ON THE discrete mean of the derivative of Hardy’s Z-function

HIROTAKA KOBAYASHI

abstract. We consider the sum of the square of the derivative of Hardy’s Z-function over the zeros of Hardy’s Z-function. If the Riemann Hypothesis is true, it is equal to the sum of $|ζ’(ρ)|^2$, where $ρ$ runs over the zeros of the Riemann zeta-function. In 1984, Gonek obtained an asymptotic formula for the sum. In this paper we prove a sharper formula. This result was obtained by Milinovich with a better error term.

1. update: 21/11/2018

This result was proved by Milinovich [4] in his PhD thesis in 2008. Moreover he obtained a better error term. He used $ζ’(s)$, but we considered $Z'(t)$.

2. Introduction

In this paper, we discuss the discrete mean of $Z'(t)$ over the zeros of $Z(t)$, where $Z(t)$ is Hardy’s Z-function. We denote the complex variable by $s = σ + it$ with $σ, t ∈ ℝ$. Throughout this article, we assume that the Riemann Hypothesis (RH) is true, and then the mean-value corresponds to that of $ζ’(ρ)$, where $ρ$ runs over the zeros of the Riemann zeta-function $ζ(s)$.

In 1984, Gonek [2] stated that

\[ \sum_{0<γ≤T} |ζ’(ρ)|^2 = \frac{1}{24π} T \log^4 \frac{T}{2π} + O(T \log^3 T), \]

where $ρ = \frac{1}{2} + iγ$ are the zeros of $ζ(s)$.

On the other hand, Conrey and Ghosh [1] showed that

\[ \sum_{0<γ≤T} \max_{γ<t≤γ^+} \left| ζ \left( \frac{1}{2} + it \right) \right|^2 \sim \frac{e^2 - 5}{4π} T \log^2 \frac{T}{2π}, \]

where $γ$ and $γ^+$ are ordinates of consecutive zeros of $ζ(s)$. This summand means the extremal value of Hardy’s Z-function. They calculated
the integral
\[ \frac{1}{2\pi i} \int_C \frac{Z_1'(s)\zeta(s)\zeta(1-s)}{Z_1(s)} ds, \]
where \( C \) is positively oriented rectangular path with vertices \( c + i, c + iT, 1 - c + iT \) and \( 1 - c + i \) where \( c = \frac{5}{8} \), and \( Z_1(s) \) is defined by

\[ Z_1(s) := \zeta'(s) - \frac{1}{2} \omega(s) \zeta(s) \]

with
\[ \omega(s) = \frac{\chi'(s)}{\chi(s)}, \]
where \( \chi(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \).

**Remark 1.** Actually, they considered the rectangle whose imaginary part is \( T \leq \Im s \leq T + T^{3/4} \) for convenience.

In this article, by considering the integral
\[ \frac{1}{2\pi i} \int_C \frac{\zeta'(s)Z_1(s)Z_1(1-s)}{\zeta(s)} ds, \]
we will prove the following theorem.

**Theorem 2.1.** If the RH is true, then for any sufficiently large \( T \),

\[ \sum_{0 < \gamma \leq T} |Z'\gamma)|^2 = \frac{1}{24\pi} T \log^4 \frac{T}{2\pi} + \frac{2\gamma_0 - 1}{6\pi} T \log^3 \frac{T}{2\pi} + a_1 T \log^2 \frac{T}{2\pi} + a_2 T \log \frac{T}{2\pi} + a_3 T + O(T^\frac{3}{2} \log^2 T), \]

where \( \gamma_0 \) is the Euler constant and \( a_i \) (\( i = 1, 2, 3 \)) are constants that can be explicitly expressed by the Stieltjes constants and the summation is over the zeros \( \gamma \) of \( Z(t) \) with the multiplicity.

Actually, the sum on the left-hand side is equal to the sum on the left-hand side of (1). Hence this theorem is an improvement on Gonek's result.

In the proof, we apply Hall's result [3] that for each \( k = 0, 1, 2, \ldots \), and any sufficiently large \( T \),

\[ \int_0^T Z^{(k)}(t)^2 dt = \frac{1}{4^k (2k+1)} TP_{2k+1} \left( \log \frac{T}{2\pi} \right) + O(T^{\frac{3}{2}} \log^{2k+\frac{3}{2}} T), \]

where \( P_{2k+1}(x) \) is the monic polynomial of degree \( 2k + 1 \) given by

\[ P_{2k+1}(x) = W_{2k+1}(x) + (4k + 2) \sum_{h=0}^{2k} \binom{2k}{h} (-2)^h \gamma_h W_{2k-h}(x), \]
in which

\[ W_g(v) = \frac{1}{e^v} \int_0^{e^v} \log^g u \, du, \quad \zeta(s) = \frac{1}{s - 1} + \sum_{h=0}^{\infty} \frac{(-1)^h \gamma_h}{h!} (s - 1)^h. \]

The \( \gamma_h \) are called the Stieltjes constants.

3. Preliminary lemmas

We prepare some lemmas for our proof.

**Lemma 3.1.** \( Z_1(s) \) has the following properties.

(i) If \( s = \frac{1}{2} + it \), then \( |Z_1(s)| = |Z'(t)| \).

(ii) \( Z_1(s) \) satisfies the functional equation \( -Z_1(s) = \chi(s) Z_1(1 - s) \) for all \( s \).

*Proof.* See the proof of the lemma in [1]. □

By Stirling’s formula, we can show that

**Lemma 3.2.** For \( |\arg s| < \pi \) and \( t \geq 1 \), we have

(6) \( \chi(1 - s) = e^{-\frac{\pi}{4} \left( \frac{t}{2\pi} \right)^{\sigma - \frac{1}{2}}} \exp \left( it \log \frac{t}{2\pi e} \right) \left( 1 + O \left( \frac{1}{t} \right) \right) \)

and

(7) \( \frac{\chi'}{\chi}(s) = -\log \frac{t}{2\pi} + O \left( \frac{1}{t} \right) \).

If RH is true, then the Lindelöf Hypothesis is also true. Therefore we can obtain the following estimates.

**Lemma 3.3.** If the RH is true, then for \( |t| \geq 1 \),

\[ \zeta(s) \ll \begin{cases} 1 & 1 < \sigma, \\ |t|^\varepsilon & \frac{1}{2} \leq \sigma \leq 1, \\ |t|^{\frac{1}{2} - \sigma + \varepsilon} & \sigma < \frac{1}{2}. \end{cases} \]

As in the paper of Conrey and Ghosh [1], we apply the following lemma by Gonek [2],

**Lemma 3.4** (Gonek). Let \( \{b_n\}_{n=1}^{\infty} \) be a sequence of complex numbers such that \( b_n \ll n^\varepsilon \) for any \( \varepsilon > 0 \). Let \( a > 1 \) and let \( m \) be a non-negative integer. Then for any sufficiently large \( T \),

\[ \frac{1}{2\pi} \int_1^T \left( \sum_{n=1}^{\infty} b_n n^{-a - it} \right) \chi(1 - a - it) \left( \log \frac{t}{2\pi} \right)^m \, dt = \sum_{1 \leq n \leq T/2\pi} b_n (\log n)^m + O(T^{a - \frac{1}{2}} (\log T)^m). \]
4. The proof of the theorem

It is sufficient to prove the theorem in the case where $T$ does not coincide with the ordinate of the zeros of $\zeta(s)$. By the assumption of RH, we can consider $m|Z'(\gamma)|^2$ as the residue of $\frac{\zeta'}{\zeta}(s)Z_1(s)Z_1(1-s)$ at the zero $\frac{1}{2} + i\gamma$ of the Riemann $\zeta$-function, where $m$ is the multiplicity of the zero. Hence when we denote our sum by $M(T)$, by the residue theorem, we see that

$$M(T) = \frac{1}{2\pi i} \int_C \frac{\zeta'}{\zeta}(s)Z_1(s)Z_1(1-s)ds.$$  

By Lemma 3.3 the integral on the horizontal line can be estimated as $O(T^{\frac{1}{2}+\varepsilon})$. Therefore it satisfies that

$$M(T) = \frac{1}{2\pi i} \int_{c+i}^{c+iT} \frac{\zeta'}{\zeta}(s)Z_1(s)Z_1(1-s)ds + \frac{1}{2\pi i} \int_{1-c+i}^{1-c+iT} \frac{\zeta'}{\zeta}(s)Z_1(s)Z_1(1-s)ds + O(T^{\frac{1}{2}+\varepsilon})$$  

$$= I_1 + I_2 + O(T^{\frac{1}{2}+\varepsilon}),$$

say. On the integral $I_2$,

$$I_2 = -\frac{1}{2\pi i} \int_{1-c+i}^{1-c+iT} \frac{\zeta'}{\zeta}(s)Z_1(s)Z_1(1-s)ds$$  

$$= -\frac{1}{2\pi i} \int_{1-c+i}^{1-c+iT} \left( \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(1-s) \right) Z_1(s)Z_1(1-s)ds$$  

$$= -\frac{1}{2\pi i} \int_{1-c+i}^{1-c+iT} \frac{\chi'}{\chi}(s)Z_1(s)Z_1(1-s)ds$$  

$$+ \frac{1}{2\pi i} \int_{1-c+i}^{1-c+iT} \frac{\zeta'}{\zeta}(1-s)Z_1(s)Z_1(1-s)ds.$$  

When we replace $s$ by $1-s$, the second integral is

$$-\frac{1}{2\pi i} \int_{c-i}^{c-iT} \frac{\zeta'}{\zeta}(s)Z_1(s)Z_1(1-s)ds = \overline{T}_1.$$  

Now we see that

$$M(T) = -\frac{1}{2\pi i} \int_{1-c+i}^{1-c+iT} \frac{\chi'}{\chi}(s)Z_1(s)Z_1(1-s)ds + 2\Re I_1 + O(T^{\frac{1}{2}+\varepsilon}).$$  

By Cauchy’s integral theorem, the first integral is equal to

$$-\frac{1}{2\pi i} \int_{\frac{1}{2}+i}^{\frac{1}{2}+iT} \frac{\chi'}{\chi}(s)Z_1(s)Z_1(1-s)ds + O(T^{\frac{1}{2}+\varepsilon}).$$
This error term is derived from the integral on the horizontal line. By (7) and Lemma 3.3 we see that the above integral is

\[
\frac{1}{2\pi} \int_1^T \log \frac{t}{2\pi} Z'(t)^2 dt + O(T^\varepsilon).
\]

Therefore when we put

\[
I(t) = \int_1^t Z'(x)^2 dx,
\]

using integration by parts and Hall’s result, we can show that the integral in (8) is equal to

\[
\frac{1}{2\pi} \log \frac{T}{2\pi} I(T) - \frac{1}{2\pi} \int_1^T t^{-1} I(t) dt
\]

\[
= \frac{1}{24\pi} T \log \frac{T}{2\pi} P_3 \left( \frac{\log \frac{T}{2\pi}}{2\pi} \right)
\]

\[
- \frac{1}{24\pi} \int_1^T P_3 \left( \frac{\log \frac{t}{2\pi}}{2\pi} \right) dt + O(T^{\frac{3}{4}} \log^{\frac{3}{2}} T),
\]

and \(P_3(x)\) is explicitly written as

\[
P_3(x) = x^3 + 3(2\gamma_0 - 1)x^2
\]

\[
- 6(2\gamma_0 + 4\gamma_1 - 1)x + 6(2\gamma_0 + 4\gamma_1 + 4\gamma_2 - 1).
\]

Then we find

\[
- \frac{1}{2\pi i} \int_{1-c+i}^{1-c+iT} \frac{\chi'(s)}{\chi} Z_1(s) Z_1(1-s) ds
\]

\[
= \frac{1}{24\pi} T \log^4 \frac{T}{2\pi} + \frac{3\gamma_0 - 2}{12\pi} T \log^3 \frac{T}{2\pi}
\]

\[
+ b_1 T \log^2 \frac{T}{2\pi} + b_2 T \log \frac{T}{2\pi} + b_3 T + O(T^{\frac{3}{4}} \log^{\frac{3}{2}} T),
\]

where \(b_i (i = 1, 2, 3)\) are constants derived from the coefficients of (9).
We calculate $I_1$ to complete the proof. By Cauchy’s integral theorem,

$$I_1 = -\frac{1}{2\pi i} \int_{b+iT}^{b+iT} \frac{\zeta'}{\zeta}(s)\chi(1-s)Z_1(s)^2 ds + O(T^{b-\frac{1}{2}+\varepsilon})$$

$$= -\frac{1}{2\pi i} \int_{b+i}^{b+iT} \frac{\zeta'}{\zeta}(s)\chi(1-s)\zeta'(s)^2 ds$$

$$+ \frac{1}{2\pi i} \int_{b+i}^{b+iT} \omega(s)\zeta'(s)^2\chi(1-s) ds$$

$$- \frac{1}{8\pi i} \int_{b+i}^{b+iT} \omega(s)^2\zeta(s)\zeta'(s)\chi(1-s) ds + O(T^{b-\frac{1}{2}+\varepsilon})$$

$$= I_I + I_{II} + I_{III} + O(T^{b-\frac{1}{2}+\varepsilon}),$$

say, where $b = \frac{9}{2}$ and the error term is derived from the integral on the horizontal line.

By applying Lemma 3.4,

$$I_I = \frac{1}{2\pi i} \int_{b+i}^{b+iT} \left( \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s} \right) \left( \sum_{n=1}^{\infty} \frac{D(n)}{n^s} \right) \chi(1-s) ds$$

$$= \frac{1}{2\pi} \int_{1}^{T} \left( \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^{b+iT}} \right) \left( \sum_{n=1}^{\infty} \frac{D(n)}{n^{b+iT}} \right) \chi(1-b-it) dt$$

$$= \sum_{1 \leq mn \leq \frac{T}{2\pi}} \Lambda(m)D(n) + O(T^{b-\frac{1}{2}}),$$

where

$$D(n) = \sum_{d|n} \log d \log \frac{n}{d}.$$ 

Using Perron’s formula and the residue theorem,

$$\sum_{mn \leq x} \Lambda(m)D(n) = -\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{\zeta'}{\zeta}(s)\zeta'(s)^2 \frac{x^s}{s} ds + O(x^{\varepsilon}) + R$$

$$= -\text{Res}_{s=1} \frac{\zeta'}{\zeta}(s)\zeta'(s)^2 \frac{x^s}{s} ds - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta'}{\zeta}(s)\zeta'(s)^2 \frac{x^s}{s} ds$$

$$+ \frac{1}{2\pi i} \int_{b+iT}^{c+iT} \frac{\zeta'}{\zeta}(s)\zeta'(s)^2 \frac{x^s}{s} ds$$

$$+ \frac{1}{2\pi i} \int_{c-iT}^{b-iT} \frac{\zeta'}{\zeta}(s)\zeta'(s)^2 \frac{x^s}{s} ds + O(x^{\varepsilon}) + R$$

$$= -\text{Res}_{s=1} \frac{\zeta'}{\zeta}(s)\zeta'(s)^2 \frac{x^s}{s} + O(x^{\varepsilon}cT^c + x^{bT-1+\varepsilon}) + R,$$
where $R$ is the error term appearing in Perron’s formula (see [5, p.139]) and satisfies that

$$
R = \frac{1}{\pi} \sum_{x/2 < mn < x} \Lambda(m)D(n) \text{ si } \left( T \log \frac{x}{mn} \right) - \frac{1}{\pi} \sum_{x < mn < 2x} \Lambda(m)D(n) \text{ si } \left( T \log \frac{x}{mn} \right) + O \left( \frac{(4x)^b}{T} \sum_{mn=1}^{\infty} \frac{|\Lambda(m)D(n)|}{(mn)^b} \right)
$$

$$
\ll \sum_{x/2 < mn < 2x \atop mn \neq x} |\Lambda(m)D(n)| \min \left( 1, \frac{x}{T|x-mn|} \right) + (4x)^b \sum_{mn=1}^{\infty} \frac{|\Lambda(m)D(n)|}{(mn)^b}
$$

with

$$
\text{si}(x) = - \int_{x}^{\infty} \frac{\sin u}{u} du.
$$

By introducing the Dirichlet convolution and considering $x$ as a half integer, the error term $R$ can be estimated as follows;

$$
R \ll \frac{x}{T} \sum_{x/2 < n < 2x} \left| \frac{\Lambda * D(n)}{x-n} \right| + \frac{(4x)^b}{T} \sum_{n=1}^{\infty} \frac{|\Lambda * D(n)|}{n^b}
$$

$$
\ll \frac{x^\epsilon}{T} \sum_{x/2 < n < 2x} \left| \frac{1}{1-n/x} \right| + \frac{x^b}{T}
$$

$$
\ll \frac{x^\epsilon}{T} \log x + \frac{x^b}{T}.
$$

Therefore we obtain

$$
\sum_{mn \leq x} \Lambda(m)D(n) = - \text{Res}_{s=1} \zeta'(s) \zeta(s)^2 \frac{x^s}{s} + O(x^bT^{-1+\epsilon} + x^cT^c).
$$

This residue is

$$
- \frac{1}{4!} x (\log^4 x - 4 \log^3 x + 12 \log^2 x - 24 \log x + 24)
$$

$$
+ \frac{\eta_0}{3!} x (\log^3 x - 3 \log^2 x + 6 \log x - 6)
$$

$$
+ \left( \eta_1 + \frac{\eta_1}{2} \right) x (\log^2 x - 2 \log x + 2)
$$

$$
+ (\eta_2 + 4 \gamma_2 - 2 \gamma_1 \eta_0) x (\log x - 1)
$$

$$
+ (\eta_3 + 6 \gamma_3 - \gamma_1^2 - 2 \eta_1 \gamma_1) x,
$$

where $\eta_k$ are defined by

$$
\frac{\zeta'}{\zeta}(s) = - \frac{1}{s-1} + \sum_{k=0}^{\infty} \eta_k (s-1)^k
$$
and can be expressed by the Stieltjes constants. The other integrals can be calculated in the same way. Taking $x = \frac{T}{2\pi}$, we obtain

$$2\Re I_1 = \frac{\gamma_0}{12\pi} T \log^3 \frac{T}{2\pi} + c_1 T \log^2 \frac{T}{2\pi} + c_2 T \log \frac{T}{2\pi} + c_3 T + O(T^{c+\varepsilon}),$$

where $c_i$ ($i = 1, 2, 3$) can be expressed by the Stieltjes constants. Hence

$$M(T) = \frac{1}{24\pi} T \log^4 \frac{T}{2\pi} + \frac{2\gamma_0 - 1}{6\pi} T \log^3 \frac{T}{2\pi} + a_1 T \log^2 \frac{T}{2\pi} + a_2 T \log \frac{T}{2\pi} + a_3 T + O(T^{3/4} \log^{7/2} T),$$

where $a_i$ ($i = 1, 2, 3$) is the sum of $b_i$ and $c_i$, and the proof is completed.

ACKNOWLEDGEMENT

I would like to thank my supervisor Professor Kohji Matsumoto for useful advice. I am grateful to the seminar members for indications by which my argument can be sophisticated. I would also like to thank Micah Baruch Milinovich of making me aware of his result.

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Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya 464-8602, Japan
E-mail address: m17011z@math.nagoya-u.ac.jp