Near optimal bounds on quantum communication complexity of single-shot quantum state redistribution

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Abstract

We show near optimal bounds on the worst case quantum communication of single-shot entanglement-assisted one-way quantum communication protocols for the quantum state redistribution task and for the sub-tasks quantum state splitting and quantum state merging. Our bounds are tighter than previously known best bounds for the latter two sub-tasks.

A key technical tool that we use is a convex-split lemma which may be of independent interest.

1 Introduction

A fundamental quantum information theoretic primitive is the following quantum state redistribution task (please refer to Section 2 for definitions of various information theoretic quantities used in this section).

Alice, Bob and Referee share a pure state $|\Psi\rangle_{RABC}$, with $AC$ belonging to Alice, $B$ to Bob and $R$ to Referee. Alice wants to transfer the system $C$ to Bob, such that the final state $\Phi_{RABC} \in B^\epsilon(\Psi_{RABC})$, for a given $\epsilon \geq 0$.

This problem has been well studied in the literature both in the asymptotic and the single-shot settings (see e.g. [HOW07, DY08, Opp08, YBW08, YD09, Ber09, ADHW09, BRWW09, Dup10, BD10, BCR11, DHO14] and references therein). A special case of this task in which system $A$ is not present is referred to as quantum state merging [HOW07]. Another special case when system $B$ is not present is referred to as quantum state splitting [ADHW09]. In the asymptotic setting, it was shown by Devatkar and Yard [DY08, YD09] (see also Luo and Devatkar [LD09]) that the worst case communication per copy, using one-way communication and shared-entanglement, required for quantum state redistribution is tightly characterized by $I(C : R|B)$ (the conditional mutual information). Subsequently, it was shown by Oppenheim [Opp08] (see independent and related work [YBW08]) that quantum state redistribution can be realized with two applications of a protocol for quantum state merging.
Recently, in independent works by Berta, Christandl, Touchette \[BCT14\] and Datta, Hsieh, Oppenheim \[DHO14\], single-shot entanglement-assisted one-way protocols for quantum state redistribution have been proposed with the worst case communication upper bounded by:
\[
\frac{1}{2} \left( H_{\max}^\varepsilon (C|B)_{RABC} - H_{\min}^\varepsilon (C|BR)_{RABC} \right) + \mathcal{O} \left( \log \left( \frac{1}{\varepsilon} \right) \right).
\]

These works also provide several lower bounds with gaps between the upper and lower bounds and the question of closing these gaps has been left open in these works. The upper bound of \[BCT14\] and \[DHO14\] has recently been used by Touchette \[Tou14\] to obtain a direct-sum result for bounded-round entanglement-assisted quantum communication complexity.

Please note that state merging can be viewed as the ‘time reversed’ version of state splitting and hence both of them require the same communication. In \[BCR11\], Berta, Christandl, and Renner have provided upper and lower bounds for state splitting and state merging. We state them below for state splitting and same bounds (with $A$ replaced by $B$) hold for state merging.

**Theorem 1.1** (\[BCR11\]). Let $|\Psi\rangle_{RAC}$ be a pure state. There exists an entanglement-assisted one-way state splitting protocol $P$, which takes as input $|\Psi\rangle_{RAC}$ shared between two parties Referee ($R$) and Alice ($AC$), uses embezzling states as shared entanglement, and outputs a state $\Phi_{RAC}$ shared between Referee ($R$), Bob ($C$) and Alice ($A$) such that $\Phi_{RAC} \in \mathcal{B}^{3\varepsilon}(\Psi_{RAC})$. The communication cost of the protocol is upper bounded by
\[
\frac{1}{2} I_{\max}^\varepsilon (R : C)_{\Psi_{RC}} + \log \log (|C|) + 4 + 2 \left( \log \left( \frac{1}{\varepsilon} \right) \right).
\]

Furthermore, let $Q$ be any entanglement-assisted one-way protocol (with communication from Alice to Bob), which takes as input $|\Psi\rangle_{RAC}$ shared between two parties Referee ($R$) and Alice ($AC$) and outputs a state $\Phi_{RAC}$ shared between Referee ($R$), Bob ($C$) and Alice ($A$) such that $\Phi_{RAC} \in \mathcal{B}^{\varepsilon}(\Psi_{RAC})$. The number of qubits communicated by Alice to Bob in $Q$ is lower bounded by,
\[
\frac{1}{2} I_{\max}^\varepsilon (R : C)_{\Psi_{RC}}.
\]

**Our results**

In this work, we obtain nearly tight upper and lower bounds on the worst case communication required by single-shot entanglement-assisted one-way protocols for quantum state redistribution. Following is the key information theoretic quantity that we define and consider.

**Definition.** Let $\varepsilon \geq 0$ and $|\Psi\rangle_{RABC}$ be a pure state. Define,
\[
Q_{|\Psi\rangle_{RABC}}^\varepsilon \overset{\text{def}}{=} \inf_{T, U_{BCT}, \sigma_T', \kappa_{RBCT}} I_{\max}^\varepsilon (RB : CT)_{\kappa_{RBCT}} = \inf_{T, U_{BCT}, \sigma_T', \kappa_{RBCT}} D_{\max}^\varepsilon (\kappa_{RBCT} \parallel \kappa_{RB} \otimes \sigma_{CT})
\]
with the conditions $U_{BCT} \in \mathcal{U}(BCT), \sigma_T' \in \mathcal{D}(\mathcal{H}_T), \sigma_{CT} \in \mathcal{D}(\mathcal{H}_{CT})$ and
\[
(I_R \otimes U_{BCT}) \kappa_{RBCT} (I_R \otimes U_{BCT}^\dagger) \in \mathcal{B}^\varepsilon (\Psi_{RBC} \otimes \sigma_T'), \kappa_{RB} \in \mathcal{B}^\varepsilon (\Psi_{RB}).
\]

\[^1\]It may be noted that these works and several others, optimize over sub-normalized states in their definitions of various information theoretic quantities, while in our definitions, we only optimize over normalized states. This changes the quantities up to an additive $\pm \varepsilon$ factor.
We show the following upper bound.

**Theorem** (Upper bound). Let \( \varepsilon \in (0, 1/3) \) and \( |\Psi\rangle_{RABC} \) be a pure state. There exists an entanglement-assisted one-way protocol \( \mathcal{P} \), which takes as input \( |\Psi\rangle_{RABC} \) shared between three parties Referee (\( R \)), Bob (\( B \)) and Alice (\( AC \)) and outputs a state \( \Phi_{RABC} \) shared between Referee (\( R \)), Bob (\( BC \)) and Alice (\( A \)) such that \( \Phi_{RABC} \in \mathcal{B}^{3\varepsilon}(\Psi_{RABC}) \). If \( Q_{\Psi_{RABC}}^{\varepsilon} \leq \varepsilon^2/2 \), there is no communication from Alice to Bob in \( \mathcal{P} \). Otherwise the number of qubits communicated by Alice to Bob in \( \mathcal{P} \) is upper bounded by:

\[
\frac{1}{2} Q_{\Psi_{RABC}}^{\varepsilon} + \log \left( \frac{6}{\varepsilon^2} Q_{\Psi_{RABC}}^{\varepsilon} \right) + O \left( \log \left( \frac{1}{\varepsilon} \right) \right).
\]

We show the following lower bound.

**Theorem** (Lower bound). Let \( \varepsilon > 0 \) and \( |\Psi\rangle_{RABC} \) be a pure state. Let \( \mathcal{Q} \) be an entanglement-assisted one-way protocol (with communication from Alice to Bob), which takes as input \( |\Psi\rangle_{RABC} \) shared between three parties Referee (\( R \)), Bob (\( B \)) and Alice (\( AC \)) and outputs a state \( \Phi_{RABC} \) shared between Referee (\( R \)), Bob (\( BC \)) and Alice (\( A \)) such that \( \Phi_{RABC} \in \mathcal{B}^{\varepsilon}(\Psi_{RABC}) \). The number of qubits communicated by Alice to Bob in \( \mathcal{Q} \) is lower bounded by:

\[
\frac{1}{2} Q_{\Psi_{RABC}}^{\varepsilon}.
\]

As a corollary of the above we show the following upper and lower bounds for state splitting.

**Corollary 1.2.** Let \( \varepsilon \in (0, 1/3) \) and \( |\Psi\rangle_{RAC} \) be a pure state. There exists an entanglement-assisted one-way protocol \( \mathcal{P} \), which takes as input \( |\Psi\rangle_{RAC} \) shared between two parties Referee (\( R \)) and Alice (\( AC \)) and outputs a state \( \Phi_{RAC} \) shared between Referee (\( R \)), Bob (\( C \)) and Alice (\( A \)) such that \( \Phi_{RAC} \in \mathcal{B}^{\varepsilon}(\Psi_{RAC}) \). If \( I_{\max}(R : C)_{\Psi_{RC}} \leq \varepsilon^2/2 \), then there is no communication from Alice to Bob in \( \mathcal{P} \). Otherwise the number of qubits communicated by Alice to Bob in \( \mathcal{P} \) is upper bounded by:

\[
\frac{1}{2} I_{\max}(R : C)_{\Psi_{RC}} + \log \left( \frac{6}{\varepsilon^2} I_{\max}(R : C)_{\Psi_{RC}} \right) + O \left( \log \left( \frac{1}{\varepsilon} \right) \right).
\]

Furthermore, let \( \mathcal{Q} \) be any entanglement-assisted one-way protocol (with communication from Alice to Bob), which takes as input \( |\Psi\rangle_{RAC} \) shared between two parties Referee (\( R \)) and Alice (\( AC \)) and outputs a state \( \Phi_{RAC} \) shared between Referee (\( R \)), Bob (\( C \)) and Alice (\( A \)) such that \( \Phi_{RAC} \in \mathcal{B}^{\varepsilon}(\Psi_{RAC}) \). The number of qubits communicated by Alice to Bob in \( \mathcal{Q} \) is lower bounded by,

\[
\frac{1}{2} I_{\max}(R : C)_{\Psi_{RC}}.
\]

Same bounds hold for state merging.

**Corollary 1.3.** Let \( \varepsilon \in (0, 1/3) \) and \( |\Psi\rangle_{RAC} \) be a pure state. There exists an entanglement-assisted one-way protocol \( \mathcal{P} \), which takes as input \( |\Psi\rangle_{RBC} \) shared between three parties Referee (\( R \)), Alice (\( C \)) and Bob (\( B \)) and outputs a state \( \Phi_{RBC} \) shared between Referee (\( R \)) and Bob (\( BC \)) such that \( \Phi_{RBC} \in \mathcal{B}^{\varepsilon}(\Psi_{RBC}) \). If \( I_{\max}(R : C)_{\Psi_{RC}} \leq \varepsilon^2/2 \), then there is no communication from Alice to Bob in \( \mathcal{P} \). Otherwise the number of qubits communicated by Alice to Bob in \( \mathcal{P} \) is upper bounded by:

\[
\frac{1}{2} I_{\max}(R : C)_{\Psi_{RC}} + \log \left( \frac{6}{\varepsilon^2} I_{\max}(R : C)_{\Psi_{RC}} \right) + O \left( \log \left( \frac{1}{\varepsilon} \right) \right).
\]
Furthermore, let $Q$ be any entanglement-assisted one-way protocol (with communication from Alice to Bob), which takes as input $|\Psi\rangle_{RBC}$ shared between three parties Referee ($R$), Alice ($C$) and Bob ($B$), and outputs a state $\Phi_{RBC}$ shared between Referee ($R$) and Bob ($BC$), such that $\Phi_{RBC} \in B^e(\Psi_{RBC})$. The number of qubits communicated by Alice to Bob in $Q$ is lower bounded by,

$$\frac{1}{2} I_{\max}^e(R : C)_{\Psi_{RC}}.$$

Please note that our bounds above are an improvement over the bounds of Berta, Christandl, and Renner [BCR11] as stated in Theorem 1.1 since $I_{\max}^e(R : C)_{\Psi_{RC}} \leq 2 \log |C|$ (Fact 2.8).

Our techniques

Central to our approach for the upper bound is the following convex-split lemma. This lemma may be of independent interest.

**Lemma (Convex-split lemma).** Let $\rho_{PQ}$ be a state on two registers $P, Q$. Let $\sigma_Q$ be a state on register $Q$. Let $k \equiv D_{\text{max}}(\rho_{PQ}||\rho_P \otimes \sigma_Q)$, $\delta \in (0, 1/6)$. If $k \leq 3\delta$ let $n = 1$, else let $n \equiv \frac{8k \cdot \log(k/\delta)}{\delta^4}$. Define the following state (please also refer to Figure 7),

$$\tau_{PQ_1Q_2...Q_n} \equiv \frac{1}{n} \sum_{j=1}^{n} \rho_{PQ_j} \otimes \sigma_{Q_1} \otimes \ldots \otimes \sigma_{Q_{j-1}} \otimes \sigma_{Q_{j+1}} \otimes \ldots \otimes \sigma_{Q_n},$$

on $n + 1$ registers $P, Q_1, Q_2 \ldots Q_n$. Then,

$$I(P : Q_1Q_2 ... Q_n)_{\tau} \leq 3\delta \quad \text{and} \quad F^2(\tau_P \otimes \tau_{Q_1Q_2...Q_n}, \tau_{PQ_1Q_2...Q_n}) \geq 1 - 6\delta.$$

We use this lemma for state redistribution as follows. Suppose Alice and Referee share a pure state $\rho_{PAQ}$ with Alice holding registers $A$ and Referee holding register $P$. Their objective is that Referee and Bob together end up with $\rho_{PQ}$. Let $|\theta\rangle_{SQ_1...Q_n}$ be a purification of $\tau_{Q_1Q_2...Q_n}$ with Alice holding register $S$ and Bob holding registers $Q_1 \ldots Q_n$. Let $\delta > 0$ be small. Since $\tau_P \otimes \tau_{Q_1Q_2...Q_n}$ and $\tau_{PQ_1Q_2...Q_n}$ are close states and $\tau_P = \rho_P$, the state $\rho_{PAQ} \otimes |\theta\rangle_{SQ_1...Q_n}$ can be thought of as (nearly) a purification of $\tau_{PQ_1Q_2...Q_n}$. Alice does appropriate measurement on her registers to realize the convex split as in Eq. [1]. After this, Alice tells Bob $j \in [n]$ (employing super-dense coding using fresh entanglement) indicating which state in the convex split has occurred. Bob then picks up the register $Q_j$ to obtain the state $\rho_{PQ_j}$ jointly between Referee and Bob. The number of qubits communicated by Alice is

$$\log(n) \leq \frac{1}{2} D_{\text{max}}(\rho_{PQ}||\rho_P \otimes \sigma_Q) + \log \left( \frac{1}{\delta} D_{\text{max}}(\rho_{PQ}||\rho_P \otimes \sigma_Q) \right) + O \left( \log \left( \frac{1}{\delta} \right) \right).$$

The precise arguments for our upper bound appear in the proof of Theorem 4.2 (in which $P$ also involves Bob’s register $B$).

To show our lower bound we sketch a general form of any entanglement-assisted one-way protocol for state-redistribution. The unitary transformations, number of qubits transferred and shared registers appearing in such a protocol correspond to the quantities that have been used in the definition of $Q_{\Psi_{RABC}}^4$. Precise argument appear in the proof of Theorem 4.3.

Protocols similar in spirit to ours, however for classical-quantum states, have appeared previously for example in [JRS02, JRS05, JRS08, AJM+14].
Organization

In Section 2 we present some definitions, facts and lemmas that are needed for our proofs. In Section 3 we state and prove the convex-split lemma. In Section 4 we prove the upper and lower bounds on communication for state redistribution. We prove the communication bounds for state splitting and state merging in Section 5. We conclude with some miscellaneous discussion and open questions in Section 6.

2 Preliminaries

In this section we present some notations, definitions, facts and lemmas that we will use later in our proofs. Readers may refer to [CT91, NC00, Wat11] for good introduction to classical and quantum information theory.

Information theory

For a natural number $n$, let $[n]$ represent the set $\{1, 2, \ldots, n\}$. We let log represent logarithm to the base 2 and ln represent logarithm to the base e. The $\ell_1$ norm of an operator $X$ is $\|X\|_1 \overset{\text{def}}{=} \text{Tr}\sqrt{X\dagger X}$ and $\ell_2$ norm is $\|X\|_2 \overset{\text{def}}{=} \sqrt{\text{Tr}XX\dagger}$. A quantum state (or just a state) is a positive semi-definite matrix with trace equal to 1. It is called pure if and only if the rank is 1. Let $|\psi\rangle$ be a unit vector. We use $\psi$ to represent the state and also the density matrix $|\psi\rangle\langle\psi|$, associated with $|\psi\rangle$. 

Figure 1: Convex split
A sub-normalized state is a positive semidefinite matrix with trace less than or equal to 1. A quantum register $A$ is associated with some Hilbert space $\mathcal{H}_A$. Define $|A| \overset{\text{def}}{=} \dim(\mathcal{H}_A)$. We denote by $\mathcal{D}(A)$, the set of quantum states in the Hilbert space $\mathcal{H}_A$. State $\rho$ with subscript $A$ indicates $\rho_A \in \mathcal{D}(A)$.

For two quantum states $\rho$ and $\sigma$, $\rho \otimes \sigma$ represents the tensor product (Kronecker product) of $\rho$ and $\sigma$. Composition of two registers $A$ and $B$, denoted $AB$, is associated with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. If two registers $A, B$ are associated with the same Hilbert space, we shall denote it by $A \equiv B$. Let $\rho_{AB}$ be a bipartite quantum state in registers $AB$. We define

$$\rho_B \overset{\text{def}}{=} \text{Tr}_A(\rho_{AB}) \overset{\text{def}}{=} \sum_i (\langle i | \otimes 1_B) \rho_{AB} (| i \rangle \otimes 1_B),$$

where $\{|i\rangle\}$ is an orthonormal basis for the Hilbert space $A$ and $1_B$ is the identity matrix in space $B$. The state $\rho_B$ is referred to as the marginal state of $\rho_{AB}$ in register $B$. Unless otherwise stated, a missing register from subscript in a state will represent partial trace over that register. A quantum super-operator (a.k.a quantum map a.k.a quantum operation) $E : A \rightarrow B$ is a completely positive and trace preserving (CPTP) linear map (mapping states from $\mathcal{D}(A)$ to states in $\mathcal{D}(B)$). The identity operator in Hilbert space $\mathcal{H}_A$ (and associated register $A$) is denoted $I_A$. A unitary operator $U_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$ is such that $U_A^\dagger U_A = U_A U_A^\dagger = I_A$. An isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B$ is such that $V^\dagger V = I_A$ and $VV^\dagger = I_B$. The set of all unitary operations on register $A$ is denoted by $U(A)$.

**Definition 2.1.** We shall consider the following information theoretic quantities. Reader is referred to [TCR10, Tom12, Dat09] for many of these definitions. We consider only normalized states in the definitions below. Let $\varepsilon \geq 0$.

1. **fidelity**

   $$F(\rho, \sigma) \overset{\text{def}}{=} \|\sqrt{\rho} \sqrt{\sigma}\|_1.$$

2. **purified distance**

   $$\mathcal{P}(\rho, \sigma) = \sqrt{1 - F^2(\rho, \sigma)}.$$

3. **$\varepsilon$-ball**

   $$\mathcal{B}^\varepsilon(\rho_A) \overset{\text{def}}{=} \{\rho'_A \in \mathcal{D}(A) | \mathcal{P}(\rho_A, \rho'_A) \leq \varepsilon\}.$$

4. **entropy**

   $$H(A)_\rho \overset{\text{def}}{=} -\text{Tr}(\rho_A \log \rho_A).$$

5. **relative entropy**

   $$D(\rho||\sigma) \overset{\text{def}}{=} \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma).$$

6. **max-relative entropy**

   $$D_{\text{max}}(\rho||\sigma) \overset{\text{def}}{=} \inf\{\lambda \in \mathbb{R} : 2^\lambda \sigma \geq \rho\}.$$

7. **mutual information**

   $$I(A : B)_\rho \overset{\text{def}}{=} D(\rho_{AB}||\rho_A \otimes \rho_B) = H(A)_\rho + H(B)_\rho - H(AB)_\rho.$$
8. conditional mutual information
\[ I(A : B|C)_\rho \overset{\text{def}}{=} I(A : BC)_\rho - I(A : C)_\rho = I(B : AC)_\rho - I(B : C)_\rho. \]

9. max-information
\[ I_{\max}(A : B)_\rho \overset{\text{def}}{=} \inf_{\sigma_B \in \mathcal{D}(B)} D_{\max}(\rho_{AB} \| \rho_A \otimes \sigma_B). \]

10. smooth max-information
\[ I_{\epsilon_{\max}}(A : B)_\rho \overset{\text{def}}{=} \inf_{\rho' \in \mathcal{B}(\rho)} I_{\max}(A : B)_{\rho'}. \]

11. conditional min-entropy
\[ H_{\min}(A|B)_\rho \overset{\text{def}}{=} -\inf_{\sigma_B \in \mathcal{D}(B)} D_{\max}(\rho_{AB} \| I_A \otimes \sigma_B). \]

12. conditional max-entropy
\[ H_{\max}(A|B)_{\rho_{AB}} \overset{\text{def}}{=} -H_{\min}(A|R)_{\rho_{AB}}, \]
where \( \rho_{ABR} \) is a purification of \( \rho_{AB} \) for some system \( R \).

13. smooth conditional min-entropy
\[ H_{\epsilon_{\min}}(A|B)_\rho \overset{\text{def}}{=} \sup_{\rho' \in \mathcal{B}(\rho)} H_{\min}(A|B)_{\rho'}. \]

14. smooth conditional max-entropy
\[ H_{\epsilon_{\max}}(A|B)_\rho \overset{\text{def}}{=} \inf_{\rho' \in \mathcal{B}(\rho)} H_{\max}(A|B)_{\rho'}. \]

15. fidelity of recovery (SW14)
\[ F(A : C|B)_{\rho_{ABC}} \overset{\text{def}}{=} \sup_{E : B \rightarrow BC} F(\rho_{ABC}, E(\rho_{AB})). \]

We will use the following facts.

**Fact 2.2** (Triangle inequality for purified distance, [Tom12]). For states \( \rho_1, \rho_2, \rho_3 \),
\[ \mathcal{P}(\rho_1, \rho_3) \leq \mathcal{P}(\rho_1, \rho_2) + \mathcal{P}(\rho_2, \rho_3). \]

**Fact 2.3** (Uhlmann’s theorem). [NC00, Theorem 9.4] Let \( \rho_A \) and \( \sigma_A \) be two quantum states in a register \( A \). Let \( |\rho\rangle_{AB} \) be a purification of \( \rho_A \) and \( |\sigma\rangle_{AC} \) be a purification of \( \sigma_A \). There exists an isometry \( V : \mathcal{H}_C \rightarrow \mathcal{H}_B \) such that,
\[ F(|\theta\rangle\langle\theta|_{AB}, |\rho\rangle\langle\rho|_{AB}) = F(\rho_A, \sigma_A), \]
where \( |\theta\rangle_{AB} = (I_A \otimes V) |\sigma\rangle_{AC}. \)
Fact 2.4 (Stinespring representation). \[\text{[Wat11], Theorem 5.3}\] Let $\mathcal{E}(\cdot) : A \rightarrow B$ be a quantum operation. There exists a Hilbert space $C$ and an unitary $U : A \otimes B \otimes C \rightarrow A \otimes B \otimes C$ such that $\mathcal{E}(\omega) = \text{Tr}_{A,C}(U(\omega \otimes |0\rangle \langle 0|_{B,C})U^\dagger)$.

Fact 2.5 (Quantum operations do not increase distance). \[\text{[Wat11], Theorem 11.6, [NC00], page 406, 414}\] For states $\rho$, $\sigma$, and quantum operation $\mathcal{E}(\cdot)$,
\[\|\mathcal{E}(\rho) - \mathcal{E}(\sigma)\|_1 \leq \|\rho - \sigma\|_1, D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leq D(\rho\|\sigma) \text{ and } F(\rho, \sigma) \leq F(\mathcal{E}(\rho), \mathcal{E}(\sigma)).\]

In particular, for bipartite states $\rho_{AB}$ and $\sigma_{AB}$,
\[\|\rho_{A} - \sigma_{A}\|_1 \leq \|\rho_{AB} - \sigma_{AB}\|_1, D(\rho_{A}\|\sigma_{A}) \leq D(\rho_{AB}\|\sigma_{AB}) \text{ and } F(\rho_{AB}, \sigma_{AB}) \leq F(\rho_{A}, \sigma_{A}).\]

Fact 2.6. For a quantum states $\rho$, $\sigma$,
\[D_{\text{max}}(\rho\|\sigma) \geq D(\rho\|\sigma).\]

Fact 2.7. For a quantum state $\rho_{AB}$,
\[I(A : B)_{\rho} = \inf_{\sigma_B} D(\rho_{AB}\|\rho_A \otimes \sigma_B) = D(\rho_{AB}\|\rho_A \otimes \rho_B).\]

Fact 2.8 (BCR11, Lemma B.7). For a quantum state $\rho_{AB}$,
\[I_{\text{max}}(A : B)_{\rho} \leq 2 \cdot \min\{\log |A|, \log |B|\}.\]

Fact 2.9 (BCR11, Lemma B.14). For a quantum state $\rho_{ABC}$,
\[I_{\text{max}}(A : BC)_{\rho} \geq I_{\text{max}}(A : B)_{\rho}.\]

Fact 2.10 (Pinsker’s inequality). \[\text{[DCHR78]}\] For quantum states $\rho$ and $\sigma$,
\[F(\rho, \sigma) \geq 2^{-\frac{1}{2}D(\rho\|\sigma)}.\]

This implies,
\[1 - F(\rho, \sigma) \leq \frac{\ln 2}{2} \cdot D(\rho\|\sigma) \leq D(\rho\|\sigma).\]

Fact 2.11 (Joint convexity of relative entropy). \[\text{[Wat11], Theorem 11.2}\] Let $\rho^0, \rho^1, \sigma^0, \sigma^1$ be quantum states and let $\lambda \in [0, 1]$. Then
\[D\left(\lambda \rho^0 + (1 - \lambda)\rho^1\|\lambda \sigma^0 + (1 - \lambda)\sigma^1\right) \leq \lambda D\left(\rho^0\|\sigma^0\right) + (1 - \lambda)D\left(\rho^1\|\sigma^1\right).\]

Fact 2.12 (Chernoff bounds). Let $X_1, \ldots, X_n$ be independent random variables, with each $X_i \in [0, 1]$ always. Let $X \overset{\text{def}}{=} X_1 + \cdots + X_n$ and $\mu \overset{\text{def}}{=} \mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$. Then for any $0 \leq \varepsilon \leq 1$,
\[P_r[X \geq (1 + \varepsilon)\mu] \leq \exp\left(-\frac{\varepsilon^2}{3}\mu\right)\]
\[P_r[X \leq (1 - \varepsilon)\mu] \leq \exp\left(-\frac{\varepsilon^2}{2}\mu\right).\]
We will need the following lemmas.

**Lemma 2.13** ([NC00], Equation 11.135). Let $p$ and $q$ be probability distributions on $[m]$. Let
\[
\rho \defeq \sum_{x=1}^{m} p(x) \rho_x \quad \text{and} \quad \sigma \defeq \sum_{x=1}^{m} q(x) \sigma_x.
\]
Then $D(\rho\|\sigma) \leq D(p\|q) + \sum_{x=1}^{m} p(x) D(\rho_x\|\sigma_x)$.

**Proof.** Define $\rho' \defeq \sum_{x=1}^{m} p(x) \rho_x \otimes |x\rangle\langle x|$ and $\sigma' \defeq \sum_{x=1}^{m} q(x) \sigma_x \otimes |x\rangle\langle x|$. Consider,
\[
D(\rho\|\sigma) \leq D(\rho'\|\sigma') \quad \text{(Fact 2.5)}
\]
\[
= \sum_{x} p(x) \cdot \text{Tr}(\rho_x \otimes |x\rangle\langle x|)(\log \rho' - \log \sigma')
\]
\[
= \sum_{x} p(x) [\log(p(x)) + \text{Tr}\rho_x \log(\rho_x) - \log(q(x)) - \text{Tr}(\rho_x \log(\sigma_x))]
\]
\[
= \sum_{x} p(x) \log \left( \frac{p(x)}{q(x)} \right) + \sum_{x} p(x) D(\rho_x\|\sigma_x)
\]
\[
= D(p\|q) + \sum_{x} p(x) D(\rho_x\|\sigma_x).
\]

\[\square\]

**Lemma 2.14.** Let $\epsilon > 0$. Let $|\psi\rangle_A$ be a pure state and let $\rho_{AB}$ be a state such that $F(|\psi\rangle\langle\psi|_A, \rho_A) \geq 1 - \epsilon$. There exists a state $\theta_B$ such that $F(|\psi\rangle\langle\psi|_A \otimes \theta_B, \rho_{AB}) \geq 1 - \epsilon$.

**Proof.** Let $|\rho\rangle_{ABC}$ be a purification of $\rho_{AB}$. From Fact 2.3 we get a pure state $\theta_{BC}$ such that
\[
1 - \epsilon \leq F(|\psi\rangle\langle\psi|_A, \rho_A)
\]
\[
\quad = F(|\psi\rangle\langle\psi|_A \otimes |\theta\rangle\langle\theta|_{BC}, |\rho\rangle\langle\rho|_{ABC})
\]
\[
\quad \leq F(|\psi\rangle\langle\psi|_A \otimes \theta_B, \rho_{AB}). \quad \text{(Fact 2.5)}
\]

\[\square\]

### 3 A convex-split lemma

Following is our key lemma. It may be of independent interest.

**Lemma 3.1** (Convex-split lemma). Let $\rho_{PQ}$ be a state on two registers $P,Q$. Let $\sigma_Q$ be a state on register $Q$. Let $\delta \in (0,1/6)$ and $k \defeq D_{\max}(\rho_{PQ}\|\rho_P \otimes \sigma_Q)$. If $k \leq 3\delta$ let $n \defeq 1$, else let $n \defeq \frac{8^k \log(k/\delta)}{\delta^3}$. Define the following state (please also refer to Figure 1),
\[
\tau_{PQ_1Q_2\ldots Q_n} \defeq \frac{1}{n} \sum_{j=1}^{n} \rho_{PQ_j} \otimes \sigma_{Q_1} \otimes \ldots \otimes \sigma_{Q_{j-1}} \otimes \sigma_{Q_{j+1}} \otimes \ldots \otimes \sigma_{Q_n},
\]
on $n+1$ registers $P,Q_1,Q_2\ldots Q_n$, where $\forall j \in [n] : \rho_{PQ_j} = \rho_{PQ}$. Then,
\[
I(P : Q_1Q_2\ldots Q_n)_\tau \leq 3\delta \quad \text{and} \quad F^2(\tau_P \otimes \tau_{Q_1Q_2\ldots Q_n}, \tau_{PQ_1Q_2\ldots Q_n}) \geq 1 - 6\delta.
\]
Proof. If $D_{\max}(\rho_{PQ}\|\rho_P \otimes \sigma_Q) \leq 3\delta$, then
\[
I(P : Q_1)_\tau = I(P : Q)_\rho = D(\rho_{PQ}\|\rho_P \otimes \rho_Q) \leq D(\rho_{PQ}\|\rho_P \otimes \sigma_Q) \quad \text{(Fact 2.7)} \\
\leq D_{\max}(\rho_{PQ}\|\rho_P \otimes \sigma_Q) \quad \text{(Fact 2.6)} \leq 3\delta.
\]
Otherwise assume $k > 3\delta$. Since $k = D_{\max}(\rho_{PQ}\|\rho_P \otimes \sigma_Q)$, we have $\rho_{PQ} \leq 2^k(\rho_P \otimes \sigma_Q)$. Let
\[
\rho_P \otimes \sigma_Q = 2^{-k}\rho_{PQ} + (1 - 2^{-k})\rho'_{PQ},
\]
for some state $\rho'_{PQ}$. This implies, $\sigma_Q = 2^{-k}\rho_Q + (1 - 2^{-k})\rho'_Q$. For this proof, it is convenient to call $\rho_Q$ as $\rho'_Q$ and $\rho'_Q$ as $\rho''_Q$. Define,
\[
\sigma^{-j} \overset{\text{def}}{=} \sigma_{Q_1} \otimes \ldots \otimes \sigma_{Q_{j-1}} \otimes \sigma_{Q_{j+1}} \otimes \ldots \otimes \sigma_{Q_n}.
\]
Note that $\tau_P = \rho_P$. Consider,
\[
I(P : Q_1Q_2 \ldots Q_n)_\tau \\
= H(P)_\tau + H(Q_1Q_2 \ldots Q_n)_\tau - H(PQ_1Q_2 \ldots Q_n)_\tau \\
= -\text{Tr}(\rho_P \log(\rho_P)) - \frac{1}{n} \sum_{j=1}^{n} \text{Tr}((\rho_{Q_j} \otimes \sigma^{-j}) \log(\tau_{Q_1Q_2 \ldots Q_n})) + \frac{1}{n} \sum_{j=1}^{n} \text{Tr}((\rho_{PQ_j} \otimes \sigma^{-j}) \log(\tau_{PQ_1Q_2 \ldots Q_n})) \\
= \frac{1}{n} \sum_{j=1}^{n} \left( D(\rho_{PQ_j} \otimes \sigma^{-j}\|\rho_P \otimes \tau_{Q_1Q_2 \ldots Q_n}) - D(\rho_{PQ_j} \otimes \sigma^{-j}\|\tau_{PQ_1Q_2 \ldots Q_n}) \right) \\
\leq \frac{1}{n} \sum_{j=1}^{n} \left( D(\rho_{PQ_j} \otimes \sigma^{-j}\|\rho_P \otimes \tau_{Q_1Q_2 \ldots Q_n}) - D(\rho_{PQ_j} \|\tau_{PQ_j}) \right) \quad \text{(Fact 2.5)} \\
= D(\rho_{PQ_n} \otimes \sigma_{Q_1} \otimes \ldots \otimes \sigma_{Q_{n-1}}\|\rho_P \otimes \tau_{Q_1Q_2 \ldots Q_n}) - D(\rho_{PQ_n} \|\tau_{PQ_n}). \tag{2}
\]
The last equality above comes from the symmetry of $\tau$ under interchange of any two registers $Q_i, Q_j$. Observe that
\[
\tau_{PQ_n} = \frac{1}{n} \rho_{PQ_n} + \frac{n - 1}{n} \rho_P \otimes \sigma_{Q_n}.
\]
Since $\rho_{PQ_n} \leq 2^k(\rho_P \otimes \sigma_{Q_n})$, we have
\[
\tau_{PQ_n} \leq \left(1 + \frac{2^k}{n}\right)(\rho_P \otimes \sigma_{Q_n}).
\]
This implies, using operator monotonicity of log,
\[
D(\rho_{PQ_n}\|\tau_{PQ_n}) = \text{Tr}_{\rho_{PQ_n}} \log(\rho_{PQ_n}) - \text{Tr}_{\rho_{PQ_n}} \log(\tau_{PQ_n}) \\
\geq \text{Tr}_{\rho_{PQ_n}} \log(\rho_{PQ_n}) - \text{Tr}_{\rho_{PQ_n}} \log \left(1 + \frac{2^k}{n}\right)(\rho_P \otimes \sigma_{Q_n}) \\
= D(\rho_{PQ_n}\|\rho_P \otimes \sigma_{Q_n}) - \log \left(1 + \frac{2^k}{n}\right). \tag{3}
\]
For a string $s \in \{0,1\}^{n-1}$, define (below $|s|$ represents the number of 1s in $s$),

$$\tau(s) \overset{\text{def}}{=} \frac{2^{-k|s|/(1 - 2^{-k}n-1-|s|)}}{n} (|s|2^k + 1)$$

and

$$\sigma(s) \overset{\text{def}}{=} 2^{-k|s|/(1 - 2^{-k}n-1-|s|)}.$$

Observe that,

$$\tau_{Q_1 \ldots Q_n} = \sum_{s \in \{0,1\}^{n-1}} \tau(s) \cdot \rho_{Q_1}^s \otimes \cdots \otimes \rho_{Q_{n-1}}^s \cdot \left( \frac{2^{-k|s|} + |s|\sigma_{Q_{n-1}}}{2^{-k} + |s|} \right)$$

$$\sigma_{Q_1} \otimes \cdots \otimes \sigma_{Q_{n-1}} = \sum_{s \in \{0,1\}^{n-1}} \sigma(s) \cdot \rho_{Q_1}^s \otimes \cdots \otimes \rho_{Q_{n-1}}^s.$$

Consider,

$$D(\rho_{PQ_n} \otimes \sigma_{Q_1} \ldots \sigma_{Q_{n-1}} \| \rho_P \otimes \tau_{Q_1 \ldots Q_n})$$

$$\leq \sum_{s \in \{0,1\}^{n-1}} \sigma(s) \log \left( \frac{\sigma(s)}{\tau(s)} \right) + \sum_{s \in \{0,1\}^{n-1}} \sigma(s) D(\rho_{PQ_n} \parallel \rho_P \otimes \left( \frac{2^{-k|s|} + |s|\sigma_{Q_{n-1}}}{2^{-k} + |s|} \right))$$

(Lemma 2.13)

$$\leq \left( \sum_{s \in \{0,1\}^{n-1}} \sigma(s) \log \left( \frac{n}{|s|2^k + 1} \right) \right) + D(\rho_{PQ_n} \parallel \rho_P \otimes \sigma_{Q_n})$$

(Fact 2.7 and Fact 2.11)

$$= D(\rho_{PQ_n} \parallel \rho_P \otimes \sigma_{Q_n}) + \sum_{s:|s|\geq \frac{n-1}{2^{k-1}(1-\delta)}} \sigma(s) \log \left( \frac{n}{|s|2^k + 1} \right) + \sum_{s:|s|< \frac{n-1}{2^{k-1}(1-\delta)}} \sigma(s) \log \left( \frac{n}{|s|2^k + 1} \right)$$

$$\leq D(\rho_{PQ_n} \parallel \rho_P \otimes \sigma_{Q_n}) + \log \left( \frac{1}{1-\delta} \right) + \sum_{s:|s|< \frac{n-1}{2^{k-1}(1-\delta)}} \sigma(s) \log(n)$$

$$\leq D(\rho_{PQ_n} \parallel \rho_P \otimes \sigma_{Q_n}) + \log \left( \frac{1}{1-\delta} \right) + (n) \cdot \exp \left( -\frac{\delta^2 \cdot 2^{-k} \cdot (n-1)}{2} \right)$$

(Fact 2.12)

Recall that $k > 3\delta$. Combining, Eqs. (2), (3), (4) and choice of $n$, we get,

$$I(P : Q_1Q_2 \ldots Q_n) \leq \log \left( \frac{1}{1-\delta} \right) + \log(n) \cdot \exp \left( -\frac{\delta^2 \cdot 2^{-k} \cdot (n-1)}{2} \right) + \log \left( 1 + \frac{2^k}{n} \right) \leq 3\delta.$$

This implies,

$$F(\tau_P \otimes \tau_{Q_1Q_2 \ldots Q_n}, \tau_{PQ_1Q_2 \ldots Q_n}) \geq 1 - D(\tau_{PQ_1Q_2 \ldots Q_n} \parallel \tau_P \otimes \tau_{Q_1Q_2 \ldots Q_n})$$

(Fact 2.10)

$$= 1 - I(P : Q_1Q_2 \ldots Q_n) \geq 1 - 3\delta.$$

And hence,

$$F^2(\tau_P \otimes \tau_{Q_1Q_2 \ldots Q_n}, \tau_{PQ_1Q_2 \ldots Q_n}) \geq (1 - 3\delta)^2 \geq 1 - 6\delta. \qed$$
4 Communication bounds on state redistribution

Upper bound

We begin with the following definition.

**Definition 4.1.** Let \( \varepsilon \geq 0 \) and \(|\Psi\rangle_{RABC}\) be a pure state. Define,

\[
Q_{\varepsilon}^{\varepsilon}_{(\cdot)_{RABC}} \equiv \inf_{T,U_{BC},\sigma_T,\kappa_{RBCT}} I_{\max}(RB : CT)_{\kappa_{RBCT}}^\varepsilon
\]

with the conditions \( U_{BCT} \in \mathcal{U}(BC) \), \( \sigma_T^* \in \mathcal{D}(\mathcal{H}_T) \), \( \sigma_{CT} \in \mathcal{D}(\mathcal{H}_{CT}) \) and

\[
(I_R \otimes U_{BC})\kappa_{RBCT}(I_R \otimes U_{BC}^\dagger) \in \mathcal{B}^\varepsilon(\Psi_{RBC} \otimes \sigma_T^*), \kappa_{RB} \in \mathcal{B}^\varepsilon(\Psi_{RB}).
\]

We show the following upper bound on communication cost of quantum state redistribution.

**Theorem 4.2** (Upper bound). Let \( \varepsilon \in (0,1/3) \) and \(|\Psi\rangle_{RABC}\) be a pure state. There exists an entanglement-assisted one-way protocol \( \mathcal{P} \), which takes as input \(|\Psi\rangle_{RABC}\) shared between three parties Referee (R), Bob (B) and Alice (A) and outputs a state \( \Phi_{RABC} \) shared between Referee (R), Bob (BC) and Alice (A) such that \( \Phi_{RABC} \in \mathcal{B}^{3\varepsilon}(\Psi_{RABC}) \). If \( Q_{\varepsilon}^{\varepsilon}_{(\cdot)_{R,B,AC}} \leq \varepsilon^2/2 \), there is no communication from Alice to Bob in \( \mathcal{P} \). Otherwise the number of qubits communicated by Alice to Bob in \( \mathcal{P} \) is upper bounded by:

\[
\frac{1}{2} Q_{\varepsilon}^{\varepsilon}_{(\cdot)_{RABC}} + \log \log \left( \frac{6}{\varepsilon^2} Q_{\varepsilon}^{\varepsilon}_{(\cdot)_{RABC}} \right) + O(\log \left( \frac{1}{\varepsilon} \right)).
\]

**Proof.** We assume for simplicity that the inf in the definition of \( Q_{\varepsilon}^{\varepsilon}_{(\cdot)_{RABC}} \) is a min (this can be formally justified using standard continuity arguments) and is achieved by \((T,U_{BC},\sigma_T^*,\sigma_{CT},\kappa_{RBCT})\) along with the conditions,

\[
\rho_{RBCT} \equiv (I_R \otimes U_{BC})\kappa_{RBCT}(I_R \otimes U_{BC}^\dagger) \in \mathcal{B}^\varepsilon(\Psi_{RBC} \otimes \sigma_T^*), \kappa_{RB} \in \mathcal{B}^\varepsilon(\Psi_{RB}).
\]

Let \( k \equiv Q_{\varepsilon}^{\varepsilon}_{(\cdot)_{R,B,AC}} \) and \( \delta \equiv \varepsilon^2/6 \). If \( k \leq 3\delta \), let \( n = 1 \), else let \( n \equiv \frac{8.2^k.\log(k/\delta)}{\delta^4} \). Consider the state,

\[
\mu_{RBF_1\ldots F_n} \equiv \frac{1}{n} \sum_{j=1}^{n} \kappa_{RB_F_j} \otimes \sigma_{F_1} \otimes \ldots \otimes \sigma_{F_{j-1}} \otimes \sigma_{F_{j+1}} \otimes \ldots \otimes \sigma_{F_n},
\]

where

\[
\forall j \in [n] : F_j \equiv CT, \kappa_{RB_F_j} = \kappa_{RBCT} \text{ and } \sigma_{F_j} = \sigma_{CT}.
\]

Note that \( \kappa_{RB} = \mu_{RB} \). Consider the following purification of \( \mu_{RBF_1\ldots F_n} \),

\[
|\mu\rangle_{MSE_{E_1\ldots E_n}RB_{F_1\ldots F_n}} \equiv \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |j\rangle_M |\kappa\rangle_{SE_jRB_{F_j}} \otimes |\sigma\rangle_{E_1F_1} \otimes \ldots \otimes |\sigma\rangle_{E_{j-1}F_{j-1}} \otimes |\sigma\rangle_{E_{j+1}F_{j+1}} \otimes \ldots \otimes |\sigma\rangle_{E_nF_n}.
\]

Here, \( \forall j \in [n] : |\sigma\rangle_{E_jF_j} \) is a purification of \( \sigma_{F_j} \) and \( |\kappa\rangle_{SE_jRB_{F_j}} \) is a purification of \( \kappa_{RB_{F_j}} \). Consider the following protocol \( \mathcal{P}_1 \).
1. Alice, Bob and Referee start by sharing the state $|\mu\rangle_{MSE_1 \ldots E_n, RBF_1 \ldots F_n}$ between themselves where Alice holds registers $MSE_1 \ldots E_n$, Referee holds the register $R$ and Bob holds the registers $BF_1 \ldots F_n$.

2. Alice measures the register $M$ and sends the measurement outcome $j \in [n]$ to Bob using $\frac{\log(n)}{2}$ qubits of quantum communication. Alice and Bob employ superdense coding \cite{NC00} using fresh entanglement to achieve this.

3. Alice swaps registers $E_j$ and $E_1$ and Bob swaps registers $F_j$ and $F_1$. Note that the joint state on the registers $SE_1 RBF_1$ at this stage is $|\kappa\rangle_{SE_1 RBF_1}$.

4. Bob performs the unitary $U_{BCT}$ on registers $BF_1$. Note that the joint state on the registers $RBF_1$ at this stage is $|\rho_{RBC}\rangle$.

5. Let $|\Psi\rangle_{RABC} \otimes |\sigma'\rangle_{TT}$ be a purification of $\Psi_{RBC} \otimes \sigma'_T$. Alice applies an isometry $V: \mathcal{H}_{SE_1} \rightarrow \mathcal{H}_{AT'}$, given by Fact \ref{fact:isometry}, so that the joint state in the registers $RABCT'T$ becomes $|\rho\rangle_{RABCT'T}$ (a purification of $\rho_{RBC}$) such that:
   \begin{equation}
   F^2(|\rho\rangle\langle\rho|_{RABCT'T}, |\Psi\rangle\langle\Psi|_{RABC} \otimes |\sigma'\rangle\langle\sigma'|_{TT}) = F^2(\rho_{RBC}, \Psi_{RBC} \otimes \sigma_T) \geq 1 - \varepsilon^2. \tag{5}
   \end{equation}

6. The state $\Phi^1_{RABC} \overset{\text{def}}{=} \text{Tr}_{TT}[|\rho\rangle\langle\rho|_{RABCT'T}]$ is considered the output of the protocol $P_1$. Eq. (5) and Fact \ref{fact:purification} imply,
   \begin{equation}
   F^2(\Phi^1_{RABC}, |\Psi\rangle\langle\Psi|_{RABC}) \geq 1 - \varepsilon^2. \tag{6}
   \end{equation}

Consider the state,
   \begin{equation}
   \xi_{RBF_1 \ldots F_n} \overset{\text{def}}{=} \Psi_{RB} \otimes \mu_{F_1, F_2 \ldots F_n}.
   \end{equation}
Let $|\theta\rangle_{E_1 \ldots E_n F_1 \ldots F_n}$ be a purification of $\mu_{F_1, F_2 \ldots F_n}$. Let
   \begin{equation}
   |\xi\rangle_{RABCE_1 \ldots E_n F_1 \ldots F_n} \overset{\text{def}}{=} |\Psi\rangle_{RABC} \otimes |\theta\rangle_{E_1 \ldots E_n F_1 \ldots F_n}.
   \end{equation}
Using Lemma \ref{lem:purification} and choice of $n$ we have,
   \begin{equation}
   F^2(\kappa_{RB} \otimes \mu_{F_1, F_2 \ldots F_n}, \mu_{RBF_1 \ldots F_n}) \geq 1 - \varepsilon^2.
   \end{equation}
Recall that $F^2(\Psi_{RB}, \kappa_{RB}) \geq 1 - \varepsilon^2$. Using the triangle inequality for purified distance (Fact \ref{fact:purified_triangle}), we get
   \begin{equation}
   \sqrt{1 - F^2(\Psi_{RB} \otimes \mu_{F_1, F_2 \ldots F_n}, \mu_{RBF_1 \ldots F_n})} \leq \sqrt{1 - F^2(\kappa_{RB} \otimes \mu_{F_1, F_2 \ldots F_n}, \mu_{RBF_1 \ldots F_n}) + 1 - F^2(\kappa_{RB}, \Psi_{RB})} \leq 2\varepsilon.
   \end{equation}
This implies
   \begin{equation}
   F^2(\xi_{RBF_1 \ldots F_n}, \mu_{RBF_1 \ldots F_n}) = F^2(\Psi_{RB} \otimes \mu_{F_1, F_2 \ldots F_n}, \mu_{RBF_1 \ldots F_n}) \geq 1 - 4\varepsilon^2.
   \end{equation}
Let $|\xi\rangle_{MSE_1 \ldots E_n RBF_1 \ldots F_n}$ be a purification of $\xi_{RBF_1 \ldots F_n}$ (guaranteed by Fact \ref{fact:isometry}) such that,
   \begin{equation}
   F^2(|\xi\rangle\langle\xi|_{MSE_1 \ldots E_n RBF_1 \ldots F_n}, |\mu\rangle\langle\mu|_{MSE_1 \ldots E_n RBF_1 \ldots F_n}) = F^2(\xi_{RBF_1 \ldots F_n}, \mu_{RBF_1 \ldots F_n}) \geq 1 - 4\varepsilon^2.
   \end{equation}
Let $V': \mathcal{H}_{ACE_1 \ldots E_n} \rightarrow \mathcal{H}_{MSE_1 \ldots E_n}$ be an isometry (guaranteed by Fact \ref{fact:isometry}) such that,
   \begin{equation}
   V'|\xi\rangle_{RABCE_1 \ldots E_n F_1 \ldots F_n} = |\xi\rangle_{MSE_1 \ldots E_n RBF_1 \ldots F_n}.
   \end{equation}
Consider the following protocol $P$.  

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1. Alice, Bob and Referee start by sharing the state $|\xi\rangle_{RABCE_1\ldots E_n F_1\ldots F_n}$ between themselves where Alice holds registers $ACE_1\ldots E_n$, Referee holds the register $R$ and Bob holds the registers $BF_1\ldots F_n$. Note that $|\Psi\rangle_{RABC}$ is provided as input to the protocol and $|\theta\rangle_{E_1\ldots E_n F_1\ldots F_n}$ is additional shared entanglement between Alice and Bob.

2. Alice applies isometry $V'$ to obtain state $|\xi'\rangle_{MSE_1\ldots E_n RBF_1\ldots F_n}$, where Alice holds registers $MSE_1\ldots E_n$, Referee holds the register $R$ and Bob holds the registers $BF_1\ldots F_n$.

3. Alice and Bob simulate protocol $P_1$ from Step 2 onwards.

Let $\Phi_{RABC}$ be the output of protocol $P$. Since quantum maps (the entire protocol $P_1$ can be viewed as a quantum map from input to output) do not decrease fidelity (Fact 2.5), we have,

$$F^2(\Phi_{RABC}, \Phi^1_{RABC}) \geq F^2(|\xi'\rangle_{MSE_1\ldots E_n RBF_1\ldots F_n}, |\mu\rangle|\mu\rangle_{MSE_1\ldots E_n RBF_1\ldots F_n}) \geq 1 - 4\varepsilon^2. \quad (7)$$

Combining Eq. 6, Eq. 7 and using triangle inequality for purified distance (Fact 2.2) we have,

$$\mathcal{P}(\Phi_{RABC}, |\Psi\rangle|\Psi\rangle_{RABC}) \leq \mathcal{P}(\Phi_{RABC}, \Phi^1_{RABC}) + \mathcal{P}(\Phi^1_{RABC}, |\Psi\rangle|\Psi\rangle_{RABC}) \leq 3\varepsilon.$$

This implies $\Phi_{RABC} \in \mathcal{B}^{3\varepsilon}(\langle\Psi\rangle|\Psi\rangle_{RABC})$. If $Q_{(\Psi)_{R,B,AC}} \leq \varepsilon^2/2$, we have $n = 1$ and hence there is no communication from Alice to Bob in $P$. Otherwise the number of qubits communicated by Alice to Bob in $P$ is upper bounded by:

$$\frac{\log(n)}{2} \leq \frac{1}{2}Q_{(\Psi)_{RABC}} + \log \left( \frac{6}{\varepsilon^2 Q_{(\Psi)_{RABC}}} \right) + O\left( \log \left( \frac{1}{\varepsilon} \right) \right).$$

\qed

**Lower bound**

We prove the following lower bound.

**Theorem 4.3** (Lower bound). Let $\varepsilon > 0$ and $|\Psi\rangle_{RABC}$ be a pure state. Let $Q$ be an entanglement-assisted one-way protocol (with communication from Alice to Bob), which takes as input $|\Psi\rangle_{RABC}$ shared between three parties Referee ($R$), Bob ($B$) and Alice ($A$) and outputs a state $\Phi_{RABC}$ shared between Referee ($R$), Bob ($BC$) and Alice ($A$) such that $\Phi_{RABC} \in \mathcal{B}^{\varepsilon}(\langle\Psi\rangle_{RABC})$. The number of qubits communicated by Alice to Bob in $Q$ is lower bounded by:

$$\frac{1}{2}Q_{(\Psi)_{RABC}}.$$

**Proof.** Protocol $Q$ can be written as follows:

1. Alice and Bob get as input $|\Psi\rangle_{RABC}$ shared between Alice ($AC$), Referee ($R$) and Bob ($B$). In addition Alice and Bob share pure state $|\theta\rangle_{S_AS_B}$ where register $S_A$ is held by Alice and register $S_B$ is held by Bob.

2. Alice applies a unitary $U_{ACS_A}$ on the registers $ACS_A$. Let $\kappa_{RAMAT_A BS_B}$ be the joint state at this stage shared between Alice ($MAT_A$), Referee ($R$) and Bob ($BS_B$), where $MAT_A \equiv CS_A$. Note that $\kappa_{RB} = \Psi_{RB}$ and $\kappa_{RBS_B} = \Psi_{RB} \otimes \theta_{S_B}$. 

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3. Alice sends the message register $M$ to Bob.

4. Bob applies a unitary $V_{BS_B M}$ on the registers $BS_B M$. Let $\Phi_{RABCT_BT_B}$ be the joint state at this stage shared between Alice ($AT_A$), Referee ($R$) and Bob ($BCT_B$) where $S_B M \equiv CT_B$.

5. The state $\Phi_{RABC}$ is considered the output of the protocol $Q$.

Using Fact 2.8, we know that there exists a state $\omega_M$, such that:

$$2 \log |M| \geq D_{\text{max}}(\kappa_{RBS_B M} \parallel \omega_M) = D_{\text{max}}(\kappa_{RBS_B M} \parallel \Psi_{RB} \otimes \theta_{S_B} \otimes \omega_M).$$

We have $F^2(\Phi_{RABC}, \Psi_{RABC}) \geq 1 - \varepsilon^2$ and $|\Psi \rangle \langle \Psi|_{RABC}$ is a pure state. From Lemma 2.14 and Fact 2.5 we get a state $\sigma_{T_B}'$ such that,

$$F^2(\Phi_{RBCT_B}, \Psi_{RBC} \otimes \sigma'_{T_B}) \geq 1 - \varepsilon^2.$$

We have,

$$\Phi_{RBCT_B} = (I_R \otimes V_{BS_B M})\kappa_{RBS_B M} (I_R \otimes V_{BS_B M}^\dagger), \kappa_{RB} = \Psi_{RB}.$$  

Recall that $S_B M \equiv CT_B$. Define $\sigma_{CT_B} \overset{\text{def}}{=} \theta_{S_B} \otimes \omega_M$. Eq. (8) and Eq. (9) imply,

$$2 \log |M| \geq D_{\text{max}}(\kappa_{RBCT_B} \parallel \Psi_{RB} \otimes \sigma_{CT_B}),$$

with the conditions

$$F^2(\Phi_{RBCT_B}, \Psi_{RBC} \otimes \sigma'_{T_B}) > 1 - \varepsilon^2, \Phi_{RBCT_B} = (I_R \otimes V_{BCB_T})\kappa_{RBCT_B} (I_R \otimes V_{BCB_T}^\dagger), \kappa_{RB} = \Psi_{RB}.$$

From above and the definition of $Q_{\Psi_{RABC}}^\varepsilon$, we conclude

$$\log |M| \geq \frac{1}{2} Q_{\Psi_{RABC}}^\varepsilon.$$

\[ \square \]

5 Communication bounds on state splitting and state merging

In this section, we describe near optimal bound for communication cost of state splitting and state merging protocols.

State splitting

We show the following lemma, which along with Theorem 4.2 and Theorem 4.3 immediately implies Corollary 1.2.

**Lemma 5.1.** Assume $B$ is a trivial register $(\dim(B) = 1)$. Then $Q_{\Psi_{RAC}}^\varepsilon = \Gamma_{\text{max}}^{\varepsilon}(R : C)\Psi_{RC}.$
Proof. In the calculations below the condition

\[ \rho_{RCT} \overset{\text{def}}{=} (I_R \otimes U_{CT})(I_R \otimes U^\dagger_{CT}) \in \mathcal{B}_\varepsilon(\Psi_{RC} \otimes \sigma'_T) \]

holds. Note that the condition \( \kappa_R \in \mathcal{B}_\varepsilon(\Psi_R) \) is now redundant (is implied by above using \( \rho_R = \kappa_R \) and Fact 2.5). Consider,

\[
Q^\varepsilon_{(\psi)_{RAC}} = \inf_{T,U_{CT},\sigma_{CT},\sigma'_T,\kappa_{RCT}} D_{\text{max}}(\kappa_{RCT} \| \kappa_R \otimes \sigma_{CT})
= \inf_{T,U_{CT},\sigma_{CT},\sigma'_T,\kappa_{RCT}} \left( (I_R \otimes U^\dagger_{CT})\rho_{RCT} (I_R \otimes U_{CT}) \| \kappa_R \otimes \sigma_{CT} \right)
= \inf_{T,U_{CT},\sigma_{CT},\sigma'_T,\kappa_{RCT}} D_{\text{max}} \left( \rho_{RCT} \| \kappa_R \otimes U_{CT} \sigma_{CT} U_{CT}^\dagger \right)
= \inf_{T,\mu_{CT},\sigma'_T,\kappa_{RCT}} D_{\text{max}} \left( \rho_{RCT} \| \kappa_R \otimes \mu_{CT} \right) \quad \text{(with } \mu_{CT} \overset{\text{def}}{=} U_{CT} \sigma_{CT} U_{CT}^\dagger \text{)}
= \inf_{T,\sigma'_T,\kappa_{RCT}} D_{\text{max}} \left( \rho_{RCT} \| \kappa_R \otimes \mu_{CT} \right)
= \inf_{T,\sigma'_T} I^\varepsilon_{\text{max}}(R : CT)_{\Psi_{RC} \otimes \sigma'_T}.
\]

Now,

\[
I^\varepsilon_{\text{max}}(R : C)_{\Psi_{RC}} \geq \inf_{T,\sigma'_T} I^\varepsilon_{\text{max}}(R : CT)_{\Psi_{RC} \otimes \sigma'_T} \quad \text{(by setting } T \text{ to be trivial register)}
= \inf_{T,\sigma'_T,\rho_{RCT} \in \mathcal{B}_\varepsilon(\Psi_{RC} \otimes \sigma'_T)} I^\varepsilon_{\text{max}}(R : CT)_{\rho_{RCT}}
\geq \inf_{\rho_{RCT} \in \mathcal{B}_\varepsilon(\Psi_{RC})} I^\varepsilon_{\text{max}}(R : C)_{\rho_{RCT}} \quad \text{(using Fact 2.9)}
= I^\varepsilon_{\text{max}}(R : C)_{\Psi_{RC}}.
\]

Therefore,

\[ Q^\varepsilon_{(\psi)_{RAC}} = \inf_{T,\sigma'_T} I^\varepsilon_{\text{max}}(R : CT)_{\Psi_{RC} \otimes \sigma'_T} = I^\varepsilon_{\text{max}}(R : C)_{\Psi_{RC}}. \]

State merging

In case of state merging, the register \( A \) is trivial. It has been noted in [BCR11] that state merging can be viewed as the ‘time reversed’ version of state splitting, and the optimal cost of communication is same as the bound for state splitting with \( A \) replaced by \( B \). The following lemma formally states the same. This along with Theorem 4.2, Theorem 4.3 and Corollary 1.2 immediately imply Corollary 1.3.

**Lemma 5.2** ([BCR11]). Fix an \( \varepsilon > 0 \). Consider an entanglement assisted state splitting protocol \( \mathcal{P} \) that starts with a state \( \Psi_{RAC} \), with (\( AC \)) on Alice’s side and \( R \) on Referee’s side, and outputs a state \( \Phi_{RAC} \), with \( C \) on Bob’s side, such that \( \Phi_{RAC} \in \mathcal{B}_\varepsilon(\Psi_{RAC}) \). Let \( q \) qubits be communicated between Alice and Bob in \( \mathcal{P} \). There exists an entanglement assisted one-way state merging protocol \( \mathcal{Q} \) that starts with the state \( \Psi_{RBC} \) (\( B \equiv A \)), with \( C \) on Alice’s side and \( B \) on Bob’s side, and outputs a state \( \Phi'_{RBC} \), with (\( BC \)) on Bob’s side, such that \( \Phi'_{RBC} \in \mathcal{B}_\varepsilon(\Psi_{RBC}) \). There are \( q \) qubits communicated between Alice and Bob in \( \mathcal{Q} \).

Converse of the above holds as well. For every entanglement assisted state merging protocol \( \mathcal{Q} \) with quantum communication cost \( q \) qubits, there exists an entanglement assisted state splitting protocol \( \mathcal{P} \) with quantum communication cost \( q \) qubits.
Proof. Let the protocol \( \mathcal{P} \) start with the overall pure state \( \Psi_{RAC} \otimes \mu_E \), where the register \( E \) include shared entanglement and other ancilla registers used by \( \mathcal{P} \). Let the final pure state of the protocol be \( \Phi_{RACE} \), with \( F^2(\Phi_{RAC}, \Psi_{RAC}) \geq 1 - \varepsilon^2 \). To describe the state merging protocol, we now relabel register \( A \) with \( B \). Since protocol \( \mathcal{P} \) is a collection of unitary operations (which are invertible), it implies that there exists a protocol \( \mathcal{P}' \) (which is protocol \( \mathcal{P} \) time reversed) that starts with the state \( \Phi_{RBCE} \), and leads to the state \( \Psi_{RBC} \otimes \mu_E \) with \( F^2(\Psi_{RBC}, \Phi_{RBC}) \geq 1 - \varepsilon^2 \). From Fact 2.3, there exists a protocol \( \mu'_E \) that satisfies
\[
F^2(\Psi_{RBC} \otimes \mu'_E, \Phi_{RBC}) = F^2(\Psi_{RBC}, \Phi_{RBC}) \geq 1 - \varepsilon^2.
\]
Let \( Q \) be the protocol that starts with the pure state \( \Psi_{RBC} \otimes \mu'_E \), and then does exactly the same as protocol \( \mathcal{P}' \). Let the overall state at the end of \( Q \) be \( \Phi_{RBC} \). Then,
\[
F^2(\Psi_{RBC}, \Phi_{RBC}) \geq F^2(\Psi_{RBC} \otimes \mu'_E, \Phi_{RBC}) = F^2(\Phi_{RBC}, \Psi_{RBC} \otimes \mu'_E)) \geq 1 - \varepsilon^2.
\]
It is clear that the communication between Alice and Bob is the same in \( \mathcal{P} \) and \( Q \).

The converse can be proved using similar arguments.

6 Discussion

In this section, we relate \( Q^0_{\Psi_{RABC}} \) to other information theoretic quantities of interest, for example to the recently introduced fidelity of recovery ([SW14], see also Section 2). We have the following claims.

Claim 6.1. \( Q^0_{\Psi_{RABC}} \geq -2 \log F(R : C|B)_{\Psi_{RBC}} \).

Proof. Let \( (I_R \otimes U_{BCT})\gamma_{RBCT} = (I_R \otimes U_{BCT})_{\Psi_{RBC} \otimes \sigma_T, \kappa_{RB} = \Psi_{RB}} \). Consider,
\[
Q^0_{\Psi_{RABC}} = \inf_{T,U_{BCT}, \sigma_T, \sigma_{CT}} D_{\max}(\gamma_{RBCT} \| \kappa_{RB} \otimes \sigma_{CT})
\geq \inf_{T,U_{BCT}, \sigma_T, \sigma_{CT}} D(\gamma_{RBCT} \| \kappa_{RB} \otimes \sigma_{CT}) \quad \text{(using Fact 2.6)}
\geq \inf_{T,U_{BCT}, \sigma_T, \sigma_{CT}} -2 \log F(\kappa_{RBCT}, \kappa_{RB} \otimes \sigma_{CT}) \quad \text{(using Fact 2.10)}
= \inf_{T,U_{BCT}, \sigma_T, \sigma_{CT}} -2 \log F(\Psi_{RBC} \otimes \sigma_T(I_R \otimes U_{BCT})_{\Psi_{RB} \otimes \sigma_{CT}(I_R \otimes U_{BCT}))
\geq \inf_{T,U_{BCT}, \sigma_{CT}} -2 \log F(\Psi_{RBC}, \text{Tr}_T((I_R \otimes U_{BCT})_{\Psi_{RB} \otimes \sigma_{CT}(I_R \otimes U_{BCT}))
= \inf_{E:B \rightarrow BC} -2 \log F(\Psi_{RBC}, \mathcal{E}(\Psi_{RB})) \quad \text{(using Fact 2.4)}
= -2 \log F(R : C|B)_{\Psi_{RBC}}.
\]

We define other quantities closely related to \( Q^0_{\Psi_{RABC}} \).

Definition 6.2. Let \( \varepsilon \geq 0 \). Define,
\[
\hat{Q}^\varepsilon_{\Psi_{RABC}} \overset{\text{def}}{=} \inf_{T,U_{BCT}, \sigma_T, \sigma_{CT}} I_{\max}(RB : CT)_{\gamma_{RBCT}},
\]
along with the condition \( \gamma_{RB} = \Psi_{RB} \), where \( \gamma_{RBCT} \overset{\text{def}}{=} (I_R \otimes U_{BCT})_{\Psi_{RBC} \otimes \sigma_T}(I_R \otimes U_{BCT}) \).

\[
Q^\varepsilon_{\Psi_{RABC}} \overset{\text{def}}{=} \inf_{T,U_{BCT}, \sigma_T, \sigma_{RBCT}} \hat{Q}^\varepsilon_{\Psi_{RABC}} \quad \text{along with the condition } \gamma_{RB} = \Psi_{RB}, \quad \gamma_{RBCT} \overset{\text{def}}{=} (I_R \otimes U_{BCT})_{\Psi_{RBC} \otimes \sigma_T}(I_R \otimes U_{BCT}).
\]
The protocols of [BCT14, DHO14] can roughly be described as follows. By decoupling arguments, Alice registers $\alpha$ in Bob’s side, such that isometry $V$ lets assume that this decoupling is exact. Since $\alpha$ is ‘almost’ decoupled from $C$, there exists a unitary $U$ the marginal state in the registers $\alpha C$. Alice and Bob share an entangled pure state $\Psi_{ABC}$ such that

$$ (I_R \otimes U_{BCT})(\gamma_{RBCT})(I_R \otimes U_{BCT})^\dagger \in B^\sigma(\Psi_{RBC} \otimes \sigma_T), $$

since

$$ (I_R \otimes U_{BCT})(\gamma_{RBCT})(I_R \otimes U_{BCT})^\dagger = \Psi_{RBC} \otimes \sigma_T. $$

Hence $Q^0_{\Psi|_{RABC}} \leq \tilde{Q}^0_{\Psi|_{RABC}}$. 

The last equality follows since $I(R : C \mid B)_{\Psi_{RBC}} = I(R : B)_{\Psi_{RB}}$, which implies $I(R : C \mid B)_{\Psi_{RBC}} = I(R : B)_{\Psi_{RB}}$, and $I(R : B)_{\gamma_{RB}} = I(R : B)_{\Psi_{RB}}$, which implies $I(R : C \mid B)_{\Psi_{RBC}} = I(R : B)_{\Psi_{RB}}$.

Thus, an interesting question is to find a register $T$ and a unitary $U_{BCT}$ which when applied to $\Psi_{RBC} \otimes \sigma_T$, decouples $B$ from $CT$ as much as possible in the resulting state $\gamma_{RBCT}$, while keeping the marginal state in the registers $RB$ unchanged, that is, $\gamma_{RB} = \Psi_{RB}$. The decoupling nature of $U_{BCT}$ is also reflected in the protocols of [BCT14, DHO14], as discussed in next subsection.

### Relation with other recent protocols

The protocols of [BCT14, DHO14] can roughly be described as follows. By decoupling arguments, there exists a unitary $U_C$ that acts on system $C$ and divides it into 3 parts: $C_1, C_2, C_3$ such that $C_1$ is ‘almost’ decoupled from $BR$ and $C_2$ is ‘almost’ decoupled from $AR$. To have a simpler picture, lets assume that this decoupling is exact. Since $\Psi_{C_1} \otimes \Psi_{BR}$ is purified in $AC_2C_3$, Alice applies an isometry $V : H_{AC_2C_3} \rightarrow H_{SC_1}$ to purify $\Psi_{C_1}$ on a separate register $C_1'$ and purify $\Psi_{BR}$ on a separate register $S$. Alice and Bob share an entangled pure state $\theta_{E_1E_1'}$ on registers $E_1E_1'$, with $E_1 \equiv C_1$ on Bob’s side, such that $\theta_{E_1} = \Psi_{C_1}$. Alice swaps $E_1'$ and $C_1'$ and applies the inverse $V^{-1}$ to restore the original system with $C_1$ on Bob’s side. Alice then transfers $C_3$ to Bob. Now, $\Psi_{C_2} \otimes \Psi_{AR}$ is purified in $BC_1C_3$. Bob prepares pure state $\alpha_{D_2D_2'}$, with $\alpha_{D_2} = \Psi_{C_2}$, $D_2 \equiv C_2$ and $D_2'$ purifying the state $\alpha_{D_2}$. Bob applies a unitary $U_{BCT} : H_{BC_1C_2D_2D_2'} \rightarrow H_{C_1'C_2}$ (where $\Phi_{C_2C_1'}$ is a purification of $\Psi_{C_2}$).
In this setting consider the quantities involved in definition of $Q_{(\psi)_{RABC}}^\epsilon$. Let $T \equiv C_2'$ and define,

$$\kappa_{RBCT} \overset{\text{def}}{=} \Psi_{RBC_1C_3} \otimes \alpha_{D_2D'_2} \sigma_T' = \Phi_{C_2'}.$$

Recall that $C_1$ was decoupled from $RB$ on application of $U_C$. Hence (using Fact 2.8),

$$I_{\max}(RB : CT)_{\kappa_{RBCT}} = I_{\max}(RB : C_1D_2C_3D'_2)_{\Psi_{RBC_1C_3} \otimes \alpha_{D_2D'_2}} = I_{\max}(RB : C_3)_{\Psi_{RBC_3}} \leq 2 \log |C_3|.$$

Also, the following holds:

$$I(B : CT)_{\kappa_{BCT}} = I(RB : CT)_{\kappa_{RBCT}} - I(R : C|B)_{\Psi_{RBC}} \leq 2 \log |C_3| - I(R : C|B)_{\Psi_{RBC}}.$$

In the actual protocol the decoupling operation is not perfect. In this case, we define $\kappa_{RBCT}$ to be the state in Bob’s registers just before his unitary and $U_{BCT}$ to be the unitary that he applies. The correctness of the protocol ensures that $Q_{(\psi)_{RABC}}^\epsilon \leq 2 \log |C_3|.$

### Conclusion and open questions

In this work we have proposed a new information theoretic quantity and exhibited that it (nearly) tightly captures the worst case entanglement-assisted one-way quantum communication complexity of the quantum state redistribution task and the sub-tasks quantum state splitting and quantum state merging. A key technical tool that we use is a convex-split lemma which may be of independent interest. Some natural questions that arise are:

1. Can we understand the quantity $Q_{(\psi)_{RABC}}^\epsilon$ better? Can we relate it to other information theoretic quantities of interest studied in the literature, for example in recent works [BSW14, BCT14, FR14, SW14]? Can we put an upper bound on the size of register $T$ in the definition of $Q_{(\psi)_{RABC}}^\epsilon$?

2. Can the gap between our upper and lower bounds be further reduced? Can the dependence of communication on $\epsilon$ be improved?

3. Our protocol uses a lot of entanglement. Can we be simultaneously (nearly) optimal in communication and entanglement used?

4. What happens if we consider expected communication instead of worst case communication?

5. What is the state $\sigma_T'$ and unitary $U_{BCT}$, that achieves the optimal decoupling between $B$ and $CT$ (as discussed in Section 6)?

6. Can our bounds be used to derive tighter direct-sum results for multi-round quantum communication complexity?

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