On the classification of boundary value problems for elastic homogeneous and piecewise homogeneous spaces with a penny-shaped crack

S M Mkhitaryan\textsuperscript{1,2}, M S Mkrtchyan\textsuperscript{1,2}, and E G Kanetsyan\textsuperscript{2*}

\textsuperscript{1}Institute of Mechanics of the National Academy of Sciences of Armenia, Yerevan, Armenia
\textsuperscript{2}National University of Architecture and Construction of Armenia, Yerevan, Armenia
E-mail: *ekanetsyan@gmail.com

Abstract. The classification of the basic axisymmetric boundary value problems of the mathematical theory of elasticity for homogeneous and piecewise homogeneous elastic spaces with a penny-shaped crack is given. Using the Hankel integral transform, governing integral equations (GIEs) of these problems are derived and methods for finding their exact solutions are indicated.

1. Introduction
In [1] classification of basic boundary value problems of the theory of elasticity for an elastic plane with a collinear system of cuts (cracks) is given; their exact solutions in the form of complex potentials are obtained by the unified method of the Riemann boundary value problem of the theory of analytic functions. By the same methods using the known dependencies between the plane and axisymmetric stress states, exact solutions of several axisymmetric boundary value problems for an elastic space with a penny-shaped crack and mixed boundary value problems for an elastic half-space with a circular line of separation of boundary conditions are obtained in [2, 3]. The same problems are solved by the Hankel integral transform method in [4]. The results of numerous studies on axisymmetric boundary value problems for homogeneous and piecewise homogeneous massive elastic bodies with circular lines of separation of boundary conditions obtained by various methods and approaches are summarized in [5–8].

In the present paper, according to [1], the basic axisymmetric boundary value problems for an elastic homogeneous and piecewise homogeneous space with a circular crack are classified as follows:

I on the upper and lower edges of a circular crack, components of normal and radial stresses are given in advance;
II on the upper and lower edges of a crack, components of normal and radial displacements are given;
III on the upper edge of a crack, components of normal and radial stresses are specified, and on the lower edge, components of displacements are given in the same directions (mixed boundary value problem).
These basic boundary value problems (BBVPs) are considered by a unified method of integral equations and using the Hankel integral transform; they are described by Fredholm integral equations (IE) of the first kind or by systems of such equations with symmetric kernels expressed by Weber–Sonin integrals, their combinations, as well as by integro-differential equations (IDE).

Exact solutions of the GIEs of the discussed BBVPs are obtained by the method of Abel rotation operators and as a result, these IEs are reduced to successively solvable Abel integral equations, or, based on the fundamental relationship between Abelian operators and the operator generated by the Cauchy kernel, the GIEs are reduced to singular integral equations (SIEs) or systems of SIEs. After constructing the solutions to the GIEs all the characteristics of the BBVPs are determined.

2. Derivation of GIEs of BBVPs

Let a piecewise homogeneous space, referred to the cylindrical coordinate system $r, \vartheta, z$, consist of the upper (+) and lower (−) elastic half-spaces with elastic constants $(E_\pm, \nu_\pm)$. Let further an interphase penny-shaped crack $\omega = \{z = 0; 0 < r < a\}$ in the shape of a circle of radius $a$ be on the interface $z = 0$ of dissimilar materials. Suppose that normal and radial forces of intensities $p_\pm (r)$ and $\tau_\pm (r)$ act on the upper (+) and lower (−) edges of the crack, i.e.,

$$
\sigma_z (r, z) \bigg|_{z=\pm 0} = -p_\pm (r), \quad \sigma_{rz} (r, z) \bigg|_{z=\pm 0} = -\tau_\pm (r) \quad (0 < r < a),
$$

(1)

where $\sigma_z$ and $\sigma_{rz}$ are normal and tangential components of stresses, respectively. Now cut mentally the piecewise homogeneous elastic space on the upper $(z > 0)$ and lower $(z < 0)$ half-spaces, and introduce the following notation:

$$
\sigma_z \bigg|_{z=\pm 0} = -\Sigma_\pm (r) = \begin{cases} -p_\pm (r), & 0 < r < a, \\ -p (r), & r > a, \end{cases} \quad \tau_{rz} \bigg|_{z=\pm 0} = -T_\pm (r) = \begin{cases} -\tau_\pm (r), & 0 < r < a, \\ -\tau (r), & r > a. \end{cases}
$$

Then, based on the Lamé equations in a cylindrical coordinate system, using the Hankel integral transform for the radial $u_r$ and normal $u_z$ displacements of the boundary points of the upper (+) and lower (−) elastic half-spaces, we obtain the following formulas [9]:

$$
u̇ṅ_+(r, 0) = u_n(r, 0) = \pm \pi \vartheta_0^+ \int_0^\infty W_{11}(r, \rho) T_\pm (\rho) \rho \, d\rho - 2 \vartheta_1^+ \int_0^\infty W_{10}(r, \rho) \Sigma_\pm (\rho) \rho \, d\rho,
$$

$$
u̇ṅ_-(r, 0) = w_\pm (r, 0) = -2 \vartheta_1^+ \int_0^\infty W_{01}(r, \rho) T_\pm (\rho) \rho \, d\rho \pm \pi \vartheta_0^+ \int_0^\infty W_{00}(r, \rho) \Sigma_\pm (\rho) \rho \, d\rho \quad (0 < r < \infty),
$$

(2)

$$W_{mn}(r, \rho) = \int_0^\infty J_m(\lambda r) J_n(\lambda r) \, d\lambda \quad (m, n = 0, 1), \quad \vartheta_0^+ = \frac{2(1-\nu_\pm^2)}{\pi E_\pm}, \quad \vartheta_1^+ = \frac{(1+\nu_\pm)(1-2\nu_\pm)}{2E_\pm}.
$$

Here $W_{mn}(r, \rho)$ are the well-known Weber–Sonin integrals, $J_n(r)$ are the Bessel functions of the first kind of index $n$, and $\lambda$ is the spectral parameter of the Hankel transform. Then in (2) we introduce the following functions:

$$
\varphi_\pm (r) = \omega_\pm (r, 0) \pm w_\pm (r, 0), \quad \psi_\pm (r) = u_\pm (r, 0) \pm u_\pm (r, 0),
$$

$$\Omega_\pm (r) = \Sigma_\pm (r) \pm \Sigma_\pm (r), \quad \Lambda_\pm (r) = T_\pm (r) \pm T_\pm (r) \quad (0 < r < \infty)
$$

and apply the Hankel inverse transform to these formulas. Denote the Hankel transformants of these functions by the same letters with an overbar. After simple transformations and using well-known properties of the Hankel integrals [10], we have the following system of linear algebraic
where

\[ f_1(\lambda) = 2\theta_1^+\ddot{\Omega}_- (\lambda) - \pi \theta_0^+ \dddot{X}_- (\lambda) + \lambda \ddot{\psi}_- (\lambda), \quad f_2(\lambda) = 2\theta_1^+ \ddot{\Omega}_- (\lambda) + \pi \theta_0^+ \dddot{X}_- (\lambda) + \lambda \ddot{\psi}_- (\lambda), \]

\[ f_3(\lambda) = -\pi \theta_0^+ \dddot{X}_- (\lambda) + 2\theta_1^+ \dddot{X}_- (\lambda) + \lambda \dddot{\varphi}_- (\lambda), \quad f_4(\lambda) = \pi \theta_0^+ \ddot{\Omega}_- (\lambda) + 2\theta_1^+ \ddot{X}_- (\lambda) + \lambda \dddot{\varphi}_- (\lambda). \]

The solution to system (3) is represented by the formulas

\[
\begin{align*}
\dot{\Omega}_+(\lambda) &= C_1 \dot{\Omega}_-(\lambda) + D_1 \dddot{X}_-(\lambda) + C_2 \lambda \dot{\varphi}_-(\lambda) + D_2 \lambda \ddot{\psi}_-(\lambda), \\
\dot{X}_+(\lambda) &= D_1 \dot{\Omega}_-(\lambda) + C_1 \dddot{X}_-(\lambda) + D_2 \lambda \ddot{\psi}_-(\lambda) + C_2 \lambda \dddot{\varphi}_-(\lambda), \\
\ddot{\varphi}_+(\lambda) &= \frac{1}{\lambda} [F_1 \ddot{\Omega}_-(\lambda) + E_1 \dddot{X}_-(\lambda)] - [C_1 \ddot{\varphi}_-(\lambda) + D_1 \dddot{\psi}_-(\lambda)], \\
\ddot{\psi}_+(\lambda) &= \frac{1}{\lambda} [E_1 \ddot{\Omega}_-(\lambda) + F_1 \dddot{X}_-(\lambda)] - [D_1 \ddot{\varphi}_-(\lambda) + C_1 \dddot{\psi}_-(\lambda)] \quad (0 < \lambda < \infty),
\end{align*}
\]  

where

\[
\begin{align*}
C_1 &= \Delta^{-1} [(3 - 4\nu_-)k^2 - 3 + 4\nu_+], \quad C_2 = 8G_+ \Delta^{-1} [1 - \nu_+ + (1 - \nu_-)k], \\
D_1 &= 4k \Delta^{-1} [(1 - 2\nu_-)(1 - \nu_-) + (1 - 2\nu_+)(1 + \nu_+)], \quad D_2 = 4G_+ \Delta^{-1} [1 - 2\nu_+ - (1 - 2\nu_-)k], \\
E_1 &= (G_- \Delta)^{-1} [(1 - 2\nu_-)(3 - 4\nu_-) - (1 - 2\nu_+)(3 - 4\nu_+)], \\
F_1 &= 2(G_- \Delta)^{-1} [(1 - \nu_-)(3 - 4\nu_+) + (1 + \nu_+)(3 - 4\nu_-)], \\
\Delta &= \Delta(\nu_+, \nu_-) = [k(3 - 4\nu_-) + 1](k + 3 - 4\nu_+), \quad G_\pm = \frac{E_\pm}{2(1 + \nu_\pm)}, \quad k = \frac{G_+}{G_-}.
\end{align*}
\]

Turning to the first BBVP formulated above with the boundary conditions (1), we apply the inverse Hankel transform formula to the first two equations from (4). To this end, we first set

\[
\begin{align*}
\ddot{\Omega}_+(\lambda) &= -\lambda \ddot{\Omega}_+(\lambda), \quad \lambda \ddot{\varphi}_-(\lambda) = \varphi_-(\lambda), \quad \dddot{X}_+(\lambda) = -\lambda \dddot{X}_+(\lambda), \quad \dddot{\psi}_-(\lambda) = \lambda \dddot{\psi}_-(\lambda), \\
\{\ddot{\Omega}_+(\lambda); \varphi_-(\lambda)\} &= \int_0^\infty \{\ddot{\Omega}_+(\lambda); \varphi_-(\lambda)\} r J_1(\lambda r) \, dr, \\
\{\dddot{X}_+(\lambda); \dddot{\psi}_-(\lambda)\} &= \int_0^\infty \{\dddot{X}_+(\lambda); \dddot{\psi}_-(\lambda)\} r J_0(\lambda r) \, dr.
\end{align*}
\]

As a result, we have

\[
\begin{align*}
\ddot{\Omega}_+(\lambda) &= -C_1 \ddot{\Omega}_-(\lambda) - D_1 \dddot{X}_-(\lambda) - C_2 \dddot{\varphi}_-(\lambda) - D_2 \dddot{\psi}_-(\lambda), \\
\dddot{X}_+(\lambda) &= -D_1 \dddot{X}_-(\lambda) - C_1 \dddot{X}_-(\lambda) - D_2 \dddot{\varphi}_-(\lambda) - C_2 \dddot{\psi}_-(\lambda).
\end{align*}
\]

Further, we multiply both sides of the first equation by $\lambda J_1(\lambda r)$ and integrate the equation with respect to $\lambda$ from 0 to $\infty$, while both sides of the second equation we multiply by $\lambda J_0(\lambda r)$ and
integrate with respect to \( \lambda \) again from 0 to \( \infty \). Then we use formulas
\[
\int_0^\infty \left( \frac{d\Omega^{(1)}_+}{dr} + \frac{\Omega^{(1)}_+}{r} \right) r J_0(\lambda r) \, dr = \lambda \Omega^{(1)}_+(\lambda),
\]
\[
\int_0^\infty \frac{d\varphi_-}{dr} r J_1(\lambda r) \, dr = -\lambda \varphi_-^{(0)}(\lambda) = -\lambda \int_0^\infty \varphi_-(r) J_0(\lambda r) \, dr,
\]

the first of which is easily obtained by integration by parts and the second is given in [10] (p. 79, formula (2.33)). As a result, we arrive at the following key equations of the problem:

\[
\Omega_+(r) = \left( \frac{d}{dr} + \frac{1}{r} \right) \left[ C_1 \int_0^a W_{11}(r, \rho) \Omega_-(\rho) \, d\rho + D_1 \int_0^a W_{11}(r, \rho) X_-(\rho) \, d\rho \\ - C_2 \int_0^a W_{11}(r, \rho) \varphi'_-(\rho) \, d\rho + D_2 \psi_- (r) \right] \quad (0 < r < \infty), \tag{5}
\]

\[
X_+(r) = -\frac{d}{dr} \left[ D_1 \int_0^a W_{00}(r, \rho) \Omega_-(\rho) \, d\rho + C_1 \int_0^a W_{01}(r, \rho) X_-(\rho) \, d\rho \\ + D_2 \varphi_- (r) + C_2 \int_0^a W_{00}(r, \rho) \left( \frac{d\psi_- (\rho)}{d\rho} + \frac{\psi_- (\rho)}{\rho} \right) \, d\rho \right] \quad (0 < r < \infty). \tag{6}
\]

Note that when deriving equations (5) and (6), the property of continuity of stresses and displacements outside a circular crack on its plane of location \( z = 0 \) was used.

Considering now the key equations (5) and (6) on a circular crack and taking into account the boundary conditions (1), after simple transformations, we arrive at the following system of the GIEs:

\[
\begin{cases}
A_1 \int_0^a W_{11}(r, \rho) \varphi'_-(\rho) \, d\rho - B_1 \psi_- (r) = \tilde{f}(r), \\
B_1 \varphi_- (r) + A_1 \int_0^a W_{00}(r, \rho) \left( \frac{d\psi_- (\rho)}{d\rho} + \frac{\psi_- (\rho)}{\rho} \right) \, d\rho = \tilde{g}(r)
\end{cases} \quad (0 < r < a), \tag{7}
\]

where
\[
\tilde{f}(r) = B_2 \int_0^a W_{10}(r, \rho) \Omega_-(\rho) \, d\rho + A_2 \int_0^a W_{11}(r, \rho) X_-(\rho) \, d\rho + \frac{1}{r} \int_0^r [p_+(\rho) + p_-(\rho)] \, d\rho,
\]
\[
\tilde{g}(r) = \int_0^r [\tau_+(\rho) + \tau_-(\rho)] \, d\rho - A_2 \int_0^a W_{00}(r, \rho) \Omega_-(\rho) \, d\rho - B_2 \int_0^a W_{01}(r, \rho) X_-(\rho) \, d\rho + C,
\]
\[
A_1 = -\frac{C_2}{2G_+}, \quad B_1 = -\frac{D_2}{2G_+}, \quad A_2 = -D_1, \quad B_2 = -C_1,
\]

from which unknown functions \( \varphi'_-(r) = \varphi(r), \ d\psi_- (r)/dr + \psi_- (r)/r = \psi(r) \) are determined.

We transform the system of IEs (7) into the IE system for the unknown functions \( \varphi(r) \) and \( \psi(r) \). It is easy to see that for crack openings we have
\[
\varphi_- (r) = -\int_r^a \varphi(\rho) \, d\rho = -\int_r^a W_{01}(r, \rho) \varphi(\rho) \, d\rho,
\]
\[
\psi_- (r) = \frac{1}{r} \int_0^r \psi(\rho) \, d\rho = \int_0^a W_{10}(r, \rho) \psi(\rho) \, d\rho \quad (0 \leq r \leq a). \tag{8}
\]

In addition, the boundary conditions \( \varphi_-(a) = 0, \ \psi_-(a) = 0 \) expressing the conditions for the continuity of displacements on the boundary circle \( r = a \) of a circular crack must be satisfied.
The condition $\varphi_-(a) = 0$ is automatically satisfied according to the first relation of (8), and the condition $\psi_-(a) = 0$ is converted to the equivalent condition

$$\int_0^a \psi(\rho) d\rho = 0. \tag{9}$$

As a result, the GIE system is converted to

$$\begin{cases}
\int_0^a W_{11}(r, \rho) \varphi(\rho) \rho d\rho - \chi_1 \int_0^a W_{10}(r, \rho) \psi(\rho) \rho d\rho = f(r), \\
-\chi_1 \int_0^a W_{01}(r, \rho) \varphi(\rho) \rho d\rho + \int_0^a W_{00}(r, \rho) \psi(\rho) \rho d\rho = g(r)
\end{cases} \quad (0 < r < a), \tag{10}$$

where

$$\chi_1 = \frac{B_1}{A_1} = \frac{D_2}{C_2} = \frac{1 - 2\nu_+ - (1 - 2\nu_-)k}{2[1 - \nu_+ + (1 - \nu_-)k]}, \quad f(r) = \frac{\tilde{f}(r)}{A_1}, \quad g(r) = \frac{\tilde{g}(r)}{A_1}. \tag{11}$$

The solution of the GIE system (10) must satisfy the condition (9).

Considering the key equations outside the circular crack in its plane $z = 0$, after solving the GIE system (9)–(10), we determine the breaking normal and tangential stresses acting there:

$$\sigma(r) = \frac{1}{2} \left( \frac{d}{dr} + \frac{1}{r} \right) \left[ C_1 \int_0^a W_{10}(r, \rho) \Omega_-(\rho) \rho d\rho + D_1 \int_0^a W_{11}(r, \rho) X_-(\rho) \rho d\rho \\
- C_2 \int_0^a W_{11}(r, \rho) \varphi(\rho) \rho d\rho + D_2 \int_0^a W_{10}(r, \rho) \psi(\rho) \rho d\rho \right] \quad (r > a), \tag{12}$$

$$\tau(r) = -\frac{1}{2} \frac{d}{dr} \left[ D_1 \int_0^a W_{00}(r, \rho) \Omega_-(\rho) \rho d\rho + C_1 \int_0^a W_{01}(r, \rho) X_-(\rho) \rho d\rho \\
+ C_2 \int_0^a W_{00}(r, \rho) \psi(\rho) \rho d\rho - D_2 \int_0^a W_{01}(r, \rho) \varphi(\rho) \rho d\rho \right] \quad (r > a).$$

In the second BBVP, we have the following boundary conditions

$$u_\pm(r, 0) = f_\pm(r), \quad w_\pm(r, 0) = g_\pm(r), \quad f_\pm(a) = g_\pm(a) = 0 \quad (0 \leq r \leq a),$$

where $f_\pm(r)$ and $g_\pm(r)$ are continuous and continuously differentiable functions given in advance, while $f_\pm(r) = O(\sqrt{a - r})$, $g_\pm(r) = O(\sqrt{a - r}) \ (r \to a - 0)$. In addition, the resultant $P_\pm$ of unknown normal stresses on the upper and lower edges of the crack are specified, i.e.

$$2\pi \int_0^a p_\pm(\rho) \rho d\rho = P_\pm. \tag{13}$$

Now, proceeding in the same way as above, we obtain the following GIE system of the second BBVP from the third and fourth equations of (4):

$$\begin{align*}
\int_0^a W_{11}(r, \rho) X_-(\rho) \rho d\rho + \chi_2 \int_0^a W_{10}(r, \rho) \Omega_-(\rho) \rho d\rho &= F(r) \\
\chi_2 \int_0^a W_{01}(r, \rho) X_-(\rho) \rho d\rho + \int_0^a W_{00}(r, \rho) \Omega_-(\rho) \rho d\rho &= G(r) \quad (0 < r < a), \tag{14}
\end{align*}$$
where

\[
\chi^2 = \frac{E_1}{F_1} = \frac{1}{2} \frac{(1 - 2\nu_+)(3 - 4\nu_+) - (1 - 2\nu_+)(3 - 4\nu_-)k}{(1 - \nu_-)(3 - 4\nu_+) + (1 - \nu_+)(3 - 4\nu_-)k}, \quad k = \frac{G_+}{G_-}.
\]

\[
F(r) = \frac{1}{F_1} \left\{ C_1[f_+(r) - f_-(r) + f_+(r) + f_-(r)] - D_1 \int_0^a W_{11}(r, \rho)[g'_+(\rho) - g'_-(\rho)] \rho d\rho \right\},
\]

\[
G(r) = \frac{1}{F_1} \left\{ C_1[g_+(r) - g_-(r) + g_+(r) + g_-(r)]
+ D_1 \int_0^a W_{00}(r, \rho) \left[ \frac{d(f_+(\rho) - f_-(\rho))}{d\rho} + \frac{f_+(\rho) - f_-(\rho)}{\rho} \right] \rho d\rho \right\}.
\]

According to (11) the solution to the GIE system (12) must satisfy the condition

\[
\int_0^a \Omega_-(\rho) d\rho = \frac{P_+ - P_-}{2\pi}.
\] (13)

From the system of GIEs (12)–(13) jumps in the tangential \(X_-(r)\) and normal \(\Omega_-(r)\) stresses are determined; the sums of these stresses are determined from (5) and (6) for \(0 < r < a\), where the crack openings \(\varphi_-(r)\) and \(\psi_-(r)\) are the known functions \(\varphi_-(r) = g_+(r) - g_-(r)\), \(\psi_-(r) = f_+(r) - f_-(r)\) \((0 \leq r \leq a)\). As a result, the normal \(p_\pm(r)\) and tangential \(\tau_\pm(r)\) stresses on the crack edges, as well as these stresses outside the crack in the plane \(z = 0\) are completely determined.

Let us move on to the third BBVP with boundary conditions

\[
\begin{align*}
\sigma_z \big|_{z=0} &= -p_+(r), \quad \tau_{zr} \big|_{z=0} = -\tau_+(r) & (0 < r < a), \\
u_-(r,0) &= f_-(r), \quad w_-(r,0) = g_-(r) & (f_-(a) = g_-(a) = 0, \ 0 \leq r \leq a),
\end{align*}
\] (14)

where \(p_+(r), \tau_+(r), f_-(r)\), and \(g_-(r)\) are the functions given in advance and besides \(f_-(r)\) and \(g_-(r)\) are continuously differentiable functions. To derive the GIE system of the mixed boundary value problem (14), we use equations (4) and obvious transformations

\[
\begin{align*}
\Omega_+(r) &= p_+(r) + p_-(r) = 2p_+(r) - \Omega_-(r), \quad X_+(r) = \tau_+(r) + \tau_-(r) = 2\tau_+(r) - X_-(r), \\
U_+(r) &= u_+(r,0) + u_-(r,0) = 2f_-(r) + \psi_-(r),
W_+(r) &= w_+(r,0) + w_-(r,0) = 2g_-(r) + \varphi_-(r) & (0 < r < a).
\end{align*}
\]

Taking into account the well-known relations for the Hankel transformants from [10] and proceeding quite similarly to the above for the first BBVP, solving the mixed boundary value problem (14) for jumps in tangential stresses \(X_-(r)\), normal stresses \(\Omega_-(r)\) and dislocation densities \(\varphi(r)\), \(\psi(r)\) on the edges of a circular crack is reduced to solving the following GIE
system of four equations \(0 < r < a\):

\[
\begin{align*}
\int_0^a W_{11}(r, \rho)X_-(\rho)\rho \, d\rho - \gamma_2 \int_0^a W_{11}(r, \rho)\varphi(\rho)\rho \, d\rho + \gamma_1 \int_0^a W_{10}(r, \rho)\Omega_-(\rho)\rho \, d\rho \\
+ \delta_2 \int_0^a W_{10}(r, \rho)\psi(\rho)\rho \, d\rho = \frac{2}{D_1} \int_0^r p_+(\rho)\rho \, d\rho, \\
- \int_0^a W_{00}(r, \rho)\Omega_-(\rho)\rho \, d\rho - \gamma_2 \int_0^a W_{00}(r, \rho)\psi(\rho)\rho \, d\rho - \gamma_1 \int_0^a W_{01}(r, \rho)X_-(\rho)\rho \, d\rho \\
- \delta_2 \int_0^a W_{01}(r, \rho)\varphi(\rho)\rho \, d\rho = - \frac{2}{D_1} \int_0^r \tau_+(\rho) \rho \, d\rho + \frac{C_0}{D_1}, \\
\int_0^a W_{11}(r, \rho)X_-(\rho)\rho \, d\rho + \chi_2 \int_0^a W_{10}(r, \rho)\Omega_-(\rho)\rho \, d\rho + \alpha_1 \int_0^a W_{11}(r, \rho)\varphi(\rho)\rho \, d\rho \\
- \beta_1 \int_0^a W_{10}(r, \rho)\psi(\rho)\rho \, d\rho = \frac{2f_-(r)}{F_1}, \\
\int_0^a W_{00}(r, \rho)\Omega_-(\rho)\rho \, d\rho + \chi_2 \int_0^a W_{01}(r, \rho)X_-(\rho)\rho \, d\rho - \alpha_1 \int_0^a W_{00}(r, \rho)\psi(\rho)\rho \, d\rho \\
+ \beta_1 \int_0^a W_{01}(r, \rho)\varphi(\rho)\rho \, d\rho = \frac{2g_-(r)}{F_1},
\end{align*}
\]

where

\[
\gamma_1 = \frac{1 + C_1}{D_1}, \quad \gamma_2 = \frac{C_2}{D_1}, \quad \delta_2 = \frac{D_2}{D_1}, \quad \alpha_1 = \frac{D_1}{F_1}, \quad \beta_1 = \frac{1 + C_1}{F_1}, \quad \chi_2 = \frac{E_1}{F_1},
\]

\[
\varphi(r) = \varphi_-(r), \quad \psi(r) = \frac{d\varphi_-(r)}{dr} + \frac{\varphi_-(r)}{r} \quad (0 < r < a).
\]

The solution of GIE (15) must satisfy the conditions

\[
\varphi_-(a) = 0, \quad \int_0^a \psi(\rho)\rho \, d\rho = 0, \quad \int_0^a \Omega_-(\rho)\rho \, d\rho = \frac{P_+ - P_-}{2\pi}. \quad (16)
\]

Note that if from the first two equations of system (15) we determine the functions \(\varphi(r)\) and \(\psi(r)\) and substitute their expressions in the third and fourth equations, then after simple transformations, we obtain a system of two IE equations, similar to (10) or (12), the solution of which must satisfy the third condition in (16).

In the case of a homogeneous elastic space with a circular crack, the GIEs of the considered BBVPs are obtained from the corresponding GIEs for a piecewise-homogeneous space with a circular crack as special cases. In this case

\[
\nu_+ = \nu_-, \quad E_+ = E_-, \quad E = E,
\]

\[
\varphi_0^+ = \varphi_0^- = \varphi_0 = \frac{(1 - \nu^2)}{\pi E}, \quad \varphi_1^+ = \varphi_1^- = \varphi_1 = \frac{(1 + \nu)(1 - 2\nu)}{2E}, \quad k = 1,
\]

and, therefore, from (1) \(D_2 = E_1 = 0\) and the remaining constants from (4) will take on a simple form. Then \(\chi_1 = 0\) and the system of GIEs (10) of the first BBVP splits into two separate GIEs:

\[
\int_0^a W_{11}(r, \rho)\varphi(\rho)\rho \, d\rho = p(r) \quad (0 < r < a), \quad (17)
\]

\[
\int_0^a W_{00}(r, \rho)\psi(\rho)\rho \, d\rho = q(r) \quad (0 < r < a), \quad (18)
\]
where
\[
p(r) = -\frac{\pi \vartheta_0}{r} \int_0^r [p_+(\rho) + p_-(\rho)] \rho \, d\rho + 2\vartheta_1 \int_0^a W_{11}(r, \rho) \varphi(\rho) \rho \, d\rho,
\]
\[
g(r) = -\pi \vartheta_0 \int_0^r [\tau_+(\rho) + \tau_-(\rho)] \, d\rho - 2\vartheta_1 \int_0^a W_{00}(r, \rho) \Omega(\rho) \rho \, d\rho + C.
\]

The solutions to GIEs (17) and (18) must satisfy, respectively, the conditions
\[
\varphi_-(a) = 0, \quad \int_0^a \psi(\rho) \rho \, d\rho = 0. \tag{19}
\]

In the particular case under consideration \(\chi_2 = 0\) and the GIE system of the second BBVP (12) also splits into two separate GIEs:
\[
\int_0^a W_{11}(r, \rho) \varphi(\rho) \rho \, d\rho = P(r) \quad (0 < r < a), \tag{20}
\]
\[
\int_0^a W_{00}(r, \rho) \Omega(\rho) \rho \, d\rho = Q(r) \quad (0 < r < a), \tag{21}
\]
where
\[
P(r) = \frac{\pi \vartheta_0}{\pi^2 \vartheta_0^2 - 4\vartheta_1^2} f(r) - \frac{2\vartheta_1}{\pi^2 \vartheta_0^2 - 4\vartheta_1^2} \int_0^a W_{11}(r, \rho) \varphi(\rho) \rho \, d\rho,
\]
\[
Q(r) = \frac{\pi \vartheta_0}{\pi^2 \vartheta_0^2 - 4\vartheta_1^2} g(r) + \frac{2\vartheta_1}{\pi^2 \vartheta_0^2 - 4\vartheta_1^2} \int_0^a W_{00}(r, \rho) \psi(\rho) \rho \, d\rho,
\]
\[
f(r) = f_+(r) + f_-(r), \quad g(r) = g_+(r) + g_-(r), \quad \varphi(r) = g'_+(r) - g'_-(r),
\]
\[
\psi(r) = \frac{d(f_+(r) - f_-(r))}{dr} + \frac{f_+(r) - f_-(r)}{r}.
\]

Note that the solution of GIE (21) must satisfy condition (13).

We also note that after solving GIEs (17)–(18) and (20)–(21) under corresponding conditions, the breaking normal and shear stresses outside a circular crack in the plane \(z = 0\) will be determined by formulas (5) and (6) in case of homogeneous elastic space.

Turn to the third BBVP. In the case of a homogeneous elastic space with a circular crack after elementary transformations, the GIE system (15) takes the form \((0 < r < a)\)
\[
\begin{cases}
\Omega_-(r) + \frac{2\vartheta_1}{\pi \vartheta_0} \left( \frac{d}{dr} + \frac{1}{r} \right) \int_0^a W_{11}(r, \rho) \varphi(\rho) \rho \, d\rho - \frac{1}{\pi \vartheta_0} \left( \frac{d}{dr} + \frac{1}{r} \right) \int_0^a W_{11}(r, \rho) \varphi(\rho) \rho \, d\rho = 2p_+(r), \\
X_-(r) - \frac{2\vartheta_1}{\pi \vartheta_0} \int_0^a W_{00}(r, \rho) \Omega_-(\rho) \rho \, d\rho - \frac{1}{\pi \vartheta_0} \int_0^a W_{00}(r, \rho) \psi(\rho) \rho \, d\rho = 2\tau_+(r), \\
\pi^2 \vartheta_0^2 - 4\vartheta_1^2 \int_0^a W_{11}(r, \rho) \varphi(\rho) \rho \, d\rho + \frac{2\vartheta_1}{\pi \vartheta_0} \int_0^a W_{11}(r, \rho) \varphi(\rho) \rho \, d\rho - \varphi_-(r) = 2f_-(r), \\
\pi^2 \vartheta_0^2 - 4\vartheta_1^2 \int_0^a W_{00}(r, \rho) \Omega_-(\rho) \rho \, d\rho - \frac{2\vartheta_1}{\pi \vartheta_0} \int_0^a W_{00}(r, \rho) \psi(\rho) \rho \, d\rho = 2g_-(r). \tag{22}
\end{cases}
\]

Now we consider the first equation of system (22) as an integral equation for \(\varphi(r)\) and represent it as
\[
\int_0^a W_{11}(r, \rho) \varphi(\rho) \rho \, d\rho = 2\vartheta_1 \int_0^a W_{11}(r, \rho) \varphi(\rho) \rho \, d\rho - \frac{\pi \vartheta_0}{r} \int_0^r [p_+(\rho) + p_-(\rho)] \rho \, d\rho \quad (0 < r < a). \tag{23}
\]
We consider the second equation of (22) as an integral equation for \( \psi(r) \) and represent it as

\[
\int_0^a W_{00}(r, \rho) \psi(\rho) \rho \, d\rho = -2 \vartheta_1 \int_0^a W_{00}(r, \rho) \Omega_-(\rho) \rho \, d\rho - \pi \vartheta_0 \int_0^a \left[ \tau_+(\rho) + \tau_-(\rho) \right] d\rho + C \quad (0 < r < a). \tag{24}
\]

To invert integral equations (23) and (24), we use the well-known representations of Weber–Sonin integrals [11, 12]:

\[
W_{00}(r, \rho) = \frac{2}{\pi} \int_0^{\min(r, \rho)} \frac{dt}{\sqrt{(r^2 - t^2)(\rho^2 - t^2)}}, \quad W_{11}(r, \rho) = \frac{2}{\pi r \rho} \int_0^{\min(r, \rho)} \frac{t^2 dt}{\sqrt{(r^2 - t^2)(\rho^2 - t^2)}}. \tag{25}
\]

Substituting these kernel expressions into equations (23) and (24), solving each of these integral equations can be reduced to solving two successively solvable Abel integral equations by a known manner. Then the solution of equation (23) is expressed by the formula

\[
\sigma(r) = -2 \vartheta_1 X_-(r) + 2 \vartheta_0 \frac{d}{dr} \int_0^a K(r, \rho) [p_+(\rho) + p_-(\rho)] \rho \, d\rho \quad (0 < r < a),
\]

\[
K(r, \rho) = \int_{\max(r, \rho)}^a \frac{dt}{\sqrt{(t^2 - r^2)(t^2 - \rho^2)}},
\]

\[(0 < r < a). \tag{26}
\]

Now we integrate both sides of (26) with respect to \( r \) from \( r \) to \( a \) and get

\[
\varphi_-(r) = - \int_r^a \sigma_-(\rho) \rho \, d\rho = -2 \vartheta_1 \int_r^a X_-(\rho) \rho \, d\rho + 2 \vartheta_0 \int_r^a K(r, \rho) [p_+(\rho) + p_-(\rho)] \rho \, d\rho \quad (0 \leq r \leq a). \tag{27}
\]

In a very similar way, by inverting equation (24), we obtain

\[
\psi_-(r) = \frac{1}{r} \int_r^a \omega(\rho) \rho \, d\rho = \frac{2 \vartheta_1}{r} \int_r^a \Omega_-(\rho) \rho \, d\rho
\]

\[- \frac{2 C}{\pi r} \sqrt{a^2 - r^2} + \frac{2 \vartheta_0}{r} \int_0^a L(r, \rho) [\tau_+(\rho) + \tau_-(\rho)] \rho \, d\rho \quad (0 \leq r \leq a), \tag{28}
\]

\[
L(r, \rho) = \int_{\max(r, \rho)}^a \frac{t^2 dt}{\sqrt{(t^2 - r^2)(t^2 - \rho^2)}},
\]

\[C = \frac{\pi}{2 a} \left[ 2 \vartheta_1 \int_0^a \Omega_-(\rho) \rho \, d\rho + 2 \vartheta_0 \int_0^a \sqrt{\frac{a^2 - \rho^2}{(\tau_+(\rho) + \tau_-(\rho))}} \, d\rho \right].
\]

Then we substitute (23) and (24), as well as (27) and (28), into the third and fourth equations of system (22). Omitting the intermediate calculations, regarding the functions \( p_-(r) \) and \( \tau_-(r) \) we arrive at the following GIE system of the third BBVP for a homogeneous elastic space with a circular crack:

\[
\begin{cases}
\psi_0 \int_0^a \left[ \pi W_{00}(r, \rho) + 2 K(r, \rho) \right] \rho \, d\rho + 2 \vartheta_1 \int_0^a \left[ W_{01}(r, \rho) - \frac{r}{\rho} W_{10}(r, \rho) \right] \rho \, d\rho = G_-(r), \\
0 \leq r < a,
\end{cases}
\]

\[
F_-(r) = \vartheta_0 \int_0^a \left[ \pi r \rho W_{11}(r, \rho) - 2 L(r, \rho) \right] \rho \, d\rho - 2 \vartheta_1 \int_0^a \left[ \rho W_{01}(r, \rho) - r W_{10}(r, \rho) \right] \rho \, d\rho = F_-(r),
\]

\[
(0 < r < a),
\]

\[
C = \frac{\pi}{2 a} \left[ 2 \vartheta_1 \int_0^a \Omega_-(\rho) \rho \, d\rho + 2 \vartheta_0 \int_0^a \sqrt{\frac{a^2 - \rho^2}{(\tau_+(\rho) + \tau_-(\rho))}} \, d\rho \right].
\]
Here

\[ P_+ = 2\pi \int_0^a p_+(\rho) \rho \, d\rho, \quad T_+ = 2\pi \int_0^a \tau_+(\rho) \rho \, d\rho. \]

They are known quantities, and the solution of the GIE system (29) must satisfy the condition

\[ \int_0^a p_-(\rho) \rho \, d\rho = \frac{P_-}{2\pi}, \quad (30) \]

where \( P_- \) is a predetermined resultant of stresses with the opposite sign on the lower edge of a circular crack.

3. On solving the IE of the BBVP

Using representation (25), solving GIE (17)–(18), as well as (20)–(21) can be reduced to two successively solvable Abel integral equations or to an SIE with the Cauchy kernel. Indeed, referring, for example, to GIE (18), in which for simplicity we will consider \( a = 1 \) (otherwise, this can be achieved by replacing the variables \( r = ax, \rho = as \)) we substitute the kernel representation \( W_{00}(r, \rho) \) from (25) in it:

\[
\frac{2}{\pi} \int_0^r \psi(\rho) \rho \, d\rho \int_0^\rho \frac{dt}{\sqrt{(r^2 - t^2)(\rho^2 - t^2)}} + \frac{2}{\pi} \int_1^r \varphi(\rho) \rho \, d\rho \int_0^r \frac{dt}{\sqrt{(r^2 - t^2)(\rho^2 - t^2)}} = C - q_0(r),
\]

\[
q_0(r) = \pi \vartheta \int_0^r \{\tau_+(\rho) + \tau_-(\rho)\} \, d\rho + 2\vartheta_1 \int_0^1 W_{00}(r, \rho) \Omega_-(\rho) \rho \, d\rho.
\]

Hence, after changing the order of integration in the first term, we arrive at the following IE:

\[
\frac{2}{\pi} \int_0^r \frac{dt}{\sqrt{r^2 - t^2}} \int_t^1 \frac{\psi(\rho) \rho \, d\rho}{\sqrt{\rho^2 - t^2}} = C - q_0(r) \quad (0 < r < 1).
\]

If we now introduce the notation

\[ X_0(t) = \int_t^1 \frac{\psi(\rho) \rho \, d\rho}{\sqrt{\rho^2 - t^2}}, \]

then solving IE (31) will be reduced to solving two successively solvable Abel integral equations. Namely, from (31) we obtain the following Abel IE for the function \( X_0(t) \):

\[
\frac{2}{\pi} \int_0^r \frac{X_0(t) \, dt}{\sqrt{r^2 - t^2}} = C - q_0(r) = g_0(r).
\]

From here, by Abel’s inversion formula, we find

\[ X_0(r) = \frac{d}{dr} \int_0^r \frac{tg_0(t) \, dt}{\sqrt{r^2 - t^2}} \quad (0 < r < 1). \]

As a result, we come to the following IE for the sought-for function \( \psi(r) \)

\[
\int_t^1 \frac{\psi(\rho) \rho \, d\rho}{\sqrt{\rho^2 - t^2}} = X_0(t) \quad (0 < t < 1).
\]

Here again, using the Abel inversion formula, we determine the function \( \psi(r) \).
Solving IE (31) can be also reduced to solving a classical SIE with the Cauchy kernel. For this, we use the important relation from [13] (p. 575, formula (54.11)), whence, after simple transformations, we obtain the following fundamental relationship between the composition of two consecutive Abel operators and the singular integral operator generated by the Cauchy kernel:

$$S\varphi(r) = \int_0^1 \frac{\varphi(\rho) \rho \, d\rho}{\rho^2 - r^2} = A\varphi(r) = \frac{d}{dr} \int_0^r \frac{t \, dt}{\sqrt{r^2 - t^2}} \int_t^1 \frac{\varphi(\rho) \, d\rho}{\sqrt{\rho^2 - t^2}} \quad (0 < r < 1). \quad (34)$$

Now we apply the operator of the right-hand side of (34) to both sides of IE (33):

$$\frac{d}{dr} \int_0^r \frac{t \, dt}{\sqrt{r^2 - t^2}} \int_t^1 \frac{\psi(\rho) \rho \, d\rho}{\sqrt{\rho^2 - t^2}} = \frac{d}{dr} \int_0^r tX_0(t) \, dt \quad (0 < r < 1).$$

Taking into account the dependence (34), we come to the following SIE:

$$\int_0^1 \frac{\psi(\rho) \rho^2 \, d\rho}{\rho^2 - r^2} = h_0(r), \quad h_0(r) = \frac{d}{dt} \int_0^r tX_0(t) \, dt \quad (0 < r < 1). \quad (35)$$

where $X_0(t)$ is defined by formula (33). Further, we write SIE (35) as

$$2 \int_0^1 \frac{\psi_0(\rho) \rho \, d\rho}{\rho^2 - r^2} = 2h_0(r), \quad \psi_0(r) = r\psi(r) \quad (0 < r < 1). \quad (36)$$

We continue the function $\psi_0(\rho)$ on the interval $(-1, 0)$ in an odd way, and the function $h_0(r)$ in an even way. As a result, we come to the classical SIE with the Cauchy kernel:

$$\int_{-1}^1 \frac{\psi_0(\rho) \rho \, d\rho}{\rho - r} = 2h_0(r) \quad (-1 < r < 1, \quad \psi_0(-r) = -\psi_0(r), \quad h_0(-r) = h_0(r)). \quad (37)$$

The inversion of SIE (37) results in [13]:

$$\psi_0(r) = -\frac{2}{\pi^2 \sqrt{1 - r^2}} \int_{-1}^1 \frac{\sqrt{1 - \rho^2} h_0(\rho) \, d\rho}{\rho - r} \quad (-1 < r < 1).$$

Hence, taking into account the expression of the function $\psi_0(r)$ from (36), after simple transformations, we find for the desired function $\psi(r)$

$$\psi(r) = -\frac{4}{\pi^2 \sqrt{1 - r^2}} \int_0^1 \frac{\sqrt{1 - \rho^2} h_0(\rho) \, d\rho}{\rho^2 - r^2} \quad (0 < r < 1). \quad (38)$$

To simplify the expressions of the function $X_0(r)$ from (32) and the function $h_0(r)$ from (35), we assume that the function $q_0(r)$ is twice continuously differentiable. Then, taking into account that $q_0(r) = C - q_0(0)$, in the expressions of the functions $X_0(r)$ and $h_0(r)$ we perform integration by parts. As a result, after elementary transformations, we get

$$X_0(r) = C - q_0(0) - r \int_0^r \frac{d^2 q_0(t)}{dt^2} \, dt \quad (0 < r < 1), \quad (39)$$

$$h_0(r) = C - q_0(0) + r \int_0^r \frac{d^2 X_0(t)}{dt^2} \, dt \quad (0 < r < 1). \quad (40)$$
In the integral (39), we also perform the integration by parts, after which we have

\[ X_0(r) = C - q_0(0) - \frac{\pi}{2} r q_0'(r) + r \int_0^r \arcsin \frac{t}{r} q_0''(t) \, dt \quad (0 < r < 1), \]

\[ X_0'(r) = -\frac{\pi}{2} q_0'(r) + \int_0^r \arcsin \frac{t}{r} q_0''(t) \, dt - \int_0^r \frac{t q_0''(t) \, dt}{\sqrt{r^2 - t^2}} \quad (0 < r < 1), \tag{41} \]

Thus, under the assumptions concerning the function \( q_0(r) \), the desired functions \( X_0(r) \) and \( h_0(r) \) will be expressed by formulas (39)–(41).

Further, we will deal with the determination of the constant \( C \). For this purpose, we represent (38) in the form

\[ r \psi(r) = -\frac{2}{\pi^2 \sqrt{1 - r^2}} \int_{-1}^1 \frac{\sqrt{1 - \rho^2} h_0(\rho)}{\rho - r} \, d\rho \quad (0 < r < 1, \ h_0(-\rho) = h_0(\rho)) \]

and integrate both sides of this equality with respect to \( r \) from 0 to 1. Taking into account the expression of the integral

\[ \int \frac{dr}{\sqrt{1 - r^2}(\rho - r)} = \frac{1}{2\sqrt{1 - \rho^2}} \ln \frac{1 - r \rho + \sqrt{(1 - r^2)(1 - \rho^2)}}{1 - r \rho - \sqrt{(1 - r^2)(1 - \rho^2)}} \]

and the second condition (19), after elementary transformations, we get

\[ \frac{2}{\pi} [C - q_0(0)] + \frac{1}{\pi^2} \int_0^1 I(t) X_0'(t) \, dt = 0, \quad I(t) = 2 \int_t^1 \ln \frac{1 + \sqrt{1 - \rho^2}}{1 - \sqrt{1 - \rho^2}} \frac{\rho \, d\rho}{\sqrt{\rho^2 - t^2}}, \]

from which the constant \( C \) is determined. Here was used that

\[ \int_0^1 \ln \frac{1 + \sqrt{1 - \rho^2}}{1 - \sqrt{1 - \rho^2}} \, d\rho = \pi. \]

Passing to the GIE system of the first BBVP for a piecewise-homogeneous elastic space with a circular crack, we note that using (25) and (34), solving this system can be reduced to solving the SIE of the second kind with a Cauchy kernel of type (for \( a = 1 \)):

\[ \Omega(r) + \frac{i \chi}{\pi} \int_{-1}^1 \frac{\Omega(\rho) \, d\rho}{\rho - r} = H(r) \quad (-1 < r < 1). \]

Under condition (30), the solution of this equation is obtained by the standard method [1, 13].

The construction of solutions of all the GIEs derived above is the subject of a separate study.

**Conclusion**

The paper considers a rather extensive class of precisely solvable axisymmetric BBVPs for homogeneous and piecewise homogeneous spaces with a penny-shaped crack. This class of problems can be significantly expanded from the point of view of the application of the ideas developed here in the mechanics of composites, geomechanics, fracture mechanics, and in other applied fields.
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