EINSTEIN-SEPARABILITY, TIME RELATED HIDDEN PARAMETERS FOR CORRELATED SPINS, AND THE THEOREM OF BELL

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Abstract

We analyze the assumptions that are made in the proofs of Bell-type inequalities for the results of Einstein-Podolsky-Rosen type of experiments. We find that the introduction of time-like random variables permits the construction of a broader mathematical model which accounts for all correlations of variables that are contained in the time dependent parameter set of the backward light cone. It also permits to obtain the quantum result for the spin pair correlation, a result that contradicts Bell’s inequality. Two key features of our mathematical model are (i) the introduction of time operators that are indexed by the measurement settings and appear in addition to Bell’s source parameters and (ii) the related introduction of a probability measure for all parameters that does depend on the analyzer settings. Using the theory of B-splines, we then show that this probability measure can be constructed as a linear combination of setting dependent subspace product measures and that the construction guarantees Einstein-separability.

1 INTRODUCTION

We address the question whether the quantum result for the spin pair correlation in Einstein-Podolsky-Rosen (EPR)-type experiments can be obtained by a hidden parameter theory and show that it can in spite of the serious objections given in the work of Bell [1].

The work of Bell [1] attempts to show that a mathematical description of EPR-type experiments [2,3] by a statistical (hidden) parameter theory [4] is not possible. In EPR experiments, two particles having their spins in a singlet state are emitted from a source and are sent to spin analyzers at two spatially separated stations, $S_1$ and $S_2$. The spin
analyzers are described by Bell using unit vectors \( \mathbf{a}, \mathbf{b}, \text{etc.} \) of three dimensional Euclidean space and functions \( A = \pm 1 \) (operating at station \( S_1 \)) and \( B = \pm 1 \) (operating at station \( S_2 \)): furthermore \( A \) does not depend on the settings \( \mathbf{b} \) of station \( S_2 \), nor \( B \) on the settings \( \mathbf{a} \) of station \( S_1 \) (Einstein separability or locality). Bell permits particles emitted from the source to carry arbitrary hidden parameters \( \lambda \) of a set \( \Omega \) that fully characterize the spins and are ”attached” to the particles with a probability density \( \rho \) (we denote the corresponding probability measure by \( \mu \)) that does not depend on the settings at the stations. Bell then assumes that the values of the functions \( A \) and \( B \) are determined by the spin analyzer settings and parameters such that:

\[
A = A(\mathbf{a}, \lambda) = A_\mathbf{a}(\lambda) = \pm 1
\]

and

\[
B = B(\mathbf{b}, \lambda) = B_\mathbf{b}(\lambda) = \pm 1
\]

Thus \( A_\mathbf{a}(\lambda) \) and \( B_\mathbf{b}(\lambda) \) can be considered as stochastic processes on \( \Omega \), indexed by the unit vectors \( \mathbf{a} \) and \( \mathbf{b} \) respectively. Quantum theory and experiments show that, for a given time of measurement for which the settings are equal in both stations, we have for singlet state spins

\[
A_\mathbf{a}(\lambda) = -B_\mathbf{a}(\lambda)
\]

with probability one. He further defines the spin pair expectation value \( P(\mathbf{a}, \mathbf{b}) \) by

\[
P(\mathbf{a}, \mathbf{b}) = \int_\Omega A_\mathbf{a}(\lambda)B_\mathbf{b}(\lambda)\rho(\lambda)d\lambda = -\int_\Omega A_\mathbf{a}(\lambda)A_\mathbf{b}(\lambda)\mu(d\lambda)
\]

From Eqs.(1)-(4), Bell derives his celebrated inequality [1]

\[
1 + P(\mathbf{b}, \mathbf{c}) \geq | P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c}) |
\]

and observes that this inequality is in contradiction with the result of Quantum Mechanics:

\[
P(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}
\]

The proof of Bell’s inequality is based on the obvious fact that for \( x, y, z = \pm 1 \) we have

\[
|xz - yz| = |x - y| = 1 - xy
\]

Substituting \( x = A_\mathbf{b}(\lambda), y = A_\mathbf{c}(\lambda), z = A_\mathbf{a}(\lambda) \) and integrating with respect to the measure \( \mu \) one obtains Eq.(3) in view of Eq.(4).

In subsequent work by Clauser-Holt-Horne-Shimony (CHHS) [5] Eqs.(1)-(3) were relaxed to permit

\[
|A_\mathbf{a}| \leq 1, \ |B_\mathbf{b}| \leq 1
\]
They also introduced inequalities that do not deal with equal settings and are of the form

$$|A_a(\lambda)B_b(\lambda) + A_d(\lambda)B_b(\lambda) + A_a(\lambda)B_c(\lambda) - A_d(\lambda)B_c(\lambda)| \leq 2$$ (9)

This inequality is based on the observation that for $max(|x|, |y|, |u|, |v|) \leq 1$

$$|u||x + y| + |v||x - y| \leq |x + y| + |x - y| \leq 2$$ (10)

Thus, from the vantage point of mathematics, the Bell and CHHS inequalities are straightforward consequences of the respective set of hypotheses and assumptions that are imposed.

We present an analysis of these hypotheses and assumptions with the intent to form a new basis for a mathematical model of EPR-experiments. The starting point of our analysis is an examination of the role of time in the characterization of the set of measured data that are represented by the functions $A_a$ and $B_b$ in Eqs.(1) and (2). In a complete EPR-experiment the settings $(a, b)$ are randomly changed. However, in order to evaluate $P(a, b)$ for the purpose of checking the Bell inequality, three settings $a, b, c$ will have to be selected and considered as being fixed and the times at which the measurements are taken for the three relevant pairs $(a, b)$, $(a, c)$ and $(b, c)$ will now have to be considered as being random. Thus, if a time dependence exists, it is certainly reasonable to allow for an additional stochastic parameter $\omega$ related to time which drives the random processes taking place in the two stations. This parameter is not correlated to $\lambda$ which can, for example, derive its randomness from current and voltage fluctuations at the source.

The essence of our approach is the introduction of setting and station specific time-like parameters as well as time related setting dependent parameters $\lambda^*_a$ on one side and $\lambda^*_b$ on the other, that co-determine the functions $A, B$ in addition to the correlated source parameters $\lambda$ (all to be defined more precisely in section 2). We show in detail in the next sections that time-like parameters cannot be fully covered by Bell’s nor any of the other proofs mentioned above. We also show that these parameters lead in a natural way to setting dependent probability measures for the parameters without spooky action at a distance. This was not considered a possibility for three major reasons. First, as discussed below, a single product measure $\mu = \mu_a \times \mu_b$ (where $\mu_a$ only depends on $a$ and $\mu_b$ only depends on $b$ which would guarantee Einstein-separability of the stations) cannot lead to the quantum result of Eq.(6). Second, stochastic parameters from the stations (although investigated [6]) were not included as integration variables in the probability measure because it seemed impossible to reconcile the fact

$$A_a(\text{parameters}) = -B_a(\text{parameters})$$ (11)

with that of setting dependent statistical parameters in the spatially separated stations. The main reasoning was that the settings are changed rapidly in between the measurements and the parameters $\lambda$ can therefore carry no information on the actual settings at the time period of measurement. Which information could then possibly lead to Eq.(11) without invoking spooky action at a distance between the stations? We show that time-like parameters (derived from a global clock time for both stations) can provide this information. The third
reason is the widely held belief that if $A$ depends on $a$, $\lambda$ and $\lambda^*_a$, then by enlarging the parameter space one can rewrite $A$ as a function of $a$ and $\lambda$ with an enlarged set of parameters $\lambda$; and similarly for $B$. We will give a counterexample to this third point in appendix 1.

While all of the above will be extensively discussed, we would like to emphasize already at this point that Bell has introduced a number of assumptions on time dependencies (the use of Eqs.(3) without regard to the time-disjoint measurements) and a significant asymmetry in describing the spin properties of the particles and the properties of the measurement equipment. The spins are described by arbitrarily large sets $\Lambda$ of parameters, while the measurement apparatus is described by a vector of Euclidean space (the settings). This is true to Bohr’s [8] postulate that the measurement must be classical. Yet, the measurement apparatus must itself in some form contain particles with spins that then, if one wants to be self-consistent, also need to be described by large sets of parameters that are related to the setting $a, b, c, ...$

It is the purpose of this paper to show that a properly chosen sum of what we call setting dependent subspace product measures (SDSPM) does not violate Einstein-separability and does lead to the quantum result of Eq.(6) while still always fulfilling Eq.(3). By this we mean the following. The probability space $\Omega$ is partitioned into a finite number $M$ of subspaces $\Omega_m$

$$\Omega = \bigcup_{m=1}^{M} \Omega_m$$

A product measure $(\mu_a \times \mu_b)_m$ is defined on each subspace $\Omega_m$. This measure can be extended to the entire space $\Omega$ by setting

$$(\mu_a \times \mu_b)_m(\Omega_j) = 0 \text{ if } j \neq m$$

which we denote by the acronym SDSPM. The final measure $\mu$ is then defined on the entire space $\Omega$ by

$$\mu = \sum_{m=1}^{M} (\mu_a \times \mu_b)_m$$

As a preview we remark that the choice of subspace is determined by time and setting dependent operators that symbolize the laws of physics. In this way we introduce time-like setting dependent parameters that are in no way suspect of spooky action at a distance. At the same time the subspace measure becomes setting dependent through these time-like parameters in a natural way. This is, of course, the key element to our approach and is therefore discussed in great detail in the bulk of the paper, particularly in sections 2.1 and 3.1.

We will show in the remainder of this paper that Bell’s assumptions are too restrictive and need to be relaxed. Moreover, that this relaxation leads to a natural use of a sum of SDSPM’s which, in turn, gives the quantum mechanical result of Eq.(6). The proof of this is rather involved since Einstein-separability needs to be guaranteed which imposes stringent requirements on the possible sum of product measures; in particular on the joint density and even conditional distributions of setting dependent parameters.
2 PARAMETER SETS BEYOND BELL’s

Before discussing possible generalized parameter sets, we would like to make a case that time $t$ cannot be included as just another parameter in Bell-type proofs.

2.1 The special role of time

We state up front what we believe to be the basis for obtaining the quantum result with the use of hidden parameters: The functions $A$ and $B$ and the densities $\rho$ of hidden parameters may both relate to time without any involvement of spooky action at a distance. Time correlations, even setting dependent ones, may exist in both stations without any suspicion or hint of spooky action. The introduction of these correlations through time leads then to a probability measure that can depend on the settings at both stations although the functions $A, B$ depend only on the settings of the respective stations and the parameters, now considered as random variables, are independent when averages over long time periods are taken. A well known fact of probability theory is at the foundation of this: random variables may be conditionally dependent (e.g. for certain time periods) while they are independent when no conditions are imposed.

Bell type proofs permit any number and form of parameters as long separate integrations can be performed over the respective densities i.e. if the joint conditional densities equal the product of the individual conditional densities. The introduction of time-like parameters presents then a critical problem since other parameters in the argument of the functions $A$ and $B$ may depend on time. This happens in an enormous number of natural physical situations.

The following example is designed to define more clearly what we understand by the term "time-like parameters". These parameters may actually include space-like labels such as the settings. However, with respect to their correlations they are time-like, just as two clocks in two stations show time-like correlations even if some space like settings (e.g. the length of the pendulum) are adjusted separately in the stations. To be definite, assume that two stations have synchronized clocks with the pointer of each clock symbolized by a vector of Euclidean space and denoted by $s_1$ in station $S_1$ and by $s_2$ in station $S_2$. Adding the setting vectors in the respective stations, one obtains setting dependent time-like and correlated parameters $s_1 + a$ in station $S_1$ and $s_2 + b$ in station $S_2$. One can find a natural implementation of this example by using gyroscopes in the two stations located on the rotating earth. If such parameters affect the functions $A$ and $B$, then integration over time cannot be factorized and time cannot be introduced in Bell-type proofs without difficulty. We note, in passing, that the rotation of the earth poses also the following problem. The quantum result for the spin pair correlation $P(a, b) = -a \cdot b$ is invariant to (time dependent) rotations while the mathematical operations performed in the proof of Bells theorem are not. Thus rotational symmetry is violated in Bell-type proofs through the factorization process without assessment of its consequences.

To demonstrate the existence of hidden parameters in principle, we may permit any parameter set that can be generated involving $t$ and local setting dependent operators $O_{a,t}^1$. 
in station $S_1$ and $O^2_{b,t}$ in station $S_2$. These local operators may act on any parameters (or information) in the respective stations to create new parameters. For example, if a particle that carries the parameter $\lambda_1$ arrives from the source within a time period characterized by $\omega$ in station $S_1$, then the time operator can transform this parameter into a new "mixed" parameter $\Lambda^1_{a,t}(\lambda_1, \omega)$. Recall from section 1 that the actual time of measurement which determines $\omega$ must assumed to be random since the settings are randomly switched [4]. We distinguish this random time period $\omega$ from the time index in the time operator because that dependence on time may or may not be random. A more specific way of thinking about these operators is by imagining two computers in the two stations which have synchronized internal clocks. These computers can run any program to create new parameters out of the locally available input. Of equal importance, they can also be used to evaluate these parameters i.e. assign them a value of $\pm 1$. Both processes, creation and evaluation, may depend on the respective setting and may be correlated in time.

We summarize now why time has a special standing and cannot be included as just another parameter in Bell-type proofs.

- Time may enter in more than one way to influence the value of the functions $A$ and $B$. It influences this value through $\omega$ the randomly picked time (or time interval) of measurement at given settings. It also influences the value through the evaluation programs (the time operators). One does not need to restrict oneself to one time operator per station, several such operators could be used in any sequence depending on any information available at the station i.e. any information available in the backward light cone of a given station at the time of measurement. This means that the functions $A$ and $B$ are permitted to have an extended variable list and several variables may be time related. The settings $a$ and $b$, the time periods $\omega$ and the time-dependent evaluation programs (or time operators) all contribute and determine the parameter random variables that appear in the argument of the functions $A$ and $B$ and in the joint density $\rho$ of these parameter random variables at a given time.

- The source parameters $\lambda$ may also depend on time. It is possible that, depending on the information contained in the backward light cone, the parameters $\lambda$ as well as the frequency with which certain $\lambda$'s occur will be different during different time periods. Actually even the number of parameters that influence the measurement can be different during different time periods; at least as far as principle is concerned.

- The above facts present grave difficulties to Bell’s proof because there may exist correlations between all these time dependent parameter random variables and in fact these correlations must exist to guarantee $A_a = -B_a$. Thus the joint density $\rho$ of these parameter random variables may now show time correlations and therefore $\rho$ may itself depend on time. In particular, separate integration over the parameter values needs to be justified and, in fact, may not even be possible. We will see that this also presents great difficulty for the proof of d’Espagnat [9] and other proofs that are, mathematically, slight generalizations or special cases of Bell’s.
The possibility of the dependence of the joint density $\rho$ on time and the relationship of the functions $A, B$ to time and settings, as discussed above, make it convenient to shift the assumption of time dependence to setting dependence. As explained in sections 4 and 5, this yields the possibility to let both the probability measure $\mu$ and the density $\rho$ associated with it depend on the settings without introducing spooky action at a distance. The mathematical model will be presented in section 4. The principle for this is as follows. The evaluation of parameters by the functions $A$ and $B$ (i.e. the value of $\pm 1$ that is assigned for a certain parameter list) and the frequency with which certain parameters appear may be correlated in time including time correlations to the other station which influence the possible choices of time related parameters that can be made there for any given setting. This time correlation may also, in each station, depend on the respective settings. Physically the possibility of such a general probability measure arises because the station equipment changes the incoming information (setting dependent time operators) and evaluates then the changed information within the same process.

It is also important to realize that the computer evaluation programs may even be regarded as the "actual" setting at a given station. There is a setting that the experimenter controls and that influences the choice of computer program. However, for different times the evaluation that the program performs may be different. If both the times and settings are equal in the two stations, then the evaluation is the same (to guarantee that $A_a = -B_a$ as demanded by Eq.(3)). This view has a great significance: the number of actual different settings in any given station may be vastly greater than the number of choices that can be made by the experimenter without the experimenter knowing. In other words, the hidden parameters may not only represent the instruction set that is sent out for evaluation but also represent the evaluation programs (i.e. the "actual" settings) themselves and are correlated in time. The number of different settings that the experimenter sees or believes to be involved may therefore be much smaller than the actual number involved in the evaluation. Note that this gives a certain symmetry to measurement equipment and incoming parameters that is absent in the approach of Bell.

The important question to be answered is now whether the proofs of Bell-type theorems still can be carried out in a mathematically rigorous way using these generalized time related parameter random variables. We show in the next section for four major proofs of locality inequalities that time related parameters of the kind discussed above do not permit a logical execution of the proofs. Before doing so, we develop a more precise mathematical notation and definition of possibly involved time dependent parameter random variables.

### 2.2 Definitions of time related parameters

The starting point is a set of source parameters $\lambda = (\lambda^1, \lambda^2)$ where the superscripts indicate information carried to stations $S_1$ and $S_2$, respectively. Random internal
parameters $\lambda_a^*$ operate at station $S_1$ and $\lambda_b^*$ at station $S_2$. In other words, there is a layer of parameters below the mere settings that will affect the values of the functions $A, B$. While the observer might imagine that $A, B$ depend on settings only, the values of the functions $A, B$ are determined by stochastic processes, indexed by the unit vectors $a$ and $b$ respectively. For given vectors $a$ and $b$ we denote the joint distribution of the resulting random variables $\lambda_a^*(\cdot)$ and $\lambda_b^*(\cdot)$ by $\gamma = \gamma_{ab}$ which we allow to depend on $a$ and $b$ in order to accommodate as broad a situation as possible. A reasonable, though not a necessary, assumption on $\gamma$ is the following continuity condition: for fixed $a$

$$\lim_{b \to a} \gamma_{ab}\{\omega : \lambda_b^{**}(\omega) = \lambda_a^*(\omega)\} \to 1 = \gamma_{aa}\{\omega : \lambda_a^{**}(\omega) = \lambda_a^*(\omega)\}. \quad (15)$$

Intuitively speaking Eq. (15) says that if the vector $b$ at station $S_2$ is parallel or close to parallel to the vector characterizing the analyzer setting in station $S_1$, then for an “overwhelming majority of cases $\omega$” the corresponding parameters $\lambda^*$ and $\lambda^{**}$ are equal.

Our probability space $\Omega$ consists of all pairs $(\lambda, \omega)$, where $\lambda$ is a source parameter and $\omega$ is the element of randomness related to time and driving the station parameters $\lambda_a^*(\cdot)$ and $\lambda_b^*(\cdot)$, respectively. Furthermore, we assume that the source parameters $\lambda$ will interact with the station parameters $\lambda_a^*(\omega)$ and $\lambda_b^*(\omega)$ with built in time $t$ dependence to form the “mixed” parameters $\Lambda^1_{a,t}(\lambda, \omega)$ and $\Lambda^2_{b,t}(\lambda, \omega)$ (one could visualize this by some many body interactions). These are not free parameters, but rather stochastic processes, indexed by the pairs $(a, t)$ and $(b, t)$ at stations $S_1$ and $S_2$, respectively, and defined on $\Omega$. The transition from $(\lambda, \omega)$ to $\Lambda^1_{a,t}(\lambda, \omega)$ and $\Lambda^2_{b,t}(\lambda, \omega)$ is thought to be defined by certain rules that can be represented by station specific operators $O^1_{a,t}$ and $O^2_{b,t}$ that depend on the globally known time $t$ that is the same at the stations as well as at the source. Notice also that the time operations and mixing of parameters occur during the collapse of the wave-function (in quantum mechanical terms). The timing in left and right stations and the values of time involved in the measurement process are also quite flexible. It only needs to be guaranteed that one deals with the same correlated pair.

Thus the connection between the time operators $O$ and the mixed parameters $\Lambda$ is given by

$$O^1_{a,t}(\lambda^1, \omega) = O^1_{a,t}(\lambda^1, \omega; \lambda_a^*(\omega)) = \Lambda^1_{a,t}(\lambda^1, \omega) \quad (16)$$

and

$$O^2_{b,t}(\lambda^2, \omega) = O^2_{b,t}(\lambda^2, \omega; \lambda_b^{**}(\omega)) = \Lambda^2_{b,t}(\lambda^2, \omega). \quad (17)$$

Furthermore, the stochastic processes $A_{a,t}$ and $B_{b,t}$ satisfy

$$A_{a,t}(\lambda^1, \omega; \lambda_a^*(\omega), \Lambda^1_{a,t}(\lambda^1, \omega)) = \pm 1 \quad (18)$$

and

$$B_{b,t}(\lambda^2, \omega; \lambda_b^{**}(\omega), \Lambda^2_{b,t}(\lambda^2, \omega)) = \pm 1, \quad (19)$$

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and if $b = a$ then in analogy to Eq. (3) we have with probability 1

$$B_{a,t}(\lambda^2, \omega; \lambda^*_a(\omega), \Lambda^2_{a,t}(\lambda^2, \omega)) = -A_{a,t}(\lambda^1, \omega; \lambda^*_a(\omega), \Lambda^1_{a,t}(\lambda^1, \omega)).$$

(20)

This means that the time operations for equal settings need to be synchronized in order to lead to Eq. (21). The synchronization may be achieved by the selection of which settings are chosen to be equal in the two stations and by the fact that the stations are in the same inertial frame with identical clock-time. (Time shifts and asymmetric station distances can easily be accommodated in our model.)

Assume therefore with us that station and setting dependent parameter random variables influence EPR-type experiments. They may be arbitrarily complicated mathematical objects. In the simplest cases each parameter could, for example, be a matrix or an n-dimensional vector etc.. We also may assume arbitrarily complicated time operators that influence these parameters.

3 TIME-LIKE PARAMETERS IN PROOFS OF BELL’S THEOREM

We have selected the following proofs because they are representative for all proofs of Bell’s theorem that are known to us and are described in Bell’s book [7]. Before we present them, we would like to clearly define the starting point for the proofs. This starting point is the set of measured data which is represented by the functions $A$ and $B$ as defined (using our generalization) in Eqs. (18) and (19) with hidden parameters as also defined in this equation.

3.1 The Proof of Bell

Bell [7] defines the following parameter sets that are in the backward light cone (as defined by relativity). He lets $N$ denote the specification of all entities (called be-ables by Bell) that are represented by parameters and belong to the overlap of the backward light cones of both space-like separated stations $S_1$ and $S_2$. In addition he considers sets of be-ables or parameters $L_a$ (our notation) that are in the remainder of the backward light cone of $S_1$ and $M_b$ for $S_2$ respectively. Bell then defines the conditional probability that the function $A_a$ assumes a certain value with $|A_a| \leq 1$ (this is a generalization of $A_a = \pm 1$)

$$\{A_a|L_a, N\}$$

(21)

and similarly for $B_b$

$$\{B_b|M_b, N\}$$

(22)
Then, to derive one of the celebrated "local inequalities" Bell considers the expectation $E$ of the product $A_a B_b$:

$$E\{A_a B_b\} = \sum_{AB} A_a B_b \{A_a|L_a, N\} \{B_b|M_b, N\}$$

(23)

Here lies the crux of the problem with Bell-type derivations of the locality inequalities. The parameter sets in the backward light cones are not constant but evolve and are, certainly in principle, different for all the different times at which each single measurement is taken. In addition these parameters may be time-like as far as their correlations are concerned. Using a more precise notation one must therefore label the sets of be-ables with indices that represent time $t$ (or time periods $\omega$) e.g. by $N_t, L_{a,t}$ and $M_{b,t}$. Then it is obvious that the summations above cannot be performed in a straightforward fashion. However, to fully show that Bell’s proof does not go forward using the time dependent parameter space, one must demonstrate that time cannot be just entered as another parameter in Bell’s proof (e.g. substituting $t$ for $\lambda$). This demonstration is indeed possible because, as mentioned frequently before, time may enter the functions $A, B$ and density $\rho$ in form of two or more different variables. It is these time-like variables, some indexed by the settings, that prevent Bell-type proofs to go forward.

For example, one variable may be obtained by dividing the time-axis in the frame of reference of the stations into equal intervals, that are then randomly selected through the actual time of measurement. We have denoted these time intervals by the variable $\omega$. A second variable related to time may be obtained by partitioning the time axis in station $S_1$ into a set of intervals $\Delta t_a$ that depend on the setting $a$ which, in turn, is randomly selected by the observer.

Define now $A_a$ and $B_b$ in two stations to be functions of these different time related variables and also of an additional variable $\lambda$ i.e.

$$A_a = A_a(\omega, \Delta t_a, \lambda)$$

(24)

and

$$B_b = B_b(\omega, \Delta t_b, \lambda)$$

(25)

We also define a probability density $\rho$

$$\rho = \rho(\omega, \Delta t_a, \Delta t_b, \lambda)$$

(26)

The expectation of the spin pair correlation is then in Bell’s notation:

$$P(a, b) = \int A_a(\omega, \Delta t_a, \lambda) B_b(\omega, \Delta t_b, \lambda) \rho(\omega, \Delta t_a, \Delta t_b, \lambda) d\omega d\Delta t_a d\Delta t_b d\lambda$$

(27)

In the proofs of Bell-type theorems (also in the variation of Clauser-Holt-Horne-Shimony [5]) time-like parameters are never explicitly considered. Station specific parameter
random variables, that may be denoted by $\lambda^*_a$ in $S_1$ and by $\lambda^*_b$ in $S_2$, are assumed to be independent (considered as random variables). Bell-type proofs contain then the following equation that is considered equivalent to the validity of Einstein separability or locality for given settings $a$ and $b$:

$$\rho(\lambda^*_a, \lambda^*_b, \lambda) = \rho_1(\lambda) p_1(\lambda^*_a|\lambda)p_2(\lambda^*_b|\lambda)$$

(28)

where $p_1(\lambda^*_a|\lambda)$ and $p_2(\lambda^*_b|\lambda)$ denote conditional probability densities given $\lambda$ and the settings $a$ and $b$.

With the time-like variables as defined for Eq.(27), Eq.(28) reads:

$$\rho(\omega, \Delta t_a, \Delta t_b, \lambda) = \rho_1(\lambda) \rho_2(\omega) p_1(\Delta t_a|\lambda, \omega)p_2(\Delta t_b|\lambda, \omega)$$

(29)

However, for this equation to hold, the choices of time intervals $\Delta t_a, \Delta t_b$ would have to be conditionally independent given the time interval $\omega$ and the source parameter $\lambda$; but this cannot be the case, as they are all correlated, i.e. connected with each other. Nor do locality conditions have any consequence for the choice of time-like intervals, even though they are indexed by the settings in the respective stations, as long as the choices of the settings are made separately and independent of the other station. Of course, the dependence on time is “spatially non local” in the sense that two clocks in separate stations may be perfectly correlated at least when they are in the same inertial frame.

Some may be uncomfortable with the above choice of variables since it is not easy to think of a general physical mechanism that, in two stations, depends on various time intervals in selected ways. We have therefore chosen in the main part of the paper time operators $O^1_a$ instead of the $\Delta t^i_a$. These time operators act on station specific and source specific parameters (having in mind a simulation of many body interactions) and result in the mixed parameters $\Lambda_a(O^1_a, \omega, \lambda)$ in station $S_1$ and $\Lambda_b(O^2_b, \omega, \lambda)$ in $S_2$. The fulfillment of physical locality conditions in presence of time correlations is then not as easy to show. However, we do not invoke spooky action at a distance as we will demonstrate in section 5.2. The core of this demonstration is the mathematical fact that parameters in two stations may be conditionally dependent (e.g. during certain time periods) and simultaneously independent when no conditions are imposed.

### 3.2 The proof of Mermin

Mermin’s proof of the Bell inequalities was aimed at a broad audience and considers only the essential basis and consequences of Bell’s theorem for a specific case that can be experimentally realized. Three possible different settings where assumed to be available to the experimenter in each of the two stations $S_1, S_2$. The settings are chosen such that (slightly paraphrased and transformed to agree with our notation):

(i) If one examines only those runs in which the settings are the same in both stations, then the sign of the functions $A$ and $B$ is always opposite.
(ii) If one examines all runs without regard to what the settings are, then one finds that the pattern of signs of $A$ and $B$ is completely random. In particular, half the time the signs are the same, and half the time different.

Since we have 2 different signs and three different settings on each side, there are $2^3 = 8$ possible instructions that can be given to determine the signs for the $3^2 = 9$ different and random settings. It is then easy to see that the same signs must occur in both stations $\frac{5}{9}$ of the time. This, however, is in contradiction to requirement (ii). Mermin concludes therefore that no instruction sets can exist.

However, from the discussion in the previous section we know that the number of settings that the experimenter controls may be much smaller than the ”actual” hidden number of settings. In the example of section 2.1, a different computer evaluation program (that can be regarded as the actual setting) may be realized for each different time of measurement and a given setting $a$ chosen by the experimenter. If $N_s$ different computer evaluation programs are realized for each of the different settings that the experimenter can choose then we have $2^{3N_s}$ possible instructions that can be given to determine the signs of the $(3N_s)^2$ different settings. Most importantly, however, only the settings that the experimenter chooses are random. The computer programs that form a ”layer” below the chosen settings may have time correlations among each other. The reasoning of Mermin is based on the completely random choice of the settings. This, however, is no longer guaranteed within the framework of our time related and time-like parameters that are represented by the computer programs. These arguments present also difficulties for the proof of d’Espagnat [9]. In defense of Bell one could say that such correlations must also run into problems with spooky action. How can (ii) be true if time correlations exist ”underneath” the random switching of the experimenter. We will show, however, that this suspicion has no mathematical basis. As mentioned in section 2.1 random variables may be conditionally dependent (e.g. for certain time periods) while they are independent when no conditions are imposed. In other words one can have correlations through certain time periods and for certain given source parameters $\lambda$ without the necessity of correlations when no constraint is applied e.g. over long periods of time. Note also that we have only used local operators and there is no suspicion of spooky action as long as we have no global (unconditional) correlations of space like parameters. There may be correlations in time that extend over space.

Mermin’s proof of a Bell type inequality can therefore not proceed when time-like parameters and setting dependent (within a station)’ time operators are involved.

### 3.3 Variations of the proof of d’Espagnat

Some variations of d’Espagnats proof [8] appear at first glance different to that of Bell although they form a special case by substituting for the general measure (considered in Bell’s proof) a properly normalized sum of point masses. We give below a prototype version of such proofs.
Assume we have chosen the setting vectors $a, d$ in station $S_1$ and setting vectors $b, c$ in station $S_2$. Further assume that we can rearrange all measurements in such a way that they can be concatenated in groups of four that then fulfill the following inequality

$$|A_a(...)B_b(...) + A_d(...)B_b(...) + A_a(...)B_c(...) - A_d(...)B_c(...)| \leq 2 \quad (30)$$

with (...) denoting a certain subset of the parameters. Assume further that the union of all subsets (...) gives all parameters that can possibly describe the given set of experiments. We call the statement of this paragraph the rearrangement assumption (RA).

The inequality of Eq.(30) follows from Eq.(10). Averaging over all parameters (...), one obtains the spin pair expectation values for the various settings. The averaged Eq.(30) represents then an important "locality inequality" of which Bell’s is a special case.

Note, that this type of proof starts from the RA and the inequality of Eq.(30). Implying that all the measurements can be covered that way if parameters are inserted, one arrives at the inequality for the average spin-pair correlation. The explanation offered to show that indeed all measurements can be covered that way is based on (i) the avoidance of spooky action (AoSA), (ii) some form of inductive logic (IL) based on the fact that repeated experiments with reasonably large numbers of measurements must give about the same result.

From (i) it is deduced that the list of parameters that appears in the arguments of the functions $A$ and $B$ contains all possible combinations of all possible parameters, independent of the particular setting. The reasoning is approximately like this: if the parameters that appear in the arguments of $A$ and $B$ would depend on the setting, then by switching from one setting to the other in station $S_1$ something must happen to the parameter set in station $S_2$ which would be spooky action. This reasoning, however, is (as in the proof of Mermin) not mathematically sound. We repeat that it is well known in probability theory that random variables may be conditionally dependent but independent when no conditions (or different ones) are imposed. This opens the possibility that there exist different conditional dependencies for the various settings while the parameters are independent when viewed without condition. For example, for time periods during which certain time operators are at work and/or certain parameters $\lambda$ are emitted from the source, the parameters in station $S_1$ may be correlated to those in station $S_2$ in other words are conditionally dependent. The different conditional dependencies mean that the listings of the parameters do not contain all possible combinations of the parameters for all settings and for all time intervals.

One might think that inductive logic (point (ii)) may save this type of proof. However, long term averages can still be the same in spite of the presence of time correlations. They are guaranteed to be the same in our model.

We summarize these ideas by presenting our argument in a slightly different way. Clearly, time cannot be partitioned into a finite number of elements that randomly
repeat themselves. We also note, that for each particular setting and measurement the time interval $\omega$ may be different. Let’s enumerate then the time intervals in the measurements with setting $a, b$ by $\omega_j$ and with setting $d, b$ by $\omega_{j*}$ with $j \neq j*$ since the measurements must necessarily be at different times. The first two terms of Eq.(30) are then

$$A_a(\omega_{j*})B_b(\omega_{j*}) + A_d(\omega_{j*})B_b(\omega_{j*})$$

(31)

The rearrangement assumption (RA) means that all parameters must appear in all arguments of the functions $A$ and $B$ independent of setting and that therefore a reordering is possible to obtain Eq.(30). But how can that be proven? It is not even necessary that there exists a periodic repetition of parameters. The parameters that are available at each time of measurement in each station comprise all the information up to present that is contained in the backward light cone. Since the backward light cone is different for each different measurement a different parameter or different combination of parameters can be selected (at least in principle) each time. The above proof needs to assess then the properties and functional dependencies of the possible parameters $A_{a,t}(\lambda, \omega)$ and operators $O_{a,t}$ etc. which could permit a reordering into the sets of four shown in Eq.(30) (for the case when the continuum of time is involved and for virtually arbitrary time operators). There is no proof in the literature known to us which does even address these questions. The only arguments that are given start with Eq.(30) and use induction in the backward direction of the proof. Using small finite sets of parameters and Eq.(30) one can proceed to larger and larger sets. However, the proof needs to start from the diversity of Eq.(31) and proceed to derive Eq.(30) by reordering. Of course, we cannot directly show that this is not possible. However, if one can find a local set of parameters that gives the quantum result (as we believe we have below), then all the variations of d’Espagnat’s proof are refuted.

### 3.4 The Proof of Clauser-Holt-Horne-Shimony

The proof of Clauser-Holt-Horne-Shimony (CHHS) introduces a variation of Bell’s inequality and permits a violation of $A_a = -B_a$ i.e. of Eq.(3) to any degree. In addition, the values of the functions $A$ and $B$ can be such that $|A|, |B| \leq 1$. This violation is caused by station specific setting dependent parameters that have no correlation from one station to the other. Note that therefore these station specific parameters (see [3] and also [4]) are very different from the ones introduced by us. Our station specific parameters are correlated by clock time and do neither lead to any violation of Eq.(3) nor to absolute values of the functions $A$ and $B$ that are smaller than 1.

The advantages of CHHS lie in the fact that their inequality does not contain exactly equal settings that are experimentally difficult to achieve. They use instead the inequality shown already above

$$|A_a(...)B_b(...) + A_d(...)B_b(...) + A_a(...)B_c(...) - A_d(...)B_c(...)| \leq 2$$
They also have a natural explanation for the experimentally observed deviations from $A_a = -B_a$: random influences of parameters or any type of fluctuations at the stations similar to noise. However, these deviations come at the price of also violating the quantum result. Their random fluctuations at the stations, when fully effective, will completely destroy the quantum result. Certainly, the Bell inequalities (or CHHS inequalities) stay valid. This is, however, without logical consequence since under these circumstance it is clear that the quantum result will not be experimentally confirmed. If the noise is weak, then part of the quantum result is recovered. However, only to the extent that $A_a = -B_a$. We need to consider therefore, as far as principle is concerned, only the subset of measurements and parameters for which $A_a = -B_a$ may be implied. For this subset the CHHS inequality brings nothing new. Replace above the settings $c \to -c$ and $d \to c$ and the original Bell inequality of Eq.(5) is recovered. Therefore, CHHS need not be treated separately from the proof of Bell as far as the principle and the fundamental deviations from the quantum result are concerned. We emphasize, however, that our station specific parameters are very different from those of CHHS. Ours are time correlated and do not violate $A_a = -B_a$.

4  A THEOREM FOR HIDDEN EPR-PARAMETER SPACES

Our goal is to show that under our generalized conditions, it is possible to obtain the quantum result, the scalar product $-a \cdot b$ for the spin pair expectation value $P(a, b)$. This task will be completed in several installments in the next section. Here we formulate a theorem which provides the stepping stone for this procedure. Note that our measure deviates from a probability measure by at most $\epsilon$, which can be chosen arbitrarily small. We believe that this presents no physical limitation of the theory but include it for reasons of mathematical precision.

**THEOREM:** Let $0 < \epsilon < \frac{1}{2}$ and let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be unit vectors. Then there exists a finite measure space $(\Omega, F, \mu = \mu_{a,b})$ and two measurable functions $A$ and $B$ defined on it with the following properties:

$$\Omega \subset \mathbb{R}^2 \text{ and } F \text{ depend on } \epsilon \text{ only}$$

$\Omega$ is a compact set. Its elements are denoted by $(u, v)$.

The measure $\mu$ only depends on $a, b$ and $\epsilon$, satisfies

$$1 \leq \mu(\Omega) < 1 + \epsilon$$

and has a density $\rho_{ab}$ with respect to Lebesgue measure.

The functions $A$ and $B$ assume the values

$$A, B = \pm 1$$
and $A$ depends only on $a$ and $u$, $B$ only on $b$ and $v$.

Further

$$E\{A_aB_b\} = \int_{\Omega} A_a(u)B_b(v)\rho_{ab}(u,v)dudv = -a \cdot b$$

(35)

and for each vector $a$ the following equation holds for all $(u,v) \in \Omega$ except on a set of $\mu$-measure $< \epsilon$:

$$B_a(u,v) = -A_a(u,v)$$

(36)

(Note that mathematical precision requires the listing of all the parameters in the functions so that the integral of Eq.(35) is well defined. However, since it is important for the physics that the functions actually depend only on a subset of parameters, as mentioned after Eq.(34) and as will become clear when the meaning of $u,v$ is fully described, we list in the arguments of the functions only that subset.)

The proof of the theorem requires the following fact which follows from a basic theorem on B-splines [10]. We state the fact here in form of a lemma.

**Lemma:** Let $n \geq 4$ be an integer. Then there exist real-valued functions $N_i(x), \psi_i(y)$ with $1 \leq i \leq n$ depending on real variables $x$ and $y$, respectively, such that

$$0 \leq N_i(x) \leq 1, 0 \leq \psi_i(y) \leq 2 \text{ for } 0 \leq x, y \leq 1$$

(37)

and

$$0 \leq \sum_{i=1}^{n} \psi_i(y)N_i(x) - (y-x)^2 \leq \frac{1}{4}n^{-2} \text{ for } 0 \leq x, y \leq 1$$

(38)

The proof for this lemma is given in appendix 2. We now proceed to prove the main theorem.

**Proof of Theorem:** Choose an even integer $n > 1/\epsilon$ and for $\Omega$ the square $\Omega = [-3,3n]^2$ with side of length $3 + 3n$. We endow $\Omega$ with Lebesgue measurability, symbolized by the $\sigma$-field $\mathcal{F}$ and define:

$$A_a(u) = \begin{cases} 
\text{sign}(a_k) & \text{if } -k \leq u < -k + 1 \quad k = 1, 2, 3 \\
-1 & \text{if } 2j \leq u < 2j + \frac{1}{2} \quad j = 0, 1, \ldots, \frac{3n}{2} - 1 \\
+1 & \text{if } 2j + \frac{1}{2} \leq u < 2j + 2 \quad j = 0, 1, \ldots, \frac{3n}{2}
\end{cases}$$

(39)

Thus $A$ depends on $a$ and $u$ only. Here and throughout we set $\text{sign}(0) = 1$. Similarly, we define

$$B_b(v) = \begin{cases} 
-\text{sign}(b_k) & \text{if } -k \leq v < -k + 1 \quad k = 1, 2, 3 \\
+1 & \text{if } 2j + \frac{1}{2} \leq v < 2j + \frac{3}{2} \quad j = 0, 1, \ldots, \frac{3n}{2} - 1 \\
-1 & \text{if } 2j - \frac{1}{2} \leq v < 2j + \frac{1}{2} \quad j = 0, 1, \ldots, \frac{3n}{2} - 1 \\
-1 & \text{if } 0 \leq v < \frac{1}{2} \text{ or } 3n - \frac{1}{2} \leq v < 3n
\end{cases}$$

(40)
Thus, \( B \) depends on \( b \) and \( v \) only. Notice that on \( \bigcup_{k=1}^{3} [-k, -k + 1)^2 \), Eq.(36) is satisfied for all values of \((u, v)\).

Next, we define

\[
\sigma_a(u) =
\]

\[
|a_k| \cdot 1\{-k \leq u < -k + 1\} \quad k \in I_1
\]

\[
N_k(|a_1|) \cdot 1\{k - 1 \leq u < k\} \quad k \in I_2
\]

\[
N_{k-n}(|a_2|) \cdot 1\{k - 1 \leq u < k\} \quad k \in I_3
\]

\[
N_{k-2n}(|a_3|) \cdot 1\{k - 1 \leq u < k\} \quad k \in I_4
\]

\[
\tau_b(v) =
\]

\[
|b_k| \cdot 1\{-k \leq v < -k + 1\} \quad k \in I_1
\]

\[
\frac{1}{2} \psi_k(|b_1|) \cdot 1\{k - 1 \leq v < k\} \quad k \in I_2
\]

\[
\frac{1}{2} \psi_{k-n}(|b_2|) \cdot 1\{k - 1 \leq v < k\} \quad k \in I_3
\]

\[
\frac{1}{2} \psi_{k-2n}(|b_3|) \cdot 1\{k - 1 \leq v < k\} \quad k \in I_4
\]

The symbols \( I_1, I_2, I_3, I_4 \) stand for: \( I_1 = -3, -2, -1; I_2 = 1, ..., n; I_3 = n + 1, ..., 2n; I_4 = 2n + 1, ..., 3n \) and \( 1\{\cdot\} \) denotes the indicator function. Furthermore, let

\[
\delta_{jk} = \begin{cases} 
1 & \text{if } j = k \\
0 & \text{if } j \neq k 
\end{cases}
\]

be the Kronecker symbol. We set

\[
\nu(u, v) = \delta_{ij} \cdot 1\{i - 1 \leq u < i\} \cdot 1\{j - 1 \leq v < j\} \quad i, j = -2, -1, ..., 3n
\]

We finally define the density \( \rho_{ab} \) by

\[
\rho_{ab}(u, v) = \sigma_a(u) \tau_b(v) \nu(u, v)
\]

and the measure \( \mu_{ab} \) by having density \( \rho_{ab} \) with respect to Lebesgue measure. This definition, of course, entails that \( \mu_{ab} \) is a sum of SDSPM’s. The integrals that we have to perform will then correspond to summations over integrals of such product measures. Note that a single product measure of independent events would, of course, assuage all concerns related to spooky action but cannot yield the quantum result because the two spin measurements are not independent (see also appendix 3). There are considerable correlations possible because of the correlations of the source parameters and because of the knowledge of clock time (the time operator) in both stations. We therefore have introduced correlations by partitioning the measure into a sum of SDSPM’s. In the above equations, the correlation is expressed by the Kronecker symbols. Note, however, that the diagonal arrangement above is not necessary and leads only to one
particular sum of SDSPM’s. A large number of off-diagonal arrangements can also be included as we will see below and the actual sum of product measures that we use is a superposition of all these possibilities. This will enable us to obtain a uniform joint density and thus to avoid any sign of spooky action.

From the above definitions we obtain the following integrals for the spin pair correlation functions:

\[
\int_{[-3,0]^2} A_a(u)B_b(v)\rho_{ab}(u,v)dudv = -\sum_{k=1}^3 |a_k||b_k| \text{sign}(a_k)\text{sign}(b_k) = -a \cdot b
\]  

(45)

Furthermore, the integral over the complement of the square \([-3,0]^2\) vanishes i.e.

\[
\int_{\Omega \setminus [-3,0]^2} A_a(u)B_b(v)\rho_{ab}(u,v)dudv = 0
\]  

(46)

which proves Eq.(35).

It remains to be shown that \(\rho_{ab}\) defines a measure \(\mu\) that is close, within \(\epsilon\), to a probability measure i.e. fulfills Eq.(33). For this, we consider the mass distribution between the square \([-3,0]^2\) and its complement. The amount of mass \(M_1\) distributed over \([-3,0]^2\) is

\[
M_1 = \sum_{k=1}^3 |a_k||b_k|
\]  

(47)

The mass \(M_2\) of \(\Omega \setminus [-3,0]^2\) equals

\[
M_2 = \frac{1}{2}\sum_{k=1}^3 \sum_{i=1}^n N_i(|a_k|)\psi_i(|b_k|)
\]  

(48)

Thus the total mass distributed equals in view of Eq.(38)

\[
M_1 + M_2 = \sum_{k=1}^3 |a_k||b_k| + \frac{1}{2}\sum_{k=1}^3 \sum_{i=1}^n N_i(|a_k|)\psi_i(|b_k|)
\]

\[
M_1 + M_2 = \sum_{k=1}^3 |a_k||b_k| + \frac{1}{2}\sum_{k=1}^3 (|a_k| - |b_k|)^2 + \theta \cdot n^{-2}
\]

\[
M_1 + M_2 = 1 + \theta \cdot n^{-2} < 1 + \epsilon
\]  

(49)

where \(0 \leq \theta < 1/24\). For the case \(b = a\) we have

\[
0 \leq M_2 < \theta \cdot n^{-2} < \epsilon
\]  

(50)

As was observed right after the definitions of \(A\) and \(B\), Eq.(36) holds for all \((u,v) \in \bigcup_{k=1}^3[-k,-k+1]^2\) and thus for all \((u,v) \in \Omega\) except, perhaps, on a set of \(\mu\)-measure \(< \epsilon\). This completes the proof of the theorem.

The proof clearly shows that for \(b = a\) we can choose \(\Omega\) to be a probability space, i.e. \(\mu(\Omega) = 1\).
5 QUANTUM RESULT WITHOUT SPOOKY ACTION

5.1 Connection to EPR-experiments

Suppose now that $\Lambda_{a,t}^1, \Lambda_{b,t}^2$ are mixed parameters as defined above. Let $f$ and $g$ be real-valued bounded functions on the space of the $\Lambda_{a,t}^1$’s and $\Lambda_{b,t}^2$’s. We do not assume that these two $\lambda$-spaces are identical, nor is it necessary to specify them at this point. However, we need to assume that, for fixed $a,b$ and time operators, the mappings $f(\Lambda_{a,t}^1)$ and $g(\Lambda_{b,t}^2)$ from $\Omega \rightarrow R$ are measurable so that they can be considered as random variables. Since $f$ and $g$ are assumed to be bounded, we may assume without loss of generality that the ranges of $f(\Lambda_{a,t}^1)$ and $g(\Lambda_{b,t}^2)$ equal the interval, $[-3,3n]$. A mathematical model for EPR-experiments can now be obtained by an application of the theorem. For fixed time operators and source parameters $\lambda$ we define the joint density of $f(\Lambda_{a,t}^1)$ and $g(\Lambda_{b,t}^2)$ to equal $\rho_{ab}(u,v)$, as defined in Eq. (44). Then by Eq. (35) and by the standard transformation formula for integrals we have for fixed $\lambda$ and time operators $O_{a,t}, O_{b,t}$:

$$E\{A_{a,t}(\lambda^1, \cdot) f(\Lambda_{a,t}^1(\lambda^1, \cdot)) B_{b,t}(\lambda^2, \cdot), g(\Lambda_{b,t}^2(\lambda^2, \cdot))\} = -a \cdot b$$ (51)

Here the expectation $E$ operates on the space of $\omega$, a subspace of $\Omega$; the dummy variable $\omega$ of the integration is symbolized by $(\cdot)$.

This direct application does not address the key question whether the introduced probability measure is free of the suspicion of spooky action at a distance. To show this, we need to ensure the following. If setting $b$ at station $S_2$ is changed into setting $c$, the probability distribution governing the parameters $\Lambda_{a,t}^1$ at station $S_1$ must remain unchanged. The fact that the ratios of the relative frequencies $\lambda_{a,t}^1/\lambda_{b,t}^2$ and $\lambda_{a,t}^1/\lambda_{c,t}^2$ may be different is not of concern. The time operator defined above can easily account for this. However, the average frequencies of the parameters $f(\Lambda_{a,t}^1)$ in each of the intervals between $-3$ and $3n$ must not change when setting $b$ is changed in the other station. This can be accomplished with ease by superposition of variations of the above described element in the following two step operation.

5.2 Avoidance of spooky action

Choose any of the $(n + 1)^2$ squares $Q_{jk}$ with vertices at the points $(3j, 3k), (3(j+1), 3k), (3j, 3(k+1))$ and $(3j, 3(k+1))$ for $j,k = -1, 0, 1, 2, ..., n - 1$. Now repeat the entire construction with $Q_{jk}$ replacing $Q_{-1,-1}$. Define $A$ and $B$ to be equal to $\text{sign}(a_i)$ or $-\text{sign}(b_i)$, respectively, on each of the three vertical and horizontal strips of $Q_{jk}$ with $i = 1, 2, 3$. On the vertical and horizontal strips not containing parts of $Q_{jk}$ define $A$ and $B$ equal $\pm 1$ in an obvious modification of the above construction. More precisely, we perform the following operations. As far as the definition of $A$ is concerned,
we interchange the vertical strips \([-3,0)\times[-3,3n)\) and \([3j,3(j+1))\times[-3,3n)\). Similarly, for the definition of \(B\) we interchange the horizontal strips \([-3,3n)\times[-3,0)\) and \([-3,3n)\times[3k,3(k+1))\). Next assign mass \(M_1\) to \(Q_{jk}\) and mass \(M_2\) to the complement \(\Omega \setminus Q_{jk}\) of \(Q_{jk}\). \(M_2\) will be distributed on \(3n\) unit squares as follows: \(Q_{jk}\) and the vertical and horizontal strips associated with them take a total of \((3n+3)(3.2) - 9 = 18n + 9\) unit squares. From the remaining \(9n^2\) unit squares we choose \(3n\) and distribute the mass \(\frac{1}{2}N_i(|a_k|)\psi_i(|b_k|)\) on them (with \(1 \leq i \leq n\) and \(1 \leq k \leq 3\)). For given \(Q_{jk}\) this yields

\[ N = (n + 1)^2 \left(\frac{9n^2}{3n}\right) \]  \hspace{1cm} (52)

possible measures \(\mu_m\) with \(1 \leq m \leq N\). For each of these measures Eqs.\((52)\) through \((58)\) hold. Label the corresponding functions \(A\) and \(B\) as \(A_{(m)}\) and \(B_{(m)}\) and consider the index \((m)\) a function of the source parameter \(\lambda = (\lambda^1, \lambda^2)\) and the time operators \(O_{a,t}^1\), \(O_{b,t}^2\). With respect to the dependence on time we make the usual assumptions, such as a possible invariance with respect to certain translations. Notice that variations (with settings) of the frequencies of setting dependent parameters in certain given time intervals are permitted by the properties of the time operator and do not indicate spooky action. Then the functions \(A_{(m)}\) and \(B_{(m)}\) can be considered as functions of \(a, \lambda, \Lambda_{a,t}^1\), and \(b, \lambda, \Lambda_{b,t}^2\), respectively. Finally define a new measure \(\mu\) on \(\Omega\) by setting

\[ \mu = \frac{1}{N} \sum_{m=1}^{N} \mu_m \]  \hspace{1cm} (53)

At this point we consider \(\Omega\) as the union of \(N\) layers of the above type stacked up in three dimensions, reinterpreting \(A,B\) and \(\mu\) accordingly.

The second step in the modification of the construction is a minor variation of the first one and, depending on ones taste, may not be needed. Instead of lining up mass \(M_1\) on the diagonal of the squares \(Q_{jk}\), we assign this mass to three unit squares within \(Q_{jk}\) such that each vertical and horizontal row contains exactly one unit square with mass \(|a_i|\) \(|b_i|\), \(i = 1, 2, 3\). Moreover, these are permuted so that they yield in \(3! = 6\) ways yielding a total of \(36\) possibilities. Taking these into account we define now \(\mu\) as the average of \(36(n + 1)^2\left(\frac{9n^2}{3n}\right)\) measures \(\mu_m\) as in Eq.\((53)\) but now with \(N\) denoting the total number of measures involved. Just as before we note that \(A, B\) and \(\mu\) satisfy Eqs.\((52)\) through \((58)\) and that \(\Omega\) is now defined as the union of \(N\) layers in three dimensions.

Let us inspect the density \(\rho_{ab}\) which is now defined on the domain

\[ \rho_{ab}(u,v,m) \text{ with } -3 \leq u,v < 3n; m = 1, 2, ..., N. \]  \hspace{1cm} (54)

For fixed \(u\) and \(v\), the joint density governing the pair of parameters \(f(\Lambda_{a,t}^1)\) and \(g(\Lambda_{b,t}^2)\) is given by

\[ = \frac{1}{N} \sum_{m=1}^{N} \rho_{ab}(u,v,m) \]
\[
\frac{1}{(3n+3)^2} \left( \sum_{k=1}^{3} |a_k||b_k| + \frac{1}{2} \sum_{k=1}^{3} \sum_{i=1}^{n} N_i(|a_k|) \psi_i(|b_k|) \right)
\]

\[
= M_1 + M_2 = \frac{1 + \theta \cdot n^{-2}}{(3n+3)^2}
\]

with \( \theta \leq 1/24 \) in view of Eq.(49). This shows that the joint density of \( f(\Lambda_{a,t}^1), g(\Lambda_{b,t}^2) \) is uniform over the square \([-3,3n)^2\) and therefore \( f(\Lambda_{a,t}^1) \) and \( g(\Lambda_{b,t}^2) \) considered as random variables are stochastically independent and themselves have uniform distribution over the appropriate intervals. Therefore, if the setting \( b \) gets changed to the setting \( c \), the random variables \( f(\Lambda_{a,t}^1), g(\Lambda_{c,t}^2) \) are also independent and there is no change in the distribution of \( f(\Lambda_{a,t}^1) \) by changing from \( b \) to \( c \). This should remove all suspicions of spooky action. We also emphasize that our construction is highly flexible to introduce uniformity even for the conditional densities, provided that the time periods considered are sufficiently long. The method to show this proceeds along the very same lines as above. It is probably worth noticing that the sum over all \( \int A B d\mu_i \) resembles a sum over all possible probability amplitudes in quantum mechanics, except that everything is real-valued here and dependencies not permitted by relativity are excluded.

### 5.3 Other physical conditions

To fulfill requirements of physics, it is necessary to be able to obtain certain values \(-1 \leq \alpha \leq 1\) for measurements on one side only and therefore one needs to be able to have predetermined values for the following type of integrals

\[
\int A p dudv = \alpha
\]

(55)

It is easily seen that this can be achieved without changing the result for the pair correlation by use of functions \( A, B \) generalized in the following way.

Define new functions

\[
A_r = A_{ra}(u, z) := A_a(u) r(z)
\]

(56)

Here \( r(z) \) can be any Lebesgue measurable function that assumes only values \( \pm 1 \) and \( z \) corresponds to a parameter specific to the source (e.g. \( \lambda_1, \lambda_2 \) or time \( t \)). Similarly define \( B_r \) as

\[
B_r = B_{rb}(v, z) := B_b(v) r(z)
\]

(57)

Then the product \( AB = A_r B_r \), while the integrals of the single function can be almost arbitrarily adjusted by a proper choice of \( r(z) \). The important special case \( \alpha = 0 \) is particularly easy to achieve in a multitude of ways. For example one can choose a function \( r(t) \) (depending only on time \( t \)) that varies rapidly and symmetrically between \( \pm 1 \).
6 CONCLUSIONS

We have presented a mathematical framework that can derive the quantum result for the spin-pair correlation in EPR-type experiments by use of hidden parameters. A key-element of our approach is contained in the introduction of time-like statistical parameters and setting dependent functions of them. This leads in a natural way to a setting dependent probability measure. The construction of this probability measure is complicated by the fact that spooky action must not be introduced indirectly. This is accomplished by letting the probability measure be a superposition of SDSPM’s with two important properties: (i) the factors of the product measure depend only on parameters of the station that they describe and (ii) the joint density of the pairs of setting dependent parameters in the two stations is uniform. The mathematical basis for this factorization is the theory of B-splines.

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8 APPENDIX 1

We show via a simple example that the mixed parameters $\Lambda_{a,t}^1$ and $\Lambda_{b,t}^2$ cannot be absorbed in an enlarged parameter space. We introduce unit vectors $\mathbf{a}$ and $\mathbf{b}$ as before and define

$$\|\mathbf{a}\| = |a_1| + |a_2| + |a_3|$$

Then

$$1 \leq \|\mathbf{a}\| \leq 3^{1/2}$$

Suppose we have two systems of functions $A = \pm 1$ and $B = \pm 1$ where $A$ depends only on $a$, $\lambda$ and $\alpha$, and $B$ only on $b$, $\lambda$ and $\beta$ with $0 < \lambda, \alpha, \beta \leq 1$ and such that now $P(\mathbf{a}, \mathbf{b})$ is given by

$$P(\mathbf{a}, \mathbf{b}) = \int_{(0,1)^3} A(\mathbf{a}, \lambda, \alpha)B(\mathbf{b}, \lambda, \beta)\rho(\lambda, \alpha, \beta)d\lambda d\alpha d\beta$$
and where $\rho$ does not explicitly depend on $a$ and $b$. Suppose that the parameters $\alpha$ and $\beta$ are allowed to depend on $a$ and $b$ respectively, just as the $\Lambda_{a,t}^1$ and $\Lambda_{b,t}^2$ above are allowed to do. To fix the ideas, let

$$\alpha = x\|a\|, \beta = y\|b\|, 0 < x, y \leq 1 \quad (61)$$

Then, taking the Jacobian into account, we obtain for the integral in Eq.(60)

$$P(a, b) = \int_{(0,1]^3} A(a, \lambda, x\|a\|)B(b, \lambda, y\|b\|)\rho(\lambda, x\|a\|, y\|b\|)\|a\|\|b\|^{-1}y\|b\|^{-1}d\lambda dx dy \quad (62)$$

Of course, $A$ as well as $B$ can be rewritten as $A^* (a, \delta)$ and $B^* (b, \delta)$ where $\delta$ ranges in an enlarged parameter set. However, for $\rho$ to be independent of $a$ and $b$, $\rho$ will have to be a function of very special form. We conclude that the generalized scenario of station dependent parameters cannot be handled by Bell’s approach.

We would like to add two remarks. First, in the above discussion we have considered only the case where the parameters $\alpha$ and $\beta$ do not depend on a common parameter, unlike $\Lambda_{a,t}^1$ and $\Lambda_{b,t}^2$ which both depend on $\omega$. The resulting integrals would become line or surface integrals, making $\rho$ not only dependent on $a$ and $b$ but also on the surface defining these integrals. Second, we have skirted the issue of properly defining the integrals corresponding to Eq.(4), after the mixed parameters are added in. Since these are stochastic processes indexed by $a, t$ these integrals would have to be stochastic integrals. Since we do not know the precise nature of the mixed parameters, the question of measurability would be difficult to address.

## 9 APPENDIX 2/LEMMA

The lemma is an immediate consequence of theorem 4.21 of Schumaker [10] for the special values of $m = 3$, $l = 0$, $r = n$, and the knots chosen to be $y_\nu = \frac{\nu}{n}$ with $\nu = 0, \pm 1, \pm 2, \ldots$. Then by Schumaker’s [10] equation (4.33) we have (dropping the fixed superscript 3 of $N_i^3$):

$$(y - x)^2 = \sum_{i=-2}^{n} \phi_{i,3}(y)N_i(x) \text{ for all } 0 \leq x \leq 1 \text{ and } y \in R \quad (63)$$

Here

$$\phi_{i,3}(y) = (y - y_{i+1})(y - y_{i+2}) \quad (64)$$

and

$$0 \leq N_i(x) \leq 1 \text{ for all } x \quad (65)$$
We now restrict $y$ to $0 \leq y \leq 1$. Then for $-2 \leq i \leq n$, we have $0 \leq \phi_{i,3}(y) < 2$ unless $y \in [y_{i+1}, y_{i+2}]$. Since we must avoid negative $\phi$, we set $\phi = 0$ in this interval by defining new functions $\psi$:

$$\psi_i(y) = 0 \text{ if } y \in [y_{i+1}, y_{i+2}]$$

$$\psi_i(y) = \phi_{i,3}(y) \text{ otherwise}$$

Since for $y \in [y_{i+1}, y_{i+2}]$ we have

$$| (y - y_{i+1})(y - y_{i+2}) | \leq \frac{1}{4n^2} \quad (66)$$

we have

$$0 \leq \sum_{i=-2}^{n} (\psi_i(y)N_i(x) - \phi_{i,3}(y)N_i(x)) \leq \frac{1}{4n^2} \quad (67)$$

because for any given $y$ and for all $x$, only one term in the sum can be off by at most $\frac{1}{4n^2}$. This proves the lemma.

10 APPENDIX 3

Assume that the functions $A_a$ and $B_b$ (considered as random variables) are independent. Then by Eq.(6)

$$E\{A_a B_b\} = E\{A_a\} E\{B_b\} = F(a)G(b) \quad (68)$$

where $F$ and $G$ are functions that depend only on $a$ and $b$, respectively. But this is in contradiction with Eq.(6) since

$$-a \cdot b = F(a)G(b) \quad (69)$$

and substitution of the three pairs $(1, 0, 0)/(0, 1, 0)$, $(1, 0, 0)/(1, 0, 0)$ as well as the pair $(0, 1, 0)/(0, 1, 0)$ for $a/b$ gives a contradiction.

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