Passivity and Immersion (P&I) Approach With Gaussian Process for Stabilization and Control of Nonlinear Systems

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ABSTRACT The virtual derivatives computation and successive derivations of virtual inputs in an adaptive backstepping controller cause the explosion of complexity. Moreover, the feedback linearization has poor robustness features and necessitates exact estimation of the feedback control law’s coefficients. Due to measurement noise, the model-based estimation techniques for identifying uncertainties result in inaccurate gradient and Hessian calculations. Such limitations lead to model and measurement uncertainties that prevent effective stabilization and control of nonlinear systems. Machine learning-based data-driven approaches offer effective tools for identifying dynamical systems and uncertainties with minimal prior knowledge of the model structure. Therefore, the contribution of this research is two-fold: First, the general controller design theory is proposed which utilizes the idea of an invariant target manifold giving rise to a non-degenerate two form, through which the definition of certain passive outputs and storage functions leads to a generation of control law for stabilizing the system. Since the above concepts are linked with the Immersion and Invariance (I&I) design policy and the passivity theory of controller design, the proposed methodology is labeled as the “Passivity and Immersion (P&I) based approach”. Second, the proposed P&I approach is integrated with a Bayesian nonparametric approach, particularly the Gaussian Process for stabilization and control of the partially unknown nonlinear systems. The effectiveness of the proposed methodologies has been evaluated on an inverted pendulum using MATLAB in the presence of input-output uncertainties.

INDEX TERMS Feedback linearizable structure, Gaussian process regression, immersion and invariance, stabilization and control, uncertainties.

I. INTRODUCTION

A. EXISTING METHODOLOGIES OF STABILIZATION AND CONTROL OF FULLY KNOWN DYNAMICS

Stabilization problems intend to build a control system that stabilizes the states of a closed-loop system around an equilibrium point [1], [2]. Most of the analyzes, syntheses, and stabilization problems in nonlinear systems are tackled using the control Lyapunov functions (CLF) [3]. The stabilizing feedback designs can be easily achieved for a broader class of nonlinear systems with comparatively straightforward computations using identified CLF [4]. The Lyapunov functions are substituted with storage functions in systems with Lagrangian or Hamiltonian structures, with passivity being the desired feature [5]. The passivity-based control is restricted to systems with a relative degree of one. The classical Backstepping [6], [7] and forwarding [8] are utilized to develop sequentially designed feedback control laws for the global stabilization of nonlinear feedback and feedforward systems.
The original Backstepping (BS) methodology [7], [8] enforces the Lyapunov function with the design of feedback control, enables a recursive way of constructing a sequence of virtual systems of relative degree one, reduces the relative degree by one by choosing a virtual input, and gets the final control law in the end. For a recursive method based on BS method [6], a controller design approach proposed in [9] and [10] enforcing incremental stability as well as incremental Lyapunov function. The BS approach overcomes the relative degree of restriction. However, the calculations (i.e., virtual derivatives and successive derivations of virtual control inputs) required in the computation of the resultant nonlinear control law for stabilization, a computer’s involvement in executing the control signal calculation is inevitable [11].

The above-mentioned control design approaches are best suited to the well-known class of systems [8]. Several alternative approaches to the above methods based on Lyapunov constructions for cascade systems have been developed by treating a cascade system as a collection of subsystems. The issue, in general, is that while each subsystem may be independently stable, their interconnection might cause the entire cascade to become unstable [12], [13].

The classical Immersion and Invariance (I&I) methodology proposed in [14] is a relatively recent strategy to develop the controllers for nonlinear systems without the requirement of a control Lyapunov function in the control law design phase. It is shown that a dynamical system is stable if it can be immersed into a stable system through a so-called preserving mapping [15]. This control approach exploits the differential geometry theory to derive the (invariant) output zeroing manifold and then immerses the specified plant dynamics in a strictly lower-order target dynamical system that captures the desired behavior. A feedback law is then developed to guarantee the state trajectories’ convergence to the equilibrium, boundedness, and implicit manifold (off-the-manifold) convergence to zero.

The last crucial step of I&I ensures the manifold’s attractivity and its internal dynamics incorporate a replica of the desired (targeted) system behavior. There is no systematic approach for developing the feedback law when designing controllers using the classical I&I procedure for practical applications [15], [16]. To perform this step in a structured manner, the authors in [16] and [15] revisited the classical I&I approach by utilizing contraction theory and horizontal contraction concepts to get a control law and geometric conditions that ensure the attractiveness of the desired manifold. In I&I horizontal contraction procedure, the selection of $\Xi(x) = \Theta(x)\Theta(x)$ (refer Ex. 3 of [15]) for the Finsler Lyapunov Function (FLF) is not defined properly. Moreover, solving a matrix inequality requires certain lengthy and tedious calculations.

According to contraction theory, if a system has the Incremental Lyapunov Function and Contraction Metrics (CM) of the differential state, all NLS trajectories converge incrementally and exponentially to one single trajectory, irrespective of the initial conditions [17]. The CM and associated Lyapunov function for systems with suitable structures, such as Lagrangian and feedback linearizable systems, can be determined analytically. Several strategies for identifying CM for general NLS exploit the linear matrix inequality aspect. The authors in [4] proposed an approach for finding the CM using Sum-of-Squares (SOS) programming via the solver Mosek and the parser YALMIP [18]. A convex optimization-based steady-state tracking error minimization is proposed in [19] to identify a CM that minimizes an upper bound of the steady-state distance between unperturbed and perturbed system trajectories. However, deriving evaluation criteria and a systematic form of CM for general NLS is a tedious and long-drawn-out process. To overcome the above-mentioned issues and challenges, a constructive and systematic strategy with more apparent degrees of freedom to achieve the stabilization and control of the NLS is proposed as the first contribution in Subsection I-C.

B. LITERATURE REVIEW ON STABILIZATION AND CONTROL OF THE PARTIALLY UNKNOWN SYSTEMS

Feedback linearization enables the use of a number of well-known linear control schemes to control real-world nonlinear plants. The system is explicitly linearized by simplifying its nonlinearity via feedback linearization. However, this technique is not robust for time-varying and uncertain nonlinearity. Furthermore, the needed performance is given by weighting various variables in several linear control laws. The precise estimation or identification of the nonlinear system is essential for implementing feedback linearization.

Despite parametric uncertainties and variations, the adaptive control techniques [20], [21] can accommodate unmodeled dynamical structures utilizing an online parameter identification method for unknown coefficients. Under an appropriate control law, the MRAC procedures [20], [21] provide the stability of the closed-loop system (CLS) and asymptotic convergence of the output tracking error to zero. Furthermore, the approximate feedback linearization-based MRAC (AMI-MRAC) [22] enables adaptive controllers to be designed for a broader class of nonlinear systems [23]. In the literature, different MRAC models with parametric approaches such as CL-based [24], DREM-based MRAC [25], [26], and composite learning-based MRAC [27] have been proposed. However, when control is applied to very complex systems, these MRAC models based on the parametric approaches become more challenging, especially when the dynamics cannot be determined using the first principle methods.

The RBFN-based MRAC has grown more popular than multilayer perceptron NN because of its LIP characteristics and ability to capture uncertainty. But, its accuracy greatly depends on the choice of RBF centers [28]. Moreover, the RBFN-based MRAC is only locally effective as the approximation holds within the estimated operating domain. The ANN-based MRAC, which utilizes feedforward networks to approximate unknown functions, exhibits a poor
transient response [29]. The GP as a Bayesian nonparametric regression
- has a connection with kernel filtering methods through an RKHS
- does not need any prior knowledge of an unknown term and uncertainty
- handles the stochastic measurement noise inherently
- removes long-standing assumptions on the bounded domain of operation.

The GP-MRAC is proposed in [22] to address the aforementioned limitations of MRAC based on the parametric approaches, ANN-based MRAC, and RBFN-based MRAC. A systematic approach via GP-based P&I approach is proposed as a second contribution in Subsection I-C.

C. CONTRIBUTION THROUGH THE P&I APPROACH FOR STABILIZATION AND CONTROL

One of the significant contributions of the paper is in developing the P&I approach for accommodating systems in suitably structured (feedback, feedforward, feedback linearizable form, etc.,) and unstructured forms (without any particular system form). To make this happen, the controller design starts with blending the classical I&I [14], [15], [16] with the concept of the generation of a suitable passive output and corresponding storage function. The general controller design theory is proposed which utilizes the idea of an invariant target manifold giving rise to a non-degenerate two form, through which the definition of certain passive outputs and storage functions leads to a generation of control law and corresponding storage function. The general controller design theory is proposed which utilizes the idea of an invariant target manifold giving rise to a non-degenerate two form, through which the definition of certain passive outputs and storage functions leads to a generation of control law and corresponding storage function. The general controller design theory is proposed which utilizes the idea of an invariant target manifold giving rise to a non-degenerate two form, through which the definition of certain passive outputs and storage functions leads to a generation of control law and corresponding storage function.

II. PROBLEM FORMULATION

The normal form of a nonlinear system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
& \vdots \\
\dot{x}_n &= \psi(x) + \phi(x)u, \\
x(0) &= 0, \quad (1)
\end{align*}
\]

with single input \(u \in U \subseteq \mathbb{R}\), state-feedback linearizable structure, \(x = [x_1, x_2, \ldots, x_n]^T \in X \subseteq \mathbb{R}^n\) is considered with the following assumptions.

**Assumption 1:** The measurement of state vector \(x\) is accessible for control and estimation. Moreover, the noisy measurement data \(y = (\dot{x}_n - \phi(x)u) + \alpha \sim \mathcal{N}(0, \rho^2)\) is available.

**Assumption 2:** The functions \(\psi(x)\) and \(\phi(x)\) are bounded, known, and infinitely differentiable.

**Assumption 3:** The relative degree of the system is equal to \(n\) i.e., system order.

In Assumption 1, the time derivatives can be obtained through finite differences. The approximation error can be considered as a part of measurement noise in \(\alpha\). Assumption 2 and 3 are related to the feedback linearization and \(\phi(x) \neq 0\) for all \(x \in X\) holds since the relative degree can not be defined for \(\phi(x) = 0\).

**Remark 1:** Consider a nonlinear system

\[
\begin{align*}
\dot{x} &= \lambda(x) + \gamma(x)u = F(x, u, \Theta_p) \\
y &= \xi(x) \quad (2)
\end{align*}
\]
with the system states $x \in \mathbb{R}^n$, input $u \in \mathbb{R}$, output $y \in \mathbb{R}$, and SISO. With the proper change of coordinates [30]

$$x = \Gamma(x) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \xi(x) \\ L_{\lambda} \xi(x) \\ \vdots \\ L_{\lambda}^{n-1} \xi(x) \end{bmatrix},$$  

(3)

the nonlinear system (2) is transformed into a normal form

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_n &= L_{\lambda}^0 \xi(x) + L_{\rho} L_{\lambda}^{n-1} \xi(x) u. 
\end{align*}$$

(4)

The form (1) is recovered from (2) with diffeomorphism mapping $\Gamma(x)$, $L_{\lambda}^0 \xi(x) = \psi(x)$, and $L_{\rho} L_{\lambda}^{n-1} \xi(x) = \phi(x)$ [30].

Remark 2: An inverted pendulum [24], [25], [29], wing rock dynamics [22], Van der Pol equation [31] and an adapted pendulum system with a sigmoidal function [23] are examples of systems in normal form. This paper focuses on the control and stabilization of systems with feedback linearizable structures (1).

The first objective is to design control law systematically to stabilize and control the physical systems. The stabilization problem for a system (1) intends to build a control system via feedback law $u = u(x)$ that stabilizes the states of a closed-loop system around an equilibrium point $x^* = 0$. The control objective is to design a feedback control law that globally asymptotically converges the system states to the desired one. A constructive and systematic strategy for the stabilization of the control system’s equilibrium point based on the generation of an appropriate passive output and related storage function is proposed.

The second objective is to estimate the unknown function $\psi(x)$ through $\hat{\psi}(x)$. The aim is to utilize this approximated function $\hat{\psi}(x)$ in the proposed control law.

Assumption 4: To estimate the unknown term using GP, it is assumed that a controller $u = u_{\text{data}}$ exists that maintains the system trajectories uniformly bounded for a finite duration. Thus, $\exists T > 0$, $\theta > 0$ such that $\|x(t)\| < \theta$, $\forall t \in [0, T]$.

The implementation of such controllers for GP training is a milder assumption because it simply needs to ensure finite-time boundedness. The data set obtained using a controller $u_{\text{data}}$ is represented as

$$D = \left\{x^1, y^1\right\}_{i=1}^N.$$

(5)

The generalized idea for the proposed controller with GP for a partially unknown system is summarized as:

- The data set (5) is collected using controller $u_{\text{data}}$ by simulating the system at initial condition $x(0)$.
- With the help of Assumption 1, the GP regression (explained in Section IV) is performed to get $\hat{\psi}(x)$ as a identified version of $\psi(x)$.
- The proposed controller (described in Section III) is employed instead of $u_{\text{data}}$ to stabilize and control the given system.

The proposed P&I approach is elaborated in Section III systematically to meet the first objective. The control and stabilization of the partially unknown system are provided in Section IV to address the second objective.

### III. PROPOSED PASSIVITY AND IMMERSION (P&I) BASED APPROACH

Building ideas from the classical I&I and I&I horizontal contraction procedure, a constructive approach for system stabilization of the equilibrium point of the control system based on the generation of a suitable passive output and corresponding storage function is proposed. This involves a particular structure of tangent bundle associated with a suitable Pseudo-Riemannian/semi-Riemannian structure imposed on the control system. The procedural steps of the proposed P&I approach is divided into following four subsections (i.e., A, B, C, and D).

#### A. CONSTRUCTION OF THE IMPLICIT MANIFOLD AND A RIEMANNIAN METRIC

For example, given the system

$$\begin{align*}
\dot{x} &= f(x, \lambda) \\
\dot{\lambda} &= u
\end{align*}$$

(6)

with $(x, \lambda) \in (\mathbb{R}^{n-1}, \mathbb{R})$. For a given system,

- it is assumed that there exists a $C^\infty$ mapping $\tau(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the subsystem $\dot{x} = f(x, \tau(x))$ has a GES/GAS equilibrium at the origin.
- The function $\lambda = \tau(x)$ to get exponentially stable subsystem $\dot{x} = -x$ as it render the implicit manifold $\Upsilon(x, \lambda) = \lambda - \tau(x) = 0$ and immersion dynamics $\dot{\eta} = -\eta$ with $x = \eta$ [32], [33].

#### B. TANGENT SPACE STRUCTURE FOR CONTROL SYSTEMS

Consider an $n$-dimensional manifold $M$ with tangent bundle $T_M$, such that all $p \in M$, $T_pM$ has the following structure

$$T_pM = H_p \oplus V_p : H_p \cap V_p = 0.$$  

(7)

If $H_p$ horizontal space and $V_p$ the vertical space are considered then at all $p \in M$, $T_pM$ is direct sum of $H_p$ and $V_p$. If $M$ is coordinatized as $(x, \lambda)$ with $x \in \mathbb{R}^k$, $\lambda \in \mathbb{R}^{n-k}$, and $k < n$ then $T_pM$ for any $p \in M$ is written as

$$T_pM = H_p \oplus V_p = (x, 0) \oplus (0, \lambda) = (\dot{x}, \dot{\lambda}).$$

(8)

The interpretation of the above decomposition (8) can be given in terms of $(M, \chi)$, i.e., $M$ is a Riemannian space with identity $\chi$ as the Riemannian metric [32]. This gives

$$\langle H_p, V_p \rangle_{\chi} = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} \begin{bmatrix} I_{k \times k} & 0 \\ 0 & m-k \times n-k \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\lambda} \end{bmatrix} = 0.$$  

(9)

Remark 3: In differential geometry, a pseudo-Riemannian manifold is a differentiable manifold with a metric tensor that is everywhere nondegenerate. This is a generalization of a
Riemannian manifold in which the requirement of positive definiteness is relaxed.

Remark 4: In order to get the passive output for the next step, the metric \( \chi \) is replaced with a pseudo-Riemannian metric \( W = \nabla Y(x, \lambda)^T \nabla Y(x, \lambda) \) as a natural choice.

Let define \( W \) as a semi-Riemannian metric on space \( T_{\text{p}M} \) that enables a semi-Riemannian structure to the fibre-bundle \((x, \lambda)\), as shown below:

\[
W = \nabla Y(x, \lambda)^T \nabla Y(x, \lambda)
= \begin{bmatrix}
\frac{\partial \tau}{\partial x}^T \\
-\frac{\partial \tau}{\partial x}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \tau}{\partial x}^T \\
-\frac{\partial \tau}{\partial x}
\end{bmatrix}
\pmatrix{T}
\tag{10}
\]

For \((\mathcal{M}, W)\), the splitting is geometrically visualized as follows:

\[
(\dot{x}, \dot{\lambda}) = \left(\dot{x}, -w_{22}^{-1}w_{21}\dot{x}\right) \oplus \left(0, \dot{\lambda} + w_{22}^{-1}w_{21}\dot{x}\right)
= \tilde{H}_p \oplus \tilde{V}_p
\tag{11}
\]

Theorem 1: For a given tangent vector \( v = (\dot{x}, \dot{\lambda}) \in T_{\text{p}M} \), the orthogonality of \( H_p \) and \( V_p \) is preserved under new metric \( W \).

Refer to the Appendix A for Proof.

Remark 5: As \( \dot{\lambda} \) is along the vertical direction, the passive output is chosen as a component \( \dot{\lambda} + w_{22}^{-1}w_{21}\dot{x} \) which is in the same direction.

The geometrical interpretation for the splitting tangent vector is shown in Fig. 1. The idea is to single-out fields that are constant. For vector fields on the plane, the such field should be literally constant. For vector fields on a manifold or an arbitrary bundle, the notion should be specified. Such fields are called “Horizontal” and are also key to defining a notion of derivative, or rate or rate of change of vector field, and is chosen to have other desirable properties, such as linearity. For example, the sum of two constant fields should still be constant. That’s how the terms \( \dot{\lambda} + w_{22}^{-1}w_{21}\dot{x} \) come into the picture as a vertical component. The discussion about the passive output and storage function is provided in Appendix.

C. PASSIVE OUTPUT

The component of \( u \) tangent vector along \( \dot{\lambda} \) is used to define the passive output \( \Gamma \) as follows:

\[
\Gamma = \int_0^t (\dot{\lambda} + w_{22}^{-1}w_{21}\dot{x})dt.
\tag{12}
\]

If \( w_{22}^{-1}w_{21} \) is a constant then \( \Gamma = (\lambda + w_{22}^{-1}w_{21}x) \). However, if \( w_{22}^{-1}w_{21} \) is a function of \( x \) then \( \Gamma \) can be written as the gradient of any function \( h(x) \) i.e., \( w_{22}^{-1}w_{21} = \nabla h(x) \). Then,

\[
\Gamma = \int_0^t (\dot{\lambda} + \nabla h(x)\dot{x})dt = (\lambda + h(x))
\tag{13}
\]

Remark 6: The condition \( w_{22}^{-1}w_{21} = \nabla h(x) \) is related to the condition of integrability i.e., integrable connection in differential geometry.

D. STORAGE FUNCTION

With the passive output \( \Gamma \), the storage function \( S(x, \lambda) \) is defined as

\[
S(x, \lambda) = \frac{1}{2} \Gamma^2 = \frac{1}{2} (\lambda + h(x))^2.
\tag{14}
\]

If the condition

\[
\dot{S} \leq -\beta_n S \quad \text{with} \quad \beta_n > 0
\tag{15}
\]

is chosen, then

\[
(\lambda + h(x))(\dot{\lambda} + \frac{\partial h(x)}{\partial x}) \leq -\frac{\beta_n}{2}(\lambda + h(x))^2
\tag{16}
\]
The implicit manifold is given by
\[ \dot{x} + \frac{\partial h(x)}{\partial x} \lambda + \frac{\beta}{2} (\lambda + h(x)) = 0 \] (17)
i.e., \[ u + \frac{\beta}{2} \lambda + \frac{\beta}{2} h(x) + \frac{\partial h(x)}{\partial x} \dot{x} = 0. \] (18)

The final control law is given as
\[ u = -\frac{\beta}{2} \lambda - \frac{\beta}{2} h(x) - \frac{\partial h(x)}{\partial x} \dot{x} \] (19)

This ensures that the above-defined control law guarantees the GAS of the system to a suitable equilibrium point.

**Theorem 2:** For a Nonlinear System (NLS) (6), the proposed P&I-based control law (19) renders the closed-loop system asymptotically stable.

**Proof:** See Appendix.

The last crucial stage of manifold attractivity in the classical I&I method lacks a straightforward and systematic procedure, which hampers the implementation of the classical I&I in practical applications. The matrix \( \Sigma(x) = \Theta^T(x) \Theta(x) \) (refer Ex. 3 in [15]) is randomly chosen and not defined properly in I&I HCP [15]. Moreover, the I&I HCP necessitates various time-consuming and complex computations with inequality solutions. The examples mentioned in this section are solved systematically by the P&I approach,

- without any assumption in the selection of matrix, unlike the I&I HCP [15],
- without SOS optimization, solver mosek, and YALMIP,
- without any construction of parameter-dependent dual CM and parameter estimation, unlike CCM.

**Example 1:** Consider a version of strict feedback system [34]
\[ \begin{align*}
\dot{x}_1 &= x_1^2 + x_2 \\
\dot{x}_2 &= u, \quad x(0) = 0.
\end{align*} \] (20)

The implicit manifold is given by \( \Upsilon(x_1, x_2) = x_1^3 + x_2 + \beta_1 x_1 = 0 \) with \( \dot{\eta} = -\beta_1 \eta, \beta_1 > 0, \beta_2 > 0, \text{and } x_1 = \eta. \)

Defining
\[ W = \nabla \Upsilon(x_1, x_2)^T \nabla \Upsilon(x_1, x_2) \] (21)
as
\[ W = \begin{bmatrix} w_{11} & w_{12} \\
w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} (3x_1^2 + \beta_1)^2 & 3x_1^2 + \beta_1 \\
3x_1^2 + \beta_1 & 1 \end{bmatrix} \] (22)

From (12) and (13), the passive output
\[ \Gamma = \int_0^t (\dot{x}_3 + w_{22}^{-1} w_{21} \dot{x}_1)dt = x_2 + \beta_1 x_1 + x_3 \] (23)
and corresponding storage function
\[ S(x_1, x_2) = \frac{1}{2} (x_2 + \beta_1 x_1 + x_3)^2 \] are defined. The simplification and solution of (14), (15), and (16) provide the final control law
\[ u = -\left( (3x_1^2 + \beta_1)(x_1^3 + x_2) + \frac{\beta_2}{2} (x_2 + \beta_1 x_1 + x_3) \right) \] (24)

**Example 2:** A specific system
\[ \begin{align*}
\dot{x}_1 &= -x_1 + x_1^2 + x_1 x_2 + x_1 x_3 \\
\dot{x}_2 &= x_2 \\
x_3 &= -x_3 + u
\end{align*} \] (25)

from [15] with \( x^* = 0 \) and \( x = (x_1, x_2, x_3) \) is highlighted in order to validate the performance of the P&I approach for the system without suitable structure.

**Remark 7:** As \( x_2 \) and \( x_3 \) is available in \( \dot{x}_1 \), a sequential approach (top-down recursive procedure) like the classical backstepping can’t be applied.

Specifically, it can be observed that the selection of \( x_2 + x_3 = -x_1 \) renders the subsystem \( \dot{x}_1 = -x_1 \). Thus, the implicit manifold
\[ \Upsilon(x_1, x_2, x_3) = x_1 + x_2 + x_3 = 0 \] (26)
is obtained easily just by inspection (This inspection is a part of a designer). To fulfill the step (S2), a PR metric R is defined as
\[ \nabla \Upsilon(x)^T \nabla \Upsilon(x) = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\
w_{21} & w_{22} & w_{23} \\
w_{31} & w_{32} & w_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \end{bmatrix} \] (27)
to obtain \( w_{13} = w_{31} = 1, w_{23} = w_{32} = 1, \text{and } w_{33} = 1. \)

The definition of passive output (12)
\[ \Gamma = \int_0^t (\dot{x}_3 + w_{33}^{-1} w_{23} \dot{x}_2 + w_{33}^{-1} w_{31} \dot{x}_1)dt = x_3 + x_2 + x_1 \] (28)

from step (S3) is fulfilled.

**Remark 8:** As the input \( u \) is available in \( \dot{x}_3 \), the elements in metric R having a connection with subscript 3 is taken to obtain \( \nabla \Upsilon(x) = 0 \) (see Fig. 1).

With the passive output \( \Gamma \), the storage function
\[ S(x_1, x_2, x_3) = \frac{1}{2} \Gamma^2 = \frac{1}{2} (\frac{x_2 + x_1}{\lambda + h(x_1, x_2)})^2 \]
\[ \Gamma = \int_0^t \left( \dot{x}_3 + w_{33}^{-1} w_{23} \dot{x}_2 + w_{33}^{-1} w_{31} \dot{x}_1 \right)dt = x_3 + x_2 + x_1 \] (29)

is defined using (S4) and (14). The condition \( \dot{\hat{S}} \leq -\beta \hat{S} \) from (15) is extended as
\[ \left( \frac{\dot{\hat{x}}_3 + \dot{\hat{x}}_2 + \dot{\hat{x}}_1}{\lambda + h(x_1, x_2)} \right) + \frac{\beta}{2} (x_3 + x_2 + x_1) = 0. \] (30)

Upon substitution, the final control law
\[ u_{P&I} = x_3 - \frac{\beta}{2} (x_1 + x_3 + x_2) + x_1 - x_1^2 - x_1 x_2 - x_1 x_3 \] (31)
is obtained that ensure the convergence of system trajectory \( x(t) \) to an equilibrium point. The trajectory boundedness and GAS equilibrium of the CLS system (25) with (31) with \( \beta = 2 \) is already proved in [15].
1) P&I FOR SYSTEM IN FEEDBACK LINEARIZABLE STRUCTURE
Consider a second-order version of (1)
\[
\dot{x}_i = x_{i+1} \quad \text{for } i = 1, \ldots, n-1 \\
\dot{x}_n = \psi(x) + \phi(x)u, \quad x(0) = 0,
\]
with \((x_i, x_n) \in (\mathbb{R}^{n-1}, \mathbb{R})\). Borrowing the idea of immersion dynamics \(\dot{\xi}_i = -\beta_i \dot{\xi}_i\) with \(\beta_i > 0\) from [14] and [15], a constructive approach utilizing tangent space structure, passive output, and storage function are proposed. With \(x_i = \eta_i\), the implicit manifold is given by
\[
Y(x) = \sum_{i=1}^{n-1} (x_{i+1} + \beta_i x_i) = 0.
\]
Defining \(W = \nabla Y(x)^T \nabla Y(x)\) as
\[
W = W_1 + W_2 + \ldots + W_{n-1} = \sum_{i=1}^{n-1} W_i \quad \text{with (34)}
\]
\[
W_i = \begin{bmatrix} w_{11}(i) & w_{12}(i) \\ w_{21}(i) & w_{22}(i) \end{bmatrix} = \begin{bmatrix} \beta_1^2 & \beta_i \\ \beta_i & 1 \end{bmatrix}
\]
From (12), the passive output
\[
\Gamma = \int_0^t \left( \sum_{i=1}^{n-1} (x_{i+1} + w_{22}(i)w_{21}(i)x_i) \right) \, dt
\]
\[
= \left( \sum_{i=1}^{n-1} (x_{i+1} + \beta_i x_i) \right)
\]
and corresponding storage function
\[
S(x) = S_1(x) + S_2(x) + \ldots + S_{n-1}(x)
\]
\[
= \frac{1}{2} \left( \sum_{i=1}^{n-1} (x_{i+1} + \beta_i x_i)^2 \right)
\]
are defined. The simplification and solution of (14), (15), and (16) provide the final control law
\[
u = -\phi(x_1, x_2) (\dot{\psi}(x_1, x_2) + x_3 + \beta_1 x_2 + x_4 + \beta_2 x_3 + \ldots + \beta_{n-1} x_n + \frac{\beta_n}{2} (x_2 + \beta_1 x_1 + x_3 + \beta_2 x_2 + \ldots + x_n + \beta_{n-1} x_{n-1}))
\]
(38)

Theorem 3: The solution trajectories i.e., the time evolution of states \(x(\bullet)\) of the nonlinear system (32) are bounded and \(x^*\) is a GES equilibrium of (32) with the proposed control law (38).

Proof: The proposed control law (38) transforms the compressed (second-order) version of the nonlinear system (32) into
\[
\dot{x}_i = x_n \\
\dot{x}_n = -((\beta_1 + \beta_n)x_n - \beta_l \beta_n x_l)
\]
(39)
It is obvious that the system (39) is globally exponentially stable with \(\beta_m > 0\) for \(m = 1, \ldots, n\) and it assures the exponential convergence of \(x_m\) for \(m = 1, \ldots, n\) to zero. Moreover, all eigenvalues of a matrix consists of \(\beta_m > 0\) for \(m = 1, \ldots, n\) have negative real parts. Therefore, it is concluded that the proposed state feedback law (38) via the P&I approach renders the system (32) GES.

IV. P&I STABILIZATION AND CONTROL OF THE PARTIALLY UNKNOWN SYSTEM USING GP
Consider a system
\[
\dot{x}_i = x_{i+1} \quad \text{for } i = 1, \ldots, n-1 \\
\dot{x}_n = \psi(x) + \phi(x)u, \quad x(0) = 0,
\]
(40)
which is similar to (32) with an additional Assumption of an unknown function \(\psi(x)\). In above system (40), the exact estimation \(\hat{\psi}(x)\) of \(\psi(x)\) is required for proper stabilization and control. The GP-based P&I control law
\[
u = -\phi(x_1, x_2) (\dot{\psi}(x_1, x_2) + x_3 + \beta_1 x_2 + x_4 + \beta_2 x_3 + \ldots + \beta_{n-1} x_n + \frac{\beta_n}{2} (x_2 + \beta_1 x_1 + x_3 + \beta_2 x_2 + \ldots + x_n + \beta_{n-1} x_{n-1}))
\]
(41)
consists of the P&I in conjunction with Gaussian Process for the system (40) with the identification of an unknown term \(\psi(x)\) via GP is proposed.

Remark 9: The identification of \(\hat{\psi}(x)\) via GP is given as \(\hat{\psi}(x)\) and is reviewed in the following subsection.

A. GAUSSIAN PROCESS
When a system is exposed to an unknown environment, it is prone to disturbances such as parametric variations and measurement noise. These uncertainties could be accounted for and characterized by a stochastic process known as the GP, which is based on the idea of Bayesian inference, which emphasizes revising the current hypothesis based on new information. The Bayesian probabilistic approach exploits the prior knowledge and maximum likelihood function to evaluate the posterior probability distributions given as
\[
P(A|B) = \frac{P(B|A)P(A)}{P(B)}.
\]
(42)
Consider a set of the noisy measurements of a function \(\psi : X \rightarrow \mathbb{R}\)
\[
y_i = \psi(x_i) + \alpha_i \quad \text{with } i = 1, \ldots, N, \ x_i \in X, \ y_i \in Y
\]
(43)
where \(X = \{x_1, x_2, \ldots, x_n\}\) being the set of input data, \(Y = \{y_1, y_2, \ldots, y_n\}\) being the set of output data, and \(\alpha_i \sim \mathcal{N}(0, \sigma^2)\). The GP regression model (approximate) the function \(\psi(x)\) through the nonlinear mapping between input and output training data sets. The GP is described as the distribution over some random functions and is defined as
\[
\psi_{GP} = \hat{\psi} = \mathcal{GP}(\mu(x), \mathcal{K}(x, x'))
\]
(44)
with the mean function $\mu$ and the covariance or kernel function $\mathcal{K}$ being the two priors defining the considered GP. To incorporate the knowledge about the prior the mean function is defined; which also assists in determining the expected value of the distribution. The similarity between the two points and how their values vary is measured with the covariance function. The kernel function describes the shape of the distribution and mostly elaborates on the smoothness of EP functions.

Both of these GP priors are defined by the hyperparameters $\xi$ optimized to the data by solving the maximum log-likelihood problem given as

$$\xi^* = \arg \max_{\xi} \log p(y|X, \xi)$$

where, $\log p(y|X, \xi) = \frac{1}{2}y^T \mathcal{K} y - \log (\det \mathcal{K}) - N \log(2\pi)$, and the pairwise kernel functions evaluated from the training input data are concatenated as

$$\mathcal{K} = \begin{bmatrix} k(x_1, x_1) & \cdots & k(x_1, x_N) \\ \vdots & \ddots & \vdots \\ k(x_N, x_1) & \cdots & k(x_N, x_N) \end{bmatrix}$$

The GP regression exploits the joint distribution among the training data $D$ and the test input $x_*$, which is mathematically given as

$$\psi_{\text{GP}}(x_*) \sim \mathcal{N} \left( \left[ \mu(x_*) \right]^T, \kappa = \kappa^T + \rho^2 \mathcal{I}_N \right)$$

where, $\mu_X = [\mu(x_1), \ldots, \mu(x_N)]^T$

The posterior mean $\hat{\mu}$ and covariance $\sigma$ are inferred through conditioning the joint distribution of the training data $D$ at the test data point $x_*$ as,

$$\hat{\mu}(x_*) = \mathbb{E}[\psi_{\text{GP}}(x_*)|D]$$

$$\sigma(x_*) = \sqrt{\mathbb{V}[\psi_{\text{GP}}(x_*)|D]}$$

$$= k^* - k^T (\mathcal{K} + \rho^2 \mathcal{I}_N)^{-1} k$$

with

$$k_0 = k(x_0, x_0),$$

$$k = [k(x_1, x_0) \cdots k(x_N, x_0)]^T$$

B. GP IDENTIFICATION IN CONTROL

Going back to the defined problem statement, the goal is to estimate the unknown term $\psi(x)$ and update it into the proposed P&I control law for a stable closed-loop scheme. The above-mentioned strategies are used:

- to identify the unknown term using GP and
- to stabilize and control the partially unknown system via

GP-based P&I approach.

The prior mean function $\hat{\mu} = 0$ and the squared exponential (SE) kernel

$$k_\psi(x, x') = \rho_\psi^2 \exp \left( -\frac{\|x - x\|^2}{2l_\psi^2} \right)$$

with the hyperparameters are signal variance $\rho_\psi$, measurement noise $\rho_n$, and length scale $l_\psi$, $i = 1, \ldots, N$ are considered. The resultant posterior mean

$$\hat{\psi}(x_a) = \hat{\mu}_\psi(x_a) = k^T (\mathcal{K} + \rho^2 \mathcal{I}_N)^{-1} (y - \mu_X)$$

of the function $\psi(x)$ are set to get $\hat{\psi}(x)$. The overall proposed structure of the closed-loop control system is depicted in Fig. 2.

![FIGURE 2. Overall closed-loop framework with the proposed GP-based P&I approach.](image)

**Theorem 4:** Consider a system (40) with an unknown term $\psi(x)$, Assumptions 1, 2, 3, and data set $D$. The GP-based identified term $\hat{\psi}(x)$ with SE kernel (50) and prior mean $\hat{\mu} = 0$ is bounded and infinitely differentiable.

**Proof:** The boundedness and differentiability follow from the Squared Exponential (SE) kernel according to [35], [23], and [36].

**Remark 10:** As the estimated function $\hat{\psi}(x)$ from GP is bounded, the stabilization and trajectory convergence of the system (40) is ensured via the proposed control law (41).

V. NUMERICAL ILLUSTRATION

A generalized second-order system is considered for the implementation and validation of the proposed P&I approach. Consider a second-order version of (1)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \psi(x_1, x_2) + \phi(x_1, x_2) u, \quad x(0) = 0.$$ (52)

Borrowing the idea of immersion dynamics $\dot{\eta} = -\beta_1 \eta$ with $\beta_1 > 0$ and $\beta_2 > 0$ from [14] and [15], a constructive approach utilizing tangent space structure, passive output, and storage function is proposed. With $x_1 = \eta$, the off-the-manifold is given by $\Upsilon(x_1, x_2) = x_2 + \beta_1 x_1 = 0$. Defining $W = \Upsilon(x_1, x_2)^T \Upsilon(x_1, x_2)$ as

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} \beta_1^2 & \beta_1 \\ \beta_1 & 1 \end{bmatrix}$$

From (12), the passive output

$$\Gamma = \int_0^t (\dot{x}_2 + w_{22}^{-1} w_{21} \dot{x}_1) dt = x_2 + \beta_1 x_1$$

(54)

and corresponding storage function $S(x_1, x_2) = \frac{1}{2}(x_2 + \beta_1 x_1)^2$ are defined. The simplification and solution of (14),...
(15), and (16) provide the final control law
\[ u = -\phi(x_1, x_2)^{-1}(\psi(x_1, x_2) + \beta_1 x_2 + \frac{\beta_2}{2}(x_2 + \beta_1 x_1)) \]  

(55)

An inverted pendulum dynamics [24], [25], [29] with
\[ \psi(x_1, x_2) = \sin(x_1) - |x_2| x_2 + \frac{1}{2} e^{x_1 x_2} \]
and \( \phi(x_1, x_2) = 1 \) as a physical example of (52) is evaluated. As the contribution of this paper is twofold, two cases are considered. First, the fully known second-order inverted pendulum dynamics is investigated via the proposed P&I approach. Second, the partially unknown term \( \psi(x_1, x_2) \) is approximated as \( \hat{\psi}(x_1, x_2) \) by GP in order to incorporate the parametric and measurement uncertainties. This approximated term \( \hat{\psi}(x_1, x_2) \) is updated in control law obtained by the proposed P&I method. This section is further divided into three subsections based on considered cases.

**A. STABILIZATION VIA PROPOSED P&I APPROACH**

In this subsection, the proposed P&I approach is applied to fully known dynamics (52) with (56). The control law (V) with known parameters \( \Theta_p \) is applied to the dynamics (52) with (56). The stabilization performance without any parametric changes is depicted in Fig. 3(a). The robustness of the control law is verified by applying the parametric variations. The 20% variation in each parameter of vector \( \Theta_p \) is applied at time \( t = 2 \) sec. The fast convergence of the state trajectories is assured with a slight overshoot (peak) during the transient as the values of \( \beta_1 \) and \( \beta_2 \) increase. The effect of parametric variations is effectively suppressed by increasing the values of \( \beta_1 \) and \( \beta_2 \), as shown in Fig. 3(b) and Fig. 4.

Even after applying the parametric variations at time \( t = 2 \) sec, the state trajectory convergence is ensured by time \( t = 2.5 \) sec approximately by the proposed P&I control, as shown in Fig. 4.

**FIGURE 3.** Time evolution of state trajectories of inverted pendulum with \( \beta_1 = \beta_2 = 10 \). State trajectory (a) without parametric change and (b) with parametric change.

**FIGURE 4.** Time evolution of state trajectories of inverted pendulum with \( \beta_1 = \beta_2 = 20 \). State trajectory (a) without parametric change and (b) with parametric change.

**Remark 11:** It can be said that the controller sustains the parametric variation within a certain domain. Therefore, an indirect approach via estimation is needed.

**B. STABILIZATION FOR PARTIALLY UNKNOWN SYSTEM VIA PROPOSED P&I APPROACH WITH GP**

The proposed P&I approach with GP as a function approximator is applied to partially unknown dynamics in this subsection. The robustness of the complete closed-loop system with the P&I control law and GP is demonstrated by using parametric variations and measurement noise in data. When the GP is used to model the relationship between input and output values, it implements a nonlinear mapping between input \( x \) and output \( y \), and (48) and (49) provide predictions in terms of mean and variance functions.

The input-output data for GP training is obtained by applying \( u = u_{\text{data}} = \sin(\pi t) \) to the system (52) for \( t = 20 \) sec. The input measurements of \( x + \alpha_1 \) with \( \alpha_1 \sim \mathcal{N}(0, \rho_1^2) \) and output measurements \( y = (\dot{x}_n - \phi(x)u) + \alpha_2 \) with \( \alpha_2 \sim \mathcal{N}(0, \rho_2^2) \) is collected. The trained GP model is then utilized in the closed-loop setup to get an estimation of an unknown function \( \hat{\psi}(x_1, x_2) \). The approximated function \( \hat{\psi}(x_1, x_2) \) of \( \psi(x_1, x_2) \) from GP is updated in control law (V). The three different scenarios are discussed in the following sections based on the value of noises in input-output data.

1) NOISELESS

The perturbations and data corruption in the input and output data are not considered in this noiseless scenario. Fig. 5 shows...
the input and output data for GP training. The stabilization performance of the system (52) with unknown term \( \psi(x_1, x_2) \) for control law (V) with approximated function \( \hat{\psi}(x_1, x_2) \) from GP is shown in Fig. 6. The fast convergence of the state trajectories within finite time is assured as the values of \( \beta = [\beta_1 \, \beta_2]^T \) increase. The convergence time for different \( \beta \) is shown in Table 1.

![FIGURE 6. Time evolution of state trajectories of inverted pendulum with different \( \beta \).](image)

### TABLE 1. Convergence time of the system states for different \( \beta \).

| Sr. No | \( \beta_1 \) | \( \beta_2 \) | Conv time |
|--------|--------------|--------------|-----------|
| 1      | 40           | 40           | 2.2 sec   |
| 2      | 20           | 20           | 0.51 sec  |
| 3      | 10           | 10           | 0.24 sec  |

2) NOISE IN OUTPUT MEASUREMENT

The stochasticity in terms of additive Gaussian white noise (AWGN) is added to the output \( y \). The MATLAB command `awgn(in, snr, signalpower)` is used to introduce the noise in the pure signal (see Fig. 5(b)) with SNR 20 and signal power 0.2 dB. The GP is trained using noiseless input and corrupted output data in this scenario, as illustrated in Fig. 7. The time evolution of states and estimation of unknown term \( \psi(x_1, x_2) \) for the proposed P&I with GP and output noise is depicted in Fig. 8. The state trajectories are converging the equilibrium points as the \( \hat{\psi}(x_1, x_2) \) aligns with its actual value.

![FIGURE 7. Noisy training data for GP. (a) shows the input data vector \( x \) (noiseless) and (b) depicts the target data \( y \) (with SNR 20 and signal power 0.2 dB) for GP training.](image)

![FIGURE 8. (a) Time evolution of state trajectories of inverted pendulum with output noise (b) unknown function approximation via GP.](image)

1) the noise in the input pure signal (see Fig. 5(a)) with SNR 20 and signal power 0.01 dB.

2) the noise in the output pure signal (see Fig. 5(b)) with SNR 50 and signal power 0.1 dB.

The convergence of state trajectories of an inverted pendulum with input and output noise in training data is shown in Fig. 10. The fast convergence of the state trajectories within finite time is assured as the values of \( \beta \) increase. These analyses and results show that the GP-based P&I approach is successful in identifying the unknown term of the system without any prior knowledge and assumption.

![FIGURE 9. Noisy training data for GP. The left-hand side Fig. (a) shows the input data vector \( x \) (with SNR 20 and signal power 0.01 dB) and the right-hand side Fig. (b) depicts the target data \( y \) (with SNR 50 and signal power 0.1 dB) for GP training.](image)

![FIGURE 10. Time evolution of state trajectories of an inverted pendulum with input and output noise in training data.](image)

3) NOISE IN INPUT AND OUTPUT MEASUREMENT

The stochasticity in terms of the AWGN is added to the input \( x \) and output \( y \). As illustrated in Fig. 9, the MATLAB command `awgn(in, snr, signalpower)` is used to introduce

### C. CONTROL VIA PROPOSED P&I AND COMPARISON ANALYSIS

The objective of a control problem is to regulate the system’s trajectory to the desired (intended) one \( x_d \). The identified
error is given as follows:
\[
\epsilon_1 = x_1 - x_1^d \Rightarrow x_1 = \epsilon_1 + x_1^d
\]  
(57)
The derivative of the error term (57) with (52) is re-written as
\[
\dot{\epsilon_1} = x_2 - \dot{x}_1^d \\
\dot{x}_2 = \psi(x_1^d + \epsilon_1, x_2) + \phi(x_1^d + \epsilon_1, x_2)u.
\]  
(58)
Defining the target dynamics \( \dot{\eta} = \beta_1 \eta \) with \( \epsilon_1 = \eta \) to get an exponentially stable dynamics \( \dot{\epsilon_1} = -\beta_1 \epsilon_1 \). This selection of the target dynamics leads to the off-the-manifold \( \dot{\gamma}(\epsilon_1, x_2, x_1^d) \)
\[
x_2 + \beta_1 \epsilon_1 - x_1^d = 0.
\]  
(59)
With \( W = \nabla \dot{\gamma}(\epsilon_1, x_2, x_1^d)^T \nabla \gamma(\epsilon_1, x_2, x_1^d) \), (56), (12) and (13), the passive output \( \Gamma = x_2 + \beta_1 \epsilon_1 - x_1^d \) and corresponding storage function \( S(x_1, x_2, x_1^d) = \frac{1}{2}(x_2 + \beta_1 \epsilon_1 - x_1^d)^2 \) are defined. The simplification and solution of (14), (15), and (16) provides the final control law
\[
u = -\phi(x_1, x_2)^{-1}(\psi(x_1, x_2) + \beta_1 (x_2 - \dot{x}_1^d) - \dot{x}_1^d + \frac{\beta_2}{2}(x_2 + \beta_1 \epsilon_1 - \dot{x}_1^d)).
\]  
(60)
Remark 12: The function \( \psi(x_1, x_2) \) is approximated as \( \hat{\psi}(x_1, x_2) \) via GP and updated in control law (60) for GP-based P&I approach.

Remark 13: For the constant reference input, the first derivative \( \dot{x}_1^d \) and second derivative \( \ddot{x}_1^d \) are zero. Many times it is difficult to get \( \dot{x}_1^d \) and \( \ddot{x}_1^d \) for time-varying function \( x_1^d(t) \). The command filtered methods [37], [38] can be easily applied to reduce the computation burden.

The time evolution of states of an inverted pendulum with the control law (60), MRAC, and without GP is shown in Fig. 11. The MRAC is designed for comparative purposes and the specifications (gains and reference model) are directly taken from [29] and [25]. From Fig. 11, it is clearly seen that the convergence time of the P&I is 1.2 sec and 7.5 sec for MRAC to track the reference one. A complete comparative analysis of MRAC with different parametric (Concurrent Learning [24], DREM [39]) and non-parametric approaches (Neural networks, GP) for estimation of \( \psi(x_1, x_3) \) is already carried out by the authors in [25] and [29] (refer Fig. 6 of [29]). The estimation performance of unknown term \( \psi(x_1, x_3) \) via GP-MRAC and RBFN-MRAC is shown in [22]. The open-loop prediction and its comparison via different machine learning techniques are presented in [40]. The GP model best captures the unknown functions and improves prediction accuracy with predictive variance as depicted by the statistical measures such as RMSE, MAE, and CC in [40] and [22].

D. DISCUSSION
A complete closed-loop structure of partially unknown dynamics is explored with GP-based P&I without any prior knowledge or assumptions. The GP for identifying the unknown terms has proven effective since the model-based estimation techniques result in inaccurate gradient and Hessian calculations due to the measurement uncertainties in data. The initial controller \( u_{data} \) determines the training data distributions, which influence the GP prediction accuracy of unknown terms. The classical test signals, such as ramps and steps, cannot adequately excite the system to estimate the parameters accurately [41]. Distorted and sinusoidal signals excite the system sufficiently well. Choosing these excitation signals relies on the theory of the Persistence of Excitation [13], [39], [42], [43]. Therefore, the initial controller \( u_{data} = \sin(\pi t) \) is used for the collection of GP training data.

In many feedback linearizable systems, the function
- \( \phi(x_1, x_2) = 1 \) in an inverted pendulum [25],
- \( \phi(x_1, x_2) = 1 \) in Van der Pol equation [31], and
- \( \phi(x_1, x_2) = L_{sat} = 3 \) in wing rock dynamics [22] is generally taken as per the defined dynamics. Therefore, the non-parametric structure of \( \psi(x_1, x_2) \) is only considered since the focus is on the proposed P&I approach and the effectiveness of GP-based P&I. The non-parametric structure of both \( \psi(x_1, x_2) \) and \( \hat{\psi}(x_1, x_2) \) can be considered. These functions can be approximated by Gaussian processes using Sum of functions as performed in [23]. Clearly, there could be infinitely many possible solutions to find these functions that add up to the same function [23]. As the identification possibly fails in separating the contribution of \( \psi(x_1, x_2) \) and \( \hat{\psi}(x_1, x_2) \), the future scope can be seen in terms of
1) the identification of \( \psi(x_1, x_2) \) and \( \hat{\psi}(x_1, x_2) \), the future scope can be seen in terms of
2) and the separation of \( \phi(x_1, x_2) \) and \( \hat{\phi}(x_1, x_2) \).

VI. CONCLUSION
This paper proposes
- the Passivity and Immersion (P&I) based approach for the stabilization and control of fully known nonlinear systems by adding the concept of the generation of a suitable passive output and corresponding storage function for designing a feedback law and manifold attractivity. This involves a particular structure of tangent bundle associated with a suitable Riemannian/sub-Riemannian structure imposed on the control system.
- the GP based P&I, which integrates P&I with the GP to stabilize and control partially unknown nonlinear systems. The uncertain term is modeled for the partially

![FIGURE 11. State trajectories of an inverted pendulum with the proposed P&I (β = 20) and MRAC.](image-url)
unknown system as a distribution over functions rather
than via a known parametric function.

The GP-based P&I approach is suitable for a variety of prac-
tical and real-world applications involving noise and para-
metric variations. Furthermore, the mathematical analysis
and simulation results of the stabilization and control of an
inverted pendulum demonstrate that the proposed P&I and
GP-based P&I approaches are effective, stable, and robust to
uncertainties methods.

**APPENDIX A**

**THEOREM-PROOF**

**A. SPLITTING TANGENT SPACE**

**Proof of Theorem 1:** From (10), it is observed that the splitting
of the vectors is only dependent on $w_{12}$ or $w_{21}$, and $w_{22}$ in any
case. Now suppose the metric $\chi$ is replaced with some other
Riemannian/sub-Riemannian metric

$$ W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \tag{61} $$

with $w_{12} = w_{21}$ and $W$ is positive definite or positive semi-
definite. Then to ensure that $T_{pM} = \mathbb{H}_p \oplus \mathbb{V}_p$, the following
is the structure of $\mathbb{H}_p$ and $\mathbb{V}_p$ under $W$

$$ T_{pM} \subset (\dot{x}, \dot{\lambda}) = \left( \dot{x}, -w_{22}^{-1}w_{21}\dot{x} \right) \oplus \left( 0, \dot{\lambda} + w_{22}^{-1}w_{21}\dot{x} \right) \tag{62} $$

becomes

$$ \langle [\mathbb{H}_p, \mathbb{V}_p]_W \rangle $$

$$ = \begin{bmatrix} \dot{x} & -w_{22}^{-1}w_{21}\dot{x} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\lambda} + w_{22}^{-1}w_{21}\dot{x} \end{bmatrix} = 0 $$

$$ = w_{21}\dot{x}(\dot{\lambda} + w_{22}^{-1}w_{21}\dot{x}) - w_{21}\dot{x}(\dot{\lambda} + w_{22}^{-1}w_{21}\dot{x}) = 0 \tag{63} $$

$$ = w_{21}\dot{x}(\dot{\lambda} + w_{22}^{-1}w_{21}\dot{x}) - w_{21}\dot{x}(\dot{\lambda} + w_{22}^{-1}w_{21}\dot{x}) = 0 \tag{64} $$

**Remark 14:** The above splitting of the tangent space under
a metric $W$ is closely related to the theory of fiber bundles and
the Ehresmann connection.

**B. PASSIVE OUTPUT AND STORAGE FUNCTION**

**Theorem 5:** The obtained $y = (\lambda + h(x))$ in (13) is a pas-
sive output and associated function (14) is a storage function
with respect to new input $v$ and $y$.

**Proof:** The time-derivative of function (14)

$$ \dot{\hat{S}} = (\lambda + h(x))(\dot{\lambda} + \frac{\partial h(x)}{\partial x} \dot{x}) $$

$$ = (\lambda + h(x))(u + \frac{\partial h(x)}{\partial x} f(x, \lambda)) \tag{65} $$

is translated into

$$ \dot{\hat{S}} = y^T u + y^T \frac{\partial h(x)}{\partial x} f(x, \lambda) \tag{66} $$

with $y = (\lambda + q(x))$. Here, $y \in \mathbb{R}^{n-k}$ and the dimension of
$y$, $u$, and $v$ is same. By substituting the control law

$$ u = -\frac{\partial h(x)}{\partial x} f(x, \lambda) + v \tag{67} $$
in (66), it becomes passive with respect to new input $v$ and $y$
due to

$$ \dot{\hat{S}} \leq y^T v. \tag{68} $$

Therefore, the $\hat{S}$ is called as the storage function with $y =
(\lambda + h(x))$ as a passive output.

**Remark 15:** The storage function is far from unique.
Nonuniqueness arises from the fact that there might be vari-
sious implicit manifolds depending on the choice of target
dynamics.

**C. STABILITY AND BOUNDEDNESS**

**Proof of Theorem 2:** The storage function obtained as (14)
can be interpreted as the Lyapunov function

$$ V(x) = \frac{1}{2} \sum_{i=0}^{n-1} (\lambda_{i+1} - \tau_i(x))^2. \tag{69} $$

mentioned in [9], [10]. With this storage function, the con-
dition for exponential stability i.e., $\dot{\hat{S}} \leq -\beta \hat{S}$ is utilized to
obtain the proposed P&I based control law. The boundedness of
the trajectories and global asymptotic stability of equilib-
rium $x^*$ for (6) follows by the same argument of the proof is
adapted from the Lemma 1 in [15] and Theorem 1 in [14].

**APPENDIX B**

**NOMENCLATURE**

The following abbreviations are used in this manuscript:

| Symbol | Description               |
|--------|---------------------------|
| P&I    | Passivity and Immersion approach. |
| SISO   | Single input single output.  |
| $\mathcal{N}(0, \rho^2)$ | Gaussian distribution with mean and vari-
| GP, NN | Gaussian Process, Neural Networks. |
| AWGN   | Additive white Gaussian noise. |
| $L^2_\xi(x)$ | Lie derivative of $\xi(x)$ w.r.t. $\lambda$. |
| $L_\lambda^n(\xi(x))$ | $L_\lambda^n$-th derivative $(L_\lambda^n \xi(x))$. |
| DREM   | Dynamic Regression Extension and Mixing. |
| MRAC   | Model Reference Adaptive Control. |
| RBFN   | Radial Basis Function Networks. |
| $\nabla \Upsilon(x)$ | gradient of $\Upsilon$ w.r.t. $x$. |
| RMSE   | Root Mean Square Error. |
| MAE, CC| Mean Absolute Error, Correlation Coeffi-
| LIP    | Linear-in-Parameters. |
| RKHS   | Reproducing kernel Hilbert space |

**REFERENCES**

[1] H. K. Khalil and J. W. Grizzle, *Nonlinear Systems*, vol. 3. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.
[2] J.-J.-E. Slotine and W. Li, *Applied Nonlinear Control*, vol. 199. Englewood Cliffs, NJ, USA: Prentice-Hall, 1991.
[3] R. Freeman and P. V. Kokotovic, *Robust Nonlinear Control Design: State-
|          | Space and Lyapunov Techniques*. Berlin, Germany: Springer, 2008.
[4] I. R. Manchester and J.-J.-E. Slotine, “Control contraction metrics: Convex
|          | and intrinsic criteria for nonlinear feedback design.” *IEEE Trans. Automat.
|          | Control*, vol. 62, no. 6, pp. 3046–3053, Jun. 2017.
[5] R. Ortega, J. A. L. Perez, P. J. Nicklasson, and H. J. Sira-Ramirez, Passivity-Based Control of Euler-Lagrange Systems: Mechanical, Electrical & Electromechanical Applications. Berlin, Germany: Springer, 2013.

[6] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, Nonlinear and Adaptive Control Design. Hoboken, NJ, USA: Wiley, 1995.

[7] P. V. Kokotovic, “The joy of feedback: Nonlinear and adaptive,” IEEE Control Syst., vol. 12, no. 3, pp. 7–17, Jun. 1992.

[8] R. Sepulchre, M. Jankovic, and P. V. Kokotovic, Constructive Nonlinear Control. Berlin, Germany: Springer, 2012.

[9] M. Zamani and P. Tabuada, “Backstepping design for incremental stability,” IEEE Trans. Autom. Control, vol. 56, no. 9, pp. 2184–2189, Sep. 2011.

[10] M. Zamani and P. Tabuada, “Towards backstepping design for incremental stability,” in Proc. 49th IEEE Conf. Decis. Control (CDC), Dec. 2010, pp. 2426–2431.

[11] F. Pozo, F. Ihkhouane, and J. Rodellar, “Numerical issues in backstepping control: Sensitivity and parameter tuning,” J. Franklin Inst., vol. 345, no. 8, pp. 891–905, Nov. 2008.

[12] S. S. Nayer, S. R. Wagh, and N. M. Singh, “Towards a constructive framework for stabilization and control of nonlinear systems: Passivity and immersion (P&I) approach,” 2022, arXiv:2211.10674.

[13] S. S. Nayer, G. Revati, S. R. Wagh, and N. M. Singh, “Passivity and immersion based-modified gradient estimator: A control perspective in parameter estimation,” 2022, arXiv:2211.10674.

[14] A. Astolfi and R. Ortega, “Immersion and invariance: A new tool for stabilization and adaptive control of nonlinear systems,” IEEE Trans. Autom. Control, vol. 48, no. 4, pp. 590–606, Apr. 2003.

[15] L. Wang, F. Forni, R. Ortega, Z. Liu, and H. Su, “Immersion and invariance stabilization of nonlinear systems via virtual and horizontal contraction,” IEEE Trans. Autom. Control, vol. 62, no. 8, pp. 4017–4022, Aug. 2017.

[16] L. Wang, F. Forni, R. Ortega, and H. Su, “Immersion and invariance stabilization of nonlinear systems: A horizontal contraction approach,” in Proc. 54th IEEE Conf. Decis. Control (CDC), Dec. 2015, pp. 3093–3097.

[17] H. Tsukamoto and S.-J. Chung, “Convex optimization-based controller design for stochastic nonlinear systems using contraction analysis,” in Proc. IEEE 59th Conf. Decis. Control (CDC), Dec. 2019, pp. 8196–8203.

[18] B. T. Lopez and J.-J. E. Slotine, “Contraction metrics in adaptive nonlinear control,” 2019, arXiv:1912.13138.

[19] H. Tsukamoto and S.-J. Chung, “Robust controller design for stochastic nonlinear systems via convex optimization,” IEEE Trans. Autom. Control, vol. 66, no. 10, pp. 4731–4746, Oct. 2021.

[20] N. T. Nguyen, “Model-reference adaptive control,” in Model-Reference Adaptive Control, Berlin, Germany: Springer, 2018, pp. 83–127.

[21] P. A. Ioannou and J. Sun, Robust Adaptive Control. Courier Corporation, 2012.

[22] G. Chowdhary, H. A. Kingravi, J. P. How, and P. A. Vela, “Bayesian nonparametric adaptive control using Gaussian processes,” IEEE Trans. Neural Netw. Learn. Syst., vol. 26, no. 3, pp. 537–550, Mar. 2015.

[23] J. Umlauf, T. Beckers, M. Kimmel, and S. Hirche, “Feedback linearization using Gaussian processes,” in Proc. IEEE 56th Annu. Conf. Decis. Control (CDC), Dec. 2017, pp. 5249–5255.

[24] G. Chowdhary and E. Johnson, “Concurrent learning for convergence in adaptive control without persistency of excitation,” in Proc. 49th IEEE Conf. Decis. Control (CDC), Dec. 2010, pp. 3674–3679.

[25] S. Shadab, J. Hozefa, S. R. Wagh, and N. M. Singh, “Parameter convergence for adaptive control in nonlinear system,” in Proc. Austral. New Zealand Control Conf. (ANZCC), Nov. 2020, pp. 42–47.

[26] S. Shadab, “Persistence of excitation in an online monitoring of transformer,” in Proc. IEEE Power Energy Soc. Gen. Meeting (PESGM), Jul. 2022, pp. 1–5.

[27] Y. Pan and H. Yu, “Composite learning from adaptive dynamic surface control,” IEEE Trans. Autom. Control, vol. 61, no. 9, pp. 2603–2609, Sep. 2016.

[28] R. M. Sanner and J.-J. E. Slotine, “Gaussian networks for direct adaptive control,” IEEE Trans. Neural Netw., vol. 3, no. 6, pp. 837–863, Nov. 1992.

[29] J. Hozefa, S. Shadab, G. Revati, S. R. Wagh, and N. M. Singh, “Adaptive control of nonlinear systems: Parametric and non-parametric approach,” in Proc. 29th Medit. Conf. Control Autom. (MED), Jun. 2021, pp. 1007–1012.

[30] S. S. Sastry and A. Isidori, “Adaptive control of linearizable systems,” IEEE Trans. Autom. Control, vol. 34, no. 11, pp. 1123–1131, Nov. 1989.

[31] W. Khovdithumji and P. Santhanapipatkul, “Adaptive immersion and invariance control of the Van Der Pol equation,” in Proc. ICCA, 2005, pp. 706–709.

[32] P. Bansode, V. Chinde, S. R. Wagh, R. Pusmaryth, and N. M. Singh, “On the geometry and linear convergence of primal-dual dynamics,” 2020, arXiv:2010.02738.

[33] R. Mehra, S. G. Satpute, F. Kazi, and N. M. Singh, “Control of a class of underactuated mechanical systems obviating matching conditions,” Automatica, vol. 86, pp. 98–103, Dec. 2017.

[34] L. Wang and Q.-L.-P. Wang, “The feedback linearization based on backstepping technique,” in Proc. IEEE Int. Conf. Intel. Comput. Intell. Syst., Nov. 2009, pp. 282–286.

[35] C. E. Rasmussen, “Gaussian processes in machine learning,” in Summer School on Machine Learning. Berlin, Germany: Springer, 2003, pp. 63–71.

[36] T. Beckers and S. Hirche, “Stability of Gaussian process state space models,” in Proc. Eur. Control Conf. (ECC), Jun. 2016, pp. 2275–2281.

[37] J. A. Farrell, M. Polycarpou, M. Sharma, and W. Dong, “Command filtered backstepping,” IEEE Trans. Autom. Control, vol. 54, no. 6, pp. 1391–1395, Jun. 2009.

[38] W. Dong, J. A. Farrell, M. M. Polycarpou, V. Djalic, and M. Sharma, “Command filtered adaptive backstepping,” IEEE Trans. Control Syst. Technol., vol. 20, no. 3, pp. 566–580, May 2012.

[39] R. Ortega, S. Aranovskiy, A. A. Pyrkin, A. Astolfi, and A. A. Bobtsov, “New results on parameter estimation via dynamic regressor extension and mixing: Continuous and discrete-time cases,” IEEE Trans. Autom. Control, vol. 66, no. 5, pp. 2265–2272, May 2021.

[40] S. Shadab, J. Hozefa, K. Sonam, S. Wagh, and N. M. Singh, “Gaussian process surrogate model for an effective life assessment of transformer considering model and measurement uncertainties,” Int. J. Electric Power Energy Syst., vol. 134, Jan. 2022, Art. no. 107401.

[41] P. Schrangl, P. Tkachenko, and L. D. Re, “Iterative model identification of nonlinear systems of unknown structure: Systematic data-based modeling utilizing design of experiments,” IEEE Control Syst. Mag., vol. 40, no. 3, pp. 26–48, Jun. 2020.

[42] M. Gevers, A. S. Bazanella, D. F. Coutinho, and S. Dasgupta, “Identifiability and excitation of linearly parametrized rational systems,” Automatica, vol. 63, pp. 38–46, Jan. 2016.

[43] M. Gevers, G. Revati, S. R. Wagh, and N. M. Singh, “Finite-time parameter estimation for an online monitoring of transformer: A system identification perspective,” Int. J. Electric Power Energy Syst., vol. 145, Feb. 2023, Art. no. 108639.

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