GEODESIC FLOW OF NONSTRICTLY CONVEX HILBERT GEOMETRIES

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Abstract. In this paper we describe the topological behavior of the geodesic flow for a class of closed 3-manifolds realized as quotients of nonstrictly convex Hilbert geometries, constructed and described explicitly by Benoist. These manifolds are Finsler geometries which have isometrically embedded flats, but also some hyperbolicity and an explicit geometric structure. We prove the geodesic flow of the quotient is topologically mixing and satisfies a nonuniform Anosov closing lemma, with applications to entropy and orbit counting. We also prove entropy-expansiveness for the geodesic flow of any compact quotient of a Hilbert geometry.

1. Introduction

We study topological behavior of the geodesic flow of a class of closed 3-manifolds which are only Finsler, meaning the tangent space admits a norm which does not come from an inner product, and for which the geodesic flow is nonuniformly hyperbolic due to the presence of isometrically embedded flats of dimension two. The 3-manifolds arise as quotients of properly convex domains in real projective space by discrete groups of projective transformations. Such objects are known as Hilbert geometries or convex real projective structures. The structure of the domain and the quotient is well-described thanks to Benoist ([6], see Theorem 2.2). As such, we refer to the 3-manifolds of interest as Benoist 3-manifolds.

We prove several recurrence properties of the geodesic flow of the Benoist 3-manifolds reminiscent of hyperbolic dynamics, such as topological transitivity and a nonuniform Anosov Closing Lemma. Though stable and unstable sets are not even defined for a dense set of points, we prove that strong unstable leaves are dense for closed hyperbolic orbits, which are dense in the phase space. These results culminate in the following:

Theorem (Theorem 5.7). The geodesic flow of a Benoist 3-manifold is topologically mixing.

The geometric properties of the universal cover which Benoist verifies in dimension three are essential for the arguments, hence the results do not immediately generalize.

This paper also serves as a precursor to work of the author on the Bowen-Margulis measure of maximal entropy [9]. To that end, we verify conditions of Bowen [8] for easier computability of topological entropy which holds generally:

Theorem (Theorem 6.2). The geodesic flow of any closed Hilbert geometry satisfies Bowen’s entropy-expansive property.

Then we have the following consequence for the Benoist 3-manifolds, a corollary of which is positive topological entropy.

\footnote{A continuous dynamical system $f^T : X \to X$ is \emph{topologically mixing} if for any open $U, V \subset X$ there exists a $T > 0$ such that $f^T(U) \cap V \neq \emptyset$.}
Proposition (Proposition 7.1). The topological entropy of the geodesic flow of a Benoist 3-manifold is bounded below by the exponential growth rate of lengths of hyperbolic closed orbits.

The structure of the paper is as follows: we first introduce the objects of interest and the relevant background in Section 2. In Section 3 we study automorphisms of the universal cover, and prove that the additive subgroup of $\mathbb{R}$ generated by lengths of closed hyperbolic orbits is dense (Proposition 3.8). This result, along with transitivity (Proposition 4.3) and nonuniform Anosov Closing (Theorem 4.4) from Section 4 will be crucial for the proof of topological mixing in Section 5. In the same section we also prove a nonuniform orbit gluing lemma (Lemma 5.3) which suffices for topological mixing but requires no control over exponential contraction or expansion. Section 6 is devoted to the proof of entropy-expansiveness and Section 7 to orbit counting, with remarks on the relationships between the topological entropy, the volume entropy, and the critical exponent of the group.

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2. Background

A domain $\Omega \subset \mathbb{RP}^n$ is proper if there is an affine chart in which $\Omega$ is bounded, and properly convex if moreover $\Omega$ is convex in this affine chart, meaning the intersection of $\Omega$ with any line is connected. Similarly, $\Omega$ is strictly convex if the intersection of $\partial \Omega$ with any line in the complement of $\Omega$ contains at most one point. A Hilbert geometry on a properly convex domain $\Omega \subset \mathbb{RP}^n$ is determined by the Hilbert metric, defined on an affine chart for $\Omega$ as follows: for any $x, y \in \Omega$, there is a unique projective line $\overline{xy}$ passing through $x$ and $y$. Take $a$ and $b$ to be the intersection points with $\partial \Omega$. Then the $\Omega$-Hilbert distance between $x$ and $y$ is

$$d_{\Omega}(x, y) := \frac{1}{2} \log |a, x, y, b|,$$

where $[a, x, y, b] := \frac{|ay|}{|ax|} \frac{|bx|}{|by|}$. One can verify that $d_{\Omega}$ satisfies the properties of a metric, is complete on $\Omega$, and is well-defined for any affine representation of $\Omega$ by projective invariance of the cross-ratio. Projective lines are always geodesic in this metric, but not all geodesics are lines.

The Hilbert metric is compatible with a Finsler norm, which is Riemannian only when $\Omega$ is an ellipsoid. One can compute that for $(x, v) \in T\Omega$, the Finsler norm is given by

$$F(x, v) := \frac{1}{|v|} \left( \frac{1}{|xv^+|} + \frac{1}{|xv^-|} \right),$$

where $v^-, v^+$ are the intersection points with the topological boundary $\partial \Omega$ of the projective line determined by $v$. A properly convex domain $\Omega$ in $\mathbb{RP}^2$ is uniquely geodesic if and only if there is at most one open line segment in $\partial \Omega$ (this can be verified using the well-definedness of the cross-ratio of four lines, or see also [9]). The ellipsoid in $\mathbb{RP}^n$ is isometric to $\mathbb{H}^n$ when endowed with the Hilbert metric. In this metric, angles are defined, though distorted, since the Finsler norm is Riemannian. This model for hyperbolic space is known as the Beltrami-Klein model.
For a properly convex open $\Omega \subset \mathbb{RP}^n$, define the *automorphism group* of $\Omega$ to be

$$ \text{Aut}(\Omega) := \{ g \in \text{PSL}(n+1, \mathbb{R}) \mid g\Omega = \Omega \}. $$

Note that $\text{Aut}(\Omega) < \text{Isom}(\Omega)$, the isometry group of $(\Omega, d_{\Omega})$, since projective transformations preserve the cross-ratio. The full isometry group of $(\Omega, d_{\Omega})$ is, up to index 2, the group of collineations which preserve $\Omega$ [25]. A properly convex domain $\Omega \subset \mathbb{RP}^n$ is *divisible* if it admits a cocompact action by a discrete subgroup $\Gamma$ of $\text{PSL}(n+1, \mathbb{R})$, in which case we say $\Gamma$ *divides* $\Omega$.

As a first example, the ellipse is divisible by any Fuchsian group. The projective triangle, isometric to $\mathbb{R}^2$ with a hexagonal norm when endowed with the Hilbert metric [15], admits a $\mathbb{Z}^2$ action with quotient torus.

Suppose $\Gamma < \text{PSL}(n+1, \mathbb{R})$ acts properly discontinuously without torsion on $\Omega \subset \mathbb{RP}^n$, so that the quotient $M = \Omega/\Gamma$ is a manifold. The *geodesic flow* of $M$ is defined on $SM$, the Finsler unit tangent bundle to $M$, by flowing unit tangent vectors along projective lines at unit Hilbert speed:

$$ \varphi^t : SM \longrightarrow SM $$

$$ (x, v) \longmapsto (x + tv, v). $$

In other words, $(x, v) \in SM$ determines a unique oriented projective line $\ell_v : \mathbb{R} \to M$ parameterized at unit Hilbert speed, with $\ell_v(0) = x$ and $\varphi^t(v)$ the Finsler unit tangent vector to $\ell_v$ based at $\ell_v(t)$. In the strictly convex case, geodesics are uniquely projective lines and this definition coincides with the standard definition of geodesic flow. In our setting, geodesics are not unique so we require defining the geodesic flow in this very natural way. Note also from the definitions that the regularity of the boundary of $\Omega$ determines the regularity of the geodesic flow.

2.1. **Benoist’s dichotomy.** The following landmark theorem of Benoist for the study of divisible Hilbert geometries is equivalence of the regularity of the boundary, convexity of the boundary, and hyperbolicity of the flow based on abstract properties of the group.

**Theorem 2.1** ([5, Theorem 1.1]). Suppose $\Gamma$ is a discrete torsion-free subgroup of $\text{PSL}(n + 1, \mathbb{R})$ dividing an open properly convex domain $\Omega \subset \mathbb{RP}^n$. Then the following are equivalent:

(i) The domain $\Omega$ is strictly convex.

(ii) The boundary $\partial \Omega$ is of class $C^1$.

(iii) The group $\Gamma$ is $\delta$-hyperbolic.

(iv) The geodesic flow on the quotient manifold $M = \Omega/\Gamma$ is Anosov.

Essential to Benoist’s theorem is Benzécri’s thesis work on the $\text{PGL}$-orbits of marked properly convex sets in projective space [7]. In fact, an application of the work of Benzécri shows that in dimension two, a divisible $\Omega$ is either strictly convex with $C^1$-boundary or a projective triangle.

2.2. **The Benoist 3-manifolds.** A natural question is whether, as in dimension two, a divisible Hilbert geometry in any dimension is either strictly convex with $C^1$-boundary or a simplex. Benoist proved, in the contrary, existence of Hilbert geometries in dimension three which are nonstrictly convex and indecomposable via a modification of the Kac–Vinberg Coxeter construction [6, Proposition 1.3]. Moreover, Benoist proved geometric properties of such Hilbert geometries. Before stating the theorem, we define some terms: let $C$ be the cone in $\mathbb{R}^{n+1}$ over a properly convex domain $\Omega$ in $\mathbb{RP}^n$, and define $C$ to be properly convex if and only if $\Omega$ is. Then $\Omega$ is *decomposable* if there exist vector subspaces $V_1, V_2 \subset \mathbb{R}^{n+1}$ and
properly convex cones $C_1 \subset V_1, C_2 \subset V_2$ such that $C = C_1 + C_2$. Else, $\Omega$ is indecomposable. Note that a simplex is always decomposable.

A properly embedded triangle in $\Omega$ is a projective triangle $\triangle \subset \Omega$ such that $\partial \triangle \subset \partial \Omega$. Let $\mathcal{T}$ denote the collection of triangles $\triangle$ properly embedded in $\Omega$, and $\Gamma_\triangle := \text{Stab}_\Gamma(\triangle) = \{ \gamma \in \Gamma \mid \gamma \triangle = \triangle \}$ be the subgroup of $\Gamma$ stabilizing $\triangle \in \mathcal{T}$.

**Theorem 2.2** ([6, Theorem 1.1]). Let $\Gamma \leq \text{SL}(4, \mathbb{R})$ be a discrete torsion-free subgroup which divides an open, properly convex, indecomposable $\Omega \subset \mathbb{RP}^3$, and $M = \Omega/\Gamma$. Then

(a) Every subgroup in $\Gamma$ isomorphic to $\mathbb{Z}^2$ stabilizes a unique triangle $\triangle \in \mathcal{T}$.
(b) If $\triangle_1, \triangle_2 \in \mathcal{T}$ are distinct, then $\triangle_1 \cap \triangle_2 = \varnothing$.
(c) For every $\triangle \in \mathcal{T}$, the group $\Gamma_\triangle$ contains an index-two $\mathbb{Z}^2$ subgroup.
(d) The group $\Gamma$ has only finitely many orbits in $\mathcal{T}$.
(e) The image in $M$ of triangles in $\mathcal{T}$ is a finite collection $\mathcal{F}$ of disjoint tori and Klein bottles.

If one cuts $M$ along each $\mathcal{T} \in \mathcal{F}$, each of the resulting connected components is atoroidal.
(f) Every nontrivial line segment is included in the boundary of some $\triangle \in \mathcal{T}$.
(g) If $\Omega$ is not strictly convex, then the set of vertices of triangles in $\mathcal{T}$ is dense in $\partial \Omega$.

We will call the compact quotients of nonstrictly convex, indecomposable, divisible Hilbert geometries in dimension three Benoist 3-manifolds. The topological decomposition as in 2.2(e) is an example of a Jaco–Shalen–Johansson (JSJ) decomposition [17, 18]. Benoist remarks after Theorem 2.2 that as a consequence of Thurston’s geometrization, the atoroidal components of the quotient are diffeomorphic to finite volume hyperbolic 3-manifolds [6, pp. 4-5].

### 3. Properties of automorphisms of Benoist’s 3-manifolds

Let $\Omega \subset \mathbb{RP}^n$ be a properly convex domain. Then for any $g \in \text{Aut}(\Omega)$, we can define the translation length of $g$ by

$$\tau(g) := \inf_{x \in \Omega} d_\Omega(x, g.x).$$

An axis of $g$ is a $g$-invariant projective line in $\Omega$.

We will diverge slightly from the literature here in our terminology. We define $g \in \text{Aut}(\Omega)$ to be hyperbolic if $\tau(g) > 0$ and the infimum is attained along a unique axis of $g$. Any other $f \in \text{Aut}(\Omega)$ for which $\tau(f)$ is positive and realized, but not along an axis of $f$, will be called flat. Typically, both flat and hyperbolic automorphisms are called hyperbolic (c.f. [14]), and in the strictly convex case there would be no need for the distinction we introduce here. A projective transformation is quasi-hyperbolic if $\tau(g) > 0$ and the infimum is not attained, parabolic if $\tau(g) = 0$ and the infimum is not attained, and elliptic if $\tau(g) = 0$ and the infimum is attained.

There is an important property of the Benoist 3-manifolds which has dynamical implications for the group elements. If $\Omega/\Gamma$ is a Benoist 3-manifold, then $\Omega$ must be indecomposable, hence $\Gamma$ is irreducible [24]. A subgroup $H < \text{PSL}(4, \mathbb{R})$ is irreducible if it does not stabilize a projective point, line, or plane in $\mathbb{RP}^3$, and $H$ is strongly irreducible if every finite-index subgroup is irreducible. Since $\Gamma$ is irreducible, all elements of $\Gamma$ are positively proximal [3, Proposition 1.1], meaning their top eigenvalues are real and positive and have one-dimensional eigenspaces. Since $\Gamma$ is a group, each $g \in \Gamma$ must in fact be biproximal, meaning the top and bottom eigenvalues each have one-dimensional eigenspaces, and we note that they must also both be real and positive. In fact, for the Benoist 3-manifolds, all elements of any $\mathbb{Z}^2$ subgroup of $\Gamma$ have only real positive eigenvalues [6, Corollary 2.4].
Proposition 3.1. Let \( M = \Omega / \Gamma \) be a Benoist 3-manifold with discrete, torsion-free dividing group \( \Gamma \). Then there are no quasi-hyperbolic group elements in \( \Gamma \).

Proof. Suppose \( \tau(g) > 0 \). Let the eigenvalues of a representative of \( g \) in \( \text{SL}(4, \mathbb{R}) \) be given by \( \lambda_i(g) \) such that \( \lambda_0(g) > |\lambda_1(g)| \geq |\lambda_2(g)| > \lambda_3(g) \). It is straightforward to verify that since \( g \) is biproximal, \( \tau(g) = \log \frac{\lambda_0(g)}{\lambda_3(g)} \) and is realized along a projective line joining the eigenvectors associated to \( \lambda_0 \) and \( \lambda_3 \). Again since \( g \in \text{Aut}(\Omega) \), we may choose a maximal projective line segment in \( \overline{\Omega} \) preserved by \( g \) along which \( \tau(g) \) is realized, which we denote \( \ell_g \). To be quasi-hyperbolic, \( \ell_g \) must be contained in \( \partial \Omega \). Then by Theorem 2.8(f), \( \ell_g \subset \partial \Delta \) for some properly embedded triangle \( \Delta \) in \( \Omega \). Since \( g \) acts by projective transformations and preserves \( \Omega \), for any properly embedded triangle \( \Delta \) we have that \( g\Delta \) is also a properly embedded triangle. Then since \( g \) stabilizes \( \ell_g \) we have \( \ell_g \subset g\Delta \cap \Delta \neq \emptyset \) implying \( g\Delta = \Delta \) by Benoist’s Theorem 2.2(b). Then \( g \in \text{Stab}(\Delta) \) and thus \( \tau(g) \) is realized in \( \Delta \).

We now introduce new terminology for points in \( \partial \Omega \). We say that \( \xi \in \partial \Omega \) is proper if there is a unique supporting hyperplane to \( \Omega \) at \( \xi \), where a hyperplane \( H \subset \mathbb{R}P^n \setminus \Omega \) is a supporting hyperplane at \( \xi \) if \( \xi \in H \cap \partial \Omega \). Also, \( \xi \in \partial \Omega \) is extremal if there is no open line segment containing \( \xi \) embedded in \( \partial \Omega \); in other words, \( \xi \) is extremal if for any supporting hyperplane \( H \) at \( \xi \), we have \( H \cap \partial \Omega = \{\xi\} \). Note that by Benoist’s Theorem 2.2(f) and duality, proper extremal points form the compliment of the boundaries of properly embedded triangles.

Proposition 3.2. Let \( M = \Omega / \Gamma \) be a Benoist 3-manifold with discrete, torsion-free dividing group \( \Gamma \). Then for all \( g \in \Gamma \),

- \( g \) is hyperbolic if and only if \( g \) has exactly two fixed points \( g^- , g^+ \) in \( \overline{\Omega} \) which are proper extremal points in the boundary. These fixed points are respectively repelling and attracting under the dynamics of \( g \) on \( \Omega \).
- \( g \) is flat if and only if \( g \in \text{Stab}(\Delta) \) for some properly embedded \( \Delta \).

These are the only possible automorphisms of a divisible, indecomposable domain in \( \mathbb{R}P^3 \).

Proof. Since \( \Gamma \) is discrete and torsion-free, there are no elliptic isometries in \( \Gamma \). Since \( M \) is compact, there are no parabolic isometries in \( \Gamma \) (see also [6, Lemma 2.8]). By Proposition 3.1, there are no quasi-hyperbolic elements of \( \Gamma \). Thus, it suffices to characterize the dynamics of group elements with translation length realized in \( \Omega \). Since all elements of \( \Gamma \) are biproximal, this is straightforward.

3.1. Lengths of hyperbolic orbits. The goal of this subsection is to prove that the additive subgroup of \( \mathbb{R} \) generated by lengths of closed hyperbolic orbits is dense via Zariski density of an immersed hyperbolic surface group in \( \Gamma \).

Fact 3.3 ([21, 2]). The fundamental group of a complete, finite volume, noncompact hyperbolic 3-manifold contains a closed hyperbolic quasi-Fuchsian surface subgroup.

Let \( \Gamma_{\text{hyp}} \) denote the hyperbolic elements of \( \Gamma \), and let \( \Sigma < \text{PSL}(4, \mathbb{R}) \) be the subgroup of \( \Gamma \) which is isomorphic to the hyperbolic surface subgroup given by Fact 3.3 and Benoist’s remark following Theorem 2.2. Since \( \Sigma \) is a quasi-Fuchsian subgroup, no element of \( \Sigma \) can preserve any properly embedded triangle. Then by Proposition 3.2, \( \Sigma \) is a subgroup in \( \Gamma_{\text{hyp}} \).

Corollary 3.4. There exist infinitely many noncommuting hyperbolic group elements in \( \Gamma \).

Let \( G \) be any subset of \( \text{Aut}(\Omega) \) and \( \mathcal{L}(G) := \langle \tau(g) \rangle_{g \in G} \) the additive subgroup of \( \mathbb{R} \) generated by translation lengths of group elements in \( G \) acting on \( \Omega \). Note that if \( G \) is a subset of the group \( \Gamma \) which divides \( \Omega \) then \( \mathcal{L}(G) \) is the additive subgroup of \( \mathbb{R} \) generated by lengths of closed curves in \( \Omega / \Gamma \) associated to conjugacy classes in \( \Gamma \) of elements of \( G \).
Corollary 3.5 (of [5, Fact 5.5]). If \(\Gamma\) is a Zariski dense subgroup of \(\text{SL}(n+1, \mathbb{R})\) preserving a properly convex domain \(\Omega \subset \mathbb{RP}^n\), then \(\mathcal{L}(\Gamma)\) is dense in \(\mathbb{R}\).

If \(\Omega\) is not an ellipsoid, then the hypotheses of Corollary 3.5 hold whenever \(\Gamma\) is acting cocompactly on an indecomposable properly convex and strictly convex \(\Omega\) in projective space [4, Theorem 1.2]. In our case, \(\Omega\), the universal cover of a Benoist 3-manifold is indecomposable but is not strictly convex, so Corollary 3.5 does not apply directly to \(\Gamma\) the fundamental group of a Benoist 3-manifold.

Proposition 3.6 (restatement of [14, Proposition 6.5]). Suppose \(\Gamma\) is a strongly irreducible subgroup of \(\text{SL}(n+1, \mathbb{R})\) which preserves a properly convex \(\Omega \subset \mathbb{RP}^n\). Let \(G\) be the Zariski closure of \(\Gamma\). Then \(G\) is a Zariski-connected real semi-simple Lie group.

If \(\log(\Gamma)\) cannot be a subspace of \(\mathbb{R}\) or \(\Pi\) by duality hold whenever \(\Gamma\) is acting properly discontinuously on \(\Omega\). If \(\Gamma\) is acting discretely without torsion cannot have elliptic elements. Also, \(\Gamma\) preserves (divides) \(\Omega\) if and only if the transpose \(\Gamma^t\) preserves (divides) the projective dual \(\Omega^*\) [5, Lemma 2.8].

Lemma 3.7. The closed hyperbolic surface subgroup \(\Sigma\) is either strongly irreducible or \(\mathcal{L}(\Sigma)\) is dense in \(\mathbb{R}\).

Proof. First, since \(\Sigma\) is a surface group, every finite-index subgroup is also a surface subgroup. It suffices to show any surface group in \(\text{PSL}(4, \mathbb{R})\) preserving a domain \(\Omega\subset \mathbb{RP}^3\) is irreducible. By contradiction, suppose \(\Sigma\) fixes a point \(p\in \mathbb{RP}^3\). Clearly \(p\notin \Omega\) because \(\Gamma\) acting discretely without torsion cannot have elliptic elements. Also, \(p\notin \partial \Omega\) because elements of \(\Sigma\) do not stabilize any triangles so all fixed points of elements of \(\Sigma\) are proper and extremal, and noncommuting hyperbolic isometries cannot fix the same proper extremal point since \(\Gamma\) acts properly discontinuously on \(\Omega\). If \(p\notin \Omega\), then we consider the dual case: when \(\Sigma^t\) preserves a projective plane \(\Pi\) which intersects \(\Omega^*\). Then \(\Sigma^t\) is acting cocompactly on a totally geodesic hypersurface \(\Pi \cap \Omega^*\). By [4, Theorem 1.2], \(\Sigma^t\) is either Zariski dense and hence \(\mathcal{L}(\Sigma)\) is dense by Corollary 3.5 if \(\Pi \cap \Omega^*\) is homogeneous and \(\mathcal{L}(\Sigma^t)\) is dense in \(\mathbb{R}\) anyways. Then so is \(\mathcal{L}(\Sigma)\) since dual groups preserving dual properly convex sets are isospectral. Thus, if \(\Sigma\) preserves \(\Omega\) and fixes a point, then \(\mathcal{L}(\Sigma)\) is dense in \(\mathbb{R}\).

Now suppose \(\Sigma\) preserves a line \(l\). The case where \(l\subset \Omega\) or \(l\) is disjoint from \(\Omega\) by duality is impossible because \(\text{Aut}(l) = \mathbb{R}\). If \(l\) intersects \(\partial \Omega\) then either \(\Sigma \notin \text{Aut}(\Omega)\) or \(\Sigma \subset \text{Stab}(\Delta)\), both a contradiction.

If \(\Sigma\) stabilizes a plane, then we revisit the dual cases where \(\Sigma^t\) stabilizes a point, unless the plane intersects \(\Omega\). In this case, we have already seen that \(\mathcal{L}(\Sigma)\) is dense in \(\mathbb{R}\).

Proposition 3.8. Let \(\Omega/\Gamma = M\) be a Benoist 3-manifold. Then \(\mathcal{L}(\Gamma_{\text{hyp}})\) is dense in \(\mathbb{R}\).

Proof. By Lemma 3.7 and Proposition 3.6, the group \(\Sigma \subset \Gamma_{\text{hyp}}\) is either Zariski dense or \(\mathcal{L}(\Sigma)\) is dense in \(\mathbb{R}\). By Corollary 3.5, density of \(\mathcal{L}(\Sigma)\) holds in both cases.

4. Recurrence behavior

Recall that a point \(\xi \in \partial \Omega\) is proper if there exists a unique supporting hyperplane to \(\Omega\) at \(\xi\) and extremal if \(\xi\) is contained in no open line segment embedded in \(\partial \Omega\). For the Benoist 3-manifolds, vertices of properly embedded triangles are the only nonproper points, and all
nonextremal points are contained in the side of some properly embedded triangle. Thus, the proper extremal points are the complement of the boundaries of properly embedded triangles. We will say \( v \in S\Omega \) is forward regular if \( v^+ \) is a proper extremal point, and similarly for backward regular. If \( v^+ \) is a nonproper or nonextremal point, then \( v \) is forward singular. If \( v \) is both forward and backward regular, then we will say \( v \) is regular (and similarly for singular vectors). Let \( S\Omega_{\text{reg}} \) be the collection of regular vectors and the complement, \( S\Omega_{\text{sing}} \), the set of \( v \in \Omega \) such that \( v^+ \) or \( v^- \) is in the boundary of some properly embedded triangle. The collection of vectors tangent to projective lines contained in properly embedded triangles is denoted \( S\Omega_{\text{flat}} \). These notions descend to the quotient since \( \Gamma \) is acting by projective transformations, and we assign the analogous definitions to \( SM_{\text{reg}}, SM_{\text{sing}}, \) and \( SM_{\text{flat}} \). Lastly, a closed orbit \( \varphi \cdot v \) is hyperbolic if when \( v \) is lifted to \( \tilde{v} \) in the universal cover, \( \ell_{\tilde{v}} \) is preserved by a hyperbolic group element. Note that vector with a closed orbit which is hyperbolic must be regular (Proposition 3.2). We will also denote by \( d \) a Finsler metric on \( SM \) compatible with the topology, see [22, pp 161-206].

4.1. Stable and unstable sets. Recall that the stable and unstable sets at a point are defined to be

\[
W^{ss}(v) = \{ u \in SM \mid d(\varphi^t v, \varphi^t u) \to 0 \mid t \to +\infty \},
\]

\[
W^{su}(v) = \{ u \in SM \mid d(\varphi^{-t} v, \varphi^{-t} u) \to 0 \mid t \to +\infty \}.
\]

The weak stable and unstable sets of \( v \) (denoted \( W^{os}(v) \) and \( W^{ou}(v) \), respectively) are the points which stay bounded distance from \( v \) under the geodesic flow in positive and negative time, respectively. The strong stable and unstable sets are global if for all regular \( u \neq v \), at least one of the following are nonempty: \( W^{ss}(v) \cap W^{ou}(u) \) or \( W^{ss}(v) \cap W^{ou}(-u) \).

**Proposition 4.1.** If \( v \in SM_{\text{reg}} \) then \( W^{ss}(v) \) and \( W^{su}(v) \) are defined globally, and \( W^{os}(v), W^{ou}(v) \) admit a flow invariant foliation by strong stable (respectively, strong unstable) leaves.

**Proof.** For proper extremal points, horospheres are well-defined and the geometric description of stable and unstable sets applies as for the strictly convex case (as in [5, Lemma 3.4]): that is, for for \( v \in S\Omega_{\text{reg}} \) we have

\[
W^{ss}(v) = \{ u \in S\Omega \mid \pi u \in H_{v^+}(\pi v), \ u^+ = v^+ \},
\]

\[
W^{su}(v) = \{ u \in S\Omega \mid \pi u \in H_{v^-}(\pi v), \ u^- = v^- \},
\]

where \( H_\xi(p) \) is the horosphere based at \( \xi \in \partial\Omega \) passing through \( p \in \Omega \), and the strong stable and unstable sets foliate the weak stable and unstable sets:

\[
W^{ou}(v) = \bigcup_{t \in \mathbb{R}} W^{su}(\varphi^t v)
\]

\[
= \{ w \in S\Omega \mid w^- = v^- \},
\]

such that the foliation is both \( \Gamma \)-invariant and \( \varphi^t \)-invariant. It is then clear that for any two \( u \neq v \in S\Omega_{\text{reg}} \), \( W^{ss}(v) \cap W^{ou}(u) \neq \emptyset \) as long as \( u^- \neq v^- \). □

Conversely, nonproper and nonextremal points do not have well-defined stable and unstable sets which foliate the weak stable and unstable sets. This can be verified by basic properties of the cross-ratio. By Theorem 2.2(g), the vertices of properly embedded triangles in \( \Omega \) are dense in \( \partial\Omega \), and as such the singular points \( x \in S\Omega \in \partial\Omega \) are dense in \( S\Omega \). Since these points do not admit stable and unstable sets, the geodesic flow cannot have local product structure.
However, the Bowen bracket for regular vectors is well-defined by Proposition 4.1. The Bowen-bracket of $u$ with $v$ is the point of intersection $w$ of the strong stable and weak stable foliations of $u$ and $v$ such that $v^- \neq u^+$. Geometrically, $w$ is uniquely determined by $w^- = v^-$, $w^+ = u^+$, and $\pi w \in \mathcal{H}_u(\pi u)$.

Another consequence of the geometric definition of stable and unstable sets is that the distance between $v, u \in W^{os}(w)$ is monotone decreasing under the geodesic flow in positive time for an adapated metric [13], which can be verified by properties of the cross-ratio. Similarly, the distance between points in the same unstable set is monotone decreasing under the flow in negative time.

4.2. Topological transitivity. In this subsection we prove topological transitivity, which is equivalent to existence of a dense orbit when the phase space is compact [19, Lemma 1.4.2], as in the case of the Benoist 3-manifolds. A continuous dynamical system $f^t : X \to X$ is topologically transitive if for every pair of open sets $U, V \subset X$, there exists a time $0 < T \in \mathbb{R}$ such that $f^T(U) \cap V \neq \emptyset$. If $X$ is a metric space then the system is uniformly transitive if for all $\delta > 0$, there exists a $T > 0$ such that for all $x, y \in X$, there is some $t \leq T$ such that $f^t(B(x, \delta)) \cap B(y, \delta) \neq \emptyset$. It is clear that transitivity implies uniform transitivity when $X$ is a compact metric space.

Lemma 4.2. Hyperbolic periodic points are dense for the geodesic flow of a Benoist 3-manifold.

Proof. We want to show any $(\xi, \eta) \in \partial \Omega \times \partial \Omega \setminus \text{diag}$ can be approximated by $(g^-, g^+)$ such that $g \in \Gamma_{\text{hyp}}$. Take two noncommuting hyperbolic elements $g, h \in \Gamma$ (Corollary 3.4). Construct the sequence $k_n = g^k h^k$. Then there are fixed points $k_n^+$ and $k_n^-$ in $\partial \Omega$ of $k_n$ such that $k_n^+ \to g^+$ and $k_n^- \to h^-$ as $n \to \infty$. Using the sequence $k_n$ and minimality of the action of $\Gamma$ on $\partial \Omega$ [6, Proposition 3.10], we conclude that any $(\xi, \eta) \in \partial \Omega \times \partial \Omega \setminus \text{diag}$ is approximable by such $k_n$. If any $k_n$ admits a projective line axis, then this projective line axis corresponds to a periodic orbit for the flow and we conclude that any vector tangent to the projective line $(\xi \eta)$ is approximable by periodic points. To prove the lemma, we just need to show there necessarily exists a subsequence $k_{n_i}$ of only hyperbolic elements.

By contradiction, suppose there is no such subsequence. There exists an $N$ such that for all $n \geq N$, each $k_n$ preserves a properly embedded triangle $\Delta_n$. If we assume $k_n$’s geodesic axis of translation, which is not necessarily a projective line, is also on the triangle, we consider the accumulation of the boundary of the triangles, $\partial \Delta_n$, in $\partial \Omega$, which will contain $h^-$ and $g^+$. This set will be either the boundary of a properly embedded triangle, a line segment, or a point. None of the above are possible since $h, g$ are hyperbolic and do not commute, and $\Gamma$ acts discretely so $h^- \neq g^+$ are proper extremal points, and $(h^- g^+) \notin \partial \Omega$. \hfill \Box

Let $\varphi \cdot v$ denote the orbit of $v$.

Proposition 4.3. The geodesic flow of a Benoist 3-manifold is topologically transitive.

Proof. Take two open sets $U$ and $V$ in $SM$. By Lemma 4.2, there are hyperbolic periodic points $u \in U$ and $v \in V$. We now construct a heteroclinic orbit. Lifting to the universal cover, we have $\tilde{u} \in \tilde{U}, \tilde{v} \in \tilde{V} \subset S\Omega$ such that $u^-$ and $v^+$ are proper extremal points of $\partial \Omega$. Then the open projective line segment $(u^- v^+)$ is contained in $\Omega$ and is the footpoint projection of an orbit of the geodesic flow. Let $\check{w} \in S\Omega$ denote the Bowen bracket of $\tilde{v}$ with $\tilde{u}$:

$$\check{w} \in W^{ss}(\tilde{v}) \cap W^{su}(\varphi^\ell \tilde{u})$$
for some $t \in \mathbb{R}$. Since $u, v$ are periodic, there are hyperbolic group elements $\gamma_u, \gamma_v$ preserving $\ell_u, \ell_v$ so $d\gamma_u^n(\bar{U}) \cap \bar{v} \cdot \bar{u}$ and $d\gamma_v^n(\bar{V}) \cap \bar{v} \cdot \bar{v}$ each contain lifts of $u$ and $v$ respectively for all $n \in \mathbb{Z}$. Since $\gamma_u, \gamma_v$ are isometries and $u^- = w^-, u^+ = w^+$ are proper extremal points, there is an $N$ such that for all $n \geq N$, $d\gamma_u^n(\bar{U}) \cap \bar{v} \cdot \bar{w} \neq \emptyset$ and $d\gamma_v^n(\bar{V}) \cap \bar{v} \cdot \bar{w} \neq \emptyset$. Then choosing times $t_1, t_2$ so that $\varphi^{t_1} \bar{w} \in d\gamma_u^n(\bar{U}) \cap \bar{v} \cdot \bar{w}$ and $\varphi^{t_2} \bar{w} \in d\gamma_v^n(\bar{V}) \cap \bar{v} \cdot \bar{w}$, we can project $\varphi^{t_1} \bar{w}$ to $SM$ and obtain $T = -t_1 + t_2$ such that $w' := d\pi_\Gamma \varphi^{t_1} \bar{w} \in U$, where $\pi_\Gamma : \Omega \to M$ is the quotient map, and $\varphi^T w' \in V$ as desired.

4.3. The Anosov Closing Lemma. In this subsection, we prove Anosov closing of recurrent orbits, originally due to Anosov in the negative curvature case [1].

Define a filtration of $SM \setminus SM_{\text{flat}}$ by compact sets bounded away from flats:

$$\Lambda_\eta := \{ v \in SM \mid d(v, w) \geq \eta \text{ for all } w \in SM_{\text{flat}} \}.$$ 

We say for points $u, v \in SM$ and $\epsilon > 0$ that $u$ $\epsilon$-shadows $v$ for time $t$ if $d_s(u, v) < \epsilon$ for $s \in [0, t]$.

**Theorem 4.4.** Let $\Omega$ be an indecomposable, nonstrictly convex domain in $\mathbb{RP}^3$. Suppose $\Gamma < \text{PSL}(4, \mathbb{R})$ is a discrete, torsion free group dividing $\Omega$, with compact quotient $M = \Omega / \Gamma$. Then for all $\eta > 0$ and sufficiently small $\epsilon > 0$, there exists a $\delta > 0$ and $T > 0$ such that:

- For any $t \geq T$, $v \in \Lambda_\eta$ with $d(\varphi^t v, v) < \delta$, there exists a hyperbolic periodic orbit $v'$ of period $t' \in (-\epsilon, t + \epsilon]$ which $\epsilon$-shadows $v$ for time $\min\{ t, t' \}$.

**Proof.** We adapt a proof by contradiction following Eberlein [16] (see also [11, Theorem 7.1]). Assume we have particular $\eta, \epsilon > 0$ and a sequence of $v_n \in \Lambda_\eta$ paired with a sequence $t_n \to \infty$ such that $d(v_n, \varphi^{t_n} v_n) \to 0$, yet any $w_n$ which $\epsilon$-shadows $v_n$ for time $t_n$ is not periodic of any period $t'_n \in \{ t_n - \epsilon, t_n + \epsilon \}$.

We can assume up to extraction of subsequences that the $v_n$ converge to some $v \in \Lambda_\eta$. Lifting $SM$ to a compact fundamental domain $SD$ containing $\bar{v}$ in $S\Omega$, we have some $\tilde{v} \in S\Omega$ with points $v^+, v^-$ in $\partial \Omega$, and lifts $\tilde{v}_n$ of the $v_n$ which converge to $\tilde{v}$ in $SD$. Also, since $\varphi \cdot v_n$ almost closes up after time $t_n$, there are group elements $\gamma_n$ which take $\tilde{v}_n$ close to $\varphi^{t_n} \tilde{v}_n$.

Note that the $\gamma_n$ need not be hyperbolic a priori.

Again, the contradiction hypothesis is that if $w_n \epsilon$-shadows $v_n$ for time $t_n$, then $w_n$ cannot be periodic of any period $t'_n \in \{ t_n - \epsilon, t_n + \epsilon \}$. Eberlein’s geometric observation is that in the universal cover, if $w_n \epsilon$-shadows $v_n$ for time $t_n$, then the same $\gamma_n$ which moves $\tilde{v}_n$ close to $\varphi^{t_n} \tilde{v}_n$ must also be responsible for moving $\tilde{w}_n$ close to $\varphi^{t_n} \tilde{w}_n$. Because $\Gamma$ is acting on $\Omega$ properly discontinuously and cocompactly by isometries, the assumption that $w_n$ is not periodic of period approximately $t_n$ is realized in the universal cover as follows: if $d(\tilde{w}_n, \tilde{v}_n) < \epsilon$, then $\gamma_n \tilde{w}_n \neq \varphi^{t_n} \tilde{w}_n$ for any $t'_n \in \{ t_n - \epsilon, t_n + \epsilon \}$.

The goal of the following lemmas will be to show that nonexistence of an axis of $\gamma_n$ which is $\epsilon$-close to $\ell_{\gamma_n}[0, t_n]$ for infinitely many $n$ is mutually exclusive with the assumption that the $v_n$ and $v$ are in $\Lambda_\eta$, producing the desired contradiction.

**Lemma 4.5.** Let $p \in \Omega$ be the footpoint of $\tilde{v}$. Then $\gamma_n : p \to v^+$ and $\gamma_n^{-1} : p \to v^-$.

**Proof.** Take any convex open neighborhood $\mathcal{N}(v^+)$ in $\overline{\Omega}$. Since $\tilde{v} \to \bar{v}$, we have $v^+_n \in \mathcal{N}(v^+)$ for all sufficiently large $n$. Then as $t_n \to +\infty$, $\ell_{\gamma_n}(t_n) \in \mathcal{N}(v^+)$ by convexity of $\mathcal{N}(v^+)$. Since $\gamma_n$ is chosen so that $d(\gamma_n \tilde{v}_n, \varphi^{t_n} \tilde{v}_n) \to 0$ as $n \to \infty$, then $d(\gamma_n \tilde{v}_n, \ell_{\gamma_n}(t_n)) \to 0$ with $n$. Once $\ell_{\gamma_n} \in \mathcal{N}(v^+)$ for all large enough $n$ and $\gamma_n$-$p_n$ is sufficiently close to $\ell_{\gamma_n}$, we will have $\gamma_n : p_n \in \mathcal{N}(v^+)$. Finally, as $\tilde{v}_n \to \bar{v}$ implies $p_n \to p$ and $\gamma_n$ is an isometry, we can conclude for large $n$ that $\gamma_n : p \in \mathcal{N}(v^+)$. 


Now consider $\mathcal{N}(v^-)$ a convex open neighborhood of $v^-$ in $\Omega$. As $\gamma_n.\tilde{v}_n$ approaches $\varphi^t_n\tilde{v}_n$, the group element $\gamma_n^{-1}$ brings the line segment $\ell_{\tilde{v}_n}[-s_n + t_n, s_n + t_n]$ back very close to the line segment $\ell_{\tilde{v}_n}[-s_n, s_n]$ for some $s_n \to \infty$ with $n$. Then as $s_n$ gets very large, $\ell_{\tilde{v}_n}[-s_n]$ gets closer to $\tilde{v}_n$, as will $\gamma_n.\ell_{\tilde{v}_n}(-s_n + t_n)$ which is converging to $\gamma_n.\tilde{v}_n$ with large $s_n$. Hence, $\gamma_n^{-1}.\tilde{v}_n$ approaches $\tilde{v}_n$ in the boundary. Then as $\tilde{v}_n \to \tilde{v}$, for sufficiently large $n$, we have $\gamma_n^{-1}.\tilde{v}_n \in \mathcal{N}(v^-)$. Since $\gamma_n^{-1}.p_n$ is a point on the line $\gamma_n^{-1}.\ell_{\tilde{v}_n}$, it suffices to observe that $d_\Omega(\gamma_n^{-1}.p_n, p_n) = d_\Omega(p_n, \gamma_n.\tilde{v}_n) \sim t_n \to \infty$ as $n \to \infty$ to conclude $\gamma_n^{-1}.p \in \mathcal{N}(v^-)$ for all sufficiently large $n$.

We next define $V_k(v^+)$, $V_k(v^-)$ open neighborhoods in $\partial \Omega$ such that for any $\xi \in V_k(v^+)$, $\zeta \in V_k(v^-)$, the projective line $(\xi \zeta)$ is distance less than $\frac{1}{k}$ from $\ell_{\tilde{v}}(0)$ in the Hilbert metric. The existence of such $V_k$ is immediate in a Hilbert geometry by the definition of $d_\Omega$. The $V_k$ are also homeomorphic to open balls in $\mathbb{R}^2$. Choose $k$ large enough that $\frac{1}{k} < \frac{\epsilon}{2}$.

Lemma 4.6. For all sufficiently large $n$, $\gamma_n(V_k(v^+)) \subset V_k(v^+)$ and $\gamma_n^{-1}(V_k(v^-)) \subset V_k(v^-)$.

Proof. Note that as $\gamma_n.\tilde{v}_n^+$ is closer to $\tilde{v}_n^+$ and $\tilde{v}_n \to \tilde{v}$, then $\gamma_n.\tilde{v}_n^+ \to \tilde{v}$ (and similarly, $\gamma_n^{-1}.\tilde{v}_n^- \to \tilde{v}$). If $\gamma_n.\tilde{v}_n^+$ is very close to $\tilde{v}^+$, then $\gamma_n$ either fixes $\tilde{v}^+$, is contracting near $\tilde{v}^+$, or both. The only way that $\gamma_n(V_k(v^+)) \not\subset V_k(v^+)$ would be if $\gamma_n$ stabilized a properly embedded triangle $\Delta_n$ such that $\partial \Delta_n \cap \partial V_k(v^+) \neq \emptyset$. If this happened infinitely often, then $v^+$ would necessarily be the limit of vertices of $\Delta_n$ which are attracting eigenvectors for the $\gamma_n \in \text{Stab}(\Delta_n)$. Since $\gamma_n^{-1}.v^- \to v^-$ and $\gamma_n^{-1} \in \text{Stab}(\Delta_n)$, we can also conclude that vertices of $\Delta_n$ which are repelling eigenvectors for $\gamma_n$ must accumulate on $v^-$. Then in the quotient $SM$, for large enough $n$, $v$ must be distance less than $\eta$ from a quotient torus of one of the $\Delta_n$, contradicting the assumption that $v \in \Lambda_\eta$ for small $\epsilon$.

An analogous argument applies to show, up to extraction of subsequences, for all sufficiently large $n$, $\gamma_n^{-1}(V_k(v^-)) \subset V_k(v^-)$.

So we now have that for large $n$, $\gamma_n(V_k(v^+)) \subset V_k(v^+)$ and similarly $\gamma_n^{-1}(V_k(v^-)) \subset V_k(v^-)$, both of which are homeomorphic to open balls in $\mathbb{R}^2$. Applying Brouwer’s fixed point theorem, it follows that $\gamma_n$ fixes points in $V_k(v^-)$ and $V_k(v^+)$. Then $\gamma_n$ has an axis distance less than $\frac{1}{k} < \frac{\epsilon}{2}$ from $\ell_{\tilde{v}}(0)$, hence $\epsilon$-close to $\ell_{\tilde{v}}(0)$ for all sufficiently large $n$. We also assume that $\gamma_n.\tilde{v}_n$ is arbitrarily close to $\varphi^t_n\tilde{v}_n$, so the axis of $\gamma_n$ will eventually and thereafter be $\epsilon$-close to $\ell_{\tilde{v}_n}(0, t_n)$ and the translation length of $\gamma_n$ must be $\epsilon$-close to $t_n$. And so we have a periodic orbit of period $t'_n \in [t_n - \epsilon, t_n + \epsilon]$ which $\epsilon$-shadows $v_n$ for time $\max\{t_n, t'_n\}$, contradicting the assumption. If we obey our hypothesis that such a periodic orbit is impossible, then we would necessarily have $v \not\in \Lambda_\eta$ as proven in Lemma 4.6 – a contradiction.

Lastly, note that for small $\epsilon$, hyperbolicity of the periodic orbit is implicit, since a periodic orbit tangent to a torus is bounded away from $v_n$ by $\eta$ and thus could not $\epsilon$-shadow $v_n$ for small $\epsilon$.

5. Topological mixing

We prove the geodesic flow of a Benoist 3-manifold is topologically mixing following the strategies of Coudene [10], but without the local product structure property. Key properties will be a nonuniform orbit gluing lemma (Lemma 5.3) and density of the unstable leaves for periodic regular vectors (Proposition 5.5).
Let $W_ε^s(v) = W^s(v) \cap B(v, ε)$ and let $\langle v, u \rangle$ denote the Bowen bracket of $v$ with $u$ where $v, u$ are regular vectors.

**Proposition 5.1.** For all $ε > 0$ and $u \in SM_{reg}$, there is a $δ > 0$ such that for all $v \in B(u, δ) \cap SM_{reg}$, there exists a $|t| < ε$ such that

$$\langle v, u \rangle \in W_{ε}^{ss}(ϕ^tv) \cap W_{ε}^{ss}(u).$$

**Proof.** Lift $v$ to $\tilde{u} \in SΩ$, with $π\tilde{u} \in \text{int}(D)$ a fundamental domain for the $Γ$-action on $Ω$. For all $ε > 0$, there are neighborhoods $U^+ \cap U^-$ of $u^+, u^-$ in $∂Ω$ such that $v \in B(u, ε) \implies v^+ \in U^+, v^- \in U^-$. For any such neighborhoods, if $v$ is such that $v^+ \in U^+, v^- \in U^-$ and $πv$ is sufficiently close to $πu$, then $v \in B(u, ε)$. Make $U^-$ small enough that $U^- \subset \{w^- \in ∂Ω \mid w \in W_{ε}^{ss}(u)\}$ guarantees that any $v$ with $v^- \in U^-$ satisfies $\langle v, u \rangle \in W_{ε}^{ss}(u)$. Taking $U^+$ be as small as needed, we can make these $v$ with $v^- \in U^-$ arbitrarily close to $ϕ^tv\langle v, u \rangle$ in this local neighborhood of $u$. It suffices to choose $δ > 0$ sufficiently small as to ensure $|v| < ε$. □

### 5.1. Orbit gluing in Hilbert geometries

Uniform orbit gluing is also known as shadowing of pseudo-orbits in the literature. We introduce a weaker notion here. We can associate to any orbit segment $ϕ^{[0, t]}v$ the pair $(v, t) \in SM × R_0^+$. An $n$-length $δ$-pseudo-orbit is a collection of $n$-many finite length orbit segments $\{(v_i, t_i)\}_{i=1}^n \subset SM × R_0^+$ such that $d(ϕ^{t_i}v_i, v_{i+1}) < δ$ for $i = 1, \ldots, n − 1$.

**Definition 5.2.** The dynamics satisfies weak orbit gluing if for all $ε > 0$ and $\{v_i\}_{i=1}^n$ there exists $δ > 0$ such that for all $n$-length $δ$-pseudo orbits $\{(v_i, t_i)\}$ there is a point $w$ which $ε$-shadows the orbit segments $[v_1, t_1], \ldots, [v_n−1, t_{n−1}], [v_n, +∞]$. More explicitly: for some $|t| < ε$

$$w \in W_{ε}^{su}(ϕ^v(1)) \quad \text{and} \quad ϕ^\sum_{i=1}^{n} t_i w \in W_{ε}^{ss}(v_n),$$

and there are numbers $|t_i| < ε$ for $i = 1, \ldots, n − 2$ such that for all $k = 2, \ldots, n − 1$,

$$d(ϕ^{t_1+\cdots+t_{k−1}+s}(w), ϕ^{t_k+s}(v_k)) < ε, \quad \text{if } 0 < s < t_k,$$

$$d(ϕ^s(w), ϕ^{t_k+s}(v_k)) < ε, \quad \text{if } 0 < s < t_1.$$

**Lemma 5.3** (weak orbit gluing). The geodesic flow of the Benoist 3-manifolds satisfies weak orbit gluing for pseudo-orbits $\{(v_i, t_i)\}_{i=1}^n$ such that $v_1, \ldots, v_{n−1}$ are backward regular and $v_n$ is forward regular.

**Proof.** The proof is effectively a finite recursive application of taking Bowen brackets (Proposition 5.1). Suppose $d(ϕ^{t_i}v_i, v_{i+1}) < δ_i$ for all $i = 1, \ldots, n − 1$. For sufficiently small $δ_0 > 0$, if $ϕ^{t_i}v_i \in B(v_2, δ_1)$ then the Bowen bracket $w_1$ is in $W_{δ_2}^{su}(ϕ^{t_1+t_2}v_1) \cap W_{δ_2}^{ss}(v_2)$ for some $|t_1| < δ_2$. Then $ϕ^2w_1 \in B(v_3, δ_2 + 1)$ and we will have $w_2 \in W_{δ_2}^{su}(ϕ^{t_{2}+t_3}w_1) \cap W_{δ_2}^{ss}(v_3)$ for $|t_2| < δ_3$ if $δ_1, δ_2$ are sufficiently small. Repeating the argument, we have $ϕ^3w_2 \in B(v_4, δ_3 + δ_1)$ implying there exists $w_3 \in W_{δ_4}^{ss}(ϕ^{t_3+t_4}w_2) \cap W_{δ_4}^{ss}(v_4)$ and so on, until we find $ϕ^{n−1}_v w_{n−2} \in B(v_n, δ_{n−1} + δ_1)$ gives $w_{n−1} \in W_{δ_n}^{ss}(ϕ^{t_{n−1}+t_n}w_{n−2}) \cap W_{δ_n}^{ss}(v_n)$ for $|t_{n−1}| < δ_n$. Observe the following:

(5.1) $w_k \in W_{δ_{k+1}}^{ss}(ϕ^{t_k+t_{k+1}}w_{k−1})$ for all $k = 2, \ldots, n − 1$,

(5.2) $w_k \in W_{δ_{k+1}}^{ss}(v_{k+1})$ for all $k = 1, \ldots, n − 1$,

(5.3) $w_1 \in W_{δ_2}^{su}(ϕ^{t_1+t_2}v_1)$,

(5.4) $δ_k$ depends only on $v_{k+1}$ and $δ_{k+1}$, and $|t'_k| < δ_{k+1}$ for $k = 2, \ldots, n − 1$. 


Though $\delta_1$ depends on $\delta_2, \ldots, \delta_n$ and $v_2, \ldots, v_n$, this is still a finite amount of data. We will also need to make $\delta_1$ smaller to meet the conditions of weak orbit gluing, which we now address.

Let $w \in \varphi^{-\sum_{i=1}^{n-1} t_i} w_{n-1}$. If $\delta_n < \epsilon$, we have $\varphi^{-\sum_{i=1}^{n-1} t_i} w \in W^{ss}(v_n)$ immediately. Moreover, for $k = 2, \ldots, n - 1$ and $s \in [0, t_k]$,

$$d(\varphi^t w, \varphi^{t_1 + \cdots + t_k-1+t_k} v_k)$$

\[
= d(\varphi^{-t_k} \cdots w_{n-1}, \varphi^{t_1 + \cdots + t_k-1+t_k} v_k) \\
\leq d(\varphi^{-t_k} \cdots w_{n-1}, \varphi^{-t_{k-1}} + t_{k-1} + w_{n-2}) \\
+ d(\varphi^{-t_k} \cdots w_{n-2}, \varphi^{-t_{k-2}} + t_{k-2} + w_{n-3}) \\
+ \cdots + d(\varphi^{-t_k} + t_{k+1} + w_{k+1}, \varphi^{t_k+1} + w_{k-1}) \\
< \sum_{i=k}^{n} \delta_i + 2 \sum_{i=k}^{n-1} |t_i|
\]

by Equation (5.1) for $k + 1, \ldots, n - 1, n$ and Equation (5.2) for $k - 1$. Similarly, for $s \in [0, t_1]$, with the addition of Equation (5.3),

$$d(\varphi^s w, \varphi^{t_1 + \cdots + t_{k-1} + t_k} v_1) \leq \sum_{i=2}^{n-1} \delta_i + d(\varphi^{-t_1 + t_2 + \cdots + t_k} w_1, \varphi^{t_1 + \cdots + t_{k-1} + t_k} v_1)$$

\[
< \sum_{i=2}^{n-1} \delta_i + 2 \sum_{i=2}^{n-1} |t_i| + d(\varphi^{-t_1 + s} w_1, \varphi^{t_1 + s} v_1) < \sum_{i=2}^{n-1} \delta_i + 2 \sum_{i=2}^{n-1} |t_i| + \delta_2.
\]

Then since $w \in W^{ou}(v_1)$ is clear, the $\delta_i, |t_i|$ can be made sufficiently small to meet the definition of weak orbit gluing by the remark in Equation (5.4).

5.2. Density of unstable sets. Using Proposition 3.8, that the additive subgroup generated by translation lengths of closed hyperbolic orbits is dense in $\mathbb{R}$, we now show that unstable sets for periodic points are dense and shortly thereafter conclude the geodesic flow is topologically mixing. Let $P$ be the set of periodic points of the geodesic flow up to orbit equivalence and let $P_{hyp}$ be all the hyperbolic periodic points in $P$. Let $T_p$ denote the length of the orbit of $p \in P$. Note that $T_p = \tau(\gamma_p)$ where $\gamma_p \in \Gamma$ is the hyperbolic isometry which preserves the projective line $\ell_p$ in $\Omega$ determined by some lift $\tilde{p}$ of $p$.

The following lemma uses transitivity (Proposition 4.3), Anosov Closing (Theorem 4.4), Orbit gluing of 3 orbit segments (Lemma 5.3), and density of $\langle T_p \rangle_{p \in P_{hyp}}$ in $\mathbb{R}$ (Proposition 3.8).

Lemma 5.4. For all open $U \subset SM$, the lengths of periodic orbits passing through $U$ generate a dense subgroup of $\mathbb{R}$.

Proof. Let $p \in P_{hyp}$. Since $SM_{flat}$ is closed, it suffices to assume $U \cap SM_{flat} = \emptyset$. Choose $\eta > 0$ such that $U \subset \Lambda_\eta$ (recall that $\Lambda_\eta$ is the compliment of the $\eta$-neighborhood of the flats in $SM$). Let $\epsilon > 0$, and consider a point $v_0 \in U$ with a dense forward orbit, with $\epsilon$ small enough that $B(v_0, 2\epsilon) \subset U$. Choose $0 < \delta(\epsilon, \eta) < \epsilon$ small enough to satisfy Anosov Closing on $\Lambda_\eta$. Choose $\delta'(\eta, \delta, 3) < \delta$ as for $\delta$-fine orbit gluing for 3 orbit segments with starting points $v_0, p, v_0$. Then there exist $s_0, s_1 > 0$ such that $d(\varphi^{s_0} v_0, p) < \delta'$ and $d(\varphi^{s_0 + s_1} v_0, v_0) < \delta'$ by transitivity of $v_0$. Thus, the orbit segments $\{(v_0, s_0), (p, nT_p), (\varphi^{s_1} v_0, s_1)\}$ can be glued by
some $v_n$ with fineness $\frac{\delta}{6}$:

$$v_n \in B_{s_0}(\varphi^t v_0, \frac{\delta}{6})$$ for some $|t'_1| < \frac{\delta}{6}$,

$$\varphi^{s_0} v_n \in B_{nT_p}(\varphi^z p, \frac{\delta}{6})$$ for some $|t'_2| < \frac{\delta}{6}$,

$$\varphi^{s_0 + T_p} v_n \in B_{s_1}(\varphi^{t'} \varphi^{s_0} v_0, \frac{\delta}{6})$$ for some $|t'_3| < \frac{\delta}{6}$.

Then

$$d(v_n, \varphi^{s_0 + T_p + s_1}(v_n)) \leq d(v_n, v_0) + d(v_0, \varphi^{s_0 + s_1} v_0) + d(\varphi^{s_0 + s_1} v_0, \varphi^{s_0 + nT_p + s_1} v_n)$$

$$< 2\left(\frac{\delta}{6}\right) + \left(\frac{\delta}{3}\right) + 2\left(\frac{\delta}{6}\right) = \delta.$$

Note that $v_n \in B(v_0, \delta/3) \subset U \subset \Lambda_q$. By Anosov closing on $\Lambda_q$, there exists a regular $n$-large $w_n$ which has period length $s_0 + nT_p + s_1 + t'_n$ for $|t'_n| < \epsilon$. Since $w_n$ also $\epsilon$-shadows $v_n$, we have $w_n \in B(v_0, 2\epsilon) \subset U$. We can repeat the above argument for all $n$ with the same $\eta, \epsilon, p$ and $v_0$, hence the same $\delta$, $\delta'$ and the same $s_0, s_1$. Then we have $w_n, w_{n+1} \in U$ and thus

$$\langle T_{q,i} \rangle_{q \in U \cap \mathcal{P}_{hyp}} \ni s_0 + (n+1)T_p + s_1 + t'_{n+1} - (s_0 + nT_p + s_1 + t'_n) = T_p + t'_{n+1} - t'_n.$$

Since $|t'_{n+1} - t'_n| \leq |t'_{n+1} + |t'_n| < 2\epsilon$, letting $\epsilon$ go to zero we conclude $T_p \in \langle T_{q,i} \rangle_{q \in U \cap \mathcal{P}_{hyp}}$ for all $p \in \mathcal{P}_{hyp}$, which proves the lemma because $\langle T_{p,i} \rangle_{p \in \mathcal{P}_{hyp}} = \mathbb{R}$. 

We are now prepared to prove a key proposition.

**Proposition 5.5.** If $v \in SM$ is a hyperbolic periodic orbit, $W^{su}(v)$ is dense in $SM$.

**Proof.** Let $U \subset SM$ be open. By Lemma 5.4 there exists a $u \in \mathcal{P}_{hyp} \cap U$ such that $\langle T_u, T_u \rangle = \mathbb{R}$. Let $\epsilon > 0$ such that $B(u, \epsilon) \subset U$. Since $v, u \in SM_{reg}$ there exists a $T \in \mathbb{R}$ such that $w \in W^{su}(v) \cap W^{st}(\varphi^T(u))$. Then $\varphi^{-T} w \in W^{ss}(u)$ so choose $M \in \mathbb{N}$ large enough that for any $m \geq M, d(\varphi^{mT_u - T}w, u) < \epsilon/2$. Because $\langle T_u, T_u \rangle = \mathbb{R}$, there are large enough $m, n \in \mathbb{N}$ with $m \geq M$ such that

$$-\epsilon/2 < -nT_v + mT_u - T < \epsilon/2$$

implying that $d(\varphi^{-nT_v} w, \varphi^{mT_u - T} w) < \epsilon/2$ and hence $\varphi^{-nT_v} w \in B(u, \epsilon) \subset U$. Note that $\varphi^{-nT_v} w \in \varphi^{-nT_v} W^{su}(v) = W^{su}(v)$ to conclude the proof. 

The following lemma, the final piece preceding the proof of topological mixing, is generally taken as fact. We have included the proof for completeness.

**Lemma 5.6.** Let $f^t : X \to X$ be any continuous flow of a compact metric space. For $p$ periodic, density of $W^{ss}(p)$ implies that for all $\delta > 0$ and for all $x \in X$, there is a $T(p, \delta, x) > 0$ such that for all $t \geq T$, $f^t W^{ss}(p) \cap B(x, \delta) \neq \emptyset$.

**Proof.** By assumption, there exists some $z \in W^{su}(p) \cap B(x, \delta/2)$. Then $d(f^{-t}z, f^{-t}p) \to 0$ as $t \to +\infty$ so there exists an $S > 0$ such that $s \geq S$ implies $d(f^{-s}p, f^{-s}z) < \delta$. For all $n \in \mathbb{N}$ such that $nT_p \geq S$, then $d(p, f^{-nT_p} z) = d(f^{-nT_p} p, f^{-nT_p} z) < \delta$, and $f^{-nT_v} z \in f^{-nT_p} W^{su}(p) = W^{su}(p)$. Hence $f^{-nT_v}(z) \in W^{s}_\delta(p)$ and $z \in f^{nT_v}(W^{su}(p)) \cap B(x, \delta/2) \neq \emptyset$.

Take a finite $\delta/2$-cover $\{t_1, \dots, t_k\}$ of $[0, T_p]$. Repeat for each periodic point $f^{t_k} p$ the above argument to produce a $z_i \in W^{su}(f^{t_i} p) \cap B(x, \delta/2)$ and minimum $n_i \in \mathbb{N}$ such that if $n \geq n_i$, then $z_i \in f^{nT_v}(W^{s}_\delta(f^{t_i} p)) \cap B(x, \delta/2) \subset f^{nT_p + t_k}(W^{s}_\delta(p)) \cap B(x, \delta/2) \neq \emptyset$. 

Let \( N = \max_{1 \leq i \leq k} n_i \) and \( T = (N + 1)T_p \). Then for all \( t \geq T \), there is some \( M_t \geq N + 1 \), \( i \in \{1, \ldots, k\} \), and \( 0 \leq \epsilon \leq \delta/2 \) such that \( t = M_t T_p + t_i + \epsilon \) and thus

\[
 z_i \in f^{M_t T_p + t_i}(W^s_{\delta}(p)) \cap B(x, \delta/2)
\]

so \( f^* z_i \in f^t(W^s_{\delta}(p)) \cap B(x, \delta) \neq \emptyset \) as desired. \( \square \)

We are now ready to prove the main theorem of the section.

**Theorem 5.7.** The geodesic flow on \( SM \) is topologically mixing.

**Proof.** Let \( U, V \) be open in \( SM \) and \( p \in U \) a hyperbolic periodic point. Let \( \delta > 0 \) be small enough that \( W^s_{\delta}(p) \subset B(p, \delta) \subset U \) and \( B(x, \delta) \subset V \) for some \( x \in V \). By Lemma 5.6, a consequence of Proposition 5.5, there is a \( T(U, V) > 0 \) such that for all \( t \geq T \),

\[
 \emptyset \neq \varphi^t W^s_{\delta}(p) \cap B(x, \delta) \subset \varphi^t U \cap V.
\]

\( \square \)

6. Entropy-expansiveness of Hilbert geometries

In this section, we prove entropy-expansiveness for the geodesic flow of any compact Hilbert geometry. First, we review some preliminary notions from entropy theory. Given any metric space admitting a flow, one can define the Bowen distance by

\[
d_t(v, u) := \max_{0 \leq s \leq t} d(\varphi^s v, \varphi^s u).
\]

Then \( d_t \) is a metric on \( SM \), nondecreasing in \( t \). Metric \( d_t \)-balls are called Bowen balls, denoted \( B_t(v, \delta) \). A \((t, \delta)\)-spanning set for \( K \subset SM \) is one which is \( \delta \)-dense in \( K \) for the \( d_t \) metric. For any compact \( K \subset SM \), we can choose a minimal finite \((t, \delta)\)-spanning set and denote the cardinality by \( S(t, \delta, K) \). Then we define the topological entropy of \( \varphi^t \) on \( K \) by

\[
h_{top}(\varphi, K) := \lim_{\delta \to 0} \lim_{t \to \infty} \frac{1}{t} \log S(t, \delta, K).
\]

There are many equivalent definitions of \( h_{top} \) [19], and we include one other here. For \( K \subset SM \) compact, we define a \((t, \delta)\)-separated set \( R \subset K \) such that for all \( u, v \in R \),

\[
d_t(v, u) \geq \delta.
\]

Let \( R(t, \delta, K) \) denote the maximal cardinality for \((t, \delta)\)-separated sets, which is again finite by compactness of \( K \). Then

\[
h_{top}(\varphi, K) = \lim_{\delta \to 0} \lim_{t \to \infty} \frac{1}{t} \log R(t, \delta, K).
\]

When \( K = SM \), we abbreviate \( S(t, \epsilon) := S(t, \epsilon, SM) \), \( R(t, \epsilon) := R(t, \epsilon, SM) \), and \( h_{top}(\varphi) := h_{top}(\varphi, SM) \).

For the purposes of applying Bowen’s work, we take

\[
h_{top}(\varphi, K, \delta) := \lim_{t \to \infty} \frac{1}{t} \log S(t, \delta, K).
\]

so that \( h_{top}(\varphi, K) = \lim_{\delta \to 0} h_{top}(\varphi, K, \delta) \), and for \( K = SM \) we have \( h_{top}(\varphi) := \lim_{\delta \to 0} h_{top}(\varphi, \delta) \).

For any point \( v \) in \( SM \), we define the infinite Bowen balls about \( v \) in positive or negative time:

\[
\Phi_\epsilon(v) := \bigcap_{t \in \mathbb{R}^+} \varphi^{-t} B(\varphi^t v, \epsilon) = \{ y \in M \mid d(\varphi^t y, \varphi^t v) \leq \epsilon \text{ for all } t \in \mathbb{R}^+ \}.
\]

Intuitively, we should think of the \( \Phi_\epsilon(v) \) as the exceptions to expansivity. An expansive map (not flow) is defined by the existence of an \( \epsilon > 0 \) such that \( \Phi_\epsilon(v) = \{ v \} \) for all \( v \). An expansive
flow would satisfy that \( \Phi_\epsilon(v) = W_\epsilon^{os}(v) \) for all \( v \). There are special cases of entropy expansive systems. Define

\[
h^*(\epsilon) := \sup_{v \in SM} h_{top}(\varphi, \Phi_\epsilon(v)).
\]

Then \( \varphi \) is \( h \)-expansive with expansivity constant \( \epsilon > 0 \) if \( h^*(\epsilon) = 0 \). In other words, there is an \( \epsilon > 0 \) such that the exceptions to \( \epsilon \)-expansivity have no influence on the entropy of the system. For an \( h \)-expansive system, Bowen proved that we can bypass the cumbersome limit over \( \delta \to 0 \) of \( h_{top}(\varphi, \delta) \) to compute \( h_{top}(\varphi) \).

**Theorem 6.1** ([8, Theorem 2.4]). If \( \epsilon \) is an \( h \)-expansive constant for \( \varphi \), then

\[
h_{top}(\varphi) = h_{top}(\varphi, \epsilon).
\]

Moreover, to compute the metric entropy of a system, one can simply take a sufficiently fine measurable partition rather than an infimum over all possible partitions. An immediate consequence is existence of a measure of maximal entropy (see [26, Theorem 8.6 (2)]).

For any manifold, the injectivity radius of \( x \in M \) is defined to be

\[
\text{inj}(x) := \frac{1}{2} \inf_{\ell} \{\text{length}(\ell)\},
\]

where \( \ell \) varies over all homotopically nontrivial loops through \( x \). Then define the injectivity radius of \( M \) to be

\[
\text{inj}(M) := \inf_{x \in M} \text{inj}(x).
\]

If \( M \) is compact then \( \text{inj}(M) > 0 \).

**Theorem 6.2.** The geodesic flow \( \varphi^t \) on any compact Hilbert geometry is \( h \)-expansive.

**Proof.** Lift \( v \) to \( \tilde{v} \) in \( S\Omega \). If \( \tilde{v}^+ \) is extremal, then by properties of the Hilbert metric, \( \tilde{u}^+ \neq \tilde{v}^+ \) for any lift \( \tilde{u} \) of \( u \) implies \( u \notin \Phi_\epsilon(v) \) for \( 0 < \epsilon < \frac{\text{inj}(M)}{3} \). Then a \((0, \delta)\)-spanning set for \( \Phi_\epsilon(v) \) is a \((t, \delta)\)-spanning set for all \( t > 0 \) and \( h_{top}(\Phi_\epsilon(v)) = 0 \).

Suppose now that \( v^+ \) is not extremal. Let \( C \subset \partial \Omega \) be the intersection of all supporting hyperplanes to \( \Omega \) at \( \tilde{v}^+ \). Note that if \( \tilde{v}^+ \) is not extremal then \( C \) has nonempty interior for the subspace topology in the minimal projective subspace containing \( C \). Since \( C \) is properly convex in this projective subspace, we can extend the Hilbert metric to the interior of \( C \), which we will denote \( d_C \), with metric balls denoted by \( B_C \). Now define

\[
\Phi_C^+(\tilde{v}, \epsilon) := \{u \in B(\tilde{v}, \epsilon) \mid u^+ \in B_C(\tilde{v}^+, \epsilon)\}.
\]

Then \( \Phi_\epsilon(v) \) is contained in the quotient projection of \( \Phi_C^+(\tilde{v}, \epsilon) \). For all \( \eta \in B_C(\tilde{v}^+, \epsilon) \) let \( v_\eta \) be such that \( \pi v_\eta = \pi \tilde{v} \) and \( v_\eta^+ = \eta \). Then \( d(v_\eta, v) \leq d_C(\eta, \tilde{v}^+) \leq \epsilon \). Then for all \( w \in \Phi_C^+(\tilde{v}, \epsilon) \), there is an \( \eta = w^+ \) implying \( d(w, v_\eta) \leq d(w, \tilde{v}) + d(\tilde{v}, v_\eta) = \epsilon + \epsilon = 2\epsilon \), hence

\[
\Phi_C^+(\tilde{v}, \epsilon) \subset \bigcup_{\eta \in B_C(\tilde{v}^+, \epsilon)} \Phi^+(v_\eta, 2\epsilon).
\]

Choose a finite \( \delta/2 \)-cover of \( B_C(\tilde{v}^+, \epsilon/2) \) by \( \{\eta_i\}_{i=1}^k \) and \( v_i := v_{\eta_i} \). Then for all \( u \in \Phi^+(v_i, 2\epsilon) \), there is an \( \eta_i \) such that \( d_C(\eta, \eta_i) < \delta/2 \) and \( d(u, v_i) \leq d(u, v_\eta) + d(v_\eta, v_i) < 2\epsilon + d_C(\eta, \eta_i) < \frac{5\epsilon}{2} \) for \( \delta \) small. Describe all such \( u \) by

\[
\Phi_C^+(v_i, 5\epsilon/2, \delta/2) := \{u \in B(\tilde{v}, 5\epsilon/2) \mid u^+ \in B_C(\tilde{v}^+, \delta/2)\}.
\]

Then for \( \epsilon < \frac{\text{inj}(M)}{3} \),

\[
\Phi_\epsilon(v) \subset \bigcup_{i=1}^k \Phi_C^+(d_\Gamma v_i, 5\epsilon/2, \delta/2)
\]
Note that for each compact $\Phi^+(d\pi v_i, 5\epsilon/2, \delta/2)$, a minimal $(0, \delta)$-spanning set $E_i$ will be a $(t, \delta)$-spanning set for all $t \geq 0$. Thus,

$$h_{\text{top}}(\Phi, \delta) \leq \lim_{t \to \infty} \frac{1}{t} \log \left( \sum_{i=1}^{k} |E_i| \right) = 0.$$ 

7. Applications to counting and entropy

Let $P_t(\varphi)$ denote the collection of isolated $\varphi$-periodic orbits of period at most $t$, modulo orbit equivalence, and

$$\rho(\varphi) := \lim_{t \to \infty} \frac{1}{t} \log \# P_t(\varphi).$$

The effect of defining $P_t$ by isolated orbits is that we neglect the flat orbits in $S\mathbb{M}_{\text{flat}}$. Any periodic orbit on a flat corresponds to a continuous 1-parameter family of periodic orbits of the same homotopy type, so these are not isolated and not counted. By contrast, the hyperbolic periodic orbits are isolated and countable.

The next proposition follows from $h$-expansivity (Theorem 6.2).

**Proposition 7.1.** The geodesic flow $\varphi^t$ of a Benoist 3-manifold satisfies

$$\rho(\varphi) \leq h_{\text{top}}(\varphi).$$

**Proof.** Choose $\epsilon \leq \text{inj} M/3$ an $h$-expansivity constant for the geodesic flow on $SM$. We show that $P_t$ is a $(t, \epsilon)$-separated set. If $v, u \in P_t$ such that $d_T(v, u) < \epsilon$, then $d_t(v, u) < \epsilon$ for all $t \in \mathbb{R}$. Since $\Gamma$ acts discretely and $\epsilon < \text{inj}(M)/3$, this is only possible if $v = u$ or if $v$ and $u$ lift to tangent vectors to a properly embedded triangle $\Delta$ such that $\ell_v, \ell_u \subset \Delta$. Then $v, u$ are in a flat so they are not counted in $P_t$.

Thus, $P_t$ is $(t, \epsilon)$-separated and has cardinality at most $R(t, \epsilon)$, the cardinality of a maximal $(t, \epsilon)$-separated set. We conclude by $h$-expansivity and Bowen’s Theorem 6.1 that

$$\rho(\varphi) = \lim_{t \to \infty} \frac{1}{t} \log \# P_t(\varphi) \leq \lim_{t \to \infty} \frac{1}{t} \log R(t, \epsilon) = h_{\text{top}}(\varphi).$$

**Proposition 7.2.** The geodesic flow of a compact Benoist 3-manifold has positive topological entropy.

**Proof.** By Corollary 3.4, there exist noncommuting hyperbolic elements $g, h \in \Gamma$ which generate a free subgroup. There is a positive lower bound for the exponential growth rate of lengths of closed curves associated to this subgroup, which bounds below $\rho(\varphi)$ and hence $h_{\text{top}}(\varphi)$. 

7.1. Volume entropy. We remark in this section that A. Manning’s proof that volume entropy and topological entropy agree for compact nonpositively curved Riemannian manifolds generalizes to our context immediately [20]. Let vol be a uniform volume on $\Omega$, meaning the volumes of unit metric balls are uniformly bounded above and below by positive constants. There is no canonical such volume but there are good candidates (c.f. beginning of [22, pp 207-261]). Then

$$h_{\text{vol}}(\Omega) = \lim_{r \to \infty} \frac{1}{r} \log \text{vol}(B_\Omega(x, r))$$

is the volume entropy of $\Omega$. Let $\delta_\Gamma$ denote the critical exponent of the action of $\Gamma$ on $\Omega$, equivalently: $\delta_\Gamma = \lim \sup_{t \to \infty} \frac{1}{t} \log N_\Gamma(t)$ where $N_\Gamma(t) = \# \{ \gamma \in \Gamma | d_\Omega(x, \gamma x) \leq t \}$. 

Proposition 7.3. If $\Omega$ is any divisible properly convex domain in $\mathbb{R}P^n$, then
\[ \delta_{\Gamma} = h_{\text{vol}} = h_{\text{top}}(\varphi). \]

Proof. Whenever a discrete group of isometries acts cocompactly on a metric space with finite critical exponent, one has $\delta_{\Gamma} = h_{\text{vol}}$ (a proof is available in [23, Lemma 4.5]). The statement in [20, Theorem 1] that $h_{\text{vol}} \leq h_{\text{top}}(\varphi)$ holds as long as $M$ is compact and $(\Omega, d_{\Omega})$ is complete. The proof of the opposite inequality in Theorem 2 uses nonpositive sectional curvature to prove a technical lemma. We can bypass curvature and prove the lemma immediately in Hilbert geometries. The rest of the proof follows in the same way.

This lemma has already been proven by Crampon in the strictly convex case, but in the proof Crampon only uses strict convexity to state the lemma with a strict inequality for all geodesics, which can only be projective lines when the domain is strictly convex. Since our geodesic flow is defined to follow projective lines, the lemma suffices.

Lemma 7.4 ([12], Lemma 8.3). For any two projective lines $\sigma, \tau : [0, r] \to M$, $r > 0$,
\[ d_{\Omega}(\sigma(t), \tau(t)) \leq d_{\Omega}(\sigma(0), \tau(0)) + d_{\Omega}(\sigma(r), \tau(r)). \]

\[ \square \]

References
[1] D. V. Anosov. Geodesic flows on closed Riemann manifolds with negative curvature. Proceedings of the Steklov Institute of Mathematics, No. 90 (1967). Translated from the Russian by S. Feder. American Mathematical Society, Providence, R.I., 1969.
[2] Mark D Baker and Daryl Cooper. Finite-volume hyperbolic 3–manifolds contain immersed quasi-Fuchsian surfaces. Algebr. Geom. Topol., 15(2):1199–1228, 2015.
[3] Yves Benoist. Automorphismes des cônes convexes. Invent. Math., 141(1):149–193, 2000.
[4] Yves Benoist. Convexes divisibles. II. Duke Math. J., 120(1):97–120, 2003.
[5] Yves Benoist. Convexes divisibles. I. In Algebraic groups and arithmetic, pages 339–374. Tata Inst. Fund. Res., Mumbai, 2004.
[6] Yves Benoist. Convexes divisibles. IV. Structure du bord en dimension 3. Invent. Math., 164(2):249–278, 2006.
[7] Jean Paul Benzécri. Sur les variétés localement affines et localement projectives. 88:229–332, 1960. French.
[8] Rufus Bowen. Entropy-expansive maps. Trans. Amer. Math. Soc., 164:323–331, 1972.
[9] Harrison Bray. Ergodicity of Bowen-Margulis measure for the Benoist 3-manifolds. 2017.
[10] Yves Coudene. Topological dynamics and local product structure. J. London Math. Soc. (2), 69(2):441–456, 2004.
[11] Yves Coudene and Barbara Schapira. Generic measures for hyperbolic flows on non-compact spaces. Israel J. Math., 179:157–172, 2010.
[12] Mickaël Crampon. Entropies of strictly convex projective manifolds. J. Mod. Dyn., 3(4):511–547, 2009.
[13] Mickaël Crampon. Lyapunov exponents in Hilbert geometry. Ergodic Theory Dynam. Systems, 34(2):501–533, 2014.
[14] Mickaël Crampon and Ludovic Marquis. Le flot géodésique des quotients géométriquement finis des géométries de Hilbert. Pacific J. Math., 268(2):313–369, 2014.
[15] Pierre de la Harpe. On Hilbert’s metric for simplices. In Geometric group theory, Vol. 1 (Sussex, 1991), volume 181 of London Math. Soc. Lecture Note Ser., pages 97–119. Cambridge Univ. Press, Cambridge, 1993.
[16] Patrick B. Eberlein. Geometry of nonpositively curved manifolds. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.
[17] William Jaco and Peter B. Shalen. A new decomposition theorem for irreducible sufficiently-large 3-manifolds. In Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, Proc. Sympos. Pure Math., XXXII, pages 71–84. Amer. Math. Soc., Providence, R.I., 1978.
[18] Klaus Johannson. *Homotopy equivalences of 3-manifolds with boundaries*, volume 761 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[19] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[20] Anthony Manning. Topological entropy for geodesic flows. *Ann. of Math. (2)*, 110(3):567–573, 1979.

[21] Joseph D. Masters and Xingru Zhang. Closed quasi-Fuchsian surfaces in hyperbolic knot complements. *Geom. Topol.*, 12(4):2095–2171, 2008.

[22] Athanase Papadopoulos and Marc Troyanov. *Handbook of Hilbert geometry*. IRMA Lectures in Mathematics and Theoretical Physics No. 22. European Mathematical Society, 2014.

[23] J. F. Quint. An overview of Patterson-Sullivan theory. https://www.math.u-bordeaux.fr/~jquint/publications/courszurich.pdf.

[24] Jacques Vey. Sur les automorphismes affines des ouverts convexes saillants. *Ann. Scuola Norm. Sup. Pisa (3)*, 24:641–665, 1970.

[25] C. Walsh. Gauge-reversing maps on cones, and Hilbert and Thompson isometries. http://arxiv.org/abs/1312.7871, 2013.

[26] Peter Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.