Abstract

Let $i(r, g)$ denote the infimum of the ratio $\frac{\alpha(G)}{|V(G)|}$ over the $r$-regular graphs of girth at least $g$, where $\alpha(G)$ is the independence number of $G$, and let $i(r, \infty) := \lim_{g \to \infty} i(r, g)$. Recently, several new lower bounds of $i(3, \infty)$ were obtained. In particular, Hoppen and Wormald showed in 2015 that $i(3, \infty) \geq 0.4375$, and Csóka improved it to $i(3, \infty) \geq 0.44533$ in 2016. Bollobás proved the upper bound $i(3, \infty) < \frac{6}{13}$ in 1981, and McKay improved it to $i(3, \infty) < 0.45537$ in 1987. There were no improvements since then. In this paper, we improve the upper bound to $i(3, \infty) \leq 0.454$.

Mathematics Subject Classification: 05C15, 05C35

Key words and phrases: independence ratio, cubic graphs, independent sets.

1 Introduction

A set $S$ of vertices in a graph $G$ is independent if no two vertices of $S$ are joined by an edge. The independence number, $\alpha(G)$, is the maximum size of an independent set in $G$. The independence ratio, $i(G)$, of a graph $G$ is the ratio $\frac{\alpha(G)}{|V(G)|}$. For positive integers $r$ and $g$, $i(r, g)$ denotes the infimum of $i(G)$ over the $r$-regular graphs of girth at least $g$, and $i(r, \infty)$ denotes $\lim_{g \to \infty} i(r, g)$. The first interesting upper bounds on $i(r, \infty)$ were obtained by Bollobás [2] in 1981. In particular, he proved $i(3, \infty) < \frac{6}{13}$. Refining the method, McKay [11] in 1987 showed

Theorem 1 (McKay [11]).

$$i(3, \infty) < 0.45537. \quad (1)$$

In the next 30 years, there were no improvements of Theorem [11] but recently some interesting lower bounds on $i(r, \infty)$ and in particular on $i(3, \infty)$ were proved. Hoppen [6] showed $i(3, \infty) \geq 0.4328$. Then Kardoš, Král and Volec [10] improved the bound to 0.4352. Csóka, Gerencsér, Harangi, and Virág [5] pushed the bound to 0.4361 and Hoppen and Wormald [7] — to 0.4375. Moreover, Csóka et al [5] claimed a computer assisted lower bound $i(3, \infty) \geq 0.438$, and Csóka [4]...
later improved the bound to 0.44533. Our result is an improvement of (1) to \( i(3, \infty) \leq 0.454 \). The improvement is small, but it decreases the gap between the upper and lower bounds on \( i(3, \infty) \) by approximately 14%.

**Theorem 2.** \( i(3, \infty) \leq 0.454 \).

The proof uses the language of configurations introduced by Bollobás [3], and shows that “many” 3-regular configurations have “small” independence ratio. The proof of our improvement is based on analyzing the presence not of largest independent sets, but of larger structures, so called MAI-sets (defined in Section 3) that contain largest independent sets.

# 2 Preliminaries

## 2.1 Notation

We mostly use standard notation. The complete \( n \)-vertex graph is denoted by \( K_n \). If \( G \) is a multigraph and \( v, u \in V(G) \), then \( E_G(v, u) \) denotes the set of all edges in \( G \) connecting \( v \) and \( u \), \( e_G(v, u) \) denotes the number of edges in \( G \) connecting \( v \) and \( u \), and \( \deg_G(v) \) denotes the degree of \( v \) in \( G \). By \( \Delta(G) \) we denote the maximum degree of \( G \), and by \( g(G) \) — the girth (the length of a shortest cycle) of \( G \). For \( A \subseteq V(G) \), \( G[A] \) denotes the submultigraph of \( G \) induced by \( A \). For \( k \in \mathbb{Z}_{>0} \), \([k]\) denotes the set \( \{1, \ldots, k\} \).

## 2.2 The Configuration Model

The configuration model in different versions is due to Bender and Canfield [1] and Bollobás [3]. Our work is based on the version of Bollobás. Let \( n \) be an even positive integer and \( V_n = [n] \). Consider the Cartesian product \( W_n = V_n \times [3] \). A configuration/pairing (of order \( n \) and degree 3) is a perfect matching on the vertex set \( W_n \). There are \((3^n - 1) \cdot (3^n - 3) \cdot \ldots \cdot 1 = (3n - 1)!! \) such matchings.

Let \( F_3(n) \) denote the collection of all \((3n - 1)!! \) possible pairings on \( W_n \). We project each pairing \( F \in F_3(n) \) to a multigraph \( \pi(F) \) on the vertex set \( V_n \) by ignoring the second coordinate. Then \( \pi(F) \) is a 3-regular multigraph (which may or may not contain loops and/or multiple edges). Let \( \pi(F_3(n)) = \{\pi(F) : F \in F_3(n)\} \) be the set of 3-regular multigraphs on \( V_n \). By definition,

\[
\text{each simple graph } G \in \pi(F_3(n)) \text{ corresponds to } (3!)^n \text{ distinct pairings in } F_3(n).
\] (2)

We will call the elements of \( V_n \) - vertices, and of \( W_n \) - points.

**Definition 3.** Let \( G_g(n) \) be the set of all cubic graphs with vertex set \( V_n = [n] \) and girth at least \( g \) and \( G'_g(n) = \{F \in F_3(n) : \pi(F) \in G_g(n)\} \).

We will heavily use the following result:

**Theorem 4** (Wormald [13], Bollobás [3]). For each fixed \( g \geq 3 \),

\[
\lim_{n \to \infty} \frac{|G'_g(n)|}{|F_3(n)|} = \exp \left\{ - \sum_{k=1}^{g-1} \frac{2^{k-1}}{k} \right\}.
\] (3)

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Remark. When we say that a pairing $F$ has a multigraph property $\mathcal{A}$, we mean that $\pi(F)$ has property $\mathcal{A}$.

Since dealing with pairings is simpler than working with labeled simple regular graphs, we need the following well-known consequence of Theorem 4.

Corollary 5 ([11](Corollary 1.1), [9](Theorem 9.5)). For fixed $g \geq 3$, any property that holds for $\pi(F)$ for almost all pairings $F \in \mathcal{F}_3(n)$ also holds for almost all graphs in $\mathcal{G}_g(n)$.

Definition 6. For a graph $G$, let $I(G)$ denote the total number of all independent sets in $G$, including the empty set. For all integer $r \geq 0$, $g \geq 3$, we define

$$I(r,g) = \inf I(G) \frac{1}{|V(G)|},$$

where the infimum is over all graphs $G$ of maximum degree at most $r$ and girth at least $g$.

Recall that the Fibonacci numbers $F_n$ are defined by $F_1 = F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$, for $i \geq 3$. The exact formula for $F_i$ is

$$F_i = \frac{\phi^i - \psi^i}{\sqrt{5}},$$

where $i \geq 0$, $\phi = \frac{1+\sqrt{5}}{2}$, and $\psi = \frac{1-\sqrt{5}}{2}$.

Lemma 7 (McKay [11]). For any $g \geq 4$, $I(2,g) = (F_{s-1} + F_{s+1})^{1/2}$, where $s = 2\lfloor g/2 \rfloor + 1$.

Remark 8. The numbers $s - 1$ and $s + 1$ in Lemma 7 are even. Therefore,

$$I(2,g) = (F_{s-1} + F_{s+1})^{1/2} = \left( \frac{\phi^{s-1} + \phi^{s+1} - \phi^{1-s} - \phi^{-s-1}}{\sqrt{5}} \right)^{1/2} = \phi \cdot \left( 1 - \phi^{-2s} \right)^{1/s}.$$

Since the function $(1 - \phi^{-2s})^{1/s}$ monotonically increases for $s \geq 1$, and $\phi((1 - \phi^{-18})^{1/9} \geq 1.618002$, we conclude that for each graph $H$ with maximum degree at most 2 and girth at least 8,

$$1.618 \leq I(2, 8) \leq I(H)^{1/|V(H)|}. \tag{4}$$

3 MAI sets in cubic graphs

Definition 9. A vertex set $A$ in a graph $G$ is an AI set (an almost independent set), if every component of $G[A]$ is an edge or an isolated vertex. In other words, $A$ is an AI set if $\Delta(G[A]) \leq 1$.

Definition 10. A vertex set $A$ is a maximum almost independent set (MAI set) in a graph $G$ if all of the following hold:

M1. $A$ is an AI set;

M2. $A$ contains an independent set $A'$ of size $\alpha(G)$;

M3. $A$ is largest among all sets satisfying M1 and M2.

Let $G \in \mathcal{G}_{16}(n)$ and $A$ be a MAI set. Denote $B = V(G) - A$. 

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Lemma 11. B is an AI set.

Proof. Let \( b \in B \). We prove that \( d_{G[B]}(b) \leq 1 \). Let \( A' \) be a maximum independent set in \( A \).

If \( d_{G[B]}(b) = 3 \), then there is no edge from \( b \) to \( A \), and \( A' \cup \{b\} \) is an independent set in \( G \) with size \( |A'| + 1 = \alpha(G) + 1 \), contradicting the definition of \( \alpha(G) \).

If \( d_{G[B]}(b) = 2 \), then there is only one edge \( e \) from \( b \) to \( A \), say \( ba \). If \( d_{G[A]}(a) = 0 \), then \( G[A \cup \{b\}] \) is an AI set in \( G \) larger than \( A \) containing \( A' \). This contradicts the fact that \( A \) is a MAI set. If \( d_{G[A]}(a) = 1 \), then without loss of generality, we may assume \( a \in A - A' \). Then \( b \) has no neighbors in \( A' \), and \( A' \cup \{b\} \) is an independent set in \( G \) with size \( |A'| + 1 \), again contradicting the definition of \( \alpha(G) \).

Let \( A \) be a MAI set in \( G \in G_{16}(n) \). Denote the set of vertices with degree 1 in \( G[A] \) by \( Y \), the set of vertices with degree 1 in \( G[B] \) by \( Z \). We introduce notation for the sizes of the sets: Let \( x := |A'|, s := |Y|/2, t := |Z|/2 \), and \( i := \frac{n}{2} - |A| \). Then \( |A| = \frac{n}{2} - i \) and \( |B| = \frac{n}{2} + i \).

Lemma 12. \( i \geq 0 \) and \( t \geq s \).

Proof. We count the number of edges with one end in \( A \) and one end in \( B \) in two ways. We have

\[
2s \cdot 2 + \left( \frac{n}{2} - i - 2s \right) \cdot 3 = e[A, B] = 2t \cdot 2 + \left( \frac{n}{2} + i - 2t \right) \cdot 3,
\]

i.e.,

\[
t - s = 3i.
\]

We also know that \( x = \alpha(G) \), so

\[
x = \frac{n}{2} - i - s \geq \frac{n}{2} + i - t,
\]

i.e.,

\[
2i \leq t - s = 3i,
\]

which implies that

\[
0 \leq i \) and \( t \geq s. \]

Lemma 13. If \( G \in G_5(n) \), then

(i) each vertex in \( Z \) has degree at most one to \( Y \);
(ii) each vertex in \( Y \) has degree at most one to \( Z \).

Proof. (i) Suppose \( z \in Z \) and \( N_G(z) = \{z', y_1, y_2\} \), where \( z' \in Z \) and \( y_1, y_2 \in Y \). Since \( g(G) \geq 4 \), \( y_1 \neq y_2 \), and \( y_1y_2 \notin E(G) \), and so \( A - y_1 - y_2 \) contains an independent set \( A' \) with \( |A'| = \alpha(G) \). Thus the set \( A' + z \) is an independent set of size \( \alpha(G) + 1 \) contradicting the definition of \( \alpha(G) \).

(ii) Similarly, suppose \( y \in Y \) and \( N_G(y) = \{y', z_1, z_2\} \), where \( y' \in Y \) and \( z_1, z_2 \in Z \). Then \( A - y \) contains an independent set \( A' \) with \( |A'| = \alpha(G) \). For \( i = 1, 2 \), let \( N_G(z_i) = \{z'_i, y, a_i\} \), where \( z'_i \in Z \). By Part (i), \( a_1, a_2 \notin Y \). Since \( g(G) \geq 5 \), \( a_2 \neq a_1 \). Then \( (A - y) \cup \{z_1, z_2\} \) is an AI set containing \( A' \) and is larger than \( A \), a contradiction. \( \square \)

Let \( J = \{y_1z_1, \ldots, y_jz_j\} \) be the set of all edges connecting \( Y \) with \( Z \) in \( G \). By Lemma \( 13 \), \( J \) is a matching in \( G \). Define an auxiliary graph \( H = H(A) \) as follows: \( V(H) = J \), and \( y_iz_i \) is adjacent to \( y_iz' \) if \( y_iz_i \in E(G) \) or \( z_iz' \in E(G) \). By construction, the maximum degree of \( H \) is at most 2 and a cycle of length \( c \) in \( H \) corresponds to a cycle of length \( 2c \) in \( G \).
Lemma 14. The graph $G$ contains at least $I(H)$ distinct MAIs.

Proof. Let $J' = \{y_1z_1, \ldots, y_jz_j\}$ be an arbitrary independent set in $H$. Then the sets $Y_1 = \{y_1, \ldots, y_j\}$ and $Z_1 = \{z_1, \ldots, z_j\}$ are independent in $G$. By the definition of $Y$, $A - Y_1$ contains an independent set $A'$ with $|A'| = \alpha(G)$. Let $A_1 = (A - Y_1) \cup Z_1$. By Lemma 13, the degree in $G[A_1]$ of every vertex in $(Y - Y_1) \cup Z_1$ is at most 1. If a vertex $a \in A - Y$ is adjacent to two vertices, say $z_1, z_2$ in $Z_1$, then the set $(A' - a) \cup \{z_1, z_2\}$ is independent and is larger than $A'$, a contradiction. Thus, $A_1$ is an AI set. Since $|A_1| = |A|$, this proves the lemma. □

Remark 15. Recall that $|A| = \frac{n}{2} - i$, $|B| = \frac{n}{2} + i$, $|Y| = 2s = 2(\frac{n}{2} - i - x)$, and $|A - Y| = 2x - \frac{n}{2} + i$. By (6), we know that $t = 3i + s = \frac{n}{2} + 2i - xn$. Therefore, $|Z| = 2t = 2(\frac{n}{2} + 2i - x)$ and $|B - Z| = 2x - \frac{n}{2} - 3i$. By (5), $e[A, B] = 2x + \frac{n}{2} - i$.

4 The set up of the proof

4.1 Restating the theorem

We will use Theorem 1 of McKay in the following stronger form.

Theorem 16 (McKay [11]). For every $\varepsilon > 0$, there exists an $N > 0$ such that for each $n > N$,

$$|\{F | F \in F_3(n) : \alpha(\pi(F)) > 0.45537n\}| < \varepsilon \cdot (3n - 1)!.$$
We will show that “almost all” cubic labeled graphs of girth at least 16 have independence ratio at most 0.454. In view of Theorem 4, the following more technical statement implies Theorem 2.

**Theorem 17.** For every $\varepsilon > 0$, there is an $N > 0$ such that for each $n > N$,

$$|\{F \in \mathcal{G}_{16}(n) : \alpha(\pi(F)) > 0.454n\}| < \varepsilon (3n - 1)!.$$  

(7)

The rest of the paper is a proof of Theorem 17. By definition, every graph has a MAI set. So, for large $n$, nonnegative integers $x \geq 0.454n$ and $i \leq \frac{n}{2} - x$, and each set $A$ of size $\frac{n}{2} - i$ with a fixed matching of size $\frac{n}{2} - i - x$ we will estimate the total $x$-weight of configurations $F \in \mathcal{G}_{16}(n)$ in which $A$ forms a MAI set. The idea of the weight (used by McKay in [11]) is to decrease overcount of the configurations containing a given MAI set, but guarantee that the total weight of each configuration containing at least one MAI set with independence number $x$ would be at least 1.

### 4.2 Setup of the proof of Theorem 17

An AI-pair on $[n]$ is a pair $(A, R)$ consisting of a set $A \subset [n]$ and a matching $R$ on a subset of $A$ such that $E(G[A]) = R$. The independence number, $\alpha(A, R)$, of an AI-pair $(A, R)$ is $|A| - |R|$. Let $\mathcal{P}(n, x)$ denote the family of all AI-pairs $(A, R)$ on $[n]$ with $\alpha(A, R) = x$.

A preimage of an AI-pair $(A, R)$ on $[n]$ is a pair $(\hat{A}, \hat{R})$ where $\hat{A} = A \times [3]$ and $\hat{R}$ is a matching on a subset of $\hat{A}$ with $|\hat{R}| = |R|$ such that for each edge $(i, j)(i', j') \in \hat{R}$, $ii' \in R$. In other words, each edge $e \in R$ is obtained from an edge in $\hat{e} \in \hat{R}$ by ignoring the second coordinates of the ends of $\hat{e}$, and this mapping is one-to-one.

By the $x$-weight of a configuration $F$ we mean

$$\omega_x(F) := \frac{1}{\text{the reciprocal of the number of preimages } (\hat{A}, \hat{R}) \subseteq F \text{ of AI-pairs } (A, R) \text{ on } [n] \text{ such that } A \text{ is an AI set in } \pi(F) \text{ with } E(\pi(F)[A]) = R \text{ and } \alpha(A, R) = x.}$$

By the definition of $x$-weight, each pairing $F' \in \mathcal{G}_{16}(n)$ with $\alpha(\pi(F')) = x$ contributes exactly 1 to

$$\sigma(n, x, 16) := \sum_{(A, R) \in \mathcal{P}(n, x)} \{\omega_x(F') : F' \in \mathcal{G}_{16}(n) \text{ and } (\hat{A}, \hat{R}) \text{ is an induced subpairing of } F'\}. \quad (9)$$

It follows that

$$\sigma(n, x, 16) \geq |\{F' \in \mathcal{G}_{16}(n) \text{ with } \alpha(\pi(F')) = x\}|. \quad (10)$$

**Lemma 18.** Let $n$ be a positive even integer and $x$ be an integer with $0.454n < x \leq 0.45537n$. The number of pairings $F \in \mathcal{G}_{16}(n)$ such that $\pi(F)$ has a MAI set $A$ with $|A| = x$ is at most

$$q(x, n) := \sum_{i=0}^{\frac{n}{2}-x} \binom{n}{\frac{n}{2} - i} \cdot \frac{(\frac{n}{2} - i)! \cdot 3^{(n-2x-2i)}}{(2x + i - \frac{n}{2})! \cdot 2^{\frac{n}{2} - x - i} \cdot (\frac{n}{2} - x - i)!} \cdot \frac{(\frac{n}{2} + i)! \cdot 3^{n-2x+4i}}{(2x - 3i - \frac{n}{2})! \cdot 2^{\frac{n}{2} - x + 2i} \cdot (\frac{n}{2} - x + 2i)!} \cdot \sum_{j=0}^{n-2i-2x} \binom{n-2i-2x}{j} \cdot \binom{n-2x+4i}{j} \cdot 2^{2j} \cdot j! \cdot \left(\frac{1}{1.618}\right)^j \cdot \frac{(3(2x - \frac{n}{2} - 3i))! \cdot (3(2x - \frac{n}{2} + i))!}{(3(2x - \frac{n}{2} - 3i) - 2(n - 2i - 2x) + j)!}. \quad (11)$$
Proof. By (10), it is enough to show that \( \sigma(n, x, 16) \leq q(x, n) \). Below we describe a procedure of constructing for every AI-pair \((A, R)\) on \([n]\) with \(\alpha(A, R) = x\) all pairings in \(G'_{16}(n)\) for which \(A\) is a MAI set. Not every obtained pairing will be in \(G'_{16}(n)\) and some pairings will have independence number larger than \(x\), but every \(F \in G'_{16}(n)\) such that \(A\) is a MAI set in \(\pi(F)\) will be a result of this procedure.

0. Choose nonnegative integers \(n, x, i, j\) such that \(n\) is even, \(0 < 0.454n < x \leq 0.45537n\), and \(0 \leq i \leq n/2 - x - i\).

1. Choose a set \(A \subset [n]\) with \(|A| = \frac{n}{2} - i\). There are \(\binom{n}{\frac{n}{2}-i}\) ways to do it.

2. Choose a matching \(R\) on \(A\) with \(|R| = \frac{n}{2} - x - i\). There are
\[
\frac{(\frac{n}{2} - i)!}{(2x + i - \frac{n}{2})! \cdot 2^{x-i} \cdot (\frac{n}{2} - x - i)!}
\]
ways to do it. Then there are \(3^{n-2x-2i}\) ways to decide which point of each chosen end of an edge in \(R\) will be the end of the corresponding edge in \(F\).

3. Similarly to Step 2, we have
\[
\frac{(\frac{n}{2} + i)!}{(2x - 3i - \frac{n}{2})! \cdot 2^{x+2i} \cdot (\frac{n}{2} - x + 2i)!}
\]
ways to construct a matching \(R'\) of \(\frac{n}{2} - x + 2i\) edges on \(B := [n] - A\), since \(|B| = \frac{n}{2} + i\). After that there are \(3^{n-2x+4i}\) ways to decide which point of each chosen end of an edge in \(R'\) will be the end of the corresponding edge in \(F\).

4. Let \(Y\) (respectively, \(Z\)) be the set of vertices covered by the matching \(R\) (respectively, \(R'\)). By Lemma 13, if \(A\) is a MAI-set in \(\pi(F)\), then the set of edges connecting \(Y\) with \(Z\) is a matching. If this matching, say \(M\) has \(j\) edges, then there are \(\binom{n-2i-2x}{j}\) ways to choose the set of the ends of \(M\) in \(Y\) and \(\binom{n-2x+4i}{j}\) ways to choose the ends of \(M\) in \(Z\). Since there are \(2\) free points left for each vertex in \(Y\) and \(Z\), we have \(2^{2j}\) ways to choose which point of each vertex in \(Y\) and \(Z\) to be used to form an edge in \(M\).

5. By Lemma 14 each pairing \(F \in G'_{16}(n)\) containing a MAI set \(A\) with \(j\) edges between \(Y\) and \(Z\) contains at least \(I(2, 8)^j\) distinct MAI sets of the same cardinality. By Lemma 7, \(I(2, 8)^j \geq 1.618^j\). Hence by \(8\), \(\omega_x(F) \leq 1.618^{-j}\).

6. Now we choose for each remaining free point \(p\) from vertices in \(Y\) a free point \(q\) in a vertex in \(B - Z\) and add edge \(pq\). There are
\[
\frac{(3(2x - \frac{n}{2} - 3i))!}{(3(2x - \frac{n}{2} - 3i) - 2(n - 2i - 2x) + j)!}
\]
ways to do it.
Then certainly, we write

\[ \frac{3(2x - \frac{n}{2} + i)!}{(3(2x - \frac{n}{2} + i) - 2(n - 2x + 4i) + j)!} \]

ways to do it.

8. Finally, there are \(3(2x - \frac{n}{2} + i) - 2(n - 2x + 4i) + j = 10x - \frac{7n}{2} - 5i + j\) free points left in \(A\) and \(10x - \frac{7n}{2} - 5i + j\) free points left in \(B\). We have \((10x - \frac{7n}{2} - 5i + j)!\) ways to complete a pairing on \(W_n\). □

In the proofs below we will use Stirling’s formula: For every \(n \geq 1\),

\[
\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{1/12n}. \tag{11}
\]

We will also use the notation \(\frac{\partial}{\partial j}\) to denote the partial derivative with respect to \(j\). Moreover, we use the domain \(x \geq 0\) and define \(\ln(0) = -\infty\) when we consider \(\ln x\).

**Lemma 19.** Let \(n\) be a positive even integer and \(x\) be an integer satisfying \(0.454n < x \leq 0.45537n\). Let

\[
\Omega = \{(\chi, \zeta, \xi) : 0.454 < \chi \leq 0.45537, 0 \leq \zeta \leq \frac{1}{2} - \chi, 0 \leq \xi \leq 1 - 2\chi - 2\zeta\}. \tag{12}
\]

Let

\[
f(\chi, \zeta) := \frac{3^{\frac{1}{2} - 4\chi + 2\zeta} \cdot (1 - 2\chi - 2\zeta)^{1 - 2\chi - 2\zeta} \cdot (1 - 2\chi + 4\zeta)^{1 - 2\chi + 4\zeta} \cdot (6\chi - \frac{3}{2} + 3\zeta)^{6\chi - \frac{3}{2} + 3\zeta} \cdot (6\chi - \frac{3}{2} - 9\zeta)^{6\chi - \frac{3}{2} - 9\zeta}}{(2\chi + \zeta - \frac{1}{2})^{2\chi + \zeta - \frac{1}{2}} \cdot 2^{1 - 2\chi + \zeta} \cdot (\frac{1}{2} - \chi - \zeta)^{\frac{1}{2} - \chi - \zeta} \cdot (\frac{1}{2} - \chi + 2\zeta)^{\frac{1}{2} - \chi + 2\zeta} \cdot (2\chi - 3\zeta - \frac{1}{2})^{2\chi - 3\zeta - \frac{1}{2}},
\]

and

\[
g(\chi, \zeta, \xi) := 2^{2\xi} \cdot (\frac{1}{1 - 10^\chi})^\xi
\]

\[
h(\chi, \zeta, \xi) := f(\chi, \zeta) \cdot g(x, \zeta, \xi).
\]

Then

\[
\frac{q(x, n)}{(3n - 1)!!} = O(n^6) \cdot \max\{(h(\chi, \zeta, \xi))^n : (\chi, \zeta, \xi) \in \Omega\}. \tag{13}
\]

**Proof.** We write \(q(x, n)\) as a double sum of \(i\) and \(j\) and let \(r(x, n, i, j)\) be the function inside the double sum of \(q(x, n)\), i.e.,

\[
q(x, n) = \sum_{i=0}^{\frac{n}{2} - x} \sum_{j=0}^{n - 2x - 2i} r(x, n, i, j).
\]

Then certainly,

\[
q(x, n) \leq n^2 \cdot \max\{r(x, n, i, j) : 0 \leq i \leq \frac{n}{2} - x, 0 \leq j \leq n - 2x - 2i\}.
\]
Therefore, this proves the lemma.

We do this in the next section, and then Theorem \[17\] easily follows.
5 Proof of (14)

In order to find the maximum value of \( h(\chi, \zeta, \xi) \) for a fixed \( \chi \), we will maximize \( \ln(h(\chi, \zeta, \xi)) \). We first find the value of \( \xi \) in terms of \( \chi \) and \( \zeta \) that maximizes \( \ln(g(\chi, \zeta, \xi)) \). By definition,

\[
\ln(g(\chi, \zeta, \xi)) = \xi \ln\left(\frac{4}{1.618}\right) - (\xi \ln(\xi) + (1 - 2\zeta - 2\chi - \xi) \ln(1 - 2\zeta - 2\chi - \xi) + (1 - 2\chi + 4\zeta - \xi) \ln(1 - 2\chi + 4\zeta - \xi) + (10\chi - \frac{7}{2} - 5\zeta + \xi) \ln(10\chi - \frac{7}{2} - 5\zeta + \xi)).
\]

Hence

\[
\frac{\partial \ln(g(\chi, \zeta, \xi))}{\partial \xi} = \ln(1 - 2\chi - 2\zeta - \xi) + \ln(1 - 2\chi + 4\zeta - \xi) - \ln(10\chi - 5\zeta + \xi - \frac{7}{2}) - \ln(\xi) + \ln\left(\frac{4}{1.618}\right).
\]

In order to solve

\[
\frac{\partial \ln(g(\chi, \zeta, \xi))}{\partial \xi} = 0,
\]

we solve the equivalent equation

\[
p(\xi) := 4 \cdot (1 - 2\chi - 2\zeta - \xi) \cdot (1 - 2\chi + 4\zeta - \xi) - 1.618 \cdot \xi \cdot (10\chi - 5\zeta + \xi - \frac{7}{2}) = 0,
\]

where \( p(\xi) \) has domain \( 0 \leq \xi \leq 1 - 2\chi - 2\zeta \). By the quadratic formula, the roots are

\[
\xi_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \xi_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},
\]

where

\[
a = 2.382, \quad b = -0.18\chi + 0.09\zeta - 2.337, \quad c = 16\chi^2 - 32\zeta^2 - 16\chi + 8\zeta - 16\chi\zeta + 4.
\]

Moreover, for fixed \( \chi \) and \( \zeta \) satisfying \( 0.454 \leq \chi \leq 0.45537 \) and \( \chi + \zeta \leq \frac{1}{2} \), \( p(\xi) \) is a parabola opening upward with \( \xi_1 \leq 1 - 2\chi - 2\zeta \leq \xi_2 \) because \( p(1 - 2\chi - 2\zeta) \leq 0 \), and \( g(\chi, \zeta, \xi) \) is a continuous function on \( \xi \). Therefore, the maximum of \( g(\chi, \zeta, \xi) \) can only be attained at \( \xi = \xi_1 \).

Let \( g_1(\chi, \zeta) = g(\chi, \zeta, \xi_1(\chi, \zeta)) \). For each fixed \( \chi \), consider the maximum of

\[
h_1(\chi, \zeta) := f(\chi, \zeta) \cdot g_1(\chi, \zeta).
\]

By definition,

\[
\ln(h_1) = \left(\frac{1}{2} - 4\chi + 2\zeta\right) \ln(3) + (1 - 2\chi - 2\zeta) \ln(1 - 2\chi - 2\zeta) + (1 - 2\chi + 4\zeta) \ln(1 - 2\chi + 4\zeta) + (6\chi - \frac{3}{2} + 3\zeta) \ln(6\chi - \frac{3}{2} + 3\zeta) + (6\chi - \frac{3}{2} - 9\zeta) \ln(6\chi - \frac{3}{2} - 9\zeta) + \xi_1(\chi, \zeta) \cdot \ln\left(\frac{4}{1.618}\right).
\]
Lemma 20. When $\chi = 0.454$, the maximum of $h(\chi, \zeta, \xi)$ over $0 \leq \zeta \leq 0.046$ and $0 \leq \xi \leq 0.092 - 2\zeta$ is at most 0.999983.

Proof Fix $\chi = 0.454$. For $0 \leq \zeta \leq 0.046$, denote

$$
\xi'_1(\zeta) := \frac{\partial \xi_1(0.454, \zeta)}{\partial \zeta}, \quad \xi''_1(\zeta) := \frac{\partial^2 \xi_1(0.454, \zeta)}{\partial \zeta^2}, \quad \text{and} \quad \xi_1(\zeta) := \xi_1(0.454, \zeta).
$$

We have

$$
\frac{\partial \ln(h_1(0.454, \zeta))}{\partial \zeta} = -4 \ln(3) + \ln(2) - 3 + 2 \ln(0.408 + \zeta) - \ln(0.046 - \zeta) + 2 \ln(0.046 + 2\zeta)
$$

$$
-6 \ln(0.408 - 3\zeta) + \ln(\frac{4}{1.618}) \cdot \xi'_1(\zeta) - \xi'(\zeta) \cdot (\ln(\xi_1(\zeta)) + 1) + (2 + \xi'_1(\zeta)) \cdot (\ln(0.092 - 2\zeta - \xi_1(\zeta)) + 1)
$$

$$
+ (\xi'_1(\zeta) - 4) \cdot (\ln(0.092 + 4\zeta - \xi_1(\zeta)) + 1) + (5 - \xi'_1(\zeta)) \cdot (\ln(1.04 - 5\zeta + \xi_1(\zeta)) + 1),
$$

$$
\frac{\partial^2 \ln(h_1(0.454, \zeta))}{\partial \zeta^2} = \frac{1}{0.046 - \zeta} + \frac{4}{0.046 + 2\zeta} + \frac{2}{0.408 + \zeta} + \frac{18}{0.408 - 3\zeta}
$$

$$
+ \ln(\frac{4}{1.618}) \cdot \xi''_1(\zeta) - \xi''(\zeta) \cdot (\ln(\xi_1(\zeta)) + 1) - (\xi'_1(\zeta))^2 \cdot \frac{1}{\xi_1(\zeta)}
$$

$$
+ \xi''(\zeta) \cdot (\ln(0.092 - 2\zeta - \xi_1(\zeta)) + 1) - (2 + \xi'_1(\zeta))^2 \cdot \frac{1}{0.092 - 2\zeta - \xi_1(\zeta)}
$$

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\[ \begin{align*}
&\quad \xi''_1(\zeta) \cdot (\ln(0.092 + 4\zeta - \xi_1(\zeta)) + 1) - (\xi'_1(\zeta) - 4)^2 \cdot \frac{1}{0.092 + 4\zeta - \xi_1(\zeta)} \\
&+ \xi''_1(\zeta) \cdot (\ln(1.04 - 5\zeta + \xi_1(\zeta)) + 1) - (\xi'_1(\zeta) - 5)^2 \cdot \frac{1}{1.04 - 5\zeta + \xi_1(\zeta)},
\end{align*} \]

where
\[ \xi''_1(\zeta) = \frac{1}{2a} \cdot (b^2 - 4ac)^{-\frac{1}{2}} \left( \frac{1}{4} \cdot (b^2 - 4ac)^{-1} \cdot (2b \frac{\partial b}{\partial \zeta} - 4a \frac{\partial c}{\partial \zeta})^2 - \frac{1}{2} \cdot (2 \frac{\partial b}{\partial \zeta})^2 + 1024a \right). \quad (17) \]

We will show that
\[ \frac{\partial^2 \ln(h_1(0.454, \zeta))}{\partial \zeta^2} < 0, \text{ for all } 0 \leq \zeta < 0.046. \quad (18) \]

This will guarantee that if we find a solution \( \zeta_0 \in [0, 0.046) \) of the equation \( \frac{\partial \ln(h_1(0.454, \zeta))}{\partial \zeta} = 0 \), then the maximum of \( h_1(0.454, \zeta) \) over \( \zeta \in [0, 0.046) \) is attained at \( \zeta_0 \).

**Claim 20.1.** For each \( \zeta \in [0, 0.046) \), \(-27.336 \leq \xi''_1(\zeta) < -24.822 \).

**Proof.** By (15) and (16), for \( \chi = 0.454 \), the function \( \Delta(\zeta) := b^2 - 4ac \) is quadratic in \( \zeta \) with derivative
\[ \Delta'(\zeta) = 2b \frac{\partial b}{\partial \zeta} - 4a \frac{\partial c}{\partial \zeta}, \]
which is linear in \( \zeta \) and has minimum at \( \zeta = 0 \) and maximum at \( \zeta = 0.046 \). Therefore,
\[ -7.45 \leq \Delta'(0) \leq \Delta'(\zeta) \leq \Delta'(0.046) \leq 20.61 \]
for each \( \zeta \in [0, 0.046) \). Also for such \( \zeta \),
\[ \Delta''(\zeta) = 2(\frac{\partial b}{\partial \zeta})^2 - 4a \frac{\partial^2 c}{\partial \zeta^2} = 2 \cdot 0.09^2 - 4 \cdot 2.382 \cdot (-64) \in (609.8, 609.81), \]
so \( \Delta(\zeta) \) is a parabola opening upward with minimum attained at the unique root \( \zeta_\xi \) of the equation \( \Delta'(\zeta) = 0 \). By \( \Delta'(0.012213) < -0.00039, \Delta'(0.012214) > 0.000219 \), and the above statements,
\[ 0.012213 \leq \zeta_\xi \leq 0.012214. \]

Hence \( \Delta(\zeta_\xi) \) satisfies
\[ 5.4821 \leq 5.48214 - 0.0004 \cdot 0.000001 \leq \Delta(0.012213) + \Delta'(0.012213) \cdot 0.000001 \]
\[ \leq \Delta(\zeta_\xi) \leq \Delta(0.012213) \leq 5.4822, \]
and the maximum of \( \Delta(\zeta) \) over \( \zeta \in [0, 0.046) \) is attained at \( \zeta = 0.046 \) and satisfies
\[ 5.83019 \leq \Delta(0.046) \leq 5.8302. \]

Therefore, for each \( \zeta \in [0, 0.046) \),
\[ 0.41415 \leq \frac{1}{\sqrt{5.8302}} \leq (\Delta(\zeta))^{-\frac{1}{2}} \leq \frac{1}{\sqrt{5.4821}} \leq 0.427098, \]
0.17152 \leq (\Delta(\zeta))^{-1} \leq 0.182412.

Thus by (17),
\[ -27.336 \leq \frac{1}{2 \cdot 2.382} \cdot 0.427098 \cdot (0 - 0.5 \cdot 609.81) \leq \xi''_1(\zeta) \]
\[ = \frac{1}{2a} \cdot (\Delta(\zeta))^{-\frac{1}{2}} \cdot \left( \frac{1}{4} \cdot (\Delta(\zeta))^{-1} \cdot (\Delta'(\zeta))^2 - \frac{1}{2} \cdot \Delta''(\zeta) \right) \]
\[ \leq \frac{1}{2 \cdot 2.382} \cdot 0.41415 \cdot (0.25 \cdot 0.182412 \cdot (20.61)^2 - 0.5 \cdot 609.8) \leq -24.822. \]  

This proves Claim 20.1. \( \Box \)

**Claim 20.2.** For each \( \zeta \in [0, 0.046) \), \(-0.91445 \leq \xi'_1(0.046) \leq \xi'_1(\zeta) \leq \xi'_1(0) \leq 0.31359.\)

*Proof.* By Claim 20.1, \( \xi'_1(\zeta) \) is a decreasing function on \( 0 \leq \zeta \leq 0.046. \) \( \Box \)

To prove (18), we write \( \frac{\partial^2 \ln(h_1(0.454, \zeta))}{\partial \zeta^2} \) in a form
\[ \frac{\partial^2 \ln(h_1(0.454, \zeta))}{\partial \zeta^2} = A_1(\zeta) + A_2(\zeta) + A_3(\zeta) + A_4(\zeta) + A_5(\zeta), \]  
and then bound these expressions separately so that the sum of the upper bounds will be negative for each \( \zeta \in [0, 0.046) \). By definition
\[ \frac{\partial^2 \ln(h_1)}{\partial \zeta^2} = \frac{1}{0.046 - \zeta} - \xi''_1(\zeta) \cdot \ln(\xi_1(\zeta)) - \frac{(\xi'_1(\zeta))^2}{\xi_1(\zeta)} + \frac{4}{0.046 + 2\zeta} + \frac{2}{0.408 + \zeta} + \frac{18}{0.408 - 3\zeta} + \ln(\frac{4}{1.618}) \cdot \xi''_1(\zeta) + \xi''_1(\zeta) \cdot \ln(0.092 - 2\zeta - \xi_1(\zeta)) - \frac{(2 + \xi'_1(\zeta))^2}{0.092 - 2\zeta - \xi_1(\zeta)} \]
\[ + \xi''_1(\zeta) \cdot \ln(0.092 + 4\zeta - \xi_1(\zeta)) - \frac{(\xi'_1(\zeta) - 4)^2}{0.092 + 4\zeta - \xi_1(\zeta)} \]
\[ - \xi''_1(\zeta) \cdot \ln(1.04 - 5\zeta + \xi_1(\zeta)) - \frac{(\xi'_1(\zeta) - 5)^2}{1.04 - 5\zeta + \xi_1(\zeta)}. \]

Let
\[ A_1(\zeta) := \frac{1}{0.046 - \zeta} - (\xi'_1(\zeta))^2 \cdot \frac{1}{\xi_1(\zeta)} - \frac{(2 + \xi'_1(\zeta))^2}{0.092 - 2\zeta - \xi_1(\zeta)}, \]  
\[ A_2(\zeta) := \xi''_1(\zeta) \cdot \ln(0.092 - 2\zeta - \xi_1(\zeta)) - \xi''_1(\zeta) \cdot \ln(\xi_1(\zeta)), \]  
\[ A_3(\zeta) := \ln(\frac{4}{1.618}) \cdot \xi''_1(\zeta), \]  
\[ A_4(\zeta) := \frac{4}{0.046 + 2\zeta} + \frac{2}{0.408 + \zeta} + \frac{18}{0.408 - 3\zeta}, \]  
and
\[ A_5(\zeta) := \xi''_1(\zeta) \cdot \ln(0.092 + 4\zeta - \xi_1(\zeta)) - \frac{(\xi'_1(\zeta) - 4)^2}{0.092 + 4\zeta - \xi_1(\zeta)} \]
\[ - \xi''_1(\zeta) \cdot \ln(1.04 - 5\zeta + \xi_1(\zeta)) - \frac{(\xi'_1(\zeta) - 5)^2}{1.04 - 5\zeta + \xi_1(\zeta)}. \]

so that (21) holds.
Claim 20.3. For each $\zeta \in [0, 0.046)$, $A_1(\zeta) < 0$.

Proof. Since $0.092 - 2\zeta - \xi_1(\zeta) \geq 0$ and $\xi_1(\zeta) \geq 0$, by Claim 20.2

\[ A_1(\zeta) = \frac{1}{0.046 - \zeta} - (\xi'_1(\zeta))^2 \cdot \frac{1}{\xi_1(\zeta)} - (2 + \xi'_1(\zeta))^2 \cdot \frac{1}{0.092 - 2\zeta - \xi_1(\zeta)} \]

\[ \leq \frac{1}{0.046 - \zeta} - (\xi'_1(\zeta))^2 \cdot \frac{1}{0.092 - 2\zeta} - (2 + \xi'_1(\zeta))^2 \cdot \frac{1}{0.092 - 2\zeta} \]

\[ = \frac{1}{0.046 - \zeta} - (\xi'_1(\zeta) + 1)^2 \cdot \frac{1}{0.046 - \zeta} = -\frac{(\xi'_1(\zeta) + 1)^2}{0.046 - \zeta} < 0. \quad \square \]

Claim 20.4. For each $\zeta \in [0, 0.046)$, $A_2(\zeta) < 0$.

Proof. Let $\zeta \in [0, 0.046)$. By Claim 20.1, inequality $A_2(\zeta) < 0$ is equivalent to

\[ 0.092 - 2\zeta - \xi_1(\zeta) > \xi_1(\zeta). \]

Let $y(\zeta) = 0.092 - 2\zeta - 2\xi_1(\zeta)$. By Claim 20.2

\[ y'(\zeta) = -2 - 2\xi'_1(\zeta) < 0. \]

Therefore, $y(\zeta) > y(0.046) = 0$ for each $\zeta \in [0, 0.046)$. This proves the claim. \quad \square

Claim 20.5. For each $\zeta \in [0, 0.046)$, $A_3(\zeta) \leq -22.46$.

Proof. This follows from the definition (24), since $\xi''_1(\zeta) \leq -24.822$ by Claim 20.1. \quad \square

Claim 20.6. The function $A'_1(\zeta)$ has exactly one root $d'_\zeta$ in the interval $[0, 0.046]$. Furthermore, $d'_\zeta \in (0.0355167, 0.0355168)$, and $A_1(\zeta)$ is decreasing on $[0, d'_\zeta]$ and increasing on $[d'_\zeta, 0.046]$.

Proof. By Definition (25),

\[ A'_1(\zeta) = -\frac{2}{(\zeta + 0.023)^2} - \frac{2}{(\zeta + 0.408)^2} + \frac{6}{(\zeta - 0.136)^2} \]

and

\[ A''_1(\zeta) = \frac{4}{(\zeta + 0.023)^3} + \frac{4}{(\zeta + 0.408)^3} - \frac{12}{(\zeta - 0.136)^3}. \]

The last expression is positive for all $\zeta \in [0, 0.046]$, so function $A'_1(\zeta)$ may have at most one root on $[0, 0.046]$. On the other hand, $A'_1(0.0355167) < -0.002$ and $A'_1(0.0355168) > 0.006$. This proves the claim. \quad \square

Claim 20.7. For each $\zeta \in [0, 0.046)$, $A_4(\zeta) + A_5(\zeta) \leq 20$.

Proof. Let

\[ z_1(\zeta) = 0.092 + 4\zeta - \xi_1(\zeta) \quad \text{and} \quad z_2(\zeta) = 1.04 - 5\zeta + \xi_1(\zeta). \]

By Claim 20.2, $z'_1(\zeta) = 4 - \xi'_1(\zeta) > 0$ and $z'_2(\zeta) = -5 + \xi'_1(\zeta) < 0$ for each $\zeta \in [0, 0.046)$. So, $z_1(\zeta)$ is increasing and $z_2(\zeta)$ is decreasing on $[0, 0.046)$. (27)
Therefore,  
\[ z_1(\zeta) < z_1(0.046) < z_2(0.046) < z_2(\zeta) \]
for each \( \zeta \in [0, 0.046] \), Definitions [25] and [26] together with Claim [20.1] yield
\[
A_4(\zeta) + A_5(\zeta) = A_4(\zeta) + \xi_1''(\zeta) \cdot \ln(z_1(\zeta)) - \left( \frac{\xi_1'(\zeta) - 4}{z_1(\zeta)} \right) - \xi_1''(\zeta) \cdot \ln(z_2(\zeta)) - \left( \frac{\xi_1'(\zeta) - 5}{z_2(\zeta)} \right)
\leq A_4(\zeta) - 27.336 \cdot (\ln(z_1(\zeta)) - \ln(z_2(\zeta))) - \left( \frac{\xi_1'(\zeta) - 4}{z_1(\zeta)} \right) - \left( \frac{\xi_1'(\zeta) - 5}{z_2(\zeta)} \right) =: Q(\zeta).
\]
Since \( \zeta \in [0, 0.046] \), it belongs to the interval \([0.001 k, 0.001(k + 1)]\) for some integer \( 0 \leq k \leq 45 \).
We consider 3 cases.

**Case 1:** \( 0 \leq k \leq 34 \). Then by Claim [20.6] and [27], for each \( \zeta \in [0.001 k, 0.001(k + 1)] \),
\[
A_4(0.001 k) \geq A_4(\zeta),
\]
\[ z_1(\zeta) \geq z_1(0.001 k), \text{ and } z_2(0.001 k) \geq z_2(\zeta). \]
Therefore,
\[
Q(\zeta) \leq M_1(k) := A_4(0.001 k) - 27.336 \cdot (\ln(z_1(0.001 k)) - \ln(z_2(0.001 k)))
- \left( \frac{\xi_1'(0.001 k) - 4}{z_1(0.001(k + 1))} \right) - \left( \frac{\xi_1'(0.001 k) - 5}{z_2(0.001 k)} \right).
\]
The bounds for \( M_1(k) \) certifying that \( M_1(k) < 20 \) for each \( 0 \leq k \leq 34 \) are given in Table 2 in Appendix 1.

**Case 2:** \( k = 35 \). Similarly to Case 1,
\[
Q(\zeta) \leq \max(A_4(0.035), A_4(0.036)) - 27.336 \cdot (\ln(z_1(0.035)) - \ln(z_2(0.035)))
- \left( \frac{\xi_1'(0.035) - 4}{z_1(0.036)} \right) - \left( \frac{\xi_1'(0.035) - 5}{z_2(0.035)} \right)
< 98.404 - 27.336 \cdot (-1.5 \cdot (-0.135)) - 94 - 36.3 < 5.5 < 20.
\]

**Case 3:** \( 36 \leq k \leq 45 \). Again, similarly to Case 1,
\[
Q(\zeta) \leq M_3(k) := A_4(0.001(k + 1)) - 27.336 \cdot (\ln(z_1(0.001 k)) - \ln(z_2(0.001 k)))
- \left( \frac{\xi_1'(0.001 k) - 4}{z_1(0.001(k + 1))} \right) - \left( \frac{\xi_1'(0.001 k) - 5}{z_2(0.001 k)} \right).
\]
The bounds for \( M_1(k) \) certifying that \( M_1(k) < 20 \) for each \( 36 \leq k \leq 45 \) are given in Table 1 in Appendix 1. \( \square \)

Thus by [21] and Claims [20.3] [20.7], for each \( \zeta \in [0, 0.046] \),
\[
\frac{\partial^2 \ln(h_1(0.454, \zeta))}{\partial \zeta^2} = \sum_{i=1}^{5} A_i(\zeta) < -22.46 + 20 = -2.46 < 0.
\]
We also can check by plugging in the values that
\[
\frac{\partial \ln(h_1(0.454, 0.0228718))}{\partial \zeta} < -9 \cdot 10^{-6}, \quad \text{and} \quad \frac{\partial \ln(h_1(0.454, 0.0228719))}{\partial \zeta} > 7.54 \cdot 10^{-8}.
\]
Thus, the derivative of \( h_1(0.454, \zeta) \) equals 0 at a unique \( \zeta_1 \in (0.0228718, 0.0228719) \).

Recall that \( h_1(0.454, \zeta) > 0 \) for \( \zeta \in [0, 0.046) \). So, after comparing the value \( h_1(0.454, 0.0228719) \) with the boundary values \( h_1(0.454, 0) \) and \( h_1(0.454, 0.46) \), we conclude that the maximum of \( h_1(0.454, \zeta) \) is attained at \( \zeta_1 \). We can plug in numbers into a computer and obtain that
\[
h_1(0.454, 0.0228718) \leq 0.999982,
\]
and
\[
\frac{\partial \ln(h_1(0.454, 0.0228718))}{\partial \zeta} \leq 1 \cdot 10^{-7},
\]
and
\[
\frac{\partial h_1(0.454, 0.0228718)}{\partial \zeta} = h_1(0.454, 0.0228718) \cdot \frac{\partial \ln(h_1(0.454, 0.0228718))}{\partial \zeta} \leq 1 \cdot 10^{-7},
\]
which implies that
\[
h_1(0.454, \zeta_1) \leq h_1(0.454, 0.0228718) + 1 \cdot 10^{-7} \cdot 0.0000001 \leq 0.999983. \quad \square
\]

The proof of the next lemma is similar but significantly simpler. It is mostly a routine bounding some expressions. So, we present the proof of Lemma 21 in Appendix 2.

**Lemma 21.** For every
\[(\chi, \zeta, \xi) \in \Omega = \{(\chi, \zeta, \xi) : 0.454 < \chi \leq 0.45537, \ 0 \leq \zeta \leq \frac{1}{2} - \chi, \ 0 \leq \xi \leq 1 - 2\chi - 2\zeta\},\]
we have
\[
\frac{\partial \ln(h(\chi, \zeta, \xi))}{\partial \chi} < 0. \quad \square \tag{28}
\]

Since \( h(\chi, \zeta, \xi) > 0 \) for each \( (\chi, \zeta, \xi) \in \Omega \), Lemma 21 yields that for each fixed \( \zeta \) and \( \xi \), the maximum of \( h(\chi, \zeta, \xi) \) over \( (\chi, \zeta, \xi) \in \Omega \) is attained at \( \chi = 0.454 \). By Lemma 20, this maximum is at most 0.999983. This yields (14).

### 6 Completion of the proof of Theorem 17

By (14) and Lemma 19, for all positive integers \( n \) and \( x \) such that \( n \) is even and \( 0.454n < x \leq 0.45537n \),
\[
\frac{q(x, n)}{(3n - 1)!!} \leq O(n^6) \cdot 0.999983^n.
\]
It follows that
\[
\frac{1}{(3n - 1)!!} \sum_{x=[0.454n]}^{[0.45537n]} q(x, n) \leq O(n^7) \cdot 0.999983^n \to 0 \quad \text{as} \ n \to \infty. \tag{29}
\]
Thus by Lemma \ref{lemma:pairings}, the number of pairings $F \in \mathcal{G}_16(n)$ with $0.454n < \alpha(F) \leq 0.45537n$ is $o((3n - 1)!!)$. Together with Theorem \ref{thm:regular}, this means that almost no pairings have independence ratio larger than 0.454. Thus by Corollary \ref{cor:conclusion} we conclude that almost no $n$-vertex 3-regular graphs of girth at least 16 have independence ratio larger than 0.454. This proves Theorem \ref{thm:main} and thus also Theorem \ref{thm:main2}.

**Acknowledgment.** We thank Jan Volec for helpful discussion and bringing \cite{Csoka} to our attention.

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### Table 1: Upper bounds for expressions in $M_1(k)$.

| $k$ | $A_4(0.001k)$ | $-\ln(z_1(0.001k))$ | $\ln(z_2(0.001k))$ | $-(\xi_1(0.001k)-4)^2/z_1(0.001k+1)^2$ | $-(\xi_1(0.001k)-5)^2/z_2(0.001k)$ | $M_1(k)$ |
|-----|----------------|----------------------|---------------------|----------------------------------------|----------------------------------------|----------|
| 0   | 135.9762       | 2.553562             | 0.05277836          | -166.7356                              | -20.83335                              | 19.7     |
| 1   | 132.6679       | 2.507105             | 0.04831009          | -161.790                               | -21.1686                               | 19.6     |
| 2   | 129.6543       | 2.462392             | 0.04379588          | -157.2333                              | -21.50947                              | 19.5     |
| 3   | 126.903        | 2.419288             | 0.03923514          | -153.0194                              | -21.85745                              | 19.3     |
| 4   | 124.384        | 2.377674             | 0.03462729          | -149.1122                              | -22.20758                              | 19.1     |
| 5   | 122.0728       | 2.33745              | 0.02997174          | -145.4974                              | -22.56505                              | 18.8     |
| 6   | 119.9504       | 2.298492             | 0.02526786          | -142.0935                              | -22.92824                              | 18.5     |
| 7   | 117.9977       | 2.260743             | 0.02051009          | -138.9302                              | -23.29722                              | 18.2     |
| 8   | 116.1989       | 2.224115             | 0.01571273          | -135.9683                              | -23.67205                              | 17.8     |
| 9   | 114.5404       | 2.188538             | 0.01086022          | -133.1892                              | -24.05282                              | 17.5     |
| 10  | 113.0099       | 2.153948             | 0.005956888         | -130.5765                              | -24.43961                              | 17.1     |
| 11  | 111.5969       | 2.120286             | 0.001002109         | -128.1157                              | -24.83248                              | 16.7     |
| 12  | 110.292        | 2.0876               | -0.004004782        | -125.793                               | -25.23152                              | 16.3     |
| 13  | 109.0867       | 2.05542              | -0.009064451        | -123.5994                              | -25.63682                              | 15.8     |
| 14  | 107.9738       | 2.024367             | -0.01417756         | -121.5220                              | -26.04845                              | 15.4     |
| 15  | 106.9466       | 1.993934             | -0.01934483         | -119.5525                              | -26.4664                               | 15.0     |
| 16  | 106.0000       | 1.964204             | -0.02456692         | -117.6823                              | -26.89104                              | 14.5     |
| 17  | 105.1269       | 1.935144             | -0.02984456         | -115.9041                              | -27.32218                              | 14.0     |
| 18  | 104.3229       | 1.90673              | -0.03517846         | -114.2111                              | -27.76000                              | 13.6     |
| 19  | 103.585        | 1.878905             | -0.04056935         | -112.5970                              | -28.20460                              | 13.1     |
| 20  | 102.9088       | 1.851668             | -0.04601797         | -111.0562                              | -28.65606                              | 12.6     |
| 21  | 102.2906       | 1.824985             | -0.05152507         | -109.5836                              | -29.1144                               | 12.1     |
| 22  | 101.7273       | 1.798832             | -0.05709143         | -108.1746                              | -29.57998                              | 11.6     |
| 23  | 101.217        | 1.773185             | -0.06271781         | -106.8248                              | -30.05263                              | 11.1     |
| 24  | 100.7543       | 1.748024             | -0.06840502         | -105.5304                              | -30.53255                              | 10.7     |
| 25  | 100.3398       | 1.723329             | -0.07415386         | -104.2877                              | -31.01984                              | 10.2     |
| 26  | 99.97009       | 1.699082             | -0.07996514         | -103.0934                              | -31.51462                              | 9.7      |
| 27  | 99.64358       | 1.675265             | -0.08583972         | -101.9446                              | -32.016                                | 9.2      |
| 28  | 99.3585        | 1.651862             | -0.09177843         | -100.8383                              | -32.52708                              | 8.7      |
| 29  | 99.11297       | 1.628858             | -0.09778215         | -99.77206                              | -33.04501                              | 8.2      |
| 30  | 98.90584       | 1.606239             | -0.1038517          | -98.74333                              | -33.5708                               | 7.7      |
| 31  | 98.7358        | 1.583989             | -0.1099881          | -97.74996                              | -34.10486                              | 7.2      |
| 32  | 98.6015        | 1.562098             | -0.1161922          | -96.78986                              | -34.64704                              | 6.7      |
| 33  | 98.50187       | 1.54056              | -0.122464           | -95.86113                              | -35.19758                              | 6.3      |
| 34  | 98.43615       | 1.519339             | -0.1288073          | -94.96198                              | -35.75661                              | 5.8      |
Appendix 2: Proof of Lemma 21

By definition, the boundary, $\partial \Omega$, of $\Omega$ is

$$\partial \Omega = \{(\chi, \zeta, \xi) : \xi = 0, 2\chi + 2\zeta \leq 1, 0.454 \leq \chi \leq 0.45537, \zeta \geq 0\} \cup$$

$$\{(\chi, \zeta, \xi) : \zeta = 0, 2\chi + \xi \leq 1, 0.454 \leq \chi \leq 0.45537, \xi \geq 0\} \cup$$

$$\{(\chi, \zeta, \xi) : \chi = 0.454, 2\zeta + \xi \leq 0.092, \zeta \geq 0, \xi \geq 0\} \cup$$

$$\{(\chi, \zeta, \xi) : \chi = 0.45537, 2\zeta + \xi \leq 0.08926, \zeta \geq 0, \xi \geq 0\}. $$

We also will consider the 2-dimensional set

$$\Omega_1 = \{(\chi, \zeta) : 0.454 \leq \chi \leq 0.45537, 0 \leq \zeta \leq 0.5 - \chi\}. $$

Then the boundary of $\Omega_1$ is

$$\partial \Omega_1 = \{(\chi, \zeta) : 0.454 \leq \chi \leq 0.45537, \zeta = 0\} \cup \{(\chi, \zeta) : 0 \leq \zeta \leq 0.046, \chi = 0.454\} \cup \{(\chi, \zeta) : 0 \leq \zeta \leq 0.04463, \chi = 0.45537\} \cup \{(\chi, \zeta) : 0.454 \leq \chi \leq 0.45537, \chi + \zeta = \frac{1}{2}\}. $$

By the definition of $h$,

$$\frac{\partial \ln(h(\chi, \zeta, \xi))}{\partial \chi} = 4\ln(2\chi - 3\zeta - \frac{1}{2}) + 4\ln(2\chi + \zeta - \frac{1}{2}) - \ln(1 - 2\chi - 2\zeta) - \ln(1 - 2\chi + 4\zeta) + 2\ln(1 - 2\chi - 2\zeta - \xi) + 2\ln(1 - 2\chi + 4\zeta - \xi) - 10\ln(10\chi - 5\zeta + \xi - \frac{7}{2}). $$

Similarly to the proof of Lemma 20, we present $\frac{\partial \ln(h(\chi, \zeta, \xi))}{\partial \chi}$ in the form $\sum_{j=1}^{6} B_j$, where

$$B_1(\chi, \zeta) := 4\ln(2\chi - 3\zeta - \frac{1}{2}), \quad B_2(\chi, \zeta) := 4\ln(2\chi + \zeta - \frac{1}{2}),$$

$$B_3(\chi, \zeta, \xi) := 2\ln(1 - 2\chi - 2\zeta - \xi) - \ln(1 - 2\chi - 2\zeta), \quad B_4(\chi, \zeta) := \ln(1 - 2\chi + 4\zeta),$$

$$B_5(\chi, \zeta, \xi) := 2\ln(1 - 2\chi + 4\zeta - \xi), \quad \text{and} \quad B_6(\chi, \zeta, \xi) := \ln(10\chi - 5\zeta + \xi - \frac{7}{2}),$$

and then bound each of the terms separately.
Claim 21.1. For all \((\chi, \zeta) \in \Omega_1\), \(B_1(\chi, \zeta) < -3.55\).

Proof. For each \((\chi, \zeta) \in \Omega_1\), we have \(\chi - \frac{3}{2} \zeta - 0.25 > 0\), since \(\chi \geq 0.454\) and \(\zeta \leq 0.046\). As for each \((\chi, \zeta) \in \Omega_1\),

\[
\frac{\partial B_1(\chi, \zeta)}{\partial \chi} = \frac{4}{\chi - \frac{3}{2} \zeta - 0.25} > 0 \quad \text{and} \quad \frac{\partial B_1(\chi, \zeta)}{\partial \zeta} = \frac{-6}{\chi - \frac{3}{2} \zeta - 0.25} < 0,
\]

the maximum is attained at a corner on the boundary \(\partial \Omega_1\). Comparing the values of \(B_1\) at the four corners of \(\partial \Omega_1\), we see that the maximum is attained at \((\chi, \zeta) = (0.45537, 0)\) and \(B_1(0.45537, 0) < -3.55\). \(\square\)

Claim 21.2. For all \((\chi, \zeta) \in \Omega_1\), \(B_2(\chi, \zeta) < -3.14\).

Proof. For each \((\chi, \zeta) \in \Omega_1\), we have \(2\chi + \zeta - \frac{1}{2} > 0\). As for each \((\chi, \zeta) \in \Omega_1\),

\[
\frac{\partial B_2(\chi, \zeta)}{\partial \chi} = \frac{8}{2\chi + \zeta - \frac{1}{2}} > 0, \quad \text{and} \quad \frac{\partial B_2(\chi, \zeta)}{\partial \zeta} = \frac{4}{2\chi + \zeta - 0.5} > 0,
\]

the maximum is attained at a corner of the boundary \(\partial \Omega_1\). Comparing the values of \(B_2\) at the four corners of \(\partial \Omega_1\), we see that the maximum is attained at \((\chi, \zeta) = (0.45537, 0.04463)\), and \(B_2(0.45537, 0.04463) < -3.14\). \(\square\)

Claim 21.3. For all \((\chi, \zeta, \xi) \in \Omega\), \(B_3(\chi, \zeta, \xi) < 0\).

Proof. We can write \(B_3(\chi, \zeta, \xi)\) in the form

\[
B_3(\chi, \zeta, \xi) = \ln(1 - 2\chi - 2\zeta - \xi) + \ln\left(\frac{1 - 2\chi - 2\zeta - \xi}{1 - 2\chi - 2\zeta}\right),
\]

and observe that \(\ln(1 - 2\chi - 2\zeta - \xi) < 0\) (since \(2\chi + 2\zeta + \xi > 0\)) and \(\ln\left(\frac{1 - 2\chi - 2\zeta - \xi}{1 - 2\chi - 2\zeta}\right) \leq 0\) (since \(1 - 2\chi - 2\zeta - \xi \leq 1 - 2\chi - 2\zeta\)). \(\square\)

Claim 21.4. For all \((\chi, \zeta) \in \Omega_1\), \(B_4(\chi, \zeta) < -1.28\).

Proof. For each \((\chi, \zeta) \in \Omega_1\), \(-2\chi + 4\zeta + 1 > 0\). As for each \((\chi, \zeta) \in \Omega_1\),

\[
\frac{\partial B_4(\chi, \zeta)}{\partial \chi} = -\frac{2}{-2\chi + 4\zeta + 1} < 0, \quad \text{and} \quad \frac{\partial B_4(\chi, \zeta)}{\partial \zeta} = \frac{4}{-2\chi + 4\zeta + 1} > 0,
\]

the maximum of \(B_4\) is attained at a corner of the boundary \(\partial \Omega_1\). Comparing the values of \(B_4\) at the four corners of \(\partial \Omega_1\), we see that the maximum is attained at \((\chi, \zeta) = (0.454, 0.046)\), and \(B_4(0.454, 0.046) < -1.28\). \(\square\)

Claim 21.5. For all \((\chi, \zeta, \xi) \in \Omega\), \(B_5(\chi, \zeta, \xi) < -2.57\).

Proof. For each \((\chi, \zeta) \in \Omega - \partial \Omega\), we have \(2\chi - 4\zeta + \xi - 1 < 0\) since \(2\chi + 2\zeta + \xi < 1 \leq 1 + 4\zeta + 2\xi\). Since

\[
\lim_{2\chi + \zeta \to 1} B_5(\chi, 0, \xi) = -\infty,
\]

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the maximum of $B_5$ is not attained at $\zeta = 0, 2\chi + \xi = 1$. As for each $(\chi, \zeta, \xi) \in \Omega$,

$$\frac{\partial B_5(\chi, \zeta, \xi)}{\partial \chi} = \frac{4}{2\chi - 4\zeta + \xi - 1} < 0, \quad \frac{\partial B_5(\chi, \zeta, \xi)}{\partial \zeta} = \frac{-8}{2\chi - 4\zeta + \xi - 1} > 0,$$

and

$$\frac{\partial B_5(\chi, \zeta, \xi)}{\partial \xi} = \frac{2}{2\chi - 4\zeta + \xi - 1} < 0,$$

the maximum of $B_5$ is attained at a corner of the boundary $\partial \Omega$. Comparing the values of $B_5$ at the corners of $\partial \Omega$, we see that the maximum is attained at $(\chi, \zeta, \xi) = (0.454, 0.046, 0)$ and $B_5(0.454, 0.046, 0) = -2.57$.

Claim 21.6. For all $(\chi, \zeta, \xi) \in \Omega$, $B_6(\chi, \zeta, \xi) < 0.14$.

Proof. For each $(\chi, \zeta, \xi) \in \Omega$, we have $10\chi - 5\zeta + \xi - \frac{7}{2} > 0$ since $10\chi - \frac{7}{2} \geq 1.04$ and $5\zeta \leq 0.23$. As for each $(\chi, \zeta, \xi) \in \Omega$,

$$\frac{\partial B_6(\chi, \zeta, \xi)}{\partial \chi} = \frac{10}{10\chi - 5\zeta + \xi - \frac{7}{2}} > 0, \quad \frac{\partial B_6(\chi, \zeta, \xi)}{\partial \zeta} = \frac{-5}{10\chi - 5\zeta + \xi - \frac{7}{2}} < 0,$$

and

$$\frac{\partial B_6(\chi, \zeta, \xi)}{\partial \xi} = \frac{1}{10\chi - 5\zeta + \xi - \frac{7}{2}} > 0,$$

the maximum of $B_5$ is attained at a corner of the boundary $\partial \Omega$. Comparing the values of $B_6$ at the corners of $\partial \Omega$, we see that the maximum is attained at $(\chi, \zeta, \xi) = (0.45537, 0, 0.08926)$ and $B_6(0.454, 0, 0.08926) < 0.14$.

By Claims 21.1–21.6, for each $(\chi, \zeta, \xi) \in \Omega$,

$$\frac{\partial \ln(h(\chi, \zeta, \xi))}{\partial \chi} = B_1(\chi, \zeta) + B_2(\chi, \zeta) + B_3(\chi, \zeta, \xi) + B_4(\chi, \zeta) + B_5(\chi, \zeta, \xi) + B_6(\chi, \zeta, \xi)$$

$$< -3.55 - 3.14 + 0 - 1.28 - 2.57 + 0.14 < 0.$$