Perturbation theory for fractional evolution equations in a Banach space

Arzu Ahmadova1,2 · Ismail Huseynov2 · Nazim I. Mahmudov2

Received: 8 October 2021 / Accepted: 7 October 2022 / Published online: 24 October 2022 © The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
A strong inspiration for studying perturbation theory for fractional evolution equations comes from the fact that they have proven to be useful tools in modeling many physical processes. We study fractional evolution equations of order $\alpha \in (1, 2)$ associated with the infinitesimal generator of an operator fractional cosine (sine) function generated by bounded time-dependent perturbations in a Banach space. We show that the fractional abstract Cauchy problem associated with the infinitesimal generator $A$ of a strongly continuous fractional cosine (sine) function remains uniformly well-posed under bounded time-dependent perturbation of $A$. We also provide some necessary special cases by using the Laplace transform of the generators of the given operator families.

Keywords Perturbation theory · Fractional evolution equation · Strongly continuous fractional cosine and sine families · Well-posedness

1 Introduction
Some partial differential equations arising in the transverse motion of an extensible beam [9], the vibration of hinged bars [30], damped McKean–Vlasov equations [22]...
and many other physical phenomena can be formulated as the second-order abstract differential equations with cosine families in infinite-dimensional spaces. The most fundamental and extensive work on cosine families of linear operators is that of Fat-torini in [7, 8].

Moreover, Travis and Webb [26] have investigated the following semi-linear second-order Cauchy problem in a Banach space $X$ via the theory of strongly continuous cosine families of linear bounded operators:

$$
\begin{cases}
u''(t) = Au(t) + f(t, u(t), u'(t)), & t \in \mathbb{R}, \\
u(t_0) = x \in X, & u'(t_0) = y \in X.
\end{cases}
$$

A systematic and general treatment of the second-order semi-linear abstract Cauchy problem (1.1) from the stand point of existence, uniqueness, continuous dependence and smoothness of solutions have been provided by Travis and Webb in [26, 27]. Also, Bochenek in [4] has investigated the existence of a solution of the initial value problem for the second-order abstract differential Eq. (1.1) under more general hypotheses than in [26]. Furthermore, it is interesting to note that Vugdalić and Halilović have introduced in [29] a new definition for the cosine operator function and its infinitesimal generator, which is the solution of a Cauchy problem (1.1) (with $f = 0$) for the linear abstract differential equation of second order.

The fractional analogue of the same problem (1.1) with Caputo time-derivative is considered by Li [17] and general results are attained by using the theory of fractional cosine families.

The existence and uniqueness of the following linear non-homogeneous $\alpha$-order Cauchy problem is studied by Li et al. [18] in a Banach space $X$:

$$
\begin{cases}
\left(D^\alpha_t u\right)(t) = Au(t) + f(t), & t \geq 0, \\
\left(I^{2-\alpha}u\right)(0) = 0, & \left(I^{1-\alpha}u\right)(0) = x \in X,
\end{cases}
$$

where $1 < \alpha < 2$, $A : \mathcal{D}(A) \subseteq X \to X$ is a closed, densely-defined linear operator, $\mathcal{D}(A)$ is the domain of $A$ endowed with the graph norm $\|x\|_{\mathcal{D}(A)} = \|x\| + \|Ax\|$, and $D^\alpha_t$ is the $\alpha$-order Riemann–Liouville fractional derivative operator. Furthermore, in [5] Chen and M. Li have established the properties of three kinds of resolvent families defined by purely algebraic equations which extend the classical cosine functional equation. In [11], Henríquez et al. have studied the differentiability of mild solutions for a class of fractional abstract Cauchy problems of order $\alpha \in (1, 2)$.

The perturbation theory has long been a very useful tool to analyse some aspects of qualitative theory for multi-term abstract differential equations, e.g., functional-retarded evolution equations [14], exponential stabilization for the semi-linear stochastic control systems [1] and so on. A considerable amount of research has been done on the perturbation theory of linear operators in a Banach space, principally by Phillips [23], Lutz [20], and Travis and Webb [28].

Springer
In [23], Phillips has obtained the closed-form of the classical solution to the following linear perturbed abstract Cauchy problem in a Banach space $X$:

\[
\begin{align*}
\frac{du(t)}{dt} &= (A + B(t))u(t) + f(t), \quad t \geq 0, \\
\quad u(0) &= x \in D(A),
\end{align*}
\]

where $B : [0, \infty) \to \mathcal{L}(X)$, $f : [0, \infty) \to X$ are strongly continuously differentiable functions on $[0, \infty)$.

In [20], as a continuation of the results obtained in [23], Lutz has studied for the first time the implications for the linear homogeneous abstract Cauchy problem associated with the infinitesimal generator of an operator cosine function generated by bounded time-dependent perturbations:

\[
\begin{align*}
\frac{d^2u(t)}{dt^2} &= (A + B(t))u(t), \quad t \in \mathbb{R}, \\
\quad u(0) &= x \in D(A), \quad u'(0) = y \in D(A).
\end{align*}
\] (1.2)

In the special case, when $B(t) \equiv B \in \mathcal{L}(X)$, the analogous results of Travis and Webb in [28] are examined with various recursive formulae for strongly continuous cosine families. Moreover, Lin in [19] has studied the time-dependent perturbation theory for second-order linear non-homogeneous abstract differential equations and their applications to partial differential equations in the form of the following problem:

\[
\begin{align*}
\frac{d^2u(t)}{dt^2} &= (A + B(t))u(t) + f(t), \quad t \in [0, T], \quad T > 0, \\
\quad u(0) &= u_0, \quad u'(0) = v_0.
\end{align*}
\] (1.3)

On the other hand, Serizawa and Watanabe [25] have investigated the same problem stated in (1.3) with Neumann boundary condition.

The fractional analogue of the abstract Cauchy problem (1.2) was first posed by Bazhlekova in [2, 3]. Bazhlekova proved by means of a counterexample that the classical perturbation results do not hold in the general case for solution operators of (1.4) with $\alpha \in (0, 1)$. In contrast to the case of $\alpha \in (0, 1)$, perturbations by bounded linear operators are always possible in the case of $\alpha \in (1, 2)$. Moreover, in [3], Bazhlekova has proposed to uniquely determine a classical solution of the following linear homogeneous abstract Cauchy problem:

\[
\begin{align*}
\left( C \mathcal{D}^{\alpha}_t u \right)(t) &= (A + B(t))u(t), \quad t \geq 0, \\
\quad u(0) &= x \in D(A), \quad u'(0) = 0,
\end{align*}
\] (1.4)

where $B : [0, \infty) \to \mathcal{L}(X)$ is a continuous function in the uniform operator topology. Furthermore, in the special case $B(t) \equiv B \in \mathcal{L}(X)$, Huseynov et al. in [12] established sufficient conditions such that if $A$ is the infinitesimal generator of a fractional strongly continuous cosine (sine) family in a Banach space $X$ and $B$ is a bounded linear operator in $X$, then $A + B$ is also the infinitesimal generator of a fractional strongly continuous cosine (sine) family in $X$. 

\[ \text{Springer} \]
Motivated by the above articles, we consider an abstract Cauchy problem of fractional order $\alpha \in (1, 2)$ in a Banach space $X$:

$$\left( C^D_t u \right)^{(\alpha)}(t) = \left( A + B(t) \right)u(t) + f(t), \quad t \geq 0,$$

(1.5)

with the initial conditions

$$u(0) = x \in D(A), \quad u'(0) = y \in D(A),$$

(1.6)

which is uniformly well-posed. Here $C^D_t$ is a fractional-order Caputo differentiation operator and $A : D(A) \subseteq X \to X$ is a densely-defined, closed linear operator in a Banach space $X$. Let $D(A) \subseteq X$ denote the domain of $A$ which will be specified later, $B : [0, \infty) \to \mathcal{L}(X)$ be twice strongly continuously differentiable function and $f : [0, \infty) \to X$ strongly continuously differentiable function on $[0, \infty)$.

Therefore, the aim of this paper is to develop the perturbation theory to study fractional-order abstract Cauchy problem of order $\alpha \in (1, 2)$ which generalizes classical case (1.2). The pioneering work on (1.5) and (1.6) in classical sense (when $\alpha = 2$) was done by Lutz [20] and our development follows this approach. The presented results extend those of [3, 12, 20] in several aspects. First, we allow for more general case, in that we assume $u'(0) = y \in D(A)$, i.e., the second initial value is non-zero and as a result of this choice we derive the corresponding perturbed fractional sine family $\{S_\alpha(t; A + B), t \geq 0\}$, while this condition is zero in [3]. Second, we consider both linear homogeneous and non-homogeneous abstract Cauchy problems for fractional evolution equations, while Lutz [20] and Bazhlekova [3] have studied only homogeneous linear cases in classical and fractional senses, respectively. Furthermore, we have attained new closed-form of solutions whenever $A, B \in \mathcal{L}(X)$ are non-permutable and permutable linear bounded operators using the Laplace transform of the generators of the given operator families. Thus, in this work it is shown that the linear non-homogeneous abstract Cauchy problem with the infinitesimal generator $A$ of a strongly continuous fractional cosine families remains uniformly well-posed under bounded time-dependent perturbations.

Therefore, in this paper, we study the perturbation properties of the linear problem (1.5) with the initial conditions (1.6), generalising some facts concerning fractional cosine and sine operator functions and also distinguishing some new properties. The structure of our paper is therefore as follows. Section 2 contains the most important definitions and relations from fractional calculus, fractional-order evolution equations and operator theory for linear operators. Section 3 is devoted to the study of the linear homogeneous and non-homogeneous abstract perturbed Cauchy problems of fractional order in a Banach space $X$. To conclude this section, we study some necessary special cases of the main problem (1.5) and (1.6) and give some comparisons between existing results and ours.
2 Preliminary concept

We embark on this section by briefly introducing the essential structure of fractional calculus, fractional-order evolution equations and operator theory for linear operators. For the more salient details on these matters, see the textbooks [6, 10, 15, 16, 24].

Let \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) denote the sets of natural, real and complex numbers, respectively and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( X \) be Banach space equipped with the norm \( \| \cdot \| \). We denote by \( \mathcal{L}(X) \) the Banach algebra of all bounded linear operators on \( X \) which becomes a Banach space with regard to the norm \( \| T \| = \sup \{ \| Tx \| : \| x \| \leq 1 \} \) for any \( T \in \mathcal{L}(X) \). The identity and zero operators on \( X \) are denoted by \( I \in \mathcal{L}(X) \) and \( \Theta \in \mathcal{L}(X) \), respectively.

We will use the following functional spaces on \( [0, T] \subset \mathbb{R} \) through the paper:

- \( \mathbb{C}([0, T], X) \) denotes the Banach space of continuous \( X \)-valued functions \( u : [0, T] \to X \) equipped with an infinity norm \( \| u \|_\infty = \sup_{0 \leq t \leq T} \| u(t) \| \);

- \( \mathbb{C}^n([0, T], X) \), \( n \in \mathbb{N} \) denotes the Banach space of \( n \)-times continuously differentiable \( X \)-valued functions defined by

\[
\mathbb{C}^n ([0, T], X) = \left\{ u \in \mathbb{C}^n([0, T], X) : u^{(n)} \in \mathbb{C}([0, T], X), n \in \mathbb{N} \right\}
\]

with respect to the \( \mathbb{C}^n \)-norm for \( n \in \mathbb{N} \):

\[
\| u \|_{\mathbb{C}^n} = \| u \|_\infty + \| u' \|_\infty + \cdots + \| u^{(n)} \|_\infty = \sum_{k=0}^{n} \sup_{0 \leq t \leq T} \| u^{(k)}(t) \|.
\]

In addition, \( \mathbb{C}^n ([0, T], X) \subset \mathbb{C}([0, T], X) \), \( n \in \mathbb{N} \).

Let \( v : [0, \infty) \to \mathbb{R} \) be an integrable scalar-valued function and let \( u : [0, \infty) \to X \) be a continuous \( X \)-valued function. We denote by

\[
(v * u)(t) = \int_0^t v(t-s)u(s)\,ds, \quad t \geq 0,
\]

the convolution operator of \( v \) and \( u \). Furthermore, if \( T : [0, \infty) \to \mathcal{L}(X) \) is a strongly continuous operator-valued map, we define

\[
(T * u)(t) = \int_0^t T(t-s)u(s)\,ds = \int_0^t T(s)u(t-s)\,ds, \quad t \geq 0.
\]

**Definition 2.1** [16, 24] The Riemann–Liouville fractional integral of order \( \alpha > 0 \) for a \( X \)-valued function \( u \in \mathbb{C}([0, T], X) \) is defined by

\[
\left( T^\alpha_T u \right)(t) = (g_\alpha * u)(t) = \int_0^t g_\alpha(t-s)u(s)\,ds.
\]
We set \((I_0^1 u)(t) = u(t)\). For the sake of brevity, we use the following notation for \(\alpha \geq 0\): 

\[
g_\alpha(t) = \begin{cases} 
\frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\
0, & t \leq 0,
\end{cases}
\]

where \(\Gamma : [0, \infty) \rightarrow \mathbb{R}\) is the well-known Euler’s gamma function. Note that \(g_0(t) = 0, \ t \in \mathbb{R}\) since \(\frac{1}{\Gamma(0)} = 0\). These functions satisfy the semigroup property:

\[
(g_\alpha \ast g_\beta)(t) = g_{\alpha+\beta}(t), \quad t \in \mathbb{R}.
\]

Moreover, the Riemann–Liouville fractional integral operators \(\{I_\alpha\}_{\alpha \geq 0}\) also satisfy the semigroup property:

\[
(I_\alpha I_\beta u)(t) = (I_{\alpha+\beta} u)(t), \quad \alpha, \beta \geq 0.
\] (2.1)

**Definition 2.2** [16, 24] The Riemann–Liouville fractional derivative of order \(1 < \alpha < 2\) for a \(X\)-valued function \(u \in C^2 ([0, T], X)\) is defined by

\[
(D_\alpha^t u)(t) = D_\alpha^2 (g_{2-\alpha} \ast u)(t) = I_\alpha^2 (I_{2-\alpha}^t u)(t),
\]

where \(D_\alpha^2 = \frac{d^2}{dt^2}\).

We sometimes adopt the convention

\[
(D_{-\alpha}^t u)(t) = (I_{-\alpha}^t u)(t), \quad \alpha > 0,
\]

under which the Riemann–Liouville fractional derivative is the analytic continuation of the Riemann–Liouville fractional integral for positive \(\alpha\).

**Definition 2.3** [16, 24] The Caputo fractional derivative of a \(X\)-valued function \(u \in C^2 ([0, T], X)\) with fractional order \(1 < \alpha < 2\) is defined by

\[
(CD_\alpha^t u)(t) = (g_{2-\alpha} \ast D_\alpha^2 u)(t) = I_{2-\alpha}^t (D_\alpha^2 u)(t).
\]

**Lemma 2.1** [16, 24] The relationship between Riemann–Liouville and Caputo fractional differential operators of order \(1 < \alpha < 2\) for \(u \in C^2 ([0, T], X)\) is given by:

\[
(CD_\alpha^t u)(t) = D_\alpha^t \left( u(t) - u(0) - tu'(0) \right).
\] (2.2)

By making use of the property (2.2), we attain the corresponding relation for \(u \in C^2 ([0, T], X)\):

\[
(I_\alpha^t (CD_\alpha^t u)(t) = u(t) - u(0) - tu'(0).
\] (2.3)
Remark 2.1 Since \( u \) is an abstract function with values in \( X \), the integrals which appear in Definitions 2.1, 2.2, 2.3 and Lemma 2.1 are taken in Bochner’s sense.

Definition 2.4 [6] Let \( u : [0, \infty) \rightarrow X \) be measurable and exponentially bounded \( X \)-valued function of exponent \( \omega \in \mathbb{R} \), i.e., \( \|u(t)\| \leq Me^{\omega t} \) for all \( t \geq 0 \) and some constant \( M > 0 \). Then the Laplace transform of \( u \) \( \mathcal{L} u : \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) > \omega \} \rightarrow X \) is defined by

\[
(\mathcal{L} u)(\lambda) = \hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t) dt,
\]

where \( \text{Re}(\lambda) \) represents the real part of the complex number \( \lambda \).

Furthermore, the Laplace transform of Caputo’s fractional differentiation operator of order \( 1 < \alpha < 2 \) [16, 24] is defined by

\[
(\mathcal{L} (C D_t^\alpha u))(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \lambda^{\alpha-1} u(0) - \lambda^{\alpha-2} u'(0).
\]

The Mittag-Leffler function is a natural generalisation of the exponential function, first proposed as a single and two parameter functions of one variable by using an infinite series [10].

Definition 2.5 [10] The classical Mittag-Leffler function is defined by

\[
E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + 1)}, \quad \alpha > 0, \quad t \in \mathbb{R}.
\]

The two-parameter Mittag-Leffler function is given by

\[
E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + \beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \quad t \in \mathbb{R}.
\]

It is important to note that

\[
E_{\alpha,1}(t) = E_\alpha(t), \quad E_1(t) = \exp(t), \quad t \in \mathbb{R}.
\]

• **Strongly continuous cosine and sine families of linear operators.**

Definition 2.6 [26] (1) A one-parameter family \( C(\cdot; A) : \mathbb{R} \rightarrow \mathcal{L}(X) \) of all bounded linear operators is called a strongly continuous cosine family in the Banach space \( X \) if and only if

• \( C(0; A) = I \);
• \( C(s + t; A) + C(s - t; A) = 2C(s; A)C(t; A) \) for all \( s, t \in \mathbb{R} \);
• \( C(t; A)x \) is continuous in \( t \) on \( \mathbb{R} \) for each fixed \( x \in X \).
Theorem 2.3 Let \( \{S(t; A), t \in \mathbb{R}\} \subset \mathcal{L}(X) \) be a strongly continuous cosine family in \( X \) with infinitesimal generator \( A \). If \( f \) is called the infinitesimal generator of a strongly continuous cosine family \( \{C(t; A), t \in \mathbb{R}\} \) and \( \mathcal{D}(A) \) is a domain of \( A \).

It is known that the infinitesimal generator \( A \) is closed, densely-defined operator on \( X \). We will also make use of the set

\[
\mathcal{E} = \left\{ x \in X : C(\cdot; A)x \in C^1(\mathbb{R}, X) \right\}.
\]

The operator families \( \{C(t; A), t \in \mathbb{R}\} \) and \( \{S(t; A), t \in \mathbb{R}\} \) have the following properties:

**Theorem 2.1** [26] Let \( \{C(t; A), t \in \mathbb{R}\} \) be a strongly continuous cosine family in \( X \) with infinitesimal generator \( A \). The following assertions hold:

- There exist constants \( M \geq 1 \) and \( \omega \geq 0 \) such that \( \|C(t; A)\| \leq Me^{\omega|t|} \) for all \( t \in \mathbb{R} \);
- If \( x \in X \), then \( S(t; A)x \in \mathcal{E} \);
- If \( x \in \mathcal{E} \), then \( S(t; A)x \in \mathcal{D}(A) \) and \( \mathcal{D}C(t; A)x = AS(t; A)x \);
- If \( x \in \mathcal{D}(A) \), then \( C(t; A)x \in \mathcal{D}(A) \) and \( \mathcal{D}^2C(t; A)x = AC(t; A)x = C(t; A)Ax \);
- If \( x \in \mathcal{E} \), then \( \lim_{t \to 0} AS(t; A)x = 0 \);
- If \( x \in \mathcal{E} \), then \( S(t; A)x \in \mathcal{D}(A) \) and \( \mathcal{D}^2S(t; A)x = AS(t; A)x \);
- If \( x \in \mathcal{D}(A) \), then \( S(t; A)x \in \mathcal{D}(A) \) and \( AS(t; A)x = S(t; A)Ax \).

**Theorem 2.2** [26] Let \( \{C(t; A), t \in \mathbb{R}\} \) be a strongly continuous cosine family in \( X \) satisfying \( \|C(t; A)\| \leq Me^{\omega|t|} \) for all \( t \in \mathbb{R} \) and let \( A \) be the infinitesimal generator of \( \{C(t; A), t \in \mathbb{R}\} \). Then for \( \text{Re}(\lambda) > \omega \), \( \lambda^2 \) is in the resolvent set of \( A \) and

\[
\lambda \mathcal{R}(\lambda^2; A)x = \int_{0}^{\infty} e^{-\lambda t} C(t; A)x dt, \quad x \in X,
\]

\[
\mathcal{R}(\lambda^2; A)x = \int_{0}^{\infty} e^{-\lambda t} S(t; A)x dt, \quad x \in X.
\]

**Theorem 2.3** [26] Let \( \{C(t; A), t \in \mathbb{R}\} \) be a strongly continuous cosine family in \( X \) with infinitesimal generator \( A \). If \( f : \mathbb{R} \to X \) is continuously differentiable, \( x \in \mathcal{D}(A) \), \( y \in \mathcal{E} \), and

\[
u(t) := C(t; A)x + S(t; A)y + \int_{0}^{t} S(t-s; A)f(s)ds, \quad t \in \mathbb{R},
\]

\( \mathbb{E} \) Springer
then \( u(t) \in \mathcal{D}(A) \) for \( t \in \mathbb{R} \), \( u \) is twice continuously differentiable, and \( u \) satisfies the following abstract Cauchy problem:

\[
\begin{aligned}
\mathcal{D}^2_t u(t) &= Au(t) + f(t), \quad t \in \mathbb{R}, \\
 u(0) &= x, \quad u'(0) = y.
\end{aligned}
\]

- **Strongly continuous fractional cosine, sine and Riemann–Liouville families of linear operators.**

**Definition 2.7** [17] Let \( 1 < \alpha < 2 \). A family \( C_\alpha(\cdot; A) : [0, \infty) \to \mathcal{L}(X) \) of all bounded linear operators on \( X \) is called a strongly continuous fractional cosine family if it satisfies the following assumptions:

- \( C_\alpha(t; A)x \) is continuous in \( t \) on \( [0, \infty) \) for each fixed \( x \in X \) and \( C_\alpha(0; A) = I \);
- \( C_\alpha(s; A)C_\alpha(t; A) = C_\alpha(t; A)C_\alpha(s; A) \) for all \( s, t \geq 0 \);
- The functional equation

\[
C_\alpha(s; A)I_\alpha t C_\alpha(t; A) - I_\alpha s C_\alpha(s; A)C_\alpha(t; A) = C_\alpha(t; A)I_\alpha s C_\alpha(s; A)C_\alpha(t; A)
\]

holds for all \( s, t \geq 0 \).

The linear operator \( A : \mathcal{D}(A) \subseteq X \to X \) is defined by

\[
Ax := \Gamma(\alpha + 1) \lim_{t \to 0^+} \frac{C_\alpha(t; A)x - x}{t^\alpha}, \quad x \in \mathcal{D}(A)
\]

\[
:= \left\{ x \in X : C_\alpha(\cdot; A)x \in C^2([0, \infty), X) \right\},
\]

where \( A \) is the infinitesimal generator of the strongly continuous fractional cosine family \( \{C_\alpha(t; A), t \geq 0 \} \).

**Definition 2.8** [17] A strongly continuous fractional cosine family \( \{C_\alpha(t; A), t \geq 0 \} \) is said to be exponentially bounded if there exists constants \( M \geq 1, \omega \geq 0 \) such that

\[
\|C_\alpha(t; A)\| \leq Me^{\omega t}, \quad t \geq 0.
\]

**Theorem 2.4** [5] Let \( \{C_\alpha(t; A), t \geq 0 \} \) be a strongly continuous fractional cosine family generated by the operator \( A \). The following assertions hold:

- \( C_\alpha(t; A) \) commutes with \( A \) which means that \( C_\alpha(t; A)\mathcal{D}(A) \subset \mathcal{D}(A) \) and \( AC_\alpha(t; A)x = C_\alpha(t; A)Ax \) for all \( x \in \mathcal{D}(A) \) and \( t \geq 0 \);
- For all \( x \in X \), \( \mathcal{I}_t^\alpha C_\alpha(t; A)x \in \mathcal{D}(A) \) and

\[
C_\alpha(t; A)x = x + A\mathcal{I}_t^\alpha C_\alpha(t; A)x, \quad t \geq 0.
\]

(2.5)

- For all \( x \in \mathcal{D}(A) \),

\[
C_\alpha(t; A)x = x + \mathcal{I}_t^\alpha C_\alpha(t; A)Ax, \quad t \geq 0.
\]
• A is closed, densely-defined operator on X;

**Definition 2.9** [17] The strongly continuous fractional sine family $S_{\alpha}(\cdot; A) : [0, \infty) \to \mathcal{L}(X)$ associated with $C_{\alpha}$ is defined by

$$S_{\alpha}(t; A)x = \int_{0}^{t} C_{\alpha}(s; A)x \, ds, \quad x \in X.$$  \hfill (2.6)

This implies that

$$\mathcal{D}_{t}^{1} S_{\alpha}(t; A)x = C_{\alpha}(t; A)x, \quad x \in X.$$  \hfill (2.7)

**Definition 2.10** [17] The strongly continuous fractional Riemann–Liouville family $T_{\alpha}(\cdot; A) : [0, \infty) \to \mathcal{L}(X)$ associated with $C_{\alpha}$ is defined by

$$T_{\alpha}(t; A)x = T_{t}^{\alpha-1} C_{\alpha}(t; A)x = \int_{0}^{t} \gamma_{\alpha-1}(t - s) C_{\alpha}(s, A)x \, ds, \quad x \in X.$$  \hfill (2.8)

For a strongly continuous fractional cosine family $\{C_{\alpha}(t; A), t \geq 0\}$, we define:

$$\mathcal{E} = \left\{ x \in X : C_{\alpha}(\cdot; A)x \in C^{1}([0, \infty), X) \right\}.$$  

It is known that $T_{\alpha}(t; A)\mathcal{E} \subseteq \mathcal{D}(A)$ for all $t \geq 0$ such that

$$\mathcal{D}_{t}^{1} C_{\alpha}(t; A)x = A T_{\alpha}(t; A)x, \quad t \geq 0, \quad x \in \mathcal{E}.$$  \hfill (2.9)

From (2.9) (or (2.5) and (2.8)), for $x \in X$, $T_{t}^{1} T_{\alpha}(t; A)x \in \mathcal{D}(A)$ we have that

$$A T_{t}^{1} T_{\alpha}(t; A)x = C_{\alpha}(t; A)x - x, \quad t \geq 0.$$  \hfill (2.10)

**Theorem 2.5** [2] Let $\{C_{\alpha}(t; A), t \geq 0\}$ be a strongly continuous fractional cosine family in $X$ satisfying $\|C_{\alpha}(t; A)\| \leq Me^{\omega t}$ for all $t \geq 0$ and let $A$ be the infinitesimal generator of $\{C_{\alpha}(t; A), t \geq 0\}$. Then for $Re(\lambda) > \omega$, $\lambda^{\alpha}$ for $\alpha \in (1, 2)$ is in the resolvent set of $A$ and

$$\lambda^{\alpha-1} \mathcal{R}(\lambda^{\alpha}; A)x = \int_{0}^{\infty} e^{-\lambda t} C_{\alpha}(t; A)x \, dt, \quad x \in X,$$

$$\lambda^{\alpha-2} \mathcal{R}(\lambda^{\alpha}; A)x = \int_{0}^{\infty} e^{-\lambda t} S_{\alpha}(t; A)x \, dt, \quad x \in X,$$

$$\mathcal{R}(\lambda^{\alpha}; A)x = \int_{0}^{\infty} e^{-\lambda t} T_{\alpha}(t; A)x \, dt, \quad x \in X.$$  

**Theorem 2.6** [17] Let $\{C_{\alpha}(t; A), t \geq 0\}$ be a strongly continuous fractional cosine family in $X$ with the infinitesimal generator $A$. If $f : [0, \infty) \to X$ is continuously differentiable, $x, y \in \mathcal{D}(A)$ and

$$u(t) := C_{\alpha}(t; A)x + S_{\alpha}(t; A)y + \int_{0}^{t} T_{\alpha}(t - s; A)f(s) \, ds, \quad t \geq 0.$$
then \( u(t) \in D(A) \) for \( t \geq 0 \) and \( u \) satisfies the following abstract Cauchy problem:

\[
\begin{cases}
(CD^\alpha_t u)(t) = Au(t) + f(t), & t \geq 0, \\
u(0) = x, & u'(0) = y.
\end{cases}
\]

### 3 Main results

We first consider the following linear homogeneous perturbed fractional evolution equation in a Banach space \( X \):

\[
(CD^\alpha_t u)(t) = (A + B(t))u(t), \quad t \geq 0,
\]

which is satisfying the initial conditions

\[
u(0) = x, \quad u'(0) = y,
\]

where \( 1 < \alpha < 2 \), \( A : D(A) \subseteq X \to X \) be a densely-defined, closed linear operator in a Banach space \( X \) and \( B : [0, \infty) \to L(X) \) be 2-times strongly continuously differentiable function.

**Theorem 3.1** Let \( A : D(A) \subseteq X \to X \) be the infinitesimal generator of the fractional cosine family \( \{C_\alpha(t; A), t \geq 0\} \) with \( \|C_\alpha(t; A)\| \leq Me^{\omega t} \) for all \( t \geq 0 \), and let \( \{S_\alpha(t; A), t \geq 0\} \) denote the fractional sine family associated with \( C_\alpha \). Furthermore, suppose that \( B : [0, \infty) \to L(X) \) is a twice strongly continuously differentiable function. Then

- The Cauchy problem for the \( X \)-valued differential Eq. (3.1) is uniformly well-posed; more precisely, for each pair \( x, y \in D(A) \), there is a uniquely determined solution \( u_0 \in C^2([0, \infty); X) \) of (3.1) fulfilling initial conditions (3.2) and \( u_0 \) depends continuously (with respect to the topology of uniform convergence on compact subsets of \( [0, \infty) \)) upon \( (x, y) \).
- The strong solution \( u_0(t) \in D(A), t \geq 0 \) for each \( x, y \in D(A) \) is given by

\[
u_0(t) = C_\alpha(t; A + B)x + S_\alpha(t; A + B)y, \quad t \geq 0,
\]

where for \( x \in D(A^2) := \{x \in X : x \in D(A), Ax \in D(A)\}, t \mapsto C_\alpha(t; A + B)x \) and \( t \mapsto S_\alpha(t; A + B)x \) are 2-times continuously differentiable functions fulfilling the conditions

\[
C_\alpha(0; A + B) = I, \quad C_\alpha'(0; A + B) = \Theta, \quad (3.4)
\]

\[
S_\alpha(0; A + B) = \Theta, \quad S_\alpha'(0; A + B) = I. \quad (3.5)
\]
• \( C_{\alpha}(\cdot; A + B), S_{\alpha}(\cdot; A + B) : [0, \infty) \to \mathcal{L}(X) \) are defined by

\[
C_{\alpha}(t; A + B)x := \sum_{n=0}^{\infty} C_{\alpha,n}(t, A)x, \quad x \in X,
\]

(3.6)

\[
S_{\alpha}(t; A + B)x := \sum_{n=0}^{\infty} S_{\alpha,n}(t; A)x, \quad x \in X,
\]

(3.7)

where for \( n \in \mathbb{N} \) and \( x \in X \):

\[
C_{\alpha,0}(t; A)x := C_{\alpha}(t; A)x, \quad S_{\alpha,0}(t; A)x := S_{\alpha}(t; A)x,
\]

\[
C_{\alpha,n}(t; A)x := \int_{0}^{t} T_{\alpha}(t - s; A)B(s)C_{\alpha,n-1}(s; A)x\,ds,
\]

(3.8)

\[
S_{\alpha,n}(t; A)x := \int_{0}^{t} T_{\alpha}(t - s; A)B(s)S_{\alpha,n-1}(s; A)x\,ds.
\]

(3.9)

• With \( K_T := \sup_{0 \leq t \leq T} \left\{ \| B(t) \|, \| B'(t) \| \right\} \), we have for all \( t \in [0, T] \), the bounds

\[
\| C_{\alpha}(t; A + B) \| \leq Me^{\omega t}E_{\alpha}\left( MK_T t^{\alpha}\right),
\]

\[
\| S_{\alpha}(t; A + B) \| \leq Me^{\omega t}E_{\alpha,2}\left( MK_T t^{\alpha}\right),
\]

\[
\| C_{\alpha}(t; A + B) - C_{\alpha}(t; A) \| \leq Me^{\omega t}\left[ E_{\alpha}\left( MK_T t^{\alpha}\right) - 1\right],
\]

\[
\| S_{\alpha}(t; A + B) - S_{\alpha}(t; A) \| \leq Me^{\omega t}\left[ E_{\alpha,2}\left( MK_T t^{\alpha}\right) - 1\right].
\]

Proof First, we suppose that there are constants \( M \geq 1 \) and \( \omega \geq 0 \) such that \( \| C_{\alpha}(t; A) \| \leq Me^{\omega t} \) for any \( t \geq 0 \), it follows that \( \| S_{\alpha}(t; A) \| \leq Me^{\omega t}, t \geq 0 \). Using the formula (2.8) and the relation \( e^{\alpha s} \leq e^{\omega s}, s \in [0, t] \) for \( \omega \geq 0 \) and \( t \geq 0 \), we have:

\[
\| T_{\alpha}(t; A) \| \leq \int_{0}^{t} g_{\alpha-1}(t - s) \| C_{\alpha}(s; A) \| \,ds
\]

\[
\leq M \int_{0}^{t} \frac{(t - s)^{\alpha-2}}{\Gamma(\alpha - 1)} e^{\omega s} \,ds
\]

\[
\leq Me^{\omega t} \int_{0}^{t} \frac{(t - s)^{\alpha-2}}{\Gamma(\alpha - 1)} \,ds
\]

\[
= Me^{\omega t}g_{\alpha}(t), \quad t \geq 0.
\]

(3.10)

For simplicity, we split our proof into several steps.

(1) \( C_{\alpha,n}(t; A) \) and \( S_{\alpha,n}(t; A) \) are strongly continuous for any \( t \in [0, \infty) \) and \( n \in \mathbb{N}_0 \): This is obviously true for \( n = 0 \). With \( C_{\alpha,n}(t; A) \) and \( S_{\alpha,n}(t; A) \) also \( B(s)C_{\alpha,n}(s; A) \) and \( B(s)S_{\alpha,n}(s; A) \) for \( s \in [0, t] \) and thus by Lutz [20] also \( C_{\alpha,n+1}(t; A) \) and \( S_{\alpha,n+1}(t; A) \) are strongly continuous for any \( t \in [0, \infty) \).
(2) Put \( K_t := \sup_{0 \leq s \leq t} \{ \| B(s) \|, \| B'(s) \| \} \). Then, it is true that for \( n \in \mathbb{N}_0 \):

\[
\begin{align*}
\| C_{\alpha,n}(t; A) \| & \leq M^{n+1} K_t^n e^{\alpha t} g_{n\alpha+1}(t), \quad t \geq 0, \\
\| S_{\alpha,n}(t; A) \| & \leq M^{n+1} K_t^n e^{\alpha t} g_{n\alpha+2}(t), \quad t \geq 0.
\end{align*}
\]

(3.11) In the case of (3.11), this is true by our assumption and remark above for \( n = 0 \). Using the formulae (3.10) and (3.11), we verify by mathematical induction principle that

\[
\| C_{\alpha,n+1}(t; A)x \| \leq \int_0^t \| T_\alpha(t - s; A) \| \| B(s) \| \| C_{\alpha,n}(s; A)x \| \, ds
\]

\[
\leq M^{n+2} K_t^{n+1} \int_0^t g_\alpha(t - s) e^{\alpha(t-s)} e^{\alpha s} g_{n\alpha+1}(s) ds \| x \|
\]

\[
= M^{n+2} K_t^{n+1} e^{\alpha t} \frac{s^{\alpha}}{\Gamma(\alpha)} g_{n\alpha+1}(t) \| x \|, \quad t \geq 0,
\]

follows for \( x \in X \).

Using the formulae (3.10) and (3.12), the analogous procedure gives the bound of \( S_{\alpha,n}(t; A), t \geq 0 \) for \( n \in \mathbb{N}_0 \), as follows:

\[
\| S_{\alpha,n+1}(t; A)x \| \leq \int_0^t \| T_\alpha(t - s; A) \| \| B(s) \| \| S_{\alpha,n}(s; A)x \| \, ds
\]

\[
\leq M^{n+2} K_t^{n+1} \int_0^t g_\alpha(t - s) e^{\alpha(t-s)} e^{\alpha s} g_{n\alpha+2}(s) ds \| x \|
\]

\[
= M^{n+2} K_t^{n+1} e^{\alpha t} \frac{s^{\alpha+1}}{\Gamma(\alpha)} g_{n\alpha+2}(t) \| x \|, \quad t \geq 0,
\]

follows for \( x \in X \).

From these bounds it follows that the series representing \( C_\alpha(t; A + B) \) and \( S_\alpha(t; A + B) \) in (3.6) and (3.7), respectively are uniformly convergent on compact subsets of \([0, \infty)\) with respect to the operator norm topology. Hence, \( C_\alpha(t; A + B) \) and \( S_\alpha(t; A + B) \) are strongly continuous families on \([0, \infty)\) with values in \( \mathcal{L}(X) \). Furthermore, the bounds for \( C_\alpha(t; A + B), S_\alpha(t; A + B), C_\alpha(t; A + B) - C_\alpha(t; A), \) and \( S_\alpha(t; A + B) - S_\alpha(t; A), t \in [0, T] \) derive as the following direct consequences:
\[
\|C_\alpha(t; A + B)\| \leq \sum_{n=0}^{\infty} \|C_{\alpha,n}(t; A)\|
\]
\[
\leq M e^{\omega t} \sum_{n=0}^{\infty} M^n K^n_T \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}
\]
\[
= Me^{\omega t} E_{\alpha,1}(MK_T t^\alpha),
\]
\[
\|S_\alpha(t; A + B)\| \leq \sum_{n=0}^{\infty} \|S_{\alpha,n}(t; A)\|
\]
\[
\leq M e^{\omega t} \sum_{n=0}^{\infty} M^n K^n_T \frac{t^{n\alpha}}{\Gamma(n\alpha + 2)}
\]
\[
= Me^{\omega t} t E_{\alpha,2}(MK_T t^\alpha),
\]
and
\[
\|C_\alpha(t; A + B) - C_\alpha(t; A)\| \leq \sum_{n=1}^{\infty} \|C_{\alpha,n}(t; A)\|
\]
\[
\leq M e^{\omega t} \sum_{n=1}^{\infty} M^n K^n_T \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}
\]
\[
= Me^{\omega t} \left[ E_{\alpha,1}(MK_T t^\alpha) - 1 \right].
\]
\[
\|S_\alpha(t; A + B) - S_\alpha(t; A)\| \leq \sum_{n=1}^{\infty} \|S_{\alpha,n}(t; A)\|
\]
\[
\leq M e^{\omega t} \sum_{n=1}^{\infty} M^n K^n_T \frac{t^{n\alpha}}{\Gamma(n\alpha + 2)}
\]
\[
= Me^{\omega t} t \left[ E_{\alpha,2}(MK_T t^\alpha) - 1 \right].
\]

(4) We show that, for \(x \in D(A^2)\), the function \(t \mapsto C_\alpha(t; A + B)x\) is twice continuously differentiable. Assuming that this is true for \(t \mapsto C_{\alpha,n}(t; A)x\), then as in Lutz [20], it is easily shown to be true for \(t \mapsto B(t)C_{\alpha,n}(t; A)x\), since \(B(t)\) is twice strongly continuously differentiable for any \(t \in [0, \infty)\) and thus, for \(t \mapsto C_{\alpha,n+1}(t; A)x\). First, for any \(t \geq 0\) and \(x \in D(A)\), we have:

\[
T_\alpha'(t; A)x = \int_0^t g_{\alpha-1}(t - s)T_\alpha(s; A)Ax ds + g_{\alpha-1}(t)x
\]
\[
= \int_0^t g_{\alpha-1}(t - s)AT_\alpha(s; A)x ds + g_{\alpha-1}(t)x.
\]

(3.13)

Then, using (3.13), we first obtain the following results for the first-order derivative of \(C_\alpha(t; A)\) and \(C_{\alpha,n+1}(t; A)\), \(n \in \mathbb{N}_0\) for any \(t \geq 0\) and \(x \in D(A)\):
\[ C'_\alpha(t; A)x = T^1_tD^2_tC_\alpha(t; A)x \]
\[ = T^{\alpha-1}_t T^{2-a}_t D^2_tC_\alpha(t; A)x \]
\[ = T^{\alpha-1}_t C_\alpha(t; A)x \]
\[ = T^{\alpha-1}_t A C_\alpha(t; A)x \]
\[ = T^{\alpha-1}_t C_\alpha(t; A)Ax \]
\[ = T_\alpha(t; A)Ax \]
\[ = AT_\alpha(t; A)x, \]
\[ C'_{\alpha,n+1}(t; A)x = \int_0^t T'_\alpha(t - s; A)B(s)C_{\alpha,n}(s; A)x ds. \quad (3.14) \]

Afterwards, using (3.13), we obtain the following results for the second derivative of \( C_\alpha(t; A) \) and \( C_{\alpha,n+1}(t; A) \), \( n \in \mathbb{N}_0 \) for any \( t \geq 0 \) and \( x \in D(A^2) \):

\[ C''_\alpha(t; A)x = T'_\alpha(t; A)Ax \]
\[ = AT'_\alpha(t; A)x, \]
\[ C''_{\alpha,n+1}(t; A)x = \int_0^t T'_\alpha(t - s; A)B'(s)C_{\alpha,n}(s; A)x ds \]
\[ + \int_0^t T'_\alpha(t - s; A)B(s)C_{\alpha,n}'(s; A)x ds, \]

where \( T'_\alpha(t; A) \) is defined as in (3.13) for any \( t \geq 0 \).

Furthermore, we suppose that there are constants \( M \geq 1 \) and \( \omega \geq 0 \) such that \( \| T_\alpha(t; A) \| \leq Me^{\omega t}g_\alpha(t) \) for any \( t \geq 0 \). It follows that for any \( t \geq 0 \) and \( x \in D(A) \):

\[ \| T'_\alpha(t; A)x \| \leq \int_0^t g_{\alpha-1}(t - s) \| T_\alpha(s; A) \| ds \| Ax \| + g_{\alpha-1}(t) \| x \| \]
\[ \leq M \int_0^t g_{\alpha-1}(t - s)g_\alpha(s)e^{\omega s} ds \| Ax \| + g_{\alpha-1}(t) \| x \| \]
\[ \leq Me^{\omega t}g_{2\alpha-1}(t) \| Ax \| + g_{\alpha-1}(t) \| x \|, \quad (3.15) \]

where we have used the fact that \( e^{\omega s} \leq e^{\omega t}, s \in [0, t] \) for \( \omega \geq 0 \) and \( t \geq 0 \).

By making use of the formulae (3.10), (3.11) and (3.15), we obtain by induction the following bounds for the first derivative of \( C_{\alpha,n}(t; A) \), \( n \in \mathbb{N}_0 \) for any \( t \geq 0 \) and \( x \in D(A) \):

\[ \| C'_\alpha(t; A)x \| \leq Me^{\omega t}g_\alpha(t) \| Ax \|, \]
\[ \| C'_{\alpha,n}(t; A)x \| \leq M^{n+1}_t K^n_t e^{\omega t}g_{n\alpha}(t) \| Ax \| \]
\[ + M^n K^n_t e^{\omega t}g_{n\alpha}(t) \| x \|, \quad n \in \mathbb{N}. \]
In a similar way, we can attain the following bounds for the second derivative of \( C_{\alpha,n}(t; A) \), \( n \in \mathbb{N}_0 \) for any \( t \geq 0 \) and \( x \in \mathcal{D}(A^2) \):

\[
\parallel C''_{\alpha}(t; A)x \parallel \leq M e^{\lambda t} \parallel g_{2\alpha-1}(t) \parallel A^2 x \parallel + g_{\alpha-1}(t) \parallel Ax \parallel,
\]

\[
\parallel C''_{\alpha,1}(t; A)x \parallel \leq M K t e^{\lambda t} \parallel g_{\alpha}(t) \parallel x \parallel + M^2 K t e^{\lambda t} \parallel g_{3\alpha-1}(t) \parallel A^2 x \parallel
\]
\[
+ M K t e^{\lambda t}\left(M g_{2\alpha}(t) + g_{2\alpha-1}(t)\right) \parallel Ax \parallel,
\]

\[
\parallel C''_{\alpha,n}(t; A)x \parallel \leq M^n K^n t e^{\lambda t}\left(M g_{n\alpha+\alpha}(t) + n g_{n\alpha+\alpha-1}(t)\right) \parallel Ax \parallel
\]
\[
+ M^{n+1} K^n t e^{\lambda t} g_{n\alpha+2\alpha-1}(t) \parallel A^2 x \parallel
\]
\[
+ M^{n-1} K^n t e^{\lambda t}\left(M g_{n\alpha}(t) + g_{n\alpha-1}(t)\right) \parallel x \parallel, \quad n \geq 2.
\]

In a completely analogous manner \( t \mapsto S_{\alpha}(t; A + B)x \) is seen to be twice continuously differentiable on \([0, \infty)\) for every \( x \in \mathcal{D}(A^2) \). To show this, we will use the following calculations for any \( t \geq 0 \):

\[
S'_{\alpha}(t; A)x = C_{\alpha}(t; A)x, \quad x \in X,
\]

\[
S'_{\alpha,n+1}(t; A)x = \int_0^t T_S(t - s; A)B(s)S_{\alpha,n}(s; A)x ds, \quad x \in \mathcal{D}(A),
\]

\[
S''_{\alpha}(t; A)x = T_S(t; A)Ax = AT_S(t; A)x, \quad x \in \mathcal{D}(A),
\]

\[
S''_{\alpha,1}(t; A)x = \int_0^t T_S(t - s; A)B'(s)S_{\alpha}(s; A)x ds
\]
\[
+ \int_0^t T_S(t - s; A)B(s)S'_{\alpha}(s; A)x ds, \quad x \in \mathcal{D}(A),
\]

\[
S''_{\alpha,n+1}(t; A)x = \int_0^t T_S(t - s; A)B'(s)S_{\alpha,n}(s; A)x ds
\]
\[
+ \int_0^t T_S(t - s; A)B(s)S'_{\alpha,n}(s; A)x ds, \quad x \in \mathcal{D}(A^2), \quad n \geq 2.
\]

Using our assumption \( \parallel C_{\alpha}(t; A) \parallel \leq M e^{\lambda t} \) for \( t \geq 0 \), (3.12) and (3.15), we obtain by induction the following bounds for the first derivative of \( S_{\alpha,n}(t; A) \), \( n \in \mathbb{N}_0 \) for any \( t \geq 0 \):

\[
\parallel S'_{\alpha}(t; A)x \parallel \leq M e^{\lambda t} \parallel x \parallel, \quad x \in X,
\]

\[
\parallel S'_{\alpha,n}(t; A)x \parallel \leq M^{n+1} K^n t e^{\lambda t} g_{n\alpha+\alpha+1}(t) \parallel Ax \parallel
\]
\[
+ M^n K^n t e^{\lambda t} g_{n\alpha+1}(t) \parallel x \parallel, \quad x \in \mathcal{D}(A), \quad n \in \mathbb{N}.
\]

Similarly, we acquire by induction the following bounds for the second derivative of \( S_{\alpha,n}(t; A) \), \( n \in \mathbb{N}_0 \) for any \( t \geq 0 \):
\[ \|S'_a(t; A)x\| \leq M e^{\omega t} \|Ax\|, \quad x \in \mathcal{D}(A), \]
\[ \|S''_a(t; A)x\| \leq M^2 K t e^{\omega t} (g_{2a}(t) + g_{2a+1}(t)) \|Ax\| + M K t e^{\omega t} (g_{2a}(t) + g_{a+1}(t)) \|x\|, \quad x \in \mathcal{D}(A), \]
\[ \|S''_n(t; A)x\| \leq M^n K^n t e^{\omega t} (M g_{na+a+1}(t) + n g_{na+a}(t)) \|Ax\| + M^n K^n t e^{\omega t} g_{na+2a}(t) \|A^2 x\| + M^n K^n t e^{\omega t} g_{na}(t) + M g_{na+1}(t) \|x\|, \quad x \in \mathcal{D}(A^2), \quad n \geq 2. \]

Therefore, for \( x \in \mathcal{D}(A) \), the series \( \sum_{n=0}^{\infty} C'_{a,n}(t; A)x \) and \( \sum_{n=0}^{\infty} S'_a(t; A)x \) converge uniformly in every compact interval of \([0, \infty)\) to continuous functions which are \( C'_a(t; A)x \) and \( S'_a(t; A)x \), respectively. In a similar way, we can conclude that for \( x \in \mathcal{D}(A^2) \), the series \( \sum_{n=0}^{\infty} C''_{a,n}(t; A)x \) and \( \sum_{n=0}^{\infty} S''_a(t; A)x \) converge uniformly in every compact interval of \([0, \infty)\) to continuous functions which, by usual argument, are \( C''_a(t; A)x \) and \( S''_a(t; A)x \), respectively.

(5) Next we prove that \( u_0(t) \in \mathcal{D}(A) \) for any \( t \geq 0 \) satisfies the fractional-order abstract differential Eq. (3.1) with initial conditions \( u_0(0) = x, u'_0(0) = y \), where \( x, y \in \mathcal{D}(A) \); in other words, (3.3) satisfies the abstract Cauchy problem (3.1) and (3.2). Since
\[ C_a(0; A) = I, \quad C_{a,n}(0; A) = \Theta, \quad C'_a(0; A) = \Theta, \quad C'_{a,n}(0; A) = \Theta, \quad n \in \mathbb{N}, \]
\[ S_a(0; A) = \Theta, \quad S_{a,n}(0; A) = \Theta, \quad S'_a(0; A) = I, \quad S'_{a,n}(0; A) = \Theta, \quad n \in \mathbb{N}, \]
we have (3.4) and (3.5), i.e., the initial conditions (3.2) are satisfied. Applying (3.3), (3.6)–(3.9), it follows that
\[ u_0(t) = C_a(t; A)x + S_a(t; A)y + \sum_{n=1}^{\infty} \int_0^t T_a(t - s; A) B(s) \left( C_{a,n-1}(s; A)x + S_{a,n-1}(s; A)y \right) ds \]
\[ = C_a(t; A)x + S_a(t; A)y + \sum_{n=0}^{\infty} \int_0^t T_a(t - s; A) B(s) \left( C_{a,n}(s; A)x + S_{a,n}(s; A)y \right) ds \]
\[ = C_a(t; A)x + S_a(t; A)y + \int_0^t T_a(t - s; A) B(s) u_0(s) ds, \quad t \geq 0, \]

(3.16)

where the interchanging of the summation and integration is justified by the uniform convergence of the given series. Then integro-differentiating (3.16) term-by-term and by using the formulae (2.7)–(2.10), we obtain for \( x, y \in \mathcal{D}(A) \):
\((CD^\alpha_tC_\alpha(t;A))x = I^2_{t^{-\alpha}}(D^2_tC_\alpha(t;A))x\)
\[= AT^2_{t^{-\alpha}}(D^1_tT_\alpha(t;A))x \]
\[= AT^2_{t^{-\alpha}}(D^1_t(I_{t^{-\alpha}}C_\alpha(t;A)))x \]
\[= AT^2_{t^{-\alpha}}(I_{t^{-\alpha}}AT_\alpha(t;A)x + g_{\alpha-1}(t)x) \]
\[= A(I_{t^{-\alpha}}AT_\alpha(t;A)x + x) \]
\[= AT^1_{t^{-\alpha}}T_\alpha(t;A)x + x \]
\[= AC_\alpha(t;A)x, \quad t \geq 0. \quad (3.17) \]

and

\((CD^\alpha_tS_\alpha(t;A))y = I^2_{t^{-\alpha}}(D^2_tS_\alpha(t;A))y\)
\[= AT^2_{t^{-\alpha}}(D^1_tC_\alpha(t;A))y \]
\[= AT^2_{t^{-\alpha}}C_\alpha(t;A)y \]
\[= AT^1_{t^{-\alpha}}C_\alpha(t;A)y \]
\[= AS_\alpha(t;A)y, \quad t \geq 0. \quad (3.18) \]

By virtue of the formulae (3.17) and (3.18), we derive that

\[CD^\alpha_tu_0(t) = AC_\alpha(t;A)x + AS_\alpha(t;A)y \]
\[+ CD^\alpha_t\left(\int_0^t T_\alpha(t-s;A)B(s)u_0(s)ds\right), \quad t \geq 0. \quad (3.19)\]

Making use of (3.19) and (2.2), \((T_\alpha \ast (Bu_0))(0) = (T_\alpha \ast (Bu_0))' (0) = 0\), the property of \(I^\alpha_t(f \ast g) = (I^\alpha_t f) \ast g\) and the semigroup property for operators of Riemann–Liouville fractional integration (2.1), we have:

\[CD^\alpha_t\left(T_\alpha(t;A) \ast (B(t)u_0(t))\right) \]
\[= D^\alpha_t\left(T_\alpha(t;A) \ast (B(t)u_0(t))\right) \]
\[= D^\alpha_tI^2_{t^{-\alpha}}\left(I_{t^{-\alpha}}C_\alpha(t;A) \ast (B(t)u_0(t))\right) \]
\[= D^\alpha_t\left(I^2_{t^{-\alpha}}I_{t^{-\alpha}}C_\alpha(t;A) \ast (B(t)u_0(t))\right) \]
\[= D^\alpha_t\left(I^2_{t^{-\alpha}}C_\alpha(t;A) \ast (B(t)u_0(t))\right) \]
\[= D^\alpha_t\left(C_\alpha(t;A) \ast (B(t)u_0(t))\right) \]
\[ D_1^t C_\alpha(t; A) \ast \left( B(t)u_0(t) \right) + B(t)u_0(t) = A T_\alpha(t; A) \ast \left( B(t)u_0(t) \right) + B(t)u_0(t), \quad t \geq 0, \quad (3.20) \]

and using the closedness of \( A \) implies that \( u_0(t) \) satisfies the abstract Cauchy problem (3.1) and (3.2) for any \( t \geq 0 \).

(6) Uniqueness: In the uniqueness proof, it will be sufficient to show that if \( v : [0, \infty) \to X \) solves the fractional-order abstract differential Eq. (3.1) for \( t \geq 0 \) with zero initial conditions \( v(0) = v'(0) = 0 \), then \( v(t) \equiv 0 \) solves (3.1) for all \( t \in [0, \infty) \), since \( A \) is densely-defined in a Banach space \( X \). In other words, we need to consider a twice strongly continuously differentiable function \( v(t) \) on \([0, \infty)\) to \( X \) such that \( v(0) = v'(0) = 0 \) and \((C D_1^t v)(t) = (A + B(t))v(t)\) for any \( t \geq 0 \).

We let \( v : [0, \infty) \to X \) be a solution of (3.1) with zero initial conditions \( v(0) = v'(0) = 0 \). Then, by using the well-known property (2.3), we have \( v(t) = T_\alpha^t A v(t) + T_\alpha^t \left( B(t)v(t) \right) \) and applying the variation of parameters formula, \( v(t) \) satisfies the Volterra integral equation of second-kind:

\[ v(t) = \int_0^t T_\alpha(t-s; A)B(s)v(s)ds, \quad t \geq 0. \]

Before we get to the rest of the proof, it should be noted that the technique used here follows from Theorem 6.2 in [23]. We are now setting \( m_t = \sup_{0 \leq s \leq t} \|v(s)\| \).

By using \( e^{\omega(t-s)} \leq e^{\omega t} \), \( s \in [0, t] \) for \( \omega \geq 0 \) and \( t \geq 0 \), we see that for \( m_t > 0 \):

\[ m_t \leq \frac{MK_t}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\omega(t-s)}ds \leq MK_t m_{t} g_{\alpha+1}(t)e^{\omega t}, \]

and for chosen sufficiently small \( t > 0 \) such that \( MK_t g_{\alpha+1}(t)e^{\omega t} < 1 \). This implies that \( m_t = 0 \). Thus, \( v(t) \equiv 0 \) on \([0, t_0]\) with \( t_0 > 0 \). Iteration of this argument leads to \( v(t) \equiv 0 \) for any \( t \geq 0 \). The proof is complete. □

The following lemma plays a crucial role in the proof of Theorem 3.2.

**Lemma 3.1** Let \( A : D(A) \subseteq X \to X \) be infinitesimal generator of the strongly continuous fractional cosine family \( \{C_\alpha(t; A), t \geq 0\} \) and \( \{T_\alpha(t; A), t \geq 0\} \) be the corresponding fractional Riemann–Liouville family associated with \( C_\alpha \) which is defined as in (2.8). For \( f(t) \) strongly continuous on \([0, \infty)\) to \( X \), \( g(t) = \int_0^t T_\alpha(t-s; A)f(s)ds \) exists and is itself strongly continuous on \([0, \infty)\) to \( X \). If \( f(t) \) is strongly continuously differentiable on \([0, \infty)\) to \( X \), then \( g(t) \) is also strongly continuously differentiable for any \( t \in [0, \infty) \) and

\[ g(t) = \int_0^t T_\alpha(t-s; A)f(s)ds. \]
\[ g'(t) = T_\alpha(t; A)f(0) + \int_0^t T_\alpha(t - s; A)f'(s)ds \]
\[ = \int_0^t T_\alpha'(t - s; A)f(s)ds, \quad t \geq 0, \]

where \( T_\alpha'(t; A) \) is defined as in (3.13) for any \( t \geq 0 \).

**Proof** Suppose that there are constants \( M \geq 1 \) and \( \omega \geq 0 \) such that \( \|T_\alpha(t; A)\| \leq Me^{\omega t}g_\alpha(t) \) for any \( t \geq 0 \). It follows that \( T_\alpha(t - s; A)f(s) \) is strongly continuous in \( s \in [0, t] \) whenever the same is true of \( f(s) \). In this case, \( g(t) = \int_0^t T_\alpha(t - s; A)f(s)ds \) will exist in the strong topology and be equal to \( \int_0^t T_\alpha(s; A)f(t - s)ds \). Moreover, since \( f : [0, \infty) \rightarrow X \) is a strongly continuously differentiable function, by using the well-known integration by parts formula for the operator-valued functions, we obtain the desired result:

\[ \int_0^t T_\alpha(t - s; A)f'(s)ds = -T_\alpha(t; A)f(0) + \int_0^t T_\alpha'(t - s; A)f(s)ds, \quad t \geq 0. \]

The proof is complete. \( \square \)

**Theorem 3.2** Let \( \{C_\alpha(t; A), t \geq 0\} \) be a strongly continuous fractional cosine family with infinitesimal generator \( A \) and \( \{T_\alpha(t; A), t \geq 0\} \) be fractional Riemann–Liouville family corresponding to the \( C_\alpha \). Let \( B(t) \) be twice strongly continuously differentiable function on \([0, \infty)\) to \( L(X) \). Let \( f(t) \) be strongly continuously differentiable function on \([0, \infty)\) to \( X \). Then, there exists a unique twice strongly continuously differentiable strong solution \( u : [0, \infty) \rightarrow X \) of the abstract Cauchy problem (1.5) and (1.6) for a linear non-homogeneous fractional evolution equation which is satisfying \( u(t) \in D(A) \) for all \( t \geq 0 \) with \( x, y \in \mathcal{D}(A) \). This solution has the closed-form:

\[ u(t) = u_0(t) + u_1(t), \quad t \geq 0, \]

where \( u_0(t), t \geq 0 \) is defined as in (3.3) and

\[ u_1(t) = \sum_{n=0}^{\infty} w_n(t), \quad t \geq 0. \]

For any \( t \geq 0 \) and \( n \in \mathbb{N} \), we have:

\[ w_0(t) = \int_0^t T_\alpha(t - s; A)f(s)ds, \]
\[ w_n(t) = \int_0^t T_\alpha(t - s; A)B(s)w_{n-1}(s)ds. \]

**Proof** It follows since \( f \in C^1([0, \infty), X) \), \( w_0(t) \) is 2-times strongly continuously differentiable and hence by induction that \( w_n(t) \) is likewise for any \( n \in \mathbb{N} \). In fact, by Lemma 3.1, we have for any \( t \geq 0 \) and \( n \in \mathbb{N} \):
\[
\begin{align*}
    w_0^0(t) &= \int_0^t T_\alpha'(t - s; A) f(s) \, ds, \\
    w_0^1(t) &= T_\alpha'(t; A) f(0) + \int_0^t T_\alpha'(t - s; A) f'(s) \, ds, \\
    w_n^1(t) &= \int_0^t T_\alpha'(t - s; A) B(s) w_{n-1}(s) \, ds, \\
    w_n^2(t) &= \int_0^t T_\alpha'(t - s; A) B'(s) w_{n-1}(s) \, ds \\
    &\quad + \int_0^t T_\alpha'(t - s; A) B(s) w_{n-1}'(s) \, ds.
\end{align*}
\]

By using (3.10) and (3.15), it is easy to acquire the following estimations:

\[
\begin{align*}
    \|w_n(t)x\| &\leq Mn^{n+1}K^n_iN_t e^{c_0t}g_{n\alpha+\alpha+1}(t) \|x\|, \quad x \in X, \quad n \in \mathbb{N}_0, \\
    \|w_n'(t)x\| &\leq Mn^{n+1}K^n_iN_t e^{c_0t}g_{n\alpha+2\alpha}(t) \|Ax\| \\
    &\quad + Mn^{n}K^n_iN_t e^{c_0t}g_{n\alpha+\alpha}(t) \|x\|, \quad x \in \mathcal{D}(A), \quad n \in \mathbb{N}_0, \\
    \|w_n''(t)x\| &\leq MN_t e^{c_0t}\left[g_{2\alpha-1}(t) + g_{2\alpha}(t)\right] \|Ax\| \\
    &\quad + N_t\left[g_{\alpha-1}(t) + g_{\alpha}(t)\right] \|x\|, \quad x \in \mathcal{D}(A), \\
    \|w_n''(t)\| &\leq Mn^{n+1}K^n_iN_t e^{c_0t}g_{n\alpha+3\alpha-1}(t) \|A^2x\| \\
    &\quad + Mn^{n}K^n_iN_t e^{c_0t}\left[(n + 1)g_{n\alpha+2\alpha-1}(t) + Mg_{n\alpha+2\alpha}(t)\right] \|Ax\| \\
    &\quad + K^n_iN_t\left[M^n e^{c_0t}g_{n\alpha+\alpha}(t) + g_{n\alpha+\alpha-1}(t)\right] \|x\|, \quad x \in \mathcal{D}\left(A^2\right), \quad n \in \mathbb{N},
\end{align*}
\]

where \(N_t := \sup_{0 \leq s \leq t} \left\{ \|f(s)\|, \|f'(s)\| \right\} \).

If we set \(u_1(t) = \sum_{n=0}^{\infty} w_n(t)\), then as in Theorem 3.1, \(u_1(t)\) is 2-times strongly continuously differentiable on \([0, \infty)\) to \(X\) and \(u_1'(t) = \sum_{n=0}^{\infty} w_n'(t), u_1''(t) = \sum_{n=0}^{\infty} w_n''(t)\). Furthermore, \(u_1(0) = u_1'(0) = 0\). From the definition of \(w_n(t), n \in \mathbb{N}_0\) and uniform convergence of the series \(\sum_{n=0}^{\infty} w_n(t)\) in every compact subset of \([0, \infty)\), it follows that
\[ u_1(t) = \sum_{n=0}^{\infty} w_n(t) = \sum_{n=1}^{\infty} w_n(t) + w_0(t) \]
\[ = w_0(t) + \sum_{n=1}^{\infty} \int_{0}^{t} T_{\alpha}(t - s; A)B(s)w_{n-1}(s)ds \]
\[ = w_0(t) + \int_{0}^{t} T_{\alpha}(t - s; A)B(s)\sum_{n=1}^{\infty} w_{n-1}(s)ds \]
\[ = w_0(t) + \int_{0}^{t} T_{\alpha}(t - s; A)B(s)\sum_{n=0}^{\infty} w_n(s)ds \]
\[ = w_0(t) + \int_{0}^{t} T_{\alpha}(t - s; A)B(s)u_1(s)ds, \quad t \geq 0, \quad (3.21) \]

where the interchanging summation and integration is justified by the uniform convergence of the series. Since \( u_1(t) \) is 2-times strongly continuously differentiable, we may differentiate (3.21) term-wise in Caputo's sense by considering \( C_\alpha (0; A) = I \), using (2.2) and (3.20) as follows:

\[
\left( C D_\alpha^\alpha u_1 \right) (t) = C D_\alpha^\alpha \left( w_0(t) + \int_{0}^{t} T_{\alpha}(t - s; A)B(s)u_1(s)ds \right) 
\]
\[ = C D_\alpha^\alpha \left( T_{\alpha}(t; A) * f(t) \right) + C D_\alpha^\alpha \left( T_{\alpha}(t; A) * \left( B(t)u_1(t) \right) \right) \]
\[ = D_\alpha^\alpha \left( T_{\alpha}(t; A) * f(t) \right) + D_\alpha^\alpha \left( T_{\alpha}(t; A) * \left( B(t)u_1(t) \right) \right) \]
\[ = AT_{\alpha}(t; A) * f(t) + f(t) + AT_{\alpha}(t; A) * \left( B(t)u_1(t) \right) + B(t)u_1(t) \]
\[ = Aw_0(t) + f(t) + A \int_{0}^{t} T_{\alpha}(t - s; A)B(s)u_1(s)ds + B(t)u_1(t) \]
\[ = \left( A + B(t) \right) u_1(t) + f(t), \quad t \geq 0. \]

Note that since initial values are zero we have used Riemann–Liouville fractional derivative instead of Caputo one. This shows immediately that \( u_1(t) \) is a particular solution of linear non-homogeneous abstract Cauchy problem (1.5) and (1.6). In other words, \( u_1(t) \) is a solution of (1.5) with zero initial conditions \( u_1(0) = u'_1(0) = 0 \). Therefore, by Theorem 3.1, \( u(t) = u_0(t) + u_1(t) \) is a unique solution for linear non-homogeneous abstract Cauchy problem (1.5) and (1.6) for any \( t \geq 0 \). The uniqueness of solution follows precisely as in the uniqueness proof of Theorem 3.1. The proof is complete. \( \square \)

**Remark 3.1** In our case, we consider the strong solution of the abstract Cauchy problem (1.5) and (1.6) under the circumstances \( u(t) \in D (A), t \geq 0 \) with \( x, y \in D (A) \). If we take \( u(t) \in X, t \geq 0 \) with \( x, y \in X \), we can consider the more general case where the solution is seen in a mild sense.
Remark 3.2 Considering the linear homogeneous Eq. (3.1) with the second initial condition equal to zero, i.e., \( u(0) = x \) and \( u'(0) = 0 \), we derive the special case published in [3] by Bazhlekova. It has been shown that in her thesis (1.4), the reason is that in some models (i.e., for waves in viscoelastic media) there are some physical arguments to choose the second initial condition equal to zero in her thesis [2]. Therefore, our theory generalizes this result.

Remark 3.3 It should be noted that some interesting perturbation properties for fractional strongly continuous cosine and sine families are studied in [12] when \( B(t) \equiv B \in L(X) \) by Huseynov et al. The authors in [12] have established the sufficient conditions for \( A \) to be the infinitesimal generator of a fractional strongly continuous cosine (sine) family in a Banach space \( X \) and \( B \) to be a bounded linear operator in \( X \), then \( A + B \) is also the infinitesimal generator of a fractional strongly continuous cosine (sine) family in \( X \). Moreover, the obtained results are applied to the linear homogeneous fractional evolution equations in [12]. Our results cover this special case (in terms of the theory for fractional evolution equations) not only for linear homogeneous but also for non-homogeneous abstract differential equations of fractional order. Note that for this special case, the particular solution can also be represented in an alternative integral form as follows.

To do this, we consider the following abstract Cauchy problem for linear non-homogeneous evolution equation with zero initial conditions in a Banach space \( X \):

\[
\begin{aligned}
\left\{ \begin{array}{l}
(C D_0^\alpha u)(t) = (A + B)u(t) + f(t), \quad t \geq 0, \\
u(0) = u'(0) = 0,
\end{array} \right.
\end{aligned}
\tag{3.22}
\]

where \( 1 < \alpha < 2 \), \( A : D(A) \subseteq X \to X \) is an infinitesimal generator of the strongly continuous fractional cosine family \( \{C_\alpha(t; A), t \geq 0\} \), \( B \in L(X) \) and \( f : [0, \infty) \to X \). Then, there exists a unique 2-times continuously differentiable function \( u_1 : [0, \infty) \to X \) which is a particular solution of Eq.(3.22) with zero initial conditions \( u_1(0) = u_1'(0) = 0 \). This solution has a variation of constants formula as follows:

\[
u_1(t) = \int_0^t T_\alpha(t - s; A + B)f(s)ds, \quad t \geq 0, \tag{3.23}\]

where \( T_\alpha(\cdot; A + B) : [0, \infty) \to L(X) \) is the fractional Riemann–Liouville family corresponding to the strongly continuous fractional cosine family \( \{C_\alpha(t; A + B), t \geq 0\} \) generated by \( A + B \) which is defined by

\[
T_\alpha(t; A + B) = \int_0^t g_{\alpha-1}(t - s)C_\alpha(s; A + B)ds, \quad t \geq 0. \tag{3.24}\]

\( \square \) Springer
Proof We assume that there exist constants $M \geq 1$ and $\omega \geq 0$ such that
\[ \|C_\alpha(t; A + B)\| \leq M e^{\omega t} E_\alpha(M \| B \| t^\alpha) \] for any $t \geq 0$. By using this fact and (3.24) together, we obtain that

\[ \|T_\alpha(t; A + B)\| \leq \int_0^t g_{\alpha-1}(t - s) \|C_\alpha(s; A + B)\| \, ds \]
\[ \leq M \int_0^t \frac{(t - s)^{\alpha-2}}{\Gamma(\alpha - 1)} e^{\omega s} E_\alpha(M \| B \| s^\alpha) \, ds \]
\[ \leq e^{\omega t} \sum_{n=0}^{\infty} M^{n+1} \| B \|^n \int_0^t \frac{(t - s)^{\alpha-2}}{\Gamma(\alpha - 1)} \cdot \frac{s^{n\alpha}}{\Gamma(n\alpha + \alpha)} \, ds \]
\[ = e^{\omega t} \sum_{n=0}^{\infty} M^{n+1} \| B \|^n \frac{t^{n\alpha+\alpha-1}}{\Gamma(n\alpha + \alpha)} \]
\[ = M e^{\omega t} t^{\alpha-1} E_{\alpha,\alpha}(M \| B \| t^\alpha), \quad t \geq 0. \]

In fact, by Lemma 3.1 and the formula (3.13), we have:

\[ u_1'(t) = \int_0^t T_\alpha(t - s; A + B) f'(s) \, ds + T_\alpha(t; A + B) f(0), \quad t \geq 0, \]
\[ u_1''(t) = T_\alpha'(t; A + B) f(0) + \int_0^t T_\alpha'(t - s; A + B) f'(s) \, ds, \quad t \geq 0, \]

where for any $t \geq 0$:

\[ T_\alpha'(t; A + B) = \int_0^t g_{\alpha-1}(t - s) C_\alpha'(s; A + B) \, ds + g_{\alpha-1}(t), \]
\[ C_\alpha'(t; A + B) = A T_\alpha(t; A) + \int_0^t T_\alpha'(t - s; A) B C_\alpha(s; A + B) \, ds. \]

It is now easy to acquire the following estimations:

\[ \|w(t)\| \leq M N_t e^{\omega t} t^\alpha E_{\alpha,\alpha+1}(M \| B \| t^\alpha) \|x\|, \quad x \in X, \]
\[ \|w'(t)\| \leq M N_t e^{\omega t} \left[ t^{\alpha-1} E_{\alpha,\alpha}(M \| B \| t^\alpha) + t^\alpha E_{\alpha,\alpha+1}(M \| B \| t^\alpha) \right] \|x\|, \quad x \in X, \]
\[ \|w''(t)\| \leq M N_t e^{\omega t} \left[ g_{2\alpha-1}(t) + g_{2\alpha}(t) \right] \|Ax\| + N_t \left[ g_{\alpha-1}(t) + g_{\alpha}(t) \right] \|x\|, \quad x \in \mathcal{D}(A), \]
\[ + M^2 \| B \| N_t e^{\omega t} \left[ t^{3\alpha-2} E_{\alpha,3\alpha}(M \| B \| t^\alpha) \right] \|Ax\| \]
\[ + t^{3\alpha-2} E_{\alpha,3\alpha-1}(M \| B \| t^\alpha) \|x\|, \quad x \in \mathcal{D}(A), \]
\[ + M \| B \| N_t e^{\omega t} \left[ t^{2\alpha-1} E_{\alpha,2\alpha}(M \| B \| t^\alpha) \right] \|x\|, \quad x \in \mathcal{D}(A), \]
\[ + t^{2\alpha-2} E_{\alpha,2\alpha-1}(M \| B \| t^\alpha) \|x\|, \quad x \in \mathcal{D}(A), \]
\[ + M e^{\omega t} t^{\alpha-1} E_{\alpha,\alpha}(M \| B \| t^\alpha) \|x\|, \quad x \in \mathcal{D}(A). \]
where we have used the following estimations for $x \in \mathcal{D}(A)$:

$$
\| C'_\alpha(t; A + B)x \| \leq M^2 \| B \| e^{\omega t} t^{2\alpha - 1} E_{\alpha,2\alpha}(M \| B \| t^\alpha) \| Ax \|
+ M \| B \| e^{\omega t} t^{\alpha - 1} E_{\alpha,\alpha}(M \| B \| t^\alpha) \| x \|
+ M e^{\omega t} g_\alpha(t) \| Ax \|,
$$

$$
\| T'_\alpha(t; A + B)x \| \leq M^2 \| B \| e^{\omega t} t^{3\alpha - 2} E_{\alpha,3\alpha - 1}(M \| B \| t^\alpha) \| Ax \|
+ M \| B \| e^{\omega t} t^{2\alpha - 2} E_{\alpha,2\alpha - 1}(M \| B \| t^\alpha) \| x \|
+ M e^{\omega t} g_{2\alpha - 1}(t) \| Ax \| + g_{\alpha - 1}(t) \| x \|,
$$

with $N_t := \sup_{0 \leq s \leq t} \left\{ \| f(s) \|, \| f'(s) \| \right\}$.

Therefore, $u_1(t)$ is $2$-times strongly continuously differentiable function on $[0, \infty)$ to $X$. Next, we prove that $u_1(t)$ satisfies the Eq. (3.22) with zero initial conditions $u_1(0) = u'_1(0) = 0$. By making use of the formulae (2.2), (3.23) and (3.24), we derive that

$$
\left( C D_t^\alpha u_1 \right)(t) = C D_t^\alpha \left( \int_0^t T_\alpha(t - s; A + B) f(s) ds \right)
= C D_t^\alpha \left( T_\alpha(t; A + B) * f(t) \right)
= D_t^\alpha \left( T_\alpha(t; A + B) * f(t) \right)
= D_t^2 T_t^{2 - \alpha} \left( T_t^{\alpha - 1} C_\alpha(t; A + B) * f(t) \right)
= D_t^2 T_t^{2 - \alpha} T_t^{\alpha - 1} \left( C_\alpha(t; A + B) * f(t) \right)
= D_t^2 T_t^1 \left( C_\alpha(t; A + B) * f(t) \right)
= D_t^1 \left( C_\alpha(t; A + B) * f(t) \right)
= D_t^1 C_\alpha(t; A + B) * f(t) + C_\alpha(0; A + B) f(t), \quad t \geq 0.
$$

Since $C D_t^\alpha C_\alpha(t; A + B) = (A + B) C_\alpha(t; A + B)$, $t \geq 0$ and $C_\alpha(0; A + B) = I$, we attain that

$$
D_t^1 C_\alpha(t; A + B) = T_t^1 D_t^2 C_\alpha(t; A + B)
= T_t^{\alpha - 1} T_t^{2 - \alpha} D_t^2 C_\alpha(t; A + B)
= T_t^{\alpha - 1} C D_t^\alpha C_\alpha(t; A + B)
= T_t^{\alpha - 1} (A + B) C_\alpha(t; A + B)
= (A + B) T_\alpha(t; A + B), \quad t \geq 0.
$$
So, we derive a desired result:

\[
\left( C_D^\alpha u_1 \right)(t) = \left( (A + B)T_\alpha (t; A + B) \right) * f(t) + f(t) \\
= (A + B) \int_0^t T_\alpha (t - s; A + B) f(s) ds + f(t) \\
= (A + B) u_1(t) + f(t), \quad t \geq 0.
\]

The proof is complete. \(\square\)

**Remark 3.4** In the particular case, if \(B \equiv 0\), then we derive the following abstract initial value problem for a fractional evolution equation in a Banach space \(X\):

\[
\begin{align*}
\left\{ C_D^\alpha u \right \}(t) &= Au(t), \quad t \geq 0, \\
u(0) &= x \in D(A), \quad u'(0) = y \in D(A),
\end{align*}
\]

where \(1 < \alpha < 2\), a linear operator \(A : D(A) \subseteq X \to X\) is the infinitesimal generator of a strongly continuous fractional cosine family of bounded linear operators \(\{ C_\alpha (t; A), t \geq 0 \}\); has a unique strong solution \(u : [0, \infty) \to X\) which is satisfying \(u(t) \in D(A)\) for any \(t \geq 0\). This solution has a closed-form with \(x, y \in D(A)\) as follows:

\[
u(t) = C_\alpha (t; A)x + S_\alpha (t; A)y, \quad t \geq 0,
\]

where \(S_\alpha : [0, \infty) \to L(X)\) denotes the fractional sine function associated with \(C_\alpha\) defined as in (2.6).

It is important to note that this particular case have considered by Li in [17]. Furthermore, the same particular case has studied by Li et al. [18] with the first initial condition is zero, i.e., \(u(0) = 0\) and by Chen and M. Li in [5] with the second initial condition is zero, i.e., \(u'(0) = 0\).

**Remark 3.5** It should be noted that our results agree with the classical results when \(\alpha = 2\). To compare some arguments between fractional and classical senses, consider the following classical abstract Cauchy problem for a perturbed linear non-homogeneous second-order evolution equation on the whole real line \(\mathbb{R}\):

\[
\begin{align*}
u''(t) &= \left( A + B(t) \right) u(t) + f(t), \quad t \in \mathbb{R}, \\
u(0) &= x, \quad u'(0) = y.
\end{align*}
\]

(3.25)

Note that in the case of \(\alpha = 2\), \(T_\alpha (t; A)\) and \(S_\alpha (t; A)\) coincide with the strongly continuous sine function \(S(t; A)\), and \(C_\alpha (t; A)\) coincide with the strongly continuous cosine function \(C(t; A)\). Similarly, the perturbed families of bounded linear operators \(T_\alpha (t; A + B)\) and \(S_\alpha (t; A + B)\) coincide with the strongly continuous perturbed sine function \(S(t; A + B)\), and \(C_\alpha (t; A + B)\) coincide with the strongly continuous perturbed cosine function \(C(t; A + B)\). Furthermore, for \(\alpha = 2\), one and two parameter Mittag-Leffler type functions which we have used in the above cases are converting to the
Perturbation theory for fractional evolution equations... 609

hyperbolic cosine and sine functions, respectively:

\[ E_2(MK_t^2) = \sum_{k=0}^{\infty} \frac{M^k K_t^k t^{2k}}{(2k)!} = \cosh(t\sqrt{MK_t}), \quad t \in \mathbb{R}, \]

\[ tE_2,2(MK_t^2) = \sum_{k=0}^{\infty} \frac{M^k K_t^k t^{2k+1}}{(2k+1)!} = \frac{1}{\sqrt{MK_t}} \sinh(t\sqrt{MK_t}), \quad t \in \mathbb{R}. \]

Note that the linear homogeneous case of the abstract Cauchy problem (3.25) (in the case of \( f = 0 \)) was first considered by Lutz in [20] and our results coincide with these results whenever \( \alpha = 2 \). In addition, we have added a particular solution for the linear non-homogeneous case (3.25) as a consequence of the above results in the fractional-order sense (under \( \alpha = 2 \)).

**Theorem 3.4** Let \( A : D(A) \subseteq X \to X \) be infinitesimal generator of the strongly continuous cosine family of linear operators \( \{ C(t; A), t \in \mathbb{R} \} \). Let \( B(t) \) be twice strongly continuously differentiable function on \( [0, \infty) \) to \( L(X) \) and \( f(t) \) be strongly continuously differentiable function on \( [0, \infty) \) to \( X \). Then, there exists a unique 2-times continuously differentiable function \( u_1 : [0, \infty) \to X \) which is a particular solution of Eq. (3.25) with zero initial conditions \( u_1(0) = u_1'(0) = 0 \). This solution has a variation of constants formula as follows:

\[ u_1(t) = \sum_{n=0}^{\infty} w_n(t), \quad t \geq 0, \]

where for any \( t \geq 0 \) and \( n \in \mathbb{N} \):

\[ w_0(t) = \int_0^t S(t-s; A)f(s)ds, \]

\[ w_n(t) = \int_0^t S(t-s; A)B(s)w_{n-1}(s)ds. \]

In the special case, when \( B(t) \equiv B \in L(X) \), a particular solution of the classical abstract Cauchy problem (3.25) can also be put into a more suggestive form by including the following formula:

\[ u_1(t) = \int_0^t S(t-s; A+B)f(s)ds, \quad t \geq 0. \]

We end this section with some results on fractional analogues of uniformly continuous operator cosine and sine functions, which are operator-valued functions of Mittag-Leffler type generated by \( A + B \), where \( A, B \) are bounded linear operators in a Banach space \( X \).
It is known that if $C_\alpha(\cdot; A) : [0, \infty) \rightarrow \mathcal{L}(X)$ is a fractional uniformly continuous cosine function, then there is an $A \in \mathcal{L}(X)$ with

$$C_\alpha(t; A) = E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} A^k \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad t \geq 0.$$ 

Moreover, the corresponding fractional uniformly continuous sine and Riemann–Liouville families $S_\alpha(\cdot; A), T_\alpha(\cdot; A) : [0, \infty) \rightarrow \mathcal{L}(X)$ are defined by

$$S_\alpha(t; A) = t E_{\alpha,2}(At^\alpha) = \sum_{k=0}^{\infty} A^k \frac{t^{k\alpha+1}}{\Gamma(k\alpha + 2)}, \quad t \geq 0,$$

$$T_\alpha(t; A) = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) = \sum_{k=0}^{\infty} A^k \frac{t^{k\alpha+\alpha-1}}{\Gamma(k\alpha + \alpha)}; \quad t \geq 0.$$ 

In this particular case, we have $A = \Gamma(\alpha + 1) \lim_{t \to 0_+} \frac{E_\alpha(At^\alpha) - I}{t^\alpha}$ in the uniform operator topology and the domain $\mathcal{D}(A)$ coincides with the state space $X$, i.e., $\mathcal{D}(A) = X$. Therefore, in this case, fractional uniformly continuous families of linear bounded operators $C_\alpha(\cdot; A + B), S_\alpha(\cdot; A + B), T_\alpha(\cdot; A + B) : [0, \infty) \rightarrow \mathcal{L}(X)$ become the perturbation of a Mittag-Leffler type of operator-valued functions generated by $A + B$:

$$C_\alpha(t; A + B) = E_\alpha((A + B)t^\alpha) = \sum_{k=0}^{\infty} (A + B)^k \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad t \geq 0,$$

$$S_\alpha(t; A + B) = t E_{\alpha,2}((A + B)t^\alpha) = \sum_{k=0}^{\infty} (A + B)^k \frac{t^{k\alpha+1}}{\Gamma(k\alpha + 2)}, \quad t \geq 0,$$

$$T_\alpha(t; A + B) = t^{\alpha-1} E_{\alpha,\alpha}((A + B)t^\alpha) = \sum_{k=0}^{\infty} (A + B)^k \frac{t^{k\alpha+\alpha-1}}{\Gamma(k\alpha + \alpha)}, \quad t \geq 0.$$

It is well known that fractional cosine, sine and Riemann–Liouville families can also be characterised by the Laplace transform of their generators. It is interesting to note that depending on the commutativity condition of the linear bounded operators $A, B \in \mathcal{L}(X)$, we can derive the following elegant representation formulae for the perturbation of fractional uniformly continuous cosine and sine families by means of their inverse Laplace transform formulae, based on the results obtained in [13] and [21].

**Non-permutable case:** $(AB \neq BA)$. First, we obtain the closed-form representations for a perturbation of Mittag-Leffler type functions generated by $A + B$, where $A, B \in \mathcal{L}(X)$ are non-permutable bounded linear operators in a Banach space $X$.

In this regard, the following lemma will be helpful to derive a closed-form of the solution to (3.22).
Lemma 3.2  Let $\alpha \in (1, 2)$ and $\gamma \in \mathbb{R}$. For $A, B \in \mathcal{L}(X)$ satisfying $AB \neq BA$, we have for $m \in \mathbb{N}_0$:

$$\mathcal{L}^{-1}\left\{ \lambda^\gamma \left[ (\lambda^\alpha I - A)^{-1} B \right]^m (\lambda^\alpha I - A)^{-1} \right\}(t)$$

$$= \sum_{k=0}^{\infty} \frac{Q_{k,m}^{A,B}}{\Gamma(k\alpha + m\alpha + \alpha - \gamma)} t^{k\alpha + m\alpha + \alpha - \gamma - 1}, \quad t \geq 0,$$

where the operator family of bounded linear operators $\{Q_{k,m}^{A,B} : k, m \in \mathbb{N}_0\} \subset \mathcal{L}(X)$ is defined by

$$Q_{k,0}^{A,B} = A^k, \quad k \in \mathbb{N}_0, \quad Q_{k,m}^{A,B} = \sum_{l=0}^{k} A^{k-l} B Q_{l,m-1}^{A,B}, \quad k, m \in \mathbb{N},$$

$$Q_{0,m}^{A,B} = B^m, \quad m \in \mathbb{N}_0.$$

**Proof** This lemma is a particular case of Lemma 3.2 in [21] and [13] (for the case of free of delay, i.e., $\tau = 0$) with $\beta = 0$, which can be proved by mathematical induction principle. Therefore, the proof of this lemma is omitted here. $\square$

Then, we apply Laplace transform method and solve the Eq. (3.22) with linear operators $A, B \in \mathcal{L}(X)$. After solving the equation with respect to $U(\lambda)$, we obtain the following result:

$$U(\lambda) = \lambda^{\alpha-1} (\lambda^\alpha I - A - B)^{-1} x + \lambda^{\alpha-2} (\lambda^\alpha I - A - B)^{-1} y$$

$$+ (\lambda^\alpha I - A - B)^{-1} F(\lambda), \quad (3.26)$$

where $U(\lambda) = (\mathcal{L}u)(\lambda)$ and $F(\lambda) = (\mathcal{L}f)(\lambda)$.

Then, for non-permutable linear operators $A, B \in \mathcal{L}(X)$ and sufficiently large $\lambda$, such that $\| (\lambda^\alpha I - A)^{-1} B \| < 1$, the operator $(\lambda^\alpha I - A - B)$ is invertible and we derive that

$$(\lambda^\alpha I - A - B)^{-1} = \left[ (\lambda^\alpha I - A)(I - (\lambda^\alpha I - A)^{-1} B) \right]^{-1}$$

$$= (I - (\lambda^\alpha I - A)^{-1} B)^{-1} (\lambda^\alpha I - A)^{-1}$$

$$= \sum_{m=0}^{\infty} \left[ (\lambda^\alpha I - A)^{-1} B \right]^m (\lambda^\alpha I - A)^{-1}.$$
Taking inverse Laplace transform of (3.26), for \( x, y \in X \), we have:

\[
    u(t) = \mathcal{L}^{-1}\left\{ \sum_{m=0}^{\infty} \lambda^{\alpha-1} \left[ (\lambda^\alpha I - A)^{-1} B \right]^m (\lambda^\alpha I - A)^{-1} \right\}(t) x
    + \mathcal{L}^{-1}\left\{ \sum_{m=0}^{\infty} \lambda^{\alpha-2} \left[ (\lambda^\alpha I - A)^{-1} B \right]^m (\lambda^\alpha I - A)^{-1} \right\}(t) y
    + \mathcal{L}^{-1}\left\{ \sum_{m=0}^{\infty} \left[ (\lambda^\alpha I - A)^{-1} B \right]^m (\lambda^\alpha I - A)^{-1} F(\lambda) \right\}(t), \quad t \geq 0.
\]

According to the Lemma 3.2, we derive an explicit representation formula for the solution \( u(t) \in X \) for any \( t \geq 0 \) to (3.22) with linear operators \( A, B \in \mathcal{L}(X) \). To do this, we will use the well-known Cauchy product formula and the identity

\[
    \sum_{m=0}^{k} Q_{k-m,m}^{A,B} = (A + B)^k, \quad k \in \mathbb{N}_0
\]

in the following calculations:

\[
    u(t) = \sum_{n=0}^{\infty} \sum_{k+m=n}^{T} Q_{k,m}^{A,B} \frac{t^n \alpha}{\Gamma(n \alpha + 1)} x
    + \sum_{n=0}^{\infty} \sum_{k+m=n}^{T} Q_{k,m}^{A,B} \frac{t^n \alpha + 1}{\Gamma(n \alpha + 2)} y
    + \int_0^t \sum_{n=0}^{\infty} \sum_{k+m=n}^{T} Q_{k,m}^{A,B} \frac{(t-s)^n \alpha + \alpha - 1}{\Gamma(n \alpha + \alpha)} f(s) ds
    + \sum_{n=0}^{\infty} \sum_{k+m=n}^{T} Q_{k,m}^{A,B} \frac{t^n \alpha + \alpha + 2}{\Gamma(n \alpha + \alpha)} y
    + \int_0^t \sum_{k=0}^{\infty} \sum_{m=0}^{T} Q_{k,m}^{A,B} \frac{(t-s)^k \alpha + \alpha + 1}{\Gamma(k \alpha + m \alpha + \alpha)} f(s) ds
    + \sum_{k=0}^{\infty} \sum_{m=0}^{T} Q_{k,m}^{A,B} \frac{t^k \alpha + \alpha + 2}{\Gamma(k \alpha + m \alpha + \alpha + 1)} y
    + \int_0^t \sum_{k=0}^{T} \sum_{m=0}^{T} Q_{k,m}^{A,B} \frac{(t-s)^k \alpha + m \alpha + \alpha + 1}{\Gamma(k \alpha + m \alpha + m \alpha + \alpha + 1)} f(s) ds
    + \sum_{k=0}^{T} \sum_{m=0}^{T} Q_{k,m}^{A,B} \frac{t^k \alpha + m \alpha + m \alpha + 1}{\Gamma(k \alpha + m \alpha + m \alpha + 1)} x
    + \sum_{k=0}^{T} \sum_{m=0}^{T} Q_{k,m}^{A,B} \frac{t^k \alpha + m \alpha + m \alpha + 1}{\Gamma(k \alpha + m \alpha + m \alpha + 2)} y
    + \int_0^t \sum_{k=0}^{T} \sum_{m=0}^{T} Q_{k,m}^{A,B} \frac{(t-s)^k \alpha + m \alpha + m \alpha + 1}{\Gamma(k \alpha + m \alpha + m \alpha + 2)} f(s) ds
    + \sum_{k=0}^{T} \sum_{m=0}^{T} (A + B)^k \frac{t^k \alpha + m \alpha + m \alpha + 1}{\Gamma(k \alpha + m \alpha + m \alpha + 2)} x
    + \sum_{k=0}^{T} (A + B)^k \frac{t^k \alpha + m \alpha + m \alpha + 1}{\Gamma(k \alpha + m \alpha + m \alpha + 2)} y
\]
Perturbation theory for fractional evolution equations...

Theorem 3.5 Let $A, B \in \mathcal{L}(X)$ with non-zero commutator, i.e., $[A, B] = AB - BA \neq 0$. Then, the closed-form of a solution $u(t) \in X, t \geq 0$ of the linear Cauchy problem (3.22) with $x, y \in X$ can be expressed as

$$u(t) = E_\alpha((A + B)t^\alpha)x + tE_{\alpha,2}((A + B)t^\alpha)y + \int_0^t (t - s)^{\alpha - 1}E_{\alpha,\alpha}((A + B)(t - s)^\alpha)f(s)ds, \quad t \geq 0.$$ 

Thus, we proved the following theorem:

**Theorem 3.5** Let $A, B \in \mathcal{L}(X)$ with non-zero commutator, i.e., $[A, B] = AB - BA \neq 0$. Then, the closed-form of a solution $u(t) \in X, t \geq 0$ of the linear Cauchy problem (3.22) with $x, y \in X$ can be expressed as

$$u(t) = E_\alpha((A + B)t^\alpha)x + tE_{\alpha,2}((A + B)t^\alpha)y + \int_0^t (t - s)^{\alpha - 1}E_{\alpha,\alpha}((A + B)(t - s)^\alpha)f(s)ds, \quad t \geq 0.$$ 

**Remark 3.6** It should be noted that a family of bounded linear operators $\{Q_{k,m}^{A,B} : k, m \in \mathbb{N}_0\} \subset \mathcal{L}(X)$ is proposed to describe the properties of the solutions of multi-term fractional evolution equations and functional evolution equations in [13, 21]. Moreover, a linear operator family $\{Q_{k,m}^{A,B} : k, m \in \mathbb{N}_0\} \subset \mathcal{L}(X)$ satisfies the following corresponding properties (for more information, see [12, 13, 21]):

(i) For any (permutable and non-permutable) linear operators $A, B \in \mathcal{L}(X)$, we have:

$$Q_{k,m}^{A,B} = AQ_{k-1,m}^{A,B} + BQ_{k,m-1}^{A,B}, \quad k, m \in \mathbb{N}_0.$$ 

(ii) If $AB = BA$, then we have:

$$Q_{k,m}^{A,B} = \binom{k+m}{m}A^k B^m, \quad k, m \in \mathbb{N}_0.$$ 

(iii) For non-permutable linear operators $A, B \in \mathcal{L}(X)$, we have:

$$\sum_{m=0}^{k} Q_{k-m,m}^{A,B} = (A + B)^k, \quad k \in \mathbb{N}_0.$$
(iv) For permutable linear operators $A, B \in \mathcal{L}(X)$, we have:

$$
\sum_{m=0}^{k} \binom{k}{m} A^{k-m} B^m = (A+B)^k, \quad k \in \mathbb{N}_0.
$$

**Permutable case:** $(AB = BA)$. Secondly, we obtain the closed-form representations for a perturbation of Mittag-Leffler type functions generated by $A + B$, where $A, B \in \mathcal{L}(X)$ are permutable bounded linear operators in a Banach space $X$.

From this point of view, the following lemma will be helpful to derive a closed-form of the solution to the linear Cauchy problem (3.22).

**Lemma 3.3** Let $\alpha \in (1, 2)$ and $\gamma \in \mathbb{R}$. For $A, B \in \mathcal{L}(X)$ satisfying $AB = BA$, we have for $m \in \mathbb{N}_0$:

$$
\mathcal{L}^{-1}\left\{ \lambda^\gamma \left( \lambda^{\alpha} I - A \right)^{-(m+1)} B^m \right\}(t) = \sum_{k=0}^{\infty} \binom{k+m}{m} A^k B^m \frac{t^{k+m+\alpha-\gamma-1}}{\Gamma(k\alpha + m\alpha + \alpha - \gamma)}, \quad t \geq 0,
$$

where $\binom{k+m}{m} = \binom{k+m}{k \cdot m}$ is a binomial coefficient with $k, m \in \mathbb{N}_0$.

**Proof** This lemma is a particular case of Theorem 4.1 in [21] and [13] (for the case of free of delay, i.e, $\tau = 0$) with $\beta = 0$, which can be proved by mathematical induction principle. Therefore, the proof of this lemma is omitted here. \qed

Therefore, taking inverse Laplace transform of (3.26), for $x, y \in X$, we obtain directly:

$$
u(t) = \mathcal{L}^{-1}\left\{ \sum_{m=0}^{\infty} \lambda^{\alpha-1} \left( \lambda^{\alpha} I - A \right)^{-(m+1)} B^m \right\}(t) x
\quad + \mathcal{L}^{-1}\left\{ \sum_{m=0}^{\infty} \lambda^{\alpha-2} \left( \lambda^{\alpha} I - A \right)^{-(m+1)} B^m \right\}(t) y
\quad + \mathcal{L}^{-1}\left\{ \sum_{m=0}^{\infty} \left( \lambda^{\alpha} I - A \right)^{-(m+1)} B^m F(\lambda) \right\}(t), \quad t \geq 0.
$$

According to the Lemma 3.3, we derive an explicit representation formula for the solution $\nu(t) \in X$ for any $t \geq 0$ to (3.22) with permutable linear operators $A, B \in \mathcal{L}(X)$. To do this, we will use the well-known Cauchy product formula and the eminent binomial theorem $\sum_{m=0}^{k} \binom{k}{m} A^{k-m} B^m = (A + B)^k$ for permutable linear bounded operators.
Thus, we proved the following theorem:

\[
\begin{align*}
A, B \in \mathcal{L}(X) \text{ in the following calculations:} \\
\sum_{n=0}^{\infty} \sum_{k,m=0}^{\infty} \binom{k+m}{m} A^k B^m \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} x \\
+ \sum_{n=0}^{\infty} \sum_{k,m=0}^{\infty} \binom{k+m}{m} A^k B^m \frac{t^{n\alpha+1}}{\Gamma(n\alpha + 2)} y \\
+ \int_0^t \sum_{n=0}^{\infty} \sum_{k,m=0}^{\infty} \binom{k+m}{m} A^k B^m \frac{(t-s)^{n\alpha+\alpha-1}}{\Gamma(n\alpha + \alpha)} f(s) ds \\
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{k+m}{m} A^k B^m \frac{t^{k\alpha+ma}}{\Gamma(k\alpha + ma + 1)} x \\
+ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{k+m}{m} A^k B^m \frac{t^{k\alpha+ma+1}}{\Gamma(k\alpha + ma + 2)} y \\
+ \int_0^t \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{k+m}{m} A^k B^m \frac{(t-s)^{k\alpha+ma+\alpha-1}}{\Gamma(k\alpha + ma + \alpha)} f(s) ds \\
= \sum_{k=0}^{\infty} \binom{k}{m} A^k \frac{t^{(k-m)\alpha+ma}}{\Gamma((k-m)\alpha + ma + 1)} x \\
+ \sum_{k=0}^{\infty} \binom{k}{m} A^k \frac{t^{(k-m)\alpha+ma+1}}{\Gamma((k-m)\alpha + ma + 2)} y \\
+ \int_0^t \sum_{k=0}^{\infty} \binom{k}{m} A^k \frac{(t-s)^{(k-m)\alpha+ma+\alpha-1}}{\Gamma((k-m)\alpha + ma + \alpha)} f(s) ds \\
= E_{\alpha}((A + B)t^{\alpha}) x + t E_{\alpha,2}((A + B)t^{\alpha}) y \\
+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((A + B)(t-s)^{\alpha}) f(s) ds, \quad t \geq 0.
\end{align*}
\]
Theorem 3.6 Let \( A, B \in \mathcal{L}(X) \) with zero commutator, i.e., \([A, B] = AB - BA = 0\).
Then, the closed-form of a solution \( u(t) \in X, t \geq 0 \) of the linear Cauchy problem (3.22) with \( x, y \in X \) can be expressed as

\[
u(t) = E_\alpha((A + B)t^\alpha)x + tE\alpha,2((A + B)t^\alpha)y
\]

\[
= \sum_{n=0}^{\infty} \sum_{k+m=n, k,m \geq 0} \binom{k+m}{m} A^k B^m \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} x
\]

\[
+ \sum_{n=0}^{\infty} \sum_{k+m=n, k,m \geq 0} \binom{k+m}{m} A^k B^m \frac{t^{n\alpha+1}}{\Gamma(n\alpha + 2)} y
\]

\[
+ \int_0^t \sum_{n=0}^{\infty} \sum_{k+m=n, k,m \geq 0} \binom{k+m}{m} A^k B^m \frac{t^{n\alpha+\alpha-1}}{\Gamma(n\alpha + \alpha)} f(s)ds, \quad t \geq 0.
\]

Remark 3.7 The permutative case can also be obtained directly using the following property of the linear operator family \( \{Q_{k,m}^{A,B} : k, m \in \mathbb{N}_0\} \subset \mathcal{L}(X) \) which is valid for permutative bounded linear operators (in particular, matrices) \( A, B \in \mathcal{L}(X) \):

\[
Q_{k,m}^{A,B} = \binom{k+m}{m} A^k B^m, \quad k, m \in \mathbb{N}_0.
\]

Remark 3.8 Note that, depending on the commutativity condition for bounded linear operators, we derive the elegant explicit formulae for uniformly continuous perturbed fractional cosine and sine families using the Laplace transform of their generators. Furthermore, these results can be obtained using a recursive sequence of operator-valued functions as in [12] (for more information, see Theorem 3.3 and Theorem 3.4 in [12]).

Remark 3.9 The corresponding classical cases with non-permutable and permutative bounded linear operators \( A, B \in \mathcal{L}(X) \) can be derived from the above cases with \( \alpha = 2 \) as follows. Therefore, the closed-form of a solution \( u(t) \in X \) for any \( t \geq 0 \) of the classical linear Cauchy problem (3.25) with \( x, y \in X \) can be represented

- with non-zero commutator, i.e., \([A, B] = AB - BA \neq 0\), as follows:

\[
u(t) = C(t; A + B)x + S(t; A + B)y + \int_0^t S(t - s; A + B)f(s)ds
\]

\[
= \sum_{n=0}^{\infty} \sum_{k+m=n, k,m \geq 0} Q_{k,m}^{A,B} \frac{t^{2n}}{(2n)!} x + \sum_{n=0}^{\infty} \sum_{k+m=n, k,m \geq 0} Q_{k,m}^{A,B} \frac{t^{2n+1}}{(2n+1)!} y
\]
\[ + \int_0^t \sum_{n=0}^{\infty} \sum_{k+m=n \atop k,m \geq 0} Q_{k,m}^A B^m (t-s)^{2n+1} (2n+1)! f(s) ds, \quad t \geq 0, \]

• with zero commutator, i.e., \([A, B] = AB - BA = 0\), as below:

\[ u(t) = C(t; A + B)x + S(t; A + B)y + \int_0^t S(t-s; A + B) f(s) ds \]

\[ = \sum_{n=0}^{\infty} \sum_{k+m=n \atop k,m \geq 0} \binom{k+m}{m} A^k B^m \frac{t^{2n}}{(2n)!} x \]

\[ + \sum_{n=0}^{\infty} \sum_{k+m=n \atop k,m \geq 0} \binom{k+m}{m} A^k B^m \frac{t^{2n+1}}{(2n+1)!} y \]

\[ + \int_0^t \sum_{n=0}^{\infty} \sum_{k+m=n \atop k,m \geq 0} \binom{k+m}{m} A^k B^m (t-s)^{2n+1} (2n+1)! f(s) ds, \quad t \geq 0. \]

Moreover, in the special case, as given in [26], if \( X = \mathbb{R}, a, b > 0 \) and \( A : \mathbb{R} \to \mathbb{R}, B : \mathbb{R} \to \mathbb{R} \) is defined by \( Ax = ax, Bx = bx \) for each fixed \( x \in \mathbb{R} \), then \( C(t; A + B) = \cosh(t\sqrt{a+b}) \) and \( S(t; A + B) = \sinh(t\sqrt{a+b})/\sqrt{a+b} \). If \( A : \mathbb{R} \to \mathbb{R}, B : \mathbb{R} \to \mathbb{R} \) is defined as \( Ax = -ax, Bx = -bx \) for each fixed \( x \in \mathbb{R} \), then \( C(t; A + B) = \cos(t\sqrt{a+b}) \) and \( S(t; A + B) = \sin(t\sqrt{a+b})/\sqrt{a+b} \).

**References**

1. Ahmadova, A., Mahmudov, N.I., Nieto, J.J.: Exponential stability and stabilization of fractional stochastic degenerate evolution equations in a Hilbert space: subordination principle. Evol. Equ. Control Theory 11(6), 1997–2015 (2022)
2. Bazhlekov, E.: Fractional Evolution Equations in Banach Spaces. Ph.D. Thesis, Eindhoven University of Technology (2001)
3. Bazhlekov, E.: Perturbation properties for abstract evolution equations of fractional order. Fract. Calc. Appl. Anal. 2(4), 359–366 (1999)
4. Bochenek, J.: An abstract nonlinear second order differential equation. Ann. Pol. Math. 54, 155–166 (1991)
5. Chen, C., Li, M.: On fractional resolvent operator functions. Semigroup Forum 80, 121–142 (2010)
6. Engel, K.-J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, vol. 194. Springer, New York (2000)
7. Fattorini, H.O.: Ordinary differential equations in linear topological spaces, I. J. Differ. Equ. 5, 72–105 (1968)
8. Fattorini, H.O.: Ordinary differential equations in linear topological spaces, II. J. Differ. Equ. 6, 50–70 (1969)
9. Fitzgibbon, W.E.: Global existence and boundedness of solutions to the extensible beam equation. SIAM J. Math. Anal. 13, 739–745 (1982)
10. Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V.: Mittag-Leffler Functions. Related Topics and Applications. Springer, Berlin (2014)
11. Henríquez, H.R., Mesquita, J.G., Poza, J.C.: Existence of solutions of the abstract Cauchy problem of fractional order. J. Funct. Anal. 281, 109028 (2021)
12. Huseynov, I.T., Ahmadova, A., Mahmudov, N.I.: Perturbation properties of fractional strongly continuous cosine and sine family operators. Electron. Res. Arch. 30(8), 2911–2940 (2022)
13. Huseynov, I.T., Ahmadova, A., Mahmudov, N.I.: On a study of Sobolev type fractional functional evolution equations. Math. Methods Appl. Sci. 45(9), 5002–5042 (2022)
14. Huseynov, I.T., Mahmudov, N.I.: Perturbation theory and linear partial differential equations with delay. arXiv preprint at arXiv:2110.12515v2
15. Kato, T.: Perturbation Theory for Linear Operators. Springer, Berlin (1966)
16. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
17. Li, K.: Fractional order semilinear Volterra integrodifferential equations in Banach spaces. Topol. Methods Nonlinear Anal. 47(2), 439–455 (2016)
18. Li, K., Peng, J., Jia, J.: Cauchy problems for fractional differential equations with Riemann–Liouville derivatives. J. Funct. Anal. 263, 476–510 (2012)
19. Lin, Y.: Time-dependent perturbation theory for abstract evolution equations of second order. Studia Math. 130, 263–274 (1998)
20. Lutz, D.: On bounded time-dependent perturbations of operator cosine functions. Aequ. Math. 23, 197–203 (1981)
21. Mahmudov, N.I., Ahmadova, A., Huseynov, I.T.: A new technique for solving Sobolev type fractional multi-order evolution equations. Comput. Appl. Math. 41, 71 (2022)
22. Mahmudov, N.I., McKibben, M.A.: Abstract second-order damped McKean–Vlasov stochastic evolution equations. Stoch. Anal. Appl. 242, 303–328 (2006)
23. Phillips, R.S.: Perturbation theory for semi-groups of linear operators. Trans. Am. Math. Soc. 74, 199–221 (1954)
24. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
25. Serizawa, H., Watanabe, M.: Time-dependent perturbation for cosine families in Banach spaces. Houst. J. Math. 12(4), 579–586 (1986)
26. Travis, C.C., Webb, G.F.: Cosine families and abstract nonlinear second order differential equations. Acta Math. Acad. Sci. Hung. 32(3–4), 75–96 (1978)
27. Travis, C.C., Webb, G.F.: Compactness, regularity and uniform continuity properties of strongly continuous cosine families. Houst. J. Math. 3, 555–567 (1977)
28. Travis, C.C., Webb, G.F.: Perturbation of strongly continuous cosine family generators. Colloq. Math. 45(2), 277–285 (1981)
29. Vugdalić, R., Halilović, S.: On general cosine operator function in Banach space. Adv. Math. Sci. J. 6(1), 23–27 (2017)
30. Woinowsky-Krieger, S.: The effect of an axial force on the vibration of hinged bars. ASME J. Appl. Mech. 17, 35–36 (1950)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.