Understanding the Reconstruction of the Biased Tracer

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Received 2018 June 28; revised 2018 October 31; accepted 2018 November 18; published 2019 January 15

Abstract

Recent development in the reconstruction of the large-scale structure has seen significant improvement in restoring the linear baryonic acoustic oscillation (BAO) from at least the nonlinear matter field. This outstanding performance is achieved by iteratively solving the Monge–Ampere equation of the mass conservation. However, this technique also relies on several assumptions that are not valid in reality, namely the longitudinal displacement, the absence of shell-crossing, and the homogeneous initial condition. In particular, the conservation equation of the tracers comprises the biasing information that breaks down the last assumption. Consequently, direct reconstruction would entangle the nonlinear displacement with complicated bias parameters and further affect the BAO. In this paper, we formulate a theoretical model describing the reconstructed biased map by matching the tracer overdensity with an auxiliary fluid with vanishing initial perturbation. Regarding the performance of the reconstruction algorithm, we show that even though the shot noise is still the most significant limiting factor in a realistic survey, inappropriate treatment of the bias could also shift the reconstructed frame and therefore broaden the BAO peak. We suggest that, in principle, this bias-related BAO smearing effect could be used to independently self-calibrate the bias parameters.

Key words: large-scale structure of universe

1. Introduction

The extraction of cosmological information from the abundant galaxies survey data has long been one of the primary objectives of the large-scale structure studies. At the large scale where the density perturbations are still linear, this process is quite straightforward from measuring the two-point function of the distribution of galaxies. However, as the nonlinearity of the structure formation gradually takes effect at smaller scales, the information starts to leak into higher-order statistics (Rimes & Hamilton 2005, 2006). One well-known example is the baryonic acoustic oscillation (BAO) peak that has been used as a standard ruler to measure the expansion history of our universe (Blake & Glazebrook 2003; Eisenstein 2003; Hu & Haiman 2003; Seo & Eisenstein 2003; Eisenstein et al. 2005). This sharp feature was smeared out when particles/galaxies started to move out of their original locations (Crocce & Scoccimarro 2008) and eventually caused us to lose the constraining power on dark energy (Seo & Eisenstein 2007; Ngan et al. 2012).

The question is whether we can undo the structure formation and recover the primordial information from the data. Two different strategies emerged to tackle this problem, one is the forward modeling, and the other one is the backward reconstruction. Both approaches have made significant progress recently. The first approach samples the vast parameter space of initial fluctuations and compares its forward evolution, using the simplified Lagrangian model, the FastPM simulation, or the full N-body nonlinear dynamics, against observations. It usually involves finding the maximum a posteriori solution using Gibbs and/or Hamiltonian Markov chain Monte Carlo sampling (Wang et al. 2009, 2013, 2014; Jasche & Wandelt 2013; Feng et al. 2018), which are usually computationally expensive.

The backward reconstruction, on the other hand, directly operates on the observed data, expecting to recover the initial fluctuation. Earlier examples include the logarithmic transformation and Gaussianization (Weinberg 1992; Neyrinck et al. 2009, 2011; Wang et al. 2011), both of which attempt to Gaussianize the one-point probability function. However, as local transformations, they are incompatible with the process of the structure formation; therefore, they do not genuinely reproduce the initial condition. Eisenstein et al. (2007) focused on the nonlinear BAO degradation instead and demonstrated a simple yet powerful method to reduce the smearing of the signal. Recently, various improved algorithms, including the isobaric reconstruction (Pan et al. 2017; Wang et al. 2017; Yu et al. 2017; Zhu et al. 2017, 2018) and other iterative solutions (Schmittfull et al. 2017; Shi et al. 2018; Hada & Eisenstein 2018), have been shown to be able to almost entirely remove the degradation, and perfectly recover the linear BAO signal. Despite their divergent technical details, all these algorithms solve the displacement potential \( \phi \) from the following mass conservation equation

\[
\det \left( \frac{\partial x_i}{\partial q_j} \right) = \det (\delta_{ij}^K + \delta_{ij}^\phi) = \frac{\rho_{\text{ini}}}{\rho} = \frac{1}{1 + \delta_{ij}^\phi}, \tag{1}
\]

Here \( q \) and \( x \) are Lagrangian and Eulerian coordinates of particles, and the displacement vector and potential are defined as \( \Psi(q) = x - q = \nabla \phi(q) \), where \( \delta_{ij}^\phi \) is the Kronecker delta. By solving Equation (1), the reconstruction algorithm eliminates this nonlinear coordinate transformation and produces the \( \phi \) field on a grid that is close to the Lagrangian system, namely the isobaric frame.

Despite their excellent performance, these algorithms have at least made the following three assumptions:
(1) The displacement field is longitudinal.

(2) There is no shell-crossing.

(3) The initial perturbation is negligible, i.e., $\delta_{\text{init}} \approx 0$.

The first assumption guarantees that the problem is solvable, i.e., one unknown from one equation; assumption (2) is necessary for the uniqueness because, otherwise, any particle permutation would produce a new solution; finally, the uniform initial distribution makes sure that no extra information about the initial condition would be needed to perform the reconstruction.

Unsurprisingly, these assumptions introduce complications and errors to the real application of the algorithm. For example, the neglected transverse component, which appears until the third order in the Lagrangian perturbation theory, add extra contributions to the reconstructed potential $\phi$. Moreover, the reconstruction result is most likely meaningless below the shell-crossing scale. However, these higher-order small-scale errors have limited effects on the cosmological constraint at the BAO scale. The third assumption breaks down for any biased samples.

Of course, one could proceed and perform the reconstruction regardlessly. As demonstrated by Yu et al. (2017), the performance, characterized by the cross-correlation with the initial condition, is notably lower than that of the matter field. While the shot noise undoubtedly plays a significant role, it is unclear how the clustering bias would affect the reconstruction. In practice, we would like to understand this question because it might lead to different planning strategies for future surveys.

Theoretically, we can study a noiseless biasing model, i.e., a (deterministic) mapping from matter overdensityõ $\delta$ to the proto-halo field $\delta_t = F[\delta]$. For the given bias function $F$ or its Taylor series, we would like to understand in detail the properties of the reconstructed field. As first observed by Yu et al. (2017), the reconstructed halo field has the same large-scale bias as the halo overdensity itself. It is unsurprising because the reconstruction algorithm does not differentiate an input dark matter map from the biased tracer. However, the question remains for small-scale, nonlinear, and scale-dependent biases. How can we extend this simple intuition to describe a more complicated biasing model? Moreover, is there any practically feasible way of correcting their effects?

The reason why the reconstruction could restore the linear BAO lies in its ability to recover the Lagrangian frame $\xi$ in which the BAO features were defined. Because of those algorithmic assumptions and numerical errors, the isobaric frame $\xi$ deviates from $q$ even for matter reconstruction, even though the difference is quite small. With the clustering bias, however, it is not difficult to imagine a much larger frame shift $\xi \equiv q$ and possibly some extra smearing of the BAO.

In this paper, we would like to address these questions. In Section 2, we investigate the consequences of the direct reconstruction without any preprocessing of the bias. We formulate a theoretical model describing the reconstructed biased map by matching the tracer overdensity with an auxiliary fluid with vanishing initial perturbation. We focus on the frame shift and BAO smearing in Section 3 and then discuss the possible bias self-calibration in Section 4. Finally, we conclude in Section 5.

We sometimes denote the argument of a function as subscripts, e.g., $f(k_i, k_j) = f_{i,j}$, where the commas are used as separators. The derivative, on the other hand, uses a semicolon, $\partial \phi / \partial x_i = \phi_{,i}$.

2. Theoretical Modeling of the Isobaric Reconstruction

As demonstrated by Wang et al. (2017), at large scale the isobaric reconstruction produces the longitudinal component of the nonlinear displacement field, and Equation (1) is only solvable assuming the homogeneity of the initial density distribution. In this section, we discuss the additional effect introduced by the clustering bias and the consequences of the direct reconstruction without any preprocessing of the map.

By definition, the reconstruction of a general map $\Delta$ could be described by Equation (1), with the substitution that $\delta_t = \Delta$. Conceptually, this is equivalent to introducing an extra auxiliary fluid with a uniform initial distribution whose displacement is described by $\nabla \Delta \phi$ regardless of how $\Delta$ was formed in the first place.

Following the standard Lagrangian framework of structure formation and assuming our auxiliary fluid also obeys the Newtonian dynamics, the comoving Eulerian position $x$ of a fluid element follows the trajectory accelerated by the gravitational force $\nabla \Phi$, i.e.,

$$d^2x + \mathcal{H} dx = -\nabla \Phi,$$

where $\nabla$ denotes the gradient in Eulerian space, $d_+$ is the Lagrangian derivative with respect to the conformal time $\tau$, $\mathcal{H} = d \ln(a)/d\tau$, and $\Phi$ is the Newtonian gravitational potential that obeys the Poisson equation

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} \Delta.$$

One could then directly solve this system given the biased map $\Delta$. We present the technical details in Appendix A. The advantage of this approach is that it provides a complete framework to derive the recurrence relation of the revised displacement in all orders, e.g., following Matsubara (2015). However, because the specific growing function (as a function of $\tau$) is not relevant here as long as the reconstructed solution gives the correct value at the initial and final times, this derivation seems unnecessarily complicated.

2.1. Density Matching in Configuration Space

A much more straightforward and more intuitive approach is also available. Since the reconstruction algorithm’s only function was to find a longitudinal displacement field $\hat{\phi}_t$ that could reproduce the density map $\Delta$ as accurately as possible, one could then define $\hat{\phi}$ by matching the density field

$$1 + \hat{\delta}_{\text{rec}}[\hat{\phi}] + \varepsilon = [\det(\delta^K_{ij} + \partial^2_{xi} \hat{\phi})]^{-1} + \varepsilon = 1 + \Delta[\phi, b^{(n)}; \Psi_i; \cdots].$$

For a given model of the biased tracer $\Delta = \delta_t = F[\delta_m; b^{(n)}; \cdots]$, we could then obtain $\hat{\phi}$ by solving this equation. Here, $b^{(n)}$ is the $n$th order bias parameter and $\varepsilon$ is the reconstruction error, i.e., the density residual. Notice that the density $\Delta$ on the right-hand side of equation (4) could depend on many other ingredients as well, including the transverse component of the displacement field $\Psi_i$, which only contributes after the second loop order; therefore, we will neglect this in this paper.

We do not have to provide any mathematical proof of the existence and uniqueness of the above equation because the density residual $\varepsilon$ is unknown and will never be zero in practice. Even with the assumption that $\varepsilon = 0$, as we follow in
Because the movement of galaxies are nonlocal in the Eulerian space, the biases become scale-dependent (Matsubara 2011).

This then leads to the linear equation of the reconstructed displacement potential \( \phi^{(1)} \)

\[
\nabla_\xi^2 \phi^{(1)}(\xi(x)) = \left[ 1 + (b_1^{(1)} - 1) \frac{D(\tau_\xi)}{D(\tau)} \right] \nabla_\eta^2 \phi^{(1)}(q(x)) = (1 + b_1^{(1)}) \nabla_\eta^2 \phi^{(1)}(q(x)).
\]

Here we have once again used the fact that at the linear order \( \phi^{(1)} = -\nabla_\eta^2 \phi^{(1)} \). Following the usual convention, we have taken the limit \( \tau_\xi \to 0 \) while keeping the Lagrangian bias \( b_1^{(1)} = b_\text{LAG}(\tau_\xi)/D(\tau) \) fixed. Since the potential field is already a first-order quantity, we are safe to drop the coordinate’s difference of the Laplacian, i.e., \( \nabla_\xi^2 = \nabla_\eta^2 \), so that the linear solution of the reconstructed field is simply

\[
\phi^{(1)}(\xi(x)) = (1 + b_1^{(1)}) \phi^{(1)}(q(x)).
\]

Since the linear Eulerian bias \( b_1^{(1)} = 1 + b_1^{(1)} \), this agrees with Equation (4).

A more important aspect of deriving Equation (13) in the Lagrangian framework is that the reconstructed isobaric frame \( \xi \) will deviate from the primordial coordinate \( q \) by \( \Delta \xi \), which at the linear order equals

\[
\Delta \xi = \xi - q = (x - \Psi) - (x - \Psi) = -(b_1^{(1)} - 1) \Psi(q(x)) = -\epsilon \Psi(q(x)).
\]

It then gives \( \epsilon = b_1^{(1)} \). As will be shown in the next section, this coordinate shift could have a major consequence on the smearing of the BAO signals.

### 2.2. Higher Orders

One could extend the above calculation to higher orders. Instead of configuration space, we present in Appendix A a systematic derivation in the Fourier space. Similar to the usual Lagrangian perturbation expansion, one could define the reconstructed displacement vector

\[
\widehat{\Psi}(k) = \sum_{n \geq 1} \frac{i}{n!} \int \frac{dk_1 \cdots k_n}{(2\pi)^n} (2\pi)^3 \delta_D(k - k_1 \cdots k_n) \times \hat{L}^{(n)}(k_1 \cdots k_n) \delta_3(k_1) \cdots \delta_3(k_n),
\]

with the modified LPT kernels \( \hat{L}^{(n)} \)

\[
\hat{L}_k^{(1)} = L_k^{(1)}(1 + b_1^{(1)}),
\]

\[
\hat{L}_{k_1 k_2}^{(2)} = L_{k_1 k_2}^{(2)} + b_1^{(1)} L_k^{(1)} L_{k_2}^{(1)} + b_1^{(1)} L_k^{(1)} L_{k_2}^{(1)} - b_1^{(1)} L_k^{(1)} L_{k_2}^{(1)} - \frac{1}{2} \left[ k : L_k^{(1)} L_{k_2}^{(1)} + \text{cy.} \right].
\]

where \( L_k^{(1,2)} \) are the original LPT kernels in Equation (40).

Once again, the first-order \( \hat{L}_k^{(1)} \) agrees with Equation (13); though, here the linear Lagrangian bias could also be scale-dependent. At the second order, however, the kernel \( \hat{L}^{(2)} \) not only depends on \( L^{(2)} \) and \( b^{(2)} \), but also on lower-order \( L^{(1)} \) and \( b^{(1)} \) as well. This statement is true in that \( \hat{L}^{(n)} \) depends on all lower-order \((<n)\) LPT kernels and bias parameters.
3. The Frame Shift and BAO Damping

As mentioned previously, the reconstruction of the biased tracers inevitably induces a coordinate shift from the Lagrangian frame $q$ to isobaric frame $\xi$. In this section, we will particularly focus on its effect on the BAO.

To proceed, we recall that the BAO features are defined in the primordial Lagrangian coordinates, and one consequence of the nonlinear gravitational evolution of the matter field is the BAO smearing caused by the pairwise displacement of particles (Crocce & Scoccimarro 2006a, 2006b, 2008) $\Delta \Psi = \Psi(q) - \Psi(q')$, where the nonlinear power spectrum of the density fluctuation could, in general, be expressed as a combination of two contributions

$$P_{nl}(k) = \int \frac{d^3r}{(2\pi)^3} e^{i k \cdot r} \langle e^{i k \cdot \Delta \Psi} \rangle - 1$$

$$= G^2(k) P_{lin}(k) + P_{nlin}(k),$$

(17)

namely the propagator part which is proportional to $G^2$ and the so-called mode-coupling terms $P_{nlin}$. Here $r = q - q'$, the nonlinear propagator $G^2(k) \approx \langle e^{i k \cdot \Delta \Psi} \rangle \approx \exp(-k^2 \sigma^2)$, where $\sigma^2$ is the variance of the displacement $\sigma^2(\tau) = (1/3) \int d^3q \ P(q, \tau)/q^2$ (Crocce & Scoccimarro 2006a). The BAO signal, defined as the ratio between $P_{nl}(k)$ and the no-wiggle power spectrum $P_{nw}(k)$,

$$Wig(k) = \frac{P_{nl}(k)}{P_{nw}(k)} \approx 1 \approx \exp(-k^2 \sigma^2) \ Wig_{lin}(k),$$

(18)

will be smeared out by the variance of this displacement $\Delta \Psi$.

Similarly, the coordinate shift $\Delta \xi = \xi - q = -\epsilon \Psi(q(x))$ induces an extra smearing on the BAO. To see this, we could express our reconstructed displacement potential in Fourier space as

$$\hat{\phi}(k) = \int d^3 \xi \ e^{-i k \xi} \hskip1cm \hat{\phi}(\xi)$$

$$= \int d^3 q \ e^{-i k q} \ e^{i [k \cdot \Psi(q) - \xi]} \hat{\phi}(\xi(q)),$$

(19)

assuming the existence of the one-to-one mapping $q \to \xi$. At the leading order, the power spectrum of the reconstructed field $\hat{\phi}$ could be approximated as

$$P_{rec}(k) = \int d^3 r \ \langle e^{-i k \cdot \Delta \Psi} \hat{\phi}(0) \hat{\phi}(r) \rangle$$

$$\approx e^{-i k^2 \sigma^2} [b^2 P(k) + \cdots],$$

(20)

where $P(k)$ is the linear power spectrum and we have neglected all higher-order contributions. Consequently, without correcting the bias, one receives an extra BAO smearing proportional to $e^{-i k^2 \sigma^2}$, i.e.,

$$Wig_{rec}(k) \approx e^{-i k^2 \sigma^2} \ Wig_{lin}(k).$$

(21)

We demonstrate this effect in Figure 1. To eliminate the complications from the shot noise, we perform the reconstruction on the matter field. In this idealized example, we simply consider the matter field $\delta_m$ as a linearly biased tracer with $b = 1$. Therefore, any incorrect estimation of the bias, i.e., reconstructing $\delta_m/b$ where $\hat{b} \neq 1$, would introduce this BAO damping effect after the reconstruction. Assuming the bias estimator differs from the true value $\hat{b}$ by $\hat{b} \Delta b$, i.e.,

$$\hat{b} = \hat{b}(1 + \Delta b),$$

the coordinate shift

$$\Delta \xi = -\epsilon \Psi = -\left(\frac{\hat{b}}{b} - 1\right) \Psi$$

$$= -\left(\frac{1}{1 + \Delta b} - 1\right) \Psi \approx \Delta b \Psi,$$

(22)

so we have $\epsilon \approx \Delta b$ when $\Delta b \ll 1$.

In Figure 1, we demonstrate the reconstructed BAO after rescaling the density map $\delta_m$ with corresponding equivalent shift parameter $|\epsilon| = 1/3$, $1/2$, and $2/3$ respectively. The BAO wiggles are derived from two otherwise identical simulations, particularly the random seed, with different transfer functions, i.e., with versus without BAO. The simulations shown in this figure have $1 \ Gpc \ h^{-1}$ box with $512^3$ particles. As shown in the top panel of the figure, the larger the $\epsilon$ is (solid lines), the more suppressed the BAO wiggles appear to be after the reconstruction. In the same plot, we also illustrate the simple analytic solution, i.e., Equation (21), as dashed lines. For smaller shifting parameters $\epsilon = 1/3$ or $1/2$, this formula describes these curves reasonably well, but it starts to deviate for larger $\epsilon$. The error of the model is shown in the bottom panel of the figure.

We then perform the reconstruction on the real halo samples. Due to the shot noise contamination, the BAO wiggles, of both halo field and its reconstruction, are averaged over three realizations of $n$-body simulations with $1 \ Gpc \ h^{-1}$ box and $1024^3$ particles. The catalog comprises the largest $N$ halos with corresponding comoving number density $n = 2 \times 10^{-3}$, $5 \times 10^{-4}$, and $3 \times 10^{-4} \ (h/Mpc)^3$ respectively.

The result is presented in Figure 2. As shown in each panel, the red dashed line is the direct reconstruction from halo number density field; the solid black line corresponds to the linearly “debiased”, i.e., $\delta_m/b^{(1)}$, reconstruction; and the green dashed line is the halo reconstruction multiplied by the inverse

![Figure 1](image-url)

Figure 1. Top: the smearing of baryonic acoustic oscillation from an incorrect estimation of the linear bias. To minimize the effect of shot noise, we divided the matter density contrast by a constant $\hat{b}$ whose relation with the shift parameter $\epsilon$ is shown in Equation (22). The solid black line shows the reconstruction of the matter density field, while the solid colored lines are from the reconstruction. The dashed lines denote a simple Gaussian smearing model (Equation (18)). Bottom: Error between the reconstruction and our Gaussian damping model (Equation (18)), $\sigma[Wig] = Wig(k) - Wig_{model}(k)$. 

$\hat{b} = \hat{b}(1 + \Delta b)$, the coordinate shift

$$\Delta \xi = -\epsilon \Psi = -\left(\frac{\hat{b}}{b} - 1\right) \Psi$$

$$= -\left(\frac{1}{1 + \Delta b} - 1\right) \Psi \approx \Delta b \Psi,$$

so we have $\epsilon \approx \Delta b$ when $\Delta b \ll 1$.
of the exponential damping model (Equation (21)). Meanwhile, the BAO of the original halo field and the matter reconstructions are shown by the blue and red dotted lines respectively.

From the figure, we can see that the reconstruction produces more and clearer BAO wiggles compared to the halo map \( \delta_{\text{halo}} \) itself in every tested sample, regardless of any bias-related preprocessing details. The shot noise is still the most significant limiting factor regarding the performance of the reconstruction. A halo sample with number density \( n = 2 \times 10^{-3} \) will provide enough information for the algorithm to recover almost all BAO signals up to \( k \sim 0.3 \) h Mpc\(^{-1}\), which is enough cosmologically because the Fisher information on the sound horizon scale saturates after \( k > 0.3 \) (Wang et al. 2017). Moreover, the clustering bias of such a sample is usually very close to one, which further helps to simplify the analysis. For example, as shown in the top panel, the BAO of the reconstructed \( \delta_{\text{halo}} \) and \( \delta_{\text{halo}}/b_1 \) are almost identical.

The frame-shift effect is noticeable for high-biased samples. In the middle panel of Figure 2, where \( n = 5 \times 10^{-4} \) and linear bias \( b_1 = 1.47 \), one could see the improvement of the linearly “debiased” sample (black solid) compared to the direct reconstruction (red dashed). At least for the first four peaks \( (k < 0.25 \) h Mpc\(^{-1}\)), this improvement is consistent with our analytic solution (green dashed). For the sample with lower number density, however, such a simple estimation starts to fail, likely caused by the combination of shot noise and larger bias deviation \( \Delta b \).

### 4. Self-calibration of the Clustering Bias

#### 4.1. Linear Bias Calibration

In the standard approach, one has to measure the clustering bias before performing the reconstruction (Padmanabhan et al. 2012). However, because of the bias-related coordinate shift and wiggle smearing, in principle, we could use the sharpness of BAO peak to calibrate or even constrain the bias parameters. In Figure 3, we replot the BAO wiggles of those two high bias samples, demonstrating this idea. Since \( b_1 > 1 \) for both of the samples, one could consider the direct reconstruction (red dashed) as an underestimated bias reconstruction. We also include the wiggles from the reconstruction of an overestimated bias \( b_1^+ = 2b_1 \) (yellow solid). Compared to the correct \( b_1 \) (black solid), both of them suffer extra damping, as expected. From Equation (22), \( |\epsilon| = (1/2 - 1) = 0.5 \) for \( b_1^+ \), whereas the direct reconstruction has a shift parameter \( |\epsilon| = |b_1^+/1 - 1| \), which equals 0.47 and 0.66 for two samples respectively. So, both situations produce a similar amount of damping, which is indeed the case from the figure.
In practice, one could constrain the linear bias by repeatedly adjusting this parameter before the reconstruction. The best-fit bias would be the one producing the least amount of BAO damping. In the perspective of the configuration space correlation function, we mainly use the width information of the BAO peak, which was not utilized before. This method is independent of other types of bias measurements.

It is then interesting to see how stringent the constraint could be. For this purpose, we calculated a two-parameter Fisher matrix estimation. The Fisher matrix is expressed as Seo & Eisenstein (2003)

\[ F_{\alpha\beta} = \int \frac{\partial \ln p(k_1)}{\partial p_{\alpha}} \frac{\partial \ln p(k_2)}{\partial p_{\beta}} V_{eff}(k) \frac{k^2 dk}{(2\pi)^2}, \]

where the effective volume \( V_{eff}(k) \)

\[ V_{eff} = \left[ \frac{n P(k)}{nP(k) + 1} \right]^2 V, \]

where \( n \) is the comoving density of the sample and \( V \) is the survey volume, which we assume \( V = 10 \) (Gpc/h)^3. Here we are only interested in two parameters: the sound horizon scale \( \ln(s) \) and the bias deviation \( \Delta b_1 \). Particularly, we choose to use \( \sigma_8 \) for its nontrivial derivative

\[ \frac{\partial \ln p}{\partial (\Delta b_1)^2} = -k^2 \Delta b_1^2 \frac{\text{Wig}^{lin}(k)}{1 + \text{Wig}^{lin}(k)}. \]

The two-dimensional constraint is shown in Figure 4. We only display the upper half of the counter as \( (\Delta b_1)^2 \geq 0 \). The comoving number densities assumed here are \( 5 \times 10^{-4}, 5 \times 10^{-3}, 10^{-3} \) (h/Mpc)^3, and \( \infty \) respectively. As mentioned previously, the constraining power on the bias deviation \( \Delta b_1 \) originates from the sharpness of the BAO peak whereas the sound horizon scale from the peak location. Hence there is essentially no degeneracy among these two parameters. For a reasonably sampled survey, e.g., \( n = 5 \times 10^{-4} \) (h/Mpc)^3, the 1\( \sigma \) constraint is 0.0039 for \( \ln(s) \) and 0.059 for \( (\Delta b_1)^2 \). They then reduce to about half for a perfect shot-noiseless survey to 0.0019 and 0.023 respectively.

Compared to other bias estimation methods such as auto/cross-correlations, redshift space distortion, etc., this technique does not provide very stringent constraints. However, as an independent measurement, it serves as an additional consistency check and calibrator and helps us to realize the full potential of the BAO measurement.

4.2. Scale-dependent and Nonlinear Biases

So far we have only discussed the self-calibration and constraints on the scale-independent linear bias. For the recovery of the linear BAO, one is further encouraged to construct a matter density estimator that is as accurate as possible. For example, one could apply a k-dependent Wiener filter \( F(k) \) so that

\[ \hat{\delta}_m(k) = F(k) \delta(k), \]

with \( F(k) = \langle \delta_m \delta_k \rangle / \langle \delta_k \delta_k \rangle \). In this paper, however, we did not proceed along this direction due to technical limitations of our reconstruction solver at this moment. In practice, this filter is clearly sample dependent and needs to be carefully parameterized. The caveat is then a subtle balance between the model accuracy and the number of parameters needed.

It is also interesting to see what the higher-order corrections are if we only partially remove, say, the linear bias. To proceed, let us divide the map by some scale-dependent bias parameter \( \tilde{b}_k \) before the reconstruction. From the derivation in the Appendix, this corresponds to defining the modified LPT kernel such that

\[ \hat{L}^{(0)}_{k_1 \cdots k_n} \left[ \hat{L}^{(1) \cdots (n)} \right] = \frac{1}{\tilde{b}_k} K^{(0)}_{k_1 \cdots k_n}, \]

where \( K^{(0)} \) is the Eulerian kernel constructed from the corresponding LPT kernel \( L^{(0)} \). It is straightforward to calculate the reconstructed kernels as

\[ \hat{L}^{(1)}_k = \frac{1 + b^{(1)}_k}{\tilde{b}_k} L^{(1)}_k, \]

\[ \hat{L}^{(2)}_{k_1 k_2} = \frac{1}{\tilde{b}_k} \left[ L^{(2)}_{k_1 k_2} + b^{(1)}_k L^{(1)}_{k_1} + b^{(1)}_k L^{(1)}_{k_2} \right] + \frac{1}{\tilde{b}_k} \left[ \frac{(1 + b^{(1)}_k)(1 + b^{(1)}_k)}{b^{(2)}_{k_1 k_2}} \right] \times (k \cdot L^{(1)}_{k_1} (k \cdot L^{(1)}_{k_2} + b^{(2)}_{k_1 k_2}) \tilde{b}_k). \]

When \( \tilde{b}_k = 1 + b^{(1)}_k \) and assuming all biases are local, we could then remove the bias effect at linear order

\[ \hat{L}^{(1)}_k = L^{(1)}_k. \]
but the second-order kernel
\[
\begin{align*}
\tilde{L}_{k_1k_2}^{(2)} &= \frac{1}{b_k^2} \left[ L_{k_1k_2}^{(2)} + \left( \hat{b}_{k_1} - 1 - \frac{\hat{b}_{k_1} - \hat{b}_{k_2}}{2} (k \cdot L_{k_1}^{(1)}) \right) \right] \\
&\times L_{k_1}^{(1)} + \left( \hat{b}_{k_1} - 1 - \frac{\hat{b}_{k_1} - \hat{b}_{k_2}}{2} (k \cdot L_{k_1}^{(1)}) \right) L_{k_2}^{(1)} \\
&+ \hat{b}_{k_1k_2} \tau^{(1)},
\end{align*}
\]

is still quite complicated, and the bias-related corrections do not vanish in general. Therefore, unless we know in advance that the nonlinear and nonlocal bias parameters are accurate enough, any limited corrections would inevitably cause some complex residuals at higher orders. We defer to a more detailed study in the future.

5. Discussion and Conclusion

In this paper, we attempt to achieve a better understanding of the isobaric reconstruction of the biased tracer. Compared to the matter field, the performance is largely limited by the shot noise and clustering bias. Still, for all samples with various number densities \( n \) that we have tested, the reconstructed BAO always improves the BAO signal regardlessly. Particularly, for a reasonable spectroscopic survey with \( n \sim 10^{-4} \) (h/\text{Mpc})\(^3\), the reconstructed BAO will be washed out by the shot noise at \( k \gtrsim 0.25 \). To some extent, that will only partially affect the dark energy constraint because the information gain saturates around \( k \sim 0.3 \) (Wang et al. 2017). On the other hand, the post-reconstruction BAO degradation at low k is likely to be caused mainly by the clustering bias.

We have demonstrated that even with a simple linear debiasing, one is already able to sharpen the BAO signal. The improvement is consistent with the simple Gaussian damping model. Furthermore, it is possible that a more sophisticated nonlinear debiasing scheme might improve the reconstruction. Alternatively, assuming that all matter exists in dark matter halos, and assuming that we have a reasonably accurate estimation of the cluster mass, one could in principle construct the mass-averaged dark matter density field as a proxy to the underlying matter distribution.

Since any incorrect bias estimator would further smear the BAO wiggles, we proposed a self-calibration scheme to constrain the linear bias. A simple two-dimensional Fisher prediction showed that the constraints on the bias and sound horizon scale are orthogonal with each other, which is because they independently use the sharpness and the scale of the BAO peak respectively.

X.W. is thankful for productive discussions with Marcel Schmittfull and Matias Zaldarriaga.

Appendix A
Revised Newtonian Dynamics

In this appendix, we start from Equations (2) and (3) and directly solve the revised dynamical equations of the auxiliary fluid. For the biased reconstruction, the gravitational potential \( \Phi \) here is instead determined by the galaxy’s overdensity, i.e., \( \Delta = \delta = n_g/n_s - 1 \), where \( n_s \) is the number density. By definition, this auxiliary fluid element starts from the isobaric frame, i.e., \( x(\xi_0) = \xi \), and could be described by
\[
x_i = \xi_i + \partial_{\xi_i} \tilde{\phi}(\tau) \quad \text{along its movement, where } \tilde{\phi} \text{ is our reconstructed displacement potential. Since we only need the longitudinal part of Equation (2), combining with Equation (3), one simply has}
\]
\[
\nabla x \cdot [T(\partial_{\xi_i} \tilde{\phi})] = -4\pi G \bar{\rho} a^2 \delta_\bar{\rho}.
\]

Here we have defined the nonlinear operator \( T = d_x^2 + \mathcal{H} d_x \). We could further rewrite the above equation with the Jacobian matrix between \( \mathbf{x} \) and the isobaric frame \( \xi \),
\[
\tilde{J}_{\bar{\xi}i} = \frac{\partial x_i}{\partial \xi_j} = \delta_{\bar{\xi}i} + \tilde{\phi}_{,ij},
\]

where \( \delta_{\bar{\xi}i} \) is the Kronecker delta. Following Matsubara (2015), one could expand the above equation and have
\[
\varepsilon_{ijk} \varepsilon_{pqr} \tilde{J}_{\bar{\xi}i} \tilde{J}_{\bar{\xi}j} (\mathbf{T} \tilde{\phi}_{,q}) = -8\pi G \bar{\rho} a^2 F[\delta_\bar{\rho} J, \tilde{J}].
\]

Remember that both sides of the equation are evaluated at \( \mathbf{x} \) even though \( \tilde{J} \) and \( J \) is the determinant of the Jacobian matrix \( \tilde{J}_{\bar{\xi}i} \). We have also assumed that the galaxy overdensity \( \delta_\bar{\rho} \) is some nonlinear/nonlocal function \( F \) of the matter density \( \delta_m = 1/J - 1 \), and here \( J \) is the Jacobian from Lagrangian \( \mathbf{q} \) to the Eulerian coordinate \( \mathbf{x} \). Let us now only keep the linear bias, denoted as \( b \), we have
\[
\varepsilon_{ijk} \varepsilon_{pqr} \tilde{J}_{\bar{\xi}i} \tilde{J}_{\bar{\xi}j} (\mathbf{T} - \frac{4}{3} \pi G \bar{\rho} a^2 b) \tilde{J}_{\bar{\xi}r} = -8\pi G \bar{\rho} a^2 b \tilde{J}.
\]

To proceed, we need the dynamical equation of \( J \) as well. At the linear order, however, the homogeneous part of Equation (35) is simply
\[
\mathcal{T} \tilde{\phi}_{,ii} - 4\pi \bar{\rho} a^2 b \phi_{,ii} = 0.
\]

Here \( \phi_{,ii}(\tau) = \delta(\tau) = g(\tau) \delta_\tau(\tau = 0) \) is just the linear density perturbation, which is the solution of the equation \( \mathcal{T} - 4\pi \bar{\rho} a^2 g(\tau) = 0 \). Hence, it is straightforward to see that \( \phi_{,ii} \) has the linear solution \( \phi_{,ii}(\tau) = b \bar{g}(\tau) \). Notice that one property of this approach is that the specific growth function (as a function of \( \tau \)) is not relevant as long as it gives correct values at the boundary, i.e., at the initial and final time.

Appendix B
Nonlinear Reconstruction Kernels

In this appendix, we will apply the density matching technique to derive the higher-order LPT kernels of the reconstructed field. We start by considering the following
conservation equation (Matsubara 2011)
\[
\rho_g(x) = \rho_0 \int d^3q \, F[\delta_m(q)] \, \delta_D(x - q - \Psi).
\] (37)

Here \( \Psi \) is the displacement vector and we have introduced the Lagrangian nonlinear bias function \( \delta_e(q) = F[\delta_m(q)] \) with the property that \( \langle F[\delta_m(q)] \rangle = 1 \), so it has the perturbative expression \( F[\delta_m(q)] = 1 + b^{(1)} \delta_m + b^{(2)} (\delta_m)^2 + \cdots \). The Lagrangian bias \( b^{(0)} \) here could also be nonlocal, so generally we have \( b^{(k)}_k \) as a function of \( k \) in Fourier space. Transforming Equation (37) to Fourier space and expanding the bias function \( F \) perturbatively, one obtains (Matsubara 2011)
\[
\delta_{e,k} = \int d^3q \, e^{-ik \cdot q} \{ 1 + \delta_e(q) \} e^{-ik \cdot \Psi(q)} = (2\pi)^3 \delta_D(k)
\]
\[
= \sum_{\alpha \geq 0} \frac{(-i)^\beta}{\alpha!} \int \frac{d^3k_1 \cdots d^3k_\alpha}{(2\pi)^{3\alpha}} \frac{(2\pi)^3}{(2\pi)^{3\alpha}} \delta_D(k - k_{1 \cdots \alpha})
\]
\[
- k_1 \cdots k_\alpha \cdot \delta_{\alpha,m,k_1 \cdots k_\alpha} \cdot \delta_{\alpha,m,k_1 \cdots k_\alpha} \cdot [k \cdot \Psi_k] \cdot \ldots \cdot [k \cdot \Psi_k].
\] (38)

where the nonlinear displacement field \( \Psi_k \) has its own perturbative expansion
\[
\Psi_k = \sum_{\alpha \geq 1} \frac{i}{\alpha!} \int \frac{d^3k_1 \cdots d^3k_\alpha}{(2\pi)^{3\alpha}} \delta_D(k - k_{1 \cdots \alpha}) \times L^{(\alpha)}_{k_{1 \cdots k_\alpha}} \delta_{\alpha,m,k_1 \cdots k_\alpha},
\] (39)
where \( L^{(\alpha)} \) is the \( \alpha \)th LPT kernel. The first two are
\[
L^{(1)}_{k,k} = \frac{k}{k^2},
\]
\[
L^{(2)}_{k,k,k} = \frac{3}{7} \frac{k}{k^2} \left[ 1 - \frac{k \cdot k_2}{k k_2} \right],
\] (40)
where \( k = k_1 + k_2 \).

On the other hand, one could expand \( \delta_e \) directly in Eulerian space
\[
\delta_e = \sum_{\alpha \geq 1} \frac{i}{\alpha!} \int \frac{d^3k_1 \cdots d^3k_\alpha}{(2\pi)^{3\alpha}} \delta_D(k - k_{1 \cdots \alpha}) \times K^{(\alpha)}_{k_1 \cdots k_\alpha} \delta_{\alpha,m,k_1 \cdots k_\alpha},
\] (41)

Here we have neglected some intermediate steps that express the \( \delta_e \) as a function of nonlinear matter density and condense all processes into the compact kernel \( K^{(\alpha)}_{k_1 \cdots k_\alpha} \). Further combining Equation (38) and (41) produces a perturbative relation between these two types of kernels (Matsubara 2011)
\[
K^{(1)}_{k,k} = k \cdot L^{(1)}_{k,k} + \delta^{(1)}_{k,k},
\]
\[
K^{(2)}_{k,k,k} = k \cdot L^{(2)}_{k,k,k} + b^{(2)}_{k,k,k} + \{ k \cdot L^{(1)}_{k,k} \}^2 \cdot \delta^{(1)}_{k,k,k} + \delta^{(1)}_{k,k,k} \cdot [k \cdot L^{(1)}_{k,k}].
\] (42)

When there is no bias, these kernels \( K^{(\alpha)} \) then reduce to the Eulerian matter perturbation kernels, which are usually denoted as \( F^{(\alpha)}_{k^{(\alpha)}_k} \), with a similar relation (Matsubara 2011)
\[
F^{(1)}_{k,k} = k \cdot L^{(1)}_{k,k},
\]
\[
F^{(2)}_{k,k,k} = k \cdot L^{(2)}_{k,k,k} + [k \cdot L^{(1)}_{k,k}] \cdot [k \cdot L^{(1)}_{k,k}].
\] (43)

Since the algorithm assumes the input map to be a matter field whose initial distribution is homogeneous, the reconstruction of the biased map \( \delta_e \) is equivalent to finding the effective LPT kernels \( \tilde{L}^{(\alpha)} \) such that the corresponding \( \tilde{F}^{(\alpha)} \) satisfy the following relation
\[
\tilde{F}^{(\alpha)}_{k_1 \cdots k_\alpha} = K^{(\alpha)}_{k_1 \cdots k_\alpha},
\] (44)

where \( \tilde{F}^{(\alpha)} \) is a function of \( \tilde{L}^{(\alpha)} \) as shown in Equation (43). To the second order, it is straightforward to derive the effective LPT kernels
\[
\tilde{L}^{(1)}_{k,k} = \frac{k}{k^2},
\]
\[
\tilde{L}^{(2)}_{k,k,k} = \frac{3}{7} \frac{k}{k^2} \left[ 1 - \frac{k \cdot k_2}{k k_2} \right],
\] (45)

The relation becomes tedious at higher \( n \) and we stop at the second order. Finally, the reconstructed displacement field of biased tracer could be described by
\[
\tilde{\Psi}_k = \sum_{\alpha \geq 1} \frac{i}{\alpha!} \int \frac{d^3k_1 \cdots d^3k_\alpha}{(2\pi)^{3\alpha}} \delta_D(k - k_{1 \cdots \alpha}) \times L^{(\alpha)}_{k_1 \cdots k_\alpha} \delta_{\alpha,m,k_1 \cdots k_\alpha},
\] (46)

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