Electron-positron pair creation in a vacuum by an electromagnetic field in 3+1 and lower dimensions

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We calculate the probability of electron-positron pair creation in vacuum in 3+1 dimensions by an external electromagnetic field composed of a constant uniform electric field and a constant uniform magnetic field, both of arbitrary magnitudes and directions. The same problem is also studied in 2+1 and 1+1 dimensions in appropriate external fields and similar results are obtained.

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I. INTRODUCTION

Pair creation of charged particles in vacuum by an external electric field was first studied by Schwinger several decades ago [1]. Related problems have been discussed by many authors, for example, in Refs. [2-6]. Though half a century has passed since the publication of Schwinger’s classic work, the subject of pair creation remains a widely discussed one in string theory and black hole theory in these days.

It seems that most authors of the cited works dealt with the problem only in electric fields. A magnetic field, on the other hand, was left out in the cold. This may be due to the fact that a pure magnetic field does not lead to creation of particle-antiparticle pairs. In spite of this fact, the presence of a magnetic field does change the probability of pair creation by a pure electric field. This can be easily seen when the magnetic field \( B \) is perpendicular to the electric field \( E \). The situation may be changed by a Lorentz boost to another one with a pure electric field, provided \(|E| > |B|\). The result is easily obtained, and is obviously different from the case where \( B \) is absent. When \( B \) is not perpendicular to \( E \), the problem is more complicated. The result for the general case cannot be easily obtained from previous ones and needs further study. As an exact and nonperturbative result can be achieved as in the case of a pure electric field, the study is worthwhile and may be of some interest.

In this paper we consider a magnetic field \( B \) as well as an electric field \( E \), both being constant and uniform, but with arbitrary magnitudes and directions. The field of electrons (or other charged fermions of spin \( \frac{1}{2} \)) is second quantized, while the electromagnetic field is treated classically as a background field for the electrons. The probability of electron-positron pair creation in vacuum is calculated exactly. More specifically, we first deal with the relatively simple case where \( B \parallel E \) (\( B \) points at the same or opposite direction of \( E \)). For the general case, one can always find by making a Lorentz boost an inertial frame \( K' \) where the transformed fields satisfy \( B' \parallel E' \). In the system \( K' \) the result is obtained in terms of \( E' \) and \( B' \). As the probability should be invariant under Lorentz boost, we obtain the result in the original system by using the relation between \((E', B')\) and \((E, B)\). When \( E = 0 \) the probability vanishes while for \( B = 0 \) it reduces to Schwinger’s result, as expected. This is done in Sec. II.

In previous works main attention was paid to the problem in ordinary 3+1 dimensions. In lower spatial dimensions the result might be expected to be somewhat different. In Sec. III we turn to the problem in 2+1 dimensions. In this case the magnetic field has only one component and the electric field has two, thus the problem is simpler than in 3+1 dimensions. In Sec. IV we calculate the result in 1+1 dimensions. In this case there is no magnetic field and the electric field has only one component, so the problem is still simpler. The results in lower dimensions are similar to that in 3+1 dimensions, but cannot be trivially

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obtained from the latter. The difference among them can be easily seen from the appearances of the corresponding results. These are summarized in Sec. V.

II. 3+1 DIMENSIONS

We use natural units where $\hbar = c = 1$ throughout this paper. Consider the electron with mass $m$ and charge $e < 0$ (the following results are applicable to other charged fermions of spin $\frac{1}{2}$), moving in a background electromagnetic field described by the vector potential $A_\mu$. The field of the electron is second quantized, while $A_\mu$ is treated classically. The vacuum-vacuum transition amplitude can be shown to be [1, 2]

$$S_0 = \exp \left[ -\text{Tr} \ln \frac{\gamma \cdot P - m + i\epsilon}{\gamma \cdot (P - eA) - m + i\epsilon} \right],$$

(1)

where $\epsilon = 0^+$, and the Tr indicates a complete diagonal summation over the space-time coordinates as well as the spinorial indices. In this section we consider the problem in ordinary 3+1 dimensions. But Eq. (1) holds in 2+1 and 1+1 dimensions as well. In Eq. (1) $X_\mu$ (the independent variables of $A_\mu$) and $P_\mu$ are now operators satisfying

$$X_\mu |x\rangle = x_\mu |x\rangle, \quad \langle x | P_\mu |\varphi\rangle = i\partial_\mu \langle x | \varphi\rangle,$$

(2)

where $|\varphi\rangle$ is an arbitrary state. Consequently

$$[X_\mu, P_\mu] = -ig_{\mu\nu},$$

(3)

where $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and similarly in lower dimensions. Using the charge conjugation matrix and the fact that the trace of an operator is invariant under matrix transposition we have

$$S_0 = \exp \left[ -\text{Tr} \ln \frac{\gamma \cdot P + m - i\epsilon}{\gamma \cdot (P - eA) + m - i\epsilon} \right].$$

(4)

This holds in lower dimensions as well. Multiplying Eqs. (1) and (4) we have

$$S_0^2 = \exp \left\{ -\text{Tr} \ln \frac{P^2 - m^2 + i\epsilon}{\gamma \cdot (P - eA)^2 - m^2 + i\epsilon} \right\}.$$  

(5)

Taking the module we obtain the vacuum-vacuum transition probability in the form

$$|S_0|^2 = \exp \left[ -\int dx \, w(x) \right],$$

(6)

where

$$w(x) = \text{Re} \left\{ x | \text{tr} \ln \frac{P^2 - m^2 + i\epsilon}{\gamma \cdot (P - eA)^2 - m^2 + i\epsilon} | x \right\} = \text{Re} W(x)$$

(7)

is to be interpreted as the probability, per unit time and per unit volume, at the space-time position $x$, of electron-positron pair creation by the external electromagnetic field. For constant uniform electromagnetic field it is expected to be independent of $x$. In the above equation the tr indicates ordinary diagonal summation over spinorial indices. Using the identity

$$\ln \frac{a + i\epsilon}{b + i\epsilon} = \int_0^\infty \frac{ds}{s} \left[ e^{is(b + i\epsilon)} - e^{is(a + i\epsilon)} \right]$$

(8)

and the relation

$$[\gamma \cdot (P - eA)]^2 = (P - eA)^2 - \frac{e^2}{2} g^{\mu\nu} F_{\mu\nu}$$

(9)

where
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \]  

we have

\[
\ln \frac{P^2 - m^2 + i\epsilon}{[\gamma \cdot (P - eA)]^2 - m^2 + i\epsilon} = \int_0^\infty \frac{ds}{s} e^{-is(m^2 - i\epsilon)} \left[ e^{i(P-eA)s} \exp \left( -\frac{i}{2} e S \sigma^{\mu\nu} F_{\mu\nu} \right) - e^{isP^2} \right].
\]

(11)

Up to this point we have only reviewed some results obtained by previous authors [1,2]. These are necessary for further discussions. We emphasize that Eqs. (7) and (11) are valid in lower dimensions as well as in 3+1 dimensions. The following calculations depend on the spatial dimension, thus we will henceforth deal only with the (3+1)-dimensional case in this section. We will return to the lower-dimensional case in the following sections.

Now we consider a constant uniform electromagnetic field where \( B \parallel E \). Without loss of generality we take

\[
E = E e_x, \quad B = Be_x,
\]

(12)

where \( e_x \) is the unit vector in the \( x^1 \) direction, \( E \) and \( B \) are constants which may be positive or negative. We have then for \( F_{\mu\nu} \) the nonvanishing components \( F_{01} = E, F_{23} = -B, \) and

\[
-\frac{1}{2} i \sigma^{\mu\nu} F_{\mu\nu} = \gamma^0 \gamma^1 E - \gamma^2 \gamma^3 B.
\]

Using the properties of the \( \gamma \) matrices, we have

\[
(-\frac{1}{2} i \sigma^{\mu\nu} F_{\mu\nu})^2 = E^2 - B^2 + i2EB\gamma_5,
\]

(13)

where \( \gamma_5 = \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \). As \( \gamma_5^2 = 1 \), the eigenvalues of \( \gamma_5 \) are \( \pm 1 \), both being double degenerate. Thus the eigenvalues of \( (-\sigma^{\mu\nu} F_{\mu\nu})/2 \) are \( (E \pm iB)^2 \), both being double degenerate, and the four eigenvalues of \( -i\sigma^{\mu\nu} F_{\mu\nu}/2 \) are \( E \pm iB, -(E \pm iB) \). Therefore we have

\[
\text{tr} \exp \left( -\frac{1}{2} i e S \sigma^{\mu\nu} F_{\mu\nu} \right) = 4 \cosh(eEs) \cos(eBs),
\]

(14)

and

\[
\text{tr} \ln \frac{P^2 - m^2 + i\epsilon}{[\gamma \cdot (P - eA)]^2 - m^2 + i\epsilon} = 4 \int_0^\infty \frac{ds}{s} e^{-is(m^2 - i\epsilon)} \left[ \cosh(eEs) \cos(eBs) e^{is(P-eA)^2} - e^{isP^2} \right].
\]

(15)

The next step is to calculate the matrix elements \( \langle x | e^{is(P-eA)^2} | x \rangle \) and \( \langle x | e^{isP^2} | x \rangle \). The second is easy. We have

\[
\langle x | e^{isP^2} | x \rangle = \int dk \left| \langle x | e^{i\mu k} | k \rangle \right|^2,
\]

(16)

where

\[
\langle x | k \rangle = \frac{1}{(2\pi)^2} e^{-ik \cdot x}, \quad \langle k | x \rangle = \frac{1}{(2\pi)^2} e^{ik \cdot x}.
\]

(17)

Using Eq. (2) it is easy to show that

\[
P_\mu | k \rangle = k_\mu | k \rangle, \quad \langle k | P_\mu = k_\mu | k \rangle.
\]

(18)

Substituting Eqs. (17) and (18) into Eq. (16) we obtain

\[
\langle x | e^{isP^2} | x \rangle = \int dk e^{i\mu k} \langle x | k \rangle \langle k | x \rangle = \frac{1}{(2\pi)^4} \int dk e^{i\mu k} = -\frac{i}{16\pi^2 s^2}.
\]

(19)

The calculation of another matrix element is somewhat complicated. We denote

\[
X^\mu = (X_0, X, Y, Z), \quad P^\mu = (P_0, P_x, P_y, P_z).
\]

(20)
and choose
\[ A^\mu = (A_0, A_x, A_y, A_z) = (-EX, 0, 0, BY), \] (21)
which corresponds to the field strengths (12), then
\[ (P - eA)^2 = (P_0 + eEX)^2 - (P_z - eBY)^2 - P_x^2 - P_y^2. \] (22)

It is not difficult to show that
\[ (P_0 + eEX)^2 = e^{iP_0 P_x/eE} e^{2E^2 X^2} e^{-iP_0 P_x/eE}, \] (23a)
\[ (P_z - eBY)^2 = e^{-iP_y P_z/eB} e^{2B^2 Y^2} e^{iP_y P_z/eB}. \] (23b)

Substituting these relations into Eq. (22) we have
\[ (P - eA)^2 = e^{-iP_0 P_x/eE} (P_0 - eE^2 X^2) e^{-iP_0 P_x/eE} - e^{-iP_y P_z/eB} (P_y^2 + e^2 B^2 Y^2) e^{iP_y P_z/eB}, \] (24)
and thus
\[ e^{is(P - eA)^2} = e^{iP_0 P_x/eE - iP_y P_z/eB} e^{-i(P_z^2 - e^2 E^2 X^2)} e^{-is(P_y^2 + e^2 B^2 Y^2)} e^{-iP_0 P_x/eE + iP_y P_z/eB}. \] (25)

With these preparations we write down
\[ \langle x | e^{is(P - eA)^2} | x \rangle = \int dk \, dk' \langle x | k' \rangle e^{is(P - eA)^2} | k \rangle | k | x \rangle. \] (26)

Using Eqs. (25), (17), (18), and \( \delta[(k_x' - k_x)/eE] = |eE| \delta(k_x' - k_x) \) etc., after some algebras we arrive at
\[ \langle x | e^{is(P - eA)^2} | x \rangle = \frac{|eE| |eB|}{4\pi^2} \text{tr} e^{-is(P_0^2 - e^2 E^2 X^2)} \text{tr} e^{-is(P_y^2 + e^2 B^2 Y^2)}, \] (27)
where
\[ \text{tr} e^{-is(P_0^2 - e^2 E^2 X^2)} = \int dk_x \langle k_x | e^{-is(P_0^2 - e^2 E^2 X^2)} | k_x \rangle \]
and similarly for another trace, where \( k_x \) is the first spatial component of \( k^\mu \). By comparison with the harmonic oscillator or by using the technique of path integral, one can find the following results.
\[ \text{tr} e^{-is(P_0^2 - e^2 E^2 X^2)} = \frac{1}{2 \sinh(|eE|s)}, \] (28a)
\[ \text{tr} e^{-is(P_y^2 + e^2 B^2 Y^2)} = -\frac{i}{2 \sin(|eB|s)}. \] (28b)

Substituting these into Eq. (27) we obtain
\[ \langle x | e^{is(P - eA)^2} | x \rangle = -\frac{i|eE| |eB|}{16\pi^2 \sinh(|eE|s) \sin(|eB|s)}. \] (29)

Combining the results (7), (15), (19), and (29) we arrive at
\[ W(x) = \frac{1}{4\pi^2} \int_0^\infty ds \frac{e^{-im^2 s}}{s} \left[ |eE| |eB| \coth(|eE|s) \cot(|eB|s) - \frac{1}{s^2} \right], \] (30)
where the \( s \)-dependent term in the exponential has been dropped as it is no longer necessary. This is obviously independent of \( x \) as expected. Using the identity \( \text{Re}W = (W + W^*)/2 \) and making the change of variable \( s \to -s \) in \( W \) we obtain
\[ w = -\frac{1}{8\pi^2i} \int_{-\infty}^{+\infty} ds \left( e^{i\pi/4} \frac{e^{im^2s}}{s} \right) \left[ eE|eB| \coth(|eE|s) \cot(|eB|s) - \frac{1}{s^2} \right]. \] (31)

It is easy to see that the integrand has singularities (simple poles) at 0 and ±nπ/|eB| (n = 1, 2, . . .) in the integration path. Thus the integral in the above equation is not well defined. An appropriate prescription should be employed. There are mainly three different prescriptions. The first is to replace the integration path by a straight line a bit above the real axis on the complex s plane, i.e., a straight line from \(-\infty + i\epsilon\) to \(+\infty + i\epsilon\) where \(\epsilon = 0^+\). The second is to replace the integration path by one from \(-\infty - i\epsilon\) to \(+\infty - i\epsilon\). The third is to keep the original path but take the Cauchy principal value. It turns out that only the first prescription gives a physically acceptable result. The second or the third will give a negative result for \(w\) when \(B = 0\), and thus are not acceptable. When \(B = 0\) the integrand has one singularity in the integration path, the simple pole at \(s = 0\). An appropriate prescription is also needed in this case. It is the first prescription described above that was implicitly used in Ref. [2]. The validity of the prescription was confirmed by the coincidence of the result with that of Schwinger obtained by a different method. We adopt this prescription, and close the contour of integration at infinity by a semicircle in the upper half plane. The integrand has simple poles at \(n\pi i/|eE|\) (n = 1, 2, . . .) in the upper half plane. The integral can be evaluated by using the residue theorem, and the result turns out to be

\[ w|| = \frac{e^2|EB|}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \coth\left( n\pi \frac{|B|}{|E|} \right) \exp\left( -\frac{n\pi m^2}{|eE|} \right). \] (32)

This is the result for the simple case \(B \parallel E\), as indicated by the subscript. The convergence of the series is obvious. It is easy the see that \(w = 0\) when \(E = 0\), which means that a pure magnetic field cannot lead to pair creation as expected. For \(B = 0\), on the other hand, we use \(\coth u \to 1/u\) \((u \to 0)\), and obtain

\[ w_E = \frac{e^2E^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left( -\frac{n\pi m^2}{|eE|} \right), \] (33)

where the subscript indicates a pure electric field. This is just Schwinger’s result. When \(|B| = |E|\), we have

\[ w_{|B|=|E|} = \frac{e^2E^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \coth(n\pi) \exp\left( -\frac{n\pi m^2}{|eE|} \right). \] (34)

Since \(\coth(n\pi) > 1\) and \(1/n \geq 1/n^2\), the series in Eq. (34) is obviously larger than that in Eq. (33), and the overall factor in Eq. (34) is also larger than the one in Eq. (33) by a factor \(\pi\), hence \(w_{|B|=|E|} > \pi w_E\). However, since \(|eE| \ll m^2\) as currently available, the dominant term in both Eqs. (33) and (34) is the first term, and thus \(w_{|B|=|E|} \approx \pi w_E\). If \(E\) increases, the result may be still larger. However, if \(E\) cannot be significantly raised, the addition of a magnetic field cannot raise the probability greatly.

We can now turn to the general case where both \(E\) and \(B\) have arbitrary magnitudes and directions. We can always find another frame of reference \(K'\) by a Lorentz boost where the transformed fields \(E'\) and \(B'\) satisfy \(B' \parallel E'\). In the system \(K'\) we can find the probability in terms of \(E'\) and \(B'\) by using Eq. (32). We know that the probability is a Lorentz scalar. This is because the total probability \(\int dx w(x)\) is a Lorentz scalar as it only involves a process of number counting, and the space-time volume \(dx\) is also invariant under Lorentz boost. Hence the probability in the original system can be obtained by using the relation between \((E, B)\) and \((E', B')\). Another approach to the result is also available. Since \(w(x)\) is a Lorentz scalar, it can only involve Lorentz invariants constructed from \(E\) and \(B\). In the special case \(B \parallel E\), it must reduces to the result (32). This enables us to obtain the general result more easily. We define the Lorentz invariants

\[ \mathcal{F} = E^2 - B^2, \quad \mathcal{G} = 2E \cdot B, \] (35a)

\[ \mathcal{E} = \left( \frac{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}}{2} \right)^{\frac{1}{2}}, \quad \mathcal{B} = \left( \frac{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}}{2} \right)^{\frac{1}{2}}, \] (35b)

then the result reads
\[
\sum_{n=1}^{\infty} \frac{1}{n \coth \left( \frac{n\pi B}{E} \right)} \exp \left( -\frac{n\pi|m|^2}{|e|E} \right).
\]

Now we can examine the result by another special case where \( \mathbf{B} \perp \mathbf{E} \). In this case we have \( \mathcal{G} = 0 \), and

\[
\mathcal{E} = \sqrt{\mathbf{E}^2 - \mathbf{B}^2}, \quad B = 0
\]

if \(|\mathbf{E}| > |\mathbf{B}|\), or

\[
\mathcal{E} = 0, \quad B = \sqrt{\mathbf{B}^2 - \mathbf{E}^2}
\]

if \(|\mathbf{E}| < |\mathbf{B}|\). The case (37a) is equivalent to that of a pure electric field and the result is given by Eq. (33) where \(|\mathbf{E}|\) is replaced by \( \mathcal{E} = \sqrt{\mathbf{E}^2 - \mathbf{B}^2} \). The case (37b) is equivalent to that of a pure magnetic field and \( w = 0 \). These are all expected results and further confirm the result (36).

III. 2+1 DIMENSIONS

In this section we calculate the probability of pair creation in vacuum in 2+1 dimensions. This cannot be trivially obtained from the result in 3+1 dimensions. In two spatial dimensions the magnetic field has only one component, while the electric field has two. There is nothing like \( \mathbf{E} \cdot \mathbf{B} \). The only Lorentz invariant constructed from \( \mathbf{E} \) and \( \mathbf{B} \) is \( \mathbf{E}^2 - \mathbf{B}^2 \). Consider an electromagnetic field \( \mathbf{E}, \mathbf{B} \) where both \( \mathbf{E} \) and \( \mathbf{B} \) are constant and uniform. There are two different situations to be distinguished. The first one is characterized by the inequality \(|\mathbf{E}| > |\mathbf{B}|\), while the second by \(|\mathbf{E}| < |\mathbf{B}|\). The first situation is equivalent to one with a pure electric field. One can calculate the result for the simple case with a pure electric field, and get the result for the general case by the method of Sec. II. This will be discussed in detail in the following. The second situation is equivalent to one with a pure magnetic field. Similar calculations give a vanishing probability in this case. This is an expected result and we will not discuss it in detail.

In Sec. II we have emphasized that Eqs. (7) and (11) are valid in lower dimensions. We will begin with these equations. Consider a pure electric field \( \mathbf{E} \) which is constant and uniform. Without loss of generality we choose

\[
\mathbf{E} = E\mathbf{e}_x,
\]

where \( E \) is a constant which may be positive or negative. We have then for \( F_{\mu\nu} \) the nonvanishing component \( F_{01} = E \), and

\[
\exp \left( -\frac{1}{2}ie\sigma^{\mu\nu}F_{\mu\nu} \right) = \exp(eE\gamma^0\gamma^1).
\]

As \((\gamma^0\gamma^1)^2 = 1\), and \( \text{tr}(\gamma^0\gamma^1) = 0 \), the trace of the above expression can be evaluated directly with the result

\[
\text{tr} \exp \left( -\frac{1}{2}ie\sigma^{\mu\nu}F_{\mu\nu} \right) = 2 \cosh(eEs).
\]

Note that in 2+1 dimensions the \( \gamma \) matrices are \( 2 \times 2 \) ones and thus \( \text{tr} 1 = 2 \). Combining Eqs. (40) and (11) we have

\[
\text{tr} \ln \left( \frac{P^2 - m^2 + ie}{\gamma \cdot (P - e\mathbf{A})^2 - m^2 + ie} \right) = 2 \int_{0}^{\infty} \frac{ds}{s} e^{-is(m^2 - e^2)} \left[ \cosh(es) \exp(is(P - e\mathbf{A})^2) - \exp(isP^2) \right].
\]

The next step is to evaluate the matrix elements \( \langle x | e^{is(P - e\mathbf{A})^2} | x \rangle \) and \( \langle x | e^{isP^2} | x \rangle \). The second one can be easily worked out with the result

\[
\langle x | e^{isP^2} | x \rangle = \frac{1 - ie}{4(2\pi)^2 s^2}.
\]

This is rather different from the corresponding result in 3+1 dimensions, but the calculation is similar. For the first one, we denote

\[
1 - \frac{ie}{4(2\pi)^2 s^2}.
\]
\[ X^\mu = (X_0, X, Y), \quad P^\mu = (P_0, P_x, P_y), \] (43)

and choose
\[ A^\mu = (A_0, A_x, A_y) = (-EX, 0, 0), \] (44)

which corresponds to the field strength (38), and results in
\[ (P - eA)^2 = (P_0 + eEX)^2 - P_x^2 - P_y^2. \] (45)

Using Eq. (23a) we have
\[ e^{is(P - eA)^2} = e^{-iP_0P_x/eE^2} e^{-is(P_x^2/e^2 + x^2)} e^{-iP_0P_y/eE} e^{-isP_y^2}. \] (46)

The subsequent calculations are similar to those carried out in obtaining Eq. (27) but simpler, the result reads
\[ \langle x | e^{is(P - eA)^2} | x \rangle = \frac{(1 - i)|eE|}{2(2\pi)^{\frac{3}{2}} \sqrt{s}} \text{tr} e^{-is(P_x^2/e^2 + x^2)}. \] (47)

Using Eq. (28a) we obtain
\[ \langle x | e^{is(P - eA)^2} | x \rangle = \frac{(1 - i)|eE|}{4(2\pi)^{\frac{3}{2}} \sqrt{s} \sinh(|eE|s)}. \] (48)

Combining Eqs. (7), (41), (42), and (48) we arrive at
\[ W(x) = \frac{1 - i}{2(2\pi)^{\frac{3}{2}}} \int_0^\infty ds \frac{e^{-im^2s}}{s^{\frac{3}{2}}} [ |eE| \coth(|eE|s) - \frac{1}{s} ]. \] (49)

If we are going to use the residue theorem for contour integrals, we must treat the integrand carefully since it is a multivalued function in the complex \( s \) plane. We cut the \( s \) plane from 0 to \(-i\infty\) along the imaginary axis, and define \( s = 0 \) in the positive real axis, then the integrand is single valued in the cut plane. We use the identity \( \text{Re}W = (W + W^*)/2 \) and making the change of variable \( s \to s' = e^{i\pi}s \) in \( W \) to yield
\[ w = \frac{1 + i}{4(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} ds' \frac{e^{im^2s'}}{s'^{\frac{3}{2}}} [ |eE| \coth(|eE|s') - \frac{1}{s'} ], \] (50)

Now the integrand has one singularity in the integration path, the origin \( s = 0 \). This is a branch point of the integrand. Moreover, the integrand tends to infinity like \( 1/\sqrt{s} \) when \( s \to 0 \). As the lower half plane has been cut, we have now two different prescriptions for the integral: to replace the integration path by one from \(-\infty + i\epsilon \) to \(+\infty + i\epsilon \), or to take the Cauchy principal value. It turns out that the two prescriptions give the same result. We adopt the first prescription, close the contour of integration at infinity by a semicircle in the upper half plane and use the residue theorem to evaluate the integral. Note that there are simple poles \( n\pi e^{i\pi/2}/|eE| \) \( (n = 1, 2, \ldots) \) of the integrand in the upper half plane. The result turns out to be
\[ w_E = \frac{|eE|^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n\pi} \exp \left( - \frac{n\pi m^2}{|eE|} \right), \] (51)

where the subscript indicates a pure electric field.

The case of a pure magnetic field \( B \) can be treated in a similar way. It turns out that the probability vanishes as expected.

To conclude this section we consider the general case of an electromagnetic field \((\mathbf{E}, \mathbf{B})\) where \( \mathbf{E} \) has an arbitrary direction. As pointed out at the beginning of this section, one must distinguish between the two different cases \(|\mathbf{E}| > |\mathbf{B}|\) and \(|\mathbf{E}| < |\mathbf{B}|\). The results can be easily obtained by the method of Sec. II. When \(|\mathbf{E}| > |\mathbf{B}|\) we have
\[ w = \frac{|e|^{\frac{3}{2}} \mathbf{E}^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n\pi} \exp \left( - \frac{n\pi m^2}{|e|E} \right), \] (52a)
where
\[ E = \sqrt{E^2 - B^2}. \] (52b)

When \(|E| < |B|\) we have a vanishing result, since the case is equivalent to one with a pure magnetic field.

**IV. 1+1 DIMENSIONS**

In 1+1 dimensions there is no magnetic field, and the electric field has only one component. Thus the problem is still simpler. As before, we begin with Eqs. (7) and (11). Consider the electric field
\[ E = E e_x, \] (53)
where \(E\) is a constant which may be positive or negative. The nonvanishing component of \(F_{\mu\nu}\) is \(F_{01} = E\), and
\[ \exp \left( -\frac{i}{2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} \right) = \exp(e E s \gamma^0 \gamma^1). \] (54)
This is exactly the same as Eq. (39). In 1+1 dimensions the \(\gamma\) matrices are also 2×2 ones, thus Eq. (40) remains valid. With Eqs. (40) and (11), we have a result of the same form as Eq. (41) where \(P^2\), say, is of course different in different dimensions. It is easy to show that
\[ \langle x | e^{is P^2} | x \rangle = \frac{1}{4\pi s}. \] (55)
As before, we denote
\[ X^\mu = (X_0, X_1), \quad P^\mu = (P_0, P_1), \] (56)
and choose
\[ A^\mu = (A_0, A_1) = (-EX, 0), \] (57)
which corresponds to the field strength (53), and results in
\[ (P - eA)^2 = (P_0 + eEX)^2 - P_1^2. \] (58)

Using Eq. (23a) we obtain
\[ e^{i\epsilon (P - eA)^2} = e^{ip_0p_1/eE} e^{-is(P_0^2 - e^2 E^2 X^2)} e^{-ip_0p_1/eE}. \] (59)

It is now quite easy to show that
\[ \langle x | e^{i\epsilon (P - eA)^2} | x \rangle = \frac{|eE|}{2\pi} \text{tr} e^{-is(P_0^2 - e^2 E^2 X^2)}. \] (60)

Using Eq. (28a) we have
\[ \langle x | e^{i\epsilon (P - eA)^2} | x \rangle = \frac{|eE|}{4\pi \sinh(|eE|s)}. \] (61)

Combining Eqs. (7), (41), (55), and (61) we arrive at
\[ W(x) = \frac{1}{2\pi} \int_0^\infty ds \frac{e^{-im^2 s}}{s} \left[ |eE| \coth(|eE|s) - \frac{1}{s} \right]. \] (62)

Using the identity \(\text{Re}W = (W + W^*)/2\) and making the change of variable \(s \to -s\) in \(W\) we obtain
\[ w = \frac{1}{4\pi} \int_{-\infty}^{+\infty} ds \frac{e^{im^2 s}}{s} \left[ |eE| \coth(|eE|s) - \frac{1}{s} \right]. \] (63)
The integrand is regular everywhere in the integration path, thus no prescription is necessary here. We close the contour of integration at infinity by a semicircle in the upper half plane and evaluate the integral by the residue theorem. The result turns out to be

\[ w = \frac{|eE|}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp \left( -\frac{n\pi m^2}{|eE|} \right), \]  

(64)

The convergence of the series is obvious. This can also be written in a closed form

\[ w = -\frac{|eE|}{2\pi} \ln \left[ 1 - \exp \left( -\frac{\pi m^2}{|eE|} \right) \right]. \]  

(64’)

V. SUMMARY AND DISCUSSIONS

In this paper we calculate the probability of electron-positron pair creation in vacuum in external constant uniform electromagnetic fields in 3+1 and lower dimensions. The results are also applicable to other charged fermions of spin \( \frac{1}{2} \). In 3+1 dimensions the most general result is given by Eq. (36), where \( E \) and \( B \) may have arbitrary magnitudes and directions. The result for the relatively simple case where \( B \parallel E \) is given by Eq. (32). For a pure magnetic field the probability vanishes, which means that pair creation cannot occur in this case, as expected physically. For a pure electric field, our result reduces to that of Schwinger, Eq. (33). In 2+1 dimensions the magnetic field has only one component while the electric field has two. When \(|E| > |B|\) the result is given by Eq. (52). For the special case of a pure electric field this reduces to Eq. (51). When \(|E| < |B|\) the result vanishes since the situation is equivalent to one with a pure magnetic field. In 1+1 dimensions there is no magnetic field, and the electric field has only one component. The result is given by Eq. (64), or by Eq. (64’) in closed form.

For a pure electric field, the results in the several cases of different dimensions studied can be written in a unified form

\[ w_E = (1 + \delta_d) \frac{|eE|^{(d+1)/2}}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{1}{n^{(d+1)/2}} \exp \left( -\frac{n\pi m^2}{|eE|} \right), \]  

(65)

where \( d = 1, 2, 3 \) is the spatial dimension, and \( \delta_3 = 1 \) when \( d = 3 \) and vanishes in other cases. The additional factor 2 in 3+1 dimensions is due to the double spin states. More specifically, in 3+1 dimensions, with a given momentum there are two linearly independent solutions of positive energy and two of negative energy to the free Dirac equation, while in 2+1 or 1+1 dimensions there is only one solution for each energy.

Pair creation of charged scalar particles in vacuum in external electromagnetic fields can also be studied in a similar way. In this case, however, the probability is reduced by the presence of a magnetic field, thus we do not discuss the problem in detail.

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NOTE ADDED IN PROOF

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