Another Conjecture about M(atrix) Theory

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The current understanding of M(atrix) theory is that in the large N limit certain supersymmetric Yang Mills theories become equivalent to M-theory in the infinite momentum frame. In this paper the conjecture is put forward that the equivalence between M and M(atrix) theory is not limited to the large N limit but is valid for finite N. It is argued that a light cone description of M-theory exists in which one of the light like coordinates is periodically identified. In the light cone literature this is called Discrete Light Cone Quantization (DLCQ). In this framework an exact light cone description exists for each quantized value N of longitudinal momentum. The new conjecture states that the sector of the DLCQ of M-theory is exactly described by a U(N) matrix theory. Evidence is presented for the conjecture.

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1. Introduction

According to our present understanding of M(atrix) theory, the connection between supersymmetric matrix field theories and M-theory is only precise in the large $N$ limit where it becomes the infinite momentum limit of M-theory. No exact significance is given to the matrix model for finite $N$. In this paper arguments will be given for an exact connection even at finite $N$.

In [1] the tool that was used to relate M and M(atrix) theory was the Infinite Momentum Limit [2]. The theory is first compactified in a spacelike direction $X^{11}$ with compactification radius $R$. The momentum $p_{11}$ is quantized in units of $1/R$. Thus an integer $N = p_{11}R$ is defined. It is then argued that in the $N \to \infty$ objects with vanishing and negative $p_{11}$ decouple. Since the only objects in IIA string theory which carry $p_{11}$ are the D0-branes, M-theory in the Infinite Momentum Limit must be the theory of $N$ D0-branes in the limit of large $N$.

The Infinite momentum method and the method of light cone quantization are usually considered to be the same thing. However there is a subtle difference especially when the longitudinal direction is compactified. In the method of Discrete Light Cone Quantization (DLCQ) [3] the coordinate which is compactified is not the space-like $X^{11}$ but rather the lightlike coordinate $X^-$. In this case the discrete momentum is $p_-$ which is quantized as $p_- = N/R$. In the limit $N \to \infty$ the IML and DLCQ are expected to become identical. For finite $N$ the two are different. For example when $N$ is finite, negative and vanishing $p_{11}$ does not decouple, nor does the system have Galilean invariance. By contrast in DLCQ the Galilean invariance, decoupling of negative $p_-$ and the simplicity of the vacuum is exact for all $N$. In this paper the proposal is put forward that the DLCQ of M-theory is exactly described by $U(N)$ super Yang Mills theory.

The evidence for the conjecture is of two kinds. Both involve the transversely compactified versions of the theory. The first piece of evidence involves the compactification on a circle in the limit of vanishing radius. In this limit M-theory becomes free type IIA string theory. As we shall see this theory admits DLCQ and its spectrum is easily worked out. Furthermore, following the work of Motl [4], Banks and Seiberg [5] and Dijkgraaf, Verlinde and Verlinde [6] we find that the spectrum agrees with that of super Yang Mills theory in the appropriate limit. Furthermore there is strong evidence from [6] that the agreement is
not limited to the free string limit.

The second kind of evidence has to do with the constraints imposed on the super Yang Mills theory by the U-duality of M-theory. In the original proposal of [1] there was no particular reason why these conditions should be satisfied other than at infinite $N$. In the present interpretation the dualities should be correct for all $N$. That meshes very well with the fact that super Yang Mills theory has the required dualities for finite $N$.

The plan of the paper is as follows. In section 2 a brief explanation of the difference between the infinite momentum frame and the light cone frame is given.

In section 3 free string theory is formulated in the light cone frame with the lightlike coordinate $X^-$ compactified on a circle of radius $R$. As usual, periodic identification of a coordinate entails two steps. The first step is to quantize the conjugate momentum $p_-$ in units of $1/R$. The second step is to introduce twisted sectors describing strings which are wound around the compact direction. The theory consists of sectors in which the total momentum $p_-$ is set equal to $N/R$. The spectrum is easily computed.

M(atrix) theory, in the limit in which a transverse coordinate is compactified to zero size is expected to yield type IIA string theory. The transversely compactified theory is described by strongly coupled 1+1 dimensional super Yang Mills theory with gauge group $U(N)$. Using the method of [4] [5] [6] we find that the spectrum of this field theory exactly matches the spectrum of the string theory at momentum $p_- = N/R$. It is also possible to see that perturbations away from the free string theory are in agreement with the strong coupling expansion of the super Yang Mills theory.

In section 4 we review the constraints on super Yang Mills theory implied by the dualities of M-theory. From the point of view of [1] these constraints are required to be true only in the limit $N \to \infty$. What we find is that they are true for all $N$. This fact now finds a natural interpretation in the context of DLCQ.
2. Light Cone Versus Infinite Momentum

The infinite momentum description given in [1] begins by considering a system of particles in a conventional reference frame. The energy of a collection of free particles is

\[ E = \sum_a \sqrt{p_a^2 + m_a^2} \]  

(2.1)

where \( p_a \) is the 3-momentum of the \( a \)th particle and \( m_a \) is its mass. The system is now boosted along the spatial “longitudinal” direction \( X^{11} \) until the longitudinal momentum of every particle is positive and much larger than any other mass scale in the problem. We will call such a state a “proper” state. In this limit the energy takes the form

\[ E = p_{11}^{\text{total}} + \sum_a \frac{P^2 + m_a^2}{2p_{11}} \]  

(2.2)

where \( p_{11}^{\text{total}} \) is the total longitudinal momentum and \( P \) stands for the transverse spatial momentum of a particle. Note that the typical energy difference between two states of the same total longitudinal momentum goes to zero like \( 1/p_{11} \). The leading term in \( E \) can be dropped for most purposes since it does not influence energy differences.

Let us consider a state with total longitudinal momentum \( p_{11}^{\text{total}} \) but which happens to contain a particle with negative \( p_{11} \) of the same order as \( p_{11}^{\text{total}} \). Such an “improper” state differs in energy from proper states by a large amount of order \( p_{11} \). For this reason as \( p_{11}^{\text{total}} \to \infty \), improper states with backward moving quanta decouple in the IMF.

Some features of quantization in the IMF make it especially attractive [2].

1. Physics in the IMF has a galilean invariance which makes its structure similar to that of nonrelativistic quantum mechanics. Thus the concepts of wave function, bound state and mass conservation have their naive nonrelativistic meaning. Eq (2.2) is an example of this.

2. All objects carrying negative \( p_{11} \) decouple for the reason just stated.

3. In the limit of infinite momentum the complex structure of the vacuum decouples from all proper systems. This means that for all practical purposes, the vacuum is trivial as it is in nonrelativistic quantum mechanics. None of the these simplifications take place as long as \( p_{11} \) is not essentially infinite.
In [1] the longitudinal spacelike coordinate $X^{11}$ was assumed to be compactified on a large circle of radius $R$ which is allowed to become infinite eventually. However for finite $R$ the effect of the compactification is to quantize the momentum $p_{11}$ in integer multiples of $1/R$. Thus a quantum number $N$ is defined

$$N = p_{11}(\text{total})R$$

(2.3)

Similarly the momentum of any constituent subsystem $a$ satisfies

$$N(a) = p_{11}(a)R$$

(2.4)

All of the simplifications described above still apply but only in the limit $N \to \infty$.

Now let us consider a different approach which is usually called light cone quantization. To make the discussion concrete, we will work out the light cone quantization of $\Phi^3$ theory. The action is given by

$$I = \int d^D x \left[ \frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{m^2 \Phi^2}{2} - \lambda \Phi^3 \right]$$

(2.5)

Let us introduce the lightlike coordinates $x^+, x^-$ and the transverse coordinates $X^i$. Light cone quantization is defined by choosing the direction $x^+$ to play the role of time. In order to indicate this choice we will relabel the coordinate $x^+$ and call it $\tau$. The action takes the form

$$I = \int d\tau dx^- dx^i \left[ 2 \partial_+ \Phi \partial_- \Phi - \frac{1}{2} \left( \partial_+ \Phi \partial_+ \Phi + m^2 \Phi^2 \right) - \lambda \Phi^3 \right]$$

(2.6)

The canonical momentum conjugate to $\Phi$ is given by $2 \partial_- \Phi$ and the canonical commutation relations are

$$[\Phi(X, x^-), \partial_- \Phi(X', x'^-) ] = \frac{i}{2} \delta(X - X') \delta(x^- - x'^-)$$

(2.7)

Eq (2.7) is easily solved by writing $\Phi$ in terms of creation and annihilation operators $a^+(p_-, X), a^-(p_-, X)$. The operator $a^+(p_-, X)$ creates a particle with longitudinal light-like momentum $p_-$ at transverse location $X$. The allowable values of $p_-$ are strictly positive.

$$\Phi(X, x_-) \sim \int_0^\infty \frac{dp_-}{\sqrt{p_-}} \left[ a^+(p_-, X) e^{ip_-x^-} + a^-(p_-, X) e^{-ip_-x^-} \right]$$

(2.8)

Note that there are no creation or annihilation operators for particles of negative $p_-$. The Fock space is composed of particles of positive longitudinal momentum from the outset.
The Hamiltonian is easily worked out

\[
H = \int_0^\infty dp_- \int dX \left( \frac{P^2 + m^2}{2p_-} \right) a^+ (p_-, X) a^-(p_-, X) + \text{interactions}
\] (2.9)

The interactions conserve longitudinal momentum and are transversely local. Since there are no creation operators for negative \( p_- \) it follows that the naive Fock space vacuum is the ground state.

Now it is very easy to define DLCQ for this system. It consists of compactifying and periodically identifying the coordinate a circle of radius \( R \). The effect is to replace every \( p_- \) by a discrete variable \( N/R \) with \( N \) being integer valued. In order to eventually pass to the uncompactified limit we allow \( N \) to go to infinity.

To illustrate the procedure we will work out the simplest case of 1+1 dimensional real scalar field theory. First consider a free field of mass \( m \). The Lagrangian is given by

\[
L = \int_0^{2\pi R} dx^- \left( 2\partial_\tau \Phi \partial_- \Phi - \frac{m^2}{2} \Phi^2 \right)
\] (2.10)

The field is assumed periodic in \( x^- \) with period \( R \) so that it can be expanded in a Fourier series.

\[
\Phi = \Phi_0 + \sum_0^{+\infty} \Phi_n e^{inx^-} + \sum_0^{+\infty} \Phi^*_n e^{-inx^-}
\] (2.11)

where \( \Phi_0 \) is the zero momentum mode of \( \Phi \). Inserting eq (2.11) into (2.10) one finds the commutation relations

\[
[\Phi_n, \Phi^*_m] = \frac{1}{n} \delta_{nm} \quad (n \neq 0)
\] (2.12)

In other words the Fock space is composed of quanta with positive discrete longitudinal momenta. In addition there is a zero mode \( \Phi_0 \) whose time derivative does not enter the action. When (nonderivative) interactions are added the zero mode remains nondynamical and may be integrated out. Typical interaction terms such as \( V(\Phi) \) induce transitions between different number of quanta but always in a way that conserves the integer valued momentum. The total momentum defines superselection sectors characterized by an integer \( N \).
In sectors of low $N$ the dynamics is extremely simple. For example the Fock space vacuum can not make a transition to any other state. This follows from the positivity of the momentum spectrum of the quanta. For the case $N = 1$ the dynamics is equally trivial. The only state with $N = 1$ is the state with a single quantum so it too can not mix with any other state. For $N = 2$ there are two states, the state with a single quantum of 2 units of momentum and the state with two quanta, each with 1 momentum unit. The only allowed processes are transitions from 1 to 2 and back. As $N$ increases the number of states and variety of processes increases and at $N = \infty$ the usual full set of light cone processes can occur. However, the DLCQ setup enjoys all of the advantages of galilean invariance, positivity of longitudinal momentum and simplicity of vacuum structure not only as $N \to \infty$ but for every finite $N$! Of course for finite $N$ the breaking of the full Lorentz invariance by the boundary conditions is felt.

3. DLCQ of String Theory

The principles of compactification of string theory are familiar for the usual situation in which a spacelike coordinate is periodically identified. In this section we will be interested in string theory with periodic identification of the lightlike direction $x^-$. In other words we want to study the sum over world sheets of arbitrary topology embedded in such a periodically identified geometry. It is natural to carry this out in a coordinate system in which $x^-$ is formally treated as a spatial coordinate and $x^+$ is treated as time. The procedure of string quantization in such a light cone frame are well known. The new features required by compactification of $x^-$ are straightforward.

Compactification by periodic identification generally involves two modifications. The first is to eliminate all states which are not invariant under translations by $2\pi R$. In the present case this is simply accomplished by retaining only the subspace of the string Fock space which is composed of strings carrying lightlike momentum $p_- = N/R$. In terms of the usual light cone description of first quantized strings, this means that the usual $\sigma$ coordinate of the world sheet must have length equal to $N/R$. That is, the length of the $\sigma$ axis of any string is quantized. Thus a string of length $2/R$ can split into two strings, each of length $1/R$. The resulting strings can not further split but can rejoin. As in field theory the set of perturbative processes is severely limited in each $N$ sector. Perturbation theory for $N = 2$
simply consists of repeated splitting and joining transitions between the 1-string and 2-string sectors.

The second and more interesting step is to introduce the twisted sectors corresponding to strings wound around the periodically identified coordinate. In the present case this means strings satisfying
\[
\int_{0}^{2\pi R} d\sigma \frac{\partial x^{-}(\sigma)}{\partial \sigma} = 2\pi \nu R \quad (3.1)
\]
where \(\nu\) is an integer representing the winding number of the string around \(x^{-}\). What makes this condition different than compactification of a transverse direction \(X^{i}\) is that in light cone quantization, the coordinate \(x^{-}\) is not an independent variable. In fact the constraints of the theory in light cone gauge relate \(\frac{\partial x^{-}(\sigma)}{\partial \sigma}\) to derivatives of transverse coordinates.

\[
\frac{\partial x^{-}}{\partial \sigma} = \frac{\partial X^{i}}{\partial \sigma} \frac{\partial X^{i}}{\partial \tau} + \text{fermionic terms} \quad (3.2)
\]

Integrating eq (3.2) over \(\sigma\) and using (3.1) gives
\[
2\pi \nu R = \frac{1}{p_{-}} (N_{L} - N_{R}) = \frac{2\pi R}{N} (N_{L} - N_{R}) \quad (3.3)
\]
where \(N_{L}, N_{R}\) are the usual oscillator level numbers of the string. In other words the usual condition \(N_{L} - N_{R} = 0\) is replaced by
\[
N_{L} - N_{R} = \nu N \quad (3.4)
\]

In the limit \(N \to \infty\) the strings wound around \(x^{-}\) can easily be seen to decouple. For example consider the case \(\nu = 1\). Eq. (3.3) can be written as
\[
(N_{L} - N_{R}) = 2\pi R p_{-} \quad (3.5)
\]
Furthermore, since \(N_{L}\) and \(N_{R}\) are strictly positive, \(N_{L} + N_{R}\) must be at least of order \(p_{-}R\). Since the light cone energy of a string with \(P_{i} = 0\) is
\[
H = \frac{N_{L} + N_{R}}{2\alpha' p_{-}} \quad (3.6)
\]
we see that the energy is of order \(\left(\frac{R}{\alpha'}\right)\). On the other hand, the energy of an unwound string is of order \(\frac{1}{\alpha' p_{-}} = \frac{R}{\alpha' N}\). Thus in the large \(N\) limit the energy of wound strings diverges relative
to the energy of strings with vanishing winding number. For finite $N$ and nonvanishing string coupling, the wound strings can be produced in intermediate states even if the total winding number vanishes.

If one of the transverse coordinates $X^1$ is compactified eq. (3.4) must be modified to include the effects of Kaluza Klein and winding charges for this direction. If $n$ and $m$ are the integer valued momentum and winding quantum numbers, eq. (3.4) is replaced by

$$N_L - N_R = \nu N + nm$$

Interactions can be added to the DLCQ of string theory. As shown by Green and Seiberg the form of the interactions is at least in part determined by supersymmetry and finiteness of the perturbation series [7]. For example the trilinear vertex for string splitting and joining has the same form as in ordinary light cone string theory except that the strings are restricted to have quantized longitudinal momentum. Nothing in the DLCQ setup breaks either the Galilean symmetry or the spacetime supersymmetry of light cone string theory. Of course not all symmetries of the large $N$ limit will be manifested at finite $N$. In particular the symmetries connected with the Lorentz transformations which rotate the longitudinal direction into transverse directions will be broken by compactifying $x^-$. 

One question which will be very important in what follows is the role of string theory dualities for the finite $N$ version of the theory. Let us begin by considering T-duality which interchanges transverse winding and momentum modes. It should be stressed that we are not considering dualities which act on the longitudinal winding and momentum quantum numbers. From eq. (3.7) we see that the spectrum of free strings is invariant under interchange of $n$ and $m$. Thus the free limit is manifestly T-dual at finite $N$. Interactions at the tree level do not change this. For example, in the scattering of ordinary strings T-duality is satisfied. To see this note that the tree level amplitudes at finite $N$ are identical to the corresponding amplitudes in conventional string theory except for the fact that the external momenta are quantized. A more complete analysis including loop diagrams will be published elsewhere, hopefully by someone else. One final point about T-duality is that although it has only been proved in string perturbation theory it is generally assumed to be an exact nonperturbative symmetry of string theory.
It can also be argued that S-duality should hold in the finite $N$ sectors of the theory. Suppose for example we are studying type IIB string theory. This theory contains both “fundamental” strings (F-strings) and D-strings which are related by S-duality. We can imagine building a DLCQ based on extrapolation of weakly interacting F-string perturbation theory or the equally valid extrapolation of D-string perturbation theory. If the results are not identical there would be two entirely different versions of DLCQ for string theory. Thus the assumption that there exists a unique quantization of string theory in longitudinal compactified spacetime requires S-duality to operate at the level of finite $N$.

4. DLCQ for M(atrix) Theory

In this section a stronger form of the conjecture of [1] will be put forward. The conjecture is this [16].

1. Discrete light cone quantization of M-theory exists with the following properties:
   a) The theory exists for every finite value of $N$. Here $N$ is to be interpreted as the total longitudinal momentum $p_-$ in units of $1/R$.
   b) Galilean invariance and supersymmetry is manifest for all $N$
   c) When the theory is transversely compactified S and T-dualities are satisfied for all $N$.
   d) In the limit $N \to \infty$ the theory tends to the usual M(atrix) description.

2. The DLCQ of M-theory is just the finite $N$ version of the super Yang Mills theory used to describe M(atrix) theory in the large $N$ limit. For example, in the case of no compact transverse directions the description is the matrix quantum mechanics of Kabat and Pouliot [8] and Danielsson, Ferretti and Sundborg [9], describing $N$ D0-branes. If $K$ directions are compactified on a K-torus the description is $K + 1$ dimensional super Yang Mills theory with gauge group $U(N)$.

The evidence for the conjecture is both perturbative and nonperturbative. Let us begin with the perturbative story. Recently a number of papers have demonstrated that weakly coupled type IIA string theory can be derived [4] [5] [6] from the standard method of compactifying M(atrix) theory [1] [10] [11]. Let us review the arguments. The starting point is M(atrix) theory with one transverse dimension $X^9$ compactified on a circle of radius $L$.  

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(We reserve the symbol \( R \) for the longitudinal compactification radius). According to [10] the matrix quantum mechanics is replaced by the 1+1 dimensional super Yang Mills theory which is obtained by dimensional reduction of 9+1 dimensional super Yang Mills theory. The details of the construction have appeared in several papers and will not be repeated here. We will follow the notations and arguments of ref [6] in the following. The super Yang Mills theory has the field content of 8 scalars \( X \), a vector potential and fermionic superpartners. The field theory is defined on a periodic spatial coordinate \( \sigma \) which varies from 0 to 2\( \pi \). The time coordinate is called \( \tau \). The action is

\[
S = \frac{1}{2\pi} \int tr \left( (D_\mu X^i)^2 + g_s^2 F_{\mu\nu}^2 - \frac{1}{g_s^2} [X^i, X^j]^2 + \text{fermionic terms} \right)
\]

The string coupling \( g_s \) is given in terms of the compactification radius \( L \) by \( g_s = L^{3/2} \). Note that the Yang Mills coupling constant is inversely proportional to the string coupling \( g_s \). Thus the Yang Mills theory tends to infinite coupling as \( L \to 0 \).

In the limit of infinite coupling, the super Yang Mills theory is described by a superconformal fixed point theory [6] [13]. It is evident from eq (4.1) that the \( X \)'s must all commute in this limit. They can therefore be described in terms of their \( 8N \) eigenvalues \( X_1^1(\sigma), X_2^1(\sigma), \ldots, X_N^1(\sigma) \). The superconformal fixed point theory is identified in [6] as a free field theory of the \( X \)'s except that the permutation group acting on the \( N \) eigenvalues must be modded out. The only effect of this is to modify the periodic boundary conditions on the \( X \)'s. The fields \( X \) must be periodic up to a permutation. The result is that the theory has disconnected sectors, each describing a set of noninteracting “slinkys” [5] of length \( 2\pi N_1, 2\pi N_2, \ldots, 2\pi N_n \) where \( N_1 + N_2 + \ldots = N \). Each slinky behaves like a IIA string and has exactly the spectrum of the previous section. In particular the individual strings are not restricted to have \( N_L = N_R \). The quantization condition on \( N_L - N_R \) is obtained as follows: The field theoretic (world sheet) momentum is quantized in integers as a result of the fact that \( \sigma \) varies from 0 to 2\( \pi \). However, the effective length of the slinky (\( a \)) is \( 2\pi N_a \). The effective quantum of world sheet momentum for that slinky is \( 1/N_a \). This kind of fractionation of quantum numbers is known from black hole physics [14]. If the true field theoretic momentum is \( \nu \), the slinky carries effective momentum \( N_a \nu \) in units of \( 1/N_a \). Hence

\[
N_L - N_R = N_a \nu \quad (4.2)
\]

This is identical to eq (3.7). Thus we see that the spectrum of the strongly coupled super
Yang Mills theory with gauge group $U(N)$ coincides with the DLCQ spectrum of free string theory.

For the interacting theory, it is shown in [6] that the leading corrections to free string theory in powers of $g_s$ match the corrections to the superconformal fixed point theory. The leading dimensional operator causes string splitting and joining in precise correspondence with the type IIA string vertex. Furthermore nothing in this argument uses the large $N$ limit. Higher order corrections (string genus expansion) have not been calculated either for DLCQ string theory or for super Yang Mills theory but it seems reasonable that the finiteness and unitarity of each theory will force them to be perturbatively equivalent.

Next we come to the nonperturbative theory. On the string theory side there is no nonperturbative theory but there are many things which are believed to be true. The most important of these are the dualities such as S and T duality. T duality is of course provable in string perturbation theory but the real issue is whether it survives nonperturbative corrections. The recent work of the last two years on duality assumes S and T-dualities are exact. In fact the success in unifying the different string theories strongly suggests that the definition of the nonperturbative theory should include the constraints of duality.

In the previous section it was explained that the dualities should also be present in the finite $N$ version of DLCQ of string theory. Thus we may assume that equivalence between finite $N$ M(atrix) theory and DLCQ of string theory requires super Yang Mills theory to have certain dualities for all $N$. Let us review what is known starting with T-duality. In [11], [12] the simplest constraint of T-duality was derived for M(atrix) theory compactified on a 3-torus. It was shown that T-duality is equivalent to the electromagnetic S-duality of 3+1 dimensional super Yang Mills theory. This is one of the oldest known dualities and is generally believed to be exact even for finite $N$. This proves that type IIA theory compactified on a 2-torus is nonperturbatively self dual under the T-duality which inverts both cycles of the torus and that the T-duality is true for finite $N$.

A constraint on the behavior of 2+1 dimensional super Yang Mills theory follows from the T-duality which maps type IIA theory to type IIB theory in 9 noncompact dimensions. In [15] it was shown that this duality requires 2+1 dimensional super Yang Mills theory to have a strongly coupled fixed point with a non-manifest $O(8)$ invariance. A proof of this invariance was given in [15] and an independent argument based on the structure of
the superconformal algebra was given by Seiberg [5]. Once again the arguments make no use of the large $N$ limit.

As for S-duality, the simplest example is the self duality of type IIB theory. In the M(atrix) description of IIB theory given in [15] the theory is identified as 2+1 dimensional super Yang Mills theory on a 2 torus. In this formulation S-duality is merely the symmetry under interchange of the 2 cycles of the torus. It is completely manifest for all $N$.

Evidently then, finite $N$ M(atrix) theory is not only perturbatively equivalent to string theory but also, in the case of toroidal compactification, manifests the nonperturbative dualities that have been conjectured in recent years. The only properties of the theory which require extrapolation to infinite $N$ are those which involve Lorentz transformations which are explicitly broken by $x^-$ compactification [17].

Additional nonperturbative evidence for the equivalence of string and M(atrix) theory can be obtained from the spectrum of D-branes. Recall that weakly coupled string theory has massive D0-branes with mass given by

$$m^2 = \frac{1}{g_s^2 \alpha'}$$

These excitations which are nonperturbative can also be found in the strongly coupled 1+1 dimensional super Yang Mills theory. They correspond [13] [6] to a single unit of abelian $U(1)$ electric flux along the compact spatial direction of the field theory.

5. Conclusion

In this paper, three lines of argument have been related. The first is that string theory, compactified on a periodically identified lightlike coordinate can be formulated. This leads to the existence of sectors of the theory characterized by a quantum number $N$ defined by $N = p_- R$. The limit $N \to \infty$ defines the uncompactified light cone quantization of string theory but the finite $N$ version of the theory is a consistent string theory exhibiting all the dualities of the theory.

The second line of argument is that M(atrix) theory compactified on a vanishingly small transverse circle is equivalent to perturbative string theory [16] [5] [6].
Finally, the dualities of string theory require related super Yang Mills theory dualities which surprisingly are satisfied for finite $N$. Together, these point to a clear conclusion. Finite $N$ M(atrix) theory is the DLCQ of string and M-theory.

In fact all of this points to a way to confirm the conjectured equivalence of the matrix model and M-theory without ever having to deal with the large $N$ limit. By arguments similar to [6] we can hope to prove that all of the weakly coupled string-theoretic corners of moduli space are described by an appropriate super Yang Mills M(atrix) model. In addition, dualities of super Yang Mills theory would then confirm the nonperturbative connections between the different corners. The manifest hamiltonian form of the super Yang Mills theory would prove the unitarity of the theory. Furthermore all of this can be done without ever having to deal with the large $N$ limit.

Finally, the connection between super Yang Mills theory and DLCQ suggests the existence of a mathematical structure, more comprehensive than ordinary gauge theory. We do not normally view each value of total momentum as defining a separate system. The usual view is that there is a single dynamical system which can have any value of momentum. It seems inevitable that we will eventually think of the super Yang Mills theories for all $N$ as being embedded in a single dynamical system. Since in M(atrix) theory we have generally identified momentum with gauge flux, it would seem likely that $N$ itself will be understood as a flux in some sort of master gauge theory.

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   the large N limit. Joseph Polchinski and Philippe Pouliot, Membrane Scattering with
   M-Momentum Transfer, hep-th/9704029.