On the growth of modular symbols

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Introduction

Let $f$ be a holomorphic cusp form of weight two for the group $\Gamma = \Gamma_0(N)$. Then $f(z)dz$ is a $\Gamma$-invariant holomorphic differential on the upper half plane $\mathbb{H}$ in $\mathbb{C}$. For $\gamma \in \Gamma$ define the modular symbol

$$\langle \gamma, f \rangle \overset{\text{def}}{=} -2\pi i \int_{z_0}^{\gamma z_0} f(z) \, dz,$$

which is independent of the choice of the point $z_0 \in \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$. The modular symbol for fixed $f$ is a group homomorphism from $\Gamma$ to the additive group of complex numbers. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ let $\|\gamma\| = \max(|a|, |b|, |c|, |d|)$.

In this paper we show that for given $f$ the modular symbol as logarithmic growth, i.e.,

$$|\langle \gamma, f \rangle| \leq A \log \|\gamma\| + B$$

for some $A, B \geq 0$. In [2, 3], D. Goldfeld conjectured that the modular symbol has moderate growth if also $f$ is allowed to vary among the normalized newforms. The result of the present paper reduces Goldfeld’s conjecture to a statement on the growth of the modular symbol on a set of generators of the group $\Gamma_0(N)$.

Note that, as $f$ is a cusp form, the modular symbol vanishes on parabolic elements, that is, $\langle p, f \rangle = 0$ for every parabolic element $p$ of $\Gamma$. 

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1 Growth of an additive homomorphism

Let $SL_2(\mathbb{R})$ denote the group of real $2 \times 2$ matrices of determinant one. Let $G$ be the group $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\pm 1$.

For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ let $\|x\| = \max(|a|, |b|, |c|, |d|)$. Let $M$ be a subset of $G$. A function $f: M \to \mathbb{C}$ is said to be of logarithmic growth, if there are constants $A, B \geq 0$ such that

$$|f(x)| \leq A \log \|x\| + B$$

holds for every $x \in M$.

**Theorem 1.1** Let $\Gamma$ be a lattice in $PSL_2(\mathbb{R})$ and let $\psi: \Gamma \to \mathbb{C}$ be a group homomorphism with $\psi(p) = 0$ for every parabolic element of $\Gamma$. Then $\psi$ is of logarithmic growth.

**Proof:** Let $\mathbb{H}$ denote the upper half plane in $\mathbb{C}$ equipped with the hyperbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. Choose $z_0 \in \mathbb{H}$ which is not a fixed point of an elliptic element of $\Gamma$, and let $\mathcal{F}$ denote the corresponding Dirichlet fundamental domain, also called the Dirichlet polygon [1], as it is a hyperbolic polygon with finitely many sides. It is defined as

$$\mathcal{F} = \{z \in \mathbb{H} : d(z, z_0) < d(\gamma z, z_0) \forall \gamma \in \Gamma \setminus \{1\}\},$$

where $d(z, w)$ denotes the hyperbolic distance of two points in the upper half plane $\mathbb{H}$. Let $S = S^{-1}$ be a finite set of generators of the group $\Gamma$. For $\gamma \in \Gamma$ we write $\gamma = s_1 \cdots s_n$ as a shortest word, so $n = l_S(\gamma)$, the word length of $\gamma$ with respect to $S$. Then,

$$|\psi(\gamma)| = |\psi(s_1 \cdots s_n)| = \left| \sum_{j=1}^{n} \psi(s_j) \right| \leq C_S l_S(\gamma),$$

where $C_S \geq 0$ is the maximum of the values $|\psi(s)|$ for $s \in S$.

We first consider the case of a uniform lattice $\Gamma$, i.e., the quotient $\Gamma \setminus G$ is compact. Then the closure $\bar{\mathcal{F}}$ of $\mathcal{F}$ in $\mathbb{H}$ is compact. According to Theorem IV
23 of [4], for every \( z \in \mathbb{H} \) there exists \( \lambda \geq 1, C \geq 0 \) with \( l_S(\gamma) \leq \lambda d(\gamma z, z) + C \) for every \( \gamma \in \Gamma \). For \( z = i \) this implies

\[
|\psi(\gamma)| \leq C_S \lambda d(\gamma i, i) + C_S C.
\]

By Theorem 7.2.1 of [1] one has for \( z, w \in \mathbb{H} \),

\[
d(z, w) = \log \left( \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|} \right).
\]

Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) we get

\[
d(\gamma i, i) = \log \left( \frac{|a i + b c + d i| + |a i + b + c - d i|}{|a i + b c + d i| - |a i + b + c - d i|} \right)
\]

\[
= \log \left( \frac{\sqrt{(b - c)^2 + (a + d)^2} + \sqrt{(b + c)^2 + (a - d)^2}}{\sqrt{(b - c)^2 + (a + d)^2} - \sqrt{(b + c)^2 + (a - d)^2}} \right).
\]

Now \( (b - c)^2 + (a + d)^2 - ((b + c)^2 + (a - d)^2) = -4bc + 4ad = 4 \), as the determinant of \( \gamma \) is 1. Therefore,

\[
d(\gamma i, i) = \log \left( \left( \frac{\sqrt{(b - c)^2 + (a + d)^2} + \sqrt{(b + c)^2 + (a - d)^2}}{2} \right)^2 \right) - 2 \log 2
\]

\[
\leq 2 \log \left( \sqrt{8 \| \gamma \|^2} + \sqrt{8 \| \gamma \|^2} \right) - 2 \log 2
\]

\[
= 2 \log \| \gamma \| + 3 \log 2.
\]

This gives the Theorem in the case of \( \Gamma \) being a uniform lattice.

Next assume that \( \Gamma \) is not uniform. Then the fundamental domain \( \mathcal{F} \) has cusps. We assume that \( z_0 \) is chosen in a way that no two cusps of \( \mathcal{F} \) are equivalent under \( \Gamma \). So the cusps of \( \mathcal{F} \) are a set of representatives of the set of cusps of \( \Gamma \) in the boundary \( \partial \mathbb{H} \) of \( \mathbb{H} \) modulo \( \Gamma \)-equivalence. For each cusp \( c \in \partial \mathbb{H} \) fix some \( \sigma_c \in G \) with \( c = \sigma_c \infty \). We do so in a \( \Gamma \)-compatible way, i.e., for \( \gamma \in \Gamma \) we suppose that \( \sigma_{\gamma c} \sigma_c^{-1} \) lies in \( \Gamma \).
Let $T > 1$ and set

$$\mathcal{F}_T = \{ z \in \mathcal{F} : \text{Im}(\sigma_c^{-1}z) \leq T \text{ for every cusp } c \}.$$ 

We choose $T$ so large that $\mathcal{F}_T$ equals $\mathcal{F}$ minus cusp sections. Let $\mathbb{H}_T$ be the union of all sets $\gamma \mathcal{F}_T$ where $\gamma$ ranges over $\Gamma$. Then $\mathbb{H}_T$ equals $\mathbb{H}$ minus a countable number of open horoballs. Thus $\mathbb{H}_T$ is a Riemannian manifold with boundary. Let $d_T$ denote the distance function on $\mathbb{H}_T$. Note that if $z, w \in \mathbb{H}_T$, and the geodesic in $\mathbb{H}$ joining them lies completely in $\mathbb{H}_T$, then $d_T(z, w) = d(z, w)$.

For a cusp $c$ let

$$H_{c,T} = \sigma_c(\{ \text{Im}(z) > T \})$$

be the $T$-horoball attached to $c$. Increasing $T$ if necessary, we can make sure, that the geodesic $z_0, \gamma z_0$ in $\mathbb{H}$ is disjoint to $\mathcal{F} \cap H_{c,T}$ for every cusp $c$ of $\mathcal{F}$ and every $\gamma \in \Gamma$. Note that for every cusp $c$ of $\Gamma$, which is not a cusp of $\mathcal{F}$, the intersection $\mathcal{F} \cap H_{c,T}$ is empty.

Let $\gamma \in \Gamma$ and suppose that the geodesic $z_0, \gamma z_0$ in $\mathbb{H}$ does not completely lie in $\mathbb{H}_T$. Then this geodesic meets some horoball $H_{c,T}$. After applying $\sigma_c^{-1}$, one can assume $c = \infty$. Then there is a generator $p_c$ of the stabilizer group $\Gamma_c$ of the cusp $c$, such that the distance $d(z_0, p_c \gamma z_0)$ is then strictly less than $d(z_0, \gamma z_0)$. From this it follows that if $d(z_0, \gamma z_0) \leq d(z_0, p \gamma z_0)$ for every parabolic $p \in \Gamma$, then the geodesic $z_0, \gamma z_0$ lies completely in the set $\mathbb{H}_T$.

By Theorem IV 23 in [4], there are $\lambda \geq 1$ and $C \geq 0$ with $l_S(\gamma) \leq \lambda d_T(z_0, \gamma z_0) + C$ for every $\gamma \in \Gamma$.

**Lemma 1.2** Let $\gamma \in \Gamma \setminus \{1\}$ be given. There are parabolic elements $p_1, \ldots, p_n$ of $\Gamma$ such that with $\gamma_s = p_n \cdots p_1 \gamma$ one has $d(z_0, \gamma_s z_0) \leq d(z_0, \gamma z_0)$ and the geodesic $z_0, \gamma_s z_0$ lies in $\mathbb{H}_T$.

**Proof:** Assume first that for every parabolic element $p \in \Gamma$ the distance $d(z_0, \gamma z_0)$ is less than or equal to $d(z_0, p \gamma z_0)$. Then the geodesic $z_0, \gamma z_0$ lies in $\mathbb{H}_T$. We set $\gamma_s = \gamma$ and we are done.

Now if there exists a parabolic $p_1 \in \Gamma$ such that $d(z_0, p_1 \gamma z_0) < d(z_0, \gamma z_0)$, then replace $\gamma$ with $p_1 \gamma$. After that, either the condition above is satisfied or we find a parabolic $p_2$ such that $d(z_0, p_2 p_1 \gamma z_0) < d(z_0, p_1 \gamma z)$.

Iteration yields
a sequence $p_1, p_2, \cdots \in \Gamma$. This process terminates, as for a given radius $r$ there are only finitely many $\Gamma$-conjugates of $z_0$ in distance $\leq r$. □

To finish the proof of the theorem let $\gamma \in \Gamma$ and consider $\gamma_s$ as in the lemma. Then $|\psi(\gamma)| = |\psi(\gamma_s)| \leq C_S l_S(\gamma_s) \leq C_S \lambda d_T(z_0, \gamma_s z_0) + C_S C$ and

$$d_T(z_0, \gamma_s z_0) = d(z_0, \gamma_s z_0) \leq d(z_0, \gamma z_0) \leq d(z_0, i) + d(i, \gamma i) + d(\gamma i, \gamma z_0) \leq 2 \log ||\gamma|| + 3 \log 2 + 2d(z_0, i).$$

□

The value of these results with respect to the Goldfeld conjecture hinges on the control over the constants $C, C_S, \lambda$ as the group $\Gamma$ shrinks. The constant $C_S$ depends on the group $\Gamma$ and on the homomorphism $\psi$, i.e., if $\psi$ is a modular symbol, on the cusp form $f$. The constants $C$ and $\lambda$, however, do not depend on $\psi$, therefore they are easier to control. The following explicit estimate might be useful.

Let $R$ be the diameter of $\mathcal{F}_T$ and let $B$ be the closed ball in $\mathbb{H}_T$ around $z_0$ of radius $R$. Let $S = \{s \in \Gamma : sB \cap B \neq \emptyset\}$. Then $S = S^{-1}$ is a finite set of generators of $\Gamma$. Let

$$r = \inf\{d(B, \gamma B) : \gamma \in \Gamma \setminus S\}.$$

**Lemma 1.3** The number $r$ is $> 0$ and

$$|\psi(\gamma)| \leq C_S \left( \frac{2}{r} \log ||\gamma|| + \frac{3}{r} \log 2 + 1 \right).$$

**Proof:** Let $r_T = \inf\{d_T(B, \gamma B) : \gamma \in \Gamma \setminus S\}$. The proof of Theorem IV 23 of [4] together with the proof of our theorem yields

$$|\psi(\gamma)| \leq \frac{C_S}{r_T} \left( 2 \log ||\gamma|| + 3 \log 2 + 2d(z_0, i) \right) + C_S$$

Now $r \geq r_T$ and the argument for $r_T > 0$ also implies $r > 0$. Further, varying $z_0$ the distance $d(z_0, i)$ can be chosen arbitrarily small. □
References

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