TOPICAL REVIEW

From geometry to numerics: interdisciplinary aspects in mathematical and numerical relativity

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Abstract
This paper reviews some aspects in the current relationship between mathematical and numerical general relativity. Focus is placed on the description of isolated systems, with a particular emphasis on recent developments in the study of black holes. Ideas concerning asymptotic flatness, the initial-value problem, the constraint equations, evolution formalisms, geometric inequalities and quasi-local black hole horizons are discussed in light of the interaction between numerical and mathematical relativists.

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1. Introduction

1.1. Objectives

In this review, we focus on specific problems which are of interest both for numerical relativists and geometers. A number of review papers have been devoted in the last few years to the technical aspects and state of the art of each respective domain—e.g. Andersson [12], Ashtekar and Krishnan [52], Bartnik and Isenberg [65], Baumgarte and Shapiro [71], Berger [88], Booth [117], Cook [187], Friedrich [261], Friedrich and Rendall [259], Gourgoulhon [288, 290], Gundlach and Martín-García [299], Klainerman and Nicolò [360], Krishnan [371], Lehner [375], Lehner and Reula [374], Rendall [437], Reula [439], Shinkai and Yoneda [469], Thornburg [493], Winicour [518] and also [2]. More theme-specific reviews will be referred to in the corresponding sections. The aim here is not the exhaustive description of the respective specific tools, but rather to identify and discuss the conceptual or structural challenges in
general relativity which represent good candidates for a close collaboration between geometers and numerical relativists.

This review is inspired by a workshop with a similar name that took place in the context of the General Relativity Trimester held at the Institut Henri Poincaré, Paris, from 20 to 24 November, 2006 [1], which brought together specialists from both the mathematical and numerical ends of general relativity (GR). Due to the broad range of topics covered in the workshop, we have taken the methodological decision to restrict our attention to the numerical and mathematical aspects of the description of isolated systems in GR. This somehow reflects the underlying ultimate interests of the organizers, biased towards astrophysically motivated problems not including cosmology. Unavoidably, not all the topics discussed at the time are covered in this review and, on the other hand, some of the topics discussed in the review did not have a counterpart talk in the workshop.

1.2. General thoughts

1.2.1. Why from geometry to numerics? General relativity is a theory that describes the gravitational interaction as a manifestation of the geometry of spacetime. It is therefore natural to expect that a geometrical perspective may be helpful, and even sometimes fundamental, for the further development of the theory at both conceptual and practical levels; the latter is understood as the explicit construction of solutions. Does this point of view live up to the raised expectations?

In our opinion, the answer to this last question has two facets. On the one hand, as discussed in [261], understanding GR can be seen as tantamount to understanding the properties of its solutions, and in this sense ‘getting qualitative and quantitative (respectively, analytical, theoretical and numerical) control on the long-time evolution of gravitational fields under general assumptions is still the most important open problem in classical general relativity’. In this context, there seems to be a plenty of occasions for the successful interaction between geometry and numerics, not only in the direction suggested by the title of the review but also on the converse one, from numerics to geometry.

Regarding the second facet, arguably, the other great challenge for the theory is to make full contact with observations—in particular, nowadays, in the area of gravitational wave physics. The nature of this specific endeavour requires the efficient calculation of spacetimes describing isolated systems. This calculation, to be physically realistic has to be performed using numerical simulations of some particular formulation of the Einstein equations. In this context, the fruitfulness of using geometry as a guideline in numerical applications seems much harder to assess, since the understanding of a particular analytical aspect of a geometric problem does not ensure its successful numerical implementation. Furthermore, it should not be forgotten that one is talking about the (potential) interaction of two communities with a different history, traditions, languages and in great measure, objectives.

1.2.2. Numerical relativity and relativistic astrophysics. As somehow hinted in the previous paragraphs, it is natural to assume that a ‘generic’ relativist would have an interest in studying both the nature of solutions of theory, and hence improving his or her understanding of the theory, and also of making a contact with the observation—which would then help us to set the context and validity of the theory. Given the broad spectrum of research topics that relativists deal with, it is to be expected that the particular choice of research problem will set the emphasis on either conceptual or observational aspects. In particular, this applies to the case of a numerical relativist. For example, from a simplified point of view, a ‘gravitational collapse code’ can potentially be read as a tool for either understanding conceptual issues like
cosmic censorship or as a tool for predicting astrophysical phenomena such as supernovae explosions. In the first case, it is natural for the numerical relativist to gain an insight and even try to simplify the code through a geometric viewpoint. In the second case, she or he has no option but to attempt to include all physical ingredients that are believed to play a role. Therefore, geometric insights may be comparatively less relevant. In any case, whether a geometry point of view is used or not reflects, ultimately, in the way the code is constructed and read. In summary, a given problem in numerical relativity can be labelled as either geometric or astrophysical numerical relativity. The tenet of this review is that the effects of the interaction between mathematical and numerical relativists are bound to be more fruitful in the former case, while in the latter the rewards would be more indirect. This duality will be a recurrent theme in the review.

1.3. Tensions and synergy between geometry and numerics

As a first contact with the topic of the review, we present some general reflexions which illustrate the possible tensions and synergies to be expected as a result of the interaction between geometry and numerics.

1.3.1. Global versus local. The global issues dominate modern mathematics. On the other hand, most numerical simulations are local in time. This leads to a tension between the tools and goals in each community. An example of how different the points of view could be is given by the notion of black hole. The classical definition of a black hole makes use of global ingredients—e.g. [177]. In contrast, everyday numerical simulations make use of quasi-local characterizations related to the notion of trapped region. This tension reaches its climax in statements like ‘there are no results about the existence of asymptotically flat vacuum black hole spacetimes with radiation’—see the discussion in section 2.2—by a mathematical relativist that can provoke the smile of a numerical relativist. However, this tension is not as dramatic as it appears at first sight. On the one hand, it is very much possible that numerical simulations of binary black holes could provide the missing clues for an eventual global existence proof of the existence of dynamical black hole spacetimes [428]. On the other hand, the global framework provided by the cosmic censorship conjecture acts precisely as a guarantor of some of the technical tools used in numerical simulations—for example, excision or punctures.

1.3.2. Good geometric or analytic properties as a hope for good numerics. To a geometer, who is inexperienced in the field of numerics, one of the first reality checks he or she has to confront is that the analytic well posedness and the convergence of a numerical solution to the analytic solution do not guarantee the long-term stability of the simulation—this issue has been clearly discussed in [53]. This last point, raised before the big breakthroughs in the simulation of black hole binaries, is still valid as regards to the realistic and accurate extraction of physics [59, 141, 425]. For this, it is possible that an analytic or geometric insight may provide the crucial ingredient.

1.3.3. Geometry as a way of prescribing and extracting physics. Numerics relies on the use of coordinates. This results, unavoidably, in ambiguities in the extraction of physics if geometric notions are not employed. However, these methods are global and hard to implement numerically. In practical applications, one could use, for example, perturbative or post-Newtonian notions to extract the physics. Nevertheless, this approach already relies on the acceptance of a certain global behaviour of the spacetime. But more importantly, the
recent advances in the numerical simulations of spacetimes could provide the opportunity for the reassessment of geometrical objects as tools for the extraction of physics.

We conclude the section presenting some examples which give a taste of both the difficulties and rewards of the interaction between geometry and numerics.

- **Application of ideas from dynamical horizons to the extraction of physical parameters.** The development of some quasi-local approaches to black horizons—see section 6.2—has been directly inspired by the needs of the numerical implementations. In particular, dynamical horizons have already provided first examples of successful interaction in the calculation of mass, angular momentum and associated parameters—cf Krishnan’s IHP talk and [57, 146, 372, 456, 457].

- **Semiglobal evolution from small data.** The existence results of the development of hyperboloidal initial data close to Minkowski by Friedrich [248] have been illustrated by the numerical simulations of Hübner—see [329].

- **Hawking mass.** The straightforward implementation of some natural geometric notions often meets with numerical difficulties. This has been exemplified for the case of the Hawking mass in Schnetter’s IHP talk [1].

- **Calculation of high-order derivatives.** Any numerical simulation contains noise which tends to be amplified in the numerical evaluation of derivatives—see again Schnetter’s IHP talk [1]. This, for instance, complicates the implementation of constructions involving derivatives of the curvature tensor—like in the case of the metric equivalence problem in GR [356, 357].

- **Handling of coordinates.** Sometimes the use of coordinates, which are easy to handle from a numerical point of view, could lead to complications such as coordinate singularities and/or singular equations—see, e.g., Schnetter’s IHP talk [1]. In some circumstances, a detailed geometric analysis does suggest a reformulation of the problem which would solve this particular tension [28, 29].

- **Penrose inequality.** Classical bounds on energy loss in the head-on collision of black holes using the geometric Penrose inequality—see, e.g. [277]—have been shown to be overtly optimistic by numerical simulations. On the other hand, lines of thought based on the Penrose inequality have been useful in the discussion of initial data [210].

- **Geometry-inspired initial data sets.** Attractive geometric properties have been employed and prescribed in the construction of initial data sets for black hole spacetimes—see among others, e.g. [197, 199, 350, 438]. However, the numerical usefulness of these geometric inputs is still to be assessed.

- **Geometry-inspired evolution systems.** The close interaction between geometry and numerics is illustrated in the application of geometry as a guideline in the development and implementation of certain GR evolution systems—e.g. Z4 formalism in [107] and also [442, 446, 526, 527]. However, the role of geometry is not so evident in most numerical codes—see section 5.

It may be of interest for the reader to look at the list of successes given in [12].

### 1.4. Cautionary notes

Although this review stems from a workshop with a similar name [1], it should not be regarded as a proceedings or a systematic account of the former. We have used the diverse contributions to inspire a reflection on the interaction between the geometric or analytic methods and numerical tools in the context of GR. In this sense, unavoidably, it projects our personal prejudices and is certainly not comprehensive. We are indebted to all the participants for the
effort put on their presentations and apologize beforehand in the case of any misrepresentation, or omission, of their ideas. The presentations during the workshop will be referenced in a bundle as IHP talks.

It is clear—although it is already evident from the introductory paragraphs—that the word geometry has been used in a loose sense that encompasses more than differential geometry and includes, for example, global and local analysis, PDE theory or group theory.

As already mentioned, our discussion will be centred in the numerical and mathematical aspects of isolated systems in GR. There are other streams of relativity which have experienced a fruitful interaction between the analytical and numerical communities: namely, on the one hand higher dimensional spacetimes, and on the other the study of cosmological spacetimes. In the latter case, this interaction has been reported in review [88]—see also the discussion in [12]. Without going into too much detail, we just mention that the analysts (geometers) working in mathematical cosmology have gained much from insights obtained through numerical experiments in which, for example, the behaviour of Gowdy and Bianchi IX spacetimes is explored towards the initial singularity—for a recent work on the subject, see [93]. In particular, it was using these sorts of numerical experiments that the existence of the so-called spikes—non-smooth behaviour and big gradients of certain field quantities as one approaches the singularity—was first attested in the Gowdy spacetimes [88]. The analytical existence and nature of these spikes has since been thoroughly analysed. This is a prime example of the use of numeric techniques as a tool to explore the nonlinear behaviour of the gravitational field.

We start from the premise that mathematical relativity and numerical relativity are research areas on their own right. This review is addressed to those researchers working in any of these two communities who feel that there is a place for a fruitful interaction. We assume a broad general knowledge of GR. We will not be extremely rigorous in our mathematical presentation and will refer the reader to the specialized literature.

The review is structured as follows. Section 2 reviews some conceptual issues concerning the notion of isolated bodies in GR, including asymptotic flatness and the black holes. In section 3, the initial-value problem in GR is presented as the appropriate approach to the generic construction of spacetimes. In particular, section 3.1 reviews the different types of initial-value problems. Section 4 explores some aspects of the construction of solutions to the constraint equations which we believe are, or could be, of interest for numerical applications—the topology of initial hypersurfaces, the conformal method, the thin sandwich and the recent developments in gluing constructions. Section 5 discusses various aspects of evolution formalisms. Here, particular emphasis is given to the initial-value problems of Cauchy type, although some observations concerning the characteristic and hyperboloidal problems are made. There is, also, a discussion of the conformal field equations. Section 6 reviews some aspects of the modern description of black holes including recent results about uniqueness and rigidity of the Kerr spacetime, quasi-local definitions of black holes, geometric inequalities, helical Killing vectors and aspects of ‘puncture’ black hole evolutions. Section 7 discusses miscellaneous topics that did not find an adequate place in the structure of the review, but that we believe could not go unmentioned. Finally, section 8 provides some conclusions and a brief list of topics where, we believe, the interaction between geometry and numerics will acquire particular relevance in the next few years.

2. Isolated systems in general relativity

As mentioned before, this review will be concerned with the solutions of general relativity describing isolated systems. GR is formulated as a gauge theory of gravity, usually in terms
of the Einstein field equations (EFE)

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \]  

(1)

where \( g_{\mu\nu} \) denotes the metric of the spacetime, \( R_{\mu\nu} \) is its corresponding Ricci tensor, \( R \) is the Ricci scalar and \( T_{\mu\nu} \) is the so-called stress–energy tensor—in geometric units \( c = G = 1 \).

Solutions to the EFE correspond to geometries rather than metrics given in a specific coordinate system, since the metrics related to spacetime diffeomorphisms are physically equivalent. As a consequence of this, equation (1) understood as a partial differential equations (PDE) system for the metric components \( g_{\mu\nu} \) constitutes an underdetermined (incomplete) system that cannot produce a unique solution. An extra ingredient, namely a gauge fixing procedure, must be provided to establish a standard PDE problem—note this remark applies to any geometrically introduced PDE. Here we adopt a broad notion of gauge which includes not only a coordinate choice but could also contemplate, for example, a tetrad choice and even perhaps a choice of conformal factor.

We are interested in solving equation (1) subject to boundary conditions describing an isolated system for which the effects of the cosmological expansion are neglected. The first way of approaching this problem is the construction of exact solutions to equation (1) by making assumptions on the symmetries of the spacetime and/or on the algebraic structure of their curvature tensors—the most comprehensive treatise on exact solutions is [480]; reviews concerned with the physical aspects of exact solutions can be found in [97, 98]. This way of doing things has provided a deep conceptual and physical insight, as it has supplied the tools of most of the observational predictions of GR. Furthermore, it is the basis of perturbative analysis.

However, if one is interested in making statements about generic spacetimes having the right kind of boundary conditions and on how to construct them in a systematic manner, then one is led to consider an initial-value problem or more generally an initial boundary-value problem. Initial-value problems can be of Cauchy type, that is, the initial data are prescribed on a Cauchy hypersurface\(^4\), or not, like in the case of the characteristic and the hyperboloidal [245] initial-value problems—for a more complete discussion on these ideas, see section 3.1. Only Cauchy initial-value problems allow, in principle, the reconstruction of the whole spacetime, while the characteristic and hyperboloidal problems allow, at most, the reconstruction of the domain of influence of the initial hypersurface.

The overall strategy is to prescribe suitable initial data on the initial hypersurface and then to evolve them by means of the EFE and the appropriate equations for the matter part. This is very much in the spirit of the approach taken by numerical relativists. There are, however, a number of caveats to this approach—such as the existence and uniqueness of solutions, stability and global issues including cosmic censorship—which are the concern of mathematical relativists (see [451] for a review addressing some open problems in mathematical relativity and articulated around the notion of Cauchy hypersurface). In particular, the satisfactory resolution of these caveats by the mathematical community is the guarantor of the consistency of the strategy employed by the numerical relativists.

Very often, the needs of the numerical community are well ahead of the developments of the mathematical community. This applies, in particular, to the study of the motion of isolated bodies in GR, a problem of clear interest for astrophysicists which has been dealt with numerically for a long time now—see, e.g., [76]. The existence issue, which is taken for granted in the numerical community, is a challenging mathematical problem. We emphasize that this last point represents a crucial aspect in the overall coherence of the problem. The

\(^4\) For a discussion of the relationship between global hyperbolic spacetimes and the existence and properties of smooth Cauchy surfaces, in particular the existence of Cauchy time functions, see [90–92].
The physical intuition suggests that the spacetime describing an isolated body far away from it should be Minkowskian in some appropriated sense. Such a behaviour is generically referred to as asymptotic flatness. As already mentioned, checks on the consistency of the theory require a rigorous formulation of this physical intuition.

The metric of an asymptotically flat spacetime, when expanded in terms of some convenient distance parameter, should coincide with the flat Minkowski metric at leading order. The physics of the problem is, on the other hand, encoded in the low-order terms. In particular, one should expect to find there the information about the mass and radiative properties of the system. Now, it is natural to ask, first, whether it is possible to rephrase the above statements in a coordinate-independent way; and second, if one can make generic statements about the properties of the low-order terms in the asymptotic expansions of the metric of an isolated system. To this end, Penrose introduced the more specific notion of asymptotic simplicity—see, e.g. [413, 419] for a precise definition. Roughly speaking, a spacetime $(\mathcal{M}, g_{\mu\nu})$ is said to be asymptotically simple if there exists a positive scalar $\Omega$ (the conformal factor) and a spacetime with boundary $(\hat{\mathcal{M}}, \hat{g}_{\mu\nu})$—the compactified, unphysical spacetime—with $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ such that the boundary of $\mathcal{M}$, defined by $\Omega = 0$, has a structure similar to that of the standard compactification of Minkowski spacetime—that is, the conformal boundary consists, at least, of a null hypersurface (null infinity) $\mathcal{I} = \mathcal{I}^− \cup \mathcal{I}^+$. In the particular case of the Minkowski spacetime, the conformal boundary consists, in addition, of three points: $i^0$, spatial infinity, and $i^−$ and $i^+$, respectively, the past and future timelike infinities. In general, these additional points will not be present in the conformal completion of a generic asymptotically simple spacetime. However, if the spacetime is constructed as the development of asymptotically Euclidean initial data—see equation (2)—then $i^0$ will be a part of the conformal boundary by construction. The situation with respect to $i^−$ and $i^+$ is more complicated. If the relevant spacetimes contain black holes, then there is no reason to expect that $i^±$ be present in the conformal completion, or if they are present, then their structure may not be that of a point—this is a difficult open question which will benefit from the input from numerical simulations. In any case, and to summarize, the definition of asymptotic simplicity gets around these issues by only being concerned with null infinity. The definition of asymptotic simplicity is geometric in that it avoids the use of coordinates. Penrose’s original definition requires the boundary of the unphysical spacetime to be smooth ($C^\infty$). Penrose’s original idea was to use the notion of asymptotic simplicity as a way of providing a characterization of isolated systems in GR: the far fields of spacetimes describing isolated systems should be asymptotically simple in the sense that they admit a smooth conformal compactification to null infinity. The latter suggestion is known in the literature by the name of Penrose’s proposal.

The conceptual and practical advantages of the use of the compactified picture to describe isolated systems have been discussed at length in the literature—see, e.g. [36, 250, 251, 253, 254, 276]. Here we just mention that (i) the use of the compactified picture allows us to rephrase questions concerning the asymptotic decay of fields by questions of their differentiability at
the conformal boundary; (ii) if the spacetime is asymptotically simple, then it is possible to deduce a certain asymptotic behaviour for the components of the Weyl tensor known as the peeling behaviour—see, e.g. [237]. If the spacetime is such that it would not admit a smooth compactification—say, just $C^k$ instead of being $C^\infty$—then a part of this formalism can still be recovered, but one has to be more careful—see, e.g. [183].

The idea of asymptotic flatness and the notion of asymptotic simplicity are inspired by the analysis of exact solutions to the EFE. If one wants to establish more general properties of this class of spacetimes, one has to resort to a formulation of the problem based on an initial-value problem and then make use of analytic techniques to obtain qualitative information about the solution or construct the solution numerically. Here, again, intuition suggests that the right initial conditions for obtaining an asymptotically flat spacetime are asymptotically Euclidean ones. More precisely, one could require an initial hypersurface $\Sigma$ with at least one asymptotic end (a region which is topologically $\mathbb{R}^3$ minus a ball) in which one can introduce the coordinates $x^i$ such that

$$\gamma_{ij} = \left(1 + \frac{2m}{|x|}\right) \delta_{ij} + O\left(\frac{1}{|x|^2}\right), \quad K_{ij} = O\left(\frac{1}{|x|^2}\right),$$

where $|x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, and $\gamma_{ij}$ and $K_{ij}$ are, respectively, the first and second fundamental forms of $\Sigma$, and $m$ is a constant—see section 4 for more on this. More general notions of asymptotic Euclideanity have been considered in the literature—see, e.g. [152, 153, 173]. The one given here, has a well-defined ADM mass at the asymptotic end in the question.

The first results concerning the existence of solutions to the vacuum EFE under the above boundary conditions can be traced back to [232] in which local existence in time was shown to hold—well posedness. Choquet-Bruhat and Geroch introduced the notion of a maximal future development of an initial data set, and showed that for each given initial data set there is a unique maximal future development [162]. The work by Penrose has given a negative answer to the question of completeness in his singularity theorem [414]: given initial data where the initial Cauchy hypersurface is non-compact and complete, if the hypersurface contains a closed trapped surface—see section 6.2.2—then the corresponding maximal future development is geodesically incomplete. A further milestone was the resolution of the so-called boost problem in GR given in [160, 173]. Along these lines, the existence results of Friedrich [248] where it has been shown that the development of sufficiently small initial data given on an hyperboloid is complete. It should be noted that a hyperboloid is not a Cauchy hypersurface, and in this sense the results of [248] are semiglobal. The analysis of the nonlinear stability of the Minkowski spacetime of [172] has shown, among other things, the existence of vacuum radiative spacetimes which are asymptotically flat. More precisely, they show that every asymptotically flat initial data which is globally close to the trivial data gives rise to a solution which is a complete spacetime tending to the Minkowski spacetime along any geodesic. There are no additional restrictions on the data. The result gives a precise description of the asymptotic behaviour at null infinity. Remarkably, however, the analysis is insufficient to guarantee the peeling behaviour of the solutions. An alternative proof has been provided by Lindblad and Rodnianski [379]. This provides a global stability result of Minkowski spacetime for the vacuum EFE in the wave coordinate gauge—see section 5.1. The initial data coincides with the Schwarzschild solution in the neighbourhood of spacelike infinity. This result is less precise as far as the asymptotic behaviour is concerned. More recently, following the spirit of Christodoulou and Klainerman’s proof, Bieri—see her IHP talk [1]—has considered more general asymptotically Euclidean initial data—i.e. with less decay and one less derivative—and shown the existence of a solution which is a geodesically complete spacetime, tending to the Minkowski spacetime at infinity along any geodesic. The
solutions have finite ADM energy, but the angular momentum on the initial hypersurface is not necessarily well defined. The question answered by Bieri’s analysis was what is the most general non-trivial asymptotically flat initial data set giving rise to a maximal development that is complete. The remaining open question in this programme is to find sharp criteria for non-trivial asymptotically flat initial data sets to give rise to complete maximal developments.

In what concerns the existence of spacetimes which satisfy the peeling behaviour, Klainerman and Nicolo [361, 362] have shown that the peeling behaviour is satisfied by the development of a big class of asymptotically Euclidean spacetimes. This result is, however, not sharp for it can be seen that initial data for the Kerr spacetime—which satisfies the peeling behaviour as it is a stationary spacetime, see, e.g. [198, 213]—are not contained in the class of initial data sets they considered.

Regarding the related question of the existence of asymptotically simple spacetimes, it is now known that there is indeed a big class of spacetimes with this property—see [179]. The essential ingredients for the latter result are the semiglobal results of [248], and a refined version of the initial data sets which can be constructed using the Corvino–Schoen method. This construction allows us to glue fairly arbitrary asymptotically Euclidean initial data sets to a Schwarzschild asymptotic end—for more on this gluing construction, see section 4.5. It has to be mentioned that the class of spacetimes obtained in [179] are very special because by construction, they do not contain any radiation near spatial infinity. The fact that the initial data are Schwarzschildian near infinity, implies that the Newman–Penrose constants of the spacetime vanish—the Newman–Penrose constants are absolutely conserved quantities defined in the cuts on null infinity [405, 406]. This value propagates along the generators of null infinity, and allows us to conclude that the Weyl tensor vanishes at future timelike infinity $i^+$—see [257, 260].

Given the state of affairs described in the last paragraph, the challenge is now to find the sharp criteria on an initial data set to obtain an asymptotically simple spacetime. Seminal work in this direction is [252]—see also [256]—where a convenient framework to discuss the behaviour of the region of spacetime where null and spatial infinity meet—the so-called cylinder at spatial infinity—has been introduced. It is important to point out that this work and related ones—see, e.g. [500, 501, 503, 504]—are carried out in the conformally rescaled spacetime (unphysical spacetime) and make use of the so-called conformal field equations, a generalization of the EFE which exploits the extra gauge freedom—a conformal one—that has been introduced via the notion of asymptotic simplicity. The programme started in [252] and continued in [256, 500, 501, 503, 504] has showed that there is a hierarchy of obstructions to the smoothness of infinity which control the smoothness of the spacetimes at null infinity—see also Valiente Kroon’s IHP talk [1]. These obstructions are expressible in terms of parts of the initial data of the spacetime. A further analysis of the obstructions point towards the following conjecture: if the development of asymptotically Euclidean initial data admits a smooth conformal compactification (in the sense of Penrose) at both past and future null infinities, then the initial set is stationary near infinity—see [500, 501, 503, 504]. If shown to be true, the latter conjecture would give a prominent role to the initial data sets which can be constructed by means of the Corvino–Schoen gluing construction. It is worth mentioning that the developments of conformally flat initial data sets such as the Brill–Lindquist [129], Misner [401], Bowen–York [124] and Brandt–Brügmann [125], as well as the data used systematically in the numerical simulations of binary black hole mergers—see the end of section 6.4.1—will have a null infinity (if any) which is not smooth. However, the differentiability of the conformal boundary seems to be enough to guarantee the peeling behaviour [505].

Given an arbitrary asymptotically Euclidean initial data set $\left( \Sigma, g_{ij}, K_{ij} \right)$, using the Corvino–Schoen construction one could construct another initial data set $\left( \Sigma', g'_{ij}, K'_{ij} \right)$ which
coincides with the former inside a compact set, but which is exactly stationary near infinity. The development of \((\Sigma, \gamma_{ij}, K_{ij})\) will possess a non-smooth but possibly peeling null infinity. On the other hand, the development of \((\Sigma', \gamma'_{ij}, K'_{ij})\) will be smooth. A natural question is the following: is there any substantial difference between the physics of the two developments? This is a complicated question which will only be answered in a satisfactory manner by a close collaboration between numerical and mathematical relativists. It is very likely that the latter issue will not have any relevance for numerical relativists interested in astrophysical applications. On the other hand, they provide a whole unexplored area for those who want to use the numerical methods to understand the nature of GR.

2.2. Black holes

A black hole is a region of no escape which does not extend out to infinity. Traditionally—see, e.g. [311]—the definition is made using the conformal compactification of spacetime. Namely, if the spacetime admits a suitable null infinity and the causal past of null infinity is globally hyperbolic—i.e. it admits a Cauchy hypersurface—then one defines the black hole region like the spacetime minus the causal past of future null infinity. The boundary of the black hole is the event horizon. A spacetime containing a single black hole contains two natural boundaries, one is null infinity and the other is the event horizon—an inner boundary. The original picture arose from an analysis of spherical symmetry. According to the so-called weak cosmic censorship picture—cf section 6.3—the end state of any generic asymptotically Euclidean initial data set will be a particular stationary black hole spacetime: the Kerr solution. Naked singularities can actually form, but the resulting spacetimes are unstable [157, 170, 171] and non-generic. The rigorous definition of a black hole makes use of global ingredients: in particular it requires the existence of a complete \(I^+.\) As already mentioned, there are no global existence proofs of dynamical black hole vacuum spacetimes. The latter is a point deserving further clarification. There are results concerning the global existence of dynamical black holes in, for example, the spherically symmetric Einstein-scalar field theory—e.g. [194] for the formation of black holes from a regular past—or in the vacuum five-dimensional setting. Regarding the latter example, see [195] where the orbital nonlinear stability—i.e. with respect to perturbations respecting symmetry—of the Schwarzschild–Tangherini black hole is discussed, and thus the existence of a global, non-stationary black hole is established. Unfortunately, these examples are not relevant for astrophysics. The available examples of four-dimensional vacuum dynamical black holes suffer from some pathology—like the Robinson–Trautman spacetimes discussed in [174] with a complete future null infinity, but where the past null infinity is incomplete—or are local—like in the discussion of ‘multi-black hole’ spacetimes of [184] or that of spacetimes with isolated horizons of [376].

Among other things, a proof of the nonlinear stability of the Kerr spacetime would show the existence of a wide class of spacetimes with non-stationary black holes. The idea behind such a proof is to show that initial data which are in some sense close to Kerr initial data will give rise to a black hole spacetime whose global structure will be similar to that of the Kerr spacetime. In addition to proving the existence of dynamical black hole spacetimes, such a result would be an important step towards a proof of the (strong) cosmic censorship conjecture—see section 3. A particular situation in which the latter conjecture has been proved in an asymptotically flat context is the spherically symmetric Einstein-scalar field system studied by Christodoulou—see [166–169, 171]. A crucial property of stationary black holes is the so-called rigidity: that is, the fact that stationarity together with a couple of further assumptions on the behaviour of the horizon imply further symmetries in the spacetime—i.e. axial symmetry. The ideas behind the rigidity theorems have been recently revisited by
Isenberg and Moncrief with the aim of obtaining generalizations which are valid for higher dimensions—cf Isenberg’s IHP talk [1] and section 6.1.1.

2.3. Critique and practicalities

There has been a number of critiques to the notion of asymptotic flatness, most notoriously the ones articulated by Ellis [223, 224]. This critique has cosmological motivations and arose from the desire of analysing the nature of the interaction between local physics and boundary conditions in the expanding Universe. In particular, there was a desire to understand what difference does the use of particular boundary conditions make on results depending in an essential manner on asymptotic flatness, such as the peeling behaviour, the positivity of mass—cf section 6.3—and black hole uniqueness. Ellis argues, for example, that it is unrealistic when considering local gravitational collapse to worry about observations with infinite lifetimes. The latter is what is implied in the usual definition of a black hole—see, however, section 6.2. The alternative to null infinity as a natural boundary of spacetime given in [223] is to consider a worldtube—\textit{finite infinity}—with a timelike boundary located at a spatial radius which is sufficiently far away from the sources. If brought to the context of an initial-value problem, Ellis proposal implies the use of an initial boundary-value problem to describe the physics of isolated systems in which the effects of the Universe are put in by appropriate boundary conditions. The well posedness of the initial boundary-value problem has been first discussed in [258]. The analysis given there shows that if one makes use of a finite infinity then there is no covariant way of prescribing the boundary information. This, in turn, would lead to ambiguities in the extraction of physics. One of the rationales behind the introduction of the \textit{null infinity formalism} was precisely to avoid this sort of problems. For a neat discussion of these issues, with an emphasis to the extraction of radiation, see [235].

This critique is remarkable in that if one takes it away from its cosmological context it could also be employed when addressing the relevance of global issues for numerical relativity. Most of the numerical simulations make an implicit use of the notion of finite infinity by calculating numerically the solution to an initial boundary problem. In this approach, there is always the potential of ‘messing up’ the physics of the problem by setting boundary conditions which are not appropriate—see section 5.4.1. There are, however, also simulations which compactify in space, see [428]. Moreover, the numerical simulations do not cover an infinite time although given the current advances in the field there seems to be no essential difficulty in letting the simulations ‘run as long as it is necessary’.

It is worth noticing that there have been some semiglobal numerical calculations using the conformal field equations which exemplify the hyperboloidal existence results of [247]—see [327–330]. There are some further attempts to calculate portions of the Schwarzschild from hyperboloidal data—[331] unpublished. For a critical review of the successes and problems of this programme, see [333]. More recently, there have been some global calculations of the Schwarzschild spacetime [526, 527]. These calculations make use of a more general version of the conformal field equations and conformal Gauss coordinates. These numerical simulations exemplify the analysis of the type of coordinates given in [255]. We also mention the recent fully pseudo-spectral scheme [319] for solving hyperbolic equations on conformally compactified spacetimes.

The discussion in the previous paragraphs leads to a situation where besides a number of simulations tailor-made for exemplifying some analytical aspects of solutions of the EFE, the majority of the state-of-the-art numerical codes used to simulate spacetimes containing dynamical black holes are not designed for addressing global issues in GR. So, what is
the relevance for the numerical community of the work on global issues carried out by mathematical relativists? As mentioned in the introduction, the role is to ensure that some of the approaches implemented in numerical simulations are justified. For example, if the conjecture on the nonlinear stability of Kerr turned out to be false, then it could happen that the development of an asymptotically Euclidean black hole initial data would not have the asymptotic structure of the Kerr spacetime, and thus measurements of gravitational radiation at finite radius would have to be rethought. Similarly, if the no-hair theorems were not valid, there would be no justification for employing perturbative methods on a Kerr background to analyse the late stages of the evolution of a black hole binary merger. These examples are extreme but highlight the need of having the theoretical framework on sound ground.

Finally, we would like to retake the point raised in [12] that the problem of the stability of the Kerr spacetime opens a natural arena for the interaction of numerical and mathematical relativity. Indeed, it is easy to imagine that tailor-made simulations could provide crucial insights concerning the asymptotic decay of the gravitational field near the horizon. This interaction would be, notwithstanding, on the lines of from numerics to geometry.

3. Initial-value problems

If one wants to discuss generic properties of solutions to the Einstein field equations which describe in some appropriate sense isolated systems, one has to find a systematic way of obtaining these solutions. The only approach to the construction of solutions which seems general enough is that of an initial-value problem (IVP). This mathematically sound approach is perhaps not ideal from a physical point of view, and indeed a number of objections have been raised. One could argue that all the major observational predictions of GR have been obtained by means of the analysis of exact solutions.

A first (conceptual) objection consists of making sure that one is able to reconstruct, starting from the initial data, the whole maximal extension of the spacetime—the strong cosmic censorship [11, 417, 418]. This has implications on causality violations and the predictability of the theory—see, e.g. [451].

Another objection questions to what extent an approach to GR based on an initial-value problem is appropriate to obtain predictions which can be contrasted with measurements. One of the crucial problems behind it is that for a certain system of physical interest there may be many possible ways of constructing physically plausible initial data. This is particularly the case when attempting to construct initial data for spacetimes containing dynamical black holes. In order to construct data for a black hole spacetime, one should be able to find a way of identifying the black hole in the initial data. For these purposes, the notion of event horizon is of no use for it is a global quantity which is only known once the whole spacetime—up to null infinity—is known. Instead, one can use the notion of apparent horizon: for if there is an apparent horizon on the data it must be contained in the black hole region—see section 6.2. In other words, the presence of an apparent horizon indicates the existence of a black hole—if the weak cosmic censorship is true. Furthermore, data with an apparent horizon will be geodesically incomplete as a consequence of Penrose’s singularity theorem [414]—thus, indicating the formation of a singularity. It should be emphasized that a black hole data does not need to contain an apparent horizon, but if it contains one, it must be a black hole. Similarly, it is not clear whether data with two apparent horizons will contain two black holes. However, this is a reasonable assumption to begin with. To make things worse, note that many of the approaches for constructing initial data for black holes are explained more in terms of mathematical advantages, like the simplicity of the equations—like in the case of conformal flatness—than in physically based considerations—see, however, section 6.2.
for an account of the ideas introduced by the quasi-local approach to black holes. To summarize, physics seems to be hard to encode in initial data sets for the EFE.

There have been a number of works devoted to comparing the results of numerical simulations in the search of robust aspects—that is, structures in the numerically calculated spacetimes which are independent of the particular type of initial data being used—see, e.g. [60, 477, 494] and IHP talks by Hannam and O’Murchadha. Crucially, the usefulness of numerical simulations as a source of waveform templates for the detection of gravitational radiation in the current generation of interferometric gravitational wave observatories depends on the assumption of the robustness. From the point of view of a mathematical relativist or a geometer, this issue is essentially open and unexplored. The problem is certainly complicated, and possibly one will have to wait several years before some qualitative statement can be made. The resolution of the nonlinear stability of the Kerr spacetime may provide the first insights and rigorous answers to the question of robustness of initial data sets.

3.1. On the different types of initial-value problems

Initial-value problems can be classified according to the nature of the hypersurface on which the initial data are prescribed. Usually, the initial hypersurface $\Sigma$ is taken to be spacelike. As mentioned in section 2.1, when discussing initial-value problems for isolated systems the spacelike initial hypersurface $\Sigma$ is naturally assumed to be asymptotically Euclidean. This particular type of the initial-value problem in GR will be referred to as a Cauchy initial-value problem as it would allow, in principle, the recovery of the whole spacetime—see [311, 507] for a rigorous discussion of these issues, and [259] for a recent review on the subject. The seminal work of Choquet-Bruhat [232] has shown the well posedness of the Cauchy initial-value problem in GR: an existence theorem local in time—that is, existence is guaranteed for a small time interval, in a neighbourhood of the initial hypersurface. The portion of spacetime recovered by this local existence theorem is called the development of the initial data. At this level, it is not possible to discuss the global existence of solutions without more assumptions or information on the initial data—for further discussion on this, see, for example, review [360]. Nevertheless, it has been shown that the development has a unique maximal extension [162].

Initial-value problems on spacelike hypersurfaces are not necessarily Cauchy initial problems. For example, the hyperboloidal initial-value problem first introduced by Friedrich in [245], on which initial data are prescribed on a spacelike hypersurface—which could be thought of as intersecting null infinity—is not of Cauchy type. An initial-value problem on a hyperboloidal hypersurface would not allow us to recover the whole spacetime. Moreover, it is not possible to find out where the development of hyperboloidal data lies in a globally hyperbolic spacetime, more specifically, one cannot relate the hyperboloidal data with hypothetical Cauchy data.

As pointed out in [481], the Cauchy problem in GR is in some sense natural for theoretical discussions, but strictly speaking it does not necessarily correspond to the way things are done in numerical relativity, where one usually aims to solve the EFE only within a compact domain—cf sections 2.3 and 5.4.1. The latter requires the solution of an initial boundary-value problem, where in addition to the data to be prescribed on a spacelike domain, one also has to prescribe extra data on a timelike boundary. The presence of this extra data—the boundary data—raises the question of which data can and must be specified on the timelike boundary in order to have a solution—and further, whether there are any compatibility conditions required between the initial data and the boundary data. A first rigorous discussion of the boundary-value problem in GR, including a local existence result, has been provided by...
Friedrich and Nagy [258]. A recent IVBP well-posedness result in the context of the harmonic formulation—see section 5.1—has been presented in [368, 369].

The initial data must not necessarily be prescribed on spacelike hypersurfaces. One could also prescribe it on null hypersurfaces, in which case we speak of a characteristic initial-value problem. However, as the domain of influence of a non-singular null hypersurface—i.e. a null hypersurface without caustics—is empty, one needs, for example, to consider data prescribed on two intersecting null hypersurfaces in order to obtain a non-trivial development. A local existence theorem for this sort of problem has been given in [436]. A variant of the problem is to prescribe data on a light cone. There is no existence result for the Einstein equations in this case—see, however, [247].

The characteristic initial-value problem in GR can be traced back to the seminal work of Bondi and collaborators on gravitational radiation in the early 1960s—see [116, 449]. An early systematic discussion of the characteristic problem can be found in [450]. From the point of view of asymptotics, one can formulate the so-called asymptotic characteristic initial-value problem in which initial data are prescribed on both an outgoing light cone and null infinity. The portion of spacetime to be recovered from this initial value problem lies at the past of these null hypersurfaces. Existence results for this type of characteristic initial-value problem have been given in [242–244, 351] in the context of the conformal field equations. An analysis of the well posedness of the problem more along the lines of the original work of Bondi and collaborators can be found in [263]. Another type of characteristic initial-value problem is the $2 + 2$ initial-value problem—see, e.g. [218]—and the initial-value problems with data prescribed at past null infinity of [247, 249].

On a characteristic initial-value problem, a part of the EFE reduces to interior (transport) equations on the null hypersurface which can be recast as ordinary differential equations along the generators of the null hypersurface. Furthermore, the data are free—i.e. not subject to elliptic constraints like in the case of spacelike hypersurfaces, see section 4. This feature was crucially exploited in the seminal work on gravitational radiation. The characteristic problem is naturally adapted to the discussion of gravitational radiation, however, it tends to run into problems when caustics form. Furthermore, physics is hard to prescribe on null hypersurfaces.

Finally, it is noted that an IVP of a mixed type on which data are prescribed on a combination of spacelike and null hypersurfaces has been considered: the Cauchy-characteristic or matching problem—see [94] and review [518].

3.2. Gauge reduction

As has been discussed in the beginning of section 2, the formulation of the EFE as a standard PDE problem requires the adoption of a gauge choice. This reduction process—see, e.g. [259, 261]—involves four different differential systems:

(i) The main evolution—also, reduced or relaxed—system, whose solution provides the spacetime geometry in a certain coordinate system.

(ii) The system of constraints, which controls the permanence in the submanifold of solutions of the theory.

(iii) The gauge system, which allows the fixation of a coordinate system and therefore casts the main evolution system as a genuine PDE system.

(iv) The subsidiary system, consisting of the evolution equations of the auxiliary (subsidiary) constraints introduced in the previous steps and guaranteeing the overall consistency.

The different manners of dealing with the previous points define a variety of evolution formalisms in GR, where relative stress to be placed on systems (i)–(iv) depends on the
specific aspect of the EFE one wants to address. In section 5, we will briefly review some of the more relevant evolution formalisms considered in numerical relativity with a focus on formulations based in the GR Cauchy IVP. Prior to this, one must discuss the system of constraints on a spacelike hypersurface.

4. Initial data for the GR Cauchy problem

The initial data to be prescribed in this IVP problem are given by a pair of symmetric tensor fields \((\gamma_{ij}, K_{ij})\) on the initial \(\Sigma\), respectively, corresponding to the induced Riemannian metric and the extrinsic curvature under the embedding of \(\Sigma\) into \((M, g_{\mu\nu})\). Such an interpretation of \(\Sigma\) such as a spacelike hypersurface of \((M, g_{\mu\nu})\) demands the fulfilment of the so-called Hamiltonian and momentum constraints, respectively,

\begin{align*}
(3) \quad & R^{ij} - K_{ij}K^{ij} + K^2 = 16\pi \rho, \\
(4) \quad & D_j(K^{ij} - \gamma^{ij}K) = 8\pi J^j,
\end{align*}

where \(R^{ij}\) and \(D_i\) are, respectively, the Ricci scalar and the Levi-Civita connection associated with \(\gamma_{ij}\); \(K\) is the trace of \(K_{ij}\), \(\rho\) is the energy density and \(J^j\) is the current vector. An initial data set is said to be maximal if \(K = 0\). If in addition \(K_{ij} = 0\), then it is called time symmetric.

We now proceed to discuss some aspects concerning the way the Einstein constraints (3) and (4) are solved, putting some emphasis on the relation of these ideas to numerical implementations—a major review on general aspects of the constraint equations can be found in [65]; see also [187, 290, 421] for a numerical counterpart. For concreteness we shall concentrate our discussion on the vacuum constraint equations.

4.1. Topology of the initial hypersurface

As mentioned already in section 2, from a Cauchy initial-value problem point of view, the natural requirement to obtain a spacetime describing an isolated body is to have an initial hypersurface \(\Sigma\) that has at least one asymptotically Euclidean end—see equation (2). The theory of hyperboloidal hypersurfaces is less developed, although some of the essential ideas of the asymptotically Euclidean setting can be retaken—see again [13–16, 65]. Initial hypersurfaces with more than one asymptotic end are said to have a non-trivial topology. Note, however, that the introduction of multiple asymptotic regions is not the only way of having initial data sets with non-trivial topology. The introduction of a wormhole connecting two points in a hypersurface would also result in a non-trivial topology. This was first exemplified in [400]; with the use of the Isenberg–Mazzeo–Pollack (IMP) gluing techniques—see section 4.4—it is now possible to construct this type of initial data in a systematic way.

Initial hypersurfaces with non-trivial hypersurfaces have been routinely used in the mathematical literature as a way of manufacturing initial data sets for black hole collisions. The simplest example of the latter are the usual time-symmetric initial data for the Schwarzschild spacetimes in isotropic coordinates, which contain two asymptotic regions connected by a so-called Einstein–Rosen bridge, throat or wormhole. This example can be readily generalized to contain \(n + 1\) asymptotic regions rendering the so-called Brill–Lindquist initial data [129]. Another approach, which renders time-symmetric initial data on a hypersurface \(\Sigma\) containing two asymptotic regions and \(n\) wormholes and can be interpreted as the data for \(n\) black holes—the Misner initial data—was given in [401].

The first step towards the construction of non-time-symmetric data for black holes was given by Bowen and York [124], where an analytic solution of the momentum constraint in the
conformally flat case was presented, the Bowen–York second fundamental form—a discussion of the general solution of the so-called Euclidean momentum constraint can be found in, e.g. [85, 206]. The so-called Brandt–Brügmann puncture initial data [125] can be in some ways regarded as the non-time-symmetric generalization of the Brill–Lindquist data. It is important to mention that all the initial data sets discussed in this last paragraph are conformally flat—see section 4.2. But this is merely a simplifying assumption and in no way essential.

The fall-off conditions for $\gamma_{ij}$ and $K_{ij}$ to be asymptotically Euclidean have already been discussed in section 2.1—see equation (2). As in the discussion of boundary conditions for the whole spacetime, the discussion of the fall-off conditions on $\Sigma$ can be geometrically reformulated in terms of a conformally compactified three-dimensional manifold $\hat{\Sigma}$—see [83, 84, 247, 252]. The compactification procedure is, in this circumstance, analogous to that induced by the introduction of stereographic coordinates to compactify $\mathbb{R}^2$ to render the 2-sphere $S^2$. From this point of view, the compactified initial hypersurface $\hat{\Sigma}$ contains, say, $m$ singled out points $\{i_1, \ldots, i_m\}$ corresponding to the $m$ asymptotic ends of the physical initial hypersurface. For example, the standard Schwarzschild initial data render a $\hat{\Sigma}$ with the topology of the 3-sphere $S^3$ and two singled out points. The Brill–Lindquist data for two black holes would render a $\hat{\Sigma}$ with three singled out points, while in the case of the Misner data the resulting $\hat{\Sigma}$ has the topology of a torus with two singled out points.

From a mathematical point of view, working with $\hat{\Sigma}$ instead of $\Sigma$ has several technical advantages. It is simpler to prove the existence of solutions for an elliptic equation on a compact manifold than on a non-compact one. The price one has to pay is that the equations will be singular at the points at infinity $i_1, \ldots, i_m$. However, these singularities are mild. It is also simpler to analyse the fields in terms of local differentiability in a neighbourhood of the points at infinity than in terms of fall-off expansions at infinity in $\Sigma$. This approach to discussing the theory of the constraint equations has been introduced and used in various applications in [82, 197–199, 247, 252].

The puncture initial data used in many of the simulations of the coalescence of black holes, as introduced in [125], is a sort of compromise between using the physical hypersurface $\Sigma$ and the compactified one $\hat{\Sigma}$. For punctured data, one of the asymptotic ends—the so-called reference, or physical asymptotic end—is not compactified. The resulting initial hypersurface $\hat{\Sigma}$ will have the topology of $S^3 \setminus i_1$. The singular behaviour of the constraints near the punctures reflects in a singular behaviour of their solutions that is mild and well understood—it is related to the singular behaviour of the Green functions of the elliptic operators appearing in the constraint equations—see [83, 84]. Numerical relativists interpret this phenomenon as a coordinate singularity at the puncture—see [83, 84, 306, 307]. The attraction of puncture initial data sets for numerical relativists is that they represent black holes in $\mathbb{R}^3$ without excision, and it is well understood how to construct the puncture initial data for any number of boosted, spinning black holes [125, 200].

The relevance for numerical relativists of initial data sets with non-trivial topology stems from the fact that they allow us to model black hole spacetimes by only making use of the vacuum EFE: indeed, if there is a wormhole in the initial data, then it can be concluded that the spacetime will be geodesically incomplete [266]. Note, however, that in all of this argument one is indirectly assuming that the cosmic censorship holds and that the resulting spacetime is really a black hole spacetime. In particular, from a physical point of view one is not interested in what happens in the extra asymptotic ends—see, however, the discussions in section 6.4.2 on why the puncture methods work. From the point of view of a physicist, it is fair to say that the use of initial data sets with non-trivial topology is a trick—there have been, however, some works on trying to analyse the possible physical differences between initial data sets with different topology but describing the same system—see, e.g. [147, 284, 285, 470, 499].
4.2. The conformal method

A standard approach to studying the space of solutions of the constraint equations is the so-called conformal method—or conformal ansatz. The method not only provides a systematic approach to making statements about the existence of solutions to the constraint equations, but also gives a starting point for the numerical computation of the solutions to the equations. A recent account of what is known about solutions to the constraint equations is given in [65]—see also [187, 290, 421].

The conformal method is based on the conformal rescaling of the metric

$$\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij},$$

and on the combined conformal rescaling and tensor splitting of the traceless part $A^{ij}$ of the extrinsic curvature $K^{ij}$ into a transverse-traceless and longitudinal part. Different, non-equivalent, constructions follow from the non-commutativity of the conformal rescaling and tensor splitting process. A key point in all of them is the need of a specific rescaling for the (conformal) traceless part of the extrinsic curvature in order to set the constraint equations in a form which is optimal for their analytic study. In particular, if $K$ is constant, the conformal method reduces the problem of solving the constraint equations to a decoupled elliptic problem.

Much has been learned about solutions of the Einstein constraint equations using the conformal method. However, many other interesting theoretical questions cannot be addressed using these methods. The main problem being that the conformal method produces solutions of the constraint equations, essentially from scratch. It turns out that it is very difficult to encode physics into the conformal metric $\tilde{\gamma}_{ij}$ and in the symmetric transverse-traceless tensor $A^{TT}_{ij}$—which act as seeds of the method. For example, there seems to be no way of constructing systematically initial data for stationary solutions, although there is an a posteriori check for the case of time-symmetric data [201]. Most of the natural simplifying assumptions that are directly suggested by the structure of the conformal method happen to have no essential physical or geometrical content—most notably the assumption of conformal flatness. This is bluntly illustrated by results showing that there are no conformally flat hypersurfaces in the Kerr spacetime [268, 501, 502]. Some examples of solutions to the Einstein constraints where it has been possible to put some desirable physical properties with some level of success can be found in, for example, [197, 198, 210]. In these last references, either a non-flat metric or a different choice of the Bowen–York ansatz for the extrinsic curvature—see [124]—has been used. Another approach to prescribing physics is the use of the tensors $\tilde{\gamma}_{ij}$ and $A_{ij}$ motivated by the post-Newtonian analysis as in [409].

It is noted that given the relation of $K_{ij}$ with the canonical momentum in the Hamiltonian formulation of the theory [35]

$$\pi^{ij} = \sqrt{\gamma}(K\gamma^{ij} - K^{ij}),$$

this view of the constraints can be referred to as a Hamiltonian point of view [422, 523]. No reference to the slicing is present and in particular no reference to the lapse function or shift vector emerges in the formulation of the constraints. This formulation reflects the character of the initial data problem as a one-hypersurface embedding problem. In this context, in particular in the numerical community, the conformal ansatz is also referred to as the extrinsic curvature approach.

4.3. The conformal thin sandwich method

The idea of a thin-sandwich approach to GR—in the spirit of the Jacobi principle in mechanics—seems to have been first introduced by Baierlein, Sharp and Wheeler (BSW) [56].
They conjectured the possibility of specifying the spacetime metric $g_{\mu\nu}$ on two hypersurfaces and then recovering the spacetime in between. In particular, when applied to two infinitesimally closed spatial slices—the BSW thin sandwich (TS) conjecture—this approach would imply a manner of providing initial data for a Cauchy evolution of the Einstein system. The TS conjecture, through its two-hypersurfaces ansatz, proposes a different perspective to the constraint problem as compared to the one-hypersurface embedding of the extrinsic curvature approach. In this setting, one aims at prescribing the induced metric and its time derivative $(\gamma_{ij}, \dot{\gamma}_{ij})$, in what can be seen as a Lagrangian formulation of the problem. However, although this conjecture can be seen to work for particular examples, the analysis in [67] shows that the conjecture fails in generic cases.

More recently—see also previous works [189, 341, 342, 517] and section 6.4.1—a new interpretation of the Einstein constraint equations, partly on the spirit of the TS viewpoint, has been given in [523]. In this approach, the freely specifiable data on the initial hypersurface $\Sigma$ consists of a conformal metric $\tilde{\gamma}_{ij}$ and its time variation $\tilde{u}_{ij} \equiv \dot{\tilde{\gamma}}_{ij}$—which is imposed to be traceless. The latter can be shown to amount to the choice of the conformal metric in two infinitesimally close slices. Using the kinematic relation between the time derivative of the physical 3-metric and the extrinsic curvature, the traceless part $A_{ij}$ can be written explicitly in terms of the lapse function $N$ and shift vector $\beta^i$—cf notation in section 5.2. A function $\tilde{N}$ conformally related to the physical lapse is identified as the naturally freely specifiable parameter. This conformal rescaling, together with the absence of rescaling of the shift vector, permits us to derive the key conformal rescaling for $A_{ij}$ introduced by Lichnerowicz [377] and leading to the decoupling of Hamiltonian and momentum constraints in the constant $K$ case. The specification of the free data $(\tilde{\gamma}_{ij}, \dot{\tilde{\gamma}}_{ij}; K, \tilde{N})$ transforms the constraints into an elliptic system on $\Psi$ and $\beta^i$ similar in form to the one obtained in the extrinsic curvature approach [523]. In this conformal TS approach (CTS), the role of every part of the metric is explicit. The formulation has been extended and improved in [422], which in particular reconciles the Hamiltonian and Lagrangian points of view. The new key feature is the introduction of a weight function $\sigma$ which permits the recovery of the commutativity between conformal rescalings and tensor splittings. Moreover, with the choice $\sigma = 2N$, the vanishing of the transverse-traceless part of the extrinsic curvature $\tilde{A}_{ij}^{TT}$ characterizes stationarity, in agreement with the general (intuitive) understanding of $\tilde{A}_{ij}^{TT}$ as the piece encoding the radiative—or dynamical—degrees of freedom. Regarding the free data, the approach in [422] pushes forward the Lagrangian analogy. Understanding $K$ as a configuration variable, it is proposed the prescription of $\dot{K}$ instead of $\tilde{N}$ at the (first) initial slice. This translates into a fifth elliptic equation for the conformal lapse $\tilde{N}$ which together with the CTS equations gives rise to the extended conformal thin sandwich (XCTS) system—see, e.g. [70, 421, 422] for the list of equations.

One of the advantages of the CTS approach is that when using it, it seems easier to prescribe physics than in the standard approach based on the extrinsic curvature approach. In particular, as discussed in [187, 290], the CTS method motivates definite ansätze for the free part in the construction of stationary initial data—at least in the axisymmetric case. More generally, this CTS approach has also been used to construct binary initial data sets in quasi-equilibrium—see, e.g. [69, 114, 292, 498] or Uryū’s IHP talk [1] for the case of neutron stars; [151, 188, 291, 296, 520] for the case of binary black holes; and [72, 295, 490] for the mixed black hole–neutron star binary systems.

4.3.1. Uniqueness issues in the XCTS system. In contrast to the CTS and closely related elliptic systems—see, e.g. [411, 412, 521] and also further references in [422, 523]—there is to date a lack of systematic analyses of the mathematical properties of XCTS system. In particular, no definite results are available about existence conditions in this system, and very
little is known about uniqueness. Numerical evidence of the non-uniqueness of solutions of the XCTS elliptic system has been found [423]. This evidence is of relevance as the XCTS system or somewhat related ideas are used to construct initial data for binary neutron star and black hole binary systems—cf the literature listed in the previous paragraph. Moreover, it also has implications for the groups implementing constrained evolution schemes—cf section 5.2.2.

The first analysis of the non-uniqueness issue, looking in particular at the question of genericity, has been given in [70]. The most complete analysis available of the non-uniqueness of the solutions to the XCTS system, using Lyapunov–Schmidt bifurcation theory, has been given in [510]—see also Walsh’s IHP talk [1]. It is found that if the linearized system has a kernel for sufficiently large data, then there exists a parabolic branching of the solutions as the one happening in [423]. The prosaic reason for the existence of the non-trivial kernel is the presence of a wrong sign in one of the equations of the XCTS system—more precisely, in the equation for the lapse. Thus, standard methods based on the use of a maximum principle cannot be employed. The numerical evidence shows that the assumption of a non-trivial kernel is certainly not far-fetched, and actually occurs in numerical implementations. The situation gets even worse if a treatment of boundary conditions is required—see, e.g. [346].

The issue of the non-uniqueness of solutions to the XCTS equations is an example of the tension underlying the relation between numeric and mathematical relativists: a successful method of prescribing physics into numerical simulations with non-desired mathematical properties. In the absence of better alternatives, the numerical community seems prepared to take the risk.

4.4. Construction of initial data sets using the Isenberg–Mazzeo–Pollack gluing

In recent years, it has been possible to address a number of issues concerning properties of initial data sets using a tool of geometric analysis: gluing—cf Pollack’s IHP talk. While the conformal method is a procedure to construct solutions to the constraint equations from scratch, gluing constructions allow us to obtain new solutions by means of direct sums of two already known solutions. Analytic gluing techniques have played a prominent role in many areas of differential geometry: some notable applications include the study of the topology of 4-manifolds and the study of minimal and constant mean curvature surfaces in Euclidean 3-space.

In [182, 338, 339] a particular gluing construction for solutions to the Einstein constraint equations was developed under the assumption of a constant $K$—the Isenberg–Mazzeo–Pollack (IMP) gluing. This construction allows us to demonstrate how spacetimes can be joined by means of a geometric connected sum, or how a wormhole can be added between two points in a given spacetime—at the level of the initial data. This construction makes it possible to address a number of issues concerning the relation of the spatial topology to the geometry of solutions of the constraints and the constructibility of multi-black hole solutions [184].

The sharpest possible gluing theorem for the Einstein constraint equations has been developed in [182]. Using this, it is possible to show that for a generic solution of the constraint equations and any pair of points in this solution, one can add a wormhole connecting these points to the solution with no change in the data away from the neighbourhood of each of these points. Further, one can show that for almost any pair of initial data sets—including, say, a pair of black hole data sets, or a cosmological data set paired with a set of black hole data—one can construct a new set which joins them.

In order to assess the relevance of the IMP gluing construction for numerical relativity, we present a rough overview on how the IMP gluing method works. Consider two initial
data sets \((\Sigma_1, \gamma_{ij}^1, K_{ij}^1)\) and \((\Sigma_2, \gamma_{ij}^2, K_{ij}^2)\) for the EFE. Let \(p_1 \in \Sigma_1\) and \(p_2 \in \Sigma_2\) be two arbitrary points. The idea of the IMP construction is to find an initial data set \((\Sigma_{12}, \gamma_{ij}^{12}, K_{ij}^{12})\) such that \(\Sigma_{12}\) has the topology of \(\Sigma_1 \# \Sigma_2\), where \# denotes the connected sum of manifolds\(^5\). The pair \((\gamma_{ij}^{12}, K_{ij}^{12})\) is such that on \(\Sigma_1 \setminus \text{ball around } p_1\) it coincides with \((\gamma_{ij}^1, K_{ij}^1)\) and on \(\Sigma_2 \setminus \text{ball around } p_2\) with \((\gamma_{ij}^2, K_{ij}^2)\).

In order to construct the pair \((\gamma_{ij}^{12}, K_{ij}^{12})\) satisfying the Einstein constraints, one considers a conformal metric \(\hat{\gamma}_{ij}^{12}\) with the property that \(\Sigma_1 \setminus B(p_1)\) and \(\Sigma_2 \setminus B(p_2)\) coincides with \(\gamma_{ij}^1\) and \(\gamma_{ij}^2\), respectively, and on the cylinder \(\mathcal{I}\) it interpolates between them. Similarly, one constructs a \(\hat{\gamma}_{ij}^{12}\)-traceless tensor \(A_{ij}^{12}\) which again interpolates between \(A_{ij}^1\) and \(A_{ij}^2\). By construction, the resulting pair renders a solution to the constraints on \(\Sigma_1 \setminus B(p_1)\) and \(\Sigma_2 \setminus B(p_2)\) but not on the cylinder \(\mathcal{I}\). The machinery of the conformal method—see section 4.2—is used then to find a solution to the constraints on \(\Sigma_1 \# \Sigma_2\) using the pair \((\hat{\gamma}_{ij}^{12}, A_{ij}^{12})\) as seeds for the procedure. Some amount of technical work has to be invested, once the solution \((\gamma_{ij}^{12}, K_{ij}^{12})\) has been obtained to guarantee that the solution coincides with the original \((\gamma_{ij}^1, K_{ij}^1)\) and \((\gamma_{ij}^2, K_{ij}^2)\) outside the gluing neighbourhoods \(B(p_1)\) and \(B(p_2)\).

The conditions on the initial data sets to be glued by the afore-discussed method seem, at first sight, relatively mild. Essentially, it is only required that the initial data sets are such that the domain of influence of the neighbourhoods \(B(p_1) \subset \Sigma_1\) and \(B(p_2) \subset \Sigma_2\) have no Killing vectors. The latter condition can be reformulated in an intrinsic way in terms of the so-called Killing initial data (KID)—see, e.g. [79]. More precisely, the condition for gluings is that there are no KIDs in \(B(p_1)\) and \(B(p_2)\)—in particular, the latter implies that there are no Killing vectors in the neighbourhoods. The condition on the non-existence of KIDs has been shown to be generic in a very specific sense—see [80].

Every step in the procedure as described in, say, [182, 339] is completely constructive. However, it would require the solving of elliptic equations on manifolds with non-trivial topology. This complication is aggravated by the fact that, because of the requirement of non-existence of Killing vectors in the initial data sets to be glued, it is not possible to use initial data sets with some symmetry—like the axial one. Furthermore, the standard simplifying assumption of conformal flatness may complicate the matters further. The reason is that the only conformally flat initial data with trivial topology is the Minkowski data. Therefore, the use of conformally flat initial data to glue would require gluing data which already have non-trivial topology.

### 4.5. The Corvino–Schoen gluing method

An alternative gluing method for the construction of solutions to the constraint equations, which may be of relevance for numerical applications, is the so-called Corvino–Schoen (CS) gluing construction. This method has been developed in [179, 180, 192, 193] and allows us to smoothly glue any interior region of an asymptotically Euclidean initial data to the asymptotic region of a slice of a stationary spacetime—e.g. Kerr initial data. For the time-symmetric asymptotically Euclidean solutions of the constraints, this method glues any interior region to an exterior region of a slice of the Schwarzschild spacetime.

The procedure is best explained by considering the time-symmetric situation in which one tries to glue a time-symmetric solution of the constraints to an Schwarzschild exterior. As

\(^5\) The connected sum is performed by removing \(B(p_1) = \text{ball around } p_1\) and \(B(p_2) = \text{ball around } p_2\) and then joining \(\Sigma_1 \setminus B(p_1)\) and \(\Sigma_2 \setminus B(p_2)\) at \(\partial B(p_1)\) and \(\partial B(p_2)\) with a cylinder, \(\mathcal{I}\), of topology \(\partial B(p_1) \times [0, 1]\).
is well known, for time-symmetric initial data sets, the constraint equations reduce to
\[(3) R[\gamma_{ij}] = 0, \tag{7}\]
where \([\gamma_{ij}]\) in the last expression highlights the fact that in what follows the Ricci scalar will be regarded as a mapping between the spaces of 3-metrics on \(\Sigma\) and scalars on \(\Sigma\). Under certain circumstances this mapping happens to be an isomorphism. That is, given a metric \(\gamma_{ij}\) and a scalar \(f\) on \(\Sigma\) such that
\[(3) R[\gamma_{ij}] = f, \tag{8}\]
then if a further scalar field \(g\) happens to be close enough to \(f\)—in a functional sense—then there is a metric \(\gamma'_{ij}\) close to \(\gamma_{ij}\) such that
\[(3) R[\gamma'_{ij}] = g. \tag{9}\]
The CS method makes use of precisely this property of the Ricci scalar mapping.

Let \(\gamma_{ij}\) be a metric defined on an asymptotically Euclidean 3-manifold \(S\) satisfying the time-symmetric constraints. Consider a subset \(K \subset \Sigma\) which is obtained by removing from \(\Sigma\) one of the asymptotic ends. The boundary of \(\partial K\), regarded as a subset of \(S\), will be required to lie in the asymptotic region and to have the topology of the 2-sphere. Next, introduce some radial coordinate \(r\) so that the locus of \(\partial K\) is given by \(r = r_0\), with \(r_0\) a constant. It is noted that \(K\) is not necessarily compact, as it could well be the case that there is another asymptotic end inside \(K\). Denote by \(A\) the asymptotic end of the time-symmetric Schwarzschildian data. This asymptotic end is characterized by four parameters. Namely, the mass, and the location of the centre of mass, \(c^i\):
\[
\gamma^S_{ij} = \left(1 + \frac{m}{2|x - c|}\right)^4 \delta_{ij}. \tag{10}\]
As a part of the gluing construction, one connects the regions \(K\) and \(A\) by means of an annular region \(C\) to obtain a new asymptotically Euclidean 3-manifold \(\Sigma'\). A positive definite symmetric tensor \(\hat{\gamma}_{ij}\) is defined on the resulting 3-manifold by requiring it to be identical to \(\gamma_{ij}\) on \(K\) and to \(\gamma^S_{ij}\) on \(A\). On \(C\), it is chosen so that it interpolates smoothly between \(\gamma_{ij}\) and \(\gamma^S_{ij}\). Now, if \(3) R[\hat{\gamma}_{ij}]\) is close enough to the constant vanishing scalar field—this can be achieved by moving \(\partial K\), regarded as a subset of \(\Sigma\), suitably further into the asymptotic region—the result about the isomorphism of the Ricci scalar mapping guarantees that there is a small—again in a functional sense—tensor \(\delta \gamma_{ij}\) such that \(\hat{\gamma}_{ij} + \delta \gamma_{ij}\) is a 3-metric and more importantly
\[(3) R[\hat{\gamma}_{ij} + \delta \gamma_{ij}] = 0, \tag{11}\]
on \(\Sigma'\). The theorems ensuring the possibility of performing the above construction allow us to show that the support of \(\delta \gamma_{ij}\) is \(C\), so that \(\hat{\gamma}_{ij} + \delta \gamma_{ij}\) coincides with \(\gamma_{ij}\) on \(K\) and with \(\gamma^S_{ij}\) on \(A\).

The precise details of the gluing construction require the linearization of the Ricci scalar mapping. In order to obtain a differential equation to be solved, one has to consider the composition of this linearization and its adjoint to render a fourth-order elliptic problem. The nonlinear problem is then solved by a Newton iteration method. It turns out that in a similar way to what happens in the IMP gluing, the presence of KIDs in the gluing region \(C\) is an obstruction to the construction. The situation here is actually more delicate as the mere presence of asymptotic KIDs could be enough to cause problems. Note that as one moves the gluing region further and further into the asymptotic region, the problem could worsen. In order to get around these complications, one uses the freedom in the choice of the Schwarzschild data one is gluing to. Indeed, it has been shown in [192] that by a clever choice of the parameters \(m\) and \(c^i\) one can get around the problems with the asymptotic
Unfortunately, it seems that precisely this procedure is non-constructive. Thus, a numerical implementation of the CS gluing method would have to devise an alternative way of determining the parameters of the Schwarzschild exterior.

Although the above discussion has been limited to time-symmetric initial data sets which are to be glued to a Schwarzschild exterior, it has been shown that generic metrics can be glued to a wide class of exteriors—including stationary ones. In this case, instead of just considering the Ricci scalar mapping, one has to consider the general constraint mapping which sends a pair of symmetric tensors on $S$ to a scalar and a 3-vector—see [180, 193].

The gluing constructions for initial data sets are remarkable geometric and analytical results which have provided new insights into the structure of solutions to the Einstein constraint equations. Important to analyse and discuss is the hypothetical use of these constructions in the simulation of black hole spacetimes. There are various issues at stake here: of course one has the conceptual ones; but there are also the computational aspects. Is it possible to calculate everything one requires? As mentioned above, a straightforward implementation of the CS gluing technique may not be possible as it contains non-constructive aspects. Nevertheless, the existence of the gluing constructions has, at least, an indirect connection with the discussion of the robustness of the simulations of black hole spacetimes.

In theory, one could use the CS techniques on the Brill–Lindquist (BL) initial data, and construct from it a new initial data set which will be exactly BL inside a compact set, but in addition it will be exactly Schwarzschildean in the reference asymptotic end. As already mentioned in section 2.1, a set of conserved constants for the Schwarzschild spacetime—the Newman–Penrose constants—is zero. On the other hand, for BL data these are nonzero—see [207]. As these constants are conserved along null infinity, what one finds is that each spacetime will have a different radiation content close to the timelike infinity. This could be interpreted saying that each spacetime approaches the asymptotic state in a different way—i.e. the Schwarzschild spacetime.

5. Evolution formalisms

As discussed in section 3.2, different strategies in the gauge reduction process lead to distinct evolution formalisms. A natural classification of evolution formalisms in GR is according to the type of initial-value problem they solve. Following the discussion in section 3.1, they can be broadly classified as being Cauchy, hyperboloidal or characteristic. In this review, we focus our attention on the Cauchy-type formalisms and will briefly touch on some aspects of hyperboloidal and characteristic schemes.

Following [107], we will distinguish between evolution formalisms, understood as the schemes devised to deal with the reduction process—involving, e.g. the choice of variables or the strategy of resolution of the constraints—and evolution systems, as the concrete PDE systems set in a particular gauge and actually explicitly solved during the evolution. PDE evolution systems must include the reduced (or main evolution) system of section 3.2, but can also incorporate additional PDEs—e.g. fixing the gauge system. In general terms, a line can be drawn between (i) formalisms addressing specific mathematical issues—such as existence, uniqueness, completeness—and (ii) schemes aiming at the numerical construction of explicit solutions. We focus on some evolution formalisms where the interaction between analysts and numerical relativists has been, or is expected to be, particularly fruitful. Excellent reviews on different aspects of this problem can be found in the literature [259, 261, 374, 428, 439, 440, 469]. The following two sections are devoted to Cauchy schemes: manifestly covariant schemes in section 5.1 and to the so-called 3 + 1 formalisms in section 5.2. Non-Cauchy approaches are considered in section 5.3.
5.1. Cauchy manifestly covariant schemes: generalized harmonic (or general wave gauge) formulations and Z4 formalism

A classical approach to the study of the EFE consists in casting it as a wave equation for which there exists a well-developed theory. Using the deDonder expression for the Ricci tensor \([220, 230]\) and keeping on the left-hand side only the principal part, equation (1) is written as

\[-\frac{1}{2} \Gamma^\rho_\sigma \partial_\rho g_\mu_\nu + \partial_\mu \Gamma_\nu^\rho = S_\mu_\nu,\]

(12)

where \(\Gamma^\mu_\rho\) does not contain second derivatives of the metric. Prescribing the gauge \(\Gamma^\mu_\rho = 0\), equation (12) becomes a wave equation. Since \(\Box^1 x^\mu = -\Gamma^\mu\) (with \(\Box^1\) denoting the scalar wave operator associated with the metric \(g_\mu_\nu\)), this choice corresponds to the use of harmonic coordinates. These coordinates were used in the first results on well posedness \([229, 232]\). More recently, the elucidation of the role of \(\Gamma^\mu\) [246]—see also [270]—led to the introduction of general gauge source functions \(H^\mu\), leading to generalized harmonic (GH) or general wave gauge conditions

\(\Box^1 x^\mu = -\Gamma^\mu = -H^\mu\).

(13)

In the language of section 3.2, this constitutes the gauge system in this covariant evolution formalism, and the reduced system follows from inserting (13) into equation (12). The local Cauchy problem is well posed for this reduced system when appropriate initial conditions consistent with constraints (3) and (4) are employed—see, e.g. [259] and references therein for details and rigour. Consistency between the reduced and gauge systems—namely, propagation of the gauge condition \(Q^\mu = \Gamma^\mu - H^\mu = 0\) along the evolution—is controlled by the subsidiary system. This follows from Bianchi identities, which imply the hyperbolic equation for \(Q^\mu\)

\[\nabla_\mu \nabla^\rho Q_\rho + R^\rho_\mu Q_\rho = 0.\]

(14)

Imposing the gauge condition \(Q^\mu = 0\) and constraints (3) and (4) on the initial slice \(\Sigma\)—which imply \(\partial_t Q^\mu = 0\), cf [259]—the vanishing of \(Q^\mu\) along the evolution and the overall consistency are guaranteed.

This subsidiary system issue exemplifies the different nature of the problems an analyst and a numerical relativist must address. In the analytical setting, the above reasoning shows the consistency of the whole scheme. In particular, one no longer needs to consider the gauge system (13) while studying the properties of the reduced system. The situation gets more complicated in the numerical setting, since numerical errors trigger constraint violation modes that invalidate the analytic argument—see also [261, 262] for a discussion of the consequences of the intrinsic nonlinearities in equation (14). The Z4 formalism proposed by Bona et al [107–109, 111] is particularly interesting in this context. Developed as a full spacetime generalization of previous numerically motivated 3+1 schemes [106, 110, 112], its manifest covariant formulation permits a general analysis along the lines of the harmonic schemes. The key elements in this formalism are the introduction of a spacetime form \(Z^\mu\) as a new dynamical field, and the modification of the EFE to

\[R^\mu_\nu + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 8\pi T^\mu_\nu - \frac{1}{2} T g_\mu_\nu.\]

(15)

The solution space of this system extends that of GR, which is recovered for \(Z^\mu = 0\) in fact, the EFE is recovered under the milder (non-generic) condition of \(Z^\mu\) being a Killing vector. The evolution of the new field is driven by the Hamiltonian and momentum GR constraints (3) and (4). Consequently, the \(Z^\mu\) field can be used as a measurement of GR constraint violations—see section 2.2 in [107, 109] for a discussion involving the Bianchi identity and the constraint structure of the theory. As a further and key development for numerical applications, [300] introduces additional damping terms in the Z4 system—and also in the GH scheme—to drive the solution towards the GR solution submanifold, \(Z^\mu = 0\).
Together with the early and ongoing application in analytic studies of GR equations—see, e.g. the work of Lindblad and Rodnianski in section 2.1, and also the Kerr uniqueness proof by Klainerman and Ionescu discussed in section 6.1.1—these covariant formalisms have led to evolution systems with a successful application in numerical implementations. Harmonic coordinates have been employed in numerical studies of the IBVP in GR [53, 487, 488] and related evolution issues [53, 54], or in the study of generic singularities [270]—the latter also discusses GH coordinates. In the current numerical relativity scenario, the evolution of binary black holes through the merger with gravitational wave extraction by Pretorius [425], for which he employed a generalized harmonic scheme [427]—see also [428]—using the damping terms based on the scheme in [300], is especially relevant. First-order symmetric hyperbolic forms of the generalized harmonic codes have been developed in [381]—cf also [7, 229]—where the appropriate choice of gauge source functions $H^\mu$ for preserving hyperbolicity of the whole evolution system is studied. The latter now includes not only the reduced system, but also the $H^\mu$ evolution equations.

5.2. Cauchy 3+1 formalisms

The Cauchy IVP involves the use of 3+1 spacetime spatial slicings. This applies in particular to the case of the harmonic systems in the previous section. We now consider a family of evolution formalisms developed in the 3+1 framework of GR [35, 215, 232, 377, 512], which provides the natural setting for the canonical or Hamiltonian approach to GR—notably inspired by GR quantization considerations [35, 37, 89, 217, 219, 491]—and is extended in its application to numerical relativity—e.g. [288, 471, 521, 522]. In this 3+1 framework, in addition to the splitting of spacetime in terms of a foliation by spatial hypersurfaces $(\Sigma_t)$, the basic variables are fields naturally living on those slices $\Sigma_t$. In this section, we use the expression ‘3+1’ in this restricted sense, customary in numerical relativity. We denote the lapse function, measuring the proper distance between slices $\Sigma_t$ in the normal $n^\mu$ direction, by $N$ and the shift vector, governing the choice of coordinate system on each slice, by $\beta^\mu$.

The evolution vector is then written as $t^\mu = N n^\mu + \beta^\mu$. In this 3+1 decomposition, the EFE splits into two sets. The first one consists of the constraints in equations (3) and (4), whereas the second one—together with the kinematic relationship between the 3-metric $\gamma_{ij}$ and the extrinsic curvature $K_{ij}$—is frequently referred to as the ADM evolution system (but is actually due to York [521]):

$$\begin{align*}
(\partial_t - \mathcal{L}_{\beta})\gamma_{ij} &= -2NK_{ij}, \\
(\partial_t - \mathcal{L}_{\beta})K_{ij} &= -D_iD_j N + N (\gamma^{kl} R_{ij} + K K_{ij} - 2K^K_{ij} + 4\pi [(S - E)\gamma_{ij} - 2S_{ij}]).
\end{align*}$$

(16)

The ADM system is not uniquely defined, since the terms homogeneous in constraints (3) and (4) can be added. Moreover, it becomes an actual PDE evolution system once the prescriptions for the lapse $N$ and shift $\beta^\mu$ are actually supplied. The resulting evolution systems have suffered from the presence of numerical instabilities and the closely related violation of the constraints. A variety of 3+1 formalisms have been developed over the years with the aim of coping with these problems—see [469] for a (non-updated) review. Depending on the strategy adopted to handle the constraints, evolution formalisms can be broadly classified as: (i) free evolution formalisms if constraints (3) and (4) are not enforced during the evolution, and (ii) constrained evolution formalisms, that can be partially- or fully-constrained, according to the prescription of some or all the constraints, respectively.

The underlying intrinsic hyperbolicity [261] of the EFE is an ingredient exploited in the task of obtaining a specific PDE evolution system. However, this does not mean that all the resulting evolution systems are necessarily hyperbolic. A clear example is provided by
constrained schemes that, due to the elliptic nature of the constraints, lead to mixed elliptic–hyperbolic systems—see, however, [275]. Among the many possible criteria, in the present general discussion we will classify 3+1 evolution systems according to the presence or absence of elliptic equations in the system.

5.2.1. Evolution systems not containing elliptic equations. A significant effort has been devoted to casting the EFE—under the appropriate gauge conditions—as a PDE hyperbolic system. The reason behind this is the analytic control these systems provide on existence, uniqueness and stability issues [439, 440]—see LeFloch’s IHP talk for some examples illustrating the richness of this approach in the interplay between nonlinear PDEs and geometry. This is the case of strongly hyperbolic systems—with their well-posedness results on the IVP—and of symmetric-hyperbolic systems, where energy estimates enhance the analytic control particularly in the IBVP permitting the use of maximally dissipative boundary conditions. In the generalized harmonic formalisms discussed in section 5.1, hyperbolicity is manifest. With regards to systems derived from the ADM formulation [469], hyperbolicity analysis is more complex due to the loss of covariance. This also applies to certain 3+1 formalisms that can be derived from covariant formalisms by performing a 3+1 splitting of the fields—e.g. 3+1 versions of the Z4 scheme in [107]. A first class of systems in the 3+1 context consists of first-order, in time and space, evolution systems. Among these, the KST system [359] provides a very general class from which, for example, the Fritelli–Reula [264, 265] or the Einstein–Christoffel [8] systems can be recovered. The Bona–Massó systems [110, 112] include some additional variables—expressible in terms of the connection coefficients of the 3-metric, as the BSSN system below—and can be related by variable reduction to KST systems [106]. A second class of 3+1 systems has been studied in which the second-order-in-space aspect of the ADM system is explicitly maintained. In this sense, [298] presents a very general second-order evolution system that includes the successful BSSN [75, 467] or the NOR [404] systems. In this context of second-order systems, we also highlight the work in [106], as a predecessor of the Z4 formalism discussed in section 5.1. We refer the reader to [469] for the presentations of a third subclass, the asymptotically constrained systems aiming at the construction of an evolution system where the constraint surface is an attractor—cf in particular the discussion of the λ-system and adjusted system approaches. Many evolution systems can be obtained from the previous systems—or closely related ones—by adopting distinct prescriptions for the lapse and shift. An exhaustive account of all the resulting 3+1 systems is beyond the scope of this review. The assessment of analytic well posedness—strongly dependent on gauge choices—and of the relations between the first- and second-order formulations, can be found in [106, 298, 404, 453].

5.2.2. Mixed elliptic–hyperbolic systems. Constrained formalisms are not the only avenue to mixed elliptic–hyperbolic systems. There are geometric or physically well-motivated choices for N and βi which are elliptic in nature and thus lead to a mixed system. We present some of the elliptic–hyperbolic PDE systems discussed in the literature.

(a) Free systems. They involve elliptic gauge choices for the slicing and/or the shift. An example of this is provided by the first of the axisymmetric 2+1+1 systems considered in [442]. A paradigmatic example is presented by Andersson and Moncrief [19], where the authors establish the well posedness of the IVP for a system containing elliptic equations for the lapse and shift. These equations follow, respectively, from a constant mean curvature condition and a spatial harmonic gauge. To the best of our knowledge, no numerical counterpart has been implemented. Other recent analytic studies can be found in [163–165].
(b) \textit{Partially constrained systems.} The enforcement of the Hamiltonian constraint as an equation for a conformal factor has been widely used in axisymmetric codes since the 1980s \cite{64, 228, 479}. More recently, maximal slicing and the Hamiltonian constraint have been employed in \cite{272}, whereas the third constrained 2+1+1 system in \cite{442} implements maximal slicing and imposes the momentum constraint—all this, still in the axisymmetric setting.

(c) \textit{Fully constrained systems.} Early systems of this kind were implemented in 2D codes \cite{227}, both for non-rotating \cite{466} and rotating spacetimes \cite{3}. More recently, axisymmetric 2+1+1 codes \cite{158, 159} have been used in the analysis of critical collapse. In this line, \cite{442} provides again an example of constrained system of this type, where ill-posedness is concluded after a maximum principle analysis of the involved scalar elliptic equations—see discussion in section 4.3.1, but also \cite{444}, where a new scheme not suffering from these problems is presented. Regarding the full 3D case, fully-constrained schemes have been discussed in \cite{9, 113} and Moncrief’s IHP talk \cite{1}. The fully-constrained scheme proposed by the Meudon group \cite{113} makes use of a 3+1 conformal decomposition of spacetime based on slices reaching spatial infinity $i^0$. The elliptic part of the PDE system includes equations for the lapse, the conformal factor and the shift, following from maximal slicing $K = 0$, the Hamiltonian constraint and a combination of the momentum constraint and a Dirac-like gauge, respectively. Only two hyperbolic scalar modes are evolved, and the rest of the components are reconstructed by using the Dirac gauge—cf Novak’s IHP talk \cite{1}—and a unimodular condition on the conformal metric. The scheme has been numerically tested for gravitational wave spacetimes. Regarding the scheme proposed in Moncrief’s IHP talk \cite{1}, it is ultimately devised to bypass the limitations in the extraction of gravitational radiation at finite distances by using instead the natural boundary of the problem—i.e. future null infinity. Even though important features are shared with \cite{113}—the use of a conformal 3+1 decomposition, the resolution of Hamiltonian and momentum constraints along the evolution, elliptic gauges for the shift—it also represents a sound shift. Namely, it is not a Cauchy formalism but rather of hyperboloidal type: hyperboloidal 3-slices reaching up to $\mathcal{I}^+$ are chosen by means of a constant mean curvature $K$ condition, implemented as an elliptic equation for the lapse. Finally, it is worth noting an interesting proposal within this scheme for the determination of the conformal class representative $\tilde{\gamma}_{\mu\nu}$ by solving an elliptic Yamabe equation.

5.2.3. \textit{Current status of 3+1 systems.} Most numerical groups make use of codes based on 3+1 formalisms derived from the ADM scheme. Codes based on harmonic formulations have produced excellent results \cite{425, 486}, but their use is still limited to a smaller part of the community. Among the 3+1 formalisms, the mainstream is represented by free systems not involving the resolution of elliptic equations, and they have provided the longest lived evolutions—in particular BSSN. In comparison, mixed elliptic–hyperbolic systems have offered limited applicability or are still at a preliminary stage—cf \cite{113, 191} and Moncrief \textit{et al} scheme \cite{1}. On the one hand, constrained systems are in principle expected to provide a better control on instabilities related to the violation of the constraints. On the other hand, well posedness in mixed systems is difficult to establish and, in particular, characteristic fields in the hyperbolic part are difficult to determine since a part of the dynamics is encoded in fields solved through the (non-causal) elliptic part. In addition, non-uniqueness issues in the XCTS system discussed in section 4.3.1 could have strong implications on the well posedness of some fully-constrained evolution schemes enforcing a maximal condition for the slicing. The numerical consequences of this non-uniqueness in the elliptic sector are still unclear and behaviour near the critical value of the parabolic branching discussed in
section 4.3.1 can be very much dependent on the details of the numerical implementation. As a significant example, authors in [446] have renounced the use of mixed systems in [442], opting for a 2+1+1 version of the $Z_4$ formalism. The lesson here contained should certainly not be underestimated. At the same time, it should not preclude the development of such an alternative line of research that can only result in overall benefits.

5.3. Other approaches to evolution

We proceed to discuss some alternative approaches to the Cauchy formulation. We focus our attention on some aspects of characteristic evolutions and later on a particular version of the hyperboloidal problem.

5.3.1. Evolution formalisms based on a characteristic approach to GR. The characteristic initial-value problem presented in section 3.1 provides an avenue to the numerical construction of spacetimes—alternative to the Cauchy approach—that is particularly well suited for the study of gravitational radiation. Since the pioneer work by Bondi and Penrose [115, 116, 413] on the characterization of gravitational radiation in terms of null hypersurfaces, developments based on the use of Bondi–Sachs metric [116, 449] have resulted in successful numerical evolution—e.g. [96]—and radiation extraction—e.g. [55]—in a number of non-trivial cases, including single black hole spacetimes. An excellent account can be found in the review by Winicour [518]. The main drawback of this approach is the need to cope with caustics formation during the evolution. A second obstacle lies in the difficulty of encoding physics. Alternatively, mixed Cauchy-characteristic formulations [94] (Cauchy-characteristic matching) offer a compromise between stable evolutions and accurate treatment of gravitational radiation. A full application to astrophysical scenarios seems currently limited as compared to the standard Cauchy approach. On the other hand, regarding applications in more geometric settings—in particular, the study of global solutions—characteristic or mixed Cauchy-characteristic approaches provide a stimulating framework for the fruitful collaboration between geometers and numerical relativists. In this sense, we highlight the ongoing activity in this line of research—see, e.g. [95, 435, 286] and references therein.

5.3.2. Conformal field equations. The conformally regular approach in numerical relativity is based on analytic studies by Friedrich [243, 245, 248] and started with numerical studies by Hübner and Frauendiener [233–236, 239, 327–330]. In this approach, one solves numerically a hyperboloidal initial-value problem for the conformally regular field equations—for reviews see [236, 237, 333]. The use of hyperboloidal foliations is promising as hyperboloidal surfaces combine advantages of Cauchy and characteristic surfaces. Instead of approaching spatial infinity as Cauchy surfaces do, they reach null infinity which makes them suitable for radiation extraction. Contrary to characteristic surfaces, these spacelike surfaces are as flexible as Cauchy surfaces and can be used in numerical calculations within the 3+1 approach. A difficulty with the conformally regular field equations is that equations are significantly larger than usual formulations of Einstein equations. Due to the large number of constraint equations, numerical errors require a stronger control on constraint propagation properties of the system. As there is not enough numerical experience with the equations, one cannot use established methods easily to deal with the encountered instabilities. Another difficulty is that the equations include, among others, evolution equations for the conformal factor. The representation of null infinity depends, in general, on the solution, so that the numerical boundary does not coincide with the conformal boundary. As a consequence, a numerical boundary treatment outside

6 The content of this section is essentially due to A Zenginoğlu. We are thankful for his enthusiastic input.
the physical spacetime is required and the calculation of the unphysical part of the conformal extension wastes computational resources. While these problems are not of a principal nature, they have made progress in the conformally regular approach difficult. The conformally regular approach as described above has been based on the hyperboloidal initial-value problem. As such, spatial infinity is not a part of the computational domain and one cannot calculate the maximal development of the Cauchy data. This has various drawbacks. In the hyperboloidal approach, it is not clear how the cut of the initial hyperboloidal surface at null infinity is related to timelike or spatial infinity. One would also like to be able to relate asymptotic quantities such as mass or momentum defined at null infinity to corresponding quantities at spatial infinity. An alternative conformally regular system was proposed by Friedrich that allows one to construct a regular finite initial-value problem at spatial infinity [252]. The study of this problem led to analytic results concerning the applicability of the Penrose proposal and smoothness properties of null infinity [256, 500, 501, 503, 504]. A major advantage of the underlying system, called the reduced general conformal field equations, is that the representation of the conformal factor is known a priori in terms of initial data and that the system consists mainly of ordinary differential equations except the Bianchi equation admitting a symmetric hyperbolic reduction. The finite initial-value problem at spatial infinity has been studied numerically in [527]. The Cauchy problem could be solved for the entire Schwarzschild–Kruskal solution including timelike, spatial and null infinity [526]. Also, certain radiative spacetimes could be studied near spatial and null infinity. This approach is currently the only approach that allows the numerical study of both spatial and null infinity in a single finite picture so that one has, in principle, numerical access to the maximal development of Cauchy data and can calculate global spacetimes including their entire asymptotics. A further development of this approach might give a valuable input from numerics to geometry as most open problems in mathematical relativity concern global questions.

A valuable input from geometry to numerics is the idea of null infinity. Having null infinity in the computational domain would solve both the outer boundary problem and the radiation extraction problem in numerical relativity. Unfortunately, this idea could not yet be implemented in astrophysically motivated numerical calculations based on the Einstein equations. The main difficulty is due to the appearance of formally singular terms arising from conformal compactification of the metric. A reasonable approach is to choose a gauge for the Einstein equations in which each formally singular term attains a regular limit in a spacetime that admits a smooth conformal compactification at null infinity. The construction of such a gauge in the context of the characteristic approach has been known for a long time [489]. This method has been quite successful within the characteristic approach. The underlying coordinates, however, do not allow the simulation of highly dynamical strong fields due to formation of caustics in the light rays generating the coordinate hypersurfaces [518]. The implementation of this idea within the 3+1 approach has turned out to be exceptionally difficult numerically as well as analytically. In view of the recent breakthrough within numerical relativity, it seems clear that yet another formulation of the Einstein equations to deal with this problem cannot be regarded as a practicable solution. What is needed is a novel numerical treatment of null infinity that does not alter the successful simulation of the sources in the interior.

An attempt to use a common reduction of the Einstein equations to study null infinity has been made in [10, 403]. A scri-fixing gauge has been suggested in the context of a hyperboloidal initial-value problem for the ADM reduction of the Einstein equations. In this gauge the spatial coordinate location of null infinity is independent of time. Such a gauge has first been constructed in [233] in the context of a frame-based conformally regular field equations. A similar approach is followed in Moncrief’s IHP talk [1].
A scri-fixing gauge is based on a mixed elliptic–hyperbolic system. A constant mean curvature condition fixes the hyperboloidal foliation by an elliptic gauge condition for the lapse. In such attempts, an important question is how to fix the conformal factor. It has been shown that the representation of the conformal factor in terms of coordinates can be given \textit{a priori} [527]. In this case, the well posedness of the scri-fixing gauge has been proven. Further, each formally singular term arising from conformal compactification attains a regular limit at $\scri$.

Preliminary numerical experiments suggest that the method has some promise to be used in astrophysically motivated calculations. In [528] scri-fixing gauges have been studied in the Minkowski and Schwarzschild spacetimes. A discussion of the tail behaviour of Schwarzschild is used to demonstrate the astrophysical relevance of the notion of null infinity for numerical applications.

5.4. Some specific issues regarding evolution formalisms

5.4.1. Outer boundaries and the initial boundary-value problem. The treatment of infinity poses diverse geometrical, numerical and physical challenges. More specifically, among the different manners of coping numerically with this issue we can mention compactifications of spatial infinity [425], the use of hyperboloidal slices together with compactification of null infinity—[233–236, 239, 327–330], Moncrief’s IHP talk [1] and section 5.3.2—or implementation of a characteristic scheme. Alternatively, one can remove infinity from the problem and consider the evolution of the EFE in a region bounded by an outer timelike boundary. This leads to the discussion of the well posedness of the IBVP presented in section 3. A part of the community no longer regards this issue as a fundamental problem in practice [2], since current simulations can push outer boundaries sufficiently far away. However, the relevance of the topic is in particular reflected in the quantity of works on the subject—also manifest in the number of IHP talks [1] dealing directly or indirectly with this topic; cf IHP talks by Buchman, Rinne, Tiglio and Winicour. The work by Friedrich and Nagy [258] provided the first formalism in which well posedness has been fully shown. However, its formulation in terms of tetrads and the Weyl tensor makes it difficult to implement using standard numerical techniques/infrastructures. More recently Kreiss and Winicour [369] have presented a system based on the harmonic formalism for which the IBVP is well posed in the generalized sense, and the result has been extended to well posedness in the classical sense in [368]. Generally speaking, symmetric hyperbolic systems play a critical role in the analysis of the well posedness in the IBVP problem, through the use of maximally dissipative boundary conditions for getting rid of the possible constraint violations—cf [300] for an alternative approach making use of point-like damping terms. In this context, a pseudo-differential theory of strongly well-posed systems in the generalized sense and techniques from semi-group theory have been used to analyse IBVP well posedness—cf [109, 139, 140, 381, 443, 445, 454, 461, 481] for some other recent references on the subject. Absorbing boundary conditions have been intensely studied in the last few years [136, 137, 410, 448]—cf the review [452].

5.4.2. Gauge conditions. The discussion about numerically and physically appropriate choices of coordinates is a vast topic. Here, we limit ourselves to briefly account for some recent developments. First, regarding sources functions $H^\mu$ in GH formalisms, choices must be done in such a way that the hyperbolicity of the GH formulation is not spoiled. References [425–427] make use of a wave evolution equation for the determination of $H^\mu$, but more research is needed on this specific topic [428]. In this sense, [380] has provided new gauge drivers preserving hyperbolicity in GH systems, and including a wider class of conditions motivated by successful 3+1 implementations. The further discussion of stability as well
as the study of intrinsically GH-motivated slicings constitute an open line of research. In a
different setting, the appropriate choice of coordinates has been crucial in the successful binary
black hole implementations using the BSSN formalism [59, 141]. In particular, modifications
of the 1+log slicings and gamma-driver condition for the shift, have proved to be fundamental
ingredients in the moving puncture approach to black hole evolution, where the punctures are
advected in the integration domain—cf section 6.4.2. An analysis of the well posedness of the
resulting evolution system has been carried out in [298], and in particular has given indications
of a breakdown for large shifts. This binary black hole case shows the relevance of using
symmetry seeking coordinates [273]. Recent contributions in this sense are the almost-Killing
condition derived from a variational principle in [105, 111], and its compatibility with the
adoption of singularity avoiding coordinates in schemes not implementing excision [104].

5.5. Final discussion on evolution formalisms
The research on evolution formalisms for the Einstein equation provides a fruitful area for
the collaboration between mathematical and numerical relativists. However, the analytic
issues on which we have focused in this section should not shadow the full complexity of the
problem, in particular the fact that current numerical successes are a direct consequence of
decades of steady and systematic numerical experimentation. The most successful codes—e.g.
generalized harmonic or BSSN—represent a compromise between well-posed but complicate
schemes—like the Friedrich–Nagy system—and formalisms easy to implement but potentially
ill-behaved, like the ADM approach. The situation could be deceiving for the analyst, but is just
a reflection of the overall complexity involved here. Analytic well posedness is not a guaranty
of numerical stability—cf discussion in [53]—but rather a necessary condition for numerical
convergence, once the consistency of the scheme and numerical stability are satisfied. In
sum, analytic issues are only one aspect in the problem of choosing the appropriate numerical
evolution scheme. Together with them, the numerical relativist must cope with computational
and physical issues [53]. In particular, a presentation of the formalisms employed in numerics
cannot be complete without a discussion of the numerical techniques involved, a task that
would require a full review in itself.

For all these reasons, the discussion about evolution formalisms is perceived by a good
part of the numerical community as an essentially technical discussion that, in a good measure,
has lost its critical character after the breakthroughs in the binary black hole problem. After
decades of struggle, analytic and computational issues are considered to be under reasonable
control and, following this line of thought, focus should consequently be shifted to the more
physical aspects of the problem—cf the special New Frontiers in Numerical Relativity issue
[2] and in particular the transcription of the final discussion. As far as astrophysical aspects
in numerical relativity are concerned, this attitude seems to be an appropriate one—although
surprises should be expected when addressing other physical problems. However, regarding the
application of numerical techniques to deepen our understanding of the geometric structures of
GR—what we have called geometric numerical relativity in section 1.2.2—it seems reasonable
to consider that further research in this subject is not only fully justified, but will probably be
needed as new geometric and/or physical goals are set.

6. State of the art of black holes
If one wants to make use of numerical methods to calculate dynamical black hole spacetimes
then, in one way or the other, one has to resort to an initial-value formulation—see section 3.
It is not obvious at all that an initial-value formulation of the merge of two black holes is the
most adequate way of discussing this problem\textsuperscript{7}—one is dealing with an idealized astrophysical problem whose initial conditions cannot be set in a laboratory. The whole approach hinges very heavily on the assumption that there are certain \textit{robust aspects} of the output of the numerical simulations—essentially the waveforms—which are not dependent on the detailed way the initial data are constructed. The only real dependence of the waveforms should come from the physical parameters of the problem—again, the masses and spin of the black holes—but not from parameters like the initial separation—as the holes rather approach from infinity undergoing an adiabatic process—or if the initial data are conformally flat or not.

The issue of the robustness of certain features in the simulation of dynamical black hole spacetimes provides a rich arena for the interaction between geometry and numerics. This is a very challenging mathematical problem whose resolution will probably take a long time. For the time being one will have to content oneself with the evidence coming from the numerical simulations. Still, the assessment of this evidence is in itself also a challenge, and may also provide further ground for interaction.

In defence of the use of an initial-value point of view, one can say that it offers a framework to look at the problems. By framework it is understood as a repertoire of mathematical techniques and theorems which, for example, allows us to state the well posedness of the differential equations governing the problem, so one knows that the problem has been formulated in a consistent way. From a strict mathematical point of view, it is not known if there exist spacetimes with a black hole and gravitational radiation—this is essentially the problem of the nonlinear stability of Schwarzschild/Kerr, cf the discussion in section 2.2.

6.1. Mathematical black holes

6.1.1. Rigidity and uniqueness of black holes. A cornerstone of what has been called the \textit{establishment’s point of view on black holes}—see, e.g. \cite{416}—is the evidence showing that the Kerr solution describes all stationary, vacuum solutions to the EFE describing black holes. This uniqueness assertion is also called a \textit{no-hair theorem}. The relevance of this result stems from the fact that it characterizes all possible asymptotic states of the general evolution of isolated systems in vacuum. Thus, for the community of numerical relativists it justifies the analysis of the late stages of black hole spacetimes by means of perturbation theory on a Kerr background—see, e.g. \cite{364}.

The problem of the uniqueness of black holes has been resolved in two different manners. Starting from a stationary, asymptotically flat black hole spacetime it follows that the stationary Killing vector field must be tangent to the event horizon, \( E \), of the black hole. If the Killing vector is null at the horizon—i.e. it is a \textit{Killing horizon}—then it is possible to conclude that the spacetime is actually static \cite{482}. For static black hole spacetimes, the uniqueness results state that the spacetime has to be a Schwarzschild spacetime—see, e.g. \cite{138, 175, 343}. If the stationary Killing vector is spacelike on the horizon then from Hawking’s area law one can conclude the existence of another vector, \( \eta^\mu \), tangent to the generators of the horizon which is a Killing vector on the horizon \cite{181, 311, 340}. This extra Killing vector field of the horizon can be used to define the notion of \textit{surface gravity} which is of great relevance in the discussion of the thermodynamics of black holes. In particular if the surface gravity is nonzero, then the horizon is said to be \textit{non-degenerate}. In the case of a non-degenerate horizon, it is customary to assume that the horizon is a smooth null hypersurface consisting of two components—a \textit{bifurcate horizon}—\( E = E^+ \cup E^- \) intersecting at a 2-surface with the topology of the 2-sphere—this assumption is supported by \cite{431}.

\textsuperscript{7} We thank S Dain for a discussion on this point.
Given that it is known that the Kerr solution with \(0 \leq a = J/M \leq M\) is the only stationary, axially symmetric, vacuum black hole with non-degenerate and connected horizon—see [150, 447]—it is suggestive to try to extend the vector \(\eta^{\mu}\) in the horizon to a Killing vector of the region exterior to the black hole—the domain of outer communication. The problem with this approach is that it requires posing a boundary-value problem on characteristic hypersurfaces (the horizon of the black hole): one constructs a wave equation for the vector \(\eta^{\mu}\) with data prescribed on \(E\). This problem is ill posed—i.e. it is not possible to establish local existence and uniqueness by standard methods—and only admits a solution if the spacetime is taken to be analytic. Thus, under the hypothesis of analyticity it is possible to show that \(\eta^{\mu}\) extends to a Killing vector on the domain of outer communication—see [176, 311]. This type of result is known in the literature as a rigidity theorem for it shows that a certain assumption on the symmetry of the spacetime—stationarity—together with some further technical assumptions imply further symmetries—axial symmetry. The rigidity of black holes has been recently revisited with the particular aim of extending it from the (3+1)-dimensional to the \(n+1\) setting—cf Isenberg’s IHP talk [1] and also [324].

The above argument for the uniqueness of the Kerr black hole has been criticized on the grounds that the analyticity assumption is overly restrictive: general spacetimes are at most smooth, and it is not obvious why a smooth spacetime will become analytic after it has emitted gravitational radiation. In order to get around this problem, an alternative approach to uniqueness has been given in [335]—see also Klainerman’s IHP talk [1]. The idea behind this approach is that although the boundary-value problem for wave-like equations on \(E\) is ill posed, if one knows that solutions do exist, then it is possible to show uniqueness [336]. In [391, 392] a certain tensor—the Mars–Simon tensor—whose vanishing implies that the vacuum spacetime under consideration is locally isometric to the Kerr solution has been discussed. This tensor requires, for its definition, the existence of a stationary Killing vector. In [335] it has been shown that the Mars–Simon tensor obeys a wave equation and that it vanishes on \(E\) under a further technical assumption—which does not require analyticity. In general, one does not know whether a solution to the boundary problem for the wave equation satisfied by the Mars–Simon tensor with vanishing data on \(E\) exists. However, if a solution exists, then it is possible to argue that the Mars–Simon tensor vanishes not only in a neighbourhood of the horizon but on the whole domain of outer communication—this is equivalent to showing the uniqueness of the ill-posed boundary-value problem. It is important to stress that this is a global result and not only local to the horizon, like the one based on the hypothesis of analyticity. Note further that this proof bypasses completely the use of rigidity results, and renders the existence of axial symmetry as a part of the main result.

6.2. Quasi-local black holes

In spite of the success and strength of the traditional formalization of the black hole notion in the asymptotic flatness setting—cf section 2.2—the black hole characterization in terms of elements not involving global aspects of the spacetime has attracted significant efforts. This area of research is particularly well suited for a fruitful collaboration between mathematical and numerical relativists.

6.2.1. Motivations for quasi-local black holes. Quasi-local characterizations of black holes have been approached from different communities addressing distinct problems. Here, we highlight geometric motivations from the mathematical community, on the one hand, and ‘practical’ needs for numerical relativists, on the other. Regarding the former, the difficulties discussed in section 2.2 related to the understanding of the full implications of asymptotic
flatness, and illustrated in the absence of known examples of non-stationary ‘strict’ black holes spacetimes, depicts a state of affairs which is non-satisfactory at a conceptual level. More dramatic are the issues raised from a numerical perspective, where the required global asymptotic information is simply not accessible during the evolution. Quasi-local characterizations of black holes are needed both for ‘technical’ reasons—e.g. need of tracking the black holes along the evolution—and for physical interpretative reasons—e.g. the need to disentangle information about ‘individual black holes’ in an interacting scenario.

Other motivations have played a crucial role in the historical development of quasi-local notions of black holes: (i) characterization of black holes in cosmological settings where asymptotic flatness notions do not apply; (ii) extension of black hole thermodynamical results to situations where global notions are not under control; (iii) studies of black holes in the context of quantum gravity, such as the microscopic evaluation of black hole entropy; and (iv) conceptual compliance with the consensus’s notion of ‘black hole’ in the astrophysical community which is completely foreign to global issues. In a general sense, it is fair to say that numerical and thermodynamical issues have been the main drivers in the development of quasi-local black holes as a line of research on its own.

6.2.2. Seminal concepts and objects: trapped surfaces. As discussed in section 2.2, the idea of a black hole as a region of no escape can be formalized in terms of the set points not being able to send a signal to infinity—as long as we have a sensible notion of infinity. Local convergence of all the light rays emitted from the given sets of points provides an alternative approach to capture the notion of no-escape. This second approach does not resort to a notion of infinity and provides the rationale for the quasi-local approaches to black holes to be discussed in this section.

The seminal idea behind this characterization is the concept of trapped surface. This notion has played a fundamental role in the context of the singularity theorems [313, 414]. Given a surface \( S \) which is spacelike orientable and closed—i.e. compact and without boundary—one can unambiguously define at each point \( p \in S \) two null directions \( \ell^\mu \) and \( k^\mu \), which span the normal plane at \( p \). That is, the vector fields \( \ell^\mu \) and \( k^\mu \) span the normal bundle \( T^\bot S \). The idea behind the notion of a trapped surface \( S \) is that either all light rays emitted in the normal null directions do locally converge or, at least, light rays which would be naturally expected to expand, do actually converge. In the first case, the trapped character is an intrinsic property of the surface \( S \), whereas in the second case some extra structure is needed to make sense of the intuition of naturally expanding light rays. Let us define the expansions \( \theta^{(\ell)} \) and \( \theta^{(k)} \), respectively associated with the null congruences \( \ell^\mu \) and \( k^\mu \), as

\[
\theta^{(\ell)} = q^{\mu\nu} \nabla_\mu \ell_\nu, \quad \theta^{(k)} = q^{\mu\nu} \nabla_\mu k_\nu, \tag{17}
\]

where \( q_{\mu\nu} \) is the spacelike metric induced on \( S \) from the ambient spacetime metric. Penrose [414] presents an intrinsic notion of trapped surface irrespective of any embedding. A closed surface \( S \) is a trapped surface (TS) if and only if

\[
\theta^{(\ell)} < 0, \quad \theta^{(k)} < 0. \tag{18}
\]

More generally, in order to cover cosmological settings, a TS is defined as satisfying \( \theta^{(\ell)} \theta^{(k)} > 0 \), but we only discuss here the cases related to black holes. The limiting case in which one of the expansions—say \( \theta^{(\ell)} \)—vanishes, characterizes \( S \) as a marginally trapped surface (MTS):

\[
\theta^{(\ell)} = 0, \quad \theta^{(k)} < 0. \tag{19}
\]

In the context of isolated systems, it is natural to consider surfaces \( S \) embedded in an asymptotically flat (Euclidean) spacelike hypersurface \( \Sigma \). In this case, we can naturally
define the outer null direction, say $\ell^\mu$, as the one ‘pointing’ towards infinity. Following Hawking \cite{311}, we say that a surface $S$ is an outer trapped surface (OTS) if and only if
\begin{equation}
\theta^{(\ell)} < 0. \tag{20}
\end{equation}

The vanishing limiting case defines a marginally outer trapped surface (MOTS):
\begin{equation}
\theta^{(\ell)} = 0. \tag{21}
\end{equation}

Dropping the global condition, the latter requirement characterizes the marginal surfaces defined by Hayward \cite{316}. In order simplify the discussion, we will refer generically to (21) as the MOTS condition, and the context will make it clear whether global conditions are being taken into account or not—this is, in fact, the common practice in the literature, e.g. in \cite{18}. These are not the only attempts to classify and characterize the idea that a surface is trapped. See in this sense \cite{394, 463} for an alternative nomenclature, \cite{308} for the introduction of average trapped surfaces—cf in this sense also \cite{385}—and \cite{464} for a geometric formulation of the hoop conjecture in terms of the notion of trapped circle.

Trapped surfaces open an avenue to control quasi-locally the notion of black hole. Given a spacelike hypersurface $\Sigma$, the trapped region $T_\Sigma$ is defined in \cite{311} as the set of points $p \in \Sigma$ belonging to some OTS, $S \subset \Sigma$. The apparent horizon (AH) is defined as the outer boundary of the trapped region $T_\Sigma$. It attempts to encapture quasi-locally the notion of black hole horizon at the hypersurface $\Sigma$. A crucial result is the characterization, under the appropriate regularity conditions, of the AH as the outermost MOTS \cite{311}. Even though these quasi-local notions were originally developed in the mathematical studies of black holes, particularly in the context of the gravitational collapse and singularity theorems, they were employed in numerical implementations very early on.

First numerical implementations using geometric quasi-local black hole ideas. Using their characterization as MOTS, AHs have played a key role in the relation between mathematical and numerical relativists. First, for numerical relativists, condition (21) represents a tentative characterization of ‘black holes’ that can be efficiently calculated in terms of data on a compact region of a three-dimensional slice. Important in this context is the development of AH-finders, i.e. algorithms to locate AHs in spatial slices by solving equation (21)—see \cite{493} for a review and \cite{378} for a recent work; cf also \cite{398} for the related work with a strong geometric input. Moreover, under appropriate energy conditions and assuming the weak cosmic censorship conjecture, AHs are geometrically defined surfaces that are guaranteed to lie inside the event horizon—see, e.g. \cite{186, 311}—and therefore are causally disconnected from the rest of the spacetime. Although the extrapolation of this last feature from the continuum description to the discretized level is not straightforward—see, e.g. \cite{131}—this geometric idea has acted as a positive criterion in the development of black hole evolution codes—see section 6.4.2 for a brief discussion of the different manners of exploiting this idea.

Together with the numerical control of the black hole during the evolution, AHs were applied early as inner boundary conditions in the construction of initial data for black hole spacetimes—following an idea proposed by Unruh. In this case, condition (21) is used to complete the elliptic system defined by constraints (3) and (4) when an interior sphere has been removed—cf section 6.4.2 for further comments on this excision approach. This idea was first numerically implemented by Thornburg \cite{492}, later becoming a standard technique. This problem has also received attention from the mathematical community \cite{208, 397, 472}, but unfortunately the interaction among the communities seems to be scarce. More recently, the idea of AHs inner boundaries has also been proposed in the evolution context by Eardley \cite{222}. The difficulties in implementing these ideas exemplify the needs of interaction between
geometric and numerical communities. See also [344, 345] for an alternative geometric solution to the inner excision problem in evolutions using, instead of AHs, locally area-preserving evolutions of a given initial trapped surface.

The need of a spacetime point of view. In spite of their conceptual and practical interest, AHs do not provide a spacetime characterization of black holes. The AH notion depends on the spacelike slice $\Sigma$ and this feature limits its applicability. The latter has been illustrated in examples given by Wald and Iyer [509] of Schwarzschild slicings where Cauchy hypersurfaces come arbitrarily close to the singularity but do not contain any OTS. In particular, if we understand the evolution of an AH, say $S_0$, as the ‘hypersurface’ formed by piling up the AHs—denoted by $S_t$—found in the 3-slices $\Sigma_t$ used in a Cauchy evolution—that is, $\Sigma_t \supset S_t$—the resulting AH-worldtube $\bigcup_t S_t$ explicitly depends on the chosen 3+1 slicing. Furthermore the geometric, dynamical and thermodynamical properties of this AH-worldtube are not under control—in particular non-continuous jumps can occur. This contrasts with event horizons, that are smooth null hypersurfaces under the appropriate conditions [311].

An intrinsic spacetime formulation can be obtained by defining the trapped region without any reference to a particular slice $\Sigma$. In this context, the trapped region $T$ of a spacetime $M$ is the set of points $p \in M$ belonging to some TS, $S \subset M$. The outer boundary of this region is sometimes also referred to in the literature as the AH, but following [314] we will denote it as the trapping boundary. This offers an intrinsic quasi-local notion for the horizon where no reference to asymptotic quantities is needed. A natural question to address consists of clarifying its relation with the event horizon—if the latter can be defined. As in the AH case, assuming cosmic censorship, the trapping boundary does not extend beyond the event horizon. Finding out if it actually extends up to it, or if it remains strictly in the interior, is an open issue which can be of relevance in the understanding of cosmic censorship conjecture and in which mathematical and numerical collaborations can prove to be very useful—cf ideas and results in [87, 222, 314, 370, 456]. However, in spite of its conceptual interest this notion also entails ‘practical’ difficulties. Namely, the trapping boundary is a complicated object to locate, in particular in a numerical evolution. And second, regularity issues may arise on this horizon [314].

6.2.3. Evolution of marginally trapped surfaces: quasi-local black holes. The need for an operational quasi-local notion of a black hole horizon in an evolution has led to the formalization of the idea of worldtubes of AHs or, more generally, worldtubes of MOTS—understood here as marginal surfaces in the sense of [314]. This is done at the price of losing uniqueness.

The first steps were taken by Hájíček in [302–304] where he introduced perfect horizons as non-expanding null hypersurfaces devised to model stationary black hole horizons. These totally geodesic null hypersurfaces [302] were the first examples of worldtubes of AHs where not only the expansion $\theta^{(0)}$ vanishes but also the shear of the null congruence does. A more systematic and general approach was initiated by Hayward in [314], where he introduced future outer trapping horizons (FOTH) to model quasi-local black holes in generic situations. A parallel line of research in quasi-local black holes as AH worldtubes has been developed by Ashtekar and collaborators. This has led to isolated horizons (IH) [43–50] and, together with Krishnan, to dynamical horizons (DH) [40, 41]. FOTHs, on the one hand, and IHs together with DHs, on the other hand, offer complementary approaches to the quasi-local black hole problem. Whereas FOTHs have the virtue of providing a general single characterization which encompasses both stationary and dynamical situations, the asset of IHs and DHs lies in their adaptation to the specific and rather different geometric structures associated with null
(stationary) and spacelike (dynamical) AH worldtubes. Both approaches underline different aspects of the problem and conceptual compatibility is guaranteed since equivalence has been shown in generic conditions [18, 51, 123]—see also [366] and later in this section. Finally, Booth and Fairhurst [120, 123, 358] have introduced the notion of slowly evolving dynamical horizons to address the physical issues happening in the transition from the stationary to the dynamical regimes. General review papers on these quasi-local approaches can be found in [52, 117, 293, 371]. We briefly review the ideas contained in them.

**Trapping horizons.** A trapping horizon is (the closure of) a hypersurface $\mathcal{H}$ foliated by MOTS [314]—this notion has applicability beyond the context of black holes, in cosmological contexts. Here the MOTS condition $\theta^{(\ell)} = 0$ must be read in its local sense as a marginal surface. White and black hole situations are distinguished by the sign of $\theta^{(k)}$, whereas the (local) outer or innermost character is captured by the sign of $\delta_k \theta^{(\ell)}$—where $\delta_k$ denotes formally a variation in the direction of the vector $k^\mu$. Trapping horizons are originally formulated in a double-null foliation of spacetime [314], where $\delta$ becomes a Lie derivative. In this sense, trapping horizons are

(i) future if $\theta^{(k)} < 0$, or past if $\theta^{(k)} > 0$,

(ii) outer if $\delta_k \theta^{(\ell)} < 0$, or inner if $\delta_k \theta^{(\ell)} > 0$.

Note that the extremal case $\delta_k \theta^{(\ell)} = 0$ is not treated—see section 6.3. Inner trapping horizons are of interest in cosmological scenarios. But in quasi-local black hole horizon settings, inner light rays are expected to converge—that is, $\theta^{(k)} < 0$—whereas outer light rays should converge just inside the black hole and diverge just outside—that is, $\delta_k \theta^{(\ell)} < 0$. Therefore, the adequate characterization is in terms of future outer trapping horizons, hypersurfaces $\mathcal{H}$ foliated by MTSs $S_t$ on which the stability condition $\delta_k \theta^{(\ell)} < 0$ holds.

We comment on some fundamental features of FOTHs. Under the dominant energy condition, the MTS slices $S_t$ of a FOTH have spherical topology—the topology law. Under the null energy condition: (i) the vector tangent to the FOTH and normal to the MTS slices can be either spacelike or null—the pointwise signature law—the latter case happening if both the shear $\sigma^{(\ell)}$ and $T^{(\ell, \ell)}$ vanish, and (ii) the area of the MTS slices remains constant if $\mathcal{H}$ is null and grows otherwise (area law). Property (i) follows from $\delta_k \theta^{(\ell)} < 0$ and formalizes the idea of AH worldtubes as being stationary (null) if nothing falls into the horizon and superluminal (spatial) otherwise, whereas property (ii) is the quasi-local counterpart of Hawking’s area theorem [309, 310].

A main application of FOTHs has been the derivation of balance laws for physical quantities on the horizon, and its subsequent application to black hole dynamics—see Hayward’s IHP talk [1] and below. Regarding the conceptual characterization of quasi-local black holes, Hayward has also shown that under a regularity assumption—the existence of foliation of $\mathcal{H}$ by limit sections [314]—the trapping boundary is actually a trapping horizon, in fact the outermost.

**Isolated horizons and dynamical horizons.** The important geometric differences between null and spatial hypersurfaces suggest the use of different strategies for addressing specific issues in the stationary and dynamical regime.

The framework of isolated horizons—see [52] for a very comprehensive account—provides a hierarchy of geometric structures built on a null hypersurface, and devised in order to capture different stationarity levels of a black hole placed in an otherwise dynamical environment.
(a) **Non-expanding horizons** (NEH) provide the minimal notion of stationarity. They are defined as null hypersurfaces $\mathcal{H}$ of $S^2 \times \mathbb{R}$ topology where (i) the null generator expansion $\theta^{(\ell)}$ vanishes and (ii) the vector $-T^\mu_{\nu} \ell^\nu$ is future-directed and causal. Using the (null) Raychaudhuri equation, a NEH $\mathcal{H}$ is a hypersurface foliated by MOTS and characterized by the vanishing of both $\sigma^{(\ell)} = 0$ and $T(\ell, \ell)$. The geometry of a NEH is given in terms of $(q_{\mu\nu}, \hat{\nabla})$, where $q_{\mu\nu}$ is the spatial metric induced on any compact slice $S$ and $\hat{\nabla}$ is a connection uniquely induced from the ambient spacetime connection $\nabla$. The NEH condition expresses the evolution invariance of the intrinsic geometry: $L_\ell q_{\mu\nu} = 0$, and essentially coincides with the notions of perfect horizons and null trapping horizons.

(b) **Weakly isolated horizons** (WIH) are NEHs with some additional structure needed for the analysis of equilibrium black hole dynamics: an equivalence class of null normals $[\ell]$ for which the surface gravities $\kappa_\ell$, defined from $\ell^\rho \nabla_\rho \ell^\mu = \kappa_\ell \ell^\mu$, are constant on $H$—the zeroth law. A Hamiltonian analysis of the symplectic space of solutions of the EFE containing a WIH leads to a (Gibbs-like) quasi-local first law of black hole thermodynamics, providing in particular a quasi-local expression for the mass of a black hole with a stationary horizon.

(c) **Isolated horizons** (IH) are NEHs where the extrinsic geometry $\hat{\nabla}$ is also invariant along the evolution: $[L_\ell, \hat{\nabla}] = 0$. They represent the strongest quasi-local notion of stationarity and crucially allow the definition of mass and angular momentum—in the axisymmetric case—multipoles [48]. The richness of the isolated horizon framework has led to applications in very diverse areas of black hole physics, such as the microscopic evaluation of entropy in the context of loop quantum gravity [38, 39, 42], or the study of properties of hairy black holes and solitonic solutions in theories involving the coupling with additional fields—e.g. Yang–Mills, dilaton, Higgs, Proca and Skyrme fields [52].

Regarding the dynamical case, Ashtekar and Krishnan introduced in [40, 41] the notion of dynamical horizons as spatial hypersurfaces $\mathcal{H}$ that can be foliated by MTSs—surfaces $S_t$ where (19) holds. Note the difference with the FOTH definition, where the local outermost condition $\delta_\ell \theta^{(\ell)} < 0$ has been substituted by the condition on the spatial character of $\mathcal{H}$. FOTHs and DHs are equivalent in generic circumstances [51, 123]. A FOTH for which $\delta_\ell \theta^{(\ell)} \neq 0$ for at least one point on each MTS section, is a DH. For the converse, a genericity condition—namely $\delta_\ell \theta^{(\ell)} \neq 0$ everywhere on $\mathcal{H}$—and the null energy condition are required. Examples of DHs not satisfying the genericity condition, and thus failing to be FOTHs, are presented in [462]—see also comments in section 6.3 in the context of extremal DHs. In contrast with the double null formalism of FOTHs, DHs are formulated in a 3+1 framework. Therefore they are specially suited for their use in numerical relativity. Although black hole mechanics relations can and have been derived in this setting, a strong emphasis is put on the application of the associated balance equations to the analysis and test of numerical simulations in the strong field regime.

**Quasi-local black hole thermodynamics.** The extension of classical black hole thermodynamical results [63] from stationary spacetimes to more generic situations has been one of the main motivations for the development of quasi-local black holes—see Hayward’s IHP talk [1]. Different approaches and definitions of the physical parameters have led to distinct existing versions of the generalized thermodynamics laws, whose detailed review goes beyond the scope of this review. We comment in general terms: (i) **Zeroth law:** constancy of the surface gravity, defined as the non-affinity parameter associated with the generating null vector in the context of isolated horizons, it is shown to characterize a WIH [52]—and therefore holds on an IH. A generalized zeroth law in terms of an inequality bounding the mean...
of the (trapping) gravity is proposed in [314], where the constancy of $\kappa$ is characterized by the saturation of the inequality—see also [408] for a general discussion on the surface gravity in the quasi-local context. (ii) The second law, non-decrease of the area, follows from the spatial (or null character) of the MOTS-worldtubes, together with the condition $\theta^{(k)} < 0$. Therefore, the second law is built into the definition of quasi-local black holes. (iii) The discussion of the first law, as a Gibbs-like expression relating the variation of the energy, area and angular momentum of the horizon, is more problematic due to the ambiguities in defining quasi-local physical parameters—cf section 7.2. In the spherically symmetric and axisymmetric IH cases, this has been successfully addressed—see [45, 50]—and has led to a first law in terms of variations in the space of physical states—Gibbs’ version. However, the attempt to derive a first law relating the time variations of physical quantities (Clausius’ version) has led to different versions according to the distinct choices of evolution vector or quasi-local energy. For these reasons, the status of such a version of the first law in the dynamical regime is unclear. In this context, the ensemble of balance or conservation laws obtained from the restriction of different components of the Einstein equation on the MOTS-worldtubes $\mathcal{H}$ acquire a particular relevance. We highlight the balance equation relating the variation of the area and angular momentum to the flux of energy [40, 41, 120, 121, 314–316]—see [123] for a discussion of the comparison between some of them. A set of evolution equations for the area [41, 52, 120, 294, 314], angular momentum [40, 120, 121, 289, 317] or the charge [317] have also been derived. In particular, [289, 293, 294] discuss a quasi-local version of the membrane paradigm [211, 212, 430, 496] based on a hydrodynamical analogy between horizons and viscous fluids—whose full analysis will plausibly involve the use of concepts and techniques from PDE theory of hyperbolic systems.

We must emphasize that some of the ambiguities present in the fully dynamical regime can be controlled when considering slight deviations from equilibrium, in the setting of Booth and Fairhurst’s slowly evolving dynamical horizons. Finally, regarding the impossibility of reaching extremal black holes (third law), a discussion of extremality in this quasi-local context has been recently developed in [118, 122].

6.2.4. Numerical implementations of dynamical trapping horizons. The implementation of ideas from dynamical trapping horizons offers an example of a fruitful relation between geometry and numerics. The application of some prescriptions derived from the framework of dynamical trapping horizon has been used to extract physical information about numerically constructed black holes. Conversely, numerical experiments have provided key insights into the geometric structure of MOTS-worldtubes.

DHs and a posteriori analyses. The main application of DHs in numerics is the physical and geometric analysis of MOTS-worldtubes located by an AH-finder along a numerical evolution. Krishnan’s IHP presentation [1] provides an excellent account of this kind of analysis. Regarding the extraction of physics, DHs have been used to extract individual black hole masses and spins in black hole spacetimes [57, 146, 372, 457]. More specifically, deformation of black holes can be analysed by computing mass and angular momentum DH multipoles [457]. DHs can be employed to characterize the rate of approach to Kerr [457] or to study spin–orbit effects in binary black hole simulations [146]. Heuristic approaches can be formulated in this setting for determining a quasi-local linear momentum of a black hole, very much relevant in the astrophysical study of recoil velocities after mergers [372]. From a geometric point of view, the results in [457] support the picture that the AHs jumps occurring in black hole evolutions correspond to a situation in which the involved DHs are
actually connected by a worldtube of MOTS which violates some of the FOTH conditions. The resulting single MOTS-worldtube presents timelike or signature-mixed sections where topology change is possible. This picture is also supported by analytic results in [119] on the Tolman–Bondi collapse—see also [407]. A prediction of this picture is that in binary black hole evolutions modelled by two MOTS-worldtubes $\mathcal{H}_1$ and $\mathcal{H}_2$, right after the formation of the outer common horizon $S_o$, the latter should \textit{bifurcate} into two MOTS-worldtubes: an exterior spacelike one $\mathcal{H}_{\text{outer}}$ of growing area, and an interior one $\mathcal{H}_{\text{inner}}$ of mixed signature evolving into a timelike MOTS-worldtube. This is actually seen in numerical experiments [457]. Under evolution, this $\mathcal{H}_{\text{inner}}$ should annihilate with the ghost horizons $\mathcal{H}_1$ and $\mathcal{H}_2$, but the details of this process are unknown. In this context, the assessment of the recent results in [486] showing the overlap of the inner horizons $\mathcal{H}_1$ and $\mathcal{H}_2$ is of special relevance.

$\textbf{DHs as a priori ingredient.}$ An alternative application of quasi-local black holes to numerics is their use as a constitutive element in the PDE evolution system. This is exemplified by the use of NEHs to prescribe inner boundary conditions in the construction of initial data describing spacetimes containing black holes in instantaneous equilibrium—see, e.g. [28, 151, 188, 209, 293, 346, 349]. A scheme for using MOTS—and not MTS—as inner boundary conditions in the context of constrained evolution formalisms has been presented in [348] and at Gourgoulhon and Jaramillo IHP talk [1]. This essentially recasts Eardley programme [222] in the dynamical trapping horizon framework. Its feasibility must still be assessed.

$\textbf{6.2.5. Geometric analysis.}$ A fundamental shift in the research on quasi-local horizons has occurred with the application of tools of geometric analysis to the understanding of dynamical horizons properties. Besides the continuing efforts to characterize DH physical parameters [367] and derive appropriate \textit{conservation equations} for them, some effort has been put into the use of maximum principles and related notions to the study of the properties of elliptic equations defined on horizon sections.

The methods of geometric analysis have been put into work to show that, under appropriate conditions, FOTHs can be fully partitioned into NEH and DH sections—see [18, 123]. This result rules out the possibility of finding MOTS sections in FOTHs that are partially null and partially spacelike—a possibility which was not discarded in early works. This means that the transition from equilibrium to the dynamical regime happens \textit{all at once}. This shows the complete equivalence, in generic circumstances, between the two main approaches to quasi-local black holes—at least in what regards equilibrium and the dynamical stages of the quasi-local horizons.

Two results deserve special mention. First, the \textit{foliation uniqueness theorem} for dynamical horizons by Ashtekar and Galloway [51]: given a dynamical horizon $\mathcal{H}$, its foliation by MTSs is unique. Second, a \textit{local existence theorem} by Andersson, Mars and Simon [18] stating that, given a 3+1 slicing ($\Sigma_i$) and an initial MTS $S_o \subset \Sigma_o$ satisfying a stability condition closely related to the outer condition of FOTHs, there exists a unique worldtube foliated by MTSs $S_i$, such that $S_i \subset \Sigma_i$—at least as long as the stability condition is satisfied.

Some more recent developments in the geometric study of dynamical trapping horizons include the refinement of previous work [18] on the stability and existence analysis of MOTS-worldtubes by Andersson, Mars and Simon in [17]; the derivation of estimates for the curvature in MOTS with application on DHs [20]; studies by Bartnik and Isenberg [66] of necessary and sufficient conditions for the existence of DHs in spherically symmetric spacetimes; the analysis of the asymptotics of MOTS-worldtubes in spherically symmetric spacetimes [516]; the formulation of a conjecture about the \textit{peeling behaviour} of DHs [464] in the
context of the geometric discussion of the hoop conjecture; or the study of area estimates for outermost MOTS and the characterization of the boundary of the trapped region [21]—see also [148, 399]. Finally, we mention an approach towards a general proof of the Penrose inequality à la Huisken–Ilmanen [332] using a spacetime generalization of the inverse mean curvature flows in terms of uniformly expanding flows [128]. These developments are very close in spirit to some of the possible applications of dynamical trapping horizons [12].

6.2.6. General perspective. The recent numerical and geometric insights have enriched the research in quasi-local horizons, which was previously focused mainly on black hole thermodynamical aspects. From the DH existence and foliation uniqueness theorems [18, 51], it follows that the question about the evolution of a given initial MTS, $S_0$, is not well defined in generic situations. The initial $S_0$ can evolve into different DHs depending on the chosen foliation—i.e. on the chosen lapse. Although this can be completely harmless in most ‘practical’ numerical situations, it is conceptually important and was not sufficiently stressed in the numerical community. In numerical evolutions, black holes characterized by MOTSs worldtubes are treated as other standard compact objects. In particular, this assumes a well-defined unique evolution independent of the observer. The clarification of this point permits us to turn the argument around, opening the possibility of setting a preferred choice of lapse function in terms of a geometrically singled out DH—the first attempts in this direction are discussed in [294]. In a different line of work, numerical results by Schnetter and Krishnan—see also [119]—have shed light on the global structure of tubes of MOTS, and on the changes of signature along the evolution and the geometric criteria to control them. Other issues, such as the study of generic dynamical trapping horizon asymptotics towards the event horizon, and its possible application to fundamental problems such as Kerr stability [12] or Penrose inequality will probably require the combination of analytic, geometric and numerical skills.

6.3. Geometric inequalities involving black hole horizons

As in the case of other physical theories, geometric inequalities in GR are often the reflect of a fundamental underlying physical principle. A prime example of this is the positivity of mass in GR [458, 459, 519]. The fundamental nature of this result is manifest from its role in many crucial developments in GR. Furthermore, its failure would put the physical consistency of the theory under question.

In this section, we will briefly comment on some geometric inequalities in the context of black hole spacetimes which, in particular, constrain the gravitational collapse process. Our present understanding of the gravitational collapse is based on a chain of results and conjectures. First, the singularity theorems [311–313, 414] guarantee that the appearance of a trapped surface during the gravitational collapse leads to the development of a singular spacetime. Second, the singularity should be hidden behind a black hole event horizon so as to avoid the lack of predictability—this physically motivated hypothesis excluding the formation of naked singularities was proposed by Penrose [415] and is known as the weak cosmic censorship conjecture. Third, the black hole spacetime should reach a stationary state—this assumption is justified by the finiteness of the amount of radiation that an isolated system can emit. And fourth, assuming that all fields have fallen into the black hole after some finite time, the black hole uniqueness theorems [322] then guarantee that the spacetime settles down to a Kerr black hole. Thus, barring some technical assumptions, if enough matter concentrates in a sufficiently compact region, then the system evolves to a final Kerr provided weak cosmic censorship conjecture holds and a final stationary state is reached.
Using a chain of heuristic arguments based on the previous standard picture—the so-called establishment picture—Penrose proposed [416] a lower bound for the total (ADM) mass of a black hole spacetime in terms of the square root of the area of the black hole. This Penrose inequality provides a lower bound for the black hole contribution to the total mass. It conjectures, in particular, a significant strengthening of positive mass theorems in the black hole context. In its first version—that can be referred to as global—this Penrose inequality conjectures that the area $A_E$ of any section of the event horizon $E$ satisfies $M_{ADM} \geq \sqrt{A_E}/16\pi$. Remarkably, it can be formulated as a problem for initial data on a Cauchy surface $\Sigma_1$, providing a version local in time of the Penrose inequality. Given complete, asymptotically flat Cauchy data on $\Sigma$ satisfying the dominant energy condition, the Penrose inequality—in the formulation of [326]—conjectures

$$A_{\min} \leq 16\pi M_{ADM}^2,$$  \hspace{1cm} (22)

where $A_{\min}$ is the minimal area enclosing the apparent horizon—cf [86] for an explicit construction illustrating the need of using the area of that minimal surface, rather than the outermost MOTS in $\Sigma$. There is a rigidity side to the conjecture. Namely, that the equality is only attained in the spherically symmetric (Schwarzschild) case. The Penrose inequality was proposed in an attempt to provide evidence of the violation of weak cosmic censorship. Other inequalities involving minimal surfaces in [277]—employed to provide lower bounds for areas of event horizon sections, as well as upper bounds for the efficiency in the emission of gravitational radiation—were also constructed with the aim of providing evidence against cosmic censorship. However, growing evidence has accumulated in time on the generic validity of weak cosmic censorship [508] and the effort has shifted to the construction of a proof of Penrose inequality. Although the latter clearly does not imply the correctness of the standard gravitational collapse picture, it would actually provide a strong support for it. The spherically symmetric case has been proved in [389], Huisken and Ilmanen [332] and Bray [127] have provided independent proofs for the so-called Riemannian case—where $K_{ij}=0$—but a general result has not yet been obtained. The Penrose inequality has evolved into a problem in its own right, becoming one of the important challenges in GR and differential geometry. Its alternative name as the isoperimetric inequality for black holes [278, 281, 282] underlines its intrinsic geometric relevance and might ‘lead to the importation into black hole theory of further useful ideas and techniques from global analysis’ [278]—see, e.g. [99, 100, 301, 383–385] for references related to isoperimetric inequalities in this context. Reviews discussing the original Penrose argument, historical developments, main results and open questions can be found in [68, 393]. For early attempts to probe numerically the Riemannian case see [352, 354], and for some more recent numerical studies see, for example, [210, 347, 353, 355].

There exist some generalizations of the Penrose inequality which involve linear momentum [388], charges inside the apparent horizon [280, 321, 511]—cf also [390] and related works [337, 389]—or a cosmological constant—cf [185, 279] for asymptotically anti-de Sitter spacetimes. Here we comment further on a particular sharpened version involving the angular momentum $J$ in axially symmetric spacetimes [210, 310]:

$$A \leq 8\pi \left( M_{ADM}^2 + \sqrt{M_{ADM}^4 - J^2} \right).$$ \hspace{1cm} (23)

A rigidity property—Dain’s rigidity conjecture—has been explicitly formulated in [210]: the equality holds if and only if the (exterior) initial data corresponds to (exterior) Kerr data. Defining $\epsilon_A := A/\left[8\pi \left( M_{ADM}^2 + \sqrt{M_{ADM}^4 - J^2} \right) \right]$, inequality (23) is expressed as $\epsilon_A \leq 1$ and Dain’s conjecture, $\epsilon_A = 1$, characterizes Kerr data by the evaluation of a single real
number—see [210, 347] for possible numerical applications. In the case of being true, this would strengthen current results on characterizations of Kerr [391, 392] and Schwarzschild [269] involving the evaluation of tensor quantities. The non-triviality of Dain’s proposal can be appreciated in inequality (37) of [393]. The latter would provide the counterpart for a variational characterization of Reissner–Nordström data, but has been found to be false [511].

A necessary condition for inequality (23) to make sense is the positivity of the quantity under the square root symbol. This leads to the consideration of mass-angular momentum inequalities in vacuum—in the presence of matter such an inequality is easily violated. The data has to be subextremal. In a series of papers [196, 202–205], Dain has proved the validity of $|J| \leq M_{\text{ADM}}^2$ for maximal, vacuum, asymptotically flat, axisymmetric initial data. Equality is only reached for extremal Kerr. This theorem can be seen as a first step in the study of the nonlinear stability of Kerr. These results have been discussed by Chruściel in his IHP talk, and further developed in [178]. In spite of its characterization in terms of initial data, these mass-angular momentum inequalities are spacetime properties. There has also been an interest in studying some quasi-local versions of these inequalities involving, in particular, the local characterization of extremality. In a first attempt, Ansorg and Petroff [27, 31, 420] have considered the substitution of the ADM mass by the Komar mass $M_{\text{Komar}}$ evaluated on the black hole horizon of stationary and axially symmetric spacetimes. This has led to the numerical construction of stationary configurations of a black hole surrounded by a matter torus where the quotient $|J|/M_{\text{Komar}}^2$ could reach arbitrary high values [31, 420] or even $M_{\text{Komar}}$ could become negative [27]. Instead, and in order to refer only to intrinsic quantities on the horizon, one could consider using the irreducible mass. This has led Petroff and Ansorg—in Ansorg’s IHP talk [1]—to conjecture an inequality for axisymmetric stationary spacetimes only involving the area: $8\pi |J| \leq A$. This conjecture has been further developed—including the charged case—in [32], where use is made of the Christodoulou–Ruffini mass and it is analytically shown that extremality is characterized by the saturation of the area-angular momentum-charge inequality. In parallel, Booth and Fairhurst [118, 122] have undertaken an analysis of the local characterization of extremality of black holes based on the dynamical trapping horizon framework of section 6.2. After also considering the use of geometric inequalities involving the area $A$—or alternatively, the vanishing of a locally defined surface gravity—they have opted for a characterization of extremality in terms of the absence of trapped surfaces inside the apparent horizon. In terms of the outer/inner trapping horizon characterization [314], this reads $\delta_k^\theta^{(i)} = 0$, and leads to the introduction of a quasi-local parameter $e$ satisfying $e \leq 1$, such that extremality corresponds to $e = 1$. DHs satisfying the genericity conditions [51, 123] referred to in section 6.2, are found to be subextremal in this sense. In fact, dropping these DH genericity conditions is equivalent to the extremal characterization by $\delta_k^\theta^{(i)} = 0$—precisely the feature exploited in [462] to construct examples of spacetimes containing DHs but without trapped surfaces.

Before concluding this section on geometric inequalities involving black holes, we must briefly comment on the so-called hoop conjecture. It proposes that black hole horizons happen whenever matter gets sufficiently compacted in all spatial directions. Stated in an intentionally vague manner, ‘black holes with horizons form when, and only when, a mass $M$ gets compacted into a region whose circumference $C$ in every direction satisfies $C \lesssim 4\pi M$’ [402, 495]. In particular, the hoop conjecture offers an interesting example of interaction between geometry and numerics—see, e.g. [154–156, 465, 525] for some numerical studies. Recently, a reformulation of this conjecture as a genuine and mathematically sound—see also [385]—geometric inequality has been presented in [464]—see also references therein for a review of the original conjecture.
Black hole geometric inequalities offer a link between conceptual issues in GR such as cosmic censorship, positivity of mass or black hole dynamics—the latter, through the role of the area in the second law. Furthermore, its constraining role in the gravitational collapse process is of potential interest in numerical constructions.

6.4. Binary black holes

The binary black hole problem has been the main challenge for numerical relativists in the last few decades. The relevance of this problem lies, on the one hand, on its conceptual richness—this two-body problem provides a probe into the strong field nonlinear regime of general relativity—and, on the other hand, on its astrophysical interest—stellar and supermassive black holes constitute some of the main candidates for the detection of gravitational wave signals by interferometric antennae. Here, we emphasize the role of the binary black hole problem as a laboratory for the development and test of new numerical, analytical and geometrical ideas in GR. The developments in the numerical evolutions of binary black hole spacetimes through the inspiral [135], merger and ringdown phases together with the extraction of gravitational radiation [59, 141, 425] have constituted a milestone. Since then, quite a large number of groups have succeeded in developing binary black hole evolution codes. The last couple of years have witnessed a rush in scientific activity. Given the volume of the associated literature, we refer the reader to review [428]. We limit ourselves here to discuss some points of interest in the interaction between geometry and numerics.

6.4.1. Helical Killing vectors and binary black hole initial data. A gravitational radiation reaction drives relativistic systems into inspiral motion, circularizing the orbits very efficiently at least for comparable mass systems [102]. Therefore, for two bodies sufficiently separated, it is natural to approximate the spiral orbits by closed circular ones. This physical image is geometrically en-captured by the existence of a one-parameter helical symmetry $\chi_\lambda : M \rightarrow M$ [241] of the spacetime whose infinitesimal generator will be denoted by $h^\mu$—that is, $\frac{d}{d\lambda} \chi^\mu = h^\mu$. Near the binary system, this helical Killing vector $h^\mu$ is timelike. Sufficiently far away $h^\mu$ becomes spacelike [241], but a number $T > 0$ exists such that the separation between a given point $p \in M$ and its image by the isometry flow $\chi_T(p)$, is timelike—see also [288] for details. The integral lines associated with $h^\mu$ are helices.

Helical symmetry is exact in theories with no gravitational radiation, such as the Newtonian gravity (second-order), post-Newtonian gravity and the Isenberg–Wilson–Mathews approximation to GR [341, 342, 517]. Helical symmetry can be exact in full GR for non-axisymmetric systems with standing gravitational waves [101, 216], although the spacetime cannot be asymptotically flat, in the sense that the ADM mass cannot be defined [283]—more precisely, it cannot have a smooth null infinity if there is no additional stationary Killing vector close to null infinity. Helical Killing vectors have been used in the numerical literature to model slow-motion adiabatic configurations of binary systems [429, 513–515]. This leads to the study of mixed-type PDEs. The light cylinder, characterized by the null character of the helical vector $h^\mu$, separates an inner domain where the PDE is elliptic from an outer one where it becomes hyperbolic—cf also the so-called periodic standing-wave approximation [23, 77, 78, 130, 373] and other recent numerical works [240, 524]. Further theoretical developments making use of the helical Killing symmetry are given in [81, 363]. In particular, the work by Klein—cf his IHP talk [1]—aims at setting a consistent framework for numerics by taking full advantage of the presence of a Killing vector. He studied the EFE in the presence of a helical Killing vector for a vacuum spacetime with two disconnected Killing horizons of spherical topology. The use of a projection formalism on the space of orbits of the Killing vector,
permits then the casting of the problem in terms of the three-dimensional gravity equations with a $SL(2,\mathbb{R})/SO(1, 1)$ sigma model as a material source.

We must briefly comment on the current constructions of binary black hole initial data. In the context of the XCTS construction—see section 4.3—the choice of the evolution vector as an (approximate) helical Killing vector, i.e. $t^\mu = h^\mu$, has been used in the literature for motivating the ansatz $\dot{\gamma}_{\mu\nu} = 0 = \dot{K}$ for the free XCTS data. The helical Killing vector idea was applied for the first time to the construction of binary black hole initial data in [291, 296] and has subsequently been used in [28, 29, 151, 188]. All these data make use of an excision technique to deal with the singularity—see section 6.4.2. Problems arise when combining XCTS and helical symmetry with a puncture approach to initial data, as shown in [305]. For this reason, punctured initial data make use of the extrinsic curvature approach—see section 4.2—which turns out to be difficult to reconcile with a helical symmetry idea. However, probably due to their relatively large initial separations, linear superpositions of two Bowen–York solutions—first used in [30, 58, 73]—have led to good results in recent punctured binary black evolutions in [59, 62, 141–144, 506].

6.4.2. On the issue of moving punctures. A key issue in numerical black hole simulations is the manner in which black holes are modelled in the calculations. The two main techniques employed are: excision, where a spatial neighbourhood of the singularity is removed from the numerical grid on the initial hypersurface and then subsequently, also on each spacelike hypersurface constructed during the evolution [26, 425, 455, 460, 478, 486, 492]—see also [386, 387] for particular analyses of global existence in this setting; and punctures, where one begins with punctured initial data in the sense discussed in section 4.1 and then evolves the data including the singular point—see, for example, [61, 134, 141, 146, 287, 320, 476, 494, 497]. A third alternative replaces the singular black hole interior at each spacelike slice by a regular one, leading to the idea of stuffed black holes [33, 34] that has been recently brought back—see [131, 226]. All these approaches—excision, punctures and stuffing—rely on the intuitive idea that no information escapes from the black hole interior, and thus, assume from the onset some sort of cosmic censorship. In particular, a detailed analysis of this last, non-trivial, point has been undertaken in [131].

There has been a recent interest in understanding some geometric aspects of the moving punctures picture—see, e.g. [133, 306, 307, 494]. In particular, it is of interest to know if they still represent a compactified infinity. More importantly, it is crucial to know whether the method relies on numerical errors near an under-resolved puncture, in which case it could fail when probed at higher resolutions or if the evolutions are let to run much longer than up to now.

Assuming the establishment point of view on black holes, the asymptotic state of the evolution will be described in some sense by a Kerr spacetime. So, after the system has evolved long enough it is natural to expect the existence of an approximate stationary Killing vector that could be used to drive the evolution to an eventually stationary slice. A crucial element in the evolution of punctured data is the use of the so-called symmetry-seeking choices of lapse and shift—the $1+\log$ lapse and the gamma freezing shift [5, 298, 506]. A symmetry-seeking gauge is the one that tends to align itself with this approximate Killing vector as the evolution proceeds. In this context, for a stationary slice it is understood a hypersurface such that for the chosen gauge drivers, the evolution vector is parallel to the stationary Killing vector—so that the evolution of the spacetime effectively freezes. The discussion of stationary slices in the Schwarzschild spacetime has been elaborated further in [74, 225, 306, 307, 434]. Clearly, it is of interest to carry out an analogous analysis for the Kerr spacetime.
The symmetry-seeking nature of the gauges used in the evolution of dynamical black hole spacetimes is supported by numerical evidence. According to this evidence, the evolution using symmetry-seeking gauges seems to pile pointwise in almost-stationary slices of the gauge. It would be of great interest, both practical and theoretical to have a rigorous analysis of this symmetry-seeking behaviour of particular gauges. The resolution of the problem of the nonlinear stability of the Kerr spacetime is likely to clarify this issue.

Numerical evolutions—together with analytical considerations—of punctured Schwarzschild initial data using symmetry-seeking gauges [306, 307] show that the solution approaches pointwise a stationary slice that does not hit the physical singularity. Most importantly, the limiting slice does not reach the inner asymptotically flat spatial infinity, but rather ends on a cylinder of finite areal radius whose throat has an infinite proper distance as measured from the apparent horizon. Notably, a change in the asymptotic behaviour of the conformal factor has been observed—from a $1/r$ behaviour near the puncture to $1/\sqrt{r}$; see also the discussion in the IHP talks [133] by Hannam and O’Murchadha. General results on the topology of hypersurfaces in asymptotically flat spacetimes—see e.g. [266, 267]—preclude the change of topology during the evolution, unless the slices touch the singularity—see also the discussion in [402], section 31.6. In the particular case of these punctured Schwarzschild evolutions, the absence of topology change follows straightforwardly from the employed slicing conditions, since their left–right symmetry behaviour in the Kruskal diagram implies that anything happening to the (right) physical asymptotic end also occurs to the (left) inner flat asymptotic region. This point has also been discussed in [133], where it is emphasized that the puncture actually remains at the inner spacelike infinity along the whole evolution and, in particular, the punctured evolution using positive lapse is never stationary in the limit of infinite resolution. The apparent tension between an evolution towards a limiting stationary slice not reaching the inner asymptotic end, on the one hand, and the fact that the puncture actually stays at this inner end, on the other hand, can be reconciled in terms of the differences between uniform and pointwise convergence: the limit to the final cylindrical stationary slice seems to be pointwise, but not uniform. In actual numerical simulations, the punctured region is under-resolved and there is no practical need to distinguish between both kinds of convergence. In particular, this lack of resolution is not likely to cause problems for finite difference codes at any reasonable—practically realistic—resolution. From this point of view the moving puncture evolution can be seen as an effective natural excision method [133]. Moreover, the blow-up of the metric in the middle of the slice—apparent from the infinite proper distance between the throats at the end of the punctured evolution—indicates that this excision mechanism is not a numerical artifact. Though important by themselves, these issues on convergence and the fate of the puncture should not undermine the non-trivial findings regarding the actual existence of a (pointwise) limiting stationary slice and, more specifically, its cylindrical asymptotics. This last point strongly suggests to start from the very beginning with asymptotically cylindrical data—rather than the original punctured ones—as a representation of black holes, since this would permit to avoid coping from scratch with the non-physical quadrants of the Kruskal diagram [306, 307]. Finally, it is expected that the mechanisms operating in the case of simulations for Schwarzschild initial data are in
essence the same ones working in numerical evolutions of dynamical black hole spacetimes—see comments in [133] about the presence of spin and momentum.

6.4.3. Discussion. The recent intense activity in numerical evolutions of binary black holes has meant a very rapid advance in the understanding of the physics of binary black holes—with a particular emphasis on astrophysical applications. Although more systematic studies are needed, a large number of results are already available, for example, concerning the effect that different configurations of physical parameters—such as mass ratios, spins or orbit eccentricities—have on the final state of the resulting black hole, in particular on the recoil velocities of the final black hole. Again we refer to [428] for a bibliographic account of these advances—cf also the IHP talks by Lousto, Lindblom and van Meter.

In what concerns astrophysical applications, one should mention the complementarity of numerical approaches with the others of analytic nature. On the one hand, given the computational cost of the numerical simulations, it is natural to explore analytic or semi-analytical methods to cover the whole parameter space of the problem. On the other hand, analytical approximations often offer a (quick) hint on the physics of the problem. The need of such a synergy between numerical and analytical efforts was advocated by Damour and Blanchet in their respective IHP talks [1], where different post-Newtonian approaches were confronted with numerical results [103, 214]. Black hole perturbation theory offers a further instance of potentially fruitful combination of analytical and numerical approaches, for example, in the analysis of the extreme mass ratio case—cf Nagar’s IHP talk [1].

Astrophysical aspects of the binary black hole problem offer a plenty of room for the collaboration between analysts and numerical relativists. In addition, the very success of the simulations already offers some indirect evidence of geometric issues such as cosmic censorship or Kerr stability. However, the numerical control of this nonlinear problem opens an outstanding possibility of gaining insights into some of the key geometric and conceptual problems of GR, if specific efforts are geared in this direction. Geometric ideas and results, such as cosmic censorship or black hole uniqueness theorems, have built a solid conceptual framework for the study of gravitational collapse. In turn, it is to be expected that strong numeric tests to some of these geometric ideas could be devised, offering a window to the assessment of nonlinear features in GR. In current simulations, nonlinear effects in the binary black hole system have resulted unexpectedly mild. But, as discussed, for example, in Lehner’s IHP talk [1], ‘uncharted trails’ remain in the road and much should be learnt from the studies of generic configurations—cf in this sense [145, 382] where multi-black hole encounters are studied.

7. Miscellaneous topics

There are some important topics in the relation between geometry and numerics that have not been discussed in the previous sections. Here we briefly present, non-exhaustively, some of those aspects and lines of research.

7.1. Critical collapse

The study of critical phenomena\(^9\) in gravitational collapse is one of the paradigms of the interaction between numerical and mathematical relativity: a type of phenomena which was discovered by means of numerical experiments [157] which then, in turn, have been understood

\(^9\) The content of this section is essentially due to J M Martín-García. We are thankful for his enthusiastic input.
by analytical methods—see [88, 299] for a review. In 1992 Choptuik [157] found that it is possible to form arbitrarily small black holes in the process of gravitational collapse of a spherical scalar field, by fine tuning any parameter which affects the self-gravity of the initial configuration. Before the black hole is formed the spacetime becomes selfsimilar and universal, forgetting the initial condition except for a single length scale which determines the mass of the black hole to be formed. This interesting behaviour has been found in many other matter systems, and perturbative arguments suggest that it could be also present without spherical symmetry, being therefore intrinsic to the strong regime of GR dynamics. Finally, the perfect fine tuning generates a naked singularity with infinite curvature, visible to observers at infinity, with important consequences for cosmic censorship and quantum gravity. These critical phenomena in gravitational collapse were completely unexpected before 1992 and are now considered the best example of how numerical relativity can contribute new physics to GR—see Aichelburg’s and Harada’s IHP talks for examples of ongoing research. A posteriori, they can be understood as the discovery of codimension 1 exact solutions in the infinite-dimensional phase space of general relativity—Minkowski, black holes and stars being global attractors of codimension 0—opening a new way of interpreting GR from the point of view of the theory of dynamical systems.

7.2. Quasi-local physical parameters

The determination of physical parameters in a finite region of spacetime is of clear practical importance in numerical implementations. More generally, it is of conceptual relevance in the theory as a whole and, more concretely, in some particular geometric constructions such as certain geometric flows [68, 128, 278, 279, 332] or in the study of geometric inequalities—cf section 6.3. The determination of the amount of energy in a compact domain is a classical problem in GR and an exhaustive list of proposals regarding this goes beyond the scope of this review—see [483]. The relevance of this subject as a boost for the interaction between geometry and numerics is illustrated, for example, in the use of quasi-local dynamical trapping horizons—section 6.2—for the extraction of physics in current numerical simulations. We highlight the attempts of defining a quasi-local angular momentum by means of an integral on a closed surface \( S \) involving an axial vector \( \phi^i \) defined on \( S \)—e.g. [41, 45, 52, 121, 132, 289, 317, 367, 483, 484], see also the related work in [485]—and in particular the recent efforts for deriving an unambiguous prescription for \( \phi^i \) [190, 221, 367, 484]. Note that the divergence-free character of \( \phi^i \) is not guaranteed in all the latter schemes, a relevant point if a slice-independent definition—i.e. intrinsic to \( S \)—of the angular momentum is desired. In addition to determining the magnitude of the spin, this vector \( \phi^i \) can be used to estimate the direction of the angular momentum vector—cf [146] for preliminary results.

7.3. Spacetime singularities and extensions of GR

The study of the limits and extensions of GR offers stimulating perspectives of research. On the one hand, numerical studies of singularities [88, 271, 323, 424]—e.g. Cauchy horizon instabilities and mass inflation near the Cauchy horizon, naked and null singularities, approach to spatial singularities and analysis of BKL conjecture—explore the internal consistency of the theory and offer an insight into new conceptual developments in it. Numerical implementations of higher dimensional spacetimes as well as the coupling of the geometry to additional fundamental fields—e.g. in the Einstein–Yang–Mills theory—probe extensions of the theory that can be relevant in the quantum context.
7.4. Some astrophysical considerations

Ultimately motivated by the astrophysical study of relativistic compact objects, this review has focused on isolated systems in GR. Moreover, we have centred the discussion on black holes, since they offer a particularly rich context for the dialogue between geometers and numerical relativists. This methodological choice could shadow the extraordinary developments in astrophysical numerical relativity regarding systems with matter. Examples of the latter are the results relative to compact objects, such as neutron or strange stars—notably the first computations of the merger of binary neutron stars by Shibata and Uryū [468]; see also Uryū’s and Saijo’s IHP talks [1]—and other advances in relativistic hydrodynamics [22, 231, 395]. This constitutes presumably an extremely active area of research in the future. Moreover, the study of problems such as the Einstein–Vlasov system presents an analytic as well as numeric interest [24, 25].

7.5. Geometrical spacetime discretizations

In section 3, the initial-value problem formulations of GR have been presented as a particularly powerful approach to the construction of generic spacetimes. A radically different approach consists of adopting a formulation of GR which avoids the use of coordinates in the spirit of the Regge calculus—see, e.g. [274, 433]. For a current programme of research along these lines, see [238, 441]—and also Frauendiener’s IHP talk [1]—which is based on Cartan’s method of frames and the use of discrete differential forms.

7.6. Numerical techniques

As we have commented in section 5, a comprehensive presentation of the evolution formalisms used in numerical relativity should be complemented with a discussion of the employed numerical techniques. Topics offering plenty of occasions for the collaboration between analysts and numerical relativists are, among others, adaptive mesh refinement, high-order methods (cf Tiglio’s IHP talk) spectral methods [297]—including their application to evolutions [319]—and spectral elements (see also Maday’s IHP talk), finite elements (e.g. [4, 325, 365, 473–475]) finite volumes (e.g. [6]) or advances in high-resolution shock capturing methods.

7.7. Algebraic symbolic calculus

The increase in computational capabilities and the development of powerful new computer algebra systems has had a great impact in many areas of GR. Many problems which for a long time seemed to be out of reach merely on the grounds of computational complexities are becoming feasible. Algebraic symbolic methods were first used in GR as a tool in the general research area of exact solutions. Besides its evident utility in the derivation of exact solutions, computer algebra systems had a significant application in the metric equivalence problem—which consists of deciding whether two metrics given in arbitrary coordinates are locally isometric to each other. The equivalence of two metrics is a classical problem in the differential geometry, and a solution was given by Cartan [149]. Early discussions of the equivalence problem in GR can be found in [126]—see also [356, 357]. A particular implementation of these ideas is the system CLASSI—see [432]. Considerations involving the equivalence problem can be of utility in the comparison of numerical simulations.

In recent years, algebraic symbolic calculus has been increasingly used as a systematic tool for analysing the analytic properties of evolution systems—e.g. in the construction of
symmetric systems, analysing characteristics, in the construction of propagation systems, construction of asymptotic expansions and in perturbative analyses. Among the systems explicitly constructed with this purpose in mind one has \texttt{xAct} [396]. Of relevance is also the use of computer algebra systems for the automatic generation of computer codes like in the case of \texttt{Kranc} [334].

8. Conclusions

The extraordinary results in the numerical evolution of black hole binaries have had, and will continue to have, a deep impact on the relation between mathematical and numerical relativists. In particular, it forces a reflection on the long-term scientific objectives of the research of both communities. On one hand it questions the potential pay-offs of certain lines of investigation, and on the other hand it offers the possibility of addressing problems which for \textit{technological} reasons were considered out of reach. In particular, theory will have the unique opportunity to confront observation by means of the accurate simulation of relativistic astrophysical systems. It is to be expected that a big proportion of the numerical community will be involved in this endeavour. Arguably, the implementation of the latter will not require many geometrical insights, although the invariant extraction of physical content would certainly benefit from it—e.g. through the development of more efficient AH-finders, quasi-local characterizations of physical quantities, or invariant algorithms for the extraction of gravitational radiation. However, if one wants to study the other fundamental aspect of the theory, namely the structural ones—exploiting the state-of-the-art numerical possibilities—then evidently a geometrical perspective is fundamental. Even more, one could argue that the study of certain crucial geometrical aspects of GR will require numerical approaches to come to fruition.

In this review, we have focused on the latter geometric aspects of numerical relativity and also on genuinely mathematical results which, we believe, are of relevance for numerical applications. There is an intrinsic aesthetic appeal in the study of intrinsic geometric aspects of GR, but the benefit goes much further. We have tried to emphasize the role—and necessity—of analytical studies as guarantors of the internal consistency of the theory. Geometric-oriented lines of research can and do offer general conceptual frameworks in which physically motivated problems can be unambiguously formulated. The gravitational collapse paradigm is an example of this. On a second stage, it is also undeniable that the geometry also provides powerful tools and insights into calculating things. Following the premise that a crucial aspect of ‘understanding a theory consists of understanding its solutions’, \textit{geometrical numerical relativity} is one of the most powerful tools available to study the space of solutions of GR. Numerical GR can, potentially—i.e. assuming enough computer resources—solve to any desired finite precision any well-formulated problem concerning the property of a specific solution in a concrete problem. When the problem involves an infinite number of solutions, then only the generic behaviour can be analysed. This is however where numerical GR can be most useful, finding new and unexpected results\textsuperscript{10}.

We finalize by naming a number of problems for which, we believe, the interaction between numerical and mathematical relativists is of particular relevance—this, of course, notwithstanding the other issues that have been suggested in the main text. Cosmic censorship remains in the eyes of most relativists the most important open problem in classical GR. It is to be expected that numerical investigations of global spacetimes will prove of utmost value in the strive towards a proof of the conjectures. In the closely related issue of the nonlinear stability of the Kerr spacetime, numerical investigations have already provided information about the

\textsuperscript{10} We acknowledge J M Martín-García for bringing out this point.
decays of various fields. This interaction is bound to become even closer in the future. A related topic, the discussion of the robustness aspects—with respect to changes in the initial data—in the production of gravitational radiation by isolated systems will require a strong input from numerics—even to obtain a rigorous formulation of the problem. The assessment of the physical relevance of certain aspects of the characterization of isolated systems through the notion of asymptotic simplicity—like the peeling behaviour—will also require a close numerical examination. Something similar can be said about the relevance of the construction of initial data sets by means of gluing. A close interaction between analytics and numerics will be required in the study of dynamical trapping horizons and their asymptotic behaviour close to the event horizon and in the relation between the global and local characterizations of the black hole notion—extension of the trapping boundary to the event horizon. Finally, a study of the solution space of GR—in the spirit of the theory of dynamical systems—using ideas developed in the study of critical phenomena should also be based on a close interaction between numerics and geometry.

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