TOR AS A MODULE OVER AN EXTERIOR ALGEBRA

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Abstract. Let $S$ be a regular local ring with residue field $k$ and let $M$ be a finitely generated $S$-module. Suppose that $f_1, \ldots, f_c \in S$ is a regular sequence that annihilates $M$, and let $E$ be an exterior algebra over $k$ generated by $c$ elements. The homotopies for the $f_i$ on a free resolution of $M$ induce a natural structure of graded $E$-module on Tor$_S^*(M, k)$. In the case where $M$ is a high syzygy over the complete intersection $R := S/(f_1, \ldots, f_c)$ we describe this $E$-module structure in detail, including its minimal free resolution over $E$.

Turning to Ext$^*_R(M, k)$ we show that, when $M$ is a high syzygy over $R$, the minimal free resolution of Ext$_R^*(M, k)$ as a module over the ring of CI operators is the Bernstein-Gel'fand-Gel'fand dual of the $E$-module Tor$_S^*(M, k)$.

For the proof we introduce higher CI operators, and give a construction of a (generally non-minimal) resolution of $M$ over $S$ starting from a resolution of $M$ over $R$ and its higher CI operators.

1. Introduction

Throughout this paper we write $S$ for a regular local ring with maximal ideal $m$ and residue field $k$, and we let $f_1, \ldots, f_c \in S$ be a regular sequence. Set $I := (f_1, \ldots, f_c) \subset S$ and consider the complete intersection $R := S/I$. Let $M$ be a finitely generated $S$-module annihilated by $I$. We denote by $E$ the exterior algebra

$$E := \bigwedge_k(I/mI) =: k(e_1, \ldots, e_c).$$

The finite-dimensional graded vector space Tor$_S^*(M, k)$ has a natural $E$-module structure induced by the action of homotopies for the $f_i$ on the minimal $S$-free resolution of $M$ (Section 2). For some modules $M$, the action of $E$ on Tor$_S^*(M, k)$ is trivial but, in the case where $M$ is a high $R$-syzygy in the sense of [EP1] (explicit bounds are given in [EP1] and [EP2]) we prove that it is highly nontrivial:

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(i) We prove that the $E$-module $\text{Tor}^S(M, k)$ is generated by $\text{Tor}_0^S(M, k)$ and $\text{Tor}_1^S(M, k)$, and its (Castelnuovo-Mumford) regularity is 1 (Corollary 5.1 and Theorem 5.3).

(ii) Let 

$$T' := E \cdot \text{Tor}_0^S(M, k) \subset \text{Tor}^S(M, k)$$

and let 

$$T'' := \text{Tor}^S(M, k)/T'$$

be the quotient. Assuming that $k$ is infinite and the generators of $(f_1, \ldots, f_c)$ are chosen generally, we compute vector space bases of $T'$ and $T''$, and show that, as $E$-modules, $T'$ and $T''$ have Gröbner deformations to direct sums of copies of $E/(e_p, \ldots, e_c)$ for $p = 1, \ldots, c$ (Theorem 4.6). It follows that, even when $k$ is finite, $T'$ and $T''$ have linear $E$-free resolutions, given explicitly in (iv) below.

(iii) We prove that the Betti numbers of the 0-linear strand of the minimal $E$-free graded resolution of $\text{Tor}^S(M, k)$ are given by the even Betti numbers of $M$ over $R$, and the Betti numbers of the 1-linear strand are given by the odd Betti numbers of $M$ over $R$. That is:

$$\beta^E_{i, i}(\text{Tor}^S(M, k)) = \beta^R_{2i}(M)$$

$$\beta^E_{i, i+1}(\text{Tor}^S(M, k)) = \beta^R_{2i+1}(M).$$

(Theorem 4.8).

(iv) We show that the numerical statement in (iii) is a consequence of the structure of the minimal $E$-free resolution of $\text{Tor}^S(M, k)$ by proving that the resolution is the mapping cone:

$$\cdots \to \text{Tor}^R_1(M, k) \otimes_R E \xrightarrow{t_2} \text{Tor}^R_2(M, k) \otimes_R E \xrightarrow{t_2} \text{Tor}^R_0(M, k) \otimes_R E \xrightarrow{t_2} \text{Tor}^R_5(M, k) \otimes_R E \xrightarrow{t_2} \cdots \oplus \xrightarrow{t_3} \oplus \xrightarrow{t_3} \oplus \cdots$$

(Theorem 9.2, see also Theorem 4.6 (iii)) where the two rows are themselves minimal linear free resolutions of the $E$-submodule $T'$ and the quotient $T''$. The maps labeled $t_2$ are the CI (=Complete Intersection) operators (also called Eisenbud operators), while the maps labeled $t_3$ between the two strands are some of the higher CI-operators, introduced in Section 7.
A curious consequence of (iv) is that if \( M \) is a high syzygy over \( R \), then the first syzygy over \( E \) of the \( E \)-module \( \text{Tor}^S(M, k) \) is \( \text{Tor}^S(L, k) \), where \( L \) is the second syzygy of \( M \) as an \( R \)-module. For a slightly sharper statement see Corollary 9.3.

Next we focus on \( \text{Ext}_R(M, k) \). The action of the CI operators makes the graded vector space \( \text{Ext}_R(M, k) \) into a finitely generated module over the ring

\[
\mathcal{R} := \text{Sym}_k \left( (I/mI)^\vee \right) =: k[\chi_1, \ldots, \chi_c].
\]

In Theorem 9.4 we prove that when \( M \) is a high \( R \)-syzygy, the minimal \( \mathcal{R} \)-free resolution of \( \text{Ext}_R^{\text{even}}(M, k) \) is obtained by the Bernstein-Gel’fand-Gel’fand (BGG) correspondence from the \( E \)-module structure of \( T'^{\vee} \), and similarly for \( \text{Ext}_R^{\text{odd}}(M, k) \) and \( T''^{\vee} \).

Corollary 9.5 doesn’t even require the definition of \( T' \). Write

\[
\mu : E_1 \otimes_k \text{Tor}^R_0(M, k) \to \text{Tor}^R_1(M, k)
\]

for the multiplication map and

\[
\mu^{\vee} : \text{Ext}_S^1(M, k) \to \text{Ext}_S^1(M, k) \otimes \mathcal{R}_1
\]

for its vector space dual. The \( \mathcal{R} \)-module \( \text{Ext}_R^{\text{even}}(M, k) \) then has the (non-minimal) linear free presentation

\[
\text{Ext}_S^1(M, k) \otimes \mathcal{R}(-1) \xrightarrow{\tau} \text{Hom}(M, k) \otimes \mathcal{R} \to \text{Ext}_R^{\text{even}}(M, k) \to 0
\]

where \( \tau \) is the map of free modules whose linear part is \( \mu^{\vee} \). This follows from Theorem 9.4 because \( \mu^{\vee} \) is 0 on the submodule \( T''^{\vee} \).

An essential ingredient in the proofs in Section 9 is a new theory of higher CI operators, introduced in Section 7. Just as the Eisenbud-Shamash construction allows one to describe an \( R \)-free resolution of any \( R \)-module from the higher homotopies on an \( S \)-free resolution, one can describe an \( S \)-free resolution from the higher CI-operators on an \( R \)-free resolution. This construction was discovered independently by Jessie Burke [Bu]. The differentials in the \( E \)-free resolution of \( \text{Tor}_S(M, k) \) are related, as above, to the higher CI-operators.

We also use the “layered” structures of the minimal \( S \)-free and \( R \)-free resolutions of \( M \) [EP2], which come from the higher matrix factorizations of [EP1]. We review the necessary definitions and results about layered resolutions in Section 3.

\textbf{Remark.} One could often replace the hypothesis that \( S \) is regular with a hypothesis that \( S \) is Gorenstein and \( M \) has finite projective dimension over \( S \), or that \( M \) is the module of a minimal higher matrix factorization. Moreover, the hypothesis that \( M \) is the module of a minimal higher matrix factorization could be replaced by the (possibly) more general hypothesis that the layered resolutions described in Section 3 are minimal. We leave these refinements to the interested reader.
The following example is computed using Macaulay2 code, which may be found in the documentation for the function “exteriorTorModule” in the Macaulay2 package CompleteIntersectionResolutions.m2 [M2, Version 1.9.1 and higher].

**Example 1.1.** Let \( S = k[[x_1, x_2, x_3]] \), and \( R = S/(x_1^3, x_2^3, x_3^3) \). Denote by \( N_i \) the \( i \)-th syzygy of \( k \) as an \( R \)-module. The minimal \( S \)-free resolution of \( N_0 = k \) is the Koszul complex on \( x_1, x_2, x_3 \), and \( x_i^2 \) times the homotopy for \( x_i \) is a homotopy \( \sigma_i \) for \( f_i := x_i^3 \). Thus the action of \( E \) on \( \text{Tor}^S(k, k) \) is trivial.

By contrast, the action of \( E \) on \( \text{Tor}^S(N_i, k) \) is nontrivial for \( i \geq 1 \). The beginnings of the Betti tables of the minimal \( E \)-free resolutions of \( \text{Tor}^S(N_i, k) \) for \( i = 1, 2, 3 \) are:

| Betti table of \( \text{Tor}^S(N_1, k) \) : |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| total :         | 10              | 27              | 52              | 85              | 126             |
| 0 :            | 3               | 9               | 18              | 30              | 45              |
| 1 :            | 6               | 15              | 28              | 45              | 66              |
| 2 :            | 1               | 3               | 6               | 10              | 15              |

| Betti table of \( \text{Tor}^S(N_2, k) \) : |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| total :         | 16              | 36              | 64              | 100             | 144             |
| 0 :            | 6               | 15              | 28              | 45              | 66              |
| 1 :            | 10              | 21              | 36              | 55              | 78              |

| Betti table of \( \text{Tor}^S(N_3, k) \) : |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| total :         | 25              | 49              | 81              | 121             | 169             |
| 0 :            | 10              | 21              | 36              | 55              | 78              |
| 1 :            | 15              | 28              | 45              | 66              | 91              |

From the first table we see that, as an \( E \)-module, \( \text{Tor}^S(N_1, k) \) is generated in degrees 0,1,2. Since \( N_1 \) is artinian, we have \( \text{Tor}^S_3(N_1, k) \neq 0 \). Hence, the \( E \)-module structure of \( \text{Tor}^S(N_1, k) \) is nontrivial.

The smallest \( i \) for which \( N_i \) is a high syzygy (in the sense of [EP1]) is \( i = 3 \), but in fact the *layered resolution* of \( N_2 \) with respect to \( f_1, f_2, f_3 \) described in Section 3 is also minimal. The \( E \)-module \( \text{Tor}^S(N_i, k) \) has a free resolution with just 2 linear strands for \( i = 2, 3 \), illustrating assertion (i) above.

Further, Macaulay2 computes the Betti table of \( N_2 \) as an \( R \)-module as

| total : | 6     | 10    | 15    | 21    | 28    | 36    | 45    | 55    | 66    | 78    | 91    | 105   | 119   |
|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0 :     | 6     | 10    | 15    | 21    | 28    | 36    | 45    | 55    | 66    | 78    | 91    | 105   |       |
| 1 :     |       |       |       |       |       |       |       |       |       |       |       |       |       |

and we see that, for \( s = 0, 1 \),

\[
\beta_{i,s}^E(\text{Tor}^S(N_2, k)) = \beta_{2i+s}^R(N_2),
\]

which illustrates (iii).

We can also illustrate Theorem 9.4 in this context. It turns out that the homogeneous components of \( T' = E \cdot \text{Tor}^S_0(N_2, k) \) are \( T'_0 = E^6 \), \( T'_1 = E^3 \), \( T'_2 = E^1 \), and it
follows that the minimal $\mathcal{R}$-free resolution of $\text{Ext}^\text{even}_R(N_2,k)$ has the form
\[
0 \rightarrow \mathcal{R}^1(-2) \xrightarrow{d_2} \mathcal{R}^3(-1) \xrightarrow{d_1} \mathcal{R}^6 \rightarrow \text{Ext}^\text{even}_R(N_2,k) \rightarrow 0.
\]
The differentials are easily computed from the action of $E$ on $\text{Tor}^S(N_2,k)$. The map
\[
\langle f_1, \ldots, f_3 \rangle \otimes \text{Tor}^S_0(N_2,k) \rightarrow \text{Tor}^S_1(N_2,k),
\]
given by the homotopies induces the map $E_1 \otimes T'_0 \rightarrow T'_1$, whose dual induces the map $d_1 : \mathcal{R}^3(-1) \rightarrow \mathcal{R}^6$. Computation shows that, in suitable bases, this map is given by the matrix
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \chi_1 & \chi_2 \\
-\chi_1 & 0 & \chi_3 \\
-\chi_2 & -\chi_3 & 0
\end{pmatrix}
\]
where the $\chi_i$ form a dual basis to $f_1, f_2, f_3$. From this presentation matrix we see that $\text{Ext}^\text{even}_R(N_2,k)$ is the direct sum of $\mathcal{R}^3$ and one copy of the maximal ideal $(\chi_1, \chi_2, \chi_3) \subset \mathcal{R}$, shifted so that it is generated in degree 0. Similar conclusions hold for $\text{Ext}^\text{odd}_R(N_2,k)$.

**Related Work.** Avramov and Buchweitz made use of the simple classification of modules over an exterior algebra on 2 generators to study free resolutions of modules over complete intersections of codimension 2 in [AB], and this study is carried further in [AY]. For other points of view on the module structure of Tor see [Da, HW]. For further results on resolutions over exterior algebras, see for example [AI, Ei2, Fl].

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2. **Homotopies and the action of the exterior algebra on Tor**

In this section we review the action of the exterior algebra on Tor. We will use the notation at the beginning of the Introduction.

For each $i$ we choose a homotopy $\sigma_i$ for $f_i$ on a free resolution $\mathbf{F}$ of the module $M$. Up to homotopy, the homotopies $\sigma_i$ anticommute and square to 0—see for example [EP1, Proposition 3.4.2]. Though the $\sigma_i$ are not maps of complexes, they become maps of complexes when tensored with an $S$-module $N$ annihilated by $f_1, \ldots, f_c$, so
σ_i⊗1 takes cycles in the complex F⊗S N to cycles, while raising the homological degree by 1. Thus the action of the σ_i gives H_i(F⊗N) = Tor^S(M, N) the structure of a graded module over the exterior algebra ∧_S(I). The action factors through an action of ∧_S(I/I^2) because I·Tor^S(M, N) = 0.

As an example, consider the Koszul complex K = K(f_1, ..., f_c). Denote by e_i the basis element of K_1 that maps to f_i. An immediate computation shows that multiplication by e_i is a homotopy for f_i. These homotopies anti-commute and square to 0, making K a free module over the exterior algebra ∧_S(I/I^2) because I·Tor^S(M, N) = 0.

The action of ∧(I/I^2) on Tor^S(M, N) is functorial by Lemma 2.1.

**Lemma 2.1.** Let (F, ∂) : · · · −→ F_1 −→ F_0 and (G, d) : · · · −→ G_1 −→ G_0 be complexes of S modules, and suppose that F and G admit homotopies σ for some element f ∈ S, respectively. Let ϕ : F −→ G be a map of complexes. If the F_i are free and the complex G is acyclic, then there are maps α_i : F_i −→ G_{i+2} such that ϕσ − τϕ = dα − α∂.

Thus, if N is a module annihilated by f, the maps ϕσ and τϕ are homotopic maps of complexes; in particular, the maps

\[
\begin{align*}
H(F ⊗ N) &\longrightarrow H(G ⊗ N) \\
H(\text{Hom}(G, N)) &\longrightarrow H(\text{Hom}(F, N))
\end{align*}
\]

induced by ϕ commute with the action of the homotopies.

The proof is an immediate computation.

For example, taking ϕ to be the identity on a free resolution of M, we see that the induced action of ∧_S(I/I^2) on Tor^S(M, N) is independent of the choice of homotopies.

We note that ∧_R(I/I^2) = ∧Tor^S(R, R), and one can see the above action on Tor as being, up to sign, induced by the action of the algebra Tor^S(R, R) on Tor^S(M, N). This is a special case of the natural product Tor^S(A, B) ⊗ Tor^S(M, N) −→ Tor^S(A ⊗ B, M ⊗ N) defined in the book of Cartan and Eilenberg [CE, Chapter XI, Section 1], as one can prove from the fact that homotopies on a tensor product complex can be defined from homotopies on one factor. In particular, the E-module structure on Tor^S(M, N) computed from homotopies on a resolution of M is, up to sign, the same as that computed from a resolution of N. We will not use these facts.

In this paper we focus on the structure of Tor^S(M, k), where S is a regular local ring with residue field k, and M is annihilated by f_1, ..., f_c ∈ S. Since Tor^S(M, k) is annihilated by the maximal ideal, it may be regarded as a graded module over the exterior algebra

\[
E := ∧_k(I/mI) = k(e_1, ..., e_c).
\]
3. Layered resolutions

Continuing with the notation of the introduction, we consider a regular local ring \( S \) and a complete intersection \( R = S/(f_1, \ldots, f_c) \) of codimension \( c \). Throughout the paper all the modules are assumed finitely generated.

Let \( M \) be a Cohen-Macaulay \( S \)-module of codimension \( c \) and let \( f := f_1, \ldots, f_c \) be a regular sequence in the annihilator of \( M \). In [EP2] we construct an \( S \)-free resolution \( L^\uparrow S(M,f) \) and an \( R \)-free resolution \( L \downarrow_R(M,f) \), called layered resolutions of \( M \). In this section we recall the features of the construction that will play a role in this paper.

The importance of the layered resolutions comes from the following result:

**Theorem 3.1.**

(i) If \( M \) is the module of a minimal higher matrix factorization with respect to \( f_1, \ldots, f_c \) in the sense of [EP1, Definition 1.2.2], then \( L^\uparrow S(M,f) \) and \( L \downarrow_R(M,f) \) are minimal resolutions.

(ii) Suppose that the ground field \( k \) is infinite. If \( M \) is a sufficiently high \( R \)-syzygy of an \( R \)-module \( N \) and the elements \( f'_1, \ldots, f'_c \) are sufficiently general among generators of \( (f_1, \ldots, f_c) \), then \( M \) is the module of a minimal higher matrix factorization with respect to \( f'_1, \ldots, f'_c \).

**Proof.** See [EP1, Theorems 1.3.1, 3.1.4, 5.1.2] and [EP2]. \( \square \)

**The \( S \)-free layered resolution.** First we will review that \( L^\uparrow S(M,f) \) has a filtration by acyclic free subcomplexes that are resolutions of maximal Cohen-Macaulay modules over the intermediate rings \( R(p) := S/(f_1, \ldots, f_p) \). Let \( M'(p) \to M \) be the maximal Cohen-Macaulay \( R(p) \)-approximation of \( M \) in the sense of [AB]. We may write \( M'(p) = M(p) \oplus R(p)^{m_p} \), where \( M(p) \) has no free summand. Following [EP1, Definition 7.3.1] we call \( M(p) \) the essential MCM approximation of \( M \) over \( R(p) \). Let \( L(p) := L^\uparrow S(M(p), f_1, \ldots, f_p) \) be the layered \( S \)-free resolution of \( M(p) \). By [EP1, Corollary 7.3.4] the essential Cohen-Macaulay approximation of \( M(p) \) over \( R(p-1) \) is \( M(p-1) \), and thus we have maps

\[
0 = M(0) \to M(1) \to \ldots \to M(c) = M.
\]

They induce inclusions of complexes

\[
0 = L(0) \subset L(1) \subset \cdots \subset L(c) =: L,
\]

with quotients

\[
L(p)/L(p-1) = K(f_1, \ldots, f_{p-1}) \otimes_S B(p),
\]
where $K(f_1, \ldots, f_{p-1})$ is the Koszul complex on $f_1, \ldots, f_{p-1}$, and $B(p)$ is a two-term free complex of the form

$$B(p) : \begin{array}{c} B_1(p) \\ \downarrow b_p \end{array} \rightarrow B_0(p).$$

Thus, as free $S$-modules,

$$L(p) = L(p-1) \oplus S(e_1, \ldots, e_{p-1}) \otimes S B(p).$$

In particular

$$A_0(p) := \bigoplus_{q=1}^p B_0(q) = L(p)_0,$$

while

$$A_1(p) := \bigoplus_{q=1}^p B_1(q) \subset L(p)_1$$

is a summand of $L(p)_1$.

Let

$$E(p) := E/(e_{p+1}, \ldots, e_c) = k\langle e_1, \ldots, e_p \rangle.$$

From Lemma 2.1 we deduce:

**Lemma 3.6.** For $1 \leq p \leq c-1$, the inclusion $L(p) \subset L$ induces an inclusion

$$\text{Tor}_S(M(p), k) \subset \text{Tor}_S(M, k)$$

of $E(p)$-modules.

We will make use of the following property:

**Proposition 3.7.** If $M$ is the module of a minimal higher matrix factorization with respect to $f_1, \ldots, f_c$, then the homotopy $\sigma_p$ for $f_p$ on $L(p)$ can be chosen so that its component

$$h_p : L(p)_0 = \bigoplus_{q=1}^p B_0(q) \longrightarrow \bigoplus_{q=1}^p B_1(q) \subset L(p)_1$$

is minimal.

**Proof:** By [EP1, Theorem 5.3.1], the higher matrix factorization $(d, h)$ for $M$ and the homotopy $\sigma_p$ for $f_p$ can be chosen so that $h_p$ is a component of $h$. Furthermore, [EP1, Theorem 5.1.2] shows that $R \otimes h$ is the second differential in the minimal $R$-free resolution of $M$ over $R$. Hence, $h$ is minimal. \hfill $\square$

As a complex, $L(p)$ is a *Koszul extension* of $L(p-1)$ in the sense of [EP1, Definition 3.1.1]. By definition, this is the mapping cone of a map of complexes

$$\Psi : K(f_1, \ldots, f_{p-1}) \otimes S B(p)[-1] \longrightarrow L(p-1)$$

that is zero on $K(f_1, \ldots, f_{p-1}) \otimes S B_0(p)$.

**Example 3.8.** We illustrate the constructions above in the codimension 2 case. When $c = 2$, the resolution $L^\uparrow S(M, f_1, f_2)$ may be represented by the diagram:
In the notation above $L(1)$ is the 2-term complex

$$B_1(1) \xrightarrow{b_1} B_0(1)$$

and $K(f_1) \otimes B(2)$ is the complex

$$B_1(2) \xrightarrow{b_2} B_0(2),$$

where $e_1$ denotes the basis element of $I/mI$ corresponding to $f_1$ and we have written three underlying free modules of $K(f_1) \otimes B(2)$, in homological degrees 2,1,0, as

$$e_1 B_1(2) \rightarrow B_1(2) \oplus e_1 B_0(2) \rightarrow B_0(2).$$

**The $R$-free layered resolution.** The maps (3.2) induce inclusions of complexes

$$0 = T(0) \subset R \otimes T(1) \subset \cdots \subset R \otimes T(c) =: T,$$

where $T(p)$ is the layered resolution of $M(p)$ over $R(p)$. By [EP2], $T(p+1)$ is obtained from $T(p)$ by the Shamash construction applied to the box complex

$$\cdots \rightarrow T(p)_2 \xrightarrow{\partial_2} T(p)_1 \xrightarrow{\partial_1} T(p)_0$$

where $B(p)$ is the two-term complex from (3.4). In particular, the following property holds.
Proposition 3.9. The CI operator $t_j : T 	o T$ for $f_j$ on the layered resolution $T$ can be chosen so that, for $j \leq p$, it preserves the box complex $T'(p)$ as a subcomplex of $T$, and its components

\[ R \otimes T(p)_2 \to R \otimes B_0(p) \]
\[ R \otimes T(p)_3 \to R \otimes B_1(p) \]

are zero.

The dual maps $\chi_j : \text{Hom}(T, R) \to \text{Hom}(T, R)$ then vanish on $\text{Hom}_R(B_0(p), R)$ and $\text{Hom}_R(B_1(p), R)$.

4. The structure of Tor

Throughout this section, as in the introduction, we assume that $S$ is a regular local ring, $R = S/(f_1, \ldots, f_c)$ is a complete intersection of codimension $c$, and modules are finitely generated. Write $f := f_1, \ldots, f_c$. We consider a module $M$ that is the module of a minimal higher matrix factorization with respect to $f$. We write $T' := E \cdot \text{Tor}_0^S(M, k)$ for the $E$-submodule of $\text{Tor}_0^S(M, k)$ generated by $\text{Tor}_0^S(M, k)$ and set $T'' := \text{Tor}_0^S(M, k)/T'$. For each module in the short exact sequence

\[ 0 \to T' \to \text{Tor}_0^S(M, k) \to T'' \to 0, \]

we will identify a vector space decomposition, the minimal generators as an $E$-module, a Gröbner basis for the relations, and the ranks of the free modules in a minimal $E$-free resolution. In Section 9 we will determine the structure of the resolutions themselves.

Notation 4.1. In addition to the notation and hypotheses above, we adopt the notations $R(p), M(p), L(p), B_0(p), B_1(p), E(p)$ of Section 3. We set

\[ A_s(p) = \bigoplus_{q=1}^p B_s(q) \]

for $s = 0, 1$, as in Section 3, and $A_s := A_s(c)$. We write $\overline{-}$ for $k \otimes -$.

Since the differential $B_1(p) \to B_0(p)$ is minimal, we may regard $\overline{B(p)}$ as the direct sum $\overline{B_1(p)} \oplus \overline{B_0(p)}$.

The next result is the key to the $E$-module structure of Tor:

Theorem 4.2. Let the notation and hypotheses be as in 4.1. For $0 \leq p \leq c - 1$ there is an isomorphism of $E(p)$-modules,

\[ \text{Tor}_0^S(M(p + 1), k) \cong \text{Tor}_0^S(M(p), k) \oplus \left( E(p) \otimes_k \overline{B_0(p + 1)} \right) \oplus \left( E(p) \otimes_k \overline{B_1(p + 1)} \right), \]

where the action of $E(p)$ on the second and third summands is by multiplication on the tensor factor $E(p)$. 

10
Proof: The minimal free resolution $L(p + 1)$ of $M(p + 1)$ is a Koszul extension of $L(p)$ by $K(f_1, \ldots, f_p) \otimes_S B(p + 1)$, that is, the mapping cone of a map

$$\psi : K(f_1, \ldots, f_p) \otimes S B(p + 1)[-1] \to L(p)$$

such that the induced map $K(f_1, \ldots, f_p) \otimes_S B_0(p + 1) \to L(p)$ is zero, as in [EP1, 3.1.1].

It follows that we may also regard $L(p + 1)$ as the mapping cone of a map

$$\psi' : K(f_1, \ldots, f_p) \otimes S B_1(p + 1)[-1] \to L(p) \oplus K(f_1, \ldots, f_p) \otimes S B_0(p + 1)[-1].$$

where the target complex is a direct sum, as complexes. We can define homotopies for $f_1, \ldots, f_p$ on $K(f_1, \ldots, f_p) \otimes S B_0(p + 1)$ and on $K(f_1, \ldots, f_p) \otimes S B_1(p + 1)$ by simple multiplication on the tensor factor $K(f_1, \ldots, f_p)$.

We can apply Lemma 2.1 to each of the maps in the exact sequence of the mapping cone

$$0 \to L(p) \oplus (K(f_1, \ldots, f_p) \otimes_S B_0(p + 1)) \to L(p + 1)$$

$$\to K(f_1, \ldots, f_p) \otimes_S B_1(p + 1) \to 0.$$

Thus, tensoring over $S$ with the residue field $k$, we get an exact sequence of $E(p)$-modules

$$0 \to \overline{L(p)} \oplus (E(p) \otimes_k B_0(p + 1)) \to \overline{L(p + 1)}$$

$$\to E(p) \otimes_k \overline{B_1(p + 1)} \to 0.$$

Since $E(p) \otimes_k \overline{B_1(p + 1)}$ is a free $E(p)$-module, the sequence splits, as claimed. \qed

Note that Theorem 4.2 does not assert that the “obvious” copy of $E(p) \otimes_k \overline{B_1(p + 1)}$ in $\operatorname{Tor}^S(M, k)$ is a submodule, but only that there is a submodule isomorphic to it.

Taking $p = c$, we get:

**Corollary 4.3.**

$$\operatorname{Tor}^S(M, k) = \bigoplus_{p=1}^c E(p - 1) \otimes_k \overline{B(p)}$$

as vector spaces. The subspace $E(p - 1) \otimes_k \overline{B(p)}$ is an $E(p - 1)$-submodule and the action of $E(p - 1)$ is via the left tensor factor.

In particular, $\operatorname{Tor}^S(M, k)$ is generated as an $E$-module in degrees 0 and 1, by

$$\overline{A_0 \oplus A_1} = \bigoplus_{p=1}^c \overline{B(p)}.$$

We can now give a Gröbner basis of relations for $\operatorname{Tor}^S(M, k)$:
Theorem 4.4. Suppose that $M$ is the module of a minimal higher matrix factorization for a regular sequence $f_1, \ldots, f_c$. Let $1 \leq p \leq c$. For every 
\[ a \in \overline{B(p)} \]
and every $r \geq p$ there is a homogeneous relation on $\text{Tor}^S(M, k)$ of the form 
\[ e_r a - b \]
with $b \in \langle e_1, \ldots, e_{r-1} \rangle (\overline{A_0} \oplus \overline{A_1})$. These relations form a Gröbner basis for the relations on $\text{Tor}^S(M, k)$ as an $E$-module with respect to any term order that refines the lexicographic order on the monomials of $E$ with $e_c \succ \cdots \succ e_1$. The $E$-module defined by the leading terms of these relations is 
\[ \bigoplus_{p=1}^c E / \langle e_p, \ldots, e_c \rangle \otimes_k \overline{B(p)} . \]

Proof: By Lemma 3.6 it suffices to consider the $E(r)$-module $H(\overline{L(r)})$. Corollary 4.3 shows that this module can be written as 
\[ \text{Tor}^S(M(r), k) \cong H(\overline{L(r)}) = E(r-1) \cdot (\overline{A_1(r)} \oplus \overline{A_0(r)}) . \]
It follows that there exists a relation on $\text{Tor}^S(M, k)$ of the form $e_r a - b$ with $b \in E(r-1) (\overline{A_0(r)} \oplus \overline{A_1(r)})$.

If $b$ had a non-zero component in $\overline{A_0} \oplus \overline{A_1}$, then we would have $a \in \overline{B_0(p)}$, which contradicts Proposition 3.7. Thus $e_r a$ is the leading term of the relation in the monomial order $\succ$.

If we factor out these leading terms from the free $E$-module generated by $\overline{A_0} \oplus \overline{A_1}$ we obtain the module 
\[ \bigoplus_{1 \leq p \leq c} E(p - 1) \otimes_k \overline{B(p)} . \]
By Corollary 4.3, this has the same vector space dimension as the $E$-module $\text{Tor}^S(M, k)$. Therefore, the given relations form a Gröbner basis for the module of relations.

We can now prove assertions (i) and (ii) of the Introduction:

Theorem 4.6. Suppose that $M$ is the module of a minimal higher matrix factorization for a regular sequence $f_1, \ldots, f_c$.

(i) The submodule $T' := E \cdot \text{Tor}^S_0(M, k)$ has underlying vector space 
\[ T' = \bigoplus_{p=1}^c E(p - 1) \otimes_k \overline{B_0(p)} , \]
and thus the quotient $T'' := \text{Tor}^S(M, k) / T'$ has underlying vector space 
\[ T'' \cong \bigoplus_{p=1}^c E(p - 1) \otimes_k \overline{B_1(p)} , \]
where, for $s = 0, 1$, the action of $E(p - 1)$ on the summand $E(p - 1) \otimes_k \overline{B}_s(p)$ is by multiplication on the left tensor factor.

(ii) The module $T'$ is generated by $\overline{A}_0$, in degree 0, while $T''$ is generated by $\overline{A}_1$, in degree 1. Both $T'$ and $T''$ have linear $E$-free resolutions.

(iii) The minimal free resolution of $\text{Tor}^S(M,k)$ as an $E$-module is the mapping cone of a map from the minimal free resolution of $T''$ (shifted by -1) to the minimal free resolution of $T'$.

(iv) The relations given in (4.5) with $a \in A_0$ form a Gröbner basis of the relations on the $E$-module $T'$, and those with $a \in A_1$ form a Gröbner basis of the relations on the $E$-module $T''$.

We will make use of the following well-known lemma:

**Lemma 4.7.** The minimal $E$-free resolution of $E(p) = E/(e_{p+1}, \ldots, e_c)$ has underlying free module

$$E \otimes_k \text{Hom}_g(k[x_{p+1}, \ldots, x_c],k),$$

where $k[x_{p+1}, \ldots, x_c]$ denotes the polynomial ring on $c - p$ variables of homological and internal degree 1 generating a vector space that is dual to $\text{span}(e_{p+1}, \ldots, e_c)$.

**Proof:** The case $p = 0$ is the resolution of the residue field $k$. This resolution is the “generalized Koszul complex” of Priddy and others—see for example [Ei3, Exercise 17.22].

The minimal resolution of $E(p)$ as an $E$-module is easily seen to be the tensor product, over $k$, of $E(p)$ with the minimal resolution of $k$ as a module over the exterior algebra $k(e_{p+1}, \ldots, e_c)$. □

**Proof of Theorem 4.6.** By Theorem 4.3,

$$U := \bigoplus_{p=1}^c E(p-1) \otimes_k \overline{B}_0(p) \subseteq T'.$$

The homogeneous relations (4.5) with $a \in \overline{A}_0$ must have $b \in \overline{A}_0$ as well, so they are relations on the free $E$-module $E \otimes_k \overline{A}_0$. If we factor out their leading terms $e_r a$ we obtain the $E$-module

$$\widehat{T}' := \bigoplus_{p=1}^c E/(e_p, \ldots, e_c) \otimes_k \overline{B}_0(p).$$

It has the same dimension as the vector space $U$, proving both that $U = T'$ and that we have a Gröbner basis for $T'$.

As for $T''$, if we factor out the leading terms of the relations (4.5) with $a \in \overline{A}_1$ from the free $E$-module $E \otimes_k \overline{A}_1$, we obtain the $E$-module

$$\widehat{T}'' := \bigoplus_{p=1}^c E/(e_p, \ldots, e_c) \otimes_k \overline{B}_1(p),$$

13
which has the same vector space dimension as $T''$, proving that these relations form a Gröbner basis for the relations on $T''$ as claimed. This concludes the proofs of parts (i) and (iv) of the Theorem.

To prove part (ii), observe first that, by Lemma 4.7, the minimal free resolutions of the $E$-modules $E/(e_p, \ldots, e_c)$ are linear. It follows that the minimal $E$-free resolutions of $T'$ and $T''$ are linear, and thus the minimal free resolutions $\mathbf{F}'$ of $T'$ and $\mathbf{F}''$ of $T''$ are also linear.

It remains to prove part (iii). From the short exact sequence
\[ 0 \to T' \to \text{Tor}^S(M, k) \to T'' \to 0 \]
we see that $\text{Tor}^S(M, k)$ has a free resolution that is the mapping cone of some map of complexes $\alpha : \mathbf{F}''[-1] \to \mathbf{F}'$. The $j$-th term $F_j$ of $\mathbf{F}'$ is generated in degree $j$, while the $j$-th term $F'_j$ of $\mathbf{F}''$ is generated in degree $j + 1$ since the generators $A_1$ of $T''$ have degree 1. Hence, the matrices in the map $\alpha$ have entries of degree 2. In particular the mapping cone is a minimal resolution of the form
\[ \cdots \to F'_j \oplus F''_j \to \cdots \to F'_0 \oplus F''_0 = E \otimes_k (A_0 \oplus A_1). \]

In Section 9 we will identify the resolutions $\mathbf{F}'$ and $\mathbf{F}''$ and the map $\alpha$ in terms of the minimal free resolution of $M$ as an $R$-module. We already have enough information to interpret the Betti numbers:

**Theorem 4.8.** Suppose that $M$ is the module of a minimal higher matrix factorization for a regular sequence $f_1, \ldots, f_c$ and for $s = 1, 2$ and $p = 1, \ldots, c$, let $b_s(p) = \text{rank } B_s(p)$. With notation as above, the graded Betti numbers of $\text{Tor}^S(M, k)$ as an $E$-module are:

\[
\beta_{i,i}^E(\text{Tor}^S(M, k)) = \sum_{p=1}^c \binom{c - p + i}{c - p} b_0(p) = \dim_k \text{Ext}^{2i}_R(M, k)
\]

\[
\beta_{i,i+1}^E(\text{Tor}^S(M, k)) = \sum_{p=1}^c \binom{c - p + i}{c - p} b_1(p) = \dim_k \text{Ext}^{2i+1}_R(M, k),
\]

and these two formulas give the graded Betti numbers of $T'$ and $T''$ individually.

**Proof:** The minimal graded $E$-free resolutions of $T'$ and $T''$ are linear, and so their Betti numbers are equal to the Betti numbers of the modules $\widehat{T}'$ and $\widehat{T}''$ used in the proof of Theorem 4.6. These Betti numbers can be obtained from Lemma 4.7. Furthermore, the minimality of the layered resolution $L_{\downarrow R}(M, f)$ implies the identical formula for $\dim \text{Ext}^{2i}_R(M, k) = \beta_{2i}^R(M)$ and $\dim \text{Ext}^{2i+1}_R(M, k) = \beta_{2i+1}^R(M)$ (see [EP1, Corollary 1.3.3]).

In experiments, we have observed that the sequence of $E$-modules
\[ 0 \to T' \to \text{Tor}^S(M, k) \to T'' \to 0, \]
often splits. Here is the simplest example we know where this is not the case:

**Example 4.9.** Let \( S = k[a, b, c] \), \( R = S/(a^4, b^4, c^4) \),
\[
N = R \otimes_S \text{Coker} \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix},
\]
and let \( M \) be a sufficiently high syzygy of the \( R \)-module \( N \). Computation using Macaulay2 shows that the dual \( E \)-module, which up to a shift in grading is \( \text{Ext}^S(M, k) \), has smaller Betti numbers than does the direct sum of the duals of \( T' \) and \( T'' \). In particular, the \( E \)-submodule \( T' \subset \text{Tor}_S(M, k) \) is not a direct summand.

5. **Regularity**

If \( V \) is a finite dimensional \( \mathbb{Z} \)-graded vector space, we set \( \max V = \max \{ j \mid V_j \neq 0 \} \). We define the **regularity** of a graded \( E \)-module \( L \) to be
\[
\text{reg}_E(L) := \sup_i \{ \max \text{Tor}^E_i(L, k) - i \}.
\]
The minimal \( E \)-free resolution \( U \) of \( k \) is linear, so \( \text{Tor}^E_i(L, k) \cong H(L \otimes U) \) gives \( \max \text{Tor}^E_i(L, k) \leq i + \max L \). Thus
\[
\text{reg}_E(L) \leq \max L.
\]

From Theorem 4.6 we get:

**Corollary 5.1.** If \( M \) is the module of a minimal higher matrix factorization for a regular sequence \( f_1, \ldots, f_c \), then
\[
\text{reg}_E(\text{Tor}^S(M, k)) = 1.
\]

We will provide a short alternate proof of this result. To do this, we compare the regularity of an \( E \)-module \( L \) with the regularity of \( L \) regarded as a module over \( E(p) := k\langle e_1, \ldots, e_p \rangle \), regarded as a subalgebra of \( E \).

**Theorem 5.2.** If \( L \) is a finitely generated graded \( E \)-module, then
\[
\text{reg}_E(L) \leq \text{reg}_{E(p)}(L) \leq \text{reg}_E(L) + c - p.
\]

**Proof:** First, we will prove the left inequality. Take \( F \) to be the tensor product over \( E(p) \) of a minimal free resolution \( G \) of \( L \) as an \( E(p) \)-module with the minimal free resolution \( D \) of \( E(p) \) as an \( E \)-module. Since the latter is split exact as a sequence of \( E(p) \)-modules, \( F \) is a (possibly non-minimal) \( E \)-free resolution of \( L \) as an \( E \)-module. By Lemma 4.7, the resolution \( D \) is linear. Therefore, for each \( i \),
\[
\max F_i \otimes k \leq \max_{q \leq i} \{ \max G_q \otimes k \} + (i - q) \leq \text{reg}_{E(p)}(L) + i.
\]
For the second inequality, note that \( E = E(p) \otimes_k k[e_{p+1}, \ldots, e_c] \) is a free \( E(p) \)-module with generators in degrees \( \leq c - p \), so a minimal \( E \)-free resolution is a (possibly non-minimal) \( E(p) \)-free resolution with regularity \( \text{reg}_E L + c - p \).

We now return to the situation of Notation 4.1.

**Theorem 5.3.** Suppose that \( M \) is the module of a minimal higher matrix factorization for a regular sequence \( f_1, \ldots, f_c \) with \( c \geq 1 \). The regularity of \( \text{Tor}_E^S(M, k) \) as an \( E \)-module is 1.

**Proof:** By [EP1, Theorem 3.1.4] the projective dimension of the \( S \)-module \( M \) is \( c \). The description of its minimal resolution \( L := L^S(M, f) \) given in Section 3 shows that the \( c \)-th free module in \( L \) is \( E(c - 1) \otimes B_1(c) \). Hence, \( B_1(c) \neq 0 \).

By Proposition 3.7 it follows that the \( E \)-module \( \text{Tor}_E^S(M, k) \) requires generators of degree 1 from \( B_1(c) \). Thus its regularity cannot be < 1, and we need only prove that it is \( \leq 1 \). We will prove this by induction on \( p \).

If \( p = 1 \) then \( M(1) \) is a maximal Cohen-Macaulay module over the hypersurface \( S/(f_1) \) and the resolution \( L(1) \) has projective dimension 1. Thus, we have \( \text{reg}_{E(1)} \text{Tor}_E^S(M(1), k) = 1 \).

By induction on \( p \), the direct sum in Corollary 4.3 shows that

\[
\text{reg}_{E(p-1)} \text{Tor}_E^S(M(p), k) \leq 1.
\]

Applying the left inequality in Theorem 5.2, we conclude

\[
\text{reg}_{E(p)} \text{Tor}_E^S(M(p), k) \leq 1.
\]

\[\square\]

6. **A Gröbner basis for the relations on \( \text{Ext}_R(M, k) \)**

As in the Introduction, we write \( \mathcal{R} := k[\chi_1, \ldots, \chi_c] \) for the ring of CI-operators acting on \( \text{Ext}_R(M, k) \). Note that the \( \chi_i \) have degree 2.

Throughout this section we will suppose that \( M \) is the module of a minimal higher matrix factorization for the regular sequence \( f_1, \ldots, f_c \). We will provide results for \( \text{Ext}_R(M, k) \) as an \( \mathcal{R} \)-module that are analogous to results proved above for \( \text{Tor}_E^S(M, k) \) as an \( E \)-module.

We use the notation and hypotheses of 4.1. Furthermore, we write \(-^\vee\) for \( \text{Hom}(-, k) \). Since the differential \( B_1(p) \rightarrow B_0(p) \) is minimal, we may think of \( B(p)^\vee \) as the direct sum \( B_1(p)^\vee \oplus B_0(p)^\vee \). We set

\[
\mathcal{R}(p) := k[\chi_p, \ldots, \chi_c] \subset \mathcal{R}.
\]

The following result is the analogue of Corollary 4.3.
Corollary 6.1. ([EP1, Corollary 5.1.6]) There is an isomorphism
\[
\text{Ext}_R(M, k) \cong \bigoplus_{p=1}^{c} k[\chi_p, \ldots, \chi_c] \otimes_k B(p)^\vee = \bigoplus_{p=1}^{c} \mathcal{R}(p) \otimes_k B(p)^\vee
\]
of graded vector spaces. The subspace
\[
\mathcal{R}(p) \otimes B(p)^\vee
\]
is an \( \mathcal{R}(p) \)-submodule and \( \mathcal{R}(p) \) acts on it via the action on the first factor.

The result above can be used to prove an analogue to Theorem 4.4:

Theorem 6.2. Suppose that \( M \) is the module of a minimal higher matrix factorization for a regular sequence \( f_1, \ldots, f_c \). Let \( 1 \leq p \leq c \). For every
\[
a \in (A_0 \oplus A_1)^\vee
\]
and every \( r < p \) there is a homogeneous relation on \( \text{Ext}_R(M, k) \) of the form
\[
\chi_r a - b
\]
with \( b \in (\chi_{r+1}, \ldots, \chi_c)(A_0 \oplus A_1)^\vee \). These relations form a Gröbner basis for the relations on \( \text{Ext}_R(M, k) \) as an \( \mathcal{R} \)-module with respect to any term order that refines the lexicographic order on the monomials of \( \mathcal{R} \) with \( \chi_1 \succ \cdots \succ \chi_c \). The module defined by the leading terms of these relations is
\[
\bigoplus_{p=1}^{c} \mathcal{R}/(\chi_1, \ldots, \chi_{p-1}) B(p)^\vee.
\]

PROOF: The existence of the desired relations follows from Proposition 3.9. The leading term of the relation \( \chi_r a - b \) in the monomial order \( \succ \), is \( \chi_r a \). If we factor out these leading terms from the free \( \mathcal{R} \)-module generated by \( (A_0 \oplus A_1)^\vee \) we obtain the module
\[
\bigoplus_{1 \leq p \leq c} \mathcal{R}(p) \otimes_k B(p)^\vee.
\]
By Corollary 6.1, this has the same Hilbert function as the \( \mathcal{R} \)-module \( \text{Ext}_R(M, k) \). Therefore, the given relations form a Gröbner basis, and in particular they generate the module of all relations.

Finally, we provide an analogue to Corollary 5.1.

Corollary 6.4. Suppose that \( M \) is the module of a minimal higher matrix factorization for a regular sequence \( f_1, \ldots, f_c \). The \( \mathcal{R} \)-module \( \text{Ext}^{\text{even}}_R(M, k) \) has regularity 0, and the \( \mathcal{R} \)-module \( \text{Ext}^{\text{odd}}_R(M, k) \) has regularity 1. ☐
7. Higher CI-operators and an inverse Eisenbud-Shamash construction

The Eisenbud-Shamash Construction (see [Sh] for the codimension 1 case and [Ei1, Section 7] for the general case) allows one to construct a (generally nonminimal) $R$-free resolution of an $R$-module from an $S$-free resolution together with a system of higher homotopies on the $S$-free resolution. In this section we will explain a construction that goes the other way: from an $R$-free resolution of an $R$-module together with a system of higher CI-operators $\{t_i\}$ as defined below, we will construct a (generally nonminimal) $S$-free resolution.

The classic CI-operators were first defined on $\text{Ext}_R(M, k)$ by Gulliksen [Gu], and then in the form used here by Eisenbud [Ei1]. The material in this section was discovered independently by Jesse Burke, and a more general version will appear in his paper [Bu].

**Proposition 7.1.** Let $S$ be a commutative ring, let $f_1, \ldots, f_c$ be a regular sequence, and let $R = S/(f_1, \ldots, f_c)$. Let

$$K := K(f_1, \ldots, f_c) : \cdots \xrightarrow{t_0^c} \wedge^2 S^c \xrightarrow{t_1^c} S^c \xrightarrow{t_2^c} S$$

be the Koszul complex resolving $R$. Let $\overline{G}$ be a complex of free $R$-modules, and suppose that

$$G : \cdots \xrightarrow{t_1} G_p \xrightarrow{t_0} G_{p-1} \xrightarrow{t_1} \cdots \xrightarrow{t_1} G_0$$

is a lifting of $\overline{G}$ to a sequence of maps of free $S$-modules. There exist operators

$$t_i = \sum_{p,q} t_{i}^{p,q} : G \otimes K \rightarrow (G \otimes K)[-1]$$

that commute with the natural action of $\wedge S^c$ on $K$, having components

$$t_{i}^{p,q} : G_p \otimes K_q \rightarrow G_{p-i} \otimes K_{q+i-1}$$

for $i, q \geq 0, p \geq i$ and satisfying the conditions

$$t_0^{p,q} = 1 \otimes (-1)^p t_{0}^{p,q}$$

$$t_1 = t_1' \otimes 1,$$

and

$$\sum_{i+j=n} t_it_j = 0$$

for all $n$. The maps $R \otimes t_i$ are determined uniquely by these conditions.

The positions of the maps $t_0, \ldots, t_3$, for example, are shown in the following figure where, for clarity, the upper indices are not shown and not all the maps have been labeled:
PROOF: We construct the $t_{n}^{p,q}$ by induction on $n$. The condition $\sum_{i+j=n}t_{i}t_{j}$ holds for $n = 0$ because $K$ is a complex. The condition $\sum_{i+j=n}t_{i}t_{j}$ holds for $n = 1$ by our choice of signs.

Thus we assume that $t_{j}^{p,q}$ has been defined for all $j < n$. We next construct the maps $t_{n}^{p,0}$, and we then define $t_{n}^{p,q}$ for $q > 0$ to be the unique maps that make

$$\sum_{q}t_{n}^{p,q}: G_{p} \otimes S^{c} \rightarrow G_{p-n} \otimes S^{c}$$

into a map of free $\wedge S^{c}$ modules.

Because $t_{0}^{p,0} = 0$, the desired condition for $t_{n}^{p,0}$ is

$$\sum_{i+j=n}t_{i}^{p-j}t_{j}^{0} = 0.$$

To simplify the notation, we drop the upper indices, which are functions of $n,p$ and $j$, and write the condition as

$$t_{0}t_{n} = - \sum_{i+j=n, i,j>0} t_{i}t_{j}.$$

Since $K$ is acyclic, both existence of $t_{n}$ and the uniqueness of $R \otimes t_{n}$ will follow if we show that

$$t_{0} \sum_{i+j=n, i,j>0} t_{i}t_{j} = 0.$$

Using the induction hypothesis

$$t_{0}t_{i} = - \sum_{\ell+m=i, \ell>0} t_{\ell}t_{m}$$

for $i < n$, we get

$$t_{0} \sum_{i+j=n, i,j>0} t_{i}t_{j} = - \sum_{\ell+m+j=n, j,\ell>0} t_{\ell}t_{m}t_{j} = - \sum_{\ell>0} t_{\ell} \sum_{m+j=n-\ell, j>0} t_{m}t_{j}.$$
Since $\ell > 0$ we can use the induction hypothesis again, and we see that each sum
\[ \sum_{m+j=n-\ell} t_m t_j \]
is 0, yielding the desired vanishing. \hfill \Box

**Corollary 7.2.** With hypothesis as in Proposition 7.1, the sequence
\[ \text{GK} : \quad \cdots \rightarrow \sum_{i+j=n} G_i \otimes_S K_j \xrightarrow{t_n} \sum_{i+j=n-1} G_i \otimes_S K_j \rightarrow \cdots \rightarrow G_0 \otimes K_0 \]
with
\[ T_n = \begin{pmatrix} t_{n,0} & t_{n-1,1} & 0 & \cdots & 0 \\ t_{n-1,0} & t_{n-1,1} & t_{n-2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{1,0} & t_{1-1,1} & t_{1-2,2} & \cdots & t_{0,n} \end{pmatrix} \]
is a complex. \hfill \Box

**Theorem 7.3.** Let $S$ be a commutative ring, let $f_1, \ldots, f_c$ be a regular sequence, and let $R = S/(f_1, \ldots, f_c)$. Let $K = K(f_1, \ldots, f_c)$ be the Koszul complex. If $G$ is a sequence of maps of free $S$-modules
\[ G : \quad \cdots \rightarrow G_p \xrightarrow{t'_1} G_{p-1} \xrightarrow{t'_1} \cdots \xrightarrow{t'_1} G_0. \]
such that $R \otimes G$ is an $R$-free resolution of an $R$-module $M$, then the complex $GK$ of Corollary 7.2 is an $S$-free resolution of $M$. \hfill \Box

**Proof:** To see that the complex is a resolution, we note that it is filtered by the subcomplexes involving just the $G_i \otimes K_j$ with $i \leq m$: all the $t_i$ except $t_0$ decrease the index of $G_i$, while $t_0$ keeps the index of $G_i$ the same. The associated graded complex is thus the direct sum of the $G_i \otimes K_j$, with differentials $t_0$; that is, the direct sum of copies of the resolution $K$ of $R$. The homology is thus $H_i(\text{gr}(GK)) = R \otimes G_i$.

In the spectral sequence converging from the homology of the associated graded complex $\text{gr}(GK)$ to the homology of $GK$, the $E_1$ page thus has nonzero terms $E_1^{(i,0)} = R \otimes G_i$ in position $(i, 0)$, and differentials induced by the differential of $GK$. But the only differential that reduces the first index by only 1 is $t_1$; thus the $E_1$ differential is the differential of the complex $R \otimes G$, which is a resolution of $M$. \hfill \Box

For any $R$-modules $M, N$ there is a spectral sequence
\[ \text{Tor}^R_i(\text{Tor}^S_j(M, R), N) \Rightarrow \text{Tor}^S_{i+j}(M, N). \]
that comes from the double complex $G \otimes_S (K \otimes_S N)$, and that allows the computation of a certain associated graded module of $\text{Tor}^S(M, N)$. It is natural to expect that the $t_i$ are special liftings of the differentials in this spectral sequence; Burke [Bu] shows that this is indeed the case. The complex $GK$ allows the computation of $\text{Tor}^S(M, N)$ itself.
8. The Bernstein-Gel’fand-Gel’fand correspondence (BGG)

The results in the next section depend on properties of the Bernstein-Gel’fand-
Gel’fand correspondence (BGG) from [EFS], and in this short section we review what
is necessary.

Let $W$ be the vector space generated by the regular sequence $f_1, \ldots, f_c$ so that
$E = \wedge W$. We set $W = V^\vee$ and $\mathcal{R} := \text{Sym}(W)$. In this section, for simplicity, we
regard both $V$ and $W$ as having degree 1, though in the next section we will need to
adjust to the situation where $W$ has degree 2.

The BGG correspondence establishes equivalences between the category of $\mathbb{Z}$-
graded $\mathcal{R}$-modules and linear free complexes over $E$, and also between $\mathbb{Z}$-graded
$E$-modules and linear free complexes over $\mathcal{R}$.

For example, giving a graded vector space $U = \bigoplus U_i$ the structure of a graded $\mathcal{R}$-
module is the same as giving multiplication maps $\mu_i : W \otimes_k U_i \to U_{i+1}$ that satisfy
the commutativity and associativity conditions. But giving the map $\mu_i$ is equivalent
to giving a map $\delta_i : U_i \to \text{Hom}_k(V, U_{i+1})$, and this is equivalent, in turn, to
giving a linear map of free $E$-modules

$$\text{Hom}_k(E, U_i)(-1) \to \text{Hom}_k(E, U_{i+1}).$$

It turns out that the associative and commutative conditions on the $\mu_i$ are equivalent
to the conditions $\delta_{i+1}\delta_i = 0$ for all $i$. We write $\mathcal{R}(U)$ for the resulting linear $E$-free
complex with $i$-th term $\text{Hom}(E, U_i)(i)$.

Similarly, given a graded $E$-module $T = \bigoplus T_i$ we construct a linear $\mathcal{R}$-free complex
$\mathbb{L}(T)$ having $i$-the term $(\mathcal{R} \otimes T_i)(i)$. Here are the results we need:

**Theorem 8.1.** ([EFS, Theorem 3.7 (Reciprocity)]) Let $U, T$ be finitely generated
graded modules over $\mathcal{R}$ and $E$, respectively. The complex $\mathbb{L}(T)$ is a free resolution of
$U$ if and only if the complex $\mathcal{R}(U)$ is an injective resolution of $T$.

Since the complex $\mathbb{L}(T)$ is linear, the equivalent conditions of the Theorem can
only be satisfied if the Castelnuovo-Mumford regularity of $U$ is 0. In fact this is
sufficient:

**Corollary 8.2.** ([EFS, Corollary 2.4]) Suppose that $U$ is a finitely generated graded
$\mathcal{R}$-module. The complex $\mathcal{R}(U)$ is acyclic —that is, the only homology of $\mathcal{R}(U)$ is
$H^0$ —if and only if $U$ has Castelnuovo-Mumford regularity 0.

9. Free resolutions of $\text{Tor}^S(M, k)$ and $\text{Ext}_R(M, k)$

We will make use of the BGG correspondence in two ways: first, if $F$ is any free
complex of $R = S/(f_1, \ldots, f_c)$-modules, where $f_1, \ldots, f_c$ is a regular sequence, then
the CI operators on $F$ define an $R$-module structure on $H_*(F \otimes_R k)$, and thus, since the CI operators have degree 2, we get a linear free complex of $E$ modules

$$
\ldots \xrightarrow{t_2} H_{2i+s}(F \otimes_R k) \otimes_k E \xrightarrow{t_2} H_{2i+s-2}(F \otimes_R k) \otimes_k E \xrightarrow{t_2} \ldots .
$$

for $s = 0$ and for $s = 1$. When $M$ is a high $R$-syzygy and $F$ is its minimal free resolution, we shall see that this is a resolution of $\text{Tor}^S(M,k)$.

Second, given an $S$-module $M$, the action of $E$ on the sub and quotient modules $T'$ and $T''$ of $\text{Tor}^S(M,k)$ gives us two linear complexes of $R$ modules; when $M$ is a high $R$-syzygy, we shall see that these are minimal $R$-free resolutions of Ext$^e_R(M,k)$ and Ext$^{od}_R(M,k)$, respectively.

To prove these results, we will use the complexes constructed in Corollary 7.2. With notation and hypothesis as in Proposition 7.1, we may regard $t^0_2$ as a map $G \rightarrow G \otimes S^c$ whose components $t_{2,i}$ satisfy $\sum f_i t_{2,i} = t^2_1$; that is, the $R \otimes t_{2,i}$ are the same as the CI-operators defined in [Ei1].

**Corollary 9.1.** With hypotheses as in Corollary 7.2, suppose that $R \otimes G$ is a minimal complex. The induced maps

$$
t_2 : G_{i+2} \rightarrow G_i \otimes k^c
$$

$$
t_3 : G_{i+3} \rightarrow G_i \otimes \wedge^2 k^c
$$

yield a complex of the form:

$$
\ldots \xrightarrow{t_2} G_{i+2} \otimes E \xrightarrow{t_2} G_{i+1} \otimes E \xrightarrow{t_2} \ldots \\
\ldots \xrightarrow{t_3} \bigoplus G_i \otimes E \xrightarrow{t_3} \bigoplus G_i \otimes E \xrightarrow{t_3} \ldots
$$

PROOF: By minimality, $t_1 \otimes k = 0$, so $H_1 G = G_i \otimes k$. Thus by Corollary 9.1 each row of the diagram is a complex. Further, Proposition 7.1 gives the identity $\sum_{i=0}^5 t_{i} t_{5-i} = 0$, and tensoring with $k$ we get

$$(t_2 t_3 + t_3 t_2) \otimes k = 0$$

as required. 

Note that, if $R \otimes G$ is the minimal resolution of an $R$-module $M$, then $G_i \otimes E = \text{Tor}_i^R(M,k)$.

**Theorem 9.2.** Let $f_1, \ldots, f_c \subset S$ be a regular sequence in a regular local ring with maximal ideal $m$ and residue field $k$. Let $I = (f_1, \ldots, f_c)$ and let $R = S/I$. Let $R := k[\chi_1, \ldots, \chi_c]$ be the ring of CI operators. If $\text{reg Ext}^e_R(M,k) = 0$ and
reg Ext^odd_R(M, k) = 1 as R-modules, where χ_i acts on Ext_R(M, k) = Hom_k(Tor^R(M, k), k) via the action of t_{2i} on Tor^R(M, k), then the complex T(M):

\[
\begin{array}{ccccccccc}
\cdots & & t_2 & & \longrightarrow & & Tor_4^R(M, k) \otimes E & & t_2 & & \longrightarrow & & Tor_2^R(M, k) \otimes E & & t_2 & & \longrightarrow & & Tor_0^R(M, k) \otimes E \\
& & \oplus & & t_3 & & \oplus & & t_3 & & \oplus \\
\cdots & & t_2 & & \longrightarrow & & Tor_5^R(M, k) \otimes E & & t_2 & & \longrightarrow & & Tor_3^R(M, k) \otimes E & & t_2 & & \longrightarrow & & Tor_1^R(M, k) \otimes E \\
\end{array}
\]

is a minimal free resolution of Tor^S(M, k) as a module over E = Tor^S(R, k). Moreover, the upper row is a minimal free resolution of the submodule T' := E·Tor_0^S(M, k) ⊂ Tor^S(M, k), and the lower row is a minimal free resolution of the quotient T'' = Tor^S(M, k)/T'.

Note that, if M is a minimal higher matrix factorization module then, by Corollary 6.4, the R-modules Ext^even_R(M, k) and Ext^odd_R(M, k) satisfy the regularity hypothesis.

We think of the minimal E-free resolution of Tor^S(M, k) as having two “strands”: the resolution of T', and the resolution of T''.

Proof: We first show that T(M) is acyclic. By Corollary 8.2, the complexes corresponding to Ext^even_R(M, k) and Ext^odd_R(M, k) are acyclic. Since Tor^R(M, k) is the graded dual of Ext_R(M, k) and E is injective as an E-module, the rows of the complex in the Theorem are acyclic. The total complex T(M) is the mapping cone of the map t_3 between these complexes, so it is acyclic as well.

Now let G be a sequence of maps of free S-modules such that G ⊗ R is a minimal R-free resolution of M. By Theorem 7.3 the homology of the complex GK ⊗ k is Tor^S(M, k). In particular, Tor_0^S(M, k), Tor_1^S(M, k) and Tor_2^S(M, k) are the homology of the following complexes at the middle position:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Tor_0^R(M, k) \otimes E_0 & \longrightarrow & 0 \\
Tor_2^R(M, k) \otimes E_0 & t_2 & \longrightarrow & Tor_0^R(M, k) \otimes E_1 & \oplus & Tor_1^R(M, k) \otimes E_0 & \longrightarrow & 0 \\
\oplus & t_3 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
Tor_2^R(M, k) \otimes E_1 & t_2 & \longrightarrow & Tor_0^R(M, k) \otimes E_2 \\
\oplus & t_3 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
Tor_3^R(M, k) \otimes E_0 & t_2 & \longrightarrow & Tor_1^R(M, k) \otimes E_1 & \oplus & Tor_2^R(M, k) \otimes E_0 & t_2 & \longrightarrow & Tor_0^R(M, k) \otimes E_1 .
\end{array}
\]
Under the regularity hypothesis of the Theorem the rows of the diagram \( T(M) \) are exact. In particular the map \[ \text{Tor}_2^R (M, k) \otimes E_0 \rightarrow \text{Tor}_0^R (M, k) \otimes E_1 \] in the sequence for \( \text{Tor}_2^S (M, k) \) above is injective. Thus \( H_0 (T(M)) \) coincides with \( \text{Tor}_2^S (M, k) \) in degrees \( \leq 2 \). Since \( \text{Tor}_2^S (M, k) \) is 1-regular by Theorem 4.6, this implies that \( H_0 (T(M)) \) coincides with \( \text{Tor}_2^S (M, k) \) in all degrees. Together with the exactness of the two strands, this proves the Theorem.

**Corollary 9.3.** With hypotheses and notation as in Theorem 4.6, let \( M_1 \) be the first syzygy of \( M \) over \( R \). The degree 0 strand of the minimal resolution of \( \text{Tor}_2^S (M_1, k) \) is equal to the degree 1 strand of the minimal resolution of \( \text{Tor}_2^S (M, k) \); and the degree 1 strand of the minimal resolution of \( \text{Tor}_2^S (M_1, k) \) is equal to the degree 0 strand of the minimal resolution of \( \text{Tor}_2^S (M, k) \), truncated at homological degree 1.

We now turn to the free resolution of \( \text{Ext}_R (M, k) \). Since the generators of \( R \) have degree 2, we have

\[ \text{Ext}_R^2 (M, k) = \text{Ext}_R^\text{even} (M, k) \oplus \text{Ext}_R^\text{odd} (M, k) \]

as \( R \)-modules. We treat only the even part in detail, as the odd part is analogous.

**Theorem 9.4.** With the hypotheses of Theorem 4.6, let

\[ \sigma_i^j : E_1 \otimes T_{i-1}^j \rightarrow T_i^j \]

be the multiplication maps. The \( i \)-th differential in the minimal \( R \)-free resolution of \( \text{Ext}_R^\text{even} (M, k) \) is the map

\[ \tau_i^j : T_i^j \otimes R(-i) \rightarrow T_{i-1}^j \otimes_k R(-i + 1) \]

whose linear part

\[ T_i^j \otimes_k R_1 \rightarrow T_{i-1}^j \]

is the vector space dual of \( \sigma_i^j \).

The corresponding statement holds for \( \text{Ext}_R^\text{odd} (M, k) \) and \( T'' \) as well.

**Proof.** By Theorem 9.2, the minimal \( E \)-free resolutions of \( T' \) is given by the \( R \)-module structure of the even part of \( \text{Tor}_2^R (M, k) \). Since \( \omega_E := \text{Hom}(E, k) \cong E(c) \) is an injective \( E \)-module, the vector space dual of this resolution is the injective resolution of the \( E \)-modules \( \text{Hom}(T', k) \). Furthermore, the differentials in this injective resolution come, via the BGG correspondence, from the module structure of the even part of the graded vector space dual of \( \text{Tor}_2^R (M, k) \), which is the \( R \)-module \( \text{Ext}_R^\text{even} (M, k) \).

By Theorem 8.1, the resolution of \( \text{Ext}_R^\text{even} (M, k) \) is the BGG dual of \( \text{Hom}(T', k) \). If the module structure of \( T' \) is given by maps \( \mu_i : E_1 \otimes T_i^j \rightarrow T_{i+1}^j \), then the module structure of \( \text{Hom}(T', k) \) is given by maps

\[ \mu_i^j : E_1 \otimes \text{Hom}(T_{i+1}^j, k) \rightarrow \text{Hom}(T_i^j, k) \]
and the BGG dual complex
\[ \cdots \to E \otimes \text{Hom}(T'_{i+1}, k) \to \cdots \to E \otimes \text{Hom}(T'_1, k) \to E \otimes \text{Hom}(T'_0, k) \]
is induced by the maps \( \mu''_i : \text{Hom}(T'_{i+1}, k) \to \text{Hom}(E_1, k) \otimes \text{Hom}(T'_1, k) \). Identifying \( \text{Hom}(E_1, k) \otimes \text{Hom}(T'_i, k) \) with \( \text{Hom}(E_1 \otimes T'_i, k) \), we see that \( \mu''_i \) is, up to change of basis, the same as \( \text{Hom}(\mu_i, k) \), proving the theorem.

Since \( T'_1 = \bigoplus_p E_1(p-1) \otimes B_0(p) \), the minimal \( R \)-free presentation of \( \text{Ext}^\text{even}_R(M, k) \), with the hypotheses in Theorem 9.4, can be written as
\[
R(-1) \otimes \left( \sum_{p=1}^c \text{span}_k \langle e_1, \ldots, e_{p-1} \rangle \otimes B_0(p) \right) \to R \otimes \left( \sum_{p=1}^c B_0(p) \right) \to \text{Ext}^\text{even}_R(M, k) \to 0,
\]
where the map is induced by the appropriate components of the homotopies.

There is an even more direct way of getting a free presentation for the even part of \( \text{Ext} \):

**Corollary 9.5.** With the hypothesis of Theorem 4.6, the module \( \text{Ext}^\text{even}_R(M, k) \) has an \( R \)-free presentation as the cokernel of the map
\[
\tau : \text{Ext}^1_S(M, k) \otimes_k R(-1) \to \text{Hom}(M, k) \otimes_k R
\]
whose linear part
\[
\mu^\vee : \text{Ext}^1_S(M, k) \to \text{Hom}(M, k) \otimes_k R_1
\]
is the vector space dual of the multiplication map
\[
\mu : E_1 \otimes_k \text{Tor}^S_0(M, k) \to \text{Tor}^S_1(M, k)
\]

**Proof.** With notation as above, \( \text{Tor}^S_0(M, k) = T'_0 \), and by Theorem 9.4 the even \( \text{Ext} \) module has minimal \( R \)-free presentation as the cokernel of the map
\[
(T'_1)^\vee \otimes_k R(-1) \to (\text{Tor}^S_0(M, k))^\vee \otimes_k R.
\]
We have
\[
\text{Tor}^S_1(M, k) = T'_1 \oplus T''_1,
\]
so it suffices to show that the image of \( \mu \) is contained in \( T'_1 \). This follows from Proposition 3.7. \( \square \)

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