Refined Restricted Inversion Sequences

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Abstract. Recently, the study of patterns in inversion sequences was initiated by Corteel–Martinez–Savage–Weselcouch and Mansour–Shattuck independently. Motivated by their works and a double Eulerian equidistribution due to Foata (1977), we investigate several classical statistics on restricted inversion sequences that are either known or conjectured to be enumerated by Catalan, Large Schröder, Baxter and Euler numbers. One of the two highlights of our results is a fascinating bijection between 000-avoiding inversion sequences and Simsun permutations, which together with Foata’s V- and S-codes, provide a proof of a restricted double Eulerian equidistribution. The other one is a refinement of a conjecture due to Martinez and Savage that the cardinality of $I_n(\geq, \geq, >)$ is the $n$-th Baxter number, which is proved via the so-called obstinate kernel method developed by Bousquet-Mélou.

Keywords. Inversion sequences, Ascents, Distinct entries, Last entry, Schröder numbers, Baxter numbers.

1. Introduction

For each $n \geq 1$, the set of inversion sequences of length $n$, denoted by $I_n$, is the set $I_n = \{(e_1, e_2, \ldots, e_n) \in \mathbb{N}^n : 0 \leq e_i < i \}$. It serves as various kind of codings for $S_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$. By a coding of $S_n$, we mean a bijection from $S_n$ to $I_n$. For example, the map $\Theta : S_n \to I_n$ defined for $\pi = \pi_1 \pi_2 \ldots \pi_n \in S_n$ as

$$\Theta(\pi) = e = (e_1, e_2, \ldots, e_n), \quad \text{where} \quad e_i := |\{j < i : \pi_j > \pi_i \}|,$$

is a natural coding of $S_n$. Clearly, the sum of the entries of $\Theta(\pi)$ equals the number of inversions of $\pi$, i.e., the number of pairs $i < j$ such that $\pi_i > \pi_j$. This is the reason why $I_n$ is named inversion sequences here.

Pattern avoidance in permutations has already been extensively studied in the literature (see the book by Kitaev [16]), while the systematic study of patterns in inversion sequences was initiated only recently in [8] and [21].
Since both permutations and inversion sequences will be regarded as words over $\mathbb{N} = \{0, 1, \ldots\}$, their patterns can be defined in a unified way as follows.

For two words $W = w_1w_2 \cdots w_n$ and $P = p_1p_2 \cdots p_k$ ($k \leq n$) on $\mathbb{N}$, we say that $W$ contains the pattern $P$ if there exist some indices $i_1 < i_2 < \cdots < i_k$ such that the subword $w_{i_1}w_{i_2} \cdots w_{i_k}$ of $W$ is order isomorphic to $P$. Otherwise, $W$ is said to avoid the pattern $P$. For example, the word $W = 32421$ contains the pattern 231, because the subword $w_2w_3w_5 = 241$ of $W$ has the same relative order as 231. However, $W$ is 101-avoiding. For a set of words $W$, the set of words in $W$ avoiding patterns $P_1, \ldots, P_r$ is denoted by $W(P_1, \ldots, P_r)$.

One well-known enumeration result in this area, attributed to MacMahon and Knuth (cf. [16]), is that $|S_n(123)| = C_n = |S_n(132)|$, where $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the $n$-th Catalan number.

In [8, 21], inversion sequences avoiding patterns of length 3 are studied, where a number of familiar combinatorial sequences, such as large Schröder numbers (denoted by $S_n$) and Euler numbers (denoted by $E_n$), arise. Martinez and Savage [23] further considered a generalization of pattern avoidance to a fixed triple of binary relations $(\rho_1, \rho_2, \rho_3)$. For each triple of relations $(\rho_1, \rho_2, \rho_3) \in \{<, >, \leq, \geq, =, \neq, -\}^3$, they studied the set $I_n(\rho_1, \rho_2, \rho_3)$ consisting of those $e \in I_n$ with no $i < j < k$ such that $e_i \rho_1 e_j$, $e_j \rho_2 e_k$ and $e_i \rho_3 e_k$. Here the relation “−” on a set $S \times S$, i.e., $x''−''y$ for all $x, y \in S$. For example, $I_n(<, >, <) = I_n(021)$ and $I_n(\geq, -, \geq) = I_n(000, 101, 110, 100, 201, 210)$. In Fig. 1, we summarize some of their enumeration results and conjectures, as well as corresponding classical facts in permutation patterns.

Based on these results, we will investigate more connections between restricted permutations and inversion sequences by considering several classical statistics that we recall below.

For each $\pi \in S_n$ and each $e \in I_n$, let

$$\text{DES}(\pi) := \{i \in [n-1] : \pi_i > \pi_{i+1}\} \quad \text{and} \quad \text{ASC}(e) := \{i \in [n-1] : e_i < e_{i+1}\}$$

be the descent set of $\pi$ and the ascent set of $e$, respectively. Another important property of the coding $\Theta$ is that $\text{DES}(\pi) = \text{ASC}(\Theta(\pi))$ for each $\pi \in S_n$.

| $C_n$ | $S_n$ | $B_n$ | $E_{n+1}$ |
|------|------|------|----------|
| $S_n(132), S_n(321)$: classical result [16]; $I_n(\geq, -, \geq)$: conjectured in [23] | $S_n(2413, 3142), S_n(2413, 4213), S_n(3124, 3214)$: classical result [18] | $S_n(2413, 3142)$: classical result [6]; $I_n(\geq, \geq, >)$: conjectured in [23] | $\text{Simsun permutations of } [n]$: classical result [27]; $I_n(000)$: proved in [8] |

**Figure 1.** Sets enumerated by Catalan number $C_n$, Schröder number $S_n$, Baxter number $B_n$ or Euler number $E_{n+1}$
Thus,
\[ \sum_{\pi \in \mathcal{S}_n} t^{\text{DES}(\pi)} = \sum_{e \in \mathcal{I}_n} t^{\text{ASC}(e)} , \] (1.1)
where \( t^S := \prod_{i \in S} t_i \) for any set \( S \) of positive integers. Throughout this paper, we use the convention that if “ST” is a set-valued statistic, then “st” is the corresponding numerical statistic. For example, \( \text{des}(\pi) \) is the cardinality of \( \text{DES}(\pi) \) for each \( \pi \). It is known that \( A_n(t) := \sum_{\pi \in \mathcal{S}_n} t^{\text{des}(\pi)} \) is the classical \( n \)-th Eulerian polynomial [11] and each statistic whose distribution gives \( A_n(t) \) is called an Eulerian statistic. In view of (1.1), “asc” is an Eulerian statistic on inversion sequences. Let \( \text{dist}(e) \) be the number of distinct positive entries of \( e \). This statistic was first introduced by Dumont [9], who also showed that it is an Eulerian statistic on inversion sequences. Amazingly, Foata [11] later invented two different codings of permutations called \( V \)-code and \( S \)-code to prove the following extension of (1.1).

**Theorem 1.1** (Foata 1977). For each \( \pi \in \mathcal{S}_n \) let \( \text{ides}(\pi) := \text{des}(\pi^{-1}) \) be the number of inverse descents of \( \pi \). Then,
\[ \sum_{\pi \in \mathcal{S}_n} s^{\text{ides}(\pi)} t^{\text{DES}(\pi)} = \sum_{e \in \mathcal{I}_n} s^{\text{dist}(e)} t^{\text{ASC}(e)} , \] (1.2)

Partial results regarding the statistics “asc” and “dist” on restricted inversion sequences have already been obtained in [8,21,23]. In particular, the ascent polynomial
\[ S_n(t) := \sum_{e \in \mathcal{I}_n(021)} t^{\text{asc}(e)} \]
was shown to be palindromic via a connection with some black-white rooted binary trees in [8]. Inspired by Foata’s result, we will consider the joint distribution of “asc” and “dist” on restricted inversion sequences and prove several restricted versions of (1.2). Another interesting statistic for \( e \in \mathcal{I}_n \) is the last entry of \( e \), denoted by \( \text{last}(e) \). This statistic turns out to be useful in solving some real root problems in [25] and will also lead us to solve two enumeration conjectures.

The rest of this paper deals with refinements of Catalan, Schröder, Baxter and Euler numbers. Two highlights of our results are: (i) a bijection between 000-avoiding inversion sequences and Simsun permutations (see Sect. 5), which is constructed in the spirit of Schützenberger’s jeu de taquin; (ii) a refinement of a conjecture due to Martinez and Savage [23] that asserts the cardinality of \( \mathcal{I}_n(\geq, \geq, >) \) is the \( n \)-th Baxter number (denoted by \( B_n \)), which is proved via Bousquet-Mélou’s obstinate kernel method (see Sect. 4).

### 2. Catalan Numbers

Let \((\rho_1, \rho_2, \rho_3)\) be a relation triple in \( \{(\geq, -, \geq), (\geq, -, >),(\geq, \geq, >)\} \). We introduce the parameter \( \text{cri}(e) \) for each \( e \in \mathcal{I}_n(\rho_1, \rho_2, \rho_3) \), that we call the critical value of \( e \), as the minimal integer \( c \) such that \((e_1, \ldots, e_n, c) \in \mathcal{I}_n(\rho_1, \rho_2, \rho_3) \)
\( I_{n+1}(\rho_1, \rho_2, \rho_3) \). Note that "cri" depends on the relation triple \((\rho_1, \rho_2, \rho_3)\). For example, if we consider \( e = (0, 1, 0, 2, 2, 4) \) as an inversion sequence in \( I_6(\geq, -, >) \), then \( \text{cri}(e) = 2 \). However, \( \text{cri}(e) = 3 \) when \( e \) is considered as an inversion sequence in \( I_6(\geq, -, \geq) \). The reason to introduce "cri" is that if \( e \in I_n(\rho_1, \rho_2, \rho_3) \), then \((e_1, \ldots, e_n, k)\) is in \( I_{n+1}(\rho_1, \rho_2, \rho_3) \) if and only if \( \text{cri}(e) \leq k \leq n \). This parameter will play an important role in our study of the Catalan, Schröder and Baxter triangles induced by the statistic "last".

As a warm-up, we first show how the critical value can help to prove that the cardinality of \( I_n(\geq, -, \geq) \) is \( C_n \), which was conjectured in [23]. Let us define the refinement

\[
C_{n,k} := |\{ e \in I_n(\geq, -, \geq) : \text{last}(e) = k \} |.
\]

The following recurrence shows that the numbers \( C_{n,k} \) generate the Catalan triangle that has already been widely studied (see OEIS: A009766).

**Proposition 2.1.** For \( 0 \leq k \leq n - 1 \), we have the three-term recurrence

\[
C_{n,k} = C_{n,k-1} + C_{n-1,k}.
\]

Consequently, \( C_{n,k} = \frac{n-k}{n} \binom{n-1+k}{k} \) generate the Catalan triangle.

**Proof.** Let \( \mathcal{C}_{n,k} := \{ e \in I_n(\geq, -, \geq) : \text{last}(e) = k \} \). We divide \( \mathcal{C}_{n,k} \) into the disjoint union \( \mathcal{A}_{n,k} \cup \mathcal{B}_{n,k} \), where

\[
\mathcal{A}_{n,k} = \{ e \in \mathcal{C}_{n,k} : \text{cri}(e_1, e_2, \ldots, e_{n-1}) = k \}
\]

and \( \mathcal{B}_{n,k} = \mathcal{C}_{n,k} \setminus \mathcal{A}_{n,k} \). Since \( \text{cri}(e_1, e_2, \ldots, e_{n-1}) \leq k-1 \) for \( e \in \mathcal{B}_{n,k} \), the mapping that sends \((e_1, e_2, \ldots, e_{n-1}, k)\) to \((e_1, e_2, \ldots, e_{n-1}, k-1)\) is a bijection from \( \mathcal{B}_{n,k} \) to \( \mathcal{C}_{n,k-1} \). Therefore, the cardinality of \( \mathcal{B}_{n,k} \) is \( C_{n,k-1} \) and so it remains to show that \( |\mathcal{A}_{n,k}| = C_{n-1,k} \).

Now, we are going to construct a bijection \( g : \mathcal{A}_{n,k} \to \mathcal{C}_{n-1,k} \), which will complete the proof of the recurrence for \( C_{n,k} \). For each \( e \in \mathcal{A}_{n,k} \), with \( \text{cri}(e_1, e_2, \ldots, e_{n-1}) = k \), there is a unique index \( i \) such that \( e_i = k-1 \) and \( e_{i+1} \leq k-1 \). Define \( g(e) \) to be the inversion sequence obtained from \( e \) by deleting \( e_{n-1} \), if \( e_{n-1} = n-2 \); by deleting \( e_i \), otherwise. For example, we have \( g(0, 1, 1, 3, 2) = (0, 1, 1, 2) \) while \( g(0, 1, 1, 2, 2) = (0, 1, 2, 2) \). In the former case, \( g(e) \) still has a critical value \( k \) and we can retrieve \( e \) by adding \( n-2 \) just before the last entry of \( g(e) \). In the latter case, \( g(e) \) has a critical value smaller than \( k \) and we can retrieve \( e \) by adding \( k-1 \) just before the rightmost entry of \( g(e) \) that is smaller than \( k \). This proves that \( g \) is actually a bijection. \( \square \)

The beginning of the Catalan triangle \( C_{n,k} \) is

\[
\begin{array}{ccccccccccc}
1 \\
1 & 1 \\
1 & 2 & 2 \\
1 & 3 & 5 & 5 \\
1 & 4 & 9 & 14 & 14 \\
1 & 5 & 14 & 28 & 42 & 42 \\
\end{array}
\]

For each \( \pi \in \mathcal{S}_n \), let \( \text{last}(\pi) \) be the last letter of \( \pi \). Connolly et al. [7, Corollary 1] showed that \( |\{ \pi \in \mathcal{S}_n(123) : \pi^{-1}(n) = n - k \}| = C_{n,k} \). Since
\[
\pi \in \mathcal{S}_n(123) \iff \pi^{-1} \in \mathcal{S}_n(123), \text{ it then follows that } |\{ \pi \in \mathcal{S}_n(123) : \text{last}(\pi) = n - k \}| = C_{n,k}. \text{ This is equivalent to the following statement.}
\]

**Corollary 2.2.** For \( n \geq 1 \) and \( 0 \leq k \leq n - 1 \), we have
\[
|\{ e \in I_n(\geq, -, \geq) : \text{last}(e) = k \}| = |\{ \pi \in \mathcal{S}_n(321) : \text{last}(\pi) = k + 1 \}|.
\]

The following stronger equidistribution involving the pair (dist, last) is also true, which will be proved using generating functions.

**Theorem 2.3.** For each \( \pi \in \mathcal{S}_n \), let \( \text{asc}(\pi) := n - 1 - \text{des}(\pi) \) be the ascent number of \( \pi \). Then,
\[
\sum_{\pi \in \mathcal{S}_n(321)} t^{\text{asc}(\pi)} u^{\text{last}(\pi)} = \sum_{e \in I_n(\geq, -, \geq)} t^{\text{dist}(e)} u^{\text{last}(e)+1}. \tag{2.1}
\]

**Remark 2.4.** It would be interesting to construct a bijective proof of (2.1).

We first compute the generating function for the left-hand side of (2.1). We will apply a simple bijection from Dyck paths to 321-avoiding permutations. Recall that a Dyck path of length \( n \) is a lattice path in \( \mathbb{N}^2 \) from \((0,0)\) to \((n,n)\) using the east step \((1,0)\) and the north step \((0,1)\), which does not pass above the line \( y = x \). The height of an east step in a Dyck path is the number of north steps before this east step. It is clear that a Dyck path can be represented as \( d_1 d_2 \ldots d_n \), where \( d_i \) is the height of its \( i \)-th east step. See Fig. 2 for the Dyck path 000344566. Denote by \( D_n \) the set of all Dyck paths of length \( n \).

For our purpose, we will give a new description of a bijection\(^1\) \( \psi : D_n \to \mathcal{S}_n(321) \) that has been previously used by Elizalde in [10, Section 3]. For a Dyck path \( D = d_1 d_2 \ldots d_n \in D_n \), we define \( \psi(D) = \pi = \pi_1 \pi_2 \ldots \pi_n \), where
- \( \pi_i = d_i + 1 \), if \( d_i \neq d_{i+1} \) or \( i = n \); otherwise,
- \( \pi_i \) is the \( j \)-th smallest integer in \( \{ d_1 + 1, d_2 + 1, \ldots, d_n + 1 \} \), when \( i \) is the \( j \)-th smallest integer in \( \{ k \in [n-1] : d_k = d_{k+1} \} \).

See Fig. 2 for a visualization of this bijection for the Dyck path 000344566.

It is known that a permutation is 321-avoiding if and only if both the subsequence formed by its excedance values and the one formed by the remaining non-excedance values are increasing. Using this characterization, one can check easily that \( \psi \) is in fact a bijection since each \( d_i \) of \( D \) with \( d_i \neq d_{i+1} \) or \( i = n \) becomes a non-excedance value of \( \psi(D) \) (see the blue crosses in Fig. 2).

We introduce the following two statistics:
- \( \text{last}(D) = d_n \), the height of last east step of \( D \);
- \( \text{segm}(D) \), the number of segments of \( D \) with length greater than 1, where a segment is a maximal string of consecutive east steps of the same height.

Continuing with our Dyck path in Fig. 2, we have \( \text{last}(D) = 6 \) and \( \text{segm}(D) = 3 \). The bijection \( \psi \) has the following property.

\(^1\)This bijection is essentially due to Krattenthaler [17]. An alternative description of \( \psi \) using zigzag strips was presented in [20].
Lemma 2.5. The bijection \( \psi : \mathcal{D}_n \rightarrow \mathfrak{S}_n(321) \) transforms \((\text{segm, last})\) to \((\text{des, last } - 1)\).

Let

\[
C = C(t, u, x) := \sum_{n \geq 1} x^n \sum_{D \in \mathcal{D}_n} t^{\text{segm}(D)} u^{\text{last}(D)} = x + (u + t)x^2 + \cdots.
\]

We have the following expression for \( C(t, u, x) \).

Proposition 2.6. The function \( C(t, u, x) \) is algebraic and has the expression

\[
C(t, u, x) = \frac{2x(tx - x + 1)}{1 + 2x(ux - tx - 1) + \sqrt{1 + 4ux(ux - tx - 1)}}.
\] (2.2)

Proof. Let \( \mathcal{B}_n \) be the set of Dyck paths in \( \mathcal{D}_n \) that begin with an east step followed immediately by a north step. If we introduce

\[
B(t, u, x) := \sum_{n \geq 1} x^n \sum_{D \in \mathcal{B}_n} t^{\text{segm}(D)} u^{\text{last}(D)},
\]

then clearly

\[
B(t, u, x) = x + ux C(t, u, x).
\] (2.3)

Each Dyck path \( D = d_1 d_2 \cdots d_n \in \mathcal{D}_n \setminus \mathcal{B}_n \) with \( k = \min\{i \geq 2 : d_{i+1} = i \text{ or } i = n\} \) can be decomposed uniquely into a pair \((D_1, D_2)\) of Dyck paths, where \( D_1 = d_2 d_3 \cdots d_k \in \mathcal{D}_{k-1} \) and \( D_2 = (d_{k+1} - k)(d_{k+2} - k) \cdots (d_n - k) \in \mathcal{D}_{n-k} \) (possibly empty). This decomposition is reversible and satisfies the following properties:

\[
\text{last}(D) = \chi(D_2 \neq \emptyset) \cdot (k + \text{last}(D_2)) + \chi(D_2 = \emptyset) \cdot \text{last}(D_1)
\]

and

\[
\text{segm}(D) = \text{segm}(D_1) + \text{segm}(D_2) + \chi(D_1 \in \mathcal{B}_{k-1}),
\]

where \( \chi(S) \) equals 1, if the statement \( S \) is true; and 0, otherwise. Turning this decomposition into generating functions, we obtain
\[ C - B = txB + utxB(t, 1, ux)C + x(C - B) + ux(C(t, 1, ux) - B(t, 1, ux))C. \] (2.4)

Setting \( u = 1 \) in (2.4) and (2.3), we can solve the two equations to get
\[ C(t, 1, x) = \frac{1 + 2x(x - 1 - tx) - \sqrt{1 - 4x(1 - x + tx)}}{2x(tx + 1 - x)}. \]
Substituting this into (2.4), we get (2.2).

Next we calculate the generating function for the right-hand side of (2.1). Define
\[ \tilde{C} = \tilde{C}(t, u, x) := \sum_{n \geq 1} x^n \sum_{e \in I_n(\geq, -, \geq)} t^{n-1-\text{dist}(e)} u^{\text{last}(e)} = x + (u + t)x^2 + \ldots. \]
A decomposition of \((\geq, -, \geq)\)-avoiding inversion sequences similar to that of Dyck paths enables us to obtain the following expression for \( \tilde{C}(t, u, x) \).

**Proposition 2.7.** The function \( \tilde{C}(t, u, x) \) is algebraic and has the expression
\[ \tilde{C}(t, u, x) = \frac{(tx - x + 1)(1 - \sqrt{1 + 4ux(ux - tx - 1)})}{(tx + 1 - ux)(2u - 1 + \sqrt{1 + 4ux(ux - tx - 1)})}. \] (2.5)

**Proof.** Let \( e = (e_1, \ldots, e_n) \in I_n(\geq, -, \geq) \). We distinguish the following two cases:

- If \( e_n = n - 1 \), then \( \text{dist}(e) = \text{dist}(e_1, \ldots, e_{n-1}) + \chi(n \neq 1) \) and \( \text{last}(e) = n - 1 \).
- If \( k = \max\{i : e_i = i - 1\} < n \), then it is straightforward to show that \( e \) can be decomposed into two smaller inversion sequences: \( f = (e_1, \ldots, e_{k-1}, e_{k+1}) \) in \( I_k(\geq, -, \geq) \) and \( g = (e_{k+2} - k, e_{k+3} - k, \ldots, e_n - k) \) in \( I_{n-1-k}(\geq, -, \geq) \) (possibly empty). This decomposition is reversible and satisfies the following properties:
  \[ \text{last}(e) = \chi(g = \emptyset) \cdot \text{last}(f) + \chi(g \neq \emptyset) \cdot (\text{last}(g) + k) \]
  and
  \[ \text{dist}(e) = \text{dist}(f) + \chi(e_{k+1} \neq k - 1) + \chi(g \neq \emptyset) \cdot (\text{dist}(g) + 1). \]

For convenience, we introduce
\[ \tilde{B} = \tilde{B}(t, u, x) := \sum_{n \geq 1} x^n \sum_{\substack{e \in I_n(\geq, -, \geq) \\text{ s.t. } e_n = n - 1}} t^{n-1-\text{dist}(e)} u^{\text{last}(e)}. \]
It is clear that
\[ \tilde{B}(t, u, x) = x + x\tilde{C}(t, 1, ux). \] (2.6)
Translating the above decomposition of \((\geq, -, \geq)\)-avoiding inversion sequences into generating function yields
\[
\tilde{C} - \tilde{B} = tx\tilde{B} + x(\tilde{C} - \tilde{B}) + tx\tilde{B}(t, 1, ux)\tilde{C} + x(\tilde{C}(t, 1, ux) - \tilde{B}(t, 1, ux))\tilde{C}.
\]

(2.7)

With a similar method applied to (2.4), we can solve (2.7) to obtain (2.5). \qed

Proof of Theorem 2.3. To check that the right-hand side of (2.2) equals that of (2.5) is routine (by Maple). Therefore, we conclude that \(C(t, u, x) = \tilde{C}(t, u, x)\), which is equivalent to equidistribution (2.1). \qed

3. Schröder Numbers

3.1. A New Schröder Triangle

Theorem 3.1. For \(n \geq 1\) and \(0 \leq k \leq n - 1\), we have

\[
|\{e \in I_n(\geq, -, >) : \text{last}(e) = k\}| = |\{e \in I_n(021) : \text{last}(e) \equiv k + 1(\text{mod } n)\}|.
\]

(3.1)

Note that this result is obviously true for \(k = n - 1, n - 2, n - 3\). We define the Schröder triangle \(S_{n,k} := |\{e \in I_n(\geq, -, >) : \text{last}(e) = k\}|.\) The first values of \(S_{n,k}\) are

\[
\begin{array}{ccccccccc}
1 & 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 \\
1 & 2 & 6 & 16 & 40 & 90 & 180 & 360 & 720 \\
2 & 6 & 22 & 68 & 90 \\
4 & 16 & 22 \\
8 & 16 & 22 \\
16 & 40 & 68 & 90 \\
\end{array}
\]

We have the following simple recurrence for \(S_{n,k}\).

Lemma 3.2. For \(0 \leq k \leq n - 3\), we have the four-term recurrence

\[
S_{n,k} = S_{n,k-1} + 2S_{n-1,k} - S_{n-1,k-1}.
\]

Proof. As in the Catalan case, we divide the set \(S_{n,k} := \{e \in I_n(\geq, -, >) : \text{last}(e) = k\}\) into the disjoint union \(A_{n,k} \cup B_{n,k}\), where

\[
A_{n,k} := \{e \in S_{n,k} : \text{cri}(e_1, \ldots, e_{n-1}) = k\}
\]

and \(B_{n,k} = S_{n,k} \setminus A_{n,k}\). Clearly, there is a natural bijection from \(B_{n,k}\) to \(S_{n-1,k-1}\), which maps \((e_1, \ldots, e_{n-1}, k)\) to \((e_1, \ldots, e_{n-1}, k - 1)\). Therefore, the cardinality of \(B_{n,k}\) is \(S_{n-1,k-1}\) and so it remains to show \(|A_{n,k}| = 2S_{n-1,k} - S_{n-1,k-1}\), assuming \(k \leq n - 3\). To do this, we further divide \(A_{n,k}\) into the disjoint union \(C_{n,k} \cup D_{n,k}\), where

\[
C_{n,k} := \{e \in A_{n,k} : e_{n-1} = n - 2, \text{cri}(e_1, \ldots, e_{n-2}) = k\}
\]

and \(D_{n,k} = A_{n,k} \setminus C_{n,k}\). Obviously, we have \(\{(e_1, \ldots, e_{n-2}, e_n) : e \in C_{n,k}\} = A_{n-1,k}\). Thus,

\[
|C_{n,k}| = |A_{n-1,k}| = |S_{n-1,k} - |B_{n-1,k}| = S_{n-1,k} - S_{n-1,k-1},
\]

which will end the proof once we can define a bijection from \(D_{n,k}\) to \(S_{n-1,k}\).
For each \( e \in D_{n,k} \), if \( e_i \) is the leftmost entry that equals \( \text{cri}(e) = k \), then the entries \( e_i, e_{i+1}, \ldots, e_{n-1} \) of \( e \) must satisfy:

(i) \( e_i = k \) and \( e_{i+1} \leq k \);
(ii) \( k \leq e_{i+2} \leq e_{i+3} \leq \cdots \leq e_{n-1} \), where the inequalities after the entries greater than \( k \) are strict.

Now removing the rightmost entry \( e_j \), such that \( e_j = k \) and \( i \leq j \leq n - 1 \), from \( e \) results in an inversion sequence in \( S_{n-1,k} \) (since \( e_{n-1} \leq n - 3 \)) that we denote \( f(e) \). For example, we have \( f(0,1,2,0,2,2) = (0,1,2,0,2) \), \( f(0,1,0,2,2,2) = (0,1,0,2,2) \) and \( f(0,1,2,1,3,2) = (0,1,1,3,2) \). We claim that the map \( f : D_{n,k} \rightarrow S_{n-1,k} \) is a bijection. \( \square \)

**Proof of Theorem 3.1.** It is not hard to show that the right-hand side of (3.1) satisfies the same recurrence relation as \( S_{n,k} \), which completes the proof of the theorem. \( \square \)

One may ask if there is any other interpretation of \( S_{n,k} \) in terms of pattern-avoiding permutations. The following conjecture will answer this question completely, if true.

**Conjecture 3.3.** \(^2\) Let \((\sigma, \pi)\) be a pair of patterns of length 4. Then,

\[ S_{n,k} = |\{\pi \in \mathbb{S}_n(\sigma, \pi) : \text{last}(\pi) - 1 = k\}| \]

for any \( 0 \leq k < n \) if and only if \((\sigma, \pi)\) is one of the following nine pairs:

\((4321, 3421), (3241, 2341), (2431, 2341), (4231, 3241), (2431, 2341), (2341, 3241), (3421, 2431), (3421, 3241)\).

Moreover, if \((\sigma, \pi)\) is one of the last six pairs (i.e., these in second line above), then

\[ \sum_{\pi \in \mathbb{S}_n(\sigma, \pi)} t^{\text{asc}(\pi)} u^{\text{last}(\pi)} = \sum_{e \in I_n(\geq, -, >)} t^{\text{dist}(e)} u^{\text{last}(e)+1}. \]

3.2. Double Eulerian Equidistributions

3.2.1. Statistics. Let \( \pi \in \mathbb{S}_n \) be a permutation. The set of \textit{values of inverse descents} of \( \pi \) is

\[ \text{VID}(\pi) := \{2 \leq i \leq n : \pi_i + 1 \text{ appears to the left of } \pi_i\}, \]

which is an important set-valued extension of “ides”. The set of \textit{positions of left-to-right maxima} of \( \pi \) is \( \text{LMA}(\pi) := \{i \in [n] : \pi_i > \pi_j \text{ for all } 1 \leq j < i\} \). Similarly, we can define the set of \textit{positions of left-to-right maxima} \( \text{LMI}(\pi) \), the set of \textit{positions of right-to-left maxima} \( \text{RMA}(\pi) \) and the set of \textit{positions of right-to-left minima} \( \text{RMI}(\pi) \) of \( \pi \).

Let \( e \in I_n \) be an inversion sequence. The set of positions of the \textit{last occurrence} of \textit{distinct positive entries} of \( e \) is \( \text{DIST}(e) := \{2 \leq i \leq n : e_i \neq 0 \} \) and \( e_i \neq e_j \) for all \( j > i \). The set of \textit{positions of zeros} in \( e \) is \( \text{ZERO}(e) := \{i \in [n] : e_i = 0\} \). The set of \textit{positions of the entries} of \( e \) that achieve

\(^2\)This conjecture has been proved recently by Mansour and Shattuck [22].
maximum is $\text{EMA}(e) := \{i \in [n] : e_i = i - 1\}$ and the set of positions of right-to-left minima of $e$ is $\text{RMI}(e) := \{i \in [n] : e_i < e_j$ for all $j > i\}$.

3.2.2. A Sextuple Equidistribution. Note that an inversion sequence avoids 021 if and only if its positive entries are weakly increasing. Permutations avoiding the patterns 2413 and 3142 are called separable permutations (cf. [16]). Separable permutations and 021-avoiding inversion sequences are all enumerated by the large Schröder numbers. Moreover, the work by Corteel et al. [8] and Fu et al. [14] show that

$$\sum_{e \in \mathcal{I}_n(021)} t^{\text{asc}}(e) = \sum_{\pi \in \mathcal{S}_n(2413, 4213)} t^{\text{des}}(\pi).$$

It is this observation that inspires us to find the following sextuple equidistribution, which is an extension of a restricted version of Theorem 1.1.

**Theorem 3.4.** There exists a bijection $\Psi : \mathcal{I}_n(021) \rightarrow \mathcal{S}_n(2413, 4213)$ such that

$$(\text{DIST, ASC, ZERO, EMA, RMI, EXPO})e = (\text{VID, DES, LMA, LMI, RMA, RMI})\Psi(e)$$

for each $e \in \mathcal{I}_n(021)$, where EXPO is the exposed positions of $e$.

The details of constructing $\Psi$, as well as its two interesting applications, is provided in [19]. In the rest of this section, we will show two more restricted versions of Theorem 1.1.

3.2.3. Two More Equidistributions. Based on calculations, Martinez and Savage [23] suspected that

$$\sum_{e \in \mathcal{I}_n(021)} t^{\text{asc}}(e) = \sum_{\pi \in \mathcal{S}_n(2413, 3142)} t^{\text{des}}(\pi).$$

This follows from Theorem 3.4, the palindromicity of $\mathcal{S}_n(t)$ and two more multivariate equidistributions (Theorems 3.5 and 3.8) stated below.

First, we introduce a set-valued extension of “dist” different from “DIST”:

$$\text{ROW}(e) := \{e_1, e_2, \ldots, e_n\} \setminus \{0\},$$

for each $e \in \mathcal{I}_n$.

**Theorem 3.5.** For $n \geq 1$, we have

$$\sum_{e \in \mathcal{I}_n(\geq, \neq, \geq)} s^{\text{ROW}(e)} t^{\text{ASC}(e)} u^{\text{last}(e)} = \sum_{e \in \mathcal{I}_n(>,-,\geq)} s^{\text{ROW}(e)} t^{\text{ASC}(e)} u^{\text{last}(e)}.$$

**Proof.** We will construct a bijection $\mathcal{R} : \mathcal{I}_n(\geq, \neq, \geq) \rightarrow \mathcal{I}_n(>,-,\geq)$, which preserves the triple statistics (ROW, ASC, last). Notice that $\mathcal{I}_n(\geq, \neq, \geq) = \mathcal{I}_n(110, 101, 201, 210)$, while $\mathcal{I}_n(>,-,\geq) = \mathcal{I}_n(100, 101, 201, 210)$. The idea is to replace iteratively occurrences of pattern 100 in an inversion sequence in $\mathcal{I}_n(\geq, \neq, \geq)$ \setminus $\mathcal{I}_n(>,-,\geq)$ with those of patterns 110.
Our \( \mathcal{R} \) when restricted to \( \mathbf{I}_n(\geq, \neq, \geq) \cap \mathbf{I}_n(\geq, - , \geq) \) is simply the identity. Therefore, we only need to define the mapping \( \mathcal{R} \) from \( \mathbf{I}_n(\geq, \neq, \geq) \) to \( \mathbf{I}_n(\geq, - , \geq) \). Let \( e = (e_1, \ldots, e_n) \in \mathbf{I}_n(\geq, \neq, \geq) \setminus \mathbf{I}_n(\geq, - , \geq) \). Clearly, \( e \) must contain the pattern 100. Find the (unique) greatest entry \( e_i \) such that there exists \( i < j < k \) and \( e_i > e_j = e_k \). It is routine to check that \( e \) has the structure

\[
e = (e_1, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e_{j'}, e_{j'+1}, \ldots, e_{k'}, e_{k'+1}, \ldots, e_n),
\]

where \( i < j' \leq k' \leq n \) and

- \( \max\{e_1, \ldots, e_{i-1}\} < e_i > e_{i+1} = \cdots = e_{j'} = e_{k'} \),
- \( \min\{e_{j'+1}, \ldots, e_{k'-1}, e_{k'+1}, \ldots, e_n\} > e_i \).

Replace the entries \( e_{i+1}, \ldots, e_{j'} \) of \( e \) by \( (j' - i) \) copies of \( e_i \) and keep other entries unchanged. The resulting inversion sequence, that we denote by \( e' \), avoids all patterns inside \( \{101, 201, 210\} \) but contains the pattern 110. If \( e' \) avoids the pattern 100, then define \( \mathcal{R}(e) = e' \). Otherwise, repeat the same operation on \( e' \) as we have done on \( e \) until we get an inversion sequence inside \( \mathbf{I}_n(\geq, - , \geq) \) which is defined to be \( \mathcal{R}(e) \). For example, if \( e = (0, 1, 0, 2, 0, 0, 3, 0, 4) \), then we have the following two steps of replacements:

\[
e = (0, 1, 0, 2, 0, 0, 3, 0, 4) \rightarrow (0, 1, 0, 2, 2, 2, 3, 0, 4) \rightarrow (0, 1, 1, 2, 2, 2, 3, 0, 4) = \mathcal{R}(e).
\]

Since each step of replacement is reversible, the mapping \( \mathcal{R} \) is bijective. It is obvious that each step of replacement preserves the triple statistics (ROW, ASC, last), and so does \( \mathcal{R} \), which completes the proof. \( \square \)

Recently, Baril and Vajnovszki [2] constructed a new coding \( b : \mathfrak{S}_n \rightarrow \mathbf{I}_n \) satisfying

\[
(VID, DES, LMA, LMI, RMA)\pi = (DIST, ASC, ZERO, EMA, RMI)b(\pi)
\]

for each \( \pi \in \mathfrak{S}_n \). Their coding can be applied to give an interpretation of \( S_n(t) \) in terms of ascent polynomial on \( (\geq, \neq, \geq) \)-inversion sequences. For this purpose, we will review briefly the construction of \( b \) below.

An integer interval (or interval for short) \([m, n] \), \( m < n \), is the set \( \{x \in \mathbb{N} : m \leq x \leq n\} \). A labeled interval is a pair \((I, \ell)\), where \( I \) is an interval and \( \ell \) is an integer. For a given permutation \( \pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n \) and an integer \( i \), \( 0 \leq i < n \), define the \( i \)-th slice of \( \pi \), denoted by \( U_i(\pi) \), to be a sequence of labeled intervals constructed recursively by the following process. Set \( U_0(\pi) = ([0, n], 0) \). For \( i \geq 1 \), if

\[
U_{i-1}(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k)
\]

is the \((i - 1)\)-th slide of \( \pi \), \( 1 \leq v \leq k \), is the index such that \( \pi_i \in I_v \), then \( U_i(\pi) \) is constructed according to the following four possible cases:

- If \( \min(I_v) < \pi_i = \max(I_v) \), then \( U_i(\pi) \) equals

\[
(I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (J, \ell_{v+1}), (I_{v+1}, \ell_{v+2}), \ldots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),
\]

where \( J = [\min(I_v), \pi_i - 1] \).
• If \( \min(I_v) < \pi_i < \max(I_v) \), then \( U_i(\pi) \) equals
\[
(I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (H, \ell_v), (J, \ell_{v+1}), (I_{v+1}, \ell_{v+2}), \ldots, (I_{k-1}, \ell_k), (I_k, \ell_k + 1),
\]
where \( H = [\pi_i + 1, \max(I_v)] \) and \( J = [\min(I_v), \pi_i - 1] \).

• If \( \min(I_v) = \pi_i < \max(I_v) \), then \( U_i(\pi) \) equals
\[
(I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (H, \ell_v), (I_{v+1}, \ell_{v+1}), \ldots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1),
\]
where \( H = [\pi_i + 1, \max(I_v)] \).

• If \( \min(I_v) = \pi_i = \max(I_v) \), then \( U_i(\pi) \) equals
\[
(I_1, \ell_1), \ldots, (I_{v-1}, \ell_{v-1}), (I_{v+1}, \ell_{v+1}), \ldots, (I_{k-1}, \ell_{k-1}), (I_k, \ell_k + 1).
\]

Now, let \( b(\pi) = (b_1, b_2, \ldots, b_n) \in I_n \), where for each \( i, 1 \leq i \leq n, b_i = \ell_v \)
if \( v \) satisfies that \( (I_v, \ell_v) \) is a labeled interval in the \( (i - 1) \)-th slice of \( \pi \) with \( \pi_i \in I_v \).

**Example 3.6.** For \( \pi = 341652 \in \mathcal{S}_6 \), we compute
\[
U_0(\pi) = ([0, 6], 0);
U_1(\pi) = ([4, 6], 0), ([0, 2], 1);
U_2(\pi) = ([5, 6], 0), ([0, 2], 2);
U_3(\pi) = ([5, 6], 0), ([2, 2], 2), ([0, 0], 3);
U_4(\pi) = ([5, 5], 2), ([2, 2], 3), ([0, 0], 4);
U_5(\pi) = ([2, 2], 3), ([0, 0], 5).
\]
Therefore, we get \( b(\pi) = (0, 0, 2, 0, 2, 3) \).

We say that an interval \( I \) is lower than another interval \( J \) if \( \max(I) < \min(J) \). From the construction of \( b \), it is easily checked by induction that the following properties hold.

**Lemma 3.7.** Let \( \pi \in \mathcal{S}_n \) and \( 0 \leq i < n \). If \( U_i(\pi) = (I_1, \ell_1), (I_2, \ell_2), \ldots, (I_k, \ell_k) \), then the intervals \( I_1, I_2, \ldots, I_k \) are in decreasing order, while their labelings \( \ell_1, \ell_2, \ldots, \ell_k \) are strictly increasing. Moreover, the labelings \( \ell_1, \ell_2, \ldots, \ell_{k-1} \) must appear as entries of \( b(\pi) \) after its \( i \)-th entry.

**Theorem 3.8.** The coding \( b \) restricted to \( \mathcal{S}_n(3124, 3142) \) is a bijection from \( \mathcal{S}_n(3124, 3142) \) to \( I_n(\geq, \neq, \geq) \). In particular,
\[
\sum_{\pi \in \mathcal{S}_n(3124, 3142)} s^{\text{VID}(\pi)} \epsilon^{\text{DES}(\pi)} = \sum_{\epsilon \in I_n(\geq, \neq, \geq)} s^{\text{DIST}(\epsilon)} \epsilon^{\text{ASC}(\epsilon)}.
\]

**Proof.** Since \( \mathcal{S}_n(3124, 3142) \) and \( I_n(\geq, \neq, \geq) \) have the same cardinality (see Fig. 1), we only need to show that if \( \pi \notin \mathcal{S}_n(3124, 3142) \), then \( b(\pi) \notin I_n(\geq, \neq, \geq) \).

Suppose \( \pi = \pi_1 \pi_2 \cdots \pi_n \) is a permutation containing at least one pattern of 3124 or 3142, then there exists \( i, j, k, l, 1 \leq i < j < k < l \leq n \), such that \( \pi_i \pi_j \pi_k \pi_l \) is order isomorphic to 3124 or 3142. Let \( b(\pi) = (b_1, b_2, \ldots, b_n) \in I_n \).

As \( \pi_i \) plays the role of 3 in \( \pi_i \pi_j \pi_k \pi_l \) and \( U_i(\pi) \) is obtained from \( U_{i-1}(\pi) \) by removing \( \pi_i \), we have that \( \max\{\pi_k, \pi_l\} \) lies in an interval different from and
lower than the interval containing \( \pi_j \) in \( U_i(\pi) \). Also, \( \pi_k \) and \( \pi_l \) lie in different intervals of \( U_j(\pi) \). Therefore, in view of Lemma 3.7, in \( U_j(\pi) \) the two intervals containing \( \pi_k \) or \( \pi_l \), which are different from the interval containing 0, have labelings smaller than or equal to \( b_j \). Since these two labelings are not the labeling of the interval containing 0, they will appear in \( b(\pi) \) after \( b_j \), which together with \( b_j \) form a \((\geq, \neq, \geq)\)-pattern in \( b(\pi) \). This completes the proof of the theorem. □

**Remark 3.9.** Interestingly, we can also show that \( b \) restricts to a bijection between \( S_n(2413, 4213) \) and \( I_n(021) \). Due to cardinality reason, we only need to show that if \( \pi \notin S_n(2413, 4213) \), then \( b(\pi) \notin I_n(021) \). To see this, suppose that \( \pi_j \pi_l \), have been removed from intervals of \( U_{j-1}(\pi) \), the two different intervals contain \( \pi_k \) or \( \pi_l \) have positive labelings in \( U_{j-1}(\pi) \). Therefore, in \( U_k(\pi) \), the labeling of the interval contains \( \pi_l \) must positive and smaller than \( b_k \). In view of Lemma 3.7, this labeling must appear after \( b_k \), which together with \( b_l \) and \( b_1 \) form a \( 021 \)-pattern of \( b(\pi) \). This shows the restricted mapping \( b : S_n(2413, 4213) \rightarrow I_n(021) \) is a bijection.

Note that this restricted \( b \) does not transform “RMI” to “EXPO”, while our bijection \( \Psi \) in Theorem 3.4 does.

### 4. Baxter Numbers

A permutation avoiding both vincular patterns (see [16] for the definition) 2413 and 3142 is called a **Baxter permutation**. It is a result of Chung et al. [6] that

\[
B_n = |S_n(2413, 3142)| = \frac{1}{\binom{n+1}{1} \binom{n+1}{2}} \sum_{k=0}^{n-1} \binom{n+1}{k} \binom{n+1}{k+1} \binom{n+1}{k+2}.
\]

The number \( B_n \) is known as the \( n \)-th **Baxter number**. Martinez and Savage [23] conjectured that \( |I_n(\geq, \geq, >)| = B_n \), which can be refined as follows.

**Theorem 4.1.** For \( n \geq 1 \), we have the equidistribution

\[
\sum_{e \in I_n(\geq, \geq, >)} u^{n+1-\cri(e)} = \sum_{\pi \in S_n(2413, 3142)} u^{\lma(\pi)+\rma(\pi)}.
\] (4.1)

In view of Theorem 4.1, the \((\geq, \geq, >)\)-avoiding inversion sequences will be named **Baxter inversion sequences**, which are the only pattern-avoiding inversion sequences known to be counted by Baxter numbers. For \( 0 \leq k \leq n-1 \), define the **Baxter triangle** as

\[
B_{n,k} := \{|e \in I_n(\geq, \geq, >) : \text{last}(e) = k\} |.
\]
The first values of the Baxter triangle $B_{n,k}$ are

\[
\begin{array}{cccccc}
1 & 1 & 1 & 2 & 2 & 2 \\
4 & 6 & 6 & 6 & 8 & 18 \\
8 & 18 & 22 & 22 & 22 & 32 \\
16 & 50 & 80 & 92 & 92 & 92 \\
32 & 130 & 268 & 378 & 422 & 422 \\
\end{array}
\]

Note that the second column appears as sequence OEIS: A048495.

**Corollary 4.2.** For $0 \leq k \leq n - 1$, we have

\[
B_{n,k} = |\{\pi \in \mathfrak{S}_{n-1}(2413, 3142) : \text{lma}(\pi) + \text{rma}(\pi) \geq n - k\}|
\]

The rest of this section is devoted to a proof of Theorem 4.1. For each $e \in I_n(\geq, \geq, >)$, introduce the parameters $(p, q)$ of $e$, where

\[
p = \max(e) + 1 - \text{cri}(e) \quad \text{and} \quad q = n - \max(e)
\]

with $\max(e) := \max\{e_1, \ldots, e_n\}$. For example, if $e = (0, 1, 0, 2, 2, 4) \in I_6(\geq, \geq, >)$, then $\max(e) = 4$ and $\text{cri}(e) = 2$, and so the parameters of $e$ is $(3, 2)$. After a careful discussion we can obtain the following new rewriting rule.

**Lemma 4.3.** Let $e \in I_n(\geq, \geq, >)$ be a Baxter inversion sequence with parameters $(p, q)$. Exactly $p + q$ Baxter inversion sequences in $I_{n+1}(\geq, \geq, >)$ will become $e$, when we remove their last entries, and their parameters are, respectively:

\[
(p - 1, q + 1), (p - 2, q + 1), \ldots, (1, q + 1),
\]

\[
(1, q + 1), (p + 1, q), (p + 2, q - 1), \ldots, (p + q, 1).
\]

The order in which the parameters are listed corresponds to the inversion sequences with last entries from $c$ to $n$, where $c = n + 1 - (p + q)$.

**Proof.** It is clear from the definition of critical value of $e$ that $f = (e_1, \ldots, e_n, b)$ is a Baxter inversion sequence if and only if $\text{cri}(e) \leq b \leq n$. We distinguish three cases:

- If $\text{cri}(e) \leq b < \max(e)$, then $\text{cri}(f) = b + 1$ and $\max(f) = \max(e)$. These Baxter inversion sequences contribute the parameters $(p - 1, q + 1), (p - 2, q + 1), \ldots, (1, q + 1)$.
- If $b = \max(e)$, then $\text{cri}(f) = \text{max}(f) = \max(e)$. This Baxter inversion sequence contributes the parameter $(1, q + 1)$.
- If $\max(e) < b \leq n$, then $\text{cri}(f) = \text{cri}(e)$ and $\max(f) = b$. These Baxter inversion sequences contribute the parameters $(p + 1, q), (p + 2, q - 1), \ldots, (p + q, 1)$.

Summing over all the above cases gives the desired rewriting rule for Baxter inversion sequences. □
According to the above rewriting rule, we can construct a generating tree (actually an infinite rooted tree) for Baxter inversion sequences by representing each element as its parameters like this: the root is (1, 1) and the children of a vertex labeled \((p, q)\) are those that are generated according to the rewriting rule in Lemma 4.3. See Fig. 3 for the first few levels of this generating tree. Note that the number of vertices in the \(n\)-th level of this tree is the cardinality of \(I_n(\geq, \geq, >)\).

Define the formal power series

\[
F(u, v) := \sum_{p,q \geq 1} F_{p,q}(t)u^p v^q,
\]

where \(F_{p,q}(t)\) is the size generating function for Baxter inversion sequences with parameters \((p, q)\). We can turn the above lemma into a functional equation as follows.

**Proposition 4.4.** We have the following equation for \(F(u, v)\):

\[
\left(1 + \frac{tv}{1 - u} + \frac{tv}{1 - v/u}\right)F(u, v) = tuv + tuv \left(1 + \frac{1}{1 - u}\right)F(1, v) + \frac{tv}{1 - v/u}F(u, u).
\]

**Proof.** In the generating tree for Baxter inversion sequences, each vertex other than the root \((1, 1)\) can be generated by a unique parent. Thus, we have

\[
F(u, v) = tuv + t \sum_{p,q \geq 1} F_{p,q}(t) \left(v^{q+1} \sum_{i=1}^{p-1} u^i + uv^{q+1} + \sum_{i=0}^{q-1} u^{p+1+i}v^{q-i}\right)
\]

\[
= tuv + t \sum_{p,q \geq 1} F_{p,q}(t) \left(\frac{u - u^p}{1 - u} v^{q+1} + uv^{q+1} + \frac{u^{p+q}v - u^p v^{q+1}}{1 - v/u}\right)
\]

\[
= tuv + tuv \left(1 + \frac{1}{1 - u}\right)F(1, v) - \frac{tv}{1 - u}F(u, v) + \frac{tv}{1 - v/u}(F(u, u) - F(u, v)),
\]

which is equivalent to (4.2). \(\square\)

Let \(G(u, v) := \sum_{n \geq 1} t^n \sum_{\pi \in \mathcal{S}_n(2413, 3142)} u^\text{lma}(\pi)v^\text{rma}(\pi)\). This formal power series \(G(u, v)\) was first introduced and studied by Bousquet-Mélou [4]. Now, Theorem 4.1 is equivalent to \(G(u, u) = F(u, u)\), which will be established by solving \(4.2\).
Proof of Theorem 4.1. It will be convenient to set \( w = v/u \) in (4.2). The equation then becomes
\[
\left(1 + \frac{tuw}{1-u} + \frac{tuw}{1-w}\right)F(u, wu) = tu^2w + tu^2w\left(1 + \frac{1}{1-u}\right)F(1, wu) \\
+ \frac{tuw}{1-w}F(u, u).
\]
Setting \( u = 1 + x \) and \( w = 1 + y \) in the above equation yields
\[
\frac{xy - t(1+x)(1+y)(x+y)}{t(1+x)(1+y)}F(1+x, (1+x)(1+y)) \\
= xy(1 + x) - (1 - x^2)yF(1, (1 + x)(1 + y)) - \tilde{F}(x), \quad (4.3)
\]
where \( \tilde{F}(x) := xF(1+x, 1+x) \). We call the numerator \( K(x, y) \) of the coefficient of \( F(1 + x, (1 + x)(1 + y)) \) the kernel of the above equation:
\[
K(x, y) = xy - t(1 + x)(1 + y)(x + y).
\]
We are going to apply the so-called kernel method (cf. [4]) to this equation.

As a polynomial in \( y \), the kernel has two roots:
\[
Y(x) = \frac{1 - t(1 + x)(1 + \bar{x}) - \sqrt{1 - 2t(1 + x)(1 + \bar{x}) - t^2(1 - x^2)(1 - \bar{x}^2)}}{2t(1 + \bar{x})},
\]
\[
Y'(x) = \frac{1 - t(1 + x)(1 + \bar{x}) + \sqrt{1 - 2t(1 + x)(1 + \bar{x}) - t^2(1 - x^2)(1 - \bar{x}^2)}}{2t(1 + \bar{x})},
\]
where \( \bar{x} = 1/x \). Only the first root can be substituted for \( y \) in (4.3), because the term \( F(1 + x, (1 + x)(1 + Y')) \) is not a well-defined power series in \( t \) (the Taylor expansion of \( Y' \) in \( t \) does not exist).

Now, we will adopt the obstinate kernel method that was invented by Bousquet-Mélou [4, Section 2.2] for producing all the pairs \((x, y)\) that can be legally substituted in (4.3): those are the pairs \((x, Y), (\bar{x}Y, Y), (\bar{x}Y, \bar{x})\) and their dual \((Y, x), (Y, \bar{x}Y), (\bar{x}, \bar{x}Y)\), thanks to the symmetry of the kernel \( K(x, y) \). Substituting the pairs \((x, Y)\) and \((Y, x)\) for \((x, y)\) in (4.3) yields
\[
\begin{cases} 
  xY(1 + x) - (1 - x^2)YF(1, (1 + x)(1 + Y)) - \tilde{F}(x) = 0, \\
  xY(1 + Y) - (1 - Y^2)xF(1, (1 + x)(1 + Y)) - \tilde{F}(Y) = 0.
\end{cases}
\]
Eliminating \( F(1, (1 + x)(1 + Y)) \), we get
\[
(x - xY^2)\tilde{F}(x) - (Y - x^2Y)\tilde{F}(Y) = (Y - Y^3)(x^2 + x^3) - (x - x^3)(Y^2 + Y^3). \quad (4.4)
\]
Similarly, substitute \((\bar{x}Y, Y), (Y, \bar{x}Y)\) and \((\bar{x}Y, \bar{x}), (\bar{x}, \bar{x}Y)\) into (4.3) and after some computation we get two equations, which together with (4.4) give the
system of equations:

\[
\begin{align*}
(x - xY^2)\hat{F}(x) &- (Y - x^2Y)\hat{F}(Y) = (Y - Y^3)(x^2 + x^3) \\
-(x - x^3)(Y^2 + Y^3),
\end{align*}
\]

\[
\begin{align*}
(Y\ddot{x} - Y^3\ddot{x})\hat{F}(Y\ddot{x}) &- (Y - Y^3\ddot{x}^2)\hat{F}(Y) = (Y - Y^3)(Y^2\ddot{x}^2 + Y^3\ddot{x}^3) \\
-(Y\ddot{x} - Y^3\ddot{x}^3)(Y^2 + Y^3),
\end{align*}
\]

\[
\begin{align*}
(Y\ddot{x} - Y\ddot{x}^3)\hat{F}(Y\ddot{x}) &- (\ddot{x} - Y^2\ddot{x}^3)\hat{F}(\ddot{x}) = (\ddot{x} - \ddot{x}^3)(Y^2\ddot{x}^2 + Y^3\ddot{x}^3) \\
-(Y\ddot{x} - Y^3\ddot{x}^3)(\ddot{x}^2 + \ddot{x}^3).
\end{align*}
\]

By eliminating \(\hat{F}(Y)\) and \(\hat{F}(Y\ddot{x})\), we get a relation between \(\hat{F}(x)\) and \(\hat{F}(\ddot{x})\):

\[
\hat{F}(x) + \hat{F}(\ddot{x}) = \frac{Y(1 + x)(x^4 - 2Yx^3 + 2Y^2x - 2Y + 1)}{x^2(Y - 1)(Y - x)}. \tag{4.5}
\]

But \(\hat{F}(x) = x\hat{F}(1 + x,1 + x)\) is a formal power series in \(t\) with coefficients in \(xN[x]\), while \(\hat{F}(\ddot{x})\) is a formal power series in \(t\) with coefficients in \(\ddot{x}N[\ddot{x}]\). Therefore, the positive part in \(x\) of the right-hand side of (4.5) is exactly \(\hat{F}(x)\).

On the other hand, it has been shown in [4, Corollary 3] that if we let \(\tilde{G}(x) := xG(1 + x,1 + x)\), then

\[
\frac{x - 2t(1 + x)^2}{t(1 + x)^2}\tilde{G}(x) = x^2 - 2R(x), \tag{4.6}
\]

where \(R(x) = xG(1 + x,1)\). Combining with the relation between \(R(x)\) and \(R(\ddot{x})\) proved in [4, Eq. (8)]:

\[
R(x) + R(\ddot{x}) = \ddot{x}^2Y(1 + x^3 - xY),
\]

we have

\[
\tilde{G}(x) + \tilde{G}(\ddot{x}) = \frac{t(1 + x)^2}{x - 2t(1 + x)^2}(x^2 + \ddot{x}^2 - 2(R(x) + R(\ddot{x})))
\]

\[
= \frac{t(1 + x)^2}{x - 2t(1 + x)^2}(x^2 + \ddot{x}^2 - 2\ddot{x}^2Y(1 + x^3 - xY)).
\]

To check that \(\frac{t(1 + x)^2}{x - 2t(1 + x)^2}(x^2 + \ddot{x}^2 - 2\ddot{x}^2Y(1 + x^3 - xY))\) equals the right-hand side of (4.5) is routine by Maple, which proves that \(\hat{F}(x) = \tilde{G}(x)\). This completes the proof of the theorem. \(\square\)

Since the proof of equidistribution (4.1) uses the obstinate kernel method based on the formal power series heavily, it is natural to ask for a bijective proof.

5. Euler Numbers

The Euler numbers \(E_n\) can be defined by the Taylor expansion of \(\tan(x) + \sec(x)\):

\[
\sum_{n \geq 0} E_n \frac{x^n}{n!} = \tan(x) + \sec(x) = 1 + x + \frac{x^2}{2!} + 2 \frac{x^3}{3!} + 5 \frac{x^4}{4!} + 16 \frac{x^5}{5!} + 61 \frac{x^6}{6!} + \cdots.
\]
The fundamental combinatorial interpretation of $E_n$ is due to Andrè [1], who showed that $E_n$ enumerates permutations $\pi_1 \pi_2 \cdots \pi_n \in S_n$ having the down-up property

$$\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots.$$ 

There are several other families known to be counted by Euler numbers, including Simsun permutations and 0-1-2-increasing trees.

For $\pi \in S_n$, an index $i \geq 2$ is called a double descents of $\pi$ if $\pi_{i-1} > \pi_i > \pi_{i+1}$. As introduced by Simion and Sundaram [27], a permutation in $S_n$ is called a Simsun permutation if it has no double descents, even after removing $n, n-1, \ldots, k$ for any $k$. Note that Simsun permutations are slight variants of the André permutations of Foata and Schüzenberger [13] (see also [15]), which were invented to interpret the cd-index of symmetry groups. Let $RS_n$ be the set of all Simsun permutations in $S_n$. Corteel et al. [8, Corollary 2] showed

$$\sum_{\pi \in RS_n} t^{\text{asc}(\pi)} = \sum_{e \in I_n(000)} t^{\text{dist}(e)}$$

via recurrence relations and posed the question of finding a natural bijection for this result. In this section, we will prove bijectively two different refinements of (5.1).

5.1. The Entriger–Eulerian Statistics on $I_n(000)$

Using the statistic “last”, we first refine (5.1) with a bijective proof.

Theorem 5.1. There exists a bijection $\Omega : I_n(000) \rightarrow RS_n$ such that

$$(\text{dist}, \text{last} + 1)(e) = (\text{asc}, \text{last})\Omega(e)$$

for each $e \in I_n(000)$. Consequently,

$$\sum_{\pi \in RS_n} t^{\text{asc}(\pi)} u^{\text{last}(\pi)} = \sum_{e \in I_n(000)} t^{\text{dist}(e)} u^{\text{last}(e)+1}.$$ 

The bijection $\Omega$ is the combination of the simple bijection in [8, Theorem 7] from $I_n(000)$ to 0-1-2-increasing trees with $n + 1$ vertices and a special ordering of the increasing tree representation of permutations due to Maria Monks (see [26, Page 198]).
It is convenient to introduce some necessary definitions about trees. A rooted tree with vertices labeled by a set of distinct integers is called an *increasing tree* if the labels of the vertices are increasing along any path from the root to a leaf. A *binary increasing tree* of order \( n \) is an ordered increasing tree with vertex set \( \{1, 2, \ldots, n\} \) in which every vertex has at most two children. In a binary increasing tree, we distinguish each child of a vertex by left or right. A 0-1-2-*increasing tree* of order \( n + 1 \) is an unordered increasing tree on the vertices \( \{0, 1, 2, \ldots, n\} \) such that every vertex has zero, one or two children. We do not consider the positions of the children of each vertex in a 0-1-2-increasing tree. See Fig. 4 for a 0-1-2-increasing tree (in left) and a binary increasing tree (in right, where we only consider the blue labels).

For \( e = e_1 e_2 \ldots e_n \in I_n(000) \), let \( C(e) \) be the unique 0-1-2-increasing tree such that \( i \) is the child of \( e_i \), for all \( 1 \leq i \leq n \). See an example of \( C \) when \( e = (0, 1, 2, 2, 1, 0, 6, 5) \in I_8(000) \) in the left-side of Fig. 4. It is clear that \( e \mapsto C(e) \) is a bijection between \( I_n(000) \) and 0-1-2-increasing tree of order \( n + 1 \). This tree representation of 000-avoiding inversion sequences was found recently by Corteel et al. [8].

Let \( w = w_1 w_2 \ldots w_n \) be a word on \( \mathbb{N} \) with no repeated letters. Define a rooted ordered tree \( T(w) \) recursively as follows. If \( w = \emptyset \), then \( T(w) = \emptyset \). Otherwise, suppose \( w_i \) is the smallest letter of \( w \) and define \( T(w) = (T(w_1 \ldots w_{i-1}), w_i, T(w_{i+1} \ldots w_n)) \), the tree with the left subtree \( T(w_1 \ldots w_{i-1}) \) and the right subtree \( T(w_{i+1} \ldots w_n) \) attached to the root \( w_i \). The mapping \( \pi \mapsto T(\pi) \) is a bijection between \( \mathfrak{S}_n \) and binary increasing trees of order \( n \), which is one classical tree representation of permutations (cf. [26, Section 1.5]). For example, the tree representation of the permutation \( 78153426 \in \mathfrak{S}_8 \) is the binary increasing tree of order 8 in the right-side of Fig. 4 (with the rightmost vertex removed).

In a binary increasing tree, the path from the root to the rightmost vertex is called the *rightmost path* of this tree. The following result about tree representation of Simsun permutations is important.

**Proposition 5.2.** A permutation \( \pi \) is Simsun if and only if in \( T(\pi) \) the smallest child of every vertex not on the rightmost path must be a right child.

**Proof.** It is clear that a permutation \( \pi \) has no double descent if and only if the tree \( T(\pi) \) has no vertex whose only child is a left child, except maybe for the rightmost vertex. The result then follows from this property and the definition of Simsun permutations. \( \square \)

A binary increasing tree \( T \) is called a *Simsun tree* if \( T^{-1}(T) \) is a Simsun permutation. Given a 0-1-2-increasing tree \( T \) of order \( n + 1 \), we can give specified position, left or right, to each child, so that

1. the path from the root 0 to \( n \) moves to the right;
2. for every vertex which is not a leaf and not on the path from the root 0 to \( n \), its smallest child is a right child while another child (if any) becomes a left child.

We then delete the vertex \( n \) (which must be on the rightmost) from this ordered tree and increase each label of other vertices by one. In view of Proposition 5.2,
the resulting ordered tree, that we denote by $\mathcal{M}(T)$, is a Simsun tree of order $n$. The mapping $T \mapsto \mathcal{M}(T)$ can be easily shown to be a bijection between 0-1-2-increasing trees of order $n + 1$ and Simsun trees of order $n$. See Fig. 4 for an example of the mapping $\mathcal{M}$.

**Proof of Theorem 5.1.** Define $\Omega$ to be the composition $T^{-1} \circ \mathcal{M} \circ C : I_n(000) \rightarrow RS_n$. Since $C$, $\mathcal{M}$ and $T$ are bijections, $\Omega$ is a bijection. See an example of $\Omega$ in Fig. 4. It is almost obvious from the construction that $\Omega$ transforms the pair $(\text{dist}, \text{last} + 1)$ to $(\text{asc}, \text{last})$. □

**Remark 5.3.** It also follows from the simple tree representation $C$ of 000-avoiding inversion sequences and a result of Poupard [24, Proposition 1] that the statistic “last + 1” is Entringer, namely

$$|\{\pi \in \text{Alt}_{n+1} : \pi_1 = k + 1\}| = |\{e \in I_n(000) : \text{last}(e) + 1 = k\}|,$$

where $\text{Alt}_n$ is the set of all down-up permutations in $\mathfrak{S}_n$. Can we calculate the generating function for this Entringer–Eulerian pair on $I_n(000)$? Interested readers are referred to [12] for the André permutation calculus for pairs of Entringer statistics.

### 5.2. Double Eulerian Distribution on $I_n(000)$

In the rest of this section, we will prove a double Eulerian equidistribution (see Theorem 5.5) involving the pair $(\text{asc}, \text{iasc})$ on $RS_n$. We begin with set-valued extensions of iasc and asc. For each $\pi \in \mathfrak{S}_n$, we introduce the set-valued statistics

$$\text{IASC}(\pi) := \{\pi_i : \pi_i \text{ appears on the left of } \pi_i + 1\}$$

and

$$\text{BOT}(\pi) := \{\pi_i : \pi_i < \pi_{i+1}\}.$$

We call $\text{BOT}(\pi)$ the set of **bottom values** of ascents of $\pi$, whose cardinality is $\text{asc}(\pi)$. For example, if $\pi = 78153426 \in \mathfrak{S}_8$, then $\text{IASC}(\pi) = \{1, 3, 5, 7\}$ and $\text{BOT}(\pi) = \{1, 2, 3, 7\}$.

Next we construct a new coding $\Upsilon : \mathfrak{S}_n \rightarrow I_n$ which transforms the statistic $\text{BOT}$ to $\text{ROW}$. For each $\pi \in \mathfrak{S}_n$, define $\Upsilon(\pi) = (e_1, e_2, \ldots, e_n)$, where $e_i$ equals the letter closest to $i$ in $\pi$, smaller than $i$ and left to $i$ (by convention $\pi_0 = 0$ is in position 0). For example, if $\pi = 78153426 \in \mathfrak{S}_8$, then $\Upsilon(\pi) = (0, 1, 1, 3, 1, 2, 0, 7)$. It is clear that if $\pi_i < \pi_{i+1}$, then $\pi_i$ in an entry of $\Upsilon(\pi)$. On the other hand, if $\pi_i > \pi_{i+1}$, then $\pi_i$ is never an entry of $\Upsilon(\pi)$. Therefore, we have $\text{BOT}(\pi) = \text{ROW}^{\dagger}(\Upsilon(\pi))$. To see that $\Upsilon$ is a bijection, we construct its inverse recursively. For each $e = (e_1, \ldots, e_n) \in I_n$, suppose the image permutation $\pi' = \Upsilon^{-1}(e_1, \ldots, e_{n-1})$ of $(e_1, \ldots, e_{n-1}) \in I_{n-1}$ is known. Then $\Upsilon^{-1}(e)$ is obtained from $\pi'$ by inserting $n$ immediately to the right of the letter equal to $e_n$ in $\pi'$.

It turns out that $\Upsilon(\pi) = V(\pi^{-1})$ for each permutation $\pi$, where $V : \mathfrak{S}_n \rightarrow I_n$ is the coding named V-code in Foata [11]. As was shown in [11, Théorème 2], there exists another coding $S : \mathfrak{S}_n \rightarrow I_n$ called S-code, satisfying

- $S(\pi)$ is word rearrangement of $V(\pi)$
Theorem 5.4. The bijection $F : \mathfrak{S}_n \rightarrow \mathfrak{I}_n$ that maps $\pi$ to $S(\pi^{-1})$ has the property

$$(\text{ROW}, \text{ASC})F(\pi) = (\text{BOT}, \text{IASC})(\pi).$$

Consequently,

$$\sum_{\pi \in \mathfrak{S}_n} s^\text{BOT}(\pi)t^\text{IASC}(\pi) = \sum_{e \in \mathfrak{I}_n} s^\text{ROW}(e)t^\text{ASC}(e),$$

or equivalently,

$$\sum_{\pi \in \mathfrak{S}_n} s^\text{IDB}(\pi)t^\text{DES}(\pi) = \sum_{e \in \mathfrak{I}_n} s^\text{ROW}(e)t^\text{ASC}(e),$$

where $\text{IDB}(\pi) := \{\pi_i^{-1} : \pi_{i+1}^{-1} < \pi_i^{-1}\}$ is the set of inverse descent bottoms of $\pi$.

Proof. By the properties of $S$-code, the bijection $F$ satisfying

- $F(\pi) = S(\pi^{-1})$ is word rearrangement of $V(\pi^{-1}) = \Upsilon(\pi)$
- $\text{ASC}(F(\pi)) = \text{ASC}(S(\pi^{-1})) = \text{ASC}(\pi^{-1}) = \text{IASC}(\pi)$.

The result then follows. \qed

Even though the bijection $F : \mathfrak{S}_n \rightarrow \mathfrak{I}_n$ in Theorem 5.4 does not restrict to a bijection between $RS_n$ and $\mathfrak{I}_n(000)$ (as $\Upsilon$ does not), we still have the following restricted version of (5.2).

Theorem 5.5. There exist a bijection $\Lambda : RS_n \rightarrow \mathfrak{I}_n(000)$ such that

$$(\text{IASC}, \text{BOT})(\pi) = (\text{ASC}, \text{ROW})\Lambda(\pi)$$

for each $\pi \in RS_n$. Consequently,

$$\sum_{\pi \in RS_n} s^\text{asc}(\pi)t^\text{iasc}(\pi) = \sum_{e \in \mathfrak{I}_n(000)} s^\text{dist}(e)t^\text{asc}(e).$$

Our bijection $\Lambda$ will be a combination of $S$-code, $\Upsilon$-code and an intriguing bijection $K$ from 0-1-2-increasing trees of order $n + 1$ to Simsun trees of order $n$, which is inspired by the jeu de taquin of Schützenberger.

Given a 0-1-2-increasing tree $T$ of order $n + 1$, remove the label 0 of the root and then successively move up the child with smallest label of the vertex without label. This procedure ends until the label of a leaf, say $l$, has been moved up. We remove this leaf without a label, which results in an unordered increasing tree on $\{1, 2, \ldots, n\}$. There is a unique way to order the children of this unordered increasing tree to turn it to be a Simsun tree such that the rightmost path of which is exactly the path from the root 1 to $l$. Denote the resulting Simsun tree by $K(T)$. For instance, if $T$ is the 0-1-2-increasing tree $T$ in left-side of Fig. 5, then the label 0 is removed and the moving procedure is: (i) 1 is moved up since $1 < 2$; (ii) 3 is moved up; (iii) 6 is moved up since $6 < 8$. Here $l = 6$. Finally, $K(T)$ becomes the Simsun tree in the right-side
of Fig. 5 (ignore the dashed line and red labels). This procedure on trees is similar to the classical jeu de taquin on skew standard Young tableaux. Since the inverse of $K$ can be constructed easily, the mapping $K$ is in fact a bijection between 0-1-2-increasing tree $T$ of order $n + 1$ and Simsun trees of order $n$.

**Lemma 5.6.** The mapping $K = T^{-1} \circ K \circ C : I_n(000) \to RS_n$ is a bijection satisfying

$$\text{ROW}(e) = \text{BOT}(K(e))$$

for each $e \in I_n(000)$. Moreover,

$$\text{IASC}(\Upsilon^{-1}(e)) = \text{IASC}(K(e)).$$

**Proof.** Since \text{ROW}(e) equals the set of the labels of all non-leaf and non-root vertices of $T^{-1}(e)$ and \text{BOT}(K(e)) equals the set of the labels of these vertices with a right child in $K \circ C(e)$, property (5.5) follows. Property (5.6) is less obvious but can be proved by induction on $n$.

Recall that $\Upsilon^{-1}(e)$ can be constructed recursively: suppose the image permutation $\pi' = \Upsilon^{-1}(e')$ of $e' = (e_1, \ldots, e_{n-1}) \in I_{n-1}$ is known, then $\Upsilon^{-1}(e)$ can be obtained from $\pi'$ by inserting $n$ immediately to the right of the letter equal to $e_n$ in $\pi'$. For example, if $e = (0, 0, 1, 2, 2, 3, 5, 3)$ in Fig. 5, then $\Upsilon^{-1}(e) = 25741386$. We call an index $0 \leq i \leq n$ an *available inserting position* of $\Upsilon^{-1}(e)$ if $i$ appears in $e$ less than 2 times. Let $\text{AVA}(\Upsilon^{-1}(e))$ be the set of all available inserting positions of $\Upsilon^{-1}(e)$. We will focus on the order of the letters in $\text{AVA}(\Upsilon^{-1}(e))$ that appear in $\Upsilon^{-1}(e)$. For our running example, the letters in $\text{AVA}(\Upsilon^{-1}(e)) = \{1, 4, 5, 6, 7, 8\}$ appear in $\Upsilon^{-1}(e)$ in the order $5, 7, 4, 1, 8, 6$.

On the other hand, Simsun trees also can be constructed recursively. Let $i$ be a vertex with less than 2 children of a Simsun tree $T$ of order $n$. Suppose the parent of $i$ is $j$ (by convention, the parent of 1 is 0). We have four different cases, depending on how we attach $n + 1$ to the vertex $i$ so that $T$ becomes a Simsun tree of order $n + 1$:

(a) If $i$ is not on the rightmost path of $T$ and $i$ is a leaf, then we can attach $n + 1$ as a right child of vertex $i$. We mark this position by $i$.

(b) If $i$ is not on the rightmost path of $T$ and $i$ has a right child, then we can attach $n + 1$ as a left child of vertex $i$. We mark this position by $i$.  

![Figure 5. An example of the bijection $K$](image-url)
(c) If $i$ is on the rightmost path of $T$ and has a right child, then we can attach $n + 1$ as a left child of vertex $i$. We mark this position by $j$.

(d) Otherwise, $i$ is the rightmost vertex in $T$. We further disguising two cases:

(d1) If $i$ has a left child, then we can attach $n + 1$ as a right child of vertex $i$ and mark this position by $i$.

(d2) Otherwise, $i$ is the rightmost leaf of $T$. In this case, we can either attach $n + 1$ as a left child or a right child to vertex $i$. We mark the position in right by $i$, while the position in left by $j$.

See Fig. 5 (right-side) for a Simsun tree with its potential positions marked (by red integers): $i = 4$, 7 or 8 is in case (a); $i = 5$ is in case (b); $i = 3$ is in case (c); $i = 6$ is in case (d1). One can check case by case that attaching $n + 1$ to a position marked $k$ in the Simsun tree $\mathcal{K} \circ \mathcal{C}(e)$ makes it the Simsun tree $\mathcal{K} \circ \mathcal{C}(e')$, where $e' = (e_1, \ldots, e_n, k) \in \mathbb{I}_{n+1}(000)$. It is clear that the set of marked positions of $\mathcal{K} \circ \mathcal{C}(e)$ equals $\text{AVA}(\Upsilon^{-1}(e))$.

Now, property (5.6) is an easy consequence of the following key observation.

**Observation:** The topological order of the marked positions of $\mathcal{K} \circ \mathcal{C}(e)$ is the same as the order (from left to right) of the letters in $\text{AVA}(\Upsilon^{-1}(e))$ appearing in $\Upsilon^{-1}(e)$.

This observation can be proved easily from the recursive constructions of $\mathcal{K} \circ \mathcal{C}$ and $\Upsilon^{-1}$ by induction on $n$, which ends the proof. $\square$

**Proof of Theorem 5.5.** Let $\widetilde{RS}_n := \{\Upsilon^{-1}(e) : e \in \mathbb{I}_n(000)\}$. It follows from Lemma 5.6 that $K \circ \Upsilon : \widetilde{RS}_n \to RS_n$ is a bijection such that

$$(\text{IASC, BOT})(\pi) = (\text{IASC, BOT})K \circ \Upsilon(\pi)$$

for each $\pi \in \widetilde{RS}_n$. Now simply set $\Lambda = \mathcal{F} \circ \Upsilon^{-1} \circ K^{-1}$, which completes the proof in view of Theorem 5.4. $\square$

Let $E_n(t) := \sum_{\pi \in RS_n} t^{\text{iasc}(\pi)}$ be the inverse ascent polynomial on Simsun permutations. We could not find any appearance of this $t$-extension of Euler numbers in the literature. The first values of $E_n(t)$ are

- $E_2(t) = 1 + t$,
- $E_3(t) = 4t + t^2$,
- $E_4(t) = 4t + 11t^2 + t^3$,
- $E_5(t) = 2t + 32t^2 + 26t^3 + t^4$,
- $E_6(t) = t + 52t^2 + 161t^3 + 57t^4 + t^5$.

Chow and Shiu [5] showed that the ascent polynomials on Simsun permutations are real-rooted. It seems that this property also holds for the inverse ascent polynomials.

**Conjecture 5.7.** The polynomial $E_n(t)$ is real-rooted for each $n \geq 2$. In particular, $E_n(t)$ is log-concave and unimodal.
6. Final Remarks

Because of Theorems 5.1 and 2.3 and Conjecture 3.3, one may wonder if the same equidistribution holds for the whole sets $\mathfrak{S}_n$ and $\mathfrak{I}_n$ without restriction. This is in fact true as we will show in the following.

**Theorem 6.1.** For $n \geq 1$, we have the equidistribution:

$$\sum_{\pi \in \mathfrak{S}_n} t^{\text{asc}}(\pi) u^{\text{last}}(\pi) = \sum_{e \in \mathfrak{I}_n} t^{\text{dist}}(e) u^{\text{last}(e)+1}. \quad (6.1)$$

**Proof.** Since the natural coding $\Theta$ transforms the pair $(\text{des}, n-\text{last})$ on $\mathfrak{S}_n$ to $(\text{asc}, \text{last})$ on $\mathfrak{I}_n$, we have

$$\sum_{e \in \mathfrak{I}_n} t^{\text{asc}}(e) u^{\text{last}(e)} = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}}(\pi) u^{n-\text{last}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} t^{\text{asc}}(\pi) u^{\text{last}(\pi)-1},$$

where the second equality follows from the simple involution on $\mathfrak{S}_n$

$$\pi_1 \pi_2 \cdots \pi_n \mapsto (n+1-\pi_1)(n+1-\pi_2) \cdots (n+1-\pi_n).$$

Therefore, equidistribution (6.1) is equivalent to

$$\sum_{e \in \mathfrak{I}_n} t^{\text{asc}}(e) u^{\text{last}(e)} = \sum_{e \in \mathfrak{I}_n} t^{\text{dist}}(e) u^{\text{last}(e)}. \quad (6.2)$$

We proceed to show (6.2) by induction on $n$. Obviously, the result is true for $n = 1$. Suppose that the result is true for $n = k$. We need to show that for a fixed $j$, $0 \leq j \leq k$,

$$\sum_{e \in \mathfrak{I}_{k+1}} t^{\text{asc}}(e) \left|_{\text{last}(e)=j} \right. = \sum_{e \in \mathfrak{I}_{k+1}} t^{\text{dist}}(e) \left|_{\text{last}(e)=j} \right..$$

Since $\sum_{e \in \mathfrak{I}_k} t^{\text{asc}}(e) = \sum_{e \in \mathfrak{I}_k} t^{\text{dist}(e)}$ by induction hypothesis, it will be sufficient to show

$$\sum_{e \in \mathfrak{I}_k} t^{\text{asc}}(e) \left|_{\text{last}(e)<j} \right. = \sum_{e \in \mathfrak{I}_k} t^{\text{dist}(e)} \left|_{j \notin \text{ROW}(e)} \right.. \quad (6.3)$$

Now consider the three different boards in Fig. 6. The second board is obtained from the first board by deleting its $j$-th row, while the third board is obtained from the second one by moving the $j$-th column to the right. By a configuration inside a board $B$ we mean a filling of the boxes of $B$ with balls each of whose column has exactly one box receiving a ball. In a configuration of $B$, a row of $B$ is said to be occupied if at least one box in this row receives a ball. Note that counting the distinct entries of inversion sequences in $\{e \in \mathfrak{I}_k : j \notin \text{ROW}(e)\}$ is equivalent to counting the occupied rows in configurations inside the first board, or alternatively inside the second or third board. It then follows that

$$\sum_{e \in \mathfrak{I}_k} t^{\text{dist}}(e) \left|_{j \notin \text{ROW}(e)} \right. = \sum_{e \in \mathfrak{I}_k} t^{\text{dist}(e)} \left|_{\text{last}(e)<j} \right.,$$

which is equivalent to (6.3) by the induction hypothesis. This completes the proof of the theorem by induction. \qed
Besides Conjecture 5.7, the palindromic polynomial $S_n(t)$ was also conjectured in [14] to be real-rooted. It would be interesting to investigate systematically the real-rootedness of all the Eulerian polynomials, i.e., the distribution polynomials of ascents or distinct positive entries, on restricted inversion sequences appearing in this paper.

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