Local polynomial estimation of time-varying parameters in nonlinear models∗

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August 25, 2023

Abstract

We develop a novel asymptotic theory for local polynomial (quasi-) maximum-likelihood estimators of time-varying parameters in a broad class of nonlinear Markov models. Under weak regularity conditions, we show the proposed estimators are consistent and follow normal distributions in large samples. Our conditions impose weaker smoothness and moment conditions on the data-generating process and its likelihood compared to existing theories. Furthermore, the bias terms of the estimators take a simpler form. We demonstrate the usefulness of our general results by applying our theory to local (quasi-)maximum-likelihood estimators of time-varying VAR’s, ARCH models, and Poisson autoregressions. For the first two models, we are able to substantially weaken the conditions found in the existing literature. For the Poisson autoregression, existing theories cannot be applied while our novel approach allows us to analyze it. An empirical study of US default counts demonstrates the usefulness of the proposed estimators and sheds new lights on their dynamics.

Keywords: local polynomial estimation, local stationarity, quasi-likelihood, time-varying parameters

1 Introduction

There is ample empirical evidence of time-varying parameters in many econometric models; see, e.g., Inoue and Rossi (2011), Giacomini and Rossi (2016), Caldara et al. (2012), Ghysels and Hall (1990) and Christoffersen et al. (2012). Most studies aiming at accommodating this feature assume a fully parametric model for this time variation; one example of this is structural break models. This has the advantage that the time-varying version of a given model stays parametric and can be estimated using existing methods. The disadvantage is that the researcher runs the risk of choosing a misspecified model for the time variation.

∗The authors would like to thank Sokbae Lee and participants at various seminars and conferences for valuable comments and suggestions.
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To reduce this risk, methods that treat the problem of time–varying parameters as a structured nonparametric one have been developed: They assume that the sequence of time-varying parameters arise as values of an underlying function which is then estimated nonparametrically; one popular class of estimators that falls in this category are local estimators which includes the so-called rolling-window estimator. However, the existing theory for such estimators mostly focuses on regression models and ARCH models and restrict themselves to local constant estimators; see, e.g., Robinson (1989), Dahlhaus and Subba Rao (2006), Fryzlewicz et al. (2008) and Kristensen (2012). In particular, there is very little theory for local estimators when applied to the non-linear models used in the papers cited in the previous paragraph.

We provide a general asymptotic theory for local polynomial (quasi-) maximum-likelihood estimators of time-varying parameters that applies to a very broad class of non-linear Markov models. The theory allows for a wide range of different underlying models and estimators thereby significantly broadens the scope for inference based on rolling-window type estimators compared to existing papers.

When we apply our general results to models and estimators already analyzed in the existing literature, we find that ours impose substantially weaker smoothness and moment conditions on the model compared to existing results. This feature is particularly important in financial applications where variables often have fat tails. Moreover, again compared to existing theories, our results hold under much weaker restrictions on the bandwidth sequence used in the estimation thereby allowing for standard bandwidth selection procedures to be used. Finally, our asymptotic limit results take a simpler form compared to existing theories and more closely resemble those found in the literature on local maximum likelihood estimation in a cross-sectional setting.

The class of estimators includes as special cases the local constant estimator and the local linear estimator. We find that the local constant estimator requires stronger regularity conditions to be well-behaved in large samples and will generally suffer from additional biases in the interior of the domain compared to the local linear estimator. These additional biases involve the so-called derivative process of the stationary approximation to data which is not present in the biases of the local linear one. Moreover, the local linear estimator enjoys the well-known automatic boundary adjustment property meaning that at the beginning and end of the sample, this estimator will perform better than the local constant one. This feature is important since often the main interest is on the values that the time-varying parameters take at the end of the sample.

The most closely related paper to ours is Dahlhaus et al. (2019) who develop a general theory for local constant estimators. Compared to their analysis, our proof techniques are different which in turn enable us to derive our results under weaker conditions compared to theirs, see discussion above for details. In particular, our results apply to estimators of models that cannot be handled by their existing theory such as Poisson autoregressions with time-varying parameters.

Our theory also contributes to the literature on asymptotic analysis of local polynomial estimators of varying-coefficient models by extending existing results (as in Fan et al. (1995) and Loader (2006)) to cover situations where the objective functions are non-concave. This is an important
extension since the quasi-likelihoods of most non-linear models are non-concave, and the analysis
of such models requires some new technical tools.

As an empirical application, we revisit the empirical study of Agosto et al. (2016) where a Poisson
autoregressive model with additional co–variates was used to model and analyze US defaults.
Using the proposed methodology, we find substantial time–variation in the model parameters that
the original study was unable to capture. In particular, we find that the ability of macroeconomic
and financial variables to predict defaults have varied substantially over time. A battery of informal
tests of the time–varying model against the time–invariant version finds strong support for the
former.

The remainder of the paper is organized as follows: Framework and estimators are introduced in
Section 2. Section 3 presents the asymptotic theory of the estimators. Section 4 provides examples
of the theory when applied to particular models. We present the results of two simulation studies
and the empirical application in Sections 5 and 6, respectively. All lemmas and proofs have been
relegated to the Appendix.

2 Framework

We are given \( n \) observations, \( Z_{n,t}, t = 1, \ldots, n \), from a nonlinear time-series model characterized
by a vector of unknown parameters \( \theta \in \Theta \subset \mathbb{R}^{d_{\theta}} \) to be estimated. We take as given a (quasi–)
log–likelihood function \( \ell_n (\theta) = \ell (Z_{n,t}; \theta) \in \mathbb{R} \) as chosen by the researcher. If \( \theta \) indeed was time–
invariant, the researcher would estimate it by \( \hat{\theta} = \arg \max_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^{n} \ell_{n,t} (\theta) \). Note here that we
here require the quasi–likelihood only to depend on \( Z_{n,t} \). This rules out non–Markov models where
the likelihood contribution at time \( t \) depends on the full past of \( Z_{n,t} \) which is, for example, the case
for GARCH–type models.

Suppose that in fact \( \theta \) is varying over time so that data is generated by \( \theta_{n,t} = \theta (t/n), t = 1, \ldots, n \), where \( \theta : [0,1] \mapsto \Theta \) is an unknown function that characterizes the time–variation in
the parameters.\(^1\) We then propose to estimate \( \theta (u) \) at any given value \( u \in [0,1] \) using local polynomial
estimators: First, we approximate \( \theta (t/n) \) for \( t/n \) in a neighbourhood of \( u \) by the
following polynomial of order \( m \geq 0 \),

\[
\theta_{u,\beta}^* (t/n) := \beta_1 + \beta_2 (t/n - u) + \cdots + \beta_{m+1} (t/n - u)^m / m! = D (t/n - u) \beta, \tag{1}
\]

where \( \beta = (\beta_1, \ldots, \beta_{m+1})' \in \mathbb{R}^{(m+1)d_{\theta}} \) with \( \beta_{i+1} = \theta^{(i)} (u) = \partial^{i} \theta (u) / \partial u^i \in \mathbb{R}^{d_{\theta}} \) and

\[
D (u) = \left( 1, u, u^2 / 2, \ldots, u^m / m! \right) \otimes I_{d_{\theta}} \in \mathbb{R}^{d_{\theta} \times (m+1)d_{\theta}}.
\]

Next, to control the approximation error, \( \theta (t/n) - \theta_{u,\beta}^* (t/n) \), we introduce a kernel weighted version

\(^1\)Data points now depends on sample size \( n \) through \( \theta (t/n) \), which is why we write \( Z_{n,t} \) instead of of \( Z_t \).
of the "global" quasi-log-likelihood evaluated at the polynomial approximation,
\[
L_n (\beta | u) = \frac{1}{n} \sum_{i=1}^{n} K_b (t/n - u) \ell_{n,t} (\theta^*_u, \beta (t/n)),
\]
where \( K_b (\cdot) = K (\cdot / b) / b, K : \mathbb{R} \to \mathbb{R} \) is a kernel function, and \( b = b_n > 0 \) a bandwidth. The kernel weights ensure that when \( t/n - u \) is "large", the corresponding observations are down weighted in the estimation, thereby controlling for the aforementioned approximation error. We then estimate the polynomial coefficients by
\[
\hat{\beta} (u) = \arg \max_{\beta \in \mathcal{B}} L_n (\beta | u),
\]
where
\[
\mathcal{B} = \{ \beta \in \mathbb{R}^{(m+1)d_B} : \theta^*_u, \beta (v) \in \Theta \text{ for } v \in [0,1] \}.
\]
The estimated \( \beta \) coefficients are used as estimates of \( \theta (u) \) and its first \( m \) derivatives, \( \hat{\theta}^{(i)} (u) = \hat{\beta}_{i+1} (u), i = 0, ..., m \). When \( m = 0 \), we recover the standard local-constant estimator. The above class of estimators is similar to the ones considered in Fan et al. (1995) for so-called varying-coefficient models, except that we consider time series models with the "regressor" that we smooth over being normalized time, \( t/n \).

Our framework includes models of the following form:
\[
Y_{n,t} = G (Y_{n,t-1}, X_{n,t-1}, \varepsilon_t; \theta_{n,t}), \quad \theta_{n,t} = \theta (t/n),
\]
for \( t = 1, 2, ..., n \) where \( G(y, x, e; \theta) \) is a known function, \( X_{n,t-1} \) contains exogenous regressors and \( \varepsilon_t \) is a sequence of errors. For a given specification of \( G \) and the distribution of \( \varepsilon_t \), we can then derive the corresponding log-likelihood for the model with time-invariant parameters, \( \ell (Z_{n,t}; \theta) \) where \( Z_{n,t} = (Y_{n,t}, Y_{n,t-1}, X_{n,t-1}) \). We illustrate this in the following two examples; see Section 4 for more examples.

**Example 1** (Zhou and Wu, 2009; Truquet, 2019): Suppose \( Y_{n,t} \in \mathbb{R} \) solves the following time-varying threshold autoregressive (tv-TAR) model with two regimes,
\[
Y_{n,t} = \theta_1 (t/n) Y_{n,t-1}^+ + \theta_2 (t/n) Y_{n,t-1}^- + \theta_3 (t/n)' X_{n,t-1} + \varepsilon_t,
\]
where \( y^+ := \max \{ y, 0 \}, y^- := \min \{ y, 0 \} \). Here, \( X_{n,t-1} \) contains additional predictors and \( \varepsilon_t \) is i.i.d. with \( \mathbb{E} [\varepsilon_t] = 0 \) and \( \mathbb{E} [\varepsilon_t^2] < \infty \). A natural estimator of \( \theta = (\theta_1, \theta_2, \theta_3)' \) is the least-squares one so that \( \ell_{n,t} (\theta) = - (Y_{n,t} - \theta_1 Y_{n,t-1}^+ - \theta_2 Y_{n,t-1}^- - \theta_3 ' X_{n,t-1})^2 \).

**Example 2** (Agosto et al., 2016): Suppose \( Y_{n,t} \in \mathbb{Z}_+ \) satisfies
\[
Y_{n,t} | F_{n,t-1} \sim \text{Poisson} (\lambda_{n,t}), \quad \lambda_{n,t} = \omega (t/n) + \alpha (t/n) Y_{n,t-1} + \gamma (t/n)' X_{n,t-1},
\]
where \( \text{Poisson}(\lambda) \) denotes a Poisson distribution with intensity parameter \( \lambda \), and \( X_{n,t-1} \) contains
additional predictors. The log-likelihood function takes the form \( \ell_{n,t}(\theta) = Y_{n,t} \log(\lambda_{n,t}(\theta)) - \lambda_{n,t}(\theta) \) where \( \lambda_{n,t}(\theta) = \omega + \alpha Y_{n,t-1} + \gamma' X_{n,t-1} \) with \( \theta = (\omega, \alpha, \gamma)' \).

3 Asymptotic theory

We here provide an asymptotic theory for \( \hat{\beta} \). One complication of this analysis is that the components of \( \hat{\beta} \) converge with different rates. We follow the existing literature and handle this issue by introducing a re–scaled version of \( \hat{\beta} \); see (see, e.g., Han and Kristensen, 2014) for a similar approach: Define \( \hat{\alpha} = U_n \hat{\beta} = \left( \hat{\theta}(u)', b\hat{\theta}^{(1)}(u)', ..., b^m\hat{\theta}^{(m)}(u)' \right)' \), where

\[
U_n = \text{diag} \{1, b, ..., b^m\} \otimes I_{d_\theta} \in \mathbb{R}^{(m+1)d_\theta \times (m+1)d_\theta}
\]

is a weighting matrix containing their relative convergence rates, Given that \( U_n \) is non-singular, the estimation problem (2) is equivalent to solving

\[
\hat{\alpha} = \arg \max_{\alpha \in \mathcal{A}} Q_n(\alpha|u), \quad Q_n(\alpha|u) = \frac{1}{n} \sum_{t=1}^n K_b(t/n - u) \ell_{n,t}(D_b(t/n - u) \alpha),
\]

where \( D_b(u) = D(u/b) \) and the parameter space \( \mathcal{A} \) is specified below. We will then analyze the properties of \( \hat{\alpha} \).

Due to the time–varying parameters, \( Z_{n,t} \) will generally be non–stationary. To develop an asymptotic theory that allows for this feature, we will rely on the concept of local stationarity as introduced by Dahlhaus (1997); see also Dahlhaus and Subba Rao (2006) and Dahlhaus et al. (2019). We first generalize this concept to sequences of random functions:

**Definition 1** A triangular family of random sequences \( W_{n,t}(\theta), \theta \in \Theta, t = 1, 2, ..., n \) and \( n \geq 1 \), is uniformly locally stationary on \( \Theta \) (ULS\( (p,q,\Theta) \)) for some \( p,q > 0 \) if there exists a family of processes \( W^*_t(\theta|u), u \in [0,1] \), such that: (i) The process \( \{W^*_t(\theta|u)\} \) is stationary and ergodic for all \( (\theta, u) \in \Theta \times [0,1] \); (ii) for some \( C < \infty \) and \( \rho < 1 \),

\[
\mathbb{E} \left[ \sup_{\theta \in \Theta} \|W_{n,t}(\theta) - W^*_t(\theta|u)\|_p \right]^{1/p} \leq C \left( \left| \frac{t}{n} - u \right|^q + \frac{1}{n^q} + \rho^t \right).
\]

If \( W_{n,t}(\theta) = W_{n,t} \) does not depend on any parameters, we write LS\( (p,q) \). The above condition states that while \( W_{n,t}(\theta) \) may be a non–stationary time series it is locally in time well–approximated by a stationary version \( W^*_t(\theta|u) \). Compared to existing definitions of local stationarity, we allow for an additional term \( \rho^t \) to appear in the approximation error. This is needed in order to allow for the initial value of \( W_{n,t}(\theta) \) to be arbitrary. In contrast, most of the existing literature implicitly assumes that \( W_{n,t}(\theta) \) has been initialized at \( W_{n,0}(\theta) = W^*_0(\theta|u) \).
assumption. In contrast, the above definition allows for the observed process to be initialized at any given value – as long as the impact of this dies out over time.

To see how the additional error term appears in Markov models, we refer the reader to Theorem 1 in Appendix A.1 which allow for an arbitrary initialization of the data-generating process. The additional error term due to different initializations is here assumed to decay geometrically and so our definition rules out long-memory type processes. This is mostly for simplicity and we expect that most of our results can be generalized to allow for slower decay rates. Appendix A.1 contains a number of other novel results for kernel weighted averages of parameter-dependent locally stationary processes which will be used in the following analysis of our polynomial estimators.

We will then require that \( \ell_{n,t}(\theta) \) is ULS\((p,q,\Theta)\) with stationary approximation \( \ell^*_{t}(\theta|u) = \ell(Z^*_t(u), \theta) \) where \( Z^*_t(u) \) is the stationary solution to the model being estimated when \( \theta_t = \theta(u) \) is constant. To illustrate, consider again (3). Under regularity conditions on \( G \) and \( \varepsilon_t \) (see Appendix A.1 for details), the stationary solution will in this case take the form

\[
Y^*_t(u) = G(Y^*_{t-1}(u), X^*_{t-1}(u), \varepsilon_t; \theta(u)),
\]

where we impose the high–level condition that the exogenous co–variates are locally stationary. If the data-generating process is locally stationary, it follows under great generality that the likelihood and its derivatives are also locally stationary as shown in the following theorem:

**Theorem 1** Suppose that \( W_{n,t}(\theta) \in \mathcal{W} \) is ULS\((p,q,\Theta)\) with stationary approximation \( W^*_t(\theta|u) \) satisfying \( \mathbb{E}[\sup_{\theta \in \Theta} \| W^*_t(\theta|u) \|^p] < \infty \); (ii) \( \varepsilon_t \) is i.i.d. and independent of \( (W_{n,t}(\theta), W^*_t(\theta|u)) \); and (iii) \( \mathbb{E}[\| f(w, \varepsilon_t; \theta) - f(w', \varepsilon_t; \theta) \|] \leq C(1 + \|w\|^r + \|w'\|^r) \|w - w'\| \) for some \( r > 0 \), all \( \theta \in \Theta \) and \( w, w' \in \mathcal{W} \). Then \( f(W_{n,t}(\theta), \varepsilon_t; \theta) \) is ULS\(p/ (r + 1), q, \Theta\).

This result generalizes Proposition 2.5 in Dahlhaus et al. (2019) in two directions: First, it allows for \( W_{n,t}(\theta) \) to be parameter dependent; second, it allows for an i.i.d. component, \( \varepsilon_t \), to enter the transformation. Allowing for parameter dependence means above result is applicable to, for example, GARCH-type models and so should be of independent interest. The reason why we allow for the presence of the additional component \( \varepsilon_t \) is best illustrated by again considering (3): In this model, we can rewrite \( Z_{n,t} = (Y_{n,t}, Y_{n,t-1}, X_{n,t}) \), and thereby the log–likelihood \( \ell(Z_{n,t}; \theta) \), as a function of \( Y_{n,t-1}, X_{n,t} \) and the error term \( \varepsilon_t \). Doing so allows for easier verification of local stationarity of the likelihood and its derivatives; see Section 4 for details.

The next step in our proof is to establish a uniform law of large numbers (ULLN) for the stationary approximation of \( Q_n(\alpha|u) \), \( Q^*_n(\alpha|u) = \frac{1}{n} \sum_{t=1}^{n} K_b(t/n - u) \ell^*_{t}(D_b(t/n - u) \alpha|u) \). A sufficient condition for a ULLN to hold is that \( \theta \mapsto \ell^*_{n,t}(\theta|u) \) is \( L_p \)-continuous:

**Definition 2** A stationary process \( W^*_t(\theta|u) \) is said to be \( L_p \)-continuous w.r.t. \( \theta \) if \( \mathbb{E}[\| W^*_t(\theta|u) \|^p] < \infty \) for all \( \theta \in \Theta \) and

\[
\forall \epsilon > 0 \exists \delta > 0 : \mathbb{E}\left[ \sup_{\theta \in \Theta: \| \theta - \theta' \| < \delta} \| W^*_t(\theta|u) - W^*_t(\theta'|u) \|^p \right]^{1/p} < \epsilon.
\]
Imposing $L_p$-continuity w.r.t. $\theta$ is weaker than almost surely continuity: If $\theta \mapsto W_i^* (\theta | u)$ is almost surely continuous with $\mathbb{E} [\sup_{\theta \in \Theta} \| W_i^* (\theta | u) \|_p^p] < \infty$ the process is also $L_p$-continuous since $D_1 (\delta) = \sup_{\| \theta - \theta' \| \leq \delta} \| W_i^* (\theta | u) - W_i^* (\theta' | u) \|_p^p$, $\delta > 0$, will then satisfy $\lim_{\delta \to 0} D_1 (\delta) = 0$ almost surely and so, by dominated convergence, $\lim_{\delta \to 0} \mathbb{E} [D_1 (\delta)] = 0$. It is easily verified that $L_p$-continuity w.r.t. $\theta$ implies stochastic equicontinuity of $Q^*_{\alpha} (\alpha | u)$ and so a ULLN holds, c.f. Lemma 3(i) in Appendix A.1.

We are now ready to state the regularity conditions used to show consistency:

**Assumption 1** (i) $K (\cdot ) \geq 0$ has compact support $K$ and $\int_{-\infty}^{+\infty} K (v) \, dv = 1$; (ii) for some $\Lambda < \infty$, $|K(v) - K(\bar{v})| \leq \Lambda |v - \bar{v}|$, $v, \bar{v} \in \mathbb{R}$.

**Assumption 2** $\mathcal{A} = \{ \alpha \in \mathbb{R}^{(m+1)d_\theta} : D (v) \alpha \in \Theta, \forall v \in \mathcal{K} \}$ where $\Theta$ is compact. The true value $\theta (u) \in \Theta$ and $v \mapsto \theta (v)$ is continuous at $u$.

**Assumption 3** (i) $\ell_{n,t} (\theta)$ is ULS$(p, q, \Theta)$ for some $p \geq 1$ and $q > 0$ with stationary approximation $\ell_t^* (\theta | u)$; (ii) $\theta \mapsto \ell_t^* (\theta | u)$ is $L_p$-continuous for some $p \geq 1$; (iii) $\theta \mapsto \mathbb{E} [\ell_t^* (\theta | u)]$ has a unique maximum at $\theta (u) \in \Theta$.

Assumption 1(i) imposes stronger than usual assumptions on $K$ and excludes, among others, the Gaussian kernel and higher-order kernels. It includes, on the other hand the Epanechnikov and the triangular kernel. The restriction that $K (\cdot ) \geq 0$ is used to ensure identification of the parameters when $m > 0$; without this, identification is not necessarily guaranteed; see below for further discussion. The compact support assumption greatly simplifies our analysis of local polynomial estimation of non-concave models: In order to establish uniform convergence of the likelihood we require $\Theta$ to be compact as is standard in the literature. But under this restriction, it is easily checked that $D_b (v) \alpha \notin \Theta$ as $b \to 0$ for any given $\alpha = (\alpha_1, ..., \alpha_{m+1})$ with $\alpha_i \neq 0$ for some $i \geq 1$ and any $v \neq 0$. Thus, to allow for kernels with unbounded support, we would generally need the parameter space $\mathcal{A}$ to collapse at $\{ (\alpha_1, 0, ..., 0) : \alpha_1 \in \Theta \}$ as $b \to 0$. Such shrinking behaviour in turn means that a formal Taylor expansion of $\ell_{n,t} (D_b (v) \alpha)$ w.r.t. $\alpha$ is difficult to obtain and so standard arguments to establish asymptotic normality of $\hat{\alpha}$ cannot be applied. On the other hand, by restricting the support $K$ to be compact and with $\mathcal{A}$ defined in Assumption 2, $K_b (v) \ell_{n,t} (D_b (v) \alpha)$ is well-defined for all $\alpha \in \mathcal{A}$ and $v \in \mathbb{R}$ (where we set $K_b (v) \ell_{n,t} (D_b (v) \alpha) = 0$ for $v/b \notin \mathcal{K}$). Moreover, $(\alpha_1, 0, ..., 0)$ is an interior point of $\mathcal{A}$ and so in our analysis of $\hat{\alpha}$ we can employ standard arguments involving a Taylor expansion of the score function around this point.

Assumption 3(ii)-(iii) are standard in the analysis of "global" extremum estimators of stationary models on the form $\hat{\theta} (u) = \arg \max_{\theta \in \Theta} \sum_{t=1}^n \ell_t^* (\theta | u)$. In particular, for a given time series model, we can import existing results for verification of Assumption 3(ii)-(iii); see Section 4 for more details. Assumption 3(iii) in conjunction with $K (\cdot ) \geq 0$ ensures that the local polynomial estimator identifies $\theta (u)$. If we allow for kernels that take negative values, we have to replace 3(iii) with the following more abstract identification condition: The function $Q^* (\alpha | u) = \int K (v) \mathbb{E} \ell_t^* (D (v) \alpha | u) \, dv$ satisfies $Q^* (\alpha | u) < Q^* ((\theta (u), 0, ..., 0) | u)$ for any $\alpha \neq (\theta (u), 0, ..., 0)$. We have not been able to
provide primitive conditions for this to hold when $K$ can take negative values and so instead impose the positivity constraint on $K$.

If the objective function $\theta \mapsto \ell_{n,t}(\theta)$ is concave and $\Theta$ is convex, we can replace Assumption 3(i)-(ii) with the following pointwise versions: For any $\theta \in \Theta$, $\ell_{n,t}(\theta)$ is locally stationary and $\mathbb{E} [\ell_u^2(\theta|u)] < \infty$; see Theorem 2.7 in Newey and McFadden (1994). Under the above assumptions, the following consistency result holds:

**Theorem 2** Let Assumptions 1-3 hold. Then, as $b \to 0$ and $nb \to \infty$, $\hat{\alpha} \to_p (\theta(u), 0, \ldots, 0)'$. In particular, $\hat{\theta}(u) \to_p \theta(u)$.

Note that the above theorem only shows consistency of $\hat{\theta}(u)$ and so at this stage we cannot make any statements regarding $\hat{\theta}^{(i)}(u)$, $i = 1, \ldots, m$. This is similar to other results for nonlinear extremum estimators where parameters associated with components appearing in the objective function that grow (shrink) with a slower (faster) rate than the leading one will not be identified; see, e.g., Theorem 9 in Han and Kristensen (2014) where a global consistency result is only provided for the component with the fastest rate.

However, with some further regularity conditions on the quasi-likelihood function, we can provide a more precise analysis of the estimators. With $s_{n,t}(\theta) = \partial \ell_{n,t}(\theta) / (\partial \theta) \in \mathbb{R}^{d_\theta}$ and $h_{n,t}(\theta) = \partial^2 \ell_{n,t}(\theta) / (\partial \theta \partial \theta') \in \mathbb{R}^{d_\theta \times d_\theta}$, $D_{n,t}(u) = D_b(t/n - u)$ and $K_{n,t}(u) = K_b(t/n - u)$, we introduce the score and hessian,

$$
S_n(\alpha|u) = \frac{1}{n} \sum_{t=1}^{n} K_{n,t}(u) D_{n,t}(u)' s_{n,t}(D_{n,t}(u) \alpha) \in \mathbb{R}^{(m+1)d_\theta},
$$

$$
H_n(\alpha|u) = \frac{1}{n} \sum_{t=1}^{n} K_{n,t}(u) D_{n,t}(u)' h_{n,t}(D_{n,t}(u) \alpha) D_{n,t}(u) \in \mathbb{R}^{(m+1)d_\theta \times (m+1)d_\theta}.
$$

It is easily checked that $\alpha_0 := U_n \beta_0$, where $\beta_0 = (\theta(u)', \theta^{(1)}(u)', \ldots, \theta^{(m)}(u)')'$, belongs to the interior of $\mathcal{A}$ for all $n$ large enough due to Assumption 4(ii) below in conjunction with Assumption 2 and, due to the consistency result, so will $\hat{\alpha}$ w.p.a. 1. Thus, $\hat{\alpha}$ will satisfy the first-order condition of (5) which combined with the mean-value theorem yields

$$
0 = S_n(\hat{\alpha}|u) = S_n(\alpha_0|u) + H_n(\hat{\alpha}|u)(\hat{\alpha} - \alpha_0),
$$

where $\bar{\alpha}$ is situated on the line segment connecting $\hat{\alpha}$ and $\alpha_0$. We then decompose the score function into a bias and variance component, $S_n(\alpha_0|u) = B_n(u) + S_n(u)$, where

$$
B_n(u) = \frac{1}{n} \sum_{t=1}^{n} K_{n,t}(u) D_{n,t}(u)' b_{n,t}, S_n(u) = \frac{1}{n} \sum_{t=1}^{n} K_{n,t}(u) D_{n,t}(u)' s_{n,t},
$$

$s_{n,t} = s_{n,t}(\theta(t/n))$, and $b_{n,t} = s_{n,t}(\theta^*_u(t/n)) - s_{n,t}(\theta(t/n))$ with $\theta^*_u(t/n)$ defined in eq. (1). This decomposition is different from the one usually employed in the analysis of kernel estimators of time-varying coefficients where $s_{n,t}(\theta(t/n))$ is replaced by the stationary version of the score function.
evaluated at $\theta (u)$, $s_t^* (\theta (u) | u)$; see, e.g., Dahlhaus et al. (2019) and Dahlhaus and Subba Rao (2006). The usual choice has as consequence that the corresponding bias term in their case generally involves the time derivative process of the score function and so the resulting analysis tends to impose stronger regularity conditions on the model being estimated. By instead centering the analysis around $s_{n,t}$, we can obtain the leading term of the bias $B_n (u)$ through a standard Taylor expansion w.r.t. $\theta$,

$$b_{n,t} \equiv h_{n,t} (\theta_u^* (t/n)) \{\theta_{u}^* (t/n) - \theta (t/n)\} \cong - h_{n,t} (\theta (u)) \frac{\theta^{(m+1)} (u)}{(m+1)!} \{t/n - u\}^{m+1}. \tag{8}$$

Thus, our approach allows for a simpler derivation of the leading bias and variance terms under the following weak regularity conditions where here and in the following we write $N (u, \epsilon) := \{\theta \in \Theta : \|\theta - \theta (u)\| < \epsilon\}$ for some arbitrarily small $\epsilon > 0$. Furthermore, throughout $p$ and $q$ have to satisfy $p \geq 1$ and $q > 0$, but can otherwise vary depending on the particular application.

**Assumption 4** (i) $\ell_{n,t} (\theta)$ is twice continuously differentiable; (ii) $\theta (u)$ lies in the interior of $\Theta$ and is $m + 1$ times continuously differentiable.

**Assumption 5** (i) $s_{n,t}$ is a martingale difference (MGD) array w.r.t. $\mathcal{F}_{n,t} = \mathcal{F} \{Z_{n,t}, Z_{n,t-1}, \ldots\}$; (ii) $\omega_{n,t} (\theta) = s_{n,t} (\theta) s_{n,t} (\theta)'$ is ULS($p,q,N (u, \epsilon)$) with $\omega_t^* (\theta | u)$ being $L_1$-continuous at $\theta = \theta (u)$.

**Assumption 6** $h_{n,t} (\theta)$ is ULS($p,q,N (u, \epsilon)$) with continuous stationary approximation $h_t^* (\theta (u))$ and $H (u) \equiv \mathbb{E} [h_t^* (\theta (u) | u)]$ is non-singular.

Assumption 5 is non-standard compared to the existing literature (as discussed above) and allows us to apply a martingale central limit theorem for locally stationary sequences (see Lemma 3(iii) in Appendix A.1) to $S_n (u)$. The MGD assumption amounts to assuming that the time-varying model is correctly specified and has to be verified on a case-by-case basis. Finally, Assumption 6 together with the expansion in eq. (8) is used to derive the limits of $B_n (u)$ and $H_n (\hat{\alpha} | u)$,

$$\sqrt{nb} S_n (u) \rightarrow^d N \left(0, \mathbb{K}_2 \otimes \Omega (u)\right), \quad \Omega (u) = \mathbb{E} [\omega_t^* (\theta (u) | u)], \tag{9}$$

$$H_n (\hat{\alpha} | u) \rightarrow^p \mathbb{K}_1 \otimes H (u), \quad B_n (u) = b^{m+1} \mu_1 \otimes H (u) \frac{\theta^{(m+1)} (u)}{(m+1)!} + o_P (b^{m+1}), \tag{10}$$

where $\mu_i = \int K (v) v^{m+i} D (v) dv$ and $\mathbb{K}_i = \int K^i (v) D (v) D (v)' dv$, $i \geq 1$. Combining these limit results, we obtain:

**Theorem 3** Suppose that Assumptions 1-6 hold. Then, as $b \rightarrow 0$ and $nb \rightarrow \infty$,

$$\sqrt{nb} U_n \left\{\hat{\beta} - \beta_0 - R_n \text{Bias} (u)\right\} \rightarrow^d N \left(0, \mathbb{K}_1^{-1} \mathbb{K}_2 \mathbb{K}_1^{-1} \otimes V (u)\right),$$

where $R_n = \text{diag} \{b^{m+1}, b^m, \ldots, b\} \otimes I_{d_\theta}$, $V (u) = H (u)^{-1} \Omega (u) H (u)^{-1}$ and $\text{Bias} (u) = \mathbb{K}_1^{-1} \mu_1 \otimes \frac{\theta^{(m+1)} (u)}{(m+1)!}$. 

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In particular, for \( i = 0, 1, \ldots, m, \)
\[
\sqrt{nb^{2i+1}} \left\{ \dot{\theta}^{(i)}(u) - \theta^{(i)}(u) - b^{m+1-i} \text{Bias}_i(u) \right\} \xrightarrow{d} N(0, \kappa_{2,i}V(u)),
\]
(11)
where \( \text{Bias}_i(u) = \kappa_{1,i} \frac{\theta^{(m+1)}(u)}{(m+1)!} \) and \( \kappa_{1,i} \) and \( \kappa_{2,i} \) denote the \( i \)th element of \( K_1^{-1} \mu_1 \) and \( (i, i) \)th element of \( K_1^{-1}K_2K_1^{-1} \), respectively.

Similar to existing results for local polynomial estimators in a cross-sectional setting, the leading bias term in (11) only depends on \( \theta^{(m+1)}(u) \) and so the estimators adapt to the curvature of \( \theta(u) \). The asymptotic variance in Theorem 3 can be estimated using plug-in methods: It follows from the proof of Theorem 3 that
\[
\hat{W}(u) = \frac{1}{n} \sum_{t=1}^{n} K_{n,t}^2(u) D_{n,t}(u) s_{n,t}(D_{n,t}(u) \hat{\alpha}) s_{n,t}(D_{n,t}(u) \hat{\alpha})' D_{n,t}(u)
\]
satisfies \( \hat{W}(u) \xrightarrow{p} \mathbb{K}_2 \otimes \Omega(u) \) while \( H_n(\hat{\alpha}|u) \xrightarrow{p} \mathbb{K}_1 \otimes H(u) \).

Comparing the above limit results and the conditions under which they are derived with the corresponding ones found in Dahlhaus et al. (2019) and the references therein, we note that our bandwidth restrictions are much weaker than theirs. In particular, standard bandwidth selection rules can be employed here but not in their set-up. Moreover, the existing literature requires so-called time derivatives of the stationary score function to exist and be well-behaved with these entering the bias expressions. Our conditions and results, on the other hand, do not require these and are analogous to the ones found in the literature on local polynomial likelihood estimators; see, e.g., Theorem 1b of Fan et al. (1995).

Equation (11) holds for any value of \( m \geq 0 \) and \( i = 0, \ldots, m \). However, when \( m - i \) is even, \( \kappa_{1,i} = 0 \) since all odd moments of \( K \) are zero due to the symmetry assumption. For example, for the local constant estimator \( (m = i = 0) \), Theorem 3 only informs us that the bias component of \( \hat{\theta}(u) \) is \( o_p(b) \) which is not a sharp rate. To obtain the leading bias term in this case, a higher-order expansion of \( b_{n,t} \) in eq. (7) is necessary. This expansion requires additional assumptions involving aforementioned time derivatives and standard derivatives w.r.t. \( \theta \) of \( h_1^*(\theta(u)|u) \). To present these, we need the following additional concept:

**Definition 3** A stationary process \( W_t^*(\theta|u) \) is said to be \( L_p \)-differentiable w.r.t. \( u \) if there exists a stationary and ergodic process \( \partial_u W_t^*(\theta|u) \) with
\[
\mathbb{E} [\|\partial_u W_t^*(\theta|u)\|^p] < \infty
\]
\[
\mathbb{E} [\|W_t^*(\theta|u + \Delta) - W_t^*(\theta|u) - \partial_u W_t^*(\theta|u) \Delta\|^p]^{1/p} = o(\Delta), \; \Delta \to 0.
\]

Our definition of time differentiability is slightly weaker compared to the one found in Dahlhaus et al. (2019) and other papers where differentiability w.r.t. \( u \) has to hold almost surely. With this definition in hand, we are ready to introduce the following additional regularity conditions in order to derive the leading bias term of the local constant estimator:
Assumption 7 $h_t^r (\theta | u)$ is $L_1$-differentiable w.r.t. $u$ at $(\theta (u), u)$ with time-derivative $\partial_u h_t^r (\theta (u) | u) \in \mathbb{R}^{d_q \times d_\theta}$.

Assumption 8 $\partial h_{n,t} (\theta) / \partial \theta_i$ exists and is $ULS(1,q,\mathcal{N}(\theta(u), \epsilon))$ with $L_1$-continuous stationary approximation $\partial h_t^* (\theta | u) / \partial \theta_i$, $i = 1,\ldots,d_\theta$.

Assumption 9 $\sum_{s=1}^{\infty} \left| \text{ Cov} \left( h_{ij,t}^*, (\theta (u) | u), h_{ij,t+s}^* (\theta (u) | u) \right) \right| < \infty$, $i, j = 1,\ldots,d_\theta$.

The time-derivative $\partial_u h_t^r (\theta (u) | u)$ will generally involve time-derivatives of the underlying stationary approximation of data: If $h_t^r (\theta (u) | u) = h (Z_t^r (u) ; \theta)$ is differentiable w.r.t. $Z_t^r (u)$ then it takes the form $\partial_u h_t^r (\theta | u) = \frac{\partial h (Z_t^r (u) ; \theta)}{\partial \theta} \partial_u Z_t^r (u)$, where $\partial_u Z_t^r (u)$ is the time derivative of $Z_t^r (u)$. The short memory condition imposed in Assumption 9 is used to control the variance component of the first-order bias term derived in Theorem 3. A sufficient condition for this assumption to hold is that $h_{ij,t}^r (\theta (u) | u)$ is a so-called geometric moment contraction (see Appendix A.1 for the definition) and $\mathbb{E} \left[ \left| h_{ij,t}^r (\theta (u) | u) \right|^2 \right] < \infty$, c.f. Proposition 2 in Wu and Shao (2004). This property holds again holds if $h (Z_t^r (u) ; \theta)$ is differentiable w.r.t. $Z_t^r (u)$ and $Z_t^r (u)$ itself is geometric moment contraction, c.f. Lemma 2 in Appendix A.1.

Under the above additional assumptions, we obtain the following higher-order expansion of the bias component which provides us with the leading bias term when $m-i$ is even:

**Theorem 4** Suppose Assumptions 1-9 hold and $\theta (\cdot)$ is $m+2$ times continuously differentiable. Then, as $b \to 0$ and $nb \to \infty$, the bias $B_n (u)$ defined in (7) satisfies

$$B_n (u) = b^{m+2} \left[ B_1 (u) + o_p (1) \right] + b^{2m+2} \left[ B_2 (u) + o_p (1) \right]$$

$$+ b^{m+1} \left[ \mu_1 H (u) \theta^{(m+1)} (u) \frac{1}{(m+1)!} + O_P \left( 1/n^{\min (1,q)} \right) + o_p \left( \frac{1}{\sqrt{nb}} \right) \right], \quad (12)$$

where $q$ given in Assumption 8 and, with $\partial_u H (u) = \mathbb{E} \left[ \partial_u h_t^\prime (\theta (u) | u) \right]$ and $\partial_\theta H (u) = \mathbb{E} \left[ \partial h_t^\prime (\theta (u) | u) / \partial \theta_i \right]$,

$$B_1 (u) = \mu_2 H (u) \theta^{(m+2)} (u) \frac{1}{(m+2)!} + \mu_2 \left[ \partial_u H (u) + \sum_{i=1}^{d_\theta} \frac{\theta_i^{(1)} (u) \theta_i H (u)}{(m+1)!} \right] \frac{\theta^{(m+1)} (u)}{(m+1)!},$$

$$B_2 (u) = - \frac{\mu_{m+2}}{2 \times (m+1)!} \sum_{i=1}^{d_\theta} \theta_i^{(m+1)} (u) \partial_\theta H (u) \theta^{(m+1)} (u).$$

As a special case, we obtain the following result for the local linear estimator:

**Corollary 1** The local constant estimator $\hat{\theta} (u)$ (corresponding to $m = 0$) satisfies, as $b \to 0$, $nb^3 \to \infty$ and $n^{\min (q,1)} b \to \infty$,

$$\sqrt{nb} \left\{ \hat{\theta} (u) - \theta (u) - b^2 \{ \text{Bias}_0 (u) + o_p (1) \} \right\} \to^d N \left( 0, \kappa_2 b V (u) \right),$$

where $\text{Bias}_0 (u) = B_1 (u) + B_2 (u)$ with $B_1 (u)$ and $B_2 (u)$ defined in Theorem 4.
To our knowledge this is the first complete characterization of the leading bias term of local constant estimators in general time-varying parameter models. Compared to existing results for specific models (see, e.g., Dahlhaus and Subba Rao, 2006), we see that our bias expression takes a different form and only involves the first-order time derivative process, \( \partial_u h^*_i (\theta (u) | u) \), while existing results involve higher-order derivatives. This is due to the aforementioned different proof techniques. One can show that our and their bias expressions are identical under their stronger regularity conditions.

Comparing Theorems 3 and 4, we see that the local linear and local constant estimators share the same convergence rate and asymptotic variance, but that the local constant estimator suffers from additional biases. This is consistent with the theory found for local constant and local linear estimators in a cross-sectional setting. However, compared with the theory in a cross-sectional setting (as in Fan et al., 1995), the bias in our framework takes a slightly different form. This is due to the fact that the data-generating process in our setting is non-stationary with the stationary approximation generating additional biases. Similar to the results found in a cross-sectional regression context, c.f. Fan (1993), we expect the additional biases of the local constant estimator to translate into reduced precision and efficiency compared to the local linear one.

Moreover, as is well-known, local linear estimators have the advantage of exhibiting automatic boundary carpentering. This property also holds in our setting where the boundaries are \( u = 0 \) and \( u = 1 \). Since the results for \( u = 1 \) are similar, we here only analyze the properties of the local constant (\( m = 0 \)) and the local linear (\( m = 1 \)) estimators at \( u = cb \) for some constant \( c > 0 \). Combining the intermediate bias–variance analysis carried out in the proofs of Theorem 3 and 4 with the arguments of Fan et al. (1995), we find that the two estimators remain asymptotically normally distributed but their asymptotic biases and variances take different forms:

**Corollary 2** Let \( \hat{\theta}_0 (u) \) and \( \hat{\theta}_1 (u) \) be the local constant and local linear estimators, respectively, of \( \theta (u) \). Under Assumptions 1-6, as \( b \to 0 \), \( nb^3 \to \infty \) and \( n^{\min \{ q, 1 \}} b \to \infty \), and for \( m \in \{ 0, 1 \} \),

\[
\sqrt{nb} \left( \hat{\theta}_m (cb) - \theta (cb) - b^{m+1} \text{Bias}_m \right) \overset{d}{\to} N (0, a_m V (0_+)),
\]

where \( V (0_+) = \lim_{u \downarrow 0} V (u) \) and

\[
\text{Bias}_0 = \left( \kappa_{1,0}^c \right)^{-1} \kappa_{1,1}^c \theta' (0_+); \quad \text{Bias}_1 = \frac{1}{2} \left( \kappa_{1,2}^c \right)^2 \kappa_{1,0}^c \kappa_{1,1}^c \theta'' (0_+); \quad \text{Bias}_2 = \frac{1}{2} \left( \kappa_{1,2}^c \right)^2 \kappa_{2,0}^c \kappa_{1,1}^c \theta'' (0_+);
\]

\[
a_0 = \frac{\kappa_{2,0}^c}{\kappa_{1,0}^c} \kappa_{1,1}^c ; \quad a_1 = \frac{\left( \kappa_{1,2}^c \right)^2 \kappa_{2,0}^c \kappa_{1,1}^c \kappa_{2,1}^c + \left( \kappa_{1,1}^c \right)^2 \kappa_{2,2}^c}{\left( \kappa_{1,0}^c \kappa_{1,2}^c \right)^2 - \left( \kappa_{1,1}^c \right)^2}.
\]

At the boundary, both the biases and variances of the local constant and local linear estimators are now different. While the difference between two asymptotic variances is a constant scale, compare \( a_0 \) and \( a_1 \) above, the biases are now of a different order: The local linear estimator still enjoys a bias of order \( O (b^2) \) while the bias of the local linear estimator blows up and becomes of order \( O (b) \). We refer to Fan et al. (1995) for the precise expressions of the normalizing constants \( \kappa_{i,j}^c \),
\(i = 1, 2\) and \(j = 0, 1, 2\). Thus, the local constant estimator will generally suffer from significantly larger biases at the boundary compared to the local linear one.

4 Examples

To demonstrate the usefulness of our general results, we here apply our theory to some particular models. For simplicity, we focus on the local constant and local linear versions of the estimators, but all results are easily generalized to the general polynomial version. Moreover, in each example we rule out the presence of exogenous predictors \(X_{n,t-1}\); all results are easily extended to allow for this under the high–level condition that \(X_{n,t-1}\) is locally stationary. Throughout this section the following assumptions are maintained where \(\theta (u)\) and \(\Theta\) are specified in each of the following examples: Assumption 1, holds, \(\theta (u) \in \text{Int} (\Theta)\) and \(\theta (\cdot)\) is twice continuously differentiable.

We first consider the time–varying TAR model in Example 1 where we assume that the errors are i.i.d. Under this assumption, one can show that the time–derivative of the locally stationary version takes the form

\[
\partial_u Y^*_t (u) = \left[ \theta_1 (u) 1 \left( Y^*_{t-1} (u) > 0 \right) + \theta_2 (u) 1 \left( Y^*_{t-1} (u) < 0 \right) \right] \partial_u Y^*_{t-1} (u) + \theta'_1 (u) Y^*_{t-1} (u) + \theta'_2 (u) Y^*_{t-1} (u).
\]

**Corollary 3 (Example 1)** For the tv-TAR(1) model given by (1) assume that

\[
\sup_{u \in [0,1]} \max \{ |\theta_1 (u)|, |\theta_2 (u)| \} < 1 \tag{13}
\]

and \(\mathbb{E} [\varepsilon_t^2] < \infty\). Then the local linear estimator satisfies Theorem 3 with

\[
H (u) = \mathbb{E} \left[ \left\{ \text{diag} \left( Y_t^{*+} (u), Y_t^{*-} (u) \right) \right\}^2 \right], \quad \Omega (u) = \mathbb{E} [\varepsilon_t^2] \: H (u).
\]

If in addition \(\mathbb{E} [\varepsilon_t^4] < \infty\), then the local constant estimator satisfies Theorem 4 with \(\partial_u H (u) = 0\), \(\partial_u H (u) = 2 \mathbb{E} \left[ \text{diag} \left( Y_t^{*+} (u), Y_t^{*-} (u) \right) \right] \: \partial_u Y_t^* (u)\) and \(\partial_u Y_t^* (u)\) given above.

This appears to be the first result for the threshold AR model in the literature. Condition (13) ensures that the model indeed has a locally stationary solution. Instead of the widely used geometrically mixing condition, which requires that \(\varepsilon_t\) has a continuous density, we employ the geometric-moment contraction condition for the theory for the local constant estimator to apply, c.f. Assumption 9 and its discussion; for the same reason, only the additional moment condition is imposed on the error term for the local constant estimator in the remaining examples below.

Next, we consider the following \(d\)-dimensional tv-VAR(q) model,

\[
Y_{n,t} = \sum_{i=1}^{q} \Phi_i (t/n) Y_{n,t-i} + \Sigma (t/n) \varepsilon_t = X_{n,t} \theta (t/n) + \Sigma (t/n) \varepsilon_t, \tag{14}
\]
where \( \varepsilon_t \in \mathbb{R}^d \) is i.i.d. with \( \mathbb{E}[\varepsilon_t] = 0 \) and \( \mathbb{E}[\varepsilon_t \varepsilon_t'] = I_d, \Phi_i(\cdot) \in \mathbb{R}^{d \times d}, i = 1, \ldots, p, \) and \( \Sigma(\cdot) \in \mathbb{R}^{d \times d}. \) We collect the VAR coefficients in

\[
\theta(u) = \left(\text{vec}' \left(\Phi_1(u)\right), \ldots, \text{vec}' \left(\Phi_p(u)\right)\right)' \in \Theta = \mathbb{R}^{d^2},
\]

and the regressors in \( X_{n,t} = (Y_{n,t-1}'', \ldots, Y_{n,t-q}'')' \otimes I_d. \) Under regularity conditions, its stationary approximation is given by \( Y^*_t(u) = \theta(u) X^*_t(u) + \Sigma(\varepsilon_t), \) where \( X^*_t(u) = (Y^*_t(u), \ldots, Y^*_{t-q}(u))' \otimes I_d, \) and its derivative process takes the form

\[
\partial_u Y^*_t(u) = \theta(u) \partial_u X^*_t(u) + \theta^{(1)}(u) X^*_{t-1}(u) + \Sigma^{(1)}(u) \varepsilon_t
\]

where \( \partial_u X^*_t(u) = (\partial_u Y^*_{t-1}(u)', \ldots, \partial_u Y^*_{t-q}(u)')' \otimes I_d. \) We estimate \( \theta(u) \) using the least-squares criterion, \( \ell_{n,t}(\theta) = \|Y_{n,t} - X_{n,t} \theta\|^2. \) Applying our asymptotic theory, we obtain the following novel result for the estimation of this time-varying VAR\( (p) \) model:

**Corollary 4** Suppose that \( Y_{n,t} \) satisfies (14) with \( \Sigma(\cdot) \) being twice continuously differentiable and \( \Phi(u) = I_d - \sum_{i=1}^p \Phi_i(u) \varepsilon^i \) having all its eigenvalues outside the unit circle, \( u \in [0,1]. \) Then the local linear estimators satisfy Theorem 3 with

\[
H(u) = \mathbb{E} \left[ X^*_t(u) X^*_t(u)' \right], \quad \Omega(u) = \mathbb{E} \left[ X^*_t(u) \Sigma(\varepsilon_t) \Sigma(\varepsilon_t)' X^*_t(u)' \right].
\]

If in addition \( \mathbb{E} \left[ \|\varepsilon_t\|^4 \right] < \infty, \) then the constant local estimators satisfy Theorem 4 with \( \partial_u H(u) = 0, \) and \( \partial_u H(u) = 2\mathbb{E} \left[ X^*_t(u) \partial_u X^*_t(u)' \right]. \)

Suppose \( Y_{n,t} \in \mathbb{R}^d \) solves the following tv-ARCH\( (q) \) model,

\[
Y_{n,t} = \lambda_{n,t} \varepsilon^2_t, \quad \lambda_{n,t} = \theta(t/n)' X_{n,t}, \quad (15)
\]

where \( \varepsilon_t \) is i.i.d. with \( \mathbb{E}[\varepsilon_t^2] = 1, X_{n,t} = (1, Y_{n,t-1}', \ldots, Y_{n,t-q}')' \) and \( \theta(u) = (\omega(u), \alpha_1(u), \ldots, \alpha_q(u))' \in \mathbb{R}^{q+1}. \) We estimate the time-varying parameters using the Gaussian quasi-loglikelihood, \( \ell_{n,t}(\theta) = -\log(\lambda_{n,t}(\theta)) - Y_{n,t}' \lambda_{n,t}(\theta) / \lambda_{n,t}(\theta), \) where \( \lambda_{n,t}(\theta) = \theta' X_{n,t}, \) with \( \Theta = \left\{ \theta \in [\delta_L, \delta_U]^{q+1}, \sum_{i=1}^q \alpha_i \leq 1 - \delta_\alpha \right\} \) for some \( 0 < \delta_L < \delta_U < \infty \) and \( \delta_\alpha > 0. \) Here, the stationary version and its time–derivative is given by \( Y^*_t(u) = \lambda^*_t(u) \varepsilon^2_t \) and \( \partial_u Y^*_t(u) = \partial_u \lambda^*_t(u) \varepsilon^2_t, \) respectively, where \( \lambda^*_t(u) = (u)^' X^*_t(u) \) and

\[
\partial_u \lambda^*_t(u) = \theta(u)' \partial_u X^*_t(u) + \theta^{(1)}(u)' X^*_t(u),
\]

with \( \partial_u X^*_t(u) = (0, \partial_u Y^*_{t-1}(u), \ldots, \partial_u Y^*_{t-q}(u))'. \)

**Corollary 5** Assume that \( \mathbb{E}[\varepsilon_t^4] < \infty \) and \( \sum_{i=1}^q \alpha_i(u) < 1 \) in the tv-ARCH\( (q) \) model (15). Then the local linear estimators satisfy Theorem 3 with

\[
H(u) = -\mathbb{E} \left[ \frac{\partial_u \lambda^*_t(u) (\partial_u \lambda^*_t(u)')'}{\lambda^*_t(u)^2} \right], \quad \Omega(u) = -\text{Var}(\varepsilon^2_t) \cdot H(u).
\]

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If in addition $\mathbb{E} [\| \varepsilon \|^{4+\delta}] < \infty$, for some $\delta > 0$, then the local constant estimators satisfy $\mathcal{L}$ with

$$
\partial_u H (u) = 2 \mathbb{E} \left[ \frac{\partial_u \lambda^* (u) (\partial_u \lambda^* (u))'}{\lambda^*_t (u)^3} + \frac{\partial^2 \lambda^* (u) (\partial_u \lambda^* (u))'}{\lambda^*_t (u)^2} \right],
$$

$$
\partial_\theta H (u) = 2 \mathbb{E} \left[ \frac{\partial_\theta \lambda^* (u) (\partial_u \lambda^* (u))'}{\lambda^*_t (u)^3} \right].
$$

Our conditions are substantially weaker compared to Dahlhaus and Subba Rao (2006) who require $\mathbb{E} [\varepsilon_t^{12}]^{1/6} \sum_{j=1}^q \alpha_j (u) < 1 - \rho$ which rules out most empirically relevant situations. For example, if $\varepsilon_t \sim N (0, 1)$ then their requirement becomes $\sum_{j=1}^q \alpha_j (u) < 0.22$. This strong condition is a by-product of their proof strategy which requires mixing and stronger moment conditions of the derivative process. Furthermore, their bias component for the local constant estimator involves the so-called second-order derivative process while ours only involves the first-order derivative.

As our final example, consider the following Poisson autoregression, c.f. Example 2,

$$
Y_{n,t} | \mathcal{F}_{n,t-1} \sim \text{Poisson} (\lambda_{n,t}), \quad \lambda_{n,t} = \theta (t/n)' X_{n,t},
$$

where Poisson($\lambda$) denotes a Poisson distribution with intensity parameter $\lambda$, and $X_{n,t} = (1, Y_{n,t-1}, \ldots, Y_{n,t-q})'$ and $\theta (t/n) = (\omega (t/n), \alpha_1 (t/n), \ldots, \alpha_q (t/n))' \in \mathbb{R}^{q+1}$. For some $0 < \delta_L < \delta_U \leq \infty$ and $\delta_\alpha > 0$, let $\Theta = \{ \theta \in [\delta_L, \delta_U]^{q+1} | \sum_{i=1}^q \alpha_i \leq 1 - \delta_\alpha \}$. We use the log-likelihood for estimation,

$$
\ell_{n,t} (\theta) = Y_{n,t} \log (\lambda_{n,t} (\theta)) - \lambda_{n,t} (\theta).
$$

**Corollary 6** Assume that $\sum_{i=1}^\infty \alpha_i (u) < 1$. Then the local constant and local linear estimators of $\theta (u)$ in (16) satisfy Theorem 3 with

$$
\Omega (u) = \mathbb{E} \left[ \frac{(\partial_\theta \lambda^* (u)) (\partial_u \lambda^* (u))'}{\lambda^*_t (u)} \right] = - H (u),
$$

where $\lambda^*_t (u) = \theta (u)' X^*_t (u)$ and $\partial_\theta \lambda^*_t (u) = X^*_t (u)$.

Note here that the derivative process of $Y_{n,t}$ is not well-defined due to it being discrete-valued and so existing results, such as the ones in Dahlhaus et al. (2019), cannot be used to analyze the estimators. We are here able to provide a complete theory for the local linear estimator. However, for the local constant estimator we are only able show that its leading bias is of order $o (b)$; an expression of this bias is not available to us.

## 5 Simulation study

In this section, we examine the finite-sample performances of the local constant and local linear estimators. All reported results are based on 1000 simulated data sets. The over–all performance of the estimators is evaluated using the mean absolute deviation error (MADE), $MADE := \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ | \hat{\theta}_i (t/n) - \theta_i (t/n) | \right]$, as well as their integrated bias, variance, and mean squared er-
Table 1: Performance of the local constant (LC) and local linear (LL) estimators for tvARCH model: Integrated squared bias (IBias2), integrated variance (IVar), integrated mean squared errors (IMSE), and MADE.

| n   | $\omega(u)$ | $\alpha(u)$ |
|-----|-------------|-------------|
|     | WLS LC      | WLS LC      |
| 250 | 0.0212      | 0.0217      |
|     | 0.0222      | 0.0279      |
|     | 0.0202      | 0.0178      |
|     | 0.0204      | 0.0213      |
|     | 0.0091      | 0.0076      |
|     | 0.0060      | 0.0038      |
| 500 | 0.0071      | 0.0017      |
|     | 0.0018      | 0.0002      |
|     | 0.0028      | 0.0003      |
| 1000| 0.0067      | 0.0045      |
|     | 0.0045      | 0.0043      |
|     | 0.0060      | 0.0072      |
|     | 0.0038      | 0.0049      |
|     | 0.0022      | 0.0017      |
|     | 0.0024      | 0.0016      |
|     | 0.0039      | 0.0016      |
|     | 0.0072      | 0.0007      |

ror. All results are based on the Epanechnikov kernel and with the bandwidth chosen using the cross-validation method proposed in Richter and Dahlhaus (2019).

5.1 Time-varying ARCH

We first consider the time-varying ARCH(1) in eq. (15) where $\varepsilon \sim i.i.d.N(0,1)$, $\omega(u) = 0.7 - 0.5 \sin (4\pi u)$ and $\alpha(u) = 0.45 + 0.4 \sin (4\pi u)$. We estimate $\omega(u)$ and $\alpha(u)$ using both Gaussian log-likelihood and the WLS method of Fryzlewicz et al. (2008). Table 1 reports the performance of the estimators. For all sample sizes, the local MLE’s perform better than the local WLS estimators. For sample sizes of $n = 250$ and $n = 500$, the local constant MLE performs as well as the local linear MLE in terms of the global measures, but the latter performs best in terms of IMSE and MADE for $n = 1000$. Thus, the over–all superiority of the local linear estimator indicated by the theory appears to be a large–sample property.

To compare the performance of the estimators near the end of the sample, we also evaluate the bias of the estimators for the first and last 2.5% of time periods corresponding to $u \in [0, 0.025] \cup [0.975, 1]$. This is reported as IBias2BD in Table 1. As predicted by the theory, we find that relative to the local constant versions the local linear WLS and ML estimators enjoy significantly smaller biases near the boundaries of $[0, 1]$ for all sample sizes.
Table 2: Performance of the local linear estimators for tvPARX models: Integrated squared bias (IBias2), integrated variance (IVar), integrated mean squared errors (IMSE), and median of MADE

| n   | DGP1            | DGP2            |
|-----|-----------------|-----------------|
|     | ω(u)            | α(u)            | γ(u)            |
|     | 0.0330          | 0.0037          | 0.0004          | 0.0339          | 0.0035          | 0.0003          |
| 250 | IVar            | 0.1550          | 0.0078          | 0.0422          | 0.1409          | 0.0077          | 0.0383          |
|     | IMSE            | 0.1880          | 0.0115          | 0.0426          | 0.1748          | 0.0111          | 0.0386          |
|     | MADE            | 0.2407          | 0.0839          | 0.1463          | 0.2371          | 0.0832          | 0.1447          |
| 500 | IBias2          | 0.0111          | 0.0016          | 0.0002          | 0.0137          | 0.0016          | 0.0002          |
|     | IVar            | 0.0528          | 0.0039          | 0.0216          | 0.0648          | 0.0042          | 0.0208          |
|     | IMSE            | 0.0639          | 0.0055          | 0.0218          | 0.0784          | 0.0057          | 0.0209          |
|     | MADE            | 0.1645          | 0.0586          | 0.1080          | 0.1717          | 0.0593          | 0.1079          |
| 1000| IBias2          | 0.0044          | 0.0007          | 0.0001          | 0.0044          | 0.0008          | 0.0001          |
|     | IVar            | 0.0266          | 0.0023          | 0.0115          | 0.0286          | 0.0022          | 0.0111          |
|     | IMSE            | 0.0310          | 0.0030          | 0.0116          | 0.0330          | 0.0030          | 0.0112          |
|     | MADE            | 0.1216          | 0.0426          | 0.0798          | 0.1199          | 0.0428          | 0.0798          |

5.2 Time-varying Poisson autoregression with exogenous covariates (PARX)

We here report simulation results for the local linear MLE of the following PARX(1) model with an additional exogenous regressor $X_{n,t}$,

$$
\lambda_{n,t} = \omega \left(\frac{t}{n}\right) + \alpha \left(\frac{t}{n}\right) Y_{n,t-1} + \gamma \left(\frac{t}{n}\right) X_{n,t-1}^2,
$$

where $\omega (u) = 0.6-0.3u+0.3 \sin(2\pi u)$, $\alpha (u) = 0.3+0.3u-0.3 \sin(2\pi u)$ and $\gamma (u) = 1-0.5 \cos(\pi u)$.

The dynamics of the exogenous regressor was chosen as

$$
X_{n,t} = \sqrt{\sigma \left(\frac{t}{n}\right) + \beta \left(\frac{t}{n}\right) X_{n,t-1}^2} \epsilon_t, \quad \epsilon_t \sim \text{i.i.d.} \mathcal{N}(0, 1).
$$

where either

- DGP1: $\sigma (u) = 0.8$ and $\beta (u) = 0.4$ so that $X_{n,t} = X_t$ is strictly stationary.

- DGP2: $\sigma (u) = 0.8 + 0.4 \cos (2\pi u)$ and $\beta (u) = 0.4 - 0.2 \sin (2\pi u)$ so $X_{n,t}$ is locally stationary.

Table 2 reports the over-all performance of the estimators in terms of integrated squared bias, variance, MSE and MADE. The table shows that, in all sample sizes, the local linear estimator behaves well for both DGP1 and DGP2. All bias, variance, and MADE decreases as the sample size increases. Finally, similar to the case of the tvARCH model, the local linear estimator performs well near the boundaries; we leave out these results since they are similar to the ones reported for the tv-ARCH model above.
6 Empirical application

We here revisit the empirical analysis of US corporate defaults carried out in Agosto et al. (2016) with the aim of examining whether there is evidence of structural instability in the time series. The data set consists of monthly number of bankruptcies among Moody’s rated industrial firms in the United States for the period 1982–2011 (n = 360 observations), collected from Moody’s Credit Risk Calculator (CRC). Figure 1 shows the time series of default counts together with its sample autocorrelation function, which reveals high temporal dependence in default counts and existence of default clusters over time.

With \( Y_t \in \{0, 1, 2, \ldots \}, \ t \geq 1 \) denoting the number of defaults in a given month, we use a log-PARX model to gain better understanding of the dynamics of \( Y_t \) as a function of its own past, \( Y_{n,t-m}, \ m \geq 1 \), but also in terms of additional covariates \( X_{n,t} \in \mathbb{R}^d \), which include relevant macroeconomic and financial factors as considered in Agosto et al. (2016). We model \( Y_{n,t} \) as a conditional Poisson distribution with time-varying intensity \( \lambda_{n,t} \):

\[
Y_{n,t} | F_{n,t-1} \sim \text{Poisson} \left( \lambda_{n,t} \right), \quad t = 1, 2, \ldots, n.
\]  

(17)

The logarithm of \( \lambda_{n,t} \) is specified as a linear function of log-past counts as proposed in Fokianos and Tjøstheim (2011), which is here augmented by the set of exogenous variables, \( X_{n,t} \), while allowing for time-varying parameters:

\[
\log \lambda_{n,t} = \omega \left( t/n \right) + \sum_{m=1}^{p} \alpha_m \left( t/n \right) \log \left( Y_{n,t-m} + 1 \right) + X'_{n,t-1} \beta \left( t/n \right),
\]

(18)

We here deviate from Agosto et al. (2016) in the modelling of \( \lambda_{n,t} \) with this other paper specifying \( \lambda_{n,t} \) to be a linear function of past counts and factors. The reasons for us favouring the above log-specification over the linear one are three-fold: First, (18) do not impose positivity constraints on the parameters which facilitates the numerical computation of the estimators; second, it allows us to include any predictors we wish without the need of first transforming them to ensure that
each component of the resulting $X_{n,t-1}$ is positive; third, when we estimate both models using the US default data, we found that the log–specification delivers a better fit.

Similar to Agosto et al. (2016), we use $\sum_{m=1}^{p} \alpha_m (t/n)$ as a measure of contagion in the financial markets: If $\sum_{m=1}^{p} \alpha_m (t/n)$ is large then firms defaulting today will lead to a large increase in the risk of other firms defaulting next period everything else equal.

As exogenous covariates, we consider the same financial, credit market, and macroeconomic variables as in Agosto et al. (2016): Realized volatility ($RV$) computed using daily squared return on the S&P 500 index, the Leading Index released by the Federal Reserve ($LI$), year-to-year change in Industrial Production Index ($IP$), one-year return on the S&P 500 index ($SPX$), the three-month Treasury bill rate ($TB3$), and BAA Moody’s rated to 10–year Treasury spread ($SP$).\(^2\) Agosto et al. (2016) decompose $LI$ and $IP$ into their positive and negative parts to deal with above–mentioned issue of $X_{n,t-1}$ having to be positive in the linear specification. In contrast, no such transformations are needed for our log–specification. Moreover, as well as $RV$, we also consider the logarithm of $RV$, log ($RV$), to evaluate if the latter is a better predictor; again, this would not be possible in the linear specification of Agosto et al. (2016).

The lag length $p$ is chosen using BIC where, for a given model, the log–likelihood is evaluated at the estimated time-varying parameters. According to this version of BIC, the preferred specification is $p = 2$. Importantly, by allowing for the parameters to be time–varying, a much more parsimonious model is selected by BIC: If we do model selection where for each model we restrict the estimated parameters to be constant over time, the preferred model is $p = 6$. Moreover, the persistence, or contagion, of the estimated time–invariant version, as measured by $\sum_{m=1}^{p} \hat{\alpha}_m$, is substantially higher than for the time-varying version, as measured by $\sum_{m=1}^{p} \hat{\alpha}_m (u), u \in [0,1]$. This is consistent with the findings reported in Hillebrand (2005) for GARCH models: Neglecting structural changes in parameters causes the estimates of these to be suffering from a strong upward bias which in turn leads BIC to selecting a bigger model.

As benchmark, we first estimate the time–invariant version of (18) with $p = 2$. Table 3 shows the estimation results for four different specifications of the time–invariant LLPARX(2) model: Column (1) contains the results when only ($RV, LI$) are included; column (2) when only (log $RV, LI$) are included; column (3) when all covariates are included except for log ($RV$); and column (4) when all covariates except $RV$ are included. As in Agosto et al. (2016), once we control for the information contained in $RV$ and $LI$, none of the other four covariates are found to be relevant in predicting future defaults. We also observe that the two specifications using log ($RV$) appear to perform better than the ones using $RV$. Based on these results, our favoured specification of the time-varying version is to use $LI$ and log ($RV$) as exogenous variables:

$$
\log \lambda_{n,t} = \omega (t/n) + \alpha_1 (t/n) \log (Y_{n,t-1} + 1) + \alpha_2 (t/n) \log (Y_{n,t-2} + 1) + \beta_{RV} (t/n) \log (RV_{n,t-1}) + \beta_{LI} (t/n) LI_{n,t-1}.
$$

\(^2\)These covariates are tested for the existence of default covariates in Das et al. (2007), Duffie et al. (2007), and Lando and Nielsen (2010).
Table 3: Estimation results of different LLPARX models. Standard errors are in parentheses.

|      | (1)   | (2)   | (3)   | (4)   |
|------|-------|-------|-------|-------|
| \( \omega \) | 0.2186 | 1.0425 | 0.1003 | 1.3015 |
|       | (0.1022) | (0.2491) | (0.2118) | (0.4188) |
| \( \alpha_1 \) | 0.3040 | 0.2777 | 0.3093 | 0.2836 |
|       | (0.0492) | (0.0496) | (0.0497) | (0.0501) |
| \( \alpha_2 \) | 0.5033 | 0.4876 | 0.5082 | 0.4956 |
|       | (0.0491) | (0.0486) | (0.0499) | (0.0492) |
| \( RV \) | 5.8496 | 5.4864 |       |       |
|       | (3.9738) |       |       |       |
| \( \log (RV) \) | 0.1230 |       | 0.1484 |       |
|       | (0.0373) |       | (0.0437) |       |
| \( LI \) | -0.1767 | -0.1482 | -0.1864 | -0.1860 |
|       | (0.0310) | (0.0308) | (0.0490) | (0.0487) |
| \( IP \) |       |       | 0.0029 | -0.0167 |
|       |       |       | (0.0480) | (0.0481) |
| \( SPX \) |       | 0.2074 | 0.2412 |       |
|       |       | (0.2408) | (0.2413) |       |
| \( TB3 \) |       | 0.0065 | 0.0007 |       |
|       |       | (0.0134) | (0.0136) |       |
| \( SP \) |       | 0.0300 | -0.0426 |       |
|       |       | (0.0585) | (0.0615) |       |

Figure 2 shows the time–series of the local linear estimates of the time-varying parameters of (19) together with the time–invariant estimates reported in column (2) of Table 3. Pointwise confidence bands are computed based on the asymptotic distribution derived in Theorem 4. The Leading Index is pointwise significant for most of the sample period which highlights the link between macroeconomic activity and corporate defaults also found in Agosto et al. (2016). At the same time, this link exhibits substantial time–variation, in particular at the end of the sample during the Great Recession. The link between realized volatility and defaults of industrial firms is less significant but also appears to be changing over time. Similar to Agosto et al. (2016), the realized volatility and the Leading Index are strong explanatory variables during the Great Recession (2007–2011). However, differently, both of them are still relevant in the late 1980s and early 1990s. Especially, the effect of \( \log (RV) \) tends to be negative in this period which cannot be captured by the aforementioned linear version of the PARX model.

Finally, judging from \( \hat{\alpha}_1 (t/n) + \hat{\alpha}_2 (t/n) \), there is very little contagion in the financial markets from 1990s and onwards, which is somewhat consistent with the findings in Agosto et al. (2016). However, at the end of the sample, we actually find a negative effect of today’s \( \log \text{–defaults} \) on tomorrow’s default risk. Also note that the time–varying estimates, \( \hat{\alpha}_1 (t/n) \) and \( \hat{\alpha}_2 (t/n) \), remain below the corresponding time–invariant ones, \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \) as marked by the horizontal red lines, throughout the sample period. Again, this seems to indicate that ignoring time–variation in the parameters of PARX models lead to over estimation of the level of persistence/contagion.
To assess in–sample fit and whether the reported time–variation in the parameters is statistically significant, we carry out an array of graphical and quantitative diagnostic tools for time series. First, we plot in the left panel of Figure 3 the actual default counts together with the predicted defaults $\hat{Y}_{n,t} := \hat{\lambda}_{n,t} = \lambda_{n,t} \left( \hat{\theta} (t/n) \right)$. As can be seen from this plot, the time–varying LLPARX model captures the default counts dynamics well. In the right panel of Figure 3, the sample autocorrelation function of the standardized Pearson residuals $\hat{e}_{n,t} = \hat{\lambda}_{n,t}^{-1/2} \left( Y_{n,t} - \hat{\lambda}_{n,t} \right)$ is plotted. Under correct specification, $e_{n,t}$ should be white noise – the plotted sample autocorrelation function supports this.

We also evaluate the adequacy of fit using the probability integral transform (PIT). We follow
Davis and Liu (2016) and compute the PIT’s by

\[ \hat{u}_{n,t} := F_{n,t}(Y_{n,t} - 1) + \nu_t [F_{n,t}(Y_{n,t}) - F_{n,t}(Y_{n,t} - 1)], \]

where \( \{\nu_t\} \) is a sequence of i.i.d. random variables from a standard uniform distribution, and \( F_{n,t} \) is the CDF of a Poisson(\( \hat{\lambda}_{n,t} \)) distribution. Under correct model specification, \( \hat{u}_{n,t} \) is a sequence of i.i.d. random variables from the standard uniform distribution. Figure 4 depicts the histogram of PIT which show that the tvLLPARX(2) model provides a better in–sample fit than the corresponding time–invariant model.

Finally, in Table 4, we report the log-likelihood, AIC and BIC values and the \( p \)-value from a Kolmogorov-Smirnov test of the PIT’s being uniformly distributed of each of four specifications in columns (1)–(4) of Table 3 together with the time-varying versions of (2) and (4), labelled tv (2) and tv(4), respectively. From these, we see that allowing for time–varying parameters increase the in–sample fit dramatically. While this is not a formal statistical test of time–variation, it provides strong informal evidence of such. As mentioned earlier, we also see that specification (2) is favoured over (1), (3) and (4) in the time–invariant case, and that (2) is favoured over (4) in the time–varying case.

We complete the analysis by conducting a final sensitivity analysis of (19). This is done by including the remaining exogenous covariates in addition to realized volatility and Leading Index...
in the time–varying version of the model:

$$\log \lambda_{n,t} = \omega (t/n) + \alpha_1 (t/n) \log (Y_{n,t-1} + 1) + \alpha_2 (t/n) \log (Y_{n,t-2} + 1)$$

$$+ \beta_{RV} (t/n) \log (RV_{n,t-1}) + \beta_{LI} (t/n) LI_{n,t-1} + \beta_{IP} (t/n) IP_{n,t-1}$$

$$+ \beta_{SP} (t/n) SPX_{n,t-1} + \beta_{TB3} (t/n) TB3_{n,t-1} + \beta_{SP} (t/n) SP_{n,t-1}. \quad (20)$$

The estimation results are provided in Figure 5. The estimates, except for the realized volatility, are consistent with our baseline findings. The Leading Index remains highly significant, which shows that macroeconomic factors are relevant in predicting future defaults. The link between short–term interest rates and defaults of industrial firms changes over time. During the late 1980s and the Great Recession (2007–2011), interest rates play a role in determining the interest expense of firms. In the 1990s, similar to the finding in Duffie et al. (2007), the sign of the coefficient for the short–term rate is consistent with the fact that the US Federal Reserve often increases the short–term rates to control business expansions. Controlling for LI and TB3, other covariates are estimated to be insignificant for most of the sample period except for the period of the Great Recession (2007–2011), in which financial, credit market, and macroeconomic variables are significant explanators of the default intensity. These are novel findings that the original analysis of Agosto et al. (2016) were unable to uncover.
Figure 5: Local linear estimate of time-varying parameter in eq. (20): Shaded areas are the 95% confidence intervals.
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A Appendix

A.1 Auxiliary results

Let $Y_{n,t}$ solve

$$Y_{n,t} = G(Y_{n,t-1}, \varepsilon_t, \theta(t/n)), \ t = 1, \ldots, n,$$

where $G: \mathcal{Y} \times \mathcal{E} \times \Theta \mapsto \mathcal{Y}$ is some mapping, $\varepsilon_t \in \mathcal{E} \subseteq \mathbb{R}^d$ is a sequence of i.i.d. errors, and $\theta(\cdot) \in \Theta$. Importantly, the initial value $Y_{n,0}$ can be arbitrarily chosen. Under regularity conditions, its corresponding stationary approximation $Y^*_t(u)$ will solve

$$Y^*_t(u) = G(Y^*_{t-1}(u), \varepsilon_t; \theta(u)), \ u \in [0, 1].$$

We impose the following assumptions:

**Assumption 10** (i) $\sup_{\theta \in \Theta} \mathbb{E}[\|G(y_0, \varepsilon_t; \theta)\|^p] < \infty$ for some $y_0 \in \mathcal{Y}$ and $p > 0$; (ii) there exists $\rho < 1$ so that for all $y, y' \in \mathcal{Y}$,

$$\mathbb{E}\left[\|G(y, \varepsilon_t; \theta) - G(y', \varepsilon_t; \theta)\|^p\right]^{1/p} \leq \rho \|y - y'\|;$$

(iii) there exist $\hat{p} \geq 1$, $q > 0$ and $r \geq 0$ so that for all $\theta, \theta' \in \Theta$,

$$\mathbb{E}\left[\|G(y, \varepsilon_t; \theta) - G(y, \varepsilon_t; \theta')\|^\hat{p}\right]^{1/\hat{p}} \leq C (1 + \|y\|^r) \|\theta - \theta'\|^q$$

and (iv) $\mathbb{E}[\|Y_{n,0}\|^{\hat{p}}] < \infty$.

This assumption is similar to the one found in Dahlhaus et al. (2019), but we here allow for $p \neq \hat{p}$. Assuming we can verify $\mathbb{E}[\|Y^*_t(u)\|^{\hat{p}^r}] < \infty$ this allows us to show higher-order local stationarity ($\hat{p} > p$). Furthermore, part (iv) only requires suitable moments of $Y_{n,0}$ but otherwise the process can be initialized at any given value while the existing literature assumes $Y_{n,0} = Y^*_0(u)$.

**Lemma 1** Under Assumptions 10(i)-(ii), there exists a stationary and ergodic solution, $\{Y^*_t(u)\}$ to (21) which is geometric moment contracting with $\sup_{u \in [0,1]} \mathbb{E}[\|Y^*_t(u)\|^p] < \infty$. If furthermore 10(iii)-(iv) hold, $\sup_{u \in [0,1]} \mathbb{E}[\|Y^*_t(u)\|^{\hat{p}^r}] < \infty$ and $\theta(\cdot) \in \Theta$ is continuously differentiable, then $Y_{n,t}$ is LS($\hat{p}, q$) with $\sup_{n,t} \mathbb{E}[\|Y_{n,t}\|^{\hat{p}}] < \infty$.

**Proof.** The first part of the result follows from Proposition 4.4 in Dahlhaus et al. (2019). For the
Continuing the above two recursions yield the desired results.

Our nonlinear Markov models in eq. (21) is a special class of stationary processes of the form

\[ Y^*_t = H \ldots, \epsilon_{t-1}, \epsilon_t, \]

where \((\epsilon_t)_{t \in \mathbb{Z}}\) is a sequence of i.i.d. random variables and \(H\) is a measurable such that \(Y^*_t\) is well-defined. To verify Assumption 9 for the local constant estimator, we can apply the geometric-moment contraction (GMC) condition stated in Theorem 1. Let \((\tilde{\epsilon}_t)_{t \in \mathbb{Z}}\) be an i.i.d. copy of \((\epsilon_t)_{t \in \mathbb{Z}}\) and let \(\tilde{Y}^*_t = H \ldots, \tilde{\epsilon}_{-1}, \tilde{\epsilon}_0, \epsilon_1, \ldots, \epsilon_t\) be a coupled version of \(Y^*_t\). We say that \(Y^*_t\) is GMC(\(\alpha\)), \(\alpha > 0\), if there exist \(C > 0\) and \(0 < \rho = \rho(\alpha) < 1\) such that, for all \(t \in \mathbb{N}\),

\[ \mathbb{E} \left[ \left\| Y^*_t - Y^*_t \right\|^{\alpha + \rho} \right] \leq C \rho^t. \]

The GMC for \(Y^*_t\) is also valid for transformations \(g(Y^*_t)\) of \(Y^*_t\) with the following lemma:

**Lemma 2** If \(Y^*_t\) is GMC(\(\alpha\)) for some \(0 < \beta \leq 1\) with \(\mathbb{E} \left[ \left\| Y^*_t \right\|^{\alpha + \rho} \right] < \infty\) and a function \(g : \mathbb{R}^d \to \mathbb{R}^{d'}\) satisfies

\[ \sup_{y \neq y'} \frac{\|g(y) - g(y')\|}{\|y - y\|^{\beta} (1 + \|y\|^{p} + \|y\|^{p})} \leq C \tag{22} \]

for some \(C > 0\), then \(g(Y^*_t)\) is GMC(\(\alpha/\beta(p+1)\)).

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Proof. The proof is immediate from Hölder’s inequality (with \( p+1 \) and \( (p+1)/p \)) and Loi’s inequality:

\[
\mathbb{E} \left[ \| g(\tilde{Y}_t^*) - g(Y_t^*) \|^{\alpha/\beta(p+1)} \right] \\
\leq C^{\alpha/\beta(p+1)} \left[ \mathbb{E} \left( \| \tilde{Y}_t^* - Y_t^* \|^{\alpha} \right) \right]^{1/(p+1)} \left[ \mathbb{E} \left( 1 + \| \tilde{Y}_t^* \|^p + \| Y_t^* \|^p \right)^{\alpha/p\beta} \right]^{p/(p+1)} \\
\leq C^{\alpha/\beta(p+1)} \left[ \mathbb{E} \left( \| \tilde{Y}_t^* - Y_t^* \|^{\alpha} \right) \right]^{1/(p+1)} 3^{(\alpha-p\beta)/p\beta} \left( 1 + \mathbb{E} \| \tilde{Y}_t^* \|^{\alpha/\beta} + \mathbb{E} \| Y_t^* \|^{\alpha/\beta} \right)^{p/(p+1)}
\]

In addition, if \( Y_t^* \) is GMC(\( \alpha_0 \)) for some \( \alpha_0 > 0 \) with \( \mathbb{E} \| Y_t^* \|^p < \infty \) \((p > 0)\), then \( Y_t^* \) is GMC(\( \alpha \)) for all \( \alpha \in (0, p) \) (Wu and Shao (2004), Lemma 1).

We note from Proposition 2 in Wu and Shao (2004) that Assumption 9 is satisfied if \( h_{ij,t}^* (\theta(u)|u) \) is GMC(2) with \( \mathbb{E} \left[ \| h_{ij,t}^* (\theta(u)|u) \|^2 \right] < \infty \). Instead of showing GMC(2) of \( h_{ij,t}^* (\theta(u)|u) \) directly, it is possible to apply Lemma 1 in Wu and Shao (2004) by showing \( \mathbb{E} \left[ \| h_{ij,t}^* (\theta(u)|u) \|^{2+\gamma} \right] < \infty \) for some \( \gamma > 0 \) together with GMC(\( \alpha \)).

In the following, assume that \( L \) satisfies: (i) \( L(\cdot) \) has a compact support; (ii) for some \( \Lambda < \infty \), \(|L(v) - L(v')| \leq \Lambda |v - v'|\), \(v, v' \in \mathbb{R}\). We denote \( L_b(\cdot) := L(\cdot)/b \).

Lemma 3 The following hold as \( b \to 0 \) and \( nb \to \infty \):

(i) Suppose \( \{W_{n,t}(\theta)\} \) is ULS(\( p, q, \Theta \)) with its stationary approximation \( \{W_t^* (\theta|u)\} \) being \( L_p \) continuous for some \( p \geq 1, q > 0 \) and \( \Theta \) is compact. Then, with \( A \) defined in Assumption 2 and for any \( u \in (0, 1) \),

\[
\sup_{\alpha \in A} \left\| \frac{1}{n} \sum_{t=1}^{n} L_b(t/n - u) W_{n,t} (D_b(t/n - u) \alpha) - \int L(v) \mathbb{E} [W_t^* (D(v) \alpha|u)] dv \right\| = o_p(1). 
\]

(ii) Suppose \( \{W_{n,t}(\theta(t/n))\}, \{F_{n,t}\} \) is a MGD array; \( V_{n,t}(\theta) = W_{n,t}(\theta) W_{n,t}'(\theta) \) is ULS(\( p, q, \{\theta: \|\theta - \theta(u)\| < \epsilon\} \)) for some \( p \geq 1 \) and \( q, \epsilon > 0 \) with its stationary approximation \( V_t^* (\theta|u) \) being \( L_p \) continuous; and \( v \mapsto \theta(v) \) is continuous at \( v = u \). Then, for any \( u \in (0, 1) \),

\[
\sqrt{\frac{b}{n}} \sum_{t=1}^{n} L_b(t/n - u) W_{n,t}(\theta(t/n)) \to^d N \left( 0, \int L^2(v) dv \times \mathbb{E} [V_t^* (\theta(u)|u)] \right);
\]

\[
\sqrt{\frac{b}{n}} \sum_{t=1}^{n} L_b(t/n - cb) W_{n,t}(\theta(t/n)) \to^d N \left( 0, \int_{-c}^{+c} L^2(v) dv \times \mathbb{E} [V_t^* (\theta(u)|u)] \right).
\]

(iii) Suppose \( W_t^* \) is stationary and ergodic with \( \sum_{s=0}^{\infty} |\text{cov} (W_{t,s}^*, W_{t,s}^*)| < \infty \). Then, for any \( u \in (0, 1) \),

\[
\left| \frac{1}{n} \sum_{t=1}^{n} L_b(t/n - u) W_t^* - \int L(v) \mathbb{E} [W_t^*] \right| = o_p \left( \frac{1}{\sqrt{nb}} \right).
\]
Proof. Proof of (i). We first show that for all \( \theta \in \Theta, \)

\[
\frac{1}{n} \sum_{t=1}^{n} L_b \left( \frac{t}{n} - u \right) W_{n,t} (\theta) \to^p \int L(v) \, dv \times \mathbb{E} [W^*_t (\theta|u)].
\]

Note that \( L(v) = 0 \) for \(|v| \geq \bar{v} \) for some \( \bar{v} > 0 \). Then, Minkowski’s inequality implies that

\[
\mathbb{E} \left[ \left\| \frac{1}{n} \sum_{t=1}^{n} L_b \left( \frac{t}{n} - u \right) \{ W_{n,t} (\theta) - W^*_t (\theta|u) \} \right\|^p \right]^{1/p}
\]

\[
\leq \frac{1}{n} \sum_{t=1}^{n} |L_b \left( \frac{t}{n} - u \right)| \mathbb{E} \left[ \left| W_{n,t} (\theta) - W^*_t (\theta|u) \right|^p \right]^{1/p}
\]

\[
\leq \frac{C}{n} \sum_{t=1}^{n} |L_b \left( \frac{t}{n} - u \right)| \left( b^q \left| \frac{t}{n} - u \right|^q + 1/n^q + p^q \right)
\]

\[
\leq \frac{C}{n} \sum_{t=1}^{n} |L_b \left( \frac{t}{n} - u \right)| \times (b^q \bar{v}^q + 1/n^q + p^q) = O\left(b^q\right) + O\left(n^{-q}\right) + O\left(\frac{1}{\sqrt{nb}}\right),
\]

where we have used that

\[
\frac{1}{n} \sum_{t=1}^{n} |L_b \left( \frac{t}{n} - u \right)| \rho^{qt} \leq \frac{1}{\sqrt{nb}} \sqrt{\frac{1}{n} \sum_{t=1}^{n} (L^2)_b \left( \frac{t}{n} - u \right)} \sum_{t=1}^{n} n^2 \rho^{2qt} = O\left(\frac{1}{\sqrt{nb}}\right).
\]

Next, with \( \tilde{W}_t = W^*_t (\theta|u) - \mathbb{E} [W^*_t (\theta|u)], \)
\[
\frac{1}{n} \sum_{t=1}^{\bar{t}} L_b \left( \frac{t}{n} - u \right) \tilde{W}_t = \frac{1}{nb} \sum_{t=1}^{\bar{t}} L_b \left( t/n - u \right) \tilde{W}_t
\]

for sufficiently large \( n, \) where \( \bar{t} = \lfloor n (u + \bar{v}b) \rfloor \) and \( \bar{t} = \lfloor n (u - \bar{v}b) \rfloor \). Here, \([x]\) denotes the integer part of any real number \( x \). By summation by parts, we have, with \( S_{n,t} = \sum_{j=1}^{t} \tilde{W}_j, \)

\[
\frac{1}{n} \sum_{t=1}^{\bar{t}} L_b \left( \frac{t}{n} - u \right) \tilde{W}_t = \frac{1}{n} \sum_{t=\bar{t}}^{\bar{t}} L_b \left( \frac{t}{n} - u \right) (S_{n,t} - S_{n,t-1})
\]

\[
= \frac{1}{n} \sum_{t=\bar{t}}^{\bar{t}-1} [L_b \left( \frac{t}{n} - u \right) - L_b \left( (t+1)/n - u \right)] S_{n,t} + \frac{1}{n} L_b \left( \frac{\bar{t}}{n} - u \right) S_{n,\bar{t}}.
\]

Since \( \{\tilde{W}_t\} \) is stationary, \( S_{n,t} \) has the same distribution as \( \tilde{S}_{n,t} = \sum_{j=1}^{\bar{t}+1} \tilde{W}_j. \) Thus, for some constant \( M, \)
\[
\left| \frac{1}{n} \sum_{t=1}^{\bar{t}} L_b \left( \frac{t}{n} - u \right) \tilde{W}_t \right| \leq \frac{M}{nb} \sup_{1 \leq \bar{t}\leq \bar{t}+1} \left| \tilde{S}_{n,\bar{t}} \right|.
\]

The ergodic theorem yields \( \tilde{S}_{n,t}/t \to 0 \) which in turn implies that \( \frac{1}{n} \sum_{t=1}^{n} L_b \left( \frac{t}{n} - u \right) \tilde{W}_t \) tends to zero almost surely. Finally, using the
mean value theorem, there exists \( v_{n,t} \in \left[ \frac{t-1}{n}, \frac{t}{n} \right] \) so that with \( L = \sup_v L(v) \),

\[
\left| \frac{1}{n} \sum_{t=1}^{n} L_b \left( \frac{t}{n} - u \right) - \int L_b (x-u) \, dx \right| = \left| \frac{1}{nb} \sum_{t=1}^{n} L_b \left( \frac{t}{n} - u \right) - \sum_{t=1}^{n} \int_{(t-1)/n}^{t/n} L_b (x-u) \, dx \right|
\]

\[
\leq \frac{1}{nb} \sum_{t=1}^{n} \left| L_b \left( \frac{t}{n} - u \right) - L_b \left( v_{n,t} - u \right) \right|
\]

\[
\leq \frac{1}{nb} \sum_{t=1}^{n} A \left| \frac{t/n - v_{n,t}}{b} \right| = O \left( \frac{1}{nb} \right),
\]

which shows that \( \frac{1}{n} \sum_{t=1}^{n} L_b \left( \frac{t}{n} - u \right) \mathbb{E} \left[ W_t^* (\theta | u) \right] = \int L_b (x-u) \, dx \mathbb{E} \left[ W_t^* (\theta | u) \right] + O(1/(nb)) \).

For the uniform convergence, we note that by definition of \( A \), \( D_b \left( v - u \right) \alpha \in \Theta \) for all \( v \in \text{supp}(L) \) and \( \alpha \in A \). Thus, \( \frac{1}{n} \sum_{t=1}^{n} K_b \left( t/n - u \right) W_{n,t} \left( D_{n,t} (u) \alpha \right) \), where \( D_{n,t} (u) = D_b \left( t/n - u \right) \), is well-defined for \( \alpha \in A \), and

\[
\mathbb{E} \left[ \sup_{\alpha \in A} \| W_{n,t} \left( D_{n,t} (u) \alpha \right) - W_t^* (D_{n,t} (u) \alpha|u) \|^p \right] \leq \mathbb{E} \left[ \sup_{\theta \in \Theta} \| W_{n,t} (\theta) - W_t^* (\theta|u) \|^p \right]
\]

\[
\leq C \left( \| t/n - u \|^q + 1/n^q + \rho^q \right)^p.
\]

Using Hölder’s and Minkowski’s inequality,

\[
\mathbb{E} \left[ \sup_{\alpha \in A} \left\| \frac{1}{n} \sum_{t=1}^{n} L_b \left( t/n - u \right) \left\{ W_{n,t} (D_{n,t} (u) \alpha) - W_t^* (D_{n,t} (u) \alpha|u) \right\} \right\| \right]
\]

\[
\leq \frac{1}{n} \sum_{t=1}^{n} \left| L_b \left( t/n - u \right) \right| \mathbb{E} \left[ \sup_{\alpha \in A} \| W_{n,t} (D_{n,t} (u) \alpha) - W_t^* (D_{n,t} (u) \alpha|u) \|^p \right]^{1/p}
\]

\[
\leq C b^q \frac{1}{n} \sum_{t=1}^{n} \left| L_b \left( t/n - u \right) \right| \left( \left| \frac{t/n - u}{b} \right|^q + 1/n^q + \rho^q \right) = O \left( b^q \right).
\]

Next,

\[
\sup_{\alpha \in A} \left\| \frac{1}{n} \sum_{t=1}^{n} L_b \left( t/n - u \right) \left\{ W_t^* (D_{n,t} (u) \alpha|u) - \mathbb{E} [W_t^* (D_{n,t} (u) \alpha|u)] \right\} \right\|
\]

\[
\leq \sup_{\theta \in \Theta} \left\| \frac{1}{n} \sum_{t=1}^{n} L_b \left( t/n - u \right) \left\{ W_t^* (\theta|u) - \mathbb{E} [W_t^* (\theta|u)] \right\} \right\| + o_p(1)
\]

where \( \frac{1}{n} \sum_{t=1}^{n} L_b \left( t/n - u \right) \left\{ W_t^* (\theta|u) - \mathbb{E} [W_t^* (\theta|u)] \right\} = o_p(1) \) for all \( \theta \in \Theta \). Thus, the result will follow if we can show stochastic equicontinuity of \( \theta \mapsto \frac{1}{n} \sum_{t=1}^{n} L_b \left( t/n - u \right) W_t^* (\theta|u) \) but this follows from the assumption of \( \theta \mapsto W_t^* (\theta|u) \) being \( L_p \) continuous: For a given \( \theta \in \Theta \) and \( \epsilon > 0 \) there
exists δ > 0 so that
\[
\mathbb{E} \left[ \sup_{\theta' : \|\theta - \theta'\| < \delta} \left\| \frac{1}{n} \sum_{t=1}^{n} L_b(t/n - u) W^*_t(\theta | u) - \frac{1}{n} \sum_{t=1}^{n} L_b(t/n - u) W^*_t(\theta' | u) \right\| \right] \\
\leq \frac{1}{n} \sum_{t=1}^{n} |L_b(t/n - u)| \mathbb{E} \left[ \sup_{\theta' : \|\theta - \theta'\| < \delta} \left\| W^*_t(\theta | u) - W^*_t(\theta' | u) \right\| \right] \\
= \frac{\epsilon}{n} \sum_{t=1}^{n} |L_b(t/n - u)| = O(\epsilon) .
\]

Proof of (ii). Observe that \(\sqrt{b/n} \sum_{t=1}^{n} L_b(t/n - u) W_{n,t}(\theta(t/n))\) is a martingale with quadratic variation \(Q = \frac{b}{n} \sum_{t=1}^{n} L^2_b(t/n - u) V_{n,t}(\theta(t/n))\). To derive the limit of \(Q\), write
\[
Q_n = \frac{b}{n} \sum_{t=1}^{n} L^2_b(t/n - u) \mathbb{E}[V^*_t(\theta(t/n) | u)] + \frac{b}{n} \sum_{t=1}^{n} L^2_b(t/n - u) \{V_{n,t}(\theta(t/n)) - V^*_t(\theta(t/n) | u)\} \\
+ \frac{b}{n} \sum_{t=1}^{n} L^2_b(t/n - u) \{V^*_t(\theta(t/n) | u) - \mathbb{E}[V^*_t(\theta(t/n) | u)]\} .
\]

For the first term, employing standard results for kernel averages together with the fact that \(\theta \mapsto \mathbb{E}[V^*_t(\theta | u)]\) is continuous (because \(V^*_t(\theta | u)\) is \(L_1\)-continuous),
\[
\frac{b}{n} \sum_{t=1}^{n} L^2_b(t/n - u) \mathbb{E}[V^*_t(\theta(t/n) | u)] \to \int L^2(x) d\mathbb{E}[V^*_t(\theta(u) | u)] .
\]

Applying arguments similar to those in the proof of Theorem 3(i) together with continuity of \(v \mapsto \theta(v)\), \(L_1\)-continuity of \(\theta \mapsto V^*_t(\theta | u)\) and \(L\) having compact support, we have for all \(n\) large enough,
\[
\frac{b}{n} \sum_{t=1}^{n} L^2_b(t/n - u) \mathbb{E}[\|V_{n,t}(\theta(t/n)) - V^*_t(\theta(t/n) | u)\|] \\
\leq \frac{b}{n} \sum_{t=1}^{n} L^2_b(t/n - u) \sup_{\|\theta - \theta(u)\| < \epsilon} \mathbb{E}[\|V_{n,t}(\theta) - V^*_t(\theta(u) | u)\|] = o(1) ,
\]
and
\[
\frac{b}{n} \sum_{t=1}^{n} L^2_b(t/n - u) \{V^*_t(\theta(t/n) | u) - \mathbb{E}[V^*_t(\theta(t/n) | u)]\} \\
\leq \frac{b}{n} \sum_{t=1}^{n} L^2_b(t/n - u) \sup_{\|\theta - \theta(u)\| < \epsilon} \mathbb{E}[\|V^*_t(\theta(u) | u) - \mathbb{E}[V^*_t(\theta(u) | u)]\|] = o(1) .
\]
The result now follows if the Lindeberg condition is satisfied, c.f. Brown (1971). But, as \(nb \to \infty\),
with $m_{n,t}(\theta) = \sqrt{b/n}L_b(t/n - u)W_t^*(\theta|u)$,

$$
\sum_{t=1}^{n} \|m_{n,t}(\theta(t/n))\|^2 1(\|m_{n,t}(\theta(t/n))\| > \varepsilon)
\leq \sum_{t=1}^{n} \left(\|m_{n,t}(\theta(t/n))\|^2 - \|m_t^*(\theta(u)|u)\|^2\right) 1(\|m_{n,t}(\theta(t/n))\| > \varepsilon)
+ \sum_{t=1}^{n} \|m_t^*(\theta(u)|u)\|^2 1(\|m_{n,t}(\theta(t/n))\| > \varepsilon; \|m_t^*(\theta(u)|u)\| \leq \varepsilon/\sqrt{2})
+ \sum_{t=1}^{n} \|m_t^*(\theta(u)|u)\|^2 1(\|m_t^*(\theta(u)|u)\| > \varepsilon/\sqrt{2}).
$$

Recycling the arguments used in the analysis of $Q_n$, it follows that the first and third terms are $o_p(1)$. Similarly, the convergence of the second term is obtained with the following inequality and Markov’s inequality:

$$
\sum_{t=1}^{n} \|m_t^*(\theta(u)|u)\|^2 1(\|m_{n,t}(\theta(t/n))\| > \varepsilon; \|m_t^*(\theta(u)|u)\| \leq \varepsilon/\sqrt{2})
\leq \frac{\varepsilon^2}{2} \sum_{t=1}^{n} 1(\|m_{n,t}(\theta(t/n))\|^2 - \|m_t^*(\theta(u)|u)\|^2 > \varepsilon^2/2).
$$

**Proof of (iii).** Assume w.l.o.g. that $E[W_t^*] = 0$ and then use

$$
Var(A_n) \leq \frac{1}{n} \sum_{t_1, t_2=1}^{n} |L_b(t_1/n - u)| |L_b(t_2/n - u)| |cov(W_{t_1}^*, W_{t_2}^*)|
\leq \frac{\bar{L}}{(nb)^2} \sum_{t_1, t_2=1}^{n} \left|L\left(t_1/n - u\right)\right| |cov(W_{t_1}^*, W_{t_2}^*)| = O\left(\frac{1}{nb}\right).
$$

\[ \blacksquare \]

**A.2 Proofs: Main results**

**Proof of Theorem 1.** We first note that $f(Z_t^*(\theta|u), \varepsilon_t; \theta)$ is stationary and ergodic because $f$ is a measurable function of $(Z_t^*(\theta|u), \varepsilon_t)$. Moreover, with $p_Z = p/(r + 1)$,

$$
E\left[\sup_{\theta \in \Theta} \|f(Z_{n,t}(\theta), \varepsilon_t; \theta) - f(Z_t^*(\theta|u), \varepsilon_t; \theta)\|^{p_Z}\right]^{1/p_Z}
\leq CE\left[\left(1 + \|Z_{n,t}(\theta)\|_{pr/(r+1)} + \|Z_t^*(\theta|u)\|_{pr/(r+1)}\right) \|Z_{n,t}(\theta) - Z_t^*(\theta|u)\|_{p_Z}\right]^{1/p_Z}
\leq CE\left[\|Z_{n,t}(\theta) - Z_t^*(\theta|u)\|^{p}\right]^{1/p} \leq C\left(|t/n - u|^q + 1/n^q + \rho^r\right).
$$

where we have employed Hölder’s inequality. \[ \blacksquare \]
Proof of Theorem 2. By Theorem 3(i), $\sup_{\alpha \in \mathcal{A}} |Q_n(\alpha | u) - Q^* (\alpha | u)| = o_P(1)$, where $Q^*(\alpha | u) = \int K(v) \mathbb{E}[\ell_v^*(D(v) \alpha | u)] \, dv$. Now, observe that for any $\alpha = (\alpha_1, ..., \alpha_m)$ with $\alpha_i \neq 0$ for some $i \geq 2$, the polynomial $v \mapsto D(v) \alpha$ is non-constant almost everywhere. Thus, for any $\alpha \neq \alpha^* = (\theta(u), 0, ..., 0)$, $D(v) \alpha \neq \theta(u) = D(v) \alpha^*$ for almost all $v \in [0, 1]$ and so by Assumption 3(iii) $\mathbb{E}[\ell_v^*(D(v) \alpha | u)] < \mathbb{E}[\ell_v^*(\theta(u) | u)] = \mathbb{E}[\ell_v^*(D(v) \alpha^* | u)]$ for almost every $v$. Since $K(\cdot) \geq 0$ this in turn implies that $Q^*(\alpha | u) \leq Q^*(\alpha^* | u)$. Finally, by the dominated convergence theorem together with Assumption 3(ii) $\alpha \mapsto Q^*(\alpha | u)$ is continuous. This proves $\hat{\alpha} \to P \alpha^*$, c.f. Theorem 2.1 in Newey and McFadden (1994).

Proof of Theorem 3. From Theorem 2 we know that $\hat{\alpha} \to P \alpha^* := (\theta(u), 0, ..., 0)$. It is easily checked that the limit is situated in the interior of $\mathcal{A}$ and so w.p.a.1. so will $\hat{\alpha}$. As a consequence, $\hat{\alpha}$ will satisfy (6) w.p.a.1. Adding and subtracting $S_n(u)$ and then rearranging yields

$$0 = \sqrt{n}b S_n(u) + H_n(\hat{\alpha} | u) \sqrt{n}b (\hat{\alpha} - \alpha_0 - H_n^{-1}(\hat{\alpha} | u) \{S_n(\alpha_0 | u) - S_n(u)\}).$$

Here, $H_n^{-1}(\hat{\alpha} | u)$ is well-defined w.p.a.1 since, as shown below, it converges towards an invertible matrix. The claimed asymptotic result now follows if we can verify the claims of eqs. (9)-(10):

Proof of eq. (9). First note that, with $L(u) = K(u)D(u)$, $\sqrt{n}b S_n(u) = \sqrt{\frac{b}{n}} \sum_{t=1}^n L(t/n - u) \otimes s_{n,t}(\theta(t/n))$. The result now follows from Theorem 3(ii) under Assumption 5.

Proof of first claim of eq. (10). With $L(u) = K(u)D(u)$, we can write $H_n(\beta | u) = \frac{1}{n} \sum_{t=1}^n L(t/n - u) \otimes h_{n,t}(D_{n,t}(u) \beta)$. It follows from Theorem 3(i) and Assumption 6 that

$$\sup_{\alpha \in \mathcal{B}(\epsilon)} \|H_n(\alpha | u) - \mathbb{K}_1 \otimes H(D(v) \alpha | u)\| = o_p(1),$$

where $H(\theta | u) = \mathbb{E}[h_v^*(\theta | u)]$ is continuous w.r.t. $\theta$ and $\mathcal{B}(\epsilon) = \{\alpha : \|\alpha - \alpha^*\| < \epsilon\}$ for some small $\epsilon > 0$. Thus, given that $\hat{\alpha} \to P \alpha^*$, $H_n(\hat{\alpha} | u) \to P \mathbb{K}_1 \otimes H(\theta(u) | u)$. Finally, note here that since $K$ is a probability density function, $\mathbb{K}_1$ is invertible, while $H(\theta(u) | u) = H(u)$ is invertible by assumption.

Proof of second claim of eq. (10). First observe that $D_{n,t}(u) \alpha^* = \theta_n^*(t/n)$ where $\theta_n^*(t/n)$ was defined in (1). Now, employ the mean-value theorem twice to obtain that, for some $\tilde{\theta}_{n,t}$ lying between $\theta_n^*(t/n)$ and $\theta(t/n)$ and some $u_{n,t} \in [t/n, u]$,

$$b_{n,t} = h_{n,t}(\tilde{\theta}_{n,t}) \{\theta_n^*(t/n) - \theta(t/n)\} = -h_{n,t}(\tilde{\theta}_{n,t}) \theta^{(m+1)}(u_{n,t}) (t/n)^{m+1} (m+1)!

= - (t/n - u)^{m+1} h_{n,t}(\theta(t/n)) \theta^{(m+1)}(\frac{t}{n}) (m+1)!

+ \left\{h_{n,t}(\theta(t/n)) \theta^{(m+1)}(\frac{t}{n}) - h_{n,t}(\tilde{\theta}_{n,t}) \theta^{(m+1)}(u_{n,t})\right\} (\frac{t}{n} - u)^{m+1} (m+1)!.$$

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The first term is locally stationary and so by the same arguments as in the proof of Theorem 3(ii),

\[
\frac{b^{m+1}}{n} \sum_{t=1}^{n} K_{n,t}(u) \frac{d}{du} L_{n,t}(u) \left( \frac{t/n - u}{b} \right)^{m+1} h_{n,t}(\theta(t/n)) \frac{\theta^{(m+1)}(t/n)}{(m+1)!} = b^{m+1} \left\{ \mu_1 \otimes H(u) \frac{\theta^{(m+1)}(u)}{(m+1)!} + o_p(1) \right\}.
\]

Next, observe that for \(|t/n - u| \leq Cb\), \(\|\bar{\theta}_{n,t} - \theta(t/n)\| \leq \|\theta^*_u(t/n) - \theta(t/n)\| \leq \tilde{C}b^{m+1}\) and so, using the ULS property of \(h_{n,t}(\theta)\),

\[
\sup_{n,t} E \left[ \left| h_{n,t} \left( \theta \left( \frac{t}{n} \right) \right) - h_{n,t}(\bar{\theta}_{n,t}) \right| \right] \leq C \left( b^q \left| \frac{t}{n} - \frac{u}{b} \right|^q + 1/n^q \right) + \sup_{\|\theta - \theta^0\| \leq \tilde{C}b^{m+1}} E \left[ \left| h^*_u(\theta|u) - h^*_u(\theta'|u) \right| \right] \to 0,
\]

as \(n \to \infty\). Similarly, \(\sup_{n,t} \|\theta^{(m+1)} \left( \frac{t}{n} \right) - \theta^{(m+1)}(u_{n,t})\| \to 0\) as \(n \to \infty\). These two results combined show that the remainder term is \(o_p(1)\).

**Proof of Theorem 4.** The proof proceeds exactly as the one of Theorem 3 except that the first-order expansion of \(b_{n,t}\) is replaced by a second-order expansion w.r.t. \(\theta\) followed by a second-order Taylor expansion w.r.t. \(u\) so that

\[
b_{n,t} = -h_{n,t}(\theta(t/n)) \left[ \frac{\theta^{(m+1)}(u)}{(m+1)!} \{t/n - u\}^{m+1} + \frac{\theta^{(m+2)}(u_{n,t})}{(m+2)!} \{t/n - u\}^{m+2} \right] + \frac{1}{2} \sum_{i=1}^{d_\theta} \frac{\theta^{(m+1)}(u_{n,t})}{(m+1)!} \frac{\partial h_{n,t}(\bar{\theta}_{n,t})}{\partial \theta_i} \frac{\theta^{(m+1)}(u_{n,t})}{(m+1)!} \{t/n - u\}^{2m+2}.
\]

For the first term, with \(L_{n,t}^{(m)}(u) = K_{n,t}(u) D_{n,t}(u) \left( \frac{t/n - u}{b} \right)^m\),

\[
\frac{1}{n} \sum_{t=1}^{n} L_{n,t}^{(m+1)}(u) h_{n,t}(\theta(t/n)) = \frac{1}{n} \sum_{t=1}^{n} L_{n,t}^{(m+1)}(u) [h_{n,t}(\theta(t/n)) - h_{n,t}(\theta(u))] + \frac{1}{n} \sum_{t=1}^{n} L_{n,t}^{(m+1)}(u) h_{n,t}^*(\theta(u)|u) + \frac{b}{n} \sum_{t=1}^{n} L_{n,t}^{(m+2)}(u) \partial u h_{n,t}^*(\theta(u)|u)
\]

\[
+ \frac{1}{n} \sum_{t=1}^{n} \left[ L_{n,t}^{(m+1)}(u) [h_{n,t}(\theta(u)) - h_{n,t}^*(\theta(u)|u)] - \partial u h_{n,t}^*(\theta(u)|u) \right] \{t/n - u\}
\]

where we write \(h_{n,t}(u) = h_{n,t}(\theta(u))\) and similar for other terms. By Theorem 3(iii) together with
Assumption 9 and Theorem 3(i),

\[
\frac{1}{n} \sum_{t=1}^{n} L_{n,t}^{(m+1)}(u) [h_{n,t}(\theta(t/n)) - h_{n,t}(u)] = b\mu_2 \sum_{i=1}^{d_0} \theta_i^{(1)}(u) \partial_{\theta_i} H(u) + o_P(b),
\]

\[
\frac{1}{n} \sum_{t=1}^{n} L_{n,t}^{(m+1)}(u) h_i^*(u) = \mu_1 H(u) + o_P\left(1/\sqrt{n}b\right),
\]

\[
\frac{1}{n} \sum_{t=1}^{n} L_{n,t}^{(m+2)}(u) \partial_u h_i^*(u) = \mu_2 \partial_u H(u) + o_P(1),
\]

while, using Assumption 7,

\[
\frac{1}{n} \sum_{t=1}^{n} \left| L_{n,t}^{(m+1)}(u) \right| \mathbb{E} \left[ \left| h_{n,t}(u) - h_i^*(u) - \partial_u h_i^*(u) \{t/n - u\} \right| \right] \\
\leq \frac{1}{n} \sum_{t=1}^{n} \left| L_{n,t}^{(m+1)}(u) \right| C \left(1/n^q + \rho^\ell\right) \\
+ \frac{1}{n} \sum_{t=1}^{n} \left| L_{n,t}^{(m+1)}(u) \right| \mathbb{E} \left[ \left| h_i^*(\theta(u) | t/n) - h_i^*(u) - \partial_u h_i^*(u) \{t/n - u\} \right| \right] \\
= O \left(n^{-q}\right) + O \left(1/\sqrt{n}b\right) + o(b).
\]

For the second and the third terms, similar to the proof of eq. (10) and using Assumption 8,

\[
\sum_{t=1}^{n} L_{n,t}^{(m+2)}(u) h_{n,t}(\hat{\theta}_{n,t}) \frac{\theta_i^{(m+2)}(u_{n,t})}{(m+2)!} = \mu_2 H(u) \frac{\theta_i^{(m+2)}(u)}{(m+2)!} + o_P(1),
\]

\[
\sum_{t=1}^{n} L_{n,t}^{(m+2)}(u) \frac{\theta_i^{(m+1)}(u_{n,t})}{(m+1)!} \frac{\partial h_{n,t}(\hat{\theta}_{n,t})}{\partial \theta_i} \frac{\theta_i^{(m+1)}(u_{n,t})}{(m+1)!} \\
= \mu_{m+2} \frac{\theta_i^{(m+1)}(u)}{(m+1)!} \frac{1}{\{m+1\}!} \partial_{\theta_i} H(u) \theta_i^{(m+1)}(u) + o_P(1).
\]

Collecting terms now yield the claimed result.  ■

A.3 Proofs: Examples

Proof of Corollary 3. The local stationarity of the least-squares criterion and its first two derivatives in Section 3, and Theorem 3 applies to the local linear estimator. We analyze the local constant estimator. Note that our proof of local stationarity also implies that \(Y_t^*(u)\) is a GMC(4) and so we can apply Proposition 2 in Wu and Shao (2004) to obtain that the process is short-range dependent. With \(\mathbb{E}(\xi_i^4) < \infty\), \(Y_t^*(u)\) is GMC(4) and \(\mathbb{E}(\|Y_t^*(u)\|^4) < 0\), which in turn imply from Lemma 2 that \(h_i^*(\theta|u) = \{\text{diag}(Y_t^{*(-)}(u), Y_t^{*(+)}(u))\}^2\) is GMC(2), and hence Assumption 9 is

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satisfied. We consider a derivative process defined by
\[
\partial_u Y^*_t (u) = [\theta_1 (u) 1 (Y^*_{t-1} (u) > 0) + \theta_2 (u) 1 (Y^*_{t-1} (u) < 0) + \theta'_1 (u) 1 (Y^*_{t-1} (u) > 0) + \theta'_2 (u) 1 (Y^*_{t-1} (u) < 0)] Y^*_{t-1} (u).
\]

The joint process \(Z_t^*(u) = (Y^*_t (u), \partial_u Y^*_t (u))^T\) solves a bivariate AR model whose stability condition is satisfied due to \(\max \{|\theta_1(u)|, |\theta_2(u)|\} < 1\). It now follows by Theorem 4.8 in Dahlhaus et al. (2019) that \(Y^*_t\) is time-differentiable at \((\theta(u), u)\) with \(\partial_u Y^*_t (u)\). Then, the derivative process of the hessian, \(\partial_u h^*_t (\theta | u) = 2 \text{diag} (Y^*_t (u), Y^*_t (u)) \partial_u Y^*_t (u)\), satisfies Assumption 7. Finally, we note that the third-order derivatives of the log-likelihood are zero and so Assumption 8 is trivially satisfied. Thus, with \(\mathbb{E} (\varepsilon_t^4) < \infty\), Theorem 4 applies to the local constant estimator.

**Proof of Corollary 4.** We apply our theory with \(\Theta = \mathbb{R}^{d_q}\) since the least-squares criterion used for estimation is concave in \(\theta\), c.f. the comments following Assumptions 1-3. We first show that \(X_{n,t}\) is locally stationary with \(p \geq 2p\) moments when \(\mathbb{E} (||\varepsilon_t||^2) < \infty\). Without loss of generality, we here only provide a proof for \(Y_{n,t} = \Phi (t/n) Y_{n,t-1} + \Sigma (t/n) \varepsilon_t\) to be locally stationary under the following conditions: \(\Phi (u)\) and \(\Sigma (u)\) are twice continuously differentiable; and all eigenvalues of \(\Phi (u)\) lie inside the unit circle for \(u \in [0, 1]\). We first verify the conditions of Theorems 1 for \(G (x, \varepsilon, \vartheta) := \Phi x + \Sigma \varepsilon\) where \(\vartheta = (\Phi, \Sigma)\) with \(\Phi\) having all eigenvalues inside the unit circle. First,
\[
\mathbb{E} (||G(0, \varepsilon, u)||^p) \leq \||\Sigma (u)||^p \mathbb{E} (||\varepsilon_t||^p) < \infty;
\]
second, for all \(x, x' \in \mathbb{R}^d\),
\[
\mathbb{E} (||G(x, \varepsilon, \vartheta) - G(x', \varepsilon, \vartheta)||^p)^{1/p} \leq ||\Phi|| ||x - x'|| \leq \rho ||x - x'||,
\]
where \(\rho = \sup_{x \neq 0} \frac{||\Phi x||}{||x||} < 1\) since all eigenvalues of \(\Phi\) lie inside the unit circle; and for all \(\vartheta, \vartheta'\),
\[
\mathbb{E} (||G(x, \varepsilon, \vartheta) - G(x, \varepsilon, \vartheta')||^p)^{1/p} = ||\Phi - \Phi'|| ||x|| + ||\Sigma - \Sigma'|| \mathbb{E} (||\varepsilon_t||^p)^{1/p} \leq C (1 + ||x||) ||\vartheta - \vartheta'||.
\]

Next, we verify that the log-likelihood and its derivatives are ULS: Observe that
\[
\left\| \frac{\partial \ell_{n,t} (\vartheta)}{\partial Y_{n,t}} \right\| \leq 2 \left( ||Y_{n,t}|| - ||\theta|| ||X_{n,t}|| \right), \quad \left\| \frac{\partial \ell_{n,t} (\vartheta)}{\partial X_{n,t}} \right\| = 2 \left( ||Y_{n,t}|| - ||\theta|| ||X_{n,t}|| \right) ||\theta||,
\]
and so 1 applies with \(r = 1\). For the score function, observe that \(s_{n,t} = 2X_{n,t} \Sigma (t/n) \varepsilon_t\) which is a Martingale difference with \(\omega_{n,t} = 4X_{n,t} \Sigma (t/n) \varepsilon_t \varepsilon_t' (\Sigma (t/n))' X_{n,t}'\). Then,
\[
s_{n,t} = s_{n,t} (\theta (t/n)), \quad \left\| \frac{\partial \omega_{n,t}}{\partial X_{n,t}} \right\| = 4 ||\Sigma (t/n) \varepsilon_t||^2 ||X_{n,t}||,
\]
and so 1 applies with \(r = 1\). The hessian is also ULS by similar arguments. Thus, with \(\mathbb{E} (||\varepsilon_t||^2) < \infty\), Theorem 4 applies to the local constant estimator.
follows by Theorem 4.8 in Dahlhaus et al. (2019) that the derivative process of the hessian, 
ability condition is satisfied due to all eigenvalues of $\Phi$ $(u)$, the derivatives of the log-likelihood are zero and so Assumption 8 is trivially satisfied. Thus, with 
the derivative process takes the form $Y_t^* (u)$, which in turn imply that $h_t^*(\theta|u) = Y_t^* (u) X_t^* (u)'$ solves another VAR model whose sta-
tility condition is satisfied due to all eigenvalues of $\Phi (u)$ lying inside the unit circle. It now follows by Theorem 4.8 in Dahlhaus et al. (2019) that the derivative process of the hessian, 
that $Y_t^* (u)$ is GMC(2), c.f. Lemma 2. Hence, Assumption 9 is satisfied. The derivative process takes the form $\partial_u Y_t^* (u) = \Phi (u) Y_{t-1}^* (u) + \Phi (u) \partial_u Y_{t-1}^* (u) + \Sigma^{(1)} (u) \varepsilon_t$. The joint process $Z_t^* (u) = (Y_t^* (u)', \partial_u Y_t^* (u)')'$ solves another VAR model whose sta-
tation of the log-likelihood and its derivatives can be shown by verifying the conditions of Theorem 1. We have 
where 
Thus, $\ell_{n,t} (\theta) = \theta' Y_{n,t}$ and $V_{n,t} = (1, Y_{n,t-1}, ..., Y_{n,t-q})'$. Here, $\lambda_{n,t} (\theta)$ is trivially ULS(1,1,$\Theta$) while 
$$\frac{Y_{n,t}}{\lambda_{n,t} (\theta)} = \frac{\theta (t/n)' V_{n,t} \varepsilon_{t}^2}{\theta' V_{n,t} \varepsilon_{t}^2} \leq \sup_u \omega (u) + \sum_{i=1}^{p} \sup_u \alpha_i (u) \varepsilon_{t}^2.$$ 
(23)
Thus, $\ell_{n,t} (\theta)$ satisfies the conditions of Theorem 1 with $r = 0$ and $q = 1$. Next, we verify Assumption 5 with the score function $s_{n,t} (\theta) = (1 - Y_{n,t}/\lambda_{n,t} (\theta)) \partial_\theta \lambda_{n,t} (\theta) / \lambda_{n,t} (\theta)$. Here, $\partial_\theta \lambda_{n,t} (\theta) = V_{n,t}$ is trivially ULS(1,1,$\Theta$). The process $\{s_{n,t} (\theta (t/n)) , F_{n,t-1}\}$ is a MGD and $\omega_{n,t} (\theta)$ takes the form 
$$\omega_{n,t} (\theta) = \frac{\partial_\theta \lambda_{n,t} (\theta) (\partial_\theta \lambda_{n,t} (\theta))'}{\lambda_{n,t}^2 (\theta)} (1 - Y_{n,t}/\lambda_{n,t} (\theta))^2.$$

∞, all conditions for Theorem 3 hold.

To analyze the local constant estimator, first note that our proof of local stationarity also implies that $Y_t^* (u)$ is a GMC(p) and so we can apply Proposition 2 in Wu and Shao (2004) to obtain that the process is short-range dependent. If $\mathbb{E} [\|\varepsilon_t\|^4] < \infty$ then $Y_t^* (u)$ is GMC(4) and $\mathbb{E} [\|Y_t^* (u)\|^4] < \infty$, which in turn imply that $h_t^*(\theta|u) = X_t^* (u) X_t^* (u)'$ is GMC(2), c.f. Lemma 2. Hence, Assumption 9 is satisfied. The derivative process takes the form $\partial_u Y_t^* (u) = \Phi (1) (u) Y_{t-1}^* (u) + \Phi (u) \partial_u Y_{t-1}^* (u) + \Sigma^{(1)} (u) \varepsilon_t$. The joint process $Z_t^* (u) = (Y_t^* (u)', \partial_u Y_t^* (u)')'$ solves another VAR model whose sta-
tation of the log-likelihood and its derivatives can be shown by verifying the conditions of Theorem 1. We have 
where 
Thus, $\ell_{n,t} (\theta)$ satisfies the conditions of Theorem 1 with $r = 0$ and $q = 1$. Next, we verify Assumption 5 with the score function $s_{n,t} (\theta) = (1 - Y_{n,t}/\lambda_{n,t} (\theta)) \partial_\theta \lambda_{n,t} (\theta) / \lambda_{n,t} (\theta)$. Here, $\partial_\theta \lambda_{n,t} (\theta) = V_{n,t}$ is trivially ULS(1,1,$\Theta$). The process $\{s_{n,t} (\theta (t/n)) , F_{n,t-1}\}$ is a MGD and $\omega_{n,t} (\theta)$ takes the form 
$$\omega_{n,t} (\theta) = \frac{\partial_\theta \lambda_{n,t} (\theta) (\partial_\theta \lambda_{n,t} (\theta))'}{\lambda_{n,t}^2 (\theta)} (1 - Y_{n,t}/\lambda_{n,t} (\theta))^2.$$
By combining (23) with \(\sup_{(\theta,||\theta-c\|<\epsilon)}\|\partial_\theta \lambda_{n,t}(\theta) / \lambda_{n,t}(\theta)\| \leq p/\delta_L\),

\[
\left\| \frac{\partial \omega_{n,t}(\theta)}{\partial Y_{n,t}} \right\| = 2 \left\| \frac{\partial_\theta \lambda_{n,t}(\theta)}{\lambda^3_{n,t}(\theta)} \right\| |1 - Y_{n,t}/\lambda_{n,t}(\theta)| \leq C(1 + \epsilon_t^2),
\]
\[
\left\| \frac{\partial \omega_{n,t}(\theta)}{\partial \lambda_{n,t}(\theta)} \right\| = \frac{2 \left\| \frac{\partial_\theta \lambda_{n,t}(\theta)}{\lambda^3_{n,t}(\theta)} \right\|}{\lambda^3_{n,t}(\theta)} \left(1 - \frac{3Y_{n,t}}{\lambda_{n,t}(\theta)} + \frac{4Y_{n,t}^2}{\lambda^2_{n,t}(\theta)} \right) \leq C\left(1 + \epsilon_t^2 + \epsilon_t^4\right),
\]
\[
\left\| \frac{\partial \omega_{n,t}(\theta)}{\partial (\partial_\theta \lambda_{n,t}(\theta))} \right\| = \frac{2 \left\| \frac{\partial_\theta \lambda_{n,t}(\theta)}{\lambda^3_{n,t}(\theta)} \right\|}{\lambda^2_{n,t}(\theta)} (1 - Y_{n,t}/\lambda_{n,t}(\theta))^2 \leq C\left(1 + \epsilon_t^2 + \epsilon_t^4\right),
\]

and so \(\omega_{n,t}(\theta)\) satisfies the conditions of Theorem 1 with \(r = 0\) and \(q = 1\). The hessian takes the form

\[
h_{n,t}(\theta) = \frac{\partial_\theta \lambda_{n,t}(\theta) \partial_\theta \lambda_{n,t}(\theta)'}{\lambda^2_{n,t}(\theta)} \left[ 2Y_{n,t} \lambda_{n,t}(\theta) - 1 \right],
\]

and recycling the inequalities established above it follows that the hessian is also ULS\((1,1,\Theta)\). This verifies the conditions for Theorem 3.

For the analysis of the local constant estimator, to verify Assumption 9, observe that

\[
h_{ij,t}(\theta(u)|u) = \frac{\partial_1 \lambda^*_{t}(\theta(u)|u) \partial_2 \lambda^*_{t}(\theta(u)|u) / \lambda^*_{t}(\theta(u)|u)}{\lambda^*_{t}(\theta(u)|u)^2} \left( \frac{2Y^*_t(u)}{\lambda^*_{t}(\theta(u)|u)} - 1 \right).
\]

Here, \(Y^*_t(u)\) is GMC\((1)\) with \(E\left(Y^*_t(u)\right) < \infty\) under \(\sum_{k=1}^q \alpha_k(u) < 1\), and \(\partial_1 \lambda^*_t(\theta(u)|u) / \lambda^*_t(\theta(u)|u) \leq 1/\delta_L\) for \(i = 1, \ldots, q + 1\). Thus, \(h_{ij,t}(\theta(u)|u)\) satisfies the conditions of Lemma 2 with \((\alpha, \beta, p) = (1, 1, 0)\). With \(E\left[\|\xi_i\|^{2+\gamma}\right] < \infty\), \(E\left[\|h_{ij,t}(\theta(u)|u)\|^{2+\gamma/2}\right] < \infty\). This in turn implies from Lemma 1 in Wu and Shao (2004) that \(h_{ij,t}(\theta(u)|u)\) is GMC\((2)\), and hence Assumption 9 is satisfied by applying Proposition 2 in Wu and Shao (2004). Next, to verify Assumption 7, we apply Theorem 4.8 in Dahlhaus et al. (2019): Under their assumptions, the derivative process takes the form

\[
\partial_\theta h^*_t(\theta|u) = \frac{\partial h^*_t(\theta|u)}{\partial Y^*_t(u)} \partial_\theta Y^*_t(u) + \frac{\partial h^*_t(\theta|u)}{\partial \lambda^*_t(\theta|u)} \partial_\theta \lambda^*_t(\theta|u) + \frac{\partial h^*_t(\theta|u)}{\partial (\partial_\theta \lambda^*_t(\theta|u))} \partial_\theta \lambda^*_t(\theta|u),
\]

where \(\partial_\theta \lambda^*_t(\theta|u) = \theta' \partial_\theta Y^*_t(u)\), \(\partial_\theta \lambda^*_t(\theta|u) = \partial_\theta Y^*_t(u) = (0, \partial_\theta Y^*_t(u-1), \ldots, \partial_\theta Y^*_t(u-q))'\) with the derivative process \(\partial_\theta Y^*_t(u)\) defined in the main text. We note that the first partial derivative is bounded by a constant \(C\) and the remaining two partial derivatives are bounded by \(C(1 + \epsilon_t^2)\). Also, Proposition 3.1 in Subba Rao (2006) implies that \(W^*_t(u)\) is time-differentiable in the \(L_1\)-sense at \(u\) and so, by another application of Theorem 1, \(h^*_t(\theta(u)|u)\) is also time-differentiable. Finally, the third-order derivatives take the form

\[
\frac{\partial h_{n,t}(\theta)}{\partial \theta_i} = -2 \frac{\partial_\theta \lambda_{n,t}(\theta) \partial_\theta \lambda_{n,t}(\theta) \partial_\theta \lambda_{n,t}(\theta)'}{\lambda^3_{n,t}(\theta)} \left[ \frac{3Y_{n,t}}{\lambda_{n,t}(\theta)} - 1 \right];
\]

\(^3\) Alternatively, we impose the slightly weaker condition on the moment of \(\epsilon_t\) and stronger assumption on the parameter set to verify Assumption 9: \(E\left(\epsilon_t^4\right) < \infty\) and \(\sqrt{E(\epsilon_t^4)} \left(\sum_{k=1}^q \alpha_k(u)\right) < 1\). Then, \(Y^*_t(u) \in \mathcal{L}^2\) and is GMC\((2)\), which implies from Lemma 2 with \((\alpha, \beta, p) = (2, 1, 0)\) that \(h^*_t(\theta(u)|u)\) is GMC\((2)\).
with \(p\) and so which has a well-defined solution while \(X\). This shows that

\[
E \left[ G(x_0, N_t; \theta) \right] = \mathbb{E} \left[ N_t \left( \omega + \sum_{i=1}^{q} \alpha_i x_{0,i} \right) \right] = \omega + \sum_{i=1}^{q} \alpha_i x_{0,i} < \infty;
\]

and for all \(x, x' \in \mathbb{R}^q_+\),

\[
\mathbb{E} \left[ \left| G(x, N_t; \theta) - G(x', N_t; \theta) \right| \right] \leq \mathbb{E} \left[ \left| N_t \left( \sum_{i=1}^{q} \alpha_i |x_i - x'_i| \right) \right| \right] = \sum_{i=1}^{q} \alpha_i |x_i - x'_i|,
\]

where \(\sum_{i=1}^{q} \alpha_i < 1\). Finally,

\[
\mathbb{E} \left[ \left| G(x, N_t; \theta) - G(x, N_t; \theta') \right| \right] = |\omega - \omega'| + \sum_{i=1}^{q} |\alpha_i - \alpha'_i| \mathbb{E} \left[ N_t (x) \right] \leq C \left( 1 + x \right) \| \theta - \theta' \|.
\]

This shows that \(X_{n,t} := (Y_{n,t-1}, ..., Y_{n,t-q})\) is LS(1, 1) which in turn implies that \(Y_{n,t}\) is LS(1, 1), c.f. Theorem 1. However, later we need the existence of higher-order moments, and so we demonstrate by induction that \(E[\lambda_r^*(u)] < \infty\) for all \(r < \infty\): First, \(E[\lambda_r^*(u)] = \omega(u) + \sum_{i=1}^{q} \alpha_i \omega(u) E[\lambda_r^*(u)]\)

which has a well-defined solution while

\[
(\lambda_r^*(u))^r = \sum_{j=0}^{r} \binom{r}{j} \left( \sum_{i=1}^{q} \alpha_i \omega(u) Y_{t-i}^*(u) \right)^j \omega^{r-j}(u),
\]

and so

\[
E[(\lambda_r^*(u))^r] = \sum_{j=0}^{r} \binom{r}{j} E \left[ \left( \sum_{i=1}^{q} \alpha_i \omega(u) Y_{t-i}^*(u) \right)^j \right] (\omega(u))^{r-j}
= \omega(u)^r + E \left[ \left( \sum_{i=1}^{q} \alpha_i \omega(u) Y_{t-i}^*(u) \right)^r \right] + E[p_{r-1}(X_{t-r}^*(u))],
\]

with \(p_{r-1}(x)\) being an \((r - 1)\)th order polynomial. By induction, \(E[p_{r-1}(X_{t-r}^*(u))] < \infty\), and we
are left with considering terms of the form, for some constants $c_{ij}$,

$$E \left[ \left( \sum_{i=1}^{q} \alpha_i (u) Y_{t-i}^* (u) \right)^r \right] = \sum_{i=1}^{q} \sum_{j=0}^{r} c_{ij} \alpha_i^r (u) E \left[ Y_{t-i}^* (u)^r \right]$$

$$= \sum_{i=1}^{q} \alpha_i^r (u) E \left[ Y_{t-i}^* (u)^r \right] + C_r$$

$$= \sum_{i=1}^{q} \alpha_i^r (u) E \left[ \lambda_i^r (u)^r \right] + C_r$$

where, again by induction, $C_r < \infty$. Collecting terms, $E \left[ (\lambda_i^r (u))^r \right] = \sum_{i=1}^{q} \alpha_i^r (u) E \left[ \lambda_i^r (u)^r \right] + \tilde{C}_r$ which has a well-defined solution since $\sum_{i=1}^{q} \alpha_i^r (u) < 1$. This in turn implies that $E \left[ \lambda_i^r (u)^r \right] < \infty$ for all $r < \infty$. We can now apply Theorem 1 to obtain that $\lambda_{n,t}$ and $Y_{n,t}$ are $\text{LS}(r,1)$ with $E[\lambda_{n,t}^r] < \infty$ and $E[Y_{n,t}^r] < \infty$.

Next, we observe that $\lambda_{n,t} (\theta), \partial_\theta \lambda_{n,t} (\theta)$ and $\partial_{\theta_\theta} \lambda_{n,t} (\theta)$ are on the same form as in the GARCH model, except that $Y_{n,t-1}^2$ has been replaced by $Y_{n,t-1}$. In particular, it is easily checked that $\lambda_{n,t} (\theta), \partial_\theta \lambda_{n,t} (\theta)$ and $\partial_{\theta_\theta} \lambda_{n,t} (\theta)$ are $\text{ULS}(1,1,\Theta)$ and with all polynomial moments since $Y_{n,t-1}$ has all polynomial moments. Thus, it only remains to show that the log-likelihood and its derivatives w.r.t. $\theta$ satisfy the conditions of Theorem 1. First,

$$\left| \frac{\partial \ell_{n,t} (\theta)}{\partial Y_{n,t}} \right| = |\log \{\lambda_{n,t} (\theta)\}| \leq \max \{ |\log \delta_L |, \lambda_{n,t} (\theta) \},$$

where $\lambda_{n,t} (\theta)$ has all relevant moments. Second,

$$\left| \frac{\partial \ell_{n,t} (\theta)}{\partial \lambda_{n,t} (\theta)} \right| = \frac{Y_{n,t}}{\lambda_{n,t} (\theta)} + 1 \leq \frac{Y_{n,t}}{\delta_L} + 1,$$

where again the right-hand side has all relevant moments. The score function takes the form $s_{n,t} (\theta) = (Y_{n,t}/\lambda_{n,t} (\theta) - 1) \partial_\theta \lambda_{n,t} (\theta)$ which satisfies the Martingale difference condition with conditional variance $\omega_{n,t} (\theta) = \partial_\theta \lambda_{n,t} (\theta) \partial_\theta \lambda_{n,t} (\theta)' / \lambda_{n,t} (\theta)$. As before, due to all polynomial moments existing, it is easily checked that the conditional variance satisfies the conditions of Theorem 1 and similarly for the hessian which is on the form

$$h_{n,t} (\theta) = \frac{Y_{n,t}}{\lambda_{n,t}^2 (\theta) \partial_\theta \lambda_{n,t} (\theta) \partial_\theta \lambda_{n,t} (\theta)'} - \left( \frac{Y_{n,t}}{\lambda_{n,t} (\theta)} - 1 \right) \partial_{\theta_\theta} \lambda_{n,t} (\theta).$$

The analysis of the third-order derivatives is similar and so is left out. ■