SHARP ENDPOINT ESTIMATES FOR SCHRÖDINGER GROUPS
ON HARDY SPACES

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Abstract. Let $L$ be a non-negative self-adjoint operator acting on $L^2(X)$ where $X$ is a space of homogeneous type with a dimension $n$. Suppose that the heat kernel of $L$ satisfies the Davies-Gaffney estimates of order $m \geq 2$. Let $H^1(X)$ be the Hardy space associated with $L$. In this paper we show sharp endpoint estimate for the Schrödinger group $e^{itL}$ associated with $L$ such that

$$\| (I + L)^{-n/2} e^{itL} f \|_{H^1(X)} + \| (I + L)^{-n/2} e^{itL} f \|_{H^1(X)} \leq C(1 + |t|)^{n/2} \| f \|_{H^1(X)}, \quad t \in \mathbb{R}$$

for some constant $C = C(n, m) > 0$ independent of $t$. By a duality and interpolation argument, it gives a new proof of a recent result of [13] for sharp endpoint $L^p$-Sobolev bound for $e^{itL}$:

$$\| (I + L)^{-s} e^{itL} f \|_{L^p(X)} \leq C(1 + |t|)^s \| f \|_{L^p(X)}, \quad t \in \mathbb{R}, \quad s \geq n \frac{1}{2} - \frac{1}{p},$$

for every $1 < p < \infty$ when the heat kernel of $L$ satisfies a Gaussian upper bound, which extends the classical results due to Miyachi ([39, 40]) for the Laplacian on the Euclidean space $\mathbb{R}^n$.

1. Introduction

Consider the Laplace operator $\Delta = -\sum_{i=1}^n \partial_{x_i}^2$ on the Euclidean space $\mathbb{R}^n$ and the Schrödinger equation

$$\begin{cases}
    i\partial_t u + \Delta u = 0, \\
    u|_{t=0} = f
\end{cases}$$

with initial data $f$. Its solution can be written as

$$u(x, t) = e^{it\Delta} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)e^{i(x, \xi) + it|\xi|^2} d\xi,$$

where $\hat{f}$ denotes the Fourier transform of $f$. It is well-known that the operator $e^{it\Delta}$ acts boundedly on $L^p(\mathbb{R}^n)$ if and only if $p = 2$; see Hörmander [30]. For $p \neq 2$, it was shown (see for example, [8, 35, 47]) that for $s > n|1/2 - 1/p|$, the operator $e^{it\Delta}$ maps the Sobolev space $L^p_{2s}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, in other words, $(I + \Delta)^{-s} e^{it\Delta}$ is bounded on $L^p(\mathbb{R}^n)$. For $s < n|1/2 - 1/p|$, it is known that the operator $(I + \Delta)^{-s} e^{it\Delta}$ is unbounded on $L^p(\mathbb{R}^n)$. In [39], Miyachi obtained the sharp endpoint estimate for $e^{it\Delta}$ on Hardy and Lebesgue spaces, and showed that for every $0 < p < \infty$,

$$\| (I + \Delta)^{-s} e^{it\Delta} f \|_{H^p(\mathbb{R}^n)} \leq C(1 + |t|)^s \| f \|_{H^p(\mathbb{R}^n)}, \quad t \in \mathbb{R}, \quad s \geq n \frac{1}{2} - \frac{1}{p},$$

\[ (1.1) \]
where $H^p(\mathbb{R}^n)$ is the classical Hardy space ([26]) on $\mathbb{R}^n$ and $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ if $1 < p < \infty$. See also Fefferman-Stein’s work [26, Section 6].

The Schrödinger semigroup $\{e^{it\Delta}\}_{t \geq 0}$ can be defined in terms of the spectral resolution of the self-adjoint Laplace operator $\Delta$. A natural question is to determine a sufficient condition so that (1.1) holds when the Laplace $\Delta$ is replaced by a non-negative self-adjoint operator $L$. For this purpose we suppose that $(X, d, \mu)$ is a metric measure space with a distance $d$ and a measure $\mu$, and $L$ is a non-negative self-adjoint operator on $L^2(X)$. Such an operator $L$ admits a spectral resolution

$$L = \int_0^\infty \lambda dE_L(\lambda),$$

where $E_L(\lambda)$ is the projection-valued measure supported on the spectrum of $L$. The operator $e^{itL}$, $t \in \mathbb{R}$, is defined by

$$e^{itL}f = \int_0^\infty e^{it\lambda}dE_L(\lambda)f$$

for $f \in L^2(X)$, and forms the Schrödinger group. By the spectral theorem ([38]), the operator $e^{itL}$ is continuous on $L^2(X)$. Our main interest will be in the mapping properties of families of operators derived from the Schrödinger group on Hardy and Lebesgue spaces.

Depending on the nature of the assumptions regarding the assumption of $e^{-itL}$, there are various nuances of the mapping properties of the Schrödinger group $e^{-itL}$ on $L^p$ spaces presently available in the literature. For example, on Lie groups with polynomial growth and manifolds with non-negative Ricci curvature, similar results as in (1.1) for $s > n|1/2 - 1/p|$ and $1 < p < \infty$ have been first announced by Lohoué in [37], then obtained by Alexopoulos in [1]. In the abstract setting of operators on metric measure spaces, Carron, Coulhon and Ouhabaz [12] showed that for every $1 < p < \infty$,

$$\| (I + L)^{-s} e^{itL}f \|_p \leq C(1 + |t|^s)\|f\|_p, \quad t \in \mathbb{R}, \quad s > n\left|\frac{1}{2} - \frac{1}{p}\right|,$$

provided the semigroup $e^{-itL}$, generated by $-L$ on $L^2(X)$, has the kernel $p_t(x, y)$ which satisfies the Gaussian upper bound, i.e.

$$(\text{GE}_m) \quad |p_t(x, y)| \leq \frac{C}{V(x, t^{1/m})} \exp \left( -c \left( \frac{d(x, y)^m}{t} \right)^{\frac{1}{m-1}} \right)$$

for every $t > 0, x, y \in X$, where $c, C$ are positive constants and $m \geq 2$. Such estimate $(\text{GE}_m)$ is typical for elliptic or sub-elliptic differential operators of order $m$ (see for example, [1, 12, 18, 21, 22, 32, 33, 42, 46, 47] and the references therein). See also related results in [9, 17, 26, 32, 33].

The question whether estimate (1.3) holds with $s = n|1/2 - 1/p|$ was recently solved in [13]. More specifically, if $L$ satisfies the Gaussian estimate $(\text{GE}_m)$, then for every $p \in (1, \infty)$ there exists a constant $C = C(n, m, p) > 0$ independent of $t$ such that

$$\| (I + L)^{-s} e^{itL}f \|_p \leq C(1 + |t|^s)\|f\|_p, \quad t \in \mathbb{R}, \quad s = n\left|\frac{1}{2} - \frac{1}{p}\right|.$$

However, this result does not give any end-point estimate on the Hardy space $H^1_L(X)$ when $p = 1$. 


This paper continues a line of study in [13] to show that the operator $(I + L)^{-n/2}e^{itL}$ is bounded on Hardy spaces $H^1_t(X)$ under the assumption that $L$ satisfies $m$-th order Davies-Gaffney estimates, that is, there exist constants $C, c > 0$ such that for all $t > 0$, and all $x, y \in X$,

$$(DG_m) \quad \left\| P_{B(x,t^{1/m})} e^{-itL} P_{B(y,t^{1/m})} \right\|_{2 \rightarrow 2} \leq C \exp \left( -c \left( \frac{d(x,y)}{t^{1/m}} \right)^{\frac{1}{m}} \right)$$

where $P_{B(x,t^{1/m})}$ denotes the characteristic function on the ball $B(x, t^{1/m})$ and $H^1_t(X)$ denotes the Hardy space associated with $L$ ([2, 23, 34], see Section 2 below). We then apply the duality argument and the complex interpolation result (see Lemma 4.1 below) to obtain a new proof of estimate (1.4) in [13] in the case that the operator $L$ satisfies a Gaussian upper bound $(GE_m)$.

Note that the $m$-th order Davies-Gaffney estimate $(DG_m)$ is much more general than the Gaussian estimate $(GE_m)$. Indeed, if an operator $L$ satisfies the $(GE_m)$ estimate, then $L$ satisfies the $(DG_m)$ estimate. However, there are large classes of operators which satisfy the $(DG_m)$ estimate but not the $(GE_m)$ estimate. This happens, e.g., for Schrödinger operators with rough potentials [44], second order elliptic operators with rough lower order terms [36], or higher order elliptic operators with bounded measurable coefficients [19]. See also [4, 5, 14, 34].

Our result can be stated as follows.

**Theorem 1.1.** Suppose that $(X, d, \mu)$ is a space of homogeneous type with a dimension $n$. Suppose that $L$ satisfies the property $(DG_m)$. Then there exists a constant $C = C(n, m) > 0$ independent of $t$ such that

$$(1.5) \quad \left\| (I + L)^{-n/2}e^{itL}f \right\|_{L^1(X)} \leq \left\| (I + L)^{-n/2}e^{itL}f \right\|_{H^1_t(X)} \leq C(1 + |t|)^{n/2}\|f\|_{H^1_t(X)}, \quad t \in \mathbb{R}.$$  

By interpolation and duality argument, we have that for $1 < p \leq 2$,

$$(1.6) \quad \left\| (I + L)^{-s}e^{itL}f \right\|_{L^p(X)} \leq C(1 + |t|)^s\|f\|_{H^p_t(X)}, \quad t \in \mathbb{R}, \quad s = n\left[ \frac{1}{2} - \frac{1}{p} \right]$$

and for $2 < p < \infty$,

$$(1.7) \quad \left\| (I + L)^{-s}e^{itL}f \right\|_{H^p_t(X)} \leq C(1 + |t|)^s\|f\|_{L^p(X)}, \quad t \in \mathbb{R}, \quad s = n\left[ \frac{1}{2} - \frac{1}{p} \right].$$

**Remark 1.2.** (i) First, we would like to remark that our main result, Theorem 1.1, implies the main result in [13] which states that under the generalised Gaussian estimate, we can obtain the sharp estimate for the Schrödinger group on Lebesgue spaces. For the convenience of the reader, we recall that the semigroup $e^{-tL}$ satisfies the generalized Gaussian $(p_0, p'_0)$-estimate of order $m$ (in which $1 \leq p_0 < 2$ and $\frac{1}{p_0} + \frac{1}{p'_0} = 1$), if there exist constants $C, c > 0$ such that

$$(GGE_{p_0, p'_0, m}) \quad \left\| P_{B(x,t^{1/m})} e^{-itL} P_{B(y,t^{1/m})} \right\|_{p_0 \rightarrow p'_0} \leq CV(x, t^{1/m})^{-\frac{1}{p_0} - \frac{1}{p'_0}} \exp \left( -c \left( \frac{d(x,y)^m}{t} \right)^{\frac{1}{m}} \right)$$

for every $t > 0$ and $x, y \in X$. In [13], sharp estimate for the Schrödinger group on $L^p(X)$ was obtained for the range $p_0 < p < p'_0$ under the assumption of $(GGE_{p_0, p'_0, m})$. 
Observe that by Hölder’s inequality, the generalized Gaussian \( (p_0, p'_0) \)-estimate implies the generalized Gaussian \( (p_1, p'_1) \)-estimate for \( 1 \leq p_0 < p_1 \leq 2 \), hence implies the Davies-Gaffney estimate \( \text{DG}_m \) (which is precisely the generalized Gaussian \( (p_0, p'_0) \)-estimate when \( p_0 = 2 \)), see for example [4]. Note also that under the generalized Gaussian \( (p_0, p'_0) \)-estimate, the Hardy space associated to operator \( H^p_L(X) \) coincides with \( L^p(X) \) for \( p_0 < p \leq 2 \) (See for example [34]). Hence this paper gives a new proof for the main result in [13].

(ii) This paper gives a new end-point estimate on the Hardy space \( H^1_L(X) \) for large classes of operators which only require the \( m \)-th order Davies-Gaffney estimate \( \text{DG}_m \). Our result gives the sharp endpoint estimate (1.5) for the Schrödinger group \( e^{itL} \) on the Hardy space, namely with the optimal number of derivatives and the optimal time growth for the factor \( (1 + |t|)^s \) in (1.5). While our endpoint estimate is obtained in terms of the Hardy space \( H^1_L(X) \) associated to the operator \( L \) instead of the classical Hardy space in the sense of Coifman and Weiss, it is known that if we assume stronger standard conditions on the operator \( L \) such as the Gaussian estimate \( \text{GE}_m \) and Hölder continuity on the heat kernel, and the conservation property \( e^{-itL}1 = 1 \), then the Hardy space \( H^1_L(X) \) associated to the operator \( L \) coincides with the classical Hardy space.

Note that when \( L \) is the Laplace operator \( \Delta \) on the Euclidean spaces \( \mathbb{R}^n \), our Theorem 1.1 gives a direct proof of the following result:

\[
(1.8) \quad \| (1 + \Delta)^{-n/2} e^{it\Delta}f \|_{H^1(\mathbb{R}^n)} \leq C(1 + |t|)^{n/2} \| f \|_{H^1(\mathbb{R}^n)}.
\]

In [39], Miyachi proved the above estimate (1.8) by using interpolation between \( H^p(\mathbb{R}^n) \) for \( p < 1 \) and \( L^2(\mathbb{R}^n) \) for the Schrödinger group \( e^{it\Delta} \) on the Euclidean space \( \mathbb{R}^n \).

(iii) We also remark that the results in [26, 39, 40] relies on Fourier analysis (e.g., Plancherel’s Theorem), which is not available in the setting of space of homogeneous type in this paper. In the proof of Theorem 1.1, the main tool is to use the Phragmén-Lindelöf theorem to show that the \( m \)-th order Davies-Gaffney estimate \( \text{DG}_m \) implies the following off-diagonal estimate of the operator \( e^{zL} \) with \( z = (i\tau - 1)R^{-1}, \tau, R > 0 \):

\[
(1.9) \quad \| P_B e^{(i\tau - 1)R^{-1}L_jP_{2j/B'}} \|_{2 \to 2} \leq C \exp \left( -c \left( \frac{\sqrt{R^2/r}}{\sqrt{1 + \tau^2}} \right)^{m/\tau} \right), \quad j = 2, 3, \ldots
\]

for all balls \( B \subseteq X \) (see Lemma 3.3 below). This new estimate (1.9) is crucial in the proof of Theorem 1.1.

(iv) In Section 5 we apply Theorem 1.1 to the Schrödinger group of the Kohn Laplacian \( \square_b \) on polynomial model domains treated by Nagel-Stein [41], where \( e^{-t\square_b} \) satisfies \( m \)-th order Davies-Gaffney estimates \( \text{DG}_m \) with \( m = 2 \). We note that in general polynomial model domains, \( e^{-t\square_b} \) does not satisfy the generalized Gaussian \( (p_0, p'_0) \)-estimate hence the result in [13] is not applicable to the Schrödinger group of the Kohn Laplacian \( \square_b \). The reason for \( e^{-t\square_b} \) not satisfying the generalized Gaussian estimate is that \( e^{-t\square_b} \) could have singularity on the diagonal since the null space of \( \square_b \) may not be \{0\}. It is worth pointing out that if the null space of \( \square_b \) is \{0\}, then \( e^{-t\square_b} \) satisfies the standard Gaussian upper bound, see for example [7].
The paper is organized as follows. In Section 2 we provide some preliminary results on Hardy spaces and spectral multipliers. In Section 3 we apply the Phragmén-Lindelöf theorem to give off-diagonal bounds for (1.9) and the operator \( F(L) \) for some compactly supported function \( F \). This plays a crucial role in the proof of Theorem 1.1 which will be given in Section 4. In Section 5 we give an application of Theorem 1.1 in a study of the Schrödinger group for the Kohn Laplacian on polynomial model domains.

2. Notations and preliminaries on Hardy spaces

We start by introducing some notation and assumptions. Throughout this paper, unless we mention the contrary, \((X, d, \mu)\) is a metric measure space where \( \mu \) is a Borel measure with respect to the topology defined by the metric \( d \). Next, let \( B(x, r) = \{ y \in X, d(x, y) < r \} \) be the open ball with centre \( x \in X \) and radius \( r > 0 \). To simplify notation we often just use \( B \) instead of \( B(x, r) \) and given \( \lambda > 0 \), we write \( \lambda B \) for the \( \lambda \)-dilated ball which is the ball with the same centre as \( B \) and radius \( \lambda r \). Let \( B^c \) be the set \( X \setminus B \). We set \( V(x, r) = \mu(B(x, r)) \) the volume of \( B(x, r) \) and we say that \((X, d, \mu)\) satisfies the doubling property (see Chapter 3, [15]) if there exists a constant \( C > 0 \) such that

\[
V(x, 2r) \leq CV(x, r) \quad \forall r > 0, \ x \in X.
\]

If this is the case, there exist \( C, n \) such that for all \( \lambda \geq 1 \) and \( x \in X \)

\[
V(x, \lambda r) \leq C\lambda^n V(x, r).
\]

In the Euclidean space with Lebesgue measure, \( n \) corresponds to the dimension of the space.

For \( 1 \leq p \leq +\infty \), we denote the norm of a function \( f \in L^p(X, d\mu) \) by \( \|f\|_p \), by \( \langle \cdot, \cdot \rangle \) the scalar product of \( L^2(X, d\mu) \), and if \( T \) is a bounded linear operator from \( L^p(X, d\mu) \) to \( L^q(X, d\mu) \), \( 1 \leq p, q \leq +\infty \), we write \( \|T\|_{p \to q} \) for the operator norm of \( T \). Given a subset \( E \subseteq X \), we denote by \( \chi_E \) the characteristic function of \( E \) and by \( P_E \) the projection \( P_E f(x) := \chi_E(x)f(x) \). We denote the dilation of a function \( F \) by \( \delta_r F(\cdot) := F(r \cdot) \) and \( \widehat{f} \) denotes the Fourier transform, i.e. of \( f \),

\[
\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-ix\xi} \, dx, \quad \xi \in \mathbb{R}^n.
\]

Sometimes we also use \( \widehat{f} \) for \( \mathcal{F} f \).

2.1. Hardy spaces associated with operators. A theory of Hardy spaces associated with certain operators was introduced and developed in [2, 20, 24, 25, 28, 34] and the references therein, similar to the way that classical Hardy spaces are adapted to the Laplacian. We present some main features of this theory in this section for reader’s convenience.

Suppose that \( L \) is a non-negative self-adjoint operator on \( L^2(X) \) which satisfies \( m \)-th order Davies-Gaffney estimates (DG\(_m\)) with \( m \geq 2 \). Following [28], we define the \( L^2 \) adapted Hardy space

\[
H^2(X) := H^2_\mu(X) := \overline{R(L)},
\]

that is, the closure of the range of \( L \) in \( L^2(X) \). Then \( L^2(X) \) is the orthogonal sum of \( H^2(X) \) and the null space \( N(L) \). Consider the following quadratic operators associated to \( L \).
where $f \in L^2(X)$. We shall write $S_h$ in place of $S_{h,1}$. For each $K \geq 1$ and $1 \leq p < \infty$, we now define

$$D_{K,p} = \{ f \in H^2(X) : S_h,Kf \in L^p(X) \}, \quad 1 \leq p < \infty.$$  

**Definition 2.1.** Let $L$ be a self-adjoint positive definite operator on $L^2(X)$ satisfying the Davies-Gaffney estimate (1.4).

(i) For each $1 \leq p \leq 2$, the Hardy space $H^p_{L,ps}(X)$ associated to $L$ is the completion of the space $D_{1,p}$ in the norm

$$\| f \|_{H^p_{L,ps}(X)} = \| S_h f \|_{L^p(X)}.$$  

(ii) For each $2 < p < \infty$, the Hardy space $H^p_L(X)$ associated to $L$ is the completion of the space $D_{K_0,p}$ in the norm

$$\| f \|_{H^p_{L,ps}(X)} = \| S_h f \|_{L^p(X)}, \quad K_0 = \left[ \frac{n}{4} \right] + 1.$$  

The Hardy spaces associated to $L$ are known to possess nice properties, for example, they form a complex interpolation scale (see Lemma 2.6 below). Note that, in the framework of the present paper, we only assume the Davies-Gaffney estimates on the heat kernel of $L$, and hence for $1 < p < \infty$, $p \neq 2$, $H^p_{L,ps}(X)$ may or may not coincide with the space $L^p(X)$. However, it can be verified that $H^2_{L,ps}(X) = H^2(X)$. It remains an open problem, in this general context, to determine whether $H^1_{L,ps}(X) \subseteq L^1(X)$ (see [28, p. 70] and [3]).

Let us describe the notion of a $(1,2,M,\varepsilon)$-molecule associated to an operator $L$ on spaces $(X,d,\mu)$. Denote by $\mathcal{D}(T)$ the domain of an operator $T$. For every ball $B$, we set

$$U_0(B) = B, \quad \text{and} \quad U_j(B) = 2^j B \setminus 2^{j-1} B \quad \text{for} \quad j = 1, 2, \ldots.$$  

**Definition 2.2.** Let $\varepsilon > 0$ and $M \in \mathbb{N}$. A function $m(x) \in L^2(X)$ is called a $(1,2,M,\varepsilon)$-molecule associated with $L$ if there exist a function $b \in \mathcal{D}(L^M)$ and a ball $B$ such that

(i) $m = L^M b$;

(ii) For every $k = 0,1,2,\ldots,M$ and $j = 0,1,2,\ldots$, there holds

$$\|(r^k B L)^j b\|_{L^2(U_j(B))} \leq 2^{-j \varepsilon} \lambda^k B V(2^j B)^{-1/2},$$

where the annuli $U_j(B)$ are defined in (2.4).

Next we give the definition of the molecular Hardy spaces associated with $L$.

**Definition 2.3.** We fix $\varepsilon > 0$ and $M \in \mathbb{N}$. The Hardy space $H^1_{L, mol,M,\varepsilon}(X)$ is defined as follows. We say that $f = \sum \lambda_j m_j$ is a molecular $(1,2,M,\varepsilon)$-representation (of $f$) if $\{ \lambda_j \}_{j=0}^\infty \in l^1$, each $m_j$ is a $(1,2,M,\varepsilon)$-molecule, and the sum converges in $L^2(X)$. Set

$$H^1_{L, mol,M,\varepsilon}(X) = \{ f : f \text{ has a molecular} (1,2,M,\varepsilon)\text{-representation} \},$$
with the norm given by
\[ \|f\|_{L^1_{mol,M,e}(X)} = \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j| : f = \sum_{j=0}^{\infty} \lambda_j m_j \text{ is a molecular (1, 2, M, \epsilon)-representation} \right\}. \]

The space \( L^1_{mol,M,e}(X) \) is then defined as the completion of \( \mathcal{H}^1_{L^1_{mol,M,e}}(X) \) with respect to this norm.

As a direct consequence of the definition, we note that \( L^1_{mol,M_1,\epsilon}(X) \subset L^1_{mol,M_2,\epsilon}(X) \) for \( \epsilon > 0 \) and \( M_1, M_2 \in \mathbb{N} \) with \( M_1 \leq M_2 \). We have the following characterization. For its proof, see [20, Section 3].

**Lemma 2.4.** Suppose \( M \geq n/4 \). Then we have \( L^1_{mol,M,e}(X) = L^1_{L^1}(X) \). Moreover,
\[ \|f\|_{L^1_{mol,M,e}(X)} \approx \|f\|_{L^1_{L^1}(X)}, \]
where the implicit constants depend only on \( M, m \) and \( n \) in (2.2) only.

We have the following dual result.

**Lemma 2.5.** Assume that the operator \( L \) satisfies \( m \)-th order Davies-Gaffney estimates (DG\(_m\)) with \( m \geq 2 \). Then for \( 1 < p < \infty \), we have
\[ (H^p_{L^1}(X))^* = H^{p'}_{L^1}(X), \]
where \( p' \) is the conjugate index of \( p \) such that \( 1/p' + 1/p = 1 \).

Similar to the classical Hardy spaces, Hardy spaces associated with operators form a complex interpolation scale. Let \([·, ·]_\theta\) stand for the complex interpolation bracket. Then we have the following result.

**Lemma 2.6.** Assume that the operator \( L \) satisfies \( m \)-th order Davies-Gaffney estimates (DG\(_m\)) with \( m \geq 2 \). Then for every \( 0 < \theta < 1 \) and \( 1 < p_0 < \infty \), we have
\[ [H^1_{L^1}, H^{p_0}_{L^1}]_{\theta} = H^p_{L^1}, \quad \frac{1}{p} = (1 - \theta) + \frac{\theta}{p_0}, \]

**Proof.** The proof can be verified that by viewing these spaces via the framework of tent spaces and by using the interpolation properties of tent spaces (see for example, [34, Lemma 4.20]).

### 2.2. Spectral multipliers on the Hardy space

The following result is a standard known result in the theory of spectral multipliers of non-negative self-adjoint operators.

**Proposition 2.7.** Let \( m \geq 2 \). Suppose that \( (X, d, \mu) \) is a space of homogeneous type with a dimension \( n \). Assume that the operator \( L \) satisfies the \( m \)-th order Davies-Gaffney estimates (DG\(_m\)) with \( m \geq 2 \). Assume in addition that \( F \) is an even bounded Borel function such that \( \sup_{R>0} \|\eta \delta_R F\|_{C^\alpha} < \infty \) for some integer \( \alpha > (n + 1)/2 \) and some non-trivial function \( \eta \in C_c(0, \infty) \). Then the operator \( F(L) \) is bounded on \( L^1_{L^1}(X) \),
\[ \|F(L)\|_{L^1_{L^1}(X)} \lesssim C \left( \sup_{R>0} \|\eta \delta_R F\|_{C^\alpha} + F(0) \right). \ Eq(2.5)
Proof. For the proof, see for example, [34, Theorem 1.4] and [25, Theorem 1.1].

3. Off-diagonal bounds for compactly supported spectral multipliers

Let us start with stating the Phragmén-Lindelöf Theorem for sectors in the complex plane $\mathbb{C}$. For its proof, we refer to [49, Lemma 4.2].

**Theorem 3.1.** Let $S$ be the open region in $\mathbb{C}$ bounded by two rays meeting at an angle $\pi/a$ for some $a > 1/2$. Suppose that $F$ is analytic on $S$, continuous on $\overline{S}$ and satisfies $|F(z)| \leq C \exp(c|z|^b)$ for some $b \in [0, a)$ and for all $z \in S$. Then the condition $|F(z)| \leq B$ on the two bounding rays implies that $|F(z)| \leq B$ for all $z \in S$.

The following result is a consequence of Theorem 3.1.

**Lemma 3.2.** Suppose that $F$ is an analytic function on $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Re} z > 0\}$, the open right half-plane. Assume that, for given numbers $M_1, M_2, \gamma > 0$, $0 < \alpha \leq 1$,

\begin{equation}
|F(z)| \leq M_1, \quad \forall z \in \mathbb{C}^+ \tag{3.1}
\end{equation}

and

\begin{equation}
|F(t)| \leq M_2 \exp\left(-\frac{\gamma t}{\alpha}\right), \quad \forall t \in \mathbb{R}^+. \tag{3.2}
\end{equation}

Then for every $z \in \mathbb{C}^+$,

\begin{equation}
|F(z)| \leq \max\{M_1, M_2\} \exp\left(-\alpha \gamma \frac{\text{Re} z}{|z|^\alpha+1}\right). \tag{3.3}
\end{equation}

**Proof.** Lemma 3.2 was proved in [19, Lemma 9]. See also [16, Proposition 2.2] and [42, Lemma 6.18]. We give a brief argument of this proof for completeness and convenience for the reader.

Consider the function

\begin{equation}
u_+(\zeta) := F\left(\frac{1}{\zeta}\right) \exp\left(\gamma e^{i(\pi/2-\pi\alpha/2)} \zeta^\alpha\right), \tag{3.4}
\end{equation}

which is also defined on $\mathbb{C}^+$. By (3.1),

\begin{equation}|u_+(\zeta)| \leq M_1 \exp(\gamma |\zeta|^\alpha), \quad \forall \zeta \in \mathbb{C}^+. \tag{3.5}
\end{equation}

Again by (3.1) we have, for any $\epsilon > 0$ and $\zeta = \epsilon + iy =: C \epsilon e^{i\theta}$,

\begin{equation} |u_+(\zeta)| = |F\left(\frac{1}{\zeta}\right) \exp\left(\gamma e^{i(\pi/2-\pi\alpha/2)} \zeta^\alpha\right)| \leq M_1 \exp\left(C_\epsilon \gamma \sin\left(\frac{\pi\alpha}{2} - \alpha \theta\right)\right). \tag{3.6}
\end{equation}

For $y \geq 0$, it follows from $0 < \alpha \leq 1$ that

\begin{equation} |u_+(\zeta)| \leq M_1 \exp\left(C_\epsilon \gamma \sin\left(\frac{\pi\alpha}{2} - \alpha \theta\right)\right) \leq M_1 \exp(\gamma e^\alpha), \tag{3.7}
\end{equation}

which implies that

\begin{equation} \sup_{\text{Re} \zeta = \epsilon, \text{Im} \zeta \geq 0} |u_+(\zeta)| \leq M_1 e^{\gamma e^\alpha}. \tag{3.8}
\end{equation}
By (3.2),

\[ \sup_{\zeta \in [\epsilon, \infty)} |u_+(\zeta)| \leq M_2. \]  

Hence, by Phragmén-Lindelöf theorem 3.1 with angle \( \pi/2 \) and \( b = \alpha \), applied to

\[ S_\epsilon^+ = \{ z \in \mathbb{C} : \text{Re} z > \epsilon \quad \text{and} \quad \text{Im} z > 0 \}, \]

we obtain

\[ \sup_{\text{Re} \zeta \geq \epsilon, \text{Im} \zeta \geq 0} |u_+(\zeta)| \leq \max\{M_2, M_1 e^{\gamma \epsilon} \}, \quad \forall \epsilon > 0. \]

Next we consider the function

\[ u_-(\zeta) := F \left( \frac{1}{\zeta} \right) \exp \left( (\gamma - \pi/2 + \pi \alpha/2) \zeta^\alpha \right). \]

A similar argument shows that

\[ \sup_{\text{Re} \zeta \geq \epsilon, \text{Im} \zeta \leq 0} |u_-(\zeta)| \leq \max\{M_2, M_1 e^{\gamma \epsilon} \}, \quad \forall \epsilon > 0. \]

Letting \( \epsilon \to 0 \) we obtain

\[ \sup_{\text{Re} \zeta > 0, \text{Im} \zeta \geq 0} |u_+(\zeta)| \leq \max\{M_1, M_2\} \]

and

\[ \sup_{\text{Re} \zeta > 0, \text{Im} \zeta \leq 0} |u_-(\zeta)| \leq \max\{M_1, M_2\}. \]

Putting \( \zeta = \frac{1}{z} \), we obtain for all \( \text{Re} z > 0 \)

\[ |F(z)| \leq \max\{M_1, M_2\} \exp \left( -\alpha \sin (\pi/2 + |\theta_z|) \frac{\gamma}{|z|^\alpha} \right), \]

where \( \theta_z = \text{arg} z \). From this, (3.3) follows readily. \( \square \)

**Lemma 3.3.** Suppose that \( L \) satisfies the \( m \)-th order Davies-Gaffney estimates (DG\(_m\)) with \( m \geq 2 \). There exist two positive constants \( C \) and \( c \) such that for every \( j = 2, 3, \ldots \)

\[ \|P_{B^j} e^{(i-1)R^{-1}} L P_{U_j(B)}\|_{2 \to 2} \leq C \exp \left( -c \left( \frac{\sqrt{R^2/j}}{\sqrt{1 + \tau^2}} \right)^m \right) \]

for all balls \( B \subseteq X \).

**Proof.** For any open sets \( U \) and \( V \), and \( \text{Re} z > 0 \), we define a function

\[ F(z) := \langle e^{-zL} f_1, f_2 \rangle, \]

where \( \text{supp} f_1 \subset U \) and \( \text{supp} f_2 \subset V \). Then \( F(z) \) is an analytic function on the complex half plane \( \text{Re} z > 0 \). It is seen that

\[ |F(z)| \leq \|e^{-zL} f_1\|_2 \|f_2\|_2 \leq \|e^{-zL}\|_{L^1 \to L^\infty} \|f_1\|_2 \|f_2\|_2 \leq \|f_1\|_2 \|f_2\|_2, \quad \forall z \in \mathbb{C}_+, \]

and it follows from the \( m \)-th order Davies-Gaffney estimates (DG\(_m\)) and [6, Theorem 1.2] that

\[ |F(t)| \leq C \exp \left( -c \frac{d(U, V)^m}{t^{\frac{m}{2}}} \right) \|f_1\|_2 \|f_2\|_2, \quad \forall t \in \mathbb{R}_+. \]
Let $M_1 = \|f_1\|_2 \|f_2\|_2$, $M_2 = C\|f_1\|_2 \|f_2\|_2$, $\gamma = cd(U, V)^{m/(m-1)}$ and $\alpha = 1/(m - 1)$. We apply Lemma 3.2 to get
\[
|e^{-xL}f_1, f_2| \leq C \exp \left( -c \text{Re} \left( \frac{d(U, V)^m}{|z|^m + 1} \right) \|f_1\|_2 \|f_2\|_2. \right)
\]
From it, we have that
\[
\|P_B e^{(i\tau-1)R^{-1}L} P_{U,(B)}\|_{2 \to 2} \leq C \exp \left( -c \left( \frac{\sqrt{R}^2 r_B}{\sqrt{1 + \tau^2}} \right)^{\frac{m}{1}} \right).
\]
This ends the proof of Lemma 3.3. □

Next we define a Besov type norm of $F$ by
\[
\|F\|_{B^s} := \int_{-\infty}^{\infty} |\hat{F}(\tau)|(1 + |\tau|)^s d\tau,
\]
where $\hat{f}$ denotes the Fourier transform of $f$. Since for every functions $F$ and $G$, it can be checked that
\[
\|FG\|_{B^s} = \int_{-\infty}^{\infty} |(\hat{F}\hat{G})(\tau)|(1 + |\tau|)^s d\tau
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\hat{F}(\tau - \eta)\hat{G}(\eta))|(1 + |\tau - \eta|)^s(1 + |\eta|)^s d\eta d\tau
\]
\[
\text{and so by the Fubini theorem,}
\|FG\|_{B^s} \leq \|F\|_{B^s} \|G\|_{B^s}.
\]
Finally, we can show the following result.

**Proposition 3.4.** Suppose that $L$ satisfies the Gaussian upper bounds $(DG_m)$ with $m \geq 2$. Then for every $s \geq 0$, there exists a constant $C > 0$ such that for every $j = 2, 3, \ldots$
\[
(3.8) \quad \|P_B F(L) P_{U,(B)}\|_{2 \to 2} \leq C \left( \sqrt{R}^2 r_B^{-3}|F(R)| \right) \|F\|_{B^s},
\]
for all balls $B \subseteq X$, and all Borel functions $F$ such that $\text{supp } F \subseteq [-R, R]$.

**Proof.** Let $G(\lambda) = F(R)e^{\lambda}$. In virtue of the Fourier inversion formula
\[
F(L) = G(L/R)e^{-L/R} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(\tau-1)R^{-1}L} \hat{G}(\tau) d\tau
\]
we have that
\[
\|P_B F(L) P_{U,(B)}\|_{2 \to 2} \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{G}(\tau)| \|P_B e^{(i\tau-1)R^{-1}L} P_{U,(B)}\|_{2 \to 2} d\tau.
\]
Then it follows from Lemma 3.3 for every $s \geq 0$,
\[
\|P_B e^{(i\tau-1)R^{-1}L} P_{U,(B)}\|_{2 \to 2} \leq C \exp \left( -c \left( \frac{\sqrt{R}^2 r_B}{\sqrt{1 + \tau^2}} \right)^{\frac{m}{1}} \right)
\leq C_s \left( \frac{\sqrt{R}^2 r_B}{\sqrt{1 + \tau^2}} \right)^{-s}.
\]
Therefore (compare [22, (4.4)])
\[
\|P_{\beta}F(L)P_{U_{\beta}(B)}\|_{2 \to 2} \leq C(\sqrt[n]{R}2^{|r|})^{-s} \int_{\mathbb{R}} |\hat{G}(\tau)| (1 + |\tau|)^s d\tau \\
\leq C(\sqrt[n]{R}2^{|r|})^{-s} \|G\|_{B^r}.
\]
Note that supp \(F \subset [-R, R]\) and so supp \(F(R) \subset [-1, 1]\). Thus taking a function \(\psi \in C^\infty_c\) such that supp \(\psi \subset [-2, 2]\) and \(\psi(\lambda) = 1\) for \(\lambda \in [-1, 1]\), we have
\[
G(\lambda) = F(\lambda)e^t = F(\lambda)\psi(\lambda)e^t
\]
and so
\[
\|G\|_{B^r} \leq C\|F(R)\|_{B^r} \|\psi(\lambda)e^t\|_{B^r} \leq C\|F(R)\|_{B^r}.
\]
This ends the proof of Proposition 3.4. □

**Remark 3.5.** In [12, Proposition 4.1], Carron, Coulhon and Ouhabaz used some techniques introduced by Davies ([19]) to show that the upper Gaussian estimate (GE\(_m\)) on \(e^{-tL}, t > 0\), extends to a similar estimate on \(e^{-zL}\) where \(z\) belongs to the whole complex right half-plane and all \(x, y \in X\),
\[
|p_z(x, y)| \leq \frac{C}{(V(x, (\frac{|x|}{\cos \theta})^{1/m})V(y, (\frac{|y|}{\cos \theta})^{1/m}))^{1/2}} \exp \left(-c \left(\frac{d(x, y)^m}{|z|}\right)^{\frac{1}{m}} \cos \theta\right) \frac{1}{(\cos \theta)^n}
\]
where \(\theta = \text{Arg } z\). It follows that for every \(j = 2, 3, \ldots\)
\[
(3.9) \quad \|P_{\beta}e^{(ir-1)R^{-1}L}P_{U_{\beta}(B)}\|_{2 \to 2} \leq C2^{jn} \exp \left(-c \left(\frac{\sqrt[n]{R}2^{|r|}}{\sqrt{1 + \tau^2}}\right)^{\frac{n}{m}}\right)
\]
for all balls \(B = B(x_B, r) \subseteq X\). In our Lemma 3.3, we made an important improvement in obtaining the upper bound on the right hand side of (3.9) without the factor \(“2^{jn}”\). This plays an essential role in estimate (3.8) of Proposition 3.4 and in the proof of Theorem 1.1 in Section 4.

### 4. Proof of Theorem 1.1

To prove (1.5), let us show that
\[
(4.1) \quad \|(I + L)^{-n/2}e^{itL}f\|_{H^1_L(X)} \leq C(1 + |t|)^{n/2}\|f\|_{H^1_L(X)}, \quad t \in \mathbb{R}.
\]
The proof of estimate of \(\|(I + L)^{-n/2}e^{itL}f\|_{L^1(X)}\) uses similar ideas, but it is much simpler. In the following, \(\phi\) denotes a non-negative \(C^\infty_c\) function on \(\mathbb{R}\) such that supp \(\phi \subseteq (1/4, 1)\) and let \(\phi_\lambda(\lambda)\) denote the function \(\phi(\lambda\tau)\). Also we let \(\psi \in C^\infty_c\) supported in \(\psi \subset [1/8, 2]\) and \(\psi(\lambda) = 1\) for \(\lambda \in [1/4, 1]\). To prove (4.1), it follows by Lemma 2.4 and a standard argument (see for example, [25, 28, 29, 34]) that it suffices to show that for every \((1, 2, M, \varepsilon)\)-molecule \(a\) associated to a ball \(B\),
\[
(4.2) \quad \left\| \left( \int_0^\infty \left( \int_{d(x, y) < 2^{-1}m} |\phi(\tau L)(1 + L)^{-n/2}e^{itL}a(y)\|_{L^1}^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \right)^{1/2} \right\|_{L^1} \leq C(1 + |t|)^{n/2}, \quad t \in \mathbb{R}
\]
where \(M \in \mathbb{N}\) is large enough so that \(M > n/2\).
Recall that if \( a \) is a \((1,2,M,\varepsilon)\)-molecule associated to a ball \( B = B(x_B, r) \), then there exists a function \( b \) such that \( a = L^M b \) and for every \( k = 0, 1, 2, \ldots, M \) and \( j = 0, 1, 2, \ldots, \), there holds
\[
\|(r^m L)^k b\|_{L^2(U_j(B))} \leq 2^{-j} r^m M V(2^j B)^{-1/2},
\]
where the annuli \( U_j(B) \) were defined in (2.4). Following [29], we write
\[
I = m \left( r^{-m} \int_r^{\sqrt{2} r} s^{-1} d s \right) I
\]
(4.4)
\[
= m r^{-m} \int_r^{\sqrt{2} r} s^{-1} (1 - e^{-s^m L})M d s + \sum_{y=1}^{M} C_{y,M} r^{-m} \int_r^{\sqrt{2} r} s^{-1} e^{-\nu r^m L} d s,
\]
where \( C_{y,M} \) are some constants depending on \( \nu \) and \( M \) only. However, \( \partial_s e^{-\nu r^m L} = -mv s^{-1} L e^{-\nu r^m L} \) and therefore,
\[
mvL \int_r^{\sqrt{2} r} s^{-1} e^{-\nu r^m L} d s = e^{-\nu r^m L} = e^{-\nu r^m L} \sum_{l=0}^{1} e^{-\nu l}.
\]
(4.5)
In the following, we set \( F_\tau(\lambda) := \phi(\tau, \lambda)(1 + \lambda)^{-n/2}e^{it \lambda}, \ t > 0 \). Applying the procedure outline in (4.4)-(4.5) \( M \) times, we have for every \( x \in X \),
\[
F_\tau(L)a(x) = (1 + L)^{-n/2} e^{it L} a(x)
\]
\[
= m \left( r^{-m} \int_r^{\sqrt{2} r} s^{-1} (1 - e^{-s^m L})M d s \right) F_\tau(L)a(x)
\]
\[
+ \sum_{k=1}^{M} (1 - e^{-r^m L})^k \left( r^{-m} \int_r^{\sqrt{2} r} s^{-1} (1 - e^{-s^m L})M d s \right)^{M-k} \times
\]
\[
\sum_{y=1}^{(2M-1)k} C(\nu, k, M)e^{-\nu r^m L} F_\tau(L)(r^{-mk} L^{M-k} b)(x)
\]
\[
= \sum_{k=0}^{(2M-1)M} \int_r^{\sqrt{2} r} s^{-1} (1 - e^{-s^m L})M G_{k,r,M}(L)F_\tau(L)(r^{-mk} L^{M-k} b) d s
\]
\[
+ \sum_{y=1}^{(2M-1)M} C(\nu, k, M)e^{-\nu r^m L} F_\tau(L)(1 - e^{-r^m L})^M (r^{-mk} b)(x)
\]
\[
=: \sum_{k=0}^{M-1} E_k(x) + E_M(x),
\]
where for \( k = 1, 2, \ldots, M - 1 \)
\[
G_{k,r,M}(\lambda) := (1 - e^{-r^m L})^k \left( r^{-m} \int_r^{\sqrt{2} r} s^{-1} (1 - e^{-s^m L})M d s \right)^{M-k} \sum_{y=1}^{(2M-1)k} C(\nu, k, M)e^{-\nu r^m L}
\]
and for $k = 0$

$$G_{0,r,M}(\lambda) := m^M \left( r^{-m} \int_{r}^{\infty} s^{m-1} (1 - e^{-s^m \lambda})^M ds \right)^{M-1}.$$ 

We will establish an adequate bound on each $E_k$, $k = 0, 1, \cdots, M$, by considering two cases $k = 0, 1, \cdots, M - 1$ and $k = M$.

Next define

$$F_{r,s}(\lambda) := \phi_r(\lambda)(1 - e^{-s^m \lambda})^M G_{k,r,M}(\lambda) F(\lambda).$$

**Case 1.** $k = 0, 1, \cdots, M - 1$. In this case, we see that

$$\left\| \left( \int_0^{\infty} \int_{d(x,y)<\tau^{1/m}} |E_k(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \right)^{1/2} \right\|_{L^1} \leq C \sup_{\tau \in [r, \sqrt{r}]} \left\| \left( \int_0^{\infty} \int_{d(x,y)<\tau^{1/m}} |F_{r,s}(L)(r^{-mk} L^{M-k} b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \right)^{1/2} \right\|_{L^1} \leq C \sum_{j \geq 0} \sup_{\tau \in [r, \sqrt{r}]} \left\| \left( \int_0^{\infty} \int_{d(x,y)<\tau^{1/m}} |F_{r,s}(L)P_{U_j(B)}(r^{-mk} L^{M-k} b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \right)^{1/2} \right\|_{L^1}$$

$$= C \sum_{j \geq 0} \left\| (E(k, j, s))_{L^1(X)} \right\|,$$

where

$$E(k, j, s) = \left( \int_0^{\infty} \int_{d(x,y)<\tau^{1/m}} |F_{r,s}(L)P_{U_j(B)}(r^{-mk} L^{M-k} b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \right)^{1/2}.$$

Let us estimate the term $\|E(k, j, s)\|_{L^1(X)}$. Note that $\|G_{k,r,M}\|_{L^\infty} + \|F\|_{L^\infty} \leq C$. We apply estimate (4.3), the $L^2$-boundedness of the area square function and the doubling condition (2.2),

$$\left( \frac{\mu((1 + t)^{2j} B)}{\mu(2^j B)} \right) \leq C(1 + t)^n, \quad j \geq 0,$$

to get

$$\|E(k, j, s)\|^2_{L^1((1+t)^{2j}B)} = \left\| \left( \int_0^{\infty} \int_{d(x,y)<\tau^{1/m}} |F_{r,s}(L)P_{U_j(B)}(r^{-mk} L^{M-k} b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \right)^{1/2} \right\|_{L^1((1+t)^{2j}B)}^2 \leq \mu(4(1 + t)^{2j} B) \leq C(\|G_{k,r,M}\|_{L^\infty} + \|F\|_{L^\infty}) r^{-2mk} \left\| P_{U_j(B)} L^{M-k} b \right\|_{L^\infty}^2 \mu((1 + t)^{2j} B) \leq C2^{-2j(1+t)} \mu(2^j B)^{-1} \mu((1 + t)^{2j} B) \leq C2^{-2}\mu(2^j B)^{-1} \mu((1 + t)^{2j} B)$$

(4.8) $\leq C2^{-2}(1 + t)^n.$
Next we show that for some \( \epsilon' > 0 \),
\[
\|E(k, j, s)\|_{L^1((4(1+t)/2)/B^c)} \leq C2^{-j\epsilon'}(1+t)^{m/2},
\]
and this is the major one. We have a decomposition according to the frequency,
\[
E(k, j, s) = \left( \int_0^\infty \int_{d(x,y)<\tau^{1/m}} |F_{\tau,s}(L)P_{U_{\ell}(B)}(r^{-mk}L^{M-k}b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \right)^{1/2},
\]
\[
= \left( \sum_{\ell \in \mathbb{Z}} \int_0^{2^{-t+1}} \int_{d(x,y)<\tau^{1/m}} |F_{\tau,s}(L)P_{U_{\ell}(B)}(r^{-mk}L^{M-k}b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \right)^{1/2},
\]
\[
\leq \sum_{\ell \in \mathbb{Z}} \int_0^{2^{-t+1}} \int_{d(x,y)<\tau^{1/m}} |F_{\tau,s}(L)P_{U_{\ell}(B)}(r^{-mk}L^{M-k}b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \right)^{1/2}.
\]
If \( \ell > 0 \), let \( v_0^+ \in \mathbb{Z}_+ \) be a positive integer such that
\[
2 \leq 2^{v_0^++j=\ell(m-1)/m}r \leq 4 \quad \text{if} \quad 2^{j-\ell(m-1)/m}r < 1;
\]
\[
(4.10) \quad v_0^+ = 1 \quad \text{if} \quad 2^{j-\ell(m-1)/m}r \geq 1.
\]
If \( \ell \leq 0 \), let \( v_0^- \in \mathbb{Z}_+ \) be a positive integer such that
\[
2 \leq 2^{v_0^-+j+\ell/m}r \leq 4 \quad \text{if} \quad 2^{\ell/m+j}r < 1;
\]
\[
(4.11) \quad v_0^- = 1 \quad \text{if} \quad 2^{\ell/m+j}r \geq 1.
\]
Then
\[
\|E(k, j, s)\|_{L^1((4(1+t)/2)/B^c)} \leq \sum_{\ell \geq 0} \sum_{v_0^+} \|E(k, j, s, \ell)\|_{L^1((U_{v_0^+}(B_{(1+t)/2}))} + \sum_{\ell \geq 0} \|E(k, j, s, \ell)\|_{L^1(B(x,2(1+t)/2(m-1)/m))}
\]
\[
+ \sum_{\ell \leq 0} \sum_{v_0^-} \|E(k, j, s, \ell)\|_{L^1((U_{v_0^-}(B_{(1+t)/2}))} + \sum_{\ell \leq 0} \|E(k, j, s, \ell)\|_{L^1(B(x,2(1+t)/2(m-1)/m))}
\]
\[
=: I^+(k, j, s) + \Pi^+(k, j, s) + \Pi^-(k, j, s) + II^+(k, j, s).
\]
We first estimate terms \( II^+(k, j, s) \) and \( I^+(k, j, s) \). Note that there is no term \( II^+(k, j, s) \) if \( 2^{j-\ell(m-1)/m}r \geq 1 \) and \( \ell > 0 \). When \( 2^{j-\ell(m-1)/m}r \leq 1 \) and \( \ell > 0 \), for the term \( II^+(k, j, s) \), we note that from the doubling condition
\[
V(x, \tau^{1/m}) \sim V(y, \tau^{1/m}), \text{ when } d(x, y) < \tau^{1/m}
\]
and then it follows from estimate (4.3) that
\[
\|E(k, j, s, \ell)\|_{L^1(X)}^2
\]
\[
= \int_X \int_0^{2^{-t+1}} \int_{d(x,y)<\tau^{1/m}} |F_{\tau,s}(L)P_{U_{\ell}(B)}(r^{-mk}L^{M-k}b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \frac{d\mu(x)}{V(y, \tau^{1/m})} \frac{d\tau}{\tau}
\]
\[
= \int_0^{2^{-t+1}} \int_X |F_{\tau,s}(L)P_{U_{\ell}(B)}(r^{-mk}L^{M-k}b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\mu(x)}{V(y, \tau^{1/m})} \frac{d\tau}{\tau}
\]

\[ \int_{2^{-\ell}}^{2^{-\ell+1}} \| F_{\tau,s}(L)P_{U_j(B)}(r^{-mk}L^{M-k}b) \|_2^2 \frac{d\tau}{\tau} \]
\[ \leq \int_{2^{-\ell}}^{2^{-\ell+1}} \| F_{\tau,s} \|_2^2 \| P_{U_j(B)}(r^{-mk}L^{M-k}b) \|_2^2 \frac{d\tau}{\tau} \]
\[ \leq \int_{2^{-\ell}}^{2^{-\ell+1}} C \min \{1, (\tau^{-1/m}r)^{2\ell/}\ell} 2^{-j/2} \nu V(x_B, 2^{j/2})^{-1} \frac{d\tau}{\tau} \]
\[ \leq C \min \{1, (2^{j/2}r)^{2\ell/\ell} 2^{-j/2} \nu V(x_B, 2^{j/2})^{-1} \]

and thus

\[ H^+ (k, j, s) \]
\[ \leq \sum_{\ell > 0} \| E(k, j, s, \ell) \|_{L^2(U_{y+j}(1+tB))} V(x_B, 2(1+t)2^{(\ell-1)/m})^{1/2} \]
\[ \leq \sum_{\ell > 0} C \min \{1, (2^{j/2}r)^M \} 2^{-\ell/\ell} 2^{-j/2} \nu V(x_B, 2^{j/2})^{-1} V(x_B, (1+t)2^{(\ell-1)/m})^{1/2} \]
\[ \leq C 2^{-j/2} \sum_{\ell > 0} \min \{1, (2^{j/2}r)^M \} (2^{j/2}r)^{-1/2} (1+t)^{j/2} \]
\[ \leq C 2^{-j/2} (1+t)^{j/2}. \]

To estimate term \( I^+(k, j, s) \), we first note that it follows from (4.10) that for \( \tau \in [2^{-\ell}, 2^{-\ell+1}] \) and \( \ell > 0 \),
\[ \tau^{1/m} \leq 2^{1/m} 2^{-\ell/m} \leq 1 \leq 2^{(\ell-1)/m} \leq 2^{\nu+1}(1+t)r. \]

So if \( d(x, y) < \tau^{1/m} \) and \( x \in U_{y+j}(1+tB) \), then \( y \in U'_{y+j}(1+tB) \) where

\[ U'_{y+j}(1+tB) := B(x_B, 2^{\nu+1}(1+t)r) \setminus B(x_B, 2^{\nu+1}(1+t)r). \]

Then we have

\[ \| E(k, j, s, \ell) \|_{L^2(U_{y+j}(1+tB))}^2 \]
\[ = \int_{U_{y+j}(1+tB)} \int_{2^{-\ell+1}}^{2^{-\ell}} \int_{d(x, y) < \tau^{1/m}} |F_{\tau,s}(L)P_{U_j(B)}(r^{-mk}L^{M-k}b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \]
\[ = \int_{2^{-\ell}}^{2^{-\ell+1}} \int_{U'_{y+j}(1+tB)} |F_{\tau,s}(L)P_{U_j(B)}(r^{-mk}L^{M-k}b)(y)|^2 \int_{d(x, y) < \tau^{1/m}} \frac{d\mu(x)}{V(y, \tau^{1/m})} \frac{d\tau}{\tau} \]
\[ \leq \int_{2^{-\ell}}^{2^{-\ell+1}} \| F_{\tau,s}(L)P_{U_j(B)}(r^{-mk}L^{M-k}b) \|_{L^2(U'_{y+j}(1+tB))}^2 \frac{d\tau}{\tau} \]
\[ \leq \int_{2^{-\ell}}^{2^{-\ell+1}} \| F_{\tau,s}(L) \|_{L^2(U_j(B))}^2 \| P_{U_j(B)}(r^{-mk}L^{M-k}b) \|_{L^2(U'_{y+j}(1+tB))}^2 \frac{d\tau}{\tau} \]
\[ \leq \int_{2^{-\ell}}^{2^{-\ell+1}} \| F_{\tau,s}(L) \|_{L^2(U_j(B))}^2 \| P_{U_j(B)}(r^{-mk}L^{M-k}b) \|_{L^2(U'_{y+j}(1+tB))} \frac{d\tau}{\tau} \]
\[ \leq \int_{2^{-\ell}}^{2^{-\ell+1}} \| F_{\tau,s}(L) \|_{L^2(U_j(B))}^2 \| P_{U_j(B)}(r^{-mk}L^{M-k}b) \|_{L^2(U'_{y+j}(1+tB))} \frac{d\tau}{\tau} \]
\[ \leq \int_{2^{-\ell}}^{2^{-\ell+1}} \| F_{\tau,s}(L) \|_{L^2(U_j(B))}^2 \| P_{U_j(B)}(r^{-mk}L^{M-k}b) \|_{L^2(U'_{y+j}(1+tB))} \frac{d\tau}{\tau} \]

By Proposition 3.4,

\[ \| P_{U'_{y+j}(1+tB)}F_{\tau,s}(L)P_{U_j(B)} \|_{L^2(U_j(B))} \leq C \left( 2^\nu \tau^{-1/m} (1+t) \right) \| \delta_{\tau^{-1}} F_{\tau,s} \|_{B^s} \]

\[ \| P_{U'_{y+j}(1+tB)}F_{\tau,s}(L)P_{U_j(B)} \|_{L^2(U_j(B))} \leq C \left( 2^\nu \tau^{-1/m} (1+t) \right) \| \delta_{\tau^{-1}} F_{\tau,s} \|_{B^s} \]

\[ \| \delta_{\tau^{-1}} F_{\tau,s} \|_{B^s} \]
for every $\alpha > 0$. To go on, we claim that for every $\tau > 0$
\[
\|\delta_{\tau^{-1}}F_{\tau,s}\|_{B^s} = \int_{-\infty}^{\infty} |\delta_{\tau^{-1}}F_{\tau,s}(\xi)|(1 + |\xi|)^{\alpha} d\xi
\]  
(4.14)  
\[ \leq C \max\{1, (\tau^{n/2-\alpha})(1 + t)^{\alpha}\} \min\{1, (\tau^{-1/m} r)^M\}. \]

Let us show the claim (4.14). Recall that $\psi \in C_c^\infty$ supported in $\psi \subset [1/8, 2]$ and $\psi(\lambda) = 1$ for $\lambda \in [1/4, 1]$. We have that for $\tau > 0$,
\[
\|\delta_{\tau^{-1}}F_{\tau,s}\|_{B^s} = \|\phi(\lambda)(1 - e^{-\xi^2/\tau^{-1}\lambda})^M G_{k,r,M}(\tau^{-1}\lambda) F(\tau^{-1}\lambda)\|_{B^s}
\leq \|\psi(\lambda)(1 - e^{-\xi^2/\tau^{-1}\lambda})^M\|_{B^s} \|\psi(\lambda)G_{k,r,M}(\tau^{-1}\lambda)\|_{B^s} \|\phi(\lambda) F(\tau^{-1}\lambda)\|_{B^s}
\leq C \|\psi(\lambda)(1 - e^{-\xi^2/\tau^{-1}\lambda})^M\|_{C^{\alpha+2}} \|\psi(\lambda)G_{k,r,M}(\tau^{-1}\lambda)\|_{C^{\alpha+2}} \|\phi(\lambda) F(\tau^{-1}\lambda)\|_{B^s}.\]

Note that for every $s \in [r, \sqrt{2}r],$
\[
\|\psi(\lambda)(1 - e^{-\xi^2/\tau^{-1}\lambda})^M\|_{C^{\alpha+2}} \leq C \min\{1, (\tau^{-1/m} r)^M\}
\]
and
\[
\|\psi(\lambda)G_{k,r,M}(\tau^{-1}\lambda)\|_{C^{\alpha+2}} \leq C
\]
with $C$ independent of $k, \tau$ and $r$. Let us estimate $\|\phi(\lambda) F(\tau^{-1}\lambda)\|_{B^s}$. It follows from the Fourier transform $F(\phi F(\tau^{-1}\cdot))$ of $\phi F(\tau^{-1}\cdot)$ that
\[
F(\phi F(\tau^{-1}\cdot))(\xi) = \int_{\mathbb{R}} \phi(\lambda) \frac{e^{i(\tau^{-1}\cdot - \xi)\lambda}}{(1 + \tau^{-1}\lambda)^{n/2}} d\lambda.
\]
Integration by parts gives for every $N \in \mathbb{N},$
\[
|F(\phi F(\tau^{-1}\cdot))(\xi)| \leq C_N \min\{1, (\tau^{n/2})(1 + |\tau^{-1} t - \xi|)^{-N},
\]
which yields
\[
|\phi(\lambda) F(\tau^{-1}\lambda)|_{B^s} \leq C \min\{1, (\tau^{n/2})\} \int_{\mathbb{R}} (1 + |\tau^{-1} t - \xi|)^{-N} (1 + |\xi|)^{\alpha} d\xi
\leq C \max\{1, (\tau^{n/2-\alpha})(1 + t)^{\alpha}\}.
\]
Hence, for every $\tau > 0$,
\[
\|\delta_{\tau^{-1}}F_{\tau,s}\|_{B^s} \leq C \max\{1, (\tau^{n/2-\alpha})(1 + t)^{\alpha}\} \min\{1, (\tau^{-1/m} r)^M\}.
\]
This proves our claim (4.14).

Next letting $\alpha$ be a fixed number such that $0 < \alpha - n/2 < \varepsilon$, we apply (4.3) and the doubling condition (2.2) and (4.12), (4.13) and (4.14) to get
\[
I^+(k, j, s) \leq \sum_{L>0} \sum_{0 \leq \ell \leq \ell_0} \|E(k, j, s, \ell)\|_{L^2(U_{\ell+1}(1+t)(B_j))} V(x_B, 2^{\ell+j}(1 + t) r)^{1/2}
\]
\[
\leq \sum_{L>0} \sum_{0 \leq \ell \leq \ell_0} \left( \int_{2^{\ell+1}}^{2^{\ell+2}} \|F_{\tau,s}(L)\|^2_{L^2(U_{\ell+1}(B_j))} d\tau \right)^{1/2} V(x_B, 2^{\ell+j}(1 + t) r)^{1/2}
\]
\[ \sum_{\ell>0} \sum_{\nu \geq 0} \sum_{\ell_0} \sum_{\nu_0} (2^{\nu} 2^{\ell m} (1 + t)^{r})^{-\alpha} 2^{(\alpha-n/2)\ell} (1 + t)^{\alpha} \min\{1, (2^{\ell/m} r)^{M} \} 2^{-j} e \left( \frac{V(x_B, 2^{\nu+j+1} (1 + t)r)}{V(x_B, 2^{r})} \right)^{1/2} \]
\[ \leq C 2^{-j} e (1 + t)^{n/2} \sum_{\ell>0} \sum_{\nu \geq 0} (2^{\nu} 2^{\ell m} (1 + t)^{r})^{-\alpha} 2^{(\alpha-n/2)\ell} \min\{1, (2^{\ell/m} r)^{M} \} \]
\[ \leq C 2^{-j} e (1 + t)^{n/2} \sum_{\ell>0} (2^{\nu} 2^{\ell m} (1 + t)^{r})^{\alpha-n/2} (2^{\ell/m} r)^{-\alpha} 2^{(\alpha-n/2)\ell} \min\{1, (2^{\ell/m} r)^{M} \} \]
\[ \leq C 2^{-j} (\epsilon - \alpha + n/2) (1 + t)^{n/2} \sum_{\ell>0} (2^{\ell/m} r)^{-n/2} \min\{1, (2^{\ell/m} r)^{M} \} \]
\[ \leq C 2^{-j} (\epsilon - \alpha + n/2) (1 + t)^{n/2} \]

with \( \epsilon' = \epsilon - \alpha + n/2 \).

For terms \( I^{-}(k, j, s) \) and \( II^{-}(k, j, s) \), the estimates are similar to terms \( I^{+}(k, j, s) \) and \( II^{+}(k, j, s) \) and simpler. Note that there is no term \( II^{-}(k, j, s) \) if \( 2^{j+\ell/m} r \geq 1 \) and \( \ell \leq 0 \). So when \( 2^{j+\ell/m} r \leq 1 \) and \( \ell \leq 0 \), then for the term \( II^{-}(k, j, s) \), we note that it follows from estimate (4.3) that

\[ ||E(k, j, s, \tau)||_{L^2(X)}^2 \]
\[ = \int_X \int_0^{2^{-\ell+1}} \int_{d(x,y)<\tau^{1/m}} |F_{\tau s}(L)P_{U_s(B)}(r^{-mk} L^{M-k} b)(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \]
\[ \leq \int_0^{2^{-\ell+1}} \frac{||F_{\tau s}(L)P_{U_s(B)}(r^{-mk} L^{M-k} b)||^2_{L^2}}{\tau^2} \frac{d\tau}{\tau} \]
\[ \leq \int_0^{2^{-\ell+1}} \frac{||F_{\tau s}||^2_{L^2} ||P_{U_s(B)}(r^{-mk} L^{M-k} b)||^2_{L^2}}{\tau^2} \frac{d\tau}{\tau} \]
\[ \leq \int_0^{2^{-\ell+1}} C \min\{1, (\tau^{-1/m})^{2M} \} 2^{-j/2} e V(x_B, 2^r)^{-1} \frac{d\tau}{\tau} \]
\[ \leq C \min\{1, (2^{\ell/m} r)^{2M} \} 2^{-j/2} e V(x_B, 2^r)^{-1} \]

and thus

\[ II^{-}(k, j, s) \]
\[ \leq \sum_{\ell \leq 0} ||E(k, j, s, \tau)||_{L^2(B(x_B, (1 + t)2^{-\ell/m})))V(x_B, (1 + t)2^{-\ell/m})}^{1/2} \]
\[ \leq \sum_{\ell \leq 0} C \min\{1, (2^{\ell/m} r)^{M} \} 2^{-j} e V(x_B, 2^r)^{-1/2} V(x_B, (1 + t)2^{-\ell/m})^{1/2} \]
\[ \leq C 2^{-j} \sum_{\ell \leq 0} \min\{1, (2^{\ell/m} r)^{M} \} (2^{\ell/m} r)^{-n/2} (1 + t)^{n/2} \]
\[ \leq C 2^{-j} (1 + t)^{n/2}. \]

To estimate term \( I^{-}(k, j, s) \), we first note that it follows from (4.11) that for \( \tau \in [2^{-\ell}, 2^{-\ell+1}] \)

\[ \tau^{1/m} \leq 2^{1/m} 2^{-\ell/m} \leq 2^{\nu+j-1}(1 + t)r. \]
Then, letting $E$ be a fixed number such that $0 < \alpha - n/2 < \varepsilon$, by (4.14) for $\ell \leq 0$ and similarly as in (4.15),

$$
\begin{align*}
I^-(k, j, s) & \leq \sum_{t \leq 0} \sum_{\nu \geq 0} \|E(k, j, s, t)\|_{L^2(U_{r,t}, (1+t)B)} V(x_B, 2^{\nu+j}(1+t) r)^{1/2} \\
& \leq \sum_{t \leq 0} \sum_{\nu \geq 0} \left( \int_{2^{-t+1}}^{2^{-t}} \|F_{r,t}(L)\|_{L^2(U_B)}^{2} \|P_{U_B}(r^{-m_k}L^{M-k} b)\|_{L^2} \|L^{2/\tau} \right)^{1/2} V(x_B, 2^{\nu+j}(1+t) r)^{1/2} \\
& \leq C 2^{-j\varepsilon}(1+t)^{n/2} \sum_{t \leq 0} \sum_{\nu \geq 0} 2^{-(\alpha+n/2)\nu} (2^{\nu+j} r)^{-\alpha} \min\{1, (2^{\nu+j} r)^{M}\} \\
& \leq C 2^{-j\varepsilon}(1+t)^{n/2} \sum_{t \leq 0} (2^{\nu+j} r)^{-\alpha} \min\{1, (2^{\nu+j} r)^{M}\} \\
& \leq C 2^{-j\varepsilon}(1+t)^{n/2}.
\end{align*}
$$

Combining two estimates of $I^+(k, j, s, j, s), I^+(k, j, s), I^-(k, j, s)$ and $I^-(k, j, s)$ we obtain (4.9). This, in combination with (4.8) and (4.7), shows that

$$
\begin{align*}
\sum_{k=0}^{M-1} \left\| \int_{0}^{\infty} \int_{d(x,y) < 1/m} |E_k(y)|^{2} \frac{d\mu(y)}{V(x, 1/m)} \frac{d\tau}{\tau} \right\|_{L^1}^{1/2} & \leq C \sum_{k=0}^{M-1} 2^{-j\varepsilon'}(1+t)^{n/2} \leq C(1+t)^{n/2}.
\end{align*}
$$

**Case 2.** $k = M$.

In this case, we write

$$
\begin{align*}
\left\| \left( \int_{0}^{\infty} \int_{d(x,y) < 1/m} |E_M(y)|^{2} \frac{d\mu(y)}{V(x, 1/m)} \frac{d\tau}{\tau} \right)^{1/2} \right\|_{L^1}^{1/2} & \leq C \sum_{j=1}^{2M-1} \left\| \left( \int_{0}^{\infty} \int_{d(x,y) < 1/m} |(1-e^{-r_m L})^M e^{-j\rho L} F_{r,t}(L)(r^{-m_k} b)(y)|^{2} \frac{d\mu(y)}{V(x, 1/m)} \frac{d\tau}{\tau} \right)^{1/2} \right\|_{L^1}.
\end{align*}
$$

Similar to the proof of $E_k$ as in **Case 1**, we have that

$$
\begin{align*}
\left\| \left( \int_{0}^{\infty} \int_{d(x,y) < 1/m} |E_M(y)|^{2} \frac{d\mu(y)}{V(x, 1/m)} \frac{d\tau}{\tau} \right)^{1/2} \right\|_{L^1} & \leq C \sum_{j=0}^{2} 2^{-j\varepsilon'}(1+t)^{n/2} \leq C(1+t)^{n/2}.
\end{align*}
$$
Hence, we have proved estimate (4.2), and then concluded the proof of (4.1).

Now we turn to prove (1.6). To do this, we need to state a complex interpolation result. Fix a pair of Banach spaces $E_0, E_1$ continuously embedded in some Banach space $V$ such that $E_0 \cap E_1$ contains a dense subspace $D$ of both $E_0, E_1$ under the corresponding norms. Let $S = \{ z : 0 < \text{Re} z < 1 \}$ and $\tilde{S} = \{ z : 0 \leq \text{Re} z \leq 1 \}$. Following [11], we define $\mathcal{F}(E_0, E_1)$ to be the set of all functions $F$ on $\tilde{S}$ with values in $E_0 + E_1$, analytic in $S$ and such that $F(it) \in E_0$ is $E_0$-continuous and tends to 0 as $|t| \to \infty$ and $F(1 + it) \in E_1$ is $E_1$-continuous and tends to 0 as $|t| \to \infty$. $\mathcal{F}(E_0, E_1)$ becomes a Banach space under the norm

$$\|F\|_{\mathcal{F}} = \sup_{z \in \mathbb{R}} (\|F(it)\|_{E_0}, \|F(1 + it)\|_{E_1}).$$

Given a real number $\theta, 0 < \theta < 1$, Calderón constructed a subspace $[E_0, E_1]_\theta$ of $E_0 + E_1$ as follows:

$$[E_0, E_1]_\theta = \{ F(\theta) : F \in \mathcal{F}(E_0, E_1) \}.$$

By introducing the norm

$$\|F\|_{[E_0, E_1]_\theta} = \inf \{ \|F\|_{\mathcal{F}} : F \in \mathcal{F}(E_0, E_1), F(\theta) = f \},$$

$[E_0, E_1]_\theta$ becomes a Banach space continuously embedded in $E_0 + E_1$. We next define analytic families of operators. Let $\{ T_z \}$ be a family of linear operators indexed by $z \in S$ so that for each $z$, $T_z$ is a mapping of functions in $D$ to measurable functions on $E$. Following [43], $\{ T_z \}$ is called an analytic family if for any $g \in D$ and for almost all $y \in E$, $(T_z(g))(y)$ is analytic in $S$ and continuous on $\tilde{S}$. The analytic family $\{ T_z \}$ is of admissible growth if for all $y \in D$ there exists a constant $C_\theta$ and a constant $a \in \pi$ such that

$$\sup_{z \in \tilde{S}} \log |(T_z g)(y)| \leq C_\theta e^{a|\text{Im} z|}$$

for almost all $y \in E$. Then we have the following result, for its proof, we refer it to [43], [27, Theorem 3].

**Lemma 4.1.** Let $E_0, E_1, D$ as before, $0 < p_0, p_1 \leq \infty$, and let $\{ T_z \}$ be an analytic family of linear operators which is of admissible growth. If for all $f \in D \| T_z f \|_{L^p} \leq c_j(z) \| f \|_{E_j}$ when $\text{Re} z = \theta, j = 0, 1$ for some constants $c_j(z)$ that satisfy $\log c_j(z) \leq A e^{a|\text{Im} z|}, A > 0, 0 \leq a < \pi$, then for all $z \in S$ there exist $A_\zeta > 0$ such that for $f \in D$,

$$\|T_z f\|_{L^p} \leq A_\zeta \| f \|_{[E_0, E_1]_\theta}, \quad \text{when} \quad \text{Re} z = \theta$$

where

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$

We now apply Lemma 4.1 to prove (1.6). Let $E_0 = H^1_L(X), E_1 = H^2(X)$ and $D = H^1_L(X) \cap H^2(X)$, and $D$ is dense in both $H^1_L(X)$. Since

$$[H^1_L(X), H^2(X)]_\theta = H^p_L(X), \quad \frac{1}{p} = 1 - \frac{\theta}{2},$$

Consider the analytic family of operators

$$T_z := e^{z(1 + |t|)} e^{-z/2} (1 + L)^{-z/2} e^{iLt}, \quad 0 \leq \text{Re} z \leq 1.$$
Note that $T_z$ is a holomorphic function of $z$ in the sense that
\[ z \mapsto \int_X T_z f(x) g(x) d\mu(x) \]
for $f, g \in L^2(X)$. If $y \in \mathbb{R}$, then
\[ T_y(L) = e^{-y^2} (1 + |t|)^{-iy/2} (1 + L)^{-iy/2} e^{itL}. \]

Since $H^2_L(X) \subseteq L^2(X)$, we have
\[ \|T_y(f)\|_{L^2(X)} = \|e^{-y^2} (1 + |t|)^{-iy/2} (1 + L)^{-iy/2} e^{itL} f\|_2 \]
\[ \leq C\|1 + \lambda\|^{-iy/2} e^{it}\|\|f\|_2 \]
\[ \leq C\|f\|_{L^2(X)} \leq C\|f\|_{H^2_L(X)} \]
with $C$ independent of $t$ and $y$. On the other hand, it follows from Proposition 2.7 that
\[ \|(1 + L)^{-iy/2}\|_{H^2_L(X) \to H^2_L(X)} \leq C(1 + |y|)^{n/2+1}. \]

This, together with (1.5), shows that $(1 + L)^{-iy/2} e^{itL}$ is bounded from $H^2_L(X)$ to $L^1(X)$ and
\[ \|T_{1+i\theta}(f)\|_{L^1(X)} \leq C e^{1-y^2} (1 + |t|)^{-n/2} \|(1 + L)^{-iy/2} e^{itL}\|_{H^2_L(X) \to L^1(X)} \|(1 + L)^{-iy/2} f\|_{H^2_L(X)} \leq C\|f\|_{H^2_L(X)} \]
with $C$ independent of $t$ and $y$. Then by Lemma 4.1, we have that for $\theta = 1 - 2/p$ and $s = n(1/2 - 1/p)$
\[ \|(1 + L)^{-s} e^{itL} f\|_{L^p(X)} = \|e^{-\theta^2} (1 + |t|)^{\theta n/2} T_{1+i\theta} f\|_{L^p(X)} \leq C(1 + |t|)^{\theta s}\|f\|_{H^2_L(X)} \]
as desired for $1 < p \leq 2$. This proves (1.6).

By duality, estimate (1.7) holds for $2 < p < \infty$. This completes the proof of Theorem 1.1. \qed

5. Application: Schrödinger groups for the Kohn Laplacian

In this section, we give an application of Theorem 1.1 to the Kohn Laplacian $\Box_b$ on polynomial model domains treated by Nagel-Stein [41].

Let $M$ be the boundary of an unbounded polynomial domain $\Omega := \{(z, w) \in \mathbb{C}^2 : \text{Im}(w) > P(z)\}$, where $P$ is a real, subharmonic, nonharmonic polynomial of degree $m$ (see [41]). Let $\bar{\partial}_b$ be the tangential Cauchy-Riemann operator on $M$ which maps functions to $(0, 1)$-forms, and let $(\bar{\partial}_b)^* \Box_b$ be the formal adjoint which maps $(0, 1)$-forms to functions. As in [41], choose real vector fields $X_1, X_2$ on $M$ so that we can identify $\bar{\partial}_b f$ with
\[ (X_1 + iX_2) f \]
by identifying functions and $(0, 1)$ forms on $M$. Then we define Kohn Laplacian $\Box_b$ acting on functions by $\Box_b := (\bar{\partial}_b)^* \bar{\partial}_b$. Since $\Box_b$ is a self-adjoint operator, it admits a spectral decomposition $E(\lambda)$; in particular, $E(0) = \pi$, where $\pi$ is the Szegö projection from $L^2(M)$ to the null space of $\Box_b$. It is known (see [48]) that the heat kernel $K_{e^{-\Omega_b}}(x, y)$ of $e^{-\Omega_b}$ is in terms of Carnot-Carathéodory distance $d$ on $M$, and there exist two positive constants $C$ and $c$ such that
\[ |K_{e^{-\Omega_b}}(x, y)| \leq \frac{C}{V(x, d(x, y))} \exp \left(-cd(x, y)^2/s\right), \]
where $V(x, d(x, y))$ is the volume of ball $\{z \in B(x, d(x, y)) : d(z, x) \leq V(x, d(x, y))\}$. \[ \Box_b f \]
\[ \frac{C}{V(x, d(x, y))} \exp \left(-cd(x, y)^2/s\right), \]
where $V(x, \delta)$ denotes the volume of ball of radius $\delta$ in the $d$ metric, centered at $x$. Note that there exist $C$ and $Q$ such that

$$V(x, \lambda \delta) \leq C \lambda^Q V(x, \delta), \quad \lambda \geq 1,$$

and so

$$V(x, r) \leq C \left(1 + \frac{d(x, y)}{r}\right)^Q V(y, r),$$

uniformly for all $x, y \in M$ and $r > 0$. It is worth pointing out that the heat kernels $K_{e^{-t\Box_b}}(x, y)$ of the Kohn Laplacian $\Box_b$ on the boundary $M$ do not satisfy standard Gaussian upper bounds ($GE_m$) with $m = 2$. However, the Kohn Laplacian $\Box_b$ satisfies the finite speed property of propagation for the corresponding wave equation (see Theorem 2.3, [48]). Equivalently, according to [45, Theorem 2], the Kohn Laplacian $\Box_b$ satisfies $m$-th order Davies-Gaffney estimates ($DG_m$) with $m = 2$. Thus we would like to apply Theorem 1.1 and relation between $H^\beta_{e^{t\Box_b}}(M)$ and $L^p(M)$ to get the endpoint $L^p$-boundedness of the Schrödinger group $e^{it\Box_b}$. Before that we state the following proposition for $\Box_b$, which is used to derive estimates on $L^p(M)$.

**Proposition 5.1.** We have the following results:

1. $F(\Box_b)f = F(\Box_b)(f - \pi f) + F(0)\pi f$, where $\pi$ is the Szegö projection operator and is bounded on $L^p(M)$.
2. Suppose $m \in \mathcal{S}([0, \infty))$, $m(0) = 0$ and $r > 0$, where $\mathcal{S}([0, \infty))$ is the Schwartz class on $[0, \infty)$. Then the kernel of $m(r^2 \Box_b)$ satisfies

$$|K_{m(r^2 \Box_b)}(x, y)| \leq C_N \left(1 + \frac{\rho(x, y)}{r}\right)^{-N} \frac{1}{V(x, \rho(x, y) + r)}.$$

3. Let $\psi \in C^\infty_c(1, 4)$ and $\sum_j \psi(2^j \lambda) = 1$ for $\lambda > 0$. Then there exists $\tilde{\psi} \in C^\infty_c(1, 4)$ such that for $\pi f = 0$

$$f(x) = \sum_j \psi(2^j \Box_b)\tilde{\psi}(2^j \Box_b)f(x).$$

4. Define the discrete square function by

$$G_{d, \Box_b}(f)(x) = \left(\sum_j |\psi(2^j \Box_b)f(x)|^2\right)^{1/2}.$$

Then for $\pi f = 0$,

$$\|G_{d, \Box_b}(f)\|_{L^p} \sim \|f\|_{L^p}.$$  

**Proof.** The properties (1)-(4) are from Proposition 7.4 and estimates (16)–(18) in p. 880 of [48].

Recall that $S_{\Box_b}f$ is the area function of $\Box_b$ given in (2.3). We have the following result.

**Proposition 5.2.** If $\pi f = 0$, then for $1 < p < \infty$

$$\|S_{\Box_b}f\|_{L^p(M)} \leq C\|f\|_{L^p(M)}.$$
Proof. First, if \( m, \phi \in S((0, \infty)) \), \(|m(\lambda)| + |\phi(\lambda)| \leq C \lambda^\epsilon \) around \( \lambda = 0 \) for some \( \epsilon > 0 \), then applying (2) of Proposition 5.1, the kernel of \( m(r^2 \Box_b) \phi(s^2 \Box_b) \) satisfies

\[
|K_{m(r^2 \Box_b) \phi(s^2 \Box_b)}(x, y)| \leq C_N \left( \min \{ s, r \} \right)^\epsilon \left( 1 + \frac{\rho(x, y)}{\max\{s, r\}} \right)^{-N} V(x, \max\{s, r\})^{-N}
\]

for all \( N > 0 \).

Let \( \psi \) and \( \tilde{\psi} \) be functions in (3) of Proposition 5.1 and thus we have that for \( \pi f = 0 \)

\[
f(x) = \sum_{j=-\infty}^{\infty} \psi(2^j \Box_b) \tilde{\psi}(2^j \Box_b) f(x).
\]

Define the square function

\[ G_{c,\Box_b}(f)(x) = \left( \int_0^\infty |\psi(t^2 \Box_b) f(x)|^2 \frac{dt}{t} \right)^{1/2} . \]

Then for \( \pi f = 0 \), we apply (5.5) and (5.4) to obtain

\[
G_{c,\Box_b}(f)(x) \leq C \left( \int_0^\infty \sum_j |\psi(t^2 \Box_b) \psi(2^j \Box_b) \tilde{\psi}(2^j \Box_b) f(x)|^2 \frac{dt}{t} \right)^{1/2}
\]

\[
\leq C \left( \sum_j \int_{2^{j-1}}^{2^{j+1}} |\psi(t^2 \Box_b) \tilde{\psi}(2^j \Box_b) f(x)|^2 \frac{dt}{t} \right)^{1/2}
\]

\[
\leq C \left( \sum_j \int_{2^{j-1}}^{2^{j+1}} \left| \int_M K_{\psi(t^2 \Box_b) \tilde{\psi}(2^j \Box_b)}(x, y) \psi(2^j \Box_b) f(y) dy \right|^2 \frac{dt}{t} \right)^{1/2}
\]

\[
\leq C \left( \sum_j \left| M \left( f(2^j \Box_b) \right) \right|^2 \right)^{1/2}
\]

where \( M \) denotes the Hardy-Littlewood maximal function, that is

\[ M f(x) = \sup_{y \in B} \frac{1}{V(B)} \int_B |f(y)| dy. \]

Hence,

\[ \|G_{c,\Box_b}(f)\|_{L^p} \leq C \left( \sum_j \left| M \left( f(2^j \Box_b) \right) \right|^2 \right)^{1/2} \leq C \|G_{d,\Box_b}(f)\|_{L^p} \leq C \|f\|_{L^p} . \]

Next directly computation shows that for all \( \alpha > 0 \),

\[ S_{\Box_b} f(x) \leq C_\alpha \left( \int_0^\infty \left| M_{\psi(2^j \Box_b)}(f)(x, t) \right|^2 \frac{dt}{t} \right)^{1/2} \]

where \( M_{\psi(2^j \Box_b)}(f) \) is the Peetre type maximal function (see for example [10, 31]) given by

\[ M_{\psi(2^j \Box_b)}(f)(x, t) := \sup_{y \in M} \frac{|t^2 \Box_b e^{-t^2 \Box_b} f(y)|}{(1 + t^{-1} d(x, y))^\alpha} . \]
Then applying (5.4) gives
\[
|t^2 \Box_b e^{-t^2 \Box_b} f(x)| \leq \sum_j \left| \psi(2^j r^2 \Box_b) \overline{\psi}(2^j r^2 \Box_b) t^2 \Box_b e^{-t^2 \Box_b} f(y) \right|
\]
\[
\leq \sum_j \int_M \left| K_{\psi}(2^j r^2 \Box_b)^2 e^{-t^2 \Box_b} (y, z) \psi(2^j r^2 \Box_b) f(z) \right| dz
\]
\[
\leq \sum_{j \geq 0} 2^{-j} \int_M \left( 1 + \frac{\rho(y, z)}{2^{j/2} t} \right)^{-Q} \frac{1}{V(z, 2^{j/2} t)} \left| \psi(2^j r^2 \Box_b) f(z) \right| dz
\]
\[
+ \sum_{j < 0} 2^j \int_M \left( 1 + \frac{\rho(y, z)}{t} \right)^{-Q} \frac{1}{V(z, t)} \left| \psi(2^j r^2 \Box_b) f(z) \right| dz.
\]
Using (5.3), we have
\[
\frac{|t^2 \Box_b e^{-t^2 \Box_b} f(y)|}{(1 + t^{-1}d(x, y))^p} \leq \sum_{j \geq 0} 2^{-j} \int_M \left( 1 + \frac{\rho(x, z)}{2^{j/2} t} \right)^{-Q} \frac{1}{V(x, 2^{j/2} t)} \left| \psi(2^j r^2 \Box_b) f(z) \right| dz
\]
\[
+ \sum_{j < 0} 2^j \int_M \left( 1 + \frac{\rho(x, z)}{t} \right)^{-Q} \frac{1}{V(x, t)} \left| \psi(2^j r^2 \Box_b) f(z) \right| dz
\]
\[
\leq \sum_{j \geq 0} 2^{-j} \| \Re(\psi(2^j r^2 \Box_b) f) \| + \sum_{j < 0} 2^j \| \Re(\psi(2^j r^2 \Box_b) f) \|.
\]
Now
\[
S_{\Box_b} f(x) \leq C_{\alpha} \left( \int_0^\infty \| \Re(\psi(2^j r^2 \Box_b) f) \| t \frac{dt}{t} \right)^{1/2}
\]
\[
\leq C_{\alpha} \sum_{j \geq 0} 2^{-j} \left( \int_0^\infty \| \Re(\psi(2^j r^2 \Box_b) f) \| t \frac{dt}{t} \right)^{1/2}
\]
\[
+ C_{\alpha} \sum_{j < 0} 2^j \left( \int_0^\infty \| \Re(\psi(2^j r^2 \Box_b) f) \| t \frac{dt}{t} \right)^{1/2}
\]
\[
\leq C_{\alpha} \left( \int_0^\infty \| \Re(\psi(2^j r^2 \Box_b) f) \| t \frac{dt}{t} \right)^{1/2}.
\]
Then for $1 < p < \infty$,
\[
\| S_{\Box_b} f \|_{L^p} \leq C \left( \int_0^\infty \| \Re(\psi(2^j r^2 \Box_b) f) \| t \frac{dt}{t} \right)^{1/2} \leq C \| \Re(\psi(2^j r^2 \Box_b) f) \|_{L^p} \leq C \| f \|_{L^p}.
\]
The proof of Proposition 5.2 is complete.

Note that the Kohn Laplacian $\Box_b$ on $M$ satisfies the $m$-th order Davies-Gaffney estimate (DG$_m$) with $m = 2$. Recall that $Q$ is the “dimension” of $M$ in (5.2). We can apply Theorem 1.1 to prove the following result.
Theorem 5.3. There exists a constant $C > 0$ independent of $t$ such that for $1 < p < \infty$,
\begin{equation}
\| (1 + \Box_b)^{-s} e^{i t \Delta_b} f \|_{L^p(M)} \leq C (1 + |t|)^s \| f \|_{L^p(M)}, \quad t \in \mathbb{R}, \quad s \geq \frac{Q}{2} - \frac{1}{p}.
\end{equation}

Proof. Let $g = f - \pi f$. We have that $\pi g = 0$. Note that $F(\Box_b) f = F(\Box_b) g + F(0) \pi f$. For $1 < p \leq 2$, we apply Theorem 1.1 and Proposition 5.2 to obtain
\begin{align*}
\| (1 + \Box_b)^{-s} e^{i t \Delta_b} f \|_{L^p(M)} & \leq \| (1 + \Box_b)^{-s} e^{i t \Delta_b} g \|_{L^p(M)} + \| \pi f \|_{L^p(M)} \\
& \leq C (1 + |t|)^s \| g \|_{L^p_{\infty}(M)} + C \| f \|_{L^p(M)} \\
& = C (1 + |t|)^s \| \mathcal{S}_{\Delta_b} g \|_{L^p(M)} + C \| f \|_{L^p(M)} \\
& \leq C (1 + |t|)^s \| g \|_{L^p(M)} + C \| f \|_{L^p(M)} \\
& = C (1 + |t|)^s \| f - \pi f \|_{L^p(M)} + C \| f \|_{L^p(M)} \\
& \leq C (1 + |t|)^s \| f \|_{L^p(M)}.
\end{align*}

By duality, we have the result (5.6) for $p > 2$. This completes the proof of Theorem 5.3. \hfill \Box

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