On the classification of Stanley sequences

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Abstract

An integer sequence is said to be 3-free if no three elements form an arithmetic progression. A Stanley sequence \( \{a_n\} \) is a 3-free sequence constructed by the greedy algorithm. Namely, given initial terms \( a_0 < a_1 < \cdots < a_k \), each subsequent term \( a_n > a_{n-1} \) is chosen to be the smallest such that the 3-free condition is not violated. Odlyzko and Stanley conjectured that Stanley sequences divide into two classes based on asymptotic growth: Type 1 sequences satisfy \( a_n = \Theta(n \log_2^3) \) and appear well-structured, while Type 2 sequences satisfy \( a_n = \Theta(n^2 / \log n) \) and appear disorderly. In this paper, we define the notion of regularity, which is based on local structure and implies Type 1 asymptotic growth. We conjecture that the reverse implication holds. We construct many classes of regular Stanley sequences, which include all known Type 1 sequences as special cases. We show how two regular sequences may be combined into another regular sequence, and how parts of a Stanley sequence may be translated while preserving regularity. Finally, we demonstrate the surprising fact that certain Stanley sequences possess proper subsets that are also Stanley sequences.

Keywords: Stanley sequence, 3-free set, arithmetic progression, Roth’s theorem, greedy algorithm

1. Introduction

A set of nonnegative integers is 3-free if no three elements form an arithmetic progression. Given a 3-free set \( A \) with elements \( a_0 < a_1 < \cdots < a_k \), we define the Stanley sequence \( S(A) = \{a_n\} \) according to the greedy algorithm, as follows: Assuming \( a_{n-1} \) has been defined, let \( a_n \) be the smallest integer greater than \( a_{n-1} \) such that \( \{a_0, \ldots, a_n\} \) is 3-free. For convenience, we shall often write \( S(a_0, a_1, \ldots, a_k) \) for \( S(\{a_0, a_1, \ldots, a_k\}) \).

The simplest Stanley sequence is \( S(0) = 0, 1, 3, 4, 9, 10, 12, 13, 27, \ldots \), the elements of which are exactly those integers with no 2’s in their ternary representation. Odlyzko and Stanley \([5]\) offered similar closed-form descriptions of the sequences \( S(0, 3^n) \) and \( S(0, 2 \cdot 3^n) \), for \( n \) any nonnegative integer. Their work also suggested an overarching dichotomy among Stanley sequences, in which the more “well-structured” sequences (such as \( S(0) \)) follow one asymptotic growth pattern, while more “disorderly” sequences follow another.

**Conjecture 1.1** (based on work by Odlyzko and Stanley \([5]\)). Let \( S(A) = \{a_n\} \) be a Stanley sequence. Then, for all \( n \) large enough, one of the following two patterns of growth is satisfied.

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• Type 1. For some value $\alpha = \alpha(A)$, the following limits exist and are bounded as follows:

$$\frac{\alpha}{2} \leq \lim \inf a_n/n^{\log_2 3} \leq \lim \sup a_n/n^{\log_2 3} \leq \alpha,$$
or

• Type 2. $a_n = \Theta(n^2/\log n)$.

**Remark.** The original paper [5] considered the first type of growth in the case of $\alpha = 1$ only. However, if $\alpha$ is so restricted, the conjecture is certainly false, with $S(0, 1, 7)$ being one counterexample, requiring $\alpha = 10/9$. (This assertion is simple to prove with machinery we present in §2 and §3.)

The closed-form descriptions given in [1] for $S(0, 3^n)$ and $S(0, 2 \cdot 3^n)$ demonstrate that these sequences do indeed follow Type 1 growth, but they are by no means the only such sequences. The justification given in [5] for conjecturing Type 2 growth is a non-constructive probabilistic method that suggests, but does not prove, that a “random” Stanley sequence should follow Type 2 growth. However, no particular sequence has yet been shown to be of Type 2. Gerver [2] has computed the sequence $S(0, 4)$ up to $a_n \approx 2.5 \times 10^6$ and has verified the conjectured growth, also observing interesting fluctuations in the density of the sequence, which empirically occur according to a geometric series. Lindhurst [3] also provides data to support the notion that $S(0, 4)$ follows Type 2 growth. No simple closed form for $S(0, 4)$ is known.

Erdős et al. [1] posed several problems similar to Conjecture 1.1 regarding the density of Stanley sequences. In a recent paper, Moy [4] solved one of these questions by showing that all Stanley sequences $\{a_n\}$ satisfy the asymptotic bound

$$a_n \leq n^2/(2 + \epsilon).$$

Another problem posed in [1] is that of finding a Stanley sequence $\{a_n\}$ for which $\lim_{n \to \infty}(a_n - a_{n-1}) = +\infty$. This remains open; however, Savchev and Chen [8] answered a related question of [1] in the affirmative: there does exist a sequence $\{a_n\}$ for which $\lim_{n \to \infty}(a_n - a_{n-1}) = +\infty$ and such that $\{a_n\}$ is a maximal 3-free set - that is, a 3-free set that is not a proper subset of any other 3-free set. Erdős et al. [1] and Moy [4] appear to have assumed that Stanley sequences are maximal 3-free sets, though in Corollary 4.2 we will show that this assumption is false.

In this paper, we approach the conjectured dichotomy among Stanley sequences from the perspective of local structure, rather than asymptotic behavior. We begin by defining the independent Stanley sequences and the more general class of regular Stanley sequences. In §2 we show that all regular Stanley sequences follow Type 1 growth and conjecture that every Stanley sequence that is not regular follows Type 2 growth.

**Definition** (see Example 2.2). We say that a Stanley sequence $S(A) = \{a_n\}$ is independent if there exists a constant $\lambda = \lambda(A)$, called the character, such that, for all sufficiently large $k$, the equations

\begin{align*}
a_{2^k+i} &= a_{2^k} + a_i, \quad (1) \\
a_{2^k} &= 2a_{2^k-1} - \lambda + 1 \quad (2)
\end{align*}

hold whenever $0 \leq i < 2^k$. We say that an integer $k_0$ is adequate if (i) these equations are satisfied for all $k \geq k_0$ and (ii) the minimal set $A$ generating $S(A)$ does not contain the element $a_{2^k_0}$.
Definition (see Example 2.5). We say that a Stanley sequence \( S(A) = \{a_n\} \) is regular if there exist constants \( \lambda, \sigma \) and an independent Stanley sequence \( \{a'_n\} \), having character \( \lambda \), such that:

- The character of \( \{a'_n\} \) equals \( \lambda \).
- For large enough \( k \), the equations

\[
\begin{align*}
a_{2^k - \sigma + i} &= a_{2^k - \sigma} + a'_i, \\
a_{2^k - \sigma} &= 2a_{2^k - \sigma - 1} - \lambda + 1
\end{align*}
\]

hold whenever \( 0 \leq i < 2^k \).

We refer to \( \lambda, \sigma, \) and \( \{a'_n\} \) respectively as the character, shift index, and core of \( S(A) \). We say that an integer \( k_0 \) is adequate if (i) equations (3) and (4) are satisfied for all \( k \geq k_0 \) and (ii) the minimal set \( A \) generating \( S(A) \) does not contain the element \( a_{2^k_0} \).

In [3], we consider methods for constructing independent Stanley sequences. We begin by describing a class of independent Stanley sequences that includes as a special case the sequences \( S(0,3^n) \) and \( S(0,2 \cdot 3^n) \) detailed in [5].

Theorem 1.2 (see Example 3.1). Let \( k \) be a positive integer and \( A \) be a monotone decreasing family of subsets of \( \{0,1,\ldots,k-1\} \). Let

\[
A = \{3^{a_1} + 3^{a_2} + \cdots + 3^{a_n} : \{a_1, a_2, \ldots, a_n\} \subseteq A\}.
\]

Then, \( S(A \cup \{3^k\}) \) and \( S(A \cup \{2 \cdot 3^k\}) \) are independent. (In Section 3, we give closed-form descriptions of these sequences.)

We next describe an operation that combines a regular with an independent sequence to yield a regular sequence.

Theorem 1.3 (see Example 3.3). Let \( S(A) = \{a_n\} \) be independent and \( S(B) = \{b_n\} \) be regular. Let \( k \) be adequate with respect to \( S(A) \). Let \( A^* = \{a_0, a_1, \ldots, a_{2^k-1}\} \) and define

\[
A \otimes_k B = \{a_{2^k}b + a : a \in A^*, b \in B\}.
\]

Then, \( S(A \otimes_k B) \) is a regular Stanley sequence, independent if and only if \( B \) is, having description

\[
S(A \otimes_k B) = \{a_{2^k}b + a : a \in A^*, b \in S(B)\},
\]

with character \( \lambda(A \otimes_k B) = a_{2^k} \cdot \lambda(B) + \lambda(A) \) and shift index \( \sigma(A \otimes_k B) = 2^k \cdot \sigma(B) \).

Using this operation, we describe another class of independent Stanley sequences.

Theorem 1.4 (see Example 3.4). Let \( k \) be a positive integer. Let \( T_1, T_2 \) be disjoint subsets of \( \{0,1,\ldots,k\} \) such that no \( t \in T_1 \) satisfies \( t - 1 \in T_2 \). Let

\[
A = \left(3^{a_1} + 3^{a_2} + \cdots + 3^{a_m}\right) + 2\left(3^{b_1} + 3^{b_2} + \cdots + 3^{b_n}\right) : \{a_1, a_2, \ldots, a_m\} \subseteq T_1, \{b_1, \ldots, b_n\} \subseteq T_2
\]

Then, \( S(A) \) is an independent Stanley sequence.
In [4], we turn to dependent Stanley sequences, regular sequences that are not independent. We show that, for some regular sequences, an element may be removed without changing the Stanley sequence property of the other elements (see Example 4.1). This is quite remarkable, as it alters the indexing of elements in the sequence so that behavior associated with \(a_{2^n}\) may become associated with \(a_{2^n-1}\), or indeed with \(a_{2^n-\sigma}\) for an arbitrarily large constant \(\sigma\).

Finally, given an independent sequence \(S(A)\), we construct a dependent sequence by translating a portion of \(S(A)\) and recomputing subsequent elements. The resulting sequence has \(S(A)\) as its core. Specifically, for a Stanley sequence \(S(A) = \{a_n\}\) and nonnegative integers \(k\) and \(c\), we define

\[
A^c_k := \left\{a_i \mid 0 \leq i < 2^k - \sigma(A) \right\} \cup \left\{c + a_i \mid 2^k - \sigma(A) \leq i < 2^{k+1} - \sigma(A) \right\}.
\]

**Theorem 1.5** (see Example 4.5). Let \(S(A) = \{a_n\}\) be an independent sequence with character \(\lambda\). Let \(\ell\) be the minimum adequate integer for \(S(A)\), and pick \(k \geq \ell\). Let \(c\) be such that

\[
\lambda \leq c \leq a_{2^k-2\ell} - \lambda.
\]

Then, \(A^c_k\) is 3-free and \(S(A^c_k)\) is a regular Stanley sequence with core \(S(A)\).

## 2. Regular Stanley sequences

We begin by introducing some useful terminology and notation. If \(S(A)\) is a Stanley sequence, we say that \(A\) is a nucleating set of \(S(A)\). Note that a given Stanley sequence has infinitely many nucleating sets, corresponding to all sufficiently large prefixes of the sequence. We define the minimum nucleating set of a Stanley sequence to be that which is of minimum cardinality.

We use the shorthand \(A + n\) to denote the set \(\{a + n \mid a \in A\}\), for any set \(A\) and integer \(n\). It is easy to see that if \(S(A)\) is a Stanley sequence and \(n\) is a nonnegative integer, then

\[
S(A + n) = S(A) + n.
\]

In other words, translating the nucleating set translates the entire sequence. For the remainder of this paper, we will assume without loss of generality that every Stanley sequence begins at 0.

We say an integer \(x\) is covered by a set \(S\) of integers if there exist \(s, t \in S\) such that \(s < t\) and \(2t - s = x\). Then, the Stanley sequence \(S(a_0, a_1, \ldots, a_k)\) is the unique increasing sequence \(S = \{a_n\}\) where each integer \(x > a_k\) is covered by \(S\) if and only if \(x \not\in S\). Given a Stanley sequence \(S(A)\), we define the omitted set \(O(A)\) to be the set of nonnegative integers that are neither in \(S(A)\) nor are covered by \(S(A)\). For \(O(A) \neq \emptyset\), we let \(\omega(A)\) denote the largest element of \(O(A)\). It is immediate that \(\omega(A)\) is less than the largest element of \(A\).

We say that an integer \(x\) is jointly covered by sets \(S\) and \(T\) if there exist \(s \in S, t \in T\) such that \(s < t\) and \(2t - s = x\). Thus, an integer jointly covered by \(S\) and \(S\) is covered by \(S\). We say that a set \(X\) is covered (or jointly covered) by a set \(S\) (or pair of sets \(S\) and \(T\)) if every element of \(X\) is so covered. The following lemma is trivial to prove but will be extremely useful hereafter.

**Lemma 2.1** (Cover-shift Lemma). If \(x\) is jointly covered by \(S\) and \(T\), and if \(n_1 \leq n_2\) are integers, then \(x + (2n_2 - n_1)\) is jointly covered by \(S + n_1\) and \(T + n_2\).

### 2.1. Independent sequences

For \(\{a_n\}\) an independent Stanley sequence, we refer to the set \(\{a_i \mid 2^k \leq i < 2^{k+1}\}\) as the \(k\)th block \(\Gamma_k\).
Example 2.2. The Stanley sequence $S(0, 2, 5)$ is independent with character $\lambda = 4$.

$$S(0, 2, 5) = 0, 2, 5, 6, 9, 11, 14, 15, 27, 29, 32, 33, 36, 38, 41, 42, \ldots$$

Note that, in moving from the last element of one block to the first element of the next block, the value of the sequence doubles and subtracts $(\lambda - 1)$. The next block is then a translate of the preceding terms. Thus, for instance,

$$2 \cdot a_3 - \lambda + 1 = 2 \cdot 6 - 4 + 1 = 9 = a_4$$

and $\{9, 11, 14, 15\} = \{0, 2, 5, 6\} + 9$. Likewise, $2 \cdot 15 - 4 + 1 = 27$, and $\{27, 29, 32, 33, 36, 38, 41, 42\} = \{0, 2, 5, 6, 9, 11, 14, 15\} + 27$.

The following proposition shows that the criterion “for all sufficiently large $k$” in the definition of independent sequences can be replaced by “for a single sufficiently large $k$.” This will be useful in proving that sequences are independent.

Proposition 2.3. Let $S(A) = \{a_n\}$ be a Stanley sequence with $\omega = \omega(A)$, and suppose integers $\lambda$ and $k$ are such that $a_{2k-1} \geq \lambda + \omega$ and that equations (1) and (2) hold whenever $0 \leq i < 2^k$. Then, $S(A)$ is independent with character $\lambda$.

Proof. It suffices to show that (1) and (2) must hold for all $k' > k$, and hence to show that they hold if $k$ is replaced with $k + 1$. Let $\Lambda = \{a_i \mid 0 \leq i < 2^k\}$ and $\Gamma = \{a_i \mid 2^k \leq i < 2^{k+1}\}$, so that $\Gamma = \Lambda + a_{2k}$ (see Figure 1). Let $B$ be the set of integers in the interval $[0, a_{2k-1}]$ that are covered by $\Lambda$.

Our strategy will be to describe the integers covered by $\Lambda \cup \Gamma$ by breaking up this set into (i) the integers covered by $\Lambda$ alone, (ii) the integers covered by $\Gamma$ alone, (iii) the integers jointly covered by $\Lambda$ and $\Gamma$. We additionally break up the set in (iii) into the sets $\{2y - x \mid x \in \Lambda, y \in \Gamma, y > x + a_{2k}\}$, $\{2y - x \mid x \in \Lambda, y \in \Gamma, y = x + a_{2k}\}$, and $\{2y - x \mid x \in \Lambda, y \in \Gamma, y < x + a_{2k}\}$.

We begin by observing that $\Lambda$ covers the following integers:

- $B$, by definition.
- The open interval $(a_{2k-1}, a_{2k})$, since these integers are not in $S(A)$ and hence must be covered by $S(A)$.
- The set $O(A) + a_{2k}$. To see this, observe that each element $s \in O(A) + a_{2k}$ must be covered by $S(A)$, and yet cannot be covered by $\Lambda + a_{2k} = B$ by the definition of $O(A)$. Hence, there must be $x \in \Lambda$ and $w \in S(A)$ such that $2w - x = s$. Since $x \leq a_{2k-1}$ and $s \leq \omega + a_{2k}$, we conclude that

$$2w \leq a_{2k-1} + \omega + a_{2k} < 2a_{2k-1} - \lambda + 1 + a_{2k}$$

(where the second inequality follows from $a_{2k-1} \geq \lambda + \omega$). The right side of this equals $2a_{2k}$, implying that $w < a_{2k}$ and hence that $w \in \Lambda$. We conclude that $s$ and hence $O(A) + a_{2k}$ is covered by $\Lambda$.

It is easy to see that the union

$$B \cup (a_{2k-1}, a_{2k}) \cup (O(A) + a_{2k})$$
of these three sets in fact constitutes exactly the integers covered by $\Lambda$. Hence, by the Cover-shift Lemma (Lemma 2.1), the set $\Gamma = \Lambda + a_{2k}$ must cover exactly the union

$$(B + a_{2k}) \cup (a_{2k+1-1}, 2a_{2k}) \cup (O(A) + 2a_{2k}).$$

Similarly, the set $\{2y - x \mid x \in \Lambda, y \in \Gamma, y > x + a_{2k}\}$ equals

$$(B + 2a_{2k}) \cup (a_{2k-1} + 2a_{2k}, 3a_{2k}) \cup (O(A) + 3a_{2k}),$$

because $\{2y - x \mid x \in \Lambda, y \in \Lambda, y > x\} = B \cup (a_{2k-1}, a_{2k}) \cup (O(A) + a_{2k})$.

We now note that

$$\{2y - x \mid x \in \Lambda, y \in \Gamma, y = x + a_{2k}\} = \Lambda + 2a_{2k}.$$  

Letting

$$C = \{2y - x \mid x \in \Lambda, y \in \Gamma, y < x + a_{2k}\},$$

we see that all elements of $C$ are less than

$$2a_{2k+1-1} - a_{2k-1} = 2a_{2k} + a_{2k-1} < 3a_{2k}.$$  

Hence, $C \subseteq [0, 3a_{2k})$.

Summing up our results, we find that the integers covered by $\Lambda \cup \Gamma$ are exactly the union

$$B \cup (a_{2k-1}, a_{2k}) \cup (O(A) + a_{2k}) \cup (B + a_{2k}) \cup (a_{2k+1-1}, 2a_{2k}) \cup (O(A) + 2a_{2k})$$

$$\cup (B + 2a_{2k}) \cup (a_{2k-1} + 2a_{2k}, 3a_{2k}) \cup (O(A) + 3a_{2k}) \cup (\Lambda + 2a_{2k}) \cup C.$$

Restricting to integers greater than $a_{2k+1-1}$, we obtain the set

$$(a_{2k+1-1}, 2a_{2k}) \cup (O(A) + 2a_{2k}) \cup (B + 2a_{2k}) \cup (a_{2k-1} + 2a_{2k}, 3a_{2k})$$

$$\cup (O(A) + 3a_{2k}) \cup (\Lambda + 2a_{2k}) \cup (C \cap (a_{2k+1-1}, \infty)).$$

Since the union $(O(A) + 2a_{2k}) \cup (B + 2a_{2k}) \cup (\Lambda + 2a_{2k})$ comprises the entire interval $[2a_{2k}, a_{2k-1} + 2a_{2k}]$, the preceding expression simplifies to

$$(a_{2k+1-1}, 3a_{2k}) \cup (O(A) + 3a_{2k}) \cup (C \cap (a_{2k+1-1}, \infty)).$$
Because \( C \) is a subset of \([0, 3a_{2k})\), the last term is already included in the first, giving
\[
(a_{2k+1-1}, 3a_{2k}) \cup (O(A) + 3a_{2k}).
\]

This shows that
\[
a_{2k+1} = 3a_{2k} = 2a_{2k} + 2a_{2k-1} - \lambda + 1 = 2a_{2k+1-1} - \lambda + 1
\]
and more generally that the terms of \( S(A) \) that follow \( a_{2k+1-1} \) are exactly the elements of the set \( A + 3a_{2k} \), followed by as many terms of \( S(A) + 3a_{2k} \) as occur before \( 2a_{2k+1} - a_0 = 6a_{2k} \). Since the elements of \( S(A) + 3a_{2k} \) that occur before \( 6a_{2k} \) are exactly the elements
\[
a_0 + 3a_{2k}, a_1 + 3a_{2k}, \ldots, a_{2k+1-1} + 3a_{2k},
\]
we conclude that equations \([1]\) and \([2]\) hold with \( k + 1 \) substituted for \( k \). Hence, these equations must hold for all \( k' > k \) and \( S(A) \) is independent.

**Proposition 2.4.** Let \( S(A) = \{a_n\} \) be an independent sequence. Then, there exists a constant \( \alpha \) such that, for \( k \) large enough,
\[
a_{2k} = \alpha \cdot 3^k.
\]

**Proof.** For sufficiently large \( k \), we verify:
\[
a_{2k+1} = 2(a_{2k} + a_{2k-1}) - \lambda + 1 = 2 \left( a_{2k} + \frac{1}{2}(a_{2k} + \lambda) \right) - \lambda + 1 = 3a_{2k}.
\]

\[\square\]

### 2.2. Regular sequences

**Example 2.5.** The sequence \( \{a_n\} = S(0, 1, 4) \) is regular with \( \lambda = 0 \) and \( \{a'_n\} = S(0) \). As with most regular sequences we will consider, the shift index \( \sigma \) equals 0.

As with independent sequences, we can break up the sequence into blocks \( \Gamma_k \) as follows, where the length of each block is a power of 2:

\[
\begin{align*}
\{a_n\} &= 0, 1, \boxed{4, 5}, 11, 12, 14, 15, \boxed{31, 32, 34, 35, 40, 41, 43, 44,} \ldots \\
& \quad \text{\( r_1 \)} \quad \text{\( r_2 \)} \quad \text{\( r_3 \)} \\
\{a'_n\} &= 0, 1, \boxed{3, 4}, 9, 10, 12, 13, \boxed{27, 28, 30, 31, 36, 37, 39, 40,} \ldots \\
& \quad \text{\( r_1 \)} \quad \text{\( r_2 \)} \quad \text{\( r_3 \)}
\end{align*}
\]

Note that, in moving from the last element of one block to the first element of the next block, the value of the sequence doubles and subtracts \( (\lambda - 1) \). The next block is then a translate of the corresponding preceding terms in the sequence \( \{a'_n\} \). Note that
\[
2 \cdot a_3 - \lambda + 1 = 2 \cdot 5 - 0 + 1 = 11 = a_4
\]
and \(\{11, 12, 14, 15\} = \{0, 1, 3, 4\} + 11\). Likewise, \(2 \cdot 15 + 1 = 31\) and \(\{31, 32, 34, 35, 40, 41, 43, 44\} = \{0, 1, 3, 4, 9, 10, 12, 13\} + 31\).

**Proposition 2.6.** Let \(S(A) = \{a_n\}\) be a regular sequence with character \(\lambda\), shift index \(\sigma\), and core \(\{a'_n\}\). Let \(\alpha\) be the constant implied by Proposition 2.4 such that \(a_{2k} = \alpha \cdot 3^k\) for large \(k\). Then, there exists a constant \(\beta\) such that, for \(k\) large enough,

\[
a_{2k-\sigma} = \alpha \cdot 3^k + \beta \cdot 2^k.
\]

**Proof.** Pick some adequate \(k\). Observe that

\[
a_{2k+1-\sigma} - 2a_{2k-\sigma} = (2a_{2k+1-\sigma} - 1) - 2a_{2k-\sigma} = 2(a_{2k+1-\sigma} - 1) - 2a_{2k-\sigma} = 2(a_{2k+1-\sigma} - 1) - \lambda + 1 = 2a_{2k+1-\sigma} - \lambda + 1 = a_{2k+1-\sigma} = \alpha \cdot 3^k,
\]

which proves the proposition.

**Proposition 2.7.** If \(S(A) = \{a_n\}\) is regular, then there is a unique choice of constants \(\lambda\), \(\sigma\) and independent Stanley sequence \(\{a'_n\}\) such that the above definition of regularity is satisfied.

**Proof.** Observe that uniqueness of \(\sigma\) implies uniqueness of \(\lambda\) and \(\{a'_n\}\). Suppose for the sake of contradiction that \(\sigma\) can take on distinct values \(\sigma_1 < \sigma_2\) (for correspondingly distinct pairs \((\lambda, \{a'_n\})\)). Then, for sufficiently large \(k\),

\[
a_{2k-\sigma_1} \approx 2a_{2k-\sigma_1 - 1} \geq 2a_{2k-\sigma_2} \approx 4a_{2k-\sigma_2 - 1} \geq 4a_{2k-1 - \sigma_1}.
\]

However, Proposition 2.6 implies that \(a_{2k-\sigma_1} = \Theta(3^k)\), a contradiction. Hence, \(\sigma\) is unique.

For \(S(A)\) regular, we write \(\lambda\), \(\sigma\), and \(\{a'_n\}\) as \(\lambda(A)\), \(\sigma(A)\), and \(S'(A)\), respectively and refer to them as the character, shift index, and core of the Stanley sequence \(S(A)\). It is evident that the independent Stanley sequences \(S(A)\) are exactly the regular Stanley sequences that satisfy \(\sigma(A) = 0\) and \(S'(A) = S(A)\). We say that a sequence is dependent if it is regular but not independent.

Proposition 2.6 allows us to define parameters \(\alpha(A)\) and \(\beta(A)\) for each regular sequence \(S(A)\). Note that while \(\alpha(A)\) and \(\beta(A)\) must evidently be rational, they need not be integers, as in the case of \(A = \{0, 1, 7\}\), where \(\alpha(A) = 10/9\). It is clear that \(\alpha(A)\) must be positive; a similar condition on \(\beta(A)\) appears true from data.

**Conjecture 2.8.** \(\beta(A) \geq 0\) for all regular Stanley sequences \(S(A)\).

As a corollary to Proposition 2.6, we obtain the following welcome result.

**Corollary 2.9.** All regular Stanley sequences follow Type 1 growth.

We say that a Stanley sequence that is not regular is irregular. No known Type 1 sequence is irregular, and a computer algorithm found no examples after checking randomly generated Stanley sequences. This suggests that the dichotomy between regular and irregular sequences corresponds precisely with the dichotomy hypothesized in 5 between sequences with Type 1 and Type 2 growth.
Conjecture 2.10. All irregular Stanley sequences follow Type 2 growth.

2.3. The character

We conclude this section by examining the possible values of $\lambda(A)$ for regular sequences $S(A)$.

Proposition 2.11. Let $S(A)$ be a regular Stanley sequence. Then $\lambda(A) \geq 0$, with $\lambda \neq 1, 3$.

Proof. Let $\lambda(A) = \lambda$. We may assume without loss of generality that $S(A)$ is independent, since the core of $S(A)$ has the same character as $S(A)$ itself. Then, consider some adequate $k$, so that $a_{2k} = 2a_{2k-1} - \lambda + 1$ holds. We note that since $a_{2k} - 1 = 2a_{2k-1} - \lambda$ is not in $S(A)$, it must be covered by the set $T = \{a_0, a_1, \ldots, a_{k-1}\}$ and hence can be at most $2a_{2k-1}$. We conclude that $\lambda \geq 0$. Further, we note that since $2a_{2k-1}$ is certainly covered by $T$, the character $\lambda$ cannot be 1.

Suppose for the sake of contradiction that $\lambda = 3$. If $1 \in S(A)$, then $a_{2k+1} = a_{2k} + a_1 = a_{2k} + 1$ by regularity. Because $a_{2k} = 2a_{2k-1} - 2$, we conclude that $a_{2k+1} = 2a_{2k-1} - 1$, which is a contradiction since then $1, a_{2k-1}, a_{2k+1}$ form an arithmetic progression. We conclude that $1 \notin S(A)$, which means that $2a_{2k-1} - 1$ is covered by $T$. Suppose $2t - s = 2a_{2k-1} - 1$ for $s, t \in T$. Since the greatest element of $T$ is $a_{2k-1}$, we have $t = a_{2k-1}$, because smaller $t$ would force $s$ to be negative. But then $s = 1$, a contradiction since we know $1 \notin S(A)$. We conclude $\lambda \neq 3$. 

For further investigation of forbidden character values, the following lemma is useful.

Lemma 2.12. If $S(A)$ is independent, then $\omega(A) < \lambda(A)$.

Proof. Take some sufficiently large integer $k$ and let $T = \{a_0, a_1, \ldots, a_{k-1}\}$. Let $x = \omega(A) + a_{2k}$. We know $x$ is covered by $S(A)$, so let $s, t \in S(A)$ be such that $s < t$ and $2t - s = x$. If neither $s$ nor $t$ is in $T$, then $s' = s - a_{2k}$ and $t' = t - a_{2k}$ must be in $S(A)$ and must satisfy $2t' - s' = x - a_{2k}$. Since $x - a_{2k} \in O(A)$ and thus cannot be covered by $S(A)$, this is impossible, so at least one of $s, t$ must be in $T$. If only $s$ is in $T$, then

$$2t - s \geq 2a_{2k} - a_{2k-1} = a_{2k} + a_{2k-1} - \lambda(A) + 1,$$

which is larger than $x$ because $k$ is large. Hence, both $s, t$ must be in $T$. Since the maximum integer covered by $T$ is $2a_{2k-1} = a_{2k} + \lambda(A) - 1$, the lemma follows. 

Corollary 2.13. Every regular sequence of character 0 has $S(0)$ as its core.

Proof. If $S(A)$ is an independent sequence with character 0, then $\omega(A) < 0$ by the lemma, implying that $\omega(A)$ is not defined and so $O(A)$ is empty. Hence, $S(A) = S(0)$ and the result follows. 

Corollary 2.14. At most finitely many independent Stanley sequences exist for any character $\lambda$.

Proof. Suppose $S(A) = \{a_n\}$ is independent with $\lambda = \lambda(A)$ and $\omega = \omega(A)$, such that $A = \{a_0, a_1, \ldots, a_m\}$ is the minimum nucleating set for the sequence. We may assume that $\omega > 0$, since otherwise $S(0)$ is the only possible sequence.

Since $A$ is minimal, $a_{m-1} < \omega$. Observe that either (i) $\omega = a_m - 1$ or else (ii) $a_m - 1$ is itself covered by $A$. In case (ii), we must have $a_m - 1 \leq 2a_{m-1}$. Since $a_{m-1} < \omega$, this implies that $a_m < 2a_{m-1} + 1$. Clearly, this inequality also holds in case (i).

Now, Lemma 2.12 tells us that $\omega < \lambda$. Hence, $a_m < 2\lambda + 1$, implying the desired result. 

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This corollary tells us that whether or not a given character is possible for an independent (and hence regular) $S(A)$ can be ascertained by checking a finite number of potential nucleating sets $A$. We have examined (by computer) these possible nucleating sets for many character values; our data suggest that $1, 3, 5, 9, 11, 15$ are impossible for the character function. However, this result is not certain since it assumed the irregularity of various Stanley sequences, while as yet no Stanley sequence has been shown definitively to be irregular. For all other characters up to 76, we have found corresponding regular sequences. (See the appendix for sample data.) A method we will outline in the next section suggests that all sufficiently large values are possible for the character function. We therefore offer the following conjecture.

**Conjecture 2.15.** The range of the character function is exactly the set of integers $n$ that are at least 0 and are not in the set $\{1, 3, 5, 9, 11, 15\}$.

3. Constructing independent sequences

Heretofore, the only sequences shown to follow Type 1 growth have been the sequences $S(0, 3^k)$ and $S(0, 2 \cdot 3^k)$, for which complete descriptions were given in [5]. It is easily checked that these sequences are independent for any $k$. In this section we present several novel methods for constructing independent sequences, while in the next section we construct dependent sequences. The sequences we describe include $S(0, 3^k)$ and $S(0, 2 \cdot 3^k)$ as special cases. We begin with Theorem 1.2

**Example 3.1.** Take $k = 3$ and $A = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 2\}\}$. Then,

$$A = \{0, 1, 10, 100, 101\} \quad \text{in base 3}$$

$$= \{0, 1, 3, 9, 10\} \quad \text{in base 10}.$$

**Theorem 1.2** implies that $S(0, 1, 3, 9, 10, 27)$ and $S(0, 1, 3, 9, 10, 54)$ are independent. Indeed,

$$S(0, 1, 9, 10, 27) = 0, 1, 10, 100, 101, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, 2011, 2110, 2111, 10000, 10001, 10010, 10011, 10100, 10101, 10110, 10111, 11000, 11001, 11010, 11011, 11100, 11101, 11110, 11111, 12011, 12110, 12111, 100000, \ldots \quad \text{in base 3}$$

$$= 0, 1, 3, 9, 10, 27, 28, 30, 31, 36, 37, 39, 40, 58, 66, 67,$$

$$\begin{array}{c}
81, 82, 84, 90, 91, 108, 109, 111, 112, 117, 118, 120, 121, 139, 147, 148,
\end{array}$$

**243, \ldots \quad \text{in base 10}**

is independent with character $\lambda = 54$ satisfying $2 \cdot 67 - \lambda + 1 = 81$ and $2 \cdot 148 - \lambda + 1 = 243$.

We define the functions $t_i$ on nonnegative integers $x$ by letting $t_i(x)$ equal the digit in the $3^i$’s place in the ternary representation of $x$. We will show that the sequences $S(A \cup \{3^k\})$ and $S(A \cup \{2 \cdot 3^k\})$ defined in Theorem 1.2 admit the following closed-form descriptions:

1. $S(A \cup \{3^k\})$ contains exactly those integers $x \geq 0$ such that
   - $t_i(x) = 0$ or 1 for $i \neq k$.
   - If $t_k(x) = 0$, then $\sum_{i=0}^{k-1} t_i(x) 3^i \in A$.  

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• If \( t_k(x) = 2 \), then \( \sum_{i=0}^{k-1} t_i(x)3^i \not\in A \).

2. \( S(A \cup \{2 \cdot 3^k\}) \) contains exactly those integers \( x \geq 0 \) such that
- \( t_i(x) = 0 \) or 1 for \( i \neq k, k + 1 \).
- \( t_k(x) = 0 \) or 2.
- If \( t_k(x) = t_{k+1}(x) = 0 \), then \( \sum_{i=0}^{k-1} t_i(x)3^i \in A \).
- If \( t_{k+1}(x) = 2 \), then \( t_k(x) = 0 \) and \( \sum_{i=0}^{k-1} t_i(x)3^i \not\in A \).

**Proof of Theorem 7.2.** We will prove the theorem for \( S(A \cup \{3^k\}) \) (the proof for \( S(A \cup \{2 \cdot 3^k\}) \) is very similar). Pick some \( k \) and \( A \) according to the theorem statement, let \( A \) be defined from \( A \) as in the theorem, and let \( S \) be the sequence consisting of those nonnegative integers \( x \) which satisfy the three desired conditions on ternary digits. We must prove that \( S = S(A \cup \{3^k\}) \), for which we need (i) that \( S \) is 3-free, and (ii) that \( x > 3^k \) is covered by \( S \) if \( x \not\in S \).

We first prove (i). Suppose for the sake of contradiction that there exist \( x, y, z \in S \) with \( y, z < x \) such that \( 2y - z = x \). Since \( t_i(x), t_i(y), t_i(z) \) must be either 0 or 1 for each \( i \neq k \), we can conclude that \( t_i(x) = t_i(y) = t_i(z) \) for \( i \neq k \). Now, if \( t_k(x), t_k(y), t_k(z) \) are not identical, they must take on all values 0, 1, 2 in some order. However, if \( t_k \) is 0, the previous ternary digits must form an element of \( A \), whereas if \( t_k \) is 2, the previous ternary digits cannot form an element of \( A \). Since we know that \( t_i(x) = t_i(y) = t_i(z) \) for \( 0 \leq i \leq k - 1 \), we conclude that \( t_k \) is identical for \( x, y, z \). Hence, \( x = y = z \), a contradiction. We conclude that \( S \) must be 3-free.

We now prove (ii). Suppose that \( x > 3^k \) with \( x \not\in S \). We construct \( y, z \in S \) digit-wise so that \( y, z < x \) and \( x = 2y - z \). For each \( i < k \), we set
- \( t_i(y) = t_i(z) = 0 \) if \( t_i(x) = 0 \).
- \( t_i(y) = t_i(z) = 1 \) if \( t_i(x) = 1 \).
- \( t_i(y) = 1 \) and \( t_i(z) = 0 \) if \( t_i(x) = 2 \).

Before assigning the remaining digits \( t_i(y) \) and \( t_i(z) \), we define the numbers \( y_0 \) and \( z_0 \) to be the ternary subwords of \( y \) and \( z \), respectively, formed by considering only digits 0 through \( k - 1 \). We note that the nonzero digits of \( y_0 \) are a subset of those of \( y_0 \). Hence, if \( y_0 \) is in \( A \) then \( z_0 \) is also, since \( A \) is monotone decreasing.

This observation made, we now proceed to define the remaining digits.

**Case 3.1.1.** \( t_k(x) \neq 0 \).

We begin by assigning \( t_i(y) \) and \( t_i(z) \) for \( i > k \) following the same rules as for \( i < k \). Next, we define \( t_k(y) \) and \( t_k(z) \), as follows. If \( t_k(x) = 1 \), we set \( t_k(y) = t_k(z) = 1 \). By the definition of \( S \), the \( y \) and \( z \) thus constructed will be in \( S \), showing \( x \) is covered by \( S \). If \( t_k(x) = 2 \) and \( z_0 \in A \), then we set \( t_k(z) = 0 \) and \( t_k(y) = 1 \). Again \( y, z \in S \), so \( x \) is covered by \( S \). On the other hand, if \( t_k(x) = 2 \) and \( z_0 \not\in A \), then we have \( y_0 \not\in S \). We here set \( t_k(y) = t_k(z) = 2 \), and conclude again that \( y, z \in S \).

**Case 3.1.2.** \( t_k(x) = 0 \) and \( y_0 \in A \).

We begin by assigning \( t_i(y) \) and \( t_i(z) \) for \( i > k \) following the same rules as for \( i < k \). Next, we set \( t_k(y) = t_k(z) = 0 \). Since \( y_0 \) is in \( A \), \( z_0 \) must be as well, so \( y, z \in S \), as desired.

**Case 3.1.3.** \( t_k(x) = 0 \) and \( y_0 \not\in A \).
We begin by assigning \( t_i(y) \) and \( t_i(z) \) for \( i > k \) following the same rules as for \( i < k \), except with \( t_i(x) \) replaced by \( t_i(x - 3^{k+1}) \) throughout. (Since \( x > 3^k \) and \( t_k(x) = 0 \), we know that \( x - 3^{k+1} \) is a nonnegative integer.) Next, we set \( t_k(y) = 2 \) and \( t_k(z) = 1 \). It is simple to verify that \( y, z \in S \).

We conclude that in all cases \( y, z \in S \) and hence all \( x \notin S \) satisfying \( x > 3^k \) are covered by \( S \). Hence, \( S = S(A \cup \{3^k\}) \), as desired. From the definition of \( S \), the sequence is independent. \( \square \)

Remark. The characters of \( S(A \cup \{3^k\}) \) and \( S(A \cup \{2 \cdot 3^k\}) \) can easily be shown to equal \( 2 \cdot 3^k \) and \( 4 \cdot 3^k \), respectively, provided that \( A \) does not contain all subsets of \( \{0, 1, \ldots, k - 1\} \). If \( A \) does contain all subsets of \( \{0, 1, \ldots, k - 1\} \), then the sequence \( S(A \cup \{2 \cdot 3^k\}) \) has character \( 2 \cdot 3^k \), whereas \( S(A \cup \{3^k\}) \) is simply \( S(0) \) and has character 0 for any \( k \).

The construction in Theorem 1.2 may be modified as follows. Given an independent sequence \( S(A) = \{a_n\} \), define the \( k \)-reversal \( R_k(A) \) of \( S(A) \) as follows: For \( x = a_{2^k-1} \), set

\[
R_k(A) = S(x - a_{2^k-1}, x - a_{2^k-2}, \ldots, x - a_1, x - a_0).
\]

Note that this nucleating set is indeed 3-free and starts with 0. We say that an independent sequence is reversible if for every adequate \( k \), the \( k \)-reversal of the sequence is independent.

Proposition 3.2. Let \( A \) be as in Theorem 1.2. The sequences \( S(A \cup \{3^k\}) \) and \( S(A \cup \{2 \cdot 3^k\}) \) are reversible.

The proof of this result is similar to that of Theorem 1.2.

We now consider Theorem 1.3, which allows us to combine regular Stanley sequences.

Example 3.3. Take \( A = \{0\} \) and \( B = \{0, 2, 5\} \). Then, \( a_2 = 3 \) and

\[
A \otimes_2 B = \{3 \cdot b + a \mid a \in \{0, 1\}, b \in \{0, 2, 5\}\} = \{0, 1, 6, 7, 15, 16\}.
\]

Then, the sequence

\[
S(0, 1, 6, 7, 15, 16) = 0, 1, 6, 7, 15, 16, 18, 19, 27, 28, 33, 34, 42, 43, 45, 46,
\]

\[
\underline{81, 82, 87, 88, 96, 97, 99, 100, 108, 109, 114, 115, 123, 124, 126, 127}
\]

\[
243, \ldots
\]

is independent with character \( 3 \lambda(B) + \lambda(A) = 3 \cdot 4 + 0 = 12 \).

Proof of Theorem 1.3. Let \( S = \{a_{2^k}b + a \mid a \in A^*, b \in S(B)\} \) be the proposed form of the sequence \( S(A \otimes_k B) \), and let \( x_0 \) be the largest element of \( A \otimes_k B \). It suffices to show (i) that \( S \) is 3-free, and (ii) that every integer \( x > x_0 \) not in \( S \) is covered by \( S \).

We first prove that \( S \) is 3-free. Suppose for the sake of contradiction that \( x, y, z \in S \) exist with \( y, z < x \) and \( 2y - z = x \). Since no three distinct elements of \( A^* \) form an arithmetic progression modulo \( a_{2^k} \), we conclude that \( x, y, z \) must all be identical modulo \( a_{2^k} \) to some common \( a_i \). But then the elements \( (x - a_i)/a_{2^k}, (y - a_i)/a_{2^k}, (z - a_i)/a_{2^k} \) of \( S(B) \) must form an arithmetic progression - a contradiction. Hence, \( S \) must be 3-free.
Now suppose that \( x > x_0 \) is not in \( S \). We must show it is covered by \( S \). Let \( m, r \) be such that \( x = m \cdot a_{2k} + r \). There are two possibilities:

**Case 3.3.1.** \( r \) is in \( A^* \) or is covered by it.

Pick \( a_i, a_j \in A^* \) such that \( 2a_i - a_j = r \). Since \( x > x_0 \), \( m \) must either be in \( S(B) \) or else be covered by it. Picking \( b_g, b_h \) such that \( 2b_g - b_h = m \), we see that

\[
x = 2(a_{2k}b_g + a_i) - (a_{2k}b_h + a_j).
\]

**Case 3.3.2.** \( r \in O(A) \).

In this case, there exist \( a_i, a_j \in A^* \) such that \( 2a_i - a_j = a_{2k} + r \). Then, \( m - 1 \) is either in \( S(B) \) or covered by it. Picking \( b_g, b_h \) such that \( 2b_g - b_h = m - 1 \), we see that

\[
x = 2(a_{2k}b_g + a_i) - (a_{2k}b_h + a_j).
\]

We conclude that, in both possible cases, \( x \) is covered by \( S \) and hence that \( S \) is indeed \( S(A \otimes_k B) \). That \( S(A \otimes_k B) \) is regular, with character and shift index as stated, follows routinely from the explicit description of \( S \).

**Remark.** Theorem 1.3 proves that a great number of integers are attainable as characters of regular Stanley sequences. For example,

\[
\lambda(\{0\} \otimes_1 A) = 3\lambda(A)
\]

\[
\lambda(\{0, 2\} \otimes_1 A) = 3\lambda(A) + 2
\]

It appears possible that similar reasoning could show the attainability of all character values above a certain constant; more research in this area is called for.

We will refer to the operation \( A \otimes_k B \) just described as the \( k \)-product of \( A \) and \( B \). We note that \( k \)-multiplication of independent sequences is associative; this follows immediately from our closed-form description of the terms of \( A \otimes_k B \). Theorem 1.3 allows for the construction of many novel regular Stanley sequences, such as those described in Theorem 1.4.

**Example 3.4.** Let \( k = 3 \), let \( T_1 = \{0, 3\} \) and \( T_2 = \{1\} \). We have:

\[
A = \{0, 1, 20, 21, 1000, 1001, 1020, 1021\} \text{ in base 3}
\]

\[
= \{0, 1, 6, 7, 27, 28, 33, 34\} \text{ in base 10}
\]

\[
S(A) = 0, 1, 20, 21, 1000, 1001, 1020, 1021, 1100, 1101, 1120, 1121, 2100, 2101, 2120, 2121,
\]

\[
10000, 10001, 10020, 10021, 11000, 11001, 11020, 11021, 11100, 11101, 11120, 11121,
\]

\[
12100, 12101, 12120, 12121,
\]

\[
100000, \ldots \text{ in base 3}
\]

\[
= 0, 1, 6, 7, 27, 28, 33, 34, 36, 37, 42, 43, 63, 64, 69, 70,
\]

\[
\underline{81, 82, 87, 88, 108, 109, 114, 115, 117, 118, 123, 124, 144, 145, 150, 151,}
\]

\[
\Gamma_4
\]

\[
243, \ldots \text{ in base 10}
\]
As predicted by Theorem 1.4, $S(A)$ is independent with character $\lambda = 60$.

Before proving Theorem 1.4, we shall present two weaker versions of this theorem. Their proofs follow routine case analysis and are omitted.

**Lemma 3.5.** Let $k$ be a positive integer, and $j$ an integer such that $0 \leq j \leq k$. Set

$$B_1 = \{3^{b_1} + 3^{b_2} + \ldots + 3^{b_m} \mid j \leq b_1 < b_2 < \ldots < b_m \leq k\}.$$

Then, $S(B_1)$ is independent, with $k + 1$ adequate. Specifically, $S(B_1)$ consists of all integers $x \geq 0$ satisfying

- $t_i(x)$ equals $0$ or $1$ for all $i$ not in the interval $[j,k]$.
- If $t_j(x), t_{j+1}(x), \ldots, t_k(x)$ are all $0$ or $1$, but are not all $1$, then $t_i = 0$ for all $i < j$.
- If $t_j(x), t_{j+1}(x), \ldots, t_k(x)$ are not all either $0$ or $1$, then (i) not all $t_i(x)$ are $0$ for $i < j$, and (ii) $t_i(x)$ equals $1$ or $2$ for $j \leq i \leq k$.

**Lemma 3.6.** Let $k$ be a positive integer, and $j$ an integer such that $0 \leq j \leq k$. Set

$$B_2 = \{2 \left(3^{b_1} + 3^{b_2} + \ldots + 3^{b_m}\right) \mid j \leq b_1 < b_2 < \ldots < b_m \leq k\}.$$

Then, $S(B_2)$ is independent, with $k + 2$ adequate. To describe the elements of $S(B_2)$ in closed form, we first define a function $\zeta$ on the set of nonnegative integers $x$ such that $t_i(x)$ equals $0$ or $2$ for all $j \leq i \leq k$. Set $\zeta(x)$ equal to

- $x$ itself, if $t_i(x) = 0$ for all $i < j$
- otherwise, the integer obtained from $x$ be switching to $1$ all digits $t_i(x) = 2$ such that $j \leq i \leq k$ and at least one of $t_j(x), t_{j+1}(x), \ldots, t_{i-1}(x)$ is zero.

Then, $S(B_2)$ consists of all integers $\zeta(x)$ such that $x$ satisfies

- $t_i(x)$ equals $0$ or $2$ for all $i$ in the interval $[j,k+1]$.
- $t_i(x)$ equals $0$ or $1$ for all $i$ not in the interval $[j,k+1]$.
- If $t_j(x), t_{j+1}(x), \ldots, t_k(x)$ are all $0$ or $2$, but are not all $2$, then $t_i = 0$ for all $i < j$.
- If $t_{k+1} = 0$, then $t_j(x), t_{j+1}(x), \ldots, t_k(x)$ are all $0$ or $2$.
- If $t_{k+1} = 2$, then $x' \notin S$, where $x'$ is obtained from $x$ by switching $t_{k+1}$ to $0$.

**Proof of Theorem 1.4.** We observe that $A$ can be expressed as the product

$$A_1 \otimes_k A_2 \otimes_{k_2} \cdots \otimes_{k_{m-1}} A_m,$$

where each $A_i$ is either of the form $B_1$ (see Lemma 3.5) or of the form $B_2$ (see Lemma 3.6), with each $k_i$ a corresponding adequate integer to $A_i$. Then, applying Theorem 1.3 finishes the proof. \(\square\)
4. Constructing dependent sequences

In this section we will demonstrate two methods for constructing dependent sequences from existing regular sequences.

4.1. Deletions

For some dependent Stanley sequences, it is possible to remove one or more elements while preserving the Stanley sequence property. This claim is made clearer by the next example.

Example 4.1. We have already noted that the sequence

\[ S(0, 1, 4) = 0, 1, 4, 5, 11, 12, 14, 15, 31, 32, 34, 35, 40, 41, 43, 44, 89, \ldots \]

is dependent, with core \( S(0) = 0, 1, 3, 4, \ldots \). Removing \( 11 \) from \( S(0, 1, 4) \) yields the sequence

\[ 0, 1, 4, 5, 12, 14, 15, 31, 32, 34, 35, 40, 41, 43, 44, 89, \ldots, \]

which may be expressed as \( S(0, 1, 4, 5, 12, 14, 15, 31) \). This Stanley sequence is dependent, with core \( S(0) \) and shift index \( \sigma = 1 \), since one element was removed.

Likewise, removing both \( 31 \) and \( 32 \) from \( S(0, 1, 4) \) yields the dependent Stanley sequence

\[ S(0, 1, 4, 5, 12, 14, 15, 31, 34, 35, 40, 41, 43, 44, 89), \]

which has core \( S(0) \) and shift index \( \sigma = 2 \), since two elements were removed.

Corollary 4.2. There exist (regular) Stanley sequences that are not maximal 3-free sets.

Remark. It follows from Theorem 1.3 that the shift index \( \sigma \) can be arbitrarily large. If \( A = \{0\} \), \( B = \{0, 1, 4, 5, 12, 14, 15, 31\} \), and \( k \) is sufficiently large, the sequence \( S(A \otimes_k B) \) is a dependent sequence satisfying

\[ \sigma(A \otimes_k B) = 2^k \cdot \sigma(B) = 2^k. \]

Given a dependent Stanley sequence \( S(A) \), we say that an element of \( S(A) \) is deletable if deleting it yields another (dependent) Stanley sequence. We have as yet been unable to derive a general formulation for which elements of a Stanley sequence are deletable.

Conjecture 4.3. Every dependent Stanley sequence contains infinitely many deletable elements.

4.2. Translations

We conjecture the following stronger statement of Theorem 1.5.

Conjecture 4.4. Let \( S(A) = \{a_n\} \) be a regular sequence with core \( \{a'_n\} \), shift index \( \sigma \), and character \( \lambda \). Let \( k \) be an adequate integer such that for all \( 0 \leq i < 2^{k-1} \),

\[ a_{2^{k-1} - \sigma + i} = a_{2^{k-1} - \sigma} + a'_i. \]

Let \( \ell \) be the minimum adequate integer for \( \{a'_n\} \). Let \( c \) be such that

\[ \lambda \leq c \leq a_{2^{k-2} - \ell - \sigma} + a'_\ell - a_{2^{k-2} - \sigma} - \lambda. \quad (6) \]

Then, \( A^c_k \) is 3-free and \( S(A^c_k) \) is a regular Stanley sequence with core \( S(A) \).
Example 4.5. Let \( \{a_n\} = S(0) \) and \( k = 2 \). Then \( \lambda = 0 \), and the minimal adequate integer is \( \ell = 0 \). Theorem 1.5 implies that \( S(\{0\}_2) \) is regular for all \( c \) such that
\[
0 \leq c \leq a_{2^{2-2^0}} - 0 = 4.
\]
Picking \( c = 3 \), we compute \( S(\{0\}_2^3) \) by taking the highlighted block of \( S(0) \), adding 3 to the block, then recomputing the subsequent terms of the sequence.

\[
S(0) = 0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, 36, 37, 39, 40, 81, 82, 84, 85, 90, 91, 93, 94, 108, 109, 110, 112, 117, 118, 120, 121, \ldots
\]

This results in the sequence

\[
S(0, 1, 3, 4, 12, 13, 15, 16) = 0, 1, 3, 4, 12, 13, 15, 16, 33, 34, 36, 37, 42, 43, 45, 46, 93, 94, 96, 97, 102, 103, 105, 106, 120, 121, 123, 124, 129, 130, 132, 133, \ldots
\]

As predicted by Theorem 1.5, this sequence is dependent with core \( S(0) \). In fact, it is possible to construct \( S(\{0\}_2^3) \) from \( S(0) \) as follows. Add 3 to the block \( \{9, 10, 12, 13\} \), add \( 2 \cdot 3 \) to the next block, add \( 2^2 \cdot 3 \) to the next block, etc.

We may now translate the block \( \{33, 34, 36, 37, 42, 43, 45, 46\} \) within the sequence \( S(\{0\}_2^3) = \{b_n\} \), corresponding to \( k = 3 \). Note that for this sequence, \( \lambda = 0 \), \( \sigma = 0 \), and the core sequence \( \{b'_n\} \) is \( S(0) \), for which the minimal adequate integer is \( \ell = 0 \). According to Conjecture 4.4, the sequence \( S(\{0\}_2^3) \) is regular for all \( c \) such that
\[
0 \leq c \leq a_{2^{3-2^0}} - a_{2^3 - 0} - 0 = 16 + 27 - 33 = 10.
\]
Picking \( c = 10 \), we have

\[
S(0, 1, 3, 4, 12, 13, 15, 16, 43, 44, 46, 47, 52, 53, 55, 56) = 0, 1, 3, 4, 12, 13, 15, 16, 43, 44, 46, 47, 52, 53, 55, 56, 113, 114, 116, 117, 122, 123, 125, 126, 140, 141, 143, 144, 149, 150, 152, 153, \ldots
\]

This sequence is dependent with core \( S(0) \) and may be constructed from \( S(\{0\}_2^3) \) as follows. Add 10 to the block \( \{33, 34, 36, 37, 42, 43, 45, 46\} \), add \( 2 \cdot 10 \) to the next block, add \( 2^2 \cdot 10 \) to the next, etc.

To prove Theorem 1.5, we begin with the following lemma.

**Lemma 4.6.** Let \( S(A) = \{a_n\} \) be independent. Pick \( m \) adequate, and set \( \Lambda_m = \{a_i \mid 0 \leq i < 2^m\} \).
Let nonnegative integers $d, e$ be such that $a_{2^{m-1}} + d < e$ (so that $\Lambda_m + d$ and $\Lambda_m + e$ occupy disjoint intervals). Then, $\Lambda_m + d$ and $\Lambda_m + e$ jointly cover

$$([2e - d, 2e - d + a_{2^m}) \setminus (O(A) + 2e - d)) \cup (O(A) + 2e - d + a_{2^m}).$$

**Proof.** It is evident that $\Lambda_m + d$ and $\Lambda_m + e$ jointly cover the set

$$\{2y - x \mid x \in \Lambda_m + d, y \in \Lambda_m + e, y = x + (e - d)\} = \Lambda_m + 2e - d. \tag{7}$$

Furthermore, since $\Lambda_m$ covers $[0, a_{2^m}) \setminus (O(A) \cup \Lambda_m)$, the Cover-shift Lemma implies that $\Lambda_m + d$ and $\Lambda_m + e$ jointly cover

$$[2e - d, 2e - d + a_{2^m}) \setminus ((O(A) + 2e - d) \cup (\Lambda_m + 2e - d)) \tag{8}$$

Now, consider some large $n$, and pick $s \in O(A) + a_{2^n}$. Then $s \not\in S(A)$, so there exist $x, y \in S(A)$ with $y > x$ and $2y - x = s$. We cannot have $x, y \geq a_{2^n}$, because otherwise the elements $x - a_{2^n}$ and $y - a_{2^n}$ of $S(A)$ would satisfy $2(y - a_{2^n}) - (x - a_{2^n}) = s - a_{2^n} \in O(A)$, a contradiction. Hence $x < a_{2^n}$. Then, since $n$ is large and $2y - x = s$, we conclude that $y < a_{2^n}$.

Because $m$ is adequate, $2a_{2^n-1} - \lambda + 1 = a_{2^n} > a_{2^n-1}$, so $a_{2^n-1} \geq \lambda$. Therefore,

$$s \geq 2a_{2^n-1} - \lambda + 1 > 2a_{2^n-1} - a_{2^n-1}. \tag{9}$$

Since $x, y \leq a_{2^n-1}$ and $2y - x = s$, we may conclude from (9) that $x \leq a_{2^n-1}$ and $y \geq a_{2^n-1} - a_{2^n-1} = a_{2^n-2^n}$, implies that $x \in \Lambda_m$ and $y \in \Lambda_m + a_{2^n-2^n}$. This implies that $\Lambda_m$ and $\Lambda_m + a_{2^n-2^n}$ jointly cover $O(A) + a_{2^n}$. Thus, by the Cover-shift Lemma, $\Lambda_m + d$ and $\Lambda_m + e$ jointly cover

$$O(A) + a_{2^n} + 2(e - a_{2^n-2^n}) - d = O(A) + 2e - d + a_{2^n} - 2(a_{2^n-1} - a_{2^n-1}) = O(A) + 2e - d + a_{2^n} - (a_{2^n} + \lambda - 1) + (a_{2^n} + \lambda - 1) = O(A) + 2e - d + a_{2^n}.$$\

Combining this result with (7) and (8), we see that $\Lambda_m + d$ and $\Lambda_m + e$ jointly cover

$$([2e - d, 2e - d + a_{2^m}) \setminus (O(A) + 2e - d)) \cup (O(A) + 2e - d + a_{2^m}),$$

as desired. \hfill \Box

We now use Lemma 4.6 to prove a stronger version of itself.

**Lemma 4.7.** Let $S(A) = \{a_n\}$ be independent with character $\lambda$, and let $\ell$ be the minimum adequate integer for $S(A)$. Pick $k \geq \ell$ and set $\Lambda_k = \{a_i \mid 0 \leq i < 2^k\}$. Let nonnegative integers $d, e$ be such that $a_{2^{k-1}} + d < e$ (so that $\Lambda_k + d$ and $\Lambda_k + e$ occupy disjoint intervals). Then, $\Lambda_k + d$ and $\Lambda_k + e$ jointly cover

$$[2e - d - a_{2^{k-2\ell}} + \lambda, 2e - d + a_{2^k}) \cup (O(A) + 2e - d + a_{2^k}).$$
Proof. Applying Lemma 4.6 we see that $\Lambda_k + d$ and $\Lambda_k + e$ must jointly cover

$$\left(\left\{2e - d, 2e - d + a_{2k}\right\} \cup (O(A) + 2e - d)\right) \cup (O(A) + 2e - d + a_{2k}).$$

(10)

Note that by Lemma 2.12 the maximum element of $O(A) + 2e - d$ is at most $2e - d + \lambda$, which proves the lemma in the case $k = \ell$.

Assume now that $k > \ell$. Pick $m$ with $\ell \leq m < k$ and let $\Lambda_m = \{a_i \mid 0 \leq i < 2^m\}$. Again by Lemma 4.6 the sets $\Lambda_k + d$ and $\Lambda_k + e$ jointly cover

$$\left\{2(a_j + e) - (a_i + d) \mid 2^k - 2^m \leq i \leq 2^k - 1, 0 \leq j \leq 2^m - 1\right\}$$

$$= \left\{2y - x \mid x \in \Lambda_m + a_{2k-2m} + d, y \in \Lambda_m + e\right\}$$

$$\supseteq \left(2e - d - a_{2k-2m}, 2e - d + a_{2m} - a_{2k-2m}\right) \setminus \left(O(A) + 2e - d - a_{2k-2m}\right)$$

$$\cup \left(O(A) + 2e - d + a_{2m} - a_{2k-2m}\right)$$

$$= \left(2e - d - a_{2k-2m}, 2e - d - a_{2k-2m+1}\right) \setminus \left(O(A) + 2e - d - a_{2k-2m}\right)$$

$$\cup \left(O(A) + 2e - d - a_{2k-2m+1}\right),$$

(11)

where the last step follows from

$$a_{2k-2m} - a_{2m} = a_{2k-1} - a_{2m-1} - a_{2m} = a_{2k-1} - a_{2m+1} = a_{2k-2m+1}.$$  

(12)

We take the union of the expression (11) over all possible $m$ ($\ell \leq m < k$) and observe that it “telescopes,” becoming the expression

$$\left(2e - d - a_{2k-2\ell}, 2e - d\right) \setminus \left(O(A) + 2e - d - a_{2k-2\ell}\right) \cup (O(A) + 2e - d),$$

which must in turn be jointly covered by $\Lambda_k + d$ and $\Lambda_k + e$.

Combining (10) and (13), we see that $\Lambda_k + d$ and $\Lambda_k + e$ must jointly cover

$$\left(2e - d - a_{2k-2\ell}, 2e - d + a_{2k}\right) \setminus \left(O(A) + 2e - d - a_{2k-2\ell}\right) \cup (O(A) + 2e - d + a_{2k})$$

$$\subseteq \left[2e - d - a_{2k-2\ell} + \lambda, 2e - d + a_{2k}\right) \cup \left(O(A) + 2e - d + a_{2k}\right).$$

(14)

where the last step follows since the maximum element of $O(A)$ is at most $\lambda$.  

Proof of Theorem 1.5. Let $c$ be such that (5) is satisfied. We must first prove that $A_k^c := \{a_i \mid 0 \leq i < 2^k\} \cup \{c + a_i \mid 2^k \leq i < 2^{k+1}\}$ is 3-free. Note that all elements of $\{c + a_i \mid 2^k \leq i < 2^{k+1}\}$ are at least

$$a_{2^k} + c \geq a_{2^k} + \lambda \geq 2a_{2^k-1} + 1,$$

so no 3-term AP can exist with its elements split between $\{a_i \mid 0 \leq i < 2^k\}$ and $\{c + a_i \mid 2^k \leq i < 2^{k+1}\}$. These sets are individually 3-free; therefore, their union $A_k^c$ is also 3-free.

Let $S(A_k^c) = \{a_i^*\}$. For each $j$, let

$$\Lambda_j = \{a_i \mid 0 \leq i < 2^j\}$$

$$\Lambda_j^* = \{a_i^* \mid 0 \leq i < 2^j\}$$

$$\Gamma_j = \{a_i \mid 2^j \leq i < 2^{j+1}\}$$

$$\Gamma_j^* = \{a_i^* \mid 2^j \leq i < 2^{j+1}\}.$$
Thus, \( A_k^* = \Lambda_k \cup (\Gamma_k + c) = \Lambda_k^* \cup \Gamma_k^* \). We claim that \( \Gamma_k^* = \Gamma_j + 2^{j-k}c \) for each \( j \geq k \). By construction, we know already that \( \Gamma_k^* \) is of this form. In our proof, we consider the block \( \Gamma_{k+1}^* \) and then we use induction to prove the result for all \( j \geq k + 1 \).

We first consider what is covered by \( \Gamma_k + c \). Since \( \Lambda_k \) covers the set \([a_{2k-1} + 1, a_{2k}) \cup (O(A) + a_{2k})\), we conclude that \( \Gamma_k + c = \Lambda_k + a_{2k} + c \) must cover the set
\[
[a_{2k-1} + a_{2k} + c + 1, 2a_{2k} + c) \cup (O(A) + 2a_{2k} + c) = [a_{2k+1} + c + 1, 2a_{2k} + c) \cup (O(A) + 2a_{2k} + c). \tag{15}
\]

We now apply Lemma 4.7 to \( \Lambda_k \) and \( \Gamma_k + c = \Lambda_k + a_{2k} + c \). This implies that \( \Lambda_k \) and \( \Gamma_k + c \) must jointly cover
\[
[2a_{2k} + 2c - a_{2k-2}\ell + \lambda, 3a_{2k} + 2c) \cup (O(A) + 3a_{2k} + 2c). \tag{16}
\]
By our choice of \( c \),
\[
2a_{2k} + 2c - a_{2k-2}\ell + \lambda \leq 2a_{2k} + c + (a_{2k-2}\ell - \lambda) - a_{2k-2}\ell + \lambda = 2a_{2k} + c.
\]

Now, we can combine (15) and (16) to conclude that \( A_k^* \) covers
\[
[a_{2k+1} + c + 1, 3a_{2k} + 2c) \cup (O(A) + 3a_{2k} + 2c) = [a_{2k+1} + c + 1, a_{2k+2} + 2c) \cup (O(A) + a_{2k+1} + 2c). \tag{17}
\]
where we used the fact that \( 3a_{2k} = a_{2k+1} \) (see Proposition 2.4).

Let \( Q \) be the set of integers covered by \( A_k^* \). We claim that
\[
Q \cap (\Gamma_{k+1} + 2c) = \emptyset. \tag{18}
\]

Pick some \( q \in Q \). Suppose for the sake of contradiction that \( q \) is covered by \( \Gamma_k + c \). Since the largest element of \( \Gamma_k + c \) is \( a_{2k+1} + c \) and the smallest is \( a_{2k} + c \), we know that
\[
2a_{2k+1} - a_{2k} + c \geq q \geq a_{2k+1} + 2c = 2a_{2k+1} - \lambda + 1 + 2c
\]
and therefore that \( c \leq \lambda - 1 - a_{2k} \), an impossibility since \( a_{2k} \geq \lambda \).

We conclude that \( q \) is not covered by \( \Gamma_k + c \) and therefore must be jointly covered by \( \Lambda_k \) and \( \Gamma_k + c \). From the original Stanley sequence \( S(A) \), we know that no integers jointly covered by \( \Lambda_k \) and \( \Gamma_k \) are in the set \( \Gamma_{k+1} \), from which (18) follows by the Cover-shift Lemma. Recall that we wished to prove \( \Gamma_{k+1}^* = \Gamma_{k+1} + 2c \). This now follows immediately from (17) and (18).

Consider some \( j \geq k + 1 \), and assume towards induction that we have \( \Gamma_j^* = \Gamma_j + 2^{j-k}c \). Assume further that \( \Lambda_j^* \) and \( \Gamma_j^* \) jointly cover the set
\[
[2a_{2j-1} - a_{2j-2}\ell + \lambda, a_{2j}^*) \cup (O(A) + a_{2j}^*). \tag{19}
\]
The base case of \( j = k + 1 \) follows from (16).
Now, let
\[
\Lambda_j^1 = \{ a_i^* \mid 0 \leq i < 2^j - 1 \} \\
\Lambda_j^2 = \{ a_i^* \mid 2^{j-1} \leq i < 2^j \} \\
\Gamma_j^1 = \{ a_i^* \mid 2^j \leq i < 2^j + 2^{j-1} \} \\
\Gamma_j^2 = \{ a_i^* \mid 2^j + 2^{j-1} \leq i < 2^{j+1} \},
\]
so that \( \Lambda_j^* = \Lambda_j^1 \cup \Lambda_j^2 \) and \( \Gamma_j^* = \Gamma_j^1 \cup \Gamma_j^2 \).

Note that \( \Lambda_j^2 = \Lambda_{j-1} + a_{2^j-1}^* \) and \( \Gamma_j^1 = \Lambda_{j-1} + a_{2^j}^* \). Then, applying Lemma 4.7, we conclude that \( \Lambda_j^* \) and \( \Gamma_j^* \) jointly cover
\[
[2a_{2^j}^* - a_{2^j-1}^* - a_{2^j-1-2}^* + \lambda, 2a_{2^j}^* - a_{2^j-1}^* + a_{2^j-1}) \\
\cup (O(A) + 2a_{2^j}^* - a_{2^j-1}^* + a_{2^j-1})
\supseteq [2a_{2^j}^* - a_{2^j-1}^* - a_{2^j-1-2}^* + \lambda, 2a_{2^j}^* - a_{2^j-1}^* + a_{2^j-1}).
\]

Similarly, \( \Lambda_j^2 \) and \( \Gamma_j^2 = \Gamma_j^1 + a_{2^j-1} \) jointly cover
\[
[2a_{2^j}^* - a_{2^j-1}^* + 2a_{2^j-1} - a_{2^j-1-2}^* + \lambda, 2a_{2^j}^* - a_{2^j-1}^* + 3a_{2^j-1}).
\]

By our inductive hypothesis, we know that \( \Lambda_{j-1}^* \) and \( \Gamma_{j-1}^* \) jointly cover (19). Since \( \Lambda_j^1 = \Lambda_{j-1}^* \) and \( \Gamma_j^1 = \Gamma_{j-1}^* + a_{2^j-1} - a_{2^j-1}^* \), the Cover-shift Lemma implies that \( \Lambda_j^1 \) and \( \Gamma_j^1 \) jointly cover
\[
[2a_{2^j-1}^* - a_{2^j-1-2}^* + \lambda + 2(a_{2^j}^* - a_{2^j-1}^*), a_{2^j}^* + 2(a_{2^j}^* - a_{2^j-1}^*)) \\
\cup (O(A) + a_{2^j}^* + 2(a_{2^j}^* - a_{2^j-1}^*)) \\
\supseteq [2a_{2^j}^* - a_{2^j-1-2}^* + \lambda, 3a_{2^j}^* - 2a_{2^j-1}^*).
\]

Similarly, \( \Lambda_j^1 = \Lambda_{j-1}^* \) and \( \Gamma_j^2 = \Gamma_{j-1}^* + a_{2^j-1} \) must jointly cover
\[
[2a_{2^j}^* + 2a_{2^j-1} - a_{2^j-1-2}^* + \lambda, 3a_{2^j}^* - 2a_{2^j-1}^* + 2a_{2^j-1}) \cup (O(A) + 3a_{2^j}^* - 2a_{2^j-1}^* + 2a_{2^j-1}).
\]

Combining (21), (23), (22), and (24), we conclude that \( \Lambda_j^* \) and \( \Gamma_j^* \) must jointly cover
\[
[2a_{2^j}^* - a_{2^j-1}^* - a_{2^j-1-2}^* + \lambda, 2a_{2^j}^* - a_{2^j-1}^* + a_{2^j-1}) \\
\cup [2a_{2^j}^* - a_{2^j-1-2}^* + \lambda, 3a_{2^j}^* - 2a_{2^j-1}^*) \\
\cup [2a_{2^j}^* + 2a_{2^j-1} - a_{2^j-1-2}^* + \lambda, 2a_{2^j}^* - a_{2^j-1}^* + 3a_{2^j-1}) \\
\cup [2a_{2^j}^* + 2a_{2^j-1} - a_{2^j-1-2}^* + \lambda, 3a_{2^j}^* - 2a_{2^j-1}^* + 2a_{2^j-1}) \\
\cup (O(A) + 3a_{2^j}^* - 2a_{2^j-1}^* + 2a_{2^j-1}).
\]

We note that
\[
a_{2^j-1-2}^* - \lambda \geq 2^{j-1-k}(a_{2^j-2}^* - \lambda) \geq 2^{j-1-k} \epsilon = a_{2^j-1}^* - a_{2^j-1}.
\]

Therefore,
\[
2a_{2^j}^* - a_{2^j-1}^* + a_{2^j-1} \geq 2a_{2^j}^* - a_{2^j-1-2}^* + \lambda
\]

20
and

\[ 2a_{2j}^* - a_{2j-1}^* + 3a_{2j-1} \geq 2a_{2j} + 2a_{2j-1} - a_{2j-1-2\ell} + \lambda. \]

These two inequalities allow us to simplify \((25)\) to

\[
[2a_{2j}^* - a_{2j-1}^* - a_{2j-1-2\ell} + \lambda, 3a_{2j}^* - 2a_{2j-1}^*] \\
\cup [2a_{2j}^* - a_{2j-1}^* + 2a_{2j-1} - a_{2j-1-2\ell} + \lambda, 3a_{2j}^* - 2a_{2j-1}^* + 2a_{2j-1}] \\
\cup (O(A) + 3a_{2j}^* + 2a_{2j-1} - 2a_{2j-1}^*). \tag{26}
\]

We observe that

\[
a_{2j}^* - a_{2j-1}^* = \left(3a_{2j-1} + 2j^{-k}c\right) - \left(a_{2j-1} + 2j^{-1-k}c\right) \\
\geq 2a_{2j-1} \\
\geq 2a_{2j-1} - a_{2j-1-2\ell} + \lambda.
\]

Therefore,

\[ 3a_{2j}^* - 2a_{2j-1}^* \geq 2a_{2j}^* - a_{2j-1} + 2a_{2j-1} - a_{2j-1-2\ell} + \lambda. \]

Then, we can simplify \((26)\) to conclude that \(\Lambda_j^*\) and \(\Gamma_j^*\) jointly cover

\[
[2a_{2j}^* - a_{2j-1}^* - a_{2j-1-2\ell} + \lambda, 3a_{2j}^* - 2a_{2j-1}^* + 2a_{2j-1}] \cup (O(A) + 3a_{2j}^* - 2a_{2j-1}^* + 2a_{2j-1}).
\]

We see that

\[
3a_{2j}^* - 2a_{2j-1}^* + 2a_{2j-1} = 3\left(a_{2j} + 2j^{-k}c\right) - 2j^{-k}c = a_{2j+1} - 2j^{1-k}c,
\]

so \(\Lambda_j^*\) and \(\Gamma_j^*\) jointly cover

\[
[2a_{2j}^* - a_{2j-1}^* - a_{2j-1-2\ell} + \lambda, a_{2j+1} - 2j^{1-k}c] \cup (O(A) + a_{2j+1} - 2j^{1-k}c). \tag{27}
\]

Finally, we observe that \(\Gamma_j^* = \Lambda_j + a_{2j}^*\). Since \(\Lambda_j\) covers \([a_{2j-1} + 1, a_{2j}]\), we conclude that \(\Gamma_j^*\) must cover

\[
[a_{2j}^* + a_{2j-1} + 1, a_{2j}^* + a_{2j}] \tag{28}
\]

Note that

\[
a_{2j-1}^* + a_{2j-1-2\ell} - \lambda \geq 2j^{-k}(a_{2k-2\ell} - \lambda) \geq 2j^{-k}c = a_{2j}^* - a_{2j}.
\]

This implies that

\[
a_{2j}^* + a_{2j} \geq 2a_{2j}^* - a_{2j-1} - a_{2j-1-2\ell} + \lambda,
\]

so we can combine \((28)\) with \((27)\) to conclude that \(\Lambda_j^* \cup \Gamma_j^*\) must cover

\[
[a_{2j}^* + a_{2j-1} + 1, a_{2j+1} - 2j^{1-k}c] \cup (O(A) + a_{2j+1} - 2j^{1-k}c).
\]

Hence, \(\Gamma_{j+1}^* = \Gamma_{j+1} + 2j^{1-k}c\), which, together with \((27)\), completes the induction.

Thus, \(\Gamma_j^* = \Gamma_j + 2j^{-k}c\) for each \(j \geq k\). This shows that \(S_k(c, A)\) is a regular Stanley sequence with core \(S(A)\).
**Corollary 4.8.** For each nonnegative integer $\lambda$, there are either no regular Stanley sequences with character $\lambda$, or else infinitely many.

**Proof.** If any regular sequence has character $\lambda$, then its core must be an independent sequence $\{a_n\}$ with character $\lambda$. Theorem 1.5 shows that it is possible to construct infinitely many dependent Stanley sequences with $\{a_n\}$ as their core. \qed

5. Concluding remarks

In this paper, we have rigorously identified and explored the notion of regularity in Stanley sequences and have constructed many new classes of regular sequences. Our research suggests several avenues for further exploration. Most significant, perhaps, among these is the problem of whether all irregular sequences satisfy Type 2 growth. An improved upper bound on the asymptotic density of Stanley sequences would also be welcome. Roth’s theorem \cite{roth} implies that $a_n$ cannot grow linearly with $n$. A result by Sanders \cite{sanders} strengthens this bound slightly to $\Omega(n \log^{\frac{1}{1-o(1)}} n)$. However, no explicit bound of the form $\Omega(n^{1+\epsilon})$ has been found (see Problem 2 of Erdős et al. \cite{erdos}).

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Appendix: Sequences with small character

We have found independent Stanley sequences $S(A)$ for each possible character $\lambda$ such that $0 \leq \lambda \leq 76$, with the exception of the values 1, 3, 5, 9, 11, 15. The following table gives examples.

| $\lambda$ | $A$ | $\lambda$ | $A$ | $\lambda$ | $A$ |
|-----------|-----|-----------|-----|-----------|-----|
| 0         | {0} | 26        | {0, 5, 9, 12} | 52        | {0, 1, 10, 13, 14, 23} |
| 1         | None | 27        | {0, 10, 11, 17} | 53        | {0, 23, 24, 30} |
| 2         | {0, 2} | 28        | {0, 3, 4, 7, 22, 25} | 54        | {0, 4, 16, 21, 25} |
| 3         | None | 29        | {0, 3, 5, 8, 21, 24, 26} | 55        | {0, 3, 28} |
| 4         | {0, 2, 5} | 30        | {0, 6, 15} | 56        | {0, 5, 9, 17, 24} |
| 5         | None found | 31        | {0, 5, 11, 13, 16} | 57        | {0, 3, 19, 22, 29} |
| 6         | {0, 3} | 32        | {0, 6, 8, 15} | 58        | {0, 1, 3, 4, 29} |
| 7         | {0, 1, 7} | 33        | {0, 3, 7, 10, 21, 24, 28, 30} | 59        | {0, 3, 21, 30} |
| 8         | {0, 3, 5} | 34        | {0, 8, 17} | 60        | {0, 7, 19, 27} |
| 9         | None found | 35        | {0, 9, 10, 13, 19, 22} | 61        | {0, 5, 13, 18, 24, 28} |
| 10        | {0, 1, 4, 6, 10} | 36        | {0, 18} | 62        | {0, 8, 12, 20, 27} |
| 11        | None found | 37        | {0, 1, 19} | 63        | {0, 4, 9, 13, 30, 33} |
| 12        | {0, 6} | 38        | {0, 3, 11, 18} | 64        | {0, 3, 9, 12, 31, 34} |
| 13        | {0, 2, 7, 9, 13} | 39        | {0, 11, 15, 16, 20, 26, 28} | 65        | {0, 5, 17, 22, 28, 30, 33} |
| 14        | {0, 3, 8} | 40        | {0, 2, 7, 15, 16, 20} | 66        | {0, 10, 22, 27, 30} |
| 15        | None found | 41        | {0, 3, 11, 14, 21, 24, 30} | 67        | {0, 11, 23, 24, 28, 34} |
| 16        | {0, 4, 7} | 42        | {0, 9, 12, 13, 21} | 68        | {0, 3, 11, 12, 23, 27, 30} |
| 17        | {0, 4, 5, 9, 15, 17} | 43        | {0, 1, 9, 10, 25} | 69        | {0, 3, 4, 19, 22, 23, 28} |
| 18        | {0, 9} | 44        | {0, 14, 18, 21} | 70        | {0, 7, 9, 19, 27, 34} |
| 19        | {0, 3, 10} | 45        | {0, 1, 16, 17, 19, 20, 29} | 71        | {0, 8, 9, 17, 30, 33, 38} |
| 20        | {0, 1, 10} | 46        | {0, 1, 4, 12, 19} | 72        | {0, 13, 25, 27, 33} |
| 21        | {0, 1, 3, 4, 21} | 47        | {0, 20, 21, 27} | 73        | {0, 4, 5, 9, 15, 17, 20, 28} |
| 22        | {0, 8, 9, 14} | 48        | {0, 1, 12, 13, 21} | 74        | {0, 14, 17, 26, 27, 33} |
| 23        | {0, 7, 9, 10, 16} | 49        | {0, 9, 25} | 75        | {0, 4, 13, 17, 25, 29, 38} |
| 24        | {0, 9, 12} | 50        | {0, 2, 12, 14, 21} | 76        | {0, 1, 7, 8, 21, 28} |
| 25        | {0, 2, 3, 5, 23, 25} | 51        | {0, 5, 13, 16, 18, 24, 28} |