Series expansion for the sound field of a ring source

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Abstract

An exact series expansion for the field radiated by a monopole ring source with angular variation in source strength is derived from a previously developed expression for the field from a finite disk. The derived series can be used throughout the field, via the use of a reciprocity relation, and can be readily integrated to find the field radiated by arbitrary circular sources of finite extent, and differentiated to find the field due to higher order sources such as dipoles and quadrupoles.

1 Introduction

Many problems in acoustics are related to the sound radiated or scattered by systems with axial symmetry. These systems include rotors such as cooling fans and aircraft propellers, circular ducts, vibrating bodies such as baffled speakers, and bodies of rotation. In each case, to predict the radiated noise, there is a requirement to compute the field produced by an elemental ring source of given radius, frequency and angular dependence. This calculation is an essential part of many noise prediction methods and there is a need for efficient techniques to perform it.

A second motivation for the study of ring sources is their use as a model problem for propeller and rotor noise. There are a number of approximations to the ring source field, developed, in the main, to examine the nature of the field and its variation with operating parameters[1], or to give a far-field approximation for use in noise prediction[2]. These approximations have proven useful for industrial noise prediction, such as in aircraft propeller noise, and form the basis of many practical prediction techniques.

An approach which does not seem to have found much favour is the use of exact series expansions for the field of the ring source. There are numerous such expansions for disc sources of finite extent with examples covering a number of different configurations[3, 4, 5, 6]. The number of published expansions for a ring source is quite small, however. One is the method of Matviyenko[7] which gives a five term recursion for the ring source of a given azimuthal order. A second, very recent paper, is that of Conway and Cohl[8] which gives series expansions for the ring source, in terms of Bessel and Hankel functions and associated Legendre functions. These series are accurate and easily implemented but they are expressed in terms of modified variables, of the type used in elliptic integral solutions of ring potential problems, or of toroidal type. This makes the series hard to interpret and complicates the issue of differentiating them to find the field due to higher order sources such as the dipoles and quadrupoles employed in rotor noise problems. It also makes it awkward to integrate the series over radius to give an expansion for a finite disc source.

In this paper, we take a previously published[4], quite simple, series for a finite disc source and use it to derive an expansion for the field from a ring source. The derivation depends on routine use of mathematical tables and yields an expansion expressed in physical variables which can, if necessary, be integrated to give a series for the field of a finite source with arbitrary radial variation in source strength.

2 Analysis

The problem to be considered is shown in Figure[1] In cylindrical coordinates \((r, \theta, z)\), we require the field radiated by a ring monopole source at radius \(a\) in the plane \(z = 0\), with source strength \(\exp[\text{jn}\theta]\). The field
for wavenumber $k$ is then $\exp[jn\theta]R_n(k, a, r, z)$:

$$R_n(k, a, r, z) = \int_0^{2\pi} \frac{\exp[ikR'(n\theta_1)]}{4\pi R'} d\theta_1,$$

$$R' = [r^2 + a^2 - 2ra \cos \theta_1 + z^2]^{1/2}. \tag{1}$$

We note that there is a reciprocity relation such that $R_n$ is unchanged if $a$ and $r$ are switched.

A simple, exact series expansion for $R_n$ can now be derived, using previously developed results for a finite disc source. The starting point is the integral expression for the field radiated by a disc source $r \leq a$:

$$I_n(k, a, r, z) = \int_0^a R_n(k, r_1, r, z) r_1 \, dr_1. \tag{2}$$

which has an exact series expansion\(^4\):

$$I_n = (\pi a^2)^{1/2} \left(\frac{-ka}{kR}\right)^{n+1/2} \sum_{m=0}^{\infty} A_m H_{n+2m+1/2}^{(1)}(kR) P_{n+2m}^m(\cos \phi)$$

$$\times {}_1F_2 \left[ \begin{array}{c} n + 2m + 2 \\ 2 \\ \end{array} ; \frac{n + 2m + 4}{2}, n + 2m + 3/2; -\left(\frac{ka}{2}\right)^2 \right] \left(\frac{ka}{2}\right)^{2m+1/2} \tag{3}$$

$$A_m = (-1)^m \frac{2^{2m-1}}{n + 2m + 2} \frac{(2m - 1)!!}{(2n + 2m)!!(2n + 4m - 1)!!}$$

where $R = [r^2 + z^2]$ is the distance of the observer from the origin, $\phi = \tan^{-1} r/z$ is the polar angle of the observer, $H_{\nu}^{(1)}$ is the Hankel function of the first kind of order $\nu$, $P_{n}^m$ is the associated Legendre function, $_1F_2(\cdot)$ is a generalized hypergeometric function\(^9\):

$$1_1F_2(a; b, c; x) = \sum_{n=0}^{\infty} B_n x^n, \tag{4}$$

$$B_n = \frac{(a)_n}{(b)_n(c)_n n!},$$

and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is Pochammer’s symbol\(^10\).

Differentiating with respect to $a$ gives an expression for the field radiated by a ring source of radius $a$:

$$\frac{1}{a} \frac{\partial I_n}{\partial a} = R_n(k, a, r, z). \tag{5}$$
Likewise, differentiating Equation 3:

\[
\frac{\partial I_n}{\partial a} = j^{2n+1} \frac{\pi^{1/2}}{(kR)^{1/2}} \sum_{m=0}^{\infty} A_m H^{(1)}_{n+2m+1/2}(kR) P_n^m(\cos \phi) \frac{(ka)^{2m+n+1}}{2^{2m-1/2}} \\
\times \left[ n + 2m + 2 \frac{1}{1F_2} - \left( \frac{ka}{2} \right)^2 1F_2' \right],
\]

Equation 6 can be rewritten:

\[
\frac{\partial I_n}{\partial a} = j^{2n+1} \frac{\pi^{1/2}}{(kR)^{1/2}} \sum_{m=0}^{\infty} A_m H^{(1)}_{n+2m+1/2}(kR) P_n^m(\cos \phi) \frac{(ka)^{2m+n+1}}{2^{2m+1/2}} \\
\times (n + 2m + 2) 1F_2 \left[ \frac{n + 2m + 4}{2}; \frac{n + 2m + 4}{2}, n + 2m + 3/2; - \left( \frac{ka}{2} \right)^2 \right],
\]

noting the change in the first parameter of the hypergeometric function.

Using the cancellation property of hypergeometric functions:

\[
1F_2(a_1; b_1, b_2; x) a_1 + 1F_2'(a_1; b_1, b_2; x)x = 1F_2(a_1 + 1; b_1, b_2; x)a_1,
\]

Equation 6 can be rewritten:

\[
\frac{\partial I_n}{\partial a} = j^{2n+1} \frac{\pi^{1/2}}{(kR)^{1/2}} \sum_{m=0}^{\infty} A_m H^{(1)}_{n+2m+1/2}(kR) P_n^m(\cos \phi) \frac{(ka)^{2m+n+1}}{2^{2m+1/2}} \\
\times (n + 2m + 2) 1F_2 \left[ \frac{n + 2m + 4}{2}; \frac{n + 2m + 4}{2}, n + 2m + 3/2; - \left( \frac{ka}{2} \right)^2 \right],
\]

noting the change in the first parameter of the hypergeometric function.

Using the cancellation property of hypergeometric functions:

\[
1F_2(a_1; a_1, b_2; x) = aF_1(; b_2; x)
\]

and the relation 11:

\[
0F_1 \left[ ]; \nu + 1; - \left( \frac{z}{2} \right)^2 \right] = \Gamma(\nu + 1) \left( \frac{2}{z} \right)^\nu J_\nu(z),
\]

where \( J_\nu \) is a Bessel function of the first kind,

\[
1F_2 \left[ \frac{n + 2m + 4}{2}; \frac{n + 2m + 4}{2}, n + 2m + 3/2; - \left( \frac{ka}{2} \right)^2 \right] =
\]

\[
\pi^{1/2} \frac{(2n + 4m + 1)!}{2^{n+2m+1}} \left( \frac{2}{ka} \right)^{n+2m+1/2} J_{n+2m+1/2}(ka),
\]

we find an expansion for the field radiated by a ring source:

\[
R_n(k, a) = \frac{1}{a} \frac{\partial I_n}{\partial a} = j^{2n+1} \frac{\pi}{4 (aR)^{1/2}} \left( \frac{1}{(2n + 2m)!} \right) \sum_{m=0}^{\infty} (-1)^m \frac{(2n + 4m + 1)(2m - 1)!}{(2n + 2m)!} \\
\times H^{(1)}_{n+2m+1/2}(kR) P_n^m(\cos \phi) J_{n+2m+1/2}(ka)
\]

Equation 8 is the main result of the paper. It is an exact series expansion for the field radiated by an oscillating ring source of radius \( a \) to any point with \( R \geq a \). If it is required to compute the field at points \( R < a \), this can be done using the reciprocity relation which allows switching of \( r \) and \( a \) in the ring source integral of Equation 5. The expansion is remarkably simple, containing only one special function, \( P_n^m(\cos \phi) \), given that the Bessel and Hankel functions can be evaluated as finite sums of elementary functions 10:

\[
J_{n+1/2}(x) = \left( \frac{2}{\pi x} \right)^{1/2} \sum_{q=0}^{n} \frac{1}{q! (n-q)!} \frac{(n+q)! \cos \left( x - (n-q+1) \pi/2 \right)}{(2x)^n},
\]

\[
H^{(1)}_{n-1/2}(x) = j^{-n} \left( \frac{2}{\pi x} \right)^{1/2} e^{ix} \sum_{q=0}^{n-1} \frac{(-1)^q}{q!} \frac{(n+q-1)!}{(2ix)^q}.
\]

We further note that this expansion has a form very similar to that of the expansion of the Helmholtz Green’s function for a point source using the “summation theorem” for Bessel functions 10.
3 Results and applications

As a check on the accuracy of Equation 8, some calculations were performed for arbitrary values of the parameters and compared to direct numerical evaluation of $R_n$. Figure 2 shows the real and imaginary parts of $R_n$ computed numerically and using the series expansion for $k = 10$, $n = 7$, $a = 2^{1/2}$, along a ray $0 \leq R \leq 4$, $\phi = \pi/6$. For points $R < a$, the series was evaluated employing the reciprocity relation and switching $r = R \sin \phi$ and $a$. As is clear from the plots, the two results are identical and the series is seen to be an accurate method of evaluating the field. Figure 3 shows data for the same case but in the source plane, $\phi = \pi/2$. Again, the results are practically identical to those from numerical evaluation. It is worth noting that the singularity near the source radius $a = 2^{1/2}$ has been accurately captured. In the discussion of their series expansion, Conway and Cohl[8] note that one of their series, which uses Hankel functions, is accurate far from the ring but not close to it, whereas their Legendre function series is accurate near the ring, since the Legendre function has a built-in singularity there, but is slow to converge in the far field. Equation 8, being based on both Hankel and Legendre functions, is able to capture the behaviour in both regions and is accurate and rapidly convergent over the full range considered.
### 3.1 Finite disc source

A major application of a ring source evaluation method is in rotor acoustics where the radiated field is given by integrals of the form:

\[ p_n(k, a, r, z) = \int_0^a s(r_1) R_n(k, r_1, r, z) r_1 \, dr_1, \quad (9) \]

where \( s(r_1) \) is a radial source function whose value depends on the rotor geometry and/or loading.

For points lying outside the sphere containing a rotor of radius \( a \), substitution of Equation 8 into Equation 9 gives a series expansion for the acoustic field of the rotor:

\[
p_n = j^{2n+1} \frac{\pi}{4} R^{1/2} \frac{1}{R^{1/2}} \sum_{m=0}^{\infty} (-1)^m \frac{(2n + 4m + 1)(2m - 1)!!}{(2n + 2m)!!} 
\times H_{n+2m+1/2}^{(1)}(kR) P_n^{2m}(\cos \phi) s_{n+2m}, \quad (10)\]

where \( s_{n+2m} = \int_0^a s(r_1) J_{n+2m+1/2}(kr_1) r_1^{1/2} \, dr_1, \)

so that the coefficients \( s_{n+2m} \) are given by a Hankel transform of the radial source term. It has been known for many years that the far-field noise from a rotor is given by a Hankel transform of the radial source term\([2, 12]\), but using a single Bessel function of integer order rather than those of order integer plus one half used in Equation 10 which is exact at all points in the field \( R > a \), including the near field.

### 4 Conclusions

A simple exact series expansion for the acoustic field radiated by a monopole ring source has been developed, derived from a previous result for a finite disc. The series has been tested numerically and compared to another recently published expansion for the Green’s function for a Helmholtz problem in cylindrical coordinates. Since it is based on physical variables, the series is easily integrated to give an expansion for finite sources, such as rotors, with arbitrary radial variation in source strength, and is also easily differentiated to find the fields due to higher order sources such as dipoles and quadrupoles.

### References

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