Matrix biorthogonal polynomials on the unit circle and non-Abelian Ablowitz–Ladik hierarchy

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Abstract
Adler and van Moerbeke (2001 Commun. Pure Appl. Math. 54 153–205) described a reduction of the 2D-Toda hierarchy called the Toeplitz lattice. This hierarchy turns out to be equivalent to the one originally described by Ablowitz and Ladik (1975 J. Math. Phys. 16 598–603) using semidiscrete zero-curvature equations. In this paper, we obtain the original semidiscrete zero-curvature equations starting directly from the Toeplitz lattice and we generalize these computations to the matrix case. This generalization leads us to the semidiscrete zero-curvature equations for the non-Abelian (or multicomponent) version of the Ablowitz–Ladik equations (Gerdzhikov and Ivanov 1982 Theor. Math. Phys. 52 676–85). In this way, we extend the link between biorthogonal polynomials on the unit circle and the Ablowitz–Ladik hierarchy to the matrix case.

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1. Introduction

Ablowitz–Ladik hierarchy was introduced in 1975 [1] as a spatial discretization of the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy. As described by Suris in [2] Ablowitz and Ladik replaced the celebrated Zakharov–Shabat spectral problem
\[
\begin{align*}
\partial_t \Psi &= L \Psi \\
\partial_x \Psi &= M \Psi
\end{align*}
\]
with a discretized version of it; namely
\[
\begin{align*}
\Psi_{k+1} &= L_k \Psi_k \\
\partial_x \Psi_k &= M_k \Psi_k.
\end{align*}
\]
Here $\Psi$ and $\Psi_k$ are two-component vectors while $L, M, L_k$ and $M_k$ are $2 \times 2$ matrices. In particular
\[
L := \begin{pmatrix} z & x \\ y & -z \end{pmatrix}
\]
and
\[
L_k := \begin{pmatrix} z & x_k \\ y_k & -z \end{pmatrix}.
\]

We can consider, as usual, the periodic case \((k \in \mathbb{Z}_n)\), the infinite case \((k \in \mathbb{Z})\) or the semi-infinite case \((k \in \mathbb{N})\). The standard zero-curvature equations for the AKNS hierarchy are replaced with the semidiscrete zero-curvature equations
\[
\partial_t L_k = M_{k+1} L_k - L_k M_k.
\]

As an example, one of the most important equations of this hierarchy is the discrete complexified version of the nonlinear Schrödinger equations
\[
\begin{align*}
\partial_t x_k &= x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\
\partial_t y_k &= -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}).
\end{align*}
\]

Quite recently different authors (see [13, 15]) worked on the link between biorthogonal polynomials on the unit circle and the semi-infinite Ablowitz–Ladik hierarchy. This is an analogue of the celebrated link between the Toda hierarchy and orthogonal polynomials on the real line. Many articles have been written about orthogonal polynomials and the Toda hierarchy and it is almost impossible to recall all of them. Let us just mention the seminal paper of Moser [8] and the monograph [9]; there interested readers can find many references. The more recent articles [11–13] should also be cited. In these articles the connection between the theory of orthogonal polynomials and Sato’s theory of infinite Grassmannians appeared for the first time. In particular, in [13] the case of Toda and Ablowitz–Ladik hierarchies is treated in a similar way to reductions of 2D-Toda. In this way a clear and unified explanation of the role played by orthogonal polynomials in the theory of integrable equations is given. As noted by the authors their approach is quite different from the original one; actually in their paper [13] they always refer to the Toeplitz lattice and the coincidence with the Ablowitz–Ladik hierarchy is only stated in the introduction. Nevertheless, as explained in section 2 of this paper, it is very easy to deduce semidiscrete zero-curvature equations starting from Adler–van Moerbeke’s equations. With this approach in mind we addressed a question arising from the following facts.

- Time evolution for orthogonal polynomials on the real line leads to the Toda hierarchy.
- Time evolution for biorthogonal polynomials on the unit circle leads to the Ablowitz–Ladik hierarchy.
- Time evolution for matrix orthogonal polynomials on the real line leads to the non-Abelian Toda hierarchy (see for instance [17] and references therein; our approach is close to [20]).

What about time evolution for matrix biorthogonal polynomials on the unit circle?

In other words, our goal was to replace the question mark in the table below with the corresponding hierarchy.

| Orthogonal polynomials on \(\mathbb{R}\) | Biorthogonal polynomials on \(S^1\) |
|---------------------------------|---------------------------------|
| Scalar case | Toda | Ablowitz–Ladik |
| Matrix case | Non-Abelian Toda | ? |
In this paper, we prove that the relevant hierarchy is the non-Abelian version of the Ablowitz–Ladik hierarchy. This hierarchy has been studied by different authors since 1983 (see [3–5]) but, to the best of our knowledge, the connection with matrix biorthogonal polynomials on the unit circle has never been established before. We also remark that this hierarchy is usually called the matrix, vector or multicomponent Ablowitz–Ladik. Here we prefer to call it non-Abelian to stress the analogy with Toda. In our setting, instead of having one Lax operator $L_k$ of size $2 \times 2$, we have two Lax operators $L_l^k$ and $L_r^k$ of size $2n \times 2n$. These operators depend on matrices $x_l^k, x_r^k, y_l^k$ and $y_r^k$ of size $n \times n$. Semidiscrete zero curvature equations are given by

$$\partial_t L_l^k = M_l^k L_l^k - L_l^k M_l^k,$$

$$\partial_t L_r^k = L_r^k M_r^k - M_r^k L_r^k,$$

where $M_l^k$ and $M_r^k$ are also block matrices. For instance we have, in this hierarchy, two versions of the non-Abelian complexified discrete nonlinear Schrödinger:

$$\begin{aligned}
\partial_t x_l^k &= x_{l+1}^k - 2x_l^k + x_{l-1}^k - x_{l+1}^k y_l^k y_{l+1}^k - x_{l-1}^k y_l^k y_{l-1}^k \\
\partial_t y_l^k &= -y_{l+1}^k + 2y_l^k - y_{l-1}^k + y_{l+1}^k x_l^k y_{l+1}^k + y_{l-1}^k y_{l+1}^k y_{l-1}^k \\
\partial_t x_r^k &= x_{r+1}^k - 2x_r^k + x_{r-1}^k - x_{r+1}^k y_r^k y_{r+1}^k - x_{r-1}^k y_r^k y_{r-1}^k \\
\partial_t y_r^k &= -y_{r+1}^k + 2y_r^k - y_{r-1}^k + y_{r+1}^k x_r^k y_{r+1}^k + y_{r-1}^k y_{r+1}^k y_{r-1}^k
\end{aligned}$$

(see [6] for a review about these equations).

The sections are organized as follows.

- In the second section, we recall some basic facts about 2D-Toda and the connection with biorthogonal polynomials. We use the approach developed in [13].
- The third section starts with the description of the Toeplitz lattice (see [13]) and shows how to deduce semidiscrete zero-curvature equations for the Ablowitz–Ladik hierarchy.
- In the fourth section, we extend the Toeplitz lattice to the case of block Toeplitz matrices.
- The fifth section gives recursion relations for matrix biorthogonal polynomials on the unit circle; these formulae slightly generalize formulae contained in [19, 22] for matrix orthogonal polynomials on the unit circle.
- In the sixth section, we derive block semidiscrete zero-curvature equations defining the non-Abelian Ablowitz–Ladik hierarchy. As an example we write the non-Abelian analogue of the discrete nonlinear Schrödinger.

2. 2D-Toda; linearization and biorthogonal polynomials

In this section, we recall some basic facts about the 2D-Toda hierarchy as presented in [7]. Moreover we describe the connection with biorthogonal polynomials as originally presented in [13]. We are interested in the semi-infinite case; we start denoting with $\Lambda$ the shift matrix

$$\Lambda := (\delta_{i,j+1})_{i,j \geq 0}.$$

For the transpose we use the notation $\Lambda^T = \Lambda^{-1}$. Then we define two Lax matrices

$$\begin{aligned}
L_1 &:= \Lambda + \sum_{i \leq 0} a_i^{(1)} \Lambda^i \\
L_2 &:= a^{(2)}_{-1} \Lambda^{-1} + \sum_{i \geq 0} a_i^{(2)} \Lambda^i
\end{aligned}$$
where \( \{a_i^{(s)}, s = 1, 2\} \) are some diagonal matrices. 2D-Toda equations, expressed in Lax form, arise as compatibility conditions for the following Zakharov–Shabat spectral problem:

\[
\begin{align*}
L_1 \Psi_1 &= z \Psi_1, \\
L_2^* \Psi_2^* &= z^{-1} \Psi_2^*, \\
\partial_n \Psi_1 &= (L_1^n)^* \Psi_1, \\
\partial_n \Psi_2^* &= -(L_2^n)^* \Psi_2^*, \\
\partial_t \Psi_1 &= (L_1^n) \Psi_1, \\
\partial_t \Psi_2^* &= -(L_2^n)^T \Psi_2^*.
\end{align*}
\]

Here we introduced two infinite sets of times \( \{t_i, i \geq 0\} \) and \( \{s_i, i \geq 0\} \). We denoted with \( N_+ \) the upper triangular part of a matrix \( N \) (including the main diagonal) and with \( N_- \) the lower triangular part (excluding the main diagonal). \( \Psi_1 \) and \( \Psi_2^* \) are semi-infinite column vectors of type

\[
\Psi_1(z) = (\Psi_{1,0}(z), \Psi_{1,1}(z), \ldots)^T, \\
\Psi_2^*(z) = (\Psi_{2,0}^*(z), \Psi_{2,1}^*(z), \ldots)^T.
\]

For every \( k \) the two expressions \( e^{-\xi(t,z)} \Psi_{1,k}(z) \) and \( e^{-\xi(s,z)} \Psi_{2,k}^*(z^{-1}) \) are polynomials in \( z \) of order \( k \). Lax equations are written as

\[
\begin{align*}
\partial_t L_i &= [(L_1^n)_+, L_i], \\
\partial_s L_i &= [(L_2^n)^*_-, L_i], \\
i &= 1, 2.
\end{align*}
\]

2D-Toda equations can be linearized as explained in [7]. We start with an initial value matrix \( M(0, 0) = (M_{ij}(0, 0))_{i,j \geq 0} \) and we define its time evolution through the equation

\[
M(t; s) := \exp(\xi(t, \Lambda)) M(0, 0) \exp(-\xi(s, \Lambda^{-1})).
\]

We assume that there exists a factorization

\[
M(0, 0) = S_1(0, 0)^{-1} S_2(0, 0).
\]

Here \( S_1 \) is lower triangular while \( S_2 \) is upper triangular. We assume that both \( S_1 \) and \( S_2 \) have non-zero elements on the main diagonal and we normalize them in such a way that every element on the main diagonal of \( S_1 \) is equal to 1. Moreover we consider values of \( t \) and \( s \) for which we can write

\[
M(t, s) = S_1(t, s)^{-1} S_2(t, s)
\]

with \( S_1 \) and \( S_2 \) having the same properties as above. It can be proven that such factorization exists if and only if all the principal minors of \( M(t, s) \) do not vanish. In particular, this condition is satisfied when \( M(t, s) \) is the matrix of the moments of a positive-definite measure. Now we denote with \( \chi(z) \) the infinite vector \( \chi(z) := (1, z, z^2, \ldots)^T \). Wave vectors for 2D-Toda and Lax matrices are constructed in the following way.

**Theorem 2.1 ([7]).** The wave vectors

\[
\Psi_1(z) := \exp(\xi(t, z)) S_1(0, z) \chi(z), \\
\Psi_2^*(z) := \exp(-\xi(s, z^{-1})) (S_2^{-1})^T \chi(z^{-1}).
\]

and the two Lax operators \( L_1 := S_1 S_1^{-1} \) and \( L_2 := S_2^{-1} S_2^{-1} \) satisfy the 2D-Toda Zakharov–Shabat spectral problem.
Proof. We only sketch the proof and make reference to the article [7]. It is clear that the matrix $M(t, s)$ satisfies differential equations

$$
\partial_t M = \Lambda^t M \quad \partial_s M = -M \Lambda^{-i}.
$$

Then it is easy to deduce Sato’s equations

$$
\partial_t S_1 = -(L^n_1) S_1, \quad \partial_t S_2 = (L^1_1) S_2,
$$

$$
\partial_s S_1 = (L^n_2) S_1, \quad \partial_s S_2 = -(L^2_1) S_2
$$

and Zakharov–Shabat’s equations can be deduced from the expression of wave vectors in terms of $S_1$ and $S_2$. □

The last thing we need is the link between the factorization of $M$ and biorthogonal polynomials. We introduce a bilinear pairing on the space of polynomials in $z$

$$
\langle \cdot, \cdot \rangle_M := M_{ij}.
$$

The following proposition is a direct consequence of (1).

Proposition 2.2 ([13]).

$$
q^{(1)}(\gamma) := S_1 \chi(z), \quad q^{(2)}(\gamma) := (S_2^{-1})^T \chi(z)
$$

are biorthonormal polynomials with respect to the pairing $\langle \cdot, \cdot \rangle_M$; i.e.

$$
\langle q^{(1)}(\gamma), q^{(2)}(\gamma) \rangle_M = \delta_{ij} \quad \forall i, j \in \mathbb{N}.
$$

3. From the Toeplitz lattice hierarchy to the semidiscrete zero-curvature equations for the Ablowitz–Ladik hierarchy

In this section, we briefly recall the reduction from 2D-Toda to the Toeplitz lattice as described in [13]. Then we show how the Ablowitz–Ladik equations are easily obtained from the Toeplitz lattice. Suppose that our initial value $M(0, 0)$ is a Toeplitz matrix; i.e. we have

$$
M(0, 0) = T(\gamma) = \begin{pmatrix}
\gamma^{(0)} & \gamma^{(-1)} & \gamma^{(-2)} & \ldots \\
\gamma^{(1)} & \gamma^{(0)} & \gamma^{(-1)} & \ldots \\
\gamma^{(2)} & \gamma^{(1)} & \gamma^{(0)} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

for some formal power series $\gamma(z) = \sum_{n \in \mathbb{Z}} \gamma^{(n)} z^n$. Since $\Lambda = T(z^{-1})$ is an upper triangular Toeplitz matrix it follows easily (see for instance [23]) that

$$
M(t, s) = \exp(\xi(t, \Lambda)) M(0, 0) \exp(-\xi(s, \Lambda^{-1})) = T(\exp(\xi(t, z^{-1})) \gamma(z) \exp(-\xi(s, z))).
$$

This means that the Toeplitz form is conserved along 2D-Toda flow, hence we deal with a reduction of it. This reduction is called the Toeplitz lattice in [13]. In that article the authors noticed, in the introduction, that this is nothing but the Ablowitz–Ladik hierarchy. Now we...
will describe how to obtain the original formulation of the Ablowitz–Ladik equations starting from Adler–van Moerbeke’s formulation. The key observation is that, in this case, the bilinear pairing \( \langle p, q \rangle_M \) between two arbitrary polynomials is given by

\[
\langle p, q \rangle_M = \oint p(z)q^\ast(z) \frac{dz}{2\pi i z}.
\]

Here the symbol of integration means that we are taking the residue of the formal series \( p(z)q^\ast(z) \) and \( q^\ast(z) = q(z^{-1}) \). In other words, \( q^{(1)}_j \) and \( q^{(2)}_j \) are nothing but orthonormal polynomials on the unit circle. We also define monic biorthogonal polynomials

\[
p^{(1)}_n = (p^{(1)}_n)_{j \geq 0} := S_1 x(z)\]

\[
p^{(2)}_n = (p^{(2)}_n)_{j \geq 0} := h(S_2^{-1})^T x(z)
\]

with \( h = \text{diag}(h_0, h_1, h_2, \ldots) \) some diagonal matrix. Given an arbitrary polynomial \( q(z) \) of degree \( n \) we define its reversed polynomial \( \tilde{q}(z) := z^nq^\ast(z) \) and reflection coefficients

\[
x_n := p^{(1)}_n(0) \quad y_n := p^{(2)}_n(0).
\]

We can state the standard recursion relation associated with biorthogonal polynomials on the unit circle. The equation below was already known to Szegő [18]. It has been used in [14] in relation to the theory of integrable equations.

**Proposition 3.1.** The following recursion relation holds:

\[
\left( \begin{array}{c} p^{(1)}_{n+1}(z) \\ \tilde{p}^{(2)}_{n+1}(z) \end{array} \right) = L_n \left( \begin{array}{c} p^{(1)}_n(z) \\ \tilde{p}^{(2)}_n(z) \end{array} \right) = \left( \begin{array}{cc} z & x_{n+1} \\ zy_{n+1} & 1 \end{array} \right) \left( \begin{array}{c} p^{(1)}_n(z) \\ \tilde{p}^{(2)}_n(z) \end{array} \right)
\]

(2)

Using this recursion relation in [13] Adler and van Moerbeke wrote the peculiar form of Lax operators for the Toeplitz reduction.

**Proposition 3.2 ([13]).** Lax operators of the Toeplitz lattice are of the following form:

\[
h^{-1}L_1h = \begin{pmatrix}
-x_1y_0 & 1 - x_1y_1 & 0 & \cdots & \cdots \\
-x_2y_0 & -x_2y_1 & 1 - x_2y_2 & 0 & \cdots \\
-x_3y_0 & -x_3y_1 & -x_3y_2 & 1 - x_3y_3 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

\[
L_2 = \begin{pmatrix}
-x_0y_1 & -x_0y_2 & -x_0y_3 & \cdots & \cdots \\
1 - x_1y_1 & -x_1y_2 & -x_1y_3 & \cdots & \cdots \\
0 & 1 - x_2y_2 & -x_2y_3 & \cdots & \cdots \\
\vdots & \vdots & 0 & 1 - x_3y_3 & \cdots & \cdots \\
\end{pmatrix}
\]

From this proposition the following corollary follows easily.

**Corollary 3.3.** Reflection coefficients satisfy the following equation

\[
\frac{h_{n+1}}{h_n} = 1 - x_{n+1}y_{n+1}.
\]
Now we can state the theorem relating the Toeplitz lattice to the original form of the
Ablowitz–Ladik hierarchy.

**Theorem 3.4.** The Toeplitz lattice flow can be written in the form

$$
\partial_t \begin{pmatrix} p^{(1)}_n(z) \\
p^{(2)}_n(z) \end{pmatrix} = M_{t,n} \begin{pmatrix} p^{(1)}_n(z) \\
p^{(2)}_n(z) \end{pmatrix}
$$

(3)

$$
\partial_s \begin{pmatrix} p^{(1)}_n(z) \\
p^{(2)}_n(z) \end{pmatrix} = M_{s,n} \begin{pmatrix} p^{(1)}_n(z) \\
p^{(2)}_n(z) \end{pmatrix}
$$

(4)

for some matrices $M_{t,n}, M_{s,n}$ depending on $\{x_j, y_j, z\}$.

**Proof.** We prove it for the set of times denoted by $t$. We denote $d([z]) = \text{diag}(1, z, z^2, z^3, \ldots)$. We have the identities

$$
\Psi_1(z) = \exp(\xi(t, z)) p^{(1)}(z)
$$

$$
\Psi_2(z) = h^{-1}d([z^{-1}]) \exp(-\xi(s, z^{-1})) \tilde{p}^{(2)}(z)
$$

giving the following time evolution for orthogonal polynomials:

$$
\partial_t p^{(1)}(z) = -zp^{(1)}(z) + (L^1)_{++} p^{(1)}(z)
$$

$$
\partial_s \tilde{p}^{(2)}(z) = -hd([z]) (L^1)_{++} h^{-1}d([z^{-1}]) \tilde{p}^{(2)}(z).
$$

Here $(L^1)_{++}$ denotes the strictly upper diagonal part of $L^1_n$. The formulae above are obtained from a straightforward computation and using the fact, proven in [13], that

$$
\partial_t \log(h_n) = (L^1)_{nn}.
$$

Hence we have that, for every $k$, $\partial_t p^{(1)}_k$ is a linear combination of $\{p^{(1)}_k, p^{(1)}_{k+1}, p^{(1)}_{k+2}, \ldots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$. In the same way, for every $k$, $\partial_s \tilde{p}^{(2)}_k$ is a linear combination of $\{\tilde{p}^{(2)}_k, \tilde{p}^{(2)}_{k-1}, \tilde{p}^{(2)}_{k-2}, \ldots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$. Using the recursion relation (2) and its inverse

$$
\begin{pmatrix} p^{(1)}_n(z) \\
p^{(2)}_n(z) \end{pmatrix} = L^{-1} \begin{pmatrix} p^{(1)}_{n+1}(z) \\
\tilde{p}^{(2)}_{n+1}(z) \end{pmatrix} = h_n \begin{pmatrix} z^{-1} & -z^{-1}x_{n+1} \\
y_{n+1} & 1 \end{pmatrix} \begin{pmatrix} p^{(1)}_{n+1}(z) \\
\tilde{p}^{(2)}_{n+1}(z) \end{pmatrix}
$$

we can obtain the desired matrices $M_{t,n}$.

**Corollary 3.5 (Ablowitz–Ladik semidiscrete zero-curvature equations).** The matrices $L_n$ satisfy the following time evolution:

$$
\partial_t L_n = M_{t,n+1} L_n - L_n M_{t,n}
$$

(5)

$$
\partial_s L_n = M_{s,n+1} L_n - L_n M_{s,n}
$$

(6)

**Proof.** These equations are nothing but the compatibility conditions of recursion relation (2) with time evolution (3) and (4).

**Remark 3.6.** Actually our Lax operator $L_n$ is slightly different from the Lax operator $L_n$ used in [1] by Ablowitz and Ladik and written in the introduction above. Nevertheless, as shown in [16], these two Lax operators are linked through a simple change of spectral parameter.
Example 3.7 (the first flows; discrete nonlinear Schrödinger). The first matrices $\mathcal{M}_{t_1,n}$ and $\mathcal{M}_{s_1,n}$ are easily computed. We have

$$\partial_t p^{(1)}_k = -z p^{(1)}_k - x_{k+1} y_k p^{(1)}_k + x_k y_{k+1} + p^{(2)}_k$$

$$\partial_t \tilde{p}^{(2)}_k = -z \frac{h_{k+1}}{h_k} \tilde{p}^{(2)}_k = z y_k p^{(1)}_k - z \tilde{p}^{(2)}_k$$

giving immediately

$$\mathcal{M}_{t_1,k} = \begin{pmatrix} -x_{k+1} y_k & x_k y_{k+1} \\ z y_k & -z \end{pmatrix}$$

An analogue computation for $s_1$ can be easily done obtaining

$$\mathcal{M}_{s_1,k} = \begin{pmatrix} z^{-1} - z^{-1} x_k \\ -y_{k+1} & x_k y_{k+1} \end{pmatrix}.$$ 

We can already write, with $t_1$ and $s_1$, the well-known integrable discretization of nonlinear Schrödinger. We just need two more trivial rescaling times introduced with substitutions

$$p^{(1)} \mapsto \exp(t_0) p^{(1)}$$

$$\tilde{p}^{(2)} \mapsto \exp(-s_0) \tilde{p}^{(2)}$$

and corresponding to matrices

$$\mathcal{M}_{t_0,k} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M}_{s_0,k} := \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Now we can construct the matrix

$$\mathcal{M}_{\tau,k} = \mathcal{M}_{t_1,k} + \mathcal{M}_{s_1,k} - \mathcal{M}_{t_0,k} - \mathcal{M}_{s_0,k}$$

$$= \begin{pmatrix} z^{-1} - 1 - x_{k+1} y_k & x_k y_{k+1} + z^{-1} x_k \\ z y_k - y_{k+1} & x_k y_{k+1} + 1 - z \end{pmatrix}$$

associated with the time $\tau = t_1 + s_1 - t_0 - s_0$ so that semidiscrete zero-curvature equation

$$\partial_\tau L_k = \mathcal{M}_{\tau,k+1} L_k - L_k \mathcal{M}_{\tau,k}$$

is equivalent to the system

$$\begin{cases} \partial_\tau x_k = x_{k+1} - 2 x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \partial_\tau y_k = -y_{k+1} + 2 y_k - y_{k-1} + x_k y_{k+1} (y_{k+1} + y_{k-1}) \end{cases}.$$ 

This is exactly the complexified version of the discrete nonlinear Schrödinger. Rescaling $\tau \mapsto i\tau$ and imposing $y_k = \pm x_k^*$ we obtain

$$-i \partial_\tau x_k = x_{k+1} - 2 x_k + x_{k-1} \mp \|x_k\|^2 (x_{k+1} + x_{k-1}).$$ 

(7)

4. Toda flow for block Toeplitz matrices and the related Lax operators

Now we generalize the Toeplitz lattice’s equations to the block case. We start with a matrix-valued formal series

$$\gamma(z) = \sum_{k \in \mathbb{Z}} \gamma^{(k)} z^k.$$ 

Here every element $\gamma^{(k)}$ is a $n \times n$ matrix. Then we define its time evolution as

$$\gamma(t, s; z) := \exp(-\xi(s, z^{-1} I)) \gamma(z) \exp(\xi(t, z I)).$$ 

(8)
where $I$ is the $n \times n$ identity matrix. Differently from the scalar case we do not consider just one Toeplitz matrix but the two block Toeplitz matrices, right and left, given by

$$T^r(\gamma) := \begin{pmatrix} \gamma^{(0)} & \gamma^{(-1)} & \gamma^{(-2)} & \cdots \\ \gamma^{(1)} & \gamma^{(0)} & \gamma^{(-1)} & \cdots \\ \gamma^{(2)} & \gamma^{(1)} & \gamma^{(0)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$T^l(\gamma) := \begin{pmatrix} \gamma^{(0)} & \gamma^{(1)} & \gamma^{(2)} & \cdots \\ \gamma^{(-1)} & \gamma^{(0)} & \gamma^{(1)} & \cdots \\ \gamma^{(-2)} & \gamma^{(-1)} & \gamma^{(0)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In this way, we obtain the following linear time evolution for our block Toeplitz matrices (in the following we will omit the symbol $\gamma$):

$$\begin{align*}
\partial_t T^l &= \Lambda^l T^l \\
\partial_s T^r &= \frac{1}{\Lambda^r} T^r \\
\partial_t T^r &= -\Lambda^t T^r \\
\partial_s T^l &= -\frac{1}{\Lambda^l} T^l
\end{align*}$$

(9) (10)

where, in this case, we have $\Lambda = T^r(z^{-1}I)$. Then we assume that there exist two factorizations

$$T^l = S_1^{-1} S_2 \quad T^r = Z_2 Z_1^{-1}.$$  

Here $S_1, Z_2$ are block-lower triangular while $S_2, Z_1$ are block-upper triangular. We assume that all these matrices have non-degenerate blocks on the main diagonal (i.e. these blocks must have non-zero determinants). Normalizations are chosen in such a way that every element on the main block-diagonal of $S_1$ and $Z_2$ is equal to the identity matrix $I$. As we did previously we assume that these conditions hold when every time is equal to 0 and then we consider just the values of $t$ and $s$ for which these conditions still hold. In the matrix case, we can define two bilinear pairings given by the following definition.

**Definition 4.1.**

$$\langle P, Q \rangle_r := \oint P(\zeta)\gamma(\zeta)Q(\zeta) \frac{d\zeta}{2\pi i \zeta} \quad \langle P, Q \rangle_l := \oint P(\zeta)\gamma^{*}(\zeta)Q^{*}(\zeta) \frac{d\zeta}{2\pi i \zeta}$$

where $P$ and $Q$ are two arbitrary matrix polynomials and $P^*(\zeta) := (P(z^{-1}))^T$.

Our two factorizations give exactly biorthonormal polynomials for $\langle \cdot, \cdot \rangle_r$ and $\langle \cdot, \cdot \rangle_l$. In the following we denote $\chi(\zeta) := (I, zI, z^2 I, z^3 I, \ldots)^T$.

**Proposition 4.2.**

$$Q_r^{(1)} := \begin{pmatrix} Q_0^{(1)} \\ Q_1^{(1)} \\ \vdots \end{pmatrix} = S_1 \chi(z)$$

(11)

$$Q_l^{(2)} := \begin{pmatrix} Q_0^{(2)} \\ Q_1^{(2)} \\ \vdots \end{pmatrix} = (S_2^{-1})^T \chi(z)$$

(12)
\[ Q^{(1)r} := (Q^{(1)r}_0 \ Q^{(1)r}_1 \ldots) = \chi(z)^T Z_1 \]
\[ Q^{(2)r} := (Q^{(2)r}_0 \ Q^{(2)r}_1 \ldots) = \chi(z)^T (Z_2^{-1})^T \]

are the biorthonormal polynomials associated with the pairing \( \langle \cdot, \cdot \rangle_l \) and \( \langle \cdot, \cdot \rangle_r \). Hence for every \( i, j \) we have
\[ \langle Q^{(1)r}_i, Q^{(2)r}_j \rangle_r = \delta_{ij} \]
\[ \langle Q^{(2)r}_i, Q^{(1)r}_j \rangle_l = \delta_{ij}. \]

**Proof.** We only prove, as an example, the proposition for the right polynomials. On the other hand, the one for left polynomials is identical to the usual proof for 2D-Toda. We have
\[ (\langle Q^{(2)r}_i, Q^{(1)r}_j \rangle_r)_{i,j \geq 0} = \left( \sum_{k,l \geq 0} (Z_1^{-1})_{kl} \langle z^k I, z^l I \rangle_r (Z_1)_{ij} \right) \]
\[ = Z_1^{-1} T^r Z_1 = I \iff T^r = Z_1 Z_2^{-1}. \]

(It should be noted that, in this case, subscripts of type \((Z_1)_{ij}\) denote the block in position \((i, j)\) and not the element \((i, j)\).) \( \square \)

Now we can write the related Sato’s equations for \( S_i \) and \( Z_i \). It is convenient to introduce the following Lax operators.

**Definition 4.3.**
\[ L_1 := S_1 \Lambda_1 S_1^{-1}, \quad L_2 := S_2 \Lambda_1^{-1} S_2^{-1} \]
\[ R_1 := Z_1^{-1} \Lambda_1^{-1} Z_1, \quad R_2 := Z_2^{-1} \Lambda Z_2. \]

**Proposition 4.4.** The following Sato’s equations are satisfied:
\[ \partial_t S_1 = -(L_1^n)_{-} S_1 \quad \partial_t Z_1 = -Z_1 (R_1^n)_{-}, \]
\[ \partial_s S_1 = (L_1^n)_{+} S_1 \quad \partial_s Z_1 = Z_1 (R_1^n)_{+}, \]
\[ \partial_t S_2 = -(L_2^n)_{-} S_2 \quad \partial_t Z_2 = -Z_2 (R_2^n)_{-}, \]
\[ \partial_s S_2 = (L_2^n)_{+} S_2 \quad \partial_s Z_2 = Z_2 (R_2^n)_{+}. \]

**Proof.** We will only prove, as an example, the equations involving the \( t \)-derivative of \( Z_1 \) and \( Z_2 \). We assume, as an ansatz, that we have
\[ \partial_t Z_1 = Z_1 A \]
\[ \partial_t Z_2 = Z_2 B \]
for some matrices \( A \) and \( B \). Then exploiting the time evolution of \( T^r \) we can write
\[ T^r A^{-n} = \partial_t T^r = \partial_t (Z_2 Z_1^{-1}) \]
\[ = Z_2 B Z_1^{-1} - Z_2 Z_1^{-1} Z_1 A Z_1^{-1} = Z_2 (B - A) Z_1^{-1} \]
hence we must have \( (B - A) Z_1^{-1} = Z_1^{-1} A^{-n} \). We rewrite it as
\[ B - A = Z_1^{-1} A^{-n} Z_1 = R_1^n. \]
Moreover, we have that $B$ must be strictly block-lower triangular and $A$ block-upper triangular. This is because $Z_1$ is block-upper triangular and $Z_2$ is block-lower triangular with constant entries on the main diagonal. Hence we conclude that $A = -(R_n^1)_-$ and $B = (R_n^1)^+$.  

Now it is just a matter of trivial computations to write down the corresponding Lax equations for $L_i$ and $R_i$.

**Proposition 4.5.** The following Lax equations are satisfied:

\[
\partial t_i L_i = \left[ (L_1^1)^+, L_i \right], \quad \partial t_i R_i = \left[ R_i, (R_1^i)^+ \right].
\]

(21)

\[
\partial s L_i = \left[ (L_2^1)^-, L_i \right], \quad \partial s R_i = \left[ R_i, (R_2^i)^- \right].
\]

(22)

The definition of our Lax operators will give us eigenvalue equations for suitably defined wave vectors.

**Definition 4.6.**

\[
\Psi_1(z) := \exp(\xi(t, z)) S_1 \chi(z)
\]

(23)

\[
\Phi_1(z) := \exp(\xi(t, z)) [\chi(z)]^T Z_1
\]

(24)

\[
\Psi_2^*(z) := \exp(-\xi(s, z^{-1} I)) (S_2^{-1})^T \chi(z^{-1})
\]

(25)

\[
\Phi_2^*(z) := \exp(-\xi(s, z^{-1} I)) \chi(z^{-1})^T (Z_2^{-1})^T
\]

(26)

**Proposition 4.7.** The following equations hold true:

\[
L_1 \Psi_1(z) = z \Psi_1(z), \quad \Phi_1(z) R_1 = z \Phi_1(z).
\]

(27)

\[
L_2^T \Psi_2^*(z) = z^{-1} \Psi_2^*(z), \quad \Phi_2^*(z) R_2^T = z^{-1} \Phi_2^*(z).
\]

(28)

**Proof.** We will only prove the last equation, all the other ones are proved in a similar way. From its very definition we have

\[
\Phi_2^*(z) R_2^T = z^{-1} I \Phi_2^*(z) \iff [\chi(z^{-1})]^T (Z_2^{-1})^T R_2^T = z^{-1} [\chi(z^{-1})]^T (Z_2^{-1})^T
\]

\[
= [\chi(z^{-1})]^T \Lambda^{-1} (Z_2^{-1})^T \iff R_2^T = Z_2^T \Lambda^{-1} (Z_2^{-1})^T \iff R_2 = Z_2^{-1} \Lambda Z_2.
\]

The proof of the following proposition is straightforward.

**Proposition 4.8.** The Lax equations (21) and (22) are the compatibility conditions of the eigenvalue equations (27) and (28) with the following equations:

\[
\partial s \Psi_1 = (L_1^1)^+ \Psi_1, \quad \partial s \Phi_1 = \Phi_1 (R_1^1)^-
\]

(29)

\[
\partial s \Psi_1 = (L_2^1)^- \Psi_1, \quad \partial s \Phi_1 = \Phi_1 (R_2^1)^+
\]

(30)

\[
\partial s \Psi_2^* = -(L_1^2)^T \Psi_2^*, \quad \partial s \Phi_2^* = -\Phi_2^* (R_1^2)^T
\]

(31)

\[
\partial s \Psi_2^* = -(L_2^2)^T \Psi_2^*, \quad \partial s \Phi_2^* = -\Phi_2^* (R_2^2)^T.
\]

(32)

**Remark 4.9.** Actually our Lax equations as well as equations (27)–(32) can be deduced from the equations of multicomponent 2D-Toda [10].
5. Recursion relations for matrix biorthogonal polynomials on the unit circle

In order to generalize the scalar theory we have to construct an analogue of the recursion relation given by proposition 3.1. Recursion relations for the matrix orthogonal polynomial on the unit circle are already known, see [19, 22]. Here we slightly generalize to the case of matrix biorthogonal polynomials on the unit circle.

We define the following important $n \times n$ matrices.

Definition 5.1.

\[ h_N^r := SC(T_{N+1}^r) \quad h_N^l := SC(T_{N+1}^l) \]

where SC denotes the $n \times n$ Schur complement of a block matrix with respect to the upper left block (see for instance [21]). For example we have

\[ SC(T_{N+1}^r) = \gamma(0) - (\gamma^{(1)}) T_{N}^{r} \cdots \gamma^{(N)} \gamma^{(-N)} \]

(here and below $T_{N}^{r} := (T_{N}^{r})^{-1}$ and similarly for $h_N^r, h_N^l$ and $T_{N}^{l}$).

Proposition 5.2. Monic biorthogonal polynomials such that

\[ \langle P_{N}^{(1)r}, P_{N}^{(1)l} \rangle = \delta_{kj} h_{N}^{r} \quad \langle P_{N}^{(1)l}, P_{N}^{(2)r} \rangle = \delta_{kj} h_{N}^{l} \]

are given by the following formulae:

\[ P_{N}^{(1)r} = SC \begin{pmatrix} \gamma(0) & \cdots & \cdots & \cdots & \gamma^{(N-1)} & \gamma^{(-N)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma^{(N-1)} & \cdots & \gamma(0) & \gamma^{(-1)} \\ I & z I & \cdots & z^{N-1} I & z^{N} I \end{pmatrix} \]

\[ (P_{N}^{(1)r})^T = SC \begin{pmatrix} \gamma(0) & \cdots & \gamma^{(N-1)} & I \\ \cdots & \cdots & \cdots & \cdots \\ \gamma^{(N-1)} & \cdots & \gamma(0) & z^{N-1} I \\ \gamma^{(N)} & \cdots & \gamma^{(1)} & z^{N} I \end{pmatrix} \]

\[ P_{N}^{(1)l} = SC \begin{pmatrix} \gamma(0) & \cdots & \cdots & \cdots & \gamma^{(-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma^{(-N)} & \cdots & \gamma^{(0)} & \gamma^{(-1)} \\ z^{(-N)} I & \cdots & z^{(-1)} I & z^{N} I \end{pmatrix} \]

\[ (P_{N}^{(1)l})^T = SC \begin{pmatrix} \gamma(0) & \cdots & \gamma^{(N-1)} & \gamma^{(-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma^{(N)} & \cdots & \gamma^{(0)} & z^{N-1} I \\ z^{(-N)} I & \cdots & z^{(-1)} I & z^{N} I \end{pmatrix} \]
Proof. We will only prove the first formula, the second one is proved similarly. First of all we have \( 0 \leq m \leq N - 1 \)

\[
\langle z^m I, P_N^{(1)}(z) \rangle_r = \oint z^{-m} \gamma(z) \left( z^N I - (I \cdots z^{N-1} I) T^{-r}_N \left( \begin{array}{c} \gamma^{(-N)} \\ \vdots \\ \gamma^{(-1)} \end{array} \right) \right) \frac{dz}{2\pi i z} = \gamma^{(m-N)} - \gamma^{(m-N)} = 0.
\]

In the same way \( 0 \leq m \leq N - 1 \)

\[
\langle P_N^{(2)}(z), z^m I \rangle_r = \oint z^{-m} (\gamma(z) z^m) \frac{dz}{2\pi i z} = \gamma^{(N-m)} - \gamma^{(N-m)} = 0.
\]

Finally

\[
\langle P_N^{(2)}(z), P_N^{(1)}(z) \rangle_r = \langle z^N I, P_N^{(1)}(z) \rangle_r = \gamma^{(0)} - \gamma^{(N)} \cdots \gamma^{(1)} T^{-r}_N \left( \begin{array}{c} \gamma^{(-N)} \\ \vdots \\ \gamma^{(-1)} \end{array} \right) = h_N.
\]

This completes the proof of the first formula, the second one is proved similarly. \(\square\)

Remark 5.3. Note that imposing

\[
Q_N^{(1)} := P_N^{(1)}(0), \quad Q_N^{(2)} := (h_N^{-1})^T P_N^{(2)}(0),
\]

we obtain biorthonormal polynomials.

Now we will write a long list of relations among these polynomials and reflection coefficients. Given any matrix polynomial \( Q(z) \) of degree \( n \) we define the associated reversed polynomial as

\[
\tilde{Q}(z) = z^n Q^*(z).
\]

The reflection coefficients are defined as follows:

\[
x_N^I := P_N^{(1)}(0), \quad x_N^r := P_N^{(1)}(0),
\]

\[
y_N^r := (P_N^{(2)}(0))^T, \quad y_N^r := (P_N^{(2)}(0))^T.
\]

Proposition 5.4. The following formulae hold true:

\[
P_N^{(1)} = z P_N^{(1)}(I - y_N^r y_N^l), \quad \tilde{P}_N^{(1)} = x_N^l \tilde{P}_N^{(1)}, \quad (33)
\]

\[
P_N^{(2)} = z P_N^{(2)}(I - y_N^r y_N^l), \quad \tilde{P}_N^{(2)} = x_N^l \tilde{P}_N^{(2)}, \quad (34)
\]

\[
P_N^{(1I)} = z P_N^{(1)}(I - y_N^r y_N^l), \quad \tilde{P}_N^{(1I)} = x_N^l \tilde{P}_N^{(1)}(I - y_N^r y_N^l), \quad (35)
\]

\[
P_N^{(2I)} = z P_N^{(2)}(I - y_N^r y_N^l), \quad \tilde{P}_N^{(2I)} = x_N^l \tilde{P}_N^{(2)}(I - y_N^r y_N^l), \quad (36)
\]

\[
P_N^{(1)} = z P_N^{(1)}(I - y_N^r y_N^l), \quad \tilde{P}_N^{(1)} = x_N^l \tilde{P}_N^{(1)}(I - y_N^r y_N^l), \quad (37)
\]

\[
P_N^{(1)} = z P_N^{(1)}(I - y_N^r y_N^l), \quad \tilde{P}_N^{(1)} = x_N^l \tilde{P}_N^{(1)}(I - y_N^r y_N^l), \quad (38)
\]

13
\[
\tilde{P}_{N+1}^{(2)r} = (I - y_{N+1}^i x_{N+1}^j) \tilde{P}_{N}^{(2)r} + y_{N+1}^i P_{N+1}^{(1)r},
\]
(39)
\[
\tilde{P}_{N+1}^{(2)l} = \tilde{P}_{N}^{(2)l} (I - x_{N+1}^i y_{N+1}^j) + P_{N+1}^{(1)l} y_{N+1}^j,
\]
(40)
\[
x_N^i h_N^r = h_N^r x_N^i,
\]
(41)
\[
y_N^i h_N^l = h_N^l y_N^i,
\]
(42)
\[
h_N^i h_{N+1} = I - y_{N+1}^i x_{N+1}^j,
\]
(43)
\[
h_N^l h_{N+1} = I - x_{N+1}^i y_{N+1}^j.
\]
(44)

**Proof.** The first four formulae are proved observing, for instance for the first case, that \( \forall 1 \leq i \leq N \) we have
\[
0 = \langle P_{N+1}^{(1)l} - z P_{N}^{(1)l} , z I \rangle_r = \langle \tilde{P}_{N}^{(2)r} , z I \rangle_I
\]
so that \( P_{N+1}^{(1)l} - z P_{N}^{(1)l} \) and \( \tilde{P}_{N}^{(2)r} \) must be proportional. Setting \( z = 0 \) you also find the constant of proportionality. In particular, when proving (34) and (36), we find a formula and then we have to take the reversed one. Equation (37) is proved substituting (36) into (35) and similarly for (38), (39), (40).

Equations (41) and (42) are proven respectively observing that we have
\[
\langle \tilde{P}_{N}^{(2)r} , P_{N}^{(1)l} \rangle_r = \langle P_{N}^{(1)l} , \tilde{P}_{N}^{(2)r} \rangle_I
\]
and then doing explicit computations. Finally (43) is obtained rewriting (37) as
\[
\frac{P_{N+1}^{(1)r}}{z} = \frac{P_{N}^{(1)r}}{z} \left( I - y_{N+1}^i x_{N+1}^j \right) + \left( P_{N+1}^{(2)l} \right)^* x_{N+1}^j,
\]
multiplying from the left for \( P_{N+1}^{(1)r} \gamma \) and then taking the residue. Equation (44) is proved similarly.

Now we define two sets of block matrices \( \{ L_N^l \}_{N \geq 0} \) and \( \{ L_N^r \}_{N \geq 0} \). They will have, in the matrix case, the same role played by \( \{ \mathcal{L}_N^l \}_{n \geq 0} \) in the scalar case.

**Definition 5.5.**
\[
L_N^l := \begin{pmatrix} z I & x_N^i \\ y_N^i & I \end{pmatrix}
\]
(45)
\[
L_N^r := \begin{pmatrix} z I & y_N^i \\ x_N^i & I \end{pmatrix}.
\]
(46)

**Corollary 5.6.** The following block matrices recursion relations are satisfied:
\[
\begin{pmatrix} P_{N+1}^{(1)l} \\ \tilde{P}_{N+1}^{(2)r} \end{pmatrix} = L_N^l \begin{pmatrix} P_{N}^{(1)l} \\ \tilde{P}_{N}^{(2)r} \end{pmatrix},
\]
(47)
\[
\begin{pmatrix} P_{N+1}^{(1)r} \\ \tilde{P}_{N+1}^{(2)l} \end{pmatrix} = \begin{pmatrix} P_{N}^{(1)r} \\ \tilde{P}_{N}^{(2)l} \end{pmatrix} L_N^r.
\]
(48)

**Proof.** These are nothing but (33)–(36).
6. Explicit expressions for Lax operators and related semidiscrete zero-curvature equations

Using our recursion relations we want to find explicit expressions for \( L_i \) and \( R_i \) in terms of our reflection coefficients \( x_k^l, x_k^r, y_k^l, y_k^r \). First of all we underline a remarkable symmetry that will allow us to reduce the amount of our computations. Doing the following three substitutions:

\[
\begin{align*}
  z &\rightarrow z^{-1} \\
  t &\rightarrow -s \\
  s &\rightarrow -t 
\end{align*}
\]

we immediately obtain the following proposition.

**Proposition 6.1.** Under the symmetry above the dressings, the orthogonal polynomials, the Lax operators and the reflection coefficients change as follows:

\[
\begin{align*}
  T^r &\rightarrow T^l \\
  T^l &\rightarrow T^r \\
  S_1 &\rightarrow Z_2^{-1} \\
  S_2 &\rightarrow Z_1^{-1} \\
  L_1 &\rightarrow R_2 \\
  L_2 &\rightarrow R_1 \\
  Q^{(1)} &\rightarrow (Q^{(2)})^* \\
  Q^{(2)} &\rightarrow (Q^{(1)})^* \\
  P^{(1)} &\rightarrow (P^{(2)})^* \\
  P^{(2)} &\rightarrow (P^{(1)})^* \\
  x^l &\rightarrow y^r \\
  y^l &\rightarrow x^r \\
  h_k^l &\rightarrow h_k^r.
\end{align*}
\]

Hence we can write only the left theory and we will have the right one as well. Actually every computation made above for the right theory can be deduced from the left theory and this symmetry will be called \( t - s \) symmetry in the following. In the theorem below the symbol \( \prod_{j=N+2}^{M-j} \) means that the terms in the product must be written from the smallest index to the biggest, going from left to right. The symbol \( \prod_{j=N+2}^{M} \) means that the product must be taken in the opposite direction.

**Theorem 6.2 (Lax operators for block Toeplitz lattice).** Lax operators \( L_i \) and \( R_i \) are expressed in terms of reflection coefficients according to the following formulae:

\[
\begin{align*}
(L_1)_{N,M+1} &= -x_{N+1}^l \left( \prod_{j=N+2}^{M-j} (I - y_j^l x_j^l) \right) y_{M+1}^r \\
(R_2)_{N,M+1} &= -y_{N+1}^r \left( \prod_{j=N+2}^{M-j} (I - x_j^l y_j^l) \right) x_{M+1}^l \\
(L_2)_{M+1,N} &= -h_{M+1}^l x_{M+1}^r \left( \prod_{j=N+2}^{M-j} (I - y_j^l x_j^l) \right) y_{N+1}^r h_N^l \\
(R_1)_{M+1,N} &= -h_{M+1}^l y_{M+1}^l \left( \prod_{j=N+2}^{M-j} (I - x_j^l y_j^l) \right) x_{N+1}^l h_N^l.
\end{align*}
\]
Moreover
\[(L_1)_{N,N+1} = (R_2)_{N,N+1} = I \quad (53)\]
\[(L_2)_{N+1,N} = h^r_{N+1} h^{-r}_{N} \quad (R_1)_{N+1,N} = h^r_{N+1} h^{-r}_{N} \quad (54)\]

**Proof.** Equations (53) and (54) follow trivially from the expressions of dressings \(S_i, Z_i\). In fact, because of our normalization, we have that \(S_1\) and \(Z_2\) are equal to the identity matrix plus a strictly lower triangular matrix which is what is stated in (53). Equation (54) is obtained by observing that the block on the main diagonal of \(S_2\) and \(Z_i\) is given respectively by matrices \(h^r_{N+1}\) and \(h^{-r}_{N+1}\) as follows from proposition 4.2 and remark 5.3. Now let us begin with (49) and (50); the important point is that we have
\[\Psi_1 = \exp(\xi(t, z)) P^{(1f)} \]
Hence as done in [13] we can find that \(\forall N > M \geq -1\) we have
\[(L_1)_{N,M+1} = -x^r_{N+1} h^{-r}_{N} h^r_{M+1} y^r_{M+1} \quad (55)\]
In fact \(\forall N > M \geq -1\)
\[\{P^{(1f)}_{N+1} - z P^{(1f)}_{N}, P^{(2f)}_{M+1} - z P^{(2f)}_{M}\}_{l} = -\{z P^{(1f)}_{N}, P^{(2f)}_{M+1}\}_{l} \]
\[= -\{P^{(1f)}_{N} + \cdots + (L_1)_{N,M+1} P^{(1f)}_{M+1} + \cdots, P^{(2f)}_{M+1}\}_{l} = -(L_1)_{N,M+1} h^r_{M+1} \]
On the other hand, using recursion relations, we also have \(\forall N > M \geq -1\)
\[\{P^{(1f)}_{N+1} - z P^{(1f)}_{N}, P^{(2f)}_{M+1} - z P^{(2f)}_{M}\}_{l} = [x^l_{N+1} \tilde{P}^{(2f)}_{N}, (y^f_{M+1})^T \tilde{P}^{(1f)}_{M}]_{l} \]
\[= x^l_{N+1} \left( z^{N-M} P^{(2f)}_{N} x^f_{M+1} \frac{dz}{2\pi i z} \right) y^f_{M+1} \]
and comparing them we find (55). Now we use \(t-\)symmetry to simplify this expression. We obtain
\[(R_2)_{N,M+1} = -y^r_{N+1} h^{-r}_{N} h^r_{M+1} x^l_{M+1} \]
Now we apply the recursion (44) to the piece \(h^r_{N} h^{-r}_{M+1}\) several times and we get (50). Equation (49) is obtained using \(t-s\) symmetry. For (51) and (52) we start by defining \(\tilde{R}_1\) such that \(z P^{(1f)} = P^{(1f)} \tilde{R}_1\); then we will have \(R_1 = h^r \tilde{R}_1 h^{-r}\) and computations for \(\tilde{R}_1\) are carried out similarly as for \(L_1\) \(\square\)

**Remark 6.3.** Our equations (49), (50), (51) and (52) extend to the matrix biorthogonal setting the equations written in [22] for matrix orthogonal polynomials (see equations (4.2) and (4.3)). In that article, the properties of \(M\) are applied to study some problems in computational mathematics (multivariate time series analysis and multichannel signal processing) and no relation is established with the Lax theory and integrable systems.

The theorem above describes the block-analogue of the Toeplitz lattice; we are now in the position to prove the analogue of theorem 3.4.

**Theorem 6.4.** Block Toeplitz lattice flow can be written in the form
\[\partial_{t_1/s_1} \left( P^{(1f)}_{N} (z) \right) = M^{d_{t_1/s_1,N}} \left( P^{(2f)}_{N} (z) \right) \]
\[\tilde{P}^{(1f)}_{N} (z) \quad (56)\]
The matrices

\[
C_{ij}(x, y) = \frac{1}{2} \left( \begin{array}{cc}
1 & \delta_{ij} \\
\delta_{ij} & 1
\end{array} \right)
\]

for some block matrices \( M_{x,y} \) depending on the matrices \( \{x_j, y_j\} \) and the spectral parameter \( z \).

**Proof.** As we did for the scalar case we prove it only for \( t \) times. The relevant equations linking biorthogonal polynomials with wave vectors are

\[
\Psi_1(z) = \exp(\xi(t, \xi I)) P^{(1)}(z),
\]

\[
\Phi_2(z) = \exp(-\xi(s, z^{-1} I))(P^{(2)}(z)T d([z^{-1}]),
\]

\[
\Phi_1(z) = \exp(\xi(t, \xi I)) P^{(1)} h^r,
\]

\[
\Psi_2(z) = \exp(-\xi(s, z^{-1} I))(h^{-1} T d([z^{-1}])) (P^{(2)}(z) T).
\]

Trivial computations give the following time evolution:

\[
\partial_t P^{(1)} = (L_1^n) n P^{(1)} - z^n P^{(1)} h,
\]

\[
\partial_t \tilde{P}^{(2)} = -d([z]) (R^n) r d([z^{-1}]) \tilde{P}^{(2)},
\]

\[
\partial_t P^{(1)} = P^{(1)} (h^{-1} R^n h' + h'^{-1} (\partial_t h') - z^n I),
\]

\[
\partial_t \tilde{P}^{(2)} = \tilde{P}^{(2)} (-h^{-1} (L_1^n) h' + h'^{-1} (\partial_t h'))
\]

the last two can be simplified giving

\[
\partial_t P^{(1)} = (L_1^n) n P^{(1)} - z^n I P^{(1)} h,
\]

\[
\partial_t \tilde{P}^{(2)} = -d([z]) (R^n) r d([z^{-1}]) \tilde{P}^{(2)},
\]

\[
\partial_t P^{(1)} = P^{(1)} (h'^{-1} - z^n I),
\]

\[
\partial_t \tilde{P}^{(2)} = -\tilde{P}^{(2)} d([z]) h^{-1} L_1^n h' d([z])
\]

where \((R^n) r\) means the lower triangular part including the main diagonal and \((L_1^n) n\) means the strictly upper diagonal part. This simplification can be obtained evaluating the terms \( h^{-1} (\partial_t h') \) and \( h'^{-1} (\partial_t h') \) through Sato’s equations. Also they can be obtained by observing that \( P^{(1)} \) and \( P^{(2)} \) are monic so that the derivative of the leading term is equal to 0. Then the proof is obtained as we did in the scalar case using forward and backward recursion relations (33), (35), (39) and (40).

**Corollary 6.5 (non-Abelian Ablowitz–Ladik semidiscrete zero-curvature equations).** The matrices \( \mathcal{L}_n \) and \( \mathcal{L}^l_n \) satisfy the following time evolution:

\[
\partial_{t1/n} \mathcal{L}^l_n = \mathcal{M}^l_{1/n,n+1} \mathcal{L}^l_n - \mathcal{M}^l_{1/n,n} \mathcal{L}^l_n
\]

\[
\partial_{t1/n} \mathcal{L}_n = \mathcal{M}^l_{1/n,n+1} \mathcal{L}_n - \mathcal{M}^l_{1/n,n} \mathcal{L}_n
\]

**Proof.** These equations are nothing but compatibility conditions of recursion relations (47) and (48) with time evolution (56) and (57).

**Remark 6.6.** It should be noted that, with respect to the equations originally written in [3], here we have two coupled non-Abelian Ablowitz–Ladik equations.

**Example 6.7 (the first flows; non-Abelian analogue of discrete nonlinear Schrödinger).** As we did for the scalar case we will compute the first matrices \( \mathcal{M}^{l/f}_{1/1} \) and use them to
We start with \( \mathcal{M}_{t_1,k} \); (58) gives us immediately
\[
\partial_t P_{k}^{(1)} = P_{k+1}^{(1)} - x_{k+1}^{l} y_{k+1}^{r} P_{k}^{(1)} - z P_{k}^{(1)}.
\]
which we use recursion relation (39) combined with (52)\( (\text{this one comes from recursion relation (41) combined with } t-s \text{ symmetry}) \) to arrive to
\[
\partial_t P_{k}^{(2)} = -z P_{k}^{(2)}.
\]
Then we use recursion relation (39) combined with
\[
h_{k}^{-1} h_{k-1} = (I - y_{k}^{l} x_{k}^{r})
\]
(this one comes from recursion relation (41) combined with \( t-s \) symmetry) to arrive to
\[
\partial_t \tilde{P}_{k}^{(2)} = z y_{k}^{l} P_{k}^{(1)} - z P_{k}^{(2)}.
\]
These computations give us
\[
\mathcal{M}_{t_1,k}^{l} = \begin{pmatrix} -x_{k+1}^{l} y_{k}^{r} & x_{k+1}^{l} \\ z y_{k}^{r} & -z I \end{pmatrix}.
\]
Exploiting \( t-s \) symmetry we can immediately write
\[
\mathcal{M}_{t_1,k}^{s} = \begin{pmatrix} z^{-1} I & -z^{-1} x_{k}^{l} \\ -y_{k+1}^{l} & y_{k+1}^{l} x_{k}^{l} \end{pmatrix}.
\]
Analogue computations for \( \mathcal{M}_{t_1}^{r} \) give us
\[
\partial_t P_{k}^{(1)r} = P_{k+1}^{(1)r} - x_{k+1}^{l} y_{k+1}^{r} P_{k}^{(1)r} - z P_{k}^{(1)r}.
\]
which we use recursion relation (39) combined with (52)\( (\text{this one comes from recursion relation (41) combined with } t-s \text{ symmetry}) \) to arrive to
\[
\partial_t \tilde{P}_{k}^{(2)r} = -z P_{k}^{(2)r}.
\]
and
\[
\partial_t \tilde{P}_{k}^{(2)} = -z P_{k}^{(2)r} (h_{k}^{-1} h_{k-1}^{-1}) = -\tilde{P}_{k}^{(2)r} z + P_{k}^{(1)r} y_{k}^{l}.
\]
(here we started from (60) and (61) and we used recursion relations (37), (40), (42), the last one combined with \( t-s \) symmetry). Then we arrive at
\[
\mathcal{M}_{t_1,k}^{l} = \begin{pmatrix} -y_{k}^{l} x_{k+1}^{l} & z y_{k}^{r} \\ x_{k+1}^{l} & -z I \end{pmatrix}
\]
and using \( t-s \) symmetry again we also get
\[
\mathcal{M}_{t_1,k}^{s} = \begin{pmatrix} z^{-1} I & -y_{k+1}^{l} \\ -z^{-1} x_{k}^{l} & x_{k+1}^{l} y_{k+1}^{l} \end{pmatrix}.
\]
As we did for the scalar case we introduce times \( t_0 \) and \( s_0 \) that give matrices
\[
\mathcal{M}_{t_0,k}^{l} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M}_{t_0,k}^{s} = \begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix}.
\]
Then we construct the matrices
\[
\mathcal{M}_{t_1,k}^{l} = \mathcal{M}_{t_0,k}^{l} + \mathcal{M}_{t_1,k}^{l} - \mathcal{M}_{t_0,k}^{l} - \mathcal{M}_{t_0,k}^{l} = \begin{pmatrix} z^{-1} I - I & -x_{k+1}^{l} y_{k}^{r} \\ y_{k}^{l} y_{k+1}^{l} & x_{k+1}^{l} x_{k}^{l} + I - z I \end{pmatrix}
\]
\[
\mathcal{M}_{t_1,k}^{s} = \mathcal{M}_{t_0,k}^{s} + \mathcal{M}_{t_1,k}^{s} - \mathcal{M}_{t_0,k}^{s} - \mathcal{M}_{t_0,k}^{s} = \begin{pmatrix} z^{-1} I - I & -y_{k}^{l} x_{k+1}^{l} \\ x_{k+1}^{l} - z^{-1} x_{k}^{l} & x_{k+1}^{l} x_{k}^{l} + z I + I \end{pmatrix}
\]
associate with the time $\tau = t_1 + s_1 - t_0 - s_0$. Semidiscrete zero-curvature equations

$$\partial_\tau L_k^t = M_{t,k+1}^\tau L_k^t - L_k^{t+1}M_{t,k}^\tau$$

$$\partial_\tau L_k^r = L_k^rM_{r,k+1}^\tau - M_{r,k}^\tau L_k^r$$

are equivalent to the systems

$$\begin{cases}
\partial_\tau x_k^t = x_{k+1}^t - 2x_k^t + x_{k-1}^t - x_{k+1}^r y_k^r y_k^t - x_k^t y_k^r x_{k-1}^t \\
\partial_\tau y_k^t = -y_{k+1}^t + 2y_k^t - y_{k-1}^t + y_{k+1}^r y_k^r y_k^t + y_k^r x_k^t y_{k-1}^t
\end{cases} \quad (69)$$

$$\begin{cases}
\partial_\tau x_k^r = x_{k+1}^r - 2x_k^r + x_{k-1}^r - x_{k+1}^r y_k^r x_k^t - x_k^r y_k^r x_{k-1}^t \\
\partial_\tau y_k^r = -y_{k+1}^t + 2y_k^t - y_{k-1}^t + y_{k+1}^r y_k^r y_k^t + y_k^r x_k^t y_{k-1}^t
\end{cases} \quad (70)$$

Note that both of them are equivalent to the discrete matrix nonlinear Schrödinger as written, for instance, in [6]. Using (69) and (70) together we perform the reduction to the Hermitian case in a different way from [6]. First of all we rescale $\tau \mapsto i\tau$ and then we impose

$$y_k^r = \pm (x_k^r)^*$$

$$y_k^t = \pm (x_k^t)^*.$$ 

Note that this reduction (with the plus sign) corresponds to studying the theory of matrix orthogonal polynomials on the unit circle as described in [19, 22], hence it is very natural. This reduction gives us the two coupled equations

$$\begin{cases}
-i\partial_\tau x_k^t = x_{k+1}^t - 2x_k^t + x_{k-1}^t \mp x_{k+1}^r (x_k^r)^* y_k^r x_k^t \\
-i\partial_\tau x_k^r = x_{k+1}^r - 2x_k^r + x_{k-1}^r \mp x_{k+1}^r (x_k^r)^* x_k^t
\end{cases} \quad (71)$$

already studied in [4] and generalized in [5].

**Remark 6.8.** In [16] the authors studied finite gap solutions of the Ablowitz–Ladik hierarchy. It could be interesting to generalize their results to the non-Abelian case. We will consider this problem in a subsequent publication.

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