A critical branching process with immigration in random environment

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Abstract
A Galton-Watson branching process with immigration evolving in a random environment is considered. Its associated random walk is assumed to be oscillating. We prove a functional limit theorem in which the process under consideration is normalized by a random coefficient depending on the random environment only. The distribution of the limiting process is described in terms of a strictly stable Levy process and a sequence of independent and identically distributed random variables which is independent of this process.

Keywords: Branching process in random environment, branching process with immigration, functional limit theorem

1. Introduction and statement of main result

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\Delta$ be the space of probability measures on $\mathbb{N}_0 := \{0, 1, \ldots\}$ equipped with the metric of total variation. A random environment is a sequence of random elements $Q_1, Q_2, \ldots$, mapping the space $(\Omega, \mathcal{F}, P)$ into $\Delta^2$. Thus, $Q_n$ for each $n \in \mathbb{N}$ has the form $(F_n, G_n)$, where $F_n, G_n$ are probability measures on $\mathbb{N}_0$. A branching process with immigration in random environment ((BPIRE)) is a stochastic process possessing the following properties. For a fixed random environment $\{Q_n, n \in \mathbb{N}\}$ this is an inhomogeneous branching Galton-Watson process with immigration (see [1], Chapter 6, §7). Here, for each $n \in \mathbb{N}$, the number of immigrants joining the $(n-1)$th gen-

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eration has the distribution $G_n$ and the offspring reproduction law of particles of the $(n - 1)$th generation is $F_n$.

Let $Z_n$ be the size of $n$th generation without the immigrants which joined this generation (we assume that $Z_0 = 0$), $\eta_n$ be the number of immigrants which joined the $n$th generation. Let $f_n(\cdot)$ and $g_n(\cdot)$ be generating functions of distributions $F_n$ and $G_n$ respectively.

We consider this model under the assumption that the random elements $Q_1, Q_2, \ldots$ are independent and identically distributed. A more detailed definition of the BPIRE can be found in [2].

Set for $i \in \mathbb{N}$

$$X_i = \ln f_i'(1), \quad \mu_i = g_i'(1)$$

(suppose that $0 < f_1'(1) < +\infty$, $0 < g_1'(1) < +\infty$ a.s.). Introduce the so-called associated random walk:

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \ n \in \mathbb{N}.$$  

It is clear that the random vectors $(X_1, \mu_1), (X_2, \mu_2), \ldots$ are independent and identically distributed under our assumptions.

We impose the following restriction on the distribution of $X_1$.

**Hypothesis A.** The distribution of $X_1$ belongs without centering to the domain of attraction of some stable law with index $\alpha \in (0, 2]$ and the limit law is not a one-sided stable law.

Under Hypothesis A the Skorokhod functional limit theorem is valid (see, for instance, [3], Chapter 16): there are such positive normalizing constants $C_n$ that, as $n \to \infty$,

$$W_n \xrightarrow{D} W,$$

where $W_n = \{C_n^{-1}S_{[nt]}, t \geq 0\}$, the process $W = \{W(t), t \geq 0\}$ is a strictly stable Levy process with index $\alpha \in (0, 2]$ and the symbol $\xrightarrow{D}$ means convergence in distribution in the space $D[0, +\infty)$ with Skorokhod topology. Moreover,

$$C_n = n^{1/\alpha} l(n),$$

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where \( \{ l(n), n \in \mathbb{N} \} \) is a slowly varying sequence. It is known that the finite-dimensional distributions of the process \( W \) are absolutely continuous. Note that \( \rho := \mathbb{P}(W(1) > 0) \in (0, 1) \) given Hypothesis A. Thus, the Spitzer-Doney condition is satisfied:

\[
\lim_{n \to \infty} \mathbb{P}(S_n > 0) = \rho \in (0, 1).
\]

The Spitzer-Doney condition means that the random walk \( \{ S_n \} \) is oscillating. As result, the absolute values of its strict descending ladder heights constitute a renewal process with the corresponding renewal function \( v(x), x \geq 0 \) (see [4] for a detailed definition of the function \( v(\cdot) \)). Similarly, weak ascending ladder heights of the random walk \( \{ S_n \} \) generate a renewal process with the corresponding renewal function \( u(x), x \geq 0 \).

The aim of this paper is to prove a functional limit theorem for the process \( \{ Z_{\lfloor nt \rfloor}, t \geq 0 \} \), as \( n \to \infty \) (see Theorem 1).

We need some notation and definitions to formulate the theorem. Let for \( n \in \mathbb{N} \)

\[
M_n = \max_{1 \leq i \leq n} S_i, \quad L_n = \min_{0 \leq i \leq n} S_i.
\]

It is known (see, for instance, [4], Lemma 2.5) that, if the Spitzer-Doney condition (2) is satisfied, then, as \( n \to \infty \),

\[
\{(Q_i, S_i, \mu_i), i \in \mathbb{N} \mid L_n \geq 0\} \overset{D}{\rightarrow} \{(Q_i^+, S_i^+, \mu_i^+), i \in \mathbb{N}\},
\]

\[
\{(Q_i, S_i, \mu_i), i \in \mathbb{N} \mid M_n < 0\} \overset{D}{\rightarrow} \{(Q_i^-, S_i^-, \mu_i^-), i \in \mathbb{N}\},
\]

where \( \{(Q_i^+, S_i^+, \mu_i^+)\}, \{(Q_i^-, S_i^-, \mu_i^-)\} \) are some random sequences. Moreover:

a) the sequences \( \{Q_i^+, i \in \mathbb{N}\}, \{Q_i^-, i \in \mathbb{N}\} \) can be viewed as some random environments; b) the sequences \( \{S_i^+, i \in \mathbb{N}\}, \{S_i^-, i \in \mathbb{N}\} \) are the corresponding associated random walks \( (S_0^+ = S_0^- = 0) \); c) the sequences \( \{\mu_i^+, i \in \mathbb{N}\} \) and \( \{\mu_i^-, i \in \mathbb{N}\} \) are positive and constructed by \( \{Q_i^+, i \in \mathbb{N}\} \) and \( \{Q_i^-, i \in \mathbb{N}\} \), respectively, the same as the sequence \( \{\mu_i, i \in \mathbb{N}\} \) is constructed by \( \{Q_i, i \in \mathbb{N}\} \).

Suppose that the sequences \( \{Q_i^+, i \in \mathbb{N}\}, \{Q_i^-, i \in \mathbb{N}\} \) are defined on the same probability space \( (\Omega^*, \mathcal{F}^*, \mathbb{P}^*) \) and are independent (below we denote the expectation on this probability space by \( \mathbb{E}^* \)).
We now come back to our initial BPIRE. Set \( N_i = \{ i, i+1, \ldots \} \) for \( i \in \mathbb{Z} \).

Fix \( i \in N_0 \) and, for \( n \in N_i \), denote by \( Z_{i,n} \) the total number of particles in the \( n \)th generation which are the descendants of the immigrants joined the \( i \)th generation (we assume that \( Z_{i,n} = 0 \) for \( i \geq n \) and \( i < 0 \)). Note that the random sequence \( \{ \eta_i; Z_{i,n}, n \in N_{i+1} \} \) is a usual (without immigration) branching process in the random environment \( \{ G_{i+1}; F_n, n \in N_{i+1} \} \). In particular, if the random environment is fixed, then \( G_{i+1} \) is the distribution of the random variable \( \eta_i \) which should be interpreted as the number of particles in the initial generation. Set for \( n \in N_i \)

\[
a_{i,n} = e^{-(S_n - S_{i})}.
\]

The sequence \( \{ \eta_i; a_{i,n}Z_{i,n}, n \in N_{i+1} \} \) is a nonnegative martingale if the random environment \( \{ G_{i+1}; F_n, n \in N_{i+1} \} \) is fixed. Hence (without assuming that the random environment is fixed), there is a finite limit \( \lim_{n \to \infty} a_{i,n}Z_{i,n} \) \( P \)-a.s.

Set

\[
Q_i^* = \begin{cases} 
Q_i^+, & i \in N, \\
Q_{-i+1}^-, & i \in \mathbb{Z} \setminus N,
\end{cases}
\]

\[
S_i^* = \begin{cases} 
S_i^+, & i \in N_0, \\
S_{-i}^-, & i \in \mathbb{Z} \setminus N_0,
\end{cases}
\]

\[
\mu_i^* = \begin{cases} 
\mu_i^+, & i \in N, \\
\mu_{-i+1}^-, & i \in \mathbb{Z} \setminus N.
\end{cases}
\]

The sequence \( \mathcal{E}^* := \{ Q_k^*, k \in \mathbb{Z} \} \) can be considered as a random environment (we denote the components of \( Q_k^* \) by \( G_k^* \) and \( F_k^* \)). We assume that the probability space \( (\Omega^*, \mathcal{F}^*, P^*) \) is reach enough for we are able to define on it a branching process with immigration in the random environment \( \mathcal{E}^* \). Fix \( i \in \mathbb{Z} \) and, for \( j \in N_i \), denote by \( Z_{i,j}^* \) the total number of particles in the \( j \)th generation being descendants of immigrants which joined the \( i \)th generation (we denote the number of such immigrants as \( \eta_i^* \)). Note that the sequence \( \{ \eta_i^*; Z_{i,j}^*, j \in N_{i+1} \} \) is a branching process in the random environment \( \{ G_{i+1}^*; F_j^*, j \in N_{i+1} \} \) with the initial value \( \eta_i^* \). The sequence \( \{ S_j^* - S_i^*, j \in N_i \} \) is the associated random
walk and the random variable $\mu^*_i$ is under fixed environment the mean of the random variable $\eta^*_i$. Set

$$a^*_{i,j} = e^{-(s^*_j - s^*_i)}.$$  

In accordance with the above the limit

$$\lim_{j \to \infty} a^*_{i,j} Z^*_{i,j} =: \zeta^*_i$$  

exists $P^*$-a.s. and $P^*$ ($\zeta^*_i > 0$) $> 0$ for $i \in \mathbb{N}_0$ (see [4], Proposition 3.1).

Introduce the following random series:

$$\Sigma_1 := \sum_{i \in \mathbb{Z}} \mu^*_{i+1} e^{-S^*_i}, \quad \Sigma_2 := \sum_{i \in \mathbb{Z}} \zeta^*_i e^{-S^*_i}.$$  

It is clear that $\Sigma_1 > 0$ $P^*$-a.s. and $P^*$ ($\Sigma_2 > 0$) $> 0$. Both series converge $P^*$-a.s. under certain restrictions (see Lemma 4).

Let $W$ be a strictly stable Levy process with index $\alpha$ (in the sequel we call $W$ simply the Levy process). By the Levy process we specify the (lower) level $L = \{L(t), t \geq 0\}$ of the Levy process as

$$L(t) = \inf_{s \in [0,t]} W(s).$$  

Let, further, $\gamma_1, \gamma_2, \ldots$ be an independent of $W$ sequence of independent random variables distributed as the random variable $\Sigma_2/\Sigma_1$.

By these ingredients we define finite-dimensional distributions of a random process $Y = \{Y(t), t \geq 0\}$ which plays an important role in the sequel. First we set $Y(0) = 0$. Consider an arbitrary $m \in \mathbb{N}$ and arbitrary moments $t_1, t_2, \ldots, t_m$: $0 = t_0 < t_1 < t_2 < \ldots < t_m$. The random vector $\{Y(t_1), \ldots, Y(t_m)\}$ coincides in distribution with the following vector $\hat{Y} := \{\hat{Y}_1, \ldots, \hat{Y}_m\}$. We describe at first the possible values of the vector $\hat{Y}$. Its first several coordinates coincide with $\gamma_1$, the next several coordinates coincide with $\gamma_2$ and so on up to the $m$th coordinate. The coordinates of the vector $\hat{Y}$ are specified according to the level $L$ of the Levy process $W$. The first coordinate $\hat{Y}_1$ is equal to $\gamma_1$. Let the coordinate $\hat{Y}_k$ for some $k < m$ be known. For instance, $\hat{Y}_k = \gamma_l$ for some $l \in \mathbb{N}$. If the level of the Levy process at the moment $t_{k+1}$
remains the same as at moment $t_k$, i.e. $L(t_{k+1}) = L(t_k)$, then $\hat{Y}_{k+1} = \gamma_l$. If the level of the Levy process at the moment $t_{k+1}$ is changed, i.e. $L(t_{k+1}) < L(t_k)$, then $\hat{Y}_{k+1} = \gamma_{l+1}$.

Set for $n \in \mathbb{N}_0$

$$a_n = e^{-S_n}, \quad b_n = \sum_{i=0}^{n-1} \mu_{i+1} e^{-S_i} \quad (b_0 = 0).$$

Introduce for each $n \in \mathbb{N}$ the random process $Y_n = \{Y_n(t), t \geq 0\}$, where

$$Y_n(0) = 0, \quad Y_n(t) = \frac{a_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} Z_{\lfloor nt \rfloor}.$$  

Note that for $k \in \mathbb{N}$ the ratio $b_k/a_k$ is equal to the mean of $Z_k$ for a fixed random environment.

Let the symbol $\Rightarrow$ means convergence of random processes in the sense of finite-dimensional distributions and $\ln^+ x = \max(0, \ln x)$ for $x > 0$.

**Theorem 1.** If Hypothesis A is valid and $E(\ln^+ \mu_1)^{\alpha + \varepsilon} < +\infty$ for some $\varepsilon > 0$, then, as $n \to \infty$,

$$Y_n \Rightarrow Y.$$  

A detailed description of the theory of critical (when Hypothesis A is valid) branching processes in random environment is available in [4] and [5].

A particular case of a subcritical BPIRE (when the offspring generating function $f_n(\cdot)$ is fractional-linear and $g_n(s) \equiv s$ for each $n \in \mathbb{N}$) was considered in [6]. The main attention there was paid to obtaining an exponential estimate for the tail distribution of the so-called life period of this process (i.e., the time until the first extinction). A more general class of subcritical BPIRE was analyzed in [7] where a limit theorem describing the population size at a distant moment was proved and an exponential estimate for the tail distribution of the life period was established. A strong law of large numbers and a central limit theorem for a wide class of subcritical BPIRE were proved in [8].

A critical BPIRE was considered in [9] where sufficient conditions of transience and recurrence were obtained. The author of [10], studying a random
walk in random environment, proved a particular case of Theorem 1 (when the
offspring generating function $f_n(\cdot)$ is fractional-linear and $g_n(s) \equiv s$ for each
$n \in \mathbb{N}$). We would like to stress that the proof used in the present paper differs
significantly from that one given in [10]. We also mention the papers [11], [12]
and [13] in which critical and supercritical processes (with stopped immigration)
are considered under some restrictions on their lifetime.

Recent papers [2] and [14] contain exact asymptotic formulae for the tail
distribution of the life period for critical and subcritical BPIRE.

2. Auxiliary statements

Let $\tau_n$ be the first moment when the minimum of the random walk $S_0, \ldots, S_n$
is attained:

$$\tau_n = \min \{i : S_i = L_n, 0 \leq i \leq n\}.$$

Set for $n \in \mathbb{N}$

$$S_{i,n}' = \begin{cases} S_{\tau_n+i} - S_{\tau_n}, & i \in \mathbb{N}_{(-\tau_n)}, \\ 0, & i \in \mathbb{Z} \setminus \mathbb{N}_{(-\tau_n)}. \end{cases}$$

For positive integers numbers $n_1 < n_2$ set

$$L_{n_1,n_2} = \min_{n_1 \leq i \leq n_2} S_i.$$

**Lemma 1.** If the Spitzer-Doney condition (2) is satisfied, then, as $n \to \infty$,

$$\{S_{i,n}', i \in \mathbb{Z}\} \overset{D}{\to} \{S_{i}^*, i \in \mathbb{Z}\}. \quad (6)$$

**Proof.** We demonstrate for simplicity only convergence of one-dimensional
distributions. Fix $i \in \mathbb{N}_0$. Let $A$ be a one-dimensional $S_i^*$-continuous (relative
to the measure $\mathbf{P}^*$) Borel set. Then for $n \geq i$

$$\mathbf{P} \left( S_{i,n}' \in A, \tau_n + i \leq n \right)$$

$$= \sum_{k=0}^{n-i} \mathbf{P} \left( S_{i,n}' \in A, \tau_n = k \right)$$
and by the Markov property of random walks we have that

\[
\begin{align*}
\mathbb{P}(S_{k+i} - S_k \in A, S_k < L_{k-1}, S_k \leq L_{k+1,n})
&= \sum_{k=0}^{n-i} \mathbb{P}(S_k < L_{k-1} \mid S_k \in A, S_k \leq L_{k+1,n}) \\
&= \mathbb{P}(S_k < L_{k-1}) \mathbb{P}(S_{k+i} - S_k \in A, S_k \leq L_{k+1,n}) \\
&= \mathbb{P}(S_k < L_{k-1}) \mathbb{P}(S_i \in A, L_n-k \geq 0) \\
&= \mathbb{P}(S_i \in A \mid L_n-k \geq 0) \mathbb{P}(S_k < L_{k-1}) \mathbb{P}(L_n-k \geq 0) \\
&= \mathbb{P}(S_i \in A \mid L_n-k \geq 0) \mathbb{P}(\tau_n = k).
\end{align*}
\]

Thus,

\[
\mathbb{P}(S_{i}^n \in A, \tau_n + i \leq n) = \sum_{k=0}^{n-i} \mathbb{P}(S_i \in A \mid L_n-k \geq 0) \mathbb{P}(\tau_n = k).
\]

If the Spitzer-Doney condition is satisfied, then the following generalized arcsine law is valid (see, for instance, [15], Chapter 8, Theorem 8.9.9): for \(x \in [0,1]\)

\[
\lim_{n \to \infty} \mathbb{P}\left(\frac{\tau_n}{n} \leq x\right) = \frac{\sin(\pi \rho)}{\pi} \int_{0}^{x} u^{\rho-1} (1-u)^{-\rho} \, du.
\]

(8)

We pass to the limit in formula (7), as \(n \to \infty\). Due to (8)

\[
\lim_{n \to \infty} \mathbb{P}(\tau_n + i \leq n) = \lim_{n \to \infty} \mathbb{P}(\tau_n/n \leq 1 - i/n) = 1.
\]

Therefore the limit of the left-hand side of (7) coincides with the limit of probability \(\mathbb{P}(S_{i}^n \in A)\), as \(n \to \infty\), if at least one of these limits exists.

If \(\varepsilon \in (0,1)\) and \(n\) is large enough, then by (7)

\[
\mathbb{P}(S_{i}^n \in A, \tau_n + i \leq n) = P_1(n, \varepsilon) + P_2(n, \varepsilon),
\]

(9)

where

\[
P_1(n, \varepsilon) = \sum_{k=0}^{\lfloor (1-\varepsilon)n \rfloor} \mathbb{P}(S_i \in A \mid L_n-k \geq 0) \mathbb{P}(\tau_n = k),
\]

\[
P_2(n, \varepsilon) = \sum_{k=\lfloor (1-\varepsilon)n \rfloor + 1}^{n-i} \mathbb{P}(S_i \in A \mid L_n-k \geq 0) \mathbb{P}(\tau_n = k).
\]
Clearly,

\[ P_2(n, \varepsilon) \leq \sum_{k=\lfloor (1-\varepsilon)n \rfloor +1}^{n} P(\tau_n = k) = P(\tau_n \geq \lfloor (1-\varepsilon)n \rfloor) \]

\[ \lim_{n \rightarrow \infty} 1 - \sin(\pi \rho) \int_{0}^{1-\varepsilon} u^{\rho-1} (1-u)^{-\rho} du \approx 0. \]

Therefore

\[ \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} P_2(n, \varepsilon) = 0. \] \hspace{1cm} (10)

In view of (3) the probability \( P(S_i \in A \mid L_n \geq k) \) tends, as \( n \rightarrow \infty \), to \( P^*(S_i^+ \in A) = P^*(S_i^* \in A) \) uniformly over \( 0 \leq k \leq \lfloor (1-\varepsilon)n \rfloor \). Consequently,

\[ \lim_{n \rightarrow \infty} P_1(n, \varepsilon) = \lim_{n \rightarrow \infty} \sum_{k=0}^{\lfloor (1-\varepsilon)n \rfloor} P(\tau_n = k) = \lim_{n \rightarrow \infty} P(S_i^* \in A) = \sin(\pi \rho) \int_{0}^{1-\varepsilon} u^{\rho-1} (1-u)^{-\rho} du \]

\[ \lim_{\varepsilon \rightarrow 0} P^*(S_i^* \in A) \]

implying

\[ \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} P_1(n, \varepsilon) = P^*(S_i^* \in A). \] \hspace{1cm} (11)

It follows from relations (9)-(11) that for \( i \in N_0 \)

\[ \lim_{n \rightarrow \infty} P(S_{i,n}' \in A, \tau_n + i \leq n) = P^*(S_i^* \in A). \]

Thus,

\[ \lim_{n \rightarrow \infty} P(S_{i,n}' \in A) = P^*(S_i^* \in A). \] \hspace{1cm} (12)

We now fix \( i \in N \). Let \( A \) be a one-dimensional \( S^*_{i,n} \)-continuous (relative to the measure \( P^* \)) Borel set. Then for \( n \geq i \)

\[ P(S_{i,n}' \in A, \tau_n - i \geq 0) \]

\[ = \sum_{k=i}^{n} P(S_{i,n}' \in A, \tau_n = k) \]
\[ = \sum_{k=i}^{n} P(S_{k-i} - S_k \in A, S_k < L_{k-1}, S_k \leq L_{k+1,n}) \]

and by the Markov property and the duality property of random walks we have that

\[
P(S_{k-i} - S_k \in A, S_k < L_{k-1}, S_k \leq L_{k+1,n}) = P(S_{k-i} - S_k \in A, S_k < L_{k-1}) P(S_k \leq L_{k+1,n})
\]

\[
= P(-S_i \in A, M_k < 0) P(S_k \leq L_{k+1,n})
\]

\[
= P(-S_i \in A | M_k < 0) P(M_k < 0) P(S_k \leq L_{k+1,n})
\]

\[
= P(-S_i \in A | M_k < 0) P(\tau_n = k).\]

Thus,

\[
P(S_{-i,n} \in A, \tau_n - i \geq 0) = \sum_{k=i}^{n} P(-S_i \in A | M_k < 0) P(\tau_n = k), \tag{13}\]

therefore, if \( \varepsilon \in (0, 1) \) and \( n \) is large enough, then

\[
P(S_{-i,n} \in A, \tau_n - i \geq 0) = P_3(n, \varepsilon) + P_4(n, \varepsilon),
\]

where

\[
P_3(n, \varepsilon) = \sum_{k=\lfloor \varepsilon n \rfloor}^{\lfloor \varepsilon n \rfloor} P(-S_i \in A | M_k < 0) P(\tau_n = k),
\]

\[
P_4(n, \varepsilon) = \sum_{k=\lfloor \varepsilon n \rfloor + 1}^{n} P(-S_i \in A | M_k < 0) P(\tau_n = k).
\]

It is not difficult to show (see our proof of relation (10)) that

\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} P_3(n, \varepsilon) = 0.
\]

Due to (4) the probability \( P(-S_i \in A | M_k < 0) \) tends, as \( n \to \infty \), to

\[
P^*(S_{-i} \in A) = P^*(S_{i}^* \in A) \] uniformly over \( \lfloor \varepsilon n \rfloor < k \leq n \). Therefore

\[
\lim_{\varepsilon \to 0, n \to \infty} P_4(n, \varepsilon) = P^*(S_{i}^* \in A).
\]

As a result, we obtain that
\[ \lim_{n \to \infty} P \left( S'_{i,n} \in A, \tau_n - i \geq 0 \right) = P^\ast \left( S^\ast_{i} \in A \right) \]

proving (12) for \( i \in \mathbb{Z} \setminus \mathbb{N}_0 \). Thus, convergence of one-dimensional distributions in (6) is established.

The lemma is proved.

**Remark 1.** It is not difficult to verify (see [16], Lemma 1) that relation (6) admits the following generalization: for any \( a \leq 0 \) and \( b > 0 \), as \( n \to \infty \),

\[ \left\{ S'_{i,n}, i \in \mathbb{Z} \mid \frac{L_n}{C_n} \leq a, \frac{S_n - L_n}{C_n} \leq b \right\} \mathcal{D} \to \left\{ S^\ast_{i}, i \in \mathbb{Z} \right\}. \]

Recall that \( (\Omega, \mathcal{F}, P) \) is the underlying probability space. Set

\[ I_n^{(2)} := \{(i,j) : i,j \in \{0,\ldots,n\} \text{ and } i \leq j\}. \]

Let \( \mathcal{F}_n, n \in \mathbb{N} \), denote the \( \sigma \)-algebra generated by the segment of the random environment \( Q_1, \ldots, Q_n \) and the random variables \( Z_{i,j} \) for \((i,j) \in I_n^{(2)} \). We now introduce a probability measure \( P^+ \) on the \( \sigma \)-algebra \( \mathcal{F}_\infty := \sigma \left( \bigcup_{n=1}^{\infty} \mathcal{F}_n \right) \), defined for each \( n \in \mathbb{N}_0 \) and each \( \mathcal{F}_n \)-measurable nonnegative random variable \( \beta \) by the formula

\[ E^+ \beta = E \left( \beta v \left( S_n \right) ; L_n \geq 0 \right). \] (14)

This may require a change of the underlying probability space (see [4] for more details). Similarly, we also introduce a probability measure \( P^- \) on the \( \sigma \)-algebra \( \mathcal{F}_\infty \), defined for each \( n \in \mathbb{N}_0 \) and each \( \mathcal{F}_n \)-measurable nonnegative random variable \( \beta \) by the formula

\[ E^- \beta = E \left( \beta u \left( -S_n \right) ; M_n < 0 \right). \] (15)

Recall that the functions \( v(\cdot) \) and \( u(\cdot) \) in formulae (14) and (15) are defined after relation (2). Thus, three measures \( P, P^+, P^- \) are defined on one and the same measurable space \( (\Omega, \mathcal{F}_\infty) \). To explicitly indicate the measure on
(\(\Omega, \mathcal{F}_\infty\)) according to which we consider this or those random elements we use the measure symbol as a lower index.

For instance, it is shown in Lemma 2.5 from [4] that

\[
\{ (Q^+_i, S^+_i, \mu^+_i) \mid i \in \mathbb{N} \} \overset{D}{=} \{ (Q_i, S_i, \mu_i) \mid i \in \mathbb{N} \}_{P^+},
\]  

(16)

(the lower index \(P^+\) for a random sequence shows here that the measure \(P^+\) is used on the space \((\Omega, \mathcal{F}_\infty)\)). Similarly,

\[
\{ (Q^-_i, S^-_i, \mu^-_i) \mid i \in \mathbb{N} \} \overset{D}{=} \{ (Q_i, S_i, \mu_i) \mid i \in \mathbb{N} \}_{P^-},
\]  

(17)

Due to (16), (17) and our assumption about the independence of the left-hand sides of these relations, the product of probability spaces \((\Omega, \mathcal{F}_\infty, P^+)\) and \((\Omega, \mathcal{F}_\infty, P^-)\) may be considered as a probability space \((\Omega^*, \mathcal{F}^*, P^*)\) and, consequently, the direct product of the measures \(P^+\) and \(P^-\) may be treated as the measure \(P^*\).

**Remark 2.** If a random element \(\xi\) is given on the space \((\Omega, \mathcal{F}_\infty, P^+)\) we can define the random element \(\xi^+\), specified on the product of the spaces \((\Omega, \mathcal{F}_\infty, P^+)\) and \((\Omega, \mathcal{F}_\infty, P^-)\) by means of the formula \(\xi^+(\omega_1, \omega_2) = \xi(\omega_1)\) for \((\omega_1, \omega_2) \in \Omega \times \Omega\). It is clear that \(P^*(\xi^+ \in A) = P^+(\xi \in A)\) for an arbitrary one-dimensional Borel set \(A\). Similarly, if a random element \(\xi\) is given on the space \((\Omega, \mathcal{F}_\infty, P^-)\) we can define the random element \(\xi^-\), specified on the product of the spaces \((\Omega, \mathcal{F}_\infty, P^+)\) and \((\Omega, \mathcal{F}_\infty, P^-)\) by means of the formula \(\xi^- (\omega_1, \omega_2) = \xi(\omega_2)\) for \((\omega_1, \omega_2) \in \Omega \times \Omega\), and \(P^*(\xi^- \in A) = P^-(\xi \in A)\) for an arbitrary one-dimensional Borel set \(A\). In accordance with the agreement we can consider the random elements standing in the left-hand sides of formulae (16) and (17) as generated by the random elements \(\{(Q_i, S_i, \mu_i)\mid i \in \mathbb{N}\}_{P^+}\) and \(\{(Q_i, S_i, \mu_i)\mid i \in \mathbb{N}\}_{P^-}\) respectively.

**Lemma 2.** If the Spitzer-Doney condition (2) is satisfied, then, as \(n \to \infty\),

\[
\{ a_{i,n}Z_{i,n} \mid i \in \mathbb{N}_0 \mid L_n \geq 0 \} \overset{D}{=} \{ \zeta^+_i \mid i \in \mathbb{N}_0 \},
\]  

(18)

where \(\{ \zeta^+_i \mid i \in \mathbb{N}_0 \}\) is the random sequence defined by relation (5).
Proof. By virtue of the first part of Lemma 2.5 from [4] for \( k \in \mathbb{N}, \) as \( n \to \infty, \)
\[
\left\{ (a_{i,j}, Z_{i,j}) , \ (i,j) \in I^{(2)}_k \mid L_n \geq 0 \right\} \overset{D}{\to} \left\{ (a_{i,j}, Z_{i,j}) , \ (i,j) \in I^{(2)}_k \right\}_{\mathbb{P}^+}.
\]
Note (see [4], Section 3) that in view of (14) for a fixed \( i \in \mathbb{N}_0 \) the random sequence \( \{\eta_i; Z_{i,j}, j \in \mathbb{N}_{i+1}\}_{\mathbb{P}^+} \) given on the probability space \((\Omega, \mathcal{F}_\infty, \mathbb{P}^+)\) is a branching process in the random environment \( \{G_{i+1}; F_n, n \in \mathbb{N}_{i+1}\}_{\mathbb{P}^+}. \) Hence, if the random environment is fixed, the sequence \( \{\eta_i; a_{i,j}Z_{i,j}, j \in \mathbb{N}_{i+1}\}_{\mathbb{P}^+} \) is a non-negative martingale. Because of this (without assuming that the random environment is fixed) there is \( \mathbb{P}^+-\text{a.s.} \) the finite limit
\[
\lim_{n \to \infty} a_{i,n}Z_{i,n} =: \zeta_i.
\]
It means, in view of the second part of Lemma 2.5 from [4], that, as \( n \to \infty, \)
\[
\left\{ a_{i,n}Z_{i,n}, i \in \mathbb{N}_0 \mid L_n \geq 0 \right\} \overset{D}{\to} \left\{ \zeta_i, i \in \mathbb{N}_0 \right\}_{\mathbb{P}^+}.
\]
To prove relation (18), it remains to note that in view of Remark 2
\[
\left\{ \zeta_i, i \in \mathbb{N}_0 \right\}_{\mathbb{P}^+} \overset{D}{=} \left\{ \zeta^*_i, i \in \mathbb{N}_0 \right\}.
\]
The lemma is proved.

Set for \( n \in \mathbb{N} \)
\[
Z'_{i,n} = \begin{cases} 
Z_{\tau_n+i,n}, & i \in \mathbb{N}_{(-\tau_n)}, \\
0, & i \in \mathbb{Z} \setminus \mathbb{N}_{(-\tau_n)},
\end{cases}
\]
\[
a'_{i,n} = \begin{cases} 
a_n/a_{\tau_n+i}, & i \in \mathbb{N}_{(-\tau_n)}, \\
1, & i \in \mathbb{Z} \setminus \mathbb{N}_{(-\tau_n)}.
\end{cases}
\]

**Lemma 3.** If the Spitzer-Doney condition (2) is satisfied, then, as \( n \to \infty, \)
\[
\left\{ a'_{i,n}Z'_{i,n}, i \in \mathbb{Z} \right\} \overset{D}{\to} \left\{ \zeta^*_i, i \in \mathbb{Z} \right\},
\]
where \( \{ \zeta^*_i, i \in \mathbb{N}_0 \} \) is the random sequence defined by relation (5).
Proof. We demonstrate for simplicity only convergence of one-dimensional distributions. Fix $i \in \mathbb{N}_0$. Let $A$ be an arbitrary one-dimensional $\zeta^*_i$-continuous (relative to the measure $P^*$) Borel set. Then for $n \geq i$

$$P \left( a'_{i,n} Z'_{i,n} \in A, \tau_n + i \leq n \right)$$

$$= \sum_{k=0}^{n-i} P \left( a'_{i,n} Z'_{i,n} \in A, \tau_n = k \right)$$

$$= \sum_{k=0}^{n-i} P \left( \frac{a_n}{a_{k+i}} Z_{k+i,n} \in A, S_k < L_{k-1}, S_k \leq L_{k+1,n} \right)$$

and

$$P \left( \frac{a_n}{a_{k+i}} Z_{k+i,n} \in A, S_k < L_{k-1}, S_k \leq L_{k+1,n} \right)$$

$$= P (S_k < L_{k-1}) P \left( \frac{a_n}{a_{k+i}} Z_{k+i,n} \in A, S_k \leq L_{k+1,n} \right)$$

$$= P (S_k < L_{k-1}) P \left( \frac{a_n-k}{a_i} Z_{i,n-k} \in A, L_{n-k} \geq 0 \right)$$

$$= P (a_{i,n-k} Z_{i,n-k} \in A \mid L_{n-k} \geq 0) P (S_k < L_{k-1}) P (L_{n-k} \geq 0)$$

$$= P (a_{i,n-k} Z_{i,n-k} \in A \mid L_{n-k} \geq 0) P (\tau_n = k).$$

Thus,

$$P \left( a'_{i,n} Z'_{i,n} \in A, \tau_n + i \leq n \right)$$

$$= \sum_{k=0}^{n-i} P (a_{i,n-k} Z_{i,n-k} \in A \mid L_{n-k} \geq 0) P (\tau_n = k).$$

Therefore, if $\varepsilon \in (0, 1)$ and $n$ is large enough, then

$$P \left( a'_{i,n} Z'_{i,n} \in A, \tau_n + i \leq n \right) = P_1 (n, \varepsilon) + P_2 (n, \varepsilon),$$

where

$$P_1 (n, \varepsilon) = \sum_{k=0}^{\lfloor (1-\varepsilon)n \rfloor} P (a_{i,n-k} Z_{i,n-k} \in A \mid L_{n-k} \geq 0) P (\tau_n = k),$$

$$P_2 (n, \varepsilon) = \sum_{k=\lfloor (1-\varepsilon)n \rfloor + 1}^{n-i} P (a_{i,n-k} Z_{i,n-k} \in A \mid L_{n-k} \geq 0) P (\tau_n = k).$$
It is easy to show (see the proof of relation (10)) that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} P_2 (n, \varepsilon) = 0.$$  

By Lemma 2 the probability $P (a_{i,n-k} Z_{i,n-k} \in A \mid L_{n-k} \geq 0)$ tends, as $n \to \infty$, to $P^* (\zeta^*_i \in A)$ uniformly over $0 \leq k \leq \lfloor (1 - \varepsilon) n \rfloor$. Therefore

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} P_1 (n, \varepsilon) = P^* (\zeta^*_i \in A).$$

As result, we obtain that

$$\lim_{n \to \infty} P (a'_{i,n} Z'_{i,n} \in A, \tau_n + i \leq n) = P^* (\zeta^*_i \in A).$$

This justifies the one-dimensional convergence in (19) for $i \in \mathbb{N}_0$.

Now fix $i \in \mathbb{N}$. Then for $x \geq 0$ and $n \geq i$

$$\begin{align*}
P (a'_{i,n} Z'_{i,n} \leq x, \tau_n - i \geq 0) \\
= \sum_{k=i}^{n} P (a'_{i,n} Z'_{i,n} \leq x, \tau_n = k) \\
= \sum_{k=i}^{n} P \left( \frac{a_n}{a_{k-i}} Z_{k-i,n} \leq x, S_k < L_{k-1}, S_k \leq L_{k+1,n} \right).
\end{align*}

(20)

Note that the random sequence \{(Z_{k-i,n}, a_{k-i,n}) \mid n \in \mathbb{N}_{k-i}\} is Markovian.

Denote by $Z_{k,n} (l)$ the number of particles of $n$th generation being descendants of $l$ particles of $k$th generation. Since $Z_{k-i,n} \overset{D}{=} Z_{k,n} (l)$ given $Z_{k-i,k} = l$, it follows that

$$\begin{align*}
P \left( \frac{a_n}{a_{k-i}} Z_{k-i,n} \leq x, S_k < L_{k-1}, S_k \leq L_{k+1,n} \right) \\
= \mathbb{E} \left( U (Z_{k-i,k}, a_{k-i,k}) ; S_k < L_{k-1} \right),
\end{align*}

$$

where

$$U (l, y) = P \left( a_{k,n} Z_{k,n} (l) \leq \frac{x}{y}, S_k \leq L_{k+1,n} \right).$$

Clearly,

$$U (l, y) = P \left( a_{0,n-k} Z_{0,n-k} (l) \leq \frac{x}{y}, L_{n-k} \geq 0 \right).$$
As result, we obtain that

\[
\begin{align*}
P(\frac{a_n}{a_{k-i}}Z_{k-i,n} \leq x, S_k < L_{k-1}, S_k \leq L_{k+1,n}) &= E(H_{n-k}(Z_{k-i,k}, \frac{x}{a_{k-i,k}}) \mid S_k < L_{k-1}) P(S_k < L_{k-1}) P(L_{n-k} \geq 0) \\
&= E(H_{n-k}(Z_{k-i,k}, x/a_{k-i,k}) \mid S_k < L_{k-1}) P(\tau_n = k), \quad (21)
\end{align*}
\]

where

\[
H_n(l,x) = P(a_{0,n}Z_{0,n}(l) \leq x \mid L_n \geq 0)
\]

for \(l \in \mathbb{N}_0\) and \(x \geq 0\).

Set \(Q_{k,l} = (Q_k,\ldots,Q_l)\) for \(k,l \in \mathbb{N}\). In the sequel, we will to explicitly include a random environment in the notation. For example, we will write \(Z_{k-i,k} \langle Q_{k-i+1,k} \rangle\) instead of \(Z_{k-i,k}\) Set

\[
Q_k = \tilde{Q}_1,\ldots,Q_{k-i+1} = \tilde{Q}_i,\ldots,Q_1 = \tilde{Q}_k
\]

and consider a branching process with immigration in the random environment \(\tilde{Q}_1,\ldots,\tilde{Q}_k\). Then

\[
E(H_{n-k}(Z_{k-i,k} \langle Q_{k-i+1,k} \rangle, x/a_{k-i,k}) \mid S_k < L_{k-1})
\]

\[
= E(H_{n-k}(Z_{0,i} \langle \tilde{Q}_{i,1} \rangle, x/\tilde{a}_{0,i}) \mid \tilde{M}_k < 0),
\]

where the symbols \(\tilde{a}_{0,i}, \tilde{M}_k, \tilde{Q}_{i,1}\) have the same meaning for the random environment \(\tilde{Q}_1,\ldots,\tilde{Q}_k\) as the symbols \(a_{0,i}, M_k, Q_{i,1}\) mean for the random environment \(Q_1,\ldots,Q_k\). Further, the random environments \(\tilde{Q}_1,\ldots,\tilde{Q}_k\) and \(Q_1,\ldots,Q_k\) are identically distributed. Therefore

\[
E(H_{n-k}(Z_{0,i} \langle \tilde{Q}_{i,1} \rangle, x/\tilde{a}_{0,i}) \mid \tilde{M}_k < 0)
\]

\[
= E(H_{n-k}(Z_{0,i} \langle Q_{i,1} \rangle, x/a_{0,i}) \mid M_k < 0).
\]

As result, we obtain that

\[
E(H_{n-k}(Z_{k-i,k} \langle Q_{k-i+1,k} \rangle, x/a_{k-i,k}) \mid S_k < L_{k-1})
\]

\[
= E(H_{n-k}(Z_{0,i} \langle Q_{i,1} \rangle, x/a_{0,i}) \mid M_k < 0). \quad (22)
\]
Set $\psi_i = Z_{0,i} \langle Q_{i,1} \rangle$. We have from (20)-(22) that
\[
\mathbb{P} \left( a'_i Z''_{i,n} \leq x, \tau_n - i \geq 0 \right) = \sum_{k=i}^{n} \mathbb{E} \left( H_{n-k} (\psi_i, x/a_{0,i}) \mid M_k < 0 \right) \mathbb{P} \left( \tau_n = k \right).
\]
Therefore, if $\varepsilon \in (0, 1)$ and $n$ is large enough, then
\[
\mathbb{P} \left( a'_i Z''_{i,n} \leq x, \tau_n - i \geq 0 \right) = P_3 (n, \varepsilon) + P_4 (n, \varepsilon) + P_5 (n, \varepsilon),
\]
where
\[
P_3 (n, \varepsilon) = \sum_{k=i}^{\lfloor \varepsilon n \rfloor} \mathbb{E} \left( H_{n-k} (\psi_i, x/a_{0,i}) \mid M_k < 0 \right) \mathbb{P} \left( \tau_n = k \right),
\]
\[
P_4 (n, \varepsilon) = \sum_{k=\lfloor (1-\varepsilon)n \rfloor + 1}^{n} \mathbb{E} \left( H_{n-k} (\psi_i, x/a_{0,i}) \mid M_k < 0 \right) \mathbb{P} \left( \tau_n = k \right),
\]
\[
P_5 (n, \varepsilon) = \sum_{k=\lfloor (1-\varepsilon)n \rfloor + 1}^{\lfloor (1-\varepsilon)n \rfloor} \mathbb{E} \left( H_{n-k} (\psi_i, x/a_{0,i}) \mid M_k < 0 \right) \mathbb{P} \left( \tau_n = k \right).
\]
Similar to relation (10) we conclude that
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} P_3 (n, \varepsilon) = 0,
\]
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} P_4 (n, \varepsilon) = 0.
\]
Let $l \in \mathbb{N}_0$ be fixed. It is not difficult to demonstrate that
\[
\lim_{n \to \infty} a_{0,n} Z_{0,n} (l) =: \zeta_0 (l)
\]
exists a.s. on the probability space $(\Omega, \mathcal{F}_\infty, \mathbb{P}^+)$. By the arguments to those used in Lemma 2 one can show, as $n \to \infty$,
\[
\{ a_{0,n} Z_{0,n} (l), i \in \mathbb{N}_0 \mid L_n \geq 0 \} \stackrel{D}{\to} \{ \zeta_0 (l), i \in \mathbb{N}_0 \}_{\mathbb{P}^+}.
\]
For $x \geq 0$ set
\[
H (l, x) = \mathbb{P}^+ (\zeta_0 (l) \leq x).
\]
It follows from (27) that
\[
\lim_{n \to \infty} H_n (l, x) = H (l, x)
\]
if \( x \geq 0 \) belongs to the set of continuity points of \( H(l, \cdot) \) (with respect to the second argument). By Lemma 2.5 in [4]

\[
\{ Z_{0,i} \langle Q_{1,1} \rangle, a_{0,i} \mid M_n < 0 \} \xrightarrow{D} (Z_{0,i} \langle Q_{1,1} \rangle, a_{0,i})_{P}^{-}
\]
as \( n \to \infty \). Therefore

\[
\{ \psi_{1,i} a_{0,i} \mid M_n < 0 \} \xrightarrow{D} \left( Z_{1,i,0}^*, a_{1,i,0}^* \right).
\] (29)

We show that, for fixed \( l \in \mathbb{N}_0 \) and \( K > 0 \)

\[
\lim_{n \to \infty} \mathbb{E} \left( H_{n-k} \left( l, x/a_{0,i} \right) I_{\{ \psi_{1,i} = l, x/a_{0,i} \leq K \}} \mid M_k < 0 \right) = \mathbb{E}^* \left( H \left( l, x/a_{*,-i,0} \right) ; Z_{*,-i,0}^* = l, x/a_{*,-i,0} \leq K \right)
\] (30)

uniformly over \( \lfloor \varepsilon n \rfloor < k \leq \lfloor (1 - \varepsilon) n \rfloor \) (here \( I_A \) is the indicator of the event \( A \)). Let \( 0 = x_0 < x_1 < \ldots < x_m = K \) for some \( m \in \mathbb{N} \). The monotonicity of the function \( H(l, \cdot) \) with respect to the second argument gives

\[
\mathbb{E} \left( H_{n-k} \left( l, x/a_{0,i} \right) I_{\{ \psi_{1,i} = l, x/a_{0,i} \leq K \}} \mid M_k < 0 \right) = \sum_{j=1}^{m} \mathbb{E} \left( H_{n-k} \left( l, x/a_{0,i} \right) I_{\{ \psi_{1,i} = l, x_{j-1} < x/a_{0,i} \leq x_j \}} \mid M_k < 0 \right) \leq \sum_{j=1}^{m} H_{n-k} \left( l, x_j \right) \mathbb{P}^* \left( \psi_{1,i} = l, x_{j-1} < x/a_{0,i} \leq x_j \mid M_k < 0 \right).
\] (31)

In view of (28) and (29) the right-hand side of (31) converges, as \( n \to \infty \), to

\[
\sum_{j=1}^{m} H \left( l, x_j \right) \mathbb{P}^* \left( Z_{*,-i,0}^* = l, x_{j-1} < x/a_{*,-i,0} \leq x_j \right)
\]
uniformly over \( \lfloor \varepsilon n \rfloor < k \leq \lfloor (1 - \varepsilon) n \rfloor \), if the selected \( x_1, \ldots, x_m \) are simultaneously the continuity points of \( H(l, \cdot) \) with respect to the second argument and of \( \mathbb{P}^* \left( Z_{*,-i,0}^* = l, x/a_{*,-i,0} \leq y \right) \) with respect to \( y \). Thus, if \( \delta > 0 \) and \( n \) is large enough, the following inequality holds

\[
\mathbb{E} \left( H_{n-k} \left( l, x/a_{0,i} \right) I_{\{ \psi_{1,i} = l, x/a_{0,i} \leq K \}} \mid M_k < 0 \right) \leq \sum_{j=1}^{m} H \left( l, x_j \right) \mathbb{P}^* \left( Z_{*,-i,0}^* = l, x_{j-1} < (a_{*,-i,0}^*)^{-1} x \leq x_j \right) + \delta
\] (32)
for \(|\varepsilon n| < k \leq [(1 - \varepsilon) n]\). Similarly, if \(\delta > 0\) and \(n\) is large enough, then

\[
E \left( H_{n-k} (l, x/a_{0,i}) I_{\{\psi_i = l, x/a_{0,i} \leq K}\} \mid M_k < 0 \right) \\
\geq \sum_{j=1}^{m} H (l, x_{j-1}) P^* (Z_{-i,0}^* = l, x_{j-1} < x/a_{-i,0} \leq x_j) - \delta 
\]  
(33)

for \(|\varepsilon n| < k \leq [(1 - \varepsilon) n]\). Since \(0 \leq H (l, x) \leq 1\) for \(x \geq 0\), the sums in the right-hand sides of (32) and (33) converge, as \(\max_{1 \leq j \leq m} (x_j - x_{j-1}) \to 0\), to (see [17], Chapter 2, § 6, Section 11)

\[
E^* [H (l, x/a_{-i,0}^*); Z_{-i,0}^* = l, x/a_{-i,0}^* \leq K].
\]

Hence, if \(\delta > 0\) and \(n\) is large enough, then

\[
E^* (H (l, x/a_{-i,0}^*); Z_{-i,0}^* = l, x/a_{-i,0}^* \leq K) - \delta \\
\leq E (H_{n-k} (l, x/a_{0,i}) I_{\{\psi_i \leq l, x/a_{0,i} \leq K\}} \mid M_k < 0) \\
\leq E^* (H (l, x/a_{-i,0}^*); Z_{-i,0}^* = l, x/a_{-i,0}^* \leq K) + \delta 
\]

for \(|\varepsilon n| < k \leq [(1 - \varepsilon) n]\). Since \(\delta > 0\) is arbitrary, we obtain the required relation (30).

Now we show that

\[
\lim_{n \to \infty} E (H_{n-k} (\psi_i, x/a_{0,i}) \mid M_k < 0) = E^* H (Z_{-i,0}^*, x/a_{-i,0}^*) 
\]  
(34)

uniformly over \(|\varepsilon n| < k \leq [(1 - \varepsilon) n]\). For \(N \in \mathbb{N}\) and \(K > 0\) we write

\[
E (H_{n-k} (\psi_i, x/a_{0,i}) \mid M_k < 0) = E_1 (k, n, N, K) + E_2 (k, n, N, K), 
\]  
(35)

where

\[
E_1 (k, n, N, K) = E (H_{n-k} (\psi_i, x/a_{0,i}) I_{\{\psi_i \leq N, x/a_{0,i} \leq K\}} \mid M_k < 0), \\
E_2 (k, n, N, K) = E (H_{n-k} (\psi_i, x/a_{0,i}) I_{\{\psi_i > N\} \cup \{x/a_{0,i} > K\}} \mid M_k < 0).
\]

Since

\[
E_2 (k, n, N, K) \leq P (\{\psi_i > N\} \cup \{x/a_{0,i} > K\} \mid M_k < 0),
\]

it follows by (29) that

\[
\lim_{K \to \infty} \lim_{N \to \infty} \limsup_{n \to \infty} E_2 (k, n, N, K) = 0 
\]  
(36)
uniformly over $\lfloor \varepsilon n \rfloor < k \leq \lfloor (1 - \varepsilon) n \rfloor$. Clearly,

$$E_1(k, n, N, K) = \sum_{l=0}^{N} E \left( H_{n-k}(l, x/a_{0,i}) I_{\{\psi_i=l, x/a_{0,i} \leq K\}} \mid M_k < 0 \right).$$

Hence, using (30) we conclude that

$$\lim_{K \to \infty} \lim_{N \to \infty} \lim_{n \to \infty} E_1(k, n, N, K) = E^* H \left( Z_{-i,0}^*, x/a_{-i,0}^* \right). \quad (37)$$

uniformly over $\lfloor \varepsilon n \rfloor < k \leq \lfloor (1 - \varepsilon) n \rfloor$. Combining (35)-(37) we obtain the desired relation (34).

It follows from (34) that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} P_5(n, \varepsilon) = E^* H \left( Z_{-i,0}^*, x/a_{-i,0}^* \right). \quad (38)$$

Now (23)-(25) and (38) imply

$$\lim_{n \to \infty} P \left( a_{-i,n}^* Z_{-i,n}^* \leq x, \tau_n - i \geq 0 \right) = E^* H \left( Z_{-i,0}^*, x/a_{-i,0}^* \right). \quad (39)$$

We now analyze a branching process with immigration in the random environment $\{Q_k^*, k \in \mathbb{Z}\}$. The random sequence $\{(Z_{-i,n}^*, a_{-i,n}^*), n \in \mathbb{N}_{-i}\}$ is Markovian. Denote by $Z_{k,n}^*(l)$ the number of particles in the $n$th generation which are descendants of $l$ particles of the $k$th generation. Note that

$$\{(Z_{0,n}^*(l), a_{0,n}^*), n \in \mathbb{N}_0\} \overset{D}{=} \{(Z_{0,n}(l), a_{0,n}), n \in \mathbb{N}_0\}_{P^+}. \quad (40)$$

Since $Z_{-i,n}^* \overset{D}{=} Z_{0,n}^*(l)$ given $Z_{-i,0}^* = l$, it follows that, for any bounded and continuous function $f : \mathbb{R} \to \mathbb{R}$ and $n \in \mathbb{N}_{-i},$

$$E^* f \left( a_{-i,n}^* Z_{-i,n}^* \right) = E^* V_n \left( Z_{-i,0}^*, 1/a_{-i,0}^* \right), \quad (41)$$

where

$$V_n(l, y) = E^* f \left( a_{0,n}^* Z_{0,n}^* (l) / y \right).$$

By (40)

$$V_n(l, y) = E^+ f \left( a_{0,n} Z_{0,n} (l) / y \right). \quad (42)$$
In view of (5), as \( n \to \infty \),
\[
a_{-i,n}^* Z_{-i,n} \overset{D}{\to} \zeta_{-i}^*,
\]
and in view of (26)
\[
(a_{0,n} Z_{0,n} (l))_{P^+} \overset{D}{\to} (\zeta_0 (l))_{P^+}.
\]
Using (42), (44) and applying the dominated convergence theorem we see that
\[
\lim_{n \to \infty} V_n (l, y) = V (l, y),
\]
where
\[
V (l, y) = \mathbb{E}^+ f (\zeta_0 (l) / y).
\]
Applying the dominated convergence theorem again we obtain from (41), (43) and (45) that
\[
\mathbb{E}^* f (\zeta_{-i}^*) = \mathbb{E}^* V (Z_{-i,0}^*, 1/a_{-i,0}^*).
\]
Fix \( x \geq 0 \). As relation (46) is valid for any bounded and continuous function \( f \), it is valid, even when a function \( f \) is the indicator of the semi-axis \((-\infty, x]\). It means that
\[
\mathbb{P}^* (\zeta_{-i}^* \leq x) = \mathbb{E}^* H (Z_{-i,0}^*, x/a_{-i,0}^*)
\]
(we take into account that \( V (l, y) = H (l, xy) \) for the specified function \( f \)).

Equalities (39) and (47) imply the one-dimensional convergence in relation (19) for \( i \in \mathbb{Z} \setminus N_0 \).

The lemma is proved.

**Remark 3.** It is not difficult to verify that (19) admits the following generalization: for any \( a \leq 0 \) and \( b > 0 \), as \( n \to \infty \),
\[
\left\{ a'_{i,n} Z'_{i,n}, i \in \mathbb{Z} \mid \frac{L_n}{C_n} \leq a, \frac{S_n - L_n}{C_n} \leq b \right\} \overset{D}{\to} \{ \zeta_i^*, i \in \mathbb{Z} \}.
\]

**Lemma 4.** If the conditions of Theorem 1 are satisfied, then \( \mathbb{P}^* \)-a.s.
\[
\Sigma_1 < +\infty, \quad \Sigma_2 < +\infty.
\]
Proof. It is shown in Lemma 2.7 from [4] that, if the conditions ofTheorem 1 are satisfied, then the series \( \sum_{i=0}^{\infty} \mu_{i+1} e^{-S_i} \) converges \( P^+ \)-a.s. Hence, the series \( \sum_{i=0}^{\infty} \mu_{i+1}^* e^{-S_i^*} \) converges \( P^* \)-a.s. Similarly we can prove that the series \( \sum_{i=1}^{\infty} \mu_i^* e^{S_i} \) converges \( P^* \)-a.s. As result, we obtain that the series \( \sum_{i=1}^{\infty} \mu_i^* e^{S_i} \) converges \( P^* \)-a.s. Thus, \( \Sigma_1 < +\infty \) \( P^* \)-a.s.

Fix \( i \in \mathbb{Z} \). If the random environment \( E^* \) is fixed, the random sequence \( \{ \eta_i^*; a_{i,j}^* Z_{i,j}^*; j \in N_{i+1} \} \) is a martingale. Therefore

\[
E^* \left( a_{i,j}^* Z_{i,j}^* \mid E^* \right) = \mu_{i+1}^* 
\]

for \( j \in N_{i+1} \). By (5), (48) using Fatou’s lemma we obtain that

\[
E^* \left( \zeta_i^* e^{-S_i^*} \mid E^* \right) \leq \liminf_{j \to \infty} E^* \left( a_{i,j}^* Z_{i,j}^* \mid E^* \right) = \mu_{i+1}^* 
\]

and, consequently,

\[
E^* \left( \zeta_i^* e^{-S_i^*} \mid E^* \right) = e^{-S_i^*} E^* \left( \zeta_i^* \mid E^* \right) \leq \mu_{i+1}^* e^{-S_i^*}. \tag{49}
\]

We have proved that the series \( \sum_{i \in \mathbb{Z}} \mu_{i+1}^* e^{-S_i^*} \) converges \( P^* \)-a.s. This fact combined with (49) implies convergence of the series \( \sum_{i \in \mathbb{Z}} E^* \left( \zeta_i^* e^{-S_i^*} \mid E^* \right) \) \( P^* \)-a.s. Since the random variables \( \zeta_i^* e^{-S_i^*} \) are nonnegative, it follows that the series \( \sum_{i \in \mathbb{Z}} \zeta_i^* e^{-S_i^*} \) converges a.s. for any fixed environment \( E^* \). Hence, \( \Sigma_2 < +\infty \) \( P^* \)-a.s.

The lemma is proved.

Set

\[
\Sigma_1^{(1)} = \sum_{i=0}^{\infty} \mu_{i+1}^* e^{-S_i^*} = \sum_{i \in N_0} \mu_{i+1}^* e^{-S_i^*}, \\
\Sigma_1^{(2)} = \sum_{i=1}^{\infty} \mu_i e^{S_i^*} = \sum_{i \in Z \setminus N_0} \mu_i^* e^{-S_i^*}.
\]

Clearly,

\[
\Sigma_1 = \Sigma_1^{(1)} + \Sigma_1^{(2)} \tag{50}
\]

and by virtue of Lemma 4 \( P^* \)-a.s.

\[
\Sigma_1^{(1)} < +\infty, \quad \Sigma_1^{(2)} < +\infty. \tag{51}
\]
Lemma 5. If the conditions of Theorem 1 are satisfied, then $P^*$-a.s., as $n \to \infty$,
\[
\left\{ \sum_{i=0}^{n-1} \mu_{i+1} e^{-S_i} \left| L_n \geq 0 \right. \right\} \overset{D}{\to} \Sigma_1^{(1)}, \\
\left\{ \sum_{i=1}^{n-1} \mu_i e^{S_i} \left| M_n < 0 \right. \right\} \overset{D}{\to} \Sigma_1^{(2)}.
\]

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. By virtue of (3) for fixed $k \in \mathbb{N}$
\[
\left\{ f \left( \sum_{i=0}^{k} \mu_{i+1} e^{-S_i} \right) \left| L_n \geq 0 \right. \right\} \overset{D}{\to} f \left( \sum_{i=0}^{k} \mu_{i+1} e^{-S_i} \right)
\]
as $n \to \infty$. Recalling (51) we conclude that
\[
\lim_{k \to \infty} f \left( \sum_{i=0}^{k} \mu_{i+1} e^{-S_i} \right) = f \left( \Sigma_1^{(1)} \right)
\]
$P^*$-a.s. From these two facts, in view of Lemma 2.5 of [4], it follows that
\[
\left\{ f \left( \sum_{i=0}^{n-1} \mu_{i+1} e^{-S_i} \right) \left| L_n \geq 0 \right. \right\} \overset{D}{\to} f \left( \Sigma_1^{(1)} \right).
\]
Thus, relation (52) is true. Relation (53) can be proved by similar arguments.

The lemma is proved.

Remark 4. It is not difficult to verify that if we combine the left-hand sides of relations (3) and (52) (or (4) and (53)), then the respective statements concerning convergence in distribution of the four dimensional tuples of the random elements given $L_n \geq 0$ (or $M_n < 0$) are still force.

Set for $n \in \mathbb{N}$
\[
\mu_{i,n} = \begin{cases} 
\mu_{\tau_n + i}, & i \in \mathbb{N}_{(-\tau_n)}, \\
0, & i \in \mathbb{Z} \setminus \mathbb{N}_{(-\tau_n)}.
\end{cases}
\]
Let
\[
\Sigma_1^{(1)}(n) = \sum_{j=0}^{n-1-\tau_n} \mu_{j+1,n} e^{-S_{j,n}}, \quad \Sigma_1^{(2)}(n) = \sum_{j=1}^{\tau_n} \mu_{-j+1,n} e^{-S_{j,n}}.
\]
Lemma 6. If the conditions of Theorem 1 are satisfied, then $P^*$-a.s., as $n \to \infty$,

\[
\begin{align*}
\left\{(\mu_{i,n}', S_{i,n}'), i \in \mathbb{N}_0\right\}, \Sigma_1^{(1)}(n) & \overset{D}{\to} \left\{(\mu_i^*, S_i^*), i \in \mathbb{N}_0\right\}, \Sigma_1^{(1)} \right. \tag{54}
\end{align*}
\]

\[
\left\{(\mu_{-i,n}', S_{-i,n}'), i \in \mathbb{N}\right\}, \Sigma_1^{(2)}(n) & \overset{D}{\to} \left\{(\mu_{-i}', S_{-i}^*), i \in \mathbb{N}\right\}, \Sigma_1^{(2)} \right. \tag{55}
\]

Moreover, the left-hand sides of these relations are asymptotically independent.

Proof. We prove for simplicity only convergence in distribution (for a fixed $i$) of the random sequences $\left(\mu_{i,n}', S_{i,n}', \Sigma_1^{(1)}(n)\right)$ and $\left(\mu_{-i,n}', S_{-i,n}', \Sigma_1^{(2)}(n)\right)$, as $n \to \infty$.

Fix $i \in \mathbb{N}_0$. Similarly to relation (7), we can show that, for any bounded and continuous function $f : \mathbb{R}^3 \to \mathbb{R}$,

\[
\mathbb{E} \left[ f \left( \mu_{i,n}', S_{i,n}', \sum_{j=0}^{n-1-\tau_n} \mu_{j+1,n} e^{-S_{j,n}} \right) ; \tau_n + i \leq n \right] = \sum_{k=0}^{n-i} \mathbb{E} \left( f \left( \mu_i, S_i, \sum_{j=0}^{n-1-k} \mu_{j+1} e^{-S_j} \right) \bigg| L_{n-k} \geq 0 \right) \mathbb{P}(\tau_n = k) \tag{56}
\]

for $n \geq i$. Repeating the arguments of Lemma 1 and using Lemma 5 and Remark 4, we can deduce from (56) that

\[
\lim_{n \to \infty} \mathbb{E} f \left( \mu_{i,n}', S_{i,n}', \sum_{j=0}^{n-1-\tau_n} \mu_{j+1,n} e^{-S_{j,n}} \right) = \mathbb{E}^* f \left( \mu_i^*, S_i^*, \sum_{j \in \mathbb{N}_0} \mu_{j+1}^* e^{-S_j^*} \right) = \mathbb{E}^* f \left( \mu_i^*, S_i^*, \Sigma_1^{(1)} \right). \]

Thus, relation (54) is proved.

Now fix $i \in \mathbb{N}$. It is easy to show (see the proof of relation (13)) that for $n \geq i$

\[
\begin{align*}
\mathbb{E} \left[ f \left( \mu_{-i,n}', S_{-i,n}', \sum_{j=1}^{\tau_n} \mu_{j+1,n} e^{-S_{j,n}} \right) ; \tau_n - i \geq 0 \right] & = \sum_{k=i}^{n} \mathbb{E} \left( f \left( \mu_{i+1} - S_i, \sum_{j=1}^{k} \mu_j e^{S_j} \right) \bigg| M_k < 0 \right) \mathbb{P}(\tau_n = k)
\end{align*}
\]
and therefore (see Lemma 5 and Remark 4)

\[
\lim_{n \to \infty} E f \left( \mu'_{i,n}, S'_{i,n} \sum_{j=1}^{\tau_n} \mu_{-j+1,n} e^{-S_{-j,n}} \right) = E^* f \left( \mu^*_{i+1}, -S^+_i \sum_{j=1}^{\infty} \mu_j e^{S^-_j} \right) = E^* f \left( \mu^*_{i}, S^-_i, \Sigma_1^{(2)} \right).
\]

This proves (55). The asymptotic independence of the left-hand sides of relations (54) and (55) is obvious.

The lemma is proved.

**Remark 5.** It is not difficult to verify that statement (54) admits the following generalization: for any \( a \leq 0 \) and \( b > 0 \), as \( n \to \infty \),

\[
\left( \{ (\mu'_{i,n}, S'_{i,n}) : i \in \mathbb{N}_0 \}, \Sigma_1^{(1)} (n) \left| \frac{L_n}{C_n} \leq a, \frac{S_n - L_n}{C_n} \leq b \right. \right) \xrightarrow{D} \left( \{ (\mu^*_i, S^*_i) : i \in \mathbb{N}_0 \}, \Sigma_1^{(1)} \right).
\]

Statement (55) allows for a similar generalization.

**Lemma 7.** If the conditions of Theorem 1 are satisfied, then, as \( n \to \infty \),

\[
\left\{ \frac{b_n - b_{\tau_n+i}}{b_n}, i \in \mathbb{N}_0 \right\} \xrightarrow{D} \left\{ \frac{\sum_{j=0}^{\tau_n} \mu_j e^{-S^+_j}}{\Sigma_1}, i \in \mathbb{N}_0 \right\},
\]

\[
\left\{ \frac{b_{\tau_n-i}}{b_n}, i \in \mathbb{N} \right\} \xrightarrow{D} \left\{ \frac{\sum_{j=i+1}^{\infty} \mu_j e^{S^-_j}}{\Sigma_1}, i \in \mathbb{N} \right\},
\]

\[
\left\{ \frac{a_{\tau_n+i}}{b_n}, i \in \mathbb{N}_0 \right\} \xrightarrow{D} \left\{ \frac{\exp (-S^+_i)}{\Sigma_1}, i \in \mathbb{N}_0 \right\},
\]

\[
\left\{ \frac{a_{\tau_n-i}}{b_n}, i \in \mathbb{N} \right\} \xrightarrow{D} \left\{ \frac{\exp (S^-_i)}{\Sigma_1}, i \in \mathbb{N} \right\}.
\]

**Proof.** To simplify the presentation we check the first statement only. Moreover, we prove only convergence of one-dimensional distributions. Fix \( i \in \mathbb{N}_0 \).

Note that for \( \tau_n + i \leq n \)

\[
\frac{b_{\tau_n+i}}{b_n} = \frac{\sum_{j=0}^{\tau_n+i-1} \mu_{j+1} e^{-S_j}}{\sum_{j=0}^{\tau_n+i-1} \mu_{j+1} e^{-S_j}} = \frac{\sum_{j=0}^n \mu_{j+1} e^{-S_j}}{\sum_{j=0}^n \mu_{j+1} e^{-S_j}}
\]

25
\[ \sum_{j=0}^{i-1} \mu_{j+1,n}^+ \exp (-S^+_{j,n}) + \Sigma_1^{(2)}(n) \]

Since the last expression is a bounded continuous function of the random element mentioned in Lemma 6, it follows that

\[ \frac{b_{\tau_n+i}}{b_n} \xrightarrow{D} \frac{\sum_{j=0}^{i-1} \mu_{j+1,n}^+ \exp (-S^+_{j,n}) + \Sigma_1^{(2)}(n)}{\Sigma_1^{(1)}(n) + \Sigma_1^{(2)}(n)} \]

as \( n \to \infty \). Whence, taking into account (50) we obtain the required relation. The remaining three statements may be proved by similar arguments.

The lemma is proved.

**Remark 6.** We can construct a new random element by combining the left-hand sides of all the relations included in Lemmas 3 and 7. It is not difficult to prove convergence in distribution of the sequence of these random elements to a random element constructed by the right-hand sides of the corresponding relations of Lemmas 3 and 7. Moreover, a random element constructed by the left-hand sides is asymptotically independent, as \( n \to \infty \), of the random event

\[ \{ C_n^{-1} L_n \leq a, C_n^{-1} (S_n - L_n) \leq b \} \]

for any \( a \leq 0 \) and \( b > 0 \).

**3. Proof of the main result**

**First part.** We establish convergence of one-dimensional distributions: if \( t > 0 \), then, as \( n \to \infty \),

\[ \frac{a_{[nt]}}{b_{[nt]}} Z_{[nt]} \xrightarrow{D} \frac{\Sigma_2}{\Sigma_1}. \] (57)

Set for \( r \in \mathbb{N} \)

\[ U_{r}^{(i)} = \sum_{j=\tau_r-i}^{\tau_r+i-1} Z_{j,r}, \]

\[ V_{r}^{(i)} = \sum_{j=0}^{\tau_r-i-1} Z_{j,r} + \sum_{j=\tau_r+i}^{\tau_r-1} Z_{j,r}. \]

It is clear that for \( i \in \mathbb{N} \)

\[ Z_{[nt]} = \sum_{j=0}^{[nt]-1} Z_{j,[nt]} = U_{[nt]}^{(i)} + V_{[nt]}^{(i)}. \] (58)
Note that
\[ \mathbb{E} \left( a_{j,\lfloor nt \rfloor} Z_{j,\lfloor nt \rfloor} \mid Q_{1,\lfloor nt \rfloor} \right) = \mu_{j+1}, \] (59)
if \(1 \leq j < \lfloor nt \rfloor\). Observing that \(a_{\lfloor nt \rfloor} = a_j a_{\lfloor nt \rfloor}\) for \(1 \leq j < \lfloor nt \rfloor\) we obtain by (59) that
\[
\begin{align*}
\mathbb{E} \left( \frac{a_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} V_{\lfloor nt \rfloor}^{(i)} \right) & \equiv \mathbb{E} b_{\lfloor nt \rfloor}^{-1} \left( \tau_{\lfloor nt \rfloor} - i - 1 \sum_{j=0}^{\lfloor nt \rfloor-1} a_j a_{\lfloor nt \rfloor} Z_{j,\lfloor nt \rfloor} + \sum_{j=\tau_{\lfloor nt \rfloor}+i}^{\lfloor nt \rfloor-1} a_j a_{\lfloor nt \rfloor} Z_{j,\lfloor nt \rfloor} \right) \\
& = \mathbb{E} b_{\lfloor nt \rfloor}^{-1} \left( \tau_{\lfloor nt \rfloor} - i + \sum_{j=0}^{\lfloor nt \rfloor-1} (\mu_{j+1} a_j + \sum_{j=\tau_{\lfloor nt \rfloor}+i}^{\lfloor nt \rfloor-1} \mu_{j+1} a_j) \right) \\
& = \mathbb{E} \frac{b_{\lfloor nt \rfloor}^{-i} + (b_{\lfloor nt \rfloor} - b_{\lfloor nt \rfloor}^{\tau_{\lfloor nt \rfloor}+i})}{b_{\lfloor nt \rfloor}}.
\end{align*}
\] (60)
Applying Lemma 7 to the right-hand side of (60), we conclude that
\[
\lim_{n \to \infty} \mathbb{E} \left( \frac{a_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} V_{\lfloor nt \rfloor}^{(i)} \right) = \sum_{i=1}^{\infty} \mathbb{E} \frac{\mu_{i+1}^{+}}{\Sigma_{1}} \exp \left( -S_{j}^{+} \right) + \sum_{i=1}^{\infty} \mathbb{E} \frac{\mu_{i}^{-}}{\Sigma_{1}} \exp \left( S_{j}^{-} \right)
\]
and, therefore (see Lemma 4),
\[
\lim_{i \to \infty} \lim_{n \to \infty} \mathbb{E} \left( \frac{a_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} V_{\lfloor nt \rfloor}^{(i)} \right) = 0.
\] (61)
By Markov inequality for any \(\varepsilon > 0\)
\[
\mathbb{P} \left( \frac{a_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} V_{\lfloor nt \rfloor}^{(i)} \geq \varepsilon \right) \leq \varepsilon^{-1} \mathbb{E} \left( \frac{a_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} V_{\lfloor nt \rfloor}^{(i)} \right).
\]
Hence, taking into account (61) we obtain that
\[
\lim_{i \to \infty} \lim_{n \to \infty} \mathbb{P} \left( \frac{a_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} V_{\lfloor nt \rfloor}^{(i)} \geq \varepsilon \right) = 0.
\] (62)
Observe that we may assume in the sequel that \(i \leq \tau_{\lfloor nt \rfloor} < \lfloor nt \rfloor - i\) (see the proof of Lemma 1). Note that
\[
U_{\lfloor nt \rfloor}^{(i)} = \sum_{j=-i}^{i-1} Z_{\tau_{\lfloor nt \rfloor}+j,\lfloor nt \rfloor} = \sum_{j=-i}^{i-1} Z_{j,\lfloor nt \rfloor}'
\]
and, therefore,
\[
\frac{a_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} U_{\lfloor nt \rfloor}^{(i)} = \sum_{j=-i}^{i-1} \frac{a_{\tau_{\lfloor nt \rfloor}+j}}{b_{\lfloor nt \rfloor}} a_{j,\lfloor nt \rfloor} Z_{j,\lfloor nt \rfloor}'.
\] (63)
Applying Lemmas 3, 7 and Remark 6 to relation (63), we obtain that, as $n \to \infty$,

$$\frac{a_{\lfloor nt \rfloor} U_{\lfloor nt \rfloor}^{(i)}}{b_{\lfloor nt \rfloor}} \to \frac{D}{\Sigma_1} \sum_{j=-i}^{i-1} \zeta_j^* e^{-S_j^*}. \quad (64)$$

Hence, for all but a countable set of $x \geq 0$

$$\lim_{n \to \infty} \mathbf{P}\left( \frac{a_{\lfloor nt \rfloor} U_{\lfloor nt \rfloor}^{(i)}}{b_{\lfloor nt \rfloor}} \leq x \right) = \mathbf{P}\left( \frac{1}{\Sigma_1} \sum_{j=-i}^{i-1} \zeta_j^* e^{-S_j^*} \leq x \right). \quad (65)$$

In view of Lemma 4

$$\lim_{i \to \infty} \mathbf{P}\left( \frac{1}{\Sigma_1} \sum_{j=-i}^{i-1} \zeta_j^* e^{-S_j^*} \leq x \right) = \mathbf{P}\left( \frac{\Sigma_2}{\Sigma_1} \leq x \right). \quad (66)$$

We obtain by (65) and (66) that

$$\lim_{i \to \infty} \lim_{n \to \infty} \mathbf{P}\left( \frac{a_{\lfloor nt \rfloor} U_{\lfloor nt \rfloor}^{(i)}}{b_{\lfloor nt \rfloor}} \leq x \right) = \mathbf{P}\left( \frac{\Sigma_2}{\Sigma_1} \leq x \right). \quad (67)$$

It follows from (58), (62) and (67) that for all but a countable set of $x \geq 0$

$$\lim_{n \to \infty} \mathbf{P}\left( \frac{a_{\lfloor nt \rfloor} Z_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} \leq x \right) = \mathbf{P}\left( \frac{\Sigma_2}{\Sigma_1} \leq x \right).$$

This proves (57).

**Remark 7.** It is not difficult to verify that relation (64) admits the following generalization: for any $a \leq 0$ and $b > 0$, as $n \to \infty$,

$$\left\{ \frac{a_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} U_{\lfloor nt \rfloor}^{(i)} \left| \frac{L_{\lfloor nt \rfloor}}{C_n} \leq a, \frac{S_{\lfloor nt \rfloor} - L_{\lfloor nt \rfloor}}{C_n} \leq b \right\} \to \frac{D}{\Sigma_1} \sum_{j=-i}^{i-1} \zeta_j^* e^{-S_j^*}. \right.$$ 

**Second part.** Now we establish convergence of two-dimensional distributions.

Select $0 < t_1 < t_2$, fix an $\varepsilon > 0$ and introduce the following random events:

$$A_{n, \varepsilon} = \left\{ L_{\lfloor nt_1 \rfloor} > L_{\lfloor nt_1 \rfloor, \lfloor nt_2 \rfloor} + \varepsilon C_n \right\},$$

$$B_{n, \varepsilon} = \left\{ L_{\lfloor nt_1 \rfloor} < L_{\lfloor nt_1 \rfloor, \lfloor nt_2 \rfloor} - \varepsilon C_n \right\},$$

$$D_{n, \varepsilon} = \left\{ \left| L_{\lfloor nt_1 \rfloor} - L_{\lfloor nt_1 \rfloor, \lfloor nt_2 \rfloor} \right| \leq \varepsilon C_n \right\}.$$
We show that, as \( n \to \infty \),
\[
\begin{align*}
\left\{ \frac{a_{[nt_1]}}{b_{[nt_1]}} Z_{[nt_1]}, \frac{a_{[nt_2]}}{b_{[nt_2]}} Z_{[nt_2]} \bigg| A_{n,\varepsilon} \right\} & \overset{D}{\to} (\gamma_1, \gamma_2), \\
\left\{ \frac{a_{[nt_1]}}{b_{[nt_1]}} Z_{[nt_1]}, \frac{a_{[nt_2]}}{b_{[nt_2]}} Z_{[nt_2]} \bigg| B_{n,\varepsilon} \right\} & \overset{D}{\to} (\gamma_1, \gamma_1),
\end{align*}
\]
where \( \gamma_1, \gamma_2 \) are independent random variables and \( \gamma_1 \overset{D}{=} \gamma_2 \overset{D}{=} \Sigma_2 / \Sigma_1 \).

First we establish (68). To this aim we prove that, for any fixed \( i \in \mathbb{N} \) and for all but a countable set of \((x_1, x_2)\) with \( x_1, x_2 \geq 0 \),
\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{a_{[nt_1]}}{b_{[nt_1]}} U_{[nt_1]}^{(i)} \leq x_1, \frac{b_{[nt_2]}}{a_{[nt_2]}} U_{[nt_2]}^{(i)} \leq x_2 \bigg| A_{n,\varepsilon} \right) = \mathbb{P} \left( \frac{1}{\sum_{j=-i}^{i+1}} \zeta_j e^{-S_j^*} \leq x_1 \right) \mathbb{P} \left( \frac{1}{\sum_{j=-i}^{i+1}} \zeta_j e^{-S_j^*} \leq x_2 \right). \tag{70}
\]

Provided the random event \( A_{n,\varepsilon} \) occurred, it follows that, as \( n \to \infty \),
\[
\frac{b_{[nt_2]}}{a_{[nt_2]}} \sim \frac{b_{[nt_2]-[nt_1]}}{a_{[nt_2]-[nt_1]}},
\]
where the values \( \tilde{a}_{[nt_2]-[nt_1]} \) and \( \tilde{b}_{[nt_2]-[nt_1]} \) are constructed by the random environment \( \tilde{Q}_i := Q_{[nt_1]+i}, i = 1, \ldots, [nt_2]-[nt_1] \), just as the values \( a_{[nt_2]-[nt_1]} \) and \( b_{[nt_2]-[nt_1]} \) are constructed by the random environment \( Q_{[nt_2]-[nt_1]} \).

Further, given \( A_{n,\varepsilon} \), the inequality \( \tau_{[nt_2]} > \tau_{[nt_1]} \) is true (we may assume that \( [nt_2] - i > \tau_{[nt_2]} > \tau_{[nt_1]} + i \)). Thus, if the random environment \( \{Q_n, n \in \mathbb{N}\} \) is fixed, the distribution of the random variable \( U_{[nt_1]}^{(i)} \) is completely determined by the random environment \( Q_{[nt_1]} \) and the distribution of the random variable \( U_{[nt_2]}^{(i)} \) is completely determined by the random environment \( Q_{[nt_1]+1,[nt_2]} \). Moreover, \( U_{[nt_2]}^{(i)} = \tilde{U}_{[nt_2]-[nt_1]}^{(i)} \), where \( \tilde{U}_{[nt_2]-[nt_1]}^{(i)} \) has the same meaning for the environment \( \tilde{Q}_i \), \( i = 1, \ldots, [nt_2] - [nt_1] \), as \( U_{[nt_2]-[nt_1]}^{(i)} \) has for the environment \( Q_{1,[nt_2]-[nt_1]} \).

Summarizing the arguments above, we see that to prove (70) it is sufficient to show that
\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{a_{[nt_1]}}{\tilde{b}_{[nt_1]}} U_{[nt_1]}^{(i)} \leq x_1, \frac{\tilde{a}_{[nt_2]-[nt_1]}}{\tilde{b}_{[nt_2]-[nt_1]}} \tilde{U}_{[nt_2]-[nt_1]}^{(i)} \leq x_2 \bigg| A_{n,\varepsilon} \right)
\]
\[
= \mathbb{P}\left( \frac{1}{\Sigma_1} \sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \leq x_1 \right) \mathbb{P}\left( \frac{1}{\Sigma_1} \sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \leq x_2 \right). \tag{71}
\]

Note that
\[
\begin{align*}
\mathbb{P}\left( \frac{a_{[nt_1]} U_{[nt_1]}^{(i)}}{b_{[nt_1]}} \leq x_1, \frac{\tilde{a}_{[nt_2]-[nt_1]} U_{[nt_2]-[nt_1]}^{(i)}}{b_{[nt_2]-[nt_1]}} \leq x_2, A_{n,\varepsilon} \right) \\
= \int_{-\infty}^{0} \int_{0}^{+\infty} \mathbb{P}\left( \frac{a_{[nt_1]} U_{[nt_1]}^{(i)}}{b_{[nt_1]}} \leq x_1, \frac{L_{[nt_1]} - L_{[nt_1]}}{C_n} \in da, \frac{S_{[nt_1]} - L_{[nt_1]}}{C_n} \in db \right) \\
\times \mathbb{P}\left( \frac{a_{[nt_2]-[nt_1]} U_{[nt_2]-[nt_1]}^{(i)}}{b_{[nt_2]-[nt_1]}} \leq x_2, \frac{L_{[nt_2]-[nt_1]}}{C_n} < b - a - \varepsilon \right).
\end{align*}
\]

Hence, taking into account Remark 7 we deduce that, as \( n \to \infty \),
\[
\begin{align*}
\mathbb{P}\left( \frac{a_{[nt_1]} U_{[nt_1]}^{(i)}}{b_{[nt_1]}} \leq x_1, \frac{\tilde{a}_{[nt_2]-[nt_1]} U_{[nt_2]-[nt_1]}^{(i)}}{b_{[nt_2]-[nt_1]}} \leq x_2, A_{n,\varepsilon} \right) \\
\sim \mathbb{P}\left( \frac{1}{\Sigma_1} \sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \leq x_1 \right) \mathbb{P}\left( \frac{1}{\Sigma_1} \sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \leq x_2 \right) \\
\times \int_{-\infty}^{0} \int_{0}^{+\infty} \mathbb{P}\left( \frac{L_{[nt_1]}}{C_n} \in da, \frac{S_{[nt_1]} - L_{[nt_1]}}{C_n} \in db \right) \\
\times \mathbb{P}\left( \frac{L_{[nt_2]-[nt_1]}}{C_n} < b - a - \varepsilon \right).
\end{align*}
\]

Since the last integral is equal to \( \mathbb{P}(A_{n,\varepsilon}) \), we obtain (71) and, as result, the required relation (70).

It follows from (70) that (see (67))
\[
\lim_{i \to \infty} \lim_{n \to \infty} \mathbb{P}\left( \frac{a_{[nt_1]} U_{[nt_1]}^{(i)}}{b_{[nt_1]}} \leq x_1, \frac{b_{[nt_2]} U_{[nt_2]}^{(i)}}{a_{[nt_k]}} \leq x_2 \mid A_{n,\varepsilon} \right) \\
= \mathbb{P}\left( \frac{\Sigma_2}{\Sigma_1} \leq x_1 \right) \mathbb{P}\left( \frac{\Sigma_2}{\Sigma_1} \leq x_2 \right). \tag{72}
\]

Applying now the same arguments which we have used in First part of the proof to establish (57) from (67), we obtain (68) from (72).

We now prove (69). To this aim we check that, for any fixed \( i \in \mathbb{N} \) and for all but a countable set of \( (x_1, x_2) \) with \( x_1, x_2 \geq 0 \),
\[
\lim_{n \to \infty} \mathbb{P}\left( \frac{a_{[nt_1]} U_{[nt_1]}^{(i)}}{b_{[nt_1]}} \leq x_1, \frac{b_{[nt_2]} U_{[nt_2]}^{(i)}}{a_{[nt_k]}} \leq x_2 \mid B_{n,\varepsilon} \right)
\]

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\[
\begin{align*}
&= P\left( \frac{1}{\Sigma_1} \sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \leq \min(x_1, x_2) \right). \\
\end{align*}
\]

(73)

Set
\[
Z'_{i,n}(m) = Z_{\tau_n+i,m},
\]
\[
U'_{n}^{(i)} (m) = \sum_{j=-i}^{i+1} Z'_{j,n} (m).
\]

Given that the random event \(B_{n,\varepsilon}\) occurred, \(\tau_{\lfloor nt_2 \rfloor} = \tau_{\lfloor nt_1 \rfloor}\) and
\[
\frac{b_{[nt_2]}}{a_{[nt_2]}} \sim \frac{b_{[nt_1]}}{a_{[nt_1]}}
\]
as \(n \to \infty\). Therefore
\[
U'_{[nt_1]}^{(i)} = \sum_{j=-i}^{i+1} Z'_{i,[nt_1]} \left( \lfloor nt_1 \rfloor \right) = U'_{[nt_1]} \left( \lfloor nt_1 \rfloor \right),
\]
\[
U'_{[nt_2]}^{(i)} = \sum_{j=-i}^{i+1} Z'_{i,[nt_2]} \left( \lfloor nt_2 \rfloor \right) = U'_{[nt_1]} \left( \lfloor nt_2 \rfloor \right).
\]

Thus, to prove (73) it is sufficient to show that
\[
\lim_{n \to \infty} P \left( \frac{a_{[nt_1]}}{b_{[nt_1]}} U'_{[nt_1]} \left( \lfloor nt_1 \rfloor \right) \leq x_1, \frac{a_{[nt_2]}}{b_{[nt_1]}} U'_{[nt_1]} \left( \lfloor nt_2 \rfloor \right) \leq x_2 \mid B_{n,\varepsilon} \right)
\]
\[
= P \left( \frac{1}{\Sigma_1} \sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \leq \min(x_1, x_2) \right).
\]

(74)

Applying the arguments similar to those used to establish relation (19), we can show that
\[
\left\{ a'_{i,m} Z'_{i,n} (m), i \in \mathbb{Z} \right\} \overset{D}{\to} \left\{ \zeta_i^*, i \in \mathbb{Z} \right\},
\]
as \(m \geq n \to \infty\). Moreover,
\[
\left\{ (a'_{i,n} Z'_{i,n} (n), a'_{i,m} Z'_{i,n} (m)), i \in \mathbb{Z} \right\} \overset{D}{\to} \left\{ (\zeta_i^*, \zeta_i^*), i \in \mathbb{Z} \right\}
\]
(75)

and the left-hand side of this relation is asymptotically independent from the random event \(\left\{ C_n^{-1} L_n \leq a, C_n^{-1} (S_n - L_n) \leq b \right\}\) for any \(a \leq 0\) and \(b > 0\). It follows from (75) that (see the proof of (64))
\[
\left( \frac{a_n}{b_n} U^{(i)} (n), \frac{a_m}{b_n} U^{(i)} (m) \right) \overset{D}{\to} \frac{1}{\Sigma_1} \left( \sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*}, \sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \right),
\]
(76)
as \( m \geq n \to \infty \). From (76) we obtain the desired relation (74) and, as result, (73). Now statement (69) follows from (73) in a standard way.

Finally, according to the Skorokhod functional limit theorem (see (1))

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbb{P}(A_{n,\varepsilon}) = \mathbb{P}(L(t_1) > L(t_1, t_2)) = \mathbb{P}(L(t_1) > L(t_2)),
\]

(77)

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbb{P}(B_{n,\varepsilon}) = \mathbb{P}(L(t_1) < L(t_1, t_2)) = \mathbb{P}(L(t_1) = L(t_2)),
\]

(78)

where \( L(t_1, t_2) = \inf_{t \in [t_1, t_2]} W(t) \), and

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbb{P}(D_{n,\varepsilon}) = 0.
\]

(79)

By the total probability formula

\[
\begin{align*}
\mathbb{P} & \left( \frac{a_{\lfloor nt_1 \rfloor}}{b_{\lfloor nt_1 \rfloor}} Z_{\lfloor nt_1 \rfloor} \leq x_1, \frac{a_{\lfloor nt_2 \rfloor}}{b_{\lfloor nt_2 \rfloor}} Z_{\lfloor nt_2 \rfloor} \leq x_2 \right) \\
= & \quad \mathbb{P} \left( \frac{a_{\lfloor nt_1 \rfloor}}{b_{\lfloor nt_1 \rfloor}} Z_{\lfloor nt_1 \rfloor} \leq x_1, \frac{a_{\lfloor nt_2 \rfloor}}{b_{\lfloor nt_2 \rfloor}} Z_{\lfloor nt_2 \rfloor} \leq x_2 \ \bigg| \ A_{n,\varepsilon} \right) \mathbb{P}(A_{n,\varepsilon}) \\
& + \mathbb{P} \left( \frac{a_{\lfloor nt_1 \rfloor}}{b_{\lfloor nt_1 \rfloor}} Z_{\lfloor nt_1 \rfloor} \leq x_1, \frac{a_{\lfloor nt_2 \rfloor}}{b_{\lfloor nt_2 \rfloor}} Z_{\lfloor nt_2 \rfloor} \leq x_2 \ \bigg| \ B_{n,\varepsilon} \right) \mathbb{P}(B_{n,\varepsilon}) \\
& + \mathbb{P} \left( \frac{a_{\lfloor nt_1 \rfloor}}{b_{\lfloor nt_1 \rfloor}} Z_{\lfloor nt_1 \rfloor} \leq x_1, \frac{a_{\lfloor nt_2 \rfloor}}{b_{\lfloor nt_2 \rfloor}} Z_{\lfloor nt_2 \rfloor} \leq x_2 \ \bigg| \ D_{n,\varepsilon} \right) \mathbb{P}(D_{n,\varepsilon}).
\end{align*}
\]

Combining (68), (69) and (77)-(80) we deduce that

\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{a_{\lfloor nt_1 \rfloor}}{b_{\lfloor nt_1 \rfloor}} Z_{\lfloor nt_1 \rfloor} \leq x_1, \frac{a_{\lfloor nt_2 \rfloor}}{b_{\lfloor nt_2 \rfloor}} Z_{\lfloor nt_2 \rfloor} \leq x_2 \right) \\
= \quad \mathbb{P} \left( \gamma_1 \leq x_1, \gamma_2 \leq x_2 \right) \mathbb{P}(L(t_1) > L(t_2)) \\
& + \mathbb{P}(\gamma_1 \leq x_1, \gamma_1 \leq x_2) \mathbb{P}(L(t_1) = L(t_2)),
\]

This gives the desired convergence of two-dimensional distributions.

**Third part.** The proof of convergence of multidimensional distributions (for dimensions exceeding two) is carried out by induction using the reasonings of Second part of the proof.

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