FULLY NONLINEAR PARABOLIC EQUATIONS IN TWO SPACE VARIABLES

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Abstract. Hölder estimates for spatial second derivatives are proved for solutions of fully nonlinear parabolic equations in two space variables. Related techniques extend the regularity theory for fully nonlinear parabolic equations in higher dimensions.

1. Introduction

Elliptic equations in two variables are very well understood, and the regularity theory for such equations is significantly stronger than that available for elliptic equations in higher dimensions. In particular, Morrey [M] and Nirenberg [N] proved Hölder estimates for the first derivatives of solutions of uniformly elliptic equations in two variables, depending only on bounds for the coefficients:

Theorem 1. Let \( \Omega \subset \mathbb{R}^2 \), and set \( d_\Omega(z) = d(z, \partial \Omega) \) for all \( z \in \Omega \). Let \( u \) be a bounded \( C^2(\Omega) \) solution of

\[
(1) \quad au_{xx} + 2bu_{xy} + cu_{yy} = f
\]

where \( a, b, c \) are measurable functions on \( \Omega \) with \( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \geq \lambda \sqrt{ac - b^2} I \) and \( \delta = \sqrt{ac - b^2} > 0 \) everywhere. Then for any \( \alpha \in (0, \sqrt{\lambda}) \) there exists \( C = C(\lambda, \alpha) \) such that for all points \( p \neq q \) in \( \Omega \) with \( d = \min\{d_\Omega(p), d_\Omega(q)\} > 0 \),

\[
\frac{|Du(q) - Du(p)|}{|p - q|^\alpha} \leq Cd^{-\alpha} \sup_\Omega (|Du| + d_\Omega \delta^{-1}|f|).
\]

These estimates can also be applied to fully nonlinear uniformly elliptic equations in two variables, to give Hölder estimates for second derivatives. in this case the equations have the form

\[
(2) \quad F[u] = F(D^2u, Du, u, x) = 0
\]

where \( u : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \), and \( F: S_2 \times \mathbb{R}^2 \times \mathbb{R} \times \Omega \to \mathbb{R} \) is Lipschitz in all variables (here \( S_2 \) is the space of symmetric \( 2 \times 2 \) matrices) and uniformly monotone in the first argument, so that there exist constants \( \Lambda \geq \lambda > 0 \) such that

\[
(3) \quad \lambda I \leq [F_{ij}] \leq \Lambda I
\]

where \( F_{ij} = \frac{\partial F(r,p,z,x)}{\partial r_{ij}} \).

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Theorem 2. Let \( u \in C^3(\Omega) \) satisfy \( F[u] = 0 \) in \( \Omega \subseteq \mathbb{R}^2 \), and suppose (2) holds. Then there exists \( \alpha = \alpha(\lambda/\Lambda) \) such that for any points \( p \neq q \in \Omega \) with \( d = \min\{d(\Omega(p), d(\Omega(q)) \} > 0 \),

\[
\frac{\|D^2 u(p) - D^2 u(q)\|}{|p - q|^\alpha} \leq C d^{-\alpha} \sup_{\Omega} (\|D^2 u\| + |Du| + 1)
\]

where \( C \) depends only on \( \lambda/\Lambda \) and \( \sup |DF| \).

In contrast, the situation in higher dimensions is much worse: For fully nonlinear equations there is no Hölder estimate known for the second derivatives of solutions, unless the equation satisfies a concavity condition with respect to the components of the second derivatives. The best result available is the following, due to Evans [E1–2] and Krylov [Kr] (I follow the treatment in [GT]):

Theorem 3. Let \( u \in C^4(\Omega) \) satisfy \( F[u] = 0 \) in \( \Omega \) where \( F \) is a \( C^2 \) function of the form (2) which is uniformly elliptic (so that \( M \leq F'' \leq M \Lambda \) for some \( \Lambda \geq \lambda > 0 \)) and concave with respect to the first argument. Then for any \( \Omega' \subset \subset \Omega \),

\[
\sup_{p \in \Omega'} \frac{\|D^2 u(p) - D^2 u(q)\|}{|p - q|^\alpha} \leq C
\]

where \( \alpha \) depends on \( n, \lambda \) and \( \Lambda \), and \( C \) depends on \( n, \lambda, \Lambda, |u|_{C^2(\Omega)} \) and \( d(\Omega', \partial \Omega) \), and on bounds for the first and second derivatives of \( F \) (other than the second derivative in the first argument).

In the parabolic case, there are results similar to Theorem 3 (due to Krylov [Kr]):

Theorem 4. Let \( u \in C^4(\Omega \times (0, T]) \) satisfy

\[
\frac{\partial u}{\partial t} = F(D^2 u, Du, u, x, t)
\]

where \( F \) is \( C^2 \), \( M \leq |F''| \leq M \Lambda \) for some \( 0 < \lambda \leq \Lambda \), and \( F \) is concave in the first argument. Then for any \( \tau > 0 \) and \( \Omega' \subset \subset \Omega \),

\[
\sup_{s,t \in [\tau, T], p,q \in \Omega'} \left( \frac{|D^2 u(p, t) - D^2 u(q, t)|}{|p - q|^\alpha + |s - t|^\alpha/2} + \frac{|\partial_t u(p, t) - \partial_t u(q, t)|}{|p - q|^\alpha + |s - t|^\alpha/2} \right) + \sup_{p \in \Omega', \tau \leq s \leq T} \frac{|Du(p, t) - Du(p, s)|}{|s - t|^{(1+\alpha)/2}} \leq C
\]

where \( \alpha \) depends on \( n, \lambda \) and \( \Lambda \), and \( C \) depends on \( n, \lambda, \Lambda, \sup_{[\tau, T]} |D^2 u|, \sup_{[\tau, T]} |\partial_t u|, d(\Omega', \partial \Omega) \), \( \tau \) and bounds for the first and second derivatives of \( F \) (other than the second derivative in the first argument).

The Morrey and Nirenberg estimates rely either on quasiconformal mapping estimates or on the fact that in two dimensions the first derivatives of solutions satisfy divergence-form elliptic equations. Neither of these methods seems to generalise readily to the parabolic setting. There are, however, special estimates known for parabolic equations in one space variable, due largely to Kruzhkov [Kz].

In this paper I will prove an analogue of Theorem 2 for parabolic equations in two space variables. I do not know whether an analogue of Theorem 1 holds.
Theorem 5. Let $\Omega$ be a domain in $\mathbb{R}^2$. Let $u \in C^4(\Omega \times (0, T])$ be a solution of the fully nonlinear equation
\[
\frac{\partial u}{\partial t} = F(D^2u, Du, u, x, t)
\]
where $F$ is Lipschitz in all arguments and uniformly monotone in the first argument, so that $\lambda \leq [\hat{F}^{ij}] \leq \Lambda$ for some $\Lambda \geq \lambda > 0$. Then for any $\tau \in (0, T)$ and $\Omega' \subset \subset \Omega$,
\[
\sup_{s,t \in [\tau, T], p,q \in \Omega'} \left( \frac{|D^2u(p, t) - D^2u(q, t)|}{|p-q|^\alpha + |s-t|^\alpha/2} + \frac{\partial_t u(p, t) - \partial_t u(q, t))}{|p-q|^\alpha + |s-t|^\alpha/2} \right)
\]
\[+ \sup_{p \in \Omega', \tau \leq s,t \leq T} \frac{|Du(p, t) - Du(p, s)|}{|s-t|^{(1+\alpha)/2}} \leq C
\]
where $\alpha$ depends on $\lambda$, $\Lambda$, and $C$ depends on $\lambda$, $\Lambda$, $\sup_{\Omega \times (0, T)}(|D^2u| + |Du|)$, $\sup_{\Omega \times (0, T)} |\partial_t u|$, $d(\Omega', \partial\Omega)$, $\tau$, and bounds for the first derivatives of $F$.

This gives a result of similar strength to Theorem 2 for fully nonlinear parabolic equations in two space variables.

I will also apply similar ideas to parabolic equations in higher dimensions, to relax the requirement of concavity to allow just convexity of level sets.

Theorem 6. Let $\Omega$ be a domain in $\mathbb{R}^n$. Suppose $u \in C^4(\Omega \times (0, T])$ satisfies
\[
\frac{\partial u}{\partial t} = F(D^2u, Du, u, x, t)
\]
where $F$ is $C^2$ and $\lambda \leq [\hat{F}^{ij}] \leq \Lambda$ for some $\Lambda \geq \lambda > 0$, and $\hat{F}^{ij,kl} M_{ij} M_{kl} \leq 0$ for all matrices $[M_{ij}]$ for which $F^{ij} M_{ij} = 0$. Then for any $\Omega' \subset \subset \Omega$ and $\tau \in (0, T)$,
\[
\sup_{s,t \in [\tau, T], p,q \in \Omega'} \left( \frac{|D^2u(p, t) - D^2u(q, t)|}{|p-q|^\alpha + |s-t|^\alpha/2} + \frac{\partial_t u(p, t) - \partial_t u(q, t))}{|p-q|^\alpha + |s-t|^\alpha/2} \right)
\]
\[+ \sup_{x \in \Omega', \tau \leq s,t \leq T} \frac{|Du(x, t) - Du(x, s)|}{|s-t|^{(1+\alpha)/2}} \leq C
\]
where $\alpha$ depends on $n$, $\lambda$, $\Lambda$, and $C$ depends on $\lambda$, $\Lambda$, $\sup_{\Omega \times (0, T)}(|D^2u| + |Du|)$, $\sup_{\Omega \times (0, T)} |\partial_t u|$, $d(\Omega', \partial\Omega)$, $\tau$, $K$, and bounds for first and second derivatives of $F$.

In the elliptic case, this extension is trivial because the equation $F = 0$ is the same as the equation $\psi(F) = 0$ if $\psi$ is an increasing function with $\psi(0) = 0$. However, in the parabolic case there is a big difference between the two equations
\[
\frac{\partial u}{\partial t} = F
\]
and
\[
\frac{\partial u}{\partial t} = \psi(F).
\]
If $F$ is concave in $D^2u$ and $\psi$ is increasing, then the latter equation is covered by Theorem 6 but not in general by Theorem 4.

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2. Some background results

In this section I will recall some results and notation to be used in the proofs of Theorems 5 and 6.

2.1 Function spaces.

Let \( \Omega \) be a domain in \( \mathbb{R}^n \). As usual I will denote by \( C^{k,\alpha} \) the space of functions which have all derivatives up to order \( k \) Hölder-continuous of with exponent \( \alpha \in (0,1) \).

In the parabolic setting, where we are considering solutions on a space-time region \( Q = \Omega \times (0,T] \), it is useful to introduce the spaces \( P^{k,\alpha}(Q) \) consisting of functions \( u \) on \( Q \) which have the following norm bounded:

\[
|u|_{P^{k,\alpha}(Q)} = \sum_{|\alpha| + 2|b| \leq k} \sup_{(x,t) \in Q} |\partial_t^\alpha D^b u(x,t)| + \sum_{|\alpha| + 2|b| = k} \sup_{(x,t) \neq (y,s) \in Q} \frac{|\partial_t^\alpha D^b u(x,t) - \partial_t^\alpha D^b u(y,s)|}{|x-y|^{1+\alpha} + |s-t|^{1+\alpha}}.
\]

2.2 The Krylov-Safonov Hölder estimate.

The Krylov-Safonov Harnack inequality, first proved in [KS], provides an oscillation estimate for solutions of elliptic or parabolic equations. The following Hölder estimate is a consequence of this.

**Theorem 7.** Denote by \( Q_r \) the region \( B_r \times [-r^2,0] \subset \mathbb{R}^n \times \mathbb{R} \). Let \( u : Q_R \to \mathbb{R} \) be a smooth solution of an equation of the form

\[
\frac{\partial u}{\partial t} = a^{ij} D_i D_j u + b^i D_i u + c u + f
\]

where the coefficients are bounded and measurable, and \( \lambda I \leq [a^{ij}] \leq \Lambda I \). Then for some \( C > 0 \) and \( \alpha \in (0,1) \) depending only on \( n, \lambda/\Lambda, \) and bounds for coefficients,

\[
|u|_{P^{0,\alpha}(Q_R/2)} \leq C \left( |u|_{P^\infty(Q_R)} + |f|_{L^\infty(Q_R)} \right).
\]

2.3 A characterisation of \( C^{1,\alpha} \) functions.

I will make use of the following characterisation of \( C^{1,\alpha} \) functions in terms of difference quotients (see [T]):

**Theorem 8.** Let \( u \) be a smooth function on \( B_R \subset \mathbb{R}^n \), and suppose there exists a constant \( C_0 \) such that for all \( \xi \in S^{n-1}, h > 0, \) and \( x \in B_R \setminus \partial B_R \),

\[
|u(x + h\xi) + u(x - h\xi) - 2u(x)| \leq Ch^{1+\alpha},
\]

where \( \alpha \in (0,1] \). Then

\[
\sup_{x \neq y \in B_R/2} \frac{|Du(x) - Du(y)|}{|x-y|^{1+\alpha}} \leq C(\alpha)C_0.
\]
Assume (by rescaling if necessary) that \( u \) is a \( C^3 \) solution on the domain \( Q_1 = B_1(0) \times (-1,0] \subset \mathbb{R}^2 \times \mathbb{R} \) of a fully nonlinear parabolic equation

\[
\frac{\partial u}{\partial t} = F(D^2 u, Du, u, x, t)
\]

where \( F \) is defined on a convex set \( S \) in \( S_2 \times \mathbb{R}^2 \times \mathbb{R} \times B_1 \times (-1,0] \) containing \( \{J = (D^2 u(x,t), Du(x,t), u(x,t), x,t) : (x,t) \in Q_1\} \). Assume that \( F \) is Lipschitz on \( S \), and satisfies

\[
\lambda \leq |\dot{F}(\tau)| \leq \Lambda
\]

at each point of \( S \), for some \( \Lambda \geq \lambda > 0 \). We will obtain estimates on regions \( Q_r = B_r(0) \times (-r^2,0] \) for suitably small \( r \).

3.1 Regularity of the time derivative.

Let \( \tau \in (0,1) \), and define \( v_\tau(x,t) = \frac{1}{\tau} (u(x,t) - u(x,t-\tau)) \). Then \( v_\tau \) satisfies a parabolic equation on \( B_1 \times (\tau - 1,0] \):

\[
\frac{\partial v_\tau}{\partial t} = \frac{F(J u(x,t)) - F(J u(x,t-\tau))}{\tau} = \frac{1}{\tau} \int_0^1 DF \left| sJ u(x,t) + (1-s) J u(x,t-\tau) \right| (J u(x,t) - J u(x,t-\tau)) \, ds
\]

\[
= a^{ij} D_i D_j v_\tau + b_i D_i v_\tau + c v_\tau + f.
\]

Here, writing \( J(s) = sJ u(x,t) + (1-s) J u(x,t-\tau) \), the coefficients are given by \( a^{ij} = \int_0^1 F^{ij} | J(s) |^2 \, ds, b_i = \int_0^1 F^i | J(s) | \, ds, c = \int_0^1 F^2 | J(s) | \, ds \) and \( f = \int_0^1 \partial F / \partial t | J(s) | \, ds \).

Note that \( \lambda \leq a^{ij} \leq \Lambda \), and \( |b_i| + |c| + |f| \leq C \text{Lip}(F) \).

Theorem 7 applies to give an oscillation estimate for \( v_\tau \) independent of \( \tau \), and hence also for \( u_\tau \). In particular, \( u_\tau \) is Hölder continuous in both space and time, and for \( (x_1, t_1) \) and \( (x_2, t_2) \) in \( Q_1/2 \)

\[
|u_\tau(x_1, t_1) - u_\tau(x_2, t_2)| \leq C \left( (|x_2 - x_1|^2 + |t_2 - t_1|)^{\alpha/2} \left( \left| \frac{\partial F}{\partial t} \right|_{L^\infty(Q_1)} + |u_\tau|_{L^\infty(Q_1)} \right) \right).
\]

3.2 Spatial regularity of second space derivatives.

This is the key estimate: When restricted to each time slice, the function \( u \) has Hölder-continuous second derivatives. This follows by combining the Morrey-Nirenberg estimates for elliptic equations with either a perturbation argument or an argument using difference quotients. I present the latter argument.

Fix \( t \in (-1/4,0] \). Write \( w(x) = u(x,t) \), and \( \phi(x) = u_\tau(x,t) \). Then the following elliptic equation holds:

\[
G[D^2 w(x), Dw(x), w(x), x] = \phi(x),
\]

where \( G[r, p, z, x] = F[r, p, z, x, t] \) and \( \phi(x) = u_\tau(x,t) \). Note that \( \lambda I \leq \hat{G} \leq \Lambda I \), \( \text{Lip}(G) \leq \text{Lip}(F) \), and \( \phi \) satisfies the oscillation estimate derived in Section 3.1.

Let \( \xi \) be a unit vector, and for \( h > 0 \) let \( \delta_h w \) be the difference quotient of \( w \) in the direction \( \xi \) with step \( h \):

\[
\delta_h w(x) = h^{-1} (w(x+h\xi) - w(x)).
\]
\( \delta_h w \) is defined on \( B_{1-h}(0) \), and satisfies an elliptic equation: Denote \( D_w(x) = (D^2 w(x), D_x w(x), w(x)) \). Then
\[
0 = h^{-1} (G[D_w(x+h\xi)] - G[D_w(x)]) - \delta_h \phi \\
= h^{-1} \int_0^1 \delta_x D(D_w(x+\xi h) - D_w(x)) \, ds - \delta_h \phi \\
= \tilde{a}^ij D_i D_j \delta_h w + \varphi
\]

Here \( D(s) = sD_w(x+h\xi) + (1-s)D_w(x) \) and \( \varphi = \tilde{b}^i D_i \delta_h w + \tilde{c} \delta_h w + \tilde{f} - \delta_h \phi \), where \( \tilde{f} = \xi \int_0^1 \frac{\partial G}{\partial x}(D(s) \, ds, \tilde{a}^ij = \int_0^1 G^i_j \, |D_s| \, ds, \tilde{b}^i = \int_0^1 G^i \, |D_s| \, ds, \text{and } \tilde{c} = \int_0^1 G^2(D(s) \, ds \text{. It follows that } \lambda |v|^2 \leq \tilde{a}^ij v_i v_j \leq \Lambda |v|^2 \text{ for all } v \neq 0, \text{ and } |\varphi(x)| \leq CLip(F)(1 + |u|_{P^2(Q_1)}) + Ch^{\alpha-1}(Lip(F) + |u|_{P^2(Q_1)}) \text{ for } x \in B_{1/4} \text{ and } h < 1/4. \)

The Morrey-Nirenberg estimates of Theorem 1 apply to the above equation to give the following estimate: If \( x \in B_{1/8}(0) \) and \( h < r \leq 1/16 \), then
\[
|D\delta_h w(x+h\xi) - D\delta_h w(x)| \leq C(h/r)^{\alpha} \left( |u|_{P^2(Q_1)} + r|\varphi| \right).
\]

where \( C \) depends on \( \Lambda/\lambda \). In particular, if we choose \( r = h^{\beta} \) for \( \beta \in (0,1) \) then we have for \( x \in B_{1/8} \) and \( h < 2^{-4/\beta} \)
\[
|\delta_h \delta_h Dw| \leq Ch^{-1}(1 + |u|_{P^2(Q_1)}) \left( h^{(1-\beta)} + h^{\alpha-(1-\beta)}(1-\alpha) \right).
\]

If we choose \( 1 > \beta > \max\{0, \frac{1-\alpha}{2}\} \) then we find
\[
|\delta_h \delta_h Dw| \leq Ch^{-1+\varepsilon}(1 + |u|_{P^2(Q_1)}),
\]
for some \( \varepsilon \in (0,1] \) depending on \( \alpha \) and \( \beta \), and it follows from the characterisation of \( C^{1,\alpha} \) functions in section 2.4 that the second derivatives of \( w \) are Hölder-continuous with exponent \( \varepsilon \), and
\[
|w|_{C^{2,\varepsilon}(B_{1/8})} \leq C(1 + |u|_{P^2(Q_1)})
\]
where \( C \) depends on \( \lambda \) and \( Lip(F) \). This proves the required spatial regularity of spatial second derivatives on the region \( Q_{1/8} \).

3.3 Time regularity of first space derivatives.

The spatial \( C^{2,\alpha} \) estimate established in the previous section can be used to deduce an estimate on the continuity of first spatial derivatives in time, using the parabolic maximum principle.

Let \( \xi \) be a unit vector. The function \( \delta_h u \) defined by \( \delta_h u(x) = (u(x+h\xi) - u(x))/h \) satisfies an useful evolution equation:
\[
\frac{\partial}{\partial t} \delta_h u = a^{ij} D_i D_j \delta_h u + \varphi
\]

where \( \sup_{Q_{1/16}} |\varphi| \leq C(Lip(F), |u|_{P^2(Q_1)}) \). The result of Section 3.2 shows that \( \delta_h u \) is \( C^{1,\alpha} \) (uniformly in \( h \)), so that
\[
|\delta_h u(z', t) - \delta_h u(z, t) - D\delta_h u(z, t) \cdot (z' - z)| \leq C |z' - z|^{1+\alpha}
\]
on the region $Q_{1/8-h}$. Young’s inequality gives the estimate
\[ |z' - z|^{1+\alpha} \leq \varepsilon + C\varepsilon^{-\frac{1}{1+\alpha}}|z' - z|^2, \]
for any $\varepsilon > 0$, and therefore
\[ \delta_h u(z', t) \leq \delta_h u(z, t) + D\delta_h u(z, t) \cdot (z' - z) + \varepsilon + C\varepsilon^{-\frac{1}{1+\alpha}}|z' - z|^2. \]
By the bounds on $a^{ij}$ and $\varphi$, the function
\[ \Psi_+(z', t') = \delta_h u(z, t) + D\delta_h u(z, t) \cdot (z' - z) + \varepsilon + C\varepsilon^{-\frac{1}{1+\alpha}}|z' - z|^2 \]
+ (sup $|\varphi| + 4CA\varepsilon^{-\frac{1}{1+\alpha}})(t' - t) \]
is a supersolution of Equation (3.3.1) on $Q_{1/16}$ provided $h < 1/32$, and if $z \in B_{1/32}$ and $\varepsilon < C(A, \alpha)|u|^{1+\alpha}_{P^2(Q_{1})}$ then this supersolution is above $\delta_h u$ on the boundary $\partial B_{1/16} \times (-1/256, 0]$. Therefore by the parabolic maximum principle $\delta_h u$ is bounded by $\Psi_+$. After evaluating at $z' = z$, and optimizing in $\varepsilon$, this becomes
\[ \delta_h u(z, t') \leq \delta_h u(z, t) + C(t' - t)\frac{1+\alpha}{1+\alpha} + C|u|^{1+\alpha}_{P^2(Q_{1})}(t' - t). \]
A similar estimate from below follows by comparison with a suitable subsolution. The desired continuity in time of the first spatial derivatives in $Q_{1/32}$ follows on sending $h$ to zero.

### 3.4 Time regularity of second space derivatives.

The proof of the parabolic $C^{2,\alpha}$ estimate can now be completed by deducing appropriate continuity of the second spatial derivatives in time. We will deduce this as a consequence of the previous two estimates.

On $Q_{1/32}$ we have the estimates
\[ |D^2u(x, t) - D^2u(y, t)| \leq C|x - y|^{\alpha} \]
and
\[ |Du(x, t) - Du(x, t - \tau)| \leq C|\tau|^{\frac{1}{2}} \]
promoted $t - \tau \geq -R^2/4$ and $x, y \in B_{R/2}$. Fix $x \in B_{1/64}$ and $s, t \in (-1/256, 0]$. Let $\xi$ be an arbitrary unit vector. Then we have (provided $x + h\xi \in B_{1/32}$)
\[ D_\xi D_\xi u(x, t) = \frac{D_\xi u(x + h\xi, t) - D_\xi u(x, t)}{h} + \frac{1}{h} \int_0^h (D_\xi D_\xi u(x + r\xi, t) - D_\xi D_\xi u(x, t)) \, dr \]
and so
\[ |D_\xi D_\xi u(x, t) - D_\xi D_\xi u(x, s)| \leq h^{-1} \int_0^h \left| (D_\xi D_\xi u(x + r\xi, t) - D_\xi D_\xi u(x, t)) \right| \, dr \]
+ $h^{-1} \int_0^h \left| (D_\xi D_\xi u(x + r\xi, s) - D_\xi D_\xi u(x, s)) \right| \, dr$
+ $h^{-1} \left| D_\xi u(x + h\xi, t) - D_\xi u(x + h\xi, s) \right|$
+ $h^{-1} \left| D_\xi u(x, t) - D_\xi u(x, s) \right|$
\[ \leq Ch^{\alpha} + Ch^{-1}|s - t|^{\frac{1+\alpha}{2}}. \]
Since $|s - t| < 1/256$ and $x \in B_{1/64}$, we can safely choose $h = |s - t|^{1/2} < 1/64$ and still ensure that $x + h\xi \in B_{1/32}$. This choice gives
\[ |D_\xi D_\xi u(x, t) - D_\xi D_\xi u(x, s)| \leq C|s - t|^{\alpha/2}, \]
proving the desired estimate.
4. Higher dimensions

In higher dimensions, a similar argument can be used to extend the class of fully nonlinear parabolic equations for which second derivative Hölder estimates hold: Instead of concavity as a function of the second derivatives, it suffices that the level sets of \( F(r, p, z, x, t) \) as a function of \( r \) (for fixed \( p, z, x \) and \( t \)) are convex.

Hölder regularity of \( u_t \) follows exactly as in Section 3.1. The key step is to show spatial regularity of the second space derivatives in analogy with Section 3.2.

The argument in Section 3.1 gives the estimate

\[
|u_t|^{p_n \cdot Q_1/2} \leq C \left( \text{Lip}(F) + |u_t|_{L^\infty(Q_1)} \right).
\]

As in Section 3.2, for any fixed \( t \in (-1/4, 0] \) the uniformly elliptic equation \( G[D^2w(x), Dw(x), w(x), x] = φ(x) \) holds, where \( w(x) = u(x, t) \), \( G[r, p, z, x] = F[r, p, z, x, t] \) and \( φ(x) = u_t(x, t) \). Note that \( Λ \leq G \leq Λ \) and \( \text{Lip}(G) \leq \text{Lip}(F) \).

Since the level sets of \( F \) in the first argument are convex, and \( F \) is uniformly monotone and \( C^2 \), there exists a constant \( K \) such that

\[
\tilde{F}^{klmn} M_{kl} M_{mn} \leq K \tilde{F}^{kl} \tilde{F}^{mn} M_{kl} M_{mn}
\]

for any symmetric matrix \( M \). But then \( \tilde{G} = -\exp(-KG) \) is uniformly monotone and concave in the first argument (with ellipticity constants depending on \( \sup_{Q_1} |D^2u| \)), and we have \( \tilde{G}[D^2w, Dw, w, x] = \tilde{φ} \) where \( \tilde{φ} = -\exp(-Kφ) \) is Hölder-continuous on \( B_{1/2} \). A perturbation result (see [C], Theorem 3) then implies the estimate \( |D^2w(y) − D^2w(x)| \leq C|y − x|^α \) for \( x \) and \( y \) in \( B_{1/8} \). The arguments of Sections 3.3 and 3.4 now apply unchanged to give the full result.

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