Resonant, non-resonant and anomalous states of Dirac electrons in a parabolic well in the presence of magnetic fields

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Abstract

We report on several new basic properties of a parabolic dot in the presence of a magnetic field. The ratio between the potential strength and the Landau level (LL) energy spacing serves as the coupling constant of this problem. In the weak coupling limit the energy spectrum in each Hilbert subspace of an angular momentum consists of discrete LLs of graphene. In the intermediate coupling regime non-resonant states form a closely spaced energy spectrum. We find, counter-intuitively, that resonant quasi-bound states of both positive and negative energies exist in the spectrum. The presence of resonant quasi-bound states of negative energies is a unique property of massless Dirac fermions. As the strong coupling limit is approached resonant and non-resonant states transform into anomalous states, whose probability densities develop a narrow peak inside the well and another broad peak under the potential barrier. These properties may investigated experimentally by measuring optical transition energies that can be described by a scaling function of the coupling constant.

(Some figures may appear in colour only in the online journal)

1. Introduction

Two-dimensional parabolic quantum dots of semiconductor heterostructures have been studied widely [1, 2], both experimentally and theoretically, because they are excellent candidates for single electron transistors. They can effectively confine electrons and the number of electrons in them can be controlled using a gate potential. Their Hamiltonian is

\[ H = \frac{1}{2m} \left( \vec{p} + \frac{e}{c} \vec{A} \right)^2 + \frac{1}{2} m \omega_c^2 r^2 \]  

(1)

with a magnetic field \( \vec{B} \) applied perpendicular to the two-dimensional (2D) plane (vector potential \( \vec{A} \) is given in a symmetric gauge). The characteristic length scale of the problem is given by \( \lambda^2 = \frac{\hbar}{m \sqrt{4 \omega_c^2 + \omega_B^2}} \), where \( \omega_c = \frac{e B}{mc} \) is the cyclotron frequency. This problem can be solved exactly [3], and the eigenenergies are all positive and their spectrum is discrete.

Massless Dirac electrons [4] moving in a 2D parabolic potential display several different features in comparison to electrons with mass. They are described by the Dirac Hamiltonian

\[ H = \gamma_1 \vec{p} \cdot \vec{\sigma} + \frac{e}{c} \vec{A} \]  

(2)

No exact solutions of this problem are known in graphene, and several fundamental properties are still unknown, such as the existence of resonant and non-resonant states. These basic properties may affect the experimentally relevant optical spectrum in a profound way. The dimensionless coupling constant of this problem is the ratio between the strength of the potential \( \frac{1}{2} \kappa \ell^2 \) and the LL energy separation \( E_C = \frac{h \nu_F}{\ell} \)

\[ \alpha = \frac{\kappa \ell^2}{E_C} = \frac{\kappa \ell^3}{h \nu_F}, \]  

(3)

where the magnetic length is \( \ell = \sqrt{\hbar c / \mu B} \). One is in the strong coupling regime \( \alpha \gg 1 \) for small values of \( B \) or large values of potential strength \( \kappa \). Parabolic dots in magnetic
fields have been investigated numerically in the weak coupling regime $\alpha < 1$. The energy spectrum is found to be discrete, and eigenstates are quasi-bound states with long oscillating tails under the barrier [5–7]. Also some of these states exhibit anticrossings [5, 7]. The parabolic potential acts as a singular perturbation [8] because eigenstate wavefunctions are qualitatively different from those in the absence of a parabolic potential. The problem has not been investigated away from the weak coupling regime. It is a highly non-trivial problem. This can be seen as follows. One of the special features of graphene LLs is the presence of negative energy states under the potential barrier [7]. The first order energy correction of a LL state $\psi_{n,m}(r)$ is, for sufficiently large $n$,

$$\langle \psi_{n,m}|V(r)|\psi_{n,m}\rangle \sim \kappa \left(r^2\right) \sim \kappa \ell^2 |n|. \quad (4)$$

This result suggests that a LL state with a large negative energy, $-E_C\sqrt{2|n|}$ with $|n| \gg 1$, corresponding to having a large average radius $\sqrt{\left(r^2\right)}$, acquires a significant positive energy correction, which can make the renormalized energy positive. In dimensionless units this energy correction is $\kappa \ell^2 |n|/E_C = \alpha |n|$, which suggests that even for small values of $\alpha$ the correction can be significant for $|n| \gg 1$. Moreover, it is unclear how eigenstates evolve from weak to strong coupling regimes. A simple dimensional analysis suggests that the energy scale of the problem in the strong coupling limit of $B \rightarrow 0$ or $\alpha \rightarrow \infty$ is $\kappa^{1/3}(\hbar v_p)^{2/3}$. In units of $E_C$ this energy scale is $\alpha^{1/3}$. It indicates that the dimensionless energy level spacing increases from $\sim 1$ to $\sim \alpha^{1/3}$ as one moves from a weak to a strong coupling regime. However, studies in ordinary semiconductors suggest that the crossover regime may be non-trivial.

We have investigated these issues by solving large Hamiltonian matrices. Let us give a brief summary of our results in a Hilbert subspace of angular momentum $J$. We have studied how eigenvalues and eigenstates evolve as $\alpha$ increases and find that they change in a non-trivial way. In the weak coupling limit of $\alpha \rightarrow 0$ the spectrum consists of discrete LLs. In the intermediate coupling regime $\alpha \sim 1$ non-resonant states form a closely spaced energy spectrum (see figure 1). We find, counter-intuitively, that resonant quasi-bound states of both positive and negative energies exist. The presence of resonant quasi-bound states of negative energies is a unique property of massless Dirac fermions, but they are well defined only for $\alpha < 1$. In the strong coupling regime $\alpha \gg 1$ both resonant and non-resonant states transform into anomalous states, and a sharp distinction between resonant and non-resonant states no longer exists. Probability densities of anomalous states develop a narrow peak inside the well and decay slowly with small oscillations under the barrier. The energy level spacing between them is proportional to the value $\kappa^{1/3}(\hbar v_p)^{2/3}$ and is independent of $\ell$. We show that optical transition energies between resonant quasi-bound states can be described by a scaling function of $\alpha$.

2 Reference [5] contains an interesting investigation of confinement and definement transitions in various potentials. For a Comment on [6] see [7].

3 A similar problem has been investigated for an impurity in an ordinary semiconductor from zero to finite magnetic fields; see [9].

Figure 1. Schematic energy spectrum of a parabolic dot in the intermediate coupling regime $\alpha \sim 1$. The energy spectrum is closely spaced, and resonant states of positive and negative energies are present.

2. Basis states and the Hamiltonian matrix

In our Hamiltonian matrix approach the basis states are chosen as graphene LL states $\psi_{n,m}(\vec{r})$ with two components $A$ and $B$:

$$\psi_{n,m}(\vec{r}) = c_n \left(-\text{sgn}(n) i\phi_{|n|,1-m}(\vec{r})\right). \quad (5)$$

Here $\text{sgn}(n) = -1, 0, 1$ for $n < 0, n = 0, n > 0$ with $n$ and $m$ integers ($m \geq 0$), and $c_n = 1$ for $n = 0$ and $1/\sqrt{2}$ otherwise. These basis states can have positive or negative LL energies:

$$E_n = \text{sgn}(n)E_C\sqrt{2|n|}. \quad (6)$$

The wavefunctions $\phi_{n,m}(\vec{r})$ are the LL wavefunctions of ordinary 2D systems [10]

$$\phi_{n,m}(\vec{r}) = A_{n,m} \exp\left(i(n-m)\theta - \frac{r^2}{2\ell^2}\right) \left(\frac{r}{\ell}\right)^{|m-n|} \times L_{|n-m|/2}^{|m-n|}(\frac{r^2}{2\ell^2}), \quad (7)$$

where $A_{n,m}$ are the normalization constants and $L_{n}^{m}(x)$ are Laguerre polynomials. In the presence of a parabolic potential $J = |n| - m - \frac{1}{2}$ remains a good quantum number. The average radius of $\phi_{n,m}(\vec{r})$ is given by

$$\left<r^2\right> = 2\ell^2(n + m + 1) = 2\ell^2(n + |n| - J + 1/2). \quad (8)$$

To investigate the strong coupling effects a large number of basis states $\psi_{n,m}(\vec{r})$ are required. It is convenient to divide the Hilbert space into subspaces of angular momentum $J = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \ldots$ We diagonalize the Hamiltonian matrix in each Hilbert subspace $J$. For given $J$, the matrix elements of the parabolic potential can be written as sum of two components:

$$\frac{\langle \psi_{n,m}|V(r)|\psi_{n',m'}\rangle}{E_C} = c_n c_{n'} \text{sgn}(m') \langle \phi_{|n|,1-m}|V(r)|\phi_{|n'|,1-m'}\rangle + \frac{V(r)}{E_C} \langle \phi_{|n|,1-m}|\psi_{n',m'}\rangle \langle \phi_{|n'|,1-m'}|\psi_{n,m}\rangle. \quad (9)$$
Using the following property of Laguerre polynomials
\[
L_n^\alpha(x) = \frac{1}{\Gamma(\alpha + 1)} (n + \alpha + 1) L_{n-1}^{\alpha-1}(x) - (n + 1) L_{n+1}^{\alpha-1}(x),
\]
and the orthogonality
\[
\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) \, dx = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{n,m},
\]
we evaluate the matrix elements. The resulting matrix is a sparse matrix (see figure 2). The dimension of the Hamiltonian matrix is denoted by \( N_c \). When the value of \( N_c \) is sufficiently large the states investigated in this paper do not exhibit dependence on \( N_c \).

### 3. Eigenstates of a Hilbert subspace

Eigenstates of a Hilbert subspace are obtained by diagonalizing the Hamiltonian matrix. They may be written as a linear combination of LL wavefunctions with same angular momentum:
\[
\Psi_N^J(r) = \sum_n C_n \psi_{n,m}(r).
\]
Here quantum number \( N \) is chosen to be the value \( n \) for which \( |C_n| \) is maximum\(^4\).

The following exact results [11] are useful in checking numerical results. The value of wavefunctions at \( r = 0 \) is non-zero only for \( J = -\frac{1}{2} \) and \( \frac{1}{2} \):
\[
\Psi_N^J(0) = \begin{cases} 
0 & \text{for } J \neq \pm \frac{1}{2} \\
\text{finite} & \text{for } J = \pm \frac{1}{2}.
\end{cases}
\]

\(^4\) In the strong coupling limit many levels with different values of \( n \) are heavily mixed.

The \( B \) and \( A \) components of \( |\Psi_N^{-1/2}\rangle \) and \( |\Psi_N^{1/2}\rangle \) are non-zero and are of s-wave type. Their values at \( r = 0 \) can be written as
\[
|\Psi_N^{-1/2}\rangle_B(0) = \sum_{\alpha \neq 0} A_{\alpha|\alpha|} C_n / \sqrt{2} + C_0 A_{00},
\]
\[
|\Psi_N^{1/2}\rangle_A(0) = -i \sum_n \text{sgn}(n) A_{|\alpha|\alpha} - 1 C_n / \sqrt{2}.
\]

We will only concentrate on optical transitions involving states with \( J = 1/2, -1/2 \), or \(-3/2\) since they give the strongest optical strengths. Unless stated otherwise the results reported in this paper are for the potential strength \( \kappa = 0.1 \text{ meV nm}^{-2} \).

#### 3.1. Resonant quasi-bound states

The computed energy spectrum of \( J = -1/2 \) is shown in figure 3(a) for the range \( 0.029 < \alpha < 0.47 \) (3.14 \( T < B < 20 \) T). Spectra for other values of \( J = 1/2 \) and \(-3/2\) are similar, as shown in section 4. Lines labeled by \( N \) represent the energies of resonant quasi-bound states and other lines represent non-resonant states. As shown in figure 3(b), in the intermediate coupling regime \( \alpha \sim 1 \), the energy levels are closely spaced due to negative energy LL states whose energies get strongly perturbed upward by the parabolic potential (see equation (4)). A resonant quasi-bound state anticrosses other states and becomes strongly mixed with the adjacent states (see figure 3(b)). For example, the resonant quasi-bound state \( |\Psi_0^{-1/2}\rangle \) is strongly mixed (see footnote 1) at \( B = 3.14 \text{ T} \) (\( \alpha = 0.47 \)), and, as shown in figure 3(b), there are three states that could be identified as \( |\Psi_0^{-1/2}\rangle \). In these states with the energies \( E = 0.025, 0.019, \) and \( 0.010 \text{ eV} \) the expansion coefficients \( C_0 \) of equation (12) take the maximum value at \( n = 0 \) with the values \( C_0 = 0.622, 0.595, \) and \( 0.287 \), respectively (since \( C_0 \) is largest for \( E = 0.025 \text{ eV} \) this state is labeled as \( |\Psi_0^{-1/2}\rangle \)). Figure 3(c) displays the probability densities of these states. Together they form a resonance with the approximate resonant energy \( E = 0.025 \text{ eV} \). In the weak coupling regime the width of a resonance is small and, in order to resolve it, the energy level spacing must be smaller than the width of a resonance, which requires a large matrix dimension.

Note also that resonant quasi-bound states of negative energies exist. This is a unique property of massless Dirac fermions. An example is shown in figure 4. They are well defined only for sufficiently large \( B \), i.e. only in the weak coupling regime. The appearance of a second peak away from \( r = 0 \) in the probability density is different from the usual behavior of the wavefunction a resonant state.

We see that as \( \alpha \) increases, or as \( B \) decreases, resonant quasi-bound states disappear into the closely spaced energy spectrum. For \( N = 0 \) states this happens around \( B \sim 3 \text{ T} \). For larger values of \( N \) this happens at smaller values of \( B \). We will show in section 3.3 that, as \( \alpha \) increases, the peak at \( r = 0 \) increases and the state becomes anomalous.

#### 3.2. Non-resonant states

The energy spectrum of \( J = -1/2 \) in figure 3(a) also displays non-resonant states. The probability density of
Figure 3. (a) Eigenenergy spectrum of a Hilbert subspace of $J = -\frac{1}{2}$. Lines labeled by $N$ represent resonant quasi-bound states. A 2001 × 2001 matrix is used. (b) Enlarged energy spectrum for $J = -\frac{1}{2}$. Resonant quasi-bound state $|\psi_{-1/2}^{-}\rangle$ anticrosses strongly at $B = 3.14$ T ($\alpha = 0.47$). Four circles represent these coupled states. (c) Probability densities of these four states are displayed. They form together a resonance with the approximate resonant energy $\epsilon_{-1/2}^{-}\sim 3.14 = 0.025$ eV.

A non-resonant state looks qualitatively different from that of the corresponding unrenormalized LL state. The probability density of a non-resonant state $|\psi_{-1/2}^{-}\rangle$ at $B = 2.04$ T ($\alpha = 0.9$) is shown in figure 5(a). Its wavefunction has a large peak at $r = 0$, which is different from the usual behavior of the wavefunction for a non-resonant state. As $\alpha$ increases the peak at $r = 0$ increases even more and the state becomes anomalous. Non-resonant states are unique to graphene parabolic wells and do not exist in ordinary parabolic wells. In the absence of the parabolic potential its energy is $E = -\sqrt{2}\epsilon_{C} = -0.051$ eV, while in the presence of the potential it is $\epsilon_{-1/2}^{-}\sim 3.14 = 1.26\epsilon_{C} = 0.054$ eV. Figure 6 displays the energy level spacing of non-resonant states as a function of $N_{c}$ at $\alpha = 1$ ($B = 1.9$ T). We observe that the level spacing decreases rather slowly for large $N_{c}$. However, the energies of resonant quasi-bound states converge rather quickly (see figure 9).

3.3. Anomalous states

The energy spectrum of $J = -1/2$ in figure 3(a) also displays anomalous states at low magnetic fields of the strong coupling regime $\alpha \gg 1$. In this regime both resonant and non-resonant states transform into anomalous states, and a sharp distinction between resonant and non-resonant states no longer exists. Such a state is shown in figure 7(a) with the energy $E = 0.177$ eV (in units of $\epsilon_{C}$ it is 12.6). We see in figure 7(a) that the peak value of probability density at $r = 0$ is much smaller than in the resonant case.
Figure 6. Average energy level spacing of non-resonant states near $E = 0$ as a function of $N_c$ at $B = 1.9$ T ($\alpha = 1$). The dimension $N_c$ varies from 101 to 18001. Matrix sizes $N_c$ are 101, 201, 401, 801, 2001, 4001, 9001, 11001, 15001, and 18001.

Figure 7. (a) Total probability density of $|\Psi_{1/2}^0\rangle$ with energy 0.177 eV at $B = 0.312$ T ($\alpha = 15$) (solid). The corresponding state at $\alpha = 0$ is shown as a dotted line. The eigenstate is obtained by diagonalizing a $7001 \times 7001$ Hamiltonian matrix. Energy levels are also shown at $B = 0.312$ T ($\alpha = 15$) for $N_c = 7001$ and 9001. (b) Expansion coefficients $C_n = \langle n | \Psi_{1/2}^0 \rangle$ of the anomalous eigenstate $|\Psi_{1/2}^0\rangle$.

Figure 8. Dimensionless average energy level spacing of the energy spectrum of Hilbert subspace $J = -1/2$ in the strong coupling regime. We have used $N_c = 9001$.

(a) Total probability density of $|\Psi_{1/2}^0\rangle$ with energy 0.177 eV at $B = 1.9$ T ($\alpha = 1$). The dimension $N_c$ varies from 101 to 18001. Matrix sizes $N_c$ are 101, 201, 401, 801, 2001, 4001, 9001, 11001, 15001, and 18001.

(b) Expansion coefficients $C_n = \langle n | \Psi_{1/2}^0 \rangle$ of the anomalous eigenstate $|\Psi_{1/2}^0\rangle$.

larger than the unperturbed value of $\frac{1}{15} \simeq 0.16$. For this state the penetration into the barrier should start from the turning point $r_0$ satisfying $\frac{1}{2} \kappa r_0^2 = E$. From this we find that the value $r_0/\ell$ is 1.3, which is rather different from the estimate of about 0.3 obtained from numerical results shown in figure 7(a) (note that the probability density under the barrier oscillates). An anomalous state is a strong coupling effect and can only be obtained correctly by computing large Hamiltonian matrices.

Figure 7(b) displays the expansion coefficient $C_n$ as a function of $n$. Note that $C_n$ has a long oscillating tail for $n < 0$. This is intimately related to the probability density having a long oscillating tail under the barrier. The sum of $C_n$ for $n < 0$ is approximately zero, while the sum for $n \geq 0$ is finite and makes $\Psi_{1/2}^0(0)$ large (see equation (14)). Note that the probability density under the barrier is somewhat smaller than that of $\alpha = 0$.

When $B \to 0$ or $\alpha \to \infty$ the natural length and energy scales of the problem are $\xi = (\frac{\hbar \bar{\kappa}}{\alpha})^{1/3}$ and $\kappa^{1/3} (\hbar \bar{\kappa})^{2/3}$ (note $\xi/\ell = \alpha^{-1/3}$). In units of $E_C$ this energy scale is $\alpha^{1/3}$, which should be proportional to the dimensionless energy level spacing of the Hilbert subspace of $J$ in the strong coupling regime. Our numerical results in the strong coupling regime $\alpha \gg 1$ are indeed consistent with this (see figure 8).

4. Scaling of optical transitions

4.1. Scaling results

In the previous section we showed that energies of the resonant quasi-bound states depend on both $\kappa$ and $B$. Here we will show that their energies, when measured in units of $E_C$ in the limit of large $N_c$, follow a scaling function of a single dimensionless variable, namely, the dimensionless coupling constant $\alpha$ (see figure 9). Figure 10(a) displays the dimensionless energies of resonant quasi-bound states $\epsilon_{N'}(\alpha)$ as a function of $\alpha$ for $J = -1/2$ and $1/2$. Figure 10(b) displays similar results for $J = -3/2$ and $-1/2$.

Since dimensionless energies of resonant quasi-bound states $\epsilon_{N'}(\alpha)$ satisfy a scaling function the transition energies $E = \epsilon_{N'}(\alpha) - \epsilon_{N}(\alpha)$ between them also obey a scaling

$$
\frac{E}{E_C} = f_{N \to N'}(\alpha).
$$

(15)
absorption selection rules are computing them. First, in the absence of a parabolic potential transitions let us first mention some useful results in

This scaling relation holds as long as quasi-bound states are well defined.

4.2. Optical transition energies and selection rules

Before we compute strengths and selection rules of optical transitions let us first mention some useful results in computing them. First, in the absence of a parabolic potential absorption selection rules are \( \epsilon_{N}^{1/2}(\alpha) \rightarrow \epsilon_{N+1}^{1/2}(\alpha) \) for \( N \geq 0 \) and \( \epsilon_{N}^{-1/2}(\alpha) \rightarrow \epsilon_{N+1}^{-1/2}(\alpha) \) for \( N < 0 \). These selection rules are displayed schematically in figure 11 (see also table II in [11]). Due to mixing of different LL states by the parabolic potential the selection rule \( \Delta N = \pm 1 \) must be relaxed. Second, the energies of the \( N = 0 \) LL states are split by the perturbing parabolic potential and they increase as \( J \) decreases: \( \epsilon_{0}^{-1/2}(\alpha) < \epsilon_{0}^{-3/2}(\alpha) < \epsilon_{0}^{-5/2}(\alpha) < \cdots \). This is also true for other LL states. Third, the optical transitions with the greatest strength occur between \( J = 1/2 \) and \( -1/2 \). Other transitions, for example transitions between \( J = -3/2 \) and \( -1/2 \), are weaker: the transition \( \epsilon_{0}^{-1/2}(0.25) \rightarrow \epsilon_{1}^{-1/2}(0.25) \) has a strength of 0.37 with transition energy 1.44\( E_{c} \) while the transition \( \epsilon_{0}^{3/2}(0.25) \rightarrow \epsilon_{1}^{-1/2}(0.25) \) has the strength 0.21 with transition energy 1.34\( E_{c} \) (the involved energy levels are shown in figure 10). These strengths and the corresponding transition energies decrease as the value \( J \) of initial states of optical transitions decreases.

Let us use these results to understand what types of optical transitions are possible. For this purpose we display in figure 12, for a relatively small value of \( N_{c} \), eigenenergies for \( J = -1/2 \) and \( 1/2 \) (results are qualitatively similar to those of a large value of \( N_{c} \)). The absorption transition strength between initial and final states \( \Psi_{1} \) and \( \Psi_{2} \) is \( |\langle \Psi_{1}|\sigma_{\alpha}|\Psi_{2}\rangle|^{2} \). Note that there are possible transitions from resonant quasi-bound states of negative energies, which is a new feature. An example of this transition is shown as (f) in figure 12. Its strength is 0.386. Also, transitions between resonant quasi-bound states of positive energies are possible. An example is shown as (e) with the strength 0.461. Energy of a resonant quasi-bound state will split into several values at small \( B \) due to anticrossing with other levels. This will lead to a splitting of transitions. An example is shown as (g) and (h). Some examples of absorption transitions involving non-resonant states are also shown. For these transitions, due to mixing of different LL states by the parabolic potential, the selection rule
The energy spectra in a magnetic field are discrete without the Coulomb interaction, while the energy spectra for a parabolic dot in a magnetic field evolve as $\alpha \rightarrow 0$ one recovers the discrete LL spectrum of graphene. In the absence of a parabolic potential the selection rules are $\varepsilon_{N+1/2}(\alpha) \rightarrow \varepsilon_{N+1/2}(\alpha)$ for $N \geq 0$ and $\varepsilon_{N+1/2}(\alpha) \rightarrow \varepsilon_{N+1/2}(\alpha)$ for $N < 0$. Similar selection rules hold for other possible $(J, J')$ with $\Delta J = \pm 1$.

5. Summary

In each Hilbert subspace of angular momentum $J$ we have studied how eigenvalues and eigenstates of a parabolic dot in a magnetic field evolve as $\alpha$ increases. We have found that they change in a non-trivial way. In the weak coupling limit of $\alpha \rightarrow 0$ one recovers the discrete LL spectrum of graphene. In the intermediate coupling regime non-resonant states form a closely spaced energy spectrum (see figure 3(a)). In the absence of a parabolic potential the selection rules are $\varepsilon_{N+1/2}(\alpha) \rightarrow \varepsilon_{N+1/2}(\alpha)$ for $N \geq 0$ and $\varepsilon_{N+1/2}(\alpha) \rightarrow \varepsilon_{N+1/2}(\alpha)$ for $N < 0$. Similar selection rules hold for other possible $(J, J')$ with $\Delta J = \pm 1$.

Figure 12. To display clearly possible optical transitions we plot together, for small $N_0 = 101$ and 100, the energy spectra for $J = -\frac{1}{2}$ (black dots) and $\frac{1}{2}$ (white dots). The value of the strength of the potential $k = 0.1$ meV nm$^{-2}$ corresponds to $\alpha = 1.424$ and $\alpha = 0.116$ at $B = 1.5$ T and $B = 8$ T, respectively. Quasi-bound state energies $\varepsilon^J_0(\alpha)$ are labeled by $(N, J)$.

$\Delta N = \pm 1$ must be relaxed. Examples of these transitions are shown as (a)–(d) in figure 12. Their absorption strengths are 0.255 (a), 0.316 (b), 0.202 (c), and 0.207 (d).

The next dominant absorption transitions occur for $(J, J') = (-3/2, -1/2)$. In the absence of a parabolic potential the selection rules are $\varepsilon_{N+3/2}(\alpha) \rightarrow \varepsilon_{N+1/2}(\alpha)$ for $N \geq 0$ and $\varepsilon_{N-1/2}(\alpha) \rightarrow \varepsilon_{N+3/2}(\alpha)$ for $N < 0$. Similar selection rules hold for other possible $(J, J')$ with $\Delta J = \pm 1$.

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6 Properties of a single antidot and an array of periodic antidots are investigated in the papers in [12].

7 Ferromagnetism of a one-dimensional electron gas of an armchair ribbon is investigated in this paper.