Random fields on model sets with localized dependency and their diffraction

Received: date / Accepted: date

Abstract For a random field on a general discrete set, we introduce a condition that the range of the correlation from each site is within a predefined compact set $D$. For such a random field $\omega$ defined on the model set $\Lambda$ that satisfies a natural geometric condition, we develop a method to calculate the diffraction measure of the random field. The method partitions the random field into a finite number of random fields, each being independent and admitting the law of large numbers. The diffraction measure of $\omega$ consists almost surely of a pure-point component and an absolutely continuous component. The former is the diffraction measure of the expectation $E[\omega]$, while the inverse Fourier transform of the absolutely continuous component of $\omega$ turns out to be a weighted Dirac comb which satisfies a simple formula. Moreover, the pure-point component will be understood quantitatively in a simple exact formula if the weights are continuous over the internal space of $\Lambda$. Then we provide a sufficient condition that the diffraction measure of a random field on a model set is still pure-point.

PACS and mathematical subject classification numbers as needed.

Keywords Diffraction · Pure point spectrum · Absolutely continuous spectrum · Quasicrystal · Model set

PACS 61.05.cc · 61.05.cp

The preliminary version is presented at the fifth Asian International Workshop on Quasicrystals, 2009, Tokyo Japan.
1 Introduction

A physical quasicrystal is a material which has (1) a diffraction pattern with Bragg peaks and (2) a symmetry that ordinary crystals cannot have. The set of the atomic positions in a quasicrystal is mathematically modelled by a model set \([17]\), which is defined by introducing an extra space (“internal space”), and a relatively compact subset (“window”) of the internal space. The topological properties of the window cause the pure-point diffraction measure (see \([22]\) and \([7]\) for example) of the model set, which explain the aforementioned properties (1) of the quasicrystal.

Although model sets are proved to have necessarily a pure-point diffraction measure, real quasicrystals have diffraction measures with not only Bragg peaks (pure-point component) but also diffuse scattering (absolutely continuous component). The phenomenon is explained from a physical point of view with a probabilistic effect in \([14]\), and in \([5]\) where to the sites of the model set associated are independent random variables. In \([12]\), Hof regarded the thermal motion of atoms as i.i.d. random displacements, and then studied the influence on the diffraction measure by aperiodic monoatomic crystals. Since correlations (\([10]\)) are present in a quasicrystal, we equip model sets with a localized probabilistic dependency, to quantitatively study the ability of diffuse scattering to characterize local structures and defects in materials. In \([15]\), Lenz employed a dynamical system of point sets to study the diffraction measures of percolation and the random displacement models based on aperiodic order. Recently, in \([20]\), Müller and Richard also made a rigorous approach on these models by using sets of \(\sigma\)-algebras.

For a model set \(\Lambda\), we consider a complex-valued random field \(\{X_s\}_{s \in \Lambda}\) with dependency localized as follows: there is a finite patch \(D\) such that each site \(s\) has correlation on, at most, sites belonging to the patch \(D\) relative to \(s\). This localized dependency condition seems essentially the same as the so-called “finite range condition” of stochastic analysis. We call a random field on a model set subject to the localized dependency condition a finitely randomized model set. We develop a method to calculate the diffraction measure of such complex-valued random field (Section 3). For the diffraction measures of finitely randomized model sets, we determine quantitatively the pure-point component (in Section 5) and the absolutely continuous component (in Section 4). As a consequence, if the fourth noncentral moments \(\{E[|X_s|^4]\}_{s \in \Lambda}\) is bounded, and if the expectation \(E[X_s]\) at each site \(s\) as well as the covariance \(X_s\) and \(X_{s-g}\) of sites \(s\) and \(s-g\) in the finitely randomized model set are given by bounded piecewise continuous functions \(e(s^*)\) and \(c_g(s^*)\), where \(s^*\) is the value of \(s\) by the star map, then

1. the inverse Fourier transform of the absolutely continuous component is a Dirac comb such that the support \(\{g_1, \ldots, g_n\}\) is the smallest \(D\) and the weight of the \(\delta_{g_i}\) is the average strength of the covariances between all the \(g_i\)-distant points.
2. the pure-point component is the diffraction measure of a Dirac comb 
\[ \sum_{s \in A} E[X_s] \delta_s \], the expectation of the random field.

This type of theorems are also seen in some other models with i.i.d. conditions. See [2] and references therein.

On the other hand, from the viewpoint of stochastic processes, we provide a sufficient condition for a randomly weighted Dirac comb on a model set to have diffraction measure whose expectation is still pure-point. The sufficient condition is satisfied when the set of the weights \( X \) to have diffraction measure whose expectation is still pure-point. The sufficient condition for a randomly weighted Dirac comb on a model set. The quantitative estimate will be established with the help of the so-called torus parametrization which was introduced in [3], then was extended in [22] to the locally compact, \( \sigma \)-compact Abelian Hausdorff groups (LCAH for short), and was finally fully exploited in [4]. Our approach is mostly based on the finite local complexity (FLC) of model sets, as in [22].

2 Basic properties of model sets: review

Throughout this paper, \( \mathfrak{G} \) and \( \mathfrak{G}_{\text{int}} \) are locally compact, \( \sigma \)-compact Abelian Hausdorff groups (LCAH for short).

**Definition 1** A cut-and-project scheme (c.-p. scheme, for short) is a triple \( \mathcal{S} = (\mathfrak{G}, \mathfrak{G}_{\text{int}}, \hat{L}) \) such that (1) \( \mathfrak{G} \) and \( \mathfrak{G}_{\text{int}} \) are called a physical space and an internal space respectively; (2) \( \hat{L} \) is a lattice of \( \mathfrak{G} \times \mathfrak{G}_{\text{int}} \), that is, a discrete subgroup of \( \mathfrak{G} \times \mathfrak{G}_{\text{int}} \) with \( (\mathfrak{G} \times \mathfrak{G}_{\text{int}})/\hat{L} \) being compact; (3) The canonical projection \( \Pi : \mathfrak{G} \times \mathfrak{G}_{\text{int}} \to \mathfrak{G} \) is injective on \( \hat{L} \), and the image of \( \hat{L} \) by the other canonical projection \( \Pi_{\text{int}} : \mathfrak{G} \times \mathfrak{G}_{\text{int}} \to \mathfrak{G}_{\text{int}} \) is dense in the internal space \( \mathfrak{G}_{\text{int}} \). For each \( s \in \Pi(\hat{L}) \), we write \( s^\star \) for \( \Pi_{\text{int}} \circ (\Pi|_{\hat{L}})^{-1}(s) \), \( L \) for \( \Pi(\hat{L}) \), and \( L^\star \) for \( (L)^\star \). The \((-)^\star \) is called the star map. Define

\[ A_S(W) := \{ \Pi(x) : x \in \hat{L}, \Pi_{\text{int}}(x) \in W \}. \]

We will often omit the subscript \( S \) when it is clear.

For sets \( A, B \subset U \ni x \), let \( A \pm B = \{ a \pm b : a \in A, b \in B \} \) and \( x + A \) be \( \{ x + a : a \in A \} \). A set \( A \subset \mathfrak{G} \) is said to be uniformly discrete, if \( (A-A) \cap U = \{ 0 \} \) for some open neighborhood \( U \) of 0, while \( A \) is said to be relatively dense, if \( \mathfrak{G} = A + K \) for some compact set \( K \). For a set \( U \), \( \text{Int}(U) \), \( \text{Cl}(U) \) and \( \partial U \) stand for the interior, the closure and the boundary \( \text{Cl}(U) \setminus \text{Int}(U) \), respectively.

**Lemma 1** If \( A \subset \mathfrak{G} \) is uniformly discrete, there is a compact neighborhood \( U \) of 0 such that \( U = -U \) and \( (s+U) \cap (s'+U) = \emptyset \) for all distinct \( s, s' \in A \).

**Definition 2** (Model set) Let \( (\mathfrak{G}, \mathfrak{G}_{\text{int}}, \hat{L}) \) be a c.-p. scheme. By a window, we mean a nonempty, measurable relatively compact subset of the internal space \( \mathfrak{G}_{\text{int}} \). If \( W \subset \mathfrak{G}_{\text{int}} \) is a window, \( A(W) \) is called a model set.
It is well-known that any model set is uniformly discrete. See [19, Proposition 2], for example. Every LCAG has a unique Haar measure up to normalization. Throughout this paper, we fix Haar measures of the LCAGs $\mathfrak{G}$ and $\mathfrak{G}_{\text{int}}$. The Haar measure of $\mathfrak{G}_{\text{int}}$ is denoted by $\theta$, and the integration of a function with respect to the Haar measure of $\mathfrak{G}$ ($\mathfrak{G}_{\text{int}}$ resp.) is denoted by $\int \cdots dx$ ($\int \cdots dy$ resp.) as usual. The Haar measure of a set $A$ is just denoted by $|A|$ if no confusion occurs. By a \textit{van Hove sequence} of $\mathfrak{G}$, we mean an increasing sequence $\{D_n\}_{n \in \mathbb{N}}$ of compact subsets of $\mathfrak{G}$ such that $|D_n| > 0$ for every $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and for every compact subset $K \subset \mathfrak{G}$, $\lim_{n \to \infty} |\partial^K(D_n)|/|D_n| = 0$, where for a compact set $A \subset \mathfrak{G}$,

$$\partial^K(A) := \left( (A + K) \setminus \text{Int}(A) \right) \cup \left( (\text{Cl}(\mathfrak{G} \setminus A) - K) \cap A \right). \quad (1)$$

In [22], the existence of a van Hove sequence for any LCAG is derived from

**Proposition 1** ([11, Theorem 9.8]) For every locally compact, compactly generated Abelian Hausdorff group $H$, there is $l, m \in \mathbb{Z}_{\geq 0}$, a compact Abelian Hausdorff group $K$, and an isomorphism $\phi$ from $H$ to $\mathbb{R}^l \times \mathbb{Z}^m \times K$.

Below we fix $\{D_n\}_n$. For a lattice $\hat{L} \subset \mathfrak{G} \times \mathfrak{G}_{\text{int}}$, the measure of the fundamental domain is denoted by $|\hat{L}|$, where the measure is the product measure of the Haar measures of $\mathfrak{G}$ and $\mathfrak{G}_{\text{int}}$.

For each discrete set $A$ of $\mathfrak{G}$, the density of $A$ with respect to the van Hove sequence $\{D_n\}_n$ is denoted by $\text{dense}_{\{D_n\}_n}(A) := \lim_{n \to \infty} \sum_{s \in A : \text{bp} \, D_n} |D_n|^{-1}$.

Let $C(A, B)$ be the set of continuous functions from $A$ to $B$. We say a class $\{V_1, \ldots, V_n\}$ of pairwise disjoint, relatively compact sets is a \textit{Riemann admissible partition} of relatively compact $W$, if $\bigcup_{i=1}^n V_i = W$ and the Haar measure of each boundary $\partial V_i$ is 0. Let $1_V : \mathfrak{G}_{\text{int}} \to \{0, 1\}$ be the indicator function of a set $V$, that is, $1_V(y) = 1$ for $y \in V$, and 0 otherwise.

**Definition 3** For a relatively compact set $W \subset \mathfrak{G}_{\text{int}}$, we say a function $f : \mathfrak{G}_{\text{int}} \to \mathbb{R}$ is \textit{bounded piecewise continuous} on $W$, if it is bounded and there is a Riemann admissible partition $\{W_i\}_{i=1}^n$ of $W$ such that $f|_{W_i}$ is continuous with respect to the relative topology induced by $\mathfrak{G}_{\text{int}}$. Let $C_{\text{bp}}(\mathfrak{G}_{\text{int}})$ be the set of functions $f$ which is bounded piecewise continuous on some relatively compact $W \subset \mathfrak{G}_{\text{int}}$ such that $\sup f \subset \text{Cl}(W)$.

We can slightly generalize [6, Proposition 6.2] as follows (see also [19]):

**Proposition 2** (Weyl’s theorem for model sets) For any relatively compact window $W \subset \mathfrak{G}_{\text{int}}$ such that $|\partial W| = 0$ and for any $f \in C_{\text{bp}}(\mathfrak{G}_{\text{int}})$, we have

$$\lim_{n \to \infty} \frac{1}{|D_n|} \sum_{s \in A(W) \cap D_n} f(s^*) = \frac{1}{|L|} \int_W f(y) \, dy.$$

It is proved by applying Proposition 6.2 of [6] to each $W_i$ of the Riemann admissible partition $\{W_i\}_{i=1}^n$. 
2.1 Diffraction

Mathematical diffraction theory, introduced by Hof [13], is reviewed below according to [7]. We say a countable set $S \subseteq S$ is FLC, if $S - S$ is closed and discrete. Let a bounded complex sequence $\{w_s\}_{s \in S}$ be indexed over an FLC set $S \subseteq S$ such that the corresponding Dirac comb $\omega := \sum_{s \in S} w_s \delta_s$ defines a regular Borel measure on $S$. Here $\delta_s$ is a Dirac measure of $S$ such that $\delta_s(A) = 1$ if $s \in A$ and 0 otherwise for any $A \subseteq S$. We often identify $\omega$ with $\{w_s\}_{s \in S}$. For a van Hove sequence $\{D_n\}_n$ on $S$, set $\omega_n := \omega|_{D_n} := \sum_{s \in S \cap D_n} w_s \delta_s$. For any complex measure $\mu$ on $S$, let $\tilde{\mu}$ be $\tilde{\mu}(A) = \mu(-A)$. Set $\gamma_{\omega}^{(n)} := \omega_n * \omega_n/|D_n|$, where * is the convolution. Actually we have $\gamma_{\omega}^{(n)} = \sum_{g \in S - S} \eta_{\omega}(g) \delta_g$ where $\eta_{\omega}(g)$ is the summation of $w_s \omega_s / |D_n|^{-1}$ over $s, t \in S \cap D_n$ such that $s - t = g$. Since $S$ is discrete and $D_n$ is compact, $\eta_{\omega}$ is well-defined, and $\gamma_{\omega}^{(n)}$ is so.

Let $\text{Cc}(S)$ be the set of complex continuous functions on $S$ with compact support in $S$. Let the autocorrelation measure of $\omega$ be the limit $\gamma_{\omega}$ of $\gamma_{\omega}^{(n)}$ in the vague topology. Then $\gamma_{\omega}$ is written as

$$
\gamma_{\omega} = \sum_{g \in S - S} \eta_{\omega}(g) \delta_g, \quad \eta_{\omega}(g) = \lim_{n \to \infty} \eta_{\omega}^{(n)}(g).
$$

For any lcag $H$, the dual group of $H$ is denoted by $\hat{H}$. When $h$ is a Haar measure of $H$, the Fourier transform of a function $f : H \to \mathbb{C}$ is $\hat{f} : \hat{H} \to \mathbb{C}$ such that $\hat{f}(\chi) := \int_H f(x) \chi(x) dh(x)$. The diffraction measure of $\omega$ is, by definition, the Fourier transform $\hat{\omega}$ of the autocorrelation measure $\gamma_{\omega}$. A measure on $S$ has Fourier transform as a measure on $\hat{S}$ as follows:

**Proposition 3 ([1, Theorem 2.1, Theorem 4.1])** Suppose $\lambda$ is a measure on $S$. The Fourier transform (if it exists) of $\lambda$ is a unique measure $\hat{\lambda}$ on $\hat{S}$ such that for all $\varphi \in C_c(S)$

$$
\int_S (\varphi \ast \check{\varphi})(x) d\lambda(x) = \int_{\hat{S}} |\check{\varphi}(\chi)|^2 \check{d}\lambda(\chi).
$$

Moreover if $\lambda$ is positive definite (i.e., $\int_S (\varphi \ast \check{\varphi})(x) d\lambda(x) \geq 0$ for all $\varphi \in C_c(S)$), then $\lambda$ indeed has the Fourier transform $\hat{\lambda}$. Here $\varphi : S \to \mathbb{C}$ be $x \in S \mapsto \varphi(-x)$.

The Haar measure on $\hat{S}$ is $\check{\delta}_0$ where $\delta_0$ is the Dirac measure at 0 on $S_{\text{int}}$. Then the equation of Proposition 3 amounts to a Plancherel formula. The integral of a function with respect to the Haar measure of $\hat{S}$ is denoted by $\int_{\hat{S}} f \check{d}\lambda$. Because $\check{\gamma}_{\omega} = \gamma_{\omega}$, we have $\check{\gamma}_{\omega} = \gamma_{\omega}$ and thus $\check{\gamma}_{\omega}$ is positive by [1, Proposition 4.1].

A measure $\mu$ on $S$ is said to be translationally bounded, if for every $\varphi \in C_c(S)$ the set $\{\int_S \varphi(x + t) d\mu(x) : t \in S\}$ is bounded.

Baake-Moody established the pure-point diffraction of weighted Dirac comb on model sets, by using Weyl’s theorem ([6, Proposition 6.2]) for model sets and an ingenious topological space.
Proposition 4 ([7, Theorem 2]) For any model set \( A(W) \) with \( W \subset \Theta_{\text{int}} \)
being a window and \(|\partial W| = 0\), and for any \( f : \Theta_{\text{int}} \to \mathbb{C} \) supported and continuous on \( \text{Cl}(W) \), the diffraction measure \( \hat{\tau}_\omega \) of the Dirac comb \( \omega = \sum_{s \in A(W)} f(s^*) \delta_s \) is translationally bounded, nonnegative and pure-point.

3 Finitely randomized model sets

For a random field on discrete sets, if the range of the correlation from each site is within a predefined compact set, we can easily find an independent subset of the random field (Subsection 3.1). For such a random field defined on a model set that satisfies a natural geometric condition (Subsection 3.2), we prove that such a random field can be partitioned into a finite number of random fields, each being independent, in Subsection 3.3. Hereafter, a “random variable” is abbreviated as “rv,” and the cardinality of a finite set \( A \) is denoted by \(#A\).

3.1 Independence in random field

Definition 4 (Dependency set) Let \( \{X_s\}_{s \in S} \) be a random field on a discrete set \( S \). A dependency set (d-set for short) is a set \( D = -D \subset S - S \) such that for any finite sets \( P, Q \subset S \), if a set \( (P - Q) \) is disjoint from \( D \), then a pair of a #\( P \)-dimensional random vector \( \{X_s\}_{s \in P} \) and a #\( Q \)-dimensional random vector \( \{X_s\}_{s \in Q} \) is independent. A d-set necessarily has 0 as an element. If a random field has a dependency set, we can replace it with an arbitrary superset of it.

A d-set is a patch such that each site \( s \) has correlation on, at most, sites belonging to the patch relative to \( s \). Recall that a sequence \( \{X_1, X_2, \ldots\} \) of rv’s is independent, if so are any finite subsequences.

Lemma 2 (Independence) Let \( D \) be a d-set of a random field \( \{X_s\}_{s \in S} \) on an FLC subset \( S \) of an LCA group. If \( N \subset S - S \) and \( D \cap ((s + N) - (t + N)) = \emptyset \) for any distinct \( s, t \in S \), then a sequence \( \left\{ \prod_{t \in (s + N) \cap S} X_t : s \in S \right\} \) is independent. Furthermore, the random field is independent, if and only if the random field has \( \{0\} \) as a d-set.

Proof We show that a sequence \( \left\{ \prod_{t \in (s + N) \cap S} X_t : s \in S' \right\} \) of rv’s is independent for any finite subset \( S' = \{s_1, \ldots, s_{\nu}\} \) of \( S \). The proof is by induction on \( \nu = \#S' \). When \( \nu = 1 \), it is trivial, so assume \( \nu > 1 \). A set \( \{(s_1, \ldots, s_{\nu-1}) + N\} - (s_{\nu} + N) = \bigcup_{i=1}^{\nu-1} (s_i + N - (s_{\nu} + N)) \) is disjoint from \( D \) by the premise. So, a random vector \( \{X_t : t \in ((s_1, \ldots, s_{\nu-1}) + N) \cap S\} \) is independent from a random vector \( \{X_t : t \in (s_{\nu} + N) \cap S\} \). Thus a \( (\nu - 1) \)-dimensional random vector \( \left\{ \prod_{t \in (s_i + N) \cap S} X_t : 1 \leq i \leq \nu - 1 \right\} \) is independent from an RV \( \prod_{t \in (s_{\nu} + N) \cap S} X_t \). Because the \( (\nu - 1) \)-dimensional random vector is independent by the induction hypothesis on \( \nu \), we are done. The if part
of the last sentence is proved by taking \( N = \{0\} \), while the only-if part is immediate.

**Definition 5** A finitely randomized model set (FRMS for short) on a model set \( \Lambda(W) \) is a random field \( \{X_s\}_{s \in \Lambda(W)} \) with a finite \( d \)-set \( D \).

The FRMS can be regarded as a Dirac comb with random weights, and when each \( X_s \) is an indicator (i.e., a \( \{0, 1\} \)-valued rv), we intend that \( X_s > 0 \) if and only if \( s \in \Lambda(W) \) indeed appears.

**Example 1** Let \( \Gamma = \Lambda(W) \) be a model set, as in Definition 1, with the star map \((-)^*\) being injective, and let \( C \) be \( \{p_1, \ldots, p_n\} \subseteq L \) such that \( (W + p_i^t) \cap (W + p_j^t) \cap L^* = \emptyset \) (\( i \neq j \)). Then \( \Lambda = \Lambda(W + C^t) \) is a model set, and for every \( s \in \Lambda \) there are unique \( t \in \Gamma \) and unique \( i \in \{1, \ldots, n\} \) such that \( s = t + p_i \). Let \( \{Y_t\}_{t \in \Gamma} \) be an infinite independent sequence of \( n \)-dimensional random vectors taking values in \( \{0, 1\}^n \). Define a random field \( \{X_s\}_{s \in \Lambda} \) by putting \( X_{t+p} \) as the \( i \)-th component of \( Y_t \). Then we can prove that \( \{X_s\}_{s \in \Lambda} \) is indeed a random field by Kolmogorov’s consistency theorem [23, p.129]. Below we explain that \( D := C - C \) is a \( d \)-set of this random field. Let \( P, Q \) be finite subsets of \( A \) such that the pair of a \#\( P \)-dimensional random vector \( (X_s)_{s \in P} \) and a \#\( Q \)-dimensional random vector \( (X_s)_{s \in Q} \) is not independent. Take minimal subsets \( P', Q' \) of \( \Gamma \) such that \( P \subseteq P' + C \) and \( Q \subseteq Q' + C \). By the premise, the pair of random vectors \( (Y_t)_{t \in P'} \) and \( (Y_t)_{t \in Q'} \) is not independent. But \( (Y_t)_{t \in P'} \) is independent, so there is \( t \in P' \cap Q' \). Then, by the minimality of \( P', Q' \), there are \( p, q \in C \) such that \( t + p \in P \) and \( t + q \in Q \). Therefore \( p - q \in P - Q \). Thus \( (P - Q) \cap D \neq \emptyset \).

**Example 2** (FRMS caused by random shift of windows) For a model set \( \Lambda(W) \) with both \( \mathfrak{G} \) and \( \mathfrak{G}_{\text{int}} \) being Euclidean vector spaces, physicists often associate to each site \( s \in L \) its own window \( W_s = W + s \), where the “shift” \( y_s \) is an rv ranging over \( R \subseteq \mathfrak{G}_{\text{int}} \), a window with nonempty interior. Then \( W + R \) is again a window. For \( s \in \Lambda(W + R) \), define an indicator rv \( X_s \) to be 1 for \( s^* \in W_s \) and 0 otherwise. If the sequence \( \{y_s\}_{s \in \Lambda(W)} \) of the rv’s is independent, then the random field \( \{X_s\}_{s \in \Lambda(W + R)} \) is independent, so it is a FRMS on a model set \( \Lambda(W + R) \).

However, if \( y_s(\omega) = y_t(\omega) \) for any \( s, t \in L \) and any \( \omega \) of the probability space, then the random field is not a FRMS, because no finite \( d \)-set can be taken owing to the existence of a relatively dense subset \( \Gamma := \Lambda(W + R) \setminus W \subseteq \Lambda(W + R) \) such that a sequence \( \{X_s\}_{s \in \Gamma} \) of the rv’s is not independent. Here the relative density follows from \( \text{Int}((W + R) \setminus W) \neq \emptyset \), and the proof is in the appendix.

3.2 Finitely periodic model sets and internal space

**Definition 6** For any lcag \( H \), any set \( A \subset H \), any \( x \in A - A \) and any \( s \in A \), define \( \ell_A(x : s) \) as \( \ell_A(0 : s) := 0 \) and \( \ell_A(x : s) \) being the maximum \( k \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \) such that \( \{s - nx : 0 \leq n \leq k\} \subset A \). Set \( \ell_A(x) := \max_{s \in A} \ell_A(x : s) \). If \( \ell_A(x) \) is finite for any \( x \in A - A \), we say \( A \) is finitely periodic. We say a FRMS \( \{X_s\}_{s \in A} \) is finitely periodic, if so is \( A \).
Lemma 3 A model set is finitely periodic, if the star map is injective and the internal space is isomorphic to $\mathbb{R}^l \times \mathbb{Z}^m \times F$ for some $l, m \in \mathbb{Z}_{\geq 0}$ and some finite Abelian group $F$.

Proof Assume the model set $A(W)$ is over a c.-p. scheme $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \tilde{L})$ and let $N = \#F$. Suppose $x \in A(W) - A(W)$ satisfies that for all $\ell \in \mathbb{N}$ there is $s_\ell \in A(W)$ such that $\#\{s_\ell - nx \in A(W) : 0 \leq n < N\ell\} = \ell N$. Let $\varphi$ be the isomorphism from $\mathfrak{G}_{\text{int}}$ to the $\mathbb{R}^l \times \mathbb{Z}^m \times F$ and let $\psi_1$ be a homomorphism $L \to L \hookrightarrow \mathfrak{G}_{\text{int}} \cong \mathbb{R}^l \times \mathbb{Z}^m \times F$ and let $\psi_2$ be the other homomorphism from $L$ to $F$. Because $F$ is a finite group, $N\psi_2(x) = 0$. Since the star map is injective, $\psi_1(Nx) \neq 0$. So the compact set $\pi(\varphi(Cl(W))) \subset \mathbb{R}^l \times \mathbb{Z}^m$ contains $\{\psi_1(s_\ell) - n\psi_1(Nx) : 0 \leq n < \ell\}$. In fact, we can take $\ell$ greater than $d/d'$ where $d$ is the diameter of the compact set $\pi(\varphi(Cl(W)))$ and $d'$ is the norm of $\psi_1(Nx)$. Contradiction.

Example 3 1. The vertex sets of the rhombic Penrose tilings are finitely periodic model sets with $\mathfrak{G}_{\text{int}} = \mathbb{C} \times (\mathbb{Z}/5\mathbb{Z})$ and injective star maps. See [18, Section 3.2].

If an internal space is the compact Abelian group of $p$-adic integers, we can find a model set [5] which is not finitely periodic.

2. Let the internal space be the ring $\hat{\mathbb{Z}}_p$ of $p$-adic integers, which is a compact Abelian group. When the physical space is $\mathbb{R}$, the star map is the canonical injection from $\mathbb{Z}$ to $\hat{\mathbb{Z}}_p$, and the window $W$ is $\hat{\mathbb{Z}}_p$, the model set is not finitely periodic.

3.3 Decomposition of finitely randomized model sets

Definition 7 A cut-and-project subscheme of a c.-p. scheme $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \tilde{L})$ is a c.-p. scheme $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \tilde{M})$ such that $\tilde{M} \subset \tilde{L}$, $\mathfrak{G}_{\text{int}} \subset \mathfrak{G}_{\text{int}}$, and the topology of $\mathfrak{G}_{\text{int}}$ is the relative topology induced from $\mathfrak{G}_{\text{int}}$.

Lemma 4 If an lcag $\mathfrak{G}_{\text{int}}$ is a subgroup of an lcag $\mathfrak{G}_{\text{int}}$, then the Haar measure $\theta$ of $\mathfrak{G}_{\text{int}}$ restricted to $\mathfrak{G}_{\text{int}}$ is a Haar measure $\vartheta$ of $\mathfrak{G}_{\text{int}}$ such that

$$\theta(\partial_{\mathfrak{G}_{\text{int}}}(B)) = 0 \Rightarrow \vartheta(\partial_{\mathfrak{G}_{\text{int}}}(B \cap \mathfrak{G}_{\text{int}})) = 0.$$
Lattices in $\mathfrak{G} \times H$ with $H$ being an LCAG are written as $\tilde{L}, \tilde{M}, \ldots$, and their canonical projections to $\mathfrak{G}$ are written as $L, M, \ldots$. For an LCAG $H$, let $\partial_H(A)$ be the boundary of $A$ with respect to the topology of $H$.

**Lemma 5** For any c.-p. scheme $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \tilde{L})$ with $\tilde{L}$ being finitely generated and for any finite subset $D \subset L$ there are a c.-p. subscheme $(\mathfrak{G}, \mathfrak{S}_{\text{int}}, \tilde{M})$ of $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \tilde{L})$ and a finite complete representation system $R$ of $L/M$ with $D \subset R$.

**Proof** By the structure theorem of a finitely generated Abelian group, $\tilde{L}$ is isomorphic to $\mathbb{Z}^r \times \mathbb{Z}/\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}/\mathbb{Z}_{n_u}$ for some $u, v \in \mathbb{Z}_{\geq 0}$ and for some integers $n_1, \ldots, n_u \geq 2$. For an integer $k$ dividing all of $n_1, \ldots, n_u$, define $M = kL$. Then we can find a finite complete representation system $R \supseteq D$. Let $M := \{(t, t^*); t \in M\}$ and $\mathfrak{S}_{\text{int}} := \text{Cl}(M^*) \subset \mathfrak{G}_{\text{int}}$. Then $\mathfrak{S}_{\text{int}}$ is a $\sigma$-compact LCAG with the relative topology induced from $\mathfrak{G}_{\text{int}}$, and $(\mathfrak{G}, \mathfrak{S}_{\text{int}}, \tilde{M})$ is a c.-p. scheme with the star map being a restriction of that of $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \tilde{L})$.

We consider the following condition:

**Condition 1** $\Lambda_C(W)$ is a finitely periodic model set over a c.-p. scheme $\mathcal{O} = (\mathfrak{G}, \mathfrak{G}_{\text{int}}, \tilde{L})$ with $\tilde{L}$ being finitely generated.

Recall that $\theta$ is a Haar measure of $\mathfrak{G}_{\text{int}}$.

**Theorem 1 (Decomposition)** Under the Condition 1, there are a c.-p. subscheme $\mathcal{S} = (\mathfrak{G}, \mathfrak{S}_{\text{int}}, M)$ of $\mathcal{O}$ and a finite complete representation system $R = \{r_C \in L; C \in L/M\}$ of $L/M$ such that for each $g \in \Lambda_{\mathcal{O}}(W) - \Lambda_{\mathcal{O}}(W)$ with $g \equiv r \in R$ mod $M$, we have

1. $\Lambda_{\mathcal{O}}(W)$ is the disjoint union of the sets $S_{C,k} := r_C + A_S(V_{C,k}), \quad (C \in L/M, \quad 0 \leq k \leq \ell := \ell_{\Lambda_{\mathcal{O}}(W)}(g-r))$ for some relatively compact sets $V_{C,k}$;

2. the sequence $\{X, X_{s-g}; s \in S_{C,k}, s-g \in \Lambda_{\mathcal{O}}(W)\}$ is independent; and

3. If $\theta(\partial W) = 0$, dense_{(\Lambda_{\mathcal{O}})}, (S_{C,k}) exists for all $C \in L/M$ and $0 \leq k \leq \ell$.

**Proof** Let $\mathcal{S}$ and $R$ be as in Lemma 7 applied for a d-set $D$ of the FRMS. Since $0 \notin D$, if $g = 0$ then $r = 0$. Because $\Lambda_{\mathcal{O}}(W)$ is finitely periodic, $\ell$ is well-defined and we have and relatively compact sets $W_k := \{y \in W; \ell_{\mathcal{O}}(g^* - r^*; y) = k\} \quad (0 \leq k \leq \ell)$.

1. Then $\Lambda_{\mathcal{O}}(W) = \coprod_{k=0}^{\ell} \{s \in L; \quad s^* \in W_k\}$ where $\coprod$ is a disjoint union. Since $L = \coprod_{C \in L/M} (r_C + M)$, we have $\Lambda_{\mathcal{O}}(W) = \coprod_{k=0}^{\ell} \coprod_{C \in L/M} \{s \in r_C + M; \quad s^* - r_C^* \in (W_k - r_C^*) \cap \mathfrak{S}_{\text{int}}\}$ because $M^* \subset \mathfrak{S}_{\text{int}}$. Therefore $\Lambda_{\mathcal{O}}(W) = \coprod_{k=0}^{\ell} \coprod_{C \in L/M} \left( r_C + A_S((W_k - r_C^*) \cap \mathfrak{S}_{\text{int}}) \right)$. So $V_{C,k} = (W_k - r_C^*) \cap \mathfrak{S}_{\text{int}}$. (2)
(2) By Lemma 2, it suffices to prove the following claim: for any distinct
$s, t \in S_{C,k}$, \( \{ s, s - g \} \cap \{ t, t - g \} \cap D = \emptyset \). In other words, for any \( g' \in D \) we have 
(i) \( s - t = (s - g) - (t - g) \neq g' \); (ii) \( s - (t - g) \neq g' \); and (iii) 
\((s - g) - t \neq g'\).

It is proved as follows: By Lemma 5, \( M \cap D = \{ 0 \} \), which derives the 
assertion (i) from \( s - t \in M \setminus \{ 0 \} \). If \( g = 0 \), then (ii) and (iii) follow from (i). So 
let \( g \neq 0 \). Assume (ii) is false. Then \( g' - g = s - t \in M \), so \( g' \equiv g \equiv r \) mod \( M \) 
for a unique \( r \in R \). Then \( g' = r \), because \( g' \in D \) belongs to \( R \). Therefore 
t - s = g - r. But \( t - (g')^* = s^* \in W_k \ni t^* \) leads to a contradiction. 
Therefore (ii) holds. The assertion (iii) follows from (ii) since \( D = -D \).

(3) The premise \(|\partial_{\Theta_{\text{int}}} W| = 0\) implies
\[ \partial(\partial_{\Theta_{\text{int}}} V_{C,k}) = 0. \] (3)

To see it first observe that \( W_k = W \cap (W + g - r) \cap \cdots \cap (W + (k-1)(g-r)) \setminus \)
\( (W + (k+1)(g-r)) \). So the premise implies \(|\partial_{\Theta_{\text{int}}} (W_k - rC)| = 0\) because of fact \( \partial_{\Theta_{\text{int}}} (A \cap B) \subset \partial_{\Theta_{\text{int}}} (A) \cup \partial_{\Theta_{\text{int}}} (B) \) and fact \( \partial_{\Theta_{\text{int}}} (A) = \partial_{\Theta_{\text{int}}} (\Theta_{\text{int}} \setminus A) \). 
Thus (3) follows from Lemma 4.

Since \((D_n - rC)_n\) is a van Hove sequence too, \( \text{dense}(D_n - rC)_n \cap \text{int}(V_{C,k}) \) 
converges by Proposition 2. But it is dense \((D_n)_n \cap (S_{C,k}) \) by Theorem 1(1).

4 Absolutely continuous component of diffraction and covariance

If a complex-valued FRMS \( \omega = \{ X_s \}_{s \in A} \) satisfies Condition 1 and all of the 
extpectation \( E[X_s] \) and the covariances between \( X_s \) and \( X_{s-g} \) are “continu-
ous” with respect to \( s^* \in \Theta_{\text{int}} \) for any \( g \in A - A \), then we quantitatively give 
the diffraction measure of \( \omega \) as follows:

- the inverse Fourier transform of the absolutely continuous component is 
a Dirac comb whose support is the smallest d-set; and
- the pure-point component is the diffraction measure of a Dirac comb 
which is the expectation of the FRMS \( \omega \).

Here

Definition 8 The expectation of a FRMS \( \omega = \{ X_s \}_{s \in A} \) is, by definition, a 
Dirac comb \( E[\omega] = \{ E[X_s]\}_{s \in A} \), that is, \( \sum_{s \in A} E[X_s] \delta_s \).

We use Kolmogorov’s strong law of large numbers. By the variance of an 
RV \( X \), we mean \( V[X] = E[|X - E[X]|^2] = E[|X|^2] - E[X]^2 \).

Proposition 5 ([23, Corollary 1.4.9]) Suppose \( \{ b_m ; m \in \mathbb{N} \} \) be a non-
decreasing sequence of positive numbers which tends to infinity, and that a set 
\( \{ X_n \}_{n \in \mathbb{N}} \) of square integrable RV’s is independent. If \( \sum_{m=1}^{\infty} V[X_m] b_m^{-2} < \infty \), then 
\[ \lim_{m \to \infty} \frac{1}{b_m} \sum_{i=1}^{m} (X_i - E[X_i]) = 0 \quad (\text{almost surely}). \]
Lemma 6 If \( A \subset \mathcal{G} \) is a nonempty discrete set and \( \{Y_s\}_{s \in A} \) is an independent set of RV’s with the variances \( \mathbb{V}[Y_s] \) bounded uniformly from above, then
\[
\lim_{n \to \infty} \frac{1}{\#(A \cap D_n)} \sum_{s \in A \cap D_n} (Y_s - \mathbb{E}[Y_s]) = 0 \quad \text{(almost surely)}.
\]

Proof There is an enumeration \( \{s_i\}_{i \in \mathbb{N}} \) of \( A \) without repetition which exhausts \( A \cap D_1 \) first, then \( A \cap D_2, A \cap D_3, \) and so on, because \( \#(A \cap D_n) < \infty \) follows from the discreteness of \( A \) and the compactness of \( D_n \). For \( m \in \mathbb{N} \) let \( b_m \) be \( \#(A \cap D_n) \) if \( s_m \in A \cap D_n \). Then \( b_m \geq m \) and the equality holds for \( m = \#(A \cap D_n) \), because the sequence \( \{D_n\}_n \) is increasing. So \( \sum_{m=1}^{\infty} b_m^{-2} \leq \sum_{m=1}^{\infty} m^{-2} < \infty \), and \( \sum_{m=1}^{\infty} \mathbb{V}[Y_{s_m}]m^{-2} < \infty \) by the premise. Moreover the sequence \( \{Y_{s_i}\}_{i \in \mathbb{N}} \) is independent from the premise. So the sequence \( \{\sum_{s \in A \cap D_n} (Y_s - \mathbb{E}[Y_s]) / \#(A \cap D_n)\}_{n \in \mathbb{N}} \) converges to 0 almost surely, by Proposition 5. Hence a subsequence \( \{\sum_{s \in A \cap D_n} (Y_s - \mathbb{E}[Y_s]) / \#(A \cap D_n)\}_{n \in \mathbb{N}} \) does so almost surely.

By the covariance between complex-valued RV’s \( X_s \) and \( X_t \), we mean
\[
\mathbb{C}[X_s, X_t] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[X_i \overline{X}_j] - \mathbb{E}[X_i] \mathbb{E}[\overline{X}_j].
\]

**Condition 2** A FRMS \( \{X_s\}_{s \in A(W)} \) has functions \( e, c_g \in C_{bp}(\mathcal{G}_{\text{int}}) \) such that
\[
\mathbb{E}[X_s] = e(s^*), \quad (s \in A(W));
\]
\[
\mathbb{C}[X_s, X_{s-g}] = c_g(s^*), \quad (g \in A(W) - A(W), \ s \in A(W) \cap (A(W) + g)).
\]

**Example 4** In Example 1, suppose that there are \( m \in C(\mathcal{G}_{\text{int}}, \mathbb{R}^n) \) and \( S \in C(\mathcal{G}_{\text{int}}, \mathbb{R}^{n \times n}) \) such that \( m(t^*) \) is the mean \( \mathbb{E}[Y_t] \) and \( S(t^*) \) is the covariance matrix \( \mathbb{C}[(Y_t - \mathbb{E}[Y_t])(Y_t - \mathbb{E}[Y_t])^\top] \) for all \( t \in \Gamma \). Then functions \( e, c_g \ (g \in A - A) \) indeed belong to \( C_{bp}(\mathcal{G}_{\text{int}}) \). For the FRMS of Example 2, assume further that each shift \( y_s \) is subject to a continuous probabilistic density function \( h \) with suph \( h = R \). Then, for \( s \in L \), \((1_W * \mathbb{H})(s^*) = \int_{\mathcal{G}_{\text{int}}} 1_W(s^* - y)h(y)dy = \int_{W \cap (s^* - y)} h(y)dy = P(s^* \in W + y_s) \), the probability for \( s \in A(W + R) \) to indeed appear. So \( e = 1_W \ast h \).

**Theorem 2** Let \( \omega = \{X_s\}_{s \in A(W)} \) be a FRMS such that Condition 2 holds, \( \{\mathbb{E}[|X_s|^4]\}_{s \in A(W)} \) is bounded, and \( W \) is compact but \( \partial W \neq \emptyset \). Then the diffraction measure \( \hat{\gamma}_\omega \) of \( \omega \) is almost surely \( \hat{\gamma}_{\mathbb{E}[\omega]} + A \), where

1. \( \hat{\gamma}_{\mathbb{E}[\omega]} \) is a pure-point diffraction measure.
2. \( A \) is an absolutely continuous, real-valued measure on \( \hat{\mathcal{G}} \). In fact, there is some \( d \)-set \( D \) of the FRMS \( \omega \) such that the Radon-Nikodym derivative of \( \mathbb{E}[\omega] \) with respect to the Haar measure \( d\mathbb{E} \) of \( \hat{\mathcal{G}} \) is \( \sum_{g \in D} A_g \mathbb{E}(-g) \) where
\[
A_g = \int_{W \cap (W + g^*)} c_g(y)dy = \lim_{n \to \infty} \sum_{s \in A(W) \cap (A(W) + g) \cap D_n} \mathbb{C}[X_s, X_{s-g}] / |D_n|.
\]

Proof We use the notation of Theorem 1. Let \( g \in A_{\mathcal{G}}(W) - A_{\mathcal{G}}(W) \).
Claim 1. \( s, s - g \in A_G(W) \cap D_n \) if and only if there is a unique \( (C, k) \in (L/M) \times \{0, \ldots, \ell \} \) such that \( s \in S_{C,k} \cap (A_G(W) + g) \cap D_n \cap (D_n + g) \).

2. There are \( r_D \in R \) and \( m \in M \) such that \( S_{C,k} \cap (A_G(W) + g) = r_C + A_S(W_{C,k}) \), \( W_{C,k} = V_{C,k} \cap ((W - r_D + m*) \cap \mathcal{F}_{\text{int}}) \) and \( g + r_D = r_C + m \).

3. \( \lim_{n \to \infty} \sum_{s \in A_G(W) \cap D_n} (\cdots) \cdot |D_n|^{-1} \) is summation of

\[
\lim_{n \to \infty} \sum_{s \in (r_C + A_S(W_{C,k})) \cap D_n \cap (D_n + g)} (\cdots) \cdot |D_n|^{-1}
\]

over \( (C, k) \in (L/M) \times \{0, \ldots, \ell \} \).

Proof of Claim 4 (1) By Theorem 1(1) and the finiteness of \((L/M)\). (2) Because of Theorem 1(1). (3) is due to (1) and (2).

Suppose \( C \in L/M, 0 \leq k \leq \ell \) and \( S_{C,k} \neq \emptyset \). The variances \( V[\overline{X_s X_{s-g}}] \leq E[|X_s X_{s-g}|^2] \leq E[|X_s|^4]/2 + E[|X_{s-g}|^4]/2 \) are uniformly bounded, because of the premise. By this, Theorem 1(2) and Lemma 6, it holds almost surely that as \( n \) goes to infinity, the deviation of

\[
\sum_{s \in (r_C + A_S(W_{C,k})) \cap D_n \cap (D_n + g)} \frac{X_s X_{s-g}}{\#((r_C + A_S(W_{C,k})) \cap D_n \cap (D_n + g))}
\]

from the expectation tends to 0 in absolute value.

Claim Let \( f \in C_{bp}(\mathcal{G}_{\text{int}}) \). Then, as \( n \to \infty \), both of the summation of \( f(s^*)|D_n|^{-1} \) over \( s \in (r_C + A_S(W_{C,k})) \cap D_n \cap (D_n + g) \) and that over \( s \in (r_C + A_S(W_{C,k})) \cap D_n \) converge to the same value.

Proof The absolute value of the difference between the two summations is not greater than \( \sum_{s \in (r_C + A_S(W_{C,k})) \cap D_n \cap (D_n + g)} |f(s^*)| / |D_n| \), where \( r_C + A_S(W_{C,k}) \) is uniformly discrete by Claim 4(2). Thus by Lemma 1, there is a compact neighborhood \( U \) of 0 such that for all \( n \), \( \#((r_C + A_S(W_{C,k})) \cap (D_n \setminus (D_n + g))) \) is not greater than \( |D_n \setminus (D_n + g)| / |U| \). Since \( f(y) \in C_{bp}(\mathcal{G}_{\text{int}}) \), \( |f(y)| \) is uniformly bounded by a positive number \( b \), so the absolute value of the difference is less than or equal to \( |D_n \setminus (D_n + g)| \cdot b|U|^{-1} / |D_n| = b|U|^{-1} |D_n \setminus (D_n + g)| / |D_n| \leq b|U|^{-1} |D_n| / |D_n| \to 0 \). Moreover, by Proposition 2, the summation of \( f(s^*) / |D_n| \) over \( s \in (r_C + A_S(W_{C,k})) \cap D_n \) converges as \( n \) goes to infinity. To see it, Claim 4(2) implies \( \partial_{D_{\text{int}}}(V) \) is contained by \( \partial_{D_{\text{int}}}(W_{C,k}) \cup \partial_{D_{\text{int}}}(W - r_D + m^*) \cap \mathcal{F}_{\text{int}} \) which has null measure with respect to the Haar measure \( \vartheta \) of \( \mathcal{F}_{\text{int}} \), where \( \vartheta(\partial_{D_{\text{int}}}(W_{C,k})) = 0 \) by (3), while \( |W| = 0 \) implies that \( (W - r_D + m^*) \cap \mathcal{F}_{\text{int}} \) has \( \vartheta \)-null boundary by Lemma 4.

By applying Claim 4 for \( f(y) := 1 \), \( \#(S_{C,k} \cap D_n \cap (D_n + g)) / |D_n| \) tends to \( \text{dense}(D_n)_n (r_C + A_S(W_{C,k})) = \text{dense}(D_n)_n A_S(V) \), which is convergent.

Thus, it holds almost surely that as \( n \) goes to \( \infty \), the deviation of

\[
\sum_{s \in (r_C + A_S(W_{C,k})) \cap D_n \cap (D_n + g)} \frac{X_s X_{s-g}}{|D_n|}
\]
from the expectation tends to 0 in absolute value. The expectation of (5) tends to following sum of two convergent limits

\[
\lim_{n \to \infty} \frac{1}{|D_n|} \sum_{s \in (r_C + A_S(W_C,k)) \cap D_n \cap (D_n + g)} e(s^*) e(s^* - g) + \lim_{n \to \infty} \sum_{s \in (r_C + A_S(W_C,k)) \cap D_n} c_g(s^*)
\]

which summand is \(E[X_s X_{s-g}].\) Hence it holds almost surely that (5) tends to (6) as \(n\) goes to infinity.

By taking the summation (5) and (6) respectively over any \((C,k) \in (L/M) \times \{0, \ldots, \ell\}\) such that \(S_{C,k} \neq \emptyset,\) by Claim 4(3), we have almost surely \(\eta_s(g) = \eta_E[\omega](g) + A_g,\) and thus \(\tilde{\eta}_\omega = \tilde{\gamma}_E[\omega] + \sum_{g \in S - S} A_g \chi(-g).\)

1. By Proposition 4 and Condition 2, \(E[\omega]\) is indeed pure-point.
2. The Radon-Nikodym derivative \(A(\chi) = \sum_{g \in S - S} A_g \chi(-g)\) is actually \(A(\chi) = \sum_{g \in D} A_g \chi(-g).\) Indeed \(\text{Cov}[X_s, X_{s-g}]\) is equal to 0 for any \(g \in (S - S) \setminus D\) and for any \(s \in A_O(W) \cap (A_O(W) + g),\) because a pair of \(X_s\) and \(X_{s-g}\) is independent for this \(g\) by Definition 4.

**Corollary 1** Under the same assumption as in Theorem 2, the smallest d-set of the FRMs is the support of the Dirac comb which is obtained by the inverse Fourier transform of the absolutely continuous component \(A\) of the diffraction measure \(\tilde{\gamma}_\omega.\)

**Proof** The inverse Fourier transform of \(A\) is \(\int_{\hat{\Theta}} \sum_{g \in D} A_g \chi(-g) \chi(x) d\chi\) which is \(\sum_{g \in D} \int_{\hat{\Theta}} A_g \chi(-g + x) d\chi = \sum_{g \in D} A_g \delta_g(x).\)

**Example 5** If the FRMs in Theorem 2 is independent, it has a d-set \(D = \{0\}\) by Lemma 2 and the absolutely continuous component of the diffraction measure \(\tilde{\gamma}_\omega\) is \(A = \lim_n \sum_{s \in A(W) \cap D_n} V[X_s]/|D_n|.\)

If we add a mild condition “\(C(I(\text{Int}(W)) = W)\)” to the theorem, we can quantitatively provide the pure-point component \(\tilde{\gamma}_E[\omega]\) by using a following theorem (Theorem 3) (and can dispense with Proposition 4.)

### 5 Diffraction of weighted Dirac comb and torus parametrization

Let \(\omega\) be a weighted Dirac comb on \(A(W),\) which is a c.p. set over a c.p. scheme \(S = (\Theta, \Theta_{\text{int}}, L).\) If \(\omega\) satisfies mild conditions, then the diffraction measure \(\tilde{\gamma}_\omega\) is a pure-point measure on \(\Theta.\) To describe the support of \(\tilde{\gamma}_\omega,\)
we use the dual c.p.scheme $(\hat{\mathcal{S}}, \hat{\mathcal{S}}_{\text{int}}, \hat{\mathcal{L}})$ for the c.p. scheme $\mathcal{S}$. Here $\hat{\mathcal{L}}$ is the 
LCAG of $(\chi, \eta) \in \hat{\mathcal{S}} \times \hat{\mathcal{S}}_{\text{int}}$ such that $\chi(s)\eta(s') = 1$ for all $(s, s') \in \hat{\mathcal{L}}$. Then $(\hat{\mathcal{S}}, \hat{\mathcal{S}}_{\text{int}}, \hat{\mathcal{L}})$ is indeed a c.p. scheme. See [18, Section 5] for the proof. Let $\mathcal{L}$ ($\mathcal{L}^*$ resp.) stand for the canonical projection of $\hat{\mathcal{L}}$ to $\hat{\mathcal{S}}$ ($\hat{\mathcal{S}}_{\text{int}}$ resp.), and let the star map be $(-)^* : \mathcal{L} \to \mathcal{L}^*$. Then

$$\chi(s) = \overline{\chi^*(s')} \quad (\chi \in \mathcal{L}, s \in L).$$

(7)

Let $\mathfrak{T} = \mathcal{S} \times \mathcal{S}_{\text{int}}/\hat{\mathcal{L}}$. Then the lattice

$$\tilde{\mathcal{L}} \simeq \hat{\mathfrak{T}}.$$  

(8)

Recall that $|\hat{\mathcal{L}}| = \int_{\mathfrak{T}} dx dy$.

We say an FLC set $P \subset \mathfrak{T}$ is repetitive if for every compact $K \subset \mathfrak{T}$ there exists compact $K' \subset \mathfrak{T}$ such that for all $t_1, t_2 \in \mathfrak{T}$ there exists $s \in K'$ such that $(t_1 + P) \cap K = (s + t_2 + P) \cap K$. According to [22], we say a window $W \subset \mathcal{S}_{\text{int}}$ has no nontrivial translation invariance if $\{e \in \mathcal{S}_{\text{int}} : e + W = W\} = \{0\}$.

**Theorem 3** Assume that $\Lambda(W)$ is a repetitive model set over a c.p. scheme $(\hat{\mathcal{S}}, \hat{\mathcal{S}}_{\text{int}}, \hat{\mathcal{L}})$ where $\Lambda(W) - \Lambda(W)$ generates $\mathcal{L}$, $W = \text{Cl}(\text{Int}(W))$, $W$ has no non-trivial translation invariance, and $|\partial W| = 0$.

Then, for any $b \in C_{\text{op}}(\mathcal{S}_{\text{int}})$, supp $b \subset W$,

$$\omega = \sum_{s \in \Lambda(W)} b(s^*) \delta_s \implies \gamma_\omega = \sum_{\chi \in \mathcal{L}} \left| \frac{\hat{b}(\chi)}{|L|^2} \right|^2 \delta_\chi.$$ 

Here $\delta_\chi$ is a Dirac measure of $\hat{\mathcal{S}}$ such that $\delta_\chi(A) = 1$ if $s \in A$ and 0 otherwise, for any $A \subset \hat{\mathfrak{T}}$.

The theorem with the physical space $\mathfrak{T}$ being $\mathbb{R}^n$ was proved by Hof [13], the theorem for Dirac combs with constant weights was by Schlottmann [22], and the theorem for weighted Dirac comb with the weights arising from an “admissible” Radon-Nikodym derivative of the $L$-invariant measure was studied in Lenz-Richard [16, Theorem 3.3]. Our theorem is another form of weaker version of Lenz-Richard’s theorem.

In order to prove the theorem, we employ a uniquely ergodic dynamical system $X_{\Lambda(W)}$ made from $\Lambda(W)$, and connection of the autocorrelation measure $\gamma_\omega$ to a complex Hilbert space over $X_{\Lambda(W)}$. Then we prove lemmas about the so-called *torus parametrization* of $\Lambda(W)$, introduced in [3] and generalized by [22].

**Proposition 6** ([22])

1. For every FLC set $\Lambda \subset \mathfrak{T}$, the closure $X_\Lambda$ of the $\mathfrak{T}$-orbit $\{\Lambda + g : g \in \mathfrak{T}\}$ of $\Lambda$ with some uniform topology is a complete, compact Hausdorff space.

2. Suppose a model set $\Lambda := \Lambda(W)$ satisfies the same assumption as Theorem 3. Then $X_\Lambda$ will be a minimal and uniquely ergodic dynamical system, with the $\mathfrak{T}$-action $(x, P) \in \mathfrak{T} \times X_\Lambda \mapsto P + x \in X_\Lambda$. 

Hereafter, the uniquely ergodic probability measure of $X_A$ will be denoted by $\nu$, and the complex Hilbert space over $X_A$ with the inner product
\[
\langle \Phi_1, \Phi_2 \rangle_{\nu} := \int_{X_A} \Phi_1(P) \overline{\Phi_2(P)} d\nu(P) \quad (\Phi_i \in L^2(X_A, \nu))
\]
will be denoted by $L^2(X_A, \nu)$.

**Proposition 7** (Torus parametrization) Assume the same conditions as in Theorem 3, and let $\mathfrak{T}$ be an lcag $(\mathfrak{G} \times \mathfrak{G}_{\text{int}})/\hat{L}$ with the Haar probability measure $\tau$, and let $\mathfrak{G}$ act on $\mathfrak{T}$ by $(x, t) \in \mathfrak{G} \times \mathfrak{T} \mapsto t + (x, 0) \in \mathfrak{T}$. Then there is a continuous surjection
\[
\beta : X_A \to \mathfrak{T}
\]
and a full measure subset $X_A'$ of $X_A$ such that $\beta$ preserves the $\mathfrak{G}$-action, $\beta(\Lambda) = \hat{L}$ and $\beta' := \beta|_{X_A'}$ is injective with the range $\beta(X_A')$ disjoint from $\beta(X_A \setminus X_A')$.

From the proposition, we can derive the following:

**Lemma 7** Let $L^2(\mathfrak{T}, \tau)$ be a complex Hilbert space with the inner product $\langle \alpha_1, \alpha_2 \rangle_{\tau} := \int_{\mathfrak{T}} \alpha_1(t) \overline{\alpha_2(t)} d\tau(t)$ for $\alpha_i \in L^2(\mathfrak{T}, \tau)$. Then $\iota : L^2(X_A, \nu) \to L^2(\mathfrak{T}, \tau) ; \Theta \mapsto \Theta \circ (\beta')^{-1}$ is a bijective isometry.

By using [1, Proposition 1.4], we can prove the following:

**Fact 1** If $\lambda$ and $\mu$ are translationally bounded, nonnegative measures on $\mathfrak{G}$ and $\{D_n\}_{n}$ is a van Hove sequence on $\mathfrak{G}$, then in the vague topology
\[
\lim_{n \to \infty} \frac{\langle \lambda \cdot D_n \rangle \ast \mu|_{D_n} - \tilde{\lambda} \ast (\mu|_{D_n})}{|D_n|} = 0.
\]

**Lemma 8** Suppose the same assumption as Theorem 3 holds. Then, for all $\varphi_1, \varphi_2 \in C_c(\mathfrak{G})$, there are unique $\Phi_1, \Phi_2 \in C_c(X_A)$ such that for any $x \in \mathfrak{G}$,
\begin{align*}
\Phi_1(A - x) &= \langle \varphi_i \ast \overline{\varphi_2} \rangle(x), \quad \text{(9)}
\end{align*}
\begin{align*}
\langle \varphi_1 \ast \varphi_2 \ast \overline{\varphi_3} \rangle(0) &= \langle \Phi_1, \Phi_2 \rangle_{\nu}. \quad \text{(10)}
\end{align*}

**Proof** By using the surjection $\beta : X_A \to \mathfrak{T}$ of the torus parametrization, define
\[
\Phi_i(P) := \sum_{(x, y) \in \beta(P)} \varphi_i(-x) b(y) \quad (P \in X_A).
\]

Then it is indeed a finite sum, because each $\beta(P) \in \mathfrak{T}$ is discrete and the function $g_i(x, y) := \varphi_i(-x)b(y)$ of $(x, y) \in \mathfrak{G} \times \mathfrak{G}_{\text{int}}$ has a compact support $K_i := \text{supp} \varphi_i \subset \text{supp} \varphi_i \times W$ for $\varphi_i \in C_c(\mathfrak{G})$.

To establish $\Phi_i \in C_c(X_A)$, it is sufficient to verify the continuity of a function $f_i : \mathfrak{T} \to \mathbb{C} : t \mapsto \sum_{(x, y) \in \mathfrak{T}} g_i(x, y)$, because $\Phi_i$ is the composition of the continuous function $\beta$ and $f_i$. 
By the assumption, for all $\varepsilon > 0$ there exists a compact neighborhood $U \subset \mathcal{G} \times \mathcal{G}_{\text{int}}$ of 0 such that for all $z, z' \in \mathcal{G} \times \mathcal{G}_{\text{int}}$ with $z' \in z + U$, 
$|g_t(z') - g_t(z)| < \varepsilon$. Then $\{z + \mathcal{L} : z \in U\}$ is a compact neighborhood of 0 in $\mathcal{T}$. Let $t', t \in \mathcal{T}$ such that $t' - t$ belongs to the compact neighborhood $\{z + \mathcal{L} : z \in U\}$. Then $t' = t + (u, v)$ for some $(u, v) \in U$. So $|f_t(t') - f_t(t)| \leq \sum_{x \in \mathcal{T}} |g_t(z + (u, v)) - g_t(z)|$. We show it is less than $\#((K_i - U) \cap t) \times \varepsilon$. If $z \in t$ contributes to the summation, then $z \in (K_i - U) \cap t$. Moreover $\#((K_i - U) \cap t)$ is finite since $t$ is a translation of the lattice $\mathcal{L}$ and $K_i - U$ is compact. Thus $f_t$ is continuous.

To prove (9), observe $\Phi_i(A - x) = \sum_{s \in L} \varphi;(-s + x)b(s^*)$ follows from $\beta(A - x) = \mathcal{L} - (x, 0)$. Here $b(s^*) = 0$ for any $s^* \notin W$. So the range $L$ of $s$ in the summation can be replaced with $A$. Thus (9) holds.

The left-hand side of (10) is $\lim_n \left( \frac{\varphi_1 \ast \widetilde{\varphi}_2 \ast \nu}{|D_n|} \right)$ by definition of $\gamma$. But by Fact 1, it is $\lim_n \left( \frac{(\varphi_2 \ast \omega)}{|D_n|} \right)$, which is equal to $\lim_n \int_{\mathcal{D}_n} \Phi_2(A - x) \Phi_1(A - x) dx : |D_n|^{-1}$ by (9). By the pointwise ergodic theorem [21], it is $\int_{\mathcal{A}_i} \Phi_2(P) \Phi_1(P) dv(P) = \langle \Phi_1, \Phi_2 \rangle_{\nu}$.

Here is a technical lemma concerning van Hove sequences and uniformly discrete sets.

**Lemma 9** For any uniformly discrete subset $A$ of $\mathcal{G}$, any bounded complex sequence $\{w_s\}_{s \in A}$, any $\chi \in \mathcal{G}$, and any $\varphi \in C_c(\mathcal{G})$, $
\frac{1}{|D_n|} \left| \sum_{s \in A} w_s \int_{D_n} \chi(x) \varphi(x - s) dx - \sum_{s \in A \cap D_n} w_s \int_{\mathcal{G}} \chi(x) \varphi(x - s) dx \right|$

tends to 0, as $n$ goes to $\infty$.

**Proof** The numerator is bounded from above by

$$\sum_{s \in A \cap D_n} |w_s| \int_{\mathcal{G}} |\varphi(x - s)| dx + \sum_{s \in A \setminus D_n} |w_s| \int_{D_n} |\varphi(x - s)| dx. \quad (11)$$

The summands in the former summation and the latter summation are bounded, because so are the sequence $\{w_s\}_{s \in A}$ and $\varphi \in C_c(\mathcal{G})$. So it is sufficient to show that the set of $s$ that “contributes” to (11) has density 0 with respect to $\{D_n\}_{n \in \mathbb{N}}$.

Let $s \in A$ “contribute” to (11). If $s \in A$ “contributes” to the former summation, then $s \in D_n$ for some $x \notin D_n$ such that $x - s \notin \text{supp} \varphi$. So $s \in [\mathcal{C}] (\mathcal{G} \setminus D_n) - \text{supp} \varphi \cap D_n \subset \partial^{\text{supp} \varphi} (D_n)$, by (1). On the other hand, if $s \in A$ “contributes” to the latter summation then $s \notin D_n$ for some $x \in D_n$ such that $x - s \notin \text{supp} \varphi$, so $s \in (D_n - \text{supp} \varphi) \setminus \text{Int}(D_n) \subset \partial^{-\text{supp} \varphi} (D_n)$. Then $s \in \partial^{-\text{supp} \varphi} (D_n) \cup \partial^{-\text{supp} \varphi} (D_n) \subset \partial^K (D_n)$ for some compact set $K \subset \mathcal{G}$ such that $K = -\text{supp} \varphi$.

Thus we have only to verify $\lim_{n \to \infty} \# \{s \in A : s \in \partial^K (D_n) \} / |D_n| = 0$. 
This is proved as follows: Because of Lemma 1, we have $U + \{ s \in A : s \in \partial^K(D_n) \} \subset \partial^K(U(D_n))$. The Haar measure of the left-hand side of the inclusion is $|U| \cdot \# \{ s \in A : s \in \partial^K(D_n) \}$ because $(A - A) \cap (U - U) = \{0\}$ implies $(s + U) \cap (t + U) = \emptyset$ for any distinct points $s, t \in A$. Thus $|U| \cdot \lim_{n \to \infty} \# \{ s \in A : -s \in \partial^K(D_n) \}/|D_n| \leq \lim_{n \to \infty} |\partial^K(U(D_n))/|D_n| = 0$ by the definition of van Hove sequence.

Lemma 10 Suppose the same assumption as in Theorem 3. Then

1. the isometry $\iota$ followed by Fourier transform is an isometry from $L^2(\Xi, \tau)$ to a complex Hilbert space $L^2(\hat{\Xi})$ with the inner product $\langle \kappa_1, \kappa_2 \rangle_{\hat{\Xi}} := \sum_{\xi \in \hat{\Xi}} \kappa_1(\xi)\kappa_2(\xi)$ for $\kappa_i \in \hat{\Xi}$.

2. A correspondence $\xi \in \hat{\Xi} \mapsto \chi := \xi((\bullet, 0) + \hat{L}) \in L$ is a bijection from $\hat{\Xi}$ onto $L$ such that for $\varphi_1, \varphi_1$ of Lemma 8,

$$\langle (\varphi_1)(\xi) = \frac{1}{|L|} \iota(\xi) \rangle \chi(x^*) \quad (\xi \in \hat{\Xi}).$$

Proof (1) By (8), the inner product of the Hilbert space $L^2(\hat{\Xi})$ is in fact a summation. Use that Fourier transform is an isometry.

(2) First we will show $\chi \in L$. Let $\eta(y)$ be $\xi((0, y) + \hat{L})$ $(y \in \Theta_{\text{int}})$. Then $\chi \in \Theta$ and $\eta \in \Theta_{\text{int}}$. Moreover, for any $(s, s^*) \in \hat{\Xi}$, $\chi(s^*)$ is $\xi((s, 0) + \hat{L}) = \xi((s, 0) + \hat{L} + (0, s^*) + \hat{L}) = \xi((s, s^*) + \hat{L}) = \xi(\hat{L})$ which is 1 because $\xi \in (\Theta \times \Theta_{\text{int}}/\hat{L})$. Thus $\xi \in \hat{\Xi}$. Next we will show the correspondence from $\xi$ to $\chi$ is injective. Let $\xi_i \in \hat{\Xi}$ ($i = 1, 2$) correspond to a same element of $\hat{\Xi}$. Then for any $x \in \Theta$, $\xi_1(\beta(x + A)) = \xi_2((x, 0) + \hat{L}) = \xi_2(\beta(x + A))$. Because the $\Theta$-orbit of $\Lambda$ is dense in $X_{\Lambda}$ and $\beta : X_{\Lambda} \to \Xi$ is continuous surjection, the continuous mappings $\xi_i$ agree on a dense subset of $\Xi$, from which $\xi_1 = \xi_2$ follows. We will show that any $\chi \in L$ is mapped from some $\xi \in \hat{\Xi}$ such that $\xi((x, 0) + \hat{L}) = \chi(x)\chi^*(y)$ for any $(x, y) + \hat{L}$. This $\xi$ is well-defined because for any $(x', y') = (x, y) + (s, s^*)$ with $(s, s^*) \in \hat{\Xi}$, we have $\chi(s^*) = 1$ by $\chi, \chi^* \in \hat{\Xi}$ and $\chi(x') = \chi(x)\chi^*(y') = \chi(x)\chi^*(y) \times \chi(s^*) = \chi(x)\chi^*(y)$. This $\xi$ indeed corresponds to $\chi$ because $\xi((x, 0) + \hat{L}) = \chi(x)\chi^*(0) = \chi(x)$.

Now, the left-hand side of the statement is $\int_{\Xi} \xi(t) \tau(t) d\tau(t)$, which is $\int_{X_A} \xi_{\theta}(\beta(P)) \Phi_1(P) d\nu(P)$ by Lemma 7. By the pointwise ergodic theorem [21], the integral is $\lim_{n \to \infty} \int_{D_n} \xi(\beta(A - x)) \Phi_1(A - x) dx / |D_n|$. Since $\beta(A - x) = \hat{L} - (x, 0)$ by Proposition 7, we have $\xi(\beta(A - x)) = \chi(x)$. By (9) and $\omega = \sum_{s \in A(W)} b(s^*)$, we have

$$\langle (\varphi_1)(\xi) = \lim_{n \to \infty} \int_{D_n} \chi(x) \sum_{s \in A(W)} \varphi_1(x - s) b(s^*) dx.$$
We can apply Lemma 9 to above, since \( A(W) \) is uniformly discrete, the sequence \( \{ b(s^*) \}_{s \in A(W)} \) of weights is bounded, and \( \tilde{\varphi}_1 \in C_c(\mathfrak{S}) \). Thus

\[
\langle \varphi \rangle^2(\xi) = \lim_{n \to \infty} \frac{1}{|D_n|} \sum_{s \in A(W) \cap D_n} b(s^*) \int_{\mathfrak{S}} \chi(x) \varphi_1(x - s) dx.
\]

Here \( \int_{\mathfrak{S}} \chi(x) \varphi_1(x - s) dx = \chi(s) \tilde{\varphi}_1(-s) \) by (7). Therefore

\[
\langle \varphi \rangle^2(\xi) = \tilde{\varphi}_1(-s) \lim_{n \to \infty} \frac{1}{|D_n|} \sum_{s \in A(W) \cap D_n} b(s^*) \chi^*(s^*).
\]

Since \( b \in C_{bp}(\mathfrak{S}_{int}) \) has well-defined \( \tilde{b} \), Proposition 2 for model sets implies that the limit in (12) is \( \int_W b(y) \chi_1(y) dy/|\tilde{L}| \), which is \( b(-\chi^*)/|\tilde{L}| \) by supp \( b \subset W \). Therefore (12) implies the desired consequence.

**Proof** of Theorem 3

By Proposition 3, it is sufficient to verify

\[
\int_{\mathfrak{S}} \langle \varphi \ast \tilde{\varphi} \rangle(x) d\gamma_\omega(x) = \int_{\mathfrak{S}} |\tilde{\varphi}(-\gamma)|^2 d \left( \sum_{\chi \in \mathcal{L}} \frac{|\tilde{b}(\chi^*)|^2}{|L|^2} \delta_\chi \right) (\gamma) \geq 0. \tag{13}
\]

Since \( \gamma_\omega = \gamma_\nu \), the leftmost integral is \( \langle (\varphi \ast \tilde{\varphi}) \ast \gamma_\omega \rangle(0) \), which is \( \langle \Phi, \Phi \rangle_\nu \) by (10). By Lemma 10, it is \( \sum_{\chi \in \mathcal{L}} |\tilde{\varphi}(\chi)b(-\chi^*)|^2 \cdot |\tilde{L}|^{-2} \), the right-hand side of (13).

By the Theorem we have proved, we can see that the pure-point diffraction is still observed as long as the sample path and the expected displacement of random weights are both continuous on the internal space of the model set. The condition is comparable Baake-Moody’s sufficient condition for deterministic model sets to have pure-point diffraction; their condition demands the continuity with respect to the internal space.

**Theorem 4** Suppose the same assumptions as in Theorem 3. If a complex-valued stochastic process \( \{ B_y(\omega) \}_{y \in W} \) is such that the sample path \( b(y) = 1_W(y)B_y(\omega) \) is continuous on \( W \) almost every \( \omega \) and \( E[B_yB_z] \) is continuous on \( y \) and \( z \), then a Dirac comb \( \sum_{s \in A(W)} B_s \cdot \delta_s \) has a diffraction measure which expectation is pure-point

\[
\sum_{\chi \in \mathcal{L}} \frac{E[|\tilde{b}(\chi^*)|^2]}{|L|^2} \delta_\chi.
\]

For example, if \( W = [0, 1] \), \( \{ B_y \}_{y \in W} \) is an Ornstein-Uhlenbeck process.

In [9], for particle gases over FLC sets with Gibbs random field under a suitable interaction potential restrictions, Baake-Zint proved that the diffraction measures do not have singular continuous components and explicitly described the pure-point component and the absolutely continuous component by using the covariance of the random field.

**Acknowledgements** The first author thanks Prof. Michael Baake and anonymous referee.
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Forms of this assertion have already appeared in Schlottmann, Lagarias and so on. Since $\bar{L}$ is a lattice of $\mathcal{G} \times \mathcal{G}$, there is a complete system $C$ of representatives of a compact, quotient set $(\mathcal{G} \times \mathcal{G})/\bar{L}$ such that $C$ is relatively compact. Then $\mathcal{G} \times \mathcal{G} = \bar{L} + C$. Since the window $W$ has nonempty interior and $L^*$ is dense in $\mathcal{G}$, it follows that $\mathcal{G} = L^* - W$. Moreover, since $\Pi_{\text{int}}(\text{Cl}(C))$ is compact, there is a finite subset $F$ of $L$ such that $\Pi_{\text{int}}(\text{Cl}(C)) \subset F^* - W$. Let $K := \Pi(\text{Cl}(C)) - F$, which is compact in $\mathcal{G}$. For each $x \in \mathcal{G}$, there are $(t, t^*) \in \bar{L}$ and $(c, d) \in C$ such that $(x, 0) = (t, t^*) + (c, d)$. Since $\Pi_{\text{int}}(C) \subset F^* - W$, there are $f \in F$ and $w \in W$ such that $d = f^* - w$. Then

$$(x, 0) = (t, t^*) + (c, f^* - w) = (t + c, t^* + f^* - w).$$

So $t^* + f^* = w \in W$, and thus $t + f \in A(W)$. Since $x = t + c = (t + f) + (c - f)$ and $c - f \in H(C) - F \subset K$, we have $x \in A(W) + K$. Therefore $\mathcal{G} = A(W) + K$. 

A Proof of “If Int$(W) \neq \emptyset$ then $A(W)$ is relatively dense.”

