NONEXISTENCE OF DECREASING EQUISINGULAR APPROXIMATIONS WITH LOGARITHMIC POLES

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Abstract. In this article, we present that for any complex manifold whose dimension is bigger than one, there exists a multiplier ideal sheaf such that there don’t exist equisingular weights with logarithmic poles, which are not smaller than the orginal weight. A direct consequence is the nonexistence of decreasing equisingular approximations with logarithmic poles.

1. Introduction

Let $\varphi$ be a plurisubharmonic function (see \cite{3}) on a complex manifold $X$. Following Nadel \cite{9}, one can define the multiplier ideal sheaf $I(\varphi)$ (with weight $\varphi$) to be the sheaf of germs of holomorphic functions $f$ such that $|f|^2 e^{-2\varphi}$ is locally integrable (see also \cite{11}, \cite{12}, \cite{3}, \cite{4}, etc.).

In \cite{2} (see also \cite{3}), Demailly shows that for any given quasi-plurisubharmonic function $\varphi$ (i.e., locally can be expressed by $\psi + v$, where $\psi$ is plurisubharmonic function and $v$ is smooth) on compact Hermitian manifold $M$, there exist quasi-plurisubharmonic functions $\varphi_{S,j}$ ($j = 1, 2, \cdots$) on $M$ with smooth poles satisfying

$$I(\varphi) = I(\varphi_{S,j})$$

($j = 1, 2, \cdots$) ("equisingularity"), which are decreasing convergent to $\varphi$, when $j$ goes to $\infty$.

It is called that a quasi-plurisubharmonic function $\varphi_A$ has logarithmic poles if there exist holomorphic functions $g_k$ ($k = 1, \cdots, N$) such that

$$\varphi_A = c \log \sum_{k=1}^{N} |g_k|^2 + O(1),$$

where $c \in \mathbb{R}$ (see \cite{2}, \cite{3}). In \cite{2} (see also \cite{3}), Demailly asked

**Question 1.1.** For any given quasi-plurisubharmonic function $\varphi$ on $M$, can one choose equisingular quasi-plurisubharmonic functions $\varphi_{A,j}$ ($j = 1, 2, \cdots$) on $M$ with logarithmic poles, which are decreasing convergent to $\varphi$ ($j \to \infty$)?

In this article, we give negative answers to Question 1.1 for any dimension $n \geq 2$ by the following theorem

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Theorem 1.2. For any complex manifold $M$ (compact or noncompact) $\dim M \geq 2$ and $z_0 \in M$, there exists a quasi-plurisubharmonic function $\varphi$ on $M$ such that for any plurisubharmonic function $\varphi_A \geq \varphi$ near $z_0 \in M$ with logarithmic poles,

$$c_{z_0}(\varphi) < c_{z_0}(\varphi_A)$$

holds, where $c_{z_0}(\varphi) := \sup \{ \mathcal{I}(c\varphi)_{z_0} = \mathcal{O}_{z_0} \}$ is the complex singularity exponent of $\varphi$.

We prove Theorem 1.2 by considering the following

Remark 1.3. Let

$$\varphi_1 := \log(\max\{|z_1|, \cdots, |z_{n-1}|, |z_n|^a\}),$$

where $a \in (1, 1 + a/2)$ is a irrational number, and $(z_1, \cdots, z_n)$ are coordinates on $\mathbb{C}^n$. Let

$$\varphi_2 := \max\{\varphi_1 - 18n, 6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n\}.$$

Let

$$\varphi := -M_\eta(\varphi_2, 0),$$

where $M_\eta(t_1, t_2)$ is in Lemma (5.18) in [4], which satisfying

1. $M_\eta(t_1, t_2)$ is smooth on $\mathbb{R}^2$;
2. $M_\eta(t_1, t_2)|_{t_2 + z_0 \leq t_1} = t_1$ and $M_\eta(t_1, t_2)|_{t_1 + z_0 \leq t_2} = t_2$,

and $\eta := (\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 = \frac{1}{1000}$.

In following two remarks present that $\varphi$ in Remark 1.3 is quasi-plurisubharmonic, which can be can be extended to $M$.

Let

1. $A_1 := \{z|\log(\max_{j=1,\cdots,n} |z_j|) < 0\}$;
2. $A_2 := \{6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n < -2\varepsilon_0\}$;
3. $A_3 := \{z|\log(\max_{j=1,\cdots,n} |z_j|) < -6\log n\}$.

It is clear that $A_3 \subset A_1 \subset A_2$.

The following remark shows that $\varphi$ in Remark 1.3 is quasi-plurisubharmonic.

Remark 1.4. As

$$6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n \geq 12\log(\max_{j=1,\cdots,n} |z_j|) - 6n \geq a\log(\max_{j=1,\cdots,n} |z_j|) - 6n \geq \varphi_1$$

on $A_1^\circ$, then

$$\varphi_2(z)|_{A_1^\circ} = 6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n.$$ (1.3)

By (1) in Remark 1.3 it follows that $\varphi$ is smooth on $(A_1^\circ)^o$.

As

$$\varphi_1|_{A_2} < 6\log(|z_1|^2 + \cdots + |z_n|^2) - 18n < 6n - 2\varepsilon_0 - 18n < -2\varepsilon_0$$

and

$$(6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n)|_{A_2} < -2\varepsilon_0,$$

then it follows that $\varphi_2|_{A_2} < -2\varepsilon_0$. By using (2) in Remark 1.3 it follows that $\varphi|_{A_1^\circ} = \varphi_2$ is plurisubharmonic on $A_2$.

Note that

$$(A_1^\circ)^o \cup A_2 = \mathbb{C}^n.$$

Then $\varphi$ in Remark 1.3 is quasi-plurisubharmonic.
The following remark shows that $\varphi$ in Remark 1.3 can be extended to $M$.

**Remark 1.5.** By equality 1.3 and (2) in Remark 1.3, then it is clear that

$$\varphi|\left\{6 \log(|z_1|^2 + \cdots + |z_n|^2) - 6n > 2\varepsilon_0\right\} = -M\eta(-6 \log(|z_1|^2 + \cdots + |z_n|^2) + 6n, 0) \equiv 0.$$  \hspace{1cm} (1.4)

The following remark present the singularity of $\varphi$ in Remark 1.3

**Remark 1.6.** As

$$6 \log(|z_1|^2 + \cdots + |z_n|^2) - 6n \leq 12 \log(\max_{j=1,\cdots,n} |z_j|) + 6 \log n - 6n \leq a \log(\max_{j=1,\cdots,n} |z_j|) - 6n \leq \varphi_1$$  \hspace{1cm} (1.5)

on $A_3$, then

$$\varphi_2|_{A_3} = \varphi_1.$$  

By Remark 1.4 ($\varphi|_{A_2} = \varphi_2$) and $A_3 \subset \subset A_1$, it follows that

$$\varphi|_{A_3} = \varphi_1.$$  

Using Theorem 1.2, we answer Question 1.1 by contradiction

**Remark 1.7.** If not, then for the plurisubharmonic function $\varphi_1 = \varphi|_{A_3}$ in Remark 1.3, there exists a plurisubharmonic function $\varphi_A$ with logarithmic poles near $o$ satisfying $c_0(\varphi_1)\varphi_A \geq c_0(\varphi_1)\varphi_1$, such that $e^{-2c_0(\varphi_1)\varphi_A}$ is not integrable near $o$. By Berndtsson’s solution of the openness conjecture (11) posed by Demailly and Kollar (7), it follows that $c_0(\varphi_A) \leq c_0(\varphi_1)$, which contradicts Theorem 1.2.

2. Some Preparations

In this section, we recall some known results and present some observations.

2.1. A sharp lower bound for the log canonical threshold for dimension 2 case. In [6], Demailly and Hiep present the following

**Theorem 2.1.** (6) Let $\varphi_A \geq \varphi_1$ be a plurisubharmonic function near $o \in \mathbb{C}^2$ with logarithmic poles, then

$$c_0(\varphi_A) \geq \frac{1}{e_1(\varphi_A)} + \cdots + \frac{e_{n-1}(\varphi_A)}{e_n(\varphi_A)},$$  \hspace{1cm} (2.1)

where $e_k(\varphi_A) := \nu((dd^c\varphi_A)^k, o)$ ($e_1(\varphi_A) = \nu(\varphi_A, o)$).

As $\varphi_A \geq \varphi$, then one can obtain

$$e_n(\varphi_A) \leq e_n(\varphi) = a$$  \hspace{1cm} (2.2)

and

$$e_k(\varphi_A) \leq e_k(\varphi) = 1 \quad (k \in \{1, \cdots, n-1\})$$  \hspace{1cm} (2.3)

(by using Second comparison theorem (7.8) and Example (6.11) in chapter III of [4])
2.2. Observations. Note that \( c_o(\log \sum_{k=1}^N |g_k|^2) \) is a rational number (see [7]), and the Lelong number \( \nu(\log \sum_{k=1}^N |g_k|^2, o) \) is a integer (see [6]), where \( g_k \) are holomorphic functions near \( o \in \mathbb{C}^n \). Then it is clear that

**Lemma 2.2.** Let plurisubharmonic function \( \varphi_A := c \log \sum_{k=1}^N |g_k|^2 + O(1) \) near \( o \), where \( c \in \mathbb{R}^+ \), and \( g_k \) are holomorphic functions near \( o \). Then

\[
c_o(\varphi_A) \nu(\varphi_A, o) = c_o(\log \sum_{k=1}^N |g_k|^2) \nu(\log \sum_{k=1}^N |g_k|^2, o)
\]

is a rational number.

We prove Theorem 1.2 by using the following lemma:

**Lemma 2.3.** Let \( \varphi_A \geq \varphi_1 \) (as in Remark 1.3) be a plurisubharmonic function near \( o \in \mathbb{C}^n \) with logarithmic poles, where \( a > 1 \) is an irrational number. Assume that \( c_o(\varphi_A) = c_o(\varphi_1)(= n - 1 + \frac{1}{a}) \) (as in [7]). Then \( \nu(\varphi_A, o) < \nu(\varphi_1, o)(= 1) \).

**Proof.** As \( \varphi_A \geq \varphi_1 \), then it is clear that \( \nu(\varphi_A, o) \leq \nu(\varphi_1, o) \).

We prove Lemma 2.3 by contradiction: if not, then \( \nu(\varphi_A, o) = \nu(\varphi_1, o) (= 1) \). By Lemma 2.2 it follows that \( c_o(\varphi_A) \nu(\varphi_A, o) \) is a rational number, which contradicts \( \nu(\varphi_A, o)c_o(\varphi_A) = (n - 1 + \frac{1}{a}) = n - 1 + \frac{1}{a} \).

\[ \Box \]

3. Proof of Theorem 1.2

We prove Theorem 1.2 by contradiction: if not, then there exists a plurisubharmonic function \( \varphi_A \geq \varphi_1 \) near \( o \) with logarithmic poles such that

\[
c_o(\varphi_1) = c_o(\varphi_A)
\]

(\( \varphi_A \geq \varphi_1 \Rightarrow c_o(\varphi) \leq c_o(\varphi_A) \)).

By inequalities 2.2 and 2.3 it follows that

\[
c_o(\varphi_A) \geq \frac{1}{e_1(\varphi_A)} + \cdots + \frac{e_{n-2}(\varphi_A)}{e_{n-1}(\varphi_A)} + \frac{e_{n-1}(\varphi_A)}{e_n(\varphi_A)}
\]

\[
\geq \frac{n - 1}{e_n(\varphi_A)} + \frac{e_{n-1}(\varphi_A)}{e_n(\varphi_A)} \quad (3.2)
\]

Note that function \( f(t) := \frac{n-1}{t + \frac{1}{a}} + \frac{t}{a} \) \((t \in (0, a^{-1}])\) is strictly decreasing with respect to \( t \). If \( e_{n-1}(\varphi_A) \leq 1 \), then we have

\[
\frac{n - 1}{e_{n-1}(\varphi_A)} + \frac{e_{n-1}(\varphi_A)}{a} \geq n - 1 + \frac{1}{a} = c_o(\varphi), \quad (3.3)
\]

moreover " = " in inequality 3.3 holds if and only if \( e_{n-1}(\varphi_A) = 1 \). If \( e_{n-1}(\varphi_A) < 1 \), then it follows that \( c(\varphi_A) > n - 1 + \frac{1}{a} \) (by inequality 3.2), which contradicts equality 3.1. Then it suffices to consider the case \( e_{n-1}(\varphi_A) = 1 \).

Note that the second " = " of inequality 3.2 is " = " if and only if \( e_1(\varphi_A) = \cdots = e_{n-1}(\varphi_A) = 1 \) (by \( e_{n-1}(\varphi_A) = 1 \)). By Lemma 2.3 it follows that \( e_1(\varphi_A) < 1 \),
which implies that the second " $\geq$ " of inequality 3.2 is " $>$ ". Using inequality 3.3 we obtain that
\[
c_o(\varphi_A) > n - 1 + \frac{1}{a},
\]
which contradicts equality 3.1.

Then Theorem 1.2 has been proved.

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