Non-Unitary evolution of quantum time-dependent non-Hermitian systems

Mustapha Maamache*

Laboratoire de Physique Quantique et Systèmes Dynamiques, Faculté des Sciences, Université Ferhat Abbas Sétif 1, Sétif 19000, Algeria.

Abstract

We provide a new perspective on non-Hermitian evolution in quantum mechanics by emphasizing the same method as in the Hermitian quantum evolution. We first give a precise description of the non unitary evolution, and collecting the basic results around it and postulating the norm preserving. This cautionary postulate imposing that the time evolution of a non Hermitian quantum system preserves the inner products between the associated states must not be read naively. We also give an example showing that the solutions of time-dependent non Hermitian Hamiltonian systems given by a linear combination of SU(1,1) and SU(2) are obtained thanks to time-dependent non-unitary transformation.

PACS: 03.65.Ca, 03.65.-w

Keywords: Non-Hermitian quantum mechanics, Time-dependent Hamiltonian systems, non-unitary time-dependent transformation.

1 Introduction

One of the postulates of quantum mechanics is that the Hamiltonian is Hermitian, as this guarantees that the eigenvalues are real. This postulate result from a set of postulates representing the minimal assumptions needed to develop the theory of quantum mechanics. One of these postulates concerns the time evolution of the state vector $|\psi(t)\rangle$ governed by the Schrödinger equation which describe how a state changes with time:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

where $H$ is the Hamiltonian operator corresponding to the total energy of the system. The time dependent Schrödinger equation is the most general way of describing how a state changes with time.

*E-mail: maamache@univ-setif.dz
The time evolution of the state of a quantum system described by $|\psi(t)\rangle = U(t,t_0) |\psi(t_0)\rangle$ preserves the normalization of the associated ket for some unitary operator $U$. The Hermiticity of the Hamiltonian $H$ guarantees that the energy spectrum is real and the time evolution of the system is unitary. A direct consequence of the hermiticity of $H$ is that the norm $\langle \psi(t) | \psi(t) \rangle$ is time independent.

For explicitly time-dependent systems, solving the time-dependent Schrödinger equation is in general very difficult and it is very rare to be able to find an exact solution. Various methods have been used to obtain approximate solutions for such time-dependent problems. The usual methods are the adiabatic approximation, the sudden approximation, and time-dependent perturbation techniques. The existence of invariants constants of the motion or first integral is one of central importance in the study of such a system, be it classical or quantum. In the quantum case, Lewis and Riesenfeld [3] first exploited the invariant operators to solve quantum-mechanical problems. In particular, they have derived a simple relation between eigenstates of invariants and solutions of the time-dependent Schrödinger equation and have applied it to the case of a quantal oscillator with time-dependent frequency. Two models have been studied extensively in the literature by several authors, for instance [1, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. One of them is the time-dependent generalized harmonic oscillator with the symmetry of the SU(1,1) dynamical group, the other is the two-level system possessing an SU(2) symmetry. With the help of the appropriate time-dependent unitary transformation instead of the invariant operator, the solutions of SU(1, 1) and SU(2) time-dependent quantum systems as well as the time-evolution operator are obtained in [15]. Time-dependent Hamiltonian systems are also of importance in quantum optics.

The quantum mechanics is capable of working for some non-Hermitian quantum systems. However, the Hermiticity is relaxed to be pseudo-Hermiticity or PT symmetry in non-Hermitian quantum mechanics, where is a linear Hermitian or an anti-linear anti-Hermitian operator, and P and T stand for the parity and time-reversal operators, respectively. The theories of non-Hermitian quantum mechanics have been developed quickly in recent decades, the reader can consult the articles [17, 18] and references cited therein.

Systems with time-dependent non-Hermitian Hamiltonian operators have been studied in [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. The most recent monograph [36] can be consulted for introduction of the non-stationary theory.

Thus, we adapt the method based on a time-dependent unitary transformation of time-dependent Hermitian Hamiltonians [15] to solve the Schrödinger equation for the time-dependent non-Hermitian Hamiltonian systems given by a linear combination of SU(1, 1) and SU(2) generators.

Because of the existence of a pseudo invariant operator, an SU(1, 1) and SU(2) time-dependent non Hermitian quantum system must be integrable. For an integrable system, the Hamiltonian can be transformed into a sum of time-independent commuting operators through a time-dependent non-unitary transformation. For this we introduce, in section 2, a formalism based on the time-dependent non-unitary transformations and we show that the time-dependent non-Hermitian Hamiltonian is related to an associated time-independent Hamiltonian multiplied by an overall time-dependent factor. In section 3, we illustrate our formalism introduced in the previous section by treating a non-Hermitian SU(1, 1) and SU(2) time-dependent quantum problem and finding the exact solution of the Schrödinger equation without making recourse to the pseudo-invariant operator theory as been done in [34, 35] or to the technique presented in [28, 29].
4, concludes our work.

2 Formalism

Consider the time-dependent Schrödinger equation

\[ i \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle, \]  

(2)

with \( \hbar = 1 \) and \( H(t) \) is the time-dependent non-Hermitian Hamiltonian operator. Suppose that is a time-dependent non unitary transformation \( V(t) \),

\[ |\phi(t)\rangle = V(t) |\psi(t)\rangle \]  

(3)

transform the state to \( |\phi(t)\rangle \), which obeys the Schrödinger equation with the accordingly trans- 
formed non-Hermitian Hamiltonian \( \mathcal{H}(t) \)

\[ i \frac{\partial}{\partial t} |\phi(t)\rangle = \mathcal{H}(t) |\phi(t)\rangle. \]  

(4)

In the above equation, the new Hamiltonian operator,

\[ \mathcal{H}(t) = V(t)H(t)V^{-1}(t) + i \frac{\partial V(t)}{\partial t}V^{-1}(t), \]  

(5)

is non-Hermitian. With the help of the appropriate non- unitary transformation \( V(t) \), we require that the solution of Schrödinger equation looks like a product of a state satisfying a stationary 
equation times a time-dependent function. For this, we may make the transformed Hamiltonian \( \mathcal{H}(t) \) possessing the following form:

\[ \mathcal{H}(t) = g(t)H_0, \]  

(6)

where the operator \( H_0 \) is time independent and \( g(t) \) is a function of time. The implication of these results is clear. The original time-dependent non-Hermitian quantum problem posed through the Hamiltonian \( H(t) \) and related to an associated time independent Hamiltonian multiplied by an overall time-dependent factor is completely solved. To get the form of the solution, let \( |\zeta_n\rangle \) be the eigenstate of \( H_0 \) with the eigenvalue \( \lambda_n \) i.e.

\[ H_0 |\zeta_n\rangle = \lambda_n |\zeta_n\rangle, \quad \lambda_n = cte. \]  

(7)

Then, suppose that the system described by the Schrödinger equation (4) is in the nth eigenstate of \( H_0 \) initially, at time \( t \) it evolves to the state

\[ |\phi_n(t)\rangle = \exp \left( i\lambda_n \int_0^t g(t')dt' \right) |\zeta_n\rangle. \]  

(8)

Of note it follows immediately that the time evolution of a quantum system described by \( |\phi_n(t)\rangle \) doesn’t preserve the normalization i.e., the inner product of evolved states \( |\phi_n(t)\rangle \) depend
on time:
\[
\langle \phi_n(t) | \phi_n(t) \rangle = \exp \left( \text{Im} \left\{ \lambda_n \int_0^t g(t') dt' \right\} \right) \langle \phi_n(0) | \phi_n(0) \rangle
\]  

(9)

At this stage, we will postulate, like the Hermitian case, that the time evolution of a quantum system preserves, not just the normalization of the quantum states, but also the inner products between the associated states \( \langle \phi_n(t) | \phi_n(t) \rangle = \langle \phi_n(0) | \phi_n(0) \rangle \), implying that \( g(t) \) and \( \lambda_n \) are reals and consequently the Hamiltonian \( \mathcal{H}(t) \) should be Hermitian.

As we are only changing our description of the system by changing basis, we must preserve the inner product between vectors. Explicitly, from preserving of this inner product between states \( | \phi_n(t) \rangle \), we can now define the inner product between states \( | \psi_n(t) \rangle = V^{-1}(t) | \phi_n(t) \rangle \) as
\[
\langle \psi_n(t) | V^+(t)V(t) | \psi_n(t) \rangle = \langle \psi_n(0) | V^+(0)V(0) | \psi_n(0) \rangle
\]  

(10)

which has both a positive definite signature and leaves the norms of vectors stationary in time.

3 Evolution of SU(1, 1) and SU(2) non-Hermitian time-dependent systems

The SU(1, 1) and SU(2) non-Hermitian time-dependent systems that we consider are described by the Hamiltonian
\[
H(t) = 2\omega(t)K_0 + 2\alpha(t)K_- + 2\beta(t)K_+,
\]  

(11)

where \((\omega(t), \alpha(t), \beta(t)) \in C\) are arbitrary functions of time. \( K_0 \) is a Hermitian operator, while \( K_+ = (K_-)^+ \). The commutation relations between these operators are
\[
\left\{ 
\begin{array}{c}
[K_0, K_+] = K_+ \\
[K_0, K_-] = -K_- \\
[K_+, K_-] = DK_0
\end{array}
\right.
\]  

(12)

The Lie algebra of SU(1, 1) and SU(2) consists of the generators \( K_0, K_- \) and \( K_+ \) corresponding to \( D = -2 \) and 2 in the commutation relations \([12]\), respectively.

With the requirement that the solution looks like a product of a wavefunction satisfying a stationary equation multiplied by a time dependent function, we perform the time-dependent non-unitary transformation
\[
V(t) = \exp \left[ 2\epsilon(t)K_0 + 2\mu(t)K_- + 2\mu^*(t)K_+ \right],
\]  

(13)

where \( \epsilon, \mu \) are arbitrary real and complex time-dependent parameters respectively. The group element in \([13]\) can be decomposed according to \([37, 38]\)
\[
V(t) = e^{\vartheta_+(t)K_+} e^{\ln \vartheta_0(t)K_0} e^{\vartheta_-(t)K_-},
\]  

(14)
We require here the variant (14) of our ansatz to be able to compute the time derivatives of $V(t)$. The time dependent coefficients read

\[ \vartheta_0(t) = \left( \cosh \theta - \frac{\epsilon}{\theta} \sinh \theta \right)^{-2} \]

\[ \theta = \sqrt{\epsilon^2 + 2D|\mu|^2} \]

\[ \vartheta_+(t) = \frac{2\mu^\ast \sinh \theta}{\theta \cosh \theta - \epsilon \sinh \theta} \]

\[ \vartheta_-(t) = \frac{2\mu \sinh \theta}{\theta \cosh \theta - \epsilon \sinh \theta} \]  

(15)

The notation may be simplified even further by introducing some new quantities (16)

\[ z = \frac{2\mu}{\epsilon} = |z| e^{i\varphi} \]

\[ \phi = \frac{|z|}{1 - \frac{2}{\theta} \coth \theta} \]

\[ \chi(t) = -\frac{\cosh \theta + \frac{2}{\theta} \sinh \theta}{\cosh \theta - \frac{2}{\theta} \sinh \theta} \]

With this adopted notation, the coefficients in (15) simplify to

\[ \vartheta_\pm = -\phi e^{\mp i\varphi} \]

\[ \vartheta_0 = -\frac{D}{2} \phi^2 - \chi \]  

(17)

Using the relations :

\[ \left\{ \begin{array}{l}
\exp [\vartheta_- K_-] K_0 \exp [-\vartheta_- K_-] = K_0 + \vartheta_- K_-
\exp [\vartheta_+ K_+] K_0 \exp [-\vartheta_+ K_+] = K_0 - \vartheta_+ K_+
\end{array} \right. \]

(18)

\[ \left\{ \begin{array}{l}
\exp [\ln \vartheta_0 K_0] K_- \exp [-\ln \vartheta_0 K_0] = \frac{K_-}{\vartheta_0}
\exp [\vartheta_+ K_+] K_- \exp [-\vartheta_+ K_+] = K_- + D \vartheta_+ K_0 - \frac{D^2}{2} \vartheta_+^2 K_+
\end{array} \right. \]

(19)

\[ \left\{ \begin{array}{l}
\exp [\ln \vartheta_0 K_0] K_+ \exp [-\ln \vartheta_0 K_0] = \vartheta_0 K_+
\exp [\vartheta_- K_-] K_+ \exp [-\vartheta_- K_-] = K_+ - D \vartheta_- K_0 - \frac{D^2}{2} \vartheta_-^2 K_-
\end{array} \right. \]

(20)

we obtain, after some algebra, the transformed Hamiltonian

\[ \mathcal{H}(t) = 2\mathcal{W}(t) K_0 + 2\mathcal{Q}(t) K_- + 2\mathcal{Y}(t) K_+ \]  

(21)

where the coefficient functions are

\[ \mathcal{W}(t) = \frac{1}{\vartheta_0} \left[ \omega \left( \frac{D}{2} \vartheta_+ \vartheta_- - \chi \right) + D (\vartheta_+ \alpha + \vartheta_- \beta \chi) + \frac{i}{2} \left( \vartheta_0 + D \vartheta_+ \vartheta_- \right) \right] \]

(22)

\[ \mathcal{Q}(t) = \frac{1}{\vartheta_0} \left[ \omega \vartheta_- + \alpha - \frac{D}{2} \beta \vartheta_-^2 + \frac{i}{2} \vartheta_- \right] \]

(23)

\[ \mathcal{Y}(t) = \frac{1}{\vartheta_0} \left[ \omega \chi \vartheta_+ - \frac{D}{2} \alpha \vartheta_+^2 + \beta \chi^2 + \frac{i}{2} \left( \vartheta_0 \vartheta_+ + \vartheta_0 \vartheta_- - \frac{D}{2} \vartheta_+^2 \vartheta_- \right) \right] \]

(24)
The central idea in this procedure is to simplify the transformed Hamiltonian $H(t)$ governing the evolution of $|\phi_n(t)\rangle$ by cancelling the terms $K_-$ and $K_+$ and requiring that the time evolution of a quantum system preserves the inner products between the associated states $\langle \phi_n(t) | \phi_n(t) \rangle = \langle \phi_n(0) | \phi_n(0) \rangle$. This is achieved by requiring $Q(t) = 0$, $V(t) = 0$ and $\text{Im} W(t) = 0$. These conditions impose, by using Eqs. (17) and after some algebra, the following constraints

$$\dot{\varphi} = 2 |\omega| \cos \varphi - 2 |\alpha| \cos(\varphi_\alpha - \varphi) + D\phi |\beta| \cos(\varphi + \varphi_\beta),$$

$$\dot{\vartheta} = -2\dot{\phi} |\omega| \sin \varphi + 2 |\alpha| \sin(\varphi_\alpha - \varphi) - D\phi^2 |\beta| \sin(\varphi + \varphi_\beta),$$

$$\dot{\theta}_0 = \frac{2\theta_0}{\varphi} \left[-2\dot{\phi} |\omega| \sin \varphi + |\alpha| \sin(\varphi_\alpha - \varphi) + (\chi - D\phi^2) |\beta| \sin(\varphi + \varphi_\beta)\right],$$

by which $\vartheta_-$, $\vartheta_+$ and $\theta_0$ are determined for given values of $\omega(t)$, $\alpha(t)$ and $\beta(t)$. It is important to note here that when considering the time-dependent coefficient $\mu$ to be real function instead of complex one, i.e., the polar angles $\varphi$ vanish, the auxiliary equations (25)–(27) that appear automatically in this process are identical to equations (28)–(30) for Maamache et al [35] who used the general method of Lewis and Riesenfeld to derive them. Then the transformed Hamiltonian $H(t)$ becomes

$$H(t) = 2 \text{Re} (W(t)) K_0$$

$$\text{Re} (W(t)) = |\omega| \cos \varphi_\omega + D\phi |\beta| \cos(\varphi + \varphi_\beta)$$

The implication of the results is clear. The original time-dependent quantum-mechanical problem posed through the Hamiltonian $H(t)$ is completely solved if the wave function for the related transformed Hamiltonian $H(t)$ defined in Eq. (3) is obtained. The exact solution of the original equation (2) can now be found by combining the above results. We finally obtain

$$|\psi_n(t)\rangle = \exp \left(i\lambda_n \int_0^t 2 |\omega| \cos \varphi_\omega + D\phi |\beta| \cos(\varphi + \varphi_\beta) \, dt'\right) V^{-1}(t) |\zeta_n\rangle.$$  

Now, we consider the SU(1,1) case first where $D = -2$. The SU(1,1) Lie algebra has a realization in terms of boson creation and annihilation operators $a^\dagger$ and $a$ such that

$$K_0 = \frac{1}{2} \left(a^\dagger a + \frac{1}{2}\right), \quad K_- = \frac{1}{2} a^\dagger, \quad K_+ = \frac{1}{2} a^2.$$  

Then, the Hamiltonian $H(t)$ describes the generalized time dependent Sawson Hamiltonian [29]. If $\omega(t)$, $\alpha(t)$ and $\beta(t)$ are reals constant, this Hamiltonian has been studied extensively in the literature by several authors, for instance [39, 40, 41, 42, 43, 44, 45, 46]. Substitution of $D = -2$, and $\lambda_n = \frac{1}{2}(n + \frac{1}{2})$ into (29) yields

$$|\psi_n(t)\rangle = \exp \left(i(n + \frac{1}{2}) \int_0^t |\omega| \cos \varphi_\omega - 2\dot{\phi} |\beta| \cos(\varphi + \varphi_\beta) \, dt'\right) V^{-1}(t) |n\rangle,$$
where $|\zeta_n\rangle = |n\rangle$ are the eigenvectors of $K_0$.

For $D = 2$, Hamiltonian (11) possesses the symmetry of the dynamical group SU(2). A spin in a complex time-varying magnetic field is a practical example in this case [47, 48, 49, 50, 51, 52, 53]. Let $K_0 = J_z$ and $K_\mp = J_\mp$ . Let $|\zeta_n\rangle = |j, n\rangle$ are the eigenvectors of $J_z$ , i.e. $J_z |j, n\rangle = n |j, n\rangle$.

The next step is the calculation of the solutions (29) which are given by

$$\langle \psi_n(t) \rangle = \exp \left( \frac{\int_0^t [\omega \sin \theta + 2 \beta \cos(\varphi + \varphi_\beta)] dt'}{V^{-1}(t)} \right) |j, n\rangle,$$

(32)

## 4 Conclusion

We adapted the method based on a time-dependent unitary transformation of time-dependent Hermitian Hamiltonians [15] to solve the Schrödinger equation for the time-dependent non-Hermitian Hamiltonian. Starting with the original time-dependent non-Hermitian Hamiltonian $H(t)$ and through a non-unitary transformation $V(t)$ we derive the transformed $\mathcal{H}(t)$ as time independent Hamiltonian multiplied by a time-dependent factor. Then, we postulate that the time evolution of a non Hermitian quantum system preserves, not just the normalization of the quantum states, but also the inner products between the associated states, which allows us to identify this transformed Hamiltonian $\mathcal{H}(t) = 2 \text{Re}(W(t)) K_0$ as Hermitian. Thus, our problem is completely solved.

Evidently, we then have presented to illustrate this theory: the SU(1, 1) and SU(2) non-Hermitian time-dependent systems described by the Hamiltonian (11) when applying the non-unitary transformation $V(t)$ we obtain the transformed Hamiltonian $\mathcal{H}(t)$ as linear combination of $K_0$ and $K_\mp$. Consequently, we must disregard the prefactors of the operators $K_\mp$. To this end, we next require that the coefficients $Q(t) = 0$ and $Y(t) = 0$ defined in Eqs. (23)-(24). Then, by using the postulate that the inner products between the associated states is preserved allows us to require that $\text{Im} W(t) = 0$ and to identify the transformed Hamiltonian $\mathcal{H}(t) = 2 \text{Re}(W(t)) K_0$ as Hermitian. Finally, we also found the exact solutions of the generalized Swanson model and a spinning particle in a time-varying complex magnetic field.

## References

[1] H. R. Lewis, Phys. Rev. Lett. 18, 636 (1967).
[2] H. R. Lewis, J. Math Phys. 9, 1976 (1968).
[3] H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. 10, 1458 (1969).
[4] C. M. Cheng and P. C. W. Fung, J. Phys. A 21, 4115 (1988).
[5] D. A. Molares, J. Phys. A 21, L889 (1988)
[6] J. M. Cervero and J. D. Lejarreta, J. Phys. A 22, L663 (1989).
[7] N. Datta and G. Ghosh, Phys. Rev. A 40, 526 (1989).
[8] X. Gao, J. B. Xu and T. Z. Qian, Ann. Phys., NY 204, 235 (1990).
[9] D. B. Monteliva, H. J. Korsch, and J. A. Nunez, J. Phys. A 27, 6897 (1994)
[10] S. S. Mizrahi, M. H. Y. Moussa, and B. Baseia, Int. J. Mod. Phys. B 8, 1563 (1994).
[11] J. Y. Ji, J. K. Kim, S. P. Kim and K; S. Soh Phys. Rev. A 52, 3352 (1995)
[12] M. Maamache, Phys. Rev. A 52, 936 (1995); J. Phys. A 29, 2833 (1996); Phys. Scr. 54, 21 (1996).
[13] Y. Z. Lai, J. Q. Liang, H. J. W. Müller-Kirsten and J. G. Zhou, Phys. Rev. A 53, 3691 (1996); J. Phys. A 29, 1773 (1996).
[14] Y. C. Ge and M. S. Child, Phys. Rev. Lett. 78, 2507 (1997).
[15] M. Maamache, J. Phys. A 31, 6849 (1996).
[16] F. G. Scholz, H. B. Geyer, F. J. Hahne, Ann. Phys. 213, 74 (1992).
[17] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
[18] A. Mostafazadeh, Int. J. Geom. Methods Mod. Phys. 07, 1191 (2010).
[19] C. Figueira de Morisson Faria and A. Fring, J. Phys. A: Math. Theor. 39, 9269 (2006).
[20] C. Figueira de Morisson Faria and A. Fring, Laser Physics 17, 424
[21] A. Mostafazadeh, Phys. Lett. B 650, 208 (2007).
[22] M. Znojil, Phys. Rev. D 78, 085003 (2008).
[23] M. Znojil, SIGMA 5. 001 (2009) (e-print overlay: arXiv:0901.0700).
[24] H. B´ıla, “Adiabatic time-dependent metrics in PT-symmetric quantum theories”, eprint arXiv: 0902.0474.
[25] J. Gong and Q. H. Wang, Phys. Rev. A 82, 012103 (2010)
[26] J. Gong and Q. H. Wang, J. Phys. A 46, 485302 (2013).
[27] M. Maamache, Phys. Rev. A 92, 032106 (2015)
[28] A. Fring and M. H. Y. Moussa, Phys. Rev. A 93, 042114 (2016).
[29] A. Fring and M.H. Y. Moussa, Phys. Rev. A 94, 042128 (2016).
[30] B. Khantoul, A. Bounames and M. Maamache, Eur. Phys. J. Plus 132: 258 (2017).
[31] A. Fring and T. Frith, Phys. Rev. A 95, 010102(R) (2017).
[32] F. S. Luiz, M. A. Pontes and M. H. Y. Moussa, Unitarity of the time-evolution and observability of non-Hermitian Hamiltonians for time-dependent Dyson maps. arXiv:1611.08286

[33] F. S. Luiz, M. A. Pontes and M. H. Y. Moussa, Gauge linked time-dependent non-Hermitian Hamiltonians. arXiv:1703.01451

[34] M. Maamache, O-K. Djeghiour, N. Mana and W. Koussa, Quantum Evolution of the Time-Dependent Non-Hermitian Hamiltonians: Real Phases. arXiv:1705.06341

[35] M. Maamache, O-K. Djeghiour, W. Koussa and N. Mana, Time evolution of quantum systems with time-dependent non-Hermitian Hamiltonian and the pseudo Hermitian invariant operator, arXiv:1705.08298

[36] F. Bagarello, J. P. Gazeau, F. H. Szafraniec and M. Znojil, “Non-selfadjoint Operators in Quantum Physics: mathematical aspects” Wiley (2015).

[37] A. B. Klimov and S. M. Chumakov, A Group-Theoretical Approach to Quantum Optics: Models of Atom-Field Interactions (Wiley-VCH, Weinheim, 2009).

[38] S. M. Barnett and P. Radmore, Methods in Theoretical Quantum Optics (Oxford University Press, New York, 1997).

[39] Z. Ahmed, Phys. Lett. A 294, 287 (2002).

[40] M. S. Swanson, J. Math. Phys. 45, 585 (2004).

[41] H. F. Jones, J. Phys. A 38, 1741 (2005).

[42] B. Bagchi, C. Quesne and R. Roychoudhury, J. Phys. A 38, L647 (2005).

[43] D.P. Musumbu, H.B. Geyer and W.D. Heiss, J. Phys. A 40, F75 (2007).

[44] C. Quesne, J. Phys. A 40, F745 (2007).

[45] A. Sinha and P. Roy, J. Phys. A 40, 10599 (2007).

[46] Eva-Maria Graefe, Hans Jurgen Korsch, Alexander Rush and Roman Schubert, J. Phys. A 48, 055301 (2015).

[47] J. C. Garrison and E. M. Wright, Phys. Lett. A 128, 177 (1988).

[48] G. Dattoli, R. Mignani, and A. Torre, J. Phys. A 23, 5795 (1990).

[49] C. Miniature, C. Sire, J. Baudon, and J. Bellissard, Europhys. Lett. 13, 199 (1990).

[50] A. Mondragon and E. Hernandez, J. Phys. A 29, 2567 (1996);

[51] A. Mostafazadeh, Phys. Lett. A 264, 11 (1999).

[52] X.-C. Gao, J.-B. Xu, and T.-Z. Qian, Phys. Rev. A 46, 3626 (1992).

[53] H. Choutri, M. Maamache, and S. Menouar, J. Korean Phys. Soc. 40, 358 (2002).