Assessment of refined higher order theories for the static and vibration analysis of laminated composite cylindrical shells

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ABSTRACT – In the present study, a generalized shell theory is presented and applied for the analysis of laminated composite cylindrical shells. A theoretical unification of the several refined shell theories is presented. The principle of work done is employed to derive five differential equations corresponding to five unknowns involved in the present generalized shell theory. Five differential equations are solved by an analytical procedure suggested by the Navier. The numerical results for simply supported laminated composite cylindrical shells are presented and compared with 3D elasticity solutions. Displacements, stresses and fundamental frequencies are obtained for isotropic, orthotropic, \([0/90/0]\) and \([90/0/90]\) laminated cylindrical shells. The numerical results are obtained for \(h/a=0.1\), \(a/b=1\) and different values of \(R/a\) ratio. Displacements and stresses of laminated cylindrical shells are estimated under sinusoidal transverse load. In the case of free vibration analysis, first five natural frequencies are presented. It is observed that refined theories predicts displacements and stresses in close agreement with 3D elasticity solutions whereas the CST and the FST underpredict the displacements and stresses. It is also observed that the CST overestimates the natural frequencies due to neglect of shear deformation effect.

INTRODUCTION

Laminated composite cylindrical shells are widely used in many engineering industries due to their attractive structural properties such as high strength-to-weight and stiffness-to-weight ratios. Among equivalent single layer theories, layerwise theories and zig-zag theories; equivalent single layer theories are more popularly used for the analysis of laminated composite shells due to simplicity of mathematics. The classical shell theory (CST) of Kirchhoff [1] and first order shear deformation theory (FST) of Mindlin [2] have been widely used theories for the analysis of laminated composite thin shells. However, both the theories have certain drawbacks and not capable enough to predict accurate static and dynamic behaviour of thick laminated shells. This forces researchers to formulate refined models which accurately predict the global response of the laminated shells. In these higher order shell theories, the displacements are expanded using polynomial or non-polynomial type strain functions. In the open literature, Bhimaraddi [3] and Reddy [4] have presented polynomial type shear deformation theories whereas Levy [5], Soldatos [6] and Karama et al. [7] have developed nonpolynomial type shear deformation theories such as trigonometric, hyperbolic and exponential shear deformation theories, respectively. Ayyogdu [8] has developed a new shear deformation theory for the bending, buckling and free vibration analysis of laminated composite plates which is considered as an improvement over exponential shear deformation theory. Akavci [9] has developed two hyperbolic models for the analysis laminated composite plates. Mantari et al. [10] have developed higher order shear deformation theory considering trigonometric and exponential functions for the analysis of laminated composite plates and shells. Neves et al. [11, 12], have developed sinusoidal and hyperbolic shear deformation theories considering the effects of transverse normal strain for the analysis of laminated composite plates. Ghugal and Sayyad [13], and Sayyad and Ghugal [14] have developed a trigonometric shear deformation theory for the bending, buckling and free vibration analysis of laminated composite and sandwich plates. Recently, Sayyad and Naik [15] have developed a fifth order shear deformation theory considering the effects of transverse shear and normal deformation for the interlaminar stress analysis of laminated plates. Detailed discussions on these theories have been presented by Sayyad and Ghugal [16], Liew et al. [17] and Qatu et al. [18].

A literature on static and free vibration analysis of cylindrical shells using various classical and higher order shell theories is published by many researchers. Reddy [19] have applied the third order shear deformation theory for the analysis of laminated composite shells. Soldatos and Timarci [20] and; Timarci and Soldatos [21] have presented an unified formulation of higher order shell theories for the free vibration analysis of laminated composite cylindrical shells. Zenkour and Fares [22] have developed a refined first order theory for the thermal bending analysis of laminated composite cylindrical shells. Khdeir [23] have presented static and vibration analysis of cross-ply shells using thick shell theory Asadi et al. [24] have presented three-dimensional solutions along with application of various shear deformation theories for the static and vibration analyses of laminated composite cylindrical shells. Khalili et al. [25] have presented free vibration analysis of homogeneous isotropic circular cylindrical shells based on a three-dimensional refined higher-order theory. Carrera and Brischetto [26] and Carrera et al. [26, 27] have presented analysis of laminated shells using
various shell theories recovered from the Carrera’s unified formulation. Mantari et al. [28] have developed a new higher order shear deformation theory for the static and free vibration analysis of laminated composite shells. However, from the aforementioned literature it is found that most of the researchers have presented only transverse deflection quantities during the static analysis of laminated shells. Values of in-plane normal stresses and transverse shear stresses for laminated composite shells under mechanical/thermal loads are not reported by many researchers in their papers. It is well known that equivalent single layer shell theories fail to satisfy the continuity at the layer interface in case of laminated shells. Therefore, transverse shear stresses are recovered by using equilibrium equations of theory of elasticity to ascertain continuity at the layer interface. Detail procedure of this method is given by Tornabene et al. [29–35] and; Sayyad and Ghugal [36].

In this work, a generalized displacement model is presented to recover several equivalent single layer higher order and classical shell theories such as the classical shell theory (CST) of Kirchhoff [1], first order shell theory (FST) of Mindlin [2], parabolic shell theory (PST) of Reddy [4], trigonometric shell theory (TST) of Levy [5], hyperbolic shell theory (HST) of Soldatos [6] and exponential shell theory (EST) of Karama et al. [7]. Further, these theories are applied for the static and free vibration analysis of laminated composite cylindrical shells. Five differential equations of the present generalized displacement model are derived using the principle of virtual work. Solutions for static and free vibration problems of simply supported cylindrical shells are obtained using the Navier’s technique. A computer code is developed in Fortran 77 to determine displacements, stresses and frequencies. Numerical results are compared with three-dimensional elasticity solutions given by Bhimaraddi and Chandrashekhara [37] for static analysis. Bhimaraddi [38] has also provided three-dimensional elasticity solutions for doubly curved laminated shells. Since, 3D elasticity solution for the free vibration analysis is not available in the literature; the present results are compared with other works available in the literature. In this work, transverse shear stresses are recovered from 3D stress equilibrium equations of elasticity to ascertain continuity at layer interface/s of the laminated cylindrical shells.

MATHMATICAL FORMULATION

A laminated composite cylindrical shell of rectangular planform \((a \times b)\) and thickness \(h\) shown in Figure 1 is considered for the mathematical formulation. \(R\) denotes the principal radius of curvature of the middle surface. Cylindrical shell is composed of a \(N\) number of orthotropic layers perfectly bonded together.

In the present generalized displacement model, in-plane displacements are presented in three components (extension, bending and shear) whereas the transverse displacement is assumed to be function of \(x\) and \(y\) coordinates. In the Eq. (1) \(u, v\) and \(w\) represent the displacements of any point of the shell domain whereas \(u_0, v_0, w_0\) represent the displacements of any point on the middle surface of the shell in the \(x\)-, \(y\)- and \(z\)-directions, respectively; \(\zeta (z)\) represent the shape functions used to recover various refined higher order and classical shell theories from the present generalized displacement model.

\[
u (x, y, z, t) = v_0 (x, y, t) - \frac{\partial w_0}{\partial y} + \zeta (z) \psi (x, y, t)
\]

\[
u (x, y, z, t) = u_0 (x, y, t) - \frac{\partial w_0}{\partial x} + \zeta (z) \phi (x, y, t)
\]

\[
u (x, y, t) = w_0 (x, y, t)
\]
Following are the strain components obtained using Eqs. (1a)-(1c) and the strain–displacement relations from the theory of elasticity [39].

\[
\varepsilon_x = \varepsilon_x^0 + z \varepsilon_x^1 + \zeta(z) \varepsilon_x^2 \tag{2a}
\]

\[
\varepsilon_y = \varepsilon_y^0 + z \varepsilon_y^1 + \zeta(z) \varepsilon_y^2 \tag{2b}
\]

\[
\gamma_{xy} = \gamma_{xy}^0 + z \gamma_{xy}^1 + \zeta(z) \gamma_{xy}^2 \tag{2c}
\]

\[
\gamma_{xz} = \zeta'(z) \gamma_{xz}^0 \tag{2d}
\]

\[
\gamma_{yz} = \zeta'(z) \gamma_{yz}^0 \tag{2e}
\]

Where

\[
\varepsilon_x^0 = \frac{\partial u_0}{\partial x} + \frac{w_0}{R}, \quad \varepsilon_y^0 = \frac{\partial u_0}{\partial y}, \quad \varepsilon_z^0 = \frac{\partial v_0}{\partial z}, \quad \varepsilon_x^1 = \frac{\partial^2 w_0}{\partial x^2}, \quad \varepsilon_y^1 = \frac{\partial^2 w_0}{\partial y^2}, \quad \varepsilon_z^1 = \frac{\partial^2 w_0}{\partial z^2}, \tag{3a}
\]

\[
\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}, \quad \gamma_{xy}^1 = -2 \frac{\partial^2 w_0}{\partial x \partial y}, \quad \gamma_{xy}^2 = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}, \quad \gamma_{xy}^0 = \phi, \quad \gamma_{xy}^0 = \psi \tag{3b}
\]

In the Eqs. (2a)-(2e), prime indicates derivative with respect to \( z \). The stresses in the \( k \text{th} \) layer of the laminated cylindrical shells are obtained using the following constitutive relationship [39].

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy} \\
\tau_{xz} \\
\tau_{yz}
\end{bmatrix}
^{(k)} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 & 0 & 0 \\
Q_{12} & Q_{22} & 0 & 0 & 0 \\
0 & 0 & Q_{66} & 0 & 0 \\
0 & 0 & 0 & Q_{55} & 0 \\
0 & 0 & 0 & 0 & Q_{44}
\end{bmatrix}
^{(k)}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix}
\tag{4}
\]

where \( Q_{ij} \) are the stiffness coefficients; \( x, y, z \) represent laminate axes and \( 1, 2, 3 \) represent material axes. Stiffness coefficients are expressed in-terms of engineering constants as follows [39].

\[
Q_{11} = \frac{E_1}{1 - \mu_{12} \mu_{21}}, \quad Q_{12} = \frac{\mu_{21} E_1}{1 - \mu_{12} \mu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \mu_{21} \mu_{12}}, \quad Q_{66} = G_{12}, \quad Q_{55} = G_{13}, \quad Q_{44} = G_{23}
\tag{5}
\]

The principle of virtual work is employed to formulate five differential equations associated with the present generalized displacement model. Following is the analytical form of the principle of virtual work where \( \delta \) denotes the virtual operator.

\[
\int_V \left( \sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz} \right) dV - \int_A q(x, y) \delta w dA - \rho \int_V \left( \frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w \right) dV = 0
\tag{6}
\]

By putting Eqs. (2a)-(2e) into the Eq. (6), one can get,

\[
\int_V \left\{ \sigma_x \left[ \varepsilon_x^0 + z \varepsilon_x^1 + \zeta(z) \varepsilon_x^2 \right] + \sigma_y \left[ \varepsilon_y^0 + z \varepsilon_y^1 + \zeta(z) \varepsilon_y^2 \right] + \tau_{xy} \left[ \gamma_{xy}^0 + z \gamma_{xy}^1 + \zeta(z) \gamma_{xy}^2 \right] + \tau_{xz} \left[ \gamma_{xz}^0 + z \gamma_{xz}^1 + \zeta(z) \gamma_{xz}^2 \right] + \tau_{yz} \left[ \gamma_{yz}^0 + z \gamma_{yz}^1 + \zeta(z) \gamma_{yz}^2 \right] \right\} dV - \int_A q(x, y) \delta w dA + \rho \int_V \left( \frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w \right) dV = 0
\tag{7}
\]

The force and moment resultants can be introduced after performing integrations with respect to \( z \) coordinate where superscript \( b \) is used for the resultants due to bending whereas superscript \( s \) is used for the resultants due to shear [39].
\[ \{N_x, M_y, M_z\}^T = \sum_{i=1}^{N_{x,y,z}} \{1, z, \zeta(z)\}^T \sigma_{i}^* dz \]  
\[ (8a) \]

\[ \{N_y, M_x, M_z\}^T = \sum_{i=1}^{N_{x,y,z}} \{1, z, \zeta(z)\}^T \sigma_{i}^* dz \]  
\[ (8b) \]

\[ \{N_z, M_x, M_y\}^T = \sum_{i=1}^{N_{x,y,z}} \{1, z, \zeta(z)\}^T \tau_{i}^* dz \]  
\[ (8c) \]

\[ \{Q_x, Q_y\}^T = \sum_{i=1}^{N_{x,y,z}} \{\tau_{i,x}, \tau_{i,y}\}^T \zeta'(z) dz \]  
\[ (8d) \]

Force and moment resultants of Eqs. (8a)-(8d) can be expressed in terms of unknown variables as follows.

\[ \left[ \begin{array}{c}
N_x \\
N_y \\
M_{x,y} \\
M_{y,z} \\
M_{z,x} \\
M_{x,y} \\
M_{y,z} \\
M_{z,x}
\end{array} \right] = \left[ \begin{array}{cccccc}
A_1 & B_{11} & A_{12} & B_{21} & A_{12} & 0 & 0 & 0 \\
A_2 & B_{12} & A_{22} & B_{22} & A_{22} & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 & 0 & A_{66} & B_{66} & A_{66} \\
& B_{11} & D_{11} & B_{12} & D_{12} & B_{12} & 0 & 0 & 0 \\
& B_{12} & D_{12} & B_{22} & D_{22} & B_{22} & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 & 0 & B_{66} & D_{66} & B_{66} \\
& A_{11} & B_{11} & A_{12} & B_{12} & A_{12} & 0 & 0 & 0 \\
& A_{12} & B_{12} & A_{22} & B_{22} & A_{22} & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 & 0 & A_{66} & B_{66} & A_{66}
\end{array} \right] \left[ \begin{array}{c}
\varepsilon \varepsilon' \varepsilon'' \\
\varepsilon' \varepsilon'' \\
\varepsilon'' \\
\varepsilon'' \\
\varepsilon'' \\
\varepsilon'' \\
\varepsilon'' \\
\varepsilon''
\end{array} \right] \]
\[ (9) \]

and

\[ \left[ \begin{array}{c}
Q_x \\
Q_y
\end{array} \right] = \left[ \begin{array}{ccc}
A_{cc,5} & 0 & \phi
\end{array} \right] \left[ \begin{array}{c}
\phi
\psi
\end{array} \right] \]
\[ (10) \]

Using Eqs. (8a)-(8d), one can modify Eq. (7) as follows

\[ \int_0^b \int_0^a \{N_x \delta e_x + M_y \delta e_y + M_z \delta e_z + N_y \delta e_y + M_z \delta e_z + N_z \delta e_z + N_y \delta \gamma_y + M_z \delta \gamma_z + M_x \delta \gamma_x + + Q_x \delta \gamma_x + Q_y \delta \gamma_y - q \delta w_0\} dx dy + \int_0^b \int_0^a \left[ + \left( I_3 + \frac{I_2}{R} \right) \frac{\partial^3 u_0}{\partial t^3} \frac{\partial \psi}{\partial t} - \left( I_3 + \frac{I_2}{R} \right) \frac{\partial^3 w_0}{\partial t^3} \frac{\partial \psi}{\partial t} - \right] \right] dx dy = 0 \]
\[ (11) \]

After performing integration of the Eq. (11) by parts, collecting the terms of \( \delta u_0, \delta v_0, \delta w_0, \delta \phi \) and \( \delta \psi \) and setting them equal to zero; one can write the following differential equations associated with the present generalized displacement model.

\[ \delta u_0 : \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = \left( I_3 + \frac{I_2}{R} \right) \frac{\partial^3 u_0}{\partial t^3} - \left( I_3 + \frac{I_2}{R} \right) \frac{\partial^3 w_0}{\partial t^3} + \left( I_4 + \frac{I_5}{R} \right) \frac{\partial^3 \phi}{\partial t^3} \]
\[ (12a) \]

\[ \delta v_0 : \frac{\partial N_y}{\partial x} + \frac{\partial N_z}{\partial y} = I_3 \frac{\partial^3 v_0}{\partial t^3} - I_3 \frac{\partial^3 w_0}{\partial t^3} + I_4 \frac{\partial^3 \psi}{\partial t^3} \]
\[ (12b) \]
\[ \delta w_0 : \frac{\partial^2 M^b}{\partial x^2} + 2 \frac{\partial^2 M^b}{\partial x \partial y} + \frac{\partial^2 M^b}{\partial y^2} - N_x - q = \left( I_4 + \frac{I_5}{R} \right) \frac{\partial^2 u_0}{\partial x^2} - I_4 \frac{\partial^2 w_0}{\partial x^2} \]  
(12c)

\[ + I_5 \frac{\partial^2 \psi}{\partial x^2} + I_2 \frac{\partial^2 v_0}{\partial y^2} - I_4 \frac{\partial^2 \psi}{\partial y^2} + I_1 \frac{\partial^2 w_0}{\partial y^2} + I_5 \frac{\partial^2 \psi}{\partial y^2} + I_1 \frac{\partial^2 w_0}{\partial y^2} + I_6 \frac{\partial^2 \psi}{\partial y^2} \]  
(12d)

\[ \delta \phi : \frac{\partial^2 M^x}{\partial x^2} + \frac{\partial^2 M^x}{\partial y^2} - Q_x = \left( I_4 + \frac{I_5}{R} \right) \frac{\partial^2 u_0}{\partial x^2} - I_4 \frac{\partial^2 w_0}{\partial x^2} + I_6 \frac{\partial^2 \phi}{\partial x^2} \]  
(12e)

where

\[ \{A_{ij}, B_{ij}, D_{ij}\} = \sum_{k=1}^{N} Q^k \int_{h_k} \{1 \ z \ z^2\} dz, \quad (i, j = 1, 2, 3, 6) \]  
(13a)

\[ \{A_{ij}, B_{ij}, Ass_{ij}\} = \sum_{k=1}^{N} Q^k \int_{h_k} \{1 \ z \ z^2\} \zeta(z) dz, \quad (i, j = 1, 2, 3, 6) \]  
(13b)

\[ Acc_{ij} = \sum_{k=1}^{N} Q^k \int_{h_k} \left[ \zeta'(z) \right]^2 dz, \quad (i, j = 4, 5) \]  
(13c)

\[ \{I_1, I_2, I_3\} = \sum_{k=1}^{N} \rho \int_{h_k} \{1 \ z \ z^2\} dz, \]  
(13d)

\[ \{I_4, I_5, I_6\} = \sum_{k=1}^{N} \rho \int_{h_k} \zeta(z) \{1 \ z \ z^2\} dz \]  
(13e)

The boundary conditions along the four edges of the shell are presented in Table 1.

**Table 1.** The boundary conditions along the four edges of the shell

| along \(x = 0\) and \(x = a\) | along \(y = 0\) and \(y = b\) |
|-----------------------------|-----------------------------|
| \(N_x = 0\) or \(u_0 = 0\) | \(N_y = 0\) or \(u_0 = 0\) |
| \(N_x = 0\) or \(v_0 = 0\) | \(N_y = 0\) or \(v_0 = 0\) |
| \(V_x = 0\) or \(w = 0\) | \(V_y = 0\) or \(w = 0\) |
| \(M^b_x = 0\) or \(\psi = 0\) | \(M^b_y = 0\) or \(\psi = 0\) |
| \(M^b_x = 0\) or \(\phi = 0\) | \(M^b_y = 0\) or \(\phi = 0\) |
| \(M^b_y = 0\) or \(\psi = 0\) | \(M^b_y = 0\) or \(\psi = 0\) |

where

\[ V_x = \partial M^b_x / \partial x + 2 \partial M^b_y / \partial y, \quad V_y = \partial M^b_y / \partial x + 2 \partial M^b_y / \partial x \]  
(14)

Above mentioned five differential equations can be also written as follows in terms of unknown variables in the generalized displacement model

\[ \delta u_0 : A_{11} \frac{\partial^2 u_0}{\partial x^2} + A_{10} \frac{\partial^2 u_0}{\partial y^2} + A_{10} \frac{\partial^2 u_0}{\partial y^2} + (A_{21} + A_{20}) \frac{\partial^2 v_0}{\partial x \partial y} + A_{11} \frac{\partial^2 v_0}{\partial x \partial y} + B_{11} \frac{\partial^2 w_0}{\partial x^2} - (B_{10} + 2B_{10}) \frac{\partial^2 w_0}{\partial x \partial y} + A_{10} \frac{\partial^2 \phi}{\partial x^2} + A_{20} \frac{\partial^2 v_0}{\partial y^2} - (B_{10} + 2B_{10}) \frac{\partial^2 w_0}{\partial x \partial y} + A_{10} \frac{\partial^2 \phi}{\partial y^2} \]  
(15)
\[
\delta v_0 : \quad (A_{12} + A_{66}) \frac{\partial^2 u_0}{\partial x \partial y} + A_{22} \frac{\partial^2 v_0}{\partial y^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} - B_{22} \frac{\partial^3 w_0}{\partial x \partial y^2} - (B_{12} + 2B_{66}) \frac{\partial^3 w_0}{\partial x \partial y^2} + A_{12} \frac{\partial w_0}{R} \frac{\partial}{\partial y} \\
+ (A_{12} + A_{66}) \frac{\partial^2 u_0}{\partial x \partial y} + A_{22} \frac{\partial^2 v_0}{\partial y^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} - B_{22} \frac{\partial^3 w_0}{\partial x \partial y^2} - (B_{12} + 2B_{66}) \frac{\partial^3 w_0}{\partial x \partial y^2} + A_{12} \frac{\partial w_0}{R} \frac{\partial}{\partial y} = 0
\]
\]
(16)

\[
\delta w_0 : \quad B_{11} \frac{\partial^3 u_0}{\partial x^3} + (B_{11} + 2B_{66}) \frac{\partial^4 v_0}{\partial y^4} + A_{11} \frac{\partial u_0}{R} \frac{\partial}{\partial x} + B_{22} \frac{\partial^3 v_0}{\partial y^3} + (B_{12} + 2B_{66}) \frac{\partial^3 v_0}{\partial y^3} - B_{22} \frac{\partial^3 v_0}{\partial y^3} + \frac{2B_{12}}{R} \frac{\partial^3 w_0}{\partial y^3} \frac{\partial}{\partial x} - A_{11} \frac{\partial w_0}{R} \frac{\partial}{\partial x} \\
- \frac{B_{11}}{R} \frac{\partial^3 u_0}{\partial x^3} - 2(B_{11} + 2B_{66}) \frac{\partial^4 w_0}{\partial y^4} - B_{22} \frac{\partial^4 w_0}{\partial y^4} + \frac{2B_{12}}{R} \frac{\partial^3 w_0}{\partial y^3} \frac{\partial}{\partial x} - A_{11} \frac{\partial w_0}{R} \frac{\partial}{\partial x} \\
+ (B_{12} + 2B_{66}) \frac{\partial^3 \phi}{\partial x \partial y^2} - A_{11} \frac{\partial \phi}{\partial x} + B_{22} \frac{\partial^3 \psi}{\partial y^3} + (B_{12} + 2B_{66}) \frac{\partial^3 \psi}{\partial y^3} - A_{11} \frac{\partial \psi}{\partial x} - q = 0
\]
\]
(17)

\[
\delta \phi : \quad A_{33} \frac{\partial^2 u_0}{\partial x^2} + A_{33} \frac{\partial^2 u_0}{\partial y^2} + (A_{33} + A_{33}) \frac{\partial^2 v_0}{\partial y^2} = B_{33} \frac{\partial^3 w_0}{\partial x \partial y^2} + \frac{A_{33}}{R} \frac{\partial w_0}{\partial x} \\
+ A_{33} \frac{\partial^2 \phi}{\partial x^2} + (A_{33} + A_{33}) \frac{\partial^2 \psi}{\partial y^2} - A_{33} \frac{\partial \phi}{\partial x} + A_{33} \frac{\partial \psi}{\partial y} - \frac{A_{33}}{R} \frac{\partial \phi}{\partial x} = 0
\]
\]
(18)

\[
\delta \psi : \quad (A_{33} + A_{33}) \frac{\partial^2 u_0}{\partial x^2} + A_{33} \frac{\partial^2 u_0}{\partial y^2} + A_{33} \frac{\partial^2 v_0}{\partial y^2} = B_{33} \frac{\partial^3 w_0}{\partial x \partial y^2} + \frac{A_{33}}{R} \frac{\partial w_0}{\partial x} \\
+ (A_{33} + A_{33}) \frac{\partial^2 \phi}{\partial x^2} + A_{33} \frac{\partial^2 \psi}{\partial y^2} + A_{33} \frac{\partial^2 \psi}{\partial y^2} - A_{33} \frac{\partial \phi}{\partial x} + A_{33} \frac{\partial \psi}{\partial y} - \frac{A_{33}}{R} \frac{\partial \phi}{\partial x} - \frac{A_{33}}{R} \frac{\partial \psi}{\partial y} = 0
\]
\]
(19)

**SOLUTIONS FOR SIMPLY-SUPPORTED SHELLS**

In this section, solution procedure of the five differential equations is explained when those are applied for the simply supported laminated cylindrical shells. Navier’s technique is the most widely adopted technique for the analysis of simply supported shells [39]. In this technique, unknowns are presented in the trigonometric form to satisfy the following boundary conditions of the simply supported edges.

at edges \( x = 0 \) and \( x = a \) : \( N_x = v_0 = w = \psi = M_x = M_y = 0 \)
\]
(20a)

at edges \( y = 0 \) and \( y = b \) : \( N_y = u_0 = w = \phi = M_x = M_y = 0 \)
\]
(20b)

The unknowns and transverse load are presented in the following trigonometric form.

For static problem

\[
\{u_0 \ \phi\} = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \{u_{mn} \ \phi_{mn}\} \cos(\alpha_x x) \sin(\alpha_y y)
\]
\]
(21a)

\[
\{v_0 \ \psi\} = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \{v_{mn} \ \psi_{mn}\} \sin(\alpha_x x) \cos(\alpha_y y)
\]
\]
(21b)

\[
\{w_0 \ \phi\} = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \{w_{mn} \ \phi_{mn}\} \sin(\alpha_x x) \sin(\alpha_y y)
\]
\]
(21c)

For free vibration problem \((q=0)\)

\[
\{u_0 \ \phi\} = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \{u_{mn} \ \phi_{mn}\} \cos(\alpha_x x) \sin(\alpha_y y) e^{i\omega t}
\]
\]
(22a)
\[
\{v_0, \psi'\} = \sum_{m=1,3,5}^{n} \sum_{n=1,3,5}^{\infty} \{v_{mn}, \psi_{mn}\} \sin(\alpha_i x) \cos(\alpha_j y) e^{i \omega t}
\] (22b)

\[
\{w_0\} = \sum_{m=1,3,5}^{n} \sum_{n=1,3,5}^{\infty} \{w_{mn}\} \sin(\alpha_i x) \sin(\alpha_j y) e^{i \omega t}
\] (22c)

where, \(\alpha_i = m\pi / a, \alpha_j = n\pi / b\); \(q_{mn}\) represents the Fourier coefficient of the transverse load, \((m, n)\) are odd integers i.e. \(m=n=1,3,5\ldots\infty\); in the case of sinusoidal transverse load, \(q_{mn} = 1\); \(\omega\) represents the natural frequency, \(t\) represents time, \(i = \sqrt{-1}\) and \(u_{mn}, v_{mn}, w_{mn}, \phi_{mn}, \psi_{mn}\) are the unknown coefficients to be determine. Substitution of Eqs. (21a)-(21c) and Eqs. (22a)-(22c) into the differential Eqs. (15)-(19) leads to the following systems of equations for static and free vibration problems respectively.

\[
[K] \{\Delta\} = \{F\}
\] (23)

\[
[[K] - \omega^2 [M]] \{\Delta\} = \{0\}
\] (24)

Displacements and stresses in homogenous and laminated cylindrical shells can be determined using solution of Eq. (23) whereas natural frequencies can be determined using solution of Eq. (24). The elements of stiffness matrix \([K]\), force vector \([F]\), mass matrix \([M]\) and vector of unknowns \([\Delta]\) are given below.

Elements of stiffness matrix \([K]\)

\[
K_{11} = A_{11} \alpha_1^2 + A_{66} \alpha_2^2, \quad K_{12} = (A_{12} + A_{66}) \alpha_1 \alpha_2, \quad K_{13} = -\left[\frac{A_{11}}{R} \alpha_1 + B_{11} \alpha_1^3 - (B_{12} + 2B_{66}) \alpha_2 \alpha_1^2\right],
\]

\[
K_{14} = A_{12} \alpha_1^2 + A_{66} \alpha_2^2, \quad K_{15} = (A_{12} + A_{66}) \alpha_1 \alpha_2, \quad K_{22} = A_{66} \alpha_1^2 + A_{22} \alpha_2^2,
\]

\[
K_{23} = -\left[\frac{A_{12}}{R} \alpha_2 + B_{22} \alpha_2^3 - (B_{12} + 2B_{66}) \alpha_2^2 \alpha_1\right], \quad K_{24} = (A_{12} + A_{66}) \alpha_1 \alpha_2, \quad K_{25} = A_{66} \alpha_2^2 + A_{22} \alpha_2^2,
\]

\[
K_{33} = D_1 \alpha_1^4 + 2(D_{12} + 2D_{66}) \alpha_2 \alpha_1^2 + D_{22} \alpha_2^4 + \frac{2B_{11}}{R} \alpha_1^2 + \frac{2B_{12}}{R} \alpha_2^2 + \frac{A_{12}}{R^2},
\]

\[
K_{34} = B_{11} \alpha_1^4 + (B_{12} + 2B_{66}) \alpha_2 \alpha_1^2 + \frac{A_{12}}{R} \alpha_1, \quad K_{35} = \frac{B_{22} \alpha_2^4 + (B_{12} + 2B_{66}) \alpha_2^2 \alpha_1 + \frac{A_{12}}{R} \alpha_2}{2},
\]

\[
K_{44} = A_{11} \alpha_1^2 + A_{66} \alpha_2^2 + 2 \mu \alpha_1 \alpha_2, \quad K_{45} = (A_{12} + A_{66}) \alpha_1 \alpha_2, \quad K_{55} = A_{12} \alpha_1 \alpha_2 + A_{22} \alpha_2^2 + A_{66} \alpha_2^2 + 2 \mu \alpha_2
\]

Elements of mass matrix \([M]\)

\[
M_{11} = \left(I_{1} + \frac{I_{2}}{R} + \frac{I_{3}}{R^2}\right), \quad M_{12} = 0, \quad M_{13} = -\left(I_{2} + \frac{I_{3}}{R}\right) \alpha, \quad M_{14} = \left(I_{4} + \frac{I_{5}}{R}\right), \quad M_{15} = 0,
\]

\[
M_{22} = I_{1}, \quad M_{23} = -\left(I_{2} + \frac{I_{3}}{R}\right) \beta, \quad M_{24} = 0, \quad M_{25} = I_{4}, \quad M_{33} = I_{2} \alpha^2 + I_{1} \beta^2 + I_{1},
\]

\[
M_{34} = -I_{5} \alpha, \quad M_{35} = -I_{5} \beta, \quad M_{44} = I_{6}, \quad M_{45} = 0, \quad M_{55} = I_{6}
\]

Elements of force vector \([F]\)

\[
[F] = \begin{bmatrix} 0 & 0 & q_{mn} & 0 & 0 \end{bmatrix}^T
\] (27)

Elements of vector of unknowns \([\Delta]\)

\[
[\Delta] = \begin{bmatrix} u_{mn} & v_{mn} & w_{mn} & \phi_{mn} & \psi_{mn} \end{bmatrix}^T
\] (28)

Since the stiffness matrix \([K]\) and mass matrix \([M]\) are symmetric matrices, \(K_{ij}=K_{ji}\) and \(M_{ij}=M_{ji}\)
RESULTS AND DISCUSSION

In this paper, the present generalized displacement model is applied for the static and free vibration analysis of simply supported isotropic, laminated composite cylindrical shells. Numerical results for displacements, stresses and natural frequencies are obtained and compared with 3D elasticity solutions and other studies available in the literature. The following higher order and classical shell theories are recovered using the present generalized displacement model.

\[
PST [4]: \zeta(z) = z \left[ 1 - (4/3)(\bar{z})^2 \right], \quad \bar{z} = z / h
\]

\[
TST [5]: \zeta(z) = (h / \pi) \sin(\pi \bar{z})
\]

\[
HST [6]: \zeta(z) = z \cosh(1/2) - h \sinh(\bar{z})
\]

\[
EST [7]: \zeta(z) = z e^{-\alpha z^2}
\]

FST [2]: \zeta(z) = z

CST [1]: \zeta(z) = 0

Recovery of Transverse Shear Stresses

In multilayered shells, transverse shear stresses leads to the discontinuity at the layer interface when obtained using the constitutive relations. Therefore, in the present study, the transverse shear stresses are recovered by direct integration of stress equilibrium equations of theory of elasticity neglecting the body forces. The transverse shear stresses are recovered by layerwise integration of Eq. (29). The integration constants are determined by imposing the shear stress boundary conditions.

\[
\tau_{\nu}^{(i)} = -\sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left( \frac{\partial \sigma_{\nu}^{(k)}}{\partial x} + \frac{\partial \tau_{\nu}^{(k)}}{\partial y} \right) dz + C_1 \quad \text{and} \quad \tau_{\nu}^{(i)} = -\sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left( \frac{\partial \sigma_{\nu}^{(k)}}{\partial y} + \frac{\partial \tau_{\nu}^{(k)}}{\partial x} \right) dz + C_2
\]

Numerical Results

The non-dimensional numerical results are presented in Tables 3 through 8 and graphically in Figures 2 through 4. The material properties and non-dimensional parameters considered in the numerical examples are presented in Table 2.

| Examples | Material properties | Non-dimensional parameters |
|----------|---------------------|---------------------------|
| 1        | \( E = 210 \text{ GPa}, \mu = 0.3 \) | \( \bar{w} = \frac{w E}{q_0 a}, \bar{\sigma}_x = \frac{\sigma_x}{q_0}, \bar{\tau}_{xx} = \frac{\tau_{xx}}{q_0} \) |
| 2, 3     | \( E_1 = 36.0885, E_2 = 26.2818, G_{12} = 4.9033, G_{13} = 4.4130, G_{23} = 4.0208, \mu_{12} = \mu_{13} = \mu_{23} = 0.105 \) | \( \bar{\omega} = \omega \left( a^2 / h \right) \sqrt{\rho / E_2} \) |
| 4        | \( E_1 / E_2 = 25, E_2 = 26.2818, G_{12} / E_2 = 0.2, G_{12} / G_{13} = 0.5, \mu_{12} = \mu_{13} = \mu_{23} = 0.25 \) | \( \bar{w} = \frac{w E_1}{q_0 a}, \bar{\sigma}_x = \frac{\sigma_x}{q_0}, \bar{\tau}_{xx} = \frac{\tau_{xx}}{q_0} \) |

| Table 3. | Displacements and stresses in isotropic cylindrical shells with different \( R/a \) values (\( h/a = 0.1, a/b = 1 \)) |
|----------|-------------------|-------------------|-------------------|
| \( R/a \) | Model | Theory | \( \bar{w} \) | \( \bar{\sigma}_x \) | \( \bar{\tau}_{xx} \) |
| 5        | Present | PST     | 28.756  | 20.808  | 2.3132 |
| Present  | TST     | 28.753  | 20.820  | 2.3125 |
| Present  | HST     | 28.756  | 20.807  | 2.3133 |
| Present  | EST     | 28.760  | 20.860  | 2.3162 |
| Present  | FST     | 28.508  | 20.620  | 2.3193 |
| Bhimaraddi and Chandrashekhara [37] | 3D Elasticity | 29.003  | 21.138  | 2.3455 |
Table 3. Displacements and stresses in isotropic cylindrical shells with different $R/a$ values ($h/a=0.1$, $a/b=1$) (cont.)

| $R/a$ | Model                        | Theory | $\bar{w}$ | $\bar{\sigma}_x$ | $\bar{\tau}_{xz}$ |
|-------|------------------------------|--------|------------|-------------------|---------------------|
| 10    | Present                      | PST    | 29.390     | 20.532            | 2.3642              |
|       | Present                      | TST    | 29.386     | 20.544            | 2.3635              |
|       | Present                      | HST    | 29.390     | 20.532            | 2.3642              |
|       | Present                      | EST    | 29.394     | 20.585            | 2.3672              |
|       | Present                      | FST    | 29.130     | 20.342            | 2.3699              |
|       | Bhimaraddi and Chandrashekhara [37] | 3D Elasticity | 29.379 | 20.719            | 2.3787              |
| 20    | Present                      | PST    | 29.552     | 20.277            | 2.3773              |
|       | Present                      | TST    | 29.549     | 20.288            | 2.3766              |
|       | Present                      | HST    | 29.552     | 20.277            | 2.3773              |
|       | Present                      | EST    | 29.557     | 20.330            | 2.3803              |
|       | Present                      | FST    | 29.290     | 20.088            | 2.3830              |
|       | Bhimaraddi and Chandrashekhara [37] | 3D Elasticity | 29.445 | 20.413            | 2.3847              |

Example 1: Static Analysis of Homogenous Isotropic Cylindrical Shells

Table 3 shows comparison of transverse displacements and stresses of homogenous isotropic cylindrical shell subjected to sinusoidal transverse load. The non-dimensional results are presented for various $R/a$ (=5, 10, 20) values. Table 2 shows material properties and non-dimensional parameters for this example. The numerical results are presented by using higher order shell theories recovered from the present generalized displacement model and compared with three-dimensional (3D) elasticity solutions presented by Bhimaraddi and Chandrashekhara [37]. From Table 3, it is observed that the trigonometric shell theory (TST) shows transverse displacements in close agreement with 3D elasticity solution for $R/a=10$ and 20, whereas, exponential shell theory (EST) shows transverse displacements slightly on higher side. The parabolic shell theory (PST) and hyperbolic shell theory (HST) show identical results of displacements and stresses. The first order shell theory underestimates the transverse displacements and stresses for all $R/a$ values.

Table 4. Displacements and stresses in orthotropic cylindrical shells with different $R/a$ values ($h/a=0.1$, $a/b=1$)

| $R/a$ | Model                        | Theory | $\bar{w}$ | $\bar{\sigma}_x$ | $\bar{\tau}_{xz}$ |
|-------|------------------------------|--------|------------|-------------------|---------------------|
| 5     | Present                      | PST    | 54.434     | 27.914            | 2.5736              |
|       | Present                      | TST    | 54.425     | 27.944            | 2.5721              |
|       | Present                      | HST    | 54.480     | 27.831            | 2.5656              |
|       | Present                      | EST    | 54.446     | 28.052            | 2.5798              |
|       | Present                      | FST    | 53.609     | 27.449            | 2.5876              |
|       | Present                      | CST    | 49.442     | 27.515            | ---                 |
|       | Bhimaraddi and Chandrashekhara [37] | 3D Elasticity | 53.298 | 28.466            | 2.6283              |
| 10    | Present                      | PST    | 55.384     | 27.820            | 2.6185              |
|       | Present                      | TST    | 55.374     | 27.850            | 2.6170              |
|       | Present                      | HST    | 55.432     | 27.735            | 2.6104              |
|       | Present                      | EST    | 55.396     | 27.960            | 2.6249              |
|       | Present                      | FST    | 54.530     | 27.348            | 2.6320              |
|       | Present                      | CST    | 50.224     | 27.423            | ---                 |
|       | Bhimaraddi and Chandrashekhara [37] | 3D Elasticity | 54.958 | 28.154            | 2.5473              |
| 20    | Present                      | PST    | 55.627     | 27.649            | 2.6300              |
|       | Present                      | TST    | 55.617     | 27.680            | 2.6285              |
|       | Present                      | HST    | 55.675     | 27.564            | 2.6218              |
|       | Present                      | EST    | 55.639     | 27.790            | 2.6364              |
|       | Present                      | FST    | 54.765     | 27.179            | 2.6434              |
|       | Present                      | CST    | 50.424     | 27.267            | ---                 |
|       | Bhimaraddi and Chandrashekhara [37] | 3D Elasticity | 55.023 | 27.908            | 2.6554              |
Example 2: Static Analysis of Homogenous Orthotropic Cylindrical Shells

Non-dimensional displacements and stresses of homogenous orthotropic cylindrical shells subjected to sinusoidal load are presented in Table 4 and compared with 3D elasticity solution of Bhimaraddi and Chandrashekhara [37]. The numerical results are presented for different $R/a$ values. Material properties and non-dimensional parameter used to obtain numerical results for this example are presented in Table 2. Examination of Table 4 reveals that the transverse displacements predicted by FST are in good agreement with those presented by Bhimaraddi and Chandrashekhara [37] in 3D elasticity solution whereas EST predicts stresses in excellent agreement with those presented in 3D elasticity solution. Non-dimensional value of transverse displacements increases with respect to increase in $R/a$ values whereas stresses are decreases with increase in $R/a$ values.

| Model                  | Theory | $\bar{w}$ | $\bar{x}$ | $\bar{z}$ | $\bar{w}$ | $\bar{x}$ | $\bar{z}$ |
|------------------------|--------|------------|------------|------------|------------|------------|------------|
| Present                | PST    | 35.358     | 17.198     | 1.4140     | 35.097     | 21.246     | 1.6189     |
| Present                | TST    | 35.362     | 17.213     | 1.4134     | 35.094     | 21.264     | 1.6179     |
| Present                | HST    | 35.343     | 17.761     | 1.4139     | 35.079     | 21.300     | 1.6190     |
| Present                | EST    | 35.223     | 17.229     | 1.4128     | 35.103     | 21.333     | 1.6176     |
| Present                | CST    | 33.247     | 17.030     | 1.4284     | 34.743     | 21.031     | 1.6382     |
| Bhimaraddi and Chandrashekhara [37] | 3D Elasticity | 41.340 | 23.700 | 1.7763 | 40.823 | 23.987 | 1.8290 |

Example 3: Static Analysis of Laminated Cylindrical Shells

In this example, all the higher order shell theories recovered from the present generalized displacement model are applied for the static analysis of laminated composite cylindrical shells on rectangular planform. Material properties and non-dimensional parameters are presented in Table 2. In case of laminated shells, it is assumed that all layers are of equal thickness i.e. for two-ply ($0^0/90^0$) laminated shells thickness of each layer is $h/2$ whereas for three-ply ($0^0/90^0/0^0$) laminated shells thickness of each layer is $h/3$. The present results are compared with 3D elasticity solution presented by Bhimaraddi and Chandrashekhara [37]. The numerical results are presented in Table 5 for two-ply ($0^0/90^0$) and three-ply ($0^0/90^0/0^0$) laminated shells for $R/a$ =1, $h/a=0.1$ and $a/b=1$. For both the lamination schemes, TST predicts transverse displacement in close agreement with 3D elasticity solution. The FST and CST underpredict the displacements and stresses for both the types of laminated shells. Through-the-thickness distributions of in-plane normal and transverse shear stresses are plotted in Figures 2 and 3 for two-ply ($0^0/90^0$) and three-ply ($0^0/90^0/0^0$) laminated shells respectively. Both the stresses are observed to be maximum in 0 degree layers and minimum in 90 degree layer. The maximum in-plane normal stresses are observed at the bottom of shell i.e. ($z=h/2$) whereas maximum transverse shear stresses are observed at the middle surface of the shell.

![Figure 2. Through-the-thickness profiles of stresses for two-ply ($0^0/90^0$) laminated shells](link)
Table 6. Natural frequencies in cylindrical shells with different $R/a$ values ($h/a=0.1, a/b=1$)

| $R/a$ | Model      | Theory | Isotropic | Orthotopic | $0^0/90^0$ | $0^0/90^0/0^0$ |
|-------|------------|--------|-----------|------------|------------|---------------|
| 0.5   | Present    | PST    | 10.0200   | 14.0875    | 14.2221    | 15.2534       |
|       | Present    | TST    | 10.0199   | 14.0847    | 14.2240    | 15.2282       |
|       | Present    | HST    | 10.0205   | 14.0975    | 14.2275    | 15.2557       |
|       | Present    | EST    | 10.0211   | 14.1178    | 14.2400    | 15.2417       |
|       | Present    | FST    | 10.0302   | 14.2902    | 14.2538    | 15.6721       |
|       | Asadi et al. [24] | FSDTQ | 9.59610   | --         | 13.7710    | 15.2500       |
|       | Asadi et al. [24] | 3D-FEM | 7.55300   | --         | 13.7720    | 14.8400       |
| 1     | Present    | PST    | 7.34706   | 13.0226    | 10.9307    | 12.9914       |
|       | Present    | TST    | 7.34689   | 13.0188    | 10.9345    | 12.9541       |
|       | Present    | HST    | 7.34782   | 13.0368    | 10.9390    | 12.9948       |
|       | Present    | EST    | 7.34898   | 13.0655    | 10.9585    | 12.9742       |
|       | Present    | FST    | 7.36499   | 13.3080    | 10.9594    | 13.6064       |
|       | Asadi et al. [24] | FSDTQ | 7.09000   | --         | 10.6660    | 13.1870       |
|       | Asadi et al. [24] | 3D-FEM | 7.08100   | --         | 10.6860    | 12.5900       |
| 2     | Present    | PST    | 6.22746   | 12.6228    | 9.57321    | 12.1277       |
|       | Present    | TST    | 6.22725   | 12.6185    | 9.57804    | 12.0849       |
|       | Present    | HST    | 6.22841   | 12.6386    | 9.58313    | 12.1316       |
|       | Present    | EST    | 6.22990   | 12.6707    | 9.60642    | 12.1080       |
|       | Present    | FST    | 6.25008   | 12.9410    | 9.60041    | 12.8294       |
|       | Asadi et al. [24] | FSDTQ | 6.09130   | --         | 9.45770    | 12.4430       |
|       | Asadi et al. [24] | 3D-FEM | 6.09210   | --         | 9.48550    | 11.7690       |

Table 7. First five natural frequencies in two-ply ($0^0/90^0$) laminated cylindrical shells with different $R/a$ values ($h/a=0.1, a/b=1$)

| $R/a$ | Model      | Theory | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|-------|------------|--------|-------------|-------------|-------------|-------------|-------------|
| 0.5   | Present    | PST    | 14.2221     | 22.6425     | 29.8696     | 32.5766     | 40.3561     |
|       | Present    | TST    | 14.2240     | 22.6697     | 29.8837     | 32.6103     | 40.4125     |
|       | Present    | HST    | 14.2275     | 22.7138     | 29.9156     | 32.6829     | 40.6236     |
|       | Present    | EST    | 14.2400     | 22.9038     | 30.0395     | 32.9716     | 41.4923     |
|       | Present    | FST    | 14.2538     | 22.7374     | 29.9785     | 32.7823     | 40.4045     |
|       | Asadi et al. [24] | FSDTQ | 13.7710     | 21.0370     | 29.5710     | 31.2000     | 38.0730     |
|       | Asadi et al. [24] | 3D-FEM | 13.7720     | 21.0400     | 29.6390     | 31.4110     | 38.2660     |
Example 4: Free Vibration Analysis of Homogenous and Laminated Cylindrical Shells

In this example, the present theories are applied for the free vibration analysis of homogenous and laminated cylindrical shells. Natural frequencies are presented in Tables 6 through 8. Material properties and non-dimensional form are presented in Table 2. Isotropic material properties are similar to example 1. Table 6 shows first natural frequencies in isotropic, orthotropic and laminated composite cylindrical shells whereas Tables 7 and 8 represents first five natural frequencies of two-ply ($0^\circ/90^\circ$) and three-ply ($0^\circ/90^\circ/0^\circ$) laminated cylindrical shells. Since 3D elasticity solution for frequency analysis is not available in the literature, the present results are compared with those presented by Asadi et al. [24], Tables 6 through 8 show that the natural frequencies predicted by the present models are in good agreement with those presented by Asadi et al. [24] for all cases. Examination of these Tables reveals that the natural frequencies are decreases with increase in $R/a$ values.

**Table 7. First five natural frequencies in two-ply ($0^\circ/90^\circ$) laminated cylindrical shells with different $R/a$ values ($h/a=0.1$, $a/b=1$) (cont.)**

| $R/a$ | Model      | Theory | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|-------|------------|--------|-------------|-------------|-------------|-------------|-------------|
| 1     | Present    | PST    | 10.9307     | 22.3397     | 24.5762     | 31.1716     | 40.3531     |
|       | Present    | TST    | 10.9345     | 22.3691     | 24.5999     | 31.2102     | 40.4138     |
|       | Present    | HST    | 10.9390     | 22.4141     | 24.6417     | 31.2855     | 40.6210     |
|       | Present    | EST    | 10.9585     | 22.6127     | 24.8177     | 31.5948     | 41.2947     |
|       | Present    | FST    | 10.9594     | 22.4171     | 24.6705     | 31.3472     | 40.3429     |
|       | Asadi et al. [24] | FSDTQ | 10.6660     | 21.7050     | 24.0900     | 30.3680     | 38.7220     |
|       | Asadi et al. [24] | 3D-FEM | 10.6860     | 21.7670     | 24.1910     | 30.6140     | 38.8960     |
| 2     | Present    | PST    | 9.57321     | 22.1518     | 22.6620     | 30.6846     | 40.2263     |
|       | Present    | TST    | 9.57804     | 22.1818     | 22.6901     | 30.7248     | 40.2888     |
|       | Present    | HST    | 9.58313     | 22.2267     | 22.7350     | 30.8001     | 40.4933     |
|       | Present    | EST    | 9.60642     | 22.4266     | 22.9309     | 31.1130     | 41.2947     |
|       | Present    | FST    | 9.60041     | 22.2207     | 22.7412     | 30.8446     | 40.1893     |
|       | Asadi et al. [24] | FSDTQ | 9.45770     | 21.6760     | 22.1500     | 29.9590     | 38.6080     |
|       | Asadi et al. [24] | 3D-FEM | 9.48550     | 21.7430     | 22.2460     | 30.1930     | 38.7450     |

**Table 8. First five natural frequencies in three-ply ($0^\circ/90^\circ/0^\circ$) laminated cylindrical shells with different $R/a$ values ($h/a=0.1$, $a/b=1$)**

| $R/a$ | Model      | Theory | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|-------|------------|--------|-------------|-------------|-------------|-------------|-------------|
| 0.5   | Present    | PST    | 15.2534     | 27.8837     | 28.6315     | 34.0363     | 42.8207     |
|       | Present    | TST    | 15.2282     | 27.8650     | 28.4831     | 33.9086     | 42.7873     |
|       | Present    | HST    | 15.2557     | 27.8850     | 28.6434     | 34.0458     | 42.8216     |
|       | Present    | EST    | 15.2417     | 27.8754     | 28.7184     | 34.1171     | 42.8107     |
|       | Present    | FST    | 15.6721     | 28.2423     | 31.9990     | 37.0829     | 43.4225     |
|       | Asadi et al. [24] | FSDTQ | 15.2500     | 17.7890     | 29.4910     | 34.7950     | 39.1390     |
|       | Asadi et al. [24] | 3D-FEM | 14.8400     | 17.4680     | 29.0940     | 32.4640     | 33.0460     |
| 1     | Present    | PST    | 12.9914     | 21.8328     | 29.1613     | 32.2639     | 34.8957     |
|       | Present    | TST    | 12.9541     | 21.8041     | 29.0050     | 33.1243     | 34.8644     |
|       | Present    | HST    | 12.9948     | 21.8352     | 29.1734     | 33.2742     | 34.8980     |
|       | Present    | EST    | 12.9742     | 21.8213     | 29.2523     | 33.3525     | 34.8878     |
|       | Present    | FST    | 13.6064     | 22.3960     | 32.7008     | 35.7787     | 36.5792     |
|       | Asadi et al. [24] | FSDTQ | 13.1870     | 18.5240     | 30.5640     | 32.2320     | 34.5230     |
|       | Asadi et al. [24] | 3D-FEM | 12.5900     | 18.0050     | 29.7320     | 30.1890     | 32.0370     |
| 2     | Present    | PST    | 12.1277     | 19.4390     | 29.3036     | 29.3036     | 33.0494     |
|       | Present    | TST    | 12.0849     | 19.4049     | 29.1453     | 29.1453     | 32.9065     |
|       | Present    | HST    | 12.1316     | 19.4419     | 29.3160     | 29.3160     | 33.0598     |
|       | Present    | EST    | 12.1080     | 19.4260     | 29.3960     | 29.3960     | 33.1402     |
|       | Present    | FST    | 12.8294     | 20.1085     | 32.8892     | 32.8892     | 36.4397     |
|       | Asadi et al. [24] | FSDTQ | 12.4430     | 18.6770     | 30.8390     | 31.3230     | 34.4560     |
|       | Asadi et al. [24] | 3D-FEM | 11.7690     | 18.1590     | 28.6000     | 30.4710     | 31.9280     |
CONCLUSION

In this study, several higher order and classical shell theories are recovered using a generalized displacement model for the static and free vibration analysis of laminated composite cylindrical shells on rectangular planform. The differential equations of the present generalized displacement model are derived by using the principle of virtual work and further solved analytically using the Navier’s solution technique. Transverse shear stresses are recovered from 3D equilibrium equations of theory of elasticity. Numerical results obtained using all the present models are compared with 3D elasticity solution and available literature and are found in good agreement. The numerical results and their comparison proved the validity and accuracy of the present generalized displacement model.

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