Quantization of a relativistic particle on the $SL(2, \mathbb{R})$ manifold based on Hamiltonian reduction

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Abstract. A quantum theory is constructed for the system of a relativistic particle with mass $m$ moving freely on the $SL(2, \mathbb{R})$ group manifold. Applied to the cotangent bundle of $SL(2, \mathbb{R})$, the method of Hamiltonian reduction allows us to split the reduced system into two coadjoint orbits of the group. We find that the Hilbert space consists of states given by the discrete series of the unitary irreducible representations of $SL(2, \mathbb{R})$, and with a positive-definite, discrete spectrum.

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1. Introduction: In the past few years the method of Hamiltonian reduction [1] has become increasingly popular and has been used, most notably, in the field of $\mathcal{W}$-algebras [2] and integrable models [3]. The basic idea of the method is to construct a system with certain properties out of a much simpler Hamiltonian system with symmetry by a reduction using constraints. For example, a large class of $\mathcal{W}$-algebras can be constructed from the Kac-Moody (current) algebra [4], whose field theoretic version is the reduction of the (generalized) Toda theories from the Wess-Zumino-Novikov-Witten (WZNW) models [5]. Other applications to two dimensional field theories, including the model of non-abelian chiral bosons, have also been reported [6].

In the present paper we investigate the problem of the motion of relativistic particles on Lie group manifolds, both classically and quantum mechanically. This problem is of interest from the point of view of constraint theory [7] because the motion of a free relativistic particle on a manifold involves a constraint analogous to the mass-shell condition $p^2 = m^2$ in Minkowski space, and the question is how it should be handled, especially with regard to quantization. The problem is also of interest from the point of view of reparametrization invariance and indeed is the particle analogue of two dimensional conformal field theory. The reasons for considering group manifolds in particular are that they are among the simplest curved manifolds and that their analogues in two dimensional conformal field theory are the WZNW models, where the method of Hamiltonian reduction has been particularly useful. Indeed, we shall see that Hamiltonian reduction allows us to quantize the system in a rather trivial manner, at least when the group is $SL(2, \mathbb{R})$.

The paper is organized as follows: We first consider general manifolds and summarize how the Lagrangian and Hamiltonian formalism is implemented for the reparametrization invariant theory. Then we specialize to manifolds corresponding to semi-simple Lie groups $G$, where there is a left-right Noether symmetry analogous to that in the WZNW models with conserved currents $L$ and $R$. We shall find that on these manifolds the constraint corresponding to the Minkowski mass-shell condition is just $\text{Tr} L^2 \equiv \text{Tr} R^2 = m^2$ — which stipulates our Hamiltonian reduction — and also provide the general solution of the reduced classical equations. We then consider the special group $G = SL(2, \mathbb{R})$ which is a three dimensional Lorentzian manifold. This group has the property that the above Hamiltonian reduction leads to a split reduced system consisting of two chiral (‘left’ and ‘right’) sectors, which are both coadjoint orbits of the group specified by the constraint. An important consequence is that the quantization of the system is then reduced to finding unitary irreducible representations of the group $SL(2, \mathbb{R})$. The time-like nature of the constraint, $m^2 > 0$, restricts these representations to the discrete series. As a result, we find that
the energy levels are positive definite and integrally spaced, while the angular momentum takes integer values only\(^1\).

2. Relativistic particle on a manifold as a constrained system: Let \(M\) be a (pseudo-)Riemannian manifold with metric \(g_{\mu\nu}(x)\) where \(x^\mu\) is a local coordinate system on \(M\). Take the familiar action describing a relativistic point particle of mass \(m > 0\) moving freely on the manifold \(M\),

\[
I_0 = -m \int dt \sqrt{g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu},
\]

where \(t\) is a parameter along the trajectory \(x^\mu(t)\) and \(\dot{x}^\mu := dx^\mu/dt\). We assume that \(t\) increases monotonically, say, from \(t = 0\) to \(t = T\), and that paths under consideration satisfy \(g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu > 0\). It is known that, at the classical level, one can replace (2.1) by the quadratic action \([9]\)

\[
I = -\frac{1}{2} \int dt \left[ \frac{1}{\lambda} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \lambda m^2 \right],
\]

with \(\lambda = \lambda(t) > 0\) being a Lagrange multiplier. Indeed, if we substitute \(\lambda\) by using its equation of motion, the action \(I\) reduces to \(I_0\). Like \(I_0\), the action \(I\) is invariant under reparametrizations \(t \rightarrow f(t)\) with

\[
\lambda(t) \longrightarrow \lambda'(f(t)) = \left( \frac{df}{dt} \right)^{-1} \lambda(t), \quad x^\mu(t) \longrightarrow x'^\mu(f(t)) = x^\mu(t),
\]

where we assume \(\dot{f}(t) > 0\) to preserve the monotonic property.

The Hamiltonian that corresponds to the action \(I\) is found to be

\[
H = -\frac{\lambda}{2} (g^{\mu\nu} p_\mu p_\nu - m^2),
\]

where \(p_\mu\) is the momentum conjugate to \(x^\mu\). Since the momentum \(\pi\) conjugate to \(\lambda\) vanishes, following Dirac’s approach \([7]\) to constrained systems we must have the consistency condition \(\pi = \{\pi, H\} \approx 0\). This leads to

\[
\phi := g^{\mu\nu} p_\mu p_\nu - m^2 \approx 0,
\]

i.e., the Hamiltonian (2.4) be zero. Being first class, the constraint (2.5) generates a local gauge symmetry, which is none other than the reparametrization of the system.

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\(^1\) The irreducible representations of \(SL(2, \mathbb{R})\) were used earlier \([8]\) in constructing a quantum theory of a relativistic particle in flat three dimensional Minkowski space, with curvature and torsion of a particle world trajectory.
Accordingly, the reduced phase space is given by factorizing the constrained surface with respect to the gauge symmetry.

3. Group manifolds: Now we shall consider the case where $M$ is the manifold of a semi-simple Lie group $G$, which possesses the nondegenerate metric

$$g_{\mu\nu}(x) := \text{Tr}\left(g^{-1}\partial_{\mu}g g^{-1}\partial_{\nu}g\right), \quad (3.1)$$

where $g = g(x) \in G$ is a group element. The ‘Tr’ in (3.1) is defined by the matrix trace ‘tr’ in some irreducible representation multiplied by a constant $c$, so as to provide an inner product $\langle X, Y \rangle := \text{Tr}(XY) = c \text{tr}(XY)$ with a proper sign in the Lie algebra $\mathcal{G}$ of the group. (The constant $c$ possesses a typical scale factor, which we set to unity for brevity.) Choosing a basis $\{T_m\}$ in $\mathcal{G}$, we have the ‘flat’ metric in the Lie algebra, $\eta_{mn} := \langle T_m, T_n \rangle$, which is the metric in the tangent space on the group manifold$^2$. With (3.1) the action (2.1) can be written as

$$I_0 = -m \int dt \sqrt{\text{Tr}(g^{-1}\dot{g})^2}, \quad (3.2)$$

which is coordinate free and hence globally well-defined over the group manifold. The equations of motion derived from (3.2) are

$$\frac{d}{dt} \left(\frac{g^{-1}\dot{g}}{\rho}\right) = 0, \quad \text{where } \rho := \sqrt{\text{Tr}(g^{-1}\dot{g})^2}. \quad (3.3)$$

Similarly, the action (2.2) admits the global form,

$$I = -\frac{1}{2} \int dt \left[\frac{1}{\lambda} \text{Tr}(g^{-1}\dot{g})^2 + \lambda m^2\right]. \quad (3.4)$$

A salient feature of $M$ being a group manifold is that, in addition to the reparametrization invariance, the system acquires a chiral invariance. In fact, both of the actions, (3.2) and (3.4), are manifestly invariant under the rigid left-right transformations,

$$g(x) \rightarrow hg(x), \quad g(x) \rightarrow g(x)\tilde{h}, \quad (3.5)$$

for arbitrary elements $h, \tilde{h} \in G$.

$^2$ As usual, $X_m := \langle T_m, X \rangle$ for $X \in \mathcal{G}$ and the indices are raised/lowered as $X^m = \eta^{mn}X_n$ using the inverse $\eta^{mn}$ of the metric $\eta_{mn}$, whence $\langle X, Y \rangle = \eta_{mn}X^mY^n$. In terms of the vielbein $e^m_\mu := \langle T^m, g^{-1}\partial_\mu g \rangle$ one has $g_{\mu\nu} = e^m_\mu e^n_\nu \eta_{mn}$. 

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To provide a globally defined Hamiltonian description, let us recall the free Hamiltonian system that can be defined to a semi-simple Lie group $G$, that is, the system whose phase space $\mathcal{M}$ is given by the cotangent bundle $[1, 3]$,

$$\mathcal{M} = T^* G \cong G \times G = \{(g, R) \mid g \in G, R \in G\},$$

(3.6)
on which the symplectic 2-form is given by

$$\omega = d\theta, \quad \text{with } \theta = -\text{Tr} R(g^{-1}dg),$$

(3.7)
while the Hamiltonian is

$$H_F = \frac{1}{2} \text{Tr} R^2.$$ (3.8)
(We again set a scale constant to unity in (3.8) for simplicity.) The non-vanishing Poisson brackets derived from (3.7) are

$$\{R_m, R_n\} = f_{mn}^l R_l, \quad \{R_m, g_{ij}\} = (gT_m)_{ij},$$

(3.9)

where $f_{mn}^l$ are the structure constants appearing in the basis: $[T_m, T_n] = f_{mn}^l T_l$. Then, the Hamiltonian system we are after is furnished by imposing the constraint (2.5), which now reads

$$\phi = \text{Tr} R^2 - m^2 \approx 0.$$ (3.10)
Thus the (total) Hamiltonian can be written as $H = -\frac{1}{2} \phi$ with $\lambda$ a Lagrange multiplier, which yields the equations of motion,

$$\dot{g} \approx \{g, H\} = \lambda g R, \quad \dot{R} \approx \{R, H\} = 0.$$ (3.11)
Since the constraint (3.10) together with the first equation of (3.11) imply $\lambda^2 = (\frac{\phi}{m})^2$, the equations motion (3.11) reproduce (3.3).

The conserved ‘right’ current $R$ appearing in (3.11) is in fact the Noether current associated with the global right symmetry in (3.5) for the action (3.2). Analogously, the ‘left’ current

$$L := -g R g^{-1},$$ (3.12)
is the conserved Noether current associated with the left symmetry in (3.5), which forms the Poisson brackets,

$$\{L_m, L_n\} = f_{mn}^l L_l, \quad \{L_m, g_{ij}\} = -(T_m g)_{ij},$$ (3.13)
and commutes with the right current, \( \{L_m, R_n\} = 0 \). Both of the two currents commute with the constraint (3.10) and are hence gauge (reparametrization) invariant.

Although unnecessary so far in the present group manifold case, a local coordinate system may be useful when we wish to find a physical interpretation for the currents. Consider, for example, the normal coordinates

\[
g(x) = e^{x^n T_m},
\]

(3.14)

where \( x^m \) are the ‘flat’ coordinates specifying the position of the particle. Then, the momentum \( p_m \) conjugate to \( x^m \) reads

\[
p_m = -(g^{-1} \partial_m g)_R = (\partial_m g g^{-1})_L R^n.
\]

(3.15)

If we now define the ‘vector current’ \( V_m \) by subtracting the two chiral currents, we get

\[
V_m = \frac{1}{2}(L_m - R_m) = p_m + \mathcal{O}(x^2),
\]

(3.16)

where \( \mathcal{O}(x^2) \) denotes a polynomial which is at least quadratic in \( x^m \). This shows that in the vicinity of the origin \( g = 1 \) the vector current \( V_m \) reduces to \( p_m \), but since \( V_m \) are conserved (while \( p_m \) are not), and since \( V_m \) are gauge invariant and survive the reduction, we may regard \( V_m \) as the ‘momentum’ (hence \( V_0 \) is the ‘energy’) of the particle in the chronological gauge \( x^0(t) = t \). On the other hand, the ‘axial vector current’ \( A_m \) defined by adding the two currents becomes

\[
A_m = \frac{1}{2}(L_m + R_m) = \frac{1}{2} f_{mn} l x^n p_l + \mathcal{O}(x^2).
\]

(3.17)

As we shall see shortly, for \( G = SL(2, \mathbb{R}) \) the current \( A_m \) will be interpreted as the generator of three dimensional Lorentz transformations (hence \( A_0 \) is the ‘angular momentum’). The orthogonality \( \langle V, A \rangle = 0 \), which follows from (3.12), is consistent with this interpretation.

We wish to remark at this point on the general solution for the equations of motion (3.3). Thanks to the reparametrization invariance, the general solution can readily be found by choosing the invariant length for the parameter \( t \) so that \( \rho = 1 \). Indeed, the equations of motion (3.3) then reduce to \( \frac{d}{dt}(g^{-1} \dot{g}) = 0 \), which can be integrated at once to be \( g(t) = g(0)e^{-tR/m} \), where \( R \in \mathcal{G} \) is a constant satisfying (3.10). The general solution for

\( ^3 \) This parametrization is available only for a neighbourhood of the identity \( g = 1 \), but this is not important for our purpose here.
(3.3) can be obtained simply by returning to the generic parameter by a reparametrization transformation $t \to f(t)$:

$$g(t) = g(0)e^{-f(t)R/m}.$$  \hspace{1cm} (3.18)

The constant $R$ is in fact the conserved right current determined from the initial condition, $g(0)$ and $\dot{g}(0)$. (The solution can also be given in terms of the left current as $g(t) = e^{f(t)L/m}g(0)$.) Thus, in the normal coordinates (3.14) the particle’s trajectory is just a straight line for the initial condition $g(0) = 1$.

4. Hamiltonian reduction for $G = SL(2, \mathbb{R})$: We now specialize to the case $G = SL(2, \mathbb{R})$ which is a three dimensional Lorentzian manifold isomorphic to $S^1 \times \mathbb{R}^2$. We shall work with the following basis $\{T_m\}$ in the algebra $G = sl(2, \mathbb{R})$,

$$T_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \hspace{1cm} (4.1)$$

Choosing $c = -\frac{1}{2}$ we find that the flat metric becomes

$$\eta_{mn} = \langle T_m, T_n \rangle = -\frac{1}{2} \text{tr}(T_m T_n) = \text{diag} (+1, -1, -1). \hspace{1cm} (4.2)$$

Since the basis elements satisfy the relation,

$$T_m T_n = -\eta_{mn} \cdot 1 + \epsilon_{mn} T_l, \hspace{1cm} (4.3)$$

with $\epsilon_{012} = +1$, we have for $X, Y \in sl(2, \mathbb{R})$ the useful formula,

$$XY = -\langle X, Y \rangle \cdot 1 + \frac{1}{2}[X, Y], \hspace{1cm} (4.4)$$

and, in particular, $XX = -|X|^2 \cdot 1$ where $|X|^2 := \langle X, X \rangle$. It is then easy to show that, if we write $X = \alpha \hat{X}$ with a ‘normalized’ vector (i.e., $|\hat{X}|^2 = \pm 1$ or 0), we have

$$e^X = \begin{cases} 
\cos \alpha \cdot 1 + \sin \alpha \cdot \hat{X}, & \text{if } |\hat{X}|^2 = +1; \\
\cosh \alpha \cdot 1 + \sinh \alpha \cdot \hat{X}, & \text{if } |\hat{X}|^2 = -1; \\
1 + \alpha \hat{X}, & \text{if } |\hat{X}|^2 = 0.
\end{cases} \hspace{1cm} (4.5)$$

We note that the orthochronous Lorentz group $SO^+(2, 1)$ in three dimensions is realized by the adjoint action of $SL(2, \mathbb{R})$,

$$X \rightarrow g X g^{-1}, \quad \text{with } g \in SL(2, \mathbb{R}). \hspace{1cm} (4.6)$$
More explicitly, the transformations in components induced by the adjoint action (4.6) read

\[ X_m \rightarrow \Lambda^m_n X_n, \quad \text{with} \quad \Lambda^m_n = \text{Tr}(T_m g T^n g^{-1}), \]  

(4.7)

where the matrices \( \Lambda^m_n \) belong to the group \( SO(2,1) \), whereas the property \( \Lambda^0_0 \geq 1 \) can be seen by a direct computation. Clearly, the axial vector current (3.17), which now takes the form \( A_m = \epsilon_{mn}^l x^n p_l \), is the generator of the Lorentz transformation (4.7), and in particular \( A_0 \) is the angular momentum.

We now carry out the reduction of the Hamiltonian system explicitly by means of the constraint (3.10) in the \( SL(2,\mathbb{R}) \) case. The first point to be noted is that the reduced phase space \( \mathcal{M}_{\text{red}} \) splits up into two coadjoint orbits of the group. To see this, let us first write the variable \( R \in G \) in (3.6) used for the phase space \( \mathcal{M} \) as

\[ R = h^{-1} K h, \quad \text{where} \quad h \in G, \quad K \in \mathcal{G}, \]  

(4.8)

where \( K \) is some fixed vector. The parametrization (4.8) is based on the observation that any element in \( \mathcal{G} = sl(2,\mathbb{R}) \) can be reached from \( K \) by an \( SO^+(2,1) \) transformation (4.6) with \( h \), if we provide three types of \( K \), that is, time-like \( ||K||^2 > 0 \), space-like \( ||K||^2 < 0 \) and null \( ||K||^2 = 0 \). Since one can write \( K = r \hat{K} \) with \( r > 0 \) and a normalized vector \( \hat{K} \), one sees that the phase space \( \mathcal{M} \) can be parametrized by the (redundant) set \( \{g, h, r; s\} \), where \( s := ||\hat{K}||^2 = \pm 1, 0 \) indicates the type of \( K \). Substituting (4.8) back into (3.7) and renaming \( gh^{-1} \) as \( g \), we obtain

\[ \theta = \theta_K(g) + \theta_{-K}(h^{-1}) , \quad \text{where} \quad \theta_K(g) := -\text{Tr} K(g^{-1}dg) . \]  

(4.9)

If \( K \) is constant but not null, then \( \theta_K \) is just the standard canonical 1-form associated with the coadjoint orbit \( \mathcal{O}_K \) of the group \( G \) passing through \( K \). But since the constraint (3.10) does indeed render \( K \) time-like constant with \( r = m \), we see that the reduced phase space is given by the direct product of the two coadjoint orbits,

\[ \mathcal{M}_{\text{red}} \simeq \mathcal{O}_K \times \mathcal{O}_{-K} , \]  

(4.10)

where the symplectic structure is carried over to those on the orbits. Accordingly, natural variables parametrizing the reduced phase space \( \mathcal{M}_{\text{red}} \) are the currents on the coadjoint orbits,

\[ L = -g K g^{-1} \quad \text{and} \quad R = h^{-1} K h , \]  

(4.11)

which form independently an \( sl(2,\mathbb{R}) \) algebra under the Poisson brackets derived from (4.9).
Before going over to the quantization of the system, we point out that for $SL(2, \mathbb{R})$ one can express the symplectic 2-form $\omega_K = d\theta_K$ (or $\omega_{-K} = d\theta_{-K}$) for the coadjoint orbit solely in terms of the chiral current $L$ (or $R$) in (4.11). For example, in terms of the left current the corresponding symplectic 2-form can be written as

$$\omega_K(g) = -\frac{1}{4m^2} \epsilon^{mnl} L_m dL_n \wedge dL_l. \quad (4.12)$$

To see this, we introduce a parameter $\delta \geq 0$ by

$$\langle \hat{K}, \hat{L} \rangle := -\cosh \delta$$

with the normalized left current $\hat{L} := L/\sqrt{|L|^2}$, and construct the three vectors,

$$T_0 := \frac{\hat{K} - \hat{L}}{2 \cosh (\delta/2)}, \quad T_1 := -\frac{\hat{K} + \hat{L}}{2 \sinh (\delta/2)}, \quad T_2 := \frac{[\hat{K}, \hat{L}]}{2 \sinh \delta}. \quad (4.13)$$

For fixed $\hat{K}$ and $\hat{L}$, these vectors form a new orthonormal basis of the $sl(2, \mathbb{R})$ algebra,

$$\langle T_m, T_n \rangle = \eta_{mn} \quad \text{and} \quad [T_m, T_n] = 2 \epsilon_{mn}^l T_l. \quad (4.14)$$

With this basis we consider the Euler angle representation of $SL(2, \mathbb{R})$ elements,

$$g = g(\alpha, \beta, \gamma) = e^{\alpha T_2} e^{\beta T_0} e^{\gamma T_2}. \quad (4.15)$$

Note that among the three parameters is a bounded one $0 \leq \beta < 2\pi$, which is the parameter in the cyclic direction $S^1$ of the group manifold $SL(2, \mathbb{R})$ (see (4.5)). Observe also that the Lorentz transformation on the vector $\hat{K}$ by the adjoint action of $g(\alpha) = e^{\alpha T_2}$ is a ‘rotation’ in the plane spanned by $\hat{K}$ and $\hat{L}$,

$$\hat{K} \rightarrow g(\alpha) \hat{K} g^{-1}(\alpha) = \frac{\sinh (2\alpha + \delta)}{\sinh \delta} \hat{K} + \frac{\sinh 2\alpha}{\sinh \delta} \hat{L}. \quad (4.16)$$

One then finds that for $\alpha = -\delta/2$ the vector $\hat{K}$ is rotated to $-\hat{L}$, and for $\alpha = -\delta/4$ it is rotated halfway to $-L$, i.e., it directs to $T_0$. But since the parametrization (4.15) consists of two rotations of the type (4.16) with $g(\alpha)$ and $g(\gamma)$, interrupted by the rotation with $e^{\beta T_0}$, the parameters fulfilling the relation $L = -gKg^{-1}$ in (4.11) are found to be

$$\alpha = \gamma = -\frac{\delta}{4}, \quad \beta = \text{arbitrary}. \quad (4.17)$$

(The appearance of the free parameter $\beta$ is expected from the counting of degrees of freedom — $SL(2, \mathbb{R})$ is three dimensional while its coadjoint orbit is two dimensional for $m \neq 0$.) If we now express the canonical 1-form $\theta_K$ in (4.9) using (4.15) and (4.17), we get

$$\theta_K(g) = -md\beta + \frac{\langle [\hat{K}, \hat{L}], dL \rangle}{4(m - \langle \hat{K}, \hat{L} \rangle)}. \quad (4.18)$$
Choosing, e.g., \( \hat{K} = -T_0 \) we find that the corresponding symplectic 2-form \( \omega_K \) is just the one given in (4.12). Note that from (4.11) this choice implies

\[ L_0 > 0 \quad \text{and} \quad R_0 < 0, \quad (4.19) \]

that is, the left current lies in the coadjoint orbit given by an upper hyperboloid in the algebra \( sl(2, \mathbb{R}) \) whereas the right current lies in the coadjoint orbit given by a lower one.

5. Quantization: We are now going to discuss the quantization of the system. However, having seen that the reduced phase space consists of the two coadjoint orbits (4.10), the problem actually reduces to the quantization of the system of coadjoint orbits. In other words, the quantization amounts to finding unitary, irreducible representations of the algebra \( sl(2, \mathbb{R}) \) formed by the chiral currents on the coadjoint orbits, \( O_K \) and \( O_{-K} \). On account of the constraint (3.10) which requires the Casimir \( q = \frac{1}{4} \text{Tr} L^2 \) to be positive constant \( \frac{m^2}{4} \), the irreducible representations [10] (see also, [11]) relevant for our purpose are the discrete series \( D_j^\pm \) with \( 2j = 3, 4, \ldots \), for which \( q = j(j-1) > 0 \). Further, the conditions (4.19) require that the representations for the left sector should be given by \( D_j^+ \) while those for the right sector are \( D_j^- \). A simple realization for these representations can be provided by the Holstein-Primakoff method, in which one uses creation/annihilation operators \( [a, a^\dagger] = 1 \) as a basic building block. For instance, for the left sector we have [12],

\[ L_- := L_1 + iL_2 = 2\sqrt{a^\dagger a + 2j} \cdot a, \]
\[ L_+ := L_1 - iL_2 = 2a^\dagger \cdot \sqrt{a^\dagger a + 2j}, \]
\[ L_0 := 2(a^\dagger a + j). \quad (5.1) \]

It is straightforward to check that the left current given in (5.1) satisfies the constraint (3.10) as well as the (quantum) commutation relations,

\[ [L_m, L_n] = 2i \epsilon_{mn}^l L_l. \quad (5.2) \]

In the familiar Fock space consisting of the states \( |n_L \rangle \) for \( n_L = 0, 1, 2, \ldots \) with

\[ a|n_L \rangle = \sqrt{n_L} |n_L - 1 \rangle, \quad a^\dagger|n_L \rangle = \sqrt{n_L + 1} |n_L + 1 \rangle, \quad (5.3) \]

we find

\[ L_- |n_L \rangle = 2\sqrt{(n_L - 1 + 2j)n_L} |n_L - 1 \rangle, \]
\[ L_+ |n_L \rangle = 2\sqrt{(n_L + 2j)(n_L + 1)} |n_L + 1 \rangle, \]
\[ L_0 |n_L \rangle = 2(n_L + j) |n_L \rangle. \quad (5.4) \]
Analogously, one can construct representations for the right sector using another pair of creation/annihilation operators for the right current. Actually, this is equivalent to the formal replacement 
\[
\{L^+, L^-, L_0\} \rightarrow \{-R^-, -R^+, -R_0\}
\]
for the above construction, which leads to the Fock space consisting of \(|n_R\rangle\) for \(n_R = 0, 1, 2, \ldots\), for which
\[
\begin{align*}
R^- |n_R\rangle & = -2\sqrt{(n_R + 2j)(n_R + 1)}|n_R + 1\rangle, \\
R^+ |n_R\rangle & = -2\sqrt{(n_R - 1 + 2j)n_R}|n_R - 1\rangle, \\
R_0 |n_R\rangle & = -2(n_R + j)|n_R\rangle.
\end{align*}
\]

The full Hilbert space is spanned by the states given by the direct product of the two representations, \(D^+_j\) and \(D^-_j\), sharing the same value for the Casimir. The states are thus labeled by two integers, \(|n_L, n_R\rangle = |n_L\rangle \otimes |n_R\rangle\), on which the energy \(V_0\) in (3.16) and the angular momentum \(A_0\) in (3.17) act as
\[
\begin{align*}
V_0 |n_L, n_R\rangle & = (n_L + n_R + 2j)|n_L, n_R\rangle, \\
A_0 |n_L, n_R\rangle & = (n_L - n_R)|n_L, n_R\rangle.
\end{align*}
\]

The above result shows that the energy levels are positive definite and spaced integrally — which is in fact expected because of our identification of \(x^0 \in S^1\) being ‘time’ — while the angular momentum takes integer values only. The allowed mass of the particle at the quantum level is
\[
m = 2\sqrt{j(j-1)}, \quad \text{with} \quad 2j = 3, 4, \ldots
\]

As we have seen in this paper, the basic ingredient underlying the simplicity of the quantization is the chiral split of the reduced system, that is, the split into two coadjoint orbits. In this respect, it is worth mentioning that essentially the same split was discussed recently (for compact groups) in [13] for the system of the cotangent bundle. This suggests that the Hamiltonian reduction and the subsequent quantization considered for \(SL(2, \mathbb{R})\) may be generalized to any higher rank group \(G\) with the simplicity intact, by specifying all the Casimir elements of the group in the form of constraints. Whether this yields a physically interesting model or not is however unclear except for \(G = SL(2, \mathbb{R})\).

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