VECTORSVALUED SHRODINGER OPERATORS IN L^p-SPACES

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Abstract. In this paper we consider the vector-valued operator div(Q∇u) − Vu of Schrödinger type. Here V = (vij) is a nonnegative, locally bounded, matrix-valued function and Q is a symmetric, strictly elliptic matrix whose entries are bounded and continuously differentiable with bounded derivatives. Concerning the potential V, we assume an that it is pointwise accretive and that its entries are in L^∞_{loc}(R^d). Under these assumptions, we prove that a realization of the vector-valued Schrödinger operator generates a C_0-semigroup of contractions in L^p(R^d; C^m). Further properties are also investigated.

1. Introduction. Recently, there is an increased interest in systems of parabolic equations with unbounded coefficients. Such systems appear in connection with Nash equilibria in stochastic differential games, in the study of backward-forward differential games, in the analysis of the weighted ∂̅-problem in C^d, in time dependent Born–Openheimer theory and also in the study of the Navier–Stokes equation. For more information we refer the reader to [1, Section 6], [11], [5], [4], [14], [13] and [10].

While the scalar theory of such equations is by now well understood (see [17] and the references therein), so far there are only few articles concerned with systems. We mention the article [12] where systems of parabolic equations involving both, a potential term and a drift term, were considered in the L^p-setting. In [1, 3, 6], the authors choose a different approach. Indeed, they first constructed solutions in the space of bounded and continuous functions and only afterwards the obtained

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semigroup is extrapolated to the $L^p$-scale. We should point out that in the presence of an unbounded drift term the differential operator does not always generate a strongly continuous semigroup on $L^p$-spaces with respect to Lebesgue measure, see [20]. Thus, in some cases appropriate growth conditions need to be imposed on the coefficients to ensure generation of a semigroup on $L^p$ with respect to Lebesgue measure.

In this article we will consider systems of parabolic equations which are coupled only through a potential term. To be more precise, consider the differential operator

$$\mathcal{A}u = \text{div}(Q \nabla u) - Vu =: \Delta u - Vu$$

acting on vector-valued functions $u = (u_1, \ldots, u_m) : \mathbb{R}^d \to \mathbb{C}^m$. Here $Q$ is a bounded, symmetric and strictly elliptic matrix with continuously differentiable entries that have bounded derivatives. The expression $\text{div}(Q \nabla u)$ should be understood componentwise, i.e. $\text{div}(Q \nabla u_1), \ldots, \text{div}(Q \nabla u_m)$. The matrix-valued function $V : \mathbb{R}^d \to \mathbb{R}^{m \times m}$ is assumed to be pointwise accretive and to have locally bounded coefficients. In contrast to the situation where an unbounded drift is present, no additional growth assumptions on the potential $V$ are needed to ensure generation of a strongly continuous semigroup on $L^2(\mathbb{R}^d; \mathbb{C}^m)$. Indeed, following Kato [15], who considered the scalar situation, we shall construct a densely defined, $m$-dissipative realization $A$ of the operator $\mathcal{A}$ in $L^2(\mathbb{R}^d; \mathbb{R}^m)$. By virtue of the Lumer–Phillips theorem $A$ generates a strongly continuous semigroup. Subsequently, we prove that this semigroup extrapolates to a consistent family of strongly continuous contraction semigroups $\{T_p(t)\}_{t \geq 0}$ on $L^p(\mathbb{R}^d; \mathbb{R}^m)$ for $1 < p < \infty$. We also give a description of the generator $A_p$ of $\{T_p(t)\}_{t \geq 0}$ and prove that the test functions form a core for the operator $A_p$.

We should point out that in our recent article [16] we were studying a similar setting. However, in [16] we were interested in proving that the domain of the vector-valued Schrödinger operator is the intersection of the domain of the diffusion part and the potential part. To that end, we had to impose growth conditions on the potential part. In the present article, we allow general potential terms without such a growth condition. The price to pay is that we can only characterize the domain of the $L^p$-realization of our operator as the maximal $L^p$-domain.

This article is organised as follows. In Section 2 we prove a version of Kato’s inequality for vector-valued functions which is crucial in all subsequent sections. In Section 3 we construct a realization of the operator $\mathcal{A}$ in $L^2(\mathbb{R}^d; \mathbb{R}^m)$ which generates a strongly continuous contraction semigroup. In Section 4, we extrapolate the semigroup to $L^p$-spaces, where $p \in (1, \infty)$. In the concluding Section 5 we characterize the domain of the generator as maximal domain.

**Notation.** Let $d, m \geq 1$. By $| \cdot |$ we denote the Euclidean norm on $\mathbb{C}^j$, $j = d, m$ and by $\langle \cdot, \cdot \rangle$ the Euclidean inner product. By $B(r) = \{x \in \mathbb{R}^d : |x| \leq r\}$ we denote the Euclidean ball of radius $r > 0$ and center 0. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^d; \mathbb{C}^m)$ is the $\mathbb{C}^m$-valued Lebesgue space on $\mathbb{R}^d$. For $1 \leq p < \infty$, the norm is given by

$$\|f\|_p := \left( \int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^d} \left( \sum_{j=1}^m |f_j|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}, \quad f \in L^p(\mathbb{R}^d; \mathbb{C}^m),$$

whereas in the case $p = \infty$ we use the essential supremum norm

$$\|f\|_{\infty} := \text{ess sup}\{|f(x)| : x \in \mathbb{R}^d\}.$$
For $1 < p < \infty$, $p'$ refers to the conjugate index, i.e. $1/p + 1/p' = 1$. Thus $L^p(\mathbb{R}^d; \mathbb{C}^m)$ is the dual space of $L^p(\mathbb{R}^d; \mathbb{C}^m)$ and the duality pairing $\langle \cdot, \cdot \rangle_{p,p'}$ is given by
\[
\langle f, g \rangle_{p,p'} = \int_{\mathbb{R}^d} \langle f(x), g(x) \rangle dx, \quad \text{for } f \in L^p(\mathbb{R}^d; \mathbb{C}^m), \ g \in L^{p'}(\mathbb{R}^d; \mathbb{C}^m).
\]

By $C_0^\infty(\mathbb{R}^d; \mathbb{C}^m)$, we denote the space of all test functions, i.e. functions $f : \mathbb{R}^d \to \mathbb{C}^m$ which have compact support and derivatives of any order. $W^{k,p}(\mathbb{R}^d; \mathbb{C}^m)$ refers to the classical Sobolev space of order $k$, that is the space of all functions $f \in L^p(\mathbb{R}^d; \mathbb{C}^m)$ such that the distributional derivative $\partial^\alpha f$ belongs to $L^p(\mathbb{R}^d; \mathbb{C}^m)$ for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ with $|\alpha| = \sum_{j=1}^d \alpha_j \leq k$. For $1 \leq p \leq \infty$ we define $W^{k,p}_0(\mathbb{R}^d; \mathbb{R}^m)$ as the space of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}^m$ such that $\chi_{B(f)} f \in L^p(\mathbb{R}^d; \mathbb{R}^m)$ for all $r > 0$. Here $\chi_{B(f)}$ is the indicator function of the ball $B(r)$. The space $W^{k,p}_0(\mathbb{R}^d; \mathbb{R}^m)$ is the space of all functions $f \in L^p(\mathbb{R}^d; \mathbb{R}^m)$ such that for $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ the distributional derivative $\partial^\alpha f$ belongs to $L^p(\mathbb{R}^d; \mathbb{R}^m)$. We write $H^k(\mathbb{R}^d; \mathbb{C}^m) := W^{k,2}(\mathbb{R}^d; \mathbb{C}^m)$ and $H^k_0(\mathbb{R}^d; \mathbb{C}^m) := W^{k,2}_0(\mathbb{R}^d; \mathbb{C}^m)$.

2. Preliminaries. Throughout this article we make the following assumptions:

**Hypotheses 2.1.**
1. The map $Q : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is such that $q_{ij} = q_{ji}$ is bounded and continuously differentiable with bounded derivative for all $i, j \in \{1, \ldots, d\}$ and there exist positive real numbers $\eta_1$ and $\eta_2$ such that
\[
\eta_1 |\xi|^2 \leq \langle Q(x)\xi, \xi \rangle \leq \eta_2 |\xi|^2
\]
for all $x, \xi \in \mathbb{R}^d$.

2. The map $V : \mathbb{R}^d \to \mathbb{R}^{m \times m}$ has entries $v_{ij} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ for all $i, j \in \{1, \ldots, m\}$ and
\[
\text{Re} \langle V(x)\xi, \xi \rangle \geq 0,
\]
for all $x \in \mathbb{R}^d$, $\xi \in \mathbb{C}^m$.

To simplify notations, we write for $\xi, \eta \in \mathbb{R}^d$
\[
\langle \xi, \eta \rangle_Q := \sum_{i,j=1}^d q_{ij} \eta_i \eta_j \quad \text{and} \quad |\xi|_Q := \sqrt{\langle \xi, \xi \rangle_Q}.
\]

We define the operator $\Delta_Q : W^{1,1}_{\text{loc}}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)$ by setting
\[
\langle \Delta_Q u, \varphi \rangle = -\int_{\mathbb{R}^d} \langle \nabla u, \nabla \varphi \rangle_Q dx.
\]
for any test function $\varphi \in C_0^\infty(\mathbb{R}^d)$, where $\mathcal{D}(\mathbb{R}^d)$ denotes the space of distributions. As usual, we will say that $\Delta_Q u \in L^1_{\text{loc}}(\mathbb{R}^d)$, if there is a function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that
\[
\langle \Delta_Q u, \varphi \rangle = \int_{\mathbb{R}^d} f \varphi dx
\]
for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. In this case we will identify $\Delta_Q u$ and the function $f$.

The following lemma, taken from [19, Lemma 2.4], generalizes Stampacchia’s result concerning the weak derivative of the absolute value of an $W^{1,2}$-function, see [9, Lemma 7.6], to vector-valued functions.
Lemma 2.2. Let \(1 < p < \infty\) and \(u = (u_1(x), \ldots, u_m(x)) \in W^{1,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^m)\). Then, \(|u| \in W^{1,p}_{\text{loc}}(\mathbb{R}^d)\) and
\[
\nabla|u| = \frac{1}{|u|} \sum_{j=1}^{m} \text{Re}(\bar{u}_j \nabla u_j) \chi_{\{u \neq 0\}}.
\]
Moreover,
\[
|\nabla|u||_Q^2 \leq \sum_{j=1}^{d} |\nabla u_j|_Q^2.
\] (2)

We can now prove a vector-valued version of Kato’s inequality.

Proposition 2.3. Let \(u = (u_1, \ldots, u_m) \in H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)\) be such that \(\Delta_Q u_j \in L^1_{\text{loc}}(\mathbb{R}^d)\) for \(j = 1, \ldots, m\). Then
\[
\Delta_Q |u| = \chi_{\{u \neq 0\}} \frac{1}{|u|} \left( \sum_{j=1}^{m} u_j \Delta_Q u_j + \sum_{j=1}^{m} |\nabla u_j|_Q^2 - |\nabla|u||_Q^2 \right).
\] (3)

Thus, the Kato inequality
\[
\Delta_Q |u| \geq \chi_{\{u \neq 0\}} \frac{1}{|u|} \sum_{j=1}^{m} u_j \Delta_Q u_j
\] (4)
holds in the sense of distributions.

Proof. Let us consider the function \(a_\varepsilon(u) = (|u|^2 + \varepsilon^2)^{\frac{1}{2}} - \varepsilon\). It was proved in [19, Lemma 2.4] that \(\lim_{\varepsilon \to 0} a_\varepsilon(u) = |u|\) in \(H^1_{\text{loc}}(\mathbb{R}^d)\), so that \(\lim_{\varepsilon \to 0} \Delta_Q a_\varepsilon(u) = \Delta_Q |u|\) in \(\mathcal{D}(\mathbb{R}^d)\).

Since \(\nabla a_\varepsilon(u) = \frac{1}{a_\varepsilon(u) + \varepsilon} \sum_{j=1}^{m} u_j \nabla u_j\), it follows that for \(\varphi \in C^\infty_c(\mathbb{R}^d)\) we have
\[
\langle \Delta_Q a_\varepsilon(u), \varphi \rangle = -\int_{\mathbb{R}^d} (\Delta_Q a_\varepsilon(u), \nabla \varphi) \, dx = -\sum_{j=1}^{m} \int_{\mathbb{R}^d} \frac{u_j}{a_\varepsilon(u) + \varepsilon} (\nabla Q u_j, \nabla \varphi) \, dx
\]
\[
= -\sum_{j=1}^{m} \int_{\mathbb{R}^d} (\nabla u_j, \nabla ((a_\varepsilon(u) + \varepsilon)^{-1} u_j \varphi)) \, dx
\]
\[
+ \sum_{j=1}^{m} \int_{\mathbb{R}^d} (\nabla u_j, \nabla ((a_\varepsilon(u) + \varepsilon)^{-1} u_j) \varphi) \, dx
\]
\[
= \sum_{j=1}^{m} \int_{\mathbb{R}^d} \frac{u_j}{a_\varepsilon(u) + \varepsilon} (\Delta_Q u_j) \varphi \, dx + \sum_{j=1}^{m} \int_{\mathbb{R}^d} \frac{1}{a_\varepsilon(u) + \varepsilon} (\nabla Q u_j, \nabla u_j) \varphi \, dx
\]
\[
- \sum_{j=1}^{m} \int_{\mathbb{R}^d} \frac{u_j}{(a_\varepsilon(u) + \varepsilon)^2} (\nabla Q u_j, \nabla a_\varepsilon(u)) \varphi \, dx
\]
\[
= \sum_{j=1}^{m} \int_{\mathbb{R}^d} \frac{u_j}{a_\varepsilon(u) + \varepsilon} (\Delta_Q u_j) \varphi \, dx + \sum_{j=1}^{m} \int_{\mathbb{R}^d} \frac{1}{a_\varepsilon(u) + \varepsilon} (\nabla Q u_j, \nabla u_j) \varphi \, dx
\]
\[
- \int_{\mathbb{R}^d} \frac{1}{a_\varepsilon(u) + \varepsilon} (\nabla Q a_\varepsilon(u), \nabla a_\varepsilon(u)) \varphi \, dx.
\]
Recall that \(\lim_{\varepsilon \to 0} a_\varepsilon(u) = |u|\) in \(L^2_{\text{loc}}(\mathbb{R}^d)\) and \(\lim_{\varepsilon \to 0} \nabla a_\varepsilon(u) = \nabla|u|\) in \(L^2_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)\). Noting that \(\frac{u_j}{a_\varepsilon(u) + \varepsilon}\) is uniformly bounded by 1 we see that we can apply the dominated convergence theorem in the first integral above. For the other two integrals,
we apply the monotone convergence theorem, using that $Q$ is strictly elliptic and observing that $(u, (u + \varepsilon)^{-1})$ decreases to $|u|^{-1}$. Note that in all integrals it is sufficient to integrate over the set $\{u \neq 0\}$. For the first and third integral, this is obvious due to the presence of $u_j$ which vanishes on $\{u = 0\}$. For the second one we infer from Stampacchia’s lemma that $\nabla u_j = 0$ on $\{u = 0\}$. Thus, by letting $\varepsilon \to 0$, we obtain

$$
\langle \Delta Q |u|, \varphi \rangle = \int_{\{u \neq 0\}} \sum_{j=1}^{m} \left( \frac{u_j}{|u|} \Delta Q u_j + \frac{1}{u} |\nabla u_j|^2_{Q} \right) \varphi \, dx - \int_{\{u \neq 0\}} \frac{1}{|u|} |\nabla |u||^2_{Q} \varphi \, dx.
$$

This proves (3). Using (2) and (3), also (4) follows.

3. Generation of a semigroup in $L^2(\mathbb{R}^d; \mathbb{C}^m)$. Let us consider the differential operator $\mathcal{A}u = \Delta Q u - Vu$, where $u = (u_1, \ldots, u_m)$. Here $\Delta Q$ acts entrywise on $u$, i.e., $\Delta Q u = (\Delta Q u_1, \ldots, \Delta Q u_m)$. We define $A$ to be the realization of $\mathcal{A}$ on $L^2(\mathbb{R}^d; \mathbb{C}^m)$ with domain

$$D(A) = \{u \in H^1(\mathbb{R}^d; \mathbb{C}^m) : \mathcal{A}u \in L^2(\mathbb{R}^d; \mathbb{C}^m)\}.$$ 

In this section we prove that $A$ generates a $C_0$-semigroup of contractions on the space $L^2(\mathbb{R}^d; \mathbb{C}^m)$. In view of the Lumer–Phillips theorem, cf. [7, Theorem II-3.15], it suffices to prove that $-A$ is maximal accretive, i.e., for $u \in D(A)$ we have $\langle -Au, u \rangle \geq 0$ and $I - A$ is surjective.

To that end, we follow the strategy from [15] and introduce some other realizations of the operator $\mathcal{A}$ on the space $H^{-1}(\mathbb{R}^d; \mathbb{C}^m)$, the dual space of $H^1(\mathbb{R}^d; \mathbb{C}^m)$. We define the operator $L_0$ by setting

$$L_0 u = \mathcal{A}u, \quad u \in D(L_0) := C_\infty^e(\mathbb{R}^d; \mathbb{C}^m)$$

and the operator $L$ by

$$Lu = \mathcal{A}u, \quad u \in D(L) := \{u \in H^1(\mathbb{R}^d; \mathbb{C}^m) : \mathcal{A}u \in H^{-1}(\mathbb{R}^d; \mathbb{C}^m)\}.$$ 

We let $\mathcal{A}^* = \Delta Q - V^*$ be the formal adjoint of $\mathcal{A}$, where $V^*$ is the conjugate matrix of $V$. We then define the operators $\tilde{L}$ and $\tilde{L}_0$ in analogy to the operators $L$ and $L_0$, using the potential $V^*$ instead of $V$.

We now collect some properties of the operators $L_0$ and $L$ and the adjoint $L_0^*$. We denote the duality pairing between $H^{-1}(\mathbb{R}^d; \mathbb{C}^m)$ and $H^1(\mathbb{R}^d; \mathbb{C}^m)$ by $\langle \cdot, \cdot \rangle$.

**Proposition 3.1.** Assume Hypotheses 2.1. Then the following hold

1. $\tilde{L} = L_0^*$ and $L = \tilde{L}_0^*$. Consequently, $\tilde{L}$ and $L$ are closed.
2. $L_0$ is closable and its closure is equal to $L_0^{**}$.

**Proof.** (1) Let $f \in D(\tilde{L})$ and $g \in C_\infty^e(\mathbb{R}^d, \mathbb{C}^m)$. Using integration by parts, we see that

$$[\tilde{L} f, g] = \int_{\mathbb{R}^d} \langle \text{div}(Q \nabla f)(x), g(x) \rangle \, dx - \int_{\mathbb{R}^d} \langle V^*(x) f(x), g(x) \rangle \, dx = \int_{\mathbb{R}^d} \langle f(x), \text{div}(Q \nabla g)(x) \rangle \, dx - \int_{\mathbb{R}^d} \langle f(x), V(x) g(x) \rangle \, dx = \langle L_0 g, f \rangle.$$ 

Thus $\tilde{L} = L_0^*$ and hence $\tilde{L}$ is closed. In a similar way one shows that $L = \tilde{L}_0^*$ and thus $L$ is also closed.

(2) Since $L_0^*$ is densely defined, $L_0$ is closable with closure $L_0^{**}$ by general theory, cf. [21, Theorem 3, pp. 196].

\[ \square \]
We can now prove the main result of this section.

**Theorem 3.2.** The operator \(-L\) is maximal monotone.

**Proof.** Step 1. We first show that \(-L_0^*\) is maximal monotone. It is easy to see that \(-L_0\) is monotone. Indeed, for \(\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)\) we have

\[
\text{Re}[-L_0\varphi, \varphi] = \int_{\mathbb{R}^d} |\nabla \varphi(x)|^2_\varphi(x) \, dx + \text{Re} \int_{\mathbb{R}^d} (V(x)\varphi(x), \varphi(x)) \, dx \geq 0.
\]

It follows that also the closure of \(-L_0\), i.e. the operator \(-L_0^*\), is maximal monotone. As \(-L_0^*\) is monotone, \(\text{rg}(1 - L_0^*)\), the range of \((1 - L_0^*)\), is a closed subset of \(H^{-1}(\mathbb{R}^d; \mathbb{C}^m)\), cf. [7, Proposition II-3.14]. Therefore, to prove that \(-L_0^*\) is maximal, it suffices to show that \(1 - L_0^*\) has dense range. We prove that \((1 - L_0^*) C_c^\infty(\mathbb{R}^d; \mathbb{C}^m)\) is dense in \(H^{-1}(\mathbb{R}^d; \mathbb{C}^m)\). Since the coefficients of \(A\) are real, it suffices to prove that \((1 - L_0^*) C_c^\infty(\mathbb{R}^d; \mathbb{C}^m)\) is dense in \(H^{-1}(\mathbb{R}^d; \mathbb{C}^m)\). To that end, let \(u \in H^1(\mathbb{R}^d; \mathbb{R}^m)\) be such that \((1 - L_0) \varphi, u) = 0\) for all \(\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)\). Then

\[u - \Delta_Q u + V^* u = 0,\]

and hence,

\[\Delta_Q u_j = u_j + \sum_{l=1}^m v_{lj} u_l,\]

for every \(j \in \{1, \ldots, m\}\) in the sense of distributions. Applying (4), we obtain

\[
\Delta_Q |u| \geq \frac{1}{|u|} \sum_{j=1}^m u_j \Delta_Q u_j \chi_{\{u\neq 0\}} \\
\geq \frac{\chi_{\{u\neq 0\}}}{|u|} \left( \sum_{j=1}^m u_j^2 + \sum_{j,l=1}^m v_{lj} u_l u_j \right) \\
\geq \frac{\chi_{\{u\neq 0\}}}{|u|} |u|^2 = |u|.
\]

Thus, \(\Delta_Q |u| \geq |u|\) in the sense of distributions. Now, let \((\phi_n)_n \subset C_c^\infty(\mathbb{R}^d)\) be such that \(\phi_n \geq 0\) and \(\phi_n \to |u|\) in \(H^1(\mathbb{R}^d)\). Then

\[0 \leq [\Delta_Q |u|, \phi_n] - [|u|, \phi_n] = - \int_{\mathbb{R}^d} \langle \nabla |u|, \nabla \phi_n \rangle_Q \, dx - \int_{\mathbb{R}^d} |u| \phi_n \, dx.
\]

Upon \(n \to \infty\), we find \(-||\nabla |u||_Q||^2 - ||u||^2 \geq 0\) which implies that \(u = 0\). This proves that the range of \(I - L_0^*\) is dense.

Step 2. We now prove that \(L = L_0^*\). We know that \(L\) is a closed extension of \(L_0\) hence \(L_0^* \subset L\). In order to get the converse inclusion, it suffices to show that \(\rho(L) \cap \rho(L_0^*) \neq \emptyset\). As \(L_0^*\) is maximal monotone we have \(1 \in \rho(L_0^*)\). On the other hand, \((1 - L) D(L) \supset (1 - L_0^*)D(L_0^*) = H^{-1}(\mathbb{R}^d; \mathbb{C}^m)\). Hence, \(1 - L\) is surjective. To prove that \(1 - L\) is injective, note that \(\ker(1 - L) = \text{rg}((1 - L)^*)\). Applying Step 1 with \(V^*\) instead of \(V\) we see that \(-L^* = -L_0^*\) is maximal. Repeating the above argument, it follows that \(\text{rg}(1 - L^*) = H^{-1}(\mathbb{R}^d; \mathbb{C}^m)\) and thus \(\ker(1 - L) = \{0\}\). This proves that \(1 \in \rho(L)\) and ends the proof.

We can now infer that \(A\) generates a strongly continuous contraction semigroup.

**Corollary 3.3.** Assume Hypotheses 2.1. Then the operator \(A\) generates a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) of contractions on \(L^2(\mathbb{R}^d; \mathbb{C}^m)\).
Proof. Since \(-L\) is monotone, so is \(-A\), the part of \(-L\) in \(L^2(\mathbb{R}^d; \mathbb{C}^m)\). As \(-L\) is maximal monotone, so is \(-A\). Indeed, given \(f \in L^2(\mathbb{R}^d; \mathbb{C}^m) \subset H^{-1}(\mathbb{R}^d; \mathbb{C}^m)\) we find \(u \in D(L)\) such that \(u - Au = f\). But then \(Au = u - f\) belongs to \(L^2(\mathbb{R}^d; \mathbb{C}^m)\), proving that \(u \in D(A)\) and \(u - Au = f\). The claim now follows from the Lumer–Phillips theorem.

4. Extension of the semigroup to \(L^p(\mathbb{R}^d; \mathbb{C}^m)\). In this section we extrapolate the semigroup \(\{T(t)\}_{t \geq 0}\) to the spaces \(L^p(\mathbb{R}^d; \mathbb{C}^m)\), \(1 < p < \infty\). As a first step, we prove that \(\{T(t)\}_{t \geq 0}\) is given by the Trotter–Kato product formula

\[
T(t)f = \lim_{n \to \infty} \left[ e_{\frac{t}{n}} \Delta e^{-\frac{t}{n}V} \right]^n f,
\]

for all \(t > 0\) and \(f \in L^2(\mathbb{R}^d; \mathbb{C}^m)\). Here \(\{e^{it\Delta}\}_{t \geq 0}\) is the semigroup generated by \(\Delta\) in \(L^2(\mathbb{R}^d; \mathbb{C}^m)\) and \(\{e^{-tV}\}_{t \geq 0}\) is the multiplication semigroup generated by the potential \(-V\), i.e. \(e^{-tV}\) is multiplication with the matrix given pointwise by

\[
\sum_{k=0}^{\infty} \frac{(-tV(x))^k}{k!}.
\]

To prove that the semigroup \(\{T(t)\}_{t \geq 0}\) is given by the Trotter–Kato formula (5) we use the following result which is also of independent interest.

**Proposition 4.1.** Assume Hypotheses 2.1. Then \(C^\infty_c(\mathbb{R}^d; \mathbb{C}^m)\) is a core for \(A\).

Proof. Since \(-A\) is maximal accretive and has real coefficients, it suffices to show that \((1 - A)C^\infty_c(\mathbb{R}^d; \mathbb{R}^m)\) is dense in \(L^2(\mathbb{R}^d; \mathbb{R}^m)\). Let \(u \in L^2(\mathbb{R}^d; \mathbb{R}^m)\) be such that \(\langle (1 - A)\varphi, u \rangle = 0\), for all \(\varphi \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^m)\). Thus, \(u - \Delta Qu + V^*u = 0\) in the sense of distributions. Hence,

\[
\Delta Qu_j = u_j + \sum_{l=1}^{m} v_{lj} u_l.
\]

In particular, \(\Delta Qu_j = \text{div}(Q \nabla u_j) \in L^2_{\text{loc}}(\mathbb{R}^d)\) for each \(j \in \{1, \ldots, m\}\). Then, by local elliptic regularity, see [2, Theorem 7.1], \(u_j \in H^2_{\text{loc}}(\mathbb{R}^d)\).

Therefore, \(|u| = \lim_{\varepsilon \to 0} \frac{|u|^2 + \varepsilon^2}{2}\) belongs to \(H^2_{\text{loc}}(\mathbb{R}^d)\). In particular, Equation (4) holds true as an inequality of functions in \(L^2_{\text{loc}}(\mathbb{R}^d)\), i.e.

\[
\Delta Q |u| \geq \frac{\chi_{\{u \neq 0\}}}{|u|} \sum_{j=1}^{m} u_j \Delta Q u_j
\]

almost everywhere. Consequently, we also have

\[
\Delta Q |u| \geq \frac{\chi_{\{u \neq 0\}}}{|u|} \left( |u|^2 + (V u, u) \right) \geq |u|
\]

almost everywhere. Here both \(\Delta Q |u|\) and \(|u|\) are functions in \(L^2_{\text{loc}}(\mathbb{R}^d)\).

Now, let \(\zeta \in C^\infty_c(\mathbb{R}^d)\) be such that \(\chi_{B(1)} \leq \zeta \leq \chi_{B(2)}\) and define \(\zeta_n(x) = \zeta(x/n)\) for \(x \in \mathbb{R}^d\) and \(n \in \mathbb{N}\). We multiply both two sides of the inequality \(\Delta Q |u| \geq |u|\) by \(\zeta_n |u|\) and integrate by parts. We obtain

\[
\begin{align*}
0 & \geq \int_{\mathbb{R}^d} |u(x)|^2 \zeta_n(x) dx - \int_{\mathbb{R}^d} \Delta Q |u(x)| \zeta_n(x) |u(x)| dx \\
& = \int_{\mathbb{R}^d} |u(x)|^2 \zeta_n(x) dx + \int_{\mathbb{R}^d} \langle \nabla (\zeta_n |u|)(x), Q(x) \nabla |u| (x) \rangle dx \\
& = \int_{\mathbb{R}^d} |u(x)|^2 \zeta_n(x) dx + \int_{\mathbb{R}^d} |\nabla |u(x)|^2 Q(x) \zeta_n(x) dx + \int_{\mathbb{R}^d} \langle \nabla \zeta_n(x), \nabla |u(x)| Q(x) |u(x)| dx
\end{align*}
\]
\[ \int_{\mathbb{R}^d} |u(x)|^2 \zeta_n(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} \langle Q(x) \nabla \zeta_n(x), \nabla |u|^2(x) \rangle dx \]

\[ = \int_{\mathbb{R}^d} |u(x)|^2 \zeta_n(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \Delta Q \zeta_n(x) |u(x)|^2 dx. \]

Here we have used in the fourth line that \( \nabla |u|^2 = 2u|\nabla u| \). A straightforward computation shows

\[ \Delta Q \zeta_n(x) = \frac{1}{n} \sum_{i,j=1}^m \partial_i q_{ij} \partial_j \zeta(x/n) + \frac{1}{n^2} \sum_{i,j=1}^m q_{ij} \partial_i \partial_j \zeta(x/n). \]

It follows that \( \| \Delta Q \zeta_n \|_\infty \to 0 \) as \( n \to \infty \). Hence, letting \( n \to \infty \) in the above inequality, we obtain \( \| u \|_2 \leq 0 \), and thus \( u = 0 \). This finishes the proof.

**Proposition 4.2.** Assume Hypotheses 2.1. Then the semigroup \( \{T(t)\}_{t \geq 0} \) is given by the Trotter-Kato product formula (5).

**Proof.** Since \( C^\infty_c(\mathbb{R}^d; \mathbb{C}^m) \subset D(\Delta Q) \cap D(V) \), where \( D(\Delta Q) = H^2(\mathbb{R}^d; \mathbb{C}^m) \) and \( D(V) = \{ u \in L^2(\mathbb{R}^d; \mathbb{C}^m) : Vu \in L^2(\mathbb{R}^d; \mathbb{C}^m) \} \), the claim follows from [7, Corollary III-5.8].

We can now extend \( \{T(t)\}_{t \geq 0} \) to \( L^p(\mathbb{R}^d; \mathbb{C}^m) \).

**Theorem 4.3.** Let \( 1 < p < \infty \) and assume Hypotheses 2.1. Then \( \{T(t)\}_{t \geq 0} \) can be extrapolated to a \( C_0 \)-semigroup \( \{T_p(t)\}_{t \geq 0} \) on \( L^p(\mathbb{R}^d; \mathbb{C}^m) \). Moreover, if we denote by \( (A_p, D(A_p)) \) its generator, then \( A_p u = Au \), for all \( u \in C^\infty_c(\mathbb{R}^d; \mathbb{C}^m) \).

**Proof.** Let \( 1 < p < \infty \) and \( f \in L^2(\mathbb{R}^d; \mathbb{C}^m) \cap L^p(\mathbb{R}^d; \mathbb{C}^m) \). Assumption (1) yields \( |e^{-tv}(x)f(x)| \leq |f(x)| \) for all \( x \in \mathbb{R}^d \) and \( t \geq 0 \). So \( \| e^{-tv}f \|_p \leq \| f \|_p \), for all \( t \geq 0 \).

On the other hand, it is well-known that \( \{e^{t\Delta Q}\} \geq 0 \) extends to a contractive \( C_0 \)-semigroup on \( L^p(\mathbb{R}^d; \mathbb{C}^m) \). Consequently, for every \( t > 0 \), both \( e^{t\Delta Q} \) and \( e^{-tv} \) leave the set

\[ B_p := \{ f \in L^2(\mathbb{R}^d; \mathbb{C}^m) \cap L^p(\mathbb{R}^d; \mathbb{C}^m) : \| f \|_p \leq 1 \} \]

invariant. Since \( B_p \) is a closed subset of \( L^2(\mathbb{R}^d; \mathbb{C}^m) \) as a consequence of Fatou’s lemma, it follows from the Trotter–Kato formula (5), that \( T(t)B_p \subset B_p \). So, \( \|T(t)f\|_p \leq \| f \|_p \) for all \( f \in L^2(\mathbb{R}^d; \mathbb{C}^m) \cap L^p(\mathbb{R}^d; \mathbb{R}^m) \). By density, we can extend \( T(t) \) to a contraction \( T_p(t) \) on \( L^p(\mathbb{R}^d; \mathbb{C}^m) \). The semigroup law for \( \{T_p(t)\}_{t \geq 0} \) can be obtained immediately.

Let us prove that \( \{T_p(t)\}_{t \geq 0} \) is strongly continuous. To that end, pick \( p^* \in (1, \infty) \) and \( \theta \in (0, 1) \) be such that \( 1/p = (1 - \theta)/2 + \theta/p^* \). By the interpolation inequality, we find

\[ \| T(t)f - f \|_p \leq \| T(t)f - f \|_2^{1-\theta} \| T(t)f - f \|_{p^*}^{\theta} \leq 2^\theta \| T(t)f - f \|_2^{1-\theta} \| f \|_{p^*}^{\theta}, \]

Thus, \( \lim_{t \to 0} T(t)f = f \) in \( L^p(\mathbb{R}^d; \mathbb{C}^m) \) for all \( f \in C^\infty_c(\mathbb{R}^d; \mathbb{C}^m) \). By density, the strong continuity of the semigroup \( \{T_p(t)\}_{t \geq 0} \) follows.

Let us now turn to the generator of \( \{T_p(t)\}_{t \geq 0} \). Fix \( t > 0 \) and \( f \in C^\infty_c(\mathbb{R}^d; \mathbb{C}^m) \). Then \( f \in D(A) \) and

\[ T(t)f - f = \int_0^t AT(s)f ds = \int_0^t T(s)Af ds, \]
where the integral is computed in \( L^2(\mathbb{R}^d; \mathbb{C}^m) \). However, \( Af \) has compact support whence \( Af \in L^p(\mathbb{R}^d; \mathbb{C}^m) \) and the map \( t \mapsto T_p(t)Af \) is continuous from \([0, \infty)\) into \( L^p(\mathbb{R}^d; \mathbb{C}^m) \). Hence, (6) holds true in \( L^p(\mathbb{R}^d, \mathbb{C}^m) \), i.e.

\[
T_p(t)f - f = \int_0^t T_p(s)Af \, ds.
\]

This implies that \( t \mapsto T_p(t)f \) is differentiable in \([0, \infty)\). It follows that \( f \in D(A_p) \) and \( A_p f = Af \).

**Remark 4.4.** It is also possible to extend \( T \) to a consistent contraction semigroup \( \{T_1(t)\}_{t \geq 0} \) on \( L^1(\mathbb{R}^d; \mathbb{C}^m) \). Mutatis mutandis, the proof is that of [16, Theorem 3.7].

5. **Maximal domain of \( A_p \) and further properties.** In this section we characterize the domain \( D(A_p) \) of the generator of \( \{T_p(t)\}_{t \geq 0} \). More precisely, we prove that it is the maximal domain in \( L^p \). We first show that the space of test functions is a core for \( A_p \).

**Proposition 5.1.** Let \( 1 < p < \infty \) and assume Hypotheses 2.1. Then,

(i) the set of test functions \( C_c^\infty(\mathbb{R}^d; \mathbb{C}^m) \) is a core for \( A_p \),

(ii) the semigroup \( \{T_p(t)\}_{t \geq 0} \) is given by the Trotter–Kato product formula.

**Proof.** (i) Fix \( 1 < p < \infty \). Since \( -A_p \) is \( m \)-accretive and the coefficients of \( A \) are real, it suffices to show that \((1 - A_p)C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)\) is dense in \( L^p(\mathbb{R}^d; \mathbb{R}^m) \). Let \( u \in L^{p'}(\mathbb{R}^d; \mathbb{R}^m) \) be such that \( \langle (1 - A)\varphi, u_p \rangle = 0 \) for all \( \varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m) \). So,

\[
u = Ax = \Delta Q u + V^* u = 0
\]

in the sense of distributions. In particular,

\[
\Delta Q u_j = u_j + \sum_{l=1}^{m} v_{lj} u_l \in L^{p'}(\mathbb{R}^d)
\]

for all \( j \in \{1, \ldots, m\} \). By local elliptic regularity, see [2, Theorem 7.1], \( u_j \in W^{2,p'}(\mathbb{R}^d) \) for all \( j \in \{1, \ldots, m\} \). Then, (7) holds true almost everywhere on \( \mathbb{R}^d \).

Consider \( \zeta \in C_c^\infty(\mathbb{R}^d) \) such that \( \chi_{B(1)} \leq \zeta \leq \chi_{B(2)} \) and define \( \zeta_n(\cdot) = \zeta(\cdot/n) \) for \( n \in \mathbb{N} \). For \( p' < 2 \) we multiply equation (7) by \( \zeta_n(|u|^2 + \varepsilon^2)^{\frac{p'-2}{2}} u \in L^p(\mathbb{R}^d; \mathbb{R}^m) \) for \( \varepsilon > 0, n \in \mathbb{N} \). Integrating by parts, we obtain

\[
0 = \int_{\mathbb{R}^d} \zeta_n(|u|^2 + \varepsilon^2)^{\frac{p'-2}{2}} |u|^2 \, dx + \sum_{j=1}^{m} \int_{\mathbb{R}^d} \left\langle \nabla u_j, \nabla \left( \zeta_n(|u|^2 + \varepsilon^2)^{\frac{p'-2}{2}} u_j \right) \right\rangle_Q \, dx
\]

\[
+ \int_{\mathbb{R}^d} \zeta_n(|u|^2 + \varepsilon^2)^{\frac{p'-2}{2}} \langle V^* u, u \rangle \, dx
\]

\[
\geq \int_{\mathbb{R}^d} \zeta_n(|u|^2 + \varepsilon^2)^{\frac{p'-2}{2}} |u|^2 \, dx + \sum_{j=1}^{m} \int_{\mathbb{R}^d} |\nabla u_j|^2_Q \zeta_n(|u|^2 + \varepsilon^2)^{\frac{p'-2}{2}} \, dx
\]

\[
+ \sum_{j=1}^{m} \int_{\mathbb{R}^d} \langle \nabla u_j, \nabla \zeta_n \rangle_Q (|u|^2 + \varepsilon^2)^{\frac{p'-2}{2}} u_j \, dx
\]

\[
+ (p' - 2) \sum_{j=1}^{m} \int_{\mathbb{R}^d} \langle \nabla u_j, \nabla |u| \rangle_Q u_j |u| \zeta_n(|u|^2 + \varepsilon^2)^{\frac{p'-2}{2}} \, dx
\]
Upon \( \epsilon \to 0 \), we find

\[
\int_{\mathbb{R}^d} \zeta_n |u|^p' \, dx - \frac{1}{p'} \int_{\mathbb{R}^d} |\Delta Q \zeta_n| |u|^p' \, dx \leq 0.
\]

As in the proof of Proposition 4.1, upon \( n \to \infty \), we conclude that

\[
\int_{\mathbb{R}^d} |u|^p' \, dx \leq 0.
\]

Therefore, \( u = 0 \).

In the case when \( p' > 2 \), one multiplies in (7) by \( \zeta_n |u|^{p'-2} u \) and argues in a similar way.

(ii) This is an immediate consequence of (i) and [7, Corollary III-5.8]. \( \square \)

Using a similar strategy as in [8] or [18] we show in the next result that the domain \( D(A_p) \) is equal to the \( L^p \)-maximal domain of \( A \).

**Proposition 5.2.** Let \( 1 < p < \infty \). Assume Hypotheses 2.1. Then

\( D(A_p) = \{ u \in L^p(\mathbb{R}^d; \mathbb{C}^m) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^m) : Au \in L^p(\mathbb{R}^d; \mathbb{C}^m) \} =: D_{p,\text{max}}(A). \)

**Proof.** Let us show first that \( D(A_p) \subseteq D_{p,\text{max}}(A) \). Take \( u \in D(A_p) \). Since \( C^\infty_c(\mathbb{R}^d; \mathbb{C}^m) \) is a core for \( A_p \), it follows that there exists \( (u_n)_n \subset C^\infty_c(\mathbb{R}^d; \mathbb{C}^m) \) such that \( u_n \to u \) and \( Au_n \to Au \) in \( L^p(\mathbb{R}^d; \mathbb{C}^m) \), and in particular in \( L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^m) \). As \( V \in L^\infty_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^m) \), we deduce that \( V u_n \to V u \) in \( L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^m) \). Consequently,

\[
\Delta Q u = A_p u + V u = \lim_{n \to \infty} Au_n + V u_n \in L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^m).
\]

So, by local elliptic regularity, we obtain \( u \in W^{2,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^m) \). Hence, \( Au = A_p u \) belongs to \( L^p(\mathbb{R}^d; \mathbb{C}^m) \), which shows that \( u \in D_{p,\text{max}}(A) \).

In order to prove the other inclusion it suffices to show that \( \lambda - A \) is injective on \( D_{p,\text{max}}(A) \), for some \( \lambda > 0 \). To this end, let \( u \in D_{p,\text{max}}(A) \) be such that \( (\lambda - A) u = 0 \). Assume that \( p \geq 2 \). Multiplying by \( \zeta_n |u|^{p-2} u \) and integrating (by part) over \( \mathbb{R}^d \) one obtains

\[
0 = \lambda \int_{\mathbb{R}^d} \zeta_n(x) |u(x)|^p \, dx + \int_{\mathbb{R}^d} \sum_{j=1}^m \langle Q \nabla u_j, \nabla (|u|^{p-2} u_j \zeta_n) \rangle \, dx
\]
So, as in the proof of the above proposition, we conclude that $u = 0$.

Let us consider the matrix potential

$$V(x) := \begin{pmatrix} w(x) & -v(x) \\ v(x) & w(x) \end{pmatrix} = v(x) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + w(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $v \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ and $0 \leq w \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. Then Hypotheses 2.1 are satisfied and we can deduce from Theorem 4.3 and Proposition 5.2, that $A_p$, the $L^p$-realization of the operator

$$A = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} - V$$

with domain $D(A_p) := \{ u \in L^p(\mathbb{R}^d; \mathbb{C}^2) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^2) : Au \in L^p(\mathbb{R}^d; \mathbb{C}^2) \}$ generates a $C_0$-semigroup on $L^p(\mathbb{R}^d; \mathbb{C}^2)$. Moreover $C_c^\infty(\mathbb{R}^d) \subset \{ A_p \}$ is a core for $A_p$.

Diagonalizing the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we see that $A_p$ is similar to a diagonal operator. More precisely, with $P = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ we have

$$P^{-1}A_pP = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} - \begin{pmatrix} iv + w & 0 \\ 0 & -iv + w \end{pmatrix}.$$

It follows that the Schrödinger operators $\Delta \pm iv - w$ with domain

$$\{ f \in L^p(\mathbb{R}^d) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^d) : \Delta f \pm ivf - wf \in L^p(\mathbb{R}^d) \}$$

generate $C_0$-semigroups on $L^p(\mathbb{R}^d)$. Moreover $C_c^\infty(\mathbb{R}^d)$ is a core for these operators. In general, these semigroups can not be expected to be analytic, see [16, Example 3.5]. However, imposing additional assumptions on the potential $V$, e.g. that the numerical range is contained in a sector, one can also prove analyticity of the
Proposition 5.4. Assume Hypotheses 2.1 and there is a positive constant $C$ such that
\[
\Re (V(x)\xi, \xi) \geq C|\Im (V(x)\xi, \xi)|, \quad \forall x \in \mathbb{R}^d, \xi \in \mathbb{C}^m.
\]
Then \(\{T_p(t)\}_{t \geq 0}\) can be extended to an analytic semigroup on \(L^p(\mathbb{R}^d, \mathbb{C}^m)\).

Using this, we see that these semigroups are analytic provided that there is a constant \(C > 0\) such that \(|v(x)| \leq Cv(x)|\) for a.e. \(x \in \mathbb{R}^d\).

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