Real-valued gridless DOA estimation in massive ULA using a single snapshot

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Basis mismatch challenges the conventional direction-of-arrival estimation constrained by sparse representation, especially in the case of massive antennas limited to a single snapshot. In this letter, we develop a real-valued gridless direction-of-arrival estimation method to improve angular precision in the aforementioned circumstances. A new data model is first established through real-valued transformation and then estimate the large low-rank matrix with nuclear norm minimization. A fast iterative algorithm is designed for this underlying matrix, faithfully and efficiently by the alternating direction method of multipliers. Numerical examples validate the performance improvement of the proposed method in the massive uniform linear array. This work also shows the potential to apply in measured data of the millimeter-wave multiple-input multiple-output system.

Introduction: Inspired by the massive multiple-input multiple-output (MIMO) technology in wireless communications, radar systems are easier to provide better estimation performance, especially in millimeter-wave (mm-Wave) automotive radars [1–3]. Since the super-resolution imaging problem in automotive radars has evolved into a multi-parameter estimation problem, direction-of-arrival (DOA) estimation algorithms can be used to achieve high-precision angle measurement of targets. And with the rise of sparse signal representation (SSR) theory, scholars have proposed some state-of-the-art SSR-based DOA estimation algorithms [4–6] to reduce angular estimation error. Exploiting the incident signal is sparse in the spatial domain, algorithm based on l1-norm minimization [4] and its variants [5, 6] are proposed for DOA estimation in the case of limited snapshots. These algorithms perform well in terms of DOA estimation precision. However, all of the above SSR-based algorithms discretize the signal in the continuous parameter space, which inevitably leads to the basis mismatch problem. To overcome the basis mismatch issue, the gridless compressed sensing framework is proposed and gradually applied to DOA estimation in the single snapshot. It can be summarized as two kinds: atomic norm minimization (ANM) [7] and Hankel-based nuclear norm minimization [8–10]. Note that literature [8] first casts the signal into the Hankel matrix form and then proposes the enhanced matrix completion (EMaC) algorithm based on nuclear norm minimization. The superiority of these two algorithms compared with SSR-based algorithms is verified in [11] and its theoretical results prove the connection between ANM and EMaC. Nevertheless, it is time-consuming to extend the existing gridless algorithms to the massive antenna scenarios.

In this letter, we propose a real-valued gridless method to guarantee angular precision from single-snapshot observation. Moreover, the proposed method is suitable for massive ULA. The primary contributions in this letter are twofold. One is that we develop a fast iterative algorithm based on the alternating direction method of multipliers (ADMM) to faithfully and efficiently generate a large low-rank matrix. The other is that we provide sufficient measurement and simulation results to prove the superiority of the proposed method in the MIMO system and the massive ULA.

Hankel-based signal model: Consider a massive ULA consisting of N > 1 omnidirectional antennas, which receives signals from K narrowband far-field sources with unknown complex amplitude sk and distinct DOA θk, k = 1, ..., K, simultaneously. In the single snapshot case, the observed array data y = [y1, ..., yK]T ∈ CN with noise in the massive ULA is determined as

\[ y = x + e, \text{ with } x = A(θ)y \]  

where \( x = [x_1, ..., x_K] \in C^K \), \( s = [s_1, ..., s_K] \in C^K \) and \( e = [e_1, ..., e_K] \in C^K \) stand for the noiseless observed signal, the complex vector of source signals and the circularly symmetric Gaussian white noise vector drawn from \( CN(0, σ^2I) \), respectively. \( (·)^T \) denotes the transpose operator and \( A(θ) = [a(θ_1), ..., a(θ_K)] \in C^{K×N} \) is the array manifold matrix with the k-th steering vector \( a(θ_k) \).

Given y, x and e as in Equation (1), let us define the Hankel matrices \( h(y) \), \( h(x) \) and \( h(e) \)

\[ h(y) = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \\ y_2 & y_3 & \cdots & y_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m-1} & y_m & \cdots & y_{m+N-1} \end{bmatrix} \in C^{m×N}, \quad m + n − 1 = N \]  

where \( h(x) \) and \( h(e) \) are similar forms as \( h(y) \). Without loss of generality, let \( m = \lceil N/2 \rceil + 1 \) to maximize the number of acceptable sources. Then it is convenient to transform the primal model (1) into the model based on Hankel matrix [8]

\[ h(y) = h(x) + h(e) \]  

Suppose that \( K < \min(m, n) \), one can now see that \( h(x) \) is low-rank and \( \text{rank}(h(x)) = K \).

Algorithm analysis and design: Since the above Hankel-based signal model is all in the complex domain and a complex multiplication is actually the sum of four real multiplications, a new real-valued data model based on the Hankel matrix is established to improve noise robustness and reduce the computational complexity.

Concerning the definition in [5], the unitary transformation matrix \( U \) is expressed as

\[ U_{2l} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_l & J_l \\ J_l & -I_l \end{bmatrix} \in C^{2l×2l} \]  

and

\[ U_{2l+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{l+1} \cdot 0_{2l} & \sqrt{2} \cdot J_{l+1} \cdot 0_{2l} \\ J_{l+1} \cdot 0_{2l} & -I_{l+1} \cdot 0_{2l} \end{bmatrix} \in C^{2(l+1)×2(l+1)} \]  

where \( j = \sqrt{-1} \) and subscript \( l \) denotes the matrix dimension. \( I_l \) and \( J_l \) are l × l identity and exchange matrix with all ones on its anti-diagonal elements and zeros elsewhere respectively.

To simplify the description, let \( m \) be odd and \( n \) be even. First, we perform the unitary transformation on \( h(x) \) through different matrix \( U_m \) in Equation (5) and \( U_n \) in Equation (6)
This means that \( U_n h(x) U_n^T \) is still in the complex domain because of \( s_1, \ldots, s_K \in C \). Then, \( U_n h(x) U_n^T \) is natural to split into real part \( \text{real}(U_n h(x) U_n^T) \) and imaginary part \( \text{imag}(U_n h(x) U_n^T) \). Note that, we obtain a real-valued matrix by combining two parts,

\[
H(x) = [\text{real}(U_n h(x) U_n^T) \ \text{imag}(U_n h(x) U_n^T)] \in \mathbb{R}^{m \times 2n}
\]  

where this combination maintains the invariance of rank.

Finally, we convert Equation (4) into a new real-valued data model based on the above transform,

\[
H(y) = H(x) + H(e)
\]  

where \( H(y) \) and \( H(e) \) are similarly defined as \( H(x) \). In particular, since the real element in the imaginary or real part of \( H(e) \) is under the assumption of \( N(0, \sigma^2/2) \), any element of \( H(e) \) approximately obeys a zero-mean real Gaussian distribution with variance \( \sigma^2/2 \). Therefore, the proposed model is expected to enhance noise robustness, thereby improving estimation precision in all signal-to-noise ratio (SNR) regions.

In order to quickly search for a large low-rank matrix that best fits Equation (9), we make the following optimization inspired by noisy matrix completion \[12\]

\[
H(x) = \arg \min_{H(x)} \frac{1}{2} \| H(y) - H(x) \|_F^2 + \mu \| H(x) \|_*
\]  

\[
s.t. P_{\Omega}(H(x)) = P_{\Omega}(M)
\]  

where \( \mu > 0 \) denotes regularization parameter and \( \| \cdot \|_* \) represents the nuclear norm. The observation operator \( P_{\Omega} : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n} \) is defined as:

\[
P_{\Omega}(M)_{pq} = \begin{cases} 
M_{pq}, & (p, q) \in \Omega \\
0, & \text{otherwise}
\end{cases}
\]

where \( P_{\Omega}(M) \) is the subsampled estimated matrix from the noiseless matrix \( M \). This approach not only implements the constraints on the low-rank structure of the matrix, but also has advantages in robustness to noise.

To quickly solve the high-dimensional optimization problem in Equation (10), we design a fast iterative algorithm through ADMM \[13\], we introduce a new variable \( z \) and rearrange Equation (10) as:

\[
H(x) = \arg \min_{H(x)} \frac{1}{2} \| H(y) - H(x) \|_F^2 + \mu \| H(z) \|_*
\]  

\[
s.t. z = P_{\Omega}(H(x)) = P_{\Omega}(M)
\]

whose augmented Lagrangian can be denoted as:

\[
\mathcal{L}(H(x), H(z), \Pi) = \min_{H(x)} \frac{1}{2} \| H(y) - H(x) \|_F^2 + \mu \| H(z) \|_* + \| \Pi (H(x) - H(z)) \|_F^2
\]

\[
+ \rho \frac{1}{2} \| H(z) - H(x) \|_F^2
\]  

s.t. \( P_{\Omega}(H(x)) = P_{\Omega}(M) \)

where \( \rho \) is the penalty parameter and \( \Pi \) is the Lagrangian multiplier. Then the implementation of ADMM is derived by the following iterative updates

\[
H(x)_{i+1} = \arg \min_{H(x) \in \mathcal{L}(x, H(z), \Pi)} \mathcal{L}(H(x), H(z), \Pi)
\]  

\[
H(z)_{i+1} = \arg \min_{H(z) \in \mathcal{L}(x, H(z), \Pi)} \mathcal{L}(H(x), H(z), \Pi)
\]

\[
\Pi_{i+1} = \Pi_i + \rho (H(x)_{i+1} - H(z)_{i+1})
\]

The sub-problem (14) corresponding to \( H(x) \) is

\[
\min_{H(x)} \frac{1}{2} \| H(y) - H(x) \|_F^2 + \| \Pi (H(x) - H(z)) \|_F^2
\]

\[
+ \rho \frac{1}{2} \| H(z) - H(x) \|_F^2 \text{ s.t. } P_{\Omega}(H(x)) = P_{\Omega}(M)
\]

Note that Equation (17) can be equivalently converted into

\[
\min_{H(x)} \frac{1}{2} \| H(x) - H(y) + \rho H(z) - \Pi \|_F^2 \text{ s.t. } P_{\Omega}(H(x)) = P_{\Omega}(M)
\]

To execute Equation (18), we can solve it column by column

\[
\min_{\rho H(z)} \frac{1}{2} \| H(y) + \rho H(z) - (1 + \rho) \Pi \|_F^2 \text{ s.t. } P_{\Omega}(H(x)) = P_{\Omega}(M)
\]

where \( H(y) \) is the \( r \)-th column of \( H(x) \) and \( \Omega' \) is the index set corresponding to a known element of \( H(y) \)’s. The updates for \( H(x) \) can be obtained by linear solution

\[
H(x)_{i+1} = \frac{1}{2} (B^T B + \rho I)^{-1} \left\{ \left( H(y) + \rho H(z)_{i+1} - \Pi \right) + P_{\Omega}(M) \right\}
\]

where \( B = \sum_{n} \text{diag}(\Omega_n) \otimes E_n \) and \( E_n \) denotes the Kronecker products. Then we reformulate the terms of \( \mathcal{L}(H(x)_{i+1}, H(z), \Pi) \), it is worth noting that Equation (15) is equivalent to

\[
H(z)_{i+1} = \arg \min_{H(z) \in \mathcal{L}(x, H(z), \Pi)} \frac{1}{2} \| H(z) \|_* + \frac{1}{2} \| H(z) - \left( H(x)_{i+1} + \frac{\Pi}{\rho} \right) \|_F^2
\]

Using singular value thresholding operator \[14\], the updates for \( H(z) \) in the \((i+1)\)-th iteration is computed as

\[
H(z)_{i+1} = S_{\rho/\mu}(H(z)_{i+1} + \Pi/\rho)
\]

where \( S_{\cdot} (\cdot) \) is the matrix shrinkage operator

\[
S_{\Pi}(Y) = U_l \text{Diag}(\gamma/2) Y U_l^T + Y - U_l \text{Diag}(\gamma/2) Y U_l^T
\]

\[
\gamma = \begin{cases} 
\gamma - \mu, & \text{if } \gamma - \mu > 0 \\
0, & \text{otherwise}
\end{cases}
\]

When the primal and dual residuals are less than the preset threshold, the fast algorithm will stop iterating. Based on searching for spectral peaks, we acquire the final estimation of DOAs as:

\[
\theta_i = \arg \max_{\theta} P_{\theta}, \quad s.t. P_{\theta} = b(\theta) H(x) H(x)^T b(\theta),
\]

\[
b(\theta) = [\sqrt{2} \cos(\frac{\pi}{2} \theta_1) \ldots \sqrt{2} \sin(\frac{\pi}{2} \theta_K)]^T
\]

The overall implementation of the proposed algorithm for DOA estimation is summarized as Algorithm 1.

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**Algorithm 1: The Proposed Algorithm for DOA estimation**

**Input:** \( y, K \)

**Initialization:** set \( m, \rho, \Pi_0 \), max iteration \( I_{max} \), primal residual, and dual residual, let iteration index \( i = 0 \), obtain \( H(y) \) by Equation (9).

**Repeat Steps 1-4**

1. **Step 1:** Update \( H(x)_{i+1} \) by utilizing (20).
2. **Step 2:** Update \( H(z)_{i+1} \) by utilizing (22).
3. **Step 3:** Update \( \Pi_{i+1} \) by utilizing (16).
4. **Step 4:** \( i = i + 1 \).

Until primal and dual residuals are less than the preset threshold or \( i > I_{max} \).

**Output:** Estimated the DOAs by utilizing Equation (24).

**Simulation and verification:** This section illustrates the superiority of the proposed method with numerical examples. For the simulation scenario, we consider the massive ULA composed of \( N = 128 \), and its element space is half-wavelength. The observed uncorrelated signals are a mixture of complex sinusoids with the single snapshot and the directions of signals are \( \{2.98^\circ, 7.13^\circ, 24.65^\circ\} \). All the simulation results are performed on 100 Monte Carlo trials.

The first simulation experiment is shown in Figure 1, which plots the CPU time versus different SNRs for different algorithms. The results
demonstrate that the proposed method costs less running time than ANM, EMaC, and Real I1-svd [5]. Besides, the running time of the proposed algorithm is drastically reduced compared with EMaC by drawing support from real-valued transformation and fast iterative algorithm.

To further elaborate the performances of the proposed algorithms in terms of estimation accuracy, we choose the root mean square error (RMSE) as a measure to evaluate estimation precision and let the step size in search be 0.01°. Figure 2 depicts the resulting RMSEs versus different SNRs. It is apparent that the resulting RMSEs in the gridless methods are significantly lower than Real I1-svd. Further, since the real-valued operation effectively suppresses noise in all SNR regions, the proposed method performs somewhat better than EMaC and ANM.

Then we extend our work to the measured scenario in Figure 3, in which the 77 GHz mm-Wave MIMO array with 2-transmit and 4-receive antennas achieves the DOA estimation of a corner reflector. The distance between MIMO array and corner reflector satisfies the far-field condition. And it is convenient to transfer MIMO into 8-ULA array equivalently. Here, we set the step size to 1°. Figure 4 shows that the spectral peak of the proposed method is quite sharp, and sidelobe suppression is low compared with ANM and EMaC. It is indicated that the proposed method can be well implemented in the conventional MIMO systems, which provided the application feasibility for the massive MIMO field.

**Conclusion:** In this letter, we propose a real-valued gridless DOA estimation for massive ULA in a single snapshot case. To reduce the computational complexity that increases with antenna multiplication, we first use the real-valued transformation and then develop the fast iterative algorithm based on ADMM. The experimental results indicate that the proposed method can perform much better estimation precision than the traditional algorithms. For future work, it will be worthy to extend the measured results to the massive MIMO array with multiple targets.

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