ANGLES IN HYPERBOLIC LATTICES: THE PAIR CORRELATION DENSITY

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Abstract. It is well known that the angles in a lattice acting on hyperbolic $n$-space become equidistributed. In this paper we determine a formula for the pair correlation density for angles in such hyperbolic lattices. Using this formula we determine, among other things, the asymptotic behavior of the density function in both the small and large variable limits. This extends earlier results by Boca, Paşol, Popa and Zaharescu and Kelmer and Kontorovich in dimension 2 to general dimension $n$. Our proofs use the decay of matrix coefficients together with a number of careful estimates, and lead to effective results with explicit rates.

1. Introduction

Let $\Gamma$ be a discrete cofinite subgroup of $G = \text{SO}_0(n,1)$ and let $z_0 \in \mathbb{H}^n$. In its most basic form the hyperbolic lattice point counting problem seeks to estimate the size of the orbit $\Gamma z_0$ inside some expanding region. This problem has been studied—when the region is a hyperbolic ball—by several people [6,8,10,13,17,18,22,23], and precise asymptotics are known. By now it is also well established that the angles in a hyperbolic lattice are equidistributed [2,8,9,20,26].

More refined angle statistics have been studied only recently: Boca, Paşol, Popa and Zaharescu [1,3] studied the pair correlation statistics for hyperbolic angles when $\Gamma = \text{PSL}_2(\mathbb{Z})$ and $z_0 = i$ or $z_0 = e^{i\pi/3}$, and gave conjectures for all lattices $\Gamma \subseteq \text{PSL}_2(\mathbb{R}) \cong \text{SO}_0(2,1)$ and all base points $z_0$. These conjectures were later resolved by Kelmer and Kontorovich [15].

In this paper, among other things, we generalize and extend these results to general lattices acting on $n$-dimensional hyperbolic space. To be precise: Let $z_0 = e_{n+1} \in \mathbb{H}^n$ be the origin of (the hyperboloid model of) $\mathbb{H}^n$. Consider

$$\|g\|^2 = 2 \cosh(d(ge_{n+1}, e_{n+1})),$$

where $d(z, w)$ denotes the hyperbolic distance between $z, w \in \mathbb{H}^n$, and let $v(g, g')$ denote the hyperbolic angle between $ge_{n+1}$ and $g'e_{n+1}$ (see Section 2.3). Let $K = \text{Stab}_G(e_{n+1})$. For simplicity we assume that $\Gamma \cap K$ is trivial, although this is not a
serious restriction. We define

\[ N_\Gamma(Q) := \Gamma \cap B_Q, \]

where \( B_Q := \{ g \in G : \|g\| \leq Q \} \). For the hyperbolic lattice point problem the main interest is in the asymptotic behavior of \( \#N_\Gamma(Q) \) as \( Q \to \infty \). In the above terminology Selberg proved (unpublished; see also [17, Thm. 1], [18, Thm. II], [7, Thm. 4.1], and Remark 2.6) that

\[ (1.1) \quad \#N_\Gamma(Q) = \frac{\text{vol}(B_Q)}{\text{vol}(\Gamma \backslash G)} + O(\text{vol}(B_Q)^{1-\delta}), \quad \text{as} \quad Q \to \infty. \]

Here \( \text{vol}(B_Q) \) is the (appropriately normalized) Haar measure of \( B_Q \subseteq G \) (see Section 2.4). The size of \( \delta \) usually depends either directly or indirectly on a spectral gap, i.e. the size of the least non-zero element in the spectrum of the automorphic Laplacian.

Consider an element \( \gamma' \in \Gamma \) and a real number \( \xi > 0 \). By a short heuristic argument based on known equidistribution and point counting results (see Section 2.5), we expect about \( \xi n^{-1} \) elements in the set

\[ \left\{ \gamma \in N_\Gamma(Q) \setminus \{\gamma'\} : v(\gamma, \gamma') < \frac{2k_{n,\Gamma}}{Q^2} \xi \right\}. \]

Here \( k_{n,\Gamma} \) is an explicit constant (see (2.5)) which we compute as part of the heuristics. Taking averages over \( \gamma' \) with \( \|\gamma'\| \leq Q \), we are led to investigate the pair correlation counting function

\[ (1.2) \quad R_{2,Q}(\xi) := \frac{\#N_{2,Q}(\xi)}{\#N_\Gamma(Q)}, \quad \text{where} \]

\[ (1.3) \quad N_{2,Q}(\xi) := \left\{ \gamma, \gamma' \in N_\Gamma(Q) : \gamma^{-1}\gamma' \notin K, v(\gamma, \gamma') < \frac{2k_{n,\Gamma}}{Q^2} \xi \right\}. \]

We note that without the condition \( \gamma^{-1}\gamma' \notin K \) in the definition of \( N_{2,Q}(\xi) \), the number \( R_{2,Q}(\xi) \) increases by exactly 1 since \( \Gamma \cap K \) is trivial.

We want to investigate how much \( R_{2,Q}(\xi) \) deviates from \( \xi^{n-1} \) in the limit as \( Q \to \infty \). Our first result is the following theorem which asserts that the limit as \( Q \to \infty \) does indeed exist.

**Theorem 1.1.** Let \( n \geq 2 \) and let \( \Gamma \subseteq G \) be a lattice as above. Then, as \( Q \to \infty \), the function \( R_{2,Q}(\xi) \) converges to a differentiable limit \( R_2(\xi) \) whose derivative satisfies

\[ g_2(\xi) = \frac{d}{d\xi} R_2(\xi) = \sum_{M \in \Gamma} F_\xi(d(Me_{n+1}, e_{n+1})), \]

where \( F_\xi \) is given explicitly in (6.5). Moreover, there exists \( \nu > 0 \) such that

\[ R_{2,Q}(\xi) = R_2(\xi) + O(\xi^{-\nu}) \]

**Remark 1.2.** The limit function \( R_2(\xi) \) is called the pair correlation function, and its derivative \( g_2(\xi) \) is called the pair correlation density. An explicit estimate on the rate of convergence (i.e. \( \nu \) in Theorem 1.1) can be given in terms of a spectral gap. We refer to Theorem 6.1 for details. For \( n = 2 \) our convergence rate is identical to that proved in [15].
The fact that the function $F_\xi$ in Theorem 1.1 can be given explicitly (see (6.5), (4.5), and Remark 4.3) allows us to determine the asymptotic behavior of the pair correlation density:

**Theorem 1.3.** Fix $s_0$ as above. We have

$$g_2(\xi) = (n - 1)\xi^{n-2} + O\left(\xi^{n-2+\frac{2(n-n+1)}{n+1}}\right), \quad \text{as } \xi \to \infty.$$

**Remark 1.4.** Integrating the asymptotic formula in Theorem 1.3 we immediately find that

$$R_2(\xi) = \xi^{n-1} + O\left(\xi^{n-1+\frac{2(s_0-n+1)}{n+1}}\right), \quad \text{as } \xi \to \infty.$$  

Turning now instead to the limit $\xi \to 0$, Kelmer and Kontorovich [15] proved that in the 2-dimensional case the pair correlation density tends to a non-zero value. Using the above explicit description of $g_2$, we show that this happens only in this case.

**Theorem 1.5.** The pair correlation density converges to zero as $\xi$ tends to zero if and only if $n \neq 2$.

We observe that the pair correlation function $R_2(\xi)$ depends heavily both on the discrete group $\Gamma$ and the choice of base point for the lattice point problem [1]. However, once the group and the base point are fixed, the pair correlation function is uniform in the following sense: Let $U$ denote the hyperbolic unit sphere centered at $e_{n+1}$. Let $S \subset U$ be a spherical cap and define $C$ to be the hyperbolic cone specified by the vertex $e_{n+1}$ and the cross-section $S$. Then, if we restrict our attention in (both the numerator and the denominator of) (1.2) to elements in $\Gamma$ corresponding to points in the orbit $\Gamma e_{n+1}$ lying in $C$, then the limit as $Q \to \infty$ still exists and equals the same function $R_2(\xi)$ achieved in Theorem 1.1. In order to give a precise statement, we define

$$N_{\Gamma,C}(Q) := \{\gamma \in N_\Gamma(Q) : \gamma e_{n+1} \in C\}$$

and

$$N_{2,C,Q}(\xi) := \left\{\gamma, \gamma' \in N_{\Gamma,C}(Q) : \gamma^{-1}\gamma' \notin K, v(\gamma, \gamma') < \frac{2k_{n,\Gamma}}{Q^2}\xi\right\}.$$  

**Theorem 1.6.** Let $n \geq 2$ and let $\Gamma \subset G$ be a lattice as above. Let $S \subset U$ be a spherical cap and denote the hyperbolic cone specified by the vertex $e_{n+1}$ and the cross-section $S$ by $C$. Then

$$\lim_{Q \to \infty} \frac{\#N_{2,C,Q}(\xi)}{\#N_{\Gamma,C}(Q)} = R_2(\xi).$$

In particular the limit exists, is differentiable with derivative $g_2(\xi)$, and is independent of the cone $C$.

**Remark 1.7.** It is clear that our techniques can also handle pair correlation functions corresponding to cones $C$ specified by more general sets $S \subset U$. However, for simplicity, we have chosen not to give the most general statement possible.

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1Recall that a change of base point in our problem can be achieved by conjugating the group $\Gamma$.  

Remark 1.8. The function $F_\xi(l)$—which by Theorem 1.1 and Theorem 1.6 determines the pair correlation density $g_2$ both in the whole space and in sectors—depends, apart from $\xi$ and $l$, only on $n$ and $\text{vol}(\Gamma \setminus G)$. It follows that $g_2$ depends on the group $\Gamma$ only through the sequence $$\{d(\gamma e_{n+1}, e_{n+1}) : \gamma \in \Gamma\}.$$ In fact, it turns out that this sequence determines and is determined by $g_2$ and the volume $\text{vol}(\Gamma \setminus G)$. Using this we show, in Section 7, that $g_2$ determines and is determined by certain spectral data. We refer to Section 7 for precise statements.

The idea of the proof of Theorem 1.1 (which is the basis for most of the subsequent results) is as follows: Considering the results in [1,3,15], we expect the pair correlation density to be expressible as a sum over $M = \gamma \gamma'$. We therefore write

$$\# N_{2,Q}(\xi) \quad \text{(see (1.3))}$$

as

$$\# N_{2,Q}(\xi) = \sum_{M \in \Gamma} \# \Gamma \cap \mathcal{R}_M(Q, k_{n,\Gamma} \xi),$$

where

$$\mathcal{R}_M(Q, \xi) := \left\{ g \in B_Q : \|gM\| \leq Q, v(g, gM) < \frac{2\xi}{Q^2} \right\}.$$ 

Our goal (following [15]) is then to show that the number $\# \Gamma \cap \mathcal{R}_M(Q, k_{n,\Gamma} \xi)$ can be approximated by $\text{vol}(R_M(Q, k_{n,\Gamma} \xi))/\text{vol}(\Gamma \setminus G)$ and to compute approximations for $\text{vol}(R_M(Q, k_{n,\Gamma} \xi))$.

The structure of the paper is as follows: In Section 2 we review known theory and results needed in the proofs of the main theorems. In Section 3 we show several stability results for angles and norms, which are later used for certain approximation arguments. In Section 4 we find expressions for $\text{vol}(R_M(Q, k_{n,\Gamma} \xi))$ given in terms of the functions $F_\xi(d(\gamma e_{n+1}, e_{n+1}))$, and in Section 5 we show how these volumes are related to $\# \Gamma \cap \mathcal{R}_M(Q, k_{n,\Gamma} \xi)$. In Section 6 we tie these investigations together and complete the proofs of all the main theorems. Several of these results use basic properties of the function $F_\xi$; the relevant properties are stated and proved in [25, Appendix A].

In a very recent paper, Marklof and Vinogradov [19] show how the mixing property of the geodesic flow can be used to obtain information about the distribution of directions in a hyperbolic lattice. They show convergence of all mixed moments of the appropriate counting functions, which in particular allows them to conclude convergence of pair correlations. In fact their results capture all local statistics (e.g. gap or nearest neighbor distributions). In the present paper, we focus on the pair correlation and use information on the decay of matrix coefficients to get more explicit and precise results.

Notation. Throughout this manuscript we consider $n$ and $\Gamma$ as fixed. In all estimates, the implied constants may depend on $n$ and $\Gamma$; any other dependence will be specified.

2. Prerequisites

2.1. The hyperboloid model of hyperbolic $n$-space. Let $n \geq 2$. We begin by recalling that

$$\text{SO}(n, 1) = \{ g \in \text{SL}_{n+1}(\mathbb{R}) : g^t J g = J \}.$$
where $J = \text{diag}(I_n, -1)$ and $I_n$ is the $n \times n$ identity matrix. By definition this is the subgroup of $\text{SL}_{n+1}(\mathbb{R})$ leaving the (symmetric and non-degenerate) bilinear form

$$\langle x, y \rangle = x^t J y \quad (x, y \in \mathbb{R}^{n+1})$$

invariant. In the present paper we will be mainly interested in the group $G := \text{SO}_0(n, 1)$ defined as the connected component of $\text{SO}(n, 1)$ containing the identity.

The group $G$ acts transitively by matrix multiplication on the set

$$H^n := \{ x \in \mathbb{R}^{n+1} : \langle x, x \rangle = -1, x_{n+1} > 0 \},$$

which is the upper sheet of a two-sheeted hyperboloid. If we define the metric $d$ on $H^n$ by the relation

$$\cosh d(x, y) = -\langle x, y \rangle,$$

then $H^n$ is a model of hyperbolic $n$-space, i.e. a maximally symmetric, simply connected Riemannian manifold of dimension $n$ and constant sectional curvature $-1$. In this model, the group $G$ acts as the full group of orienting preserving isometries on hyperbolic $n$-space.

2.2. Cartan decomposition. Consider the groups

$$K := \left\{ \begin{pmatrix} k' & \\ 1 \end{pmatrix} : k' \in \text{SO}(n) \right\}$$

and

$$A := \left\{ a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\}.$$

The Cartan decomposition of $G$ gives that $G = KA^+$, where $A^+ := \{ a_t \in A : t \geq 0 \}$. In particular every element $g = (g_{i,j}) \in G$ can be written as

$$g = k_g a_{t(g)} k'_g \quad (2.1)$$

where $k_g, k'_g \in K$ and $a_{t(g)} \in A^+$. Here$^2$

$$t(g) = \cosh^{-1} (g_{n+1, n+1}) = d(g e_{n+1}, e_{n+1}).$$

We note that $K$ is the stabilizer of $e_{n+1} \in H^n$.

The decomposition (2.1) is not unique as the centralizer of $A$ in $K$, $M := Z_K(A)$, is non-trivial. In concrete terms,

$$M = \left\{ \begin{pmatrix} 1 & m' \\ m' & 1 \end{pmatrix} : m' \in \text{SO}(n-1) \right\} \subset K.$$

An element $g \in G$ can be written as $g = k_g a_{t(g)} k'_g$ and $g = \bar{k}_g a_{\bar{t}(g)} \bar{k}'_g$ if and only if $t(g) = \bar{t}(g)$, $\bar{k}_g = k_g m$ and $\bar{k}'_g = m^{-1} k'_g$ for some $m \in M$.

$^2$Here and throughout, we let $e_j$ denote the $j$th standard basis vector in $\mathbb{R}^{n+1}$. 
2.3. Basic properties of hyperbolic angles. The formula for the hyperbolic angle between two geodesic segments intersecting at a point \( z \in \mathbb{H}^n \) is in general quite complicated (see, e.g., [24, Sect. 3.2]). However, we will restrict our attention to the case where the vertex of the angle is located at the point \( e_{n+1} \in \mathbb{H}^n \), which simplifies the formulas considerably.

For \( x, y \in \mathbb{H}^n \backslash \{ e_{n+1} \} \), we define the corresponding (unsigned) angle \( v(x, y) \in [0, \pi] \), based at the point \( e_{n+1} \), via the relation

\[
\cos(v(x, y)) = \frac{u \cdot v}{\sqrt{u \cdot u} \sqrt{v \cdot v}},
\]

where \( x = (u, t), y = (v, s) \) for appropriate choices of \( u, v \in \mathbb{R}^n, s, t \in \mathbb{R} \), and \( \cdot \) is the usual Euclidean inner product on \( \mathbb{R}^n \). In addition, for \( g, g' \in G \setminus K \), we define, by an abuse of notation, the angle between them by

\[
v(g, g') := v(ge_{n+1}, g'e_{n+1}).
\]

Fixing the point \( N := (1, 0, \ldots, 0, \sqrt{2}) \in \mathbb{H}^n \), it is straightforward to verify that if \( g = katk' \) with \( k, k' \in K, t > 0 \) and \( k = (k_{i,j}) \), then

\[
\cos(v(ge_{n+1}, N)) = k_{1,1}.
\]

Furthermore, we will find it useful to fix the matrix

\[
g_N := \begin{pmatrix}
\sqrt{2} & 1 \\
1 & \sqrt{2} \\
\end{pmatrix} \in G
\]

satisfying \( g_N e_{n+1} = N \).

It is an exercise in linear algebra to verify the following (non-unique) decomposition of \( K \).

**Lemma 2.1.** Let \( n > 2 \). Every \( k = (k_{i,j}) \in K \) can be written as

\[
k = m_1 k^\theta m_2,
\]

where \( m_i \in M \) and

\[
k^\theta := \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta \\
\end{pmatrix} I_{n-1}
\]

is uniquely determined by \( \theta = v(ka_t, g_N) \) for any \( t > 0 \). We also have \( \cos \theta = k_{1,1} \).

**Remark 2.2.** Note that in the case \( n = 2 \), we only get the trivial statement

\[
K = \{ k^\theta : \theta \in [-\pi, \pi) \}.
\]

From now on, whenever we use the decomposition in Lemma 2.1, we will only give statements and provide calculations for the case \( n > 2 \). However, using the above observation it should always be clear how to change a statement (calculation) in order to arrive at a valid statement (calculation) also when \( n = 2 \).

Note that if \( g \in G \) has Cartan decomposition \( g = m_1 k^\theta m_2 a_t \varphi m_3 k^\varphi m_4 \) with \( m_i \in M, t(g) > 0 \) and \( \theta(g), \varphi(g) \in [0, \pi] \), then

\[
g^{-1} = m_1 k^{-\varphi} \tilde{m}_2 a_t \varphi \tilde{m}_3 k^{-\theta} \tilde{m}_4
\]

for some \( \tilde{m}_i \in M \).

The following elementary properties of \( v(\cdot, \cdot) \) are very useful and yet straightforward to verify.
Proposition 2.3. Let $n > 2$ and let $g, g' \in G \setminus K$. Then the following hold:

(i) $v(kg, kg') = v(g, g')$ for every $k \in K$,
(ii) $v(gk, g') = v(g, g')$ for every $k \in K$,
(iii) $v(g, a_t) = v(g, gN)$ for every $t > 0$,
(iv) $v(g, a_{-t}) = \pi - v(g, a_t)$ for every $t > 0$,
(v) $v(g, g') = v(g', g)$,
(vi) $v(g, g') \leq v(g, g'') + v(g'', g')$ for every $g'' \in G \setminus K$,
(vii) $v(g, gN) = \cos^{-1}(g_{1,n+1}/\sinh t(g))$,
(viii) $v(g, gN) = \theta(g)$ and $v(g^{-1}, gN) = \pi - \varphi(g),$
(ix) $|\theta(g) - \theta(g')| \leq v(g, g')$,
(x) $|\varphi(g) - \varphi(g')| \leq v(g^{-1}, g'^{-1})$.

We will find it convenient to define the angle $v(g, g')$ also when at least one of $g, g' \in K$. We let

$$v(g, g') := \begin{cases} v(gN, g') & \text{if } g \in K \text{ but } g' \notin K, \\ v(g, gN) & \text{if } g' \in K \text{ but } g \notin K, \\ 0 & \text{if } g, g' \in K. \end{cases}$$

In other words, if one (both) of the elements $g$ and $g'$ occurring in the expression $v(g, g')$ belongs to $K$, then we exchange that element (those elements) with $gN$ in order to interpret the angle $v(g, g')$. We admit that the above extension is rather arbitrary. However, we note that our choice is natural in the sense that Proposition 2.3 will continue to hold also for this extended concept of angles.

2.4. Normalization of Haar measure and integration formulas. We normalize the Haar measure $dk$ on $K$ so that

$$\text{vol}(K) = \int_K dk = 1.$$ 

Furthermore, recalling the identification $G/K \simeq \mathbb{H}^n$, we normalize the Haar measure $dg$ on $G$ in such a way that the induced measure on $G/K$ corresponds to the standard (hyperbolic) measure $d\mu_{\mathbb{H}^n}$ on $\mathbb{H}^n$. In particular, for any cofinite $\Gamma \subset G$ and any (nice) fundamental domain $F_\Gamma$ of $\Gamma$, we find that $\mu_{\mathbb{H}^n}(F_\Gamma) = \text{vol}(\Gamma \setminus G)$.

With these normalizations, we get the following integration formula (see e.g. [11, Prop. 1.17 (p. 381)]).

Proposition 2.4. Let the Haar measures $dk$ on $K$ and $dg$ on $G$ be normalized as above. Then, for any function $f \in L^1(G)$, we have

$$\int_G f(g) dg = \omega_n \int_K \int_0^\infty \int_K f(k_{1,a_1}k_2)(\sinh t)^{n-1} dk_1 dt dk_2,$$

where $\omega_n$ denotes the $(n-1)$-dimensional volume of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

We will also need the following closely related formula.

Proposition 2.5. Let $n > 2$. Let the Haar measure $dk$ on $K$ be normalized as above and let the Haar measure $dm$ on $M$ be normalized so that $\text{vol}(M) = \omega_n^{-1}$. Then, for any function $f \in L^1(K)$, we have

$$\int_K f(k) dk = \omega_n \omega_{n-1} \int_M \int_0^\pi \int_M f(m_{1,2}(\sin \theta)^{n-2} dm_1 d\theta dm_2,$$

where $\omega_j$ denotes the $(j-1)$-dimensional volume of the unit sphere $S^{j-1} \subset \mathbb{R}^j$. 


Proof. To begin, we recall that $K/M \simeq S^{n-1}$. Furthermore, using the explicit form of the volume element of the $(n - 1)$-sphere in terms of spherical coordinates, we easily find a relation of measures which in integrated form becomes

$$\omega_n = \omega_{n-1} \int_0^\pi (\sin \theta)^{n-2} \, d\theta.$$  

We conclude that $dk$ can be written as a positive multiple of $(\sin \theta)^{n-2} \, dm_1 \, d\theta \, dm_2$. Finally, integrating the constant function $f(k) \equiv 1$, we find that the positive multiple equals $\omega_n \omega_{n-1}$. \qed

We note that if $g = k_g a_{t(g)} k'_g$, then $\|g\|^2 = 2 \cosh t(g)$, and with the above normalizations

$$(2.2) \quad \text{vol}(B_Q) \sim \frac{\omega_n}{2^{n-1}(n-1)} Q^{2(n-1)} \quad \text{as } Q \to \infty.$$  

Indeed, this fact follows from Proposition 2.4, since we readily get

$$\text{vol}(B_Q) = \omega_n \int_0^{\cosh^{-1}(Q^2/2)} (\sinh t)^{n-1} \, dt$$

$$\sim \frac{\omega_n}{2^{n-1}} \int_0^{2 \log Q} e^{(n-1)t} \, dt \sim \frac{\omega_n}{2^{n-1}(n-1)} Q^{2(n-1)}$$

as $Q \to \infty$.

Remark 2.6. In relation to the asymptotic formula (2.2), let us recall that Selberg proved (unpublished; see also [7,17,18]) that for cofinite $\Gamma \subset G$, we have

$$\# N_\Gamma(Q) \sim \frac{\text{vol}(B_Q)}{\text{vol}(\Gamma \backslash G)} \sim \frac{\omega_n}{2^{n-1}(n-1) \text{vol}(\Gamma \backslash G)Q^{2(n-1)}}$$

as $Q \to \infty$ (cf. (1.1)).

2.5. The pair correlation counting function. We now give a brief discussion of the normalized counting function

$$(2.3) \quad R_{2,Q}(\xi) = \frac{1}{\# N_\Gamma(Q)} \# \left\{ \gamma, \gamma' \in N_\Gamma(Q) : \gamma^{-1}\gamma' \notin K, v(\gamma, \gamma') < \frac{2k_{n,\Gamma}Q}{\xi} \right\}.$$  

Our purpose is to give a short heuristic determination of the value of $k_{n,\Gamma}$ that—if equidistribution were uniform in all parameters—would make $R_{2,Q}(\xi)$ tend to $\xi^{n-1}$ as $Q \to \infty$.

We recall (see, e.g., [20 Thm. 2]) that angles in hyperbolic lattices are equidistributed in the sense that, for every fixed $g \in G$ and angle $\theta \in [0, \pi]$, we have

$$\frac{\# \{ \gamma \in N_\Gamma(Q) : v(\gamma, g) < \theta \}}{\# N_\Gamma(Q)} \to \frac{\text{vol}(S_{\theta}^{n-1})}{\text{vol}(S^{n-1})}, \quad \text{as } Q \to \infty.$$  

Here $S_{\theta}^{n-1} := \{ x \in S^{n-1} \subset \mathbb{R}^n : x \cdot e_1 > \cos \theta \}$ is a spherical cap of opening angle $\theta$. Hence, using Remark 2.6 and spherical coordinates, we get

$$\# \{ \gamma \in N_\Gamma(Q) \setminus \{g\} : v(\gamma, g) < \theta \} \sim \# \{ \gamma \in N_\Gamma(Q) : v(\gamma, g) < \theta \}$$

$$\sim \frac{\text{vol}(S_{\theta}^{n-1})}{\text{vol}(S^{n-1})} \# N_\Gamma(Q)$$

$$\sim \frac{V_{n-1}}{\text{vol}(\Gamma \backslash G)} \int_0^\theta (\sin t)^{n-2} \, dt \left( \frac{Q^2}{2} \right)^{n-1}$$

as $Q \to \infty$. \qed
as \( Q \to \infty \), where \( V_n = \omega_n / n \) denotes the volume of the unit ball in \( \mathbb{R}^n \). Since we are interested in small values of \( \theta \) (in fact, in our case, \( \theta \to 0 \) as \( Q \to \infty \)), we furthermore note that

\[
\int_0^\theta (\sin t)^{n-2} \, dt \sim \frac{\theta^{n-1}}{n-1}
\]
as \( \theta \to 0 \). Thus, combining these two asymptotic formulas, we expect

\[
\# \{ \gamma \in N_\Gamma(Q) \setminus \{g\} : v(\gamma, g) < \theta \} \approx \frac{V_{n-1}}{(n-1) \text{vol}(\Gamma \setminus G)} \left( \frac{Q^2 \theta}{2} \right)^{n-1}
\]
as \( Q \to \infty \) and \( \theta \to 0 \).

Recall that we are interested in understanding the function \( R_{2,Q}(\xi) \) in (2.3). In particular, we are interested in finding the value of \( \theta \) such that the right-hand side in (2.4), on average over \( g = \gamma' \in N_\Gamma(Q) \) and as \( Q \) tends to infinity, is expected to be close to \( \xi^{n-1} \). Thus, we solve the equation

\[
\frac{V_{n-1}}{(n-1) \text{vol}(\Gamma \setminus G)} \left( \frac{Q^2 \theta}{2} \right)^{n-1} = \xi^{n-1},
\]
and we immediately find the solution

\[
\theta = \frac{2 \xi}{Q^2} \left( \frac{(n-1) \text{vol}(\Gamma \setminus G)}{V_{n-1}} \right)^{\frac{1}{n-1}}.
\]

Based on the above discussion, we define

\[
k_{n,\Gamma} := \left( \frac{(n-1) \text{vol}(\Gamma \setminus G)}{V_{n-1}} \right)^{\frac{1}{n-1}};
\]
this is the constant that we will use throughout to normalize the angles in the counting function \( R_{2,Q}(\xi) \) (see (2.3)).

### 2.6. Decay of matrix coefficients.

A basic ingredient in our argument is the decay properties of matrix coefficients of the right regular representation of \( G \) on the space \( L^2(\Gamma \setminus G) = L^2(\Gamma \setminus G, dg) \). That is, for \( \Phi_1, \Phi_2 \in L^2(\Gamma \setminus G) \), we are interested in the properties of

\[
g \mapsto \langle \pi(g)\Phi_1, \Phi_2 \rangle_{\Gamma \setminus G},
\]
where \( \langle f_1, f_2 \rangle_{\Gamma \setminus G} = \int_{\Gamma \setminus G} f_1(g) f_2(g) \, dg \) and \( (\pi(g)f)(h) = f(hg) \). We refer to [21] and the references therein for recent developments on various counting problems using precise information on the decay of matrix coefficients.

Since we assume \( \Gamma \) to be a lattice in \( G \), we have \( L^2(\Gamma \setminus G) = C \oplus L^2_0(\Gamma \setminus G) \). It is well known that there exists \( s_0 \) in \((n-1)/2, n-1\) such that \( L^2_0(\Gamma \setminus G) \) does not contain any complementary series representation with parameter \( s \geq s_0 \). Recall that this implies that the non-trivial spectrum of the automorphic Laplacian is contained in the interval \((s_0(n - 1 - s_0), \infty)\), i.e. it implies a spectral gap. Furthermore, for \( \Phi_1, \Phi_2 \in L^2_0(\Gamma \setminus G) \) smooth and \( K \)-finite, we have

\[
|\langle \pi(g)\Phi_1, \Phi_2 \rangle_{\Gamma \setminus G}| \leq C e^{(s_0-n+1)t(g)} \prod_{i=1,2} (\dim(\pi(K)\Phi_i))^{1/2} \|\Phi_i\|_{L^2}
\]
for some positive constant \( C \) depending only on \( s_0 \) (see e.g. [16] Eq. (5.4)) for the case \( n = 3 \); the argument given there readily extends to general \( n \geq 2 \). Using Fourier decomposition on the compact group \( K \), it is possible to remove the
assumption of $K$-finiteness at the expense of a Sobolev norm. For $\Phi \in C^\infty(\Gamma \setminus G)$ and $l \in \mathbb{N}$, we define the $l$th Sobolev norm of $\Phi$ by
\begin{equation}
S_l(\Phi) := \sum \|X(\Phi)\|_{L^2},
\end{equation}
where the sum is taken over all monomials of degree at most $l$ in some fixed basis for the Lie algebra of $G$. The above argument leads to the following theorem (see e.g. \cite{21} Thm. 3.1 and the references therein).

**Theorem 2.7.** There exist $n-\frac{1}{2} < s_0 < (n-1)$ and $l \in \mathbb{N}$ such that for all $\Phi_1, \Phi_2 \in L^2(\Gamma \setminus G) \cap C^\infty(\Gamma \setminus G)$, we have
\begin{equation}
\langle (g)\Phi_1, \Phi_2 \rangle_{\Gamma \setminus G} = \frac{\langle \Phi_1, 1 \rangle_{\Gamma \setminus G} \langle 1, \Phi_2 \rangle_{\Gamma \setminus G}}{\text{vol}(\Gamma \setminus G)} + O\left(e^{(s_0-n+1)t(g)}S_l(\Phi_1)S_l(\Phi_2)\right).
\end{equation}

**Remark 2.8.** We note that the argument removing the $K$-finiteness above is done independently for the two functions $\Phi_1$ and $\Phi_2$. In particular, this implies that in the case where $\Phi_1$ ($i = 1$ or 2) is $K$-invariant, we can replace the Sobolev norm $S_l(\Phi_1)$ in (2.7) by the corresponding $L^2$-norm $\|\Phi_1\|_{L^2}$. In addition we mention that, in the case $n = 2$ where it is possible to take $l = 1$, Venkatesh \cite{27} Sect. 9.1.2 has given an interpolation argument that replaces the Sobolev norm $S_l(\Phi_1)$ in (2.7) by $S_l(\Phi_1)^{1/2+\epsilon}\|\Phi_1\|_{L^2}^{1/2-\epsilon}$ for any $0 < \epsilon < \frac{1}{2}$.

**Remark 2.9.** The implied constant in (2.7) depends on $s_0$. However, since we will work with a fixed admissible $s_0$, we choose not to indicate this dependence.

3. Stability

We will need some information about how $\|g\|$ and $\theta(g)$ change when we multiply from the right with $M \in G$. This is addressed in the following proposition.

**Proposition 3.1.** Let $g, M \in G$. Then
\begin{equation}
\|gM\|^2 = 2(\cosh t(g)\cosh t(M) + \cos v \sinh t(g) \sinh t(M)).
\end{equation}
Assume further that $t(g) > t(M)$. Then $v(gM, g) < \pi/2$ and
\begin{equation}
\tan v(gM, g) = \frac{\sin v \sinh t(M)}{\cosh t(M) \sinh t(g) + \cos v \cosh t(g) \sinh t(M)}.
\end{equation}
In both cases $v = \pi - v(g^{-1}, M)$.

**Proof.** For $g \in K$ or $M \in K$ everything is clear. Assume this not to be the case. Using the Cartan decompositions of $g$ and $M$, and that $d(x, y)$ is a point-pair invariant, we see that
\begin{equation}
\|gM\|^2 = 2\cosh d(a_{t(g)}k_g^t k_M a_{t(M)}e_{n+1}, e_{n+1}).
\end{equation}
It follows from the definition of $d$ that $\cosh d(a_{t(g)}k_g^t k_M a_{t(M)}e_{n+1}, e_{n+1})$ equals the lower entry of $a_{t(g)}k_g^t k_M a_{t(M)}e_{n+1}$, from which (3.1) follows easily by inspection if $\cos v$ equals the upper left entry of $k_g^t k_M$. However, the upper left entry of $k_g^t k_M$ equals $\cos(v(k_g^t k_M a_{t(M)}, g_N))$ for any $t > 0$, and by Proposition 2.3 we see that
\begin{equation}
v(k_g^t k_M a_{t(M)}, g_N) = v(k_g^t k_M a_{t(M)}, a_{t(g)}) = \pi - v(k_M a_{t(M)}, k_g^{(-1)} a^{-t(g)}) = \pi - v(M, g^{-1}).
\end{equation}
To prove (3.2), we note that
\begin{equation}
v(gM, g) = v(a_{t(g)}k_g^t k_M a_{t(M)}, a_{t(g)}) = v(a_{t(g)}k_g^t k_M a_{t(M)}, g_N).
\end{equation}
Using that the upper left entry of $k'_g k_M$ equals $\cos v$, and that, by orthogonality, the sum of the squares of the elements in the rest of the first column in $k'_g k_M$ equals $\sin^2 v$, a direct computation from the definition of $v$ shows that

$$\cos v(gM, g) = \cos v(a_{t(g)} k'_g k_M a_{t(M)}, g_N)$$

(3.3)

$$\frac{\sinh t(g) \cosh t(M) + \cos v \cosh t(g) \sinh t(M)}{\sqrt{(\sinh t(g) \cosh t(M) + \cos v \cosh t(g) \sinh t(M))^2 + \sin^2 v \sinh^2 t(M)}}.$$

We note that $t(g) > t(M)$ implies that the numerator of (3.3) is positive, so the angle $v(gM, g)$ is at most $\pi/2$. From this the result follows easily.

For any (small) $\delta > 0$, we define

$$A_\delta := \{ a_t : |t| \leq \delta \}.$$

Then clearly

$$B_{\delta_1} = KA_\delta K,$$

where $\delta_1^2 = 2 \cosh \delta$.

**Lemma 3.2.** Let $g \in G$ with $\|g\| > 3$, and let $h \in B_{\delta_1}$. Then, for $\delta > 0$ sufficiently small, we have

(i) $\|gh\| = \|g\| (1 + O(\delta))$,

(ii) $\nu(g, gh) = O\left(\frac{\delta}{\|g\|^2}\right).$

**Proof.** From Proposition 3.1 we see that

$$\|gh\|^2 = 2 \cosh(t(g))(1 + O(\delta^2)) + O(\delta \|g\|^2) = \|g\|^2 (1 + O(\delta)),$$

which implies (i).

Similarly, from Proposition 3.1 we see that for $\delta$ sufficiently small

$$|\tan \nu(g, gh)| = O\left(\frac{\delta}{\|g\|^2 (\tanh t(g) \cosh t(h) - \sinh t(h))}\right) = O\left(\frac{\delta}{\|g\|^2}\right),$$

which implies (ii) since $|v| \leq |\tan v|$ for $|v| \leq \pi/2$.

**Remark 3.3.** By using that $t(g^{-1}) = t(g)$, Lemma 3.2 implies $\|hg\| = \|g\| (1 + O(\delta))$ and $\nu(g^{-1}, g^{-1}h^{-1}) = O\left(\frac{\delta}{\|g\|^2}\right)$. We note also that combining the above angle bounds with Proposition 2.3, we obtain

$$|\theta(g) - \theta(gh)| = O\left(\frac{\delta}{\|g\|^2}\right), \quad |\varphi(g) - \varphi(hg)| = O\left(\frac{\delta}{\|g\|^2}\right)$$

for $\|g\| > 3$ and $h \in B_{\delta_1}$.

We now define

$$D_\delta := K_\delta A_\delta K_\delta,$$

(3.5)

where $K_\delta$ is defined by

$$K_\delta := \{ k \in K : |ka - a| < \delta, \text{ for all } a \in S^n \subseteq \mathbb{R}^{n+1} \}.$$

Note that the elements of $K_\delta$ rotate any given direction in $\mathbb{R}^{n+1}$ by at most a small amount.
**Lemma 3.4.** For $\delta > 0$ sufficiently small the following holds: Let $g \in G$, with $\|g\|^2 > 3$, and let $g_1 = h_1 g h_2 \in B_{\delta_1} g D_{\delta}$. Then

(i) $t(g_1) - t(g) = O(\delta)$,
(ii) $v(g_1^{-1}, g^{-1}) = O(\delta)$.

**Proof.** To prove (i), we use Lemma 3.2, Remark 3.3 and the mean value theorem. For $\delta > 0$ sufficiently small, $\|g_1\|^2 / 2$ and $\|g\|^2 / 2$ are larger than say $5/4$. Also, $r \leq 2\sqrt{r^2 - 1}$ for $r > 5/4$. Hence, for some $r$ between $\|g_1\|^2 / 2$ and $\|g\|^2 / 2$, we obtain

$$|t(g_1) - t(g)| = \left| \cosh^{-1} \left( \frac{\|g_1\|^2 / 2}{\|g\|^2 / 2} \right) - \cosh^{-1} \left( \frac{\|g\|^2 / 2}{\|g\|^2 / 2} \right) \right|$$

$$= \frac{1}{\sqrt{r^2 - 1}} \left| \frac{\|g_1\|^2 / 2 - \|g\|^2 / 2}{\sqrt{r^2 - 1}} \right| = O\left( \frac{\delta \|g\|^2}{r} \right) = O(\delta).$$

To prove (ii), we observe that by Lemma 3.2, Remark 3.3 and Proposition 2.3 we have, since $h_1^{-1}, h_2^{-1} \in B_{\delta_1}$, that

$$0 \leq v(g_1^{-1}, g_1^{-1}) \leq v(h_2^{-1} g_1^{-1} h_1^{-1}, h_2^{-1} g_1^{-1}) + v(h_2^{-1} g_1^{-1}, g_1^{-1})$$

$$= O\left( \frac{\delta}{\|h_2^{-1} g_1^{-1}\|^2} \right) + v(h_2^{-1} g_1^{-1}, g_1^{-1}) = O(\delta) + v(h_2^{-1} g_1^{-1}, g_1^{-1}).$$

Let $C > 0$ be a fixed constant. We claim that, for any $h \in G$ satisfying

$$\max_{i,j} \left| (h - I)_{ij} \right| \leq C\delta,$$

we have—for $g \in G$ with $\|g\|$ bounded away from 1—that

$$v(hg, g) = O(\delta).$$

(Note that, for $\delta$ sufficiently small, we have $hg, g \notin K$.) From this and the above considerations, (ii) follows directly since $h_2^{-1}$ satisfies (3.7).

To prove (3.8), we first note that for any angle $0 \leq v \leq \pi$ we have

$$v \leq 2\sqrt{2(1 - \cos v)}.$$

For $g = k_g a_t k_g'$ we have, by Proposition 2.3 that

$$v(hg, g) = v(h g a_t, a_t),$$

where $h_g = k_g^{-1} h k_g$. Note that

$$\max_{i,j} \left| (h_g - I)_{ij} \right| = O(\delta).$$

Let $u : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the mapping which drops the last coordinate in the standard basis representation. From (3.10) it follows, using equivalence of norms on finite-dimensional vector spaces, that

$$|u(h_{g,1}) - u(e_1)| = O(\delta), \quad |u(h_{g,n+1})| = O(\delta),$$

where $h_{g,i}$ is the $i$th column of $h_g$. 

\footnote{Note that the right-hand side of (3.9) is twice the Euclidean distance between $(\cos v, \sin v)$ and $(1, 0).
Now, by definition \( \cos v(hg, g) \) is the first entry in \( u(h_g a_t e_{n+1})/|u(h_g a_t e_{n+1})| \). Since this vector has norm 1, the sum of squares of the remaining entries equals \( \sin^2 v(hg, g) \). It follows that

\[
(3.12) \quad \frac{|u(h_g a_t e_{n+1})|}{|u(h_g a_t e_{n+1})|} - u(e_1) = 2(1 - \cos v(hg, g)).
\]

We note that, since \( a_t e_{n+1} = \sin(t) \cdot e_1 + \cosh(t) \cdot e_{n+1} \), we have

\[
u(h_g a_t e_{n+1}) = \sinh(t) \cdot u(h_g, 1) + \cosh(t) \cdot u(h_g, n+1).
\]

Therefore, using (3.11) and the fact that \( t \) is bounded away from zero, we have

\[
(3.13) \quad |u(h_g a_t e_{n+1})| = \sinh(t(1 + O(\delta))).
\]

Finally, combining (3.9), (3.12), (3.13), and (3.11) we find, after a small computation, that

\[
v(hg, g) \leq 2\sqrt{2(1 - \cos v(hg, g))} = 2 \frac{|u(h_g a_t e_{n+1})|}{|u(h_g a_t e_{n+1})|} - u(e_1) = O(\delta),
\]

which proves the claim. \( \square \)

**Lemma 3.5.** For any \( \delta > 0 \) sufficiently small the following holds: Let \( g, M \in G, M \notin K \). Assume \( \|g\| \geq 10 \|M\| \). Then, for every \( g_1 \in B_\delta, gD_\delta \), we have \( v(g_1, g_1 M), v(g, gM) < \pi/2 \) and

(i) \( \|g_1 M\|^2 = \|gM\|^2 + O(\|g\|^2 \|M\|^2) \),

(ii) \( \tan v(g_1, g_1 M) = \tan v(g, gM) + O\left(\frac{\|M\|^4}{\|g\|^4}\right)\).

**Proof.** The proof closely follows the proof of [15] Lemma 2.20. Consider the functions (denoted by \( F_1, F_2 \) in [15])

\[
G_1(t, v) := 2(\cosh t(M) \cosh t + \cos(\pi - v) \sinh t(M) \sinh t),
\]

\[
G_2(t, v) := \frac{\sin(\pi - v) \sinh t(M) \sinh t + \cos(\pi - v) \cosh t \sinh t(M)}{\cosh t(M) \cosh t + \cos(\pi - v) \cosh t \sinh t(M)}.
\]

The assumption \( \|g\| \geq 10 \|M\| \) implies that \( t(g) > t(M) + 1 \). Therefore, using Lemma 3.4 we find that \( t(g_1), t(g) > t(M) \), which by Proposition 3.1 implies that \( v(g_1, g_1 M), v(g, gM) < \pi/2 \).

By Proposition 3.1 we have

\[
\|g_1 M\|^2 - \|gM\|^2 = G_1(t(g_1), v(g_1^{-1}, M)) - G_1(t(g), v(g^{-1}, M)),
\]

\[
\tan v(g_1, g_1 M) - \tan v(g, gM) = G_2(t(g_1), v(g_1^{-1}, M)) - G_2(t(g), v(g^{-1}, M)).
\]

Furthermore, using Lemma 3.4 and Proposition 2.3 we find

\[
|t(g_1) - t(g)|, |v(g_1^{-1}, M) - v(g^{-1}, M)| = O(\delta).
\]

Finally, using the mean value theorem and the Cauchy-Schwarz inequality, the result follows from the following bounds on the partial derivatives of \( G_1 \) and \( G_2 \) valid when \( t > t(M) + 1 \):

\[
\frac{\partial G_1}{\partial t}(t, v), \frac{\partial G_1}{\partial v}(t, v) = O(\|M\|^2 \cosh t),
\]

\[
\frac{\partial G_2}{\partial t}(t, v), \frac{\partial G_2}{\partial v}(t, v) = O(\|M\|^4 / \cosh t)
\]

(see [15] Lemma 2.20 for the proofs of these estimates). \( \square \)
4. Volume computations

In this section we consider the problem of asymptotically determining the volume of the set

\[ R_M(Q, \xi) = \left\{ g \in B_Q : \|gM\| \leq Q, v(g, gM) < \frac{2\xi}{Q^2} \right\} \]

for \( M \in \Gamma, M \notin K \). In order to formulate our results we introduce some more notation. We let

\[ A := A(l) = \cosh l, \quad B := B(l) = \sinh l, \quad C := C(l) = 2\sinh(l/2). \]

For \( \xi \leq B \), we define the real numbers

\[ \alpha := \alpha(\xi, l) = \sqrt{1 - \frac{\xi^2}{B^2}}, \]

(4.1)

\[ \lambda_{\pm} := \lambda_{\pm}(\xi, l) = \frac{-\xi^2A}{B} \pm \sqrt{1 - \frac{\xi^2}{B^2}}. \]

(4.2)

Furthermore, we introduce the set

\[ I(\xi, l) := \begin{cases} [-1, \lambda_-] \cup (\alpha, 1] & \text{if } \xi \leq C, \\ [-1, \lambda_-) \cup (\lambda_+, -\alpha) \cup (\alpha, 1] & \text{if } C < \xi \leq B, \\ [-1, 1] & \text{if } B < \xi. \end{cases} \]

(4.3)

\[ f_{\xi}(l) = \frac{1}{\xi^n} \int_{I(\xi, l)} \left( 1 - y^2 \right)^{n-2} (y + \coth l)^{-(n-1)} \, dy \]

(4.5)

(\text{where } I(\xi, l) \text{ is defined by (4.3)}) will play a central role in the rest of the paper. It enters our discussion as part of the following result.

**Theorem 4.1.** For every \( M \in \Gamma, M \notin K \), and for \( \xi/Q^2 \) sufficiently small, we have

\[ \text{vol} (R_M(Q, \xi)) = \frac{\omega_{n-1}Q^{2(n-1)}}{2n-1} \int_0^\xi f_{\xi}(t(M)) \, d\zeta + O \left( g(\xi)\|M\|^{2(n-1)}Q^{2\left(\frac{n-1}{n+1}\right)} \right) \]

(4.6)

as \( Q \to \infty \), where

\[ g(\xi) = \xi^{n-1} + \xi^{-(n-1)}. \]

(4.7)

**Remark 4.2.** Note that we need \( \|M\| = o \left( Q^{2/(n+1)} \right) \) for the error term in (4.6) to be non-trivial.
Remark 4.3. For any given value of $n$, we can perform the integration in (4.5) and obtain completely explicit expressions for the function $f_\xi$. When $n = 2$ and $n = 3$, we have

$$f_\xi(l) = \frac{2}{\xi^2} \times \begin{cases} 
 l - \log \left( A + \sqrt{B^2 - \xi^2} \right) & \text{if } \xi \leq C, \\
 l + \log(1 + \xi^2) - 2 \log \left( A + \sqrt{B^2 - \xi^2} \right) & \text{if } C < \xi \leq B, \\
 l & \text{if } B < \xi,
\end{cases}$$

and

$$f_\xi(l) = \frac{4}{\xi^3} \times \begin{cases} 
 \frac{A}{B} \log \left( \frac{A+B}{A+\sqrt{B^2-\xi^2}} \right) + \frac{(\xi^2+2)\sqrt{B^2-\xi^2}+A\xi^2}{2B(\xi^2+1)} & \text{if } \xi \leq C, \\
 \frac{A}{B} \log \left( \frac{(A+B)(\xi^2+1)}{(A+\sqrt{B^2-\xi^2})^2} \right) + \frac{(\xi^2+2)\sqrt{B^2-\xi^2}}{B(\xi^2+1)} & \text{if } C < \xi \leq B, \\
 \frac{A}{B} l - 1 & \text{if } B < \xi,
\end{cases}$$

respectively. When $n$ gets larger such expressions become very cumbersome. Plotting $f_\xi(l)$ for fixed $\xi$ or $l$ seems to indicate, however, that the ‘complexity’ of $f_\xi(l)$ doesn’t grow significantly when $n$ gets larger (cf. Figure 1 and Figure 2).

We will prove Theorem 4.1 in two steps. The general strategy of the proof closely follows the one in [15, Sect. 3]. However, since some parts of the proof are computationally different, we have chosen to give a detailed proof for completeness.
Proposition 4.4. For every $M \in \Gamma$, $M \notin K$, and for $\xi/Q^2$ sufficiently small, we have

\begin{equation}
\text{vol}(\mathcal{R}_M(Q, \xi)) = \frac{\omega_n \pi^{2(n-1)} Q^{2(n-1)}}{2^{n-1}} \int_{-1}^{1} (1 - y^2)^{(n-3)/2} \int_{J_{\xi}(y)} x^{n-2} \, dx \, dy
+ O\left(g(\xi)\|M\|^{2(n-1)}Q^{2(n-1)/n+1}\right)
\end{equation}

as $Q \to \infty$, where $g(\xi)$ is given by (1.7) and the interval $J_{\xi}(y)$ is defined by

$$J_{\xi}(y) := \left\{ x \in [0, 1] : \frac{B_M \sqrt{1 - y^2}}{\xi (A_M + B_M y)} \leq x \leq \frac{1}{A_M + B_M y} \right\}.$$ 

Proof. To begin, we note that Remark 4.2 implies that we may assume

\begin{equation}
\|M\| \sim Q^{2/(n+1)}.
\end{equation}

We introduce a large positive parameter $X = X(M, Q)$ satisfying

\begin{equation}
5 \|M\|^2 < X < Q^2,
\end{equation}

and use it to truncate the region $\mathcal{R}_M(Q, \xi)$ as follows:

$$\tilde{\mathcal{R}}_M(Q, \xi) := \mathcal{R}_M(Q, \xi) \cap \{ g \in G : \| g \|^2 > X \}.$$ 

At the end of this proof (see (4.22)), we will determine a value of $X$ that balances the sizes of our error terms. For now, we will confine ourselves to observe that (2.2) immediately implies that

\begin{equation}
\text{vol}(\mathcal{R}_M(Q, \xi)) = \text{vol}((\tilde{\mathcal{R}}_M(Q, \xi)) + O(X^{n-1}).
\end{equation}

Next, we describe the conditions determining the set $\tilde{\mathcal{R}}_M(Q, \xi)$ in terms of the $KA^+K$ coordinates. Writing $g = k_g a_t k_g^\prime$ (recall 3.1), we first note that $\tilde{\mathcal{R}}_M(Q, \xi)$ is left $K$-invariant, i.e. it does not impose any restriction on the first $K$-component $k_g$. Furthermore, using Proposition 3.1 together with (4.10), we find that $\tilde{\mathcal{R}}_M(Q, \xi)$ is determined by the following inequalities once $\xi/Q^2$ is sufficiently small:

\begin{align}
X < 2 \cosh t \leq Q^2, \\
2(A_M \cosh t + B_M \cos v \sinh t) \leq Q^2, \\
\frac{B_M \sin v}{A_M \sinh t + B_M \cos v \cosh t} < \tan \left(\frac{2\xi}{Q^2}\right),
\end{align}

where $v = \pi - v(g^{-1}, M) = v(k_g^\prime k_M a_t(M), g_N)$. We let $k = k_g^\prime k_M$ and note that $k = m_1 k \cdot m_2$ for some appropriate $m_1, m_2 \in M$. We also let $\chi_{\tilde{\mathcal{R}}_M(Q, \xi)}$ denote the indicator function of the set $\tilde{\mathcal{R}}_M(Q, \xi)$. However, since $\tilde{\mathcal{R}}_M(Q, \xi)$ is left $K$-invariant, we will also let $\chi_{\tilde{\mathcal{R}}_M(Q, \xi)}$ denote the indicator function of the set of $(t, k) \in [0, \infty) \times K$ satisfying the conditions (4.12), (4.13) and (4.14). Now, using Proposition 2.1 we obtain

\begin{align*}
\text{vol}(\tilde{\mathcal{R}}_M(Q, \xi)) &= \omega_n \int_K \int_0^\infty \int_K \chi_{\tilde{\mathcal{R}}_M(Q, \xi)}(k_g a_t k_g^\prime) (\sinh t)^{n-1} dk_g \, dt \, dk_g^\prime \\
&= \omega_n \int_K \int_0^\infty \chi_{\tilde{\mathcal{R}}_M(Q, \xi)}(a_t k_g^\prime) (\sinh t)^{n-1} dt \, dk_g^\prime.
\end{align*}
Furthermore, changing variables $k'_g \mapsto k$ and applying Proposition 2.5 we arrive at

\begin{equation}
\text{vol}(\mathcal{R}_M(Q, \xi)) = \omega_{n-1} \int_0^\pi \int_0^\infty \chi_{\mathcal{R}_M(Q, \xi)}(t, k) \frac{1}{k^2} dtdk.
\end{equation}

In order to evaluate this integral, we make the following change of variables:

\[ x = \frac{2 \cosh t}{Q^2}, \quad y = \cos v. \]

We find that the inequalities (4.12), (4.13) and (4.14) are transformed into

\begin{align*}
\frac{X}{Q^2} < x & \leq 1, \\
x(M + B_M yz) & \leq 1, \\
\frac{B_M \sqrt{1 - y^2}}{x(Mz + B_M y)} & < \frac{Q^2}{2} \tan \left( \frac{2\xi}{Q^2} \right),
\end{align*}

where we have used the notation

\[ z := \tanh t = \sqrt{1 - \frac{4}{Q^4 x^2}}. \]

Next, for any $y \in [-1, 1]$, we let $\tilde{J}_{Q, \xi}(y)$ denote the set of all real numbers $x$ satisfying (4.10), (4.17) and (4.18). Then, it follows from (4.13) that

\begin{equation}
\text{vol}(\mathcal{R}_M(Q, \xi)) = \omega_{n-1} \int_0^1 \int_{-1}^{1 - y^2} \int_{-1}^{1 - y^2} x^2 \left( \frac{Q^4 x^2}{4} - 1 \right)^{n/2 - 1} dx dy + \mathcal{E}(Q),
\end{equation}

where

\[ \mathcal{E}(Q) = \begin{cases} 0 & \text{if } n = 2, \\ O_*(Q^4) & \text{if } n = 3, \\ O(Q^{2(n-3)}) & \text{if } n \geq 4. \end{cases} \]

We continue by noting that

\[ \frac{1 - y^2}{A_M + B_M y} \leq 1, \quad 1 - z = O(X^{-2}), \]

\[ \frac{Q^2}{2} \tan \left( \frac{2\xi}{Q^2} \right) = \xi + O\left( \frac{\xi^3}{Q^4} \right). \]

\footnote{The calculation in (4.19) uses Proposition 2.5 and hence is valid only for $n > 2$. However, exchanging the use of Proposition 2.5 with an easy symmetry argument, one easily finds that the last line of (4.13) is correct also in the case $n = 2$ (cf. [15, Prop. 3.1]).}
where the last estimate holds for $\xi/Q^2$ sufficiently small. We also recall from \cite[Prop. 3.1]{[15]} that
\[
\frac{1}{A_M + B_M y} \frac{1}{A_M + B_M y} = O(\|M\|^2).
\]
It follows immediately from these observations that the inequality \eqref{4.17} can be replaced by
\[
(4.20) \quad x \leq \frac{1}{A_M + B_M yz} + O\left( \|M\| \frac{6}{X^2} \right).
\]
Similarly, we find that \eqref{4.18} can be replaced by
\[
(4.21) \quad \frac{B_M \sqrt{1 - y^2}}{\xi(A_M + B_M y)} + O\left( \frac{\xi \|M\|^4}{Q^4} + \frac{\|M\|^6}{\xi X^2} \right).
\]

To finish the proof of the proposition, we investigate how the error terms in \eqref{4.20} and \eqref{4.21} affect the error term in \eqref{4.19}. By an elementary calculation, we find that
\[
\frac{\omega_{n-1} Q^{2(n-1)}}{2^{n-1}} \int_{-1}^{1} \left( 1 - y^2 \right)^{(n-3)/2} \frac{x^{n-2}}{J_{\xi}(y)} \, dxdy = O\left( X^{-1} - \xi^{-(n-3)} \right) \frac{\|M\|^{2n} Q^{2(n-3)}}{X^2}.
\]
Combining this estimate with \eqref{4.19} and \eqref{4.11}, we obtain
\[
\text{vol} (\mathcal{R}_M (Q, \xi)) = \frac{\omega_{n-1} Q^{2(n-1)}}{2^{n-1}} \int_{-1}^{1} \left( 1 - y^2 \right)^{(n-3)/2} \frac{x^{n-2}}{J_{\xi}(y)} \, dxdy + E(Q)
\]
\[
+ O\left( X^{-1} - \xi^{-(n-3)} \right) \frac{\|M\|^{2n} Q^{2(n-3)}}{X^2} + O\left( \|M\|^2 \frac{Q^{2(n-1)}}{X^2} \right).
\]
In order to balance the error terms above, we choose\footnote{Note that, by \eqref{4.9} and taking $Q$ large enough, this is an admissible choice of $X$ (i.e. $X$ satisfies \eqref{4.10}).}
\[
(4.22) \quad X = \|M\|^2 Q^{\frac{n-1}{n-1}}
\]
and arrive at
\begin{equation}
\text{vol}(\mathcal{R}_M(Q, \xi)) = \frac{\omega_{n-1}Q^{2(n-1)}}{2^{n-1}} \int_{-1}^{1} \int_{J_t(y)} x^{n-2} \, dx \, dy + O\left((\xi^{-(n-3)} + \xi^{-(n+1)}) M^{2n} Q^{2(n-3)} + (1 + \xi^{-(n-1)}) \|M\|^{2(n-1)} Q^{2(n-3)}\right).
\end{equation}

Finally, the desired result follows from the simple observation that, for \( M \) satisfying (4.9), the first error term is subsumed by the second error term in (4.23). \( \square \)

In order to further investigate the main term in (4.3), we introduce the sets
\begin{align*}
I_1(\xi, M) &:= \left\{ y \in [-1, 1] : \frac{B_M \sqrt{1 - y^2}}{\xi (A_M + B_M y)} < 1 \leq \frac{1}{A_M + B_M y} \right\}, \\
I_2(\xi, M) &:= \left\{ y \in [-1, 1] : \frac{B_M \sqrt{1 - y^2}}{\xi (A_M + B_M y)} > \frac{1}{A_M + B_M y} \leq 1 \right\}.
\end{align*}

\( I_1(\xi, M) \) and \( I_2(\xi, M) \) are unions of intervals, and we recall from [15, Lemma 3.18] the following more explicit description of these sets.
\begin{align*}
I_1(\xi, M) &:= \begin{cases} 
[-1, \lambda_{-, M}(\xi)) & \text{if } \xi \leq C_M, \\
[-1, \lambda_{-, M}(\xi)) \cup (\lambda_{+, M}(\xi), \frac{1 - A_M}{B_M}] & \text{if } C_M < \xi \leq B_M, \\
[-1, \frac{1 - A_M}{B_M}] & \text{if } B_M < \xi,
\end{cases} \\
I_2(\xi, M) &:= \begin{cases} 
(\alpha_M(\xi), 1] & \text{if } \xi \leq C_M, \\
[\frac{1 - A_M}{B_M}, -\alpha_M(\xi)) \cup (\alpha_M(\xi), 1] & \text{if } C_M < \xi \leq B_M, \\
[\frac{1 - A_M}{B_M}, 1] & \text{if } B_M < \xi.
\end{cases}
\end{align*}

Note in particular that
\begin{equation}
I(\xi, t(M)) = I_1(\xi, M) \cup I_2(\xi, M)
\end{equation}
(see (4.3)). We are now ready to finish the proof of Theorem 4.1.

Proof of Theorem 4.1 We call the integral in the main term of Proposition 4.4 \( F_M(\xi) \). Given the information about the intervals \( I_1(\xi, M) \) and \( I_2(\xi, M) \) above, it is immediate to note that
\begin{equation}
F_M(\xi) = \frac{1}{n - 1} \int_{I_1(\xi, M)} (1 - y^2)^{(n-3)/2} \left( 1 - \left( \frac{B_M \sqrt{1 - y^2}}{\xi (A_M + B_M y)} \right)^{n-1} \right) dy + \frac{1}{n - 1} \int_{I_2(\xi, M)} (1 - y^2)^{(n-3)/2} \left( \left( \frac{1}{A_M + B_M y} \right)^{n-1} - \left( \frac{B_M \sqrt{1 - y^2}}{\xi (A_M + B_M y)} \right)^{n-1} \right) dy.
\end{equation}

Furthermore, we note that \( F_M(\xi) \rightarrow 0 \) as \( \xi \rightarrow 0 \). Indeed, using (4.1) and (4.2) in the explicit formulas for \( I_1(\xi, M) \) and \( I_2(\xi, M) \), we find that both intervals are of length \( O(\xi^2) \) as \( \xi \rightarrow 0 \), from which the claim follows.

Next, we compute the derivative \( F'_M(\xi) \) in each of the three regimes of \( I_1(\xi, M) \) and \( I_2(\xi, M) \). We first consider the case \( B_M < \xi \), where the computations are easier due to the fact that none of the endpoints of \( I_1(\xi, M) \) and \( I_2(\xi, M) \) depend

\[ \text{Recall the definitions of } \alpha_M \text{ and } \lambda_{\pm, M} \text{ from (4.1), (4.2), and (4.4).] \]
on $\xi$. Using the explicit integral description (4.25) of $F_M(\xi)$, we immediately find that
\begin{equation}
F_M'(\xi) = \frac{1}{\xi^n} \int_{I_1(\xi,M) \cup I_2(\xi,M)} (1 - y^2)^{n-2} \left( y + \frac{A_M}{B_M} \right)^{-(n-1)} dy.
\end{equation}

We continue by considering the case $\xi < C_M$. Again, it follows from (4.25) that
\begin{align*}
F_M'(\xi) &= (1 - \lambda_{-M}(\xi)^2)^{(n-3)/2} \left( 1 - \left( \frac{B_M \sqrt{1 - \lambda_{-M}(\xi)^2}}{\xi(A_M + B_M \lambda_{-M}(\xi))} \right)^{n-1} \right) \frac{\lambda'_{-M}(\xi)}{n-1} \\
&\quad - (1 - \alpha_M(\xi)^2)^{(n-3)/2} \left( \frac{1}{A_M + B_M \alpha_M(\xi)} \right)^{n-1} \frac{\alpha'_{M}(\xi)}{n-1} \\
&\quad + \frac{1}{\xi^n} \int_{I_1(\xi,M) \cup I_2(\xi,M)} (1 - y^2)^{n-2} \left( y + \frac{A_M}{B_M} \right)^{-(n-1)} dy.
\end{align*}

It is now straightforward to verify, using (4.12) and (4.11) respectively, that
\begin{align*}
&\frac{B_M \sqrt{1 - \lambda_{-M}(\xi)^2}}{\xi(A_M + B_M \lambda_{-M}(\xi))} = 1, \\
&\frac{B_M \sqrt{1 - \alpha_M(\xi)^2}}{\xi(A_M + B_M \alpha_M(\xi))} = \frac{1}{A_M + B_M \alpha_M(\xi)}.
\end{align*}

Hence, since the first two terms in the expression for $F_M'(\xi)$ vanish, we arrive at
\begin{equation}
F_M'(\xi) = \frac{1}{\xi^n} \int_{I_1(\xi,M) \cup I_2(\xi,M)} (1 - y^2)^{n-2} \left( y + \frac{A_M}{B_M} \right)^{-(n-1)} dy.
\end{equation}

Furthermore, by essentially the same argument, we find that also in the case $C_M < \xi < B_M$, we have
\begin{equation}
F_M'(\xi) = \frac{1}{\xi^n} \int_{I_1(\xi,M) \cup I_2(\xi,M)} (1 - y^2)^{n-2} \left( y + \frac{A_M}{B_M} \right)^{-(n-1)} dy.
\end{equation}

Finally, it follows from (4.24) that the right-hand sides of (4.26), (4.27) and (4.28) all give the desired expression for $F_M'(\xi)$ (i.e. $F_M'(\xi) = f_\xi(t(M))$, where $f_\xi$ is the function defined in (4.15)). Therefore, since $F_M(\xi)$ and $f_\xi(t(M))$ are continuous functions of $\xi$, and $F_M(\xi) \to 0$ as $\xi \to 0$, the theorem follows from the fundamental theorem of calculus.

We end this section by pointing out that the result corresponding to Theorem 4.1 needed in the proof of Theorem 1.6 can be established by essentially the same proof as Theorem 4.1. Recall that $S \subset U$ is a spherical cap (recall also that $U$ denotes the hyperbolic unit sphere centered at $e_{n+1}$) and that the hyperbolic cone specified by the vertex $e_{n+1}$ and the cross-section $S$ is denoted by $C$. We are interested in determining the volume, asymptotically as $Q \to \infty$, of the set
\begin{equation}
R_{M,C}(Q,\xi) := \left\{ g \in B_Q : \|gM\| \leq Q, g e_{n+1}, g M e_{n+1} \in C, v(g,M) < \frac{2\xi}{Q^2} \right\}
\end{equation}
for $M \in \Gamma, M \notin K$. We denote the volume measure on $S^{n-1}$ by $\mu_{S^{n-1}}$ and let $\phi : S^{n-1} \to U \subset \mathbb{H}^n$ denote an embedding of the Euclidean sphere $S^{n-1}$ into $\mathbb{H}^n$ preserving all angles based at the center of the sphere.
Theorem 4.5. Let $S \subset U$ be a spherical cap and denote the hyperbolic cone specified by the vertex $e_{n+1}$ and the cross-section $S$ by $C$. Then, for every $M \in \Gamma$, $M \notin K$, and for $\xi/Q^2$ sufficiently small, we have

\[(4.30) \quad \text{vol} (\mathcal{R}_{M,C}(Q,\xi)) = \frac{\omega_{n-1} \mu_{S^{n-1}}(\phi^{-1}(S)) Q^{2(n-1)}}{\omega_n 2^{n-1}} \int_0^\xi f_\xi(t(M)) \, d\zeta \]

\[+ O \left( \xi Q^{2(n-2)} + g(\xi) \|M\|^{2(n-1)} Q^{2(n-1)^2} \right)\]

as $Q \to \infty$, where the functions $f_\xi$ and $g$ are defined by (4.5) and (4.7) respectively.

Proof. To prove (4.30), we first replace $\mathcal{R}_{M,C}(Q,\xi)$ by the more tractable set $S_{M,C}(Q,\xi) := \left\{ g \in B_Q : \|g\| \leq Q, g e_{n+1} \in C, v(g,gM) < \frac{2\xi}{Q^2} \right\}$ and notice that

\[
\text{vol}(S_{M,C}(Q,\xi)) - \text{vol}(\mathcal{R}_{M,C}(Q,\xi)) = O(\xi Q^{2(n-2)}).
\]

Finally, we determine an asymptotic formula for $\text{vol}(S_{M,C}(Q,\xi))$ in the same way as we found the formula for $\text{vol}(\mathcal{R}(M,Q,\xi))$ in the proof of Theorem 4.1. \qed

5. Approximating counts by volumes

We are now in a position where we can relate the counting of terms in the sum (1.4) to the volumes $\text{vol}(\mathcal{R}(M,Q,\xi))$ which have been calculated in Theorem 4.1. Let $X = X(M,Q)$ be a truncation parameter satisfying

\[(5.1) \quad 1 < X < \frac{Q}{20 \|M\|}.
\]

We define

\[
\mathcal{R}(M,Q,\xi,X) := \{ g \in \mathcal{R}(M,Q,\xi) : \|g\| > Q/X \}
\]

and observe that

\[(5.2) \quad \text{vol}(\mathcal{R}(M,Q,\xi) \setminus \mathcal{R}(M,Q,\xi,X)) = O \left( \frac{Q^{2(n-1)}}{X^{2(n-1)}} \right),
\]

since this complement is contained in $\{ g \in G : \|g\| \leq Q/X \}$.

5.1. Fattening and slimming. Recall the definitions of $B_{\delta_1}$ and $D_{\delta}$ from (3.4) and (3.5) and note that these sets are invariant under inversion. We consider the fattening and slimming of $\mathcal{R}(M,Q,\xi,X)$ by $B_{\delta_1} \times D_\delta$. More generally: For any sets $S,C_1,C_2 \subset G$, we define the $C_1 \times C_2$-fattening $S^+$ of $S$, and the $C_1 \times C_2$-slimming $S^-$ of $S$, by

\[
S^+ := \bigcup_{(h_1,h_2) \in C_1 \times C_2} h_1 \cdot S \cdot h_2,
\]

\[
S^- := \bigcap_{(h_1,h_2) \in C_1 \times C_2} h_1^{-1} \cdot S \cdot h_2^{-1}.
\]

It is easy to see that

\[(5.3) \quad S^- \subseteq (S^+)^+ \subseteq S \subseteq (S^-)^- \subseteq S^+
\]

and that if $A \subseteq B$, then

\[(5.4) \quad A^+ \subseteq B^+, \quad A^- \subseteq B^-.
\]
The next two lemmas verify that, for small values of the parameter \(0 < \delta = \delta(M, Q, X)\), the \(B_\delta \times D_\delta\)-fattening (respectively \(B_\delta \times D_\delta\)-slimming) of \(R_M(Q, \xi, X)\) doesn’t grow (or shrink) too drastically.

**Lemma 5.1.** Let \(\xi_0 > 0\). For \(\delta\) and \(\delta \|M\|^2\) sufficiently small, there exists a constant \(c > 0\) such that for any \(\xi \geq \xi_0\) we have

\[
R_M^+(Q, \xi, X) \subseteq R_M(\delta_2 Q, \delta_3 \xi, 2X),
\]

\[
R_M(Q, \xi, X) \subseteq R_M^-(\delta_2 Q, \delta_3 \xi, 2X),
\]

where

\[
\delta_2 := 1 + c\delta \|M\|^2, \quad \delta_3 := 1 + 7c\delta X^2 \|M\|^4.
\]

**Proof.** We start by noticing that, using (5.3) and (5.4), any of the two inclusions implies the other. To prove the first inclusion, we let \(g_1 = h_1 gh_2\) with \(g \in R_M(Q, \xi, X)\) and \((h_1, h_2) \in B_\delta \times D_\delta\). Note that since \(\|g\| > Q/X\), we have \(\|g\| > 20 \|M\|\) by assumption (5.1), so we are free to apply Lemma 3.2 and Lemma 3.5.

First, we observe that, by Lemma 3.2(i) and Remark 3.3 (recall also that \(D_\delta \subseteq B_\delta\)), there exist an absolute constant \(c_1 > 0\) such that

\[
\|g_1\|^2 < \|g\|^2 (1 + c_1 \|M\|^2 \delta)^2 \leq Q^2 (1 + c_1 \|M\|^2 \delta)^2.
\]

We observe in a similar way, using Lemma 3.3(ii), that

\[
\|g_1 M\|^2 < Q^2 (1 + c_2 \|M\|^2 \delta)^2
\]

for another absolute constant \(c_2 > 0\).

Next, we use Lemma 3.5(iii), basic properties of \(\arctan\), and \(\|g\| > Q/X\) to see that

\[
v(g_1, g_1 M) < v(g, g M) + c_3 \delta \|M\|^4 / \|g\|^2 < Q^{-2} (2\xi + c_3 \delta X^2 \|M\|^4).
\]

Letting \(c = \max(c_1, c_2, (2\xi_0)^{-1} c_3)\), we observe that for \(\delta \|M\|^2\) sufficiently small, we have \(\delta_2^2 \leq 1 + 3c\delta \|M\|^2\) and

\[
(1 + c\delta X^2 \|M\|^4)\delta_2^2 \leq 1 + 7c\delta X^2 \|M\|^4,
\]

which, together with (5.7), implies

\[
v(g_1, g_1 M) < \frac{2\xi \delta_3}{(Q \delta_2)^2}.
\]

We now observe that the inequalities (5.5), (5.3) and (5.8) show \(g_1 \in R_M(\delta_2 Q, \delta_3 \xi)\).

To prove that \(g_1 \in R_M(\delta_2 Q, \delta_3 \xi, 2X)\), we note by Lemma 3.2(i), Remark 3.3 and the above choice of \(c\) that we have

\[
\|g_1\|^2 > \frac{(1 - c\delta)^2 Q^2}{X^2} > \frac{(\delta_2 Q)^2}{(2X)^2},
\]

where the last inequality holds for \(\delta\) and \(\delta \|M\|^2\) sufficiently small. This finishes the proof.

**Remark 5.2.** Concerning the numbers \(\delta_2, \delta_3\) in Lemma 5.1, we have \(1 \leq \delta_2 = O(1)\), whereas a priori we only have \(1 \leq \delta_3 = O(X^2 \|M\|^2)\).
Lemma 5.3. Let \( \epsilon > 0 \) and fix \( \xi > 0 \). For \( ||M|| \geq m_0 > 1, \delta ||M||^2 \) sufficiently small, and \( \delta_3 = 1 + 7c\delta X^2 ||M||^4 \) bounded, we have
\[
\text{vol}(R_M(Q\delta_2^{\pm 1}, \xi\delta_3^{\pm 1})) = \text{vol}(R_M(Q, \xi)) + O(\epsilon, \delta(\xi/\delta_3)^2(\delta_2 - 1))
\]

Proof. Clearly \( \xi/Q^2, \xi\delta_3/(Q\delta_2)^2 \) all become sufficiently small for \( Q \) sufficiently large, so we may apply Theorem 4.1 to see that up to an error of
\[
O\left((\xi - \delta_3\xi) ||M||^{-n+\epsilon} + \xi(\delta_3 - 1) ||M||^{-n+\epsilon}\right)
\]

Bounding the integrals using [25, Lemma A.1(iii)], we see that (5.9) is
\[
O\left(Q^{2(n-1)}\left((\delta_2 - 1)\xi ||M||^{-n+\epsilon} + \xi(\delta_3 - 1) ||M||^{-n+\epsilon}\right)\right)
\]

Substituting \( Q/\delta_2 \) for \( Q \) and \( \xi/\delta_3 \) for \( \xi \) in the above (notice that this is allowed since \( (\xi/\delta_3)^2((Q/\delta_2)^2 \) becomes small when \( Q \) grows sufficiently large), we find that, up to an error of
\[
O\left((\xi/\delta_3) + \xi(\delta_3^{n-1})\right)
\]

the difference
\[
\text{vol}(R_M(Q, \xi)) - \text{vol}(R_M(Q/\delta_2, \xi/\delta_3))
\]
is also bounded by \( O(\xi Q^{2(n-1)}(\delta X^2 ||M||^{4-n+\epsilon})\). The result now follows easily using
\[
g(\xi) + g(\delta_3^{n-1}) = O(\xi(\delta_3^{n-1}),
\]

which by assumption equals \( O(\xi(1)\right).

\]

5.2. Test functions. In this short section, we introduce two functions \( \Psi_1 \) and \( \Psi_2 \) on \( \Gamma\setminus G \) that will be of fundamental importance when we relate counts to volumes in Section 5.3. However, we begin by determining the asymptotic order of decay of the volumes of the sets \( B_\delta \) and \( D_\delta \) as \( \delta \to 0 \). Using Proposition 2.4 we immediately find that
\[
\text{vol}(B_\delta) = \omega_n \int_0^\delta (\sinh t)^{-n-1} \, dt \asymp \delta^n
\]

for all sufficiently small \( \delta \). Moreover, with a little more effort, we can also establish the following lemma.

Lemma 5.4. We have \( \text{vol}(D_\delta) \asymp \delta^{n^2} \) for all sufficiently small \( \delta \).
Proof. Using Proposition 2.4, we obtain
\[
\text{vol}(D_\delta) = \omega_n \left( \int_{K_\delta} dk \right)^2 \int_0^\delta (\sinh t)^{n-1} dt \asymp \text{vol}(K_\delta)^2 \delta^n
\]
for all small enough \(\delta\). It remains to determine the asymptotic order of decay of \(\text{vol}(K_\delta)\). Recalling the definition of \(K_\delta\) in (3.6), we find that there exists a constant \(C > 1\) such that, for all sufficiently small \(\delta\), the pre-image of \(K_\delta\) (in the Lie algebra \(\text{Lie}(K)\) of \(K\)) under the exponential map satisfies
\[
B_{\delta/C} \subset \exp^{-1}(K_\delta) \subset B_{C\delta},
\]
where \(B_\epsilon\) denotes the Euclidean ball of radius \(\epsilon\) centered at the origin in \(\text{Lie}(K)\).

Using this fact, together with \([12, \text{Thm. 1.14}]\) and possibly shrinking the size of the admissible set of parameters \(\delta\), we obtain
\[
\text{vol}(K_\delta) \asymp \delta^n \left( \frac{n}{2} \right),
\]
where we have used \(\dim(\text{Lie}(K)) = n \left( \frac{n}{2} \right)\). Finally, combining (5.11) and (5.12), we arrive at the desired result. \(\Box\)

We now let \(\delta > 0\) be small enough to guarantee that the asymptotics in (5.10) and Lemma 5.4 are valid. We introduce a smooth and non-negative test function \(\psi_1\) satisfying \(\psi_1(k_1 g k_2) = \psi_1(g)\) (i.e. \(\psi_1\) is spherically symmetric) and
\[
\text{supp } \psi_1 \subset B_{\delta_1}, \quad \int_G \psi_1(g) \, dg = 1.
\]
Furthermore, we introduce a smooth and non-negative test function \(\psi_2\) satisfying
\[
\text{supp } \psi_2 \subset D_\delta, \quad \int_G \psi_2(g) \, dg = 1.
\]
We can, as usual, use the test functions \(\psi_i\) \((i = 1, 2)\) to construct \(\Gamma\)-automorphic functions
\[
\Psi_i(g) := \sum_{\gamma \in \Gamma} \psi_i(\gamma g)
\]
in \(L^2(\Gamma \backslash G)\) satisfying \(\Psi_i(\gamma g) = \Psi_i(g)\) for all \(\gamma \in \Gamma\). It is well known that we can choose the test functions \(\psi_1\) and \(\psi_2\) in such a way that \(\Psi_1\) and \(\Psi_2\) also satisfy
\[
\|\Psi_1\|_{L^2} \asymp \text{vol}(B_{\delta_1})^{-1/2} \asymp \delta^{-n/2}, \quad \|\Psi_2\|_{L^2} \asymp \text{vol}(D_\delta)^{-1/2} \asymp \delta^{-n^2/2}
\]
and
\[
S_l(\Psi_1) \asymp \delta^{-l} \text{vol}(B_{\delta_1})^{-1/2} \asymp \delta^{-l - n/2}, \quad S_l(\Psi_2) \asymp \delta^{-l} \text{vol}(D_\delta)^{-1/2} \asymp \delta^{-l - n^2/2}
\]
for any \(l \in \mathbb{N}\) (recall (2.6), (5.10) and Lemma 5.4). From now on we fix such an admissible pair of test functions \(\psi_1\) and \(\psi_2\). The asymptotics in (5.16) and (5.17), together with Theorem 2.7 and Remark 2.8 imply the following two corollaries.

**Corollary 5.5.** Let \(s_0\) be as in Theorem 2.7 and let \(\Psi_1\) be defined by (5.15) with our fixed test function \(\psi_1\). Then
\[
\langle \pi(g) \Psi_1, \Psi_1 \rangle_{\Gamma \backslash G} = \frac{1}{\text{vol}(\Gamma \backslash G)} + O\left( \|g\|^{2(s_0-n+1)} \delta^n \right)
\]
for all sufficiently small \(\delta > 0\).
Corollary 5.6. Let \( s_0 \) be as in Theorem 2.7 and let \( \Psi_1 \) and \( \Psi_2 \) be defined by (5.15) with our fixed test functions \( \psi_1 \) and \( \psi_2 \). Then there exists an integer \( c_n > \frac{n(n+1)}{2} \) such that
\[
\langle \pi(g) \Psi_1, \Psi_2 \rangle_{\Gamma \setminus G} = \frac{1}{\text{vol}(\Gamma \setminus G)} + O\left( \|g\|^{2(s_0-n+1)} \delta^{-c_n} \right)
\]
for all sufficiently small \( \delta > 0 \).

5.3. Relating counts to volumes. We are now ready to show that the number of elements in \( \Gamma \cap R_M(Q, \xi) \) can be approximated by \( \text{vol}(R_M(Q, \xi)) / \text{vol}(\Gamma \setminus G) \). Recall the constant \( c_n \) from Corollary 5.6.

Lemma 5.7. Fix \( \xi > 0 \) and fix \( s_0 \) as in Theorem 2.7. For \( \|M\| \geq m_0 > 1 \), we have
\[
\# \Gamma \cap R_M(Q, \xi) = \frac{\text{vol}(R_M(Q, \xi))}{\text{vol}(\Gamma \setminus G)} + O_{\xi}(Q^{a_n} \|M\|^{b_n}),
\]
where
\[
a_n = 2(n-1) \left( 1 - \frac{n-1-s_0}{n(1+c_n)-1} \right), \quad b_n = (n-1) \frac{4c_n}{n(1+c_n)-1}.
\]

Proof. Let \( A \subseteq G \) be a bounded set and consider
\[
F_A(g_1, g_2) := \sum_{\gamma \in \Gamma} 1_A(g_1^{-1} \gamma g_2).
\]
We note that this function is \( \Gamma \)-invariant in both variables under multiplication from the left. We claim that
\[
(5.18) \quad \# \Gamma \cap A^- \leq \langle F_A, \Psi_1 \otimes \Psi_2 \rangle_{\Gamma \setminus G \times \Gamma \setminus G} \leq \# \Gamma \cap A^+.
\]
Here \( \Psi_i \) is defined in (5.15) and \( A^+ \) (resp. \( A^- \)) is the \( B_{\delta_1} \times D_{\delta} \)-fattening (resp. slimming) of \( A \).

First, we unfold the functions \( \Psi_i \) in the middle expression and find that this inner product equals
\[
(5.19) \quad \sum_{\gamma \in \Gamma} \int_G \int_G 1_A(g_1^{-1} \gamma g_2) \psi_1(g_1) \psi_2(g_2) \, dg_1 dg_2.
\]
To see the left inequality in (5.18), we now note that for every \( \gamma \in \Gamma \cap A^- \) we have, since \( B_{\delta_1} \) is symmetric under inversion, that \( g_1^{-1} \gamma g_2 \in A \) whenever \( (g_1, g_2) \in B_{\delta_1} \times D_{\delta} \). Therefore, using (5.13) and (5.14), we find that every term in the sum (6.19) corresponding to such a \( \gamma \) contributes by 1.

To see the right inequality in (5.18), we note that for \( \gamma \) to give a non-zero contribution to the sum (5.19), the requirements (5.13), (5.14) imply that there must exist a pair \( (g_1, g_2) \in B_{\delta_1} \times D_{\delta} \) such that \( g_1^{-1} \gamma g_2 \in A \). But this implies, since \( D_{\delta} \) is symmetric under inversion, that \( \gamma \in A^+ \). Moreover, given \( \gamma \in A^+ \) the corresponding integral in (5.19) can be at most 1, again by (5.13), (5.14).

On the other hand, analyzing the inner product in (5.18) by unfolding the \( \Gamma \)-sum defining \( F_A \) and making the change of variables \( g = g_1^{-1} g_2 \), we find
\[
(5.19) \quad \langle F_A, \Psi_1 \otimes \Psi_2 \rangle_{\Gamma \setminus G \times \Gamma \setminus G} = \int_A \int_{\Gamma \setminus G} \Psi_1(g_1) \Psi_2(g_1 g) \, dg_1 dg = \int_A \langle \pi(g) \Psi_2, \Psi_1 \rangle_{\Gamma \setminus G} \, dg,
\]
where $\pi$ denotes the right regular representation. Assume now that the parameters $\delta$ and $X$ are satisfying

\begin{equation}
\delta \|M\|^2 \text{ small, Eq. (5.1), and } \delta_3 = 1 + 7c\delta X^2 \|M\|^4 = O(1).
\end{equation}

Here the constant $c$ is as in Lemma 5.1 and Lemma 5.3. It follows from the above discussion and Lemma 5.1 that\footnote{Notice that we are free to apply Lemma 5.1 since $\xi/\delta_3$ by assumption (5.20) is bounded from below.}

\[
\int_{R_M(Q/\delta_2, \xi/\delta_3, X/2)} (\pi(g)\Psi_2, \Psi_1)_{\Gamma \setminus G} dg \leq \#\Gamma \cap R_M(Q, \xi, X) \\
\leq \int_{R_M(Q\delta_2, \xi\delta_3, 2X)} (\pi(g)\Psi_2, \Psi_1)_{\Gamma \setminus G} dg.
\]

Once this has been established, we only need to use the decay of matrix coefficients (Corollary 5.6) to approximate the integrals by volumes, and then the volume estimates from Lemma 5.3 to estimate the relevant count. To be more precise:

Using Corollary 5.6 and $R_M(\delta_2 Q, \delta_3 \xi, 2X) \subset B_{2Q}$ for $\delta \|M\|^2$ sufficiently small, we find that since

\[
\int_{R_M(Q\delta_2, \xi\delta_3, 2X)} \|g\|^{2(n-1)} dg = O(Q^{2s_0})
\]

we have

\[
\frac{\text{vol}(R_M(Q/\delta_2, \xi/\delta_3, X/2))}{\text{vol}(\Gamma \setminus G)} + O(\delta^{-c_\epsilon} Q^{2s_0}) \leq \#\Gamma \cap R_M(Q, \xi, X)
\]

and

\[
\#\Gamma \cap R_M(Q, \xi, X) \leq \frac{\text{vol}(R_M(Q\delta_2, \xi\delta_3, 2X))}{\text{vol}(\Gamma \setminus G)} + O(\delta^{-c_\epsilon} Q^{2s_0}).
\]

Furthermore, using (5.2) and Lemma 5.3 with a fixed small $\epsilon$, we see that

\[
\frac{\text{vol}(R_M(Q\delta_2, \xi\delta_3, 2X))}{\text{vol}(\Gamma \setminus G)} = \frac{\text{vol}(R_M(Q\delta_2, 2X))}{\text{vol}(\Gamma \setminus G)} + O \left( \frac{Q^{2(n-1)}}{X^{2(n-1)}} \right)
\]

\[
= \frac{\text{vol}(R_M(Q, \xi))}{\text{vol}(\Gamma \setminus G)} + O_\epsilon \left( Q^{2(n-1)} \delta X^2 \|M\|^{4-n+\epsilon} + \|M\|^{2(\gamma(n)-1)} Q^{2(n-1)} \frac{Q^{2(n-1)}}{X^{2(n-1)}} \right).
\]

In order to control the error terms above, we first balance $Q^{2(n-1)} \delta X^2 \|M\|^4$ with $Q^{2(n-1)} / X^{2(n-1)}$ and find

\begin{equation}
\delta = X^{-2n} \|M\|^{-4}.
\end{equation}

We omit the extra decay in $\|M\|^{-n+\epsilon}$ in order to be able to verify that $\delta_3$ is bounded; with the above choice of $\delta$ we have $\delta_3 = 1 + 7cX^{-2(n-1)}$, and by (5.1) this is indeed bounded. Using

\[
\#\Gamma \cap R_M(Q, \xi, X) = \#\Gamma \cap R_M(Q, \xi) + O \left( Q^{2(n-1)} \frac{Q^{2(n-1)}}{X^{2(n-1)}} \right),
\]
Lemma 5.1. To be more precise: Let
we find, with 

\[ \# \Gamma \cap R_M(Q, \xi) = \frac{\text{vol}(R_M(Q, \xi))}{\text{vol}(\Gamma \setminus G)} + O_\xi \left( \frac{Q^{2(n-1)}}{X^{2(n-1)}} + \delta^{-n}Q^{2s_0} + \|M\|^2(\frac{Q^n}{n+1}) \right). \]

We now balance \( X \) between the first two error terms and find
\[ X = Q^{\frac{n-1-\epsilon_0}{n+1+\epsilon_0}} \|M\|^{-\frac{2\epsilon_0}{n+1+\epsilon_0}}. \]

We can certainly assume that \( \|M\| < Q^{\frac{2(n-1)-\epsilon_m}{\epsilon_m}} = Q^{\frac{n-1-\epsilon_0}{2\epsilon_0}} \), since otherwise the claim of the lemma is trivial. With these choices of parameters, a computation, using also that \( c_n > \frac{n}{n+1} \), verifies that for \( Q \) sufficiently large \( \|M\| \) is indeed satisfied. It is also straightforward to verify, again using \( c_n > \frac{n}{2n+1} \), that
\[
\|M\|^2(\frac{Q^n}{n+1}) \leq Q^{a_n} \|M\|^{b_n}.
\]
Inserting these values of \( X \) and \( \delta \) in (5.22), we arrive at the claim. \( \square \)

5.4. More on the relation between counts and volumes. In this section we briefly discuss a modified version of Lemma 5.3 needed in the proof of Theorem 1.7. We recall that \( U \) denotes the hyperbolic unit sphere centered at \( e_{n+1} \). Let \( S \subset U \) be a spherical cap with opening angle \( \theta < \pi \), and denote the hyperbolic cone specified by the vertex \( e_{n+1} \) and the cross-section \( S \) by \( C \).

We are interested in counting the number of points in the intersection of \( \Gamma \) with the set \( R_{M,C}(Q, \xi) \) defined in (4.29). As in the case studied above we consider, for positive numbers \( X \) satisfying (5.1), the truncation
\[
R_{M,C}(Q, \xi, X) := \{ g \in R_{M,C}(Q, \xi) : \|g\| > Q/X \}.
\]

We note that in contrast to \( R_M(Q, \xi, X) \), this set is not left \( K \)-invariant and hence we have to adapt the fattening and slimming described in Lemma 5.1 slightly. We need to consider, for small parameters \( \delta > 0 \), the \( D_\delta \times D_\delta \)-fattening (respectively \( D_\delta \times D_\delta \)-slimming) of \( R_{M,C}(Q, \xi, X) \). It turns out that both the result and the proof of Lemma 5.3 carries over to the present situation except for one detail. If we let \( g_1 = h_1 \cdot g h_2 \) with \( g \in R_{M,C}(Q, \xi, X) \) and \( h_1, h_2 \in D_\delta \), then \( g_1 e_{n+1} \) and \( g_1 M e_{n+1} \) need not be contained in the cone \( C \). In order to compensate for this fact, we have to enlarge the cone on the right-hand sides of the statements corresponding to Lemma 5.1. To be more precise: Let \( C = \{ x \in \mathbb{H}^n : v(x, g e_{n+1}) < \theta \} \) for a suitable \( g' \in G \) not fixing the base point \( e_{n+1} \). Then, using Lemma 3.4(6) and the triangle inequality (Proposition 2.3(7)), it is possible to show that
\[
R_{M,C}(Q, \xi, X) \subseteq R_{M,C}(\delta_2 Q, \delta_3 e_{n+1}, 2X),
\]
\[
R_{M,C}(Q, \xi, X) \subseteq R_{M,C}(\delta_2 Q, \delta_3 e_{n+1}, 2X),
\]
where \( C_1 = \{ x \in \mathbb{H}^n : v(x, g' e_{n+1}) < \theta + \delta_4 \} \) and \( \delta_4 = \kappa \delta + \frac{2\delta_3}{Q \delta_2} \) for an absolute constant \( \kappa \) (here \( \delta_2 \) and \( \delta_3 \) are as in Lemma 5.1). Recall that we may assume that \( \delta_3 \) is bounded and that \( \xi/Q^2 \) is sufficiently small; hence \( \delta_4 \) is small. The fact that we have to consider \( C_1 \) on the right-hand sides above introduces an extra approximation step when we generalize Lemma 5.3. We first compare (for example) \( \text{vol}(R_{M,C}(\delta_2 Q, \delta_3 e_{n+1})) \) to \( \text{vol}(R_{M,C}(\delta_2 Q, \delta_3 e_{n+1})) \) using an elementary estimate and then
compare \( \text{vol}(R_{M,C}(\delta_2 Q, \delta_3 \xi)) \) to \( \text{vol}(R_{M,C}(Q, \xi)) \) using Theorem 4.5. The details are straightforward.

Turning to the generalization of Lemma 5.7, we need to replace the test function \( \Psi_1 \otimes \Psi_2 \) with \( \Psi_2 \otimes \Psi_2 \). It follows that we need to consider the matrix coefficient \( \langle \pi(g)\Psi_2, \Psi_2 \rangle_{\Gamma \backslash G} \). Using Theorem 2.7 and (5.17), we find that there exists an integer \( d_n > n^2 + 1 \) such that

\[
\langle \pi(g)\Psi_2, \Psi_2 \rangle_{\Gamma \backslash G} = \frac{1}{\text{vol}(\Gamma \backslash G)} + O\left( \|g\|^2(s_0 - n + 1)\delta - d_n \right)
\]

for all sufficiently small \( \delta > 0 \). Noticing that the rest of the proof can be generalized with only minor changes, we arrive at the following result.

**Lemma 5.8.** Fix \( \xi > 0 \) and fix \( s_0 \) as in Theorem 2.7. Let \( d_n \) be as in (5.23). Let \( S \subset \mathcal{U} \) be a spherical cap and denote the hyperbolic cone specified by the vertex \( e_{n+1} \) and the cross-section \( S \) by \( C \). Then, for \( \|M\| \geq m_0 > 1 \), we have

\[
\#\Gamma \cap R_{M,C}(Q, \xi) = \frac{\omega_n Q^2(n-1)}{2(n-1)(n-1)\text{vol}(\Gamma \backslash G)} R_2(\xi) + O_{\xi}(Q^{2(n-1) - \nu}).
\]

6. **Proofs of the main theorems**

In this section we finish the proofs of the main results stated in the introduction.

6.1. **Proof of Theorem 1.1.** Recall from (1.2) and (1.4) that the basic object we need to investigate is the sum

\[
\#N_{2,Q}(\xi) = \sum_{M \in \Gamma \backslash G} \#\Gamma \cap \mathcal{R}_M(Q, k_n, \Gamma)\xi).
\]

Recall also the constant \( k_{n,\Gamma} \) (see (2.5)) and the function

\[
f_\xi(t) = \frac{1}{\xi^n} \int_{I(\xi, t)} (1 - y^2)^{n-2} (y + \coth t)^{-(n-1)} dy
\]

defined in (4.5) (see also (4.3)). We will prove the following precise version of Theorem 1.1 establishing the existence and properties of the limit

\[
R_2(\xi) := \lim_{Q \to \infty} R_{2,Q}(\xi).
\]

**Theorem 6.1.** Let \( n \geq 2 \) and let \( \Gamma \subset G \) be a lattice. The limit (6.2), defining the pair correlation function \( R_2 \), exists and is differentiable. In fact, the pair correlation density function \( g_2 \) is given by

\[
g_2(\xi) = \frac{(n-1)\omega_{n-1}k_{n,\Gamma}}{\omega_n} \sum_{M \in \Gamma} f_{\xi k_{n,\Gamma}}(t(M)).
\]

Furthermore, there exists a real number \( \nu > 0 \), depending only on \( n \) and the spectral gap for the group \( \Gamma \), satisfying, for fixed \( \xi > 0 \) and \( Q \to \infty \), the relation

\[
\#N_{2,Q}(\xi) = \frac{\omega_n Q^{2(n-1)}}{2(n-1)(n-1)\text{vol}(\Gamma \backslash G)} R_2(\xi) + O_{\xi}(Q^{2(n-1) - \nu}).
\]
Remark 6.2. It follows immediately from (6.3) that the function $F_\xi$ in Theorem 1.1 is given by

\begin{equation}
F_\xi(t) := \frac{(n-1)k_n,\Gamma}{\omega_n} f_{\xi k_n,\Gamma}(t).
\end{equation}

Remark 6.3. The proof of Theorem 6.1 shows that (6.4) holds with any exponent $\nu$ satisfying

$$
\nu < \frac{(n-1)(n-1-s_0)}{n(1+c_n) + c_n - 1}.
$$

We begin by proving an elementary lemma.

**Lemma 6.4.** For each $g' \in G$, we have

$$
\# \left\{ \gamma \in N_\Gamma(Q) : v(\gamma, g') < \frac{2\xi}{Q^2} \right\} = O_\xi(\log Q).
$$

**Proof.** We let $3 < Q_1 < Q$ and consider the quantity

$$
\mathcal{L} := \# \left\{ \gamma \in \Gamma : Q_1 < \|\gamma\| \leq 2Q_1, v(\gamma, g') < \frac{2\xi}{Q^2} \right\}.
$$

By choosing a fixed $\delta > 0$, depending only on $\Gamma$ and small enough to ensure that $\gamma_1 B_{\delta_1} \cap \gamma_2 B_{\delta_1} = \emptyset$ for all $\gamma_1 \neq \gamma_2$ in $\Gamma$ (recall the definition of the ball $B_{\delta_1}$ in (3.3)), we write

$$
\mathcal{L} = \frac{1}{\text{vol}(B_{\delta_1})} \sum_{\gamma \in \Gamma, \|\gamma\| \leq 2Q_1} \text{vol}(\gamma B_{\delta_1}).
$$

Using Lemma 3.2 and Proposition 2.3 possibly decreasing the value of $\delta$, we obtain

$$
\mathcal{L} \leq \frac{1}{\text{vol}(B_{\delta_1})} \text{vol} \left( \left\{ g \in G : \|g\| < 3Q_1, v(g, g') < \frac{2\xi + c\delta}{Q_1^2} \right\} \right),
$$

with an absolute constant $c > 0$. Estimating the volume in the numerator, we find that $\mathcal{L} = O(1 + \xi^{n-1}) = O_\xi(1)$ independently of $g'$ (recall that all our implied constants are allowed to depend on $\Gamma$). Hence, the desired result follows from a dyadic decomposing of the condition $\|\gamma\| \leq Q$ (estimating the contribution from elements $\gamma$ satisfying, say, $\|\gamma\| \leq 6$ trivially). \qed

We continue by giving an upper bound on the tail of the sum (6.1). To be more precise, we consider

$$
\mathcal{E}_{Q,T}(\xi) := \sum_{\|M\| \geq T} \# \Gamma \cap R_M(Q, \xi),
$$

where $0 < T = T(Q) < Q$ is a parameter tending to infinity with $Q$.

**Lemma 6.5.** Fix $\xi$ and let $T < Q$. Then, for $T$ sufficiently large, we have

$$
\mathcal{E}_{Q,T}(\xi) = O_\xi \left( \frac{Q^{2(n-1)} \log Q}{T^{2(n-1)}} \right).
$$

**Proof.** Note that $\mathcal{E}_{Q,T}(\xi)$ equals the cardinality of the set

$$
S := \left\{ (\gamma, \gamma') \in N_\Gamma(Q)^2 : \|\gamma^{-1}\gamma'\| \geq T, v(\gamma, \gamma') < \frac{2\xi}{Q^2} \right\}.
$$
For \((\gamma, \gamma') \in \mathcal{S}\) we find, using Proposition 3.1 and the relation \(\cos(\pi - \nu(\gamma, \gamma')) = -1 + O(\xi^2/Q^3)\), that
\[
T^2 \leq \|\gamma^{-1}\gamma'\|^2 = 2\left(\cosh t(\gamma) \cosh t(\gamma') + \cos(\pi - \nu(\gamma, \gamma')) \sinh t(\gamma) \sinh t(\gamma')\right)
= 2 \cosh (t(\gamma) - t(\gamma')) + O_\xi(1).
\]
Assuming \(t(\gamma) - t(\gamma') \geq 0\), it follows that for \(T\) sufficiently large (depending only on \(\xi\)), we have \(e^{t(\gamma) - t(\gamma')} \geq T^2/2\), and therefore, since \(t(\gamma) \leq 2 \log Q\), we also have \(t(\gamma') \leq 2 \log(Q/T) + \log 2\). In particular, we readily have \(\|\gamma'\|^2 \leq 3Q^2\). Hence, we conclude that
\[
\mathcal{E}_{Q,T}(\xi) \leq 2\# \left\{(\gamma, \gamma') \in \Gamma^2 : \|\gamma\| \leq Q, \|\gamma'\| \leq \sqrt{3Q/T}, v(\gamma, \gamma') < \frac{2\xi}{Q^2}\right\}
= 2 \sum_{\gamma' \in \Gamma} \# \left\{\gamma \in N_\Gamma(Q) : v(\gamma, \gamma') < \frac{2\xi}{Q^2}\right\},
\]
and the result follows immediately from Lemma 6.5 \(\square\)

We are now ready to complete the proof of Theorem 6.1.

**Proof of Theorem 6.1.** Recall from (6.3) that our goal is an asymptotic expansion of \(#N_{2,Q}(\xi)\), with a power saving error term. We introduce a positive parameter \(T < Q\) (at the end of the proof we will determine a value of \(T\) that balances our error terms; see (6.4)) and apply Lemma 6.5 to get
\[
#N_{2,Q} \left(\frac{\xi}{k_n}\right) = \sum_{M \in \Gamma} \sum_{M \not\in K, \|M\| < T} \# R_M(Q, \xi) + O_\xi \left(\frac{Q^{2(n-1)} \log Q}{T^{2(n-1)}}\right).
\]
Also applying Lemma 5.7 and Theorem 4.1 yields
\[
#N_{2,Q} \left(\frac{\xi}{k_n}\right) = \frac{\omega_{n-1}Q^{2(n-1)}}{2n-1 \text{vol}(\Gamma \backslash G)} \sum_{M \in \Gamma, \|M\| < T} \int_0^\xi f_\xi(t(M)) d\zeta
+ O_\xi \left(T^{4(n-1)}Q^{2(n-1)^2/n} + T^{2(n-1)+b_n}Q^{a_n} + \frac{Q^{2(n-1)} \log Q}{T^{2(n-1)}}\right),
\]
where \(a_n\) and \(b_n\) are as in Lemma 5.7. Furthermore, using [25 Lemma A.1(ii)], we find that we may drop the condition \(\|M\| < T\) in the above summation; the error term introduced in this step is subsumed in the error term \(O_\xi((Q/T)^{2(n-1)} \log Q)\). Thus
\[
#N_{2,Q} \left(\frac{\xi}{k_n}\right) = \frac{\omega_{n-1}Q^{2(n-1)}}{2n-1 \text{vol}(\Gamma \backslash G)} \sum_{M \in \Gamma} \int_0^\xi f_\xi(t(M)) d\zeta
+ O_\xi \left(T^{4(n-1)}Q^{2(n-1)^2/n} + T^{2(n-1)+b_n}Q^{a_n} + \frac{Q^{2(n-1)} \log Q}{T^{2(n-1)}}\right).
\]
We balance the last two error terms in (6.6) by choosing
\[
T = Q^{\frac{2(n-1)-a_n}{4(n-1)+b_n}},
\]
and the theorem follows immediately from Lemma 6.5.
and with this choice of $T$, also using that $c_n > \frac{n(n+1)}{2}$, we readily verify that

$$T^{4(n-1)}Q^2\frac{(n-1)^2}{n^2} \leq \frac{Q^{2(n-1)}}{T^{2(n-1)}}.$$ 

Hence we can drop the first error term in (6.6) and arrive at

(6.8) $\#N_{2,Q}\left(\frac{\xi}{k_{n,\Gamma}}\right) = \omega_{n-1}Q^{2(n-1)}\sum_{M \in \Gamma} f_\xi(t(M)) \xi d\zeta + O_\xi,\epsilon \left(\frac{Q^{2(n-1)}(\frac{2(n-1)}{4(n-1)+s_n+b_n})+\epsilon}{\xi} \right)$. 

It is now straightforward to verify, using (1.2), (6.2) and Remark 2.6, that this confirms the claim in (6.4) with any exponent $\nu$ satisfying

$$\nu < \frac{(n-1)(n-1-s_0)}{n(1+c_n)+c_n-1}.$$ 

Also, since the remaining part of Theorem 6.1 follows immediately from (6.8), the proof is complete.

Remark 6.6. Let us point out that the proof of Theorem 6.1 can, with only minimal changes (e.g. replacing the use of Lemma 5.7 and Theorem 4.1 by applications of Lemma 5.8 and Theorem 4.5 respectively), be turned into a proof of Theorem 1.6.

6.2. Proof of Theorem 1.3. We recall from (6.3) that

(6.9) $g_2\left(\frac{\xi}{k_{n,\Gamma}}\right) = \frac{(n-1)\omega_{n-1}k_{n,\Gamma}}{\omega_n} \sum_{M \in \Gamma} f_\xi(t(M))$. 

Our main task is to prove the following asymptotic formula.

Lemma 6.7. Let $\epsilon > 0$ and let $s_0$ be as in Theorem 2.7. Then we have

$$\sum_{M \in \Gamma} f_\xi(t(M)) = \frac{1}{\text{vol}(\Gamma \setminus G)} \int_G f_\xi(t(g)) dg + O_{\epsilon} \left(\xi^{n-2+\frac{2(s_0-n+1)}{n+1}} + \epsilon\right)$$

as $\xi \to \infty$.

Proof. Let $\delta > 0$ be a small parameter. As in the proof of Lemma 3.4, we find that $t(g_1^{-1}Mg_2) - t(M) = O(\delta)$ for all $g_1, g_2 \in B_{\delta_1}$. Using this fact, together with [25, Lemma A.5] and Remark 2.6, we obtain

(6.10) $\left| \sum_{M \in \Gamma} f_\xi(t(g_1^{-1}Mg_2)) - \sum_{M \in \Gamma} f_\xi(t(M)) \right| \leq \sum_{M \in \Gamma} \left| f_\xi(t(g_1^{-1}Mg_2)) - f_\xi(t(M)) \right|$

$$\ll \sum_{\|M\| \ll \xi^{1/2}} \delta^{1/2}\xi^{-n} + \sum_{\|M\| \ll \xi} \delta\xi^{-n} + \sum_{\|M\| \gg \xi} \delta\xi^{n-2}\|M\|^{-4(n-1)}$$

$$\ll \delta^{1/2}\xi^{-1} + \delta\xi^{n-2} + \delta\xi^{-n} \ll \delta^{1/2}\xi^{-1} + \delta\xi^{n-2}.$$ 

For $\delta > 0$ sufficiently small, we again consider the spherically symmetric test function $\psi_1$ (with support contained in $B_{\delta_1}$) and the corresponding $\Gamma$-automorphic function $\Psi_1$ defined in Section 5.2. We recall in particular that $\int_G \psi_1(g) dg = 1$.
and that $\|\Psi_1\| \asymp \delta^{-n/2}$. Using the functions $\psi_1$ and $\Psi_1$, together with the estimate (6.10), we get

$$\sum_{M \in \Gamma} f_{\xi}(t(M)) = \left< \sum_{M \in \Gamma} f_{\xi}(t(g_1^{-1}Mg_2)), \Psi_1 \otimes \Psi_1(g_1, g_2) \right>_{\Gamma \setminus G \times \Gamma \setminus G} + O(\delta^{1/2} + \delta \xi^{n-2}).$$

Unfolding the summation in $\sum_{M \in \Gamma} f_{\xi}(t(g_1^{-1}Mg_2))$, making the change of variables $g = g_1^{-1}g_2$, and applying Corollary 5.5 and [25, Thm. A.3(ii)], we find that

$$\sum_{M \in \Gamma} f_{\xi}(t(g_1^{-1}Mg_2)), \Psi_1 \otimes \Psi_1(g_1, g_2) \right>_{\Gamma \setminus G \times \Gamma \setminus G}
= \int_{\Gamma} f_{\xi}(t(g)) \langle \pi(g)\Psi_1, \Psi_1 \rangle_{\Gamma \setminus G} \, dg
= \frac{1}{\text{vol}(\Gamma \setminus G)} \int_{\Gamma} f_{\xi}(t(g)) \, dg + O\left( \delta^{-n} \int_{\Gamma} f_{\xi}(t(g)) \|g\|^{2(s_0-n+1)} \, dg \right)
= \frac{1}{\text{vol}(\Gamma \setminus G)} \int_{\Gamma} f_{\xi}(t(g)) \, dg + O(\delta^{-n} \xi^{2s_0-n+\epsilon}),$$

where again $\pi$ denotes the right regular representation on $G$. We balance the error terms in (6.11) and (6.12) by choosing

$$\delta = \xi^{2(s_0-n+1)/(n+1)},$$

and arrive at the asymptotic formula

$$\sum_{M \in \Gamma} f_{\xi}(t(M)) = \frac{1}{\text{vol}(\Gamma \setminus G)} \int_{\Gamma} f_{\xi}(t(g)) \, dg + O(\xi^{-2+2(s_0-n+1)/(n+1)+\epsilon}),$$

which is the desired result.

We are now in a position to finish the proof of Theorem 1.3.

**Proof of Theorem 1.3** Using (6.9), Lemma 6.7 and [25, Thm. A.3(i)], we find that

$$g_2(\frac{\xi}{k_{n,\Gamma}}) = \frac{\omega_{n-1}k_{n,\Gamma}}{(n-1)\text{vol}(\Gamma \setminus G)} \xi^{n-2} + O_\epsilon \left( \xi^{-1+\epsilon} + \xi^{-n+\epsilon} + \xi^{-2+2(s_0-n+1)/(n+1)+\epsilon} \right).$$

Hence

$$g_2(\xi) = \frac{\omega_{n-1}k_{n,\Gamma}}{(n-1)\text{vol}(\Gamma \setminus G)} \xi^{n-2} + O_\epsilon \left( \xi^{-1+\epsilon} + \xi^{-n+\epsilon} + \xi^{-2+2(s_0-n+1)/(n+1)+\epsilon} \right)
= (n-1)\xi^{n-2} + O_\epsilon \left( \xi^{-2+2(s_0-n+1)/(n+1)+\epsilon} \right).$$

Finally, noting that the condition specifying $s_0$ is an open condition, we find that the conclusion in (6.13) also holds with $s_0$ replaced by any smaller number still admissible in the statement of Theorem 2.7. In particular we can, by allowing the implied constant to depend on $s_0$, drop the $\epsilon$ in the above error term. This finishes the proof. \qed
6.3. Proof of Theorem 1.5. Kelmer and Kontorovich prove in [15, p. 8] that in the case $n = 2$ the pair correlation density tends to the strictly positive value

$$g_2(0) = \frac{\text{vol}(\Gamma \setminus G)}{\pi} \sum_{\begin{smallmatrix} M \in \Gamma \\ t(M) > 0 \end{smallmatrix}} \frac{1}{e^{2t(M)} - 1}$$

as $\xi \to 0$ (see also [3, Eq. (1.3)]). Theorem 1.5 now follows directly from Theorem 1.1 and [25, Lemma A.1(ii)].

7. Geometric and spectral information contained in $g_2$

For a lattice $\Gamma = \{ M_i : i \in \mathbb{N} \} \cup \{ I \} \subseteq G$, we call the sequence

$$0 = t(I) < t(M_1) \leq t(M_2) \leq t(M_3) \leq \ldots$$

denote the lattice length spectrum of $\Gamma$ (not to be confused with the length spectrum). By [11], this set determines $n$ and $\text{vol}(\Gamma \setminus G)$ and, by Theorem 1.1 it therefore also determines the pair correlation density function.

In fact, the converse is also true. Given a pair correlation density function $g_2(\xi)$ and a volume $\text{vol}(\Gamma \setminus G)$, we can find the lattice length spectrum as follows: We find $n$ from Theorem 1.3. By [25, Lemma A.6], we know that $g_2$ is non-differentiable precisely at the points $2k_n^{-1} \sinh(t(M)/2)$ and $k_n^{-1} \sinh(t(M))$ (here $M$ runs through the non-trivial elements of $\Gamma$). The smallest value of $\xi$ for which $g_2(\xi)$ is non-differentiable will therefore determine $t(M_1)$. Subtracting the term in $g_2$ coming from $t(M_1)$, we repeat the process and find $t(M_2)$, $t(M_3)$, etc.

For $\Gamma$ a uniform lattice, the lattice length spectrum is related to the spectrum of the automorphic Laplacian in the following way. Let

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

denote the eigenvalues of $-\Delta$ and let $\{ \varphi_j \}$ be a (fixed) corresponding complete orthonormal set of eigenfunctions. For $\lambda \geq 0$, we set

$$B(\lambda) := \sum_{j: \lambda_j = \lambda} |\varphi_j(e_{n+1})|^2.$$  

We call the set

$$\{(\lambda_j, B(\lambda_j)) : B(\lambda_j) \neq 0\}$$

the pre-spectrum of $\Gamma$ at $e_{n+1}$. Then, using Selberg’s pre-trace formula (see, e.g., [5, Ch. XI Sect. 2]) in a way similar to the one in the proof of Huber’s theorem (see [4, Thm. 9.2.9]), one shows that the pre-spectrum determines and is determined by the lattice length spectrum: If two uniform hyperbolic lattices have the same pre-spectrum, then the right-hand sides of their pre-trace formulas will be the same for every choice of test function, and if they have the same lattice length spectra, then the left-hand sides of their pre-trace formulas will be the same for every choice of test function.

We summarize the above discussion as follows.

**Theorem 7.1.** Two uniform hyperbolic lattices in $G$ have the same pair correlation densities and covolumes if and only if they have the same lattice length spectra if and only if they have the same pre-spectra at $e_{n+1}$.

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*Note that we get the same value for $g_2(0)$ as Kelmer and Kontorovich, even though our normalization of the pair correlation function is slightly different from the one in [15].*
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