The de Sitter space-time as attractor solution in higher order gravity

Sabine Kluske
Universität Potsdam,
Mathematisch-naturwissensch. Fakultät,
D-14415 Potsdam, Germany

1. Introduction

Hermann Weyl [1] was the first to think about gravitational field equations of higher than second order. 1951 Buchdahl [2] gave a general method for deriving the field equation from a Lagrangian, written as an arbitrary invariant of the curvature tensor and its covariant derivatives up to an arbitrarily high order. He studied the example \( R \Box R \). \( \Box \) denotes the D’Alembertian. This example leads to a field equation of sixth order. Other generalisations of the Lagrangian are \( R^k \), \( R^k \Box R \), \( R^k \Box R \), and \( R \frac{1}{1-\Box} R \). The example \( R + aR^2 + bR \Box R \) corresponds for finely tuned initial conditions to a cosmological solution with double inflation. The attempt to find more typical cosmological solutions with double inflations led to \( R^k \Box R \) [3], [4]. This Lagrangian gives unfortunately a theory with unstable weak field behaviour. The effort to connect the gravitation theory with quantum theory leads to studies about the Lagrangian \( R \frac{1}{1-\Box} R \), [5]. This Lagrangian can be approximat at the \( k \)-th step by the sum

\[
\sum_{i=0}^{k} R \Box^i R . \quad (1.1)
\]

This gives a \( 2k+4 \)-th order field equation. The Lagrangian \( c_0 R + c_1 R \Box R + c_2 R \Box R \Box R \) is discussed in [6]. In this paper we will deduce the stability properties of the de Sitter solution for \( R^k \), \( R \Box^k R \) and \( \sum_{i=0}^{k} c_i R \Box^i R \).

2. The field equation

The Lagrangian \( L = F(R, \Box R, \ldots, \Box^k R) \) leads to a field equation of \( 4k + 4 \)-th order in general:
\[ 0 = GR^{ij} - \frac{1}{2} F g^{ij} - G^{ij} + g^{ij} \Box G + \]
\[ + \sum_{A=1}^{k} \frac{1}{2} g^{ij} \left[ F_A(\Box^{A-1} R)^k \right]_{;k} - F_A(\Box^{A-1} R)^{ij}. \]  
(2.1)

The abbreviations are:

\[ F_A := \sum_{j=A}^{k} \Box^j A \frac{\partial F}{\partial \Box^j R} \]  
(2.2)

and

\[ G := F_0. \]  
(2.3)

The operator \( \Box \) denotes the D’Alembertian, “,” the partial derivation and “;” the covariant derivation. For the \( D \)-dimensional \((D = n + 1 \geq 2)\) de Sitter space-time with \((H \neq 0)\) we use

\[ ds^2 = dt^2 - e^{2Ht} \sum_{i=1}^{n} (dx^i)^2. \]  
(2.4)

The relation between \( H \) and \( R \) is

\[ R = -n(n + 1)H^2. \]  
(2.5)

We restrict \( R \) to the interval \( R < 0 \) subsequently. Other important relations are

\[ R^{ij} = \frac{R}{n+1} g^{ij} \]  
(2.6)

and

\[ \Box^k R = 0 \quad \text{for} \quad k > 0. \]  
(2.7)

We get the field equation:

\[ 0 = GR^{ij} - \frac{1}{2} F g^{ij} \]
\[ = g^{ij} \left( \frac{1}{n+1} RG - \frac{1}{2} F \right) \]  
(2.8)

for the de Sitter space-time.

If we choose the Lagrangian \((-R)^u\) with \( u \in \mathbb{R} \) the \( D \)-dimensional de Sitter space-time satisfies the field equation iff \( u = \frac{n+1}{2} = \frac{D}{2} \).

An other important Lagrangian is \( R \Box^k R \) for \( k > 0 \). For this we get

\[ F = R \Box^k R, \quad G = 2 \Box^k R \]  
(2.9)

and the solubility condition

\[ R \Box^k R = \frac{u}{n+1} R \Box^k R \]  
(2.10)

for the field equation. It is automatically satisfied because of equation (2.7)

The \( D \)-dimensional \((D > 2)\) de Sitter space-time is an exact solution of the field equation following from the Lagrangian \( R \Box^k R \) iff \( D \neq 4 \) and \( k > 0 \).
3. **The attractor property of the de Sitter space-time**

We will examine the attractor property of the de Sitter space-time in the set of the spatially flat Friedmann-Robertson-Walker model (FRW model). We use the metric

$$ds^2 = dt^2 - e^{2\alpha(t)} \sum_{i=1}^{n}(dx^i)^2$$

(3.1)

for the spatial flat FRW model. For $\alpha(t) = Ht, H > 0$ we get the de Sitter space-time metric. to find the dynamically behaviour in the neighbourhood of the de Sitter space-time we make the ansatz

$$\dot{\alpha}(t) = H + \beta(t)$$

(3.2)

for the linearisation of the field equation. This is justified, because the field equation does not depend on $\alpha$ itself, but on its derivatives only.

We say, the de Sitter space-time is an attractor solution of the differential equation if the solutions $\alpha(t)$ of the around the de Sitter space-time linearized differential equation satisfies

$$\lim_{t \to \infty} \frac{\alpha(t)}{t} = \tilde{H} = \text{const}.$$ 

(3.3)

It is enough to discuss the special de Sitter space-time with $H = 1$, because homothetic and coordinate transformations transfer de Sitter space-time of the same dimension into each other.

4. **The Lagrangian $F= (-R)^u$**

We will examine the attractor property of the de Sitter space-time in the set of the FRW models for the Lagrangian $(-R)^u$ with $2u = D = n + 1 > 2$. From this Lagrangian follows

$$F_A = 0$$

(4.1)

and

$$G = -u(-R).$$

(4.2)

We get the field equation

$$0 = -u(-R)^{u-1}R^{ij} - \frac{1}{2}g^{ij}(-R)^{u} + u \left[(-R)^{u-1}\right]^{ij} - g^{ij}u \Box \left[(-R)^{u-1}\right].$$

(4.3)

It is enough to examine the 00-component of the field equation, because all the other components are fulfilled, if the 00-component is fulfilled. We make the ansatz

$$\dot{\alpha}(t) = 1 + \beta(t)$$

(4.4)
and get

\[ R^{00} = -n\dot{\beta} - 2n\beta - n \]
\[ R = -2n\dot{\beta} - 2(n^2 + n)\beta - (n^2 + n) \]
\[ (-R)^m = 2nm(n^2 + n)^m \dot{\beta} + 2m(n^2 + n)^m \beta + (n^2 + n)^m. \]  

It follows the field equation

\[ 0 = -2n^2u(u + 1)(n^2 + n)^{u-2}\ddot{\beta} - 2n^3u(u - 1)(n^2 + n)u - 2\ddot{\beta} + \\
+ nu(2u - n - 1)(n^2 + n)^{u-1}\beta + \left( nu - \frac{1}{2}(n^2 + n) \right) (n^2 + n)^{u-1}. \]  

Using the condition 2u = n + 1 we get

\[ 0 = \ddot{\beta} + n\dot{\beta}. \]  

All solutions of the linearized field equation are

\[ \beta(t) = c_1 + c_2 e^{-nt}. \]  

It follows

\[ \alpha(t) = t + \tilde{c}_1 t + \tilde{c}_2 e^{-nt} + \tilde{c}_3 \]  

and

\[ \lim_{t \to \infty} \frac{\alpha(t)}{t} = 1 + \tilde{c}_1. \]  

The D-dimensional de Sitter space-time is for the Lagrangian \( F = (-R)^D \) an attractor solution.

5. The Lagrangian \( F=R\square^k R \)

The Lagrangian \( (-R)^D \) leads only to a field equation of fourth order for \( D > 2 \). The Lagrangian \( R\square^k R \) with \( k > 0 \) give a field equation of higher then fourth order.

For the 00-component of the field equation we need

\[ F_A = \square^{k-A} R \]  

and

\[ G = 2\square^k R \]  

and get

\[ 0 = \square^k R \left( 2R^{00} - \frac{1}{2} R \right) + 2n\dot{\alpha}\square^k R_{,0} + \\
+ \sum_{A=1}^{k} (\square^{k-A} R)(\square^A R) - \frac{1}{2}(\square^{k-A} R)_{,0}(\square^{A-1} R)_{,0}. \]
The ansatz
\[ \dot{\alpha}(t) = 1 + \beta(t) \] (5.4)
leads to
\[ R_{00} = -n\dot{\beta} - 2n\beta - n \]
\[ R = -2n\dot{\beta} - 2(n^2 + n)\beta - (n^2 + n) , \] (5.5)
and
\[ \Box(R^k) = (\Box^k R)_{00} + n(\Box^k R)_{,0} . \] (5.6)
We get the linearized field equation
\[ \Box^k R = (\Box^k R)_{,0} . \] (5.7)
For \( k = 1 \) we have
\[ 0 = \beta^{(4)} + 2n\ddot{\beta} + (n^2 - n - 1)\dot{\beta} + (-n^2 - n)\dot{\beta} \] (5.8)
with the characteristic polynomial
\[ P(t) = x^4 + 2nx^3 + (n^2 - n - 1)x^2 + (-n^2 - n)x \] (5.9)
with the roots \( x_1 = 1, \ x_2 = 0, \ x_3 = -n \) and \( x_4 = -n - 1 \). We get the solutions
\[ \beta(t) = c_1 + c_2 e^t + c_3 e^{-nt} + c_4 e^{-(n+1)t} \] (5.10)
and
\[ \alpha(t) = \tilde{c}_1 t + \tilde{c}_2 e^t + \tilde{c}_3 e^{-nt} + \tilde{c}_4 e^{-(n+1)t} \] (5.11)
and
\[ \lim_{t \to \infty} \frac{\alpha(t)}{t} = \infty . \] (5.12)
The \( D \)-dimensional de Sitter space-time is for the Lagrangian \( R \Box^k R \) not an attractor solution of the field equation. The formula
\[ 0 = (\Box^{k+1} R)_{,0} - \Box^{k+1} R = (\Box^k R,0 - \Box^k R)_{,00} + n((\Box^k R),0 - \Box^k R)_{,0} \] (5.13)
for the linearized field equation for \( k + 1 \) leads to the recursive formula for the characteristic polynomial:
characteristic polynomial for \( k + 1 = x(x + n) \cdot \) characteristic polynomial for \( k \).
The characteristic polynomial for \( k \) has the roots:
\[ x_1 = 1 \quad \text{simple} \]
\[ x_2 = 0 \quad \text{k-fold} \]
\[ x_3 = -n \quad \text{k-fold} \]
\[ x_4 = -n - 1 \quad \text{simple} . \] (5.14)
We get the solutions
\[
\beta(t) = S(t) + T(t)e^{-nt} + c_1e^t + c_2e^{(-n-1)t} \quad (5.15)
\]
and
\[
\alpha(t) = \tilde{S}(t) + \tilde{T}(t)e^{-nt} + \tilde{c}_1e^t + \tilde{c}_2e^{(-n-1)t} \quad (5.16)
\]
with \( S, T, \tilde{T} \) polynomials at most \( k \)-th degree and \( \tilde{S} \) polynomial at most \( k + 1 \)-th degree. For the most solutions is
\[
\lim_{t \to \infty} \frac{\alpha(t)}{t} = \infty \quad (5.17)
\]
fulfilled and the de Sitter space-time is not an attractor solution for the field equation derived from Lagrangian \( R\Box^k R \).

6. The generalized Lagrangian

The results of the last section have shown, that for the Lagrangian \( R\Box^k R \) with \( k > 1 \) the de Sitter space-time is not an attractor solution. The Lagrangian \( (-R)\Box^k \) gives only a fourth order differential equation. We will try to answer the following question:

Are there generalized Lagrangians so, that the de Sitter space-time is an attractor solution of the field equation?

First we make the ansatz
\[
F = \sum_{k=1}^{m} c_k R\Box^k R \quad \text{with} \quad c_m \neq 0 . \quad (6.1)
\]
In this case is the de Sitter space-time not an attractor solution, because for each term is \(+1\) a root of the characteristic polynomial of the linearized field equation.

Now we make the ansatz
\[
F = c_0 (-R)\Box^k + \sum_{k=1}^{m} c_k R\Box^k R \quad \text{with} \quad c_m \neq 0 . \quad (6.2)
\]
for the generalized Lagrangian. It follows the characteristic polynomial
\[
P(x) = x(x + n) \left[ c_0 + \sum_{k=1}^{m} c_k x^{k-1}(x + n)^{k-1}(x - 1)(x + n - 1) \right] \quad (6.3)
\]
for the linearized field equation. The solutions \( x_1 = 0 \) and \( x_2 = -n \) do not depend on the coefficients \( c_i \) of the Lagrangian. It is sufficient to look for the roots of the polynomial
\[
P(x) = c_0 + \sum_{k=1}^{m} c_k x^{k-1}(x + n)^{k-1}(x - 1)(x + n - 1) . \quad (6.4)
\]
If the above polynomial only has solutions with negative real part, then is the de Sitter space-time an attractor solution for the field equation. The transformation

\[ z = x^2 + nx + \frac{n^2}{4} \quad (6.5) \]

gives

\[
P(x) = Q(z) = c_0 + \sum_{k=1}^{m} c_k \left( z - \frac{n^2}{4} \right)^{k-1} \left( z - \frac{n^2}{4} - n - 1 \right) = \left[ c_l + \sum_{k=l+1}^{m} c_k \left( -\frac{n^2}{4} \right)^{k-l-1} \right] \cdot \left[ \left( k-1 \right) \left( -\frac{n^2}{4} \right) - \left( k-1 \right) \left( \frac{n^2}{4} + n + 1 \right) \right] z^l + c_m z^m \quad (6.6)
\]

Now let be

\[
a_{ll} = 1 \quad l = 0, \ldots, m
\]
\[
a_{0k} = -\left( -\frac{n^2}{4} \right)^{k-1} \left( \frac{n^2}{4} + n + 1 \right) \quad k = 1, \ldots, m
\]
\[
a_{lk} = \left( -\frac{n^2}{4} \right)^{k-l-1} \left[ \left( k-1 \right) \left( -\frac{n^2}{4} \right) - \left( k-1 \right) \left( \frac{n^2}{4} + n + 1 \right) \right] \quad l < k \leq m
\]
\[
a_{kl} = 0 \quad \text{else} \quad . (6.7)
\]

This gives the equation

\[
\begin{pmatrix}
d_0 \\
\vdots \\
d_m
\end{pmatrix} = A \begin{pmatrix}
c_0 \\
\vdots \\
c_m
\end{pmatrix} \quad \text{with} \quad A \quad \text{regular} \quad . (6.8)
\]

The roots of \( P(x) \) have a negative real part iff the roots of \( Q(z) \) are from the set

\[
M := \left\{ x + iy : x > \frac{n^2}{4} \land |y| < n\sqrt{x - \frac{n^2}{4}} \right\} \quad . (6.9)
\]

If the roots \( z_k \) of the polynomial \( Q(z) \) are elements of \( M \), then the coefficients \( d_k \) are determined by

\[
Q(z) = \sum_{k=0}^{m} d_k z^k = \prod_{k=1}^{m} (z - z_k) \quad . (6.10)
\]

The coefficients

\[
\begin{pmatrix}
c_0 \\
\vdots \\
c_m
\end{pmatrix} = A^{-1} \begin{pmatrix}
d_0 \\
\vdots \\
d_m
\end{pmatrix} \quad . (6.11)
\]
belongs to a Lagrangian, that gives a field equation with a de Sitter attractor solution. The above considerations have shown that for every $m$ there exists an example for coefficients $c_k$, so that the de Sitter space-time is an attractor solution for the field equation derived from the Lagrangian $c_0(-R)^D + \sum_{k=1}^{m} c_k R \Box^k R$ with $c_m \neq 0$.

Acknowledgements

The author would like to thank H.-J. Schmidt and K. Peters for discussions and valuable comments.

References

[1] Weyl, H.: Sitz. Ber. Akad. Wiss. Berlin (1918) 465
[2] Buchdahl, H.: Acta Math. 85 (1951) 63
[3] Amendola, L., Mayer, A. B., Cappoziello, S., Gottlöber, S., Müller, V., Occhionero, F. and Schmidt, H.-J.: Class. Quantum Grav. 10 (1993) L43
[4] Gottlöber, S., Müller, V. and Schmidt, H.-J.: Astron. Nachr. 312 (1991) 291
[5] Gottlöber, S., Schmidt, H.-J. and Starobinsky, A. A.: Class. Quantum Grav. 7 (1990) 893
[6] Schwarz, D. J. and Kummer, W.: Class. Quantum Grav. 10 (1993) 235
[7] Mayer, A. B. and Schmidt, H.-J.: Class. Quantum Grav. 10 (1993) 2441