DYNAMICAL MODEL OF AN EXPANDING SHELL

ASAF PE’ER

Harvard-Smithsonian Center for Astrophysics, MS-51, 60 Garden Street, Cambridge, MA 02138, USA

Received 2012 March 23; accepted 2012 April 20; published 2012 May 22

ABSTRACT

Expanding blast waves are ubiquitous in many astronomical sources, such as supernova remnants, X-ray emitting binaries, and gamma-ray bursts. I consider here the dynamics of such an expanding blast wave, both in the adiabatic and the radiative regimes. As the blast wave collects material from its surroundings, it decelerates. A full description of the temporal evolution of the blast wave requires consideration of both the energy density and the pressure of the shocked material. The obtained equation is different from earlier works in which only the energy was considered. The solution converges to the familiar results in both the ultrarelativistic and the sub-relativistic (Newtonian) regimes.

Key words: hydrodynamics – ISM: jets and outflows – plasmas – shock waves

Online-only material: color figures

1. INTRODUCTION

Expanding blast waves are one of the most common phenomena in many astronomical transients. Interaction of the expanding shells of supernova remnants (SNRs) with their environment has been studied since more than three decades ago (Chevalier 1976, 1982). Similarly, in X-ray emitting binaries (XRBs), mildly relativistic (Lorentz factor \( \Gamma \sim \text{few} \)) expanding radio blobs have been observed for nearly two decades now (Mirabel & Rodríguez 1994; Hjellming & Rupen 1995; Fender et al. 1999). In gamma-ray bursts (GRBs) relativistic blast waves are an inherent part of the GRB “fireball” model (Goodman 1986; Paczynski 1986; Shemi & Piran 1990; Rees & Meszaros 1992; Narayan et al. 1992) and are the source of the frequently seen afterglow emission.

These expanding blast waves originate from a stellar explosion (such as in GRBs or supernovae), or a rapid ejection of material (as in XRBs). As they propagate through the ambient medium, they collect material and decelerate. The expansion may be adiabatic, or highly radiative. The dynamics of the blast wave expansion in both these scenarios were extensively studied in the past (Blandford & McKee 1976; Katz & Piran 1997; Chiang & Dermer 1999; Piran 1999; Piran 1999; Huang et al. 1999; van Paradijs et al. 2000). These formulations are still used until now as a basis for calculating the expected radiation during the deceleration phase of the blast wave evolution (e.g., Narayan et al. 2011; Granot & Piran 2012; van Eerten & MacFadyen 2012; Shen & Matzner 2012).

The dynamical calculations in the above-mentioned works are based on a “toy model,” in which a basic assumption is that the interaction between the blast wave and the interstellar material (ISM) can be described by a series of inelastic collisions (Katz & Piran 1997; Huang et al. 1999). While this prescription asymptotes to the known analytical solutions in the ultrarelativistic limit \( \Gamma \gg 1 \) (Meszaros & Rees 1997) and in the non-relativistic limit (Sedov 1959), it neglects the contribution of the swept-up material to the internal pressure. This, in turn, affects the evolution of the blast wave dynamics. In this Letter, I revise the basic assumptions of the blast-wave–ISM interaction scenario, and re-derive the equations that govern the blast wave evolution. As will be shown below, the obtained equations are different from the ones previously used. Nonetheless, they asymptote to the known results at the ultrarelativistic as well as the sub-relativistic limits.

2. DYNAMICAL MODEL

Consider an explosion that ejects mass \( M \) and creates a shell that propagates into the cold ISM. In front of the expanding shell is a blast wave. The system is thus composed of three regions: (1) the unshocked ISM, (2) the shocked ISM, and (3) the ejected shell material. For simplicity, I assume that a reverse shock wave (if created) had long passed, hence the ejected shell material is cold. Furthermore, I assume that the thermodynamical quantities of the gas: \( n_i, p_i, \) and \( e_i \) (particle number density, pressure, and internal energy density) are steady in each region. The pressure in each region is given by \( p = (\gamma - 1)(e - \rho) \), where \( \rho = n m_p e^\gamma \) is the rest mass density and \( \gamma \) is the adiabatic index. Here and below, primed quantities are in the comoving frame, and unprimed quantities are in the observer’s frame in which the unshocked ISM is at rest.

The evolution of the blast wave as it propagates through the ISM is calculated by considering energy conservation in the observer’s frame. Let us assume that at time \( t \) the plasma propagates with Lorentz factor \( \Gamma \).\(^2\) Neglecting radiative losses
and assuming that the gas can be described as a prefect fluid, the
energy density in region (2) as is measured by a distant observer is
given by
\[
e_{2} = T^{00} = (\gamma + 1)\epsilon_{2} + p_{2} + \rho_{2} v_{2}^{2} = [\gamma \Gamma^{2} - (\gamma - 1)]\epsilon_{2} + (\gamma - 1)(1 - \Gamma^{2})n_{2}m_{p}c^{2}.
\]  
(1)

Here, \(T^{00}\) is the 00 component of the stress-energy tensor. Due to
Lorentz contraction, the comoving volume of region (2) is \(V' = \Gamma V\). The total
energy of the gas contained in this region (in the adiabatic case), as viewed by a distant observer, is thus
\[
E_{2}(\text{ad.}) = [\gamma \Gamma^{2} - (\gamma - 1)]\epsilon_{2}V' + (\gamma - 1)(1 - \Gamma^{2})n_{2}^{2}m_{p}c^{2}V'
= [\gamma \Gamma^{2} - (\gamma - 1)(1 + \Gamma^{2})]N_{2}m_{p}c^{2},
\]  
(2)

where \(N_{2} = n_{2}V\) is the number of particles in region (2), \(\beta \equiv (1 - \Gamma^{-2})^{1/2}\) is the normalized bulk velocity of the plasma
in this region. In going to the second line, use was made in the relation \(\epsilon_{2}'/n_{2}' = \Gamma_{mp}c^{2}\), which is exact for any value of \(\Gamma\) as
long as the unshocked ISM in region (1) is cold (Blandford & McKee 1976).

The calculation in Equation (2) assumes no radiative losses.
In order to allow the possibility that part of the thermal energy
 gained by the ISM as it crosses the shock wave is radiated, Equation (2) is modified as follows. The energy calculated in
Equation (2) is the sum of three separate components: (1) the
rest mass energy of the shocked ISM, (2) its kinetic energy, and (3) its thermal energy. The first two components sum up
as \(\Gamma N_{2}m_{p}c^{2}\). Only the energy in the third component can in principle be radiated. If a fraction \(\epsilon\) of the thermal energy is
radiated, then the energy of the gas in region (2) is given by
\[
E_{2} = (\Gamma + (1 - \epsilon)[\gamma \Gamma^{2} - (\gamma - 1)(1 + \Gamma^{2})])mc^{2},
\]  
(3)

where \(m \equiv N_{2}m_{p}\) is the mass of the shocked ISM. The material
in region (3) is assumed to be cold, and so its (kinetic + rest
mass) energy is \(E_{3} = \Gamma M c^{2}\).

A differential equation for the evolution of the plasma velocity is derived in the following way. Between times \(t\) and \(t + \delta t\), the plasma propagates a distance \(\beta c \delta t\), and an ISM
of mass \(dm\) crosses the forward shock and gains kinetic and
thermal energy. A fraction \(\epsilon\) of the gained thermal energy is
assumed to be radiated, hence the radiated energy is \(\delta E_{\text{rad}} = \epsilon (\gamma \Gamma^{2} - \Gamma - (\gamma - 1)(1 + \Gamma^{2}))dm c^{2}\). As it collects material, the plasma decelerates; at time \(t + \delta t\) its Lorentz factor is \(\Gamma = - d\Gamma / d\tau\) (corresponding velocity \(\rho - d\beta\)). Conservation of energy at times \(t\) and \(t + \delta t\) is written as
\[
\{\Gamma + (1 - \epsilon)[\gamma \Gamma^{2} - (\gamma - 1)(1 + \Gamma^{2})]\}m + \Gamma M
= -dm + \{[\Gamma - (\gamma - 1)(1 + \Gamma^{2})\}m + \Gamma M
\]
\[
- (\gamma - 1)(1 + \Gamma^{2})\}m + \epsilon [\gamma \Gamma^{2} - (\gamma - 1)(1 + \Gamma^{2})]dm.
\]  
(4)

Rearranging the terms in Equation (4), it can be written as a
dynamical equation for the evolution of the bulk motion Lorentz factor,
\[
\frac{d\Gamma}{dm} = - \frac{\gamma \Gamma^{2} - (\gamma - 1)\Gamma^{2}}{M + \epsilon m + (1 - \epsilon)m [2\gamma \Gamma - (\gamma - 1)(1 + \Gamma^{-2})]},
\]  
(5)

Equation (5) thus describes the evolution of the plasma Lorentz
factor due to its interaction with the ISM. This equation holds
both in the ultrarelativistic (\(\Gamma \gg 1\)) as well as the sub-relativistic
(\(\beta \ll 1\)) limits. It can be compared to Equation (7) in Huang
et al. (1999), which, as described above, was derived based
on the assumption of continuous inelastic collisions between
the blast wave and the ISM, and hence does not contain the
contribution of the shock-heated ISM to the pressure in
region (2).\(^{3}\)

2.1. Asymptotic Limits

It is possible to obtain analytical solutions to the dynamical
Equation (5) in the limits of ultrarelativistic and sub-relativistic
outflows. These are useful for comparison with former results.

In the adiabatic scenario, \(\epsilon = 0\), and Equation (5) reduces to
\[
\frac{d\Gamma}{dm} = - \frac{\gamma \Gamma^{2} - (\gamma - 1)\Gamma^{2}}{M + m [2\gamma \Gamma - (\gamma - 1)(1 + \Gamma^{-2})]}.
\]  
(6)

In the ultrarelativistic limit, \(\Gamma \gg 1\), Equation (6) can be re-written as
\[
\frac{d\Gamma}{dm} \approx - \frac{\beta}{M + m [2 - \beta^{2}]}.
\]  
(7)

Denoting by \(\Gamma_{0}\) the initial Lorentz factor of the flow, the
solution of the adiabatic Lorentz factor can be separated into
two regimes. (1) Initially, \(M/\gamma \gg \Gamma_{0}m\). In this regime, \(\Gamma \simeq \Gamma_{0}\) (2)
For \(\Gamma m \gg M/\gamma\) one obtains the familiar solution, \(\Gamma \simeq m^{1/2}\),
which, for constant density ISM \((\pi \propto r^{0}, m \propto r^{2})\) results in the
well-known solution \(\Gamma \propto r^{-3/2}\).\(^{5}\) Interestingly, Equation (7) is
similar to Equation (8) of Huang et al. (1999), with \(M\) replaced by
\(M/\gamma \simeq (3/4)M\). This discrepancy has only a minor effect on the
blast wave evolution.

On the other extreme, the sub-relativistic limit \(\beta \ll 1\), Equation (6) reduces to
\[
\frac{d(\Gamma\beta)}{dm} \approx - \frac{\beta}{M + m [2 - \beta^{2}]}.
\]  
(8)

which admits the solution \(\beta \propto m^{-1/2}\), for \(m \gg M\). Thus,
the general solution describing the blast wave evolution in the
adiabatic scenario (Equation (6)) for \(\Gamma m \gg M/\gamma\) can be written as
\(\Gamma \propto m^{-1/2}\).

In the radiative scenario, \(\epsilon = 1\). In the ultrarelativistic limit
\(\Gamma \gg 1\), the dynamical equation (Equation (5)) becomes
\[
\frac{d\Gamma}{dm} \approx - \frac{\gamma \Gamma^{2}}{M + m}.
\]  
(9)

Initially, \(M \gg m\), resulting in a steady Lorentz factor, \(\Gamma \simeq \Gamma_{0}\).
At a later stage, \(\Gamma_{0} m \gg M\) and one obtains the decay law
\(\Gamma(m) \approx \gamma m^{1} M / \gamma m\) (as long as \(m \ll M\)). For constant density ISM,
\(n \propto r^{0}\), this leads to the familiar decay law, \(\Gamma \propto r^{-3}\). This result
is different from the result of Huang et al. (1999) by a factor
\(\gamma^{-1} \propto 3/4\).

In the sub-relativistic limit, \(\beta \ll 1\), Equation (5) becomes
\[
\frac{d(\Gamma\beta)}{dm} \approx - \frac{\beta}{M + m},
\]  
(10)

with the solution \(\beta \propto m^{-1}\) (for \(M \gg m\)). This solution is similar
to the classical "snowplow" evolution of expanding SNRs in the
radiative regime (Spitzer 1968).

\(^{3}\) Note that Equation (7) of Huang et al. (1999) is obtained by setting \(\gamma = 1\) in
Equation (5).

\(^{4}\) Equation (7) does not admit a third regime, \(\Gamma_{0} m \gg M \gg \Gamma m\).
3. NUMERICAL SOLUTION

A full solution of the dynamic Equation (5) can easily be obtained numerically. In solving this equation, one first needs to determine the value of the adiabatic index $\eta$, which depends on the gas temperature. In calculating $\eta$, I assume that the gas in region (2) maintains a Maxwellian distribution with normalized temperature $\theta \equiv k_B T/m_p c^2$. The average energy per particle in this region is thus given by $\langle e' / n_i m_p c^2 \rangle = K_1(\theta^{-1}) / K_2(\theta^{-1}) + 3\theta$. Here, $K_1, K_2$ are modified Bessel $K$ functions of the second kind (Lightman et al. 1975, 5.34).

The ratio $\langle e' / n_i m_p c^2 \rangle \equiv \Gamma$ is determined by the shock jump conditions and is thus known at any given instance. For a given $\Gamma$, a good fit to the normalized temperature is

$$\theta \sim \left( \frac{\Gamma \beta}{3} \right) \left( \frac{\Gamma \beta + 1.07(\Gamma \beta)^2}{1 + \Gamma \beta + 1.07(\Gamma \beta)^2} \right).$$ (11)

This fit asymptotes to the exact solution in the limits $\Gamma \gg 1$ and $\beta \ll 1$. The maximum error found is less than $3 \times 10^{-2}$, for $\Gamma \beta \simeq 1$. Once the gas temperature is calculated, I use the polynomial fit given by Service (1986), to calculate $\theta \simeq (5 - 1.21937z + 0.18203z^2 - 0.96583z^3 + 2.32513z^4 - 2.39332z^5 + 1.07136z^6)/3$, where $z \equiv \theta/(0.24 + \theta)$. This fit is accurate to $10^{-3}$.

Equation (5) is solved using fourth-order Runge-Kutta method. I consider two sets of parameters, one representing GRB and one XRB. In both scenarios, the blast wave is assumed to propagate into a constant density ISM, and hence the collected mass is

$$dm = 4\pi r^2 n m_p dR,$$ (12)

where $n$ is the number density of the ISM and $m_p$ is the proton rest mass. Photons emitted as the plasma propagates a distance $dR$ are observed at time $dt$, given by

$$dR = \Gamma \beta c (\Gamma + \beta) dt.$$ (13)

While Equation (13) is derived under the assumption that the observed photons are emitted from a plasma that propagates toward the observer, a more comprehensive calculation which considers the integrated emission from different angles to the line of sight results in a similar solution, up to a numerical factor of a few (Waxman 1997; Pe’er & Wijers 2006).

The evolution of the Lorentz factor $\Gamma$, the momentum $\Gamma \beta$, and the radius $R$ are presented in Figures 1–3. In solving the dynamical equation, an ISM density of $n = 1 \text{ cm}^{-3}$ is taken. For the initial explosion conditions, two sets of parameters are used. The first set is representative for GRBs: I take $E = 10^{52}$ erg and $M = 2 \times 10^{-5} M_\odot$, resulting in $\Gamma_0 = 278$ (Huang et al. 1999). The second set is representative for XRBs and is chosen as follows. As an initial Lorentz factor I take a fiducial value $\Gamma_0 = 3$ (Miller-Jones et al. 2005). Observed XRB radio blobs are typically emitted when the luminosity is close to the Eddington luminosity, and the flux rise time is a few days, hence the total energy released is of the order of $\sim 10^{45}$ erg (e.g., Fender et al. 2004). With $\Gamma_0 = 3$ this leads to an ejected mass of $M = 3 \times 10^{23}$ g.
The Astrophysical Journal Letters, 752:L8 (4pp), 2012 June 10

The results in Figures 1–3 are given for both the adiabatic scenario and the radiative scenario. As expected, the results asymptote to the known solutions, which can be divided into three regimes. (I) Initially, $\Gamma \simeq \Gamma_0$, and $R \propto t$; (II) $\Gamma t \gg M$, and $\Gamma \gg 1$. In the adiabatic scenario, this leads to $R(t) \propto t^{1/4}$ and $\Gamma(t) \propto t^{-3/8}$, while in the radiative scenario, $R(t) \propto t^{1/7}$ and $\Gamma(t) \propto t^{-3/7}$. (III) For $\beta \ll 1$, in the adiabatic scenario $R(t) \propto t^{2/5}$ and $\beta(t) \propto t^{-3/5}$, while in the radiative scenario, $R(t) \propto t^{1/4}$ and $\beta(t) \propto t^{-3/4}$.

4. SUMMARY AND DISCUSSION

In this Letter, I have revisited the dynamics of a blast wave propagating into an ISM, as is expected in many astronomical objects, such as GRBs, XRBs, and supernovae. I derived Equation (5), which determines the evolution of the blast wave Lorentz factor as the plasma collects material from the ISM and decelerates. Analytical solutions in both the adiabatic and the radiative regimes are found in the asymptotic limits $\Gamma \gg 1$ and $\beta \ll 1$ (Section 2.1). Numerical integration is easily carried, and the resulting dynamics valid in the full regime is presented in Section 3.

The dynamical equation is different from the dynamical equations derived earlier by several authors (Katz & Piran 1997; Huang et al. 1999; Chiang & Dermer 1999; Piran 1999). This is due to a conceptual difference: earlier works considered a “toy model,” in which the basic assumption is that the interaction between the blast wave and the ISM can be described by a series of inelastic collisions. As opposed to that, here I consider the full energy-momentum tensor, which takes into account the contribution of the collected material to both the energy and the pressure in the shocked region. Such an inclusion was neglected in earlier works.

While the results of earlier works retrieve the correct asymptotic behavior in the limits $\Gamma \gg 1$ and $\beta \ll 1$ (Huang et al. 1999), the evolution of the Lorentz factor derived in these works is different from the evolution derived here by a numerical factor of tens of percents. In the radiative scenario, the difference in the derived value of the Lorentz factor at a given radius is up to $\gtrsim 30\%$. This result is not surprising, given that the main difference between the dynamics derived here and the dynamics derived in former works lies in the inclusion of $\gamma \simeq 4/3$ (for $\Gamma \gg 1$). Interestingly, in the adiabatic scenario also the numerical difference is larger than 12%.

In recent years, a renewed interest in calculating the expected flux from the interaction of an expanding blast wave with its environment had emerged. Works had been carried in the context of GRB afterglow emission (Granot & Ramirez-Ruiz 2010; Granot & Piran 2012; van Eerten & MacFadyen 2012; Xu et al. 2012), the evolution of the observed radio blobs seen in XRBs (Shen & Matzner 2012; Narayan & McClintock 2012), as well as the evolution of emission from supernovae (Chakraborti & Ray 2011). The dynamical calculations presented here are an obvious essential step in these calculations.

I thank Ramesh Narayan, Lorenzo Sironi, and Ralph Wijers for useful discussions.

REFERENCES

Bianco, C. L., & Ruffini, R. 2004, ApJ, 605, L1
Bianco, C. L., & Ruffini, R. 2005a, ApJ, 633, L13
Bianco, C. L., & Ruffini, R. 2005b, ApJ, 620, L23
Blandford, R. D., & McKee, C. F. 1976, Phys. Fluids, 19, 1130
Chakraborti, S., & Ray, A. 2011, ApJ, 729, 57
Chevalier, R. A. 1976, ApJ, 207, 872
Chevalier, R. A. 1982, ApJ, 259, 302
Chiang, J., & Dermer, C. D. 1999, ApJ, 512, 699
Fender, R. P., Belloni, T. M., & Gallo, E. 2004, MNRAS, 355, 1105
Fender, R. P., Garrington, S. T., McKay, D. J., et al. 1999, MNRAS, 304, 865
Goodman, J. 1986, ApJ, 308, L47
Granot, J., & Piran, T. 2012, MNRAS, 421, 570
Granot, J., & Ramirez-Ruiz, E. 2010, arXiv:1012.5101
Hjellming, R. M., & Rupen, M. P. 1995, Nature, 375, 464
Huang, Y. F., Dai, Z. G., & Lu, T. 1999, MNRAS, 309, 513
Katz, J. I., & Piran, T. 1997, ApJ, 490, 772
Lightman, A. P., Press, W. H., Price, R. H., & Teukolsky, S. A. (ed.) 1975, Problem Book in Relativity and Gravitation (Princeton, NJ: Princeton Univ. Press)
Meszaros, P., & Rees, M. J. 1997, ApJ, 476, 232
Miller-Jones, J. C. A., McCormick, D. G., Fender, R. P., et al. 2005, MNRAS, 363, 867
Mirabel, I. F., & Rodríguez, L. F. 1994, Nature, 371, 46
Narayan, R., Kumar, P., & Tchekhovskoy, A. 2011, MNRAS, 416, 2193
Narayan, R., & McClintock, J. E. 2012, MNRAS, 419, L69
Narayan, R., Paczynski, B., & Piran, T. 1992, ApJ, 395, L83
Paczynski, B. 1986, ApJ, 308, L43
Pe’er, A., & Wijers, R. A. M. J. 2006, ApJ, 643, 1036
Piran, T. 1999, Phys. Rep., 314, 575
Rees, M. J., & Meszaros, P. 1992, MNRAS, 258, 41
Sedov, L. I. (ed.) 1959, Similarity and Dimensional Methods in Mechanics (New York: Academic)
Service, A. T. 1986, ApJ, 307, 60
Shemi, A., & Piran, T. 1990, ApJ, 365, L55
Shen, R., & Matzner, C. D. 2012, ApJ, 744, 36
Spitzer, L. (ed.) 1968, Diffuse Matter in Space (New York: Interscience)
van Eerten, H. J., & MacFadyen, A. I. 2012, ApJ, 747, L30
van Paradijs, J., Kouveliotou, C., & Wijers, R. A. M. J. 2000, ARA&A, 38, 379
Waxman, E. 1997, ApJ, 491, L19
Xu, M., Nagataki, S., Huang, Y. F., & Lee, S.-H. 2012, ApJ, 746, 49