Theory of orthogonality of eigenfunctions of the characteristic equations as a method of solution boundary problems for model kinetic equations

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Abstract

We consider two classes of linear kinetic equations: with constant collision frequency and constant mean free path of gas molecules (i.e., frequency of molecular collisions, proportional to the modulus molecular velocity). Based homogeneous Riemann boundary value problem with a coefficient equal to the ratio of the boundary values dispersion function, develops the theory of the half-space orthogonality of generalized singular eigenfunctions corresponding characteristic equations, which leads separation of variables.

And in this two boundary value problems of the kinetic theory (diffusion light component of a binary gas and Kramers problem about isothermal slip) shows the application of the theory orthogonality eigenfunctions for analytical solutions these tasks.

Key words: kinetic equation, collision frequency, boundary value problems, eigenfunctions, dispersion function, analytical solution.

PACS numbers: 05.60.-k Transport processes, 51.10.+y Kinetic and transport theory of gases,

1. Introduction

Construction of precise solutions of boundary value problems mathematical physics is a great success. This fully applies to the boundary value problems for kinetic equations.

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In 1960 K. Case in his work [1] for the first time proposed a method of analytical solutions of boundary value problems for the model equation neutron transport

\[ \mu \frac{\partial h}{\partial x} + h(x, \mu) = \frac{c}{2} \int_{-1}^{1} h(x, \mu') d\mu'. \quad (1.1) \]

The general method of Fourier’s separation of variables leads to the substitution

\[ h_\eta(x, \mu) = \exp \left( -\frac{x}{\eta} \right) \Phi(\eta, \mu). \quad (1.2) \]

Substituting (1.2) reduces equation (1.1) to the characteristic equation

\[ (\eta - \mu) \Phi(\eta, \mu) = \eta \frac{c}{2} \int_{-1}^{1} \Phi(\eta, \mu') d\mu'. \quad (1.3) \]

K. Case’s brilliant hunch was that it offered seek a solution of the characteristic equation (1.3) in space of generalized functions [2]

\[ \Phi(\eta, \mu) = \eta \frac{c}{\sqrt{\pi}} P \frac{1}{\eta - \mu} + \lambda(\eta) \delta(\eta - \mu), \quad (1.4) \]

where \( \lambda(z) \) is the dispersion function,

\[ \lambda(z) = 1 + \frac{z}{2} \int_{-1}^{1} \frac{d\tau}{\tau - z}, \]

\( P x^{-1} \) is the generalized function (principal value of the integral in the integration \( x^{-1} \)), \( \delta(x) \) is the Dirac delta function.

Properties of the eigenfunctions (1.4), expansion of the solutions of equations (1.1) and their generalizations in eigenfunctions were investigated in works [3]–[9].

One of the first boundary value problems for a model kinetic BGK equation (Bhatnagar, Gross, Krook), for which was exact solution is obtained, has been linearized problem of the Kramers isothermal slip. This problem was solved analytically in 1962 C. Cercignani [10].
After Cercignani’s work, there have been numerous attempts to solve analytically the Smoluchowski problem of the temperature jump and low evaporation. An overview of such attempts is presented in the works [11]–[13]. These attempts continued until the appearance of work [14], which developed an analytical method of solving boundary value problems for this class of kinetic equations, which can be reduced to the solution of vector integro-differential equations of the type transport equations.

C. Cercignani [10] reduced the solution of the isothermal slip problem to solving the following boundary value problem

\[
\mu \frac{\partial h}{\partial x} + h(x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} h(x, \mu') d\mu', \quad x > 0, -\infty < \mu < +\infty,
\]

\[
h(0, \mu) = 0, \quad \mu > 0,
\]

\[
h(x, \mu) = h_{as}(x, \mu) + o(1), \quad x \to +\infty.
\]

Here \( h_{as}(x, \mu) \) is the Chapman–Enskog asymptotic distribution function,

\[
h_{as}(x, \mu) = 2U_0 + 2G_v(x - \mu),
\]

\( U_0 \) is the unknown dimensionless slip velocity gas, subject to finding, \( G_v \) is the specified far from the wall dimensionless mass velocity gradient, \( \mu = C_x, \quad C = v/v_T, \quad v_T = 1/\sqrt{\beta} \) is the thermal velocity of the gas, \( \beta = m/(2kT), \quad m \) is the mass of gas molecule, \( k \) is the Boltzmann constant, \( T = \text{const} \) is the gas temperature.

In the problem of evaporation of the binary gas light component (see, e.g., [15]) investigated the one-parameter family of equations

\[
\mu \frac{\partial h}{\partial x} + h(x, \mu) = \frac{c}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} h(x, \mu') d\mu', \quad (1.5)
\]

where \( c \) is the numeric parameter, \( 0 < c < 1, \quad x > 0, -\infty < \mu < +\infty. \)

In works [16] and [17] in solving boundary value problems for a model kinetic equation with the collision frequency, proportional to the
modulus of molecular speed, consider the equation
\[
\mu \frac{\partial h}{\partial x} + h(x, \mu) = \frac{3}{4} \int_{-1}^{1} (1 - \mu'^2) h(x, \mu') d\mu', \quad x > 0, -1 < \mu < +1. \tag{1.6}
\]

In this paper we develop the theory of orthogonality eigenfunctions of the characteristic equations corresponding equations (1.5) and (1.6). Underlying this theory is the solution of the boundary Riemann problem \textsuperscript{[18]} from the theory of complex variable functions. This theory is then applied to the solution of boundary value problems for equations (1.5) and (1.6).

2. Eigenfunctions in the problem of the diffusion of the binary gas light component and their orthogonality

We consider the equation (1.5). The general method of Fourier’s separation of variables, as already mentioned, leads to the substitution
\[
h_\eta(x, \mu) = e^{-x/\eta} \Phi(\eta, \mu), \tag{2.1}
\]
where \(\eta\) is the complex-valued spectral parameter.

Substituting (2.1) in (1.5), immediately obtain the characteristic equation
\[
(\eta - \mu) \Phi(\eta, \mu) = \eta \frac{c}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} \Phi(\eta, \mu') d\mu'. \tag{2.2}
\]

We denote
\[
n(\eta) = \int_{-\infty}^{\infty} e^{-\mu'^2} \Phi(\eta, \mu') d\mu' \tag{2.3}
\]
and rewrite (2.2) in the form
\[
(\eta - \mu) \Phi(\eta, \mu) = \eta \frac{c}{\sqrt{\pi}} n(\eta). \tag{2.4}
\]

By the homogeneity of the equation (1.5) without loss of generality, we can assume that
\[
n(\eta) \equiv \int_{-\infty}^{\infty} e^{-\mu'^2} \Phi(\eta, \mu') d\mu' = 1. \tag{2.5}
\]
From equations (2.3) and (2.5) in the space of generalized functions we find the eigenfunctions corresponding to the continuous spectrum
\[ \Phi(\eta, \mu) = \eta c \sqrt{\pi} P_{1, \eta} - \mu + e^{\eta^2 \lambda_c(\eta)} \delta(\eta - \mu). \] (2.6)

Where \( \lambda_c(\eta) \) is the dispersion function,
\[ \lambda_c(\eta) = 1 + z c \sqrt{\pi} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{\tau - z}, \]

The basic theory of orthogonality we set scalar product with weight \( \rho(\mu) = e^{-\mu^2} \gamma(\mu) \), where
\[ \gamma(\mu) = \mu \frac{X^+(\mu)}{\lambda^+_c(\mu)}. \]

Here \( X(z) \) is the solution of the homogeneous Riemann boundary value problem from [15]
\[ \frac{X^+(\mu)}{X^-(\mu)} = \frac{\lambda^+_c(\mu)}{\lambda^-_c(\mu)}, \quad \mu > 0. \]

The solution of this problem (see [15]) defined by the equalities
\[ X(z) = \frac{1}{z} e^{V(z)}, \quad V(z) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\theta(\mu) - \pi}{\mu - z} d\mu, \]

\[ \theta(\mu) = \arg \lambda^+_c(\mu) = \arctg \frac{\text{Re} \lambda^+_c(\mu)}{\text{Im} \lambda^+_c(\mu)}, \quad \theta(0) = 0, \]

\[ \lambda^+_c(\mu) = \text{Re} \lambda^+_c(\mu) + i \text{Im} \lambda^+_c(\mu) = \lambda_c(\mu) + ic \sqrt{\pi \mu e^{-\mu^2}}, \]

\[ \lambda_c(\mu) = 1 - 2c^2 \mu^2 e^{-\mu^2} \int_{0}^{1} e^{\mu^2 \tau^2} d\tau. \]

Scalar product on the set of functions, that depend on the speed variable \( \mu \), we introduce by equality
\[ (f, g) = \int_{0}^{\infty} e^{-\mu^2 \gamma(\mu)} f(\mu) g(\mu) d\mu. \]
For convenience the eigenfunctions $\Phi(\eta, \mu)$ we denote by $\Phi_\eta(\mu)$.

**Theorem 1.** Scalar product number one and eigenfunction of the continuous spectrum is equal to the spectral parameter, i.e.

$$(1, \Phi_\eta) = \eta, \quad \eta > 0. \quad (2.8)$$

**Proof.** By the definition of scalar product, we have

$$(1, \Phi_\eta) = \int_0^\infty e^{-\tau^2} \gamma(\tau) \Phi_\eta(\tau) d\tau. \quad (2.8)$$

We represent this expression in explicit form

$$(1, \Phi_\eta) = \int_0^\infty e^{-\tau^2} \gamma(\tau) \left[ \frac{c\eta}{\sqrt{\pi}} P \frac{1}{\eta - \tau} + e^{\eta^2} \lambda_c(\eta) \delta(\eta - \tau) \right] d\tau =$$

$$= -\frac{c\eta}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\tau^2} \gamma(\tau) d\tau}{\tau - \eta} + \gamma(\eta) \lambda_c(\eta) \theta_+(\eta),$$

where $\theta_+(\eta)$ is the Heaviside step function.

Now we use the integral representation (see [15])

$$X(z) = 1 + \frac{c}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\tau^2} \gamma(\tau) d\tau}{\tau - z}. \quad (2.8)$$

Using this representation, we obtain

$$(1, \Phi_\eta) = -\eta X(\eta) + \eta + \eta \frac{X^+(\eta) \lambda_c^+(\eta) + \lambda_c^-(\eta)}{\lambda_c(\eta)} = \eta,$$

Q.E.D.

**Theorem 2.** Eigenfunctions $\Phi_\eta(\mu)$ form an orthogonal family and we have the equality

$$(\Phi_\eta, \Phi_{\eta'}) = N(\eta) \delta(\eta - \mu), \quad (2.7)$$

where

$$N(\eta) = e^{\eta^2} \gamma(\eta) \lambda_c^+(\eta) \lambda_c^-(\eta). \quad (2.8)$$
**Proof.** By the definition of scalar product, we have

\[
(\Phi_\eta, \Phi_\eta') = \int_0^\infty e^{-\tau^2} \gamma(\tau) \Phi_\eta(\tau) \Phi_\eta'(\tau) d\tau.
\]

We represent this expression in explicit form

\[
(\Phi_\eta, \Phi_\eta') = \int_0^\infty e^{-\tau^2} \gamma(\tau) \left[ \frac{c\eta}{\sqrt{\pi}} P \frac{1}{\eta - \tau} + e^{\eta^2} \lambda_c(\eta) \delta(\eta - \tau) \right] \times
\]

\[
\times \left[ \frac{c\eta'}{\sqrt{\pi}} P \frac{1}{\eta' - \tau} + e^{\eta'^2} \lambda_c(\eta') \delta(\eta' - \tau) \right] d\tau = J_1 + J_2 + J_3 + J_4.
\]

Here

\[
J_1 = e^{2\eta \eta'} \frac{\eta}{\pi} \int_0^\infty \frac{e^{-\tau^2} \gamma(\tau) d\tau}{(\eta - \tau)(\eta' - \tau)},
\]

\[
J_2 = \frac{c\eta}{\sqrt{\pi}} e^{\eta^2} \lambda_c(\eta') \int_0^\infty \frac{e^{-\tau^2} \gamma(\tau) \delta(\eta' - \tau)}{\eta - \tau} d\tau,
\]

\[
J_3 = \frac{c\eta'}{\sqrt{\pi}} e^{\eta'^2} \lambda_c(\eta) \int_0^\infty \frac{e^{-\tau^2} \gamma(\tau) \delta(\eta - \tau)}{\eta' - \tau} d\tau,
\]

\[
J_4 = e^{\eta^2 + \eta'^2} \lambda_c(\eta) \lambda_c(\eta') \int_0^\infty e^{-\tau^2} \gamma(\tau) \delta(\eta - \tau) \delta(\eta' - \tau) d\tau.
\]

Second, third and fourth integrals are easily calculated as convolution with the Dirac delta function

\[
J_2 = \frac{c\eta \lambda_c(\eta') \gamma(\eta')}{\sqrt{\pi}(\eta - \eta')},
\]

\[
J_3 = \frac{c\eta' \lambda_c(\eta) \gamma(\eta)}{\sqrt{\pi}(\eta' - \eta)},
\]

\[
J_4 = e^{\eta^2} \lambda_c^2(\eta) \gamma(\eta) \delta(\eta - \eta').
\]

Calculate the first integral. We use the expansion into elementary fractions

\[
\frac{1}{(\eta - \tau)(\eta' - \tau)} = \frac{1}{\eta - \eta'} \left( \frac{1}{\tau - \eta} - \frac{1}{\tau - \eta'} \right),
\]
and the Poincaré–Bertrand formula

\[
P \frac{1}{\eta - \mu} P \frac{1}{\eta' - \mu} = P \frac{1}{\eta - \eta'} \left( P \frac{1}{\eta' - \mu} - P \frac{1}{\eta - \mu} \right) + \pi^2 \delta(\eta - \mu) \delta(\eta' - \mu).
\]

As a result, we obtain

\[
J_1 = c^2 \frac{\eta \eta'}{\sqrt{\pi}} \left[ \frac{1}{\eta - \eta'} \int_0^\infty \frac{e^{-\tau^2} \gamma(\tau) d\tau}{\tau - \eta} + \frac{1}{\eta - \eta'} \int_0^\infty \frac{e^{-\tau^2} \gamma(\tau) d\tau}{\tau - \eta'} \right] + \pi^2 \int_0^\infty e^{-\tau^2} \gamma(\tau) \delta(\eta - \tau) \delta(\eta' - \tau) d\tau.
\]

Now we use the integral representation [15]

\[
X(z) = 1 + \frac{c}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\tau^2} \gamma(\tau) d\tau}{\tau - z}.
\]

With its help the integral \(J_1\) is equal to

\[
J_1 = c \frac{\eta \eta'}{\sqrt{\pi}} \left( \frac{X(\eta) - X(\eta')}{\eta - \eta'} + \pi c^2 \eta^2 e^{-\eta^2} \gamma(\eta) \delta(\eta - \eta') \right).
\]

We find the sum

\[
J_1 + J_2 = \frac{c}{\sqrt{\pi}} \frac{\eta \lambda_c(\eta') \gamma(\eta') - \eta' \lambda_c(\eta) \gamma(\eta)}{\eta - \eta'}.
\]

We use the definition of the function \(\gamma(\tau)\). Then we see that

\[
J_2 + J_3 = \frac{c \eta \eta'}{\sqrt{\pi} (\eta - \eta')} \left[ \lambda_c(\eta') \frac{X^+(\eta')}{\lambda_c^+(\eta')} - \lambda_c(\eta) \frac{X^+(\eta)}{\lambda_c^-(\eta)} \right].
\]

Value of the dispersion function at the cut in this equality we will replace half the sum of its boundary values, as well as we use homogeneous Riemann boundary value problem. The result is that

\[
J_2 + J_3 = \frac{c \eta \eta'}{\sqrt{\pi} (\eta - \eta')} \left[ \frac{X^+(\eta') + X^-(\eta')}{2} - \frac{X^+(\eta) + X^-(\eta)}{2} \right] = \]
\[ = \frac{c\eta'}{\sqrt{\pi(\eta - \eta')}}[X(\eta') - X(\eta)].\]

Adding expressions \( J_1, J_2 + J_3 \) and \( J_4 \), we see that
\[
(\Phi_\eta, \Phi_{\eta'}) = \left[ e^{\eta^2} \lambda_c^2(\eta) + c^2\pi\eta^2 e^{-\eta^2} \right] \gamma(\eta)\delta(\eta - \eta') =
\]
\[ = [\lambda_c(\eta) + i\sqrt{\pi}c\eta e^{-\eta^2}] [\lambda_c(\eta) - i\sqrt{\pi}c\eta e^{-\eta^2}] e^{\eta^2} \gamma(\eta)\delta(\eta - \eta') =
\]
\[ = \lambda_c^+(\eta)\lambda_c^-(\eta)e^{\eta^2}\gamma(\eta)\delta(\eta - \eta') = |\lambda_c^+(\eta)|^2 e^{\eta^2}\gamma(\eta)\delta(\eta - \eta') =
\]
\[ = N(\eta)\delta(\eta - \eta'), \]

Q.E.D.

We apply the theorem to solve the problem of the diffusion of the binary gas light component. In \[15\] shows that the solution of this problem reduces to the solution of the integral equation
\[
\frac{G_n}{1 - c} = \int_0^\infty \Phi(\eta', \mu)a(\eta')d\eta'. \tag{2.8}
\]

Here \( G_n = g_n l, g_n = \frac{d\ln n(y)}{dy}, l = v_\tau \tau \) is the mean free path of the gas molecules, \( \tau = 1/(\nu_1 + \nu_2), c = \frac{\nu_1}{\nu_1 + \nu_2}, \nu_1 \) and \( \nu_2 \) is the frequency of collisions between molecules of the first and the second gas component.

We multiply equation (2.8) at the expression \( e^{-\mu^2}e^{\eta^2}\gamma(\eta)\Phi(\eta, \mu) \) and integrate the resulting equation by \( \mu \). The result is that
\[
\frac{G_n}{1 - c} \int_0^\infty e^{-\mu^2}e^{\eta^2}\gamma(\mu)\Phi(\eta, \mu)d\mu =
\]
\[ = \int_0^\infty a(\eta')d\eta' \int_0^\infty e^{-\mu^2}e^{\eta^2}\gamma(\mu)\Phi(\eta, \mu)\Phi(\eta', \mu)d\mu,
\]

or, using the above notation and theorem,
\[
\frac{G_n}{1 - c} (1, \Phi_\eta) = \int_0^\infty a(\eta')N(\eta - \eta')\delta(\eta - \eta')d\eta'.
\]
Hence, by the theorem 1 we have
\[ a(\eta) = \frac{G_n}{1 - c N(\eta)} \frac{\eta}{1 - c N(\eta)} = \frac{G_n}{1 - c} \frac{\eta}{e^{\eta^2} \gamma(\eta) \lambda_c^+(\eta) \lambda_c^-(\eta)} = \]
\[ = \frac{G_n}{1 - c} \frac{e^{-\eta^2}}{X^+(\eta) \lambda_c^-(\eta)}. \]
which coincides exactly with the result of \[15\].

We introduce another scalar product on the set of their eigenfunctions with the integration of the spectral parameter and weight \( r(\eta) = 1/N(\eta) \)
\[ \langle f, g \rangle = \int_0^\infty \frac{1}{N(\eta)} f(\eta) g(\eta) d\eta. \]

Similarly can we prove

**THEOREM 3.** Eigenfunction of the characteristic equation corresponding to the continuous spectrum, orthogonal and have the relation
\[ \langle \Phi_\eta(\mu) \Phi_\eta(\mu') \rangle = \frac{1}{\rho(\mu)} \delta(\mu - \mu'). \] (2.10)

We represent in explicit form the equality (2.10):
\[ \langle \Phi_\eta(\mu) \Phi_\eta(\mu') \rangle \equiv \int_0^\infty \frac{e^{-\eta^2}}{\lambda_c^+(\eta) \lambda_c^-(\eta) \gamma(\eta)} \Phi_\eta(\mu) \Phi_\eta(\mu') d\eta = \]
\[ = \frac{e^{\mu^2}}{\gamma(\mu)} \delta(\mu - \mu'). \]

From theorems 2 and 3 shows that in the transition to orthogonality spectral parameter swapped weight and normalization integral.

**3. Kinetic equation with collision frequency proportional to absolute velocity of the molecules**

We now consider the equation (1.6). Substitution (2.1) reduces this equation to the characteristic
\[ (\eta - \mu) \Phi(\eta, \mu) = \frac{3}{4} \eta \] (3.1)
with single normalization
\[ n(\eta) \equiv \int_{-1}^{1} (1 - \mu'^2) \Phi(\eta, \mu') d\mu' \equiv 1. \] (3.2)

From the equations (3.1) and (3.2) find the eigenfunctions of the characteristic equation
\[ \Phi(\eta, \mu) = \frac{3}{4} \eta P \frac{1}{\eta - \mu} + \frac{\lambda(\eta)}{1 - \eta^2} \delta(\eta - \mu), \] (3.3)
where \( \lambda(z) \) is the dispersion function,
\[
\lambda(z) = 1 + \frac{3}{4} z \int_{-1}^{1} \frac{1 - \tau^2}{\tau - z} d\tau = \frac{3}{4} \int_{-1}^{1} \frac{\tau(1 - \tau^2)}{\tau - z} d\tau =
\]
\[
= -\frac{1}{2} + \frac{3}{2} (1 - z^2) \lambda_0(z),
\]
\[
\lambda_0(z) = 1 + \frac{z}{2} \int_{-1}^{1} \frac{d\tau}{\tau - z} = 1 + \frac{z}{2} \ln \frac{1 - z}{1 + z}.
\]

Discrete spectrum of the characteristic equation, as shown in [11, 15], consists of one point \( \eta_i = \infty \) multiplicity two. This point corresponds to the eigenfunction \( \Phi_\infty = 1 \), corresponding normalization \( n(\eta) = \frac{4}{3} \).

Homogeneous Riemann boundary value problem
\[
\frac{X^+(\mu)}{X^-(\mu)} = \frac{\lambda^+(\mu)}{\lambda^-(\mu)}, \quad 0 < \mu < 1,
\]
as shown in [11, 15], has a solution
\[ X(z) = \frac{1}{z} e^{V(z)}, \] (3.4)
where
\[ V(z) = \frac{1}{\pi} \int_{0}^{1} \frac{\theta(\mu) - \pi}{\mu - z}. \]
Here \( \theta(\mu) = \arg \lambda^+(\mu) \), or
\[
\theta(\mu) = \arctg \frac{4\lambda(\mu)}{3\pi \mu (1 - \mu^2)}.
\]

Introduce the scalar product
\[
(f, g) = \int_0^1 (1 - \mu^2) \gamma(\mu) f(\mu) g(\mu) d\mu,
\]
wherein
\[
\gamma(\mu) = \frac{\mu X^+(\mu)}{\lambda^+(\mu)}.
\]

Exactly the same way as theorem 1, we prove the following theorem.

**THEOREM 4.** The following relations hold
\[
(\Phi_\infty, \Phi_\infty) = -\frac{4}{3},
\]
and
\[
(\mu, \Phi_\infty) = -\frac{4}{3} V_1,
\]
where
\[
V_1 = -\frac{1}{\pi} \int_0^1 \theta(\mu) - \pi] d\mu \approx 0.581946 \cdots,
\]

and
\[
(\mu, \Phi_\eta) = \eta, \quad \eta > 0.
\]

**PROOF.** We prove the equalities (3.5) and (3.6). We use the integral representation [15]
\[
X(z) = \frac{3}{4} \int_0^1 \frac{(1 - \tau^2) \gamma(\tau)}{\tau - z} d\tau.
\]

We expand the function \( X(z) \) in a neighborhood of infinity. We use the equations (3.8) and (3.4). As result we have
\[
X(z) = -\frac{1}{z} \cdot \frac{3}{4} \int_0^1 (1 - \mu^2) \gamma(\mu) d\mu -
\]
\[-\frac{1}{z^2} \cdot \frac{3}{4} \int_0^1 \mu(1 - \mu^2)\gamma(\mu)d\mu - \cdots, \quad z \to \infty, \quad (3.9)\]

and
\[X(z) = \frac{1}{z} + \frac{V_1}{z^2} + \cdots, \quad z \to \infty. \quad (3.10)\]

From a comparison of the coefficients of series (3.9) and (3.10) implies the equalities
\[\frac{3}{4} \int_0^1 (1 - \mu^2)\gamma(\mu)d\mu = -1\]

and
\[\frac{3}{4} \int_0^1 \mu(1 - \mu^2)\gamma(\mu)d\mu = -V_1,\]

which proves the equalities (3.5) and (3.6).

The other equalities are proved similarly to theorem 1.

**Theorem 1.** Eigenfunctions of the continuous spectrum are orthogonal to each other and have the following orthogonality relations
\[(\Phi_\infty, \Phi_\eta) = 0, \quad (3.11)\]
\[(\Phi_\eta, \Phi_\eta') = N(\eta)\delta(\eta - \eta'), \quad (3.12)\]

where
\[N(\eta) = \gamma(\eta)\frac{\lambda^+(\eta)\lambda^-(\eta)}{1 - \eta^2}.\]

Theorem 5 is proved similarly to theorem 2.

We apply the developed theory to the solution of the Kramers problem. In [11, 15] shown that the solution of the Kramers problem reduces to the solution of the integral equation
\[2U_0 - 2Gv\mu + \int_0^\infty \Phi(\eta', \mu)a(\eta')d\eta'. \quad (3.13)\]

Where \(U_0\) is the unknown dimensionless sliding speed, and \(G_v\) is the specified far from the wall dimensionless mass velocity gradient.
To find the sliding speed multiply equation (3.13) to $\rho(\mu) = (1 - \mu^2)\gamma(\mu)$ and integrate by $\mu$ from 0 to 1. As a result, we obtain the equation

$$2U_0(1, 1) - 2G_v(1, \mu) + \int_0^\infty a(\eta')(1, \Phi_{\eta'})d\eta' = 0. \quad (3.14)$$

According to theorem 5 $(1, \Phi_{\eta'}) = 0$. Therefore from the equation (3.14) in view of theorem 5 we derive known of [11, 15] result

$$U_0 = \frac{(1, \mu)}{(1, 1)}G_v = V_1G_v.$$

To find the coefficient of the continuous spectrum $a(\eta)$ multiply (3.14) by $(1 - \mu^2)\gamma(\mu)\Phi(\eta', \mu)$ and integrate by $\mu$ from 0 to 1. As a result, we obtain the equation

$$2U_0(1, \Phi_{\eta'}) - 2G_v(\mu, \Phi_{\eta'}) + \int_0^\infty a(\eta')(\Phi_{\eta}, \Phi_{\eta'})d\eta' = 0. \quad (3.15)$$

According to theorem 5

$$(1, \Phi_{\eta'}) = 0, \quad (\mu, \Phi_{\eta'}) = \eta, \quad (\Phi_{\eta}, \Phi_{\eta'}) = N(\eta)\delta(\eta - \eta').$$

Therefore from the equation (3.15) that

$$a(\eta) = \frac{\eta}{N(\eta)}(2G_v) = \frac{1 - \eta^2}{X^+(\eta)\lambda^-(\eta)}(2G_v),$$

that exactly coincides with the known result of [11, 15, 16].

4. Conclusion

In this paper we develop a theory of of the orthogonality eigenfunctions of the characteristic equations corresponding to two kinetic equations.
This theory is developed on the positive real axis (and in the range of $0 < \eta < 1$) using the Riemann boundary value problem [18] with a coefficient equal to the ratio of the boundary values of the dispersion function on the cut. Orthogonality applied to solving boundary value problems for the equations considered.

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