On Reverse Laplacian Energy of a Graph

Gowtham Kalkere Jayanna 1

1 Department of Mathematics, University College of Science, Tumkur University, Tumakuru, Karnataka State, India
* Correspondence: gowtham_k_j@yahoo.com;

Scopus Author ID 57226685931

Received: 23.09.2021; Accepted: 15.11.2021; Published: 9.01.2022

Abstract: Let G be a simple undirected graph with n vertices and m edges. The Laplacian matrix $L(G)$ of graph G is defined as $L(G) = (l_{ij})$, where $(l_{ij})$ is equal to $−1$ if $v_i$ and $v_j$ are adjacent, 1 if $v_i$ and $v_j$ are not adjacent and $d(v_i)$ if $i = j$, where $d(v_i)$ is the vertex degree of $v_i$. This paper defines the reverse Laplacian matrix, reverse Laplacian energy and finds the same for some standard graphs. Further, we calculate upper for reverse Laplacian energy.

Keywords: reverse vertex degree; reverse Laplacian matrix; reverse Laplacian energy.

© 2022 by the authors. This article is an open-access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Let $G$ be a simple graph with $n$ vertices and $m$ size. Let $\Delta(G)$ denote the maximum degree among the vertices of $G$. The reverse vertex degree of a vertex $v_i$ in $G$ is defined as, $c_{v_i} = \Delta(G) - d(v_i) + 1$, where $d(v_i)$ is the degree of vertex $v_i$. Let $A = (a_{ij})$ be the adjacency matrix of the graph $G$. In 1978 [1] I. Gutman introduced the concept of energy of a graph. If $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $A$, assumed in non-increasing order. Then, the energy $E(G)$ of the graph $G$ is defined to be the sum of the absolute values of the eigenvalues of $G$. That is, $E(G) = \sum_{i=1}^{n} |\mu_i|$.

In 2006,[2] I. Gutman and B. Zhou defined the Laplacian energy of a graph $G$. Let $G$ be a graph with $n$ vertices and $m$ size. The Laplacian matrix of the graph $G$, denoted by $L = (l_{ij})$, is a $n \times n$ matrix whose elements are defined as, $l_{ij} = -1$ if $v_i$ and $v_j$ are adjacent, 1 if $v_i$ and $v_j$ are not adjacent, $d(v_i)$ if $i = j$. If $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $L(G)$, then, Laplacian energy $LE(G)$ is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

The basic properties, upper and lower bounds for Laplacian energy, have been found in the literature[3-8]. Motivated by the definition of Laplacian matrix and reverse vertex degree, we define reverse Laplacian matrix and reverse Laplacian energy of a graph. We found reverse Laplacian energy for some standard graphs and further found bounds for the graph $G$.

Let $G$ be a graph with $n$ vertices and $m$ size. Then, the reverse Laplacian matrix $L_R(G)$ is defined as $L_R(G) = (l_{r_{ij}})$ where, $l_{r_{ij}} = -1$ if $v_i$ and $v_j$ are adjacent, 1 if $v_i$ and $v_j$ are not adjacent, $c(v_i)$ if $i = j$. If $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $L_R(G)$, then, reverse Laplacian energy $EL_R(G)$ is defined as $EL_R(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.
2. Materials and Methods

In literature, the reverse vertex degree of a vertex \( v_i \) in \( G \) is defined as, \( c_{v_i} = \Delta(G) - d(v_i) + 1 \), where \( d(v_i) \) is a degree of the vertex \( v_i \)\(^9\) and found many results in \([12-13]\). The energy of a graph plays an important role in chemistry. Based on this, several authors have worked on computing the polynomials of various chemical graphs and nanostructures, thereby finding their respective indices\(^{14-15}\).

Likewise, we defined the new energy of the graph called reverse Laplacian energy and found some results.

3. Results and Discussion

3.1. Reverse Laplacian energy.

3.1.1. Theorem 1.

Let \( G \) be a simple graph with \( n \) vertices and \( m \) edges. If \( \mu_1, \mu_2 \ldots \mu_n \) are the eigenvalues of the Reverse Laplacian matrix \( L_R(G) \), then

\[
\text{i. } \sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} c_{v_i} \\
\text{ii. } \sum_{i=1}^{n} \mu_i^2 = 2|E| + \sum_{i=1}^{n} c_{v_i}^2
\]

Proof:

(i) Since the sum of eigenvalues of the matrix \( L_R(G) \) is the trace of the matrix \( L_R(G) \). But from the definition, the principle diagonal elements are reverse vertex degree \( c_{v_i} \), \( i = 1,2,3,\ldots n \). So we have, \( \sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} l_{r_{ii}} = \sum_{i=1}^{n} c_{v_i} \)

(ii) Similar to above, the sum of the square of the eigenvalues of the matrix \( L_R(G) \) is same as trace of \([L_R(G)]^2\). Therefore we have,

\[
\sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} l_{r_{ij}} l_{r_{ji}} \\
= \sum_{i \neq j} l_{r_{ij}} l_{r_{ji}} + \sum_{i=1}^{n} l_{r_{ii}^2} \\
= 2 \sum_{1 \leq i < j \leq n} l_{r_{ij}}^2 + \sum_{i=1}^{n} l_{r_{ii}^2} \\
= 2|E| + \sum_{i=1}^{n} c_{v_i}^2
\]

3.1.2. Theorem 2.

For \( n \geq 3 \), the reverse Laplacian energy of the star graph \( S_n \) is \( EL_R(S_n) = n^3 - 4n^2 + 8n - 4 \)

Proof:

Consider the Star graph \( S_n \) with vertex \( v_1, v_2, \ldots v_n \). Then, the maximum reverse degree matrix of the graph \( S_n \)
Characteristic polynomial is
\[
1 - \mu & -1 & -1 & \ldots & -1 \\
-1 & n-1 & 0 & \ldots & 0 \\
-1 & 0 & n-1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \ldots & n-1 \\
\]

Characteristic equation is \((\mu)(\mu - (n-1))^{n-2}(\mu - n) = 0\). The eigenvalues of reverse Laplacian matrix \(L_R(G)\) of the graph are:
\[
\mu = 0 \text{ (one times), } \mu = (n-1)\text{ (n-2 times) }, \mu = n\text{ (one time)}
\]
Number of vertices=\(n\), Number of edges=\(n-1\)
\[
\therefore \text{ Average degree}=\frac{2(n-1)}{n}
\]
Hence the reverse Laplacian energy is,
\[
EL_R(S_n) = \left| \begin{array}{c} 0 - \frac{2(n-1)}{n} \\ (1) + (n-1) - \frac{2(n-1)}{n} \\ (n-2) + n - \frac{2(n-1)}{n} \end{array} \right|
\]
\[
= \frac{2(n-1)}{n} + \frac{n^2 - 3n + 2}{n} (n-2) + \frac{n^2 - 2n^2}{n}
\]
\[
= n^3 - 4n^2 + 8n - 4
\]

3.1.3. Theorem 3.

For \( n \geq 3 \), the reverse Laplacian energy of the Complete graph \(K_n\) is \(EL_R(S_n) = n^2 - 2n\)

Proof:
Consider the Complete graph \(K_n\) with vertex \(v_1, v_2, \ldots v_n\). Then, the reverse Laplacian matrix of the graph \(K_n\)
\[
M_R(K_n) = \begin{bmatrix}
1 & -1 & -1 & \ldots & -1 \\
-1 & 1 & -1 & \ldots & -1 \\
-1 & -1 & 1 & \ldots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \ldots & 1 \\
\end{bmatrix}
\]

The characteristic polynomial is
\[
1 - \mu & -1 & -1 & \ldots & -1 \\
-1 & 1 - \mu & -1 & \ldots & -1 \\
-1 & -1 & 1 - \mu & \ldots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \ldots & 1 - \mu \\
\]

The characteristic equation is \((\mu - 2)^{n-1}(\mu - (n-2)) = 0\). The eigenvalues of the maximum reverse degree matrix \(L_R(G)\) of the graph are:
\[
\mu = 2 \text{ (n - 1 times), } \mu = -(n - 2)\text{ (one time)}
\]
Number of vertices=\(n\), Number of edges=\(nC_2 = \frac{n(n-1)}{2}\)
\[
\therefore \text{ Average degree}=\frac{2m}{n} = \frac{2(n-2)}{n} = n - 1
\]
Hence the reverse Laplacian energy is,
\[ EL_R(K_n) = |2 - (n - 1)|((n - 1) + |-(n - 2) - (n - 1)| \]
\[ = |n - 3|(n - 1) + 2n - 3 \]
\[ = n^2 - 2n \]

3.1.4. Theorem 4.

For \( n \geq 3 \), the reverse Laplacian energy of the complete bipartite graph \( K_{n,n} \) is
\[ EL_R(K_{n,n}) = 2n^2 - 2n + 2 \]

Proof:

Consider the complete bipartite graph \( K_{n,n} \) with vertex \( v_1, v_2, \ldots, v_n \). Then, the reverse Laplacian matrix of the graph \( K_{n,n} \) is given by,
\[
M_R(K_{n,n}) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\
0 & 1 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\
0 & 0 & 1 & \cdots & 0 & -1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 - \mu & -1 & -1 & \cdots & -1 \\
-1 & 1 - \mu & -1 & \cdots & 1 & 1 - \mu & 0 & \cdots & 0 \\
-1 & -1 & -1 & \cdots & -1 & 0 & 1 - \mu & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & 1 - \mu & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & 0 & 1 - \mu \\
\end{bmatrix}_{(2n \times 2n)}
\]

The characteristic polynomial is
\[
1 - \mu & 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\
0 & 1 - \mu & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\
0 & 0 & 1 - \mu & \cdots & 0 & -1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 - \mu & -1 & -1 & \cdots & -1 \\
-1 & 1 - \mu & -1 & \cdots & 1 & 1 - \mu & 0 & \cdots & 0 \\
-1 & -1 & -1 & \cdots & -1 & 0 & 1 - \mu & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & 1 - \mu & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & 0 & 1 - \mu \\
\end{bmatrix}
\]

The characteristic equation is \((\mu - 1)^{2n-2}(\mu - (n + 1))(\mu + (n - 1)) = 0\)

The eigenvalues of the maximum reverse degree matrix \( M_R(K_{n,n}) \) of the graph are:
\[
\mu = 1 \quad (2n-2 \text{ times}), \quad \mu = (n + 1) \quad (\text{one time}), \quad \mu = -(n - 1) \quad (\text{one time})
\]

Number of vertices=2n, Number of edges=n²
\[
\therefore \text{Average degree} = \frac{2m}{n} = \frac{2n^2}{2n} = n
\]

Hence the reverse Laplacian energy is,
\[
EL_R(K_{n,n}) = |1 - (n)|(2n - 2) + |(n + 1) - (n)|(1) + |-(n - 1) - (n)|
\]
\[ = 2n^2 - 2n + 2 \]

Theorem 3.1.5.

For \( n \geq 3 \), the reverse Laplacian energy of the Crown graph \( S_n^0 \) is
\[ EL_R(S_n^0) = 2n^2 - 4n + 2 \]

Proof:

Consider the crown graph \( S_n^0 \) with vertex \( v_1, v_2, \ldots, v_n \). Then, the reverse Laplacian matrix of the graph \( S_n^0 \)
The reverse Laplacian matrix $M_R(S^n_0)$ of the graph are:

\[
M_R(S^n_0) = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 \\
0 & 1 & 0 & \ldots & 0 & -1 & 0 & -1 & \ldots & -1 \\
0 & 0 & 1 & \ldots & 0 & -1 & -1 & 0 & \ldots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & -\mu & -1 & -1 & -1 & \ldots & 0 \\
0 & -1 & -1 & \ldots & -1 & 1 - \mu & 0 & 0 & \ldots & 0 \\
-1 & 0 & -1 & \ldots & -1 & 0 & 1 - \mu & 0 & \ldots & 0 \\
-1 & -1 & 0 & \ldots & -1 & 0 & 0 & 1 - \mu & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 - \mu
\end{bmatrix}
\]

Characteristic polynomial is
\[
1 - \mu & 0 & 0 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 \\
0 & 1 - \mu & 0 & \ldots & 0 & -1 & 0 & -1 & \ldots & -1 \\
0 & 0 & 1 - \mu & \ldots & 0 & -1 & -1 & 0 & \ldots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 - \mu & -1 & -1 & -1 & \ldots & 0 \\
0 & -1 & -1 & \ldots & -1 & 1 - \mu & 0 & 0 & \ldots & 0 \\
-1 & 0 & -1 & \ldots & -1 & 0 & 1 - \mu & 0 & \ldots & 0 \\
-1 & -1 & 0 & \ldots & -1 & 0 & 0 & 1 - \mu & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 - \mu
\]

The characteristic equation is $\mu^{(n-1)}(\mu - 2)^{(n-1)}(\mu + (n - 2))(\mu - n) = 0$

The eigenvalues of reverse Laplacian matrix $M_R(S^n_0)$ of the graph are:

$\mu = 0$ ((n-1) times), $\mu = 2$ ((n-1) times), $\mu = -(n-2)$ (one time), $\mu = n$ (one time)

Number of vertices=2n, Number of edges=n(n - 1)

\[
\therefore \text{Average degree} = \frac{2m}{n} = \frac{2n(n-1)}{2n} = n - 1
\]

Hence the reverse Laplacian energy is,

\[
EL_R(S^n_0) = |0 - (n - 1)|(n - 1) + |2 - (n - 1)|(n - 1) + |-(n - 2) - (n - 1)|(1) + |n - (n - 1)|(1)
\]

\[
= (n - 1)^2 + (n - 3)(n - 1) + (2n - 3) + 1
\]

\[
= 2n^2 - 4n + 2
\]

3.2. Bounds for the reverse Laplacian energy.

In this section, lower and upper bounds for the reverse Laplacian energy of graphs are calculated. To do this, we need the following known inequality.

3.2.1 Lamma (Cauchy–Schwarz inequality).

For all sequences of real number $a_i$ and $b_i$, $(\sum_{i=1}^{n} a_i b_i)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$, equality holds if and only if $a_i = kb_i$ for a non zero constant $k \in \mathbb{R}$.

3.2.2. Theorem (upper bound).

Let $G$ be a simple graph with $n$ vertices, $m$ size. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the reverse Laplacian matrix $L_R(G)$ then, $EL_R(G) \leq \sqrt{n \left(2|E| + \sum_{i=1}^{n} c_i^2\right) + 2|E|}$

Proof:

Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the Laplacian edge dominating matrix $L_R(G)$.
In Cauchy–Schwarz inequality, put \( a_i = 1 \) and \( b_i = |\mu_i| \) we get,
\[
\left( \sum_{i=1}^{n} |\mu_i| \right)^2 \leq \left( \sum_{i=1}^{n} 1 \right) \left( \sum_{i=1}^{n} |\mu_i|^2 \right)
\]
Using Theorem 3.1.1, we have
\[
\left( \sum_{i=1}^{n} |\mu_i| \right)^2 \leq n \left( 2|E| + \sum_{i=1}^{n} c_{\mu_i}^2 \right)
\]
\[
\Rightarrow \left( \sum_{i=1}^{n} |\mu_i| \right) \leq \sqrt{n \left( 2|E| + \sum_{i=1}^{n} c_{\mu_i}^2 \right)}
\]
By Triangle inequality, \(|\mu_i - \frac{2m}{n}| \leq |\mu_i| + | \frac{2m}{n} | \) \( \forall i = 1, 2, 3, \ldots n \)
\[
\Rightarrow \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}| \leq \sum_{i=1}^{n} |\mu_i| + \sum_{i=1}^{n} | \frac{2m}{n} |
\]
\[
EL_R(G) \leq \sqrt{n \left( 2|E| + \sum_{i=1}^{n} c_{\mu_i}^2 \right) + 2m}
\]
\[
\Rightarrow EL_R(G) \leq \sqrt{n \left( 2|E| + \sum_{i=1}^{n} c_{\mu_i}^2 \right) + 2|E|}
\]
3.2.3. Theorem (Upper bound).
Let \( G \) be a simple graph with \( n \) vertices, \( m \) size. Let \( \mu_1, \mu_2, \ldots \mu_n \) be the eigenvalues of the reverse Laplacian matrix \( L_R(G) \). Then, \( EL_R(G) \leq \sqrt{2|E| \left( n - 2 \sum_{i=1}^{n} c_{\mu_i} \right) + 4|E|^2 + n \sum_{i=1}^{n} c_{\mu_i}^2} \)
Proof:
Let \( \mu_1, \mu_2, \ldots \mu_n \) be the eigenvalues of the Laplacian edge dominating matrix \( L_R(G) \).
In Cauchy–Schwarz inequality, put \( a_i = 1 \) and \( b_i = |\mu_i - \frac{2m}{n}| \) we get,
\[
\left( \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| \right)^2 \leq \left( \sum_{i=1}^{n} 1 \right) \left( \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|^2 \right)
\]
\[
(EL_R(G))^2 \leq n \left( \sum_{i=1}^{n} \mu_i^2 + \sum_{i=1}^{n} \frac{4m^2}{n^2} - \frac{4m}{n} \sum_{i=1}^{n} \mu_i \right)
\]
\[
= n \left( 2|E| + \sum_{i=1}^{n} \frac{c_{\mu_i}^2}{n} + \frac{4m^2}{n} - \frac{4m}{n} \sum_{i=1}^{n} c_{\mu_i} \right)
\]
\[
= 2n|E| + n \sum_{i=1}^{n} c_{\mu_i}^2 + 4m^2 - 4m \sum_{i=1}^{n} c_{\mu_i}
\]
\[
\Rightarrow EL_R(G) \leq \sqrt{2n|E| + n \sum_{i=1}^{n} c_{\mu_i}^2 + 4|E|^2 - 4|E| \sum_{i=1}^{n} c_{\mu_i}}
\]
\[
\Rightarrow EL_R(G) \leq \sqrt{2|E| \left( n - 2 \sum_{i=1}^{n} c_{\mu_i} \right) + 4|E|^2 + n \sum_{i=1}^{n} c_{\mu_i}^2}
\]
3.2.4. Theorem.

Let $G$ be a graph with $n$ vertices, $m$ size. If the sum of the absolute eigenvalues of $L_R(G)$ is rational, then it must be an integer, satisfying the following relation:

$$\sum_{i=1}^{n} |\mu_i| \equiv \sum_{i=1}^{n} c_{v_i} \pmod{2}$$

Proof:

Let $\mu_1, \mu_2, ..., \mu_n$ be the eigenvalues of the reverse Laplacian matrix $L_R(G)$ such that $\mu_1, \mu_2, ..., \mu_t$ are positive and rest are non-positive. we have,

$$\sum_{i=1}^{n} |\mu_i| = (\mu_1 + \mu_2 + ... + \mu_t) - (\mu_{(t+1)} + \mu_{(t+2)} + ... + \mu_n)$$

$$= 2(\mu_1 + \mu_2 + ... + \mu_t) - (\mu_1 + \mu_2 + ... + \mu_n)$$

$$= 2(\mu_1 + \mu_2 + ... + \mu_t) - \sum_{i=1}^{n} \mu_i$$

By the result of, the partial sum $\mu_1 + \mu_2 + ... + \mu_t$ eigenvalues of a matrix whose characteristic polynomial has integer coefficients. If $\sum_{i=1}^{n} |\mu_i|$ is rational then $\mu_1 + \mu_2 + ... + \mu_t$ is rational, and hence it must be an integer.

$$\therefore \sum_{i=1}^{n} |\mu_i| \equiv \sum_{i=1}^{n} c_{v_i} \pmod{2}$$

4. Conclusions

In this paper, we had defined reverse Laplacian matrix, reverse Laplacian energy and studied the same for some standard graphs. We found upper for reverse Laplacian energy. Further, we proved if the sum of the absolute eigenvalues of $L_R(G)$ is rational then it must be an integer.

Funding

This research received no external funding.

Acknowledgments

The author would like to thank Dr. B.J. Gireesha, Department of Mathematics, Kuvempu University, who motivated and inspired me to research.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Gutman, I. The Energy of a Graph. Ber. Math. Stat. Sekt. Forschungszent Graz 1978, 103, 1-22.
2. Gutman, B. Z. Laplacian energy of a graph. Lin. Algebra Appl. 2006, 414, 29-37.
3. T. Aleksić, Upper bounds for Laplacian energy of graphs. MATCH Commun. Math. Comput. Chem. 2008, 60, 435-439.
4. Fath-Tabar, G. H., Ashrafi, A. R., Gutman, I. Note on Laplacian energy of graphs. Bull. Acad. Serbe Sci. Arts (Cl. Math. Natur.) 2008, 137, 1-10.
5. M. Fiedler, Additive compound matrices and an inequality for eigenvalues of symmetric stochastic matrices, Czech. Math. J., 1974, 24, 392-402.
6. Gutman, N. M. M. de Abreu; Vinagre, C. T. M.; Bonifacio, A. S.; Radenković, S. Č. Relation between energy and Laplacian energy. *MATCH Commun. Math. Comput. Chem.* 2008, 59, 343-354.
7. Liu, J.; Liu, B. On relation between energy and Laplacian energy. *MATCH Commun. Math. Comput. Chem.* 2009, 61, 403-406.
8. Gowtham, K. J.; Narahari N. S. “On Somboor energy of graphs”, *Nanomaterials: Physics, Chemistry, Mathematics* 2021, 411-417.
9. Ediz, S.; Cananc, M. Reverse Zagreb indices of Cartesian product of graphs. *International Journal of Mathematics and Computer Science* 2016, 11, 51-58.
10. Kulli, V. R. A new multiplicative arithmetic-geometric index. *International Journal of Fuzzy Mathematical Archive* 2017, 12, 49-53.
11. Kulli, V. R. Multiplicative connectivity indices of certain nanotubes. *Annals of Pure and Applied Mathematics* 2016, 12, 169-176, https://dx.doi.org/10.22457/apam.v12n2a8.
12. Kulli, V. R. Reverse Zagreb and reverse hyper-Zagreb indices and their polynomials of rhombus silicate networks. *Annals of Pure and Applied Mathematics* 2018, 16, 47-51.
13. Kulli, V. R. Computing F-reverse index and F-reverse polynomial of certain networks. *International Journal of Mathematical and Applied Archive* 2018, 9, 27-33.
14. Virk, A. U. R.; Jhanggeer, M. N.; Rehman, M. A. Reverse Zagreb And Reverse Hyper-Zagreb Indices For Silicon Carbide $Si_nC_sI[r,s]$ and $Si_nC_sII[r,s]$. *Eng. Appl. Sci. Lett.* 2018, 1,37-50, https://doi.org/10.30538/psrp-ea2018.0010.
15. Gowtham, K. J.; Narahari, N. S.; Multiplicative Reverse Geometric-Arithmetic Indices and Arithmetic-Geometric of Silicate Network, *Biointerface applied research in chemistry* 2022, 12, 4192 – 4199.
16. Rodriguez, J. M.; Sánchez, J. L.; Sigarreta, J. M.; Tourís, E. Bounds on the Arithmetic-Geometric Index. *Symmetry* 2021, 13, 689, https://doi.org/10.3390/sym13040689.
17. Cananc, M.; Ediz, S.; Alaeiyan, M.; Farahani, M. R. On Ve-degree molecular properties of copper oxide. *Journal of Information and Optimization Sciences* 2020, 41, 949-957, https://doi.org/10.1080/02522667.2020.1747191.
18. Mukhtar, A.; Murtaza, R.; Rehman, S. U.; Usman, S.; Baig, A. Q. Computing the size of zero divisor graphs. *Journal of Information and Optimization Sciences* 2020, 41, 855-864, https://doi.org/10.1080/02522667.2020.1745378.
19. Bishan, W. Optimized model of energy industry chain considering low-carbon development mechanism. *Energy Sources, Part A: Recovery, Utilization, and Environmental Effects* 2020, 42, 2593-2602, https://doi.org/10.1080/15567036.2019.1612485.
20. Divnić, T.; Milivojević, M.; Pavliović, L. Extremal graphs for the geometric–arithmetic index with given minimum degree. *Discrete Applied Mathematics* 2014, 162, 386-390, https://doi.org/10.1016/j.dam.2013.08.001.
21. Wang, X.-L.; Liu, J.-B.; Jahanbani, A.; Siddiqui, M. K.; Rad, N. J.; Hasni, R. On Generalized Topological Indices of Silicon-Carbon. *Journal of Mathematics* 2020, 2020, 2128594, https://doi.org/10.1155/2020/2128594.
22. Ashrafi, A. R.; Nikzad, P. Connectivity index of the family of dendrimer nanostars, *Digest Journal of Nanomaterials and Bionanotechnologies* 2009, 4, 269-273.
23. Kulli, V. R.; Chaluvuraraj, B.; Boregowda, H. S. Some degree-based connectivity indices of Kulli cycle windmill graphs. *South Asian Journal of Mathematics* 2016, 6, 263-268.
24. Munir, M.; Nazeer, W.; Rafique, S.; Kang, S. M. M-Polynomial and Degree-Based Topological Indices of Polyhex Nanotubes. *Symmetry* 2016, 8, https://doi.org/10.3390/sym8120149.
25. Gao, W.; Younas, M.; Farooq, A.; Virk, A. U.; Nazeer, W. Some Reverse Degree-Based Topological Indices and Polynomials of Dendrimers. *Mathematics* 2018, 6, https://doi.org/10.3390/math6100214.
26. Kulli, V. R.; Gutman, I. Computation of Somboor indices of certain networks. *SSRG International Journal of Applied Chemistry* 2021, 8, 1-5, https://doi.org/10.14445/23939133/IJAC-V8I1P101.
27. Kulli, V. R. Inverse sum temperature index and multiplicative inverse sum temperature index of certain nanotubes.*International journal of recent scientific research* 2021, 12, 40635-40639, https://doi.org/10.24327/ijrsr.2021.1201.5734.
28. Kulli, V. Multiplicative connectivity status neighborhood indices. *International journal of science engineering and technology research* 2020,9, 59-68, https://doi.org/10.29121/ijersr.v9i12.2020.8.
29. Kulli, V., Multiplicative status neighborhood indices of graphs. *International journal of engineering sciences and research technology* 2020, 9, 110-120, https://doi.org/10.29121/ijesrt.v9i6.2020.18.
30. Hosoya, H. Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons. *Bull. Chem. Soc. Jpn.* 1971, 44, 2332-2339, https://doi.org/10.1246/bcsj.44.2332.

31. Asok, A.; Kureethara, J.V. QSPR Analysis of certain degree and eccentricity based topological indices and butane derivatives. *Rasayan Journal of Chemistry* 2019, 12, 464-470, https://dx.doi.org/10.31788/RJC.2019.1225093.

32. Shirakol, S.; Kalyanshetti, M.; Hosamani, S.M. QSPR analysis of certain distance based topological indices. *Applied Mathematics and Nonlinear Sciences* 2019, 4, 371-386, https://doi.org/10.2478/amns.2019.2.00032.

33. Javaid, M.; Ali, A.; Milovanović, I.; Milovanović, E. On the extremal cactus graphs for variable sum exdeg index with a fixed number of cycles. *AKCE International Journal of Graphs and Combinatorics* 2020, 17, 920-923, https://doi.org/10.1016/j.akcej.2019.08.007.

34. Baig, A.Q.; Naeem, M.; Gao, W.; Liu, J.-B. General fifth M-Zagreb indices and fifth M-Zagreb polynomials of carbon graphite. *Eurasian chemical communications* 2020, 2, 634-640, https://doi.org/10.33945/SAMI/ECC.2020.5.10.

35. Kulli, V.R. General fifth M-Zagreb indices and fifth M-Zagreb polynomials of PAMAM dendrimers. *International Journal of Fuzzy Mathematical Archive* 2017, 13, 99-103, https://doi.org/10.22457/ijfma.v13n1a10.