Lorentz-covariant quantum transport and the origin of dark energy

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Abstract
A possible explanation for the enigma of dark energy, responsible for about 76% of the mass–energy of the universe, is obtained by requiring only that the rigorous continuity equation (the Boltzmann transport equation) for quanta propagating through space should have the form of a Lorentz-covariant and dispersion-free wave equation. This requirement implies (i) properties of space–time that an observer would describe as uniform expansion in agreement with Hubble’s law and (ii) that the quantum transport behaves like in a multiplicative medium with multiplication factor $\nu = 2$. This inherent, essentially explosive multiplicity of vacuum, caused by the requirement of Lorentz covariance, is suggested as a potential origin of dark energy. In addition, it is shown (iii) that this requirement of Lorentz-covariant quantum transport leads to an apparent accelerated expansion of the universe.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Physics entered the new millennium with the enigma that the universe had just been found to be subjected to an accelerated expansion [1, 2], as if caused by some dark energy. This added to the mystery inherent in the earlier discoveries by Fritz Zwicky [3], and later by Very Rubin [4], of large amounts of dark matter around galaxies, and these discoveries then finally began to be taken seriously.

Thus, surprisingly, it became clear that most of the mass in the universe must be in the form of something hitherto unknown. The mass–energy of the universe is now considered to consist of about 76% of dark energy and 20% of dark matter [5]. Regarding the nature and origin of both, there still seems to be no clear consensus [6] despite this being a field of intense interest [7–9].

At present, the approach that seems most favourable to describe dark energy is a revival (one more time) of Einstein’s cosmological constant $\Lambda$. The cosmological constant, it should be recalled, was originally introduced by Einstein as a ‘fudge factor’ to permit steady-state solutions to his gravitational equations (which it actually didn’t—since they were unstable—and which also turned out to be unnecessary anyway after Hubble had discovered that the universe indeed expands).

However, invoking the cosmological constant again might at least this time be supported theoretically by the fact pointed out by Zel’dovich [10] that the cosmological constant is mathematically equivalent to the stress-energy of vacuum, which in quantum field theory is filled with virtual particles. Unfortunately, however, calculations along these lines give estimates that are at least 60 orders of magnitude wrong [5]. The physical mechanism behind cosmic acceleration thus still remains a deep mystery.

Recent results from the seven-year Wilkinson Microwave Anisotropy Probe (WMAP) and other experiments indicate that on cosmological scales the universe is flat, i.e. it has a density parameter $\Omega_{\text{tot}} = 1$ (corresponding to a cosmological constant $\Lambda = -1$), at least to within a very narrow error margin ($\Omega_{\text{tot}} = 1.0023 \pm 0.0056$) [11]. However, it should be remarked that on ‘smaller’ scales—up to the size of galaxies—gravitational effects are still important and dominate over the effects studied in this paper. Matter can then clump together under the influence of gravity, and these clumps—e.g. galaxies—will not expand individually, even

1 http://en.wikipedia.org/wiki/Lambda-CDM_model.
though they recede from each other due to expansion on the cosmological scale as discussed in this paper.

The problem of the origin of cosmic acceleration and dark energy will be studied here—from first principles and in an assumed flat universe—by requiring only that the exact transport equation (the Boltzmann transport equation) for quanta propagating through space should be Lorentz covariant and dispersion free, and equivalent to a wave equation as given e.g. in electromagnetism.

It will be shown that this simple and natural assumption leads to a condition that an observer would interpret as an accelerated expansion of the universe, and also a condition in which huge amounts of quanta are seemingly released, resembling the multiplication process in a nuclear fission explosive—albeit on a quite different time-scale. It is suggested in this paper that this could be a possible origin of dark energy, and could explain the huge amounts of dark energy now present in the universe.

It should be emphasized that the mechanism studied here implies that dark energy and the accelerated expansion of the universe are thus two independent consequences of the requirement of Lorentz-covariant quantum transport. Dark energy and accelerated expansion are hence not dynamically connected to each other—accelerated expansion is not driven by any pressure from dark energy.

The first calculations as outlined above were given in detail in a paper [12], published some 35 years ago; however, because the predicted quantum multiplication process seemed to have no relationship to the astronomical picture at the time, the paper had very little impact. Now, however, the problem of the origin of dark energy and the observed accelerated expansion of the universe may perhaps make it meaningful to carry the ideas in the 1975 paper to their logical conclusion. In order to give the basis for the discussion later in this paper, the 1975 paper will now be recapitulated (although the reader is encouraged to consult the original paper for the more detailed derivation given there).

2. Boltzmann’s transport equation

The time-dependent propagation of neutral quanta (such as e.g. in gamma radiation) moving with the velocity of light \( c \) through a medium, with which they interact by localized collisions, is rigorously described by the Boltzmann transport equation [13],

\[
\frac{df(r, t, \Omega)}{c \, dt} = -\Omega \cdot \nabla f(r, t, \Omega)
+ \int \Sigma(r, t, \Omega') K(r, t, \Omega' \rightarrow \Omega) f(r, t, \Omega') d\Omega'
- \Sigma(r, t, \Omega) f(r, t, \Omega) + S(r, t, \Omega),
\]

(1)

where \( S(r, t, \Omega) \) is a source term, and \( f(r, t, \Omega) \) is the angular flux in direction \( \Omega = (\Omega_x, \Omega_y, \Omega_z) \) at point \( r = (x, y, z) \) and time \( t \). Possible interactions with the medium through which the quanta propagate are described by the interaction cross-section \( \Sigma(r, t, \Omega) \), and where the kernel \( K(r, t, \Omega' \rightarrow \Omega) \) then describes how quanta may become scattered from direction \( \Omega' \) to direction \( \Omega \), and/or partially absorbed or multiplied (like neutrons in fission) in the process.

It should be emphasized that the Boltzmann transport equation is a rigorous continuity equation for the angular flux, and is exact as long as the angular flux is sufficiently low so that the effects of particle–particle interactions between the propagating quanta themselves can be neglected.

In comparison, the diffusion equation is an approximate equation for the total flux \( \Phi(r, t) \), defined as

\[
\Phi(r, t) = \int f(r, t, \Omega) d\Omega.
\]

(2)

The diffusion equation can be derived from the Boltzmann transport equation (1) above, but by necessity contains serious approximations, mainly because it does not involve the angular distribution of the flux, and the diffusion equation also describes an infinite propagation velocity. Nevertheless, for quanta undergoing isotropic scattering in a homogeneous medium, the quantum propagation as described by the flux \( \Phi(r, t) \) in (2) can be derived [12] rigorously from the Boltzmann transport equation (1) to take the form of the ‘telegrapher’s equation’ [14],

\[
\Delta \Phi - \frac{\partial^2 \Phi}{c^2 \partial t^2} - \left( \frac{1}{3D} + \Sigma_a \right) \frac{\partial \Phi}{\partial t} - \left( \frac{\Sigma_a}{3D} + \frac{\partial \Sigma_a}{c \, \partial t} \right) \Phi
+ \left( \frac{1}{3D} + \frac{\partial}{c \, \partial t} \right) S = 0,
\]

(3)

where

\[
D = \frac{1}{\Sigma},
\]

(4)

\[
\Sigma_a = (1 - v) \Sigma,
\]

(5)

with

\[
v = \int K(r, t, \Omega' \rightarrow \Omega) d\Omega',
\]

(6)

and where e.g. the value \( v = 0 \) corresponds to pure absorption, \( v = 1 \) to pure scattering and \( v > 1 \) to a multiplying medium. The condition of isotropic scattering in a homogeneous medium implies \( \Sigma(r, t, \Omega) = \Sigma(t) \), so that \( \Sigma_a = \Sigma_a(t) \).

3. Special case—wave equation

Due to the third and fourth terms on the left-hand side of the telegrapher’s equation (3) above, this equation will not be Lorentz covariant and it will also display dispersion—two properties that would make it incompatible with a wave equation derived from e.g. electromagnetism. However, we note that the telegrapher’s equation is compatible with a Lorentz-covariant and dispersion-free quantum propagation as described by a wave equation if the third and fourth terms in (3) satisfy the following two conditions:

\[
\frac{1}{3D} + \Sigma_a = 0,
\]

(7)

\[
\frac{\Sigma_a}{3D} + \frac{\partial \Sigma_a}{c \, \partial t} = 0,
\]

(8)

i.e.

\[
\frac{\partial \Sigma_a}{c \, \partial t} - \Sigma_a^2 = 0,
\]

(9)
which has the solution \( \Sigma_0 = -\frac{1}{R + ct} \). (10)

From (4) and (7), we see that \( \Sigma_0 = -\Sigma \), and from (10) and (5), respectively, we thus obtain

\[
\Sigma = \frac{1}{R + ct}, \quad v = 2.
\]

Note that the second-order time derivative in the basic wave equation (3) (after setting (7) and (8)) may lead to a wavelength with an arbitrarily much shorter characteristic length than the parameter \( R \) in (10). This wavelength corresponding to the time derivative may easily be in e.g. the optical region, despite the fact that the parameter \( R \) may possibly be up to the order of the extension of the observable universe.

4. The Pareto distribution

The Lorentz-covariant and dispersion-free quantum transport derived above is thus a Markov process with a multiplication factor of 2, and a cumulative path-length distribution function given by the following expression [15]:

\[
F(s) = 1 - \exp\left[-\int_0^s \frac{ds'}{R + s'}\right],
\]

or evaluated as

\[
F(s) = 1 - \frac{R}{R + s}, \quad f(s) = \frac{R}{(R + s)^2},
\]

that is a Pareto distribution of the second kind [16], a distribution more commonly encountered in economics and sociology. The frequency distribution corresponding to the above cumulative path-length distribution in (14) is

\[
\int_0^\infty s f(s) \, ds = \int_0^\infty \frac{sR}{(R + s)^2} \, ds = \infty.
\]

A typical Pareto frequency distribution is illustrated in figures 1 and 2.

The path-length distribution above is thus a rational function in contrast to the exponential path-length distribution normally encountered in transport theory. In particular—and quite different from normal particle propagation in a medium—the mean-free path corresponding to the Pareto distribution above will be infinite.

\[
\int_0^\infty s f(s) \, ds = \int_0^\infty \frac{sR}{(R + s)^2} \, ds = \infty.
\]

Figures 3 and 4 show the result of a computer simulation, in which a particle is started at the origin and then followed through successive collisions as described by the Pareto distribution, and where each collision becomes the starting point for two new trajectories.
shows the radial distribution of particles around the origin. We see how (19) (22) (24) (20) (21) (23) (17) (20)

As a prelude to the discussion below in connection with figures 6–8, figure 5 shows the radial distribution of particles as a function of time from a Monte Carlo simulation of Pareto transport and particle doubling, as discussed earlier, here with 200 particles started at radius \( r = 0 \) and time \( t = 0 \).

5. Exponentially accelerated expansion

The somewhat peculiar transport process described with time-varying cross-sections and quantum creation will now be further analysed by demonstrating how it can be considered as a particular representation of a much simpler transport process.

The simplest non-trivial transport is linear transport in an infinite, homogeneous and isotropic medium with pure isotropic scattering described by a scattering cross-section \( \Sigma_0 \), constant in space and time, and a multiplication factor \( v_0 = 1 \) (i.e. with the kernel \( K = 1/4\pi \) above). As discussed earlier, the quantum transport will then be exactly described by the following Boltzmann equation for the angular flux \( \phi(\rho, \tau, \omega) \) in space coordinates \( \rho \), time \( \tau \), and direction \( \omega \),

\[
\frac{\partial \phi(\rho, \tau, \omega)}{c \partial t} = -\omega \cdot \nabla \phi(\rho, \tau, \omega) + \int \frac{\Sigma_0}{4\pi} \phi(\rho, \tau, \omega) \, d\omega - \Sigma_0 \phi(\rho, \tau, \omega) + S(\rho, \tau, \omega),
\]

(17)

which, as discussed earlier, will not lead to a Lorentz-covariant and dispersion-free wave equation in the world \( \rho \tau \).

However, we may transform the above Boltzmann equation (17) into one that does represent a Lorentz-covariant and dispersion-free transport by making the following variable transformations:

\[
d\rho = \alpha \, d\tau,
\]

(18)

\[
d\tau = \alpha \, dt,
\]

(19)

\[
\phi(\rho, \tau, \omega) = \alpha f(r, t, \Omega),
\]

(20)

where

\[
\alpha = \frac{2}{\Sigma_0 (R + ct)^{-1}}.
\]

(21)

Making these transformations in the simple, non-Lorentz-covariant transport equation (17), we obtain

\[
\frac{\partial f(r, t, \Omega)}{c \partial t} + f(r, t, \Omega) \frac{\partial \alpha}{c \alpha \partial t} = -\Omega \cdot \nabla f(r, t, \Omega)
\]

\[
+ \int \frac{1}{R + ct} \frac{2}{4\pi} f(r, t, \Omega) \, d\Omega - \frac{2}{R + ct} f(r, t, \Omega),
\]

(22)

which since \( \partial \alpha / \partial t = -c \alpha / (R + ct) \) simplifies to

\[
\frac{\partial f(r, t, \Omega)}{c \partial t} = -\Omega \cdot \nabla f(r, t, \Omega) + \int \frac{1}{R + ct} \frac{2}{4\pi} f(r, t, \Omega) \, d\Omega
\]

\[
- \frac{1}{R + ct} f(r, t, \Omega),
\]

(23)

which we can identify from (1) and (6) to have \( \Sigma = 1/(R + ct) \) and \( v = 2 \), and hence corresponds to a Lorentz-covariant and dispersion-free transport as seen in (11) and (12).

We note from the two transformations \( d\rho = \alpha \, d\tau \) and \( d\tau = \alpha \, dt \) in (18) and (19) that the velocity of light is equal to \( c \) in both the \( \rho \tau \) and \( rt \) systems, and that by using (21) the relationship between \( \tau \) and \( t \) can be derived from the following equation:

\[
d\tau = \alpha \, dt = \frac{2 \, dt}{\Sigma_0 (R + ct)},
\]

(24)

which integrates to

\[
\tau = \frac{2}{c \Sigma_0} \ln \left( 1 + \frac{ct}{R} \right),
\]

(25)
or in units so that $2/(c\Sigma_0) = 1$ and $c/R = 1$,

\[ \tau = \ln(1+t), \] (26)

i.e.

\[ t = e^\tau - 1. \] (27)

Differentiating (27), i.e.

\[ dt = e^\tau d\tau, \] (28)

and comparing with the transformation $d\tau = \alpha dt$ in (19), we thus obtain $\alpha = e^{-\tau}$, and hence $d\rho = \alpha d\tau = e^{-\tau}d\tau$, i.e.

\[ dr = e^\tau d\rho. \] (29)

Integrating (29) (with suitable origins) we thus obtain

\[ r = e^\tau \rho. \] (30)

As defined above, the $\rho\tau$ system is a system with ‘classical’ space–time. Compared with the simple, classical transport in the $\rho\tau$ system, the Lorentz-covariant system $rt$, with which the $\rho\tau$ system coincides for $t = \tau = 0$, is thus subjected to an exponentially accelerated expansion as given by (29) and (30), showing how a line element $dr$ in the Lorentz-covariant $rt$ system increases exponentially with time $\tau$ (and correspondingly for a time element $dt$ as shown in (28)).

6. Discussion of the $\rho\tau$ and $rt$ systems

It is illustrative to consider Monte Carlo simulations of particle transport in the $\rho\tau$ and $rt$ systems discussed. First of all, it is necessary to consider what happens to figure 5 with better statistics, i.e. with more particles started at $r = 0$ and $t = 0$ in the $rt$ system, as shown in figure 6.

We see that particles then accumulate in the wave front, whereas behind it the particle distribution becomes smeared out to an essentially homogeneous background. This thus illustrates how the combination of the Pareto distribution in (14) and the particle doubling in (12) indeed leads to what looks like propagation of a wave front, despite the fact that it is very much the result of a process of particle scattering.

It will now be illustrated how this wave-type transport in the $rt$ system can be regarded as the result of a transformation as in (18)–(20) of the simpler transport in the $\rho\tau$ system mentioned, i.e. as a transformation of a pure isotropic scattering with multiplication factor $\nu = 1$ in an infinite, homogeneous and isotropic medium. In the $\rho\tau$ system we then have a particle distribution as in figure 7 as a function of radius $\rho$ and time $\tau$, where a Monte Carlo simulation gives a slowly expanding blob of collision points and connecting trajectories.

Transforming the $\rho\tau$ system in figure 7 to the $rt$ system according to (27) and (30), we then get a picture as in figure 8, which apart from minor statistical scatter agrees well with figure 6, thus giving a numerical illustration of the transformation defined by (27) and (30) as discussed in the previous section.

Figure 6. Monte Carlo simulation as in figure 5, but with 20 000 particles started. The wave front $r = t$ is now the dominating feature.

Figure 7. Monte Carlo simulation with 15 000 particles started at $\rho = 0$, $\tau = 0$ and with $c = 1$, and followed for 50 collisions in the simple $\rho\tau$ system: an infinite, homogeneous and isotropic medium with pure isotropic scattering and a multiplication factor $\nu = 1$.

Figure 8. Transformation according to (27) and (30) of the simple transport in the $\rho\tau$ system in figure 7, giving a result essentially as in figure 6.
are dependent e.g. on the and the shows a which according to (\(\text{particles started at time zero, 15 generations of particle doubling according to (12). The solid curve is an example of a corresponding, exponentially accelerated expansion as a function of time as given by (30). In this example, there is an initial period when the exponentially accelerated inflation dominates, which is then replaced by an epoch when the collisions dominate, then being replaced again by an epoch when the exponentially accelerated expansion dominates.}

7. Quantum multiplication versus expansion

It might perhaps superficially look as if the exponentially accelerated expansion derived would be a dynamical effect of the multiplication of the quantum flux described earlier in this paper. However, it should be emphasized that this flux multiplication and the accelerated expansion are completely unrelated phenomena dynamically. In this case, they are just two essentially independent consequences of the change of metric required by Lorentz covariance of the transport equation as discussed above.

The flux multiplication discussed in section 3 and the exponentially accelerated expansion discussed in section 5 will, in general, display different behaviour as a function of time, as is illustrated by the example in figure 9 which gives the results of a Monte Carlo simulation based on the above derivation. The example in figure 9 shows a short initial period when exponentially accelerated inflation dominates the evolution of the universe. This short period of initial exponential inflation is then followed by a long period when the exponential expansion is hidden behind a collision-dominated world and then again followed by a recent period dominated by the exponential acceleration.

Thus the general behaviour in this simulated example is in crude qualitative agreement with the current cosmological picture. However, it should be emphasized that the details of the simulation in figure 9 are dependent e.g. on the assumed number of generations and on the exponential parameter used in the accelerated expansion. As will be further discussed, it is possible that the late exponential expansion \(e^t = 2^{t/\ln(2)}\) given in (30) could in the present epoch more or less exactly match the quantum duplication function of type \(2^t\) as given by (12), and together give an essentially constant mass–energy in the universe as a function of time.

8. Is energy conserved?

The mechanism described to ensure that particle transport in the \(\rho t\) system is relativistically covariant thus requires an accelerated expansion and quantum duplication to be forced upon the system. This raises a serious question regarding the conservation of mass–energy in the \(\rho t\) system in this process. This question will now be addressed.

Noether’s symmetry theorem (see e.g. the summary in [17]), which can be shown to be valid under very general conditions, states that symmetry properties of a system lead to conservation laws: symmetry under translation corresponds to conservation of momentum, symmetry under rotation corresponds to conservation of angular momentum, symmetry in time corresponds to conservation of energy, etc.

Although conservation of energy is normally ascertained by symmetry with respect to time, such symmetry may be in doubt on cosmological time scales and for objects of galactic dimensions subjected to accelerated expansion. In particular for phenomena like the combined accelerated expansion and particle duplication described in this paper, there seems to be no guarantee that the first law of thermodynamics in its normal formulation should any longer be strictly valid for the system.

On the other hand, we also considered the \(\rho t\) system with an assumed infinite, homogeneous and isotropic medium with pure isotropic scattering and a constant scattering probability in space and time. Such a system will be symmetric in time over cosmological time scales and galactic dimensions, and the validity of the first law of thermodynamics will thus be ascertained in the \(\rho t\) system.

The relationship between the \(\rho t\) system and the \(\rho t\) system as given in (27) and (30) will then also put stringent conditions on the energy content of the \(\rho t\) system as a function of time, even though energy may not be strictly conserved in the \(\rho t\) system over cosmological times. Note also the possibility discussed in the previous section that the exponential expansion and quantum duplication in the \(\rho t\) system may match each other to produce a constant mass–energy in the universe as a function of time despite its expansion.

9. Paradigm shift?

What is left out of the discussion in this paper is the nature of the ‘classical, non-Hubble world’ \(\rho t\), which in principle could be an essentially eternal, steady-state world, and compared to which we see distant galaxies in the Lorentz-covariant world \(\rho t\) like in perspective distortion due to the requirement of Lorentz covariance.

More critically, what is also left out of the discussion is the nature of the quanta assumed to constitute dark energy, and the related question of the physical mechanism behind the duplication process by which these quanta get multiplied and so pervade the universe. This latter question will now be briefly addressed.
The theory of relativity has led to a paradigm shift: we no longer question what detailed dynamical mechanisms cause the tension in fast-moving objects to make them shrink in the direction of motion, or influence the working of the balance-wheel in clocks to make them go more slowly at very high speed; we know that these phenomena are due to basic properties of space–time, not to any particular mechanical effects. After over ten years of fruitless struggle to understand dark energy, maybe we now have to accept that dark energy, and its relationship to the accelerated expansion of the universe, similarly cannot be described in mechanical terms, but are other observational effects of the relativistic properties of space–time as discussed in this paper.

10. Summary

By assuming only that the time-dependent deep-space propagation of quanta is governed by a rigorous, Lorentz-covariant continuity equation, the following observational characteristics of the universe can be deduced.

1. The universe is subjected to an apparent exponentially accelerated expansion as given in (29) and (30).
2. The quantum propagation from distant objects is subjected to an apparent duplication process as given in (12), and assumed to be the source of dark energy. As a result, the dark energy part of the total mass–energy content of the universe increases as $2^t$ with time $t$ (suitably scaled).
3. After sufficiently long time, dark energy may thus constitute a dominating part of the mass–energy content of the universe.
4. The accelerated expansion and the amount of dark energy created are two independent consequences of the Lorentz-covariant transport and are not dynamically connected to each other.
5. The quantum duplication process producing dark energy may possibly more or less balance the exponentially accelerated expansion to give an essentially constant mass–energy in the universe as a function of time despite the expansion.

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References

[1] Perlmutter S et al 1999 *Astrophys. J.* **517** 565
[2] Riess A G et al 1998 *Astron. J.* **116** 1009
[3] Zwicky F 1937 *Astrophys. J.* **86** 217
[4] Rubin V, Burstein D, Ford W K and Thonnard N 1985 *Astrophys. J.* **289** 81
[5] Frieman J, Turner M and Huterer D 2008 *Annu. Rev. Astron. Astrophys.* **46** 385
[6] Copeland E J, Sami M and Tsujikawa S 2006 *Int. J. Mod. Phys. D* **15** 1753
[7] Shapiro C, Dodelson S, Hoyle B, Samuschia L and Flaugher B 2010 *Phys. Rev. D* **82** 043520
[8] Dutta S and Scherrer R 2010 *Phys. Rev. D* **82** 043526
[9] Ziaeepour H 2010 *Phys. Rev. D* **81** 103526
[10] Zel’dovich Y B 1968 *Sov. Phys.—Usp.* **11** 381
[11] Jarosik N et al 2011 *Astrophys. J. Suppl.* **192** 14
[12] Bergstrom A 1975 *Nuovo Cimento B* **27** 145
[13] Weinberg A M and Wigner E P 1958 *The Physical Theory of Neutron Chain Reactors* (Chicago, IL: University of Chicago Press) p 223, 235
[14] Stratton J A 1941 *Electromagnetic Theory* (New York: McGraw-Hill) p 550
[15] Bell G I and Glasstone S 1970 *Nuclear Reactor Theory* (Princeton, NJ: Van Nostrand Reinhold) p 54
[16] Johnson N L, Kotz S and Balakrishnan N 1994 *Continuous Univariate Distributions* vol 1 (New York: Wiley)
[17] Courant R and Hilbert D 1989 *Methods of Mathematical Physics* vol 1 (New York: Wiley) p 262