Inference based on Kotlarski’s Identity

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Abstract

This paper presents the nonparametric inference problem about the probability density function of a latent variable in the measurement error model with repeated measurements. We construct a system of linear complex-valued moment restrictions by Kotlarski’s identity, and then establish a confidence band for the density of the latent variable. Our confidence band controls the asymptotic size uniformly over a class of data generating processes, and it is consistent against all fixed alternatives. Simulation studies support our theoretical results.

Keywords: deconvolution, Hermite function, Kotlarski, measurement error, uniform confidence band.

1 Introduction

Empirical researchers are often interested in recovering features of unobserved variables in economic models. Kotlarski’s identity (Kotlarski, 1967) – see also Rao (1992) – is one of the most popular tools used to identify probability density functions of unobserved latent variables. Since its first introduction to econometrics by Li and Vuong (1998), Kotlarski’s identity has been widely used in economics when data admit repeated measurements. Examples of research topics that use Kotlarski’s identity include, but are not limited to, empirical auctions (e.g., Li, Perrigne, and Vuong 2002; Krasnokutskaya 2011), income dynamics (e.g., Bonhomme and Robin 2010), and labor economics (e.g., Cunha, Heckman, and Navarro 2005; Cunha, Heckman, and Schennach 2010; Bonhomme and Sauder 2011; Kennan and Walker 2011). In these applications, researchers are interested in identifying the probability density function $f_X$ of a latent variable $X$ among others. The variable $X$ of interest is not observed in data, but two measurements $(Y_1, Y_2)$ are available in data with additive independent errors, $U_1 = Y_1 - X$ and $U_2 = Y_2 - X$. Kotlarski’s identity is a nonparametric identifying restriction for the probability density function $f_X$ of $X$ implied by this setup.

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Existing econometric methods and econometric theories on Kotlarski’s identity focus on identification and consistent estimation of $f_X$ (e.g., Li and Vuong 1998; Schennach 2004; Bonhomme and Robin 2010). On the other hand, satisfactory inference methods for $f_X$ are missing in this literature – indeed some empirical papers implement nonparametric bootstrap without a theoretical guarantee. In light of the demands by the empirical researchers for an inference method and given the current unavailability of theoretically supported methods of inference, we propose a method of inference based on Kotlarski’s identity in this paper. Specifically, we develop confidence bands for $f_X$.

Our construction of confidence bands works as follows. First, we derive complex-valued moment restrictions based on Kotlarski’s identity. Second, we let the Hermite polynomial sieve (cf. Chen 2007) approximate unknown probability density functions. Third, for a given sieve dimension and for a given class for probability density functions, we compute a bias bound for the complex-valued moment restrictions, and slack the complex-valued moment restrictions by this bias bound. Fourth, applying Chernozhukov, Chetverikov, and Kato (2017), we compute the uniform norm of the self-normalized process of the slacked complex-valued moment restrictions as the test statistics for each point in a set of sieve coefficients. Fifth, inverting this test statistic in the spirit of Anderson and Rubin (1949) yields a confidence set of sieve approximations to possible probability density functions. Sixth, for a given sieve dimension and for a given class for probability density functions, we compute a bias bound for sieve approximations of probability density functions, and the desired confidence band is obtained by uniformly enlarging the set of sieve approximations by this bias bound.

The process of identifying $f_X$ in additive measurement error models is called deconvolution – for solving convolution integral equations. There are a number of existing papers on nonparametric inference in deconvolution. Bissantz, Dümbgen, Holzmann, and Munk (2007), Bissantz and Holzmann (2008) and van Es and Gugushvili (2008) develop uniform confidence bands for $f_X$ under the assumption of known error distributions. In most economic applications, however, it is not plausible to assume that the error distributions are known. More recently, Kato and Sasaki (2017a) and Adusumilli, Otsu, and Whang (2017) develop uniform confidence bands for $f_X$ and the distribution function, respectively, without assuming that the error distributions are known, but they both assume that at least one error distribution is symmetric. Kotlarski’s identity is a powerful device for new identification results which require neither the known error distribution assumption nor the symmetric error distribution assumption. This useful feature attracts many economic applications including those listed above, but no econometrician has developed a method of inference in this framework for twenty years ever since its first introduction by Li and Vuong (1998) until our present paper.

It is not surprising that such an inference method has been missing for long in the literature, given the technical difficulties of the problem. Deconvolution is an ill-posed inverse problem, and inference under this problem is known to be challenging – see Horowitz and Lee (2012); Hall and Horowitz (2013); Kato and Sasaki (2017a,b); Babai (2018); Chen and Christensen (2018) for existing papers developing confidence bands in ill-posed inverse problems for example. It appears that Kotlarski’s identity entails an even more sophisticated form of an inverse problem to solve. As such, in the present paper, we chose the path of not solving the difficult inverse problem. Instead, we take a robust inference approach à la Anderson and Rubin (1949), and directly work with the moment restrictions based on Kotlarski’s identity. A positive side product of taking this approach is that we do not need to assume the non-vanishing characteristic functions (i.e., we...
do not need the completeness), which is commonly assumed for nonparametric identification or inversion.

It is also worth mentioning that we chose to use the Hermite polynomial sieve among other sieves in this paper. The Hermite polynomial sieve has been in fact already known in the literature to be useful to approximate “smooth density with unbounded support” (Chen, 2007) – also see her discussion of Gallant and Nychka (1987) therein. In addition to this known advantage, we also find this sieve particularly useful for the deconvolution problem. Note that the deconvolution problem involves applications of the Fourier transform operation and the inverse Fourier transform operation. To our convenience, the Hermite functions are eigen-functions of the Fourier transform operator. While we deal with simultaneous restrictions in terms of density and characteristic functions, we can use the Hermite polynomial sieve to approximate both the density and characteristic functions without having to apply the Fourier transform or the Fourier inverse because of the eigen-function property. This convenient property saves computational time and resources as costly numerical integration within each iteration of a numerical optimization routine would be necessary if any other sieve were used. Furthermore, we find that a couple of properties of the Hermite functions (namely the Schrödinger equation for a harmonic oscillator and a pair of recursive equations) can be exploited to obtain informative bias bounds of the Hermite polynomial sieve, which in turn contributes to informative inference we establish in this paper.

The rest of the paper is organized as follows. Section 2 derives complex-valued moment restrictions based on Kotlarski’s identity. Section 3 presents how to compute the confidence band. Section 4 presents asymptotic properties of the confidence band. Section 5 discusses practical considerations. Section 6 illustrates simulation studies. The paper concludes in Section 7. All mathematical derivations and details are delegated to the appendix.

2 Complex-Valued Moment Restrictions

Consider the repeated measurement model

\[
\begin{align*}
Y_1 &= X + U_1 \\
Y_2 &= X + U_2
\end{align*}
\]

(1)

where \(Y_1\) and \(Y_2\) are observed, but none of \(X\), \(U_1\), or \(U_2\) is observed. We equip this model with the following assumption.

**Assumption 1.**

(i) \(X, U_1,\) and \(U_2\) are continuous random variables with finite first moments, and \(U_1\) has mean zero.

(ii) \(X, U_1,\) and \(U_2\) are mutually independent.

This assumption is standard in the literature on identification and estimation based on Kotlarski’s identity (e.g., Li and Vuong, 1998). In fact, this assumption is weaker than the standard assumption. Specifically, we do not invoke an identification assumption since the object of our interest is inference as opposed to identification – see Remark 1 ahead. Throughout, we let \(i = \sqrt{-1}\) denote the imaginary unit. For the set of absolutely integrable functions, \(L^1\), we define the transformation \(F\) on \(L^1\) by \(|Ff|(t) = \int_{-\infty}^{\infty} e^{itz} f(x) dx\), and its inverse transform is \(F^{-1}\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itz} \phi(t) dt\) – see Folland (2007). In light of Assumption 1 (i), we let \(f_X, f_{U_1},\) and \(f_{U_2}\) denote the density functions of \(X, U_1,\) and \(U_2,\) respectively. Further, we denote the characteristic functions of them by \(\phi_X = Ff_X, \phi_{U_1} = Ff_{U_1},\) and \(\phi_{U_2} = Ff_{U_2}.\)

We state the following result providing a restriction based on which inference will be conducted.
Theorem 1 (Complex-Valued Moment Restrictions). For every joint distribution \( P \) of \( (Y_1, Y_2) \) satisfying Assumption 7 for \( \mathcal{I} \),
\[
\mathbb{E}_P [(iY_1 \phi_X(t) - \phi'_X(t)) \exp(iY_2)] = 0
\]
holds for every real \( t \), where \( \mathbb{E}_P \) is the expectation under \( P \).

A proof is provided in Appendix A.1

Remark 1. Taking additional steps beyond the claim in Theorem 7 will lead us to the identification of the density function \( f_X \) under additional assumptions, namely the invertibility or non-vanishing characteristic functions – see Li and Vuong (1998). For the purpose of inference, however, it is not essential to solve the inverse problem, and thus we only use the moment condition (2).

3 Construction of the Confidence Band

Our objective is to construct a confidence band for the probability density function \( f_X \) of \( X \) on an interval \( I \subset \mathbb{R} \). The construction procedure is based on the complex-valued moment restriction (2). Throughout, we focus on the set of probability density functions given by
\[
\mathcal{L} \subset \{ f \in \mathcal{L}^1 \cap \mathcal{L}^2 : f \text{ is a probability density function and } \mathcal{F} f \in \mathcal{L}^1 \}.
\]
For this set of candidate probability density functions, we use \( \mathcal{L}^1 \) for applying the Fourier transform and the inverse, whereas \( \mathcal{L}^2 \) is used to approximate \( \mathcal{L} \) by an orthonormal basis \( \Psi = \{ \psi_j : j = 0, 1, \ldots \} \) of \( \mathcal{L}^2 \) – see Section 2 for the example of the Hermite basis.

We use a \((q + 1)\)-dimensional sieve basis \( \{ \psi_0, \ldots, \psi_q \} \subset \Psi = \{ \psi_j : j = 0, 1, \ldots \} \), with \( \psi_j \in \mathcal{L}^1 \cap \mathcal{L}^2 \) and \( \mathcal{F} \psi_j \in \mathcal{L}^1 \) for each \( j \in \mathbb{N} \), to approximate the probability density function \( f_X \). Let \( \Theta^{q+1} \subset \mathbb{R}^{q+1} \) be a compact set, and write \( \psi = (\psi_0, \ldots, \psi_q)' \). With a uniform tolerance level \( \eta > 0 \), each \( f \in \mathcal{L} \) is approximated by \( x \mapsto \psi(x)^T \theta \) for some \( \theta = (\theta_0, \ldots, \theta_q)' \in \Theta^{q+1} \), i.e., \( \sup_{x \in I} |f(x) - \psi(x)^T \theta| \leq \eta \). The set of values of the sieve coefficients \( \theta \in \Theta^{q+1} \) approximating a probability density function \( f \in \mathcal{L} \) in this manner is denoted by
\[
\mathcal{B}_{q+1, \eta}(f) = \left\{ \theta \in \Theta^{q+1} : \sup_{x \in I} |f(x) - \psi(x)^T \theta| \leq \eta \right\}.
\]

We next incorporate the complex-valued moment restrictions (2) in this sieve framework. For every function \( \psi \in \mathcal{L} \) and for every frequency \( t \in \mathbb{R} \), define
\[
R_{\psi, t}(y_1, y_2) = -\cos(t y_2)(y_1 \text{Im}(\phi(t)) + \text{Re}(\phi^{(1)}(t))) - \sin(t y_2)(y_1 \text{Re}(\phi(t)) - \text{Im}(\phi^{(1)}(t))) \quad \text{and} \quad (4)
\]
\[
I_{\psi, t}(y_1, y_2) = \cos(t y_2)(y_1 \text{Re}(\phi(t)) + \text{Im}(\phi^{(1)}(t))) - \sin(t y_2)(y_1 \text{Im}(\phi(t)) - \text{Re}(\phi^{(1)}(t))), \quad (5)
\]
where \( \phi = \mathcal{F} \psi \). Further, stack these functions across \( \psi \in \{ \psi_0, \ldots, \psi_q \} \) to define the vector-valued functions
\[
R_t(y_1, y_2) = (R_{\psi_0, t}(y_1, y_2), \ldots, R_{\psi_q, t}(y_1, y_2))' \quad \text{and} \quad (6)
\]
\[
I_t(y_1, y_2) = (I_{\psi_0, t}(y_1, y_2), \ldots, I_{\psi_q, t}(y_1, y_2))'.
\]

With these notations, we now represent the complex-valued moment restrictions (2) for the sieve approxi-
Properties of the Confidence Band

where $Y$ which the joint distribution of $(Y_1, Y_2)$ defined in (3) is given by

$$
\left( Y_1, Y_2 \right) \sim \left( \text{Normal} \left( 0, \Sigma \right) \right)
$$

for all $t \in [-T, T]$ for $T \in (0, \infty)$, where $\delta(t) > 0$ is the tolerance level of sieve approximation error for each $t \in [-T, T]$.

Our construction of the confidence band is based on a test statistic that quantifies the extent of deviation from the moment inequalities (6). To construct a feasible test statistic, we use a grid $\{t_1, \ldots, t_L\} \subset [-T, T]$ of $L$ frequencies. Define the test statistic by

$$
T(\theta) = \sqrt{n} \max_{1 \leq i \leq L} \left\{ \frac{\left| \mathbb{E}_n[R_{t_i}(Y_1, Y_2)]^T \theta \right| - \delta(t_i)}{\sqrt{\theta^T \mathbb{V}_n[R_{t_i}(Y_1, Y_2)] \theta}} , \frac{\left| \mathbb{E}_n[I_{t_i}(Y_1, Y_2)]^T \theta \right| - \delta(t_i)}{\sqrt{\theta^T \mathbb{V}_n[I_{t_i}(Y_1, Y_2)] \theta}} \right\}
$$

for each $\theta \in \Theta_{q+1}$. Let $\alpha \in (0, 1/2)$ be fixed. We define the critical value $c(\alpha, \theta)$ of this statistic $T(\theta)$ by the conditional $(1 - \alpha)$-th quantile of the multiplier process

$$
\sqrt{n} \max_{1 \leq i \leq L} \left\{ \frac{\left| \mathbb{E}_n[\epsilon(R_{t_i}(Y_1, Y_2) - \mathbb{E}_n[R_{t_i}(Y_1, Y_2)])]^T \theta \right|}{\sqrt{\theta^T \mathbb{V}_n[R_{t_i}(Y_1, Y_2)] \theta}} , \frac{\left| \mathbb{E}_n[\epsilon(I_{t_i}(Y_1, Y_2) - \mathbb{E}_n[I_{t_i}(Y_1, Y_2)])]^T \theta \right|}{\sqrt{\theta^T \mathbb{V}_n[I_{t_i}(Y_1, Y_2)] \theta}} \right\}
$$

given the data, where $\epsilon_1, \ldots, \epsilon_n$ are independent standard normal random variables independent of the data. As a more conservative yet simpler alternative following Chernozhukov et al. (2017 eq. (19)), we may define the critical value as

$$
c(\alpha) = \frac{\Phi^{-1}(1 - \alpha/4L)}{1 - \Phi^{-1}(1 - \alpha/4L)^2/n},
$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution. Our confidence band for the density function of $X$ is given by the $I$-restriction of

$$
\mathcal{C}_n(\alpha) = \{ f \in \mathcal{L} : T(\theta) \leq c(\alpha, \theta) \text{ for some } \theta \in \mathcal{B}_{q+1, \eta}(f) \},
$$

where $\mathcal{B}_{q+1, \eta}(f)$ is defined in (3).

4 Properties of the Confidence Band

In this section, we present theoretical properties of the confidence band (7). Let $\mathcal{P}$ denote a known space to which the joint distribution of $(Y_1, Y_2)$ belongs. For every $P \in \mathcal{P}$, define the identified set

$$
\mathcal{L}_0(P) = \{ f \in \mathcal{L} : \phi = \mathcal{F}f \text{ and } \mathbb{E}_P [ (iY_1 \phi(t) - \phi'(t)) \exp(itY_2) ] = 0 \text{ for every } t \in \mathbb{R} \}
$$

as the set of density functions for which the complex-valued moment restriction (2) is satisfied. Furthermore, we define the sieve-approximation counterpart of $\mathcal{L}_0(P)$ by

$$
\mathcal{L}_0^\delta(P) = \left\{ f \in \mathcal{L}_0(P) : \inf_{\theta_\ast \in \mathcal{B}_{q+1, \eta}(f)} \max_{1 \leq i \leq L} \left( \max\{ ||\mathbb{E}_P[R_{t_i}(Y_1, Y_2)]^T \theta_\ast|| , ||\mathbb{E}_P[I_{t_i}(Y_1, Y_2)]^T \theta_\ast|| \right) - \delta(t_i) \right\} < 0 \right\},
$$

where $\mathcal{B}_{q+1, \eta}(f)$ is defined in (3). Here, the infimum over the empty set is understood to be the infinity.

The current section is structured as follows. First, we establish the size control for the confidence band
to contain the approximation set \( \mathcal{L}_0(P) \) in Section 4.1. Second, we establish the containment of the identified set by the approximation set (i.e., \( \mathcal{L}_0(P) \subset \mathcal{L}_0^*(P) \)) in Section 4.2. These two pieces of the results together show the validity of the confidence band to contain the identified set \( \mathcal{L}_0(P) \). Lastly, Section 4.3 presents power properties with local alternatives. Throughout, we assume to observe \( n \) i.i.d. copies of \( (Y_1, Y_2) \) drawn from \( P \in \mathcal{P} \).

### 4.1 Size Control

We make the following assumption for a uniform size control.

**Assumption 2.** (i) \( \mathbb{E}_P[Y_1^2] < \infty \). (ii) There are positive numbers \( c_1 < 1/2 \) and \( C_1 \) such that

\[
(M_{L,q}^2(\theta, P) \vee M_{L,q}^2(\theta, P) \vee B_{L,q}(\theta, P))^2 \log^{7/2}(4Ln) \leq C_1 n^{1/2 - c_1}
\]

for all \( P \in \mathcal{P} \) and \( \theta \in \Theta^{q+1} \), where

\[
M_{L,q,k}(\theta, P) = \max_{1 \leq i \leq L} \max \left\{ \mathbb{E}_P \left[ \frac{(R_{ij}(Y_1, Y_2) - \mathbb{E}_P[R_{ij}(Y_1, Y_2)])^2}{1/k} \right] \right\}
\]

and

\[
B_{L,q}(\theta, P) = \mathbb{E}_P \max_{1 \leq i \leq L} \max \left\{ \frac{(R_{ij}(Y_1, Y_2) - \mathbb{E}_P[R_{ij}(Y_1, Y_2)])^2}{1/k} \right\}
\]

**Theorem 2** (Size Control). Suppose that Assumption 3 holds. Then, there are positive constants \( c \) and \( C \) such that

\[
\inf_{P \in \mathcal{P}} \inf_{f \in \mathcal{L}_0^*(P)} \mathbb{P}(f \in \mathcal{C}_n(\alpha)) \geq 1 - \alpha - Cn^{-c}.
\]

A proof is provided in Appendix A.2. This theorem guarantees the size control for the event of the inclusion of the density function in the approximate identified set \( \mathcal{L}_0^*(P) \) as opposed to the identified set \( \mathcal{L}_0(P) \). The next section presents conditions under which the approximate identified set \( \mathcal{L}_0^*(P) \) contains the identified set \( \mathcal{L}_0(P) \) for every possible joint distribution \( P \in \mathcal{P} \).

### 4.2 Bounds of Approximation Errors

Throughout, we equip \( \mathcal{L}^2 \) with the inner product \( \langle \cdot, \cdot \rangle \) defined by

\[
\langle f_1, f_2 \rangle = \int_{\mathbb{R}} f_1(x)f_2(x)dx.
\]

**Assumption 3.** (i) \( \Psi = \{\psi_j : j = 0, 1, \ldots\} \) is an orthonormal basis of \( \mathcal{L}^2 \), i.e., orthonormal and complete in \( \mathcal{L}^2 \). (ii) \( \langle f, \psi_0 \rangle, \ldots, \langle f, \psi_q \rangle \) is \( \Theta^{q+1} \) for all \( f \in \mathcal{L}_0(P) \) for all \( P \in \mathcal{P} \). In Section 5.1, we propose a concrete orthonormal basis \( \Psi \) and the set \( \Theta^{q+1} \) of coefficients to satisfy Assumption 3. The following theorem provides a guide for choices of \( \eta \) and \( \delta(t_1), \ldots, \delta(t_L) \) such that \( \mathcal{L}_0^*(P) \) contains \( \mathcal{L}_0(P) \) for every possible \( P \in \mathcal{P} \).
Theorem 3 (Approximation). Suppose that Assumption 3 is satisfied. If

\[
\sup_{f \in \mathcal{L}} \sup_{t \in I} \sum_{j=q+1}^{\infty} \langle f, \psi_j \rangle \cdot \psi_j(t) \leq \eta \quad \text{and} \quad (10)
\]

\[
\sup_{f \in \mathcal{L}} \sum_{j=q+1}^{\infty} \langle f, \psi_j \rangle \cdot (i \phi_j(t_1) \cdot \mathbb{E}_P[Y_1 \exp(it_2)] - \phi_j'(t_1) \cdot \mathbb{E}_P[\exp(it_2)]) \leq \delta(t_1) \quad (11)
\]

for every \( l \in \{1, \ldots, L\} \), then \( \mathcal{L}_0(P) \subset \mathcal{L}_0^*(P) \) holds for all \( P \in \mathcal{P} \).

A proof is provided in Appendix A.3. In Section 5.1, we provide concrete evaluations of the left-hand side of (10) and (11) under a concrete orthonormal basis \( \Psi \). Putting Theorems 2 and 3 together, we obtain the following result on the validity of the confidence band.

Corollary 1 (Validity of the Confidence Band). Suppose that the conditions of Theorems 2 and 3 are satisfied. Then, there are positive constants \( c \) and \( C \) such that

\[
\inf_{P \in \mathcal{P}} \inf_{f \in \mathcal{L}_0(P)} \mathbb{P}(f \in \mathcal{C}_n(\alpha)) \geq 1 - \alpha - Cn^{-c}.
\]

4.3 Power

We introduce the following short-hand notation for the random variable defined as the maximum deviation of the sample variance from the population variance:

\[
B_V = \sup_{\theta \in \mathcal{B}_{\alpha+\eta}(f)} \max_{t=1, \ldots, L} \left\{ \theta^T (V_n(\mathbf{R}_t(Y_1, Y_2)) - \mathbb{V}_P(\mathbf{R}_t(Y_1, Y_2))) \theta, \theta^T (V_n(\mathbf{I}_t(Y_1, Y_2)) - \mathbb{V}_P(\mathbf{I}_t(Y_1, Y_2))) \theta \right\}.
\]

The following theorem shows a power property of our proposed inference method.

Theorem 4 (Power). Suppose that Assumption 2 (i) holds. For every \( P \in \mathcal{P} \), every \( f \in \mathcal{L} \), every \( \nu > 0 \), and every \( b \in (0, \infty) \), if there is \( t_* \in \{t_1, \ldots, t_L\} \) such that at least one of the following statements holds:

\[
\sqrt{n} \inf_{\theta \in \mathcal{B}_{\alpha+\eta}(f)} \frac{\mathbb{E}_P[\mathbf{R}_t(Y_1, Y_2)]^T \theta - \delta(t_*)}{\sqrt{\theta^T \mathbb{V}_P(\mathbf{R}_t(Y_1, Y_2)) \theta + \nu}} \geq (1 + b) \cdot \mathbb{E}_P \left[ \sup_{\theta \in \mathcal{B}_{\alpha+\eta}(f)} \frac{|\mathbb{G}_n[\mathbf{R}_t(Y_1, Y_2)]^T \theta|}{\sqrt{\theta^T \mathbb{V}_n(\mathbf{R}_t(Y_1, Y_2)) \theta}} \right] + \sqrt{2 \log(4L)} + \sqrt{2 \log(1/\alpha)} \quad (12)
\]

\[
\sqrt{n} \inf_{\theta \in \mathcal{B}_{\alpha+\eta}(f)} \frac{-\mathbb{E}_P[\mathbf{I}_t(Y_1, Y_2)]^T \theta - \delta(t_*)}{\sqrt{\theta^T \mathbb{V}_P(\mathbf{I}_t(Y_1, Y_2)) \theta + \nu}} \geq (1 + b) \cdot \mathbb{E}_P \left[ \sup_{\theta \in \mathcal{B}_{\alpha+\eta}(f)} \frac{|\mathbb{G}_n[\mathbf{I}_t(Y_1, Y_2)]^T \theta|}{\sqrt{\theta^T \mathbb{V}_n(\mathbf{I}_t(Y_1, Y_2)) \theta}} \right] + \sqrt{2 \log(4L)} + \sqrt{2 \log(1/\alpha)} \quad (13)
\]

\[
\sqrt{n} \inf_{\theta \in \mathcal{B}_{\alpha+\eta}(f)} \frac{\mathbb{E}_P[\mathbf{I}_t(Y_1, Y_2)]^T \theta - \delta(t_*)}{\sqrt{\theta^T \mathbb{V}_P(\mathbf{I}_t(Y_1, Y_2)) \theta + \nu}} \geq (1 + b) \cdot \mathbb{E}_P \left[ \sup_{\theta \in \mathcal{B}_{\alpha+\eta}(f)} \frac{|\mathbb{G}_n[\mathbf{I}_t(Y_1, Y_2)]^T \theta|}{\sqrt{\theta^T \mathbb{V}_n(\mathbf{I}_t(Y_1, Y_2)) \theta}} \right] + \sqrt{2 \log(4L)} + \sqrt{2 \log(1/\alpha)} \quad (14)
\]
\[
\sqrt{n} \inf_{\theta \in B_{q+1,n}(f)} \frac{-E_P[I_{t_*(Y_1,Y_2)}]^T \theta - \delta(t_*)}{\sqrt{\mathcal{V}_n(I_{t_*(Y_1,Y_2)}^T \theta) + \nu}} \geq (1 + b) \cdot E_P \left[ \sup_{\theta \in B_{q+1,n}(f)} \frac{|G_n[I_{t_*(Y_1,Y_2)}]^T \theta|}{\sqrt{\mathcal{V}_n(I_{t_*(Y_1,Y_2)}^T \theta) + \nu}} \right]
\]

\[
+ \sqrt{2 \log(4L)} + \sqrt{2 \log(1/\alpha)}, \quad (15)
\]

then

\[
P_P(f \notin C_n(\alpha)) \geq P_P(B_V \leq \nu) - \frac{1}{1 + b}.
\]

A proof is provided in Appendix A.4. This theorem implies the consistency against all fixed alternatives.

5 Practical Considerations

The current section presents a guide to practice. Construction of the confidence band and its theoretical properties are presented in Sections 3 and 4 with abstract objects. These objects in particular include an orthonormal basis \( \Psi = \{\psi_j : j = 0, 1, \ldots\} \), the sieve dimension \( q \), the set \( \Theta^{q+1} \) of sieve coefficients, and the tolerance levels, \( \eta, \delta(t_1), \ldots, \delta(t_L) \), of approximation errors. Section 5.1 presents concrete choices of \( \Psi = \{\psi_j : j = 0, 1, \ldots\} \) and \( \Theta^{q+1} \). Section 5.2 presents a concrete data-driven procedure for selecting the tolerance levels \( \eta, \delta(t_1), \ldots, \delta(t_L) \). Section 5.3 presents a concrete implementation procedure for constructing the confidence band with these choices of the objects.

5.1 The Hermite Orthonormal Basis

There is a large extent of freedom of choice for an orthonormal basis \( \Psi \) – see Chen (2007) for a list of options. We recommend the Hermite orthonormal basis in particular for its convenient properties and its nice compatibility with the deconvolution framework – a Hermite function is an eigenfunction of the Fourier transform and the Fourier inverse. The Hermite functions take the form

\[
\psi_j(x) = \frac{1}{\sqrt{2^j j! \sqrt{\pi}}} \cdot \exp(-x^2/2) \cdot H_j(x), \quad (16)
\]

where \( H_j \) is the Hermite polynomial defined by

\[
H_j(x) = (-1)^j \cdot \exp(x^2) \cdot \frac{d^j}{dx^j} \exp(-x^2).
\]

The Hermite functions are the eigenfunctions of the Fourier transform operator, and specifically, \( \phi_j = \mathcal{F} \psi_j = i^j \sqrt{2\pi} \psi_j \) holds. For any \( q \in \mathbb{N} \) to be chosen below, have the set of sieve coefficients satisfy

\[
\Theta^{q+1} \supset \left[ -1.086435\pi^{-1/4}, 1.086435\pi^{-1/4} \right]^{q+1} \quad \text{or} \quad \Theta^{q+1} = \left[ -1.086435\pi^{-1/4}, 1.086435\pi^{-1/4} \right]^{q+1} \quad (17).
\]

These concrete choices are made for the sake of satisfying Assumption 3 so we can use Theorem 3.

Proposition 1 (Sufficient Condition for Assumption 3). If \( \Psi = \{\psi_j : j = 0, 1, \ldots\} \) is the sequence of the Hermite functions given in (16), then it satisfies Assumption 3 with the coefficient set given in (17).

A proof is provided in Appendix B.1. We also present the following proposition which provides approximation bounds for the condition of Theorem 3 to guarantee the containment of the identified set \( \mathcal{L}_0(P) \) by the approximate identified set \( \mathcal{L}_0^*(P) \) for every possible joint distribution \( P \in \mathcal{P} \).
Proposition 2 (Approximation Bounds). Suppose that $\Psi = \{\psi_j : j = 0, 1, \ldots\}$ is the sequence of the Hermite functions given in (16) and $\Theta^{q+1}$ satisfies (17). If
\[
\int \left( \frac{d^2}{dx^2}(f''(x) + x^2 f(x)) + x^2 \cdot (f''(x) + x^2 f(x)) \right)^2 dx \leq M,
\] then, for each $q = 1, 2, \ldots,$
\[
\sup_{f \in \mathcal{L}} \sup_{x \in \mathcal{I}} \left| \sum_{j=q+1}^{\infty} \langle f, \psi_j \rangle \cdot \psi_j(x) \right| \leq \frac{1.086435 \pi^{-1/4}}{\sqrt{2q + 3}} \sqrt{M \sum_{j=q+1}^{\infty} (2j + 1)^{-3}},
\] (19)
\[
\sup_{f \in \mathcal{L}} \sup_{t \in \mathbb{R}} \left| \sum_{j=q+1}^{\infty} \langle f, \psi_j \rangle \cdot \phi_j(t) \right| \leq \frac{1.086435 \pi^{-1/4}}{\sqrt{2q + 3}} \sqrt{M \sum_{j=q+1}^{\infty} (2j + 1)^{-3}}, \quad \text{and}
\] (20)
\[
\left| \sum_{j=q+1}^{\infty} \langle f, \psi_j \rangle \cdot \left( i\phi_j(t) \cdot \mathbb{E}_P [Y_1 \exp(itY_2)] - \phi_j'(t) \cdot \mathbb{E}_P \left[ \exp(itY_2) \right] \right) \right| \leq \frac{1.086435 \pi^{-1/4}}{\sqrt{2\pi}} \cdot \sqrt{M \sum_{j=q+1}^{\infty} (2j + 1)^{-3} \left( \frac{\mathbb{E}_P \|Y_1\|}{\sqrt{2q + 3}} + 1 \right)} \quad \text{for all } t.
\] (21)

A proof is provided in Appendix B.2. An admissible function class in terms of smoothness restriction can be specified by (18). Note that this is analogous to the standard practice in the literature to work with Sobolev classes of functions. With the function class specified in this way, equations (19) and (21) prescribe possible choices of the tolerance levels $\eta, \delta(t_1), \ldots, \delta(t_L)$ which admit $\mathcal{L}_0(P) \subset \mathcal{L}^n_0(P)$ for all $P \in \mathcal{P}$ per Theorem 3. See Section 5.2 for concrete choice procedures. Equation (20) in addition suggests the worst approximation error for the characteristic function, which will be useful when we impose natural restrictions on the characteristic functions – see Section 5.3.

5.2 Choice of Tuning Parameters

In this section, we provide example procedures of choosing the tolerance levels $\eta, \delta(t_1), \ldots, \delta(t_L) \in (0, \infty)$ in finite sample. We consider the framework where the remaining tuning parameters, namely the sieve dimension $q \in \mathbb{N}$ and the smoothness bound $M \in (0, \infty)$, are imposed by a researcher in the spirit of honest inference. In fact, the tolerance level $\eta$ can be seen as a bias bound that is implied by $q$ and $M$ – see (22) below.

**Tolerance Level $\eta$:** In light of Proposition 2 we can choose the tolerance level $\eta$ in the following manner. By (21), we can satisfy condition (11) of Theorem 3 if
\[
\frac{1.086435 \pi^{-1/4}}{\sqrt{2q + 3}} \sqrt{M \sum_{j=q+1}^{\infty} (2j + 1)^{-3}} \leq \eta.
\]
Thus, having selected the smoothness bound $M$ and the sieve dimension $q$, one can set $\eta$ to
\[
\eta = \frac{1.086435 \pi^{-1/4}}{\sqrt{2q + 3}} \sqrt{M \sum_{j=q+1}^{\infty} (2j + 1)^{-3}}.
\] (22)

**Tolerance Levels $\delta(t_1), \ldots, \delta(t_L)$:** In light of Proposition 2 we can choose the tolerance levels $\delta(t_1), \ldots, \delta(t_L)$ of approximation errors in the following manner. By (21), we can satisfy condition (11) of Theorem 3 if
\[
\frac{1.086435 \pi^{-1/4}}{\sqrt{2\pi}} \cdot \sqrt{M \sum_{j=q+1}^{\infty} (2j + 1)^{-3} \left( \frac{\mathbb{E}_P \|Y_1\|}{\sqrt{2q + 3}} + 1 \right)} \leq \delta(t_l) \quad \text{for all } l \in \{1, \ldots, L\}.
\]
Thus, having selected the
smoothness bound $M$ and the sieve dimension $q$, one can set $\delta(t_l)$ to

$$
\delta(t_l) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{M \sum_{j=q+1}^{\infty} (2j+1)^{-3}} \left( \frac{\mathbb{E}_n |Y_1|}{\sqrt{2q+3}} + 1 \right)
$$

for all $l \in \{1, \ldots, L\}$, where we replaced $1.086435\pi^{-1/4}$ by one for slackness accounting for estimation of $\mathbb{E}_P |Y_1|$ by $\mathbb{E}_n |Y_1|$.

### 5.3 Implementation

When we implement the Anderson-Rubin-type inference, we would generally sweep across the parameter set $\Theta^{q+1}$ of sieve coefficients, and this operation may demand long computational time. However, we do not need to conduct the test at every point in $\Theta^{q+1}$, because properties of probability density functions and characteristic functions together with the sieve approximation property rule out substantially many elements of $\Theta^{q+1}$. From the property $f \geq 0$ of probability density functions and the approximation bound (19), we can impose the restriction

$$
\psi^T(x)\theta \geq -\frac{1.086435\pi^{-1/4}}{\sqrt{2q+3}} \sqrt{M \sum_{j=q+1}^{\infty} (2j+1)^{-3}} \quad \text{for all } x \in I.
$$

Similarly, from the property $[\mathcal{F}f](0) = 1$ of characteristic functions and the approximation bound (20), we can impose the restriction

$$
\left| [\mathcal{F}\psi^T \theta] (0) - 1 \right| \leq \frac{1.086435\pi^{-1/4}\sqrt{2\pi}}{\sqrt{2q+3}} \sqrt{M \sum_{j=q+1}^{\infty} (2j+1)^{-3}}.
$$

Since the Hermite function is an eigenfunction of $\mathcal{F}$, (25) can be simplified when the Hermite orthonormal basis $\Psi$ (see Section 5.1) is used. Specifically, (25) reduces to

$$
\left| \sqrt{2\pi}\psi^T (0) \text{diag} (1, 0, -1, 0, \ldots) \theta - 1 \right| \leq \frac{1.086435\pi^{-1/4}\sqrt{2\pi}}{\sqrt{2q+3}} \sqrt{M \sum_{j=q+1}^{\infty} (2j+1)^{-3}}.
$$

Note that the left-hand side of (26) does not require to compute an integral unlike that of (25), which is a major advantage of using the Hermite orthonormal basis in the context of deconvolution. Use of the constraints (24) and (25) and (26) is motivated by the definition of the confidence band (7) consisting only of “density functions” $f \in \mathcal{L}$ which indexes the set $B_{q+1, \eta}(f)$ of possible values of $\theta$.

We also remark that we do not need to conduct a grid search for the purpose of drawing confidence bands. In fact, solving $\min_{\theta \in \Theta^{q+1}} \psi(x)^T \theta$ subject to $T(\theta) \leq c(\alpha)$ as well as (17), (24), and (25)/ (26) yields the lower bound of the confidence band up to the approximation error $\eta$. Similarly, solving $\max_{\theta \in \Theta^{q+1}} \psi(x)^T \theta$ subject to $T(\theta) \leq c(\alpha)$ as well as (17), (24), and (25)/ (26) yields the upper bound of the confidence band up to the approximation error $\eta$. Accounting for these points, we propose the following implementation algorithm.

**Algorithm 1.**

1. Choose the tuning parameters according to Section 5.2.
2. For each $x \in I$, compute $f^L(x) = \arg \min_{\theta \in \Theta^{\eta+1}} \psi(x)^T \theta$ subject to $T(\theta) \leq c(\alpha, \theta)$, \cite{17}, \cite{24}, \& \cite{26}.

3. For each $x \in I$, compute $f^U(x) = \arg \max_{\theta \in \Theta^{\eta+1}} \psi(x)^T \theta$ subject to $T(\theta) \leq c(\alpha, \theta)$, \cite{17}, \cite{24}, \& \cite{26}.

4. The confidence band is set to $[f^L(x) - \eta, f^U(x) + \eta]$, $x \in I$.

We remark that, in computing the test statistic $T(\theta)$ in steps 2 and 3 of the algorithm above, the use of the Hermite orthonormal basis element $\psi = \psi_j$, $j = 0, 1, \ldots$ simplifies \cite{1} and \cite{5} to

$$R_{\psi, t}(y_1, y_2) = \sqrt{2\pi} \left\{ -\cos(ty_2)(y_1 \text{Im}(i^j \psi_j(t)) + \text{Re}(i^j \psi_j^{(1)}(t))) \\
-\sin(ty_2)(y_1 \text{Re}(i^j \psi_j(t)) - \text{Im}(i^j \psi_j^{(1)}(t))) \right\}$$

and

$$I_{\psi, t}(y_1, y_2) = \sqrt{2\pi} \left\{ \cos(ty_2)(y_1 \text{Re}(i^j \psi_j(t)) + \text{Im}(i^j \psi_j^{(1)}(t))) \\
-\sin(ty_2)(y_1 \text{Im}(i^j \psi_j(t)) - \text{Re}(i^j \psi_j^{(1)}(t))) \right\},$$

respectively. As such, one need not compute an integral to obtain the test statistic $T(\theta)$. This convenient property again follows from the fact that the Hermite function is an eigenfunction of $F$.

## 6 Simulation Studies

In this section, we present and discuss finite-sample performance of the proposed method by simulation studies. Simulation outcomes that we present include the size under the null of the true distribution, the power under alternative distributions, and the lengths of confidence bands. The lengths will be further decomposed into the bias bound $\eta$ and the remaining lengths due to the stochastic part.

### 6.1 Simulation Setting

We employ three distribution families to generate the latent variable $X$ – the normal distribution, the skew normal distribution, and the $t$ distribution. We employ the skew normal distribution and the $t$ distribution to see whether our method is effective for asymmetric distributions and super-Gaussian tails, respectively. Specifically, we generate a random sample of $(X, U_1, U_2)$ mutually independently according to the marginal laws:

- **Model 1:** $X \sim \mathcal{N}(\xi_1, \xi_2^2)$, $U_1 \sim \mathcal{N}(0, \sigma_{U_1}^2)$, $U_2 \sim \mathcal{N}(0, \sigma_{U_2}^2)$
- **Model 2:** $X \sim \mathcal{SN}(\xi_1, \xi_2, \xi_3)$, $U_1 \sim \mathcal{N}(0, \sigma_{U_1}^2)$, $U_2 \sim \mathcal{N}(0, \sigma_{U_2}^2)$
- **Model 3:** $X \sim \mathcal{t}_{\xi_4}$, $U_1 \sim \mathcal{N}(0, \sigma_{U_1}^2)$, $U_2 \sim \mathcal{N}(0, \sigma_{U_2}^2)$

Here, $\mathcal{N}(\xi_1, \xi_2^2)$ denotes the normal distribution with mean $\xi_1$ and variance $\xi_2^2$, $\mathcal{SN}(\xi_1, \xi_2, \xi_3)$ denotes the skew normal distribution with location $\xi_1$, scale $\xi_2$, and shape $\xi_3$, and $\mathcal{t}_{\xi_4}$ denotes the $t$ distribution with $\xi_4$ degrees of freedom. The distribution parameters for the latent variable $X$ are set to $(\xi_1, \xi_2) = (0, 1)$ for Model 1, $(\xi_1, \xi_2, \xi_3) = (0, 1, 1)$ for Model 2, and $\xi_4 = 5$ for Model 3. The choice of the normal error distribution, which is an instance of super-smooth distributions, imposes a difficult case in deconvolution – see Li and Vuong \cite{1998}. The error variance parameters are set to $\sigma_{U_1} = \sigma_{U_2} = 0.5$ in each of the three models. We conduct experiments with three sample sizes $n = 1,000, 2,000, \text{and} 4,000$, and run 2,500 Monte Carlo iterations for each set of simulations.
We conduct inference with the tuning parameters \((M,q) = (5, 5)\) in the baseline case, but will also experiment with each of \(q = 4, 5, \text{and } 6\). The frequency bound is set to \(T = 5\) and the number of frequency grid points is set to \(L = 50\). The interval on which the confidence band is formed is set to \(I = \left[ E[X] - 2\sqrt{\text{Var}(X)}, E[X] + 2\sqrt{\text{Var}(X)} \right] \), where \(E[X]\) and \(\text{Var}(X)\) are the theoretical mean and the theoretical variance, respectively, of \(X\) under the relevant model. To enjoy favorable speed of computation for numerous Monte Carlo iterations, we use the conservative critical value \(c(\alpha)\). The level is set to \(\alpha = 0.05\) throughout.

### 6.2 Simulation Results

Figure 1 (A) shows the simulated frequencies that the confidence band formed under Model 1 covers alternative probability density functions for \(N(\xi_1, \xi_2^2)\) indexed by location parameter values \(\xi_1 \in [0.0, 0.5]\) while the scale parameter is fixed at the true value \(\xi_2 = 1.0\). The coverage frequency under \(\xi_1 = 0.0\) indicates (the complement of) the size, whereas the coverage frequencies under \(\xi_1 \in (0.0, 0.5]\) indicate (the complement of) the power. Similarly, Figure 1 (B) shows the simulated frequencies that the confidence band formed under Model 1 covers alternative probability density functions for \(N(\xi_1, \xi_2^2)\) indexed by scale parameter values \(\xi_2 \in [1.0, 1.5]\) while the location parameter is fixed at the true value \(\xi_1 = 0.0\). These results show the correct size and increasing power. The size entails over-coverage, which is still consistent with our theory on size control.

Figures 2 (A) and 2 (B) show analogous results to Figure 1 (A) except that Model 2 and Model 3, respectively, are used instead of Model 1. For Model 2, the shape parameter is fixed at the true value \(\xi_3 = 1.0\). These results evidence that the proposed method is similarly effective for the cases where the latent variable \(X\) follows asymmetric distributions or distributions with super-Gaussian tails.

Figures 3 (A), 3 (B), and 3 (C) show comparative results across alternative sieve dimensions \(q \in \{4, 5, 6\}\). These results show that the power improves with the sieve dimension \(q\) for closer-to-null alternatives. This phenomenon is consistent with the fact that the bias bound \(\eta\) decreases with the sieve dimension \(q\).

We next present average lengths of the confidence bands on \(I\), and their decomposition into the bias bound \(\eta\) and the remaining length due to the stochastic part. Table 1 summarizes results on the lengths. There are a couple of features in these results that deserve discussions. First, observe that the confidence bands shrink as sample size increases when the sieve dimension \(q\) is held fixed. On the other hand, the bias bound \(\eta\) remains invariant across sample sizes while the sieve dimension \(q\) is fixed. These results are natural features of our approach. Second, observe that the bias bound \(\eta\) decreases as the sieve dimension \(q\) increments.

Finally, we display instances of confidence bands in Figure 4. The gray shades indicate the confidence bands including the bias bound and the stochastic parts together. The internal dark gray shades include only the stochastic parts. We also plot the true density functions and Li-Vuong estimates as solid and dashed curves, respectively. While such instances of confidence bands will not tell us any evidence on the statistical properties, they at least inform how a confidence band may look in applications.

### 7 Conclusion

Since its introduction to econometrics by [Li and Vuong (1998)](https://www.jstor.org/stable/3487601), Kotlarski’s identity ([Kotlarski, 1967](https://projecteuclid.org/euclid.sanken/1218141276)) – see also [Rao (1992)](https://www.tandfonline.com/doi/abs/10.1080/01621459.1992.10475883) – has been widely used in empirical economics. Examples include applications to empirical auctions (e.g., [Li et al. 2002](https://www.jstor.org/stable/1391703)) [Krasnokutskaya 2011](https://www.jstor.org/stable/20155389), income dynamics (e.g., [Bonhomme and Robin 2010](https://www.jstor.org/stable/43891351)).
(A) Coverage frequencies with \((M, q) = (5, 5)\) under location alternatives in Model 1

(B) Coverage frequencies with \((M, q) = (5, 5)\) under scale alternatives in Model 1

Figure 1: The simulated frequencies that the confidence band formed under Model 1 covers alternative probability density functions for \(N(\xi_1, \xi_2^2)\) with the tuning parameters \((M, q) = (5, 5)\). Panel (A) runs across alternative location parameter values \(\xi_1 \in [0.0, 0.5]\) while the scale parameter is fixed at the true value \(\xi_2 = 1.0\). Panel (B) runs across alternative scale parameter values \(\xi_2 \in [1.0, 1.5]\) while the location parameter is fixed at the true value \(\xi_1 = 0.0\).
(A) Coverage frequencies with $(M, q) = (5, 5)$ under location alternatives in Model 2

Figure 2: (A) The simulated frequencies that the confidence band formed under Model 2 covers alternative probability density functions for $SN(\xi_1, \xi_2, \xi_3)$ indexed by the alternative location parameter values $\xi_1 \in [0, 0.5]$ while the scale and shape parameters are fixed at $(\xi_2, \xi_3) = (1.0, 1.0)$. (B) The simulated frequencies that the confidence band formed under Model 3 covers alternative probability density functions for the (non-)central t-distributions indexed by the alternative locations in $[0.0, 0.5]$ while the degrees of freedom is fixed at the true value 5. Results in both panel (A) and panel (B) are based on the tuning parameters $(M, q) = (5, 5)$.
(A) Coverage frequencies with $(M, q) = (5, 4)$ under location alternatives in Model 1

(B) Coverage frequencies with $(M, q) = (5, 5)$ under location alternatives in Model 1

(C) Coverage frequencies with $(M, q) = (5, 6)$ under location alternatives in Model 1

Figure 3: The simulated frequencies that the confidence band formed under Model 1 covers alternative probability density functions for $N(\xi_1, \xi_2)$ indexed by the alternative location parameter values $\xi_1 \in [0.0, 0.5]$ while the scale parameter is fixed at the true value $\xi_2 = 1.0$. The tuning parameters are (A) $(M, q) = (5, 4)$, (B) $(M, q) = (5, 5)$, and (C) $(M, q) = (5, 6)$. 
| Model  | $q$ | $n$  | Average Length | Supremum Bias ($\eta$) | Stochastic Length |
|--------|-----|------|----------------|-------------------------|------------------|
| Model 1 | 4   | 1,000 | 0.149          | 0.055                   | 0.094            |
|        | 4   | 2,000 | 0.121          | 0.055                   | 0.066            |
|        | 4   | 4,000 | 0.092          | 0.055                   | 0.037            |
| Model 1 | 5   | 1,000 | 0.161          | 0.042                   | 0.119            |
|        | 5   | 2,000 | 0.127          | 0.042                   | 0.085            |
|        | 5   | 4,000 | 0.097          | 0.042                   | 0.055            |
| Model 1 | 6   | 1,000 | 0.135          | 0.034                   | 0.101            |
|        | 6   | 2,000 | 0.097          | 0.034                   | 0.064            |
|        | 6   | 4,000 | 0.063          | 0.034                   | 0.029            |
| Model 2 | 4   | 1,000 | 0.144          | 0.055                   | 0.089            |
|        | 4   | 2,000 | 0.115          | 0.055                   | 0.061            |
|        | 4   | 4,000 | 0.092          | 0.055                   | 0.037            |
| Model 2 | 5   | 1,000 | 0.152          | 0.042                   | 0.110            |
|        | 5   | 2,000 | 0.118          | 0.042                   | 0.076            |
|        | 5   | 4,000 | 0.091          | 0.042                   | 0.049            |
| Model 2 | 6   | 1,000 | 0.133          | 0.034                   | 0.099            |
|        | 6   | 2,000 | 0.097          | 0.034                   | 0.063            |
|        | 6   | 4,000 | 0.063          | 0.034                   | 0.030            |
| Model 3 | 4   | 1,000 | 0.147          | 0.055                   | 0.093            |
|        | 4   | 2,000 | 0.116          | 0.055                   | 0.061            |
|        | 4   | 4,000 | 0.092          | 0.055                   | 0.038            |
| Model 3 | 5   | 1,000 | 0.153          | 0.042                   | 0.111            |
|        | 5   | 2,000 | 0.109          | 0.042                   | 0.067            |
|        | 5   | 4,000 | 0.071          | 0.042                   | 0.029            |
| Model 3 | 6   | 1,000 | 0.146          | 0.034                   | 0.112            |
|        | 6   | 2,000 | 0.102          | 0.034                   | 0.068            |
|        | 6   | 4,000 | 0.074          | 0.034                   | 0.040            |

Table 1: Average lengths of confidence bands, the supremum biases ($\eta$), and the lengths for stochastic parts of confidence bands.
Figure 4: Instances of confidence bands. The gray shades indicate the confidence bands. The internal dark gray shades indicate the stochastic parts of the confidence bands. The solid and dashed curves indicate the true density functions and Li-Vuong estimates, respectively.
and labor economics (e.g., Cunha et al., 2005, 2010; Bonhomme and Sauder, 2011; Kennan and Walker, 2011). Despite its popular use in applications, a method of inference based on Kotlarski’s identity has long been missing in the literature. After twenty years since Li and Vuong (1998), we now propose a method of inference based on Kotlarski’s identity. Specifically, we develop confidence bands for the probability density function \( f_X \) of \( X \) in the repeated measurement model where two measurements \((Y_1, Y_2)\) of unobserved variable \( X \) are available in data with additive independent errors, \( U_1 = Y_1 - X \) and \( U_2 = Y_2 - X \).

Our construction of confidence bands can be summarized as follows. First, we derive complex-valued moment restrictions based on Kotlarski’s identity. Second, we let the Hermite polynomial sieve approximate unknown probability density functions. Third, for a given sieve dimension and for a given class for probability density functions, we compute a bias bound for the complex-valued moment restrictions, and slack the complex-valued moment restrictions by this bias bound. Fourth, we compute the uniform norm of the self-normalized process of the slacked complex-valued moment restrictions as the test statistics for each point in a set of sieve coefficients. Fifth, inverting this test statistic yields a confidence set of sieve approximations to possible probability density functions. Sixth, for a given sieve dimension and for a given class for probability density functions, we compute a bias bound for sieve approximations of probability density functions, and the desired confidence band is obtained by uniformly enlarging the set of sieve approximations by this bias bound.

We not only provide a method that works, but also care for its practicality. The Fourier transform and the inverse Fourier transform operations are known to be computationally costly in the deconvolution literature. By exploiting the property of the Hermite functions as eigen-functions of the Fourier transform operator, we propose to let the Hermite polynomial sieve approximate both the density and characteristic functions without having to implement numerical integrations within each iteration of a numerical optimization routine. This convenient feature of the proposed method saves computational resources. Furthermore, we also exploit a couple of other convenient properties of the Hermite functions (namely the Schrödinger equation for a harmonic oscillator and a pair of recursive equations), and consequently obtain informative bias bounds and thus informative inference. With these practical features of our method, simulation studies indeed conclude reasonably fast with informative inference results. The results evidence the efficacy of the proposed method. Since Kotlarski’s identity is one of the most popular methods in a number of applied fields, including empirical auctions, income dynamics, and labor economics, we hope that our method will contribute to the practice of economic analyses in these and other topics.
Appendix

A Proofs for the Main Theorems

A.1 Proof of Theorem 1 (Complex-Valued Moment Restrictions)

Proof. By (1) and Assumption 1 (ii), we have

\[ E_P[\exp(it_1Y_1 + it_2Y_2)] = \phi_X(t_1 + t_2)\phi_{U_1}(t_1)\phi_{U_2}(t_2) \]

for every \((t_1, t_2) \in \mathbb{R}^2\). Note that random variables with finite first moments have continuously differentiable characteristic functions. Since \(\phi_{U_1}(0) = 1\) and \(\phi'_{U_1}(0) = 0\) by Assumption 1 (i), it follows that

\[ E_P[\exp(itY_2)] = \phi_X(t)\phi_{U_2}(t) \]

and

\[ \frac{\partial}{\partial t_1} E_P[\exp(it_1Y_1 + itY_2)] \bigg|_{t_1=0} = \phi'_{X}(t)\phi_{U_2}(t) \]

for every real \(t\) under Assumption 1 (i). Therefore,

\[ E_P[(iY_1\phi_X(t) - \phi'_X(t))\exp(itY_2)] = E_P[iY_1 \exp(itY_2)] \phi_X(t) - E_P[\exp(itY_2)] \phi'_X(t) = \phi'_X(t)\phi_{U_2}(t) - \phi_X(t)\phi_{U_2}(t)\phi'_X(t) = 0 \]

for every real \(t\), and the claim of the lemma follows.

A.2 Proof of Theorem 2 (Size Control)

Proof of Theorem 2. Let \(P \in \mathcal{P}\) and \(f \in L_0^\ast(P)\). We have

\[ \mathbb{P}_P(f \in C_\alpha) = P_P(T(\theta) \leq c(\alpha, \theta) \text{ for some } \theta \in B_{q+1, \eta}(f)) \]

\[ \geq P_P(T(\theta_{**}) \leq c(\alpha, \theta_{**})) \]

where \(\theta_{**} \in B_{q+1, \eta}(f)\) satisfies

\[ \max_{1 \leq i \leq L} (\max\{|E_P[R_{t_1}(Y_1, Y_2)]^T\theta_{**}|, |E_P[I_{t_1}(Y_1, Y_2)]^T\theta_{**}|\} - \delta(t_i)) < 0 \]

as \(f \in L_0^\ast(P)\). Note that

\[ \sqrt{n} \max_{1 \leq i \leq L} \left\{ \frac{|E_P[R_{t_1}(Y_1, Y_2)]^T\theta_{**}| - \delta(t_i)}{\sqrt{\theta_{**}^T\text{Var}_P(R_{t_1}(Y_1, Y_2))\theta_{**}}}, \frac{|E_P[I_{t_1}(Y_1, Y_2)]^T\theta_{**}| - \delta(t_i)}{\sqrt{\theta_{**}^T\text{Var}_P(I_{t_1}(Y_1, Y_2))\theta_{**}}} \right\} \leq 0. \]
By Chernozhukov, Chetverikov, and Kato (2017, Threom 6.1), there exist positive constants $c$ and $C$ depending on $\alpha$, $c_1$ and $C_1$ under Assumption 2 such that

$$P_T(T(\theta_\ast)) \leq c(\alpha, \theta_\ast) \geq 1 - \alpha - Cn^{-c}.$$ 

Therefore, the statement of the theorem follows.

A.3 Proof of Theorem 3 (Approximation)

Proof. Let $f \in L_0(P)$ and $\phi = \mathcal{F}f$. Assumption 3 implies that

$$f = \sum_{j=0}^{\infty} \langle f, \psi_j \rangle \cdot \psi_j \quad \text{and} \quad \phi = \sum_{j=0}^{\infty} \langle f, \psi_j \rangle \cdot \phi_j$$

– see Folland (2007, Theorem 5.27). Define

$$\psi_{0:q} = \sum_{j=0}^{q} \langle f, \psi_j \rangle \cdot \psi_j \quad \text{and} \quad \phi_{0:q} = \sum_{j=0}^{q} \langle f, \psi_j \rangle \cdot \phi_j.$$

Since $f \in L_0(P)$ and $\phi = \mathcal{F}f$,

$$|\mathbb{E}_P \left[ (iY_1 \phi_{0:q}(t) - \phi'_{0:q}(t)) \exp(itY_2) \right] | = |\mathbb{E}_P \left[ (iY_1 (\phi_{0:q}(t) - \phi(t)) - (\phi'_{0:q}(t) - \phi'(t))) \exp(itY_2) \right] | = \sum_{j=q+1}^{\infty} \langle f, \psi_j \rangle \cdot \mathbb{E}_P \left[ (iY_1 \phi_j(t) - \phi_j'(t)) \exp(itY_2) \right].$$

Similarly,

$$\sup_{t \in I} |f(t) - \phi_{0:q}(t)| \leq \sup_{t \in I} \left| \sum_{j=q+1}^{\infty} \langle f, \psi_j \rangle \cdot \psi_j(t) \right|.$$

Therefore, the statement of the theorem follows.

A.4 Proof of Theorem 4 (Power)

Proof. This proof focuses on the case in (12). The proofs for the cases of (13)–(15) are similar. By the definition of $\mathcal{C}_n(\alpha)$, we can write

$$P_P(f \notin \mathcal{C}_n(\alpha)) = P_P(T(\theta) > c(\alpha, \theta) \text{ for every } \theta \in B_{q+1, \eta}(f)) \geq P_P(T(\theta) > \sqrt{2\log(4L)} + \sqrt{2\log(1/\alpha)} \text{ for every } \theta \in B_{q+1, \eta}(f)) = P_P \left( \inf_{\theta \in B_{q+1, \eta}(f)} T(\theta) > \sqrt{2\log(4L)} + \sqrt{2\log(1/\alpha)} \right),$$

where the inequality follows from

$$c(\alpha, \theta) \leq \sqrt{2\log(4L)} + \sqrt{2\log(1/\alpha)}.$$
for each $f$ with $\Theta$

\[ \text{Proof.} \]

B.1 Proof of Proposition 1 (Sufficient Condition for Assumption 3)

By (12). Thus, we obtain

\begin{align*}
\inf_{\theta \in B_{q+1,0}(f)} T(\theta) &\geq \inf_{\theta \in B_{q+1,0}(f)} \frac{\sqrt{n} \mathbb{E}_n[R_t(1, Y_2)]^T \theta - \delta(t_*)}{\sqrt{\mathbb{V}_n(R_t(1, Y_2)) \theta}} \\
&\geq \inf_{\theta \in B_{q+1,0}(f)} \frac{G_n[R_t(1, Y_2)]^T \theta}{\sqrt{\mathbb{V}_n(R_t(1, Y_2)) \theta}} + \sup_{\theta \in B_{q+1,0}(f)} \frac{\mathbb{E}_n[R_t(1, Y_2)]^T \theta - \delta(t_*)}{\sqrt{\mathbb{V}_n(R_t(1, Y_2)) \theta}} \\
&\geq \sup_{\theta \in B_{q+1,0}(f)} \frac{|G_n[R_t(1, Y_2)]^T \theta|}{\sqrt{\mathbb{V}_n(R_t(1, Y_2)) \theta}} + 2 \log(4L) + \sqrt{2 \log(1/\alpha)} + (1 + b) \cdot \mathbb{P}_P \left[ \sup_{\theta \in B_{q+1,0}(f)} \frac{|G_n[R_t(1, Y_2)]^T \theta|}{\sqrt{\mathbb{V}_n(R_t(1, Y_2)) \theta}} \right]
\end{align*}

by (12). Thus, we obtain

\[ \mathbb{P}_P(f \notin C_\alpha(\alpha)) \geq \mathbb{P}_P \left( \sup_{\theta \in B_{q+1,0}(f)} \frac{|G_n[R_t(1, Y_2)]^T \theta|}{\sqrt{\mathbb{V}_n(R_t(1, Y_2)) \theta}} < 1 + b \right) \cap \{ B_V \leq \nu \} \]

\[ \geq \mathbb{P}_P \left( \sup_{\theta \in B_{q+1,0}(f)} \frac{|G_n[R_t(1, Y_2)]^T \theta|}{\sqrt{\mathbb{V}_n(R_t(1, Y_2)) \theta}} < 1 + b \right) - (1 - \mathbb{P}_P(B_V \leq \nu)) \]

\[ \geq 1 - \frac{1}{1 + b} - (1 - \mathbb{P}_P(B_V \leq \nu)), \]

where the last inequality is due to Markov’s inequality. Therefore, the statement of the theorem follows.

\[ \square \]

B Additional Proofs

B.1 Proof of Proposition 1 (Sufficient Condition for Assumption 3)

Proof. First, it follows from Blanchard and Bruening [2002] Theorem 16.3.1 that $\Psi = \{ \psi_j : j = 0, 1, \ldots \}$ satisfies Assumption 3 (i). Furthermore, since $|\psi_j| \leq 1.086435 \pi^{-1/4}$ for each $j = 0, 1, \ldots$ (see e.g., Erdélyi, Magnus, Oberhettinger, and Tricomi [1953] p. 208), we have

\[ |(f, \psi_j)| \leq (f(x), |\psi_j|) \leq 1.086435 \pi^{-1/4} \cdot \int f(x)dx = 1.086435 \pi^{-1/4} \]

for each $f \in L_0(P)$ for each $P \in \mathcal{P}$ and for each $j = 0, 1, \ldots$. This shows that Assumption 3 (ii) is satisfied with $\Theta^{q+1} = [-1.086435 \pi^{-1/4}, 1.086435 \pi^{-1/4}]^q$.

\[ \square \]

B.2 Proof of Proposition 2 (Approximation Bounds)

Proof. First, note that the Hermite function $\psi_j$ in (10) satisfies the Schrödinger equation:

\[ \psi_j''(x) = -(2j + 1 - x^2) \cdot \psi_j(x) \]

(27)
for each \( j = 0, 1, \ldots \) \cite{Folland2009} Theorem 6.14 (6.41)). Second, note that the Hermite functions \cite{Folland2009} also satisfy the recurrence relation:

\[
\psi'_j = \sqrt{\frac{j}{2}} \psi_{j-1} - \sqrt{\frac{j+1}{2}} \psi_{j+1}
\]

for each \( j = 1, 2, \ldots \) \cite{Folland2009} Theorem 6.14 (6.39)–(6.40)). We will use these properties of the Hermite functions in the proof below.

By Proposition 1, we can write

\[
f = \sum_{j=0}^{\infty} \langle f, \psi_j \rangle \psi_j.
\]

– see \cite{Folland2007} Theorem 5.27. Taking the second derivatives of both sides, we obtain

\[
f''(x) = \sum_{j=0}^{\infty} \langle f, \psi_j \rangle \psi''_j(x) = -\sum_{j=0}^{\infty} \langle f, \psi_j \rangle (2j + 1 - x^2) \psi_j(x),
\]

where the second equality is due to \cite{45}. Rearranging, we have

\[
f''(x) + x^2 f(x) = -\sum_{j=0}^{\infty} \langle f, \psi_j \rangle (2j + 1) \psi_j(x).
\]

Further taking the second derivatives of both sides yields

\[
\frac{d^2}{dx^2} (f''(x) + x^2 f(x)) = -\sum_{j=0}^{\infty} \langle f, \psi_j \rangle (2j + 1)^2 \psi_j(x)
\]

where the second equality is again due to \cite{45}. Rearranging terms, we obtain

\[
\frac{d^2}{dx^2} (f''(x) + x^2 f(x)) + x^2 \cdot (f''(x) + x^2 f(x)) = \sum_{j=0}^{\infty} \langle f, \psi_j \rangle (2j + 1)^2 \psi_j(x)
\]

Combining \cite{46} and \cite{47} together, we have \(|\sum_{j=0}^{\infty} \langle f, \psi_j \rangle (2j + 1)^2 \psi_j(\cdot)\|_2^2 \leq M\), and hence

\[
\sum_{j=0}^{\infty} ((\langle f, \psi_j \rangle)^2 (2j + 1)^4 \leq M.
\]

Define \( D = \sum_{j=q+1}^{\infty} (2j + 1)^{-3} \). The above inequality implies

\[
D \sum_{j=q+1}^{\infty} \frac{(2j + 1)^{-3}}{D} ((\langle f, \psi_j \rangle)^2 (2j + 1)^7 \leq M.
\]

Using Jensen’s inequality, we can write

\[
\sqrt{D} \sum_{j=q+1}^{\infty} \frac{(2j + 1)^{-3}}{D} \sqrt{((\langle f, \psi_j \rangle)^2 (2j + 1)^7 \leq \sqrt{D} \sum_{j=q+1}^{\infty} \frac{(2j + 1)^{-3}}{D} ((\langle f, \psi_j \rangle)^2 (2j + 1)^7 \leq \sqrt{M},
\]

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which implies
\[
\sum_{j=q+1}^{\infty} \sqrt{2j + 1} |\langle f, \psi_j \rangle| \leq \sqrt{MD}.
\] (30)

Thus, we obtain
\[
\sup_{f \in \mathcal{L}} \sup_{x \in I} \left| \sum_{j=q+1}^{\infty} \langle f, \psi_j \rangle \cdot \psi_j(x) \right| \leq \left( \sup_{f \in \mathcal{L}} \sum_{j=q+1}^{\infty} \sqrt{2j + 1} |\langle f, \psi_j \rangle| \right) \cdot \sup_{j=q+1,...} \frac{\sup_{x \in I} |\psi_j(x)|}{\sqrt{2j + 1}}
\leq \frac{1.086435\pi^{-1/4}}{\sqrt{2q + 3}} \sqrt{MD}.
\]

Next, we note that Hermite function is the eigenfunction of the Fourier transform operator. Specifically, 
\[|\phi_j| = \sqrt{2\pi} |\psi_j| \leq 1.086435\pi^{-1/4}\sqrt{2\pi} \] holds. Thus, similar lines of calculations to those above yield
\[
\sup_{f \in \mathcal{L}} \sup_{t \in \mathbb{R}} \left| \sum_{j=q+1}^{\infty} \langle f, \psi_j \rangle \cdot \phi_j(t) \right| \leq \left( \sup_{f \in \mathcal{L}} \sum_{j=q+1}^{\infty} \sqrt{2j + 1} |\langle f, \psi_j \rangle| \right) \cdot \sup_{j=q+1,...} \frac{\sup_{x \in I} |\phi_j(x)|}{\sqrt{2j + 1}}
\leq \frac{1.086435\pi^{-1/4}\sqrt{2\pi}}{\sqrt{2q + 3}} \sqrt{MD}
\]

Finally, if \(q \in \mathbb{N}\), then we also obtain
\[
\sup_{f \in \mathcal{L}} \left| \sum_{j=q+1}^{\infty} \langle f, \psi_j \rangle \cdot (i\phi_j(t) \cdot \mathbb{E}_P [Y_1 \exp(itY_2)] - \phi_j'(t) \cdot \mathbb{E}_P [\exp(itY_2)]) \right|
\leq \sup_{f \in \mathcal{L}} \sum_{j=q+1}^{\infty} |\langle f, \psi_j \rangle| \cdot \left| (\phi_j(t) \cdot \mathbb{E}_P |Y_1| + |\phi_j'(t)|) \right|
\leq \sup_{f \in \mathcal{L}} \sum_{j=q+1}^{\infty} \sqrt{2j + 1} |\langle f, \psi_j \rangle| \cdot \frac{|\phi_j(t) \cdot \mathbb{E}_P |Y_1| + \sqrt{j/2} |\phi_{j-1}(t)| + \sqrt{(j+1)/2} |\phi_{j+1}(t)|}{\sqrt{2j + 1}}
\leq \sup_{f \in \mathcal{L}} \sum_{j=q+1}^{\infty} \sqrt{2j + 1} |\langle f, \psi_j \rangle| \cdot \sup_{j=q+1,...} \frac{|\phi_j(t) \cdot \mathbb{E}_P |Y_1| + \sqrt{j/2} |\phi_{j-1}(t)| + \sqrt{(j+1)/2} |\phi_{j+1}(t)|}{\sqrt{2j + 1}}
\leq \sqrt{MD} \left( \frac{\mathbb{E}_P |Y_1|}{\sqrt{2q + 3} + 1} \right) \cdot \frac{1.086435\pi^{-1/4}}{\sqrt{2\pi}}
\]
where the second inequality is due to (28). This completes a proof of the proposition. \(\square\)

### C Identification and Estimation from the Previous Literature

This appendix section presents the identification and estimation for the characteristic function \(\varphi_X\) and the density function \(f_X\) of \(X\) based on [Li and Vuong (1998)] and its extensions. Moreover, a choice of the tuning parameter based on [Delaigle and Gijbels (2004)] is also reviewed. Although the main text of this paper is focused on inference, one would also want to present estimates along with confidence bands as we presented in Figure 4. This appendix section provides a method of obtaining estimates for convenience of readers.
C.1 Identification and Estimation of the Characteristic Functions

For a joint distribution $P$ of $(Y_1, Y_2)$, Li and Vuong (1998) show that the characteristic functions of $X$ and $U_1$ are identified by

$$\varphi_X(t) = \exp \left( \int_0^t \frac{iE_P[Y_1 e^{itY_2}]}{E_P[e^{itY_2}]} \, d\tau \right)$$ and

$$\varphi_{U_1}(t) = \frac{E_P[e^{itY_1}]}{\exp \left( \int_0^t \frac{iE_P[Y_1 e^{itY_2}]}{E_P[e^{itY_2}]} \, d\tau \right)}$$

respectively, under the assumption of nonvanishing characteristic function of $Y_2$ in addition to Assumption 1. The sample-counterpart estimator of (31) reads

$$\hat{\varphi}_X(t) = \exp \left( \int_0^t \frac{iE_n[Y_1 e^{itY_2}]}{E_n[e^{itY_2}]} \, d\tau \right). \quad (32)$$

Similarly,

$$\hat{\varphi}_{U_1}(t) = \frac{E_n[e^{itY_1}]}{\exp \left( \int_0^t \frac{iE_n[Y_1 e^{itY_2}]}{E_n[e^{itY_2}]} \, d\tau \right)}.$$

C.2 Tuning Parameter

To estimate the probability density function $f_X$ of $X$ using the characteristic function estimator (31), we need to impose a regularization by limiting the integration for the Fourier transform to a contact interval $[-h^{-1}, h^{-1}]$ for some “bandwidth” $h$. Finite-sample choice methods of choosing the limit frequency $h$ are proposed in the literature of deconvolution kernel density estimation. One of the most widely used approaches is to minimize the MISE (Stefanski and Carroll, 1990) or its asymptotically dominating part (Delaigle and Gijbels, 2004):

$$AMISE(h) = \frac{1}{2\pi nh} \int \left| \frac{\phi_K(t)}{\varphi_{U_1}(t/h)} \right|^2 \, dt + \frac{h^4}{4} \int u^2 K(u) \, du \cdot \int f_X''(x)^2 \, dx.$$

where $\varphi_K$, supported on $[-1, 1]$, is $FK$ for some kernel function $K$.

There are alternative ways to compute $\int f_X''(x)^2 \, dx$. Based on Parseval’s identity, Delaigle and Gijbels (2004) suggest

$$\int f_X''(x)^2 \, dx = \frac{1}{2\pi h^5} \int t^4 \frac{\left| \varphi_X(t/h) \right|^2 |\varphi_K(t)|^2}{|\varphi_{U_1}(t/h)|^2} \, dt.$$

Combining the above two equations together yields

$$AMISE(h) = \frac{1}{2\pi nh} \int \left| \frac{\phi_K(t)}{\varphi_{U_1}(t/h)} \right|^2 \, dt + \frac{1}{8\pi h} \int u^2 K(u) \, du \cdot \int t^4 \frac{\left| \varphi_X(t/h) \right|^2 |\varphi_K(t)|^2}{|\varphi_{U_1}(t/h)|^2} \, dt.$$

With this formula, one may choose $h$ to minimize the plug-in counterpart of $AMISE(h)$, replacing the unknown characteristic functions $\varphi_X$ and $\varphi_{U_1}$ by the sample counterparts $\hat{\varphi}_X$ and $\hat{\varphi}_{U_1}$, respectively, in Appendix C.1.

Since the set $[-h^{-1}, h^{-1}]$ of frequencies is used for estimation, it is also a natural idea to use this set
[\[-h^{-1}, h^{-1}\]] of frequencies for inference as well, although our theory for inference does not require such a finite limit unlike the estimation which requires regularization.

C.3 Estimation of the Density Function

With the estimated characteristic function (32) and the bandwidth parameter $h$ chosen in Section C.2, the density function may be estimated by

$$
\hat{f}_X(x) = \frac{1}{2\pi} \int e^{-itx} \varphi_K(th) \hat{\varphi}_X(t) dt.
$$

The “Li-Vuong estimates” shown in Section 6 are based on the above formula together with the tuning parameter chosen according to the procedure outlined in Appendix C.2.
References

ADUSUMILLI, K., T. OTSU, AND Y.-J. WHANG (2017): “Inference on Distribution Functions under Measurement Error,” STICERD - Econometrics Paper Series 594.

ANDERSON, T. W. AND H. RUBIN (1949): “Estimation of the Parameters of a Single Equation in a Complete System of Stochastic Equations,” The Annals of Mathematical Statistics, 20, 46–63.

BABIH, A. (2018): “Honest Confidence Sets in Nonparametric IV Regression and Other Ill-posed Models,” Working paper.

BISSANTZ, N., L. DÜMBGEN, H. HOLZMANN, AND A. MUNK (2007): “Nonparametric Confidence Bands in Deconvolution Density Estimation,” Journal of the Royal Statistical Society: Series B (Statistical Methodology), 69, 483–506.

BISSANTZ, N. AND H. HOLZMANN (2008): “Statistical Inference for Inverse Problems,” Inverse Problems, 24, 034009.

BLANCHARD, P. AND E. BRUENING (2002): Mathematical Methods in Physics: Distributions, Hilbert Space Operators, and Variational Methods, Birkhäuser, 1 ed.

BONHOMME, S. AND J.-M. ROBIN (2010): “Generalized Non-Parametric Deconvolution with an Application to Earnings Dynamics,” The Review of Economic Studies, 77, 491–533.

BONHOMME, S. AND U. SAUER (2011): “Recovering Distributions in Difference-in-Differences Models: A Comparison of Selective and Comprehensive Schooling,” Review of Economics and Statistics, 93, 479–494.

CARROLL, R. J. AND P. HALL (1988): “Optimal Rates of Convergence for Deconvolving a Density,” Journal of the American Statistical Association, 83, 1184–1186.

CHEN, X. (2007): “Large Sample Sieve Estimation of Semi-Nonparametric Models,” in Handbook of Econometrics, ed. by J. J. Heckman and E. E. Leamer, Elsevier, chap. 76, 5549–5632.

CHEN, X. AND T. M. CHRISTENSEN (2018): “Optimal Sup-norm Rates and Uniform Inference on Nonlinear Functionals of Nonparametric IV Regression,” Quantitative Economics, 9, 39–84.

CHERNOZHUKOV, V., D. CHETVERIKOV, AND K. KATO (2017): “Testing Many Moment Inequalities.” Working paper.

COMTE, F. AND C. LACOUR (2011): “Data-Driven Density Estimation in the Presence of Additive Noise with Unknown Distribution,” Journal of the Royal Statistical Society: Series B (Statistical Methodology), 73, 601–627.

CUNHA, F., J. HECKMAN, AND S. NAVARRO (2005): “Separating Uncertainty from Heterogeneity in Life Cycle Earnings,” Oxford Economic Papers, 57, 191–261.

CUNHA, F., J. J. HECKMAN, AND S. M. SCHENNACH (2010): “Estimating the Technology of Cognitive and Noncognitive Skill Formation,” Econometrica, 78, 883–931.

DELAIGLE, A. AND I. GUJBELS (2004): “Practical Bandwidth Selection in Deconvolution Kernel Density Estimation,” Computational Statistics & Data Analysis, 45, 249–267.
Delaigle, A. and P. Hall (2015): “Methodology for Non-Parametric Deconvolution When the Error Distribution Is Unknown,” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 78, 231–252.

Delaigle, A., P. Hall, and A. Meister (2008): “On Deconvolution with Repeated Measurements,” *Annals of Statistics*, 36, 665–685.

Diggle, P. J. and P. Hall (1993): “A Fourier Approach to Nonparametric Deconvolution of a Density Estimate,” *Journal of the Royal Statistical Society. Series B (Methodological)*, 55, 523–531.

Efromovich, S. (1997): “Density Estimation for the Case of Supersmooth Measurement Error,” *Journal of the American Statistical Association*, 92, 526–535.

Erdélyi, A., W. Magnus, F. Oberhettinger, and F. G. Tricomi (1953): *Higher Transcendental Functions*, vol. 2, McGraw-Hill.

Fan, J. (1991a): “Asymptotic Normality for Deconvolution Kernel Density Estimators,” *Sankhy: The Indian Journal of Statistics, Series A (1961-2002)*, 53, 97–110.

——— (1991b): “On the Optimal Rates of Convergence for Nonparametric Deconvolution Problems,” *The Annals of Statistics*, 19, 1257–1272.

Folland, G. B. (2007): *Real Analysis: Modern Techniques and Their Applications*, Wiley, 2 ed.

——— (2009): *Fourier Analysis and Its Applications*, American Mathematical Society.

Gallant, A. and D. W. Nychka (1987): “Semi-nonparametric Maximum Likelihood Estimation,” *Econometrica*, 55, 363–90.

Hall, P. and J. Horowitz (2013): “A Simple Bootstrap Method for Constructing Nonparametric Confidence Bands for Functions,” *Annals of Statistics*, 41, 1892–1921.

Horowitz, J. L. and S. Lee (2012): “Uniform Confidence Bands for Functions Estimated Nonparametrically with Instrumental Variables,” *Journal of Econometrics*, 168, 175–188.

Horowitz, J. L. and M. Markatou (1996): “Semiparametric Estimation of Regression Models for Panel Data,” *The Review of Economic Studies*, 63, 145–168.

Johannes, J. (2009): “Deconvolution with Unknown Error Distribution,” *Annals of Statistics*, 37, 2301–2323.

Kato, K. and Y. Sasaki (2017a): “Uniform Confidence Bands in Deconvolution with Unknown Error Distribution,” ArXiv:1608.02251.

——— (2017b): “Uniform Confidence Bands for Nonparametric Errors-in-Variables Regression,” ArXiv:1702.03377.

Kennan, J. and J. R. Walker (2011): “The Effect of Expected Income on Individual Migration Decisions,” *Econometrica*, 79, 211–251.

Kotlarski, I. (1967): “On Characterizing the Gamma and the Normal Distribution.” *Pacific Journal of Mathematics*, 20, 69–76.
Krasnokutskaya, E. (2011): “Identification and Estimation of Auction Models with Unobserved Heterogeneity,” The Review of Economic Studies, 78, 293–327.

Li, T., I. Perrigne, and Q. Vuong (2002): “Structural Estimation of the Affiliated Private Value Auction Model,” The RAND Journal of Economics, 33, 171–193.

Li, T. and Q. Vuong (1998): “Nonparametric Estimation of the Measurement Error Model Using Multiple Indicators,” Journal of Multivariate Analysis, 65, 139–165.

Neumann, M. H. and O. Hössjer (1997): “On the Effect of Estimating the Error Density in Nonparametric Deconvolution,” Journal of Nonparametric Statistics, 7, 307–330.

Rao, B. (1992): Identifiability in Stochastic Models: Characterization of Probability Distributions, Probability and mathematical statistics, Academic Press.

Schennach, S. M. (2004): “Estimation of Nonlinear Models with Measurement Error,” Econometrica, 72, 33–75.

Stefanski, L. A. and R. J. Carroll (1990): “Deconvolving Kernel Density Estimators,” Statistics, 21, 169–184.

van Es, B. and S. Gugushvili (2008): “Weak Convergence of the Supremum Distance for Supersmooth Kernel Deconvolution,” Statistics & Probability Letters, 78, 2932–2938.