Infinite Bar-Joint Frameworks

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ABSTRACT
Some aspects of a mathematical theory of rigidity and flexibility are developed for general infinite frameworks and two main results are obtained. In the first sufficient conditions, of a uniform local nature, are obtained for the existence of a proper flex of an infinite framework. In the second it is shown how continuous paths in the plane may be simulated by infinite Kempe linkages.

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Infinite Equation Sets, Materials Analysis, Geometric Constraints

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bar-joint framework, rigidity matrix, rigidity operator, compactness, Kempe linkage

1. INTRODUCTION
We describe some results and work in progress in the analysis of infinite bar-joint frameworks, their constraint systems and their solution spaces. In particular we are interested in forms of flexibility and rigidity.

The behaviour of some physical systems, such as flexible materials (e.g. foam [3]) or the positioning of large arrays of components in an engineering design, may be approximated by a large number of polynomial equations and effectively modeled by an infinite equation set. Recently Deshpande et al [4], Guest and Hutchinson [5], Donev and Torquato [6] and others have considered rigidity issues for infinite systems of geometric constraints. Examples are important. We give several contrasting examples together with a range of concepts and terminology aimed at differentiating some of the rich variety of framework types. We follow this with two main results. The first, with full proof, illustrates one way in which topological compactness in function spaces can be useful. Here it assists in establishing sufficient conditions for the continuous (real) flexibility of an infinite framework. The second result is inspired by the celebrated 1876 linkage construction of Kempe who showed that a finite linkage (a two-dimensional bar-joint framework with one degree of flexibility) can be designed to simulate a given algebraic curve. See also Gao et al [7]. Here we show how infinite frameworks can simulate continuous functions, once again, with zero error, and we provide outline proofs.

For diverse discussions of finite framework rigidity and constraint systems see, for example, [1, 5, 13, 14, 15, 16, 17, 20].

2. EXAMPLES
We define a (countable) infinite (bar-joint) framework in \( \mathbb{R}^d \) to be a pair \( (G, p) \) where \( G = (V, E) \), the abstract graph of \( G \), has countable vertex set \( V \) and edge set \( E \), and where \( p = (p_1, p_2, \ldots) \) with \( p_i \in \mathbb{R}^d \) for all \( i \), is the framework vector of \( G \) associated with an enumeration \( V = \{v_1, v_2, \ldots\} \). The framework edges of \( G \) are the unordered straight line segments \([p_i, p_j]\) for each \( i, j \) with \((v_i, v_j)\) an
Let $G$ be the infinite framework in $\mathbb{R}$ with framework vector $p = (p_1, p_2, \ldots)$ and framework edges $[p_n, p_{n+1}]$ for all $n$. The abstract graph $G$ here is a tree with a single branch and a single vertex of degree 1. Two such linear frameworks $(G, p)$ and $(G, q)$ are equivalent if $[q_n - q_{n+1}] = [p_n - p_{n+1}]$ for $n = 1, 2, \ldots$, and are congruent if for some isometry of $T$ of $\mathbb{R}$, we have $q_n = T p_n$ for all $n$. Recall the fact that for every real number $\alpha \in \mathbb{R}$ there is a sequence $a_1, a_2, \ldots$ with $a_n = 1$ or $-1$ for all $n$, such that $\alpha = \sum_{n=1}^{\infty} a_n n^{-1}$. Thus the framework with vector $p = (0, 1, 1-1/2, 1-1/2+1/3, \ldots)$, has uncountably many pairwise noncongruent equivalent frameworks (obtained by flipping edge directions). From this, and analogous infinitely folding frameworks in higher dimensions, we also easily see that a continuously rigid framework (formally defined below) may possess uncountably many pairwise noncongruent equivalent frameworks that are ε-close (in the sense of Definition 3.2).

### 2.1 Diminishing Rectangles

Let $G_1 = (G, p)$ be the infinite planar framework in Figure 1. We may label it with $p_0 = (1, -1/4)$, $p_1 = (1, 0)$, $p_2 = (1, 1)$, $p_3 = (1, 0)$, $p_4 = (1, 0)$, $p_5 = (1, 0)$, and so on, with edges $[p_i, p_{i+1}]$ for $i$ odd, with edges $[p_i, p_{i+2}]$ for $i \geq 1$, and with the indicated edges to $p_0$ which have the effect of "rigidifying" the $x$-axis edges.

![Figure 1: An inflexible framework.](image)

Suppose for the moment that $p_i(t), i = 0, 1, 2, \ldots$ are continuous functions from $[0, 1]$ to $\mathbb{R}^2$, with $|p_i(t) - p_j(t)| = |p_i - p_j|$ for all $t$ and all framework edges $[p_i, p_j]$. We may suppose moreover that $p_i(t) = p_i$ for all $t \in [0, 1]$ and $i$ odd. Then it can be proven that $p_i(t) = p_i$ for all $t$ and all $i$ even. That is, the continuous flex $p(t) = (p_1(t), p_2(t), \ldots)$ must be constant. The reason for this, roughly speaking, is that the flexible rectangular subframework determined by $p_i, p_{i+1}, p_{i+2}, p_{i+3}$, for $i = 1, 3, 5, \ldots$, has a limited flexibility, tending to zero as $i$ tends to infinity, and since flexes propagate linearly no continuous flex of $p_2$ (and hence of any $p_i$) is admissible.

### 2.2 Cobweb Graph Frameworks

Let $G_1$ be the square frame framework with framework points

$$\{p_1, \ldots, p_4\} = \{(1, 1), (1, -1), (-1, -1), (-1, 1)\}.$$

Let $G_\infty$ be the framework which, roughly speaking, consists of the union $G_1 \cup \frac{1}{2} G_1 \cup \frac{1}{4} G_1 \cup \ldots$ together with connected edges between the corresponding corners of consecutive squares. Then we call $G_\infty$ the dyadic cobweb framework and we have $G_\infty = (G_\infty, p)$ where the abstract graph $G_\infty$ is a cobweb graph. It can be shown that while every finite subframework of $G_\infty$ is continuously flexible, $G_\infty$ itself is not, again for reasons of vanishing flexibility, although in this (less intuitive) case some geometric analysis is needed.

![Figure 2: The dyadic cobweb framework.](image)

The similar framework $G_\infty$ which is constructed on the framework points of $G_1 \cup 2G_1 \cup 4G_1 \cup \ldots$ is continuously flexible, while the two-way infinite framework $G_\infty = G_\infty \cup G_\infty$ is rigid.

From a mathematical perspective (and perhaps also from other perspectives) the cobweb framework $G_\infty$ is interesting in that it admits a proper flex which is increasingly negligible towards infinity. We see an opposite amplifying effect in the next example.

### 2.3 Lattice Flexing

It is straightforward to construct a finite framework with one degree of flexibility which 'simulates' two rigid bars jointed at their midpoints. For example take four equal length framework edges jointed at a common central framework point and add two "extraneous" vertices and six edges to force them to be collinear in two pairs.

Similarly we can simulate two rigid bars jointed at any interior points. Catenating infinitely many such 'tweezer' components leads to frameworks with one degree of flexibility. Catenating identical components leads to the infinite wine rack in the diagram. (The open circles in Figure 3 indicate interior jointing of rigid bars.)

Note that any proper flex $p(t) = (p_1(t), p_2(t), \ldots)$ of the infinite winerack is unbounded in the sense that for each $t > 0$ the sequence $p(t) - p(0)$ is not a bounded sequence.
One can assemble infinitely many tweezer components in all manner of interesting ways. In particular one may arrange the total edge length sum to be finite while maintaining flexibility (despite the presence of arbitrarily small rectangles). One can also arrange tree structured assemblages with Cantor set topological boundaries which exhibit interesting dynamics under framework flexing. An example of this is the Cantor tree framework in Figure 4.

![Figure 3: The infinite winerack framework.](image)

![Figure 4: Cantor tree tweezer framework.](image)

### 2.4 Periodic Frameworks

Spatially periodic frameworks are, of course, ubiquitous, appearing, for example, in the mathematical models underlying crystallography and polymer frameworks, in the real finite world of space structures, and in the pure mathematical realm of planar tilings. Simply enumerating periodic tetrahedral frameworks (of interest for hypothetical tetrahedral SiO₂) is a major project (for which see Treacy et al). However, as Donev and Torquato and others have observed there has been little development of rigidity theory for truly periodic (and hence infinite) frameworks.

We do not comment further on this here except to refer the reader to Deshpande et al, Guest and Hutchinson, and Donev and Torquato, for examples of interest in materials analysis, and to remark that some actual space structures are, in a manner of speaking, almost infinite. (The dome of the Sports Palace Sant Jordi in Barcelona was assembled from 9070 bars and 2343 joints.)

### 3. Rigidity and Rigidity Operators

**Definition 3.1.** Frameworks $\mathcal{G} = (G, p)$ and $\mathcal{G}' = (G', p')$ are equivalent if there is a graph isomorphism $\pi: G \rightarrow G'$ such that $|p_i - p_{\pi(i)}| = |p_i' - p_{\pi'(i)}|$ for all edges $(v_i, v_j)$ of $G$ (where $v_i' = \pi(v_i)$). The frameworks are congruent if $T_{p_i} = T_{p_{\pi(i)}}$ for all $i$ for some permutation $\pi$ and isometry $T$ of $\mathbb{R}^d$.

For a useful discussions of equivalence in the finite case, including the problem of unique rigidity (or global rigidity), in which equivalent frameworks are necessarily congruent, see Connelly.

**Definition 3.2.** A framework $(G, p)$ is $\epsilon$-rigid whenever $(G', p')$ is an equivalent framework (with equivalence map $\pi = \text{identity}$) and $|p_i - p_i'| \leq \epsilon$ for all $i$, then $(G, p)$ and $(G', p')$ are congruent. A framework $(G, p)$ is perturbation-ally rigid if it is $\epsilon$-rigid for some $\epsilon > 0$.

The concept of $\epsilon$-rigidity was introduced in the pioneering paper of Gluck for finite frameworks. For finite frameworks it was shown by Gluck to be equivalent to continuous rigidity, as expressed in the next definition, and also, in the case of generic frameworks, to infinitesimal rigidity, as expressed in the subsequent one. (A generic finite framework is one whose framework point coordinates are algebraically independent over the rational numbers.) It is convenient to restrict to two-dimensional frameworks.

**Definition 3.3.** Let $(G, p)$ be a (possibly infinite) framework in $\mathbb{R}^2$ with connected abstract graph $G = (V, E)$. Let $V = \{v_1, v_2, \ldots\}$ and $p = (p_1, p_2, \ldots)$. Then $(G, p)$ is said to be flexible, or more precisely, continuously flexible, with a (proper) continuous flex $p(t)$ if there exists a function $p(t) = (p_1(t), p_2(t), \ldots)$ from $[0, 1]$ to $\prod_i \mathbb{R}^2$ with the following five properties.

(i) $p(0) = p$,
(ii) each coordinate function $p_i(t)$ is continuous,
(iii) for some base edge $(a, b)$ with $|p_a - p_b| \neq 0$, $p_a(t) = p_a(0)$ and $p_b(t) = p_b(0)$ for all $t$,
(iv) each edge distance is conserved: $|p_i(t) - p_j(t)| = |p_i(0) - p_j(0)|$ for all edges $(v_i, v_j)$, and all $t$,
(v) $p(t)$ is not a constant function.

(b) The framework $(G, p)$ is rigid (or continuously rigid) if it is not flexible, that is, if it has no (proper) continuous flex.

We have already seen from our elementary linear examples that perturbational rigidity may fail rather spectacularly for a continuously rigid framework. This can also be seen in a similar way for the simple infinite framework suggested by Figure 5 (and this framework is "regular" in the terminology below).

If (instantaneous velocity) vectors $u_1, u_2, \ldots$ in $\mathbb{R}^2$ have the property $(p_i - p_j)(u_i - u_j) = 0$ for all $i, j$ then the vector $u = (u_1, u_2, \ldots)$ in the infinite dimensional vector space $\mathcal{H}_u = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \ldots$ is called an infinitesimal flex. We maintain this traditional terminology for infinite frameworks even though $u$ may be an unbounded sequence. Every framework in $\mathbb{R}^2$ has a three-dimensional subspace of infinitesimal flexes coming from isometric motions (spatial isometries). Any nonzero infinitesimal flex not in this space is a proper infinitesimal flex.
Figure 5: Rigid but not $\epsilon$-rigid for all $\epsilon$.

Definition 3.4. An infinite framework $(G, p)$ is infinitesimally rigid if it has no proper infinitesimal flexes and is infinitesimally flexible otherwise.

The dyadic cobweb framework has infinitely many infinitesimal motions because of its symmetries (which allow evident "local infinitesimal rotations"), but even a generic cobweb framework retains a proper infinitesimal flex. (By generic here we mean merely that each finite subgraph is generic.) So, in contrast with the finite case, generic infinite frameworks may be continuously rigid without being infinitesimally rigid. Also infinitesimal rigidity and perturbational rigidity differ, so the three definitions are pairwise inequivalent in general.

We now indicate briefly below how the three definitions also have conditional forms that are appropriate to infinite frameworks.

The following terminology is useful. Let $G = (G, p)$ be an infinite framework with countably many edges and vertices, and let $e_1, e_2, \ldots$ be an enumeration of the edges. Let $d_e = d_{ij}$ be the separation $|p_i - p_j|$ for the edge $e = (v_i, v_j)$. Say that $G$ is edge vanishing (respectively edge unbounded) if the sequence $(d_e)$ has no lower bound $\delta > 0$ (respectively no upper bound), and say that $G$ is regular if a lower bound $\delta > 0$ and an upper bound $M > 0$ exist. We also say that $G$ is bounded or unbounded if the sequence $\delta$ has this property. Also $G$ is locally finite if each vertex of $V$ has finite degree.

In particular, periodic frameworks, with a repeating finite cell $(G[i, 0, 1], 0)$, are regular and locally finite.

In the case of edge vanishing frameworks in many respects it is appropriate to take into account local scales when considering a perturbed or nearby framework. To quantify this let $m_e = \inf_i d_{ij}, M_e = \sup_i d_{ij}$. We say that a locally finite framework $G = (G, p)$ is relatively $\epsilon$-rigid if every equivalent framework $G = (G, p')$ (with $\pi = \text{id}$) such $|p'| - |p| \leq \epsilon m_i$ for all $i$ is congruent to $G$. It is natural then (particularly in the light of the simple one-dimensional example above) to determine conditions for relatively $\epsilon$-rigidity.

Similarly, one can consider conditional forms of infinitesimal rigidity (resp. continuous rigidity) by restricting attention to specific subspaces of infinitesimal (resp. continuous) flexes which, for example, may decay at an appropriate local rate. Or, if one is concerned with decaying flexes in a regular framework, one may impose square summable decay. These and similar perspectives amount to considering the rigidity matrix (and related matrices of the framework equation system) as a bounded linear operator, the rigidity operator between appropriate sequence spaces.

Recall that the rigidity matrix $R(G, p)$ of a framework $(G, p)$ is the Jacobian of the system of framework edge-length equations multiplied by 1/2 and evaluated at the framework points. We take the same definition for countable frameworks. Thus rows are indexed by edges and columns by the coordinates $x_i, y_i$ of $p_i, i = 1, 2, \ldots$ (The entry for the edge $(v_i, v_j)$ and the coordinate $x_i$ is $(x_i - x_j)$ etc.)

One may consider "conditioning" infinitesimal flexes $u = (u_1, u_2, \ldots)$ by requiring that they lie in the vector space $\ell_2^\infty$ of bounded sequences. (This rules out infinitesimal rotations of unbounded frameworks for example.) For a regular framework the rigidity matrix actually determines a bounded linear transformation $R(G, p)_{\infty, \infty}$ from $\ell_2^\infty$ to $\ell_2^\infty$. Moreover we may require bounded displacements of framework points which amounts to a further restriction on the domain of $R(G, p)$. This applies in particular to the infinite winerack framework; the natural "squeeze" infinitesimal flex, while being a bounded sequence does not give bounded displacements of framework points. In this sense the winerack is a boundedly isostatic framework.

Finally, note that the framework of Figure 5 while being infinitesimally rigid and continuously rigid has the "flavour" of an infinitesimally flexible structure. The following definition gives a natural notion of approximate flexibility to capture this and which we expect to lead to a useful form of strong rigidity.

Definition 3.5. A framework $(G, p)$ is approximately flexible if for every $\epsilon > 0$ there is a proper flex $u$ with

$$|(u_i - u_j)(p_i - p_j)| \leq \epsilon(|u_i| + |u_j|)|p_i - p_j|$$

for all edges $e_i, e_j$. A framework $(G, p)$ is strongly rigid if it is not approximately flexible.

4. COMPACTNESS AND PROPER FLEXES

If an infinite framework has flexible finite subframeworks then under what conditions might one conclude the existence of a proper (continuous) flex? The dyadic cobweb framework $G_\infty$ which is inflexible, with all its finite subframeworks flexible, shows that some care is needed here. In this section we give a sample theorem which resolves this question. It is stated and discussed for planar frameworks but holds for higher dimensions with the same proof. The proof makes use of the Ascoli-Arzelà compactness theorem in the following form. A bounded equicontinuous sequence of functions $f_k : [0, 1] \to \mathbb{R}^n, k = 1, 2, \ldots$ has a convergent subsequence. (See [15] or [4] for example.)

Definition 4.1. A continuous flex $p(t)$ of a normalised framework is a smooth flex if each coordinate $p_i(t)$ is differentiable on $[0, 1]$ with continuous derivative $p_i'(t)$, where $p_i'(0)$ and $p_i'(1)$ are right and left derivatives respectively. Furthermore a smooth flex is a boundedly smooth flex, or $M$-smooth flex, if for some $M > 0$ and for every pair $p_i, p_j$, the distance function

$$d_{ij}(t) = |p_i(t) - p_j(t)|$$

has bounded derivative, with $|d_{ij}'(t)| \leq M$ for all $t$ in $[0, 1]$.

Let $(G, p)$ be an infinite locally finite framework in $\mathbb{R}^2$ with connected graph and with normalised framework vector $p$, in the sense that $p_1 = (0, 0), p_2 = (d_{ij}, 0)$. Let us say that a standard chain for $(G, p)$ is any sequence of vertex induced connected subframeworks $(G_1, p_1) \subseteq (G_2, p_2) \subseteq \ldots$ whose union is $(G, p)$. Denote the separation distance $|p_i - p_j|$ by $d_{ij}$.

The following theorem, in paraphrase, says that there will be a proper continuous flex of the infinite framework if there
are two framework points such that every finite framework containing them has at least one smooth flex which changes the separation of these points, and these separation changes are bounded away from zero. In general these smooth flexes need not be related and indeed the entire framework could have, loosely speaking, many (and even infinitely many) degrees of freedom.

**Theorem 4.2.** Let \((G, p)\) be an infinite locally finite planar framework with connected graph, let 
\[(G_1, p) \subseteq (G_2, p) \subseteq \ldots, \]
be a standard chain and let \(v_i, v_j\) be vertices of \(G_1\). Suppose that there exist \(M > 0\) and \(c > 0\) and a sequence of \(M\)-smooth (normalised) flexes \(p^{(r)}(t)\) of \((G_r, p)\), for \(r = 1, 2, \ldots\), such that for all \(r\)
\[|d^{(2)}_{v_i}(1) - d^{(r)}_{v_i}(0)| \geq c.\]
Then \((G, p)\) is continuously flexible.

The proof is constructed as an iterated application of the Ascoli-Arzelà theorem and a standard diagonal selection to create a sequence of coordinate functions
\[(q^{(1)}(1), q^{(2)}(1), \ldots)\]
which (although not infinite flexes) converge (uniformly in coordinates) to a proper flex \(q^*(t)\) as \(k(n, n)\) tends to infinity (with \(n\)). The inequality ensures that the resulting limit flex is not trivial.

**Proof.** Let \(\mathcal{F}_1\) be the set of all \(M\)-smooth flexes \(q : [0, 1] \to \mathbb{R}^{2|V|}\), for \((G_1, p)\). This is a nonempty family of continuous vector-valued functions which are, moreover, equicontinuous. Let \(q^{(r)}(t), r = 1, 2, \ldots\) be the given sequence of \(M\)-smooth flexes. Each of these flexes restricts to a flex of the first subframework \((G_1, p)\). We can write these restrictions as \(P_1q^{(r)}(t)\) where \(P_1\) is the natural projection from the space of infinite framework vectors to the space determined by the coordinates for \(G_1\). This set of restrictions is a bounded set of equicontinuous vector-valued functions in \(\mathcal{F}_1\). This follows from the hypotheses on derivatives. By the Arzelà-Ascoli theorem there is a uniformly convergent subsequence, determined by some subsequence \(k(1, n), n = 1, 2, \ldots\) of \(k = 1, 2, \ldots\). That is we have obtained a subsequence \(q^{(k(n, n))}(t), n = 1, 2, \ldots\) with the \(G_1\) coordinates actually converging to a flex of the subframework \((G_1, p)\).

Likewise considering the restrictions \(P_2q^{(k)}(t)\), for \(k = k(1, n), n = 1, 2, \ldots\), there is a subsequence of this subsequence, say \(k(2, n), n = 1, 2, \ldots\) such that the restrictions \(P_2q^{(k(2, n))}(t)\) converge uniformly to a continuous flex of \((G_2, p)\) as \(n \to \infty\). Continue in this manner for the entire standard chain, and select the diagonal subsequence \(k(n, n)\). This has the property that for each coordinate location, \(m\), say, the sequence of coordinate function \(q^*_m(k(n, n))\) for \(n = 1, 2, \ldots\) converges uniformly to a continuous function \(q^*_m(t)\) as \(n \to \infty\). Moreover the function \(q^*(t) = (q^*_1(t), q^*_2(t), \ldots)\) is the desired flex. Note in particular that this limit is a proper flex since the inequality persists in the limit, that is,
\[||q^*_1(1) - q^*_1(0)| - |q^*_2(0) - q^*_2(0)|| \geq c.\]

In fact stronger forms of this theorem hold. For example it is enough to require that for \(r = 1, 2, \ldots\), there are smooth flexes \(p^{(r)}(t)\) of \((G_r, p)\) such that for each \(l\) the set of restrictions of \(p^{(r)}(t)\) to \((G_l, p)\), for \(r \geq l\), are uniformly boundedly smooth. This scheme is appropriate for flexible frameworks similar to or containing an infinite winerack.

**5. INFINITE KEMPE LINKAGES**

We state a theorem due to Kempe [12] and follow this with a discussion of exactly what the theorem means and the principal ideas behind the proof.

**Theorem 5.1.** Every finite algebraic curve in the plane has a linkage realisation.

Although Kempe does not define a linkage as a mathematical construct one may view it, in the spirit of Asimow and Roth [1], as a bar-joint framework whose (normalised) positions give a real variety which is one dimensional (at regular points). We take the following more convenient linkesque view which also serves for infinite frameworks. We let \(\langle \cdot, \cdot \rangle\) denote the usual inner product of real vectors.

**Definition 5.2.** A plane linkage (resp. infinite plane linkage) is a finite (resp. infinite) connected framework \(G = (G, p)\) in \(\mathbb{R}^2\) with a degree two vertex \(v_1\) with edges \((v_1, v_2)\), \((v_1, v_3)\) and a continuous flex \(p(t)\) such that

(i) the cosine angle function
\[g(t) = \langle p_2(t) - p_1(t), p_3(t) - p_1(t) \rangle\]
is strictly increasing and

(ii) \(p(t)\) is the unique flex \(q(t)\) of \(G\) with \(q_i(t) = p_i(t), i = 1, 2, 3\).

Make a partial normalisation by requiring that \(p_1(t)\) and all \(p_i(t)\) are equal to the origin \((0, 0)\). We may think of a finite linkage articulating a motion as the points \(p_2(t), p_3(t)\) make changing angles \(\theta, \phi\), respectively, with the \(x\) axis. The framework points move smoothly if \((\theta, \phi)\) move smoothly. Identifying a specific "end-point" \(p_n\) of the framework, if \((\theta, \phi)\) moves smoothly in a one-dimensional (real) algebraic variety then the endpoint \(p_n(\theta, \phi)\) describes an algebraic curve. In particular, with \(\phi\) fixed, a "circular input" via \(\theta\) gives an algebraic curve \(p_n(\theta, \phi)\) with \(\theta\) ranging in some interval. Kempe solved the inverse problem by showing that any particular finite algebraic curve may be realised as such a linkage curve for some linkage.

The convenience of the double angle parametrisation comes from the use of parallelogram and quadrilateral linkages in the assembly of composite linkages.

Kempe’s original construction (which simulates an algebraic output curve from a linear input) may be conceived of as a combination of the following four stages.

1. A parallelogram linkage \(L_1 = (R, q)\) with \(q_1\) rooted at the origin, \(q_4\) on a given algebraic curve, provides a 
   (virtual) curve; \(\Phi(\theta, \phi) = 0\).
2. The observation that \(\Phi(\theta, \phi) = 0\) translates into a multiple angle equation of the form
   \[C = \sum A_n \cos(n_\theta + s_n \phi + t_n).\]
   \(f(\theta, \phi)\) for the function given by this finite sum.
3. The construction of a linkage \(L_2\) so that for input angles \(\theta, \phi\) the \(x\) coordinate of the endpoint \(p_n(\theta, \phi)\) is \(f(\theta, \phi)\).
It is in this stage that Kempe uses an assembly argument, indicating basic component linkages (translator, multiplier, etc) and how they may be combined. See also Gao et al [7].

4. $L_1, L_2$ are joined together at the origin and their respective edges, incident to the origin, joined appropriately. Thus the output angles $(\theta, \phi)$ from $L_1$ become input angles for $L_2$. As $q_4$ moves on the curve $p_4$ move on the vertical line $x = \xi$, and vice versa. (One must also add framework structure to this join to fix the origin to a "base edge" parallel to the line $x = \xi$ and so create a free standing linkage.)

If an infinite linkage $(G, p)$ is such that a subsequence $p_{n_k}, k = 1, 2, \ldots$ is convergent to $p$, say, then the flex gives rise to a continuous plane curve $p^*(t)$. A sample inverse result, in the spirit of Kempe’s theorem, is given in the following. We say that the infinite framework $(G, p)$ is pointed if $p$ is a convergent sequence and if the sequence of edge lengths tends to zero.

**Theorem 5.3.** Let $f(t), t \in [0, 2\pi]$ be a continuous real-valued function with absolutely summable Fourier series. Then the graph of $f$ has an infinite linkage realisation by a pointed locally finite linkage $(G, p)$.

The proof follows a similar format to the breakdown above, although now the sum is infinite, and some modified assembly components are needed to ensure that edge lengths diminish to zero.

A consequence of the theorem is that the motion of limit points of normalised infinite linkages may fail to be continuously differentiable in every finite interval.

In fact, more generally, we have found an assembly scheme, based on uniform approximation rather than Fourier series, which creates an infinite linkage which realises (with no error) a given continuous curve. Moreover, if one admits non locally finite linkages, possessing a single framework point with infinite degree, then we can arrange that this point coincides with the curve tracing limit point above. In this way we can obtain the following theorem. Recall that a continuous planar curve (with parametrisation) is a continuous function from $[0, 1]$ to $\mathbb{R}^2$. In particular such a curve can be space filling and so these mathematical linkages are distinctly curious: with a single input flex a distinguished framework point may visit every point in a region of positive area!

**Theorem 5.4.** Every continuous planar curve has an infinite linkage realisation.

6. REFERENCES

[1] L. Asimow and B. Roth, The rigidity of graphs, Trans. Amer. Math. Soc., 245 (1978) 279-289.

[2] R. Connelly, On generic global rigidity, DIMACS Series in Discrete Mathematics and Th. Computer Sci., 4 (1991) 147-155.

[3] K.R. Davidson and A. P. Donsig, Real Analysis with Real Applications, Prentice Hall, 2002.

[4] V.S. Deshpande, M.F. Ashby, N.A. Fleck, Foam topology: Bending versus stretching dominated architectures, Acta mater. 49 (2001), 1035-1040.

[5] A. Donev and S. Torquato, Energy-efficient actuation in infinite lattice structures, J. Mech. Phys. Solids, 51 (2003) 1459-1475.

[6] H. Gluck, Almost all simply connected closed surfaces are rigid, in Geometric Topology, Lecture Notes in Math., no. 438, Springer-Verlag, Berlin, 1975, pp. 225-239.

[7] X-S Gao, C-C Zhu, S-C Chou, J-X Ge, Automated generation of Kempe linkages for algebraic curves and surfaces, Mechanism and Machine Th., 36 (2001), 1019-1033.

[8] J. Graver, B. Servatius and H. Servatius, Combinatorial rigidity, Graduate Texts in Mathematics, vol 2, Amer. Math. Soc., 1993.

[9] S. D. Guest and J. W. Hutchinson, On the determinacy of repetitive structures, Journal of the Mechanics and Physics of Solids 51 (2003) 383391.

[10] Grunbaum, B., Shephard, G.C., Tilings and patterns. W.H. Freeman, New York, 1987.

[11] C. Jermann, B. Neveu and G. Trombettoni, A new structural rigidity for geometric constraint systems, Proceedings of the Fourth International Workshop on Automated Deduction in Geometry, September, 2002, Springer Lecture Notes in Artificial Intelligence, vol 2930, 2004.

[12] A. B. Kempe, On a general method of describing plane curves of the nth degree by linkwork, Proc. London Math. Soc., 7 (1876) 213-216.

[13] G. Laman, On graphs and the rigidity of plane skeletal structures, J. Engineering Mathematics, 4 (1970), 331-340.

[14] J.C. Owen, Constraints on simple geometry in two and three dimensions. Geometric constraints: theory and practice (Tempe, AZ, 1993). Internat. J. Comput. Geom. Appl. 6 (1996) 421-434.

[15] J.C. Owen and S.C. Power, The non-solvability by radicals of generic 3-connected planar Laman graphs, Trans. Amer. Math. Soc., 359 (2007), 2269-2303.

[16] G. A.R. Parke and C. M. Howard (Editors), Space Structures 4, volumes 1 and 2: Proceedings of the Fourth International Conference On Spaces Structures, Guilford, UK, 5-10 September 1993, Published by Thomas Telford, 1993.

[17] B. Roth, Rigid and flexible frameworks, American Math. Monthly, 1981.

[18] W. Rudin, Functional Analysis, McGraw and Hill, NY, 1991.

[19] Treacy, M.M.J, Rivin, I, Balkovsky, E, Randall, K.H., Foster, M.D., Enumeration of periodic tetrahedral frameworks. II. Poly nodal graphs, Microporous and Mesoporous Materials, 74 (2004), 121-132.

[20] W. Whiteley, in Matroid Applications ed. N. White, Encyclopedia of Mathematics and its applications 40 (1992), 1-51.