Speed of Sound in Hadronic matter using Non-extensive Tsallis Statistics

Arvind Khuntia¹, Pragati Sahoo¹, Prakhar Garg¹,a, Raghunath Sahoo¹,b, and Jean Cleymans²

¹ Discipline of Physics, School of Basic Science, Indian Institute of Technology Indore, Khandwa Road, Simrol, M.P. India-453552
² UCT-CERN Research Centre and Department of Physics, University of Cape Town, Rondebosch 7701, South Africa

Received: date / Revised version: September 27, 2016

Abstract. The speed of sound (cₚ) is studied to understand the hydrodynamical evolution of the matter created in heavy-ion collisions. The quark-gluon plasma (QGP) formed in heavy-ion collisions evolves from an initial QGP to the hadronic phase via a possible mixed phase. Due to the system expansion in a first order phase transition scenario, the speed of sound reduces to zero as the specific heat diverges. We study the speed of sound for systems, which deviate from a thermalized Boltzmann distribution using non-extensive Tsallis statistics. In the present work, we calculate the speed of sound as a function of temperature for different q-values for a hadron resonance gas. We observe a similar mass cut-off behaviour in the non-extensive case for cₚ by including heavier particles, as is observed in the case of a hadron resonance gas following equilibrium statistics. Also, we explicitly show that the temperature where the mass cut-off starts varies with the q-parameter which hints at a relation between the degree of non-equilibrium and the limiting temperature of the system. It is shown that for values of q above approximately 1.13 all criticality disappears in the speed of sound, i.e. the decrease in the value of the speed of sound, observed at lower values of q, disappears completely.

PACS. 25.75.Dw, 25.75.Nq, 24.10.Pa, 51.30.+i

1 Introduction

In high energy hadronic and nuclear collisions, the space-time evolution of the created hot and dense matter is controlled by the initial energy density and the created temperature. Because of the very high initial pressure, the system expansion takes place through the decrease of its temperature and energy density. The change in pressure with energy density is related to a physical observable called the speed of sound inside the system, which can be used to probe the degrees of freedom, as it is related to the equation of state (EoS) of the system. This also helps in looking for any critical behaviour the system could undergo during its space-time evolution. The study of the speed of sound in hadronic matter hence becomes a subject of paramount importance in heavy-ion collisions. There have been several experimental and theoretical studies in this direction [1,2,3,4,5,6,7,8]. The speed of sound has also been discussed in a slightly different context in Ref. [9]. It should be noted here that the speed of sound in a medium can be derived from Euler’s law as applied to a continuous medium without having to use thermodynamics [10].

The temperature dependence of the speed of sound in a medium is well established but the effect of temperature fluctuations is less explored, particularly in the case of high-energy collisions, where the non-extensive parameter q lies in 1 < q < 1.2 [11]. As q = 1 corresponds to an equilibrium system described by Boltzmann-Gibbs statistics, the systems with small values of (q − 1) are assumed to undergo an infinitesimal perturbation and tend to approach equilibrium. In non-extensive statistics, the Tsallis parameter (q) is related to the temperature fluctuations [12] and therefore we estimate the speed of sound using Tsallis statistics in hadronic medium, and study it as a function of q, for systems undergoing small perturbations. Further, it has been observed that the transverse momentum spectra of the secondaries created in high energy p + p(¯p) [9,13,14,15,16,17,18,19], e⁺ + e⁻ collisions [20,21] are better described by the non-extensive Tsallis statistics [22]. In addition, non-extensive statistics with radial flow successfully describes the spectra at intermediate pₚ in heavy-ion collisions [22,23,24,25,26,27]. Tsallis non-extensive statistics in Boltzmann Transport Equation with relaxation time approximation successfully describes the nuclear modification factor of heavy flavors in heavy-ion collisions at RHIC and LHC energies [28]. Tsallis distributions have also been used to study the conserved number susceptibilities in heavy-ion collisions [29]. The present paper ehu-
candidates on the speed of sound in the framework of Tsallis non-extensive statistics taking a hadron resonance gas.

In this paper, first we discuss the thermodynamics of a hadron resonance gas in Section 2, taking a grand canonical partition function and using Boltzmann-Gibbs statistics. In Section 3, we derive the necessary formalism for calculating the speed of sound using Tsallis non-extensive statistics. In addition, we discuss the speed of sound in a hadron resonance gas in Section 2, taking a hadron resonance gas in the framework of non-extensive statistics. In Section 3. We summarize the results and conclude in Section 4.

2 Thermodynamics of a Hadron Resonance Gas

As we make a direct comparison of the thermodynamic quantities with those of an ideal pion gas, obeying Boltzmann statistics, for completeness we derive the necessary tools following Ref. [8]. The grand canonical partition function for an ideal pion gas having three degrees of freedom is given by

$$\ln Z(T, V) = \frac{3VTm_0^3}{2\pi^2} K_2(m_0/T),$$

where \(V\) is the system volume at temperature \(T\) and \(m_0\) is the mass of a pion. The logarithm of the partition function gives

$$\frac{1}{T^2} - 3 = \frac{3m_0^2K_2(m_0/T)}{2\pi^2Ts_0},$$

and

$$s_0(T) = \frac{\epsilon_0(T) + P_0(T)}{T} \equiv \frac{3m_0^2T}{2\pi^2} \left[ 4K_2(m_0/T) + (m_0/T) K_1(m_0/T) \right],$$

where \(K_1\) and \(K_2\) are the well-known modified Bessel functions of second kind.

For an ideal gas with zero chemical potential, the temperature dependent speed of sound, \(c_s(T)\), is given by

$$c_s^2(T) = \frac{\left( \frac{\partial P}{\partial T} \right)_V}{\frac{\partial\epsilon}{\partial T}_V} = \frac{s(T)}{C_V(T)},$$

where

$$s = \left( \frac{\partial P}{\partial T} \right)_V$$

is the entropy density and

$$C_V(T) = \left( \frac{\partial\epsilon}{\partial T} \right)_V$$

is the specific heat at constant volume. The specific heat at constant volume for an ideal pion gas using Eqs. 4 and 8 is given by

$$C^0_V(T) = 3s_0(T) + \frac{3m_0^4}{2\pi^2T}K_2(m_0/T).$$

Using Eqn. 6 we get the speed of sound, \(c_s\) as

$$\frac{1}{c_s^2} - 3 = \frac{3m_0^2K_2(m_0/T)}{2\pi^2Ts_0},$$

and

$$s_0(T) = \frac{\epsilon_0(T) + P_0(T)}{T} \equiv \frac{3m_0^2T}{2\pi^2} \left[ 4K_2(m_0/T) + (m_0/T) K_1(m_0/T) \right],$$

where \(K_1\) and \(K_2\) are the well-known modified Bessel functions of second kind.

3 Speed of Sound in a Physical Hadron Resonance Gas

A hadron resonance gas consists of mesons and baryons obeying Bose-Einstein (BE) and Fermi-Dirac (FD) statistics, respectively. The Tsallis form of the Fermi-Dirac and Bose-Einstein distributions as proposed in Refs. [30,31,32,33,34,35] is

$$f_T(E) \equiv \frac{1}{\exp_q\left( \frac{E-\mu}{T} \right) + 1}.$$
shown in Fig. 1 as a function of temperature scaled with the Hagedorn limiting temperature, $T_H$. It is observed that for $q = 1.15$, there is no distinction between both the distributions, i.e. between the Tsallis-Boltzmann and the Tsallis-BE gas, and hence one considers Tsallis-Boltzmann distributions as a good approximation for describing the thermodynamics of a pion gas, as is done in the previous section. However, as expected, there is a clear distinction between equilibrium and Tsallis non-extensive statistics, for which the degree of deviation is higher towards lower temperatures.

For a hadron resonance gas, the energy density, pressure and specific heat at constant volume in non-extensive statistics are given by

$$\epsilon = \frac{g}{2\pi^2} \int_0^\infty dp \; p^2 \sqrt{p^2 + m_0^2}$$

$$\times \left[ 1 + \left( \frac{q-1}{q} \sqrt{\frac{p^2}{T^2} + m_0^2} \right)^{\frac{1}{q-1}} + 1 \right]^{-q} \quad (14)$$

$$P = \frac{g}{2\pi^2} \int_0^\infty dp \; p^4 \frac{1}{3\sqrt{p^2 + m_0^2}}$$

$$\times \left[ 1 + \left( \frac{q-1}{q} \sqrt{\frac{p^2}{T^2} + m_0^2} \right)^{\frac{1}{q-1}} + 1 \right]^{-q} \quad (15)$$

and,

$$C_V = \frac{qg}{2\pi^2 T^2} \int_0^\infty dp \; p^2 (p^2 + m_0^2)$$

Hence the speed of sound can be calculated from these relations using Eqn. 6.

Figure 2 shows the speed of sound for a hadron resonance gas and the effect of taking different mass cut-offs of hadrons for a very small value of $q$ (=1.005). The mass cut-off $M$ is introduced as the highest mass of the resonances contributing to the hadron resonance gas. Ref. [9] has also considered the case of contributions going beyond such a cut-off. It is evident from Fig. 2 that in the limit of $q\rightarrow 1$, a hadron resonance gas in Tsallis framework assuming quantum statistics gives identical results as shown in Ref. [8] for a similar system.

An increase in the value of $q$ above one means a deviation from equilibrium statistics. To study the effect of higher values of $q$ on the speed of sound, we have taken $q = 1.05$ and 1.15 with different mass cut-offs by inclusion of heavy resonances.

Figures 2 and 3 show that, for small values of $q$ and large mass cut-off, the speed of sound at first increases as a function of temperature, reaches a maximum value and then decreases, which is indicative of a phase transition. In a first order phase transition it is expected that the speed of sound drops to zero at the phase transition point. This behavior is no longer present for larger values of $q$ as can be seen from Figure 4 where the speed of sound increases monotonically with temperature. This basically is the behavior for any value of the mass cut-off. A comparison of Figures 2 and 3 with Figure 4 opens the possibility that there exists a special value of $q$ where a phase transition is no longer present. To explore this, we have studied the square of the speed of sound as a function of the scaled temperature for different values of $q$, taking a mass cut-off of 2.5 GeV. This is shown in Fig. 5. It could be inferred...
from the figure that with the progressive increase of the \( q \)-values, when the system goes away from equilibrium, the speed of sound slowly decreases to a minimum value up to \( q = 1.13 \). This behaviour, however, vanishes for higher values of \( q \). This indicates the softening of the equation of state and a possible phase transition. Interestingly, this critical value of \( q \) is close to the value one obtains in the analysis of \( pT \) spectra in \( p + p \) collisions at high energies [30].

In order to further explore the nature of the phase transition, we have studied the energy density \( (\epsilon) \) scaled with \( T^4 \), which is a measure of the degrees of freedom of the system, as a function of the temperature, taking the same mass cut-off for the physical resonance gas. This is shown in Fig. 4. With the increase of the \( q \)-values, \( \epsilon/T^4 \) increases faster, showing a consistent sharp increase in its value for some temperatures, which may indicate a possible transition to higher degrees of freedom. Then after reaching a maximum value for all \( q \)-values, it starts decreasing for higher temperature and goes to a saturation region. This behavior can be understood as follows. As the temperature increases more and more heavy resonances start contributing to the energy density and hence one notices a strong increase in the energy density. At infinite temperature, the system starts behaving like a gas of massless particles and \( \epsilon/T^4 \) ultimately goes to a constant value. The rapid increase with the parameter \( q \) occurs because the distribution deviates more and more from a Boltzmann distribution and the tails at large momentum contribute more and more as \( q \) increases. For higher \( q \)-values the critical behaviour of \( c_s^2 \) diminishes by including heavy resonances.

The speed of sound as a function of upper mass cut-off for a hadron resonance gas at temperature, \( T = 170 \) MeV, with different \( q \)-values is shown in Fig. 3. We choose a limiting temperature, \( T_H \sim T_c = 170 \) MeV motivated by the lattice QCD calculations [37] and also the Hagedorn limiting temperature \( T_H \) and \( T_c \) is the critical temperature for a deconfinement transition. This shows a monotonic decrease of speed of sound with the inclusion of higher resonances to the system and a systematic trend in the increase of the speed of sound is seen with the increase in \( q \)-value of the system.
which stay away from thermal equilibrium. Event temperature fluctuation specifically for the systems and higher energies, where one expects to study event-by-event temperature fluctuation in a hadron resonance gas, we have studied the sound shifts towards the lower temperature in the phase transition in non-extensive systems is achieved at lesser temperatures as compared to the systems following extensive statistics. Taking a mass cut-off of 2.5 GeV in the physical temperature, for different non-extensive parameters. Our results leave open the possibility that there exists a special value of $q$ as a function of the speed of sound in a hadronic medium using non-extensive Tsallis statistics is calculated for a hadron resonance gas, as the Tsallis statistics allows for the exploration of systems which are away from thermal equilibrium. Although the speed of sound has been estimated previously as a function of the temperature in a hadronic medium by several authors, it has not been studied for a hadron resonance gas with temperature fluctuations. Tsallis parameter “$q$” is related to the temperature fluctuation in a thermodynamic system. We study $c_s^2$ as a function of the non-extensive parameter $q$ in order to explore the effect of temperature fluctuations.

In the present work, the effect of different mass cut-offs on $c_s^2$ by adding massive resonances to the system is consistent with the earlier results calculated in the framework of extensive statistics [8]. However, by taking higher $q$-values in non-extensive statistics, $c_s^2$ increases near the limiting temperature as compared to the extensive Boltzmann statistics. Also, it is observed that the cut-off effect appears at lower temperature for higher values of “$q$”. This indicates that if there is any temperature fluctuation inside the system then the criticality in the speed of sound shifts towards the lower temperature in the phase diagram. In other words, the phase transition in non-extensive systems is achieved at lesser temperatures as compared to the systems following extensive statistics. Our results leave open the possibility that there exists a special value of $q$ where a phase transition is no longer present. Taking a mass cut-off of 2.5 GeV in the physical resonance gas, we have studied the $c_s^2$ as a function of system temperature, for different $q$-values and found that for $q$-values higher than 1.13, all criticality disappears. The present study bears significant implications at the LHC and higher energies, where one expects to study event-by-event temperature fluctuation specifically for the systems which stay away from thermal equilibrium.

## 4 Summary and Conclusion

The speed of sound in a hadronic medium using non-extensive statistics is calculated for a hadron resonance gas, as the earlier results calculated in the framework of extensive statistics [8]. However, by taking higher $q$-values in non-extensive statistics, $c_s^2$ increases near the limiting temperature as compared to the extensive Boltzmann statistics. Also, it is observed that the cut-off effect appears at lower temperature for higher values of “$q$”. This indicates that if there is any temperature fluctuation inside the system then the criticality in the speed of sound shifts towards the lower temperature in the phase diagram. In other words, the phase transition in non-extensive systems is achieved at lesser temperatures as compared to the systems following extensive statistics. Our results leave open the possibility that there exists a special value of $q$ where a phase transition is no longer present. Taking a mass cut-off of 2.5 GeV in the physical resonance gas, we have studied the $c_s^2$ as a function of system temperature, for different $q$-values and found that for $q$-values higher than 1.13, all criticality disappears. The present study bears significant implications at the LHC and higher energies, where one expects to study event-by-event temperature fluctuation specifically for the systems which stay away from thermal equilibrium.

## References

1. M. Chojaacki and W. Florkowski, Acta Phys. Polon. B 38, 3249 (2007).
2. J. Steinheimer and M. Bleicher, Eur. Phys. J. A 48, 100 (2012).
3. A. N. Tawilk and H. Magdy, Int. J. Mod. Phys. A 29, no. 27, 1450152 (2014).
4. B. Mohanty and J. e. Alam, Phys. Rev. C 68, 064903 (2003).
5. L. N. Epele, H. Fanchiotti, C. A. Garcia Canal and E. Roulet, Phys. Rev. D 36, 1508 (1987).
6. M. Plumer, S. Raha and R. M. Weiner, Phys. Lett. B 139, 198 (1984).
7. R. V. Gavai, S. Gupta and S. Mukherjee, Phys. Rev. D 71, 074013 (2005).
8. P. Castorina, J. Cleymans, D. E. Miller and H. Satz, Eur. Phys. J. C 66, 207 (2010).
9. A. Deppman, J. Phys. G 41, 055108 (2014).
10. L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Pergamon Press, Second Edition, Chapter 8, page-251.
11. C. Beck, Physica A 286, 164 (2000).
12. G. Wilk and Z. Wlodarczyk, Phys. Rev. Lett. 84, 2770 (2000).
13. A. Deppman, Physica A 391, 6380 (2012).
14. I. Sena and A. Deppman, Eur. Phys. J. A 49, 17 (2013).
15. M. D. Azmi and J. Cleymans, J. Phys. G 41, 055001 (2014).
16. C. Y. Wong and G. Wilk, Phys. Rev. D 87, 114007 (2013).
17. J. Cleymans, G. I. Lykasov, A. S. Parvan, A. S. Sorin, O. V. Teryaev and D. Worku, Phys. Lett. B 723, 351 (2013).
18. C. Y. Wong and G. Wilk, Acta Phys. Polon. B 43, 2047 (2012).
19. J. Cleymans and D. Worku, J. Phys. G 39, 025006 (2012).
20. I. Bediaga, E. M. F. Curado and J. M. de Miranda, Physica A 286, 156 (2000).
21. K. Urnossy, G. G. Barnafoldi and T. S. Biro, Phys. Lett. B 701, 111 (2011).
22. C. Tsallis, J. Statist. Phys. 52, 479 (1988).
23. Z. Tang, Y. Xu, L. Ruan, G. van Buren, F. Wang and Z. Xu, Phys. Rev. C 79, 051901 (2009).
24. T. Bhattacharyya, J. Cleymans, A. Khuntia, P. Pareek and R. Sahoo, Eur. Phys. J. A 52, 30 (2016).
25. B. De, S. Bhattacharyya, G. Sau and S. K. Biswas, Int. J. Mod. Phys. E 16, 1687 (2007).
26. B. De, Eur. Phys. J. A 50, 138 (2014).
27. B. De, Eur. Phys. J. A 50, 70 (2014).
28. S. Tripathy, T. Bhattacharyya, P. Garg, P. Kumar, R. Sahoo and J. Cleymans, Eur. Phys. J. A 52, 289 (2016).
29. D. K. Mishra, P. Garg, P. K. Netrakanti and A. K. Mohanty, J. Phys. G 42, 105105 (2015).
30. J. M. Conroy, H. G. Miller, A. R. Plastino, Phys. Lett. A 374, 4581 (2010).
31. F. Buyukkilic, D. Demirhan, Phys.Lett. A181, 24 (1993).
32. F. Pennini, A. Plastino, A. R. Plastino, Phys. Lett. A 208, 4, 309 (1995).
33. A. M. Teweldeberhan, A. R. Plastino, H. G. Miller, Phys.Lett. A 343, 71 (2004).
34. J. M. Conroy, H. Miller, Phys. Rev. D 78, 054010 (2008).
35. J. Chen et al., Phys. Lett. A 300, 1, 65 (2002).
36. M. D. Azmi and J. Cleymans, Eur. Phys. J. C 75, 430 (2015).
37. F. Karsch, E. Laermann and A. Peikert, Nucl. Phys. B 605, 579 (2001).
38. R. Hagedorn, Supplemento al Nuovo Cimento, 3, 147 (1965).
39. R. Hagedorn, Nuovo Cimento 35, 395 (1965).
40. R. Hagedorn, Nuovo Cimento A 56, 1027 (1968).