Post-Processed Posteriors for Sparse Covariances and Its Application to Global Minimum Variance Portfolio

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Abstract

We consider Bayesian inference of sparse covariance matrices and propose a post-processed posterior. This method consists of two steps. In the first step, posterior samples are obtained from the conjugate inverse-Wishart posterior without considering the sparse structural assumption. The posterior samples are transformed in the second step to satisfy the sparse structural assumption through the hard-thresholding function. This non-traditional Bayesian procedure is justified by showing that the post-processed posterior attains the optimal minimax rates. We also investigate the application of the post-processed posterior to the estimation of the global minimum variance portfolio. We show that the post-processed posterior for the global minimum variance portfolio also attains the optimal minimax rate under the sparse covariance assumption. The advantages of the post-processed posterior for the global minimum variance portfolio are demonstrated by a simulation study and a real data analysis with S&P 400 data.

1 Introduction

Suppose \(X_1, \ldots, X_n\) are independent and identically distributed from a \(p\)-dimensional multivariate normal distribution with mean zero and a covariance matrix. We consider the estimation of the covariance matrix in this paper. When \(p\) is larger than \(n\), we refer to this situation as the high-dimensional settings, in which traditional covariance estimation methods such as the sample covariance estimator and the Bayesian inference by the inverse-Wishart prior, are not consistent. Marčenko and Pastur (1967) showed that the eigenvalues derived from the sample covariance are not consistent if \(p/n \rightarrow c \in (0, 1)\). For the Bayesian inference on the high-dimensional covariance, Lee and Lee (2018) showed that an obviously inappropriate
degenerate prior is a minimax optimal prior for the class of unconstrained covariances. Thus, structural assumptions on the covariance matrix are commonly employed for the consistent covariance estimation.

Among the structural assumptions on the covariance matrix, we focus on the sparse covariance assumption, which imply that a large portion of elements of the covariance is (near) zero. It is a natural assumption when quite a few pairs of random variables in a random vector $X_i$ are perceived to be marginally independent. Sparse covariance estimation methods were introduced in both frequentist and Bayesian perspectives. Bickel and Levina (2008a) proposed the thresholded sample covariance estimator, which is constructed by hard-thresholding elements of the sample covariance matrix, and Rothman et al. (2009) generalized the thresholded sample covariance estimator so that other thresholding transformations such as soft-thresholding and SCAD are covered. Cai et al. (2012) have shown the minimax optimality of the thresholded sample covariance estimator under a class of sparse covariances. A Bayesian method for sparse covariances is also proposed by Lee, Jo and Lee (2021). They proposed the Beta-mixture shrinkage prior distribution for the covariance matrix and showed the convergence rate of the posterior distribution. According to this posterior convergence rate, this Bayesian method is justified only when $n$ is smaller than $p$. Thus, a Bayesian method having theoretical support under high-dimensional settings is required.

We propose a post-processed posterior as a non-traditional Bayesian method for the class of sparse covariances. The procedure of post-processed posterior consists of two steps as follows. First, the initial posterior sample is generated from the initial posterior distribution, the conjugate inverse-Wishart distribution. The sparse covariance assumption is not considered in this step. Second, the initial posterior samples are transformed via the hard-thresholding function used in Cai et al. (2012). We call the post-processed posterior (PPP) the thresholding post-processed posterior (thresholding PPP) when we need to clarify the post-processing function. We justify this procedure by showing that the post-processed posterior has the minimax optimal convergence rate under the sparse covariance assumption.

The main feature of the post-processed posterior is that posterior samples are transformed to fit in the constrained parameter space. This idea has been employed in various constrained parameter spaces. For example, Dunson and Neelon (2003) and Lin and Dunson (2014) used this idea for the ordered finite dimensional space and the monotone continuous function space, respectively. Chakraborty and Ghosal (2020) also used this idea for the monotone measurable function space and showed the posterior consistency of their methods. This idea also was used for constrained covariance parameter spaces other than the sparse covariance structure in Lee et al. (2020) and Lee, Lee and Lee (2021). The present work expands the application of this idea to the sparse covariance space with the decision-theoretic support.

We apply the post-processed posterior to the estimation of the global minimum variance portfolio (GMVP). Suppose that there are $p$ assets and the returns of $p$ assets are expressed as a $p$-dimensional random vector $X$ with covariance $\Sigma_0$. We consider a portfolio as an asset allocation vector $w \in \mathbb{R}^p$ with $w^T \mathbf{1} = 1$, where $\mathbf{1}$ is the $p$-dimensional column vector of ones. The global minimum variance portfolio is
defined as the portfolio vector $w$ such that the expected risk of return is minimized. Given a portfolio $w$, the return is $w^T X$ and the expected risk is defined as $Var(w^T X) = w^T \Sigma_0 w$. Thus, the global minimum variance portfolio is $\arg\min_w w^T \Sigma_0 w$, which is given as

$$w_{GMVP}(\Sigma_0) := \frac{\Sigma_0 1}{1^T \Sigma_0 1},$$

(See Merton; 1972). Since the GMVP is derived from $\Sigma_0$, covariance estimation methods are commonly employed for the estimation of the global minimum variance portfolio.

The traditional estimator for the GMVP is the plug-in estimator with the sample covariance as

$$w_{GMVP}(S_n) = \frac{S_n 1}{1^T S_n 1},$$

where $S_n = n^{-1} \sum_{i=1}^n X_i X_i^T$ is the sample covariance matrix. Okhrin and Schmid (2006) also derived the sampling distribution of this plug-in estimator. However, since the sample covariance matrix is singular when $p$ is larger than $n$, this estimator is not feasible for the high-dimensional settings. To overcome the difficulty of high-dimensional settings, shrinkage type estimators were suggested. This type of estimators reduce the risk of estimators by allowing bias of estimators. For example, Frahm and Memmel (2010) and Bodnar et al. (2018) proposed the dominating estimator and Bona fide estimator, respectively, as the shrinkage type estimators.

There are also researches on the Bayesian inference on the global minimum variance portfolio. Once the posterior distribution on the covariance matrix is obtained, the posterior distribution on the GMVP is easily derived. The Bayesian inference on the covariance matrix with inverse-Wishart prior is commonly used for the GMVP and has been largely investigated and used, for instance, in Barry (1974), Klein and Bawa (1977) and Stambaugh (1997), to name a few. Instead of assigning a prior distribution on the covariance matrix, Bodnar et al. (2017) proposed a Bayesian method to assign prior distributions on the GMVP parameter $w_{GMVP}(\Sigma_0)$ directly. However, this method can not be used for the high-dimensional settings since the corresponding posterior distributions are not proper when $p$ is larger than $n$.

By applying the thresholding post-processed posterior to the GMVP, we derive the posterior distribution on the GMVP parameter. This posterior distribution has a minimax optimal convergence rate under the sparse covariance assumption. Since the result of minimax analysis is valid even in high-dimensional settings, the proposed method can be used when lots of assets are considered.

The main contributions of this paper are summarized as follows:

- We suggest a Bayesian method for high-dimensional sparse covariances with the optimal minimax rate. Note that the previous work (Lee, Jo and Lee; 2021) gives the convergence rate under the assumption $p < n$. 

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We suggest a Bayesian method for estimating the global minimum variance parameter under the high-dimensional sparse covariance assumption with the optimal minimax rate. This aspect of our work is new since there are no results on the minimax rate for the GMVP parameter under the high-dimensional settings.

Our method makes the interval estimation for the GMVP parameter under the high-dimensional sparse covariance assumption. Although Bodnar et al. (2017) showed the performance of the interval estimation for the GMVP parameter by the simulation study, they considered the case when \( p < n \) in the simulation study.

The rest of this paper is structured as follows. In Section 2, the algorithm of the thresholding post-processed posterior is given, and we show the minimax optimality of the method. In Section 3, the application of the thresholding post-processed posterior to the global minimum variance portfolio analysis is introduced with the result of minimax optimality. In Section 4, the thresholding post-processed posterior is demonstrated via a simulation study and data analysis of S&P 400 data. The proofs of theorems are given in the supplementary material.

2 Thresholding Post-Processed Posterior

2.1 Notation

For any positive sequences \( a_n \) and \( b_n \), we denote \( a_n = o(b_n) \) if \( a_n/b_n \to 0 \) as \( n \to \infty \), and \( a_n \lesssim b_n \) if there exists a constant \( C > 0 \) such that \( a_n \leq Cb_n \) for all sufficiently large \( n \). We denote \( a_n \asymp b_n \) if \( a_n \lesssim b_n \) and \( b_n \lesssim a_n \). For a given matrix \( A \in \mathbb{R}^{p \times p} \), \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) denote the minimum and maximum eigenvalues of \( A \) and \( [A]_i \) denote \( i^{th} \) row of \( A \). Let \( ||A|| = \{\lambda_{\max}(B^TB)\}^{1/2} \) be the spectral norm of \( A \in \mathbb{R}^{p \times p} \) and \( ||A||_q \) be the matrix \( q \)-norm defined as

\[
\arg\max_{x \neq 0} \frac{||Ax||_q}{||x||_q},
\]

where \( ||x||_q \) represents the vector \( q \)-norm, i.e. \( ||x||_q = \left( \sum_{i=1}^{p} |x_i|^q \right)^{1/q} \) for \( x = (x_1, x_2, \ldots, x_q) \). For any positive integer \( p \), let \( [p] = \{1, 2, \ldots, p\} \).

2.2 Algorithm of the post-processed posterior

Suppose \( X_1, \ldots, X_n \) are independent and identically distributed from the \( p \)-dimensional multivariate normal distribution with zero mean vector and covariance matrix \( \Sigma_0 \in \mathbb{R}^{p \times p}, N_p(0, \Sigma_0) \). We assume \( \Sigma_0 \) is an element of a class of sparse covariances \( \mathcal{G}_q(c_{n,p}, M_0, M_1) \) defined as

\[
\mathcal{G}_q(c_{n,p}, M_0, M_1) = \{ \Sigma = (\sigma_{ij}) \in \mathbb{C}_p : \sigma_{-j,j} \in B_{q}^{p-1}(c_{n,p}), \sigma_{jj} \leq M_0, 1 \leq j \leq p, \lambda_{\min}(\Sigma) > M_1 \},
\]
where $M_0, M_1$ and $c_{n,p}$ are positive real numbers, $C_p$ is the set of all $p \times p$-dimensional positive definite matrices, and

\[ B_q^{-1}(c) = \{ \xi \in \mathbb{R}^{p^{-1}} : |\xi|_{(k)}^{q} \leq ck^{-1}, \text{ for all } k = 1, \ldots, p \}, \]

where $|\xi|_{(k)}$ is the $k$th largest element in the absolute values of the elements of $\xi$. This class of sparse covariances is the same as the class in Cai et al. (2012) except the minimum eigenvalue condition.

We propose a post-processed posterior for the class of sparse covariances.

(a) (Initial posterior sampling step) We take the inverse-Wishart prior $\text{IW}_p(B_0, \nu_0)$ as the initial prior, whose density function is

\[ \pi_i(\Sigma) \propto |\Sigma|^{-\nu_0/2} e^{-\frac{1}{2}tr(\Sigma^{-1}B_0)}, \quad \Sigma \in C_p, \]

where $B_0 \in C_p$ and $\nu_0 > 2p$. The initial posterior distribution is then given as

\[ \Sigma | \mathcal{X}_n \sim \text{IW}_p(B_0 + nS_n, \nu_0 + n), \]

where $\mathcal{X}_n = (X_1, \ldots, X_n)^T$. Initial posterior samples $\Sigma^{(1)}, \ldots, \Sigma^{(N)}$ are generated from the initial posterior distribution.

(b) (Post-processing step) The initial posterior samples are transformed via the positive-definite adjusted thresholding function defined as

\[ H_{\gamma}(\Sigma) = \begin{cases} H_{\gamma}(\Sigma) + \left[ \epsilon_n - \lambda_{\min}\{H_{\gamma}(\Sigma)\} \right] I_p & \text{if } \lambda_{\min}\{H_{\gamma}(\Sigma)\} < \epsilon_n, \\ H_{\gamma}(\Sigma) & \text{otherwise}, \end{cases} \]

for positive constants $\gamma$ and $\epsilon_n$, where $H_{\gamma}(\Sigma)$ is the element-wise hard thresholding function as

\[ H_{\gamma}(\Sigma) = \left( \sigma_{ij} I\left( |\sigma_{ij}| \geq \gamma \sqrt{\frac{\log p}{n}} \right) \right), \]

where $\sigma_{ij}$ is the $(i,j)$ element of $\Sigma$.

2.3 Minimax analysis of the post-processed posterior

The post-processed posterior is justified in the decision-theoretic perspective. We use the extended P-risk framework, which was introduced by Lee et al. (2020). This framework is an extension of the P-risk framework in Lee and Lee (2018) to incorporate post-processed posteriors. When there is no confusion, we refer to the extended P-risk framework as the P-risk framework and review the extended P-risk framework in this section. In the framework, we consider the parameter space and the data as $\mathcal{G}_q$ and $\mathcal{X}_n$, respectively. The action of decision theory is post-processed posterior $\pi'_i(\mathcal{X}_n; f)$ for a post-processing function $f$ and an initial prior $\pi'_i$, and the decision rule is a pair of initial prior and post-processing function. Note that a decision rule is defined as a mapping from the data space to the action space, and
the post-processed posterior is obtained by combining a pair of initial prior and post-processing functions with data. The posterior-loss (P-loss) and posterior-risk (P-risk) are defined as

$$
\mathcal{L}(\Sigma_0, \pi^{pp}(\cdot | \mathcal{X}_n; f)) = E^{\pi^i}(||\Sigma_0 - f(\Sigma)||| \mathcal{X}_n),
$$

$$
\mathcal{R}(\Sigma_0, \pi^{pp}) = E_{\Sigma_0} \mathcal{L}(\Sigma_0, \pi^{pp}(\cdot | \mathcal{X}_n; f))
= E_{\Sigma_0} E^{\pi^*} (||\Sigma_0 - f(\Sigma)||| \mathcal{X}_n),
$$

where $E_{\Sigma_0}$ and $E^{\pi^*}(\cdot; \mathcal{X}_n)$ denote the expectations with respect to $\mathcal{X}_n$ and the initial posterior distribution, respectively.

Given the definition of P-risk, we define the P-risk minimax rate. For a real-valued sequence $r_n$, if

$$
\inf_{\Sigma_0} \sup_{\Sigma \in G} E_{\Sigma_0} E^{\pi^*}(||\Sigma_0 - f(\Sigma)||| \mathcal{X}_n) \asymp r_n,
$$

as $n \rightarrow \infty$, then $r_n$ is the minimax convergence rate. A pair of prior and post-processing function $(\pi^*, f^*)$ is said to attain the P-risk minimax rate $r_n$, if

$$
\sup_{\Sigma_0 \in G} E_{\Sigma_0} E^{\pi^*}(||\Sigma_0 - f^*(\Sigma)||| \mathcal{X}_n) \asymp r_n,
$$

as $n \rightarrow \infty$.

Using the extended P-risk framework, we show that the post-processed posterior has the optimal minimax rate. First, we show the upper bound of the convergence rate of the method under the spectral norm using Theorem 2.1. If we set $\epsilon_n$ such that $\epsilon_n^2 \lesssim c_{n,p}(\log p/n)^{(1-q)} + (\log p)/n$, then the convergence rate is $c_{n,p}(\log p/n)^{(1-q)} + (\log p)/n$, which is the same as the lower bound of the frequentist minimax risk (Cai et al.; 2012). Based on the second remark in Lee and Lee (2018), the minimax rate of P-risk is larger than or equal to the frequentist minimax rate, i.e. the lower bound of P-risk minimax rate is also $c_{n,p}(\log p/n)^{(1-q)} + (\log p)/n$. Thus this rate turns out to be the P-risk minimax rate. Thus, the post-processed posterior has minimax optimal convergence rate.

**Theorem 2.1.** Let the prior $\pi^i$ of $\Sigma$ be $\mathcal{I}W_p(A_n, \nu_n)$. If $(\nu_n - 2p) \vee ||A_n||_2 \vee \log p = o(n)$ and $\gamma$ is a sufficiently large constant, then, there exists a positive constant $C$ such that

$$
E_{\Sigma_0} E^{\pi^*}(||H^{(\epsilon_n)}(\Sigma) - \Sigma_0||_2^2 | \mathcal{X}_n) \leq C \left( c_{n,p}^2 \left( \frac{\log p}{n} \right)^{(1-q)} + \frac{\log p}{n} + \epsilon_n^2 \right),
$$

for all sufficiently large $n$.

The proof of Theorem 2.1 is given in the supplementary material.
3 Application to Bayesian Inference of the Global Minimum Variance Portfolio

We apply the thresholding post-processed posterior to the Bayesian inference on the global minimum variance portfolio. Suppose \( p \)-dimensional return vectors \( X_1, \ldots, X_n \) are generated from \( N_p(0, \Sigma_0) \) and assume \( \Sigma_0 \in G_q \). Given \( N \) samples of the post-processed posterior \( H_\gamma^{(\epsilon_n)}(\Sigma^{(1)}), \ldots, H_\gamma^{(\epsilon_n)}(\Sigma^{(N)}) \), we suggest the post-processed posterior on the GMVP parameter \( w_{\text{GMVP}}(\Sigma_0) \) as

\[
H_\gamma^{(\epsilon_n)}(\Sigma^{(1)}), \ldots, H_\gamma^{(\epsilon_n)}(\Sigma^{(N)}).
\]

We show that the post-processed posterior for the global minimum variance portfolio has the minimax optimal convergence rate. For the minimax analysis, we define a P-risk on the \( \mathbb{R}^p \), in which portfolio weights reside, as

\[
E_{\Sigma_0} \left( \mathbb{E}^\pi \left( \frac{||w_{\text{GMVP}}(\Sigma) - w_{\text{GMVP}}(\Sigma_0)||^2}{||w_{\text{GMVP}}(\Sigma)||^2} \ | \ X_n \right) \right).
\]

We use the loss function \( ||w_{\text{GMVP}}(\Sigma) - w_{\text{GMVP}}(\Sigma_0)||^2 / ||w_{\text{GMVP}}(\Sigma)||^2 \) instead of \( ||w_{\text{GMVP}}(\Sigma) - w_{\text{GMVP}}(\Sigma_0)||^2 \), since arguments on the consistency based on the loss function \( ||w_{\text{GMVP}}(\Sigma) - w_{\text{GMVP}}(\Sigma_0)||^2 \) are not rational when \( p \) goes to infinity. We briefly explain the problem here. We have

\[
||w_{\text{GMVP}}(\Sigma) - w_{\text{GMVP}}(\Sigma_0)||^2 \leq 2||w_{\text{GMVP}}(\Sigma)||^2 + 2||w_{\text{GMVP}}(\Sigma_0)||^2 \\
\leq 2||w_{\text{GMVP}}(\Sigma)|| \cdot ||w_{\text{GMVP}}(\Sigma)||_\infty + 2||w_{\text{GMVP}}(\Sigma_0)||^2 \\
\leq 2||w_{\text{GMVP}}(\Sigma)||_\infty + \frac{2||\Sigma_0||^2 \ ||\Sigma_0^{-1}||^2}{p}
\]

This inequality shows that if \( ||w_{\text{GMVP}}(\Sigma)||_\infty \) goes to zero, then \( w_{\text{GMVP}}(\Sigma) \) is a consistent estimator. However, even the trivial estimator, equal weight portfolio \( (1/p, \ldots, 1/p) \), satisfies the condition. Hence, this loss function makes most estimators consistent, which does not seem to be.

We show the convergence rate of the post-processed posterior for the global minimum variance portfolio and its minimax optimality with the P-risk. In Theorem 3.1, we show the upper bound of the convergence rate.

**Theorem 3.1.** Let the prior \( \pi^* \) of \( \Sigma \) be \( IW_p(A_n, \nu_n) \). If \( (\nu_n - 2p) \vee ||A_n||_2 \vee \log p = o(n) \), \( \epsilon_n < \lambda_{\min}(\Sigma_0) \), \( c_{n,p}(\log p)/n)^{(1-q)/2} + 4\gamma((\log p)/n)^{1/2} \to 0 \) as \( n \to \infty \) and \( \gamma \) is a sufficiently large constant, then there exists a positive constant \( C \) such that

\[
E_{\Sigma_0} \left( \mathbb{E}^\pi \left( \frac{||w_{\text{GMVP}}(\Sigma) - w_{\text{GMVP}}(\Sigma_0)||^2}{||w_{\text{GMVP}}(\Sigma)||^2} \ | \ X_n \right) \right) \leq C \left( \frac{1}{n} + c_{n,p} \left( \frac{\log p}{n} \right)^{1-q} \right)
\]

for all sufficiently large \( n \).
The proof of Theorem 3.1 is given in the supplementary material.

Next, we give the minimax lower bound of the P-risk using Assouad’s lemma (Lemma 2 in Cai et al.; 2012) and an extended Assouad’s lemma introduced in Lemma 3 of Cai et al. (2012). First, we apply Assouad’s lemma to space \(G^{(1)}\) defined as

\[
G^{(1)} = \{ \Sigma(\theta) : \Sigma(\theta)^{-1} = M^{-1}I_p + \sum_{m=1}^{p'} \theta_m \frac{\tau}{\sqrt{n}} I(i = j = m), \theta = (\theta_1, \theta_2, \ldots, \theta_{p'}) \in \{0,1\}^{p'} \},
\]

where \(p' = \lfloor p/2 \rfloor\) and \(M = M_0 + M_1\). Since \(G^{(1)} \subset G_q(c_n,p,M_0,M_1)\) for all sufficiently large \(n\) when \(\tau\) is a constant, Assouad’s lemma gives

\[
\sup_{\Sigma_0 \in G_q} 2^2 E_{\Sigma_0} \| w_{GMV}(\Sigma(\theta')) - w_{GMV}(\Sigma(\theta)) \|^2 \\
\geq \max_{\theta \in \{0,1\}^{p'}} 2^2 E_{\Sigma(\theta)} \| w_{GMV}(\Sigma(\theta')) - w_{GMV}(\Sigma(\theta)) \|^2 \\
\geq \min_{H(\theta,\theta') \geq 1} H(\theta,\theta') \| w_{GMV}(\Sigma(\theta')) - w_{GMV}(\Sigma(\theta)) \|^2 \frac{p'}{2} \min_{H(\theta,\theta') = 1} \| P_\theta \wedge P_{\theta'} \|.
\]

Using Lemma 3.2, we obtain the first minimax lower bound as

\[
\sup_{\Sigma_0 \in G_q} 2^2 E_{\Sigma_0} \| w_{GMV}(\Sigma(\theta')) - w_{GMV}(\Sigma(\theta)) \|^2 \geq \frac{C}{np},
\]

for a positive constant \(C\).

**Lemma 3.2.** If \(\tau/\sqrt{n} \leq M/3\) and \(\tau/M \leq 1/3\), then there exists a positive constant \(C\) such that

\[
\min_{H(\theta,\theta') \geq 1} \frac{\| w_{GMV}(\Sigma(\theta')) - w_{GMV}(\Sigma(\theta)) \|^2 \frac{p'}{2} \min_{H(\theta,\theta') = 1} \| P_\theta \wedge P_{\theta'} \| \geq \frac{C}{np},
\]

for all sufficiently large \(n\).

The proof of Lemma 3.2 is given in the supplementary material.

Next, we apply the extended Assouad’s lemma (Cai et al.; 2012, Lemma 3). Before using this lemma for the GMVP problem, we review this lemma here. Let \(B \subset \mathbb{R}^p \setminus \{0\}\) be a finite set and let \(\Lambda \subset B^r\) with a positive integer \(r\). Then, we define a finite set \(\Theta\) as

\[
\Theta = \{0,1\}^r \otimes \Lambda.
\]

We define projections \(\gamma : \Theta \mapsto \{0,1\}^r\) and \(\lambda : \Theta \mapsto \Lambda\) as \(\gamma(\theta_0) = \gamma_0\) and \(\lambda(\theta_0) = \lambda_0\), respectively, for \(\theta_0 = (\gamma_0,\lambda_0)\). \(\gamma_0 \in \{0,1\}^r\) and \(\lambda_0 \in \Lambda\). We also let \(\gamma_i(\theta_0)\) and \(\lambda_i(\theta_0)\) be \(i\)th element of \(\gamma_0\) and \(i\)th row-vector of \(\lambda_0\), respectively. The extended Assouad’s lemma (Lemma 3 in Cai et al. (2012)) gives

\[
\max_{\theta \in \Theta} 2^2 E_{\Sigma(\theta)} d^2(T,\psi(\Sigma(\theta))) \geq \min_{H(\gamma(\theta),\gamma(\theta')) \geq 1} \frac{d^2(\psi(\Sigma(\theta)),\psi(\Sigma(\theta')))}{H(\gamma(\theta),\gamma(\theta'))} \frac{r}{2} \min_{1 \leq i \leq r} \| \bar{P}_{i,0} \wedge \bar{P}_{i,1} \|,
\]

where

\[
\bar{P}_{i,j} = \frac{1}{2^{r-1} Card(\Lambda)} \sum_{\theta \in \Theta_{i,j}} P_\theta,
\]

\[
\Theta_{i,j} = \{ \theta \in \Theta : \gamma_i(\theta) = j \},
\]
for \( j = 0,1 \) and \( i = 1,2,\ldots, r \).

For the application of the extended Assouad’s lemma, the elements \( r, \Lambda \) and \( \psi(\Sigma(\theta)) \) need to be specified. We set \( r = \lfloor p/4 \rfloor \) and \( \psi(\Sigma(\theta)) = w_{GMVP}(\Sigma(\theta)) \). Given a \( p \)-dimensional vector \( \lambda_i \), let \( A_i(\lambda_i) \) be a \( p \times p \) symmetric matrix with \( i \)th row and column are equal to \( \lambda_i \) and zero for the other elements. We set \( \Lambda \) as

\[
\Lambda = \Lambda_k := \{ (\lambda_1, \lambda_2, \ldots, \lambda_r) \in B^r : (\lambda_i)_j = 0 \text{ or } 1, ||\lambda_i||_0 = k, ||(\lambda_i)_{1:(p-r)}||_1 = 0, \\
\quad \quad ||\sum_{i=1}^r A_i(\lambda_i)||_1 \vee ||\sum_{i=1}^r A_i(\lambda_i)^T||_1 \leq 2k, \text{ for } i \in [r], j \in [p] \}.
\]

Given the notations, we define a covariance space \( \mathcal{G}^{(2)} \) as

\[
\mathcal{G}^{(2)}(k, \epsilon_{n,p}) = \{ \Sigma(\theta) : \Sigma(\theta) = MI_p + \epsilon_{n,p} \sum_{i=1}^r \gamma_i(\theta) A_i(\lambda_i(\theta)), \theta = (\gamma, \lambda) \in \{0,1\}^r \otimes \Lambda_k \}. \tag{3}
\]

When \( k = \max(\lfloor c_{n,p} \epsilon_{n,p}^{-q} \rfloor, 0) \), \( c_{n,p} \leq C_0 n^{(1-q)/2}(\log p)^{-(3-q)/2} \) for a positive constant \( C_0 \) and \( \epsilon_{n,p} = (0.25(\log 2) \min(1, M_1) C_0^{-1})^{1/(1-q)}((\log p)/n)^{1/2} \), we have

\[
2k\epsilon_{n,p} \leq 0.5(\log 2) \min(1, M_1),
\]

thus, \( \mathcal{G}^{(2)}(k, \epsilon_{n,p}) \subset \mathcal{G}_q \). The extended Assouad’s lemma gives

\[
\sup_{\Sigma_0 \in \mathcal{G}_q} 2^2 E_{\Sigma_0} ||w_{GMV}(\Sigma(\theta')) - w_{GMV}(\Sigma(\theta))||^2 \\
\geq \min_{H(\gamma(\theta), \gamma(\theta')) \geq 1} \frac{||w_{GMV}(\Sigma(\theta')) - w_{GMV}(\Sigma(\theta))||^2}{2 \min_{1 \leq i \leq r} ||\bar{\Sigma}_{i,0} \wedge \bar{\Sigma}_{i,1}||}.
\]

Using Lemma 3.3, we obtain

\[
\sup_{\Sigma_0 \in \mathcal{G}_q} 2^2 E_{\Sigma_0} ||w_{GMV}(\Sigma(\theta')) - w_{GMV}(\Sigma(\theta))||^2 \geq \frac{C_{\epsilon_{n,p}^2}}{p} \left( \frac{\log p}{n} \right)^{1-q}, \tag{4}
\]

for a positive constant \( C \).

**Lemma 3.3.** Suppose the notation of \( \mathcal{G}^{(2)} \) in (3). Assume \( c_{n,p} \leq C_0 n^{(1-q)/2}(\log p)^{-(3-q)/2} \) for some positive constant \( C_0 \). There exists a positive constant \( C \) such that

\[
\min_{H(\gamma(\theta), \gamma(\theta')) \geq 1} \frac{||w_{GMV}(\Sigma(\theta)) - w_{GMV}(\Sigma(\theta'))||^2}{2 \min_{1 \leq i \leq r} ||\bar{\Sigma}_{i,0} \wedge \bar{\Sigma}_{i,1}||} \geq \frac{C_{\epsilon_{n,p}^2}}{p} \left( \frac{\log p}{n} \right)^{1-q},
\]

for all sufficiently large \( n \).

The proof of Lemma 3.3 is given in the supplementary material.

Using Lemmas 3.2 and 3.3, we obtain Theorem 3.4, the theorem of the minimax lower bound. The minimax lower bound is the same as the upper bound given in Theorem 3.1; thus, the post-processed posterior is minimax optimal in respect of the global minimum variance portfolio.
**Theorem 3.4.** Assume \( c_{n,p} \leq C_0 n^{(1-q)/2} (\log p)^{(3-q)/2} \) for some positive constant \( C_0 \). There exists a positive constant \( C \) such that

\[
\inf_{\hat{\Sigma}} \sup_{\Sigma \in \mathcal{G}_n} E_{\Sigma_0} \frac{||w_{GMV}(\hat{\Sigma}) - w_{GMV}(\Sigma_0)||^2}{||w_{GMV}(\Sigma_0)||^2} \geq C \left( \frac{1}{n} + c_{n,p}^2 \left( \frac{\log p}{n} \right)^{1-q} \right),
\]

for all sufficiently large \( n \).

**Proof.** Since

\[
||w_{GMV}(\Sigma_0)||^2 \leq \frac{||\Sigma_0^{-1}|| \ ||\Sigma_0||^2}{p},
\]

the proof is completed by collecting the first and second lower bounds in (2) and (4). \( \square \)

In the comparison of the minimax rate with that under the spectral norm (Theorem 2.1), while the convergence rate under the loss function \( p||w_{GMV}(\Sigma) - w_{GMV}(\Sigma_0)||^2 \) is \( c_{n,p}^2 (\log p)/n + 1/n \), the convergence rate under the spectral norm is \( c_{n,p}^2 ((\log p)/n)^{1-q} + (\log p)/n \). The term \( 1/n \) is replaced with \( (\log p)/n \) for the convergence rate of the global minimum variance portfolio.

### 4 Simulation Studies

#### 4.1 Simulation Study

In this section, we compare the post-processed posterior with the thresholded sample covariance (Cai et al.; 2012), the Bona fide estimator (Bodnar et al.; 2018), and the conventional Bayesian method by the inverse-Wishart prior and Beta mixture shrinkage prior (Lee, Jo and Lee; 2021). Let \( p = 100 \) and we consider two \( 100 \times 100 \) true covariances \( \Sigma_0^{(1)} \) and \( \Sigma_0^{(2)} \). We define \( \Sigma_0^{(1)} = (\sigma_{0,1}^{(1)}) + 0.1I_p \) with

\[
\sigma_{0,1}^{(1)} = \begin{cases} 
0.1, & 10k + 1 \leq i, j \leq 10k + 10, k \text{ is even numbers} \\
4, & 10k + 1 \leq i, j \leq 10k + 10, k \text{ is odd numbers} \\
0, & \text{otherwise}, 
\end{cases}
\]

and \( \Sigma_0^{(2)} = (\sigma_{0,1}^{(2)}) + 0.1I_p \) with

\[
\sigma_{0,1}^{(2)} = \begin{cases} 
0.25, & 10k + 1 \leq i, j \leq 10k + 10, k = 0, 5 \\
0.5, & 10k + 1 \leq i, j \leq 10k + 10, k = 1, 6 \\
1, & 10k + 1 \leq i, j \leq 10k + 10, k = 2, 7 \\
2, & 10k + 1 \leq i, j \leq 10k + 10, k = 3, 8 \\
4, & 10k + 1 \leq i, j \leq 10k + 10, k = 4, 9 \\
0, & \text{otherwise}, 
\end{cases}
\]
For each true covariance, the true global minimum variance portfolio is derived, and true covariances and the corresponding global minimum variance portfolios are visualized in Figure 1. We generate the data $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N_p(0, \Sigma_0^{(t)})$ for $t = 1$ and 2, and $n = 50, 500$ and 2000. For the Bayesian methods, we need to specify the hyperparameters and the number of posterior samples. When we use the inverse-Wishart prior for the inverse-Wishart method and the initial prior of PPP, we set the shape parameter $\nu_0 = 2p + 2$, the scale matrix as $\bar{s}_{ii}I_p$ where $\bar{s}_{ii} = p^{-1} \sum_{i=1}^{p} s_{ii}$ and $s_{ii}$ is the $i$th diagonal element of the sample covariance. We generate 2000 posterior sample for the inverse-Wishart posteriors. For the beta-mixture shrinkage prior, we set the hyperparameter as suggested in Lee, Jo and Lee (2021) and generate 3000 posterior sample including 1000 burn-in sample.

For the thresholded sample covariance and the post-processed posterior, tuning parameters of threshold parameter and positive-definite adjustment parameter need to be specified. We use the cross-validation idea for this purpose. We split the data into train and validation data and measure the prediction error using the validation data and the estimator based on the train data. Let $X^{train}$ and $X^{val}$ denote train data and validation data, respectively, and let $\hat{\Sigma}(X^{train})$ be the covariance estimator based on the training data. We measure the prediction error as $f(\hat{\Sigma}(X^{train}), X^{val})$ for a loss function $f$, which is chosen in accordance with the goal of data analysis. We present two examples of this loss function in this section.
If $\hat{\Sigma}$ is given as a set of posterior sample, $\Sigma^{(1)}, \ldots, \Sigma^{(N)}$, then we measure the prediction error as the posterior mean of prediction errors $N^{-1} \sum_{i=1}^{N} f(\Sigma^{(i)}, X^{val})$.

We conduct the comparison study in the respect of estimating covariance itself. For the tuning parameter selection, we set the prediction error function $f(\hat{\Sigma}(X^{train}), X^{val})$ as

$$||\hat{\Sigma}(X^{train}) - (X^{val})^T X^{val}/N_{val}||_2,$$

where $N_{val}$ is the number of observations in $X^{val}$. This error function was used for the covariance estimation in Bickel and Levina (2008b). For 50 sets of simulated data, we calculate the error of each covariance estimators as

$$\frac{1}{50} \sum_{s=1}^{50} ||\Sigma_0 - \hat{\Sigma}^{(s)}||_2 / ||\Sigma_0||_2,$$

where $\hat{\Sigma}^{(s)}$ is an point estimator based on the $s$-th simulated data set. For Bayesian methods, we use posterior mean as the point estimator. Table 1 gives the simulation errors. The thresholding post-processed posterior and the thresholded sample covariance have the smallest errors in all settings.

|                | $\Sigma_0^{(1)}$ |               | $\Sigma_0^{(2)}$ |               |
|----------------|------------------|---------------|------------------|---------------|
| $n = 50$       | 0.40             | 0.11          | 0.05             | 0.31          | 0.11          | 0.07          |
| $n = 500$      | 0.85             | 0.77          | 0.73             | 0.82          | 0.75          | 0.69          |
| $n = 2000$     | 0.55             | 0.17          | 0.06             | 0.40          | 0.13          | 0.07          |
| Thres          | 0.39             | 0.12          | 0.07             | 0.30          | 0.10          | 0.07          |
| Sample cov     | 0.55             | 0.17          | 0.09             | 0.40          | 0.13          | 0.07          |

Table 1: Errors of point estimators for the covariance under the spectral norm. PPP represents the our method. CGM and IW represent the Bayesian methods with the Beta-mixture shrinkage prior and the inverse-Wishart prior, respectively. Thres and Sample cov represent the frequentist methods with the thresholded sample covariance and the sample covariance, respectively.

Next, we conduct the comparison study in terms of estimation of the global minimum variance portfolio. For the tuning parameter selection, we set the prediction error function $f(\hat{\Sigma}(X^{train}), X^{val})$ as

$$\hat{\sigma}(w_{GMVP}(\hat{\Sigma}), X^{val}) := \hat{\text{Var}}\{w_{GMVP}(\hat{\Sigma})^T [X^{val}]_1, \ldots, w_{GMVP}(\hat{\Sigma})^T [X^{val}]_{N_{val}}\},$$

where $\hat{\text{Var}}A$ is the sample variance of set $A \subset \mathbb{R}$, and $[X^{val}]_i$ represents $i$th observation of $X^{val}$. This loss function is constructed based on the fact that the true GMVP is $\text{argmin}_w \text{Var}(w^T X)$. We obtain the GMVP estimators using the simulated data and visualize the result of estimation of the Bayesian methods in Figures 2 and 3. In these figures, posterior means and the 95% credible intervals are represented.
Figure 2: When the true covariance is \( \Sigma_0^{(1)} \), the posterior mean and the 95% credible intervals of elements in the global minimum variance portfolio are represented for the Bayesian methods: the post-processed posterior, conventional Bayesian methods with the Beta-mixture shrinkage prior and the inverse-Wishart prior.
Figure 3: When the true covariance is $\Sigma^{(2)}_0$, the posterior mean and the 95% credible intervals of elements in the global minimum variance portfolio are represented for the Bayesian methods: the post-processed posterior, conventional Bayesian methods with the Beta-mixture shrinkage prior, and the inverse-Wishart prior.
We repeat generating the simulation data 50 times and give the summarized performance in both point and interval estimation aspects. We measure the point estimation error of the GMVP as

$$\frac{1}{50} \sum_{s=1}^{50} ||w_{GMVP}(\Sigma_0) - \hat{w}_{GMVP}^{(s)}||_2/||w_{GMVP}(\Sigma_0)||_2,$$

where $\hat{w}_{GMVP}^{(s)}$ is the point estimator for the GMVP in the $s$-th simulated data set. The performances are summarised in Table 2. Table 2 shows that the thresholding post-processed method has the smallest errors in all settings.

|       | $\Sigma_0^{(1)}$ |       | $\Sigma_0^{(2)}$ |       |       |
|-------|-----------------|-------|-----------------|-------|-------|
|       | n = 50 | n = 500 | n = 2000 | n = 50 | n = 500 | n = 2000 |
| PPP   | 0.22    | 0.15    | 0.09    | 0.29    | 0.24    | 0.14    |
| CGM   | 1.06    | 0.28    | 0.22    | 2.14    | 0.38    | 0.40    |
| IW    | 1.95    | 1.47    | 0.72    | 3.88    | 2.69    | 1.30    |
| Thres | 0.30    | 0.18    | 0.12    | 0.38    | 0.27    | 0.16    |
| Bona fide | 2.01 | 1.51 | 0.72 | 2.91 | 2.37 | 1.26 |

Table 2: Errors of point estimators for the global minimum variance portfolio. PPP represents the our method. CGM and IW represent the Bayesian methods with the Beta-mixture shrinkage prior and the inverse-Wishart prior, respectively. Thres and Bona fide represent the frequentist methods with the thresholded sample covariance and the Bona fide estimator, respectively.

We also show the performances in the respect of interval estimation for the GMVP. Given the posterior sample $w_{GMV}^{(1)}, \ldots, w_{GMV}^{(N)}$, let $[l_i, u_i]$ denote 95% credible interval of $i$th asset weight. We measure the coverage probability as

$$100 \frac{1}{P} \sum_{i=1}^{P} I(w_{GMVP}(\Sigma_0)_i \in [l_i, u_i]).$$

Table 3 gives the average of coverage probabilities. The coverage probabilities by the thresholding post-processed posterior are nearer to the desired probability 95% than those by the other methods.

4.2 Real Data Analysis

We apply the GMVP estimators to S&P 400 data, consisting of the stock price of 400 mid-cap companies in the United States. We collect the data from State Street Global Advisors (2021) and the collection period is from 2nd May 2011 to 30th April 2021. We process this data to obtain a monthly return and discard data of stocks that have missing values. As a result of preprocessing, we obtain $120 \times 327$ data matrix $X$ in which the rows represent observations and the columns represent variables.
To compare the performances of the GMVP estimators, we extract the train and test data form the S&P 400 data as follows:

1. Sample an index $i$ from $\{48, 49, \ldots, 120 - 12\}$
2. Let $[X]_{(i-48+1):i}$ and $[X]_{(i+1):i+12}$ denote the train and test data, respectively, where $[X]_{j:k}$ is the submatrix of $X$ from $j$th row to $k$th row.

By setting the pair of train and test data, we use 4 years data for the next 1 year.

We repeat generating the pair of train and test data 20 times and let $X^{(\text{train}),i}$ and $X^{(\text{test}),i}$ be the train and test data in the $i$th iteration. Using these pairs of train and test data, we estimate the portfolio variance for the GMVP estimators as

$$\frac{1}{20} \sum_{i=1}^{20} 100 \sqrt{\text{Var}(\hat{w}(X^{(\text{train}),i}), X^{(\text{test}),i})},$$

which is represented in Table 4. Table 4 shows that the thresholded sample covariance and the post-

processed posterior give the smallest variance.

|          | $\Sigma_{0}^{(1)}$ | $\Sigma_{0}^{(2)}$ |
|----------|--------------------|--------------------|
|          | n = 50             | n = 2000           |
| PPP      | 95.8%              | 97.0%              |
| CGM      | 100%               | 99.9%              |
| IW       | 87.8%              | 83.1%              |

Table 3: Average coverage probabilities of the Bayesian methods. PPP represents the our method. CGM and IW represent the Bayesian methods with the Beta-mixture shrinkage prior and the inverse-Wishart prior, respectively.

|          | PPP | Thres | Bona fide | IW |
|----------|-----|-------|-----------|----|
| Var      | 3.50| 3.49  | 4.07      | 3.80|

Table 4: The portfolio vector is estimated with the train data, and the portfolio variance is estimated with the test data and the estimated portfolio. The average value of estimated portfolio variance is represented. PPP represents the our method. IW represent the Bayesian method with the inverse-Wishart prior. Thres and Bona fide represent the frequentist methods with the thresholded sample covariance and the Bona fide estimator, respectively.
5 Discussion

We have proposed a post-processed posterior for the Bayesian inference on sparse covariances and applied this method to estimate the global minimum variance portfolio. The main advantage of the method over the frequentist methods is on interval estimations for functionals of covariance. The simulation study shows that the interval estimators on the global minimum variance parameter have the frequentist property, attaining nominal coverage probability on the true value. We have also shown that the proposed method has the minimax optimal convergence rates for the covariance and the global minimum variance portfolio parameter.

The present work extends the post-processing method to the class of sparse covariances and the minimax analysis on the global minimum variance portfolio parameter. Since the minimax analysis is limited to the sparsity assumption on the covariance, we expect to conduct the minimax analysis to other covariance assumptions as to future works. For example, for the unconstrained covariance space, we can check the attainability of consistent estimation for the global minimum variance portfolio parameter as Lee and Lee (2018) did for the covariance itself.

References

Barry, C. B. (1974). Portfolio analysis under uncertain means, variances, and covariances, The Journal of Finance 29(2): 515–522.

Bickel, P. J. and Levina, E. (2008a). Covariance regularization by thresholding, The Annals of Statistics pp. 2577–2604.

Bickel, P. J. and Levina, E. (2008b). Regularized estimation of large covariance matrices, The Annals of Statistics pp. 199–227.

Bodnar, T., Mazur, S. and Okhrin, Y. (2017). Bayesian estimation of the global minimum variance portfolio, European Journal of Operational Research 256(1): 292–307.

Bodnar, T., Parolya, N. and Schmid, W. (2018). Estimation of the global minimum variance portfolio in high dimensions, European Journal of Operational Research 266(1): 371–390.

Cai, T. T., Zhou, H. H. et al. (2012). Optimal rates of convergence for sparse covariance matrix estimation, The Annals of Statistics 40(5): 2389–2420.

Chakraborty, M. and Ghosal, S. (2020). Convergence rates for Bayesian estimation and testing in monotone regression, arXiv preprint arXiv:2008.01244.
Dunson, D. B. and Neelon, B. (2003). Bayesian inference on order-constrained parameters in generalized linear models, *Biometrics* **59**(2): 286–295.

Frahm, G. and Memmel, C. (2010). Dominating estimators for minimum-variance portfolios, *Journal of Econometrics* **159**(2): 289–302.

Golub, G. H. and Van Loan, C. F. (1996). Matrix computations. johns hopkins studies in the mathematical sciences.

Klein, R. W. and Bawa, V. S. (1977). The effect of limited information and estimation risk on optimal portfolio diversification, *Journal of Financial Economics* **5**(1): 89–111.

Lee, K., Jo, S. and Lee, J. (2021). The beta-mixture shrinkage prior for sparse covariances with posterior minimax rates, *arXiv preprint arXiv:2101.04351*.

Lee, K. and Lee, J. (2018). Optimal Bayesian minimax rates for unconstrained large covariance matrices, *Bayesian Analysis* **13**(4): 1211–1229.

Lee, K., Lee, K. and Lee, J. (2020). Post-processed posteriors for banded covariances, *arXiv preprint arXiv:2011.12627*.

Lee, K., Lee, K. and Lee, J. (2021). Minimax analysis of the conditional mean linear operator under the bandable covariance structure, *unpublished manuscript*.

Lin, L. and Dunson, D. B. (2014). Bayesian monotone regression using gaussian process projection, *Biometrika* **101**(2): 303–317.

Marčenko, V. A. and Pastur, L. A. (1967). Distribution of eigenvalues for some sets of random matrices, *Mathematics of the USSR-Sbornik* **1**(4): 457.

Merton, R. C. (1972). An analytic derivation of the efficient portfolio frontier, *Journal of financial and quantitative analysis* **7**(4): 1851–1872.

Okhrin, Y. and Schmid, W. (2006). Distributional properties of portfolio weights, *Journal of econometrics* **134**(1): 235–256.

Press, S. J. (2012). *Applied multivariate analysis: using Bayesian and frequentist methods of inference*, Courier Corporation.

Rothman, A. J., Levina, E. and Zhu, J. (2009). Generalized thresholding of large covariance matrices, *Journal of the American Statistical Association* **104**(485): 177–186.
Stambaugh, R. F. (1997). Analyzing investments whose histories differ in length, *Journal of Financial Economics* 45(3): 285–331.

State Street Global Advisors (2021).

**URL:** https://www.ssga.com/. *Accessed May 1, 2021*