Dilatonic Probe, Force Balance and Gyromagnetic Ratio

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Following the Papapetrou-Dixon-Wald procedure we derive the equation of motion for a dilatonic test body(probe) with the dilaton coupling $\alpha = \sqrt{p/(2 + p)}$ in four dimension. Since the dilatonic freedom sometimes comes from extra dimensions, it is best to derive the EOM by a dimensional reduction from $(p + 4)$-dimensions. We discuss about the force balance up to the gravitational spin-spin interactions via the probe technique. The force balance condition yields the saturation of a Bogomol’nyi bound and the gyromagnetic ratio of the test body.

I. INTRODUCTION

BPS solitons are an important object and has comprehensive features from the view points of superstring theory and black hole physics. Great interest is its force cancellation. By its virtue multi-soliton solution can exist and the existence helps us to construct new solutions which might be related to a string state. The simplest example is the Majumdar-Papapetrou(MP) solution \( \square \) in general relativity or \( N = 2 \) supergravity \( \square \). What we should do in order to check the force balance is to investigate the motion of a charged test body on an isolated soliton, for example, an extremal Reissner-Nordström solution. We should observe that the electrostatic repulsive force is balanced by the gravitational attractive one when \( q = m \).

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It is, however, natural to consider rotating solitons in general case. It is well known that the Israel-Wilson-Perjes(IWP) solution [3] is stationary one which becomes the MP solution at the static limit. The balance between the gravitational spin-spin interaction and magnetic dipole-dipole force was confirmed by using a spinning test body [4]. The gyromagnetic ratio turns out to be \( g = 2 \) which is same with the Kerr solution. Since the body has the spin it does not follow geodesic motion. Unfortunately, the ISW solution is not extreme and has naked singularities except in the static limit [5].

In the heterotic string theory on a torus, similar \( D \)-dimensional stationary charged solution has been found through \( O(d-1,d-1) \) transformation from an uncharged Kerr solution [6] [7]. For \( D > 5 \) the extreme limit saturates a Bogomol’nyi bound without naked singularities and then we can obtain a regular multi-soliton solution [6]. Here we remind you that the extreme limit of non-dilatonic Kerr solution does not exists for even \( D > 5 \). But now the solution contains the dilaton sector and the situation is changed. We also be able to obtain a four dimensional solution by taking doubly periodic arrays of extremal black holes in six dimensions. Thus it is worth to study the force balance in dilatonic or stringy multi-black hole systems.

In this paper, as a first step, we derive the equation of motion for a dilatonic test body with the dilaton coupling \( \alpha = \sqrt{p/(2 + p)} \) in four dimension. This theory can be easily obtained from the dimensional reduction of \( (p + 4) \)-dimensional Maxwell-Einstein theory. The motion and force balance of a test body without any moments, i.e. test particle, has been investigated in ref. [8].

The rest of this paper is organised as follows. In Sec. II we briefly review the Papapetrou-Dixon-Wald procedure [9] [10] [11] in higher dimensions and give the equation of motion for a test body. In Sec. III, we rewrite the above formal equation in terms of four dimensions and discuss about the force balance up to the gravitational spin-spin interaction and give the force balance condition. It is well known that the force balance between the gravitational, dilatonic and electrostatic forces yields the saturation of the Bogomol’nyi bound. Moreover the condition supplies the gyromagnetic ratio of the test body balancing with the central soliton. Finally we give summary and discussion in Sec. IV.
II. EQUATION OF MOTION

The \((p + 4)\)-dimensional Einstein-Maxwell theory gives the Einstein-Maxwell-Dilaton theory with the coupling \(\alpha = \sqrt{p/(p + 2)}\) in four dimension. We follow the argument given in ref. [12]. Let the metric \(G_{MN}\) of \((p + 4)\)-dimensional space-time to be

\[
\begin{align*}
&ds_4^{2+p} = G_{MN}dx^Mdx^N = e^{2\alpha \phi}g_{\mu
u}dx^\mu dx^\nu + e^{2(\alpha - \alpha^{-1})\phi}(d\chi_1^2 + \cdots + d\chi_p^2), \\
&\quad (2.1)
\end{align*}
\]

where \(\alpha = \sqrt{p/(p + 2)}\) and \(\phi\) is a scalar function of \(\{x^\mu\}\). The Geek indices runs from 0 to 3. As you can easily see, the Einstein-Hilbert action in \((p + 4)\)-dimension yields

\[
S = \frac{1}{16\pi G} \int d^4x\sqrt{-g}[R_g - 2(\nabla\phi)^2 - e^{-2\alpha \phi}F^2] \quad (2.2)
\]

in terms of four dimension. Here \(R_g\) is the Ricci scalar of the metric \(g_{\mu\nu}\).

Following the Papapetrou-Dixon-Wald procedure [9] [10] [11], we can instantly written down the equation of motion from the conservation law of the energy-momentum tensor in higher dimension, \(\nabla^MT_{MN} = 0\). The result is as follows,

\[
\begin{align*}
v^M\nabla_MP^N &= \frac{1}{2}(4+p)R_{MIJ}^Nv^MS^{IJ} + qv^MF_M^N + \frac{9q}{4m}S^{IJ}\nabla^NF_{IJ} \\
&\quad (2.3)
\end{align*}
\]

\[
\begin{align*}
v^M\nabla_MS^{IJ} &= 2p[I^Mu^J] + \frac{9q}{2m}S^M|J^F^M| \\
&\quad (2.4)
\end{align*}
\]

where \(g\) is the gyromagnetic ratio of the spinning test body. \(p^M\) and \(v^M\) are \(M\)-momentum and \(M\)-velocity of a test body which has mass \(m\), charge \(q\) and angular momentum tensor \(S^{MN}\). In the above we imposed the “supplementary condition” on the test body,

\[
p^MS_{MN} = 0 \quad (2.5)
\]

which determines the motion of the test body and specifies ‘the center of mass’ [11]. We find the fact that \(p^M\) is not always proportional to \(v^M\) because of the existence of spin. In fact the relation turns out in the form

\[
v^M = f\left[p^M - \frac{1}{p^I p^J} \left[ \frac{4R_{NIJK}S^{JK} - qF_{NI}}{4R_{KLPQ}S^{LPQ} - \frac{1}{2}qS^{KL}F_{KL}} \right] \right], \quad (2.6)
\]

where \(f = (v^Mp^M)/(p^Np_N)\). However, \(p^M \propto v^M\) if one ignores the higher order terms at the spacelike asymptote.

From eq. (2.3) we obtain
\[ u^M \nabla_M Q_\xi = -\frac{gq}{4m} S^{MN} \mathcal{L}_\xi F_{MN} \]  

(2.7)

where

\[ Q_\xi = (p^M + qA^M)\xi_M + \frac{1}{2} S^{MN} \nabla_N \xi_M. \]  

(2.8)

Hence \( Q_\xi \) is conserved if \( \mathcal{L}_\xi F_{MN} = 0 \) holds. The condition \( \mathcal{L}_\xi F_{MN} = 0 \) holds on the stationary exact solution.

As a next step we will rewrite the above equation in terms of four dimension. We will do that for examples given in the next section.

### III. FORCE BALANCE

The force balance between solitons is important features because it guarantees the existence of multi-soliton and this gives a new black hole solution by taking some periodic arrays of solitons which might be related to a string state.

First of all we derive the equation of motion for a test body without spin on a static charged dilatonic solution \[13\].

#### A. Non-Rotating Case

In this case we can take \( S^{IJ} = 0 \) and the equation of motion is simply given by the geodesic motion in \((p + 4)\)-dimension;

\[ \frac{d}{d\tau} p^M = -\Gamma^N_{MI} v^M p^I + q F^N_I v^I. \]  

(3.1)

Here we are interest in the motion in four dimensional space-time and it is natural to assume that the extra dimensions components of the velocity \( v^M \) vanish. From eq. \[2.6\] we see that \( p^M \) is proportional to \( v^M \):

\[ p = p^I \partial_I = p^\mu \partial_\mu = f^{-1} v, \]  

(3.2)

where \( v^\mu = dz^\mu(\tau)/d\tau \) and \( g(v, v) = -1 \). From the above equation of motion

\[ v^M \nabla_M (p^I p_I) = -\partial_\tau (f^{-2} e^{2\alpha\phi}) = 0 \]  

(3.3)

holds. Then \( f \propto e^{\alpha\phi} \) and

\[ p^I = m e^{-\alpha\phi} v^I. \]  

(3.4)
After arranging of several terms we obtain the final form as
\[ v^\mu \nabla_\mu (e^{\alpha \phi} v^\nu) + \nabla_\nu e^{\alpha \phi} = \frac{q}{m} F^\nu_\mu v^\mu. \]  
(3.5)

This equation is also derived from the variational principle of the action given by [8]:
\[ S_4 = \int d\tau \left[ m \sqrt{-e^{2\alpha \phi} g_{\mu\nu} dx^\mu dx^\nu} + e A_\mu dx^\mu \right]. \]  
(3.6)

We have the constant of motion like energy per mass, \( E \);
\[ E := - (e^{\alpha \phi} v^\mu + \frac{q}{m} A^\mu) \xi_\mu \]  
(3.7)

and
\[ \mathcal{L}_v E = 0 \]  
(3.8)

The metric, vector potential and dilation field of the static black hole solution for the action (2.2) are given by [13]
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = - \frac{\Delta}{\sigma^2} dt^2 + \sigma^2 \left[ \frac{1}{\Delta} dr^2 + r^2 d\Omega_2^2 \right] \]  
(3.9)

\[ A = - \frac{Q}{r} dt \quad \text{and} \quad e^{2\alpha \phi} = \sigma^2, \]  
(3.10)

where
\[ \Delta = \left( 1 - \frac{r_-}{r} \right) \left( 1 - \frac{r_+}{r} \right) \quad \text{and} \quad \sigma^2 = \left( 1 - \frac{r_-}{r} \right)^{2\alpha^2/(1+\alpha^2)}. \]  
(3.11)

\( r_\pm \) are related to the ADM mass and total charge as follows,
\[ 2M = r_+ + \frac{1-\alpha^2}{1+\alpha^2} r_- \quad \text{and} \quad Q^2 = \frac{r_+ r_-}{1+\alpha^2}. \]  
(3.12)

Inserted metric [3.9] into eq. (3.7) we obtain
\[ E = \frac{\Delta}{\sigma^2} e^{\alpha \phi} \frac{dt}{d\tau} + \frac{q}{m} \frac{Q}{r} \]  
(3.13)

and then
\[ \left( \frac{dt}{d\tau} \right)^2 = \frac{\sigma^2}{\Delta^2} \left( E - \frac{qQ}{m r} \right)^2. \]  
(3.14)

From \( g(v, v) = -1 \) and eq. (3.14)
\[
\left( \frac{dr}{d\tau} \right)^2 + V(r) = 0,
\] (3.15)

where
\[
V(r) = \frac{1}{\sigma^2} \left[ \frac{r_+ + r_- - 2qQ/m}{r} + \frac{r_+ r_- - (qQ/m)^2}{r^2} \right]
\] (3.16)

When \( r_+ = r_- \), that is,
\[
Q = (1 + \alpha^2)^{1/2} M \quad \text{and} \quad q = (1 + \alpha^2)^{1/2} m,
\] (3.17)

the potential exactly vanishes, that is, \( V(r) = 0 \) for arbitrary \( r \). In the dilatonic gravity theory, the Bogomol’nyi type bound \( M \geq (1 + \alpha^2)^{-1/2}|Q| \) has been proven [14]. So we realise again that the extreme limit saturates the Bogomol’nyi bound. We also be able to check the force balance from the equation of motion directly. For simplicity, hereafter, we take the extreme limit. The metric is written as
\[
\text{ds}^2 = g_{\mu\nu} dx^\mu dx^\nu = -V(R)^{-1} dt^2 + V(R) dX^2,
\] (3.18)

where \( V(R) = (1 + r_+/R)^2(1+\alpha^2) = (1 + r_+/R)^{(p+2)/(p+1)} \) and \( R = r - r_+ = \sqrt{\delta_{ij} X^i X^j} \). Then
\[
\frac{d^2 x^i}{d\tau^2} = -\Gamma^i_{00} - \alpha \partial_i \phi + \frac{q}{m} e^{-\alpha \phi} \partial_i A_0 + O(\frac{1}{R^3}) = O(\frac{1}{R^3})
\] (3.19)

because of \( \Gamma^i_{00} \approx \frac{r_+}{1+\alpha^2} \frac{X^i}{R^2} \), \( \partial_i \phi \approx \frac{\alpha r_+}{1+\alpha^2} \frac{X^i}{R^2} \) and \( \frac{q}{m} \partial_i A_0 \approx r_+ \frac{X^i}{R^2} \). This tells us the cancellation between the gravitational, dilatonic and electrostatic forces.

We remember that the above extreme dilatonic solution with odd \( p \) can be derived from dense arrays of infinite multi-black holes solution with the same mass along the \( \chi \)-direction in the \( (p + 4) \)-dimensional Einstein-Maxwell theory [15]. The metric of multi-black holes solution is given by
\[
\text{ds}^2_{4+p} = -U^{-2} dt^2 + U^{2/(p+1)} dx^2 + U^{2/(p+1)} (d\chi_1^2 + \cdots + d\chi_p^2),
\] (3.20)

where \( U = 1 + \sum_{i=1}^p m_i/|(\vec{x} - \vec{x}_i)^{p+1} + |\vec{x} - \vec{X}_i|^{p+1} \). The dense arrays along \( p \)’s \( \chi \) direction leads us
\[
U = 1 + \sum_{i=1}^p \frac{m}{|\vec{x}_i|^{p+1} + |\vec{x} - \vec{X}_i|^{p+1}} = 1 + \frac{M}{r}.
\] (3.21)

Note that the metric (3.18) can be written as
\[
\text{ds}^2_{4+p} = U^{-p/(1+p)} \left[ -U^{-(p+2)/(p+1)} dt^2 + U^{(p+2)/(p+1)} dx^2 \right] + U^{2/(p+1)} (d\chi_1^2 + \cdots + d\chi_p^2)
= U^{-p/(1+p)} g_{\mu\nu} dx^\mu dx^\nu + U^{2/(p+1)}(d\chi_1^2 + \cdots + d\chi_p^2).
\] (3.22)
This is black $p$-brane. Thus the motion of test body is equivalent with the geodesic motion for perpendicular direction to the surface where arrays(black $p$-brane) are.

**B. Rotating Case**

For purpose of this paper it is enough to pick up the first order of the angular momentum (slow rotating approximation). The general solution is given by [16]

$$ds^2 = -\frac{\Delta}{\sigma^2} + \sigma^2 \left[ \frac{1}{\Delta} dr^2 + r^2 d\Omega_2^2 \right] - 2af(r)\sin^2\theta dt d\phi \quad (3.23)$$

$$f(r) = \frac{(1 + \alpha^2)^2}{(1 - \alpha^2)(1 - 3\alpha^2)} \left( \frac{r}{r_-} \right)^2 \left( 1 - \frac{r_-}{r} \right)^{2\alpha^2/(1+\alpha^2)}$$

$$- \left( 1 - \frac{r_-}{r} \right)^{(1-\alpha^2)/(1+\alpha^2)} \left[ 1 + \frac{(1 + \alpha^2)^2}{(1 - \alpha^2)(1 - 3\alpha^2)} \left( \frac{r}{r_-} \right)^2 + \frac{1 + \alpha^2}{1 - \alpha^2} \frac{r}{r_-} - \frac{r_+}{r} \right] \quad (3.24)$$

and

$$A = \frac{Q}{r} dt - \sin^2\theta \frac{Q}{r} d\phi \quad (3.25)$$

As, at first sight, the function $f(r)$ seems to have a bad behaviour at $p = 1$, we will consider the case with $p = 1$ separately.

The apparent bad behaviour of the function $f(r)$ does not yield any troubles and we obtain the expression

$$f(r) = -\left( 1 - \frac{r_-}{r} \right)^{1/2} \left[ \left( 1 - \frac{r_+}{r} + \frac{2r}{r_-} \right) - \frac{2r^2}{r_-^2} \ln \left( 1 - \frac{r_+}{r} \right) \right] \quad (3.26)$$

at the limit $p = 1$. Since we have $f(r) \simeq \frac{5r_+}{3R}$ at the extreme limit($r_+ = r_-$), the metric and vector potential become

$$ds^2 = -(1 + r_+/R)^{-3/2} dt^2 + (1 + r_+/R)^{1/2} dX^2 + \frac{20}{3} \frac{1}{R^3} \epsilon_{ijk} J^j X^k dt dX^i \quad (3.27)$$

and

$$A \simeq \frac{Q}{R} dt + \frac{2Q}{r_+ R} \epsilon_{ijk} J^j X^k dX^i, \quad (3.28)$$

where $|J| = \frac{ar_+}{2}$. As

$$R_{\alpha M\alpha M} v^M S^{IJ} \simeq (4+p) R_{\alpha ij} S^{ij} \quad (3.29)$$

and
we can evaluate the first and third terms of the right-hand side in eq. (2.3) as follows,

\[ F^i_{\text{spin-spin}} = -\frac{1}{2} (4) R^i_{0jk} S^{jk} = \frac{5}{3} \frac{\partial}{\partial X^i} \left[ -\frac{J \cdot S + 3(J \cdot \hat{X})(S \cdot \hat{X})}{R^3} \right] + O(1/R^5), \]  

(3.31)

and

\[ F^i_{\text{dipole-dipole}} = -\frac{ggQ}{4m} \frac{\partial}{\partial X^i} \left[ -\frac{J \cdot S + 3(J \cdot \hat{X})(S \cdot \hat{X})}{R^3} \right] + O(1/R^5), \]  

(3.32)

where \( \hat{X} = X/R \). By summing up those we obtain

\[ F^i_{\text{spin-spin}} + F^i_{\text{dipole-dipole}} = \left( \frac{5}{3} - \frac{ggQ}{m r^3} \right) \frac{\partial}{\partial X^i} \left[ -\frac{J \cdot S + 3(J \cdot \hat{X})(S \cdot \hat{X})}{R^3} \right] + O(1/R^5), \]  

(3.33)

As we see in the previous subsection, monopole components are balanced by each other when \( Q = 2M/\sqrt{3} \) and \( q = 2m/\sqrt{3} \). Then the balance between the above forces holds when the gyromagnetic ratio is \( g = 5/3 \).

Next we consider cases with \( p \neq 1 \). The metric and vector potential are approximately given by

\[ ds^2 = -\frac{\Delta}{\sigma^2} + \sigma^2 \left[ \frac{1}{\Delta} dr^2 + r^2 d\Omega_2^2 \right] + \frac{4}{R^3} \epsilon_{ijk} J^j X^k dt dX^i \]  

(3.34)

and

\[ A \simeq \frac{Q}{R} dt + \frac{3(1 + \alpha^2)}{3 + \alpha^2} \frac{Q}{r^3} \epsilon_{ijk} J^j X^k dX^i \]  

(3.35)

where \( |J| = \frac{(3+\alpha^2)mr^3}{3(1+\alpha^2)} \). As the same way as \( p = 1 \) case, we obtain

\[ F^i_{\text{spin-spin}} + F^i_{\text{dipole-dipole}} = \left( 1 - \frac{3(1 + \alpha^2)}{2(3 + \alpha^2)} \frac{ggQ}{m r^3} \right) \frac{\partial}{\partial X^i} \left[ -\frac{J \cdot S + 3(J \cdot \hat{X})(S \cdot \hat{X})}{R^3} \right] + O(1/R^5), \]  

(3.36)

Then the balance between the above forces holds when the gyromagnetic ratio is \( g = \frac{2(3+\alpha^2)}{3(1+\alpha^2)} \). Here we realise that the previous value of the gyromagnetic ratio in the case with \( p = 1 \) can be combined to the present expression. It is remarkable that the value is different from that of the background space-time. According to ref. [16], the background space-time has \( g = \frac{6}{3+\alpha^2} \). That is, the test body is not like the background space-time. The discrepancy indicates the non-existence of spinning multi-soliton solutions in the Einstein-Maxwell-Dilaton theory. If and only if \( \alpha = p = 0 \), all gyromagnetic ratios are same as \( g = 2 \) and the exact IWP solutions exists as we expect.
IV. SUMMARY AND DISCUSSION

In this paper we derived the equation of motion for a spinning charged dilatonic probe (test body) by using the Papapetrou-Dixon-Wald procedure and discuss about the force balance. We found that the force balance holds up to the spin-spin interaction when $|Q| = \sqrt{1 + \alpha^2} M$, $|q| = \sqrt{1 + \alpha^2} m$ and the gyromagnetic ratio $g = \frac{2(3+\alpha^2)}{3(1+\alpha^2)}$. The former condition saturates the Bogomol’nyi bound and its fact was well known. On the other hand, the latter condition has a discrepancy with the value of the background space-times except in the $p = 0$ case. This fact might mean that the Einstein-Maxwell-Dilaton theory does not have the exact spinning multi-soliton solutions. We are aware of the multi-soliton solution in $N = 4, D = 4$ supergravity [17] or heterotic string theory [6] [7]. In those theory the additional ingredient is the three form field $H_{\mu\nu\alpha}$ which can be expressed by the ‘axion’ field, $a$, so that $H_{\mu\nu\alpha} = e^{4\phi} \epsilon_{\mu\nu\alpha\beta} \partial^\beta a$ in four dimension. Hence, the resolution to the discrepancy may be essential to investigate the force balance in so-called ‘Einstein-Maxwell-Dilaton-Axion’ theory. The investigation on the stringy probe, that is, test body with the moment associated to the three form field might be important and will be reported in the next study [18]. In this paper our consideration was concentrated on the bosonic sector. Obviously, the deeper investigation of the fermion sector is also great interest although we can expect a parallel issue naively.

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