ON ALGEBRAIC GROUP VARIETIES

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Abstract. Several results on presenting an affine algebraic group variety as a product of algebraic varieties are obtained.

This note explores possibility of presenting an affine algebraic group variety as a product of algebraic varieties. As starting point served the question of B. Kunyavsky [6] about the validity of the statement formulated below as Corollary of Theorem 1. For some special presentations, their existence is proved in Theorem 1 and, on the contrary, nonexistence in Theorems 2–5.

Let $G$ be a connected reductive algebraic group over an algebraically closed field $k$. The derived subgroup $D$ and the connected component $Z$ of the identity element of the center of the group $G$ are respectively a connected semisimple algebraic group and a torus (see [3, Sect. 14.2, Prop. (2)]). The algebraic groups $D \times Z$ and $G$ are not always isomorphic; the latter is equivalent to the equality $D \cap Z = 1$, which, in turn, is equivalent to the property that the isogeny of algebraic groups $D \times Z \to G$, $(d,z) \mapsto dz$, is their isomorphism.

Theorem 1. There is an injective algebraic group homomorphism

$$\iota: Z \hookrightarrow G$$

such that $\varphi: D \times Z \to G$, $(d,z) \mapsto d\iota(z)$, is an isomorphism of algebraic varieties.

Corollary 1. The underlying varieties of (generally nonisomorphic) algebraic groups $D \times Z$ and $G$ are isomorphic.

Remark 1. The proof of Theorem 1 contains more information than its statement (the existence of $\iota$ is proved by an explicit construction).

Example 1 ([9, Thm. 8, Proof]). Let the group $G$ be $\text{GL}_n$. Then $D = \text{SL}_n$, $Z = \{\text{diag}(t, \ldots , t) \mid t \in k^\times\}$, and one can take $\text{diag}(t, \ldots , t) \mapsto \text{diag}(t, 1, \ldots , 1)$ as $\iota$. In this Example, $G$ and $D \times Z$ are nonisomorphic algebraic groups.

Proof of Theorem 1. Let $T_D$ be a maximal torus of the group $D$, and let $T_G$ be a maximal torus of the group $G$ containing $T_D$. The torus $T_D$ is a direct factor of the group $T_G$: in the latter, there is a torus $S$ such
that the map $T_D \times S \to T_G$, $(t, s) \mapsto ts$, is an isomorphism of algebraic groups (see [3 8.5, Cor.]). We shall show that

$$\psi: D \times S \to G, \quad (d, s) \mapsto ds,$$

is an isomorphism of algebraic varieties.

As is known (see [3 Sect. 14.2, Prop. (1), (3)]),

(a) $Z \subseteq T_G$,
(b) $DZ = G$. 

Let $g \in G$. In view of (2)(b), we have $g = dz$ for some $d \in D, z \in Z$, and in view of (2)(a) and the definition of $S$, there are $t \in T_D, s \in S$ such that $z = ts$. We have $dt \in D$ and $\psi(dt, s) = dts = g$. Therefore, the morphism $\psi$ is surjective.

Consider in $G$ a pair of mutually opposite Borel subgroups containing $T_G$. The unipotent radicals $U$ and $U^-$ of these Borel subgroups lie in $D$. Let $N_D(T_D)$ and $N_G(T_G)$ be the normalizers of tori $T_D$ and $T_G$ in the groups $D$ and $G$ respectively. Then $N_D(T_D) \subseteq N_G(T_G)$ in view of (2)(b). The homomorphism $N_D/T_D \to N_G/T_G$ induced by this embedding is an isomorphism of groups (see [3 IV.13]), by which we identify them and denote by $W$. For each $\sigma \in W$, fix a representative $n_\sigma \in N_D(T_D)$. The group $U \cap n_\sigma U^- n_\sigma^{-1}$ does not depend on the choice of this representative, since $T_D$ normalizes $U^-$; we denote it by $U'_\sigma$.

It follows from the Bruhat decomposition that for each $g \in G$, there are uniquely defined $\sigma \in W, u \in U, u' \in U'_\sigma$ and $t_G \in T_G$ such that $g = u'n_\sigma ut_G$ (see [5 Sect. 28.4, Thm.]). In view of the definition of $S$, there are uniquely defined $t_D \in T_D$ and $s \in S$ such that $t_G = t_D s$, and in view of $u', n_\sigma, u, t_D \in D$, the condition $g \in D$ is equivalent to the condition $s = 1$. It follows from this and the definition of the morphism $\psi$ that the latter is injective.

Thus $\psi$ is a bijective morphism. Therefore, to prove that it is an isomorphism of algebraic varieties, it remains to prove its separability (see [3 Sect. 18.2, Thm.]). We have $\text{Lie } G = \text{Lie } D + \text{Lie } T_G$ (see [3 Sect. 13.18, Thm.]) and $\text{Lie } T_G = \text{Lie } T_D + \text{Lie } S$ (in view of the definition of $S$). Therefore,

$$\text{Lie } G = \text{Lie } D + \text{Lie } S.$$

On the other hand, it is obvious from (1) that the restrictions of the morphism $\psi$ to the subgroups $D \times \{1\}$ and $\{1\} \times S$ in $D \times S$ are isomorphisms respectively with subgroups $D$ and $S$ in $G$. Since $\text{Lie } (D \times S) = \text{Lie } (D \times \{1\}) + \text{Lie } (\{1\} \times S)$, it follows from (3) that the differential of morphism $\psi$ at the point $(1, 1)$ is surjective. Therefore (see [3 Sect. 17.3, Thm.]), the morphism $\psi$ is separable.
Since \( \psi \) is an isomorphism, it follows from (1) that \( \dim G = \dim D + \dim S \). On the other hand, (2)(b) and finiteness of \( D \cap Z \) imply that \( \dim G = \dim D + \dim Z \). Therefore, \( Z \) and \( S \) and equidimensional, and hence isomorphic tori. Whence, as \( \iota \) we can take the composition of any isomorphism of tori \( Z \to S \) with the identity embedding \( S \hookrightarrow G \). □

**Theorem 2.** An algebraic variety on which there is a nonconstant invertible regular function, cannot be a direct factor of a connected semisimple algebraic group variety.

**Proof of Theorem 2.** If the statement of Theorem 2 were not true, then the existence the nonconstant invertible function specified in it would imply the existence of such a function \( f \) on a connected semisimple algebraic group. Then, according to [10, Thm. 3], the function \( f/f(1) \) would be a nontrivial character of this group, despite the fact that connected semisimple groups have no nontrivial characters. □

In Theorems 3, 5 below we assume that \( k = \mathbb{C} \); according to the Lefschetz principle, then they are valid for fields \( k \) of characteristic zero. Below, topological terms refer to the Hausdorff \( \mathbb{C} \)-topology, homology and cohomology are singular, and the notation \( P \simeq Q \) means that the groups \( P \) and \( Q \) are isomorphic.

**Theorem 3.** If a \( d \)-dimensional algebraic variety \( X \) is a direct factor of a connected reductive algebraic group variety, then \( H_d(X, \mathbb{Z}) \simeq \mathbb{Z} \) and \( H_i(X, \mathbb{Z}) = 0 \) for \( i > d \).

**Proof.** Suppose that there are a connected algebraic reductive group \( R \) and an algebraic variety \( Y \) such that the algebraic variety \( R \) is isomorphic to \( X \times Y \). Let \( n := \dim R \); then \( \dim Y = n - d \). The algebraic varieties \( X \) and \( Y \) are irreducible, smooth, and affine. Therefore (see [7, Thm. 7.1]),

\[
H_i(X, \mathbb{Z}) = 0 \text{ for } i > d, \quad H_j(Y, \mathbb{Z}) = 0 \text{ for } j > n - d. \tag{4}
\]

By the universal coefficient theorem, for any algebraic variety \( V \) and every \( i \), we have

\[
H_i(V, \mathbb{Q}) \simeq H_i(V, \mathbb{Z}) \otimes \mathbb{Q}, \tag{5}
\]

and by the Künnehm formula,

\[
H_n(R, \mathbb{Q}) \simeq H_n(X \times Y, \mathbb{Q}) \simeq \bigoplus_{i+j=n} H_i(X, \mathbb{Q}) \otimes H_j(Y, \mathbb{Q}). \tag{6}
\]

Therefore, it follows from (4) that

\[
H_n(R, \mathbb{Q}) \simeq H_d(X, \mathbb{Q}) \otimes H_{n-d}(Y, \mathbb{Q}). \tag{7}
\]

On the other hand, if \( K \) is a maximal compact subgroup of the real Lie group \( R \), then the Iwasawa decomposition shows that \( R \), as
a topological manifold, is a product of $K$ and a Euclidean space, and therefore, the manifolds $R$ and $K$ have the same homology. Since the algebraic group $R$ is the complexification of the real Lie group $K$, the dimension of the latter is $n$. Therefore, $H_n(K, \mathbb{Q}) \simeq \mathbb{Q}$ because $K$ is a closed connected orientable topological manifold. Whence, $H_n(R, \mathbb{Q}) \simeq \mathbb{Q}$. This and (7) imply that $H_d(X, \mathbb{Q}) \simeq \mathbb{Q}$. In turn, in view of (5), this implies that $H_d(X, \mathbb{Z}) \simeq \mathbb{Z}$ because $H_d(X, \mathbb{Z})$ is a finitely generated (see [4, Sect. 1.3]), torsion free (see [1, Thm. 1]) Abelian group.

**Corollary 2.** A contractible algebraic variety (in particular, $\mathbb{A}^d$) of positive dimension cannot be a direct factor of a connected reductive algebraic group variety.

**Theorem 4.** An algebraic curve cannot be a direct factor of a connected semisimple algebraic group variety.

**Proof.** Suppose an algebraic curve $X$ is a direct factor a connected semisimple algebraic group $R$ variety. Then $X$ is irreducible, smooth, affine, and there is a surjective morphism $\pi: R \to X$. In view of rationality of the algebraic variety $R$ (see [2, 14.14]), the existence of $\pi$ implies unirationality, and therefore, by Lüroth’s theorem, rationality $X$. Hence $X$ is isomorphic to an open subset $U$ of $\mathbb{A}^1$. The case $U = \mathbb{A}^1$ is impossible due to Theorem 3. If $U \neq \mathbb{A}^1$, then on $X$ there is a nonconstant invertible regular function, which is impossible due to Theorem 2.

**Theorem 5.** An algebraic surface cannot be a direct factor of a connected semisimple algebraic group variety.

**Proof.** Suppose there are a connected semisimple algebraic group $R$ and the algebraic varieties $X$ and $Y$ such that $X$ is a surface and $X \times Y$ is isomorphic to the algebraic variety $R$. We keep the notation of the proof of Theorem 3. Since $R$ is semisimple, $K$ is semisimple as well. Therefore, $H^1(K, \mathbb{Q}) = H^2(K, \mathbb{Q}) = 0$ (see [8, §9, Thm. 4, Cor. 1]). Since $R$ and $K$ have the same homology, and $\mathbb{Q}$-vector spaces $H^i(K, \mathbb{Q})$ and $H_i(K, \mathbb{Q})$ are dual to each other, this yealds

$$H_1(R, \mathbb{Q}) = H_2(R, \mathbb{Q}) = 0.$$  \hfill (8)

Since $R$ is connected, $X$ and $Y$ are also connected. Therefore,

$$H_0(X, \mathbb{Q}) = H_0(Y, \mathbb{Q}) = \mathbb{Q}.$$ \hfill (9)

It follows from (6), (8), and (9) that $H_2(X, \mathbb{Q}) = 0$. In view of (5), this contradicts Theorem 3 which completes the proof.

**Remark 2.** It seems plausible that, using, in the spirit of [2], étale cohomology in place of singular homology and cohomology, one can
adapt the proofs of Theorems 3 and 5 to the case of field $k$ of any characteristic.

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