Gravity in conformal superspace.

Henrique Gomes

Perimeter Institute for Theoretical Physics
31 Caroline Street, ON, N2L 2Y5, Canada

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Abstract

Motivated by well-known obstacles to quantum gravity, I look for the most general geometrodynamical symmetries compatible with a reduced physical configuration space for metric gravity. I argue that they lead either to a completely static Universe, or one embodying spatial conformal diffeomorphisms. Demanding locality for an action compatible with these principles determines the sort of terms one can add, both for the purely gravitational part as well as matter couplings. The symmetries guarantee that there are two gravitational propagating physical degrees of freedom (and no explicit refoliation-invariance). The simplest such system has a geometric interpretation as a geodesic model in infinite dimensional conformal superspace. An approximate solution to the equations of motion corresponds to a static Bianchi IX spatial ansatz. The unique coupling to electromagnetism forces its propagation equations to be hyperbolic, enabling us to “build” a standard space-time causal structure. There are, however, deviations from the standard Maxwell equations when space-time anisotropies become too large. Regarding quantization, with the geometric interpretation and the lack of refoliation invariance, the path integral treatment of the symmetries becomes much less involved than the similar approaches to GR. The symmetries form an (infinite-dimensional) Lie algebra, and no BFV treatment is necessary. We find that the propagator around the flat solution has up to 6-th order spatial derivatives, giving it plausible regularization properties as in Horava-Lifschitz.

1 Introduction

1.1 Motivation

Renormalization and Lorentz invariance

The main technical obstacle to obtaining a theory of quantum gravity, well-documented in the literature, lies in its renormalization properties. Gravity is a non-linear theory, which means that geometrical disturbances around a flat background can act as sources for the geometry itself. The problem is that, unlike what is the case in other non-linear theories, due to the mass dimension of the gravitational coupling constant the ‘charges’ carried by the non-linear terms in linearized general relativity become too ‘heavy’, generating a cascade of ever increasing types of interactions once one goes to high enough energies.

But there is another problem, one that will also appear in almost any non-perturbative approach. Quantum field theory is formulated in a fixed spacetime geometry, but in general relativity, spacetime is dynamical. To combine the two, we need to understand the superposition of causal structures. Without a fixed definition of time or an a priori distinction between past and future, it is hard to impose causality or interpret probabilities in quantum mechanics. For instance, observing which causal structure do space-like separated operators commute?

One way to overcome this issue is to a priori base the theory not on an underlying covariant spacetime, but on a more dynamical account. It turns out that indeed, such breaking of covariance can also help with the normalization properties, as is the case with Horava-Lifshitz [2, 3]. But these theories have other problems. Namely, they often carry scalar gravitational degrees of freedom, due to their reduced symmetry content.

Quantum gravity and the Problem of Time

The main idea behind a dynamical point of view is to set up initial conditions and construct the spacetime geometry by deterministically evolving in a given auxiliary definition of time. Having a notion of evolution already makes a dynamical setting for gravity more compatible with quantum mechanics and its physical transformations from one physical state to another.

We don’t need to reinvent this account, as it is already standard in the study of general relativity, going by the acronym ADM (after Arnowitt, Deser and Misner [4], see also [5] for a modern review of the methods).
Indeed most of the work in numerical GR requires the use of the dynamical approach. The hope is then that one can single out the physical instantaneous degrees-of-freedom which one would like to quantize, and then evolve in a more standard quantum mechanical framework.

However, for the dynamical account — or canonical, or Hamiltonian, framework — the required slicing of spacetime into equal-time surfaces is merely an auxiliary structure, and physics must be independent of such choices. Indeed, the canonical system comes with a conserved quantity — the Hamiltonian constraint — which has an associated symmetry transformation, and that gives us back the freedom of choosing such artificial slicings. In this way, local time translations become part of the symmetries of the theory. Since physical observables should not vary under the action of a symmetry transformation, they are required to be non-local in time.\(^1\) This is the source of the Problem of Time (see e.g.\(^{[7,8]}\)).

The problem is also the reason one cannot define the canonical configuration degrees-of-freedom which are invariant with respect to all of the local symmetries of general relativity. This should be the space of physically relevant and instantaneous configuration degrees-of-freedom of the theory. For example, in standard quantum mechanics, if one has a (uniformly dense) ring with spherical symmetry, we are able to parametrize the physical degrees-of-freedom by radius only, and quantum transition amplitudes tell us how to go from the ring at one radius (in an initial time) to another (at a final time). See figure 1 for an illustration of a reduced relational configuration space in the simple example of three-particles.

![Figure 1: An illustration of the physical configuration space in the case of relations between three particles (triangles). Vertical motion (generated by the symmetries \(G\)) changes the representation of the triangle in Cartesian space \(Q^N\), while horizontal motion changes the shape of the triangle (e.g., its internal angles). A slice is given by a lift of the reduced configuration space into the extended configuration space. It gives a split between physically different configurations and just different representations of the same physical configuration. For GR, one cannot form this picture for extended configuration space being the space of 3-metrics. See also figure 4. This figure taken from arxiv [9].](image1)

Ideally, quantum evolution could correspond to a transition between instantaneous configuration degrees-of-freedom also for gravity. The problem here is that the space of instantaneous configurations of gravity — the space of Riemannian 3-metrics — does not itself carry an intrinsic representation of the local space-time symmetries. The action of the constraints on the gravitational phase space (\(T^*\text{Riem}\)) does not project down to an intrinsic action on gravitational configuration space (\(\text{Riem}\)). This should come as no surprise. The instantaneous physical observables need to correspond to the entire space-time history, and so require information about the initial rate of change of the spatial metric, i.e. information about the gravitational momenta (which live on \(T^*\text{Riem}\)).

Of course, not having access to the physical instantaneous configuration degrees-of-freedom is not a fundamental limitation. Indeed, one can prove a “slice theorem” for GR; a way in which we can separate space-times which are related by a symmetry from those that are physically independent from each other. In the

\(^1\) Another constraint — spatial diffeomorphisms — generates spatial re-labeling of points. The non-locality in space required by spatial diffeomorphism symmetry is still entirely compatible with quantum mechanics. Moreover, it is a slight exaggeration to say that local time translations are a canonical symmetry. In fact, as stated below, refoliations have a well defined Hamiltonian representation only for space-times that satisfy Einstein equations [6].
case of general relativity, we only know how to do this when the entire space-time admits a particular choice of slicing into ‘auxiliary’ equal-time surfaces [10]. This choice — called Constant Mean Curvature (CMC) — corresponds to synchronizing clocks so that they measure the same expansion rate of space everywhere (i.e. same local Hubble parameter). More importantly for us, it turns out this choice has special properties, for it allows a description of the evolution of spatial geometry without reference to local spatial scale. I.e. it allows one to describe gravitational evolution in this auxiliary time in terms of spatially ‘conformally invariant’ geometries [9,11,12].

Indeed, most of the formal proofs for existence and uniqueness of solutions of the general relativistic dynamics requires the use of this particular choice of time labels (CMC). Within the dynamical framework, which I will here for simplicity refer to as ‘the York method’ (after J. York, who developed the method together with N. O’Murchadha [11]), the phase space variables are constrained so that they describe only such types of foliations. The constraints they are obligated to satisfy in order to be CMC, as per Noether’s theorem, generate canonical transformations. Such transformations are exactly changes of spatial scale, i.e. the local conformal transformations.² Surprisingly, both the York method [11] and the slice theorem [10] show that although GR is not fundamentally concerned with spatial conformal geometries, it is deeply related to them.

But what if we go the opposite way: instead of assuming the canonical symmetries associated to general relativity — spatial diffeomorphisms and local time reparametrizations — we investigate what kind of local symmetries still allow a description of the invariant — i.e. physical — and instantaneous configuration degrees-of-freedom? Such symmetries would be compatible with standard quantum evolution between physically different instantaneous configurations.

This how I will start the paper. By looking at the possible symmetry content such a theory can carry. In the case of gravity, under some assumptions, it can be shown that the non-trivial allowed symmetries that do act as a group in the metric configuration space are precisely spatial diffeomorphisms and scale transformations. The remaining physical independent degrees-of-freedom — the quotient of the space of spatial metrics by this full group of local symmetries — belong to ‘conformal superspace’, the space of conformal spatial geometries. Unlike what is the case for the canonical configuration degrees-of-freedom of GR, this invariant quotient is well-defined and has a simple geometric interpretation: it describes spatial angles, with two degrees-of-freedom per point. A conformal geometry is the level of structure in-between the merely topological and the fully geometrical. It is a rather nice thought that quantum mechanical evolution should refer to transitions between such simple geometrical degrees-of-freedom.

²In fact, the momenta are set to be transverse and constant trace, not trace-free; which would be the true generator of conformal transformations. However, one can interpret the constant trace part as defining an auxiliary quantity, the York time. The transverse traceless choice sets the gravitational momenta to be purely tensorial (spin-2), with no scalar (spin-0) or vector (spin-1) components.

Figure 2: The convergent arguments for fundamental spatial conformal symmetry. In the first part of the paper, I will argue for the bottom sequence of arguments, and explore a few consequences.

Indeed, shape dynamics [9,12] amounts to taking these hints — coming from the initial value formulation of general relativity and from the consideration of symmetries compatible with a quantum mechanical evolution between instantaneous physical configuration states — seriously. It does this by considering dynamical systems whose natural habitat is conformal superspace. However, in the way it was constructed to obtain a very explicit correspondence to GR dynamics, shape dynamics required a non-local Hamiltonian. Here I will consider local constructions which only require compatibility with the symmetry. The existence and properties of such theories is a considerably important point to settle in the entire spatially relational approach to which shape dynamics belongs.
I will show that the simplest of these theories, when treated in the most naive way, can yield Bianchi IX types of solutions, which, when coupled to other sources, can give a standard notion of space-time. I will then investigate some of the properties of the quantum theory, such as its BRST structure and its propagator around a conformally flat solution, showing that it has some of the desirable properties of Horava-Lifschitz in this respect.

I will employ an infinite dimensional principal fiber bundle view of the field configuration space of the theory. The path integral is then a summation over all paths (or field histories) in the physical quotient space (configuration space quotiented by the symmetries). The interesting thing about this formulation is that we don’t have to provide a section of the bundle, as is usually done when defining gauge-fixings. I.e. we don’t have to provide a unique representative for each orbit of configuration space. We only have to provide a unique representative path in configuration space for each path in the physical quotient space. This allows us to introduce a gauge-fixing corresponding to a unique way of lifting the curves from physical space to the full configuration space. This type of velocity, or ‘history’, gauge-fixing is not subject to the Gribov problem [14].

Preferred surfaces of simultaneity? Contrary to common wisdom, such an essentially 3+1 formulation does not imply a “preferred simultaneity surface”, even without the explicit presence of a constraint generating refoliations. The misguided intuition comes from imagining such surfaces as being embedded in space-time, in which case indeed, they do define a notion of simultaneity. As standard general relativity is a space-time theory, its formulation as a dynamical system is predicated on an auxiliary definition of complete surfaces of simultaneity (a foliation). The lapse and shift come from time components of the metric, and they carry the information about how to glue spatial geometries along.

However, without the prior existence of space-time, one would first require a recipe to construct it. From a smooth 1-parameter family (e.g. in the classical limit of extremal curves in configuration space) one has no given definition of duration, and a priori no lapse or shift. Therefore, a definition of ‘simultaneity’ must be an empirical one, using the dynamics of clocks and rods and fields (e.g. \( g_{ab}(l) \)), and need not be achievable in any physically absolute sense. In fact, for shape dynamics – a theory that can be defined intrinsically from the ‘spatial’ point of view — Carlip et al, and Koslowski [15,16] have shown that an absolute definition of simultaneity is no more possible than in standard GR.

This route to quantum gravity — furnishing an a priori non-relativistic theory with a notion of operational time — provides an alternative, principled route to constructing modified gravitational models. Nowadays a large industry of modified gravity models exists, in attempts to address problems in cosmology. However, unlike the more ad-hoc Horndeski-type models (see references in [17]), in the present, principled construction, we can easily get rid of the extra scalar gravitational degree of freedom that plagues both these theories and Horava-Lifshitz [2], and which, in view of the stringent constraints placed on these modes by the recent observation of gravitational waves, is a highly desirable feature.

# 2 The Classical Theory

Here I will show how to construct the most general action compatible with local symmetries in configuration space.

**Basic structure of Riem.** In the present context, I will let \( M \) denote a spatial, closed (i.e. compact without boundary) 3-dimensional manifold. For the non-relativistic gravitational systems in consideration here, I will take configuration space to be the space \( \text{Riem}(M) =: Q \) of positive-definite sections of the symmetric covariant tensor bundle \( \mathbb{C}_\infty^+(T^*M \otimes S T^*M) \) over \( M \), which forms a subspace (a cone) of the Banach vector space \( \mathbb{B} := \mathbb{C}_\infty^+(T^*M \otimes S T^*M) \). This subspace has a one-parameter family of natural Riemannian structures, induced pointwise by the metric \( g_{ab} \):

\[
(v, w)_g := \int d^3x \sqrt{g} C^{abcd}_\lambda v_{ab} w_{cd}
\]

(1)

where \( C^{abcd}_\lambda := g^{ae} g^{bd} - \lambda g^{ab} g^{cd} \) (when acting on symmetric tensor fields, \( v_{ab} = v(ab) \) and so on), and \( 0 \leq \lambda < 1/3 \). These are called the DeWitt supermetrics (with DeWitt value \( \lambda \)), and they are often useful for explicit calculation, such as in the proof of slice theorems [19,20].

\[\text{Here I have chosen the degree of differentiability to be infinity, i.e. smooth functions. However, it is well-known that in this case the space of sections above will form instead an inverse limit Banach manifold (Hilbert, once we have introduced a complete metric) by the Sobolev construction. I will ignore these more technical obstructions here, assuming they do not interfere with what I would like to achieve. For more on these matters, see [18–20].}\]
2.1 Finding the most general symmetries compatible with our framework.

If one takes configuration space to be fundamental, one must consider symmetries that act intrinsically on it. Here, I will argue that the most general such symmetries are conformal diffeomorphism transformations.

To find them, we first look at the Hamiltonian vector field associated to a smeared functional $F[g, \pi, \lambda]$, polynomial in its variables. These functions generate transformations in canonical variables through the action of their Poisson bracket, i.e. they have vector field $\chi_F$ generating flow along phase space as in $t_F \Omega = dF$, where $\Omega$ is the canonical symplectic structure, and $t$ is the inner derivative. See figure 3. For example, in general relativity, the scalar (super-Hamiltonian) constraints are given by:

$$H^\perp (x) := \left( \frac{\pi^{ab} \pi_{ab} - \frac{1}{2} \tau^2}{\sqrt{g}} - R \sqrt{g} \right) (x) = 0 \quad (2)$$

and, given a particular linear combination of these constraints, defined by a smearing $\lambda^\perp$, it generates the following transformation:

$$\delta_{\lambda^\perp} g_{ab} (x) = \frac{2 \lambda^\perp (\pi_{ab} - \frac{1}{2} \pi g_{ab})}{\sqrt{g}} (x) \quad (3)$$

depends not only on the metric, but also on the momenta (see figure 4). As a counter-example, the spatial
diffeomorphisms, generated by $H^a = -\nabla_b \pi^{ab} = 0$, act with the smearing $\lambda^a$ as:

$$\delta_{\lambda^a} g_{ab} (x) = \mathcal{L}_{\lambda^a} g_{ab} (x)$$

Figure 3: Relation between a scalar function and its associated symplectic vector field.

Figure 4: Many-fingered time issue: a single solution to Einstein’s equations (the slab of spacetime on the left) between two Cauchy hypersurfaces (in purple) where two initial and final 3-geometries $[g_1], [g_2]$ (the [ ] brackets stand for ‘equivalence class under diffeomorphisms’) are specified, does not correspond to a single curve in superspace (on the right). Instead, for each choice of foliation of the spacetime on the left that is compatible with the boundary conditions $[g_1], [g_2]$ there is a different curve in superspace. Here I have represented two different choices of foliation in red and blue. The point being that one cannot find the “superspace” equivalent to the full set of local symmetries of GR. This section shows that the maximal local symmetry acting on the space of 3-metrics for which this is possible is conformal diffeomorphisms, giving conformal superspace. Figure taken from arxiv [9].
which has pointwise dependence in \(Q\) (and \(M\)).

Indeed, for the associated symmetry to have an action on configuration space that is independent of the momenta, \(F[g, \pi, \lambda]\) must be linear in the momenta. This already severely restricts the forms of the functional to
\[
F[g, \pi, \lambda] = \int \hat{F}(g, \lambda)_{ab}(x) \pi^{ab}(x)
\]
so that the infinitesimal gauge transformation for the gauge-parameter \(\lambda\) gives:
\[
\delta_\lambda g_{ab}(x) = \hat{F}(g, \lambda)_{ab}(x)
\]
A Poisson bracket here results in
\[
\{F[g, \pi, \lambda_1], F[g, \pi, \lambda_2]\} = \int d^4x \left( \frac{\delta \hat{F}(g, \lambda_1)_{ab}}{\delta g_{cd}} \pi^{cd} \hat{F}(g, \lambda_2)_{cd} - \frac{\delta \hat{F}(g, \lambda_2)_{ab}}{\delta g_{cd}} \pi^{cd} \hat{F}(g, \lambda_1)_{cd} \right)
\]
and this must close in order that it has any chances of being a symmetry generator (i.e. it must be a first class constraint).

Let us for now assume that \(\hat{F}(g, \lambda)_{ab}\) is covariant tensor of rank two. This possibly requires integrating away covariant derivatives from \(\pi^{ab}\). If \(\hat{F}\) has no derivatives of the metric, \(F[g, \pi, \lambda_1], F[g, \pi, \lambda_2]\) will straightforwardly commute. But with no derivatives the only objects we can form are:
\[
F(g, \lambda)_{ab} = \lambda g_{ab} \quad \text{and} \quad \hat{F}(g, \lambda)_{ab} = \lambda_{ab}
\]
In the first case, \(F\) generates spatial conformal transformations, since:
\[
\{F[g, \pi, \lambda], g_{ab}(x)\} = \lambda(x) g_{ab}(x) \quad \{F[g, \pi, \lambda], \pi^{ab}(x)\} = -\lambda(x) \pi^{ab}(x)
\]
In the second case, they would imply that \(\pi^{ab} = 0\), a constraint killing any possibility of dynamics.

Suppose \(\hat{F}\) is a tensor with two or more derivatives of the metric. For example,
\[
\hat{F}(g, \lambda)_{ab} = \lambda (a R_{ab} + \beta R g_{ab})
\]
A substantial amount of computations show that the algebra of equation (5) does not close for any values of \(a\) and \(\beta\). To see this, one needs to first calculate the general bracket, integrate away derivatives from one of the smearings, \(\lambda_1\) which is then set to a Dirac-delta function, \(\lambda_1(x) = \delta(x, y)\). This gets rid of the integral signs and gives us a local density which must vanish, for all values of the other smearing function, \(\lambda_2(x)\). To finish the proof, one shows that there are terms that cannot be proportional to \(F[g, \pi, \lambda]\).

As a specific example, the Ricci flow, or the conformal Ricci flow do not have a local group action in Riem, in the sense that commuting the flow with respect to two different smearings (usually the Ricci-flow is taken with respect to a unit smearing), will not give another Ricci-flow. That is, the local generator of Ricci-flow on phase space is \(\hat{F}(g, \lambda)_{ab}(x) = \lambda R_{ab}(x)\), since this is what will generate \(\delta_\lambda g_{ab}(x) = \lambda R_{ab}(x)\). Performing the algorithm described above (see auxiliary Mathematica file), one obtains, integrating away all derivatives away from \(\lambda_2\) and setting it as a Dirac delta, a single term containing derivatives of the momenta:
\[
\nabla^a \lambda_1 (2 R^{bc} (\nabla_a \pi_{bc} - \nabla_b \pi_{ac}))
\]
which therefore cannot vanish by combination with the other terms.\(^4\)

If one instead chose a term of the form \(\beta R \lambda_{ab}\) one can show that the rank of this constraint is not constant along phase space, and moreover it would imply that \(\pi^{ab} = 0\) almost everywhere, as before. Taking (4) to contain covariant derivatives of \(\pi_{ab}\), with terms of the form e.g.: \(\lambda \nabla_a R \nabla_b \pi^{ab}\), is tantamount to taking a generator of the form \(\lambda \nabla^2 R + \partial_\alpha \lambda \nabla^\alpha R\). It can be shown that only the momentum constraint, generator of diffeomorphisms, \(F(\lambda, g, \pi) = \int d^4x \lambda_a \nabla_b \pi^{ab}\) survives as the generators of a closed algebra [21].^5

These conclusions hold order by order in number of derivatives of the metric (see [22] for a method of proof that would need to be pursued here, and accompanying auxiliary Mathematica file for the calculations). The outcome is that the only relevant geometrical structure between the differentiable one and the metric one is conformal geometry.

Based on these considerations, I will take the gauge symmetries in Riem\((M) = Q\) compatible with the foundation of the theory to be at most conformal diffeomorphism transformations.

\(^4\)Unless we further stipulate symmetries, which in this case would be overly restrictive in any case.

\(^5\)In fact, the constraint \((\nabla^2 - \frac{1}{2} R) \pi = 0\) can also survive. This is the conformal Laplacian in 3-dimensions acting on the trace of the momentum (which is a conformally invariant scalar density). By exploiting conformal transformations one can show that this reduces to the conformal constraint \(\pi = 0\) almost everywhere on phase space as well [21].
2.2 The structure of conformal superspace.

Theories with gauge symmetry are defined by possessing redundant descriptions. In the present context, this refers to 3-metrics, which are related by a symmetry transformation. Usually, a symmetry is only defined once we have an action functional, by those transformations which leave such a functional invariant. Here I am pursuing a different strategy: defining the local symmetries by demanding that configuration space have a well-defined quotient. The appropriate mathematical tool for defining and exploring such a quotient is called “a slice”. Its use shows us that the reduced configuration space has a rich geometrical structure [20,23]; a structure we can co-opt in order to build a wave-function of the Universe, as e.g. in the no-boundary proposal [24].

Basic structure Conformal superspace is the name given to the quotient of configuration space with respect to the group of conformal transformations. First, let’s explain a bit further what this group is. I will here mostly follow the nomenclature of [20].

Spatial diffeomorphisms, \( f \in \text{Diff}(M) \), act on the metric through pull-back, \( \text{Diff} \times Q \ni (f,g) \mapsto f^*g \in Q \). The infinitesimal action is given by the Lie derivative, i.e. for \( \xi \), the vector field flow of \( f, \frac{d}{dt}(f^*g) = \mathcal{L}_\xi g \), where \( f_0 = \text{Id} \). The orbits of the spatial diffeomorphisms in \( Q \) are along Killing directions wrt to the metric \( g \). Conformal transformations are designated by \( P \), the group of positive scalar functions on \( M \), acting pointwise through multiplication, \( P \times Q \ni (\rho,g) \mapsto \rho g \in Q \).

With these two groups, we can form the semi-direct product \( \mathcal{C} := \text{Diff} \ltimes P \), acting on \( Q \). As Diff, the set of conformal diffeomorphisms \( \mathcal{C} \) forms an infinite-dimensional Lie group, with Lie algebra given by the semi-direct sum of smooth vector fields \( C^\infty(TM) \) and scalar functions \( C^\infty(M) \). That is, let, \( f_1, f_2 \) be diffeomorphisms, and \( \rho_1, \rho_2 \) two scalar transformations, conformal diffeomorphisms have group structure \((f_1, \rho_1) \cdot (f_2, \rho_2) = (f_1 \circ f_2, \rho_2(\rho_1(f_2))) \) where \( \rho_2(\rho_1(f_2)) \) just means scalar multiplication at each \( x \in M \) as \( \theta_2(x)(\rho_1(f_2(x))) \). As with Diff, thus \( \mathcal{C} \) is an infinite-dimensional regular Lie group and it acts on \( Q \) on the right as a group of transformations by:

\[
(\text{Diff} \ltimes P) \times Q \rightarrow Q \\
(f, \rho, g) \mapsto \rho(f^*g)
\]

The slice theorem and stratified manifolds. Roughly speaking, a slice for the action of a group \( G \) on a manifold \( \mathcal{X} \) at a point \( x \in \mathcal{X} \) is a manifold \( S_x \), transversal to the orbit of \( x, O_x \) (see appendix A.1 for a rough and brief sketch of definitions, corollaries and how proofs of existence work). If the isotropy group \( G_x \) of \( x \) is trivial, then \( S_x \) gives a local chart for the space \( \mathcal{X}/G \) near \( x \). I.e. it gives \( \mathcal{X} \) a local product structure, with one of the factors being isomorphic to the group, \( U \approx S_x \times G \), with \( U \) being a proper subset of \( \mathcal{X} \) containing \( x \) (see corollary 2 in appendix A.1). In this case, one can use the slice to parametrize the physically distinct configurations. But complications arise when the isotropy groups of \( x \in \mathcal{X} \) become non-trivial, because the symmetry group in question may act qualitatively differently on different orbits.

In that case, \( \mathcal{X}_x = \{ x \in \mathcal{X} \mid I_{i_x} \text{ is conjugate to } I_{i_0} \} \). Then according to corollary 2, a slice theorem shows that \( \mathcal{X}_x/G \) is a manifold (since \( I_{i_x} \) doesn’t change dimension). Each such manifold defines a ‘stratum’, containing the orbit \( O_x \). Then (see [10] for a review):

Corollary 1 (Stratification) There exists an isomorphism \( \mathcal{X}/G = S_1 \cup S_2 \cdots S_n \) with \( S_i \) a stratum as above, where \( i \) characterizes the dimensionality of an isotropy group, with \( S_1 = \mathcal{X} \setminus \{ x \in \mathcal{X} \mid I_x = \text{Id} \} \), and \( S_{i+1} \subset \partial S_i \). I.e. the strata are ordered by increasing symmetry and decreasing dimensionality.

Let us take the diffeomorphism group \( \text{Diff} = \text{Diff}(M) \), acting on the metrics through pull-back \( f^*g_{ab} \), \( f \in \text{Diff} \). For metrics that have non-trivial isotropy groups, \( I_x \in \text{Diff} \), we have a degenerate action of the diffeomorphism group. For this reason, the quotient wrt the diffeomorphisms of the space of metrics \( Q \) over \( M \), has a structure with different strata.

Stratified manifolds have nested “corners” – each stratum corresponds to a dimension of the stabilizer group, and has as boundaries a lesser dimensional stratum. The larger the stabilizer group, the lower the strata. Let \( Q_o \) be the set of metrics without isometries. This is a dense and open subset of \( Q \), the space of smooth metrics over \( M \). Let \( I_n \) be the isotropy group of the metrics \( g_{\mu n} \), such that the dimension of \( I_n \) is \( d_n \). Then the quotient space of metrics with isotropy group \( I_n \) forms a manifold with boundaries, \( Q_n/\text{Diff}(M) = S_n \).

The boundary of \( S_n \) decomposes into the union of \( S_{n'} \) for \( n' > n \) (see [23]).

A useful picture to have in mind for this structure is a “bottomless” tetrahedron (seen as a manifold with boundaries). The interior of the tetrahedron has boundaries which decomposes into faces, whose boundaries decompose into lines, whose boundaries are the single vertex at the top. The single vertex at the top of the tetrahedron is geometrically singled out, and we will use it to construct our path integral.
Preferred strata. In general relativity, there is no intrinsic action of the local symmetries on configuration space (i.e. the action of the refoliations don’t project down to configuration space, as illustrated by equation (3)). Thus, having specifying conditions on the quotient space of configurations doesn’t have any meaning. Here, we have the opposite state of affairs, and the stratification above has a salient physical meaning.

In particular, one can single out configurations with the highest possible dimension of the stabilizer subgroup, the most homogeneous configurations. Let:

\[ \Phi_0 = \{ \mathcal{O}_g \subset \mathcal{Q} \mid I_g \subset \mathcal{C} \text{ has maximum dimension} \} \]

Such a set is composed of the least dimensional, and simplest — in the sense that they correspond to the most homogeneous configurations — corners of reduced configuration space. It is these preferred singular points of configuration space that I will define as an origin of the transition amplitude, or the “anchor” of the path integral, as is done in Hartle-Hawking [24], Vilenkin [25], and Linde [26], below. Here, however, such boundary conditions on \( \mathcal{Q}/\mathcal{P} \) are physical.

Depending on the symmetries acting on configuration space, and on the topology of \( M \), one can have different such preferred configurations. For the case at hand — in which we have both scale and diffeomorphism symmetry and \( M = S^3 \) — there exists a unique such preferred point. It is easy to show that in this case \( \Phi_0 = \{ [Ω_o] \} \) where \( Ω_o \) is the round sphere metric (see e.g. [27]).

### 2.3 The form of the Lagrangian

Now that we have explored the gauge structure of the symmetries which have the appropriate action on metric configuration space, \( \mathcal{Q} \), we move on, to calculate the possible form of Lagrangians which possess such symmetries.

To start the calculation, we note that if the 3-metric \( g_{ab} \) has conformal weight 4, i.e. \( δ_ξ g_{ab} = e^{4η} g_{ab} \), the symmetric 2-tensor \( g_{ab} \), also has conformal weight 4, and the undensitized totally anti-symmetric 3-tensor \( e_{abc} \) has conformal weight -6.

Thus, as a necessary condition to have a conformal diffeomorphism invariant action, we must match the following tensor indices and conformal weights:

\[
\begin{align*}
-3N_ε + 2N_γ + 2N_δ - 2N_δ - 1 + N_ψ &= 0 \quad (6) \\
-3N_ε + 2N_γ + 2N_δ - 2N_δ - 1 &= -3 \quad (7) \\
N_ψ &= -3 \\
\Rightarrow N_ψ &= -3
\end{align*}
\]

It turns out that the only polynomial invariant we can form with three derivatives of the metric is the Chern-Simons functional [28],

\[
CS[g] = \int d^3x (dΓ ∧ Γ + \frac{2}{3} Γ^3)
\]

where \( Γ(g_{ab}) \) is the Levi-Civita connection one form associated to \( g_{ab} \).

Since \( CS[g] \) is a conformal diffeomorphism invariant, From functionally differentiating it, we obtain the symmetric tensor:

\[
\frac{δCS[g]}{δg_{ab}} = √gC^{ab}
\]

where the (undensitized) Cotton tensor is defined as:

\[
C^{ab} := e^{acd} \nabla_c \left( R^b_d - \frac{1}{4} δ^b_d R \right)
\]

where here we are using the undensitized totally anti-symmetric pseudo-tensor \( e^{abc} \).

Since it is a functional derivative of a conformal diffeomorphism invariant functional, under an infinitesimal diffeomorphism \( δ_ξ g_{ab} = L_ξ g_{ab} \), or infinitesimal conformal transformation, \( δ_ρ g_{ab} = ρ g_{ab} \), it must remain invariant. Thus besides being symmetric, we obtain equations telling us that the Cotton tensor is also transverse and traceless:

\[
C^{ab} = C^{(ab)} , \quad C^{ab} _{;b} = 0 , \quad C^a_a = 0
\]

Thus, under a conformal transformation \( g_{ab} \rightarrow e^{4η} g_{ab} \)

\[
\sqrt{g}C^{ab} [e^{4η} g] = e^{4η} \frac{δCS[e^{4η} g]}{δ(e^{4η} g_{ab})} = e^{-4η} \frac{δCS[g]}{δg_{ab}} = e^{-4η} \sqrt{g}C^{ab}[g]
\]

\footnote{For this analysis I assumed that no degenerate metrics are allowed. If they are allowed, then the analysis differs. The completely degenerate metric would be a natural candidate in this extended configuration space, as it possesses the full group of diffeomorphisms as a stabilizer. However, it is not clear that the completely degenerate metric is even represented in reduced configuration space, since there is no section of the bundle \( \mathcal{Q}/\mathcal{P} \).}
we get $C_{ab} \to e^{-10\rho} C_{ab}$.

The Cotton is the unique tensor which transforms covariantly under conformal diffeomorphism in 3-dimensions, and its determinant completely specifies the conformal geometry of a metric. With it, we can form scalars by contraction, $C^a_i \cdots C^a_l$, transforming as $e^{(-6n)\rho}$, since we need to use $n$ metrics to contract all the tensors. Since this is a scalar, we can take the $n$-th roots, so that we match the conformal factor of $\sqrt{g}$ to make a conformal invariant density. For $n = 1$, since the Cotton is traceless, we get just zero. Thus we arrive at the simplest conformally invariant density function:

$$\sqrt{g} C_{ab}$$

Overall, it contains 3 powers of spatial derivatives ($6/2$).

The simplest candidate for a conformal-diffeomorphism invariant Lagrangian is thus:

$$\mathcal{L} = \int d^3 x \sqrt{g} \left\{ C_{ab} \left( \dot{g}^{cd} - (\mathcal{L}_g)_{,cd} - \rho g_{,cd} \right) \right\}$$  (11)

This Lagrangian is quadratic in the velocities (multiplied by a positive conformal factor), and is invariant wrt the gauge transformations; it thus defines a conformally invariant metric in $\mathcal{Q}$ (see (64) in appendix A.2)

$$(v, w)_g := \int d^3 x \sqrt{g} C_{ij} v_{ab} w^{ab}$$  (12)

for $v, w \in T_g \mathcal{Q}$. The Lagrange multipliers $\xi^a$ and $\rho$ have been inserted in (11) to guarantee invariance under time-dependent gauge transformations, as we explain below. But the same principles could generate:

$$\mathcal{L} = \int d^3 x \sqrt{g} \left\{ C_{ab} \right\} \left[ \left( \dot{g}^{cd} - (\mathcal{L}_g)_{,cd} - \rho g_{,cd} \right) \right\} (\dot{g}_{cd} - (\mathcal{L}_g)_{,cd} - \rho g_{,cd}) \right\}$$  (13)

where $\Lambda_m$ are a generalization of a cosmological constant, and $f(\mathcal{CS}[g])$ is a general function of the Chern-Simons functional and $\alpha_n$ are arbitrary constants.

The least amount of derivatives of the metric we can get (besides zero), with an ultralocal (tensorial) supermetric and no other ingredients, gives:

$$\mathcal{L} = \int d^3 x \sqrt{g} \left\{ C_{ab} \right\} \left[ (\dot{g}^{cd} - (\mathcal{L}_g)_{,cd} - \rho g_{,cd}) \right\} (\dot{g}_{cd} - (\mathcal{L}_g)_{,cd} - \rho g_{,cd}) \right\} + f(\mathcal{CS}[g])$$  (14)

The transformations of $\xi^a$ and $\rho$ Under a time dependent diffeomorphism, $f_t \xi_{ab}$, and under a conformal transformation $e^\rho \xi_{ab}$, maintaining invariance would require the transformations (at $t = 0$)

$$\delta \xi = \dot{\xi} = \xi_{,\alpha} \xi_{\alpha} - e^\rho \xi_{,\alpha} \xi_{\alpha}$$  (15)

where $\xi := e^\rho \frac{\partial}{\partial \xi}$ is the vector field flow of $f_t$ at $t = 0$ and $f_0 = 1/d$ (we use boldface to simplify the notation for equations involving the commutator). Thus, $\xi^a$ and $\rho$ transform in a manner to cancel the extra terms. In the transformations for $e^\rho$:

$$\delta e_{ab} = L e_{ab} \quad \delta \xi_{ab} = L \xi_{ab} + \xi_{ab}$$  (16)

so

$$\delta L \xi_{cd} = L \xi_{cd} - (\xi_{,e}) \xi_{,e} \xi_{cd} + L \xi \xi_{cd} + L \xi \xi_{cd} + L \xi \xi_{cd}$$

The transformations for $\rho$ follow a similar pattern. Putting them together, we get:

$$\delta (\dot{g}_{cd} - (\mathcal{L}_g)_{,cd} - \rho g_{,cd}) = L \xi (\dot{g}_{cd} - (\mathcal{L}_g)_{,cd} - \rho g_{,cd})$$  (17)

---

7 We could also add $L(g) = \nabla^2 - \frac{1}{2} R$, the conformal Laplacian, sandwiched between functions of appropriate homogeneous conformal weight. That is, for a function $f$ which does not transform under our conformal change, we have $L(e^{\rho}) f = e^{-\rho} L(g)(e^\rho f)$, which won’t be invariant unless $f \to e^{\rho f}$, and we have another term cancelling the $e^{\rho}$ on the lhs of the operator. Similarly, but with more constraints on the coupling, we could add the higher order Paneitz operator [29], which also transforms conformally covariantly, provided, again, we wanted to couple functions with another appropriate conformal weight. These can be added and studied in a case by case basis, but they will not be considered here. See [30] for a manner of constructing these types of operators.
which means that this combination transforms covariantly wrt to conformal diffeomorphisms, a non-trivial result. The transformation of the shift, $\xi^a$, can be checked to be inherited by a space-time formulation, when identified with components of the spacetime metric under the ADM decomposition.

But in the present case, in which one is abdicating the space-time view, they must come from a different source, and indeed they do; they arise from the transformation properties of connection one forms in principle fiber bundles (as I elaborate in appendix A). In other words, instead of appealing to a spacetime picture, we merely take the true gauge structure of Riem under conformal diffeomorphisms. It forms a principal fiber bundle [18], and thus we can use connection 1-forms to covariantize derivatives wrt field-dependent gauge transformations (see figure 4).

In particular, from (61) (derived in all generality in [31]), for spatial diffeomorphisms we would obtain, for the connection 1-form (in field space), $\omega$, under a vertical field-dependent (i.e. time-dependent) transformation:

$$\delta_\xi(\omega(g)) = \xi - [\omega(g), \xi]$$  \hfill (18)

which matches the first of (15), for $\omega^a(g)$ identified with the shift, $\xi^a$.

For instance, if we define the connection form by orthogonality to the group orbits with respect to the conformally invariant metric (12), the defining equations for $\xi^a$ and $\rho$ would be:

$$\nabla_a \left( \sqrt{C^{cd}C_{ab}}(g^{ab} - C^{cd}_{a}\xi^d) - \rho g^{ab} \right) = 0$$ \hfill (19)

$$\dot{g}_{ab} g^{ab} - \text{div}(\xi) - 3\rho = 0$$ \hfill (20)

Upon a Legendre transformation, these would equal the first class constraints of the theory, which form a Lie algebra, as we show below in (23).

**Hamiltonian version** Using (19) to get rid of extra terms that would arise from taking the variational derivative of the connection form wrt $\dot{g}_{ab}$, we have

$$\delta L \over \delta \dot{g}_{cd} =: \pi^{cd} \equiv \sqrt{g} \sqrt{C^{ab}C_{ab}}(g^{cd} - (C_{a}^{c}g_{d}^{a} - \rho g^{cd})$$ \hfill (21)

where the hat means we have used (19).

The Hamiltonian form of (14) is easily worked out to be:

$$\mathcal{H} = \int d^3x \left( \frac{\pi^{ab} \pi_{ab}}{4\sqrt{g}} + g_{ab} \pi_{a}^{b} - \rho \pi_{a}^{b} \pi^{ab} - \sqrt{g} \sqrt{C^{ab}C_{ab}} \Lambda \right) - \alpha CS[g]$$ \hfill (22)

The transformations (15) guarantee that the action:

$$S = \int \pi^{ab} \dot{g}_{ab} - \mathcal{H}$$

is invariant wrt to generalized field-space dependent conformal diffeomorphisms.

If we furthermore want the action to be invariant wrt the choice of field-space connection 1-form, we obtain the usual form of the constraints. As is the case in general relativity, their content is, as expected, that the gravitational momentum is transverse and traceless: \(^8\)

$$D_b(x) := \nabla_a \pi^a_b(x) = 0 \quad \text{and} \quad D(x) := \pi^{ab} \dot{g}_{ab}(x) = 0$$ \hfill (23)

The commutator relations, with smearings $\xi^a$ and $\rho$ are:

$$\{D_b(\xi^b_1), D_b(\xi^b_2)\} = D_b(\{\xi^b_1, \xi^b_2\}) \quad \{D_b(\rho^b_1), D_b(\rho^b_2)\} = D_b(\{\rho^b_1, \rho^b_2\}) = 0$$ \hfill (24)

The equations of motion of the metric and the momenta are, calling $\sqrt{C^{ab}C_{ab}} = \Xi$:

$$\dot{g}_{ab} = \frac{\pi_{ab}}{2\sqrt{\Xi}} - C_{a}^{c}g_{d}^{a} - \rho g_{ab}$$ \hfill (25)

$$\pi^{ab} = \frac{1}{2\sqrt{\Xi}} \left( \pi^{ac} \pi^{c}_{b} - \frac{\pi_{ij} \pi^{ij}}{2\Xi} (C^{ac}C_{b}^{c} + C^{cd}C_{ab}^{cd}) \right)$$

$$- \Lambda \sqrt{\Xi} \left( \frac{\pi^{ab}}{2\sqrt{\Xi}} + \frac{1}{2\Xi} C^{cd} \frac{\partial C^{cd}}{\partial g_{ab}} \right) - \alpha C^{ab} - L_{\xi} \pi^{ab} - \rho \pi^{ab}$$ \hfill (26)

The functional derivative of the Cotton tensor is rather involved, and we thus leave its form implicit.

\(^8\)An interesting project is to define the theory wrt a particular connection form, i.e. with a particular horizontal lift. One would have a non-local action which is diffeomorphism invariant, but no associated free Lagrange multiplier. This has not been explored.
2.4 Coupling to matter

The coupling of matter is usually determined by minimal coupling in general relativity. It is an important issue to settle under the new management of different symmetry principles. However, it is too complicated an issue to be discussed to completion here. I leave a devoted study for the near future. For now, I will only explicitly investigate electromagnetism.

The solution employed by York et al. [32] and Isenberg et al. [33] for the conformal weight of extra matter fields, was to make them zero. That is, the extra fields are then only carried along by the transformations of the metric. This automatically makes the usual Yang-Mills Lagrangian conformally covariant at least.

Under a conformal transformation \( g_{ab} \rightarrow e^{\phi} g_{ab} \), terms that depend on the derivative of \( \phi \) would appear in the Christoffel symbols. Thus, for (possibly Lie-algebra valued) one forms \( A_a \), we need to have anti-symmetry in spatial derivatives, as in \( \nabla [A_1] \nabla [aA^b] \), so that the corresponding term can be conformally covariant. That is because anti-symmetrized covariant derivatives of vector fields don’t register different Christoffel symbols (for Levi-Civita connections),

\[
\nabla [aA_1] = \partial [aA_1] + \Gamma^c_{ab} A_c = \partial [aA_1]
\]

In other words, no symmetrized spatial derivatives of spin-1 forms can appear, since otherwise there would be no way to make the coupling conformally covariant without the introduction of a further Weyl-connection [34], and this we would like to avoid (see [35]).

Since the potential for the field can only appear with anti-symmetric spatial derivatives, the symmetrized part is not dynamical. There must thus be a redundancy of description. This is just the symmetry \( A_a \rightarrow A_a + \partial_a \phi \), in the case of \( U(1) \) for some scalar function \( \phi \). By the usual arguments of gauge theory, this redundancy elicits the introduction of a spatial Gauss constraint. Let the conjugate density to \( A_1 \) be no way to make the coupling conformally covariant without the introduction of a further Weyl-constraint [34], and this we would like to avoid (see [35]).

From these two equations we can get back to the Lagrangian equations of motion (setting \( \xi^a = 0 = \lambda \)):

\[
\dot{A}_a = \frac{\alpha_i \beta_i}{\sqrt{8} \Omega^{1/3}} \left( \nabla^b \nabla [aA_1] - \frac{1}{3} \nabla [aA_1] \nabla b \ln \Xi - \dot{A}_a \left( \frac{\dot{\xi}}{2} + \frac{d}{3 d t} \ln \Xi \right) \right) - \dot{A}_a \left( \frac{\dot{\xi}}{2} + \frac{d}{3 d t} \ln \Xi \right)
\]
where $\dot{g} = g^{ab}\dot{g}_{ab}$. In the adiabatic limit for the geometry rewriting the wave-equation in terms of the vector potential, assuming that $\nabla_a \ln Z$ is small when compared to the gradients of the magnetic field, and in the divergence-free gauge, we obtain:

$$\ddot{A}_a = \frac{\alpha}{27} (\nabla^2 A_a - R^b_a A_b)$$

which is the standard form of the wave-equation for the vector potential in curved space-time, where

$$\left[ \frac{\alpha}{27} \right] \sim \left[ H_0 c_0 \right] \sim \left[ c^2 \right]$$

One possible line of investigation regards the degree to which local inhomogeneities in the gravitational field (parametrized by $\nabla_a \ln Z$) influence the standard curved spacetime electrodynamics equation. Note that, having been gauge-fixed, these equations of motion need no longer be conformally invariant.

**A simple approximate classical solution: Bianchi IX.** Spherical symmetry implies that the Cotton tensor vanishes, and so any expansion around such a metric for the Hamiltonian (22) is subtle. Nonetheless, there are cosmological models, mostly based on Bianchi IX, which are anisotropic and carry shape degrees of freedom (see [36] for a discussion of these models in the context of shape dynamics). The Bianchi IX spatial metric is of the (abstract-index) form:

$$g_{ab} = a^2 \delta_1 \otimes \sigma_1 + b^2 \delta_2 \otimes \sigma_2 + c^2 \delta_3 \otimes \sigma_3$$

where $d\sigma_i = \frac{1}{2} e_{ijk} \sigma_j \wedge \sigma_k$ are left-invariant one-forms, defined on the generic 3-surfaces $M$ (here considered to be 3-spheres). In principle the coefficients $a, b, c$ can have arbitrary dependence on time, however, here I will limit myself to the regime where this dependence is very slow. Since we have a conformally invariant model, the parametrization (32) just forms “a section” for our metric; we can multiply this metric arbitrarily by a factor $\phi$. We will fix this factor as follows.

Let us take such a history of 3-metrics, $g_{ab}(t)$ in a gravitational adiabatic limit, so that, roughly,$^{11}$ $O(\pi^{ab}) \sim \epsilon$. It is easy to see that this is an approximate solution for the equations of motion (25)-(26) for $\alpha = \rho = \xi = \Lambda = 0$.

For traceless momenta and divergence-free $\xi$, setting $\rho = 0$ means that the trace of $\dot{g}$ is constant. We thus set the conformal factor to be that of the unit round 3-sphere, since eventually we want the curve of geometries to cross that point,$^{12}$ $g_{ab}(0) = \Omega_0$ (see section 2.2 above), and thus $\det g = \det \Omega_0$.

If we moreover use the characteristics of the electromagnetic wave-equation (31) to build a light-cone, we have a 4-metric of the standard Bianchi IX form (with zero shift):

$$ds^2 = -N^2 dt^2 + g_{ab}$$

where $N^2 = \frac{\alpha}{27} \dot{\xi}$ is of the same dimension as $c^2$. It should be noted that, away from the adiabatic limit of the geometry, we would not have such a simple solution at hand. This shows some hope in modifying quantum cosmology in a regime-dependent manner, where inhomogeneities in space and time affect the speed of light.

### 3 The path integral

For completion, we will now see how much simpler the formal (and abstract) study of symmetries under the path integral quantization may become.

The action of the present simple model is given through (22) simply as

$$S = \int dt \left( \int d^3x \dot{g}_{ab} \pi^{ab} - \mathcal{H} \right)$$

The local gauge transformations, under the parameters $\epsilon^a(x)$ and $\epsilon(x)$ are given by (from (15)-(16)):

$$\delta g_{ab}(x) = \mathcal{L}_\epsilon g_{ab}(x) + \epsilon g_{ab}(x)$$

$$\delta \pi^{ab} = \mathcal{L}_\epsilon \pi^{ab}(x) - \epsilon \pi_{ab}(x)$$

$$\delta \xi = \dot{\epsilon} - [\xi, \epsilon]$$

$$\delta \rho = \dot{\epsilon} - \dot{\xi} \partial_a \epsilon - c^2 \partial_a \rho$$

---

11Here, gravitationally adiabatic means that, if coupled to another source, e.g. electromagnetism, as above, the rate of change $\dot{g}_{ab}$ is small when compared with $E^i$. Moreover, we are glossing over the fact that it is only the order of the transverse traceless components of the momenta that are gauge-invariant.

12The limit is indeed very delicate, since there the Cotton tensor vanishes. This would also have extreme implications for the quantum theory, together with our “past hypothesis” (see section 2.2).
3.1 Gauge fixation

In order to path integrate the exponential of $iS$, we must select a unique representative among each field history related by iteration of the infinitesimal transformations (35). The simplest possible gauge condition one can choose is

$$\xi^a = 0 \quad \rho = 0 \quad (36)$$

These could be implemented by adding the Gaussian term

$$S_{el} = \frac{1}{\sigma} \int dt \int d^3x \sqrt{\mathcal{S}} \left( \rho^2 + \xi^a g_{ab} \xi^b \right) \quad (38)$$

and taking the limit $\sigma \to 0$.

Note that in GR, it is not possible to choose the gauge for the lapse as $N = 0$ (or in our notation of equation (3), $\lambda^\perp = 0$), since this would also “kill the dynamics” (again, here a problem of mixing between dynamics and gauge freedom).

In whichever manner we rebuild spacetime, condition (36) tells us that the line joining two points with the same spatial coordinates lying on two neighboring surfaces will be normal to the first surface. This condition, together with (23), (25) and condition (37), tell us that the metric won’t change its local volume form along time. In fact, these conditions are equivalent to having the choice of curve in $Q$ representing the curve in $Q/C$ be given by a horizontal lift, according to a connection in the principal fiber bundle, as described briefly in appendix A (see figure 5). Of course, any horizontal lift still leaves the freedom to have a global transformation, identical at the endpoints of each curve. For any given choice on the particular initial metric $g_1$ representing the initial conformal geometry $[g_1]$, the horizontal lift $\gamma_H$ of the given path $[\gamma]$ will determine the form of the final metric as well $g_2 = \gamma_H(t_2)$. Of course, not all the paths need end at the same representative, i.e. in principle, for two different base space paths $[\gamma], [\tilde{\gamma}]$:

$$\gamma_H(t_2) = g_2 \neq \tilde{g}_2 = \tilde{\gamma}_H(t_2)$$

In fact, this non-trivial holonomy is a representation of the curvature of the connection used to horizontally lift the paths. While it is known that the purely conformal lift has trivial holonomy, the diffeomorphism part usually has a non-trivial one [18, 38–40].

13 It is possible nonetheless, to choose such a gauge for the perturbation fields in the background field formalism [3].

14 In [37], Teitelboim seeks to keep the arbitrariness in coordinate system independent at each end. For this purpose, he introduces a second interval in the path integral, with different gauge conditions. This severely complicates the analysis, as now one must also worry about folding (gluing) properties of the path integral.
We are interested in a wave-function of the Universe, a la Hartle-Hawking but here with an initial state given by the orbit $[\Omega_o]$, of the least dimensional stratum of $Q/\mathcal{C}$. Choosing a specific representative for $[\Omega_o]$, schematically we have:

$$\Psi([g]) := \int \mathcal{D}f \int_{\Omega_o} f^g \mathcal{D}\gamma \exp iS[\gamma_H(\Omega_o, f^g g)]/\hbar \quad (39)$$

where here $f \in \text{Diff}$ acts on an arbitrary representative of the final point of the transition amplitude and is integrated over with some measure (which we discuss). A Haar measure is not required here; unlike the standard case, we are not doing a group averaging procedure, each path on the base space corresponds to at most one $f$. If the curvature of the field-space connection form is zero, there is no relative holonomy on the fiber for two paths $\gamma_1, \gamma_2$, interpolating between a choice of $\Omega_o$ and the orbit of $[g]$. Thus all the lifts for the paths would end up in a single height of the orbit. Then the path integral will acquire a functional delta, cancelling the integral over $Df$. This is what happens with the purely conformal transformations, since they are Abelian and our choice above (37) has no associated curvature [38].

Alternatively, we could have an integration over the diffeomorphic representatives of $\Omega_o$, fixing the representative $g$. In this case, the wavefunction would be given by $\Psi(g)$, which would obey the respective conservation laws:\footnote{We note that in the case of odd-dimensional Weyl symmetries, there is no conformal anomaly, and thus the path measure can be suitably made Weyl-invariant in conjunction with the action [26]. This is also true in the Hamiltonian setting if the anomalies have a local representation [41].}

$$S_{\text{ghost}} \frac{\delta \Psi(g)}{\delta g_{ab}} = 0, \quad \nabla_a \frac{\delta \Psi(g)}{\delta g_{ab}} = 0 \quad (40)$$

### 3.2 BRST invariance

#### The ghost action

The ghost action is obtained by replacing the infinitesimal parameters in the gauge variation of the gauge-fixings (36) and (37), given by (35), by anti-commuting parameters, the ghosts, $\eta^a, \eta$, and contracting them with the anti-ghosts, $\bar{\eta}^a, \bar{\eta}$. That is, taking the BRST variations (which we define in the next paragraph)

$$\delta_B \xi^a = \eta - [\xi, \eta] \quad (41)$$
$$\delta_B \rho = \eta - \xi^a \partial_a \eta - \eta^a \partial_a \rho \quad (42)$$

where $\xi = \xi^a \frac{\partial}{\partial \xi^a}$ and the square brackets stand for the usual vector field commutator. One thus obtains:

$$S_{\text{ghost}} = -i \int dt \int d^3 x \sqrt{\tilde{g}} \left( \eta_a (\eta^a - [\xi, \eta]^a) + \bar{\eta} (\bar{\eta} - \xi^a \partial_a \bar{\eta} - \eta^a \partial_a \eta) \right) \quad (43)$$

which on the gauge-fixing surface becomes simply:

$$S_{\text{ghost}} \hat{=} -i \int dt \int d^3 x \sqrt{g} \left( \eta_a \eta^a + \bar{\eta} \eta \right) \quad (44)$$

where the hat indicates restriction to the gauge-fixing surface. The reduction in complexity, when compared to the similar term in the unitary gauge-fixing of ADM gravity, [37] is considerable.

#### BRST transformations

I mentioned above the BRST variations. They are defined as above for the fields $\xi^a$ and $\rho$. For the metric, they are defined in the same way, replacing $\nu^a$ and $\theta$ by $\eta^a$ and $\eta$. This guarantees that the classical part of the action, i.e. (34), remains BRST invariant. For $\eta$ and $\eta^a$, the BRST transformations are defined as follows:

$$\delta_B \eta^a = \frac{1}{2} [\eta, \eta]^a, \quad \delta_B \eta = \eta^a \partial_a \eta \quad (45)$$

The first equation does not vanish due to the anti-commuting nature of the ghosts, indeed $[\eta, \eta]^a = 2\eta^b \partial_b \eta^a$. Defining $\xi^a$ as the time-derivative of $\eta^a$, from the commutation property of the BRST differential with the time derivative:

$$\delta_B \xi^a := \frac{d}{dt} \delta_B \eta^a = \frac{d}{dt} \left( \eta^a \partial_a \eta + \eta^a \partial_a (\eta^b \partial_b \eta) \right) = \frac{d}{dt} \left( \eta^a \partial_a \eta + \eta^a \partial_a (\eta^b \partial_b \eta) \right) \quad (46)$$

Now, $\delta_B \xi$ can be seen to vanish when acting on $\eta^a$ due to the Jacobi identity. On $\eta$ we have

$$\delta_B^2 \eta = \frac{1}{2} [\eta, \eta]^a \partial_a \eta + \eta^a \partial_a (\eta^b \partial_b \eta) \quad (47)$$

which vanishes since $\eta^a \eta^b \partial_a \partial_b \eta = 0$ due to anti-symmetry.
For the metric we have:

\[ \delta^2 \eta_{ab} = \frac{1}{2} \mathcal{L}_{[\eta,\eta]^{ab}} + \eta^a \partial_b \eta_{ab} - \mathcal{L}_{\eta} (\mathcal{L}_{\eta^{ab}} + \eta \mathcal{G}_{ab}) \]

this vanishes since: i) \( \eta \eta = 0 \), ii) \( \mathcal{L}_{\eta} (\mathcal{G}_{ab}) = \eta^a \partial_a \eta_{ab} - \mathcal{L}_{\eta} \mathcal{G}_{ab} \) and iii) \( \mathcal{L}_{\eta} \mathcal{G}_{ab} = \frac{1}{2} \mathcal{L}_{\eta \eta} \mathcal{G}_{ab} \). The actions on \( \mathcal{G}_{ab} \) and on the other time derivatives of the fields necessarily vanish from (46) and the constituent transformations.

To see it directly, we require the Jacobi identity for Lie superalgebras. We have that, for ghost number one fields, \( x, y \):

\[ [x, [y, z]] = [[x, y], z] - [y, [x, z]] \]

Thus, for example, for \( \delta^2 \tau^a \):

\[ \delta_B (\delta_B \xi) = \delta_B ([\eta, \xi]) = \frac{1}{2} ([\eta, \eta], \xi) - [\eta, [\eta, \xi]] = 0 \]

since from (47), for \( x = y = \eta^a \), and \( z = \zeta^a \),

\[ 2[\eta, [\eta, \zeta]]^a = [[\eta, \eta], \zeta]^a \]

Lastly, we apply the BRST transformation to the transformation of the Lagrange multiplier (41) (equation (42) follows a similar pattern). We have:

\[ \delta_B (\eta - [\xi, \eta]) = [\eta, \eta] - [\eta + [\xi, \eta], \eta] - \frac{1}{2} ([\xi, [\eta, \eta]] = 0 \]

If the constraints are first class and the structure functions are constants, i.e. the algebra of constraints is not “soft”, then the BRST charge is of rank one and comes in the following form:

\[ Q = \eta^a \chi_a - \frac{1}{2} \eta^a \eta^b \mathcal{G}_{ab} P \]

for \( \xi \) the first class constraints, \( \mathcal{U} \) the structure functions, and \( P \) the ghost momenta. The rank of a system can be identified with the order of ghost momenta required for constructing a nilpotent BRST charge. The nilpotent, rank 1 Hamiltonian generator of the corresponding BRST symmetry in the conformal diffeomorphism case is then:

\[ Q = \int d^3 x \left( \eta^a \mathcal{G}_{ab} \nabla_b \pi^{bc} - P_b \eta^a \partial_b \eta^b \right) \]

with the ghost momenta given according to (44), by \( P_a = \sqrt{3} \eta_a \), \( P = \sqrt{3} \eta \).

### 3.3 Invariance of the path integral

Here, we will sketch the construction of [13], showing that in our case the constructed wave-function indeed satisfies the conservation equations (40). The linearity of the constraints greatly simplifies aspects of the proof, however.

For coordinate variables \( q_i \), momentum coordinates \( p^i \) and Lagrange multipliers \( \lambda_a \), encompassed by the variables \( z^A \), for an action functional \( S[q^i, p^i, \lambda_a] = \Sigma[z^A] \) with certain invariances, Hartle and Halliwell want to show that wave-function constructed by a sum over paths satisfy certain constraints.

The sum over paths is

\[ \Psi(q^{i''}) = \int Dz^A \delta (q^i (t'') - q^{i''}) \Delta_C [z^A] \delta [C^a (z^A)] \exp (i S[z^A]) \]

where \( C^a \) are the gauge-fixing conditions, \( \Delta_C [z^A] \) are weight factors associated to the gauge-fixing conditions. \( C \) is the class of paths being summed over, including the integration over the final values, \( q^{i''} (t'') \). It is the surface delta function that ensures that paths end up at the argument of the wave-function, \( q^{i''} \).

Their claim relies on the following assumptions:

---

16Note that the manner in which I wrote the super Jacobi identity, (47) doesn’t depend on the parity of \( z \). The full form for general grading is: \([x, [y, z]] = [[x, y], z] = (-1)^{|x||y|} [y, [x, z]]\).

17Note, however, that if one were to insert a second delta function, determining the anchor of the wave-function, the gauge-fixing (36) would not be legitimate (although (37) would be). One would have to adopt two different sets of gauge-fixings, as is done for GR in [37] and there further greatly complicate the analysis. This is the reason that one integrates over the orbit of the initial point.
1. Under a given transformation $\delta q^a = \epsilon^a h_a(p_i, q^i)$, for some function $h_a$ depending solely on the $q^i, p_i$, parametrized linearly by $\epsilon^a$, the action functional $S[z^A]$ changes at the most by a surface term, i.e. $\delta S = [\epsilon^a F_a(p_i, q^i)]^\text{ii'}$.

2. The class of paths $\mathcal{C}$ is invariant under the given transformation.

3. The path integral is independent of the gauge-fixing conditions, at least for a class

$$C^a_1[z^A] = C^a[z^A + \delta_c z^A]$$

4. The combination of measure $Dz^A$ and gauge-fixing weight factor transform according to

$$Dz^A \Delta_c[z^A] \rightarrow Dz^A \Delta_c[z^A]$$

5. Path integrals weighted by functions of $q^i, p^i$ on the final surface are equal to (appropriately ordered) operators acting on the wave-function $\Psi(q^{\text{ii'}})$. In other words, for a given $H(p_i, q^i)$,

$$\int Dz^A H(p_i(t'', q^i(t'')) \delta(q^i(t'') - q^{ii''}) \Delta_c[z^A] \delta(C^a(z^A)) \exp(iS[z^A]) = H(i\frac{\delta}{\delta q^{ii''}}, q^{ii''}) \Psi(q^{ii''})$$

Consequences or not of the path integral construction, these are taken to be the minimal criteria under which invariance of the wave-function follows. A non-trivial presence of the boundary term of item 1 slightly complicates the application of the Fradkin-Vilkoswisky theorem to the path integral, as it becomes dependent on a manipulation of non-BRST invariant boundary conditions (see sec III of [13]). The further issue which complicates the application of the Fradkin-Vilkowsky theorem to the path integral, as it becomes dependent on a manipulation of non-BRST invariant boundary conditions (see sec III of [13]). The further issue which complicates the verification of invariance in ADM is that its commutation relations do not form a Lie algebra, and therefore one cannot interpret the weights of item 4 as the simple Fadeev-Popov determinants, which are the Jacobians which emerge for integration along the gauge-fixing surfaces, and thus arises the requirement 4.

There, one must use the full apparatus of the BFV approach. In our case, all these complications are avoided, items 3 and 4 become the same, and we will not require item 5 at all.

Let us reproduce our version of the items.

1’ As given in (35), $\delta G_{ab}(x) = L_x G_{ab}(x) + \epsilon G_{ab}(x)$ the action functional $S[z^A]$ does not change at all, i.e. $\delta S = 0$. This simplifies many aspects of the proof.

2’ The class of paths $\mathcal{C}$ is invariant under the given transformation. Here the class of paths are all those in $\mathcal{Q}$. Since the group acts intrinsically in configuration space (which is how we found it), this class is invariant.18

3’ The path integral is independent of the gauge-fixing conditions.

where we have joined the previous items 3 and 4, and there is no substitute for item 5.

The only item above which still requires explanation is item 3. The easiest way to see it is first to notice that the manner in which we have arrived at the ghost action (44) is predicated on its interpretation as a functional determinant (I sketch the proof of this part in appendix D). In the presence of a principal fiber bundle structure, this is easy to prove (see [42] for a nice geometrical interpretation and proof using the PFB structure).

Using the Nakanishi-Laudrup trick, we can join the gauge-fixing action (38) with the ghost action (43), in the following way (I’ll do it for the diffeomorphisms, the conformal transformations follow suit): let $B_a$ be a Bosonic (i.e. commuting, or of ghost number zero) variable such that $\delta_B \delta_a = B_a$ (and we define $\delta_B B_a = 0$). Then, the term:

$$\Theta = \eta^a \left( \frac{1}{2} \sigma B_a - \xi_a \right)$$

has a BRST variation of the form:

$$\delta_B \Theta = B_a \gamma_{ab} \left( \frac{1}{2} \sigma B_a - \xi_a \right) - \eta^a \left( \eta_a - [\xi, \eta]_a \right)$$

which is what we want to add to the classical action. But now it is easy to see that if we solve the equations of motion for the auxiliary variable $B_a$, we obtain $\sigma B_a = \xi_a$. Inputting this back into (53) we obtain two terms: (43) and (38). From the nilpotency of $\delta_B$, the entire gauge-fixed action,

$$S_a + S_d + S_{\phi} = S_a + \delta_B \Omega$$

18In the case of Hartle-Hawking, the original definition [24] only uses paths in $\mathcal{Q}$ which have $\det(g) > 0$. This is, of course, not a spacetime condition, and thus slice-dependent. So it is not clear to me why such a class of paths is invariant under the assumed symmetry. Of course, this presents no problem in the usual minisuperspace approximation.
is BRST invariant.

For the measure, as mentioned, we are secretly using the Liouville measure and projecting to configuration space. But, under a BRST transformation, the Liouville measure transforms by a total derivative term which is just a canonical transformation (see e.g. appendix A of [13]).

Finally, given these items, we proceed to prove the invariance of the wave-function (51). The only term that transforms in (51) under a gauge transformation is the delta function, in our language $\delta(\hat{g}_{ab}(t''') - \hat{g}_{ab}'') = \delta(\hat{g}_{ab}^1 - \hat{g}_{ab}^2)$, which transforms under an infinitesimal diffeomorphism acting on $g_{ab}^1$. But we have, shifting variables, according to the standard property of functional delta, (81), up to first order,

$$
\delta(\hat{g}_{ab}^1 + \mathcal{L}_e \hat{g}_{ab}^1 - \hat{g}_{ab}^2) = \delta(\hat{g}_{ab}^1 - (\hat{g}_{ab}^2 + \mathcal{L}_e \hat{g}_{ab}^2)(x)) \left( \det (\mathrm{Id} + \frac{\delta}{\delta \hat{g}_{ab}^1(y)} \mathcal{L}_e \hat{g}_{ab}^1(x)) \right)^{-1}
$$

The identity for $\delta$ implicitly uses a redefinition of the integration variable to produce the determinant. Now $\det(1 + \epsilon) = 1 + tr(\epsilon)$, where the trace is an integration over $x = y$ and summation over internal indices. Since the structure constants of our group are traceless,\(^{19}\) we get

$$
\delta(\hat{g}_{ab}^1 + \mathcal{L}_e \hat{g}_{ab}^1 - \hat{g}_{ab}^2) = \delta(\hat{g}_{ab}^1 - \hat{g}_{ab}^2) - \int d^3x \mathcal{L}_e \hat{g}_{ab}^2(x) \frac{\delta}{\delta \hat{g}_{ab}^1(x)} \delta(\hat{g}_{ab}^1 - \hat{g}_{ab}^2) \tag{54}
$$

Finally, we will implement this result in (51), by shifting the integration variables, and using 1’, 2’ and 3’, above. Noticing that none of the variables depend on (the equivalent of) $g_{ab}^2$, and thus we can apply the functional derivative on the rhs of (54) to everything:

$$
0 = \int \mathcal{D}g_{ab}^1 \int_M d^3x \mathcal{L}_e \hat{g}_{ab}^2(x) \frac{\delta}{\delta \hat{g}_{ab}^1(x)} \left( \delta(\hat{g}_{ab}^1 - \hat{g}_{ab}^2) \Delta_C \hat{C}_{[\hat{g}_{ab}]} \right) \exp \left( -\sigma S_{\hat{g}_{ab}} \right) \tag{55}
$$

\[ 0 = -\int d^3x \right. c^4 \nabla^2 \frac{\delta}{\delta \hat{g}_{ab}^1(x)} \left. \Psi_{g_{ab}} \right|_{g_2} \tag{56}

where $\nabla^2$ refers to the metric $g_{ab}^2$. This shows the second equation of (40) holds. The first follows from the same procedure and is much simpler as the conformal group is Abelian.\(^{20}\)

**Perturbation theory** In fact, for theories that have preferred foliations, a similar gauge-fixing of the path integral is performed in [43]. There, one chooses gauge-fixings of the space-time diffeomorphisms only through gauge-fixings of the perturbations of the lapse and shift, not for the metric and its perturbations. That is also what is being done here: conditions on the propagation of coordinates are being chosen, rather than the coordinates themselves. The gauge-fixing terms become much simplified — one can have just terms that set the perturbations of the shift to zero for example — as opposed to choosing e.g. harmonic coordinate conditions for the full perturbations of the metric tensor.

To actually perform calculations for perturbation theory, we need to find a background of all the fields (including Lagrange multipliers if there are any), and then perform the second variation of the gauge-fixed action. The problem that shows up here, and not elsewhere, is that for perturbative renormalizability we need to choose a background more amenable to the constraints of the theory than the usual Minkowski background; a Minkowski background is not directly suitable for the calculations, since it is conformally flat and thus has vanishing Cotton tensor.

As mentioned before, the physical degrees of freedom of the conformal geodesic theory are the same as in general relativity: both have the transverse traceless gravitational momenta, as per equation (23). But we need to find a linearized version of the theory if we would like to obtain a linearized wave equation, as in general relativity. Ideally, we would have an approximation to the calculation on the full Bianchi IX background. This will be pursued in a further paper.

For now, to get an idea for the structure of the equations, we can use the preferred configuration $\Omega_0$, the 3-sphere metric, to construct a density which is not affected by conformal transformations. This allows us to write a different action:

$$
S = \int dt \int_M d^3x \sqrt{g} \left( g^{\phi \phi} \dot{g}_{ab}^{\phi} + \Lambda \right) \left( \sqrt{g} C_{ab \phi} + \circ \sqrt{g_0} \right) = \int dt \int d^3x TV
$$

where I wrote the action as the multiplication of a kinetic term $T$ and potential one $V$. Now it is possible to calculate the Euler-Lagrange equations of motion around a spherically symmetric background, i.e. $g_{ab} \to$ \(^{19}\)This is in essence the same reason the Fadeev-Popov procedure works simply in our case. See appendix D. This is not the case with the standard ADM algebra.

\(^{20}\)See [41], for a study of the appearance of conformal anomalies for the conformal symmetry in the Hamiltonian setting appropriate to this foliation-preserving group, $\hat{g}_{ab}$.
\( g^{\alpha}_{ab} + \epsilon h_{ab} \), with \( \Omega_{ab}^o \) having radius of curvature \( r \). It is easy to see that \( \frac{\delta V}{\delta g_{ab}} |_{g_{ab} = \Omega_{ab}^o} = 0 \). After some algebra, one can evaluate the only surviving terms from the Euler-Lagrange equation:

\[
\bar{h}_{ab}(x) = \frac{\delta}{\delta g_{ab}} \int d^3y \, \bar{g}_{klm} \bar{g}_{nij} \, \frac{\delta C(y)}{\delta g_{ab}(y)} \, h_{ef}(y) \frac{\delta g_{ef}(z)}{\delta g_{ab}(x)} \bigg|_{g_{ab} = \Omega_{ab}^o} = \frac{1}{\omega^2} \left( 4 \nabla^2 h_{ab} - \frac{\partial^2}{\partial t^2} \nabla^4 h_{ab} + r^4 \nabla^6 h_{ab} \right)
\]

This wave equation has the interesting property that it possesses higher order spatial derivatives. This will be a general property of specific solutions to the equation (26). It provides modified dispersion relations that may be fruitful in regularizing gravitational divergencies, as is the case of Horava-Lifschitz [2].

The study of perturbation theory around a Bianchi IX static solution has only just begun. For future work, we can follow the algorithm set in [3] as applied to this theory.

### 4 Conclusions

#### 4.1 Summary

I started the paper looking for gravitational theories which admit a reduced configuration space in metric variables. That is, superspace \( Q/\text{Diff}(M) \) is not the reduced space of physical configurations for GR; one still has an action of refoiliations. Thus the question arises: what is the maximal local symmetry group acting on metric configuration space that admits a respective “superspace”?

The answer is: conformal diffeomorphisms, and the reduced configurations space it forms is called ‘conformal superspace’. I then went on to investigate the local gravitational actions that were compatible with this principle. Surprisingly, minimal coupling to vector fields yields a \( U(1) \) gauge theory with hyperbolic equations of motion, a surprising consequence from our starting point.

This field and symmetry content guarantee that the emerging theory has the correct transverse traceless gravitational physical degrees of freedom, a desirable result, vis a vis the great struggle to modify gravity while keeping its two degrees of freedom intact. Lastly, one can use the structure of conformal superspace to establish a unique preferred orbit. This is, in a very precise sense, the most homogeneous of all configurations, and it has a preferred geometrical status on reduced configuration space; the analogy I used was ‘the vertex of an bottomless tetrahedron’. This preferred point serves to establish a preferred initial point in the path integral kernel, \( \phi^* = g^* \), as above, thereby defining the static wavefunction, in a similar fashion as what is achieved through the Hartle-Hawking boundary conditions [24].

This principled derivation of the theory sets it apart from the standard modifications of gravity already at a classical level; it is not ad hoc (see e.g. [17]) and it contains no spin-0 (or scalar) gravitational degree of freedom [2]. Regarding the quantum mechanical theory, departing from the same principles, one would not be able to define different natural ‘initial conditions’ for the wave-function of the Universe, \( \Psi(g) \), as the nested boundary structure in conformal superspace has a unique, asymmetric structure. In my opinion, for GR there is more ambiguity (see e.g. [24–26]) because one does not have access to the true configuration degrees of freedom (see also footnote 18).

Further addressing the scientific merit of the theory; it seems vulnerable to falsification. For cosmology, one could study what is known for Bianchi IX cosmology (see e.g. [44]) and study the modification affected by breaking the adiabatic limit (see last paragraphs of section 2.4). It is encouraging that the Yang-Mills form of the Hamiltonian for (Lie-algebra valued) one-forms emerges very naturally, from the requirements of conformal covariance and conformal weight matching, in (27). The emergent hyperbolic equations for YM mean that the electromagnetic sector would have a light cone. Nonetheless, as can be seen from equation (30), there are discrepancies at lower derivatives, of the form

\[
- \frac{1}{3} \frac{a^f b^g}{\Omega^{\alpha\beta}} \nabla^a A_b \nabla^c \ln \Omega - \dot{A}_a \left( \frac{\dot{\phi}}{2} + \frac{1}{3} \frac{d}{dt} \ln \Omega \right)
\]

This shows some hope in modifying quantum cosmology in a regime-dependent manner, where inhomogeneities in space and time affect the propagation of light. It would be extremely interesting for instance to study when such terms become relevant in standard Bianchi IX type of cosmologies, something I leave for further study.

The distinct extra coupling of all matter fields and gravitons to the geometry through multiplication by powers of \( C^{a\beta}_c \) can give an extra, dynamical hierarchy to the different interactions. It seems that all of our couplings to the gravitational sector have to be in some sense non-minimal, meaning that different regimes of gravity can suppress or amplify different types of interactions.\(^{21}\)

\(^{21}\) In effect, such non-minimal coupling terms seem to also emerge in the usual renormalization group flow of the Einstein-Hilbert action when coupled to the standard model sector [45].
With regards to technical questions surrounding quantum gravity, the theory seems to have several advantages. To be specific, they are seven-fold: i) it is second order in time derivatives (in space-time covariant field space, Lagrangians extending general relativity require more time derivatives, creating possible problems for unitarity, since the propagators around a flat background acquire imaginary poles), ii) we can apply spatial regulators on the exact renormalization group equation (finding regulators that depend on the background metric with indefinite signature is a challenge), iii) the theory explicitly maintains two, transverse traceless, propagating degrees of freedom per space point (this is an issue with other theories, such as Horava-Lifshitz and Hořava-type theories), iv) The treatment of the path integral is very simplified because of simple constraint (Lie) algebra, for the GR case it is much lengthier (it would correspond to parts of both [13, 37]). The theory possesses simple BRST symmetry (hard to achieve for the ADM Dirac algebra for example, which requires a BFV treatment [13]). v) There is no Gribov problem for the horizontal lift gauge-fixing (a non-perturbative problem for basically all non-abelian theories based on space-time fields, see appendix B), vi) superposition can readily be made sense of, simply as the interference between (coarse-grained) paths in configuration space (see [1]), there is no need to define superposition of causal structures, vii) the linearized equations of motion contain higher order spatial (but not temporal) derivatives. This property is extremely desirable in the context of regularizing ultraviolet divergencies (as is done in e.g. [2]), a topic we will explore in further work. The study of perturbation theory around a Bianchi IX static solution has only just begun. For future work, we can follow the algorithm set in [3] as applied to this theory.

Indeed, I believe that these features already make these theories extremely interesting as a toy model, irrespective of having the correct classical limit.22

4.2 Relation to previous work

In a previous paper, [1], I described properties of a general relational theory in the non-relativistic field configuration space \( Q = \{ \phi \} \), with \( \phi(x) \) a given field on a Riemannian closed manifold, \( M \). Starting from an action for paths in configuration space, \( S[\phi(t)] \), I defined a single static wave-function, \( \psi[\phi] \), given by the path integral transition from a preferred configuration – the most homogeneous one, \( \phi^* \) – to any other, i.e. \( \psi[\phi] = \int D\phi(t) \exp[i S[\phi(t)]/\hbar] \), for spatial field histories such that \( \phi(0) = \phi^* \) and \( \phi(1) = \phi \). I found that even in the absence of an explicit time variable, one could extract a notion of time from structures of the wavefunction itself. I called such structures "records".

The most interesting property of a configuration holding a record is that its Born probability becomes a conditional probability, conditional on the configuration representing the record. Even in timeless configuration space, observables of the record-holding configuration behave as if they are the culmination of a history, a history which included elements of the recorded configuration.

In [46], I used the structures introduced in [1] to discuss a dynamical origin for locality. It is notoriously difficult to disentangle locality from theories that possess gauge degrees of freedom, and general relativity has an honorable tradition in this respect [47]. I proposed criteria that would determine locality of subsystems from properties of solution curves in configuration space, a notion particularly useful for theories with non-local Hamiltonians (such as shape dynamics [12]).

However, the mechanisms discussed in [46] were not viable for theories that possessed refoliation invariance, such as ADM gravity [4]. Moreover, I used tools that required the action of the theory to be of Jacobi-type, i.e. expressible as a distance functional in configuration space. I did not provide an example of a theory that would satisfy all of those criteria in either paper.

Here I closed this gap, by providing a theory that possesses the following features: i) it is completely spatially relational. For a gravitational theory in metric variables, item i is maximally realized by the local symmetry group of (foliation-preserving) conformal diffeomorphisms. The simplest form of theories obeying these symmetries has the interpretation of ii) being of Jacobi type, a theory about "geodesic" field histories in conformal superspace – which is the physical quotient-space, or the configuration space quotiented by the symmetries.

We should remark that it is not because the theory does not appear to be general relativity – since it has an anisotropic number of spatial and time derivatives – that it cannot match it phenomenologically.23 The time appearing in the equations of motion is merely the path parameter in configuration space, and might have little to do with actual experienced duration.

---

22Most quantum gravity theories have also the problem of matching known physics. String theory doesn’t explicitly recover the full Standard Model and Loop Quantum gravity and spin foams don’t recover realistic Einstein space-times in the appropriate limits. Asymptotic safety is on better grounds on this respect, since it is more in line with an effective field theory approach.

23As Petr Horava is fond of saying, preferred foliation theories do not “break Lorentz symmetry”, any more than general relativity breaks “Unicorn symmetry”. In both cases, it is a symmetry that the theory never had, so there is nothing to break, and naive analysis of effects of “Lorentz breaking”, such as those of [46, 49] – which show that even if these effects appear at the Planck scale, they percolate to the IR through the RG flow – are not applicable.
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APPENDIX

A Gauge theory in Riem

The results of this appendix are used in section 2.2.

A.1 Slice theorem for Riem

Definition 1 (Slice) Let $G \supset I_x = \{g \in G \mid gx = x\}$ be the isotropy subgroup of $G$ at $x$. A slice at $x \in X$ for the action of $G$ is a submanifold $S_x$ such that $x \in S_x \subset X$ and:

- if $g \in I_x$, then $gS_x = S_x$;
- if $g \in G$ and $(gS_x) \cap S_x \neq \emptyset$, then $g \in I_x$;
- there is a map $\mu : G/I_x \rightarrow G$, called a local cross-section, defined in a neighborhood $U$ of the identity of $G$, such that the map $\mathcal{F} : U \times S_x \rightarrow X$, defined by $\mathcal{F}(g, y) = \mu(g)y$ is a diffeomorphism onto a neighborhood $U$ of $x$.

Corollary 2 From the existence of a slice $S_x$ defined as above, it is easy to show that for a given neighborhood $U$ of $x$. i) for $y \in U \cap S_x$ then $I_y \subset I_x$ (follows from the second item in definition 1); and ii) for $y \in U$ then $I_y$ is conjugate to $I_x$ (follows from successively applying the third property of definition 1 and item i)).

From a slice, we also trivially obtain the local product structure generically, i.e. for points in $\hat{X} = \{x \in X \mid I_x = \text{Id}\}$, we have $\mathcal{U} \simeq S_x \times G$, with $\mathcal{U}$ being a proper subset of $X$ containing $x$. It follows that the quotient $\hat{X}/G$ has a manifold structure.

Ebin and Palais have shown that $\mathcal{Q}$ has a local slice [19, 50]. They did this through the use of the normal exponential map to the orbit along a given point. That is, for an arbitrary given metric $\bar{g}_{ab}$, the orbits $O_{\bar{g}}$ were shown to be embedded submanifolds, which therefore have a well-defined tubular neighborhood. Given the tangent of the orbit, $T_gO_{\bar{g}} =: V_g$ and a $G$-invariant supermetric (e.g. (1) with $\lambda = 0$), one can then define the normal exponential map: $\text{Exp}_g : W \subset V_g \rightarrow \mathcal{Q}$, which was shown to be a local diffeomorphism onto its image for a given open set $W$, where, as can easily be seen from the form of $V_g$ and (1),

$$V_g^\perp = \{u_{ab} \in T_g \mathcal{Q} \mid \nabla_u u_{ab} = 0\}$$  (57)

By then showing that the tubular bundle around this orbit was locally diffeomorphic to $\mathcal{Q}$, one has, for $\bar{g}_{ab} \in \pi^{-1}(\pi(\text{Im}(\text{Exp}_g(W))))$, a unique $f_g \in \text{Diff}(\mathcal{M})$ such that

$$f_g \bar{g}_{ab} = \text{Exp}_g(w_g)$$

for a unique $w_g \in W$,

$$w_g = \text{Exp}_g^{-1}(f_g \bar{g}_{ab})$$

Thus

$$\bar{g}_{ab} = f_g(\text{Exp}_g(w_g))$$

Furthermore, for $\bar{g}_{ab}^2 = h^* \bar{g}_{ab}$, we then have $f_{g^2} = h^{-1} \circ f_g$. Thus $w_g = w_{|g}$ and for any $\bar{g}_{ab} \in [\bar{g}_{ab}] \subset \pi^{-1}\pi(U)$, the section is given by

$$\chi(\pi(\bar{g}_{ab})) = \text{Exp}_g(w_{|g})$$  (58)

In more heuristic terms, $\chi$ takes any metric along the orbits and translates it along the orbit until it hits the orthogonal exponential section at the height of $\bar{g}_{ab}$. This intersection gives us the value of $\chi$ for the given equivalence class.
A.2 Principal fiber bundles and connections

A principal fiber bundle is a manifold \( P \), on which a Lie group \( G \) acts freely: \( P \times G \to P \), here we will assume it acts on the left, \( (p, g) \to g \cdot p = L_g(p) \). The space \( \{ g \cdot p \mid g \in G \} \) is called the fiber through \( p \). Moreover, \( P \) is assumed to have a locally trivializing section, but we will not need these details here.

Given a slice theorem for the generic subspaces where \( I_k = Id \), we can form such a principal fiber bundle, and make use of all the structure that comes with it. Given \( v \in g \), where \( g := T_0G \), we define the fundamental vector field associated to it as \( T_v P \ni v^g_p := \frac{d}{dt} \exp_t(tv) \cdot p \). The vertical subspace \( V_p \subset T_p P \) is the tangent space to the fiber at \( p \), i.e. \( V_p = \{ v^g_p \mid v \in g \} \). A horizontal distribution is a smooth equivariant tangential distribution which complements the vertical spaces, i.e \( H_p \subset T_p P \) such that:

\[
i H_p \oplus V_p = T_p P \quad \text{and} \quad \mathcal{L}_\xi (H_p) = H_p
\]

Such horizontal spaces are uniquely identified by a connection form on \( TP - g \)-valued one-form \( \omega \), satisfying:

\[
i \omega(v^g_p) = v \quad \text{and} \quad \mathcal{L}_\xi \omega = \text{ad}(g)\omega
\]

The identification is obtained by the vertical projection, for a given \( X_p \in T_p P \), it is the fundamental vector field (or vector tangent to the fiber) \( \hat{V}_p(X_p) = (\omega(X_p))^g \). The horizontal projection is its complement, \( \hat{H}_p = Id - \hat{V}_p(X_p) \).

We can rewrite the property corresponding to ii) of (60) for a field-dependent transformation as (see [31]):

\[
\delta_{\xi}(\omega(u))^a = \delta_{\xi}^a - [\omega(u), \xi]^a
\]

for \( u \) a vector at \( T_p P \).

Connections in Riem. For this geometric approach to gravitational theories proposed here, one can think of configuration space as an infinite-dimensional principal fiber bundle \( P = G \), foliated by the group orbits of the diffeomorphisms \( G = \text{Diff} \), acting by pull-backs (and scalar transformations, acting by pointwise multiplication).

In this case, best-matching, or the gauge connection in Riem, is given by a linear action \( \hat{H} : TP \to \mathfrak{g} \), where \( \mathfrak{g} \) is the corresponding (infinite-dimensional) Lie algebra (for instance \( C^\infty(TM) \) with the Lie bracket for diffeomorphisms) and \( P = G \).

As in usual formulations of gauge symmetry in principal fiber bundle language, this gives a projection \( \hat{H} : \mathfrak{g} ab \to \mathfrak{g} ab \) into a 'horizontal' component of the metric velocity (see [52] for a quick introduction to principal fiber bundles). That is, in the Lagrangian formulation, a "preferred shift" defining an equilocality relation – the 'best-matched coordinates – emerges from a connection form (see [18] for details on how these objects are constructed in this infinite-dimensional context).

In plain words, the role of the connection form (i.e. of best-matching) is to project out the pure 'coordinate-change' component of a metric velocity. The inverse of the (conformal) thin-sandwich differential operator (see [53]) is obtainable by such a form which projects out orthogonal components to the fibers, defining \( \hat{H} \). If a given supermetric is positive-definite and covariant with respect to the gauge transformations, we can define \( \hat{H} \) by the projection orthogonal to the fibers.

This is what is done for instance in [19] (see also [20]), using the canonical (positive-definite) supermetric in Riem, \( G^{abcd} := \frac{1}{2} \left( g^{ac} g^{bd} + g^{ad} g^{bc} \right) \) (i.e. the canonical supermetric with zero DeWitt parameter). Since fundamental vector fields at \( g_{ab} \) (i.e. tensors tangent to the fibers) are given by \( \mathcal{L}_\xi g_{ab} = \nabla_a \xi^b + \nabla_b \xi^a \), one obtains that horizontal subspaces are the ones that obey \( \nabla_c g_{ab} = 0 \). The vertical projection is given by the vector field-valued one form (i.e. a functional acting linearly on metric velocities with values on the Lie algebra of the diffeomorphisms) \( \xi^a \) such that:

\[
\nabla_a (g_{ab} - \mathcal{L}_\xi(g_{ax})g_{ab}) = 0
\]

i.e. it is defined by the horizontal projection being orthogonal to the fibers according to the supermetric. Because the supermetric is positive definite, we note that \( H_p \cap V_p = 0 \), so that, barring metrics with non-trivial isometry group (for which the principal fiber bundle picture needs mending),

\[
\nabla_a (\mathcal{L}_\xi(g_{ax})g_{ab}) = (\nabla^2 \delta^a_b + R^a_b)(\xi^c | g | x)_a - \nabla_b (\text{div}(\xi^c | g | x)) = 0
\]

24This is not strictly true, because metrics with isometries possess non-trivial stabilizer subgroups of the diffeomorphisms, one obtains a stratified manifold for the quotient \( G/\text{Diff} \) (see [23]). However one can make sense of this as a principal fiber bundle if one considers only the diffeomorphisms which fix one point of \( M \), and a linear frame on it. [51].

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only for $\zeta(\dot{g}; x)_b = 0$, which is the property corresponding to $\omega(v^a_b) = 0$.

Furthermore, since the supermetric is equivariant wrt to the action of the symmetry group, we automatically obtain that the horizontal projections obey (59) and therefore the connection form obeys (60). These properties (which will also hold for the conformal connection defined below) guarantee that this functional shift vector transform as (15).

Conformal diffeomorphism connections given by a supermetric in Riem. In [18], one uses the supermetric:

$$ (v, w)_g := \int d^3x \sqrt{g} \sqrt{C_h C_f} g^{abcd} v_{ab} w_{cd} \quad (64) $$

where the vectors $v_{ab}$ are taken to transform as $v_{ab} \to e^{\rho} v_{ab}$ under conformal transformations $g_{ab} \to e^{2\rho} g_{ab}$, for $\rho \in C^\infty(M)$ a scalar function. The supermetric (64) is positive definite and is equivariant with respect to conformal diffeomorphisms. The integrand in (64) is thus invariant under conformal transformations. Thus, using (64), one forms a genuine gauge connection from the orthogonal projection,

$$ g^{ab}_{\Omega} = (g^{ab} - \nabla^{(b} \omega_x^{a)}(\dot{g}; x) + \omega_x(\dot{g}; x) g^{ab}) \quad (65) $$

This equation defines $g^{ab}_{\Omega}$ as a generalized ‘transverse traceless’ metric velocity, corrected by infinite-dimensional gauge connections on Riem, $\omega_x^{[a}$ and $\omega_x^{]a}$, along the diffeomorphism and conformal fibers respectively, which project the velocities into the space orthogonal to the conformal diffeomorphism fibers, as we did before just with the diffeomorphisms. We note that here too, the horizontal projections obey (59) and therefore the connection form obeys (60), and these properties guarantee that this functional shift vector transform as (15), without any need for any ad hoc definitions of transformation properties.

To be more specific, rewriting the full group of diffeomorphisms as a product of the incompressible diffeomorphisms and the pure dilatations. These are generated by divergenceless vector fields and their complements. In that case, we have, - from an orthogonality condition to the fiber, according to (64):

$$ \nabla_a \left( \sqrt{C^{cd} C_{cd} \omega_x^b} \right) = 0 \quad (66) $$
$$ v_{ab} g^{ab} - 3\rho = 0 \quad (67) $$

Given a metric velocity, $g_{ab} \in T_g P$, we want to horizontally correct it, as is the job of the usual covariant derivative:

$$ \nabla_a \left( \sqrt{C^{cd} C_{cd} (g^{ab} + \nabla^{(a} \dot{\omega}^{b)} + \rho g^{ab})} \right) = 0 \quad (68) $$
$$ \dot{g}_{ab} g^{ab} - \text{div}(\dot{\xi}) - 3\rho = 0 \quad (69) $$

Since we want to invert this for $\xi^a$ and $\rho$, we look at equations (66) and (67) as equations to be inverted for $\xi^a$ and $\rho$ in terms of sources given by the metric and its time derivatives.

The condition for the existence of an element of $T_g Q$ which is both orthogonal to a fiber and parallel to it (i.e. is both horizontal and vertical) is that a non-zero $\xi^a$ exist such that

$$ \nabla_a \left( \sqrt{C^{cd} C_{cd} (\nabla^{(a} \dot{\omega}^{b)})} \right) = 0 \quad (70) $$

where we are taking the divergence-free Killing form. But in our case we the supermetric (64) is positive definite, and thus the only solution is $\nabla^{(a} \dot{\omega}^{b)} = 0 = L_{\dot{\xi}_a} \dot{g}_{ab}$.

Thus equation (70) only has non-trivial solutions in case the metric has conformal Killing vector fields (since we are taking the transverse part of the vector fields, and so on).

We thus construct the two operators on $TQ$, i.e. $\omega_x^a : T_g Q \to C^\infty(TM)$ and $\omega_x : T_g Q \to C^\infty(M)$, such that:

$$ \nabla_a \left( \sqrt{C^{cd} C_{cd} (\nabla^{(a} \dot{\omega}^{b)} - \omega_x(\dot{g}; x) g^{ab})} \right) = 0 \quad (71) $$
$$ \frac{1}{3} \dot{g}_{ab} g^{cd} - \omega_x(\dot{g}; x) = 0 \quad (72) $$
covariantly wrt to the diffeomorphisms $f$ and by virtue of the connection forms in (65), under a conformal diffeomorphism much detail, the gauge-fixed Yang-Mills Euclidean path integral can be written as:

$$B_{\text{Gribov problem}}$$

Fortonian time parameter $t$ and using "clock" subsystems. [54]

for the more formal mathematical proof that one can indeed obtain a connection form from this orthogonality criterion, one needs to show that the criterion can be expressed as the kernel of an elliptic operator, and use the Fredholm alternative in the infinite-dimensional Banach space case. This is done to full detail in [18].

Thus for the geodesic action:

$$S = \int dt \sqrt{\sum_{x} C^{ab} C_{ab}(\xi^{\mu})}$$

and by virtue of the connection forms in (65), under a conformal diffeomorphism $f$ and using the Fredholm alternative in the infinite-dimensional Banach space case. This is done to full detail in [18].

For the more formal mathematical proof that one can indeed obtain a connection form from this orthogonality criterion, one needs to show that the criterion can be expressed as the kernel of an elliptic operator, and use the Fredholm alternative in the infinite-dimensional Banach space case. This is done to full detail in [18].

B Gribov problem

In this section, we will conform to the more usual notation of gauge field theory. Without getting into too much detail, the gauge-fixed Yang-Mills Euclidean path integral can be written as:

$$Z_{YM} = \int_{\mathcal{F}} [DA] \det \left( \frac{\delta F[A]}{\delta \xi} \right) e^{-S_{YM}}$$

where $F[A] = 0$ is the gauge-condition,

$$F[A] = \partial_{\mu} A_{\mu} = 0$$

which under an infinitesimal transformation, behaves as:

$$A_{\mu}^{\xi} = A_{\mu} - D_{\mu}^{ab} \xi^{b}$$

where $D_{\mu}^{ab} = \delta_{a}^{b} \partial_{\mu} - \epsilon_{a}^{bc} A_{\mu}^{c}$

where we used $\epsilon$ for the coupling (as we already have far too many $g$’s). Thus the condition for the invertibility of the propagator – the same condition for the non-degeneracy of the Fadeev-Popov matrix – is that the following operator:

$$\mathcal{M}_{\mu}^{ab} := (\delta_{a}^{b} \partial_{\mu} - \epsilon_{a}^{bc} A_{\mu}^{c}) \xi_{b}$$

has a trivial kernel. A kernel of this operator will belong both to the gauge-section and the orbits. In the Euclidean case, for small product $eA$ the operator is indeed invertible. However, in the non-perturbative regime, eventually (77) develops zeros, since $eA$ is not non-negative, meaning the section has become tangential to the orbits.

That this must be the case can be seen from very general arguments [14], which we reproduce here. Suppose one disregards the reducible gauge potentials $A$, and the ones that have non-trivial stability group. Recall the space of such gauge-potentials $\mathcal{F}$. This is a contractible space, i.e. one can deform any $n$-dimensional sphere (by which we mean a map $S^{n} \to \mathcal{F}$) to a point within it, and thus all of its homotopy groups are trivial, $\pi_{j}(\mathcal{F}) = 0$. The same is true for the image of a given section $\sigma : \mathcal{F} \to \mathcal{G}$. Now, if field space decomposed into a product, $\mathcal{F} \simeq \text{Im}(\sigma) \times \mathcal{G}$, where $\mathcal{G}$ is the group of gauge transformations by the Lie group $G$, then this would imply that $\pi_{j}(\mathcal{G}) = 0$. However, as can be shown, for some $j \neq 0$, and underlying space-time manifold given by $S^{4}$ (or $S^{3}$), and $G = \text{SU}(n)$, $n \geq 2$, there always exists a $j$ for which $\pi_{j}(\mathcal{G}) \neq 0$. Therefore the space $\mathcal{F}$ can’t be decomposed onto a product, and therefore there exists no global section $\sigma : \mathcal{F} \to \mathcal{F}$.

This result implies that there exists no flat connection on $\mathcal{F}$! For if there were, then the horizontal distribution would be integrable, and it would itself form a section.

C Arc-length parametrization

Equation (34) is an energy functional in Riem with just one global lapse and thus one global notion of time.29 Its extrema give arc-length parametrized geodesics wrt the conformal supermetric (64). If we also want reparametrization invariance, we arrive at the conformal-diffeomorphism invariant geodesic action (as opposed to the energy action, (34)):

$$S_{\text{mod}} = \int dt \sqrt{\sum_{x} C^{ab} C_{ab}(\xi^{\mu}) + \nabla_{\nu} (c^{\mu})} \left( \dot{\xi}_{\mu} + \nabla_{\nu} (c^{\mu}) \right)$$

28The original argument in [14] does not require these assumptions, but its core is simplified with them.

29This says nothing about the experienced duration of the degrees of freedom. The distinction is analogous to that between the Hamiltonian time parameter $t$ and using "clock" subsystems. [54]
Then, we still need to gauge-fix the time-reparametrizations, which we choose to be an arc-length one:

$$G_t := \sqrt{\int_M \sqrt{g} \mathcal{D}^3x \sqrt{\mathcal{C}^a_{ab} (\mathcal{G}^{cd}(t)) \mathcal{G}^{cd}(t)}} - 1 = 0 \quad (79)$$

The infinitesimal version, i.e. for $\delta_\epsilon t = c + \epsilon t + O(\epsilon^2)$,

$$G_\epsilon = |\epsilon| \sqrt{\int_M \sqrt{g} \mathcal{D}^3x \sqrt{\mathcal{C}^a_{ab} (\mathcal{G}^{cd}(t))} - 1}$$

which gives the variation:

$$\frac{\delta G_\epsilon}{\delta \epsilon} |_{G_\epsilon = 0} = \pm 1 \quad (80)$$

Coming from (78), this gauge choice could be responsible for the absence of the global square root in (34).

In other words, with this gauge choice one can express the action as an energy functional (in the Riemannian geometry sense) as opposed to a length functional:

$$\int ds \| \dot{g} \|^2$$

as opposed to

$$\int ds \| \dot{g} \|$$

The main feature however of the energy functional is that extremals wrt it automatically implement arc-length parametrization as one of the extremum conditions. In other words, since the length and energy functional coincide for arc-length parametrization, and this is implied by the extrema of the energy functional, in the semi-classical approximation one can directly use the energy functional as opposed to the length. In a way, this gauge fixing ‘trades’ global time reparametrization invariance for locality of the action functional.

### D Fadeev-Popov

Morally, the Fadeev-Popov determinant is a simple functional extension of the following argument: for a function with a single root, $f(x_0) = 0$, the Dirac delta function obeys the identity:

$$\delta(f(x)) = \frac{\delta(x - x_0)}{\vert \det f'(x_0) \vert} \quad (81)$$

Since the integral of $\delta(x - x_0)$ over $x$ gives unity, then

$$\vert \det f'(x_0) \vert \int dx \delta(f(x)) = 1$$

The functional setting works morally in the same way.

Let a gauge-field $A^\mu_a(x)$, where $\mu$ are internal indices, $a$ are spacetime ones, and we will suppress both, on which a gauge-group $G$ acts as $A \rightarrow A^g$. For the gauge-fixing $C^a(A) = 0$, we define again a partition of unity by:

$$1 = \Delta_C [A] \int d\mu_\tau(g) \delta C^a(A^g) \quad (82)$$

where $d\mu_\tau$ is the right-invariant measure of the group $G$. The previous equation can in fact be seen as a definition of $\Delta_C [A]$, which implies it is invariant under a gauge transformation, $\Delta_C [A^g] = \Delta_C [A]$. Fadeev-Popov now tells us that an ensuing path integral decomposes into an integral over the gauge-group, and another, gauge-fixed invariant one, containing $\delta(C^a(A)) \Delta_C [A]$.

Now, if $g_o$ is such that $C[A^g] = 0$, i.e., it satisfies the gauge-fixing condition, then following equation (81) we have that for an infinitesimal gauge transformation,

$$\delta g_0 = M_C \cdot \epsilon$$

where $M_C = \frac{\delta C[A^g]}{\delta g_0}$. Now, separating the total derivative along a vertical (i.e. along the gauge-direction) and horizontal (along the gauge-section, infinitesimally), $\delta = \delta_v + \delta_h$ (see [31] and appendix A.2) since then $\delta_h C[A] = 0$, we have:

$$\int d\mu_\tau(g) \delta C^a(A^g) = \det M_C(A) \quad (83)$$

From (83) and (82), if the two measures coincide, we of course obtain $\Delta_C = \det M_C^{-1}$, as before, and thus $\det M_C$ is gauge-invariant. But this is not necessarily the case. Let $\tau^a$ be the canonical generators of the Lie
algebra $g$. Then for any $g$, we have $\Lambda_a(g)$ such that $g = \exp(\Lambda_a \tau^a)$. Then the difference between the left and the right measure is given by

$$d\mu_r(g) = d\mu_l(g) \exp \varpi(g),$$

where $\varpi(g) = \Lambda_b(g)U_b^a$. Then we obtain from (82)

$$1 = \Delta_C[A] \int d\mu_l(g) \exp \varpi(g) \delta C^a(\Lambda^\delta)$$

and thus the correction to the relation between $\det M_C^{-1}$ and $\Delta_C$ becomes:

$$\delta_c \det M_C^{-1} = (1 + \delta_c \Lambda_b(g)U_b^a) \det M_C^{-1}$$

which only vanishes when the structure constants are field independent and traceless. This is our case with the algebra of $C$, but it is not the case with the ADM algebra.

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