INFORMATION OF STRUCTURES IN GALAXY DISTRIBUTION

FAN FANG

Spitzer Science Center, California Institute of Technology, Mail Stop 314-6, 1200 East California Boulevard, Pasadena, CA 91125

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ABSTRACT

We introduce an information-theoretic measure, the Rényi information, to describe the galaxy distribution in space. We discuss properties of the information measure and demonstrate its relationship with the probability distribution function and multifractal descriptions. Using the First Look Survey galaxy samples observed by the Infrared Array Camera on board the Spitzer Space Telescope, we present measurements of the Rényi information, as well as the counts-in-cells distribution and multifractal properties of galaxies in mid-infrared wavelengths. Guided by a multiplicative cascade simulation based on a binomial model, we verify our measurements and discuss the spatial selection effects on measuring information of the spatial structures. We derive structure scan functions at scales where selection effects are small for the Spitzer samples. We discuss the results and the potential of applying the Rényi information to the measurement of other spatial structures.

Subject headings: galaxies: clusters: general — large-scale structure of universe — methods: statistical

1. INTRODUCTION

The large-scale spatial distribution of galaxies is an important topic for modern cosmology. The cosmic structure as revealed by the observed galaxy spatial distribution is believed to originate from primordial density fluctuations. Gravitation amplifies these fluctuations and is the main driver for the formation and evolution of the cosmic structures. In the current popular scenario, galaxies form inside the previously collapsed “dark” gravitational wells in a process joined and modified by gas dynamics, radiative cooling, and photoionization. The coalescence of these dark halos brings galaxies together to merge in a hierarchical manner.

The large-scale distribution of the galaxies can be characterized by various statistical and topological methods. In particular, the two-point correlation function has been extensively used. It measures the second moment of the probability distribution and statistically completely describes a Gaussian density field, which is believed to represent the primordial density fluctuations. However, the density field smoothed over the observed galaxy spatial distribution is highly non-Gaussian. The evolved cosmic structure as probed by the galaxy distribution contains highly dense regions crowded by galaxies delineating spatial voids where few galaxies are located. We generally need the probability distribution function, or its moments, to completely characterize such a galaxy distribution in space. A counts-in-cells method has been used to establish the galaxy probability distribution function. Theory (Saslaw & Hamilton 1984; Saslaw & Fang 1996) and models (Carruthers & Minh Duong-Van 1983; Fry 1984, 1985) have been developed to interpret the probability distribution.

A multifractal description for galaxy spatial distribution has been studied both theoretically (Pietronero 1987; Jones et al. 1988; Borgani 1993) and numerically (Valdarnini et al. 1992) and applied to several galaxy samples (Martinez et al. 1990; Borgani et al. 1993). In particular, Borgani (1993) studied the multifractal behavior of various hierarchical probability distribution functions and derived the behavior of multifractal dimensions for extreme underdense and overdense regions. Indeed, the geometrical concepts of fractal and multifractal are appealing given the ubiquitous presence of such structures in various natural and social phenomena (Mandelbrot 1983). Less well perceived has been the statistical origin of multifractals as characterizing the moments of a probability distribution. (For a review of multifractal applications in large-scale structure, see Coleman & Pietronero [1992] and Borgani [1995].)

The purpose of this paper is to introduce Rényi information as a valid characterization of any spatial structure, including galaxy distribution. We show that Rényi information, being closely related to probability distribution and multifractal measures, probe the statistical moments sensitive to any levels of under- and over-dense spatial structures. At scales where the information contents are well preserved and can be accurately quantified, statistical moments are jointly described by Rényi information and dimensions, for which the underlying generator has a physical origin. We also illustrate the procedure by applying the Rényi information, along with the probability distribution and multifractal measures, to observed galaxy samples in the infrared wavelengths, as well as a simulation.

In the next section, we introduce Rényi information and their properties, the relations to the moments of the probability distribution function and to the multifractal measurements. In § 3 we present the results of the probability distribution and Rényi information for the infrared samples observed by the Infrared Array Camera (IRAC) on board the Spitzer Space Telescope. We discuss a multiplicative cascade simulation in § 4, which provides means of validating our methods of measurement and shows the effects of spatial selections in our galaxy samples. We further derive the functions scanning the structure of the moments for the samples based on simulation results. We discuss our results and potential applications of the information measure in § 5.

2. RÉNYI INFORMATION, RÉNYI DIMENSIONS, AND STRUCTURE SCAN FUNCTIONS

Shannon & Weaver (1948) derived an information measure to describe the amount of information needed in order to know the occurrence of an event with a given probability. In an important development, Rényi (1970) expanded Shannon’s information measure to arbitrary orders. Suppose we have \( N_c \) cells placed to cover a distribution of \( N_g \) galaxies. This can either be a two-dimensional angular or three-dimensional spatial distribution. The probability \( p_k \) of finding a galaxy in a given cell \( k \) containing
$N_b$ galaxies is $p_k = N_b/N_g$. The Rényi information are defined as

$$I_\beta = \frac{1}{\beta - 1} \log \sum_{k=1}^{N_g} P_k^\beta,$$  \hspace{1cm} (1)

where $\beta$ is the information order, which in principle can be any real number (although in our application we consider integers only). At positive orders the overdense structures dominate the information estimate, whereas the underdense structures contribute the most to the information measure at negative orders. At $\beta = 1$, the Rényi information reduce to the Shannon information.

The summation term for the probabilities $p_k$ to order $\beta$ can also be written as

$$\sum_{k=1}^{N_g} P_k^\beta = \sum_{N_i=0}^{N_g} N_i \left( \frac{N_i}{N_g} \right)^\beta,$$  \hspace{1cm} (2)

where $f(N_i)$ is the galaxy probability distribution function. Therefore, the Rényi information of order $\beta$ is related to the $\beta$-moment of the probability distribution as

$$I_\beta = -\frac{1}{\beta - 1} \left[ \log_2 N_g + \log \sum_{N_i=1}^{N_g} N_i^\beta f(N_i) \right], \hspace{1cm} \beta \neq 1,$$  \hspace{1cm} (3)

$$I_\beta = \frac{N_e}{N_g} \sum_{N_i=1}^{N_g} N_i \log_2 \left( \frac{N_i}{N_g} \right) f(N_i), \hspace{1cm} \beta = 1,$$  \hspace{1cm} (4)

which is in turn related to the volume-averaged $\beta$-point correlation function (Peebles 1980). The relation is intuitively easy to understand as $\sum P_k^\beta$ is simply the total probability of finding $\beta$ galaxies in a cell. At positive orders of integral $\beta$ the Rényi information characterizes the amount of information corresponding to the event of finding $\beta$ galaxies in the cells covering the discrete galaxy distribution.

Some properties of the Rényi information indicate the behavior of the moments of the galaxy spatial distribution. It can be proved (Beck 1990) that

$$\left( \sum P_k^{\beta_1} \right)^{1/(\beta_1 - 1)} \leq \left( \sum P_k^{\beta_2} \right)^{1/(\beta_2 - 1)}, \hspace{1cm} \beta_1 < \beta_2,$$  \hspace{1cm} (5)

$$\left( \sum P_k^{\beta_1} \right)^{1/\beta_1} \geq \left( \sum P_k^{\beta_2} \right)^{1/\beta_2}, \hspace{1cm} \beta_1 < \beta_2 \text{ and } \beta_1 \beta_2 > 0.$$  \hspace{1cm} (6)

Taking the logarithm we get

$$I_{\beta_1} \leq I_{\beta_2}, \hspace{1cm} \beta_1 < \beta_2,$$  \hspace{1cm} (7)

$$\frac{\beta_1 - 1}{\beta_1} I_{\beta_1} \geq \frac{\beta_2 - 1}{\beta_2} I_{\beta_2}, \hspace{1cm} \beta_1 < \beta_2 \text{ and } \beta_1 \beta_2 > 0.$$  \hspace{1cm} (8)

Since $0 < p_k \leq 1$, we have $\sum P_k^\beta \leq \sum P_k = 1$ for $\beta > 1$ and $\sum P_k^\beta \geq \sum P_k = 1$ for $\beta < 1$. Therefore, there is an upper limit $I_\beta \leq 0$ for all $\beta$. We need zero information, or have perfect knowledge for an event, when $I_\beta = 0$. The bounds are also reflected by

$$I_{\beta_1} \leq \frac{\beta_2}{\beta_2 - 1} - \frac{\beta_1}{\beta_1 - 1} I_{\beta_1}, \hspace{1cm} 1 < \beta_1 < \beta_2 \text{ and } \beta_1 < \beta_2 < 0,$$  \hspace{1cm} (9)

$$I_{\beta_1} \geq \frac{\beta_2}{\beta_2 - 1} - \frac{\beta_1}{\beta_1 - 1} I_{\beta_1}, \hspace{1cm} 0 < \beta_1 < \beta_2 < 1.$$  \hspace{1cm} (10)

The Rényi information depend on the cell size $l$ and diverge as $l \rightarrow 0$. One property that remains finite at this limit is the so-called Rényi dimension:

$$D(\beta) = \lim_{l \rightarrow 0} \frac{I_\beta}{\log_2 l}.$$  \hspace{1cm} (11)

Any galaxy distribution becomes discontinuous at the scale of the typical galaxy separation. The above limit is not achieved in a discrete distribution or in practical measurements. A more practical definition for galaxy distribution is the “effective” Rényi dimension, for which we calculate the slope of $I_\beta$ versus $\log_2 l$. There is no reason a priori to expect the slope for a given order to be a constant over all scales for a given structure. In fact, this is not implied in equation (11) for a continuous multifractal distribution. We call it a simple multifractal if the effective Rényi dimension for any given order has a single slope across all scales.

Examining Rényi information and dimensions over information orders is identical to inspecting the structure of statistical moments of a distribution. Studying such scan functions has the advantage of summarizing an infinite amount of parameters (moments) in just a few relations for a statistical distribution. Here we relate Rényi dimensions to a scan function defined in a continuous multifractal field. Suppose $\langle \epsilon_k \rangle$ is the field density measured and ensemble-averaged at scale $l$. The function $K(\beta)$ is the scaling exponent for moments of the field (also called the structure function) $\langle \epsilon^\beta \rangle \propto l^{-K(\beta)}$. Now that the Rényi dimension is practically $D(\beta) = dl/d \log_2 l$, since $\langle \epsilon^\beta \rangle \propto \langle N^\beta \rangle l^{-D(\beta)} \propto \sum P_k^{\beta} l^{-D(\beta) + D}$, where $N$ is the counts in the cells of size $l$ and $D$ is the dimension of the space in which the distribution is embedded (e.g., $D = 2$ in our applications below), we obtain

$$K(\beta) = (\beta - 1)[D - D(\beta)].$$  \hspace{1cm} (12)

The function $K(\beta)$ is therefore also called the codimension (Schartzer & Lovejoy 1987). Here we use a general name, the structure scan function, for the Rényi information and dimensions as functions of $\beta$, as well as for functions such as $K(\beta)$.

The multifractals are usually defined by using the generalized correlation integral (Hentschel & Procaccia 1983; Grassberger & Procaccia 1983). Many measurements of the multifractal properties of galaxy distribution have been based on measuring the generalized correlation integral, which uses cells of varying sizes centered at selected galaxies. Such a procedure is not valid for estimating Rényi information, since neighboring cells are bound to cross each other above a certain scale. Below this scale there is a nonzero probability that some of the galaxies are not covered by the ensemble of cells. Either case changes the normalization for the probabilities, and the Rényi information are not accurately quantified for the original structure. This is further explained in § 4. We want to emphasize here
that not only the slope of the Rényi information versus scale (Rényi dimensions) but also the Rényi information itself is a physical measurement, both being open to physical interpretations. We further discuss this point in § 5.

We note that the differential form of the second-order correlation integral is called the conditional density, which had been used to characterize galaxy distribution in early surveys (Coleman et al. 1988; Lemson & Sanders 1991).

The Rényi dimensions basically show the scaling properties of Rényi information. Loosely speaking, a multifractal galaxy distribution has a position-dependent scaling exponent \( \alpha_n \) in \( p_k \sim k^{\alpha_n} \). It can be shown strictly (Schuster 1995) that the spectra of these scaling exponents \( f(\alpha_n) \) and the Rényi dimensions (multiplied by a factor of \( k \)) are related by a Legendre transformation,

\[
f(\alpha_n) = -\tau(\beta) + \beta \alpha_n,
\]

where \( \tau(\beta) \equiv (\beta - 1)D(\beta), d\tau/d\beta = \alpha_n \), and \( df/d\alpha_n = \beta \). A number of interesting properties of \( D(\beta) \) and \( f(\alpha_n) \) are discussed in Beck (1990), including \( D(\beta) \) being a decreasing function of \( \beta \) and bounded as \( \beta \to \pm \infty \), \( D(\pm \infty) = \alpha_{\text{min}} \), and \( D(-\infty) = \alpha_{\text{max}} \). These limits and the ways \( D(\beta) \) and \( f(\alpha_n) \) approach the limits show the properties of the moments of the spatial distribution from a scan function perspective.

3. MEASUREMENTS AND RESULTS

The IRAC instrument on board the Spitzer Space Telescope provides a fresh view into the cosmos in the mid-infrared wavelengths of 3.6, 4.5, 5.8, and 8.0 \( \mu \)m. The Spitzer First Look Survey (FLS) using IRAC provides uniform coverage of a 4 deg\(^2\) field centered at R.A. = \( 17^\text{h} 18^\text{m} \), decl. = \( 59^\circ 30^\prime \) with a total 60 s exposure time for each pixel in the 256 \( \times \) 256 arrays (Lacy et al. 2005). For our present purpose, we use the full galaxy samples established for an earlier two-point correlation analysis (Fang et al. 2004) across the IRAC wavelengths.

We divide the two-dimensional area covered by an IRAC sample into square cells of varying sizes. The cells are nonoverlapping and contiguous for the purpose of accurately estimating the Rényi information. The boundary of the sample area and the usage of a cell are determined by the mask files used to establish the galaxy sample (Fang et al. 2004). We always have >500 “good” cells at the largest scales of measurement to ensure good statistics. At smaller scales the cell numbers are much greater.

For each sample and each cell size we count the number of galaxies in the cells and establish histograms in Figure 1. The histograms represent the estimates of the probability distribution of galaxy counts.
function, from which the moments of the distribution can also be measured. For each histogram, we plot the fit of the theoretical gravitational quasi-equilibrium distribution function (GQED; Saslaw & Hamilton 1984; Saslaw & Fang 1996). The single fitting parameter \( b \), the average ratio of the gravitational correlation potential energy to twice the kinetic energy, is also shown in the plots. For comparison, we also draw the Poisson distributions with the same mean galaxy counts in cells of given sizes. Apparently, the galaxy distribution deviates more from the Poisson distribution at larger cell sizes, indicating the effect of galaxy clustering in IRAC wavelengths. The GQED, on the other hand, describes the distributions of IRAC galaxies remarkably well over all scales.

We follow the same procedures as in the counts-in-cells experiment to divide the IRAC sample areas into square cells to calculate the Rényi information. Figure 2 shows the relation between the cell sizes and the measured Rényi information at orders from 1 to 20. The Rényi information scale with the cell sizes, but the relation is not linear for our galaxy samples. We discuss below the effects that can potentially change the scaling relation. The apparent crowding of the curves at high information orders implies an upper limit, which is 0 based on the above discussion, for the information measures. The limit constrains the behavior of the moments of galaxy distribution in the information space.

Intuitively, exclusion of structures, such as galaxies not covered by the cells or regions that cells avoid due to masking, changes the information content of the structure. Although we intend to cover the sample using contiguous cells, changing the cell size causes some galaxies in the sample not to be covered by cells of a new size due to the sample boundary and masked areas. Until these effects can be fully accounted for, we are in fact measuring the information of slightly different structures at each scale, even though the probability is normalized by the total number of galaxies covered by cells. This introduces noise in the information measurements. In the next section, we study these effects using a simulation of a known structure.

4. SIMULATION AND FURTHER RESULTS

To verify our results, we generate a multiplicative cascade simulation based on a binomial model. The binomial model was found to describe well the multifractal scaling in the dissipation field of fully developed turbulence (Meneveau & Sreenivasan 1987). We use the binomial model for its simplicity and analytically derivable relations for Rényi information and multifractal properties. The multiplicative cascade method was formulated to study energy transfers at different scales in turbulence (Mandelbrot 1974; Frisch 1995). It is by far the most effective method for simulating a multifractal field.
We use a discrete multiplicative cascade simulation, consistent with our purpose to study counts at multiple length scales. The simulation aims to create distributions of counts at 10 different scales within a given area with a conserved overall number density $D_0$. At the first level, the area is divided into four quadrants of the same size, two of which hold a fraction $p/2$ of the total number, and the other two have the fraction $(1-p)/2$ (so the overall number density is conserved as $D_0$; also called the canonical process). There are many ways to distribute the two number densities equally among four cells. We choose to have a fixed pattern of distribution here. We tested different patterns, and the results are the same.

At the second level, each quadrant area is further divided into four identical (smaller) cells, with the distribution of the same probabilities of the same pattern. The number counts in a cell is the product of the probability assigned at this level multiplied by the probability (of the quadrant covering the cell) at the previous level (and by an arbitrary total source number, for which we choose to use 1).

We continue the process to generate smaller and smaller cells and their number counts. We stop at the 10th level, where we have data over 10 scales (of ratio 2) for statistics. At level $n$, a cell has a number count proportional to $(p/2)^k[(1-p)/2]^{n-k}$, where $k$ is an integer between 0 and $n$. Therefore, it is called the binomial model. The resulting structure, although modulated sharply by cell edges, is a simple multifractal. Based on Halsey et al. (1986) and Meneveau & Sreenivasan (1987), we derive the Rényi information, the Rényi dimensions, and the spectra of the multifractal scaling exponent for the two-dimensional binomial field as

$$I_\beta = n \frac{\log p}{\beta - 1} + \log_2 \left[ 2 \left( \frac{p}{2} \right)^\beta + (1-\frac{p}{2})^\beta \right], \quad (14)$$

$$D_\beta = 1 + \frac{1}{1 - \beta} \log_2 \left[ \left( \frac{p}{2} \right)^\beta + (1-\frac{p}{2})^\beta \right], \quad (15)$$

$$f(\alpha) = 1 + \log_2 \left[ \frac{\log_2 (1-p) - \log_2 p}{\log_2 (1-p) + \alpha} \right] + \frac{\log_2 \frac{p + \alpha}{\log_2 (1-p) - \log_2 p}}{\log_2 \frac{\log_2 \frac{p + \alpha}{\log_2 (1-p) - \log_2 p}}{\log_2 p + \alpha}}, \quad (16)$$

where $n$ is the level number in the cascade (e.g., the smallest scale being $n = 10$) and $\beta$ is the information order.

Figure 3 shows the scaling of Rényi information in the binomial field. The measurements, using the same methods and algorithm used for the infrared samples, are indicated by points in the figure. The lines are calculated based on equation (14). The agreement is nearly perfect for all orders.

Next we compare measurements of the Rényi dimensions and multifractal spectra with those predicted by equations (15) and (16). Since the scaling of the measured Rényi information in Figure 3 is well represented by lines, we use a linear least-squares fit for each information order in that figure to obtain the Rényi dimensions. We confirm from the fit that the information values at the smallest scale (where $\log r = 0$) and the slope are both within $10^{-3}\%$ of those predicted by equations (14) and (15). The slope values from the fit for each order are plotted as dots in the left panel of Figure 4. The line is the scan function for Rényi dimensions based on equation (15). To obtain the multifractal spectra, we use a cubic spline fit to model the measured Rényi dimensions and derive the $\alpha$- and $f(\alpha)$-values within the range of information orders. These values are again represented by dots in the right panel of Figure 4. The line in the figure is based on equation (16). In both panels the values based on the measured Rényi information agree well with the predicted values. This shows the reliability of the methods and algorithms we use.

In Figure 5 we show measurements of a generalized correlation integral superimposed on the plane of Rényi information versus scale for our simulation. We generate 1000 cell positions in the $1024 \times 1024$ field and vary the size of the cells centered at these positions. Cells can overlap, but they are ignored if they cross the field boundary. The generalized correlation integral is calculated using the remaining cells, following the standard algorithm (Martinez et al. 1990). In the figure the dashed lines show generalized correlation integral measurements, and the solid lines show the predicted Rényi information. It is clear that although the Rényi dimensions can be approximately maintained, the generalized correlation integral does not measure Rényi information. We also find that the values of the generalized correlation integral depend on the number of cells in the experiment, further strengthening this point.

To investigate the effects of spatial selection of structures, we include the mask files on which the IRAC samples are based. Each of the mask files is an image representation of the mask field with a dimension of $6200 \times 6600$ pixels. Since our simulation has a different dimension, we first project these masks onto a $1024 \times 1024$ field and vary the size of the cells centered at these positions. Cells can overlap, but they are ignored if they cross the field boundary. The generalized correlation integral is calculated using the remaining cells, following the standard algorithm (Martinez et al. 1990). In the figure the dashed lines show generalized correlation integral measurements, and the solid lines show the predicted Rényi information. It is clear that although the Rényi dimensions can be approximately maintained, the generalized correlation integral does not measure Rényi information. We also find that the values of the generalized correlation integral depend on the number of cells in the experiment, further strengthening this point.

The results, using the mask files for the four IRAC samples, are shown in Figure 6, where the lines show the predictions, superimposed on the measured data points connected by dotted lines. There is an obvious effect of spatial selection on measuring both Rényi information and dimensions. Notably, involving the IRAC masks introduces an apparent scale dependence of Rényi dimensions, particularly at greater scales, where both the Rényi information and the dimensions are higher than predicted. At smaller scales, there is a systematic offset to higher (negative) information values, although the slopes for the Rényi dimensions
are approximately maintained. Masking reduces the number of structures in the original binomial field, and a smaller amount of information (identical to the absolute value of the measured Rényi information) is needed to know whether an event occurs (such as \( \beta \) sources in a cell) with a given probability. As cell size increases, the chance of including a masked pixel in a given cell increases, and the cells covering the unmasked regions sample a smaller amount of the original structure. This confirms our intuition that Rényi information is an intrinsic property of a spatial structure. Any modifications of the structure modify its information content. Other derived properties such as the Rényi dimensions can also be affected if not measured properly. While the geometry and pattern of the four mask files vary, the effects on the Rényi information are remarkably similar.

Although it changes the information content of the original structure, it appears that the IRAC masks preserve the scaling of the information at smaller scales. The graininess of the galaxy distribution, however, introduces the Poisson limit for these smaller scales (remember that our simulation is not grainy), below which a cell contains either one or no galaxy in most of the regions. Any multifractal behavior breaks down at this limit. At smaller scales, the number of cells contributing to the Rényi information is roughly identical to the total number of galaxies, and the Rényi information reach a (lower) limit and also flatten out (see Fig. 8). Both of these effects at large and small scales can make the Rényi information curves of a multifractal become concave. This curvature is observed in Figure 2.

Based on the IRAC sample sizes and the unmasked areas for the samples, we estimate the mean separations of any two galaxies in the samples, which are roughly 20" for the IRAC channel 1 and 2 and 55" for the IRAC channel 3 and 4 samples, assuming uniform distributions. We use these as the lower scale limit for a reliable multifractal estimate. For the upper limit, Figure 6 implies a linear scale of \( \sim1\% \) of the field size, assuming that the scale ratio applies to the FLS field. This is only slightly higher than the lower limit of the IRAC channel 3 and 4 samples. Basically, the smaller number of galaxies in these samples combined with the amount of masking prevented us from reliably estimating the multifractal behavior for these two samples.

For the purpose of illustration, we perform a cubic spline fit to each of the Rényi information relations in Figure 2 and derive

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**Fig. 4.**—Rényi dimensions as a function of information order and the spectra of scaling exponents in the simulated binomial field. The dots show measurements, and the lines show predictions based on the binomial model.
Fig. 6.—Rényi information for orders –20 to 20 (bottom to top) as a function of scale in the simulated binomial field modified by FLS IRAC masks scaled to the size of the simulation field. The plus signs show measurements, connected by dotted lines. The solid lines show predictions as in Fig. 3.

Fig. 7.—Rényi dimensions as a function of information order and the spectra of scaling exponents, estimated at a scale of 4500 for IRAC channel 1 and 2 galaxies. Left: Results of a cubic spline fit for each curve in Fig. 2 for estimating the Rényi dimensions at different information orders. Right: Results of a second cubic spline fit used to estimate the scaling exponents and their spectra from the scan function.
where underdense structures dominate, the graininess of the galaxy distribution
the Rényi dimensions and the scan function for the IRAC channel 1 and 2 samples at a scale of 45\degree. We perform another cubic
spline fit to the scan functions (like we did for the binomial field) and obtain $\alpha$ and $f(\alpha)$ throughout the range of Rényi dimensions. In Figure 7 we show these relations for the two FLS samples. The figure illustrates how Rényi dimensions decrease with increasing information order and converge to a limit. In addition, $f(\alpha)$ appears to be a convex function of $\alpha$. Where $f(\alpha) = 0$, the $\alpha$-values represent the Rényi dimension limit when $\beta \to \infty$. All are typical behaviors of multifractals.

In Figure 8, we plot the Rényi information as a function of order, a different type of scan function, measured for orders $-20$ to $20$ at scales of 20\', 32\', 44\', 55\', and 68\' for all IRAC samples. At most of these scales the masking effect is small, where the information can be measured accurately for the galaxy distribution. For samples at IRAC channels 3 and 4, however, Poisson effects dominate the three smaller scales. This is told by the scan curves converging to the Poisson limit below or near information order 0. For all IRAC samples, the limit is shown by the scan curve behavior at negative information orders, where the information measure is sensitive to and dominated by underdense regions in the samples. At positive orders and at scales where information can be measured accurately, the scan curves tell of structures of high moments of the galaxy distribution.

5. DISCUSSION

We have shown that the Rényi information, the effective Rényi dimensions, their structure scan functions, and the multifractal spectra contain the properties of the high moments of a spatial distribution. These measurements can be used to scan properties of these high moments. These properties detect the amount of deviation from Gaussian densities and are highly constrained in the parameter space in these measurements.

Our experiments also show that spatial selection effects are important and can bias these measurements. Any selection modifies the original structure and the amount of Rényi information the structure contains. Depending on the amount of selection, the Rényi dimensions can be maintained over a limited range of scales above the Poisson limit for discrete distributions. One needs to conduct controlled experiments such as simulations to verify at these scales. For IRAC channel 1 and 2 samples, there is indication in Figure 2 that the information-scale relation is still not linear within the range. It is yet uncertain how much of this is caused by masking, as well as by approaching the Poisson limit, both effects leading systematically to a concave curve, or whether there is scale dependence for the Rényi dimensions in our IRAC samples, which would imply a more complex structure than a simple multifractal distribution at these scales.

Whether galaxy spatial distribution is a multifractal, or whether homogeneity can be reached at large scales, as cosmological principle states, has been observationally a controversial issue (Peebles 1993; Coleman & Pietronero 1992; Avnir et al. 1998; Martinez 1999). Our analyses show that caution needs to be exercised when extrapolating a multifractal structure to small and large scales, particularly if spatial selection exists for a galaxy sample, even if multifractality is observed at scales more reliable for multifractal measurements.

It may be possible to recover the lost information in a galaxy sample by “filling in” the masks based on known properties of galaxy distribution. Such known properties may come from minimally masked samples of galaxies of the same type or from $N$-body simulations, for example. Like the $\delta$-function for generating the probability in our multiplicative cascade simulation, there is a variety of statistical functions that can serve as the generating functions for simulating full-scale multifractal fields (Gupta & Waymire 1993). Among these generating functions, the log-Lévy distribution is of particular interest due to the unique position of the Lévy distribution in replacing a Gaussian in the generalized central-limit theorem where variances of the component distributions can be infinite and also due to its applications to a “universal class” of geophysical structures (Scherzer & Lovejoy 1987). The structure scan functions are uniquely determined by probability-generating functions that are of physical origin. The generating function would be a significant property to know if galaxy distribution is a multifractal to large scales.

Another way to seek a physical interpretation of the Rényi information is to use the moments of the probability distribution via equations (3) and (4), which are not restricted to a multifractal structure. Since $N_c \propto I^{-D}$, where $D$ is the dimension of the space in which the distribution is embedded, we can also derive from equation (3)

$$D_{\beta} = -\frac{D}{\beta - 1} + \frac{1}{\beta - 1} \frac{d}{d \log \beta} m_{\beta},$$

where $m_{\beta}$ is the $\beta$-moment of the probability distribution function. It is clear from the relation that we have a simple multifractal distribution across all scales only if $d \log m_{\beta}/d \log I$ is not a function of scale. This is not the case for QED, for example. On the other hand, any physically derived probability distribution can interpret the Rényi information and dimensions via these relations.

Independent of the multifractality of a structure, the Rényi information and dimensions are general characterizations of statistical properties of the structure. A simple multifractal is a special and very restrictive type of structure in its practical definition. The Rényi information and dimensions and their corresponding scan functions can describe any type of structure, whether or not they are multifractals. The Rényi information is extensive, whereas its scaling, or “information rate” with changing scales, is an intensive parameter. Both are important for a given structure. As we collect galaxy samples from surveys with greater area coverage and increasing depth, as well as in more wavelength channels,
we are collecting increasingly more information about the large-scale structure and the absolute values of the measured Rényi information increase at a given scale. Any variations of the Rényi dimensions, on the other hand, are of different origin.

For spatially confined structures, such as a giant molecular cloud, the extensivity of Rényi information also depends on resolution. A more resolved observation reveals more detailed structure and therefore more information content. While the Rényi information and dimensions can be identically applied to continuous and discrete spatial fields, it is important to recognize what properties are used for measurement. It is clear that we want to characterize the moments of a spatial structure, and that we can use spatial densities for measuring. An astronomical observation is usually a radiation measurement, however, and the proportionality between the two is only an assumption. For non-astronomical structures, the meaning of the measurements can be more clear-cut.

The Rényi information and dimensions can also be applied to one-dimensional time series. In the temporal domain, the amount of time delay serves as scaling, and the information content and rate describe the temporal structure built by distributions of the change of the observed properties over certain and different time spans. An information measure is a measure about the knowledge of a structure or system and therefore its predictability. It would be desirable to quantify the predictability of a statistical distribution or a time series using Rényi information and dimensions. So far research on this topic remains limited.

The relation between the Rényi information and dimensions measured in two dimensions and those in three-dimensional space for the same structure can be straightforward. The two-dimensional cells used to cover a structure can also be three-dimensional cells with the third dimension extended to cover the same structure. When properties such as spatial density can be accounted for by measurements when projecting the structure onto a two-dimensional area, the information is not lost. The only uncertainty is the correspondence between the two-dimensional and three-dimensional scales. What scales are measured by cells of nonidentical dimensions is, however, a generally interesting question. For galaxy spatial distribution, the evolutionary effects of galaxies in the third dimension need to be disentangled from projection before the structure can be analyzed in three dimensions.

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REFERENCES
Avnir, D., Biham, O., Lidar, D., & Malcai, O. 1998, Science, 279, 39
Beck, C. 1990, Physica D, 41, 67
Borgani, S. 1993, MNRAS, 260, 537
———. 1995, Phys. Rep., 251, 1
Borgani, S., Plionis, M., & Valdarnini, R. 1993, ApJ, 404, 21
Carruthers, P., & Minh Duong-Van. 1983, Phys. Lett. B, 131, 116
Coleman, P. H., & Pietronero, L. 1992, Phys. Rep., 213, 311
Coleman, P. H., & Sanders, R. H. 1988, A&A, 200, L32
Fang, F., et al. 2004, ApJS, 154, 35
Frisch, U. 1995, Turbulence: The Legacy of A. N. Kolmogorov (Cambridge: Cambridge Univ. Press)
Fry, J. N. 1984, ApJ, 277, L5
———. 1985, ApJ, 289, 10
Grassberger, P., & Procaccia, I. 1983, Phys. Rev. Lett., 50, 346
Gupta, V. K., & Waymire, E. C. 1993, J. Appl. Meteorol., 32, 251
Halsey, T. C., Jensen, M. H., Kadanoff, L. P., Procaccia, I., & Shraiman, B. I. 1986, Phys. Rev. A, 33, 1141
Hentschel, H. G. E., & Procaccia, I. 1983, Physica D, 8, 435
Jones, B. J. T., Martinez, V. J., Saar, E., & Einasto, J. 1988, ApJ, 332, L1
Lacy, M., et al. 2005, ApJS, 161, 41
Lemson, G., & Sanders, R. H. 1991, MNRAS, 252, 319
Mandelbrot, B. B. 1974, J. Fluid Mech., 62, 331
———. 1983, The Fractal Geometry of Nature (New York: Freeman)
Martinez, V. J. 1999, Science, 284, 445
Martinez, V. J., Jones, B. J. T., Domínguez-Tenreiro, R., & Van de Weygaert, R. 1990, ApJ, 357, 50
Meneveau, C., & Sreenivasan, K. R. 1987, Phys. Rev. Lett., 59, 1424
Peebles, P. J. E. 1980, The Large Scale Structure of the Universe (Princeton: Princeton Univ. Press)
———. 1993, Principles of Physical Cosmology (Princeton: Princeton Univ. Press)
Pietronero, L. 1987, Physica A, 144, 257
Rényi, A. 1970, Probability Theory (Amsterdam: North-Holland)
Saslaw, W. C., & Fang, F. 1996, ApJ, 460, 16
Saslaw, W. C., & Hamilton, A. J. S. 1984, ApJ, 276, 13
Schertzer, D., & Lovejoy, S. 1987, J. Geophys. Res., 92, 9693
Schuster, H. G. 1995, Deterministic Chaos (Weinheim: VCH)
Shannon, C., & Weaver, W. 1948, The Mathematical Theory of Communication (Urbana: Univ. Illinois Press)
Valdarnini, R., Borgani, S., & Provenzale, A. 1992, ApJ, 394, 422