MATROIDS, MOTIVES AND A CONJECTURE OF KONTSEVICH

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Abstract. Let $G$ be a finite connected graph with $\#E$ edges and with first betti number $b_1$. The Kirchhoff polynomial $P_G$ is a certain homogeneous polynomial of degree $b_1$ in $\#E$ variables. These polynomials appear in the study of electrical circuits and in the evaluation of Feynman amplitudes. Motivated by work of D. Kreimer and D. J. Broadhurst associating multiple zeta values to certain Feynman integrals, Kontsevich conjectured that the number of zeros of $P_G$ over the field with $q$ elements is a polynomial function of $q$. We show that this conjecture is false by relating the schemes $V(P_G)$ to the representation spaces of matroids. Moreover, using Mnëv’s universality theorem, we show that the schemes $V(P_G)$ essentially generate all arithmetic of schemes over $\mathbb{Z}$.

1. Introduction

1.1. Kontsevich’s Conjecture. Let $G$ be a finite graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Considering such graphs as finite CW-complexes, the betti numbers $b_0(G)$ and $b_1(G)$ are both defined. Recall that a graph $T$ is called a tree if $b_0(T) = 1$ and $b_1(T) = 0$. A subgraph $T \subset G$ is called a spanning tree if $T$ is a tree and $V(T) = V(G)$. In a connected graph, a tree is a spanning tree if and only if it is maximal.

For each edge $e$, let $x_e$ denote a formal variable. Consider the polynomial

$$P_G = \sum_T \prod_{e \notin T} x_e$$

where the sum runs through all spanning trees of $G$. If $G$ is not connected, $P_G = 0$ because the sum is empty. Otherwise, $P_G$ is a homogeneous polynomial of degree $b_1(G)$.

Let $V(P_G)$ denote the scheme of zeros of $P_G$ over $\mathbb{Z}$ — a hypersurface in $\mathbb{A}^E$. Let $Y_G$ denote the complement of $V(P_G)$ in $\mathbb{A}^E$. Motivated by computer calculations of the counterterms appearing in the renormalization of Feynman integrals \[3\], Kontsevich speculated that the
periods of $Y_G$ are multiple zeta values (MZVs). (See [10] for a discussion of MZVs and their relationship to certain Feynman amplitudes). Under this assumption on the periods, it is natural to expect that the zeta functions associated to the $Y_G$ are the zeta functions of motives of mixed Tate type [26].

Based on this idea, M. Kontsevich made a conjecture about the number of points of $Y_G$ over a finite field [9]. To describe this conjecture, we first make a notational convention: For any scheme $X$ of finite type over $\mathbb{Z}$, let $[X]$ denote the function $q \mapsto \#X(\mathbb{F}_q)$. Thus $[X]$ is a function from the set $\mathbb{Q}$ of prime powers to $\mathbb{Z}$. We say that $X$ is polynomially countable if $[X]$ is a polynomial in $\mathbb{Z}[q]$.

**Conjecture 1.1** (Kontsevich). For all graphs $G$, $[Y_G] \in \mathbb{Z}[q]$.

Since $[V(P_G)] + [Y_G] = q^#E$, this conjecture is equivalent to the conjecture that $V(P_G)$ is polynomially countable.

Stembridge [21] verified this conjecture for all graphs with fewer than 12 edges. For certain graphs it is relatively easy to see that the conjecture holds. For example, for $G$ a cycle of length $n$, $V(P_G)$ is simply $\mathbb{A}^{n-1}$ and, thus, $[Y_G] = q^n - q^{n-1}$.

We will show, however, that Conjecture 1.1 is false. In fact, contrary to the extremely strong restrictions on the arithmetic nature of the schemes $Y_G$ claimed by the conjecture, they are, from an arithmetic point of view, the most general schemes possible.

1.2. Motives and the Main Theorem. To make this last statement precise we introduce some notation. Let $\text{Mot}^+$ denote the group generated by all functions of the form $[X]$ for $X$ a scheme of finite type over $\mathbb{Z}$. We think of $\text{Mot}^+$ as a coarse version of the ring of motives over $\mathbb{Z}$ [[1]]. As $[X \times Y] = [X][Y]$, $\text{Mot}^+$ is a ring. And, as $[\mathbb{A}^1] = q$, $\text{Mot}^+$ is a $\mathbb{Z}[q]$ module. Let $S$ be the saturated multiplicative system in $\mathbb{Z}[q]$ generated by the functions $q^n - q$ for $n > 1$. Set $\text{Mot} = S^{-1}\text{Mot}^+$. We remark that, since the functions in $S$ are nonvanishing on $\mathbb{Q}$, elements of $\text{Mot}$ give everywhere-defined functions from $\mathbb{Q}$ to $\mathbb{Q}$.

Let $R = S^{-1}\mathbb{Z}[q]$. It is interesting to note that the $K$-theory of $R$ turns up in questions about dynamical systems, and, in a study of this $K$-theory [[1]], Grayson showed that $R$ is a principal ideal domain.

Let $\text{Graphs}$ denote the $R$-module generated by all functions of the form $[Y_G]$. We can now state our main theorem.

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1 In fact, all statements in this paper involving $\text{Mot}^+$ do remain valid in the finest possible setting. That is we can replace $\text{Mot}^+$ with the Grothendieck group defined by imposing the relation $[X] = [U] + [X - U]$ for $X$ a scheme of finite type over $\mathbb{Z}$, $U$ an open subset, and $[X]$ a formal symbol associated to $X$. 
Theorem 1.2. Graphs $= \text{Mot}$.

The theorem immediately implies that Conjecture 1.1 is false. For, if the conjecture were true, all functions of the form $[X]$ would be in $\mathbb{R}$. In particular, they would be rational functions. However, if we let $X$ be the closed subscheme of $\mathbb{A}^1_z$ defined by $px = 0$ for $p$ a given prime, then $[X](q) = q$ if $p|q$ and 0 otherwise. Thus $[X]$ can not be a rational function. Of course, other more interesting examples of $X$ such that $[X]$ is not rational exist.

1.3. Stanley’s Reformulation of Conjecture 1.1. The proof of Theorem 1.2 is based on Stanley’s reformulation of Kontsevich’s conjecture in terms of a polynomial $Q_G$ which is, roughly speaking, dual to $P_G$. In [19], Stanley sets

$$Q_G = \sum_T \prod_{e \in T} x_e.$$  \hfill (1.2)

where the sum again runs through all spanning trees. For $G$ connected, $Q_G$ is homogeneous of degree $\#E(G) - b_1(G)$. Let $X_G = \mathbb{A}^E - V(Q_G)$. Stanley showed that Kontsevich’s conjecture is equivalent to the following analogous conjecture:

Conjecture 1.3. For all graphs $G$, $[X_G]$ is a polynomial.

In fact, we will see in Theorem 2.2 that the $\mathbb{R}$-submodule of $\text{Mot}$ generated by the $[X_G]$ is exactly the same as the one generated by the $[Y_G]$.

The schemes $X_G$ are, however, more tractable than the $Y_G$ — particularly when the graph $G$ is simple (i.e., has neither loops nor multiple edges) and has an apex. This is because, when $G$ is simple, the polynomial $Q_G$ has a simple expression as a determinant via the Matrix-Tree theorem (see section 4). This expression simplifies even further when $G$ has an apex. On the other hand, while $P_G$ can also be expressed as a determinant, this expression is combinatorially complicated.

A vertex $v$ is said to be an apex if there is an edge from $v$ to every other vertex in $G$. Suppose that $G$ is an arbitrary simple graph with vertex set $V = \{v_1, \ldots, v_n\}$. Then we form a graph $G^*$ with apex by simply adding a vertex $v_0$ and connecting it by an edge to all other vertices. All graphs with apex can be obtained through this process.

Using the Matrix-Tree theorem, Stanley showed that, for any field $K$, $X_{G^*}(K)$ is isomorphic to the set of $n \times n$ nondegenerate, symmetric matrices $M$ satisfying the condition that

$$M_{ij} = 0 \text{ if } i \neq j \text{ and there is no edge from } v_i \text{ to } v_j.$$  \hfill (1.3)

Here $i, j \in [1, n]$. 

We then let $Z^o_G$ be the scheme of all $n \times n$ nondegenerate, symmetric matrices $M$ satisfying condition 1.3. (See section 4.) Stanley’s observation essentially shows that $Z^o_G \cong X_G^\ast$. Thus, the following conjecture, stated by Stembridge as Conjecture 7.1 [21], would follow from Conjecture 1.3.

**Conjecture 1.4.** For every simple graph $G$, $[Z^o_G]$ is a polynomial.

Note that, while Conjectures 1.1 and 1.3 are trivial when $G$ is disconnected, Conjecture 1.4 is not. This is related to the fact that the operation $G \mapsto G^\ast$ always produces a connected graph.

However, we will see that Conjecture 1.4 is also false.

For any subgraph $H$ of $G$, let $G - H$ be the graph obtained by removing the edges in $H$ but leaving all vertices. Note that $(G - H)^\ast = G^\ast - H$. If $G$ is a simple graph with $n$ vertices, then $G$ is contained in the complete graph $\Gamma_n$. We define the complement $G^o$ of $G$ to be the graph $\Gamma_n - G$. Note that $(G^o)^\ast = (DG)^o$ where $D$ is the operation of adding a disjoint vertex.

It becomes convenient at this point to shift attention from $G$ to its complement. We therefore define $Z_G = Z^o_G$. When $G$ has vertices $\{v_1, \ldots, v_n\}$ as above, $Z_G$ is then the scheme of all $n \times n$ matrices $M$ satisfying the condition

$$M_{ij} = 0 \text{ if there is an edge from } v_i \text{ to } v_j. \tag{1.4}$$

As partial justification that the schemes $Z_G$ are more natural than the scheme $Z^o_G$, we mention that many of the results obtained thus far on Conjecture 1.3 are most easily stated in terms of the $[Z_G]$. For example, in Theorem 5.4 of [13], Stanley showed that Conjecture 1.3 holds when $G = \Gamma_n - K_{1,s}$ where $K_{1,s}$ is a star (one vertex connected by edges to all other vertices) and $s \leq n - 2$. In the case $n = s + 2$, $G = \Gamma^s$ with $\Gamma = K_{s+1} - K_{1,s}$. Thus $\Gamma = K^o_{1,s}$ and $X_G = Z^\ast \Gamma = Z^o_{K_{1,s}}$. It follows that Stanley’s Theorem 5.4 is equivalent to the statement that $[Z_{K_{1,s}}] \in \mathbb{Z}[q]$.

1.4. **Overview.** Let $\text{Graphs}_s$ be the $\mathbb{R}$-module generated by all functions of the form $[Z_G]$ for $G$ a simple graph. Since $[Z_G] = [Z^o_G] = [X_{(G^o)^\ast}]$, it is clear that $\text{Graphs}_s \subset \text{Graphs}$. Therefore the following theorem implies Theorem 1.3:

**Theorem 1.5.** $\text{Graphs}_s = \text{Mot}$.

The proof of Theorem 1.5 involves two steps. In the first, we study certain incidence schemes $A_G(s, r, k)$. These schemes are defined so that, when $K$ is a field, the $K$ points of $A_G(s, r, k)$ are the set of pairs $(Q, f)$ with $Q$ a symmetric bilinear form on $K^s$ of rank $r$ and $f$ a
function from $V(G)$ to $K^s$ whose span is of dimension $k$. The pair $(Q, f)$ is also subject to the incidence condition that

$$Q(f(v_i), f(v_j)) = 0 \text{ if there is an edge from } v_i \text{ to } v_j. \quad (1.5)$$

If $G$ has $n$ vertices, then $[A_G(n, n, n)] = [Z_G][GL_n]$. Since $[GL_n] \in \mathbb{R}$, this implies that $[A_G(n, n, n)] \in \text{Graphs}_s$. Moreover, there are important relations between the $A_G(s, r, k)$ for varying $s, r$ and $k$, and between the $A_G(s, r, k)$ for varying $G$. By exploiting these relations, we will see that the $\mathbb{R}$-module generated by the $[A_G(s, r, k)]$ is exactly $\text{Graphs}_s$.

This fact allows us to shift our focus from the symmetric form $Q$ to the function $f$. In particular, for each $s$ we consider the scheme,

$$J_G(s) = \cup_k A_G(s, s, k).$$

Again, it turns out that the $\mathbb{R}$-module generated by the $J_G(s)$ is exactly $\text{Graphs}_s$. And the $J_G(s)$ turn out to be quite manageable schemes because the dimension of the span of $f$ is allowed to vary.

The second step in our proof of Theorem 1.5 involves comparing the $J_G(s)$ to the representation spaces of matroids. For any matroid $M$, we define a scheme $X(M, s)$. For $K$ a field, $X(M, s)(K)$ is the set of all possible representations of $M$ in $K^s$. We then let $\text{Matroids}$ denote the $\mathbb{R}$-module generated by all functions $[X(M, s)]$. As we will see in Section 11, it essentially follows from Mnev’s Universality Theorem \cite{Mnev} that $\text{Matroids} = \text{Mot}$. On the other hand, we prove that, for each matroid $M$, there is a finite set of graphs $\{G_i\}$ and rational functions $a_i \in \mathbb{R}$ such that

$$[X(M, s)] = \sum a_i [J_G(s)]. \quad (1.6)$$

This equation proves that $\text{Matroids} \subset \text{Graphs}$ and, thus, it proves Theorem 1.5. Moreover, as we will see, (1.6) can be used even without Mnev Universality to produce a contradiction to Conjecture 1.4. This is because there are matroids $M$, for example the Fano matroid, which are representable only over fields of characteristic 2. Thus, for such matroids, $[X(M, r)]$ (with $r$ equal to the rank of $M$) could not possibly be a rational function as Conjecture 1.4 and (1.5) would demand. As Conjecture 1.4 implies Conjecture 1.3, this shows that Conjecture 1.3 is false.

1.5. Forest Complements. A considerable amount of work has been done to find examples of graphs for which Conjecture 1.3 (resp. Conjecture 1.5, Conjecture 1.4) holds and to compute the functions $[Y_G]$ (resp. $[X_G]$, $[Z_G]$) explicitly \cite{4, 14, 21, 25}. It remains an interesting question to determine the largest classes of graphs for which these conjectures remain true.
The class of graphs for which Conjecture 1.3 is true is already known to include various interesting graphs. Stanley showed that \( X_{K_n - K_m} \) is polynomially countable. Chung and Yang then computed the polynomial \([X_{K_n - K_m}] \) explicitly \([4]\). Yang showed that \( X_G \) is polynomially countable when \( G \) is an outplanar graph. And, as mentioned above, Theorem 5.4 of \([19]\) shows that \( Z_{K_1,s} \) is polynomially countable. In fact, the theorem is equivalent to the statement that \( Z_G \) is polynomially countable when \( G \) is the union of a star and a discrete graph. (Stanley also computes \([Z_G] \) explicitly in this case.)

Recall that a forest is a graph with no cycles. In section 12, we show that \( Z_F \) is polynomially countable whenever \( F \) is a forest. This generalizes Stanley’s Theorem 5.4 and implies that Conjecture 1.4 holds for forest complements. The result is essentially a consequence of the manageability of the schemes \( J_F(s) \) which allows us to compute \([J_F(s)]\) inductively in terms of the \([J_{F'}(s)]\) for smaller forests \( F' \).

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2. Preliminary Results

In this section we carry out two minor adjustments to two theorems of Stanley.

2.1. The Module of all Graphs. The first adjustment is an amplification of Proposition 2.1 of \([13]\). It concerns the relation between the schemes \( Y_G \) and the schemes \( X_G \).

**Proposition 2.1.** The subgroup of \( \text{Mot}^+ \) generated by the functions \( [X_G] \) is equal to the subgroup generated by the functions \( [Y_G] \).
We remark that the proof of this proposition is completely contained in Stanley’s proof of his Proposition 2.1. However, for the convenience of the reader, we translate Stanley’s proof into our own setting.

Proof. Let $S$ be a subset of $E = E(G)$. Let $A^S$ be the image of the obvious inclusion of $i^S : A^S \to A^E$. Let $G^S_m = i^S(G^S_m)$. Note that, as $S$ varies over all subsets of $E$, the subschemes $G^S_m$ stratify $A^E$.

For any subscheme $X \subset A^E$, let $X^S = X \cap A^E - S$ (resp. $X^+_S = X \cap G^E - S$). Thus $X_S$ is the intersection of $X$ with the hyperplanes defined by the equations $x_e$ for $e \in S$. Note that $X^S_\emptyset = X$ and, as $S$ varies over the subsets of $E$, the subschemes $X^+_S$ stratify $S$. We therefore have,

$$[X_S] = \sum_{T \supset S} [X^+_T]$$

and, by the Inclusion-Exclusion Principle,

$$[X^+_S] = \sum_{T \supset S} (-1)^{\#(T - S)} [X_T].$$

By inspecting the $Q_G$, it is easy to see that $X_{G,S} \cong X_{G - S}$ and $X^+_G \cong X^+_S$. Dually, if $S$ is a forest, $Y_{G,S} \cong Y_{G/S}$ (resp. $Y^+_G \cong Y^+_G$) where $G/S$ is the graph obtained by contracting each component of $S$ to a point. On the other hand, if $S$ is not a forest, it is easy to see that $Y_{G,S}$ is empty.

Now, as Stanley notes, $Q_G(x) = P_G(1/x) \prod_{e \in E} x_e$. Thus

$$X^+_G \cong Y^+_G \emptyset$$

through the map $x \mapsto 1/x$.

Putting our equations together we obtain the following:

$$[Y_G] = \sum_{S \subseteq E} \sum_{T} (-1)^{\#T}[X_{(G/S) - T}],$$

$$[X_G] = \sum_{S \subseteq E} \sum_{T} (-1)^{\#T}[Y_{(G - S)/T}].$$

Together, these two equations, the first of which appears (in a different notation) as Proposition 4.1 of [21], prove the proposition.

The proposition implies the following theorem as a corollary.

**Theorem 2.2.** Graphs is equal to the $R$-submodule of Mot spanned by the $[X_G]$.

We remark that, as $[Y_G] = q^n - q^{n-1}$ for $G$ a cycle of length $n$, $R$ is itself a submodule of Graphs.
2.2. An Observation on Polynomial Countability. Our second adjustment to Stanley's results is to Proposition 2.2 of [19]. This proposition, which Stanley deduces from the Weil conjectures, essentially states that, if $X$ is a scheme of finite type over $\mathbb{Z}$, then the knowledge that $[X] \in \mathbb{Q}[q]$ implies that, in fact, $[X] \in \mathbb{Z}[q]$.

In Section 12, we require a result which is analogous to Stanley's Proposition 2.2 but easier to prove. While the result is not strictly weaker than Stanley's proposition, it does not require the Weil conjectures. Rather, it is a consequence of the Euclidean algorithm and the infinitude of the primes.

**Proposition 2.3.** If $f \in \mathbb{R}$ and $f(q) \in \mathbb{Z}$ for all $q \in \mathbb{Q}$, then $f \in \mathbb{Z}[q]$.

We will use the proposition in the case $f = [X]$ for $X$ a scheme of finite type over $\mathbb{Z}$.

**Proof.** Write $f = a/s$ with $a \in \mathbb{Z}[q]$ and $s \in S$. Since $s$ is monic, we can write $f = d + r/s$ with $d, r \in \mathbb{Z}[q]$ and $\deg(r) < \deg(s)$. But this implies that $r(q)/s(q) \in \mathbb{Z}$ for all $q$ which implies that $r = 0$. Thus $f = d$. \qed

We remark that it is possible, using the rationality of the Zeta function, proved by Dwork in [5], to prove a much stronger result which implies both Stanley's Proposition 2.2 and our Proposition 2.3 above.

**Theorem 2.4.** Let $p$ be a prime and $X$ and $Y$ schemes of finite type over $\mathbb{F}_p$. For all $n > 0$, let $f_X(p^n) = \#X(\mathbb{F}_{p^n})$ (resp. $f_Y(p^n) = \#Y(\mathbb{F}_{p^n})$). Let $g \in \mathbb{Q}(q)$ be a rational function.

(a) If $f_X(p^n) = g(p^n)$ for almost all $n$, then $g \in \mathbb{Z}[q]$, and $f_X(p^n) = g(p^n)$ for all $n$.

(b) If $f_X(p^n) = f_Y(p^n)$ for almost all $n$, then $f_X(p^n) = f_Y(p^n)$ for all $n$.

We do not need this result because the denominators we consider always lie in the multiplicative set $S$. The theorem would be useful, however, if this were not the case. The proof is left to the interested reader.

3. Determinantal Schemes

In this section, we collect certain basic properties of determinantal schemes which are necessary for the rigorous definition of the incidence schemes $A_G(s, r, k)$ discussed in the introduction and for some of the discussion in Section 11. We first describe the general theory of determinantal schemes in functorial language and then restrict to the
specific case of determinantal schemes over \( \mathbb{Z} \) that are the focus of the paper.

The results of this section are not strictly necessary for the proof that Conjecture 1.1 is false. In fact, the proof of Theorem 1.5 does not require the fact that the \( A_G(s, r, k) \) are schemes. It is enough to consider the \( A_G(s, r, k) \) as functions from finite fields to sets, and for this the definition given in the introduction suffices. Therefore, the reader only interested in the proof that Conjecture 1.1 is false can safely skim this section.

3.1. **General Theory.** Let \( S \) be a scheme and let \( E \) and \( F \) be locally free \( \mathcal{O}_S \)-modules of ranks \( e \) and \( f \) respectively. Write \( \text{Hom}_{\mathcal{O}_S}(E, F) \) for the abelian group of homomorphisms from \( E \) to \( F \). The scheme of homomorphisms \( \text{Hom}(E, F) \) is then an abelian group scheme over \( S \) representing the functor

\[
T \mapsto \text{Hom}_{\mathcal{O}_T}(E_T, F_T).
\]

(3.1)

Now let \( r \in \mathbb{N} \), and write \( \text{Hom}_{\mathcal{O}_S,r}(E, F) \) for the set of morphisms

\[
\phi \in \text{Hom}_{\mathcal{O}_S}(E, F)
\]

such that \( \text{coker} \, \phi \) is locally free of rank \( f - r \). The functor

\[
T \mapsto \text{Hom}_{\mathcal{O}_T,r}(E_T, F_T).
\]

(3.2)

is representable by a scheme \( \text{Hom}_r(E, F) \).

To see this, first note that we can assume without loss of generality that the base \( S \) is affine and the sheaves \( E \) and \( F \) are both free. Let \( S = \text{Spec} \, A \) and let \( T = \text{Spec} \, B \). Then \( \text{Hom}_{\mathcal{O}_T,r}(E_T, F_T) \) is equal to the set of maps \( \phi : B^e \to B^f \) such that \( \text{coker} \, \phi \) is a projective \( B \)-module of rank \( f - r \). Now, \( \text{coker} \, \phi \) is projective of rank \( f - r \) if and only if for every prime \( \mathfrak{p} \in \text{Spec} \, B \) the rank of \( \phi \otimes (B_\mathfrak{p}/\mathfrak{p}) \) is \( r \). And this will be the case if and only if every \((r + 1) \times (r + 1)\) minor in \( \phi \) vanishes, but some \( r \times r \) minor is invertible.

Let \( \{ y_{ij} \}_{i,j=1}^{e,f} \) be a set of formal variables, and consider each \( y_{ij} \) as an entry in an \( e \times f \) matrix. Let \( A[y] \) be the polynomial ring in all variables \( y_{ij} \). For each \( k \), let \( m^k_i \in \mathbb{Z}[y_{ij}] \) be a complete list of the \( k \times k \) minors, and let \( I_k \) be the ideal generated by the \( m^k_i \). In this notation, \( \text{Hom}_r(\mathcal{O}_S^e, B^f) \) is the locally closed subscheme of \( \text{Hom}(\mathcal{O}_S^e, \mathcal{O}_S^f) \) given by the union of the affine schemes

\[
\bigcup_i \text{Spec}(A[y]/I_{r+1}(m_i)).
\]

(3.3)

It follows that the set of points associated to \( \text{Hom}_r(\mathcal{O}_S^e, \mathcal{O}_S^f) \) is simply \( \bigcap_i V(m^r_i) - \bigcap_i V(m^{r+1}_i) \).
Thus the schemes \( \text{Hom}_r(E, F) \) form a stratification of \( \text{Hom}(E, F) \). To see this stratification in a coordinate free manner, we consider the \( \text{determinantal sets} \) \( \text{Hom}_{\leq r}(E, F) \). This is the set of all maps \( \phi \in \text{Hom}(E, F) \) such that the stalk of \( \text{coker} \, \phi \) has rank less than or equal to \( r \) at every point \( x \in S \). The functor

\[
T \mapsto \text{Hom}_{\leq r}(E_T, F_T)
\]

is then represented by the \( \text{determinantal scheme} \) \( \text{Hom}_{\leq r}(E, F) \). These scheme have been studied extensively. [8, 17].

In the case that \( S = \text{Spec} \, A \), \( E = A^e \) and \( F = A^f \) considered above, \( \text{Hom}_{\leq r}(E, F) \) is simply \( \text{Spec} \, A/I^{r+1} \). Thus

\[
\text{Hom}_r(E, F) = \text{Hom}_{\leq r}(E, F) - \text{Hom}_{\leq r-1}(E, F).
\]  

(3.4)

3.1.1. Maps to the Grassmanian. Write \( \text{Gr}(r, E) \) for the Grassmanian of \( r \) planes in \( E \). \( \text{Hom}_r(E, F) \) is that this scheme is equipped with two maps to Grassmanians. We have a map \( p : \text{Hom}_r(E, F) \to \text{Gr}(r, F) \) given essentially by sending \( \phi \) to its image. And we have a map \( q : \text{Hom}_r(E, F) \to \text{Gr}(e - r, F) \) given by sending \( \phi \) to its kernel.

3.1.2. Function Spaces. When \( V \) is a finite set we write \( \text{Fun}(V, E) \) for \( \text{Hom}(O^V_S, E) \) (resp. \( \text{Fun}_r(V, E) \)) for \( \text{Hom}_r(O^V_S, E) \).

3.1.3. Symmetric Bilinear Forms. Let \( E^\vee \) denote the dual of \( E \). There is a natural transpose automorphism

\[
t : \text{Hom}(E, E^\vee) \to \text{Hom}(E, E^\vee)
\]

and we define \( \text{Sym} \, E \) to be the subscheme fixed by \( t \). We then write \( \text{Sym}_r \, E \) (resp. \( \text{Sym}_{\leq r} \, E \)) for the scheme-theoretic intersection of \( \text{Sym} \, E \) with \( \text{Hom}_r(E, E^\vee) \) (resp. \( \text{Hom}_{\leq r}(E, E^\vee) \)).

3.2. A Specific Case. We will be primarily interested in the case \( S = \text{Spec} \, Z \), \( E = O^e_S \) and \( F = O^f_S \). In this case, \( \text{Hom}(E, F) \) is \( \text{Spec} \, Z[y] \). \( \text{Hom}_{\leq r}(E, F) \) is the closed subscheme in \( \text{Hom}(E, F) \) defined by the \( (r + 1) \times (r + 1) \) minors. And \( \text{Hom}_r(E, F) \) is the Zariski open subset of \( \text{Hom}_{\leq r}(E, F) \) defined by requiring at least one \( r \times r \) minor to be invertible.

These equalities can be used without reference to the preceding general theory to define the \( \text{Hom}_r(E, F) \). It follows directly that for any field \( K \), \( \text{Hom}_r(E, F)(K) \) is the set of maps from \( K^e \) to \( K^f \) of rank \( r \).

Similarly, when \( E = O^e_S \) with \( S = \text{Spec} \, Z \), \( \text{Sym} \, E \) can be viewed as the closed subscheme of \( Z[y] \) defined by the equations \( y_{ij} = y_{ji} \). \( \text{Sym}_{\leq r} \, E \) is then the closed subscheme of \( \text{Sym} \, E \) defined by the \( (r + 1) \times (r + 1) \) minors. And \( \text{Sym}_r \, E \) is the Zariski open subset of \( \text{Sym}_{\leq r} \, E \).
defined by requiring at least one $r \times r$ minor to be invertible. The $K$ points of $\text{Sym}_r E$ are the bilinear forms on $K^e$ of rank $r$.

3.3. Polynomial Countability. With $E$ trivialized over $\text{Spec} \mathbb{Z}$ as above, we write $\text{GL}_e$ for $\text{Hom}_e(E, E)$, $\text{Gr}(r, e)$ for $\text{Gr}(r, E)$, and $\text{Sym}_r^e$ for $\text{Sym}_r E$.

We now list a few results concerning the polynomial countability of the schemes just discussed.

$$[\text{GL}_n] = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$$ (3.6)

$$[\text{Gr}(a, b)] = \frac{[\text{GL}_b]}{[\text{GL}_a][\text{GL}_{b-a}][q^{a(b-a)}]}$$ (3.7)

$$[\text{Hom}_r(E, F)] = [\text{Gr}(r, e)][\text{Gr}(r, f)][\text{GL}_r]$$ (3.8)

The first two of the above equalities are well known and the last is easy. Note that each of the functions given is a polynomial lying in the multiplicative set $S$.

A more difficult formula is the following, taken from [E3].

$$[\text{Sym}_n^r] = \left\{ \prod_{i=1}^{2s-1} \frac{q^{2i}}{q^{2i-1}} \cdot \prod_{i=0}^{2s-1} (q^{n-i} - 1), 0 \leq r = 2s \leq n, \frac{\prod_{i=1}^{2s} q^{2i-1}}{\prod_{i=0}^{2s} (q^{n-i} - 1), 0 \leq r = 2s + 1 \leq n} \right\}$$ (3.9)

Note again that $[\text{Sym}_n^r] \in S$.

4. The Matrix Tree Theorem

Stanley’s positive results mentioned in the introduction were mainly consequences of the Matrix-Tree Theorem of Kirchhoff, Borchardt and Sylvester which gives an expression of the polynomial $Q_G$ as the determinant of a symmetric matrix. As this theorem is also basic to our results, we describe it in this section after fixing some useful notation.

4.1. Notation. When $G$ is a simple graph, an assumption we will make for the remainder of this paper, $E$ can be considered as a subset of $\text{Sym}^2 V$. For $v, w \in V$, we write $e_{vw}$ for the set $\{v, w\}$. Thus the statement $e_{vw} \in E$ means that there is an edge in $G$ connecting $v$ to $w$.

It is convenient to pick an ordering $V = \{v_1, \ldots, v_{n_G}\}$ of the edges $V$. Thus $n_G = \#V(G)$. We write $n$ for $n_G$ when there is only one graph under consideration.

Set $e_{ij} = e_{v_i v_j}$. We write $x_{ij}$ for the variable $x_{e_{ij}}$ when $e_{ij} \in E$, and we extend this notation by setting $x_{ij} = 0$ when $e_{ij} \notin E$. 
4.2. The Laplacian. Let $L = L_{ij}$ be the $n \times n$ matrix defined by

$$L_{ij} = \begin{cases} \sum_{k=1}^{n} x_{ik} & \text{if } i = j \\ -x_{ij} & \text{if } i \neq j \end{cases}$$

Let $L_0$ be $L$ with first row and the first column removed. $L$ is called the generic Laplacian matrix of $G$ and $L_0$ the reduced generic Laplacian.

The following theorem can be found in the work of Cayley, Kirchhoff, Maxwell and Sylvester. For a proof, see [20].

**Theorem 4.1** (The Matrix-Tree Theorem). $Q_G = \det L_0$.

Now, as in the introduction, let $Z^0_G$ be the scheme of all $n \times n$ symmetric, non-degenerate bilinear forms $M_{ij}$ such that $M_{ij} = 0$ whenever $i \neq j$ and $e_{ij} \notin E$. In the notation of section 3, $Z^0_G$ is simply the closed subscheme of $\text{Sym}_n^n$ defined by the equations $y_{ij} = 0$ for all $i \neq j$ with $e_{ij} \notin E$.

Our use of Theorem 4.1 is based on the following important consequence, recognized by Stanley.

**Theorem 4.2.** $X_{G^*} \cong Z^0_G$.

**Proof.** Let $Z[x]$ be the ring generated by the variables $x_{ij}$ for $0 \leq i < j \leq n$. Let $I$ be the ideal generated by the variables $x_{ij}$ for all pairs $i < j$ with $e_{ij} \notin E$. Then $X_{G^*} = \text{Spec } A$ with $A = (Z[x]/I)_{Q_G}$.

On the other hand, let $Z[y]$ be the ring generated by all $y_{ij}$ for $i, j \in \{1, \ldots, n\}$, and let $J$ be the ideal generated by all expressions of the form $y_{ij} - y_{ji}$ for $i \neq j$ and $y_{ij}$ for $i \neq j$ and $e_{ij} \notin E$. Then, letting $D$ be the determinant of the matrix of $y_{ij}$'s, $Z^0_G = \text{Spec } B$ with $B = (Z[y]/J)_D$.

Let $p : Z[y] \to Z[x]$ be the map

$$y_{ij} \mapsto \begin{cases} \sum_{k<i} x_{ki} + \sum_{i<k} x_{ik} & i = j \\ -x_{ij} & i < j \\ -x_{ji} & j < i \end{cases}$$

(4.1)

Let $q : Z[x] \to Z[y]$ be the map

$$x_{ij} \mapsto \begin{cases} \sum_{k} y_{jk} & i = 0 \\ -y_{ij} & i > 0 \end{cases}$$

(4.2)

It is easy to verify that $p(I) \subset J$, that $q(J) \subset I$, and that $p$ and $q$ give inverse isomorphisms between the rings $Z[x]/I$ and $Z[y]/J$. It then follows from the Matrix-Tree theorem that $p(Q_G) = D$. Thus $p$ and $q$ give inverse isomorphisms between the rings $A$ and $B$. 

As mentioned in the introduction, it is convenient to shift our attention from the simple graph $G$ to its complement. We therefore set...
Thus $Z_G = Z^a_G$. Thus $Z_G$ is the subscheme of $\text{Sym}_n$ defined by the equations $y_{ij} = 0$ for every pair $i, j$ with $e_{ij} \in E$. And $Z_G = X^{(G^o)_*} = X_{(DG)^o}$ where $D$ is the operation of adding a disjoint vertex.

**Example 4.3.** Let $G$ be a graph with $n$ vertices and no edges. Then $Z_G \cong \text{Sym}_n$. This is recognized in [19]. By Equation 3.9, it follows that $[Z_G] \in Z[q]$. In fact, $[Z_G] \in S$, and this shows that $R \subseteq \text{Graphs}_s$.

### 5. Incidence Schemes

We now introduce the incidence schemes mentioned in the introduction. At first, we work in full generality over a base scheme $S$. But our main interest is the case $S = \text{Spec} \mathbb{Z}$.

**Definition 5.1.** Let $W$ be a locally free $\mathcal{O}_S$-module, and let $G$ be a graph. We write $A_G(W)$ for the closed subscheme of $\text{Sym} W \times \text{Fun}(V, W)$ consisting of pairs $(Q, f)$ satisfying the condition that $Q(f(v), f(w)) = 0$ if $e_{vw} \in E(G)$. (5.1)

If $r$ and $k$ are integers, we write $A_G(W, r, k)$ for $A_G(W) \cap (\text{Sym}_r(W) \times \text{Fun}_k(V, W))$.

When $S = \text{Spec} \mathbb{Z}$ and $W = \mathcal{O}_S$, we write $A_G(s)$ for $A_G(W)$ and $A_G(s, r, k)$ for $A_G(W, r, k)$.

The $A_G(W, r, k)$ form a stratification of $A_G(W)$ by locally closed subschemes. Note that $A_G(s, r, k)$ is empty unless $0 \leq k < n$ and $0 \leq r, k \leq s$. Also note that $A_G(s, 0, 0) = \text{Sym}_r^a$, and $A_G(W, 0, k) = \text{Fun}_k(V, W)$. Thus $[A_G(s, r, 0)]$ and $[A_G(s, 0, k)]$ are both in $Z[q]$.

Now assume that $V(G) = \{v_1, \ldots, v_n\}$ as in paragraph 4.1. Recall from the introduction that $\text{Graphs}_s$ is the $R$-submodule of $\text{Mot}$ spanned by the functions $Z_G$.

**Theorem 5.2.** (a) $A_G(n, n, n) \cong Z_G \times \text{GL}_n$.
(b) $\text{Graphs}_s$ is exactly equal to the $R$-module generated by the functions $[A_G(n, n, n)]$.

**Proof.** We first remark that (b) follows directly from (a) and the fact that $[\text{GL}_n] \in S$.

To prove (a) we let $W = \mathcal{O}_S^a$ with $S = \text{Spec} \mathbb{Z}$. Then $\text{Fun}_r(V, W) = \text{GL}_n$. The map $(Q, f) \mapsto (f^t Q f, f)$ then identifies $A_G(n, n, n)$ with $Z_G \times \text{GL}_n$. (Here $f^t$ denotes the transpose of $f$).

**Remark 5.3.** Let $Z_G(r)$ be the scheme consisting of all $n \times n$ symmetric bilinear forms of rank $r$ such that $M_{ij} = 0$ whenever $e_{ij} \in E$. These schemes have been studied implicitly in [4, 19]. In Stanley’s notation,
\(h(G, r) = [Z_G(r)]\), and Chung and Yang call a graph \(G\) strongly admissible if \(Z_G(r)\) is polynomially countable for all \(r\). An easy modification of the proof above shows that \(A_G(n, r, n) \cong Z_G(r) \times GL_n\).

6. Extensions of Bilinear Forms

In this section, we review a result of MacWilliams \cite{13} counting the number of ways to extend a bilinear form of rank \(r_1\) to a bilinear form of rank \(r_2\). This count will be important in the next section for finding relations among the \(A_G(s, r, k)\).

Let \(Q\) be a fixed bilinear form on \(F_q^{d_1}\) with rank \(r_1\). Let \(C_Q(d_2, r_2, d_1, r_1)\) be the number of ways to extend \(Q\) to a form on \(F_q^{d_2}\) of rank \(r_2\). The following result is Lemma 4 of \cite{13}.

**Theorem 6.1.**

\[
C_Q(d_1 + 1, r_2, d_1, r_1) = \begin{cases} 
q^{r_1} & r_2 = r_1 \\
q^{r_1+1} - q^{r_1} & r_2 = r_1 + 1 \\
q^{d_1+1} - q^{r_1+1} & r_2 = r_1 + 2 \\
0 & \text{otherwise.}
\end{cases}
\]

Note that \(C_Q(d_1 + 1, r_2, d_1, r_1)\) only depends on \(d_1, r_2\) and \(r_1\). By induction on \(d_2 - d_1\), we can show that \(C_Q(d_2, r_2, d_1, r_1)\) only depends on the integer parameters \(d_2, r_2, d_1\) and \(r_1\). Thus we simply write \(C(d_2, r_2, d_1, r_1)\) for this number. We can also see by induction that the following recursion is satisfied

\[
C(d_2, r_2, d_1, r_1) = \sum_{j=0}^{2} C(d_2, r_2, d_1 + 1, r_1 + j)C(d_1 + 1, r_1 + j, d_1, r_1). \tag{6.1}
\]

**Corollary 6.2.**

(a) \(C(d_2, r_2, d_1, r_1)\) is a polynomial in \(q\).

(b) \(C(d_2, r_2, d_1, r_1) \neq 0\) iff \(d_2 \geq r_2, \ d_1 \geq r_1,\) and \(0 \leq r_1 \leq r_2 \leq r_1 + 2(d_2 - d_1)\).

**Proof.** (a) follows directly from the recursion formula \(6.1\).

The necessity of the first two inequalities of (b) are obvious for dimension reasons (the rank of a bilinear form can not be greater than the dimension of the ambient space). Necessity of the third inequality follows from formula \(6.1\) by induction.

We prove the sufficiency of the the inequalities in (b) by induction on \(i = d_2 - d_1\) using formula \(6.1\). We do not actually need this for the rest of the paper so the reader may safely skip the proof.
For $i = 0$ sufficiency is obvious. For $i = 1$ the sufficiency results from the fact that $C(d_1 + 1, r_2, d_1, r_1) \neq 0$ iff $r_2 \in [r_1, r_1 + 2]$ when $d_1 \neq r_1$ and iff $r_2 \in [r_1, r_1 + 1]$ when $d_1 = r_1$.

Now suppose sufficiency is known for $d_2 - d_1 < i$ and assume that $(d_2, r_2, d_1, r_1)$ satisfies the conditions in (b) with $d_2 = d_1 + i$ and $r_2 = r_1 + k$. By formula 6.1, $C(d_2, r_2, d_1, r_1) \neq 0$ if there is a $j$ such that both

1. $C(d_1 + i, r_1 + k, d_1 + 1, r_1 + j) \neq 0$ and
2. $C(d_1 + 1, r_1 + j, d_1, r_1) \neq 0$.

One computes that (2) is satisfied whenever $j \leq d_1 - r_1 + 1$. Using the induction hypothesis, we see that (1) is satisfied for $k - 2i + 2 \leq j \leq \min(k, d_1 - r_1 + 1)$.

So we need only show that $k - 2i + 2 \leq \min(k, d_1 - r_1 + 1)$. That $k - 2i + 2 \leq k$ only says that $i \geq 1$ which we are of course assuming. And $k - 2i + 2 \leq d_1 - r_1 + 1$ iff $(d_2 - d_1) + (d_2 - r_2) \geq 1$ which then follows from the fact that $d_2 \geq r_2$.

7. Reduction Formulae

In this section we give three formulae which allow us to reduce questions about $A_G(s, r, k)$ for given $s, r$ or $k$ to questions where $s, r$ or $k$ is smaller. We also give a formula that allows us to connect the $A_G(s, r, k)$ to the $A_{DG}(s, r, k)$ where $DG$, as in the introduction, is the graph obtained from $G$ by adding a disjoint vertex.

**Theorem 7.1.**

$$[A_G(s, r, k)] = [Gr(k, s)] \sum_j C(s, r, k, j)[A_G(k, j, k)].$$

**Proof.** In this proof and the next one we will pick a base field $\mathbf{F}_q$ at the beginning and then, for any scheme $X$ we encounter, write $X$ instead of $X(\mathbf{F}_q)$.

Write $W = \mathbf{F}_q^s$. For every map $f \in W^V$ let $\langle f \rangle$ denote the span of the $f(v_i)$. The map $(Q, f) \mapsto \langle f \rangle$ fibers the set $A_G(s, r, k)$ over $Gr(k, s)$. The fiber over a subspace $U \subset W$ is then the set $A_G(s, r, U)$ of $(Q, f) \in A_G(s, r, k)$ such that $\langle f \rangle = U$. The transitivity of the $GL_s$ action on $Gr(k, s)$ shows that the fibers all have the same number of points. Thus for any given $U$

$$\#A_G(s, r, k) = \#Gr(k, s) \cdot \#A_G(s, r, U). \tag{7.1}$$

Now let $A_G(s, r, U, j)$ be the set of $(Q, f) \in A_G(s, r, U)$ such that $Q|_U$ has rank $j$. This decomposes $A_G(s, r, U)$ into disjoint subsets. Consider
the map

\[ p_U : A_G(s, r, U, j) \to A_G(U, j, k) \]

\[(Q, f) \mapsto (Q|_U, f).\]

The fiber of \( p_U \) above a given \((\overline{Q}, f) \in A_G(U, j, k)\) is \(C_{\overline{Q}}(s, r, j, k)\). Thus

\#A_G(s, r, U, j) = C(s, r, k, j) \cdot #A_G(k, j, k) \quad (7.2)

Summing over all the \( j \) in Equation 7.2 and substituting the result into Equation 7.1 we obtain the desired result.

**Theorem 7.2.** \([A_G(s, r, k)] = [Gr(r, s)] \sum_l [Gr(n-k, n-l)] [Gr(k-l, s-r)] [GL(k-l)] q^{l(s-r)} [A_G(r, r, l)]\]

**Proof.** Write \( W = F_p^s \) and let \( \Psi : A_G(W, r, k) \to Gr(s - r, W) \) be the map associating to every \((Q, f)\) the kernel \( \ker Q \) of \( Q \). The fiber of \( \Psi \) over a subspace \( U \subset W \) is the set \( A_G(W, r, k) \) consisting of all \((Q, f) \in A_G(W, r, k)\) with \( Q|_U = 0 \). The transitivity of the action of \( \text{GL}(W) \) on \( \text{Gr}(s - r, W) \) show then that

\#A_G(W, r, k) = \#Gr(s - r, W) \cdot #A_G(W, r, k)_U \quad (7.3)

Let \( T = W/U \) and let \( \pi : W \to T \) be the quotient map. \( Q \) reduces in an obvious way to a form \( \overline{Q} \) on \( T \). In fact \( Q \mapsto \overline{Q} \) is a one-one correspondence between bilinear forms on \( W \) with kernel \( U \) and non-degenerate bilinear forms on \( T \).

Now stratify \( A_G(W, r, k)_U \) by the dimension of \( \langle \pi \circ f \rangle \). This stratum corresponding to span = \( l \) maps to \( A_G(T, r, l) \) by sending \((Q, f)\) to \((\overline{Q}, \pi \circ f)\). The fiber above a pair \((\overline{Q}, g)\) is identified with set of \( f : V \to W \) such that \( \pi \circ f = g \) and \( \langle f \rangle \) is of dimension \( k \). It is elementary linear algebra to verify that this is given by

\[ [Gr(n-k, n-l)][Gr(k-l, s-r)][GL(k-l)] q^{l(s-r)}.\]

Hence putting these fibrations together we get the desired result. \( \square \)

It is worth recording an important special case of this result.

**Corollary 7.3.**

\[ [A_G(s, r, s)] = [Gr(n-s, n-r)][Gl(s-r)] q^{r(s-r)} [A_G(r, r, r)].\]

**Proof.** To get a non-zero contribution corresponding to \( l \) in the previous theorem need
1. \( r \leq s \).
2. \( l \leq r \).
3. \( l \leq k \).
4. \( l \geq k + r - s \).

In the case of the corollary \( s = k \), so we get \( l \leq r \) and \( l \geq r \). Hence \( l = r \), and the formula reduces to exactly the above.

We now give a reduction theorem relating the incidence schemes of \( DG \) to those of \( G \).

**Theorem 7.4.**

\[
[A_{DG}(s, r, k)] = q^k[A_G(s, r, k)] + (q^s - q^{k-1})[A_G(s, r, k - 1)].
\]

**Proof.** Let \( W = \mathbb{F}_q^s \). Let \( f(V(DG)) \to W \) with \( \langle f \rangle \) a \( k \)-dimensional subspace. The span of \( f|_{V(G)} \) is either a \( k \) or a \( k - 1 \) dimensional subspace. If \( \{v\} = V(DG) - V(G) \), counting the possibilities for \( f(v) \) proves the lemma.

---

**8. The Module of a Graph**

For a simple graph \( G \) with \( n \) vertices, let \( M(G) \) be the \( \mathbb{R} \)-submodule of \( \text{Mot} \) generated by the \( [A_G(s, r, k)] \). Let \( M(G)_t \) be the submodule of \( M(G) \) generated by the \( [A_G(s, r, k)] \) for \( s \leq t \). Theorem 7.4 shows that \( [A_G(s, r, k)] \in M(G)_k \). Thus we have a finite filtration

\[
M(G) = M(G)_n \supset M(G)_{n-1} \supset \ldots \supset M(G)_0 = \mathbb{R}.
\]

The goals of this section are to compute the structure of \( M(G) \) and to show that, in fact, \( M(G) \subset \text{Graphs}_n \). To do this we introduce three special schemes: \( K_G(s) = A_G(s, s, s), J_G(s) = \cup_k A_G(s, s, k) \) and \( H_G(s) = A_G(n, s, n) \). Note that \( J_G(s) \) consists of the scheme of all pairs \( (Q, f) \in A_G(s) \) with \( Q \in \text{Sym}^s; \) that is, there is no restriction on the rank of \( f \). Note also that \( K_G(n) = H_G(n) \).

**Theorem 8.1.** (a) \([A_G(s, r, k)] \in M(G)_d \) for \( d = \min(s, r, k) \).

(b) \( M(G)_t \) is spanned as an \( \mathbb{R} \)-module by the \([K_G(s)]\) for \( s \leq t \).

(c) \( M(G)_t \) is spanned as an \( \mathbb{R} \)-module by the \([J_G(s)]\) (resp. by \([H_G(s)]\)) for \( s \leq t \).

**Proof.** (a) and (b). Apply Theorem 7.1 to obtain an expression for \([A_G(s, r, k)]\) as a \( \mathbb{Z}[q] \)-linear combination of terms of the form \([A_G(k, j, k)]\) with \( j \leq \min(r, k) \leq s \). Then apply Corollary 7.3 to obtain an expression for each \([A_G(k, j, k)]\) as a \( \mathbb{Z}[q] \)-linear combination of terms of the form \([K_G(j)]\).

(c) To see that the \([J_G(s)]\) span, note that (a) implies that \([J_G(s)] \equiv [K_G(s)] \mod M(G)_{s-1} \). To see that the \([H_G(s)]\) span, use the fact that \([H_G(s)] = \sigma[K_G(s)]\) for \( \sigma \in S \), a consequence of Corollary 7.3. \( \Box \)
Our interest in the $H_G(s)$ is based on the following lemma, which allows us to compare the $[H_G(s)]$ to the $[H_{DG}(s)]$. The lemma is essentially a translation of Theorem 5.1 of [19] into our language. As there are two graphs involved in the lemma, we write $n_G$ for the cardinality of $V(G)$.

**Lemma 8.2.** For $r \leq n_G + 1$,

$$[H_{DG}(r)] = a_G(r)[H_G(r)] + b_G(r)[H_G(r - 1)] + c_G(r)[H_G(r - 2)]$$  \hspace{1cm} (8.1)

with

$$
\begin{align*}
    a_G(r) &= q^{n_G r}(q^{n_G + 1} - 1) \\
    b_G(r) &= q^{n_G r - 1}(q^{n_G + 1} - 1)(q - 1) \\
    c_G(r) &= q^{n_G}(q^{n_G + 1} - 1)(q^{n_G + 1} - q^{r - 1})
\end{align*}
$$

all polynomials in $S$.

**Proof.** By the Theorem 7.4,

$$[A_{DG}(n_G + 1, r, n_G + 1) = (q^{n_G + 1} - q^{n_G})[A_G(n_G + 1, r, n_G)].$$

Now applying Theorem 7.1 to $[A_G(n_G + 1, r, n_G)]$ and expanding out $[\text{Gr}(n_G, n_G + 1)]$ in terms of $q$ gives the result.

The polynomials $a_G, b_G$ and $c_G$ in the theorem are clearly in $S$ as long as they are nonzero. Inspection shows that this is the case under the assumption that $r \leq n_G + 1$. \hfill \Box

There is a simpler identity relating $J_G(s)$ to $J_{DG}(s)$.

**Proposition 8.3.** $[J_{DG}(s)] = q^s[J_G(s)]$.

**Proof.** The obvious map $J_{DG}(s)(\mathbb{F}_q) \rightarrow J_G(s)(\mathbb{F}_q)$ restricting $f$ from $V(DG)$ to $V(G)$ has fiber $\mathbb{F}_q^s$. \hfill \Box

A direct consequence of Proposition 8.3 and Theorem 8.1 (c) is the following.

**Theorem 8.4.** $M(DG) = M(G)$.

We are now ready to prove the main theorem of this section.

**Theorem 8.5.** $M(G)$ is equal to the $R$-module spanned by the functions $[Z_{DG}]$ for $k \geq 0$. In particular, $M(G) \subset \text{Graphs}_s$.

**Proof.** For the proof, let $N(G)$ be the $R$-module spanned by the functions $[Z_{DG}]$ for $k \geq 0$. Since $[K_G(n_G)] = [Z_G][\text{GL}_{n_G}]$, $[Z_G] \in M(G)$. Thus it follows from Theorem 8.4 that $N(G) \subset M(G)$. To prove that $M(G) \subset N(G)$ we use Lemma 8.2 and an inductive argument.
By Theorem 8.1 (c), it will be enough to show that $H_G(s) \in N(G)$ for all $s$. Since $H_G(n_G) = K_G(n_G)$ this is obvious for $s = n_G$. Now by Lemma 8.2

$$H_{DG}(n_G + 1) = b_G[H_G(n_G)] + c_G[H_G(n_G - 1)]$$

with $b_G, c_G \in S$. (The first term on the right hand side of (8.1) vanishes because $H_G(n_G + 1)$ is empty). We know that $H_{DG}(n_G + 1)$ and $H_G(n_G)$ are in $N(G)$. Thus $H_G(n_G - 1) \in S$.

We then assume inductively that $H_G(n_G - i) \in N(G)$ for all $i \leq a$ and for all graphs $G$. Another application of Lemma 8.2 shows us that

$$H_{DG}(n_G - (a - 1)) = a_G[H_G(n_G - (a - 1))] + b_G[H_G(n_G - a)] + c_G[H_G(n_G - (a + 1))].$$ (8.3)

By induction, the left-hand side and the two first terms on the right hand side are in $N(G)$. Thus, as $c_G \in S$, $H_G(n_G - (a + 1)) \in N(G)$ as well.

9. Matroid Theory

A matroid $M$ consists of a finite set $E$ called the edges of the matroid and a rank function $\rho : 2^E \to \mathbb{N}$ satisfying the following axioms

1. $\rho(E) \leq \#E$.
2. For $X \subset Y \subset E$, $\rho(X) \leq \rho(Y)$.
3. For any $X, Y \subset E$,

$$\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y).$$ (9.1)

The integer $\rho(E)$ is said to be the rank of the matroid.

Matroids were introduced by H. Whitney [24] as a simultaneous generalization of matrices and graphs. An excellent modern reference for matroid theory is [17].

9.0.1. Representability. A matroid $M$ of rank $r$ is said to be representable over a field $K$ if there is a function $f : E \to K^r$ such that the dimension of the span of the set $f(X)$ is equal to $\rho(X)$ for all $X \subset E$.

9.0.2. Matroids from matrices. To every subset $E \subset K^r$ there is naturally a matroid $M$ representable over $K$ given by setting $\rho(X) = \dim \langle X \rangle$ for every $X \subset E$.

Let $K^{n+1} - \{0\} \to \mathbb{P}^n(K)$ be the natural map taking a nonzero vector in $v \in K^{n+1}$ to the line $Kv$. Suppose $E \subset \mathbb{P}^n(K)$. Then any set-theoretic splitting $\sigma : \mathbb{P}^n(K) \to K^{n+1}$ gives a subset $\sigma(E)$ of $K^{n+1}$ and, thus, defines a matroid. It is easy to see that this matroid is independent of the splitting $\sigma$. Thus, since such splitting always exist, $E$ defines a matroid.
9.0.3. **Representation schemes.** For any matroid $M$ and a locally free sheaf $W$ over a base $S$, let $X(M, W)$ be the subscheme of $\text{Fun}(E, W)$ consisting of all $f$ whose restrictions to $\text{Fun}(X, W)$ lie in $\text{Fun}_{\rho(X)}(X, W)$ for all $X \subset E$. This is the scheme of representations of $M$ in $W$. When $S = \text{Spec} \mathbb{Z}$ and $W = \mathcal{O}_S$, we write $X(M, s)$ for $X(M, W)$ as in the introduction. For a field $K$, $X(M, s)(K)$ is the set of all $f : E \to K$ such that $\dim \langle f(X) \rangle = \rho(X)$ for all $X \subset E$. That is, $X(M, s)(K)$ is the set of all representations of $M$ in $K$. When $r$ is the rank of $M$, we write $X(M)$ for $X(M, r)$. $X(M)(K)$ is non-empty if and only if $M$ is representable over $K$. Clearly $X(M, s)(K)$ is non-empty if and only if $s > r$ and $X(M)(K)$ is non-empty.

**Definition 9.1.** Let $\text{Matroids}$ be the $R$-module generated by all functions of the form $[X(M)]$.

**Remark 9.2.** It is easy to see that $[X(M, s)] = [\text{Gr}(r, s)][X(M)]$. Thus $\text{Matroids}$ is the same as the $R$-module generated by all functions of the form $[X(M, s)]$.

In the next section we show that $\text{Matroids} \subset \text{Graphs}_*$.

10. **A Counterexample to Kontsevich’s Conjecture**

Let $G$ be a graph, $V$ the set of its vertices, $U \subset 2^V$. A function $\pi : U \mapsto \mathbb{N}$ will will be called a *partially defined rank function* for $V$. Notice that the data of a partially defined rank function $\pi$ determines $U = \text{dom}(\pi)$. Associated to every such function we have a scheme defined as follows:

**Definition 10.1.** $J_G(s, \pi)$ is the scheme of all of all $(Q, f) \in J_G(s)$ such that $f|_H$ has rank $\rho(H)$ for all $H \in \text{dom} \pi$.

**Theorem 10.2.** For every $G$ and every partially defined rank function $\pi$ for $V(G)$, $[J(s, \pi)] \in \text{Graphs}_*$.

**Proof.** The proof is by induction on the cardinality of $\text{dom}(\pi)$. If $\text{dom}(\pi)$ is empty, $J_G(s, \pi) = J_G(s)$. Thus the result follows from Theorem 8.3.

Now assume the result holds for all graphs $G$ and all $\pi$ such that $\# \text{dom} \pi \leq a$. Let $W \subset 2^V$ be a set of subsets with $a + 1$ elements, let $H \in W$ and let $U = W - \{H\}$. Let $\pi : U \mapsto \mathbb{N}$ be a partially defined rank function, and let $\pi_i : W \mapsto \mathbb{N}$ be the extension of $\pi$ to $W$ such that $\pi_i(H) = s - i$. Clearly, any partially defined rank function with domain $W$ is of the form $\pi_i$ for some $\pi : U \mapsto \mathbb{N}$ and some $i \in [0, s]$. 
Now for each \( t \in \mathbb{N} \) we define a graph \( G_t \) as follows: \( G_t \) is the graph obtained from \( G \) by adjoining \( t \) disjoint vertices \( y_1, \ldots, y_t \) and connecting each of the \( y_i \) by edges only to the vertices in \( H \). Thus \( V(G_t) = V(G) \cup Y \) where \( Y = \{y_1, \ldots, y_t\} \), and
\[
E(G_t) = E(G) \cup \{e_{hy} \mid h \in H \setminus y \in Y\}.
\]

Since \( V(G) \subset V(G_t) \), \( U \subset 2^{V(G_t)} \). Thus we can consider \( \pi \) as a partially defined rank function for \( V(G_t) \).

The result will follow from the following equation:
\[
[J_{G_t}(s, \pi)] = \sum_{i=0}^{s} q^i [J_G(s, \pi_i)] \quad (10.1)
\]

To see that the equation holds, note that we can stratify the \( \mathbf{F}_q \) points of \( J_{G_t}(s, \pi) \) according to the dimension of the span of \( f(H) \). Let \( J_{G_t}(s, \pi) \) be the stratum where this dimension is \( s - i \). This stratum maps to \( J_G(s, \pi_i) \) by restricting \( f \) from \( V(G_t) \) to \( V(G) \). The fiber of map above any point \((Q, f)\) is an affine space \( \mathbf{A}^i \). This is because the only condition on the \( f(y_i) \) is that they be orthogonal to the span of \( f(H) \). Thus, as the bilinear form \( Q \) is always nondegenerate, they must lie in a linear subspace of dimension \( i \).

To complete the proof, note that by varying the \( t \) from 0 to \( s \) we obtain a system of equations for the \( [J_{G_t}(s, \pi)] \) in terms of the \( [J_G(s, \pi)] \). Solving this system for the \( J_G(s, \pi_i) \) using Cramer’s rule, we have to invert a Vandermonde determinant which lies in \( \mathbf{S} \). Thus, as we assumed by induction that \( [J_{G_t}(s, \pi)] \) lies in \( \text{Graphs}_s \), it follows that each \( [J_G(s, \pi_i)] \) lies in \( \text{Graphs}_s \) as well.

Let \( G \) be a discrete graph (that is \( E(G) \) is empty). In this case, if \( \pi \) is a partially defined rank function then
\[
[J(s, \pi)] = [\text{Sym}^s \mathbf{L}(s, \pi)] \quad (10.2)
\]
where \( L(s, \pi) \) is the scheme consisting of all \( f \in \text{Fun}(V, \mathcal{O}_{\text{Spec} \mathbb{Z}}^s) \) such that \( f \) restricts to \( \text{Fun}_{\pi(H)}(H, \mathcal{O}_{\text{Spec} \mathbb{Z}}^s) \) for all \( H \in \text{dom}(\pi) \). To see this, note that the definition of \( J(s, \pi) \) makes it clear that \( Q \) does not enter in the definition of the \( J \)'s for the discrete graph. And only the vertex set \( V \) is needed for the definition of the \( L \)'s since \( G \) is discrete.

As \( \text{Sym}^s \in \mathbf{S} \), it follows that the \( L \)'s are all in \( \text{Graphs}_s \). Now note that, if \( M \) is a matroid, with rank function \( \rho : 2^E \to \mathbb{N} \), then \( X(M, s) = L(s, \rho) \). Thus we obtain the following theorem:

**Theorem 10.3.** Matroids \( \subset \text{Graphs}_s \).
It is now possible to see directly that Conjecture 1.4 and thus Conjecture 1.1 are false. Let $M$ be the Fano matroid. This is a rank 3 matroid whose edge set $E$ is the set $\mathbf{P}^2(\mathbb{F}_2)$. This matroid is representable over a field $\mathbb{F}_q$ if and only if $2|q$ (see [23] Chapter 9).

Thus the function $[X(M)]$ is supported on the set of $q$ such that $2|q$. It follows that $[X(M)]$ can not be a rational function. And this contradicts Conjecture 1.4 by Theorem 10.3.

11. Representation problem of Matroids

Our objective in this section is to show that Matroids = Mot and thus that Graphs, = Mot. This will follow from the known results on the Matroid representation problem.

We saw in the previous section that $[L(k, \pi)]$ were in Graphs, even if $\pi$ is only partially defined. It suffices therefore to show that the R-module generated by all functions of the form $[L(k, \pi)]$ is all of Mot+. This was in essence proved by Mnëv [14, 15] as the unoriented matroid component of a more difficult theorem concerning the representation spaces of oriented matroids (see also [7, 18]). It was independently proved by Bokowski and Sturmfels [2, 22]. Moreover, the idea of the proof using von Staudt’s “algebra of throws” goes back at least to [12] (see [11] for an enlightening explication). However, as we have been unable to extract a proof of the exact statement we need from the literature, we give a sketch of the proof in our context.

**Theorem 11.1** (Mnëv,Sturmfels). Let $X$ be a quasi-projective scheme of finite type over $\mathbb{Z}$, then there is a set $V$, a set of subsets $W$ of $V$, a function $\pi : W \mapsto \mathbb{Z}$, and an element $\sigma \in S$ so that

$$\sigma[X] = [L(3, \pi)].$$

**Remark 11.2.**

1. The theorems in Matroid theory are not in such a direct form because, in Matroid theory we are committed to declare the rank of all the subsets of $V$. Our partially defined $\pi$ does not have this problem. By inclusion-exclusion principles the R-module generated by all functions of the form $[L(k, \pi)]$ where $\pi$ may only be partially defined is same as the R-module generated by all functions of the form $[L(k, \pi)]$ where $\pi$ is defined on all subsets of $V$.

2. Note that any scheme of finite type/$\mathbb{Z}$ is a finite disjoint union of quasiprojective schemes/$\mathbb{Z}$.

**Proof.** The proof follows essentially from the following observations...
1. 4 elements in $P^2$ such that any 3 are linearly independent can by a unique automorphism of $P^2$ in PGL(2), be assumed to be $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ and $(1, 1, 1)$.

2. if given two points $(x, 0, 1)$ and $(x', 0, 1)$ on the $X$-axis, then by drawing lines alone through the 4 points above and these two points, we can locate $(x + x', 0, 1), (xx', 0, 1), (-x, 0, 1)$. Finding intersection of lines can be translated as a vector which lies on both lines, and hence as a condition on linear dependence. These constructions can be found for example in the proof of Theorem 2.2 of [2].

3. Iterating these constructions, given $(x_1, x_2, \ldots, x_n)$ we can determine the points $(f(x_1, x_2, \ldots, x_n), 0, 1)$, where $f$ is a polynomial with integer coefficients by just drawing lines starting from the configuration of the four given points and the points $(x_i, 0, 1)$. Setting $f(x_1, \ldots, x_n)$ either equal to zero or not equal to zero is just another spanning condition: A condition on whether $(f(x_1, x_2, \ldots, x_n), 0, 1), (0, 0, 1)$ is linearly independent or not.

4. The cone over any quasi-projective scheme/$\mathbb{Z}$ can be written as a set of equalities and a set of nonequalities in a finite set of variables $(x_1, \ldots, x_n)$. Note that we can also have conditions of the form $n = 0$ in the list.

12. Forests

In this section we prove that $[J_G(s)] \in Z[q]$ whenever $G$ is a forest. It follows that $M(G) = Z[q]$ for such graphs. To do so we need to introduce a two operations on graphs.

Let $v \in V(G)$. We obtain a graph $I_v(G)$ by adding one edge $e$ connected to $v$ and one new vertex $w$ connected to $e$. That is, we insert an edge at $v$. Clearly, a graph is tree if and only if it can be obtained from the graph with one vertex by successive applications of $I_v$ for various $v$. A graph is a forest if and only if it can be obtained from the empty graph $\emptyset$ by successive applications of $I_v$ and the operation $D$. We write $D_n$ for the graph $D^n\emptyset$.

We define $R_v$ to be the graph obtained from $G$ by deleting $v$ and all edges meeting it. Note that if $G$ is a forest and $v$ is any vertex in $G$, $R_v G$ is also a forest.

**Theorem 12.1.** Let $G$ be a graph with $v \in V(G)$.

(a) $[J_{D_n}(s)] = q^n[s\operatorname{Sym}_n^*]$. 
(b) \[ [J_{DG}(s)] = q^s J_G(s). \]
(c) \[ [J_{IV}(s)] = q^s - 1 (J_G(s) + (q - 1) J_{RGv}(s)) \]

Proof. For (b) let \( w \) be the vertex in \( V(DG) - V(G) \). Then the map \( J_{DG}(s) \to J_G(s) \times F_q^s \) given by \( (Q, f) \mapsto (Q, f|_{V(G)}, f(w)) \) is an isomorphism. For (a) assume first that \( n = 0 \). Then tracing through the definitions one sees that \( J_\emptyset(s) = \text{Sym}^s \). The rest of (a) follows by induction from (b).

For (c) we work over \( F_q \) and consider the map \( \pi : J_{IV}(G)(s) \to J_G(s) \) given by \( (Q, f) \mapsto (Q, f|_{V(G)}). \) The fiber of \( \pi \) above a point \( (Q, g) \in J_G(s) \) depends on whether \( g(v) \) is 0 or not. Let \( J^0_G(s) \) (resp. \( J_G(s) \)) be the set where \( g(v) = 0 \) (resp. \( g(v) \neq 0 \)). Above a point \( (Q, g) \in J^0_G(s) \) the fiber will have \( q^s \) points. Above a point \( (Q, g) \in J_G(s) \) the fiber will have \( q^s-1 \) points since \( Q \) is non-degenerate. Thus

\[
|J_{IV}(G)(s)| = q^s-1 |J^0_G(s)| + q^s |J_G(s)| \quad (12.1)
\]

The result now follows from the observation that \( |J^0_G(s)| = |J_{RGv}(s)| \).

Corollary 12.2. For \( F \) a forest, \( [Z_F] \in \mathbb{Z}[q] \).

Proof. An easy induction using Theorem 12.1 shows that \( [J_F(s)] \in \mathbb{Z}[q] \) for any \( s \). Thus \( M(F) = \mathbb{R} \). It follows from Theorem 8.5 that \( [Z_F] \in \mathbb{R} \). But this implies that \( [Z_F] \in \mathbb{Z}[q] \) by Proposition 2.3.

The next corollary follows from Theorem 12.1 and the results in Section 11.1.

Corollary 12.3. Let \( F \) be a forest with \( r \) vertices contained in a complete graph \( K_s \). Let \( G = K_s - F \).

1. \( [Z^0_G] \in \mathbb{Z}[q] \).
2. If \( s > r \), then \( [X_G] \in \mathbb{Z}[q] \).

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