WITTEN–VENEZIANO from GREEN–SCHWARZ

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Abstract

We consider the $U(1)$ problem within the AdS/CFT framework. We explain how the Witten–Veneziano formula for the $\eta'$ mass is related to a generalized Green–Schwarz mechanism. The closed string mode, that cancels the anomaly of the gauged $U(1)$ axial symmetry, is identified with the $\eta'$ meson. In a particular set-up of D3-branes on a $\mathbb{C}^3/(\mathbb{Z}_3 \otimes \mathbb{Z}_3)$ orbifold singularity, the $\eta'$ meson is a twisted-sector R-R field.
1 Introduction

The $U(1)$ problem was considered as one of the major issues in particle physics in the 70's. The dynamical breaking of the $U_L(3) \times U_R(3)$ chiral symmetry to the diagonal $U_V(3)$ should result in nine Goldstone bosons, whereas only a light octet is observed in nature. The $\eta'$ meson that correspond to the breaking of the $U_A(1)$ symmetry is too heavy to be a Goldstone.

It was understood that, at the qualitative level, the resolution of the puzzle should involve the ABJ anomaly. It was later suggested by 't Hooft that the $\eta'$ meson becomes massive because of instantons [1]. A different solution, which will be reviewed here, was suggested by Witten [2] and by Veneziano [3]. They showed that the $\eta'$ should be massless in the 't Hooft large-$N$ limit (the planar theory), since the anomaly is a $1/N$ effect, and that a $1/N$ mass for the $\eta'$ is generated when the anomaly is taken into account. Their analysis resulted in the celebrated “Witten–Veneziano formula” (WV) [2, 3]:

$$M_{\eta'}^2 = \frac{4N_f}{f_\pi^2} \frac{1}{(16\pi^2)^2} \int d^4x \langle \text{tr} F^\ast(x) \text{tr} F(0) \rangle|_{N_f=0}. \quad (1)$$

In this note we would like to derive the WV formula (1) from the AdS/CFT correspondence [4]. Aspects of chiral symmetry breaking within the “AdS/CFT with flavor” approach were discussed recently in [5]. As we shall see the derivation links the WV formula to the Green–Schwarz (GS) mechanism [6]: the closed string that is needed to cancel the $U_A(1)$ anomaly will be identified with the $\eta'$ meson, see fig. 1.

The set-up that we will use is D3-branes on orbifold singularities. Although we will use a specific orbifold, we suggest that this phenomenon is common to a generic set-up of D-branes on orbifold singularities. In particular it does not require supersymmetry. In fact, it should be even more general: in string theory there are no global symmetries. If the set-up involves the field-theory $U(1)$ axial symmetry as a symmetry of string theory\(^2\), it must be local. Local symmetries cannot be anomalous and hence the anomaly must be canceled via a generalized GS mechanism. The intermediate (odd parity R-R) closed string mode, which is involved in the anomaly cancellation, is the $\eta'$ meson. Another remark is that we are taking the standard AdS/CFT

\(^1\)Note that in those models the $\eta'$ is massless, since the limit $N_f/N_c \to 0$ is assumed.

\(^2\)The $U(1)$ axial symmetry might be an “accidental symmetry” as well.
U(1) + = 0

SU(N) SU(N) SU(N) SU(N)

Figure 1: a. The Green–Schwarz mechanism. b. The string theory diagram: the R-R closed string mode is identified with the $\eta'$ meson.
decoupling limit: $\alpha' \to 0$, while keeping $R^2/\alpha'$ fixed. The proposed GS mechanism "survives" this limit, as the anomaly is a field theory effect.

While our analysis is general, it would be interesting to make an explicit calculation of the $\eta'$ mass in the resolved $\mathbb{C}^3/(\mathbb{Z}_3 \otimes \mathbb{Z}_3)$ orbifold model (in units of the string tension).

The rest of this note is organized as follows: in section 2 we describe our model and in section 3 we show how, within our specific set-up, we can derive the WV formula.

## 2 D3 branes on $\mathbb{C}^3/(\mathbb{Z}_3 \otimes \mathbb{Z}_3)$ orbifold singularity

In this section we set the general framework. Consider, as in [7], the type IIB string with a stack of $N$ D3-branes placed on a $\mathbb{C}^3/\Gamma$ orbifold singularity. In particular we will be interested in the $\mathbb{C}^3/(\mathbb{Z}_3 \otimes \mathbb{Z}_3)$ orbifold. This model, as well as the general case of D3-branes on $\mathbb{C}^3/(\mathbb{Z}_k \otimes \mathbb{Z}_{k'})$ orbifold singularities, were analyzed in [8].

The conformal field theory on the D3-branes is a supersymmetric chiral $U^9(N)$ gauge theory with bi-fundamental matter. Therefore, there are mixed anomalies of the form $U_i(1)SU^2_j(N)$ (in fact, some combinations of $U(1)$’s are anomaly-free).

The resolution of this local anomaly problem in the $\mathbb{C}^3/(\mathbb{Z}_3 \otimes \mathbb{Z}_3)$ model is via a generalized GS mechanism [9]. The anomalous $U(1)$’s receive a mass of the string scale and therefore they are infinitely massive from the field theory point of view. The low-energy gauge group is thus $SU^9(N)$ (plus additional anomaly-free $U(1)$’s). In addition the local anomalies are canceled via an exchange of massless closed strings, see fig. 1. In the above model the closed strings modes are twisted-sector R-R fields. The terms that give mass to the $U(1)$’s and cancel the anomaly are WZ-terms, which are localized on the world-volume of the D3-branes:

$$\int d^4x \, C \wedge \exp F.$$  \hfill (2)

Indeed, the result of the detailed analysis of ref.[8] is that there are six R-R (and NS-NS) twisted-sector moduli. They live on $\mathbb{C} \subset \mathbb{C}^3$, hence they are six-dimensional.
Table 1: The simplified model: a $U(N_c)$ gauge theory with fundamental and anti-fundamental matter.

In the AdS/CFT framework (the near-horizon limit of the configuration), we consider type IIB string theory on $AdS_5 \times S^5/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. The R-R twisted sector moduli that cancel that $U(1)$ anomalies live on $AdS_5 \times S^1$.

The above model is satisfactory for most of our purposes; however, in order to have a clearer picture, we would still like to make two simplifications. We wish to turn off the coupling of all but one local $U(N_c)$ group. In the string theory set-up, combinations of gauge couplings are controlled by NS-NS moduli. By varying the six-dimensional twisted NS-NS moduli, we can achieve the limit where the coupling of eight of the nine gauge factors is much smaller than the coupling of the ninth. This is clearly seen in the T-dual picture [8].

The second simplification that we wish to make is to change the rank of the symmetry group so that we will have a $U_L(n_f) \times U(N_c) \times U_R(n_f)$ group. String theory allows us to do so by adding fractional branes at the orbifold singularity. A different way of achieving an arbitrary number of flavors, including the situation with no flavors at all, is to start with the T-dual picture of D5-branes and NS5-branes and to vary the number of D5-branes [8]. Thus we will consider the model in table 1.

Since the two groups $U_L(1)$ and $U_R(1)$ are global (or very weakly coupled), the twisted-sector R-R fields will not lift their masses. Only the mass of the center of the $U(N_c)$, which can be identified with the $U_B(1)$ baryon number, is lifted by the mixed anomalies $SU_L^2(n_f)U_B(1)$ and $SU_R^2(n_f)U_B(1)$. Thus the model we consider here is a $SU(N_c)$ gauge theory with $N_f \equiv 3n_f$ fundamental fermions (and scalars) and $N_f$ anti-fundamentals fields. Actually, when all (but the gauge) interactions are turned off, the global symmetry is enhanced to $SU(N_f)$. The GS action, in terms of two R-R fields, denoted by $C_L$ and $C_R$, takes the following form (see ref. [10] for an explicit derivation).
\[ S = \int d^4x \frac{1}{\alpha'} \left( \frac{1}{2}(\partial C_L - \partial C_R)^2 + \frac{1}{2}(\partial C_L + \partial C_R - \text{tr} A)^2 \right) \]
\[ + \int d^4x \frac{N_f}{8\pi^2}(C_L - C_R)\text{tr} F\tilde{F}. \]  

Note that the anomaly is proportional to \( N_f \). The action \( C_L \to C_L + \alpha L \) (\( C_R \to C_R + \alpha_R \)) corresponds to a chiral rotation of the fundamental (anti-fundamental) fermions. A somewhat simpler way of writing (3) is in terms of the combinations \( C_A \equiv C_L - C_R \) and \( C_V \equiv C_L + C_R \)

\[ S = \int d^4x \frac{1}{\alpha'} \left( \frac{1}{2}(\partial C_A)^2 + \frac{1}{2}(\partial C_V - \text{tr} A)^2 \right) + \frac{N_f}{8\pi^2} C_A \text{tr} F\tilde{F}. \]  

The symmetry \( C_A \to C_A + \alpha \), which became global once the gauge couplings of the \( U_L(n_f), U_R(n_f) \) gauge groups were turned off, is the axial symmetry. Note the similarity of (4) to the QCD effective action for the \( \eta' \) [11] (see also [12]), upon the identification \( C_A = -\eta' \).

### 3 A derivation of the Witten–Veneziano formula

We now wish to derive the WV formula (1) from the AdS/CFT correspondence. In the AdS/CFT framework, color singlets are described by closed strings [13]. We suggest that the \( \eta' \) meson of the \( \mathbb{C}^3/(\mathbb{Z}_3 \otimes \mathbb{Z}_3) \) model, namely the lightest pseudo-scalar meson, is the twisted R-R field \( C_A \). The arguments in favor of this suggestion are clear: the action \( C_A \to C_A + \alpha \) is the generator of the axial symmetry. Moreover, the field \( C_A \) is a pseudo-scalar and it couples to the anomaly with a strength \( N_f/N_c \), as we expect from the \( \eta' \)

\[^3\] In our set-up, the ratio \( N_f/N_c \) is kept fixed, while \( N_c \to \infty \); the \( \eta' \) is therefore expected to have a non-vanishing mass (in contrast to the set-ups in refs.[5]).
strings that couple to the boundary operator $O(x)$ [4]. There are two bulk fields that couple to $\text{tr} F\tilde{F}$: the twisted sector R-R field $C_\lambda$ (3) and the ten dimensional R-R 0-form $C$ (the “axion”). In fact, it is well known that the axion describes pseudo-scalar glueballs [14]. Thus, according to the AdS/CFT correspondence

$$\frac{1}{(16\pi^2)^2} \langle \text{tr} F\tilde{F}(x) \text{tr} F\tilde{F}(0) \rangle = \xi^2 \sum_{C \text{ modes}} K_C(x,0) + \lambda^2 \sum_{C_\lambda \text{ modes}} K_{C_\lambda}(x,0),$$

(5)

where $\xi, \lambda$ are the couplings of the ten-dimensional R-R 0-form and the twisted sector R-R 0-form to $O = \text{tr} F\tilde{F}$, respectively. $K(x,0)$ is a boundary-to-boundary propagator. In the following we will identify the $C$-modes and the $C_\lambda$-modes with glueballs and flavor singlet mesons respectively.

The propagator of a massless bulk field in a confining supergravity background is the same as that of a free massive field in a flat space [13]. It is seen by taking the ansatz $K(x; r) = e^{iqx} F(r)$ and by solving the 5-d bulk wave equation. Indeed, for a background of the form

$$ds^2 = dr^2 + f(r)dx_\mu^2,$$

(6)

The bulk wave-equation takes the form

$$f^{-1}(r)\partial_r \left( f^2(r)\partial_r F(r) \right) - q^2 F(r) = 0.$$

(7)

The eigenvalues $q^2$ are interpreted as the 4-d mass of the color-singlet hadron and thus the boundary-to-boundary propagators, in momentum space, take the form

$$K_C = \frac{1}{q^2 - M_C^2} ; \quad K_{C_\lambda} = \frac{1}{q^2 - M_{C_\lambda}^2}.$$

(8)

Since the theory contains massless quarks

$$\int d^4x \langle \text{tr} F\tilde{F}(x) \text{tr} F\tilde{F}(0) \rangle = 0.$$

(9)

(There is no meaning to a theta vacuum, since we can use a chiral rotation to set the value of theta to an arbitrary value). Let us proceed as in [2]. In the

4The gauginos, on the other hand, are assumed to have a small mass. It can easily be realized in the AdS/CFT framework by an appropriate perturbation of the super-gravity solution.
version of our theory without matter, namely when \( \frac{N_f}{N_c} = 0 \), the quantity 
\[
\int d^4x \langle \text{tr} \, F \tilde{F}(x) \text{tr} \, F \tilde{F}(0) \rangle
\]
is non-vanishing. We therefore must conclude that in eq.(5) the contribution from the mesons cancels that from the glueballs:
\[
\frac{1}{(16\pi^2)^2} \int d^4x \, e^{iqx} \langle \text{tr} \, F \tilde{F}(x) \text{tr} \, F \tilde{F}(0) \rangle|_{\frac{N_f}{N_c}=0, q^2=0} = \frac{\lambda^2}{M_{CA}^2} \quad (10)
\]
In fact, when \( \frac{N_f}{N_c} \to 0 \) the sum over mesons (\( C_A \)-modes) is dominated by the lightest mode whose mass tends to zero since in this limit the anomaly vanishes and \( C_A \) becomes a true Goldstone. In this limit \( C_A \) does not couple to the boundary field theory and (7) is solved by \( F(r) = \text{const.} \).

The second ingredient that is needed to complete our derivation is the anomaly equation. From (4) we can read the equation of motion for \( C_A \)
\[
\frac{1}{\alpha'} \partial_\mu \partial^\mu C_A = \frac{N_f}{8\pi^2} \text{tr} \, F \tilde{F} \quad (11)
\]
We therefore identify \( \frac{1}{\alpha'} \partial_\mu \partial^\mu C_A \) with the axial current \( J^\mu \). The coupling of the r.h.s. of (11) to \( C_A \) is by definition \( 2N_f \lambda \). Following ref.[2] we argue that the coupling of \( \partial_\mu J^\mu \) to \( C_A \) is proportional to \( M_{CA}^2 \), with a proportionality parameter \( f_A \). The reason is that, in momentum space, the coupling of \( J^\mu \) should be \( \sim q^\mu \) (the momentum is the only vector in the problem) and therefore the coupling of \( q_\mu J^\mu \) is \( \sim q^2 = M_{CA}^2 \). Thus
\[
2N_f \lambda = f_A M_{CA}^2 \quad (12)
\]
Inserting (12) in (10)
\[
M_{CA}^2 = \frac{4N_f^2}{f_A^2} \frac{1}{(16\pi^2)^2} \int d^4x \, e^{iqx} \langle \text{tr} \, F \tilde{F}(x) \text{tr} \, F \tilde{F}(0) \rangle|_{\frac{N_f}{N_c}=0, q^2=0} \quad (13)
\]
we arrive at the WV formula (1), provided that \( C_A \) is identified with \( \eta' \) and its coupling square \( f_A^2 \) with \( N_f f_\pi^2 \).

Note added: I have been informed by J. Barbon of a related work [15], where the WV formula is discussed within the framework of “AdS/CFT with flavor”.

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