Multiplicative controllability of the reaction-diffusion equation on a parallelepiped with finitely many zero hyperplanes.

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Abstract

We study the global approximate controllability of the reaction-diffusion equation in a parallelepiped $\Omega = (a_1, b_1) \times \ldots (a_n, b_n) \subset \mathbb{R}^n$, governed by a multiplicative control in a reaction term. It is assumed that the initial state $u_0$ admits zeros only on the intersections of $\Omega$ with finitely many hyperplanes, parallel to the sides of $\Omega$, and that $u_0$ changes its sign after crossing such hyperplanes (we further refer to them as the “hyperplanes of change of sign” or “zero hyperplanes”). This paper can be viewed as a continuation of work presented in [2, 3] for the controllability of the one dimensional reaction-diffusion equation with solutions admitting finitely many zeros. However, the methods of [2, 3] are intrinsically one dimensional, while in this paper we introduce a novel approach to deal with the case of multiple spatial variables.

1. Introduction. The traditional linear operator methods, based on the duality pairing, developed to study controllability of linear evolution systems with additive controls, do not apply to nonlinear control problems arising in the context of multiplicative controls. The latter also represent a principally different class of applications (such as, e.g., chain-reactions, see [6] and the references therein).

Among early works on “multiplicative” controllability, let us mention a pioneering work [1] (1982) by Ball, Marsden and Slemrod, establishing the approximate controllability of the rod and wave equations, based on an implicit nonharmonic Fourier series approach, adapted for the time-dependent (only) multiplicative controls.

In turn, the multiplicative controllability of linear and semilinear parabolic equations in several spatial dimensions was originated in the series of papers by Khapalov [4]-[5], further summarized in monograph [6]. The approach of [6] makes use of explicit asymptotic qualitative
methods that employ piecewise constant-in-time multiplicative controls of both spatial and time variables.

In more recent papers [2], [3] we discussed the multiplicative controllability properties of the one dimensional reaction-diffusion equation with the initial and target states admitting finitely many “matching” changes of sign. The results were extended in [3] to a two dimensional Heat equation on a disk in the case when its solutions are radially symmetrical. Nonetheless, the methods of [2], [3] deal with simultaneous control of the motion of isolated zero points and are intrinsically one dimensional (with respect to the spatial variable) as they focus on the controlling the sign of $u_t$ at zero points.

In the case of several spatial dimensions, the sets of points of change of sign of solutions to the pde at hand are no longer finite sets of points. Such principal change of setup demands a new methodology, which is the subject of this paper. Namely, we intend to exploit the fact that the choice of multiplicative control affects the selection of eigenfunctions of the spectral problem at hand. Respectively, our novel strategy is based on dealing with multiplicative controls for which the desirable target state becomes the first “essentially” non-zero eigenfunction in the Fourier series expansion of the resulting solution.

Controlled evolution system. In this paper we consider the following reaction-diffusion equation:

$$u_t = \Delta u + v(x,t)u \quad \text{in} \quad Q_T = \Omega \times (0,T), \quad T > 0, \quad x = (x_1, \ldots, x_n) \quad (1.1)$$

$$u |_{\partial \Omega} = 0, \quad t \in (0,T), \quad u |_{t=0} = u_0 \in H^2(\Omega) \bigcap H^1_0(\Omega),$$

where $\Omega = (a_1,b_1) \times \ldots (a_n, b_n) \subset \mathbb{R}^n$. The symbol $v \in L^\infty(Q_T)$ stands for the multiplicative control function, which we further assume to be piecewise constant-in-time.

It is known that, for any $T > 0$, system (1.1) admits a unique solution in $H^{2,1}(Q_T) \bigcap C([0,T]; H^2(\Omega) \bigcap H^1_0(\Omega))$. Here and below, we use the standard notations for Sobolev spaces, in particular, $H^1_0(\Omega) = \{ \phi \mid \phi, \phi_{x_i} \in L^2(\Omega), \quad \phi |_{\partial \Omega} = 0, \quad i = 1, \ldots, n \}$, and $H^{2,1}(Q_T) = \{ \phi \mid \phi, \phi_x, \phi_{x_i, x_j}, \phi_t \in L^2(Q_T) \quad i, j = 1, \ldots, n \}$, $H^2(\Omega) = \{ \phi \mid \phi, \phi_{x_i}, \phi_{x_i, x_j} \in L^2(\Omega) \quad i, j = 1, \ldots, n \}$.

Let us remind the reader that, in the classical sense, an evolution system is called globally approximately controllable in a given space $H$ at time $T > 0$, if it can be steered in $H$ from any
initial state into any neighborhood of any desirable target state at time $T$ by making use of a suitable available control. However, this type of controllability is out of question for boundary problem (1.1) (e.g., the zero-function is the fixed point of the right-hand side in (1.1)).

The layout of the paper is as follows. In the next section we discuss some critical “observations” exposing the principal difficulties which one can encounter in the framework of multiplicative controllability, particularly, due to the maximum principle for solutions of parabolic equations. In Section 3, we describe the proposed methodology and state the main results of this paper. In Sections 4-7, we prove the main results.

2. Maximum principle and multiplicative controllability of linear parabolic PDE’s: some critical observations. In the following observations and examples we assume that $\Omega = (0, 1) \times (0, 1)$ (though, of course, it is just for illustration purposes).

Observation 1: Assume that in (1.1) $u_0(x) \geq 0$. Then, the classical maximum principle requires the respective solution to (1.1) to stay nonnegative at any moment of time $t > 0$, regardless of the choice of $v$. This means that system (1.1) cannot be steered from any such $u_0$ to a target state which is negative on a nonzero measure set in $\Omega$. However, one can hope to approximately steer it to any non-negative state in $L^2(\Omega)$ and it was shown in [4]-[6] for a rather general class of semilinear parabolic equations in several spatial dimensions.

Observation 2: Let us assume now that $u_0$ admits changes of sign across finitely many curves, say, $S_i(0), i = 1, \ldots, M$, in domain $\Omega$, splitting $\Omega$ into $N$ simply-connected open subdomains $A_i(0), i = 1, \ldots, N$, in which $u_0$ does not change sign. A straightforward adaptation of reasoning in Observation 1 implies that for any control $v$ the respective solution $u$ to (1.1) at any moment of time $t > 0$ cannot have more than as many curves of sign change (while some curves of change of sign can “merge”). Furthermore, the “geometrical topology” of these evolving curves should be compatible with that of $u_0$. Namely, the respective $A_i(t)$’s are the result of “continuous transformations” of the original $A_i(0)$’s, see the illustrating examples below on Figures 0-5 (the change of sign is indicated with “±” symbols).
Observation 3: Due to the maximum principle of solutions to parabolic equations, no new zero-curves of change of sign can emerge inside any open set where, prior to that, we had $u(x, t) \geq 0$ or $u(x, t) \leq 0$ for all spatial points.

Let us illustrate the above ideas with the following two ‘hypothetical” examples.

Example 2.1. In (1.1) with $\Omega = (0, 1) \times (0, 1)$ assume that $u_0(x)$ is a function, describing the initial temperature distribution $u_0(x)$, is positive on the left of a single vertical line and
is negative on the right of it as shown on the left square on Fig. 1. The change of sign is indicated with “±” symbols.

On the following two squares we can see two possible positions of the zero line which moves, as indicated with arrows, to the right towards the right vertical part of the boundary $\partial \Omega$ and finally it can “merges” with it (or get arbitrarily close to it).

![Figure 1](image)

**Example 2.2.** On Fig. 2 below we again consider a squared domain $\Omega$ but now with two vertical and two horizontal zero-lines of change of sign for some initial temperature distribution $u_0(x)$. The change of sign is again indicated with “±”.

On the three squares below we illustrate a possible evolution of positions of these zero lines compatible with the above-mentioned maximum principle.
3. Methodology and main results. In this paper we will introduce a novel approach to multiplicative controllability exploiting the idea to employ multiplicative controls which would make the desirable target states to be co-linear to the first “essentially non-zero” eigenfunction in the Fourier series expansion of the resulting solutions.

This new strategy will become possible due to the “supporting” qualitative techniques, introduced in [4], [5] (see also [6]) in the context of multiplicative controllability, exploiting the idea of extracting various qualitative dynamics from system (1.1), namely, by making use of:

- either “long-term” static controls (i.e., not depending on the time-variable), which will drive system (1.1) to the first non-zero term in the Fourier series expansion of the respective solution (while the subsequent terms are dissipating in time),

- or static controls acting on “vanishingly small” time-intervals to ensure that the reaction dynamics in (1.1) will dominate over the diffusion process, when such controls are applied.

In our main results below we will use the following definition detailing the concept of change of sign of a function in a domain $\Omega \subset \mathbb{R}^n$, $n = 2, \ldots$ that of our interest in this paper.

**Definition 3.1.** Everywhere in this paper, when we talk about “finitely many $(n - 1)$-dimensional surfaces of change of sign” or “zero-surfaces/hyperplanes” for a given function,
we mean that the total measure of these surfaces is zero and that they can split \( \Omega \) into finitely many subdomains (simply-connected open sets) \( S^+_i, i = 1, \ldots, l \) and \( S^-_j, j = 1, \ldots, m \), bounded by the respective aforementioned surfaces and, possibly, by parts of boundary \( \partial \Omega \). In addition, they are such that the function at hand is positive inside of any of \( S^+_i, i = 1, \ldots, l \) and is negative inside of any of \( S^-_i, i = 1, \ldots, m \) (except, possibly, on sets of zero measure). We also assume that this function must change its sign (in the above sense, that is, almost everywhere) across these surfaces.

**Main methodological question.** In the previous section we pointed out at some principal restrictions on the target states which can be reached from a given initial state \( u_0 \) along the dynamics of (1.1). Respectively, in this paper, we intend to attack this problem from the “opposite direction”, namely, we ask:

*From what initial states \( u_0 \) system (1.1) can be steered to a given desirable target state \( u_1 \)?*

Our first result deals with a special case when curves of change of sign do not need to be moved.

**Theorem 3.1.** Assume that the initial state \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) in (1.1) and the desirable target states \( u_1 \in L^2(\Omega) \) have the same surfaces of change of sign (see Definition 3.1). Then system (1.1) can be steered from the former to the latter as close as we wish in \( L^2(\Omega) \) at some time \( T > 0 \).

Theorem 3.1 is proven in Appendix. It is instrumental for the proofs of Theorems 3.3-3.6, particularly, due to the following remark.

**Remark 3.1.** In Theorem 3.1 the steering is achieved as \( T \to 0^+ \).

**Remark 3.2.** We assume that solutions of all introduced below spectral problems are orthonormalized in the respective \( L^2 \)-spaces.

Let \( \{\lambda_k, \omega_k\}_{k=1}^{\infty} \) denote the eigenelements of the spectral problem associated with (1.1):

\[
\Delta \omega_k + v_0(x) \omega_k = \lambda_k \omega_k, \quad \lambda_1 \geq \lambda_2 \geq \ldots, \quad \lambda_k \to -\infty \text{ as } k \to -\infty. \tag{3.1}
\]
Then, the solution to (1.1) with \( v = v_0 \) admits the following Fourier series representation:

\[
\begin{align*}
  u(x, t) &= \sum_{k=1}^{\infty} \left( \int_{\Omega} u_0 \omega_k \, dx \right) e^{\lambda_k t} \omega_k(x). \\
  &\quad \text{(3.2)}
\end{align*}
\]

For each positive integer \( k_* \) introduce the following “\( k_* \)-momentum problem”.

**\( k_* \)-Momentum problem.** Let in (3.1) \( \lambda_{k_*} > \lambda_{k_*+1} \). Find \( u_0^* \) such that:

\[
\begin{align*}
  \int_{\Omega} u_0^* \omega_k \, dx &= 0, \quad k = 1, \ldots, k_* - 1, \\
  \int_{\Omega} u_0^* \omega_{k_*} \, dx &= c_0 \neq 0
\end{align*}
\]

for some \( c_0 \).

**Remark 3.3.** We would like to emphasize here the strict nature of the inequality \( \lambda_{k_*} > \lambda_{k_*+1} \).

The next result, Theorem 3.2, describes one of the central ideas of our controllability strategy in this paper.

**Theorem 3.2.** Consider any \( v_0 \in L^\infty(\Omega) \) and a positive integer \( k_* > 1 \). Let, for this \( v_0 \), \( \lambda_{k_*} > \lambda_{k_*+1} \) in (3.1). Assume that the \( k_* \)-momentum problem (3.3a-b) admits a solution for some \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), that is,

\[
  u_0 = \sum_{k=k_*}^{\infty} a_k \omega_k, \quad a_{k_*} \neq 0.
\]

Then any of the target states \( \alpha \omega_{k_*}, \alpha > 0 \) can be approximately reached in \( L^2(\Omega) \) from this \( u_0 \) at some time \( T \geq T_1 \) along the dynamics of (1.1) with \( v = v_0(x) \).

**Remark 3.4** If \( k_* = 1 \), then the first eigenfunction is \( \omega_{k_*} \) is not multiple (due to the aforementioned classical maximum principle for solutions of parabolic equations) and it does not change its sign in \( \Omega \), say, is non-negative). Hence, the respective \( k_* = 1 \)-momentum problem (3.4a-b) becomes trivial. In this case, with the help of Theorem 3.1, the result of Theorem 3.2 holds for any non-negative target state and, thus, we have the “non-negative” approximate controllability (to all non-negative target states) in \( L^2(\Omega) \) as it was shown earlier in [4]-[5].
Remark 3.5. Theorems 3.1 and 3.2 do not use the specific geometric structure of \( \Omega \) and, thus, apply to general domains.

Let us assume that our initial state \( u_0(x), x = (x_1, \ldots, x_n) \) changes its sign in \( \Omega \) on \( k_i - 1 \geq 1 \) hyperplanes \( Q_{ij} \) perpendicular to the \( x_i \)-axis

\[
Q_{ij} = \{ x = (x_1, \ldots, x_n) | x_i = x_{ij}^0 \}, \ a_i < x_{i1}^0 < \ldots < x_{ik_i-1}^0 < b_i, \ i = 1, \ldots, n, j = 1, \ldots, k_i - 1. \tag{3.4a}
\]

Select any set of functions \( u_i(x_i) \in L^2(a_i, b_i), i = 1, \ldots, n \) such that:

- each \( u_i(x_i) \) has exactly \( k_i - 1 \) points of change of sign at \( a_i < x_{i1} < \ldots < x_{ik_i-1}^0 < b_i \)
- and the function

\[
u^*(x) = \Pi_{i=1}^n u_i(x_i) \tag{3.5a}
\]

has the same sequence of change sign as \( u_0 \).

In the above and below, if \( k_i = 1 \), then there is no zero-hyperplane perpendicular the the \( i \)-dimension.

In turn, assume that our target state \( u_1(x), x = (x_1, \ldots, x_n) \) changes its sign in \( \Omega \) on \( k_i - 1 \geq 0 \) hyperplanes \( P_{ij} \) (or “zero-hyperplanes”) perpendicular to the \( x_i \)-axis

\[
P_{ij} = \{ x = (x_1, \ldots, x_n) | x_i = x_{ij} \}, \ a_i < x_{i1} < \ldots < x_{ik_i-1} < b_i, \ i = 1, \ldots, n, j = 1, \ldots, k_i - 1. \tag{3.4b}
\]

Select a set of functions \( w_i(x_i), i = 1, \ldots, n \) such that they are either constant (say, identically equal to 1) or:

- \[
\begin{align*}
w_i(x_i) & \in H^2(a_i, b_i) \cap H^1_0(a_i, b_i) \cap C^2[a_i, b_i], \ i = 1, \ldots, n; \\
each \ w_i(x_i) & \ has \ exactly \ k_i - 1 \ points \ of \ change \ of \ sign \ at \ a_i < x_{i1} < \ldots < x_{ik_i-1} < b_i; \\
selection \ of \ w_i(x_i) \equiv 1 & \ takes \ place \ if \ there \ are \ no \ zero-hyperplanes \ perpendicular \ to \ the \ i-th \ axis,
\end{align*}
\]

• selection of \( w_i(x_i) \equiv 1 \) takes place if there are no zero-hyperplanes perpendicular to the \( i \)-th axis,
and the function
\[ w(x) = \prod_{i=1}^{n} w_i(x_i) \]  
has the same sequence of change sign as \( u_1 \) in (3.4b);

- each \( w_i \) is linear near \( x_{ij} \)’s (whence, \( w_{ix_i} x_i = 0 \) near these points).

Select control \( v = v_0 \) for (1.1) as follows:

\[ v_0(x) = \sum_{i=1}^{n} v_i(x_i), \quad v_i(x_i) = -\frac{w_{ix_i}(x_i)}{w_{ix_i}} \cap \mathbb{C}[0, 1], \]  
at finitely many points (see Remark 5.2) where the denominator is not zero, and set \( v_i = 0 \) otherwise.

Denote by \( \omega_{ij} \)'s and \( \lambda_{ij} \)'s the (orthonormalized) solutions of the following spectral problem in \( H^2(a_i, b_i) \cap H^1_0(a_i, b_i) \cap C^2[a_i, b_i] \):

\[ \omega_{ijx_i}(x_i) + v_i(x_i)\omega_{ij}(x) = \lambda_{ij}\omega_{ij}(x_i), \quad i = 1, \ldots, n; \quad j = 1, \ldots \]  
(3.7a)

Remark 3.6: Instrumental observations.

- All “one-dimensional” eigenvalues \( \lambda_{ij} \)'s are simple:
  \[ \lambda_{i1} < \lambda_{i2} < \ldots, \lambda_{ij} \to \infty \quad \text{as} \quad j \to -\infty, \quad i = 1, \ldots, n. \]

- Note that \( w_i(x_i) \) is co-linear to \( \omega_{ik_i}(x_i) \) and \( \lambda_{ik_i} = 0 \) (see (3.6)). Indeed, \( \omega_{ik_i}(x_i) \) has \( k_i - 1 \) zeros in \( (a_i, b_i) \), \( \omega_{ik_i-1}(x_i) \) has \( k_i - 2 \) zeros and so forth ... (see Remark 5.2 below).

Theorems 3.1 and 3.2 do not use the specific geometric structure of \( \Omega \) and, thus, apply to general domains. However, our next result fully exploits the fact that \( \Omega \) is a parallelepiped in order to apply the method of separation of variables to reduce the multidimensional moment problem in (3.3a-b) to a set of one dimensional moment problems. It also combines the results of Theorems 3.1 and 3.2.

Theorem 3.3. Let \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) have zero-hyperplanes (of change of sign) as in (3.4a) and \( u_1 \in L^2(\Omega) \) have zero-hyperplanes as in (3.4b). Let \( w_i(x_i), i = 1, \ldots, n \) be any set of functions as in (3.5b) (there are infinitely many sets like that). Assume that there exist functions \( u_i, i = 1, \ldots, n \) from the closure in \( L^2(\Omega) \) of the set of functions as in (3.5a) such that:
\[ \int_{a_i}^{b_i} u_i(x_i) \omega_{ij}(x_i) dx_i = 0, \quad j = 1, \ldots, k_i - 1, \quad i = 1, \ldots, n, \quad (3.7b) \]

or, alternatively, we can find sequences \( \{u_i^s \in L^2(a_i, b_i)\}_{s=1}^{\infty}, i = 1, \ldots, n \) as in (3.5a) such that

\[ \lim_{s \to \infty} \int_{a_i}^{b_i} u_i^s(x_i) \omega_{ij}(x_i) dx_i = 0, \quad j = 1, \ldots, k_i - 1, \quad i = 1, \ldots, n, \quad (3.7c) \]

• while

\[ \Pi_{i=1, \ldots, n} \int_{a_i}^{b_i} u_i(x_i) \omega_{ik}(x_i) dx_i = 1, \quad (3.8a) \]

or, respectively,

\[ \Pi_{i=1, \ldots, n} \int_{a_i}^{b_i} u_i^s(x_i) \omega_{ik}(x_i) dx_i = 1, \quad \text{as } s \to \infty, \quad (3.8b) \]

independently of the choice of the “degree of closedness to zero” of terms in (3.7c).

Then, system (1.1) can be steered from \( u_0 \) to \( u_1 \) as close as we wish in \( L^2(\Omega) \) at some time \( T > 0 \).

**Discussion of conditions (3.7b)-(3.8b).** These conditions form a set of moment problems in one spatial dimension for each \( i = 1, \ldots \) on a cone (that is, these are not standard linear moment problems in a Hilbert space), derived from the moment problem (3.3a-b). Namely, for each dimension \( i \) we want to find a sequence of functions \( \{u_i^s\}_{s=1}^{\infty} \) which:

1. in general, “approaches” a subspace of \( L^2(a_i, b_i) \) perpendicular to the span of \( \{\omega_{ij}\}_{j=1}^{k_i-1} \) (it may not be a converging sequence), namely, containing functions of the form

\[ \sum_{k=k_i}^{\infty} a_k \omega_{ij}(x_i). \]

In other words, \( u_i^s \)'s tend to become “perpendicular to the aforementioned span (see (3.7b-c));

2. but not to become perpendicular to \( \omega_{ik} \) (see (3.8a-b));

3. functions \( u_i^s \)'s should all have the same zeros in the dimension \( i \) as the given initial condition \( u_0 \) in (1.1) with the same sequence of change of sign.
4. If (3.7b-c)-(3.8a-b) hold, then the first “essentially non-zero” terms in the Fourier series representations of respective solutions to (1.1), (3.6), with any of the initial conditions described in Theorem 3.3, have the same sequence of zero-hyperplanes as any of the targets in this theorem, that is, as \( u_1 \).

We will show below that Theorem 3.3, in particular, implies following straightforward implication in the case when (3.7b-c)-(3.8a-b) become trivial.

**Theorem 3.4.** System (1.1) can be steered from any initial state \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), with at most one, per each dimension \( i \), hyperplane of change of sign as in (3.4a), to any target state \( u_1 \in H^2(\Omega) \cap H^1_0(\Omega) \), which has a respectively equal amount of hyperplanes of change of sign, that is, as in (3.4b), as close as we wish in \( L^2(\Omega) \) at some time \( T > 0 \).

In the general case of finitely many hyperplanes of change of sign we have the following two results.

**Theorem 3.5.** Consider any target state \( u_1 \in H^2(\Omega) \cap H^1_0(\Omega) \) with given finitely many hyperplanes of change of sign (as in (3.4b)). Then system (1.1) can be steered to \( u_1 \), as close as we wish in \( L^2(\Omega) \), from any initial state \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), which has respectively the same amount of similarly oriented hyperplanes of change of sign (as in (3.4a)), provided that the points \( x^0_{ij} \)'s in (3.4a) are selected arbitrarily in their respective segments \( (a_i, b_i) \), except for, possibly, a set of at most countably many points.

The proof of Theorem 3.5 deals with the following two assumptions.

**Assumption 3.1.** The following sets of vectors are linear independent in the respective space \( R^{k_i-1} \):

\[
\{(\omega_{ij}(x^0_{i1}), \ldots, \omega_{ij}(x^0_{i(k_i-1)}))\}, \ i = 1, \ldots, n, \ j = 1, \ldots, k_i - 1, \tag{3.9}
\]

where \( x^0_{ij} \) are from (3.4a).

**Assumption 3.2.** If Assumption 3.1 does not hold for some (dimension) \( i \), that is, the respective set in (3.9) is linear dependent, assume that

\[
\{(\omega_{ik}(x^0_{i1}), \ldots, \omega_{ik}(x^0_{ik}))\} \not\subseteq \text{span} \{\{(\omega_{ij}(x^0_{i1}), \ldots, \omega_{ij}(x^0_{i(k_i-1)}))\} \ j = 1, \ldots, k_i - 1\}. \tag{3.10}
\]
Theorem 3.6. The statements of Theorem 3.5 hold under Assumptions 3.1 and 3.2.

Discussion of Assumptions 3.1. Note that in case of Theorem 3.4, Assumption 3.1 holds in a trivial way for \( k_i - 1 = 1 \) - we just have one non-zero vector in (3.9).

Let us assume that Assumption 3.1 does not hold for some dimension \( i \). We claim that the set of \( x_i \)'s for which it fails cannot have interior points in \((a_i, b_i)\). To this end, we will investigate the linear dependence of vectors forming the columns of matrix generated by the vectors in (3.9).

Let, for some \( \alpha_j, j = 1, \ldots, k_i - 1 \), in some open interval \((c, d) \subset (a_i, b_i)\) he have:

\[
g(x_i) = \sum_{j=1}^{k_i-1} \alpha_j \omega_{ij}(x_i) \equiv 0.
\]  

(3.11)

Apply the elliptic operator in (3.7a) to (3.11) \( k_i - 1 \) times to obtain the following linear algebraic system in \( \alpha_j \omega_{ij}(x_i), j = 1, \ldots, k_i - 1 \) with Vandermonde matrix:

\[
\sum_{j=0}^{k_i-1} \alpha_j \omega_{ij}(x_i) \equiv 0, \ldots, \sum_{j=0}^{k_i-1} \lambda_{ij} \omega_{ij}(x_i) \equiv 0
\]

Since all \( \lambda_{ij} \)'s are single, it has only the trivial solution in \((c, d)\), that is,

\[
\alpha_j \omega_{ij}(x_i), j \equiv 0, \ldots, k_i - 1, x_i \in (c, d).
\]

However, \( \omega_{ij}(x_i) \)'s can have only finitely many zero's in \((a_i, b_i)\) (see Remark 5.1). Hence, all \( \alpha_j = 0 \). Contradiction.

Thus, the set of \( x_i \)'s for which Assumption 3.1 fails, at most, consists of points that are separated by open segments of points for which Assumption 3.1 holds. In other words, the set of \( x_i \)'s, for which Assumption 3.1 fails, is at most countable.

Furthermore, if the functions \( \omega_{ij}(z) \)'s are analytic in \([a_i, b_i]\), then the set of \( x_i \)'s, for which Assumption 3.1 fails, is finite. Indeed, otherwise, the analytic function \( g(x_i) \) in (3.11) would be vanishing on a set of real points that have a limit point in \([a_i, b_i]\). Hence, \( g(x) \equiv 0 \) in \([a_i, b_i]\), which contradicts to Remark 5.1.
Example to Theorems 3.5-3.6: The case of two zero-hyperplanes. Let the initial state $u_0$ and target state $u_1$ have two hyperplanes of change of sign perpendicular to the $x_1$-axis, as described, respectively, in (3.4a) and (3.4b). Then, according to Remark 5.1, we have the following layout of the zero-points to change of sign for $\omega_{11}, \omega_{12}$ and $\omega_{13}$ (in the notations of (3.4b)):

$$a_1 < x_{11} < x_{12}^1 < x_{12} < b_1,$$

where $x_{11}$ and $x_{12}$ are zeros of $u_1(x_1)$ and $\omega_{13}(x_1), x_{12}^1$ is the only zero-point of change of sign of $\omega_{12}(x_1)$, while $\omega_{11}(x_1)$ does not have such zero-points. Without loss of generality, we can assume that immediately on the right of $a_1$, the above eigenfunctions are positive.

Then, Assumption 3.1 holds if, e.g., the zeros of $u_0$ in (3.4a) are located as follows:

$$x_{11}^0 \in (a_i, x_{12}^1), \ x_{12}^0 \in (x_{12}^1, b_i).$$

Indeed, in this case components of vector $(\omega_{11}(x_{11}^0), \omega_{11}(x_{12}^0))$ are of the same sign, while components of vector $(\omega_{12}(x_{11}^0), \omega_{12}(x_{12}^0))$ have the opposite signs.

Alternatively, Assumption 3.1 can (theoretically) fail in the areas where two aforementioned vectors have both coordinates of the same sign, while these vectors have to be co-linear, say,

$$x_{11}^0, x_{12}^0 \in (a_1, x_{12}^1).$$

In this case, if

$$x_{11}^0 \in (a_i, x_{11}), \ x_{12}^0 \in (x_{11}, x_{12}^1),$$

then vector $(\omega_{13}(x_{11}^0), \omega_{13}(x_{12}^0))$ has coordinates of the opposite sign and condition (3.10) holds, i.e., Assumption 3.2 holds.

Remark 3.7: Selection of zero-hyperplanes for $u_0$ in Theorem 3.5. From the above discussion it follows that, for each dimension $i$, we can select the zero points $x_{ij}^0, j = 1, \ldots, k_i - 1$ for $u_o$ in Theorem 3.5 as follows:

- Select $x_{ij}^0$ on (3.4a) arbitrarily in $(a_i, b_i)$;

- in order to make vectors

$$\{(\omega_{ij}(x_{ij}^0), \ldots, \omega_{ij}(x_{ij}^0))\}, \ j = 1, 2,$$
linear independent we can select the point \( x^0 \) everywhere in \((a_i, b_i)\) except of a set of at most countably many points;

- and so on ...

**Remark 3.8: About extension of Theorem 3.5.** At a first glance, it may seem that one can apply some “density argument” to get rid of Assumption 3.1 in Theorems 3.5 and 3.6. However, if we select a sequence of auxiliary \( u_{0s} \), satisfying Assumption 3.1, to approximate the actual \( u_0 \), not satisfying this assumption, we may have a divergent sequence of controls associated with each \( u_{0s} \), that is, in the framework of our methods used to prove Theorem 3.6. These methods determine a suitable sequence of controls as solutions to a suitable linear algebraic system, with matrices constructed out of vectors in (3.9), i.e., with non-zero determinants, guaranteed by Assumption 3.1. In a “conventional density argument”, if zeros of \( u_{0s} \) approximate those of \( u_0 \), these determinants may converge to a degenerate one, associated with zeros of \( u_0 \) in (3.9).

**Remark 3.9: Geometry of zero surfaces during the steering.** For the one dimensional case, the method of his paper can be viewed as an alternative method to proof the main results in [2]-[3] under the Assumptions 3.1 and 3.2. These assumptions are not required in the aforementioned works. The main results in [2]-[3] were achieved by finitely many (continuous) incremental moves of zero points. In particular, this fact be used to move zero points into open sets for which Assumption 3.1 holds, which would allow one to assume Assumption 3.1 without loss of generality. In the multidimensional case, considered in this paper, an analogous approach would be to move the zero-hyperplanes, which would also need to somehow maintain the strict geometry of these hyperplanes at every moment of time. In the arguments of this paper we only require the zero surfaces to be hyperplanes perpendicular to the respective axes at the initial and final moments of steering.

The following two figures illustrate Theorems 3.5 and 3.6 for the case of \( \Omega = (0,1) \times (0,1) \).
4. Proof of Theorem 3.1. Consider any target state $u_1 = \alpha \omega_{k^*}$, where $\alpha > 0$ is fixed.

Solution to (1.1) with $v = v_0(x)$ and $u_0$ solving the $k^*$-momentum problem admits the following Fourier series representation as in (3.2):

$$u(x, t) = \sum_{k=1}^{k^*+1} \left( \int_{\Omega} u_0 \omega_k dx \right) e^{\lambda_k t} \omega_k(x) + \left( \int_{\Omega} u_0 \omega_{k^*+1} dx \right) e^{\lambda_{k^*+1} t} \omega_{k^*+1}(x) + \sum_{k=k^*+1}^{\infty} \left( \int_{\Omega} u_0 \omega_k dx \right) e^{\lambda_k t} \omega_k(x)$$
\begin{equation}
\left( \int_{\Omega} u_0 \omega_k^* \, dx \right) e^{\lambda_k^* t} \omega_k^* (x) + \sum_{k=k+1}^{\infty} \left( \int_{\Omega} u_0 \omega_k \, dx \right) e^{\lambda_k t} \omega_k (x) \ldots \tag{4.1}
\end{equation}

In the definition of $k^*$-momentum problem we assumed that $\lambda_{k^*} > \lambda_{k^*+1}$. Therefore, we can select a control of the following form:

$$v(x) = v_0(x) - \lambda_{k^*} + a,$$

where $a > 0$ will be selected below in (4.4).

With this new control, the solution to (1.1) will take the form similar to (4.1), with the same eigenfunctions and the eigenvalues shifted by $\lambda_{k^*} - a$ units to the right:

$$u(x, t) = e^{at} \left( \int_{\Omega} u_0 \omega_k^* \, dx \right) \omega_k^* (x) + e^{at} \sum_{k=k+1}^{\infty} \left( \int_{\Omega} u_0 \omega_k \, dx \right) e^{(\lambda_{k^*} - \lambda_{k^*}) t} \omega_k (x). \tag{4.2}$$

Without loss of generality we can assume that

$$0 < \alpha \neq \int_{\Omega} u_0 \omega_k \, dx = c_0 > 0,$$ 

see (3.3b) for $c_0$. (Alternatively, we would select $a = 0$ in (4.2)).

Select an arbitrary sequence of $0 < T_k \to \infty$ as $k \to \infty$ and set

$$a = a_i = \frac{1}{T_i} \ln\left( \frac{\alpha}{\int_{\Omega} u_0 \omega_k \, dx} \right) \to 0 + \text{ as } i \to \infty \tag{4.4}$$

in which case:

$$e^{aT_i} = \frac{\alpha}{\int_{\Omega} u_0 \omega_k \, dx}.$$ 

In view of (4.3), the respective sequence of solutions to (1.1) at times $T_i$’s will converge to

$$u_1 = \alpha \omega_{k^*} \text{ in } L^2(\Omega) \text{ as } i \to \infty.$$ 

This ends the proof of Theorem 3.2.

**Remark 4.1.** Note that, in the above proof, the result of Theorem 3.2 is achieved as the time of steering tends to zero.

5. **Proof of Theorems 3.3 and 3.4:** The case of $\Omega \subset \mathbb{R}^2$ and a single line of change of sign. Without loss of generality (and for the sake of simplicity of notations), we can assume that

$$\Omega = (0, 1) \times (0, 1).$$
Let $u_0$ have just one vertical line of change of sign. We intend to show how one can steer system (1.1) in $L^2(\Omega)$ from any such $u_0$ to any state $u_1$, which also has a single vertical line of change of sign at any desirable position within $\Omega$, e.g., as shown, e.g., on Fig. 5:

Let $u_1$ have a sign change on a vertical line positioned at $x_1 = x_{11} (x = (x_1, x_2))$, while $u_0$ changes its sign on a vertical line positioned at $x_1 = x_{11}^0 \neq x_{11}$ (otherwise, we can apply Theorem 3.1). The plan of proof is as follows:

- In **Step 1** we select a stationary control $v(x) = v_0(x_1)$ such that one of the eigenfunctions $\omega_{k_*}$ of the respective spectral problem as in (3.1) has a single vertical line of sign change at $x_1 = x_{11}$, the same as the desirable target state $u_1(x)$.

- In **Step 2**, making use of Theorem 3.1, we will steer (1.1) on some $(0, T_*)$, where $T_*$ can be selected as small as we wish, to some intermediate auxiliary target state $u_*(x)$, which can “approximately” solve the $k_*$-momentum problem (3.2)-(3.3a-b) for the eigenfunctions associated with $v(x) = v_0(x_1)$. Thus, $u_*$ will have the same zero-line as $u_0$.

The aforementioned steering in Step 2 will ensure that, for $t > T_*$, the term containing $\omega_{k_*}$ will be the first “substantially nonzero, dominating” term in the respective expansion of solution to system (1.1), *if control* $v_0(x_1)$ *from Step 1 is engaged.*

- In **Step 3**, making use of the argument of Theorem 3.2, we will show that we can steer
system (1.1) from $u_*$ to a state co-linear with $\omega_{k_*}$ as close as we wish in $L^2(\Omega)$.

- In **Step 4** we will apply Theorem 3.1 again to further (approximately in $L^2(\Omega)$) steer system (1.1) to the target state $u_1(x)$ without change of the position of the zero-line at $x_{11}$.

**Step 1**: Selection of an auxiliary target state $w(x_1)$ and associated control $v = v_0(x)$.

Select, in notations of (3.5b), any function $w_1 \in H^2(0,1) \cap H_0^1(0,1) \cap C^2[0,1]$, $w_2(x_2) \equiv 1$, (see (3.5b)) such that

\[
\begin{align*}
  w_1(x_1) &= \begin{cases} 
    0, & \text{for } x = 0, x_{11}, 1, \\
    \neq 0, & \text{for } x \neq 0, x_{11}, 1, \\
  \end{cases} \\
  ||w||_{L^2(0,1)} &= 1,
\end{align*}
\]

whence:

\[
w(x) = w_1(x_1)w_2(x_2) = w_1(x_1). \tag{5.1a}\]

Select (as in (3.6)):

\[
v_0(x) = v_0(x_1) = v_1(x_1) + v_2(x_2) = -\frac{w_{1x_1x_1}(x_1)}{w_1(x_1)} - \frac{w_{2x_2x_2}(x_2)}{w_2(x_2)}
\]

\[
= -\frac{w_{1x_1x_1}(x_1)}{w_1(x_1)} + 0, x \neq 0, x_{11}, 1, v_0 \in L^\infty(0,1). \tag{5.1b}\]

This can be achieved if, for example, we select $w$ such that is linear near the boundary and $x_{11}$, where it vanishes.

Note that $w_1(x_1)$, satisfying to (5.1b), also solves the following one dimensional Dirichlet spectral problem in $H^2(0,1) \cap H_0^1(0,1)$ (*with simple eigenvalues*):

\[
\omega_{1k_{x_1x_1}}(x_1) + v_0(x_1)\omega_{1k}(x_1) = \lambda_{1k}\omega_{1k}(x_1), \quad k = 1, \ldots, \lambda_{11} > \lambda_{12} > \ldots \tag{5.2}\]

\[
\parallel \omega_{1k} \parallel_{L^2(0,1)} = 1, \quad k = 1, \ldots
\]

and for some $k = k^*$ we have

\[
\lambda_{1k^*} = 0, \quad \omega_{1k^*} = w_1 = w, \quad \lambda_k > 0, k = 1, \ldots, k^* - 1.
\]
The choice of \( w_1 \) in (5.1a-b) implies that
\[
w(x_{11}, x_2) = w_1(x_{11}) = \omega_{1k^*}(x_{11}) = 0,
\]
and thus, without loss of generality, \( \omega_{1k^*}(x_1) \) is positive on the left of \( x_{10} \) and negative on the right of it (it also vanishes at \( x_1 = 0, 1 \)).

**Remark 5.1: Instrumental properties of solutions to Sturm-Liouville problem.**

Let us recall along these lines ([8], page 272) that all eigenvalues of the Sturm-Liouville problem of our interest below, namely:
\[
z_{ix_{1}x_{1}} + q(x_{1})z_{i}(x_{1}) \quad x \in (0, 1), t > 0, z_{i}|_{x_{1}=0,1} = 0, \quad i = 1, \ldots,
\]
where \( \beta_1 < \beta_2 < \ldots, q \in C[0, 1] \), can have only finitely many zeros and \( z_{i}(x_{1}) \) has exactly \( i - 1 \) zeros. Furthermore, between two successive zeros of \( z_{i} \) and also between \( x_1 = 0 \) and its first zero and between its last zero and \( x_1 = 1 \) there is exactly one zero of \( z_{i+1} \).

Due to Remark 5.1, \( \omega_{12}(x_{1}) = w_1(x_{1}) = w(x) \) is the 2nd eigenfunction for the spectral problem (5.2) (indeed, the first eigenfunction does not have zeros in \( (0, 1) \) and the next eigenfunction has one zero in \( (0, 1) \)), and
\[
\lambda_{1k^*} = \lambda_{12}. \tag{5.3}
\]

**The form of eigenelements of (1.1).** The eigenvalues and orthonormalized (in \( L^2(0, 1) \)) eigenfunctions of the respective spectral Dirichlet problem for (1.1) with \( v(x_{1}, x_{2}) = v_{0}(x_{1}) \), namely:
\[
\Delta \omega_{l}(x) + v_{0}(x_{1})\omega_{l}(x) = \lambda_{l}\omega_{l}(x), \tag{5.4}
\]
will be of the following form:
\[
\omega_{l}(x) = \omega_{1k}(x_{1})\sqrt{2}\sin \pi mx_{2}, \quad \lambda_{l} = \lambda_{1k} - (\pi m)^2, \quad l, k, m = 1, \ldots, \quad \lambda_1 \geq \lambda_2 \geq \ldots \tag{5.5}
\]

**Step 2: Auxiliary steering to \( u^* \).** In terms of the momentum problem in (3.3a-b), we want to have:
\[
\omega_{k_{*}}(x) = \sqrt{2}\sin \pi x_{2}\omega_{12}(x_{1}), \quad \lambda_{k_{*}} = \lambda_{12} - (\pi)^2 = -(\pi)^2. \tag{5.6}
\]
Due to (5.5) and Remark 5.2,
• \( \omega_{k^*} \) is the only eigenfunction from (5.4) which has a single, more precisely, vertical line of change of sign and no horizontal lines of change of sign.

• In turn, there is only one eigenfunction in (5.4) with a single horizontal zero-line, namely:

\[
\sqrt{2}\omega_{11}(x_1)\sin 2\pi x_2,
\]

• while \( \omega_1(x) = \omega_{11}(x_1)\sin \pi x_2 \) does not have internal zero lines in \( \Omega \).

We intend, in this Step 2, to apply the method of Theorem 3.1 to approximately steer system (1.1), on some \((0, T^*), \) to a state \( u|_{t=T^*} = u_*(x) \) that “approximately” solves the momentum problem in (3.3a-b) for \( k_* \) as in (5.6), that is, with control \( v \) as in (5.1). We will to show below that it is possible.

**Description of desirable \( u^* \) in (3.5a) and \( u_* \).** If the aforementioned desirable \( u_* \) is chosen as the new initial condition for (1.1), (5.1) on \((T_*, \infty), \) it will “approximately” eliminate (on some \((T_*, T^*), \) all the terms in the respective generalized Fourier series representation (5.7) of solution to (1.1), (5.1), \( u|_{t=T_*} = u_* \), preceding (in order of decrease of \( \lambda_l \)'s) the term containing the eigenfunction \( \omega_{k_*} \), namely:

\[
u(x, t) = \sum_{m=1}^{m_*} \left( \int_{\Omega} u_\omega_{11}(x_1)\sqrt{2}\sin \pi mx_2 dx \right) e^{-(\pi m)^2+\lambda_{11})(t-T_*)} \omega_{11}(x_1)\sqrt{2}\sin \pi mx_2
+ \left( \int_{\Omega} u_\omega_{k_*} dx \right) e^{-\pi^2(t-T_*)}\omega_{k_*}(x) + \ldots, \quad t > T_* \tag{5.7}\]

for some \( m_* \geq 1 \) (if exists) that

\[-(\pi m)^2 + \lambda_{11} = \lambda_1 > \lambda_{k_*} = -(\pi)^2 + \lambda_{12} = -\pi^2, l = 1, \ldots, l_* - 1.\]

The other terms, not explicitly present in (5.7), in view of (5.5) are associated with eigenvalues strictly smaller that \(-\pi^2\).

Therefore, if \( u_* \) is such that it “approximately eliminates” the 1st sum on the right of (5.7), that is, if, e.g.,

\[
\int_{\Omega} u_\omega(x_1, x_2)\omega_{11}(x_1)dx_1
\]

can be made “as small as we wish” in \( L^2(0, 1) \) by applying Theorem 3.1, while satisfying a respective condition of type (3.3b), then we will be in a position to apply Theorem 3.2 to steer system (1.1), (5.1) to a term in (5.7) containing \( \omega_{k_*} \) at some \( t = T^* \).
Ideally, this would be the case, if, for example, \( u^* \in L^2(\Omega) \) is such that it has the same zero-line as \( u_0 \) (this can be achieved by Theorem 3.3) and, simultaneously, in notations of (3.5b)

\[
u^*(x) = u^*(x) = u_1(x_1)w_2(x_2), \quad u_2(x_2) \equiv 1, \quad \int_\Omega u_1(x_1)dx_1 = 0, \quad \int_\Omega u_1(x_1)w_1dx_1 = c^0 \neq 0,
\]
or, which is the same, \( u^* \) formally solves the \( k^* \)-momentum problem (3.3a-b) for our \( v_0 \) in (5.1). Unfortunately, Theorem 3.1 deals with approximate steering in \( L^2(\Omega) \) only.

Nonetheless, we can use Theorem 3.1 to get rid of the 1st sum in (5.7) with any pre-assigned accuracy.

Selection of \( u^*(x) \) as in (3.5a) and \( u^* \). Indeed, since \( \omega_{11}(x_1) \) does not change sign in \((0, 1)\), and \( \omega_{12}(x_1) \) admits only one change of sign in \((0, 1)\), we can easily select a function \( u_1 \in L^2(0, 1) \) with the same order of change of sign along the \( x_1 \)-axis as \( u_0 \) in (1.1) such that

\[
\int_\Omega u_1(x_1)\omega_{11}(x_1)dx_1 = 0, \quad \int_\Omega u_1(x_1)\omega_{12}(x_1)dx_1 = c^0 > 0.
\]

In terms of (3.5a), we set

\[
u^*(x) = u_1(x_1) \times 1 = w_1(x_1)w_2(x_2).
\]

Let us consider any sequence \( \{T_{**}\}_{i=1}^{\infty} \) (it will be further defined in more detail) such that

\[
\lim_{i \to 0^+} T_{**} = 0.
\]

Theorem 3.1 allows us to steer system (1.1) on each of the intervals \((0, T_{**})\) to states \( u_{i*} = u(\cdot, T_{**}) \), with the same single line of change of sign as the original \( u_0 \), such that desirable \( u_{i*} \)'s admit the following presentation:

\[
u_{i*}(x) = u(x, T_{**}) = u^*(x) + r(x, T_{**}) = u_1(x_1) + r(x, T_{**}), \quad (5.8')
\]

where

\[
\| r(\cdot, T_{**}) \|_{L^2(\Omega)} \to 0 \text{ as } T_{**} \to 0^+.
\]

For \( t > T_{**} \), in view of (5.7), if we will use control \( v(x) = v_0(x_1) \), the respective solution to system (1.1) will have the following representation:

\[
u(x, t) = \left( \int_\Omega u_1(x_1)\omega_{k*}dx_1 \right) e^{-\pi^2 (t-T_{**})} \left( \int_0^1 \sqrt{2 \sin \pi x_2}dx_2 \right) \omega_{k*}(x)
\]
\[ \sum_{\lambda_l < -\frac{\pi}{2}} \left( \int_{\Omega} u_1(x)\omega_l(x)dx \right) e^{\lambda_l(t-T_{i*})}\omega_l(x) + p(x, t - T_{i*}), \quad t > T_{i*}, \tag{5.10a} \]

where \( \omega_l \)'s do not contain \( \omega_{11} \), due to (5.8), and \( \omega_l \)'s and \( \lambda_l \)'s are defined in (5.5) and

\[ p(x, t - T_{i*}) = \sum_{l=1}^{\infty} \left( \int_{\Omega} r(x, T_{i*})\omega_l(x)dx \right) e^{\lambda_l(t-T_{i*})}\omega_l(x), \tag{5.10b} \]

\[ \| p(\cdot, t - T_{i*}) \|_{L^2(\Omega)} \leq e^{\lambda_l(t-T_{i*})} \| r(\cdot, T_{i*}) \|_{L^2(\Omega)}. \tag{5.10c} \]

**Step 3: Steering to \( \omega_{k*} \).** Now we apply the argument of Theorem 3.2 in Section 4 for \( t \in (T_{i*}, T_t) \), where \( \{T_i\}_{i=1}^{\infty} \) is any monotone increasing sequence such that \( T_i > T_{i*} \) and

\[ \lim_{i \to \infty} T_i = \infty, \]

with control

\[ v(x) = v_0(x_1) - \lambda_{k*} + a_i = v_0(x_1) + \pi^2 + a_i, \quad i = 1, \ldots, \]

and with

\[ a_i = \frac{1}{T_i - T_{i*}} \ln \left( \frac{1}{\int_{\Omega} u_1(x_1)\omega_{k*}dx} \right) \to 0 \text{ as } i \to \infty, \tag{5.11} \]

see (5.1a-b) and (4.4). Note that (5.11) implies that

\[ \lambda_l - \lambda_{k*} + a_i < 0, \quad \lambda_l < \lambda_{k*} \quad \text{when} \quad T_i - T_{i*} \to \infty \text{ as } i \to \infty. \tag{5.12} \]

This will result in the following formula in place of (4.2) and (5.10a-c):

\[ u(x, T_i) = \omega_{k*}(x) + \sum_{\lambda_l < -\pi^2} \left( \int_{\Omega} u_1(x)\omega_l(x)dx \right) e^{(\lambda_l - \lambda_{k*} + a_i)(T_i - T_{i*})}\omega_l(x) \]

\[ + \sum_{l=1}^{\infty} \left( \int_{\Omega} r(x, T_{i*})\omega_l(x)dx \right) e^{(\lambda_l - \lambda_{k*} + a_i)(T_i - T_{i*})}\omega_l(x), \quad i = 1, \ldots, \tag{5.13} \]

where, again \( \omega_l \)'s do not contain \( \omega_{11} \), due to (5.8).

In view of (5.11)-(5.12), the 1st series on the right in (5.13) tends to zero in \( L^2(\Omega) \) as \( T_i - T_{i*} \to \infty \).

The same, by the same reasoning, will happen to the tail of the 2nd series beginning for \( \lambda_l < -\pi^2 \), see (5.9). In turn, for finitely many other terms for the 2nd series in (5.13) (namely, \( k* - 1 \) terms) we have the following estimate:

\[ \| \sum_{\lambda_l > -\pi^2} \left( \int_{\Omega} r(x, T_{i*})\omega_l(x)dx \right) e^{(\lambda_l - \lambda_{k*} + a_i)(T_i - T_{i*})}\omega_l \|_{L^2(\Omega)} \]
\[ \leq (k_\ast - 1)e^{(\lambda_1 - \lambda_{k_\ast} + a_i)T_i} \parallel r(\cdot, T_{i*}) \parallel_{L^2(\Omega)} \to 0 \text{ as } i \to \infty, \]

if (for a selected sequence of \( T_i \)'s) \( T_{i*} \)'s are, additionally to their convergence to zero, selected to ensure the rate of convergence to zero in (5.9) to be higher than that of convergence of \( e^{(\lambda_1 - \lambda_{k*} + a_i)T_i} \) to \( \infty \) as \( i \to \infty \).

Combining the above yields that \( u(\cdot, T_i) \to \omega_{k*}(\cdot) \) in \( L^2(\Omega) \) as \( i \to \infty \).

**Step 4: “Magnitude adjustment” steering.** This can be achieved by applying Theorem 3.1 to system (1.1) on some intervals \((T_i, T_{(i)})\) to steer it to a sequence of states converging to \( u_1 \).

This ends the proof of Theorems 3.3-3.4 in the case of squared domain and a single line of change of sign.

6. **Proof of Theorems 3.3 and 3.4 in the general case:** “Separation of variables”.

To this end, we only need to modify the arguments in Steps 1-2 from Section 5, while the Steps 3 and 4 would be nearly identical. Namely, we need to achieve an auxiliary steering, on some \((0, T_*\)) from the original initial state to an intermediate state \( u_*(x) \) with the same zero-hyperplanes as \( u_0 \) such that the the term containing the desirable \( \omega_{k*} \) (that is, with the same zero-hyperplanes as the target state \( u_1(x) \)) will be the 1st “substantially nonzero” term in the respective expansion of solution like in (5.7) after \( T_* \) when a suitable control applied .

Without loss of generality and for simplicity of notations, we can assume that \( \Omega \) is a unit \( n \)-dimensional cube.

**Separation of variables.** Let us assume that our target state \( u_1(x) \) has \( k_\ast - 1 \) hyperplanes of change of sign \( P_{ij}, i = 1, \ldots, n, j = 0, \ldots, k_i - 1 \) as given in (3.4b).

Select functions \( u_i(x_i) \)'s, \( u_* \), \( w_i(x_i) \)'s, \( w \), and control \( v_0 \) as in (3.4a) -(3.6). In turn, in this general case, instead of (5.5)-(5.6), we have (see (3.7a) for notations):

\[ \omega_{k*}(x) = \Pi_{i=1}^n \omega_{i k_i}(x_i). \quad (6.1) \]

Respectively, to make the term with \( \omega_{k*}(x) \) be the first (in the order of decrease of \( \lambda_i \)'s) essentially non-zero terms, making use of Theorem 3.1, we need to steer (1.1) on some \((0, T_*)\)
(\(T_\ast \to 0^+\) as in Step 2 in Section 5) to

\[ u_\ast(x) = \Pi_{i=1,...,n,j_i=1,...,u_i(x_i)}, \]

which has the same zero-hyperplanes as the initial state \(u_0\), such that (3.7b)-(3.8b) holds, as it is assumed in Theorem 3.4. This can be achieved by dealing separately with each \(i\)-th one dimensional problem, \(i = 1, \ldots, n\) (similar to how it was done in Section 5 for \(i = 1\)) and with the same \(T_\ast(\to 0^+)\) for all \(i\)'s. Note that the latter conditions are straightforward for Theorem 3.3 as we discussed it in Section 5.

This ends the proof of Theorems 3.3 and 3.4.

7. Proof of Theorems 3.5 and 3.6. Let us show how one can find \(u_i(x_i)\)'s, satisfying (3.7b)-(3.8b).

7.1. Conversion of (3.7b)-(3.8b) to a problem associated with a system of linear algebraic equations. For simplicity of notation assume again that \(\Omega = (0,1) \times \ldots \times (0,1)\).

Pick any \(i \in \{1,\ldots,n\}\). Without loss of generality we can further assume that \(i = 1\) and \(k_1 - 1 \geq 1\), that is, there is at least one zero-hyperplane perpendicular to the \(x_1\)-axis. Select one more point \(s\) in \((0,1)\) different from \(x_{1j}^0\)'s in (3.4a) (we will refine this selection later).

Next, select \(u_1(x_1)\) in (3.4a) to be a piecewise constant function defined by \(2k_1 + 1\) real values \(S, l_j, m_j, j = 1, \ldots, k_1\) as follows:

\[
\begin{align*}
  u_1(x_1) &= \begin{cases} 
  l_j, & x_1 \in (x_{1j}^0 - h, x_{1j}^0), \\
  m_j, & x_1 \in (x_{1j}^0, x_{1j}^0 + h), \\
  S, & x_1 \in (s, s + h), \\
  0, & \text{elsewhere in } (0,1).
  \end{cases} \\
\end{align*}
\]

(7.1)

This \(u_1(x_1)\) can be reached for (1.1) as close as we wish in \(L^2(0,1)\) due to Theorem 3.1.

Remark 7.1: Values of \((l_j + m_j)'s and of S. Note that \(l_j m_j \leq 0, j = \ldots, k_1\) as \(x_{1j}^0\)'s are points of change of sign for \(u_1(x_1)\). The signs of \(S, l_j \)'s and \(m_j \)'s are defined by the sequence of change of sign of \(u_1(x_1)\), or, which is the same, of \(u_0(x)\) along the \(x_1\)-axis. Hence, the range of available \((l_j + m_j)'s in (7.1) is \(R\), while \(S\) cannot change its sign.

Our goal below is to investigate (7.1)) when \(h\) tends to zero. Therefore, without loss of generality, we can assume that intervals \((x_{1j}^0 - h_j, x_{1j}^0 + h_j)\) do not overlap.
The use of (7.1) will generate the following linear algebraic system of $k_1 - 1$ equations in $l_j, m_j$ out of conditions (3.7b)-(3.7c) in Theorem 3.4:

$$\sum_{j=1}^{k_1-1} A_{kj}l_j + \sum_{j=1}^{k_1-1} B_{kj}m_j + C_kS = 0, \ k = 1, \ldots, k_1 - 1, \quad (7.2)$$

where

$$A_{kj} = \int_{x_{1j}^0}^{x_{1j}^0 - h} \omega_{1k}(x_1)dx_1,$$

$$B_{kj} = \int_{x_{1j}^0 + h}^{x_{1j}^0} \omega_{1k}(x_1)dx_1,$$

$$C_k = \int_s^{s+h} \omega_{1k}(x_1)dx_1, \ k = 1, \ldots, k_1 - 1.$$

Since

$$\omega_{1k}(x_1) = \omega_{1k}(x_{1j}^0) + \omega'_{1k}(x_{1j}^0)(x_1 - x_{1j}^0) + O((x_1 - x_{1j}^0)^2)$$

in $(x_{1j}^0 - h, x_{1j}^0 + h)$ as $h \to 0$, and

$$\omega_{1k}(x_1) = \omega_{1k}(s) + \omega'_{1k}(s)(x_1 - x_{1j}^0) + O((x_1 - s_j)^2)$$

in $(s, s + h), \ k = 1, \ldots, k_1 - 1$, we can re-write (7.2) as follows:

$$h \sum_{j=1}^{k_1-1} (l_j + m_j)\omega_{1k}(x_{1j}^0) + hS\omega_{1k}(s) = +O(h^2)[\max |(l_j| + |m_j|) + |S|], \ j = 1, \ldots, k_1 - 1. \quad (7.3)$$

**An auxiliary linear algebraic system on a cone.** Let us consider the following “limit” system (7.4) of $(k_1 - 1)$ linear algebraic equations in $k_1$ real-valued variables $(V_1, \ldots, V_{k_1 - 1}, P)$:

- $P$ of certain sign only, defined by the location of $s$ between $x_{1j}^0$’s and the sequence of change of sign of $u_0$ along the $x_1$-axis,
- and $V_j \in R, j = 1, \ldots, k_1 - 1$ (see Remark 7.1),

generated by (7.3) as $h \to 0$:

$$\sum_{j=1}^{k_1-1} V_j \omega_{1k}(x_{1j}^0) + P\omega_{1k}(s) = 0, \ k = 1, \ldots, k_1 - 1, \quad (7.4)$$

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Since the number of equations exceeds the number of variables, without loss of generality, we can say that (7.4) admits non-trivial solutions under the above restrictions of variables forming a cone in $R^{k_1}$. Furthermore, under Assumption 3.1 we can consider $P$ to be a free parameter, while under Assumption 3.2 we can set $P = 0$.

Indeed, we can solve this system in $R^{k_1}$, and then if it gives solution $(V_1, \ldots, V_{k_1-1}, P)$ with $P$ of a “wrong” sign, we can replace (7.4) with:

$$
\sum_{j=1}^{k_1-1} V_j^* \omega_{1k}(x_{1j}^0) + P^* \omega_{1k}(s) = 0, \quad k = 1, \ldots, k_1 - 1,
$$

(7.4)'

and further deal with solution $(V_1^* = -V_1, \ldots, V_{k_1-1} = -V_{k_1-1}, P^* = -P)$, where $P^* = -P$ will be of a “correct” sign.

**Construction of $u_1^h(x_1)$ in (3.7c).** Consider any solution $(V_1, \ldots, V_{k_1-1}, P)$ to (7.4) with “correct sign” of $P$ and fix it. Then set:

$$
S = \frac{P}{h}, \ V_j = \frac{l_j}{h}, \ m_j = 0 \text{ or } V_j = \frac{m_j}{h}, \ l_j = 0 \quad j = 1, \ldots, k_1 - 1,
$$

depending on the sign of $V_j$’s.

Construct next $u_1(x_1)$ as in (7.1) with the just described choice of $S$ and $l_j$’s and $m_j$’s. This will give us, due to (7.3), (7.4) the relation in (3.7c) for $u_1^h$ in the following form:

$$
\int_0^1 u_1^h(x_1) \omega_{1j}(x_1) dx_1 = O(h), \ j = 1, \ldots, k_1 - 1 \text{ as } h \to 0.
$$

(7.5)

**Remark 7.2** Note that relation (7.5) will hold uniformly of any aforementioned $(V_1, \ldots, V_{k_1-1}, P)$ lying in a fixed bounded set, e.g., in a fixed ball intersected with a cone describing the restriction on sign of $S$.

**Discussion of condition in (3.8b).** To ensure a suitable “contribution of $u_1^h(x_1)$’s in (3.8b), it suffices to show that we can ensure that we can find a sequence $\{h_l\}_{l=1}^\infty$ such that for $u_1^{h_l}(x_1)$ satisfying (7.5) we can have:

$$
| \int_0^1 u_1^{h_l}(x_1) \omega_{1k_1}(x_1) dx_1 | = 1, \text{ as } h_l \to 0, \ l = 1, \ldots
$$

(7.6)

In view of Remark 7.1, to achieve (7.6), it is sufficient to show that vector

$$
(\omega_{1k_1}(x_{11}^0), \ldots, \omega_{1k_1}(x_{k_1-1}^0), \omega_{1k_1}(s))
$$

(7.7)
does not belong to the span of vectors
\[(\omega_{1k}(x_{11}^0), \ldots, \omega_{1k}(x_{1k_1-1}^0), \omega_{1k}(s)), \; k \leq k_1 - 1, \quad (7.8a)\]
that is,
\[(\omega_{1k_1}(x_{11}^0), \ldots, \omega_{1k_1}(x_{1k_1-1}^0), \omega_{1k_1}(s)) \not\in \text{span} \{ (\omega_{1k}(x_{11}^0), \ldots, \omega_{1k}(x_{1k_1-1}^0), \omega_{1k}(s)) \mid k \leq k_1 - 1 \}. \quad (7.8b)\]
If this is true for the “cut-off” vectors composed of the vectors in (7.7) and (7.8), by removing the the last coordinates, as in Assumption 3.2, then we are done.

If Assumption 3.2 does not hold, then, under Assumption 3.1, the vector in (7.7) would be a unique linear combination of the vectors in (7.8) because the vectors
\[(\omega_{1k}(x_{11}^0), \ldots, \omega_{1k}(x_{1k_1-1}^0)), \; k = 1, \ldots, k_1 - 1, \quad (7.9)\]
composed of the first \(k_1 - 1\) coordinates in (7.8), are linear independent.

Recall now that we are free to select a point \(s\) to avoid (7.8b). Indeed, if this is not possible for any \(s\) in \((0, 1)\), then the last coordinate in (7.7) or (7.8), namely, \(\omega_{1k_1}(x_1)\) will be a unique linear combination of the functions \(\omega_{1k}(x_1), k = 1, \ldots, k_1 - 1\) when \(x_1\) ranges over all \((0, 1)\) (see the previous paragraph). But \(\omega_{1k}(x_1), k = 1, \ldots, k_1\) are orthonormal to each other in \(L^2(0, 1)\) and, hence, are linear independent. Moreover, the set of such “bad” points \(s\) for which (7.8b) does not hold, can at most be countable as we discussed it after Assumptions 3.1 and 3.2 in Section 3.

This completes the proof of Theorems 3.5 and 3.6.

**Appendix: Proof of Theorem 3.1.** We intend to adapt the scheme, previously used for system (1.1) in [6] (Chapter 3) in several spatial dimension, assuming that its solutions do not change sign, or in [3] in the case in one spatial dimension, assuming that system’s solutions can change sign finitely many times. Our goal here is to apply a suitable static control \(v = v(x)\) which will steer system (1.1) from a given \(u_0\) to any desirable state \(u(\cdot, T)\) (satisfying conditions stated in Theorem 3.1) as close in \(L^2(\Omega)\) to \(u_1\) as we wish. We split the proof into two cases.
Step 1. **A special case.** In this step we assume the following condition:

**Assumption A.1.** Suppose that \( |u_1(x)| < |u_0(x)| \) in the sets \( S_i^\pm \)'s and \( S_i^- \)'s in Definition 3.1, and the function

\[
v_0(x) = \begin{cases} 
\ln \left( \frac{u_1(x)}{u_0(x)} \right), & \text{where } u_0(x) \neq 0 \text{ in } \Omega, \\
0, & \text{elsewhere in } \Omega, \end{cases}
\]

lies in \( L^\infty(\Omega) \).

**Step 1.1: Selection of bilinear control.** Note that the function \( v_0(x) \) in (A.1) satisfies \( v_0(x) \leq 0 \) in \( \Omega \). Let

\[
v(x, t) = \frac{1}{T} v_0(x).
\]

Then, the corresponding solution to (1.1), treated as an ordinary differential equation in time in a respective Banach space, admits the following representation a.e. in \( \Omega \):

\[
u(x, t) = e^{v_0(x) \frac{t}{T}} u_0(x) + \int_0^t e^{v_0(x) \frac{(t-\tau)}{T}} \Delta u(x, \tau) d\tau.
\]

At time \( t = T \) we have a.e. in \( \Omega \):

\[
u(x, T) = u(x) + \int_0^T e^{v_0(x) \frac{(T-\tau)}{T}} \Delta u(x, t) dt.
\]

**Step 1.2: Evaluation of \( \|\Delta u\|_{L^2(\Omega)} \).** Let us show that the 2-nd term in the right-hand side of (A.3) tends to zero in \( L^2(\Omega) \) as \( T \to 0^+ \), which would mean that \( u(\cdot, T) \to u_1 \) in \( L^2(\Omega) \) at the same time. Note first that, since \( v_0(x) \) is nonpositive,

\[
\int_\Omega \left( \int_0^T e^{v_0(x) \frac{(T-\tau)}{T}} \Delta u(x, \tau) d\tau \right)^2 \, dx \leq T \|\Delta u\|_{L^2(\Omega)}^2.
\]

Without loss of generality, we can further assume that \( v_0 \in C^2(\overline{\Omega}) \).

**Remark A.1.** Indeed, if \( v_0 \notin C^2(\overline{\Omega}) \), then we could consider instead a sequence of uniformly bounded controls \( \{v_0_l\}_{l=1}^\infty \), \( v_0_l \in C^2(\overline{\Omega}) \), approximating \( v_0 \) in \( L^2(\Omega) \), making use of the following limit relation:

\[
e^{v_0_l(x) t/T} u_0(x) \big|_{t=T} \to e^{v_0(x) t/T} u_0(x) \big|_{t=T} = u_1(x) \quad \text{in } L^2(\Omega) \quad \text{as } l \to \infty.
\]
Multiplying (1.1) by \( u_{xx} \) with \( v = \frac{1}{T} v_0 \leq 0 \) and integrating by parts over \( Q_T \), we have:

\[
\| \Delta u \|^2_{L^2(\Omega \times (0, T))} = \int_0^T \int_{\Omega} u_t \Delta u \, dx \, dt - \frac{1}{T} \int_0^T \int_{\Omega} v_0 \Delta u \, dx \, dt
\]

\[
= -\frac{1}{2} \int_0^T \int_{\Omega} \| \nabla u \|^2 \, dx \, dt + \frac{1}{2T} \int_0^T \int_{\Omega} \nabla v_0 \cdot \nabla \| u \|^2 \, dx \, dt + \frac{1}{T} \int_0^T \int_{\Omega} v_0 \| \nabla u \|^2 \, dx \, dt
\]

\[
= -\frac{1}{2} \int_0^T \int_{\Omega} \| \nabla u \|^2 \, dx \, dt - \frac{1}{2T} \int_0^T \int_{\Omega} \nabla v_0 \cdot \nabla \| u \|^2 \, dx \, dt + \frac{1}{T} \int_0^T \int_{\Omega} v_0 \| \nabla u \|^2 \, dx \, dt.
\]

Thus, we obtain:

\[
\| \Delta u \|^2_{L^2(\Omega \times (0, T))} + \frac{1}{2} \int_{\Omega} \| \nabla u(x, 0) \|^2 \, dx - \frac{1}{T} \int_0^T \int_{\Omega} v_0 \| \nabla u \|^2 \, dx \, dt
\]

\[
= -\frac{1}{2} \int_{\Omega} \| \nabla u(x, 0) \|^2 \, dx - \frac{1}{2T} \int_0^T \int_{\Omega} \Delta v_0 \, u^2 \, dx \, dt. \quad (A.5)
\]

In particular, recalling that \( v_0(x) \leq 0 \),

\[
\| \Delta u \|^2_{L^2(\Omega \times (0, T))} \leq \frac{1}{2} \int_{\Omega} \| \nabla u_0 \|^2 \, dx + \frac{1}{2T} \max_{x \in \Omega} | \Delta v_0 | \int_0^T \int_{\Omega} u^2 \, dx \, dt.
\]

Now, since \( v_0(x) \leq 0 \), multiplication of (1.1) by \( u \) and integration by parts over \( Q_T \) yield:

\[
\int_0^T \int_{\Omega} u^2(x, t) \, dx \, dt \leq T \int_{\Omega} u_0^2(x) \, dx.
\]

Hence, combining this with (A.5), we derive that

\[
\| \Delta u \|^2_{L^2(\Omega \times (0, T))} \leq \frac{1}{2} \int_{\Omega} \| \nabla u_0 \|^2 \, dx + \frac{1}{2T} \max_{x \in \Omega} | \Delta v_0 | \int_0^T \int_{\Omega} u^2 \, dx \, dt. \quad (A.6)
\]

In turn, making use of (A.4) and (A.6), we further obtain that

\[
\int_{\Omega} \left( \int_0^T e^{\frac{\tau}{T+t}} \Delta u(x, \tau) \, d\tau \right)^2 \, dx \leq \frac{T}{2} \int_{\Omega} \| \nabla u_0 \|^2 \, dx + \frac{T}{2} \max_{x \in \Omega} | \Delta v_0 | \int_0^T \int_{\Omega} u^2 \, dx. \quad (A.7)
\]
Note now that the right-hand side in (A.7) tends to zero as $T \to 0$, which combined with (A.3) and (A.4) yields the desirable approximate controllability result under Assumption A.1.

**Step 2: Assumption A.1 does not hold.** In this case, we will apply, first, an auxiliary constant control $v(x, t) = m > 0$ on some interval $(0, t_*)$, which generates the following solution to (1.1):

$$u(x, t_*) = e^{mt_*} \sum_{k=1}^{\infty} 2e^{-\pi k^2 t_*} \left( \int_{0}^{1} u_0(r) \omega_k(r) \, dr \right) \omega_k(x)$$

$$= e^{mt_*} \sum_{k=1}^{\infty} 2(e^{-\pi k^2 t_*} - 1) \left( \int_{0}^{1} u_0(r) \omega_k(r) \, dr \right) \omega_k(x) + e^{mt_*} u_0(x).$$

Consider any $L > 1$. Then, by selecting $m = (\ln L)/t_*, t_*>0$, we have that

$$e^{mt_*} = L,$$

and $u(\cdot, t_*) \to Lu_0$ in $L^2(Q_T)$ as $t_* \to 0+$.

Hence, for any positive integer $i$ we can find a suitably large parameter $L_i$ and respective moment $t_i > 0$ such that the inequality in the first line of Assumption A.1 will hold for $u(x, t_i)$, regarded as a new initial condition for a future action as in Step 1, everywhere except, possibly, some some (measurable) set $A_i \subset \Omega$ whose measure tends to zero as $n$ increases.

Now, in place of the control in (A.2), we can select its modified version $v_{0n} \in L^\infty(0,1)$ as follows:

$$v_{0n}(x) = \begin{cases} 
\ln \left( \frac{u_1(x)}{u(x, t_n)} \right), & x \in \Omega \setminus A_i, \\
0, & x \in A_i,
\end{cases} \quad i = 1, 2, \ldots \quad (A.8)$$

The argument of Step 1 yields (see (A.3) and (A.7)) that we can steer the solution of (1.1) from $u(\cdot, t_i)$ to an auxiliary target state

$$u_{1i}(x) = \begin{cases} 
u_1, & x \in \Omega \setminus A_i, \\
0, & x \in A_i,
\end{cases} \quad i = 1, 2, \ldots$$

as close in $L^2(\Omega)$ as we wish at some moment $t_{1i} > t_i$ (see the formula between (A.2) and (A.3), and (A.6)). Since $u_1 \in L^2(Q_T)$ (and is fixed), selecting appropriately large value for $i$ will provide that same approximate controllability result as in Step 1.

This ends the proof of Theorem 3.1.
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