LONG-WAVELENGTH UNSTABLE MODES IN THE FAR UPSTREAM OF RELATIVISTIC COLLISIONLESS SHOCKS

ITY RABINAK, BOAZ KATZ, AND ELI WAXMAN
Department of Particle Physics and Astrophysics, The Weizmann Institute of Science, Rehovot 76100, Israel; itay.rabinak@weizmann.ac.il
Received 2010 June 22; accepted 2011 May 23; published 2011 July 19

ABSTRACT

The growth rate of long-wavelength kinetic instabilities arising due to the interaction of a collimated beam of relativistic particles and a cold unmagnetized plasma are calculated in the ultrarelativistic limit. For sufficiently culminated beams, all long-wavelength modes are shown to be Weibel-unstable, and a simple analytic expression for their growth rate is derived. For large transverse velocity spreads, these modes become stable. An analytic condition for stability is given. These analytic results, which generalize earlier ones given in the literature, are shown to be in agreement with numerical solutions of the dispersion equation and with the results of novel particle in cell simulations in which the electromagnetic fields are restricted to a given k-mode. The results may describe the interaction of energetic cosmic rays, propagating into the far upstream of a relativistic collisionless shock, with a cold unmagnetized upstream. The long-wavelength modes considered may be efficient in deflecting particles and could be important for diffusive shock acceleration. It is shown that while these modes grow in relativistic shocks propagating into electron–positron pair plasmas, they are damped in relativistic shocks propagating into electron–proton plasmas with moderate Lorentz factors $\Gamma_{\text{sh}} \lesssim (m_e/m_p)^{1/2}$. If these modes dominate the deflection of energetic cosmic rays in electron–positron shocks, it is argued that particle acceleration is suppressed at shock frame energies that are larger than the downstream thermal energy by a factor of $\gtrsim \Gamma_{\text{sh}}$. Key words: acceleration of particles – cosmic rays – shock waves

Online-only material: color figures

1. INTRODUCTION

Current understanding of gamma-ray burst (GRB) “afterglows,” the delayed low energy emission following the prompt $\gamma$-ray emission, suggests that the radiation observed is the synchrotron emission of energetic non-thermal electrons in the downstream of an ultrarelativistic collisionless shock driven into the surrounding interstellar medium (ISM) or stellar wind (Zhang & Mészáros 2004; Piran 2004).

This model requires a strong magnetic field and a large population of energetic electrons to be present in the downstream. Observations suggest that the fraction of post-shock thermal energy density carried by non-thermal electrons, $\epsilon_e$, is large, $\epsilon_e \approx 0.1$ (e.g., Zhang & Mészáros 2004; Frail et al. 2001; Freedman & Waxman 2001; Berger et al. 2003). The fraction of post-shock thermal energy carried by the magnetic field, $\epsilon_B$, is less well constrained by observations. However, in cases where $\epsilon_B$ can be reliably constrained by multi-waveband spectra, values close to equipartition, $\epsilon_B \sim 0.01–0.1$, are inferred (e.g., Frail et al. 2000).

The non-thermal energetic electron (and proton) population is believed to be produced by the diffusive (Fermi) shock acceleration (DSA) mechanism (for reviews see Drury 1983; Blandford & Eichler 1987; Malkov & O’C Drury 2001). The required magnetic fields in the shock frame in the downstream (e.g., Frail et al. 2000) and upstream (Li & Waxman 2006) regions are much larger than the ambient field, and thus require substantial amplification. The accelerated particles are likely to have an important role in generating and maintaining the inferred magnetic fields.

The main challenge associated with the downstream magnetic field is that the field amplitude must remain close to equipartition deep into the downstream, over distances $\sim 10^{10} l_{\text{sd}}$ (Gruzinov & Waxman 1999; Gruzinov 2001a). While near-equipartition fields on skin-depth scale are likely to be produced in the vicinity of the shock by electromagnetic (e.g., Weibel-like) instabilities (e.g., Blandford & Eichler 1987; Gruzinov & Waxman 1999; Medvedev & Loeb 1999; Wiermsa & Achterberg 2004), they are expected to decay within a few skin depths downstream (Gruzinov 2001a; Chang et al. 2008a, 2008b; Keshet et al. 2009). This suggests that the correlation length of the magnetic field far downstream and possibly upstream must be much larger than the skin depth, $D \gg l_{\text{sd}}$, perhaps even of the order of the distance from the shock (Gruzinov & Waxman 1999; Gruzinov 2001a; Medvedev et al. 2005; Katz et al. 2007).

The search for a self-consistent theory of collisionless shocks has led to extensive numerical studies using the particle in cell (PIC) based algorithms (e.g., Gruzinov 2001a, 2001b; Silva et al. 2003; Nishikawa et al. 2003; Feiderksen et al. 2004; Jaroschek et al. 2004; Spitkovsky 2005, 2008a; Martins et al. 2009). Such simulations have provided compelling evidence for acceleration of particles and generation of long lasting near-equipartition magnetic fields. However numerically simulating the long-term behavior is challenging and is currently restricted to pair ($e^+e^-$) plasmas in two dimensions (e.g., Spitkovsky 2008b; Keshet et al. 2009; Medvedev & Zakutnyaya 2009).

Large-scale magnetic fields may possibly be generated in the upstream by the interaction of the beam of cosmic rays (CRs) propagating ahead of the shock and the upstream plasma (e.g., Katz et al. 2007; Keshet et al. 2009). In particular, high-energy CRs naturally introduce large scales due to their large Larmor radius, and the large distances to which they propagate into the upstream. Instabilities arising from the interaction of relativistic beams and cold plasmas have long been studied (Akhiezer 1975, and references therein) and are suspected of amplifying the magnetic field in the shock transition layer (Gruzinov & Waxman 1999; Medvedev & Loeb 1999; Wiermsa & Achterberg 2004; Bret et al. 2005; Lyubarsky & Eichler...
2. ANALYSIS

Consider a homogeneous, anisotropic distribution of particles consisting of a cold plasma and a beam of ultrarelativistic CRs moving in the positive x-axis direction with a narrow, axisymmetric, velocity distribution. The analysis is carried out in the rest frame of the cold plasma which initially has zero magnetic and electric fields. All quantities considered below are taken in this frame.

The plasma frequencies of the cold plasma and the beam are denoted by $\omega_0$ and $\omega_{\text{CR}}$, respectively, where the plasma frequency of a plasma with species $i$ is defined by

$$\omega_p^2 = \sum_i \frac{4\pi q_i^2}{m_i} \int \frac{d^3p'}{\gamma(p')} f_i(p'),$$

where $q_i$, $m_i$, $f_i(p)$ are the species’ charge, mass, and momentum distribution. The momentum distribution function of the cold plasma is $f_0(p) = n_0 \delta(p)$, where $n_0$ is the density of the plasma. The axisymmetric momentum distribution function of the narrow CR beam,

$$f_{\text{CR}}(p) = f(p_x, p_\perp),$$

is non-zero only in a narrow range of angles $p_\perp/p_x < \theta_{\text{max}}$, where $p_x$ and $p_\perp$ are the components of the momentum parallel and perpendicular to the $x$-axis, respectively. It is also assumed that $\omega_0 \gg \omega_{\text{CR}}$.

In this section we analyze the linear growth of unstable modes with long wavelengths, $k \ll \omega_0$. We start by considering a beam with no transverse velocity spread in Section 2.1. We show that the entire $k$-space regime considered is unstable and provides a simple analytic expression of the instability growth rate. The effects of a spread in the velocity directions of the CRs are discussed in Section 2.2.

2.1. No Spread

Consider the simplest case in which all the particles in the beam propagate in the same direction, $f_{\text{CR}}(p) = f(p_x) \delta(p_x) \delta(p_\perp)$, and are ultrarelativistic (with an arbitrary energy distribution). It is straightforward to write the full dispersion equation, which turns out to be a sixth-order polynomial equation for $\omega$ with real coefficients (cf. Appendix A and, e.g., Akhiezer 1975, Section 6.4.2). Four of the six solutions for $\omega$ are small perturbations, of order $\omega_0^2/\omega^2$ of the four cold plasma oscillating modes $\omega = \pm \omega_0 \pm (\omega_0^2 + k^2)^{0.5}$, and are real (stable). The two remaining solutions have a non-zero imaginary part and therefore are complex conjugates of each other. Hence, for each $k$ there is one unstable mode. Below we derive, directly from the Maxwell–Vlasov equations, an approximate expression for the growth rate of this mode, Equations (12) and (13). More details and a numerical solution for the dispersion equation are given in Appendix A.

For any axisymmetric distribution of particles the linear modes can be separated into modes having an electric field in the $x-k$ plane, where $x$ is the axis of symmetry (and magnetic field perpendicular to this plane), and modes with an electric field perpendicular to $x$ and $k$ (cf. Appendix A). The unstable mode has an electric field, $E$, in the $x-k$ plane, and a magnetic field, $B$, perpendicular to this plane.

The electrical currents carried by the cold plasma and the CRs, as derived by the Vlasov equations (see Appendix A and also, e.g., Akhiezer 1975, Section 6.4.1), are respectively given by

$$4\pi J_0 = \frac{\omega_0^2}{-i\omega} E,$$

and

$$4\pi J_{\text{CR},\perp} = i\omega_{\text{CR}}^2 (B - E_{\perp})/\Omega,$$

$$4\pi J_{\text{CR},||} = -i\omega_{\text{CR}}^2 (B - E_{\perp}) k_{||}/\Omega^2,$$

where subscripts $\perp$ and $||$ correspond to components that are perpendicular to the beam and parallel to the beam, respectively, $\Omega \equiv \beta_{\text{CR}} k - \omega = k_{||} - \omega$, and the ultrarelativistic approximation $\beta = 1$ for the CRs was used. In addition, terms with $E_{\perp}$, which are smaller by a factor of $\gamma^{-2}$ than the other terms, were neglected (cf. Equation (A1)). Below (following Equation (13)), we also give the growth rate when the terms with $E_{\perp}$ are not neglected.

By neglecting the displacement current $\partial_t E$, compared with the current carried by the cold plasma (using $\omega \ll \omega_0$ which is self consistently implied by the result, Equations (12) and (13)), the Maxwell equations read

$$ik_{\perp} B = 4\pi (J_{\text{CR},||} + J_{\text{CR},\perp});$$

$$-ik_{||} B = 4\pi (J_{\text{CR},\perp} + J_{\text{CR},||});$$

$$i\omega B = -ik_{\perp} E_{||} + ik_{||} E_{\perp}.$$  

Equation (8) can be written as

$$i\omega (B - E_{\perp}) = -ik_{\perp} E_{||},$$

by self consistently neglecting $\Omega E_{\perp}$ as follows. In regimes where $\Omega \ll k_{||}$, this term is negligible compared to the term $k_{||} E_{\perp}$. Otherwise, where $\Omega \gg k_{||}$, Equation (13), implies that $\Omega \sim \omega$ and Equations (12), (4), and (7) imply that $E_{\perp} \ll B$, making the term $\Omega E_{\perp}$ negligible compared with the term $\omega B$. In a similar manner, the result $E_{||} \ll B$ can be obtained in the former regime.

Substituting for $J_{\text{CR},||}$, $J_{\text{CR},\perp}$, $E_{\perp}$, Equations (5), (3), and (9) respectively, and neglecting $E_{||}$ with respect to $B$, Equation (6) becomes

$$k_{\perp} B = -\omega_{\text{CR}}^2 k_{\perp} B / \Omega^2 - \omega_0^2 B / k_{\perp}.$$  

2006; Achterberg & Wiersma 2007; Achterberg et al. 2007; Bret 2009). A systematic study of these instabilities for the lowest energy CRs, with energies comparable to the thermal energy of the shocked plasma, and their application for Fermi acceleration is given in Lemoine & Pelletier (2010).

In this paper we analyze long-wavelength plasma instabilities resulting from the counterstreaming flow of high-energy CRs, $\gamma \gg \Gamma_{\text{sh}}$, running far ahead of the shock and a non-magnetized upstream plasma. The analysis is restricted to long-wavelength modes, $k \ll \omega_0/c$, which are expected to deflect particles efficiently. For simplicity it is assumed that the particle distribution is homogenous. The paper is organized as follows. In Section 2 we calculate the growth rate of long-wavelength modes. We separately discuss highly collimated beams and beams with a significant transverse velocity spread, and derive a condition for the stability of these modes. In Section 3 we discuss the possible implications of these results to collisionless shocks. In Section 4 we summarize the main results and conclusions. An estimate of the saturation level of the modes is beyond the scope of this paper. Throughout this paper, units with $c = 1$ are assumed ($c$ is retained in some of the expressions).
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2.2. With Spread

For the case where the particles in the beam propagate in different directions, \( f_{\text{CR}}(p) \neq f(p)\delta(p)\delta(p_c) \), Equation (11) should be replaced with

\[
i \omega_{\text{CR}}^2 B k_\perp \left\langle \frac{1}{(\Omega - k_\perp \beta_\perp)^2} \right\rangle = -i \omega_0^2 B / k_\perp
\]

or

\[
i \omega_{\text{CR}}^2 B k_\perp \left\langle \frac{1}{(\Omega - k_\perp \beta_\perp)^2} \right\rangle = - \frac{1}{\eta_0}
\]

where \( \Omega \equiv k_\perp \beta_\perp - \omega, \eta_0 \) is the growth rate for a delta function momentum distribution Equation (12), and in addition to the approximations leading to Equation (11) we have also neglected self consistently terms proportional to \( \beta_\perp \) in the equation for the current Equation (A1) before plugging it to Equation (6). These terms are neglected because they are smaller by a factor of the small parameter \( \alpha_{\text{CR}} / \omega_0 \) with respect to the rest of the terms (cf. Equations (25), (31), and the following remarks). The averaging above is carried over the velocity distribution function,

\[
\left\langle \frac{1}{(\Omega - k_\perp \beta_\perp)^2} \right\rangle = \int_{-\infty}^{\infty} \frac{f_\beta(\beta_\perp)}{(\Omega - k_\perp \beta_\perp)^2} d\beta_\perp,
\]

where

\[
f_\beta(\beta_\perp) = \sum_i \frac{4\pi q_i^2}{m_i} \int d^3p \gamma(p') f_i(p') \delta(\beta_\perp - \beta_\perp).\]

It is to be understood that whenever a singularity is encountered, the expression should be evaluated at \( \Omega \to -i \epsilon \) in the limit \( \epsilon \to 0^+ \).

Consider first the following one-dimensional rectangular distribution: \( \beta_\parallel \equiv \beta_\parallel = \beta \cos(\theta), \beta_\perp \equiv \beta_\perp = \beta \sin(\theta), \beta_\parallel = 0 \), with \( \theta \) uniformly distributed between \( -\Delta \theta \), and \( \beta = \sqrt{1 - \gamma^2} \). This distribution is similar to the one discussed by Silva et al. (2002). The analysis here includes modes with \( k_\parallel \neq 0 \), not included in (Silva et al. 2002). For a discussion of a distribution with two identical counterstreaming beams see Achterberg & Wiersma (2007).

For small angular spread the growth rate can be approximated analytically by neglecting the variations of \( \beta_\parallel \) (assuming \( \beta_\parallel = \beta \)). Under this assumption Equation (18) reads

\[
\left\langle \frac{1}{(\Omega - k_\perp \beta_\perp)^2} \right\rangle = \int_{-\Delta \theta}^{\Delta \theta} \frac{d\beta_\perp}{(\Omega - k_\perp \beta_\perp)^2}
\]

\[
= \frac{1}{(k_\perp \beta \Delta \theta)^2 - \Omega^2},
\]

the dispersion relation (Equation (17)) reads

\[
\Omega^2 = (k_\perp \beta \Delta \theta)^2 - \eta_0^2
\]

(see also Silva et al. 2002, Equation (6)).

For illustration, the growth rates of the unstable modes are shown in Figure 2 as a function of the spread \( \Delta \theta \) for the

Figure 1. Growth rate of unstable modes for a delta function momentum distribution of beam particles with plasma frequency \( \omega_{\text{CR}} / \omega_0 = 0.01 \) and Lorentz factor \( \gamma = 5000 \). The solid lines are the exact solutions of the dispersion relation, evaluated numerically (cf. Appendix A). The dashed line (covering the solid line with \( k_\parallel = 0.05 \omega_0 \)) is the analytical approximation given in Equation (12). The curvature at low values of \( k_\perp \) is due to the electrostatic mode.
one-dimensional rectangular distribution considered above. The solid lines in this figure are the results of a semi-analytical calculation in which the velocity integrals in the equation for the beam susceptibility (cf. Equation (A2) and the following remarks) were evaluated analytically, and a continuous solution for the dispersion function as a function of the spread was found numerically, starting from the solution for a delta function distribution. The obtained solution is verified to be the fastest growing one at a given \( k_\perp / \omega_0 \), by comparing it to the results of a “1-mode PIC simulation” (cf. Appendix B), respectively. Dotted lines show the growth rate for a delta function momentum distribution (no spread, \( \Delta \theta = 0 \)).

The last equality follows from the fact that

\[
\int_{-\infty}^{\infty} \frac{1}{(x+i\epsilon)^2} \, dx = 0.
\]

The following features of Equation (17) generalize the features of the one-dimensional distribution considered above with the condition for instability,

\[
\langle k_\perp \rangle (\beta_\perp^{-2})^{-1/2} < \eta_0,
\]

or

\[
\langle \beta_\perp^{-2} \rangle^{-1/2} < \omega_{\text{CR}}/\omega_0.
\]

generalizing the condition (24).

1. For a small spread, \( \langle \beta_\perp^{-2} \rangle^{-1/2} < \eta_0/k_\perp \) (equivalent to \( k_\perp^2 \beta_\perp^2 > 1/\eta_0^2 \)), there exists an unstable mode, with \( \Re\{\omega\} = k_\parallel \beta \). To see this, note that the term \( \langle (\Omega-k_\perp \beta)^{-2} \rangle \) continuously changes from \( -k_\perp^2 \beta^{-2} \) to 0 as \( \eta \) changes from 0 to \( \infty \), and must be equal to \( -\eta_0^2 \) for some positive \( \eta \).

2. Assuming that the term \( \langle (\Omega-k_\perp \beta)^{-2} \rangle \) is monotonically increasing with \( \eta \), for a marginal spread \( \langle \beta_\perp^{-2} \rangle^{-1/2} \to \eta_0/k_\perp \) the mode becomes stable \( \eta \to 0 \). This suggests that for larger spreads the mode is stable.

As for the one-dimensional distribution considered above, the criterion for instability given in Equation (30) can roughly be interpreted as the requirement that the particles in the beam do not move in the direction perpendicular to the beam a distance exceeding the wavelength of the mode during one e-folding time. Note however that the simple expression, \( \Delta \theta^2 \beta \), in the one-dimensional case, which is equal to the maximal velocity of the particles in the direction perpendicular to the beam, is replaced by the non trivial average, \( \langle \beta_\perp^2 \rangle^{-1/2} \) (the velocity average defined in Equation (27)), which has a less obvious meaning.

3. APPLICATION TO COLLISIONLESS SHOCKS

We next discuss the possible application of the above results to the study of long-wavelength magnetic field generation in the far upstream of collisionless shocks, adopting the following scenario. Consider a shock with Lorentz factor \( \Gamma_{\text{sh}} \gg 1 \) propagating into a cold non-magnetized plasma with particle density \( n_0 \) and plasma frequency \( \omega_0 \). We assume that high-energy shock-accelerated CRs carry a fraction \( \sim \epsilon_p \) of the post-shock energy, and that in the shock frame the CRs are not
highly beamed and have an energy distribution \( n_{e,p}(>\gamma_s) \sim \epsilon_p \Gamma_{sh} \eta_0 (\gamma_s/\Gamma_{sh})^{-p+1} \) with \( p \approx 2 \), where \( \gamma_s \) is the Lorentz factor of the CRs in the shock frame.

We analyze below the generation of long-wavelength magnetic fields in a region surrounding a point \( x \), lying deep in the upstream, due to the interaction of the CRs that reach this point and the incoming upstream particles. We assume that this point is reached by a substantial fraction of the CRs that have Lorentz factors larger than a space-dependent minimum \( \gamma_s(x) \), and study the instabilities in a simplified homogenous model of the upstream frame.

We first show in Section 3.1.1 that for shocks propagating into an electron–positron plasma, the instability grows in upstream regions where the minimal shock frame CR Lorentz factor is sufficiently low, \( \gamma_s < \Gamma_{sh,1/2}^{1/2} \), suggesting that if these instabilities dominate the particle deflections, the acceleration of CRs to shock frame Lorentz factors exceeding \( \Gamma_{sh,1/2}^{1/2} \) may be obstructed. Next, we show in Section 3.1.2 that for shocks propagating into an electron–proton plasma, the instabilities considered grow only for high Lorentz factor shocks, \( \gamma_s > 100 \). Limitations of the applicability of our results due to a finite magnetization of the upstream plasma and due to a finite size of the system are discussed in Section 3.2.

It should be noted that the application of our results, obtained from a linear analysis of homogenous plasma distributions, to the non-homogenous and non-linear problem of collisionless shocks is far from trivial. Furthermore, the long-wavelength modes studied here are not the fastest growing modes, and can be affected by the faster growing, short-wavelength modes (e.g., the oblique mode; Lemoine & Pelletier 2010) once the latter reach the nonlinear stages. Nevertheless, the analysis of linear growth of long-wavelength modes is an important step in the study of long-wavelength magnetic field generation and can be used as a basis for comparison once more accurate calculations are made (e.g., PIC simulations). In addition, it is possible that some of the main features of the linear modes also appear in the more complicated shock scenario.

### 3.1. Instabilities in the Far Upstream of a Collisionless Shock

#### 3.1.1. Electron–Positron Plasma

Let us first assume that all particles in the plasma have the same mass (e.g., electrons–positrons). The CRs in the upstream frame are beamed into an angular spread of \( \sim 1/\Gamma_{sh} \), their Lorentz factor is boosted to \( \sim \Gamma_{sh} \gamma_s \), and they have a plasma frequency

\[
\omega_{CR} \sim \epsilon_p^{1/2} (\gamma_s/\Gamma_{sh})^{-p/2} \omega_0,
\]

where \( \omega_0 \) is the upstream plasma frequency and different values of \( \gamma_s \) represent different positions in the simplified picture.\(^1\)

Deep in the upstream, where \( \gamma_s(x) \ll \Gamma_{sh} \), the plasma frequency of the cosmic rays is much smaller than that of the upstream, and the analysis of Section 2 holds. Equations (12) and (32) imply that long-wavelength modes will grow with a growth rate of approximately

\[
\eta_0 = \Gamma_{sh} \epsilon_p^{1/2} (\gamma_s/\Gamma_{sh})^{-p/2}.
\]

Far away from the shock in the upstream, where \( \omega_{CR} \) is sufficiently small, the instability will be suppressed due to the \( 1/\Gamma_{sh} \) spread in the cosmic rays transverse velocities. Using

\[\text{Figure 3. Growth rate for a specific Weibel mode, with } k_1/\omega_0 = 0.05 \text{ and } k_2/\gamma_s = 0.5, \text{ shown as a function of the shock frame Lorentz factor } \gamma_s \text{ of the beam particles. The beam particles have a one-dimensional rectangular phase space distribution (cf. Section 2.2) with parameters as discussed in the text in Section 3. The solid line, dots, and } \times \text{ symbols give the growth rates obtained using the semi-analytical calculation (cf. Section 2.2), Equation (23), and a "1-mode PIC simulation" (cf. Appendix B), respectively. The dotted line is the growth rate obtained for a delta function distribution, } \eta_0. \text{ The red circle shows the value of } \gamma_{cr,s} \text{ given by Equation (34), beyond which the modes are predicted to be stable. (A color version of this figure is available in the online journal.)}
\]

Equations (30) (or (24)) and (33), the modes are unstable only at locations in the upstream where

\[
\gamma_s(x) \lesssim \Gamma_{sh} \left( \epsilon_p^{1/2} \Gamma_{sh} \right)^{2/p}.
\]

This implies that if these instabilities dominate the particle deflections, the acceleration of CRs to shock frame Lorentz factors exceeding \( \Gamma_{sh,1/2}^{1/2} \) may be obstructed.

For illustration, the growth rate of a specific unstable mode is shown in Figure 3 as a function of \( \gamma_s \). For simplicity, the momentum distribution is assumed to be the same as in Section 2.2 with parameters chosen in terms of shock parameters as: \( \pm \Delta \theta = \pm 1/\Gamma_{sh} \) (with \( \Gamma_{sh} = 25 \)), upstream Lorentz factor \( \gamma_u = 2 \gamma_s \Gamma_{sh} \), and beam plasma frequency, \( \omega_{CR} \) as given in Equation (32) with \( p = 2 \), and \( \epsilon_p = 0.1 \). In the figure we also show the results of the semi-analytical calculation (cf. Section 2.2), the growth rate estimate of Equation (23) (dots) and the results of the 1-mode PIC simulation (cf. Appendix B, \( \times \) signs). For comparison the growth rate \( \eta_0 \) of the delta function distribution (cf. Section 2.1, dotted line), and the maximal \( \gamma_s \) from Equation (34) (red circle, \( y \)-axis value arbitrarily set to 0) are also shown. As can be seen in the figure, the delta function momentum distribution result is a good approximation at small angular spread, and the angular spread suppresses the growth rate at high cosmic ray Lorentz factors in accordance with the estimate in Equation (34).

#### 3.1.2. Electron–Proton Plasma

Next, consider a shock propagating into an electron–proton plasma. The equation for the plasma frequency of the beam (32) should be replaced by

\[
\omega_{CR} \sim \epsilon_p^{1/2} (m_e/m_p)^{1/2} (\gamma_{s,p}/\Gamma_{sh})^{-p/2} \omega_0,
\]

where \( \gamma_{s,p} \) is the minimal CR proton shock frame Lorentz factor at the position considered and where the contribution of the

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\(^1\) Note that the energy carried by the cosmic rays in this frame greatly exceeds that of the upstream particle rest-mass energy density, \( n_0 m c^2 \), for \( \gamma_s \gg \epsilon_p^{1/4} \).
electron CRs at a given particle energy was neglected. Given the transverse velocity spread, 1/Γ_{sh}, of the CRs, Equations (31) and (25) imply that the long-wavelength modes do not grow for moderate shock Lorentz factors (see also Lyubarsky & Eichler 2006)

$$\Gamma_{sh} \lesssim \epsilon_p^{1/2}(m_e/m_p)^{-1/2}. \quad (36)$$

For higher shock Lorentz factors, the long-wavelength modes grow at locations where the minimal shock frame CR energy, $$\epsilon_s(x) = \gamma_s mc^2$$ (equal for CR electrons and protons), is sufficiently low (cf. Equation (34))

$$\epsilon_s(x) \lesssim \Gamma_{sh} m_p (\epsilon_p^{1/2} \Gamma_{sh}^{2/3}) (m_e/m_p)^{1/p}. \quad (37)$$

In the case where the shock Lorentz factors is low, the long-wavelength modes are suppressed as long as the electrons are not heated to relativistic velocities (Lemoine & Pelletier 2011).

3.2. Limitations

Various factors, independent of deflection by the modes considered here, limit the upstream distance $$L(\gamma_s)$$ to which a CR can propagate ahead of the shock. The analysis presented above is not applicable if $$L(\gamma_s)/c$$ is smaller than the $$e$$-folding time $$\tau_0^{-1}$$ of the instability. Here we consider two factors limiting $$L(\gamma_s)$$.

In the presence of a large-scale upstream magnetic field $$B_0$$, $$L(\gamma_s)$$ is limited to $$L_B$$, the distance a CR can propagate ahead of the shock before being deflected by an angle 1/Γ_{sh} and swept up by the shock. This distance is given by $$(1 - \beta_{sh})R_L/\Gamma_{sh}$$, where $$R_L \sim \gamma_s \Gamma_{sh} mc^2/eB_0$$ is the Larmor radius in the upstream frame. Using Equation (33) we find

$$L_B \eta_0/c = \sim \left( \frac{k_L}{\omega_0} \right) \epsilon_p^{1/2} (m_e/m_p)^{1/2} \epsilon_{B,0}^{1/2} (\gamma_s/\Gamma_{sh})^{1-p/2}, \quad (38)$$

where $$\epsilon_{B,0} \sim B_0^2/(8\pi n_0 mc^2)$$. Given that $$k_L < \omega_0$$, $$\gamma_s > \Gamma_{sh}$$, and Equation (36), we find that long-wavelength instabilities do not grow, i.e., $$L_B \eta_0/c < 1$$, for

$$\epsilon_{B,0} \gtrsim \epsilon_p^{3} (m_e/m_p)^{2} \sim 3 \times 10^{-9} \epsilon_p^{2, -1}. \quad (39)$$

where $$\epsilon_p = 0.1 \epsilon_{p, -1}$$. For smaller values of the magnetic field, there will be a range of $$k$$ vectors,

$$k_L \gtrsim \omega_0 \Gamma_{sh} \epsilon_p^{1/2} (m_e/m_p)^{-1/2} \epsilon_{B,0}^{1/2}. \quad (40)$$

for which the instabilities will have sufficient time to grow significantly. Note that in the presence of a magnetic field in the upstream plasma, other instabilities may exist that are not discussed here (e.g., Pelletier et al. 2009).

$$L(\gamma_s)$$ is also limited to $$L_R = (1 - \beta_{sh})R_{sh}$$ due to the finite age of the system, $$R_{sh}/c$$, where $$R_{sh}$$ is the radius of the blast wave. This can be written as

$$L_R \eta_0/c \sim t_{\text{obs}} \eta_0. \quad (41)$$

where $$t_{\text{obs}}$$ is the time measured by a distant observer. This limitation suppresses instability growth for

$$t_{\text{obs}} \lesssim 2 \times 10^{-5} \left( \frac{n_{p,0} \epsilon_p m_e}{m_p} \right)^{-1/2} \left( \frac{\gamma_s}{\Gamma_{sh}} \right)^{-1+p/2} \tau_0, \quad (42)$$

where $$n_{p,0} = n_{p,0} \text{ cm}^{-3}$$ is the upstream proton density. For larger $$t_{\text{obs}}$$ there will be a range of $$k$$ vectors,

$$k_L \gtrsim t_{\text{obs}} \epsilon_p^{1/2} (m_e/m_p)^{-1/2} (\gamma_s/\Gamma_{sh})^{-1+p/2}. \quad (43)$$

for which the instability will have sufficient time to grow significantly.

Let us consider, for example, a GRB afterglow shock propagating into the interstellar medium for which (e.g., Piran 2004)

$$\Gamma_{sh} \sim 300 (E_{\text{iso}}/10^{54} \text{ erg})^{1/8} (t_{\text{obs}}/30 \text{ s})^{3/8} (n_{p,0})^{-1/8}, \quad (44)$$

where $$E_{\text{iso}}$$ is the isotropic equivalent energy. In this case both the finite size of the system and the upstream magnetization are not expected to restrict the applicability of our analysis (see Equations (42) and (39), and note that for typical interstellar magnetic fields $$\epsilon_B \sim 2 \times 10^{-10} (B_0/3 \mu G)^2 R_{sh}$$.

4. DISCUSSION

In this paper, the growth rates of the long-wavelength unstable modes arising from the interaction of a beam of ultrarelativistic CRs and an unmagnetized cold plasma were calculated. We have shown that in the ultrarelativistic limit all long-wavelength modes are unstable with a growth rate $$\eta_0 = k_L \omega_{\text{CR}}/\omega_0$$ (Equation (12)), as long as the spread in the transverse velocity distribution of the beam is sufficiently small. An extension of this result, for finite Lorentz factors and $$k_L \gtrsim \omega_0$$, is given in Equation (14). For large transverse velocity spreads, the instability is suppressed. The condition for instability was derived for a large class of velocity distributions (Equations (31) and (27)).

The possible application of these results to the interaction of CRs with the incoming plasma in the far upstream of unmagnetized collisionless shocks was addressed in Section 3, where we neglected the effect of other modes, some of which with higher growth rates. In a shock propagating into an electron–positron plasma, the instability grows in upstream regions where the minimal shock frame CR Lorentz factor is sufficiently low, $$\gamma_s < \Gamma_{sh} \epsilon_p^{1/2}$$ (cf. Equation (34)). Farther upstream of the shock the instability is suppressed due to the low density of the CRs. If these instabilities dominate the particle deflections responsible for particle acceleration, the acceleration of CRs to shock frame Lorentz factors exceeding $$\Gamma_{sh} \epsilon_p^{1/2}$$ (cf. Equation (34)) may be obstructed.

The long-wavelength instabilities considered do not grow in the upstream of a shock propagating into an electron–proton plasma having Lorentz factors $$\lesssim 100$$ (cf. Equation (36); see also Lyubarsky & Eichler 2006). This implies that: (1) the modes considered are not important in the relativistic shocks responsible for the observed GRBs afterglow emission except possibly at the earliest stages, when $$\Gamma_{sh} \gtrsim 100$$, for which observations suggest that electrons are accelerated to energies exceeding the post-shock proton thermal energy; (2) particle acceleration in electron–positron plasmas suggested by the results of PIC simulations (Spitkovsky 2008b), where these modes can grow, may differ from acceleration in electron–proton plasmas.

We thank Anatoly Spitkovsky, Uri Kesher, and Avi Loeb for useful discussions. This research was partially supported by ISF, AEC, and Minerva grants.

Note that the “thermal” electrons, which have energies similar to the post-shock thermal protons, emit synchrotron radiation with typical photon energies of $$h\nu \sim 10 (\Gamma_{sh}/10^4) (\eta_{0,0}/0.1) 1.5 \text{ eV}$$ and that the synchrotron interpretation of X-ray observations on timescales of a day and of GeV photons observations at earlier stages (Abdo et al. 2009) implies that electrons are accelerated to energies exceeding the “thermal” energy.
APPENDIX A

FULL SOLUTION OF THE DISPERSION EQUATION

Consider the case, described in Section 2, where the velocity distribution of the particles is axisymmetric (around x). In this case averages over the velocity distributions of the type $(\beta\perp)$ and $(\beta\parallel)$, where $\perp$ and $\parallel$ are directions perpendicular to x, are zero.

As a result, terms in the beam susceptibility (cf. Melrose 1986), $\chi_{\text{CR}}$ containing such averages cancel out, and the susceptibility will only have non-diagonal term in the plane defined by beam direction and the wavenumber vector, $k$ (the $x-k$ plane). In this plane, the electric current carried by a beam with velocity $\beta$ linearly perturbed by an unstable mode with electric field $E$, in the plane, and magnetic field, $B$, perpendicular to the plane, is

$$4\pi j_{\text{CR}} = 4\pi \left( \frac{j_{\text{CR}||}}{j_{\text{CR}\perp}} \right) = \frac{i\omega}{\Theta} \left( B(k\beta^2 + \beta\omega) + E(-k\beta + \omega(1 - \beta^2)) \right),$$

$$\times \left( \frac{B(k\beta^2 - \beta\omega) + E(-k\beta - \omega - \beta^2\omega) + E((k\beta - \beta\omega)\omega)}{B(k\beta^2 - \beta\omega) + E(-k\beta - \omega - \beta^2\omega) + E((k\beta - \beta\omega)\omega)} \right),$$

(A1)

where $\Theta \equiv k_i|\beta| + k_\perp$ and, averaging is over $q^2_{\text{CR}}(p)/(m_{\text{CR}}\gamma)$ (cf. Equation (1)). The beam susceptibility has the matrix form of

$$\chi_{\text{CR}} = \frac{i\omega}{\Theta} \left( \begin{array}{c} k^2\beta^2 - 2k\beta\omega - \omega^2 - k^3\omega^2 \cr -k\beta\omega + k\beta\omega + \beta\omega(k\beta - \beta\omega) \cr k^2\beta^2 - 2k\beta\omega - \omega^2 - k^3\omega^2 \end{array} \right),$$

(A2)

and the electromagnetic field susceptibility and the cold plasma susceptibility have matrix forms which are, respectively,

$$\chi_{\text{EM}} = -\left( \begin{array}{c} \omega^2 - k^2\beta^2 \cr k\beta \cr \omega^2 - k^2\beta^2 \end{array} \right); \quad \chi_0 = \omega_0^2.$$  

(A3)

In the remaining axis, outside the plane, only the plasma frequency term remains and this axis gives rise only to the plasma oscillation modes. In the case where all the particles in the beam propagate in the same direction (as considered in Section 2.1), the beam susceptibility simplifies and has the matrix form of

$$\chi_{\text{CR}} = \omega_{\text{CR}}^2 \left( \begin{array}{c} 1 - \frac{2k\beta}{\omega} + \frac{(k^2 - \omega^2)/2}{\omega} - \frac{k\beta}{\omega} \cr \frac{k\beta}{\omega} \cr 1 \end{array} \right).$$

(A4)

In the absence of the beam the dispersion equation, \text{det}\{\chi_0 + \chi_{\text{EM}}\} = 0, is a forth-order polynomial equation in $\omega$ with real coefficients which has the following four solutions, $\omega = \pm \omega_0, \pm (\omega_0^2 + k^2)^{0.5}$, that represent respectively the plasma oscillation and the electromagnetic mode. In the presence of the beam the dispersion equation becomes

$$\text{det}\{\chi_0 + \chi_{\text{CR}} + \chi_{\text{EM}}\} = 0, \quad \text{(A5)}$$

which is a sixth-order polynomial equation in $\omega$ with real coefficients. This equation is a slight perturbation of the original dispersion equation with six distinct solutions. Four of these solutions are slight deviation from the solutions of the original dispersion equation, while the other two solutions, for the $k$-space regime that is considered above, are the solutions discussed in Section 2.1. The full dispersion equation can be solved numerically for any $k$ and these solutions are shown in Figure 1.

APPENDIX B

ONE-MODE PIC SIMULATIONS

The solution described above was numerically verified to be the fastest growing one at several wavevectors, $k$. This was done by performing an efficient PIC simulation in which only one $k$ mode is treated while the rest of the modes are neglected. In this simulation, the electric field, magnetic field, and electric current are sinusoidal with a given $k$ value and a time-dependent amplitude, while the CRs are treated as particles with continuous position (one dimension along $k$) and momenta (three dimensions). The upstream, which is assumed to have a delta function momentum distribution, is written in terms of fluid quantities in the linear approximation and is likewise sinusoidal with the given $k$. The time-dependent amplitude of the electric current carried by the CRs is derived from the distribution of the particles by

$$j_{\text{CR}} = \sum_j \frac{q_j \beta_j \exp(k \cdot x_j)}{1!}, \quad \text{(B1)}$$

where $q_j$, $\beta_j$, and $x_j$ are the charge, velocity, and position of the particle $j$. As such, the simulation is only accurate for the linear regime (the amplitude at which the mode becomes nonlinear can in principle be identified). The growth rate is calculated by fitting the time evolution of the amplitude of the mode with an exponential function in time. With this method only the fastest growing mode at the given $k$ is accounted for. This is a direct, physically transparent method of studying the linear regime of a single wavevector, with no restriction on the three-dimensional velocity distribution, which is time and memory efficient.

For illustration, the results of the simulation used for Figure 3 are presented in Figure B1. As an underlaying quantity for the fitting we use the amplitude of the electric current, $j_0$, which is normalized to the current $j_0 = q n_0 c$, where $n_0$ is the upstream density and $q$ is the electron charge. As can be seen in the figure, the current $j_k$ grows exponentially with time and the growth rate

Figure B1. Growth of $|k_0/\omega_0| = 0.05, k_\parallel/\omega_0 = 0.5$ Weible mode in a 1-mode PIC simulation described in the text in Appendix B. The parameters chosen for this simulation were taken to reproduce the $\gamma_1 = 25$ ($\omega_{\text{CR}} = 0.1$, $\gamma_1 = 1250$) point given in Figure 3. The green curve is the logarithm of the amplitude of the current vector, $j_k$, obtained from the simulation, and the blue line is a linear fit to this curve in the range 19.6 < $\omega_0 c < 44.5$. Its slope, 0.131, represents the growth rate.

(A color version of this figure is available in the online journal.)
of the instability is easily obtained from the fit. The accuracy of the growth rate obtained numerically is high despite the fact that only 10^6 particles (CRs) were used. The saturation level of j_k can also be obtained from this figure. However, since all quantities, except those related to the CRs, are treated in the linear regime, this saturation is representative only if in reality the saturation is governed by the CRs.

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