Mean field ground state of a spin-1 condensate in a magnetic field

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Abstract. We revisit the topic of the mean field ground state of a spin-1 atomic condensate inside a uniform magnetic field (B) under the constraints that both the total number of atoms (N) and the magnetization (M) are conserved. In the presence of an internal state (spin component) independent trap, we also investigate the dependence of the so-called single spatial mode approximation on the magnitude of the magnetic field and M. Our results indicate that the quadratic Zeeman effect is an important factor in balancing the mean field energy from elastic atom–atom collisions that are known to conserve both N and M.

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1. Introduction

Atomic Bose–Einstein condensates (BEC) have provided a successful testing ground for theoretical studies of quantum many-body systems [1]. In earlier BEC experiments, atoms were spatially confined with magnetic traps, which essentially freeze the atomic internal degrees of freedom [2]. Most studies were thus focused on scalar models, i.e. single component quantum
degenerate gases [3]. More recently, the emergence of spin-1 condensates [4]–[6] (of atoms with hyperfine quantum number \( F = 1 \)) has created opportunities for understanding degenerate gases with internal degrees of freedom [7]–[11], [14].

In this paper, we investigate the mean field ground state structures of a spin-1 atomic condensate in the presence of an external magnetic field (\( B \)). We focus on several aspects of the ground state properties strongly affected by the requirement that elastic atom–atom collisions conserve both the total number of atoms (\( N \)) and the magnetization (\( M \)). Several earlier studies have focused on the global ground state structures when the conservation of \( M \) was ignored, or in the limiting case of a vanishingly small magnetic field (\( B = 0 \)) [6]–[11], [14]. As we show in this study, in the presence of a non-zero magnetic field, the conservation of \( M \) leads to ground state population distributions significantly different from that of the global ground state.

Our system is described by the Hamiltonian (repeated indices are summed) [7]

\[
\mathcal{H} = \int d\vec{r} \psi_i^\dagger (\mathcal{L}_{ij} + \mathcal{H}_{ZM}) \psi_j + \frac{c_0}{2} \int d\vec{r} \psi_i^\dagger \psi_j^\dagger \psi_j \psi_i + \frac{c_2}{2} \int d\vec{r} \psi_k^\dagger \psi_l^\dagger (F_{\eta ij})(F_{\eta kl}) \psi_l \psi_i, \tag{1}
\]

where \( \psi_j(\vec{r}) \) is the field operator that annihilates an atom in the \( j \)th (\( j = +, 0, - \)) internal state at location \( \vec{r} \), \( \mathcal{L}_{ij} \equiv [-\hbar^2 \nabla^2/2M + V_{ext}(\vec{r})] \delta_{ij} \) with \( M \) the mass of each atom and \( V_{ext}(\vec{r}) \) an internal state independent trap potential. Terms with coefficients \( c_0 \) and \( c_2 \) in equation (1) describe elastic collisions of the spin-1 atom (\( |F = 1, M_F = +, 0, - \rangle \)), expressed in terms of the scattering length \( a_0 \) (or) for two spin-1 atoms in the combined symmetric channel of total spin 0 (2), \( c_0 = 4\pi \hbar^2(a_0 + 2a_2)/3M \) and \( c_2 = 4\pi \hbar^2(a_2 - a_0)/3M \). \( F_{\eta=x,y,z} \) are spin-1 matrices with

\[
F_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

The external magnetic field \( B \) is taken to be along the quantization axis (\( \hat{z} \)); it induces a Zeeman shift on each atom given by

\[
H_{ZM}(B) = \begin{pmatrix} E_+ & 0 & 0 \\ 0 & E_0 & 0 \\ 0 & 0 & E_- \end{pmatrix}.
\]

According to the Breit–Rabi formula [12], the individual level shift can be expressed as

\[
E_+ = -\frac{E_{\text{HFS}}}{8} - g_I \mu_I B - \frac{1}{2} E_{\text{HFS}} \sqrt{1 + \alpha + \alpha^2}, \tag{2}
\]

\[
E_0 = -\frac{E_{\text{HFS}}}{8} - \frac{1}{2} E_{\text{HFS}} \sqrt{1 + \alpha^2},
\]

\[
E_- = -\frac{E_{\text{HFS}}}{8} + g_I \mu_I B - \frac{1}{2} E_{\text{HFS}} \sqrt{1 - \alpha + \alpha^2},
\]

where \( E_{\text{HFS}} \) is the hyperfine splitting [12], and \( g_I \) is the Lande \( g \)-factor for the atomic nuclei with nuclear spin \( I \). \( \mu_I \) is the nuclear magneton and \( \alpha = (g_I \mu_I B + g_I \mu_B B) / E_{\text{HFS}} \) with \( g_I \) the Lande \( g \)-factor for the valence electron with total angular momentum \( J \). \( \mu_B \) is the Bohr magneton.

2. Mean field approximation

At near zero temperatures and when the total number of condensed atoms is large, the ground state is essentially determined by the mean field term \( \Phi_i = (\psi_i) \). Neglecting all quantum
fluctuations we arrive at the mean field energy functional from equation (1) \[8, 13\]

\[
H[\{\Phi_i\}] = H_S + E_0 N + \frac{c_2}{2} \langle \vec{F} \rangle^2 - \eta_0 \langle F_z \rangle + \delta \langle F_z^2 \rangle, \tag{3}
\]

where the symmetric part

\[
H_S = \int d\vec{r} \left[ \Phi_i^* \mathcal{L}_{ij} \Phi_j + \frac{c_0}{2} \Phi_i^* \Phi_j^* \Phi_j \Phi_i \right] \tag{4}
\]
is invariant under the exchange of spin component indices, and thus is independent of the external \(B\)-field. The Zeeman shift as given by the Breit–Rabi formula (2) can be described by two positive parameters \[13\]

\[
\begin{align*}
2\eta_0 &= E_- - E_+ , \\
2\delta &= E_+ + E_- - 2E_0 , 
\end{align*} \tag{5}
\]

which measure approximately the linear and quadratic Zeeman effects. The \(B\)-field dependence of \(\eta_0\) and \(\delta\) for a \(^{87}\text{Rb}\) atom are displayed in figure 1.

The elastic atomic collisions as described by the \(c_0\) and \(c_2\) parts of the Hamiltonian (1) conserve both \(N\) and \(M\), which in the mean field approximation are given by

\[
\begin{align*}
N &= \sum_{j=\pm 0} \int d\vec{r} \langle \psi_j^\dagger(\vec{r}) \psi_j(\vec{r}) \rangle 
\approx \sum_{j=\pm 0} \int d\vec{r} |\Phi_j(\vec{r})|^2, \\
M &= \int d\vec{r} \left[ (\psi_+^\dagger(\vec{r}) \psi_+(\vec{r})) - (\psi_-^\dagger(\vec{r}) \psi_-(\vec{r})) \right] 
\approx \int d\vec{r} [|\Phi_+(\vec{r})|^2 - |\Phi_-(\vec{r})|^2].
\end{align*} \tag{6}
\]

Before continuing our discussion of the ground state structures, we shall first briefly comment on the importance of the above two constraints. In a typical experiment, the last stage before condensation consists of atomic evaporations, during which neither \(N\) nor \(M\) is conserved. For a scalar condensate, typically the ground state is obtained from a minimization of equation (3) subjected to the constraint of only \(N\) conservation. This gives rise to the Gross–Pitaevskii equation (GPE) and the associated condensate chemical potential, which mathematically is simply the Lagrange multiplier of the constrained minimization. A spin-1 condensate requires the introduction of two Lagrange multipliers during the minimization subjected to both the \(N\) and \(M\) conservation constraints, as was first performed in [13].

When atomic interactions are ferromagnetic (\(c_2 < 0\) as for \(^{87}\text{Rb}\) atoms) and when the external \(B\)-field is negligible, we have shown previously that the ground state structure is simply

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a state where all individual atomic spins are aligned in the same direction [14]. In this case, the conservation of \( \mathcal{M} \) can be simply satisfied by tilting the quantization axis away from the direction of the condensate spin. This can always be done if a system described by (1) is rotationally symmetric, and thus contains the \( SO(3) \) symmetry [7]. The presence of a non-zero \( B \)-field, on the other hand, breaks the rotational symmetry (e.g. the linear Zeeman shift reduces the \( SO(3) \) to \( SO(2) \) symmetry), and thus the conservation of \( \mathcal{M} \) has to be directly included in the minimization process.

The global ground state phase diagram including both the linear and quadratic Zeeman effect was first investigated by Stenger \textit{et al} [6]. In this early study, although the \( \mathcal{M} \) conservation was included in their formulation, it was not separately discussed; consequently their results do not easily apply to systems with fixed values of \( \mathcal{M} \). The ground state structures as given in [6] correspond to the actual ground state as realized through an \( \mathcal{M} \)-non-conserving evaporation process (with/without a \( B \)-field) due to an \( \mathcal{M} \) conserving evaporation process. Although more limited, as our results can be traced to linear trajectories of \( \mathcal{M} = \) constant in the phase diagram of [13], we expect them to be useful, especially in predicting ground state structures when a ready-made spinor condensate is subjected to external manipulations that conserve both \( N \) and \( \mathcal{M} \).

When atomic interactions are antiferromagnetic (\( c_2 > 0 \)), the global ground state was first determined to be a total spin singlet [8]. More elaborate studies, including quantum fluctuations, were performed by Ho and Yip [17] and Koashi and Ueda [18]. Unfortunately, these results [17, 18] do not correspond to actual ground states as realized in current experiments, because of the presence of background magnetic fields. For instance, the states as found in [17] are only possible if the magnetic field \( B \) is less than 70 \( \mu \)G at the condensate density as realized in the MIT experiments [13]. The Zeeman shift (see figure 1) due to the presence of even a small magnetic field can overwhelm the atomic mean field interaction and typical atomic thermal energy. Thus if it were not for the conservation of \( \mathcal{M} \), the ground state would simply correspond to all atoms condensing into the lowest Zeeman sublevel of \( |M_F = 1 \rangle \).

We now minimize \( H \) of equation (3) by denoting \( \Phi_j(\vec{r}) = \sqrt{N_j} \phi_j(\vec{r}) \exp(-i\theta_j) \), with a real mode function \( \phi_j(\vec{r}) \) \( (\int \phi_j^2(\vec{r}) \, d\vec{r} = 1) \) and phase \( \theta_j \). It is easy to check that the phase convention of ferromagnetic/antiferromagnetic interactions as obtained previously [10] in the absence of a \( B \)-field still remains true, i.e.

\[
\begin{align}
\theta_+ + \theta_- - 2\theta_0 &= 0, \quad c_2 < 0 \quad \text{(ferromagnetic)}, \\
\theta_+ + \theta_- - 2\theta_0 &= \pi, \quad c_2 > 0 \quad \text{(antiferromagnetic)}. 
\end{align}
\]

3. Ground state in a homogeneous system

In a homogeneous system such as a box-type trap (of volume \( V \)), on adopting the above phase convention the resulting ground state energy functional becomes (+/− for \( c_2 < 0 \) and \( c_2 > 0 \) respectively)

\[
H[\{N_i\}] = H_0 + E_0 N + \frac{c_2}{2V} \left[ (N_+ - N_-)^2 + 2N_0 (\sqrt{N_+} \pm \sqrt{N_-})^2 \right] \\
- \eta_0 (N_+ - N_-) + \delta(N_+ + N_-).
\]

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Figure 2. The dependence of fractional population for different spin components on $m$ and $B$ for a spin-1 $^{87}\text{Rb}$ homogeneous condensate with $N/V = 5 \times 10^{14} \text{cm}^{-3}$.

Expressing everything in terms of fractional populations and fractional magnetization $n_i = N_i/N$ and $m = M/N$, and noting that $n_+ + n_- = 1 - n_0$, $n_+ - n_- = m$, equation (9) becomes

$$\frac{H[(n_i)]}{N} = \frac{H_S}{N} + E_0 + c [(n_+ - n_-)^2 + 2n_0 (\sqrt{n_+} \pm \sqrt{n_-})^2] - \eta_0 (n_+ - n_-) + \delta (n_+ + n_-),$$

with an interaction coefficient $c = c_2 N/2V$, tunable through a change of condensate density.

We now minimize equation (10) under the two constraints $n_+ + n_0 + n_- = 1$ and $n_+ - n_- = m$. We restrict our discussion to the region $-1 < m < 1$ as the special cases of $m = \pm 1$ are trivial. Because $H_S$, $E_0$, $c$, $\eta_0$ and $m$ are all constants for given values of $B$, $N$ and $V$, the only part left to be minimized is

$$\mathcal{F} = 2cn_0 (\sqrt{n_+} \pm \sqrt{n_-})^2 + \delta (n_+ + n_-).$$

In the special case of $c = 0$, equation (11) reduces to

$$\mathcal{F} = \delta (n_+ + n_-).$$

The ground state is then very simple. When $\delta > 0$, which seems to be always the case for the quadratic Zeeman shift, the minimum is reached by having as large an $n_0$ (thus as small an $n_+ + n_-$) as possible, namely

$$n_0 = 1 - |m|, \quad n_+ = \begin{cases} |m|, & m \geq 0 \\ 0, & m < 0 \end{cases}, \quad n_- = \begin{cases} 0, & m \geq 0 \\ |m|, & m < 0. \end{cases} \quad (13)$$

When $\delta = 0$, we have (in general) three condensate components with $n_\pm = (1 - n_0 \pm m)/2$ and $0 \leq n_0 \leq 1 - |m|$. For ferromagnetic interactions with $c < 0$, we define $x = n_+ + n_-$. The ground state is then determined by the minimum of

$$\mathcal{F} = g_+(x) + \delta x,$$

with $g_+(x) \equiv 2c(1-x)(x + \sqrt{x^2 - m^2})$. When $\delta = 0$, we find

$$n_\pm = \frac{1}{4}(1 \pm m)^2, \quad n_0 = \frac{1}{2}(1 - m^2). \quad (15)$$
which is the same as obtained in [11, 14]. However, with a non-zero $\delta > 0$, we find in general

$$n_\pm = \frac{1}{2}(x_0 \pm m), \quad n_0 = 1 - x_0 \geq \frac{1}{2}(1 - m^2),$$  \hspace{1cm} (16)

with $x_0$ being the root of the equation $g'_1(x) + \delta = 0$, and it turns out that there always exists one and only one solution to the equation. The equilibrium value for $n_0$ is larger than the result of equation (15) because the quadratic Zeeman effect causes a lowering of the total energy if two $|M_F = 0\rangle$ atoms are created when an $|M_F = +1\rangle$ atom collides with an $|M_F = -1\rangle$ atom.

Figure 2 displays the results of equation (16) for a typical $^{87}\text{Rb}$ condensate, for which the atomic parameters are $E_{\text{HFS}} = (2\pi)6.8347$ GHz [12], $a_0 = 0.101 a_B$ and $a_2 = 0.404 a_B$ ($a_B$ is the Bohr radius) [15]. At weak magnetic fields, typically a condensate contains all three spin components. With increasing $B$-field, the quadratic Zeeman effect becomes important, which energetically favours the $|0\rangle$ component, so typically only two components survive: the $|0\rangle$ component and the larger (initial population) of the $|+\rangle$ or $|-\rangle$ component, so the ground state becomes (for $m > 0$) $n_+ \simeq m$ and $n_0 \simeq 1 - m$.

Finally we consider the case of antiferromagnetic interactions for $c > 0$. We have then

$$\mathcal{F} = g_-(x) + \delta x,$$  \hspace{1cm} (17)

with $g_-(x) = 2c(1 - x)(x - \sqrt{x^2 - m^2})$. For $\delta = 0$, we again recover the standard result

$$n_0 = 0, \quad n_\pm = \frac{1}{2}(1 \pm m),$$  \hspace{1cm} (18)

if $m \neq 0$. When $m = 0$, the ground state is underdetermined as many solutions are allowed as long as they satisfy $n_+ = n_- = 1 - n_0$ with $n_0 \in [0, 1]$.

In an external $B$-field when $\delta > 0$, we first consider the special case of $m = 0$. It can be easily seen from equation (17) that $n_0 = 1$ is the ground state. For $m \neq 0$, we obtain the following result: when $\delta > 2c[1 - \sqrt{1 - m^2}]$, the ground state will have three condensate components with

$$n_\pm = (x_0 \pm m)/2, \quad n_0 = 1 - x_0,$$  \hspace{1cm} (19)

where $x_0$ is the root of equation $g'_2(x) + \delta = 0$; when $\delta \leq 2c[1 - \sqrt{1 - m^2}]$, only $|+\rangle$ and $|-\rangle$ components exist, i.e. $n_\pm = (1 \pm m)/2$.

Figure 3 is the typical result for a spin-1 $^{23}\text{Na}$ condensate. The atomic parameters are $E_{\text{HFS}} = (2\pi)1.7716$ GHz [12], $a_0 = 50 a_B$ and $a_2 = 55 a_B$ [16]. At $B = 0$ there are only two condensate components, $|+\rangle$ and $|-\rangle$. For $B > 0$ but not very strong, there are two possibilities: three non-zero condensate components if $m < m_c$ and two non-zero condensate components if $m \geq m_c$, with $\delta(B) = 2c(1 - \sqrt{1 - m^2})$. When the $B$-field gets stronger, i.e. $\delta(B) \geq 2c$, there are always three condensate components.

Figure 4 summarizes the ground state structures of a homogeneous spin-1 condensate in a $B$-field for different $c$ and $m$.

4. Ground state inside a harmonic trap

In the previous section, we investigated in detail mean field ground state structures for a spin-1 condensate in a homogeneous confinement. For the case of a harmonic trap, as in most experiments, there is no reason to believe $\textit{a priori}$ that the above conclusions still hold. In fact, the structures and phase diagrams as discussed previously are only meaningful if the spatial mode function $\phi_j(\vec{r})$ for different spin components is identical. Otherwise, it would be impossible to classify the rich variety of possible solutions. When the spatial mode functions are the same,
Figure 3. The same as in figure 2, but now for a spin-1 $^{23}$Na condensate.

Figure 4. The ground state phase diagram for a homogeneous spin-1 condensate. Dashed curve and lines denote gradual transitions across the boundaries, solid lines denote discontinuous jumps. $x_0$ is the solution to the equation $g'_\pm(x) + \delta = 0$ and the curve for $c > 0$ is determined by $\delta(B) = 2c[1 - \sqrt{1 - m^2}]$. The open circle at $B = 0, m = 0$ for $c > 0$ denotes the family of degenerate ground state $(\frac{1-n_0}{2}, n_0, \frac{1-n_0}{2})$.

The spatial confinement simply introduces an average over the inhomogeneous density profile of the mode function.

The aim of this section is therefore to determine the validity of the single mode approximation (SMA) in the presence of an external $B$-field and a harmonic trap. For simplicity, we assume the trap to be spherically symmetric. We employ numerical methods to find the ground state solutions directly from the coupled GPE

$$i\hbar \frac{\partial}{\partial t} \Phi_+ = [\mathcal{H} + E_+ - \eta + c_2(n_+ + n_0 - n_-)] \Phi_+ + c_2 \Phi_0^* \Phi_-,$$

$$i\hbar \frac{\partial}{\partial t} \Phi_0 = [\mathcal{H} + E_0 + c_2(n_+ + n_-)] \Phi_0 + 2c_2 \Phi_0^* \Phi_+ \Phi_-,$$

$$i\hbar \frac{\partial}{\partial t} \Phi_- = [\mathcal{H} + E_- + \eta + c_2(n_- + n_0 - n_+)] \Phi_- + c_2 \Phi_0^2 \Phi_+,$$

subject to the conservations of both $N$ and $\mathcal{M}$ (equations (6)). $\mathcal{H} = -\hbar^2 \nabla^2 / 2M + V(\vec{r}) + c_0 n$. 

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Figure 5. Typical densities of spatial mode functions for each component of a $^{87}\text{Rb}$ (a) and a $^{23}\text{Na}$ (b) condensate. The solid curve denotes the $|+\rangle$ component, the dashed curve the $|-\rangle$ component, and the dash–dotted curve the $|0\rangle$ component. The parameters are $N = 10^6$, $\omega = 2\pi \times 100 \text{ Hz}$, $B = 1.0 \text{ G}$ and $m = 0.5$.

$n_j = |\Phi_j|^2$, $V_r(\vec{r}) = M\omega^2 r^2/2$ and $n = n_+ + n_0 + n_-$. $\eta$ is the Lagrange multiplier introduced numerically to enable the conservation of $\mathcal{M}$.

It was shown previously that in the absence of an external $B$-field, and for ferromagnetic interactions, the SMA is rigorously valid despite the presence of a harmonic trap [14]. We can also show that in the presence of a non-zero $B$-field, the linear Zeeman shift does not affect the validity of the SMA because it can be simply balanced by the external Lagrange multiplier $\eta$. The quadratic Zeeman effect, on the other hand, cannot be simply balanced, as it favours the production of two $|0\rangle$ atoms by annihilating one $|+\rangle$ and one $|-\rangle$ atom during a collision. Such unbalanced elastic collisions thus break the $SO(3)$ symmetry of the freedom for an arbitrary quantization axis. Therefore, we do not in general expect the SMA to remain valid inside a non-zero $B$-field.

Numerically, we find the ground state solutions of equation (20) by propagating the equations in imaginary time. We typically start with an initial wavefunction such as that of a complex Gaussian with a constant velocity: $\exp[-(x^2/2q_x^2 + y^2/2q_y^2 + z^2/2q_z^2) - i\vec{k} \cdot \vec{r}]$. $q_x$, $q_y$, $q_z$ and $\vec{k}$ are adjustable parameters which are checked to ensure that their choices do not affect the final converged ground state [14].

For $c_2 = 0$ or $c = 0$, it is easy to check that the SMA is always valid since the energy functional is symmetric with respect to the spin component index. The fractional populations for each component are therefore the same as for a homogeneous system, i.e. given by $((1 - n_0 + m)/2, n_0, (1 - n_0 - m)/2)$ if $B = 0$, and $(m, 1 - m, 0)$ if $B > 0$.

For $^{87}\text{Rb}$ and $^{23}\text{Na}$ condensates, which are believed to be ferromagnetic $c_2 < 0$ ($c < 0$) and antiferromagnetic $c_2 > 0$ ($c > 0$) respectively, figure 5 gives typical density distributions of spatial mode function, $\rho(\vec{r}) = |\phi_j(\vec{r})|^2$. Both the diagrams clearly indicate that the SMA is no longer valid. To get an overall idea of the validity of the SMA we plot in figure 6 the overlap integrals of our mode functions with respect to the SMA mode function $\phi_{\text{SMA}}(\vec{r})$ as determined from a scalar GPE with a non-linear coefficient proportional to $c_0$ (due to the symmetric $H_S$ only) [14]. For a $^{87}\text{Rb}$ condensate, we see that the overlap is close to unity when $B$ is small and therefore the SMA remains approximately applicable. But it becomes increasingly poor with increase of $B$. We thus conclude that the SMA remains reasonable in a weak magnetic field but it is clearly invalid in a strong $B$-field. In fact, our numerical results confirm that the stronger the $B$-field, the worse the SMA gets. For typical system parameters, the dividing line occurs...
Figure 6. The overlap between the SMA mode function and the mode function for the $|−\rangle$ component. Left diagram is for a $^{87}$Rb condensate. Right diagram is for a $^{23}$Na condensate. The atomic parameters are the same as in figure 5.

Figure 7. The same as in figure 6, but now comparing the spin asymmetric energy $E_a = c_2\langle \vec{F} \rangle^2/2 - (\eta_0 + \eta)\langle \hat{F}_z \rangle + \delta\langle \hat{F}_z^2 \rangle$ with the spin symmetric one $H_s$. at a $B$-field of a fraction of a gauss when the system magnetization $\mathcal{M}$ is not too small or too large. For a condensate with antiferromagnetic interactions, it was found earlier that the SMA is violated in the limit of both large $N$ and $\mathcal{M}$ even without an external $B$-field, while the case of $\mathcal{M} = 0$ presents an exception where the SMA remains strictly valid for $B = 0$ [14]. Figure 6 shows the overlap integral for a $^{23}$Na condensate; indeed we see that the SMA is invalid except at $\mathcal{M} = 0$ where all atoms are in the $|0\rangle$ component. Remarkably, despite the seemingly large deviations from the SMA (as in figure 6), the spin asymmetric energy term remains very small in comparison with the spin symmetric term as evidenced in figure 7.

Figure 8 shows the dependence of fractional populations on the fractional magnetization for a $^{87}$Rb (left column) and a $^{23}$Na condensate (right column) at different $B$-fields. For $^{87}$Rb atoms, these curves resemble the same dependence as for a homogeneous system where the SMA is strictly valid. Nevertheless, we find that the densities of mode functions can become quite different, i.e. the SMA is not valid in general. For $^{23}$Na atoms, the fractional component populations at different $B$-fields again follow the results as obtained previously for the homogeneous case. When $B = 0$ (as in figure 8(d)), the ground state distribution clearly obeys the same earlier (homogeneous) result $((1 + m)/2, 0, (1 - m)/2)$, including the special case when $\mathcal{M} = 0$ where it becomes $((1 - n_0)/2, n_0, (1 - n_0)/2)$ with $n_0 \in [0, 1]$. For $B \neq 0$ (as in figures 8(e) and (f)), our numerical solutions again reveal two distinct regions: one for $m < m_c$ where all three components coexist, and another for $m > m_c$ where only two components ($|+\rangle$ and $|−\rangle$) coexist. We find that $m_c$ increases with the $B$-field, and is of course limited to $m_c < 1$. We conclude that despite the fact that a harmonic trap induces a spatially
Figure 8. Fractional population for each spin component of a $^{87}$Rb (left column) and a $^{23}$Na (right column) condensate. The values of the $B$-field from the top row to the bottom are $B = 0, 0.1, 1.0$ G. The atomic parameters are the same as in figure 5. The solid lines with plus signs denote the $|+\rangle$ component, the lines with triangles are for the $|-\rangle$ component and the lines with open circles for the $|0\rangle$ component. The vertical dashed lines in (e) and (f) indicate the critical value $m_c$, the boundary between the two distinct regions discussed in the text. In (d), $m_c = 0$.

inhomogeneous distribution to condensate density, thus breaking the SMA in general, the overall ground state properties as measured by the fractional component distributions closely follow the results as obtained previously for the homogeneous case. Physically, we believe the above results can be understood as fractional populations related to integrals of wavefunctions over all spaces, during which differences between wavefunctions can be averaged out. When only the $|+\rangle$ and $|-\rangle$ components coexist, in fact, the two constraints on $N$ and $\mathcal{N}$ always give the fractional population $n_{\pm} = (1 \pm m)/2$ if $N_0 = 0$.

5. Conclusion

We have revisited the question of the mean field ground state structures of spin-1 condensate in the presence of a uniform magnetic field. For a homogeneous system, when $c = 0$, there exists in general only two non-zero components, $|+\rangle$ and $|0\rangle$, except when $B = 0$ where the ground state solution becomes indefinite; for ferromagnetic interactions when $c < 0$, the ground state in general has three non-zero components; when $c > 0$ as for antiferromagnetic interactions, except for $m = 0$, there are two regions: one for $\delta > 2c[1 - \sqrt{1 - m^2}]$ where three non-zero components coexist and one for $\delta \leq 2c[1 - \sqrt{1 - m^2}]$ where only two components coexist. Inside a harmonic trap, these results remain largely true, although the SMA becomes generally invalid. We find interestingly (see figure 9) that the $B$-field (or the $\delta$) dependence of the critical
Figure 9. The $B$-field dependence of the critical fractional magnetization $m_c$ as computed numerically for a $^{23}$Na condensate in a harmonic trap. The smooth curve corresponds to the result of $\delta = 2c[1 - \sqrt{1 - m_c^2}]$ (as from the homogeneous case) with an appropriately adjusted coefficient $c$ (or density). The atomic parameters are the same as in figure 5.

value $m_c$ separating the two and three component condensate regions remains almost identical to that given by the analytical formulae $\delta = 2c[1 - \sqrt{1 - m_c^2}]$ for a homogeneous system. In a sense, this also points to the validity of the use of a mean field description, as the number of atoms is really large ($10^6$).

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