ON THE GIT STRATIFICATION OF PREHOMOGENEOUS VECTOR SPACES III

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Abstract. We determine all orbits of the prehomogeneous vector space \((GL_5 \times GL_4, \wedge^2 \text{Aff}^5 \otimes \text{Aff}^4)\) rationally over an arbitrary perfect field in this paper.

1. Introduction

This is part three of a series of four papers. In Part I, we determined the set \(\mathcal{B}\) of vectors which parametrizes the GIT (geometric invariant theory) stratification [18] of the four prehomogeneous vector spaces (1)–(4) in [17]. Even though the set \(\mathcal{B}\) was determined, there may be strata \(S_\beta (\beta \in \mathcal{B})\) which are the empty set. In Part II, we determined which strata \(S_\beta\) are non-empty for the prehomogeneous vector spaces (1), (2) in [17].

In this part, we consider the following prehomogeneous vector space:

\[(1.1)\quad G = GL_5 \times GL_4, \quad V = \wedge^2 \text{Aff}^5 \otimes \text{Aff}^4.\]

This is the prehomogeneous vector space (3) in [17].

For a general introduction to this series of papers, see the introduction of [17]. Throughout this paper, \(k\) is a fixed field. We shall assume that \(k\) is a perfect field in the main theorem and in Sections 11, 12. In that case the algebraic closure \(\overline{k}\) coincides with the separable closure \(k_{\text{sep}}\). If \(X, Y\) are schemes, algebraic groups, etc., over \(k\) then \(X, Y\) are said to be \(k\)-forms of each other if \(X \times_k k_{\text{sep}} \cong Y \times_k k_{\text{sep}}\).

Let \(Q_0(v) \in \text{Sym}^2 \text{Aff}^4\) be the quadratic form \(v_1v_4 - v_2v_3\).

Definition 1.2. (1) \(\text{Ex}_n(k)\) is the set of conjugacy classes of homomorphisms from \(\text{Gal}(\overline{k}/k)\) to \(S_n\).
(2) \(\text{Prg}_2(k)\) is the set of \(k\)-isomorphism classes of \(k\)-forms of \(\text{PGL}_2\).
(3) \(\text{QF}_4(k)\) is the set of \(k\)-isomorphism classes of algebraic groups of the form \(\text{GO}(Q)^o\) where \(Q \in \text{Sym}^2 \text{Aff}^4\). Let \(\text{IQF}_4(k) \subset \text{QF}_4(k)\) be the subset consisting of inner forms of \(\text{GO}(Q_0)^o\).

Note that if \(n = 2, 3\) then \(\text{Ex}_n(k)\) coincides with the set of \(k\)-isomorphism classes of separable extensions of \(k\) of degree up to \(n\).

The following theorem is our main theorem in this part.

Theorem 1.3. (Main Theorem) Suppose that \(k\) is a perfect field.
(1) For the prehomogeneous vector space (1.1), there are 61 non-empty strata \(S_\beta\).
(2) Suppose that \( \text{ch}(k) \neq 2 \). If \( S_\beta \neq \emptyset \) then \( G_k \backslash S_{\beta k} \) is either (i) a single point (abbreviated as SP from now on) (ii) \( \text{Ex}_2(k) \) (iii) \( \text{Ex}_3(k) \) (iv) \( \text{Prg}_2(k) \) or (v) \( \text{IQF}_4(k) \). Moreover the number of \( S_\beta \)'s for (i)–(v) are as follows.

| Type       | SP | \( \text{Ex}_2(k) \) | \( \text{Ex}_3(k) \) | \( \text{Prg}_2(k) \) | \( \text{IQF}_4(k) \) |
|------------|----|---------------------|---------------------|---------------------|---------------------|
| Number of \( S_\beta \)'s | 43 | 12                  | 3                   | 2                   | 1                   |

As we pointed out in Part I, the orbit decomposition of these prehomogeneous vector spaces is known over \( \mathbb{C} \) (see [15]). Our approach answers to rationality questions and provide the inductive structure of orbits rationally over \( k \). The prehomogeneous vector space \([1, 1]\) is the “quintic case” and it was proved in [20] that the set of generic rational orbits is in bijective correspondence with \( \text{Ex}_5(k) \). Integral orbits of this case was considered by Bhargava in [1].

In the process of proving Theorem \([1, 3]\) we end up with determining the set of generic rational orbits of the following prehomogeneous vector spaces \((G, V)\).

1. \( G = \text{GL}_5 \times \text{GL}_3, V = \wedge^2 \text{Aff}^5 \otimes \text{Aff}^3 \)
2. \( G = \text{GL}_2^3 \times \text{GL}_2, V = \wedge^2 \text{Aff}^3 \oplus \text{Aff}^3 \otimes \text{Aff}^2 \).
3. \( G = \text{GL}_3 \times \text{GL}_2^2, V = \text{Aff}^4 \otimes \text{Aff}^2 \oplus \wedge^2 \text{Aff}^4 \otimes \text{Aff}^2 \).
4. \( G = \text{GL}_3 \times \text{GL}_3, V = \wedge^2 \text{Aff}^4 \otimes \text{Aff}^2, \wedge^2 \text{Aff}^2 \otimes \text{Aff}^2 \).
5. \( G = \text{GL}_4 \times \text{GL}_2 \times \text{GL}_4, V = \wedge^2 \text{Aff}^3 \otimes \text{Aff}^2 \oplus \text{Aff}^3 \otimes \text{Aff}^2 \otimes \text{Aff}^2 \).
6. \( G = \text{GL}_2^3, V = \text{Aff}^2 \otimes \text{Aff}^2 \oplus \text{Aff}^2 \otimes \text{Aff}^2 \).
7. \( G = \text{GL}_3^3, V = \text{Aff}^2 \otimes \text{Aff}^2 \oplus \text{Aff}^2 \otimes \text{Aff}^2 \).
8. \( G = \text{GL}_2^3, V = \text{Aff}^2 \otimes \text{Aff}^2 \oplus \text{Aff}^2 \otimes \text{Aff}^2 \).

For details on the precise definitions of actions and interpretations of rational orbits, see Sections [5, 10] and the case (18) in Section [11].

These cases are either irreducible, 2-simple or 3-simple prehomogeneous vector spaces. The case (1) was considered in [11, 22] over \( k \) such that \( \text{ch}(k) = 0 \). Prehomogeneous vector spaces which are 2-simple or 3-simple were considered in [8, 9, 10, 12, 11]. The second case of (4) is the case (3) of [8, p.396]. Relative invariants of this case are constructed in the case (4a) of Theorem 5.8 (the case (3) of Theorem 8.1) [12, pp.465, 478]. In this case, there are two relative invariant polynomials \((P_1(x), P_2(x)\) in Proposition [8,18] and [8,20]). They should be more or less \( P_1(v), P_2(v) \) in [12, p.465]. Note that the case (4a) in [12, p.465] is bigger than the second case of the above (4), but \( P_1(v), P_2(v) \) in [12, p.465] give rise to relative invariant polynomials on the second case of the above (4) by restriction. We define \( P_1(x), P_2(x) \) in a superficially different manner in Section [8]. The cases (2), (7), (8) are the cases (5), (2), (3) of [11, p.187]. The case (5) is in the form of prehomogeneous vector spaces in THEOREM 1.1 [11, p.161]. The case (3) contains a 2-simple prehomogeneous vector space of trivial type and is not contained in the list of [11]. The case (6) is not reduced in the sense that the component \( \text{Aff}^3 \otimes \text{Aff}^2 \otimes \text{Aff}^2 \) is not reduced and is not contained in the list of [11] either.

Since the ground field is \( \mathbb{C} \) for the above papers and \( k \) is an arbitrary perfect field in this paper, we carry out some Lie algebra computations to make sure that some cases are regular prehomogeneous vector space regardless of \( \text{ch}(k) \).

The organization of this paper is as follows. In Section [2] we discuss notations used in this part. We have to deal with many reducible prehomogeneous vector spaces.
in this part and we shall discuss the notion of “regularity” formulated in Section 2 of [7]. In [7], the regularity was used mainly for irreducible prehomogeneous vector spaces. Even though the definition of regularity is the same as the one in [7], it is necessary to consider the relation between the number of independent characters and the number of independent relative invariant polynomials. We shall discuss this issue in Section 3. In Section 4, we review known results ([20], [19]) on rational orbits for some prehomogeneous vector spaces. In Sections 5–10, we determine the set of generic rational orbits of some prehomogeneous vector spaces which appear as orbits for some prehomogeneous vector spaces. In Section 11, we show that there are 61 non-empty strata $S_{\beta}$. Assuming $\text{ch}(k) \neq 2$, we determine $G_k \backslash S_{\beta}$ for such $\beta$. In Section 12, we show that the remaining strata $S_{\beta}$ are the empty set. The method is similar to the one in [19].

The prehomogeneous vector space (1.1) is a rather big prehomogeneous vector space. If a prehomogeneous vector space $(G', V')$ appears as $(M_{\beta}, Z_{\beta})$ of a prehomogeneous vector space $(G, V)$, then $(G', V')$ is “smaller” than $(G, V)$ in some sense. In that case, the GIT stratification of $(G', V')$ should follow from that of $(G, V)$. In Section 13, we prove a proposition by which the consideration of most strata of $(G', V')$ is reduced to that of $(G, V)$ with possible relatively easy computer computations. We shall consider GIT stratifications which follow from this series of papers in the future.

2. Notation

We discuss notations used in this part.

Let $\text{ch}(k)$ be the characteristic of $k$. We often have to refer to a set consisting of a single point. We use the notation $\text{SP}$ for such a set.

If $X$ is a variety over $k$ and $x \in X$ then $T_x(X)$ is the tangent space at $x$ of $X$. Let $k[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers. We identify $T_x(X)$ with the space of $k[\varepsilon]/(\varepsilon^2)$-rational points of $X$ which reduce to $x$.

If $G$ is an algebraic group over $k$ then $G^0$ is the identity component of $G$, i.e., the connected component containing the unit element of $G$. Unless otherwise stated, $e_G$ is the unit element of $G$ (since we use the notation $e_{ijk}$ later, it is necessary to use a slightly different notation for unit elements of groups).

If $V$ is a vector space then we denote the dual space of $V$ by $V^*$. If $V$ is a representation of $G$ then $V^*$ becomes a representation of $G$ where the action of $g \in G$ is defined by $V^* \ni f \mapsto (V \ni v \mapsto f(g^{-1}v))$.

If $\sigma, \tau \in \text{Gal}(k_{\text{sep}}/k)$ then we define $\sigma \tau(x) = \tau(\sigma(x))$ for $x \in k_{\text{sep}}$. So the action of $\text{Gal}(k_{\text{sep}}/k)$ is a right action. We use the notation such as $x^\sigma$. If $G$ is an algebraic group over $k$ then we denote by $H^1(k, G)$ the Galois cohomology set with coefficients in $G$. We use the notation such as $h = (h_\sigma)_\sigma$ for 1-cochains. The cocycle condition is $h_{\sigma \tau} = h_\tau h_\sigma^\tau$ and $h_1 = (h_{1, \sigma})_\sigma$, $h_2 = (h_{2, \sigma})_\sigma$ are equivalent if there exists $g \in G_{k_{\text{sep}}}$ such that $h_{1, \sigma} = g^{-1}h_{2, \sigma}g^\sigma$ for all $\sigma$. We denote the trivial 1-cocycle (i.e., $h_\sigma = 1$ for all $\sigma$) and its cohomology class by 1.

Note that $H^1(k, G)$ may not have a group structure if $G$ is not commutative. Let $X^*(G)$ (resp. $X_*(G)$) be the group of characters (resp. the group of one parameter subgroups) of $G$. We abbreviate “one parameter subgroup” of algebraic groups as “1PS” from now on. Suppose that $\chi$ is a non-trivial character of $G$. It is said to be primitive if $\psi$ is a character of $G$ and $\chi = \psi^a$ for an integer $a$ then $a = \pm 1$. 


If \( \rho : G \to \text{GL}(V) \) is a representation of \( G \) then \( \wedge^2 \rho, \text{Sym}^3 \rho, \) etc., are the induced representations on \( \wedge^2 V, \text{Sym}^3 V, \) etc.

Let \( \text{GL}_n \) (resp. \( \text{SL}_n \)) be the general linear group (resp. special linear group), \( M_{n,m} \) the space of \( n \times m \) matrices and \( M_n = M_{n,n}. \) Let \( E_{ij} \) be the matrix whose \((i,j)\)-entry is 1 and other entries are 0. We use this notation only in the situation where the Pfaffian has the property that \( \text{Pfaff}(A) = \det(A) \) for \( A \) being an alternating matrix.

Let \( \text{GL}_n(\mathbb{R}) \) be \( n \times n \) matrices with real entries. We often use the following matrices:

\[
\begin{pmatrix}
0 & a & b \\
* & 0 & c \\
* & * & 0
\end{pmatrix}
\]

because entries in * are determined by other entries. We denote the set of \( k \)-rational points of \( \text{GL}_n, \) etc., by \( \text{GL}_n(\mathbb{Q}) \), etc. We sometimes use the notation \([x_1, \ldots, x_n]\) to express column vectors to save space. We denote the unit matrix of dimension \( n \) by \( I_n. \) We use the notation \( \text{diag}(g_1, \ldots, g_m) \) for the block diagonal matrix whose diagonal blocks are \( g_1, \ldots, g_m. \) We often use the following matrices:

\[
\tau_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

For \( A = (a_{ij}) \) is a \( 2n \times 2n \) alternating matrix then \( \text{Pfaff}(A) \) is the Pfaffian of \( A. \) It has the property that \( \text{Pfaff}(A)^2 = \det A, \text{Pfaff}(gA^t g) = (\det g) \text{Pfaff}(A) \) for \( g \in \text{GL}_{2n}. \) We choose the sign so that \( \text{Pfaff}(A) = 1 \) for the matrix \( A = \sum_{i=1}^{n} (E_{2i-1,2i} - E_{2i,2i-1}). \)

For \( u = (u_{ij}) \in \text{Aff}^{n(n-1)/2} \) \((1 \leq j < i \leq n), \) let \( n_n(u) \) be the lower triangular matrix whose diagonal entries are 1 and the \((i,j)\)-entry is \( u_{ij} \) for \( i > j. \) If \( v_1, \ldots, v_m \) are elements of a vector space \( V \) then \( \langle v_1, \ldots, v_m \rangle \) is the subspace spanned by \( v_1, \ldots, v_m. \)

For the rest of this paper, tensor products are always over \( k. \)

Let \( n_1 = 5, n_2 = 4. \) We use parabolic subgroups which consist of lower triangular blocks. Let \( i = 1, 2 \) and \( j_{i0} = 0 < j_{i1} < \cdots < j_{i,n_i} = n_i. \) We use the notation \( P_{i,[j_1,\ldots,j_{i},N_{i}]} \) (resp. \( M_{i,[j_1,\ldots,j_{i},N_{i}]} \)) for the parabolic subgroup (resp. reductive subgroup) of \( \text{GL}_{n_i} \) in the form:

\[
\begin{pmatrix}
P_{11} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
P_{N_{i},1} & \cdots & P_{N_{i},N_{i}}
\end{pmatrix},
\begin{pmatrix}
M_{11} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M_{N_{i},N_{i}}
\end{pmatrix}
\]

where the size of \( P_{kl}, M_{kl} \) is \( (j_{ik} - j_{ik-1}) \times (j_{il} - j_{il-1}). \) If \( N_i = 1 \) then we use the notation \( P_{i,0}, M_{i,0}. \)

We put

\[
P_{[j_{11},\ldots,j_{1},N_{1}],[j_{21},\ldots,j_{2},N_{2}]} = P_{1,[j_{11},\ldots,j_{1},N_{1}]} \times P_{2,[j_{21},\ldots,j_{2},N_{2}]};
M_{[j_{11},\ldots,j_{1},N_{1}],[j_{21},\ldots,j_{2},N_{2}]} = M_{1,[j_{11},\ldots,j_{1},N_{1}]} \times M_{2,[j_{21},\ldots,j_{2},N_{2}]}.
\]

If \( N_i = 1 \) then we replace \([j_{i1}, \ldots, j_{iN_i-1}]\) by \( \emptyset. \) Let

\[
M_{[j_{11},\ldots,j_{1},N_{1}],[j_{21},\ldots,j_{2},N_{2}]} = (\text{SL}_5 \times \text{SL}_4) \cap (M_{1,[j_{11},\ldots,j_{1},N_{1}]} \times M_{2,[j_{21},\ldots,j_{2},N_{2}]}).
\]
and $M^a_{[j_1,\ldots,j_{N_1-1}],[j_{N_1},\ldots,j_{N_2-1}]}$ be the semi-simple part of $M_{[j_1,\ldots,j_{N_1-1}],[j_{N_1},\ldots,j_{N_2-1}]}$.

We consider many representations of groups of the form $M_{[j_1,\ldots,j_{N_1-1}],[j_{N_1},\ldots,j_{N_2-1}]}$ in later sections. We use notations such as

$$
\Lambda^{m,i}_{j,[c,d]}.
$$

The meaning of this notation is that this is $\wedge^i \text{Aff}^m$ as a vector space where $\text{Aff}^m$ is the standard representation of $\text{GL}_m$ and the indices $j$, $[c, d]$ mean that the block from the $(c, c)$-entry to the $(d, d)$-entry of the $j$-th factor $\text{GL}_{n_j}$ of $M_{[j_1,\ldots,j_{N_1-1}],[j_{N_1},\ldots,j_{N_2-1}]}$ ($n_1 = 5, n_2 = 4$) is acting on this vector space. For example, $M_{[1],[2]}$ consists of elements of the form $(\text{diag}(t_1, g_1), \text{diag}(g_2, g_3))$ where $t_1 \in \text{GL}_1$, $g_1 \in \text{GL}_4$, $g_2, g_3 \in \text{GL}_2$. Then $\Lambda^{2,1}_{2,[1,2]}$ is the standard representation of $g_2 \in \text{GL}_2$ identified with the element $(I_5, \text{diag}(g_2, I_2))$. We denote by 1 the trivial representation of $M^a_{[j_1,\ldots,j_{N_1-1}],[j_{N_1},\ldots,j_{N_2-1}]}$.

For the prehomogeneous vector space $(\mathbb{R}^5, \mathbb{R}^4)$, let $T = \{(t_1, t_2) \in G \mid t_1, t_2 \text{ diagonal}\}$. We choose

$$
T_0 = \{(t_1 I_5, t_2 I_4) \mid t_1, t_2 \in \text{GL}_4\}
$$

(\text{the center of } G) \text{ and } G_{st} = \text{SL}_5 \times \text{SL}_4. \text{ The subscript “st” stands for “stability” since } G_{st} \text{ is the subgroup by which we measure the stability.}

When we consider the GIT stratification, $k$ is assumed to be a perfect field. Regarding the GIT stratification, we use notations such as

$$
\mathfrak{t}^*, \mathfrak{t}^*_{+}, \mathfrak{t}^*_{\mathfrak{q}}, (\cdot)^* M_\beta, P_\beta, U_\beta, Z_\beta, W_\beta, Y_\beta, S_\beta, \lambda_\beta, \chi_\beta
$$

of [18] and Section 2 of [17]. There is a slight ambiguity on the domain of definition of $\chi_\beta$ in [18]. In this paper, $\chi_\beta$ is an indivisible character on $M_\beta$, proportion to $\beta$. If $M_\beta$ is in the form (2.2), $M_\beta^1$ is defined to be the group (2.3). Note that $M_\beta^1 = M_\beta \cap G_{st}$. Let $M_\beta^a$ be the semi-simple part of $M_\beta$. Let $G_{st,\beta} = \{g \in M_\beta^1 \mid \chi_\beta(g) = 1\}^o$. This $G_{st,\beta}$ is $G_{st}^1$ of [18].

The space $\mathfrak{t}^*$ is

$$
\left\{(a_{11}, \ldots, a_{15}, a_{21}, \ldots, a_{24}) \in \mathbb{R}^9 \mid \sum_{i=1}^5 a_{1i} = \sum_{i=1}^4 a_{2i} = 0 \right\}.
$$

Let $e_i$ be the coordinate vector of $\text{Aff}^5$ with respect to the $i$-th coordinate and $f_i$ the coordinate vector of $\text{Aff}^4$ with respect to the $i$-th coordinate. We put $e_{i_1i_2i_3} = (e_{i_1} \wedge e_{i_2}) \otimes f_{i_3}$ for $i_1, i_2 = 1, \ldots, 5, i_3 = 1, \ldots, 4$. The numbering used in [17] for [11] is as follows.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|----|
| $e_{112}$ | $e_{113}$ | $e_{114}$ | $e_{115}$ | $e_{121}$ | $e_{122}$ | $e_{123}$ | $e_{124}$ | $e_{131}$ | $e_{132}$ |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $e_{122}$ | $e_{123}$ | $e_{124}$ | $e_{125}$ | $e_{126}$ | $e_{127}$ | $e_{128}$ | $e_{129}$ | $e_{130}$ | $e_{131}$ |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| $e_{132}$ | $e_{133}$ | $e_{134}$ | $e_{135}$ | $e_{136}$ | $e_{137}$ | $e_{138}$ | $e_{139}$ | $e_{140}$ | $e_{141}$ |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| $e_{142}$ | $e_{143}$ | $e_{144}$ | $e_{145}$ | $e_{146}$ | $e_{147}$ | $e_{148}$ | $e_{149}$ | $e_{150}$ | $e_{151}$ |

We denote the $i$-th coordinate vector by $a_i$. For example, $a_1 = e_{121}$.
3. Regularity

In [7], we discussed the notion of regularity mainly for irreducible prehomogeneous vector spaces. The notion of regularity was defined for not necessarily irreducible prehomogeneous vector spaces in [7]. In this paper, we have to deal with many reducible prehomogeneous vector spaces and it will be necessary to clarify the relation between the number of independent characters and the number of independent relative invariant polynomials.

Let $G$ be a connected reductive group, $Z$ the identity component of the center of $G$, $V_1, \ldots, V_N \neq \{0\}$ finite dimensional representations of $G$ over $k$ and $\chi_1, \ldots, \chi_N$ characters of $Z$ over $k$. We assume that $t \in Z$ acts on $V_i$ by multiplication by $\chi_i(t)$. Let $\Gamma \subset X^*(Z)$ be the subgroup generated by $\chi_1, \ldots, \chi_N$. We assume that $\Gamma \cong \mathbb{Z}^N$. This implies that $\{\chi_1, \ldots, \chi_N\}$ is a basis of $\Gamma$. Let $V = V_1 \oplus \cdots \oplus V_N$.

**Definition 3.1.** In the above situation, $(G, V)$ is called a prehomogeneous vector space if it satisfies the following properties.

1. There exists a Zariski open orbit.
2. There exists a non-constant polynomial $\Delta(x) \in k[V]$ and a character $\chi$ such that $\Delta(gx) = \chi(g)\Delta(x)$.
3. $\Delta(x)$ in (2) is called a relative invariant polynomial.

**Lemma 3.2.** If $\Delta(x)$ is a relative invariant polynomial then it is homogeneous with respect to each of $V_1, \ldots, V_N$.

*Proof.* We only consider $V_1$. Suppose that $P(gx) = \chi(g)P(x)$. Since $\Gamma \cong \mathbb{Z}^N$, there exists a $1$PS $\lambda(t) \subset Z$ such that $\chi_1(\lambda(t)) = t^a$ ($a > 0$) and $\chi_2(\lambda(t)) = \cdots = \chi_N(\lambda(t)) = 1$. Then for $v_1 \in V_1, \ldots, v_N \in V_N$ and $t \in \text{GL}_1$,

$$\Delta(\lambda(t)(v_1, \ldots, v_N)) = \chi(\lambda(t))\Delta(v_1, \ldots, v_N) = \Delta(t^av_1, v_2, \ldots, v_N).$$

There exists an integer $b$ such that $\chi(\lambda(t)) = t^b$. Then

$$\Delta(t^av_1, v_2, \ldots, v_N) = t^b\Delta(v_1, \ldots, v_N).$$

Therefore, $\Delta(x)$ is homogeneous with respect to $V_1$. □

The following proposition will be useful to show the existence of an open orbit. One can prove it by the same argument as in [7, p.321].

**Proposition 3.3.** Suppose that $G$ is an algebraic group, $V$ a finite dimensional representation of $G$ defined over $k$, $x \in V$ and that $\dim T_eG(Gx) = \dim G - \dim V$. Then $Gx \subset V$ is Zariski open and the group scheme $G_x$ is smooth over $k$.

**Proposition 3.4.** Let $(G, V)$ be as above (we are not assuming that $(G, V)$ is a prehomogeneous vector space). Suppose that there exists a point $w \in V_k$ such that $U = Gw$ is Zariski open and $G_w$ is reductive (and so smooth as a group scheme). Then

1. $U$ is affine,
2. $U_{\text{sep}}$ is a single $G_{\text{sep}}$-orbit.
3. There exists a relative invariant polynomial and so $(G, V)$ is a prehomogeneous vector space.
Proof. (1), (2) are proved in Proposition 2.2 [7] p.310.

(3) Let $F_1, \ldots, F_m$ be all irreducible components of codimension 1 of $V \setminus U$. Since $\bigcup_i F_i$ is $G$-invariant and $G$ is irreducible, $F_1, \ldots, F_m$ are $G$-invariant. Let $W = V \setminus (\bigcup_i F_i)$. Then $U \subset W$ are both affine and the codimension of $W \setminus U$ is greater than 2. Since $W$ is regular, it is normal. Therefore, any regular function on $U$ extends to $W$. So $W = U$.

Since the origin 0 does not belong to $U$, $V \setminus U \neq \emptyset$. Suppose that $\Delta_i(x) \in k[V]$ and $F_i$ is the zero set of $P_i(x)$ for $i = 1, \ldots, m$. Since $F_i$ is $G$-invariant, it is also the zero set of $\Delta_i(gx)$ for all $g \in G$. Therefore, there exists a character $\psi_i$ of $G$ such that $\Delta_i(gx) = \psi_i(g)\Delta_i(x)$. Therefore, there exists a relative invariant polynomial. This proves (3). \hfill \square

**Definition 3.5.** If $(G, V)$ is a prehomogeneous vector space which satisfies the condition of the above proposition then $(G, V)$ is said to be regular.

**Proposition 3.6.** Suppose that $(G, V)$ is regular prehomogeneous vector space as in Proposition 3.5 and $V \setminus U = F_1 \cup \cdots \cup F_m$ is the irreducible decomposition $(F_1, \ldots, F_m$ are distinct). Then $m \leq N$.

**Proof.** Suppose that $m > N$ and that $F_i$ is the zero set of $\Delta_i$ for all $i$. Since $\Delta_i(x)$ is homogeneous with respect to $V_1, \ldots, V_N$, there exist integers $c_1, \ldots, c_m$ such that $(c_1, \ldots, c_m) \neq (0, \ldots, 0)$ and $\prod_{i=1}^m \Delta_i(tx)^{c_i} = \prod_{i=1}^m \Delta_i(x)^{c_i}$ for all $t \in Z$. Since the derived subgroup $[G, G]$ is connected semi-simple, $\psi_1, \ldots, \psi_m$ are trivial on $[G, G]$. Since $G = Z \cdot [G, G]$, $\prod_{i=1}^m \Delta_i(x)^{c_i} \in k(V)$ is invariant under the action of $G$. However, since $U = Gw$ is Zariski open, $\prod_{i=1}^m \Delta_i(x)^{c_i} \in k^\times$. This is a contradiction since $k[V]$ is UFD and $F_1, \ldots, F_m$ are distinct. \hfill \square

**Corollary 3.7.** In the situation of Proposition 3.5, if $m = N$ then $\{x \in V_{\sep} \mid \Delta_1(x), \ldots, \Delta_N(x) \neq 0\} = G_{\sep} w$.

There is an alternative definition of prehomogeneous vector spaces, specifying a character. Let $G, V$ be as in the beginning of this section. We fix a non-trivial rational character $\chi$ of $G$. The following is an alternative definition of prehomogeneous vector spaces.

**Definition 3.8.** In the above situation, $(G, V, \chi)$ is called a prehomogeneous vector space if it satisfies the properties (1) of Definition 3.1 and there exists a non-constant polynomial $\Delta(x) \in k[V]$ and a positive integer $a$ such that $\Delta(gx) = \chi(g)^a \Delta(x)$.

$\Delta(x)$ in the above definition is called a relative invariant polynomial. If $(G, V, \chi)$ is a prehomogeneous vector space then $\Delta(x)$ in the above definition is essentially unique, i.e., if $\Delta_1(x), \Delta_2(x)$ are relative invariant polynomials then there exist positive integers $a, b$ and $c \in k^\times$ such that $\Delta(x)^a = c \Delta_2(x)^b$. We define $V_{\ss} = \{x \in V \mid \Delta(x) \neq 0\}$. This definition does not depend on the choice of $\Delta(x)$.

The situation where we consider the GIT stratification is as follows. There is a split torus $T_0 \subset Z$ of positive dimension where $Z$ is the center of $G$ and a connected reductive subgroup $G_1 \subset G$ such that $G = T_0G_1$, $T_0 \cap G$ is finite and that $T_0$ acts on $V$ by scalar multiplication. If $G$ is in this form and $(G, V, \chi)$ is a prehomogeneous vector space in the above sense then $G_1 \subset \text{Ker}(\chi)$. So $\Delta(x)$ in Definition 3.8 is indeed an invariant polynomial with respect to the action of $G_1$. Therefore, points of $V_{\ss}$ are pull backs from semi-stable points of $\mathbb{P}(V)$ in the sense of GIT.
Suppose that the condition of Corollary 3.7 is satisfied. If moreover, a positive power of \( \chi \) is \( \chi_1^{a_1} \cdots \chi_m^{a_m} \) with \( a_1, \ldots, a_m > 0 \) then \( \Delta(x) \) is a constant multiple of a positive power of \( \Delta_1(x)^{a_1} \cdots \Delta_m(x)^{a_m} \). The converse is true, i.e., if \( \Delta(x) \) is a constant multiple of a positive power of \( \Delta_1(x)^{a_1} \cdots \Delta_m(x)^{a_m} \) then a positive power of \( \chi \) is \( \chi_1^{a_1} \cdots \chi_m^{a_m} \). Therefore, \( V_{\text{ss}} = \{ x \in V \mid \Delta_1(x), \ldots, \Delta_m(x) \neq 0 \} \). So \( V_{\text{ss}}^k = G_{k, \text{ss}}w \) is a single \( G_{k, \text{ss}} \)-orbit. It turns out later that in our case the condition of Corollary 3.7 and this additional condition are satisfied for all \((M_\beta, Z_\beta)\) where we have to verify \( Z_{\text{ss}}^{k, \text{ss}} \) is a single \( M_{\beta, \text{ss}} \)-orbit.

For the rest of this section, we review the notion of universally generic element in [6]. We consider a slightly general situation than in [6].

Let \( S \) be a finite set of primes in \( \mathbb{Z} \). Let \( R \) be the ring generated over \( \mathbb{Z} \) by \( \{ p^{-1} \mid p \in S \} \). Let \( G \) be a smooth group scheme over \( R \) with connected geometric fibers, \( V \) a free \( R \)-module of finite rank and \( G \to \text{GL}_R(V) \) a homomorphism. If \( k \) is an algebraically closed field and \( R \to k \) is a homomorphism then \( G \times_R k \) is an algebraic group over \( k \) and \( V \times_R k \) is a representation of \( G \times_R k \) over \( k \). Suppose that \( w \in V_R \). We denote the image of \( w \) in \( V \times_R k \) by \( w(k) \).

**Definition 3.9.** If for any \( k \) as above \((G \times_R k)w(k) \subset V \times_R k \) is Zariski open then we say that \( w \) is universally generic outside \( S \). If \( S = \emptyset \) then we say \( w \) is universally generic.

The following proposition can be proved in the same manner as in Proposition 1 [6] p. 279] and so we do not provide the proof.

**Proposition 3.10.** Suppose that \( S, R, G, V, w \) are as above and that \( w \) is universally generic outside \( S \). Suppose that \((\pi, W)\) is a finite dimensional representation of \( G \times_R \mathbb{Q} \) defined over \( R \). If \( \Phi : V \times_R \mathbb{Q} \to W \times_R \mathbb{Q} \) is a \( G \times_R \mathbb{Q} \)-equivariant morphism and that \( \Phi(w) \in W_R \). Then \( \Phi \) is defined over \( R \).

We make a remark on the notion of Castling transform. Suppose that \( 0 < m < n \) are integers (then \( n > 1 \) ). Let \( G \) be an algebraic group and \( V \) an \( n \)-dimension representation of \( G \) over \( k \). Let \( \text{Aff}^m, \text{Aff}^{n-m} \) be the standard representations of \( \text{GL}_m, \text{GL}_{n-m} \) respectively. Let \( G_1 = G \times \text{SL}_m, G_2 = G \times \text{SL}_{n-m} \). Then \((G_1, V \otimes \text{Aff}^m)\) and \((G_2, V^* \otimes \text{Aff}^{n-m})\) are called Castling transforms of each other. Let

\[
U_1 = \{ x = (x_1, \ldots, x_m) \in V \otimes \text{Aff}^m \mid x_1, \ldots, x_m \text{ are linearly independent} \},
\]

\[
U_2 = \{ y = (y_1, \ldots, y_{n-m}) \in V \otimes \text{Aff}^{n-m} \mid y_1, \ldots, y_{n-m} \text{ are linearly independent} \}.
\]

Then the actions of \( \text{SL}_m, \text{SL}_{n-m} \) on \( U_1, U_2 \) are free respectively, \( \text{SL}_m \setminus U_1, \text{SL}_{n-m} \setminus U_2 \) are affine and the coordinate rings are generated by Plücker coordinates (see Theorem 2.1 [3] p. 20) for the proof over any field

Since the codimensions of \( U_1, U_2 \) are greater than 1, regular functions on \( U_1, U_2 \) can be regarded as elements of \( k[V \otimes \text{Aff}^m], k[V^* \otimes \text{Aff}^{n-m}] \) respectively. Since Plücker coordinates on \( V \otimes \text{Aff}^m, V^* \otimes \text{Aff}^{n-m} \) coincide except possibly with the choice of signs, \( \text{SL}_m \setminus U_1, \text{SL}_{n-m} \setminus U_2 \) are isomorphic. Since \( \text{SL}_m, \text{SL}_{n-m} \) have no non-trivial characters, relative invariant polynomials on \( V \otimes \text{Aff}^m, V^* \otimes \text{Aff}^{n-m} \) correspond bijectively and \( G_1, G_2 \)-orbits in \( U_1, U_2 \) correspond bijectively also. Since \( H^1(k, \text{SL}_m), H^1(k, \text{SL}_{n-m}) \) are trivial, the correspondence of orbits is bijective rationally over \( k \). By the above consideration, \((G_1, V \otimes \text{Aff}^m)\) is a prehomogeneous vector space if and only if \((G_2, V^* \otimes \text{Aff}^{n-m})\) is.
Aff$^{n-m}$) is a prehomogeneous vector space. If so, since the stabilizers of corresponding points are isomorphic, the regularity of these prehomogeneous vector spaces coincide. The reader should see Proposition 18 [16, p.68] for details (the argument there works over any field).

If both $(G_1, V \otimes \text{Aff}^m)$, $(G_2, V^* \otimes \text{Aff}^{n-m})$ are prehomogeneous vector spaces then the number, say $l$, of distinct invariant hypersurfaces for these prehomogeneous vector spaces coincide. If $\Delta_1, \ldots, \Delta_l$ are distinct relative invariant polynomials on $V \otimes \text{Aff}^m$ and $\Delta_1^*, \ldots, \Delta_l^*$ are the corresponding relative invariant polynomials on $V^* \otimes \text{Aff}^{n-m}$ then $G_1$-orbits in \( \{ x \in V \otimes \text{Aff}^m \mid \Delta_1(x), \ldots, \Delta_l(x) \neq 0 \} \) and $G_2$-orbits in \( \{ y \in V^* \otimes \text{Aff}^{n-m} \mid \Delta_1^*(y), \ldots, \Delta_l^*(y) \neq 0 \} \) correspond bijectively. In particular \( \{ x \in V \otimes \text{Aff}^m \mid \Delta_1(x), \ldots, \Delta_l(x) \neq 0 \} \) is a single $G_1$-orbit if and only if \( \{ y \in V^* \otimes \text{Aff}^{n-m} \mid \Delta_1^*(y), \ldots, \Delta_l^*(y) \neq 0 \} \) is a single $G_2$-orbit. If $l = 1$ then $G_1 \backslash (V \otimes \text{Aff}^m)_k^{ss}$ and $G_2 \backslash (V^* \otimes \text{Aff}^{n-m})_k^{ss}$ are in bijective correspondence.

Note that any relative invariant polynomial on $V \otimes \text{Aff}^m$ is invariant for the action of $\text{SL}_m$. The situation is similar for $V^* \otimes \text{Aff}^{n-m}$. The only issue we have to be careful is the correspondence of rational orbits. Let $N$ be the binomial number $\binom{n}{m}$. If $R$ is the quotient of $k[z_1, \ldots, z_N]$ by Plücker relations, both $\text{GL}_m U_1, \text{GL}_{n-m} U_2$ are isomorphic to $\text{Proj}(R)$ this time rather than $\text{Spec}(R)$. Since $H^1(k, \text{GL}_m), H^1(k, \text{GL}_{n-m})$ are trivial, this bijection is rational over $k$ also. Note that if $x = (x_1, \ldots, x_m) \in U_1$ then $H = (x_1, \ldots, x_m) \subset V$ is a subspace of dimension $m$. Then \( \{ f \in V^* \mid \forall x \in H, f(x) = 0 \} \subset V^*$ is a subspace of dimension $n - m$. By taking a basis $\{y_1, \ldots, y_{n-m}\}$, the orbit of $y = (y_1, \ldots, y_m)$ is the orbit corresponding to the orbit of $x$. By this correspondence, $(G_k \times \text{GL}_m(k))$-orbits in \( \{ x \in V_k \otimes k^m \mid \Delta_1(x), \ldots, \Delta_m(x) \neq 0 \} \) and $(G_k \times \text{GL}_{n-m}(k))$-orbits in \( \{ y \in V^*_k \otimes k^{n-m} \mid \Delta_1^*(y), \ldots, \Delta_m^*(y) \neq 0 \} \) correspond bijectively also.

4. Rational orbits (1)

In this section and the next six sections, we consider rational orbits of some prehomogeneous vector spaces which appear as $(M_{\alpha}, Z_{\beta})$. In this section we quote some known cases. In the next six sections, we consider cases which require some labor.

For the rest of this paper, \( \{p_{n,1}, \ldots, p_{n,n}\} \) is the standard basis of $\text{Aff}^n$. We put $p_{n, i_1 \ldots i_m} = p_{n, i_1} \wedge \cdots \wedge p_{n, i_m}$ for $1 \leq i_1, \ldots, i_m \leq n$. We identify $\wedge^n \text{Aff}^n$ with $\text{Aff}^1$, so that $p_{n, 12 \ldots n}$ corresponds to 1. Let $\text{Ex}_n(k), \text{Pr}_n(k), \text{QF}_n(k)$ be as in Definition 1.2.

For the proof of the following two propositions, see [20, pp.303].

**Proposition 4.1.** We consider the natural action of $\text{GL}_2^3$ on $\text{Aff}^2 \otimes \text{Aff}^2 \otimes \text{Aff}^2$. There is a homogeneous degree 4 polynomial $P(x)$ on $\text{Aff}^2 \otimes \text{Aff}^2 \otimes \text{Aff}^2$ such that $P((g_1, g_2, g_3)x) = (\det g_1 \det g_2 \det g_3)^2 P(x)$ and $P(q_{111} + q_{222}) = 1$ where $q_{111} = p_{2,1} \otimes p_{2,1} \otimes p_{2,1}$, etc.

**Proposition 4.2.** We consider the natural action of $\text{GL}_3^2 \times \text{GL}_2$ on $\text{Aff}^3 \otimes \text{Aff}^3 \otimes \text{Aff}^2$. There is a homogeneous degree 12 polynomial $P(x)$ on $\text{Aff}^3 \otimes \text{Aff}^3 \otimes \text{Aff}^2$ such that $P((g_1, g_2, g_3)x) = (\det g_1 \det g_2)^4 (\det g_3)^6 P(x)$ and $P(q_{111} - q_{221} + q_{222} - q_{332}) = 1$ where $q_{111} = p_{3,1} \otimes p_{3,1} \otimes p_{2,1}$, etc.

For the proof of the following proposition, see [20, pp.306] (the point considered in Proposition 4.2 is slightly different from the one in [20], but in the same $G_k$-orbit).
Proposition 4.3. We consider the natural action of $\GL_4 \times \GL_2$ on $(\wedge^2 \Aff^4) \otimes \Aff^2$. There is a homogeneous degree 4 polynomial $P(x)$ on $(\wedge^2 \Aff^4) \otimes \Aff^2$ such that $P((g_1, g_2)x) = (\det g_1)^2(\det g_2)^2 P(x)$ and $P(q_{121} + q_{342}) = 1$ where $q_{121} = (\mathfrak{p}_{4,1} \wedge \mathfrak{p}_{4,2}) \otimes \mathfrak{p}_{2,1}$, etc.

Let $(G, V)$ be either of the above three representations. The set $G_k \backslash V^\ss_k$ of rational orbits were considered in §3, §4. One can use the notion of regularity and simple Lie algebra computations to show that $V^\ss_{k,v}$ is a single $G_{k,v}$-orbit. This makes the assumption on $\ch(k)$ unnecessary. Therefore, the consideration in [20, 21] implies the following proposition without any assumption on $\ch(k)$.

Proposition 4.4. Let $(G, V)$ be the representation of Propositions 4.1, 4.2 or 4.3. Then $G_k \backslash V^\ss_k$ is in bijective correspondence with $\Ex_2(k), \Ex_3(k)$ or $\Ex_2(k)$ respectively.

Let $V, W$ be vector spaces over $k$ and $0 < m < n$ integers. We define a map

$$\Phi : V \otimes \Aff^m \oplus W \otimes \Aff^{n-m} \to V \otimes W \otimes \wedge^n \Aff^n \cong V \otimes W,$$

linear with respect to each component, so that $\Phi(v \otimes x, w \otimes y) = v \otimes w \otimes (x \wedge y)$ for $v \in V, w \in W, x \in \Aff^m, y \in \Aff^{n-m}$. As we pointed out earlier, we identify $\wedge^n \Aff^n$ with $\Aff^1$ so that $p_{n,12\ldots}$ corresponds to 1.

The following lemma is obvious.

Lemma 4.5. In the above situation,

$$\Phi(g_1v \otimes g_3x, g_2w \otimes g_3y) = (\det g_3)(g_1, g_2)\Phi(v \otimes x, w \otimes y)$$

for $g_1 \in \GL(V), g_2 \in \GL(W), g_3 \in \GL_n$ where the action of $(g_1, g_2)$ on $V \otimes W$ is the natural action.

The following proposition follows by applying Lemma 4.5 twice. Also it is proved explicitly in Proposition 3.5 [19] during the consideration of $S_{36}$ in Section 3 [19]. Note that the formulation in [19] is slightly different and there are extra $\GL_1$-factors, but the proof works.

Proposition 4.6. Let $G = \GL_2^2 \times \GL_1$, $V = \Aff^2 \otimes \Aff^2 \oplus \Aff^2 \oplus \Aff^2$. We define a linear action of $G$ on $V$ so that $(g_1, g_2, t)(v_1 \otimes v_2, v_3, v_4) = (g v_1 \otimes g v_2, t g v_3, g v_4)$.

1. There is a homogeneous degree 3 polynomial $P_1(x)$ on $V$, linear with respect to each of $v_1 \otimes v_2, v_3, v_4$ such that $P_1(gx) = t(\det g_1)(\det g_2)P_1(x)$ for $g = (g_1, g_2, t) \in G, x \in V$ and that $P(q_{11} + q_{22}, \mathfrak{p}_{2,1}, \mathfrak{p}_{2,1}) = 1$ where $q_{11} = \mathfrak{p}_{2,1} \otimes \mathfrak{p}_{2,1}$, etc.

2. For $x = (A, v_1, v_2) \in V$ where $A \in \Aff^2 \otimes \Aff^2$, let $P_2(x) = \det A$ identifying $\Aff^2 \otimes \Aff^2 \cong M_2$. Then $P_2(gx) = (\det g_1)(\det g_2)P_2(x)$ and $\{ x \in V_k \mid P_1(x), P_2(x) \neq 0 \} = \mathbb{G}_k(q_{11} + q_{22}, \mathfrak{p}_{2,1}, \mathfrak{p}_{2,1})$.

For the following proposition, see the consideration of $S_{36}$ in Section 3 [19] (take the determinant of $\Phi(x)$ in (3.3) [19] identifying the dual space of $M_2$ with $M_2$).

Proposition 4.7. We consider the natural action of $G = \GL_3 \times \GL_2^2$ on $V = \Aff^3 \otimes \Aff^2 \otimes \Aff^2$. We put

$$R_{322} = \mathfrak{p}_{3,1} \otimes (-q_{11} + q_{22}) + \mathfrak{p}_{3,2} \otimes q_{12} + \mathfrak{p}_{3,3} \otimes q_{21}$$

(4.8)
where \( q_{11} = \mathbb{P}_{2,1} \otimes \mathbb{P}_{2,1} \), etc. Then there is a homogeneous degree 6 polynomial \( P(x) \) on \( V \) such that

\[
P((g_1, g_2, g_3)x) = (\det g_1)^2(\det g_2)^3(\det g_3)^3 P(x)
\]

for \((g_1, g_2, g_3) \in G\) and that \( P(R_{322}) = 1 \). Moreover, \( \{x \in V_k \mid P(x) \neq 0\} = G_k R_{322} \).

In the above proposition, if we identify \( V \) with \( \text{Aff}^3 \otimes M_2 \) then the action of \( g = (g_1, g_2, g_3) \in G \) on \((A_1, A_2, A_3) (A_1, A_2, A_3 \in M_2)\) is by the standard representation of \( g_1 \) and \( A_i \mapsto g_2 A_i g_3 \). Also if we put

\[
(4.9) \quad B_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

then we can regard that

\[
(4.10) \quad R_{322} = (B_1, B_2, B_3).
\]

**Remark 4.11.** In the situation of Proposition 4.7, we obtain the same result even if we change the action of \( g \in \text{GL}_3 \) to \( \text{Aff}^3 \ni x \mapsto t g^{-1} x \in \text{Aff}^3 \). This is because \( \{t g^{-1} \mid g \in \text{GL}_3(k)\} = \text{GL}_3(k) \). Similarly we can change the action of \( g \in \text{GL}_2 \) to \( \text{Aff}^2 \ni x \mapsto t g^{-1} x \in \text{Aff}^2 \).

5. **Rational orbits (2)**

In the section and the subsequent five sections, we consider representations which appear as \((M_\beta, Z_\beta)\). We have to consider two questions. One is the existence of relative invariant polynomials and the other is the interpretation of generic rational orbits. For the sake of the GIT stratification, it is enough to determine whether or not \( S_\beta \neq \emptyset \). For that purpose we only have to construct non-zero relative invariant polynomials and \((M_\beta, Z_\beta)\) does not have to be a prehomogeneous vector space. However, to describe rational orbits in \( S_\beta \), we need more information and it is convenient to show that \((M_\beta, Z_\beta)\) is a regular prehomogeneous vector space. For that purpose, we sometimes have to assume that \( \text{ch} (k) \neq 2 \).

In this section let \( G = \text{GL}_5 \times \text{GL}_3 \), \( V = \wedge^2 \text{Aff}^3 \otimes \text{Aff}^3 \). Rational orbits of this case was considered in [22] under the assumption \( \text{ch} (k) = 0 \). The proof in [22] works as long as \( \text{ch} (k) \neq 2,3 \). However, we would like to minimize the number of primes excluded for \( \text{ch} (k) \). So we make some modification of the argument so that the only prime excluded is \( p = 2 \). Some of the considerations in this section is due to [14] if \( k = \mathbb{C} \).

We shall construct a non-zero relative invariant polynomial without any assumption on \( \text{ch} (k) \). However, we have to assume \( \text{ch} (k) \neq 2 \) to describe rational orbits in a satisfactory manner. We sometimes consider over \( \mathbb{Z}, \mathbb{Q} \) and then deduce certain assertions over any \( k \) by the natural homomorphism \( \mathbb{Z} \rightarrow k \).

We first assume \( k = \mathbb{Q} \). Let \( W = \text{Aff}^2 \) be the standard representation of the ring \( \mathbb{M}_2 \), \( W_4 = \text{Sym}^4 W \) and \( W_2 = \text{Sym}^2 W \). Let \( \{p_{2,1}, p_{2,2}\} \) be the standard basis of \( W \) as before and \( l_0 = p_{2,1}^2, l_1 = p_{2,1} p_{2,2}, \ldots, l_4 = p_{2,2}^2 \in W_4 \). Then \( S_4 = \{l_0, l_1, 3l_2, l_3, l_4\} \) is a basis of \( W_4 \) over \( \mathbb{Q} \). For \( g \in \text{GL}_2 \), let \( \rho(g) \) be the matrix of the action of \( g \in \text{GL}_2 \).
on \( W_4 \) with respect to the basis \( S_4 \). We put

\[
\begin{align*}
  w_1 &= \begin{pmatrix}
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & -1 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 \\
  -1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
  \end{pmatrix},
  w_2 = \begin{pmatrix}
  0 & 0 & 0 & 0 & -1 \\
  0 & 0 & 0 & 2 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & -2 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 & 0
  \end{pmatrix},
  w_3 = \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & -1 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & -1 & 0 & 0 & 0
  \end{pmatrix},
  w = \begin{pmatrix}
  w_1 \\
  w_2 \\
  w_3
  \end{pmatrix} \in \bigwedge^2 \mathrm{Aff}^5 \otimes \mathrm{Aff}^3 = V.
\end{align*}
\]

Let \( L = \langle w_1, w_2, w_3 \rangle \subset \bigwedge^2 W_4 \) be the subspace spanned by \( w_1, w_2, w_3 \) and \( L_\mathbb{Z} \) the \( \mathbb{Z} \)-module spanned by \( w_1, w_2, w_3 \).

**Lemma 5.1.**

1. If \( g \in M_2(\mathbb{Z}) \) then entries of \( \rho(g) \) are polynomials of entries of \( g \) with coefficients in \( \mathbb{Z} \).
2. If \( g_1, g_2 \in M_2(\mathbb{Z}) \) then \( \rho(g_1g_2) = \rho(g_1)\rho(g_2) \).

**Proof.** (1) We only have to verify that the coefficient of \( \mathbf{p}^2_{2,1}\mathbf{p}^2_{2,2} \) in \( (\text{Sym}^4 g)_i \), say \( C_i \), is divisible by 3 for \( i = 0, 1, 3, 4 \). If \( g = (g_{ij}) \) then

\[
C_0 = 6g_{11}^2g_{21},
C_1 = 3g_{12}^2g_{21} + 3g_{11}^2g_{21}g_{12}.
\]

So \( C_0, C_1 \) are divisible by 3. Similarly, \( C_3, C_4 \) are divisible by 3. (2) is obvious. \( \square \)

Note that if \( u \in \mathbb{Z} \) then

\[
(5.2) \quad \rho(n_2(u)) = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 \\
  4u & 1 & 0 & 0 & 0 \\
  2u^2 & u & 1 & 0 & 0 \\
  4u^3 & 3u^2 & 6u & 1 & 0 \\
  u^4 & u^3 & 3u^2 & u & 1
  \end{pmatrix},
  \rho(t_2(n_2(u))) = \begin{pmatrix}
  1 & u & 3u^2 & u^3 & u^4 \\
  0 & 1 & 6u & 3u^2 & 4u \\
  0 & 0 & 1 & u & 2u^2 \\
  0 & 0 & 0 & 1 & 4u \\
  0 & 0 & 0 & 0 & 1
  \end{pmatrix}.
\]

By Lemma 5.1 the natural homomorphism \( \mathbb{Z} \to k \) induces a map \( \rho : M_2(k) \to M_5(k) \) (we use the same notation \( \rho \)) such that \( \rho(g_1g_2) = \rho(g_1)\rho(g_2) \) for \( g_1, g_2 \in M_2(k) \). Since \( \rho(I_2) = I_5 \) over \( \mathbb{Z} \), the same is true over \( k \). Therefore, \( \rho(g) \in \text{GL}_5(k) \) if \( g \in \text{GL}_2(k) \).

**Lemma 5.3.** If \( g \in \text{GL}_2(k) \) and \( A \in L \) then \( \bigwedge^2 \rho(g)A \in L \).

**Proof.** It is enough to consider \( n_2(u), t_2(n_2(u)) \) and diagonal matrices. If \( g = \text{diag}(t_1, t_2) \) then the assertion is obvious. Explicitly,

\[
\bigwedge^2 \rho(g)w_1 = t_1^3t_2^3w_1,
\bigwedge^2 \rho(g)w_2 = t_1^4t_2^4w_2,
\bigwedge^2 \rho(g)w_3 = t_1^3t_2^5w_3.
\]

If \( g = n_2(u) \) then

\[
\bigwedge^2 \rho(g)w_1 = w_1 - uw_2 + u^2w_3,
\bigwedge^2 \rho(g)w_1 = w_2 - 2uw_3,
\bigwedge^2 \rho(g)w_3 = w_3
\]

and so \( \bigwedge^2 \rho(g)L \subset L \). The consideration is similar for \( g = t_2(n_2(u)) \). \( \square \)
For $g \in \text{GL}_2$, let $\rho_1(g) = (\det g)^{-2} \rho(g)$. Then $\rho_1(g) = I_3$ if $g$ is a scalar matrix. For $g \in \text{GL}_2$, let $\rho_2(g) = (\rho_2(g)_{ij}) \in \text{GL}_3$ be the matrix such that

$$(\wedge^2 \rho_1(g) w_1 \wedge^2 \rho_1(g) w_2 \wedge^2 \rho_1(g) w_3) = (w_1 \ w_2 \ w_3) \rho_2(g)$$

treating $(w_1 \ w_2 \ w_3)$ as a row vector.

**Lemma 5.4.** The image of $(\rho_1, t \rho_2^{-1})$ fixes $w$.

**Proof.** The action of $(g_1, g_2) \in \text{GL}_5 \times \text{GL}_3$ on $x = [x_1, x_2, x_3] \in V$ is $g_2 \begin{pmatrix} \wedge^2 g_1 x_1 \\ \wedge^2 g_1 x_2 \\ \wedge^2 g_1 x_3 \end{pmatrix}$
treating $[\wedge^2 g_1 x_1, \wedge^2 g_1 x_2, \wedge^2 g_1 x_3]$ as a column vector. So if $g \in \text{GL}_2$ then

$$(\wedge^2 \rho_1(g), t \rho_2(g)^{-1})w = t \rho_2(g)^{-1} \begin{pmatrix} \wedge^2 \rho_1(g) w_1 \\ \wedge^2 \rho_1(g) w_2 \\ \wedge^2 \rho_1(g) w_3 \end{pmatrix} = t \rho_2(g)^{-1} t (w_1 \ w_2 \ w_3) \rho_2(g) = t \rho_2(g)^{-1} t \rho_2(g) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = w.$$

$\square$

We show that $G_w \cong \text{PGL}_2 \times \text{GL}_1$. For that purpose we first determine the Lie algebra of $G_w$.

Let $k[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers, $X = (x_{ij}) \in M_5$ and $Y = (y_{ij}) \in M_3$. If $(e_G + \varepsilon(X, Y))w = w + \varepsilon(A, B, C)$ then

$$A = Xw_1 + w_1^t X + y_{11} w_1 + y_{12} w_2 + y_{13} w_3,$$

$$B = Xw_2 + w_2^t X + y_{21} w_1 + y_{22} w_2 + y_{23} w_3,$$

$$C = Xw_3 + w_3^t X + y_{31} w_1 + y_{32} w_2 + y_{33} w_3.$$

So $e_G + \varepsilon(X, Y) \in T_{e_G}(G_w)$ if and only if the right hand sides of (5.5) are 0. By long but straightforward computations, we obtain the following proposition. We do not provide the details.

**Proposition 5.6.** Suppose that $\text{ch}(k) \neq 2$. Then $e_G + \varepsilon(X, Y) \in T_{e_G}(G_w)$ if and only if $X, Y$ are in the following form:

$$X = \begin{pmatrix} x_{11} & x_{12} & 0 & 0 & 0 \\ 4x_{32} & x_{22} & 6x_{12} & 0 & 0 \\ 0 & x_{32} & -x_{11} + 2x_{22} & x_{12} & 0 \\ 0 & 0 & 6x_{32} & -2x_{11} + 3x_{22} & 4x_{12} \\ 0 & 0 & 0 & x_{32} & -3x_{11} + 4x_{22} \end{pmatrix},$$

$$Y = \begin{pmatrix} x_{11} - 3x_{22} & x_{32} & 0 \\ 2x_{12} & x_{11} - 4x_{22} & 2x_{32} \\ 0 & x_{12} & 3x_{11} - 5x_{22} \end{pmatrix}.$$
Since \( \dim T_{eG}(G_w) = 4 = 34 - 30 = \dim G - \dim V \), the following corollary follows from Proposition 3.3.

**Corollary 5.7.** If \( \text{ch}(k) \neq 2 \) then \( G_w \subset V \) is Zariski open and \( G_w \) is smooth over \( k \).

**Proposition 5.8.** If \( \text{ch}(k) \neq 2 \) then \( G_w = G_w^0 \cong \text{PGL}_2 \times \text{GL}_1 \).

**Proof.** Let \( H = \text{GL}_2 \times \text{GL}_1 \). We define a homomorphism \( \phi : H \to G \) by \( \phi(g,t) = (t \rho_1(g), t^{-2} \rho_2(g)^{-1}) \). By Lemma 5.4, \( \phi(g,t) \in G_w \). Let \( \psi \) be the differential of \( t \rho_1(g) \). We compute the image of \( \psi \). Note that \( T_{eH}(H) \) consists of elements of the form \((I_2 + \varepsilon A, 1 + c \varepsilon)\) where \( A = (a_{ij}) \in M_2, c \in \text{Aff}^1 \).

If \( A = E_{21}, E_{12} \) (see Section 2) then (5.2) implies that \( (\det(I_2 + \varepsilon A))^{-2} \rho(I_2 + \varepsilon A) = \rho(I_2 + \varepsilon A) = I_5 + \varepsilon X \) where \( X \) is as in Proposition 5.6 with \( x_{11} = x_{22} = 0 \) and \( x_{12} = 0, x_{32} = 1 \) or \( x_{12} = 1, x_{32} = 0 \). If \( A = \text{diag}(a_1, a_2) \) then \( \det(I_2 + \varepsilon A)^{-2} = 1 - 2(a_1 + a_2) \varepsilon \). So if we put \( a_1 - a_2 = b \) then
\[
\psi(I_2 + \varepsilon A, 1 + c \varepsilon) = I_5 + \varepsilon \text{diag}(2b + c, b + c, -b + c, -2b + c).
\]

If \( x_{11} = 2b + c, x_{22} = b + c \) then \( c = -x_{11} + 2x_{22}, -b + c = -2x_{11} + 3x_{22}, -2b + c = -3x_{11} + 4x_{22} \). So, the dimension of the image of \( \psi \) is \( 4 = \dim T_{eG}(G_w) \). Therefore, \( \psi : T_{eH}(H) \to T_{eG}(G_w) \) is surjective.

Assuming that \( k = \mathbb{R} \), we show that \( \text{Ker}(\phi) = \{ (tI_2, 1) \mid t \in \text{GL}_1 \} \) set theoretically. Then the dimension of the image of \( \phi \) is \( 5 - 1 = 4 = \dim G_w \). This implies that \( \text{Im}(\phi) = G_w^0 \).

Suppose that \( (g,t) \in \text{Ker}(\phi) \). Then \( \rho(g) \) is a scalar matrix. Since \( \rho(g) \mathbb{P}^1, \rho(g) \mathbb{P}^2 \) are scalar multiples of \( \mathbb{P}^1, \mathbb{P}^2 \) respectively, \( g \) is a diagonal matrix, say \( \text{diag}(t_1, t_2) \). Since \( \rho(g) \) is a scalar matrix, \( t_1^4 = t_2^4 t_2 \), which implies that \( t_1 = t_2 \). Since \( \rho_1(t_1 I_2) = I_5, t = 1 \). Therefore, \( \text{Ker}(\phi) = \{ (tI_2, 1) \mid t \in \text{GL}_1 \} \).

As we pointed out above, \( \text{Im}(\phi) = G_w^0 \). Since \( \text{Im}(\psi) = T_{eG}(G_w), \phi : H \to G_w^0 \) is smooth surjective. Since the fibers are smooth by the Jacobian criterion, \( \phi^{-1}(e_G) = \{ (tI_2, 1) \mid t \in \text{GL}_1 \} \) as schemes. Therefore, \( G_w^0 \cong \text{PGL}_2 \times \text{GL}_1 \).

Suppose that \( g = (g_1, g_2) \in G_w \). Since all automorphisms of \( \text{PGL}_2 \) are inner, by multiplying an element of \( \text{PGL}_2 \) to \( g \), we may assume that \( g_1 \) commutes with elements of \( \text{PGL}_2 \).

If \( \text{ch}(k) \neq 3 \), it is easy to show that \( \rho \) is an irreducible representation (we are assuming \( \text{ch}(k) \neq 2 \)). Therefore, by Schur’s lemma, \( g_1 \) must be a scalar matrix.

Suppose that \( \text{ch}(k) = 3 \). We show that the only non-trivial invariant subspace is \( U = \langle [0,0,1,0,0,0] \rangle \). By (5.2), \( U \) is an invariant subspace since \( \text{ch}(k) = 3 \). Suppose that \( H \subset \text{Aff}^5 \) is an invariant subspace. Then it is invariant by the Lie algebra of \( \text{SL}_2, \tau_0 \) in (2.1) and \( \text{diag}(t, t^{-1}) \).

Coordinate vectors of \( \text{Aff}^5 \) are weight vectors with respect to \( \text{diag}(t, t^{-1}) \) with distinct weights \( 4, 2, 0, -2, -4 \). Therefore, \( H \) is spanned by coordinate vectors contained in \( H \). Let \( e_0 = [1, 0, 0, 0, 0, 0] \), \( e_1 = [0, 0, 0, 1, 1, 1] \). By the action of the Lie algebra element \( I_2 + n_2(e) \),
\[
e_0 \mapsto 4e_1, \ e_1 \mapsto e_2, \ e_3 \mapsto e_4
\]
and similarly for \( I_2 + 4n_2(e) \). Since \( \tau_0 \) exchanges \( e_0 \leftrightarrow e_4 \) and \( e_1 \leftrightarrow e_3 \), if \( H \) contains any one of \( e_0, e_1, e_3, e_4 \) then \( H = \text{Aff}^5 \).
The above consideration implies that $g_1$ fixes the subspace $U$ and acts on $\text{Aff}^5/U$ by scalar multiplication by Schur’s lemma. Therefore, $g_1$ is in the form

$$g_1 = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{pmatrix}$$

where $a, b_3 \neq 0$. Since $g \in G_w$, $g_1$ fixes the subspace $\langle w_1, w_2, w_3 \rangle$.

By computation,

$$g_1w_1^tg_1 = \begin{pmatrix} 0 & 0 & ab_4 & a^2 & 0 \\ 0 & 0 & -ab_3 & 0 & 0 \\ -ab_4 & ab_3 & 0 & ab_1 & 0 \\ -a^2 & 0 & -ab_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g_1w_2^tg_1 = \begin{pmatrix} 0 & 0 & -ab_5 & 0 & -a^2 \\ 0 & 0 & 2ab_4 & 2a^2 & 0 \\ ab_5 & -2ab_4 & 0 & 2ab_2 & -ab_1 \\ 0 & -2a^2 & -2ab_2 & 0 & 0 \\ a^2 & 0 & ab_1 & 0 & 0 \end{pmatrix}.$$

By the $(1,3), (3,4)$-entries of $g_1w_1^tg_1$, $b_4 = b_1 = 0$. By the $(1,4), (2,3)$-entries of $g_1w_1^tg_1$, $a^2 = ab_3$ and so $a = b_3$. By the $(1,3), (3,4)$-entries of $g_1w_2^tg_1$, $b_5 = b_2 = 0$. Therefore, $g_1 = aI_5$.

In all cases $g_1$ is a scalar matrix. Since $g \in G_w$, $g_2$ is a scalar matrix also. If $g = (t_1I_5, t_2I_3)$ then $t_2 = t_1^{t_-2}$ and so $g \in \text{GL}_4$. Therefore, $G_w = G_w^0$. □

Since $G_w$ is reductive, the following corollary follows.

**Corollary 5.9.** If $\text{ch}(k) \neq 2$ then $(G, V)$ is a regular prehomogeneous vector space.

If $\text{ch}(k) \neq 2$ then Corollary 5.7 implies the existence of a relative invariant polynomial. However, we would like to construct a relative invariant polynomial explicitly without assuming $\text{ch}(k) \neq 2$. Since the argument is similar as in [22], we will be brief. We also would like to find an element for which the value of the relative invariant polynomial is no-zero even if $\text{ch}(k) = 2$.

We put $w'_1 = w_1, w'_3 = w_3$,

$$w'_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad w' = (w'_1, w'_2, w'_3).$$

Let $u_i = p_{3,i}$ for $i = 1, 2, 3$. We regard elements of $V$ as $5 \times 5$ alternating matrices $x = (x_{ij}(u))$ with entries which are linear combinations of $u = (u_1, u_2, u_3)$. Let
Let $R = k[u_1, u_2, u_3]$. We regard $x \in \wedge^2_R R^5$. Then $x \wedge x \in \wedge^4_R R^5 \cong \text{Hom}_{R}(R^5, R)$. We can regard that $\text{Pfaff}_1(x), \ldots, \text{Pfaff}_5(x)$ are the 5-components of $\text{Hom}_{R}(R^5, R)$ and

$$\text{Pfaff}((g_1, g_2)x(u)) = (\det g_1)^4 g_1^{-1} \text{Pfaff}(x(ug_2)).$$

Let

$$\Phi(x) = \text{Pfaff}_1(x) \wedge \cdots \wedge \text{Pfaff}_5(x) \in \wedge^5 \text{Sym}^2 \text{Aff}^3 \cong (\text{Sym}^2 \text{Aff}^3)^*.$$ 

Then with respect to a basis of $\text{Sym}^2 \text{Aff}^3$ and its dual basis,

$$\Phi(gx) = (\det g_1)^4 (\det g_2)^4 (\text{Sym}^2 g_2)^{-1} \Phi(gx).$$

Let

$$l_1 = u_1^2, l_2 = u_1 u_2, l_3 = u_1 u_3, l_4 = u_2^2, l_5 = u_2 u_3, l_6 = u_3^2.$$ 

Then $\{l_1, \ldots, l_6\}$ is a basis of $\text{Sym}^2 \text{Aff}^3$. Let $\{l_1^*, \ldots, l_6^*\}$ be the dual basis. We identify $\wedge^5 \text{Sym}^2 \text{Aff}^3$ with $(\text{Sym}^2 \text{Aff}^3)^*$ so that $l_1^* = l_2 \wedge \cdots \wedge l_6$, $l_2^* = -l_1 \wedge l_3 \wedge \cdots \wedge l_6$, etc.

Easy computations show that

$$\begin{pmatrix}
\text{Pfaff}_1(w) \\
\text{Pfaff}_2(w) \\
\text{Pfaff}_3(w) \\
\text{Pfaff}_4(w) \\
\text{Pfaff}_5(w)
\end{pmatrix} =
\begin{pmatrix}
-l_6 \\
-l_5 \\
-l_3 - 2l_4 \\
-l_2 \\
-l_1
\end{pmatrix}, 
\begin{pmatrix}
\text{Pfaff}_1(w') \\
\text{Pfaff}_2(w') \\
\text{Pfaff}_3(w') \\
\text{Pfaff}_4(w') \\
\text{Pfaff}_5(w')
\end{pmatrix} =
\begin{pmatrix}
-l_6 \\
l_5 \\
l_3 - l_4 \\
l_2 \\
l_1
\end{pmatrix},$$

so that

$$\Phi(w) = l_1^* - 2l_3^*, \Phi(w') = l_1^* - l_3^*.$$ 

For $g \in \text{GL}_3$, with respect to the above bases, we put $(\text{Sym}^2 g)^* = (\det g)^4 (\text{Sym}^2 g)^{-1}$.

For $y = \sum_{i=1}^6 y_i l_i^*$, we put

$$P_1(y) = -\det
\begin{pmatrix}
y_1 & y_2 & y_3 \\
y_2 & y_4 & y_5 \\
y_3 & y_5 & y_6
\end{pmatrix}.$$

**Proposition 5.12.** Over any field $k$, $P_1((\text{Sym}^2 g)^* y) = (\det g)^{10} P_1(y)$.

**Proof.** $P_1((\text{Sym}^2 g)^* y), (\det g)^{10} P_1(y)$ are polynomials of $y, g, (\det g)^{-1}$ defined over $\mathbb{Z}$. So if we can prove the lemma over $\mathbb{Z}$ then by considering the natural homomorphism $\mathbb{Z} \to k$, the lemma follows for $k$ also. So we consider the lemma over $\mathbb{Q}$.

For $a \in \text{Aff}^3 \otimes \text{Aff}^3, b \in (\text{Aff}^3)^* \otimes (\text{Aff}^3)^*$, let $(a, b)$ be the natural pairing. We can naturally identify $(\text{Aff}^3 \otimes \text{Aff}^3)^*$ with $(\text{Aff}^3)^* \otimes (\text{Aff}^3)^*$ by this pairing. The map $\iota : \text{Sym}^2 \text{Aff}^3 \ni v_1 v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1) \in \text{Aff}^3 \otimes \text{Aff}^3$ is equivariant with respect to the group action. There is a similar map $\iota^*$ for $\text{Sym}^2 (\text{Aff}^3)^*$ also.
Let \( \{v_1, v_2, v_3\} \) be the dual basis of \( \{u_1, u_2, u_3\} \). Then \( (\iota(u_i, u_j), \iota^*(v_j, v_j)) = 0 \) unless \( \{i_1, i_2\} = \{j_1, j_2\} \) and

\[
(\iota(u_i^2), \iota^*(v_i^2)) = 1 \quad (i = 1, 2, 3), \quad (\iota(u_i u_j), \iota^*(v_i v_j)) = \frac{1}{2} \quad (1 \leq i < j \leq 3).
\]

By the above consideration, we can identify \( l_1^* \), \( \ldots \), \( l_6^* \) with \( v_1^2, 2v_1v_2, v_2^2, 2v_2v_3, v_3^2 \). For \( z = z_1v_1^2 + z_2v_1v_2 + z_3v_2v_3 + z_4v_2^2 + z_5v_2v_3 + z_6v_3^2 \), we put

\[
P_2(z) = \det \begin{pmatrix} 2z_1 & z_2 & z_3 \\ z_2 & 2z_4 & z_5 \\ z_3 & z_5 & 2z_6 \end{pmatrix}.
\]

Then it is well known that \( P_2(z) \) is a relative invariant polynomial. So \( P_2((\text{Sym}^2 g)^* z) = (\det g)^10 P_2(z) \). Since \( y \) can be identified with \( z = y_1v_1^2 + 2y_2v_1v_2 + 2y_3v_1v_3 + y_4v_2^2 + 2y_5v_2v_3 + 2y_6v_3^2 \) and \( P_2(z) = -8P_1(y) \) if \( \text{Sym}^2 g \) is a relative invariant polynomial. So \( (\text{Sym}^2 g)^* y) = (\det g)^10 P_1(y) \). We can divide by 8 over \( \mathbb{Q} \) and \( P_1((\text{Sym}^2 g)^* y) = (\det g)^10 P_1(y) \).

Let \( P(x) = P_1(\Phi(x)) \). The following corollary follows from (5.10), (5.11) and Proposition 5.12.

**Corollary 5.13.**

1. \( P((g_1, g_2)x) = (\det g_1)^{12}(\det g_2)^{10}P(x) \).
2. \( P(w) = 4 \), \( P(w') = 1 \).

By Proposition 5.12 and Corollary 5.13, \( P(x) \) is a non-zero relative invariant polynomial without any assumption on \( \text{ch}(k) \). Even though \( P(w') \neq 0 \) regardless of \( \text{ch}(k) \), \( \dim T_{\mathbb{Q}}(G_{w'}) \neq 4 \) if \( \text{ch}(k) = 2 \) and \( G_{w'} \) may not be isomorphic to \( \text{PGL}_2 \times \text{GL}_1 \) even if \( \text{ch}(k) \neq 2 \). So we had to use \( w \) instead.

We are in the situation of of Section 3 where \( m = N = 1 \). So if \( \text{ch}(k) 
eq 2 \), by Corollary 3.7, \( V_{\text{ss}} = G_{k_{\text{sep}}}w \). Therefore, we can use the standard argument of Galois cohomology. Since \( \text{PGL}_2 \) is the automorphism group of \( \text{PGL}_2 \), \( H^1(k, \text{PGL}_2) \) is in bijective correspondence with \( k \)-forms of \( \text{PGL}_2 \). However, we would like to be more explicit.

Let \( \widetilde{T} = \{(tI_5, t^{-1}I_3) \mid t \in \text{GL}_1\} \).

**Proposition 5.14.** Suppose that \( \text{ch}(k) 
eq 2 \). Then \( G_k \backslash V_{\text{ss}} \) is in bijective correspondence with \( \text{Prg}_2(k) \). If \( x \in V_{k_{\text{ss}}} \) then the corresponding \( k \)-form of \( \text{PGL}_2 \) is \( G_x/\widetilde{T} \).

**Proof.** The first statement has already been shown. Note that \( \widetilde{T} \) is contained in the center of \( G \) and \( G_w = \phi(\text{GL}_2)\widetilde{T} \). Since \( H^1(k, \text{GL}_1) = \{1\} \), \( (G_x/\widetilde{T})_k \cong G_x/\widetilde{T} \). Let \( x \in V_{k_{\text{ss}}} \). Then there exists \( g_x \in G_{k_{\text{sep}}} \) such that \( x = g_xw \). So

\[
G_{x,k_{\text{sep}}} = g_x\phi(\text{GL}_2(k_{\text{sep}}))g_x^{-1}\widetilde{T}_{k_{\text{sep}}}.
\]

If \( g \in g_x\phi(\text{GL}_2(k_{\text{sep}}))g_x^{-1}, t \in \widetilde{T}_{k_{\text{sep}}} \) and \( \sigma \in \text{Gal}(k_{\text{sep}}/k) \) then \( g_xgtg_x^{-1} \in G_k \) if and only if \( t \in \widetilde{T}_k \) and \( g_x^\sigma g^\sigma (g_x^\sigma)^{-1} = g_xg^{-1}_x \). This is equivalent to \( (g^{-1}g_x^\sigma)g^\sigma (g^{-1}g_x^\sigma)^{-1} = g \). Let \( h_\sigma = g^{-1}g_x^\sigma \) and \( h = \{h_\sigma\} \). Then the class of the 1-cocycle \( h \) is the element of \( H^1(k, G_w) \) corresponding to \( x \). The above condition is equivalent to \( h_\sigma g^\sigma h_\sigma^{-1} = g \) for all \( \sigma \). The set of such \( g \) is precisely the \( k \)-form of \( \text{PGL}_2 \) corresponding to \( x \). Therefore, \( G_x/\widetilde{T} \) is the \( k \)-form of \( \text{PGL}_2 \) corresponding to \( x \).
6. Rational orbits (3)

In this section let $G_1 = G_2 = \text{GL}_3$, $G_3 = \text{GL}_2$, $W_1 = W_2 = \text{Aff}^2$. We regard $W_1, W_2$ the standard representations of $G_1, G_2$ respectively. Let $\text{Aff}^2$ be the standard representation of $G_3 = \text{GL}_2$. We denote the standard basis of $W_1, W_2, \text{Aff}^2$ by $\{p_1, p_2, p_3\}, \{q_1, q_2, q_3\}, \{x_1, x_2\}$ respectively (we use these notations instead of $\mathbb{P}_3, \text{etc.}$, to distinguish two $\text{GL}_3$’s). Let $G = G_1 \times G_2 \times G_3 \times \text{GL}_4$, $V = \wedge^3 W_2 \oplus W_1 \otimes W_2 \otimes \text{Aff}^2$. We denote elements of $G, V$ by $g = (g_1, g_2, g_3, t), x = (x_1, x_2)$ respectively. We define an action of $t \in \text{GL}_1$ on $V$ by $V \ni (x_1, x_2) \mapsto (tx_1, x_2) \in V$. With the above representations of $G_1, G_2, G_3, V$ is a representation of $G$.

Let

$$w_1 = q_1 \wedge q_2 - q_1 \wedge q_3 + q_2 \wedge q_3 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

(6.1)

$$w_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad w_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and $w_2 = (w_{21}, w_{22}), w = (w_1, w_2) \in V$. We regard $w_{21} = p_1 \otimes q_1 \otimes x_1 - p_2 \otimes q_2 \otimes x_1, w_{22} = p_2 \otimes q_2 \otimes x_2 - p_3 \otimes q_3 \otimes x_2$.

We determine the Lie algebra of $T_{eG}(G_w)$.

Lemma 6.2. $\dim T_{eG}(G_w) = 2$.

Proof. Let $A = (a_{ij}), B = (b_{ij}) \in M_3, C = (c_{ij}) \in M_2, d \in \text{Aff}^1$. Then $(e_G + \varepsilon(A, B, C, d))w_2 = w_2$ if and only if

$$Aw_{21} + w_{21}^tB + c_{11}w_{21} + c_{12}w_{22} = 0,$$

$$Aw_{22} + w_{22}^tB + c_{21}w_{21} + c_{22}w_{22} = 0.$$

This implies that $A, B, C$ are diagonal matrices and

$$a_{11} + b_{11} + c_{11} = 0, \quad a_{22} + b_{22} + c_{11} = 0,$$

$$a_{22} + b_{22} + c_{22} = 0, \quad a_{33} + b_{33} + c_{22} = 0$$

(we do not provide the details).

Then $(e_G + \varepsilon(A, B, C, d))w_1 = w_1$ if and only if

$$Aw_1 + w_1^tA + dw_1 = 0.$$ 

This implies that $a_{11}a_{22} = a_{11}a_{33} = a_{22}a_{33}$ and so $a_{11} = a_{22} = a_{33}$. By (6.3), $A, B, C$ are scalar matrices. We put $A = aI_3, B = bI_3, C = cI_2$. Then $c = -a - b, d = -2a$. □

It is easy to see that elements of the form $(t_1I_3, t_2I_3, (t_1t_2)^{-1}I_2, t_2^{-2})$ fix $w$. By Lemma 6.2, $\dim T_{eG}(G_w) = 2 = \dim G - \dim V$. So Proposition 3.3 implies that $G_w \subset V$ is Zariski open and $G_w$ is smooth over $k$. Therefore, we obtain the following corollary.

Corollary 6.4. (1) $G_w^* = \{(t_1I_3, t_2I_3, (t_1t_2)^{-1}I_2, t_2^{-2}) \mid t_1, t_2 \in \text{GL}_1\} \cong \text{GL}_2$

(2) $G_w$ is smooth reductive over $k$ and so $(G, V)$ is a regular prehomogeneous vector space.
Since $k$ is arbitrary and $G, V, w$ are defined over $\mathbb{Z}, w$ is universally generic (see Definition 3.9).

Let
\[
\tau_1 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
-1 & 0 \\
1 & 1 \\
-1
\end{pmatrix}, \quad (6.5)
\]
\[
\tau_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, \quad (6.7)
\]

Then $\tau_1, \tau_2$ fix $w_1, w_2$ and generate a subgroup of $G_k$ isomorphic to $\mathfrak{S}_3$. We denote the subgroup of $G_k$ generated by $\tau_1, \tau_2$ by $H$.

**Proposition 6.6.** $G_w \cong H \times G_w^0 \cong \mathfrak{S}_3 \times \text{GL}_1^2$.

**Proof.** Suppose that $g \in G_w$. Let $x = (x_1, x_2) \in V$. Regarding $x_2$ as a $3 \times 3$ matrix with entries in linear forms in two variables $v = (v_1, v_2)$, let $F_x(v)$ be its determinant. Then $F_w = v_1v_2(v_1 - v_2)$. So its zero set in $\mathbb{P}^1$ is $\{(1,0), (0,1), (1,1)\}$. Since $F_gx(v) = (\det g_1)(\det g_2)F_x(vg_3)$, $g_3$ induces a permutation of $(1,0), (0,1), (1,1) \in \mathbb{P}^1$.

Multiplying an element of $H$, we may assume that $g_3$ fixes $(1,0), (0,1), (1,1) \in \mathbb{P}^1$. Then $g_3$ is a scalar matrix. This implies that $g_1w_2^tg_2$ is a scalar multiple of $w_3$, for $i = 1, 2$.

By considering the $(1,3), (2,3), (3,1), (1,2), (1,3), (2,1), (3,1)$-entries of $g_1w_2^tg_2$ (resp. $g_1w_2^tg_2$), $a_{12} = a_{13} = a_{31} = a_{32} = b_{12} = b_{13} = b_{31} = b_{32} = 0$. Then by the $(1,2), (2,1)$-entries (resp. $(2,3), (3,2)$-entries) of $g_1w_2^tg_2$ (resp. $g_1w_2^tg_2$), $a_{21} = a_{23} = b_{21} = b_{23} = 0$. By considering $g_1w_1^tg_1$, $a_{11} = a_{22} = a_{33}$ and so $g_1$ is a scalar matrix. Then it is easy to see that $g_2$ is a scalar matrix also and $g \in G_w^0$.

Let $P_1(x)$ be the degree 12 polynomial on $W_1 \otimes W_2 \otimes \text{Aff}^2$ obtained by Proposition 4.2.

Then
\[
P_1(gx) = (\det g_1)^4(\det g_2)^4(\det g_3)^6P_1(x).
\]

Since $P_1(w)$ is the discriminant of $F_w$ in the proof of Proposition 6.6, $P_1(w) = 1$.

We construct another relative invariant polynomial on $V$. We first construct an equivariant map from $W_1 \otimes W_2 \otimes \text{Aff}^2$ to $W_2 \otimes W_2 \otimes W_2$. Let $\Phi_1 : W_1 \otimes W_2 \otimes \text{Aff}^2 \to (W_1 \otimes W_2 \otimes \text{Aff}^2)^{\otimes 6}$ be the map defined by

\[
\Phi_1(x) = x \otimes \cdots \otimes x.
\]

We identify $\wedge^3 W_1, \wedge^3 W_2 \cong \text{Aff}^1, \wedge^2 \text{Aff}^2 \cong \text{Aff}^1$ in the usual manner. We define a linear map $\Phi_2 : (W_1 \otimes W_2 \otimes \text{Aff}^2)^{\otimes 6} \to (W_2 \otimes \text{Aff}^2)^{\otimes 6}$ so that

\[
\Phi_2((v_{11} \otimes v_{12} \otimes v_{13}) \otimes \cdots \otimes (v_{61} \otimes v_{62} \otimes v_{63}))
\]
\[
= (v_{11} \wedge v_{21} \wedge v_{31})(v_{41} \wedge v_{51} \wedge v_{61})(v_{12} \otimes v_{13}) \otimes \cdots \otimes (v_{62} \otimes v_{63}).
\]

We define a linear map $\Phi_3 : (W_2 \otimes \text{Aff}^2)^{\otimes 6} \to W_2^{\otimes 6}$ so that

\[
\Phi_3((v_{12} \otimes v_{13}) \otimes \cdots \otimes (v_{62} \otimes v_{63}))
\]
\[
= (v_{13} \wedge v_{43})(v_{23} \wedge v_{53})(v_{33} \wedge v_{63})v_{12} \otimes \cdots \otimes v_{62}.
\]
We define a linear map $\Phi_4 : W_2^{6\otimes} \to W_2^{3\otimes}$ so that
\[
\Phi_4(v_{12} \otimes \cdots \otimes v_{62}) = -(v_{12} \land v_{42} \land v_{52})v_{32} \otimes v_{52} \otimes v_{62}.
\]
Let $\Phi = \Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1$.

We define a polynomial $P_2(x)$ on $V$ by
\[
P_2(x_1, x_2) = \frac{1}{6}(x_1 \otimes x_1 \otimes x_1) \land \Phi(x_2)
\]
for $x_1 \in \wedge^2 W_2, x_2 \in W_1 \otimes W_2 \otimes \text{Aff}^2$ ($\land$ is applied to all three components of $\Phi(x_2) \in W_2^{3\otimes}$). We shall explain in Proposition 6.8 that the coefficient $\frac{1}{6}$ can be justified.

**Proposition 6.8.**
(1) $P_2$ is a polynomial, homogeneous of degrees 3, 6 with respect to $x_1, x_2$ respectively and $P_2(gx) = t^3(\det g_1)^2(\det g_2)^4(\det g_3)^3 P_2(x)$.
(2) $\Phi(w_2) = \sum_{\sigma \in S_3} q_{\sigma(1)} \otimes q_{\sigma(2)} \otimes q_{\sigma(3)}$.
(3) $P_2(w) = 1$.

**Proof.** (1) The first part of the statement of (1) is obvious. It is easy to see that
\[
\Phi(gx) = (\det g_1)^2(\det g_2)(\det g_3)^3(g_2 \otimes g_2 \otimes g_2)\Phi(x).
\]
Since $\Phi(x)$ has three components of $W_2$ and $(tx) \otimes (tx) \otimes (tx) = t^3 x \otimes x \otimes x$, we obtain the second statement of (1).

(2) Let
\[
a = -(q_1 \otimes r_1) \otimes q_2 \otimes (-r_1 + r_2) \otimes (q_3 \otimes r_2)
+ (q_1 \otimes r_1) \otimes (q_3 \otimes r_2) \otimes q_2 \otimes (-r_1 + r_2)
+ q_2 \otimes (-r_1 + r_2) \otimes (q_1 \otimes r_1) \otimes (q_3 \otimes r_2)
- q_2 \otimes (-r_1 + r_2) \otimes (q_3 \otimes r_2) \otimes (q_1 \otimes r_1)
- (q_3 \otimes r_2) \otimes (q_1 \otimes r_1) \otimes q_2 \otimes (-r_1 + r_2)
+ (q_3 \otimes r_2) \otimes q_2 \otimes (-r_1 + r_2) \otimes (q_1 \otimes r_1).
\]
Then $\Phi_2 \circ \Phi_1(w_2) = a \otimes a$.

Straightforward computations show that
\[
\Phi_3(a \otimes a) = q_1 \otimes q_2 \otimes q_3 \otimes q_2 \otimes q_3 \otimes q_1 - q_1 \otimes q_2 \otimes q_3 \otimes q_3 \otimes q_1 \otimes q_2
+ q_1 \otimes q_3 \otimes q_2 \otimes q_2 \otimes q_1 \otimes q_3 - q_1 \otimes q_3 \otimes q_2 \otimes q_3 \otimes q_2 \otimes q_1
- q_2 \otimes q_1 \otimes q_3 \otimes q_3 \otimes q_2 \otimes q_1 + q_2 \otimes q_1 \otimes q_3 \otimes q_3 \otimes q_2 \otimes q_1
- q_2 \otimes q_3 \otimes q_1 \otimes q_2 \otimes q_3 \otimes q_1 + q_2 \otimes q_3 \otimes q_1 \otimes q_2 \otimes q_3 \otimes q_1
+ q_3 \otimes q_1 \otimes q_2 \otimes q_1 \otimes q_3 \otimes q_2 - q_3 \otimes q_2 \otimes q_1 \otimes q_2 \otimes q_1 \otimes q_3.
\]
Then
\[
\Phi(w_2) = \Phi_4 \circ \Phi_3(a \otimes a)
= 0 + q_3 \otimes q_1 \otimes q_2 + q_2 \otimes q_1 \otimes q_3 + 0 + 0 + q_3 \otimes q_2 \otimes q_1
+ q_1 \otimes q_2 \otimes q_3 + 0 + 0 + q_2 \otimes q_3 \otimes q_1 + q_1 \otimes q_3 \otimes q_2 + 0.
\]
(3) Note that $w_1 \land q_i = q_1 \land q_2 \land q_3$ for $i = 1, 2, 3$. Therefore, $6P_2(w) = 6$. Since $w$ is universally generic, Proposition 3.10 implies that $P_2$ is defined over $\mathbb{Z}$, $P_2(w) = 1$ and that $6P_2(x_1, x_2) = (x_1 \otimes x_1 \otimes x_1) \land \Phi(x_2)$. \(\square\)
So we are in the situation of Section 3 where \( m = N = 2 \). We put \( U = \{ x \in V \mid P_1(x), P_2(x) \neq 0 \} \). Then by Corollary 3.7, \( U_{\text{sep}} = G_{\text{sep}}w \). So we can use the standard argument of Galois cohomology.

**Proposition 6.9.** The map \( U_k \ni (x_1, x_2) \mapsto x_2 \in \text{Aff}^3 \otimes \text{Aff}^3 \otimes \text{Aff}^2 \) induces a bijection

\[
G_k \backslash U_k \cong (\text{GL}_3(k)^2 \times \text{GL}_2(k)) \backslash (\text{Aff}^3 \otimes \text{Aff}^3 \otimes \text{Aff}^2)^{\text{ss}} \cong \text{Ex}_3(k).
\]

**Proof.** Since \( H^1(k, G) = \{1\} \), \( G_k \backslash U_k \cong H^1(k, H \times G_w^0) \cong H^1(k, \mathfrak{G}_3) \cong \text{Ex}_3(k) \).

Suppose that \( z^3 + a_1z^2 + a_2z + a_3 \in k[z] \) has distinct roots \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \). Let \( \alpha = (a_1, a_2, a_3), F = k(\alpha) = k(\alpha_1, \alpha_2, \alpha_3) \) and \( D(\alpha) = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) \).

We put

\[
P_\alpha = \begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{pmatrix}, \quad Q_\alpha = D(\alpha)^{-1} \begin{pmatrix} -(\alpha_3 - \alpha_2) & -(\alpha_1 - \alpha_2) \\ \alpha_1(\alpha_3 - \alpha_2) & \alpha_2(\alpha_1 - \alpha_2) \end{pmatrix},
\]

\[
g_\alpha = (P_\alpha, P_\alpha, Q_\alpha, D(\alpha)), \quad x_a = g_\alpha w.
\]

Then \( g(\alpha_2, \alpha_3, \alpha_3) = g_{\alpha_1} \tau_1, g(\alpha_1, \alpha_2, \alpha_2) = g_{\alpha_1} \tau_2 \) (see (5.5)). Therefore, for any \( \sigma \in \text{Gal}(F/k) \), \( g_{\alpha_1}^{-1} g_{\alpha_1}^\sigma \in H \). This implies that \( x_a \in U_k \).

For \( \sigma \in \text{Gal}(F/k) \) let \( h_\sigma = g_{\alpha_1}^{-1} g_{\alpha_1}^\sigma \). If \( \sigma_1, \sigma_2 \in \text{Gal}(F/k) \) then \( h_{\sigma_1} h_{\sigma_2} = h_{\sigma_2} h_{\sigma_1} \).

The action of \( \text{Gal}(F/k) \) on \( \alpha \) enables us to identify \( \text{Gal}(F/k) \) with a subgroup of \( \mathfrak{G}_3 \cong H \).

Since any element of \( \mathfrak{G}_3 \) is a finite product of transpositions \((1 2), (2 3)\), \( h_{\sigma} \in G_k \) and so \( h_{\sigma_1} h_{\sigma_2} = h_{\sigma_2} h_{\sigma_1} \). If \( \sigma \in H \cong \mathfrak{G}_3 \) and \( \alpha = \sigma_1 \cdots \sigma_n \) where \( \sigma_1, \ldots, \sigma_n = (1 2) \) or \( (2 3) \) then \( h_{\sigma} = h_{\sigma_n} \cdots h_{\sigma_1} \), and \( h_{\sigma_1} \) is either \( \tau_1 \) or \( \tau_2 \). This implies that \( h_{\sigma} \) can be identified with \( \sigma^{-1} \) Therefore, the field corresponding to \( x_a \) is \( F \).

If we associate \( x_2 \) to \( x = (x_1, x_2) \) then the element corresponding to \( x_a \) is the projection to the second component. If \( x_a = (x_{a,1}, x_{a,2}) \) then \( x_{a,2} = g_{\alpha} w_2 \). So the cohomology class corresponding to \( x_{a,2} \) is the same as that of \( x_a \). Therefore, \( x = (x_1, x_2) \mapsto x_2 \) induces a bijection of rational orbits.

\[\square\]

**7. Rational orbits (4)**

In this section let \( G = \text{GL}_4 \times \text{GL}_2^2, W = \text{Aff}^4, V = W \otimes \text{Aff}^2 \oplus \wedge^2 W \otimes \text{Aff}^2 \). We consider the natural action of \( \text{GL}_4 \) on \( W, \wedge^2 W \). The first and the second \( \text{GL}_2 \)-factor act on the first and the second \( \text{Aff}^2 \) respectively. We identify \( W \otimes \text{Aff}^2 \cong M_{4,2} \) and \( \wedge^2 W \otimes \text{Aff}^2 \) with the space of pairs of \( 4 \times 4 \) alternating matrices. We express elements of \( G, V \) as \( g = (g_1, g_2, g_3), x = (A, B_1, B_2) (A \in M_{4,2}, B_1, B_2 4 \times 4 \) alternating). Then the action of \( g \) on \( x \) is \( g(A, B_1, B_2) = (g_1 A^t g_2, B_1^t, B_2^t) \) where

\[
\begin{pmatrix} B_1^t \\ B_2^t \end{pmatrix} = g_3 \begin{pmatrix} g_1 B_1^t g_1 \\ g_1 B_2^t g_1 \end{pmatrix}
\]

treating \([g_1 B_1^t g_1, g_1 B_2^t g_1]\) as a column vector. If \( x = (A, B_1, B_2) \) then we may denote \( A, B_1, B_2 \) by \( A(x), B_1(x), B_2(x) \), \( B(x) = (B_1(x), B_2(x)) \).

Let \( \{p_{4,1}, \ldots, p_{4,4}\} \) be the standard basis of \( W \) and \( \{p_{4,1}^*, \ldots, p_{4,4}^*\} \) its dual basis. We use the notation such as \( p_{4,12} = p_{4,1} \wedge p_{4,2} \), etc. We identify \( p_{4,234}, p_{4,134}, p_{4,124}, p_{4,123} \) with \( p_1^*, -p_2^*, p_3^*, -p_4^* \) respectively. To distinguish standard bases of the two \( \text{Aff}^2 \)'s, we denote the standard representation and its standard basis of the first (resp. second) \( \text{GL}_2 \) by \( L_1, \{p_{2,1}, p_{2,2}\} \) (resp. \( L_2, \{q_{2,1}, q_{2,2}\} \)).
Lemma 7.2. \( \dim T_e(G_w) = 4. \)

Proof. Let \( A = (a_{ij}) \in M_4, B = (b_{ij}), C = (c_{ij}) \in M_2. \) We consider the condition \( (e_G + \varepsilon(A, B, C))w = w. \) It is easy to see that \( (e_G + \varepsilon(A, B, C))w = w + \varepsilon(X, Y, Z) \) where

\[
X = A \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} tB = \begin{pmatrix} a_{12} + a_{14} + b_{12} & a_{11} + a_{13} + b_{22} \\ a_{22} + a_{24} + b_{11} & a_{21} + a_{23} + b_{21} \\ a_{32} + a_{34} + b_{12} & a_{31} + a_{33} + b_{22} \\ a_{42} + a_{44} + b_{11} & a_{41} + a_{43} + b_{21} \end{pmatrix} = 0,
\]

\[
Y = A \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} tA + \begin{pmatrix} 0 & c_{11} & 0 & 0 \\ -c_{11} & 0 & 0 & 0 \\ 0 & 0 & c_{12} & 0 \\ 0 & 0 & -c_{12} & 0 \end{pmatrix} = 0,
\]

\[
Z = A \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} tA + \begin{pmatrix} 0 & c_{21} & 0 & 0 \\ -c_{21} & 0 & 0 & 0 \\ 0 & 0 & c_{22} & 0 \\ 0 & 0 & -c_{22} & 0 \end{pmatrix} = 0.
\]

So \( A, B, C \) are in the following form

\[
A = \begin{pmatrix} -b_{22} & -b_{12} & 0 & 0 \\ -b_{21} & -b_{11} & 0 & 0 \\ 0 & 0 & -b_{22} & -b_{12} \\ 0 & 0 & -b_{21} & -b_{11} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad C = \begin{pmatrix} b_{11} + b_{22} \\ 0 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{22} \end{pmatrix}.
\]

\( \square \)
Let \( \tau_0 \) be as in [2.1] and
\[
\tau_1 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \tau = (\tau_1, \tau_0).
\]
Then \( \tau \) fixes \( w \).

**Proposition 7.3.**
1. \( Gw \subset V \) is Zariski open and \( G_w \) is smooth over \( k \).
2. \( G_w^\circ \cong \text{GL}_2 \).
3. \( G_w \cong G_w^\circ \times \{1, \tau\} \cong \text{GL}_2 \times \mathbb{Z}/2\mathbb{Z} \).

**Proof.**
(1) Since \( \dim \text{Te}_G(G_w) = 4 = 24 - 20 = \dim G - \dim V \), \( Gw \subset V \) is Zariski open and \( G_w \) is smooth over \( k \).

(2) If \( h \in \text{GL}_2 \) we define
\[
f(h) = (\text{diag}(\tau_0 h^{-1} \tau_0, \tau_0 h^{-1} \tau_0), h, (\det h)I_2) \in G.
\]
Then \( f(\text{GL}_2) \) fixes \( w \). So \( f(\text{GL}_2) \subset G_w^\circ \). Since \( \dim G_w^\circ = 4 \) and \( f \) is an imbedding, \( G_w = f(\text{GL}_2) \cong \text{GL}_2 \).

(3) It is easy to see that \( \tau \) commutes with elements of \( G_w^\circ \). We may assume that \( k = \overline{k} \). Suppose that \( g = (g_1, g_2, g_3) \in G_w \). Conjugation by \( g \) induces an automorphism of \( \text{SL}_2 \subset \text{GL}_2 \cong G_w^\circ \). Since all automorphisms of \( \text{SL}_2 \) are inner, by multiplying an element of \( G_w^\circ \), we may assume that \( g \) commutes with this \( \text{SL}_2 \).

Since \( W \) is a copy of two standard representations of \( \text{SL}_2 \subset \text{GL}_2 \), by Schur’s lemma, \( g_1 \) either exchanges these two copies or leave them invariant. By multiplying \( \tau \) if necessary, we may assume that \( g_1 \) leaves them invariant. Then \( g_1 \) must be in the form \( \text{diag}(\alpha I_2, \beta I_2) \). Since \( g \) fixes the first component of \( w \), \( g_2 \) is a scalar matrix, say \( \gamma I_2 \), \( \alpha = \beta \) and \( \gamma = \alpha^{-1} \). By multiplying an element of \( G_w^\circ \), we may assume that \( g_1 = I_4, g_2 = I_2 \). Since the last two components of \( w \) are linearly independent, \( g_3 = I_2 \).

**Corollary 7.4.** \( (G, V) \) is a regular prehomogeneous vector space.

We construct two relative invariant polynomials.

We define a map \( \Phi_1 : V \to W^* \otimes L_1 \otimes L_2 \), linear with respect to each of \( W \otimes L_1, \wedge^2 W \otimes L_2 \) so that
\[
\Phi_1(b_1 \otimes c_1, b_2 \otimes c_2) = (b_1 \wedge b_2) \otimes c_1 \otimes c_2 \in \wedge^3 W \otimes L_1 \otimes L_2 \cong W^* \otimes L_1 \otimes L_2
\]
for \( b_1 \in W, b_2 \in \wedge^2 W, c_1 \in L_1, c_2 \in L_2 \).

We put \( u_i = \mathfrak{p}_4^* \) for \( i = 1, \ldots, 4 \) and \( u = (u_1, \ldots, u_4) \). We identify \( W^* \otimes L_1 \otimes L_2 \) with the space of \( 2 \times 2 \) matrices with entries which are linear combinations of \( u \). So we write \( \Phi_1(x) = \Phi_1(x)(u) \). For \( y \in W^* \otimes L_1 \otimes L_2 \), we put \( \Phi_2(y)(u) = \det y(u) \in \text{Sym}^2 W^* \). It is easy to see that
\[
\Phi_1(w)(u) = \begin{pmatrix} u_3 & u_1 \\ -u_4 & -u_2 \end{pmatrix}, \quad \Phi_2 \circ \Phi_1(w)(u) = u_1 u_4 - u_2 u_3.
\]

Let \( P_1(x) \) be the discriminant of \( \Phi_2 \circ \Phi_1(x) \) for \( x \in V \). By the above computations, \( P_1(w) = 1 \). Note that \( P_1(x) \) is homogeneous of degree 8 with respect to each of \( W \otimes L_1, \wedge^2 W \otimes L_2 \).
Let $P_2(x)$ be the degree 4 polynomial of $B(x) \in \wedge^2 W \otimes L_2$ given by Proposition 4.1. Then $P_2(w) = 1$. It is easy to see that

$$
\Phi_1(gx)(u) = (\det g_1)2\Phi_1(x)(u^t g_1^{-1})\Phi_3,
$$

$$
\Phi_2 \circ \Phi_1(gx) = (\det g_1)^2 (\det g_2)(\det g_3)\Phi_2 \circ \Phi_1(x)(u^t g_1^{-1}),
$$

$$
P_1(gx) = (\det g_1)^4 (\det g_2)^4 (\det g_3)^4 P_1(gx),
$$

$$
P_2(gx) = (\det g_1)^2 (\det g_3)^2 P_2(gx).
$$

(7.5)

So we are in the situation of Section 3 where $m = N = 2$. We put $U = \{x \in V \mid P_1(x), P_2(x) \neq 0\}$. Then by Corollary 3.7, $U_{k_{sep}} = G_{k_{sep}} w$. Therefore, we can use the standard argument of Galois cohomology.

**Proposition 7.6.** The map $U_k \ni x \mapsto B(x) \in \wedge^2 W \otimes L_2$ induces a bijection

$$
G_k \backslash U_k \cong (GL_4(k) \times GL_2(k)) \backslash (\wedge^2 W \otimes L_2)_k^{ss} \cong \text{Ex}_2(k).
$$

**Proof.** Since $H^1(k, G) = H^1(k, GL_2) = \{1\}$, $G_k \backslash U_k \cong H^1(k, \mathbb{Z}/2\mathbb{Z}) \cong \text{Ex}_2(k)$.

Suppose that $z^2 + a_1z + a_2 \in k[z]$ has distinct roots $\alpha = (\alpha_1, \alpha_2)$. We put $a = (a_1, a_2), F = k(\alpha_1), \quad g_\alpha = \left(\begin{array}{cc} I_2 & I_2 \\ -\alpha_1 I_2 & -\alpha_2 I_2 \end{array}\right), (\alpha_1 - \alpha_2)^4 I_2, \left(\begin{array}{cc} 1 & 1 \\ -\alpha_1 & -\alpha_2 \end{array}\right), \quad x_a = g_\alpha w.$

Let $\sigma \in \text{Gal}(F/k)$ be the non-trivial element. Then $g_\alpha^\sigma = g_\alpha \sigma$ and so $x_\alpha \in U_k$. Since $g_\alpha^{-1} g_\alpha \sigma = \tau$, the element of $H^1(k, \mathbb{Z}/2\mathbb{Z})$ which corresponds to $x_\alpha$ is determined by a 1-cocycle which sends $\sigma$ to $\tau$.

Let $H = GL_4 \times GL_2$ and $W_1 = \wedge^2 W \otimes L_2$. It can be shown that $(H, W_1)$ is a regular prehomogeneous vector space, $B(w) \in W_1^{ss}$,

$$
H^0_{B(w)} = \{ (\text{diag}(h_1, h_2), \text{diag}((\det h_1)^{-1}, (\det h_2)^{-1}) \mid h_1, h_2 \in GL_2 \} \cong \text{GL}_2^2,
$$

$$
H_{B(w)} / H^0_{B(w)} \cong \mathbb{Z}/2\mathbb{Z},
$$

and $H_{B(w)} / H^0_{B(w)}$ is represented by $\{1, (\tau_1, \tau_0)\}$.

Since $B((g_1, g_2, g_3)x) = (g_1, g_3)B(x)$, the map $V \ni x \mapsto B(x) \in W_1$ is equivariant with respect to the natural action of $H$ on $W_1$. Since the map $G \ni (g_1, g_2, g_3) \mapsto (g_1, g_3) \in H$ sends $\tau$ to $(\tau_1, \tau_0)$, the cohomology class associated with $B(x_\alpha)$ is determined by the 1-cocycle which sends $\sigma$ to the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$. Therefore, the map $U_k \ni x \mapsto B(x) \in W_1$ induces a bijection of rational orbits.

8. Rational orbits (5)

Let $G = GL_4 \times GL_3$, $W = \text{Aff}_4$ and (a) $V = \wedge^2 W \otimes \text{Aff}^3$, (b) $V = \wedge^2 W \otimes \text{Aff}^3 \otimes W$. We consider the natural action of $G$ in both cases.

We first consider the case (a).

We put

$$
w_1 = \left(\begin{array}{cc} J & 0 \\ 0 & 0 \end{array}\right), \quad w_2 = \left(\begin{array}{cc} 0 & J \\ J & 0 \end{array}\right), \quad w_3 = \left(\begin{array}{cc} 0 & 0 \\ 0 & J \end{array}\right)
$$

and $w = (w_1, w_2, w_3)$.

Long but straightforward computations show the following proposition. We do not provide the proof.
Proposition 8.2. \( T_{e_G}(G_w) \) consists of elements of the form \( e_G + \varepsilon(A, B) \) where

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & 0 \\
a_{21} & a_{22} & 0 & a_{13} \\
a_{31} & 0 & a_{33} & a_{12} \\
0 & a_{31} & a_{21} & -a_{11} + a_{22} + a_{33}
\end{pmatrix},
\]

\( (8.3) \)

\[
B = \begin{pmatrix}
-a_{11} - a_{22} & -a_{31} & 0 \\
-2a_{13} & -a_{22} - a_{33} & -2a_{31} \\
0 & -a_{13} & a_{11} - a_{22} - 2a_{33}
\end{pmatrix}.
\]

By Proposition 8.2, \( \dim T_{e_G}(G_w) = 7 = 25 - 18 = \dim G - \dim V \), which implies that the orbit \( Gw \subset V \) is Zariski open and that \( G_w \) is smooth over \( k \). This implies that \( w \) is universally generic.

The tensor product of two standard representations of \( GL_2 \) induces a homomorphism \( \rho : GL_2 \times GL_2 \rightarrow GL(\text{Aff}^2 \otimes \text{Aff}^2) \cong GL_4 \). We show that \( G_w^e \) can be identified with the image of \( \rho \).

We identify \( \text{Aff}^2 \otimes \text{Aff}^2 \) with \( M_2 \). Then \( \rho(g_1, g_2)M = g_1 M g_2 \). We identify \( M_2 \) with \( W \) using the basis \( \{ E_{11}, E_{21}, E_{12}, E_{22} \} \) (see Section 2). The matrix representations of \( \rho(g_1, I_2), \rho(I_2, g_2) \) for \( g_2 = (\alpha \beta \gamma \delta) \) with respect to this basis are

\[
(8.4) \quad \begin{pmatrix}
g_1 & 0 \\
0 & g_1
\end{pmatrix}, \quad \begin{pmatrix}
\alpha I_2 & \beta I_2 \\
\gamma I_2 & \delta I_2
\end{pmatrix}
\]

respectively.

\( \rho \) induces a representation \( \wedge^2 \rho : GL_2 \times GL_2 \rightarrow GL(\wedge^2 \text{Aff}^4) \). Since

\[
\begin{aligned}
\begin{pmatrix}
\alpha I_2 & \beta I_2 \\
\gamma I_2 & \delta I_2
\end{pmatrix}
&= \begin{pmatrix}
aJ & bJ \\
bJ & cJ
\end{pmatrix}
\begin{pmatrix}
\alpha I_2 & \gamma I_2 \\
\beta I_2 & \delta I_2
\end{pmatrix} \\
&= \begin{pmatrix}
(\alpha a + \beta b)J & (\alpha b + \beta c)J \\
(\gamma a + \beta b)J & (\gamma b + \delta c)J
\end{pmatrix}
\begin{pmatrix}
\alpha I_2 & \gamma I_2 \\
\beta I_2 & \delta I_2
\end{pmatrix}
\end{aligned}
\]

\( (8.5) \)

\[
= \begin{pmatrix}
((\alpha + \beta b)\alpha + (ab + \beta c)\beta)J & ((\alpha + \beta b)\gamma + (ab + \beta c)\delta)J \\
((\gamma a + \beta b)\alpha + (ab + \beta c)\beta)J & ((\gamma a + \beta b)\gamma + (ab + \beta c)\delta)J
\end{pmatrix}.
\]

elements of the form \((I_2, g_2)\) leave the subspace \( \langle w_1, w_2, w_3 \rangle \) invariant. Elements of the form \((g_1, I_2)\) acts on the subspace \( \langle w_1, w_2, w_3 \rangle \) by multiplication by \( \text{det} g_1 \).

For \( g = (g_1, g_2) \in GL_2 \times GL_2 \), let \( \rho_1(g) = (\rho_1(g)_{ij}) \in GL_3 \) be the matrix such that

\[
(8.6) \quad (\wedge^2 \rho(g)w_1 \wedge^2 \rho(g)w_2 \wedge^2 \rho(g)w_3) = (w_1 \ w_2 \ w_3) \rho_1(g)
\]

(we are treating \( \langle w_1, w_2, w_3 \rangle \) as a row vector). It is easy to see that if \( g, h \in GL_2 \times GL_2 \) then \( \rho_1(gh) = \rho_1(g)\rho_1(h) \).

The proof of the following lemma is similar to that of Lemma 5.4 and we do not provide the proof.

Lemma 8.7. The image of \( (\rho, t \rho_1^{-1}) \) fixes \( w \).

Let \( \phi = (\rho, t \rho_1^{-1}) : GL_2 \times GL_2 \rightarrow GL_4 \times GL_3 \).

Lemma 8.8. \( \text{Ker}(\rho) = \{(t_1 I_2, t_2 I_2) \mid t_1 t_2 = 1\} \).
Proof. Clearly, $\ker(\rho) \supset \{(t_1I_2, t_2I_2) \mid t_1t_2 = 1\}$.

Suppose that $(g_1, g_2) \in \ker(\rho)$. If $g_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then

$$\rho(g_1, g_2) = \begin{pmatrix} \alpha g_1 & \beta g_1 \\ \gamma g_1 & \delta g_1 \end{pmatrix} = I_4.$$ 

So $\beta = \gamma = 0$, $\alpha = \delta$ and $g_1$ is a scalar matrix. Therefore, there exists $t_1, t_2 \in \text{GL}_1$ such that $(g_1, g_2) = (t_1I_2, t_2I_2)$. Then it is easy to see that $t_1t_2 = 1$. \qed

**Proposition 8.9.**

1. $\ker(\phi) = \ker(\rho)$.
2. $G_w^\circ = \text{im}(\phi) \cong \text{im}(\rho) \cong (\text{GL}_2 \times \text{GL}_2)/\ker(\rho)$.

Proof. (1) Clearly, $\ker(\phi) \subset \ker(\rho)$. Suppose that $g = (g_1, g_2) \in \ker(\rho)$. Since $\rho(g) = I_4$, $\wedge^2 \rho(g)w_i = w_i$ for $i = 1, 2, 3$. Therefore, $\rho_1(g) = I_3$.

(2) By Lemma 8.8 dim $\text{im}(\phi) = 7 = \text{dim} G_w^\circ$. Since $\text{im}(\phi)$ is closed and both $\text{im}(\phi), G_w^\circ$ are irreducible algebraic sets, $G_w^\circ = \text{im}(\phi)$. \qed

We shall prove later that $G_w = G_w^\circ$ after we construct an equivariant map from $V$ to $\text{sym}^2 W$.

**Corollary 8.10.** $(G, V)$ is a regular prehomogeneous vector space.

We construct an equivariant map $V \to \text{sym}^2 W$.

Let $\{p_{4,1}, \ldots, p_{4,4}\}$ be the standard basis of $W$ (this basis corresponds to the basis $\{E_{11}, E_{21}, E_{12}, E_{22}\}$ of $M_2$). We define a linear map $\iota: \wedge^2 W \to \wedge^2 W$ so that $\iota(a_1 \wedge a_2) = a_1 \otimes a_2 - a_2 \otimes a_1$. This map is clearly well-defined. We regard $\text{sym}^2 W$ as the space of quadratic forms on $W^*$. We identify $\wedge^4 W \cong \text{Aff}^1$ so that $p_{4,1234}$ corresponds to 1. We define a linear map $\Phi_1: W^\otimes \to \text{sym}^2 W$ so that

$$\Phi_1(a_1 \otimes a_2 \otimes \cdots \otimes a_6) = (a_1 \wedge a_2 \wedge a_3 \wedge a_5)a_4a_6.$$

We define a map $\Phi: V = \wedge^2 W \otimes \text{Aff}^3 \to \text{sym}^2 W$ so that

$$\Phi(x_1, x_2, x_3) = \frac{1}{12}\Phi_1 \left( \sum_{\sigma \in S_3} \text{sgn}(\sigma) \iota(x_{\sigma(1)}) \otimes \iota(x_{\sigma(2)}) \otimes \iota(x_{\sigma(3)}) \right).$$

Note that $12\Phi$ is defined over $\mathbb{Z}$. We shall show that $12\Phi(w)$ is divisible by 12 over $\mathbb{Z}$. Then, since $w$ is universally generic (see Definition 3.10), $\Phi$ itself is defined over $\mathbb{Z}$ by Proposition 8.10 and so $\Phi$ is defined over any field. It is easy to see that

$$\Phi((g_1, g_2)x) = (\det g_1)(\det g_2)(\text{sym}^2 g_1)\Phi(x).$$

In the following, we need notations for elements of $W^\otimes, W^\otimes, \text{sym}^2 W$. We denote $p_{4,i_1} \otimes \cdots \otimes p_{4,i_m}$ by $q_{i_1, \ldots, i_m}$ for $m = 2, 6, i_1, \ldots, i_m = 1, \ldots, 4$ and $p_{4,i}p_{4,j}$ by $v_i v_j$ in $\text{sym}^2 W$ for $i, j = 1, \ldots, 4$.

**Proposition 8.12.** $\Phi(w) = v_1v_4 - v_2v_3$. 

Lemma 8.13. \[ \sum_{\sigma \in S_3} \text{sgn}(\sigma) \iota(w_{\sigma(1)}) \otimes \iota(w_{\sigma(2)}) \otimes \iota(w_{\sigma(3)}) \]

\[= (q_{12} - q_{21}) \otimes (q_{14} - q_{41} - q_{23} + q_{32}) \otimes (q_{34} - q_{43}) \]

\[- (q_{12} - q_{21}) \otimes (q_{34} - q_{43}) \otimes (q_{14} - q_{41} - q_{23} + q_{32}) \]

\[- (q_{14} - q_{41} - q_{23} + q_{32}) \otimes (q_{12} - q_{21}) \otimes (q_{34} - q_{43}) \]

\[+ (q_{14} - q_{41} - q_{23} + q_{32}) \otimes (q_{34} - q_{43}) \otimes (q_{12} - q_{21}) \]

\[+ (q_{34} - q_{43}) \otimes (q_{12} - q_{21}) \otimes (q_{14} - q_{41} - q_{23} + q_{32}) \]

\[- (q_{34} - q_{43}) \otimes (q_{14} - q_{41} - q_{23} + q_{32}) \otimes (q_{12} - q_{21}). \]

Expanding the tensor product and applying \( \Phi \) to each term, we obtain \( 12\Phi(w) = 12(v_1v_4 - v_3v_2) \). We do not provide further details. \( \square \)

Let \( P(x) \) be the discriminant of \( \Phi(x) \). Then \( P(w) = 1 \). The following lemma follows from (8.11).

Lemma 8.13. \( P(x) \) is a non-zero homogeneous polynomial of degree 12 and if \( g = (g_1, g_2) \in G \) then \( P(gx) = (\det g_1)^6(\det g_2)^4 \Phi(x) \).

We define an action of \( (t, g_1) \in H = \text{GL}_1 \times \text{GL}_4 \) on \( \text{Sym}^2 W \) by \( \text{Sym}^2 W \ni Q \mapsto t(\text{Sym}^2 g_1)Q \in \text{Sym}^2 W \) where \( t(\text{Sym}^2 g_1)Q \) is the scalar multiplication by \( t \) to \( (\text{Sym}^2 g_1)Q \). We define a homomorphism \( f : G \to H \) by \( f(g_1, g_2) = ((\det g_1)(\det g_2), g_1) \). Then \( \Phi((g_1, g_2)x) = f(g_1, g_2)\Phi(x) \). Therefore, \( f(G_w^o) \subseteq H_{\Phi(w)}^o \).

Let

\[\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Proposition 8.15. \( G_w = G_w^o \). Moreover, \( f \) induces an isomorphism \( G_w = G_w^o \cong H_{\Phi(w)}^o \).

Proof. We may assume that \( k = \overline{k} \).

By simple Lie algebra computations, \( (H, \text{Sym}^2 W) \) is a regular prehomogeneous vector space and \( H\Phi(w) \subseteq \text{Sym}^2 W \) is Zariski open. Since \( \dim H_{\Phi(w)} = 7 \) and \( G_w^o \cong \text{Im}(\rho) \), the projection from \( G_w^o \) to \( \text{GL}_4 \) is an isomorphism and \( f(G_w^o) = H_{\Phi(w)}^o \). Let \( Z \subseteq G_w^o \) be the center of \( G_w^o \). Then \( G_w^o/Z \cong H_{\Phi(w)}^o/f(Z) \cong \text{PGL}_2 \times \text{PGL}_2 \).

Let \( t = (\text{diag}(t_1, t_1^{-1}), \text{diag}(t_2, t_2^{-1})) \in \text{GL}_2 \times \text{GL}_2 \). Then

\[\rho(t) = \text{diag}(t_1t_2, t_1^{-1}t_2, t_1t_2^{-1}, t_1^{-1}t_2^{-1}), \]

\[\tau \rho \tau = \text{diag}(t_1t_2, t_1t_2^{-1}, t_1^{-1}t_2, t_1^{-1}t_2^{-1}) = \rho((\text{diag}(t_2, t_2^{-1}), \text{diag}(t_1, t_1^{-1}))). \]

This implies that the conjugation by \( (1, \tau) \in H \) induces an outer automorphism of \( H_{\Phi(w)}^o/f(Z) \). By (3.12) \[ \text{[1], p.} 556 \), \( H_{\Phi(w)}/H_{\Phi(w)}^o \) is represented by \( \{1, (1, \tau)\} \).

If \( g = (g_1, g_2) \in G_w \) then \( f(g) \in H_{\Phi(w)} \). So \( g_1 \in \rho(\text{GL}_2 \times \text{GL}_2) \) or \( g_1 \tau \in \rho(\text{GL}_2 \times \text{GL}_2) \). Since (8.8) holds for \( g, \wedge^2 g_1 \) leaves the subspace \( \langle w_1, w_2, w_3 \rangle \) invariant. If \( g_1 \tau \in \rho(\text{GL}_2 \times \text{GL}_2) \) then \( \wedge^2 \tau \) must leave the subspace \( \langle w_1, w_2, w_3 \rangle \) invariant. However,
\[ \tau w_2 \tau \not\in \langle w_1, w_2, w_3 \rangle. \] Therefore, \( g_1 \in \rho(\text{GL}_2 \times \text{GL}_2) \). If \( g_1 = \rho(h_1, h_2) \) with \( h_1, h_2 \in \text{GL}_2 \) then \( g_2 \) has to be \( t_1 \rho_1(h_1, h_2)^{-1} \). Therefore, \( g \in G_w^\circ \).

**Proposition 8.16.** The map \( G_k \backslash \text{V}_k^{\text{ss}} \ni G_k x \mapsto \text{IQF}_4(k) \) is bijective.

**Proof.** For the definition of \( \text{IQF}_4(k) \), see Definition \[ \text{I}2 \text{H} \]. We would like to interpret \( G_k \backslash \text{V}_k^{\text{ss}} \) using the equivariant map \( \Phi \).

Since \( H^1(k, G) = \{1\} \), \( G_k \backslash \text{V}_k^{\text{ss}} \) is in bijective correspondence with \( H^1(k, G_w) = H^1(k, G_w^\circ) \cong H^1(k, H^\circ_{\Phi(w)}) \). By LEMMA (1.2) \[ 21 \text{ p.118} \], the following sequence

\[ 1 \to (\mathbb{Z}/2\mathbb{Z}) \to H^1(k, H^\circ_{\Phi(w)}) \to H^1(k, H^\circ_{\Phi(w)}) \to H^1(k, \mathbb{Z}/2\mathbb{Z}) \]

is exact where the action of \( \mathbb{Z}/2\mathbb{Z} \) on \( H^1(k, H^\circ_{\Phi(w)}) \) is by the conjugation by \( \tau \). It is known that \( H^1(k, H^\circ_{\Phi(w)}) \) is in bijective correspondence with \( \text{QF}_4(k) \). For this, see Proposition 3.15 \[ 21 \text{ p.557} \] for example. We shall show below that \( \mathbb{Z}/2\mathbb{Z} \) acts on \( H^1(k, H^\circ_{\Phi(w)}) \) trivially.

Let \( i : \text{GL}_1 \to \text{GL}_2 \times \text{GL}_2 \) be the homomorphism defined by \( i(t) = (t, t^{-1}) \). We identify \( H^\circ_{\Phi(w)} \cong (\text{GL}_2 \times \text{GL}_2) / i(\text{GL}_1) \) by \( f \circ \phi \) (see Lemma \[ 8 \text{7} \]). Let \( \psi_1 : \text{GL}_1 \to \text{GL}_1, \psi_2 : \text{GL}_2 \times \text{GL}_2 \to \text{GL}_2 \times \text{GL}_2 \) be the automorphisms such that \( \psi_1(t) = t^{-1}, \psi_2(g_1, g_2) = (g_2, g_1) \). The conjugation by \( \tau \) on \( H^\circ_{\Phi(w)} \) is denoted by \( \psi_3 \). Then the following diagram is commutative.

\[
\begin{array}{ccc}
1 & \longrightarrow & \text{GL}_1 \\
\downarrow \psi_1 & & \downarrow \psi_2 \\
1 & \longrightarrow & \text{GL}_2 \times \text{GL}_2 \\
\downarrow \psi_3 & & \downarrow \psi_3 \\
1 & \longrightarrow & H^\circ_{\Phi(w)} \\
\end{array}
\]

Let \( T_1 = \{t I_2 \mid t \in \text{GL}_1\}, T_2 = i(\text{GL}(1)) \subset \text{GL}_2 \times \text{GL}_2 \). For \( g \in \text{GL}_2 \) or \( h = (h_1, h_2) \in \text{GL}_2 \times \text{GL}_2 \), we use the notation \( \overline{g} = g \mod T_1, \overline{h} = h \mod T_2 \). We denote continuous maps from \( \text{Gal}(k^{\text{sep}}/k) \) to \( \text{GL}_2(k^{\text{sep}}) \), etc., as \( (h_\sigma)_\sigma (h_\sigma \in \text{GL}_2(k^{\text{sep}})) \). Note that we consider the discrete topology on \( \text{GL}_2(k^{\text{sep}}) \). For \( h = (h_\sigma)_\sigma (h_\sigma \in \text{GL}_2(k^{\text{sep}})) \), etc., let \( \delta h_{\sigma, \tau} = h_\tau h_\sigma^{-1} h_{\sigma, \tau} \) and \( \delta h = (\delta h_{\sigma, \tau})_\sigma, \tau \). This is a function on \( \text{Gal}(k^{\text{sep}}/k) \times \text{Gal}(k^{\text{sep}}/k) \). If \( h = (h_\sigma)_\sigma (h_\sigma \in \text{GL}_2(k^{\text{sep}})) \), \( g \in \text{GL}_2(k^{\text{sep}}) \) and \( h' = (g^{-1} h_\sigma g^\sigma)_\sigma \) then \( \delta h_{\sigma, \tau} = g^{-1} \delta h_{\sigma, \tau} g \) for all \( \sigma, \tau \). So if \( \delta h_{\sigma, \tau} \in T_1^{k^{\text{sep}}} \) for all \( \sigma, \tau \) then \( \delta h = \delta h' \).

Suppose that \( h_1 = (h_1, \sigma)_\sigma, h_2 = (h_2, \sigma)_\sigma (h_1, \sigma, h_2, \sigma \in \text{GL}_2(k^{\text{sep}})) \), \( h = (h_1, h_2) \) and \( \delta h_{\sigma, \tau} \in i(T(k^{\text{sep}})) \) for all \( \sigma, \tau \). This implies that \( (\delta h_1, \sigma, \tau)(\delta h_2, \sigma, \tau) = 1 \) for all \( \sigma, \tau \). Then if we put \( \overline{h}_1 = (\overline{h}_1, \sigma)_\sigma \), etc., then \( \overline{h} \) is trivial. So \( \overline{h} \) is a 1-cocycle with coefficients in \( H^\circ_{\Phi(w)} \). Any 1-cocycle with coefficients in \( H^\circ_{\Phi(w)} \) can be expressed in this manner. We define \( \psi_2(h), \psi_3(h) \) in the obvious manner.

Let \( h \) be as above such that \( \delta h = 1 \). We show that \( \overline{h}, \psi_3(\overline{h}) \) are in the same cohomology class. let \( q_i = (q_i, \sigma, \tau) = \delta h_i \) for \( i = 1, 2 \). Then

\[ q_i, \sigma, \tau = (\det h_i, \sigma, \tau)(\det h_i, \sigma, \tau) = 1. \]

We put \( h_{i, \sigma} = (\det h_{i, \sigma})^{-1} h_{i, \sigma}, h_{i, \sigma} = (\det h_{i, \sigma}) h_{i, \sigma}, h_{i, \sigma} = (h_{i, \sigma}) \) for \( i = 1, 2 \) and \( h_{i, \sigma} = (h_{i, \sigma}) h_{i, \sigma} \). Then \( \overline{h} = \overline{h} \). If \( \sigma, \tau \in \text{Gal}(k^{\text{sep}}/k) \) then

\[ \delta h_{i, \sigma, \tau} = (\delta h_{i, \sigma, \tau}, \delta h_{i, \sigma, \tau}), \delta h_{i, \sigma, \tau} = q_i, \sigma, \tau \delta h_{i, \sigma, \tau} = \delta h_{i, \sigma, \tau}. \]

Therefore, \( \delta h = \delta h_2(h) \).
If \( c_\sigma \in \text{GL}_2(k^\text{sep}) \), \( c = (c_\sigma)_\sigma \) and \( \delta c_\sigma, \tau \in \{ tI_k | t \in (k^\text{sep})^\times \} \) then \( \mathbf{F} \) is a 1-cocycle with coefficients in \( \text{PGL}_2 \). Since \( \delta c \) is a 2-cocycle with coefficients in \( \text{GL}_1 \), by associating the class of \( \delta c \) to the class of \( \mathbf{F} \), we obtain a map \( H^1(k, \text{PGL}_2) \to H^2(k, \text{GL}_1) \). Since \( H^1(k, \text{PGL}_2) \) can be identified with \( k \)-forms of \( M_2 \) and \( H^2(k, \text{GL}_1) \) is the Brauer group, the map \( H^1(k, \text{PGL}_2) \to H^2(k, \text{GL}_1) \) is injective. Note that if a \( k \)-form of \( M_2 \) is not split then it is isomorphic to \( D^\times \) where \( D \) is a division algebra.

Since \( \delta h^{(t)} = \delta \psi_2(h) \) and \( h^{(t)}_{i,\sigma} \) is a scalar multiple of \( h_{1,\sigma} \), there exists \( g \in \text{GL}_2(k^\text{sep}) \) and \( t \in (k^\text{sep})^\times \) such that \( g^{-1} h_{1,\sigma} g^\sigma = t_\sigma h_{2,\sigma} \) for all \( \sigma \). Since we are considering the action of \( \mathbf{Z}/2\mathbf{Z} \) on \( H^1(k, H^0_{\Phi(w)}) \), we may assume that \( h_{1,\sigma} = t_\sigma h_{2,\sigma} \) for all \( \sigma \). Since

\[
(t_\sigma h_{2,\sigma}, h_{2,\sigma}) = (t_\sigma, t^{-1}_\sigma)(h_{2,\sigma}, t_\sigma h_{2,\sigma}),
\]

\( \bar{h} = \bar{\psi}_2(h) \). This implies that \( \psi_3 \) induces the trivial action on \( H^1(k, H^0_{\Phi(w)}) \). Therefore, \( H^1(k, H^0_{\Phi}) \to H^1(k, H_\Phi) \) injective and the image is the inverse image of the trivial class of \( H^1(k, \mathbf{Z}/2\mathbf{Z}) \). This implies that the map \( G_k \setminus V^\text{ss} \to H_k \setminus (\text{Sym}^2 W)^\text{ss} \) induced by \( \Phi \) is injective.

Suppose that \( x = g_x w \in V^\text{ss}_k \). Then the class of \( (g_x^{-1} g_x^\sigma) \) is the element in \( H^1(k, G_w) = H^1(k, G^0_w) \) corresponding to the orbit of \( x \). Let \( h_{x,\sigma} = g_x^{-1} g_x^\sigma \) for \( \sigma \in \text{Gal}(k^\text{sep}/k) \). As in the proof of Proposition 5.13, \( G_{x,k} \) consists of elements \( g \in G_{w,k^\text{sep}} \) such that \( h_{x,\sigma} g^\sigma h_{x,\sigma}^{-1} = g \) for all \( \sigma \). Note that \( G_w \cong H^0_{\Phi(w)} \cong \text{GO}(\Phi(w)) \) (the identity component of the group of similitude of the quadratic form \( \Phi(w) \)). Since \( G_{w,k^\text{sep}} \supseteq g \mapsto h_{x,\sigma} g h_{x,\sigma}^{-1} \in G_{w,k^\text{sep}} \) is an inner automorphism, \( G_x \) is an inner form of \( \text{GO}(\Phi(w))^\circ \). Therefore, by associating \( G_x \) to \( x \in V^\text{ss}_k \), we obtain a bijective correspondence between \( G_k \setminus V^\text{ss}_k \) and \( \text{IQF}_4(k) \). \( \square \)

We consider the case (b) next.

Let \( G = \text{GL}_4 \times \text{GL}_3 \), \( V = \wedge^2 W \otimes \text{Aff}^3 \oplus W \). The action of \( G \) on \( V \) is the obvious one. We express elements of \( \wedge^2 \text{Aff}^4 \) by alternating matrices as in the case (a).

Let \( w_1, w_2, w_3 \) be as in (8.1), \( w_0 = [1, 0, 1, 1] \in \text{Aff}^4 \) and \( w = (w_1, w_2, w_3, w_0) \in V \). Elements of the form \( e_G + \varepsilon(A, B) \) \( A = (a_{ij}) \in M_4 \), \( B = (b_{ij}) \in M_3 \) fix \( (w_1, w_2, w_3) \) if and only if \( A, B \) are in the form (8.3). Then \( e_G + \varepsilon(A, B) \) fixes \( w_0 \) if and only if

\[
a_{11} = a_{21} + a_{13} = a_{31} + a_{12} = -a_{11} + a_{22} + a_{33} = 0.
\]

So \( A, B \) are in the following form:

\[
(8.17) \quad A = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ -a_{13} & a_{22} & 0 & a_{13} \\ -a_{12} & 0 & -a_{22} & a_{12} \\ 0 & -a_{12} & -a_{13} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -a_{22} & a_{12} & 0 \\ -2a_{13} & 0 & 2a_{12} \\ 0 & -a_{13} & a_{22} \end{pmatrix}.
\]

Therefore, \( \dim T_{e_G}(G_w) = 3 = 25 - 22 = \dim G - \dim V \). By Proposition 5.3, the orbit \( Gw \subset V \) is Zariski open and \( Gw \) is smooth over \( k \). Since this computation does not depend on \( \text{ch} (k) \), \( w \) is defined over \( \mathbf{Z} \) and is universally generic.

We define a map \( \Phi_1 : V \to \wedge^2 W \otimes \wedge^2 W \otimes \wedge^2 W \) by

\[
\Phi_1(x) = \sum_{\sigma \in \mathcal{S}_3} \text{sgn}(\sigma)x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}
\]

for \( x = (x_1, x_2, x_3, x_0)(x_1, x_2, x_3 \in \wedge^2 W, x_0 \in W) \).
For \( x_0 \in W \), we define a linear map \( \Phi_{2,x_0} : \wedge^2 W \otimes \wedge^2 W \otimes \wedge^2 W \rightarrow \wedge^4 W \otimes \wedge^4 W \) so that

\[
\Phi_{2,x_0}(a_1 \otimes a_2 \otimes (b_1 \wedge b_2)) = (a_1 \wedge b_1 \wedge x_0) \otimes (a_2 \wedge b_2 \wedge x_0) - (a_1 \wedge b_2 \wedge x_0) \otimes (a_2 \wedge b_1 \wedge x_0)
\]

for \( a_1, a_2 \in \wedge^2 W, b_1, b_2 \in W \).

We first consider over \( \mathbb{Q} \). We put \( P_1(x) = -\frac{1}{6} \Phi_{2,x_0}(\Phi_1(x)) \) for \( x = (x_1, x_2, x_3, x_0) \).

We identify two \( \wedge^4 W \)'s with \( \text{Aff}^1 \) so that \( p_{4,1234} \) corresponds to 1. Then \( P_1 \) can be regarded as a polynomial on \( V \) over \( \mathbb{Q} \). It is easy to see that \( P_1(x) \) is homogeneous of degree 3, 2 with respect to \( \wedge^2 W \otimes \text{Aff}^3, W \), respectively.

**Proposition 8.18.**

1. \( P_1(x) \) is defined over \( \mathbb{Z} \) and \( P_1(w) = 1 \).
2. \( P_1((g_1, g_2)x) = (\det g_1)^2(\det g_2)P_1(x) \) for \( (g_1, g_2) \in M_4 \times M_3 \).

**Proof.** (1) By long but straightforward computations, \( 6P_1(w) = 6 \). Since \( w \) is universally generic, \( P_1(x) \) is defined over \( \mathbb{Z} \) and \( P_1(w) = 1 \). (2) is easy. \( \square \)

By the natural homomorphism \( \mathbb{Z} \rightarrow k \), there is a polynomial \( P_1(x) \) over \( k \) such that \( P_1(w) = 1 \) and Proposition 8.18 (2) holds for \( (g_1, g_2) \in M_4 \times M_3 \). In particular, this holds for \( (g_1, g_2) \in GL_4 \times GL_3 \).

Let \( u = (u_1, u_2, u_3) \) be variables. For \( x \) as above, let \( \Phi_3(x) \) be the Pfaffian of \( u_1x_1 + u_2x_2 + u_3x_3 \). Then \( \Phi_3(x) = \Phi_3(x)(u) \in \text{Sym}^2\text{Aff}^3 \) and

\[
\Phi_3(gx)(u) = (\det g_1)^2(\det g_2)P_3(x)(ug_2)
\]

regarding \( u \) as a row vector. It is easy to see that \( \Phi_3(w) = u_1u_3 + u_2^2 \). It is known that there is a homogeneous polynomial \( \Phi_4 \) of degree 3 on the space of quadratic forms of \( u \) such that \( \Phi_4(u_1u_3 + u_2^2) = 1 \) and \( \Phi_4(hQ) = (\det h)^2\Phi_4(Q) \) for \( h \in GL_3, Q \in \text{Sym}^2\text{Aff}^3 \). We define \( P_2(x) = \Phi_4(\Phi_3(x)) \). Then

\[
P_2((g_1, g_2)x) = (\det g_1)^3(\det g_2)^2P_2(x).
\]

It is easy to see that \( P_2(x) \) is a homogeneous polynomial of degree 6. As we pointed out in Introduction, \( P_1(x), P_2(x) \) are more or less \( P_1(v), P_2(v) \) of [12, p.465].

We use the standard argument to determine the set of rational orbits. For that purpose, we have to determine the stabilizer of \( w \).

Let \( \rho : GL_2 \rightarrow GL(M_2) = GL_4 \) be the representation defined by \( GL_2 \times M_2 \ni (g, A) \mapsto gAg^{-1} \in M_2 \). We use the basis \( \{E_{11}, E_{21}, E_{12}, E_{22}\} \) as in the case (a) for \( M_2 \). This \( \rho \) coincides with \( \rho(g, t^g) \) of the case (a). So \( \wedge^2\rho \) leaves the subspace \( \langle w_1, w_2, w_3 \rangle \) invariant. We define \( \rho_1(g) \in GL_3 \) in the same manner as in Lemma 8.7. Since the coordinate of \( I_2 \in M_2 \) is \( [1, 0, 0, 1] \) and \( gI_2g^{-1} = I_2, \rho(g) \) fixes \( w_0 \). Therefore, we obtain the following lemma.

**Lemma 8.21.** The image of \( (\rho, t^g) \) fixes \( w \).

**Proposition 8.22.**

1. \( \text{Ker}(\rho, t^g) = \text{Ker}(\rho) \) is the center \( \mathbb{Z}(GL_2) \).
2. \( G_w = G_w^u = \text{Im}(\rho, t^g) \cong \text{PGL}_2 \).

**Proof.** (1) is easy. (2) By (1), \( \text{Im}(\rho, t^g) \cong \text{PGL}_2 \) and so \( \dim \text{Im}(\rho, t^g) = 3 \). We have shown in [8.17] that \( G_w \) is smooth and \( \dim G_w = 3 \). Therefore, \( G_w^u = \text{Im}(\rho, t^g) \).
To show that \( G_w = G_w^o \), we may assume that \( k = \overline{K} \). Suppose that \( g = (g_1, g_2) \in G_w \). Since \( \text{PGL}_2 \) has no outer automorphisms, by multiplying an element of \( G_w^o \), we may assume that \( g \) commutes with all elements of \( G_w^o \).

If \( t = \text{diag}(t_1, t_2) \) then
\[
(\rho(t), \rho_1(t)^{-1}) = (\text{diag}(1, t_1^{-1}t_2, t_1t_2^{-1}), \text{diag}(t_1^{-1}t_2, 1, t_1t_2^{-1})).
\]

Since \( g \) commutes with this element, \( g_2 \) must be diagonal and \( g_1 \) is in the form
\[
g_1 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}
\]
where \( h_{11}, \ldots, h_{22} \in M_2 \) and \( h_{11}, h_{22} \) are diagonal.

Since \( g \) fixes \( w \) and \( g_2 \) is diagonal, there exist scalars \( c_1, c_2, c_3 \) such that \( g_1 w_i^t g_1 = c_i w_i \). For \( i = 1, 3 \),
\[
\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} = c_1 \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},
\]
\[
\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} = c_3 \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.
\]

So \( h_{12} = h_{21} = 0 \). Therefore, \( g_1 \) is diagonal also. Now it is easy to deduce that \( g \in G_w^o \).

**Corollary 8.24.** \((G, V)\) is a regular prehomogeneous vector space.

Let \( U = \{x \in V \mid P_1(x), P_2(x) \neq 0\} \). We are in the situation of Section 8 where \( m = N = 2 \). So by Corollary 8.7, \( U_{k^\text{sep}} = G_{k^\text{sep}} w \). So we can use the standard argument of Galois cohomology. Since \( \text{PGL}_2 \) is the automorphism group of \( \text{PGL}_2 \) itself, \( H^1(k, \text{PGL}_2) \) is in bijective correspondence with \( \text{Pr}_g(k) \).

The proof of the following proposition is similar to that of Proposition 5.14 and we do not provide the proof.

**Proposition 8.25.** \( G_k \backslash U_k \) is in bijective correspondence with \( \text{Pr}_g(k) \). If \( x \in U_k \) the corresponding \( k \)-form of \( \text{PGL}_2 \) is the stabilizer \( G_x \).

9. Rational orbits (6)

Let (a) \( G = \text{GL}_3 \times \text{GL}_2^2 \times \text{GL}_1, V = \wedge^2 \text{Aff}^3 \oplus \text{Aff}^3 \otimes M_2, \) (b) \( G = \text{GL}_3 \times \text{GL}_2^3, V = \wedge^2 \text{Aff}^3 \otimes \text{Aff}^2 \oplus \text{Aff}^3 \otimes M_2. \)

We use the notation such as \( p_{3,12} \), etc.

We first consider the natural action of \( H = \text{GL}_3 \times \text{GL}_2^3 \) on \( W = \text{Aff}^3 \otimes M_2 \). Let \( H = \text{GL}_3 \times \text{GL}_2^2, W = \text{Aff}^3 \otimes M_2 \) and \( R = R_{322} \in W \) (see 4.10). We consider the Lie algebra of \( H_{\overline{R}} \). Let \( w_1, w_2, w_3 \) be \( B_1, B_2, B_3 \) of \( 4.9 \). Then \( R = (w_1, w_2, w_3) \). Let \( X = (x_{ij}) \in M_3, Y = (y_{ij}), Z = (z_{ij}) \in M_3. \) Then \( (e_H + \varepsilon(X, Y, Z)) R = R \) if and only if
\[
Y w_1 + w_1^t Z + x_{11} w_1 + x_{12} w_2 + x_{13} w_3 = 0,
\]
\[
Y w_2 + w_2^t Z + x_{21} w_1 + x_{22} w_2 + x_{23} w_3 = 0,
\]
\[
Y w_3 + w_3^t Z + x_{31} w_1 + x_{32} w_2 + x_{33} w_3 = 0.
\]

Straightforward considerations show the following lemma and so we do not provide the details.
Lemma 9.1. \( X, Y, Z \) are in the form
\[
X = \begin{pmatrix}
-y_{11} - z_{11} & -2y_{12} & -2y_{21} \\
-y_{21} & -y_{11} - z_{22} & 0 \\
y_{12} & 0 & -y_{22} - z_{11}
\end{pmatrix},
\]
\[
Y = \begin{pmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{pmatrix}, 
Z = \begin{pmatrix}
z_{11} & -y_{21} \\
-y_{12} & z_{22}
\end{pmatrix}
\]
where \( y_{11} + z_{11} = y_{22} + z_{22} \).

Since \( \dim T_{e_H}(H_R) = 5 = \dim H - \dim W \), we have the following corollary.

Corollary 9.2. \( H_R \subset W \) is Zariski dense and \( H_R \) is smooth over \( k \).

Let \( \rho : \text{GL}_2 \times \text{GL}_1 \rightarrow \text{GL}(M_2) \) be the homomorphism defined by \( \rho(g, t) A = t g A g^{-1} \) for \( g \in \text{GL}_2, t \in \text{GL}_1, A \in M_2 \). Let \( L \subset M_2 \) be the subspace spanned by \( w_1, w_2, w_3 \). Obviously, \( \dim L = 3 \).

Lemma 9.3. If \( g \in \text{GL}_2, t \in \text{GL}_1 \) then \( \rho(g, t)L \subset L \).

Proof. This lemma follows from the observation that \( L \) is the subspace of matrices of trace 0. \( \square \)

For \( (g, t) \in H \), let \( \rho_1(g, t) = (\rho_1(g, t)_{ij}) \in \text{GL}_3 \) be the matrix such that
\[
(\rho(g, t)w_1, \rho(g, t)w_2, \rho(g, t)w_3) = (w_1, w_2, w_3) \rho_1(g, t).
\]

We define a homomorphism \( \phi : \text{GL}_2 \times \text{GL}_1 \rightarrow H \) by
\[
(9.4) \quad \phi(g, t) = (t \rho_1(g, t)^{-1}, g, t g^{-1}).
\]

The proof of the following lemma is similar to that of Lemma 5.4.

Lemma 9.5. The image of \( \phi \) fixes \( R \).

Proposition 9.6. \( H_R = H_R^0 = \text{Im} (\phi) \).

Proof. It is easy to see that \( \phi \) is an imbedding and so \( \dim \phi(\text{GL}_2 \times \text{GL}_1) = 5 = \dim H_R \).

So \( H_R^0 = \text{Im} (\phi) \). Suppose that \( k = \overline{k} \) and \( g \in H_{Rk} \). Since \( \text{SL}_2 \subset \text{GL}_2 \times \text{GL}_1 \) has no outer automorphisms, by multiplying an element of \( \text{SL}_2 \) if necessary, we may assume that \( g \) commutes with elements of \( \phi(\text{SL}_2) \).

Note that if \( t \in \text{GL}_1 \) then with respect to the basis \( \{E_{11}, E_{21}, E_{12}, E_{22}\} \),
\[
\rho(\text{diag}(t, t^{-1}), 1) = \text{diag}(1, t^{-2}, t^2, 1).
\]

So
\[
\phi(\text{diag}(t, t^{-1}), 1) = \left( \text{diag}(1, t^2, t^{-2}), \text{diag}(t, t^{-1}), \text{diag}(t^{-1}, t) \right).
\]

Since \( g \) commutes with this element, all components of \( g \) are diagonal matrices. Then it is easy to show that \( g \in H_R^0 \). \( \square \)

If \( g = (g_{ij}) \in \text{GL}_2 \) then by computation,
\[
(9.7) \quad \rho_1(g, 1) = (\det g)^{-1} \begin{pmatrix}
g_{11}g_{22} + g_{12}g_{21} & g_{11}g_{21} & g_{12}g_{22} \\
2g_{11}g_{12} & g_{11}^2 & -g_{12}^2 \\
-2g_{21}g_{22} & -g_{21}^2 & g_{22}^2
\end{pmatrix}.
\]
We now consider the case (a). We express elements of $V$ as $(A,B)$ where $A = a_1p_{3,12} + a_2p_{3,13} + a_3p_{3,23}$.

\[
B_i = \begin{pmatrix} b_{i,11} & b_{i,12} \\ b_{i,21} & b_{i,22} \end{pmatrix}
\]

for $i = 1, 2, 3$ and $B = \mathbb{P}_{3,1} \otimes B_1 + \mathbb{P}_{3,2} \otimes B_2 + \mathbb{P}_{3,3} \otimes B_3$. We sometimes write $B = (B_1, B_2, B_3)$. If $x = (A, B) \in V$ then we write $A(x), B(x)$ for these $A, B$. We identify $\wedge^2 \text{Aff}^3$ with the space of $3 \times 3$ alternating matrices.

We consider the natural action of $\text{GL}_3 \times \text{GL}_2^2$ on $V$. The action of $t \in \text{GL}_1$ on $V$ is defined by $V \ni (A,B) \mapsto (tA, B)$. We shall show the existence of non-zero relative invariant polynomials first without any condition on $\text{ch}(k)$ and then show that $(G,V)$ is a regular prehomogeneous vector space assuming that $\text{ch}(k) \neq 2$.

Let

\[
\Phi(x) = a_1 B_3(x) - a_2 B_2(x) + a_3 B_1(x) \in M_2
\]

be the element obtained by applying Lemma 4.4 to $A(x), B(x)$. We put $P_1(x) = \det \Phi(x)$. Let $P_2(x)$ be the degree 6 polynomial of $B$ obtained by Proposition 4.7.

Let $g = (g_1, g_2, g_3, t) \in G$. Then $A(gx) = t(\wedge^2 g_1)A(x), B(gx) = g_1 \otimes (g_2, g_3)B(x)$ and

\[
\begin{align*}
\Phi(gx) &= t(\det g_1)(g_2, g_3)\Phi(x), \\
P_1(gx) &= t^2(\det g_1)^2(\det g_2)(\det g_3)P_1(x), \\
P_2(gx) &= (\det g_1)^2(\det g_2)^3(\det g_3)^3P_2(x).
\end{align*}
\]

(9.8)

Note that the action of $(g_2, g_3)$ on $M_2$ is $M_2 \ni M \mapsto g_1 M' g_2 \in M_2$. So $P_1(x), P_2(x)$ are relative invariant polynomials.

We put

\[
R_{322,3} = (p_{3,23}, R_{322}).
\]

(9.9)

It is easy to see that $\Phi(R_{322,3})$ is $B_1$ of (4.10). So $P_1(R_{322,3}) = -1$. By Proposition 4.4, $P_2(R_{322,3}) = 1$. We identify $p_{3,23}$ with

\[
A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\]

(9.10)

By the above consideration, $P_1, P_2 \neq 0$.

Let $T$ be the torus consisting of elements of the form

\[
t = \begin{pmatrix} (t_1 t_3)^{-1} & 0 & 0 \\ 0 & (t_1 t_2^{-1} t_3)^{-1} & 0 \\ 0 & 0 & (t_2 t_3)^{-1} \end{pmatrix}, \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \begin{pmatrix} t_3 & 0 \\ 0 & t_1 t_2^{-1} t_3 \end{pmatrix}, t_1 t_3^2.
\]

(9.11)

Elements of $T$ fix $R$.

**Proposition 9.12.** Suppose that $\text{ch}(k) \neq 2$.

(1) $(G,V)$ is a regular prehomogeneous vector space.

(2) $\{x \in V_{k^{\text{sep}}} | P_1(x), P_2(x) \neq 0\} = G_{k^{\text{sep}}} R_{322,3}$.

(3) $G^0_{R_{322,3}} = T$. 

Proof. We put $R = R_{322,3}$. We first determine the Lie algebra of the stabilizer $G_R$. Let $X = (X_{ij}) \in M_3, Y = (y_{ij}), Z = (z_{ij}) \in M_2, a \in \text{Aff}^1$. Suppose that $(1 + \varepsilon(X,Y,Z,a))R = R$. Then $X, Y, Z$ are in the form of Lemma 9.1. Since

$$X A + A^t X + a A = \begin{pmatrix} 0 & 2y_{21} & -2y_{12} \\ * & 0 & -2y_{11} - 2z_{11} + a \\ * & * & 0 \end{pmatrix} = 0,$$

$y_{12} = y_{21} = 0, a = 2y_{11} + 2z_{11}$. So

$$X = \text{diag}(-y_{11} - z_{11}, -y_{11} - z_{22}, -y_{22} - z_{11}), Y = \text{diag}(y_{11}, y_{22}), Z = \text{diag}(z_{11}, z_{22})$$

and $y_{11} + z_{11} = y_{22} + z_{22}, a = 2y_{11} + 2z_{11}$. This implies that $\dim T_{e_G}(G_R) = 3 = 18 - 15 = \dim G - \dim V$. Therefore, $GR \subset V$ is Zariski dense and $G_R$ is smooth over $k$. Since $\dim T = 3, G_R^o = T$. Therefore, $(G, V)$ is regular.

The assumption of Corollary 3.7 is satisfied and the second assertion of the proposition follows.

Let $\tau_0$ be as in (2.1) and

$$(9.13) \quad \tau_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau = (\tau_1, \tau_0, \tau_0, -1) \in G_k.$$

Then $\tau R = R$.

Lemma 9.14. $G_R/G_R^o$ is represented by $\{1, \tau\}$.

Proof. If $(g, h, h', t) \in G_R (g = (g_{ij}), h = (h_{ij}), h' = (h'_{ij}))$ then this element sends weight spaces (resp. trivial weight spaces) of $T$ to weight spaces (resp. trivial weight spaces). The subspaces spanned by $p_{3,12}, p_{3,13}, p_{3,23}$ are weight spaces and the weights of $t \in T$ in $\langle 9.11 \rangle$ are $t_1^{-1}t_2, t_1^{-1}t_2^{-1}$, 1 respectively. So $g$ leaves $\langle p_{3,12} \rangle$ invariant and either fixes or exchanges the subspaces $\langle p_{3,13} \rangle, \langle p_{3,13} \rangle$. Since $\tau$ exchanges $\langle p_{3,12} \rangle, \langle p_{3,13} \rangle$, by multiplying $\tau$ if necessary, we may assume that $g$ fixes the subspaces $\langle p_{3,12} \rangle, \langle p_{3,13} \rangle, \langle p_{3,23} \rangle$.

Since

$$gp_{3,12}^t g = \begin{pmatrix} 0 & g_{11}g_{22} - g_{12}g_{21} & g_{11}g_{32} - g_{12}g_{31} \\ -g_{11}g_{22} + g_{12}g_{21} & 0 & g_{21}g_{32} - g_{22}g_{31} \\ -g_{11}g_{32} + g_{12}g_{31} & -g_{21}g_{32} + g_{22}g_{31} & 0 \end{pmatrix},$$

$$gp_{3,13}^t g = \begin{pmatrix} 0 & g_{11}g_{23} - g_{13}g_{21} & g_{11}g_{33} - g_{13}g_{31} \\ -g_{11}g_{23} + g_{13}g_{21} & 0 & g_{21}g_{33} - g_{23}g_{31} \\ -g_{11}g_{33} + g_{13}g_{31} & -g_{21}g_{33} + g_{23}g_{31} & 0 \end{pmatrix},$$

$$gp_{3,23}^t g = \begin{pmatrix} 0 & g_{12}g_{23} - g_{13}g_{22} & g_{12}g_{33} - g_{13}g_{32} \\ -g_{12}g_{23} + g_{13}g_{22} & 0 & g_{22}g_{33} - g_{23}g_{32} \\ -g_{12}g_{33} + g_{13}g_{32} & -g_{22}g_{33} + g_{23}g_{32} & 0 \end{pmatrix},$$

we have

$$g_{11}g_{32} - g_{12}g_{31} = g_{21}g_{32} - g_{22}g_{31} = 0,$$

$$g_{11}g_{23} - g_{13}g_{21} = g_{21}g_{33} - g_{23}g_{31} = 0,$$

$$g_{12}g_{23} - g_{13}g_{22} = g_{12}g_{33} - g_{13}g_{32} = 0.$$
If $g_{21} \neq 0$ then $g_{32} = g_{21}^{-1}g_{31}g_{21}$, $g_{32} = g_{21}^{-1}g_{31}g_{22}$, $g_{33} = g_{21}^{-1}g_{31}g_{23}$, which implies that $\det g = 0$. So $g_{21} = 0$. If $g_{23} \neq 0$ then $g_{11} = g_{23}^{-1}g_{13}g_{21}$, $g_{12} = g_{23}^{-1}g_{13}g_{22}$, $g_{13} = g_{23}^{-1}g_{13}g_{23}$, which implies that $\det g = 0$. So $g_{23} = 0$. This implies that $g_{22} \neq 0$.

Since $g_{22}g_{31} = g_{13}g_{21} = 0$, $g_{31} = g_{13} = 0$. So $g_{11}, g_{33} \neq 0$. Since $g_{12}g_{33} = g_{11}g_{32} = 0$, $g_{12} = g_{32} = 0$. Therefore, $g$ is a diagonal matrix.

By Proposition 9.16 there exists $g' \in \text{GL}_2$ such that $(g, h, h') = \left(\rho_1(g')^{-1}, g', t g'^{-1}\right)$. Since $g$ is diagonal, $\rho_1(g')$ is diagonal. By (9.7), $g'$ must be diagonal. So $h = g', h' = h^{-1}$ are diagonal. Then it is easy to verify that $(g, h, h', t) \in G^*_R$.

We write $g = (g_1, g_2, g_3, t) \in G$, $x = (x_1, x_2) \in V$ where $x_2 \in \text{Aff}^3 \otimes \text{Aff}^2 \otimes \text{Aff}^2$. Then it is easy to see that

\begin{equation}
\Phi_4((g_1, g_2, g_3)x_2) = (\det g_1)(\det g_2)^2(\det g_3)(g_1, g_3)\Phi_4(x_2).
\end{equation}

We write $u_i = p_{2, i}$ for $i = 1, 2$. Note that $u_iu_j \in \text{Sym}^2\text{Aff}^2$ for all $i, j$. Long but straightforward computations show the following proposition. We do not provide the details.

**Lemma 9.16.** $\Phi_4(R_{322}) = 6p_{3, 1} \otimes u_1u_2 - 3p_{3, 2} \otimes u_2^2 + 3p_{3, 3} \otimes u_1^2$.

Let $\Phi_5 : V \to \text{Sym}^2\text{Aff}^2$ be the map which is obtained by applying Lemma 9.1 to the $\lambda^2 \text{Aff}^3$ component of $V$ and the $\text{Aff}^3$ factor of $\Phi_4$. The following proposition follows from (9.15) and Lemma 9.16.

**Proposition 9.17.** Suppose that $g = (g_1, g_2, g_3, t) \in G$ and $x \in V$.

1. $\Phi_5(gx) = t(\det g_1)^2(\det g_2)^2(\det g_3)g_5\Phi_5(x)$.
2. $\Phi_5(R) = 6u_1u_2$.

Note that $G, V, \Phi$ and $R_{322, 3}$ are defined over $\mathbb{Z}$. The element $R_{322, 3}$ is universally generic outside 2. So Proposition 3.10 implies the following corollary.

**Corollary 9.18.** If $\text{ch}(k) \neq 2$ then there is a map $\Phi : V \to \text{Sym}^2\text{Aff}^2$ over $k$ such that $\Phi(R) = u_1u_2$ and $\Phi(gx) = t(\det g_1)^2(\det g_2)^2(\det g_3)g_5\Phi(x)$.

**Proposition 9.19.** If $\text{ch}(k) \neq 2$ then $G_k \setminus \{x \in V_k \mid P_1(x), P_2(x) \neq 0\}$ is in bijective correspondence with $\text{Ex}_2(k)$ by associating to $x \in V_k$ such that $P_1(x), P_2(x) \neq 0$, the field generated over $k$ by a root of $\Phi(x)$.

**Proof.** Since $\{x \in V_{k^{\text{sep}}} \mid P_1(x), P_2(x) \neq 0\} = G_{k^{\text{sep}}}R$ and $H^1(k, G) = \{1\}$, by the standard argument of Galois cohomology,

$G_k \setminus \{x \in V_k \mid P_1(x), P_2(x) \neq 0\} \cong H^1(k, G_R)$. 

By Lemma 9.14 \( G_R/G_R^o \cong \mathbb{Z}/2\mathbb{Z} \) and \( \text{Gal}(k_{\text{sep}}/k) \) acts on \( \mathbb{Z}/2\mathbb{Z} \) trivially. So there is a natural map

\[
\alpha_V : G_k \setminus \{ x \in V_k \mid P_1(x), P_2(x) \neq 0 \} \cong H^1(k, G_R) \to H^1(k, \mathbb{Z}/2\mathbb{Z}) \cong \text{Ex}_2(k).
\]

To show that this map is bijective, for each \( F \in \text{Ex}_2(k) \) we choose a point \( x_F \in \alpha_V^{-1}(F) \) and prove that \( H^1(k, G_R^o) = \{ 1 \} \). Then LEMMA (1.8) \cite[p.120]{[21]} implies that \( \alpha_V \) is bijective. Moreover, we show that \( F \) coincides with the field generated over \( k \) by a root of \( \Phi(x_F) \).

Let \( F/k \) be a quadratic extension. Since \( \text{ch}(k) \neq 2 \), this is a Galois extension. Let \( f(z) = z^2 + a_1z + a_2 \in k[z] \) be a polynomial whose roots \( \alpha = (\alpha_1, \alpha_2) \) generate \( F \) over \( k \). Let \( \sigma \in \text{Gal}(F/k) \) be the non-trivial element. Let

\[
g_F = \begin{pmatrix}
(\alpha_1 - \alpha_2)^{-1} & 0 & 0 \\
0 & 1 & 1 \\
-\alpha_1 & -\alpha_2 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & 1 \\
-\alpha_1 & -\alpha_2 & 1 \\
-\alpha_1 & -\alpha_2 & (\alpha_1 - \alpha_2)^{-3}
\end{pmatrix}.
\]

We put \( x_F = g_FR \). Then by Corollary 9.18 \( \Phi(x_F) = (u_1 - \alpha_1 u_2)(u_1 - \alpha_2 u_2) = u_1^2 + a_1 u_1 u_2 + a_2 u_2^2 \) and so its roots generate \( F \) over \( k \).

It is easy to see that \( g_F^\sigma = g_F \tau \). Since \( \tau \in G_R \),

\[
(g_F)^\sigma R = g_F \tau R = g_F R = x_F.
\]

Therefore, \( x_F \in V_k \) and \( P_1(x_F), P_2(x_F) \neq 0 \). Since \( g_F^{-1} g_F^\sigma = \tau \), the cohomology class in \( H^1(k, \mathbb{Z}/2\mathbb{Z}) \) determined by \( x_F \) corresponds to the field \( F \).

We express \( t \) in (9.11) as \( a(t_1, t_2, t_3) \). Then the group of \( F \)-rational points of \( G_{x_F}^o \) is

\[
G_{x_F} = \{ g_F a(t_1, t_2, t_3) g_F^{-1} \mid t_1, t_2, t_3 \in F^x \}.
\]

Since \( (g_F a(t_1, t_2, t_3) g_F^{-1})^\sigma = g_F a(t_1^\sigma, t_2^\sigma, t_3^\sigma) \tau a(t_1, t_2, t_3) g_F^{-1} \in G_{x_F}^o \) if and only if \( \tau a(t_1^\sigma, t_2^\sigma, t_3^\sigma) = a(t_1, t_2, t_3) \). This is equivalent to the following conditions:

\[
t_1 t_3 \in k^x, \quad t_2 = t_1^\sigma, \quad t_1^2 t_2^{-1} t_3 = t_2^\sigma t_3, \quad t_3 = t_1 t_2^{-1} t_3.
\]

The last two conditions follow from the first two conditions. Note that the condition on the last component is \( (t_1 t_3)^2 \in \mathbb{Z}^x \), which follows from the above conditions. So the above conditions follow from the consideration of the \( \text{GL}_3 \times \text{GL}_2^\sigma \) part of the group.

By the above consideration, \( G_{x_F}^o \cong GL_1(k) \times (R_{F/k}GL_1)(k) \). We only considered \( k \)-rational points but considering rational points over all \( k \)-algebras as in \cite{[5]}, one can show that \( G_{x_F}^o \cong GL_1 \times (R_{F/k}GL_1) \) as algebraic groups. Also \( g_\alpha \tau g_\alpha^{-1} \in G_{x_F}^o \).

Therefore, \( H^1(k, G_{x_F}^o) = \{ 1 \} \). Hence \( \alpha_V \) is bijective.

Finally, we consider the case (b).

Let \( G = \text{GL}_3 \times GL_2^3 \), \( V = \wedge^2 \text{Aff}^3 \otimes \text{Aff}^2 \oplus \text{Aff}^3 \otimes \text{M}_2 \). We define an action of \( g = (g_1, g_2, g_3, g_4) \in G \) on \( V \) so that the action on \( \wedge^2 \text{Aff}^3 \otimes \text{Aff}^2 \) is by \( (\wedge^2 g_1, g_4) \) and the action on \( \text{Aff}^3 \otimes \text{M}_2 \) is the natural action of \( (g_1, g_2, g_3) \).

If \( x \in V \) then we write the \( \wedge^2 \text{Aff}^3 \otimes \text{Aff}^2 \) component of \( x \) as pairs \( A(x) = (A_1(x), A_2(x)) \) where \( A_1(x), A_2(x) \in \wedge^2 \text{Aff}^3 \). Let \( B(x) \) be the \( \text{Aff}^3 \otimes \text{M}_2 \) component of \( x \). Let \( R_1 = -p_{3,13}, R_2 = p_{3,12} \). We put

\[
(9.20) \quad R = (R_1, R_2, R_{322}).
\]
We define a map $\Phi : V \rightarrow \text{Aff}^2 \otimes M_2$, linear with respect to each of $\wedge^2 \text{Aff}^3 \otimes \text{Aff}^2$, $\text{Aff}^3 \otimes M_2$ so that
\[
\Phi(v_1 \otimes v_2, v_3 \otimes v_4) = (v_1 \wedge v_3) v_2 \otimes v_4
\]
for $v_1 \in \wedge^2 \text{Aff}^3, v_3 \in \text{Aff}^3, v_2, v_4 \in M_2$ ($p_{3,123}$ corresponds to 1).

Then for $x \in V, g = (g_1, g_2, g_3, g_4)$,
\[
(9.21) \quad \Phi(gx) = (\det g_1)(g_4, g_2, g_3)\Phi(x)
\]
where $(g_4, g_1, g_2)\Phi$ is the natural action.

Since $R_1 \wedge \mathbb{P}_{3,2} = R_2 \wedge \mathbb{P}_{3,3} = p_{3,123},$
\[
(9.22) \quad \Phi(R) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Let $\tau_0, \tau_1$ be as in (2.1), (9.13) respectively. We put $\tau = (\tau_1, \tau_0, \tau_0, \tau_0)$. It is easy to see that $\tau$ fixes $R$.

**Proposition 9.23.**  
(1) $GR \subset V$ is Zariski open and $G_R$ is smooth over $k$.

(2) $G^0_R$ consists of elements of the form
\[
\text{diag}(t_1^{-1}t_3, t_1^{-2}t_2t_3^{-1}, t_2^{-2}t_1^{-1}t_3^{-1}), \text{diag}(t_1, t_2), \text{diag}(t_3, t_1t_2^{-1}t_3), \text{diag}(t_1t_2t_3^2, t_1^{-1}t_2^{-1}t_3^2))
\]
where $t_1, t_2, t_3 \in \text{GL}_4$.

(3) $G_R/G^0_R$ is represented by $\{1, \tau\}$.

(4) $(G, V)$ is a regular prehomogeneous vector space.

**Proof.** (1) Let $e_G + \varepsilon(X, Y, Z, Q) \in T_{e_G}(G)$ where $X = (x_{ij}) \in M_3, Y = (y_{ij}), Z = (z_{ij}), Q = (q_{ij}) \in M_2$. Suppose that $e_G + \varepsilon(X, Y, Z, Q)$ fixes $R$. Since it fixes $R_{322}, X, Y, Z$ are in the form of *Lemma 9.1*. Regarding $R_1, R_2$ as $3 \times 3$ alternating matrices,
\[
XR_1 + R_1^tX + q_{11}R_1 + q_{12}R_2 = 0,
\]
\[
XR_2 + R_2^tX + q_{21}R_1 + q_{22}R_2 = 0.
\]
By computations, $x_{21} = x_{31} = 0$. By *Lemma 9.1* $Y$ is diagonal. This implies that $X, Z$ are diagonal also. By the above equalities, $Q$ is diagonal also and
\[
X = \begin{pmatrix} -y_{11} - z_{11} & 0 & 0 \\ 0 & -y_{11} - z_{22} & 0 \\ 0 & 0 & -y_{22} - z_{11} \end{pmatrix}, \quad Y = \begin{pmatrix} y_{11} & 0 \\ 0 & y_{22} \end{pmatrix}, \quad Z = \begin{pmatrix} z_{11} & 0 \\ 0 & z_{22} \end{pmatrix},
\]
\[
Q = \begin{pmatrix} y_{11} + y_{22} + 2z_{11} & 0 \\ 0 & 2y_{11} + z_{11} + z_{22} \end{pmatrix}, \quad y_{11} + z_{11} = y_{22} + z_{22}.
\]
Therefore, $\dim T_{e_G}(G_R) = 3 = 21 - 18 = \dim G - \dim V$ and (1) follows.

(2) Let $T \subset G$ be the subgroup consisting of elements of the form (2). It is easy to see that $T$ fixes $R$. Since $\dim T = 3, G^0_R = T$. So (4) follows.

(3) Suppose that $g = (g_1, g_2, g_3, g_4) \in G_R$. By (9.21), $(\det g_1)(g_4, g_2, g_3)$ fixes $\Phi(R)$. The stabilizer of $(1, 1, 1, 1)\Phi(R)$ is known and is generated by $(\tau_0, \tau_0, \tau_0)$ and elements of the stabilizer of $(1, 1, 1)\Phi(R)$ whose components are diagonal matrices. This implies that the stabilizer of $\Phi(R)$ is also generated by $(\tau_0, \tau_0, \tau_0)$ and elements whose components are diagonal matrices. So multiplying $\tau$ to $g$ if necessary, we may assume that $g_4, g_2, g_3$ are diagonal matrices. Since $(I_3, g_2, g_3)R_{322}$ is in the form $(c_1B_1, c_2B_2, c_3B_3)$ and
\[
(g_1, I_2, I_2)(c_1B_1, c_2B_2, c_3B_3) = R_{322} = (B_1, B_2, B_3),
\]
Let $P_1(x)$ be the degree 4 polynomial of $\Phi(x)$ obtained by Proposition 9.19. Then $P_1(R) = 1$. Let $P_2(x)$ be the degree 6 polynomial of $B(x)$ obtained by Proposition 4.7. Then $P_2(R) = 1$ and

\begin{align}
P_1(gx) = (\det g_1)^4(\det g_2)^2(\det g_3)^2(\det g_4)^2P_1(x), \\
P_2(gx) = (\det g_1)^2(\det g_2)^3(\det g_3)^3P_2(x).
\end{align}

(9.24)

Let $U = \{x \in V \mid P_1(x), P_2(x) \neq 0\}$.

**Proposition 9.25.** The map $\Phi$ induces a bijective correspondence $G_k \setminus U_k \cong (\text{GL}_2(k))^3/(k^2 \otimes k^2 \otimes k^2)^{\text{ss}} \cong \text{Ex}_2(k)$.

**Proof.** We proceed as in Proposition 9.19. We first show that $G_k \setminus U_k$ is in bijective correspondence with $H^1(k, \mathbb{Z}/2\mathbb{Z})$ where the action of the Galois group on $\mathbb{Z}/2\mathbb{Z}$ is trivial.

We are in the situation of Section 3 where $m = N = 2$. So by Corollary 3.7, $U_{\text{sep}} = G_{\text{sep}}R$. Since $H^1(k, G) = \{1\}$, $G_k \setminus U_k \cong H^1(k, G_R)$.

Let $F/k$ be a quadratic extension. Since $\text{ch}(k) \neq 2$, $F/k$ is a Galois extension. Suppose that $f(z) = z^2 + a_1z + a_2 \in k[z]$ is a polynomial whose roots $\alpha = (\alpha_1, \alpha_2)$ generate $F$ over $k$. Let $\sigma \in \text{Gal}(F/k)$ be the non-trivial element. Let

$$
g_{\alpha, 1} = \begin{pmatrix} (\alpha_1 - \alpha_2)^{-5} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -\alpha_1 & -\alpha_2 \end{pmatrix}, \quad g_{\alpha, 2} = \begin{pmatrix} 1 & 1 \\ -\alpha_1 & -\alpha_2 \end{pmatrix},
$$

$$
g_{\alpha} = (g_{\alpha, 1}, g_{\alpha, 2}, g_{\alpha, 2}, g_{\alpha, 2}), \ x_F = g_{\alpha}R.
$$

Since $g_{\alpha}^2 = g_{\alpha} \tau$, $x_F \in U_k$.

We determine the identity component of the stabilizer of $x_F$. Let $a(t_1, t_2, t_3)$ be the element in the statement (2) of Proposition 9.23. Then

$$
G_{x_F, F} = g_{\alpha}\{a(t_1, t_2, t_3) \mid t_1, t_2, t_3 \in F\}g_{\alpha}^{-1}.
$$

$g_{\alpha}a(t_1, t_2, t_3)g_{\alpha}^{-1} \in G_{x_F, k}$ if and only if

$$
(9.26) \quad \tau a(t_1^\sigma, t_2^\sigma, t_3^\sigma) = a(t_1, t_2, t_3).
$$

Since the first three components of $a(t_1, t_2, t_3)$ are the same as those of (9.11), $t_1t_3 \in k^\times$, $t_1^\sigma = t_2$. Then it is easy to verify that this implies (9.26). As in Proposition (2.10) [5 p.323] using Theorem [13 p.17], $G_{x_F, F} \cong GL_1 \times R_{F/k}GL_1$. Also $g_{\alpha} \tau g_{\alpha}^{-1} \in G_{x_F, k}$.

Since $g_{\alpha}^{-1}g_{\alpha}^\sigma = \tau$, the cohomology class corresponding to $x_F$ corresponds to $F$. Since $H^1(k, H_{x_F}) = \{1\}$ for all such $F$, $G_k \setminus U_k \cong H^1(k, \mathbb{Z}/2\mathbb{Z})$. By (9.21),

$$
\Phi(x_F) = (\det g_{\alpha, 1})(g_{\alpha, 2}, g_{\alpha, 2}, g_{\alpha, 2})\Phi(R) = (\alpha_1 - \alpha_2)^{-4}(g_{\alpha, 2}, g_{\alpha, 2}, g_{\alpha, 2})\Phi(R)
$$

So the cohomology class corresponding to $\Phi(x_F)$ is the class of the 1-cocycle

$$
(g_{\alpha, 2}, g_{\alpha, 2}, g_{\alpha, 2})^{-1}(g_{\alpha, 2}, g_{\alpha, 2}, g_{\alpha, 2})^\sigma = (\tau_0, \tau_0, \tau_0).
$$

Therefore, $\Phi(x_F)$ corresponds to the same cohomology class in $H^1(k, \mathbb{Z}/2\mathbb{Z})$. This proves the statement of this proposition. □
10. Rational orbits (7)

Let $G = \text{GL}_2^3$, (a) $V = \text{Aff}^2 \times \text{Aff}^2 \times \text{Aff}^2 \oplus \text{Aff}^2$ or (b) $V = \text{Aff}^2 \times \text{Aff}^2 \times \text{Aff}^2 \oplus \text{Aff}^2 \oplus \text{Aff}^2$.

We identify $\text{Aff}^2 \times \text{Aff}^2 \times \text{Aff}^2$ with the space of pairs of $2 \times 2$ matrices. We describe elements of $V$ as (a) $x = (A(x), B(x), v(x))$ where $A(x), B(x) \in M_2$ and $v(x) \in \text{Aff}^2$ is a column vector or (b) $x = (A(x), B(x), v_1(x), v_2(x))$ where $A(x), B(x) \in M_2$ and $v_1(x), v_2(x) \in \text{Aff}^2$ are column vectors. If $g_1, g_2 \in \text{GL}_2$ and $g_3 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then we define an action of $g = (g_1, g_2, g_3) \in G$ on $V$ by

(a) $gx = (\alpha g_1 A(x)^t g_2 + \beta g_1 B(x)^t g_2, \gamma g_1 A(x)^t g_2 + \delta g_1 B(x)^t g_2, g_1 v(x))$,

(b) $gx = (\alpha g_1 A(x)^t g_2 + \beta g_1 B(x)^t g_2, \gamma g_1 A(x)^t g_2 + \delta g_1 B(x)^t g_2, g_1 v_1(x), g_2 v_2(x))$.

If we only consider the action of $G$ on $W = \text{Aff}^2 \times \text{Aff}^2 \times \text{Aff}^2$, this is a regular prehomogeneous vector space and $G_k \backslash W^e$ is in bijective correspondence with $\text{Ex}_k$. Suppose that $F/k$ is a separable quadratic extension. We choose $a = (a_1, a_2) \in k^2$ so that $f_a(t) = t^2 + a_1 t + a_2 \in k[t]$ is irreducible with distinct roots $\alpha = (\alpha_1, \alpha_2)$ and that $F = k(\alpha_1)$. We put

$$w = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ x_F = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}, \ y = \begin{pmatrix} a_1 \\ a_1^2 - a_2 \end{pmatrix},$$

(10.1)

$$h_\alpha = \begin{pmatrix} 1 & -1 \\ -\alpha_1 & \alpha_2 \end{pmatrix}, \ \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $g_\alpha = (-\alpha_1 - \alpha_2)^{-1} h_\alpha, h_\alpha, h_\alpha$. Then $w$ (resp. $x_F$) corresponds to the trivial extension (resp. quadratic extension $F$) of $k$. Also $x_F = g_\alpha w$.

Let $\sigma \in \text{Gal}(F/k)$ be the non-trivial element. Then

$$G^r_{w,k} = \{(\text{diag}(t_1, t_2), \text{diag}(t_3, t_4), \text{diag}((t_1 t_3)^{-1}, (t_2 t_4)^{-1})) \mid t_1, \ldots, t_4 \in k^\times\},$$

$$G^r_{x_F,F} = g_\alpha \{(\text{diag}(t_1, t_2), \text{diag}(t_3, t_4), \text{diag}((t_1 t_3)^{-1}, (t_2 t_4)^{-1})) \mid t_1, \ldots, t_4 \in F^\times\} g_\alpha^{-1}.$$  

The above element of $G^r_{x_F,F}$ belongs to $G^r_{x_F,k}$ if and only if $t_2 = t_2^\sigma, t_4 = t_4^\sigma$.

Let $P_1(x)$ be the degree 4 polynomial of $(A(x), B(x))$ obtained by Proposition 4.1. In the case (a), let $\Phi : V \to \text{Aff}^2 \times \text{Aff}^2 \cong M_2$ be the map obtained by applying Lemma 6.5 to the first $\text{Aff}^2$-factor of $\text{Aff}^2 \times \text{Aff}^2 \times \text{Aff}^2$ and the last $\text{Aff}^2$. We put $P_2(x) = \det \Phi(x)$. In the case (b), let $\Phi_1, \Phi_2 : V \to \text{Aff}^2 \times \text{Aff}^2 \cong M_2$ be the map obtained by applying Lemma 6.5 to the first (resp. second) $\text{Aff}^2$-factor of $\text{Aff}^2 \times \text{Aff}^2 \times \text{Aff}^2$ and the next to the last (resp. last) $\text{Aff}^2$. We put $P_2(x) = \det \Phi_1(x), P_3(x) = \det \Phi_2(x)$. Then

$$P_1(gx) = (\det g_1)(\det g_2)(\det g_3)^2 P_1(x),$$

$$P_2(gx) = (\det g_1)(\det g_2)(\det g_3) P_2(x)$$

in both cases and $P_3(gx) = (\det g_1)(\det g_2)(\det g_3) P_3(x)$ in the case (b). Note that $P_1(x)$ (resp. $P_2(x)$ in the case (b)) is homogeneous of degree 2 with respect to each of $\text{Aff}^2 \times \text{Aff}^2 \times \text{Aff}^2$ and the first $\text{Aff}^2$ (the second $\text{Aff}^2$ in the case (b)). Let

$$w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ w_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ v_1 = v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
We put (a) \( w = (w_1, w_2, v_1) \) (b) \( w = (w_1, w_2, v_1, v_2) \). We determine the Lie algebra \( T_{e_G}(G_w) \). Let \( A = (a_{ij}), B = (b_{ij}), C = (c_{ij}) \in M_2 \). Suppose that \((e_G + e(A, B, C))w = w\). By considering the action on \((w_1, w_2)\), \(A, B, C\) are diagonal matrices. Then \(A, B, C\) are in the form \( A = \text{diag}(a_1, a_2), B = \text{diag}(b_1, b_2), C = \text{diag}(-a_1 - b_1, -a_2 - b_2) \).

In the case (a), \((e_G + e(A, B, C))v_1 = v_1\) if and only if \(Av_1 = 0\). This is equivalent to \(A = 0\). In the case (b), \((e_G + e(A, B, C))v_i = v_i\) for \(i = 1, 2\) if and only if \(Av_1 = 0, Bv_2 = 0\). This is equivalent to \(A = B = 0\). So \(A = 0, B = \text{diag}(b_1, b_2), C = -B\) in the case (a) and \(A = B = C = 0\) in the case (b). Therefore,

\[
\dim T_{e_G}(G_w) = \begin{cases} 2 = 12 - 10 = \dim G - \dim V & \text{case (a)}, \\ 0 = 12 - 12 = \dim G - \dim V & \text{case (b)}. \end{cases}
\]

This implies that \(Gw \subset V\) is Zariski dense in both cases and \(G_w\) is smooth over \(k\).

**Proposition 10.3.** (1) \(G_w^0 = \{(I_2, \text{diag}(t_1, t_2), \text{diag}(t_1^{-1}, t_2^{-1}))\}\) in the case (a) and \(G_w^0 = \{e_G\}\) in the case (b).

(2) \((G, V)\) is a regular prehomogeneous vector space in both cases.

**Proof.** (1) It is easy to see that the right hand side is contained in \(G_w^0\). Since the dimensions are the same, (1) follows.

(2) follows from (1). \(\square\)

Let \(\tau = (\tau_0, \tau_0, \tau_0)\) (see (2.1)). It is easy to see that \(\tau\) fixes \(w\).

**Proposition 10.4.** \(G_w/G_w^0\) is represented by \(\{1, \tau\}\).

**Proof.** Suppose that \(g = (g_1, g_2, g_3) \in G_w\). Since \(g\) fixes \((w_1, w_2)\), by multiplying \(\tau\) if necessary, we may assume that \(g_1, g_2, g_3\) are diagonal matrices. Then it is easy to show that \(g \in G_w^0\). \(\square\)

We put

\[
U = \begin{cases} \{x \in V_k \mid P_1(x), P_2(x) \neq 0\} & \text{case (a)}, \\ \{x \in V_k \mid P_1(x), P_2(x), P_3(x) \neq 0\} & \text{case (b)}. \end{cases}
\]

**Proposition 10.5.** The map \(V \ni x \mapsto (A(x), B(x)) \in \text{Aff}^2 \otimes \text{Aff}^2 \otimes \text{Aff}^2\) induces a bijection \(G_k \backslash U \rightarrow (\text{GL}_2(k)^3) \backslash (k^2 \otimes k^2 \otimes k^2)^{ss} \cong \text{Ex}_2(k)\).

**Proof.** Note that the set \(U\) is \(G_k\)-invariant. We show that if \(x \in U\) then \(G_kx\) is determined by the orbit of \((A(x), B(x))\). So we assume that \((A(x), B(x))\) is \(w\) or \(x_F\).

Since the consideration is similar, we only consider the case (b). The case (a) is easier. Suppose that \(x \in U\). We first consider the case \((A(x), B(x)) = w\). Let \(v_1(x) = [x_1, x_2], v_2(x) = [x_3, x_4]\). Since \(\Phi_1(w, v_1(x)) = \left(\begin{smallmatrix} x_2^2 & 0 \\ 0 & -x_1 \end{smallmatrix}\right)\), \(x_1, x_2 \neq 0\). Similarly, \(x_3, x_4 \neq 0\). If \(t = (\text{diag}(x_1^{-1}, x_2^{-1}), \text{diag}(x_3^{-1}, x_4^{-1}), \text{diag}(x_1x_3, x_2x_4))\) then \(tx = (w, [1, 1], [1, 1])\). So \(x\) is in the orbit of \((w, [1, 1], [1, 1])\).

We next consider the case \((A(x), B(x)) = x_F\). Let \(a, \alpha\) be as in (10.1) and \(F = k(\alpha_1)\). Let \(\sigma \in \text{Gal}(F/k)\) be the non-trivial element. We express \(v_1(x), v_2(x)\) in the form \((-\alpha_1 - \alpha_2)^{-1}h_\alpha[x_1, x_2], h_\alpha[x_3, x_4]\) respectively where \(x_1, x_2, x_3, x_4 \in F\). Then \(v_1(x), v_2(x) \in k^2\) if and only if \(x_2 = x_3, x_4 = -x_3^\sigma\). Since \(P_2(x), P_3(x) \neq 0\), \(x_1, x_3 \neq 0\). If \(t = g_\alpha(\text{diag}(x_1^{-1}, (x_1^{-1})^\sigma), \text{diag}(x_3^{-1}, (x_3^{-1})^\sigma), \text{diag}(x_1x_3, x_1x_3^\sigma))g_\alpha^{-1}\) then \(t \in G_k\) and

\[
tx = (x_F, -(\alpha_1 - \alpha_2)^{-1}h_\alpha[1, 1], h_\alpha[1, -1]) = (x_F, [0, 1], [2, a_1])\).
So \( x \) is in the orbit of \( (x_F, [0, 1], [2, a_1]) \).

11. Non-empty strata

In this section and the next two sections, we assume that \( k \) is a fixed perfect field. The set \( \mathcal{B} \) consists of 292 \( \beta \)'s. We use the table in Section 8 [17]. In this section we shall prove that \( S_{\beta_i} \neq \emptyset \) for

\[
i = 1, 3, 5, 9, 14, 15, 16, 20, 21, 33, 35, 36, 37, 40, 42, 49, 50, 64, 70, 71, 75, 95, 101, 105, 106, 107, 108, 110, 113, 121, 131, 149, 150, 151, 152, 164, 178, 202, 216, 217, 223, 224, 226, 227, 232, 254, 256, 258, 259, 270, 271, 272, 273, 280, 281, 285, 286, 287, 289, 291, 292
\]

(11.1)

for the prehomogeneous vector space \( (11.1) \). We shall prove that \( S_{\beta_i} = \emptyset \) for other \( \beta \)'s in the next section.

The following table describes \( M_{\beta_i}, Z_{\beta_i} \) as a representation of \( M_{\beta_i}^k \), the coordinates of \( Z_{\beta_i}, W_{\beta_i} \) and \( G_k \setminus S_{\beta_i} \cong P_{\beta_i} k \setminus Y_{\beta_i}^8 \).

| \( i \) | \( M_{\beta_i}, \text{coordinates of } Z_{\beta_i}, \text{coordinates of } W_{\beta_i} \) |
|---|---|
| 1 | \( M_{[0],1} \cong \text{GL}_5 \times \text{GL}_3 \times \text{GL}_1, \quad A^5_{1,1}[2,2] \otimes A^3_{1,1}[2,4] \) |
| 1 | \( \text{Ex}_2(k) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots, 20 \} \) |
| 2 | \( M_{[2],1} \cong \text{GL}_5 \times \text{GL}_3 \times \text{GL}_1, \quad A^5_{1,1}[3,3] \otimes A^3_{1,1}[2,4] \otimes A^3_{1,1}[2,4] \) |
| 3 | \( \text{Ex}_3(k) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots, 20 \} \) |
| 4 | \( M_{[4],2} \cong \text{GL}_5 \times \text{GL}_3 \times \text{GL}_1, \quad A^5_{1,1}[4,4] \otimes A^3_{1,1}[2,4] \otimes A^3_{1,1}[2,4] \) |
| 5 | \( \text{Ex}_2(k) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots, 20 \} \) |
| 6 | \( M_{[1],[3]} \cong \text{GL}_5 \times \text{GL}_3 \times \text{GL}_1, \quad A^5_{1,1}[2,2] \otimes A^3_{1,1}[2,4] \otimes A^3_{1,1}[2,4] \) |
| 7 | \( \text{Ex}_3(k) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots, 20 \} \) |
| 8 | \( M_{[2],[2]} \cong \text{GL}_5 \times \text{GL}_3 \times \text{GL}_1, \quad A^5_{1,1}[2,2] \otimes A^3_{1,1}[2,4] \otimes A^3_{1,1}[2,4] \) |
| 9 | \( \text{Ex}_2(k) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots, 20 \} \) |
| 10 | \( M_{[3],[3]} \cong \text{GL}_5 \times \text{GL}_3 \times \text{GL}_1, \quad 1 \otimes A^3_{1,1}[2,2] \otimes A^3_{1,1}[2,4] \otimes A^3_{1,1}[2,4] \) |
| 11 | \( \text{Ex}_2(k) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots, 20 \} \) |
| 12 | \( \text{Ex}_2(k) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots, 20 \} \) |
|   |   |
|---|---|
| 16 | $M_{[3],0} \cong \text{GL}_4 \times \text{GL}_4 \times \text{GL}_2$, $\Lambda^{2,1}_{[1,4]} \otimes \Lambda^{3,1}_{[1,3]} \otimes \Lambda^{2,1}_{[1,4]}$ |
| SP | $3, 4, 6, 7, 8, 9, 13, \ldots, 39$ |
|   | $10, 20, 33, 40$ |
| 20 | $M_{[3],[3]} \cong \text{GL}_3 \times \text{GL}_4$, $\Lambda^{2,1}_{[1,3]} \oplus \Lambda^{2,1}_{[1,2]} \oplus 1 \oplus \Lambda^{2,1}_{[1,3]} \oplus \Lambda^{2,1}_{[1,4]} \oplus \Lambda^{2,1}_{2,[3,4]}$ |
| SP | $9, 10$ |
|      | $14, 17, 21$ |
| Ex$_2(k)$ | $10, 13, 15, 17, 21, 27, 31, 35, 39$ |
|      | $22, 23, 25, 26, 32, 33, 35, 36$ |
|      | $28, 38, 29, 30, 39, 40$ |
| 21 | $M_{[2],[4],[1,2,3]} \cong \text{GL}_2 \times \text{GL}_4$, $\Lambda^{2,1}_{[1,4]} \oplus \Lambda^{2,1}_{1,[2,3]} \oplus \Lambda^{2,1}_{2,[2,3]} \oplus \Lambda^{2,1}_{[1,2]} \oplus \Lambda^{2,1}_{2,[1,2]} \oplus \Lambda^{2,1}_{1,1,2}$ |
| SP | $9, 10$ |
|      | $14, 17, 21, 27$ |
|      | $18, 28$ |
|      | $32, 33, 35, 36$ |
| Ex$_2(k)$ | $18, 23, 24, 34, 35, 44, 45$ |
|      | $24, 25, 26, 32, 33, 35, 36$ |
|      | $35, 36, 37, 38, 39, 40$ |
| 33 | $M_{[1],[2,4],[1,2,3]} \cong \text{GL}_2 \times \text{GL}_4$, $\Lambda^{2,1}_{[1,4]} \oplus 1 \oplus (\Lambda^{2,1}_{[1,3]} \oplus \Lambda^{2,1}_{[1,4]})$ |
| SP | $9, 10$ |
|      | $17, 18$ |
|      | $24$ |
| Ex$_2(k)$ | $23, 24, 33, 34$ |
|      | $25, 36, 37, 38, 39, 40$ |
| 36 | $M_{[1],[3],[1,2,3]} \cong \text{GL}_2 \times \text{GL}_4$, $1 \oplus \Lambda^{2,1}_{[1,2,3]} \oplus \Lambda^{2,1}_{1,1,4} \oplus \Lambda^{2,1}_{2,2,3} \oplus \Lambda^{2,1}_{1,1,2}$ |
| Ex$_2(k)$ | $6, 7, 8, 9, 10, 15, \ldots, 40$ |
|      | $23, 24, 33, 34$ |
|      | $25, 35$ |
|      | $30, 40$ |
| 37 | $M_{[1],[3],[2]} \cong \text{GL}_2 \times \text{GL}_4$, $\Lambda^{2,1}_{[1,2,3]} \oplus \Lambda^{2,1}_{1,1,4} \oplus \Lambda^{2,1}_{2,2,3} \oplus \Lambda^{2,1}_{1,1,2}$ |
| Ex$_2(k)$ | $7, 9$ |
|      | $14, 24$ |
|      | $10$ |
|      | $17, 19, 27, 29$ |
|      | $20, 30$ |
| 40 | $M_{[1],[3],[1,2,3]} \cong \text{GL}_2 \times \text{GL}_4$, $1 \oplus \Lambda^{2,1}_{[1,2,3]} \oplus \Lambda^{2,1}_{1,1,4} \oplus \Lambda^{2,1}_{2,2,3} \oplus \Lambda^{2,1}_{1,1,2}$ |
| Ex$_3(k)$ | $10$ |
|      | $13, 14, 16, 17, 18, 19, 23, \ldots, 39$ |
|      | $20, 30, 40$ |
| 49 | $M_{[1],[3],[1,2,3]} \cong \text{GL}_2 \times \text{GL}_4$, $\Lambda^{2,1}_{[1,3]} \oplus \Lambda^{2,1}_{1,2,4} \oplus \Lambda^{2,1}_{1,4} \oplus \Lambda^{2,1}_{4}$ |
| SP | $10, 17, 18, 21$ |
|      | $26, 33, 35$ |
|      | $34, 36, 37, 38, 39, 40$ |
| 50 | $M_{[1],[3],[1,2,3]} \cong \text{GL}_2 \times \text{GL}_4$, $\Lambda^{2,1}_{[1,3]} \oplus \Lambda^{2,1}_{1,2,4} \oplus \Lambda^{2,1}_{1,4} \oplus \Lambda^{2,1}_{4}$ |
| k | \( M_{[k],1} \) & \( \cong GL_3 \times GL_2 \times GL_1 \) & \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) & \( \cong GL_3 \times GL_2 \times GL_1 \) & \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) & \( \cong GL_3 \times GL_2 \times GL_1 \) & \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
|---|---|---|---|---|---|---|
| 64 | \( M_{[4],1,3} \) | \( \cong GL_4 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_4 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_4 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
| SP | 14, 17, 20, 24, \ldots, 29 | 31, 32, 33, 35, 36, 38 | 34, 37, 39, 40 |

| 70 | \( M_{[2],1,3} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
| SP | 18, 21, 24, 27, 32, 34, \ldots, 40 |

| 71 | \( M_{[3],1,2} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
| SP | 20 |

| 95 | \( M_{[1,3],1,2,2} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
| SP | 10 |

| 101 | \( M_{[2,2],1,3,1} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
| SP | 40 |

| 105 | \( M_{[3],1,3,1} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
| SP | 10 |

| 107 | \( M_{[1,1],1,1,1} \) | \( \cong GL_1 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_1 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_1 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
| SP | 10 |

| 109 | \( M_{[2],1,3,2} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_2 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
| SP | 8, 9, 18, 19 |

| 111 | \( M_{[3],1,3,1} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
| SP | 40 |

| 113 | \( M_{[1,3],1,1,2} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) | \( \cong GL_3 \times GL_2 \times GL_1 \) | \( 1 \oplus A_{1,1}^{1,1} \otimes A_{1,2}^{2,1} \otimes A_{1,3}^{3,1} \) |
| SP | 10 |
| 121 | $M_{[2,4],[2,3]} \cong \text{GL}^1_{\mathbb{A}_1, [2,3]} \otimes \text{GL}^3_{\mathbb{A}_1, [2,3]} \otimes \mathbb{A}^3_{1, [3,4]} \oplus 1$ | SP |
| 131 | $M_{[1,1]} \cong \text{GL}^4 \times \text{GL}^3 \times \text{GL}^1_{\mathbb{A}_1, [2,3]} \oplus \mathbb{A}^3_{1, [2,3]} \otimes \mathbb{A}^3_{2, [2,4]}$ | IQF\(_4(k)\) |
| 149 | $M_{[1,3,4],[1,2,3]} \cong \text{GL}^9_{\mathbb{A}_1, [1]}$ | SP |
| 150 | $M_{[1,3,4],[1,2,3]} \cong \text{GL}^9_{\mathbb{A}_1, [2]}$ | SP |
| 151 | $M_{[1,3,4],[1,2,3]} \cong \text{GL}^9_{\mathbb{A}_1, [3]}$ | SP |
| 152 | $M_{[1,3,4],[1,2,3]} \cong \text{GL}^9_{\mathbb{A}_1, [4]}$ | SP |
| 164 | $M_{[3],[2]} \cong \text{GL}^4 \times \text{GL}^3 \times \text{GL}^1_{\mathbb{A}_1, [3,4]} \oplus \mathbb{A}^3_{1, [3,4]} \otimes \mathbb{A}^3_{2, [2,3]} \otimes \mathbb{A}^3_{2, [2,4]}$ | SP |
| 178 | $M_{[1,4],[1,2]} \cong \text{GL}^4 \times \text{GL}^3_{\mathbb{A}_1, [2,3]} \oplus \mathbb{A}^3_{1, [2,4]} \otimes \mathbb{A}^3_{2, [2,3]} \oplus \mathbb{A}^3_{2, [2,4]}$ | SP |
| 202 | $M_{[2,3],[1,3]} \cong \text{GL}^4 \times \text{GL}^3_{\mathbb{A}_1, [1,3]} \oplus \mathbb{A}^3_{1, [2,3]} \otimes \mathbb{A}^3_{2, [2,3]} \oplus \mathbb{A}^3_{2, [2,4]} \otimes \mathbb{A}^3_{2, [2,4]}$ | SP |
| 216 | $M_{[1,3],[1,2]} \cong \text{GL}^4 \times \text{GL}^3_{\mathbb{A}_1, [1,2]} \oplus \mathbb{A}^3_{1, [1,3]} \otimes \mathbb{A}^3_{2, [1,2]} \otimes \mathbb{A}^3_{2, [1,3]}$ | Ex\(_2(k)\) |
| 217 | $M_{[2,4],[1,3]} \cong \text{GL}^4 \times \text{GL}^3_{\mathbb{A}_1, [1,3]} \oplus \mathbb{A}^3_{1, [2,3]} \otimes \mathbb{A}^3_{2, [1,3]}$ | SP |
| 223 | $M_{[1,4],[2,3]} \cong \text{GL}^4 \times \text{GL}^3_{\mathbb{A}_1, [2,3]} \otimes \mathbb{A}^3_{1, [3,4]} \oplus \mathbb{A}^3_{2, [2,3]} \otimes \mathbb{A}^3_{2, [2,4]}$ | SP |
| \(M_{[1,2,4],[1,3,4]} \cong GL^2_4 \times GL^2_4\), \(A^2_{1,[1,3,4]} \otimes A^2_{2,[1,3,4]} \oplus 1^{2\mathbb{B}}\) |
|---|---|---|---|
| \(9, 10, 19, 20\) \(27, 28, 31\) | \(29, 30, 32, 33, 35, 36\) | \(x_{351}, x_{352}, x_{353}, x_{354}\) |
| \(x_{351}, x_{352}, x_{353}, x_{354}\) | | \(x_{124}\) |

| \(M_{[1,2,3,4],[1,2]} \cong GL^2_3 \times GL^2_3\), \(1^{3\mathbb{B}} \oplus (A^2_{1,[2,3]} \otimes 1^{2\mathbb{B}})\) |
|---|---|---|---|
| \(9, 10, 17, 18\) \(24, 26, 34, 36, 39, 40\) | \(29, 30, 32, 33, 35, 36\) | \(x_{352}, x_{353}, x_{354}\) |
| \(x_{351}, x_{352}, x_{353}, x_{354}\) | | \(x_{244}\) |

| \(M_{[1,2,3,4],[1,2]} \cong GL^2_3 \times GL^2_3\), \(A^2_{1,[3,4]} \otimes 1^{3\mathbb{B}} \oplus A^2_{2,[3,4]} \otimes 1^{2\mathbb{B}}\) |
|---|---|---|---|
| \(9, 10, 17, 28\) \(34, 35, 36, 39, 40\) | \(37, 38, 39, 40\) | \(x_{351}, x_{352}, x_{353}, x_{354}\) |
| \(x_{351}, x_{352}, x_{353}, x_{354}\) | | \(x_{244}\) |

| \(M_{[1,3,4],[1,3]} \cong GL^2_4 \times GL^2_4\), \(1 \oplus A^2_{1,[1,2,3]} \otimes A^2_{2,[1,2,3]} \oplus 1^{2\mathbb{B}}\) |
|---|---|---|---|
| \(10, 17, 19, 23, 30\) | \(20, 22, 24, 26, 30\) | \(19, 20, 27, 30\) | \(37, 39, 40\) |
| \(x_{351}, x_{352}, x_{353}, x_{354}, x_{244}\) | | \(x_{351}, x_{352}, x_{353}, x_{354}\) |

| \(M_{[1,2,3],[1,3]} \cong GL^3_4 \times GL^3_4\), \(A^2_{1,[3,4]} \otimes A^2_{2,[3,4]} \otimes A^3_{1,[3,4]} \otimes A^3_{2,[3,4]} \oplus 1^{2\mathbb{B}}\) |
|---|---|---|---|
| \(18, 19, 20, 23, 30\) | \(32, 33, 34, 35, 36, 37\) | \(38, 39, 40\) | \(40\) |
| \(x_{351}, x_{352}, x_{353}, x_{354}, x_{355}, x_{244}\) | | \(x_{351}, x_{352}, x_{353}, x_{354}, x_{355}\) |

| \(M_{[1,2],[1,3]} \cong GL^4_3 \times GL^4_3\), \(A^2_{1,[1,3]} \otimes A^2_{1,[3,4]} \oplus 1^{2\mathbb{B}}\) |
|---|---|---|---|
| \(25, 26, 27, 28, 29, 30\) | \(32, 33, 34, 35, 36, 37\) | \(38, 39, 40\) |
| none | | \(x_{351}, x_{352}, x_{353}, x_{354}\) |

| \(M_{[1,2,3],[1,2,3]} \cong GL^2_4 \times GL^2_4\), \(1 \oplus A^2_{1,[1,2,3]} \oplus 1 \oplus A^2_{1,[1,2,3]}\) |
|---|---|---|---|
| \(20, 27, 29\) | \(34, 36, 38\) | \(30, 37, 39\) | \(40\) |
| \(x_{351}, x_{352}, x_{353}, x_{354}\) | | \(x_{351}, x_{352}, x_{353}, x_{354}\) |

| \(M_{[1,2,4],[1,4]} \cong GL^2_4 \times GL^2_4\), \(A^4_{1,[1,4]} \otimes A^4_{2,[1,4]}\) |
|---|---|---|---|
| \(4, 7, 9, 10, 14, \ldots, 40\) | none | | |
| \(x_{151}, x_{152}, x_{153}, x_{154}, \ldots, x_{454}\) | | | |

| \(M_{[1,2,3],[1,3]} \cong GL^3_4 \times GL^3_4\), \(1 \oplus A^2_{1,[1,2]} \otimes A^2_{2,[1,2]} \otimes (A^2_{1,[3,4]} \otimes 1^{2\mathbb{B}})\) |
|---|---|---|---|
| \(9, 10, 19, 20\) | \(27, 37\) | \(28, 38\) | \(29, 30, 39, 40\) |
| \(x_{351}, x_{352}, x_{353}, x_{354}\) | | \(x_{351}, x_{352}, x_{353}, x_{354}\) |

| \(M_{[2,3],[1,3]} \cong GL^2_4 \times GL^2_4\), \(A^2_{1,[1,3]} \otimes A^2_{1,[1,3]} \oplus 1\) |
|---|---|---|---|
| \(8, 9, 10, 18, \ldots, 30\) | \(34, 35, 36, 37\) | \(38, 39, 40\) |
| \(x_{351}, x_{352}, x_{353}, x_{354}, x_{355}\) | | | |

| \(M_{[2,3],[1,3]} \cong GL^2_4 \times GL^2_4\), \(1 \oplus (A^2_{1,[1,3]} \otimes 1^{2\mathbb{B}})\) |
|---|---|---|---|
| \(30, 36, 38, 37, 39\) | \(40\) | | |
| \(x_{351}, x_{352}, x_{353}, x_{354}, x_{354}\) | | | |
We now verify that $S_{β_i} \neq \emptyset$ and determine $G_k \setminus S_{β_k}$ for the above $i$’s.
In the following, we write $S_i, Z_i, W_i, Y_i, χ_i$ instead of $S_{β_i}, Z_{β_i}, W_{β_i}, Y_i, χ_{β_i}$ (but we keep writing $M_{β_i}$, etc.).

**Proposition 11.2.** If $β \in ℳ$ and $M_β = T$ then $Z^{ss}_β \neq \emptyset$.

**Proof.** By the definition of $ℳ$, there exist $1 \leq i_1 < \cdots < i_N \leq 40$ such that $β$ is the closest point to the origin of the convex hull of their weights $γ_{i_1}, \ldots, γ_{i_N} \in X^*(T \cap G_{st})$. We may assume that this convex hull is contained in the hyperplane $\{t \in t^* \mid (t, β)_* = (β, β)_*\}$ ($\langle \cdot, \cdot \rangle_*$ is the inner product on $t^*$ which is used to define $ℳ$, etc.). So $a_{i_1}, \ldots, a_{i_N} \in Z_β$ and there exist rational numbers $0 < c_1, \ldots, c_N < 1$ such that $β = c_1 γ_{i_1} + \cdots + c_N γ_{i_N}, c_1 + \cdots + c_N = 1$.

There exists an integer $d_1 > 0$ such that $d_1 β$ is actually a character $χ$ (i.e. an element of $X^*(T \cap G_{st})$). We take a common denominator $d_2$ of $c_1, \ldots, c_N$ and express $c_i$ as $b_i/d_2$ for $i = 1, \ldots, N$. Then $d_1 b_1 γ_{i_1} + \cdots + d_1 b_N γ_{i_N} = d_2 χ$. Writing multiplicatively, $\prod_{j=1}^{N} γ_{i_j}(t)^{d_{ij}} = χ(t)^{d_2}$ for $t \in T \cap G_{st}$. Since $χ$ is proportional to $β$, $χ(t) = 1$ for $t \in G_{st, β}$. Therefore, if we put $P(x) = \prod_{j=1}^{N} x_{ij}^{d_{ij}}$ then $P(x)$ is a non-constant polynomial on $Z_β$ which is invariant under the action of $G_{st, β}$. □

We consider the situation of the above proposition. We express elements of $T$ as

$$t = (\text{diag}(t_1, \ldots, t_5), \text{diag}(t_6, \ldots, t_9)).$$

Let $δ_1, \ldots, δ_9 \in X^*(T)$ be the characters such that $δ_i(t) = t_i$. Then \{δ_1, \ldots, δ_9\} is a $Z$-basis of $X^*(T)$. Suppose that $Z_β = \{a_{i_1}, \ldots, a_{i_N}\}$ and that $χ_{i_1}, \ldots, χ_{i_N} \in X^*(T)$ are
the weights of $\alpha_i, \ldots, \alpha_{i_N}$. There exists $m_{ij} \in \mathbb{Z}$ for $i = 1, \ldots, N, j = 1, \ldots, 9$ such that $\chi_i = \prod_{j=1}^9 \sigma_j^{m_{ij}}$.

The following proposition is obvious and so we do not provide the proof.

**Proposition 11.4.** In the above situation, if there exists an $(N \times N)$-minor $C$ of the matrix $(m_{ij})$ such that $\det C = \pm 1$, then $Z_{i_k}^{ss}$ is a single $T_k$-orbit.

When we consider individual cases we would like to show that $P_{i_k} \setminus Y_k^{ss} \cong M_{i_k} \setminus Z_k^{ss}$ for all $i$ such that $Z_i^{ss} \neq \emptyset$. What we have to do is to show that if $x \in Z_{i_k}^{ss}$ and $y \in W_{i_k}$ then there exists $u \in U_{i_k}$ such that $ux = (x, y) \in Z_{i_k} \oplus W_{i_k}$. We show that under a certain condition, if this is possible for a particular element $R \in Z_{i_k}$ then it is possible for all $x \in Z_{i_k}^{ss}$. Since we mention the above property very often, we give it a name.

**Property 11.5.** Let $x \in Z_{i_k}^{ss}$. If $y \in W_{i_k}$ then there exists $u \in U_{i_k}$ such that $ux = (x, y) \in Z_{i_k} \oplus W_{i_k}$.

**Condition 11.6.**

1. $Z_{i_k}^{ss}$ is a single $M_{i_k}$-orb-bit.
2. There exists $R \in Z_{i_k}^{ss}$ with the property that if $y \in W_{i_k}$ then there exists $u \in U_{i_k}$ such that $uR = (R, y)$.
3. $G_R \cap U_{i_k}$ is connected.

**Proposition 11.7.** If Condition 11.6 is satisfied then Property 11.5 holds for any $x \in Z_{i_k}^{ss}$.

**Proof.** Since $G_R \cap U_{i_k}$ is connected and $x \in M_{i_k} \cap R$, $G_x \cap U_{i_k}$ is connected also. Since $k$ is a perfect field and $G_x \cap U_{i_k}$ is unipotent, it splits over $k$ (see 15.5 Corollary (ii) [2, p.205]). This means that there is a composition series $G_x \cap U_{i_k} = G_0 \supset G_1 \supset \cdots \supset G_s = \{e_G\}$ such that $G_i/G_{i+1} \cong G_a$ (the additive group over the ground field). Therefore, $H^1(k, G_x \cap U_{i_k}) = \{1\}$.

Since $x \in M_{i_k} \cap R$, if $y \in W_{i_k}$ then there exists $u \in U_{i_k}$ such that $ux = (x, y)$. Then $u^i (x, y)$ also for all $\sigma \in \text{Gal}(k^{sep}/k)$. Therefore, $u^{-1}u^\sigma \in G_x \cap U_{i_k}$. Since $H^1(k, G_x \cap U_{i_k}) = \{1\}$, there exists $u_1 \in G_x \cap U_{i_k}$ such that $u^{-1}u_1 = u_1^{-1}u_1 = \sigma x \in Z_{i_k}^{ss}$ for all $\sigma \in \text{Gal}(k^{sep}/k)$. This implies that $uu_1^{-1} \in U_{i_k}$ and that $uu_1^{-1}x = ux = (x, y)$. \qed

Also the following proposition is obvious.

**Proposition 11.8.** If $R \in Z_{i_k}^{ss}$, $Z_{i_k}^{ss} = M_{i_k} R$ and Property 11.5 holds for $R$ then Property 11.5 holds for any $x \in Z_{i_k}^{ss}$.

It is convenient to point out certain situations where we can apply Proposition 11.5.

Suppose that $n \geq 0, m > 0$ and that $\phi : \text{Aff}^{n+m} \to \text{Aff}^m$ is a map in the form

$\phi(u) = \phi(u_1, \ldots, u_{n+m}) = (u_{n+1} + P_1(u), \ldots, u_{n+m} + P_m(u))$

where $P_i(u)$ is a polynomial of $u_1, \ldots, u_{n+i-1}$. For convenience, we call $u_1, \ldots, u_n$ “extra variables”.

**Lemma 11.9.** In the above situation we have following.

1. $\phi$ induces a surjective map $k^{n+m} \to k^m$.
2. $\phi^{-1}(0, \ldots, 0) \subset \text{Aff}^{n+m}$ is connected.

We do not provide the proof of the above lemma since it is very easy. We shall use the above lemma very often when we consider individual cases.
Remark 11.10. Suppose that $P(x)$ is a polynomial on $Z_i$ and that $\psi(g)$ is a character of $M_{\beta_1}$. If $P(gx) = \psi(g)P(x)$ for all $x \in Z_i$, $g \in M_{\beta_1}$ and $\psi$ is proportional to $\chi_i$. Then $P(gx) = P(x)$ for $g \in G_{st, \beta_1}$.

For, there exists an indivisible character $\omega$ on $M_{\beta_1}$ and integers $a, b > 0$ such that $\psi = \omega^a, \chi_i = \omega^b$ on $M_{\beta_1}$. Since $G_{st, \beta_1}$ is the identity component of $\{g \in M_{\beta_1} | \chi_i(g) = 1\}$, it is generated by the center $Z$ and $[G_{st, \beta_1}, G_{st, \beta_1}]$, which is connected semi-simple and is a normal subgroup of $G_{st, \beta_1}$. So a character of $G_{st, \beta_1}$ is trivial if and only if it is trivial on $Z$. Since $X^*(Z)$ is torsion free and $\omega^b = 1$ on $G_{st, \beta_1}, \omega = 1$ on $G_{st, \beta_1}$. Therefore, $\psi = 1$ on $G_{st, \beta_1}$ also.

We now consider individual strata. We remind the reader that $\{n,i \mid i = 1, \ldots, n\}$ is the standard basis of $\text{Aff}^n$. We use the notation such as $p_{i,12} \in \wedge^2 \text{Aff}^3$, etc.

For each $i$,

1. we show that $Z_i^{ss} \neq \emptyset$ and interpret $M_{\beta_1, k} \setminus Z_i^{ss}$,
2. choose a point $R(i) \in Z_i^{ss}$ and apply Lemma 11.9.

We shall verify the following theorem for the rest of this section.

**Theorem 11.11.**

1. $S_{\beta_1} \neq \emptyset$ for $i$ in $\{1,1,1\}$.
2. The information on the above table is correct.

For the rest of this section, $u_1 = (u_{1ij})$ where $5 \geq i > j \geq 1$ and $u_2 = (u_{2ij})$ where $4 \geq i > j \geq 1$. We put $u = (u_1, u_2)$, $n(u) = (n_5(u_1), n_4(u_2))$. For each $i$, we may consider additional conditions for $u_1, u_2$.

1. $S_1, \beta_1 = \frac{1}{12} (0, 0, 0, 0, 0, -3, 1, 1, 1)$

   In this case $G_{st, \beta_1} \cong \text{SL}_5 \times \text{SL}_3$ is semi-simple. So any relative invariant polynomial is invariant by the action of $G_{st, \beta_1}$.

   By the consideration of Section 5, $Z_i^{ss} \neq \emptyset$. Proposition 5.14 implies that if $\text{ch}(k) \neq 2$ then $M_{\beta_1, k} \setminus Z_i^{ss}$ is in bijective correspondence with $\text{Pr}_{g_2}$.

   In this case $W_i = \{0\}$ and so $P_{\beta_1, k} \setminus Y_i^{ss} \cong M_{\beta_1, k} \setminus Z_i^{ss}$ if $\text{ch}(k) \neq 2$.

2. $S_3, \beta_3 = \frac{1}{780} (-12, -12, 8, 8, 8, -15, 5, 5, 5)$

   We identify the element $(\text{diag}(g_{11}, g_{12}), \text{diag}(t_2, g_2)) \in M_{[2],[1]} = M_{\beta_3}$ with the element $g = (g_2, g_{12}, g_{11}, t_2) \in \text{GL}_5^3 \times \text{GL}_2 \times \text{GL}_1$. On $M_{\beta_3}$,

   $$\chi_3(g) = (\det g_{11})^{-12} (\det g_{12})^8 t_2^{-15} (\det g_2)^5 = (\det g_{12})^{20} (\det g_2)^{20}.$$

For $x \in Z_3$, let

$$A(x) = \begin{pmatrix} 0 & x_{341} & x_{351} \\ -x_{341} & 0 & x_{451} \\ -x_{351} & -x_{451} & 0 \end{pmatrix},$$

$$B_1(x) = \begin{pmatrix} x_{132} & x_{142} & x_{152} \\ x_{133} & x_{143} & x_{153} \\ x_{134} & x_{144} & x_{154} \end{pmatrix}, B_2(x) = \begin{pmatrix} x_{232} & x_{242} & x_{252} \\ x_{233} & x_{243} & x_{253} \\ x_{234} & x_{244} & x_{254} \end{pmatrix}$$

and $B(x) = (B_1(x), B_2(x))$. We identify $Z_3 \cong \wedge^2 \text{Aff}^3 \oplus M_3 \otimes \text{Aff}^2$ by the map $Z_3 \ni x \mapsto (A(x), B(x))$. This is the same vector space as the one considered in...
Since
\[ A(gx) = t_2(x^2 g_{12}) A(x), \quad \begin{pmatrix} B_1(gx) \\ B_2(gx) \end{pmatrix} = g_{11} \begin{pmatrix} g_2 B_1(x)^4 g_{12} \\ g_2 B_2(x)^4 g_{12} \end{pmatrix}, \]
the action of \((g_2, g_{12}, g_{11}, t_2)\) on \(V\) is the same as the one in Section III.

Let \(P_1(x), P_2(x)\) be the relative invariant polynomials defined in Section III. Then by (6.7) and Proposition 6.8 (1),
\[ P_1(gx) = (\det g_{11})^6 (\det g_{12})^4 (\det g_2)^4 P_1(x), \]
\[ P_2(gx) = (\det g_{11})^3 (\det g_{12})^4 t_2^3 (\det g_2)^2 P_2(x). \]

Let \(P(x) = P_1(x) P_2(x)^3\). Then on \(M_{3}^{1}\),
\[ P(gx) = ((\det g_{11})^6 (\det g_{12})^4 (\det g_2)^4)((\det g_{11})^3 (\det g_{12})^4 t_2^3 (\det g_2)^2)^3 P(x) = (\det g_{11})^{15} (\det g_{12})^{16} t_2^9 (\det g_2)^{10} P(x) = (\det g_{12}) (\det g_2) P(x). \]

Therefore, \(P(x)\) is invariant under the action of \(G_{st,3}^{1}\).

By Proposition 6.9, \(M_{3,k}^{0} / Z_{3,k}^{0}\) is in bijective correspondence with \(E_{3}(k)\).

Let \(R(3) \in Z_3\) be the element such that \(A(x) = w_1, B_1(x) = w_{21}, B_2(x) = w_{22}\) where \(w_1, w_{21}, w_{22}\) are the elements in (6.1). Explicitly, \(R(3) = e_{341} - e_{351} + e_{451} + e_{132} - e_{143} + e_{243} - e_{254}\). Then \(R(3) \in Z_3^{ss}\).

We assume that \(u_{1ij} = 0\) unless \(i = 3, 4, 5, j = 1, 2\) and \(u_{2ij} = 0\) unless \(j = 1\). Then the four components of \(n(u) R(3)\) are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 1 & -1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -u_{141} + u_{221} \\
0 & 0 & * & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
0 & 0 & -u_{131} + u_{132} + u_{231} & -u_{231} \\
1 & -1 & * & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & u_{241} & -u_{132} - u_{241} \\
0 & 0 & * & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & u_{241} & -u_{132} - u_{241} \\
0 & 0 & * & 0
\end{pmatrix}.
\]

Note that we obtain \(R(3)\) by substituting \(u_{1ij} = u_{2ij} = 0\) for all \(i, j\).

We can apply Lemma 11.9 to the map \(\text{Aff}^{9} \to \text{Aff}^{9}\) defined by the sequence
\[ u_{221}, u_{141} - u_{221}, u_{151} + u_{221}, u_{241}, u_{132} + u_{241}, \\
u_{142} - u_{241}, u_{231}, u_{152} - u_{151} - u_{231}, u_{131} - u_{132} - u_{231} \]
(there are no extra variables). Note that the above 9 entries exhaust coordinates of \(W_3\) (we shall not point this out in the remaining cases). So by Proposition 11.7, Property
(11.5) holds for any \( x \in Z^{ss}_{3k} \). Therefore, \( P_{\beta, k} \setminus Y^{ss}_{3k} \) is in bijective correspondence with \( \text{Ex}_2(k) \) also.

(3) \( S_5, \beta_5 = \frac{1}{180}(-2, -2, -2, -2, 8, -5, -5, 5, 5) \)

We identify the element \( (\text{diag}(g_1, t_1), \text{diag}(g_{21}, g_{22})) \in M_{[4][2]} = M_{\beta_5} \) with the element \( g = (g_1, g_{21}, g_{22}, t_1) \in \text{GL}_4 \times \text{GL}_2 \times \text{GL}_1 \). On \( M_1^{\beta_5} \),

\[
\chi_5(g) = (\det g_1)^{-2}t_1^5(\det g_{21})^{-5}(\det g_{22})^5 = t_1^{10}(\det g_{22})^{10}.
\]

For \( x \in Z_5 \), let

\[
A(x) = \begin{pmatrix} x_{151} & x_{152} \\ x_{251} & x_{252} \\ x_{351} & x_{352} \\ x_{451} & x_{452} \end{pmatrix}, \quad B_1(x) = \begin{pmatrix} 0 & x_{123} & x_{133} & x_{143} \\ -x_{123} & 0 & x_{233} & x_{243} \\ -x_{133} & -x_{233} & 0 & x_{343} \\ -x_{143} & -x_{243} & -x_{343} & 0 \end{pmatrix},
\]

\[
B_2(x) = \begin{pmatrix} 0 & x_{124} & x_{134} & x_{144} \\ -x_{124} & 0 & x_{234} & x_{244} \\ -x_{134} & -x_{234} & 0 & x_{344} \\ -x_{144} & -x_{244} & -x_{344} & 0 \end{pmatrix}, \quad B(x) = (B_1(x), B_2(x)).
\]

We identify \( Z_5 \cong M_{4,2} \oplus \wedge^2 \text{Aff}^4 \oplus \text{Aff}^2 \) by the map \( Z_5 \ni x \mapsto (A(x), B(x)) \). Since

\[
A(gx) = g_1 A(x) t_{g_{21}}, \quad \begin{pmatrix} B_1(gx) \\ B_2(gx) \end{pmatrix} = g_{22} \begin{pmatrix} g_1 B_1(x) t_{g_{21}} \\ g_1 B_2(x) t_{g_{21}} \end{pmatrix},
\]

\( Z_5 \) is the same vector space as the one considered in Section 7 and \( M_{\beta_5} \) is almost the same as the group considered in Section 7 except for the extra \( \text{GL}_1 \)-factor. The action of \( (g_1, g_{21}, g_{22}, 1) \) is the same as the one in Section 7. If \( t = (I_4, I_2, I_2, t_1) \) then \( A(tx) = t_1 A(x), B(tx) = B(x) \).

Let \( P_1(x), P_2(x) \) be the polynomials in (7.35). Since \( P_1(x) \) is homogeneous of degree \( 8 \) with respect to \( A(x) \), \( P_1(tx) = t^8 P_1(x) \). Since \( P_2(x) \) is a polynomial of \( B(x) \), \( P_2(tx) = P_2(x) \). We put \( P(x) = P_1(x)^2 P_2(x) \). Then on \( M_1^{\beta_5} \), by (7.35),

\[
P(gx) = ((\det g_1)^6 t_1^8(\det g_{21})^4(\det g_{22})^4)^2((\det g_1)^2(\det g_{22})^2) P(x) = (\det g_1)^{14} t_1^{16} (\det g_{21})^8 (\det g_{22})^{10} P(x) = t_1^2 (\det g_{22})^2 P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_5} \).

Let \( R(5) \in Z_5 \) be the element which corresponds to \( w \) in (7.4). Explicitly, \( R(5) = e_{251} + e_{451} + e_{152} + e_{352} + e_{123} + e_{344} \). Then \( R(5) \in Z^{ss}_5 \). By Proposition 7.3, \( M_{\beta_5} \setminus Z^{ss}_{5,k} \) is in bijective correspondence with \( \text{Ex}_2(k) \).

We assume that \( u_{1ij} = 0 \) unless \( i = 5 \) and \( u_{2ij} = 0 \) unless \( i = 3, 4, j = 1, 2 \). Then the first two components of \( n(u)R(5) \) are the same as those of \( R(5) \) and the remaining components are as follows:

\[
\begin{pmatrix} 0 & 1 & 0 & 0 & u_{152} + u_{232} \\ -1 & 0 & 0 & 0 & -u_{151} + u_{231} \\ 0 & 0 & 0 & 0 & u_{232} \\ 0 & 0 & 0 & 0 & u_{231} \\ * & * & * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & u_{242} \\ 0 & 0 & 0 & 0 & u_{241} \\ 0 & 0 & 0 & 1 & u_{154} + u_{242} \\ 0 & 0 & -1 & 0 & -u_{153} + u_{241} \\ * & * & * & * & 0 \end{pmatrix}.
\]
We can apply Lemma 11.9 to the map $\text{Aff}^8 \to \text{Aff}^8$ defined by the sequence

$$u_{231}, u_{232}, u_{151} - u_{231}, u_{152} + u_{232}, u_{241}, u_{242}, u_{153} - u_{241}, u_{154} + u_{242}$$

(there are no extra variables). So by Proposition 11.7, Property 11.5 holds for any $x \in Z_{5k}$. Therefore, $P_{\beta_1 k} \setminus Y_{5k}$ is in bijective correspondence with $\text{Ex}_{2}(k)$ also.

(4) $S_9, \beta_9 = \frac{3}{20}(-16, 4, 4, 4, -5, -5, -5, 15)$

We identify the element $(\text{diag}(t_1, g), \text{diag}(g_2, t_2)) \in M_{[1],[3]} = M_{\beta_9}$ with the element $g = (g_1, g_2, t_1, t_2) \in \text{GL}_4 \times \text{GL}_3 \times \text{GL}_2^2$. On $M_{\beta_9}^1$,

$$\chi_9(g) = t_1^{-16} (\det g_1)^4 (\det g_2)^{-5} t_2^{15} = (\det g_1)^{20} t_2^{20}.$$  

For $x \in Z_9$, let

$$A_i(x) = \begin{pmatrix} 0 & x_{23i} & x_{24i} & x_{25i} \\ -x_{23i} & 0 & x_{34i} & x_{35i} \\ -x_{24i} & -x_{34i} & 0 & x_{45i} \\ -x_{25i} & -x_{35i} & -x_{45i} & 0 \end{pmatrix}, \quad (i = 1, 2, 3), \quad v(x) = \begin{pmatrix} x_{124} \\ x_{134} \\ x_{144} \\ x_{154} \end{pmatrix}$$

and $A(x) = (A_1(x), A_2(x), A_3(x))$. We identify $Z_9 \cong \wedge^2 \text{Aff}^4 \otimes \text{Aff}^3 \oplus \text{Aff}^4$ by the map $Z_9 \ni (A(x), v(x))$. This is the same vector space considered in the case (b) of Section 8 and $M_{\beta_9}$ is almost the same as the group considered in Section 8 except for the extra $\text{GL}_1$-factors. Since

$$A(gx) = g_2 \begin{pmatrix} g_1 A_1(x)^t g_1 \\ g_1 A_1(x)^t g_1 \\ g_1 A_1(x)^t g_1 \end{pmatrix}, \quad v(gx) = t_1 t_2 g_1 v(x),$$

the action of $(g_1, g_2, 1, 1)$ is the same as the one in Section 8, i.e., the natural action of $\text{GL}_4 \times \text{GL}_3$.

Let $P_1(x), P_2(x)$ be the polynomials on $Z_9$ obtained in the consideration of the case (b) of Section 8 (see Proposition 8.18 and 8.20). $P_1(x)$ is homogeneous of degrees 3, 2 with respect to $A(x), v(x)$ respectively and $P_2(x)$ is a homogeneous degree 6 polynomial of $A(x)$. So by Proposition 8.18 (2) and 8.20,

$$P_1(gx) = t_1^2 (\det g_1)^2 (\det g_2)^2 t_2^2 P_1(x), \quad P_2(gx) = (\det g_1)^3 (\det g_2)^2 P_2(x).$$

Let $P(x) = P_1(x)^5 P_2(x)$. Then on $M_{\beta_9}^1$,

$$P(gx) = t_1^2 (\det g_1)^2 (\det g_2)^2 t_2^2 ((\det g_1)^3 (\det g_2)^2)$$

$$= t_1^8 (\det g_1)^{12} (\det g_2)^7 t_2^{10} P(x) = (\det g_1)^3 t_2 P(x).$$

Therefore, $P(x)$ is invariant under the action of $G_{st,\beta_9}$. By Proposition 8.25, $M_{\beta_9 k} \setminus Z_{9k}$ is in bijective correspondence with $\text{Pr}_{\mathfrak{g}_5}(k)$.

Let $R(9) \in Z_9$ be the element which corresponds to $(w_1, w_2, w_3, w_0)$ in the case (b) of Section 8. Explicitly, $R(9) = e_{231} + e_{252} - e_{342} + e_{453} + e_{124} + e_{154}$.

We assume that $u_{11j} = 0$ unless $j = 1$ and $u_{2ij} = 0$ unless $i = 4$. Then the first three components of
We identify the element \((\text{diag}(g_{11}, g_{12}), \text{diag}(g_{21}, g_{22})) \in M_{[2],[2]} = M_{\beta_{14}}\) with the element \(g = (g_{12}, g_{11}; g_{22}, g_{21}) \in \text{GL}_3 \times \text{GL}_2^3\). On \(M_{\beta_{14}}\),
\[
\chi_{14}(g) = (\det g_{11})^{-6}(\det g_{12})^4(\det g_{21})^{-5}(\det g_{22})^5 = (\det g_{12})^{10}(\det g_{22})^{10}.
\]
For \(x \in Z_{14}\), let
\[
A_1(x) = \begin{pmatrix}
0 & x_{341} & x_{351} \\
-x_{341} & 0 & x_{451} \\
-x_{351} & -x_{451} & 0
\end{pmatrix},
A_2(x) = \begin{pmatrix}
0 & x_{342} & x_{352} \\
-x_{342} & 0 & x_{452} \\
-x_{352} & -x_{452} & 0
\end{pmatrix},
B_1(x) = \begin{pmatrix}
x_{133} & x_{134} & x_{233} & x_{234}
\end{pmatrix},
B_2(x) = \begin{pmatrix}
x_{143} & x_{144} & x_{243} & x_{244}
\end{pmatrix},
B_3(x) = \begin{pmatrix}
x_{153} & x_{144} \\
x_{253} & x_{254}
\end{pmatrix},
A(x) = (A_1(x), A_2(x)), B(x) = (B_1(x), B_2(x), B_3(x)).
\]
We identify \(Z_{14} \cong \wedge^2 \text{Aff}^3 \otimes \text{Aff}^2 \oplus \text{Aff}^3 \otimes M_2\) by the map \(Z_{14} \ni x \mapsto (A(x), B(x))\). Since
\[
\begin{pmatrix}
A_1(gx) \\
A_2(gx)
\end{pmatrix} = g_{21} \begin{pmatrix}
g_{12}A_1(x)^t g_{12} \\
g_{12}A_2(x)^t g_{12}
\end{pmatrix},
\begin{pmatrix}
B_1(gx) \\
B_2(gx) \\
B_3(gx)
\end{pmatrix} = g_{12} \begin{pmatrix}
g_{11}B_1(x)^t g_{22} \\
g_{11}B_2(x)^t g_{22} \\
g_{11}B_3(x)^t g_{22}
\end{pmatrix},
\]
\((M_{\beta_{14}}, Z_{14})\) can be identified with the case (b) of Section 9.
Let \(P_1(x), P_2(x)\) be polynomials which correspond to \(P_1(x), P_2(x)\) in the case (b) of Section 9. Then by (9.24),
\[
P_1(gx) = (\det g_{11})^2(\det g_{12})^4(\det g_{21})^2(\det g_{22})^2P_1(x),
P_2(gx) = (\det g_{11})^3(\det g_{12})^2(\det g_{22})^3P_2(x).
\]
Let \(P(x) = P_1(x)^2P_2(x)\). Then on \(M_{\beta_{14}}^1\),
\[
P(gx) = ((\det g_{11})^2(\det g_{12})^4(\det g_{21})^2(\det g_{22})^2)^2((\det g_{11})^3(\det g_{12})^2(\det g_{22})^3)P(x)
= (\det g_{11})^7(\det g_{12})^{10}(\det g_{21})^4(\det g_{22})^7P(x) = (\det g_{12})^3(\det g_{22})^3P(x).
\]
Therefore, \(P(x)\) is invariant under the action of \(G_{st,\beta_{14}}\).
Proposition 9.19 implies that \(M_{\beta_{14}} \setminus Z_{14}^* \cong \text{Ex}_2(k)\).
Let \(R(14) \in Z_{14}\) be the element such that
\[
A_1(R(14)) = -p_{3,13}, \ A_2(R(14)) = p_{3,12}, \ B(R(14)) = R_{322}.
\]
\(n(u)R(9)\) are the same as those of \(R(9)\) and the last component is as follows:
\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & -u_{131} + u_{241} & -u_{141} \\
0 & * & 0 & -u_{242} \\
0 & * & * & 0
\end{pmatrix}
\]
We can apply Lemma 11.9 to the map \(\text{Aff}^7 \to \text{Aff}^6\) defined by the sequence
\[
u_{131}, \nu_{141}, \nu_{242}, \nu_{u_{131}}, \nu_{u_{121}} - u_{151} + u_{242}, \nu_{u_{243}}, \nu_{u_{141}} + u_{243}
\]
where \(u_{151}\) is an extra variable. So by Proposition 11.7, Property 11.5 holds for any \(x \in Z_{9g}\). Therefore, \(P_{\beta_{k}} \setminus V_{9g}^*\) is in bijective correspondence with \(\text{Prg}_{2}(k)\) also.

(5) \(S_{14}, \beta_{14} = \frac{3}{220}(-6, -6, 4, 4, 4, -5, -5, 5, 5)\)
We identify the element \((\text{diag}(g_{11}, g_{12}), \text{diag}(g_{21}, g_{22})) \in M_{[2],[2]} = M_{\beta_{14}}\) with the element \(g = (g_{12}, g_{11}; g_{22}, g_{21}) \in \text{GL}_3 \times \text{GL}_2^3\). On \(M_{\beta_{14}}\),
\[
\chi_{14}(g) = (\det g_{11})^{-6}(\det g_{12})^4(\det g_{21})^{-5}(\det g_{22})^5 = (\det g_{12})^{10}(\det g_{22})^{10}.
\]
(see (4.8)). Explicitly, $R(14) = -e_{351} + e_{342} - e_{133} + e_{253} + e_{144} + e_{234}$. This element corresponds to $R$ in (9.20).

We assume that $u_{ij} = 0$ unless $i = 3, 4, 5, j = 1, 2$ and $u_{2ij} = 0$ unless $i = 3, 4, j = 1, 2$. Then the first two components of $u(R(14))$ are the same as those of $R(14)$ and the remaining components are as follows:

$$
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & u_{141} + u_{232} & u_{151} + u_{132} - u_{231} \\
0 & 0 & * & 0 & u_{142} \\
0 & -1 & * & * & 0
\end{pmatrix}
$$

Explicitly, $u_{141} + u_{232}, u_{142}, u_{151}, u_{152} + u_{241}, u_{132} + u_{151} - u_{231}, u_{131} - u_{142} + u_{242}$

where $u_{231}, u_{232}, u_{241}, u_{242}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{14k}^{ss}$. Therefore, $P_{\beta_{14k}} \backslash Y_{14k}^{ss}$ is in bijective correspondence with $\text{Ex}_2(k)$ also.

$$(6) \ S_{15}, \ \beta_{15} = \frac{1}{36}(-2, -2, -2, 3, 3, -5, 0, 0, 5)$$

We identify the element $(\text{diag}(g_{11}, g_{12}), \text{diag}(t_{21}, g_{2}, t_{22})) \in M_{[3], [1, 3]} = M_{\beta_{15}}$ with the element $g = (g_{11}, g_{12}, g_{2}, t_{21}, t_{22}) \in GL_3 \times GL_2 \times GL_1$. On $M_{\beta_{15}}$, $\chi_{15}(g) = (\det g_{11})^{-2}(\det g_{12})^{3}t_{21}^{-5}t_{22}^{-5} = (\det g_{12})^{5}(\det g_{2})^{5}t_{22}^{10}$.

For $x \in Z_{15}$, let $A(x) = x_{124}p_{3, 12} + x_{134}p_{3, 13} + x_{234}p_{3, 23}$.

$$
B_1(x) = \begin{pmatrix}
x_{142} & x_{143} \\
x_{152} & x_{153}
\end{pmatrix}, \ B_2(x) = \begin{pmatrix}
x_{242} & x_{243} \\
x_{252} & x_{253}
\end{pmatrix}, \ B_3(x) = \begin{pmatrix}
x_{342} & x_{343} \\
x_{352} & x_{353}
\end{pmatrix}
$$

and $B(x) = p_{3, 1} \otimes B_1(x) + p_{3, 2} \otimes B_2(x) + p_{3, 3} \otimes B_3(x)$. We identify $Z_{15}$ with $1 \oplus \wedge^2 \text{Aff}^3 \oplus \text{Aff}^3 \otimes M_2$ by the map $Z_{15} \ni x \mapsto (x_{451}, (A(x), B(x)))$. Since

$A(gx) = t_{22}(\wedge^2 g_{11})A(x), \begin{pmatrix}
B_1(gx) \\
B_2(gx) \\
B_3(gx)
\end{pmatrix} = g_{11} \begin{pmatrix}
g_{12}B_1(x)^tg_2 \\
g_{12}B_2(x)^tg_2 \\
g_{12}B_3(x)^tg_2
\end{pmatrix}$,

$Z_{15}$ is the same vector space considered in the case (a) of Section 9 except for a component of the trivial representation of $M_{\beta_{15}}$ and the action of $(g_{11}, g_{12}, g_{2})$ is the same as that of the case (a) of Section 9. If $t = (I_3, I_2, I_2, t_{21}, t_{22})$ then the $x_{451}$-coordinate of $tx$ is $t_{21}x_{451}$ and $A(tx) = t_{22}A(x), B(tx) = B(x)$.

Let $P_1(x), P_2(x)$ be the polynomials considered in (9.8). Note that $P_2(x)$ is the homogeneous degree 6 polynomial of $B(x)$ obtained by Proposition 4.7 and that
$P_1(x)$ is homogeneous of degree 2 for each of $A(x), B(x)$. So by (9.8)

$$P_1(gx) = (\det g_{11})^2(\det g_{12})(\det g_2)t_{22}^2P_1(x),$$

$$P_2(gx) = (\det g_{11})^2(\det g_{12})^3(\det g_2)^3P_2(x).$$

We put $P(x) = P_1(x)^5P_2(x)x_{451}^6$. Then on $M_{15}^1$:

$$P(gx) = ((\det g_{11})^2(\det g_{12})(\det g_2)t_{22}^2)^5((\det g_{11})^2(\det g_{12})^3(\det g_2)^3)
\times ((\det g_{12})t_{21})^6P(x)
= (\det g_{11})^{12}(\det g_{12})^{14}t_{21}^6(\det g_2)^{10}P(x) = (\det g_{12})^2(\det g_2)^2t_{22}^4P(x).$$

Therefore, $P(x)$ is invariant under the action of $G_{st,\beta_{15}}$. Let $R(15) \in Z_{15}$ be the element such that $(A(R(15)), B(R(15))) = R_{322,3}$ (see (9.9)) and that the $x_{451}$-coordinate is 1. Explicitly, $R(15) = e_{451} - e_{142} + e_{153} + e_{243} + e_{234}$. Then $P(R(15)) = 1$ and so $R(15) \in Z_{15}^{ss}$ (there is no restriction on $ch(k)$).

Suppose that $x \in Z_{15}^{ss}$. Then $x_{451} \neq 0$. If $t = (I_3, I_2, I_2, x_{451}^{-1}, 1)$ then the $x_{451}$-coordinate of $tx$ is 1. If $x_{451} = 1$ and $g = (g_{11}, g_{12}, g_2, t_21, t_22)$ then $g$ does not change the $x_{451}$-coordinate if and only if $t_21 = (\deg g_{12})^{-1}$. If so, the action of $g$ on $(A(x), B(x))$ is the same as that of the case (a) of Section 9. Therefore, by Proposition 9.19 $M_{15,k} \setminus Z_{15,k}^{ss}$ is in bijective correspondence with $\text{Ex}_2(k)$ if $ch(k) \neq 2$.

We assume that $u_{14j} = 0$ unless $i = 4, 5, j = 1, 2, 3$ and $u_{232} = 0$. Then the four components of $n(u)R(15)$ are as follows:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & u_{141} - u_{152} + u_{231} \\
-1 & 0 & 0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & -u_{242} & u_{243} \\
0 & 0 & 1 & u_{143} + u_{243} & u_{153} \\
0 & -1 & 0 & -u_{142} & -u_{152} + u_{242} \\
* & * & * & 0 & Q(u) + u_{241} \\
* & * & * & * & 0
\end{pmatrix},
$$

where $Q(u)$ is a polynomial which does not depend on $u_{241}$.

We can apply Lemma 11.9 to the map $\text{Aff}^{11} \to \text{Aff}^9$ defined by the sequence

$$u_{142}, u_{242}, u_{243}, u_{143} + u_{243}, u_{153}, u_{152} - u_{242},
\quad u_{231} + u_{141} - u_{152}, u_{221} + u_{143} + u_{151}, u_{241} + Q(u)$$

where $u_{141}, u_{151}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{15,k}^{ss}$. Therefore, $P_{\beta_{15,k}} \setminus Y_{15,k}^{ss}$ is in bijective correspondence with $\text{Ex}_2(k)$ also if $ch(k) \neq 2$.

(7) $S_{16, \beta_{16}} = \frac{1}{36}(-2, -2, -2, 3, 3, 0, 0, 0, 0, 0)$

Since $G_{st,\beta_{16}}$ is semi-simple, any relative invariant polynomial is invariant by $G_{st,\beta_{16}}$. We identify the element $(\text{diag}(g_{11}, g_{12}), g_2) \in M_{[3],\emptyset} = M_{\beta_{16}}$ with the element $g = (g_2, g_{11}, g_{22}) \in \text{GL}_4 \times \text{GL}_3 \times \text{GL}_2$. 

Remark 4.11. Therefore, \( \mathbb{Z} \) by the map \( \mathbb{Z} \) is the tensor product of standard representations. A(16) corresponds to the element \( \{1/\mathbb{D}4\}_{k} \otimes \mathbb{D}4 \). We assume that \( \{1/\mathbb{D}4\} \) be the element such that \( f(x) = e_{141} + e_{351} + e_{152} + e_{242} + e_{343} + e_{254} \). By the Castling transform, \( R(16) \) corresponds to the element \( (-q_{11}^{*} + q_{32}^{*}, q_{12}^{*} + q_{21}^{*}) \in W^{*} \otimes \mathbb{A}^{2} \).

Let \( \{\mathbb{P}_{3,1}^{*}, \mathbb{P}_{3,2}^{*}, \mathbb{P}_{3,3}^{*}\} \) be the dual basis of \( \{\mathbb{P}_{3,1}, \mathbb{P}_{3,2}, \mathbb{P}_{3,3}\} \). Since \( (\mathbb{A}^{3} \otimes \mathbb{A}^{2})^{*} \cong (\mathbb{A}^{3})^{*} \otimes (\mathbb{A}^{2})^{*} \), the above element is

\[
-\mathbb{P}_{3,1}^{*} \otimes \mathbb{P}_{2,1}^{*} \otimes \mathbb{P}_{2,1}^{*} + \mathbb{P}_{3,1}^{*} \otimes \mathbb{P}_{2,2}^{*} \otimes \mathbb{P}_{2,2}^{*} + \mathbb{P}_{3,2}^{*} \otimes \mathbb{P}_{2,1}^{*} \otimes \mathbb{P}_{2,2}^{*} + \mathbb{P}_{3,3}^{*} \otimes \mathbb{P}_{2,2}^{*} \otimes \mathbb{P}_{2,1}^{*}.
\]

This element of \( (\mathbb{A}^{3})^{*} \otimes (\mathbb{A}^{2})^{*} \otimes \mathbb{A}^{2} \) corresponds to \( R_{322} \) in Proposition 3.7 (see Remark 4.11). Therefore, \( Z_{16}^{ss} = M_{\beta_{14}} R(16) \neq \emptyset \).

We assume that \( u_{11j} = 0 \) unless \( i = 4, 5, j = 1, 2, 3 \) and \( u_{2} = 0 \). Then the four components of \( n(u)R(16) \) are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & -u_{151} + u_{143} \\
0 & 0 & -1 & * & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & u_{152} + u_{141} \\
-1 & 0 & 0 & * & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & -u_{153} \\
0 & 0 & 0 & * & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u_{142} \\
0 & -1 & 0 & * & 0 \\
\end{pmatrix}
\]

We can apply Lemma 11.9 to the map \( \mathbb{A}^{6} \rightarrow \mathbb{A}^{4} \) defined by the sequence

\( u_{142}, u_{153}, u_{152} + u_{141}, u_{151} - u_{143} \).
where $u_{141}, u_{143}$ are extra variables. So by Proposition 11.7, Property 11.5 holds for any $x \in Z_{16}^{ss}$. Therefore, $Y_{16}^{ss} = P_{16,k}R(16)$.

(8) $S_{20}, \beta_{20} = \frac{1}{400}(-16, -16, 4, 4, 24, -25, -5, 15, 15, 20)$

We identify the element $(\text{diag}(g_{11}, g_{12}, t_1), \text{diag}(t_{21}, t_{22}, g_2)) \in M_{[2,4],[1,2]} = M_{\beta_{20}}$ with the element $g = (g_{12}, g_{11}, g_2, t_1, t_{21}, t_{22}) \in \text{GL}_2^3 \times \text{GL}_1^3$. On $M_{\beta_{20}}^1$,

$$\chi_{20}(g) = (\det g_{11})^{-16}(\det g_{12})^{t_{21}^4 t_{22}^{-25} t_{22}^{-5}(\det g_2)^{15}} = (\det g_{12})^{t_{21}^4 t_{22}^{-9}(\det g_2)^{40}}.$$

For $x \in Z_{20}$, let

$$A_1(x) = \begin{pmatrix} x_{133} & x_{233} \\ x_{143} & x_{243} \end{pmatrix}, \quad A_2(x) = \begin{pmatrix} x_{134} & x_{234} \\ x_{144} & x_{244} \end{pmatrix},$$

$$A(x) = (A_1(x), A_2(x)) \in \Lambda_{1,[3,4]}^1 \otimes \Lambda_{1,[1,2]}^1 \otimes \Lambda_{2,[3,4]}^1.$$

$$v_1(x) = \begin{pmatrix} x_{351} \\ x_{251} \end{pmatrix} \in \Lambda_{1,[3,4]}^1, \quad v_2(x) = \begin{pmatrix} x_{152} \\ x_{252} \end{pmatrix} \in \Lambda_{1,[1,2]}^1.$$

We identify $Z_{20}$ with $\text{Aff}^2 \oplus \text{Aff}^2 \oplus 1 \oplus \text{Aff}^2 \otimes \text{Aff}^2$ by the map $Z_{20} \ni (v_1(x), v_2(x), x_{342}, A(x))$. Since

$$v_1(gx) = t_1 t_{21} g_1 v_1(x), \quad v_2(gx) = t_1 t_{22} g_{11} v_2(x), \quad \begin{pmatrix} A_1(gx) \\ A_2(gx) \end{pmatrix} = g \begin{pmatrix} g_{12} A_1(x) \\ g_{12} A_2(x) \end{pmatrix},$$

the action of $(g_{12}, g_{11}, g_2, 1, 1, 1)$ on $(v_1(x), v_2(x), A(x))$ is the same as that of the case (b) of Section 10.

Let $P_1(x)$ be the degree 4 polynomial of $A(x)$ obtained by Proposition 10.1 and $P_2(x), P_3(x)$ the polynomials of $(A(x), v_1(x)), (A(x), v_2(x))$ respectively, for the case (b) of 10.2. $P_1(x)$ is homogeneous of degree 2 for each of $A(x), v_1(x)$ and $P_2(x)$ (resp. $P_3(x)$) is homogeneous of degree 2 for each of $A(x), v_1(x)$ (resp. $A(x), v_2(x)$). So by 10.2,

$$P_1(gx) = (\det g_{11})^2(\det g_{12})^2(\det g_2)^2P_1(x),$$

$$P_2(gx) = (t_{12})^2(\det g_{11})(\det g_{12})^2(\det g_2)P_2(x),$$

$$P_3(gx) = (t_{12})^2(\det g_{11})^2(\det g_{12})(\det g_2)P_3(x).$$

Also the $x_{342}$-coordinate of $gx$ is $t_{22}(\det g_{12}) x_{342}$.

Let $P(x) = P_1(x) P_2(x)^4 P_3(x)^4 x_{342}$. Then on $M_{\beta_{20}}^1$,

$$P(gx) = (\det g_{11})^2(\det g_{12})^2(\det g_2)^2((t_{12})^2(\det g_{11})(\det g_{12})^2(\det g_2))^4 \times ((t_{12})^2(\det g_{11})^2(\det g_{12})(\det g_2))^4(t_{22}(\det g_{12})) P(x)$$

$$= (\det g_{11})^{14}(\det g_{12})^{15} t_{21}^4 t_{22}^{-9}(\det g_2)^{10} P(x) = (\det g_{12}) t_{21}^2 t_{22}(\det g_2)^2 P(x).$$

Therefore, $P(x)$ is invariant under the action of $G_{st, \beta_{20}}$.

Suppose that $x \in Z_{20}^{ss}$. Then $x_{342} \neq 0$. If $t = (I_2, I_2, I_3, x_{342}, x_{342}^{-1}, x_{342}^{-1})$ then $A(tx) = A(x), v_1(tx) = v_1(x), v_2(tx) = v_2(x)$ and the $x_{342}$-coordinate of $tx$ is 1. If $g = (g_{12}, g_{11}, g_2, t_1, t_{21}, t_{22})$ then $g$ does not change the $x_{342}$-coordinate if and only if $t_{22} = (\det g_{12})^{-1}$. If we replace $g_{11}, g_{12}, g_2$ by $t_{12}^2 g_{11}, t_{12}^{-1} t_{21}^{-1} g_{12}, t_{12}^{-1} g_2$ respectively, then the action of $(g_{12}, g_{11}, g_2)$ on $(A(x), v_1(x), v_2(x))$ is the same as that in Section 10. Since $Z_{20}^{ss}$ corresponds to $U$ in Proposition 10.3, $M_{\beta_{20}} \setminus Z_{20}^{ss}$ is in bijective correspondence with $E_2(k)$. 
Let \( R(20) \in Z_{20} \) be the element such that

\[
A_1(R(20)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2(R(20)) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad v_1(R(20)) = v_2(R(20)) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

and that the \( x_{342} \)-coordinate is 1. Explicitly, \( R(20) = e_{251} + e_{451} + e_{252} + e_{252} + e_{133} + e_{344} \). Then \( P_i(R(20)) = 1, P_i(R_{20}) = -1 \) for \( i = 2, 3 \). So \( R(20) \in Z_{20}^{ss} \).

We assume that \( u_{12} = 0 \) for \( (i, j) = (2, 1), (4, 3) \) and \( u_{243} = 0 \). Then the four components of \( n(u)R(20) \) are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -u_{141} + u_{232} & Q_1(u) - u_{151} + u_{231} \\
0 & 0 & * & 0 & Q_2(u) + u_{231} \\
* & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & u_{154} + u_{242} \\
0 & 0 & 0 & u_{132} + u_{242} & Q_3(u) + u_{241} \\
0 & 0 & -1 & * & 0 \\
* & * & * & * & 0
\end{pmatrix}
\]

where \( Q_1(u), Q_2(u), Q_3(u), Q_4(u) \) are polynomials which do not depend on \( u_{151}, u_{152}, u_{231}, u_{241} \).

We can apply Lemma 11.9 to the map \( \text{Aff}^{13} \to \text{Aff}^{12} \) defined by the sequence

\[
u_{232}, u_{242}, u_{153} + u_{232}, u_{154} + u_{242}, u_{132} + u_{242}, u_{141} - u_{232}, u_{131} + u_{232} + u_{154} + u_{221}, u_{142} + u_{141} - u_{153} + u_{221}, \]

\[
u_{231} + Q_2(u), u_{151} - u_{231} - Q_1(u), u_{241} + Q_3(u), u_{152} - u_{241} - Q_4(u)
\]

where \( u_{221} \) is an extra variable. So by Proposition 11.7 Property 11.5 holds for any \( x \in Z_{20}^{ss} \). Therefore, \( \beta_{20} \setminus Y_{20}^{ss} \) is in bijective correspondence with \( \text{Ex}_2(k) \) also.

(9) \( S_{21}, \beta_{21} = \frac{1}{55}(-4, -4, 1, 1, 6, -5, 0, 0, 5) \)

We identify the element \( (\text{diag}(g_{11}, g_{12}, t_1), \text{diag}(t_{21}, g_{22}, t_{22})) \in M_{[2,4],[1,3]} = M_{\beta_{21}} \) with the element \( g = (g_{11}, g_{12}, g_{21}, t_{21}, t_{22}) \in \text{GL}_{2}^{3} \times \text{GL}_{2}^{3} \). On \( M_{[2,4]}^{1} \),

\[
\chi_{21}(g) = (\text{det} g_{11})^{-4}(\text{det} g_{12}) t_{11}^{4} t_{22}^{-5} t_{22} = (\text{det} g_{12})^{5} t_{11}^{5} (\text{det} g_{22})^{5} t_{22}^{5}.
\]

For \( x \in Z_{21} \), let

\[
A(x) = \begin{pmatrix} x_{152} & x_{153} \\ x_{252} & x_{253} \end{pmatrix} \in \Lambda_{[1,1]2}^{1} \otimes \Lambda_{2,2,3}^{1}, \quad v_1(x) = \begin{pmatrix} x_{342} \\ x_{433} \end{pmatrix} \in \Lambda_{2,2,2}^{1}.
\]

\[
B(x) = \begin{pmatrix} x_{134} & x_{144} \\ x_{234} & x_{244} \end{pmatrix} \in \Lambda_{[1,2]1}^{1} \otimes \Lambda_{2,1,3}^{1}, \quad v_2(x) = \begin{pmatrix} x_{351} \\ x_{451} \end{pmatrix} \in \Lambda_{1,2,3}^{1}.
\]

We identify \( Z_{21} \) with \( M_{2} \otimes \text{Aff}^{2} \otimes \Lambda_{2} \otimes \text{Aff}^{2} \) by the map \( Z_{21} \ni x \mapsto (A(x), v_1(x), B(x), v_2(x)) \).
It is easy to see that
\[ A(gx) = t_1 g_{11} A(x)^t g_2, \quad B(gx) = t_{22} g_{11} B(x)^t g_{12}, \]
\[ v_1(gx) = (\det g_{12}) g_{22} v_1(x), \quad v_2(gx) = t_1 t_{21} g_{12} v_2(x). \]
By applying Lemma 1.5 to \((A(x), v_1(x)), (B(x), v_2(x))\), we have maps
\[ \Phi_1 : Z_{21} \to \Lambda^{2,1}_{2,1} \otimes \Lambda^{2,1}_{2,2} \to \Lambda^{2,1}_{2,1} \cong \text{Aff}^2, \]
\[ \Phi_2 : Z_{21} \to \Lambda^{2,1}_{2,1} \otimes \Lambda^{2,1}_{2,2} \to \Lambda^{2,1}_{2,1} \cong \text{Aff}^2. \]
Let \(P_1(x)\) be the determinant of \((\Phi_1(x), \Phi_2(x)) \in M_2\). Then
\[ \Phi_1(gx) = (\det g_{12}) t_1 (\det g_2) g_{11} \Phi_1(x), \quad \Phi_2(gx) = (\det g_{12}) t_1 t_{21} t_{22} g_{11} \Phi_2(x), \]
\[ P_1(gx) = (\det g_{11})(\det g_{12})^2 t_1^2 t_2 t_{21} (\det g_2) t_{22} P_1(x). \]
Let \(P_2(x) = \det A(x), P_3(x) = \det B(x)\). Then
\[ P_2(gx) = (\det g_{11}) t_1^2 (\det g_2) P_2(x), \quad P_3(gx) = (\det g_{11})(\det g_{12}) t_1^2 t_{22} P_3(x). \]
We put \(P(x) = P_1(x)^2 P_2(x) P_3(x)\). Then on \(M_{21}^1\),
\[ P(gx) = (\det g_{11})(\det g_{12})^2 t_1^2 t_{21} (\det g_2) t_{22}^2 \]
\[ \times (\det g_{11}) t_1^2 (\det g_2)(\det g_{12}) t_{22}^2 P(x) \]
\[ = (\det g_{11})^4 (\det g_{12})^5 t_1^2 t_{21} (\det g_2) t_{22}^2 P(x) = (\det g_{12}) t_1^2 (\det g_2) t_{22}^2 P(x). \]
Therefore, \(P(x)\) is invariant under the action of \(G_{st,21}\).
Let \(R(21) \in Z_{21}^1\) be the element such that
\[ A(R(21)) = B(R(21)) = I_2, \quad v_1(R(21)) = [1,0], \quad v_2(R(21)) = [0,1]. \]
Explicitly, \(R(21) = e_{451} + e_{152} + e_{342} + e_{253} + e_{134} + e_{244}\). Then \(\Phi_1(R(21)) = [0,-1], \)
\(\Phi_2(R(21)) = [1,0]\). So \(P_i(R(21)) = 1\) for \(i = 1, 2, 3\) and so \(R(21) \in Z_{21}^k\).
We show that \(Z_{21}^{s1} = M_{21,k} R(21)\). Suppose that \(x \in Z_{21}^{s1}\). Since \(\det A(x), \)
\(\det B(x) \neq 0\), there exists \(g \in M_{21,k}\) such that \(A(gx) = B(gx) = I_2\). So we may assume that \(A(x) = B(x) = I_2\). Then \(\Phi_1(x) = [x_{343}, -x_{342}], \Phi_2(x) = [x_{451}, -x_{351}]\). Since \(P_1(x) \neq 0\), either \(x_{342} \neq 0\) or \(x_{343} \neq 0\). Let \(\tau_0\) be the element in \(2.1\). By applying \((\tau_0, \tau_0, \tau_0, 1, 1, 1)\) if necessary, we may assume that \(x_{342} \neq 0\).
Let \(g = (I_2, I_2, x_{342}^{-1} I_2, x_{342}, 1, 1)\). Then \(A(gx) = B(gx) = I_2\) and the \(x_{342}\)-coordinate
of \(gx\) is 1. So we may assume that \(x_{342} = 1\). Let \(h = n_2(-x_{343})\) (see Section 2.2) and \(g = (t^i h^{-1}, h, h, 1, 1, 1)\). Then \(A(gx) = B(gx) = I_2, v_1(x) = [1, 0]\). Since \(P_1(x) \neq 0, x_{451} \neq 0\). By applying the element \((I_2, I_2, I_2, 1, x_{451}, 1)\) to \(x\), we may assume that \(x_{451} = 1\). Let \(h = t^i n_2(-x_{351})\) and \(g = (t^i h^{-1}, h, h, 1, 1, 1)\). Then \(gx = R(21)\).
Therefore, \(Z_{21}^{s1} = M_{21,k} R(21)\).
We assume that \(u_{11j} = 0\) for \((i, j) = (2,1), (4,3)\) and \(u_{232} = 0\). Then the first component of \(n(u) R(21)\) is the same as that of \(R(21)\) and the remaining components
are as follows:
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & u_{131} + u_{154} \\
0 & 0 & -1 & u_{141} - u_{153} + u_{221} & 0 \\
-1 & 0 & * & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{132} \\
0 & 0 & 0 & 0 & u_{142} + u_{231} \\
-1 & 0 & * & * & 0
\end{pmatrix}
\]
where $Q_1(u), Q_2(u)$ are polynomials which do not depend on $u_{151}, u_{152}, u_{241}$.

We can apply Lemma 11.9 to the map $\text{Aff}^3 \to \text{Aff}^9$ defined by the sequence

\[
\begin{align*}
&u_{132}, u_{153} + u_{242}, u_{154} + u_{243}, u_{131} + u_{154}, u_{141} - u_{132} - u_{242}, \\
u_{221} + u_{141} - u_{153}, u_{142} + u_{231}, u_{151} - Q_1(u), u_{241} - u_{152} + Q_2(u)
\end{align*}
\]

where $u_{152}, u_{231}, u_{242}, u_{243}$ are extra variables. So by Proposition 11.7, Property 11.5 holds for any $x \in Z_{21k}^{ss}$. Therefore, $Y_{21k}^{ss} = P_{b_{21}k}R(21)$ also.

(10) $S_{33}, \beta_{33} = \frac{1}{340}(-16, -6, 4, 4, 14, -15, -5, 5, 15)$.

We identify the element $(\text{diag}(t_{11}, t_{12}, g_1, t_{13}), \text{diag}(t_{21}, t_{22}, t_{23}, t_{24})) \in M_{[1,2,4],[1,2,3]} = M_{b_{33}}$ with the element $g = (g_1, t_{11}, \ldots, t_{24}) \in \text{GL}_2 \times \text{GL}_1^4$. On $M_{b_{33}}$,

\[
\chi_{33}(g) = t_{11}^{-14}t_{12}^{-6t}t_{13}^{14}t_{14}^{-15}t_{21}^{-5}t_{22}^{5}t_{23}^{15}t_{24}^{10} = t_{12}^{10}(\text{det} g_1)^{20}t_{13}^{10}t_{22}^{10}t_{23}^{10}t_{24}^{10}.
\]

For $x \in Z_{33}$, let

\[
v_1(x) = [x_{351}, x_{451}], v_2(x) = [x_{233}, x_{243}], v_3(x) = [x_{134}, x_{144}].
\]

We identify $Z_{33} \cong (\text{Aff}^2)^{3\oplus} \oplus 1^{3\oplus}$ by the map $Z_{33} \ni x \mapsto (v_1(x), v_2(x), v_3(x), x_{252}, x_{342}, x_{153})$.

We put

\[
P_1(x) = \text{det}(v_1(x) v_2(x)), P_2(x) = \text{det}(v_1(x) v_3(x)), P_3(x) = \text{det}(v_2(x) v_3(x))
\]

and $P(x) = P_1(x)^2P_2(x)^5P_3(x)^5x_{252}^2x_{342}^2x_{153}$. Then on $M_{b_{33}}^1$,

\[
\begin{align*}
v_1(gx) &= t_{13}t_{21}g_1v_1(x), v_2(gx) = t_{12}t_{23}g_1v_2(x), v_3(gx) = t_{11}t_{24}g_1v_3(x), \\
P_1(gx) &= t_{12}(\text{det} g_1)t_{13}t_{21}v_1(x), P_2(gx) = t_{11}(\text{det} g_1)t_{13}t_{21}P_2(x), \\
P_3(gx) &= t_{11}t_{12}(\text{det} g_1)t_{23}t_{24}P_3(x),
\end{align*}
\]

\[
P(gx) = (t_{12}(\text{det} g_1)t_{13}t_{21}t_{23})^2(t_{11}(\text{det} g_1)t_{13}t_{21}t_{24})^5(t_{11}t_{12}(\text{det} g_1)t_{23}t_{24})^5
\times (t_{12}t_{13}t_{22})^6((\text{det} g_1)t_{22})^2(t_{11}t_{13}t_{23})^2P(x)
\]

\[
= t_{12}^{10}(\text{det} g_1)^{14}t_{13}^{10}t_{14}^{10}t_{22}^{10}t_{23}^{10}t_{24}^{10}P(x) = t_{12}(\text{det} g_1)^{20}t_{13}^{10}t_{22}^{10}t_{23}^{10}t_{24}^{10}P(x).
\]

Therefore, $P(x)$ is invariant under the action of $G_{st, b_{33}}$.

Let $R(33) \in Z_{33}$ be the element such that

\[
v_1(R(33)) = [1, 0], v_2(R(33)) = [0, 1], v_3(R(33)) = [1, 1]
\]

and that the $x_{252}, x_{342}, x_{153}$-coordinates are 1. Explicitly, $R(33) = e_{351} + e_{252} + e_{342} + e_{153} + e_{243} + e_{134} + e_{144}$. Then $P_1(R(33)) = P_2(R(33)) = 1$, $P_3(R(33)) = -1$ and so $R(33) \in Z_{33k}^{ss}$.

We show that $Z_{33k}^{ss} = M_{b_{33}k}R(33)$. Suppose that $x \in Z_{33k}^{ss}$. It is easy to see that there exists $g \in M_{b_{33}k}$ such that $v_1(gx) = [1, 0], v_2(gx) = [0, 1]$. So we may assume
that \( v_1(x) = [1, 0], v_2(x) = [0, 1] \). By assumption, \( x_{134}, x_{144}, x_{252}, x_{342}, x_{153} \neq 0 \). If
\[
t = (t_1^{-1}, x_{144}, \text{diag}(x_{134}, x_{144}), t_1^{-1} x_{134}, t_2, t_1, t_2, 1)
\]
then \( v_1(tx) = [1, 0], v_2(tx) = [0, 1], v_3(tx) = [1, 1] \) and the \( x_{252}, x_{342}, x_{153} \)-coordinates of \( tx \) are
\[
t_1^{-1} t_2 x_{134} x_{144} x_{252}, t_2 x_{134}^{-1} x_{144} x_{342}, t_1^{-1} t_2 x_{134} x_{153}.
\]
We can choose \( t_2, t_1, t_2 \) in this order so that \( x_{342}, x_{252}, x_{153} \)-coordinates of \( tx \) are 1.
So, there exists such \( t \) such that \( tx = R(33) \). Therefore, \( Z^{ss}_{33k} = M_{33k} \cdot R(33) \).

We assume that \( u_{143} = 0 \). Then the four components of \( n(u)R(33) \) are as follows:
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & u_{132} + u_{154} + u_{221} \\
0 & 0 & -1 & u_{142} - u_{153} \\
0 & -1 & * & * & 0
\end{pmatrix},
\]
where \( Q_1(u), \ldots, Q_4(u) \) are polynomials which do not depend on \( u_{131}, u_{141}, u_{151}, u_{152}, u_{231}, u_{241}, u_{242} \) and \( Q_5(u), Q_6(u) \) are polynomials which do not depend on \( u_{141}, u_{241} \).

We can apply Lemma 11.9 to the map \( \text{Aff}^{15} \to \text{Aff}^{13} \) defined by the sequence
\[
u_{121} + u_{154} + u_{232},
u_{132} + u_{232} + u_{221} + u_{132} + u_{154},u_{242} + Q_3(u),u_{131} - u_{141} - u_{242} + Q_4(u),u_{231} + Q_5(u),u_{152} - u_{142} + Q_2(u),u_{151} - Q_6(u),u_{241} - u_{151} + Q_5(u)
\]
where \( u_{141}, u_{153} \) are extra variables. So by Proposition 11.7, Property 11.5 holds for any \( x \in Z^{ss}_{33k} \). Therefore, \( Y^{ss}_{33k} = P_{33k} \cdot R(33) \) also.

(11) \( S_{35}, \beta_{35} = \frac{1}{10}(-4, 1, 1, 1, 0, 0, 0, 0) \)

Since \( G_{\beta_{35}} \) is semi-simple, any relative invariant polynomial is invariant with respect to \( G_{\beta_{35}} \).

Let \( W = \text{Aff}^{4} \). As in the case \( S_{16}, \) by the Castling transform, there is a bijective correspondence between \( (\text{GL}_4 \times \text{GL}_4)(\mathbb{A}^2 W \otimes \text{Aff}^{4})^{ss} \) and \( (\text{GL}_4 \times \text{GL}_2)(\mathbb{A}^2 W \otimes \text{Aff}^{4})^{ss} \). Since the latter is in bijective correspondence with \( \text{Ex}_2(k) \), so is \( M_{35k} \cdot Z^{ss}_{35k} \). Since \( W_{35} = \{0\} \), \( P_{35k} \cdot Y^{ss}_{35k} \) is in bijective correspondence with \( \text{Ex}_2(k) \) also.

(12) \( S_{36}, \beta_{36} = \frac{1}{30}(-6, -1, -1, 4, 4, -5, 0, 0, 5) \)
We identify the element $(\text{diag}(t_1, g_{11}, g_{12}), \text{diag}(t_{21}, g_{2}, t_{2})) \in M_{[1,3],[1,3]} = M_{\beta_{36}}$ with the element $g = (g_{12}, g_{11}, g_{2}, t_1, t_{21}, t_{2}) \in \text{GL}_2 \times \text{GL}_1$. On $M_{\beta_{36}}^1$,
\[
\chi_{36}(g) = t_1^{-6}(\det g_{11})^{-1}(\det g_{12})^4 t_2^{-5} = (\det g_{11})^5(\det g_{12})^{10}(\det g_{2})^{5} t_2^{10}.
\]
For $x \in Z_{36}$, let
\[
A_1(x) = \begin{pmatrix} x_{242} & x_{252} \\ x_{342} & x_{352} \end{pmatrix}, \quad A_2(x) = \begin{pmatrix} x_{243} & x_{253} \\ x_{343} & x_{353} \end{pmatrix}, \quad v(x) = [x_{144}, x_{154}] \in \Lambda_{2,1}^{1}\Lambda_{2,1,2,3}^{1},
\]
and $A(x) = (A_1(x), A_2(x)) \in \Lambda_{1,2,3}^{1,2,3} \otimes \Lambda_{1,2}^{1,2,3}$. We identify $Z_{36}$ with $M_2 \otimes \text{Aff}^2 \oplus \text{Aff}^2 \oplus 1^2$ by the map $Z_{36} \ni x \mapsto (A_1(x), A_2(x), v(x), x_{451}, x_{234})$.
It is easy to see that
\[
\begin{pmatrix} A_1(gx) \\ A_2(gx) \end{pmatrix} = g_2 \begin{pmatrix} g_11 A_1(x) \\ g_12 A_2(x) \end{pmatrix}, \quad v(gx) = t_1 t_2 g_12 v(x).
\]
We apply Lemma 4.4 to $(A(x), v(x))$ and obtain a map $\Phi : Z_{36} \to \Lambda_{1,2,3}^{1,2,3} \otimes \Lambda_{2,1,2,3}^{1,2,3} \cong M_2$
Then
\[
(11.12) \quad \Phi(x) = t_1 \det(g_{12}) t_2 (g_{11}, g_2) \Phi(x)
\]
where the right hand side is the natural action. We put $P_1(x) = \det \Phi(x)$. Then
\[
P_1(gx) = (t_1 \det(g_{12}) t_2)^2 (\det g_{11})(\det g_{2}) P_1(x).
\]
Let $P_2(x)$ be the degree 4 polynomial of $A(x)$ obtained by Proposition 4.4. Then
\[
P_2(gx) = (\det g_{11})^2(\det g_{12})^2(\det g_{2})^2 P_2(x).
\]
We put $P(x) = P_1(x)^2 P_2(x)x_{451}^2 x_{234}^2$. Then on $M_{\beta_{36}}^1$,
\[
P(gx) = ((t_1 \det(g_{12}) t_2)^2 (\det g_{11})(\det g_{2})^2((\det g_{11})^2(\det g_{12})^2(\det g_{2})^2)
\]
\[
\times ((\det g_{12}) t_2)^2 (\det g_{11})^2 t_2^2 P(x)
\]
\[
= t_1^4 (\det g_{11})^6 (\det g_{12})^8 (\det g_{2})^4 t_2^6 P(x)
\]
\[
= (\det g_{11})^2(\det g_{12})^4(\det g_{2})^2 t_2^4 P(x).
\]
Therefore, $P(x)$ is invariant under the action of $G_{st, \beta_{36}}$.
Suppose that $x \in Z_{36}^{ss, k}$. Then $x_{451}, x_{234} \neq 0$. If $t = (I_2, I_2, I_2, 1, x_{451}, x_{234})$ then the $x_{451}, x_{234}$-coordinates of $tx$ are 1. Then $g = (g_{12}, g_{11}, g_{2}, t_1, t_{21}, t_{2})$ does not change the $x_{451}, x_{234}$-coordinates of $x$ if and only if $t_1 = (\det g_{11}), t_{21} = (\det g_{2})^{-1}, t_{2} = (\det g_{11})^{-1}$ and its action $g$ on $(A(x), v(x))$ is the same as that in Proposition 10.5.
Therefore, $M_{\beta_{36}}^1 \backslash Z_{36}^{ss, k}$ is in bijective correspondence with $\text{Ex}_2(k)$.
Let $R(36) \in Z_{36}$ be the element such that
\[
A_1(R(36)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2(R(36)) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad v(R(36)) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]
and that the $x_{451}, x_{234}$-coordinates are 1. Explicitly, $R(36) = e_{451} + e_{242} + e_{353} - e_{144} + e_{154} + e_{234}$. Then $P_1(R(36)) = P_2(R(36)) = 1$ and so $R(36) \in Z_{36}^{ss, k}$. 

We assume that \( u_{132} = u_{154} = 0 \) and \( u_{232} = 0 \). Then the first component of \( n(u)R(36) \) is the same as that of \( R(36) \) and the remaining components are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -u_{152} + u_{221} & 0 \\
0 & 0 & 0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{143} + u_{231} \\
0 & 0 & 0 & -1 & 0
\end{pmatrix},
\]

where \( Q(u) \) is a polynomial which does not depend on \( u_{141}, u_{151}, u_{241} \).

We can apply Lemma 11.9 to the map \( \text{Aff}^{13} \to \text{Aff}^7 \) defined by the sequence

\[
u_{221} - u_{152}, u_{231} + u_{143}, u_{242} - u_{121}, u_{143}, u_{153} + u_{121}, u_{142} + u_{131}, u_{243} + u_{131} - u_{152}, u_{151} + u_{141} + u_{241} + Q(u)\]

where \( u_{121}, u_{131}, u_{141}, u_{143}, u_{152}, u_{241} \) are extra variables. So by Proposition 11.7, Property 11.5 holds for any \( x \in Z_{36}^{ss} \). Therefore, \( P_{36,k} \setminus V_{36,k}^{ss} \) is in bijective correspondence with \( \text{Ex}_2(k) \) also.

(13) \( S_{37}, \beta_{37} = \frac{1}{386}(−12, −2, −2, 8, 8, −5, −5, 5, 5) \)

We identify the element \( \chi(\text{diag}(t_1, g_{11}, g_{12}), \text{diag}(g_{21}, g_{22})) \in M_{[1,3],[2]} = M_{\beta_{37}} \) with the element \( g = (g_{11}, \ldots, g_{22}, t_1) \in \text{GL}_2 \times \text{GL}_4 \). On \( M_{\beta_{37}}^1 \),

\[
\chi_{37}(g) = t_1^{-12}(\det g_{11})^{-2}(\det g_{12})^8(\det g_{21})^{-5}(\det g_{22})^5
= (\det g_{11})^{10}(\det g_{12})^{20}(\det g_{22})^{10}.
\]

For \( x \in Z_{37} \), let

\[
A(x) = (x_{241}, \ldots, x_{352}) \in \Lambda_{1,2,3}^{2,1} \otimes \Lambda_{1,4,5}^{2,1} \otimes \Lambda_{2,1,2}^{2,1};
\]

\[
B(x) = \begin{pmatrix} x_{143} & x_{144} \\ x_{153} & x_{154} \end{pmatrix} \in \Lambda_{1,4,5}^{2,1} \otimes \Lambda_{2,3,4}^{2,1};
\]

\[
v(x) = \begin{pmatrix} x_{233} \\ x_{234} \end{pmatrix} \in \Lambda_{2,3,4}^{2,1}.
\]

We identify \( Z_{37} \) with \( \text{Aff}^2 \otimes \text{Aff}^2 \otimes \text{Aff}^2 \oplus M_2 \oplus \text{Aff}^2 \) by the map \( Z_{37} \ni x \mapsto (A(x), B(x), v(x)) \).

It is easy to see that

\[
A(gx) = (g_{11}, g_{12}, g_{21})A(x), \quad B(gx) = t_1 g_{12} B(x) g_{22}, \quad v(gx) = (\det g_{11}) g_{22} v(x)
\]

where the right hand sides are the natural actions. We apply Lemma 11.5 twice to \((A(x), B(x), v(x))\) and obtain a map

\[
\Phi : Z_{37} \to \Lambda_{1,2,3}^{2,1} \otimes \Lambda_{1,4,5}^{2,1} \otimes \Lambda_{2,1,2}^{1,2} \otimes \Lambda_{1,4,5}^{2,1} \otimes \Lambda_{2,1,2}^{2,1} \to \Lambda_{1,2,3}^{2,1} \otimes \Lambda_{2,1,2}^{2,1}.
\]

Then \( \Phi(gx) = t_1 (\det g_{11})(\det g_{12})(\det g_{22})(g_{11}, g_{21}) \Phi(x) \). Identifying \( \Lambda_{1,2,3}^{2,1} \otimes \Lambda_{2,1,2}^{2,1} \) with \( M_2 \), we put \( P_1(x) = \det \Phi(x) \). Then

\[
P_1(gx) = t_1^3 (\det g_{11})^3 (\det g_{12})^2 (\det g_{22})^2 P_1(x).
\]
Let $P_2(x)$ be the degree 4 polynomial of $A(x)$ obtained by Proposition \ref{prop:4.1}. Then

$$P_2(gx) = (\det g_{11})^2(\det g_{12})^2(\det g_{21})^2P_2(x).$$

Let $P_3(x) = \det B(x)$. Then $P_3(gx) = t_1^2(\det g_{12})(\det g_{22})P_3(x)$. We put $P(x) = P_1(x)P_2(x)P_3(x)$. Then on $M_{\beta_{37}}^1$,

$$P(gx) = (t_1^2(\det g_{11})^3(\det g_{12})^2(\det g_{22}))^3((\det g_{11})^2(\det g_{12})^2(\det g_{22})^2)^3$$

$$\times (t_1^2(\det g_{12})(\det g_{22}))^4P(x)$$

$$= t_1^{14}(\det g_{11})^{15}(\det g_{12})^{16}(\det g_{21})^9(\det g_{22})^9P(x)$$

$$= (\det g_{11})(\det g_{12})^2(\det g_{22})P(x).$$

Therefore, $P(x)$ is invariant under the action of $G_{st,\beta_{37}}$.

Let $R(37) \in Z_{\beta_{37}}$ be the element such that

$$A(R(37)) = (1,0,\ldots,0,1), \quad B(R(37)) = I_2, \quad v(R(37)) = [-1,1].$$

Explicitly,

$$R(37) = e_{241} + e_{352} + e_{143} - e_{233} + e_{154} + e_{234}.$$ Then

$$\Phi_1(R(37)) = \text{diag}(1,-1), P_1(R(37)) = -1, P_2(R(37)) = P_3(R(37)) = 1.$$ So $P(R(37)) = 1$ and $R(37) \in Z_{\beta_{37}}^{ss}$.

Suppose that $x \in Z_{\beta_{37} k}^{ss}$. Then there exists $g_{22} \in \text{GL}_2(k)$ such that $B(x)g_{22} = I_2$. So we may assume that $B(x) = I_2$. Then $g = (g_{11}, g_{12}; g_{21}, g_{22}; t_1)$ does not

change this condition if and only if $g_{22} = t_1^{-1}g_{12}^{-1}$. If $g = (g_{11}, g_{12}, g_{21}, t_1^{-1}g_{12}^{-1}, t_1)$ then $A(gx) = (g_{11}, g_{12}, g_{21})$ and $v(gx) = (\det g_{11})^{-1}t_1^{-1}g_{12}^{-1}v(x)$. Substituting $g_{12}, g_{21}$ by $(\det g_{11})t_1g_{12}, (\det g_{11})^{-1}t_1g_{21}$ respectively, the action of $g$ on $A(x), v(x)$ are the natural action and the action by $t_1^g$. Since this action by $t_1^g$ is still the standard representation, if we exchange the order of $(g_{11}, g_{12}, g_{21})$ then Proposition \ref{prop:10.5} implies that $M_{\beta_{37} k} \backslash Z_{\beta_{37} k}^{ss}$ is in bijective correspondence with $E_{x_2}(k)$.

We assume that $u_{132} = u_{154} = 0$ and $u_{221} = u_{243} = 0$. Then the four components of $n(u)R(37)$ are as follows:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -u_{152} \\
0 & 0 & 0 & * & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{143} \\
0 & 0 & -1 & * & 0
\end{pmatrix},$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & u_{121} - u_{143} + u_{231} & -u_{153} \\
0 & 1 & 0 & u_{131} + u_{142} & u_{152} + u_{232} \\
-1 & * & * & 0 & Q_1(u) - u_{151} \\
0 & * & * & * & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & u_{143} + u_{241} & u_{121} + u_{153} \\
0 & -1 & 0 & -u_{142} & u_{131} - u_{152} + u_{242} \\
0 & * & 0 & * & Q_2(u) + u_{141} \\
-1 & * & * & * & 0
\end{pmatrix}.$$ 

where $Q_1(u), Q_2(u)$ are polynomials which do not depend on $u_{141}, u_{151}$. 


We can apply Lemma 11.3 to the map \( \text{Aff}^{12} \to \text{Aff}^{12} \) defined by the sequence
\[
\begin{align*}
&u_{142}, u_{143}, u_{152}, u_{153}, u_{232} + u_{152}, u_{121} + u_{153}, u_{231} + u_{121} - u_{143}, u_{131} + u_{142}, \\
u_{241} + u_{143}, u_{242} + u_{131} - u_{152}, u_{151} - Q_1(u), u_{141} + Q_2(u)
\end{align*}
\]
with no extra variables. So by Proposition 11.7 Property 11.3 holds for any \( x \in Z_{37}^{ss} \).
Therefore, \( P_{\beta_{40}} \setminus Y_{37}^{ss} \) is in bijective correspondence with \( \text{Ex}_2(k) \) also.

(14) \( S_4 \), \( \beta_{40} = \frac{1}{180}(-7, -2, -2, 3, 8, -5, 0, 0, 5) \)
We identify the element \((\text{diag}(t_{11}, g_1, t_{12}, t_{13}), \text{diag}(t_{21}, g_2, t_{22})) \in M_{[1, 3, 4], [1, 3]} = M_{\beta_{40}}\)
with the element \( g = (g_1, g_2, t_{11}, \ldots, t_{22}) \in \text{GL}_2^g \times \text{GL}_1^5 \). On \( M_{\beta_{40}}^\dagger \),
\[
\chi_{40}(g) = (t_{11})^{-7}(\det g_1)^{-2} t_{12}^3 t_{13}^4 t_{21}^3 t_{22}^5 = (\det g_1)^{5} t_{12}^{10} t_{13}^{15} (\det g_2)^{5} t_{22}^{10}.
\]
For \( x \in Z_{40} \), let
\[
A(x) = \begin{pmatrix}
 x_{242} & x_{243} \\
x_{342} & x_{343}
\end{pmatrix} \in \Lambda_{1,[2,3]}^2 \otimes \Lambda_{2,[2,3]}^1,
\]
\[
v_1(x) = [x_{251}, x_{351}] \in \Lambda_{1,[2,3]}^1, \quad v_2(x) = [x_{152}, x_{153}] \in \Lambda_{2,[2,3]}^1.
\]
We identify \( Z_{40} \) with \( \text{Aff}^2 \otimes \text{Aff}^2 \otimes \text{Aff}^2 \oplus \text{Aff}^2 \oplus 1^{2\oplus} \) by the map \( Z_{40} \ni x \mapsto (A(x), v_1(x), v_2(x), x_{144}, x_{234}) \).

It is easy to see that
\[
A(gx) = t_{12}g_1 A(x)^t g_2, \quad v_1(gx) = t_{13}t_{21}g_1 v_1(x), \quad v_2(gx) = t_{11}t_{13}g_2 v_2(x).
\]
The representation of \( M_{\beta_{40}} \) on \((A(x), v_1(x), v_2(x))\) can be identified with the representation considered in Proposition 11.6 except for the extra \( \text{GL}_1 \)-factors.

Let \( P_1(x) \) be the degree 3 polynomial of \((A(x), v_1(x), v_2(x))\) obtained by Proposition 11.6. Since \( P_1(x) \) is linear with respect to each of \( A(x), v_1(x), v_2(x) \),
\[
P_1(gx) = t_{11}(\det g_1) t_{12}^2 t_{13}^3 t_{21} t_2 (\det g_2) P_1(x).
\]
Let \( P_2(x) = \det A(x) \). Then \( P_2(gx) = (\det g_1) t_{12}^2 (\det g_2) P_2(x) \).

We put \( P(x) = P_1(x)^8 P_2(x)^5 x_{234}^4 x_{144}^3 x_{234} \). Then on \( M_{\beta_{40}}^3 \),
\[
P(gx) = t_{11}(\det g_1) t_{12}^2 t_{13}^3 t_{21} (\det g_2)^8 (\det g_1) t_{12}^5 (\det g_2) (t_{11} t_{12} t_{22})^5 (\det g_1) t_{22}^5 P(x)
\]
\[
= t_{11}^{15} (\det g_1)^{14} t_{12}^{15} t_{13}^{16} t_{21}^8 (\det g_2)^9 t_{22}^5 P(x) = (\det g_1) t_{12}^5 t_{13}^{3} (\det g_2) t_{22}^2 P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_{40}} \).

Let \( R(40) \in Z_{40} \) be the element such that
\[
A(R(40)) = I_2, \quad v_1(R(40)) = v_2(R(40)) = [1, 0]
\]
and that the \( x_{144}, x_{234} \)-coordinates are 1. Explicitly, \( R(40) = e_{251} + e_{152} + e_{242} + e_{343} + e_{144} + e_{234} \). Then \( P(R(40)) = 1 \) and so \( R(40) \in Z_{40}^{ss} \).

We show that \( Z_{40}^{ss} = M_{\beta_{40}} R(40) \). Suppose that \( x \in Z_{40}^{ss} \). Then \( x_{144}, x_{234} \neq 0 \). By Proposition 11.6 there exists \( g \in M_{\beta_{40}} \) such that \( A(gx) = I_2, v_1(gx) = v_2(gx) = [1, 0] \). So we may assume that \( A(x) = I_2, v_1(x) = v_2(x) = [1, 0] \). If \( t = (I_2, I_2, t_{11}, 1, t_{11}^2, t_{11}, t_{22}) \) then \( A(tx) = I_2, v_1(tx) = v_2(tx) = [1, 0] \) and the \( x_{144}, x_{234} \)-coordinates of \( tx \) are \( t_{11} t_{22} x_{144}, t_{22} x_{234} \). Therefore, there exists such \( t \) such that \( tx = R(40) \). Hence, \( Z_{40}^{ss} = M_{\beta_{40}} R(40) \).
We assume that $u_{132} = 0$ and $u_{232} = 0$. Then the four components of $n(u)R(40)$ are as follows:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u_{142} \\
0 & -1 & 0 & 0 & 0
\end{pmatrix}
, 
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & u_{121} + u_{154} + u_{221} \\
0 & 0 & 0 & 0 & u_{131} \\
0 & -1 & 0 & 0 & Q_1(u) + u_{141} - u_{152} \\
-1 & * & * & * & 0
\end{pmatrix}
, 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u_{231} \\
0 & 0 & 1 & 0 & u_{154} \\
0 & -1 & 0 & Q_2(u) - u_{153} & * \\
* & * & * & * & 0
\end{pmatrix}
, 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & u_{121} + u_{143} + u_{242} & u_{154} + u_{242} \\
0 & -1 & 0 & u_{131} - u_{142} + u_{243} & Q_3(u) + u_{153} + u_{241} \\
-1 & * & * & 0 & Q_4(u) - u_{152} \\
* & * & * & * & Q_5(u) - u_{151}
\end{pmatrix}

$$

where $Q_1(u), \ldots, Q_4(u)$ are polynomials which do not depend on $u_{141}, u_{151}, u_{152}, u_{153}, u_{241}$ and $Q_5(u)$ is a polynomial which does not depend on $u_{151}$.

We can apply Lemma 11.9 to the map $\text{Aff}^{14} \to \text{Aff}^{13}$ defined by the sequence

$$
\begin{align*}
u_{131}, u_{142}, u_{154}, u_{231}, u_{242} + u_{154}, u_{243} + u_{131} - u_{142}, u_{221} + u_{121} + u_{154}, \\
u_{143} + u_{121} + u_{242}, u_{152} - Q_4(u), u_{141} - u_{152} + Q_1(u), u_{153} - Q_2(u), \\
u_{241} + u_{153} + Q_3(u), u_{151} - Q_5(u)
\end{align*}
$$

where $u_{121}$ is an extra variable. So by Proposition 11.7, Property 11.5 holds for any $x \in Z_{40k}^{ss}$. Therefore, $Y_{40k}^{ss} = P_{\beta_{40k}}R(40)$ also.

(15) $S_{12}, \beta_{12} = \frac{3}{269}(-8, -8, -8, 12, 12, -15, 5, 5, 5)$

We identify the element $(\text{diag}(g_{11}, g_{12}), \text{diag}(t_2, g_2)) \in M_{[3],[1]} = M_{\beta_{12}}$ with the element $g = (g_{11}, g_2, g_{12}, t_2) \in \text{GL}_3^2 \times \text{GL}_2 \times \text{GL}_1$. On $M_{\beta_{12}}$,

$$
\chi_{42}(g) = (\det g_{11})^{-8}(\det g_{12})^{12}t_2^{-15}(\det g_2)^5 = (\det g_{12})^{20} (\det g_2)^{20}.
$$

For $x \in Z_{42}$, let

$$
A_1(x) = \begin{pmatrix} x_{142} & x_{143} & x_{144} \\
x_{242} & x_{243} & x_{244} \\
x_{342} & x_{343} & x_{344} \end{pmatrix}, \quad A_2(x) = \begin{pmatrix} x_{152} & x_{153} & x_{154} \\
x_{252} & x_{253} & x_{254} \\
x_{352} & x_{353} & x_{354} \end{pmatrix}
$$

and $A(x) = (A_1(x), A_2(x))$. We identify $Z_{42} \cong M_3 \otimes \text{Aff}^2 \oplus 1$ by the map $Z_{42} \ni x \mapsto (A(x), x_{451})$.

It is easy to see that

$$
\begin{pmatrix} A_1(g_x) \\
A_2(g_x) \end{pmatrix} = g_{12} \begin{pmatrix} g_{11}A_1(x)^t g_2 \\
g_{11}A_2(x)^t g_2 \end{pmatrix}.
$$

Let $P_1(x)$ be the degree 12 polynomial of $A(x)$ obtained by Proposition 4.2. Then

$$
P_1(g_x) = (\det g_{11})^4(\det g_{12})^6(\det g_2)^4P_1(x)
$$
We put $P(x) = P_1(x)x_{451}$. Then on $M_{\beta_{42}}^1$,

$$P(gx) = ((\det g_{11})^4(\det g_{12})^6(\det g_2)^4)(t_2(\det g_{12}))P(x)$$

$$= (\det g_{11})^4(\det g_{12})^7t_2(\det g_2)^4P(x) = (\det g_{12})^3(\det g_2)^3P(x).$$

Therefore, $P(x)$ is invariant under the action of $G_{st,\beta_{42}}$.

Suppose that $x \in Z_{42k}^{ss}$. Then $x_{451} \neq 0$. If $g = (I_3, I_3, x_{451}^{-1})$ then the $x_{451}$-coordinate of $gx$ is 1. If $x_{451} = 1$ then $g = (g_{11}, g_2, g_{12}, (\det g_{12})^{-1})$ does not change $x_{451}$ and acts on $A(x)$ by the tensor product of standard representations. Therefore, we can apply Propositions 11.2, 11.4 to $M_{\beta_{42}} \setminus Z_{42k}^{ss}$. Hence, $M_{\beta_{42}} \setminus Z_{42k}^{ss}$ is in bijective correspondence with $Ex_3(k)$.

Let $R(42)$ be the element such that $A(R(42)) = \text{diag}(1, -1, 0), \text{diag}(0, 1, -1))$ and the $x_{451}$-coordinate is 1. Explicitly, $R(42) = e_{451} + e_{142} - e_{243} + e_{253} - e_{354}$. Then $P(R(42)) = 1$ by Proposition 11.2.

We assume that $u_{1ij} = 0$ unless $i = 4, 5, j = 1, 2, 3$ and $u_{2ij} = 0$ unless $j = 1$. Then the four components of $n(u)R(42)$ are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}
\quad
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -u_{151} + u_{221} \\
0 & 0 & 0 & * & 0 \\
\end{pmatrix}
\quad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -u_{143} + u_{241} \\
\end{pmatrix}
\quad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & * & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

We can apply Lemma 11.9 to the map $\text{Aff}^9 \rightarrow \text{Aff}^3$ defined by the sequence

$$u_{221} - u_{151}, u_{231} + u_{142} + u_{152}, u_{241} - u_{143}$$

where $u_{1ij}$ ($i = 4, 5, j = 1, 2, 3$) are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{42k}^{ss}$. Therefore, $P_{\beta_{42}} \setminus Z_{42k}^{ss}$ is in bijective correspondence with $Ex_3(k)$ also.

(16) $S_{49}, \beta_{49} = \frac{1}{100}(-8, -4, 0, 4, 8, -9, -1, 3, 7)$

(17) $S_{50}, \beta_{50} = \frac{1}{100}(-4, -2, 0, 2, 4, -3, -1, 1, 3)$

For $l = 49, 50$, $M_{\beta_l} = T$ and so $Z_{42k}^{ss} \neq \emptyset$ by Proposition 11.4. Let $R(l) \in Z_l$ be element whose coordinates in $Z_l$ are all 1.

We express elements of $T$ as (11.3). Then the matrix $(m_{ij})$ of Proposition 11.4 is
for $S_{49}, S_{50}$ respectively.

The determinant of the $7 \times 7$ minor of columns 1, 3, 4, 5, 6, 7, 8 (resp. columns 1, 2, 3, 5, 6, 8, 9) is $-1$ (resp. 1). Therefore, $Z_{i,k}^{s_{i}} = T_k R(l)$ for $i = 49, 50$.

We consider the case $S_{49}$. Explicitly, $R(49) = e_{451} + e_{252} + e_{342} + e_{153} + e_{243} + e_{144} + e_{234}$.

The four components of $n(u) R(49)$ are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Q_1(u) + u_{142} - u_{153} + u_{221} & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & u_{121} + u_{154} + u_{232} & 0 & 0 \\
0 & 0 & 0 & u_{132} + u_{232} & Q_2(u) + u_{131} & 0 & 0 \\
0 & -1 & * & * & Q_3(u) + u_{141} - u_{152} + u_{231} & * & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & u_{121} + u_{143} + u_{243} & Q_4(u) + u_{153} + u_{242} & Q_6(u) - u_{152} & 0 \\
0 & -1 & 0 & Q_5(u) + u_{131} - u_{142} + u_{242} & * & * & 0 \\
-1 & * & * & 0 & Q_7(u) - u_{151} + u_{241} & * & 0
\end{pmatrix}
\]

where $Q_1(u), Q_2(u), Q_4(u), Q_5(u)$ do not depend on $u_{131}, u_{141}, u_{142}, u_{151}, u_{152}, u_{153}, u_{221}, u_{241}, u_{242}, Q_3(u), Q_6(u)$ do not depend on $u_{141}, u_{151}, u_{152}, u_{241}$ and $Q_7(u)$ does not depend on $u_{151}, u_{241}$.

We can apply Lemma 11.9 to the map $\text{Aff}_{16}^{s_{16}} \rightarrow \text{Aff}_{12}^{s_{12}}$ defined by the sequence

\[
\begin{align*}
u_{154} + u_{132} &+ u_{232} + u_{121} + u_{154} + u_{232} + u_{143} + u_{143} + u_{121} + u_{243}, \\
u_{131} + Q_2(u), u &_{153} + u_{242} + Q_4(u), u_{142} - u_{131} - u_{242} - Q_5(u), \\
u_{221} + u_{142} - u_{153} + Q_1(u), u_{152} - Q_6(u), u_{141} - u_{152} + Q_3(u), u_{241} - u_{151} + Q_7(u)
\end{align*}
\]

where $u_{132}, u_{151}, u_{231}, u_{242}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{49,k}^{s_{49}}$. Therefore, $Y_{49,k}^{s_{49}} = P_{49,k} R(49)$ also.

We consider the case $S_{50}$. $R(50) = e_{351} + e_{252} + e_{342} + e_{153} + e_{243} + e_{144} + e_{234}$. The four components of $n(u) R(50)$ are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & Q_1(u) + u_{142} - u_{153} & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & u_{132} + u_{154} + u_{221} & 0 & 0 \\
0 & 0 & 0 & u_{143} & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & * & * & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & u_{121} + u_{154} + u_{232} & 0 & 0 \\
0 & 0 & 0 & u_{132} + u_{232} & Q_2(u) + u_{131} + u_{231} & Q_3(u) + u_{141} - u_{152} & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & u_{121} + u_{154} + u_{232} & 0 & 0 \\
0 & -1 & * & * & Q_3(u) + u_{141} - u_{152} & 0 & 0
\end{pmatrix}
\]
where $Q_4(u), Q_5(u), Q_6(u), Q_7(u)$ do not depend on $u_{131}, u_{141}, u_{151}, u_{153}, u_{231}, u_{241}, u_{242}, u_{154}$ and $Q_7(u)$ does not depend on $u_{151}$. We can apply Lemma 11.9 to the map $\text{Aff}^{16} \to \text{Aff}^{13}$ defined by the sequence

\[ u_{143}, u_{154} + u_{243}, u_{141} + u_{232} + u_{232} + u_{154}, u_{132} + u_{221} + u_{232} + u_{154}, \]

\[ u_{153} - u_{142} = Q_1(u), u_{242} + u_{131} + Q_4(u), u_{131} - u_{142} + u_{242} + Q_5(u), \]

\[ u_{231} + u_{131} + Q_2(u), u_{141} - u_{152} + u_{231} + Q_3(u), u_{241} - u_{152} + Q_6(u), u_{151} - Q_7(u) \]

where $u_{142}, u_{152}, u_{243}$ are extra variables. So by Proposition 11.7, Property 11.5 holds for any $x \in Z_{50}^s$. Therefore, $Y_{50}^s = P_{50}^s R(50)$ also.

(18) $\beta_{64} = \frac{1}{220}(-58, -58, -58, -58, 32, -55, 5, 45)$

We identify the element $(\text{diag}(g_1, t_1), \text{diag}(t_{21}, g_2, t_{22})) \in M_{[4],[1,3]} = M_{\beta_{64}}$ with the element $g = (g_1, g_2, t_1, t_{21}, t_{22}) \in \text{GL}_4 \times \text{GL}_2 \times \text{GL}_3$. On $M_{\beta_{64}}$,

\[ \chi_{\beta_{64}}(g) = (\det g_1)^{-8} t_{21}^{-55} (\det g_{12})^{5} t_{22} = t_{1}^{40} (\det g_{12})^{60} t_{22}^{100}. \]

Let $P_{4,i}, P_{4,ij}, \ldots$, be as before. For $x \in Z_{64}$, let

\[ v_1(x) = [x_{152}, x_{252}, x_{352}, x_{452}], v_2(x) = [x_{153}, x_{253}, x_{353}, x_{453}], \]

\[ A(x) = (v_1(x) v_2(x)) \in M_{4,2}, B(x) = \begin{pmatrix} 0 & x_{124} & x_{134} & x_{144} \\ -x_{124} & 0 & x_{234} & x_{244} \\ -x_{134} & -x_{234} & 0 & x_{344} \\ -x_{144} & -x_{244} & -x_{344} & 0 \end{pmatrix}. \]

We identify $Z_{64} \cong M_{4,2} \oplus \wedge^2 \text{Aff}^4$ by the map $Z_{64} \ni x \mapsto (A(x), B(x))$. We also identify $B(x)$ with $\sum_{1 \leq i < j \leq 4} x_{ij} P_{4,ij} \in \wedge^2 \text{Aff}^4$.

It is easy to see that

\[ A(g x) = t_1 g_1 A(x) g_2, B(g x) = t_{22} (\wedge^2 g_1) B(x). \]

Let $P_1(x)$ be the Pfaffian of $B(x)$ and $P_2(x)$ the polynomial such that

\[ v_1(x) \wedge v_2(x) \wedge B(x) = P_2(x)p_{4,1234}. \]

We choose the sign of $P_1(x)$ so that $P(x) = 1$ if $B(x) = p_{2,12} + p_{2,34}$. By (11.13),

\[ P_1(g x) = (\det g_1)^2 t_{22}^2 P_1(x), P_2(g x) = (\det g_1)^t t_{22}^3 P_2(x). \]

We put $P(x) = P_1(x)P_2(x)^3$. Then on $M_{\beta_{64}}$,

\[ P(g x) = ((\det g_1)^t t_{22}^2)((\det g_1)^t t_{22}^2)^3 P(x) \]

\[ = (\det g_1)^4 t_{22}^6 P(x) = t_{22}^3 (\det g_2)^3 t_{22}^5 P(x). \]

Therefore, $P(x)$ is invariant under the action of $G_{\text{st}, \beta_{64}}$. 

Let \( R(64) \in Z_{64} \) be the element such that \( v_1(R(64)) = [1, 0, 0, 0] \), \( v_1(R(64)) = [0, 0, 1, 0] \) and \( B(R(64)) = p_{4,13} + p_{4,24} \). Explicitly, \( R(64) = e_{152} + e_{353} + e_{134} + e_{244} \). Then \( P_1(R(64)) = P_2(R(64)) = -1 \). So \( P(R(64)) = 1 \) and \( R(64) \in Z_{64}^{ss} \).

Suppose that \( x \in Z_{64}^{ss} \). We show that \( x \in M_{s\beta_{64}}R(64) \). By Lemma II–4.6 and applying the permutation matrix corresponding to the transposition \((2 3)\), there exists \( g \in M_{s\beta_{64}} \) such that \( B(gx) = B(R(64)) \). So we may assume that \( B(x) = B(R(64)) \).

By assumption,

\[
\text{det} \begin{pmatrix} x_{152} & x_{153} \\ x_{352} & x_{353} \end{pmatrix} + \text{det} \begin{pmatrix} x_{252} & x_{253} \\ x_{452} & x_{453} \end{pmatrix} \neq 0.
\]

Let \( \tau_0 \) be as in \([2.7]\), \( u, a, b, c, d \in k \) and

\[
\tau_1 = \text{diag}(\tau_0, \tau_0), \quad m_1(u) = \text{diag}(n_2(u), \tau n_2(-u)),
\]

\[
m_2(a, b, c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & 1 & 0 \\ b & c & 0 & 1 \end{pmatrix}, \quad s(a, b, c, d) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{(ad} - bc = 1).
\]

Regarding that \( \tau_1, m_1(u), m_2(a, b, c), s(a, b, c, d) \in \text{GL}_4 \subset M_{64} \), these elements fix \( B(R(64)) = p_{4,13} + p_{4,24} \).

By applying \( \tau_1 \) if necessary, we may assume that

\[
\text{det} \begin{pmatrix} x_{152} & x_{153} \\ x_{352} & x_{353} \end{pmatrix} \neq 0.
\]

By applying an element of the form \( s(a, b, c, d) \) if necessary, we may assume that \( x_{152} = 1 \). By applying an element of the form \( m_1(u) \), we may assume that \( x_{252} = 0 \). By applying an element of the form \( m_2(a, b, c) \), we may assume that \( x_{352} = x_{452} = 0 \), i.e., \( v_1(x) = [1, 0, 0, 0] \).

Then \( (11.15) \) is satisfied and so \( x_{353} \neq 0 \). Let \( h = \text{diag}(1, x_{353}^{-1}) \). By replacing \( x \) by \( (I_4, h, 1, 1, 1)x \), we may assume that \( x_{353} = 1 \). By applying an element of the form \( t^m \tau_1(u) \), we may assume that \( x_{153} = 0 \). Note that \( v_1(x) \) does not change. Then by applying an element of the form \( t^m_2(a, b, c) \), \( x \) becomes \( R(64) \). Therefore, \( Z_{64}^{ss} = M_{s\beta_{64}}R(64) \).

We assume that \( u_{1ij} = 0 \) unless \( i = 5 \) and \( u_{2ij} = 0 \). Then the first three components of \( n(u)R(64) \) are the same as those of \( R(64) \) and the last component is as follows:

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & u_{153} + u_{242} \\
0 & 0 & 0 & 1 & u_{154} \\
-1 & 0 & 0 & 0 & -u_{151} + u_{243} \\
0 & -1 & 0 & 0 & -u_{152} \\
* & * & * & * & 0
\end{pmatrix}.
\]

We can apply Lemma \([11.9]\) to the map \( \text{Aff}^9 \rightarrow \text{Aff}^4 \) defined by the sequence

\[
u_{153} + u_{242}, \quad u_{154}, \quad u_{151} - u_{243}, \quad u_{152}\]

where \( u_{221}, u_{231}, u_{241}, u_{242}, u_{243} \) are extra variables. So by Proposition \([11.7]\), Property \([11.5]\) holds for any \( x \in Z_{64}^{ss} \). Therefore, \( Y_{64,k}^{ss} = P_{s\beta_{64}}R(64) \) also.

(19) \( S_{\beta_0} = \frac{1}{146}(-6, -6, 4, 4, 4, -35, 5, 15, 15) \).
We identify the element \( (\text{diag}(g_{11}, g_{12}), \text{diag}(t_{21}, t_{22}, g_2)) \in M_{[2],[1,2]} = M_{\beta_7} \) with the element \( g = (g_{12}, g_{11}, g_2, t_{21}, t_{22}) \in GL_3 \times GL_2 \times GL_1^2 \). On \( M_{\beta_{30}}^1 \),
\[
\chi_{70}(g) = (\det g_{11})^{-6}(\det g_{12})^4 t_{21}^{-35} t_{22}^5 (\det g_2)^{15} = (\det g_{12})^{10} t_{22}^{40} (\det g_2)^{50}.
\]

For \( x \in Z_{70} \), let \( A(x) = x_{342}p_{3,12} + x_{352}p_{3,13} + x_{452}p_{3,23} \),
\[
B_1(x) = \begin{pmatrix} x_{133} & x_{134} \\ x_{233} & x_{234} \end{pmatrix}, \quad B_2(x) = \begin{pmatrix} x_{143} & x_{144} \\ x_{243} & x_{244} \end{pmatrix}, \quad B_3(x) = \begin{pmatrix} x_{153} & x_{154} \\ x_{253} & x_{254} \end{pmatrix}
\]
and \( B(x) = p_{3,1} \otimes B_1(x) + p_{3,2} \otimes B_2(x) + p_{3,3} \otimes B_3(x) \). We identify \( Z_{70} \) with \( \wedge^2 \text{Aff}^3 \otimes \text{Aff}^3 \otimes M_2 \) by the map \( Z_{70} \ni x \mapsto (A(x), B(x)) \). It is easy to see that
\[
A(gx) = t_{22}(\wedge^2 g_{12})A(x), \quad \begin{pmatrix} B_1(gx) \\ B_2(gx) \\ B_3(gx) \end{pmatrix} = g_{12} \begin{pmatrix} g_{11}B_1(x) & g_2 \\ g_{11}B_2(x) & g_2 \\ g_{11}B_3(x) & g_2 \end{pmatrix}.
\]

Since \( A(gx), B(gx) \) do not depend on \( t_{21} \), if we ignore \( t_{21} \in GL_1 \), then \( (M_{\beta_{70}}, Z_{70}) \) can be identified with the case (a) of Section 9.

Let \( P_1(x), P_2(x) \) be the relative invariant polynomials in \( \mathfrak{g}(9,8) \). Then
\[
P_1(gx) = (\det g_{11})(\det g_{12})^2 t_{22}^2 (\det g_2) P_1(x),
\]
\[
P_2(gx) = (\det g_{11})^3 (\det g_{12})^2 (\det g_2)^3 P_2(x).
\]

We put \( P(x) = P_1(x)^2 P_2(x) \). Then on \( M_{\beta_{70}}^1 \),
\[
P(gx) = ((\det g_{11})(\det g_{12})^2 t_{22}^2 (\det g_2))^2 ((\det g_{11})^3 (\det g_{12})^2 (\det g_2)^3) P(x)
\]
\[
= (\det g_{11})^5 (\det g_{12})^6 t_{22}^4 (\det g_2)^5 P(x) = (\det g_{12}) ^4 t_{22}^4 (\det g_2)^5 P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_{70}} \).

Let \( R(70) \in Z_{70} \) be the element such that \( (A(R(70)), B(R(70))) = R_{322,3} \) (see (9.9)). Explicitly, \( R(70) = e_{452} - e_{133} + e_{253} + e_{144} + e_{234} \). Then \( R(70) \in Z_{70}^{st} \), which implies that \( S_{70} \neq \emptyset \). By Proposition 9.19 if \( \text{ch}(k) \neq 2 \) then \( M_{\beta_{70}} \setminus Z_{70}^{st} \) is in bijective correspondence with \( \text{Ex}_2(k) \).

We assume that and \( u_{1i} = 0 \) unless \( i = 3, 4, 5, j = 1, 2 \) and \( u_{243} = 0 \). Then the first two components of \( n(u)R(70) \) are the same as those of \( R(70) \) and the remaining components are as follows:
\[
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & u_{141} & u_{151} + u_{132} \\
0 & 0 & * & 0 & u_{142} + u_{232} \\
0 & -1 & * & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & u_{131} & u_{142} - u_{152} \\
-1 & 0 & * & 0 & -u_{151} + u_{242} \\
0 & 0 & * & 0 & 0
\end{pmatrix}.
\]

We can apply Lemma 11.9 to the map \( \text{Aff}^1 \to \text{Aff}^6 \) defined by the sequence \( u_{141}, u_{151} + u_{132}, u_{142} + u_{232}, u_{131} - u_{142}, u_{152}, u_{242} - u_{151} \) where \( u_{132}, u_{221}, u_{231}, u_{232}, u_{241} \) are extra variables. So by Proposition 11.17 Property 11.9 holds for any \( x \in Z_{70, k}^{st} \). Therefore, if \( \text{ch}(k) \neq 2 \) then \( P_{\beta_{70}} \setminus Z_{70,k}^{st} \) is in bijective correspondence with \( \text{Ex}_2(k) \) also.

(20) \( S_{71}, \beta_{71} = \frac{1}{140}(-16, -16, -16, 24, 24, -35, -15, 25, 25) \)
We identify the element \((\text{diag}(g_{11}, g_{12}), \text{diag}(t_{21}, t_{22}, g_2))\) \(\in M_{[3],[1,2]} = M_{\beta_{71}}\) with the element \(g = (g_{11}, g_{12}, g_{12}, t_{21}, u_{22}) \in \text{GL}_3 \times \text{GL}_2^2 \times \text{GL}_1^2\). On \(M^1_{\beta_{71}}\),

\[
\chi_{71}(g) = (\det g_{11})^{-4}(\det g_{12})^{24}t_{21}^{-25}t_{22}^{-15}((\det g_2)^2)^{25} = (\det g_{12})^{40}t_{22}^{20}(\det g_2)^{60}.
\]

For \(x \in Z_{71}\), let

\[
A_1(x) = \begin{pmatrix} x_{143} & x_{153} \\ x_{144} & x_{154} \end{pmatrix}, \quad A_2(x) = \begin{pmatrix} x_{243} & x_{253} \\ x_{244} & x_{254} \end{pmatrix}, \quad A_3(x) = \begin{pmatrix} x_{343} & x_{353} \\ x_{344} & x_{354} \end{pmatrix}
\]

and \(A(x) = (A_1(x), A_2(x), A_3(x))\). We identify \(Z_{71}\) with \(\text{Aff}^3 \otimes \text{M}_2 \oplus 1\) by the map \(Z_{71} \ni x \mapsto (A(x), x_{452})\).

It is easy to see that the action of \((g_{11}, g_{12}, g_{12}, 1, 1)\) on \(A(x)\) is the tensor product of standard representations. So we are in the situation of Proposition [14.7]. Let \(P_1(x)\) be the degree 6 polynomial of \(A(x)\) obtained by Proposition [14.7]. Since \(A(gx)\) does not depend on \(t_{21}, t_{22}\), \(P_1(gx) = ((\det g_{11})^2(\det g_{12})^3(\det g_2)^3 P_1(x))\).

We put \(P(x) = P_1(x)x_{452}\). Then on \(M^1_{\beta_{71}}\),

\[
P(gx) = ((\det g_{11})^2(\det g_{12})^3(\det g_2)^3)((\det g_{12})t_{22})P(x) = (\det g_{12})^2t_{22}(\det g_2)^3P(x).
\]

Therefore, \(P(x)\) is invariant under the action of \(G_{st, \beta_{71}}\).

Let \(R(71) \in Z_{71}\) be the element such that \(A(R(71)) = R_{452}\) and the \(x_{452}\)-coordinate is 1. Explicitly, \(R(71) = e_{452} - e_{143} + e_{253} + e_{154} + e_{344}\). Then \(P(R(71)) = 1\) and so \(R(71) \in Z^e_{71}\).

Suppose that \(x \in Z^e_{71,k}\). By Proposition [14.6] there exists \(g \in M_{\beta_{71}, k}\) such that \(A(x) = A(R(71))\). Let \(t = (I_3, I_2, I_2, 1, x_{452})\). Then \(A(tx) = A(R(71))\) and the \(x_{452}\)-coordinate of \(tx\) is 1. Therefore, \(Z^e_{71,k} = M_{\beta_{71}, k}R(71)\).

We assume that \(u_{iij} = 0\) unless \(i = 4, 5, j = 1, 2, 3\). and \(u_{243} = 0\). Then the first two components of \(n(u)R(71)\) are the same as those of \(R(71)\) and the remaining components are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & u_{142} + u_{151} + u_{232} & 0 \\
0 & -1 & 0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & u_{141} - u_{153} + u_{242} \\
-1 & 0 & 0 & * & 0
\end{pmatrix}.
\]

We can apply Lemma [11.9] to the map \(\text{Aff}^1 \rightarrow \text{Aff}^2\) defined by the sequence \(u_{142} + u_{151} + u_{232}, u_{141} - u_{153} + u_{242}\) where \(u_{131}, u_{132}, u_{151}, u_{152}, u_{221}, \ldots, u_{242}\) are extra variables. So by Proposition [11.7] Property [11.5] holds for any \(x \in Z^e_{71,k}\). Therefore, \(Y^e_{71,k} = P_{\beta_{71,k}}R(71)\). also.

(21) \(S_{75}, \beta_{75} = \frac{1}{60}(-4, -4, 1, 1, 6, -15, 0, 5, 10)\)

We identify the element \((\text{diag}(g_{11}, g_{12}, t_1), \text{diag}(t_{21}, \ldots, t_{24}))\) \(\in M_{[2,4],[1,2,3]} = M_{\beta_{75}}\) with the element \(g = (g_{12}, g_{11}, t_1, t_{21}, \ldots, t_{24}) \in \text{GL}_2 \times \text{GL}_5\). On \(M^1_{\beta_{75}}\),

\[
\chi_{75}(g) = (\det g_{11})^{-4}(\det g_{12})^{6}t_{21}^{-15}t_{23}^{-6}t_{24}^{10} = (\det g_{12})^{5}t_{21}^{-10}t_{22}^{15}t_{23}^{20}t_{24}^{25}.
\]
For $x \in Z_{75}$, let

$$A(x) = \begin{pmatrix} x_{134} & x_{234} \\ x_{144} & x_{244} \end{pmatrix}, \quad v_1(x) = \begin{pmatrix} x_{352} \\ x_{452} \end{pmatrix}, \quad v_2(x) = \begin{pmatrix} x_{153} \\ x_{253} \end{pmatrix}. $$

We regard that $A(x) \in \Lambda_{1,3,4}^{2,1} \otimes \Lambda_{1,1,2}^{2,1}$, $v_1(x) \in \Lambda_{1,3,4}^{2,1}$, $v_2(x) \in \Lambda_{1,1,2}^{2,1}$. We identify $Z_{75}$ with $M_2 \oplus \text{Aff}^2 \oplus \text{Aff}^2 \oplus 1$ by the map $Z_{75} \ni x \mapsto (A(x), v_1(x), v_2(x), x_{343})$. It is easy to see that

$$A(gx) = t_{24}g_{12}A(x)^t g_{11}, \quad v_1(gx) = t_1 t_{22}g_{12}v_1(x), \quad v_2(gx) = t_1 t_{23}g_{11}v_2(x).$$

The representation of $M_{\beta_{75}}$ on $(A(x), v_1(x), v_2(x))$ can be identified with the representation considered in Proposition 4.6 except for the extra $\text{GL}_1$-factors.

Let $P_1(x)$ be the degree 3 polynomial obtained by Proposition 4.6. We put $P_2(x) = \det A(x)$. Then

$$P_1(gx) = (\det g_{11})(\det g_{12})t_1^2 t_{22} t_{23} t_{24} P_1(x), \quad P_2(gx) = (\det g_{11})(\det g_{12})t_{24}^2 P_2(x).$$

We put $P(x) = P_1(x)^2 P_2(x) x_{343}$. Then on $M_{\beta_{75}}^f$,

$$P(gx) = ((\det g_{11})(\det g_{12})t_1^2 t_{22} t_{23} t_{24})^3((\det g_{11})(\det g_{12})t_{24}^2)((\det g_{12}) t_{23}) P(x) = (\det g_{11})^4 (\det g_{12})^5 t_1^6 t_{22} t_{23} t_{24}^3 P(x) = (\det g_{12}) t_{24}^2 t_{23} t_{24}^3 P(x).$$

Therefore, $P(x)$ is invariant under the action of $G_{\text{st}, \beta_{75}}$.

Let $R(75) \in Z_{75}$ be the element such that $v_1(R(75)) = v_2(R(75)) = [1,0], A(R(75)) = I_2$ and the $x_{343}$-coordinate is 1. Explicitly, $R(75) = e_{352} + e_{153} + e_{343} + e_{134} + e_{244}$. Then $P_1(R(75)) = P_2(R(75)) = 1$. So $P(R(75)) = 1$ and $R(75) \in Z_{75}^s$.

We show that $Z_{75}^s = M_{\beta_{75}} R(75)$. Suppose that $x \in Z_{75}^s$. Then $x_{343} \neq 0$. By Proposition 4.6, there exists $g \in M_{\beta_{75}}$ such that $A(gx) = I_2, v_1(gx) = v_2(gx) = [1,0]$. So we may assume that $A(x) = I_2, v_1(x) = v_2(x) = [1,0]$. If $t = (I_2, I_2, 1, 1, 1, x_{343}, 1)$ then $tx = R(75)$. Therefore, $Z_{75}^s = M_{\beta_{75}} R(75)$.

We assume that $u_{121} = u_{143} = 0$. Then the first component of $n(u)R(75)$ is 0 and the remaining components are as follows:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

where $Q_1(u), Q_2(u)$ do not depend on $u_{151}, u_{152}, u_{242}$.

We can apply Lemma 4.9 to the map $\text{Aff}^4 \rightarrow \text{Aff}^7$ defined by the sequence

$$u_{154}, u_{153} + u_{243}, u_{141} - u_{153}, u_{131} + u_{154} + u_{232},$$

$$u_{132} - u_{141} + u_{243}, u_{151} - u_{242} - Q_1(u), u_{152} - Q_2(u)$$
where \( u_{142}, u_{221}, \ldots, u_{243} \) are extra variables. So by Proposition \([11.7]\), Property \([11.3]\) holds for any \( x \in Z_{ss}^{85k} \). Therefore, \( Y_{ss}^{85k} = P_{R,75k}R(75) \) also.

\[(22) \quad S_{95}, \beta_{95} = \frac{1}{630}(-28, -8, -8, 12, 32, -35, -15, 5, 45)\]

We identify the element \((\text{diag}(t_{11}, g_1 t_{12}, t_{13}), \text{diag}(t_{21}, \ldots, t_{24})) \in M_{[1,3],[1,2,3]} = M_{\beta_{95}}\) with the element \( g = (g_1 t_{11}, \ldots, t_{24}) \in \text{GL}_2 \times \text{GL}_1^2 \). On \( M_{\beta_{95}} \),

\[
\chi_{95}(g) = t_{11}^{-8} t_{12}^{12} t_{13}^{32} t_{14}^{-35} t_{15}^{-15} t_{16}^{-1} t_{17}^{5} t_{18}^{45} = (\det g_1)^2 t_{12}^{40} t_{13}^{60} t_{22}^{20} t_{23}^{40} t_{24}^{80}.
\]

For \( x \in Z_{95} \), let

\[
v_1(x) = [x_{252}, x_{352}], \quad v_2(x) = [x_{243}, x_{343}], \quad v_3(x) = [x_{124}, x_{134}],
\]

We identify \( Z_{95} \cong (\text{Aff}^2)^3 \oplus 1^{24} \) by the map \( Z_{95} \ni x \mapsto (v_1(x), v_2(x), v_3(x), x_{451}, x_{153}) \). It is easy to see that

\[
v_1(gx) = t_{13} t_{22} g_1 v_1(x), \quad v_2(gx) = t_{12} t_{23} g_1 v_2(x), \quad v_3(gx) = t_{11} t_{24} g_1 v_3(x).
\]

Let

\[
A(x) = (v_1(x), v_2(x)), \quad B(x) = (v_1(x), v_3(x)), \quad C(x) = (v_2(x), v_3(x)).
\]

We put \( P_1(x) = \det A(x), \quad P_2(x) = \det B(x), \quad P_3(x) = \det C(x) \). Then

\[
P_1(gx) = t_{12} t_{13} t_{22} t_{23} (\det g_1) P_1(x),
\]

\[
P_2(gx) = t_{11} t_{13} t_{22} t_{24} (\det g_1) P_2(x),
\]

\[
P_3(gx) = t_{11} t_{12} t_{23} t_{24} (\det g_1) P_3(x).
\]

We put \( P(x) = P_1(x)^2 P_2(x)^5 P_3(x)^5 x_{451} x_{153} \). Then on \( M_{\beta_{95}}^1 \),

\[
P(gx) = (t_{12} t_{13} t_{22} t_{23} (\det g_1))^2 (t_{11} t_{13} t_{22} t_{24} (\det g_1))^5 (t_{11} t_{12} t_{23} t_{24} (\det g_1))^5
\]

\[
\times (t_{12} t_{13} t_{21})^6 (t_{11} t_{13} t_{23}) P(x)
\]

\[
= t_{11}^4 (\det g_1)^{12} t_{12}^{14} t_{13}^{14} t_{21}^{14} t_{22}^{14} t_{23}^{14} t_{24}^{14} P(x) = (\det g_1)^{20} t_{12}^{40} t_{13}^{60} t_{22}^{20} t_{23}^{40} t_{24}^{80} P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st,\beta_{95}} \).

Let \( R(95) \) be the element such that \( v_1(R(95)) = [1, 0], \quad v_2(R(95)) = [0, 1], \quad v_3(R(95)) = [1, 1] \) and that the \( x_{451}, x_{153} \)-coordinates are 1. Then \( P_1(R(95)) = P_2(R(95)) = 1, \quad P_3(R(95)) = -1 \). So \( P(R(95)) = -1 \) and \( R(95) \in Z_{ss}^{85k} \). Explicitly, \( R(95) = e_{451} + e_{252} + e_{153} + e_{343} + e_{124} + e_{134} \).

We show that \( Z_{ss}^{85k} = M_{\beta_{95}} R(95) \). Suppose that \( x \in Z_{ss}^{85k} \). Since \( P_1(x) \neq 0 \), there exists \( g \in M_{\beta_{95}} \) such that \( v_1(gx) = [1, 0], \quad v_2(gx) = [0, 1] \). So we may assume that \( v_1(x) = [1, 0], \quad v_2(x) = [0, 1] \). Then elements of the form

\[
t = (\text{diag}(q_1, q_2), 1, q_2^{-1}, q_1^{-1}, t_{21}, 1, 1, t_{24})
\]

do not change this condition. Since \( P_2(x), P_3(x) \neq 0, \quad x_{124}, x_{134} \neq 0 \). Since \( v_3(tx) = t_{24}[q_1 x_{124}, q_2 x_{134}] \) and the \( x_{451}, x_{153} \)-coordinates of \( tx \) are \( q_1^{-1} q_2^{-1} t_{21} x_{451}, q_1^{-1} x_{153} \), there exists such \( t \) such that \( tx = R(95) \). Therefore, \( Z_{ss}^{85k} = M_{\beta_{95}} R(95) \).
We assume that $u_{132} = 0$. Then the first component of $n(u)R(95)$ is the same as that of $R(95)$ and the remaining components are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_{142} + u_{221} & 0 \\
0 & -1 & 0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{121} + u_{232} \\
0 & 0 & 0 & 1 & u_{131} + u_{154} \\
0 & 0 & 1 & 0 & Q_1(u) + u_{141} - u_{153} + u_{231} \\
-1 & * & * & * & 0
\end{pmatrix},
\]

where $Q_1(u) = Q_1(u_{142}, u_{143}, u_{154}, u_{232}), Q_2(u) = Q_2(u_{121}, u_{142}, u_{143}), Q_4(u) = Q_4(u_{131}, u_{142}, u_{143}), Q_3(u), Q_5(u), Q_6(u)$ do not depend on $u_{151}, u_{241}, u_{242}$ and $Q_6(u)$ does not depend on $u_{241}$.

We can apply Lemma 11.9 to the map $\text{Aff}^{15} \rightarrow \text{Aff}^{12}$ defined by the sequence

\[
u_{221} + u_{142}, u_{232} + u_{121} - u_{121}, u_{154} + u_{131}, u_{143} + u_{142}, u_{141} - Q_2(u),
\nu_{231} + u_{141} - u_{153} + Q_1(u), u_{243} - u_{141} + Q_4(u), u_{152} + u_{153} + u_{243},
\nu_{151} - Q_5(u), u_{242} - u_{151} + Q_3(u), u_{241} + Q_6(u)
\]

where $u_{121}, u_{142}, u_{153}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{95}^{\beta_{95}k}$. Therefore, $Y_{95}^{\beta_{95}k} = P_{\beta_{95}k}R(95)$ also.

(23) $S_{101}, \beta_{101} = \frac{1}{129}(-48, -28, 12, 32, 32, -25, -25, 15, 35)$

We identify the element $(\text{diag}(t_{11}, t_{12}, t_{13}, g_1), \text{diag}(g_2, t_{21}, t_{22})) \in M_{[1,2,3],[2,3]} = M_{\beta_{101}}$ with the element $g = (g_1, g_2, t_{11}, \ldots, t_{22}) \in \text{GL}_2 \times \text{GL}_4$. On $M_{\beta_{101}}^1$,

$$\chi_{101}(g) = t_{11}^{48} t_{12}^{28} t_{13}^{12} (\det g_1)^{32} (\det g_1)^{-25} t_{21}^{15} t_{22}^{35} = t_{12}^{20} t_{13}^{60} (\det g_1)^{80} t_{21}^{40} t_{22}^{60}.$$

For $x \in Z_{101}$, let

$$A(x) = \begin{pmatrix} x_{341} & x_{342} \\ x_{351} & x_{352} \end{pmatrix}, B(x) = \begin{pmatrix} x_{243} & x_{144} \\ x_{253} & x_{154} \end{pmatrix}.$$ 

We identify $Z_{101} \cong M_2 \oplus M_2 \oplus 1$ by the map $Z_{101} \ni x \mapsto (A(x), B(x), x_{234})$. Note that the second $M_2$ is not an irreducible representation of $M_{\beta_{101}}$. We sometimes consider $M_2$ which is a reducible representation of $\text{SL}_2$, but shall not point this out in the following.

Let $P_1(x) = \det A(x), P_2(x) = \det B(x)$. It is easy to see that

$$A(gx) = t_{13} g_1 A(x)^t g_2, B(gx) = g_1 B(x) \begin{pmatrix} t_{12} t_{21} & 0 \\ 0 & t_{11} t_{22} \end{pmatrix},$$

$$P_1(gx) = t_{13}^2 (\det g_1) (\det g_2) P_1(x), P_2(gx) = t_{11} t_{12} (\det g_1) t_{21} t_{22} P_2(x).$$

We put $P(x) = P_1(x)^4 P_2(x)^6 x_{234}$. Then on $M_{\beta_{101}}^1$,

$$P(gx) = (t_{13}^2 (\det g_1) (\det g_2))^4 (t_{11} t_{12} (\det g_1) t_{21} t_{22})^6 (t_{12} t_{13})^2 P(x)$$

$$= t_{11}^6 t_{12}^6 t_{13}^6 (\det g_1)^{10} (\det g_2)^4 t_{21} t_{22} P(x) = t_{12} t_{13}^3 (\det g_1)^{4} t_{21} t_{22} P(x).$$
Therefore, $P(x)$ is invariant under the action of $G_{st,3\nu_1}$.

Let $R(101)$ be the element such that $A(R(101)) = B(R(101)) = I_2$ and that the $x_{234}$-coordinate is 1. Explicitly, $R(101) = e_{341} + e_{352} + e_{243} + e_{154} + e_{234}$. Then $P_1(R(101)) = P_2(R(101)) = 1$. So $P(R(101)) = 1$ and $R(101) \in Z_{101}^{ss}$.

We show that $Z_{101k}^{ss} = M_{\beta_{101k}}R(101)$. Suppose that $x \in Z_{101k}^{ss}$. Since $P_1(x), P_2(x) \neq 0$, there exists $g \in M_{\beta_{101k}}$ such that $A(gx) = B(gx) = I_2$. So we may assume that $A(x) = B(x) = I_2$. Then elements of the form $t = (I_2, I_2, 1, t_{12}, 1, t_{12}^{-1}, 1)$ do not change this condition. Since the $x_{234}$-coordinate of $tx$ is $t_{12}x_{234}$, there exists $t$ such that $tx = R(101)$. Therefore, $Z_{101k}^{ss} = M_{\beta_{101k}}R(101)$.

We assume that $u_{154} = 0$ and $u_{221} = 0$. Then the four components of $n(u)R(101)$ are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & -u_{153} & 0 \\
0 & 0 & * & 0 & u_{143}
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & * & 0 & u_{143} \\
0 & 0 & -1 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & u_{132} + u_{231} & u_{232} \\
0 & -1 & * & 0 & Q_1(u) - u_{152} \\
0 & 0 & * & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & u_{143} + u_{243} & u_{121} + u_{153} \\
0 & -1 & 0 & Q_2(u) - u_{142} + u_{241} & Q_3(u) + u_{131} - u_{152} + u_{242} \\
0 & * & * & 0 & Q_4(u) + u_{141} \\
-1 & * & * & * & 0
\end{pmatrix},
\]

where $Q_1(u), Q_2(u), Q_3(u)$ do not depend on $u_{131}, u_{141}, u_{142}, u_{152}$ and $Q_4(u)$ does not depend on $u_{141}$.

We can apply Lemma 11.9 to the map $\text{Aff}^{14} \to \text{Aff}^{10}$ defined by the sequence

\[
u_{143}, u_{153}, u_{232}, u_{132} + u_{231}, u_{243} + u_{143}, u_{121} + u_{153}, u_{152} - Q_1(u),
u_{142} - u_{241} - Q_2(u), u_{131} - u_{152} + u_{242} + Q_3(u), u_{141} + Q_4(u)
\]

where $u_{231}, u_{241}, u_{242}, u_{243}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{101k}^{ss}$. Therefore, $Y_{101k}^{ss} = P_{\beta_{101k}}R(101)$ also.

\[24\] $S_{105}, \beta_{105} = \frac{1}{140}(-56, 4, 4, 24, 24, -15, 5, 5, 5)$

We identify the element $(\text{diag}(t_1, g_{11}, g_{12}), \text{diag}(t_2, g_2)) \in M_{[13], [1]} = M_{\beta_{105}}$ with the element $g = (g_{21}, g_{11}, g_{12}, t_1, t_2) \in \text{GL}_3 \times \text{GL}_2 \times GL_1$. On $M_{\beta_{105}}^1$, $\chi_{105}(g) = t_1^{-56}(\det g_{11})^4(\det g_{12})^{24}t_2^{-15}(\det g_{22})^5 = (\det g_{11})^{60}(\det g_{12})^{80}(\det g_{22})^{20}$.

For $x \in Z_{105}$, let

\[
A_1(x) = \begin{pmatrix} x_{242} & x_{252} \\ x_{342} & x_{352} \end{pmatrix},
A_2(x) = \begin{pmatrix} x_{243} & x_{253} \\ x_{343} & x_{353} \end{pmatrix},
A_3(x) = \begin{pmatrix} x_{244} & x_{254} \\ x_{344} & x_{354} \end{pmatrix},
\]
and $A(x) = (A_1(x), A_2(x), A_3(x))$. We identify $Z_{105} \cong \text{Aff}^3 \otimes M_2 \oplus 1$ by the map $Z_{105} \ni x \mapsto (A(x), x_{451})$. The action of $(g_2, g_{11}, g_{12}, 1, 1)$ on $A(x)$ is the same as that of Proposition 4.7. Also if $t = (I_3, I_2, I_2, t_1, t_2)$ then $A(tx) = A(x)$.

Let $P_1(x)$ be the degree 6 polynomial of $A(x)$ obtained by Proposition 4.7. Then

$$P_1(gx) = (\det g_{11})^3(\det g_{12})^3(\det g_2)^2 P_1(x).$$

We put $P(x) = P_1(x)x_{451}$. Then on $M_{\beta_{105}}^3$,

$$P(gx) = ((\det g_{11})^3(\det g_{12})^3(\det g_2)^2)((\det g_2) t_2) P(x)$$

$$= (\det g_{11})^3(\det g_{12})^3 t_2(\det g_2)^2 P(x) = (\det g_{11})^3(\det g_{12})^4(\det g_2) P(x).$$

Therefore, $P(x)$ is invariant under the action of $G_{st, \beta_{105}}$.

Let $R(105)$ be the element such that $A(R(105)) = R_{322}$ and the $x_{451}$-coordinate is 1. Explicitly, $R(105) = e_{451} - e_{242} + e_{352} + e_{253} + e_{344}$. Then $P(R(105)) = 1$ and so $R(105) \in Z_{105}^{ss}$.

We show that $Z_{105k}^{ss} = M_{\beta_{105}} R(105)$. Suppose that $x \in Z_{105k}^{ss}$. Then $x_{451} \neq 0$. By Proposition 4.7, there exists $g \in M_{\beta_{105}}$ such that $A(gx) = A(R(105))$. So we may assume that $A(x) = A(R(105))$. If $t = (I_3, I_2, I_2, 1, x_{451}^{-1})$ then $gx = R(105)$. Therefore, $Z_{105k}^{ss} = M_{\beta_{105}} R(105)$.

We assume that $u_{132} = u_{154} = 0$ and $u_{2ij} = 0$ unless $j = 1$. Then the four components of $n(u) R(105)$ are as follows:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
, 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & u_{143} + u_{152} + u_{221} \\
0 & 0 & -1 & * & 0
\end{pmatrix}
, 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_{142} + u_{231} & 0 \\
0 & -1 & 0 & * & 0
\end{pmatrix}
, 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

We can apply Lemma 11.9 to the map $\text{Aff}^{11} \to \text{Aff}^3$ defined by the sequence

$$u_{143} + u_{152} + u_{221}, u_{142} + u_{231}, u_{153} - u_{241}$$

where $u_{121}, u_{131}, u_{141}, u_{151}, u_{152}, u_{221}, u_{231}, u_{241}$ are extra variables. So by Proposition 4.7 Property 11.7 holds for any $x \in Z_{105k}^{ss}$. Therefore, $Y_{105k}^{ss} = P_{\beta_{105}} R(105)$ also.

(25) $S_{106}, \beta_{106} = \frac{1}{260}(-24, -4, -4, 16, 16, -65, 15, 15, 35)$

We identify the element $(\text{diag}(t_1, g_{11}, g_{12}), \text{diag}(t_2, g_2, t_{22})) \in M_{[1,3][1,3]} = M_{\beta_{106}}$ with the element $g = (g_{12}, g_{11}, g_{12}, 1, t_1, t_{12}, t_{22}) \in \text{GL}_2^1 \times \text{GL}_1^3$. On $M_{\beta_{106}}^1$, $\chi_{106}(g) = t_1^{-24}(\det g_{11})^{-4}(\det g_{12})^{-16} t_{21}^{-65} (\det g_2)^{15} t_{22}^{35}$

$$= (\det g_{11})^{20}(\det g_{12})^{10}(\det g_2)^{80} t_{12}^{100}.$$

For $x \in Z_{106}$, let

$$A(x) = (x_{242}, x_{252}, x_{342}, x_{352}, x_{243}, \ldots, x_{353}), \quad v(x) = [x_{144}, x_{154}].$$

We regard $A(x) \in \Lambda^1_{1,4,5} \otimes \Lambda^1_{2,3} \otimes \Lambda^2_{2,3}$, $v(x) \in \Lambda^2_{1,4,5}$. We identify $Z_{106}$ with $\text{Aff}^2 \otimes \text{Aff}^2 \otimes \text{Aff}^2$, $\text{Aff}^2 \otimes \text{Aff}^2 \oplus \text{Aff}^2 \oplus 1$ by the map $Z_{106} \ni x \mapsto (A(x), v(x), x_{234})$.

Let $P_1(x), P_2(x)$ be the polynomials of $(A(x), v(x))$ defined in the case (a) of Section 11. By (10.2),

$$A(gx) = (g_{12}, g_{11}, g_2) A(x), \quad v(gx) = t_1 t_2 g_{12} v(x),$$

$$P_1(gx) = (\det g_{11})^2 (\det g_{12})^2 P_2(x),$$

$$P_2(gx) = t_1^2 (\det g_{11})^2 (\det g_{12}) t_{22}^2 P_1(x).$$

We put $P(x) = P_1(x) P_2(x) x_{234}$. Then on $M^1_{\beta_{106}}$,

$$P(gx) = ((\det g_{11})^2 (\det g_{12})^2 (t_1^2 (\det g_{11})^2 (\det g_{12}) t_{22}^2)^2$$

$$\times ((\det g_{11}) t_{22}) P(x)$$

$$= t_1^4 (\det g_{11})^5 (\det g_{12})^6 (\det g_2)^4 t_{22}^5 P(x) = (\det g_{11}) (\det g_{12})^2 (\det g_2)^4 t_{22}^2 P(x).$$

Note that $P_1(x)$ depends only on $A(x)$ and $P_2(x)$ is homogeneous of degree 2 with respect to each of $A(x), v(x)$. Therefore, $P(x)$ is invariant under the action of $G_{st, \beta_{106}}$.

If $x \in Z^s_{106}$, then applying $t = (I_2, I_2, I_2, 1, 1, x_{234}^{-1})$, we may assume that $x_{234} = 1$. Then elements of the form $g = (g_{12}, g_{11}, g_2 = (\det g_{11}), 1, (\det g_{11})^{-1})$ do not change this condition and $A(gx) = (g_{12}, g_{11}, g_2) A(x), v(gx) = g_{12} v(x)$. So by Proposition 10.5, $M_{\beta_{106}} \setminus Z^s_{106}$ is in bijective correspondence with $\text{Ex}_2(k)$.

Let $R(106) \in Z_{106}$ be the element such that

$$A(R(106)) = (1, 0, \ldots, 0, 1), \quad v(R(106)) = [1, 1]$$

and that the $x_{234}$-coordinate is 1. Explicitly, $R(106) = e_{242} + e_{353} + e_{144} + e_{154} + e_{234}$. Then $R(106) \in Z^s_{106}$ (see the case (a) of Section 10).

We assume that $u_{132} = u_{154} = 0$ and $u_{232} = 0$. Then the first component of $n(u) R(106)$ is 0 and the remaining components are as are as follows:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -u_{152} \\
0 & 0 & 0 & * & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u_{143} \\
0 & 0 & 0 & * & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & u_{121} + u_{143} + u_{242} & u_{121} + u_{153} \\
0 & 1 & u_{131} - u_{142} & u_{131} - u_{152} + u_{243} \\
-1 & * & 0 & Q(u) + u_{141} - u_{151} \\
-1 & * & * & * & 0
\end{pmatrix}$$

where $Q(u)$ does not depend on $u_{141}, u_{151}$.

We can apply Lemma 11.9 to the map $\text{Aff}^{13} \to \text{Aff}^7$ defined by the sequence

$$u_{143}, u_{152}, u_{121} + u_{143} + u_{242}, u_{153} + u_{121}, u_{131} - u_{142},$$

$$u_{243} + u_{131} - u_{152}, u_{141} - u_{151} + Q(u)$$

where $u_{142}, u_{143}, u_{151}, u_{221}, u_{241}, u_{242}$ are extra variables. So by Proposition 11.7, Property 11.5 holds for any $x \in Z^s_{106}$. Therefore, $P_{\beta_{106}} \setminus Y^s_{106}$ is in bijective correspondence with $\text{Ex}_2(k)$ also.
We regard $g_{107} = \frac{1}{10}(-4, 0, 0, 2, 2, -3, -1, -1, 5)$.

We identify the element $t_1(g_{11}, g_{12}, g_{2}, t_1, t_2, t_{21}, t_{22}, g_{2}, t_{22}) \in M_{1,1} = M_{107}$ with the element $g = (g_{11}, g_{12}, t_{21}, t_{22}) \in GL_2 \times GL_3$. On $M_{107}$, we have

$$\chi_{107}(g) = t_1^{-4}(\det g_{12})^2 t_{21}^{-3} (\det g_{2})^{-1} t_{22}^5 = (\det g_{12})^4 (\det g_{12})^6 (\det g_{2})^2 t_{22}^5.$$

For $x \in Z_{107}$, let

$$A(x) = (x_{242}, x_{252}, x_{342}, x_{352}, x_{243}, \ldots, x_{353}), \ v(x) = [x_{124}, x_{134}].$$

We regard $A(x) \in \Lambda_{1,1} \otimes \Lambda_{1,1} \otimes \Lambda_{2,2} \otimes \Lambda_{2,2} \otimes \Lambda_{2,2}$, $v(x) \in \Lambda_{2,2}$. We identify $Z_{107} \cong \text{Aff} \otimes \text{Aff} \otimes \text{Aff} \otimes \text{Aff} \otimes \text{Aff}$ by the map $Z_{107} \ni x \mapsto (A(x), v(x), x_{451})$.

Let $P_1(x), P_2(x)$ be the polynomials of $(A(x), v(x))$ defined in the case (a) of Section 11. By (10.2),

$$A(gx) = (g_{11}, g_{12}, g_{2})A(x), \ v(gx) = t_1 t_{22} g_{11} v(x),$$

$$P_1(gx) = t_1^{-4}(\det g_{12})^2 t_{21}^{-3} (\det g_{2})^{-1} t_{22}^5 P_1(x),$$

$$P_2(gx) = t_1^{-2}(\det g_{12})^2 (\det g_{12}) (\det g_{2}) t_{22}^5 P_1(x).$$

We put $P(x) = P_1(x)P_2(x)\mathcal{S}_{451}^6$. Then on $M_{107}$,

$$P(gx) = (\det g_{12})^2 (\det g_{12})^2 (\det g_{2})^2 (\det g_{2})^2 (\det g_{2})^2 t_{22}^5 = (\det g_{12})^2 (\det g_{2})^2 (\det g_{2})^2 t_{22}^5 t_{22}^5 P(x).$$

Therefore, $P(x)$ is invariant under the action of $G_{st,107}$.

Similarly as in the previous case, $M_{k,107} \setminus Z_{107}^{ss}$ is in bijection correspondence with $\text{Ex}_3(k)$. Note that the action of $GL_3^3$ can be absorbed by the action of $GL_2^3$.

Let $R(107) \in Z_{107}$ be the element such that $A(R(107)) = (1, 0, \ldots, 0, 1), \ v(R(107)) = [1, 1]$ and that the $x_{451}$-coordinate is 1. Explicitly, $R(107) = e_{451} + e_{245} + e_{353} + e_{241} + e_{134}$. Then $R(107) \in Z_{107}^{ss}$.

We assume that $u_{132} = u_{154} = 0$ and $u_{232} = 0$. Then the four components of $n(u)R(107)$ are as follows:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -u_{152} + u_{221} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{143} + u_{231} \\
0 & 0 & -1 & * & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 & 1 & u_{142} + u_{143} & u_{152} + u_{153} \\
-1 & 0 & u_{141} - u_{131} & Q_1(u) - u_{141} + u_{242} & Q_2(u) - u_{151} \\
-1 & * & 0 & Q_3(u) - u_{141} & Q_4(u) - u_{151} + u_{243} \\
0 & 0 & 0 & * & Q_5(u) + u_{241} \\
0 & * & * & * & 0
\end{pmatrix}$$

where $Q_1(u), Q_2(u), Q_3(u), Q_4(u)$ do not depend on $u_{141}, u_{151}, u_{241}, u_{242}, u_{243}$ and $Q_5(u)$ does not depend on $u_{241}$. 

We can apply Lemma 11.4 to the map $\text{Aff}^{13} \to \text{Aff}^{10}$ defined by the sequence
\[
\begin{align*}
&u_{152} - u_{221}, u_{143} + u_{231}, u_{121} - u_{131}, u_{142} + u_{143}, u_{153} + u_{152}, u_{141} - Q_3(u), \\
u_{242} - u_{141} + Q_1(u), u_{151} - Q_2(u), u_{243} - u_{151} + Q_4(u), u_{241} + Q_5(u)
\end{align*}
\]
where $u_{131}, u_{221}, u_{231}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{107}^{ss}$. Therefore, $P_{\beta_{107}} \setminus Y_{107}^{ss}$ is in bijective correspondence with $\text{Ex}_{2}(k)$ also.

(27) $S_{108}, \beta_{108} = \frac{1}{26}(-2, -2, 0, 2, 2, -1, -1, 1, 1)$

We identify the element $(\text{diag}(g_{11}, t_1, g_{12}), \text{diag}(g_{21}, g_{22})) \in M_{[2,3],[2]} = M_{\beta_{108}}$ with the element $g = (g_{11}, g_{12}, g_{21}, g_{22}, t_1) \in \text{GL}_2^4 \times \text{GL}_1$. On $M_{\beta_{108}}$,
\[
\chi_{108}(g) = (\det g_{11})^2(\det g_{12})^2(\det g_{21})^{-1}(\det g_{22}) = t_1^2(\det g_{12})^4(\det g_{22})^2.
\]
For $x \in Z_{108}$, let
\[
A(x) = (x_{143}, x_{153}, x_{243}, x_{253}, x_{144}, \ldots, x_{254}), \quad B(x) = \left( \begin{array}{cc} x_{341} & x_{342} \\ x_{351} & x_{352} \end{array} \right).
\]
We regard $A(x) \in \Lambda_{1,[1,2]}^2 \otimes \Lambda_{1,[4,5]}^2 \otimes \Lambda_{2,[3,4]}^2$, $B(x) \in \Lambda_{1,[1,4]}^2 \otimes \Lambda_{2,[1,2]}^2$. We identify $Z_{108} \cong \text{Aff}^2 \otimes \text{Aff}^2 \otimes \text{Aff}^2 \oplus M_2$ by the map $Z_{108} \ni x \mapsto (A(x), B(x))$.

It is easy to see that
\[
A(gx) = (g_{11}, g_{12}, g_{22})A(x), \quad B(gx) = t_1g_{12}B(x)^4 g_{21}.
\]
Let $P_1(x)$ be the degree 4 polynomial of $A(x)$ obtained by Proposition 11.7 we put $P_2(x) = \det B(x)$. Then
\[
P_1(gx) = (\det g_{11})^2(\det g_{12})^2(\det g_{22})^2 P_1(x), \quad P_2(gx) = t_1^2(\det g_{12})(\det g_{21})P_2(x).
\]
We put $P(x) = P_1(x)^3P_2(x)^4$. Then on $M_{\beta_{108}}$,
\[
P(gx) = ((\det g_{11})^2(\det g_{12})^2(\det g_{22})^2)^3(t_1^2(\det g_{12})(\det g_{21}))^4P(x)
\]
\[
= (\det g_{11})^6 t_1^8(\det g_{12})^{10}(\det g_{21})^4(\det g_{22})^6P(x)
\]
\[
= t_1^2(\det g_{12})^4(\det g_{22})^2P(x).
\]
Therefore, $P(x)$ is invariant under the action of $G_{st,\beta_{108}}$.

Suppose that $x \in Z_{108}^{ss}$. It is easy to see that there exists $g \in M_{\beta_{108}}$ such that $B(gx) = I_2$. If $B(x) = I_2$ then $B(gx) = B(x)$ if and only if $t_1g_{12}g_{21} = I_2$. If $g = (g_{11}, g_{12}, t_1^{-1}g_{12}^{-1}g_{12}, g_{22}, t_1)$ then $A(gx) = (g_{11}, g_{12}, g_{22})A(x)$. Therefore, $M_{\beta_{108}} \setminus Z_{108}^{ss}$ is in bijective correspondence with the set of orbits for the first case in Proposition 11.4. Therefore, $M_{\beta_{108}} \setminus Z_{108}^{ss}$ is in bijective correspondence with $\text{Ex}_{2}(k)$.

Let $R(108) \in Z_{108}$ be element such that $A(R(108)) = (1, 0, \ldots, 0, 1)$, $B(R(108)) = I_2$. Explicitly, $R(108) = e_{341} + e_{352} + e_{143} + e_{254}$. Then $P(R(108)) = 1$ and so $R(108) \in Z_{108}^{ss}$.
We assume that \( u_{121} = u_{154} = 0 \) and \( u_{2ij} = 0 \) unless \( i = 3, 4, j = 1, 2 \). Then the four components of \( n(u)R(108) \) are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & -u_{153} \\
0 & 0 & 0 & * & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{143} \\
0 & 0 & -1 & 0 & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & u_{241} & u_{132} + u_{142} \\
0 & 0 & * & 0 & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
-1 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & * \\
0 & * & * & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where \( Q_1(u), Q_2(u) \) do not depend on \( u_{142}, u_{151} \).

We can apply Lemma 11.9 to the map \( Aff^{12} \to Aff^8 \) defined by the sequence

\[
u_{143}, u_{153}, u_{232}, u_{241}, u_{131} + u_{231}, u_{132} + u_{242}, u_{151} - Q_1(u), u_{142} + Q_2(u)\]

where \( u_{141}, u_{152}, u_{231}, u_{242} \) are extra variables. So by Proposition 11.7 Property 11.5 holds for any \( x \in Z_{107}^{ss} \). Therefore, \( P_{\beta_{110}} \circ Y_{107}^{ss} \) is in bijective correspondence with \( Ex_2(k) \) also.

(28) \( S_{110} : \beta_{110} = \frac{1}{10}(-4, -4, -4, 6, 6, -15, 5, 5, 5) \)

We identify the element \( (\text{diag}(g_{11}, g_{12}), \text{diag}(t_2, g_2)) \in M_{[3],[1]} = M_{\beta_{110}} \) with the element \( g = (g_{11}, g_{21}, g_{12}, t_2) \in \text{GL}_3^2 \times \text{GL}_2 \times \text{GL}_1 \). On \( M_{\beta_{110}}^{1} \),

\[
\chi_{110}(g) = (\det g_{11})^{-4}(\det g_{12})^6 t_2^{-15} (\det g_2)^5 = (\det g_{12})^{10} (\det g_2)^{20}.
\]

For \( x \in Z_{110} \), let

\[
A_1(x) = \begin{pmatrix}
x_{142} & x_{143} & x_{144} \\
x_{242} & x_{243} & x_{244} \\
x_{342} & x_{343} & x_{344}
\end{pmatrix}, \quad A_2(x) = \begin{pmatrix}
x_{152} & x_{153} & x_{154} \\
x_{252} & x_{253} & x_{254} \\
x_{352} & x_{353} & x_{354}
\end{pmatrix},
\]

and \( A(x) = (A_1(x), A_2(x)) \). We identify \( Z_{110} \cong M_3 \otimes Aff^2 \) by the map \( Z_{110} \ni x \mapsto A(x) \).

It is easy to see that

\[
\begin{pmatrix}
A_1(gx) \\
A_2(gx)
\end{pmatrix} = g_{12} \begin{pmatrix}
g_{11} A_1(x)^t g_2 \\
g_{11} A_2(x)^t g_2
\end{pmatrix}.
\]

So \( gx \) does not depend on \( t_2 \) and this is the tensor product of standard representations of \( g_{11}, g_{12}, g_2 \).

Let \( P(x) \) be the degree 12 polynomial on \( Z_{\beta_{110}} \cong Aff^3 \otimes Aff^3 \otimes Aff^3 \) obtained by Proposition 11.2. Then on \( M_{\beta_{110}}^{1} \),

\[
P(gx) = (\det g_{11})^4 (\det g_{12})^6 (\det g_2)^4 P(x) = (\det g_{12})^2 (\det g_2)^4 P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_{110}} \).

By Proposition 11.4, \( M_{\beta_{110}} \setminus Z_{110}^{ss} \) is in bijective correspondence with \( Ex_3(k) \).

Let \( R(110) \in Z_{110} \) be the element such that

\[
A_1(R(110)) = \text{diag}(1, -1, 0), \quad A_2(R(110)) = \text{diag}(0, 1, -1).
\]
Explicitly, $R(110) = e_{142} - e_{243} + e_{253} - e_{354}$. Then $R(110) \in \mathbb{Z}_{110}^{ss}$ by Proposition 4.12.

We assume that $u_{1ij} = 0$ unless $i = 4, 5, j = 1, 2, 3$ and $u_{2ij} = 0$ unless $j = 1$. Then the first component of $n(u)R(110)$ is 0 and the remaining components are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -u_{151} \\
0 & 0 & 0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & u_{142} + u_{152} & 0 \\
0 & 0 & 0 & 0 & -u_{143}
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]

We can apply Lemma 11.9 to the map $\text{Aff}^3 \to \text{Aff}^3$ defined by the sequence $u_{143}, u_{151}, u_{142} + u_{152}$ where $u_{141}, u_{152}, u_{153}, u_{221}, \ldots, u_{241}$ are extra variables. So by Proposition 11.7, Property 11.8 holds for any $x \in \mathbb{Z}_{110}^{ss}$. Therefore, $P_{\beta_{110,k}} \setminus Y_{110,k}^{ss}$ is in bijective correspondence with $\text{EX}_3(k)$ also.

\[
(29) \quad S_{113}, \beta_{113} = \frac{1}{60}(-9, -4, -4, 6, 11, -10, 0, 5, 5)
\]

We identify the element $((\text{diag}(t_{11}, g_1, t_{12}, t_{13}), \text{diag}(t_{21}, t_{22}, g_2))) \in M_{[1,3],[1,2]} = M_{\beta_{113}}$ with the element $g = (g_1, g_2, t_{11}, \ldots, t_{22}) \in \text{GL}_2 \times \text{GL}_4$. On $M_{\beta_{113}}$,

\[
\chi_{113}(g) = t_{11}^{-9}(\det g_1)^{-4}t_{12}^{6}t_{13}^{11}t_{21}^{-10}(\det g_2)^{5} = (\det g_1)^{5}t_{12}^{15}t_{13}^{20}t_{21}^{10}(\det g_2)^{15}.
\]

For $x \in Z_{113}$, let

\[
A(x) = \begin{pmatrix} x_{243} & x_{244} \\ x_{343} & x_{344} \end{pmatrix},
\]

\[
v_1(x) = \begin{pmatrix} x_{252} \\ x_{352} \end{pmatrix},\quad v_2(x) = \begin{pmatrix} x_{153} \\ x_{154} \end{pmatrix}.
\]

We regard that $A(x) \in \Lambda^2_{1,2[3]} \otimes \Lambda^2_{2,[3,4]}, \quad v_1(x) \in \Lambda^2_{1,2[3]}, \quad v_2(x) \in \Lambda^2_{2,[3,4]}$. We identify $Z_{113} \cong \text{M}_2 \oplus \text{Aff}^2 \oplus \text{Aff}^2 \oplus 1$ by the map $Z_{113} \ni x \mapsto (A(x), v_1(x), v_2(x), x_{451}).$

It is easy to see that

\[
A(gx) = A(gx) = t_{12}g_1A(x)^t g_2, \quad v_1(gx) = t_{13}t_{22}g_1v_1(x), \quad v_2(gx) = t_{11}t_{13}g_2v_2(x).
\]

Let $P_1(x)$ be the degree 3 polynomial of $(A(x), v_1(x), v_2(x))$ obtained by Proposition 4.6 and $P_2(x) = det A(x)$. Then

\[
P_1(gx) = t_{11}(\det g_1)t_{12}t_{13}t_{22}(\det g_2)P_1(x), \quad P_2(gx) = (\det g_1)t_{12}^2(\det g_2)P_2(x).
\]

We put $P(x) = P_1(x)^3P_2(x)x_{451}$. Then on $M_{\beta_{113}}$,

\[
P(x) = (t_{11}(\det g_1)t_{12}t_{13}t_{22}(\det g_2))^3((\det g_1)t_{12}(\det g_2))(t_{12}t_{13}t_{21})P(x)
\]

\[
= t_{11}^3(\det g_1)^4t_{12}^6t_{13}^7t_{21}^3t_{22}^3(\det g_2)^4P(x) = (\det g_1)t_{12}^4t_{13}^4t_{22}^4(\det g_2)^3P(x).
\]

Therefore, $P(x)$ is invariant under the action of $G_{\text{st},\beta_{113}}$.

Let $R(113) \in Z_{113}$ be element such that

\[
A(R(113)) = I_2, \quad v_1(R(113)) = v_2(R(113)) = [1, 0]
\]

and that the $x_{451}$-coordinate is 1. Explicitly, $R(113) = e_{451} + e_{252} + e_{153} + e_{243} + e_{344}$. Then $P_1(R(113)) = P_2(R(113)) = 1$. So $P(R(113)) = 1$ and $R(113) \in \mathbb{Z}_{113}^{ss}$.

We show that $Z_{113,k}^{ss} = M_{\beta_{113,k}}R(113)$. Suppose that $x \in Z_{113,k}^{ss}$, By Proposition 4.6 there exists $g \in M_{\beta_{113,k}}$ such that $A(gx) = I_2, v_1(gx) = v_1(gx) = [1, 0]$. So we may assume that $A(x) = I_2, v_1(x) = v_2(x) = [1, 0]$. Let $t = (I_2, I_2, 1, 1, 1, t_{21}, 1)$. Then $A(tx) = I_2, v_1(tx) = v_2(tx) = [1, 0]$ and the $x_{451}$-coordinate of $tx$ is $t_{21}x_{451}$. Therefore, $Z_{113,k}^{ss} = M_{\beta_{113,k}}R(113)$. 

We assume that \( u_{132} = 0 \) and \( u_{243} = 0 \). Then the four components of \( n(u)R(113) \) are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u_{142} + u_{221} \\
0 & -1 & 0 & * & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & u_{121} + u_{154} + u_{232} & u_{131} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u_{242} \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & u_{153} + u_{241} & u_{154} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where \( Q_1(u), Q_2(u) \) do not depend on \( u_{141}, u_{152}, u_{153}, u_{231}, u_{241} \).

We can apply Lemma 11.9 to the map \( \text{Aff}^{14} \rightarrow \text{Aff}^7 \) defined by the sequence

\[
\begin{align*}
u_{131} & , u_{154} , u_{242} , u_{142} + u_{221} , u_{121} + u_{154} + u_{232} , \phantom{0}
\end{align*}
\]

\[
\begin{align*} \quad & , u_{141} - u_{152} + u_{231} + Q_1(u) , u_{153} - u_{241} - Q_2(u) \end{align*}
\]

where \( u_{143}, u_{151}, u_{152}, u_{221}, u_{231}, u_{232}, u_{241} \) are extra variables. So by Proposition 11.7 Property 11.5 holds for any \( x \in Z_{113,k}^{ss} \). Therefore, \( Y_{113,k}^{ss} = P_{\beta_{113,k}}R(113) \) also.

(30) \( S_{121}, \beta_{121} = \frac{1}{140}(-4, -4, 0, 0, 8, -3, -3, 1, 5) \)

We identify the element \((\text{diag}(g_{11}, g_{12}, t_1), \text{diag}(g_2, t_{21}, t_{22})) \in M_{[2,4],[2,3]} = M_{\beta_{121}}\) with the element \( g = (g_{11}, g_{12}, g_2, t_1, t_{21}, t_{22}) \in \text{GL}_{2} \times \text{GL}_{1}. \) On \( M_{\beta_{121}}, \)

\[
\gamma_{121}(g) = (\det g_{11})^{-4} t_1^5 (\det g_2)^{-3} t_{21} t_{22}^5 = (\det g_{12})^4 t_1^5 t_{21} t_{22}. 
\]

Let

\[
A(x) = \begin{pmatrix} x_{151} & x_{152} \\ x_{251} & x_{252} \end{pmatrix}, \quad B(x) = \begin{pmatrix} x_{134} & x_{144} \\ x_{234} & x_{244} \end{pmatrix}. 
\]

We identify \( Z_{121} \cong M_2 \oplus M_2 \oplus 1 \) by the map \( Z_{121} \ni x \mapsto (A(x), B(x), x_{343}). \)

It is easy to see that

\[
A(gx) = t_1 g_{11} A(x)' g_2, \quad B(gx) = t_{22} g_{11} B(x)' g_{12} 
\]

Let \( P_1(x) = \det A(x), \quad P_2(x) = \det B(x). \) Then

\[
P_1(gx) = (\det g_{11}) t_1^2 (\det g_2) P_1(x), \quad P_2(gx) = (\det g_{11})(\det g_{12}) t_{22}^2 P_2(x). 
\]

We put \( P(x) = P_1(x) s^2 P_2(x) x_{343}^9. \) Then on \( M_{\beta_{121}}, \)

\[
P(gx) = ((\det g_{11}) t_1^2 (\det g_2))^8 ((\det g_{11})(\det g_{12}) t_{22}^2)^5 ((\det g_{12}) t_{21})^9 P(x) 
\]

\[
= (\det g_{11})^{13} (\det g_{12})^{14} t_1^{16} (\det g_2)^8 t_{21} t_{22}^2 P(x) = (\det g_{12})^{13} t_{21} t_{22}^2 P(x). 
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_{121}}. \)

Let \( R(121) \in Z_{121} \) be element such that \( A(R(121)) = B(R(121)) = I_2 \) and the \( x_{343} \)-coordinate is 1. Explicitly, \( R(121) = e_{151} + e_{252} + e_{343} + e_{134} + e_{244}. \) Then \( P_1(R(121)) = P_2(R(121)) = 1. \) So \( P(R(121)) = 1 \) and \( R(121) \in Z_{121}^{ss}. \)

We show that \( Z_{121}^{ss} = M_{\beta_{121}, k}R(121). \) Suppose that \( x \in Z_{121}^{ss}. \) Since \( P_1(x), P_2(x) \neq 0, \) there exists \( g \in M_{\beta_{121}, k} \) such that \( A(gx) = B(gx) = I_2. \) So we may assume that
\( A(x) = B(x) = I_2 \). Let \( t = (I_2, I_2, I_2, 1, t_{21}, 1) \). Then \( A(tx) = B(tx) = I_2 \) and the \( x_{349} \)-coordinate of \( tx \) is \( t_{21}x_{343} \). Therefore, \( Z_{1211k}^{ss} = M_{\beta_{121k}} R(121) \).

We assume that \( u_{121} = u_{143} = 0 \) and \( u_{221} = 0 \). Then the four components of \( n(u) R(121) \) are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_{131} & 0 \\
0 & 0 & 0 & u_{141} & 0 \\
-1 & 0 & * & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{132} \\
0 & 0 & 0 & 0 & u_{142} \\
0 & -1 & * & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & u_{231} & 0 \\
0 & 0 & 0 & u_{232} & 0 \\
0 & 0 & 0 & 1 & Q_1(u) + u_{154} \\
0 & 0 & -1 & Q_2(u) - u_{153} & 0 \\
* & * & * & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 & 0 & u_{153} + u_{241} \\
0 & 0 & 0 & 1 & u_{154} + u_{242} \\
-1 & 0 & 0 & u_{132} - u_{141} + u_{243} & Q_3(u) - u_{151} \\
0 & -1 & * & 0 & Q_4(u) - u_{152} \\
* & * & * & * & 0
\end{pmatrix}
\]

where \( Q_1(u), Q_2(u) \) do not depend on \( u_{151}, \ldots, u_{154}, u_{241}, u_{242} \) and \( Q_3(u), Q_4(u) \) do not depend on \( u_{151}, u_{152} \).

We can apply Lemma \[\mathbf{11.9}\] to the map \( \text{Aff}^{13} \to \text{Aff}^{13} \) defined by the sequence

\[
u_{131}, u_{132}, u_{141}, u_{142}, u_{231}, u_{232}, u_{243} + u_{132} - u_{141}, u_{154} + Q_1(u),
\]

\[
u_{153} - Q_2(u), u_{241} + u_{153}, u_{242} + u_{154}, u_{151} - Q_3(u), u_{152} - Q_4(u)
\]

with no extra variables. So by Proposition \[\mathbf{11.7}\] Property \[\mathbf{11.5}\] holds for any \( x \in Z_{1211k}^{ss} \). Therefore, \( Y_{121k}^{ss} = P_{\beta_{121k}} R(121) \) also.

(31) \( S_{131}, \beta_{131} = \frac{1}{60}(-24, 6, 6, 6, -15, 5, 5, 5) \)

We identify the element \( (\text{diag}(t_1, g_1), \text{diag}(t_2, g_2)) \in M_{[1],[1]} = M_{\beta_{131}} \) with the element \( g = (g_1, g_2, t_1, t_2) \in \text{GL}_4 \times \text{GL}_3 \times \text{GL}_2 \). On \( M_{\beta_{131}}^1 \),

\[
\chi_{131}(g) = t_1^{24} (\det g_1)^6 t_2^{-15} (\det g_2)^5 = (\det g_1)^{30} (\det g_2)^{30}.
\]

We identify of \( Z_{131} \cong \wedge^2 \text{Aff}^3 \otimes \text{Aff}^3 \). Let \( P(x) \) be the degree 12 polynomial on \( Z_{131} \) obtained by Lemma \[\mathbf{8.13}\]. Since elements of the form \( g = (I_4, I_3, t_1, t_2) \) act on \( Z_{131} \) trivially, \( P(gx) = (\det g_1)^6 (\det g_2)^4 P(x) \). Therefore, \( P(x) \) is invariant under the action of \( G_{\beta_{131}} \).

\( M_{\beta_{131}} \backslash Z_{1311k}^{ss} \) can be identified with the set of rational orbits considered in the case (a) of Section \[\mathbf{8}\]. So by Proposition \[\mathbf{8.16}\] \( M_{\beta_{131}} \backslash Z_{1311k}^{ss} \) is in bijective correspondence with \( \text{IQF}_4(k) \). Since \( W_{131} = \{0\} \), \( P_{\beta_{131}} Y_{1311k}^{ss} \) is in bijective correspondence with \( \text{QF}_4(k) \) also.

(32) \( S_{149}, \beta_{149} = \frac{1}{220}(-28, -8, 2, 12, 22, -15, -5, 15) \)

(33) \( S_{150}, \beta_{150} = \frac{1}{60}(-8, -4, 0, 4, 8, -7, -3, 1, 9) \)

(34) \( S_{151}, \beta_{151} = \frac{3}{20}(-2, -1, 0, 1, 2, -2, -1, 1, 2) \)

(35) \( S_{152}, \beta_{152} = \frac{1}{30}(-4, -2, 0, 2, 4, -3, -1, 1, 3) \)

For \( l = 149 - 152, M_{\beta_l} \) = \( T \) and so \( Z_{1l}^{ss} \neq \emptyset \) by Proposition \[\mathbf{11.4}\]. Let \( R(l) \in Z_l \) be element whose coordinates in \( Z_l \) are all 1.
We express elements of $T$ as \( (11.3) \). Then the matrix \( (m_{ij}) \) of Proposition \( (11.4) \) is

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

for $S_{149}$–$S_{152}$ respectively.

The determinant of the $6 \times 6$ minor of the columns $1, 5, 6, 7, 8, 9$ is (a) $-1$ for $S_{149}$, (b) $1$ for $S_{150}$, (c) $1$ for $S_{151}$ and (d) $1$ for $S_{152}$. Therefore, $Z^{SS}_{l,k} = T_k R(l)$ for $l = 149$–152 by Proposition \( (11.4) \).

We consider the case $S_{149}$. Explicitly, $R(149) = e_{351} + e_{252} + e_{342} + e_{243} + e_{154} + e_{234}$. The four components of $n(u)R(149)$ are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{143} \\
0 & 0 & -1 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & u_{221} + u_{132} + u_{154} \\
0 & -1 & * & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & -1 & * & 0 \\
* & * & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & u_{143} + u_{243} \\
0 & -1 & 0 & Q_5(u) - u_{142} + u_{242} \\
0 & * & * & 0 \\
-1 & * & * & *
\end{pmatrix}
\]

where $Q_1(u)$ is a polynomial of $u_{143}, u_{221}, u_{154}, Q_2(u), Q_4(u), Q_5(u)$ are polynomials of $u_{132}, u_{143}, u_{154}, u_{232}, u_{243}, Q_3(u), Q_6(u)$ do not depend on $u_{131}, u_{141}, u_{152}, u_{241}$ and $Q_7(u)$ does not depend on $u_{141}$.

We can apply Lemma \( (11.9) \) to the map $\text{Aff}^{16} \to \text{Aff}^{12}$ defined by the sequence

\[
u_{143}, u_{243} + u_{143}, u_{154} + u_{232}, u_{132} + u_{232}, u_{221} + u_{132} + u_{154}, u_{142} - u_{242} - Q_5(u),
\]

\[
 u_{153} - u_{142} - Q_1(u), u_{121} + u_{153} + u_{242} + Q_4(u), u_{231} + Q_2(u), u_{152} - Q_3(u),
\]

\[
 u_{131} - u_{152} + u_{241} + Q_6(u), u_{141} + Q_7(u)
\]
where $u_{151}, u_{232}, u_{241}, u_{242}$ are extra variables. So by Proposition 11.7, Property 11.5 holds for any $x \in Z_{149}^{ss}$. Therefore, $Y_{149}^{ss} = P_{\beta_{149}}R(149)$ also.

We consider the case $S_{150}$. Explicitly, $R(150) = e_{451} + e_{352} + e_{253} + e_{343} + e_{144} + e_{234}$. The first component of $n(u)R(150)$ is the same as that of $R(150)$ and the remaining components are as follows:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_{132} + u_{154} + u_{232} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{pmatrix},
$$

where $Q_1(u), Q_2(u), Q_3(u)$ are polynomial of $u_{121}, u_{132}, u_{143}, u_{154}, u_{232}, Q_1(u)$ is a polynomial of $u_{131}, u_{132}, u_{153}, u_{154}, u_{243}$ and $Q_5(u)$ does not depend on $u_{151}, u_{241}$.

We can apply Lemma 11.9 to the map $\text{Aff}^{16} \to \text{Aff}^9$ defined by the sequence

$$
u_{154}, u_{121} + u_{143}, u_{221} + u_{143}, u_{132} + u_{154} + u_{232}, u_{142} - u_{153} + u_{231} + Q_1(u),
$$

$$u_{243} + u_{153} + Q_2(u), u_{131} - u_{142} + u_{243} + Q_3(u), u_{152} - u_{242} - Q_4(u),
$$

$$u_{151} - u_{241} - Q_5(u)
$$

where $u_{141}, u_{143}, u_{153}, u_{231}, u_{232}, u_{241}, u_{242}$ are extra variables. So by Proposition 11.7, Property 11.5 holds for any $x \in Z_{150}^{ss}$. Therefore, $Y_{150}^{ss} = P_{\beta_{150}}R(150)$ also.

We consider the case $S_{151}$. Explicitly, $R(151) = e_{451} + e_{352} + e_{153} + e_{243} + e_{144} + e_{234}$. The first component of $n(u)R(151)$ is the same as that of $R(151)$ and the remaining components are as follows:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & u_{132} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{pmatrix},
$$

where $Q_1(u), Q_3(u), Q_4(u)$ are polynomial of $u_{121}, u_{132}, u_{143}, u_{154}, u_{243}, Q_2(u)$ is a polynomial of $u_{142}, u_{143}, u_{154}, u_{232}, Q_5(u)$ does not depend on $u_{151}, u_{152}, u_{241}, u_{242}$ and $Q_6(u)$ does not depend on $u_{151}, u_{241}$.
We can apply Lemma 11.9 to the map $\text{Aff}^{16} \to \text{Aff}^{11}$ defined by the sequence
\[
\begin{align*}
&u_{132}, u_{121} + u_{154}, u_{243} + u_{154}, u_{143} + u_{121} + u_{243}, u_{221} + u_{143}, u_{131} + u_{232} + Q_1(u), \\
u_{153} + Q_3(u), u_{142} - u_{131} - Q_4(u), u_{141} - u_{152} + u_{231} + Q_2(u), \\
u_{242} - u_{152} + Q_5(u), u_{241} - u_{151} + Q_6(u)
\end{align*}
\]
where $u_{151}, u_{152}, u_{154}, u_{231}, u_{232}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{ss}^{151}$. Therefore, $Y_{ss}^{151} = P_{\beta_{151}} R(151)$ also.

We consider the case $S_{152}$. Explicitly, $R(152) = e_{451} + e_{352} + e_{253} + e_{343} + e_{154} + e_{244}$. The first component of $n(u)R(152)$ is the same as that of $R(152)$ and the remaining components are as follows:
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & u_{143} + u_{221} & 0 \\
0 & 0 & -1 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 \\
-1 & * & * & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & u_{132} + u_{243} & 0 & 0 \\
0 & 0 & u_{132} + u_{243} & Q_3(u) + u_{141} - u_{152} + u_{241} & 0 \\
-1 & * & * & * & 0
\end{pmatrix}
\]
where $Q_1(u), Q_2(u)$ are polynomials of $u_{132}, u_{143}, u_{154}, u_{232}, u_{243}$ and $Q_3(u)$ does not depend on $u_{141}, u_{152}, u_{241}$.

We can apply Lemma 11.9 to the map $\text{Aff}^{16} \to \text{Aff}^{7}$ defined by the sequence
\[
\begin{align*}
&u_{143} + u_{221}, u_{132} + u_{243}, u_{154} + u_{132} + u_{232}, u_{121} + u_{154} + u_{243}, \\
u_{142} - u_{153} + u_{231} + Q_1(u), u_{131} + u_{242} + Q_2(u), u_{141} - u_{152} + u_{241} + Q_3(u)
\end{align*}
\]
where $u_{151}, u_{152}, u_{153}, u_{221}, u_{231}, u_{232}, u_{241}, u_{242}, u_{243}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{ss}^{152}$. Therefore, $Y_{ss}^{152} = P_{\beta_{152}} R(152)$ also.

(36) $S_{164}, \beta_{164} = \frac{1}{60}(-4, -4, -4, 6, 6, -15, -15, 15, 15)$

We identify the element $(\text{diag}(g_{11}, g_{12}), \text{diag}(g_{21}, g_{22})) \in M_{[3],[2]} = M_{\beta_{164}}$ with the element $g = (g_{11}, g_{12}, g_{21}, g_{22}) \in \text{GL}_3 \times \text{GL}_3$. On $M_{\beta_{164}}$, \n\[
\chi_{164}(g) = (\det g_{11})^{-4}(\det g_{12})^6(\det g_{21})^{-15}(\det g_{22})^{15} = (\det g_{12})^{10}(\det g_{22})^{30}.
\]

We identify $Z_{164} \cong \text{Aff}^3 \otimes \text{Aff}^2 \otimes \text{Aff}^2$. If $g = (g_{11}, g_{12}, g_{21}, g_{22})$ and $x \in Z_{164}$ then $gx$ does not depend on $g_{21}$ and the action of $(g_{11}, g_{12}, I_2, g_{22})$ on $Z_{164}$ is the same as that of Proposition 4.7 Therefore, if $R(164) \in Z_{164}$ is the element which corresponds to $R_{322}$ in Proposition 4.7, then $R(164) \in Z_{ss}^{164}$ and $Z_{ss}^{164} = M_{\beta_{164}} R(164)$. Explicitly, $R(164) = -e_{143} + e_{154} + e_{244} + e_{353}$.

We assume that $u_{1ij} = 0$ unless $i = 4, 5, j = 1, 2, 3$ and $u_{221} = u_{243} = 0$. Then the first two components of $n(u)R(164)$ are 0 and the remaining components are as
by Proposition 11.7, Property 11.5 holds for any \( x \). We regard that \( \Lambda \) is the orbit of \( \Lambda \) under the action of \( Z \) on \( A \) obtained by \( \text{diag}(t_{11}, t_{12}, t_{13}) \). Then on \( M \),

\[
\chi_{178}(g) = t_{11}^{-28}(\det g_{1})^{-8} t_{12}^{12} t_{13}^{3} t_{22}^{4} (\det g_{2})^{25} = (\det g_{1})^{20} t_{12}^{4} t_{13}^{2} t_{22}^{2} (\det g_{2})^{80}.
\]

For \( x \in Z_{164} \), let

\[
A(x) = \begin{pmatrix} x_{243} & x_{244} & x_{343} & x_{344} \\ x_{243} & x_{244} & x_{343} & x_{344} \end{pmatrix}, \quad v_{1}(x) = \begin{pmatrix} x_{252} \\ x_{352} \end{pmatrix}, \quad v_{2}(x) = \begin{pmatrix} x_{153} \\ x_{154} \end{pmatrix}.
\]

We regard that \( A(x) \in \Lambda_{2,1}^{1,2} \otimes \Lambda_{2,1}^{1,2} \), \( v_{1}(x) \in \Lambda_{1,2}^{1,2} \), \( v_{2}(x) \in \Lambda_{2,1}^{1,2} \). We identify \( Z_{178} \cong M_{2} \oplus \text{Aff}^{2} \oplus \text{Aff}^{2} \) by the map \( Z_{178} \ni x \mapsto (A(x), v_{1}(x), v_{2}(x)) \).

It is easy to see that

\[
A(gx) = t_{12}g_{1}A(x)t_{22}g_{2}, \quad v_{1}(gx) = t_{13}t_{22}g_{1}v_{1}(x), \quad v_{2}(gx) = t_{11}t_{13}g_{2}v_{2}(x).
\]

Let \( P_{1}(x) \) be the homogeneous degree 3 polynomial of \( A(x), v_{1}(x), v_{2}(x) \) obtained by Proposition 11.0 and \( P_{2}(x) = \det A(x) \). Since \( P_{1}(x) \) is linear with respect to each of \( A(x), v_{1}(x), v_{2}(x) \),

\[
P_{1}(gx) = t_{11}(\det g_{1})t_{12}t_{13}t_{22}(\det g_{2})P_{1}(x), \quad P_{2}(gx) = (\det g_{1})t_{12}^{2}(\det g_{2})P_{2}(x).
\]

We put \( P(x) = P_{1}(x)^{3}P_{2}(x) \). Then on \( M_{178}^{1} \),

\[
P(gx) = (t_{11}(\det g_{1})t_{12}t_{13}t_{22}(\det g_{2}))^{3}((\det g_{1})t_{12}^{2}(\det g_{2}))P(x)
= t_{11}^{3}(\det g_{1})t_{12}^{3}t_{13}t_{22}(\det g_{2})^{4}P(x) = (\det g_{1})t_{12}^{2}t_{13}t_{22}(\det g_{2})^{4}P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, 178} \).

Let \( R(178) \in Z_{178} \) be the element such that \( A(R(178)) = I_{2}, \quad v_{1}(R(178)) = [1, 0], \quad v_{2}(R(178)) = [1, 0] \). Explicitly, \( R(178) = e_{252} + e_{153} + e_{243} + e_{344} \). By Proposition 11.0 \( P(R(178)) \neq 0 \) and \( Z_{178}^{st} \) is the orbit of \( R(178) \) by the action of \( (g_{1}, g_{2}, t_{22}) \). Adding the action of \( t_{11}, t_{12}, t_{13}, t_{21} \) does not make the orbit space bigger. Therefore, \( Z_{178}^{st} = M_{178}^{178}R(178) \).
We assume that $u_{132} = 0$ and $u_{243} = 0$. Then the first component of $n(u)R(178)$ is 0 and the remaining components are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u_{142} \\
0 & -1 & 0 & * & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & u_{121} + u_{154} + u_{232} \\
0 & 0 & 0 & 0 & u_{131} \\
0 & -1 & 0 & 0 & Q_1(u) + u_{141} - u_{152} \\
-1 & * & 0 & * & 0
\end{pmatrix},
\]

where $Q_1(u) = Q_1(u_{142}, u_{154}, u_{232})$ and $Q_2(u)$ does not depend on $u_{153}$.

We can apply Lemma [11.3] to the map $\text{Aff}^1 \rightarrow \text{Aff}^3$ defined by the sequence

\[
\begin{pmatrix}
u_{131}, u_{142}, u_{154}, u_{242}, u_{121} + u_{154} + u_{232}, u_{141} - u_{152} + Q_1(u), u_{153} - Q_2(u)\end{pmatrix}
\]

where $u_{143}, u_{151}, u_{221}, u_{231}, u_{232}, u_{241}$ are extra variables. So by Proposition [11.7], Property [11.5] holds for any $x \in Z_{178}^{ss}$. Therefore, $Y_{178}^{ss} = P_{\beta_{178}^{ss}}R(178)$ also.

(38) $S_{202}, \beta_{202} = \frac{1}{180}(-32, -32, 8, 28, -25, -5, 35)$

We identify the element $(\text{diag}(g_{11}, t_1, g_{12}), \text{diag}(t_{21}, g_{22}, t_2)) \in M_{[2,3],[1,3]} = M_{\beta_{202}}$ with the element $g = (g_{11}, g_{12}, g_{22}, t_1, t_{21}, t_2) \in \text{GL}_2 \times \text{GL}_3$. On $M_{\beta_{202}}$,

\[
\chi_{202}(g) = (\det g_{11})^{-32}t_1^{-5}t_{21}^{28}t_2^{-25}(\det g_{22})^{-5}t_{22}^{35}t_1^{40}(\det g_{12})^{60}(\det g_{2})^{20}t_{22}^{60}.
\]

For $x \in Z_{202}$, let

\[
A(x) = \begin{pmatrix} x_{342} & x_{333} \\ x_{352} & x_{353} \end{pmatrix}, \quad B(x) = \begin{pmatrix} x_{144} & x_{154} \\ x_{244} & x_{254} \end{pmatrix}.
\]

We identify $Z_{202} \cong M_2 \oplus M_2 \oplus 1$ by the map $Z_{202} \ni x \mapsto (A(x), B(x), x_{451})$.

Let $P_1(x) = \det A(x)$, $P_2(x) = \det B(x)$. We put $P(x) = P_1(x)^2P_2(x)^2x_{451}$. Then on $M_{\beta_{202}}$,

\[
A(gx) = t_1g_{12}A(x)^tg_{12}, \quad B(gx) = t_{22}g_{11}A(x)^tg_{12},
\]

\[
P_1(gx) = t_1^2(\det g_{12})(\det g_{2})P_1(x), \quad P_2(gx) = (\det g_{11})(\det g_{12})P_2(x)t_{22}^2,
\]

\[
P(gx) = (t_1^2(\det g_{12})(\det g_{2}))^2((\det g_{11})(\det g_{12})t_{22}^2)^2((\det g_{11})t_{21}t_1)P(x)
\]

\[
= (\det g_{11})^2t_1^4(\det g_{12})^5t_{21}(\det g_2)^2t_{22}^4P(x) = t_1^2(\det g_{12})^3(\det g_2)t_{22}^3P(x).
\]

Therefore, $P(x)$ is invariant under the action of $G_{\text{st}, \beta_{202}}$.

Let $R(202) \in Z_{202}$ be element such that $A(R(202)) = B(R(202)) = I_2$ and the $x_{451}$-coordinate is 1. Then $P_1(R(202)) = P_2(R(202)) = 1$. So $R(202) = I_2$ and $R(202) \in Z_{202}^{ss}$. Explicitly, $R(202) = e_{451} + e_{342} + e_{353} + e_{144} + e_{254}$.

We show that $Z_{202}^{ss} = M_{\beta_{202}^{ss}}R(202)$. Suppose that $x \in Z_{202}^{ss}$. It is easy to see that there exists $g \in M_{\beta_{202}^{ss}}$ such that $A(gx) = B(gx) = I_2$. So we may assume that $A(x) = B(x) = I_2$. Let $t = (I_2, I_2, I_2, 1, t_{21}, 1)$. Then $A(tx) = B(tx) = I_2$ and the $x_{451}$-coordinate of $tx$ is $t_{21}x_{451}$. Therefore, $Z_{202}^{ss} = M_{\beta_{202}^{ss}}R(202)$. 

We assume that \( u_{121} = u_{154} = 0 \) and \( u_{232} = 0 \). Then the first component of \( n(u)R(202) \) is the same as that of \( R(202) \) and the remaining components are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & -u_{153} + u_{221} \\
0 & 0 & 0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & u_{143} + u_{231} \\
0 & 0 & 1 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & u_{131} + u_{242} & u_{132} + u_{243} \\
-1 & 0 & * & 0 & Q(u) + u_{142} - u_{151} + u_{241} \\
0 & -1 & * & * & 0
\end{pmatrix}
\]

where \( Q(u) = Q(u_{143}, u_{153}, u_{242}, u_{243}) \).

We can apply Lemma 11.9 to the map \( \text{Aff}^{13} \rightarrow \text{Aff}^{5} \) defined by the sequence

\( u_{143} + u_{231}, u_{153} - u_{221}, u_{131} + u_{242}, u_{132} + u_{243}, u_{142} - u_{151} + u_{241} + Q(u) \)

where \( u_{141}, u_{152}, u_{221}, u_{231}, u_{241}, u_{242}, u_{243} \) are extra variables. So by Proposition 11.7 Property 11.3 holds for any \( x \in Z_{202, k}^{ss} \). Therefore, \( Y_{202, k}^{ss} = \beta_{202, k}R(202) \) also.

\((39)\) \( S_{216}, \beta_{216} = \frac{1}{30}(8, 0, 0, 4, 4, -5, -1, 3, 3) \)

We identify the element \( (\text{diag}(t_{1}, g_{11}, g_{12}), \text{diag}(t_{21}, t_{22}, g_{2})) \in M_{1, 3, 1, 2} = M_{\beta_{216}} \)

with the element \( g = (g_{11}, g_{12}, g_{2}, t_{1}, t_{21}, t_{22}) \in GL_{3}^{\times} \times GL_{1}^{\times} \). On \( M_{\beta_{216}}^{1} \),

\( \chi_{216}(g) = t_{1}^{8}(\det g_{12})^{4}t_{21}^{-5}t_{22}^{-1}(\det g_{2})^{3} = (\det g_{11})^{8}(\det g_{12})^{12}t_{22}^{4}(\det g_{2})^{8} \).

For \( x \in Z_{216} \), let

\( A(x) = (x_{243}, x_{253}, x_{343}, x_{353}, x_{244}, ..., x_{354}) \).

We identify \( Z_{216} \cong \text{Aff}^{2} \otimes \text{Aff}^{2} \otimes \text{Aff}^{2} \oplus 1 \) by the map \( Z_{216} \ni x \mapsto (A(x), x_{452}) \).

It is easy to see that \( A(gx) = (g_{11}, g_{12}, g_{2})A(x) \) (the natural action). Let \( P_{1}(x) \) be the degree 4 polynomial of \( A(x) \) obtained by Proposition 4.1. We put \( P(x) = P_{1}(x)x_{452} \). Then on \( M_{\beta_{216}}^{1} \),

\( P_{1}(gx) = (\det g_{11})^{2}(\det g_{12})^{2}(\det g_{2})^{2}P_{1}(x), \)

\( P(gx) = ((\det g_{11})^{2}(\det g_{12})^{2}(\det g_{2})^{2})((\det g_{12})t_{22})P(x) \)

\( = (\det g_{11})^{2}(\det g_{12})^{3}t_{22}(\det g_{2})^{2}P(x). \)

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_{216}} \).

By applying an element of the form \( t = (I_{2}, I_{2}, I_{2}, 1, 1, t_{22}) \) to \( x \), the \( x_{452} \)-coordinate becomes 1. Elements of the form \( (g_{11}, g_{12}, g_{2}, 1, 1, (\det g_{12})^{-1}) \) do not change this condition. Therefore, \( M_{\beta_{216}}^{1} \backslash Z_{216}^{ss} \) is in bijective correspondence with the set of rational orbits of the representation of Proposition 4.1. Therefore, by Proposition 4.1, \( M_{\beta_{216}}^{1} \backslash Z_{216}^{ss} \) is in bijective correspondence with \( \text{Ex}_{2}^{3}(k) \).

Let \( R(216) \in Z_{216} \) be the element such that \( A(R(216)) = (1, 0, ..., 0, 1) \) and the \( x_{452} \)-coordinate is 1. Explicitly, \( R(216) = e_{452} + e_{243} + e_{354} \). Then \( P(R(216)) = 1 \) and so \( R(216) \in Z_{216}^{ss} \).
We assume that $u_{132} = u_{154} = 0$ and $u_{243} = 0$. Then the first component of $n(u)R(216)$ is 0 and the remaining components are as follows:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -u_{152} + u_{232} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{143} + u_{242} \\
0 & 0 & -1 & 0 & 0
\end{pmatrix}.
$$

We can apply Lemma 11.5 to the map $\text{Aff}^2 \to \text{Aff}^2$ defined by the sequence $u_{143} + u_{242}, u_{152} - u_{232}$ where $u_{121}, u_{131}, u_{141}, u_{142}, u_{151}, u_{153}, u_{221}, u_{231}, u_{232}, u_{241}, u_{242}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{216}^{ss}$. Therefore, $P_{\beta_{216}} \setminus Y_{216}^{ss}$ is in bijective correspondence with $\text{Ex}_2(k)$ also.

(40) $S_{217}, \beta_{217} = \frac{1}{180}(-32, -12, 8, 8, 28, -45, -5, 15, 35)$

We identify the element $(\text{diag}(t_{11}, t_{12}, g_1, t_{13}), \text{diag}(t_{21}, \ldots, t_{24})) \in M_{[1,2,4], [1,2,3]} = M_{\beta_{217}}$ with the element $g = (g_1, t_{11}, \ldots, t_{24}) \in \text{GL}_2 \times \text{GL}_7$. On $M_{\beta_{217}}^{1}$,

$$
\chi_{217}(g) = (t_{11})^{-32}t_{12}^{12}(\det g_1)^{8t_{13}^{28}t_{14}^{24}t_{15}^{15}t_{16}^{35}} = t_{12}^{20}(\det g_1)^{40t_{13}^{60}t_{14}^{60}t_{15}^{80}}.
$$

For $x \in Z_{217}$, let $A(x) = (x_{352}^2, x_{343}^2, x_{244}^2)$. We identify $Z_{217} \cong M_2 \oplus 1^{3,3}$ by the map $Z_{217} \ni x \mapsto (A(x), x_{235}, x_{343}, x_{154})$.

It is easy to see that $A(gx) = g_1A(x)\text{diag}(t_{13}t_{22}, t_{12}t_{24})$. We put $P_1(x) = \det A(x)$ and $P(x) = P_1(x)^2x_{235}^2x_{343}^2x_{154}^2$. Then on $M_{\beta_{217}}^{1}$,

$$
P_1(gx) = t_{12}(\det g_1)t_{13}t_{22}t_{24}P_1(x),
P(gx) = (t_{12}(\det g_1)t_{13}t_{22}t_{24})^2(t_{12}t_{13}t_{23})(\det g_1)t_{23})^2(t_{11}t_{13}t_{24})^2P(x)
= t_{12}^2(\det g_1)^2t_{13}^2t_{22}^2t_{23}^2t_{24}^2P(x).
$$

Therefore, $P(x)$ is invariant under the action of $G_{\text{st}, \beta_{217}}$.

Let $R(217) \in Z_{217}$ be element such that $A(R(217)) = I_2$ and the $x_{235}, x_{154}, x_{344}$-coordinates are 1. Explicitly, $R(217) = e_{352} + e_{253} + e_{343} + e_{154} + e_{244}$. Then $P(R(217)) = 1$ and so $R(217) \in Z_{217}^{ss}$.

We show that $Z_{217}^{ss} = M_{\beta_{217}}^{1}R(217)$. Suppose that $x \in Z_{217}^{ss}$. It is easy to see that there exists $g \in M_{\beta_{217}}^{1}$ such that $A(gx) = I_2$. So we may assume that $A(x) = I_2$. Let $t = (\text{diag}(t_{12}, t_{13}^2), t_{11}, t_{12}, 1, 1, t_{22}, 1, 1)$. Then $A(tx) = I_2$ and the $x_{235}, x_{343}, x_{154}$-coordinates of $tx$ are $t_{12}x_{235}$, $t_{12}^{-1}t_{22}^{-1}x_{343}$, $t_{11}x_{154}$ respectively. Therefore, $Z_{217}^{ss} = M_{\beta_{217}}^{1}R(217)$.

We assume that $u_{143} = 0$. Then the first component of $n(u)R(217)$ is 0 and the remaining components are as follows:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{132} + u_{154} + u_{232} \\
0 & 0 & -1 & 0 & u_{142} - u_{153} \\
0 & 0 & -1 & * & 0
\end{pmatrix}.
$$
where $Q_1(u) = Q_1(u_{132}, u_{154}, u_{243})$ and $Q_2(u) = Q_2(u_{142}, u_{153}, u_{154}, u_{243})$.

We can apply Lemma 11.19 to the map $\text{Aff}^5 \rightarrow \text{Aff}^6$ defined by the sequence

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{121} + u_{154} + u_{243} \\
0 & 0 & 0 & u_{132} + u_{243} & Q_1(u) + u_{131} + u_{242} \\
0 & -1 & * & 0 & Q_2(u) + u_{141} - u_{152} \\
-1 & * & * & * & 0
\end{pmatrix}
\]

where $Q_1(u) = Q_1(u_{132}, u_{154}, u_{243})$ and $Q_2(u) = Q_2(u_{142}, u_{153}, u_{154}, u_{243})$.

We show that $R$ also.

Property 11.7 holds for any $x \in Z_{217k}^{88}$. Therefore, $Y_{217k}^{88} = P_{\beta_{217}k}R(217)$ also.

\begin{align*}
(41) & \quad S_{223}, \beta_{223} = \frac{1}{100}(-36, -16, 4, 4, 44, -15, -15, 5, 25) \\
\text{We identify the element } & (\text{diag}(t_{11}, t_{12}, t_{13}), \text{diag}(g_2, t_{21}, t_{22})) \in M_{1,2,4,2,3} = M_{\beta_{223}} \\
\text{with the element } & g = (g_1, g_2, t_{11}, \ldots, t_{22}) \in \text{GL}_2 \times \text{GL}_5. \text{ On } M_{\beta_{223}}, \\
\chi_{223}(g) & = t_{11}^{36}t_{12}^{16}(\det g_1)^{443}(\det g_2)^{-15}t_{21}^{25}t_{22}^{25} = t_{12}^{20}(\det g_1)^{40}t_{21}^{40}t_{22}^{40}.
\end{align*}

For $x \in Z_{223}$, let $A(x) = (x_{135x_{145}x_{452}})$. We identify $Z_{223} \cong M_{2} \oplus 1^{3\beta}$ by the map $Z_{223} \ni x \mapsto (A(x), x_{253}, x_{154}, x_{344}).$

It is easy to see that $A(gx) = t_{13}g_1A(x)^xg_2$. We put $P_1(x) = \det A(x)$ and $P(x) = (\det A(x))x_{13}^2x_{144}x_{344}^2$. Then on $M_{\beta_{223}},$

\begin{align*}
P_1(gx) & = (\det g_1)t_{13}^2(\det g_2)P_1(x), \\
P(gx) & = ((\det g_1)t_{13}^2(\det g_2))((\det g_1)t_{13}t_{23})^2((\det g_1)t_{23})^2P(x) \\
& = t_{11}t_{12}^2(\det g_1)t_{13}^2t_{13}(\det g_2)t_{21}^2t_{22}^2P(x) = t_{12}(\det g_1)t_{13}^2t_{13}t_{21}^2t_{22}P(x).
\end{align*}

Therefore, $P(x)$ is invariant under the action of $G_{\text{st}, \beta_{223}}$.

Let $R(223) \in Z_{223}$ be element such that $A(R(223)) = I_2$ and the $x_{253}, x_{154}, x_{344}$-coordinates are 1. Explicitly, $R(223) = e_{351} + e_{452} + e_{253} + e_{154} + e_{344}$. Then $P(R(223)) = 1$ and so $R(223) \in Z_{223}^{88}$.

We show that $Z_{223k}^{88} = M_{\beta_{223}k}R(223)$. Suppose that $x \in Z_{223k}^{88}$. It is easy to see that there exists $g \in M_{\beta_{223}k}$ such that $A(gx) = I_2$. So we may assume that $A(x) = I_2$.

Let $t = (I_2, t_{12}, t_{13}, t_{11}, 1, 1, t_{22})$. Then $A(tx) = I_2$ and the $x_{253}, x_{154}, x_{344}$-coordinates of $tx$ are $t_{12}x_{253}, t_{11}t_{22}x_{154}, t_{22}x_{344}$. Therefore, $Z_{223k}^{88} = M_{\beta_{223}k}R(223)$.

We assume that $u_{143} = 0$ and $u_{221} = 0$. Then the first two components of $n(u)R(223)$ is the same as those of $R(223)$ and the remaining components are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & u_{132} + u_{231} & Q_1(u) + u_{131} + u_{242} \\
0 & 0 & 0 & 0 & Q_2(u) + u_{141} - u_{152} \\
-1 & * & * & * & 0
\end{pmatrix}
\]

where $Q_1(u), Q_2(u)$ are polynomials of $u_{132}, u_{142}, u_{243}$. 

We can apply Lemma 11.9 to the map \( \text{Aff}^{14} \to \text{Aff}^5 \) defined by the sequence

\[
    u_{121} + u_{243}, u_{231} + u_{132}, u_{232} + u_{142},
    u_{131} + u_{154} + u_{241} + Q_1(u), u_{141} - u_{153} + u_{242} + Q_2(u)
\]

where \( u_{132}, u_{142}, u_{151}, u_{152}, u_{153}, u_{154}, u_{241}, u_{242}, u_{243} \) are extra variables. So by Proposition 11.7 Property 11.5 holds for any \( x \in Z^s_{223,k} \). Therefore, \( Y^s_{223,k} = P_{\beta_{223,k}} R(223) \) also.

(42) \( S_{224}, \beta_{224} = \frac{1}{580}(-32, -12, 8, 8, 28, -25, -25, -5, 55) \)

We identify the element \((\text{diag}(t_{11}, t_{12}, g_1, t_{13}), \text{diag}(g_2, t_{21}, t_{22})) \in M_{[1,2,4],[2,3]} = M_{\beta_{224}}\) with the element \( g = (g_1, g_2, t_{11}, \ldots, t_{22}) \in \text{GL}_2 \times \text{GL}_5^5 \). On \( M^1_{\beta_{224}} \),

\[
    \chi_{224}(g) = t_{11}^{-12}t_{12}^{28}(\det g_1)^8t_{13}^{25}(\det g_2)^{−5}t_{21}^{−155}t_{22} = t_{12}(\det g_1)^4t_{13}^{40}t_{21}^{60}t_{22}^{20}.
\]

For \( x \in Z_{224} \), let \( A(x) = (\frac{x_{351}}{x_{451}}, \frac{x_{352}}{x_{452}}) \). We identify \( Z_{224} = M_2 \oplus \mathbb{I}^{13} \) by the map \( Z_{224} \ni x \mapsto (A(x), x_{253}, x_{343}, x_{124}) \).

It is easy to see that \( A(gx) = t_{13}g_1A(x)^t g_2 \). We put \( P_1(x) = \det A(x) \) and \( P(x) = P_1(x)^6x_{253}x_{343}x_{124}^{10} \). Then on \( M^1_{\beta_{224}} \),

\[
    P_1(gx) = (\det g_1)^2t_{13}^2(\det g_2)P_1(x),
    P(gx) = ((\det g_1)^2t_{13}^2(\det g_2))^6(t_{12}t_{13}t_{21}((\det g_1)^2t_{21}^6t_{13}^6)^{10}P(x)
    = t_{12}(\det g_1)^{12}t_{13}^7t_{12}^{10}P(x) = t_{12}(\det g_1)^2t_{13}^3t_{21}^4t_{22}^4 P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_{224}} \).

Let \( R(224) \in Z_{224} \) be the element such that \( A(R(224)) = I_2 \) and the \( x_{253}, x_{343}, x_{124} \)-coordinates are 1. Explicitly, \( R(224) = e_{351} + e_{452} + e_{253} + e_{343} + e_{124} \). Then \( P(R(224)) = 1 \) and so \( R(224) \in Z^s_{224,k} \).

We show that \( Z^s_{224,k} = M_{\beta_{224}}k R(224) \). Suppose that \( x \in Z^s_{224,k} \). It is easy to see that there exists \( g \in M_{\beta_{224}}k \) such that \( A(gx) = I_2 \). So we may assume that \( A(x) = I_2 \). Let \( t = (t_1, t_2, t_{11}, t_{12}, 1, t_{21}, 1) \). Then \( A(tx) = I_2 \) and the \( x_{253}, x_{343}, x_{124} \)-coordinates of \( tx \) are \( t_{12}t_{21}x_{253}, t_{21}x_{343}, t_{11}t_{12}x_{124} \). Therefore, \( Z^s_{224,k} = M_{\beta_{224}}k R(224) \).

We assume that \( u_{143} = u_{221} = 0 \). Then the four components of \( n(u) R(224) \) are as follows:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & u_{132} & u_{134} & u_{142} & u_{152} \\
−1 & 0 & Q_1(u) - u_{131} & Q_2(u) - u_{141} & Q_3(u) - u_{151} & Q_5(u) + u_{243} \\
* & * & 0 & Q_4(u) + u_{243} & Q_5(u) + u_{241} & Q_6(u) + u_{242} \\
* & * & * & 0 & 0 & 0 \\
* & * & * & * & 0 & 0
\end{array}
\]

where \( Q_1(u), Q_2(u), Q_3(u) \) are polynomials of \( u_{121}, u_{132}, u_{142}, u_{152}, Q_4(u) \) is a polynomial of \( u_{131}, u_{132}, u_{141}, u_{142} \) and \( Q_5(u), Q_6(u) \) do not depend on \( u_{241}, u_{242} \).
We can apply Lemma 11.9 to the map $\text{Aff}^{14} \to \text{Aff}^{11}$ defined by the sequence
\[ u_{132}, u_{142}, u_{152}, u_{154} + u_{132}, u_{153} - u_{142} - u_{232}, u_{131} - Q_1(u), \]
\[ u_{141} - Q_2(u), u_{243} + Q_4(u), u_{151} - u_{243} - Q_3(u), u_{241} + Q_5(u), u_{242} + Q_6(u) \]
where $u_{121}, u_{231}, u_{232}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{224}^{ss}$. Therefore, $Y_{224}^{ss} = P_{\beta_{224}} R(224)$ also.

(43) $S_{226}, \beta_{226} = \frac{1}{320}(-48, -28, -8, 32, 52, -25, -5, 15, 15)$

We identify the element $(\text{diag}(t_{11}, \ldots, t_{15}), \text{diag}(t_{21}, t_{22}, g_2)) \in M_{1,2,3,4,1,2} = M_{\beta_{226}}$
with the element $g = (g_2, t_{11}, \ldots, t_{22}) \in GL_2 \times GL_7$. On $M_{\beta_{226}}$
\[ \chi_{226}(g) = t_{11}^{-48} t_{12}^{-28} t_{13}^{-8} t_{14}^{32} t_{15}^{52} t_{21}^{-25} t_{22}^{-5} (\det g_2)^{15} = t_{12}^{20} t_{13}^{40} t_{14}^{80} t_{15}^{100} t_{22}^{20} (\det g_2)^{40}. \]

For $x \in Z_{226}$, let $A(x) = (x_{131} x_{154})$. We identify $Z_{226} \cong M_2 \oplus 13^{35}$ by the map
$Z_{226} \ni x \mapsto (A(x), x_{231}, x_{252}, x_{342})$.

It is easy to see that $A(gx) = (t_{11}^{-48} \cdot 0 \cdot t_{12}^{-14}) A(x)^t g_2$. We put $P_1(x) = \det A(x)$ and
$P(x) = P_1(x)^6 x_{351} x_{252} x_{342}^4$. Then on $M_{\beta_{226}}$
\[ P_1(gx) = t_{11} t_{12} t_{14} t_{15} (\det g_2) P_1(x), \]
\[ P(gx) = (t_{11} t_{12} t_{14} (\det g_2))^6 (t_{13} t_{15} t_{21})^4 (t_{12} t_{15} t_{22}) (t_{13} t_{14} t_{22})^4 P(x) \]
\[ = t_{11}^6 t_{12}^7 t_{13}^8 t_{14}^{11} t_{15}^4 t_{21}^5 (\det g_2) P(x) = t_{12} t_{13} t_{14} (\det g_2)^2 P(x). \]

Therefore, $P(x)$ is invariant under the action of $G_{\text{st}, \beta_{226}}$. Let $R(226) \in Z_{226}$ be element such that $A(R(226)) = I_2$ and the $x_{351}, x_{252}, x_{342}$-coordinates are 1. Explicitly, $R(226) = e_{351} + e_{252} + e_{342} + e_{131} + e_{244}$. Then
$P(R(226)) = 1$ and so $R(226) \in Z_{226}^{ss}$.

We show that $Z_{226}^{ss} = M_{\beta_{224}} R(226)$. Suppose that $x \in Z_{226}^{ss}$. It is easy to see that there exists $g \in M_{\beta_{226}}$ such that $A(gx) = I_2$. So we may assume that $A(x) = I_2$. Let
$t = (t_{11}^{-1} I_2, t_{11}, t_{12}, t_{13}, 1, t_{21}, 1)$. Then $A(tx) = I_2$ and the $x_{351}, x_{252}, x_{342}$-coordinates of $tx$ are $t_{13} x_{351}, t_{11} x_{252}, t_{13} x_{342}$. Therefore, $Z_{226}^{ss} = M_{\beta_{226}} R(226)$.

We assume that $u_{243} = 0$. Then the four components of $n(u) R(226)$ are as follows:
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & u_{143} & 0 \\
0 & 0 & -1 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & u_{132} + u_{154} + u_{221} \\
0 & 0 & -1 & 0 & Q_1(u) + u_{142} - u_{153} \\
0 & 0 & 0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & u_{121} + u_{232} & 0 \\
0 & 0 & u_{232} & Q_2(u) + u_{131} + u_{231} & 0 \\
0 & 0 & * & Q_3(u) + u_{141} & 0 \\
-1 & * & * & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & u_{154} + u_{242} \\
0 & 0 & 0 & u_{132} + u_{242} & Q_4(u) + u_{241} \\
0 & 0 & -1 & 0 & Q_5(u) - u_{152} \\
0 & 0 & * & * & 0
\end{pmatrix},
\]

where $Q_1(u), Q_2(u), Q_4(u)$ are polynomials of $u_{132}, u_{143}, u_{154}, u_{221}, u_{232}, u_{242}$, $Q_3(u)$ does not depend on $u_{141}, u_{152}$ and $Q_5(u)$ does not depend on $u_{152}$.

We can apply Lemma 11.9 to the map $\text{Aff}^{15} \to \text{Aff}^{11}$ defined by the sequence
\[ u_{143}, u_{232}, u_{121} + u_{232}, u_{242} + u_{132}, u_{154} + u_{242}, u_{221} + u_{132} + u_{154}, \]
\[ u_{142} - u_{153} + Q_1(u), u_{131} + u_{231} + Q_2(u), u_{241} + Q_4(u), u_{141} + Q_3(u), u_{152} - Q_5(u) \]
where \( u_{132}, u_{151}, u_{153}, u_{231} \) are extra variables. So by Proposition 11.7 Property 11.5 holds for any \( x \in \mathbb{Z}^{ss}_{226k} \). Therefore, \( Y^{ss}_{226k} = P_{\beta_{226k}} R(226) \) also.

\[
(44) \quad S_{227}, \beta_{227} = \frac{1}{10} (-26, -6, 4, 4, 24, -15, -5, 5, 15)
\]

We identify the element \((\text{diag}(t_{11}, t_{12}, g_{1}, t_{13}), \text{diag}(t_{21}, \ldots, t_{24})) \in M_{124,123} = M_{\beta_{227}}\) with the element \( q = (g_{1}, t_{11}, \ldots, t_{24}) \in \text{GL}_2 \times \text{GL}_4 \). On \( M_{\beta_{227}} \),

\[
\chi_{227}(g) = t_{12}^{-6t_{12}}(\det g_{1})^{4}t_{13}^{24}-15t_{22}^{5}t_{23}^{4}t_{24}^{5} = t_{11}^{20}(\det g_{1})^{30}t_{13}^{40}t_{22}^{20}t_{23}^{20}.
\]

For \( x \in \mathbb{Z}_{227} \), let \( A(x) = (x_{251}, x_{254}, x_{243}, x_{154}) \). We identify \( \mathbb{Z}_{227} \cong M_{2} \oplus \mathbb{Z}^{3} \) by the map \( \mathbb{Z}_{227} \ni x \mapsto (A(x), x_{252}, x_{343}, x_{154}) \).

It is easy to see that \( A(gx) = g_{1}A(x)\text{diag}(t_{13}t_{21}, t_{12}t_{24}) \). We put \( P_{1}(x) = \det A(x) \) and \( P(x) = P_{1}(x)^{2}x_{252}x_{434}x_{154}^{3} \). Then on \( M_{\beta_{227}} \),

\[
P_{1}(gx) = t_{12}(\det g_{1})t_{13}t_{21}t_{24}P_{1}(x),
\]

\[
P(gx) = (t_{12}(\det g_{1})t_{13}t_{21}t_{24})^{2}(t_{12}t_{13}t_{22})^{3}((\det g_{1})t_{23})^{4}(t_{11}t_{13}t_{24})^{3}P(x)
\]

\[
= t_{12}^{3}t_{13}^{5}(\det g_{1})^{3}t_{13}^{6t_{12}^{2}}t_{22}^{3}t_{23}^{4}t_{24}^{5}P(x) = t_{12}^{2}(\det g_{1})^{3}t_{13}^{4}t_{22}^{3}t_{23}^{2}P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_{227}} \).

Let \( R(227) \in \mathbb{Z}_{227} \) be element such that \( A(R(227)) = I_{2} \) and the \( x_{252}, x_{343}, x_{154} \)-coordinates are extra variables. So by Proposition 11.7 Property 11.5 holds for any \( x \in \mathbb{Z}^{ss}_{227k} \). Therefore, \( \mathbb{Z}^{ss}_{227k} = M_{\beta_{227}k} R(227) \). Suppose that \( x \in \mathbb{Z}^{ss}_{227k} \). It is easy to see that there exists \( g \in M_{\beta_{227}k} \) such that \( A(gx) = I_{2} \). So we may assume that \( A(x) = I_{2} \). Let \( t = (t_{21}^{-1}I_{2}, t_{11}, t_{12}, 1, t_{12}, 1, t_{23}, 1) \). Then \( A(tx) = I_{2} \) and the \( x_{252}, x_{343}, x_{154} \)-coordinates of \( tx \) are \( t_{12}x_{252}, t_{12}^{2}t_{23}x_{343}, t_{11}x_{154} \). Therefore, \( \mathbb{Z}^{ss}_{227k} = M_{\beta_{227}k} R(227) \).

We assume that \( u_{143} = 0 \). Then the four components of \( n(u)R(227) \) are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_{132} + u_{221} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_{142} & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & * & * & 0 & 0 & 0 & 0 & 0 \\
-1 & * & * & * & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where \( Q_{1}(u), Q_{2}(u) \) are polynomials of \( u_{132}, u_{142}, u_{232} \) and \( Q_{3}(u), Q_{4}(u) \) do not depend on \( u_{131}, u_{141}, u_{152}, u_{241} \).

We can apply Lemma 11.9 to the map \( \text{Aff}^{15} \to \text{Aff}^{9} \) defined by the sequence

\[
u_{142}, u_{232}, u_{132} + u_{221}, u_{243} + u_{132}, u_{121} + u_{154} + u_{242}, u_{231} + u_{154} + Q_{1}(u), u_{153} - Q_{2}(u), u_{131} + u_{241} + Q_{3}(u), u_{141} - u_{152} + Q_{4}(u),
\]

where \( u_{151}, u_{152}, u_{154}, u_{221}, u_{241}, u_{242} \) are extra variables. So by Proposition 11.7 Property 11.5 holds for any \( x \in \mathbb{Z}^{ss}_{227k} \). Therefore, \( Y^{ss}_{227k} = P_{\beta_{227}k} R(227) \).

\[
(45) \quad S_{232}, \beta_{232} = \frac{1}{10} (-16, -6, -6, 4, 24, -15, -5, -5, 25)
\]
We identify the element \((\text{diag}(t_{11}, g_1, t_{12}, t_{13}), \text{diag}(t_{21}, g_2, t_{22})) \in M_{[1,3,4],[1,3]} = M_{[2,3]}^1\) with the element \(g = (g_1, g_2, t_{11}, \ldots, t_{22}) \in GL_2^2 \times GL_4^5\).

Then on \(M_{[2,3]}^1\),
\[
\chi_{232}(g) = t_{11}^{-16}(\det g_1)^{-6}t_{12}^{24}t_{13}^{-15}(\det g_2)^{-5}t_{22}^{25} = (\det g_1)^{10}t_{12}^{20}t_{13}^{40}(\det g_2)^{10}t_{22}^{40}.
\]

For \(x \in Z_{232}\), let \(A(x) = (x_{232} x_{234})\). We identify \(Z_{232} \cong M_2 \oplus 1^3\) by the map \(Z_{232} \ni x \mapsto (A(x), x_{451}, x_{144}, x_{234})\).

It is easy to see that \(A(gx) = t_{13}g_1 A(x)^t g_2\). We put \(P_1(x) = \det A(x)\) and \(P(x) = P_1(x)^3 x_{451}^2 x_{144}^3 x_{234}^4\). On \(M_{[2,3]}^k\),
\[
P_1(gx) = (\det g_1 t_{13}(\det g_2)) P_1(x),
\]
\[
P(gx) = ((\det g_1)^2 t_{13}(\det g_2))^3 (t_{12} t_{13} t_{21})^2 (t_{11} t_{12} t_{22})^4 ((\det g_1) t_{22})^2 P(x)
\]
\[
= t_{11}^4 (\det g_1)^2 t_{13}^2 t_{21}^2 (\det g_2)^3 t_{22}^2 P(x) = (\det g_1) t_{12}^4 t_{13}^2 (\det g_2) t_{22}^4 P(x).
\]

Therefore, \(P(x)\) is invariant under the action of \(G_{st, \beta_{232}}\).

Let \(R(232) \in Z_{232}\) be element such that \(A(R(232)) = I_2\) and the \(x_{451}, x_{144}, x_{234}\)-entries are 1. Explicitly, \(R(232) = e_{451} + e_{252} + e_{353} + e_{144} + e_{234}\). Then \(P(R(232)) = 1\) and so \(R(232) \in Z_{232}^{ss}\).

We show that \(Z_{232}^{ss} = M_{[2,3]}^k R(232)\). Suppose that \(x \in Z_{232}^{ss}\). It is easy to see that there exists \(g \in M_{[2,3]}^k\) such that \(A(gx) = I_2\). So we may assume that \(A(x) = I_2\).

Let \(t = (t_1, t_2, t_{11}, 1, 1, t_{21}, t_{22})\). Then \(A(tx) = I_2\) and the \(x_{451}, x_{144}, x_{234}\)-coordinates of \(tx\) are \(t_{21} x_{451}, t_{11} t_{22} x_{144}, t_{22} x_{234}\). Therefore, \(Z_{232}^{ss} = M_{[2,3]}^k R(232)\).

We assume that \(u_{132} = 0\) and \(u_{232} = 0\). Then the four components of \(n(u) R(232)\) are as follows:
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & u_{21} + u_{143} & Q_1(u) + u_{153} + u_{242} \\
0 & -1 & u_{31} - u_{142} & Q_2(u) - u_{152} + u_{243} \\
-1 & * & * & 0 & Q_3(u) - u_{151} + u_{241} \\
* & * & * & * & 0
\end{pmatrix}
\]

where \(Q_1(u), Q_2(u)\) are polynomials of \(u_{121}, u_{131}, u_{154}\) and \(Q_3(u)\) does not depend on \(u_{151}, u_{241}\).

We can apply Lemma \([11.9]\) to the map \(\text{Aff}^{14} \to \text{Aff}^8\) defined by the sequence
\[
u_{154}, u_{142} + u_{221}, u_{143} + u_{231}, u_{121} + u_{143}, u_{131} - u_{142}, u_{153} + u_{242} + Q_1(u), u_{152} - u_{243} - Q_2(u), u_{151} - u_{241} - Q_3(u),
\]
where \(u_{141}, u_{221}, u_{231}, u_{241}, u_{242}, u_{243}\) are extra variables. So by Proposition \([11.7]\) Property \([11.5]\) holds for any \(x \in Z_{232}^{ss} k\). Therefore, \(Y_{232}^{ss} k = P_{[2,3]}^k R(232)\).

(46) \(S_{254}, \beta_{254} = \frac{1}{30}(-3, -3, 2, 2, 2, -5, 0, 0, 5)\)

We identify the element \((\text{diag}(g_{11}, g_{12}), \text{diag}(t_{21}, g_2, t_{22})) \in M_{[2],[1,3]} = M_{[2,5]}^1\) with the element \(g = (g_{12}, g_{11}, g_2, t_{21}, t_{22}) \in GL_3 \times GL_2 \times GL_2 \). Then on \(M_{[2,5]}^1\),
\[
\chi_{254}(g) = (\det g_{11})^{-3}(\det g_{12})^2 t_{21}^{15} t_{22}^5 = (\det g_{12})^5 (\det g_2)^5 t_{22}^{10}.
\]
Let $\mathbb{P}_{3,i}, p_{3,ij}$ be as before. For $x \in Z_{254}$, let

\[
A_1(x) = x_{342}p_{3,12} + x_{352}p_{3,13} + x_{452}p_{3,23}, \quad A_2(x) = x_{343}p_{3,12} + x_{353}p_{3,13} + x_{453}p_{3,23},
\]

\[
B_1(x) = x_{134}p_{3,1} + x_{144}p_{3,2} + x_{154}p_{3,3}, \quad B_2(x) = x_{234}p_{3,1} + x_{244}p_{3,2} + x_{254}p_{3,3}
\]

and $A(x) = (A_1(x), A_2(x)), B(x) = (B_1(x), B_2(x))$. We regard that $A(x), B(x)$ are elements of $\Lambda^2_3 \otimes \Lambda^1_2 \cong \Lambda^3 \otimes \text{Aff}^2$, $\Lambda^3_1 \otimes \Lambda^1_2 \cong \Lambda^3 \otimes \text{Aff}^2$ respectively.

We identify $Z_{254} \cong (\Lambda^3 \otimes \text{Aff}^2) \otimes \text{Aff}^2 \otimes \text{Aff}^2$ by the map $Z_{254} \ni x \mapsto (A(x), B(x))$.

It is easy to see that

\[
\begin{pmatrix}
A_1(gx) \\
A_2(gx)
\end{pmatrix} = g_2 \begin{pmatrix}
(\Lambda^2_3 \Lambda^1_2 A_1(x) \\
(\Lambda^2_3 \Lambda^1_2 A_2(x)
\end{pmatrix}, \begin{pmatrix}
B_1(gx) \\
B_2(gx)
\end{pmatrix} = t_{22}g_{11} \begin{pmatrix}
g_{12}B_1(x) \\
g_{12}B_2(x)
\end{pmatrix}.
\]

We define $\Phi(x) = A(x) \wedge B(x) \in M_2$ regarding $A(x), B(x)$ as column, row vectors respectively and taking $\wedge$ of entries. Then $\Phi(gx) = (\det g_{12})t_{22}g_2\Phi(x)^t g_{11}$. Let $P(x) = \det \Phi(x)$. Then on $M_{254}^1$,

\[
P(gx) = (\det g_{11})(\det g_{12})^2(\det g_2)^t_{22}P(x) = (\det g_{12})(\det g_2)^t_{22}P(x).
\]

Therefore, $P(x)$ is invariant under the action of $G_{st, 254}$.

Let $R(254) \in Z_{254}$ be the element such that

\[
A(R(254)) = [p_{3,23}, -p_{3,13}], \quad B(R(254)) = [p_{3,1}, p_{3,2}].
\]

Explicitly, $R(254) = e_{452} - e_{353} + e_{134} + e_{244}$. Then $\Phi(R(254)) = I_2$, $P(R(254)) = 1$ and so $R(254) \in Z_{254}^{ss}$.

We show that $Z_{254}^{ss} = M_{254,k}R(254)$. Suppose that $x \in Z_{254}^{ss}$. Since $A(x) \wedge B(x)$ is a non-singular matrix, the entries of $B(x)$ are linearly independent. So there exists $g \in M_{254,k}$ such that $B(gx) = B(R(254))$. If $g = (I_3, I_2, g_2, 1, 1)$ then $B(gx) = B(x)$ and $\Phi(gx) = g_2\Phi(x)$. Therefore, we may assume that $\Phi(x) = I_2$. This implies that $x_{452} = 1, x_{453} = 0, x_{352} = 0, x_{353} = -1$. Let $g = (g_{12}, I_2, I_2, 1, 1)$ where

\[
g_{12} = \begin{pmatrix}
1 & 0 & x_{342} \\
0 & 1 & x_{343} \\
0 & 0 & 1
\end{pmatrix}.
\]

Then $B(gR(254)) = B(R(254))$ and $A(gx) = A(R(254))$. This implies that $Z_{254}^{ss} = M_{254,k}R(254)$.

We assume that $u_{13,j} = 0$ unless $i = 3, 4, 5, j = 1, 2$ and $u_{232} = 0$. Then the first component of $n(u)(R(254))$ is 0 and the remaining components are as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & u_{132} - u_{141} & -u_{151} - u_{243} \\
0 & -1 & * & 0 & -u_{152} + u_{242} \\
0 & 0 & * & * & 0
\end{pmatrix}.
\]

We can apply Lemma 11.9 to the map $\text{Aff}^3 \to \text{Aff}^2$. By the sequence $u_{132} - u_{141}, u_{151} + u_{243}, u_{152} - u_{242}$ where $u_{131}, u_{141}, u_{142}, u_{221}, u_{231}, u_{241}, u_{242}, u_{1243}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{254}^{ss}$. Therefore, $Y_{254} = P_{254,k}R(254)$ also.

(47) $S_{256}, \beta_{256} = \frac{1}{20}(-8, 2, 2, 2, -5, -5, 5, 5)$
We identify the element \((\text{diag}(t_1, g_1), \text{diag}(g_{21}, g_{22})) \in M_{[1],[2]} = M_{\beta_256}\) with the element \(g = (g_1, g_{21}, g_{22}, t_1) \in \text{GL}_4 \times \text{GL}_2 \times \text{GL}_1\). On \(M_{\beta_256}^1\):

\[
\chi_{256}(g) = t_1^{-8}(\det g_1)^2(\det g_{21})^{-5}(\det g_{22})^5 = (\det g_1)^{10}(\det g_{22})^{10}.
\]

We identify \(Z_{256} \cong \wedge^2 \text{Aff}^4 \otimes \text{Aff}^2\). Note that \(g_{21}\) acts trivially on \(Z_{256}\). Let \(P(x)\) be the degree 4 polynomial on \(Z_{\beta_256}\) obtained by Proposition 4.3. Then on \(M_{\beta_256}^1\):

\[
P(gx) = (\det g_1)^2(\det g_{22})^2P(x).
\]

Therefore, \(P(x)\) is invariant under the action of \(G_{st, \beta_256}\).

By Proposition 4.3, \(M_{\beta_256} k \setminus Z_{256}^s k\) is in bijective correspondence with \(\text{Ex}_2(k)\). Since \(W_{256} = \{0\}, P_{\beta_256} k \setminus Y_{256}^s k\) is in bijective correspondence with \(\text{Ex}_2(k)\) also.

\[(88) S_{258}, \beta_{258} = \frac{1}{20}(-8, -3, 2, 2, 7, -5, 0, 0, 5).
\]

We identify the element \((\text{diag}(t_{11}, t_{12}, g_1, t_{13}), \text{diag}(t_{21}, g_{2}, t_{22})) \in M_{[1,2],[4],[3]} = M_{\beta_258}\) with the element \(g = (g_1, g_{2}, t_{11}, \ldots, t_{22}) \in \text{GL}_2 \times \text{GL}_4\). On \(M_{\beta_258}^1\):

\[
\chi_{258}(g) = t_{11}^{-8}t_{12}^{-8}(\det g_1)^2t_{13}^{-5}t_{21}^{-5}t_{22}^{-5} = t_{12}^{-2}(\det g_1)^{10}t_{13}^{-15}(\det g_2)^{5}t_{22}.
\]

For \(x \in Z_{258}\), let \(A(x) = (x_{252}, x_{352}, x_{452})\). We identify \(Z_{258} \cong M_2 \oplus 1^{25}\) by the map \(Z_{258} \ni x \mapsto (A(x), x_{254}, x_{344})\).

It is easy to see that \(A(gx) = t_{13}g_1A(x)\). We put \(P_1(x) = \det A(x)\) and \(P(x) = P_1(x)x_{254}x_{344}\). Then on \(M_{\beta_258}^1\):

\[
P_1(gx) = (\det g_1)t_{13}^2(\det g_2)P_1(x),
\]

\[
P(gx) = (((\det g_1)t_{13}^2(\det g_2))(t_{12}t_{13}t_{22})((\det g_1)t_{22})P(x)
\]

\[
= t_{12}(\det g_1)^2t_{13}^2(\det g_2)t_{22}^2P(x).
\]

Therefore, \(P(x)\) is invariant under the action of \(G_{st, \beta_258}\).

Let \(R(258) \in Z_{258}\) be the element such that \(A(R(258)) = I_2\) and the \(x_{254}, x_{344}\)-coordinates are 1. Explicitly, \(R(258) = e_{352} + e_{453} + e_{254} + e_{344}\). Then \(P(R(258)) = 1\) and so \(R(258) \in Z_{258}^s\).

We show that \(Z_{258}\) \(M_{\beta_258} R(258)\). Suppose that \(x \in Z_{258}^s\). It is easy to see that there exists \(g \in M_{\beta_258}\) such that \(A(gx) = I_2\). So we may assume that \(A(x) = I_2\). Let \(t = (I_2, I_2, 1, t_{12}, 1, 1, t_{22})\). Then \(A(tx) = I_2\) and the \(x_{254}, x_{344}\)-coordinates of \(tx\) are \(t_{12}t_{22}x_{254}, t_{22}x_{344}\). Therefore, \(Z_{258}^s = M_{\beta_258} R(258)\).

We assume that \(u_{143} = 0\) and \(u_{232} = 0\). Then the first three components of \(n(u)R(258)\) are the same as those of \(R(258)\) and the last component is as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & u_{132} + u_{154} + u_{242} & . \\
0 & 0 & -1 & 0 & u_{142} - u_{153} + u_{243} \\
0 & -1 & * & * & 0
\end{pmatrix}
\]

We can apply Lemma 11.9 to the map \(\text{Aff}^{14} \rightarrow \text{Aff}^2\) defined by the sequence \(u_{132} + u_{154} + u_{242}, u_{142} - u_{153} + u_{243}\) where variables other than \(u_{132}, u_{142}\) are extra variables. So by Proposition 11.7 Property 11.3 holds for any \(x \in Z_{258}^s k\). Therefore, \(Y_{258}^s = P_{\beta_258} k R(258)\) also.

\[(49) S_{259}, \beta_{259} = \frac{1}{60}(-14, -4, -4, 6, 16, -15, -5, 5, 15)
\]
We identify the element
\[(\text{diag}(t_{11}, g_1, t_{12}, t_{13}), \text{diag}(t_{21}, \ldots, t_{24})) \in M_{[1,3,4],[1,2,3]} = M_{\beta_{259}}\]
with the element \(g = (g_1, t_{11}, \ldots, t_{24}) \in \text{GL}_2 \times \text{GL}^7_1\). On \(M_{\beta_{259}}^1\),
\[\chi_{259}(g) = t_{11}^{-14}(\text{det } g_1)^{-4}t_{12}^{14}t_{13}^{-15}t_{14}^{-7}t_{22}^{-5}t_{23}^{-4}t_{24}^{-15} = (\text{det } g_1)^{10}t_{12}^{20}t_{13}^{-30}t_{14}^{20}t_{24}^{-30}\]
For \(x \in Z_{259}\), let \(A(x) = (x_{153}, x_{244})\). We identify \(Z_{259} \cong M_2 \oplus 1^2\) by the map \(Z_{259} \ni x \mapsto (A(x), x_{452}, x_{154})\). It is easy to see that \(A(gx) = g_1A(x)\text{diag}(t_{13}t_{23}, t_{12}t_{24})\), We put \(P_1(x) = \text{det } A(x)\) and \(P(x) = P_1(x)^2x_{452}x_{154}\). Then on \(M_{\beta_{259}}^1\),
\[P_1(gx) = (\text{det } g_1)t_{12}t_{13}t_{23}t_{24}P_1(x),\]
\[P(gx) = ((\text{det } g_1)t_{12}t_{13}t_{23}t_{24})^2(t_{12}t_{13}t_{22})(t_{11}t_{13}t_{24})P(x)\]
\[= t_{11}(\text{det } g_1)^2t_{12}^2t_{13}^2t_{22}t_{23}^2t_{24}^2 = (\text{det } g_1)t_{12}^2t_{13}^2t_{22}^2t_{23}^2t_{24}^2.\]
Therefore, \(P(x)\) is invariant under the action of \(G_{st,\beta_{259}}\).

Let \(R(259) \in Z_{259}\) be the element such that \(A(R(259)) = I_2\) and the \(x_{452}, x_{154}\)-coordinates are 1. Explicitly, \(R(259) = e_{452} + e_{253} + e_{154} + e_{344}\). Then \(P(R(259)) = 1\) and so \(R(259) \in Z_{259}^{ss}\).

We show that \(Z_{259,k}^{ss} = M_{\beta_{259}}^1kR(259)\). Suppose that \(x \in Z_{259,k}^{ss}\). It is easy to see that there exists \(g \in M_{\beta_{259}}^1\) such that \(A(gx) = I_2\). So we may assume that \(A(x) = I_2\). Let \(t = (I_2, t_{11}, 1, 1, t_{22}, 1, 1)\). Then \(A(tx) = I_2\) and the \(x_{452}, x_{154}\)-coordinates of \(tx\) are \(t_{22}x_{452}, t_{11}x_{154}\). Therefore, \(Z_{259,k}^{ss} = M_{\beta_{259}}^1kR(259)\).

We assume that \(u_{132} = 0\). Then the first two components of \(n(u)R(232)\) are the same as those of \(R(259)\) and the remaining components are as follows:
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_{124} + u_{232} \\
0 & -1 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & Q(u) + u_{141} - u_{153} + u_{242} \\
-1 & * & * & * & 0
\end{pmatrix}
\]
where \(Q(u)\) does not depend on \(u_{111}, u_{153}, u_{242}\).

We can apply Lemma 11.9 to the map \(\text{Aff}^{15} \rightarrow \text{Aff}^4\) defined by the sequence
\[u_{142} + u_{232}, u_{121} + u_{243}, u_{131} + u_{154}, u_{141} - u_{153} + u_{242} + Q(u)\]
where variables other than \(u_{121}, u_{131}, u_{141}, u_{142}\) are extra variables. So by Proposition 11.7 Property 11.5 holds for any \(x \in Z_{259,k}^{ss}\). Therefore, \(Y_{259,k}^{ss} = P_{\beta_{259}}kR(259)\) also.

(50) \(S_{270}, \beta_{270} = \frac{3}{20}(-1, -1, -1, -1, 4, 0, 0, 0, 0)\)

Since \(G_{st,\beta_{270}}\) is semi-simple, any relative invariant polynomial is invariant. We identify \(Z_{270} \cong \Lambda^4_{1,1,1,4} \otimes \Lambda^4_{2,1,1,4} \cong M_4\) and put \(P(x) = \text{det } x\). Since \(P(x)\) is a relative invariant polynomial, it is invariant under the action of \(G_{st,\beta_{270}}\).

Let \(R(270) \in Z_{270}\) be the element which corresponds to \(I_4\). Explicitly, \(R(270) = e_{151} + e_{252} + e_{353} + e_{454}\). Then \(P(R(270)) = 1\) and so \(R(270) \in Z_{270}^{ss}\). It is easy to see that \(Z_{270,k}^{ss} = M_{\beta_{270}}^1kR(270)\). Since \(W_{270} = \{0\}, Y_{270,k}^{ss} = P_{\beta_{270}}kR(270)\).

(51) \(S_{271}, \beta_{271} = \frac{1}{60}(-24, -4, 6, 6, 16, -5, -5, 5, 5)\)
We identify the element $(\text{diag}(t_{11}, t_{12}, g_1, t_{13}), \text{diag}(g_{21}, g_{22})) \in M_{[1,2,4],[2]} = M_{\beta_{271}}$ with the element $g = (g_1, g_{21}, g_{22}, t_{11}, t_{12}, t_{13}) \in \text{GL}_2^3 \times \text{GL}_1^3$. On $M_{\beta_{271}}$

\[ \chi_{271}(g) = t_{11}^{24}t_{12}^{-4}(\det g_1)^6t_{13}^{16}(\det g_{11})^{-5}(\det g_{12})^5 = t_{12}^{20}(\det g_1)^{30}t_{13}^{40}(\det g_{12})^{10}. \]

For $x \in Z_{271}$, let

\[ A(x) = \begin{pmatrix} x_{351} & x_{352} \\ x_{451} & x_{452} \end{pmatrix}, \quad B(x) = \begin{pmatrix} x_{253} & x_{254} \\ x_{343} & x_{344} \end{pmatrix}. \]

We identify $Z_{271} \cong M_2 \oplus M_2$ by the map $Z_{271} \ni x \mapsto (A(x), B(x))$. It is easy to see that

\[ A(gx) = t_{13}g_1A(x)^tg_{21}, \quad B(gx) = \begin{pmatrix} t_{12}t_{13} & 0 \\ 0 & \det g_1 \end{pmatrix} B(x)^tg_{22}. \]

We put $P_1(x) = \det A(x), \quad P_2(x) = \det B(x)$ and $P(x) = P_1(x)P_2(x)^2$. Then on $M_{\beta_{271}}$

\[ P_1(gx) = (\det g_1)t_{13}^2(\det g_{21})P_1(x), \quad P_2(gx) = t_{12}(\det g_1)t_{13}(\det g_{22})P_2(x), \]

\[ P(gx) = ((\det g_1)^2t_{13}(\det g_{21}))(t_{12}(\det g_1)t_{13}(\det g_{22}))^2P(x) \]

\[ = t_{12}(\det g_1)^3t_{13}(\det g_{21})(\det g_{22})^2P(x) = t_{12}^2(\det g_1)^3t_{13}(\det g_{22})P(x). \]

Therefore, $P(x)$ is invariant under the action of $G_{st,\beta_{271}}$.

Let $R(271) \in Z_{271}$ be the element such that $A(R(271)) = B(R(271)) = I_2$. Explicitly, $R(271) = e_{351} + e_{452} + e_{253} + e_{344}$. Then $P_1(R(271)) = P_2(R(271)) = 1$. So $P(R(271)) = 1$ and $R(271) \in Z_{271}^{ss}$. It is easy to see that $Z_{271}^{ss} = M_{\beta_{271}} \cap R(271)$.

We assume that $u_{143} = 0$ and $u_{221} = u_{243} = 0$. Then the first two components of $n(u)R(271)$ are the same as those of $R(271)$ and the remaining components are as follows:

\[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{123} + u_{231} \\ 0 & 0 & 0 & u_{142} + u_{232} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \]

We can apply Lemma 11.9 to the map Aff$^{13} \rightarrow$ Aff$^4$ defined by the sequence

\[ u_{132} + u_{231}, u_{142} + u_{232}, u_{154} + u_{241}, u_{153} - u_{242} \]

where variables other than $u_{132}, u_{142}, u_{153}, u_{154}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{271}^{ss}$. Therefore, $Y_{271}^{ss} = P_{\beta_{271}} \cap R(271)$ also.

\[ (52) \quad S_{272}, \quad \beta_{272} = \frac{1}{220}(-28, -28, -8, 12, 52, -35, 5, 25) \]

We identify the element $(\text{diag}(g_1, t_{11}, t_{12}, t_{13}), \text{diag}(g_{21}, g_{22})) \in M_{[2,3,4],[1,3]} = M_{\beta_{272}}$ with the element $g = (g_1, g_2, t_{11}, t_{12}, t_{13}, t_{22}) \in \text{GL}_2^2 \times \text{GL}_1^3$. On $M_{\beta_{272}}$

\[ \chi_{272}(g) = (\det g_1)^{-35}t_{11}^{-8}t_{12}^{12}t_{13}^{52}t_{22}^{-5}(\det g_{21})^{25}t_{22}^{25} = t_{11}^{20}t_{12}^{40}t_{13}^{80}(\det g_2)^{40}t_{22}^{20}. \]

For $x \in Z_{272}$, let $A(x) = (x_{252}, x_{253})$. We identify $Z_{272} \cong M_2 \oplus M_3^2$ by the map $Z_{272} \ni x \mapsto (A(x), x_{451}, x_{344})$. It is easy to see that $A(gx) = t_{13}g_1A(x)^tg_{21}$. 


We put $P_1(x) = \det A(x)$ and $P(x) = P_1(x)^3 x_{451}^4 x_{344}^4$. Then on $M_{\beta_{272}}^3$,

$$
P_1(gx) = (\det g_1)^2 \det g_2 P_1(x),$$

$$
P(gx) = ((\det g_1)^2 \det g_2)^3 (t_{12} t_{13} t_{21})(t_{11} t_{12} t_{22})^4 P(x)$$

$$
= (\det g_1)^3 t_{11}^4 t_{12}^4 t_{13}^4 t_{21}(\det g_2)^2 t_{22} = t_{11}^2 t_{12}^2 t_{13}^2 (\det g_1)^2 t_{22}^2.
$$

Therefore, $P(x)$ is invariant under the action of $G_{st, \beta_{272}}$.

Let $R(272) \in Z_{272}$ be the element such that $A(R(272)) = I_2$ and the $x_{451}, x_{344}$-coordinates are 1. Explicitly, $R(272) = e_{451} + e_{152} + e_{253} + e_{344}$. Then $P(R(272)) = 1$ and so $R(272) \in Z_{272}^s$.

We show that $Z_{272}^s = M_{\beta_{272}} R(272)$. Suppose that $x \in Z_{272}^s$. It is easy to see that there exists $g \in M_{\beta_{272}}$ such that $A(gx) = I_2$. So we may assume that $A(x) = I_2$. We put $t = (I_2, I_2, 1, t_{12}, 1, 1, t_{22})$. Then $A(tx) = I_2$ and the $x_{451}, x_{344}$-coordinates of $tx$ are $t_{12} x_{451}, t_{22} x_{234}$. Therefore, $Z_{272}^s = M_{\beta_{272}} R(272)$.

We assume that $u_{121} = 0$ and $u_{232} = 0$. Then the four components of $n(u) R(272)$ are as follows:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{131} \\
0 & 0 & 0 & 0 & u_{141} + u_{221} \\
-1 & 0 & * & * & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & u_{142} + u_{231} \\
0 & 0 & 0 & 0 & u_{121} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

where $Q_1(u)$ does not depend on $u_{153}, u_{154}, u_{241}$ and $Q_2(u)$ does not depend on $u_{153}, u_{241}$.

We can apply Lemma 11.9 to the map $\text{Aff}^4 \to \text{Aff}^8$ defined by the sequence

$$
\begin{align*}
&u_{131}, u_{132}, u_{242}, u_{243}, u_{414} + u_{221}, u_{142} + u_{231}, u_{154} + Q_1(u), u_{153} - u_{241} - Q_2(u) \\
\end{align*}
$$

where $u_{143}, u_{151}, u_{152}, u_{221}, u_{231}, u_{241}$ are extra variables. So by Proposition 11.7 Property 11.5 holds for any $x \in Z_{272}^s$. Therefore, $Y_{272}^{ss} = P_{\beta_{272}} R(272)$ also.

$S_{273}$, $\beta_{273} = \frac{1}{10}(-6, -6, 4, 4, 4, -5, -5, -5, 15)$

We identify the element $(\text{diag}(g_{11}, g_{12}), \text{diag}(g_{2} , t_{2})) \in M_{[2],[3]} = M_{\beta_{273}}$ with the element $g = (g_{12}, g_{2}, g_{11}, t_{2}) \in \text{GL}_3 \times \text{GL}_2 \times \text{GL}_1$. On $M_{\beta_{273}}^1$,

$$
\chi_{273}(g) = (\det g_{11})^{-6}(\det g_{12})^4(\det g_{2})^{-5} t_{2}^{15} = (\det g_{12})^{10} t_{2}^{20}.
$$

For $x \in Z_{273}$, let

$$
A(x) = \begin{pmatrix}
x_{341} & x_{342} & x_{343} \\
x_{351} & x_{352} & x_{353} \\
x_{451} & x_{452} & x_{453}
\end{pmatrix}.
$$

We identify $Z_{273} \cong M_3 \oplus 1$ by the map $Z_{273} \ni x \mapsto (A(x), x_{124})$. It is easy to see that $A(gx) = (\wedge^2 g_{12}) A(x)^t g_{2}$. 

We put \( P_1(x) = \det A(x) \) and \( P(x) = P_1(x)^3 x_{124}^5 \). Then on \( M_{\beta_{273}}^1 \),
\[
P_1(gx) = (\det g_{12})^2 (\det g_2) P_1(x),
\]
\[
P(gx) = ((\det g_{12})^2 (\det g_2))^3 ((\det g_{11}) t_2)^5 P(x)
= (\det g_{11})^5 (\det g_{12})^6 (\det g_2)^2 t_2^3 P(x)
= (\det g_2)^2 P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_{273}} \).

Let \( R(273) \in \mathbb{Z}_{273} \) be the element such that \( A(R(273)) = I_3 \) and the \( x_{124} \)-coordinate is 1. Explicitly, \( R(273) = e_{341} + e_{352} + e_{453} + e_{124} \). Then \( P(R(273)) = 1 \) and so \( R(273) \in \mathbb{Z}_{273}^{ss} \).

We show that \( Z_{273}^{ss} = M_{\beta_{273}} R(273) \). Suppose that \( x \in Z_{273}^{ss} \). It is easy to see that there exists \( g \in M_{\beta_{273}} \) such that \( A(gx) = I_3 \). So we may assume that \( A(x) = I_3 \). Let 
\[
t = (t_3, t_3, t_2, t_3).
\]
Then \( A(tx) = I_3 \) and the \( x_{124} \)-coordinate of \( tx \) is \( t_2 x_{124} \). Therefore, 
\[
Z_{273}^{ss} = M_{\beta_{273}} R(273).
\]

We assume that \( u_{ij} = 0 \) unless \( i = 3, 4, 5, j = 1, 2 \) and \( u_{2ij} = 0 \) unless \( i = 4 \). Then the first three components of \( n(u) R(273) \) are the same as those of \( R(273) \) and the last component is as follows:
\[
\begin{pmatrix}
0 & 1 & u_{132} & u_{142} & u_{152} \\
-1 & 0 & -u_{131} & -u_{141} & -u_{151} \\
* & * & 0 & Q_1(u) + u_{241} & Q_2(u) + u_{242} \\
* & * & * & 0 & Q_3(u) + u_{243} \\
* & * & * & * & 0
\end{pmatrix}
\]
where \( Q_1(u), Q_2(u), Q_3(u) \) do not depend on \( u_{241}, u_{242}, u_{243} \).

We can apply Lemma 11.9 to the map \( \text{Aff}^9 \rightarrow \text{Aff}^9 \) defined by the sequence
\[
u_{131}, u_{132}, u_{141}, u_{142}, u_{151}, u_{152} + Q_1(u), u_{242} + Q_2(u), u_{243} + Q_3(u)
\]
with no extra variables. So by Proposition 11.7 Property 11.5 holds for any \( x \in Z_{273}^{ss} \).

Therefore, \( Y_{273}^{ss} = P_{\beta_{273}} R(273) \) also.

\[
\chi_{2730}(g) = t_1^{124} (\det g_{11})^{-4} (\det g_{12})^{16} (\det g_{22})^{-15} t_1^{25} = (\det g_{11})^{29} (\det g_{12})^{40} t_1^{20} t_2^{40}.
\]

For \( x \in Z_{280} \), let \( A(x) = \left( \begin{array}{cc} x_{144} & x_{245} \\ x_{344} & x_{355} \end{array} \right) \). We identify \( Z_{280} \) with \( M_2 \) by the map
\[
\chi_{280}(x) = x_{453}.
\]
It is easy to see that \( A(gx) = t_2 g_{11} A(x)^t g_{12} \).

We put \( P_1(x) = \det A(x) \) and \( P(x) = P_1(x) x_{453} \). Then on \( M_{\beta_{280}}^1 \),
\[
P_1(gx) = (\det g_{11}) (\det g_{12}) t_2^2 P_1(x),
\]
\[
P(gx) = ((\det g_{11}) (\det g_{12}) t_2^2 (\det g_{12}) t_2) P(x) = (\det g_{11}) (\det g_{12})^2 t_2^3 P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_{280}} \).

Let \( R(280) \in Z_{280} \) be the element such that \( A(R(280)) = I_2 \) and the \( x_{453} \)-coordinate is 1. Explicitly, \( R(280) = e_{453} + e_{244} + e_{354} \). Then \( P(R(280)) = 1 \) and so \( R(280) \in Z_{280}^{ss} \).

We show that \( Z_{280}^{ss} = M_{\beta_{280}} R(280) \). Suppose that \( x \in Z_{280}^{ss} \). It is easy to see that there exists \( g \in M_{\beta_{280}} \) such that \( A(gx) = I_2 \). So we may assume that \( A(x) = I_2 \).

Let \( t = (t_2, t_2, t_2, 1, t_{21}, 1) \). Then the \( x_{453} \)-coordinate of \( tx \) is \( t_2 x_{453} \). Therefore, 
\[
Z_{280}^{ss} = M_{\beta_{280}} R(280).
\]
We assume that \( u_{132} = 0 \) and \( u_{221} = 0 \). Then the first three components of \( n(u)R(280) \) are the same as those of \( R(280) \) and the last component is as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & u_{143} - u_{152} + u_{243} \\
0 & 0 & -1 & * & 0 \\
\end{pmatrix}
\]

Obviously we can apply Lemma 11.9 and by Proposition 11.7, Property 11.5 holds for any \( x \in Z_{280,k}^{ss} \). Therefore, \( Y_{280,k}^{ss} = P_{\beta_{280}}kR(280) \) also.

(55) \( S_{281}, \beta_{281} = \frac{1}{20}(-4, 0, 0, 0, 4, -5, -5, 3, 7) \)

We identify the element \((\text{diag}(t_{11}, g_1, t_{12}), \text{diag}(g_2, t_{21}, t_{22})) \in M_{[1,4],[2,3]} = M_{\beta_{281}} \) with the element \( g = (g_1, g_2, t_{11}, \ldots, t_{22}) \in \text{GL}_3 \times \text{GL}_2 \times \text{GL}_4 \). On \( M_{\beta_{281}} \),

\[
\chi_{281}(g) = t_{11}^{-4}t_{12}^4 (\det g_2)^{-5}t_{21}^3 t_{22}^7 = (\det g_1)^{4}t_{12}^4 t_{21}^3 t_{22}^7.
\]

For \( x \in Z_{281} \), let

\[
A(x) = x_{234}p_{3,12} + x_{244}p_{3,13} + x_{344}p_{3,23}, \quad B(x) = x_{253}p_{3,1} + x_{353}p_{3,2} + x_{453}p_{3,3}.
\]

We identify \( Z_{281} \cong \wedge^2 \text{Aff}^3 \oplus \text{Aff}^3 \oplus 1 \) by the map \( Z_{281} \ni x \mapsto (A(x), B(x), x_{154}) \). It is easy to see that \( A(gx) = t_{22}(\wedge^2 g_1)A(x), \quad B(gx) = t_{12}t_{21}g_1B(x) \).

We put \( P_1(x) = A(x) \wedge B(x) \) and \( P(x) = P_1(x)^2x_{154} \). Then on \( M_{\beta_{281}} \),

\[
P_1(gx) = (\det g_1)t_{12}t_{21}t_{22}P_1(x),
\]

\[
P(gx) = ((\det g_1)t_{12}t_{21}t_{22})^2(t_{11}t_{12}t_{22})P(x)
= t_{11}(\det g_1)^2t_{12}^2 t_{21}^3 t_{22}^3 P(x) = (\det g_1)t_{12}^2 t_{21}^3 t_{22}^3 P(x).
\]

Therefore, \( P(x) \) is invariant under the action of \( G_{st, \beta_{281}} \).

Let \( R(281) \in Z_{281} \) be the element such that \( A(R(281)) = p_{3,12}, \quad B(R(281)) = p_{3,3} \) and that the \( x_{154} \)-coordinate is 1. Explicitly, \( R(281) = e_{453} + e_{154} + e_{234} \). Then \( P(R(281)) = 1 \) and so \( R(281) \in Z_{281}^{ss} \).

We show that \( Z_{281,k}^{ss} = M_{\beta_{281}}kR(281) \). Suppose that \( x \in Z_{281,k}^{ss} \). By assumption, \( A(x) \neq 0 \). So by Lemma II-4.6, there exists \( g \in M_{\beta_{281},k} \) such that \( A(gx) = p_{3,12} \). So we may assume that \( A(x) = p_{3,12} \). Then \( x_{453} \neq 0 \) by assumption. Let

\[
g_1 = \begin{pmatrix} \mathbf{1} & 0 & -x_{453}^{-1}x_{253} \\ 0 & 1 & -x_{453}^{-1}x_{353} \\ 0 & 0 & x_{453}^{-1} \end{pmatrix}.
\]

Then \( gx = R(281) \). Therefore, \( Z_{281,k}^{ss} = M_{\beta_{281}}kR(281) \).

We assume that \( u_{11j} = 0 \) unless \( i = 5 \) or \( j = 1 \) and \( u_{221} = 0 \). Then the first three components of \( n(u)R(281) \) are the same as those of \( R(281) \) and the last component is as follows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & u_{121} + u_{153} \\
0 & -1 & 0 & 0 & u_{131} - u_{152} \\
0 & 0 & 0 & 0 & u_{141} + u_{243} \\
-1 & * & * & * & 0 \\
\end{pmatrix}
\]
we can apply Lemma 11.9 and by Proposition 11.7. Property 11.3 holds for any $x \in Z_{281,k}^{ss}$. Therefore, $Y_{281,k}^{ss} = P_{\beta_{281}} R(281)$ also.

(56) $S_{285}, \beta_{285} = \frac{1}{50}(-24, -4, -4, -4, 36, -15, 5, 5, 5)$

We identify the element $(\text{diag}(t_{11}, g_1, t_{12}), \text{diag}(t_2, g_2)) \in M_{[1,4],[1]} = M_{\beta_{285}}$ with the element $g = (g_1, g_2, t_{11}, t_{12}, t_2) \in \text{GL}_3^2 \times \text{GL}_1^3.$ On $M_{\beta_{285}}^1,$

$$\chi_{285}(g) = t_{11}^{-24} (\det g_1)^{-4} t_{12}^{36} t_2^{-15} (\det g_2)^5 = (\det g_1)^{20} t_{12}^{60} (\det g_2)^{20}.$$ 

Let

$$A(x) = \begin{pmatrix} x_{252} & x_{253} & x_{254} \\ x_{352} & x_{353} & x_{354} \\ x_{452} & x_{453} & x_{454} \end{pmatrix}.$$ 

We identify $Z_{285} \cong M_3$ by the map $Z_{285} \ni x \mapsto A(x).$ Let $P(x) = \det A(x).$ Then

$$A(gx) = t_{12} g_1 A(x)^t g_2, \quad P(gx) = (\det g_1)^3 t_{12}^2 (\det g_2) P(x).$$

Therefore, $P(x)$ is invariant under the action of $G_{st, \beta_{285}}$.

Let $R(285) \in Z_{285}$ be the element such that $A(R(285)) = I_3.$ Explicitly, $R(285) = e_{252} + e_{353} + e_{454}.$ Then $P(R(285)) = 1$ and so $R(285) \in Z_{285}^{ss}.$ It is easy to see that $Z_{285,k}^{ss} = M_{\beta_{285}} R(285).$ Since $W_{285} = \{0\}, Y_{285,k}^{ss} = P_{\beta_{285}} R(285)$.

(57) $S_{286}, \beta_{286} = \frac{1}{50}(-24, -24, 16, 16, 16, -15, 5, 5, 5)$

We identify the element $(\text{diag}(g_{11}, g_{12}), \text{diag}(t_2, g_2)) \in M_{[2],1} = M_{\beta_{286}}$ with the element $g = (g_{12}, g_2, g_{11}, t_2) \in \text{GL}_3^2 \times \text{GL}_2 \times \text{GL}_1$. On $M_{\beta_{286}}^1,$

$$\chi_{286}(g) = (\det g_{11})^{-24} (\det g_{12})^{16} t_2^{-15} (\det g_2)^5 = (\det g_{12})^{20} (\det g_2)^{20}.$$ 

For $x \in Z_{286},$ let

$$A(x) = \begin{pmatrix} x_{342} & x_{343} & x_{344} \\ x_{352} & x_{353} & x_{354} \\ x_{452} & x_{453} & x_{454} \end{pmatrix}.$$ 

We identify $Z_{286} \cong M_3$ by the map $Z_{286} \ni x \mapsto A(x).$ Let $P(x) = \det A(x).$ Then

$$A(gx) = (x^{g_{12}} A(x)^t g_2, \quad P(gx) = (\det g_{12})^2 (\det g_2) P(x).$$

Therefore, $P(x)$ is invariant under the action of $G_{st, \beta_{286}}$.

Let $R(286) \in Z_{286}$ be the element such that $A(R(286)) = I_3.$ Explicitly, $R(286) = e_{342} + e_{353} + e_{454}.$ Then $P(R(286)) = 1$ and so $R(286) \in Z_{286}^{ss}.$ It is easy to see that $Z_{286,k}^{ss} = M_{\beta_{286}} R(286).$ Since $W_{286} = \{0\}, Y_{286,k}^{ss} = P_{\beta_{286}} R(286)$.

(58) $S_{287}, \beta_{287} = \frac{1}{50}(-2, -2, 0, 0, 4, -5, 1, 1, 3)$

We identify the element $(\text{diag}(g_{11}, g_{12}, t_1), \text{diag}(t_{21}, g_2, t_{22})) \in M_{[2],[1,3]} = M_{\beta_{287}}$ with the element $g = (g_{11}, g_{12}, g_2, t_1, t_{21}, t_{22}) \in \text{GL}_3^2 \times \text{GL}_1^3.$ On $M_{\beta_{287}}^1,$

$$\chi_{287}(g) = (\det g_{11})^{-2} t_{21}^4 (\det g_2) t_{22}^3 = (\det g_{12})^2 t_{12}^6 (\det g_2)^6 t_{22}^8.$$ 

For $x \in Z_{287},$ let $A(x) = \begin{pmatrix} x_{152} & x_{153} \\ x_{252} & x_{253} \end{pmatrix}.$ We identify $Z_{287} \cong M_2 \oplus 1$ by the map $Z_{287} \ni x \mapsto (A(x), x_{344}).$ It is easy to see that $A(gx) = t_1 g_{11} A(x)^t g_2.$
We put \( P_1(x) = \det A(x) \) and \( P(x) = P_1(x)^3x_{344}^4 \). Then on \( M_{287}^1 \),
\[
\begin{align*}
P_1(gx) &= (\det g_{11})t_1^2(\det g_2)P_1(x), \\
P(gx) &= ((\det g_{11})t_1^2(\det g_2))^3(\det g_{12})t_2^4P(x) \\
&= (\det g_{11})^3(\det g_{12})t_1^4(\det g_2)^3t_2^4. = (\det g_{12})t_1^4(\det g_2)^3t_2^4.
\end{align*}
\]
Therefore, \( P(x) \) is invariant under the action of \( G_{st,287} \).

Let \( R(287) \in Z_{287} \) be the element such that \( A(R(287)) = I_2 \) and the \( x_{344} \)-coordinate is 1. Explicitly, \( R(287) = e_{152} + e_{253} + e_{344} \). Then \( P(R(287)) = 1 \) and so \( R(287) \in Z_{287} \).

We show that \( Z_{287}^{ss} = M_{287}kR(287) \). Suppose that \( x \in Z_{287}^{ss} \). It is easy to see that there exists \( g \in M_{287}k \) such that \( A(gx) = I_2 \). So we may assume that \( A(x) = I_2 \).

Let \( t = (I_2, I_2, I_2, 1, 1, t_{22}) \). Then \( A(tx) = I_2 \) and the \( x_{344} \)-coordinate of \( tx \) is \( t_{22}x_{344} \). Therefore, \( Z_{287}^{ss} = M_{287}kR(287) \).

We assume that \( u_{121} = u_{143} = 0 \) and \( u_{232} = 0 \). Then the first component of \( n(u)R(287) \) is 0 and the remaining components are as follows:
\[
\begin{pmatrix}
    0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & u_{131} & 0 \\
    0 & 0 & 0 & u_{141} & 0 \\
    -1 & 0 & * & * & 0
\end{pmatrix},
\begin{pmatrix}
    0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & u_{132} \\
    0 & 0 & 0 & u_{142} & 0 \\
    0 & -1 & 0 & 0 & 0 \\
    * & * & * & * & 0
\end{pmatrix},
\begin{pmatrix}
    0 & 0 & 0 & 0 & u_{242} \\
    0 & 0 & 0 & 0 & u_{243} \\
    0 & 0 & 0 & 0 & 1 \\
    0 & 1 & 0 & 0 & 0 \\
    Q_1(u) + u_{154} & Q_2(u) - u_{153} & 0 & 0 & 0
\end{pmatrix}
\]
where \( Q_1(u), Q_2(u) \) do not depend on \( u_{153}, u_{154} \).

We can apply Lemma 11.9 to the map \( \text{Aff}^3 \to \text{Aff}^8 \) defined by the sequence
\[
u_{131}, u_{132}, u_{141}, u_{142}, u_{242}, u_{243}, u_{154} + Q_1(u), u_{153} - Q_2(u)
\]
where \( u_{151}, u_{152}, u_{221}, u_{231}, u_{241} \) are extra variables. So by Proposition 11.7, Property 11.5 holds for any \( x \in Z_{287}^{ss} \). Therefore, \( Y_{287}^{ss} = P_{287}kR(287) \) also.

\[(59) \ S_{289}, \beta_{289} = \frac{1}{20}(-8, 2, 2, 2, -5, -5, -5, 15)\]
We identify the element \( (\text{diag}(t_1, g_1), \text{diag}(g_2, t_2)) \in M_{[1],[3]} = M_{289} \) with the element \( g = (g_1, g_2, t_1, t_2) \in \text{GL}_4 \times \text{GL}_3 \times \text{GL}_2 \). On \( M_{289}^1 \),
\[
\chi_{289}(g) = t_1^{-8}(\det g_1)^2(\det g_2)^{-5}t_2^5 = (\det g_1)^{10}t_2^{10}.
\]
For \( x \in Z_{289} \), let
\[
A(x) = \begin{pmatrix}
    0 & x_{234} & x_{244} & x_{254} \\
    -x_{234} & 0 & x_{344} & x_{354} \\
    -x_{244} & -x_{344} & 0 & x_{454} \\
    -x_{254} & -x_{354} & -x_{454} & 0
\end{pmatrix}.
\]
We identify \( Z_{289} \equiv \wedge^2 \text{Aff}^4 \) by the map \( Z_{289} \ni x \mapsto A(x) \in \wedge^2 \text{Aff}^4 \). It is easy to see that \( A(gx) = t_2g_1A(x)g_1 \). Let \( P(x) \) be the Pfaffian of \( A(x) \). Then \( P(gx) = (\det g_1)t_2^2P(x) \). Therefore, \( P(x) \) is invariant under the action of \( G_{st,289} \).

Let \( R(289) \in Z_{289} \) be the element such that \( A(R(289)) = \begin{pmatrix} t_1 & 0 \\ 0 & 0 \end{pmatrix} \) (see (2.1)). Then \( P(R(289)) = 1 \) and so \( R(289) \in Z_{289}^{ss} \). By Lemma II-4.6, \( Z_{289}^{ss} = M_{289}kR(289) \). Since \( W_{289} = \{0\}, Y_{289}^{ss} = P_{289}kR(289) \).

\[(60) \ S_{291}, \beta_{291} = \frac{1}{20}(-8, -8, 2, 2, 12, -5, -5, 5, 5, 5)\]
We identify the element \((\mathrm{diag}(g_{11}, g_{12}, t_1), \mathrm{diag}(g_{21}, g_{22})) \in M_{[2,4],[2]} = M_{\beta_{291}}\) with the element \(g = (g_{11}, g_{12}, g_{21}, g_{22}, t_1) \in \text{GL}_4 \times \text{GL}_1\). On \(M^1_{\beta_{291}}\),

\[ \chi_{291}(g) = (\det g_{12})^{-8}(\det g_{12})^2 t_1^{12}(\det g_{21})^{-5}(\det g_{22})^5 = (\det g_{12})^{10} t_1^{20}(\det g_{22})^{10}. \]

For \(x \in Z_{291}\), let \(A(x) = \left( \begin{array}{cc} x_{451} & x_{445} \\ x_{435} & x_{434} \end{array} \right) \). We identify \(Z_{291} \cong M_2\) by the map \(Z_{291} \ni x \mapsto A(x) \in M_2\). It is easy to see that \(A(gx) = t_1 g_{12} A(x)^t g_{22}\). Let \(P(x) = \det A(x)\). Then \(P(gx) = (\det g_{11}) t_1^2(\det g_{22}) P(x)\). Therefore, \(P(x)\) is invariant under the action of \(G_{\text{st},\beta_{291}}\).

Let \(R(291) \in Z_{291}\) be the element such that \(A(R(291)) = I_2\). Then \(P(R(291)) = 1\) and \(R(291) \in Z_{291}^{ss}\). It is easy to see that \(Z_{291}^{ss} = M_{\beta_{291}} R(291)\). Since \(W_{291} = \emptyset\), \(Y_{291,k}^{ss} = P_{\beta_{291}} k R(291)\).

(61) \(S_{292}, \beta_{292} = \frac{1}{20}(-8, -8, -8, 12, 12, -5, -5, 15)\)

Let \(R(292) \in Z_{292}\) be the element such that \(x_{451} = 1\). It is easy to see that \(P(x) = x_{451}\) is invariant under the action of \(G_{\text{st},\beta_{292}}\). \(R(292) \in Z_{292}^{ss}\) and that \(Z_{292,k}^{ss} = M_{\beta_{292}} k R(292)\). Since \(W_{292} = \emptyset\), \(Y_{292,k}^{ss} = P_{\beta_{292}} k R(292)\).

This completes the proof of Theorem 11.1

12. Empty strata

In this section we prove that \(S_{\beta_i} = \emptyset\) for \(i\) not in the list \(1, 11\) for the prehomogeneous vector space \(1, 1\). We use the notation \(S_i\), etc., in this section also. We assume that elements of \(Z_i\) are in the form \(x = \sum_j y_j a_j \in Z_i\) (see the end of Section 2) unless otherwise stated. The reason why we use this coordinate system instead of \(x_{ijkl}\) is that it is convenient this way to check the computation by MAPLE.

We proceed as in Sections 4.6 of [19]. Since stability does not change by replacing \(k\) by \(\overline{k}\), we assume that \(k = \overline{k}\) throughout this section.

We prove three easy lemmas before considering individual cases. We shall also use Lemmas II–4.1, . . . , II–4.6 very often.

**Lemma 12.1.** Let \(\beta \in \mathfrak{B}\). Suppose that \(x \in Z_{\beta}, M_\beta x \subset Z_\beta\) is Zariski open and that \(x\) is unstable with respect to the action of \(G_{\text{st},\beta}\). Then \(Z_{\beta}^{ss} = \emptyset\).

**Proof.** Let \(T_0\) be as in (2.5). Then \(M_\beta = M_\beta T_0\). Let \(T_1 \subset G\) be the subgroup generated by \(T_0\) and \(\{\lambda_\beta(t) | t \in \text{GL}_1\}\) (see Section 2 [17]). Then \(M_\beta = G_{\text{st},\beta} T_1\) and \(T_1\) acts on \(Z_\beta\) by scalar multiplication.

Suppose that \(Z_{\beta}^{ss} \neq \emptyset\). Since \(Z_{\beta}^{ss}, M_\beta x \subset Z_\beta\) are Zariski open, there exists \(g \in G_{\text{st},\beta}, t \in T_1\) such that \(gtx \in Z_{\beta}^{ss}\). Since \(Z_{\beta}^{ss}\) is \(G_{\text{st},\beta}\)-invariant, \(tx \in Z_{\beta}^{ss}\). Since \(tx\) is a scalar multiple of \(x, x \in Z_\beta\), which is a contradiction. \(\square\)

Let \(G = \text{GL}_2 \times \text{GL}_2 \times \text{GL}_2, V_1 = \text{Aff}^2 \otimes \text{Aff}^2 \oplus \text{Aff}^2 \otimes \text{Aff}^2 \oplus \text{Aff}^2, V_2 = \text{Aff}^2 \otimes \text{Aff}^2 \oplus \text{Aff}^2 \oplus \text{Aff}^2\). We define an action of \((g_1, g_2, g_3) \in G\) on \(V_1, V_2\) so that

\[
V_1 \ni (v_1 \otimes v_2, v_3 \otimes v_4, v_5) \mapsto (g_1 v_1 \otimes g_2 v_2, g_2 v_3 \otimes g_3 v_4, g_3 v_5) \in V_1,
\]
\[
V_2 \ni (v_1 \otimes v_2, v_3, v_4) \mapsto (g_1 v_1 \otimes g_2 v_2, g_2 v_3, g_3 v_4) \in V_2.
\]

Let \(\{p_{2,1}, p_{2,2}\}\) be the standard basis of \(\text{Aff}^2\) as before. Let \(q_{12} = p_{2,1} \otimes p_{2,2}\), etc.
Lemma 12.2. In the above situation, let \( x \in V_1 \) or \( x \in V_2 \). Then there exists \( g \in G \) such that if \( y = (y_1, y_2, y_3) = gx \) then the coefficients of \( q_{11} \) in \( y_1 \), \( q_{11} \) or \( p_{2,1} \) in \( y_2 \) and \( p_{2,1} \) in \( y_3 \) are 0.

Proof. Since the consideration is similar, we only consider \( V_1 \). By Lemma II–4.1, we may assume that the coefficient of \( p_{2,1} \) in \( y_3 \) is 0. Then elements of the form \( (g_1, g_2, I_2) \in G \) do not change this condition. By the action of \( \text{GL}_2 \) on \( M_2 \), we can always make the \((1,1)\)-entry 0. So by replacing \( y \) by \((I_2, g_2, I_2)y \) \((g_2 \in \text{GL}_2)\) if necessary, we can make the coefficient of \( q_{11} \) in \( y_2 \) 0. Then elements of the form \((g_1, I_2, I_2) \in G \) do not change these conditions. By the same argument, we can make the coefficient of \( q_{11} \) in \( y_1 \) 0.

Let \( G = \text{SL}_3 \times \text{SL}_2, V = M_{3,2} \oplus \text{Aff}^2 \). We express elements of \( V \) as \( x = (A(x), v(x)) \) where \( A(x) \in M_{3,2}, v(x) \in \text{Aff}^2 \). If \( g = (g_1, g_2) \in G \) and \( x = (A(x), v(x)) \in V \) then we define \( gx = (g_1 A(x)^t g_2, g_2 v(x)) \).

Lemma 12.3. We fix \( i = 2 \) or \( 3 \), \( j = 1 \) or \( 2 \) and \( l = 1 \) or \( 2 \). In the above situation, if \( x = (A(x), v(x)) \in V, \) then there exists \( g \in G \) such that \( gx = (B, w) \) where the first row of \( B \) is 0 and the \((i,j)\)-entry of \( B \) and the \( l \)-th entry of \( w \) are 0.

Proof. By applying Lemma II–4.2 to the action of \( \text{SL}_3 \) on \( M_{3,2} \), we may assume that the first row of \( A(x) \) is 0. Then \( \text{SL}_2 \) in the 2–3 block of \( \text{SL}_3 \) and the second component of \( G \) do not change this condition. By applying Lemma II–4.3 to the action of \( \text{SL}_2 \times \text{SL}_2 \) on the subspace consisting of elements \((A(x), v(x))\) such that the first row of \( A(x) \) is 0, we obtain the statement of the lemma. Note that because of the symmetry, \( i, j, l \) do not have to be 1.

In the following, we shall show either (i) some coordinates of \( x \in Z_\beta \) can be eliminated by the action of the semi-simple part \( M_\beta^s \) or (ii) there exists \( x \in Z_\beta \) such that \( M_\beta x \subset Z_\beta \) is Zariski open. We then list a 1PS for each case with the property that weights of non-zero coordinates of \( x \) are all positive. Note that this proves that \( Z_\beta^{ss} = \emptyset \) (using the Lemma 12.1 for the case (ii)).

For example, for the case \( i = 2 \), the top row in the following table shows coordinates in \( Z_2 \). The first small table starts with \( 4, \ldots, x_{151}, \ldots \). This means that \( a_4 = e_{151} \in Z_2 \). These numbers are grouped according as representations of \( M_\beta^s \). The second row shows that \( M_{\beta_2} = M_{\beta_4},[3] \), we can make the coefficients of \( a_4, a_9, \ldots, a_{36} \) zero, and that the 1PS we have to consider is \( \text{GL}_1 \ni t \mapsto (\text{diag}(t^{-39}, \ldots, t^{-12}), \text{diag}(t^{-32}, \ldots, t^{12})) \). We found these 1PS’s in the same way as in Section 4 19. The last row shows that \( Z_2 \cong \Lambda_{4,1}^{4,1} \otimes \Lambda_{3,1}^{3,1} \otimes \Lambda_{1,1}^{4,2} \otimes \Lambda_{1,1}^{4,2} \) and weights with respect to the above 1PS of \( a_7, \ldots, a_{38} \) are 1, \ldots, 18 respectively. Note that the image of the above 1PS is in \( \text{G}_{ss,\beta_7} \) since \(-39 + 45 + 22 + 28 = 12 = -32 + 35 - 15 + 12 = 0 \) and the inner product of \((-39, 45, -22, 28, -12, -32, 35, -15, 12) \) and \((-4, -4, -4, -4, 16, -5, -5, -5, 15) \) = \( (620/7) \beta_7 \) is 0. As was the case in Part II, this can be verified by MAPLE commands similarly as in the comments before the table in Section 4 12.

For later \( S_i \)’s, less information is required and we may change the format of the table to save space.
coordinates of $Z_{β_i}$ & zero coordinates of $Z_{β_i}$ & 1PS \\
$Z_{β_i}$ as a representation of $M_{β_i}$ & non-zero coordinates of $Z_{β_i}$ and their weights \\

| $i$ | 4, 7, 9, 10, 14, . . . , 30 | 31, 32, 33, 35, 36, 38 |
|---|---|---|
| 2 | $M_{[4],[3]}$ | \{4, 9, 10, 14, 17, 20, 24, 27, 29, 32, 35, 36\} \{−39, 45, −22, 28, −12, −32, 35, −15, 12\} |
| $\Lambda^{1,2}_{1,[1,4]} \otimes \Lambda^{1,1}_{2,[1,3]} \oplus \Lambda^{1,2}_{4,[1,4]}$ | 7 | 19 | 30 | 31 | 33 | 38 |
| | 1 | 1 | 1 | 18 | 18 | 18 |
| 4 | $M_{[4],[3]}$ | \{7, 9, 17, 35, 36\} \{3, −13, 2, 2, 6, −2, −5, 10, −3\} |
| $\Lambda^{1,2}_{1,[2,4]} \otimes \Lambda^{1,1}_{2,[1,3]} \oplus 1 \oplus \Lambda^{3,2}_{1,[2,4]}$ | 10 | 19 | 20 | 27 | 29 | 30 | 34 | 38 |
| | 6 | 3 | 3 | 18 | 18 | 6 | 1 |
| 6 | $M_{[4],[2]}$ | \{24, 25, 35\} \{−2, −12, −12, 18, 8, 5, 5, −5, −5\} |
| $\Lambda^{1,2}_{1,[2,4]} \otimes \Lambda^{1,1}_{2,[1,2]} \oplus \Lambda^{2,1}_{2,[2,4]} \oplus \Lambda^{3,2}_{1,[2,4]} \oplus \Lambda^{1,1}_{3,[2,4]}$ | 7 | 9 | 10 | 17 | 19 | 20 | 34 | 26 | 28 | 36 | 38 |
| | 1 | 1 | 31 | 1 | 1 | 31 | 1 | 1 | 1 | 1 | 1 |
| 7 | $M_{[4],[1]}$ | \{7, 9, 14, 24\} \{−12, 7, 7, −9, 3, −13, −12, 22\} |
| $\Lambda^{1,2}_{1,[2,4]} \oplus \Lambda^{1,1}_{2,[2,4]} \oplus \Lambda^{2,1}_{2,[2,4]} \otimes \Lambda^{1,1}_{3,[2,4]}$ | 10 | 34 | 15 | 16 | 18 | 25 | 26 | 25 | 35 | 36 | 38 |
| | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 36 | 36 | 36 |
| 8 | $M_{[4],[2,3]}$ | \{7, 17, 20, 26\} \{−10, −17, 26, −18, −1, 20, −22, −7, 9\} |
| $\Lambda^{1,2}_{1,[2,4]} \otimes \Lambda^{1,1}_{2,[1,2]} \oplus 1 \oplus \Lambda^{3,2}_{1,[2,4]} \otimes \Lambda^{1,2}_{4,[2,4]}$ | 9 | 10 | 19 | 24 | 25 | 28 | 31 | 32 | 33 |
| | 45 | 1 | 3 | 2 | 2 | 1 | 2 | 45 | 1 |
| 10 | $M_{[2],[3]}$ | \{8, 9, 32, 35, 36\} \{−2, −1, −13, 8, −13, 8, 8, −3\} |
| $\Lambda^{2,1}_{1,[3,5]} \otimes \Lambda^{1,1}_{2,[1,3]} \oplus \Lambda^{2,1}_{2,[1,2]} \otimes \Lambda^{3,1}_{1,[3,5]}$ | 10 | 18 | 19 | 20 | 28 | 29 | 30 | 33 | 34 | 37 |
| | 3 | 3 | 24 | 3 | 3 | 24 | 3 | 3 | 4 |
| 11 | $M_{[2],[3]}$ | \{8, 18\} \{15, 15, −80, −80, 130, −29, −29, 86, −28\} |
| $\Lambda^{2,1}_{1,[3,5]} \otimes \Lambda^{1,1}_{2,[1,3]} \oplus \Lambda^{2,1}_{2,[1,2]} \otimes \Lambda^{3,1}_{1,[3,5]} \oplus 1.$ | 9 | 10 | 19 | 20 | 22 | 23 | 24 | 25 | 26 | 27 | 31 |
| | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 |
| 12 | $M_{[3],[3]}$ | \{10, 23, 24\} \{7, 7, −17, −4, −56, 34, 23, −1\} |
| $\Lambda^{2,1}_{2,[1,2]} \oplus \Lambda^{1,1}_{1,[1,3]} \oplus \Lambda^{2,1}_{4,[5]} \oplus \Lambda^{2,1}_{1,[1,3]}$ | 20 | 26 | 27 | 28 | 29 | 31 | 32 | 35 |
| | 13 | 13 | 26 | 13 | 26 | 13 | 13 | 13 |
We change the format of the table to save space.

|   | coordinates of $Z_{\beta_1}$, $Z_{\beta_2}$ as a representation of $M_{\beta_1}$, $M_{\beta_2}$, zero coordinates of $Z_{\beta_1}$, 1PS, non-zero coordinates of $Z_{\beta_1}$ and their weights |
|---|---|
| 13 | $M_{[3],[3]}$, {10, 20, 33, 34}, $[-2, 1, 1, 0, 0, -1, -1, 2, 0]$ |
|   | $A_{2,[1,3]} \otimes A_{2,[1,3]} \otimes A_{2,[4,5]}$ |
|   | 30 | 36 | 37 | 38 | 39 |
|   | 10, 20 | 23, 24 | 26, 27, 28 | 29, 33, ..., 39 |
|   | $x_{451}, x_{452}, x_{453}, x_{454}, x_{455}, x_{244}, x_{245}, x_{246}, x_{247}, x_{248}, x_{249} |
| 17 | $M_{[3],[2]}$, {10}, [2, 2, 2, -3, -3, -13, 7, 3, 3] |
|   | $A_{2,[2,1]} \otimes A_{2,[1,3]} \otimes A_{2,[4,5]} \otimes A_{2,[3,4]}$ |
|   | 20 | 23 | 24 | 26 | 27 | 28 | 29 | 33 | 34 | 36 | 37 | 38 | 39 |
|   | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 18 | $M_{[3],[3]}$, {7, 8, 16, 18, 19, 26, 27, 29, 33, ...}, $[-2, -4, -4, 16, 16, 15, -5, -5, 15]$ |
|   | $A_{2,[2,3]} \otimes A_{2,[1,3]} \otimes A_{2,[4,5]} \otimes A_{1,[1,2]} \otimes A_{2,[4,5]} \otimes A_{1,[1,5]}$ |
|   | 6 | 9 | 17 | 28 | 34 | 35 |
| 19 | $M_{[2],[4],[3]}$, {8, 18}, [8, 8, -13, -10, 16, -17, 27, 6] |
|   | $A_{1,[1,2]} \otimes A_{2,[2,3]} \otimes A_{2,[1,3]} \otimes A_{1,[1,2]} \otimes A_{2,[1,3]}$ |
|   | 1 | 4 | 7 | 14 | 17 | 24 | 27 | 28 | 32 | 35 | 36 | 39 |
|   | 1 | 2 | 1 | 1 | 45 | 45 | 1 | 1 | 1 | 1 |
| 22 | $M_{[2],[4],[2]},$ {9, 24, 32}, $[-13, 9, -7, 15, -4, -10, 12, -4, 2]$ |
|   | $A_{2,[3,4]} \otimes A_{2,[1,2]} \otimes A_{2,[1,2]} \otimes A_{1,[2,1]} \otimes A_{1,[3,4]}$ |
|   | 10 | 19 | 20 | 27 | 28 | 33 | 35 | 36 |
|   | 1 | 1 | 23 | 1 | 4 | 4 | 4 | 26 |
| 23 | $M_{[1],[3],[2],}$ {10, 20, 26, 27, 28, 33, 34}, [1, 1, 1, -2, -1, -7, 4, 2, 1] |
|   | $A_{2,[1,3]} \otimes A_{2,[1,3]} \otimes A_{1,[3,4]} \otimes A_{2,[1,3]} \otimes A_{2,[1,3]}$ |
|   | 20 | 27 | 28 | 29 | 34 | 35 |
|   | 1 | 2 | 1 | 2 | 1 | 3 |
| 24 | $M_{[2],[4],[3]}$, {34}, [-9, 5, 2, 3, -1, 2, 1, 0, -3] |
|   | $A_{1,[3,4]} \otimes A_{2,[3]} \otimes A_{1,[3,4]} \otimes A_{1,[3,5]} \otimes A_{2,[4,5]} \otimes A_{1,[1,2]} \otimes A_{1,3]} |
|   | 9 | 10 | 19 | 20 | 29 | 30 | 37 | 38 |
|   | 1 | 4 | 2 | 3 | 1 | 2 | 1 | 2 |
| 25 | $M_{[1],[3],[2]},$ {10, 23, 31}, [-2, -7, 7, 1, 0, 0, -4, -2, 3, 1] |
|   | $A_{2,[1,2]} \otimes A_{2,[1,2]} \otimes A_{2,[1,2]} \otimes A_{2,[1,2]} \otimes A_{1,1,2]} \otimes A_{1,2]} \otimes A_{1,2]} \otimes A_{1,2]} |
|   | 20 | 21 | 25 | 32 |
|   | 1 | 2 | 3 | 4 | 1 |
| 26 | $M_{[2],[4],[1,3]}$, {9, 14, 18}, [1, 1, -7, 5, 0, -4, 2, 3, -1] |
|   | $A_{2,[1,3]} \otimes A_{1,[3,4]} \otimes A_{2,[1,2]} \otimes A_{2,[2,3]} \otimes A_{2,[2,3]} \otimes A_{2,[2,3]}$ |
|   | 10 | 17 | 24 | 27 | 28 | 31 |
|   | 1 | 3 | 4 | 4 | 1 | 1 |
| Page | Rows | Columns | Description |
|------|------|---------|-------------|
| 27   | 2    | 2       | $M_{[1,2],[3]}$: \{7, 17, 35\}, \{3, 8, -20, 18, 7, 3, 3, -9\} |
|      | 2    | 1       | $A^{3,1}_{2,[1,3]} \oplus A^{3,1}_{2,[1,3]} \oplus 1 \oplus A^{3,1}_{1,[3,4]}$ |
| 28   | 2    | 2       | $M_{[1,4],[2,3]}$: \{10, 26\}, \{3, -6, 6, 0, -3, -14, 10, 3, 1\} |
|      | 2    | 1       | $A^{2,1}_{2,[1,2]} \oplus 1 \oplus A^{2,1}_{1,[2,3]} \oplus 1^{2\oplus}$ |
| 29   | 2    | 2       | $M_{[1,3],[1,2]}$: \{16\}, \{16, -60, 50, -20, 14, 7, -29, 11, 11\} |
|      | 2    | 1       | $A^{2,1}_{2,[2,3]} \oplus A^{1,1}_{1,[2,3]} \oplus A^{2,1}_{2,[3,4]} \oplus 1^{2\oplus}$ |
| 30   | 2    | 2       | $M_{[1,3],[1,3]}$: \{14, 16\}, \{0, -10, 8, 8, -6, 0, -11, 7, 4\} |
|      | 2    | 1       | $A^{2,1}_{2,[1,2]} \oplus \left( A^{2,1}_{2,[3,4]} \right)^{2\oplus}$ |
| 31   | 2    | 2       | $M_{[1,2],[2]}$: \{7, 24, 25\}, \{-2, 1, -4, 6, -1, 0, 2, -6, 4\} |
|      | 2    | 1       | $A^{2,1}_{2,[3,4]} \oplus A^{2,1}_{1,[3,4]} \oplus A^{2,1}_{2,[3,4]} \oplus 1^{2\oplus}$ |
| 32   | 2    | 2       | $M_{[1,2],[2,3]}$: \{9, 35\}, \{3, -5, -11, 9, 4, -10, 10, 3, -3\} |
|      | 2    | 1       | $A^{2,1}_{2,[1,3]} \oplus A^{2,1}_{1,[1,2]} \oplus 1^{3\oplus} \oplus A^{2,1}_{1,[3,4]}$ |
| 34   | 2    | 2       | $M_{[1,3],[2,3]}$: \{7, 26\}, \{4, -14, 14, -7, 3, -15, 15, -5, 5\} |
|      | 2    | 1       | $A^{2,1}_{2,[1,2]} \oplus A^{2,1}_{1,[2,3]} \oplus 1 \oplus A^{2,1}_{1,[2,3]} \oplus 1^{2\oplus}$ |
| 38   | 2    | 2       | $M_{[2,4],[1]}$: \{9\}, \{6, 6, -37, 37, -12, -24, 8, 8\} |
|      | 2    | 1       | $A^{2,1}_{1,[3,4]} \oplus A^{2,1}_{1,[1,2]} \oplus A^{3,1}_{2,[2,4]} \oplus A^{2,1}_{2,[2,4]}$ |
| 41   | 2    | 2       | $M_{[2,3],[1]}$: \{24, 25\}, \{-9, 4, -8, 12, 1, 9, -3, -15, 9\} |
|      | 2    | 1       | $A^{2,1}_{3,[3,4]} \oplus 1^{2\oplus} \oplus A^{2,1}_{2,[3,4]} \oplus A^{2,1}_{1,[3,4]}$ |
| 43   | 2    | 2       | $M_{[1,2],[1]}$: \{14, 24, 15\}, \{-9, -9, -21, 27, -6, 3, -34, 15, 16\} |
|      | 2    | 1       | $A^{2,1}_{1,[2,3]} \oplus A^{2,1}_{2,[4,5]} \oplus A^{2,1}_{1,[4,5]} \oplus A^{2,1}_{2,[3,4]} \oplus A^{2,1}_{2,[2,4]}$ |

**Notes:**
- The table lists various mathematical objects and their properties.
- The notation and symbols used are specific to the field of prehomogeneous vector spaces.
- The page number 109 is indicated at the top right corner of the page.
| 57 | 10 | 14, 17, 19, 24, 27, 29 | 33, 36, 38 | $1 \oplus \Lambda^3_{1,[1,3]} \oplus \Lambda^2_{2,[2,3]} \oplus \Lambda^3_{1,[1,3]}$ |
| 58 | 8, 9, 10, 18, ..., 30 | 35, 36, 37 | $\Lambda^3_{1,[1,3]} \oplus \Lambda^3_{1,[1,3]} \oplus \Lambda^3_{2,[2,3]} \oplus \Lambda^4_{1,[3,5]}$ |
| 59 | 4, 7, 9, 14, ..., 29 | 33, 36, 38 | $\Lambda^3_{1,[1,3]} \oplus \Lambda^3_{1,[1,3]} \oplus \Lambda^3_{2,[2,3]} \oplus \Lambda^4_{1,[1,3]}$ |
| 60 | 17, 19, 20 | 24, 34 | 25, 26, 28, 35, 36, 38 | $\Lambda^3_{1,[2,4]} \oplus \Lambda^3_{1,[2,4]} \oplus \Lambda^2_{2,[2,4]} \oplus \Lambda^2_{2,[2,4]}$ |
| 61 | 8, 9, 10, 18, ..., 40 | $\Lambda^3_{1,[3,5]} \oplus \Lambda^4_{2,[1,4]}$ |
| 62 | 7, 9, 10 | 15, 16, 18, 25, 26, 28 | 34 | $\Lambda^3_{1,[2,4]} \oplus \Lambda^3_{1,[2,4]} \oplus \Lambda^3_{2,[2,3]} \oplus 1$ |
| 63 | 20 | 23, 24, 26, ..., 29 | 31, 32, 35 | $1 \oplus \Lambda^3_{1,[1,3]} \oplus \Lambda^3_{1,[4,3]} \oplus \Lambda^2_{1,[1,3]}$ |
| 64 | 7, 9, 10, 17, 19, 20 | 24 | 33, 36, 38 | $\Lambda^3_{1,[2,4]} \oplus \Lambda^3_{2,[1,2]} \oplus 1 \oplus \Lambda^3_{1,[2,4]}$ |
| 65 | 35, 36 | $\Lambda^3_{1,[2,4]} \oplus \Lambda^3_{1,[2,4]} \oplus 1 \oplus \Lambda^3_{2,[2,4]}$ |
| 66 | 7, 9, 10, 17, 19, 20 | 25, 26, 28, 31, 32, 33 | $\Lambda^3_{1,[2,4]} \oplus \Lambda^2_{1,[2,4]} \oplus \Lambda^3_{1,[2,4]} \oplus \Lambda^3_{1,[2,4]}$ |
| 67 | 10, 20 | 24, 27, 29 | 31, 32, 35 | $\Lambda^3_{1,[1,3]} \oplus \Lambda^3_{1,[1,3]} \oplus \Lambda^3_{2,[1,2]} \oplus \Lambda^2_{1,[1,3]} \oplus \Lambda^3_{1,[1,3]}$ |
| 68 | 10, 29, 30 | 31, 32, 35 | $\Lambda^3_{2,[1,3]} \oplus \Lambda^3_{1,[1,3]}$ |

$M_{[3,4],[1,3]} = \{14, 33, 36\}, \{-8, -3, 8, 0, 3, 0, 1, 6, -7\}$

$M_{[2,2],[3]} = \{8, 9, 18, 35, 36\}, \{-4, -12, 6, 6, -9, 9, -9\}$

$M_{[3,4],[3]} = \{4, 7, 14, 33, 36\}, \{-4, 2, 5, -4, 1, -2, -2, 4, 0\}$

$M_{[1,4],[1,2]} = \{24, 25, 35\}, \{-5, -14, -14, 28, 5, 0, 10, -13, 3\}$

$M_{[2,2],[9]} = \{8, 9, 10\}, \{0, 0, 0, 0, 0, -3, 1, 1, 1\}$

$M_{[1,4],[1,3]} = \{7, 9, 15\}, \{-2, -15, -19, 109, -73, -32, -82, 38, 76\}$

$M_{[3,2],[2,3]} = \{23, 31, 32\}, \{-5, 2, 1, -2, 4, 0, 0, 2, -2\}$

$M_{[1,4],[4,3]} = \{35, 36\}, \{-5, -18, 6, 6, 11, 8, 8, -5, -11\}$

$M_{[1,4],[3,3]} = \{7, 9, 10, 18, 35, 36\}, \{15, -31, 55, -27, -12, 41, -41, -18, 18\}$

$M_{[3,4],[2,3]} = \{10\}, \{4, 4, 4, -12, 0, -15, 15, -1, 1\}$
\[
\begin{array}{cccc}
20, 30 & 33, 34, 36, \ldots, 39 & \Lambda_{2,[2,3]}^{1,1} \oplus \Lambda_{1,[1,3]}^{3,1} \otimes \Lambda_{1,[1,3]}^{2,1} \\
\times 142, \times 154, \times 244, \ldots, \times 344 & & & \\
\end{array}
\]

\[M_{[3],[1,3]} = \{20, 33, 34\}, \quad [-2, 1, 1, 0, 0, 0, -1, 1, 0, 0] \]

\[
\begin{array}{cccc}
6, 7, 8, 9, 16, \ldots, 29 & 33, 34 & \Lambda_{2,[2,3]}^{1,1} \oplus \Lambda_{1,[1,3]}^{3,1} \otimes \Lambda_{1,[1,3]}^{2,1} \\
\times 141, 151, 153, 154, \times 242, \ldots, \times 343 & & & \\
\end{array}
\]

\[M_{[1,3],[3]} = \{7, 8, 16, 18, 19, 26, 27, 30, 33\}, \quad [-6, 16, -10, -13, 13, -2, -22, 30, -6] \]

\[
\begin{array}{cccc}
14, 17, 24, 27 & 18, 28 & 32, 33, 35, 36 & \Lambda_{2,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{3,1} \oplus \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{2,1} \\
\times 152, 155, 153, 154 & 144, 145, 144, 144 & x_{154}, x_{144}, x_{244}, x_{244} & \\
\end{array}
\]

\[M_{[2,4],[1,3]} = \{14, 18, 32\}, \quad [-8, 12, -13, 7, 2, 0, -11, 9, 2] \]

\[
\begin{array}{cccc}
20 & 26, 27, 28, 29 & 33, 34, 35 & 1 \oplus \Lambda_{2,[1,2]}^{1,1} \oplus \Lambda_{1,[1,3]}^{3,1} \oplus \Lambda_{1,[1,3]}^{2,1} \oplus \Lambda_{1,[1,3]}^{2,1} \\
\times 152, 155, 153, 154 & x_{154}, x_{244}, x_{343}, 343 & x_{144}, x_{154}, x_{244} & \\
\end{array}
\]

\[M_{[1,3],[1,2,3]} = \{26, 33\}, \quad [-2, -3, 5, -4, 0, 1, 0, -1] \]

\[
\begin{array}{cccc}
10, 20 & 28, 29 & 33, 34, 36, 37 & \Lambda_{2,[1,2]}^{1,1} \oplus \Lambda_{1,[1,2]}^{3,1} \oplus \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{2,1} \\
\times 152, 155, 153, 154 & x_{154}, x_{244}, x_{343}, 343 & x_{144}, x_{154}, x_{244} & \\
\end{array}
\]

\[M_{[2,3],[2,3]} = \{10\}, \quad [-3, -3, 12, -3, -3, 7, 7, 7, 7] \]

\[
\begin{array}{cccc}
9, 10 & 18 & 24, 27 & 32, 33, 35, 36 & \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{3,1} \oplus \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{2,1} \\
\times 151, x_{151}, 152, 154 & 144, 154, 144, 144 & x_{144}, x_{154}, x_{244}, x_{244} & \\
\end{array}
\]

\[M_{[2,4],[1,2,3]} = \{24, 32\}, \quad [-12, 8, -8, 8, 4, 5, 1, -11, 5] \]

\[
\begin{array}{cccc}
9, 10, 10, 19, 20 & 24, 27 & 32, 33, 35, 36 & \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{3,1} \oplus \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{2,1} \\
\times 151, x_{151}, 152, 154 & 144, 154, 144, 144 & x_{144}, x_{154}, x_{244}, x_{244} & \\
\end{array}
\]

\[M_{[2,4],[2,3]} = \{9, 24, 32\}, \quad [-9, 7, -6, 10, -2, -7, 9, -4, 2] \]

\[
\begin{array}{cccc}
9, 10, 9, 10, \ldots, 30 & 32, 33, 35, 36 & \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{3,1} \oplus \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{2,1} \\
\times 151, x_{151}, 152, 154, 153, 154 & 144, 154, 144, 144, 144 & x_{144}, x_{154}, x_{244}, x_{244} & \\
\end{array}
\]

\[M_{[2,4],[3]} = \{9, 10\}, \quad [-1, -1, 10, 10, -18, -18, 9, 9, 0, 0] \]

\[
\begin{array}{cccc}
4, 7, 14, \ldots, 37 & 8, 18, 28, 38 & \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{3,1} \oplus \Lambda_{1,[1,2]}^{2,1} \\
\times 151, x_{151}, 152, 154, 153, 154 & 144, 154, 144, 144, 144 & x_{144}, x_{154}, x_{244}, x_{244} & \\
\end{array}
\]

\[M_{[2,4],[4]} = \{4, 7, 14, 17, 8\}, \quad [0, 0, 0, 0, 0, -3, 1, 1, 1] \]

\[
\begin{array}{cccc}
9, 10, 14, 24, 27 & 18, 28 & 33, 36 & \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{3,1} \oplus \Lambda_{1,[1,2]}^{2,1} \\
\times 151, x_{151}, 152, 154, 153, 154 & 144, 154, 144, 144 & x_{144}, x_{154}, x_{244}, x_{244} & \\
\end{array}
\]

\[M_{[2,3],[4],[1,3]} = \{14, 23\}, \quad [-22, 14, 3, 7, -2, 0, -9, 25, -16] \]

\[
\begin{array}{cccc}
7, 9, 17, 19 & 24, 34 & 26, 28, 36, 38 & \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{3,1} \oplus \Lambda_{1,[1,2]}^{2,1} \oplus \Lambda_{1,[1,2]}^{2,1} \\
\times 151, x_{151}, 152, 154, 153, 154 & 144, 154, 144, 144 & x_{144}, x_{154}, x_{244}, x_{244} & \\
\end{array}
\]

\[M_{[1,3,4],[2]} = \{7, 24, 26\}, \quad [-4, -2, 10, -1, -3, -6, 6, -8, 8] \]
| Page | Description |
|------|-------------|
| 83   | \[7, 9\], \[14, 24, 34\], \[16, 18, 26, \ldots, 38\] \[A_2, 1, [2, 3] \oplus A_2, 1, [2, 4] \oplus A_2, 3, 2 \oplus A_2, 3, 4\] \[M_{[1,3,4],[1]}\] \{7, 14, 24\}, \[-15, 5, 3, 25, -18, 18, -27, -27, 36\], \[9, 34, 16, 18, 26, 28, 36, 38\] \[3, 3, 1, 3, 1, 60, 64\] |
| 84   | \[19, 20\], \[24, 27\], \[28, 31\] \[A_2, 1, [3, 4] \oplus A_2, 2, [1, 2] \oplus \mathbb{1}_{2^D}\] \[M_{[2,4],[1,2,3]}\] \{19, 24\}, \[-4, -4, -1, 3, -2, 0, 0, -1, 1\], \[20, 27, 28, 31\] \[1, 1, 1, 1\] |
| 85   | \[20\], \[24\], \[26, 28\], \[33, 35\] \[A_2, 2, [2, 3] \oplus A_2, 2, [2, 4] \oplus 1 \oplus A_2, 1, [3, 4]\] \[M_{[1,3,4],[1,2,3]}\] \{26\}, \[2, -18, 14, -6, 8, 2, 0, -7, 5\], \[20, 24, 28, 33, 35\] \[2, 3, 1, 1, 1\] |
| 86   | \[17, 27\], \[18, 28\], \[34, 35, 36\] \[A_2, 1, 2, [2, 3] \oplus A_2, 2, [2, 4] \oplus 1 \oplus A_2, 1, [3, 4]\] \[M_{[1,2,4],[1,3]}\] \{35\}, \[-2, 5, -12, 11, 4, 0, 3, 2, -5\], \[17, 27, 18, 28, 34, 36\] \[2, 1, 2, 1, 1, 1\] |
| 87   | \[17\], \[18\], \[24, 34\], \[25, 26, 35, 36\] \[A_2, 2, [3, 4] \oplus A_2, 1, [3, 4] \oplus A_2, 2, [3, 4]\] \[M_{[1,2,4],[1,2]}\] \{24, 25\}, \[0, 4, -8, 8, -4, -1, 7, -11, 5\], \[17, 18, 28, 26, 35, 36\] \[7, 7, 1, 1, 1, 17\] |
| 88   | \[10, 20, 30\], \[34, 37\], \[38\] \[A_2, 2, [2, 3] \oplus A_2, 1, [1, 2] \oplus 1\] \[M_{[2,3,4],[3]}\] \{10, 20, 34\}, \[-14, 12, 0, 13, -11, 0, 0, 0, 0\], \[30, 37, 38\] \[2, 1, 13\] |
| 89   | \[10, 20\], \[29\], \[34, 37\], \[38\] \[A_2, 1, 2, [1, 2] \oplus 1 \oplus A_2, 1, [1, 2] \oplus 1\] \[M_{[2,3,4],[2,3]}\] \{10, 34\}, \[0, 0, 4, -4, 0, -5, 5, -1, 1\], \[20, 29, 37, 38\] \[1, 3, 1, 1\] |
| 90   | \[10, 20\], \[24\], \[26, 28\], \[35\] \[A_2, 1, 2, [1, 2] \oplus 1 \oplus A_2, 1, [2, 3] \oplus 1\] \[M_{[1,3,4],[2,3]}\] \{10, 26\}, \[0, 10, 10, -20, -43, 21, 41, -19\], \[20, 34, 28, 35\] \[1, 21, 51, 1\] |
| 91   | \[9, 10, 19, 20\], \[24, 27\], \[28\] \[A_2, 1, [3, 4] \oplus A_2, 1, [1, 2] \oplus A_2, 1, [1, 2] \oplus 1\] \[M_{[2,3,4],[2,3]}\] \{9, 24\}, \[-24, 20, 6, 6, -8, 3, 3, -11, 5\], \[10, 19, 20, 27, 28, 31\] \[1, 1, 1, 1, 1\] |
| 92   | \[8, 18\], \[27\], \[34, 35, 36\] \[A_2, 2, [1, 2] \oplus 1 \oplus A_2, 1, [3, 4]\] \[M_{[1,2,4],[2,3]}\] \{8, 35\}, \[4, -8, -16, 16, 4, -1, 1, 7, -7\], \[18, 27, 34, 36\] \[1, 3, 1, 1\] |
| 93   | \[10\], \[14, 24\], \[16, 18, 26, 28\], \[35\] \[1 \oplus A_2, 1, [2, 3] \oplus A_2, 1, [2, 3] \oplus A_2, 1, [2, 3] \oplus 1\] \[M_{[1,3,4],[1,3]}\] \{14, 16\}, \[-2, -14, 10, 10, -4, 0, -17, 7, 10\], \[10, 24, 18, 26, 28, 35\] \[6, 1, 3, 3, 27, 6\] |
### ON THE GIT STRATIFICATION OF PREHOMOGENEOUS VECTOR SPACES III

|   | 9, 10, 19, 20 | 24, 27, 34, 37 | 28, 38 | $\Lambda_2^{1,3} \otimes \Lambda_2^{2,1}$, $\Lambda_2^{1,3} \otimes \Lambda_2^{1,2}$, $\Lambda_2^{3,1} \otimes \Lambda_2^{1,3}$, $\Lambda_2^{3,1} \otimes \Lambda_2^{2,4}$, $\Lambda_2^{2,1} \otimes \Lambda_2^{2,3}$, $\Lambda_1^{2,1} \otimes \Lambda_1^{3,1}$ |
|---|---|---|---|---|
| $M_{[2,4],[2]}$: | (24, 28), $[-9, 11, -3, -3, 4, 1, 1, -9, 7]$ | 10 19 20 27 34 37 38 |
|   | 10 | 14, 17, 24, ... | 18, 28, 38 |
|   | $|10, 22, 25, ...|, 34, 37 | 1 \otimes \Lambda_2^{2,1} \otimes \Lambda_2^{1,3} \otimes \Lambda_2^{3,1}$ |
| $M_{[2,4],[4]}$: | (14, 17, 24, 28), $[-38, 58, -29, 0, 9, -3, 31, -59, 31]$ | 10 27 34 38 |
|   | 6 | 8 | 2 | 98 | 2 |
| $M_{[1,4],[1]}$: | (24, 25), (1PS) $[-1, -61, -18, 158, -90, 124, -124, -95, 95]$ | 9 10 18 34 26 35 36 |
|   | 16 | 192 | 16 | 10 | 2 | 16 | 192 |
| $M_{[1,3],[1]}$: | (13, 14, 23, 25), $[-3, -3, -10, 13, 0, 7, -15, 8]$ | 10 24 33 34 15 35 |
|   | 3 | 1 | 1 | 24 | 1 |
| $M_{[2,4],[3]}$: | (9, 10, 34), $[-1, 1, 0, 0, 0, -2, 1, 1, 0]$ | 19 | 29 | 30 | 37 |
|   | 1 | 1 | 1 | 1 |
| $M_{[1,2],[3]}$: | (9, 10), $[0, 3, 0, 0, -3, -14, 4, 4, 6]$ | 19 | 20 | 29 | 30 | 37 | 38 |
|   | 1 | 1 | 1 | 1 | 6 | 6 |
| $M_{[1,2],[4],[2]}$: | (9, 35), $[0, -3, -6, 8, 1, -7, 7, 4, -4]$ | 10 19 20 27 28 36 |
|   | 2 | 2 | 16 | 2 | 6 | 1 |
| $M_{[1,3],[4],[1]}$: | (7, 16), $[1, -9, 13, 0, -5, -6, -11, 11, 6]$ | 9 18 26 28 34 35 |
|   | 2 | 2 | 24 | 2 | 10 |
| $M_{[2,4],[1],[3]}$: | (9, 18, 34), $[-1, 1, -1, 1, 0, 0, -1, 1, 0]$ | 10 28 37 |
|   | 1 | 1 | 1 |
| $M_{[1,2],[1],[1]}$: | (9, 17, 18), $[0, 0, -1, 1, 0, 0, -2, 1, 1]$ | 10 27 37 28 38 |
|   | 1 | 1 | 1 | 1 |
| $M_{[1,2],[4],[1]}$: | (9, 17), $[0, 12, -13, 13, -12, 0, -15, 1, 14]$ | 10 27 34 38 |
|   | 1 | 1 | 2 | 14 |
| 125 | 9, 10 | 17 | 24, 34 | 28, 38 | $\Lambda_{2,1}^{3,4} \oplus 1 \oplus (\Lambda_{2,1}^{3,4})^{2\otimes}$ |
|----|----|----|----|----|----|
|    | $x_{351}, x_{352}$ | $x_{153}, x_{154}$ | $x_{343}, x_{344}$ | $x_{144}, x_{244}$ | $M_{[1,2],[1,3]}^{[1,2],[1,3]}$, $\{9, 24\}$, $[1, 2, -10, 10, -3, -6, 2, 1, 3]$, $10$ $17$ $34$ $28$ $38$ |
| 126 | 10 | 16, 18 | 24 | 33 | $1 \oplus \Lambda_{2,1}^{3,3} \oplus 1^{\otimes}$ |
|    | $x_{151}, x_{242}, x_{342}$ | $x_{153}, x_{144}$ | $x_{234}$ | | $M_{[1,4],[1,2],[3]}^{[1,2],[1,3]}$, $\{16\}$, $[12, -37, 19, -18, 24, 2, 0, -28, 26]$, $10$ $18$ $24$ $33$ $35$ |
| 127 | 7, 17 | 8, 18 | 25, 26 | 34 | $(\Lambda_{2,1}^{3,1})^{2\otimes} \oplus \Lambda_{1,3}^{3,4} \oplus 1$ |
|    | $x_{351}, x_{242}, x_{342}$ | $x_{233}, x_{243}$ | $x_{154}$ | | $M_{[1,2],[4],[2],[3]}^{[1,2],[1,3]}$, $\{8, 25\}$, $[-2, 0, -20, 16, 6, 3, 5, -5, -3]$, $7$ $17$ $18$ $26$ $34$ |
| 128 | 9, 10 | 17 | 18 | 24 | $\Lambda_{2,1}^{1,3,4} \oplus 1^{\otimes} \oplus \Lambda_{1,3}^{3,4}$ |
|    | $x_{351}, x_{352}$ | $x_{152}$ | $x_{243}, x_{344}$ | $x_{144}, x_{244}$ | $M_{[1,2],[1,3],[1,2],[3]}^{[1,2],[1,3]}$, $\{35\}$, $[-18, 4, -30, 30, 14, 19, 3, 7, -29]$, $9$ $10$ $17$ $18$ $24$ $36$ |
| 129 | 7, 9 | 14 | 26, 28 | 33 | $\Lambda_{2,1}^{2,3} \oplus 1 \oplus \Lambda_{1,2}^{3,4} \oplus 1^{\otimes}$ |
|    | $x_{351}, x_{352}$ | $x_{152}$ | $x_{243}, x_{344}$ | $x_{144}, x_{244}$ | $M_{[1,2],[1,3],[1,2],[3]}^{[1,2],[1,3]}$, $\{36\}$, $[-5, -32, 26, 0, 11, 22, -5, -24, 7]$, $7$ $9$ $14$ $28$ $33$ $35$ |
| 130 | 10, 20 | 24 | 25 | 31, 32 | $\Lambda_{2,1}^{3,3} \oplus 1^{\otimes} \oplus \Lambda_{1,2}^{3,4}$ |
|    | $x_{351}, x_{352}$ | $x_{152}$ | $x_{243}, x_{344}$ | $x_{144}, x_{244}$ | $M_{[1,2],[1,3],[1,2],[3]}^{[1,2],[1,3]}$, $\{37\}$, $[-1, -4, 4, 0, 1, 0, 0, 2, -2]$, $20$ $24$ $25$ $32$ |
| 132 | 10 | 19 | 24, 27 | 28 | $\Lambda_{2,1}^{3,3} \oplus 1^{\otimes} \oplus \Lambda_{1,2}^{3,4}$ |
|    | $x_{351}, x_{352}$ | $x_{152}$ | $x_{243}, x_{344}$ | $x_{144}, x_{244}$ | $M_{[2,3],[1,2],[3]}^{[1,2],[1,3]}$, $\{34\}$, $[-24, 30, -8, -7, 9, -1, 0, 16, -15]$, $10$ $19$ $24$ $27$ $28$ $36$ |
| 133 | 17, 19 | 24 | 26, 28 | 33 | $\Lambda_{2,1}^{2,3} \oplus 1 \oplus \Lambda_{1,2}^{3,4} \oplus 1^{\otimes}$ |
|    | $x_{352}, x_{352}$ | $x_{152}$ | $x_{243}, x_{344}$ | $x_{144}, x_{244}$ | $M_{[1,2],[1,3],[1,2],[3]}^{[1,2],[1,3]}$, $\{17\}$, $[13, -18, 28, 4, -27, -6, 0, 15, -9]$, $19$ $24$ $26$ $28$ $33$ $35$ |
| 134 | 10 | 16, 17 | 23, 24 | 25 | $\Lambda_{2,1}^{1,3,4} \oplus 1 \oplus \Lambda_{1,2}^{3,4} \oplus 1^{\otimes}$ |
|    | $x_{351}, x_{242}, x_{342}$ | $x_{143}, x_{153}$ | $x_{233}$ | $x_{134}$ | $M_{[1,2],[1,3],[1,2],[3]}^{[1,2],[1,3]}$, $\{16\}$, $[-7, -12, 3, -16, 18, -1, 0, 10, -9]$, $10$ $17$ $23$ $24$ $25$ $32$ |
| 135 | 9, 19 | 24, 27 | 28 | 33 | $\Lambda_{2,1}^{3,4} \oplus 1^{\otimes} \oplus \Lambda_{1,2}^{3,4}$ |
|    | $x_{351}, x_{352}$ | $x_{152}$ | $x_{343}, x_{344}$ | $x_{144}, x_{244}$ | $M_{[2,3],[4],[2],[3]}^{[1,2],[1,3]}$, $\{9, 24\}$, $[-2, -2, -6, 11, -1, -9, 9, 7, -7]$, $19$ $27$ $28$ $33$ $36$ |
| 136 | 10 | 16, 17, 18, 19 | 25 | 33, 34 | $\Lambda_{2,1}^{3,4} \oplus 1^{\otimes} \oplus \Lambda_{1,2}^{3,4}$ |
|    | $x_{351}, x_{352}, x_{352}, x_{352}$ | $x_{233}$ | $x_{144}, x_{154}$ | | $M_{[1,3],[1,2],[3]}^{[1,2],[1,3]}$, $\{16, 33\}$, $[-16, -17, 33, -25, 25, 10, 2, -6, -6]$, $10$ $10$ $10$ $60$ $10$ $3$ |
| Page 117 |
|-----------------|-----------------|
| 137 | \( M_{[2,3],[2,3]} \): \[ \{10\}, \[0,0,2,-1,-1,-5,3,3,-1\] \] |
| | \( \Lambda_{2,1}^{2,1} \oplus \Lambda_{2,1}^{2,1} \oplus \Lambda_{2,1}^{2,1} \oplus \Lambda_{2,1}^{2,1} \oplus \Lambda_{1,1}^{2,1} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
| 138 | \( M_{[1,3],[2,3]} \): \[ \{10,27\}, \[0,-6,6,4,-4,-1,1,-1\] \] |
| | \( \Lambda_{2,1}^{2,1} \oplus \Lambda_{2,1}^{2,1} \oplus 1^{2B} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
| 139 | \( M_{[1,3],[1,2]} \): \[ \{26\}, \[-8,-39,41,-8,14,-1,30,-28,-1\] \] |
| | \( 1 \oplus \left( \Lambda_{2,1}^{2,1} \oplus \Lambda_{2,1}^{2,1} \right)^{2B} \oplus 1^{2B} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
| 140 | \( M_{[1,3],[1,2]} \): \[ \{16\}, \[0,-12,12,0,0,3,-11,4,4\] \] |
| | \( 1 \oplus \Lambda_{2,1}^{2,1} \oplus \left( \Lambda_{2,1}^{2,1} \right)^{2B} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
| 141 | \( M_{[1,2],[2,3]} \): \[ \{9,35\}, \[-2,0,-13,15,0,-14,14,7,-7\] \] |
| | \( \Lambda_{2,1}^{2,1} \oplus \left( \Lambda_{2,1}^{2,1} \right)^{2B} \oplus 1^{2B} \oplus \Lambda_{1,1}^{2,1} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
| 142 | \( M_{[1,2],[2,3]} \): \[ \{7,8\}, \[3,0,0,-2,-1,-15,3,3,9\] \] |
| | \( \left( \Lambda_{2,1}^{2,1} \right)^{2B} \oplus 1^{2B} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
| 143 | \( M_{[1,2],[2,3]} \): \[ \{18\}, \[81,-35,29,-111,36,0,-115,83,32\] \] |
| | \( 1 \oplus \left( \Lambda_{2,1}^{2,1} \right)^{2B} \oplus 1^{2B} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
| 144 | \( M_{[1,2],[2,3]} \): \[ \{18\}, \[-1,15,-16,-1,3,-1,-16,18,-1\] \] |
| | \( 1 \oplus \left( \Lambda_{2,1}^{2,1} \right)^{2B} \oplus 1^{2B} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
| 145 | \( M_{[1,2],[2,3]} \): \[ \{17\}, \[-2,-10,16,6,-10,-3,-19,23,-1\] \] |
| | \( 1 \oplus \left( \Lambda_{2,1}^{2,1} \right)^{2B} \oplus 1^{2B} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
| 146 | \( M_{[1,2],[2,3]} \): \[ \{18\}, \[-2,6,-4,-4,4,1,-9,9,-1\] \] |
| | \( \Lambda_{1,1}^{2,1} \oplus \left( \Lambda_{2,1}^{2,1} \right)^{2B} \oplus 1 \oplus \Lambda_{1,1}^{2,1} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
| 147 | \( M_{[1,3],[2]} \): \[ \{10\}, \[4,-1,-1,-1,-1,-11,3,4,4\] \] |
| | \( 1 \oplus \left( \Lambda_{2,1}^{2,1} \oplus \Lambda_{2,1}^{2,1} \right)^{2B} \) |
| | \( x_{145}, x_{245}, x_{146}, x_{246}, x_{147}, x_{247} \) |
\[
\begin{align*}
M_{[1,2,3,4],[2,3]}: & \{10\}, \{3, -5, 5, -3, 0, -4, 4, -1, 1\}, \\
& \begin{bmatrix} 20 & 24 & 28 & 33 & 35 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix} \\
M_{[1,2,3,4],[2,3]}: & \{10\}, \{3, 0, 0, 0, -3, -10, 4, 4, 2\}, \\
& \begin{bmatrix} 20 & 27 & 28 & 34 & 35 \\ 1 & 1 & 4 & 2 & 2 \end{bmatrix} \\
M_{[1,4],[1,3]}: & \{17, 19, 27, 35\}, \{-2, -11, -11, 22, 2, 0, -6, -16, -10\}, \\
& \begin{bmatrix} 20 & 29 & 30 & 36 & 38 \\ 18 & 7 & 40 & 1 & 1 \end{bmatrix} \\
M_{[1,4],[1,3]}: & \{20, 34, 37\}, \{-1, -1, 2, 0, 0, 0, -1, 1, 0\}, \\
& \begin{bmatrix} 30 & 30 \\ 1 & 2 \end{bmatrix} \\
M_{[3,4],[1,3]}: & \{14, 17, 24, 36\}, \{-10, -19, -9, -6, 6, 2, -14, 14, -2\}, \\
& \begin{bmatrix} 19 & 27 & 29 & 33 & 38 \\ 1 & 1 & 29 & 2 & 1 \end{bmatrix} \\
M_{[1,2],[1,3]}: & \{18, 28\}, \{0, 21, -24, -25, 28, -6, -2, 2, 6\}, \\
& \begin{bmatrix} 19 & 20 & 29 & 30 & 35 & 36 & 37 \\ 2 & 1 & 6 & 5 & 3 & 2 & 55 \end{bmatrix} \\
M_{[4,2],[3]}: & \{24, 27, 29\}, \{-3, -3, -3, 17, -8, 0, 0, -8, 8\}, \\
& \begin{bmatrix} 30 & 31 & 32 & 33 & 35 & 36 & 38 \\ 1 & 2 & 2 & 22 & 2 & 22 & 22 \end{bmatrix} \\
M_{[1,4],[1,3]}: & \{17, 19, 36\}, \{-23, -22, -3, 5, 0, 0, 19, -19\}, \\
& \begin{bmatrix} 20 & 24 & 35 & 38 \\ 2 & 1 & 2 & 1 \end{bmatrix} \\
M_{[3,1],[3]}: & \{20, 31, 32\}, \{-1, -1, 2, 0, 0, 0, -1, 1, 0\}, \\
& \begin{bmatrix} 30 & 35 \\ 1 & 1 \end{bmatrix} \\
M_{[0],[3]}: & \{31, 32, 33, 34\}, \{-4, 1, 1, 1, 0, 0, 0\}, \\
& \begin{bmatrix} 35 & 36 & 37 & 38 & 39 & 40 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}
\end{align*}
\]
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\[
\begin{array}{c|c|c|c|c}
30 & 33, 34, 36, \ldots, 39 & 1 \oplus A_{1, [3, 3]}^3 \otimes A_{1, [4, 5]}^2 \\
x_{453} & x_{144}, x_{154}, x_{244}, \ldots, x_{354} \\
\end{array}
\]
\[M_{[3, [2, 3]}, \{33, 34\}, \{\{-14, 4, 4, 3, 3, 0, 0, 6, -6\}, \{30, 36, 37, 38, 39\},
\]
\[12 \quad 1 \quad 1 \quad 1 \quad 1
\]

165
\[
\begin{array}{c|c|c|c}
26, 27, 28, 29 & 33, 34, 35 & A_{1, [2, 3]}^2 \otimes A_{1, [4, 5]}^2 \oplus A_{1, [4, 5]}^1 \oplus 1 \\
\end{array}
\]
\[M_{[3, [2, 3]}, \{26, 33\}, \{-2, -2, -6, 5, 3, 0, 0, 0\},
\]
\[27 \quad 28 \quad 29 \quad 34 \quad 35 \\
1 \quad 1 \quad 9 \quad 1 \quad 4
\]

166
\[
\begin{array}{c|c|c|c|c|c}
20 & 28, 29 & 33, 34, 36, 37 & 1 \oplus A_{1, [3, 4]}^2 \oplus A_{1, [1, 2]}^2 \oplus A_{1, [4, 5]}^2 \\
x_{452} & x_{144}, x_{154}, x_{244}, x_{254}, x_{344} \\
\end{array}
\]
\[M_{[2, 3], [2, 3]}, \{28, 33\}, \{-12, 12, 0, -12, 12, 0, 4, -11, 7\},
\]
\[20 \quad 29 \quad 34 \quad 36 \quad 37 \\
4 \quad 1 \quad 7 \quad 7 \quad 31
\]

167
\[
\begin{array}{c|c|c|c}
29, 30 & 34, 37 & 38 & A_{1, [3, 4]}^2 \oplus A_{1, [1, 2]}^2 \oplus 1 \\
x_{153}, x_{453} & x_{144}, x_{154}, x_{244}, x_{344} \\
\end{array}
\]
\[M_{[2, 4], [2, 3]}, \{29, 34\}, \{-1, 1, -3, 5, -2, 0, 0, -2, 2\},
\]
\[30 \quad 37 \quad 38 \\
1 \quad 1 \quad 4
\]

168
\[
\begin{array}{c|c|c|c}
27 & 28 & 34 & 35, 36 \\
x_{253} & x_{343} & x_{154} & x_{244} \\
\end{array}
\]
\[1^3 \oplus A_{1, [3, 4]}^2 \\
\]
\[M_{[1, 2, 4], [2, 3]}, \{35\}, \{2, -5, -10, 10, 3, 0, 4, -4\},
\]
\[27 \quad 28 \quad 34 \quad 36 \\
2 \quad 4 \quad 1 \quad 1
\]

169
\[
\begin{array}{c|c|c|c}
29, 30 & 34, 37 & 38 & A_{2, [2, 3]}^2 \oplus A_{1, [1, 2]}^2 \oplus 1 \\
x_{452}, x_{453} & x_{154}, x_{254} & x_{344} \\
\end{array}
\]
\[M_{[2, 3], [1, 3]}, \{34\}, \{-5, -1, 5, -4, 0, 0, 0\},
\]
\[20 \quad 30 \quad 37 \quad 38 \\
1 \quad 1 \quad 1 \quad 4
\]

170
\[
\begin{array}{c|c|c|c}
19, 20, 29, 30 & 34, 37 & 38 & A_{1, [3, 4]}^2 \oplus A_{2, [2, 3]}^2 \oplus A_{1, [1, 2]}^2 \oplus 1 \\
x_{352}, x_{452}, x_{353}, x_{453} & x_{154}, x_{254} & x_{344} \\
\end{array}
\]
\[M_{[2, 4], [1, 3]}, \{34\}, \{-9, 5, 3, 3, -2, 0, 1, 1, -2\},
\]
\[19 \quad 20 \quad 29 \quad 30 \quad 37 \quad 38 \\
2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 4
\]

171
\[
\begin{array}{c|c|c|c|c}
17 & 28 & 34 & 35, 36 & 1^3 \oplus A_{1, [3, 4]}^2 \\
x_{252} & x_{343} & x_{154} & x_{244}, x_{244} \\
\end{array}
\]
\[M_{[1, 2, 4], [1, 2, 3]}, \{35\}, \{3, -2, -16, 10, 5, 0, 7, -7\},
\]
\[17 \quad 28 \quad 34 \quad 36 \\
3 \quad 1 \quad 1 \quad 1
\]

172
\[
\begin{array}{c|c|c|c}
20, 30 & 33, 34 & 35 & A_{2, [2, 3]}^2 \oplus A_{1, [4, 5]}^2 \oplus 1 \\
x_{452}, x_{144}, x_{154} & x_{234} \\
\end{array}
\]
\[M_{[1, 3], [1, 3]}, \{20, 33\}, \{2, 0, 0, -1, -1, 0, -5, 3, 2\},
\]
\[30 \quad 34 \quad 35 \\
1 \quad 3 \quad 2
\]

173
\[
\begin{array}{c|c|c|c|c}
20 & 23, 24, 33, 34 & 25, 35 & 1 \oplus A_{1, [4, 5]}^2 \oplus A_{2, [3, 4]}^2 \oplus A_{2, [3, 4]}^2 \\
x_{452} & x_{144}, x_{154}, x_{244}, x_{254} & x_{234} \\
\end{array}
\]
\[M_{[1, 3], [1, 2]}, \{23, 25\}, \{2, -2, -2, -4, 6, 0, 0, -5, 5\},
\]
\[20 \quad 24 \quad 33 \quad 34 \quad 35 \\
2 \quad 3 \quad 3 \quad 3 \quad 1
\]

174
\[
\begin{array}{c|c|c|c|c|c}
19, 20, 29, 30 & 34 & 35, 36 & 1 \oplus A_{1, [3, 4]}^2 \oplus A_{2, [2, 3]}^2 \oplus 1 \oplus A_{1, [3, 4]}^2 \\
x_{352}, x_{452}, x_{353}, x_{453} & x_{154} & x_{244}, x_{244} \\
\end{array}
\]
\[M_{[1, 2, 4], [1, 3]}, \{19, 35\}, \{0, 0, -33, 15, 18, 0, -17, 31, -14\},
\]
\[20 \quad 29 \quad 30 \quad 34 \quad 36 \\
16 \quad 16 \quad 64 \quad 4 \quad 1
\]
\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
175 & 20 & 24, 27, 29, 34, 37, 39 & 1 \oplus \Lambda_{1,1,3}^3 \oplus \Lambda_{2,3}^3 \\
& x_{452} & x_{153}, x_{253}, x_{353}, x_{154}, x_{254}, x_{354} & \\
M_{[3,4], [1,2]} & \{24, 34\} & [-12, 6, 6, 0, 0, -2, 12, -5, -5] & 20 & 27 & 29 & 37 & 39 \\
& & & 12 & 1 & 1 & 1 & 1 & 1 \\
176 & 18, 19, 20, 28, 29, 30 & 31 & \Lambda_{1,1,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus 1 \\
& x_{452}, x_{352}, x_{453}, x_{353}, x_{154}, x_{254} & x_{343}, x_{344} & \\
M_{[2,1,3]} & \{18, 28\} & [6, 6, -16, 16, 20, 0, -3, -3, 6] & 19 & 20 & 29 & 30 & 31 \\
& & & 1 & 1 & 1 & 1 & 18 \\
177 & 20 & 24, 27, 34, 37, 28, 38 & 1 \oplus \Lambda_{1,1,2}^3 \oplus \Lambda_{2,2,3}^3 \oplus \Lambda_{2,2,4}^3 \\
& x_{452} & x_{153}, x_{253}, x_{353}, x_{154}, x_{254}, x_{354} & x_{343}, x_{344} & \\
M_{[2,3,4], [1,2]} & \{24, 28\} & [-2, 6, -3, 0, -1, -5, 5, -4, 4] & 20 & 27 & 34 & 37 & 38 \\
& & & 4 & 1 & 1 & 9 & 1 \\
179 & 14, 15, 24, \ldots, 37 & 18, 28, 38 & \Lambda_{1,1,2}^3 \oplus \Lambda_{2,2,4}^3 \oplus \Lambda_{2,2,2}^3 \\
& x_{152}, x_{252}, x_{352}, \ldots, x_{254} & x_{342}, x_{343}, x_{344} & \\
M_{[2,4], [1]} & \{14, 17, 24, 28\} & [-11, 15, -4, -4, 0, 9, -18, 9] & 27 & 34 & 37 & 18 & 38 \\
& & & 1 & 2 & 28 & 1 & 1 \\
180 & 9, 10, 19, \ldots, 30 & 35, 36 & \Lambda_{1,2,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus \Lambda_{3,1,1}^3 \\
& x_{351}, x_{451}, x_{352}, \ldots, x_{453} & x_{324}, x_{244} & \\
M_{[1,2,4], [3]} & \{9, 10, 35\} & [0, 0, -1, 1, 0, -4, 2, 2, 0] & 19 & 20 & 29 & 30 & 36 \\
& & & 1 & 3 & 1 & 3 & 1 \\
181 & 9, 10, 19, \ldots, 30 & 37 & \Lambda_{1,2,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus 1 \\
& x_{351}, x_{451}, x_{352}, \ldots, x_{453} & x_{234} & \\
M_{[1,2,4], [3]} & \{9, 10\} & [0, 1, 0, 0, -1, -16, 2, 2, 12] & 19 & 20 & 29 & 30 & 37 \\
& & & 1 & 1 & 1 & 1 & 12 \\
182 & 9, 10, 19, \ldots, 30 & 38 & \Lambda_{1,2,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus 1 \\
& x_{351}, x_{451}, x_{352}, \ldots, x_{453} & x_{344} & \\
M_{[2,4], [3]} & \{9, 10\} & [0, 0, 3, 3, -6, -14, 4, 4, 6] & 19 & 20 & 29 & 30 & 38 \\
& & & 1 & 1 & 1 & 1 & 12 \\
183 & 4, 7, 14, \ldots, 27 & 38 & \Lambda_{1,2,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus 1 \\
& x_{151}, x_{251}, x_{152}, \ldots, x_{253} & x_{344} & \\
M_{[2,4], [3]} & \{4, 7\} & [5, 5, 0, 0, -10, -30, 6, 6, 18] & 14 & 17 & 24 & 27 & 38 \\
& & & 1 & 1 & 1 & 1 & 18 \\
184 & 10 & 17, 19, 27, 29 & 36, 38 & \Lambda_{1,2,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus 1 \\
& x_{451} & x_{252}, x_{352}, x_{253}, x_{353} & x_{244}, x_{344} & \\
M_{[1,3,4], [1,3]} & \{17, 36\} & [-1, -4, 4, 0, 1, 3, -4, 4, -3] & 10 & 19 & 27 & 29 & 38 \\
& & & 4 & 1 & 1 & 9 & 1 \\
185 & 10 & 18, 19, 28, 29 & 36, 37 & \Lambda_{1,2,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus 1 \\
& x_{451} & x_{342}, x_{343}, x_{353}, x_{244}, x_{254} & \\
M_{[1,2,3], [3]} & \{18, 36\} & [-4, 0, -4, -16, 24, 0, -9, 31, -22] & 10 & 19 & 28 & 29 & 37 \\
& & & 8 & 11 & 11 & 51 & 2 \\
186 & 7, 9 & 16, 18, 26, 28 & 34 & \Lambda_{1,2,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus \Lambda_{2,2,3}^3 \oplus 1 \\
& x_{351}, x_{352} & x_{242}, x_{243}, x_{244}, x_{344} & x_{354} & \\
M_{[1,3,4], [1,3]} & \{7, 16\} & [1, -8, 8, 6, -7, 0, -12, 4, 8] & 9 & 18 & 26 & 28 & 34 \\
& & & 1 & 2 & 2 & 18 & 2 \\
\end{array}
\]
| Page | Example | Description | Details |
|------|---------|-------------|---------|
| 187 | $A_{1,3}^{2,1} \oplus 1 \oplus A_{1,3}^{2,1}$ | $M_{[2,4],[1,2,3]} \{19,34\}, \{19,34\}, \{19,34\}, \{19,34\}, \{19,34\}$ | $[6,6,2,2,0,-1,0,6,-5]$, $[0,2,28,37,6,1,2,0]$ |
| 188 | $A_{1,3}^{2,1} \oplus 1^{3\oplus}$ | $M_{[2,4],[1,2,3]} \{19\}, \{3,0,-13,13,-3,0,-9,5,4\}$ | $27,34,38,1,2,4,4$ |
| 189 | $2^{3\oplus} \oplus 1^{3\oplus}$ | $M_{[2,4],[1,2,3]} \{25\}, \{0,0,-4,4,0,2,-3,1\}$ | $17,18,26,34,2,2,1,1$ |
| 190 | $1 \oplus A_{1,3}^{2,1} \oplus 1 \oplus A_{1,3}^{2,1}$ | $M_{[2,3],[1,2,3]} \{23,32\}, \{33\}, \{9,9,-2,-1,3,0,0,7,-7\}$ | $19,24,27,28,36,1,1,19,4,1$ |
| 191 | $1 \oplus A_{1,3}^{2,1} \oplus A_{1,4,5} \oplus A_{1,3}^{2,1}$ | $M_{[2,3],[1,2,3]} \{32\}, \{22\}, \{22\}$ | $20,24,26,27,35,2,3,3,13,1$ |
| 192 | $A_{2,1}^{3,1} \oplus A_{2,1}^{2,1}$ | $M_{[1,3],[1,2,3]} \{10,20,37\} \{20,20,37\}$ | $30,39,2,1$ |
| 193 | $A_{2,1}^{2,1} \oplus 1 \oplus A_{2,1}^{2,1}$ | $M_{[2,3],[1,2,3]} \{10,34\}, \{20,20\}$ | $1,4,1$ |
| 194 | $A_{2,1}^{3,1} \oplus A_{2,1}^{2,1}$ | $M_{[2,3],[1,2,3]} \{9,17\}, \{10,17\}$ | $[0,3,-4,4,-3,0,-5,1,4]$, $[0,3,-4,4,-3,0,-5,1,4]$ |
| 195 | $A_{2,1}^{2,1} \oplus 1 \oplus A_{2,1}^{2,1}$ | $M_{[2,3],[1,2,3]} \{10,34\}, \{20,20\}$ | $1,4,1$ |
| 196 | $A_{2,1}^{3,1} \oplus A_{2,1}^{2,1}$ | $M_{[2,3],[1,2,3]} \{9,14\}, \{9,14\}$ | $[0,0,-13,19,-6,-12,8,8,-4]$, $[10,17,21,27,38,2,2,2,2]$ |
| 197 | $A_{2,1}^{3,1} \oplus A_{2,1}^{2,1}$ | $M_{[2,3],[1,2,3]} \{9,34\}, \{9,34\}$ | $[0,2,28,37,6,1,2,0]$ |
| 198 | \(20\) | \(27, 29\) | \(33\) | \(35\) | \(1 \otimes A^2, 1_{[1,2.3]} \oplus 1^2\oplus\) |
|-----|------|------|------|------|---------------------------------|
| \(M_{[1,3.4],[1,2.3]}\) | \{27\}, \([-1, -14, 14, 14, -13, -1, 0, 0, 1]\), \(20\) | \(29\) | \(33\) | \(35\) | \(1\) | \(1\) | \(14\) | \(1\) |

| 199 | \(9, 10\) | \(18, 28\) | \(37\) | \(A^2, 1_{[1,3.4]} \oplus A^2, 1_{[2,2.3]} \oplus 1\) |
|-----|------|------|------|---------------------------------|
| \(M_{[1,2.4],[1,3]}\) | \{9, 18\}, \([0, 0, -6, -6, 12, 0, -11, 13, -2]\), \(10\) | \(28\) | \(37\) | \(6\) | \(1\) | \(10\) |

| 200 | \(10\) | \(16, 18\) | \(25\) | \(34\) | \(1 \otimes A^2, 1_{[1,2.3]} \oplus 1^2\oplus\) |
|-----|------|------|------|------|---------------------------------|
| \(M_{[1,3.4],[1,2.3]}\) | \{16\}, \([-1, -5, 3, -2, 3, 0, 0, 3, -3]\), \(10\) | \(18\) | \(25\) | \(34\) | \(1\) | \(1\) | \(1\) |

| 201 | \(10\) | \(17, 19\) | \(24\) | \(36, 38\) | \(1 \otimes A^2, 1_{[1,2.3]} \oplus 1 \oplus A^2, 1_{[1,2.3]}\) |
|-----|------|------|------|------|---------------------------------|
| \(M_{[1,3.4],[1,2.3]}\) | \{36\}, \([-2, -16, 16, -6, 11, 11, -3, -8]\), \(10\) | \(17\) | \(19\) | \(24\) | \(38\) | \(2\) | \(1\) | \(33\) | \(1\) | \(4\) |

| 202 | \(19, 20\) | \(24\) | \(28\) | \(35, 36\) | \(A^2, 1_{[1,3.4]} \oplus 1^2\oplus \Lambda^2, 1_{[1,3.4]}\) |
|-----|------|------|------|------|---------------------------------|
| \(M_{[1,2.4],[1,2.3]}\) | \{35\}, \([-7, 1, -16, 16, 6, 0, 11, 2, -13]\), \(19\) | \(20\) | \(24\) | \(28\) | \(36\) | \(1\) | \(33\) | \(1\) | \(2\) |

| 203 | \(10, 20, 30\) | \(38, 39\) | \(A^2, 1_{[1,3.4]} \oplus 1^2\oplus A^2, 1_{[1,3.4]}\) |
|-----|------|------|---------------------------------|
| \(M_{[2],[3]}\) | \{10, 20, 38\}, \([0, 0, 0, -1, 1, -1, 1, 2, 0]\), \(30\) | \(39\) | \(2\) | \(1\) |

| 204 | \(10, 20\) | \(23, 24\) | \(35\) | \(A^2, 1_{[1,2.3]} \oplus A^2, 1_{[1,4.5]}\) |
|-----|------|------|------|---------------------------------|
| \(M_{[1,3],[2.3]}\) | \{10, 23\}, \([2, 0, 0, -7, 5, -3, 3, -6, 6]\), \(20\) | \(24\) | \(35\) | \(1\) | \(1\) | \(1\) |\(\) |

| 205 | \(9, 10\) | \(18\) | \(24\) | \(35, 36\) | \(A^2, 1_{[1,3.4]} \oplus 1^2\oplus A^2, 1_{[1,3.4]}\) |
|-----|------|------|------|------|---------------------------------|
| \(M_{[1,2.4],[1,2.3]}\) | \{35\}, \([-7, 0, -11, 13, 5, 7, 0, 3, -10]\), \(9\) | \(10\) | \(18\) | \(24\) | \(36\) | \(1\) | \(25\) | \(2\) | \(1\) |

| 206 | \(10, 20\) | \(25\) | \(31, 32\) | \(A^2, 1_{[1,2.3]} \oplus 1 \oplus A^2, 1_{[1,2.3]}\) |
|-----|------|------|------|---------------------------------|
| \(M_{[1,3],[2.3]}\) | \{10, 31\}, \([-2, -3, 5, 0, 0, -1, 1, 2, -2]\), \(20\) | \(25\) | \(32\) | \(1\) | \(4\) | \(1\) |

| 207 | \(7, 9, 17, 19\) | \(24\) | \(36, 38\) | \(A^2, 1_{[1,2.3]} \oplus A^2, 1_{[2,1.2]} \oplus 1 \oplus A^2, 1_{[1,2.3]}\) |
|-----|------|------|------|---------------------------------|
| \(M_{[1,3.4],[2.3]}\) | \{7, 36\}, \([0, -3, 7, -5, 1, -5, 5, 1, -1]\), \(9\) | \(17\) | \(19\) | \(24\) | \(38\) | \(3\) | \(3\) | \(13\) | \(2\) | \(1\) |

| 208 | \(9, 19\) | \(24, 27\) | \(33, 36\) | \(A^2, 1_{[1,2.3]} \oplus A^2, 1_{[1,2]} \oplus 1^2\oplus\) |
|-----|------|------|------|---------------------------------|
| \(M_{[2,3.4],[2.3]}\) | \{9\}, \([0, 0, 0, -1, -6, 2, 4, 0]\), \(19\) | \(24\) | \(27\) | \(33\) | \(36\) | \(1\) | \(3\) | \(1\) | \(1\) |
| 210 | 10, 20, 30 | 32, 35 | \( \Lambda^3_{1,1} \oplus \Lambda^2_{1,1} \oplus \Lambda^2_{2,1} \) |
|---|---|---|---|
| 17, 27 | 18, 28 | 34 | \( \Lambda^2_{1,1} \oplus \Lambda^2_{2,1} \oplus 1^{2\otimes} \) |
| 211 | 10, 17 | 24, 34 | 28, 38 | \( 1^{2\otimes} \oplus (\Lambda^2_{1,1})^{2\otimes} \) |
| 212 | 9 | 17, 27 | 18, 28 | 33 | \( 1 \oplus (\Lambda^2_{1,1})^{2\otimes} \oplus 1 \) |
| 213 | 7, 17 | 24, 34 | 28, 38 | \( \Lambda^2_{1,1} \oplus (\Lambda^2_{1,1})^{2\otimes} \) |
| 214 | 10, 20, 30 | 36, 37, 38, 39 | \( \Lambda^3_{1,1} \oplus \Lambda^2_{1,1} \oplus \Lambda^2_{1,1} \oplus \Lambda^1_{1,1} \) |
| 215 | 10, 20 | 29 | 37 | 38 | \( \Lambda^1_{1,1} \oplus 1^{2\otimes} \) |
| 216 | 220 | 27, 29 | 34 | 36, 38 | \( \Lambda^2_{1,1} \oplus \Lambda^2_{1,1} \oplus \Lambda^2_{1,1} \) |
| 217 | 10, 20 | 26, 27, 28, 29 | 35 | \( \Lambda^2_{1,1} \oplus \Lambda^2_{1,1} \oplus \Lambda^2_{1,1} \) |
| 218 | 9, 19 | 24 | 28 | 36 | \( \Lambda^2_{1,1} \oplus 1^{3\otimes} \) |
\[
M_{[1,2,3,4],[1,3]}, \quad \{19\}, \quad [0, 0, 0, 0, 0, 8, -11, 1, 2], \quad 10 \quad 29 \quad 34 \quad 36 \\
\]
\[
M_{[1,2,3,4],[2,3]}, \quad \{8\}, \quad [0, 4, -5, 2, -1, -8, 4, 0, 4], \quad 18 \quad 27 \quad 34 \quad 36
\]
\[
M_{[1,2,3,4],[3,2]}, \quad \{10\}, \quad [12, -1, -1, -11, 1, -11, 13, 7, 5], \quad 20 \quad 24 \quad 33 \quad 35
\]
\[
M_{[1,2,3,4],[2,3]}, \quad \{10\}, \quad [6, -6, 6, -3, -3, -9, 7, 0, 2], \quad 20 \quad 28 \quad 33 \quad 34 \quad 35
\]
\[
M_{[4],[3]}, \quad \{34, 37, 39\}, \quad [-1, -1, -1, 3, 0, 0, 0, 0, 0], \quad 40 \quad 3
\]
\[
M_{[3],[2,3]}, \quad \{31, 32\}, \quad [-14, 4, 4, 3, 3, 0, 3, -3], \quad 30 \quad 35
\]
\[
M_{[3],[3]}, \quad \{33, 34\}, \quad [-2, 1, 0, 0, 0, 0, 0, 0], \quad 36 \quad 37 \quad 38 \quad 39
\]
\[
M_{[2],[3,3]}, \quad \{28, 33\}, \quad [-4, 6, -2, -5, 5, 0, 0, -2, 2], \quad 29 \quad 34 \quad 36 \quad 37 \quad 13
\]
\[
M_{[2,3],[3,3]}, \quad \{34\}, \quad [-4, -4, -1, 4, -3, 0, 0, 0, 0], \quad 30 \quad 37 \quad 38 \quad 3
\]
| Page | 239 | 240 | 241 | 242 | 243 | 244 | 245 | 246 | 247 | 248 | 249 |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( M_{[1,3],[2,3]} \) | \{33\} | \([-2,1,1,-4,0,0,1,-1]\) | \(30\) | \(34\) | \(35\) | \(1 \oplus \Lambda^{2,1}_{1,[4,3]} \oplus 1\) |
| \( M_{[1,4],[2,4]} \) | \{27,37\} | \([-0,2,1,1,0,0,0,0]\) | \(29\) | \(30\) | \(39\) | \(40\) | \(1\) | \(1\) | \(1\) | \(1\) |
| \( M_{[2,2]} \) | \{28,38\} | \([0,0,-1,-1,2,0,0,0]\) | \(29\) | \(30\) | \(39\) | \(40\) | \(1\) | \(1\) | \(1\) | \(1\) |
| \( M_{[2,4],[3,4]} \) | \{24,28\} | \([-4,6,-2,-2,2,0,0,-5,5]\) | \(27\) | \(34\) | \(37\) | \(38\) | \(3\) | \(3\) | \(13\) | \(1\) |
| \( M_{[4,2]} \) | \{19,20\} | \([0,0,0,0,0,0,-2,1,1]\) | \(29\) | \(30\) | \(39\) | \(40\) | \(1\) | \(1\) | \(1\) | \(1\) |
| \( M_{[4,4],[1,3]} \) | \{19,35\} | \([1,-4,-5,5,3,0,-5,5,0]\) | \(20\) | \(29\) | \(30\) | \(36\) | \(3\) | \(3\) | \(13\) | \(1\) |
| \( M_{[4,4],[2,3]} \) | \{24,27,34,37\} | \([-1,-1,1,0,0,0,0]\) | \(29\) | \(30\) | \(39\) | \(40\) | \(1\) | \(1\) | \(1\) | \(1\) |
| \( M_{[4,2],[2,3]} \) | \{29\} | \([0,1,-4,4,-1,0,0,-2,2]\) | \(30\) | \(37\) | \(38\) | \(1\) | \(2\) | \(2\) | \(1\) | \(1\) |
| \( M_{[1,3],[4,3]} \) | \{20,37\} | \([-0,1,1,0,0,0,-1,1,0]\) | \(30\) | \(39\) | \(1\) | \(1\) | \(1\) | \(1\) | \(1\) | \(1\) |
| \( M_{[2,4],[1,3]} \) | \{34\} | \([-4,4,0,0,0,-1,2,2,-3]\) | \(19\) | \(20\) | \(29\) | \(30\) | \(37\) | \(2\) | \(2\) | \(2\) | \(1\) |
| \( M_{[2,4],[1,2]} \) | \{19\} | \([0,0,-4,4,0,1,-3,1,1]\) | \(20\) | \(27\) | \(37\) | \(28\) | \(38\) | \(1\) | \(1\) | \(1\) | \(1\) |
| 250 | 20 | 28 | 34, 37 |
|-----|----|----|--------|
| $x_{452}$ | $x_{343}$ | $x_{154}, x_{254}$ | $1^{2\oplus} \oplus A_{1,[1,2]}^{2,1}$ |

$M_{[2,4],[1,2,3]}$, (34), $[-3, 3, -1, 0, 0, 0, 3, -3]$, 20 28 37 1 2

| 251 | 20, 30 | 38, 39 |
|-----|-------|--------|
| $x_{452}, x_{453}$ | $x_{344}, x_{354}$ | $\Lambda_{2,[2,3]}^{2,1} \oplus \Lambda_{1,[4,5]}^{2,1}$ |

$M_{[2,3],[1,3]}$, (20, 38), $[0, 0, 0, -1, 1, 0, -1, 1, 0]$, 30 39 1 1

| 252 | 20 | 23, 24 | 35 |
|-----|----|--------|----|
| $x_{452}$ | $x_{143}, x_{153}$ | $x_{234}$ | $1 \oplus A_{1,[4,5]}^{2,1} \oplus 1$ |

$M_{[1,3],[1,2,3]}$, (23), $[0, 0, 0, -5, 5, 0, 1, -4, 3]$, 20 24 35 1 1

| 253 | 17, 19 | 24 | 36, 38 |
|-----|-------|----|--------|
| $x_{252}, x_{352}$ | $x_{151}, x_{234}$ | $x_{344}$ | $\Lambda_{1,[2,3]}^{2,1} \oplus 1 \oplus \Lambda_{1,[2,3]}^{2,1}$ |

$M_{[1,3],[1,2,3]}$, (36), $[-5, -11, 11, 0, 5, 0, 8, 1, -9]$, 17 19 24 38 2 1

| 255 | 17 | 24, 34 | 28, 38 |
|-----|----|--------|--------|
| $x_{252}$ | $x_{153}, x_{154}$ | $x_{344}$ | $1 \oplus (\Lambda_{2,[3,4]}^{2,1})^{2\oplus}$ |

$M_{[1,2,4],[1,2]}$, (28), $[7, -2, -5, 5, 5, 0, -2, -10, 12]$, 17 24 34 38 1 1 2 4 2

| 257 | 20, 30 | 36, 37, 38, 39 |
|-----|-------|------------------|
| $x_{452}, x_{453}$ | $x_{294}, x_{254}, x_{344}, x_{354}$ | $\Lambda_{2,[2,3]}^{2,1} \oplus \Lambda_{1,[2,3]}^{2,1} \oplus \Lambda_{1,[4,5]}^{2,1}$ |

$M_{[1,3],[1,3]}$, (20, 36), $[8, -4, -4, 0, 0, 0, -7, 1, 6]$, 30 37 38 39 1 2 2 2

| 260 | 20, 30 | 34 | 35 |
|-----|-------|----|----|
| $x_{452}, x_{453}$ | $x_{154}$ | $x_{234}$ | $\Lambda_{2,[2,3]}^{2,1} \oplus 1^{2\oplus}$ |

$M_{[1,3,4],[1,3]}$, (20), $[0, 1, 1, -3, 1, 0, -3, 3, 0]$, 30 34 35 1 1 2

| 261 | 10, 20, 30, 40 |
|-----|----------------|
| $x_{451}, \ldots, x_{454}$ | $\Lambda_{2,[1,4]}^{2,1}$ |

$M_{[3]}, (10, 20, 30), [0, 0, 0, 0, 0, -1, -1, -1, 3]$, 40 3

| 262 | 10, 20, 30 | 39 |
|-----|-------|----|
| $x_{451}, x_{452}, x_{453}$ | $x_{354}$ | $\Lambda_{2,[1,3]}^{3,1} \oplus 1$ |

$M_{[2,3],[3]}$, (10, 20), $[0, 0, 1, 0, -1, -1, -4, 2, 3]$, 30 39 1 3

| 263 | 10, 20, 30 | 35 |
|-----|-------|----|
| $x_{451}, x_{452}, x_{453}$ | $x_{234}$ | $\Lambda_{2,[1,3]}^{3,1} \oplus 1$ |

$M_{[1,3],[3]}$, (10, 20), $[4, 0, 0, -2, -2, -3, -11, 5, 9]$, 30 35 1 3

| 264 | 9, 10, 19, \ldots, 40 |
|-----|---------------------|
| $x_{351}, x_{352}, \ldots, x_{454}$ | $\Lambda_{1,[3,4]}^{2,1} \otimes \Lambda_{1,[4,1]}^{4,1}$ |

$M_{[2,4]}, [9, 10, 19, 20], [0, 0, 0, 0, 0, -1, -1, 1, 1]$, 29 30 39 40 1 1 1 1
ON THE GIT STRATIFICATION OF PREHOMOGENEOUS VECTOR SPACES III

|   | 265 | 266 | 267 | 268 | 269 |
|---|-----|-----|-----|-----|-----|
|   | \(\begin{array}{ccc}
10, 20 & 27, 29, 37, 39 & \Lambda^{2,1}_{2,[1,2]} \oplus \Lambda^{2,1}_{1,[2,3]} \oplus \Lambda^{2,1}_{2,[3,4]} \\
\end{array} \)
|   | \(\begin{array}{ccc}
M_{[1,3,4],[2]} \cdot \{10\}, & [0, 0, 0, 1, -1, -7, 1, 3, 3], & 20, 27, 29, 37, 39 \\
\end{array} \)
|   | \(\begin{array}{ccc}
10, 20 & 28, 29, 38, 39 & \Lambda^{2,1}_{2,[1,2]} \oplus \Lambda^{2,1}_{1,[4,5]} \oplus \Lambda^{2,1}_{2,[3,4]} \\
\end{array} \)
|   | \(\begin{array}{ccc}
M_{[2,3],[2]} \cdot \{10\}, & [0, 0, 2, -1, -1, -5, 3, 1, 1], & 20, 28, 29, 38, 39 \\
\end{array} \)
|   | \(\begin{array}{ccc}
10 & 19, 29 & 36 & 1 \oplus \Lambda^{2,1}_{2,[2,3]} \oplus 1 \\
\end{array} \)
|   | \(\begin{array}{ccc}
M_{[1,2,3,4],[1,3]} \cdot \{19\}, & [0, -1, 0, 3, -2, 0, -3, 3, 0], & 10, 29, 36 \\
\end{array} \)
|   | \(\begin{array}{ccc}
7, 17 & 24 & 38 & \Lambda^{2,1}_{2,[1,2]} \ominus 1 \oplus 1^2 \ominus \\
\end{array} \)
|   | \(\begin{array}{ccc}
M_{[1,2,3],[2]} \cdot \{7\}, & [0, 0, 1, -2, -7, 3, 3, 1], & 17, 24, 38 \\
\end{array} \)
|   | \(\begin{array}{ccc}
8, 18 & 24, 27, 34, 37 & \Lambda^{2,1}_{2,[1,2]} \oplus \Lambda^{2,1}_{1,[1,2]} \oplus \Lambda^{2,1}_{2,[3,4]} \\
\end{array} \)
|   | \(\begin{array}{ccc}
M_{[2,4],[2]} \cdot \{8, 24\}, & [1, 1, 0, 0, -2, -11, 1, 5, 5], & 18, 27, 34, 37 \\
\end{array} \)

We change the format of the table again.

(1) coordinates of \(Z_{\beta_i}\),  (2) \(Z_{\beta_i}\) as a representation of \(M_{\beta_i}\),  (3) \(M_{\beta_i}\),  (4) zero coordinates of \(Z_{\beta_i}\),  (5) 1PS, non-zero coordinates of \(Z_{\beta_i}\) and their weights

|   | (1), (2), (3), (4), (5) |   |   |   |   |
|---|-----------------------|---|---|---|---|
| 274 | \(\begin{array}{ccc}
37, 39, 40 & \Lambda^{2,1}_{1,[2,3]} \oplus 1 & M_{[1,4],[3]} \cdot \{37, 39\}, & [0, 0, -1, 1, 0, 0, 0, 0, 0], & 40 \\
\end{array} \)
|   | \(\begin{array}{ccc}
\end{array} \)
| 275 | \(\begin{array}{ccc}
38, 39, 40 & \Lambda^{2,1}_{1,[3,4]} \oplus 1 & M_{[2,3],[3]} \cdot \{38, 39\}, & [0, 0, -1, 0, 1, 0, 0, 0, 0], & 40 \\
\end{array} \)
|   | \(\begin{array}{ccc}
\end{array} \)
| 276 | \(\begin{array}{ccc}
34, 37 & \Lambda^{2,1}_{1,[2,3]} \oplus 1 & M_{[1,3,4],[2,3]} \cdot \{37\}, & [-5, 3, 2, 2, -2, 0, 0, 0, 0], & 37, 38 \\
\end{array} \)
|   | \(\begin{array}{ccc}
\end{array} \)
| 277 | \(\begin{array}{ccc}
30 & \Lambda^{2,1}_{1,[2,3]} \oplus 1 & M_{[1,3,4],[2,3]} \cdot \{37\}, & [0, -4, 4, -1, 1, 0, 0, 4, -4], & 37, 38 \\
\end{array} \)
|   | \(\begin{array}{ccc}
\end{array} \)
| 278 | \(\begin{array}{ccc}
30 & \Lambda^{2,1}_{1,[2,3]} \oplus 1 & M_{[1,3,4],[2,3]} \cdot \{37\}, & [0, -4, 4, -1, 1, 0, 0, 4, -4], & 30, 39 \\
\end{array} \)
|   | \(\begin{array}{ccc}
\end{array} \)
| 279 | \(\begin{array}{ccc}
28 & \Lambda^{2,1}_{1,[2,3]} \oplus 1 & M_{[1,3,4],[2,3]} \cdot \{37\}, & [-7, 5, 0, 0, 2, 0, 0, 6, -6], & 28, 37 \\
\end{array} \)
|   | \(\begin{array}{ccc}
\end{array} \)
| 282 | \(\begin{array}{ccc}
20, 30, 40 & \Lambda^{2,1}_{2,[3,4]} \oplus 1 & M_{[1,3,4],[1,1]} \cdot \{20\}, & [0, 0, 0, 0, 0, 0, 0, -1, 1], & 40 \\
\end{array} \)
|   | \(\begin{array}{ccc}
\end{array} \)
| 283 | \(\begin{array}{ccc}
20, 30 & \Lambda^{2,1}_{2,[3,4]} \oplus 1 & M_{[1,3,4],[1,1]} \cdot \{20\}, & [0, 0, 0, 0, 0, 0, 0, -1, 1], & 30, 39 \\
\end{array} \)
|   | \(\begin{array}{ccc}
\end{array} \)
| 284 | \(\begin{array}{ccc}
20, 30 & \Lambda^{2,1}_{2,[2,3]} \oplus 1 & M_{[1,3,4],[1,1]} \cdot \{20\}, & [0, 0, 0, 0, 0, 0, 0, -1, 1], & 30, 39 \\
\end{array} \)
|   | \(\begin{array}{ccc}
\end{array} \)
We verify the information (which coordinates we can eliminate) in the above table in the following, which proves the following theorem.

**Theorem 12.4.** For $i$ not in the list $\{11, 12\}$, $S_{\beta_i} = \emptyset$.

We either use Lemmas II–4.1–II–4.6, $\{12.1, 12.2\}$ with possible extra arguments to show that we can assume some coordinates are 0. The easiest cases are the following. Each factor of $M_{\beta_i}$ except for $GL_i$ have at most one standard representation in $Z_i$ and it is enough to apply Lemma II–4.1 to all such standard representations to show that $S_{\beta_i} = \emptyset$. The applicable cases are the following. The numbering is according to the order in the table.

(4) $S_7$, $\beta_7 = \frac{1}{14}(-4, 0, 0, 0, 4, -3, 1, 1, 1)$
(10) $S_{17}$, $\beta_{17} = \frac{7}{20}(-4, -4, -4, 6, -6, -5, -5, 5, 5)$
(12) $S_{19}$, $\beta_{19} = \frac{3}{128}(-16, -16, 4, 4, 24, -5, -5, -5, 15)$
(15) $S_{24}$, $\beta_{24} = \frac{23}{128}(-16, -16, 4, 4, 24, -5, -5, -5, 15)$
(16) $S_{25}$, $\beta_{25} = \frac{1}{380}(-24, -4, -4, 16, 16, -25, -25, 15, 35)$
(19) $S_{28}$, $\beta_{28} = \frac{10}{178}(-28, -8, -8, 12, 32, -25, -25, 15, 35)$
(39) $S_{56}$, $\beta_{56} = \frac{69}{69}(-14, -14, -14, 6, 36, -5, -5, -5, 15)$
(62) $S_{83}$, $\beta_{83} = \frac{1}{170}(-44, -24, -24, 36, 56, -15, 5, 5, 5)$
(63) $S_{84}$, $\beta_{84} = \frac{1}{170}(-4, -4, 1, 1, 6, -140, 40, 45, 55)$
(64) $S_{85}$, $\beta_{85} = \frac{1}{730}(-56, -16, -16, 24, 64, -185, -5, 75, 115)$
(67) $S_{88}$, $\beta_{88} = \frac{1}{170}(-104, -104, -64, 116, 156, -55, -55, -55, 165)$
(68) $S_{89}$, $\beta_{89} = \frac{1}{170}(-36, -36, 4, 14, 54, -15, -15, -5, 35)$
(69) $S_{90}$, $\beta_{90} = \frac{1}{170}(-89, -4, -4, 6, 91, -50, -50, 45, 55)$
(71) $S_{92}$, $\beta_{92} = \frac{1}{170}(-82, -42, 28, 28, 68, -25, -25, 5, 45)$
(75) $S_{97}$, $\beta_{97} = \frac{1}{170}(-12, -2, -2, 8, 8, -5, -5, -5, 15)$
(92) $S_{122}$, $\beta_{122} = \frac{1}{170}(-26, -26, 14, 14, 24, -15, -5, -5, 25)$
(94) $S_{124}$, $\beta_{124} = \frac{1}{930}(-256, -16, 4, 4, 264, -75, -55, -55, 185)$
(96) $S_{126}$, $\beta_{126} = \frac{1}{1069}(-124, -24, -24, 76, 96, -135, -15, 65, 85)$
(100) $S_{130}$, $\beta_{130} = \frac{1}{1200}(-64, -4, -4, 16, 56, -55, -55, 25, 85)$
(107) $S_{138}$, $\beta_{138} = \frac{1}{220}(-38, -8, -8, 22, 32, -25, -25, 5, 45)$
(116) $S_{147}$, $\beta_{147} = \frac{1}{390}(-16, -1, -1, 9, 9, -5, -5, 5, 5)$,
(117) $S_{148}$, $\beta_{148} = \frac{1}{170}(-82, -52, -2, 28, 108, -75, -75, 35, 115)$
(118) $S_{153}$, $\beta_{153} = \frac{1}{170}(-88, -28, 12, 32, 72, -45, -45, 15, 75)$
(121) $S_{156}$, $\beta_{156} = \frac{1}{170}(-76, -76, -76, 24, 204, -85, -5, -5, 95)$
(124) $S_{159}$, $\beta_{159} = \frac{1}{390}(-4, -4, -4, 16, 65, -65, 65, 55, 75)$
(131) $S_{167}$, $\beta_{167} = \frac{1}{220}(-48, -48, 12, 12, 72, -55, -55, 25, 85)$
(132) $S_{168}$, $\beta_{168} = \frac{1}{390}(-32, -12, 8, 8, 28, -95, -95, 85, 105)$
(134) $S_{170}$, $\beta_{170} = \frac{1}{390}(-64, -64, 16, 16, 96, -65, -5, -5, 75)$
(135) $S_{171}$, $\beta_{171} = \frac{1}{390}(-46, -6, 4, 4, 4, 85, 5, 35, 45)$
(136) $S_{172}$, $\beta_{172} = \frac{1}{520}(-168, -28, -28, 112, 112, -205, -25, -25, 255)$
(150) \( S_{187}, \beta_{187} = \frac{1}{360}(-148, -148, 52, 52, 192, -205, -45, 95, 155) \)
(151) \( S_{188}, \beta_{188} = \frac{1}{360}(-92, -32, 8, 8, 108, -95, -15, 25, 85) \)
(152) \( S_{189}, \beta_{189} = \frac{1}{360}(-64, -4, 16, 16, 36, -365, 95, 115, 155) \)
(155) \( S_{192}, \beta_{192} = \frac{1}{360}(-136, -56, -56, 44, 204, -25, -25, -25, 75) \)
(156) \( S_{193}, \beta_{193} = \frac{1}{220}(-48, -48, -28, -8, 132, -15, -15, 5, 25) \)
(157) \( S_{194}, \beta_{194} = \frac{1}{360}(-328, 32, 52, 52, 192, -45, -25, -25, 95) \)
(158) \( S_{195}, \beta_{195} = \frac{1}{360}(-148, -148, -28, 132, 192, -125, -125, 95, 155) \)
(161) \( S_{198}, \beta_{198} = \frac{1}{360}(-52, -12, -12, 28, 48, -95, -15, 25, 85) \)
(162) \( S_{199}, \beta_{199} = \frac{1}{360}(-328, -28, 112, 112, 132, -45, -25, -25, 95) \)
(163) \( S_{200}, \beta_{200} = \frac{1}{10}(-4, 0, 0, 1, 3, -3, 0, 1, 2) \)
(166) \( S_{204}, \beta_{204} = \frac{1}{360}(-136, -136, 24, 124, 124, -25, -25, -25, 75) \)
(167) \( S_{205}, \beta_{205} = \frac{1}{360}(-32, -12, -12, 28, 28, -35, -35, 25, 45) \)
(169) \( S_{207}, \beta_{207} = \frac{1}{120}(-88, 12, 12, 32, 32, -45, -45, -5, 95) \)
(172) \( S_{210}, \beta_{210} = \frac{1}{360}(-92, -92, 48, 68, 68, -45, -45, -45, 135) \)
(178) \( S_{218}, \beta_{218} = \frac{1}{130}(-56, -16, -16, 44, 44, -15, -15, -15, 45) \)
(179) \( S_{219}, \beta_{219} = \frac{1}{260}(-104, -44, 16, 36, 96, -25, -25, -5, 55) \)
(181) \( S_{221}, \beta_{221} = \frac{1}{260}(-24, 4, 4, 8, 8, -3, -3, 1, 5) \)
(182) \( S_{222}, \beta_{222} = \frac{1}{130}(-224, -64, -4, 36, 256, -125, -125, 95, 155) \)
(183) \( S_{225}, \beta_{225} = \frac{1}{60}(-16, -4, 0, 4, 16, -7, -3, -3, 13) \)
(184) \( S_{228}, \beta_{228} = \frac{1}{360}(-52, -32, 8, 28, 48, -15, -15, 5, 25) \)
(185) \( S_{229}, \beta_{229} = \frac{1}{260}(-196, -16, -16, 64, 164, -65, -65, -65, 195) \)
(186) \( S_{230}, \beta_{230} = \frac{1}{260}(-124, -64, -4, 256, -125, -125, -5, 255) \)
(188) \( S_{233}, \beta_{233} = \frac{1}{60}(-3, -3, -3, 12, -5, -5, -5, 15) \)
(189) \( S_{234}, \beta_{234} = \frac{1}{20}(-4, -4, -4, 0, 12, -5, -5, 3, 7) \)
(193) \( S_{238}, \beta_{238} = \frac{1}{60}(-4, -4, -4, 2, 6, -5, -5, 1, 9) \)
(194) \( S_{239}, \beta_{239} = \frac{1}{60}(-6, -1, -1, 4, 4, -10, -10, 5, 15) \)
(201) \( S_{246}, \beta_{246} = \frac{1}{20}(-8, -8, 2, 2, 6, -5, -5, 3, 7) \)
(202) \( S_{247}, \beta_{247} = \frac{1}{60}(-8, -3, -3, 2, 12, -5, 0, 0, 5) \)
(203) \( S_{248}, \beta_{248} = \frac{1}{20}(-4, -4, -2, -2, 12, -5, 1, 1, 3) \)
(205) \( S_{250}, \beta_{250} = \frac{1}{200}(-44, -44, -4, 36, 56, -65, -25, 35, 55) \)
(206) \( S_{251}, \beta_{251} = \frac{1}{10}(-8, -8, 2, 7, 7, -5, 0, 0, 5) \)
(207) \( S_{252}, \beta_{252} = \frac{1}{10}(-16, -6, -6, 14, 14, -85, 5, 35, 45) \)
(211) \( S_{260}, \beta_{260} = \frac{1}{260}(-64, -4, 16, 56, -65, -5, -5, 75) \)
(212) \( S_{261}, \beta_{261} = \frac{1}{60}(-2, -2, -2, 3, 3, 0, 0, 0, 0) \)
(213) \( S_{262}, \beta_{262} = \frac{1}{60}(-8, -8, 0, 4, 12, -1, -1, -1, 3) \)
(214) \( S_{263}, \beta_{263} = \frac{1}{60}(-32, 3, 3, 13, 13, -5, -5, -5, 15) \)
(216) \( S_{265}, \beta_{265} = \frac{1}{60}(-8, -2, -2, 0, 12, -1, -1, 1, 1) \)
(217) \( S_{266}, \beta_{266} = \frac{1}{60}(-8, -8, 4, 6, 6, -1, -1, 1, 1) \)
(218) \( S_{267}, \beta_{267} = \frac{1}{200}(-104, -4, 16, 36, 56, -25, -5, -5, 35) \)
(219) \( S_{268}, \beta_{268} = \frac{1}{10}(-46, -16, -6, -6, 74, -25, -25, 5, 45) \)
(221) \( S_{274}, \beta_{274} = \frac{1}{10}(-24, -4, -4, -4, 36, -15, -15, -15, 45) \)
(223) \( S_{276}, \beta_{276} = \frac{1}{130}(-16, -16, 4, 24, -35, -35, -35, 105) \)
(224) \( S_{277}, \beta_{277} = \frac{1}{130}(-56, -16, -16, 4, 84, -35, -35, 25, 45) \)
(225) \( S_{278}, \beta_{278} = \frac{1}{130}(-56, -56, 24, 44, 44, -35, -35, 25, 45) \)
(226) \( S_{279}, \beta_{279} = \frac{1}{220}(-28, -28, 12, 12, 32, -55, -55, 45, 65) \)
The next easy cases are those where we can apply Lemma II–4.1, but either it is
equipped to consider a part of standard representations or we have to choose a standard
representation for a particular component of $M_{\beta_i}$. Note that there are cases
where a GL$_2$-component of $M_{\beta_i}$ has two standard representations in $Z_i$ and eliminating
a coordinate for only one representation works. In the following, we list such cases and
the subspaces to which we apply Lemma II–4.1 to show that $S_{\beta_i} = \emptyset$. 

(18) $S_{27}$, $\beta_{27} = \frac{11}{2369}(-32, -12, 8, 28, -5, -5, -5, 15), \langle a_{19}, a_{27}, a_{35}, a_{36} \rangle$
(20) $S_{29}$, $\beta_{29} = \frac{11}{2369}(-7, -2, -2, 3, 8, -10, 0, 5, 5), \langle a_{16}, a_{18} \rangle$
(25) $S_{38}$, $\beta_{38} = \frac{14}{451}(-16, -16, 4, 4, 24, -15, 5, 5, 5), \langle a_{9}, a_{10} \rangle$
(29) $S_{144}$, $\beta_{144} = \frac{1}{5320}(-32, -12, 8, 28, -35, -15, 5, 45), \langle a_{9}, a_{10} \rangle$
(31) $S_{146}$, $\beta_{146} = \frac{1}{3025}(-14, -4, -4, 6, 16, -15, -5, 15), \langle a_{17}, a_{19} \rangle$
(32) $S_{147}$, $\beta_{147} = \frac{1}{3025}(-32, -12, 8, 28, -15, -15, 5, 25), \langle a_{7}, a_{17} \rangle$
(33) $S_{148}$, $\beta_{148} = \frac{1}{3025}(-8, -4, 0, 4, 8, -3, -3, 1, 5), \langle a_{8}, a_{18} \rangle$
(49) $S_{167}$, $\beta_{167} = \frac{1}{3025}(-26, -26, -26, -16, 94, -35, -35, -25, 95), \langle a_{10}, a_{20} \rangle$
(55) $S_{176}$, $\beta_{176} = \frac{1}{3025}(-44, -44, -14, 51, 51, -40, -40, 25, 55), \langle a_{10}, a_{20} \rangle$
(65) $S_{196}$, $\beta_{196} = \frac{1}{3025}(-56, -21, 14, 14, 49, -110, 25, 25, 60), \langle a_{35}, a_{36} \rangle$
(82) $S_{1029}$, $\beta_{1029} = \frac{1}{3025}(-44, -44, 11, 11, 66, -65, -15, 40, 40), \langle a_{19}, a_{20} \rangle$
(98) $S_{128}$, $\beta_{128} = \frac{1}{3025}(-36, -16, 9, 9, 34, -30, -5, 15, 20), \langle a_{35}, a_{36} \rangle$
(99) $S_{129}$, $\beta_{129} = \frac{1}{3025}(-9, -4, -4, 1, 16, -10, -5, 5, 10), \langle a_{26}, a_{28} \rangle$
(101) $S_{132}$, $\beta_{132} = \frac{1}{3025}(-68, -68, -38, 72, 102, -105, 5, 35, 65), \langle a_{36}, a_{36} \rangle$
(102) $S_{133}$, $\beta_{133} = \frac{1}{3025}(-7, -2, -2, 3, 8, -45, 10, 15, 20), \langle a_{17}, a_{19} \rangle$
(103) $S_{134}$, $\beta_{134} = \frac{1}{3025}(-12, -7, 3, 8, 8, -15, 0, 5, 10), \langle a_{16}, a_{17} \rangle$
(104) $S_{135}$, $\beta_{135} = \frac{1}{3025}(-96, -96, 4, 44, 144, -75, -75, 25, 125), \langle a_{9}, a_{19} \rangle$
(106) $S_{137}$, $\beta_{137} = \frac{3}{315}(-7, -7, -2, 8, 8, -10, -10, 5, 15), \langle a_{10}, a_{20} \rangle$
(108) $S_{139}$, $\beta_{139} = \frac{1}{3025}(-16, -1, -1, 4, 14, -10, -5, 5, 10), \langle a_{26}, a_{28} \rangle$
(109) $S_{140}$, $\beta_{140} = \frac{1}{3025}(-86, -6, -6, 24, 74, -75, 5, 35, 35), \langle a_{16}, a_{18} \rangle$
(112) $S_{143}$, $\beta_{143} = \frac{1}{3025}(-23, -18, 2, 7, 32, -10, -5, 5, 20), \langle a_{18}, a_{28} \rangle$
(113) $S_{144}$, $\beta_{144} = \frac{1}{3025}(-88, -58, -28, 72, 102, -105, 25, 25, 55), \langle a_{18}, a_{28} \rangle$
(114) $S_{145}$, $\beta_{145} = \frac{1}{3025}(-36, -16, -1, 19, 34, -20, -5, -5, 30), \langle a_{17}, a_{27} \rangle$
(115) $S_{146}$, $\beta_{146} = \frac{1}{3025}(-8, -3, 2, 2, 7, -5, 0, 5), \langle a_{18}, a_{28} \rangle$
(133) $S_{169}$, $\beta_{169} = \frac{1}{3025}(-48, -48, -28, 52, 72, -55, -15, -15, 85), \langle a_{34}, a_{37} \rangle$
(153) $S_{190}$, $\beta_{190} = \frac{1}{3025}(-96, -96, -36, 84, 144, -285, 35, 95, 155), \langle a_{33}, a_{36} \rangle$
(164) $S_{201}$, $\beta_{201} = \frac{1}{3025}(-136, -96, -96, 84, 244, -185, -5, 35, 155), \langle a_{36}, a_{38} \rangle$
(165) $S_{203}$, $\beta_{203} = \frac{1}{3025}(-136, -16, 4, 4, 144, -285, -5, 135, 155), \langle a_{35}, a_{36} \rangle$
(168) $S_{206}$, $\beta_{206} = \frac{1}{3025}(-24, -20, 12, 12, 20, -21, -13, 15, 19), \langle a_{35}, a_{36} \rangle$
(171) $S_{209}$, $\beta_{209} = \frac{1}{3025}(-18, -18, -8, 2, 42, -15, -15, -5, 35), \langle a_{9}, a_{19} \rangle$
(173) $S_{211}$, $\beta_{211} = \frac{1}{3025}(-136, -76, -16, 84, 144, -285, 75, 75, 135), \langle a_{18}, a_{28} \rangle$
(174) $S_{212}$, $\beta_{212} = \frac{1}{3025}(-176, -136, -16, 84, 244, -185, 35, 75, 75), \langle a_{28}, a_{38} \rangle$
(175) $S_{213}$, $\beta_{213} = \frac{1}{3025}(-24, -20, -8, 20, 32, -13, -1, -1, 15), \langle a_{17}, a_{27} \rangle$
There are cases where it is enough to apply Lemma II–4.3 once. For that purpose there has to be a factor \(GL_2 \times GL_2\) in \(M_{\beta_i}\) and a component \(M_2 \oplus \text{Aff}^2\) in \(Z_i\). If this pair is unique then we do not have to worry about to which component we apply Lemma II–4.3. We list such cases with the coordinates we can eliminate to show that \(S_{\beta_i} = \emptyset\).

**Remark 12.5.** By symmetry, we can eliminate any one entry of \(M_2\) and either entry of \(\text{Aff}^2\).

(177) \(S_{215}, \beta_{215} = \frac{1}{238}(-38, -8, 2, 2, 42, -15, -15, 15, 15), \langle a_{28}, a_{38} \rangle\)

(180) \(S_{220}, \beta_{220} = \frac{1}{380}(-92, -32, -32, 48, 108, -55, -55, 25, 85), \langle a_{10}, a_{20} \rangle\)

(187) \(S_{231}, \beta_{231} = \frac{1}{380}(-72, -52, 28, 48, -35, -35, -15, 85), \langle a_{10}, a_{20} \rangle\)

(204) \(S_{249}, \beta_{249} = \frac{1}{30}(-16, -1, 4, 4, 9, -10, 0, 5, 5), \langle a_{19}, a_{20} \rangle\)

(208) \(S_{253}, \beta_{253} = \frac{1}{200}(-24, -14, -14, 6, 46, -65, 5, 15, 45), \langle a_{36}, a_{38} \rangle\)

(209) \(S_{255}, \beta_{255} = \frac{1}{200}(-34, -24, 6, 6, 46, -65, 15, 25, 25), \langle a_{28}, a_{38} \rangle\)

There are cases where it is enough to apply Lemma II–4.2 once. For that purpose there has to be a factor \(GL_n\) in \(M_{\beta_i}\) and a component \(M_{n,m}\) with \(n > m\) in \(Z_i\). In each of the
following cases, there is a unique such component in $Z_i$ and applying Lemma II–4.2 once is enough to show that $S_{S_i} = \emptyset$.

(37) $S_{54}, \beta_{54} = \frac{1}{19}(-6, -1, -1, -1, 9, 0, 0, 0, 0)$
(44) $S_{61}, \beta_{61} = \frac{2}{3}(-3, -3, 2, 2, 0, 0, 0, 0)$
(58) $S_{79}, \beta_{79} = \frac{1}{30}(-56, -56, 34, 34, 44, -25, -25, -25, 75)$
(79) $S_{102}, \beta_{102} = \frac{1}{60}(-9, -9, -9, -9, 36, -15, 5, 5, 5)$
(89) $S_{118}, \beta_{118} = \frac{1}{220}(-88, -38, 22, 22, 82, -15, -15, -15, 45)$
(111) $S_{142}, \beta_{142} = \frac{1}{780}(-112, -52, 8, 48, 108, -15, -15, -15, 45)$
(123) $S_{158}, \beta_{158} = \frac{1}{340}(-136, 4, 44, 44, -85, 15, 15, 55)$
(128) $S_{163}, \beta_{163} = \frac{1}{340}(-56, -56, -56, 84, 84, -85, 85, 15, 155)$
(139) $S_{175}, \beta_{175} = \frac{1}{340}(-56, -56, -56, -36, 204, -85, 15, 35, 35)$
(140) $S_{176}, \beta_{176} = \frac{1}{340}(-12, -12, 8, 8, 8, -145, 35, 35, 75)$
(144) $S_{181}, \beta_{181} = \frac{1}{340}(-136, -36, 16, -16, 204, -5, -5, -5, 15)$
(145) $S_{182}, \beta_{182} = \frac{1}{340}(-136, -136, 84, 84, 104, -5, -5, -5, 15)$
(146) $S_{183}, \beta_{183} = \frac{1}{340}(-52, -52, -12, -12, 128, -25, -25, -25, 75)$
(191) $S_{236}, \beta_{236} = \frac{1}{60}(-4, -4, -4, 6, 6, -15, -15, -15, 45)$
(195) $S_{240}, S_{240} = \frac{1}{60}(-24, -4, -4, -4, 36, -15, -15, 15, 15)$
(196) $S_{241}, S_{241} = \frac{1}{60}(-24, -24, 16, 16, -15, -15, 15, 15)$
(198) $S_{243}, \beta_{243} = \frac{1}{60}(-24, -24, 6, 6, 36, -15, 5, 5, 5)$
(200) $S_{245}, \beta_{245} = \frac{1}{60}(-3, -3, -3, -3, 12, -5, -5, 5, 5)$
(215) $S_{264}, \beta_{264} = \frac{1}{10}(-4, -4, 1, 1, 6, 0, 0, 0, 0)$

We consider the remaining cases one by one. We assume that $x = \sum_j y_j a_j \in Z_i$.

(1) $S_{2}, \beta_{2} = \frac{7}{620}(-4, -4, -4, -4, 16, -5, -5, -5, 15)$
We show that $S_2 = \emptyset$ using Lemma II–12.1
For $x \in Z_2$, let

$$M_1(x) = \begin{pmatrix} x_{151} & x_{152} & x_{153} \\ x_{251} & x_{252} & x_{253} \\ x_{351} & x_{352} & x_{353} \\ x_{451} & x_{452} & x_{453} \end{pmatrix}, \quad M_2(x) = \begin{pmatrix} 0 & x_{124} & x_{134} & x_{144} \\ -x_{124} & 0 & x_{244} & x_{244} \\ -x_{134} & -x_{234} & 0 & x_{344} \\ -x_{144} & -x_{244} & -x_{344} & 0 \end{pmatrix}. $$

We can identify $Z_2$ with $Z$ by the map $Z_2 \ni x \mapsto (M_1(x), M_2(x)) \in Z$.

$M_{\beta_2}$ contains $M = \text{GL}_4 \times \text{GL}_3$ whose action on $Z = \text{Aff}^4 \otimes \text{Aff}^3 \oplus \wedge^2 \text{Aff}^4$ is the natural action. Let

$$w_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

and $w = (w_1, w_2)$.

Let $k[\varepsilon]/(\varepsilon^2)$ be the space of dual numbers, $e_M = (I_4, I_3)$ and $A = (a_{ij}) \in M_4, B = (b_{ij}) \in M_3$. Then elements of the form $e_M + \varepsilon(A, B)$ belong to $T_{e_M}(M_w)$ (the tangent space of the stabilizer $M_w$) if and only if

$$Aw_1 + w_1^t B = 0, \quad Aw_2 + w_2^t A = 0.$$
By long but straightforward computations, $A, B$ are in the following form:

$$A = \begin{pmatrix}
  a_{11} & 0 & 0 & 0
  a_{21} & -a_{11} - a_{24} & a_{23} & a_{24}
  -a_{24} & a_{32} & a_{33} & -a_{32}
  a_{21} + a_{23} & -a_{11} - a_{24} + a_{33} & a_{43} & a_{24} - a_{33}
\end{pmatrix},$$

$$B = \begin{pmatrix}
  a_{11} + a_{24} & -a_{32} & a_{11} + a_{24} - a_{33}
  -a_{23} & -a_{33} & -a_{43}
  -a_{24} & a_{32} & -a_{24} + a_{33}
\end{pmatrix}.$$  

Therefore, $\dim T_{e,M}(M_y) = 7 = 25 - 18 = \dim M - \dim Z$. By Proposition 3.3, $Mw \subset Z$ is Zariski open and $M_w$ is smooth over $k$ (we do not need this part). This implies that $M_{\beta_4}w \subset Z_2$ is Zariski open. For this $w$, the $y_i$ coordinate (the coefficient of $a_i$) is 0 unless $i = 7, 19, 30, 31, 33, 38$. The 1PS with the required property is listed in the above table. Therefore, $Z_2^w = \emptyset$. We shall not repeat this comment in the following.

(2) $S_4$, $\beta_4 = \frac{3}{17}(-4, 0, 0, 0, 4, -1, -1, -1, 3)$

Note that $\text{GL}_3$ is contained in the 2–4 block (resp. 1–3 block) of the first (resp. second) component of $M_{\beta_4}$. Let

$$A(x) = \begin{pmatrix}
  x_{251} & x_{252} & x_{253}
  x_{351} & x_{352} & x_{353}
  x_{451} & x_{452} & x_{453}
\end{pmatrix},
B(x) = \begin{pmatrix}
  0 & x_{234} & x_{244}
-x_{234} & 0 & x_{344}
-x_{244} & -x_{344} & 0
\end{pmatrix}.$$

Then $Z_4$ can be identified with $M_3 \oplus \wedge^2 \text{Aff}^3 \oplus 1$ by the map $Z_4 \ni x \mapsto (A(x), B(x), x_{154})$. The element $(g_1, g_2) \in \text{GL}_3 \times \text{GL}_3 \subset M_{\beta_4}$ acts on $Z_4$ by

$$(A, B, c) \mapsto (g_1 A^t g_2, g_1 B^t g_1, c).$$

By Lemma II–4.6, we can eliminate the (1, 2), (1, 3)-entries of $B$. Then we can eliminate the (1, 1), (1, 2), (2, 1)-entries of $A$ by using the following lemma for the action of elements of the form $(I_3, g_2)$.

**Lemma 12.6.** If $A \in M_3(k)$ then there exists $g \in \text{GL}_3(k)$ such that the (1, 1), (1, 2), (2, 1)-coordinates of $gA$ are 0. We obtain the same statement replacing the action to $A \mapsto A^t g$.

**Proof.** Since the argument is similar, we only consider the first statement. It is a standard fact in linear algebra that there exists $g \in \text{GL}_3$ such that the (2, 1), (3, 1), (3, 2)-coordinates of $gA$ are 0. Then it is enough to apply the permutation matrix corresponding to the transposition (13).

By the above consideration, we can make the $y_i$-coordinate 0 for $i = 7, 9, 17, 35, 36$.

(3) $S_6$, $\beta_6 = \frac{1}{17}(-2, 0, 0, 0, 2, -1, -1, 1, 1)$

We apply the argument of Lemma II–4.4 to the subspace $\langle a_{24}, a_{34} \rangle \oplus \langle a_{25}, \ldots, a_{38} \rangle$. The latter subspace is $(\wedge^2 \text{Aff}^3) \otimes \text{Aff}^2$ instead of $\text{Aff}^3 \otimes \text{Aff}^2$ as a representation of $\text{GL}_3 \times \text{GL}_2 \subset M_{\beta_6}$, but the consideration is similar and we may assume that $y_{24}, y_{25}, y_{35} = 0$.

(5) $S_8$, $\beta_8 = \frac{1}{36}(-4, 0, 0, 0, 4, -3, -3, 1, 5)$
We first apply Lemma II–4.2 to the subspace $\langle a_7, a_9, a_{10}, a_{17}, a_{19}, a_{20} \rangle$ and may assume that $y_7, y_{17} = 0$. Then $GL_2$’s in $M_{1, [1, 2, 4]}, M_{2, [2, 3]} \subset M_{38}$ do not change this condition. We apply Lemma II–4.3 to the action of this $GL_2 \times GL_2$ on the subspace $\langle a_9, a_{10}, a_{19}, a_{20} \rangle \oplus \langle a_{25}, a_{26} \rangle$ and eliminate $y_{20}, y_{26}$.

(6) $S_{10}, \beta_{10} = \frac{11}{750}(-12, -12, 8, 8, 8, -5, -5, -5, 15)$

By applying Lemma II–4.2 to the subspace $\langle a_{32}, \ldots, a_{37} \rangle$, we may assume that $y_{32}, y_{35} = 0$. Then $GL_2 \subset M_{1, [1, 2, 3]}$ does not change this condition. So by applying Lemma II–4.1 to the subspace $\langle a_{36}, a_{37} \rangle$, we may assume that $y_{36} = 0$. Note that $GL_3 \subset M_{2, [3, 4]}$ does not change these conditions. So by applying Lemma II–4.6 to the subspace $\langle a_8, a_9, a_{10}, \ldots, a_{20} \rangle$, we may assume that $y_8, y_{9} = 0$.

(7) $S_{11}, \beta_{11} = \frac{1}{1580}(-12, -12, 8, 8, 8, -15, -15, 5, 25)$

We apply the consideration of Lemma II–4.2 to the subspace $\langle a_8, a_9, a_{10} \rangle$, we may assume that $y_{10}, y_{23}, y_{24} = 0$.

(8) $S_{12}, \beta_{12} = \frac{9}{1580}(-8, -8, -8, -8, 12, 12, -15, -15, 5, 25)$

By applying Lemma II–4.4 to the subspace $\langle a_{10}, a_{20} \rangle \oplus \langle a_{23}, \ldots, a_{29} \rangle$, we may assume that $y_{10}, y_{23}, y_{24} = 0$.

(9) $S_{13}, \beta_{13} = \frac{10}{1750}(-8, -8, -8, -8, 12, 12, -5, -5, -5, 15)$

By applying Lemmas II–4.1, II–4.2 to the subspaces $\langle a_{10}, a_{20}, a_{30} \rangle, \langle a_{33}, \ldots, a_{39} \rangle$ respectively, we may assume that $y_{10}, y_{20} = 0, y_{33}, y_{34} = 0$.

(11) $S_{18}, \beta_{18} = \frac{7}{1420}(-24, -4, -4, 16, 16, -5, -5, -5, 15)$

Let $M = GL_3 \times GL_2, W = Aff^3 \circ M_2, Z = W \oplus Aff^2$. We define an action of $(g_2, g_3) \in GL_2$ on $M_2$ by $M_2 \ni A \mapsto g_2A^t g_3 \in M_2$. With the standard representation of $GL_3$, it induces a linear action of $M$ on $W$. We define a linear action of $g = (g_1, g_2, g_3) \in M$ on $Z$ by $Z \ni (v, \nu) \mapsto (gv, g_3 \nu) \in Z$. Let $B = (B_1, B_2, B_3) \in W$ where $B_1, B_2, B_3$ are as in (1.9) and $v_0 = [0, 1]$.

Lemma 12.7. In the above situation, $M(B, v_0) \subset Z$ is Zariski open.

Proof. By Proposition 4.4, $GB \subset W$ is Zariski dense. Let $S = \{(B, v) \mid v \in Aff^2 \setminus \{[0, 0]\}\}$. Then $GS \subset V$ is Zariski open. We show that $G(B, v_0) = GS$, which proves that $G(B, v_0) \subset V$ is Zariski open.

Let $H = \langle B_1, B_2, B_3 \rangle \subset M_2$ be the subspace spanned by $B_1, B_2, B_3$. We define an action of $g \in GL_2$ on $M_2$ by $M_2 \ni A \mapsto gAg^{-1}$. Since $H = \{A \in M_2 \mid \text{tr}(A) = 0\}$, $H$ is invariant under the action of $GL_2$. For $g \in GL_2$, let $\rho_1(g) \in GL_3$ be the matrix such that

$$(gB_1g^{-1}, gB_2g^{-1}, gB_3g^{-1}) = (B_1, B_2, B_3) \rho_1(g)$$

treating $(B_1, B_2, B_3)$ as a row vector. As in Lemma 5.1, $(\rho_1(g)^{-1}, g, g^{-1})$ fixes $B$. For any $v \in Aff^2 \setminus \{[0, 0]\}$, there exists $g \in GL_2$ such that $^tg^{-1}v = v_0$. Then $(\rho_1(g)^{-1}, g, g^{-1})(B, v) = (B, v_0)$. This proves that $G(B, v_0) = GS$.

For $x \in Z_{18}$, let

$$A_i(x) = \begin{pmatrix} x_{24i} & x_{25i} \\ x_{34i} & x_{35i} \end{pmatrix} \quad (i = 1, 2, 3), \quad v(x) = \begin{pmatrix} x_{144} \\ x_{154} \end{pmatrix}.$$

Then by the map $x \mapsto ((A_1(x), A_2(x), A_3(x)), v(x), x_{234}) \in V \oplus 1$, we can identify $Z_{18}$ with $V \oplus 1$. We use Lemma 12.1 to show that $S_{18} = \emptyset$. By the above consideration,
we only have to consider $x = \sum_i y_i a_i$ such that $A(x) = B, v(x) = v_0$. So we may assume that $y_i = 0$ for $i = 7, 8, 16, 18, 19, 26, 27, 29, 33$.

(13) $S_{22}, \beta_{22} = \frac{13}{220} (-16, -16, 4, 4, 24, -15, -15, 5, 25)$

By Lemma II–2.2 we may assume that $y_9, y_{24}, y_{32} = 0$.

(14) $S_{23}, \beta_{23} = \frac{1}{220} (-24, -4, -4, 16, 16, -15, -15, 5, 25)$

By Lemma II–2.2 we may assume that $y_{10}, y_{26}, y_{33} = 0$.

(17) $S_{26}, \beta_{26} = \frac{3}{3600} (-16, -16, 4, 4, 24, -25, -5, -5, 35)$

By applying Lemmas II–4.1, II–4.3 to the subspaces $\langle a_9, a_{10} \rangle, \langle a_{14}, \ldots, a_{27}, a_{18}, a_{28} \rangle$ respectively, we may assume that $y_9, y_{14}, y_{18} = 0$.

(22) $S_{31}, \beta_{31} = \frac{3}{625} (-16, 6, 4, 4, 14, -5, -5, 5, 5)$

By applying Lemmas II–4.1, II–4.3 to the subspaces $\langle a_7, a_{17} \rangle, \langle a_{24}, a_{34}, a_{25}, \ldots, a_{36} \rangle$ respectively, we may assume that $y_7, y_{24}, y_{25} = 0$.

(26) $S_{39}, \beta_{39} = \frac{11}{3180} (-32, -12, 8, 8, 28, -25, -5, 15, 15)$

By applying Lemma II–4.3 to the subspace $\langle a_{24}, a_{34}, a_{25}, \ldots, a_{36} \rangle$, we may assume that $y_{24}, y_{25} = 0$.

(27) $S_{41}, \beta_{41} = \frac{1}{230} (-32, -12, 8, 8, 28, -15, 5, 5, 5)$

By Lemma II–4.5, we may assume that $y_{14} = y_{24} = y_{15} = 0$.

(28) $S_{43}, \beta_{43} = \frac{7}{220} (-24, -4, -4, 16, 16, -25, -5, 15, 15)$

By applying II–4.3 to the subspace $\langle a_{23}, \ldots, a_{34}, a_{25}, a_{35} \rangle$ and then Lemmas II–4.1 to the subspace $\langle a_{16}, a_{18} \rangle$, we may assume that $y_{16}, y_{23}, y_{25} = 0$.

(30) $S_{55}, \beta_{55} = \frac{1}{80} (-12, -2, -2, 8, 8, -15, -5, 5, 15)$

By applying Lemma II–4.3 to the subspace $\langle a_{16}, \ldots, a_{19}, a_{31}, a_{32} \rangle$, we may assume that $y_{16}, y_{31} = 0$.

(34) $S_{51}, \beta_{51} = \frac{1}{220} (-8, 2, 2, 2, 2, -55, 15, 15, 25)$

We identify $\langle a_{15}, \ldots, a_{30} \rangle$ with with the space of $4 \times 4$ alternating matrices with coefficients in the space of linear forms in two variables $u = (u_1, u_2)$. Taking the Pfaffian, we obtain a quadratic form $Q_x$ of $x$ for each $x \in Z_{51}$. Since $k = \mathbb{F}$, by the action of $GL(2) \subset M_{2, [1, 3]} \subset M_{\beta_{51}}$, we may assume that $Q_x(1, 0) = 0$, which implies that the first $4 \times 4$ alternating matrix is degenerate (rank < 4). By Lemma II–4.6, we may assume that $y_{15}, \ldots, y_{19} = 0$. Then $GL_2 \subset M_{1, [1, 3, 4]} \subset M_{\beta_{31}}$ does not change this condition. By applying Lemma II–4.1 to the action of this $GL_2$ on the subspace $\langle a_{31}, a_{32} \rangle$, we may assume that $y_{31} = 0$.

(35) $S_{52}, S_{52} = \frac{1}{40} (0, 0, 0, 0, 0, -1, -1, 1, 1)$

By applying Lemma II–4.6 to the subspace $\langle a_{21}, \ldots, a_{30} \rangle$, we may assume that $y_{21}, \ldots, y_{24}, y_{26}, \ldots, y_{29} = 0$. Then $GL_2 \times GL_2 \subset M_{1, [1, 3]} \subset M_{\beta_{52}}$ does not change this condition. So by applying Lemma II–4.1 to the subspaces $\langle a_{31}, a_{32} \rangle, \langle a_{33}, a_{34} \rangle$, we may assume that $y_{31}, y_{33} = 0$.

(36) $S_{53}, \beta_{53} = \frac{1}{120} (-48, 7, 7, 7, 27, -5, -5, -5, 15)$

By applying Lemma II–4.6 to the subspace $\langle a_{35}, a_{36}, a_{38} \rangle$, we may assume that $y_{35}, y_{36} = 0$. Then $GL_2 \subset M_{2, [1, 3]} \subset M_{\beta_{53}}$ does not change this condition. By applying Lemma II–4.6 to the action of this $GL_2$ on the subspace $\langle a_{7}, \ldots, a_{30} \rangle$, we may assume that $y_{7}, y_{9}, y_{17} = 0$.

(38) $S_{55}, \beta_{55} = \frac{1}{4} (-1, 0, 0, 0, 1, -1, 0, 0, 1)$

By applying Lemma II–4.2 to the subspace $\langle a_{17}, \ldots, a_{30} \rangle$, we may assume that $y_{17}, y_{27} = 0$. Then $GL_2 \subset M_{1, [1, 2, 4]} \subset M_{\beta_{55}}$ does not change this condition. By applying Lemma II–4.1 to the subspace $\langle a_{35}, a_{36} \rangle$, we may assume that $y_{35} = 0$.
(40) \( S_{57}, \beta_{57} = \frac{1}{469}(-44, -44, -44, 36, 96, -75, 5, 5, 65) \)

By Lemma II–4.5, we may assume that \( y_{14}, y_{33}, y_{36} = 0 \).

(41) \( S_{58}, \beta_{58} = \frac{1}{120}(-48, -3, 17, 17, -5, -5, -5, 15) \)

By applying Lemma II–4.1 to the subspace \( \langle a_{35}, a_{36}, a_{37} \rangle \), we may assume that \( y_{35}, y_{36} = 0 \). Then \( GL_3 \subset M_{2,[3]} \subset M_{\beta_{58}} \) does not change this condition. By applying Lemma 12.6 to the action of this \( GL_3 \) on the subspace, \( \langle a_8, \ldots, a_{30} \rangle \), we may assume that \( y_8, y_9, y_{18} = 0 \).

(42) \( S_{59}, \beta_{59} = \frac{1}{60}(-4, -4, -4, 0, 12, -3, -3, -3, 9) \)

Similarly as in the above case, we may assume that \( y_1, y_7, y_{14}, y_{33}, y_{36} = 0 \).

(43) \( S_{60}, \beta_{60} = \frac{1}{5}(-1, 0, 0, 0, 1, -2, 0, 1, 1) \)

By applying Lemma II–4.4 to the subspace \( \langle a_{24}, a_{34}, a_{25}, \ldots, a_{38} \rangle \), we may assume that \( y_{24}, y_{25}, y_{35} = 0 \). Note that this subspace is \( Aff^2 \oplus (\wedge^2 Aff^3) \otimes Aff^2 \) instead of \( Aff^2 \oplus Aff^3 \otimes Aff^2 \) as a representation of \( GL_3 \times GL_2 \subset M_{\beta_{60}} \), but the consideration is similar.

(45) \( S_{62}, \beta_{62} = \frac{1}{540}(-56, 4, 4, 4, 44, -35, 5, 5, 25) \)

By the consideration of Lemma II–4.5, we may assume that \( y_7, y_9, y_{15} = 0 \).

(46) \( S_{63}, \beta_{63} = \frac{1}{400}(-6, -6, -6, 9, 9, -60, 5, 20, 35) \)

By the consideration of Lemma II–4.5, we may assume that \( y_{23}, y_{31}, y_{32} = 0 \).

(47) \( S_{65}, \beta_{65} = \frac{1}{240}(-31, -6, -6, -6, 49, -20, -20, 5, 35) \)

By Lemma II–4.6, we may assume that \( y_{35}, y_{36} = 0 \).

(48) \( S_{66}, \beta_{66} = \frac{1}{360}(-34, 6, 6, 6, 16, -15, -15, -5, 35) \)

By applying Lemma II–4.2 to the subspace \( \langle a_7, \ldots, a_{20} \rangle \), we may assume that \( y_7, y_{17} = 0 \). Then \( GL_2 \)'s in \( M_{1,[1,2,4], M_{2,[2,3]} \subset M_{\beta_{66}} \) do not change this condition. By applying Lemma II–4.3 to the subspace \( \langle a_9, a_{10}, a_{19}, a_{20}, a_{25}, a_{26} \rangle \), we may assume that \( y_{20}, y_{26} = 0 \) (see Remark 12.3).

(50) \( S_{68}, \beta_{68} = \frac{1}{60}(-4, -4, -4, 6, 6, -5, -5, -5, 15) \)

By applying Lemmas II–4.1, II–4.6 to the subspaces \( \langle a_{10}, a_{20}, a_{30} \rangle \), \( \langle a_{31}, a_{32}, a_{35} \rangle \) respectively, we may assume that \( y_{10}, y_{20} = 0, y_{31}, y_{32} = 0 \).

(51) \( S_{69}, \beta_{69} = \frac{1}{140}(-26, -26, -26, 39, 39, -35, -10, -10, 55) \)

By applying Lemmas II–4.1, II–4.2 to the subspaces \( \langle a_{10}, a_{20} \rangle \), \( \langle a_{33}, \ldots, a_{39} \rangle \) respectively, we may assume that \( y_{20} = 0, y_{33}, y_{34} = 0 \).

(52) \( S_{72}, \beta_{72} = \frac{1}{360}(-36, -16, -16, 34, 34, -5, -5, -15, 15) \)

Let \( M = GL_3 \times GL_2, W = Aff^3 \otimes M_2 \) and \( Z = W \oplus Aff^2 \).

We define an action of \( M \) on \( Z \) in the same way as the case (11) \( S_{18} \). Let \( B = (B_1, B_2, B_3) \) be as in (4.9) and \( v_0 = [0,1] \). By Lemma 12.7, \( M(B, v_0) \subset Z \) is Zariski open. In the same way as in the case (11) \( S_{18} \), we can identify \( Z_{72} \) with \( Z \). The element \( -a_6 + a_7 + a_{27} + a_{32} + a_{34} \) corresponds to the element \( (B, v_0) \). We use Lemma 12.1 to show that \( S_{72} = \emptyset \). By the above consideration, we may assume that \( y_i = 0 \) for \( i = 7, 8, 16, 18, 19, 26, 27, 29, 33 \).

(53) \( S_{73}, \beta_{73} = \frac{1}{260}(-4, -4, 1, 1, 6, -65, 20, 20, 25) \)

By Lemma 12.2 we may assume that \( y_{14}, y_{18}, y_{32} = 0 \).

(56) \( S_{77}, \beta_{77} = \frac{1}{108}(-16, -16, 9, 9, 14, -15, -10, 10, 15) \)

By applying Lemma II–4.3, to the subspace \( \langle a_{24}, a_{27}, a_{32}, \ldots, a_{36} \rangle \), we may assume that \( y_{24}, y_{32} = 0 \).

(57) \( S_{78}, \beta_{78} = \frac{1}{260}(-19, -19, -4, -4, 46, -20, -20, -5, 45) \)

By Lemma 12.2 we may assume that \( y_9, y_{24}, y_{32} = 0 \).
By applying Lemma II–4.2 to the subspace \( \langle a_4, \ldots, a_{37} \rangle \), we may assume that 
\( y, y_r, y_{14}, y_{17} = 0 \). Then \( GL_2 \subset M_{2,[2,3]} \subset M_{\beta_0} \) does not change this condition. By 
applying Lemma II–4.1 to the subspace \( \langle a_8, a_{18} \rangle \), we may assume that \( y_8 = 0 \).

By applying Lemma II–4.3 to the subspace \( \langle a_{14}, \ldots, a_{27}, a_{33}, a_{36} \rangle \), we may assume 
that \( y_{14}, y_{33} = 0 \).

By Lemma II–4.4, we may assume that \( y_7, y_{24}, y_{26} = 0 \).

By the consideration of Lemma II–4.4, we may assume that 
\( y_{10}, y_{21}, y_{31} = 0 \).

We apply Lemmas II–4.1, II–4.3 to the subspaces \( \langle a_{10}, a_{20} \rangle, \langle a_{23}, \ldots, a_{34}, a_{25}, a_{35} \rangle \) 
respectively and may assume that \( y_{10} = 0, y_{23}, y_{25} = 0 \).

By Lemma II–4.4, we may assume that \( y_{10}, y_{21}, y_{31} = 0 \).

By Lemma II–4.4, we may assume that \( y_{10}, y_{21}, y_{31} = 0 \).

By the consideration of Lemma II–4.4, we may assume that 
\( y_{10}, y_{21}, y_{31} = 0 \).

We apply Lemma II–4.3 to the subspace \( \langle a_{24}, \ldots, a_{37}, a_{28}, a_{38} \rangle \), we may assume 
that \( y_{24}, y_{28} = 0 \).

By Lemma II–4.3 to the subspace \( \langle a_{24}, \ldots, a_{37}, a_{28}, a_{38} \rangle \), we may assume 
that \( y_{24}, y_{28} = 0 \).

By Lemma II–4.4, we may assume that \( y_{14}, y_{17} = 0, y_{24}, y_{28} = 0 \).

By Lemma II–4.4, we may assume that \( y_{10}, y_{34} = 0 \).

We identify \( \langle a_{17}, \ldots, a_{38} \rangle \) with \( M_{3,2} \). By applying Lemmas II–4.1, II–4.2 to the 
subspaces \( \langle a_9, a_{10} \rangle, \langle a_{17}, \ldots, a_{38} \rangle \) respectively, we may assume that \( y_9 = 0, y_{17}, y_{18} = 0 \).

By applying Lemma II–4.1 to the subspaces \( \langle a_9, a_{10} \rangle, \langle a_{24}, a_{34} \rangle \), we may assume 
that \( y_9 = 0, y_{24} = 0 \).

By Lemma II–4.2 to the subspaces \( \langle a_9, a_{10} \rangle, \langle a_{24}, a_{34} \rangle \), we may assume 
that \( y_9 = 0, y_{24} = 0 \).

By the argument of Lemma II–4.1, we may assume that 
\( y_8 = 0, y_{28} = 0 \).

By the argument of Lemma II–4.1, we may assume that 
\( y_8 = 0, y_{28} = 0 \).

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\( y_8 = 0, y_{28} = 0 \).

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\( y_8 = 0, y_{28} = 0 \).

By the argument of Lemma II–4.1, we may assume that 
\( y_8 = 0, y_{28} = 0 \).
By the argument of Lemma 12.3 we may assume that $y_{17}, y_{27} = 0, y_{19}, y_{35} = 0.$

(122) $S_{157}, \beta_{157} = \frac{1}{420}(-28, -28, -28, 12, 72, -105, 15, 15, 75)$

By Lemma 12.3 we may assume that $y_{14}, y_{24} = 0, y_{17}, y_{36} = 0.$

(125) $S_{160}, \beta_{160} = \frac{1}{80}(-7, -2, -2, -2, 13, -20, 0, 5, 15)$

By applying Lemma II–4.1 to the subspace $\langle a_{17}, a_{19}, a_{20}, \rangle$, we may assume that $y_{17}, y_{19} = 0$. Then $\text{GL}_2 \subset M_{1,[1,3]} \subset M_{160}$ does not change this condition. By applying Lemma II–4.1 to the subspace $\langle a_{36}, a_{38}, \rangle$, we may assume that $y_{36} = 0.$

(126) $S_{161}, \beta_{161} = \frac{1}{380}(-32, -32, -32, 48, 48, -145, -5, 5, 155)$

By applying Lemmas II–4.1, II–4.6 to the subspaces $\langle a_{20}, a_{30}, \rangle, \langle a_{31}, a_{32}, a_{33}, \rangle$ respectively, we may assume that $y_{20} = 0, y_{31}, y_{32} = 0.$

(127) $S_{162}, \beta_{162} = \frac{1}{1}(0, 0, 0, 0, 0, -1, -1, -1, 1, 3)$

By Lemma II–4.6, we may assume that $y_{31}, y_{32}, y_{33}, y_{34} = 0.$

(142) $S_{179}, \beta_{179} = \frac{1}{420}(-48, -48, 12, 12, 72, -105, 35, 35, 35)$

By Lemma 12.3, we may assume that $y_{14}, y_{17} = 0, y_{24}, y_{28} = 0.$

(143) $S_{180}, \beta_{180} = \frac{1}{20}(-8, 0, 2, 2, 4, -1, -1, -1, 3)$

By Lemma II–4.4, we may assume that $y_{9}, y_{10}, y_{35} = 0.$

(159) $S_{196}, \beta_{196} = \frac{1}{380}(-46, -46, 4, 4, 84, -45, 5, 5, 35)$

By applying Lemma II–4.1 to the subspaces $\langle a_{9}, a_{10}, \rangle, \langle a_{14}, a_{17}, \rangle$, we may assume that $y_{9} = 0, y_{14} = 0.$

(176) $S_{214}, \beta_{214} = \frac{1}{420}(-96, -76, 24, 24, 124, -5, -5, -5, 15)$

We identify $\langle a_{7}, \ldots, a_{28}, \rangle$ with $M_{4,2}$. By Lemma II–4.2, we may assume that $y_{7}, y_{8} = 0.$

(190) $S_{235}, \beta_{235} = \frac{1}{80}(-2, -2, -2, 3, 3, -20, -20, 15, 25)$

By Lemma II–4.6, we may assume that $y_{31}, y_{32} = 0.$

(220) $S_{269}, \beta_{269} = \frac{1}{180}(-22, -22, 8, 8, 28, -5, -5, 5, 5)$, By applying Lemma II–4.1 to the subspaces $\langle a_{8}, a_{18}, \rangle, \langle a_{24}, a_{27}, \rangle, y_{8}, y_{24} = 0.$

(222) $S_{275}, \beta_{275} = \frac{1}{60}(-24, -24, 16, 16, 16, -15, -15, -15, 45)$

By Lemma II–4.6, we may assume that $y_{38}, y_{39} = 0.$

This completes the proof of Theorem 12.3.

### 13. Smaller prehomogeneous vector spaces

In this section we show that if the GIT stratification of $(G, V)$ is determined and $S_{\beta}$ is a non-empty stratum of the GIT stratification of $(G, V)$, then it requires much less work to determine the GIT stratification of $(M_{\beta}, Z_{\beta})$. We shall prove two propositions and explain what has to be done to determine the GIT stratification of $(M_{\beta}, Z_{\beta})$.

Let $G, G_{st}, V, G_{st,\beta}, T_0, \mathfrak{B}$ be as before (see [18, p.264]). We assume that $G$ is split. Note that $T_0$ for (1.1) is defined in (2.5). Also $G_{st,\beta}$ is $G_{st}$ in [18, p.264].

Suppose that $\beta \in \mathfrak{B}$, $S_{\beta} \neq \emptyset$. We consider the representation of $M_{\beta}$ on $Z_{\beta}$. Let $T_{\beta,0} \subset T$ be the subgroup generated by $T_0$ and $\text{Im}(\lambda_{\beta})$. Then elements of $T_{\beta,0}$ act on $Z_{\beta}$ by scalar multiplication and $M_{\beta} = T_{\beta,0}G_{st,\beta}$ as algebraic groups (i.e., $M_{\beta, T} = T_{\beta,0}G_{st,\beta, T}$). For the representation $(M_{\beta}, Z_{\beta})$ we let $T_{\beta,0}, G_{st,\beta}$ play the role of $T_0, G_{st}$ in the general situation. So $Z_{\beta}^{ss}$ is the pull back of $\mathbb{P}(Z_{\beta})^{ss}$ where the stability is with respect to the action of $G_{st,\beta}$.

Let $T \subset G_{st}$ be a maximal split torus and $t^* = X^*(T) \otimes \mathbb{R}$. Note that we are using a Weyl group invariant inner product $(\cdot, \cdot)_*$ on $t^*$. Then $T_{\beta} \overset{\text{def}}{=} (T \cap G_{st,\beta})^\circ$ is a maximal
split torus of $G_{st, \beta}$ and we identify $t_\beta^* \overset{\text{def}}{=} X^*((T \cap G_{st, \beta})^0) \otimes \mathbb{R}$ with the orthogonal complement in $t^*$ of $\mathbb{R}\beta$. Let $t_\beta^*$ be a Weyl chamber. We use the restriction of $(\cdot, \cdot)_*$ to $t_\beta^*$. Since it is invariant by $\mathbb{W}$, it is invariant by the Weyl group of $M_\beta$. Since the Weyl group of $M_\beta$ is a subgroup of $\mathbb{W}$, we choose a Weyl chamber $t_\beta^*\cap t_\beta^*$ so that it includes $t_\beta^*\cap t_\beta^*$. Let $Z_{M_\beta} \subset t_\beta^*$ be the set which parametrizes the GIT stratification of $(M_\beta, Z_{M_\beta})$ for the above situation.

We assume that the action of $T$ on $V$ is diagonalized. Let $e_i$ ($i = 0, \ldots, N$) be the coordinate vector corresponding to the $i$-th coordinate and $\gamma_i \in t^*$ its weight. Let $I_\beta = \{i \mid e_i \in Z_\beta\}$, $J_\beta = \{i \mid e_i \in W_\beta\}$. If $i \in I_\beta$ then $(\gamma_i, \beta)_* = (\beta, \beta)_*$. So $\gamma'_i \overset{\text{def}}{=} \gamma_i - \beta$ is orthogonal to $\beta$. Therefore, $\gamma'_i$ can be regarded as the weight of $e_i$ in $t_\beta^*$. By assumption, $\text{Conv}\{\gamma'_i \mid i \in I_\beta\}$ contains the origin. Therefore, there exists $0 \leq a_i \leq 1$ for each $i \in I_\beta$ such that $\sum_{i \in I_\beta} a_i = 1$, $0 = \sum_{i \in I_\beta} a_i \gamma'_i$. This implies that $\beta = \sum_{i \in I_\beta} a_i \gamma_i$.

Suppose that $\beta' \in Z_{M_\beta}$. Note that $(\beta', \beta)_* = 0$. Let $Z_{M_{\beta'}} \subset Z_\beta$ be the subspace determined by $\beta'$ for the action of $M_\beta$. We define $M_{\beta'}, W_{\beta'}, Y_{\beta'}, S_{\beta'},$ etc., similarly. We assume that $S_{\beta'} \neq \emptyset$. Then there exist a finite subset $X \subset I_\beta$ and $0 \leq a_i \leq 1$ for each $i \in X$ such that $\sum_{i \in X} a_i = 1$, $\beta' = \sum_{i \in X} a_i \gamma'_i = \sum_{i \in X} a_i \gamma_i - \beta$. Then $\beta'' \overset{\text{def}}{=} \beta' + \beta$ is in the convex hull of $\{\gamma_i \mid i \in X\}$. Even though $\beta''$ may not belong to $t_\beta^*$, we can define $Z_{\beta''}$, etc., for the action of $G$.

**Lemma 13.1.** $Z_{\beta''} \cap Z_\beta = Z_{\beta'}$.

**Proof.** Let $0 \leq i \leq N$. Suppose that $e_i \in Z_{\beta''} \cap Z_\beta$. Since $(\beta, \beta')_* = 0$ and $(\gamma_i, \beta)_* = (\beta, \beta)$,

$$(\beta'', \beta'')_* = (\beta' + \beta, \beta' + \beta)_* = (\beta', \beta')_* + (\beta, \beta)_*,$$

$$(\gamma_i, \beta'')_* = (\gamma_i, \beta')_* + (\beta, \beta)_* = (\gamma'_i, \beta')_* + (\beta, \beta)_*.$$  

Since $(\gamma_i, \beta'')_* = (\beta'', \beta'')_*$, we have $(\gamma'_i, \beta')_* = (\beta', \beta')_*$. Therefore, $e_i \in Z_{\beta'}$. Since $Z_{\beta''} \cap Z_\beta, Z_{\beta'}$ are spanned by coordinate vectors, $Z_{\beta''} \cap Z_\beta \subset Z_{\beta'}$.

Conversely, suppose that $e_i \in Z_{\beta'}$. Then $e_i \in Z_\beta$ of course. Since $(\gamma'_i, \beta')_* = (\beta', \beta')_* = 0$,

$$(\gamma_i, \beta'')_* = (\gamma'_i + \beta, \beta' + \beta)_* = (\gamma'_i, \beta')_* + (\beta, \beta')_* + (\beta, \beta)_* = (\beta', \beta')_* + (\beta, \beta)_* = (\beta'', \beta'').$$  

So $e_i \in Z_{\beta''}$. This implies that $Z_{\beta'} = Z_{\beta''} \cap Z_\beta$. \hfill $\Box$

There exists $w \in \mathbb{W}$ (the Weyl group of $G$) such that $\beta'' \overset{\text{def}}{=} w^{-1}\beta'' \in t_\beta^*$. Since $\beta'' = \sum_{i \in X} a_i \gamma_i$, we have $\beta'' = \sum_{i \in X} a_i w^{-1} \gamma_i$ and so $\beta'' \in \mathbb{W}$.

**Proposition 13.2.** Suppose that $Z_{\beta''} \subset Z_\beta$ and that $M_{\beta''} \subset M_\beta$. Then

1. $Z_{\beta''}^{ss} = Z_{\beta'}^{ss}$.
2. $M_{\beta''} \setminus Z_{\beta''}^{ss} \cong M_\beta \setminus Z_{\beta'}^{ss}$
3. $S_{\beta'} \neq \emptyset$ if and only if $S_{\beta''} \neq \emptyset$.
Proof. (1) If the action of \( g \in G \) fixes \( \beta, \beta' \) then it fixes \( \beta'' \). So \( \overline{M}_{\beta'} \subset M_{\beta''} \). If \( g \in M_{\beta''} \) then \( g \in M_\beta \) by assumption. So \( g \) fixes \( \beta, \beta'' \). Therefore, \( g \) fixes \( \beta' \). This implies that \( M_{\beta''} = \overline{M}_{\beta'} \).

We may assume that \( k = \overline{k} \). Let \( H_{\beta''} \) be the semi-simple part of \( M_{\beta''} = \overline{M}_{\beta'} \).

Let \( \chi_\beta, \chi_\beta'' \) be characters on \( T \) proportional to \( \beta, \beta'' \) respectively. Note that \( T_\beta = \text{Ker}(\chi_\beta)^0 \). We put \( T_{\beta''} = \text{Ker}(\chi_{\beta''})^0 \). Let \( \chi_{\beta''} \) be a character on \( T_\beta \) proportional to \( \beta' \). Then \( \overline{T}_{\beta'} = \{ t \in T_\beta \mid \chi_{\beta''}(t) = 1 \}^0 \). It is easy to see that \( G_{st, \beta''} \) (resp. \( \overline{G}_{st, \beta'} \)) is generated by \( H_{\beta''} \) and \( T_{\beta''} \) (resp. \( H_{\beta''} \) and \( \overline{T}_{\beta'} \)). Since \( \beta'' = \beta + \beta' \), \( \overline{T}_{\beta'} = \left( \text{Ker}(\chi_\beta) \cap \text{Ker}(\chi_{\beta''}) \right)^0 \). So the only difference between \( G_{st, \beta''} \) and \( \overline{G}_{st, \beta'} \) is the difference of \( \overline{T}_{\beta'} \) and \( T_{\beta''} \).

Note that \( \lambda_\beta, \lambda_{\beta''} \) act on \( Z_{\beta''} = \overline{Z}_{\beta'} \) by scalar multiplication by weight \( (\beta, \beta)_*, (\beta', \beta')_* \) respectively. So for \( a, b \in \mathbb{Z} \), \( \lambda_\beta a \lambda_{\beta''} b \) acts on \( Z_{\beta''} = \overline{Z}_{\beta'} \) by scalar multiplication by weight \( a(\beta, \beta)_* + b(\beta', \beta')_* \). Since \( \beta'' = \beta + \beta' \), the weight of \( \chi_{\beta''}(\lambda_\beta a \lambda_{\beta''} b) \) is also \( a(\beta, \beta)_* + b(\beta', \beta')_* \). Therefore, \( T_{\beta''}/\overline{T}_{\beta'} \) is represented by \( \lambda_\beta a \lambda_{\beta''} b \) such that \( a(\beta, \beta)_* + b(\beta', \beta')_* = 0 \) and this acts on \( Z_{\beta''} = \overline{Z}_{\beta'} \) trivially. This implies that a polynomial on \( Z_{\beta''} = \overline{Z}_{\beta'} \) is invariant with respect to \( G_{st, \beta''} \) if and only if it is invariant with respect to \( \overline{G}_{st, \beta'} \). Therefore, \( Z_{\beta''}^{ss} = \overline{Z}_{\beta'}^{ss} \).

Since \( M_{\beta''} = \overline{M}_{\beta'} \) (even though \( G_{st, \beta''} \neq \overline{G}_{st, \beta'} \)), (2) follows. Then (3) is obvious. \( \square \)

In the situation of the above proposition, we still have to determine if \( \overline{T}_{\beta'} k \backslash \overline{Y}_{\beta'} k \cong \overline{M}_{\beta'} k \backslash \overline{Z}_{\beta'} k \).

We review the property of the action of \( U_\beta \) on \( Y_\beta, Z_\beta \). For \( \gamma \in \mathfrak{t}^*, \lambda \in \mathfrak{t} \), let \( \langle \lambda, \gamma \rangle \) be the natural paring. The element \( \lambda(\beta) \in \mathfrak{t} \) dual to \( \beta \) is characterized by the property that \( (\beta, \gamma)_* = \langle \lambda(\beta), \gamma \rangle \). There is a positive rational number \( a > 0 \) such that the 1PS \( \lambda_\beta \) is \( a \lambda(\beta) \). So \( a(\beta, \gamma)_* = \langle \lambda_\beta(\beta), \gamma \rangle \). This implies that if \( x \in V \) is a weight vector with weight \( \gamma \) with respect to the action of \( T \) then \( \lambda_\beta(t)x = t^{a(\beta, \gamma)_*}x \) for \( t \in \text{GL}_1 \).

**Lemma 13.3.** If \( x \in Y_\beta \) is a weight vector with weight \( \gamma \) and \( u \in U_\beta \) then there exist weight vectors \( y_1, \ldots, y_n \in Y_\beta \) with weights \( \delta_1, \ldots, \delta_n \) such that \( ux = x + \sum_{i=1}^n y_i \) and \( (\beta, \gamma)_* < (\beta, \gamma)_* \) for all \( i \).

**Proof.** Since \( Y_\beta \) is spanned by weight vectors, there exist weight vectors \( y_0, \ldots, y_n \in Y_\beta \setminus \{0\} \) with distinct weights \( \gamma_0, \ldots, \gamma_n \) such that \( ux = y_0 + \cdots + y_n \). Then

\[
\lambda_\beta(t)u \lambda_\beta(t)^{-1}x = t^{-a(\beta, \gamma)_*} \lambda_\beta(t)ux = \sum_{i=0}^n t^{-a(\beta, \gamma)_*} \lambda_\beta(t)y_i = \sum_{i=0}^n t^{a(\beta, \gamma)_*} - a(\beta, \gamma)_* y_i
\]

By definition, \( U_\beta \) consists of elements \( u \in P_\beta \) such that

\[
\lim_{t \to 0} \lambda_\beta(t)u \lambda_\beta(t)^{-1} = e_G.
\]

So \( (\beta, \gamma)_* \geq (\beta, \gamma)_* \) for all \( i \) and

\[
x = \sum_{i:(\beta, \gamma)_* = (\beta, \gamma)_*} y_i.
\]

Since \( x \) is a weight vector, we may assume that \( y_0 = x \) and \( (\beta, \gamma)_* > (\beta, \gamma)_* \) for \( i = 1, \ldots, n \). \( \square \)
Corollary 13.4. If \( x \in Z_\beta \) and \( u \in U_\beta \) then \( ux = x + y \) with \( y \in W_\beta \).

Proposition 13.5. In the situation of Proposition 13.2, suppose that Condition 11.6 holds for \( \beta, \beta'' \) (with respect to \( G, V \), etc.). Then Condition 11.6 holds for \( \beta' \) (with respect to \( M_\beta, Z_\beta \), etc.).

Proof. Condition 11.6 (1) follows from Proposition 13.2 (2).

Suppose that \( R \in Z_{3p, k}^{ss} \) satisfies Condition 11.6 (2). Note that \( U_{\beta''} = \overline{U}_{\beta'} \ltimes U_\beta \) and \( \overline{U}_{\beta'} \subset M_\beta \). We show that \( G_R \cap U_{\beta''} = (G_R \cap \overline{U}_{\beta'}) \ltimes (G_R \cap U_\beta) \).

Suppose that \( u_1, u_2 \in U_{\beta'}, u_2 \in U_\beta \), \( u = u_2u_1 \) and \( uR = R \). By Corollary 13.3, there exists \( y \in W_\beta \) such that \( u_2R = R + y \). Since \( u_1 \in \overline{U}_{\beta'} \subset M_\beta \), \( u_1 \) commutes with \( \lambda_\beta(t) \). So \( u_1 \) fixes weight spaces of \( \lambda_\beta \). This implies that \( uR = u_1R + u_1y = R \) and \( u_1R \in Z_\beta, u_1y \in W_\beta \). Therefore, \( u_1R = R, u_1y = 0 \). Since \( \overline{U}_{\beta'} \) is a group, \( y = 0 \), and so \( u_2R = R \). Hence,

\[
G_R \cap U_{\beta''} = (G_R \cap \overline{U}_{\beta'}) \ltimes (G_R \cap U_\beta).
\]

By assumption, \( G_R \cap U_{\beta''} \) is connected. So \( (G_R \cap \overline{U}_{\beta'}) = M_R \cap \overline{U}_{\beta'} \) must be connected and Condition 11.6 (3) is satisfied for \( \beta' \).

Suppose that \( y \in W_{\beta'} \). By assumption, there exists \( u_1 \in \overline{U}_{\beta', k^{sep}}, u_2 \in U_{\beta', k^{sep}} \) such that \( u_1u_2R = R + y \). By Corollary 13.4, \( u_2R = R + y_1 \) with \( y_1 \in W_\beta \), \( u_1u_2R = u_1R + u_1y_1 = R + y \) and \( u_1y \in W_\beta \). Since \( u_1R, R + y \in Z_\beta, u_1R = R + y \) and \( u_1y_1 = 0 \). So Condition 11.6 (2) is satisfied for \( \beta' \). Condition 11.6 (1) follows from Proposition 13.2 (2) (by replacing \( k \) by \( k^{sep} \)). \( \square \)

We now explain what has to be done to determine the GIT stratification of \( (M_\beta, Z_\beta) \). Since \( (M_\beta, Z_\beta) \) is smaller than \( (G, V) \), it is very likely that if \( \mathcal{B} \) can be determined by computer computation then we can determine \( \mathcal{B}_\beta \) by computer computation also. If \( \beta' \in \mathcal{B}_\beta \) then it is in the form \( \beta' = w\beta'' - \beta \) where \( w \in W \) and \( \beta'' \in \mathcal{B} \). Of course \( \beta' \) has to be in the Weyl chamber \( t_{\beta warmer}^+ \). For example, if \( (G, V) \) is the prehomogeneous vector space \( (1) \), then we have determined the set \( \mathcal{B} \) which consists of 292 elements. The possibilities for \( w \) are at most 2880. So the possibilities for \( (\beta'', \beta) \) are at most 292 \( \times \) 2880. What we have to do is to compare \( \beta' \) and \( w\beta'' - \beta \). This is not such a big task for computer computation. If the condition of Proposition 13.2 is satisfied for \( \beta, \beta'' = \beta' + \beta \) then we do not have to worry about whether or not \( \overline{S}_{\beta'} \) is empty. We expect that Proposition 13.2 is applicable in most cases and we only have to determine whether or not \( \overline{S}_{\beta'} \) is empty for exceptional cases.

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