Suppressing information storage in a structured thermal bath: Objectivity and non-Markovianity

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Quantum systems interacting with environments tend to have their information lost from transmission through the correlations generated with the degrees of freedom of the environment. For situations where we have non-Markovian environments, the information contained in the environment may return to the system and such effect can be captured by non-Markovian witnesses. In the present work, we use a central qubit coupled to a spin chain with Ising interactions subject to a magnetic field, i.e., a central spin model, and solve the exact dynamics of a system subject to dephasing dynamics via Kraus operators. For such we use two witnesses to analyze the presence of non-Markovianity: the BLP trace-based measure and the conditional past-future correlator (CPF). Furthermore, we see how such behavior suppresses the classic plateau in Partial Information Plot (PIP) from the paradigm of quantum Darwinism, as well as objective information from Spectrum Broadcast Structure (SBS). In addition to the system point of view, we show the impossibility of encoding accessible information for measurement in the environment for any model limit.

I. INTRODUCTION

A quantum system in contact with an environment, i.e., an open quantum system, is subject to decoherence, which means that we lose states of quantum superposition as a result of the classical uncertainty created by the persistent loss of information to the environment. What we have, then, is that the disturbance caused by the environment promotes the loss of quantumness of the systems that would preserve this nature, in principle, when isolated [1, 2]. This situation gives us a crucial overview of the experiments, which will have to be mediated taking into account the sensitivity of states to the environment, hence the importance of studying open quantum systems. An important class of such open systems is those coupled to finite baths with finite temperatures, where the transmission of system-environment information and vice versa becomes non-trivial, introducing, for example, situations where information previously lost to the environment returns for the system of interest.

The characterization of information return from the environment to a given system can be analyzed in the light of recoherence, indicated, for example, by non-Markovianity. When we are dealing with classical systems, the system’s Markovianity reflects the divisibility of the transition maps of a stochastic process [3–5]. This implies that in a Markovian system the states defined at some point in time do not depend on previous states, that is, there is the emergence of the memoryless property. Notably, this intuitive notion allows for some new ways to characterize non-Markovianity in open quantum systems, where we can, for example, characterize such systems using state distinguishability based on the distance of distinct states in the state space, such as the Breuer-Laine-Piilo (BLP) measure, proposed by [6], or by the correlation between states measured at different points on the timeline, as described by the conditional past-future correlation (CPF) [7, 8]. For both previous programs, we are using the observation of what is happening in the quantum system of interest to identify feedback of information. We can, of course, take another point of view, where the deposit of information in the environment and its accessibility are important. For this analysis, we can use, respectively, the paradigms of quantum Darwinism [9–13], and the Spectrum Broadcast Structure (SBS) [14–16], which answer questions about encoded and accessible information for knowledge from the perspective of observers for whom the environment is accessible.

In this paper, we consider these physical descriptions to study and characterize a platform of an open quantum system based on a qubit coupled to a thermal bath structured by spins with ferromagnetic interactions between the first neighbors (with intensity $J > 0$), also subject to the action of a magnetic field $h$ in the direction $\hat{z}$. The bath is in thermal equilibrium at the temperature $T = \frac{1}{\beta}$. Along the paper we consider $k_B = \hbar = 1$.

Figure 1. (Color online) Here, we use a central qubit coupled to a thermal bath structured by spins with ferromagnetic interactions between the first neighbors (with intensity $J > 0$), also subject to the action of a magnetic field $h$ in the direction $\hat{z}$. The bath is in thermal equilibrium at the temperature $T = \frac{1}{\beta}$. Along the paper we consider $k_B = \hbar = 1$. 

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nal purpose is to relate the non-Markovian behavior (including its witnesses and measurements) as well as the storage of information in the environment (which we will indistinctly call bath to the throughout the text), aiming at an overview from the point of view of the system and the environment. For this, we will investigate the suppression of both quantum Darwinism and SBS.

The structure of the paper is arranged as follows. In Sec.II we construct a description of the system and the environment, the results about its dynamics using Kraus representation, and the closed relation between the coherence function and the Lee-Yang zeros. In Sec.III we present two non-Markovianity witnesses and its respective plots to show the limits of non-Markovianity with respect to the interacting parameters - magnetic field and Ising coupling - and temperature. The Sec.IV contains the informational description of the problem using the key ideas of quantum Darwinism and SBS, explaining the insufficiency of first to explain the objectivity, the importance of the distinguishability between the broadcast states to the emergence of objectivity, and the results for the present model, while the Sec.V presents the conclusions.

II. DESCRIPTION OF THE MODEL

Here we will defining the notation and the structure of the representative Hilbert spaces. Denoting \( \mathcal{H}_S \) the Hilbert space representative of the states of the central qubit and by \( \mathcal{H}_B \) the Hilbert space of the spin bath states. The total system is then given by the tensor product space \( \mathcal{H}_{SB} = \mathcal{H}_S \otimes \mathcal{H}_B \). For the Hilbert state with respect to the central qubit, we take a 2-dimensional complex space, \( \mathcal{H}_S = \text{span}\{|0\rangle, |1\rangle\} \cong \mathbb{C}^2 \), and this states \(|0\rangle, |1\rangle\) are the eigenstates of the Pauli matrix \(\sigma^z\) corresponds to the eigenvalues 1 and −1, respectively.

Let us consider a Hilbert subspace that can be described too by a 2-dimensional complex space, \( \mathcal{H}_k = \text{span}\{|0\rangle, |1\rangle\} \cong \mathbb{C}^2 \). This space is the subspace descriptive of the \( k \)-th spin site in the bath. As one can see, the bath can be construct as the tensor product of subspace corresponding to each site

\[
\mathcal{H}_B = \bigotimes_{k=1}^{N} \mathcal{H}_k \cong (\mathbb{C}^2)^\otimes N,
\]

and the notation \((\mathbb{C}^2)^\otimes N\) means the composition of \( N \) 2-dimensional complex spaces.

To know how the environment acquires and record information about the central system, we will use the environment point-of-view. Thereby, it is important to construct partial Hilbert spaces, that will be tensorial compositions of such number (lower than \( N \)) of subsystems. If one call this fractional Hilbert space \( \mathcal{F} \), it can be written as

\[
\mathcal{F} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N = \bigotimes_{k=1}^{fN} \mathcal{H}_k,
\]

such that \( fN \equiv \#\mathcal{F} \leq N \), and \# represents the number of composed subsystems (or the cardinality of \( \mathcal{F} \) with respect to \( \mathcal{H}_k \)).

Now, with each Hilbert space defined, we can propose that the total Hamiltonian that describes system-environment is given by

\[
H = H_S + H_B + H_{SB},
\]

where \( H_S = \omega \sigma^z \) is the Hamiltonian of the system and \( H_B \) is the Ising-like environment Hamiltonian of \( N \) spin 1/2 z-axis, described by

\[
H_B = -J \sum_{i=1}^{N} \sigma^z_i \otimes \sigma^z_{i+1} - h \sum_{i=1}^{N} \sigma^z_i,
\]

that contains Pauli matrices \( \sigma^z \) acting in each space \( \mathcal{H}_i \), \( J \) corresponds to a nearest-neighbor coupling between the spins and \( h \), the magnetic field along z-axis affecting the spin chain.

On the one hand, the initial state of the system will be considered as a general qubit state in the Bloch sphere

\[
|\psi\rangle = a|0\rangle + b|1\rangle \in \mathcal{H}_S,
\]

where \( a, b \in \mathbb{C} \) and \( |a|^2 + |b|^2 = 1 \).

On the other hand, since this qubit is subject to dynamics due to the interaction with a thermal bath at temperature \( T \), we will consider that such bath state is described by the Gibbs state

\[
\rho_B = \frac{e^{-\beta H_B}}{Z_B},
\]

where \( Z_B = \text{Tr}[e^{-\beta H_B}] \) the partition function.

Such Gibbs state will gives us a distribution in the state space and one can construct each microstate as

\[
|\chi\rangle := |\chi_1...\chi_N\rangle \equiv |\chi_1\rangle \otimes \ldots \otimes |\chi_N\rangle,
\]

that is the state that diagonalizes \( H_B \), where \( |\chi_i\rangle \in \mathcal{H}_i \) is a eigenstate of the Pauli matrix \( \sigma^z \) in computational basis \{\( |0\rangle, |1\rangle \}\), correspondent to the eigenvalues \( \sigma_i = \pm 1 \). Then, the diagonal Hamiltonian results in

\[
H_B |\chi\rangle = E(\chi) |\chi\rangle,
\]

defines the configuration correspondent to the energy

\[
E(\chi) = -J \sum_{i \in \mathcal{A}} \sigma_i \sigma_{i+1} - h \sum_{i \in \mathcal{A}} \sigma_i,
\]

for some \( \mathcal{A} \subseteq \mathcal{H}_B \). We will be interested in bath fractions, sometimes \( \mathcal{A} \) will correspond to a subspace smaller than the space corresponding to the bath \( \mathcal{H}_B \).

The diagonalization of the Ising chain Hamiltonian allows us to rewrite bath density operator in the energy basis:

\[
\rho_B = \frac{e^{-\beta H_B}}{Z_B} = \frac{1}{Z_B} \sum_{\chi} e^{-\beta E(\chi)} |\chi\rangle \langle \chi|.
\]

This system presents ground states obtained from \( \lim_{\beta \to \infty} \rho_B \), in which for \( h = 0 \) emerges a \( \mathbb{Z}_2 \) symmetry and the ground states are \( |0...0\rangle \langle 0...0| = |0\rangle \langle 0| \) and \( |1...1\rangle \langle 1...1| = |1\rangle \langle 1| \).
and for $h \neq 0$ the symmetry broken and the ground states depends on the direction of the magnetic field.

The central spin interacts with the bath with an interaction strength $\alpha \in [0, 1]$ (with this interval for realistic purposes), described using the Hamiltonian

$$H_{SB} = \alpha \sigma_z \otimes \sum_i \sigma_i^z,$$

(9)

acting in the space $\mathcal{H}_{SB} \cong (\mathbb{C}^2)^{\otimes N+1}$, and inducing the unitary time evolution operator $U(t) = e^{-iH_{SB}t}$ - considering here the situation of interaction picture with respect to the operator $H_S$; since $[H_S, H_{SB}] = 0$, arises in the setup a case of pure decoherence, this does not affect $H_B + H_{SB}$ and obviously $[H_B, \rho_B] = 0$.

This operator $H_{SB}$ generate a quantum map $\mathcal{E} : \mathcal{L}(\mathcal{H}_S) \rightarrow \mathcal{L}(\mathcal{H}_S)$ and following the calculations of the Appendix A, the dynamics just affects the off-diagonal terms

$$\rho_S(t) = \begin{pmatrix} |a|^2 & a^*b\Gamma(t) \\ ab\Gamma^*(t) & |b|^2 \end{pmatrix},$$

(10)

where $\Gamma(t)$ is given by

$$\Gamma(t) = \frac{1}{Z_B} \sum_\sigma e^{-\beta E(\chi)} e^{-2i\alpha \sum_i \sigma_i} = \frac{Z_B (h - 2i\alpha \beta)}{Z_B(h)}.$$

(11)

and the partition function $Z_B$ can be compute using the transfer matrix formalism, as described by Ref. [19]. This function $\Gamma(t)$ is the so-called decoherence function, and, for the present case, is a periodic function with period $\tau = 2\pi/4\alpha$.

Here, we have a situation with finite time reversibility implied by oscillations presented in the density operator coherences and, the irreversible process is obtained by taking the continuous limit [20], e.g., considering each mode corresponding to a state $\chi = (\chi_1, \ldots, \chi_N)$ and defining a mode density operator $\Omega(\chi)$, i.e.,

$$\Omega(\chi) = \int d\chi_1 \ldots d\chi_N \prod_{i=1}^N D(\chi_i) \delta \left( m(\chi) - \sum_{i=1}^N \sigma_i \right),$$

(12)

where $D(\chi_i)$ the density of states. The partition function can be written as

$$Z_B = \int d\chi \Omega(\chi) e^{-\beta E(\chi)}.$$

(13)

For this case, one can speak of a decoherence rate $\gamma(t) = \log \frac{1}{\Gamma(t)}$ that describes how fast the coherence vanishes.

### A. Decoherence behavior and Lee-Yang zeros

As a result, the discrete environment gives a dephasing process when the coherence is recovered periodically that, as we will show, is a signature of non-Markovianity. Decoherence theory provides an archetypal mechanism for open quantum systems, which can be summarized simply as follows: interaction between the system and the environment causes decoherence and this, in turn, causes the loss of information from the system to the environment [2]. Here, a case of pure decoherence, when there is no energy dissipation in the system, for all practical purposes, means that we have a situation without any effect in the population of the central qubit, and the effect of bath interaction in the system recover elastic scattering.

The decoherence function recalls the Loschmidt amplitude, a fundamental object of the theory of dynamical phase transitions (DQPT) [21]. The Loschmidt amplitude quantifies an overlap between an initial state and its post quench evolution, meaning that this amplitude measure how a quantum system differs from its initial state after applied an evolution operation. One can define the Loschmidt amplitude as

$$\mathcal{G}(t) = \langle \Psi_0 | \Psi_t \rangle = \langle \Psi_0 | e^{-iHt} | \Psi_0 \rangle,$$

(14)

for some initial state $\Psi_0$ and a general driven Hamiltonian $H$. Notice that the Loschmidt amplitude vanishes for the case when the states are orthogonal. Analogous to thermal phase transitions, DQPT’s occurs when $t = t_c$ if $\mathcal{G}(t_c) = 0$ which results in nonanalyticity of $\log \mathcal{G}(t)$, a dynamical analog of the thermal free energy.

Accordingly, Loschmidt amplitude have a closed relation with the partition function, which can be seen considering the boundary partition function [22], represented in the following form

$$Z = \langle \Psi_1 | e^{-R_H} | \Psi_2 \rangle,$$

(15)

in which the states $| \Psi_1 \rangle$ and $| \Psi_2 \rangle$ encoding the boundary conditions and $H$ denoting the bulk Hamiltonian, $R$ is the distance between two borders of the system, such as a situation described by our system: a qubit with two energy levels corresponding to the energy boundaries coupled to another system with a coupling strength $\alpha$. The coupling of with respect of the qubit is described by the interacting Hamiltonian, which can be re-written as $\alpha \langle 0 | 0 \rangle - \langle 1 | 1 \rangle \otimes \sum_i \sigma_i^z$, and each level of the qubit is submitted to an amount $\alpha$ of energy with respect to the bath.

Thus, we can speak of an effective Hamiltonian that computes this behavior of the system, $H(\alpha) = 2\alpha \sum_i \sigma_i^z$, introducing the overlap effect caused by the thermal bath in the central qubit, and the bath thermal state $\hat{\rho}_B$, we can use the generalization for the mixed state Loschmidt amplitude given by Refs. [23] and [24] to obtain

$$\mathcal{G}(t) = \text{Tr} \left( \hat{\rho}_B \exp \left( -2i\alpha \sum_i \sigma_i^z \right) \right) = \frac{\text{Tr} \left[ e^{-\beta H_B} e^{-iH(\alpha) t} \right]}{Z_B}.$$

(16)

i.e., the same as decoherence function $\Gamma(t)$. In the same way, one can consider another way to write the coherence function with $\Gamma(t) = \langle e^{-iH(\alpha) t} \rangle_B$ in which $\langle \bullet \rangle_B = \text{Tr} [ \bullet \hat{\rho}_B ]$ denotes the thermal average with respect to the bath.

The critical times on the Loschmidt amplitude reveal a closed relation with the zeros of the partition function, the
so-called Lee-Yang zeros. For the case of equilibrium phase transitions, the theory of Lee-Yang tells us that the zeros of the partition function determine critical points in the fugacity plan. The decomposition of a partition function in the Nth order polynomial of \( z \equiv e^{-2\beta h} \) can be obtained by

\[
Z_B = \text{Tr}[e^{\beta B N \omega}] = e^{\beta N h} \sum_{n=0}^{N} p_n z^n,
\]

(17)

where \( p_n \) is the partition function with zero magnetic field in which \( n \leq N \) spins are in the state \(-1\) and \( N \) the number of spins.

The \( N \) zeros of partition function lying on the unit circle in the complex plane of \( z \), can be written as \( z_n = e^{i\theta_n} \) with \( n \in \mathbb{N} \). We re-write the partition function in function of its zeros

\[
Z_B = p_0 e^{\beta N h} \prod_{n=1}^{N} (z - z_n).
\]

(18)

Then, the Lee-Yang zeros in the time domain, as proposed in Ref. [17], are

\[
\Gamma(t) = e^{-2\beta N \omega t} \prod_{n=1}^{N} \frac{e^{-2\beta h + i\alpha t} - z_n}{e^{-2\beta h} - z_n},
\]

(19)

which, of course, clarifies the one-to-one correspondence between the decoherence function (or the Loschmidt amplitude) and the Lee-Yang zeros. Notice that the numerator term is obtained simply by rewriting a new (time-dependent) magnetic field \( h \rightarrow h - 2i\alpha t / \beta \). When \( h = 0 \), this function vanishes at the critical times given by Lee-Yang zeros in fugacity plan. Then, the situation provides a setup in which we can map an equilibrium system to a probe decoherence system. This correspondence, beyond a mere theoretical result, guarantees the possibility of observing Lee-Yang zeros experimentally, as can be seen in Ref. [18].

### III. NON-MARKOVIAN BEHAVIOR AND ITS DETECTION

A crucial point for understanding the mechanism of decoherence is the study of how information flows from the system to the environment [2]. However, this is only part the story, because information can also flow in the opposite direction, that is, from the environment to the system. We call this non-Markovianity. While in a Markovian process the open system continuously loses information to the environment, a non-Markovian process can be characterized as a flow of information from the environment back into the open system [3–5].

As we mentioned before, important recent contributions have been made in terms of obtaining definitions that mean quantum counterparts for non-Markovianity. Classically, Markovianity is reflected in the divisibility of conditional probabilities of a stochastic process as described by Chapmann-Kolmogorov equation [25]. Quantumly, a definition characterized in the divisibility of quantum channels cannot simply be imported. The propose of Rivas, Huelga, and Plenio (RHP) [4] can be seen as a most similar to the classical concept because consider that a quantum process \( \mathcal{E}(t, t_0) \) is Markovian if it is a CP-divisible map, i.e., a trace-preserving, completely positive (CPTP) such that, for any intermediate time, it can be divisible into two CPTP maps

\[
\mathcal{E}(t, t_0) = \mathcal{E}(t, t_1)\mathcal{E}(t_1, t_0), \quad t_0 \leq t_1 \leq t.
\]

(20)

This composition between the operators frames a family of trace-preserving and completely positive maps - a semigroup with respect to Eq.(20). Any dynamics that is Markovian according to the semigroup definition is also Markovian according to the divisibility definition, and hence according to the BLP definition [26], which will be presented in the next section.

#### A. Trace distance-based witness of non-Markovianity

Markovian processes are in practice memoryless processes, i.e., processes where information is monotonically decreasing. With this in mind, an interesting way to obtain an intuitive and consistent definition can be constructed by characterizing the Markovianity from distance measures in Hilbert space [6]. The definition of non-Markovian dynamics proposed by Breuer, Laine, and Piilo (BLP) [4] can be seen as a most similar to the classical concept because consider that a quantum process \( \mathcal{E}(t, t_0) \) is Markovian if it is a CP-divisible map, i.e., a trace-preserving, completely positive (CPTP) such that, for any intermediate time, it can be divisible into two CPTP maps

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Then, from a concise definition we have an indicator for non-Markovian dynamics in general physical systems, where the trace distance can be well defined. Such witness of non-Markovianity is widely used in the literature.

In the present problem, the situation is characterized by a qubit dephasing when put it in contact to a thermal bath. Considering two initial pure states

$$\rho_1 = |+\rangle \langle +|, \quad \rho_2 = |-\rangle \langle -|,$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. The pure decoherence while keep the population terms unchanged, subject the off-diagonal terms to a modulation given by the decoherence function for the model considered here. This case results in an immediate dependence of the trace distance with the decoherence function, which can be written as

$$D(\rho_1, \rho_2; t) = |\Gamma(t)|. \quad (25)$$

Then, we find that

$$dD(\rho_1, \rho_2; t) = \frac{\Gamma(t)}{|\Gamma(t)|} \Gamma'(t) = \text{sign}[\Gamma(t)] |\Gamma'(t)|, \quad (26)$$

where \(\text{sign}[z] := z/|z|\) with \(z \in \mathbb{C}_{\geq 0}\). As a result, a decrease in the trace distance mean a non-Markovian behavior as showed in the Fig. 2.

To understand Fig. 2 for the trace distance, we can analyze two different cases, where we have the interacting \((J = 1)\) and noninteracting \((J = 0)\) regimes. Considering two function defined by \(C(t) := \cosh(\beta h - 2i\alpha t)\) and \(S(t) := [e^{-4\beta J} + \sin^2(\beta h - 2i\alpha t)]^{1/2}\), the decoherence function for the interaction regime \((J > 0)\) takes the form

$$\Gamma(t) = \frac{(C(t) + S(t))^N + (C(t) - S(t))^N}{(C(0) + S(0))^N + (C(0) - S(0))^N}. \quad (27)$$

To search the zeros of this function, one can consider the Lee-Yang zeros, explicitly given by the formula

$$z_n = -e^{-4\beta J} + (1 - e^{-4\beta J}) \cos k_n \pm$$

$$\pm \sqrt{(e^{-4\beta J} - 1) \left[ \sin^2 k_n + e^{-4\beta J} (1 + \cos k_n)^2 \right]}, \quad (28)$$

with \(k_n = \pi(2n - 1)/N\) and \(n \in \mathbb{N}\), and hence, take a transformation in magnetic field \(h \to h - 2i\alpha t/\beta\) the zeros of the decoherence function reduces to:

$$t = \frac{\beta h}{2i\alpha} + \frac{1}{4\alpha} \ln z_n \quad (29)$$

that, to results in a real times, obviously need the conditions \(h \to 0\) or \(\beta \to 0\). Here, by rewriting the decoherence function in terms of Lee-Yang zeros we have a clear correspondence for the critical times. In the case where the magnetic field is null, the decoherence function touches the time axis at an interval \(t = \tau\) the same number of times that zeros of the partition function occur in the fugacity plane, as previously shown in Ref. [17].

For the condition \(J = 0\), only a zero rises from the fugacity plane for \(z_n = -1\). The decoherence function simply reduces to

$$\Gamma(t) = \frac{C^N(t)}{C^N(0)}. \quad (30)$$

This finally results in the following zeros for the system’s decoherence function:

$$t = \frac{\beta h + i\pi/2}{2i\alpha}, \quad (31)$$

that results in real terms only for weak fields \(h \to 0\) or high temperatures \(\beta \to 0\), as the last case.

It can be seen, therefore, that this model results in a strongly non-Markovian environment, where for low temperatures the decoherence is not affected. Here, the recurrence at each interval \(\tau\) appears as a finite size effect, and during short intervals, the system has its initial information restored. This effect is crucial for the storage of information in the environment, and we see here that the information deposited quickly returns to the system, preventing there being accessible information in the environment to be measured.

### B. Conditional past-future correlation

An also intuitive idea, proposed in Refs. [7, 8], to characterize a Markov process, consider random variables relatives
to temporal-spaced events $x,y$, and $z$, such that $(x)$ is past and $(z)$ future event with respect to a given present state $(y)$ (Fig. 3). These variables represents temporal-spaced points in which one realize consecutive measures $t_x$, $t_y$, and $t_z$. We expect some sort of correlation between temporal-spaced points of states with memory and, equivalently, memoryless states without correlation.

The idea is similar to general cases of two-points correlation functions - more specifically, temporal correlation functions, where we measure correlations between two spaced time points in relation to the same system. In probabilistic language, using Bayes rule for writing the probability $P(z,x|y)$ for the occurrence of a state $(y)$ conditioned by a past and a future state

$$P(z,x|y) = P(z|y,x)P(x|y), \quad (32)$$

which $P(\cdot | \cdot)$ is the conditional probability between two events. This property can be corroborated through a conditional past-future correlation, which is defined as

$$C_{pf} \equiv \langle O_z O_x \rangle_y - \langle O_z \rangle_y \langle O_x \rangle_y = \sum_{x,z} [P(z,x|y) - P(z|y)P(x|y)]O_zO_x,$$

where $O_{x,y}$ is a quantity or property related to each system state. When $C_{pf} \neq 0$, one says that the system break CPF (Conditional Past-Future) independence. Naturally, these probability distributions are obtained from information accessible about the system from measures. For the present case, we are accessing from measurements the information contained in a qubit, which could be reconstructed, e.g., using quantum state tomography. From these considerations, we can define Markovian dynamics using the CPF measure.

**Definition.** (Markovianity - Budini) A quantum system evolves between consecutive measurement events. Its dynamics are defined as Markovian if, for arbitrary measurement processes, it does not break CPF independence.

Follow the scheme proposed by Budini, which uses measurement operators along the $x$ axis, i.e., $\Pi_x = |\pm\rangle \langle \pm|$, with $\Pi_x$ a POVM, one can compute the conditional past-future correlation in terms of decoherence function

$$C_{pf}(t,\tau) = \sum_{x,z} [P(z,x|y) - P(z|y)P(x|y)]O_zO_x = f(t,\tau) - f(t)f(\tau),$$

in which $f(t,\tau) = [f(t+\tau) + f(t-\tau)]/2$ and $f(t) = \text{Re}(\Gamma(t))$. In the present case we have a thermal bath of structured spins, and the temperature assign new features in the system as can be seen in Figs. 4 and 5.

**IV. ENVIRONMENT POINT-OF-VIEW**

When dealing with open systems, we reduce the degrees of freedom to analyze only a portion of the system-plus-environment configuration. However, the information that at some initial moment was contained in the system will be deposited in an environment that, in general, has many more degrees of freedom. So far, we have evaluated the effective amount of the action of the environment on the states of the system, from its perspective. But instead of having all the information about the environment decoded in the decoherence function, we can look from its perspective to get the behavior of the information flow between system-bath. Then, we can summarize this session with two questions: Is there information storage in the environment? If so, is the information accessible from measurements? Such questions can be answered using the paradigms of quantum Darwinism and SBS.

First, let us consider quantum Darwinism, an idea based on a heuristic idea centered on the proliferation of information from a central system to a nearby environment [9, 12]. In a situation of pure decoherence, the interaction between the system and the environment causes the destruction of superposed states. After a certain decoherence time, one has

$$\rho^\text{dec}_S \approx \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix}, \quad (33)$$
and this would be the key to understanding the emergence of classicality in quantum systems, and the time of this process is typically extremely short for every day, macroscale process. Being another way to analyze the same phenomenon, quantum Darwinism makes use of a more realistic platform: Instead of a monolithic structure for the environment, it is divided into fractions where there is information proliferation, and information redundancy/storage and its accessibility would be responsible for the classical emergence, from the idea of objectivity (Fig. 6).

Roughly speaking, objectivity is the common agreement among observers about the state of the system, which is not necessarily true for the quantum world [15, 27]. Let us define the idea more formally:

**Definition.** (Objectivity) A system state is objective if it is (1) simultaneously accessible to many observers (2) who can all determine the state independently without perturbing it and (3) all arrive at the same result.

Therefore, emergence of objectivity in quantum systems means emergence of classicality. The conditions for this important link are given recently by the framework of quantum Darwinism but criticized in Ref. [15], based on the importance of the possibility of information extraction, i.e., measurable, distinguishable and accessible information, that is not necessarily taken into account in the quantum Darwinism paradigm.

Quantum Darwinism’s approach was used to treat a wide range of systems, like spin [28–32], and photonic [33–36] environments, harmonic oscillator [37] and Brownian [38, 39] models, and experimentally in quantum dots [40], and photonic [33, 35] setups. To study quantum Darwinism, we focus on correlations between fragments of the environment and the system. The relevant reduced density matrix $\rho_{SF}$ is given by

$$\rho_{SF} = \text{Tr}_{B/F}[\psi_{SB}\langle \psi_{SB} \rangle],$$

(34)

Above, the trace is over $B$ less $F$, or $B/F$ - all of $B$ except for the fragment $F$. Being $S(\rho_{A})$ the von Neumann entropy with respect to a system $A$, quantum Darwinism gives how much $F$ knows about $S$ can be quantified using mutual information

$$I(S : F) = S(\rho_{S}) + S(\rho_{F}) - S(\rho_{SF}),$$

(35)

defined as the difference between entropies of two systems (here $S$ and $F$) treated separately and jointly. Thus, we can define quantum Darwinism from the amount of shared information that proliferates throughout the environment.

**Definition.** (Quantum Darwinism) There exists an environment fraction size $f_{0}$ such that all fractions larger than it, $f \geq f_{0}$, it holds:

$$I(S : F) = S(\rho_{S}),$$

(36)

independently of $f$.
The most direct way to check for this condition is via so-called partial information plots (PIPs), where \( I(S : F) \) is plotted as a function of \( f \). The PIPs format depends on the intrinsic characteristics of the system-environment density operator. If we take, for example, a global pure state, we will have anti-symmetric plots around \( f = 1/2 \) [29, 30, 38, 41], and this fact can be easily seen considering the marginal mutual informations for the system, i.e., mutual informations correspondents to the operators \( \rho_{SF} \) and \( \rho_{S\tilde{F}} \), in which \( \tilde{F} \) is the pointer basis of \( S \). These structures have a close relation with the possibility of quantum Darwinism and are the keys for lead the idea of strong quantum Darwinism [15, 16].

Definition. (Spectrum Broadcast Structure) The joint state \( \rho_{SF} \) of the system \( S \) and a collection of subenvironments \( F = E_1 \otimes ... \otimes E_f \) has spectrum broadcast structure if it can be written as:

\[
\rho_{SF} = \sum_n p_n |n\rangle \otimes \rho_n^{E_1} \otimes ... \otimes \rho_n^{E_f},
\]

where \( \{ |n\rangle \} \) is the pointer basis of \( S \), \( p_i \) are the probabilities and the operators \( \rho_n^{E_k} \) are perfectly distinguishable, i.e., two by two orthogonal considering each pair of fragment environments.

The basic idea for such structures is to consider states that can be faithfully broadcasted satisfying Bohr non-disturbance definition:

Definition (Bohr non-disturbance) A measurement \( \Pi^S_k \) on the subsystem \( S' \) is Bohr non-disturbing on the subsystem \( S \) iff

\[
\sum_k \Pi^S_k \rho_{SS'} \Pi^S_k = \rho_{SS'}.
\]

Therefore, these are the states such that many observers can find out the state \( S \) independently, and without perturbing it, as assigned in the definition of objectivity. Let us assume a system-environment interaction under the decoherence action, that can be written as follows

\[
\rho_{SF} = \sum_n p_n |n\rangle \otimes \bigotimes_k \rho_n^{k} + \rho^{coh},
\]

so that, the SBS structures emerges when \( \rho^{coh} = 0 \). The partially-traced density operator for the \( J > 0 \) (see appendix B),

\[
\rho_{SF}(t) = \sum_{nn'} \sum_{i=1}^{f} \sum_{m} \rho_{SS'}^{nm} |n\rangle \langle m| \bigotimes_{i=1}^{f} |\chi_i\rangle \langle \chi_i| \times \sum_{\sigma} e^{-\beta E(\chi)} e^{-i(\epsilon_n - \epsilon_m) \sum_i \sigma_i t}
\]

that not express the structure that we need to broadcast all information to environment from the system, since \( e^{z_{2i\alpha m} \chi^N} \neq 0 \) for any time and magnetization. On the other hand, in the case of our system, \( \Gamma_f(t) = 0 \) the non-interacting situation gives us (see appendix C)

\[
\rho_{SF}(t) = \sum_n p_n^{nm} |n\rangle \langle n| \bigotimes_{i \in \mathcal{F}} \frac{e^{i\sigma_i t}}{Z_B}.
\]

Notice that the partially-traced decoherence function can be written as

\[
\Gamma_f(t) = [\cos(2\alpha t) + i \tanh(\beta h) \sin(2\alpha t)]^{(1-f)N}.
\]

Then, that condition is only satisfied for situations in which \( \beta h \to 0 \) and \( t = \pi n/2 \alpha - \pi/4 \alpha \), such that \( n \in \mathbb{Z} \). However,
For the study of quantum Darwinism, we divided the Hilbert space of the bath into fractions $\mathcal{H}_i \equiv \mathbb{C}^2$. The PIPs (Partial Information Plots) method quantifies the information between a set of $fN$ fractions and the system. The idea is to consider that the complete knowledge of the environment about the system occurs when the amount of correlations is $I(S : F) = S(\rho_S)$. Here, we present (plot (b)) PIPs for $J = 0$ for different temperatures ($\beta = 0.1, \beta = 0.5, \beta = 1, \beta = 2, \beta = 4$) and an initial system state $\rho_S = \ket{+} \bra{+}$ and, respectively, $t = \tau/2$ (red dashed lines), and $t = \tau$ (blue solid lines). For the case where $t = \tau/2$, decoherence inhibits the storage of information in the environment more intensely as the temperature increases. This situation sheds light on the influence of information flow for the emergence of quantum Darwinism. In $t = \tau$, we have total recoherence and storage is not temperature dependent [39, 42]. Note, however, the symmetry around $f = 1/2$ for the case where $\beta = 4$ (and the states is pure, see Fig. (a)) for both cases $t = \tau$ and $t = \tau/2$. For comparison, we use the solid red line to represent an emergence of quantum Darwinism. We set the coupling $\alpha = 0.1$.

the second condition for these states is not satisfied, because for each pair $(n,m)$ of states with $n \neq m$, one have $\rho_n \parallel \rho_m$. However, these are clearly not surprising situations.

As shown in [15], in addition to the notion of SBS being a formalization of the emergence of objectivity in open systems, it is also a stronger condition than quantum Darwinism for the emergence of such. In the literature, there is also a proposal to witnessing non-objectivity in situations of strong quantum Darwinism [16]. Strong quantum Darwinism is an extension of the theory of quantum Darwinism that emphasizes the structure of states and their available information [15, 16, 43].

V. CONCLUSIONS

For the present model, we use two distinct perspectives to assess the information flow between the system and the bath. From the perspective of the environment, we used two non-Markovian witnesses, which resulted in compatible descriptions, where the Markovian scenario was obtained only for situations where temperatures were very small, for different couplings between first neighbors and magnetic fields. For the trace-distance based measure, the memory effects for $h \neq 0$ vanishes at $\beta = 4$ (as we show in Fig. 2), and the same behavior can be obtained for the conditional past-future correlator measure, in which $C_{PF} \approx 0$ at the same temperature (see Figs. 4 and 5). The strong non-Markovian behavior of the bath resulted in the immediate return of information to the system, even before the quantum recoherence period. Here, it is worth noting that the decoherence function, in addition to having an immediate role in the description of non-Markovianity, is also closely linked to the Lee-Yang zeros and to the description of dynamical phase transitions in the bath probing by central qubit: the trace-based measure vanishes same times as the appearance of zeros in the fugacity plan.

In turn, the return of information prevents the proliferation of system states in the bath, as well as information accessible for measurement, which returns to the system or is destroyed by the effects of temperature, showing the fragility of quantum states subjected to thermal effects. For this reason, we did not obtain the appearance of the phenomenon of quantum Darwinism, as we show in the Fig. 7b, as $\beta$ approaches 0, the environment has more difficult to store the system information. Similarly, we did not see formation of SBS in the system; even for the noninteracting situation in total decoherence Eq. (42), the broadcasting states has not total distinguishability (a condition for the formation of SBS states). Despite this, the model studied here shows an interesting situation, in which even the degrees of freedom of the environment being reasonably large in relation to the system, there is a reconstruction of the total information that is lost to the environment during the dynamics.

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Appendix A: Exact solution

In this section of the appendix we will calculate the exact solution of the system density operator. Let us start with the initial density operator of the system \( \rho_S(0) = |\psi\rangle \langle \psi| \), with \( |\psi\rangle \) as defined in Eq. (3), and fully uncorrelated with the thermal bath described by density operator in Eq. (4), i.e., \( \rho = \rho_S(0) \otimes \rho_B \). Then the evolved density operator can be given by

\[
\rho_S(t) = \mathcal{E}(\rho_S) = \text{Tr}_B \left[ U(t) \rho_S \otimes \rho_B U^\dagger(t) \right], \tag{A1}
\]

that admits a exactly representation in the Kraus form \([44, 45]\), i.e.,

\[
\mathcal{E}(\rho_S) = \sum_{xx'} K_{xx'} \rho_S K^\dagger_{xx'}, \tag{A2}
\]
Writting the trace operation explicitly using a eigenbasis $|\chi\rangle \in \mathcal{H}_B$, one can obtain the follow
\[ = \sum_{\chi'} \langle \chi' | U(t) \rho_S \sum_{\chi} e^{-\beta E(\chi)} |\chi\rangle \langle \chi | U^\dagger(t) | \chi'\rangle \]
\[ = \sum_{\chi \chi'} e^{-\frac{\beta E(\chi)}{2}} Z_B^\frac{1}{2} \langle \chi | U(t) | \chi' \rangle \rho_S e^{-\frac{\beta E(\chi')}{2}} Z_B^\frac{1}{2} \langle \chi | U(t) | \chi' \rangle, \]
where we put in a form that one can identify the Kraus operators, that are
\[ K_{\chi \chi'} = e^{-\frac{\beta E(\chi)}{2}} \frac{1}{\sqrt{Z_B}} \langle \chi | U(t) | \chi' \rangle. \]

Now, let us write the evolution operator at the energy eigenbasis to obtain the Kraus operators, i.e.
\[ U(t) = e^{-iH_{SB}t} = \sum_{n \chi} e^{-i\epsilon_n m(\chi)} |n, \chi\rangle \langle n, \chi|, \]
in which \( m(\chi) = \sum_i \sigma_i \) is the total magnetization spin with respect to the state $\chi$ and $\epsilon_n = \alpha (-1)^n$ with $n = 0, 1$ is the energy gap obtained when one diagonalize the operator $H_{SB}$ in the basis $|n, \chi\rangle = |n\rangle \otimes |\chi\rangle$. Consequently, for the Kraus operators
\[ K_{\chi \chi'} = e^{-\frac{\beta E(\chi)}{2}} \frac{1}{\sqrt{Z_B}} \langle \chi | U(t) | \chi' \rangle \quad \text{(A5)} \]
\[ = e^{-\frac{\beta E(\chi)}{2}} \frac{1}{\sqrt{Z_B}} \langle \chi | \sum_{n \gamma} e^{-i\epsilon_n m(\gamma)} |\gamma\rangle \langle \gamma | \chi' \rangle |n\rangle \langle n| \]
\[ = e^{-\frac{\beta E(\chi)}{2}} \frac{1}{\sqrt{Z_B}} \sum_{n \gamma} e^{-i\epsilon_n m(\gamma)} |n\rangle \langle n| \delta_{\chi \gamma} \delta_{\chi' \gamma'} \quad \text{(A7)} \]
\[ = \frac{1}{\sqrt{Z_B}} \sum_{\chi \chi'} e^{-\frac{\beta E(\chi)}{2}} e^{-i\epsilon_n m(\chi')} |n\rangle \langle n| \delta_{\chi \chi'}. \]

Let us decompose the initial density matrix $\rho_S = \sum_{nm} \rho_S^{nm} |n\rangle \langle m|$, where $\rho_S^{nm} = \langle n | \rho_S | m \rangle$. Then, apllying these Kraus operators, an evolved state subject to the evolution take the particular form
\[ \rho_S(t) = \frac{1}{Z_B} \sum_{m,n,o,p,\chi,\chi'} \rho_S^{nm} e^{-\beta E(\chi)} e^{-it(\epsilon_o - \epsilon_p)m(\chi')} \times \langle o | \langle a, n \rangle \langle m | p \rangle \langle p | \delta_{\chi \chi'} \delta_{om} \delta_{mp} \rangle \]
\[ = \frac{1}{Z_B} \sum_{m,n,\sigma} e^{-\beta E(\chi)} e^{-it(\epsilon_n - \epsilon_m)m(\chi)} \rho_S^{nm} |n\rangle \langle m|. \]
\[ \text{(A8)} \]
and, one can easily check $\epsilon_n - \epsilon_m = \alpha [(-1)^n - (-1)^m]$ results in null terms for $n = m$, then the dynamics just affects off-diagonal terms
\[ \rho_S(t) = \begin{pmatrix} |a|^2 & a^*b \Gamma(t) \\ a b^* \Gamma(t) & |b|^2 \end{pmatrix} \]
\[ \text{(A9)} \]
whilst the coherences (off-diagonal) are modulated by the periodic function $\Gamma(t)$, given by

$$\Gamma(t) = \frac{1}{Z_B} \sum_\sigma e^{-\beta E(\chi)} e^{-2i\alpha \sum_\sigma \sigma_i}$$

$$= \frac{Z_B(h-2i\alpha t/\beta)}{Z_B(h)},$$

where the partition function $Z_B \equiv \text{Tr}[e^{-\beta H_B}]$ is the Ising partition function.

**Appendix B: Time evolution of the operator $\rho_{SF}$ for general case**

For the present system with non-zero coupling and magnetic field, the partially-traced density operator can be obtained expanding the density operator $\rho_B$ in the energy eigenbasis (Eq. 8), in the same way as the exact solution. But, for present work, a better path to obtain the partially-traced density operator is decompose the time evolution operator as a tensor product, i.e.,

$$U(t) = e^{-i\alpha \sigma^z \otimes \sigma^z_i} \otimes \ldots \otimes e^{-i\alpha \sigma^z \otimes \sigma^z_N} = \bigotimes_{i=1}^N e^{-i\alpha \sigma^z \otimes \sigma^z_i}.$$ Hence, the operator comes

$$\rho_{SF}(t) = \text{Tr}_{\mathcal{B}/\bar{\mathcal{F}}}[U(t)\rho_S \otimes \rho_B U^\dagger(t)]$$

$$= Z_B^{-1} \text{Tr}_{\mathcal{B}/\bar{\mathcal{F}}} \left( \bigotimes_{i=1}^N e^{-i\alpha \sigma^z \otimes \sigma^z_i} \sum_\sigma e^{-\beta E(\sigma)} |\chi\rangle \langle \chi| \left( \bigotimes_{i=1}^N e^{-i\alpha \sigma^z \otimes \sigma^z_i} \right) \right),$$

Writing each term $\langle n|\rho_{SF}|m \rangle \equiv \rho_{SF}^{nm}$, one have:

$$\rho_{SF}^{nm}(t) = \rho_{SF}^{nm} Z_B^{-1} \text{Tr}_{\mathcal{B}/\bar{\mathcal{F}}} \left( \bigotimes_{i=1}^N e^{-i(\epsilon_n - \epsilon_m) \sigma^z_i} \sum_\chi e^{-\beta E(\chi)} |\chi\rangle \langle \chi| \right)$$

$$= \rho_{SF}^{nm} Z_B^{-1} \sum_\chi e^{-\beta E(\chi)} \text{Tr}_{\mathcal{B}/\bar{\mathcal{F}}} \left( \bigotimes_{i=1}^N e^{-i(\epsilon_n - \epsilon_m) \sigma^z_i} |\chi_i\rangle \langle \chi_i| \right)$$

$$= \rho_{SF}^{nm} Z_B^{-1} \sum_\chi e^{-\beta E(\chi)} \bigotimes_{i=1}^N e^{-i(\epsilon_n - \epsilon_m) \sigma^z_i} |\chi_i\rangle \langle \chi_i| \right) \text{Tr} \left[ e^{-i(\epsilon_n - \epsilon_m) \sigma^z_i} |\chi_i\rangle \langle \chi_i| \right]$$

and, finally:

$$\rho_{SF}(t) = \frac{1}{Z_B} \left( \sum_q \sum_\chi e^{-\beta E(\chi)} \bigotimes_{i=1}^N |\chi_i\rangle \langle \chi_i| \right)$$

$$= \frac{1}{Z_B} \left( \sum_q \sum_\chi \sum_{a,b} e^{-\beta E(\chi)} e^{2i\alpha \sigma^z \otimes \sigma^z_i} \bigotimes_{i=1}^N |\chi_i\rangle \langle \chi_i| \right)$$

that not express the structure that we need to broadcast all information to environment from the system. Since $e^{\pm 2i\alpha \sigma^z \otimes \sigma^z_i} \neq 0$ for any time and magnetization.

**Appendix C: Time evolution of the operator $\hat{\rho}_{SF}$ for non-interacting situation: $J = 0$**

In this appendix, we show how derivate the partially reduced state for a more simple situation (non-interacting regime). Here, the calculation is easier by the fact of the non-interacting Hamiltonian can be describe as $H_B = \sum_i H_B^i$ with $H_B^i = -h \mathbb{1}_i \otimes \ldots \otimes \mathbb{1}_{i-1} \otimes \sigma^z_i \otimes \mathbb{1}_{i+1} \otimes \ldots \otimes \mathbb{1}_N$. Density operator can be re-written as

$$\rho_B = \frac{1}{Z_B} \bigotimes_{i=1}^N e^{\beta h \sigma^z_i},$$

in which $Z_B = 2^N \cosh^N (\beta h)$. Of course, for Gibbs states, non-interagent means uncorrelated. Then, one can compute each terms of the partial reduced matrix in the follow form:
\[ \rho_{SF}^{nm}(t) = Z_B^{-1} \rho_{SF}^n \text{Tr}_{B/F} \left[ \bigotimes_{i=1}^{N} e^{-i\alpha(e_n-e_m)\sigma_i^z t} e^{\beta \hbar \sigma_i^z} \right] \]
\[ = Z_B^{-1} \rho_{SF}^n \text{Tr}_f N \ldots \text{Tr}_N \left[ e^{-i\alpha(e_n-e_m)\sigma_i^z t} e^{\beta \hbar \sigma_i^z} \otimes \ldots \otimes e^{-i\alpha(e_n-e_m)\sigma_N^z t} e^{\beta \hbar \sigma_N^z} \right] \]
\[ = Z_B^{-1} \rho_{SF}^n \bigotimes_{i \in F} e^{-i\alpha(e_n-e_m)\sigma_i^z t} e^{\beta \hbar \sigma_i^z} \prod_{i \in \overline{B/F}} \text{Tr}_f \left[ e^{-i\alpha(e_n-e_m)\sigma_i^z t} e^{\beta \hbar \sigma_i^z} \right], \]  
\( (C2) \)

with \( \rho_{SF}^{nm}(t) = \langle n | \rho_{SF} | m \rangle \). Explicitly writing the resulting density operator:

\[ \rho_{SF}^n(t) = \frac{1}{Z_F} \left( \begin{array}{cc} |a|^2 \cosh(1-f)N (\beta h) \bigotimes_{i \in F} e^{\beta \hbar \sigma_i^z} & a^* b \cosh(1-f)N (\beta h - 2i\alpha t) \bigotimes_{i \in F} e^{(\beta h - 2i\alpha t) \sigma_i^z} \\ b^* c \cosh(1-f)N (\beta h) \bigotimes_{i \in F} e^{\beta \hbar \sigma_i^z} & |b|^2 \cosh(1-f)N (\beta h) \bigotimes_{i \in F} e^{\beta \hbar \sigma_i^z} \end{array} \right) \]  
\( (C3) \)

We can do the identifications that follow

\[ \rho_F = \bigotimes_{i \in F} e^{\beta \hbar \sigma_i^z}, \quad \rho_F'(t) = \bigotimes_{i \in F} e^{(\beta h - 2i\alpha t) \sigma_i^z} \]

and the decoherence function comes

\[ \Gamma_F(t) = \left[ \frac{\cosh(\beta h - 2i\alpha t)}{\cosh(\beta h)} \right]^{(1-f)N}. \]

Where we finally were able to explicitly write the matrix for the partially traced density operator

\[ \rho_{SF}(t) = \left( \begin{array}{cc} |a|^2 \rho_F & a^* b \rho_F'(t) \Gamma_F(t) \\ b^* c \rho_F'(t) \Gamma_F(t) & |b|^2 \rho_F \end{array} \right). \]  
\( (C4) \)

In this limit the calculations for the fraction entropy can be done easily, because the fraction density operator is a tensorial composition of \( fN \) 2 \times 2 matrices.