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Global threshold dynamics of an SVIR model with age-dependent infection and relapse

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\section*{ABSTRACT}
A susceptible-vaccinated-infectious-recovered epidemic model with infection age and relapse age has been formulated. We first address the asymptotic smoothness of the solution semiflow, existence of a global attractor, and uniform persistence of the model. Then by constructing suitable Volterra-type Lyapunov functionals, we establish a global threshold dynamics of the model, which is determined by the basic reproduction number. Biologically, it is confirmed that neglecting the possibility of vaccinees getting infected will over-estimate the effect of vaccination strategies. The obtained results generalize some existing ones.

\section*{1. Introduction}
Vaccination is the administration of antigenic material (a vaccine) to stimulate an individual’s immune system to develop adaptive immunity to a pathogen. Vaccines can prevent or ameliorate morbidity from infection. Vaccination is the most effective method of preventing infectious diseases \cite{6}. In the last decades, much attention has been paid on formulating and analysing epidemic models with vaccination to gain insights into the role played by vaccination. For example, by incorporating continuous vaccination strategy into a classical susceptible-infectious-recovered (SIR) model, Liu \textit{et al.} \cite{10} proposed a vaccination model described by a system of four ordinary differential equations (ODEs). It is found that vaccination has an effect of decreasing the basic reproduction number and it is also established that the basic reproduction number governs the global dynamics of the model. In this model, individuals are divided into susceptible, vaccinated, infective, and recovered. The two underlying important assumptions are as follows. On the one hand, vaccinated individuals still have the possibility of being infected by contacting with infected individuals. On the other hand, vaccinated individuals are thought to have partial immunity and hence the effective contacts with infectious individuals may decrease compared with those of susceptibles. In this sense, the continuous vaccination strategy can be evaluated by the
basic reproduction number because of the two infection paths, susceptible infection path and vaccinated infection path.

Nowadays, it has been recognized that the transmission dynamics of certain diseases could not be correctly described by the traditional compartmental epidemic models with no age structure. Models with (continuous) age structures are described by a hybrid system of ODEs and partial differential equations (PDEs) [32]. Following the spirit of the works [10, 15, 19], Wang et al. [26] formulated a susceptible-vaccinated-exposed-infectious-recovered model with the structure of infection age (time elapsed since the infection). Under the assumption that the removal rate from the infectious class is constant instead of a function of the infection age, the model can be reformulated as a differential equation with an infinite delay. It is shown that the global threshold dynamics determined by the basic reproduction number still holds. Duan et al. [5] investigated the global stability of a susceptible-vaccinated-infectious-recovered (SVIR) model with vaccination age. Compared with the models in [10, 26], the model studied in [5] did not take into account the fact that vaccinated individuals still have the possibility of being infected by contacting with infected individuals.

Furthermore, the study of van den Driessche and Zou [24] and van den Driessche et al. [23] indicated that relapse is an important feature of some animal and human diseases such as tuberculosis and herpes. Relapse is characterized by the reactivation of removed individuals who have been previously infected and then reverting back to the infectious class. In [23, 24], relapse takes forms of a constant relapse period and of a general relapse distribution, respectively. Then, Liu et al. [11] studied a susceptible-exposed-infectious-recovered epidemic model with age-dependent latency and relapse and proved that the threshold dynamics is preserved. Here, the recurrent phenomenon is characterized by PDEs with age-dependent relapse rate (or varying relapse rate) instead of a constant relapse period or a general relapse distribution.

Following the line of Liu et al. [10] and Wang et al. [26] and assuming that before obtaining immunity the vaccinees still have the possibility of being infected by contacting with infected individuals, Wang et al. [27] investigated the dynamics of a hybrid system of the SVIR model with the infection age.

Motivated by the above discussion, in this paper, we structure infected and recovered individuals by age of infection and by age of recovery, respectively. We formulate an SVIR model with two continuous structuring variables, termed as infection age and relapse age for short. Denote by \(S(t)\) and \(V(t)\) the numbers of susceptibles and vaccinees at time \(t\), respectively. Let \(i(t, a)\) and \(r(t, b)\) be the densities of infected individuals at time \(t\) with infection age \(a\) and of recovered individuals at time \(t\) with relapse age \(b\), respectively. The model to be studied is

\[
\begin{align*}
\frac{dS(t)}{dt} &= \Lambda - \beta S(t) \int_0^\infty p(a) i(t, a) \, da - (\mu + \alpha) S(t), \\
\frac{dV(t)}{dt} &= \alpha S(t) - \beta_1 V(t) \int_0^\infty p(a) i(t, a) \, da - \mu V(t), \\
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) i(t, a) &= - (\mu + \delta(a)) i(t, a),
\end{align*}
\] (1)
\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial b} \right) r(t, b) = -(\mu + \eta(b))r(t, b)
\]

with boundary conditions

\[
i(t, 0) = (\beta S(t) + \beta_1 V(t)) \int_0^\infty p(a)i(t, a) \, da + \int_0^\infty \eta(b)r(t, b) \, db, \\
r(t, 0) = \int_0^\infty \delta(a)i(t, a) \, da,
\]

and initial conditions

\[
S(0) = S_0 \in \mathbb{R}_+ := [0, \infty), \quad V(0) = V_0 \in \mathbb{R}_+, \\
i(0, \cdot) = i_0 \in L^1_+(0, \infty), \quad r(0, \cdot) = r_0 \in L^1_+(0, \infty),
\]

where \(L^1_+(0, \infty)\) is the set of functions on \((0, \infty)\) that are nonnegative and Lebesgue integrable. The meanings of the parameters in (1) are summarized in Table 1.

We mention that if all \(p(\cdot), \delta(\cdot), \text{ and } \eta(\cdot)\) are constant functions with the constants \(p, \delta, \text{ and } \eta\), respectively, then Equation (1) reduces to the following system of ODES:

\[
\frac{dS(t)}{dt} = \Lambda - \beta pS(t)I(t) - (\mu + \alpha)S(t), \\
\frac{dV(t)}{dt} = \alpha S(t) - \beta_1 pV(t)I(t) - \mu V(t), \\
\frac{dI(t)}{dt} = (\beta pS(t) + \beta_1 pV(t))I(t) - (\mu + \delta)I(t) + \eta R(t), \\
\frac{dR(t)}{dt} = \delta I(t) - (\mu + \eta)R(t),
\]

where \(I(t) = \int_0^\infty i(t, a) \, da\) and \(R(t) = \int_0^\infty r(t, b) \, db\). The case where \(\eta = 0\) of Equation (3) is one of the special cases of the model studied by Wang et al. [26] and a threshold dynamics was established. The main purpose of this paper is to establish such a result for (1) and hence our result generalizes some existing ones.

Throughout this paper, we make the following assumption.

**Assumption 1.1:**

(i) \(\Lambda, \mu, \alpha, \beta, \beta_1 > 0\).

(ii) \(p, \delta, \eta \in L^\infty_+(0, \infty)\) with finite essential upper bounds \(\bar{p}, \bar{\delta}, \bar{\eta}\), respectively.

**Table 1.** Biological meanings of parameters in (1).

| Parameter | Meaning |
|-----------|---------|
| \(\Lambda\) | Constant recruitment rate of susceptibles |
| \(\mu\) | The natural death rate of the population |
| \(\alpha\) | The vaccination rate |
| \(\beta\) | Infection rate of susceptibles by infected individuals |
| \(\beta_1\) | Infection rate of vaccinated by infected individuals |
| \(p(a)\) | The kernel of disease transmission with infectious individuals with infection age \(a\) |
| \(\delta(a)\) | The age-dependent removal (recovery) rate of infected individuals with infection age \(a\) |
| \(\eta(b)\) | The age-dependent relapse rate of recovered individuals with relapse age \(b\) |
(iii) $p$, $\delta$, and $\eta$ are Lipschitz continuous on $\mathbb{R}_+$ with Lipschitz constants $M_p$, $M_\delta$, and $M_\eta$, respectively.

In recent years, epidemic models and viral infection models with age-dependent structures have been extensively studied. To name a few, see [13] for a two-group model with infection age, [14] for an SIR model with infection age, [11, 16] for SEIR models with infection age, [3] for an SIRS model with infection age, [30] for an SVIER model with infection age, [1, 31, 33] for models on cholera with infection age, and [2, 8, 17, 18, 28, 29] for viral infection. Generally, it is not easy to analyse a hybrid system coupled of ODEs and PDEs, especially for the global stability/attractivity of equilibria. As mentioned earlier, the goal of this paper is to establish a threshold dynamics for Equation (1).

The organization of the remaining part of this paper is as follows. Some preliminary results on Equation (1) are presented in Section 2. In Section 3, we show the asymptotic smoothness of the semiflow of Equation (1). Section 4 is devoted to the uniform persistence of the system when the basic reproduction number is larger than one. In Section 5 and Section 6, we show that the disease dies out if the basic reproduction number is less than one and the endemic equilibrium is globally attractive if the basic reproduction number is larger than one by the technique of Lyapunov functionals, respectively. The paper concludes with a brief discussion.

2. Preliminaries

2.1. The solution semiflow

Denote

$$\mathcal{Y} = \mathbb{R}_+ \times \mathbb{R}_+ \times L^1_+(0, \infty) \times L^1_+(0, \infty),$$

the positive cone of the Banach space $\mathbb{R} \times \mathbb{R} \times L^1(0, \infty) \times L^1(0, \infty)$ equipped with induced product norm

$$\|(x, y, \varphi, \psi)\| = |x| + |y| + \|\varphi\|_{L^1} + \|\psi\|_{L^1}$$

for $(x, y, \varphi, \psi) \in \mathbb{R} \times \mathbb{R} \times L^1(0, \infty) \times L^1(0, \infty)$.

If any initial value $X_0 = (S_0, V_0, i_0, r_0) \in \mathcal{Y}$ satisfies the coupling equations

$$i(0, 0) = (\beta S_0 + \beta_1 V_0) \int_0^\infty p(a) i_0(a) \, da + \int_0^\infty \eta(b) r_0(b) \, db$$

and

$$r(0, 0) = \int_0^\infty \delta(a) i_0(a) \, da,$$

then (1) is well-posed under Assumption 1.1 due to Iannelli [9] and Magal [12]. In fact, for such solutions, it is not difficult to show that $(S(t), V(t), i(t, \cdot), r(t, \cdot)) \in \mathcal{Y}$ for each $t \in \mathbb{R}_+$. In the sequel, we always assume that the initial values satisfy the coupling equations. Then,
we obtain a continuous solution semiflow \( \Phi : \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y} \) defined by

\[
\Phi(t, X_0) = \Phi_t(X_0) := (S(t), V(t), i(t, \cdot), r(t, \cdot)), \quad t \in \mathbb{R}_+, \ X_0 \in \mathcal{Y}.
\]

For the sake of convenience, we introduce the following notations:

\[
\Omega(a) = e^{-\int_0^a (\mu + \delta(\tau)) \, d\tau} \quad \text{and} \quad \Gamma(b) = e^{-\int_0^b (\mu + \eta(\tau)) \, d\tau} \quad \text{for} \ a, b \in \mathbb{R}_+.
\]

By Assumption 1.1, we have

\[
0 \leq \Omega(a) \leq e^{-\mu a} \quad \text{and} \quad 0 \leq \Gamma(b) \leq e^{-\mu b} \quad \text{for all} \ a, b \in \mathbb{R}_+.
\] (4)

For any solution of Equation (1), we define the following functions on \( \mathbb{R}_+ \):

\[
P(t) = \int_0^\infty p(a)i(t, a) \, da, \quad Q(t) = \int_0^\infty \delta(a)i(t, a) \, da,
\]

\[
R(t) = \int_0^\infty \eta(b)r(t, b) \, db \quad \text{for} \ t \in \mathbb{R}_+.
\]

Then the boundary conditions given in Equation (2) can be rewritten as

\[
i(t, 0) = \left[ \beta S(t) + \beta_1 V(t) \right] P(t) + R(t) \quad \text{and} \quad r(t, 0) = Q(t).
\]

Integrating the third and the fourth equations of (1) along the characteristic lines \( t - a = \text{const.} \) and \( t - b = \text{const.} \), respectively gives

\[
i(t, a) = \begin{cases} 
[(\beta S(t - a) + \beta_1 V(t - a))P(t - a) + R(t - a)]\Omega(a) = i(t - a, 0)\Omega(a) & \text{if} \ 0 \leq a \leq t, \\
i_0(a - t)\frac{\Omega(a)}{\Omega(a - t)} & \text{if} \ 0 \leq t \leq a,
\end{cases}
\] (5)

and

\[
r(t, b) = \begin{cases} 
Q(t - b)\Gamma(b) = r(t - b, 0)\Gamma(b) & \text{if} \ 0 \leq b \leq t, \\
r_0(b - t)\frac{\Gamma(b)}{\Gamma(b - t)} & \text{if} \ 0 \leq t \leq b.
\end{cases}
\] (6)

The boundedness of the solution semiflow of Equation (1) is given in the following result.

**Proposition 2.1:** Define

\[
\Xi := \left\{ X_0 = (S_0, V_0, i_0, r_0) \in \mathcal{Y} \ \bigg| \ \|X_0\| \leq \frac{\Lambda}{\mu} \right\}.
\]

Then \( \Xi \) is a positively invariant subset for \( \Phi \), that is,

\[
\Phi(t, X_0) \in \Xi \quad \text{for all} \ t \geq 0 \quad \text{and} \quad X_0 \in \Xi.
\]

Moreover, \( \Phi \) is point dissipative and \( \Xi \) attracts all points in \( \mathcal{Y} \).
Proof: First, we have
\[
\frac{d}{dt} ||\Phi_t(X_0)|| = \frac{dS(t)}{dt} + \frac{dV(t)}{dt} + \frac{di(t, \cdot)}{dt}L^1 + \frac{d}{dt} ||r(t, \cdot)||L^1.
\]
On the one hand, it follows from Equation (5) and changes of variables that
\[
||i(t, \cdot)||L^1 = \int_0^t i(t - a, 0) \Omega(a) da + \int_t^\infty i_0(a - t) \frac{\Omega(a)}{\Omega(a - t)} da
\]
\[
= \int_0^t i(\sigma, 0) \Omega(t - \sigma) d\sigma + \int_0^\infty i_0(\tau) \frac{\Omega(t + \tau)}{\Omega(\tau)} d\tau.
\]
Then
\[
\frac{d||i(t, \cdot)||L^1}{dt} = i(t, 0) \Omega(0) + \int_0^\infty i_0(\tau) \frac{d}{dt} \Omega(t + \tau) d\tau + \int_0^t i(\sigma, 0) \frac{d}{dt} \Omega(t - \sigma) d\sigma.
\]
Note that \( \Omega(0) = 1 \) and \( (d/da) \Omega(a) = (\mu + \delta(a)) \Omega(a) \) for almost all \( a \in \mathbb{R}_+ \).
Using Equation (5) and changes of variables, one can get
\[
\frac{d||i(t, \cdot)||L^1}{dt} = i(t, 0) - \int_0^\infty i_0(\tau) \frac{\Omega(t + \tau)(\mu + \delta(t + \tau)))}{\Omega(\tau)} d\tau
\]
\[
- \int_0^t i(\sigma, 0)(\mu + \delta(t - \sigma))) \Omega(t - \sigma) d\sigma
\]
\[
= i(t, 0) - \int_0^\infty (\mu + \delta(a)) i(t, a) da.
\]
Similarly, we can get
\[
\frac{d||r(t, \cdot)||L^1}{dt} = r(t, 0) - \int_0^\infty (\mu + \eta(b)) r(t, b) db.
\]
These, combined with the first and the second equations of (1), give us
\[
\frac{d}{dt} ||\Phi_t(X_0)|| = \Lambda - \mu ||\Phi_t(X_0)|| \quad \text{for } t \in \mathbb{R}_+.
\]
It follows that
\[
||\Phi_t(X_0)|| = \frac{\Lambda}{\mu} - e^{-\mu t} \left\{ \frac{\Lambda}{\mu} - ||X_0|| \right\} \leq \frac{\Lambda}{\mu}
\]
for \( t \in \mathbb{R}_+ \) if \( X_0 \in \mathcal{Z} \). In summary, we have shown that \( \mathcal{Z} \) is positively invariant with respect to \( \Phi \).
Lastly, it follows from Equation (7) that \( \lim \sup_{t \to \infty} ||\Phi_t(X_0)|| \leq \Lambda/\mu \) for any \( X_0 \in \mathcal{Y} \), that is, \( \Phi \) is point dissipative and \( \mathcal{Z} \) attracts all points in \( \mathcal{Y} \). This completes the proof.

The following result is a direct consequence of Proposition 2.1.

Proposition 2.2: Let \( A \geq \Lambda/\mu \) be given. If \( X_0 \in \mathcal{Y} \) satisfies \( ||X_0|| \leq A \), then the following statements hold for all \( t \in \mathbb{R}_+ \).
(i) \( S(t), V(t), \| i(t, \cdot) \|_{L^1}, \| r(t, \cdot) \|_{L^1} \leq A \);
(ii) \( P(t) \leq \bar{p}A, Q(t) \leq \bar{\delta}A \) and \( R(t) \leq \bar{\eta}A \);
(iii) \( i(t, 0) \leq \bar{\beta}A \) and \( r(t, 0) \leq \bar{\delta}A \), where \( \bar{\beta} = \beta \bar{p}A + \beta_1 \bar{p}A + \bar{\eta} \).

It follows from Proposition 2.2, Assumption 1.1 and [27, Proposition 4.1], we obtain the following basic properties of the functions \( P(t), Q(t) \) and \( R(t) \).

**Proposition 2.3:** For any solution of Equation (1), the associated functions \( P(t), Q(t) \) and \( R(t) \) are Lipschitz continuous on \( \mathbb{R}_+ \) with Lipschitz constants,

\[
L_P = (\bar{p} \bar{\beta} + \bar{p}(\mu + \bar{\delta}) + M_p)A, \quad L_Q = (\bar{\delta} \bar{\beta} + \bar{\delta}(\mu + \bar{\delta}) + M_\delta)A, \\
L_R = (\bar{\eta} \bar{\delta} + \bar{\eta}(\mu + \bar{\eta}) + M_\eta)A,
\]

respectively.

### 2.2. Equilibria and the basic reproduction number

For Equation (1), there always exists the infection-free equilibrium

\[
E^0 = (S^0, V^0, i^0, r^0) := \left( \frac{\Lambda}{\mu + \alpha}, \frac{\alpha \Lambda}{\mu(\mu + \alpha)}, 0, 0 \right).
\]

Moreover, if an equilibrium is not infection-free then it must be endemic.

An endemic equilibrium \( E^* = (S^*, V^*, i^*, r^*) \) satisfies the following algebraic equations:

\[
\begin{align*}
\Lambda - \beta S^* \int_0^\infty p(a)i^*(a) \, da - (\mu + \alpha)S^* &= 0, \\
\alpha S^* - \beta_1 V^* \int_0^\infty p(a)i^*(a) \, da - \mu V^* &= 0, \\
\frac{di^*(a)}{da} &= -(\mu + \delta(a))i^*(a), \\
\frac{dr^*(b)}{db} &= -(\mu + \eta(b))r^*(b), \\
i^*(0) &= (\beta S^* + \beta_1 V^*) \int_0^\infty p(a)i^*(a) \, da + \int_0^\infty \eta(b)r^*(b) \, db, \\
r^*(0) &= \int_0^\infty \delta(a)i^*(a) \, da.
\end{align*}
\]

From the third and fourth equations of (8), we obtain \( i^*(a) = i^*(0)\Omega(a) \) and \( r^*(b) = r^*(0)\Gamma(b) \). Substituting them into the last two equations of (8) gives

\[
1 = (\beta S^* + \beta_1 V^*) \int_0^\infty p(a)\Omega(a) \, da + \int_0^\infty \delta(a)\Omega(a) \, da \int_0^\infty \eta(b)\Gamma(b) \, db. \tag{9}
\]

In order to obtain an endemic equilibrium, we define a key threshold parameter. According to Diekmann et al. [4] and van den Driessche and Watmough [22], the expected number
of secondary cases produced by a typical infectious individual during its entire period of infectiousness is defined as the basic reproduction number $R_0$, which is given by

$$R_0 = (\beta S_0 + \beta_1 V_0)H + LK := \left( \beta + \frac{\beta_1 \alpha}{\mu} \right) \frac{\Lambda H}{\mu + \alpha} + LK,$$

where

$$H = \int_0^\infty p(a) \Omega(a) \, da, \quad L = \int_0^\infty \eta(b) \Gamma(b) \, db, \quad K = \int_0^\infty \delta(a) \Omega(a) \, da.$$

Note that $L \leq \bar{\eta}/(\mu + \bar{\eta}) < 1$ and $K \leq \bar{\delta}/(\mu + \bar{\delta}) < 1$. As we will see later, $R_0$ serves as a sharp threshold parameter for Equation (1), which completely determines the global behaviour of Equation (1).

Biologically, $\Omega(a)$ is the probability of an infected individual still in the infected class at the infection age $a$ and hence $\int_0^\infty p(a) \Omega(a) \, da$ is the total infection force. It follows that the first term in $R_0$ is the average number of secondary cases directly produced by an infected individual introduced into a population with susceptibles and vaccinees only. Furthermore, $\int_0^\infty \delta(a) \Omega(a) \, da$ represents the chance of recovery for an infected individual and $\int_0^\infty \eta(b) \Gamma(b) \, db$ is the chance of a recovered individual can be infectious after relapse. Therefore, $LK$ in $R_0$ is the average number of secondary cases produced by infected individual after recovery through relapse.

Now, from the first and second equations of (8), we have

$$S^* = \frac{\Lambda}{\mu + \alpha + \beta Hi^*(0)}, \quad V^* = \frac{\alpha \Lambda}{(\mu + \alpha + \beta Hi^*(0))(\mu + \beta_1 Hi^*(0))}.$$

Plugging them into (9) yields

$$G(i^*(0)) = a_0 (Hi^*(0))^2 + a_1 Hi^*(0) + a_2 = 0,$$

where

$$a_0 = \beta \beta_1 (1 - LK) > 0,$$

$$a_1 = \beta \mu + \beta_1 (\mu + \alpha) - \beta \beta_1 \Lambda H - (\mu + \alpha) \beta_1 LK - \mu \beta LK,$$

$$a_2 = \mu (\mu + \alpha)(1 - R_0).$$

If $R_0 \leq 1$, then $a_1 > 0$ and $G(0) = a_2 \geq 0$. It follows from the relationship between zeros and coefficients, $G$ has no positive zeros and hence there is no endemic equilibrium. Now, suppose that $R_0 > 1$. Then $a_2 < 0$. By the relationship between zeros and coefficients, $G$ has a unique positive zero and hence there is a unique endemic equilibrium

$$E^* = (S^*, V^*, i^*, r^*) := \left( \frac{\Lambda}{\mu + \alpha + \beta Hi^*(0)}, \frac{\alpha \Lambda}{(\mu + \alpha + \beta Hi^*(0))(\mu + \beta_1 Hi^*(0))}, i^*(0) \Omega(\cdot), K i^*(0) \Gamma(\cdot) \right),$$

where $i^*(0)$ is the only positive zero of $G$ defined by Equation (10). In summary, we have obtained the following result.
Theorem 2.1: (i) System (1) always has an infection-free equilibrium $E^0$.
(ii) If $\mathfrak{R}_0 > 1$, then Equation (1) also has a unique endemic equilibrium $E^* = (S^*, V^*, i^*, r^*)$ defined by Equation (11).

3. Asymptotic smoothness

A semiflow is asymptotically smooth if each forward invariant bounded closed set is attracted by a nonempty compact set. Since $L_1^+ (0, \infty )$ is a part of $\mathcal{Y}$, which belongs to infinite dimension, we cannot deduce precompactness only from boundedness. On the other hand, the global attractivity results will utilize the Lyapunov functional technique combined with the invariance principle. Due to [25, Theorem 4.2 of Chapter IV], we need the asymptotic smoothness of the orbit $\{ \Phi(t, X_0) \mid t \in \mathbb{R}_+ \}$ in $\mathcal{Y}$ in order to use the invariance principle. To this end, we first decompose $\Phi: \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y}$ into the following two operators $\Theta, \Psi: \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y}$ defined respectively by

$$\Theta(t, X_0) := (0, 0, \tilde{\varphi}_i(t, \cdot), \tilde{\varphi}_r(t, \cdot)),$$
$$\Psi(t, X_0) := (S(t), V(t), \tilde{i}(t, \cdot), \tilde{r}(t, \cdot)),$$

where

$$\tilde{\varphi}_i(t, a) = \begin{cases} 0 & \text{if } t > a \geq 0, \\
 i(t, a) & \text{if } a \geq t \geq 0; \\
 i(t, a) & \text{if } a \geq t \geq 0; \\
 0 & \text{if } a \geq t \geq 0; \\
 0 & \text{if } b \geq t \geq 0; \\
 0 & \text{if } b \geq t \geq 0; \\
 r(t, b) & \text{if } b \geq t \geq 0; \\
 0 & \text{if } b \geq t \geq 0.
\end{cases}$$

Then $\Phi(t, X_0) = \Theta(t, X_0) + \Psi(t, X_0)$ for $t \in \mathbb{R}_+$. Note that $\tilde{i}(t, a)$ and $\tilde{r}(t, b)$ can also be written as

$$\tilde{i}(t, a) = \begin{cases} (\beta S(t - a) + \beta_1 V(t - a))P(t - a) + R(t - a) \Omega(a) & \text{if } t > a \geq 0, \\
 0 & \text{if } a \geq t \geq 0,
\end{cases} \quad (12)$$

and

$$\tilde{r}(t, b) = \begin{cases} Q(t - b) \Gamma(b) & \text{if } t > b \geq 0, \\
 0 & \text{if } b \geq t \geq 0,
\end{cases}$$

respectively.

Theorem 3.1: For $X_0 \in \Xi$, the orbit $\{ \Phi(t, X_0) \mid t \geq 0 \}$ has a compact closure in $\mathcal{Y}$ if the following two conditions hold.

(i) There exists a function $\Delta: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $r > 0$, $\lim_{t \to \infty} \Delta(t, r) = 0$ and if $X_0 \in \Omega$ with $\| X_0 \| \leq r$ then $\| \Theta(t, X_0) \| \leq \Delta(t, r)$ for $t \in \mathbb{R}_+$.

(ii) For $t \in \mathbb{R}_+$, $\Psi(t, \cdot)$ maps any bounded sets of $\Xi$ into sets with compact closure in $\mathcal{Y}$. 
Lemma 3.1: For \( r > 0 \), let \( \Delta(t, r) = e^{-\mu t}r \). Then \( \Delta \) satisfies condition (i) of Theorem 3.1.

Proof: Obviously, \( \lim_{t \to \infty} \Delta(t, r) = 0 \). By Equations (5) and (6),

\[
\tilde{\varphi}_i(t, a) = \begin{cases} 
0 & \text{if } t > a \geq 0 \\
\frac{i_0(a - t)}{\Omega(a)} & \text{if } a \geq t \geq 0 
\end{cases}
\]

and

\[
\tilde{\varphi}_r(t, b) = \begin{cases} 
0 & \text{if } t > b \geq 0 \\
\frac{r_0(b - t)}{\Gamma(b)} & \text{if } b \geq t \geq 0.
\end{cases}
\]

Then, for \( X_0 \in \Xi \) satisfying \( \|X_0\| \leq r \) and for \( t \in \mathbb{R}_+ \), we have

\[
\|\Theta(t, X_0)\| = |0| + |0| + \|\tilde{\varphi}_i(t, \cdot)\|_{L^1} + \|\tilde{\varphi}_r(t, \cdot)\|_{L^1}
\]

\[
= \int_t^\infty i_0(a - t) \frac{\Omega(a)}{\Omega(a - t)} \, da + \int_t^\infty r_0(b - t) \frac{\Gamma(b)}{\Gamma(b - t)} \, db
\]

\[
= \int_0^\infty i_0(\sigma) \frac{\Omega(\sigma + t)}{\Omega(\sigma)} \, d\sigma + \int_0^\infty r_0(\sigma) \frac{\Gamma(\sigma + t)}{\Gamma(\sigma)} \, d\sigma
\]

\[
= \int_0^\infty |i_0(\sigma) e^{-\int_\sigma^\sigma+\eta(t, \mu + \eta(\tau))} \, d\tau| \, d\sigma + \int_0^\infty |r_0(\sigma) e^{-\int_\sigma^\sigma+\eta(t, \mu + \eta(\tau))} \, d\sigma| \, d\sigma
\]

\[
\leq e^{-\mu t} i_0\|_{L^1} + e^{-\mu t} r_0\|_{L^1}
\]

\[
\leq e^{-\mu t} \|X_0\|
\]

\[
\leq e^{-\mu t} r.
\]

This completes the proof.

Lemma 3.2: For \( t \in \mathbb{R}_+ \), \( \Psi(t, \cdot) \) maps any bounded sets of \( \Xi \) into sets with compact closure in \( \mathcal{Y} \).

Proof: First, \( S(t) \) and \( V(t) \) remain in the compact set \([0, A/\mu] \subset [0, A]\) from Proposition 2.1. Thus, it suffices to show that \( \tilde{i}(t, a) \) and \( \tilde{r}(t, b) \) remain in a precompact subset of \( L^1_+(0, \infty) \), which is independent of \( X_0 \in \Xi \). To this end, we verify the following conditions for \( i(t, a) \) and similar ones for \( r(t, b) \) (see, for example, [21, Theorem B.2]).

(i) The supremum of \( \|\tilde{i}(t, \cdot)\|_{L^1} \) with respect to \( X_0 \in \Xi \) is finite;

(ii) \( \lim_{h \to 0} \int_h^\infty \tilde{i}(t, a) \, da = 0 \) uniformly with respect to \( X_0 \in \Xi \);

(iii) \( \lim_{h \to 0} \int_0^\infty \|\tilde{i}(t, a + h) - \tilde{i}(t, a)\| \, da = 0 \) uniformly with respect to \( X_0 \in \Xi \);

(iv) \( \lim_{h \to 0} \int_0^h \tilde{i}(t, a) \, da = 0 \) uniformly with respect to \( X_0 \in \Xi \).
It follows from Equations (5) and (12) that

$$0 \leq \tilde{i}(t, a) = \begin{cases} 
i(t - a, 0)\Omega(a) & \text{if } 0 \leq a \leq t, \\ 0 & \text{if } 0 \leq t < a. \end{cases}$$

This, combined with Proposition 2.2 and Equation (4), produces

$$\tilde{i}(t, a) \leq \bar{\beta} A e^{-\mu a} = (\beta \bar{p} A + \beta_1 \bar{p} A + \bar{\eta}) A e^{-\mu a},$$

from which conditions (i), (ii) and (iv) directly follows.

Now, we are in a position to verify condition (iii). For sufficiently small $h \in (0, t)$, it follows from the definition of $\tilde{i}$ and Proposition 2.2 that

$$\int_0^\infty |\tilde{i}(t, a + h) - \tilde{i}(t, a)| \, da$$

$$= \int_0^{t-h} |i(t, a + h) - i(t, a)| \, da + \int_{t-h}^t |0 - i(t, a)| \, da$$

$$= \int_0^{t-h} |i(t - a - h, 0)\Omega(a + h) - i(t - a, 0)\Omega(a)| \, da + \int_{t-h}^t |i(t - a, 0)\Omega(a)| \, da$$

$$\leq \Delta_1 + \Delta_2 + \bar{\beta} Ah,$$  \hspace{1cm} (13)

where

$$\Delta_1 = \int_0^{t-h} i(t - a - h, 0)|\Omega(a + h) - \Omega(a)| \, da$$

and

$$\Delta_2 = \int_0^{t-h} |i(t - a - h, 0) - i(t - a, 0)\Omega(a)| \, da.$$  

We estimate $\Delta_1$ and $\Delta_2$ separately as follows.

We claim that $\Delta_1 \leq \bar{\beta} Ah$. In fact, it follows from the fact that $0 \leq \Omega(a) \leq 1$ and $\Omega$ is a non-increasing function, we have

$$\int_0^{t-h} |\Omega(a + h) - \Omega(a)| \, da = \int_0^{t-h} (\Omega(a) - \Omega(a + h)) \, da$$

$$= \int_0^{t-h} \Omega(a) \, da - \int_0^{t-h} \Omega(a + h) \, da$$

$$= \int_0^{t-h} \Omega(a) \, da - \int_h^t \Omega(a) \, da$$

$$= \int_0^{t-h} \Omega(a) \, da - \int_{t-h}^t \Omega(a) \, da - \int_{t-h}^t \Omega(a) \, da$$

$$= \int_0^h \Omega(a) \, da - \int_{t-h}^t \Omega(a) \, da \leq h.$$

This, combined with Proposition 2.2, gives $\Delta_1 \leq \bar{\beta} Ah$. 
Now we show that there exists $M$ such that $\Delta_2 \leq Mh/\mu$. Actually, it follows from Equation (12) that

$$
\Delta_2 = \int_{0}^{t-h} \left[ ([\beta S(t-a-h) + \beta_1 V(t-a-h)] P(t-a-h) + R(t-a-h)) - ([\beta S(t-a) + \beta_1 V(t-a)] P(t-a) + R(t-a)) \right]|\Omega(a) da \\
\leq \int_{0}^{t-h} |(\beta S(t-a-h) P(t-a-h) - (\beta S(t-a) P(t-a))] |\Omega(a) da \\
+ \int_{0}^{t-h} |(\beta_1 V(t-a-h) P(t-a-h) - \beta_1 V(t-a) P(t-a))| \Omega(a) da \\
+ \int_{0}^{t-h} |(R(t-a-h) - R(t-a))| \Omega(a) da.
$$

From Proposition 2.2 and the first equation of (1), we can conclude that $S(t)$ and $V(t)$ are Lipschitz continuous on $\mathbb{R}_+$ with Lipschitz constants $M_S = \Lambda + (\mu + \alpha)A + \beta A^2 \bar{p}$ and $M_V = \alpha A + (\mu + \alpha)A + \beta_1 \bar{p} A^2$, respectively. Furthermore, it follows from Proposition 2.3 that $P(t)$ and $R(t)$ are Lipschitz continuous with Lipschitz constants $L_P$ and $L_R$, respectively. Thus $S(t)P(t)$ and $V(t)P(t)$ are Lipschitz continuous with Lipschitz constants $M_{SP} = \alpha L_P + \bar{p} AM_S$ and $M_{VP} = \alpha L_P + \bar{p} AM_V$, respectively. Hence taking $M = \beta M_{SP} + \beta_1 M_{VP} + L_R$, we have

$$
\Delta_2 \leq M \int_{0}^{t-h} e^{-\mu a} da \leq \frac{Mh}{\mu}.
$$

It follows from Equations (13)–(14) that condition (iii) holds. \hfill \blacksquare

According to results on the existence of global attractors in [7, 21], the following result is a consequence of Proposition 2.1, Theorem 3.1, Lemmas 3.1 and 3.2.

**Theorem 3.2:** The semiflow $\Phi$ has a global attractor $\mathcal{A}$ in $\mathcal{Y}$, which attracts any bounded subset of $\mathcal{Y}$.

### 4. The uniform persistence

The aim of this section is to show that Equation (1) is uniformly persistent.

Define $\rho : \mathcal{Y} \to \mathbb{R}_+$ by

$$
\rho(S, V, i, r) = (\beta S + \beta_1 V) \int_{0}^{\infty} p(a) i(a) da + \int_{0}^{\infty} \eta(b) r(b) db \quad \text{for} \quad (S, V, i, r) \in \mathcal{Y}.
$$

Let

$$
\mathcal{Y}_0 = \{(S_0, V_0, i_0, r_0) \in \mathcal{Y} : \rho(\Phi(t_0, (S_0, V_0, i_0, r_0))) > 0 \text{ for some } t_0 \in \mathbb{R}_+\}.
$$

Obviously, if $(S_0, V_0, i_0, r_0) \in \mathcal{Y} \setminus \mathcal{Y}_0$, then $\Phi(t, (S_0, V_0, i_0, r_0)) \to E^0$ as $t \to \infty$. 

**Definition 4.1 ([21, p. 61]):** System (1) is said to be uniformly weakly $\rho$-persistent (respectively, uniformly strongly $\rho$-persistent) if there exists an $\epsilon > 0$, independent of the initial conditions, such that

$$\limsup_{t \to \infty} \rho(\Phi(t, (S_0, V_0, i_0, r_0))) > \epsilon \quad (\text{respectively, } \liminf_{t \to \infty} \rho(\Phi(t, (S_0, V_0, i_0, r_0))) > \epsilon)$$

for $(S_0, V_0, i_0, r_0) \in \mathcal{Y}_0$.

Let $\hat{i}(t) := i(t, 0)$ and $\hat{r}(t) := r(t, 0)$. Note that $\hat{i}(t) = \rho(\Phi(t, X_0))$. Then, Equations (5) and (6) can be rewritten as

$$i(t, a) = \begin{cases} 
\hat{i}(t-a)\Omega(a) & \text{if } t \geq a \geq 0, \\
i_0(a-t)\frac{\Omega(a)}{\Omega(a-t)} & \text{if } a \geq t \geq 0,
\end{cases}$$

respectively. Substitute them into the boundary condition (2) to obtain

$$\hat{i}(t) = (\beta S(t) + \beta_1 V(t)) \left\{ \int_0^t p(a)\Omega(a)\hat{i}(t-a) \, da + \int_t^\infty p(a)\frac{\Omega(a)}{\Omega(a-t)}i_0(a-t) \, da \right\}$$

$$+ \left\{ \int_0^t \eta(b)\Gamma(b)\hat{r}(t-b) \, db + \int_t^\infty \eta(b)\frac{\Gamma(b)}{\Gamma(b-t)}r_0(b-t) \, db \right\}$$

and

$$\hat{r}(t) = \int_0^t \delta(a)\Omega(a)\hat{i}(t-a) \, da + \int_t^\infty \delta(a)\frac{\Omega(a)}{\Omega(a-t)}i_0(a-t) \, da.$$  

**Lemma 4.1:** If $\mathcal{R}_0 > 1$, then Equation (1) is uniformly weakly $\rho$-persistent.

**Proof:** We first get an estimate on $\hat{i}(t)$ as follows. By Equation (16), we have

$$\hat{i}(t) \geq (\beta S(t) + \beta_1 V(t)) \int_0^t p(a)\Omega(a)\hat{i}(t-a) \, da + \int_0^t \eta(b)\Gamma(b)\hat{r}(t-b) \, db. \quad (18)$$

By way of contradiction, for any $\epsilon > 0$, there exists an $X^\epsilon \in \mathcal{Y}_0$ such that $\limsup_{t \to \infty} \rho(\Phi(t, X^\epsilon)) \leq \epsilon$. Since $\mathcal{R}_0 > 1$, there exists a sufficiently small $\epsilon_0 > 0$ such
that
\[
\left( \beta \left( \frac{\Lambda - \epsilon_0}{\mu + \alpha} - \epsilon_0 \right) + \beta_1 \left( \frac{\alpha(\Lambda - \epsilon_0)}{\mu + \alpha} - \frac{\epsilon_0}{\mu} - \epsilon_0 \right) \right) \int_0^\infty p(a) \Omega(a) \, da \\
+ \int_0^\infty \eta(b) \Gamma(b) \, db \int_0^\infty \delta(a) \Omega(a) \, da > 1. \tag{19}
\]

In particular, there exists $X^{\epsilon_0/2} \in \mathcal{Y}_0$ (for simplicity, denoted by $X_0$ in the remaining of the proof) such that
\[
\limsup_{t \to \infty} \rho(\Phi(t, X_0)) \leq \frac{\epsilon_0}{2}.
\]

A contradiction is arrived at as follows.

Firstly, there exists $T \in \mathbb{R}_+$ such that
\[
\rho(\Phi(t, X_0)) \leq \epsilon_0 \quad \text{for all } t \geq T.
\]

Without loss of generality, we assume that $T = 0$ by replacing $X_0$ with $\Phi(T, X_0)$. Then it follows from the first equation of (1) that
\[
\frac{dS(t)}{dt} \geq \Lambda - \epsilon_0 - (\mu + \alpha)S(t) \quad \text{for all } t \in \mathbb{R}_+.
\]

It follows that $\limsup_{t \to \infty} S(t) \geq (\Lambda - \epsilon_0)/(\mu + \alpha)$. As before, with possible replacing of the initial value, we can assume that
\[
S(t) \geq \frac{\Lambda - \epsilon_0}{\mu + \alpha} - \epsilon_0 \quad \text{for all } t \in \mathbb{R}_+. \tag{20}
\]

Similarly, with Equation (20) and the second equation of (1), we can get
\[
V(t) \geq \frac{\alpha(\Lambda - \epsilon_0)}{\mu + \alpha} - \epsilon_0 \quad \text{for all } t \in \mathbb{R}_+. \tag{21}
\]

Combining Equations (17), (18), (20) and (21), we have
\[
\hat{i}(t) \geq \left( \beta \left( \frac{\Lambda - \epsilon_0}{\mu + \alpha} - \epsilon_0 \right) + \beta_1 \left( \frac{\alpha(\Lambda - \epsilon_0)}{\mu + \alpha} - \frac{\epsilon_0}{\mu} - \epsilon_0 \right) \right) \int_0^t p(a) \Omega(a) \hat{i}(t - a) \, da \\
+ \int_0^t \eta(b) \Gamma(b) \int_0^{t-b} \delta(a) \Omega(a) \hat{i}(t - a - b) \, da \, db
\]

for all $t \in \mathbb{R}_+$. Taking the Laplace transforms of both sides of the above inequality, we obtain
\[
\mathcal{L}[\hat{i}] \geq \left( \beta \left( \frac{\Lambda - \epsilon_0}{\mu + \alpha} - \epsilon_0 \right) + \beta_1 \left( \frac{\alpha(\Lambda - \epsilon_0)}{\mu + \alpha} - \frac{\epsilon_0}{\mu} - \epsilon_0 \right) \right) \int_0^\infty p(a) \Omega(a) e^{-\lambda a} \, da \mathcal{L}[\hat{i}] \\
+ \int_0^\infty \eta(b) \Gamma(b) e^{-\lambda b} \, db \int_0^\infty \delta(a) \Omega(a) e^{-\lambda a} \, da \mathcal{L}[\hat{i}].
\]
Here, \( \mathcal{L}[\hat{i}] \) denotes the Laplace transform of \( \hat{i} \), which is strictly positive. Dividing both sides of the above inequality by \( \mathcal{L}[\hat{i}] \) and letting \( \lambda \to 0 \) give us

\[
1 \geq \left( \beta \left( \frac{\Lambda - \epsilon_0}{\mu + \alpha} - \epsilon_0 \right) + \beta_1 \left( \frac{\alpha(\Lambda - \epsilon_0) - \epsilon_0}{\mu} - \epsilon_0 \right) \right) \int_0^\infty p(a) \Omega(a) \, da \\
+ \int_0^\infty \eta(b) \Gamma(b) \, db \int_0^\infty \delta(a) \Omega(a) \, da,
\]

which contradicts with Equation (19). This proves the claim and hence completes the proof. \( \blacksquare \)

In order to apply a technique used by Smith and Thieme [21, Chapter 9], we consider a total \( \Phi \)-trajectory. \( \phi: \mathbb{R} \to \mathcal{Y} \) is a total \( \Phi \)-trajectory if \( \Phi(s, \phi(t)) = \phi(t + s) \) for all \( t \in \mathbb{R} \) and \( s \in \mathbb{R}_+ \). A \( \Phi \)-total trajectory \( \phi = (S, V, i, r) \) satisfies

\[
i(t, a) = i(t - a, 0) \Omega(a) = \hat{i}(t - a) \Omega(a) \quad \text{for } t \in \mathbb{R} \text{ and } a \in \mathbb{R}_+, \]

\[
r(t, b) = r(t - b, 0) \Gamma(b) = \hat{r}(t - b) \Gamma(b) \quad \text{for } t \in \mathbb{R} \text{ and } b \in \mathbb{R}_+.
\]

Therefore,

\[
\frac{dS(t)}{dt} = \Lambda - \beta S(t) \int_0^\infty p(a) \Omega(a) \hat{i}(t - a) \, da - (\mu + \alpha) S(t),
\]

\[
\frac{dV(t)}{dt} = \alpha S(t) - \beta_1 V(t) \int_0^\infty p(a) \Omega(a) \hat{i}(t - a) \, da - \mu V(t),
\]

\[
\hat{i}(t) = (\beta S(t) + \beta_1 V(t)) \int_0^\infty p(a) \Omega(a) \hat{i}(t - a) \, da + \int_0^\infty \eta(b) \Gamma(b) \hat{r}(t - b) \, db,
\]

\[
\hat{r}(t) = \int_0^\infty \delta(a) \Omega(a) \hat{i}(t - a) \, da
\]

for \( t \in \mathbb{R} \). Substituting the fourth equation of (22) into the third one, we have

\[
\hat{i}(t) = (\beta S(t) + \beta_1 V(t)) \int_0^\infty p(a) \Omega(a) \hat{i}(t - a) \, da + \int_0^\infty \eta(b) \Gamma(b) \hat{r}(t - b) \, db
\]

\[
= (\beta S(t) + \beta_1 V(t)) \int_0^\infty p(a) \Omega(a) \hat{i}(t - a) \, da \\
+ \int_0^\infty \eta(b) \Gamma(b) \int_0^\infty \delta(a) \Omega(a) \hat{i}(t - a - b) \, da \, db
\]

for \( t \in \mathbb{R} \).

**Lemma 4.2:** For a total \( \Phi \)-trajectory \( \phi \), \( S(t) \) is strictly positive on \( \mathbb{R} \) and \( \hat{i}(t) = 0 \) for all \( t \in \mathbb{R}_+ \) if \( \hat{i}(t) = 0 \) for all \( t \in (-\infty, 0] \).

**Proof:** First, we show that \( S(t) \) is strictly positive on \( \mathbb{R} \). By way of contradiction, if we suppose that \( S(t^*) = 0 \) for some number \( t^* \in \mathbb{R} \). Then \( dS(t^*)/dt = \Lambda > 0 \) follows from
the first equation of (22), which implies that \( S(t^* - \eta) < 0 \) for sufficiently small \( \eta > 0 \) and it contradicts with the fact that the total \( \Phi \)-trajectory \( \phi \) remains in \( \mathcal{V} \). This completes the proof of \( S(t) \) is strictly positive on \( \mathbb{R} \).

Next, we show that \( \hat{i}(t) = 0 \) for all \( t \in \mathbb{R}_+ \) if \( \hat{i}(t) = 0 \) for all \( t \in (-\infty, 0] \). By changing the variables, we can rewrite Equation (24) as

\[
\hat{i}(t) = (\beta S(t) + \beta_1 V(t)) \int_{-\infty}^{t} p(t - a) \Omega(t - a) \hat{i}(a) \, da
+ \int_{-\infty}^{t} \eta(t - \bar{b}) \Gamma(t - \bar{b}) \int_{-\infty}^{\bar{b}} \delta(b - \bar{c}) \Omega(b - \bar{c}) \hat{i}(c) \, dc \, db
\]

for \( t \in \mathbb{R} \). If \( \hat{i}(t) = 0 \) for all \( t \in (-\infty, 0] \) then

\[
\hat{i}(t) \leq (\beta \bar{S} + \beta_1 \bar{V}) \bar{p} \int_{0}^{t} \hat{i}(a) \, da + \bar{\eta} \bar{\delta} \int_{0}^{t} \int_{0}^{\bar{b}} \hat{i}(c) \, dc \, db \quad \text{for } t \in \mathbb{R},
\]

where \( \bar{S} \) and \( \bar{V} \) are upper bounds for \( S \) and \( V \), respectively. This is a Gronwall-like inequality and hence \( \hat{i}(t) = 0 \) for all \( t \in \mathbb{R}_+ \). In fact, let

\[
\hat{I}(t) := \int_{0}^{t} \hat{i}(a) \, da + \int_{0}^{t} \int_{0}^{\bar{b}} \hat{i}(c) \, dc \, db, \quad t \in \mathbb{R}_+.
\]

Then, for \( t \in \mathbb{R}_+ \),

\[
\frac{d\hat{I}(t)}{dt} = \hat{i}(t) + \int_{0}^{t} \hat{i}(a) \, da
\leq (\beta \bar{S} + \beta_1 \bar{V}) \bar{p} \int_{0}^{t} \hat{i}(a) \, da + \bar{\eta} \bar{\delta} \int_{0}^{t} \int_{0}^{\bar{b}} \hat{i}(c) \, dc \, db + \int_{0}^{t} \hat{i}(a) \, da
\leq \max(\bar{\eta} \bar{\delta}, (\beta \bar{S} + \beta_1 \bar{V}) \bar{p} + 1) \hat{I}(t),
\]

which implies

\[
\hat{I}(t) \leq \hat{I}(0) e^{\max(\bar{\eta} \bar{\delta}, (\beta \bar{S} + \beta_1 \bar{V}) \bar{p} + 1)t} = 0, \quad t \in \mathbb{R}_+.
\]

According to Lemma 4.2, we have the following result that a total \( \Phi \)-trajectory \( \phi \) enjoys.

**Lemma 4.3:** For a total \( \Phi \)-trajectory \( \phi \), \( \hat{i} \) is strictly positive or identically zero on \( \mathbb{R} \).

**Proof:** From Lemma 4.2, we can conclude that \( \hat{i}(t) = 0 \) for all \( t \geq t^* \) if \( \hat{i}(t) = 0 \) for all \( t \leq t^* \) by performing appropriate shifts, where \( t^* \in \mathbb{R} \) is arbitrary. Hence, we have that either \( \hat{i} \) is identically zero on \( \mathbb{R} \) or there exists a decreasing sequence \( \{t_j\}_{j=1}^{\infty} \) such that \( t_j \to -\infty \) as \( j \to \infty \) and \( \hat{i}(t_j) > 0 \). In the latter case, letting \( \hat{i}_j(t) = \hat{i}(t + t_j) \) for \( t \geq 0 \), we have from
Equation (23) that, for \( t \in \mathbb{R}_+ \),
\[
    \hat{i}_j(t) \geq (\beta_\Sigma + \beta_1 V) \int_0^t p(a) \Omega(a) \hat{i}_j(t-a) \, da + \hat{j}_j(t),
\]
where \( \Sigma = \inf_{t \in \mathbb{R}} S(t) > 0, V = \inf_{t \in \mathbb{R}} V(t) > 0 \) and
\[
    \hat{j}_j(t) = (\beta S(t + t_j) + \beta_1 V(t + t_j)) \int_t^\infty p(a) \Omega(a) \hat{i}_j(t-a) \, da \\
    + \int_0^\infty \eta(b) \Gamma(b) \int_0^\infty \delta(a) \Omega(a) \hat{i}_j(t-a-b) \, da \, db.
\]
Since \( \hat{j}_j(0) = \hat{i}(t_j) > 0 \) and \( \hat{j}_j(t) \) is continuous at 0, it follows from Corollary B.6 of Smith and Thieme [21] that there exists a number \( t^* > 0 \) (which depends only on \( (\beta_\Sigma + \beta_1 V)p(a)\Omega(a) \)) such that \( \hat{i}_j(t) > 0 \) for all \( t > t^* \) or \( \hat{i}(t) > 0 \) for all \( t > t^* + r_j \). Since \( t_j \to -\infty \) as \( j \to \infty \), we obtain that \( \hat{i}(t) > 0 \) for all \( t \in \mathbb{R} \) by letting \( j \to \infty \). This completes the proof.

According to Theorem 3.2, Lemmas 4.1–4.3, and the Lipschitz continuity of \( \hat{i} \) (which immediately follows from Proposition 2.3), we can apply results as in [21, Theorem 5.2] to \( \Phi \). The precise results are as follows.

**Theorem 4.1:** If \( \mathcal{R}_0 > 1 \), then the semiflow \( \Phi \) is uniformly (strongly) \( \rho \)-persistent.

When \( \mathcal{R}_0 > 1 \), the uniform persistence in \( \mathcal{Y}_0 \) of (1) immediately follows from Theorem 4.1. In fact, it follows from Theorem 15 that \( \|i(t, \cdot)\|_{L^1} \geq \int_0^t \hat{i}(t-a) \Omega(a) \, da \) and hence from a variation of the Lebesgue–Fatou lemma [20, Section B.2], we get
\[
    \liminf_{t \to \infty} \|i(t, \cdot)\|_{L^1} \geq \hat{\iota}^\infty \int_0^\infty \Omega(a) \, da,
\]
where \( \hat{\iota}^\infty = \liminf_{t \to \infty} \hat{i}(t) \). Under Theorem 4.1, there exists a positive constant \( \epsilon > 0 \) such that \( \hat{\iota}^\infty > \epsilon \) if \( \mathcal{R}_0 > 1 \) and hence the persistence of \( i(t, a) \) with respect to \( \|\cdot\|_{L^1} \) follows. By a similar argument, we can prove that \( S(t) \) and \( V(t) \) are persistent with respect to \( \|\cdot\| \) and \( r(t, \cdot) \) is persistent with respect to \( \|\cdot\|_{L^1} \). To summarize, we get the following result.

**Theorem 4.2:** If \( \mathcal{R}_0 > 1 \), then the semiflow \( \Phi \) is uniformly persistent in \( \mathcal{Y}_0 \), that is, there exists a constant \( \epsilon > 0 \) such that, for each \( X_0 \in \mathcal{Y}_0 \),
\[
    \liminf_{t \to \infty} S(t) \geq \epsilon, \quad \liminf_{t \to \infty} V(t) \geq \epsilon, \quad \liminf_{t \to \infty} \|i(t, \cdot)\|_{L^1} \geq \epsilon, \quad \liminf_{t \to \infty} \|r(t, \cdot)\|_{L^1} \geq \epsilon.
\]

**5. Global stability of the infection-free equilibrium**

The aim of this section is to establish the global stability of the infection-free equilibrium \( E^0 \) when \( \mathcal{R}_0 < 1 \). We first show the local stability of \( E^0 \).
Theorem 5.1: The infection-free equilibrium $E^0$ is locally asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.

Proof: Linearizing (1) at $E^0$ by introducing the perturbation variables

\[
\begin{align*}
    x_1(t) &= S(t) - S^0, \quad x_2(t) = V(t) - V^0, \quad x_3(t, a) = i(t, a), \quad x_4(t, a) = r(t, a),
\end{align*}
\]

we obtain the linearized system

\[
\begin{align*}
    \frac{dx_1(t)}{dt} &= -(\mu + \alpha)x_1(t) - \beta S^0 \int_0^\infty p(a)x_3(t, a) \, da,
    \\
    \frac{dx_2(t)}{dt} &= \alpha x_1(t) - \beta_1 V^0 \int_0^\infty p(a)x_3(t, a) \, da - \mu x_2(t),
    \\
    \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) x_3(t, a) &= -(\mu + \delta(a))x_3(t, a),
    \\
    \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial b} \right) x_4(t, b) &= -(\mu + \eta(b))x_4(t, b),
\end{align*}
\]

\[
\begin{align*}
    x_3(t, 0) &= (\beta S^0 + \beta_1 V^0) \int_0^\infty p(a)x_3(t, a) \, da + \int_0^\infty \eta(b)x_4(t, b) \, db,
    \\
    x_4(t, 0) &= \int_0^\infty \delta(a)x_3(t, a) \, da.
\end{align*}
\]

Set $x_1(t) = x_1^0 e^{\lambda t}$, $x_2(t) = x_2^0 e^{\lambda t}$, $x_3(t, a) = x_3^0(a) e^{\lambda t}$, and $x_4(t, b) = x_4^0(b) e^{\lambda t}$. After some calculation, we get the characteristic equation at $E^0$, which is

\[
C(\lambda) \triangleq (\beta S^0 + \beta_1 V^0) \int_0^\infty p(a) e^{-\lambda a} \Omega(a) \, da
\]

\[
\quad + \int_0^\infty \eta(b) e^{-\lambda b} \Gamma(b) \, db \int_0^\infty \delta(a) e^{-\lambda a} \Omega(a) \, da - 1 = 0.
\]

First, suppose that $\mathcal{R}_0 > 1$. Then $C(0) = \mathcal{R}_0 - 1 > 0$. This, combined with $\lim_{\lambda \to \infty} C(\lambda) = -1$ and the intermediate value theorem, implies that $C(\lambda)$ has a positive zero. Therefore, $E^0$ is unstable if $\mathcal{R}_0 > 1$. Now assume that $\mathcal{R}_0 < 1$. We claim that $C(\lambda)$ has no zeros with nonnegative real parts. Otherwise, $C(\lambda^*) = 0$ for some $\lambda^* \in \mathbb{C}$ with $\text{Re}(\lambda^*) \geq 0$. Then, we have

\[
1 = |C(\lambda^*)| \leq \mathcal{R}_0,
\]

a contradiction to $\mathcal{R}_0 < 1$. This proves the claim and hence $E^0$ is locally asymptotically stable if $\mathcal{R}_0 < 1$.

In the following, we obtain the global attractivity of $E^0$ by using the Lyapunov technique. During the discussion, we need the function $g : x \to x - 1 - \ln x, x \in (0, \infty)$. It is easy to check that $g$ is continuous and concave up and has the global minimum 0 only at $x = 1$.

Theorem 5.2: The infection-free equilibrium $E^0$ of Equation (1) is globally asymptotically stable if $\mathcal{R}_0 < 1$. 

Proof: By Theorem 5.1, it suffices to show that $E^0$ is globally attractive and we show it by the Lyapunov method. Consider the Lyapunov functional candidate,

$$L(t) = L_1(t) + L_2(t) + L_3(t),$$

where

$$L_1(t) = S^0g\left(\frac{S(t)}{S^0}\right) + V^0g\left(\frac{V(t)}{V^0}\right), \quad L_2(t) = \int_0^\infty \phi_1(a)i(t, a) \, da,$$

$$L_3(t) = \int_0^\infty \psi(b)r(t, b) \, db.$$ 

Here, the nonnegative kernel functions $\phi(a)$ and $\psi(b)$ will be determined later. The derivative of $L_1$ along the solutions of Equation (1) is calculated as follows:

$$\frac{dL_1(t)}{dt} = \left(1 - \frac{S^0}{S}\right)\left(\Lambda - \beta S \int_0^\infty p(a)i(t, a) \, da - (\mu + \alpha)S\right)$$

$$+ \left(1 - \frac{V^0}{V}\right)\left(\alpha S - \beta_1 V \int_0^\infty p(a)i(t, a) \, da - \mu V\right)$$

$$= \left(1 - \frac{S^0}{S}\right)(\mu + \alpha)S^0 - \beta S \int_0^\infty p(a)i(t, a) \, da - (\mu + \alpha)S$$

$$+ \left(1 - \frac{V^0}{V}\right)(\alpha S - \beta_1 V \int_0^\infty p(a)i(t, a) \, da - \mu V)$$

$$= \mu S^0\left(2 - \frac{S}{S^0} - \frac{S^0}{S}\right) + \alpha S^0\left(3 - \frac{V}{V^0} - \frac{SV^0}{S^0V} - \frac{S^0}{S}\right)$$

$$+ (\beta S^0 + \beta_1 V^0) \int_0^\infty p(a)i(t, a) \, da - (\beta S + \beta_1 V) \int_0^\infty p(a)i(t, a) \, da$$

$$= dS^0\left(2 - \frac{S}{S^0} - \frac{S^0}{S}\right) + \alpha S^0\left(3 - \frac{V}{V^0} - \frac{SV^0}{S^0V} - \frac{S^0}{S}\right) + (\beta S^0)$$

$$+ \beta_1 V^0) \int_0^\infty p(a)i(t, a) \, da - i(t, 0) + \int_0^\infty \eta(b)r(t, b) \, db.$$ 

Using integration by parts, we have

$$\frac{dL_2(t)}{dt} = \int_0^\infty \phi_1(a)\frac{\partial i(t, a)}{\partial t} \, da = -\int_0^\infty \phi_1(a)\left[(\mu + \delta(a))i(t, a) + \phi_1(a)\right] \, da$$

$$= -\phi_1(a)i(t, a)|_0^\infty + \int_0^\infty \phi_1'(a)i(t, a) \, da - \int_0^\infty \phi_1(a)(\mu + \delta(a))i(t, a) \, da$$

$$= \phi_1(0)i(t, 0) + \int_0^\infty (\phi_1'(a) - \phi_1(a)(\mu + \delta(a)))i(t, a) \, da.$$ 

Similarly,

$$\frac{dL_3(t)}{dt} = \psi(0)r(t, 0) + \int_0^\infty (\psi'(b) - \psi(b)(\mu + \eta(b)))r(t, b) \, db.$$
Now we choose
\[ \psi(b) = \int_b^\infty \eta(\mu) e^{-\int_b^\mu (\mu + \eta(\omega)) \, d\omega} \, d\mu \]
and
\[ \phi_1(a) = \int_a^\infty [(\beta S^0 + \beta_1 V^0) p(u) + \psi(0) \delta(u)] e^{-\int_a^\mu (\mu + \delta(\omega)) \, d\omega} \, d\mu. \]
Then \( \psi(0) = \int_0^\infty \eta(b) \Gamma(b) \, db = L \), \( \phi_1(0) = \Re_0 \), and \( \psi \) and \( \phi_1 \) satisfy
\[
\psi'(b) - \psi(b)(\mu + \eta(b)) + \eta(b) = 0, \\
\phi_1'(a) - \phi_1(a)(\mu + \delta(a)) + (\beta S^0 + \beta_1 V^0) p(a) + \psi(0) \delta(a) = 0.
\]
Therefore, we have
\[
\frac{d\mathcal{L}(t)}{dt} = dS^0 \left( 2 - \frac{S}{S^0} - \frac{S^0}{S} \right) + \alpha S^0 \left( 3 - \frac{V}{V^0} - \frac{S V^0}{S^0 V} - \frac{S^0}{S} \right) \\
+ (\beta S^0 + \beta_1 V^0) \int_0^\infty p(a) i(t, a) \, da - i(t, 0) + \int_0^\infty \eta(b) r(t, b) \, db \\
+ \phi(0) i(t, 0) + \int_0^\infty (\phi'(a) - \phi(a)(\mu + \delta(a))) i(t, a) \, da \\
+ \psi(0) r(t, 0) + \int_0^\infty (\psi'(b) - \psi(b)(\mu + \eta(b))) r(t, b) \, db \\
= dS^0 \left( 2 - \frac{S}{S^0} - \frac{S^0}{S} \right) + \alpha S^0 \left( 3 - \frac{V}{V^0} - \frac{S V^0}{S^0 V} - \frac{S^0}{S} \right) + (\Re_0 - 1)i(t, 0) \\
\leq 0.
\]
Notice that \( d\mathcal{L}(t)/dt = 0 \) implies that \( S = S^0 \) and \( V = V^0 \). It can be verified that the largest invariant set where \( d\mathcal{L}(t)/dt = 0 \) is the singleton \( \{E^0\} \). Therefore, by the invariance principle, \( E^0 \) is globally attractive.

6. Global attractiveness of the endemic equilibrium

As at the beginning of Section 5, we can get the characteristic equation at \( E^* \), which is
\[
0 = \begin{vmatrix}
\lambda + \mu + \alpha + \beta H^*(0) & 0 & \beta S^* \hat{H}(\lambda) & 0 \\
-\alpha & \lambda + \mu + \beta_1 H^*(0) & \beta_1 V^* \hat{H}(\lambda) & 0 \\
-\beta H^*(0) & -\beta_1 H^*(0) & 1 - (\beta S^* + \beta_1 V^*) \hat{H}(\lambda) & -\hat{L}(\lambda) \\
0 & 0 & -\hat{K}(\lambda) & 1
\end{vmatrix},
\]
where \( \hat{H}(\lambda) = \int_0^\infty e^{-\lambda a} p(a) \Omega(a) \, da \), \( \hat{K}(\lambda) = \int_0^\infty e^{-\lambda a} \delta(a) \Omega(a) \, da \), and \( \hat{L}(\lambda) = \int_0^\infty e^{-\lambda b} \eta(b) \Gamma(b) \, db \). Though we believe that when \( \Re_0 > 1 \), all roots of Equation (24) have negative real parts and hence \( E^* \) is locally asymptotically stable. Unfortunately, it is difficult to confirm it. In the following, we just show that \( E^* \) is globally attractive. To achieve this, we need the following properties of solutions to Equation (1).
Lemma 6.1: Suppose that \( \mathcal{R}_0 > 1 \). Then, for any solution \((S(t), V(t), i(t, \cdot), r(t, \cdot))\) of Equation (1) with the initial value in \( \mathcal{Y}_0 \), the following equalities hold:

\[
\int_0^\infty \delta(a)i^*(a) \left[ 1 - \frac{i(t, a)r^*(0)}{i^*(a)r(t, 0)} \right] \, da = 0, \tag{25}
\]

\[
\beta S^* \int_0^\infty p(a)i^*(a) \left[ 1 - \frac{i(t, a)i^*(0)S}{i^*(a)i(t, 0)S^*} \right] \, da + \beta_1 V^* \int_0^\infty p(a)i^*(a) \left[ 1 - \frac{i(t, a)i^*(0)V}{i^*(a)i(t, 0)V^*} \right] \, da
\]

\[
+ \int_0^\infty \eta(b)r^*(b) \left[ 1 - \frac{i^*(0)r(t, b)}{i(t, a)r^*(b)} \right] \, db = 0. \tag{26}
\]

**Proof:** The proofs of Equations (25) and (26) are quite similar and we only give that for Equation (25) as an illustration. In fact,

\[
\int_0^\infty \delta(a)i^*(a) \left[ 1 - \frac{i(t, a)r^*(0)}{i^*(a)r(t, 0)} \right] \, da = \int_0^\infty \delta(a) i^*(a) \, da - \frac{r^*(0)}{r(t, 0)} \int_0^\infty \delta(a) i(t, a) \, da.
\]

This, combined with Equations (2) and (8), immediately gives Equation (25).

Theorem 6.1: The unique endemic equilibrium \( E^* = (S^*, V^*, i^*, r^*) \) of Equation (1) defined by Equation (11) is globally attractive when \( \mathcal{R}_0 > 1 \).

**Proof:** Define

\[
G[x, y] = x - y - y \ln \frac{x}{y}.
\]

It is easy to see that \( G \) is non-negative on \((0, \infty) \times (0, \infty)\) with the minimum value 0 only when \( x = y \). Furthermore, it is easy to verify that \( xG_x[x, y] + yG_y[x, y] = G[x, y] \).

Consider a candidate Lyapunov functional,

\[
\mathcal{H}(t) = \mathcal{H}_1(t) + \mathcal{H}_2(t) + \mathcal{H}_3(t),
\]

where \( \mathcal{H}_1(t) = G[S(t), S^*] + G[V(t), V^*], \mathcal{H}_2(t) = \int_0^\infty \phi_2(a)G[i(t, a), i^*(a)] \, da, \mathcal{H}_3(t) = \int_0^\infty \psi(b)G[r(t, b), r^*(b)] \, db \). Similar to the one as in the Proof of Theorem 5.2, \( \psi(b) \) and \( \phi_2(a) \) are given as

\[
\psi(b) = \int_b^\infty \eta(u) e^{-\int_u^\infty (\mu + \eta(\omega)) \, d\omega} \, du
\]

and

\[
\phi_2(a) = \int_a^\infty \left[ (\beta S^* + \beta_1 V^*) p(u) + \psi(0) \delta(u) \right] e^{-\int_u^\infty (\mu + \delta(\omega)) \, d\omega} \, du.
\]

It is easy to see that \( \phi_2(0) = 1, \psi(0) = \int_0^\infty \eta(b) \Gamma(b) \, db = L, \) and

\[
\psi'(b) - \psi(b)(\mu + \eta(b)) = -\eta(b), \tag{27}
\]

\[
\phi_2'(a) - \phi_2(a)(\mu + \delta(a)) = -[(\beta S^* + \beta_1 V^*) p(a) + \psi(0) \delta(a)]. \tag{28}
\]

In what follows, we shall calculate the derivative of \( \mathcal{H}(t) \) along solutions of Equation (1).
Firstly, differentiating $H_1$ along solutions of Equation (1) yields

$$\frac{dH_1(t)}{dt} = \left(1 - \frac{S^*}{S}\right) \left(\Lambda - \beta S \int_0^\infty p(a)i(t, a) \, da - (\mu + \alpha)S\right)$$

$$+ \left(1 - \frac{V^*}{V}\right) \left(\alpha S - \beta_1 V \int_0^\infty p(a)i(t, a) \, da - \mu V\right)$$

$$= \left(1 - \frac{S^*}{S}\right) \left((\mu + \alpha)S^* + \beta S^* \int_0^\infty p(a)i^*(a) \, da\right)$$

$$- \beta S \int_0^\infty p(a)i(t, a) \, da - (\mu + \alpha)S$$

$$+ \left(1 - \frac{V^*}{V}\right) \left(\alpha S - \beta_1 V \int_0^\infty p(a)i(t, a) \, da - \mu V\right)$$

$$= \mu S^* \left(2 - \frac{S - S^*}{S}\right) + \mu V^* \left(3 - \frac{V}{V^*} - \frac{SV^* - S^*}{S^* V}\right)$$

$$+ \beta S^* \int_0^\infty \left(1 - \frac{S^* i^*(a) - S^* S^*}{S^* i^*(a) i^*(a) + \frac{i(a)}{t^*(a)}}\right) p(a)i^*(a) \, da$$

$$+ \beta_1 V^* \int_0^\infty \left(2 - \frac{V i(t, a)}{V^* i^*(a) - \frac{S^* SV^*}{S^* V} + \frac{i(a)}{t^*(a)}}\right) p(a)i^*(a) \, da.$$

Secondly, using Equation (5), we have

$$H_2(t) = \int_0^t \phi_2(a) G[i(t - a, 0)\Omega(a), i^*(a)] \, da$$

$$+ \int_t^\infty \phi_2(a) G[i_0(a - t) e^{-\int_{a-t}^a (\mu + \delta(\omega)) \, d\omega}, i^*(a)] \, da$$

$$= \int_0^t \phi_2(t - r) G[i(r, 0)\Omega(t - r), i^*(t - r)] \, dr$$

$$+ \int_t^\infty \phi_2(t + r) G[i_0(r) e^{-\int_{t+r}^t (\mu + \delta(\omega)) \, d\omega}, i^*(t + r)] \, dr.$$

Differentiating $H_2(t)$ and using $i^*(a) = i^*(0) e^{-\int_0^a (\mu + \delta(\omega)) \, d\omega}$ produce

$$\frac{dH_2(t)}{dt} = \phi_2(0) G[i(t, 0), i^*(0)] + \int_0^t \phi_2(t - r) G[i(r, 0) e^{-\int_{t-r}^r (\mu + \delta(\omega)) \, d\omega}, i^*(t - r)] \, dr$$

$$- \int_0^t \phi_2(t - r)(\mu + \delta(t - r))[i(r, 0) e^{-\int_0^r (\mu + \delta(\omega)) \, d\omega}$$

$$\times G_x[i(r, 0) e^{-\int_0^r (\mu + \delta(\omega)) \, d\omega}, i^*(t - r)]$$

$$+ i^*(t - r)G_y[i(r, 0) e^{-\int_0^r (\mu + \delta(\omega)) \, d\omega}, i^*(t - r)] \, dr$$
Similarly, using Equations (6) and (27), we have
\[ x = G [l_0(t) e^{-\int_r^{t+r} \mu + \delta(\omega)} d\omega}, i^*(t + r)]
\[ G_x [l_0(t) e^{-\int_r^{t+r} \mu + \delta(\omega)} d\omega}, i^*(t + r)]
\[ + i^*(t + r) G_y [l_0(t) e^{-\int_r^{t+r} \mu + \delta(\omega)} d\omega}, i^*(t + r)] \] 
\[ = G[i(t, 0), i^*(0)] + \int_0^\infty [\phi_2^*(a) - \phi_2(a) (\mu + \delta(a))] G[i(t, a), i^*(a)] \] 
\[ = (\beta S^* + \beta V^*) \int_0^\infty \bar{p}(a) \left( i^*(a) - i(t, a) + \frac{i(t, a)}{i^*(a)} \right) da \]
\[ + L \int_0^\infty \delta(a) \left( i^*(a) - i(t, a) + \frac{i(t, a)}{i^*(a)} \right) da \]
\[ + i(t, 0) - i^*(0) - i^*(0) \ln \frac{i(t, 0)}{i^*(0)} \quad \text{(using Equation (28)).} \]

The second last equality follows from (5) and the fact that \( xG_x[x, y] + yG_y[x, y] = G[x, y] \). Similarly, using Equations (6) and (27), we have
\[ \frac{dH_3(t)}{dt} = \psi(0) G[t(t, 0), r^*(0)] + \int_0^\infty [\psi'(b) - \psi(b) (\mu + \eta(b))] G[r(t, b), r^*(b)] db \]
\[ = \int_0^\infty \eta(b) \left( r^*(b) - r(t, b) + r^*(b) \ln \frac{r(t, b)}{r^*(b)} \right) db \]
\[ + L \left( r(t, 0) - r^*(0) - r^*(0) \ln \frac{r(t, 0)}{r^*(0)} \right). \]

It follows that
\[ \frac{dH}{dt} = \mu S^* \left( 2 - \frac{S}{S^*} - \frac{S^*}{S} \right) + \mu V^* \left( 3 - \frac{V}{V^*} - \frac{SV^*}{SV} - \frac{S^*}{S} \right) \]
\[ + \beta S^* \int_0^\infty \left( 2 - \frac{Si(t, a)}{S^* i^*(a)} - \frac{S^*}{S} + \ln \frac{i(t, a)}{i^*(a)} \right) \bar{p}(a) i^*(a) \] 
\[ + \beta V^* \int_0^\infty \left( 3 - \frac{Vi(t, a)}{V^* i^*(a)} - \frac{S^*}{S} + \ln \frac{i(t, a)}{i^*(a)} \right) \bar{p}(a) i^*(a) \]
\[ + L \int_0^\infty \delta(a) \left( i^*(a) - i(t, a) + \frac{i(t, a)}{i^*(a)} \right) da \]
\[ + i(t, 0) - i^*(0) - i^*(0) \ln \frac{i(t, 0)}{i^*(0)} \]
\[ + \int_0^\infty \eta(b) \left( r^*(b) - r(t, b) + r^*(b) \ln \frac{r(t, b)}{r^*(b)} \right) db \]
\[ + L \left( r(t, 0) - r^*(0) - r^*(0) \ln \frac{r(t, 0)}{r^*(0)} \right). \]
Using Equation (2) and the fifth and sixth equations of (8), we have

\[
\frac{dH(t)}{dt} = \mu S^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \mu V^* \left(3 - \frac{V}{V^*} - \frac{SV^*}{S^*V} - \frac{S^*}{S}\right)
\]

\[
+ \beta S^* \int_0^\infty \left(2 - \frac{Si(t,a)}{S^*i^*(a)} - \frac{S^*}{S} + \ln \frac{i(t,a)}{i^*(a)}\right) p(a)i^*(a) \, da
\]

\[
+ \beta_1 V^* \int_0^\infty \left(3 - \frac{Vi(t,a)}{V^*i^*(a)} - \frac{S^*}{S} + \ln \frac{i(t,a)}{i^*(a)}\right) p(a)i^*(a) \, da
\]

\[
+ \beta S \int_0^\infty p(a)i(t,a) \, da + \beta_1 V \int_0^\infty p(a)i(t,a) \, da + \int_0^\infty \eta(b)r(t,b) \, db \tag{29}
\]

\[
- \left(\beta S^* \int_0^\infty p(a)i^*(a) \, da + \beta_1 V^* \int_0^\infty p(a)i^*(a) \, da + \int_0^\infty \eta(b)r^*(b) \, db\right)
\]

\[
\times \left(1 + \ln \frac{i(t,0)}{i^*(0)}\right) + L \int_0^\infty \delta(a) \left(i^*(a) - i(t,a) + i^*(a) \ln \frac{i(t,a)}{i^*(a)}\right) \, da
\]

\[
+ L \int_0^\infty \delta(a) \left(i(t,a) - i^*(a) - i^*(a) \ln \frac{r(t,0)}{r^*(0)}\right) \, da
\]

\[
+ \int_0^\infty \eta(b) \left(r^*(b) - r(t,b) + r^*(b) \ln \frac{r(t,b)}{r^*(b)}\right) \, db.
\]

Collecting the terms of Equation (29) yields

\[
\frac{dH(t)}{dt} = \mu S^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \mu V^* \left(3 - \frac{V}{V^*} - \frac{SV^*}{S^*V} - \frac{S^*}{S}\right)
\]

\[
+ \beta S^* \int_0^\infty \left(1 - \frac{S^*}{S} + \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)}\right) p(a)i^*(a) \, da
\]

\[
+ \beta_1 V^* \int_0^\infty \left(2 - \frac{SV^*}{S^*V} + \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)}\right) p(a)i^*(a) \, da \tag{30}
\]

\[
+ L \int_0^\infty \delta(a)i^*(a) \left(\ln \frac{i(t,a)}{i^*(a)} - \ln \frac{r(t,0)}{r^*(0)}\right) \, da
\]

\[
+ \int_0^\infty \eta(b)r^*(b) \left(\ln \frac{r(t,b)}{r^*(b)} - \ln \frac{i(t,0)}{i^*(0)}\right) \, db.
\]

It follows from Lemma 6.1 that putting Equations (25) and (26) into Equation (27) gives

\[
\frac{dH(t)}{dt} = \mu S^* \left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right) + \mu V^* \left(3 - \frac{V}{V^*} - \frac{SV^*}{S^*V} - \frac{S^*}{S}\right)
\]

\[
+ \beta S^* \int_0^\infty \left(1 - \frac{S^*}{S} + \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)}\right) p(a)i^*(a) \, da
\]

\[
+ \beta_1 V^* \int_0^\infty \left(2 - \frac{SV^*}{S^*V} + \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)}\right) p(a)i^*(a) \, da
\]

\[
+ L \int_0^\infty \delta(a)i^*(a) \left(\ln \frac{i(t,a)}{i^*(a)} - \ln \frac{r(t,0)}{r^*(0)}\right) \, da
\]

\[
+ \int_0^\infty \eta(b)r^*(b) \left(\ln \frac{r(t,b)}{r^*(b)} - \ln \frac{i(t,0)}{i^*(0)}\right) \, db.
\]
\[+ \beta_1 V^* \int_0^\infty \left( 1 - \frac{S^*}{S} + \ln \frac{S^*}{S} + 1 - \frac{SV^*}{S^*V} + \ln \frac{SV^*}{S^*V} \right)
+ 1 - \frac{i(t,a)i^*(0)V}{i^*(a)i(t,0)V^*} + \ln \frac{i(t,a)i^*(0)V}{i^*(a)i(t,0)V^*} \right) \mu^* (a) \right) da
+ \frac{L}{i^*(a)i(t,0)\ln t} \int_0^\infty \delta(a) \right) \left( 1 - \frac{r(t,b)i^*(0)}{r^*(b)i(t,0)} + \ln \frac{r(t,b)i^*(0)}{r^*(b)i(t,0)} \right) db
- \left[ \beta S^* \int_0^\infty \left( 1 - \frac{i(t,a)i^*(0)V}{i^*(a)i(t,0)V^*} \right) \mu^* (a) \right) da
+ \beta_1 V^* \int_0^\infty \left( 1 - \frac{i(t,a)i^*(0)V}{i^*(a)i(t,0)V^*} \right) \mu^* (a) \right) da
+ \beta S^* \int_0^\infty \left( 1 - \frac{i(t,a)i^*(0)V}{i^*(a)i(t,0)V^*} \right) \mu^* (a) \right) da
+ \beta_1 V^* \int_0^\infty \left( 1 - \frac{i(t,a)i^*(0)V}{i^*(a)i(t,0)V^*} \right) \mu^* (a) \right) da
+ \int_0^\infty \eta(b) \mu^* (b) \left( 1 - \frac{r(t,b)i^*(0)}{r^*(b)i(t,0)} + \ln \frac{r(t,b)i^*(0)}{r^*(b)i(t,0)} \right) db
+ L \int_0^\infty \delta(a) \mu^* (a) \left( 1 - \frac{i(t,a)i^*(0)}{i^*(a)i(t,0)} + \ln \frac{i(t,a)i^*(0)}{i^*(a)i(t,0)} \right) da \right].
\]

Note that \(1 - x - \ln x \leq 0\) for all \(x > 0\) with the equality holds only when \(x = 1\). It follows that
\[
\frac{dH(t)}{dt} = \mu S^* \left( 2 - \frac{S}{S^*} - \frac{S^*}{S} \right) + \mu V^* \left( 3 - \frac{V}{V^*} - \frac{SV^*}{S^*V} - \frac{S^*}{S} \right)
- \beta S^* \int_0^\infty g \left( \frac{S^*}{S}, \frac{i(t,a)i^*(0)}{i^*(a)i(t,0)V^*} \right) \mu^* (a) \right) da
- \beta S^* \int_0^\infty g \left( \frac{i(t,a)i^*(0)}{i^*(a)i(t,0)V^*} \right) \mu^* (a) \right) da
- \beta_1 V^* \int_0^\infty g \left( \frac{i(t,a)i^*(0)V}{i^*(a)i(t,0)V^*} \right) \mu^* (a) \right) da
- \int_0^\infty \eta(b) \mu^* (b) g \left( \frac{r(t,b)i^*(0)}{r^*(b)i(t,0)} \right) db - L \int_0^\infty \delta(a) \mu^* (a) g \left( \frac{i(t,a)i^*(0)}{i^*(a)i(t,0)} \right) da \leq 0
\]
and \(dH(t)/dt = 0\) implies that \(S = S^*\), \(V = V^*\) and
\[
\frac{i(t,a)}{i^*(a)} = \frac{i(t,0)}{i^*(0)} = \frac{r(t,b)}{r^*(b)} = \frac{r(t,0)}{r^*(0)} \quad \text{for all} \ a, b \geq 0.
\]
It is not difficult to check that the largest invariant subset where \(dH(t)/dt = 0\) is the singleton \(\{E^*\}\). By the invariance principle, \(E^*\) is globally attractive. \(\blacksquare\)

7. Discussion

In this paper, we first integrate solutions along the characteristic line to obtain an equivalent integral equation, which is developed by Webb [32] and Walker [25] for age-dependent
models. Secondly, the asymptotic smoothness of the semiflow generated by the system is proved by the method in [21]. Thirdly, in order to make use of the invariance principle, we establish the uniform persistence and the existence of a compact global attractor of the system similarly as in [16, 30, 31]. Finally, the global stability of the equilibria is derived by constructing suitable Volterra-type Lyapunov functionals. Theorems 5.2 and 6.1 imply that the basic reproduction number \( \mathcal{R}_0 \) completely governs the global dynamics of system (1).

In the following, we discuss the biological implications of the continuous vaccination strategy. According to Liu et al. [10] and Wang et al. [26, 27], we also assume that \( \beta_1 \leq \beta \) as the vaccinees may have some partial immunity and hence the effective contacts with infected individuals becomes smaller.

**Case for vaccinees without infection.** Let \( \beta_1 = 0 \) and denote

\[
\mathcal{R}_1 := \mathcal{R}_0|_{\beta_1=0} = \frac{\beta \Lambda}{\mu + \alpha} H + LK.
\]

It is easy to see that \( \mathcal{R}_1 \leq \mathcal{R}_0 \). In this case,

\[
\frac{\partial \mathcal{R}_1}{\partial \alpha} < 0 \quad \text{and} \quad \lim_{\alpha \to \infty} \mathcal{R}_1 = LK < 1.
\]

According to Theorems 5.2 and 6.1, the disease always can be eradicated by some suitable vaccination strategy.

**Case for vaccinees with infection.** Note that

\[
\frac{\partial \mathcal{R}_0}{\partial \alpha} = -\frac{\Lambda \mu H(\beta - \beta_1)}{\mu(\mu + \alpha)^2} < 0,
\]

which implies that vaccination always has a positive effect on disease control by increasing the vaccination rate. It is obvious to see that \( \mathcal{R}_0 \leq \mathcal{R}_0|_{\alpha=0} := \mathcal{R}_0 \). Suppose that \( \mathcal{R}_0 > 1 \). Denote \( \lim_{\alpha \to \infty} \mathcal{R}_0 \) by \( \mathcal{R}_2 \), the average number of secondary cases produced by one infected individual introduced to a population with vaccinees only during the lifespan. If \( \mathcal{R}_2 < 1 \), then there is a unique \( \alpha^* \) such that \( \mathcal{R}_0 = 1 \) for \( \alpha = \alpha^* \) and \( \mathcal{R}_0 < 1 \) for \( \alpha > \alpha^* \), which implies that the disease can be eliminated by some suitable vaccination strategies with \( \alpha > \alpha^* \) by Theorem 5.2. If \( \mathcal{R}_2 \geq 1 \), then \( \mathcal{R}_0 > \mathcal{R}_2 \geq 1 \), which implies that disease cannot be eradicated by any vaccination strategies (any values of \( \alpha \)) according to Theorem 6.1. It follows that \( \mathcal{R}_2 < 1 \) is a key condition for disease elimination when vaccinees can be infected (with small \( \beta_1 \)). Thus, neglecting the possibility for vaccinees to be infected will over-estimate the effect of vaccination strategy.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

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