The Clifford group fails gracefully to be a unitary 4-design

Huangjun Zhu\textsuperscript{1}, Richard Kueng\textsuperscript{1}, Markus Grassl\textsuperscript{2}, and David Gross\textsuperscript{1}

\textsuperscript{1}Institute for Theoretical Physics, University of Cologne, Germany
\textsuperscript{2}Max Planck Institute for the Science of Light, Erlangen, Germany

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Abstract

A unitary $t$-design is a set of unitaries that is “evenly distributed” in the sense that the average of any $t$-th order polynomial over the design equals the average over the entire unitary group. In various fields – e.g. quantum information theory – one frequently encounters constructions that rely on matrices drawn uniformly at random from the unitary group. Often, it suffices to sample these matrices from a unitary $t$-design, for sufficiently high $t$. This results in more explicit, derandomized constructions. The most prominent unitary $t$-design considered in quantum information is the multi-qubit Clifford group. It is known to be a unitary 3-design, but, unfortunately, not a 4-design. Here, we give a simple, explicit characterization of the way in which the Clifford group fails to constitute a 4-design. Our results show that for various applications in quantum information theory and in the theory of convex signal recovery, Clifford orbits perform almost as well as those of true 4-designs. Technically, it turns out that in a precise sense, the 4th tensor power of the Clifford group affords only one more invariant subspace than the 4th tensor power of the unitary group. That additional subspace is a stabilizer code – a structure extensively studied in the field of quantum error correction codes. The action of the Clifford group on this stabilizer code can be decomposed explicitly into previously known irreps of the discrete symplectic group. We give various constructions of exact complex projective 4-designs or approximate 4-designs of arbitrarily high precision from Clifford orbits. Building on results from coding theory, we give strong evidence suggesting that these orbits actually constitute complex projective 5-designs.

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1 Introduction

1.1 Motivation: Designs and derandomizations

A $d$-dimensional complex projective design is a configuration of vectors that are “evenly distributed” on the complex unit sphere in $\mathbb{C}^d$. More precisely, a set of unit-length vectors is a complex projective $t$-design, if sampling uniformly from the set gives rise to a random vector whose first $2t$ moments agree with the moments of the uniform distribution on the sphere $[25, 45, 68, 73, 3]$. This property makes designs a useful tool for the derandomization of constructions that rely on random vectors. Unitary designs are an analog of complex projective designs on the unitary group, which are equally useful for the derandomization of constructions that rely on random unitaries $[22, 23, 96, 69]$. In addition, unitary designs provide a simple way for constructing projective designs.

Applications of designs abound, with examples including randomized benchmarking $[83, 59, 81]$, quantum state tomography $[13, 73, 92, 87]$, quantum process tomography $[74, 49]$, quantum cryptography $[2]$, data hiding $[26]$, decoupling $[11, 78]$, and tensor networks $[63]$.

1.2 Outline of result: Overcoming the “$t = 3$-barrier”

One major drawback of the program of using complex projective and unitary designs for derandomization is that there has been little progress in constructing explicit families of $t$-designs for $t > 3$. There are various constructions using “structured randomness” – most notably the random circuit model that yields approximate designs in any dimension and of any degree $[42, 11]$. While the resulting designs are sufficiently well-structured for some tasks in quantum information theory, they are arguably not as explicit as one could hope for.

The only explicit infinite family of unitary 3-designs known so far is the complex Clifford group, while the only explicit infinite family of projective 3-designs are the orbits of the Clifford group $[54, 88, 83]$ (and...
even these are very recent results). Unfortunately, it has also been shown that the Clifford group is not a unitary 4-design, and their orbits are not, in general, projective 4-designs \[54, 88, 83\].

This situation seems all the more unsatisfactory, as there are various applications – including the two examples given in Section 1.3 below – where 2-designs are essentially useless \[60, 38\], 3-designs give first non-trivial improvements \[38\], and 4-designs show already optimal behavior \[48, 3, 60\]. The case \(t = 4\) treated here is thus not another step in an infinite series of potential papers, but rather seems to constitute a fundamental threshold. Other prominent applications of 4-designs include randomized benchmarking \[81\] and quantum process tomography \[49\].

The main result of the present work is that while Clifford orbits fall short of constituting 4-designs, their 8th moments can be calculated explicitly. The results are sufficiently well-behaved that for several applications, Clifford orbits turn out to perform nearly as well as 4-designs or uniform random vectors would. Moreover, even exact 4-designs can easily be constructed from Clifford orbits. In order to establish these statements, we give an explicit description of the irreducible representations of the 4th tensor power of the Clifford group. In a precise sense, it turns out that this tensor power affords only one more invariant subspace than the 4th tensor power of the unitary group. That additional subspace is a stabilizer code – a structure extensively studied in the field of quantum error correction codes \[32, 64\]. This feature allows for an explicit analysis.

This paper contains only the representation-theoretic analysis of the 4th tensor power of the Clifford group. In two companion papers we apply this technical result to problems from signal analysis \[56\] and quantum information theory \[55\] respectively. These applications are briefly sketched below. The reason for splitting our discussion three-ways is that we target two different scientific communities that have come to employ very different languages.

### 1.3 Two applications

Here, we give a high-level description of two seemingly very different problems that originally motivated our work and to which our results can be applied.

#### 1.3.1 Application: Phase Retrieval

The signal analysis example is the problem of phase retrieval: Let \(x\) be an unknown vector in \(\mathbb{C}^d\). Assume we have access to a set of “phase insensitive linear measurements”

\[
y_i = |(a_i, x)|, \quad i = 1, \ldots, m.\tag{1}
\]

Here, the \(a_i \in \mathbb{C}^d\) are a given set of measurement vectors. The task now is to recover \(x\) given \(y_1, \ldots, y_m\).

There are many practical applications – for example in optical microscopy, where information about a sample is encoded in the electro-magnetic light field, but where only phase-insensitive intensity measurements are usually feasible. From a mathematical point of view, the absolute value in Eq. (1) means that we are facing a non-linear inverse problem – which are often difficult to solve in theory and in practice.

A recent research program has investigated the use of algorithms based on convex optimization for the purpose of solving the phase retrieval problem. First theoretical results have shown that certain convex algorithms do indeed recover \(x\) with high probability, if the measurements \(a_i\) are random Gaussian vectors or drawn uniformly from the unit sphere in \(\mathbb{C}^d\) \[18, 19\]. However, in many practical applications, such measurements cannot be realized. Therefore, we are facing the task of re-proving those guarantees for measurements that are ideally deterministic, or, if randomized, at least drawn from a “smaller” and “more structured” set of vectors than from the entire unit sphere. Such derandomized versions of convex-optimization algorithms have indeed been established for a variety of models – see e.g. Refs. \[17, 38\].

Starting from Ref. \[38\], some of the present authors have been interested in using spherical designs as a “general-purpose” tool for randomizing phase retrieval algorithms. The basic insight is that protocols that ostensibly require Gaussian vectors often only rely on certain measure-concentration estimates that can be derived already from information about finite moments. Initial results \[38\] showed that while a 2-design property alone does not give rise to non-trivial recovery guarantees, this changes from \(t = 3\) onward. Later, it has been proven that 4-designs essentially match the performance of random Gaussian measurements.
In accordance with our initial hope, the results of [48] were first proven for Gaussian measurements and then generalized – with comparatively few changes in the argument – to the design case.

In Ref. [56], we use the theory of the present paper to establish near-optimal performance guarantees for phase retrieval from measurements drawn randomly from Clifford orbits. This includes the case of stabilizer measurements. Generalizations to the recovery of low-rank matrices are also proven.

1.3.2 Application: Quantum state distinguishability

Our second example comes from quantum information theory. In quantum mechanics, the state of a $d$-level system is encoded in a positive semi-definite $d \times d$-matrix, the so-called density operator or density matrix. A measurement maps density operators to classical probability distributions over a space of outcomes. The fundamental property of quantum complementarity means that measurements necessarily entail a loss of information about the quantum system.

One way of precisely measuring this information loss is as follows: The (single-shot) statistical distinguishability of two classical probability distributions $p, q$ is measured by the total variational distance, or half their $\ell_1$-norm distance $d_c(p, q) := \frac{1}{2} \| p - q \|_{\ell_1}$. Analogously, the optimal probability of distinguishing between two quantum states $\rho, \sigma$ is given by one half the Schatten-1 norm (or trace norm or nuclear norm) distance: $d_q(\rho, \sigma) := \frac{1}{2} \| \rho - \sigma \|_1$. Quantum measurements are represented by positive-operator-valued measures (POVMs), which realize (certain) linear maps $\Lambda$ from the set of density matrices to the set of classical probability distributions. The fact that “information is lost” in such a process can e.g. be made precise by stating that $\Lambda$ is a contraction:

$$d_c(\Lambda(\rho), \Lambda(\sigma)) \leq d_q(\rho, \sigma).$$

The information loss of a given $\Lambda$ can be upper-bounded via the POVM norm constant $C_\Lambda < 1$, which is the largest constant so that

$$d_c(\Lambda(\rho), \Lambda(\sigma)) \geq C_\Lambda d_q(\rho, \sigma)$$

holds for any pair $\rho, \sigma$. It thus makes sense to ask for an optimal measurement $\Lambda$, i.e. one that maximizes $C_\Lambda$.

It has been shown that the uniform POVM achieves this goal [60]. This measurement maps quantum states to probability distributions on the complex unit sphere, where the density $p(\psi)$ at the vector $\psi$ is proportional to $\text{tr}(\rho |\psi\rangle\langle\psi|)$. The situation is now very similar to the one considered in the phase retrieval example above: The uniform POVM is optimal, but impractical to implement in large quantum experiments. However, as has been shown in Ref. [3, 60], restricting the uniform POVM to a set of vectors that form a 4-design gives rise to a quantum measurement which already matches the optimal scaling behavior. Again, an analogous statement for 2-designs does not hold [60].

Building on the theory developed below, we analyze quantum state distinguishability as measured by POVM norms of Clifford orbits in Ref. [55]. For states with high purity, near-optimal results are established for all Clifford orbits, including stabilizer measurements. As an auxiliary result, we also establish tighter entropic uncertainty relations [84, 21] for stabilizer measurements.

1.4 Note added

While finalizing this paper, we became aware of a work by Helsen, Wallman, and Wehner that analyses a closely related representation of the Clifford group with the aim to derive improved bounds for randomized benchmarking [44]. More precisely, they work with the representation $\tau^{(2,2)}$ in the sense of Eq. (39). As described below, this means that the main results of our respective papers are largely equivalent. The proof methods seem to be rather different.

2 Mathematical Background

In this section we review the mathematical background on complex projective designs, unitary designs, the Pauli group, Clifford group, and stabilizer states.
2.1 Projective t-designs

Complex projective designs are configurations of vectors that are evenly distributed on the complex unit sphere in $\mathbb{C}^d$. They are an analog of spherical designs on the real unit sphere \cite{25,45,68,10}; c.f. Appendix G. Such designs are interesting to a number research areas, such as approximation theory, combinatorics, experimental designs etc. Recently, they have also found increasing applications in many quantum information processing and signal analysis tasks, such as quantum state tomography \cite{43,73,92,87}, quantum state discrimination \cite{3,60}, and phase retrieval \cite{38}. Here we review three equivalent definitions of (complex projective) $t$-designs.

Let $\text{Hom}_{(t,t)}(\mathbb{C}^d)$ be the space of polynomials homogeneous of degree $t$ in the coordinates of $\psi \in \mathbb{C}^d$ with respect to a given basis and homogeneous of degree $t$ in the coordinates of the complex conjugate $\psi^*$ (refer to Appendix G for detailed notes on multivariate polynomials).

**Definition 1.** A set of $K$ normalized vectors $\{\psi_j\}$ in dimension $d$ is a (complex projective) $t$-design if

$$\frac{1}{K} \sum_j p(\psi_j) = \int p(\psi) d\psi \quad \forall p \in \text{Hom}_{(t,t)}(\mathbb{C}^d),$$

where the integral is taken with respect to the normalized uniform measure on the complex unit sphere in $\mathbb{C}^d$.

To derive simpler criteria on $t$-designs, we need to introduce several additional concepts. Let $\text{Sym}_t(\mathbb{C}^d)$ be the $t$-partite symmetric subspace of $(\mathbb{C}^d)^{\otimes t}$ with corresponding projector $P_{[t]}$. The dimension of $\text{Sym}_t(\mathbb{C}^d)$ reads

$$D_{[t]} = \binom{d + t - 1}{t}.$$  \hspace{1cm} (3)

The $i$th frame potential of $\{\psi_j\}$ is defined by

$$\Phi_i(\{\psi_j\}) := \frac{1}{K^2} \sum_{j,k} |\langle \psi_j | \psi_k \rangle|^{2t}. $$  \hspace{1cm} (4)

**Proposition 1.** The following statements are equivalent:

1. $\{\psi_j\}$ is a $t$-design.
2. $\frac{1}{K} \sum_j (|\langle \psi_j | \psi_j \rangle|^{\otimes t} = P_{[t]} / D_{[t]}$, where $K = |\{\psi_j\}|$.
3. $\Phi_i(\{\psi_j\}) = 1 / D_{[t]}$.

**Remark 1.** In general, $\Phi_i(\{\psi_j\}) \geq 1 / D_{[t]}$, and the lower bound is saturated iff $\{\psi_j\}$ is a $t$-design.

**Proof.** Let $L(\text{Sym}_t(\mathbb{C}^d))$ be the space of linear operators acting on $\text{Sym}_t(\mathbb{C}^d)$. There is a one-to-one correspondence (Lemma 4) between polynomials $p \in \text{Hom}_{(t,t)}(\mathbb{C}^d)$ and operators $A \in L(\text{Sym}_t(\mathbb{C}^d))$,

$$A \mapsto p_A, \quad p_A(\psi) := \text{tr}[A|\psi\rangle \langle \psi|^{\otimes t}].$$  \hspace{1cm} (5)

Therefore,

$$\frac{1}{K} \sum_j p_A(\psi_j) = \frac{1}{K} \text{tr} \left[ A \sum_j (|\langle \psi_j | \psi_j \rangle|^{\otimes t} \right], \quad \int p_A(\psi) d\psi = \text{tr} \left[ A \int (|\langle \psi | \psi \rangle|^{\otimes t} d\psi \right].$$  \hspace{1cm} (6)

It follows that $\{\psi_j\}$ is a $t$-design iff

$$\frac{1}{K} \sum_j (|\langle \psi_j | \psi_j \rangle|^{\otimes t} = \int (|\langle \psi | \psi \rangle|^{\otimes t} d\psi = \frac{P_{[t]}}{D_{[t]}}.$$  \hspace{1cm} (7)
Here the second equality follows from the fact that the $t$th symmetric subspace is irreducible under the action of the unitary group. This observation confirms the equivalence of statements 1 and 2. The equivalence of statements 2 and 3 is a consequence of the following equation,

$$\left\| \frac{1}{K} \sum_j (|\psi_j\rangle\langle\psi_j|)^{\otimes t} - \frac{P_{[t]}}{D_{[t]}} \right\|_2^2 = \Phi_t(\{\psi_j\}) - \frac{1}{D_{[t]}}, \quad (8)$$

where $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm or the Frobenius norm. This equation implies that $\Phi_t(\{\psi_j\}) \geq 1/D_{[t]}$, and the lower bound is saturated iff Eq. (7) is satisfied. \hfill \Box

Proposition 1 suggests several useful measures for characterizing the deviation of $\{\psi_j\}$ from $t$-designs. For example, two common measures are the operator norm and trace norm of the deviation operator

$$\frac{D_{[t]}}{K} \sum_j (|\psi_j\rangle\langle\psi_j|)^{\otimes t} - P_{[t]}.$$  

Another measure is the ratio of the frame potential over the minimum frame potential, that is, $D_{[t]} \Phi_t(\{\psi_j\})$.

Any $t$-design in dimension $d$ has at least

$$\left( d + \lceil t/2 \rceil - 1 \right) \left( d + \lfloor t/2 \rfloor - 1 \right) \quad \lfloor t/2 \rfloor$$

elements, where $\lceil t/2 \rceil$ denotes the smallest integer not smaller than $t/2$, and $\lfloor t/2 \rfloor$ the largest integer not larger than $t/2$ [45, 68, 73]. The bound is equal to $d, d^2, d^2(d+1)/2, d^2(d+1)^2/4$ for $t = 1, 2, 3, 4$, respectively. A $t$-design is tight if the lower bound is saturated. A 1-design is tight iff it defines an orthonormal basis; a 2-design is tight iff it defines a symmetric informationally complete measurement (SIC) [80, 68, 73, 75, 78]. Another interesting example of 2-designs are complete sets of mutually unbiased bases (MUB) [47, 85, 51, 29]. The only known explicit infinite family of 3-designs are the orbits of the (multiqubit) Clifford group, among which the set of stabilizer states is particularly prominent [64, 88, 83].

**Definition 2.** A set of weighted normalized vectors $\{\psi_j, w_j\}$ in dimension $d$ with $w_j \geq 0$ and $\sum_j w_j = 1$ is a weighted (complex projective) $t$-design if

$$\sum_j w_j p(\psi_j) = \int p(\psi) \, d\psi \quad \forall p \in \text{Hom}_{(t,t)}(C^d). \quad (11)$$

A weighted $t$-design reduces to an ordinary unweighted $t$-design when all weights are equal. In many contexts, weighted designs are equally useful as unweighted designs. In the current paper, we construct unweighted 4-designs for dimensions that are a power of two. They can easily be turned into weighted 4-designs for arbitrary dimensions $d$. Indeed, let $d \leq d$, let $P$ be a projection operator onto an arbitrary $d$-dimensional subspace of $C^d$, and let $\{\tilde{\psi}_j\}$ be a $t$-design. Then one can verify immediately that with

$$\tilde{\psi}_j = \frac{1}{\|P\tilde{\psi}_j\|_2} \tilde{\psi}_j, \quad \tilde{w}_j = \|P\psi_j\|_2,$$

the $\{\tilde{\psi}_j, \tilde{w}_j\}$ forms a weighted $t$-design. This way, the findings of the present paper have consequences for any dimension — a power of two or not.

Almost all conclusions about $t$-designs mentioned above, including Proposition 1, also apply to weighted $t$-designs provided that the operator $\frac{1}{t} \sum_j (|\psi_j\rangle\langle\psi_j|)^{\otimes t}$ is replaced by $\sum_j w_j (|\psi_j\rangle\langle\psi_j|)^{\otimes t}$, and the frame potential $\Phi_t(\{\psi_j\})$ is replaced by

$$\Phi_t(\{\psi_j, w_j\}) := \sum_{j,k} w_jw_k |\langle \psi_j | \psi_k \rangle |^{2t}. \quad (12)$$
2.2 Unitary $t$-designs

Unitary designs are configurations of unitary operators that are “evenly distributed” on the unitary group, in analogy to spherical designs and complex projective designs. They are particularly useful in derandomizing constructions that rely on random unitaries, such as randomized benchmarking [33, 50, 51], quantum process tomography [49], quantum cryptography [2], data hiding [26], and decoupling [1, 78].

Let $\text{Hom}_{t}(U(d))$ be the space of polynomials homogeneous of degree $t$ in the matrix elements of $U \in U(d)$ and homogeneous of degree $t$ in the matrix elements of $U^\dagger$ (the complex conjugate of $U$; the Hermitian conjugate of $U$ is denoted by $U^\dagger$).

**Definition 3.** A set of $K$ unitary operators $\{U_j\}$ in dimension $d$ is a unitary $t$-design if

$$\frac{1}{K} \sum_j p(U_j) = \int dU p(U) \quad \forall p \in \text{Hom}_{t}(U(d)), \quad (13)$$

where the integral is taken over the normalized Haar measure.

The above equation remains intact even if $U_j$ are multiplied by arbitrary phase factors, so what we are concerned are actually projective unitary $t$-designs. The $t$th frame potential of $\{U_j\}$ is defined as

$$\Phi_t(\{U_j\}) := \frac{1}{K^2} \sum_{j,k} |\text{tr}(U_j U_k^\dagger)|^{2t}. \quad (14)$$

As shown in the proof of Proposition 2 below,

$$\Phi_t(\{U_j\}) \geq \gamma(t, d) := \int dU |\text{tr}(U)|^{2t}, \quad (15)$$

and the lower bound is saturated iff $\{U_j\}$ is a unitary $t$-design [36, 74, 89]. The value of $\gamma(t, d)$ has been computed explicitly: it is equal to the number of permutations of $\{1, 2, \ldots, t\}$ with no increasing subsequence of length larger than $d$ [65, 67]. Here we only need the formula in two special cases [74],

$$\gamma(t, d) = \begin{cases} \frac{(2d)!}{(2t)!} & d = 2, \\ \frac{t!}{d!} & d \geq t. \end{cases} \quad (16)$$

Like projective $t$-designs, there are many equivalent definitions of unitary $t$-designs.

**Proposition 2.** The following statements are equivalent:

1. $\{U_j\}$ is a unitary $t$-design.
2. $\frac{1}{K} \sum_j \text{tr}[BU_j \otimes A(U_j^\dagger)] = \int dU \text{tr}[BU^\dagger \otimes A(U^\dagger)]$ for all $A, B \in L((\mathbb{C}^d)^\otimes t)$.
3. $\frac{1}{K} \sum_j U_j \otimes A(U_j^\dagger) = \int dU U^\dagger \otimes A(U^\dagger)$. for all $A \in L((\mathbb{C}^d)^\otimes t)$.
4. $\frac{1}{K} \sum_j U_j \otimes (U_j^\dagger)^\dagger = \int dU U^\dagger \otimes (U^\dagger)^\dagger$.
5. $\frac{1}{K} \sum_j U_j \otimes (U_j^\dagger)^\dagger = \int dU U^\dagger \otimes (U^\dagger)^\dagger$.
6. $\Phi_t(\{U_j\}) = \gamma(t, d)$.

**Proof.** Note that $\text{tr}[BU^\dagger \otimes A(U^\dagger)]$ is a homogeneous polynomial in $\text{Hom}_{t}(U(d))$ and that all polynomials of this form for $A, B \in L((\mathbb{C}^d)^\otimes t)$ span $\text{Hom}_{t}(U(d))$. Therefore, statements 1 and 2 are equivalent. The equivalence of statements 2 and 3 is obvious.

The equivalence of statements 1 and 4 follows from the following equation,

$$\text{tr}\{V(B \otimes A)[U^\otimes (U^\dagger)]\} = \text{tr}\{BU^\otimes A(U^\dagger)\}, \quad (17)$$
where \( V \) is the swap operator of parties \( 1, 2, \ldots, t \) with the parties \( t + 1, t + 2, \ldots, 2t \). The equation in statement 5 is a partial transposition of the one in statement 4.

The equivalence of statements 5 and 6 follows from the following equation

\[
\left\| \frac{1}{K} \sum_j U_j^{\otimes t} \otimes (U_j^{\otimes t})^* - \int dU U^{\otimes t} \otimes (U^{\otimes t})^* \right\|_2 = \Phi_t(\{U_j\}) = \gamma(t, d). \tag{18}
\]

Most known examples of unitary designs are constructed from subgroups of the unitary group, which are referred to as (unitary) group designs henceforth. Given a finite group \( G \) of unitary operators, the frame potential of \( G \) takes on the form

\[
\Phi_t(G) = \frac{1}{|G|} \sum_{U \in G} |\text{tr}(U)|^{2t}. \tag{19}
\]

Let \( \overline{G} \) be the quotient of \( G \) over the phase factors. Then

\[
\Phi_t(G) = \Phi_t(\overline{G}) = \frac{1}{|\overline{G}|} \sum_{U \in \overline{G}} |\text{tr}(U)|^{2t}. \tag{20}
\]

This formula is applicable whenever \( \overline{G} \) is a finite group even if \( G \) is not. Note that \( \Phi_t(G) \) is equal to the sum of squared multiplicities of irreducible components of

\[
\tau^t(G) := \{U^{\otimes t} | U \in G\}, \tag{21}
\]

which coincides with the dimension of the commutant of \( \tau^t(G) \) \[35\]. Recall that the commutant \( A' \) of a set of operators \( A \) is the algebra of all operators that commute with every element of \( A \),

\[
A' = \{B | [A, B] = 0 \ \forall A \in A\}. \tag{22}
\]

Let \( H \) be a subgroup in \( G \). It is clear that every irreducible representation of \( \tau^t(G) \) on \((\mathbb{C}^d)^{\otimes t}\) is also invariant under \( \tau^t(H) \) and thus forms a representation space of \( H \). However, these spaces need not be irreducible under the action of \( H \). As a consequence, \( \Phi_t(H) \geq \Phi_t(G) \) for any subgroup \( H \) in \( G \), and the equality is saturated if every irreducible component of \( \tau^t(G) \) is also irreducible when restricted to \( \tau^t(H) \); that is, \( \tau^t(G) \) and \( \tau^t(H) \) decompose into the same number of irreducible components.

At this point, it is instructive to review the representation theory of the unitary group \( U(d) \) on the space of all tensors \((\mathbb{C}^d)^{\otimes t}\) from the point of view of Schur-Weyl duality \[31\] \[60\]. By definition the unitary group \( U(d) \) acts on \( \mathbb{C}^d \). The action extends to the diagonal action on \((\mathbb{C}^d)^{\otimes t}\).

\[
U \mapsto \tau^t(U) : |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_t\rangle \mapsto U|\psi_1\rangle \otimes U|\psi_2\rangle \otimes \cdots \otimes U|\psi_t\rangle \ \forall |\psi_j\rangle \in \mathbb{C}^d, \forall U \in U(d). \tag{23}
\]

Meanwhile, the symmetric group \( S_t \) acts on the tensor product space \((\mathbb{C}^d)^{\otimes t}\) by permuting the tensor factors:

\[
\pi(|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_t\rangle) = |\psi_{\pi_1}\rangle \otimes |\psi_{\pi_2}\rangle \otimes \cdots \otimes |\psi_{\pi_t}\rangle \ \forall |\psi_j\rangle \in \mathbb{C}^d, \forall \pi \in S_t. \tag{24}
\]

The diagonal action of \( U(d) \) and the permutation action of \( S_t \) on \((\mathbb{C}^d)^{\otimes t}\) commute with each other. Schur-Weyl duality states that \((\mathbb{C}^d)^{\otimes t}\) decomposes into multiplicity-free irreducible representations of \( U(d) \times S_t \) \[31\]. More precisely,

\[
(\mathbb{C}^d)^{\otimes t} = \bigoplus_{\lambda} H_{\lambda} = \bigoplus_{\lambda} W_{\lambda} \otimes S_{\lambda}. \tag{25}
\]

Here the \( \lambda \)'s are non-increasing partitions of \( t \) into no more than \( d \) parts, \( W_{\lambda} \) is the Weyl module carrying the irrep of \( U(d) \) associated with \( \lambda \), and \( S_{\lambda} \) the Specht module on which \( S_t \) acts irreducibly. We denote the dimensions of \( S_{\lambda} \) and \( W_{\lambda} \) by \( d_{\lambda} \) and \( D_{\lambda} \), respectively. Note that \( d_{\lambda} \) equals the multiplicity of the Weyl module \( W_{\lambda} \), and, likewise, \( D_{\lambda} \) is the multiplicity of the Specht module \( S_{\lambda} \). As an implication, the
The commutant of the diagonal action of the unitary group is generated by all permutations of the tensor factors. If \( \lambda = [t] \) is the trivial partition, then \( W_\lambda = \text{Sym}_t(\mathbb{C}^d) \) and \( S_t \) acts trivially on \( S_\lambda \simeq \mathbb{C} \). In particular, it follows that the space \( \text{Sym}_t(\mathbb{C}^d) \) carries an irreducible representation of \( U(d) \).

The discussion above leads to a number of equivalent characterizations of \( t \)-designs constructed from groups.

**Proposition 3.** The following statements concerning \( G \subseteq U(d) \) are equivalent:

1. \( G \) is a unitary \( t \)-design.
2. \( \Phi_t(G) = \gamma(t,d) \).
3. \( \tau^t(G) \) decomposes into the same number of irreps as \( \tau^t(U(d)) \).
4. Every irreducible component in \( \tau^t(U(d)) \) is still irreducible when restricted to \( \tau^t(G) \).
5. \( \tau^t(G) \) and \( \tau^t(U(d)) \) has the same commutant.
6. The commutant of \( \tau^t(G) \) is generated by all the permutations of the tensor factors.

For example, \( G \) is a 1-design iff it is irreducible; in that case, \( G \) has at least \( d^2 \) elements, and the lower bound is saturated iff it defines a nice error basis, that is, \( \text{tr}(U_j U_k^*) = d \delta_{jk} \) for \( U_j, U_k \in G \).

The group \( G \) is a unitary 2-design iff \( \tau^2(G) \) has only two irreducible components, which correspond to the symmetric and antisymmetric subspaces of the bipartite Hilbert space. Prominent examples of unitary group 2-designs include Clifford groups and restricted Clifford groups in prime power dimensions 16, 19, 22, 29, 36.

Complex projective designs and unitary designs are connected by the following proposition.

**Proposition 4.** Any orbit of normalized vectors of a unitary group \( t \)-design forms a complex projective \( t \)-design.

**Proof.** Let \( G \) be a unitary group \( t \)-design, then \( \tau^t(G) \) acts irreducibly on \( \text{Sym}_t(\mathbb{C}^d) \). Therefore,

\[
\frac{1}{|G|} \sum_{U \in G} (U|\psi\rangle \langle \psi|U^\dagger)^{\otimes t} = \frac{1}{|G|} \sum_{U \in G} U^{\otimes t}(|\psi\rangle \langle \psi|)^{\otimes t} (U^{\otimes t})^\dagger = \frac{P_1}{D_2},
\]

for any normalized vector \( \psi \). It follows that any orbit of pure states of \( G \) forms a complex projective \( t \)-design. \( \square \)

### 2.3 Pauli group, Clifford group, and stabilizer codes

The Pauli group and Clifford group play a crucial role in quantum computation 32, 33, 64, 12, quantum error correction 32, 64, randomized benchmarking 33, 59, 81, and quantum state tomography with compressed sensing 39, 35, 49. They are also closely related to many interesting discrete structures, such as discrete Wigner functions 34, 91, 37, mutually unbiased bases 29. Many nice properties of the Clifford group are closely related to the fact that the group forms a unitary 2-design 26, 19, 22, 33, 36, 74, 98, 42, 20. Recently, it was shown that the multiquit Clifford group is actually a unitary 3-design, but not a 4-design 88, 83, 54. In the rest of this paper we assume that the dimension is a power of 2 when referring to the Pauli group or the Clifford group.

Let \( \mathbb{F}_2 = \mathbb{Z}_2 = \{0,1\} \) be the finite field of integers with arithmetic modulo 2. We label the **Pauli matrices** on a single qubit by elements of \( \mathbb{F}_2^2 \) in the following way:

\[
\sigma_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{(1,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_{(1,1)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

A **Pauli operator** on \( n \) qubits is defined as the tensor product of \( n \) Pauli matrices. Concretely, each \( a \in \mathbb{F}_2^{2n} \) defines a Pauli operator as follows,

\[
W_a := \sigma_{(a_1,a_2)} \otimes \cdots \otimes \sigma_{(a_{2^{n-1}},a_{2n})}.
\]
Every pair of Pauli operators either commute or anticommute,

\[ W_a W_b = (-1)^{\langle a, b \rangle} W_b W_a, \tag{27} \]

where \( \langle a, b \rangle = a^T J b \) is the symplectic form with \( J \) being the \( 2n \times 2n \) block-diagonal matrix over \( \mathbb{F}_2 \) with \( n \) blocks of \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) on the diagonal. Let

\[ \mathcal{P}_n = \{ W_a | a \in \mathbb{F}_2^{2n} \} \]

be the set of all \( n \)-qubit Pauli operators. The Pauli group on \( n \)-qubits is the group generated by all the Pauli operators in \( \mathcal{P}_n \),

\[ \mathcal{P}_n = \langle \mathcal{P}_n \rangle = \{ i^j W_a | a \in \mathbb{F}_2^{2n}, j \in \mathbb{Z}_4 \}. \]

In the following discussion \( \mathcal{P}_n \) is also identified as the projective Pauli group, the quotient group of \( \mathcal{P}_n \) with respect to the phase factors. As a group, \( \mathcal{P}_n \) is isomorphic to \( \mathbb{F}_2^{2n} \).

The \( n \)-qubit Clifford group is usually defined as the normalizer of the \( n \)-qubit Pauli group \( \mathcal{P}_n \). For the convenience of the following discussion, we shall define the Clifford group by specifying explicit generators. The single qubit Clifford group \( C_1 \) is generated by the Hadamard matrix \( H \) and the phase matrix \( S \), where

\[ H = \frac{1 + i}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \tag{28} \]

Here our definition of the Hadamard matrix differs from the usual definition by a phase factor of \( e^{\pi i/4} \). This convention has a crucial advantage in studying the representation of the Clifford group and symplectic group, as we shall see in Sec. 3.3. In general, the Clifford group \( C_n \) is generated by Hadamard matrices and phase matrices for respective qubits, as well as CNOT gates between all pairs of qubits, where

\[ \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{29} \]

It can be proved that the Clifford group \( C_n \) generated by these matrices is the normalizer of the Pauli group in \( U(d, \mathbb{Q}[i]) \) \[32 \, 61\], where \( \mathbb{Q}[i] \) is the extension of the rational field \( \mathbb{Q} \) by the imaginary unit \( i \) (thanks to our definition of the Hadamard matrix, we do not need the eighth roots of unity), and \( U(d, \mathbb{Q}[i]) \) is the group of unitary operators in dimension \( d \) with entries in \( \mathbb{Q}[i] \). In addition, the normalizer of \( \mathcal{P}_n \) in \( U(d) \) is generated by \( C_n \) and phase factors. The center of the Clifford group \( C_n \) is the order-4 cyclic group generated by the scalar matrix \( i \).

Let \( \text{Sp}(2n, \mathbb{F}_2) \) be the symplectic group composed of all \( 2n \times 2n \) matrices \( F \) over \( \mathbb{F}_2 \) that satisfy the following equation

\[ FF^T = J. \tag{30} \]

For every Clifford unitary \( U \in C_n \), there is a unique symplectic matrix \( F \in \text{Sp}(2n, \mathbb{F}_2) \) such that

\[ U W_a U^\dagger = (-1)^{f(a)} W_{F a} \quad \forall a \in \mathbb{F}_2^{2n}, \tag{31} \]

where \( f \) is a function from \( \mathbb{F}_2^{2n} \) to \( \mathbb{F}_2 \). Conversely, for each symplectic matrix \( F \in \text{Sp}(2n, \mathbb{F}_2) \) there exists a Clifford unitary \( U \in C_n \) and a suitable function \( f \) such that the above equation is satisfied. Note that the \( 4d^2 \) Clifford unitaries \( i^j U W_a \) for \( j = 0, 1, 2, 3 \) and \( a \in \mathbb{F}_2^{2n} \) induce the same symplectic transformation. Denote by \( \overline{C}_n \) the projective Clifford group. Then both \( C_n/\mathcal{P}_n \) and \( \overline{C}_n/\mathcal{P}_n \) are isomorphic to \( \text{Sp}(2n, \mathbb{F}_2) \).

The Clifford group \( C_n \) is a unitary 3-design, but not a 4-design \[88 \, 83 \, 54\]. Nevertheless, its fourth frame potential is not far from the value of a 4-design [c.f. Eq. \[116\]] according to the formula \[88\]

\[ \Phi_4(C_n) = \begin{cases} 15 & n = 1, \\ 29 & n = 2, \\ 30 & n \geq 3. \end{cases} \tag{32} \]
This observation indicates that the fourth tensor power of the Clifford group has only a few more irreducible components than that of the whole unitary group, which will be spelled out more precisely in the next section.

*Stabilizer codes and states* [32] are certain subspaces of $\mathbb{C}^d$ that are of fundamental importance in quantum information theory. Among other applications, they form the foundation of the theory of quantum error correction [64].

A *stabilizer group* is an abelian subgroup of the Pauli group that does not contain $-1$. A *stabilizer code* is the common $+1$-eigenspace of operators in a stabilizer group [32] [64]. Let $S \subset P_n$ be a stabilizer group. One can easily verify that

$$P = \frac{1}{|S|} \sum_{W \in S} W$$

is the orthogonal projector onto the stabilizer code associated with the group. The order of any $n$-qubit stabilizer group is a divisor of $d = 2^n$. If the stabilizer group has order $2^m$ with $m \leq n$, then the stabilizer code has dimension $2^{n-m}$. Those $n$-qubit stabilizer groups of order $d$ are called *maximal*. When the stabilizer group is maximal, the stabilizer code has dimension 1. Such codes are commonly referred to as *stabilizer states*.

Stabilizer codes can be described in terms of the geometry of the discrete symplectic vector space $\mathbb{F}_2^{2n}$. We mention this connection only briefly – c.f. Refs. [34, 40, 54] for more details. Any $n$-qubit stabilizer group $S$ is of the form

$$S = \{-1\}^{f(a)} W_a \mid a \in M \subset \mathbb{F}_2^{2n}\}$$

for some set $M \subset \mathbb{F}_2^{2n}$ and some function $f : \mathbb{F}_2^{2n} \rightarrow \mathbb{F}_2$. The fact that $S$ forms a group implies that $M$ is a subspace of $\mathbb{F}_2^{2n}$. From the fact that $S$ is abelian and Eq. (27), it follows that the symplectic inner product vanishes on $M$. Such subspaces are called *isotropic* in symplectic geometry. So there is a close correspondence between stabilizer codes and isotropic subspaces of finite symplectic vector spaces.

### 3 Decomposition of the fourth tensor power of the Clifford group

#### 3.1 A special stabilizer code

To state our main result, we need to introduce a certain stabilizer code. Whenever $k$ is even, the following set of Pauli operators

$$S_{n,k} = \{\tau^k(W_a) \mid a \in \mathbb{F}_2^{2n}\}$$

commute with each other. The set is also invariant under the diagonal action of the Clifford group. If in addition $k$ is a multiple of 4, then $S_{n,k}$ is closed under multiplication and thus forms a stabilizer group. Denote by $V_{n,k}$ the stabilizer code defined by the joint $+1$-eigenspace of operators in $S_{n,k}$. The dimension of the stabilizer code is $d^{k-2}$, and the projector onto it is given by

$$P_{n,k} = \frac{1}{|S_{n,k}|} \sum_{a \in \mathbb{F}_2^{2n}} \tau^k(W_a) = \frac{1}{2^{2n}} \sum_{a \in \mathbb{F}_2^{2n}} W_a \otimes \cdots \otimes W_a \mid k\}$$

The stabilizer code $V_{n,k}$ and projector $P_{n,k}$ are invariant under the action of the symmetric group $S_k$, which acts on $(\mathbb{C}^d)^{\otimes k}$ by permuting the $k$ tensor factors. Meanwhile, they are also invariant under the diagonal action of the Clifford group. In other words, $V_{n,k}$ affords a representation of the Clifford group $C_n$. Our main result stated in Section [3.2] in a precise sense, $V_{n,4}$ is the only subspace of $(\mathbb{C}^d)^{\otimes 4}$ stabilized by $C_n$ but not by the unitary group $U(d)$.

Given that $V_{n,k}$ is a common $+1$ eigenspace of $\tau^k(W_a)$ for all Pauli operators $W_a$ and that $i^k = 1$ when $k$ is a multiple of 4, it follows that the Pauli group $P_n$ acts trivially on $V_{n,k}$. Therefore, $V_{n,k}$ affords a representation of the symplectic group Sp$(2n,\mathbb{F}_2)$, which is isomorphic to $C_n/P_n$. The property of this representation is discussed in more detail in Sec. [3.4].

In the rest of this section, we construct an orthonormal basis for $V_{n,k}$, though this is not essential to understanding the main result. First consider the special case $n = 1$. Let $u \in \mathbb{F}_2^2$ and define $\hat{u} :=$
Likewise, appearing in Lemma 1. A major technical stepping stone for establishing our main result are explicit formulas for the dimensions G. In particular, it follows that for each λ and define the spaces G of irreducible components of likewise, D(U(4)). Theorem 1 (Main Theorem) factors of (d, respectively, as listed in Table 1. Note that d(λ) equals the multiplicity of the Weyl module W(λ), and, likewise, D(λ) is the multiplicity of the Specht module S(λ). The dimensions of S(λ) and W(λ) are denoted by d(λ) and D(λ), respectively, as listed in Table 1. Note that d(λ) equals the multiplicity of the Weyl module W(λ), and, likewise, D(λ) is the multiplicity of the Specht module S(λ). Let G be a subgroup of U(d), then the number of irreducible components of G × S(λ) on H(λ) is equal to the number of irreducible components of G on W(λ). In particular, G × S(λ) is irreducible on H(λ) iff G is irreducible on W(λ). The multiplicity of each irrep of G appearing in H(λ) is always a multiple of d(λ).

Next, we will give a more concrete formulation of the main result. To this end, recall that Schur-Weyl duality can be used to find the decomposition

\[ (\mathbb{C}^d)^{\otimes 4} = \bigoplus_{\lambda} H(\lambda) = \bigoplus_{\lambda} W(\lambda) \otimes S(\lambda) \]

of (\mathbb{C}^d)^{\otimes 4} into irreps of U(d) × S(4). Here, the λ’s are partitions of 4 into no more than d parts, W(λ) is the Weyl module carrying an irrep of U(d) and S(λ) the Specht module on which S(4) acts irreducibly; the group U(d) × S(4) acts irreducibly on each H(λ). The dimensions of S(λ) and W(λ) are denoted by d(λ) and D(λ), respectively, as listed in Table 1. Note that d(λ) equals the multiplicity of the Weyl module W(λ), and, likewise, D(λ) is the multiplicity of the Specht module S(λ). Let G be a subgroup of U(d), then the number of irreducible components of G × S(λ) on H(λ) is equal to the number of irreducible components of G on W(λ). In particular, G × S(λ) is irreducible on H(λ) iff G is irreducible on W(λ). The multiplicity of each irrep of G appearing in H(λ) is always a multiple of d(λ).

Now recall that V_{n,4} is the stabilizer code defined above. We denote its orthogonal complement by V_{n,4}^\perp, and define the spaces

\[ H(\lambda)^+ := H(\lambda) \cap V_{n,4}, \quad H(\lambda)^- := H(\lambda) \cap V_{n,4}^\perp. \]

Because V_{n,4} is invariant under the action of S(4), and because the S(λ) are irreducible under the same action, it follows that for each λ, there is a subspace W(\lambda)^+ \subset W(\lambda) such that

\[ H(\lambda)^+ = W(\lambda)^+ \otimes S(\lambda). \]

Likewise,

\[ H(\lambda)^- = W(\lambda)^- \otimes S(\lambda), \]

where W(\lambda)^- is the ortho-complement, within W(\lambda), of W(\lambda)^+. Define \[ D(\lambda)^\pm := \dim W(\lambda)^\pm, \]

then \[ \dim H(\lambda)^\pm = d(\lambda) D(\lambda)^\pm. \]

A major technical stepping stone for establishing our main result are explicit formulas for the dimensions of these spaces.

**Lemma 1.** The values of D(\lambda)^\pm for nonincreasing partitions λ of 4 are given in Table 7.
Whenever they are non-trivial, the spaces decomposes into irreps as \( C_\tau \) of certain related representations of \( C_\tau \) where \( \tau \) is the complex conjugate is taken. To verify this claim, note that and extended linearly to the general case. The transpose is to be understood in the same basis in which \( \tau \) acts on \( W_\lambda^s \) and \( I_\lambda \) is the identity on \( S_\lambda \). Therefore, the spaces \( W_\lambda^\pm \) carry representations \( U \mapsto U_\lambda^\pm \) of the Clifford group \( C_n \). We can now state a more concrete version of the main theorem.

**Proposition 5.** Whenever they are non-trivial, the spaces \( W_\lambda^\pm \) carry irreducible and inequivalent representations of the \( n \)-qubit Clifford group \( C_n \). What is more, under the action of \( C_n \times S_4 \), the space \((\mathbb{C}^d)^\otimes 4 \) decomposes into irreps as

\[
(\mathbb{C}^d)^\otimes 4 = \bigoplus_{\lambda,s=\pm \mid D_\lambda \neq 0} W_\lambda^s \otimes S_\lambda.
\]

We remark that following Ref. [80], the commutant of \( \tau^4(C_n) \) can easily be mapped to the commutant of certain related representations of \( C_n \). Indeed, consider as a first example the representation

\[
\tau^{(3,1)} : U \mapsto U \otimes U \otimes U \otimes \bar{U}.
\] (38)

Then

\[
A \in \tau^{(3,1)}(C_n)' \Leftrightarrow A^{\Gamma_4} \in \tau^4(C_n)'. \tag{39}
\]

Here, \( A^{\Gamma_4} \) is the partial transpose of \( A \) with respect to the fourth tensor factor. It is defined on product matrices as

\[
A_1 \otimes A_2 \otimes A_3 \otimes A_4 \mapsto A_1 \otimes A_2 \otimes A_3 \otimes (A_4)^T,
\]

and extended linearly to the general case. The transpose is to be understood in the same basis in which the complex conjugate is taken. To verify this claim, note that

\[
\tau^{(3,1)}(U)(A_1 \otimes A_2 \otimes A_3 \otimes A_4)^{\Gamma_4} + \tau^{(3,1)}(U)^T
= (U \otimes U \otimes U \otimes \bar{U})(A_1 \otimes A_2 \otimes A_3 \otimes A_4^T)(U^\dagger \otimes U^\dagger \otimes U^\dagger \otimes \bar{U}^T)
= [(U \otimes U \otimes U \otimes \bar{U})(A_1 \otimes A_2 \otimes A_3 \otimes A_4)(U^\dagger \otimes U^\dagger \otimes U^\dagger \otimes \bar{U}^T)]^{\Gamma_4},
\]

so that

\[
\tau^{(3,1)}(U)A^{\Gamma_4} + \tau^{(3,1)}(U)^T - A^{\Gamma_4} = 0 \Leftrightarrow [\tau^4(U)A^\Gamma_4(U)^\dagger - A]^{\Gamma_4} = 0 \Leftrightarrow \tau^4(U)A^\Gamma_4(U)^\dagger - A = 0.
\]

An analogous reasoning applies to the representations \( \tau^{(k,l)} \) for general \( k,l \). Particularly relevant are the representations \( \tau^{(k,k)} \), which are isomorphic to the \( k \)th tensor power of the adjoint representation. Based on this connection, one could work out the irreducible representations of \( \tau^{(k,l)}(C_n) \) by diagonalizing the commutant. We have not pursued this route any further in the present paper (but see [44]).
the above two equations, from which $D$ group l where permutation contains at least one cycle of odd length. Now the value of $\chi$.

Here the trace $\text{tr}$.

Proof of Lemma 1. Let $H_\lambda, W_\lambda, S_\lambda$ be the representation spaces appearing in the Schur-Weyl decomposition in Eq. (37). Let $P_\lambda$ be the projector onto $H_\lambda$. We have

$$P_\lambda = \frac{d_\lambda}{24} \sum_{\sigma \in S_4} \chi_\lambda(\sigma) U_\sigma,$$

where $U_\sigma$ is the unitary operator that realizes the permutation of the tensor factors corresponding to $\sigma$, and $\chi_\lambda$ is the character of the irrep of $S_4$ corresponding to the partition $\lambda$; see Table 2. For example, the projectors onto the symmetric and antisymmetric subspaces are respectively given by

$$P_{[4]} = \frac{1}{24} \sum_{\sigma \in S_4} U_\sigma, \quad P_{[4]^*} = \frac{1}{24} \sum_{\sigma \in S_4} \text{sgn}(\sigma) U_\sigma,$$

where $\text{sgn}(\sigma)$ is equal to 1 for even permutations and $-1$ for odd permutations.

Note that $P_\lambda$ commutes with the projector $P_{n,4}$ onto the stabilizer code, so the dimension of $H_\lambda^+ = V_{n,4} \cap H_\lambda$ is given by $d_\lambda D_\lambda^+ = \text{tr}(P_{n,4} P_\lambda)$. Therefore,

$$D_\lambda^+ = \frac{1}{d_\lambda} \text{tr}(P_{n,4} P_\lambda) = \frac{1}{d_\lambda^2} \sum_{\sigma \in S_4} \text{tr}(W_a \otimes^4 P_\lambda) = \frac{1}{d_\lambda^2} \left[ D_\lambda + \frac{1}{24} \sum_{\sigma \in S_4} \sum_{0 \neq a \in \mathbb{F}_2^n} \chi_\lambda(\sigma) \text{tr}(U_\sigma W_a \otimes^4 W_\sigma) \right].$$

Here the trace $\text{tr}(U_\sigma W_a \otimes^4 W_\sigma)$ with $a \neq 0$ can be computed using the following simple formula,

$$\text{tr}(U_\sigma W_a \otimes^4 W_\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ contains a cycle of odd length}, \\ d(\sigma) & \text{otherwise}, \end{cases}$$

where $d(\sigma)$ is the number of cycles in $\sigma$ that have even lengths. According to Table 2, the symmetric group $S_4$ has three permutations of cycle type $(2^2)$ and six permutations of cycle type $(4)$, while any other permutation contains at least one cycle of odd length. Now the value of $D_\lambda^+$ can be computed by virtue of the above two equations, from which $D_\lambda^+ = D_\lambda - D_\lambda^+$ follows immediately, as shown in Table 1.

3.3 Proof of Main Theorem

In this section, we prove Lemma 1 and conclude from it our main result. An alternative proof of Lemma 1 – which also yields orthonormal bases for $W_{[4]}^+$ and $W_{[1]}^+$ – is presented in the appendix.

Proof of Lemma 1. Let $H_\lambda, W_\lambda, S_\lambda$ be the representation spaces appearing in the Schur-Weyl decomposition in Eq. (37). Let $P_\lambda$ be the projector onto $H_\lambda$. We have

$$P_\lambda = \frac{d_\lambda}{24} \sum_{\sigma \in S_4} \chi_\lambda(\sigma) U_\sigma,$$

where $U_\sigma$ is the unitary operator that realizes the permutation of the tensor factors corresponding to $\sigma$, and $\chi_\lambda$ is the character of the irrep of $S_4$ corresponding to the partition $\lambda$; see Table 2. For example, the projectors onto the symmetric and antisymmetric subspaces are respectively given by

$$P_{[4]} = \frac{1}{24} \sum_{\sigma \in S_4} U_\sigma, \quad P_{[4]^*} = \frac{1}{24} \sum_{\sigma \in S_4} \text{sgn}(\sigma) U_\sigma,$$

where $\text{sgn}(\sigma)$ is equal to 1 for even permutations and $-1$ for odd permutations.

Note that $P_\lambda$ commutes with the projector $P_{n,4}$ onto the stabilizer code, so the dimension of $H_\lambda^+ = V_{n,4} \cap H_\lambda$ is given by $d_\lambda D_\lambda^+ = \text{tr}(P_{n,4} P_\lambda)$. Therefore,

$$D_\lambda^+ = \frac{1}{d_\lambda} \text{tr}(P_{n,4} P_\lambda) = \frac{1}{d_\lambda^2} \sum_{\sigma \in S_4} \text{tr}(W_a \otimes^4 P_\lambda) = \frac{1}{d_\lambda^2} \left[ D_\lambda + \frac{1}{24} \sum_{\sigma \in S_4} \sum_{0 \neq a \in \mathbb{F}_2^n} \chi_\lambda(\sigma) \text{tr}(U_\sigma W_a \otimes^4 W_\sigma) \right].$$

Here the trace $\text{tr}(U_\sigma W_a \otimes^4 W_\sigma)$ with $a \neq 0$ can be computed using the following simple formula,

$$\text{tr}(U_\sigma W_a \otimes^4 W_\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ contains a cycle of odd length}, \\ d(\sigma) & \text{otherwise}, \end{cases}$$

where $d(\sigma)$ is the number of cycles in $\sigma$ that have even lengths. According to Table 2, the symmetric group $S_4$ has three permutations of cycle type $(2^2)$ and six permutations of cycle type $(4)$, while any other permutation contains at least one cycle of odd length. Now the value of $D_\lambda^+$ can be computed by virtue of the above two equations, from which $D_\lambda^+ = D_\lambda - D_\lambda^+$ follows immediately, as shown in Table 1.

Proof of Proposition 5 and Theorem 1. From the discussion in Sec. 2.2, the sum of squared multiplicities of irreducible components of $\tau^4(C_n)$ is equal to the fourth frame potential of the Clifford group $C_n$. For now, we restrict to $n \geq 3$. In this case, both $H_4^+$ and $H_\lambda^+$ are nontrivial invariant subspaces of $C_n \times S_4$ for $\lambda = [4], [1, 1, 1, 1], [2, 2]$. So the frame potential of $C_n$ is at least

$$\Phi_4(C_n) \geq d_{[4]}^2 + d_{[1,1,1,1]}^2 + d_{[2,2]}^2 + \sum_\lambda d_\lambda^2 = 30,$$
with equality if and only if all the representations of $C_n$ afforded by $W^{++}_{\Delta}$ for $D^{\pm}_{\Delta} \neq 0$ are irreducible and inequivalent. However, we know from Eq. (32) that $\Phi(C_n)$ is indeed equal to 30 for $n \geq 3$. Thus, equality must hold and we have proved the first part of Proposition 5. The proofs for the special cases $n = 1, 2$ are similar.

The second part of Proposition 5 is a straight-forward combination of the first part with Schur-Weyl duality.

By the second part of Proposition 5 and Schur’s Lemma, every element $B$ of the commutant of $\tau^4(C_n)$ is of the form

$$B = \bigoplus_{\lambda, s=\pm |D^{\pm}_{\lambda} \neq 0} I^s_{\lambda} \otimes B^s_{\lambda},$$

with $I^s_{\lambda}$ the identity on $W^s_{\lambda}$ and $B^s_{\lambda}$ a suitable linear operator on $S_{\lambda}$. Thus

$$B = P_{n,4} \left( \bigoplus_{\lambda | D^{+}_{\lambda} \neq 0} I_{\lambda} \otimes B^{+}_{\lambda} \right) + (I - P_{n,4}) \left( \bigoplus_{\lambda | D^{-}_{\lambda} \neq 0} I_{\lambda} \otimes B^{-}_{\lambda} \right),$$

where $I_{\lambda}$ is the identity on $W_{\lambda}$. The expressions in parentheses commute with the diagonal representation of $U(d)$ and are thus, by Schur-Weyl duality, linear combinations of the representation of $S_4$, which permutes the tensor factors. This proves Theorem 1.

**3.4 Representations of the discrete symplectic group**

We have argued in Sec. 3.4 that whenever $k$ is a multiple of 4, the stabilizer code $V_{n,k}$ carries a representation of the symplectic group $Sp(2n, F_2)$. For $k = 4$, Proposition 5 and Table I imply that

$$V_{n,4} \simeq W^{+}_{[4]} \oplus W^{+}_{[1,1,1,1]} \oplus W^{+}_{[2,2]} \otimes \mathbb{C}^2$$

(45)

gives the decomposition of that stabilizer code into irreps of $Sp(2n, F_2)$. This decomposition is remarkably similar to the decomposition of the complex Weil character $\zeta_n$ of $Sp(2n, F_2)$ as discussed in Ref. [41] pages 4976—4977,

$$\zeta_n = \alpha_n + \beta_n + 2\zeta_{n}^1,$$

(46)

Moreover, the dimensions of $V_{n,4}$, $W_{[1,1,1,1]}$, $W_{[4]}$, and $W_{[2,2]}$ coincide with the degrees of the Weil characters $\zeta_n$, $\alpha_n$, $\beta_n$, and $\zeta_{n}^1$, respectively, according to Table I and Table I in Ref. [41]. The following proposition reveals the reason behind this coincidence.

**Proposition 6.** $V_{n,4}$ carries the complex Weil representation of $Sp(2n, F_2)$ with character $\zeta_n$ as defined in Ref. [41] pages 4976—4977. What is more, the characters of $W_{[1,1,1,1]}$, $W_{[4]}$, and $W_{[2,2]}$ are their $\alpha_n$, $\beta_n$, and $\zeta_n$, respectively.

When $n \geq 4$, Proposition 6 follows from Corollary 6.2 in Ref. [41], which states that $\alpha_n$, $\beta_n$, and $\zeta_{n}^1$ are nontrivial characters of $Sp(2n, F_2)$ of three minimal degrees. When $n = 3$, $\alpha_n$ and $\beta_n$ are still characters of the two minimal degrees [79], but there is another character of $Sp(6, F_2)$ that has the same degree of 21 as $\zeta_{3}^1$. When $n = 2$, $Sp(2n, F_2)$ is isomorphic to the symmetric group $S_6$. When $n = 1$, $Sp(2n, F_2)$ is isomorphic to the symmetric group $S_3$, in which case $W_{[1,1,1,1]}$ has dimension 0, $W_{[2,2]}$ carries the sign representation of $S_3$, and $W_{[4]}$ carries the unique two-dimensional representation. Here we shall give a simple and uniform proof of Proposition 6 which does not rely on Corollary 6.2 in Ref. [41]. Moreover, we derive an explicit formula for the character afforded by $V_{n,k}$ and determine the sum of squared multiplicities of irreducible components, assuming $k$ is a multiple of 4.

**Lemma 2.** Suppose $F \in Sp(2n, F_2)$ and $U_F \in C_n$ is a Clifford unitary that induces the symplectic transformation $F$. If $\text{tr}(U_F) \neq 0$, then

$$[\text{tr}(U_F)]^4 = (-4)^{\dim(\ker(F^{-1}))}.$$  

(47)
As an implication of this lemma,

$$\text{tr}(U_F) = \sqrt{f(F)} \cdot \begin{cases} 1, & \frac{\dim(\ker(F-1))}{2} \text{ is odd,} \\ 0, & \frac{\dim(\ker(F-1))}{2} \text{ is even.} \end{cases}$$ (48)

where \(j = 1, 2, 3, 4\), and \(f(F) := 2^{\dim(\ker(F-1))}\) is the number of fixed points of \(F\) on the symplectic space \(\mathbb{F}_2^n\) \[^{[SS]}\]. Note that \(f(F) = |\text{tr}(U_F)|^2\) if \(U_F\) is not traceless.

**Proof.** Recall that the Clifford group \(C_n\) is generated by phase gates and Hadamard gates of respective qubits as well as CNOT gates between all pairs of qubits (cf. Sec. 2.3). Therefore, the Clifford unitary \(U_F\) has the form \(U_F = A/2^j\), where \(j\) is a nonnegative integer, and \(A\) is a matrix each entry of which is a linear combination of 1 and \(i\) with integer coefficients. Consequently, \(\text{tr}(U_F)\) has the form

$$\text{tr}(U_F) = \frac{a + bi}{2^k},$$ (49)

where \(a, b\) are integers and \(k\) is a nonnegative integer. In addition, we may assume that the greatest common divisor of \(a, b\) is odd if \(k > 0\). According to Ref. \[^{[SS]}\],

$$|\text{tr}(U_F)|^2 = f(F) = 2^{\dim(\ker(F-1))}$$ (50)

whenever \(U_F\) is not traceless. If \(k > 0\), then at least one of \(a, b\) is odd, so that \(\text{tr}(U_F)|^2 = (a^2 + b^2)/4^k\) cannot be an integer. It follows that \(k = 0\), \(\text{tr}(U_F) = a + bi\), and

$$a^2 + b^2 = 2^{\dim(\ker(F-1))}.\hspace{1cm} (51)$$

If \(\dim(\ker(F-1))\) is odd, then \(a^2 = b^2 = 2^{\dim(\ker(F-1))} - 1\) given that \(a, b\) are integers, so that

$$|\text{tr}(U_F)|^4 = -4a^2b^2 = (-4)^{\dim(\ker(F-1))}.\hspace{1cm} (52)$$

If \(\dim(\ker(F-1))\) is even, then \(ab = 0\) and

$$|\text{tr}(U_F)|^4 = (a^2 + b^2)^2 = 4^{\dim(\ker(F-1))} = (-4)^{\dim(\ker(F-1))}.\hspace{1cm} (53)$$

\[\square\]

Recall that the stabilizer code \(V_{n,k}\) carries a representation of the symplectic group \(\text{Sp}(2n, \mathbb{F}_2)\) whenever \(k\) is a multiple of 4. The following lemma yields an explicit formula for the character of this representation. Note that the same formula also applies to any subgroup of \(\text{Sp}(2n, \mathbb{F}_2)\).

**Lemma 3.** Suppose \(F \in \text{Sp}(2n, \mathbb{F}_2)\) and \(U_F \in C_n\) is a Clifford unitary that induces the symplectic transformation \(F\). If \(k\) is a multiple of 4, then

$$\text{tr} \left( U_F^{\otimes k} P_{n,k} \right) = \left[ \frac{(-4)^{k/4}}{2} \right]^{\dim(\ker(F-1))}.\hspace{1cm} (54)$$

In particular,

$$\text{tr} \left( U_F^{\otimes 4} P_{n,4} \right) = (-2)^{\dim(\ker(F-1))}.\hspace{1cm} (55)$$

**Proof.**

$$\text{tr} \left( U_F^{\otimes k} P_{n,k} \right) = \frac{1}{d^2} \text{tr} \left( U_F^{\otimes k} \sum_a W_a^{\otimes k} \right) = \frac{1}{d^2} \sum_a \left[ \text{tr}(U_F W_a) \right]^k.\hspace{1cm} (56)$$

Note that the \(d^2\) operators \(U_F W_a\) for \(a \in \mathbb{F}_2^n\) induce the same symplectic transformation as \(U_F\). In addition, \(|\text{tr}(U_F W_a)|^2 = 2^{\dim(\ker(F-1))}\) when \(U_F W_a\) is not traceless and \(\sum_a |\text{tr}(U_F W_a)|^2 = d^2\). So among the \(d^2\) operators \(U_F W_a\) for \(a \in \mathbb{F}_2^n; 2^{2n-\dim(\ker(F-1))}\) of them are not traceless \[^{[SS]}\]. Now application of Lemma 2 to the above equation yields

$$\text{tr} \left( U_F^{\otimes 4} P_{n,4} \right) = 2^{-2n} \times (-4)^{k \dim(\ker(F-1))/4} \times 2^{2n-\dim(\ker(F-1))} = \left[ \frac{(-4)^{k/4}}{2} \right]^{\dim(\ker(F-1))}.\hspace{1cm} (57)$$

\[\square\]
Proof of Proposition 4. According to Lemma 3, the character afforded by \( V_{n,4} \) coincides with \( \zeta_n \) discussed in Ref. [41]. Consequently, \( V_{n,4} \) decomposes into the same irreps as \( \zeta_n \). Since the character afforded by \( W_{[2,2]} \) has multiplicity 2, it must correspond to \( \zeta_4^1 \). Comparison of the dimensions shows that the characters of \( W_{[1,1,1,1]} \) and \( W_{[4]} \) correspond to \( \alpha_4 \) and \( \beta_4 \), respectively.

Suppose \( R \leq \text{Sp}(2n, F_2) \) and let \( G_R \) be the preimage in \( C_n \) of \( R \) under the homomorphism \( C_n/\mathcal{P}_n \). Denote by \( M_k(R) \) the sum of squared multiplicities of the representation of \( R \) or \( G_R \) afforded by the stabilizer code \( V_{n,k} \), assuming \( k \) is a multiple of 4. Then \( M_k(R) \) may also be understood as the contribution of \( V_{n,k} \) to the \( k \)th frame potential \( \Phi_k(G_R) \) of \( G_R \).

Lemma 4. Suppose \( R \leq \text{Sp}(2n, F_2) \) and \( G_R \) is the preimage in \( C_n \) of \( R \) under the homomorphism \( C_n/\mathcal{P}_n \). If \( k \) is a multiple of 4, then

\[
M_k(R) = \frac{1}{|R|} \sum_{F \in R} f(F)^{k-2} = \Phi_{k-1}(G_R). \tag{58}
\]

Moreover, \( M_k(R) \) is equal to the number of orbits of \( R \) on \((F_2^{2n})^{\times (k-2)}\).

Surprisingly, the contribution of \( V_{n,k} \) to the \( k \)th frame potential of \( G_R \) is equal to the \((k-1)\)th frame potential of \( G_R \).

Proof. According to Lemma 3,

\[
M_k(R) = \frac{1}{|R|} \sum_{F \in R} \left[ \frac{(-4)^k}{2} \right]^{2 \dim(\ker(F-1))} = \frac{1}{|R|} \sum_{F \in R} 2^{(k-2) \dim(\ker(F-1))} = \frac{1}{|R|} \sum_{F \in R} f(F)^{k-2}. \tag{59}
\]

This proves the first equality in Eq. (58); the second equality follows from Lemma 1 in Ref. [88]. Now according to the well-known orbit-stabilizer relation, \( M_k(R) \) is equal to the number of orbits of \( R \) on \((F_2^{2n})^{\times (k-2)}\).

Observing that \( \text{Sp}(2n, F_2) \) has five orbits on \((F_2^{2n})^{\times 2}\) when \( n = 1 \) and six orbits when \( n \geq 2 \) [88], we conclude that

\[
M_4(\text{Sp}(2n, F_2)) = \begin{cases} 
5 & n = 1, \\
6 & n \geq 2,
\end{cases} \tag{60}
\]

which agrees with the decomposition in Eq. (45). A subgroup \( R \) of \( \text{Sp}(2n, F_2) \) has the same decomposition on \( V_{n,4} \) iff \( R \) has the same number of orbits on \((F_2^{2n})^{\times 2}\) as \( \text{Sp}(2n, F_2) \). This condition is equivalent to the condition that \( G_R \) forms a unitary 3-design [88]. Technically, this means that \( R \) is 2-transitive on \( F_2^{2n} \) when \( n = 1 \) and is a rank-3 permutation group when \( n \geq 2 \) [27, 13, 88], where \( F_2^{2n} \) is the set of nonzero vectors in \( F_2^{2n} \). However, there is no proper subgroup of \( \text{Sp}(2n, F_2) \) with this property except when \( n = 2 \), in which case there is a unique counterexample [14, 15, 88]. Therefore, any proper subgroup of \( \text{Sp}(2n, F_2) \) with \( n \neq 2 \) has more irreducible components in \( V_{n,4} \) (and also in \( V_{n,k} \) as a consequence); in other words, at least one of the characters \( \alpha_n, \beta_n, \zeta_4^1 \) becomes reducible when restricted to a proper subgroup. Similarly, any proper subgroup of \( \overline{C}_n \) with \( n \neq 2 \) has more irreducible components on \( V_{n,4} \) than \( \overline{C}_n \), and at least one of the representations \( W_{[1,1,1,1]}, W_{[4]}, \) and \( W_{[2,2]} \) becomes reducible when restricted to a proper subgroup of \( \overline{C}_n \).

### 4 t-designs from Clifford orbits

In this section we determine all Clifford covariant \( t \)-designs in the case of a single qubit. We then show that random orbits of the Clifford group in general are very good approximations to \( 4 \)-designs. Furthermore, we introduce several simple and efficient methods for constructing exact \( 4 \)-designs and approximations with arbitrarily high precision from Clifford orbits.
4.1 Clifford covariant $t$-designs for one qubit

In the case of $n = 1$, the $t$-partite symmetric subspace has dimension $t + 1$, so the frame potential of a qubit $t$-design is equal to $1/(t+1)$. Since the Clifford group is a unitary 3-design, every orbit of the Clifford group forms a complex projective 3-design. The unique shortest orbit is composed of six stabilizer states, which form a complete set of mutually unbiased bases. When represented on the Bloch sphere, the six states form the vertices of the octahedron.

To derive a simple criterion on the orbit that forms a 4-design, suppose the fiducial state has Bloch vector $(x, y, z)$ with $x^2 + y^2 + z^2 = 1$. Then the fourth frame potential of the Clifford orbit is given by

$$\Phi_4(x, y, z) = \frac{21 - 6(x^4 + y^4 + z^4) + 5(x^4 + y^4 + z^4)^2}{96}.$$ (61)

The orbit forms a 4-design iff $x^4 + y^4 + z^4 = 3/5$, in which case $\Phi_4(x, y, z)$ attains the minimum of $1/5$. One explicit solution is given by

$$x = \sqrt{\frac{5 + 2\sqrt{10}}{15}}, \quad y = z = \sqrt{\frac{5 - \sqrt{10}}{15}}.$$ (62)

It turns out that the orbit forms a 5-design under the same condition; that is, a Clifford orbit forms a 5-design iff it forms a 4-design. As explained in Section 4.6, this may not be a coincidence. By contrast, $\Phi_4(x, y, z)$ is maximized when $x^4 + y^4 + z^4 = 1$, in which case the Bloch vector corresponds to a stabilizer state.

When the condition $x^4 + y^4 + z^4 = 3/5$ is satisfied, the sixth and seventh frame potentials satisfy the following equation

$$8\Phi_7(x, y, z) - 1 = 4[7\Phi_6(x, y, z) - 1] = \frac{11(1 - 21x^2 + 105x^4 - 105x^6)}{2400} = \frac{11(1 - 21y^2 + 105y^4 - 105y^6)}{2400} = \frac{11[3 - 7(x^6 + y^6 + z^6)]}{480}.$$ (63)

The orbit forms a 6-design iff $x^2, y^2, z^2$ are distinct roots of the cubic equation $1 - 21u + 105u^2 - 105u^3 = 0$, which are given by

$$u_j = \frac{1}{3} \left(1 + 2\sqrt{\frac{2}{5} \cos \frac{\theta + 2j\pi}{3}}\right), \quad \theta = \arctan \frac{3\sqrt{10}}{20}, \quad j = 1, 2, 3.$$ (64)

Equivalently, the orbit forms a 6-design iff $x^6 + y^6 + z^6 = 3/7$ or if $x^2y^2z^2 = 1/105$ (assuming $x^4 + y^4 + z^4 = 3/5$). The same condition also guarantees that the orbit forms a 7-design. There are 48 solutions in total, which compose two Clifford orbits. When represented on the Bloch sphere, the two orbits can be converted to each other by inversion. The two orbits are not unitarily equivalent, but are equivalent under antiunitary transformations. Actually, the 48 solutions form one orbit under the action of the extended Clifford group [5], the group generated by the Clifford group and complex conjugation with respect to the computational basis. Since any qubit 8-design has at least 25 elements according to Eq. (10), no Clifford orbit can form an 8-design.

Calculation shows that a random Clifford orbit is approximately a $t$-design for $t$ up to 7. If $(x, y, z)$ is distributed uniformly on the Bloch sphere, then the ratio of the average frame potential over the minimum potential is given by

$$(t + 1)E[\Phi_t(x, y, z)] = \begin{cases} 1 & t = 3, \\ \frac{127}{126} & t = 4, \\ \frac{1795}{1796} & t = 5, \\ \frac{1381}{1382} & t = 6, \\ \frac{1289}{1290} & t = 7. \end{cases}$$ (65)
4.2 Random Clifford orbits are good approximations to 4-designs

In this section we show that random Clifford orbits are very good approximations to projective 4-designs. Recall that \(\tau^4(C_n)\) has two irreducible components \(W^\pm_{[4]}\) in the totally symmetric space \(W_{[4]} = \text{Sym}_4(\mathbb{C}^d)\). According to Table 1, the dimensions of \(W_{[4]}\) are

\[
D_{[4]} = \frac{d(d+1)(d+2)(d+3)}{24},
\]

\[
D_+ := D_{[4]}^+ = \frac{(d+1)(d+2)}{6},
\]

\[
D_- := D_{[4]}^- = \frac{(d-1)(d+1)(d+2)(d+4)}{24}.
\]

The projectors \(P_\pm\) onto the two irreps \(W^\pm_{[4]}\) read

\[
P_+ = P_{n,4} P_{[4]}, \quad P_- = (1 - P_{n,4}) P_{[4]},
\]

where \(P_{n,4}\) is the projector onto the stabilizer code \(V_{n,4}\) given in Eq. (34) and \(P_{[4]}\) is the projector onto \(W_{[4]}\).

As an implication of Theorem 1 or Proposition 3, we have

**Corollary 1.** Let \(\text{orb}(\psi)\) be the orbit of a vector \(\psi \in \mathbb{C}^{2^n}\) under the action of the Clifford group \(\overline{C}_n\). Then

\[
\frac{1}{|\text{orb}(\psi)|} \sum_{\phi \in \text{orb}(\psi)} (|\phi\rangle\langle\phi|)^{\otimes 4} = \beta_+ P_+ + \beta_- P_-,
\]

where \(\beta_+\) and \(\beta_-\) satisfy

\[
\beta_+ = \frac{1}{D_+} \text{tr}[P_+ (|\psi\rangle \langle\psi|)^{\otimes 4}] = \frac{1}{D_+} \text{tr}[P_{n,4} (|\psi\rangle \langle\psi|)^{\otimes 4}], \quad D_+ \beta_+ + D_- \beta_- = \|\psi\|^8_2.
\]

A normalized vector \(\psi\) is a fiducial vector of a 4-design iff

\[
\beta_- = \beta_+ = 1/D_{[4]}.
\]

In what follows, we will investigate the condition (69) from various points of view. To this end, we introduce a number of related measures.

Define the characteristic function (c.f. e.g. Refs. 32, 34) \(\Xi(\psi)\) as the vector composed of the \(d^2\) elements

\[
\Xi_a(\psi) = \text{tr}(W_a |\psi\rangle \langle\psi|).
\]

Recall that the \(\ell_p\)-norm of a vector is the \(p\)-th root of the sum of the \(p\)-th powers of its elements. For our study, the \(\ell_4\)-norm of the characteristic function

\[
\|\Xi(\psi)\|_{\ell_4}^4 = \sum_{a \in \mathbb{F}_2^n} |\text{tr}(W_a |\psi\rangle \langle\psi|)|^4
\]

turns out to be particularly important. It follows directly from the definition of \(P_{n,4}\) and the symmetry of \(\psi^{\otimes 4}\) that

\[
\alpha_+(\psi) := \text{tr}[P_+ (|\psi\rangle \langle\psi|)^{\otimes 4}] = \text{tr}[P_{n,4} (|\psi\rangle \langle\psi|)^{\otimes 4}] = \frac{1}{d^2} \|\Xi(\psi)\|_{\ell_4}^4.
\]

We also set

\[
\epsilon(\psi) := \frac{D_{[4]}^3}{D_+} \alpha_+(\psi) - \|\psi\|_2^8 = \frac{d(d+3)}{4} \alpha_+(\psi) - \|\psi\|_2^8.
\]

The condition (69) for a normalized vector \(\psi\) to be a 4-design fiducial can now be re-cast in three equivalent forms,

\[
\|\Xi(\psi)\|_{\ell_4}^4 = \frac{4d}{(d+3)}, \quad \alpha_+(\psi) = \frac{D_+}{D_{[4]}^3} = \frac{4}{d(d+3)}, \quad \epsilon(\psi) = 0.
\]
From now on, we will assume the normalization condition $\|\psi\|_2 = 1$. Then, $\epsilon$ quantifies the deviation of the Clifford orbit of $\psi$ from a 4-design. More precisely, $|\epsilon(\psi)|$ is the operator norm of the deviation

$$\frac{D_{[4]}}{|\text{orb}(\psi)|} \sum_{\phi \in \text{orb}(\psi)} (|\phi\rangle\langle\phi|)^{\otimes 4} - P_{[4]},$$

(73)

while $2D_+|\epsilon(\psi)| = (d+1)(d+2)|\epsilon(\psi)|/3$ is the trace norm (or nuclear norm) of the deviation. In addition, $\epsilon(\psi)$ determines the fourth frame potential of the Clifford orbit as follows,

$$\Phi_4(\text{orb}(\psi)) = \frac{\alpha_+(\psi)^2}{D_+} + \frac{\alpha_-(\psi)^2}{D_-} = \frac{1}{D_{[4]}} \left[ 1 + \frac{D_+ \epsilon(\psi)^2}{D_-} \right] = \frac{1}{D_{[4]}} \left[ 1 + \frac{4\epsilon(\psi)^2}{(d-1)(d+4)} \right],$$

(74)

where $\alpha_-(\psi) = 1 - \alpha_+(\psi)$.

We now turn to clarifying the extremal and typical values of the functions $\alpha_+$ and $\epsilon$. To this end, note that since $\{W_a\}$ forms a nice error basis and Hermitian operator basis, we have

$$\|\Xi(\psi)\|_{\ell_2}^2 = \sum_a \text{tr}[W_a^{\otimes 2}(|\psi\rangle\langle\psi|)^{\otimes 2}] = d, \quad \|\Xi(\psi)\|_{\ell_\infty} = \max_a |\text{tr}(W_a|\psi\rangle\langle\psi|)| \leq 1.$$

Consequently,

$$\frac{2d}{d+1} \leq \|\Xi(\psi)\|_{\ell_4}^4 \leq d,$$

(75)

which are equivalent to the following inequalities

$$\frac{2}{d(d+1)} \leq \alpha_+(\psi) \leq \frac{1}{d}, \quad -\frac{d-1}{2(d+1)} \leq \epsilon(\psi) \leq \frac{d-1}{4}.$$

(76)

The upper bound in Eq. (76) follows from the Hölder inequality; it is saturated iff $\Xi(\psi)$ has $d$ entries equal to 1 and all other entries equal to 0; this can happen iff $\psi$ is a stabilizer state (c.f. Lemma 9 in the appendix). The lower bound is saturated iff

$$\Xi_a(\psi) = \frac{1}{\sqrt{d+1}} \quad \forall a \neq 0,$$

(77)

in which case the $d^2$ vectors $W_a|\psi\rangle$ for $a \in \mathbb{F}_2^n$ define a symmetric informationally complete measurement (SIC) [86, 68, 75, 8], which happens to be a minimal 2-design [73]. It is known that SIC fiducial vectors of the $n$-qubit Pauli group cannot exist except for $n = 1, 3$ [30], so the lower bounds in Eqs. (76) and (77) cannot be saturated except for $n = 1, 3$. As an implication of Eqs. (74) and (77), the frame potential satisfies

$$\frac{1}{D_{[4]}} \leq \Phi_4(\text{orb}(\psi)) \leq \frac{1}{D_{[4]}} \left( 1 + \frac{d-1}{4(d+4)} \right),$$

(78)

where the lower bound is saturated iff the orbit forms a 4-design, and the upper bound is saturated iff $\psi$ is a stabilizer state.

Next, we show that random Clifford orbits are very good approximations to 4-designs. To this end, we first compute the variance of the deviation parameter $\epsilon(\psi)$. Suppose $\psi$ is distributed according to the uniform measure. Then the first and second moments of $\alpha_+(\psi)$ are given by

$$E[\alpha_+(\psi)] = \text{tr}(P_{n,4}E[(|\psi\rangle\langle\psi|)^{\otimes 4}]) = \frac{1}{D_{[4]}} \text{tr}(P_{n,4}P_{[4]}) = \frac{4}{d(d+3)},$$

(80)

$$E[\alpha_+(\psi)^2] = \frac{1}{D_{[8]}} \text{tr}(P_{n,4}^{\otimes 2}P_{[8]}) = \frac{16(d^2 + 15d + 68)}{d^2(d+3)(d+5)(d+6)(d+7)},$$

(81)

where the last equality was derived in Appendix D. The variance of $\alpha_+(\psi)$ reads

$$\text{Var}[\alpha_+(\psi)] = E[\alpha_+(\psi)^2] - E[\alpha_+(\psi)]^2 = \frac{96(d-1)}{d^2(d+3)^2(d+5)(d+6)(d+7)},$$

(82)
As an immediate consequence,
\[ E[\epsilon(\psi)] = 0, \quad E[\epsilon(\psi)^2] = \frac{\text{Var}[\alpha_+(\psi)]}{E[\alpha_+(\psi)]^2} = \frac{6(d-1)}{(d+5)(d+6)(d+7)}. \quad (83) \]

Since the function \( \epsilon(\psi) \) is continuous, the equality \( E[|\epsilon(\psi)|] = 0 \) guarantees the existence of a root (actually many roots) of \( \epsilon(\psi) \), so exact fiducial vectors of 4-designs always exist. In addition, Eq. (83) shows that the typical value of \(|\epsilon(\psi)|\) is around \( \sqrt{6/d} \) when \( d \) is large, which is much smaller than the upper bound \((d-1)/4\). Application of the Chebyshev inequality further implies that
\[ \text{Prob}\{|\epsilon(\psi)| \geq \xi\} \leq \frac{6(d-1)}{(d+5)(d+6)(d+7)\xi^2}, \quad \forall \xi > 0. \quad (84) \]

For example,
\[ \text{Prob}\{|\epsilon(\psi)| \geq 1/2\} = \text{Prob}\{|\epsilon(\psi)| \geq 1/2\} = \text{Prob}\{\alpha_+(\psi) \geq \frac{6}{d(d+3)}\} \leq \frac{24(d-1)}{(d+5)(d+6)(d+7)}. \quad (85) \]

This particular bound is of interest to studying the distinguishability of quantum states under measurements constructed from Clifford orbits. In Ref. [55], it was shown that Clifford orbits of \( \psi \) with \( \alpha_+(\psi) \geq 6/d(d+3) \) can achieve almost the same POVM norm constants as 4-designs. Therefore, random Clifford orbits are very good approximations to 4-designs in this concrete setting.

In conjunction with Equation (74), we can also determine the ratio of the average fourth frame potential (the potential for a 4-design) and bound the probability of large deviation,
\[ D_{[4]}E[\Phi_4(\text{orb}(\psi))] = 1 + \frac{4}{(d-1)(d+4)} E[\epsilon(\psi)^2] = 1 + \frac{24}{(d+4)(d+5)(d+6)(d+7)}, \quad (86) \]
\[ \text{Prob}\left\{ D_{[4]}E[\Phi_4(\text{orb}(\psi))] \geq 1 + \frac{4\xi^2}{(d-1)(d+4)} \right\} \leq \frac{6(d-1)}{(d+5)(d+6)(d+7)\xi^2}. \quad (87) \]

In the rest of this section we derive another large-deviation bound based on Levy’s lemma [57].

**Lemma 5** (Levy). Let \( f : S^{2d-1} \to \mathbb{R} \) be Lipschitz-continuous with Lipschitz constant \( \eta \), that is,
\[ |f(x) - f(y)| \leq \eta \|x - y\|, \quad (88) \]
where \( \|x - y\| \) is the Euclidean norm in the surrounding space \( \mathbb{R}^{2d} \) of \( S^{2d-1} \). Suppose \( x \) is drawn randomly according to the uniform measure on the sphere \( S^{2d-1} \). Then
\[ \text{Prob}\{|f(x) - E[f(x)]| \geq \xi\} \leq 2 \exp\left(\frac{-d\xi^2}{9\pi^2\eta^2}\right), \quad \forall \xi \geq 0. \quad (89) \]

**Lemma 6.** The functions \( \alpha_+(\psi) \) and \( \epsilon(\psi) \) are Lipschitz-continuous with Lipschitz constants \( 5.4/d \) and \( 5.4(d+3)/4 \), respectively, that is,
\[ |\alpha_+(\psi) - \alpha_+(\psi')| \leq \frac{5.4}{d} \|\psi - \psi'\|, \quad |\epsilon(\psi) - \epsilon(\psi')| \leq \frac{5.4(d+3)}{4} \|\psi - \psi'\|. \quad (90) \]

This lemma is proved in the appendix. Note that the second inequality is an immediate consequence of the first one and Eq. (74). We guess that the two Lipschitz constants can be improved to \( 1/d \) and \( d/4 \), respectively. The following proposition is an immediate consequence of Lemma 5 and Levy’s lemma.

**Proposition 7.** Suppose \( \psi \) is drawn randomly according to the uniform measure on the complex sphere \( \mathbb{C}^d \). Then
\[ \text{Prob}\{|\alpha_+(\psi) - E[\alpha_+(\psi)]| \geq \xi\} \leq 2 \exp\left(\frac{-d\xi^2}{8138}\right), \quad \text{Prob}\{|\epsilon(\psi)| \geq \xi\} \leq 2 \exp\left(\frac{-d\xi^2}{509(d+3)^2}\right), \quad \forall \xi \geq 0. \quad (91) \]
Here the bound on \( \text{Prob}\{ \epsilon(\psi) \geq \xi \} \) is tighter than that given in Eq. (83) only when \( \epsilon(\psi) \) is very large, that is, \( \epsilon(\psi) \gg \sqrt{d} \).

Although random Clifford orbits are good approximations to 4-designs with respect to a number of measures, such as the frame potential and operator-norm deviation. They are not good enough according to certain other measures. For example, the second moment of the trace norm deviation is given by

\[
(2D_{\epsilon})^2 \mathbb{E}[\epsilon(\psi)^2] = \frac{2(d-1)(d+1)^2(d+2)^2}{3(d+5)(d+6)(d+7)},
\]

where the equality follows from Eq. (83). When \( d \) is large, the typical deviation with respect to the trace norm is around \( \sqrt{2/3d} \), while it is desirable that the deviation does not grow with the dimension for some applications. This observation motivates us to search for exact 4-designs or approximations with higher precision.

### 4.3 Fiducial vectors of exact 4-designs up to five qubits

In this section we propose a method for constructing exact fiducial vectors of 4-designs of the Clifford group. Solutions up to five qubits are presented explicitly.

Recall that an \( n \)-qubit state vector \( \psi \) is a fiducial vector of a 4-design iff \( \| \Xi(\psi) \|_{\ell_4}^4 = 4d/(d+3) \); see Eq. (72). Suppose \( \psi = \psi_1 \otimes \psi_2 \) is a tensor product of an \( n_1 \)-qubit state vector and an \( n_2 \)-qubit state vector with \( n_1 + n_2 = n \). Then \( \| \Xi(\psi) \|_{\ell_4}^4 = \| \Xi(\psi_1) \|_{\ell_4}^4 \| \Xi(\psi_2) \|_{\ell_4}^4 \) since \( P_{n,4} \) decomposes in the same way \( P_{n,4} = P_{n_1,4} \otimes P_{n_2,4} \). In the case of a single qubit, let \( \psi(x, y, z) \) be a fiducial vector with Bloch vector \( (x, y, z) \) with \( x^2 + y^2 + z^2 = 1 \); then

\[
\| \Xi(\psi) \|_{\ell_4}^4 = 1 + x^4 + y^4 + z^4.
\]

The vector generates a 4-design iff \( x^4 + y^4 + z^4 = 3/5 \) as pointed out in Sec. 4.1. Let \( \psi_T \) be the magic state with Bloch vector \( (1, 1, 1)/\sqrt{3} \) (which is also a SIC fiducial). Then fiducial vectors of 4-designs for \( n = 2, 3, 4 \) can be constructed as follows,

\[
\begin{cases}
\psi_T \otimes \psi(x, y, z), & x^4 + y^4 + z^4 = 5/7, \quad n = 2; \\
\psi_T^2 \otimes \psi(x, y, z), & x^4 + y^4 + z^4 = 7/11, \quad n = 3; \\
\psi_T^3 \otimes \psi(x, y, z), & x^4 + y^4 + z^4 = 8/19, \quad n = 4.
\end{cases}
\]

Many other constructions are also available.

In dimension 8, the set of Hoggar lines forms a SIC that is covariant with respect to the three-qubit Pauli group \( \{46, 86, 90\} \). One fiducial vector of the SIC is given by

\[
\psi_{\text{Hog}} = \frac{1}{\sqrt{6}} (1 + i, 0, -1, 1, -i, -1, 0, 0)^T.
\]

According to Eq. (79), \( \| \Xi(\psi_{\text{Hog}}) \|_{\ell_4}^4 = 16/9 \) attains the minimum over all three-qubit state vectors. This observation enables us to construct fiducial vectors of 4-designs for \( n = 4, 5 \),

\[
\begin{cases}
\psi_{\text{Hog}} \otimes \psi(x, y, z), & x^4 + y^4 + z^4 = 17/19, \quad n = 4; \\
\psi_{\text{Hog}} \otimes \psi_T \otimes \psi(x, y, z), & x^4 + y^4 + z^4 = 8/19, \quad n = 5.
\end{cases}
\]

The tensor-product construction of fiducial vectors of 4-designs also has a limitation. Consider tensor powers of \( \psi_T \) and \( \psi_{\text{Hog}} \) for example,

\[
\| \Xi(\psi_T^{\otimes n}) \|_{\ell_4}^4 = \left( \frac{4}{3} \right)^n, \\
\| \Xi(\psi_{\text{Hog}}^{\otimes n/3}) \|_{\ell_4}^4 = \left( \frac{16}{9} \right)^{n/3} = \left( \frac{4}{3} \right)^{2n/3} 3^n.
\]

As \( n \) increases, \( \| \Xi(\psi_T^{\otimes n}) \|_{\ell_4}^4 \) and \( \| \Xi(\psi_{\text{Hog}}^{\otimes n/3}) \|_{\ell_4}^4 \) increase exponentially with \( n \). By contrast, the value required for a 4-design approaches the constant 4. The following proposition clarifies this limitation; see Appendix F for a proof.
Proposition 8. Suppose a 4-design fiducial vector of the n-qubit Clifford group is a tensor product of \( m \geq 2 \) vectors \( \psi = \otimes_{j=1}^{m} \psi_j \), where \( \psi_j \) is an \( n_j \)-qubit state vector with \( \sum_j n_j = n \) and \( n_1 \geq n_2 \geq \cdots \geq n_m \). Then \( m \leq 3 \) except when \( n = 4 \) and \( n_1 = n_2 = n_3 = n_4 = 1 \). If \( m = 3 \), then \( n_2 = n_3 = 1 \), except when \((n_1,n_2,n_3) = (2,2,1)\) or \((n_1,n_2,n_3) = (3,2,1)\).

More explicitly, this proposition implies that \((n_1,n_2,\ldots,n_m)\) can only admit one of the following forms \((1,1,1,1)\), \((3,2,1)\), \((2,2,1)\) \((n_1,1,1)\), and \((n_1,n_2)\).

4.4 Algorithms for constructing projective 4-designs

In this section we present two algorithms for constructing fiducial vectors of 4-designs. Also presented is a method for constructing exact weighted 4-designs from two Clifford orbits.

Let \( \psi \) be an n-qubit state vector. Recall that \( \psi \) is a fiducial vector of a 4-design iff \( \|\Xi(\psi)\|_{\ell_4}^4 = 4d/(d+3) \) or, equivalently, if \( \epsilon(\psi) = 0 \); cf. Eq. (72).

The first algorithm is based on the tensor-product construction discussed in the previous section.

Algorithm 1:
1. Generate an \((n-1)\)-qubit state vector \( \psi_{n-1} \) such that \( \|\Xi(\psi_{n-1})\|_{\ell_4}^4 \leq 3d/(d+3) \), where \( d = 2^n \).
2. Let \( c = 4d/[d+3] \|\Xi(\psi_{n-1})\|_{\ell_4}^4 \). Choose a qubit state vector \( \psi \) with Bloch vector \((x,y,z)\) which satisfies \( x^4 + y^4 + z^4 = c-1 \). Then \( \psi_{n-1} \otimes \psi \) is a fiducial vector of a 4-design.

The vector required in Step 2 can always be found since \( 1/3 \leq c-1 \leq 2(d+2)/(d+3) - 1 < 1 \), given that \( 2d/(d+2) \leq \|\Xi(\psi_{n-1})\|_{\ell_4}^4 \leq 3d/(d+3) \), where the lower bound follows from Eq. (77).

In general, it is still not clear whether there exists an \((n-1)\)-qubit state vector \( \psi_{n-1} \) which satisfies \( \|\Xi(\psi_{n-1})\|_{\ell_4}^4 \leq 3d/(d+3) \), but we believe that the answer is positive. Actually, any eigenstate of a Singer unitary might satisfy the requirement; see the next section. In addition, one may try to minimize \( \|\Xi(\psi_{n-1})\|_{\ell_4}^4 \) numerically as in the search of SICs [68, 75]. Note that here the task is much simpler since the target \( 3d/(d+3) \) is much larger than the value \( 2d/(d+1) \) required for a SIC.

Given two n-qubit state vectors \( \psi_1, \psi_2 \) with \( \epsilon(\psi_1) > 0 \) and \( \epsilon(\psi_2) < 0 \), then any continuous curve of state vectors connecting \( \psi_1 \) and \( \psi_2 \) contains a 4-design fiducial vector. The following bisecction algorithm is based on this simple observation. Suppose \( \epsilon_0 \) is the target precision.

Algorithm 2:
1. Generate two state vectors \( \psi_1, \psi_2 \) such that \( \epsilon(\psi_1) > 0 \), \( \epsilon(\psi_2) < 0 \), and \( \langle \psi_1|\psi_2 \rangle \neq 0 \). Choose suitable phase factors so that \( \langle \psi_1|\psi_2 \rangle > 0 \).
2. Let \( \psi'_3 = (\psi_1 + \psi_2)/2 \) and \( \psi_3 = \psi'_3/\sqrt{|\langle \psi'_3|\psi'_3 \rangle|} \). Stop if \( |\epsilon(\psi_3)| \leq \epsilon_0 \).
3. If \( \epsilon(\psi_3) \geq 0 \), then replace \( \psi_1 \) with \( \psi_3 \); otherwise, replace \( \psi_2 \) with \( \psi_3 \). Repeat Steps 2,3.

Remark 2. A candidate for \( \psi_1 \) is any stabilizer state, while a potential candidate for \( \psi_2 \) is an eigenstate of a Singer unitary introduced in the next section. In Step 2 we may also use a weighted sum of \( \psi_1, \psi_2 \), say

\[ \psi'_3 = \frac{\epsilon(\psi_1)\psi_1 - \epsilon(\psi_2)\psi_2}{\epsilon(\psi_1) - \epsilon(\psi_2)}. \]

Given two n-qubit state vectors \( \psi_1, \psi_2 \) with \( \epsilon(\psi_1) > 0 \) and \( \epsilon(\psi_2) < 0 \) as above, we can also construct an exact weighted 4-design from two Clifford orbits. Note that

\[ \frac{|\epsilon(\psi_2)|}{|\text{orb}(\psi_1)|} \sum_{\phi \in \text{orb}(\psi_1)} (|\phi\rangle \langle \phi |)^{\otimes 4} + \frac{|\epsilon(\psi_1)|}{|\text{orb}(\psi_2)|} \sum_{\phi \in \text{orb}(\psi_2)} (|\phi\rangle \langle \phi |)^{\otimes 4} = \frac{P_4}{D_4}, \]

according to Corollary [1] and the definition of \( \epsilon(\psi) \) [c.f. Eq. (71)]. Therefore, the union of \( \text{orb}(\psi_1) \) and \( \text{orb}(\psi_2) \) forms an exact weighted 4-design provided that the vectors in \( \text{orb}(\psi_1) \) and that in \( \text{orb}(\psi_2) \) have the following weights respectively,

\[ \frac{|\epsilon(\psi_2)|}{|\text{orb}(\psi_1)||\epsilon(\psi_1) + |\epsilon(\psi_2)||} \quad \frac{|\epsilon(\psi_1)|}{|\text{orb}(\psi_2)||\epsilon(\psi_1) + |\epsilon(\psi_2)||}. \]

(99)

Similar construction also applies to more than two Clifford orbits.
4.5 Approximate fiducial vectors of 4-designs from MUB cycler

In this section we reveal an interesting connection between approximate 4-designs and eigenstates of certain special unitary transformations in the Clifford group. While these states and unitary transformations have been found useful in a number of contexts, the connection with 4-designs seems to be unexplored. We hope our preliminary observation will stimulate further progress.

Let \( \{\psi_{r,j}^n\} \) be a set of MUB \( [29] \), where \( r \) labels the basis, and \( j \) labels each element in a basis. A balanced state \( \psi \) with respect to \( \{\psi_{r,j}^n\} \) is a state that looks the same from every basis in the set, that is, the set of probabilities \( \{\langle \psi_{r,j}^n | \psi \rangle \} \) is independent of \( r \) \( [4, 7] \). If there exists a unitary operator that cycles through all the bases, then any eigenstate of the unitary operator is a balanced state. For example, the complete set of MUB constructed by Wootters and Fields \( [33] \) has a cycler when the dimension is a power of 2, that is, \( d = 2^n \). Here each MUB cycler is a special unitary transformation in the Clifford group, which is also known as a Singer unitary \( [29] \). The group generated by a Singer unitary is called a Singer unitary group. All Singer unitary groups are conjugated to each other in the Clifford group; in particular all of them have the same order of \( d+1 \) (modular phase factors). In addition, each Singer unitary has a nondegenerate spectrum, so the eigenbasis is well-defined. In the case of a qubit, each Singer unitary has order 3, and each eigenstate of a Singer unitary is a SIC fiducial and a magic state.

When \( n \) is a power of 2, a simple construction of Singer unitaries (MUB cyclers) was presented in Ref. \( [16] \). Here we are interested in constructing approximate fiducial vectors of 4-designs from the eigenvectors of a Singer unitary. For \( n = 1, 2, 4, 8 \), numerical calculation shows that all eigenvectors \( \psi_{n,j} \) of a Singer unitary for given \( n \) have the same value of the deviation parameter \( \epsilon(\psi_n) \) [cf. Eq. \( (71) \)]. Let \( \psi_T \) be a single qubit magic state vector. Calculation shows that

\[
-\epsilon(\psi_n \otimes \psi_T) = \begin{cases} 
\frac{2}{d} & n = 1, \\
0.12 & n = 2, \\
0.0312 & n = 4, \\
0.0020 & n = 8.
\end{cases}
\]  

The magnitude of the deviation \( \epsilon(\psi_n \otimes \psi_T) \) is around \( 1/2^{n+1} \), which has the same order of magnitude as the standard deviation of \( \epsilon(\psi) \) of a random \( (n+1) \)-qubit state vector \( \psi \); cf. Eq. \( (83) \). The orbit generated from \( \psi_n \otimes \psi_T \) is a very good approximation to a 4-design. Exact 4-design fiducial vectors can be constructed using algorithm 1 in the previous section. In addition, \( \psi_n \) or \( \psi_n \otimes \psi_T \) can serve as an input to Algorithm 2 presented in the previous section.

**Conjecture 1.** Suppose \( \psi_n \) is any eigenvector of a Singer unitary operator in the \( n \)-qubit Clifford group. Then

\[
\lim_{n \to \infty} \epsilon(\psi_n \otimes \psi_T) = 0.
\]  

This conjecture implies that the orbit generated by the \( (n+1) \)-qubit Clifford group from \( \psi_n \otimes \psi_T \) converges to a 4-design with respect to the operator norm as \( n \) grows. Equation \( (101) \) has several equivalent formulations; a succinct alternative reads

\[
\lim_{n \to \infty} \|\Xi(\psi_n)\|_{\ell_4} = 3.
\]  

4.6 Harmonic invariants, connections to the real-valued theory, and 5-designs

One original motivation \( [34] \) Section 1.E] for this work came from a result on the real Clifford group \( RC_n \). This is the group generated by tensor products of the real Pauli matrices \( \sigma_{(0,0)}, \sigma_{(0,1)}, \sigma_{(1,0)} \), together with the (real) Hadamard matrix

\[
H_R = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\]  

(103)

and the CNOT matrix as in \( [29] \) between each pair of qubits. In Refs. \( [77, 72, 61, 62] \), the authors studied invariant polynomials of \( RC_n \) and real spherical designs \( [25, 9] \) that appear as the group’s orbits.
Using methods from classical invariant theory, they showed \[21\] Corollary 4.13 that there are no invariant harmonic polynomials of \( RC_n \) of degree \( 2t \) for \( t = 1, 2, 3, 5 \), and – up to scalar multiples – a single harmonic invariant for \( t = 4 \) (c.f. Appendix C). It follows that the orbit of any vector forms a real spherical design of strength \( 2 \cdot 3 + 1 \). Furthermore, the orbit of any real root of the unique harmonic invariant of degree \( 2 \cdot 4 \) forms a spherical design of strength \( 2 \cdot 5 + 1 \). The existence of real roots follows from an averaging argument similar to the one we employ in Sec. 4.2.

References \[61, 62\] also treat the complex Clifford group \( C_n \). However, it seems that these works only characterize the invariant polynomials in \( \text{Hom}_{(2d)}(\mathbb{C}^d) \) rather than the ones in \( \text{Hom}_{(t,t)}(\mathbb{C}^d) \) investigated here (c.f. Appendix C). To the present authors, these two cases seem significantly different and we are not aware of any way that would allow one to directly apply the complex results from Refs. \[61, 62\] in our setting (however, see below for corollaries of their real-valued statements).

It was therefore an initial goal of this work to see whether methods from quantum information theory (such as stabilizer codes) could be used to find similar statements to the ones summarized above. The theory developed in the previous sections largely achieves this goal. The following proposition reformulates our results in a way that emphasizes the similarities.

**Proposition 9.** The Clifford group \( C_n \) has no non-trivial harmonic invariants of degrees \((1,1),(2,2)\), or \((3,3)\). All harmonic invariants of degree \((4,4)\) are multiples of \( \epsilon \) as defined in Eq. (71). The orbit of a normalized vector \( \psi \) forms a 4-design if and only if it is a root of \( \epsilon \).

**Proof.** Because the Clifford group forms a unitary 3-design, it follows that for \( t = 1, 2, 3 \), the commutant \( L(\text{Sym}_t(\mathbb{C}^d))C_n \) of \( C_n \) acting on \( \text{Sym}_t(\mathbb{C}^d) \) is given by multiples of \( P_t \). By Eqs. (189) and (190) in Appendix C these are just the embeddings of \( H_{(0,0)} \) into \( L(\text{Sym}_t(\mathbb{C}^d)) \) (corresponding to the polynomials \( \psi \mapsto \|\psi\|_{t}^{2} \)). This proves the first part.

From this and (189), we have that
\[
L(\text{Sym}_4(\mathbb{C}^d))C_n \simeq H_{(0,0)}^{C_n} \oplus H_{(4,4)}^{C_n} = H_{(0,0)} \oplus H_{(4,4)}^{C_n}.
\]

At the same time, Theorem 1 implies that \( L(\text{Sym}_4(\mathbb{C}^d))C_n \) is spanned by the two projectors \( P_\pm \) onto \( W_4^{\pm} \) defined in Sec. 4.2. As in the first part, \( P_4 = P_+ + P_- \) spans \( H_{(0,0)} \). Clearly, the operator
\[
A := \frac{D_4}{D_+} P_+ - P_4
\]
is an element of the commutant and orthogonal to \( P_4 \). As such, \( A \) must span \( H_{(4,4)}^{C_n} \). But \( \epsilon \) is the polynomial \( p_A \) associated with \( A \) in the sense of Lemma 14.

The final statement of Proposition 9 is just Eq. (72). \( \square \)

The results on the real Clifford group mentioned above strongly suggest upper bounds on the dimensions of the spaces of harmonic invariants of \( C_n \). Indeed, up to slightly different phase conventions for the Hadamard gate [Eq. (25) vs Eq. (103)], which are inmaterial for the present discussion, the real Clifford group \( RC_n \) is a subgroup of the complex one \( C_n \). Now let \( p_A \in \text{(Harm}_{(t,t)} \text{)} C_n \) be a \( C_n \)-invariant polynomial. It is clearly also invariant under any subgroup of \( C_n \), in particular, under \( RC_n \). Let \( A = A_R + iA_I \), for \( A_R, A_I \) real matrices be the decomposition of \( A \) into its real and imaginary part. Since the action of \( RC_n \) does not mix the real and the imaginary components, it follows that the restrictions of \( p_A \) to real arguments are \( RC_n \)-invariant polynomials. Using (191), they can easily be checked to lie in the harmonic space \( H_{2d}(\mathbb{R}^d) \). (The the restriction of \( p_A \) to real arguments also gives a real polynomial – but it need not be harmonic, even if \( p_A \) was.) We can therefore convert invariant harmonic polynomials of \( C_n \) into those of \( RC_n \). Unfortunately, the resulting real polynomials may turn out to be zero: In the language of Ref. 25 (Appendix C), it could happen that the matrices \( A_I, A_R \) – while elements of \( L(\text{Sym}_t(\mathbb{R}^d)) \) – are orthogonal to the totally symmetric matrices \( \text{MSym}_t(\mathbb{R}^d) \). This technical problem prevents us from directly inferring the absence of harmonic invariants of \( C_n \) of bi-degree \((t,t)\) for \( t = 1, 2, 3, 5 \) from the absence of real harmonic invariants of \( RC_n \) of degree \( 2 \cdot t \) for the same \( t \)'s.

We conjecture, however, that this potential problem is not realized for the Clifford group, at least not for degree \((5,5)\). In this case, \[21\] Corollary 4.13 – stating the absence of harmonic invariants of \( RC_n \) of degree \( 2 \cdot 5 \) – would imply the following:
Conjecture 2. Let $\psi$ be a normalized vector. Its orbit forms a complex projective 5-design iff it is a root of $\epsilon$.

There are several pieces of evidence in favor of this conjecture:

1. Conjecture 2 holds when $n = 1$ according to the discussion in Sec. [13]. It is also supported by numerical calculation when $n = 2$.
2. The argument works for $t = 1, 2, 3$.
3. The set of matrices in $L(\text{Sym}_i(\mathbb{R}^d))$ that is orthogonal to $\text{MSym}_i(\mathbb{R}^d)$ is of measure zero.

It would be interesting to verify this conjecture, as well as to re-prove the statement of Ref. [61] using just the tools of the present paper.

Even if a simple way of turning general harmonic invariants of the complex Clifford group into those for the real Clifford group could be constructed, the results of the present work and those of Refs. [77, 72, 61, 62] would still differ in scope. On the one hand, our results are stronger, as they allow for a decomposition of the entire space $(\mathbb{C}^d)^\otimes 4$ under $C_n$, as opposed to just the totally symmetric subspace. On the other hand, the cited references are stronger by giving a characterization of the invariant polynomials of any degree (in terms of weight enumerator polynomials of certain binary codes), while we restrict attention to degree 4.

5 Summary

The most prominent unitary $t$-design considered in quantum information is the multi-qubit Clifford group, which is a unitary 3-design, but, unfortunately, not a 4-design. Accordingly, Clifford orbits are 3-designs, but generally not 4-designs. The lack of an explicit family of well structured 4-designs has been a major limitation in the applications of $t$-designs for derandomizing constructions that rely on random vectors.

In this work we showed that although Clifford orbits do not constitute 4-designs, their 4th moments are well-behaved such that for several major applications, including phase retrieval and quantum state discrimination, typical Clifford orbits turn out to perform as well as 4-designs or Gaussian random vectors would. Moreover, we gave various constructions of exact 4-designs and approximations of arbitrarily high precision to serve for more demanding applications. In order to achieve this goal, we determined all irreducible components that appear in the 4th tensor power of the Clifford group. It turns out that the structure of these representations is completely captured by Schur-Weyl duality and a special stabilizer code. In addition to the applications mentioned above, our results may help construct exact unitary 4-designs or better approximations. In the course of our study, we also discovered several results concerning the representations of the discrete symplectic group, which may be of interest to pure mathematician.

Our work also leaves several open problems, which deserve further study.

1. Is there any orbit of the Clifford group that forms a $t$-design for $t > 4$ (c.f. Conjecture 2)? The answer is positive when $n = 1$. It seems that the same could hold for larger $n$.
2. What is the maximal $t$ such that there is an orbit of the Clifford group that forms a $t$-design. The answer is 7 when $n = 1$. How about approximate $t$-designs?
3. Prove Conjecture [11]
4. Construct unitary 4-designs based on the Clifford group.

More generally, it would be desirable to give an explicit description of the commutant of higher tensor powers of the Clifford group – maybe similar to the characterization of invariant polynomials of the Clifford group in terms of weight enumerator polynomials described in Refs. [72, 61, 62]. There is a potentially simpler problem. Central to our construction was the stabilizer projector $P_{n,k}$. It belongs to a stabilizer code in $(\mathbb{C}^d)^\otimes 4$ that is not a tensor product itself. Similarly, the recent work Ref. [63] identifies an element of the commutant of a tensor power of the Clifford group, that is itself a non-factoring Clifford operation on the tensor product space. If a general explicit description of the commutant of powers of the Clifford group might not be realistically available, one could ask how far one can go by classifying those commuting elements that can themselves be expressed in terms of Clifford theory or related constructions.
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A Alternative proof of Lemma 1

In this appendix, we present an alternative approach for computing the dimensions of $W^+_\lambda$ defined in Sec. 3.2, thereby yielding an alternative proof of Lemma 1. In the course of study, we also construct explicit orthonormal bases for $W^+_4$ and $W^+_{[14]}$.

To achieve our goal, we first construct an orthonormal basis for $V_{n,4}$ and determine the orbits of basis elements under the action of the symmetric group $S_4$. When $n = 1$, one orthonormal basis of $V_{1,4}$ is composed of the following four states,

$$\phi_0 := \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$

$$\phi_1 := \frac{1}{\sqrt{2}}(|1001\rangle + |0110\rangle),$$

$$\phi_2 := \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle),$$

$$\phi_3 := \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle).$$

The symmetric group $S_4$ (permuting the four tensor factors) fixes $\phi_0$ and acts like $S_3$ on $\phi_1, \phi_1, \phi_2$. Since $V_{n,4} = V^\otimes 1,4 \otimes V^\otimes n_{1,4}$ for general $n$, one orthonormal basis of $V_{n,4}$ is composed of the following $4^n$ states,

$$|\phi_{i_1,i_2,\ldots,i_n}\rangle = |\phi_{i_1}\rangle \otimes \cdots \otimes |\phi_{i_n}\rangle, \quad i_1, i_2, \ldots, i_n \in \{0,1,3,4\}.$$ (104)

Here each state in the basis is labeled by a length-$n$ string $i_1, \ldots, i_n$ with $i_j \in \{0,1,3,4\}$. Each permutation in the symmetric group $S_4$ induces a permutation on the basis states and a corresponding permutation on the strings, which acts on all letters simultaneously. The orbits on the strings divide into three types as described as follows.

1. One orbit containing $0^n$, referred to as type I orbit below.
2. Any string in $\{0,i\}^n$ (for given $i \in \{1,2,3\}$) excluding $0^n$ generates an orbit of length 3. There are $2^n - 1$ such orbits of length 3, referred to as type II orbits below.
3. The remaining strings have either two or three distinct non-zero letters and are partitioned into orbits of length 6, referred to as type III orbits below. The number of such orbits is

$$\frac{4^n - 3 \times 2^n + 2}{6} = \frac{(2^n - 2)(2^n - 1)}{6} = \frac{(d - 2)(d - 1)}{6}.$$ (106)

The total number of orbits is

$$2^n + \frac{4^n - 3 \times 2^n + 2}{6} = \frac{4^n + 3 \times 2^n + 2}{6} = \frac{(2^n + 2)(2^n + 1)}{6} = \frac{(d + 2)(d + 1)}{6}.$$ (107)

The strings corresponding to the three types of orbits are referred to as type I, II, III strings, respectively. The stabilizer of a type I string is $S_4$, that of a type II string is a Sylow-2 subgroup of $S_4$, and that of a type III string is the unique order-4 normal subgroup of $S_4$. 

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Now we are ready to compute the dimensions of $W^+_s$. Let $\text{orb}(s)$ denote the orbit of the string $s \in \{0, 1, 2, 3\}^n$ under the action of $S_4$. According to Eq. (104),

$$P_\lambda|\phi_s\rangle = \frac{d_\lambda}{24} \sum_{\sigma \in S_4} \chi_\lambda(\sigma)|U_\sigma|\phi_s\rangle = \frac{d_\lambda}{24} \sum_{r \in \text{orb}(s)} \left( \sum_{\sigma \in S_4 \mid \sigma(s) = r} \chi_\lambda(\sigma) \right)|\phi_r\rangle,$$

(108)

where $\sum_{\sigma \in S_4 \mid \sigma(s) = r} \chi_\lambda(\sigma)$ is the sum of $\chi_\lambda(\sigma)$ over a coset of the stabilizer of $s$. For example,

$$P_\lambda|\phi_{0\cdots 0}\rangle = \begin{cases} |\phi_{0\cdots 0}\rangle & \lambda = [4], \\ 0 & \text{otherwise.} \end{cases}$$

(109)

When $\lambda = [4]$, $d_\lambda = 1$ and $\chi_\lambda(\sigma) = 1$ for all $\sigma \in S_4$. Consequently,

$$P_{[4]}|\phi_s\rangle = \frac{1}{|\text{orb}(s)|} \sum_{r \in \text{orb}(s)} |\phi_r\rangle.$$

(110)

Note that $P_{[4]}|\phi_s\rangle \in W^+_s$ only depends on $\text{orb}(s)$ and that the states corresponding to different orbits are orthogonal. Let $\mathcal{S}$ be a subset of $\{0, 1, 2, 3\}^n$ that contains exactly one string from each orbit. Then the set

$$\{ \sqrt{|\text{orb}(s)|} P_{[4]}|\phi_s\rangle \mid s \in \mathcal{S} \}$$

(111)

forms an orthonormal basis for $W^+_s$. In particular, the dimension of $W^+_s$ is equal to the total number of orbits of strings, that is,

$$D^+_s = \dim(W^+_s) = \frac{(d + 2)(d + 1)}{6}.$$  

(112)

Now consider the subspace $W^+_1$. Note that $P_{[1]}|\phi_s\rangle = 0$ when $s$ is an type I or type II string. An orthonormal basis for $W^+_1$ is given by

$$\{ \sqrt{|\text{orb}(s)|} P_{[1]}|\phi_s\rangle \mid s \in \mathcal{S} \text{ is of type III} \}.$$  

(113)

The dimension of $W^+_1$ is equal to the number of type III orbits, that is,

$$D^+_1 = \dim(W^+_1) = \frac{(d - 2)(d - 1)}{6}.$$  

(114)

It is more involved to compute the dimension of $W^+_2$. Fortunately, this task can be avoided if we can compute the dimensions of $W^+_{[2,1,1]}$ and $W^+_{[3,1]}$. It turns out that $P_{[2,1,1]}|\phi_s\rangle = 0$ and $P_{[3,1]}|\phi_s\rangle = 0$ for all strings $s \in \{0, 1, 2, 3\}^n$. This conclusion follows from Eq. (109) when $s$ is a type I string, that is $s = 0 \cdots 0$. When $s$ is a type II or III string, this conclusion follows from Eq. (108) and Lemmas 7, 8 below, recall that the stabilizer of a type II string is a Sylow-2 subgroup of $S_4$ and that of a type III string is the unique order-4 normal subgroup of $S_4$. Consequently, both $W^+_{[2,1,1]}$ and $W^+_{[3,1]}$ have dimension 0, so that

$$D^+_{[2,1,1]} + D^+_{[3,1]} = d^2/3,$$

which implies that $D^+_{[2,2]} = (d^2 - 1)/3$.

**Lemma 7.** Let $G$ be a Sylow 2-subgroup of $S_4$. Then $\sum_{\sigma \in gG} \chi_\lambda(\sigma) = \sum_{\sigma \in G} \chi_\lambda(\sigma) = 0$ for $\lambda = [2, 1, 1], [3, 1]$ and all $g \in S_4$.

**Lemma 8.** Let $H$ be the unique order-4 normal subgroup of $S_4$. Then $\sum_{\sigma \in gH} \chi_\lambda(\sigma) = \sum_{\sigma \in H} \chi_\lambda(\sigma) = 0$ for $\lambda = [2, 1, 1], [3, 1]$ and all $g \in S_4$. 

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Suppose \( \lambda = [2,1,1] \) or \( \lambda = [3,1] \). Note that \( G \) is isomorphic to the order-8 dihedral group; it has one element of cycle type \((1^4)\), three elements of cycle type \((2^2)\), two elements of cycle type \((2,1^3)\), and two elements of cycle type \((4)\). Therefore, \( \sum_{\sigma \in G} \chi_\lambda(\sigma) = 0 \) according to Table 2. Let \( g \) be any order-3 element in \( S_4 \); then \( G, gG, g^{-1}G \) are three distinct left cosets of \( G \). Since \( \sum_{\sigma \in G} \chi_\lambda(\sigma) = 0 \) and \( \sum_{\sigma \in S_4} \chi_\lambda(\sigma) = 0 \), it follows that

\[
\sum_{\sigma \in gG} \chi_\lambda(\sigma) + \sum_{\sigma \in g^{-1}G} \chi_\lambda(\sigma) = 0.
\] (116)

On the other hand, by conjugation \( G \) acts transitively on the eight order-3 elements in \( S_4 \), so there exists an element \( h \) in \( G \) such that \( hgh^{-1} = g^{-1} \), that is, \( hgGh^{-1} = g^{-1}G \). It follows that

\[
\sum_{\sigma \in gG} \chi_\lambda(\sigma) = \sum_{\sigma \in g^{-1}G} \chi_\lambda(\sigma),
\] (117)

which, together with Eq. (116), implies that

\[
\sum_{\sigma \in gG} \chi_\lambda(\sigma) = \sum_{\sigma \in g^{-1}G} \chi_\lambda(\sigma) = 0.
\] (118)

In conclusion, \( \sum_{\sigma \in gG} \chi_\lambda(\sigma) = 0 \) for \( \lambda = [2,1,1], [3,1] \) and all \( g \in S_4 \). The equality \( \sum_{\sigma \in gG} \chi_\lambda(\sigma) = 0 \) follows from the same reasoning.

**Proof of Lemma 8.** Suppose \( \lambda = [2,1,1] \) or \( \lambda = [3,1] \). Note that \( H \) has one element of cycle type \((1^4)\) and three elements of cycle type \((2^2)\). Therefore, \( \sum_{\sigma \in H} \chi_\lambda(\sigma) = 0 \) according to Table 2. The symmetric group \( S_4 \) has three Sylow 2-subgroups, each of which is the union of two cosets of \( H \). Let \( G \) be any Sylow 2-subgroup of \( S_4 \), then \( \sum_{\sigma \in G} \chi_\lambda(\sigma) = 0 \) according to Lemma 4, which implies that \( \sum_{\sigma \in G \setminus H} \chi_\lambda(\sigma) = 0 \).

Let \( g_j \) for \( j = 1,2,3,4,5,6 \) be the coset representatives of \( H \), with \( g_1 \) being the identity. Above analysis shows that \( \sum_{\sigma \in g_j H} \chi_\lambda(\sigma) = 0 \) for four of the six cosets, say, \( j = 1,2,3,4 \), so that

\[
\sum_{\sigma \in g_j H} \chi_\lambda(\sigma) = \sum_{\sigma \in g_6 H} \chi_\lambda(\sigma) = 0.
\] (119)

In addition, the two coset representatives \( g_5, g_6 \) necessarily have order 3 since otherwise they would belong to certain Sylow 2-subgroups of \( S_4 \). Observing that \( H \) is normal in \( S_4 \) and that all order-3 elements in \( S_4 \) are conjugated to each other, we conclude that

\[
\sum_{\sigma \in g_5 H} \chi_\lambda(\sigma) = \sum_{\sigma \in g_6 H} \chi_\lambda(\sigma),
\] (120)

which, together with Eq. (119), implies that

\[
\sum_{\sigma \in g_5 H} \chi_\lambda(\sigma) = \sum_{\sigma \in g_6 H} \chi_\lambda(\sigma) = 0.
\] (121)

In conclusion, \( \sum_{\sigma \in gH} \chi_\lambda(\sigma) = 0 \) for \( \lambda = [2,1,1], [3,1] \) and all \( g \in S_4 \). As an immediate consequence, \( \sum_{\sigma \in Hg} \chi_\lambda(\sigma) = 0 \) since left cosets and right cosets of \( H \) coincide. 

Remark 3. Lemma 7 follows from Lemma 8 since each coset of a Sylow 2-subgroup of \( S_4 \) is a union of two cosets of the unique order-4 normal subgroup of \( S_4 \). The two lemmas can be verified directly based on Table 2. Nevertheless, we shall present more instructive proofs below.
**B. Two natural sets of vectors in the stabilizer code \( V_{n,4} \)**

**B.1 An interesting basis for the stabilizer code**

Recall the basis-dependent vectorization map which sends matrices to tensors

\[
\text{vec}: L(\mathbb{C}^d) \rightarrow \mathbb{C}^d \otimes \mathbb{C}^d \\
\sum_{i,j} L_{i,j} |e_i \rangle \langle e_j| \mapsto \sum_{i,j} L_{i,j} |e_i \rangle \otimes |e_j \rangle.
\]

It fulfills

\[
\text{vec}(ABC) = A \otimes C^T \text{vec}(B),
\]

where the transpose is to be taken with respect to the same basis in which the vectorization map is defined.

With this notion, note that

\[
B := \{ \text{vec}(W_a) \otimes \text{vec}(W_a) \mid a \in \mathbb{F}_2^{2n} \}
\]

defines a set of \( d^2 \) orthogonal vectors in \((\mathbb{C}^d)^{\otimes 4}\). One easily verifies that \( B \) is contained in the stabilizer code \( V_{n,4} \):

\[
\tau^4(W_b) \left( \text{vec}(W_a) \otimes \text{vec}(W_a) \right) = \text{vec}(W_b W_a W_b^T) \otimes \text{vec}(W_b W_a W_b^T) \\
= (\pm \text{vec}(W_a)) \otimes (\pm \text{vec}(W_a)) \\
= \text{vec}(W_a) \otimes \text{vec}(W_a).
\]

It thus forms an orthogonal basis of the code. A similar calculation shows that the real elements of the Clifford group act on this basis by permutation

\[
\tau^4(U) \left( \text{vec}(W_a) \otimes \text{vec}(W_a) \right) = \text{vec}(W_{Fa}) \otimes \text{vec}(W_{Fa}),
\]

where \( F \in \text{Sp}(2n, \mathbb{F}_2) \) is the symplectic map associated with \( U/P_n \). Complex elements of the Clifford group \( C_n \) still act by signed permutation on the basis – i.e. they permute the elements and may multiply them with signs \( \pm 1 \). The latter fact can be verified explicitly by inspecting the action of those generators of the Clifford group as discussed in Sec. 2.3, all of which are real except for the phase gate. In particular, all Clifford unitaries act monomially. The above discussion also implies that the representation of the symplectic group \( \text{Sp}(2n, \mathbb{F}_2) \) afforded by the stabilizer code \( V_{n,4} \) is a signed permutation representation in the above basis.

We have not used this basis affording a monomial representation of the Clifford group and the symplectic group in the present paper. However, we speculate that one might use it to give a more explicit derivation of the characters described in Sec. 3.4.

**B.2 An interesting orbit**

Here, we describe a Clifford orbit of vectors in \( W^+_{[4]} \) that is naturally labeled by isotropic subspaces \( M \subset \mathbb{F}_2^{2n} \) of dimension \( \dim M = n \) (such spaces are called maximally isotropic). The authors are not aware of any application of this particular configuration of vectors. However, we feel that the construction is sufficiently canonic to deserve a mention.

Choose a maximally isotropic space \( M \subset \mathbb{F}_2^{2n} \) and define

\[
S'_n(M) := \{ W_a \otimes W_a \otimes I \otimes I \mid a \in M \}, \\
S''_n(M) := \{ W_a \otimes I \otimes W_a \otimes I \mid a \in M \}.
\]

\(^1\) Certain monomial representations of the Clifford group have been studied before in Ref. [6]. However, their results are incomparable to our findings, as they classified monomial representations that also contain a faithful representation of the Pauli group \( P_n \).
Then the union $S_{n,4} \cup S_{n,4}' \cup S_{n,4}''$ generates a maximal stabilizer group on $4n$ qubits and thus determines a stabilizer state $|\psi_M\rangle \in V_{n,4}$, where $S_{n,4}$ is defined in Eq. (33).

Consider the concrete example

$$M_Z = \{(p_1,0,p_2,0,\ldots,p_n,0) \mid p_1,p_2,\ldots,p_n \in \mathbb{F}_2\}.$$ 

Then $W_a$ for $a \in M_Z$ is an element of $\{\sigma_{(0,0)},\sigma_{(1,0)}\}^\otimes n$. In this particular case, one verifies that

$$|\psi_{M_Z}\rangle = \frac{1}{2^{n/2}} \sum_{x \in \mathbb{F}_2^n} |x\rangle^\otimes 4 = \frac{1}{2^{n/2}} |\phi_0\rangle^\otimes n \in H^+_4 \cong W^+_4,$$

where $|\phi_0\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$.

Now consider a Clifford unitary $U \in C_n$, associated with the symplectic transformation $F$. Then, for any maximally isotropic subspace $M$ of $\mathbb{F}_2^{2n}$,

$$\tau^4(U)S_n(M)\tau^4(U)^\dagger = S_n(FM)$$

and the same is true for $S_n''(M)$. We conclude that

$$\tau^4(U)|\psi_M\rangle \propto |\psi_{FM}\rangle.$$

Because the symplectic group acts transitively on maximal isotropic subspaces, the Clifford group acts monomially (i.e. by permutation and possibly multiplication with a phases) on the set

$$X := \{|\psi_M\rangle \mid M \text{ max. isotropic}\}.$$

As $X$ is – up to phases – an orbit of the Clifford group generated from $|\psi_{M_Z}\rangle \in W^+_4$, the entire set is contained in $W^+_4$.

Because the projectors $|\psi_M\rangle\langle\psi_M|$ form an orbit under the action by conjugation of the irreducible representation of $C_n$ on $W^+_4$, it follows from Schur’s Lemma that $X$ is a tight frame on that space. It is known (e.g. [34]) that

$$|X| = |\{\text{max. isotropic subspaces of } \mathbb{F}_2^{2n}\}| = \prod_{i=1}^n (2^i + 1),$$

which implies that $|X| > D^+_{[4]}$. If $n \geq 3$, then

$$|X| = 3D^+_{[4]} \prod_{i=1}^{n-2} (2^i + 1).$$

So $X$ is overcomplete as a frame. We can compute the squared inner products between elements of $X$ from the intersection of their stabilizer groups (c.f. e.g. Ref. [54]):

$$|\langle \psi_M | \psi_N \rangle|^2 = \frac{1}{2^{4n}} 2^{2n+2 \dim(M \cap N)} = \left(2^{\dim(M \cap N) - n}\right)^2.$$

That number is the square of what one would obtain for the overlap-squared between $n$ qubit stabilizer states taken from bases associated with, respectively, $M$ and $N$ [54].

**C Proof of a generalization of Eq. (76)**

Here we prove a generalization of Eq. (76), which also provides some insight on the entanglement property of the stabilizer code $V_{n,4}$. 

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Lemma 9. Suppose \( \psi_j \) for \( j = 1, 2, 3, 4 \) are four normalized state vectors in dimension \( d = 2^n \). Then

\[
0 \leq \text{tr} \left( P_{n,4} \bigotimes_{j=1}^{4} |\psi_j \rangle \langle \psi_j | \right) \leq \frac{1}{d^2}. \tag{122}
\]

If the upper bound is saturated, then \( \psi_j \) for \( j = 1, 2, 3, 4 \) are stabilizer states that belong to a same stabilizer basis.

The upper bound in Eq. (122) means that the stabilizer code \( V_{n,4} \) contains no product state. Moreover, it sets an upper bound \( 1/d \) for the fidelity between any pure state in \( V_{n,4} \) and any product state, that is, a lower bound for the geometric measure of entanglement of any pure state in \( V_{n,4} \). The upper bound can be saturated if \( \psi_j \) for \( j = 1, 2, 3, 4 \) are identical stabilizer states, but this is not necessary. For example, it can also be saturated if \( \psi_1 = \psi_2 \) and \( \psi_3 = \psi_4 \) are two orthogonal stabilizer states that belong to a same stabilizer basis. By contrast, the lower bound in Eq. (122) can be saturated if \( \psi_1 = \psi_2 = \psi_3 \) and \( \psi_4 \) are two orthogonal stabilizer states that belong to a same stabilizer basis.

Proof. The lower bound is trivial since both \( P_{n,4} \) and \( \bigotimes_{j=1}^{4} |\psi_j \rangle \langle \psi_j | \) are positive semidefinite. The upper bound can be derived as follows.

\[
d^2 \text{tr} \left( P_{n,4} \bigotimes_{j=1}^{4} |\psi_j \rangle \langle \psi_j | \right) = \sum_{a} \prod_{j=1}^{4} \langle \psi_j | W_a | \psi_j \rangle \leq \left[ \prod_{j=1}^{4} \sum_{a} \langle \psi_j | W_a | \psi_j \rangle \right]^{1/4} \leq d. \tag{123}
\]

Here the first inequality follows from repeated applications of the Cauchy inequality or the Hölder inequality. It is saturated if

\[
|\langle \psi_1 | W_a | \psi_1 \rangle| = |\langle \psi_2 | W_a | \psi_2 \rangle| = |\langle \psi_3 | W_a | \psi_3 \rangle| = |\langle \psi_4 | W_a | \psi_4 \rangle|, \quad \prod_{j=1}^{4} \langle \psi_j | W_a | \psi_j \rangle \geq 0 \quad \forall a. \tag{124}
\]

The second inequality in Eq. (123) is saturated if each \( \langle \psi_j | W_a | \psi_j \rangle \) takes on only one of the three values 0, \( \pm 1 \). In that case, each \( \Xi(\psi_j) \) (recall that \( \Xi_a(\psi_j) = \langle \psi_j | W_a | \psi_j \rangle \)) has exactly \( d \) entries equal to \( \pm 1 \), given that \( \sum_{a} (\langle \psi_j | W_a | \psi_j \rangle)^2 = d \) for \( j = 1, 2, 3, 4 \). Note that the set \( \{ a \in \mathbb{F}_2^n \mid \langle \psi_j | W_a | \psi_j \rangle = \pm 1 \} \) for given \( j \) must form a maximal isotropic subspace of the symplectic vector space \( \mathbb{F}_2^n \), so each \( \psi_j \) is an eigenvector of a stabilizer group and is thus a stabilizer state by definition. If the two inequalities in Eq. (123) are saturated simultaneously, then \( \psi_j \) for \( j = 1, 2, 3, 4 \) must be eigenvectors of a common stabilizer group due to Eq. (124). In other words, they belong to a same stabilizer basis.

D Derivation of Eq. (81)

In this appendix, we derive the second moment of \( \alpha_+(\psi) \), as presented in Eq. (81).

\[
E[\alpha_+(\psi)^2] = \frac{1}{D[8]} \text{tr}(P_{n,4}^{\otimes 2} P[8]) = \frac{1}{d^2 D[8]} \sum_{a,b} \text{tr} \left[ P[8](W_a^{\otimes 4} \otimes W_b^{\otimes 4}) \right] = \frac{16(d^2 + 15d + 68)}{d^2(d+3)(d+5)(d+6)(d+7)}. \tag{125}
\]

where \( P_{n,4} \) is the projector onto the stabilizer code \( V_{n,4} \) discussed in Sec. 3.1, \( P[8] \) is the projector onto the \( k \)-partite symmetric subspace \( \text{Sym}_k(\mathbb{C}^d) \) of \( (\mathbb{C}^d)^{\otimes k} \) with \( d = 2^n \), \( D[k] \) is the rank of \( P[k] \) or the dimension.
of odd length are listed in Table 3. Note that $A$ is balanced if each cycle involves even number of parties both in the first four parties and in the second four parties. Define $A$ as the subset of balanced permutations in $S_8$. The sets $A$ and $A_\pm$ are defined in the text. Note that $N_{3+} + N_{3-} = N_3$.

| cycle type | $(2^4)$ | $(2^2, 4)$ | $(4^2)$ | $(2, 6)$ | $(8)$ |
|------------|---------|------------|---------|---------|------|
| $N_1$      | 105     | 1260       | 1260    | 3360    | 5040 |
| $N_2$      | 9       | 252        | 684     | 1440    | 5040 |
| $N_3$      | 9       | 108        | 108     | 288     | 432  |

Table 3: Permutations of $S_8$ without cycle of odd length. $N_1$ is the number of permutations of a given cycle type; $N_2$ is the number of balanced permutations (those in $A'$) of a given cycle type; $N_3 = N_{3+} - N_{3-}$, where $N_{3+}$ is the number of permutations of a given cycle type that belong to $A'_\pm$. The sets $A'$ and $A'_\pm$ are defined in the text.

The first case in Eq. (126) is trivial. When $W_b = 1, W_a \neq 1$, the result follows from Eqs. (127), (128), and (132). To derive Eq. (126), we recall the following facts,

$$P_{[k]} = \frac{1}{k!} \sum_{\sigma \in S_k} U_\sigma, \quad \text{tr}_k P_{[k]} = \frac{D_{[k]}}{D_{[k-1]}} P_{[k-1]},$$

(127)

where $\text{tr}_k$ means the partial trace over party $k$. If $a \neq 0$, then

$$\text{tr}(U_\sigma W_a^{\otimes k}) = \begin{cases} 0 & \sigma \text{ contains a cycle of odd length}, \\ d^{l(\sigma)} & \text{otherwise}. \end{cases}$$

(128)

where $l(\sigma)$ is the number of cycles in $\sigma$ of even lengths. The cycle types of elements in $S_8$ without cycle of odd length are listed in Table 3.

The first case in Eq. (126) is trivial. When $W_b = 1, W_a \neq 1$,

$$\text{tr} \left[ P_{[8]} (W_a^{\otimes 4} \otimes W_b^{\otimes 4}) \right] = \frac{D_{[8]}}{D_{[4]}} \text{tr} \left( P_{[4]} W_a^{\otimes 4} \right) = \frac{D_{[8]} 3d^2 + 6d}{D_{[4]} 24},$$

(129)

call that the symmetric group $S_8$ has three permutations of cycle type $(2^4)$, six permutations of cycle type $(4)$, and all other permutations contain at least one cycle of odd length (cf. Sec. 2). The case $W_a = 1, W_b \neq 1$ has the same result. When $W_b = W_a \neq 1$, the result follows from Eqs. (127), (128), and Table 3.

To settle the last two cases in Eq. (126), we need to introduce some terminology. A permutation in $S_8$ is balanced if each cycle involves even number of parties both in the first four parties and in the second four parties. Define $A$ as the subset of balanced permutations in $S_8$. Each permutation in $S_8$ induces a permutation on the vector $v = (a, a, a, b, b, b, b)$. Define

$$A_+ = \{ \sigma \in A | \sigma \text{ induces even number of transpositions between } a \text{ and } b \},$$

(130)

$$A_- = \{ \sigma \in A | \sigma \text{ induces odd number of transpositions between } a \text{ and } b \}.$$  

(131)

Note that $A = A_+ \cup A_-$. If $W_b, W_a \neq 1$, $W_b \neq W_a$, and $W_b W_a = W_a W_b$, then

$$\text{tr} \left[ U_\sigma (W_a^{\otimes 4} \otimes W_b^{\otimes 4}) \right] = \begin{cases} d^{l(\sigma)} & \sigma \in A, \\ 0 & \sigma \notin A. \end{cases}$$

(132)
If \( W_b W_a = -W_a W_b \), then
\[
\text{tr} \left[ U_\sigma (W_a^\otimes 4 \otimes W_b^\otimes 4) \right] = \begin{cases} 
4^{d(\sigma)} & \sigma \in \mathcal{A}^+, \\
-4^{d(\sigma)} & \sigma \in \mathcal{A}^-, \\
0 & \text{otherwise.}
\end{cases}
\]  
(133)

Now the last two cases in Eq. (126) can be determined by virtue of Table 3 and the above two equations.

### E Proof of Lemma 6

**Proof.** Let \( X = |\psi\rangle\langle\psi| - |\varphi\rangle\langle\varphi| \) and suppose \( X \) has spectral decomposition \( X = \lambda (|\mu\rangle\langle\mu| - |\nu\rangle\langle\nu|) \) with \( 0 \leq \lambda \leq 1 \). Then \( \|X\|_1 = 2\lambda, \|X\|_2 = \sqrt{2}\lambda \), and we have
\[
\|\psi - \varphi\|_2^2 = 2 - \langle \psi|\varphi \rangle - \langle \varphi|\psi \rangle \geq 2 - 2\|\psi|\varphi \| \leq 2(\|\psi|\varphi \|) \leq \|\psi - \varphi\|_2^2,
\]
(134)
which implies that \( \lambda \leq \|\psi - \varphi\|_2 \). In addition,
\[
\alpha_+ (\psi) - \alpha_+(\varphi) = \text{tr} \left[ P_{n,4} (|\psi\rangle\langle\psi|)^\otimes 4 - \text{tr} \left[ P_{n,4} (|\varphi\rangle\langle\varphi|)^\otimes 4 \right] \right] \\
= \text{tr} \left[ P_{n,4} (|\varphi\rangle\langle\varphi| + X)^\otimes 4 \right] - \text{tr} \left[ P_{n,4} (|\varphi\rangle\langle\varphi|)^\otimes 4 \right] \\
= 4 \text{tr} \left[ P_{n,4} ((|\varphi\rangle\langle\varphi|)^\otimes 3 \otimes X) \right] + 6 \text{tr} \left[ P_{n,4} ((|\varphi\rangle\langle\varphi|)^\otimes 2 \otimes X^\otimes 2) \right] + 4 \text{tr} \left[ P_{n,4} ((|\varphi\rangle\langle\varphi|) \otimes X^\otimes 3) \right] + \text{tr} \left( P_{n,4} X^\otimes 4 \right).
\]
(135)

According to Lemma 10 below, if \( 0 \leq \lambda \leq 1/2 \), then
\[
\alpha_+ (\psi) - \alpha_+ (\varphi) \leq \frac{1}{d} \left[ 4\lambda (1 - \lambda) + 6\sqrt{2}\lambda^2 (1 + \lambda) + 8\lambda^3 (1 - \lambda) + 8\lambda^4 \right] = \frac{1}{d} \lambda f (\lambda) \leq \frac{1}{d} f (\lambda) \|\psi - \varphi\|_2,
\]
(136)
where
\[
f (\lambda) := 4 + (6\sqrt{2} - 4)\lambda + (8 + 6\sqrt{2})\lambda^2.
\]
(137)

Note that \( f (\lambda) \) increases monotonically with \( \lambda \) when \( \lambda \geq 0 \). If \( 0 \leq \lambda \leq 1/5.4 \), then \( f (\lambda) \leq f (1/5.4) < 5.4 \), so that \( \alpha_+(\psi) - \alpha_+(\varphi) \leq 5.4 \|\psi - \varphi\|_2 / d \). If \( 1/5.4 \leq \lambda \leq 1 \), then
\[
\alpha_+ (\psi) - \alpha_+(\varphi) \leq \alpha_+(\psi) \leq \frac{1}{d} \lambda \leq \frac{5.4\lambda}{d} \leq \frac{5.4}{d} \|\psi - \varphi\|_2,
\]
(138)
where we have applied Eq. (177). By symmetry we have
\[
|\alpha_+(\psi) - \alpha_+(\varphi)| \leq \frac{5.4}{d} \|\psi - \varphi\|,
\]
(139)
which confirms the first inequality in Lemma 6. The second inequality in the lemma is an immediate consequence of the first one and Eq. (171). \( \square \)

**Lemma 10.** Suppose \( \psi, \varphi \) are two normalized \( n \)-qubit state vectors, \( X = |\psi\rangle\langle\psi| - |\varphi\rangle\langle\varphi| \), and \( \lambda = \|X\|_1/2 \). Then
\[
\text{tr} \left[ P_{n,4} ((|\varphi\rangle\langle\varphi|)^\otimes 3 \otimes X) \right] \leq \left\{ \begin{array}{ll}
\frac{2\lambda(1-\lambda)}{d} & 0 \leq \lambda \leq \frac{1}{2}, \\
\frac{1}{2} & \frac{1}{2} \leq \lambda \leq 1;
\end{array} \right.
\]
(140)
\[
\text{tr} \left[ P_{n,4} ((|\varphi\rangle\langle\varphi|)^\otimes 2 \otimes X^\otimes 2) \right] \leq \min \left\{ \frac{2\lambda^2}{d}, \sqrt{2}\lambda^2 (1 + \lambda) \right\};
\]
(141)
\[
\text{tr} \left[ P_{n,4} ((|\varphi\rangle\langle\varphi|) \otimes X^\otimes 3) \right] \leq \left\{ \begin{array}{ll}
\frac{2\lambda^3(1-\lambda)}{d} & 0 \leq \lambda \leq \frac{1}{2}, \\
\frac{\lambda^2}{d} & \frac{1}{2} \leq \lambda \leq 1;
\end{array} \right.
\]
(142)
\[
\text{tr} \left( P_{n,4} X^\otimes 4 \right) \leq \frac{8\lambda^4}{d}.
\]
(143)
Proof. Suppose $X$ has spectral decomposition $X = \lambda(|\mu\rangle\langle\mu| - |\nu\rangle\langle\nu|)$. Then
\[
\text{tr} \left\{ P_{n,4} \left[ (|\varphi\rangle\langle\varphi|) \otimes X \otimes X \right] \right\} \leq \lambda^2 \text{tr} \left\{ P_{n,4} \left[ (|\varphi\rangle\langle\varphi|) \otimes (|\mu\rangle\langle\mu|) \otimes (|\nu\rangle\langle\nu|) \right] \right\} \leq \frac{2\lambda^2}{d},
\]
(144)
\[
\text{tr} \left\{ P_{n,4} X^\otimes 4 \right\} \leq \lambda^4 \text{tr} \left\{ P_{n,4} \left[ (|\mu\rangle\langle\mu|) \otimes (|\nu\rangle\langle\nu|) \otimes (|\mu\rangle\langle\mu|) \otimes (|\nu\rangle\langle\nu|) \right] \right\} \leq \frac{8\lambda^4}{d},
\]
(145)
where we have applied Lemma 9.

According to Lemma 11 below with $W$ being a Pauli operator, we also have
\[
\text{tr} \left\{ P_{n,4} \left[ (|\varphi\rangle\langle\varphi|) \otimes X \right] \right\} = \frac{1}{d^2} \sum_a \text{tr} \left\{ W_{a\otimes 4} \left[ (|\varphi\rangle\langle\varphi|) \otimes X \right] \right\} = \frac{1}{d^2} \sum_a (|\varphi\langle W_a |\varphi\rangle|^3 \text{tr}(W_a X))
\]
\[
\leq \left\{ \frac{(1+\lambda)}{d^2} \sum_a |\varphi\langle W_a |\varphi\rangle|^2 = \frac{(1+\lambda)}{d^2} \quad 0 \leq \lambda \leq \frac{1}{2}, \right. \]
\[
\left. \frac{1}{d^2} \sum_a |\varphi\langle W_a |\varphi\rangle|^2 = \frac{1}{d} \quad \frac{1}{2} \leq \lambda \leq 1. \right\}
\]
(146)
\[
\text{tr} \left\{ P_{n,4} \left[ (|\varphi\rangle\langle\varphi|) \otimes X \otimes X \right] \right\} = \frac{1}{d^2} \sum_a \text{tr} \left\{ W_{a\otimes 4} \left[ (|\varphi\rangle\langle\varphi|) \otimes X \otimes X \right] \right\} = \frac{1}{d^2} \sum_a (|\varphi\langle W_a |\varphi\rangle|^2 \text{tr}(W_a X))^2
\]
\[
\leq \frac{(1+\lambda)}{d^2} \sum_a |\varphi\langle W_a |\varphi\rangle|^2 \text{tr}(W_a X) \leq \frac{(1+\lambda)}{d^2} \left\{ \sum_a (|\varphi\langle W_a |\varphi\rangle|^2)^2 \sum_b \text{tr}(W_b X)^2 \right\}^{1/2}
\]
\[
= \frac{(1+\lambda)}{d} \|X\|_2 = \sqrt{\frac{2\lambda^2(1+\lambda)}{d}}.
\]
(147)
\[
\text{tr} \left\{ P_{n,4} \left[ (|\varphi\rangle\langle\varphi|) \otimes X \otimes X \otimes X \right] \right\} = \frac{1}{d^2} \sum_a \text{tr} \left\{ W_{a\otimes 4} \left[ (|\varphi\rangle\langle\varphi|) \otimes X \otimes X \otimes X \right] \right\} = \frac{1}{d^2} \sum_a |\varphi\langle W_a |\varphi\rangle|^3 \text{tr}(W_a X)^3
\]
\[
\leq \left\{ \frac{(1+\lambda)}{d^2} \sum_a \text{tr}(W_a X)^2 = \frac{(1+\lambda)}{d^2} \|X\|_2^2 = \frac{2\lambda^3(1-\lambda)}{d} \quad 0 \leq \lambda \leq \frac{1}{2}, \right. \]
\[
\left. \frac{1}{d^2} \sum_a |\varphi\langle W_a |\varphi\rangle|^2 = \frac{1}{d} \|X\|_2^2 = \frac{3\lambda}{d} \quad \frac{1}{2} \leq \lambda \leq 1. \right\}
\]
(148)
\[\square\]

Lemma 11. Suppose $\psi, \varphi$ are two normalized state vectors, $X = |\psi\rangle\langle\psi| - |\varphi\rangle\langle\varphi|$, $\lambda = \|X\|_1/2$, and $W$ is an operator satisfying $-I \leq W \leq I$. Then
\[- \lambda(1+\lambda) \leq \langle \varphi |W|\varphi\rangle \text{tr}(W X) \leq \left\{ \begin{align*}
\frac{(1+\lambda)}{d} & \quad 0 \leq \lambda \leq \frac{1}{2}, \\
\frac{1}{d} & \quad \frac{1}{2} \leq \lambda \leq 1.
\end{align*} \right. \]
(149)

Proof. Let $P = (I + W)/2$, then $0 \leq P \leq I$. In addition,
\[
\langle \varphi |W|\varphi\rangle \text{tr}(W X) = \langle \varphi |(2P - I)|\varphi\rangle \text{tr}([2P - I]X) = (\langle \varphi |P|\varphi\rangle - 1) \text{tr}(2PX) = (\langle \varphi |Q|\varphi\rangle - 1) \text{tr}(2QX),
\]
(150)
where $Q$ is the projection of $P$ onto the two-dimensional subspace spanned by $\psi$ and $\varphi$, which satisfies $0 \leq Q \leq 1$. Now the lemma follows from Lemma 12 below. \[\square\]

Lemma 12. Suppose $\rho_1$ and $\rho_2$ are two qubit pure states, $X = \rho_2 - \rho_1$, and $\lambda = \|X\|_1/2$. Let $Q$ be a positive operator on the qubit which satisfies $Q \leq I$. Then
\[- \lambda(1+\lambda) \leq \left[ 2\text{tr}(Q\rho_1) - 1 \right] \text{tr}(2QX) \leq \left\{ \begin{align*}
\frac{(1+\lambda)}{d} & \quad 0 \leq \lambda \leq \frac{1}{2}, \\
\frac{1}{d} & \quad \frac{1}{2} \leq \lambda \leq 1.
\end{align*} \right. \]
(151)
Here both the lower bound and the upper bound can be saturated.
Proof. With a suitable unitary transformation if necessary, \( \rho_1, \rho_2 \) can be written as follows,

\[
\rho_1 = \frac{1 + r_1 \cdot \sigma}{2}, \quad r_1 = (\sin \theta, 0, \cos \theta); \quad \rho_2 = \frac{1 + r_2 \cdot \sigma}{2}, \quad r_2 = (-\sin \theta, 0, \cos \theta),
\]

where \( \sigma \) is the vector composed of the three Pauli matrices \( \sigma_x, \sigma_y, \sigma_z \) and \( \theta = \arcsin \lambda \) with \( 0 \leq \theta \leq \pi/2 \). Similarly, the operator \( Q \) can be expanded in terms of the Pauli basis. Moreover, to achieve the maximum or minimum in Eq. (151), the coefficient of \( \sigma_y \) can be set to zero, so we have

\[
Q = \frac{1}{2}(1 + a + b\sigma_x + c\sigma_z),
\]

where \( a, b, c \) are real constants. The constraint \( 0 \leq Q \leq 1 \) amounts to the inequality

\[
|a| + \sqrt{b^2 + c^2} \leq 1,
\]

which defines the region of feasible solutions, denoted by \( R \) below. Geometrically, the region \( R \) is the intersection of two opposite cones, which is convex. In addition \( R \) is symmetric under inversion of each of the three coordinates \( a, b, c \). With this notation, we have

\[
[2 \text{tr}(Q\rho_1) - 1] \text{tr}(2QX) = -\sin \theta f(a, b, c),
\]

where

\[
f(a, b, c) := 2b(a + b \sin \theta + c \cos \theta),
\]

Since \( f(a, b, c) \) is invariant under inversion, to determine the maximum of \( f(a, b, c) \) over \( R \), we may assume that \( b \geq 0 \), and then it suffices to consider the case \( a, c \geq 0 \). In addition, for given \( b, f(a, b, c) \) is linear in \( a, c \), so the maximum of \( f(a, b, c) \) over the convex region \( R \) can be attained at the boundary of \( R \). Define \( s := \sqrt{b^2 + c^2} \) and \( \phi := \arccos(c/s) \); then \( 0 \leq s \leq 1, \ 0 \leq \phi \leq \pi/2, \ b = s \sin \phi, \ c = s \cos \phi, \) and \( a = 1 - s \) under the above assumptions. Consequently,

\[
f(a, b, c) = 2s \sin \phi[1 - s + s \cos(\phi - \theta)] = 2s \sin \phi\{s^2[1 - \cos(\phi - \theta)] + s\}.
\]

If \( \cos(\phi - \theta) \leq 1/2 \), then

\[
f(a, b, c) \leq \frac{\sin \phi}{2[1 - \cos(\phi - \theta)]} \leq \sin \phi \leq 1 + \sin \theta.
\]

If \( \cos(\phi - \theta) \geq 1/2 \), then

\[
f(a, b, c) \leq 2 \sin \phi \cos(\phi - \theta) = \sin \theta + \sin(2\phi - \theta) \leq 1 + \sin \theta.
\]

Similarly, to determine the minimum of \( f(a, b, c) \) over \( R \), we may assume that \( b \geq 0 \) and \( a, c \leq 0 \). In addition, it suffices to consider the boundary of \( R \), so that \( a = \sqrt{b^2 + c^2} - 1 \). Define \( s := \sqrt{b^2 + c^2} \) and \( \phi := \arccos(c/s) \); then \( 0 \leq s \leq 1, \ \pi/2 \leq \phi \leq \pi, \ b = s \sin \phi, \ c = s \cos \phi, \) and \( a = s - 1 \). Consequently,

\[
f(a, b, c) = 2s \sin \phi[s - 1 + s \cos(\phi - \theta)] = 2s \sin \phi\{s^2[1 + \cos(\phi - \theta)] - s\}.
\]

If \( \cos(\phi - \theta) \geq -1/2 \), that is, \( \phi \leq (2\pi/3) + \theta \), then

\[
f(a, b, c) \geq -\frac{\sin \phi}{2[1 + \cos(\phi - \theta)]} \geq -\frac{\sin(\pi - \theta)}{2[1 + \cos(\pi - \theta - \theta)]} = -\frac{1}{4 \sin \theta}.
\]

If \( \cos(\phi - \theta) \leq -1/2 \), that is, \( \phi \geq (2\pi/3) + \theta \), then

\[
f(a, b, c) \geq 2 \sin \phi \cos(\phi - \theta) = \sin \theta + \sin(2\phi - \theta) \geq \sin \theta - 1 \geq -\frac{1}{4 \sin \theta}.
\]
If $0 \leq \theta \leq \pi/6$ and $\phi \leq (2\pi/3) + \theta$ then

$$f(a, b, c) \geq -\frac{\sin \phi}{2[1 + \cos(\phi - \theta)]} \geq -\frac{\sin \left(\frac{2\pi}{3} + \theta\right)}{2[1 + \cos(2\pi/3)]} = -\sin \left(\frac{2\pi}{3} + \theta\right) \geq \sin \theta - 1. \quad (163)$$

In summary we have

$$\begin{align*}
-(1 - \sin \theta) &\leq f(a, b, c) \leq 1 + \sin \theta & 0 \leq \theta \leq \pi/6, \\
-\frac{1}{4 \sin \theta} &\leq f(a, b, c) \leq 1 + \sin \theta & \pi/6 \leq \theta \leq \pi/2,
\end{align*} \quad (164)$$

which implies Eq. (151) given Eq. (155) and the inequality $\sin \theta = \lambda$. The upper bound in Eq. (164) is saturated when $s = 1$ and $\phi = (\pi + 2\theta)/4$, in which case $a = 0$, $b = \sqrt{(1 + \sin \theta)/2}$, $c = \sqrt{(1 - \sin \theta)/2}$, and

$$Q = \frac{1}{2} \left(1 + \sqrt{\frac{1 + \sin \theta}{2}\sigma_x} + \sqrt{\frac{1 - \sin \theta}{2}\sigma_z}\right). \quad (165)$$

If $0 \leq \theta \leq \pi/6$, the lower bound in Eq. (164) is saturated when $s = 1$ and $\phi = (3\pi + 2\theta)/4$, in which case $a = 0$, $b = \sqrt{(1 - \sin \theta)/2}$, $c = -\sqrt{(1 + \sin \theta)/2}$, and

$$Q = \frac{1}{2} \left(1 + \sqrt{\frac{1 - \sin \theta}{2}\sigma_x} - \sqrt{\frac{1 + \sin \theta}{2}\sigma_z}\right). \quad (166)$$

If $\pi/6 \leq \theta \leq \pi/2$, the lower bound is saturated when $s = 1/(4\sin^2 \theta)$ and $\phi = \pi - \theta$, in which case $a = [1/(4\sin^2 \theta)] - 1$, $b = 1/(4\sin \theta)$, $c = -\cos \theta/(4\sin^2 \theta)$, and

$$Q = \frac{1}{2} \left(\frac{1}{4\sin^2 \theta} + \frac{1}{4\sin \theta}\sigma_x - \frac{\cos \theta}{4\sin^2 \theta}\sigma_z\right) = \frac{1}{8\sin^2 \theta} (1 + \sin \theta \sigma_x - \cos \theta \sigma_z). \quad (167)$$

Therefore, both the lower bound and the upper bound in Eq. (151) can be saturated. \qed

## F  Proof of Proposition 8

**Proof.** Let $d_j = 2^n$, and suppose $m = 4$. Then

$$\|\Xi(\psi)\|_{\ell_4}^4 = \prod_{j=1}^4 \|\Xi(\psi_j)\|_{\ell_4}^4 \geq \prod_{j=1}^4 \frac{2d_j}{d_j + 1} \geq \left(\frac{4}{3}\right)^3 \frac{2^{n-2}}{2^{n-3} + 1}, \quad (168)$$

where the first inequality follows from Eq. (76). If $n \geq 5$, then

$$\left(\frac{4}{3}\right)^3 \frac{2^{n-2}}{2^{n-3} + 1} - \frac{4d}{d + 3} = \frac{5 \times 2^{n+2}(2^n - 24)}{27(2^n + 3)(2^n + 8)} > 0. \quad (169)$$

So the vector $\psi$ cannot generate a 4-design, recall that $\psi$ is a 4-design fiducial iff $\|\Xi(\psi)\|_{\ell_4}^4 = 4d/(d + 3)$, where $d = 2^n$.

Now suppose $m = 3$, so that $n_1 + n_2 + n_3 = n$. If $n_3 = 2$, then

$$\|\Xi(\psi)\|_{\ell_4}^4 \geq \left(\frac{8}{5}\right)^3 = \frac{512}{125} \geq 4 > \frac{4d}{d + 3}. \quad (170)$$

so $\psi$ cannot be a 4-design fiducial. If $n_3 = 1, n_1, n_2 \geq 3$, then $n \geq 7$,

$$\|\Xi(\psi)\|_{\ell_4}^4 - \frac{4d}{d + 3} \geq \prod_{j=1}^4 \frac{2d_j}{d_j + 1} \geq \frac{4}{3} \times \frac{2^{n+2}}{2^n + 3} \times \frac{2^{n-3}}{2^{n-4} + 1} \times \frac{2^{n+2}}{2^n + 3} = \frac{2^{n+2}(5 \times 2^n - 336)}{27(2^n + 3)(2^n + 16)} > 0. \quad (171)$$

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So \( \psi \) cannot be a 4-design fiducial. If \( n_3 = 1, n_1, n_2 \geq 2 \), then \( n \geq 5 \),

\[
\| \Xi(\psi) \|^4_{\ell_4} - \frac{4d}{d+3} \geq \prod_{j=1}^{4} \frac{2d_j}{d_j+1} - \frac{2^{n+2}}{2^n+3} \geq \frac{4}{3} \times \frac{8}{5} \times \frac{2^{n-2}}{2^{n-3}+1} - \frac{2^{n+2}}{2^n+3} = \frac{2^{n+2}(2^n - 72)}{15(2^n+3)(2^n+8)}. \tag{172}
\]

If in addition \( n \geq 7 \), then \( \| \Xi(\psi) \|^4_{\ell_4} > 4d/(d+3) \), so that \( \psi \) cannot be a 4-design fiducial. This observation completes the proof of the proposition. \( \square \)

## G Notes on multivariate polynomials

### G.1 Real case

In this section, we recall several standard facts about multivariate polynomials. This is intended mainly for the benefit of readers from quantum information theory, where the notions discussed below do not seem to be commonly known. The material presented here is based on Refs. \[66\ \[71\ \[31\ \[70\ \[28\]

We start by considering the real vector space \( \mathbb{R}^d \). Let \( \text{Hom}_t(\mathbb{R}^d) \) be the set of polynomial functions on \( \mathbb{R}^d \) that is homogeneous of degree \( t \).

There is a close connection between polynomials, symmetric multilinear forms, and totally symmetric tensors. Recall that the symmetric group \( S_t \) acts on the tensor product space \( (\mathbb{R}^d)^\otimes t \) by permuting tensor factors:

\[
\pi(u_1 \otimes \cdots \otimes u_t) = u_{\pi_1} \otimes \cdots \otimes u_{\pi_k} \quad \forall u_i \in \mathbb{R}^d, \pi \in S_t.
\tag{173}
\]

Let \( \text{Sym}_t(\mathbb{R}^d) \) be the space of \textit{totally symmetric degree-} \( t \) tensors:

\[
\text{Sym}_t(\mathbb{R}^d) = \{ f \in (\mathbb{R}^d)^\otimes t \mid \pi f = f \forall \pi \in S_t \}.
\]

Every \( f \in \text{Sym}_t(\mathbb{R}^d) \) specifies a symmetric \( t \)-linear form on \( \mathbb{R}^d \) by setting

\[
f(u_1, \ldots, u_t) := \langle f, u_1 \otimes \cdots \otimes u_t \rangle \quad u_i \in \mathbb{R}^d.
\]

Conversely, every symmetric \( t \)-linear form arises in this way, and we will not distinguish between symmetric tensors and symmetric forms in the following\footnote{If, instead of \( \mathbb{R}^d \), one starts with a general linear space \( V \) for which no canonical scalar product has been specified, it would be cleaner to work with the symmetric tensor powers \( \text{Sym}_t(V^*) \) of the dual space \( V^* \). Here, however, we find it advantageous to to identify \( (\mathbb{R}^d)^* \) with \( \mathbb{R}^d \) via the standard scalar product whenever necessary.}

By restricting all \( u_i \) to be equal, we obtain a homogeneous order- \( t \) polynomial \( p_f \in \text{Hom}_t(\mathbb{R}^d) \):

\[
p_f(u) := \langle f, u \otimes \cdots \otimes u \rangle. \tag{174}
\]

It is a less trivial fact that this relation between symmetric \( t \)-linear forms and homogeneous order- \( t \) polynomials is one-one, i.e. that one can recover \( f \) from the restriction \( p_f \). The map \( p_f \mapsto f \) is called the \textit{polarization map} (c.f. \[66\ Chapter 3.2\]). To see how this works, we construct the polarization map explicitly for a basis of \( \text{Hom}_t(\mathbb{R}^d) \). Let \( \mu \in \mathbb{N}_d \) be a vector of \( d \) non-negative integers summing to \( t \). Then \( \mu \) is called a \textit{partition of} \( t \) \textit{into} \( d \) \textit{parts}. The polynomial \( x^\mu \in \text{Hom}_t(\mathbb{R}^d) \) defined by

\[
\mathbb{R}^d \ni x = (x_1, \ldots, x_d) \mapsto x^\mu := \prod_{i=1}^d x_i^{\mu_i}
\]

is the \textit{monomial associated with} \( \mu \). By definition, the degree- \( t \) monomials span \( \text{Hom}_t(\mathbb{R}^d) \) and they are easily seen to be linearly independent as functions on \( \mathbb{R}^d \) (c.f. Lemma \[14\ where we give a proof for the complex case). The symmetric vector

\[
e_\mu := \frac{1}{|S_t|} \sum_{\pi \in S_t} \pi (e_1^{\otimes \mu_1} \otimes \cdots \otimes e_d^{\otimes \mu_d}) \in \text{Sym}_t(\mathbb{R}^d)
\]

clearly fulfills

\[
\langle e_\mu, x^\otimes t \rangle = x^\mu \quad \forall x \in \mathbb{R}^d
\]

and thus constitutes the polarization of \( x^\mu \). In fact, we have:
Lemma 13. The relations
\[ x^\mu \mapsto e_\mu \quad \forall \mu \in \mathbb{N}_0^d, \mu \text{ partition of } t \] (175)
define a linear isomorphism from \( \text{Hom}_t(\mathbb{R}^d) \) to \( \text{Sym}_t(\mathbb{R}^d) \). Its inverse is
\[ f \mapsto p_f, \quad p_f(x) := \langle f, x^{\otimes t} \rangle. \]

Proof. It remains to be shown that the map (175) is onto, i.e. that the \( \{ e_\mu \} \) span \( \text{Sym}_t(\mathbb{R}^d) \). That is true because the set \( \{ e_\mu \} \) constitutes the projection onto \( \text{Sym}_t(\mathbb{R}^d) \) of the standard tensor product basis \( \{ e_i \otimes \cdots \otimes e_i \} \) \( i \in \mathbb{R}^d \).

As a corollary, we see directly that the set of tensor powers spans the totally symmetric space:
\[ \text{Sym}_t(\mathbb{R}^d) = \{ \{ u^{\otimes t} | u \in \mathbb{R}^d \} \}. \] (176)

We now turn to the relevant symmetries. By definition, the orthogonal group \( O(d) \) acts on \( \mathbb{R}^d \). The action extends to the diagonal action on degree-\( t \) tensors \( (\mathbb{R}^d)^{\otimes t} \):
\[ O : u_1 \otimes \cdots \otimes u_t \mapsto (O \otimes \cdots \otimes O) (u_1 \otimes \cdots \otimes u_t) = (O u_1) \otimes \cdots \otimes (O u_t) \quad u_i \in \mathbb{R}^d. \]
The diagonal representation of \( O(d) \) commutes with the action of the symmetric group \( S_t \) defined in Eq. (173). This implies in particular that \( \text{Sym}_t(\mathbb{R}^d) \) is an invariant subspace and thus carries a representation of \( O(d) \). By Eq. (174), the action coincides with the natural action
\[ O : p(\cdot) \mapsto p(O^{-1} \cdot) = p(O^T \cdot) \]
of \( O(d) \) on polynomials in \( \text{Hom}_t(\mathbb{R}^d) \). The representation of \( O(d) \) on \( \text{Sym}_t(\mathbb{R}^d) \sim \text{Hom}_t(\mathbb{R}^d) \) is reducible (this is an important conceptual distinction to the complex case, described below). We will now describe the irreducible representations for the case of even degree – first as subspaces of \( \text{Sym}_t(\mathbb{R}) \) and then viewed as subspaces of \( \text{Hom}_t(\mathbb{R}^d) \).

Choose an ortho-normal basis \( \{ e_1, \ldots, e_d \} \) of \( \mathbb{R}^d \). In the case of of \( t = 2 \), it follows directly from the defining property \( OO^T = I \) of the orthogonal group that
\[ v_0 = \sum_{i=1}^d e_i \otimes e_i \in \text{Sym}_2(\mathbb{R}^d) \]
is an invariant vector. A fruitful way to think about this fact lies in the relation
\[ \langle v_0, u_1 \otimes u_2 \rangle = \langle u_1, u_2 \rangle \quad \forall u_i \in \mathbb{R}^d, \] (177)

together with the fact that the orthogonal group preserves inner products like those appearing on the right hand side. If we expand tensors in coordinates
\[ u = \sum_{i,j} u_{i,j} (e_i \otimes e_j) \in (\mathbb{R}^d)^{\otimes 2}, \]
then the inner product with \( v_0 \) corresponds to a contraction of the indices
\[ \langle v_0, u \rangle = \sum_i u_{i,i}. \]

We can apply these findings to higher orders \( t > 2 \) by “contracting only two of the indices with \( v_0 \).” More precisely, define the contraction map
\[ C : (\mathbb{R}^d)^{\otimes t} \to (\mathbb{R}^d)^{\otimes (t-2)} \]
by its actions on product tensors as follows:
\[ C : u_1 \otimes \cdots \otimes u_t \mapsto \langle u_1, u_2 \rangle (u_3 \otimes \cdots \otimes u_t). \] (178)
In coordinates, it corresponds to a contraction of the first two indices

\[ u_{i_1 \ldots i_t} \mapsto \sum_i u_{i,i_3 \ldots i_t} \, . \]

From (178), it is clear that the kernel \( \ker C \) is an invariant subspace of the diagonal representation \( O \mapsto O^{\otimes t} \) of \( O(d) \). The same is true for its intersection

\[ H_t(\mathbb{R}^d) := \ker C \cap \text{Sym}_t(\mathbb{R}^d) \]

which turns out to carry an irreducible representation. We refer to \( H_t \) as the **harmonic totally symmetric space** of degree \( t \). From Eq. (176), it follows that \( C \) maps \( \text{Sym}_t(\mathbb{R}^d) \) onto \( \text{Sym}_{t-2}(\mathbb{R}^d) \) and thus

\[ \text{Sym}_{t-2}(\mathbb{R}^d) \simeq \text{Sym}_t(\mathbb{R}^d) / \ker C \simeq (\ker C)^\perp, \]

where the ortho-complement is taken within \( \text{Sym}_t(\mathbb{R}^d) \). Therefore, we can embed \( H_{t-2}(\mathbb{R}^d) \) into \( \text{Sym}_t(\mathbb{R}^d) \), as the kernel of \( C \circ C \) intersected with \( (\ker C)^\perp \). Iterating this procedure and setting

\[ H_0(\mathbb{R}^d) = \text{Sym}_0(\mathbb{R}^d) = \mathbb{R}, \]

one obtains the decomposition

\[ \text{Sym}_t(\mathbb{R}^d) \simeq \bigoplus_{i=0}^{t/2} H_{2i}(\mathbb{R}^d). \] (179)

of \( \text{Sym}_t(\mathbb{R}^d) \) into irreducible representations of \( O(d) \). The embedding \( H_{t-2j} \to \text{Sym}_t(\mathbb{R}^d) \) is given explicitly by

\[ f \mapsto P_{[t]}(v_0^{\otimes j} \otimes f), \quad P_{[t]} := \frac{1}{t!} \sum_{\pi \in S_t} \pi. \] (180)

In particular, the one-dimensional invariant space \( H_0(\mathbb{R}^d) \) is realized as a multiple of

\[ v_0^{(t)} = P_{[t]} v_0^{\otimes t/2}. \] (181)

We will use these embeddings implicitly from now on and treat \( H_{2i}(\mathbb{R}^d) \) as a subset of \( \text{Sym}_i(\mathbb{R}^d) \). The same convention will apply to related spaces for polynomials, and in the complex case. It turns out that these embeddings are much more transparent from the point of view of polynomials, as described next.

Using the language of homogeneous polynomials, one finds that up to normalization, the contraction \( C \) corresponds to the action of the **Laplacian**. Recall that the **Laplacian** differential operator

\[ \Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} : \text{Hom}_t(\mathbb{R}^d) \to \text{Hom}_{t-2}(\mathbb{R}^d). \]

Simple direct calculations yield

\[ \Delta x^\mu = \sum_{i=1}^d \mu_i(\mu_i - 1)x^{\mu-2e_i}, \]

\[ C e_\mu = \frac{1}{t(t-1)} \sum_{i=1}^d \mu_i(\mu_i - 1)e_{\mu-2e_i}. \]

Therefore, up to normalization, the contraction operator \( C \) corresponds to the Laplacian in the sense that

\[ \Delta p_f = t(t-1)p_{Cf} \quad \forall f \in \text{Sym}_t(\mathbb{R}^d). \]
Polynomials in the kernel of the Laplacian are called harmonic and we write
\[ \text{Harm}_t(\mathbb{R}^d) := \ker \Delta \cap \text{Hom}_t(\mathbb{R}^d). \]

As above, this gives rise to the decomposition
\[ \text{Hom}_t(\mathbb{R}^d) \simeq \bigoplus_{i=0}^{t/2} \text{Harm}_{2i}(\mathbb{R}^d) \quad (182) \]
of the set of all homogeneous polynomials of degree \( t \) into irreducible spaces, equivalent to harmonic polynomials of lower degrees. The embedding of \( \text{Harm}_{t-2j}(\mathbb{R}^d) \to \text{Hom}_t(\mathbb{R}^d) \) used in Eq. (182) maps \( p \mapsto p' \), where
\[ p'(u) = \|u\|_2^2 p(u) \quad \forall u \in \mathbb{R}^d, \quad (183) \]
and the one-dimensional space \( \text{Harm}_0(\mathbb{R}^d) \) is realized within \( \text{Hom}_t(\mathbb{R}^d) \) as multiples of
\[ u \mapsto \|u\|_2^t. \quad (184) \]

Equations (183) and (184) are the polynomial analogues of Eqs. (180) and (181), respectively.

We are frequently concerned with the restriction \( \text{Hom}_t(S^{d-1}) \to \text{Harm}_t(S^{d-1}) \) to the unit sphere \( S^{d-1} \subset \mathbb{R}^d \). There, the embedding in Eq. (183) becomes trivial, so that \( \text{Harm}_{2i}(S^{d-1}) \) is a subset of \( \text{Hom}_t(S^{d-1}) \). For any homogeneous polynomial \( p \) of degree \( t \), it is true by definition that
\[ p(u) = \|u\|_2^t p \left( \frac{u}{\|u\|_2} \right). \]

Thus, homogeneous polynomials are fully specified by their restriction to \( S^{d-1} \) and therefore \( \text{Hom}_t(\mathbb{R}^d) \simeq \text{Hom}_t(S^{d-1}) \).

Given an even number \( t \geq 0 \), a spherical \( t \)-design \[25\] for \( S^{d-1} \) is a set of vectors \( X \subset S^{d-1} \) such that
\[ \frac{1}{|X|} \sum_{\psi \in X} \psi^{\otimes t} \in H_0(\mathbb{R}^d). \]

It is a \((t+1)\)-design if in addition
\[ \sum_{\psi \in X} \psi^{\otimes t+1} = 0. \]
Clearly, this equation is satisfied automatically when the set \( X \) is symmetric under inversion, that is, \( -\psi \in X \) whenever \( \psi \in X \).

Let \( G \subset O(d) \) be a subgroup of \( O(d) \). For simplicity, we take \( G \) to be finite. The spaces \( H_{2i} \) are \( G \)-invariant. For each \( i \), let \( \{ b_{j}^{(2i)} \}_{j} \) be an orthonormal basis of the space of \( G \)-invariant vectors in \( H_{2i}(\mathbb{R}^d) \).

If now \( X = G \cdot \psi_0 \) is a \( G \)-orbit, then
\[ \frac{1}{|X|} \sum_{\psi \in X} \psi^{\otimes t} = \sum_{j} b_{j}^{(2i)} b_{j}^{(2i), \psi_0^{\otimes t}} = \sum_{j} b_{j}^{(2i)} p_{j}^{(2i)}(\psi_0). \]

In particular, \( G \cdot \psi_0 \) is a \( t \)-design if and only if \( p_{j}^{(2i)}(\psi_0) = 0 \) for all \( i \geq 1 \) and all \( j \). This is equivalent to saying that \( \psi_0 \) is a root of all \( G \)-invariant harmonic polynomials. If \( G \) affords no harmonic invariants of degree \( s \) for \( 1 \leq s \leq t \), then the orbit of any vector \( \psi_0 \) constitutes a spherical \( t \)-design.

### G.2 Complex case

In analogy to the real case, we define \( \text{Hom}_t(\mathbb{C}^d) \) to be the complex vector space of homogeneous polynomials in \( d \) complex variables. The definitions of \( \text{Sym}_t(\mathbb{C}^d) \), the monomials \( x^\mu \), and symmetric tensors \( e_\mu \) carry
The Wirtinger derivatives are
\begin{equation}
\text{Hom} \in \text{Sym}_d(C^d) \rightarrow \text{Sym}_d(C^d).
\end{equation}
Thus, the complex analogue
\begin{equation}
\text{Sym}_d(C^d) = \langle \{u^{\otimes t} | u \in C^d\} \rangle
\end{equation}
of Eq. (174) holds.

In the main part of this paper, we frequently work with a notion of “sesquilinear” polynomials. To define this concept, let \( p \in \text{Hom}_d(C^d) \). Then we define a function that is homogeneous of degree \( t \) in the coordinates \( x \) with respect to the complex coordinates \( x_1, \ldots, x_d \) on \( C^d \) and also homogeneous of degree \( t \) in the complex conjugates \( \bar{x}_1, \ldots, \bar{x}_d \) via
\[ x \mapsto p(\bar{x}, x). \]
We denote the set of all such polynomials of bi-degree \( (t, t) \) on \( C^d \) by \( \text{Hom}_{(t,t)}(C^d) \). The following lemma establishes the right notion of polarization for functions in \( \text{Hom}_{(t,t)}(C^d) \).

**Lemma 14.** The relations
\[ x^\mu \bar{x}^\nu \mapsto e_\mu e_\nu^* \quad \forall \mu, \nu \in \mathbb{N}_0^d \text{ partitions of } t \]
define a linear isomorphism from \( \text{Hom}_{(t,t)}(C^d) \) to the set of linear maps \( L(\text{Sym}_d(C^d)) \) on \( \text{Sym}_d(C^d) \). Its inverse is
\[ A \mapsto p_A, \quad p_A(\bar{x}, x) := \text{tr} (A(x^{\otimes t})(\bar{x}^{\otimes t})^T). \]

Our proof of the polarization relation relies on the notion of Wirtinger derivatives, introduced next. On \( C^d \), one can define real coordinates \( (x_1, \ldots, x_d, y_1, \ldots, y_d) \) by writing
\[ C^d \ni z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n). \]

The Wirtinger derivatives are
\[ \frac{\partial}{\partial z_i} z_j = \delta_{i,j}, \quad \frac{\partial}{\partial \bar{z}_i} z_j = 0, \quad \frac{\partial}{\partial z_i} \bar{z}_j = 0, \quad \frac{\partial}{\partial \bar{z}_i} \bar{z}_j = \delta_{i,j}, \]
as one can easily verify. For multi-indices \( \mu, \nu \in \mathbb{N}_0^d \), define
\[ \partial_{(\mu, \nu)} := \prod_{i=1}^d \frac{1}{\mu_i! \nu_i!} \frac{\partial^\mu_i}{\partial z_i^{\mu_i}} \frac{\partial^\nu_i}{\partial \bar{z}_i^{\nu_i}}. \]

Below, we will also use the **complex Laplacian**
\[ \Delta_C := \sum_{i=1}^d \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar{z}_i}. \]

**Proof of Lemma 14.** We first show that the set of monomials \( \{x^\mu \bar{x}^\nu\}_{\mu, \nu} \) is linearly independent, where \( \mu, \nu \) range over partitions of \( t \) into \( d \) parts. Indeed, if \( (\alpha, \beta) \) are two partitions of \( t \), then
\[ \partial_{(\alpha, \beta)} x^\mu \bar{x}^\nu = \delta_{\alpha, \mu} \delta_{\beta, \nu}, \]
because for each \( (\mu, \nu) \neq (\alpha, \beta) \), at least one of the variables has lower degree than the corresponding derivative. Consequently,
\[ \partial_{(\alpha, \beta)} \sum_{\mu, \nu} c_{\mu, \nu} x^\mu \bar{x}^\nu = c_{\alpha, \beta}, \]
and therefore, a linear combination of monomials cannot be zero unless all the coefficients are.

It follows that the linear map of Eq. (185) is well-defined. It is onto, because the \( \{e_\mu\}_\mu \) form a basis of \( \text{Sym}_d(C^d) \) and the rank-one outer products \( \{e_\mu e_\nu^*\}_{\mu, \nu} \) constitute a basis of the space of linear maps \( L(\text{Sym}_d(C^d)) \).
As in the real case, this gives rise to a decomposition in terms of irreducible [71, Chapter 12.2] representations of \( \ker \text{tr} \), that the space of linear maps on \( V \) is a tensor product space: \( L(V) \simeq V \otimes V^* \). Then the isomorphism \( L((\mathbb{C}^d)^\otimes) \to (L(\mathbb{C}^d))^\otimes \) just amounts to a re-ordering of tensor factors:

\[
(u_1 \otimes \cdots \otimes u_t) \otimes (\alpha_1 \otimes \cdots \otimes \alpha_t) \mapsto (u_1 \otimes \alpha_1) \otimes \cdots \otimes (u_t \otimes \alpha_t), \quad u_i, \alpha_i \in \mathbb{C}^d. \quad (187)
\]

We make liberal and implicit use of this identification throughout the paper (e.g. in statement [3] in Proposition [1]).

The isomorphism restricts to a map

\[
L(\text{Sym}_t(\mathbb{C}^d)) \to \text{Sym}_t(L(\mathbb{C}^d)), \quad (u \otimes \cdots \otimes u) \otimes (\alpha \otimes \cdots \otimes \alpha) \mapsto (u \otimes \alpha) \otimes \cdots \otimes (u \otimes \alpha). \quad (188)
\]

However, while the l.h.s. does span \( L(\text{Sym}_t(\mathbb{C}^d)) \), there is no reason to believe that the r.h.s. spans \( \text{Sym}_t(L(\mathbb{C}^d)) \).

In fact,

\[
\dim L(\text{Sym}_t(\mathbb{C}^d)) = \binom{d + t - 1}{t} < \binom{d^2 + t - 1}{t} = \dim \text{Sym}_t(L(\mathbb{C}^d)) \quad \forall d, t \geq 2,
\]

so the map cannot be onto. Thus, the “order of \( L \) and \( \text{Sym} \)” in the above lemma cannot be interchanged.

For \( \mathbb{C}^d \), the relevant symmetry group is \( U(d) \). We now discuss its representation on \( L(\text{Sym}_t(\mathbb{C}^d)) \), where we will identify an analogue of harmonic symmetric tensors.

The complex version of the contraction map \( C \) is the partial trace \( \text{tr}_1 \). It is defined by

\[
\text{tr}_1 : L((\mathbb{C}^d)^\otimes) \to L((\mathbb{C}^d)^{(t-1)\otimes}), \quad (u_1 \otimes \cdots \otimes u_t) \otimes (\alpha_1 \otimes \cdots \otimes \alpha_t) \mapsto \alpha_1(u_1) (u_2 \otimes \cdots \otimes u_t) \otimes (\alpha_2 \otimes \cdots \otimes \alpha_t).
\]

Using the isomorphism \( (187) \), we can equivalently define the partial trace by its action on \( (L(\mathbb{C}^d))^\otimes \), where it takes the form

\[
\text{tr}_1 : A_1 \otimes \cdots \otimes A_t \mapsto (\text{tr} A_1) A_2 \otimes \cdots \otimes A_t, \quad A_i \in L(\mathbb{C}^d).
\]

The origin of the name “partial trace” becomes most apparent in this formulation. In any case, it is clear that the space \( \ker \text{tr}_1 \) is invariant under the action by conjugation

\[
A \mapsto U^{\otimes t} A (U^*)^{\otimes t}
\]

of \( U(d) \) on \( L((\mathbb{C}^d)^\otimes) \). We now define the complex harmonic tensors of bi-degree \((t, t)\) to be

\[
H_{(t, t)}(\mathbb{C}^d) := \ker \text{tr}_1 \cap L(\text{Sym}_t(\mathbb{C}^d)).
\]

As in the real case, this gives rise to a decomposition

\[
L(\text{Sym}_t(\mathbb{C}^d)) \simeq \bigoplus_{i=0}^t H_{(i, i)}(\mathbb{C}^d) \quad (189)
\]

in terms of irreducible [71, Chapter 12.2] representations of \( U(d) \).

In the language of polynomials, the partial trace maps to the complex Laplacian up to a normalization factor:

\[
\Delta_C x^\mu x^\nu = \sum_{i=1}^d \mu_i \nu_i x^{\mu-e_i} x^{\nu-e_i},
\]

\[
\text{tr}_1 \epsilon_\mu \epsilon^*_\nu = \frac{1}{t^2} \sum_{i=1}^d \mu_i \nu_i \epsilon_\mu \epsilon^*_\nu e^{-e_i},
\]

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so that
\[ \Delta_C p_A = t^2 p_{(t, t)} \quad \forall A \in L(\text{Sym}_t(\mathbb{C}^d)). \]
The \textit{harmonic polynomials} of bi-degree \((t, t)\) are
\[ \text{Harm}_{(t, t)}(\mathbb{C}^d) := \text{ker} \Delta_C \cap \text{Hom}_{(t, t)}(\mathbb{C}^d). \]
An embedding of \(\text{Harm}_{(t-j, t-j)}(\mathbb{C}^d) \to \text{Hom}_{(t, t)}(\mathbb{C}^d)\) is given by \(p \mapsto p'\), with
\[ p'(\bar{x}, x) = ||x||_2^2 p(\bar{x}, x). \]
For \(\text{Harm}_{(t-j, t-j)} \to L(\text{Sym}_t(\mathbb{C}^d))\), this corresponds to
\[ A \mapsto P[I] (\mathbb{I}^{\otimes j} \otimes A) P[I], \tag{190} \]
where \(P[I]\) is the projector onto \(\text{Sym}_t(\mathbb{C}^d)\).

In the current language, a \textit{complex projective} \(t\)-\textit{design} for \(\mathbb{C}^d\) is a set \(X\) of normalized vectors such that
\[ \frac{1}{|X|} \sum_{\psi \in X} (|\psi\rangle \langle \psi|)^{\otimes t} \in H_{(0,0)}(\mathbb{C}^d). \]
Let \(G \subset U(d)\) be a finite subgroup of \(O(d)\). As in the real case, let \(\{B_j^{(2i)}\}_j\) be an orthonormal basis of the \(G\)-invariant linear maps in \(H_{(i,i)}(\mathbb{C}^d)\). For \(X = G \cdot \psi_0\), we get
\[ \frac{1}{|X|} \sum_{\psi \in X} (|\psi\rangle \langle \psi|)^{\otimes t} = \sum_{j,i} B_j^{(2i)} \text{tr} \left( B_j^{(2i)} (|\psi_0\rangle \langle \psi_0|)^{\otimes t} \right) = \sum_{j,i} B_j^{(2i)} p_{B_j^{(2i)}}(\tilde{\psi}_0, \psi_0). \]
In particular, \(G \cdot \psi_0\) is a complex projective \(t\)-design if and only if \(p_{B_j^{(2i)}}(\tilde{\psi}_0, \psi_0) = 0\) for all \(i \geq 1\) and all \(j\). This is equivalent to saying that \(\psi_0\) is a root of all \(G\)-invariant harmonic polynomials.

Comparing these conditions with the analogues ones of the real case, we note that a \textit{real spherical} \(t\)-\textit{design} is defined in terms of powers \(\psi^{\otimes t}\) of degree \(t\), while \textit{complex projective} \(t\)-\textit{designs} depend on the behavior of \((|\psi\rangle \langle \psi|)^{\otimes t}\) of \(t\)-degree \(t\). Therefore, complex projective \(t\)-designs are often conceptually close to real spherical designs of degree \(2t\) or even \(2t + 1\). One example of such a connection is discussed in Section 4.6.

If \(G\) affords no harmonic invariants of degree \((s, s)\) for \(1 \leq s \leq t\), then the orbit of any vector \(\psi_0\) constitutes a complex projective \(t\)-design. Using Schur’s Lemma and the identification of \(\text{Hom}_{(t,t)}(\mathbb{C}^d)\) with \(L(\text{Sym}_t(\mathbb{C}^d))\), this condition is equivalent to demanding that \(G\) acts irreducibly on \(\text{Sym}_t(\mathbb{C}^d)\).

For real actions and spherical designs, irreducibility is sufficient but \textit{not} necessary \[24\]. The crucial difference seems to be that
\[ \text{Hom}_{(t,t)}(\mathbb{C}^d) \simeq L(\text{Sym}_t(\mathbb{C}^d)), \quad \text{while} \quad \text{Hom}_t(\mathbb{R}^d) \not\simeq L(\text{Sym}_{t/2}(\mathbb{R}^d)). \]
Indeed, let \(f \in \text{Sym}_t(\mathbb{R}^d) \subset (\mathbb{R}^d)^{\otimes t}\) for \(t\) even. The standard inner product on \(\mathbb{R}^d\) induces a linear isomorphism \(\mathbb{R}^d \simeq (\mathbb{R}^d)^*\). Applying this isomorphism to the “rear \(t/2\)” factors, we obtain an isomorphism
\[ (\mathbb{R}^d)^{\otimes t} \simeq (\mathbb{R}^d)^{\otimes t/2} \otimes ((\mathbb{R}^d)^*)^{\otimes t/2} \simeq L((\mathbb{R}^{t/2})^{\otimes t/2}). \]
Explicitly,
\[ u_1 \otimes \cdots \otimes u_t \mapsto (u_1 \otimes \cdots \otimes u_{t/2})(u_{t/2+1} \otimes \cdots \otimes u_t)^*. \tag{191} \]
The image of \(\text{Sym}_t(\mathbb{R}^d)\) under this isomorphism is a subset
\[ \text{MSym}_{t/2} \subset L(\text{Sym}_{t/2}(\mathbb{R}^d)), \]
which, following \[28\], we refer to as the set of \textit{maximally symmetric matrices}. These are linear maps whose range and support are on the totally symmetric subspace, and which are in addition invariant under the partial transpose operation. With respect to a product basis, these matrices are invariant not only under permutations of covariant \textit{or} contravariant indices amongst themselves, but in addition under permutations of arbitrary indices \[28\].

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