Some properties of Fibonacci numbers, Fibonacci octonions and generalized Fibonacci-Lucas octonions

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Abstract. In this paper we determine some properties of Fibonacci octonions. Also, we introduce the generalized Fibonacci-Lucas octonions and we investigate some properties of these elements.

Key Words: quaternion algebras; octonion algebras; Fibonacci numbers; Lucas numbers.

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1. Preliminaries

Let \((f_n)_{n\geq0}\) be the Fibonacci sequence:

\[f_0 = 0; f_1 = 1; f_n = f_{n-1} + f_{n-2}, \quad n \geq 2\]

and let \((l_n)_{n\geq0}\) be the Lucas sequence:

\[l_0 = 2; l_1 = 1; l_n = l_{n-1} + l_{n-2}, \quad n \geq 2\]

Let \((h_n)_{n\geq0}\) be the generalized Fibonacci sequence:

\[h_0 = p, h_1 = q, h_n = h_{n-1} + h_{n-2}, \quad n \geq 2,\]

where \(p\) and \(q\) are arbitrary integer numbers. The generalized Fibonacci numbers were introduced of A. F. Horadam in his paper [Ho; 61]. Later A. F. Horadam introduced the Fibonacci quaternions and generalized Fibonacci quaternions (in the paper [Ho; 63]). In their work [Fl, Sh; 12], C. Flaut and V. Shpakivskyi and later in the paper [Ak, Ko, To; 14], M. Akyigit, H. H Kosal, M. Tosun gave some properties of the generalized Fibonacci quaternions. In the papers [Fl, Sa; 14] and [Fl, Sa, Io; 13], the authors introduced the Fibonacci symbol elements and Lucas symbol elements. Moreover, they proved that all these elements determine \(\mathbb{Z}\)-module structures. In the paper [Ke, Ak; 15] O. Kecioglu, I. Akkus introduced the Fibonacci and Lucas octonions and they gave some identities and properties of them.

Quaternion algebras, symbol algebras and octonion algebras have many properties and many applications, as readers can see in [Lam; 04], [Sa; 14], [Sa, Fl,
In the paper [Ke, Ak; 15 (a)], O. Kecilioglu, I. Akkus gave some properties of the split Fibonacci and Lucas octonions in the octonion algebra $O(1, 1, -1)$.

In this paper we study the Fibonacci octonions in certain generalized octonion algebras.

In the paper [Fl, Sa; 15 (a)], we introduced the generalized Fibonacci - Lucas quaternions and we determined some properties of these elements. In this paper we introduce the generalized Fibonacci - Lucas octonions and we prove that these elements have similar properties with the properties of the generalized Fibonacci - Lucas quaternions.

2. Properties of the Fibonacci and Lucas numbers

The following properties of Fibonacci and Lucas numbers are known:

**Proposition 2.1.** ([Fib.]) Let $(f_n)_{n \geq 0}$ be the Fibonacci sequence and let $(l_n)_{n \geq 0}$ be the Lucas sequence. Therefore the following properties hold:

i) $f_n + f_{n+2} = l_{n+1}, \forall n \in \mathbb{N};$

ii) $l_n + l_{n+2} = 5f_{n+1}, \forall n \in \mathbb{N};$

iii) $f_n^2 + f_{n+1}^2 = f_{2n+1}, \forall n \in \mathbb{N};$

iv) $l_n^2 + l_{n+1}^2 = l_{2n} + l_{2n+2} = 5f_{2n+1}, \forall n \in \mathbb{N};$

v) $l_n^2 = l_{2n} + 2(-1)^n, \forall n \in \mathbb{N}^*;$

vi) $l_{2n} = 5f_n^2 + 2(-1)^n, \forall n \in \mathbb{N}^*;$

vii) $l_n + f_n = 2f_{n+1}.$

**Proposition 2.2.** ([Fl, Sa; 14], [Fl, Sa, Io; 13]) Let $(f_n)_{n \geq 0}$ be the Fibonacci sequence and let $(l_n)_{n \geq 0}$ be the Lucas sequence. Then:

i) $f_n + f_{n+3} = 2f_{n+2}, \forall n \in \mathbb{N};$
Proposition 2.3. Let \((f_n)_{n \geq 0}\) be the Fibonacci sequence and \((l_n)_{n \geq 0}\) be the Lucas sequence Then:

i) 
\[ l_{n+4} + l_n = 3l_{n+2}, \forall n \in \mathbb{N}. \]

ii) 
\[ l_{n+4} - l_n = 5f_{n+2}, \forall n \in \mathbb{N}. \]

iii) 
\[ f_n + f_{n+8} = 7f_{n+4}, \forall n \in \mathbb{N}. \]

Proof. i) Using Proposition 2.1 (i) we have:

\[ l_{n+4} + l_n = f_{n+3} + f_{n+5} + f_{n-1} + f_{n+1}. \]

From Proposition 2.2 (ii) and Proposition 2.1 (i), we obtain:

\[ l_{n+4} + l_n = 3f_{n+1} + 3f_{n+3} = 3l_{n+2}. \]

ii) Applying Proposition 2.1 (ii), we have:

\[ l_{n+4} - l_n = (l_{n+4} + l_{n+2}) - (l_{n+2} + l_n) = 5f_{n+3} - 5f_{n+1} = 5f_{n+2}. \]

iii) 
\[ f_n + f_{n+8} = (f_n + f_{n+4}) + (f_{n+8} - f_{n+4}). \]

Using Proposition 2.2 (ii,iv), we have:

\[ f_n + f_{n+8} = 3f_{n+2} + l_{n+6}. \]

From Proposition 2.1 (i) and Fibonacci recurrence, we obtain:

\[ f_n + f_{n+8} = 3f_{n+2} + f_{n+5} + f_{n+7} = 3f_{n+2} + 2f_{n+5} + f_{n+6} = 3f_{n+2} + 3f_{n+5} + f_{n+4}. \]

Using Proposition 2.2 (i), we obtain:

\[ f_n + f_{n+8} = 6f_{n+4} + f_{n+4} = 7f_{n+4}. \]

\[ \square \]
3. Fibonacci octonions

Let \( O_\mathbb{R}(\alpha, \beta, \gamma) \) be the generalized octonion algebra over \( \mathbb{R} \) with basis \( \{1, e_1, e_2, \ldots, e_7\} \). It is known that this algebra is an eight-dimensional non-commutative and non-associative algebra.

The multiplication table for the basis of \( O_\mathbb{R}(\alpha, \beta, \gamma) \) is

|   | 1   | e1  | e2  | e3  | e4  | e5  | e6  | e7  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | 1   | e1  | e2  | e3  | e4  | e5  | e6  | e7  |
| e1| e1  | -a  | -ae2| e3  | -ae4| -ae5| e7  | -ae6|
| e2| e2  | -e3 | -\beta| e1  | e6  | e7  | -e4 | -\beta| e5  |
| e3| e3  | -ae2| -\beta| e1  | e7  | -ae6| \beta| e5  | -ae4|
| e4| e4  | -e5 | -e6 | -\gamma| e1  | -\gamma| e3  | -\beta| e2  |
| e5| e5  | -ae4| -e7 | ae6 | -\gamma| -\gamma| e3  | -\beta| e2  |
| e6| e6  | -e7 | \beta| -e5 | -\gamma| e3  | -\beta| -\beta| e1  |
| e7| e7  | -ae6| \beta| -e5 | -\gamma| -\gamma| e2  | -\beta| e1  |

Let \( x \in O_\mathbb{R}(\alpha, \beta, \gamma) \), \( x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \) and its conjugate \( \overline{x} = x_0 - \alpha x_1 - \beta x_2 - \gamma x_3 + x_4 e_4 + x_5 e_5 - x_6 e_6 + x_7 e_7 \), the norm of \( x \) is \( n(x) = x\overline{x} = x_0^2 + \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 + \alpha\beta x_4^2 + \alpha\gamma x_5^2 + \beta\gamma x_6^2 + \alpha\beta\gamma x_7^2 \in \mathbb{R} \).

If, for \( x \in O_\mathbb{R}(\alpha, \beta, \gamma) \), we have \( n(x) = 0 \) if and only if \( x = 0 \), then the octonion algebra \( O_\mathbb{R}(\alpha, \beta, \gamma) \) is called a division algebra. Otherwise \( O_\mathbb{R}(\alpha, \beta, \gamma) \) is called a split algebra.

Let \( K \) be an algebraic number field. It is known the following criterion to decide if an octonion algebra is a division algebra.

**Proposition 3.1.** ([Fl, St; 09]) A generalized octonion algebra \( O_K(\alpha, \beta, \gamma) \) is a division algebra if and only if the quaternion algebra \( \mathbb{H}_K(\alpha, \beta) \) is a division algebra and the equation \( n(x) = -\gamma \) does not have solutions in the quaternion algebra \( \mathbb{H}_K(\alpha, \beta) \).

It is known that the octonion algebra \( O_\mathbb{R}(1, 1, 1) \) is a division algebra and the octonion algebra \( O_\mathbb{R}(1, 1, -1) \) is a split algebra (see [Ke, Ak; 15 (a)], [Fl, Sh; 15]). In [Fl, Sh; 15] appear the following result, which we allow us to decide if an octonion algebra over \( \mathbb{R} \), \( O_\mathbb{R}(\alpha, \beta, \gamma) \) is a division algebra or a split algebra.

**Proposition 3.2.** ([Fl, Sh; 15]) We consider the generalized octonion algebra \( O_\mathbb{R}(\alpha, \beta, \gamma) \), with \( \alpha, \beta, \gamma \in \mathbb{R}^* \). Then, there are the following isomorphisms:

i) if \( \alpha, \beta, \gamma > 0 \), then the octonion algebra \( O_\mathbb{R}(\alpha, \beta, \gamma) \) is isomorphic to octonion algebra \( O_\mathbb{R}(1, 1, 1) \);

ii) if \( \alpha, \beta > 0, \gamma < 0 \) or \( \alpha, \gamma > 0, \beta < 0 \) or \( \alpha < 0, \beta, \gamma > 0 \) or \( \alpha > 0, \beta, \gamma < 0 \) or \( \alpha, \gamma < 0, \beta > 0 \) or \( \alpha, \beta < 0, \gamma > 0 \) or \( \alpha, \beta, \gamma < 0 \) then the octonion algebra \( O_\mathbb{R}(\alpha, \beta, \gamma) \) is isomorphic to the octonion algebra \( O_\mathbb{R}(1, 1, -1) \).

Let \( n \) be an integer, \( n \geq 0 \). In the paper [Ke, Ak; 15], O. Kecilioglu, I. Akkus, introduced the Fibonacci octonions:

\[ F_n = f_n + f_{n+1}e_1 + f_{n+2}e_2 + f_{n+3}e_3 + f_{n+4}e_4 + f_{n+5}e_5 + f_{n+6}e_6 + f_{n+7}e_7. \]
where $f_n$ is $n^{th}$ Fibonacci number.

Now, we consider the generalized octonion algebra $O_R(\alpha, \beta, \gamma)$, with $\alpha, \beta, \gamma$ in arithmetic progression, $\alpha = a + 1$, $\beta = 2a + 1$, $\gamma = 3a + 1$, where $a \in \mathbb{R}$.

In the following, we calculate the norm of a Fibonacci octonion in this octonion algebra.

**Proposition 3.3.** Let $a$ be a real number and let $F_n$ be the $n$-th generalized Fibonacci octonion. Then the norm of $F_n$ in the generalized octonion algebra $O_R(a + 1, 2a + 1, 3a + 1)$ is:

$$
\begin{align*}
n(F_n) &= f_{2n+6} \left( 79a^2 + 46a + \frac{174a^3 - 4a}{5} \right) + \\
&+ f_{2n+7} \left( 130a^2 + 84a + 21 + \frac{282a^3 + 8a}{5} \right) + (-1)^n \left( 4a^2 + \frac{12a^3 + 8a}{5} \right).
\end{align*}
$$

**Proof.**

$$
\begin{align*}
n(F_n) &= f_{2n}^2 + (a + 1) f_{2n+1}^2 + (2a + 1) f_{2n+2}^2 + (a + 1)(2a + 1) f_{2n+3}^2 + \\
&+ (3a + 1) f_{2n+4}^2 + (a + 1)(3a + 1) f_{2n+5}^2 + \\
&+ (2a + 1)(3a + 1) f_{2n+6}^2 + (a + 1)(2a + 1)(3a + 1) f_{2n+7}^2 = \\
&= f_{2n}^2 + f_{2n+1}^2 + f_{2n+2}^2 + f_{2n+3}^2 + f_{2n+4}^2 + f_{2n+5}^2 + f_{2n+6}^2 + f_{2n+7}^2 + \\
&+ a \left( f_{2n+1}^2 + 2f_{2n+2}^2 + 3f_{2n+3}^2 + 3f_{2n+4}^2 + 4f_{2n+5}^2 + 5f_{2n+6}^2 + 6f_{2n+7}^2 \right) + \\
&+ a^2 \left( 2f_{2n+3}^2 + 3f_{2n+5}^2 + 6f_{2n+6}^2 + 11f_{2n+7}^2 \right) + 6a^3 f_{2n+7}^2 = \\
&= S_1 + S_2 + S_3 + 6a^3 f_{2n+7}^2,
\end{align*}
$$

where, we denoted with $S_1 = f_{2n}^2 + f_{2n+1}^2 + f_{2n+2}^2 + f_{2n+3}^2 + f_{2n+4}^2 + f_{2n+5}^2 + f_{2n+6}^2 + f_{2n+7}^2$, $S_2 = a \left( f_{2n+1}^2 + 2f_{2n+2}^2 + 3f_{2n+3}^2 + 3f_{2n+4}^2 + 4f_{2n+5}^2 + 5f_{2n+6}^2 + 6f_{2n+7}^2 \right)$, $S_3 = a^2 \left( 2f_{2n+3}^2 + 3f_{2n+5}^2 + 6f_{2n+6}^2 + 11f_{2n+7}^2 \right)$.

Now, we calculate $S_1, S_2, S_3$.

Using [Ke, Ak; 15] (p.3), we have

$$S_1 = f_8 f_{2n+7} = 21 f_{2n+7}. \quad (3.2)$$

Applying Proposition 2.1 (iii) and Proposition 2.1 (i), we have:

$$S_2 = a \left( f_{n+1}^2 + 2f_{n+2}^2 + 3f_{n+3}^2 + 3f_{n+4}^2 + 4f_{n+5}^2 + 5f_{n+6}^2 + 6f_{n+7}^2 \right) =$$
From Proposition 2.2 (ii), Proposition 2.3 (ii) and Proposition 2.1 (i), we have:

\[ S_2 = 6a (f_{n+6}^2 + f_{n+17}^2) - a (f_{n+6}^2 + f_{n+17}^2) + 5a (f_{n+5}^2 + f_{n+4}^2) - 2a (f_{n+4}^2 + f_{n+3}^2) + \\
+ 4a f_{n+3}^2 + a (f_{n+2}^2 + f_{n+3}^2) + a (f_{n+1}^2 + f_{n+2}^2) = \\
a \cdot (6f_{2n+13} - f_{2n+11} + 5f_{2n+9} - 2f_{2n+7} + 4f_{n+3}^2 + f_{2n+5} + f_{2n+3}) = \\
a \cdot [6f_{2n+13} - f_{2n+11} + 7f_{2n+9} - 2(f_{2n+7} + f_{2n+9}) + 4f_{n+3}^2 + l_{2n+4}] = \\
a \cdot 6(f_{2n+9} + f_{2n+13}) + (f_{2n+9} - f_{2n+11}) + l_{2n+4} - 2l_{2n+8} + 4 \cdot \frac{l_{2n+6} - 2(-1)^{n+3}}{5} .
\]

From Proposition 2.2 (ii), Proposition 2.3 (ii) and Proposition 2.1 (i), we have:

\[ S_2 = a \cdot \left[ 18f_{2n+11} - f_{2n+10} - (l_{2n+8} - l_{2n+4}) - l_{2n+8} + 4 \cdot \frac{l_{2n+6} + 2(-1)^n}{5} \right] = \\
a \cdot \left[ 18f_{2n+11} - f_{2n+10} - 5f_{2n+6} - f_{2n+7} - f_{2n+9} + 4 \cdot \frac{l_{2n+6} + 2(-1)^n}{5} \right].
\]

Using several times the recurrence of Fibonacci sequence and Proposition 2.1 (vii), we obtain:

\[ S_2 = a \cdot \left[ 46f_{2n+6} + 84f_{2n+7} + 4 \cdot \frac{2f_{2n+7} - f_{2n+6} + 2(-1)^n}{5} \right]. \quad (3.3)
\]

Applying Proposition 2.1 (iii) and Proposition 2.1 (vi,i), we have:

\[ S_3 = a^2 \cdot (2f_{n+3}^2 + 3f_{n+5}^2 + 6f_{n+6}^2 + 11f_{n+7}^2) = \\
a^2 \cdot [2(f_{n+3}^2 + f_{n+4}^2) - 2(f_{n+4}^2 + f_{n+5}^2) + 5(f_{n+5}^2 + f_{n+6}^2) + (f_{n+6}^2 + f_{n+7}^2) + 10f_{n+7}^2] = \\
a^2 \cdot \left[ 2f_{2n+7} - 2f_{2n+9} + 5f_{2n+11} + f_{2n+13} + 2l_{2n+14} - 4(-1)^{n+7} \right] = \\
a^2 \cdot \left[ -2f_{2n+8} + 5f_{2n+11} + 3f_{2n+13} + 2f_{2n+15} + 4(-1)^n \right].
\]

From Proposition 2.3 (iii) and the recurrence of Fibonacci sequence, we have:

\[ S_3 = a^2 \cdot [-2f_{2n+7} - 2f_{2n+6} + 5f_{2n+11} + 3f_{2n+13} + 14f_{2n+11} - 2f_{2n+7} + 4(-1)^n] = \\
a^2 \cdot \left[ -4f_{2n+7} - 2f_{2n+6} + 19f_{2n+11} + 3f_{2n+13} + 4(-1)^n \right] = \\
a^2 \cdot \left[ -4f_{2n+7} - 2f_{2n+6} + 134f_{2n+7} + 81f_{2n+6} + 4(-1)^n \right].
\]

Therefore, we obtained that:

\[ S_3 = a^2 \cdot [79f_{2n+6} + 130f_{2n+7} + 4(-1)^n] . \quad (3.4)
\]

From Proposition 2.1 (vi,i), we have:

\[ 6a^3 f_{n+7}^2 = \frac{6a^3}{5} \cdot \left[ f_{2n+14} - 2 \cdot (-1)^{n+7} \right] = \frac{6a^3}{5} \cdot [f_{2n+13} + f_{2n+15} + 2 \cdot (-1)^n] .
\]
Applying Proposition 2.3 (iii) and the recurrence of Fibonacci sequence many times, we have:

\[
6a^3 f_{n+7}^2 = \frac{6a^3}{5} [7f_{2n+9} - f_{2n+5} + 7f_{2n+11} - f_{2n+7} + 2(-1)^n] = \\
\frac{6a^3}{5} [29f_{2n+6} + 47f_{2n+7} + 2(-1)^n].
\] (3.5)

From the relations (3.1), (3.2), (3.3), (3.4), (3.5), we have:

\[
n(F_n) = 21f_{2n+7} + a \cdot \left[ 46f_{2n+6} + 84f_{2n+7} + 4 \cdot \frac{2f_{2n+7} - f_{2n+6} + 2(-1)^n}{5} \right] + \\
+ a^2 \cdot [79f_{2n+6} + 130f_{2n+7} + 4(-1)^n] + \frac{6a^3}{5} [29f_{2n+6} + 47f_{2n+7} + 2(-1)^n].
\]

Therefore, we get:

\[
n(F_n) = f_{2n+6} \left( 79a^2 + 46a + \frac{174a^3 - 4a}{5} \right) + \\
+ f_{2n+7} \left( 130a^2 + 84a + 21 + \frac{282a^3 + 8a}{5} \right) + (-1)^n \left( 4a^2 + \frac{12a^3 + 8a}{5} \right)
\]

\[\square\]

We obtain immediately the following remark:

**Remark 3.1.** If \( a \) is a real number, \( a < -1 \), then, the generalized octonion algebra \( O_R(a+1, 2a+1, 3a+1) \) is a split algebra.

**Proof.** Using Proposition 3.2 (ii) and the fact that the octonion algebra \( O_R(1, 1, -1) \) is a split algebra, it results that, if \( a < -1 \), the generalized octonion algebra \( O_R(a+1, 2a+1, 3a+1) \) is a split algebra.

For example, for \( a = -4 \) we obtain the generalized octonion algebra \( O_R(-3, -7, -11) \). From Remark 3.1 it results that this is a split algebra (another way for to prove that this algebra is a split algebra is to remark that the equation \( n(x) = 11 \) has solutions in the quaternion algebra \( H_K (-3, -7) \) and then we apply Proposition 3.1.

Now, we want to determine how many Fibonacci octonions invertible are in the octonion algebra \( O_R(-3, -7, -11) \). Applying Proposition 3.3, we obtain that \( n(F_n) = -1144f_{2n+6} - 1851f_{2n+7} - 96(-1)^n \), \( n \in \mathbb{N} \). Using that \( f_{2n+6}, f_{2n+7} > 0, \forall n \in \mathbb{N} \), it results that \( n(F_n) < 0, \forall n \in \mathbb{N} \), therefore, in the split octonion algebra \( O_R(-3, -7, -11) \) all Fibonacci octonions are invertible.

For \( a = -2 \), after a few calculations, we also get that in the split octonion algebra \( O_R(-1, -3, -5) \) all Fibonacci octonions are invertible.

From the above, the following question arises: how many invertible Fibonacci
For example, for octonions zero divisors? octonions in a such octonion algebra are invertible or there are Fibonacci octonions there are in the octonion algebra $O_{\mathbb{R}}(a + 1, 2a + 1, 3a + 1)$, with $a < -1$? We get the following result:

**Proposition 3.4.** Let $a$ be a real number, $a \leq -2$ and let $O_{\mathbb{R}}(a + 1, 2a + 1, 3a + 1)$ be a generalized octonion algebra. Then, in this algebra, all Fibonacci octonions are invertible elements.

**Proof.** It is sufficient to prove that $n(F_n) \neq 0$, ($\forall$) $n \in \mathbb{N}$. Using Proposition 3.3., we have:

$$n(F_n) = f_{2n+6} \left(79a^2 + 46a + \frac{174a^3 - 4a}{5}\right) +$$

$$+ f_{2n+7} \left(130a^2 + 84a + 21 + \frac{282a^3 + 8a}{5}\right) + (-1)^n \left(4a^2 + \frac{12a^3 + 8a}{5}\right) \Leftrightarrow$$

$$n(F_n) = f_{2n+6} \cdot \frac{174a^3 + 395a^2 + 226a}{5} +$$

$$+ f_{2n+7} \frac{282a^3 + 650a^2 + 428a + 105}{5} + (-1)^n \cdot \frac{12a^3 + 20a^2 + 8a}{5}.$$ 

After a few calculations, we obtain:

$$n(F_n) = f_{2n+6} \cdot \frac{a(a + 2)(174a^2 + 47) + 132a}{5} +$$

$$+ f_{2n+7} \cdot \frac{2(a + 2)(141a^2 + 43a + 126) - 407}{5} + (-1)^n \cdot \frac{4a(a + 2)(3a - 1) + 16a}{5}.$$ 

We remark that $141a^2 + 43a + 126 > 0$ ($\forall$) $a \in \mathbb{R}$ (since $\Delta < 0$) and

$$\frac{a(a + 2)(174a^2 + 47) + 132a}{5} < 0, \ (\forall) a \leq -2, \ 2(a + 2)(141a^2 + 43a + 126) - 407 < 0, \ (\forall) a \leq -2,$$

$$\frac{4a(a + 2)(3a - 1) + 16a}{5} < 0, \ (\forall) a \leq -2.$$ 

Since $f_{2n+6}, f_{2n+7} > 0$ ($\forall$) $n \in \mathbb{N}$, we obtain that $n(F_n) < 0, (\forall) a \leq -2, n \in \mathbb{N}$ (even if $n$ is an odd number). This implies that, in the generalized octonion algebra $O_{\mathbb{R}}(a + 1, 2a + 1, 3a + 1)$, with $a \leq -2$, all Fibonacci octonions are invertible. □

Now, we wonder what happens with the Fibonacci octonions in the generalized octonion algebra $O_{\mathbb{R}}(a + 1, 2a + 1, 3a + 1)$, when $a \in (−2, -1)$? All Fibonacci octonions in a such octonion algebra are invertible or there are Fibonacci octonions zero divisors?

For example, for $a = -\frac{3}{2}$, using Proposition 3.3, we have that the norm of a Fibonacci octonion in the octonion algebra $O_{\mathbb{R}}(-\frac{1}{2}, -2, -\frac{7}{2})$ is $n(F_n) = -\frac{7}{2} f_{2n+6} + \frac{83}{2} f_{2n+7} - \frac{3}{2} \cdot (-1)^n > 0, (\forall) n \in \mathbb{N}^*$. This implies that in the generalized octonion algebra $O_{\mathbb{R}}(-\frac{1}{2}, -2, -\frac{7}{2})$ all Fibonacci octonions are invertible.

In the future, we will study if this fact is true in each generalized octonion algebra $O_{\mathbb{R}}(a + 1, 2a + 1, 3a + 1)$, with $a \in (−2, -1)$. 

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4. Generalized Fibonacci-Lucas octonions

In the paper [Fl, Sa; 15 (a)], we introduced the generalized Fibonacci - Lucas numbers, namely: if \(n\) is an arbitrary positive integer and \(p, q\) be two arbitrary integers, the sequence \((g_n)_{n \geq 1}\), where

\[ g_{n+1} = p f_n + q l_{n+1}, \quad n \geq 0 \]

is called the sequence of the generalized Fibonacci-Lucas numbers. For not make confusions, we will use the notation \(g_{n, p, q}^{p, q}\) instead of \(g_n\).

Let \(O_Q(\alpha, \beta, \gamma)\) be the generalized octonion algebra over \(\mathbb{Q}\) with the basis \(\{1, e_1, e_2, ..., e_7\}\). We define the \(n\)-th generalized Fibonacci-Lucas octonion to be the element of the form

\[ G_{p, q}^{p, q} = g_{n, p, q}^{p, q} \cdot 1 + g_{n+1, p, q}^{p, q} \cdot e_1 + g_{n+2, p, q}^{p, q} \cdot e_2 + g_{n+3, p, q}^{p, q} \cdot e_3 + g_{n+4, p, q}^{p, q} \cdot e_4 + g_{n+5, p, q}^{p, q} \cdot e_5 + g_{n+6, p, q}^{p, q} \cdot e_6 + g_{n+7, p, q}^{p, q} \cdot e_7. \]

We wonder what algebraic structure determine the generalized Fibonacci-Lucas octonions. First, we make the following remark.

**Remark 4.1.** Let \(n, p, q\) three arbitrary positive integers, \(p, q \geq 0\). Then, the \(n\)-th generalized Fibonacci-Lucas octonion \(G_{p, q}^{p, q} = 0\) if and only if \(p = q = 0\).

**Proof.** "\(\Rightarrow\)" If \(G_{p, q}^{p, q} = 0\), it results \(g_{n, p, q}^{p, q} = g_{n+1, p, q}^{p, q} = ... = g_{n+7, p, q}^{p, q} = 0\). This implies that \(g_{n+1}^{p, q} = ... = g_{n}^{p, q} = g_{1, p, q}^{p, q} = 0\). We obtain immediately that \(q = 0\) and \(p = 0\). "\(\Leftarrow\)" is trivial.

In the paper [Fl, Sa; 15 (a)], we proved the following properties of the generalized Fibonacci-Lucas numbers:

**Remark 4.2.** Let \(n, m \in \mathbb{N}^*, a, b, p, q, p', q' \in \mathbb{Z}\). Then, we have:

i) \[ ag_{n, p, q}^{p, q} + bg_{n, p', q'}^{p', q'} = g_{n}^{a, p, p', q} + g_{b}^{b, p, p', q}. \]

ii) \[ 5g_{n, p, q}^{p, q}, 5g_{p, q}^{p', q'} = 5g_{n, m+n-2, p\cdot p', q\cdot q'}^{p, q} + 5g_{m+n+1, p, q}^{p, q, 0} + 5g_{n, m+n, p\cdot p', q\cdot q'}^{p, q, 0} + 5g_{n, m-1, p\cdot p', q\cdot q'}^{p, q, 0} + 5g_{n, m+n, p, q}^{p, q}. (-1)^m. 5g_{n, m+n, p, q}^{p, q}. (-1)^m. \]

Using these remark we can prove the followings:

**Theorem 4.1.** Let \(A\) and \(B\) be the sets

\[ A = \left\{ \sum_{i=1}^{n} G_{n_i, q_i}^{p_i, q} | n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Z}, (\forall) i = 1, n \right\}, \]
\[ B = \left\{ \sum_{i=1}^{n} 5G_{p_i,q_i}^n | n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Q}, (\forall) i = \overline{1,n} \right\} \cup \{1\} . \]

Then, the following statements are true:

i) \( A \) is a free \( \mathbb{Z} \)- submodule of rank 8 of the generalized octonions algebra \( O_\mathbb{Q}(\alpha, \beta, \gamma) \);

ii) \( B \) with octonions addition and multiplication, is a unitary non-associative subalgebra of the generalized octonions algebra \( O_\mathbb{Q}(\alpha, \beta, \gamma) \).

**Proof.** (i) Using Remark 4.2, it results immediately that

\[ aG_n^p q + bG_m^p q' = G_m^p a q + G_m^p a q', \quad (\forall) m, n \in \mathbb{N}^*, a, b, p, q, p', q' \in \mathbb{Z}. \]

Moreover, applying Remark 4.1, it results that \( 0 \in A \).

These imply that \( A \) is a \( \mathbb{Z} \)- submodule of the generalized octonions algebra \( O_\mathbb{Q}(\alpha, \beta, \gamma) \). Since \( \{e_1, e_2, ..., e_7\} \) is a basis of \( A \), it results that \( A \) is a free \( \mathbb{Z} \)-module of rank 8.

(ii) From Remark 4.2 (ii), it results immediately that \( 5G_n^p q \cdot 5G_n^p q' \in B, \quad (\forall) m, n \in \mathbb{N}^*, p, q, p', q' \in \mathbb{Z} \). Using this fact and a similar reason that in the proof of (i), it results that \( B \) is a unitary non-associative subalgebra of the generalized octonions algebra \( O_\mathbb{Q}(\alpha, \beta, \gamma) \).

\[ \square \]

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