Highest weight irreducible representations of the Lie superalgebra $gl(1|\infty)$

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Two classes of irreducible highest weight modules of the general linear Lie superalgebra $gl(1|\infty)$ are constructed. Within each module a basis is introduced and the transformation relations of the basis under the action of the algebra generators are written down.

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I. INTRODUCTION

We construct two classes of irreducible representations of the infinite-dimensional general linear Lie superalgebra $gl(1|\infty)$. Both of them are classes of highest weight representations, corresponding to two different orderings of the basis in the Cartan subalgebra. Related to this it is convenient to define $gl(1|\infty)$ in two different, but certainly equivalent ways. We denote them as $gl_{0}(1|\infty)$ and $gl(\infty|1|\infty)$ (see the end of the Introduction for the notation that follow).

**Definition 1.** The Lie superalgebra $gl_{0}(1|\infty)$ is a complex linear space with a basis $\{e_{ij}\}_{i,j\in\mathbb{N}}$. The $\mathbb{Z}_{2}$-grading on $gl_{0}(1|\infty)$ is defined from the requirement that $e_{1j}$, $e_{11}$, $j = 2,3,\ldots$ are odd generators, whereas all other generators are even. The multiplication (\textdef{the supercommutator}) $[,]$ on $gl(1|\infty)$ is a linear extension of the relations:

\[ [e_{ij},e_{kl}] = \delta_{jk}e_{il} - (-1)^{deg(e_{ij})deg(e_{kl})}\delta_{il}e_{kj}, \quad i,j,k,l \in \mathbb{N}. \]  

(1)

As a basis in the Cartan subalgebra $H_{0}$ we choose $\{e_{ii}\}_{i\in\mathbb{N}}$ with a natural order between the generators: $e_{ii} < e_{jj}$, if $i < j$. Then $\mathcal{E}_{0}^{+} = \{e_{ij}\}_{i< j\in\mathbb{N}}$ (resp. $\mathcal{E}_{0}^{-} = \{e_{ij}\}_{i> j\in\mathbb{N}}$) are the positive (resp. the negative) root vectors and $\{e_{i,i+1}\}_{i\in\mathbb{N}}$ are the simple root vectors.

**Definition 2.** The Lie superalgebra $gl(\infty|1|\infty)$ is a complex linear space with a basis $\{E_{ij}\}_{i,j\in\mathbb{Z}}$. The $\mathbb{Z}_{2}$-grading on $gl(\infty|1|\infty)$ is defined from the requirement that $E_{0j}$, $E_{00}$, $0 \neq j \in \mathbb{Z}$ are odd generators, whereas all other generators are even. The supercommutator on $gl(\infty|1|\infty)$ is a linear extension of the relations:

\[ [E_{ij},E_{kl}] = \delta_{jk}E_{il} - (-1)^{deg(E_{ij})deg(E_{kl})}\delta_{il}E_{kj}, \quad i,j,k,l \in \mathbb{Z}. \]  

(2)

As a basis in the Cartan subalgebra $H$ we choose $\{E_{ii}\}_{i\in\mathbb{Z}}$ with a natural order between the generators: $E_{ii} < E_{jj}$, if $i < j$. $\mathcal{E}^{+} = \{E_{ij}\}_{i< j\in\mathbb{Z}}$ (resp. $\mathcal{E}^{-} = \{E_{ij}\}_{i> j\in\mathbb{Z}}$) are the positive (resp. the negative) root vectors in $gl(\infty|1|\infty)$ and $\{E_{i,i+1}\}_{i\in\mathbb{Z}}$ are the simple root vectors.

Both algebras are isomorphic. In order to see this let $g : \mathbb{Z} \to \mathbb{N}$ be a bijective map, defined as

\[ g(z) = 2|z| + \theta(z) \in \mathbb{N}, \quad \forall z \in \mathbb{Z}. \]  

(3)

Then it is easy to verify that the map $\varphi$, which is a linear extension of the relations

\[ \varphi(E_{ij}) = e_{g(i),g(j)}, \quad i,j \in \mathbb{Z}, \]  

(4)

is an isomorphism of $gl(\infty|1|\infty)$ on $gl_{0}(1|\infty)$. Therefore both $gl_{0}(1|\infty)$ and $gl(\infty|1|\infty)$ are two different realizations of one and the same algebra, namely $gl(1|\infty)$. Note that $\varphi$ is a map of $H$ onto $H_{0}$; it is not however a map of $\mathcal{E}^{+}$ into $\mathcal{E}_{0}^{+}$. For instance take $E_{-1,0} \in \mathcal{E}^{+}$. Then $\varphi(E_{-1,0}) = e_{21} \in \mathcal{E}_{0}^{-}$. Hence a highest weight representation of $gl(\infty|1|\infty)$ may be not a highest weight representation of $gl_{0}(1|\infty)$.

The reasons for studying representations of this particular superalgebra, namely $gl(1|\infty)$, stem from physical considerations. Our motivation originates from an attempt to introduce new quantum statistics both in quantum mechanics$^{1,2}$ (in this case the superalgebras are finite-dimensional) and in quantum field theory (QFT)$^{3,4}$. In order to see where the connection to the statistics comes from, we recall shortly the origin of the Lie superstatistics.
The starting point is based on the observation that any \( n \) pairs of Bose creation and annihilation operators (CAOs), namely (below and throughout \( [x, y] = xy - yx, \{ x, y \} = xy + yx \))

\[
[b_i^-, b_j^+] = \delta_{ij}, \quad [b_i^-, b_j^-] = [b_i^+, b_j^+] = 0,
\]

considered as odd elements, generate a representation, the Bose representation \( \rho \), of the Lie superalgebra \( osp(1|2n) \equiv B(0|n) \).\(^5\) Denote by \( B_1^\pm, \ldots, B_n^\pm \) those generators of \( B(0|n) \), which in the Bose representation coincide with the Bose operators, \( \rho(B_i^\pm) = b_i^\pm \). Similarly as the Chevalley generators do, the operators \( B_1^\pm, \ldots, B_n^\pm \) and the relations they satisfy, namely

\[
[[B_i^\xi, B_j^\eta], B_k^\epsilon] = (\epsilon - \xi)\delta_{ik}B_j^\eta + (\epsilon - \eta)\delta_{jk}B_i^\xi, \quad \xi, \eta, \epsilon = \pm \text{ or } \pm 1,
\]

define uniquely the LS \( B(0|n) \).\(^5\) The operators \( B_i^\pm \) are odd root vectors of \( B(0|n) \), whereas \( \{ B_j^+, B_j^- \} \) belong to the Cartan subalgebra. The operators (6) are known in quantum field theory: these are the para-Bose operators, generalizing the statistics of the tensor fields.\(^6\) The important conclusion is that the representation theory of \( n \) pairs of para-Bose (pB) operators is equivalent to the representation theory of the Lie superalgebra \( B(0|n) \). Certainly in QFT the algebra is \( B(0|\infty) \), it is infinite-dimensional.

The identification of the para-Bose statistics with a well known algebraic structure provides a natural background for further generalizations. In QFT the commutation relations between the CAOs are determined from the translation invariance of the field under consideration.\(^7\) In momentum space the translation invariance of a scalar (or tensor) field \( \Psi(x) \) is expressed as a commutator between the energy-momentum \( P^m, m = 0, 1, 2, 3 \) and the CAOs \( a_i^\pm \) of \( \Psi(x) \):

\[
[P^m, a_i^\pm] = \pm k^m_i a_i^\pm,
\]

where the index \( i \) replaces all (continuous and discrete) indices of the field and

\[
P^m = \frac{1}{2} \sum_j k^m_j \{ a_j^+, a_j^- \}.
\]

To quantize the field means, loosely speaking, to find solutions of Eqs. (7) and (8), where the unknowns are the CAOs \( a_i^\pm \). The Bose operators (5) and their generalization, the pB operators (6), certainly satisfy (7). By no means however they do not exhaust the set of the possible solutions.

The first possibility for finding new solutions and hence for further generalization of the statistics stems from the observation that the commutation relations between the Cartan elements and the root vectors, in particular Eq. (7), remain unaltered upon q-deformations. The deformations of the parastatistics along this line was studied in Refs. 8-11 and more generally in Ref. 12.

Another opportunity, closely related to the present paper, is based on the observation that \( B(0|n) \) belongs to the class \( B \) superalgebras in the classification of Kac.\(^13\) Therefore it is natural to try to satisfy the quantization equations (7) and (8) with CAOs, generating superalgebras from the classes \( A, C \) and \( D \) or generating other superalgebras from the class \( B \). In Refs. 3, 4 it was shown that this is possible indeed. For charged tensor fields the main quantization condition (7) can be satisfied with CAOs, which generate the LS \( gl(\infty|1|\infty) \), namely a LS from the class \( A \). Up to now however this new statistics, the \( A \)–superstatistics,
did not achieve any further development. The reason is that so far the Fock spaces corresponding to the $A$–superstatistics were not constructed. Here we come to the relation between the $A$–superstatistics and the present investigation. The Fock spaces are representation spaces of $gl(1|\infty)$. In order to study the physical consequences of the $A$–superstatistics in QFT one has to develop first the representation theory of $gl(\infty|1|\infty)$ (for charged scalar fields) and of $gl_0(1|\infty)$ (for neutral fields). This is what we do in the present paper. The reason to study only highest weight representations reflects the fact that there should exist a state with a lowest energy, a vacuum, which turns to be the highest weight vector in the corresponding $gl(1|\infty)$–module.

So far the $A$–superstatistics was tested only in finite-dimensional cases, namely in the frame of a (noncanonical) quantum mechanics. We have in mind the Wigner quantum systems, introduced in Refs. 1 and 2, which attracted recently some attention from different points of view. These systems possess quite unconventional physical features, properties which cannot be achieved in the frame of the canonical quantum mechanics. The $(n+1)$–particle WQS, based on $sl(1/3n)$, exhibits a quark like structure: the composite system occupies a small volume around the centre of mass and within it the geometry is noncommutative. The underlying statistics is a Haldane exclusion statistics, a subject of considerable interest in condensed matter physics. The $osp(3/2)$ WQS, studied in Ref. 19, leads to a picture where two spinless point particles, curling around each other, produce an orbital (internal angular) momentum $1/2$. One can expect that also in QFT the Lie superstatistics could lead to new features.

In the literature one does not find many papers dealing with representations of infinite-dimensional simple Lie superalgebras. Implicitly however such algebras and their representations were used in theoretical physics since the QFT was created. On the first place we have in mind the ordinary Fock space $W_1$ of infinitely many pairs of Bose CAOs $\{b_i^{\pm}\}_{i \in \mathbb{Z}}$. As mentioned above, the Bose operators are (representatives of) the odd generators of $B(0|\infty)$ and their Fock space $W_1$ is one particular irreducible $B(0|\infty)$–module. The Fock spaces $W_p$ of para-Bose operators $\{B_i^{\pm}\}_{i \in \mathbb{Z}}$, corresponding to order of the parastatistics $p \in \mathbb{N}$, are also irreducible and inequivalent to each other $B(0|\infty)$–modules. The Clifford construction in Ref. 21 is a generalization to the case when both bosons $\{b_i^{\pm}\}_{i \in \mathbb{Z}}$, considered as odd variables, and fermions $\{f_i^{\pm}\}_{i \in \mathbb{Z}}$, which are even generators, are involved. The assumption is that the bosons anticommute with the fermions. Then any $n$ pairs of Bose CAOs and $m$ pairs of Fermi CAOs generate (a representation of) the Lie superalgebra $B(m|n)$. Therefore the Fock representation of $\{b_i^{\pm}, f_i^{\pm}\}_{i \in \mathbb{Z}}$ is an irreducible $B(\infty|\infty)$–module. Its restriction to $gl(\infty|\infty)$ leads to a set of irreducible representations of this superalgebra.

In the paper we use essentially results from the representation theory of $gl(1|n)$. The finite-dimensional irreducible modules (fidirmods) of the latter are, one can say, well understood. A character formula for all typical and atypical modules has been constructed. The dimensions of all fidirmods are known. A basis, similar to the GZ basis of $gl(n)$, was defined and its transformation under the action of the Chevalley generators was written down. This is in contrast to the more general case of $gl(m|n)$ and $U_q[gl(m/n)]$, where most of the above problems are still waiting to be solved although partial results do exist. The irreducible highest weight representations of $gl_0(1|\infty)$, which we consider, are a generalization to the infinite-dimensional case of the finite-dimensional essentially typical representations of $gl(1|n)$ in the Gel’fand-Zetlin basis (GZ basis). In order to see where the possibility for a generalization comes from we
recall (Sect. II.A) the way the Gel’fand-Zetlin basis was introduced.\textsuperscript{31} This basis is, however, inappropriate for a generalization to the case of highest weight $gl(\infty|1|\infty)$ modules. Therefore in Sect. II.B we modify it, introducing a new basis, which we call a $C$–basis. It is an analogue of the $C$–basis for $gl_\infty$.\textsuperscript{34,35} Section III is devoted to the irreducible $gl(1|\infty)$ modules. In Sect. III.A we extend the Gel’fand-Zetlin basis to the infinite-dimensional case and apply it to $gl_0(1|\infty)$. The highest weight irreducible $gl(\infty|1|\infty)$ representations are defined in Sect. III.B. They appear as a generalization of the essentially typical representations of $gl(1|n)$ in the $C$–basis. The transformations of the basis under the action of the algebra generators are explicitly written down.

Throughout the paper we use the notation:

LS, LS’s - Lie superalgebra, Lie superalgebras;
CAOs - creation and annihilation operators;
fidirmod(s) - finite-dimensional irreducible module(s);
GZ basis - Gel’fand-Zetlin basis;
$\mathbb{N}$ - all positive integers;
$\mathbb{Z}_+$ - all non-negative integers;
$\mathbb{Z}_2 = \{0, 1\}$ - the ring of all integers modulo 2;
$\mathbb{C}$ - all complex numbers;
$[p; q] = \{p, p + 1, p + 2, \ldots, q - 1, q\}$, if $q - p \in \mathbb{Z}_+$ and $[p; q] = \emptyset$ otherwise; (9)
$[m]_k = [m_{1k}, m_{2k}, \ldots m_{kk}]$, where $m_{ik} \in \mathbb{C}$; (10)
$[M]_{2k+\theta} = [M_{-k, 2k+\theta}, M_{-k+1, 2k+\theta}, \ldots, M_{k-1, \theta, 2k+\theta}]$, $\theta \in \{0, 1\}$, $k \in \mathbb{N}$; (11)
$l_{1j} = m_{1j} + 1$, $l_{ij} = -m_{ij} + i - 1$, $i \in [2; j]$; (12)
$L_{0, 2k+\theta} = M_{0, 2k+\theta}$, $\theta \in \{0, 1\}$,
$L_{i, 2k+\theta} = -M_{i, 2k+\theta} + i + 1$, $\theta \in \{0, 1\}$, $i \in [-k; -1]$; (13)
$L_{j, 2k+\theta} = -M_{j, 2k+\theta} + j - 1$, $\theta \in \{0, 1\}$, $j \in [1; k - 1 + \theta]$;
$[m] = \{m_1, m_2, \ldots, m_k, \ldots\} = \{m_i | m_i \in \mathbb{C}, i \in \mathbb{N}\}$; (14)
$[M] = \{\ldots, M_{-p}, \ldots, M_{p+1}, M_0, M_1, \ldots, M_q, \ldots\} = \{M_i | M_i \in \mathbb{C}, i \in \mathbb{Z}\}$; (15)
$P(j, l) = \begin{cases} 1 & \text{for } j \geq l \\ -1 & \text{for } j < l \end{cases}$; (16)
$Q(j, l) = \begin{cases} 1 & \text{for } j > l \\ -1 & \text{for } j \leq l \end{cases}$; (17)
$\theta(i) = \begin{cases} 1, & \text{for } i \geq 0 \\ 0, & \text{for } i < 0 \end{cases}$. (18)

II. FINITE-DIMENSIONAL ESSENTIALLY TYPICAL REPRESENTATIONS OF $gl(1|2n)$

As in the case of $gl(1|\infty)$ it is convenient to use two different notation for the finite-dimensional superalgebras from this class. In the first notation $gl_0(1|N)$ is the same as in Definition 1, but the indices $i, j$ run from 1 to $N + 1$. 

5
Then \( e_{11}, e_{22}, \ldots, e_{N+1,N+1} \) is a basis in the Cartan subalgebra \( \mathcal{H}_0 \). Denote by \( \epsilon^1, \ldots, \epsilon^{N+1} \) the dual basis, \( \epsilon^i(e_{jj}) = \delta^i_j \). The correspondence root vector \( \leftrightarrow \) root reads: \( e_{ij} \leftrightarrow \epsilon^i - \epsilon^j, \ i \neq j = 1, \ldots, N+1; \) \( \Delta^0 = \{ \epsilon^i - \epsilon^j \}_{i \neq j \in [1:N+1]} \) is the root system; \( \Delta^0_+ = \{ \epsilon^i - \epsilon^j \}_{i < j \in [1:N+1]} \) and \( \pi^0 = \{ \epsilon^1 - \epsilon^2, \epsilon^2 - \epsilon^3, \ldots, \epsilon^N - \epsilon^{N+1} \} \)

\[
(19)
\]

are the standard systems of positive roots and simple roots, respectively. The special linear superalgebra \( sl_0(1|N) \) is a subalgebra of \( gl_0(1|N) \) spanned by all \( gl_0(1|N) \) root vectors and the Cartan elements \( e_{ii} + e_{ii} \) for all \( i \neq 1 \).

Similarly, \( gl(M|1|N) \) is the same as in Definition 2, but \( i, j = -M, -M + 1, \ldots, N \) and \( M, N \in \mathbb{Z}_+ \). In particular \( \{ E_{ii} \}_{i \in [-M:N]} \) is a basis in the Cartan subalgebra \( \mathcal{H} \) with \( \{ \mathcal{E}^i \}_{i \in [-M:N]} \) its dual. The simple root vectors are \( \{ E_{i,i+1} \}_{i \in [-M:N-1]} \). Hence

\[
\pi = \{ \mathcal{E}^{-M} - \mathcal{E}^{-M+1}, \mathcal{E}^{-M+1} - \mathcal{E}^{-M+2}, \ldots, \mathcal{E}^{-1} - \mathcal{E}^0, \mathcal{E}^0 - \mathcal{E}^1, \ldots, \mathcal{E}^{N-1} - \mathcal{E}^N \}
\]

is the system of simple roots.

We have written explicitly the systems (19) and (20) in order to underline that they contain different number of odd roots: \( \pi^0 \) has only one, \( \epsilon^1 - \epsilon^2 \), whereas the odd roots in \( \pi \) are \( \mathcal{E}^{-1} - \mathcal{E}^0, \mathcal{E}^0 - \mathcal{E}^1 \). Therefore the systems of the simple roots of \( sl_0(1|2n) \) and \( sl(n|1|n) \) are different, despite of the fact that these algebras are isomorphic. This property demonstrates one of the essential differences between the Lie algebras and the Lie superalgebras. For each simple Lie algebra there exists (up to a transformation from the Weyl group) only one system of simple roots. This is not the case for the basic Lie superalgebras, where several inequivalent simple root systems can be in general defined (for more details see Ref. 36, 37, 38). As a result one and the same irreducible \( gl(1|2n) \) module can be described with different signatures. We shall have to take this into account in the definition of the \( C \)-basis.

A. GZ basis

Let \( V([m]_{N+1}) \) be a highest weight finite-dimensional irreducible \( gl_0(1|N) \) module (fidirmod) with a highest weight

\[
[m]_{N+1} \equiv [m_{1,N+1}, m_{2,N+1}, \ldots, m_{N+1,N+1}] \equiv \sum_{i=1}^{N+1} m_{i,N+1} \epsilon^i,
\]

where

\[
m_{j,N+1} \in \mathbb{C}, \ j = 1, \ldots, N + 1, \ m_{i,N+1} - m_{i+1,N+1} \in \mathbb{Z}_+, \ i = 2, 3, \ldots, N.
\]

If \( x_{N+1} \) is the highest weight vector in \( V([m]_{N+1}) \), then \( e_{ii} x_{N+1} = m_{i,N+1} x_{N+1} \).

Consider the chain of subalgebras

\[
\text{gl}_0(1|N) \supset \text{gl}_0(1|N-1) \supset \text{gl}_0(1|N-2) \supset \ldots \supset \text{gl}_0(1|2) \supset \text{gl}_0(1|1) \supset \text{gl}_0(1|0) \equiv \text{gl}_0(1).
\]

Then \( V([m]_{N+1}) \) is said to be essentially typical, if it is completely reducible with respect to any one of the subalgebras in the chain (23). Each essentially typical module \( V([m]_{N+1}) \) carries a typical representation of the special linear superalgebra \( sl_0(1|n) \), but the inverse is in general not true.
Set
\[ l_{1,N+1} = m_{1,N+1} + 1; \quad l_{i,N+1} = -m_{i,N+1} + i - 1, \quad i = 2, 3, \ldots, N + 1. \]  \hspace{1cm} (24)

**Proposition 1.** The \( gl_0(1 \mid N) \) module \( V([m]_{N+1}) \) is essentially typical if and only if
\[ l_{1,N+1} \notin [l_{2,N+1}; l_{N+1,N+1}]. \]  \hspace{1cm} (25)

Let \( V([m]_{N+1}) \) be an essentially typical \( gl_0(1 \mid N) \) module and let
\[ V([m]_{N+1}) \supset V([m]_N) \supset V([m]_{N-1}) \supset \ldots \supset V([m]_{k+1}) \supset \ldots V([m]_2) \supset V(m_1) \]  \hspace{1cm} (26)
be a flag of \( gl_0(1 \mid k) \) fidirmods \( V([m]_{k+1}), k = 0, 1, 2, \ldots, N, \) where
\[ [m]_{k+1} \equiv [m_{1,k+1}, m_{2,k+1}, \ldots, m_{k+1,k+1}] \equiv \sum_{i=1}^{k+1} m_{i,k+1} \epsilon^i \]  \hspace{1cm} (27)
is the signature of \( V([m]_{k+1}). \) In the ordered basis
\[ \epsilon_{11}, \epsilon_{22}, \ldots, \epsilon_{k+1,k+1} \]  \hspace{1cm} (28)
of the Cartan subalgebra of \( gl_0(1 \mid k), \) \( m_{i,k+1} \) is the eigenvalue of \( \epsilon_{ii} \) on the highest weight vector \( x_{k+1} \in V([m]_{k+1}), \)
\[ \epsilon_{ii} x_{k+1} = m_{i,k+1} x_{k+1}, \quad i = 1, \ldots, k+1. \]  \hspace{1cm} (29)

Since we consider only essentially typical modules and the fidirmods of \( gl_0(1) \) are one dimensional, the flag (26) defines a vector \( |m\rangle \) in \( V([m]_{N+1}). \) It turns out this vector is uniquely defined (up to, certainly, a multiplicative constant) by the signatures \( [m]_{N+1}, [m]_N, \ldots, [m]_2, m_1. \) Therefore one can set
\[ |m\rangle = \begin{bmatrix} [m]_{N+1} \\ [m]_N \\ \vdots \\ [m]_2 \\ m_{11} \end{bmatrix} = \begin{bmatrix} m_{1,N+1} & m_{2,N+1} & \ldots & m_{N,N+1} & m_{N+1,N+1} \\ m_{1,N} & m_{2,N} & \ldots & m_{N,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{12} & m_{22} \\ m_{11} \end{bmatrix}. \]  \hspace{1cm} (30)

The vectors (30), corresponding to all possible flags (26), constitute a basis \( \Gamma([m]_{N+1}) \) in the \( gl_0(1 \mid N) \) fidirmod \( V([m]_{N+1}). \) This is the GZ basis introduced in Ref. 31 (for the more general case of \( gl(M/N). \))

**Proposition 2.** The GZ basis \( \Gamma([m]_{N+1}) \) in the essentially typical module \( V([m]_{N+1}) \) is given by all tables (30) for which
1. the numbers \( m_{i,N+1}, \) \( i = 1, 2, \ldots, N + 1 \) are fixed for all tables and satisfy (22), (24), (25).
2. \( m_{ii} - m_{1,i-1} \equiv \theta_{i-1} \in \{0, 1\}, \quad i = 2, 3, \ldots, N + 1, \)  \hspace{1cm} (31)
3. \( m_{i,j+1} - m_{ij} \in \mathbb{Z}_+; \quad m_{ij} - m_{i+1,j+1} \in \mathbb{Z}_+; \quad 2 \leq i \leq j \leq N. \)  \hspace{1cm} (32)
The transformations of the basis \( \Gamma([m]_{N+1}) \) under \( gl_0(1 \mid N) \) are completely defined from the action of the Chevalley generators

\[ 7 \]
where \( l_{ij} = m_{ij} + 1; \quad l_{ij} = -m_{ij} + i - 1, \quad i \neq 1 \) and the table \(|m\rangle_{\pm(i,j)}\) is obtained from the table \(|m\rangle_{\pm}\) by the replacement \( m_{ij} \to m_{ij} \pm 1 \).

If a vector from the r.h.s. of (35) or (36) does not belong to the module under consideration, then the corresponding term is zero even if the coefficient in front is undefined; if an equal number of factors in numerator and denominator are simultaneously equal to zero, they should be canceled out.

The \( gl_0(1|N) \) highest weight vector \( x_{N+1} \) in \( V(|m\rangle_{N+1}) \) is a vector from the GZ basis

\[
x_{N+1} = |\hat{m}\rangle, \text{ for which } m_{ii} = m_{i,i+1} = \ldots = m_{i,N+1}, \quad i = 1, 2, \ldots, N,
\]
i.e.,

\[
|\hat{m}\rangle = \begin{bmatrix}
m_{1,N+1} & m_{2,N+1} & \ldots & m_{N,N+1} & m_{N+1,N+1} \\
m_{1,N+1} & m_{2,N+1} & \ldots & m_{N,N+1} & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_{1,N+1} & m_{2,N+1} & \ldots & m_{N,N+1} \\
m_{1,N+1} & m_{2,N+1}
\end{bmatrix}.
\]

In this case

\[
e_{ii}|\hat{m}\rangle = m_{i,N+1}|\hat{m}\rangle, \quad i = 1, 2, \ldots, N + 1, \quad e_{k,k+1}|\hat{m}\rangle = 0, \quad k = 1, 2, \ldots, N.
\]
B. C-basis

Let $E_{ij}, \ i, j = -n, -n + 1, \ldots, n$ be the generators of $gl(n|1|n)$. Define a sequence of subalgebras

\[ gl(k|1|k - 1 + \theta) = \text{lin.env.}\{E_{ij} | i, j \in [-k; k + 1 + \theta]\} \quad \forall \theta \in \{0, 1\}, \ k \in [1 - \theta; n]. \tag{40} \]

As an ordered basis in the Cartan subalgebra of $gl(k|1|k - 1 + \theta)$ take

\[ E_{-k, -k}, E_{-k+1, -k+1}, \ldots, E_{k-1+\theta, k-1+\theta}. \tag{41} \]

**Proposition 3.** The map $\varphi$, which is a linear extension of the relations

\[ \varphi(E_{ij}) = e_{g(i), g(j)}, \ i, j = -n, -n + 1, \ldots, n, \tag{42} \]

is an isomorphism of $gl(n|1|n)$ on $gl_0(1|2n)$. Its restriction on $gl(k|1|k - 1 + \theta)$ is an isomorphism of $gl(k|1|k - 1 + \theta)$ on $gl_0(1|2k - 1 + \theta)$ for each $\theta \in \{0, 1\}$ and $k \in [1 - \theta; n]$. The chain of subalgebras

\[ gl(n|1|n) \supset gl(n|1|n - 1) \supset gl(n - 1|1|n - 1) \supset gl(n - 1|1|n - 2) \supset \ldots \supset gl(1|1|1) \supset gl(1|1) \supset gl(1), \tag{43} \]

(\( gl(1|1|0) \equiv gl(1|1), \ gl(0|1|0) \equiv gl(1) \)) is transformed by $\varphi$ into the chain (23)

\[ gl_0(1|2n) \supset gl_0(1|2n - 1) \supset gl_0(1|2n - 2) \supset \ldots \supset gl_0(1|2) \supset gl_0(1|1) \supset gl_0(1). \tag{44} \]

The proof is straightforward.

The isomorphism $\varphi$ allows one to turn any $gl_0(1|2k - 1 + \theta)$ irreducible module $V([m]_{2k+\theta})$ into a $gl(k|1|k - 1 + \theta)$ module:

\[ \varphi(E_{ij})x = e_{g(i), g(j)}x, \quad \forall x \in V([m]_{2k+\theta}). \tag{45} \]

The relevant for us point is that each $V([m]_{2k+\theta})$ can be labeled also with its highest weight with respect to $gl(k|1|k - 1 + \theta)$. By definition it consists of the eigenvalues of the representatives of the Cartan generators (41), namely

\[ \varphi(E_{-k, -k}), \varphi(E_{-k+1, -k+1}), \ldots, \varphi(E_{-2, -2}), \varphi(E_{-1, -1}), \varphi(E_{0, 0}), \varphi(E_{1, 1}), \ldots, \varphi(E_{k-1+\theta, k-1+\theta}) \tag{46} \]

on the $gl(k|1|k - 1 + \theta)$ highest weight vector $y_{2k+\theta} \in V([m]_{2k+\theta})$. The latter is defined from the requirements

\[ \varphi(E_{ij})y_{2k+\theta} = 0, \quad i < j = -k, -k + 1, \ldots, k - 1 + \theta, \tag{47} \]

\[ \varphi(E_{ii})y_{2k+\theta} = M_{i, 2k+\theta}y_{2k+\theta}, \quad i = -k, -k + 1, \ldots, k - 1 + \theta. \tag{48} \]

Set

\[ [M]_{2k+\theta} \equiv [M_{-k, 2k+\theta}, M_{-k+1, 2k+\theta}, \ldots, M_{k-1+\theta, 2k+\theta}]. \tag{49} \]

The new signature $[M]_{2k+\theta}$ defines, as mentioned above, uniquely $V([m]_{2k+\theta})$. Hence

\[ V([m]_{2k+\theta}) = V([M]_{2k+\theta}). \tag{50} \]
Consider now a GZ basis vector \(|m\rangle\) corresponding to the flag

\[
V([m]_{2n+1}) \supset V([m]_{2n}) \supset V([m]_{2n-1}) \supset \ldots \supset V([m]_{2k+\theta}) \supset \ldots \supset V([m]_2) \supset V(m_{11}) \leftrightarrow |m\rangle,
\]

namely the vector (30) with \(N = 2n\). In view of (50) the same flag can be written as

\[
V([M]_{2n+1}) \supset V([M]_{2n}) \supset V([M]_{2n-1}) \supset \ldots \supset V([M]_{2k+\theta}) \supset \ldots \supset V([M]_2) \supset V(M_{11})
\]

and therefore the vector \(|m\rangle\) is completely defined by the signatures \([M]_{2n+1}, [M]_{2n}, \ldots, [M]_2, M_{11}\). Therefore we can write any GZ basis vector (30) also in the form

\[
|M\rangle \equiv \begin{bmatrix}
M_{-n,2n+1} & M_{-n+1,2n+1} & \ldots & M_{-1,2n+1} & M_{0,2n+1} & M_{1,2n+1} & \ldots & M_{n-1,2n+1} & M_{n,2n+1} \\
M_{-n,2n} & M_{-n+1,2n} & \ldots & M_{-1,2n} & M_{0,2n} & M_{1,2n} & \ldots & M_{n-1,2n} \\
M_{-n+1,2n-1} & M_{-n,2n-1} & \ldots & M_{-1,2n-1} & M_{0,2n-1} & M_{1,2n-1} & \ldots & M_{n-1,2n-1} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
M_{-1,3} & M_{0,3} & M_{1,3} & \ldots & \ldots & \ldots & \ldots & \ldots \\
M_{-1,2} & M_{0,2} & M_{1} & \ldots & \ldots & \ldots & \ldots & \ldots \\
M_{0} & M_{1} & M_{2} & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

(53)

Obviously (30) (with \(N = 2n\)) and (53) are two different labelings for one and the same vector \(|m\rangle\). We call the basis, written in the notation (53), a \(C\)–basis in \(V([M]_{2n+1}) \equiv V([m]_{2n+1})\) and denote it as \(\Gamma([M]_{2n+1})\).

In order to use effectively the basis \(\Gamma([M]_{2n+1})\) we need to determine all signatures \([M]_{2k+\theta}\), namely to find the values of the entries in (53). To this end we have to determine as a first step the highest weight vector \(y_{2k+\theta}\) within each \(gl(k|1|k-1+\theta)\)–module \(V([m]_{2k+\theta})\) in the chain (51) and subsequently, using (48), to compute its \(gl(k|1|k-1+\theta)\) signature \([M]_{2k+\theta}\).

**Proposition 4.** The \(gl(k|1|k-1+\theta)\) highest weight vector \(y_{2k+\theta}\) in \(V([m]_{2k+\theta})\) (from the chain (51)) is the GZ vector \(|m\rangle_{2k+\theta}\), for which

\[
m_{1,2r+\tau} + k - r = m_{1,2k+\theta}, \quad \forall \tau \in \{0,1\}, \ r \in [1 - \tau; k - \tau];
\]

\[
m_{r-j,2k-2j+\tau} = m_{r,2k+\theta}, \quad \forall \ r \in [3 - \theta; k + 1], \ \tau \in \{0,1\}, \ j \in [1 - \theta; r - 2];
\]

\[
m_{r-j,2k-2j+\tau} = m_{r,2k+\theta}, \quad \forall \ r \in [k + 2; 2k], \ \tau \in \{0,1\}, \ j \in [1 - \theta; 2k - r + \tau].
\]

**Proof:** It is easy to verify that the conditions (54) are equivalent to

\[
\theta_{2i-1} = 1, \quad i \in [1; k],
\]

\[
\theta_{2i} = 0, \quad i \in [1; k-1+\theta],
\]

whereas the conditions (55) can be replaced by

\[
l_s,2i+1 - l_s,2i = 0, \quad i \in [1; k-1+\theta], \ s \in [2; 2i],
\]

\[
l_{s+1,2i} - l_s,2i-1 = 0, \quad i \in [2; 2k], \ s \in [2; 2i-1].
\]
We need to show that (47) holds for $y_{2k+\theta} = |m\rangle_{2k+\theta}$. It certainly suffices to verify it only for the $gl(k|k-1+\theta)$ simple root vectors, namely to prove that

\begin{align}
\varphi(E_{i,-i+1})|m\rangle_{2k+\theta} &= 0, \quad i \in [1;k], \\
\varphi(E_{i,i+1})|m\rangle_{2k+\theta} &= 0, \quad i \in [0;k-2+\theta].
\end{align}

(58) (59)

The validity of the latter follows from the observation that $\varphi(E_{1,0}) = e_{21}$, $\varphi(E_{01}) = [e_{12},e_{23}]$, $\varphi(E_{i-i+1}) = [e_{2i,2i-1},e_{2i-1,2i+2}, i \in [2;k], \varphi(E_{i,i}) = [e_{2i-1,2i},e_{2i,2i+1}]$, $i \in [2;k-1+\theta]$ and Eqs. (34)-(36). This completes the proof.

We are now ready to determine the $gl(k|k-1+\theta)$ signature of $V(|m\rangle_{2k+\theta})$ for any $\theta \in \{0,1\}$ and $k \in [1;n]$. Taking into account (54), (55) and (45) and using the transformation relation (33), one obtains

\begin{align}
\varphi(E_{i,i})|m\rangle_{2k+\theta} &= e_{2|i|,2|i|} |m\rangle_{2k+\theta} = (m_{i+k+2k+\theta}+1)|m\rangle_{2k+\theta}, \quad i \in [-k;-1], \\
\varphi(E_{00})|m\rangle_{2k+\theta} &= e_{11} |m\rangle_{2k+\theta} = (m_{1,2k+\theta}-k)|m\rangle_{2k+\theta}, \\
\varphi(E_{i,i})|m\rangle_{2k+\theta} &= e_{2i+1,2i+1} |m\rangle_{2k+\theta} = m_{i+k+1,2k+\theta}|m\rangle_{2k+\theta}, \quad i \in [1;k-1+\theta].
\end{align}

(60a) (60b) (60c)

Comparing (60) with the definition (48) we obtain the $gl(k|k-1+\theta)$ signature $[M]_{2k+\theta}$ of $V(|m\rangle_{2k+\theta})$:

\begin{align}
M_{i,2k+\theta} &= m_{i+k+2k+\theta}+1, \quad i \in [-k;-1], \\
M_{0,2k+\theta} &= m_{1,2k+\theta} - k, \\
M_{i,2k+\theta} &= m_{i+k+1,2k+\theta}, \quad i \in [1;k-1+\theta], \\
M_{01} &= m_{11}.
\end{align}

(61a) (61b) (61c) (61d)

We have added the evident relation (61d) for completeness, since it is not contained in (61a-c). The above relations hold for any $\theta \in \{0,1\}$ and $k \in [1;n]$. In particular,

\begin{align}
M_{i,2n+1} &= m_{i+n+2,2n+1} + 1, \quad i \in [-n;-1], \\
M_{0,2n+1} &= m_{1,2n+1} - n, \\
M_{i,2n+1} &= m_{i+n+1,2n+1}, \quad i \in [1,n].
\end{align}

(62a) (62b) (62c)

The $gl(n|n)$ highest weight vector $y_{2n+1} \equiv |\hat{M}\rangle$ is the one from (53), for which $M_{i,j} = M_{i,2n+1}$ for any admissible $i$ and $j$:

\begin{align}
|\hat{M}\rangle \equiv \\
\begin{bmatrix}
M_{-n,2n+1} & M_{-n+1,2n+1} & \cdots & M_{-1,2n+1} & M_{0,2n+1} & M_{1,2n+1} & \cdots & M_{n-1,2n+1} & M_{n,2n+1} \\
M_{-n,2n+1} & M_{-n+1,2n+1} & \cdots & M_{-1,2n+1} & M_{0,2n+1} & M_{1,2n+1} & \cdots & M_{n-1,2n+1} \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots & \ddots \\
& & & & & & \ddots & \ddots & \ddots \\
& & & & & & & \ddots & \ddots \\
& & & & & & & \ddots & \ddots \\
& & & & & & & & \ddots \\
\end{bmatrix}
\end{align}

(63)
From (31) and (32) one derives the "in-betweenness conditions", which define completely the new basis (53). The transformations of the $C$-basis are most easily written in terms of the following variables:

$$L_{0,2k+\theta} = M_{0,2k+\theta}$$
$$L_{i,2k+\theta} = -M_{i,2k+\theta} + i + 1, \quad i \in [-k; -1],$$
$$L_{i,2k+\theta} = -M_{i,2k+\theta} + i - 1, \quad i \in [1; k - 1 + \theta].$$

We formulate the result as a proposition.

**Proposition 5.** The $2n + 1$-tuple $[M]_{2n+1} = \{M_{-n,2n+1}, M_{-n+1,2n+1}, \ldots, M_{n,2n+1}\}$ is a signature of an essentially typical $gl(n|1)n$ module $V([M]_{2n+1})$ if and only if

$$M_{i,2n+1} \in \mathbb{C}, \quad i \in [-n; n],$$
$$M_{i,2n+1} - M_{i+1,2n+1} \in \mathbb{Z}_+, \quad i \in [-n; -2] \cup [1; n - 1],$$
$$M_{-1,2n+1} - M_{1,2n+1} \in \mathbb{N},$$
$$M_{0,2n+1} = L_{0,2n+1} \notin \{L_{-n,2n+1}; L_{n,2n+1}\}.$$

The $C$-basis $\Gamma([M]_{2n+1})$ in $V([M]_{2n+1})$ consists of all tables (53) for which the labels

$$M_{i,2k+\theta}, \quad \theta \in \{0, 1\}, \quad k \in [1 - \theta; n], \quad i \in [-k; k - 1 + \theta],$$

take all possible values consistent with the "in-betweenness conditions"

$$M_{i,2k+1} - M_{i,2k} \in \mathbb{Z}_+, \quad k \in [1; n], \quad i \in [-k; -1] \cup [1; k - 1],$$
$$M_{i,2k-1} - M_{i,2k} \in \mathbb{Z}_+, \quad k \in [2; n], \quad i \in [-k + 1; -1] \cup [1; k - 1],$$
$$M_{i-1,2k} - M_{i,2k-1} \in \mathbb{Z}_+, \quad k \in [2; n], \quad i \in [-k + 1; -1] \cup [2; k - 1],$$
$$M_{i-1,2k} - M_{i,2k+1} \in \mathbb{Z}_+, \quad k \in [1; n], \quad i \in [-k + 1; -1] \cup [2; k],$$
$$M_{-1,2k} - M_{1,2k-1} \in \mathbb{N}, \quad k \in [2; n],$$
$$M_{i-1,2k} - M_{1,2k+1} \in \mathbb{N}, \quad k \in [1; n],$$
$$M_{0,2k+1} - M_{0,2k} \equiv \psi_{2k} \in \{0, 1\}, \quad k \in [1; n],$$
$$M_{0,2k} - M_{0,2k-1} \equiv \psi_{2k-1} \in \{0, -1\}, \quad k \in [1; n].$$

The transformations of the $C$-basis under the action of the inverse images $\varphi^{-1}(e_{ii})$, $\varphi^{-1}(e_{i+1,i})$ and $\varphi^{-1}(e_{i+1,i})$ of the $gl(1|2n)$ Chevalley generators follow from (33)-(36) and (61),(62). The result reads (we write $E_{ij}$ instead of $\varphi(E_{ij})$):

$$E_{ii}|M\rangle = \left( \sum_{j=-|i|}^{[i]+\theta(i)-1} M_{j,2[i]+\theta(i)} - \sum_{j=-[i]-1}^{[i]-1} M_{j,2[i]+\theta(i)-1} \right) |M\rangle, \quad i \in [-n; n],$$
$$E_{0,-1}|M\rangle = (1 + \psi_1)|M\rangle_{(0,1)},$$

(68, 69)
We have written the transformation relations of the $C-$basis under the action of generators, which are different from the $\text{gl}(n|1|n)$ Chevalley elements. These generators however define completely all other generators. In this sense Eqs. (68)-(74) are complete. We shall use them in order to derive the transformations of the $\text{gl}(\infty|1|\infty)$ irreducible modules under the action of the Chevalley generators.

Remark. We are thankful to the referee for pointing out that Proposition 4 can be proved also without using the transformation relations (34)-(36). To this end note (see Eqs. (58)-(59)) that the $\text{gl}(k|1|k - 1 + \theta)$ highest weight vector $y_{2k+\theta} \equiv |m\rangle_{2k+\theta} \in V(|m\rangle_{2k+\theta})$ is determined from the requirement to be annihilated by the generators $\{\varphi(E_{i,-1})| i \in [1;k]\} \cup \{\varphi(E_{i,1})| i \in [0;k - 2 + \theta]\}$, i.e., by $\{e_{2k,2k-2}, e_{2k-2,2k-4}, \ldots, e_{42}, e_{21}, e_{13}, e_{35}, \ldots, e_{2k+2\theta-3,2k+2\theta-1}\}$. The roots, corresponding to the above root vectors, namely

$$\hat{\Delta}_{2k+\theta} = \{e^{2k} - e^{2k-2}, e^{2k-2} - e^{2k-4}, \ldots, e^{4} - e^{2}, e^{2} - e^{1}, e^{1} - e^{3}, e^{3} - e^{5}, \ldots, e^{2k+2\theta-3} - e^{2k+2\theta-1}\},$$

(75)

can be taken as a new system of simple roots of $\text{gl}(1|2k + \theta - 1)$ with a system of positive roots $\hat{\Delta}_{2k+\theta}$. 

13
Let \( \Lambda_{2k+\theta} \equiv [m]_{2k+\theta} \equiv \sum_{i=1}^{2k+\theta} m_i \epsilon_i \) be the standard signature (= the highest weight) of \( V([m]_{2k+\theta}) \), namely the signature corresponding to the choice of simple roots

\[
\pi_{2k+\theta} = \{ \epsilon^1 - \epsilon^2, \epsilon^2 - \epsilon^3, \ldots, \epsilon^{2k-1+\theta} - \epsilon^{2k+\theta} \}.
\]  

Denote by \( \Delta^{2k+\theta}_+ \) the corresponding to it system of positive roots. The problem is to determine the signature (= the highest weight) \( \hat{\Lambda}_{2k+\theta} \) of \( V([m]_{2k+\theta}) \) with respect to \( \Delta^{2k+\theta}_+ \). This problem can be solved on the ground of results from Refs.\(^{39,40}\) Given a subset of positive roots \( \Delta'_+ \) of \( gl(1|2k+\theta-1) \) and a simple root \( \alpha \in \Delta'_+ \), one constructs a new system of positive roots \( \Delta''_+ \) by a simple reflection \( \langle \alpha \rangle \):\(^{39,40}\)

\[
\Delta''_+ = \langle \alpha \rangle (\Delta'_+) = \left\{ \begin{array}{ll}
\rho_\alpha(\Delta'_+) & \text{if } \alpha \text{ is even;} \\
(\Delta'_+ \cup \{-\alpha\}) \setminus \{\alpha\} & \text{if } \alpha \text{ is odd},
\end{array} \right.
\]

(77)

where \( \rho_\alpha \) is an element from the Weyl group of \( gl(1|2k+\theta-1) \), corresponding to \( \alpha \).

If \( V_{2k+\theta} \) is an essentially typical \( gl(1|2k+\theta-1) \) module with a highest weight \( \lambda' \), corresponding to \( \Delta'_+ \), then the highest weight with respect to \( \Delta''_+ \) is

\[
\lambda'' = \rho_\alpha(\lambda') \quad \text{if} \quad \alpha \text{ is an even root} \quad \text{and} \quad \lambda'' = \lambda' - \alpha \quad \text{if} \quad \alpha \text{ is an odd root}.
\]

(78)

Let \( \prod_{i=1}^{N}(\alpha_i) = \langle \alpha_1 \rangle \langle \alpha_2 \rangle \ldots \langle \alpha_N \rangle \). Then

\[
\hat{\Delta}^{2k+\theta}_+ = \prod_{i=1}^{k} \prod_{j=1}^{2i-1} (\epsilon_j - \epsilon_{2i}) \Delta^{2k+\theta}_+.
\]

(79)

From (77)-(79) one derives that

\[
\hat{\Lambda}^{2k+\theta}_+ = \sum_{j=2}^{k+1} (m_{j,2k+\theta} + 1) \epsilon^{2k-2j+4} + (m_{1,2k+\theta} - k) \epsilon^1 + \sum_{j=k+2}^{2k+\theta} m_{j,2k+\theta} \epsilon^{2j-2k-1},
\]

(80)

i.e.,

\[
\begin{align*}
\epsilon^{2k-2i+4,2k-2i+4|m}_{2k+\theta} &= (m_{i,2k+\theta} + 1)|m|_{2k+\theta}, \quad i \in [2; k + 1], \\
\epsilon_{11|m}_{2k+\theta} &= (m_{1,2k+\theta} - k)|m|_{2k+\theta}, \\
\epsilon^{2i-2k-1,2i-2k-1|m}_{2k+\theta} &= m_{i,2k+\theta}|m|_{2k+\theta}, \quad i \in [k + 2; 2k + \theta].
\end{align*}
\]

(81a-c)

Eqs. (81) are the same as (60) (written in somewhat different notation). Hence one obtains the \( gl(k|1|k+\theta-1) \) signature as given in (61) and the corresponding to it highest weight \( |m|_{2k+\theta} \) (Proposition 4).

### III. IRREDUCIBLE REPRESENTATIONS OF \( gl(1|\infty) \)

Here we construct representations of \( gl_0(1|\infty) \) and \( gl(\infty|1|\infty) \), which appear as a generalization to the case \( n \to \infty \) of the results obtained in the previous section. In both cases the representations (or the corresponding modules) are labeled with infinite sequences of (in general different) complex numbers. Due to the isomorphism \( \varphi \) (see (4)) each \( gl_0(1|\infty) \) module is also a \( gl(\infty|1|\infty) \) module and vice versa. Therefore
we can also say that we describe below two classes of representations of the “abstract” Lie superalgebra $gl(1|\infty)$. For definiteness we refer to the class of representations of $gl_0(1|\infty)$ as to Gel’fand-Zetlin (GZ) representations (Sect. III.A), whereas the representations of $gl(\infty|1|\infty)$ are said to be C-representations.

A. Gel’fand-Zetlin representations

The extension of the results of Sect. II to the case $n \to \infty$ is rather evident. We collect the results in a proposition.

Proposition 6. To each sequence of complex numbers

$$[m] \equiv [m_1, m_2, \ldots, m_k, \ldots] \equiv \{m_i|m_i \in \mathbb{C}, i \in \mathbb{N}\},$$ (82)

such that

$$m_i - m_{i+1} \in \mathbb{Z}_+, \quad i = 2, 3, \ldots,$$

$$l_1 \notin \{l_2, l_2 + 1, l_2 + 2, \ldots\},$$ (83)

where

$$l_1 = m_1 + 1; \quad l_i = -m_i + i - 1, \quad i = 2, 3, \ldots,$$ (84)

there corresponds an irreducible highest weight $gl_0(1|\infty)$ module $V([m])$ with a signature (82). The basis $\Gamma([m])$ in $V([m])$, which we call a GZ basis, consists of all tables

$$[m] \equiv \begin{bmatrix} m_1 & m_2 & \ldots & m_j & \ldots & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\ m_{1j} & m_{2j} & \ldots & m_{jj} & \ldots & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\ m_{12} & m_{22} \\ m_{11} \end{bmatrix} \equiv \begin{bmatrix} [m] \\ \vdots \\ [m]_j \\ \vdots \\ [m]_2 \\ [m]_{11} \end{bmatrix},$$ (85)

characterized by an infinite number of coordinates

$$m_{ij}, \quad \forall j \in \mathbb{N}, \quad i = 1, 2, \ldots, j,$$ (86)

which are consistent with the conditions:

1. for each table $[m]$ there exists a positive (depending on $[m]$) integer $N([m]) \in \mathbb{N}$ such that

$$m_{ij} = m_i, \quad \forall j > N([m]), \quad i = 1, \ldots, j;$$ (87)

2. $m_{1i} - m_{1,i-1} \equiv \theta_{i-1} \in \{0, 1\}, \quad i = 2, 3, \ldots;$ (88)

3. $m_{i,j+1} - m_{ij} \in \mathbb{Z}_+; \quad m_{ij} - m_{i+1,j+1} \in \mathbb{Z}_+, \quad 2 \leq i \leq j \in \mathbb{N}.$ (89)

The transformation of the basis (85) is determined from the action of the Chevalley generators

$$e_{ii}[m] = \left(\sum_{k=1}^{i} m_{ki} - \sum_{k=1}^{i-1} m_{k,i-1}\right)[m], \quad i \in \mathbb{N},$$ (90)
Observation 1: Let

\[ e_{12}|m\rangle = \theta_1|m\rangle_{11}, \quad e_{21}|m\rangle = (1 - \theta_1)(l_{12} - l_{22})|m\rangle_{-(1,1)}, \]

\[ e_{i,i+1}|m\rangle = \theta_i(1 - \theta_{i-1})|m\rangle_{(1)} + \sum_{j=2}^{i} \left( -\prod_{k=2}^{i-1}(l_{k,i-1} - l_{ji} + 1) \prod_{k=2}^{i+1}(l_{k,i+1} - l_{ji}) \right)^{1/2} \frac{(l_{i,i-1} - l_{ji})(l_{i,i+1} - l_{ji} + 1)}{(l_{i+1,i})(l_{i+1,i})} |m\rangle_{(ji)}, \quad i = 2, 3, \ldots, \]

\[ e_{i+1,i}|m\rangle = \theta_{i-1}(1 - \theta_i) \prod_{k=2}^{i-1}(l_{k,i-1} - l_{k,i-1} - 1) \prod_{k=2}^{i+1}(l_{k,i+1} - l_{k,i}) |m\rangle_{-(1,i)} + \sum_{j=2}^{i} \left( -\prod_{k=2}^{i-1}(l_{k,i-1} - l_{ji}) \prod_{k=2}^{i+1}(l_{k,i+1} - l_{ji} - 1) \right)^{1/2} |m\rangle_{-(ji)}, \quad i = 2, 3, \ldots. \]

The highest weight vector \(|\hat{m}\rangle\) is the one from (85) for which

\[ m_{ij} = m_i, \quad \forall j \in \mathbb{N}, \quad i \in 1, 2, \ldots, j. \]

Proof: Let

\[ |m\rangle = \begin{bmatrix} [m] \\ \vdots \\ [m]_{N+1} \\ \vdots \\ [m]_2 \\ m_{11} \end{bmatrix} \in \Gamma([m]). \]

Then

(i) \([m]_{N+1} \equiv [m_{1,N+1}, m_{2,N+1}, \ldots, m_{N+1,N+1}], \) \(N = 1, 2, \ldots, \) is said to be the \((N+1)^{th}\)-signature of \(|m\rangle\); (ii)

\[ |m\rangle^{up}(N+1) \equiv \begin{bmatrix} [m] \\ \vdots \\ [m]_j \\ \vdots \\ [m]_{N+2} \end{bmatrix} \quad \text{and} \quad |m\rangle^{low}(N+1) \equiv \begin{bmatrix} [m]_{N+1} \\ \vdots \\ [m]_i \\ \vdots \\ [m]_2 \\ m_{11} \end{bmatrix} \]

are said to be the \((N+1)^{th}\)-upper and the \((N+1)^{th}\)-lower part of \(|m\rangle\), respectively. Consider the subalgebra

\[ gl_0(1|N) = \{ e_{ij} | i, j = 1, \ldots, N + 1 \} \subset gl_0(1|\infty). \]

Observation 1: Let \(e\) be a \(gl_0(1|N)\) generator or any polynomial of \(gl_0(1|N)\) generators. Then, for any \(|m\rangle \in \Gamma([m])\), \(e|m\rangle\) is a linear combination of vectors from \(\Gamma([m])\) with one and same \((N+1)^{th}\)-upper part \(|m\rangle^{up}(N+1)\).
Therefore the linear space
\[ V([m]_i | i \geq N+1) \subset V([m]) \]
be the linear span of \( \Gamma([m]_i | i \geq N+1) \). From (90)-(93) it follows that \( V([m]_i | i \geq N+1) \) is invariant with respect to \( gl_0(1|N) \). To each vector \( |m\rangle \in \Gamma([m]_i | i \geq N+1) \) put in correspondence its \((N+1)\)th—lower part:
\[
f(|m\rangle) = |m\rangle^{low(N+1)}, \quad \forall \ |m\rangle \in \Gamma([m]_i | i \geq N+1).
\]
(100)

Let
\[
\Gamma([m]_{N+1}) = \{ f(|m\rangle) | \ |m\rangle \in \Gamma([m]_i | i \geq N+1) \}. 
\]
(101)
Then \( f \) maps bijectively \( \Gamma([m]_i | i \geq N+1) \) on \( \Gamma([m]_{N+1}) \). Obviously \( \Gamma([m]_{N+1}) \) consists of all GZ tables of an essentially typical \( gl_0(1|N) \) module with a signature \([m]_{N+1}\). Define an action of \( gl_0(1|N) \) on \( |m\rangle \in \Gamma([m]_{N+1}) \) with the relations (33)-(36). Then the linear envelope \( V([m]_{N+1}) \) of \( \Gamma([m]_{N+1}) \) is an essentially typical \( gl_0(1|N) \) module with a signature \([m]_{N+1}\). After comparing the relations (90)-(93) with (33)-(36) and having in mind Observation 1 we have:

**Observation 2.** The subspace \( V([m]_i | i \geq N+1) \subset V([m]) \) is an essentially typical finite-dimensional \( gl_0(1|N) \) module with a signature \([m]_{N+1}\) and a GZ basis \( \Gamma([m]_i | i \geq N+1) \).

Let \( e_{ij}, e_{kl} \) be any two generators from \( gl_0(1|\infty) \) and \( |m\rangle \) be an arbitrary vector from \( \Gamma([m]) \). Consider \( e_{ij}, e_{kl} \) as elements from \( gl_0(1|N) \subset gl_0(1|\infty) \), where \( N+1 \geq \max(i,j,k,l) \). Then \( |m\rangle \) is a vector from the \( gl_0(1|N) \) fidirmod \( V([m]_i | i \geq N+1) \subset V([m]) \) and therefore (Observation 2)
\[
(e_{ij}e_{kl} - (-1)^{deg(e_{ij})deg(e_{kl})}e_{kl}e_{ij})|m\rangle = (\delta_{jk}e_{il} - (-1)^{deg(e_{il})deg(e_{kl})}\delta_{ki}e_{kj})|m\rangle.
\]
(102)
Therefore the linear space \( V([m]) \) is a \( gl_0(1|\infty) \) module.

Consider any two vectors \( x, y \in V([m]) \),
\[
x = \sum_{i=1}^{p} \alpha_i |m^i\rangle, \quad y = \sum_{i=p+1}^{q} \alpha_i |m^i\rangle, \quad |m^i\rangle \in \Gamma([m]),
\]
\[
\alpha_i \in \mathbb{C}, \quad i = 1, \ldots, q.
\]
(103)
Let
\[
\tilde{N} = \max\{N|[m^i]|i = 1, \ldots, q\}.
\]
(104)
According to (87) all vectors \( |m^i\rangle \), \( i = 1, \ldots, q \), have one and the same \( k-1 \) signatures, for every \( k-1 \geq \tilde{N} \). Therefore \( |m^i\rangle \in V([m^k-1]|k-1 \geq \tilde{N}) \subset V([m]) \). Hence \( x, y \in V([m^k-1]|k-1 \geq \tilde{N}) \). The space \( V([m^k-1]|k-1 \geq \tilde{N}) \) is a \( gl_0(1|\tilde{N}) \) fidirmod (Observation 2) and, therefore, there exist a polynomial \( P \) of the \( gl_0(1|\tilde{N}) \) generators such that \( y = Px \). Hence \( V([m]) \) is an irreducible \( gl_0(1|\infty) \) module.

Consider the vector \( \hat{m} \in \Gamma([m]) \) [see (91)]. From Eqs. (90)-(93) we have
\[
e_{ii}|\hat{m}\rangle = m_i|\hat{m}\rangle, \quad \forall i \in \mathbb{N},
\]
(105)
and
\[ e_{k,k+1}|\hat{m}\rangle = 0, \quad \forall k \in \mathbb{N}. \] (106)

Therefore the irreducible \( gl_0(1|\infty) \) module \( V([m]) \) is a highest weight module with a signature
\[ [m] \equiv [m_1, m_2, \ldots, m_k, \ldots] \] (107)

and a highest weight vector \(|\hat{m}\rangle\). This completes the proof.

**B. C-representations**

Most of the preliminary work for constructing the representations of \( gl(\infty|1|\infty) \) was done in Sect. II.B. It remains to give a precise definition of the \( C^- \) basis in the infinite-dimensional case and to write down the transformation of the basis under the action of the Chevalley generators.

Let
\[ [M] \equiv [\ldots, M_{-p}, \ldots, M_{-1}, M_0, M_1, M_2, \ldots] \equiv \{M_i\}_{i \in \mathbb{Z}} \] (108)

be a sequence of complex numbers such that
\[ M_i - M_{i+1} \in \mathbb{Z}_{+}, \quad i \in [-\infty; -2] \cup [1; \infty], \] (109a)
\[ M_{-1} - M_1 \in \mathbb{N}, \] (109b)
\[ M_0 + M_1 \notin \mathbb{Z}. \] (109c)

Here and throughout
\[ [-\infty; a] = \{a, a-1, a-2, \ldots, a-i, \ldots\} \equiv \{a-i\}_{i \in \mathbb{Z}_{+}}, \] (110)
\[ [b; \infty] = \{b, b+1, b+2, \ldots, b+i, \ldots\} \equiv \{b+i\}_{i \in \mathbb{Z}_{+}}, \] (111)

A table \([M]\), consisting of infinitely many complex numbers
\[ M_{i,2k+\theta-1}, \quad \forall k \in \mathbb{N}, \quad \theta \in \{0, 1\}, \quad i = [-k-\theta+1; k-1], \] (112)

will be called a \( C^- \) table, provided the following conditions hold:

(1) There exists a positive, depending on \([M]\), integer \(N([M])\) such that
\[ M_{i,2k+\theta-1} = M_i, \quad \forall k > N([M]), \quad \theta \in \{0, 1\}, \quad i \in [1-\theta-k,k-1]; \] (113)

(2) The coordinates \( M_{i,2k+\theta-1}, \quad \theta \in \{0, 1\}, \) take all possible values
\[ M_{i,2k+1-2\theta} - M_{i,2k} \in \mathbb{Z}_{+}, \quad k \in [1+\theta; \infty], \quad i \in [-k+\theta;-1] \cup [1; k-1], \] (114a)
\[ M_{i-1,2k} - M_{i,2k+1-2\theta} \in \mathbb{Z}_{+}, \quad k \in [1+\theta; \infty], \quad i \in [-k+1;-1] \cup [2; k-\theta], \] (114b)
\[ M_{-1,2k} - M_{i,2k+1-2\theta} \in \mathbb{N}, \quad k \in [1+\theta; \infty], \] (114c)
\[ M_{0,2k+1-\theta} - M_{0,2k-\theta} \equiv \psi_{2k-\theta} \in \{0,1-2\theta\}, \quad k \in [1; \infty]. \] (114d)
We are ready now to state our main and final result.

Proposition 7. To each sequence (108) (see also (109)) there corresponds an irreducible highest weight \( gl(\infty|1|\infty) \) module \( V([M]) \) with a signature \([M]\). The basis \( \Gamma([M]) \) in \( V([M]) \) consists of all \( C \)-tables (115). The transformations of the basis under the action of the \( gl(\infty|1|\infty) \) Chevalley generators read:

\[
E_{kk}|M\rangle = \left( \frac{|k| + \theta(k) - 1}{|k|} \sum_{i=-|k|}^{k} M_{i,k+\theta(k)} - \frac{|k|-1}{|k|+1-\theta(k)} \sum_{i=-|k|+1-\theta(k)}^{k} M_{i,k+\theta(k)} \right)|M\rangle, \quad k \in \mathbb{Z},
\]

(116)

\[
E_{0,-1}|M\rangle = (1 + \psi_1)|M\rangle_{(01)}
\]

(117)

\[
E_{-1,0}|M\rangle = -\psi_1(L_{0,2} - L_{-1,2})|M\rangle_{(-01)}
\]

(118)

\[
E_{01}|M\rangle = -\psi_2(1 + 2\psi_1)|M\rangle_{(02)}
\]

\[
+ (1 + \psi_1)(-L_{-1,3} - L_{-1,2})(L_{13} - L_{-1,2}))^{1/2} \frac{(L_{02} - L_{-1,2})(L_{02} - L_{-1,2} + 1)}{(L_{03} - L_{-1,2})(L_{01} - L_{-1,2})(L_{01} - L_{-1,2} + 1)}|M\rangle_{(-1,2)}^{(01)}
\]

(119)

\[
E_{10}|M\rangle = -(-1)^{\psi_1}(1 - \psi_2) \frac{(L_{02} - L_{-1,2} - \psi_1 - 1)(L_{03} - L_{-1,3})(L_{03} - L_{-1,2})}{(L_{03} - L_{-1,2} - 1)(L_{03} - L_{-1,2})}|M\rangle_{(-02)}^{(01)}
\]

\[
- \psi_1(-L_{-1,3} - L_{-1,2} - 1)(L_{13} - L_{-1,2} - 1))^{1/2} |M\rangle_{(-1,2)}^{(-01)}
\]

(120)

\[
E_{k,k+1}|M\rangle = -\psi_{2k+2}(1 - \psi_{2k})(1 + 2\psi_{2k+1})|M\rangle_{(02k+2)}^{(0,2k+1)}
\]

\[
+ \sum_{j \neq 0}^{k} \psi_{2k+2}\psi_{2k+1} \left( \prod_{i=0}^{k} (L_{i,2k} - L_{j,2k+1} + 1) \prod_{i=0}^{k} (L_{i,2k+2} - L_{j,2k+1} + 1) \right)^{1/2} \frac{(L_{02k+1} - L_{j,2k+1})(L_{02k+1} - L_{j,2k+1} + 1)}{(L_{02k+2} - L_{j,2k+1} + 2)(L_{02k+2} - L_{j,2k+1} + 1)(L_{02k} - L_{j,2k+1} + 1)}|M\rangle_{(0,2k+2)}^{(j,2k+1)}
\]

\[
+ \sum_{j \neq 0}^{k} (1 + \psi_{2k+1})(1 - \psi_{2k}) \left( \prod_{i=0}^{k} (L_{i,2k+1} - L_{j,2k+2} + 1) \prod_{i=0}^{k} (L_{i,2k+3} - L_{j,2k+2} + 1) \right)^{1/2} \frac{(L_{02k+1} - L_{j,2k+1})(L_{02k+1} - L_{j,2k+1} + 1)}{(L_{02k+2} - L_{j,2k+1} + 2)(L_{02k+2} - L_{j,2k+1} + 1)(L_{02k} - L_{j,2k+1} + 1)}|M\rangle_{(0,2k+2)}^{(j,2k+1)}
\]

(120)
$$\times \frac{(L_{0,2k+2} - L_{j,2k+2})(L_{0,2k+2} - L_{j,2k+2} + 1)}{(L_{0,2k+3} - L_{j,2k+3})(L_{0,2k+1} - L_{j,2k+2})(L_{0,2k+1} - L_{j,2k+2} + 1)} |M|^{(0,2k+1)}_{(j,2k+2)}$$

$$+ \sum_{i \neq 0}^{k} \sum_{j \neq 0}^{k} Q(j, l) \left( - \prod_{i \neq 0}^{k} \prod_{l \neq 0}^{k}(L_{i,2k+2} - L_{l,2k+2})(L_{i,2k+2} + 1) \right)^{1/2}$$

$$\times \left( \prod_{i \neq 0}^{k} \prod_{l \neq 0}^{k} (L_{i,2k+2} - L_{l,2k+2} + 1) \left( L_{i,2k+1} - L_{l,2k+1} + 1 \right) \right)^{1/2}$$

$$\times \left( L_{0,2k+2} - L_{i,2k+2} \right) \left( L_{0,2k+3} - L_{i,2k+3} \right) \left( L_{0,2k+1} - L_{i,2k+2} \right) \left( L_{0,2k+1} - L_{i,2k+2} + 1 \right) \left( L_{0,2k+1} - L_{i,2k+1} + 1 \right) \left( L_{0,2k} - L_{i,2k+1} + 1 \right) |M|^{(j,2k+1)}_{(i,2k+2)},$$

$$k \in [1, \infty], \quad (121)$$

$$E_{-k+1, -k} |M| = -(1 + 2\psi_{2k-1}) \left( 1 - 2\psi_{2k-2} \right) |M|^{(0,2k-2)}_{(0,2k-1)}$$

$$- \sum_{j \neq 0}^{k-2} \left( 1 + 2\psi_{2k-1} \right) \left( 1 - 2\psi_{2k-2} \right) \left( \prod_{i \neq 0}^{k-2} \prod_{l \neq 0}^{k-2} \left( L_{i,2k-3} - L_{j,2k-2} \right) \left( L_{i,2k-2} - L_{j,2k-2} + 1 \right) \right)^{1/2}$$

$$\times \left( L_{0,2k-2} - L_{i,2k-2} \right) \left( L_{0,2k-1} - L_{i,2k-1} + 1 \right) \left( L_{0,2k-1} - L_{i,2k-1} + 1 \right) \left( L_{0,2k-1} - L_{i,2k-1} + 1 \right) \left( L_{0,2k-1} - L_{i,2k-1} + 1 \right) |M|^{(j,2k-2)}_{(0,2k-1)}$$

$$- \sum_{j \neq 0}^{k-1} \psi_{2k-3} \psi_{2k-3} \left( - \prod_{i \neq 0}^{k-2} \prod_{l \neq 0}^{k-2} \left( L_{i,2k-2} - L_{j,2k-1} + 1 \right) \left( L_{i,2k-1} - L_{j,2k-1} + 1 \right) \right)^{1/2}$$

$$\times \left( L_{0,2k-1} - L_{j,2k-1} \right) \left( L_{0,2k-1} - L_{j,2k-1} + 1 \right) \left( L_{0,2k-1} - L_{j,2k-1} + 1 \right) \left( L_{0,2k-1} - L_{j,2k-1} + 1 \right) |M|^{(0,2k-2)}_{(j,2k-1)}$$

$$+ \sum_{i \neq 0}^{k-1} \sum_{j \neq 0}^{k-2} P(j, l) \left( - \prod_{i \neq 0}^{k-2} \prod_{l \neq 0}^{k-2} \left( L_{i,2k-2} - L_{j,2k-1} + 1 \right) \left( L_{i,2k-1} - L_{j,2k-1} + 1 \right) \right)^{1/2}$$

$$\times \left( \prod_{i \neq 0}^{k-2} \prod_{j \neq 0}^{k-2} \left( L_{i,2k-3} - L_{j,2k-2} \right) \left( L_{i,2k-2} - L_{j,2k-2} + 1 \right) \right)^{1/2}$$

$$\times \left( L_{0,2k-1} - L_{j,2k-1} \right) \left( L_{0,2k-1} - L_{j,2k-1} + 1 \right) \left( L_{0,2k-1} - L_{j,2k-1} + 1 \right) \left( L_{0,2k-1} - L_{j,2k-1} + 1 \right) \left( L_{0,2k-1} - L_{j,2k-1} + 1 \right) |M|^{(j,2k-1)}_{(l,2k-1)},$$

$$k \in [2, \infty], \quad (122)$$

$$E_{k+1, k} |M| = -(1 + 2\psi_{2k+1}) \psi_{2k} \left( 1 - 2\psi_{2k+2} \right)$$

$$\times \left( \prod_{i \neq 0}^{k} \left( L_{i,2k+2} - L_{i,2k+1} - 1 \right) \prod_{i \neq 0}^{k} \left( L_{i,2k+2} - L_{i,2k+1} - \psi_{2k+1} \right) \right)^{1/2}$$

$$\times \left( \prod_{i \neq 0}^{k} \prod_{j \neq 0}^{k} \left( L_{i,2k+3} - L_{j,2k+3} \right) \left( L_{i,2k+3} - L_{j,2k+2} + 1 \right) \right)^{1/2}$$

$$k \in [2, \infty], \quad (123)$$
\[- \sum_{j \neq 0 = -k}^{k} (1 + \psi_{2k+1})(1 - \psi_{2k+2}) \left( \prod_{i \neq 0 = -k}^{k-1} (L_{i,2k} - L_{j,2k+1}) \prod_{j \neq 0 = -k}^{k} (L_{i,2k+2} - L_{j,2k+1}) \right)^{1/2} \]

\[ \times \prod_{i \neq 0, j = -k}^{k} (L_{0,2k+3} - L_{i,2k+3}) \prod_{j \neq 0, i = -k}^{k+1} (L_{0,2k+3} - L_{i,2k+3}) \prod_{j \neq 0, i = -k}^{k+1} (L_{0,2k+3} - L_{i,2k+1} - 1)(L_{i,2k+1} - L_{j,2k+1}) \]

\[- \sum_{j \neq 0 = -k - 1}^{k} \psi_{2k+1} \psi_{2k+1} \left( - \prod_{i \neq 0, j = -k}^{k} (L_{i,2k+1} - L_{j,2k+2} - 1) \prod_{j \neq 0, i = -k}^{k+1} (L_{i,2k+3} - L_{j,2k+2} - 1)(L_{j,2k+2} - L_{j,2k+2}) \right)^{1/2} \]

\[ \times \prod_{i \neq 0, j = -k}^{k} (L_{0,2k+2} - L_{i,2k+2}) \prod_{j \neq 0, i = -k}^{k-1} (L_{0,2k+2} - L_{i,2k}) \prod_{j \neq 0, i = -k}^{k-1} (L_{0,2k+2} - L_{i,2k+1} + 1) \]

\[ + \sum_{j \neq 0 = -k - 1}^{k} \sum_{i \neq 0 = -k}^{k} Q(j, l) \left( - \prod_{i \neq 0, j = -k}^{k} (L_{i,2k+1} - L_{i,2k} - 1) \prod_{j \neq 0, i = -k}^{k+1} (L_{i,2k+3} - L_{i,2k+2} - 1)(L_{i,2k+2} - L_{i,2k+2}) \right)^{1/2} \]

\[ \times \left( \prod_{i \neq 0, j = -k}^{k} (L_{i,2k} - L_{j,2k+1}) \prod_{j \neq 0, i = -k}^{k} (L_{i,2k+2} - L_{j,2k+1}) \right)^{1/2} |M|_{- (j,2k+2)} \]

\[ E_{-k, -k+1} |M| = - (1 + \psi_{2k-2}) \psi_{2k-1} \]

\[ \times \prod_{i \neq 0 = -k}^{k} (L_{0,2k+1} - L_{i,2k+1} - \psi_{2k-2}) \prod_{i \neq 0 = -k}^{k} (L_{0,2k} - L_{i,2k} - \psi_{2k-2}) \]

\[ \times \prod_{i \neq 0 = -k+1}^{k-1} (L_{0,2k} - L_{i,2k-2})(L_{0,2k} - L_{i,2k}) \prod_{i \neq 0 = -k+1}^{k-1} (L_{0,2k} - L_{i,2k-1})(L_{0,2k} - L_{i,2k-1} + 1) \]

\[ + \sum_{j \neq 0 = -k + 1}^{k} \psi_{2k-2} \psi_{2k-1} \left( - \prod_{i \neq 0, j = -k + 1}^{k} (L_{i,2k-3} - L_{j,2k-2} - 1) \prod_{j \neq 0, i = -k + 1}^{k-1} (L_{i,2k-1} - L_{j,2k-2} - 1)(L_{j,2k-2} - L_{j,2k-2}) \right)^{1/2} \]

\[ \times \prod_{i \neq 0 = -k + 1}^{k} (L_{0,2k} - L_{i,2k-2}) \prod_{j \neq 0, i = -k + 1}^{k} (L_{0,2k} - L_{i,2k})(L_{0,2k} - L_{i,2k-1} + 1) \]

\[ + \sum_{j \neq 0 = -k + 1}^{k} (1 + \psi_{2k-3})(1 - \psi_{2k-2}) \left( - \prod_{i \neq 0, j = -k + 1}^{k} (L_{i,2k-2} - L_{j,2k-1}) \prod_{j \neq 0, i = -k + 1}^{k-1} (L_{i,2k-1} - L_{j,2k-1}) \right)^{1/2} \]

\[ \times \prod_{i \neq 0 = -k + 1}^{k} (L_{0,2k-1} - L_{i,2k-1}) \prod_{j \neq 0, i = -k + 2}^{k} (L_{0,2k-1} - L_{i,2k-3}) \prod_{j \neq 0, i = -k + 1}^{k} (L_{0,2k-1} - L_{i,2k-1} - 1)(L_{i,2k-1} - L_{j,2k-1}) \]

\[ + \sum_{j \neq 0 = -k + 1}^{k} \sum_{i \neq 0 = -k + 1}^{k-2} P(j, l) \left( - \prod_{i \neq 0, j = -k + 1}^{k} (L_{i,2k-2} - L_{j,2k-1}) \prod_{j \neq 0, i = -k + 1}^{k-1} (L_{i,2k-1} - L_{j,2k-1} - 1)(L_{j,2k-1} - L_{j,2k-1}) \right)^{1/2} \]

\[ \times \left( \prod_{i \neq 0, j = -k + 1}^{k} (L_{i,2k-3} - L_{j,2k-2} - 1) \prod_{j \neq 0, i = -k + 1}^{k} (L_{i,2k-1} - L_{j,2k-2} - 1)(L_{j,2k-2} - L_{j,2k-2}) \right)^{1/2} |M|_{- (j,2k-2)} \]

\[ k \in [2, \infty]. \]
The above transformation relations (116)-(124) were derived first for $gl(n|1|n)$ from (68)-(74) and the supercommutation relations. Therefore they give a representation of $gl(n|1|n)$ for any $n$. An essential requirement, when passing to $n \to \infty$, is given with the condition (113). It is straightforward to check that $V([M])$ is invariant under the action of the generators. The rest of the proof, which we skip, is rather similar to that of Proposition 6, although technically it is more involved.

IV. CONCLUDING REMARKS

We have constructed two classes of highest weight irreps of the infinite-dimensional Lie superalgebra $gl(1|\infty)$. It should be noted that the GZ representations are inequivalent to the $C$-representations. More than that: the $C$-representations, being highest weight irreps of $gl(\infty|1|\infty)$, are not highest weight representations of $gl_0(1|\infty)$ and vice versa. Indeed, assume that the $gl_0(1|\infty)$ module $V([m])$ is also a highest weight $gl(\infty|1|\infty)$ module with a highest weight vector $y$. Then $y$ has to be a highest weight vector of any of the subalgebras $gl(k|1|k-1+\theta)$. Hence Eqs. (54) and (55) have to hold for any $\theta = 0, 1, k \in [1-\theta; \infty]$. Therefore $y \notin V([m])$ (see (87)).

Our primary interest in the present investigation is related to its eventual applications in a generalization of the statistics in quantum field theory. From this point of view our results are however very preliminary. The first observation in this respect is that the algebra (for definiteness) $gl(\infty|1|\infty)$ is not large enough. It does not contain important physical observables (like the energy-momentum of the field $P^m$, see (8)), which are infinite linear combinations of the generators of $gl(\infty|1|\infty)$. In order to incorporate them one has to go to the completed central extension $a(\infty|1|\infty)$ of $gl(\infty|1|\infty)$ in a way similar as for the Lie algebra $gl_{\infty}$ or the Lie superalgebra $gl_{\infty|\infty}$. This is only the first step. The next one will be to determine those $gl(\infty|1|\infty)$ modules $V([M])$, which can be extended to $a(\infty|1|\infty)$ modules.

The most important and perhaps the most difficult step will be to express the transformations of the $gl(\infty|1|\infty)$ modules in terms of natural for the QFT variables, namely via the creation and the annihilation operators $a_i^{\pm}$ of $gl(\infty|1|\infty)$, which are just its odd generators. This is however not simple and, may be, even not necessary in the general context of the representation theory. The physical state spaces, the Fock spaces, have to satisfy several additional physical requirements. In particular any such space has to be generated from the vacuum (the highest weight vector) by polynomials of the creation operators, which are only a part of the negative root vectors. This imposes considerable restriction on the physically admissible modules. Hence in the applications one has to select first the Fock spaces from all $gl(\infty|1|\infty)$ modules and then study their transformation properties under the action of the physically relevant operators, in particular of the CAOs.

An additional problem is related to the circumstance that in QFT the indices of the CAOs are not elements form a countable set. Therefore as a test model one can try to consider first the $gl(\infty|1|\infty)$ statistics in the frame of a lattice quantum filed theory or locking the field in a finite volume.
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1 T.D. Palev, J. Math. Phys. 23, 1778 (1982).
2 T.D. Palev, Czech. Journ. Phys. B32, 680 (1982).
3 T.D. Palev, Czech. Journ. Phys. B29, 91 (1979).
4 T.D. Palev, A-superquantization, Communication JINR E2-11942 (1978).
5 A.Ch. Ganchev and T.D. Palev, J. Math. Phys. 21, 797 (1980).
6 H.S. Green, Phys. Rev. 90, 270 (1953).
7 N.N. Bogoljubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields, Moscow 1957 (English ed. Interscience Publishers, Inc., New York, 1959)
8 E. Celeghini, T.D. Palev, and M. Tarlini, Mod. Phys. Lett. B5, 187 (1991).
9 T.D. Palev, J. Phys. A: Math. Gen. 26, L1111 (1993) and hep-th/9306013.
10 L.K. Hadjiivanov, J. Math. Phys. 34, 5476 (1993).
11 T.D. Palev and J. Van der Jeugt, J. Phys. A : Math. Gen. 28, 2605 (1995) and q-alg/9501020.
12 T.D. Palev, Commun. Math. Phys. 196, 429 (1998) and q-alg/9709003.
13 V.G. Kac, Lecture Notes in Math. 676, 597 (Springer, 1979).
14 S. Okubo, J. Math. Phys. 35, 2785 (1994).
15 J. Van der Jeugt in New Trends in Quantum Field Theory (Heron Press, Sofia, 1996).
16 S. Meljanac, M. Milekovic and M. Stojic, On parastatistics defines as triple operator algebras q-alg/9712017.
17 T.D. Palev and N.I. Stoilova, J. Math. Phys. 38, 2506 (1997) and hep-th/9606011.
18 F.D.M. Haldane, Phys. Rev. Lett. 67, 937 (1991).
19 T.D. Palev and N.I. Stoilova, J. Phys. A : Math. Gen. 27, 977, 7387 (1994)
   and hep-th/9307102, hep-th/9405125.
20 V.G. Kac and J.W. van der Leur, Ann. Inst. Fourier, Grenoble 37, #4, 99 (1987).
21 V.G. Kac and J.W. van der Leur, Advanced Series in Math. Phys. 7, 369 (1988).
22 T.D. Palev, J. Math. Phys. 22, 2127 (1981).
23 J. Van der Jeugt, J.W.B. Hughes, R.C. King, J. Thierry-Mieg, Commun. Algebra 18, 3453 (1990).
24 H. Schlosser, Seminar Sophus Lie 3, 15 (1993)
25 H. Schlosser, Beiträge zur Algebra and Geometry 31, 193 (1994)
26 T.D. Palev, Funkt. Anal. Prilozh. 21, N 3, 85 (1987); Funct. Anal. Appl. 21, 245 (1987)
   (English translation).
27 T.D. Palev, J. Math. Phys. 30, 1433 (1989).
T.D. Palev and V.N. Tolstoy, Comm. Math. Phys. 141, 549 (1991).

J. Van der Jeugt, J.W.B. Hughes, R.C. King, J. Thierry-Mieg, J. Math. Phys. 31, 2278 (1990).

J.W.B. Hughes, R.C. King, J. Van der Jeugt, J. Math. Phys. 33, 470 (1992).

T.D. Palev, Funkt. Anal. Prilozh. 23, N 2, 69 (1989); Funct. Anal. Appl. 23, 141 (1989) (English translation).

J. Van der Jeugt, J. Math. Phys. 36, 605 (1995).

T.D. Palev, N.I. Stoilova and J. Van der Jeugt, Comm. Math. Phys. 166, 367 (1994).

T.D. Palev, Funkt. Anal. Prilozh. 24, N 1, 69 (1990); Funct. Anal. Appl. 24, 72 (1990) (English translation).

T.D. Palev, J. Math. Phys. 31, 579 (1990) and 31, 1078 (1990).

V.V. Serganova, Math. USSR Izv. 24, 359 (1985).

D.A. Leites, M.V. Saveliev and V.V. Serganova, Serpukhof preprint 85-81 (1985).

J.W. Van der Leur, Cotragradient Lie superalgebras of finite growth, Utrecht thesis (1985).

I. Penkov and V. Serganova, Indag. Math. 3, 419 (1992).

V.G. Kac and M. Wakimoto, Progress in Math. 123, 415 (1994).

V.G. Kac and V.G. Peterson, Proc. Natl. Acad. Sci. USA 78, 3308 (1981).

T.D. Palev, J. Math. Phys. 21, 1293 (1980).