On the twistor space of a quaternionic contact manifold

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Abstract

In this note, we prove that the CR manifold which is induced from the canonical parabolic geometry of a quaternionic contact (qc) manifold via a Fefferman-type construction is equivalent to the CR twistor space of the qc manifold defined by O. Biquard.

1 Introduction

As introduced by O. Biquard in [3], a quaternionic contact (qc) manifold is given by a 4-tuple $\mathcal{M} = (M, \mathcal{D}, [g], \mathcal{Q})$, where $M$ is a manifold of dimension $4n + 3$, $\mathcal{D} \subset TM$ a distribution of co-rank 3; $[g]$ a conformal class of positive-definite Carnot-Carathéodory metrics defined on $\mathcal{D}$; $\mathcal{Q}$ a rank 3 sub-bundle of $\text{End}(\mathcal{D})$ (all $C^\infty$); and where we assume that $\mathcal{Q}$ admits local bases $\{I_1, I_2, I_3\}$ satisfying the quaternion relations (so $I_2 = -I_d$, $I_1 I_2 = -I_2 I_1 = I_3$) and $\mathcal{D}$ is given as the kernel of local 1-forms $\eta^1, \eta^2, \eta^3$, so that the following compatibility relation holds for all $u, v \in \mathcal{D}$, $a = 1, 2, 3$ and some $g \in [g]$:

$$d\eta^a(u, v) = 2g(I_a u, v).$$

In dimension 7, i.e. for $n = 1$, the following integrability condition, due to D. Duchemin [7], will also be assumed: The local 1-forms $\eta^a$ above may be chosen so that the restrictions of the 2-forms $d\eta^a$ to $\mathcal{D}$ form a local oriented orthonormal basis of $\Lambda^2_+ \mathcal{D}^*$, and local vector fields $\xi_1, \xi_2, \xi_3$ (called the Reeb vector fields of the $\eta^a$) exist, which satisfy

$$\xi_a \cdots d\eta^b|_\mathcal{D} = - (\xi_b \cdots d\eta^a|_\mathcal{D},$$

for $a, b = 1, 2, 3$ (in higher dimensions, we always have existence of the Reeb vector fields).

A qc structure is naturally defined on the boundary of the rank one symmetric space $\mathbb{H}^{n+1} = Sp(1, n + 1)/Sp(1)Sp(n + 1)$ (the boundary is diffeomorphic to $S^{4n+3}$), and more generally qc structures can be thought of as the natural geometric structures at “conformal infinity” of asymptotically symmetric quaternionic-Kähler manifolds. Indeed, one of the central results of Biquard’s foundational study [3] (Theorem D) says that any real analytic qc manifold $\mathcal{M}$ can be realised as the conformal infinity of a unique asymptotically symmetric quaternionic-Kähler metric which is real analytic up to the boundary and defined in a neighbourhood of $\mathcal{M}$.

Quaternionic contact structures are the quaternionic analog of Cauchy-Riemann (CR) structures, and there are interesting relations between the two types of geometric structure. An important step in the proof of Biquard’s Theorem D is the construction of a natural CR structure on the total space $Z$ of a 2-sphere bundle naturally associated to a qc structure $\mathcal{M}$. The space $Z$ together with this natural CR structure is called the twistor space of the qc structure $\mathcal{M}$. (For the construction and proofs of naturality and integrability, cf. II.5 of [3]; we briefly recall the definition in Section 3.)

An alternative approach to qc structures is via parabolic geometry: Any qc manifold $\mathcal{M}$ can be canonically identified with a Cartan geometry $(\pi : \mathcal{G} \to \mathcal{M}, \omega)$ of parabolic type $(G, P)$, where $G \cong Sp(1, n + 1)/\{\pm Id\}$ and $P \subset G$ is the parabolic subgroup which is the image under the quotient of the stabiliser in $Sp(1, n + 1)$ of a light-like quaternionic line in $\mathbb{H}^{1,n+1}$. That is, $\pi : \mathcal{G} \to \mathcal{M}$ is a $P$-principal bundle, and $\omega \in \Omega^1(\mathcal{G}; \mathfrak{g})$ is

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a Cartan connection of type \((G, P)\). (This is an application of Theorem 3.1.14 of [3], to which the reader is also referred for background on parabolic geometry; some details of the parabolic structure of a qc manifold are given in Section 2 of [2].)

Using the Cartan geometry, there is an elegant way to associate a natural CR structure to the qc structure \(M\). Namely, with respect to the inclusion \(G \to \tilde{G} := SU(2,2n+2)/\{\pm Id\}\), and for the parabolic subgroup \(P \subset \tilde{G}\) which is the quotient of the stabiliser in \(SU(2,2n+2)\) of a light-like complex line in \(\mathbb{C}^{2,2n+2} \cong \mathbb{H}^{1,n+1}\), we have \(\tilde{G} \cap P \subset P\) and \(G/(\tilde{G} \cap P) = \tilde{G}/P\). These conditions allow one to execute a Fefferman-type construction (cf. 4.5 of [3] for the general procedure, which includes the application to this specific case in 4.5.5): From \((\pi: \tilde{G} \to M, \omega)\), this construction yields a canonical Cartan geometry \((\tilde{\pi}: \tilde{G} \to \tilde{M}, \tilde{\omega})\) of type \((\tilde{G}, \tilde{P})\). A Cartan geometry of the latter type (which is also parabolic) is known to induce a partially integrable CR structure of real signature \((4n+2, 2)\) on the base space \(\tilde{M}\) (some details are recounted in Section 2).

Let us refer to the result as the CR Fefferman space of \(M\). In fact, as a by-product of the proof of the main result in [1] (cf. Theorem 5.1), the Cartan geometry \((\tilde{G}, \tilde{\omega})\) of CR type is both normal and torsion-free, and hence (cf. 4.2.4 of [3]) the induced CR structure is integrable. A natural question is as to the relation between this integrable CR structure and the CR twistor space \(Z\) of \(M\). The purpose of this note is to prove that they coincide, confirming the expectation expressed in 4.5.5 of [3]:

**Theorem A.** Let \(\mathcal{M} = (M, \mathcal{D}, [g], \mathcal{Q})\) be a qc manifold (assumed integrable in dimension 7), and let \((\tilde{G} \to \tilde{M}, \tilde{\omega})\) denote the CR Fefferman space induced from the canonical parabolic geometry of \(\mathcal{M}\). Then \(\tilde{M}\) is naturally identified with the twistor space \(Z\), and the induced CR structures coincide.

We expect this result to have useful applications for studying the twistor space of a qc manifold, such as computing the Webster scalar curvature for a natural pseudo-hermitian structure on \(\mathcal{M}\). Then \(\tilde{M}\) is obviously equivalent to first carrying out the Fefferman construction of conformal type, and then carrying out a Fefferman construction of conformal type on the resulting CR structure. But in [3] it was shown that the result of the latter construction is conformally equivalent, up to a finite covering, to the classical Fefferman metric of a CR manifold.

### 2 Background on the flag structures of qc and CR manifolds

In general, for \(G\) a semi-simple Lie group, a parabolic subgroup \(P \subset G\) determines an associated \(|k\)-grading of the Lie algebra \(\mathfrak{g}\) for some \(k \in \mathbb{N}\): \(\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_k\) as a vector space, \([\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}\) and \(P\) (with Lie algebra \(\mathfrak{p} = \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k\)) consists of the elements in \(G\) whose adjoint action preserves the associated filtration \(\mathfrak{g} = \mathfrak{g}_{-k} \supset \ldots \supset \mathfrak{g}_{-k}\) (where \(\mathfrak{g}_i := \mathfrak{g}_i \oplus \ldots \oplus \mathfrak{g}_k\)). The parabolic subgroup has Levi decomposition \(P \cong G_0 \times P_+\) where \(G_0 \subset P\) is reductive and its adjoint action preserves the grading of \(\mathfrak{g}\), while \(P_+ \subset P\) is a normal, nilpotent subgroup, which is diffeomorphic under the exponential map to \(\mathfrak{p}_+ := \mathfrak{g}_1\), consisting of those elements which strictly increase the grading of elements in \(\mathfrak{g}\) under the adjoint action. An important object for understanding the underlying geometry on \(M\) (called a *flag structure*) which is induced by a Cartan geometry \((G \to M, \omega)\) of parabolic type \((G, P)\), is the bundle \(\pi_0: \mathcal{G}_0 \to M\), given by \(\mathcal{G}_0 := G/P_+\). The filtration of \(\mathfrak{g}\) induces a filtration of the tangent bundle \(TM\) via the isomorphism \(TM \cong G \times_{Ad(G)} \mathfrak{g}/\mathfrak{p}\) (which is general for Cartan geometries), and the Cartan connection \(\omega\) identifies the bundle \(\mathcal{G}_0\) as a reduction of the associated graded tangent bundle to \(G_0\) (see Chapter 3 of [3]).
Now we fix some concepts and notation for the parabolics associated to qc and CR structures, and from here on \((G, P)\) and \((\tilde{G}, \tilde{P})\) will denote these fixed parabolic pairs, as indicated in the introduction: First, let \(Q\) be the non-degenerate quaternion-hermitian form on \(\mathbb{H}^{n+2}\) defined by:
\[
Q(x) := x_0 \overline{x}_{n+1} + \sum_{a=1}^{n} x_a \overline{x}_a + x_{n+1} \overline{x}_0,
\]
where we fix the standard ordered basis \(\{d_0, \ldots, d_{n+1}\}\) of \(\mathbb{H}^{n+2}\) over \(\mathbb{H}\) and let \(x_i \in \mathbb{H}\) denote the corresponding coordinates of \(x\). A calculation yields:
\[
\mathfrak{g} := \mathfrak{sp}(Q) = \left\{ \begin{pmatrix} a & z & q \\ \overline{x} & A_0 & -\overline{x} \\ \overline{x} & -x^t & -\overline{x} \end{pmatrix} \mid a \in \mathbb{H}, A_0 \in \mathfrak{sp}(n), p, q \in \text{Im}(\mathbb{H}), x, z^t \in \mathbb{H}^n \right\},
\]
which shows the \(|2|\) grading of \(\mathfrak{g}\) associated to the parabolic subalgebra
\[
\mathfrak{p} := \text{stab}(\mathbb{H}d_0) = \left\{ \begin{pmatrix} a & z & q \\ 0 & A_0 & -\overline{z} \\ 0 & 0 & -\overline{\mathfrak{p}} \end{pmatrix} \in \mathfrak{g} \right\}.
\]
We use the form of general elements of \(\mathfrak{g}\) given by (3) in order to employ a space-saving notation for elements of the specific grading components: E.g., for \(p \in \text{Im}(\mathbb{H})\) we write \([\overline{p}]_{-2} \in \mathfrak{g}_{-2}\) to denote the matrix as in (3) with all other entries set to zero; in a similar manner, for \(x \in \mathbb{H}^n\) we write \([x]_{-1} \in \mathfrak{g}_{-1}\) and for \((a, A_0) \in \mathbb{H} \oplus \mathfrak{sp}(n) \cong \mathfrak{sp}(1)\mathfrak{sp}(n)\) we write \([a, A_0]_0 \in \mathfrak{g}_0;\) etc.

Now we let \(G := \text{Sp}(Q)/\{\pm \text{Id}\}\), which has Lie algebra \(\mathfrak{g}\), and let \(P \subset G\) be the parabolic subgroup (with Lie algebra \(\mathfrak{p}\)) which is the image of the stabiliser in \(G\) of \(\mathbb{H}d_0\). A further calculation shows that the reductive subgroup preserving the grading components of \(\mathfrak{g}\) is:
\[
G_0 = \left\{ \begin{pmatrix} s & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & s^{-1}z \end{pmatrix} \mid s \in \mathbb{R}^+, z \in \text{Sp}(1), A \in \text{Sp}(n) \right\} / \{\pm \text{Id}\},
\]
so \(G_0 \cong \text{CSp}(1)\text{Sp}(n)\) and \(P_+ \cong (\mathbb{H}^n)^* \ltimes (\text{Im}(\mathbb{H}))^*\).

In Section 2.2 of [2], we have given a detailed description of the bundle \(\pi_0 : G_0 := G/P_+ \to M\) in terms of the underlying data \((M, D, [g], \mathcal{Q})\) of a qc manifold, and the explicit action of elements \([(s, z, A)] \in G_0\) on this bundle: A point \(u \in G_0\) is given by a basis \(u = (e_1, \ldots, e_4)\) of \(D_{\pi_0}(u)\) which is symplectic with respect to a metric \(g \in [g]\) and a choice of local quaternionic basis \([I_1, I_2, I_3]\) of \(\mathcal{Q}\) near \(\pi_0(u)\). This gives an isomorphism \([u]_{-1} : D_{\pi_0}(u) \to \mathfrak{g}_1 \cong \mathbb{H}^n\), and hence for \(T^{-1}G_0 := (T\pi_0)^{-1}(D)\) we get a partially-defined 1-form \(\omega_1 \in \Gamma(\text{Lin}(T^{-1}G_0; \mathfrak{g}_{-1}))\) by \(\omega_1(\xi) := [u]_{-1}(T_u\pi_0(\xi))\) for \(\xi \in T_u^{-1}G_0\). By construction, \(\omega_1\) is \(G_0\)-equivariant with respect to the \(G_0\)-module structure \((\mathfrak{g}_{-1}, \text{Ad}_{(G_0)}) \cong (\mathbb{H}^n, \rho_{-1})\), where \(\rho_{-1}([s, z, A]) : \mathcal{F} \mapsto s^{-1}A(\mathcal{F})\mathfrak{F}\). In addition, we have a 1-form \(\omega_2 \in \Omega^1(G_0; \mathfrak{g}_{-2})\) which by construction is \(G_0\)-equivariant with respect to the \(G_0\)-module structure \((\mathfrak{g}_{-2}, \text{Ad}_{(G_0)}) \cong (\text{Im}(\mathbb{H}), \rho_{-2})\), where \(\rho_{-2}([s, z, A]) : \mathcal{F} \mapsto s^{-2}z\mathfrak{F}\).

Fixing a Carnot-Carathéodory metric \(g \in [g]\) determines a scale for the parabolic geometry \((G, \omega)\), and hence a (exact) Weyl structure, i.e. a \(G_0\)-equivariant section \(\sigma : G_0 \to G\). Under pull-back via the section \(\sigma\), the Cartan connection \(\omega\) satisfies: \((\sigma^*\omega_i)|_{T^*G_0} = \omega_i\), for \(i = -1, -2\) as described above. (See [2], where the component \(\sigma^*\omega_0\) was also computed.) A fixed \(g \in [g]\) also determines a complement \(\mathcal{V} \subset TM\) of \(\mathcal{D}\), given as the span of local Reeb vector fields (which is invariant for a fixed \(g\)).

Now let \(\tilde{Q}\) be the non-degenerate complex-hermitian form on \(\mathbb{C}^{2n+4}\) defined by:
\[
\tilde{Q}(y, z) := y_0 \overline{y}_{n+1} + \sum_{a=1}^{n} y_a \overline{y}_a + y_{n+1} \overline{y}_{n+1} + z_0 \overline{z}_{n+1} + \sum_{a=1}^{n} z_a \overline{z}_a + z_{n+1} \overline{z}_{n+1},
\]

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where we identify a vector \(y + jz \in \mathbb{H}^{n+2}\) with \((y, z) \in \mathbb{C}^{2n+4}\). We have the standard inclusion \(\varphi: \mathfrak{gl}(n + 2, \mathbb{H}) \hookrightarrow \mathfrak{gl}(2n + 4, \mathbb{C})\), given by:

\[
\varphi: U + jV \mapsto \begin{pmatrix} U & -V \\ V & U \end{pmatrix},
\]

and one can verify that this is compatible with the chosen identification \(\mathbb{H}^{n+2} \cong \mathbb{C}^{2n+4}\), i.e. that \((U + jV)(y + jz) \simeq \varphi(U + jV)(y, z)\). (In particular, for \(\bar{p} \subset \bar{\mathfrak{g}} := \mathfrak{su}(\bar{Q})\) the parabolic subalgebra given by \(\bar{p} := \text{stab}(\mathbb{C}d_0)\), we have \(\varphi^{-1}(\bar{p}) \subset p\).)

One can now calculate the decomposition of \(\bar{\mathfrak{g}}\) according to the \([2]\)-grading associated to \(\bar{p}\), but to save space we will only give the form of the component \(\bar{\mathfrak{g}}_{-1}\), because this is all we need explicitly. We have:

\[
\bar{\mathfrak{g}}_{-1} = \left\{ \begin{pmatrix}
0 & 0 & 0 \\
y & 0 & 0 \\
z_- & 0 & 0 \\
z_+ & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-z_+ & -z_+ & 0 \\
0 & 0 & 0
\end{pmatrix} \bigg| y, z \in \mathbb{C}^n, z_-, z_+ \in \mathbb{C} \right\}.
\]

(Let us denote an element as above with the row vector \((y, z_-, z, z_+) \in \bar{\mathfrak{g}}_{-1}\).) One verifies that the inclusion \(\varphi(\mathfrak{p}_+) \subset \bar{p}\), and that \(\varphi(\mathfrak{g}_{-1}) \subset \bar{\mathfrak{g}}_{-1}\). Furthermore, if we let \(x = xu + jx_v \in \mathbb{H}^n, p = pu + jpv \in \text{Im}(\mathbb{H})\) and \(a = au + jav \in \mathbb{H}, A_0 \in \mathfrak{sp}(n)\), then we can compute the following formula for the image of elements of \(p\) under the map \(\varphi^{-1} = \text{proj}_{\bar{\mathfrak{g}}_{-1}} \circ \varphi\):

\[
\begin{align*}
\varphi^{-1}: [\mathfrak{p}]_{-2} &\mapsto (0, 0, 0, -p_v) \in \bar{\mathfrak{g}}_{-1}; \\
\varphi^{-1}: [\mathfrak{p}]_{-1} &\mapsto (ax_u, 0, -x_v, 0) \in \bar{\mathfrak{g}}_{-1}; \\
\varphi^{-1}: [(a, A_0)]_0 &\mapsto (0, au, 0, 0) \in \bar{\mathfrak{g}}_{-1}.
\end{align*}
\]

Letting \(\tilde{G} := SU(\bar{Q})/\{\pm \text{Id}\}\), then the same map gives us an injective homomorphism \(\Phi: G \hookrightarrow \tilde{G}\) with differential \(\Phi_* = \varphi\). Furthermore, for the (reductive) subgroup \(\tilde{G}_0\) of elements which preserve the grading components of the \([2]\)-grading of \(\bar{\mathfrak{g}}\) associated to \(\bar{p}\), we have \(\tilde{G}_0 \cong (\mathbb{R}_+ \times U(1) \times SU(1, 2n + 1))/\{\pm \text{Id}\}\).

Now let us describe how a Cartan geometry \((\tilde{\pi}: \tilde{G} \to \bar{M}, \tilde{\omega})\) of type \((\bar{G}, \bar{P})\) induces a \((a \text{ priori} \text{ partially-integrable})\) CR structure on the base space: The Cartan connection \(\tilde{\omega}\) by definition determines a linear isomorphism \(\tilde{\omega}_u: T_u\tilde{G} \to \tilde{\mathfrak{g}}\) at each point \(u \in \tilde{G}\), so in particular this defines a distribution \(T^{-1}\tilde{G} \subset \tilde{T}\tilde{G}\) defined by \(T^{-1}\tilde{G} = \tilde{\omega}^{-1}(\tilde{\mathfrak{g}}_{-1})\). This defines a distribution \(\bar{D} \subset \bar{T}\bar{M}\) by letting, for any point \(x \in \bar{M}, \bar{D}_x := T_u\hat{\pi}(T_u^{-1}\tilde{G})\) for some \(u \in \tilde{G}_x\). Since the subspace \(\tilde{\mathfrak{g}}_{-1} := \tilde{\mathfrak{g}}_{-1} \oplus \bar{p} \subset \tilde{\mathfrak{g}}\) is \(\text{Ad}(\bar{P})\)-invariant, and the Cartan connection \(\tilde{\omega}\) is Ad(\(\bar{P}\))-equivariant by definition (i.e. \(R_{\bar{p}}\bar{\omega} = \text{Ad}(\bar{p}^{-1}) \circ \bar{\omega}\)), it follows that this distribution is well-defined. Also, \(\text{rank}_x(\bar{D}) = \dim(\tilde{\mathfrak{g}}_{-1}/\bar{p}) = \dim(\tilde{\mathfrak{g}}_{-1}) = 4n + 4\), which shows that \(\bar{D}\) is a co-rank 1 distribution on \(\bar{M}\) (since \(\dim(\bar{M}) = \dim(\tilde{\mathfrak{g}}/\bar{p}) = 4n + 5\)).

Now let us specify a natural almost complex structure \(\tilde{J}\) on \(\bar{D}\): Clearly, one can choose a \(\tilde{G}_0\)-invariant complex structure \(J_0\) on \(\tilde{\mathfrak{g}}_{-1}\) \(\cong \mathbb{C}^{2n+2}\) (e.g. scalar multiplication by \(-i\)), and in fact such a choice is unique up to sign. Since \(\tilde{\mathfrak{g}}_{-1} \cong \tilde{\mathfrak{g}}_{-1}/\bar{p}\) as \(\bar{P}\)-modules, and \(\bar{P}\) acts trivially on \(\tilde{\mathfrak{g}}_{-1}/\bar{p}\), we get a \(\bar{P}\)-invariant endomorphism of \(\tilde{\mathfrak{g}}_{-1}\) from \(J_0\) by extending trivially to \(\bar{p}\), and we’ll also denote this by \(J_0\). For \(x \in \bar{M}\) and \(X \in \bar{D}_x\), choose \(u \in \tilde{G}_x\) and \(\tilde{X} \in T_u^{-1}\tilde{G}\) such that \(T_u\hat{\pi}(\tilde{X}) = X\). Then we define

\[
\tilde{J}(X) := T_u\hat{\pi}(\tilde{\omega}_u^{-1}(J_0(\tilde{\omega}(\tilde{X})))�).
\]

Again, equivariance of \(\tilde{\omega}\) and \(\bar{P}\)-invariance of \(J_0\) may be invoked to verify that this definition is proper.

### 3 Proof of Theorem A

First let us recall the construction of the twistor space \(Z\) and its CR structure from [2] (we refer also to the exposition in Section 3 of [6]): The space \(Z \subset Q\) is defined fibre-wise, for each point \(x \in M\), to be the set of
complex structures on $D_x$ in $Q_x$:

$$Z_x := \{ I \in Q_x | I^2 = -Id_{D_x} \}.$$  

This is evidently a $S^2$-bundle over $M$, since any choice of a local quaternionic basis $\{I_1, I_2, I_3\}$ of $Q$ around $x$ determines an identification of the restriction of $Z$ to a neighbourhood of $x$ with the endomorphisms $I = a_1 I_1 + a_2 I_2 + a_3 I_3 \in Q$ such that $a_1^2 + a_2^2 + a_3^2 = 1$.

If we fix a choice of Carnot-Carathéodory metric $g \in [g]$, then we have a distinguished linear connection $\nabla$ on $M$ (cf. Theorem B, [3]), called the Biquard connection of $g$, which induces a horizontal distribution on $Z$, i.e. we have:

$$T_dZ = \text{Hor}^\nabla (Z) \oplus \text{Ver}_1(Z),$$

where $\text{Ver}_1(Z) = T_I(Z_x)$ is the vertical tangent bundle at $I$ for $I \in Z_x$. In particular, a choice of $g \in [g]$ determines in this way the horizontal lift of a vector $X \in T_x M$ to $X^\nabla \in \text{Hor}^\nabla (Z) \subset T_dZ$.

A CR distribution $\mathcal{H} \subset TZ$ is defined as follows: For $I = a_1 I_1 + a_2 I_2 + a_3 I_3 \in Z_x$, a corresponding vector $\xi_I \in V_x \subset T_x M$ is given by letting $\xi_I = a_1 \xi_1 + a_2 \xi_2 + a_3 \xi_3$, where $\xi_1, \xi_2, \xi_3$ are the Reeb vector fields defined locally around $x$ for the unique choice of 1-forms $\eta^1, \eta^2, \eta^3$ which locally define $D$ and are compatible with the local basis $\{I_1, I_2, I_3\}$ and the metric $g$ in the sense of identity (1), and whose existence is assumed in dimension 7. Declaring $\xi_1, \xi_2, \xi_3$ to be orthonormal, we also have an inner product on $V_x$, and a rank 2 subspace $\xi^\perp_I \subset V_x$ for any $I \in Z_x$. Biquard defines:

$$\mathcal{H}_I := (\xi^\perp_I)^\nabla \oplus (D_x)^\nabla \oplus \text{Ver}_1(Z).$$

Furthermore, an almost complex structure $J^Z \in \Gamma(\text{End}(\mathcal{H}))$ is defined by letting $J^Z (\xi_I)^\nabla = I^\nabla$ (the horizontal lift of $I$), and defining the restriction of $J^Z$ to $(\xi^\perp_I)^\nabla$ and to $\text{Ver}_1(Z)$ to be the natural complex structures (described explicitly below). Biquard (resp. Duchemin for $n = 1$) has proven that the CR structure thus defined is independent of a conformal change of $g \in [g]$, non-degenerate of signature $(4n+2,2)$, and integrable (Theorem II.5.1 of [3]). Once we have identified the twistor space $Z$ with the CR Fefferman space $(\hat{\mathcal{G}}, \hat{\omega})$, these properties follow automatically since $\hat{\omega}$ is normal and torsion-free.

First, let us identify $\hat{M} \cong Z$: By definition, $\hat{M} := \mathcal{G}/\Phi^{-1}(\hat{P})$ is the quotient of $\mathcal{G}$ by the subgroup $\Phi^{-1}(\hat{P}) \subset P$. Since $P_+ \subset \Phi^{-1}(\hat{P})$ in our case, and $\mathcal{G}_0 := \mathcal{G}/P_+$, we can identify $\hat{M} = \mathcal{G}_0/(G_0 \cap \Phi^{-1}(\hat{P}))$. In fact, in our case the $\mathbb{R}^+$ component of $G_0 \cong CSp(1)Sp(n) = \mathbb{R}^+ \times Sp(1)Sp(n)$ is contained in $\Phi^{-1}(\hat{P})$, and so for any reduction $\mathcal{G}_0 \rightarrow \mathcal{G}_0$ to the structure group $Sp(1)Sp(n) \subset G_0$, we get an isomorphism $\hat{M} \cong \mathcal{G}_0/(Sp(1)Sp(n) \cap \Phi^{-1}(\hat{P}))$. This can be applied, for a fixed choice of $g \in [g]$, to the reduced frame-bundle $\mathcal{G}_0$ consisting, fibre-wise, of those frames $u = (e_1, \ldots, e_4n)$ of $D_x$ which are symplectic with respect to $g$ and some local quaternionic basis $\{I_1, I_2, I_3\}$ of $Q$.

Note that the subgroup $Sp(1)Sp(n) \cap \Phi^{-1}(\hat{P})$ consists, with respect to the presentation (1), of those elements $[(1, z, A)] \in G_0$ for which $z \in Sp(1)$ is of the form $z = z_0 + z_1 i$, i.e. $z \in U(1) \subset Sp(1)$. Thus, $Sp(1)Sp(n) \cap \Phi^{-1}(\hat{P}) = U(1)Sp(n)$. Also, we see that the subgroup $Sp(1)Sp(n) \cap \Phi^{-1}(\hat{P})$ equals the stabiliser of the point $i \in S^2$ (identifying $S^2$ with the unit imaginary quaternions) under the action $\rho_0 : Sp(1)Sp(n) \rightarrow \text{Diff}(S^2)$ given by $\rho_0([(1, z, A)] : q \mapsto zq\overline{z}$.

For a point $u \in \mathcal{G}_0$ given by a symplectic basis of $D$ with respect to $g$ and $\{I_1, I_2, I_3\}$, we identify $(u, i) \simeq I_1$, $(u, j) \simeq I_2$ and $(u, k) \simeq I_3$. By definition of the $G_0$ action on $\mathcal{G}_0$ (cf. Section 2.2 of [2]), this identification is invariant under the right action by $Sp(1)Sp(n)$ on $\mathcal{G}_0 \times S^2$ given by $(u, q \cdot g) = (ug, \rho_0(g^{-1})(q))$. In particular, this gives an isomorphism $\mathcal{G}_0 \times_{\rho_0} S^2 \cong \mathcal{G}_0/(Sp(1)Sp(n) \cap \Phi^{-1}(\hat{P}))$, since $Sp(1)Sp(n) \cap \Phi^{-1}(\hat{P})$ is the stabiliser of $i$ under the action $\rho_0$.

In summary, this gives us the identification $\hat{M} \cong Z$ (and hence a submersion which we’ll denote $\phi_0 : \mathcal{G}_0 \rightarrow Z$, given by sending $u \in \mathcal{G}_0$ to the point in $Z$ identified with $[(u, i)] \in \mathcal{G}_0 \times_{\rho_0} S^2$). From the preceding argument, the following fact about this identification is evident: Fixing a local quaternionic basis $\{I_1, I_2, I_3\}$ of $Q$ around $x \in M$ and a point $u \in (\mathcal{G}_0)_x$ corresponding to this basis (and to $g \in [g]$), which
is fixed throughout), then for any point $I = a_1I_1(x) + a_2I_2(x) + a_3I_3(x) \in Z_x$ we have $I = \varphi_0(ug_I)$ where $g_I = [(1, z_I, Id)]$ for some $z_I \in S\rho(1)$ such that $\rho_0(g_I)(i) = z_I\mathbb{T} = a_1i + a_2j + a_3k$. This will be useful for subsequent calculations.

In the next step, we identify the induced CR distribution on $Z$ corresponding to $\tilde{D} \subset TM$. We will abuse notation slightly by writing $\tilde{D}_I \subset T_I Z$ for $I \in Z$. From the construction of the Fefferman space $(\mathfrak{G}, \omega)$, we have an inclusion $\iota : G \to \mathfrak{G}$ of bundles over $M$, and the Cartan connections are related by $\iota^* \omega = \varphi \circ \omega$. Moreover, the composition $\pi \circ \iota : \mathfrak{G} \to M$ equals the defining projection $p : \mathfrak{G} \to M := \mathfrak{G}/\mathfrak{P}_1(I)$. Denoting the induced projection by $\varphi : \mathfrak{G} \to M$, we thus have $\tilde{D}_I = T_{ug_I}((\varphi \circ \omega)^{-1}(\mathfrak{g}^{-1}))$ for a choice of $u \in \mathfrak{g}^{-1}(I)$. If we denote by $\sigma : G_0 \to \mathfrak{G}$ the $G_0$-equivariant section (Weyl structure) corresponding to the Carnot-Carathéodory metric $g \in [g]$, let $u \in G_0$, $I = a_1I_1 + a_2I_2 + a_3I_3$ and $g_I, z_I$ be as in the preceding paragraph, then $ug_I \in \varphi_0^{-1}(I)$ and we have

$$\tilde{D}_I = T_{ug_I} \varphi_0((\varphi \circ \sigma^* \omega)^{-1}(\mathfrak{g}^{-1})).$$

As noted in Section 2, we have $\varphi(\mathfrak{g}^{-1}) \subset \mathfrak{g}^{-1}$. Hence, for $I \in Z_x$ and any $X \in D_x \subset T_x M$, if $\tilde{X} \in T_{ug_I} G_0$ is any lift of $X$ to the point $ug_I$, then $T_{ug_I} \varphi_0(\tilde{X}) \in \tilde{D}_I$ (since $\sigma^* \omega(\tilde{X}) \in \mathfrak{g}^{-1}$). Also, any vertical (over $M$) tangent vector in $T_{ug_I} G_0$ projects to $\tilde{D}_I$. In particular, $(D_2)^{\nabla} \subset \text{Vert}(Z) \subset \tilde{D}_I$, since we can take the horisontal lift $X^h$ of any vector $X \in D_x$ to $ug_I$ with respect to the Biquard connection form on $\mathfrak{g}_0$, which clearly projects to $X^{\nabla} \subset T_I Z$.

To show the inclusion $(\xi^j)^{\nabla} \subset \tilde{D}_I$ (and hence the equality $H_I = \tilde{D}_I$), we first look closer at the image $\varphi(\mathfrak{g}^{-1}) \subset \mathfrak{g}$: Namely, one calculates that $\varphi([-j]^{-1}) \subset \mathfrak{g}_- \subset \mathfrak{g}^{-1}$. For $I \in Z_x$ and $u \in G_0$, $g_I \in S\rho(1)Sp(n)$, $z_I \in S\rho(1)$ as specified above, we define $J, K \in Z_x$ by $J := b_1I_1 + b_2I_2 + b_3I_3$ and $K := c_1I_1 + c_2I_2 + c_3I_3$, for $b_i + b_2 + b_3 := z_I\mathbb{T}$ and $c_1 + c_2 + c_3 := z_Ik\mathbb{T}$. Then $\xi_J, \xi_K \in \mathfrak{V}_x$ span the orthogonal complement of $\xi_I$ in $\mathfrak{V}_x$. By construction, $(\omega^{-2}(u))((\mathfrak{g}^{-1})^-1(z_I\mathbb{T})) = (-z_Ik\mathbb{T})^{-1}$-2 for $\xi_J, \xi_K$ any lifts of $\xi_J, \xi_K$, respectively, to the point $u$. Using $G_0$-equivariance, we get:

$$\omega^{-2}(ug_I)((R_{g_I})_*(\xi_J)) = \omega^{-2}(ug_I)((R_{g_I})_*(\xi_J)) = \text{Ad}(g_I^{-1})((\omega^{-2}(u))((\mathfrak{g}^{-1})^-1(z_I\mathbb{T})))$$

Similarly, $(\omega^{-2}(ug_I)((R_{g_I})_*(\xi_K)) = [-k]^{-1}$. Thus $(R_{g_I})_*(\xi_J), \xi_K) \in (\varphi \circ \sigma^* \omega)^{-1}(\mathfrak{g}^{-1}) \subset T_{ug_I} G_0$ (and hence any lifts of the vectors $\xi_J, \xi_K$ to the point $ug_I \in G_0$) project via $\varphi_0$ to $\tilde{D}_I$, so $(\xi^j)^{\nabla} \subset \tilde{D}_I$.

It remains to compute the induced almost complex structure $\tilde{J}$ on $\tilde{D}$. For this calculation, we only need to consider the components $(\varphi \circ \sigma^* \omega)(\tilde{X}) \in T \mathfrak{G}_0$, where $\omega := \omega_2 + \omega_1 + \omega_0$, since $\varphi(\mathfrak{p}_+) \subset \mathfrak{p}$. First, note that for any $X \in D_x$, the horizontal lifts of $X$ to vectors in $G_0$ with respect to the Biquard connection and the Weyl connection $\omega_0 \circ \sigma^* \omega$ are the same, so in particular we have $\sigma^* \omega(\xi_I^h) = \omega_1(\xi_I^h)$ for $\xi_I^h$ the horizontal lift via the Biquard connection. This follows from the computation of the Weyl connection with respect to the Weyl structure $\sigma$ induced by $g \in [g]$, cf. Theorem 3.7 of [2]. On the other hand, let us denote by $\xi^h \in T_{ug_I} G_0$ the horizontal lift of a Reeb vector field $\xi^a \in \mathfrak{V}_x$ to the point $u$ with respect to the Biquard connection (where $u$ and the bases $\{I_1, I_2, I_3\}$ are related as specified above). Then it follows from the same result that $\sigma^* \omega(\xi^h) = \omega_2(\xi^h) + \omega_0(\xi^h)$ and we have:

$$\omega(\xi^h) = [[(\tilde{s}_{\xi^h}A, \omega_A(\xi^h))]_0 \in \mathfrak{g}_0 \quad (8)$$

where $\tilde{s}_g := \text{scal}/32n(n+2)$ is the rescaled qc scalar curvature of $g \in [g]$ and $i_1 := i, i_2 := j, i_3 := k$.

Using (8) and the formulae (5) and (7), one sees that

$$(\varphi \circ \sigma^* \omega)(ug_I)((\xi^h_I(ug_I))) = (\varphi \circ \sigma^* \omega)(ug_I)((R_{g_I})_*(\xi^h_I(u))) = (0, \tilde{s}_g, 0, -1)$$

and $(\varphi \circ \sigma^* \omega)(ug_I)((\xi^h_K(ug_I))) = (0, -\tilde{s}_g, 0, i)$. Thus, if we denote by $J_0$ the complex structure on $\mathfrak{g}^{-1}$ given by component-wise multiplication by $-i$, then one computes:

$$J_0((\varphi \circ \sigma^* \omega)(\xi^h_I(ug_I))) = (\varphi \circ \sigma^* \omega)(\xi^h_K(ug_I));$$

$$J_0((\varphi \circ \sigma^* \omega)(\xi^h_K(ug_I))) = -(\varphi \circ \sigma^* \omega)(\xi^h_I(ug_I)).$$
So on $\mathcal{D}_I$, the restriction of $\tilde{J}$ to $(\xi_j^I)^V$ is given on basis vectors by:

$$\tilde{J} : \xi_j^V \mapsto \xi_j^\perp = (\xi_I \times \xi_J)^V = (\xi_{I_0} J)^V$$

where “×” denotes the cross product in $\mathcal{V}_c \cong \mathbb{R}^3$. In a similar way, one sees that the complex structure $J_0$ on $g_{-1}$ induces the natural complex structure on $\text{Ver}_I \mathcal{Z}_I : \tilde{J} : J \mapsto I \circ J = K$ and $\tilde{J} : K \mapsto I \circ K = -J$ ($J, K$ are naturally identified with vectors in $T_\ell (\mathcal{Z}_I \cong S^2)$ since they are orthogonal to $I$, and one calculates that this $\tilde{J}$ is induced from the transformation on $\text{Ver}_{ug_I} \mathcal{D}_I$ which sends the fundamental vector field of $[(0, k, 0)] \in g_0$ to the fundamental vector field of $[(0, k, 0)] \in g_0$ and sending $[(0, k, 0)]$ to $-(0, k, 0)]$.

Finally, to see the restriction of $\tilde{J}$ to $(D_x)^V$, let $X \in D_x$ be such that $\omega^{-1}(u)(X^h) = [u]_1 - (X) = [\mathcal{I}]_1 \in g_{-1}$. Then $J_0(\omega^{-1}(u)(X^h))(\omega^{-1}(u)(X^h)) = \omega^{-1}([\mathcal{I}]_1 - (X))^V = [\mathcal{I}]_1 - (X)$, since $\omega$ is clearly $\mathbb{C}$-linear. On the other hand, we have $\omega^{-1}(u)(I_1(X)^h) = [u]_1 - (I_1(X)) = [\mathcal{I}]_1 - (X)$ (cf. Section 2.2. and Appendix A of [2]), which shows that at the point $I_1 \in \mathcal{Z}_z$, the restriction of $\tilde{J}$ to $(D_x)^V \subset D_I$ is given by $I^V$. On the other hand, from the equivariance of $\omega$ it follows that $\omega^{-1}(ug_I)(I(X)^h) = [ug_I]_1 - (I(X)) = [ug_I]_1 - (X)$, which shows that the restriction of $\tilde{J}$ to $(D_x)^V \subset D_\mathcal{Z}$ is also given by $I^V$ for arbitrary $I \in \mathcal{Z}_z$. This completes the proof of Theorem A.

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