Decoding the hologram: Scalar fields interacting with gravity

Daniel Kabat, Gilad Lifschytz

1Department of Physics and Astronomy
Lehman College, City University of New York, Bronx NY 10468, USA

2Department of Mathematics and Physics
University of Haifa at Oranim, Kiryat Tivon 36006, Israel

We construct smeared CFT operators which represent a scalar field in AdS interacting with gravity. The guiding principle is micro-causality: scalar fields should commute with themselves at spacelike separation. To $O(1/N)$ we show that a correct and convenient criterion for constructing the appropriate CFT operators is to demand micro-causality in a three-point function with a boundary Weyl tensor and another boundary scalar. The resulting bulk observables transform in the correct way under AdS isometries and commute with boundary scalar operators at spacelike separation, even in the presence of metric perturbations.
1 Introduction

A long-standing problem in formulating any theory of quantum gravity is to specify the observables of the theory. For a review of various approaches see [1]. The problem becomes particularly acute in the context of the AdS/CFT correspondence [2], where correlation functions of local operators in the boundary CFT in principle provide a complete set of observables. The challenge is then deciding how local or semi-local observables in the bulk can be expressed in terms of the CFT.

The purpose of the present paper is to construct a set of smeared or non-local observables in the CFT which can be used to represent a scalar field in the bulk interacting with gravity. The approach we take builds on a long series of developments. To summarize the history, CFT operators which represent free fields in the bulk were constructed in [3, 4, 5, 6, 7, 8, 9] for scalar fields and in [10, 11] for fields with spin. These constructions proceeded essentially by solving free wave equations in AdS. The resulting observables can be used to describe bulk physics in the large $N$ limit of the CFT. The corrections needed to account for bulk interactions, corresponding to the $1/N$ expansion of the CFT, were constructed in [12] and further developed in [13]. In particular [12] advocated an approach based on bulk micro-causality and argued that the correct bulk observables could be built up order-by-order in the $1/N$ expansion, by demanding that they obey appropriate commutation relations at spacelike separation. This is the approach we will take in the present paper. Thus our work builds on the free-field representations of [11] and takes the approach to including interactions developed in [12]. Our analysis parallels the study of charged scalar fields interacting with gauge fields carried out in [14].

Since this work culminates a long series of developments, the rest of the introduction is devoted to a survey of the general approach, including the issues it addresses and the insights it has to offer. Readers familiar with the approach who wish to get to the technical details may skip to section 2.

In any approach to quantum gravity one would like to understand how local or semi-local observables arise from the formalism of the underlying theory. One can of course choose a covariant gauge where all fields look local
But in a covariant gauge the operators, although local, are not necessarily physical. On the other hand in the canonical quantization of general relativity coupled to matter via the ADM formalism one can show that in many nice gauges matter field operators obey the usual equal-time commutators with themselves and with some (but not all) of the gravity degrees of freedom. It’s easy to see that matter field operators cannot commute at spacelike separation with all of the gravity degrees of freedom. For example if there is a boundary at spatial infinity, then the Hamiltonian is a boundary term, but its commutator with matter fields should still generate a time translation. This means that matter fields are not in fact local operators: rather they’re non-local operators in the bulk. This can also be seen from the fact that physical operators must commute with the constraints. The constraints in quantum gravity generate diffeomorphisms. No locally defined quantity can commute with the constraints, since there is no local way of defining the position of the operator. However if there is a boundary at infinity then the operators and coordinates on the boundary are diffeomorphism invariant, since as usual the constraints are implemented with fall-off conditions at infinity. One can use these boundary coordinates to define an invariant position in the bulk, by following geodesics in from the boundary for some given proper length.

If there is a boundary, then the Hamiltonian of the bulk theory is non-zero and one can talk about unitarity. At the boundary there are always local observables, and since the Hamiltonian is a boundary operator, one can ask if the theory is unitary for the boundary operators by themselves or if one needs the bulk operators as well. The AdS/CFT conjecture states that the boundary operators by themselves form a unitary theory. But this can only work if the boundary operators are the only operators in the theory. This means that the bulk operators, which are usually thought of as describing the actual spacetime physics, can be expressed in terms of the boundary operators. This is a drastic (holographic) reduction in the number of degrees of freedom, but it seems to be forced on us by unitarity.

The program of building bulk operators amounts to constructing the map between the non-local bulk observables and the boundary observables. It is

\[\text{1}^{1}\text{At best one could call them quasi-local. The matter fields we will construct can be thought of as local operators in the bulk attached to Wilson lines that run off to infinity.}\]

\[\text{2}^{2}\text{For an argument that this must be the case in quantum gravity see [18].}\]
important to stress that all information and degrees of freedom live at the boundary, and that the notion of the bulk spacetime is emergent. There is no need to introduce bulk operators for the theory to be well-defined or complete. In fact there is no “good” or “bad” definition of bulk operators, until we have specified what properties we wish these operators to have. The properties one wants the bulk operators to have are tied to what we think the notion of a spacetime means, and this we can only define through our experience with semiclassical physics. However one cannot expect that a semiclassical spacetime will emerge for any state of the boundary theory. Indeed a necessary requirement is that it obeys $1/N$ factorization. As such the most reasonable starting point to define bulk operators is to look for a map in situations where we know a semiclassical spacetime emerges$^3$ such as a CFT with large central charge in its vacuum state, which is dual to an empty AdS space. A minimal requirement on the bulk operators is that they will obey micro-causality with respect to the bulk causal structure. One might also hope that they will describe bulk fields which transform correctly under AdS isometries, and that they will have the appropriate local interactions expected for fields in the bulk. It turns out the above three requirements are tied together in an interesting way.

Assuming the boundary theory is a CFT, with a central charge that we will label by $N$, then in the large $N$ limit CFT correlators factorize into a product of two-point functions. In this case the boundary-to-bulk map was constructed in the early days of AdS/CFT. The map can be written as a sum over modes in momentum space $^3$ $^4$, or as a smearing function over the entire boundary $^6$, or just the part of the boundary which is space-like to the bulk point $^7$ $^8$, or over a compact region on the complexified boundary $^9$. The key ingredient is that in the large central charge limit, commutators of operators are $c$-numbers, and after normalization the operator Fourier components can serve as creation and annihilation operators that describe free fields in the bulk. However conventional bulk perturbation theory, which is based on creation and annihilation operators, cannot simply be adopted to take interactions into account. The problem is that in $1/N$ perturbation theory the operators will no longer commute to a $c$-number, so the identification of creation and annihilation operators with operator Fourier components breaks down. Indeed if one tries to use the zeroth-order

---

$^3$For some speculation on the definition of bulk operators in a more general framework see $^9$ and section 6 of $^{11}$. 

---
smearing function to compute a bulk three-point function the result, while AdS covariant, does not respect bulk micro-causality. It turns out there are two equivalent approaches to constructing bulk operators that respect micro-causality [12]. One approach, which is simple to implement if one knows the bulk action, is to solve the bulk equations of motion in perturbation theory. This is most conveniently done with the help of a spacelike Greens function [7, 8, 13]. Another approach, more intrinsic to the CFT, is to correct the zeroth-order definition of the bulk operator by adding appropriately-smeared higher-dimension operators. One fixes the coefficients in front of these higher-dimension operators by requiring bulk micro-causality in all three-point functions. This procedure was carried out for scalars in [12, 14], where it was found that for bulk scalar fields one needs to add an infinite tower of smeared higher-dimension primary scalar operators. This construction is possible in 1/N perturbation theory, where the needed primary scalars are constructed as multi-trace operators in the CFT. The two approaches to constructing bulk operators are equivalent. In both approaches what one is constructing is the bulk Heisenberg picture field operator.

It is important to realize that the required tower of higher-dimension primary scalars is only guaranteed to exist in 1/N perturbation theory. Most likely it does not exist in a unitary CFT with finite central charge. Thus even at this level having finite central charge precludes micro-causality of bulk operators. Note that the breakdown of locality associated with a finite Planck length in the bulk does not manifest itself in correlators as the absence of a singularity at lightlike-separated or coincident points, but rather through a breakdown of micro-causality. This may seem strange, since we have not yet incorporated any gravitational degrees of freedom. But note that this breakdown comes from demanding that the boundary operators are described by a consistent unitary CFT. This is certainly not the case for scalar fields propagating on a fixed background, where the boundary data does not evolve in a unitary way by itself. Another point we wish to stress is that in this construction, the operators needed to correct the zeroth order definition of a bulk field are smeared primary scalars in the CFT. It may seem obvious that one should smear primary scalars, but as we will see, this is in fact a consequence of demanding micro-causality. Moreover, smeared primary scalars transform like a local scalar field in the bulk under AdS isometries. So we see that both the transformation properties in the bulk (the fact that the bulk operator transforms like a scalar field), and the emergence of local
bulk interactions (via the addition of multi-trace operators in the CFT), are a consequence of demanding bulk micro-causality.

The construction for interacting scalar fields was extended to the case of charged scalar fields interacting with a bulk gauge field in [14]. Since the bulk theory has a gauge redundancy, the first step is fixing a gauge. A natural choice is holographic gauge, which sets the radial component of the gauge field to zero. Solving the bulk equations of motion in holographic gauge is relatively straightforward. However the micro-causality conditions in holographic gauge are somewhat complicated. Canonical quantization of this system shows that, although matter fields obey canonical commutators with themselves, they have non-zero (and non-local) commutators with certain components of the gauge field. These matter – gauge commutators do not vanish, even at spacelike separation. These non-trivial commutators make the CFT approach to constructing bulk operators more subtle, since one cannot simply demand that operators commute at spacelike separation in the bulk.

From the CFT perspective, the obstruction to building commuting observables can be traced back to the Ward identities associated with the conserved current. Consider, for example, the CFT three-point function \( \langle O \bar{O} j_\mu \rangle \) of a charged scalar primary \( O \) and its complex conjugate \( \bar{O} \) with a conserved current \( j_\mu \). If one smears \( O \) into the bulk using the zeroth-order smearing function, the resulting mixed bulk – boundary correlator \( \langle \phi \bar{O} j_\mu \rangle \) violates micro-causality. A straightforward attempt to restore micro-causality, by adding smeared higher-dimension primary operators to the definition of the bulk field \( \phi \), goes nowhere: due to the Ward identities of the CFT, the addition of such smeared higher-dimension primaries cannot change the correlator.

Fortunately one can proceed indirectly, and demand micro-causality in a three-point function \( \langle \phi \bar{O} F_{\mu\nu} \rangle \) involving the boundary field strength \( F = dj \). Micro-causality can be restored by adding smeared operators to the definition of \( \phi \) which are higher-dimension and multi-trace but are not primary. Since

---

4Similar non-local commutators arise in electrodynamics in Coulomb gauge. They are required by the Gauss constraint, which allows the total charge to be expressed as a surface integral at infinity.

5It is the absence of the divergence of the current as an operator, i.e. \( \partial_\mu j^\mu = 0 \), which prevents one from building a primary scalar. With non-conserved currents there is no
the operators we add are not smeared primaries, we obtain a bulk operator which does not transform like a scalar under AdS isometries. Rather, in the example worked out in [14], the bulk operator transforms exactly like a charged scalar field in the bulk attached to a Wilson line that runs off to infinity. This shows that by demanding an appropriate statement of micro-causality in the bulk, one automatically obtains operators with the correct transformation properties under bulk isometries. We find this connection rather nice.

The purpose of the present paper is to generalize this construction to the case of scalar fields interacting with gravity. In section 2 we start with a canonical treatment of a scalar field coupled to gravity in the ADM formalism. We do this because the commutation relations are gauge-dependent, and although ADM showed (among many other things) that matter Poisson brackets often have the standard form, holographic gauge was not among the class of gauges they considered. So we need to check if matter fields commute at spacelike separation in holographic gauge, which turns out to be true due to conservation of the matter stress tensor. In section 3 we turn to the CFT construction and revisit the $U(1)$ vector case, extending some of the results of [14]. In particular we show that matter operators commute with each other at spacelike separation, even in the presence of a gauge field. In section 4 we consider scalar fields interacting with gravity. The CFT construction is carried out for dimensions $d \geq 4$, where the analysis is facilitated by the existence of a boundary Weyl tensor (constructed from the stress tensor of the CFT) with four distinct indices. We show how to construct a bulk scalar field which obeys micro-causality inside a three-point correlation function with a boundary scalar and a boundary Weyl tensor. As a result the bulk scalar commutes at spacelike separation with other matter fields, but not with all components of the metric. This is expected, given that the Hamiltonian can be written as a surface integral. Again the construction involves adding smeared operators which are higher-dimension and multi-trace but are not primary. Somewhat remarkably, just as in the gauge field case, the transformation of such operators under AdS isometries exactly matches the transformation expected for a bulk scalar field in holographic gauge. Some background computations are given in the appendices.

obstruction to constructing a primary scalar [11].
2 Bulk perspective

Our main interest in this paper will be to take a boundary perspective. We want to construct operators in the CFT which can mimic local observables in the bulk. Our guiding principle in the construction will be to enforce an appropriate notion of bulk micro-causality. That is, we propose that the algebra of local operators in the CFT can be lifted to an algebra of local operators in the bulk, roughly speaking by requiring that bulk operators commute at spacelike separation. In pursuing this program there are two issues which confront us.

- The bulk fields have gauge redundancy, associated with diffeomorphisms and other gauge symmetries that act trivially on the CFT. To construct bulk observables these gauge symmetries must somehow be fixed. What choice of gauge should we make?

- In a gauge theory the commutators are gauge-dependent and can be non-zero at spacelike separation. So we need a refined statement of bulk micro-causality, appropriate to our choice of gauge. What are the correct commutation relations to impose on our bulk observables?

It’s useful to gain some insight into these issues before jumping into the CFT construction. So in this section we take a bulk perspective, and study commutators and gauge fixing for a theory of gravity coupled to matter in AdS.

Of course gravity plus matter has been studied before, in particular in [17] and section 6.2 of [16]. In these works it was indeed found that matter fields continue to obey canonical brackets even when coupled to gravity. However these references adopted a particular choice of gauge, transverse – traceless gauge, which simplifies the canonical analysis but is not so natural from the point of view of AdS/CFT. So we will revisit these issues, in a gauge which is designed to make the bulk – boundary correspondence as simple as possible.

---

6From a technical point of view, compared to the work of ADM, the main additional complication we face is that we will eventually impose a condition which fixes one of the components of the shift vector.
As a prototype example we consider Einstein gravity coupled to a scalar field in an asymptotically anti-de Sitter space with dimension $D = d + 1 > 3$.

$$S = \int d^{d+1}x \sqrt{-G} \left( -\frac{1}{2} G^{MN} \partial_M \phi \partial_N \phi + \frac{1}{2\kappa^2} (R - \Lambda) \right)$$  \hspace{1cm} (1)

Notation: $x^M = (x^\mu, z)$ are bulk spacetime coordinates where $M = 0, \ldots, d$. $z$ is a radial coordinate in AdS, with $z \to 0$ at the boundary, and $x^\mu, \mu = 0, \ldots, d-1$ are coordinates in the CFT. We’ll denote bulk spatial coordinates including the radial direction by $x^i, i = 1, \ldots, d$, and bulk spatial coordinates excluding $z$ by $x^\hat{i}, \hat{i} = 1, \ldots, d-1$. The cosmological constant is related to the anti-de Sitter radius by $\Lambda = -d(d-1)/R^2$. We’ll set $2\kappa^2 = 1$ from now on.

First let’s discuss our procedure for fixing bulk diffeomorphisms. Similar issues arise for gauge symmetries, as discussed in [10, 11, 14] and appendix A. In principle in the bulk we are free to make whatever choice of gauge we like. But given the existence of a boundary CFT, there is a preferred choice of coordinates in the bulk which makes the bulk – boundary correspondence as simple as possible. The following construction was first used to discuss bulk observables in AdS/CFT by Heemskerk [10]. The boundary has a preferred set of Minkowski coordinates $x^\mu$. We can extend these coordinates into the bulk by sending in geodesics perpendicular to the boundary. Points along a given geodesic are then labeled by $x^\mu$ and the proper distance $s$ measured from the boundary. This proper distance diverges, of course, so to make it well-defined we introduce an IR cutoff in AdS and place the boundary at position $z' \to 0$. Rather than work with proper distance directly, we define our radial coordinate $z$ in AdS by setting

$$z = z' \exp(s/R)$$  \hspace{1cm} (2)

Here $R$ is the AdS radius and we have in mind that the regulator can be removed by taking the limit $z' \to 0, s \to \infty$ with $z$ fixed.

From the geodesic equation $\frac{d^2x^A}{d\tau^2} + \Gamma^A_{BC} \frac{dx^B}{d\tau} \frac{dx^C}{d\tau} = 0$ the requirement that $x^\mu$ remain constant along these radially-directed geodesics amounts to a restriction on the bulk Christoffel connection, namely that

$$\Gamma^\mu_{zz} = 0$$  \hspace{1cm} (3)
This condition can be solved by putting the bulk metric in the form

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{z^2} dz^2 \]  

(4)

These are known as Fefferman - Graham coordinates \[20\], and were also used in \[10, 11\] to make the bulk – boundary dictionary as simple as possible. We will refer to this construction as fixing holographic gauge.

The next step is to understand how we impose holographic gauge on the theory \(\Pi\), and what the resulting commutation relations are\[7\]. To this end we consider the theory in a general gauge and introduce an ADM decomposition of the metric \[16\]

\[ G_{AB} = \begin{pmatrix} -N^2 + g_{ij} N^i N^j & g_{ij} N^j \\ g_{ij} N^j & g_{ij} \end{pmatrix} \]  

(5)

\[ G^{AB} = \begin{pmatrix} -1/N^2 & N^i/N^2 \\ N^i/N^2 & g^{ij} - N^i N^j/N^2 \end{pmatrix} \]  

(6)

with \(\sqrt{-G} = N \sqrt{g}\). Spatial indices \(i, j\) are raised and lowered with the metric \(g_{ij}\). Following ADM the theory can be put in first-order form, with Lagrangian

\[ \mathcal{L} = \pi^{ij} \partial_t g_{ij} + \pi_\phi \partial_t \phi - N \mathcal{H} - N^i \mathcal{P}_i \]  

(7)

The lapse \(N\) and shift \(N^i\) are Lagrange multiplier fields which enforce the Hamiltonian and momentum constraints \(\mathcal{H} = \mathcal{P}_i = 0\).

\[ \mathcal{H} = \frac{1}{\sqrt{g}} \left( \pi^{ij} \pi_{ij} - \frac{1}{d-1} \pi^2 \right) - \sqrt{g} R(g) + \frac{1}{2} \sqrt{g} \pi_\phi \partial_t \phi \partial_t \phi \\ = \frac{1}{\sqrt{g}} \left( \pi^{ij} \pi_{ij} - \frac{1}{d-1} \pi^2 \right) - \sqrt{g} R(g) + N^2 \sqrt{g} \Gamma^{00} \]  

(8)

\[ \mathcal{P}_i = -2 \nabla_j \pi^{ij} + \pi_\phi \partial_t \phi \\ = -2 \nabla_j \pi^j - N \sqrt{g} \Gamma^{0i} \]  

(9)

\[ = -2 \nabla_j \pi^j - N \sqrt{g} \Gamma^{0i} \]  

(10)

\[ ^7\text{Although we refer to commutation relations, our analysis will be purely classical, i.e. at the level of Poisson brackets. Also we’ll stop short of determining the full set of commutators, since our goal is really just to show that matter fields have the usual equal-time commutators even in the presence of gravity.} \]
Here $\nabla_i$ is the spatial covariant derivative (compatible with the metric $g_{ij}$), $R(g)$ is the curvature scalar constructed from $g_{ij}$, and $\pi \equiv g^{ij} \pi_{ij}$. We’ve written the constraints both explicitly for a real scalar field, and more generally in terms of the matter stress tensor $T^{AB}$. Along with these constraints we impose four conditions to fix holographic gauge. The appropriate gauge conditions turn out to be

\begin{align}
  g_{iz} & = 0 \quad i = 1, \ldots, d - 1 \tag{11} \\
  g_{zz} & = \frac{R^2}{z^2} \tag{12} \\
  \pi_{zz} & = \frac{1}{d-1} g_{zz} \pi \tag{13}
\end{align}

The conditions (11), (12) aren’t surprising; they put the spatial metric $g_{ij}$ in the Fefferman-Graham form (4). The condition (13) is less obvious. It can be given several interpretations.

1. In the ADM decomposition we have\(^8\)

   \[ \Gamma_{ij}^0 = \frac{1}{N \sqrt{g}} \left( \pi_{ij} - \frac{1}{d-1} g_{ij} \pi \right) \tag{14} \]

   Thus the condition (13) sets $\Gamma_{zz}^0 = 0$, which is the time component of the condition (3) on the bulk Christoffel connection.

2. In the ADM formalism the extrinsic curvature of the equal-time hypersurfaces is given by\(^9\)

   \[ K_{ij} = -\frac{1}{\sqrt{g}} \left( \pi_{ij} - \frac{1}{d-1} g_{ij} \pi \right) \tag{15} \]

   Thus the condition (13) sets $K_{zz} = 0$.

3. As we will see below, it leads to the condition $N_z = 0$ or equivalently $G_{0z} = 0$. It therefore puts the full metric (not just the spatial components) in the Fefferman-Graham form (4).

---

\(^8\)See equation (3.9b) in [16].

\(^9\)See section 3.3 and footnote 3 in [16].
At this stage it’s useful to count degrees of freedom. The metric $g_{ij}$ and
its conjugate momentum $\pi^{ij}$ each have $d(d + 1)/2$ independent components,
for a total of $d(d + 1)$ phase space degrees of freedom. The Hamiltonian and
momentum constraints eliminate $d + 1$ degrees of freedom, while the gauge
conditions eliminate another $d + 1$. This leaves $d(d - 1) - 2$ phase space
degrees of freedom, or $\frac{d(d-1)}{2} - 1$ configuration space degrees of freedom, as
expected for a massless spin-2 particle.\textsuperscript{10}

However we still have to fix the Lagrange multiplier fields $N, N^i$. We
do this by using equations of motion and requiring that the gauge-fixing
conditions (11) – (13) are preserved under time evolution.\textsuperscript{11} For example the
equation of motion for the metric is\textsuperscript{12}

$$ \partial_t g_{ij} = 2N \sqrt{g} \left( \pi_{ij} - \frac{1}{d-1} g_{ij} \pi \right) + \nabla_i N_j + \nabla_j N_i $$ (16)

This means

$$ \partial_t g_{zz} = 2N \sqrt{g} \left( \pi_{zz} - \frac{1}{d-1} g_{zz} \pi \right) + 2\nabla_z N_z $$ (17)

The left hand side vanishes by (12), while the quantity in parenthesis vanishes
by (13), so with suitable boundary conditions as $z \to 0$ we are forced to set

$$ N_z = 0. $$ (18)

As promised, this condition on the shift puts the full spacetime metric in
Fefferman - Graham form. Then the $g_{iz}$ equation of motion reduces to

$$ \partial_t g_{iz} = \frac{2N}{\sqrt{g}} \pi_{iz} + \nabla_z N_i $$ (19)

The left hand side vanishes by (11), so this becomes an equation we can
solve to determine $N_i$. Finally we need to impose the condition that (13) is
preserved under time evolution.

$$ \partial_t \left( \pi_{zz} - \frac{1}{d-1} g_{zz} \pi \right) = 0 $$ (20)

\textsuperscript{10}A symmetric traceless tensor under the $SO(d - 1)$ little group.
\textsuperscript{11}For the analogous procedure in transverse - traceless gravity see section 4.5 of [16].
The analogous procedure for Maxwell theory in axial gauge can be found on p. 80 of [21].
\textsuperscript{12}See equation (3.15a) in [16]. Although ADM consider pure gravity, this equation of
motion should remain the same in the presence of matter.
This is a fairly complicated equation, but in principle it determines \( N \) in terms of other degrees of freedom.

Now let’s see about imposing our constraints and gauge-fixing conditions on the theory (7). To do this we break the metric up into its components \( g_{ij}, g_{iz}, \) and \( g_{zz} \). Also it’s useful to decompose \( g_{ij} \) into a conformal factor and a metric with unit determinant. We do this by setting\(^{13}\)

\[
\begin{align*}
g_{ij} &= e^\sigma \tilde{g}_{ij} \quad \text{with } \det \tilde{g}_{ij} = 1 \\
\pi^{ij} &= e^{-\sigma} \left( \tilde{\pi}^{ij} + \frac{1}{d-1} \tilde{g}^{ij} \pi_\sigma \right) \quad \text{with } \tilde{g}_{ij} \tilde{\pi}^{ij} = 0
\end{align*}
\]

The condition \( \det \tilde{g} = 1 \) implies that \( \tilde{\pi}^{ij} \partial_t \tilde{g}_{ij} = 0 \). So this is an orthogonal decomposition, and the Lagrangian (7) becomes\(^{15}\)

\[
\mathcal{L} = \pi_\sigma \partial_t \sigma + \tilde{\pi}^{ij} \partial_t \tilde{g}_{ij} + 2 \pi^{iz} \partial_t g_{iz} + \pi^{zz} \partial_t g_{zz} + \pi_\phi \partial_t \phi - N \mathcal{H} - N^i \mathcal{P}_i
\]

Our goal is to solve the constraints and gauge-fixing conditions for \( \sigma, g_{iz}, g_{zz} \) and their conjugate momenta \( \pi_\sigma, \tilde{\pi}^{iz}, \pi^{zz} \). This will leave a theory in which \( \tilde{g}_{ij}, \tilde{\pi}^{ij}, \phi, \pi_\phi \) are the dynamical variables. Now to work.

1. The conditions (11), (12) can be directly imposed on the metric, they simply set \( g_{iz} = 0 \) and \( g_{zz} = R^2/z^2 \). The Lagrangian reduces to

\[
\mathcal{L} = \pi_\sigma \partial_t \sigma + \tilde{\pi}^{ij} \partial_t \tilde{g}_{ij} + \pi_\phi \partial_t \phi - N \mathcal{H} - N^i \mathcal{P}_i
\]

Note that as a consequence of our gauge choice \( \pi^{iz} \) and \( \pi^{zz} \) drop out of the canonical \( p_q \) part of the Lagrangian.

2. The condition (13) can be rewritten as

\[
\pi^{zz} = \frac{1}{d-2} g^{zz} \pi_\sigma
\]

This eliminates \( \pi^{zz} \) as a dynamical variable, since it’s proportional to the momentum conjugate to \( \sigma \).

\(^{13}\)For \( \partial_t \pi_{ij} \) in pure gravity see (3.15b) in [16], but note that this equation is modified in the presence of matter.

\(^{14}\)This decomposition was introduced by Dirac [22]. See [21] p. 122.

\(^{15}\)For the corresponding Poisson brackets see (7.98) – (7.101) in [21].
3. The momentum constraint \( P^i = 0 \) sets
\[
-2\nabla_j \pi^j \pi^i - 2\nabla_z \pi^i \pi^z + \pi_\phi \partial_t \phi = 0 \tag{26}
\]
which can be solved to determine \( \pi^{iz} \).

4. Likewise the constraint \( P_z = 0 \), which now reads
\[
-2\nabla_i \pi^i - \frac{2}{d-2} \nabla_z \pi_\sigma + \pi_\phi \partial_z \phi = 0 \tag{27}
\]
can be solved to determine \( \pi_\sigma \).

5. Finally the Hamiltonian constraint (8), namely
\[
\frac{1}{\sqrt{g}} \left( \pi^{ij} \pi_{ij} - \frac{1}{d-1} \pi^2 \right) - \sqrt{g} R(g) + \frac{1}{2} \sqrt{g} \pi_\phi^2 + \frac{1}{2} \sqrt{g} g^{ij} \partial_i \phi \partial_j \phi = 0 \tag{28}
\]
can in principle be solved to determine \( \sigma \).

This completes our goal of eliminating all constraints and reducing the theory to the independent dynamical variables \( \tilde{g}^{ij}, \tilde{\pi}^{ij}, \phi, \pi_\phi \). However we still need to determine the commutators for these physical degrees of freedom. To do this in principle we should take the solutions for \( \pi_\sigma \) and \( \sigma \), plug into the Lagrangian (24), and read off the brackets from the resulting symplectic form [23]. This is a complicated procedure, which we will not attempt to carry out in detail. However we do want to address the question of whether matter fields still obey standard brackets when coupled to gravity – in particular whether matter fields still commute with each other at spacelike separation – since this will form the basis of our CFT construction of local bulk observables.

The issue we face is the following. Suppose we solve the system of constraints to determine \( \sigma \). We will get a (spatially non-local) expression of the form
\[
\sigma = \sigma[\phi, \pi_\phi, \tilde{g}_{ij}, \tilde{\pi}^{ij}] \tag{29}
\]
When we take the time derivative of (29) and plug the resulting expression for \( \partial_t \sigma \) into (24), it would seem that time derivatives of \( \phi \) and \( \pi_\phi \) could appear, which would modify the matter brackets.

\[\text{more precisely Poisson brackets}\]
Studying this in more detail, it becomes apparent that matter brackets are not modified by coupling to gravity\textsuperscript{17}\ The reason is that matter only appears in the constraints through its conserved stress tensor. More precisely it enters the Hamiltonian constraint (8) in the combination $N^2\sqrt{g}T^{00}$, and it enters the momentum constraint (9) in the combination $N\sqrt{g}T^0_i$. It turns out these are exactly the combinations where, with the help of stress tensor conservation, all matter time derivatives can be eliminated from $\partial_t \sigma$.

Let’s see how this works in detail. The Hamiltonian constraint (8) depends on matter only through the combination $N^2\sqrt{g}T^{00}$. When we take a time derivative of this combination we generate the expression

$$\partial_t \left( N^2\sqrt{g}T^{00} \right) = (\partial_t N) N\sqrt{g}T^{00} + N\partial_i \left( N\sqrt{g}T^{00} \right)$$

where in the second line we used stress tensor conservation, $\nabla_A T^{A0} = 0$. So far this still looks dangerous: $N$ is implicitly a function of the matter fields, and the Christoffel symbols also have matter time derivatives hidden inside them. Fortunately in the ADM decomposition of the metric the $\Gamma^0_{AB}$ Christoffel symbols are\textsuperscript{18}

$$\Gamma^0_{00} = \frac{1}{N} \partial_t N + \frac{1}{2N^2} \partial_i g_{ij} N^i N^j + \frac{1}{2N^2} N^i \partial_i \left( N^2 - N_k N^k \right)$$

$$\Gamma^0_{0i} = \frac{1}{N} \partial_i N - \frac{1}{2N^2} \partial_i (N_k N^k) + \frac{1}{2N^2} N^j \left( \partial_t g_{ij} + \partial_i N_j - \partial_j N_i \right)$$

$$\Gamma^0_{ij} = \frac{1}{N\sqrt{g}} \left( \pi_{ij} - \frac{1}{d-1} g_{ij} \pi \right)$$

Using this in the second line of (30), one finds that all terms involving $\partial_t N$ cancel. So we’re left with an expression for $\partial_t \left( N^2\sqrt{g}T^{00} \right)$ that involves time derivatives of the spatial metric $\partial_t g_{ij}$. But there are no time derivatives of matter fields, nor are there time derivatives of $\pi^{ij}$.

Likewise in the momentum constraint (9) matter only enters through $N\sqrt{g}T^0_i$. When we take a time derivative of this combination, due to the conservation equation $\nabla_A T^A_i = 0$ we get

$$\partial_t \left( N\sqrt{g}T^0_i \right) = -\partial_j \left( N\sqrt{g}T^j_i \right) + \Gamma^A_{Bi} N\sqrt{g}T^B_i$$

\textsuperscript{17}We are grateful to Stanley Deser for reassuring us on this point.

\textsuperscript{18}The first two lines are a straightforward calculation in ADM variables. The third line follows from (3.9b) in [16].
Now the relevant Christoffel symbols are, besides (32) and (33),
\[ \Gamma^i_{0j} = -\frac{1}{N} N^i \partial_j N + \frac{1}{2N^2} N^i \partial_j (N_k N^k) + \frac{1}{2} \left( g^{jk} - \frac{N^i N^k}{N^2} \right) (\partial_t g_{jk} + \partial_j N_k - \partial_k N_j) \]
\[ \Gamma^i_{jk} = \gamma^i_{jk} - \frac{1}{N \sqrt{g}} N^i \left( \pi_{ij} - \frac{1}{d-1} g_{ij} \pi \right) \]
where \( \gamma^i_{jk} \) is the connection constructed from the spatial metric \( g_{ij} \). So we’re left with an expression for \( \partial_t (N \sqrt{g} T^0_0) \) that involves time derivatives of the spatial metric \( \partial_t g_{ij} \). But again there are no time derivatives of matter fields, nor are there time derivatives of \( \pi^{ij} \).

In this way stress tensor conservation has allowed us to eliminate matter time derivatives from our formula for \( \partial_t \sigma \). Moreover, as we have seen, the coupling to matter generates no time derivatives of \( \pi^{ij} \). We still have \( \partial_t g_{ij} \) in our formulas, but note that
\[ \partial_t g_{ij} = e^\sigma \partial_t \sigma \tilde{g}_{ij} + e^\sigma \partial_t \tilde{g}_{ij} \]
\[ \partial_t g_{iz} = \partial_t g_{zz} = 0 \]
This means the equation one obtains by taking a time derivative of (29) can be solved to express \( \partial_t \sigma \) in terms of the dynamical variables \( \tilde{g}_{ij} \), \( \tilde{\pi}^{ij} \), \( \phi \), \( \pi_\phi \), but in a way that does not involve time derivatives of matter fields or \( \tilde{\pi}^{ij} \). Substituting \( \partial_t \sigma \) in (29), the symplectic form for matter retains its canonical form. We show this in detail in appendix B. This conclusion seems to hold quite generally, and applies to any theory in which the constraints (8), (9) only depend on matter through a conserved stress tensor. In such a theory, provided the gauge fixing conditions (11) – (13) do not involve matter degrees of freedom, matter fields will obey their usual canonical brackets even when coupled to gravity.

3 Gauge fields

In this section we consider fields with spin one. While we are actually interested in the massless case (gauge fields), we will keep the discussion general.

\[^{19}\text{In general we do not expect that solving the constraints will generate terms involving } \partial_t \pi^{ij}, \text{ since the Hamiltonian formalism begins from an action that only involves first-order time derivatives of the metric. We will assume this in the sequel.}\]
Consider the 3-point function of two scalars and one field strength \( F_{\mu\nu} = \partial_\mu j_\nu - \partial_\nu j_\mu \), built from a current \( j_\mu \) of dimension \( \Delta \). If the current is conserved then \( \Delta = d - 1 \), otherwise \( \Delta > d - 1 \).

\[
\langle F_{\mu\nu}(x)O_1(y_1)O_2(y_2) \rangle \sim \frac{[(x - y_1)_\mu (x - y_2)_\nu - \mu \leftrightarrow \nu]}{(y_1 - y_2)^{\Delta_1 + \Delta_2 - \Delta + 1}(x - y_1)^{\Delta + \Delta_1 - \Delta_2 + 1}(y_2 - x)^{\Delta + \Delta_2 - \Delta_1 + 1}}
\]

(37)

This can be written as a derivative operator acting on a scalar three-point function,

\[
\langle F_{\mu\nu}(x)O_1(y_1)O_2(y_2) \rangle \sim \frac{1}{(y_1 - y_2)^{\Delta_1 + \Delta_2 - \Delta + 1}(x - y_1)^{\Delta + \Delta_1 - \Delta_2 - 1}(y_2 - x)^{\Delta + \Delta_2 - \Delta_1 + 1}}
\]

(38)

where the function the operator is acting on has the form of a three-point function of three primary scalars of dimension \( \Delta, \Delta_1, \Delta_2 + 1 \) respectively. If we smear the scalar operator at the point \( y_1 \) into the bulk we will get the same derivative operator acting on the known result for a mixed bulk–boundary scalar three-point function\(^{20}\). This has singularities at bulk spacelike separation which can be canceled, provided the current is not conserved, by adding smeared higher-dimension primary scalar operators to the definition of the bulk scalar. These operators can be constructed in \( 1/N \) perturbation theory as double-trace operators built from \( j_\mu, O_2 \) and derivatives. These higher-dimension primary scalars will also cancel the unwanted singularities in a three-point function with the current and another boundary primary scalar\(^{14}\).

However when \( \Delta = d - 1 \), that is when \( j_\mu \) is a conserved current, one cannot build a higher-dimension primary scalar out of the conserved current and another primary scalar. Even if one could, it wouldn’t help: due to the Ward identity, the three-point function of a conserved current and two primary scalars is only non-zero when the scalar operators have the same dimension\(^{21}\).

Fortunately, it turns out that for any current it is possible to construct from \( j_\mu, O_2 \) and derivatives a tower of non-primary scalar operators, which

\(^{20}\)For a summary of the scalar case see appendix C.

\(^{21}\)See (33) in [14].
have correlation functions with $F_{\mu\nu}$ and $O_2$ that take the form (37) but with increasing $\Delta_1$. This is not a possibility we need to make use of for a non-conserved current. But these non-primary operators will allow us to implement an appropriate notion of bulk micro-causality in the conserved current case.

Let us still be general and consider any current, conserved or not, and construct these non-primary scalars. At leading order in $1/N$ note that the most general scalar operator made out of $j_\mu$ (but not $\partial_\mu j^\mu$) and $O_2$, together with derivative operators, which transforms under dilatons with naive dimension $\Delta_n = \Delta_2 + \Delta + 2n + 1$, is a sum of operators $A^{mlk}$ given by

$$A^{mlk} = \partial_{\mu_1\ldots\mu_m} \nabla^{2l} j_\nu \partial^{\mu_1\ldots\mu_m} \nabla^{2k} \partial^{\nu} O_2$$  

(39)

We want to find an operator $A_n$ whose correlator has the form (37) with $\Delta_1 \to \Delta_n$, namely

$$\langle F_{\mu\nu}(x) A_n(y_1) O_2(y_2) \rangle = \left[\frac{(y_1 - x)_\mu(y_2 - x)_\nu - \mu \leftrightarrow \nu}{(y_1 - y_2)^{2\Delta_2 + 2 + 2n}(x - y_1)^{2\Delta + 2 + 2n}}\right]$$  

(40)

To construct such an operator we write the right hand side of (40) as

$$\left[(x - y_1)_\mu(x - y_2)_\nu - \mu \leftrightarrow \nu\right] \sum_{m+l+k=n} d_{mlk} \frac{[(y_1 - y_2)_\alpha(x - y_1)^{\alpha}]}{(x - y_1)^{2\Delta_2 + 2 + 2l}(y_1 - y_2)^{2\Delta + 2 + 2m + 2k}}$$  

(41)

where

$$d_{mlk} = 2^m \binom{m+l+k}{m} \binom{l+k}{l}.$$  

(42)

In appendix D we show that we can construct operators $V^{mlk}$ built from linear combinations of the $A^{mlk}$, where $\langle F_{\mu\nu}(x) V^{mlk}(y_1) O_2(y_2) \rangle$ exactly matches the corresponding term in the expansion written in (41). Then we can write

$$A_n = \frac{1}{N} \sum_{m+l+k=n} d_{mlk} V^{mlk}$$  

(43)

This is true whether or not the current is conserved, since we have not used the operator $\partial^\mu j_\mu$ anywhere. For a conserved current the same formulas hold, with the following changes of notation. In the original three-point function (37) we replace $O_1 \to O$ and take $O_2$ to be its complex conjugate, $O_2 \to \bar{O}$.
We take the dimension of the current $\Delta = d - 1$, and in the expressions for $V_{mlk}$ we replace $O_2 \rightarrow \mathcal{O}$.

Now using the result from smearing a scalar operator inside a scalar three-point function [12, 14], we see that we can define a local bulk scalar field interacting with a bulk gauge field by setting

$$
\phi(z, y_1) = \int K_{\Delta_1} \mathcal{O} + \sum_{n=0}^{\infty} a_n \int K_{\Delta_n} A_n
$$

(44)

Here $K_{\Delta}(z, y|y')$ is the scalar smearing function for a dimension $\Delta$ primary scalar. With appropriately chosen $a_n$, all the unwanted space-like singularities can be canceled in a three-point function of this operator with a boundary field strength $F_{\mu\nu}$ and a boundary scalar $\bar{\mathcal{O}}$.

Note that if we had been considering a scalar field interacting with a massive vector field in the bulk, we would not need to consider the non-primary operators $A_n$. Rather we would cancel the unwanted singularities using the higher-dimension primary scalars $\sim \partial_{\mu} j_{\mu} O_2 + j_{\mu} \partial_{\mu} O_2$ that one can build from a non-conserved current $j_{\mu}$ and $O_2$. This procedure would allow us to build a bulk scalar which is local in correlation functions involving $j_{\mu}$ [14]. For a massive vector, we could use non-primary scalars if we were only interested in achieving locality in correlators involving the boundary field strength $F = dj$. But since locality would then be violated in correlators involving the current $j$ itself, this procedure is not physically sensible.

### 3.1 Special conformal transformations

In this section we show that bulk micro-causality implies that our bulk observables have the correct transformation properties under AdS isometries. The issue is that the operators $A_n$ we have constructed are not primary scalars, but in (44) they are smeared using the usual scalar smearing function. This means the bulk operator defined in (44) will not transform like an ordinary scalar field under AdS isometries. Instead, as we will show, it obeys the correct transformation rule for a charged scalar field in holographic gauge.

We start with a non-primary scalar operator $A_n$, whose three-point func-
tion with \( F_{\mu\nu} \) and a primary scalar of dimension \( \Delta_2 \) is

\[
\langle F_{\mu\nu}(x)A_n(y_1)O_2(y_2) \rangle \sim \frac{1}{(y_1 - y_2)^{\Delta_n + \Delta_2 - \Delta + 1}(y_1 - x)^{\Delta + \Delta_2 - \Delta} (y_2 - x)^{\Delta + \Delta - \Delta_n + 1}} [(y_1 - x)_\mu (y_2 - x)_\nu - \mu \leftrightarrow \nu]
\]

This restricts how \( A_n \) transforms under conformal transformation. Of course one possibility is that \( A_n \) is a primary scalar, but we will see that there is another possibility. To see how \( A_n \) does behave, we transform both sides of the equality and ask how they can match. Since \( A_n \) transforms as a scalar under rotations, and as a scalar with dimension \( \Delta_n \) under dilations, we only need to see what happens under special conformal transformations. Under a special conformal transformation with parameter \( b_\mu \), to linear order in \( b_\mu \) one has the transformation properties

\[
F'_{\rho\mu} = F_{\rho\mu}(1 - 2(\Delta_j + 1)(b \cdot x)) - 2b_\mu x^\nu F_{\rho\nu} + 2b_\rho x^\nu F_{\mu\nu} + 2x_\mu b^\nu F_{\rho\nu} - 2x_\rho b^\nu F_{\mu\nu} + 2(\Delta_j - 1)(b_\mu j_\rho - b_\rho j_\mu)
\]

\[
O'_2 = (1 - 2\Delta_2 b \cdot x)O_2
\]

Let us split the transformation of \( A_n \) into a piece \( \delta_s A_n \) which is the expected behavior if it was a primary scalar of dimension \( \Delta_n \), and an extra piece \( \delta_e A_n \). Since the right hand side of (45) transforms as if \( A_n \) were a primary scalar, there must be some cancellations on the left hand side to achieve this. Under a special conformal transformation the left hand side of (45) changes by

\[
\langle \delta F_{\mu\nu} A_n O_2 \rangle + \langle F_{\mu\nu} \delta A_n O_2 \rangle + \langle F_{\mu\nu} A_n \delta O_2 \rangle
\]

Since \( A_n \) behaves like a primary scalar in a three-point function with \( F_{\mu\nu} \) and \( O_2 \) (but not with \( j_\mu \)), the only terms which are sensitive to the fact that \( A_n \) is actually not a primary scalar are

\[
\langle 2(\Delta_j - 1)(b_\nu j_\mu - b_\mu j_\nu)A_n O_2 \rangle + \langle F_{\mu\nu} \delta_e A_n O_2 \rangle
\]

But again, since \( A_n \) has a three-point function like a primary scalar with \( F_{\mu\nu} \) and \( O_2 \), it must be the case that

\[
\langle j_\mu A_n O_2 \rangle = (\text{primary scalar result}) + \partial_\mu B
\]
where the last term has a vanishing exterior derivative and drops out of the three-point function with $F$. This means the first term in (48) has the form
\[(b_\nu \partial_\mu - b_\mu \partial_\nu)B(x, y_1, y_2)\] (50)

Now in order for (48) to vanish, it must be the case that $\delta_e A_n$ is composed of terms of the form
\[\partial_{\alpha_1 \cdots \alpha_n} (\nabla^{2n_2} b \cdot j) \partial^{\alpha_1 \cdots \alpha_n} \nabla^{2n_3} O_2,\] (51)
since only then will $\langle F_{\mu \nu} \delta_e A_n O_2 \rangle$ have the form (50). That is, the vector index of the transformation parameter $b_\rho$ must be contracted with the index on $j_\rho$, and not with one of the derivative operators. This is verified by explicit computation for the two lowest-dimensions operators in appendix E.

This means that under special conformal transformations the expression for the bulk field in (44) transforms as
\[\phi'(z', y'_1) = \phi(z, y_1) + \sum_{n=0}^{\infty} a_n \int K_{\Delta_n} \delta_e A_n\] (52)

Fortunately this is exactly the type of transformation that a charged bulk scalar field should have. To see this recall how a charged bulk field behaves under special conformal transformation. In a completely fixed gauge the degrees of freedom which are left are physical, but they may only look local in the chosen gauge. For example a charged matter field in holographic gauge $\phi_{\text{phys}}$ can be written in terms of the non-gauge-fixed variables as
\[\phi_{\text{phys}}(z, y_1) = e^{\frac{1}{N} \int_0^z A_z dz} \phi(z, y_1)\] (53)

where we have attached a Wilson line running to the boundary, and $1/N$ is the charge of the field. In holographic gauge $A_z = 0$ and the Wilson line is invisible. This expression makes it manifest that the matter degrees of freedom in holographic gauge are secretly non-local.

One can directly compute how special conformal transformations act on the right hand side of (53). Alternatively one can realize that since special conformal transformations do not preserve the gauge $A_z = 0$, they must

---

22This was discussed in [14].
be combined with a compensating gauge transformation chosen to restore holographic gauge. Only the combined transformation is a symmetry of the gauge-fixed theory. This tells us how operators in holographic gauge should behave under special conformal transformations, namely

$$\phi'_\text{phys}(z', y'_1) = e^{-\frac{i}{N} \lambda(z, y_1)} \phi_\text{phys}(z, y_1)$$

where the compensating gauge transformation is \[1\]

$$\lambda = -\frac{1}{\text{vol} (S^{d-1})} \int d^d x'' \theta(\sigma z'') 2b \cdot j$$

To leading order in $1/N$

$$\phi'_\text{phys}(z', y'_1) = \phi_\text{phys}(z, y_1) - \frac{i}{N} \lambda(z, y_1) \phi_0(z, y_1)$$

where $\phi_0 = \int K(z, y_1, x'') \mathcal{O}(x'')$ is the zeroth-order smeared operator, without any interaction corrections. The term $\lambda(z, y_1) \phi_0(z, y_1)$ is a bi-local smeared expression on the boundary involving the operators $b \cdot j(x_1) \mathcal{O}(x_2)$, and hence should have a Taylor expansion around $b \cdot j(y_1) \mathcal{O}(y_1)$ involving exactly the operators appearing in \[51\].

### 3.2 Scalar commutator

We have shown that, even when the current is conserved, one can construct the double-trace operators $\mathcal{A}_n$ given in \[43\]. These operators are scalars but are not primary. They have the feature that, even though they are not primary scalars, they behave like a primary scalar when inserted in a three-point function with $F_{\mu \nu}$ and another primary scalar. That is, the correlation function \[40\] has the same form as \[37\]. Then given \[38\] one can define an interacting bulk scalar field by smearing the operators $\mathcal{A}_n$ as though they were primary scalars and adding them to the zeroth-order definition of the bulk field with arbitrary coefficients as in \[44\]. By choosing the coefficients $a_n$ appropriately, we can make the commutator between the bulk field and the boundary primary scalar $\mathcal{O}_2$ or the boundary field strength $F_{\mu \nu}$ as small as we like at spacelike separation, at least inside the three-point function. This procedure is directly analogous to the case of interacting scalar fields.
Note that this procedure only addresses the commutator in a three-point function $\langle \phi F \bar{O} \rangle$ involving the bulk scalar, a boundary field strength $F_{\mu\nu}$, and another boundary scalar. So it does not guarantee a vanishing commutator between the bulk field and the boundary current. Indeed we expect the commutator of the bulk scalar with the boundary current to be non-zero at spacelike separation due to the Gauss constraint. But will now argue that the procedure does imply a vanishing commutator between the bulk field and the boundary scalar in a three-point function with the current. That is, we claim that for conserved currents, to leading order in $1/N$

$$\langle F_{\mu\nu}(x) [\phi(z, y_1), O_2(y_2)] \rangle = 0 \Rightarrow \langle j_\mu(x) [\phi(z, y_1), O_2(y_2)] \rangle = 0 \quad (57)$$

The argument is as follows.

For a three-point function involving the commutator and the current $\langle [\phi, \bar{O}] j \rangle$ to be non-zero at leading order in the $1/N$ expansion the commutator must be proportional to an operator linear in the current. Then for the commutator to have a vanishing two-point function with $F_{\mu\nu}$, i.e. $\langle [\phi, \bar{O}] F \rangle = 0$, the commutator must be proportional to the divergence of the current. But for a conserved current the divergence vanishes, and this implies the right hand side of (57). Thus while the addition of higher-dimension non-primary scalar operators can cancel the spacelike commutator with another boundary scalar, it cannot cancel the non-vanishing commutator with the current. This is, of course, required by the bulk Gauss constraint. Note that the same logic cannot be used to show that if the bulk scalar commutes with $F_{\mu\nu}$ it will also commute with $j_\mu$. The reason is that a vanishing commutator with $F_{\mu\nu}$ allows a non-zero commutator with the current of the form

$$[j_\mu(x), \phi(z, y_1)] \sim \partial_\mu (\int dx'' c(z, y_1, x, x'') O_2(x'')) \quad (58)$$

4 Gravity

We now turn to bulk scalar fields interacting with gravity. We will follow a similar route to the previous section, and will arrive at similar conclusions, but instead of working with a conserved current we will work with the conserved energy-momentum tensor of the CFT.
The three-point function of the energy-momentum tensor and two primary scalars of dimension $\Delta$ is given by

$$\langle T_{\mu\nu}(x)O(y_1)O(y_2) \rangle = \frac{c_{d,\Delta}}{(x - y_1)^{d-2}(x - y_2)^{d-2}(y_1 - y_2)^{2\Delta - d + 2}} \left[ \left( \frac{(x - y_1)_\mu}{(x - y_1)^2} - \frac{(x - y_2)_\mu}{(x - y_2)^2} \right) \left( \frac{(x - y_1)_\nu}{(x - y_1)^2} - \frac{(x - y_2)_\nu}{(x - y_2)^2} \right) - \frac{\eta_{\mu\nu}}{d} \left( \frac{(x - y_1)_\rho}{(x - y_1)^2} - \frac{(x - y_2)_\rho}{(x - y_2)^2} \right)^2 \right]$$

(59)

This can be written as a second-order derivative operator with respect to $x$, and some functions of $(x - y_2)$, acting on

$$\frac{1}{(x - y_1)^{d-2}(x - y_2)^{d-2}(y_1 - y_2)^{2\Delta - d + 2}}$$

This expression is the three-point function of scalar primaries of dimension $d - 2, \Delta, \Delta^2$. One can smear the operator $O(y_1)$ to move it into the bulk. Then one gets the same derivative operator acting on a smeared scalar three-point function, whose analytic structure we know. To make a local bulk scalar one would need a tower of operators of dimension $\Delta_n$, whose three-point function would resemble (59) with

$$\frac{1}{(x - y_1)^{d-2}(x - y_2)^{d-2}(y_1 - y_2)^{2\Delta_n - d + 2}} \rightarrow \frac{1}{(x - y_1)^{d-2+\Delta_n-\Delta}(x - y_2)^{d-2+\Delta-\Delta_n}(y_1 - y_2)^{\Delta+\Delta_n-d+2}}$$

(60)

But such operators do not exist. In fact, due to the Ward identity, the three-point function of the energy-momentum tensor with two primary scalars can only be non-zero if the scalars have the same dimension. (The same issue arose with a conserved current in the previous section.) The inability to construct such operators out of $T_{\mu\nu}$ and $O$ can be traced to the absence of operators associated with the divergence of the energy-momentum tensor. For a non-conserved spin-2 tensor there would be no such obstruction. This breakdown of locality is desirable, since the Hamiltonian can be written as a surface integral (a spatial integral of $T^{00}$), and the Hamiltonian cannot commute with any bulk operator that is not a constant of the motion.

Instead we wish to proceed as in the previous section, and see if we can make a bulk scalar which commutes at spacelike separation with another

\footnote{We leave aside the issue of operators with such low dimensions.}
boundary scalar when inserted in a three-point function. For this we need to find a gravity operator analogous to $F_{\mu\nu}$. It turns out the appropriate choice is the boundary Weyl tensor with all indices taken to have distinct values. This is given by

$$C_{\alpha\beta\gamma\delta} = \partial_\alpha \partial_\gamma T_{\beta\delta} - \partial_\alpha \partial_\delta T_{\beta\gamma} - \partial_\beta \partial_\gamma T_{\alpha\delta} + \partial_\beta \partial_\delta T_{\alpha\gamma}$$  \hspace{1cm} (61)

For later use it is important that the expression for the Weyl tensor is invariant under

$$T_{\mu\nu} \rightarrow T_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu.$$  \hspace{1cm} (62)

The three-point function of $C_{\alpha\beta\gamma\delta}$ with two primary scalars of dimension $\Delta$ is

$$\langle C_{\alpha\beta\gamma\delta}(x)O(y_1)O(y_2) \rangle \sim \frac{-4d\cdot4d\Delta}{(x-y_1)^{d+2}(x-y_2)^{d+2}(y_1-y_2)^{2\Delta-d+2}} \left[ (x-y_1)_\beta(x-y_1)_\gamma(x-y_2)_\alpha(x-y_2)_\delta \gamma \leftrightarrow \delta \gamma \leftrightarrow \delta \beta \leftrightarrow \alpha \right]$$  \hspace{1cm} (63)

This can be written as

$$\langle C_{\alpha\beta\gamma\delta}(x)O(y_1)O(y_2) \rangle \sim \mathcal{L}_{g} \frac{1}{(x-y_1)^{d-2}(x-y_2)^{d+2}(y_1-y_2)^{2\Delta-d+2}} \left[ (x-y_2)_\alpha(x-y_2)_\gamma \partial_\beta \partial_\delta \gamma \leftrightarrow \delta \beta \leftrightarrow \alpha \right]$$  \hspace{1cm} (64)

where the operator $\mathcal{L}_{g}$ is acting on the three-point function of primary scalars of dimensions $(d, \Delta, \Delta + 2)$. If we try to promote $O(y_1)$ to a bulk operator by smearing it we will obtain a three-point function that has singularities at bulk spacelike separation. To cancel these singularities we will need to add appropriately smeared higher-dimension scalar operators. Thus we need to find a tower of scalar operators $T_n$ which transform under dilations with increasing dimensions $\Delta_n = \Delta + d + 2 + 2n$, and whose three-point function with $C_{\alpha\beta\gamma\delta}(x)$ and $O(y_2)$ matches (64) with

$$\frac{1}{(x-y_1)^{d-2}(x-y_2)^{d+2}(y_1-y_2)^{2\Delta-d+2}} \rightarrow \frac{1}{(x-y_1)^{d-2+\Delta_n-\Delta}(x-y_2)^{d+2+\Delta-\Delta_n}(y_1-y_2)^{\Delta+\Delta_n-d+2}}$$  \hspace{1cm} (65)

\[24\] We work in dimension $d \geq 4$ so that this is possible.
Note that for $\Delta_n = \Delta + d + 2 + 2n$ this becomes
\[
\frac{(x - y_2)^{2n}}{(x - y_1)^{2d+2n}(y_1 - y_2)^{2\Delta+2n+4}}
\]
(66)

So we are looking for operators $T_n$ which obey
\[
\langle C_{\alpha\beta\gamma\delta}(x)T_n(y_1)\mathcal{O}(y_2) \rangle = \frac{(x - y_2)^{2n}}{(x - y_1)^{2d+2n+4}(y_1 - y_2)^{2\Delta+2n+4}} \times \nonumber \\
[(x - y_1)_{\beta}(x - y_1)_{\delta}(x - y_2)_{\alpha}(x - y_2)_{\gamma} - (\gamma \leftrightarrow \delta) - (\beta \leftrightarrow \alpha) + (\gamma \leftrightarrow \delta \beta \leftrightarrow \alpha)]
\]
(67)

As in the vector case it is useful to write
\[
(x - y_2)^{2n} = \sum_{m+k+l=n} d_{mlk}(x - y_1)^{2k}(y_1 - y_2)^{2l}[(y_1 - y_2)_{\alpha}(x - y_1)_{\alpha}]^m
\]
(68)

and look for operators $M_{mlk}$ obeying
\[
\langle C_{\alpha\beta\gamma\delta}(x)M_{mlk}(y_1)\mathcal{O}(y_2) \rangle = \frac{[(y_1 - y_2)_{\alpha}(x - y_1)_{\alpha}]^m}{(x - y_1)^{2d+2m+2l+4}(y_1 - y_2)^{2\Delta+2m+2l+4}} \times \nonumber \\
[(x - y_1)_{\beta}(x - y_1)_{\delta}(x - y_2)_{\alpha}(x - y_2)_{\gamma} - (\gamma \leftrightarrow \delta) - (\beta \leftrightarrow \alpha) + (\gamma \leftrightarrow \delta \beta \leftrightarrow \alpha)]
\]
(69)

To leading order in $1/N$ these operators can be constructed starting from the most general scalar operator with the correct dimension
\[
M_{mlk} = \sum_{m+l+k=m'+l'+k'} b_{mlk}^{m'l'k'} \partial_{\mu_1} \cdots \partial_{\mu_m} \nabla^{2l'} T_{\rho\nu} \partial^{\mu_1} \cdots \partial^{\mu_{m'}} \nabla^{2k'} \partial^{\rho} \partial^{\sigma} O_2
\]
(70)
and solving for the coefficients $b$. In appendix D we give an iterative construction of these coefficients. The desired scalar non-primary operators are then
\[
T_n = \sum_{m+l+k=n} d_{mlk}M_{mlk}
\]
(71)

For example when $n = 0$ we have $T_0 = T_{\rho\nu} \partial^{\rho} \partial^{\sigma} O$, and when $n = 1$ we find
\[
b_{0,0,1}^{0,1,0} = \frac{C_1}{(2\Delta + 2 + d)(2\Delta + 4)}, \quad b_{1,0,0}^{1,0,0} = \frac{C_1}{(2 + d)(2d + 4)}, \quad b_{0,1,0}^{0,1,0} = \frac{-C_1}{(2d + 4)(2\Delta + 4)}
\]
(72)
where $C_1 = \frac{1}{\Delta \Gamma \delta \Delta \Delta + 1}$. Much like the gauge field case, the fact that we cannot construct a primary scalar from $T_{\mu\nu}$ and $O_2$ is due to stress tensor conservation, $\partial_\mu T^{\mu\nu} = 0$.

Given these operators, we can define a bulk scalar field that has a micro-causal three-point function with a boundary primary scalar and the boundary Weyl tensor by setting

$$\phi(z, y_1) = \int K_{\Delta_1} O + \sum_{n=0}^{\infty} b_n \int K_{\Delta_n} T_n$$

(73)

The constants $b_n$ are chosen so that the unwanted space-like singularities in the three-point function are canceled.

### 4.1 Special conformal transformation

Since $T_n$ is not a primary scalar, the field $\phi(z, y_1)$ defined in (73) will not transform like a conventional bulk scalar field under AdS isometries. Instead, as we will see, it has the correct transformation properties to represent a scalar field in holographic gauge. Thus somewhat remarkably, just as in the gauge field case, imposing micro-causality leads to bulk fields with the correct transformation properties.

To determine how $\phi(z, y_1)$ transforms, we first need to know how $T_n$ transforms. Rather than constructing $T_n$ explicitly and finding its transformation properties, we use an alternate route. We wish to determine the transformation of a scalar operator $T_n$ whose 3-point function

$$\langle C_{\alpha\mu\beta\nu}(x) T_n(y_1) O(y_2) \rangle$$

(74)

obeys (67). We follow the same logic as in the conserved current case, and look for the behavior under infinitesimal special conformal transformations which would not be present if $T_n$ was a primary scalar.

We start with the behavior of the energy momentum tensor under a special conformal transformation. To first order in the parameter $b_\mu$

$$T'_{\mu\nu} = T_{\mu\nu} (1 - 2d(b \cdot x)) + 2b^\delta x_\nu T_{\mu\delta} - 2b_\nu x^\delta T_{\mu\delta} + 2b^\delta x_{\mu} T_{\delta\nu} - 2b_\mu x^\delta T_{\delta\nu}$$

(75)
The transformation of $C_{\alpha\mu\beta\nu}$ when all indices are distinct is
\[
C'_{\alpha\mu\beta\nu} = C_{\alpha\mu\beta\nu}(1 - 2(d + 2)b \cdot x) + 2b^\delta x_\nu C_{\alpha\mu\beta\delta} + 2b^s \delta x_\mu C_{\alpha\delta\beta\nu} + 2b^s x_\beta C_{\alpha\mu\delta\nu} + 2b^s x_\alpha C_{\delta\mu\beta\nu} \\
-2x^s b_\nu C_{\alpha\mu\beta\delta} - 2x^s b_\mu C_{\alpha\delta\beta\nu} - 2x^s b_\beta C_{\alpha\mu\delta\nu} - 2x^s b_\alpha C_{\delta\mu\beta\nu} \\
-2d[(b_\alpha \partial_\beta + b_\beta \partial_\alpha)T_{\mu\nu} + (b_\mu \partial_\nu + b_\nu \partial_\mu)T_{\alpha\beta} - (b_\alpha \partial_\nu + b_\nu \partial_\alpha)T_\mu_\beta - (b_\mu \partial_\beta + b_\beta \partial_\mu)T_\alpha\nu] \\
\] (76)

Let us look at the transformation of the left hand side of (67), namely
\[
\langle \delta C_{\alpha\mu\beta\nu}(x)T_n\mathcal{O}_2 \rangle + \langle C_{\alpha\mu\beta\nu}(x)\delta T_n\mathcal{O}_2 \rangle + \langle C_{\alpha\mu\beta\nu}(x)T_n\delta\mathcal{O}_2 \rangle \\
\] (77)

Let us also split the transformation of $T_n$ into a piece $\delta_n T_n$, which is the part that looks like the transformation of a primary scalar with dimension $\Delta_n$, and an extra piece $\delta_{\text{extra}} T_n$. Since by assumption $T_n$ obeys (64) and (65), the only terms which differ from the case that $T_n$ is actually a primary scalar are
\[
\langle \left( b_\alpha \partial_\beta + b_\beta \partial_\alpha \right) T_{\mu\nu} + \left( b_\mu \partial_\nu + b_\nu \partial_\mu \right) T_{\alpha\beta} - \left( b_\alpha \partial_\nu + b_\nu \partial_\alpha \right) T_\mu_\beta - \left( b_\mu \partial_\beta + b_\beta \partial_\mu \right) T_\alpha\nu \rangle \mathcal{O}(y_2) + \langle C_{\alpha\mu\beta\nu}(x) \delta_{\text{extra}} T_n(y_1) \mathcal{O}(y_2) \rangle \\
\] (78)

However since $\langle C_{\alpha\mu\beta\nu}(x)T_n(y_1)\mathcal{O}(y_2) \rangle$ transforms as if $T_n$ is a scalar, then
\[
\langle T_{\mu\nu}T_n\mathcal{O} \rangle = \text{scalar case} + \partial_\mu B_\nu + \partial_\nu B_\mu. \\
\] (79)

This means that for (78) to vanish one must have
\[
\langle C_{\alpha\mu\beta\nu}(x) \delta_{\text{extra}} T_n(y_1) \mathcal{O}(y_2) \rangle \sim \left( b_\alpha \partial_\beta \partial_\mu - b_\mu \partial_\beta \partial_\alpha \right) B_\nu + \left( b_\beta \partial_\alpha \partial_\nu - b_\nu \partial_\beta \partial_\alpha \right) B_\mu \\
+ \left( b_\nu \partial_\beta \partial_\mu - b_\beta \partial_\mu \partial_\nu \right) B_\alpha + \left( b_\alpha \partial_\nu \partial_\beta - b_\beta \partial_\nu \partial_\alpha \right) B_\mu. \\
\] (80)

This can only be achieved if $\delta_{\text{extra}} T_n$ is made out of terms of the form
\[
\partial_{\alpha_1\cdots\alpha_n} \left( \nabla^{2n_2} b^\delta T_\delta \right) \mathcal{O}^{\alpha_1\cdots\alpha_n} \nabla^{2n_3} \mathcal{O}_2 \ \ (81)
\]

No other types of contraction of $b_\mu$ will give the right result. Explicit computations in appendix E for $T_0$ agree with this form. Thus under special conformal transformation the bulk scalar (73) transforms as
\[
\phi'(z', y'_1) = \phi(z, y_1) + \sum_{n=0}^\infty b_n \int K_{\Delta_n} \delta_{\text{extra}} T_n \\
\] (82)

27
Let us compare this to the expected transformation of a bulk scalar field under a special conformal transformation. Again as in the vector case it is useful to understand how the physical operator $\phi_{\text{phys}}(z, y_1)$ in the gauge fixed theory is related to the degrees of freedom of the non-gauged fixed theory. For gravity the gauge symmetry is diffeomorphisms, and the question is how to label a particular spacetime point. From the bulk point of view the simplest method is to start at a boundary point (which is invariant under the diffeomorphisms we consider, that fall off quickly enough at infinity), and follow a geodesic orthogonal to the boundary for a certain proper distance. The distance to the boundary is infinite, but we can regularize this by subtracting an infinite piece which is common to all geodesics or equivalently by putting the boundary at $z = \epsilon$ and later sending $\epsilon \to 0$. In this way one can define labels $X, Z$ and a bulk scalar field $\phi(X, Z)$, where $X, Z$ are given by the procedure of starting at some point on the boundary and following an orthogonal geodesic for a certain proper distance. With this definition the position of the scalar field is invariant under diffeomorphisms. However $\phi$ is not a local operator since it depends on metric along some path. In holographic gauge these geodesics orthogonal to the boundary have fixed $x_\mu$, since the Christoffel symbol $\Gamma^\mu_{zz} = 0$. Thus in holographic gauge we can identify $x = X, z = Z$ and $\phi(x, z) = \phi(X, Z)$.

To see how this operator behaves under special conformal transformations, we use the same strategy as the gauge field case. The effect of a special conformal transformation on an operator in holographic gauge can be obtained from a standard special conformal transformation, followed by a coordinate transformation that restores holographic gauge.

$$\phi'(z', x') = \phi(z + \epsilon_z, x_{\mu} + \epsilon_{\mu})$$ (83)

Here to first order in the parameter of the special conformal transformation $b_\rho$ one has [11]

$$\epsilon_{\mu} \sim \frac{1}{N} \int d^d x'' \theta(\sigma z'') \sigma z z'' b^\alpha T_{\alpha\mu}, \quad \epsilon_z = 0$$ (84)

where the $1/N$ comes from canonically normalizing the kinetic term of the

\[25\] Note that this is not how our smearing function seems to label the bulk operator. The smearing function labels the bulk operator by specifying the points on the complexified boundary which are spacelike to the bulk point, which is also an invariant notion.
gravity fluctuations. So to first order
\[ \phi'(z', x') = \phi(z, x) + \epsilon_\mu \partial^\mu \phi(z, x). \] (85)

This involves a bi-local smearing on the boundary of the operator \( b^a T_{\alpha\mu}(x_1) \partial^\mu \mathcal{O}(x_2) \), which can be expanded around \( b^a T_{\alpha\mu} \partial^\mu \mathcal{O}(x) \) using exactly the operators in (81). Thus we see that the operators we constructed in the CFT by demanding micro-causality, have the same behavior under special conformal transformations as a bulk scalar field in holographic gauge.

### 4.2 Scalar commutator

We saw that we can add smeared non-primary scalar operators to the definition of a bulk field such that the three-point function \( \langle C_{\alpha\mu\beta\nu}(x) \phi(z, y_1) \mathcal{O}(y_2) \rangle \) does not suffer from non-analyticity at bulk space-like separation. Hence commutators with the bulk field vanish when inserted inside a three-point function. That is
\[
\begin{align*}
\langle C_{\alpha\mu\beta\nu}(x)[\phi(z, y_1), \mathcal{O}(y_2)] \rangle &= 0 \\
\langle [C_{\alpha\mu\beta\nu}(x), \phi(z, y_1)]\mathcal{O}(y_2) \rangle &= 0
\end{align*}
\] (86)

whenever the points in the commutator are spacelike separated. This does not imply that the bulk scalar commutes with the boundary stress tensor at spacelike separation, for example \( \langle [T_{\mu\nu}(x), \phi(z, y_1)]\mathcal{O}(y_2) \rangle \) does not need to vanish. But we claim it does imply that
\[
\langle T_{\mu\nu}(x)[\phi(z, y_1), \mathcal{O}(y_2)] \rangle = 0
\] (87)

at bulk spacelike separation to leading order in \( 1/N \). The argument is similar to the gauge field case. For (87) to be non-zero the commutator \( [\phi(z, y_1), \mathcal{O}(y_2)] \) must be a scalar operator that is linear in \( T_{\alpha\beta} \). For example it could be \( T_{\mu\nu}(y_1 - y_2)^\mu(y_1 - y_2)^\nu \). But for a conserved stress tensor, given the available operators, if the commutator is linear in \( T_{\alpha\beta} \) then (86) will not be zero.\(^{26}\)

\(^{26}\)Given a non-conserved spin-two operator one would have operators available such as its divergence, but in this case one would also be able to construct higher-dimension primary scalars that make the bulk field obey micro-causality.
While this argument relies on the operator $C_{\alpha\beta}(x)$ with distinct values for all indices, which is only possible in $d \geq 4$, we expect the results should also hold in $d = 3$. The expressions for the bulk scalar field can simply be analytically continued to $d = 3$.

5 Higher point functions

So far we have shown that three-point functions involving a bulk field can be made local to $O(1/N)$. We now use this result to argue that four-point functions can be made local to $O(1/N^2)$.

Consider a four-point function with one bulk scalar operator and three boundary scalar operators. We claim that the four-point function with the bulk operator constructed as above will obey micro-causality. That is

$$\langle [\phi(z,x_1), \mathcal{O}(x_2)]\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = 0 \quad (88)$$

whenever $(z, x_1)$ and $x_2$ are spacelike separated. To show this we use the OPE of $\mathcal{O}(x_3)$ and $\mathcal{O}(x_4)$

$$\mathcal{O}(x_3)\mathcal{O}(x_4) = \sum c_i(x_3, x_4)\mathcal{G}^i(x_3) \quad (89)$$

where $\mathcal{G}^i(x_3)$ includes a complete set of CFT operators, and we have suppressed any spin indices on $\mathcal{G}^i$. Then

$$\langle [\phi(z,x_1), \mathcal{O}(x_2)]\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \sum c_i(x_3, x_4)\langle [\phi(z,x_1), \mathcal{O}(x_2)]\mathcal{G}^i(x_3) \rangle \quad (90)$$

This reduces the problem to a three-point function. We have shown that if $\mathcal{G}^i$ is a scalar, then $\phi(z,y_1)$ can be constructed such that the commutator vanishes at spacelike separation in a three-point function [12, 14]. We have shown in [14] and in section 3 that if $\mathcal{G}^i$ is a spin-one current, conserved or not, then again $\phi(z,y_1)$ can be constructed so that the right hand side of (90) vanishes at spacelike separation. Finally we showed in section 4 that the same is true when $\mathcal{G}$ is the stress tensor. This covers the range of operators we expect to find in the low energy bulk theory, and strongly suggests that the commutator can be made to vanish at spacelike separation in a three-point function whatever $\mathcal{G}^i$ may be.
This suggests that a bulk scalar field can be constructed in such a way that n-point functions with one bulk operator and \( n - 1 \) boundary operators can be made causal, at least to leading non-trivial order in the \( 1/N \) expansion, even in quantum gravity.

Acknowledgements

We are deeply grateful to Stanley Deser for valuable discussions. DK is supported by U.S. National Science Foundation grant PHY-1125915 and by grants from PSC-CUNY. The work of GL was supported in part by the Israel Science Foundation under Grant No. 392/09 and under Grant No. 504/13 and in part by a grant from the GIF, the German-Israeli Foundation for scientific research and development under grant no. 1156-124.7/2011.

A Bulk gauge theory

In this appendix we present the canonical formalism for a gauge field coupled to matter, in a way that parallels our treatment of gravity in section 2. Although the method is different, the results agree with those obtained in section 2 of [14] by following Dirac’s procedure. This both illustrates our method and gives us more confidence in our approach. It also gives us the opportunity to improve on the boundary conditions which were adopted in [14].

For simplicity we consider an abelian gauge field \( A_M \) coupled to a complex scalar field \( \phi \) with action

\[
S = \int d^{d+1}x \sqrt{-G} \left( -D_M \phi^* D^M \phi - \frac{1}{4} F_{MN} F^{MN} \right)
\]

(91)

Here \( D_M = \partial_M + iq A_M \), and we work in AdS\(_{d+1}\) with metric

\[
ds^2 = \frac{R^2}{z^2} \left( -dt^2 + |d\vec{x}|^2 + dz^2 \right)
\]

(92)
The theory can be presented in first-order form.

\[ S = \int d^{d+1}x \pi_\phi \dot{\phi} + \pi_\phi^* \phi^* + \pi_i \dot{A}_i + A_0 \left( \partial_i \pi_i + i q (\pi_\phi \phi - \pi_\phi^* \phi^*) \right) \]
\[ - \left( \frac{z}{R} \right)^{d-1} \pi_\phi^* \pi_\phi - \frac{1}{2} \left( \frac{z}{R} \right)^{d-3} \pi_i \pi_i - \left( \frac{R}{z} \right)^{d-1} \left| (\nabla + i q A) \phi \right|^2 - \frac{1}{4} \left( \frac{R}{z} \right)^{d-3} F_{ij} F_{ij} \]  

(93)

A_0 is a Lagrange multiplier enforcing the Gauss constraint

\[ \partial_i \pi_i + i q (\pi_\phi \phi - \pi_\phi^* \phi^*) = 0 \]  

(94)

We must also impose a gauge condition. As in [14] we adopt the holographic (or axial) gauge condition which sets

\[ A_z = 0 . \]  

(95)

This gives us the right number of degrees of freedom. The gauge field \( A_i \) and its conjugate momentum \( \pi_i \) contain \( 2d \) degrees of freedom. The constraints (94), (95) kill two phase space degrees of freedom, leaving \( 2d - 2 \) phase space degrees of freedom or equivalently \( d - 1 \) configuration space degrees of freedom. This is appropriate for a massless spin-1 particle which is a vector under the \( SO(d - 1) \) little group.

The steps now parallel section 2. To fix the Lagrange multiplier we require that the gauge fixing condition (95) is preserved by time evolution. The equation of motion for \( A_z \), obtained by varying the action with respect to \( \pi_z \), is

\[ \partial_t A_z = \partial_z A_0 + \left( \frac{z}{R} \right)^{d-3} \pi_z \]  

(96)

The left hand side vanishes by the gauge condition, so we get an equation we can solve for \( A_0 \). With suitable boundary conditions as \( z \to 0 \) – to be discussed in more detail below – the solution is

\[ A_0 = \int_0^z dz' \left( \frac{z'}{R} \right)^{d-3} \pi_z \]  

(97)

Next we impose the constraint (94) and the gauge condition (95) on the action (93). To take the gauge condition into account we simply set \( A_z = 0 \).

\[ ^{27} \text{In the approach of [14] this equation was imposed as an additional gauge condition.} \]
Then we solve the Gauss constraint for the conjugate momentum $\pi_z$. Again imposing suitable boundary conditions as $z \to 0$ we have

$$\pi_z = \int_z^\infty dz' \left( \partial_t \pi_i + i q (\pi_\phi \dot{\phi} - \pi_\phi^* \dot{\phi}^*) \right)$$  \hspace{1cm} (98)

Let us pause to discuss our boundary conditions in more detail. With the boundary conditions adopted in (98), the electric field $\pi_z$ knows about all the charge at $z' > z$. In particular there’s no flux through the horizon since $\pi_z \to 0$ as $z \to \infty$. We can write the solution to the Gauss constraint as $(\vec{x} = (\vec{x}, z))$

$$\pi_z = \int d^d x' \ f(x, x') \left( \partial_t \pi_i + i q (\pi_\phi \dot{\phi} - \pi_\phi^* \dot{\phi}^*) \right)$$ \hspace{1cm} (99)

Using this in (97), the solution for $A_0$ becomes

$$A_0 = \int d^d x' \ g(x, x') \left( \partial_t \pi_i + i q (\pi_\phi \dot{\phi} - \pi_\phi^* \dot{\phi}^*) \right)$$ \hspace{1cm} (101)

The boundary conditions on $A_0$ have been chosen so that $A_0 \sim z^{d-2}$ as $z \to 0$, which is the expected behavior for a gauge field near the boundary of AdS.

Finally we consider the resulting brackets. Plugging the solution to the constraints back into the action (93) one obtains an unconstrained action of the form

$$S = \int d^{d+1} x \left( \pi_\dot{\phi} \dot{\phi} + \pi_\dot{\phi}^* \dot{\phi}^* + \pi_i \dot{A}_i - \mathcal{H}(\phi, \pi_\phi, \phi^*, \pi_\phi^*, A_1, \pi_i) \right)$$ \hspace{1cm} (102)

The symplectic form for the physical degrees of freedom $\phi, \pi_\phi, \phi^*, \pi_\phi^*, A_1, \pi_i$ retains its canonical form, so these degrees of freedom obey the usual brackets.

$$\{ \pi_i(x), A_j(x') \} = \delta_{ij} \delta^d(x - x') \hspace{1cm} \hat{i}, \hat{j} = 1, \ldots, d - 1$$

$$\{ \pi_\phi(x), \phi(x') \} = \delta^d(x - x')$$

$$\{ \pi_\phi^*(x), \phi^*(x') \} = \delta^d(x - x')$$

\[28\]In the AdS/CFT dictionary $A_\mu \sim z^{d-2} j_\mu$ as $z \to 0$ where $j_\mu$ is identified with a conserved current in the CFT. Slightly different boundary conditions were used in section 2 of [14].
However these fields have non-trivial brackets with $A_0$ and $\pi_z$. These brackets can be calculated from the solutions (99), (101) by using the canonical brackets for the physical degrees of freedom. In this way we obtain

\[
\begin{align*}
\{A_0(x), A_1(x')\} &= \partial_1 g(x, x') \\
\{A_0(x), \phi(x')\} &= i q g(x, x') \phi(x') \\
\{A_0(x), \pi_\phi(x')\} &= -i q g(x, x') \pi_\phi(x') \\
\{\pi_z(x), A_1(x')\} &= \partial_1 f(x, x') \\
\{\pi_z(x), \phi(x')\} &= i q f(x, x') \phi(x') \\
\{\pi_z(x), \pi_\phi(x')\} &= -i q f(x, x') \pi_\phi(x')
\end{align*}
\]

Aside from the different choice of boundary conditions for the kernel $g$, these results match section 2 of [14], where the brackets were obtained following Dirac’s procedure for constrained systems.

B Symplectic form for matter

Consider a phase space with coordinates $p_i, q_i, P_I, Q_I$ and suppose there is a first-order Lagrangian of the form

\[
L = p_i \dot{q}_i + f_I(p, q, P, Q) \dot{Q}_I - H(p, q, P, Q)
\]

(103)

This applies to gravity in holographic gauge, with $p, q$ representing matter degrees of freedom and $P, Q$ representing the physical gravity degrees of freedom $\tilde{\pi}^ij$, $\tilde{g}_{ij}$. Following [23], we introduce the canonical 1-form

\[
a = p_i dq_i + f_I dQ_I
\]

(104)

and the symplectic 2-form $\omega = da$. The canonical brackets are then given by, for example,

\[
\{p_i, q_j\} = -(\omega^{-1})_{ij}
\]

(105)

To compute this we decompose the symplectic form in blocks,

\[
\omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

(106)
where
\[
\begin{align*}
    a &= \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix}, &
    b &= \begin{pmatrix} 0 & \frac{\partial f_j}{\partial p_i} \\ 0 & \frac{\partial f_i}{\partial q_i} \end{pmatrix}, \\
    c &= \begin{pmatrix} 0 & \frac{\partial f_i}{\partial p_j} - \frac{\partial f_j}{\partial q_i} \\ -\frac{\partial f_i}{\partial p_j} & 0 \end{pmatrix}, &
    d &= \begin{pmatrix} 0 & \frac{\partial f_j}{\partial f_j} \frac{\partial f_i}{\partial p_i} - \frac{\partial f_i}{\partial q_i} \\ 0 & \frac{\partial f_j}{\partial f_i} \frac{\partial f_i}{\partial p_i} - \frac{\partial f_i}{\partial q_j} \end{pmatrix}.
\end{align*}
\]

(107)

By the blockwise inversion theorem
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix}
\]

(109)

Using this to compute \(d^{-1}\) one obtains a matrix of the form
\[
d^{-1} = \begin{pmatrix} \vdots & \vdots \\ \vdots & 0 \end{pmatrix}
\]

(110)

Using it a second time to compute \(\omega^{-1}\), one finds that the upper left block of \(\omega^{-1}\) is
\[
(a - bd^{-1}c)^{-1} = \left( a - \begin{pmatrix} 0 & \vdots \\ 0 & \vdots \end{pmatrix} \left( \begin{pmatrix} \vdots & \vdots \\ \vdots & 0 \end{pmatrix} \right) \right)^{-1}
\]

(111)

But this reduces to \(a^{-1} = \begin{pmatrix} 0 & -\delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix}\), which means the coordinates \(p_i, q_i\) obey canonical brackets \(\{p_i, q_j\} = \delta_{ij}\) independent of the functions \(f_i\).

### C Scalar three-point function

In this section for completeness we present results on the three-point function of one bulk scalar operator and two boundary primary scalar operators. For additional details see [12, 14]. The general form of the correlator, with the appropriate behavior under conformal transformations, is

\[
\langle \phi_i(x, z)O_j(y_1)O_k(y_2) \rangle = c_{ijk} \frac{1}{(y_1 - y_2)^{2\Delta_j}} \left[ \frac{z}{z^2 + (x - y_2)^2} \right]^{(\Delta_k - \Delta_i)} \times \left( \frac{1}{\chi - 1} \right)^{\Delta_0} F(\Delta_0, \Delta_0 - \frac{d}{2} + 1, \Delta_i - \frac{d}{2} + 1, \frac{1}{1 - \chi})
\]

(112)
where
\[
\chi = \frac{[(x - y_1)^2 + z^2][(x - y_2)^2 + z^2]}{z^2(y_2 - y_1)^2} \tag{113}
\]
and \(\Delta_0 = \frac{1}{2}(\Delta_i + \Delta_j - \Delta_k)\).

The analytic structure, which controls the commutator between any two of the operators, is different in different dimensions. We look for non-analyticity in the region \(0 < \chi < 1\) where all points are bulk space-like separated from each other, since any non-analyticity in this region would signal a breakdown in micro-causality. If \(d\) is an odd integer we can use the transformation formula for the hypergeometric function

\[
F(a, b, c, z) = (-z)^{-a} \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(c - a)\Gamma(b)} F(a, a - c + 1, a - b + 1, \frac{1}{z})
+ (-z)^{-b} \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(c - b)\Gamma(a)} F(b, b - c + 1, b - a + 1, \frac{1}{z}) \tag{114}
\]

The use of this formula in \((112)\) gives the three-point function an expansion about \(\chi = 1\) of the form

\[
\frac{1}{(\chi - 1)^{\frac{d}{2}}} \sum_{n=0}^{\infty} a_n (1 - \chi)^{n+1} \tag{115}
\]

Due to the square root branch cut, this implies a non-zero commutator at spacelike separation.

If \(d\) is an even integer, the above transformation formula is not valid. Instead one can use

\[
F(a, a + n, c, z) = \frac{\Gamma(c)(-z)^{-a}}{\Gamma(c - a)\Gamma(a + n)} \sum_{k=0}^{n-1} \frac{(n - k - 1)!\Gamma(1 - c + a)k}{k!} (-z)^{-k}
+ \frac{\Gamma(c)(-z)^{-a}}{\Gamma(a)\Gamma(c - a - n)} \sum_{k=0}^{\infty} \frac{(a + n)k(1 - c + a + n)k}{(n + k)!k!} [\psi(k + 1) + \psi(n + k + 1) - \psi(a + n + k) - \psi(c - a - n - k) + \ln(-z)] z^{-n-k} \tag{116}
\]

where \(\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}\) and \((n)_k = \frac{\Gamma(n+k)}{\Gamma(n)}\). Using this in \((112)\) we get an expansion about \(\chi = 1\) of the form

\[
\sum_{k=0}^{d/2} b_k (1 - \chi)^{-\frac{d}{2}+1+k} + \ln(\chi - 1) \sum_{k=0}^{\infty} a_k (1 - \chi)^k \tag{117}
\]
Again this results in a non-zero commutator at spacelike separation.

Thus whether $d$ is even or odd one finds a non-zero commutator in the region $0 < \chi < 1$. The commutator has an expansion in powers of $(\chi - 1)$. It is important to note that, as can be seen from (112), for fixed $O_j$ and $O_k$ the form of the expansion is independent of the dimension $\Delta_i$.

We wish to define our bulk operator $\phi_i(x, z)$ in order to have the smallest possible commutator with the boundary operators at spacelike separation. It should also transform as a bulk scalar under AdS isometries and have the correct boundary behavior

$$\phi_i(x, z) \xrightarrow{z \to 0} z^{\Delta_i} O_i. \quad (118)$$

From the above structure we see that if we have a tower of primary scalar operators $O_l$ with dimensions $\Delta_l$, whose three-point functions $\langle O_l O_j O_k \rangle$ are non-zero, then we can redefine the bulk operator $\phi_i(x, z)$ to have the form

$$\phi_i(z, x) = \int dx' K_{\Delta_i}(z, x|x') O_i(x') + \sum_l a_l \int dx' K_{\Delta_l}(z, x|x') O_l(x') \quad (119)$$

Here $K_{\Delta}(z, x|x')$ is the smearing function for a dimension $\Delta$ primary scalar.

Since the form of the singularity is the same for any $O_l$, we can choose the coefficients $a_l$ to cancel the commutator to any desired order in the expansion about $\chi = 1$. If we have an infinite number of suitable higher-dimension operators we can make the bulk scalar commute at spacelike separation. Fortunately in the $1/N$ expansion one can construct the necessary $O_l$ as multi-trace operators built from products of the $O_k$ and $O_j$ together with derivative operators. If $O_j$ and $O_k$ are single-trace operators this procedure begins with a double-trace operator and thus $a_l \sim 1/N^{29}$ For any three-point function involving $\phi_i(z, x)$ we will have to add a different tower of higher dimension operators to our definition of the bulk scalar. At leading order in the $1/N$ expansion these towers exist and are independent. It should be possible to correct these operators, order-by-order in the $1/N$ expansion, to preserve micro-causality in the bulk. But it seems clear that at finite $N$ the required towers of operators cannot exist and micro-causality will be violated.

\[29\] For a general discussion of large-$N$ counting in this context see p. 26 of [12].
D Constructing operators

In this appendix we give details of the construction of the operators $V_{mlk}$ and $M_{mlk}$.

D.1 Spin 1

The operators $V_{mlk}$ are defined by

$$V_{mlk} = \frac{1}{N} \sum_{m'+l'+k'=m+l+k} a_{mlk}^m A_{mlk}^m$$

where

$$A_{mlk}^m = \partial_{\mu_1} \cdots \partial_{\mu_m} \nabla^2 \partial_{\nu} \cdots \partial_{\nu} O_2$$

and the coefficients $a_{mlk}^m$ are to be chosen such that

$$\langle F_{\mu\nu}(x)V_{mlk}(y_1)O_2(y_2) \rangle = \frac{1}{(y_1 - y_2)^{2\Delta_j + 2 + 2m + 2l}}$$

The three-point function is evaluated to leading order in $1/N$ as a factorized product of two-point functions, thus

$$\langle F_{\mu\nu}(x) \partial_{\mu_1} \cdots \partial_{\mu_m} \nabla^2 \partial_{\nu} \cdots \partial_{\nu} O_2(y_1)O_2(y_2) \rangle$$

Let's study the structure of each term. Using

$$\langle F_{\mu\nu}(x)j_{\mu}(y_1) \rangle = (1 - \Delta_j) \frac{\eta_{\mu\nu}(x - y_1) - \eta_{\mu\nu}(x - y_2)}{(x - y_1)^{2\Delta_j + 2}}$$

$$\langle O(y_1)O(y_2) \rangle = \frac{1}{(y_1 - y_2)^{2\Delta}}$$

one can easily see that from symmetry consideration and counting derivatives that the right hand side of (123) is a linear combination of terms appearing
on the right hand side of (122), with \( m + l + k = m' + l' + k' \). So in principle one can just invert this. We label
\[
\langle F_{\mu\nu}(x) A^{mlk}(y_1) \mathcal{O}^*(y_2) \rangle \equiv \langle A^{mlk} \rangle \\
\langle F_{\mu\nu}(x) V^{mlk}(y_1) \mathcal{O}^*(y_2) \rangle \equiv \langle V^{mlk} \rangle
\]  
(125)

We start with the initial condition (for a conserved current \( \Delta_j = d - 1 \))
\[
\langle V^{0lk} \rangle = \frac{1}{\alpha_{lk}} \langle A^{0lk} \rangle \\
\alpha_{lk} = 2\Delta(\Delta_j - 1)\prod_{i=0}^{l-1}(2\Delta_j + 2 + 2i)(2\Delta_j - d + 2i)\prod_{r=0}^{k-1}(2\Delta + 2 - d + 2r)(2\Delta + 2 + 2r)
\]
and use the relationships
\[
\langle V^{m+1,l,k} \rangle = -\frac{1}{\beta_1} \partial_{y_2}^{\alpha} \partial_{\rho,y_2} \langle V^{mlk} \rangle + \frac{\beta_2}{\beta_1} \langle V^{m-1,l+1,k+1} \rangle \\
\langle A^{m+1,l,k} \rangle = -\partial_{y_2}^{\alpha} \partial_{\rho,y_2} \langle A^{mlk} \rangle \\
\beta_1 = (2\Delta + 2 + 2m + 2l)(2\Delta + 2 + 2m + 2k) \\
\beta_2 = m(3 + 2\Delta_j + 2\Delta + 3m + 2l + 2k - d)
\]  
(126)

to get an iterative procedure to express \( V^{mlk} \) in terms of \( A^{m'l'k'} \).

## D.2 Spin 2

Here we give a similar discussion for spin two. For simplicity we only consider operators built from a conserved stress tensor.

We would like to find operators \( \mathcal{M}_{mlk} \) that obey
\[
\langle C_{\alpha\beta\gamma\delta}(x) \mathcal{M}_{mlk}(y_1) \mathcal{O}(y_2) \rangle = \frac{[(y_1 - y_2)_\alpha(x - y_1)^\alpha]^m}{(x - y_1)^{2d+2m+2l+4}(y_1 - y_2)^{2\Delta+2m+2k+4}} \\
[\{(x - y_1)_\beta(x - y_1)_\delta(x - y_2)_\alpha(x - y_2)_\gamma - (\gamma \leftrightarrow \delta) - (\beta \leftrightarrow \alpha) + (\gamma \leftrightarrow \delta + \beta \leftrightarrow \alpha)\]
\]

We do this by writing
\[
\mathcal{M}_{mlk} = \sum_{m+l+k=m'+l'+k'} b_{mlk}^{m'l'k'} \mathcal{T}_{m'l'k'}
\]  
(127)
where
\[
T_{m'\nu'k'} = \partial_{\mu_1'\cdots\mu_{m'}} \nabla^{2\nu'} T_{\rho\nu} \partial^{\mu_1'\cdots\mu_{m'}} \nabla^{2k'} \partial^{\nu} \partial^{\rho} O
\] (128)

The three-point function is evaluated to leading order in $1/N$ as a factorized product of two-point functions.
\[
\langle C_{\alpha\beta\gamma\delta}(x) T_{\rho\sigma}(y_1) O(y_2) \rangle = \langle C_{\alpha\beta\gamma\delta}(x) \partial_{\mu_1'\cdots\mu_{m'}} \nabla^{2\nu'} T_{\rho\nu}(y_1) \rangle \times \langle \partial^{\mu_1'\cdots\mu_{m'}} \nabla^{2k'} \partial^{\nu} \partial^{\rho} O(y_1) O(y_2) \rangle
\] (129)

It is useful to note that the only term in the $\langle TT \rangle$ correlator, that contributes to the $\langle CT \rangle$ correlator when all indices on the Weyl tensor are taken to have distinct values, is
\[
\langle T_{\mu\nu}(x_1) T_{\rho\sigma}(y_1) \rangle = d(d - 1)(\eta_{\mu\sigma} \eta_{\nu\rho} + \eta_{\mu\rho} \eta_{\nu\sigma}) \frac{1}{(x - y_1)^{2d}} + \cdots.
\] (130)

We now describe an iterative procedure to get $M_{mlk}$. Let us label
\[
\langle M_{mlk} \rangle \equiv \langle C_{\alpha\beta\gamma\delta}(x) M_{mlk}(y_1) O(y_2) \rangle
\]
\[
\langle T_{mlk} \rangle \equiv \langle C_{\alpha\beta\gamma\delta}(x) T_{mlk}(y_1) O(y_2) \rangle
\] (131)

We start with the initial condition
\[
\langle M_{0lk} \rangle = \frac{1}{\beta_{lk}} \langle T_{0kk} \rangle
\]
\[
\beta_{lk} = 2d(d - 1) \Pi_i=0^{i+1}(2d + 2i) \Pi_j=1^{j+1}(d + 2j) \Pi_i=0^{i+1}(2\Delta + 2i) \Pi_j=1^{j+1}(2\Delta - d + 2j)
\]
and use the relationships
\[
\langle M_{m+1,l,k} \rangle = -\frac{1}{\gamma_2} \partial^{\mu}_x \partial_{\mu,y_2} \langle M_{mlk} \rangle + \frac{\gamma_1}{\gamma_2} \langle M_{m-1,l+1,k+1} \rangle
\]
\[
-\partial^{\mu}_x \partial_{\mu,y_2} \langle T_{mlk} \rangle = \langle T_{m+1,l,k} \rangle
\]
\[
\gamma_1 = m(5 + d + 3m + 2l + 2\Delta + 2k)
\]
\[
\gamma_2 = (2d + 2m + 2l + 4)(2\Delta + 2m + 2k + 4)
\]
to compute in an iterative procedure $M_{mlk}$ as linear combinations of $T_{m'\nu'k'}$.



development of the behavior under conformal transformations

In this appendix we give an explicit computation of the behavior under special

conformal transformations of the operators $A_n$ for $n = 0$ and $n = 1$ in
gauge theory, and for $T_0$ in gravity. For gauge theory we use the following

transformation rules. To first order in the parameter $b$,

$$
x'_\mu = x_\mu + 2b \cdot xx_\mu - b_\mu x^2
$$

$$
\partial'_\mu = \partial_\mu - 2b \cdot x \partial_\mu - 2b_\mu x^\lambda \partial_\lambda + 2x_\mu b^\lambda \partial_\lambda
$$

$$
j'_\mu = j_\mu + 2x_\mu b \cdot j - 2b_\mu x \cdot j - 2(d - 1)b \cdot x j_\mu
$$

$$
O' = (1 - 2\Delta b \cdot x)O
$$

Now $A_0 = j_\mu \partial^\mu O$, and using the above transformation to first order in $b_\rho$,

one finds [14]

$$
(j_\mu \partial^\mu O)' = j_\mu \partial^\mu O(1 - 2b \cdot x(\Delta + d)) - 2\Delta b \cdot jO
$$

(132)

This matches the property advertised in (51), that the only terms in $(A_0)'$

that differ from a primary scalar involve $b \cdot j$.

Now consider $A_1$. From the requirement that it has the correct three-

point correlation function with $F_{\mu \nu}$ and $O$ we found in [14] that

$$
A_1 \sim \left( \frac{1}{2d^2} \nabla^2 j_\rho \partial^\rho O + \frac{1}{2(\Delta + 1)(2\Delta + 2 - d)}j_\rho \nabla^2 \partial^\rho O - \frac{1}{2d(\Delta + 1)}\partial_\rho j_\mu \partial^\rho \partial^\mu O \right)
$$

(133)

Now using the above transformation one finds

$$
(\nabla^2 j_\rho \partial^\rho O)' = (\nabla^2 j_\rho \partial^\rho O)(1 - 2(\Delta + d + 2)b \cdot x) + 4b^\rho \partial_\mu j_\rho \partial^\mu O - 2db^\rho \partial_\rho j_\mu \partial^\mu O - 2\Delta b^\rho \nabla^2 j_\mu O
$$

$$
(j_\rho \nabla^2 \partial^\rho O)' = (j_\rho \nabla^2 \partial^\rho O)(1 - 2(\Delta + d + 2)b \cdot x) - 2(2\Delta + 2 - d)b^\rho j_\mu \partial^\rho \partial^\mu O - 2(\Delta + 2)b^\rho j_{\mu\nu} \nabla^2 O
$$

$$
(\partial_\rho j_\mu \partial^\rho \partial^\mu O)' = (\partial_\rho j_\mu \partial^\rho \partial^\mu O)(1 - 2(\Delta + d + 2)b \cdot x) - 2(\Delta + 1)b^\rho \partial_\rho j_\mu \partial^\mu O - 2\Delta b^\rho \partial_\rho j_\mu \partial^\rho \partial^\mu O - 2(d - 1)b \cdot x j_\mu O
$$

(134)

Putting this all together, again the only terms in $(A_1)'$ that differ from a

primary scalar are those which involve $b \cdot j$.

41
For gravity we need the transformation of $T_0 = T_{\mu\nu} \partial^\mu \partial^\nu \mathcal{O}$. We have

$$T'_{\mu\nu} = T_{\mu\nu}(1 - 2d(b \cdot x)) + 2b^\delta x_\nu T_{\mu\delta} - 2b_\mu x^\delta T_{\mu\delta} + 2b^\delta x_\mu T_{\delta\nu} - 2b_\mu x^\delta T_{\delta\nu}$$

$$(\partial^\mu \partial^\nu \mathcal{O})' = (\partial^\mu \partial^\nu \mathcal{O})(1 - 2(\Delta + 2)d(b \cdot x)) - 2(\Delta + 1)(b^\mu \partial^\nu + b^\nu \partial^\mu)\mathcal{O} - 2b^\nu x^\lambda \partial_\lambda \partial^\mu \mathcal{O} - 2b^\mu x^\lambda \partial_\lambda \partial^\nu \mathcal{O} + 2x^\nu b^\lambda \partial_\lambda \partial^\mu \mathcal{O} + 2x^\mu b^\lambda \partial_\lambda \partial^\nu \mathcal{O}$$

(135)

from which we find to first order in $b_\mu$

$$(T_{\mu\nu} \partial^\mu \partial^\nu \mathcal{O})' = (T_{\mu\nu} \partial^\mu \partial^\nu \mathcal{O})(1 - 2(\Delta + 2 + d)b \cdot x) - 4(\Delta + 1)b^\mu T_{\mu\nu} \partial^\nu \mathcal{O}$$

Again the terms that differ from a primary scalar have the form (81).

References

[1] C. J. Isham, “Canonical quantum gravity and the problem of time,” arXiv:gr-qc/9210011.

[2] J. M. Maldacena, “The large $N$ limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231–252, hep-th/9711200.

[3] T. Banks, M. R. Douglas, G. T. Horowitz, and E. J. Martinec, “AdS dynamics from conformal field theory,” arXiv:hep-th/9808016.

[4] V. Balasubramanian, P. Kraus, and A. E. Lawrence, “Bulk vs. boundary dynamics in anti-de Sitter spacetime,” Phys. Rev. D59 (1999) 046003, arXiv:hep-th/9805171.

[5] V. K. Dobrev, “Intertwining operator realization of the AdS/CFT correspondence,” Nucl. Phys. B553 (1999) 559–582, arXiv:hep-th/9812194.

[6] I. Bena, “On the construction of local fields in the bulk of AdS(5) and other spaces,” Phys. Rev. D62 (2000) 066007, arXiv:hep-th/9905186.

[7] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, “Local bulk operators in AdS/CFT: A boundary view of horizons and locality,” Phys. Rev. D73 (2006) 086003, hep-th/0506118.
[8] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, “Holographic representation of local bulk operators,” Phys. Rev. D74 (2006) 066009, hep-th/0606141.

[9] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, “Local bulk operators in AdS/CFT: A holographic description of the black hole interior,” Phys. Rev. D75 (2007) 106001, hep-th/0612053.

[10] I. Heemskerk, “Construction of bulk fields with gauge redundancy,” arXiv:1201.3666 [hep-th].

[11] D. Kabat, G. Lifschytz, S. Roy, and D. Sarkar, “Holographic representation of bulk fields with spin in AdS/CFT,” Phys. Rev. D86 (2012) 026004 arXiv:1204.0126 [hep-th].

[12] D. Kabat, G. Lifschytz, and D. A. Lowe, “Constructing local bulk observables in interacting AdS/CFT,” Phys. Rev. D83 (2011) 106009, arXiv:1102.2910 [hep-th].

[13] I. Heemskerk, D. Marolf, and J. Polchinski, “Bulk and transhorizon measurements in AdS/CFT,” arXiv:1201.3664 [hep-th].

[14] D. Kabat and G. Lifschytz, “CFT representation of interacting bulk gauge fields in AdS,” Phys. Rev. D87 (2013) 086004 arXiv:1212.3788 [hep-th].

[15] B. S. DeWitt, “Quantum theory of gravity. 2. The manifestly covariant theory,” Phys. Rev. 162 (1967) 1195–1239.

[16] R. L. Arnowitt, S. Deser, and C. W. Misner, “The dynamics of general relativity,” Gen. Rel. Grav. 40 (2008) 1997–2027, arXiv:gr-qc/0405109 [gr-qc].

[17] R. L. Arnowitt, S. Deser, and C. W. Misner, “Gravitational-electromagnetic coupling and the classical self-energy problem,” Phys. Rev. 120 (1960) 313–320.

[18] D. Marolf, “Unitarity and holography in gravitational physics,” Phys. Rev. D79 (2009) 044010 arXiv:0808.2842 [gr-qc].
[19] G. Lifschytz and V. Periwal, “Schwinger-Dyson = Wheeler-DeWitt: Gauge theory observables as bulk operators,” *JHEP* **04** (2000) 026, arXiv:hep-th/0003179

[20] C. Fefferman and C.R. Graham, *Conformal Invariants*. In *Elie Cartan et les Mathématiques d’aujourd’hui* (Astérisque, 1985), p. 95.

[21] A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems*. Accademia Nazionale dei Lincei, 1976.

[22] P. Dirac, “Fixation of coordinates in the Hamiltonian theory of gravitation,” *Phys.Rev.* **114** (1959) 924–930

[23] L. Faddeev and R. Jackiw, “Hamiltonian reduction of unconstrained and constrained systems,” *Phys.Rev.Lett.* **60** (1988) 1692