A Down to Earth Attempt at Determining the Low Energy Effective Action of \( N = 2 \) Supersymmetric Yang-Mills Theory

M. Chaichian\(^a,b\), W. F. Chen\(^a,c\) and C. Montonen\(^b\)

\(^a\) High Energy Physics Division, Department of Physics, University of Helsinki
\(^b\) Helsinki Institute of Physics, P.O. Box 9 (Siltavuorenpuisto 20 C)
FIN-00014 University of Helsinki, Finland
\(^c\) Winnipeg Institute for Theoretical Physics and Department of Physics
University of Winnipeg, Winnipeg, Manitoba, Canada R3B 2E9

Abstract

We review a detailed investigation of the perturbative part of the low-energy effective action of \( N = 2 \) supersymmetric Yang-Mills theory in a conventional effective field theory approach. With the restriction that the effective action should contain at most two derivatives and not more than four-fermion couplings, the features of the low-energy effective action obtained by Seiberg based on \( U(1)_R \) anomaly and non-perturbative \( \beta \)-function arguments are shown to emerge.

1 Introduction

The understanding to non-perturbative dynamics of supersymmetric gauge theory has made rapid progress in recent years following the seminal contribution by Seiberg and Witten\(^[1]\), combining the ideas of holomorphicity\(^[2]\) and duality\(^[3]\). The web of arguments leading to the explicit results consists of a skillful combination of perturbative and nonperturbative arguments, formal considerations and physical reasoning. It should be checked by explicit computations, whenever possible, that no unexpected failure of these arguments occurs. In a recent work we have made an investigation in this direction\(^[4]\) and this paper is intended as a review.

The starting point in Seiberg and Witten’s work is the low-energy effective action of an \( N = 2 \) supersymmetric Yang-Mills theory with the gauge group \( SU(2) \) of the following form,

\[
\Gamma = \frac{1}{16\pi} \text{Im} \int d^4x d^2\theta d^2\bar{\theta} \left[ \frac{1}{2} \tau \Psi^2 + \frac{i}{2\pi} \Psi^2 \ln \frac{\Psi^2}{\Lambda^2} + \sum_{n=1}^{\infty} A_n \left( \frac{\Lambda^2}{\Psi^2} \right)^{2n} \Psi^2 \right],
\]

(1)

where \( \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \) is the modular parameter and \( \Psi \) the \( N = 2 \) chiral superfield describing the light degrees of freedom. The logarithmic term represents the one-loop perturbative result and was first obtained by Di Vecchia et al.\(^[5]\) in a calculation where they coupled the gauge superfield to an \( N = 2 \) matter supermultiplet and integrated out the latter. Subsequently, Seiberg\(^[2]\) used the anomalous transformation behaviour under \( U(1)_R \) and holomorphicity to argue that the full low-energy effective action should take the form \((\ref{eq:1})\), where the infinite series arises from nonperturbative instanton contributions. The Seiberg-Witten solution \([\ref{eq:1}]\) gives the explicit form of this part of \( \Gamma \).

\(^[1]\) contributed paper to LP99, August, SLAC, USA
The form (1) has been confirmed by calculations in $N = 1$ superspace and in harmonic superspace, extending the result to nonleading terms in the number of derivatives $[6, 7, 8, 9, 10]$. Independent confirmation has been obtained from $M$-theory $[11]$.

Our intention is to check the perturbative part of the effective action in Wess-Zumino gauge by a very down-to-earth calculation. In the Higgs phase of the theory, the $SU(2)$ gauge symmetry breaks down to $U(1)$, and the super-Higgs mechanism splits the supermultiplet into a massive one and a massless one. The effective action of the massless fields should be obtained by integrating out the heavy fields. In comparison with other approaches, our method is quite conventional and is along the lines of the standard definition of the low-energy effective theory.

It should be emphasized that this conventional calculation is very complicated. Even this modest programme we cannot carry out fully. What we have actually accomplished is the computation of the heavy fermion determinant. Reassuringly, we find that the form (1) is reproduced. Although no unexpected surprises were unearthed by our calculation, we still hope that it has some pedagogical value in showing explicitly how the effective action arises.

The outline of this review is as follows. In section 2 we describe the model and exhibit the Higgs mechanism. Section 3 contains the computation of the heavy fermion determinant using the constant field approximation. The detailed calculations of the fermion eigenvalues and their degeneracies, which contain certain subtle points, are given in Appendix B. In section 4 we present a discussion of the results. In the pedagogical vein of this paper, we give in Appendix A the component form of the low-energy effective action (1).

## 2 Splitting of $N = 2$ Supermultiplet

The classical action of $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory is $[12]$,

$$
S = \int \! d^4x \left[ \frac{-1}{4} G^{\mu\nu a}_a G_{\mu\nu a} + D_\mu \varphi^\dagger a D^\mu \varphi^a + i \bar{\psi}^a \gamma^\mu D_\mu \psi^b \right. \\
+ \frac{ig}{\sqrt{2}} \epsilon^{abc} \overline{\psi}^c \left[(1 - \gamma^5) \varphi^a + (1 + \gamma^5) \varphi^\dagger a\right] \psi^b + \frac{g^2}{2} \epsilon^{abc} \epsilon^{ade} \varphi^b \varphi^c \varphi^d \varphi^e \bigg],
$$

(2)

where

$$
G^a_{\mu\nu} = \partial_\mu K^a_\nu - \partial_\nu K^a_\mu - g \epsilon^{abc} K^b_\mu K^c_\nu, \quad D_\mu \varphi^a = \partial_\mu \varphi^a - g \epsilon^{abc} K^b_\mu \varphi^c, \\
\varphi^a = \frac{1}{\sqrt{2}} (S^a + i P^a), \quad \varphi^\dagger a = \frac{1}{\sqrt{2}} (S^a - i P^a), \quad a = 1, 2, 3.
$$

The bosonic part of the action (2) is just the Georgi-Glashow model in the Bogomol’nyi-Prasad-Sommerfield (BPS) limit. In addition to the fermionic term and Yukawa interaction term, this action has the scalar potential

$$
V(\varphi) = -\frac{g^2}{2} \epsilon^{abc} \epsilon^{ade} \varphi^b \varphi^c \varphi^d \varphi^e \equiv g^2 \text{Tr} \left( [\varphi, \varphi^\dagger] \right)^2.
$$

(3)

The unbroken supersymmetry requires that in the ground state the scalar potential must vanish, which leads to

$$
[\varphi, \varphi^\dagger] = 0.
$$

(4)
The corresponding classical Lagrangian can be written as follows,

\[ \langle S^a \rangle = v \delta^a 3, \quad \langle P^a \rangle = 0, \quad (5) \]

where \( v \) is a real constant. For \( v \neq 0 \) the theory is in the Higgs phase and exhibits a spontaneous breaking of the gauge symmetry. In a unitary gauge

\[ S^T = (0, 0, S + v). \quad (6) \]

The corresponding classical Lagrangian can be written as follows,

\[ \mathcal{L} = \mathcal{L}_V + \mathcal{L}_S + \mathcal{L}_P + \mathcal{L}_F + \mathcal{L}_Y, \quad (7) \]

where \( \mathcal{L}_V, \mathcal{L}_S, \mathcal{L}_P, \mathcal{L}_F \) and \( \mathcal{L}_Y \) denote respectively the vector field, the scalar field, the scalar potential, the fermionic and the Yukawa interaction parts,

\[ \mathcal{L}_V = -\frac{1}{4}(\partial^\mu A^\mu - \partial^\nu A^\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{2}(\partial_\mu W_\mu^- + \partial_\nu W_\nu^+)(\partial^\nu W^- - \partial^\nu W^-) \]

\[-ig[(\partial^\mu W_\mu^+ W_\mu^- - \partial^\nu W_\nu^- W_\nu^+) A^\nu + (\partial_\mu W_\mu^- W_\mu^+ - \partial_\nu W_\nu^+ W_\nu^-) A^\nu] \]

\[+ (\partial^\mu A^\nu - \partial^\nu A^\mu)W_\mu^+ W_\nu^-] + g^2(-W_\mu^+ W^- A_\nu A^\nu + W_\mu^+ W_\nu^- A^\mu A^\nu) \]

\[\frac{g^2}{2} W^+ W^- \nu(W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+); \quad (8)\]

\[\mathcal{L}_S = \frac{1}{2} \partial_\mu P \partial^\mu P + \frac{1}{2} \partial_\mu S \partial^\mu S + \partial_\mu P^+ \partial^\mu P^- + igA^\mu(\partial_\mu P^- P^+ - \partial_\mu P^+ P^-) \]

\[+igP(\partial^\mu P^+ W_\mu^- - \partial^\nu P^- W_\nu^+) + ig\partial^\mu P(W_\mu^+ P^- - W_\mu^- P^+) + g^2 P^2 W^+ W^- \]

\[+g^2(S + v)^2 W^+ W^- + g^2 A^\nu A_\nu P^+ P^- - g^2(W_\mu^+ P^- - W_\mu^- P^+) A^\mu P \]

\[\frac{g^2}{2} (W^+ P^- - W^- P^+)^2. \quad (9)\]

\[\mathcal{L}_P = g^2(S + v)^2 P^+ P^-, \quad (10)\]

where the various quantities are defined as follows:

\[ W_\mu^+ \equiv \frac{1}{\sqrt{2}}(K_\mu^1 - iK_\mu^2), \quad W_\mu^- \equiv \frac{1}{\sqrt{2}}(K_\mu^1 + iK_\mu^2), \quad K_\mu^a \equiv A_\mu^a; \]

\[ P^+ \equiv \frac{1}{\sqrt{2}}(P^1 - iP^2), \quad P^- \equiv \frac{1}{\sqrt{2}}(P^1 + iP^2), \quad P^3 \equiv P. \quad (11)\]

The above Lagrangians clearly show that \( W_\mu^\pm \) and \( P^\pm \) become massive with mass \( m \equiv \left| gv \right| \) while \( A_\mu, S \) and \( P \) remain massless.

Up to some total derivative terms, the bosonic part of the Lagrangian can be rewritten in the following form:

\[ \mathcal{L}_B = \mathcal{L}_V + \mathcal{L}_S + \mathcal{L}_P \]

\[= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu P \partial^\mu P + \frac{1}{2} \partial_\mu S \partial^\mu S + \frac{1}{2} W^{+\mu} \left[ \eta_{\mu\nu} D^\dagger \alpha D_\alpha - D^\dagger \nu D_\nu - igF_{\mu\nu} \right] W^{-\nu} \]
\[ + \frac{1}{2} W^{-\mu} \left[ \eta_{\mu \nu} D^\alpha D^\dagger_\alpha D_{\nu} D^\dagger_\nu + ig F_{\mu \nu} \right] W^{+\nu} + g^2 [P^2 + (S + v)^2] W^{+\mu} W^{-\mu} \\
+ \frac{1}{2} P^+ (-\partial^\mu \partial_\mu + 2ig A_\mu \partial^\mu + g^2 A_\mu A^\mu) P^- + \frac{1}{2} P^- (-\partial^\mu \partial_\mu - 2ig A_\mu \partial^\mu + g^2 A_\mu A^\mu) P^+ \\
+ \frac{1}{2} W^{+\mu} (-ig P \partial^\mu + ig \partial^\mu P - g^2 A_\mu P) W^{-\mu} + \frac{1}{2} W^{-\mu} (-ig \partial^\mu P + ig P \partial^\mu - g^2 A_\mu P) W^{+\mu} \\
+ \frac{1}{2} P^+ (-2ig \partial^\mu P - ig P \partial^\mu - g^2 A_\mu P) W^{+\mu} + \frac{1}{2} W^{-\mu} (-2ig \partial^\mu P + ig P \partial^\mu - g^2 A_\mu P) W^{-\mu} \\
+ \frac{g^2}{2} W^{+\mu} W^{-\nu} (W^{+\mu} W^- - W^{-\mu} W_\nu) - \frac{g^2}{2} (W^{+\mu} P^- - W^{-\mu} P^+)^2 \]

\[ = - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \partial_\mu \phi^2 \partial^\mu \phi + \frac{1}{2} W^{+\mu} \Delta_{\mu \nu} W^{-\nu} + \frac{1}{2} W^{-\mu} \Delta_{\mu \nu} W^{+\nu} + \frac{1}{2} P^+ \Delta P^- \\
+ \frac{1}{2} P^- \Delta^+ P^+ + \frac{1}{2} W^{+\mu} \Delta_\mu W^{-\nu} + \frac{1}{2} W^{-\mu} \Delta^+_\mu W^{+\nu} + \frac{1}{2} P^+ \Delta_\mu W^{-\nu} + \frac{1}{2} W^{+\mu} \Delta^+_\mu P^+ \\
+ \frac{g^2}{2} W^{+\mu} W^{-\nu} (W^{+\mu} W^- - W^{-\mu} W_\nu) - \frac{g^2}{2} (W^{+\mu} P^- - W^{-\mu} P^+)^2, \quad (12) \]

where

\[
\Delta_{\mu \nu} = \eta_{\mu \nu} D^{+\alpha} D_{\alpha} - D^\dagger_{\nu} D_{\mu} - ig F_{\mu \nu} + g^2 |\sqrt{2} \phi + v|^2 \eta_{\mu \nu}, \\
\Delta^+_{\mu \nu} = \eta_{\mu \nu} D^\alpha D^\dagger_{\alpha} - D^\dagger_{\nu} D^\dagger_{\mu} + ig F_{\mu \nu} + g^2 |\sqrt{2} \phi + v|^2 \eta_{\mu \nu}; \\
\Delta_\mu = -ig P \partial^\mu + ig \partial^\mu P - g^2 A_\mu P, \quad \Delta^+_{\mu} = ig P \partial^\mu + ig \partial^\mu P - g^2 A_\mu P; \\
\Delta = -\partial^\mu \partial_\mu + 2ig A_\mu \partial^\mu + g^2 A_\mu A^\mu, \quad \Delta^+ = -\partial^\mu \partial_\mu - 2ig A_\mu \partial^\mu + g^2 A_\mu A^\mu; \\
D_\mu = \partial_\mu - ig A_\mu, \quad D^\dagger_\mu = \partial_\mu + ig A_\mu; \quad \phi \equiv \frac{1}{\sqrt{2}} (S + iP). \quad (13)\]

To explicitly show that the spinor fields split into massive and massless ones, we need some operations on $L_F$ and $L_Y$. The fermionic part is

\[
L_F = i \bar{\psi} \gamma^\mu \partial_\mu \psi^1 + i \bar{\psi} \gamma^\mu \partial_\mu \psi^2 + i \bar{\psi} \gamma^\mu \partial_\mu \psi^3 \\
+ \frac{g}{\sqrt{2}} \bar{\psi} (W^{+\mu} - W^{-\mu}) \gamma^\mu \psi^3 + ig \bar{\psi} A_\mu \gamma^\mu \psi^2 \\
+ ig \bar{\psi} A_\mu \gamma^\mu \psi^2 + \frac{g}{\sqrt{2}} \bar{\psi} \gamma^\mu (W^{+\mu} + W^{-\mu}) \psi^3 \\
- \frac{ig}{\sqrt{2}} \bar{\psi} \gamma^\mu (W^{+\mu} + W^{-\mu}) \psi^3 - \frac{g}{\sqrt{2}} \bar{\psi} \gamma^\mu (W^{+\mu} - W^{-\mu}) \psi^1. \quad (14)\]

As for the Yukawa part, we first write it in terms of chiral spinors,

\[
L_Y = i \sqrt{2} g f^{abc} \bar{\psi} \gamma^a \varphi^c \psi^b_R + i \sqrt{2} g f^{abc} \bar{\psi} R \varphi^a \psi^b_L, \quad (15)\]

where $\psi_L = \frac{1}{2} (1 - \gamma_5) \psi$ and $\psi_R = \frac{1}{2} (1 + \gamma_5) \psi$. In the unitary gauge, Eq.(15) becomes

\[
L_Y = ig(\sqrt{2} \phi + v)(\bar{\psi}^2 \psi^1_R - \bar{\psi}^1 \psi^2_R) + ig(\sqrt{2} \bar{\psi}^* + v)(\bar{\psi}^2 \psi^1_L - \bar{\psi}^1 \psi^2_L).
\]

4
With the combination
\[ \Psi_1 = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2), \quad \Psi_2 = \frac{1}{\sqrt{2}}(\psi^1 - i\psi^2), \quad \Psi = \psi^3, \]

(17)

\( \mathcal{L}_F \) and \( \mathcal{L}_Y \) can be formulated in these new fields,

\[
\begin{align*}
\mathcal{L}_Y &= -g\bar{\Psi}_1 \left[ \frac{1}{\sqrt{2}}(1 - \gamma_5)\phi + \frac{1}{\sqrt{2}}(1 + \gamma_5)\phi^* + v \right] \Psi_1 \\
&\quad + g\bar{\Psi}_2 \left[ \frac{1}{\sqrt{2}}(1 - \gamma_5)\phi + \frac{1}{\sqrt{2}}(1 + \gamma_5)\phi^* + v \right] \Psi_2 \\
&\quad - igP^+\bar{\Psi}\gamma_5\Psi_1 + igP^-\bar{\Psi}\gamma_5\Psi_2 - ig\bar{\Psi}_1\gamma_5\Psi P^- + ig\bar{\Psi}_2\gamma_5\Psi P^+, \
\end{align*}
\]

(18)

\[
\mathcal{L}_F = -g\bar{\Psi}_1\gamma^\mu\partial_\mu\Psi_1 + i\bar{\Psi}_2\gamma^\mu\partial_\mu\Psi_2 + i\bar{\Psi}\gamma^\mu\partial_\mu\Psi
\]

+ \[ g\bar{\Psi}_1\gamma^\mu A_\mu\Psi_1 - g\bar{\Psi}_2\gamma^\mu A_\mu\Psi_2 \]

+ \[ g\bar{\Psi}_2\gamma^\mu W^{+}_\mu\Psi - g\bar{\Psi}_1\gamma^\mu W^{-}_\mu\Psi \]

\[ - g\bar{\Psi}_\gamma^\mu W^{+}_\mu\Psi_1 + g\bar{\Psi}_\gamma^\mu W^{-}_\mu\Psi_2. \]

(19)

So now the whole classical action is given by the Lagrangian

\[
\begin{align*}
\mathcal{L} &= -\frac{1}{4}F_{\mu
u}F^{\mu\nu} + \partial_\mu\phi^*\partial^\mu\phi + i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + \frac{1}{2}W^{+\mu}\Delta_\mu W^{-\nu} + \frac{1}{2}W^{-\mu}\Delta_\mu W^{+\nu} \\
&\quad + \frac{1}{2}P^+\Delta P^- + \frac{1}{2}P^-\Delta^+ P^+ + \frac{1}{2}W^{+\mu}\Delta_\mu P^- + \frac{1}{2}P^-\Delta^+ W^{+\mu} \\
&\quad + \frac{1}{2}P^+\Delta^+ W^{-\mu} + \frac{1}{2}W^{-\mu}\Delta^+ P^+ + \bar{\Psi}_1\Delta_F\Psi_1 + \bar{\Psi}_2\Delta_F\Psi_2 \\
&\quad - igP^+\bar{\Psi}_1\gamma_5\Psi_1 + igP^-\bar{\Psi}_1\gamma_5\Psi_2 - ig\bar{\Psi}_1\gamma_5\Psi P^- + ig\bar{\Psi}_2\gamma_5\Psi P^+ \\
&\quad + g\bar{\Psi}_2\gamma^\mu W^{+}_\mu\Psi - g\bar{\Psi}_1\gamma^\mu W^{-}_\mu\Psi + g\bar{\Psi}_2\gamma^\mu W^{+}_\mu\Psi_1 + g\bar{\Psi}_1\gamma^\mu W^{-}_\mu\Psi_2 \\
&\quad + \frac{g^2}{2}W^{+\mu}W^{-\nu}(W^{+}_\mu W^{-}_\nu - W^{-}_\mu W^{+}_\nu) - \frac{g^2}{2}(W^+ P^- - W^- P^+)^2
\end{align*}
\]

(20)

with

\[
\Delta_F \equiv i\gamma^\mu D_\mu - \frac{g}{\sqrt{2}}(1 - \gamma_5)\phi - \frac{g}{\sqrt{2}}(1 + \gamma_5)\phi^* + gv,
\]

\[
\tilde{\Delta}_F \equiv i\gamma^\mu D_\mu + \frac{g}{\sqrt{2}}(1 - \gamma_5)\phi + \frac{g}{\sqrt{2}}(1 + \gamma_5)\phi^* + gv.
\]

(21)

3. Low-energy Effective Action: Calculation of the Fermionic Determinant in Constant Field Approximation

The standard definition of the low-energy effective action is given by

\[
\exp \left\{ i \Gamma_{\text{eff}}[A_\mu, \phi, \Psi, \bar{\Psi}] \right\} \equiv \int D\Psi D\bar{\Psi} D\bar{\Psi} D\Psi D\Psi D\bar{\Psi} D\bar{\Psi} D\Psi D\bar{\Psi} D\Psi \exp \left[ i \int d^4x \mathcal{L} \right].
\]

(22)
At tree level
\[
\Gamma_{\text{eff}}^{(0)} = S_{\text{tree}} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial^\mu \phi^* \partial_\mu \phi + i \overline{\psi} \gamma^\mu \partial_\mu \psi \right]. \tag{23}
\]

At one-loop level, the integration over the heavy modes will lead to the determinants of the dynamical operators. In practical calculation we cannot evaluate the determinant exactly. Here we shall employ a technique called constant field approximation to compute the determinant, which was invented by Schwinger\cite{13} and later was used in in \cite{5} and \cite{14} to extract the anomaly term in \( N = 2 \) supersymmetric Yang-Mills theory and the one-loop effective action of the supersymmetric \( CP^{N-1} \) model. To apply this method we first rewrite the the quadratic part of the classical action \cite{20} as
\[
S_{\text{quad}} = S_{\text{tree}} + \int d^4x \left( \Phi^\dagger M_{bb} \Phi + \overline{\psi} M_{fb} \phi + \Phi^\dagger M_{bf} \overline{\psi} + \overline{\psi} M_{ff} \phi \right), \tag{24}
\]
where
\[
\Phi \equiv \begin{pmatrix} W_{-\mu} \\ W_{+\mu} \\ P^- \\ P^+ \end{pmatrix}, \quad \overline{\psi} \equiv \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix};
\]
\[
M_{bb} \equiv \frac{1}{2} \begin{pmatrix} \Delta_{\mu\nu} & 0 & \Delta_\mu & 0 \\ 0 & \Delta_{\mu\nu} & 0 & \Delta_\mu \\ \Delta_\nu & 0 & \Delta & 0 \\ 0 & \Delta_\nu & 0 & \Delta \end{pmatrix};
\]
\[
M_{fb} \equiv \frac{1}{2} \begin{pmatrix} -g\gamma_\mu \Psi & 0 & -ig \gamma_5 \Psi & 0 \\ 0 & g\gamma_\mu \Psi & 0 & ig \gamma_5 \Psi \\ -g\gamma_\mu \Psi & 0 & -ig \gamma_5 \Psi & 0 \\ 0 & g\gamma_\mu \Psi & 0 & ig \gamma_5 \Psi \end{pmatrix};
\]
\[
M_{bf} \equiv \frac{1}{2} \begin{pmatrix} -g\overline{\psi} \gamma_\mu & 0 & -g\overline{\psi} \gamma_\mu & 0 \\ 0 & g\overline{\psi} \gamma_\mu & 0 & g\overline{\psi} \gamma_\mu \\ -ig \overline{\psi} \gamma_5 & 0 & -ig \overline{\psi} \gamma_5 & 0 \\ 0 & ig \overline{\psi} \gamma_5 & 0 & ig \overline{\psi} \gamma_5 \end{pmatrix};
\]
\[
M_{ff} \equiv \frac{1}{2} \begin{pmatrix} \Delta_F & 0 & 0 & 0 \\ 0 & \Delta_F & 0 & 0 \\ 0 & 0 & \Delta_F & 0 \\ 0 & 0 & 0 & \Delta_F \end{pmatrix}. \tag{25}
\]

Using the standard formulas
\[
I = \int \mathcal{D}b^\dagger \mathcal{D}b \mathcal{D}\overline{\psi} \mathcal{D}\psi \exp \left[ \int (dx) \left( b^\dagger M_{bb} b + \overline{f} M_{fb} b + b^\dagger M_{bf} \overline{f} + \overline{f} M_{ff} f \right) \right]
\]
\[
= \int \mathcal{D}b^\dagger \mathcal{D}b \mathcal{D}\overline{\psi} \mathcal{D}\psi \exp \left\{ \int (dx) \left[ b^\dagger (M_{bb} - M_{ff} M_{ff}^{-1}) M_{fb} b 
+ (\overline{f} + b^\dagger M_{bf} M_{ff}^{-1}) M_{ff} (M_{ff}^{-1} M_{fb} f + f) \right] \right\}
\]
\[
= \det M_{ff} \det^{-1} (M_{bb} - M_{bf} M_{ff}^{-1} M_{fb});
\]
\[
\det M = \exp \text{Tr} \ln M, \tag{26}
\]
\[ b \text{ and } f \text{ representing the general bosonic and fermionic fields, respectively, we get} \]
\[
Z[A, \phi, \Psi, \overline{\Psi}] = \exp \left\{ i \Gamma_{\text{eff}}[A, \phi, \Psi, \overline{\Psi}] \right\} \equiv \int D\Psi D\overline{\Psi} D\phi D\Pi D\phi D\Pi D\phi D\Pi \exp [i S] \\
= \exp [i S_{\text{tree}}] \det M_{ff} \det^{-1}(M_{bb} - M_{bf} M^{-1}_{ff} M_{fb}) \\
= \exp \left[ i S_{\text{tree}} + \text{Tr ln } M_{ff} - \text{Tr ln}(M_{bb} - M_{bf} M^{-1}_{ff} M_{fb}) \right] ; \\
\Gamma_{\text{eff}} = S_{\text{tree}} - i \left[ \text{Tr ln } M_{ff} - \text{Tr ln}(M_{bb} - M_{bf} M^{-1}_{ff} M_{fb}) \right] . \tag{27}
\]

The following task is to evaluate the above determinants. Let us first consider the fermionic part. Since \( M_{ff} \) has the form of a reducible matrix,
\[
\det M_{ff} = \frac{1}{16}(\det \Delta_{F})^{2}(\det \overline{\Delta}_{F})^{2} = \frac{1}{16}\exp[2(\text{Tr ln } \Delta_{F} + \text{Tr ln } \det \overline{\Delta}_{F})]. \tag{28}
\]

Now we switch on the constant field approximation to work out the eigenvalues and eigenvectors of the above operators and further evaluate the determinant. We choose only the third components of the electric and magnetic fields to be the constants different from zero,
\[
-E_{3} = F^{03} \neq 0, \quad B_{3} = F^{12} \neq 0, \tag{29}
\]
and \( \phi \) the non-vanishing constant field. Consequently, the potential becomes
\[
A^{1} = -F^{12}x_{2}, \quad A^{3} = -F^{30}x_{0}, \quad A^{0} = A^{2} = 0. \tag{30}
\]

To get the eigenvalues of the operators, it is necessary to rotate into Euclidean space,
\[
x^{4} = x_{4} = -ix^{0}, \quad \partial_{0} = \frac{\partial}{\partial x^{0}} = i \frac{\partial}{\partial x^{4}}, \\
f^{34} = f_{34} = iF^{30}, \quad f^{12} = f_{12} = F^{12}. \tag{31}
\]

Let us first consider \( \det \Delta_{F} \). The eigenvalue equation for \( \Delta_{F} \) is
\[
\Delta_{F}\psi(x) = \left[ i\gamma^{\mu}D_{\mu} - \frac{g}{\sqrt{2}}(1 - \gamma_{5})\phi - \frac{g}{\sqrt{2}}(1 + \gamma_{5})\phi^{*} - gv \right] \psi(x) = \omega \psi_{1}, \tag{32}
\]
where \( \psi \) is a four-component spinor wave function. In order to get normalizable eigenstates, we consider the system in a box of finite size \( L \) in the \( x_{1} \) and \( x_{3} \) directions with periodic boundary conditions, so the eigenvector should be of the following form,
\[
\psi(x) = \frac{1}{L}e^{ip_{1}x_{1}}e^{ip_{3}x_{3}}\chi(x_{2}, x_{4}), \\
p_{1} = \frac{2\pi l}{L}, \quad p_{3} = \frac{2\pi k}{L}, \quad k, l = \text{integers}. \tag{33}
\]

To find the eigenvalues and eigenvectors, we write the operators and the wave function in two-component forms,
\[
\Delta_{F} = \begin{pmatrix}
-g(\sqrt{2}\phi^{*} + v)1 & \Delta^{-} \\
\Delta^{+} & -g(\sqrt{2}\phi + v)1
\end{pmatrix}, \quad \chi = \begin{pmatrix}
\chi_{1} \\
\chi_{2}
\end{pmatrix}, \tag{34}
\]

7
where $1$ is the $2 \times 2$ identity matrix and

$$
\Delta^\pm = \partial_4 \pm i \left[ \sigma_1(\partial_1 + igf_{12}x_2) + \sigma_2(\partial_2 + igf_{34}x_4) \right].
$$

The eigenvalue equation (32) is thus reduced to the following set of equations,

$$
- g(\sqrt{2}\phi^* + v)\chi_1 + \Delta^- \chi_2 = \omega \chi_1,
\Delta^+ \chi_1 - g(\sqrt{2}\phi + v)\chi_2 = \omega \chi_2,
$$

and now

$$
\Delta^\pm = \partial_4 \pm [\sigma_1(p_1 + gf_{12}x_2) - i\sigma_2(p_2 + gf_{34}x_4)].
$$

A detailed calculation and discussion of the eigenvalues are collected in Appendix B. We obtain two series of eigenvalues,

$$
\omega_{\pm}(m,n) = -g \left[ \frac{\phi + \phi^*}{\sqrt{2}} + v \right] \pm \sqrt{\frac{1}{2}g^2(\phi - \phi^*)^2 - 2mgf_{12} - 2ngf_{34}},
$$

where for $m \geq 1$, $n \geq 1$ both eigenvalues are doubly degenerate, while $\omega_{\pm}(m,0)$ and $\omega_{\pm}(0,n)$ are nondegenerate, and for $m = n = 0$, there exists only a nondegenerate eigenvalue $\omega_{-(0,0)}$.

In a similar way we can solve the eigenvalue equation

$$
\Delta_F \psi = \left[ i\gamma^\mu D^\dagger_\mu + \frac{g}{\sqrt{2}}(1 - \gamma_5)\phi + \frac{g}{\sqrt{2}}(1 + \gamma_5)\phi^* + gv \right] \psi = \bar{\omega} \psi,
$$

and obtain the eigenvalues,

$$
\bar{\omega}_{\pm}(m,n) = g \left[ \frac{\phi + \phi^*}{\sqrt{2}} + v \right] \pm \sqrt{\frac{g^2}{2}(\phi - \phi^*)^2 - 2mgf_{12} - 2ngf_{34}}.
$$

The degeneracies of $\bar{\omega}_{\pm}(m,n)$, $\bar{\omega}_{\pm}(m,0)$ and $\bar{\omega}_{\pm}(0,n)$ with $m \geq 1$, $n \geq 1$ are the same as those of the $\omega$s. There still only exists a nondegenerate eigenvalue $\omega_{-(0,0)}$.

With the above eigenvalues $\text{Tr} \ln \Delta_F$ and $\text{Tr} \ln \Delta_F$ can be computed straightforwardly,

$$
\text{Tr ln } \Delta_F = \ln \det \Delta_F = \ln \left[ \Pi \omega_{\pm}(l,k) \right]^{r} = \sum_{l,k=-\infty}^{+\infty} \sum_{m,n=0}^{\infty} r \ln \omega_{\pm}(l,k) \omega_{\pm}(m,n),
$$

where $r$ is the degeneracy of $\omega_{\pm}(m,n)$. Due to the relation $x_2 = 2\pi l/(gf_{12}L)$ and $x_4 = 2\pi k/(gf_{34}L)$, the summation over the momenta $k$ and $l$ is actually equivalent to an integration over $x_2$ and $x_4$. Since the fields are constants, this integration will yield only a Euclidean space volume factor, which tends to infinity in the continuous limit $(L \to \infty)$,

$$
\sum_{l,k} = \frac{L^2}{4\pi^2}gf_{12}f_{34} \int dx_2dx_4 = \frac{V}{4\pi^2}gf_{12}f_{34},
$$

where $V$ is the volume of the Euclidean space.
Thus we finally obtain. Consider the degeneracy of each eigenvalue, we have

\[
\text{Tr} \ln \Delta_F = \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \omega_-(0,0) + \sum_{m=1}^{\infty} \ln \omega_+(m,0) + \sum_{n=1}^{\infty} \ln \omega_+(0,n) + 2 \sum_{m,n=1}^{\infty} \ln \omega_-(m,n) \right\}
\]

\[
+ 2 \sum_{m,n=1}^{\infty} \ln \omega_+(m,n)
\]

\[
= \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \omega_-(0,0) + \sum_{m=1}^{\infty} \ln [\omega_+(m,0)\omega_-(m,0)] + \sum_{n=1}^{\infty} \ln [\omega_+(0,n)\omega_-(0,n)] + 2 \sum_{m,n=1}^{\infty} \ln [\omega_+(m,n)\omega_-(m,n)] \right\}
\]

\[
= \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln [-g(\sqrt{2}\phi + v)] + \sum_{m=1}^{\infty} \ln [g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12}] + \sum_{n=1}^{\infty} \ln [g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2ngf_{34}] + 2 \sum_{m,n=1}^{\infty} \ln [g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12} + 2ngf_{34}] \right\}. \tag{43}
\]

Similarly, we get

\[
\text{Tr} \ln \tilde{\Delta}_F = \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \left[ g(\sqrt{2}\phi^* + v) \right] + \sum_{m=1}^{\infty} \ln [g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12}] + \sum_{n=1}^{\infty} \ln [g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2ngf_{34}] + 2 \sum_{m,n=1}^{\infty} \ln [g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12} + 2ngf_{34}] \right\}. \tag{44}
\]

Thus we finally obtain

\[
\text{Tr} \ln \Delta_F + \text{Tr} \ln \tilde{\Delta}_F = \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \left[ \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right] + \sum_{m=1}^{\infty} \ln [g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12}] + \sum_{n=1}^{\infty} \ln [g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2ngf_{34}] + 4 \sum_{m,n=1}^{\infty} \ln [g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12} + 2ngf_{34}] \right\}. \tag{45}
\]

Making use of the proper-time regularization,

\[
\ln \alpha = -\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-\alpha s} \tag{46}
\]
with $\Lambda^2$ being the cut-off to regularize the infinite sum, we have

$$\text{Tr} \ln \Delta_F + \text{Tr} \ln \tilde{\Delta}_F = \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \left[ \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right] - 2 \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \right\}$$

$$\times \left\{ \sum_{m=1}^{\infty} e^{-2mg f_{12}s} + \sum_{n=1}^{\infty} e^{-2ng f_{34}s} + 2 \sum_{m,n=1}^{\infty} e^{-(2mg f_{12} + 2ng f_{34})s} \right\}$$

$$= \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \left[ \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right] - 2 \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \right\}$$

$$+ \frac{e^{-gf_{12}s}}{\sinh(g f_{12}s)} + \frac{e^{-(gf_{12} + gf_{34})s}}{\sinh(g f_{12}s) \sinh(g f_{34}s)}$$

$$= \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left[ \ln \left( \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right) - \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \right.$$\[\left. \cosh[g(f_{12} + f_{34})s] + \cosh[g(f_{12} - f_{34})s] \right]$$

$$\left/ \cosh[g(f_{12} + f_{34})s] - \cosh[g(f_{12} - f_{34})s] \right],$$

where we have used

$$\sum_{m=1}^{\infty} e^{-2mt} = \frac{e^{-t}}{2 \sinh t}, \quad \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y. \quad (48)$$

Rotating back to Minkowski space and denoting $X \equiv H + iE$, we write (47) as

$$\text{Tr} \ln \Delta_F + \text{Tr} \ln \tilde{\Delta}_F = \frac{V}{4\pi^2} g^2 iE_z H_z \ln \left( \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right)$$

$$- \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \left[ \cosh[g(H_z + iE_z)s] + \cosh[g(H_z - iE_z)s] \right]$$

$$\left/ \cosh[g(H_z + iE_z)s] - \cosh[g(H_z - iE_z)s] \right]$$

$$= \frac{V}{4\pi^2} g^2 iE \cdot H \ln \left( \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right)$$

$$- \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \left[ \cosh[gXs] + \cosh[gX^*s] \right]$$

$$\left/ \cosh[gXs] - \cosh[gX^*s] \right],$$

(49)

To extract the divergence, we must analyze the small-$s$ behaviour of the integrand of (49) by using the identities

$$iE \cdot H = \frac{1}{4}(X^2 - X^{*2}) = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad H^2 - E^2 = \frac{1}{2}(X^2 + X^{*2}) = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad (50)$$

and the series expansion near $s \sim 0$

$$\cosh[gXs] + \cosh[gX^*s] = \frac{1}{(X^2 - X^{*2})} \left[ \frac{4}{g^2 s^2} + \frac{2}{3}(X^2 + X^{*2}) + \mathcal{O}(s^2) \right].$$

(51)
It can be easily seen from (51) that the integral in (19) has a quadratic divergence and a logarithmic one. Thus the divergence term can be extracted by writing (19) as the form,

\[
\text{Tr} \ln \Delta_F + \text{Tr} \ln \bar{\Delta}_F = \frac{V}{4\pi^2} \left\{ \frac{1}{4} g^2 F_{\mu\nu} \bar{F}^{\mu\nu} \ln \left[ \sqrt{2} \phi^* + v \right] \sqrt{2} \phi + v \right\} - \int_{1/\Lambda^2}^\infty ds \left( \frac{1 + 1}{s^3/6} g^2 F_{\mu\nu} F^{\mu\nu} \right) e^{-g^2(\sqrt{2} \phi^* + v)(\sqrt{2} \phi + v)s} - \int_0^\infty ds \frac{1}{s^3} e^{-g^2(\sqrt{2} \phi^* + v)(\sqrt{2} \phi + v)s} \left[ \frac{1}{4} g^2 s^2 F_{\mu\nu} F^{\mu\nu} \cosh(gXs) + \cosh(gX^*s) \right] \cosh(gXs) - \cosh(gX^*s) \right\}.
\]

(52)

The second term (52) is the UV divergent term, so the cut-off \(1/\Lambda^2\) is preserved to regularize the integral, while the last term is a finite term and hence the cut-off has been removed.

Now we turn to the bosonic determinant. From (27) we have

\[
M^{-1}_{ff} = 2\begin{pmatrix}
\frac{1}{\Delta_F} & 0 & 0 & 0 \\
0 & \frac{1}{\Delta_F} & 0 & 0 \\
0 & 0 & \frac{1}{\Delta_F} & 0 \\
0 & 0 & 0 & \frac{1}{\Delta_F}
\end{pmatrix}, \quad M_{bb} - M_{bf} M^{-1}_{ff} M_{fb} = 
\]

\[
\frac{1}{2} \begin{pmatrix}
\Delta_{\mu\nu} - 2g^2 \bar{\psi}_\gamma \gamma_\mu \bar{\Delta}_F \gamma_\nu \psi & 0 & \Delta_\mu - 2ig^2 \bar{\psi}_\gamma \gamma_\mu \frac{1}{\Delta_F} \gamma_\nu \psi & 0 \\
0 & \Delta_{\mu\nu} - 2g^2 \bar{\psi}_\gamma \gamma_\mu \bar{\Delta}_F \gamma_\nu \psi & 0 & \Delta_\mu - 2ig^2 \bar{\psi}_\gamma \gamma_\mu \frac{1}{\Delta_F} \gamma_\nu \psi \\
0 & 0 & \Delta_{\mu\nu} - 2g^2 \bar{\psi}_\gamma \gamma_\mu \bar{\Delta}_F \gamma_\nu \psi & 0 \\
0 & 0 & 0 & \Delta_{\mu\nu} - 2g^2 \bar{\psi}_\gamma \gamma_\mu \bar{\Delta}_F \gamma_\nu \psi
\end{pmatrix}
\]

(53)

In constant field approximation, \(\bar{\psi}\) and \(\psi\) can be regarded Grassman numbers, so we can expand the bosonic determinant only to the quartic terms in \(\bar{\psi}\) and \(\psi\). Now the key problem is how to find the eigenvalues and eigenstates of the operator matrix \(M_{bb} - M_{bf} M^{-1}_{ff} M_{fb}\). If they could be worked out, then with the eigenvalues and eigenvectors of fermionic operator, we can use the technique developed in [5] to evaluate this determinant. Unfortunately, it seems to us that in the constant field approximation it is very to find the eigenvalues and eigenstates of such a horrible operator matrix. This difficulty is waiting to be overcome.

Despite the fact that the bosonic part cannot be evaluated, we can see from (27) and (52) that the effective Lagrangian associated with the fermionic part has already shown the features of the perturbative part of the low-energy effective action. First, we believe that the quadratic divergence of Eq.(52) will be canceled owing to the nonrenormalization theorem. Second, for the logarithmic divergence of Eq.(52), with

\[
\int_{1/\Lambda^2}^\infty ds \frac{1}{s} e^{-g^2(\sqrt{2} \phi^* + v)(\sqrt{2} \phi + v)s} \sim - \ln \left[ \frac{g^2(\sqrt{2} \phi + v)(\sqrt{2} \phi^* + v)}{\Lambda^2} \right],
\]

(54)

Eq.(52) shows that the Wilson effective action has one term proportional to

\[
F_{\mu\nu} F^{\mu\nu} \ln \left[ \frac{g^2(\sqrt{2} \phi^* + v)(\sqrt{2} \phi + v)}{\Lambda^2} \right].
\]

(55)

Comparing with the component field form given by (A.13), we can conclude that the complete calculation should give the form (1) of the low-energy effective action. One can even guess
this from the requirement of supersymmetry since the constant field approximation and the proper-time regularization preserve the supersymmetry explicitly. Further, there is a finite term proportional to $F\tilde{F}\ln\left[(\sqrt{2}\phi + v)/(\sqrt{2}\phi^* + v)\right]$ in (52). As pointed out in [5], this is the reflection of the axial $U(1)_R$ anomaly in the effective action. This anomaly term had played a crucial role in the nonperturbative analysis[3].

4 Summary

In summary, we have tried to calculate the perturbative part of the low-energy effective action of $N = 2$ supersymmetric Yang-Mills theory based on a standard effective field theory technique. It is well known that the Seiberg-Witten effective action is the cornerstone for all those new developments in $N = 2$ supersymmetric gauge theory, and that this effective action has been obtained in an indirect way. Therefore, it is worthwhile to try to compute this effective action using a straightforward integration of the heavy degrees of freedom. Unfortunately, we have encountered an insurmountable difficulty in evaluating the bosonic operator adopting the constant field approximation. This prevents us from getting the complete result and giving a thorough comparison with the form of (1). However, the calculation of the fermionic determinant has indeed shown the basic features of the low-energy effective action. This gives a partial verification of the abstract symmetry analysis in extracting the low-energy effective action. The complete calculation presents an interesting problem for further investigation.

Acknowledgments: We acknowledge the financial support by the Academy of Finland under the project No. 37599 and 44129. W.F.C is partially supported by the Natural Sciences and Engineering Research Council of Canada.
A Low-energy Effective Action in Component Field Form

To compare our result with that obtained from non-perturbative analysis, in this appendix we write the perturbative part of the Seiberg-Witten low-energy effective action (4) in the form of component fields. First, (4) can be expressed in $N = 1$ superfield

$$\Gamma = \frac{1}{16\pi} \text{Im} \int d^4x \left[ \int d^2\theta F''(\Phi)W^\alpha W_\alpha + \int d^2\theta d^2\theta F'(\Phi) \right]$$

where $\Phi$ is the $N = 1$ chiral superfield

$$\Phi = \phi(x) + i\theta \sigma^\mu \partial_\mu \phi - \frac{1}{4} \theta^2 \bar{\theta}^2 \partial^2 \phi + \sqrt{2} \theta \psi - \frac{i}{\sqrt{2}} \theta^2 \partial_\mu \psi \sigma^\mu \bar{\theta} + \theta^2 F(x)$$

and

$$F'(\Phi) = \frac{1}{2} \tau \Phi^2 + \frac{i}{\pi} \Phi^2 \ln \frac{\Phi^2}{\Lambda^2}, \quad \tau = \theta + \frac{4\pi i}{g^2}.$$ (A.3)

In Wess-Zumino gauge, the Abelian vector superfield and the corresponding superfield strength are, respectively,

$$V = -\theta \sigma^\mu \bar{\theta} A_\mu + i\theta^2 (\bar{\theta} \lambda) - \bar{\theta}^2 (\theta \lambda) + \frac{1}{2} \theta^2 \bar{\theta}^2 D;$$

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - i\sigma^\mu_\alpha \theta \beta F_{\mu \nu}(y) + \theta^2 \sigma^\mu_\alpha \partial_\mu \bar{\lambda}(y),$$ (A.4)

where $y^\mu = x^\mu + i\theta \sigma^\mu \bar{\theta}$, $\sigma^\mu_\nu = \frac{1}{2}(\sigma^\mu \bar{\theta}^\nu - \sigma^\nu \bar{\theta}^\mu)$ and $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Using the expansion

$$F'(\Phi) = F'(\phi) + F'(\phi) \left[ i\theta \sigma^\mu \bar{\theta} \partial_\mu \phi - \frac{1}{4} \theta^2 \bar{\theta}^2 \partial^2 \phi + \sqrt{2} \theta \psi - \frac{i}{\sqrt{2}} \theta^2 \partial_\mu \psi \sigma^\mu \bar{\theta} + \theta^2 F(x) \right]$$

$$\times \left[ i\theta \sigma^\mu \bar{\theta} \partial_\mu \phi - \frac{1}{4} \theta^2 \bar{\theta}^2 \partial^2 \phi + \sqrt{2} \theta \psi - \frac{i}{\sqrt{2}} \theta^2 \partial_\mu \psi \sigma^\mu \bar{\theta} + \theta^2 F(x) \right]$$

and the similar expansion for $F''(\Phi)$, we obtain

$$\Gamma = \frac{1}{16\pi} \text{Im} \int d^4x \left[ -F''(\phi) \phi \partial^2 \phi - F''(\phi) \phi \partial^\mu \phi \partial_\mu \phi + 2F''(\phi) \partial_\mu \phi \partial^\mu \phi \right.$$

$$- \partial^2 \phi F'(\phi) + 2iF''(\phi) \partial_\mu \psi \sigma^\mu \bar{\theta} - 2iF''(\phi) \psi \sigma^\mu \partial_\mu \bar{\theta} + 2iF''(\phi) \phi \sigma^\mu \bar{\theta} \partial_\mu \phi$$

$$- 2F''(\phi) \psi^\lambda \partial_\mu \psi \sigma^\mu \bar{\theta} D^2$$

$$+ 4F''(\phi) (- F''^\mu F_{\mu \nu} + iF'_{\mu \nu} \bar{F}^\nu) - 2\sqrt{2} iF''(\phi) \lambda \partial_\mu \bar{\lambda} - 2F''(\phi) \partial_\mu \bar{\lambda}$$

$$\left. - F''(\phi) (\psi \lambda \partial_\mu \lambda) \right] \right).$$ (A.6)

Using (A.3),

$$F''(\phi) = \left( \tau + \frac{i}{\pi} \right) \phi + \frac{i}{\pi} \phi \ln \frac{\phi^2}{\Lambda^2}, \quad F''(\phi) = \tau + \frac{3i}{\pi} + \frac{i}{\pi} \ln \frac{\phi^2}{\Lambda^2},$$

$$F''(\phi) = \frac{2i}{\pi}, \quad F''(\phi) = \frac{2i}{\pi} \frac{1}{\phi^2}.$$ (A.7)
and rescaling the field $X \rightarrow gX$, $X = (A, \phi, \lambda, \psi)$, we write (A.6) as

$$\Gamma = \int d^4x \left\{ \left[-8\pi \phi \partial^2 \phi + 8\pi \partial_\mu \phi \partial^\mu \phi + 8\pi i \partial_\mu \psi \sigma^\mu \overline{\psi} - 8\pi i \psi \overline{\sigma} \partial^\mu \overline{\psi} - 16\pi i \lambda \sigma^\mu \partial_\mu \overline{X} ight] 
- 4\pi F_{\mu\nu} F^{\mu\nu} \right\}
+ \frac{g^2}{\pi} \left[ -4\phi \partial^2 \phi + 4\partial_\mu \phi \partial^\mu \phi + 6 i \partial_\mu \psi \sigma^\mu \overline{\psi} - 6\pi i \psi \overline{\sigma} \partial^\mu \overline{\psi} ight]
+ 4 i \psi \sigma^\mu \overline{\psi} \partial_\mu \phi + 12 i \lambda \sigma^\mu \partial_\mu \overline{X} - 3 F_{\mu\nu} F^{\mu\nu} \right]\left[ -2\phi \partial^2 \phi + 2\partial_\mu \phi \partial^\mu \phi 
+ 2 i \partial_\mu \psi \sigma^\mu \overline{\psi} - 2 i \psi \overline{\sigma} \partial_\mu \overline{\psi} + 4 i \lambda \sigma^\mu \partial_\mu \overline{X} - F_{\mu\nu} F^{\mu\nu} \right] + \frac{g^2}{8\pi^2} \frac{1}{\phi^2} (\psi \psi)(\lambda \lambda)

- \frac{1}{2} \left( 1 + \frac{3g^2}{4\pi^2} + \frac{g^2}{8\pi^2} \ln \frac{\phi^2}{\Lambda^2} \right) - \frac{\sqrt{2}g^2 i(\psi \lambda)}{4\pi^2} \frac{\rho}{\phi} - \frac{g^2}{4\pi^2} F^\dagger(\psi \psi)
+ F^\dagger F \left( 1 + \frac{3g^2}{4\pi^2} + \frac{g^2}{4\pi^2} \ln \frac{\phi^2}{\Lambda^2} \right) + \frac{g^2}{4\pi^2} (\lambda \lambda) F \left( \psi \psi \right),
\tag{A.8}$$

where the vacuum angle $\theta$ is set to zero. Eliminating the auxiliary fields $F$, $F^\dagger$ and $D$ with the equations of motion derived from (A.8),

$$F \left( 1 + \frac{3g^2}{4\pi^2} + \frac{g^2}{8\pi^2} \ln \frac{\phi^2}{\Lambda^2} \right) - \frac{g^2}{4\pi^2} \frac{\rho \psi}{\phi} = 0,
\tag{A.9}$$

and performing the algebraic manipulations

$$\int d^4x \ln \frac{g^2 \phi^2}{\Lambda^2} \phi \partial^2 \phi = -\int d^4x \left[ 2 + 2 \ln \frac{g^2 \phi^2}{\Lambda^2} \right] \partial_\mu \phi \partial^\mu \phi,
\tag{A.10}$$

$$\int d^4x \psi \overline{\sigma} \partial^\mu \overline{\psi} \frac{\partial_\mu \phi}{\phi} = - \frac{1}{2} \int d^4x \ln \frac{g^2 \phi^2}{\Lambda^2} \left[ \partial_\mu \psi \overline{\sigma} \partial^\mu \overline{\psi} + \psi \overline{\sigma} \partial^\mu \overline{\psi} \right],
\tag{A.11}$$

we get

$$\Gamma = \int d^4x \left\{ \left[ \partial_\mu \phi \partial^\mu \phi + \lambda \sigma^\mu \partial_\mu \overline{X} + i \overline{\psi} \overline{\sigma} \partial_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] 
+ \frac{3g^2}{4\pi^2} \left[ \partial_\mu \phi \phi + \overline{\lambda} \sigma^\mu \partial_\mu \overline{X} + i \overline{\psi} \overline{\sigma} \partial_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]
+ \frac{g^2}{4\pi^2} \ln \frac{g^2 \phi^2}{\Lambda^2} \left[ \partial_\mu \phi \partial^\mu \phi + i \overline{\lambda} \sigma^\mu \partial_\mu \overline{X} + i \overline{\psi} \overline{\sigma} \partial_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] + \frac{g^2}{8\pi^2} \frac{(\lambda \lambda)(\psi \psi)}{\phi^2} \right\},
\tag{A.12}$$
Considering the four-component spinor field form
\[ \Psi = \left( \begin{array}{c} \psi \\ \lambda \end{array} \right), \quad \overline{\Psi} = (\lambda, \overline{\psi}), \quad \gamma^\mu = \left( \begin{array}{cc} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{array} \right), \]
and especially using the fact that for a \( N = 2 \) Abelian supermultiplet, \( \Psi \) should be a Majorana spinor: \( \psi = \lambda \) and \( \overline{\psi} = \overline{\lambda} \) and \( (\lambda \lambda)(\psi \psi) = (\psi \psi)^2 = 1/4(\overline{\Psi} \Psi)^2 \), we finally write (A.12) as the following form
\[ \Gamma = \int d^4x \left\{ 1 + \frac{3g^2}{4\pi^2} + \frac{g^2}{4\pi^2} \ln \frac{\phi^2}{\Lambda^2} \right\} \left[ \partial_\mu \psi \partial^\mu \psi + i \overline{\Psi} \gamma^\mu \partial_\mu \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] + \frac{g^4}{32\pi^2} (\overline{\Psi} \Psi)^2 \right\}. \]
(A.13) is the perturbative part of the low-energy effective action in Wess-Zumino gauge given by Seiberg[1].

## B Eigenvalues of the Fermionic Operator

In this appendix we present a detailed calculation on the eigenvalues of fermionic operator \( \Delta_F \). First Eq.(B.1) implies
\[ \Delta^+ \Delta^- \chi_1(x_2, x_4) = \left[ \omega + g(\sqrt{2}\phi^* + v) \right] \left[ \omega + g(\sqrt{2}\phi + v) \right] \chi_1(x_2, x_4), \]
\[ \Delta^- \Delta^+ \chi_2(x_2, x_4) = \left[ \omega + g(\sqrt{2}\phi^* + v) \right] \left[ \omega + g(\sqrt{2}\phi + v) \right] \chi_2(x_2, x_4) \]
with
\[ \Delta^+ \Delta^- = -H_{12} - H_{34} + g\sigma_3(f_{12} + f_{34}), \]
\[ \Delta^- \Delta^+ = -H_{12} - H_{34} + g\sigma_3(f_{12} - f_{34}), \]
where \( H_{12} \) and \( H_{34} \) are the Hamiltonian operators of two independent harmonic oscillators,
\[ H_{12} = -\frac{\partial^2}{\partial x_2^2} + g^2 f_{12}(x_2 + \frac{p_1}{g f_{12}})^2 = -\frac{\partial^2}{\partial x_2^2} + \Omega_{12}^2 \xi_2^2, \xi_2 \equiv x_2 + \frac{p_1}{g f_{12}}, \Omega_{12} \equiv |g f_{12}|; \]
\[ H_{34} = -\frac{\partial^2}{\partial x_4^2} + g^2 f_{34}(x_4 + \frac{p_3}{g f_{34}})^2 = -\frac{\partial^2}{\partial x_4^2} + \Omega_{34}^2 \xi_4^2, \xi_4 \equiv x_4 + \frac{p_3}{g f_{34}}, \Omega_{34} \equiv |g f_{34}|. \]
Eq.(B.1) means that the eigenvalue and the eigenvector of \( \Delta_F \) must be that of \( \Delta^+ \Delta^- \) and \( \Delta^- \Delta^+ \), while the reverse may be not true. In the following we make use of the eigenvalue and the eigenvector of \( \Delta^+ \Delta^- \) and \( \Delta^- \Delta^+ \) to find the ones of \( \Delta_F \). As the usual operator method dealing with the harmonic oscillator, defining the destruction and creation operators
\[ a_2 = \frac{1}{\sqrt{2}} \left( \sqrt{\Omega_{12}} \xi_2 + \frac{1}{\sqrt{\Omega_{12}}} \frac{\partial}{\partial \xi_2} \right), a_2^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{\Omega_{12}} \xi_2 - \frac{1}{\sqrt{\Omega_{12}}} \frac{\partial}{\partial \xi_2} \right), [a_2, a_2^\dagger] = 1; \]
\[ a_4 = \frac{1}{\sqrt{2}} \left( \sqrt{\Omega_{34}} \xi_4 + \frac{1}{\sqrt{\Omega_{34}}} \frac{\partial}{\partial \xi_4} \right), a_4^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{\Omega_{34}} \xi_4 - \frac{1}{\sqrt{\Omega_{34}}} \frac{\partial}{\partial \xi_4} \right), [a_4, a_4^\dagger] = 1, \]
(B.4)
we obtain the Hamiltonian operators and their eigenstates in Fock space,

\[ H_{12} = \Omega_{12}(2a_2a_2^\dagger + 1), \quad |n_{12}\rangle = \frac{1}{\sqrt{n_{12}+1}}(a_2^\dagger)^{n_{12}}|0_{12}\rangle, \]
\[ a_2|0_{12}\rangle = 0, \quad H_{12}|n_{12}\rangle = \Omega_{12}(2n_{12} + 1)|n_{12}\rangle; \]
\[ H_{34} = \Omega_{34}(2a_4a_4^\dagger + 1), \quad |n_{34}\rangle = \frac{1}{\sqrt{n_{34}+1}}(a_4^\dagger)^{n_{34}}|0_{34}\rangle, \]
\[ a_4|0_{34}\rangle = 0, \quad H_{34}|n_{34}\rangle = \Omega_{34}(2n_{34} + 1)|n_{34}\rangle. \] (B.5)

The operators \( \Delta^+ \) and \( \Delta^- \) can be rewritten in terms of the destruction and creation operators,

\[
\Delta^+ = i\sigma_2 \frac{\partial}{\partial \xi_2} - g f_{12} \xi_2 \sigma_1 + \frac{\partial}{\partial \xi_4} - g f_{34} \xi_4 \sigma_3 \\
= \left( \sqrt{\frac{\Omega_{12}}{2}}(a_4 - a_4^\dagger) - \frac{gf_{34}}{\sqrt{2\Omega_{34}}} (a_4 + a_4^\dagger) \right) - \left( \sqrt{\frac{\Omega_{12}}{2}}(a_2 - a_2^\dagger) - \frac{gf_{12}}{\sqrt{2\Omega_{12}}} (a_2 + a_2^\dagger) \right) \\
\Delta^- = -i\sigma_2 \frac{\partial}{\partial \xi_2} + g f_{12} \xi_2 \sigma_1 + \frac{\partial}{\partial \xi_4} + g f_{34} \xi_4 \sigma_3 \\
= \left( \sqrt{\frac{\Omega_{12}}{2}}(a_4 - a_4^\dagger) + \frac{gf_{34}}{\sqrt{2\Omega_{34}}} (a_4 + a_4^\dagger) \right) - \left( \sqrt{\frac{\Omega_{12}}{2}}(a_2 - a_2^\dagger) + \frac{gf_{12}}{\sqrt{2\Omega_{12}}} (a_2 + a_2^\dagger) \right). \] (B.6)

There are four different cases that should be considered:

1. \( g f_{12} > 0, g f_{34} > 0; \ \Omega_{12} = g f_{12}, \ \Omega_{34} = g f_{34}; \)

\[
\Delta^+ = \left( -\sqrt{2\Omega_{12}}a_4 - \sqrt{2\Omega_{12}}a_4^\dagger \right) - \sqrt{2\Omega_{34}}a_4 \sigma_1, \quad \Delta^- = \left( -\sqrt{2\Omega_{34}}a_4 - \sqrt{2\Omega_{34}}a_4^\dagger \right) - \sqrt{2\Omega_{12}}a_4 \sigma_1. \] (B.7)

2. \( g f_{12} > 0, g f_{34} < 0; \ \Omega_{12} = g f_{12}, \ \Omega_{34} = -g f_{34}; \)

\[
\Delta^+ = \left( \sqrt{2\Omega_{12}}a_4 - \sqrt{2\Omega_{12}}a_4^\dagger \right) - \sqrt{2\Omega_{34}}a_4 \sigma_1, \quad \Delta^- = \left( -\sqrt{2\Omega_{34}}a_4 - \sqrt{2\Omega_{34}}a_4^\dagger \right) - \sqrt{2\Omega_{12}}a_4 \sigma_1. \] (B.8)

3. \( g f_{12} < 0, g f_{34} > 0; \ \Omega_{12} = -g f_{12}, \ \Omega_{34} = g f_{34}; \)

\[
\Delta^+ = \left( -\sqrt{2\Omega_{12}}a_4 + \sqrt{2\Omega_{12}}a_4^\dagger \right) - \sqrt{2\Omega_{34}}a_4 \sigma_1, \quad \Delta^- = \left( \sqrt{2\Omega_{34}}a_4 - \sqrt{2\Omega_{34}}a_4^\dagger \right) - \sqrt{2\Omega_{12}}a_4 \sigma_1. \] (B.9)

4. \( g f_{12} < 0, g f_{34} < 0; \ \Omega_{12} = -g f_{12}, \ \Omega_{34} = -g f_{34}; \)

\[
\Delta^+ = \left( \sqrt{2\Omega_{12}}a_4 - \sqrt{2\Omega_{12}}a_4^\dagger \right) - \sqrt{2\Omega_{34}}a_4 \sigma_1, \quad \Delta^- = \left( -\sqrt{2\Omega_{34}}a_4 - \sqrt{2\Omega_{34}}a_4^\dagger \right) - \sqrt{2\Omega_{12}}a_4 \sigma_1. \] (B.10)

Now we look for the eigenvalues of \( \Delta_F \) with aid of the eigenvalues of \( \Delta^+\Delta^- \) and \( \Delta^-\Delta^+ \). From Eq. (B.7) we have

\[
\Delta^-\Delta^+|\chi_1\rangle = [\omega + g(\sqrt{2}\phi^* + v)] [\omega + g(\sqrt{2}\phi + v)] |\chi_1\rangle, \]
\[
\Delta^+\Delta^-|\chi_2\rangle = [\omega + g(\sqrt{2}\phi^* + v)] [\omega + g(\sqrt{2}\phi + v)] |\chi_2\rangle. \] (B.11)
The eigenstates $\chi_1$ and $\chi_2$ should be the following form,

$$|\chi_i\rangle \sim \begin{pmatrix} |k,l\rangle \\ |m,n\rangle \end{pmatrix}, \quad i = 1, 2,$$

(B.12)

where $|k,l\rangle \equiv |k|l\rangle$, $|m,n\rangle \equiv |m|n\rangle$, $k, m$ are the quantum numbers of the harmonic oscillator $H_{12}$ and $l, n$ are those of $H_{34}$.

We first consider the case 1, since that

$$\Delta^- \Delta^+ \begin{pmatrix} |k,l\rangle \\ |m,n\rangle \end{pmatrix} = \begin{pmatrix} [-2k\Omega_{12} - 2(l + 1)\Omega_{34}] |k,l\rangle \\ [-2(m + 1)\Omega_{12} - 2n\Omega_{34}] |m,n\rangle \end{pmatrix},$$

$$\Delta^+ \Delta^- \begin{pmatrix} |k,l\rangle \\ |m,n\rangle \end{pmatrix} = \begin{pmatrix} (-2k\Omega_{12} - 2l\Omega_{34}) |k,l\rangle \\ [-2(m + 1)\Omega_{12} - 2(n + 1)\Omega_{34}] |m,n\rangle \end{pmatrix},$$

(B.13)

the common eigenstate of $\Delta^- \Delta^+$ and $\Delta^+ \Delta^-$ with eigenvalue $-2m\Omega_{12} - 2n\Omega_{34}$, $m, n \geq 1$ is

$$\begin{pmatrix} |\chi_1\rangle \\ |\chi_2\rangle \end{pmatrix} = \begin{pmatrix} (\alpha|m,n-1\rangle \\ (\beta|m-1,n\rangle \\ (\gamma|m,n\rangle \\ (\delta|m-1,n-1\rangle \end{pmatrix},$$

(B.14)

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are normalization parameters. With this eigenstate, we rewrite the eigenvalue equation (B.14) in Fock space,

$$\begin{pmatrix} -g(\sqrt{2}\phi^* + v)1 & \Delta^- \\ \Delta^+ & -g(\sqrt{2}\phi + v)1 \end{pmatrix} \begin{pmatrix} (\alpha|m,n-1\rangle \\ (\beta|m-1,n\rangle \\ (\gamma|m,n\rangle \\ (\delta|m-1,n-1\rangle \end{pmatrix} = \omega \begin{pmatrix} (\alpha|m,n-1\rangle \\ (\beta|m-1,n\rangle \\ (\gamma|m,n\rangle \\ (\delta|m-1,n-1\rangle \end{pmatrix},$$

(B.15)

Eq. (B.15) means that searching for the operator eigenvalue can be changed into an ordinary matrix eigenvalue problem,

$$\begin{pmatrix} -g(\sqrt{2}\phi^* + v) & 0 & \sqrt{2m\Omega_{34}} & \sqrt{2m\Omega_{12}} \\ 0 & -g(\sqrt{2}\phi^* + v) & \sqrt{2m\Omega_{12}} & \sqrt{2n\Omega_{34}} \\ -\sqrt{2m\Omega_{34}} & -\sqrt{2m\Omega_{12}} & -g(\sqrt{2}\phi + v) & 0 \\ -\sqrt{2m\Omega_{12}} & \sqrt{2n\Omega_{34}} & 0 & -g(\sqrt{2}\phi + v) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \omega \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. $$

(B.16)
Using the fact
\[
\det \begin{pmatrix} K & 0 & A & B \\ 0 & K & B & -A \\ -A & -B & L & 0 \\ -B & A & 0 & L \end{pmatrix} = (A^2 + B^2 + KL)^2, \tag{B.17}
\]
we see that the eigenvalue \(\omega\) is determined by the following equation,
\[
\left[ \omega + g(\sqrt{2}\phi^* + v) \right] \left[ \omega + g(\sqrt{2}\phi + v) \right] + 2m\Omega_{12} + 2n\Omega_{34} = 0,
\]
\[
\omega_{\pm}(m, n) = -g \left[ \frac{\phi + \phi^*}{\sqrt{2}} + v \right] \pm \sqrt{\frac{1}{2}g^2(\phi - \phi^*)^2 - 2m\Omega_{12} - 2n\Omega_{34}}. \tag{B.18}
\]
Eqs. (B.16) and (B.17) explicitly show that \(\omega_{\pm}(m, n)\) and \(\omega_{-}(m, n)\) with \(m, n \geq 1\) are doubly degenerate, since for a 4x4 matrix there should exist four eigenvalues. Special attention should be paid to the cases of \(m = 0\) or \(n = 0\) as well as both of them equal to zero, when we will see that the degeneracies of the eigenvalue are different:

- \(m \geq 1, n = 0\): in this case the eigenvalue equation (B.15) will reduce to the following form,
  \[
  \begin{pmatrix}
  0 \\
  -g(\sqrt{2}\phi^* + v)\beta + \sqrt{2m\Omega_{12}} |m - 1, 0\rangle \\
  -\sqrt{2m\Omega_{12}}\beta - g(\sqrt{2}\phi + v)\gamma |m, 0\rangle 
  \end{pmatrix} = \omega \begin{pmatrix}
  0 \\
  \beta |m - 1, 0\rangle \\
  \gamma |m, 0\rangle 
  \end{pmatrix},
  \]
  \[
  \omega_{\pm}(m, 0) = -g \left[ \frac{\phi + \phi^*}{\sqrt{2}} + v \right] \pm \sqrt{\frac{1}{2}g^2(\phi - \phi^*)^2 - 2m\Omega_{12}}. \tag{B.19}
  \]
  The eigenvalues \(\omega_{\pm}(m, 0)\) are obviously nondegenerate.

- \(m = 0, n \geq 1\): in this case we have the eigenvalue equation as follows,
  \[
  \begin{pmatrix}
  0 \\
  -g(\sqrt{2}\phi^* + v)\alpha + \sqrt{2n\Omega_{34}} |0, n - 1\rangle \\
  -\sqrt{2n\Omega_{34}}\alpha - g(\sqrt{2}\phi + v)\gamma |0, n\rangle 
  \end{pmatrix} = \omega \begin{pmatrix}
  |0, n - 1\rangle \\
  \alpha |0, n - 1\rangle \\
  \gamma |0, n\rangle 
  \end{pmatrix},
  \]
  \[
  \omega_{\pm}(0, n) = -g \left[ \frac{\phi + \phi^*}{\sqrt{2}} + v \right] \pm \sqrt{\frac{1}{2}g^2(\phi - \phi^*)^2 - 2n\Omega_{34}}. \tag{B.20}
  \]
  The eigenvalues \(\omega_{\pm}(0, n)\) are also nondegenerate.

- \(m = n = 0\): the eigenvalue equation becomes very simple,
  \[
  \begin{pmatrix}
  0 \\
  0 \\
  -g(\sqrt{2}\phi + v) |0, 0\rangle \\
  0 
  \end{pmatrix} = \omega \begin{pmatrix}
  0 \\
  0 \\
  |0, 0\rangle \\
  0 
  \end{pmatrix},
  \]
  \[
  \omega(0, 0) = -g(\sqrt{2}\phi + v) = \omega_{-}(0, 0). \tag{B.21}
  \]
  Thus there only exists one \(\omega_{-}(0, 0)\) and it is nondegenerate.
For the case 2, the common eigenstate of $\Delta^+ \Delta^-$ and $\Delta^- \Delta^+$ with eigenvalue $-2m\Omega_{12} - 2n\Omega_{34}$ is
\[
\begin{pmatrix}
|\chi_1\rangle \\
|\chi_2\rangle
\end{pmatrix} = \begin{pmatrix}
\alpha|m,n\rangle \\
\beta|m-1,n-1\rangle \\
\gamma|m,n-1\rangle \\
\delta|m-1,n\rangle
\end{pmatrix}.
\] (B.22)

In a similar way, one can see that the eigenvalues $\omega_+(m,n)$, $\omega_+(m,0)$, $\omega_+(0,n)$ with $m,n \geq 1$ and their degeneracies are the same as the case 1 except that $\omega(0,0)$ is different,
\[
\omega(0,0) = -g(\sqrt{2}\phi^* + v) = \omega_+(0,0).
\] (B.23)

As for the cases 3 and 4, the common eigenstates of $\Delta^+ \Delta^-$ and $\Delta^+ \Delta^+$ with eigenvalue $-2m\Omega_{12} - 2n\Omega_{34}$ are, respectively,
\[
3. \quad \begin{pmatrix}
|\chi_1\rangle \\
|\chi_2\rangle
\end{pmatrix} = \begin{pmatrix}
\alpha|m-1,n-1\rangle \\
\beta|m,n\rangle \\
\gamma|m-1,n\rangle \\
\delta|m,n-1\rangle
\end{pmatrix};
\] (B.24)
\[
4. \quad \begin{pmatrix}
|\chi_1\rangle \\
|\chi_2\rangle
\end{pmatrix} = \begin{pmatrix}
\alpha|m-1,n\rangle \\
\beta|m,n-1\rangle \\
\gamma|m-1,n-1\rangle \\
\delta|m,n\rangle
\end{pmatrix}.
\]

The eigenvalues $\omega_+(m,n)$, $\omega_+(m,0)$, $\omega_+(0,n)$ with $m,n \geq 1$ and their degeneracies are the same as the cases 1, 2, but $\omega(0,0)'s$ are, respectively,
\[
3. \quad \omega(0,0) = -g(\sqrt{2}\phi^* + v) = \omega_+(0,0);
4. \quad \omega(0,0) = -g(\sqrt{2}\phi + v) = \omega_-(0,0).
\] (B.25)

The eigenvalues of $\tilde{\Delta}_F$ can be determined in a similar way, and the only difference is $g \rightarrow -g$.

It should be emphasized that these four cases are not equivalent, since the eigenstates are different from each other. However, they give the identical $\det \Delta_F \det \tilde{\Delta}_F$.

References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19; Nucl. Phys. B431 (1994) 484; (E) B430 (1994) 485.
[2] N. Seiberg, Phys. Lett. B206 (1988) 75.
[3] C. Montonen and D. Olive, Phys. Lett. B72 (1977) 117; H. Osborn, Phys. Lett. B83 (1979) 321.
[4] M. Chaichian, W.F. Chen and C. Montonen, Nucl. Phys. B537 (1999) 161.
[5] P. Di Vecchia, R. Musto, F. Nicodemi and R. Pettorino, Nucl. Phys. B252 (1985) 635.

[6] B. de Wit, M.T. Grisaru and M. Roček, Phys. Lett. B374 (1996) 297; A. De Giovanni, M.T. Grisaru, M. Roček, R. von Unge and D. Zanon, Phys. Lett. B409 (1997) 251.

[7] A. Pickering and P. West, Phys. Lett. B383 (1996) 54.

[8] I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko and B.A. Ovrut, Phys. Lett. B417 (1998) 61; I.L. Buchbinder, S.M. Kuzenko and B.A. Ovrut, Phys. Lett. B443 (1998) 335.

[9] M. Matone, Phys. Rev. Lett. 78 (1997) 1412; D. Bellisai, F. Fucito, M. Matone and G. Travaglini, Phys. Rev. D56 (1997) 5218.

[10] S.V. Ketov, Phys. Rev. D57 (1998) 1277.

[11] J. de Boer, K. Hori, H. Ooguri and Y. Oz, Nucl. Phys. B518 (1998) 173.

[12] A. D’Adda, R. Horsley and P. Di Vecchia, Phys. Lett. B76 (1978) 298.

[13] J. Schwinger, Phys. Rev. 82 (1951) 664.

[14] A. D’Adda, A.C. Davis, P. Di Vecchia and P. Salomonson, Nucl. Phys. B222 (1983) 45.