Boundary layer and the structure

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Abstract. L.D. Landau’s analysis showed, the plane harmonic wave used in the small perturbation method is inapplicable as the turbulent disturbance motion form. The search for the movement form carried out in accordance with the considerations of H. Lorentz, led to replace the longitudinal wave with a spherical wave layer arising on the wall as a result of friction. This form of disturbing motion formed the basis of the turbulence wave model which reproduced the space-time field of the flow parameter pulsations. In this paper, the model is applied to describe the formation of the structure of the boundary layer and of the viscous sublayer.

Keywords: laminar and turbulent flows, acoustics, spherical wave layer, parameters pulsations, boundary layer, viscous sublayer, flow structure.

1. Introduction

The accepted description of the laminar flow in the channels (pipes) does not cause general objections. As for the turbulent flow, it is appropriate to refer to the figurative assessment of T. Karman [1]: “When I have to appear before the Creator, the first revelation I ask of Him is to reveal the secrets of turbulence.” The absence of a complete theory of the turbulence was repeatedly noted by Landau and Lifshitz in [2], §31, §33, §35.

The reason for this strange situation is explained by the lack of a correct idea about the form of the disturbing movement. Two approaches were determined from the very beginning Rayleigh [3] and Reynolds [4] took one dimensional harmonic wave $A(t) = A_0 \exp(\gamma t + i\omega t)$ of small amplitude as a perturbing motion. H. Lorenz [5], on the contrary, believed that the primary perturbation can have any form compatible with the continuity equation. Since Lorenz did not find the proper form, it was the first assumption which got spread. However, it was not possible to build a field of pulsations on this path. It was only shown that the perturbation wave amplitude tends to a finite limit [6].

$$|A|_{max} = 2\kappa/\alpha$$

where $\alpha$ is the Landau constant, $\kappa$ is the attenuation index of the disturbance wave.

It follows that the advantage of the first assumption is not justified. In the meantime, there appeared numerous theories based on different assumptions, which did not bring the desired result, since they had no correct disturbing motion, and accordingly, they did not describe the turbulence occurrence. An exception is the vortex model proposed by Richardson [7], in which large vortices can be qualified as primary perturbations. However, even in this case there is no exact analytical solution of the equations for the turbulent flow region and the boundary layer in which vortices form.
2. Flow core structure

The incompleteness of the turbulence theory when we deal with the flow of a finite aperture, for example in a pipe, lies in the fact that the actual conditions of the flow are not taken into account. Also the process is often considered in the established mode. Meanwhile, these circumstances significantly change the flow properties, moreover, the formation of the flow and its pulsations occur, most likely, simultaneously.

Indeed, every movement is created by a force. In the current environment, the force is the pressure gradient. A standard example is the Riemann wave in a pipe, running in front of the piston which compensates for frictional energy losses. The flow parameters in Riemann wave are usually considered as monotonic functions. But Riemann wave of a finite aperture does not exist as such [8], since friction and diffraction divergence of flow waves, being inevitable in the flow of an aperture bounded by walls, drastically change the wave propagation and the flow properties. Both of these factors form acoustic disturbances – spherical waves. Moving in the pipe from the walls, they create turbulent pulsations. The process of their formation and the structure of the flow core are described in [9].

In short, the process is as follows. The Reynolds criterion is the ratio of the longitudinal pulse flow to the flow of the transverse pulse. The latter means the transverse pressure gradient appearance in the medium stream, i.e.

\[- \frac{\partial P}{\partial x} > 0 \]  

(2)

The greatest increase in pressure occurs at the wall. At the same time, the reflection of diverging waves (elementary simple waves that support the flow) forms local pressure maxima at the wall, generating spherical wave layers that carry oscillations of the medium parameters. The propagation of the layer creates parameter pulsations in the flow, which are shown in scheme of Fig. 1. The set of such waves forms the turbulent flow structure.

![Figure 1](image)

*Figure 1* Distribution of a perturbation of radius a in the form of a wave layer \( \rho (\xi, \eta, \zeta) \) in a flat channel with a height \( d \) from point \( x_p \) and velocity fluctuations in an arbitrary point \( r_0, r_i \) – its imaginary positions.

3. Boundary layer structure

The boundary layer is strongly affected by the wall, which requires special consideration. The task of the boundary layer structure and, thus, completing the construction of the wave model of the turbulence was solved by the example of air flow in a channel formed by two flat plates with a gap \( d = 0.1 \) m. The velocity profile has been chosen in the form

\[ M = M_0 \left( 4 (1 - z)^2 \right)^{1/m}. \]  

(3)

Here \( M_0 = U_0/c \) is the dimensionless flow velocity in the middle of the channel, \( z = 0.5 \), and \( m \) is the degree of turbulence of the flow. Formula (3) satisfactorily reproduces the velocity profile and is convenient in that it is applicable for laminar \((m = 1)\) and turbulent \((m = 5)\) flow regimes. Profiles of the velocities and their vorticity are shown in Fig. 2 for \( m = 5 \).
4. Laminar flow features

The specificity of wave propagation in a limited aperture is due to the flow velocity profile. It deforms the wave, and the deformation is determined by the position of the test point on the wave front. Group wave velocity is given by expression

$$\mathbf{V} = \frac{\partial \omega}{\partial k},$$

where \(k\) is the wave vector, \(\omega(k)\) is the dispersion relation

$$\omega = ck + Uk.$$  \hspace{1cm} (4)

We introduce the unit vector \(s\) tangent to the trajectory. If the medium is at rest, \(U = 0\), then the test point moves with the velocity \(\mathbf{V} = c k / k\), and since \(s = r / r\), its trajectory has the form of a straight line. If the velocity is constant, \(U = Cst\), then the wave is carried away by the flow as a whole, and the straight line of the trajectory is preserved, it is only shifted along the flow. However, at the wall \(U \neq Cst\). a test point trajectory is bent, and the trajectory shape depend not on the coordinates \(r_2 - r_1\), but on the distance traveled by the point along the trajectory \(\ell_2 - \ell_1\). When \(a \ll \ell\), the element of the trajectory is

$$d\ell = c dt,$$

and the density, pressure and velocity \(u\) in the packet, \(|\ell - c t| \leq a\), vary according to formula (5):

$$\rho, p, u \approx f \left( \frac{\ell - c t}{a} \right)^{1/2}.$$  \hspace{1cm} (5)

The trajectory of the test point (wave vector) is described by equation ([2], § 68)]

$$\frac{ds}{d\ell} = \frac{1}{c} [\text{rot} \ U, s]$$

In the flat current, it has components:

$$\frac{ds_x}{d\ell} = \frac{sz}{\ell}, \quad \frac{ds_y}{d\ell} = 0, \quad s_z = \sqrt{1 - s_x^2}.$$  \hspace{1cm} (6)

When a wave occurs on the lower wall of the channel, the boundary conditions take the form

$$x(0) = z(0) = 0, \quad \frac{dx}{d\ell}(0) = \pm \cos \theta_0, \quad \frac{dz}{d\ell}(0) = \pm \sin \theta_0.$$  \hspace{1cm} (7)

Substituting relation \(s_z = dz/d\ell\) in (7) and integrating the first equation, we get

$$s_x = s_{x0} + M(z).$$  \hspace{1cm} (8)

It follows from this equation that the test point moves away from the wall, but at the same time its trajectory is bent. The direction of the bend depends on the initial angle of inclination \(\theta_0\) of the wave vector of the test point. When the wave moves against the flow and \(s_x < 0\), the trajectory bends away from the wall. When the directions of the motion of the wave and the flow coincide, \(s_x > 0\), the trajectory inclines towards the wall. And if the angle \(\theta_0\) is small enough, then the trajectory moves away from the wall not further than the value \(z_{max} = h\).

At maximum \(h\), the tangent to the trajectory is parallel to the x axis. Accordingly, \(\theta_0 = 0\) and \(s_z = 0\).

Using the identities \(s_{x0}^2 + s_{z0}^2 = 1\) and the inequalities \(M^2(z) \ll M(z) \ll 1\), we replace \(s_x\) by \(s_{x0}\) and \(s_{z0}\). As a result we obtain the dependence of height \(h\) of the trajectory maximum (in scale \(d\)) on the initial angle \(\theta_0\) and the velocity \(M_0\) for its profile (3):

$$h = \frac{1}{2} \left( 1 - \sqrt{1 - \left( \frac{1}{M_0^2 \cos \theta_0} \right)^2} \right).$$  \hspace{1cm} (9)
Laminar air flow becomes turbulent in the channel $d=0.1\text{m}$ at $M_0\approx0.001$ ($U_0=0.3\text{m/s}$). Formula (9) is presented in Fig. 3 as the dependence of the natural logarithm of $h$ on $m=\{1,5\}$, $U_0=\{0.1,1,2,4,6,8\text{m/s}\}$ and $\theta_0\in[0,10^\circ]$.

Figure 3. Dependence of parameter $h$ on angle $\theta_0$ and velocity $M_0$.

The $h$ value in the laminar flow looks like a vertical line at the smallest values of the angle $\theta_0$, i.e. as if the boundary layer does not exist. To make sure we build function $h(M_0,\theta_0)$ within the range $U_0\in[0,0.3]\text{m/s}$ shown in Fig. 4. Surface $h(M_0,\theta_0)$ has interesting features. At small $\theta_0$, it practically lies on the channel wall, $h\approx0$. Then the trajectory maximum sharply (within interval $\Delta\theta_0<0.5^\circ$) takes the value $h=1$. It seems that the boundary layer does not exist. However, by definition, the boundary layer is a flow area thereof properties highly depend on the viscosity. Just in the laminar flow, the effect of viscosity extends to the entire flow. Hence, the boundary layer in the laminar case fills the entire cross-section of the channel, and it is the Poiseuille formula that outlines the velocity profile, repeat, of the boundary layer.

Figure 4. Function $h(M_0,\theta_0)$ in laminar flow.

Surface $h(M_0,\theta_0)$ has interesting features. At small $\theta_0$ values, it practically lies on the channel wall, $h\approx0$. Then the maximum of the trajectory sharply (within interval $\Delta\theta_0<0.5^\circ$) takes the value $h=1$. It seems that there is no any boundary layer. However, by definition, the boundary layer is a flow area thereof properties highly depend on the viscosity. Just in the laminar case, the viscosity action extends over the entire flow. Hence, the boundary layer in the laminar case fills the entire cross-section of the channel, and it is the Poiseuille formula that describes the velocity profile, repeat, of the boundary layer.

5. **Boundary layer of turbulent flow and its structure**

In the turbulent flow, the rapid growth of the velocity gradient at the wall initiates important changes in the propagation of perturbation waves. The maxima appear at the trajectories of the test points of the
wave close to the wall. As can be seen from Fig. 3, the value of \( h \) is determined by angle \( \theta_0 \), though it also strongly depends on the velocity \( U_0 \). Moreover, for each of \( U_0 \) values, there exists a trajectory with an angle \( \theta_{cr} \), beyond which the trajectories do not extend in accordance with the Thomson theorem. Consider this question in detail.

Parameter \( h \) has a real value provided that the difference under the root of formula (9) is not negative. Otherwise, the trajectories have no maximum in velocity field (3), and run over the entire flow, experiencing only some deviation in the direction of the hour hand. Figure 5 illustrates the trajectory for the angles \( \theta_0 = \{13,14,15,17,20^\circ\} \) in a narrow stripe \( z \in [0, 0.1] \).

![Figure 5. Trajectories for the angles \( \theta_0 = \{13,14,15,17,20^\circ\} \) in \( z \in [0, 0.1] \) stripe](image)

As the angle \( \theta_0 \) decreases, the deviation of the trajectories increases and, starting from the angle \( \theta_{cr} \), the trajectories do not go beyond the limits of the \( \delta \) level, i.e. \( \delta \) value can be given the status of the outer boundary of the boundary layer. The trajectories with lesser angles \( \theta_0 \) are concentrated within the \( \delta \) layer, whereto the spherical wave layers reflected from the channel walls arrive as well. The wave propagation is accompanied by fluctuations in the parameters of the medium. They form the boundary layer structure. The value of \( \delta \) depends on the thickness of the spherical wave layer which is determined by the size of the primary perturbation, and ultimately by the length of the elementary flow waves. Calculations show that the size of \( \delta \) is approximately equal to the radius \( a \) of the primary perturbation.

The boundary layer structure is being formed by a sequence of flow waves and therein fluctuations, exactly the same as in the core flow. However, a thin layer adjoins the wall, which is often called the “laminar sublayer”. The view point on the interpretation of this sublayer structure is expressed in ([2], §42): “..., there is turbulent movement in it, ..., but it reveals peculiar features that have not yet been adequately interpreted”. Therefore, “viscous sublayer” term was suggested instead of the wrong term “laminar sublayer”.

The wave model gives this missing interpretation. In fact, a viscous sublayer arises when the perturbation waves are reflected from the wall. A wave packet bears translational motion energy and energy of pulsations with a specific density of \( cp \). During reflection, the velocity and its pulsations disappear at the wall, and their energy is transferred respectively to local pressure and to its oscillations. In other words, the energy of these do not disappear, it simply take another form. The viscous sublayer thickness, as well as the boundary layer, depends on the parameters in the primary perturbation. For the parameters of various distribution, the thickness of the sublayer is estimated as \( \delta L \sim (0.1–0.2)a \).

The trajectories within the boundary layer are represented in Fig. 5 by ascending branches. Each of them is limited by a point of maximum \( z(x_{max}) = h \). Equation (6) has a singularity in this point. However, it is not a simple singularity. The fact is that approximation of geometrical acoustics is unsuitable for strong interaction, in particular, in the boundary layer. The matter is that the perturbing wave moves in the field of other perturbations, in particular, vorticity formed by \( \text{curl} \ U \), and the diffraction divergence. Interaction with these fields leads to phase redistribution and to deformation of the wave flat front, i.e. to aberrations.

Vorticity changes the flow velocity profile. However, the vortex cannot be seen directly, except through the movement of some external body, such as a speck of dust, captured by the vortex motion. The trajectory of a speck of dust coincides with the vortex line, provided that it does not move by itself. If it has its own velocity, then its circular trajectory is deformed as a result of summing of the two velocities.
A test point of the wave layer is also an external body, though only in the sense of motion direction and velocity value. “It means the addition of motions in the vortex and in spherical wave layer. Here the difference between the vortex created by the obstacle and the vortex arising at the wall smooth surface should be noted. The obstacle creates some continuous perturbation, while a perturbation on a smooth wall surface occurs from time to time, and the length of its vortex line is limited by the width of the spherical wave layer $2a$. In the latter case, the vortex line breaks, that can be observed in experiments. An intermediate option happens when the channel wall surfaces are rough, and both methods work simultaneously, forming the vorticity of different types.

So, the vortices appear on the descending branches of the test point trajectories of the s. Since the curvature of the trajectories increases in this area, and equation (6) does not describe aberration, the geometrooptical approach with equation (6) must be replaced by the wave method. Searching for the influence of aberrations on the wave propagation, we use an example from optics [10, 11], for this approach has not yet been developed in acoustics.

The aberrations being considered in the classical theory deform the wavefront without changing its topology. Quite a different picture is observed when optical vortices appear in the wave beam: singular points appear on the surface of the wave front, which in many respects are similar to defects in the crystal lattice – screw dislocations. When going around the dislocation, the phase changes exactly by $2\pi$. At the very singular point, the amplitude of light oscillations is zero, and the phase is not determined.

The wavefront of light (acoustic) beams that are close in their properties to a plane wave looks like a family of disjoint surfaces (Fig. 6 a) with a distance between them equal to the wavelength $\lambda$. The screw dislocation dramatically changes the wavefront topology. The direction of propagation of light energy is given by the Umov-Poynting vector, which is known to be perpendicular to the surface of the wave front at each point. Accordingly, in the vicinity of the screw dislocation, a “swirl” of the energy flow occurs, and an equiphase surface consisting of many sheets shown in Fig. 6 a turns into a single surface with a screw structure represented in Fig. 6 b.

![Figure 6. Wave front structure in the screw dislocation absence (a) and if available (b)](image)

As you can see, the trajectory takes the form of the vortex. As a matter of fact, the used procedure is in physical interpretation of the curl operation.

It is worth to add some comment at this place, concerning Thomson theorem which restricts the vortex model application by the boundary layer, while the vortexes ill only some portion of the flow section, whereas the potential part or the flow remains free from vortexes. Meanwhile, the wave model has no such a restriction, for the wave freely crosses the potential flow boundary carrying therewith some matter from the boundary layer.

Indeed, the oscillations of hydrodynamic parameters in a spherical wave layer are described by equations ([2], § 71):

$$\begin{aligned}
p \sim \rho \sim \frac{1}{r} f(r - ct); \\
u \sim \frac{1}{r} f(r - ct) - \frac{1}{r^2} \int_{r-ct} \int_{r+ct} f(r - ct) d(r - ct).
\end{aligned}$$

(10)

It follows that the velocity expression differs from the pressure and density formula by the convective term. In addition, it changes the sign later, therefore, the wave entrains a part of the medium,
and transfers it with frozen vorticity into the potential current. However, the oscillations associated with the convective velocity term decay rapidly, \( \sim 1/r^2 \), unlike the velocity oscillations in the wave, which decay slowly, \( \sim 1/r \).

6. Conclusion
The process of vortex formation was considered in the framework of the concept of a wave model of turbulence. But this process is the starting point of the vortex model. In addition, the wave model made it possible to solve a number of issues that could not be resolved within the framework of the vortex model. These include problems such as limitations associated with Thomson’s theorem that action over the entire section is possible only after the boundary layers from all walls fill the section, that is, when the boundary layers merge. The problem of the formation and thickness of the boundary layer, the viscous sublayer, and the structure of the spatiotemporal pulsation field of these layers is solved. In this sense, we can say that the wave concept includes the model proposed by L.F. Richardson [7].

The interpretation of the problems discussed is mainly based on the geometro-optical approximation of the theory. However, the interpretation of the vortex motion in the boundary layer of turbulent flow required the use of ideas about the change in the helical dislocation of the wave structure of the front of the acoustic wave.

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