A solution of LIDAR problem in double scattering approximation

Alexey Buzdin, Sergey Leble*
Immanuel Kant Baltic Federal University, A. Nevskogo st. 14
)* now: Gdansk University of Technology,
ul. Narutowicza 11/12, 80-952, Gdansk, Poland, leble@mif.pg.gda.pl

December 15, 2011

Abstract

A problem of monoenergetic particles pulse reflection from half-infinite stratified medium is considered in conditions of elastic scattering with absorption account. The theory is based on multiple scattering series solution of Kolmogorov equation for one-particle distribution function. The analytical representation for first two terms are given in compact form for a point impulse source and cylindric symmetrical detector. This publication is an authorized translation (by S. Leble) of a bad-published paper from 20.06.1980 (available in its original version: http://www.mif.pg.gda.pl/krrizm/page/leble/publications.html, 140. A. Buzdin, S. Leble Lidar Problem Solution in Double-Scattering Approximation VINITI N2536-80dep (1980) 126 [PDF]). Reading recent articles on the LIDAR sounding of environment (e.g. Atmospheric and Oceanic Optics (2010) 23: 389-395, Kaul, B. V.; Samokhvalov, I. V.) one recovers standing interest to the related direct and inverse problems. A development of the result for the case of n-fold scattering and polarization account as well as correspondent convergence series problem solution of the Kolmogorov equation will be published in nearest future.

1 Introduction

A problem of monoenergetic particles pulse reflection from half-infinite stratified medium is considered. The particles scattering is supposed elastic, its interaction is negligible. Such problem relates a LIDAR probing of atmosphere. It is supposed that such process is described by Kolmogorov equation \[ \Pi \] for one-particle distribution function.

The expansion of the distribution function with respect to scattering multiplicity allows to obtain a closed expression for the series terms as multiple integrals.
The main purpose of this paper is a simplification of analytic expressions and investigation of its applicability range in terms of geometric parameters and macroscopic cross-sections. The point impulse source and cylindric symmetrical detector is considered.

Computations by Monte-Carlo method show a quick convergence of the scattering multiplicity expansion hence the one- and double scattering contributions are enough for the practical LIDAR response evaluation.

\section{The problem formulation}

The equation for the probability density \( f = f(t, \vec{r}, \vec{v}, t') \) has the form:

\[
\frac{1}{c} \left[ \partial_t f + \vec{v} \cdot \nabla f \right] = -\sigma_t(z)f - \int \sigma(\cos \gamma, z) f d\Omega, \tag{1}
\]

Where \( t' \)-time, \( d\Omega \) - solid angle; \( \sigma - \) bulk differential cross-section of elastic scattering to the angle \( \gamma \); \( \sigma_t \) - the sum of \( \int \sigma d\Omega \) and total cross-section of absorption, and

\[
\vec{v} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\]

In spherical coordinates: \( r, \theta, \phi \) the scattering angle is expressed as

\[
\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi'). \tag{2}
\]

We suppose that the scattering is elastic, \( |\vec{v}| \) does not change while the scattering process occur. Initial conditions are represented by distributions

\[
f(0, x, y, z) = V \delta(x) \delta(y) \delta(z) \delta(\theta). \tag{3}
\]

It means that we built a solution for the probability density as a weak limit (when \( t \to 0 \)) to \( \delta - function \) at \( t > 0 \). The distribution \( \delta(\theta) \) is chosen as

\[
(\delta(\theta), \psi(\theta, \phi)) = \int_{0}^{2\pi} \psi(0, \phi) d\phi. \tag{4}
\]

\section{Solution}

Let us denote \( t = ct' \). A solution is searched as an N-fold scattering expansion

\[
f = f_0 + f_1 + f_2 + ...
\]

We choose for \( f_0 \),

\[
L f_0 = \frac{\partial f_0}{\partial t} + \sin \theta \cos \phi \frac{\partial f_0}{\partial x} + \sin \theta \sin \phi \frac{\partial f_0}{\partial y} + \cos \theta \frac{\partial f_0}{\partial z} = -\sigma_t(z)f_0, \tag{6}
\]

and initial condition,

\[
f_0(0, x, y, z) = V \delta(x) \delta(y) \delta(z) \delta(\theta). \tag{7}
\]
For \( n \geq 1 \), the expansion coefficients are defined by

\[
Lf_{n+1} = -\sigma_1(z) f_{n+1} + \int_0^{2\pi} \int_0^{\pi} \sigma_2(\cos\gamma, z) f_n \sin\theta' d\theta' d\phi'.
\]  

(8)

with initial conditions for \( n > 0 \)

\[
f_{n+1}|_{t=0} = 0.
\]

(9)

Such expansion is usually named as expansion by folds scattering. The expression for \( f_0 \) is easily found:

\[
f_0 = \frac{1}{2\pi} \delta(x) \delta(y) \delta(z-t) \exp\left[ -\int_0^t \sigma_1(z-\tau) d\tau \right].
\]

Let us denote a function \( E \) via

\[
E(t, z, \theta) = \exp\left[ -\int_0^t \sigma_1(z-\tau \cos\theta) d\tau \right],
\]

(10)

then for \( f_{n+1} \) we obtain the recurrence

\[
f_{n+1}(x, y, z, \theta, \phi, t) = \int_0^t E(\tau, z, \theta) \int_0^{2\pi} \int_0^{\pi} \sigma(\cos\gamma, z-\tau \cos\theta) f_n(x-\tau \cos\theta \cos\phi, y-\tau \cos\theta \sin\phi, z-\tau \cos\theta, \theta', \phi', t-\tau) \sin\theta' d\theta' d\phi' d\tau.
\]

(11)

Particularly, for the \( n = 1 \), it yields

\[
f_1 = \int_0^t E(\tau, z, \theta) E(t-\tau, z-\tau \cos\theta, 0) \sigma(\cos\theta, z-\tau \cos\theta) \delta(x-\tau \sin\theta \cos\phi) \delta(y-\tau \sin\theta \sin\phi) \delta(z-\tau \cos\theta - (t-\tau)) d\tau
\]

and for \( n = 2 \)

\[
f_2 = \int_0^t E(\tau_2, z, \theta) \int_0^{2\pi} \sigma(\cos\gamma, z-\tau_2 \cos\theta) \int_0^{\pi} \sigma(\cos\gamma, z-\tau_2 \cos\theta, 0) E(t-\tau_2 - \tau_1, z-\tau_2 \cos\theta - \tau_1 \cos\theta', 0) \sigma(\cos\theta', z-\tau_2 \cos\theta - \tau_1 \cos\theta', 0) \delta(x-\tau_2 \sin\theta \cos\phi - \tau_1 \sin\theta' \cos\phi') \delta(y-\tau_2 \sin\theta \sin\phi - \tau_1 \sin\theta' \sin\phi') \delta(z-\tau_2 \cos\theta - \tau_1 \cos\theta' - (t-\tau_2 - \tau_1)) d\tau_1 \sin\theta' d\theta' d\phi' d\tau_2.
\]

(12)

It is convenient to represent the integrands as distributions in \( x, y, z \) space parametrized by \( \theta, \phi, \tau_1, \tau_2, \theta_1, \phi_1 \). Integrations by \( \tau_1, \tau_2, \theta_1, \phi_1 \) is understood as integration of the distribution by these parameters. For example, \( f_1 \) acts as on a function \( \psi \) from Schwartz space as

\[
(f_1(t, x, y, z, \theta, \phi), \psi(x, y, z)) = \int_0^t E(\tau, \tau \cos\theta + t - \tau, \theta) E(t-\tau, t-\tau, 0) \sigma(\cos\theta, t-\tau) \psi(\tau \sin\theta \cos\phi, \tau \sin\theta \sin\phi, \tau \cos\theta + t - \tau) d\tau.
\]
4 Number of particles rate

Our next problem is evaluation of number of particles which enter the round area of radius $\rho_0$ laying in the plane $xy$ with center in the origin and having velocity vectors inclined to z-axis within the angle interval $\theta \in [\pi - \theta_0, \pi]$. The angle $\theta_0$ relates the aperture angle of a receiver (e.g. lidar telescope window). By its direct sense the number is proportional to

$$I(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\pi - \theta_0}^{\pi} \int_{0}^{2\pi} (f(x, y, z, \theta, \phi, t), \psi(x, y, z)) \sin \theta d\phi d\theta,$$  \hspace{1cm} (13)

where $\psi(x, y, z) = 1$ for internal points of the domain $x^2 + y^2 \leq \rho_0^2$, $0 \leq z \leq \Delta t \cos \theta$ and zero outside.

Contribution of $n$-fold scattering $I_n(t)$ is obtained by substitution $f_n$ instead of $f$ in the Eq. (13). Let us evaluate one- and two-fold scattering.

$$I_1(t) = \lim_{\Delta t \to 0} \frac{2\pi}{\Delta t} \int_{\pi - \theta_0}^{\pi} \int_{0}^{2\pi} E(\tau, \tau \cos \theta + t - \tau, \theta) E(t - \tau, t - \tau, 0) \sigma(\cos \theta, t - \tau) \psi(\tau \sin \theta, 0, \tau \cos \theta + t - \tau) d\tau \sin \theta d\phi d\theta.$$  \hspace{1cm} (14)

The independence of the integrand on $\phi$ is taken into account.

In the problems of LIDAR probing the aperture angle is small $\theta_0 << 1$, therefore we estimate $\cos \theta$ as $-1$. Introduce new variables $z = t - \tau, \xi = \tau \sin \theta$. Intersection of the integration domain in (14) with the area in which $\psi$ is nonzero, satisfy the following inequalities (in new variables)

$$\xi \leq \rho_0,$$
$$0 \leq 2z - t \leq \Delta t,$$
$$0 \leq \xi \leq \epsilon(t - z),$$

where $\epsilon = t_0 \theta_0$. When $\frac{t}{2} > \frac{\rho_0^2}{\epsilon}$, the third inequality from (15) follows from the first ones, hence

$$I_1(t) = \lim_{\Delta t \to 0} \frac{2\pi \rho_0^2}{\Delta t} \int_{0}^{\frac{t}{2}} E(t - z, 2z - t, \pi) E(z, z, 0) \sigma(-1, z) \frac{dz}{(t - z)^2},$$

$$\frac{2\pi \rho_0^2}{t^2} E\left(\frac{t}{2}, 0, \pi\right) E\left(\frac{t}{2}, \frac{t}{2}, 0\right) \sigma(-1, \frac{t}{2}),$$

which results in LIDAR formula

$$\frac{2\pi \rho_0^2}{t^2} \exp\left[-\int_{0}^{t/2} \sigma(z) dz\right] \sigma(-1, \frac{t}{2}).$$

(18)

Contribution of the two-fold scattering (12) gives

$$I_2(t) = \lim_{\Delta t \to 0} \frac{4\pi}{\Delta t} \int_{\pi - \theta_0}^{\pi} \int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{t - \tau} \int_{0}^{\tau} E(\cos \gamma, \tau_1(\cos \theta_1 - 1) + t - \tau_2) \sigma(\cos \theta_1, t - \tau_1 - \tau_2) \psi(\tau_1 \sin \theta_1 + \tau_2 \sin \theta_2 \cos \phi, \tau_2 \sin \theta_2 \sin \phi, \tau_1 \cos \theta_1 + \tau_2 \cos \theta_2 + t - \tau_1 - \tau_2) d\theta_1 d\theta_2 d\gamma d\phi d\theta_2.$$  \hspace{1cm} (19)
where

\[
E = \exp[- \int_0^{t - \tau_2 - \tau_1} \sigma(t - \tau_2 - \tau_1 - \tau')]d\tau' \int_0^{\tau_1} \sigma((\tau_1 - \tau')\cos\theta_1 + t - \tau_2 - \tau_1)d\tau' - \int_0^{\tau_2} \sigma(\tau_1 \cos\theta_1 + (\tau_2 - \tau')\cos\theta_2 + t - \tau_2 - \tau_1)d\tau'.
\]  

(20)

It was used that the integrand depends only on \(\phi = \phi_1 - \phi_2\) and, due to definition of \(\psi\) and \(\cos\gamma\), is even in \(\phi\).

Let us introduce new variables:

\[
\begin{align*}
z_1 &= t - \tau_2 - \tau_1, \\
z_2 &= t \cos\theta_1 + t - \tau_1 - \tau_2, \\
\xi &= \tau_1 \sin\theta_1, \\
\eta &= \tau_2 \sin\theta_2, \\
\rho &= \left[\tau_1^2 \sin^2 \theta_1 + 2 \tau_1 \tau_2 \sin \theta_1 \sin \theta_2 + \tau_2^2 \sin^2 \theta_2\right]^{1/2}.
\end{align*}
\]

The sense of \(z_{1,2}, \rho\) has transparent intuitive meaning: \(z_{1,2}\) are heights of scattering points, \(\rho\) is the polar coordinate of a trajectory end. In the domain of consideration there is a one-to-one correspondence between old and new coordinates. The domain is fixed by \(\xi \geq 0, \eta \geq 0, \tau_1 = \sqrt{(z_1 - z_2)^2 + \xi^2}, \tau_2 = t - z_1 - \tau_1, \cos \theta_1 = \frac{(z_1 - z_2)^2 + \xi^2}{\sqrt{(z_1 - z_2)^2 + \xi^2}}, \sin \phi = \frac{\sqrt{(\rho^2 - (\xi - \eta)^2)(\rho^2 - (\xi + \eta)^2)}}{2 \rho}.
\]

(21)

The Jacobian evaluation gives:

\[
J \sin \theta_1 \sin \theta_2 = \frac{\rho}{(\tau_1 \tau_2)^2 |\cos \theta_1 | \sin \phi},
\]

(23)

that is obtained via transformation formulas. Below, as in derivation of the one-fold scattering, we shall approximate \(\cos \theta_1\) as 1.

Let us obtain conditions that restrict the integration domain taking into account the definition of \(\psi\). The conditions for \(\rho\) are obvious. From the problem statement it follows also that \(z_1 \geq 0, z_2 \geq 0\). Due to \(|\cos \phi| \leq 1, |\xi - \eta| \leq \rho, |\xi + \eta| \geq \rho.\)

(24)

From the choice of \(\psi\) and formulas above it follows, that

\[
0 \leq z_2 - t + z_1 + \sqrt{\xi^2 + (z_1 + z_2)^2} \leq \Delta t.
\]

(25)

While \(\pi - \theta_c \leq \theta_2 \leq \pi, \quad 0 \leq \frac{\eta}{z_2} \leq \epsilon = \tan \theta_c.\)

(26)

Besides that, the inequalities \(0 \leq \theta_2 \leq \tau_1, \quad 0 \leq \theta_1 \leq t\) holds automatically because \(\tau_1 + \tau_2 = t - z_1 \leq t\) as \(\tau_2 \geq 0, \) and \(z_1 \geq 1.\) Due to (25),

\[
\begin{align*}
0 \geq z_1 \geq \frac{t + \Delta t}{2}, \\
0 \geq z_2 \geq \frac{t + \Delta t}{2}.
\end{align*}
\]

(27)
Inequality (25) with fixed \( z_1 \) and \( z_2 \) gives restrictions for the variable \( \xi \).

\[
\xi_{\text{min}} = (t - 2z_1)(t - 2z_1) \geq \xi^2 \geq (t - 2z_1)(t - 2z_1) + 2(t - z_2 - z_1)\Delta t + (\Delta t)^2 = \xi_{\text{max}}^2.
\]

Cross-section of area of integration by the plane parallel to the plane \( z_1 = z_2 = \xi = 0 \) is presented on the Fig. 1 in the case of \( \xi < \rho_0, \quad \epsilon z_2 > \xi + \rho_0 \). It is seen that the straight line \( \eta = \epsilon z_2 \) in dependence on \( z_2, \xi, \epsilon \) crosses shaded area or not. It determines the integral (19) evaluation.

5 Evaluation and approximate formulas

Expression for \( I_3(t) \) splits to three contributions \( I_{21}, I_{22}, I_{23}; \) \( I_{22}, I_{23} \) respond the case \( \epsilon z_2 \leq \xi_{\text{max}} + \rho_0 \). For the integral \( I_{22} \rho_0 \leq \xi_{\text{max}} \), while for the \( I_{23} \) an opposite equality \( \xi_{\text{max}} \leq \rho_0 \) holds.

One can show that in conditions of \( \epsilon \sigma_{\text{max}} \rho_0 \ln \frac{\rho_0}{\rho_0} \ll 1 \) and \( \frac{\rho_0}{\epsilon} \gg \frac{\rho_0}{\epsilon} \), where \( \sigma_{\text{max}} = \sigma_{\text{max}} \sigma_{\text{max}} \sigma_{\text{max}} \sigma_{\text{max}} \); integrals \( I_{22} \) and \( I_{23} \) give contributions small compared to one of one-fold scattering. In particular even rough estimation gives:

\[
I_{22} \leq \frac{8\pi^2 \sigma_{\text{max}}^2 \rho_0^3 \epsilon}{t^2},
\]

\[
I_{23} \leq \frac{16\pi^2 \sigma_{\text{max}}^2 \rho_0^3 \epsilon}{t^2} (2 + \pi + \ln \frac{\rho_0}{\rho_0}).
\]

The integral \( I_{12} \) in the same conditions is comparable with the contribution of one-fold scattering. Limits of integration in it are found from the condition \( \epsilon z_2 > \xi_{\text{max}} + \rho_0 \). This condition determines the part of the square (27), points of which satisfy inequality

\[
z_1 \geq \frac{t}{2} - \frac{(\epsilon z_2 - \rho_0)^2}{2(t - 2z_2)} + O(\Delta t).
\]

Denoting by the simbol \( D_{\Delta t} \) the integration domain of the plane \( z_1 z_2 \), one has

\[
I_{21} = \lim_{\Delta t \to 0} \frac{4\pi}{\pi} \int_{D_{\Delta t}} d_z z_1 d_z z_2 \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi \int_0^{\rho_0} d\rho \int_{\xi - \rho}^{\xi + \rho} d\eta \frac{\xi^2}{(\xi^2 - \xi^2)(t - \xi - \sqrt{(z_1 - z_2)^2 + \xi^2})(\xi - \xi^2)(\xi - \xi^2)}.
\]

where \( E \) is defined above (20), \( \cos \theta_1 \) is given by (22). If one suppose that in a range of small \( \cos \gamma \) (scattering angles close to \( \frac{\pi}{2} \)) the cross-section \( \sigma \) changes not very quickly, the expression \( \sigma \sigma \sigma \sigma \) may be approximated by \( \sigma \sigma \sigma \sigma \sigma \), or, as in (5), by \( \sigma \sigma \sigma \sigma \sigma \). In such conditions the integration with respect to \( \eta, \rho, \xi \) can be performed in explicit form. After transition to limit of \( \Delta t \to 0 \), \( \cos \theta_1 = \frac{z_1 - z_2}{z_1 - z_2} \) and the integral under consideration takes the form

\[
I_{21} = 2\pi^2 \rho_0^2 \int_{D_0} d_z z_1 d_z z_2 \frac{E \sigma(-\cos \theta_1, z_2)\sigma(\cos \theta_1, z_1)}{z_1^2(t - z_1 - z_2)},
\]

(33)
where $D_0$ is defined by the following inequalities

$$0 \leq z_1 \leq \frac{t}{2}, \quad 0 \leq z_2 \leq \frac{t}{2}, \quad z_1 \geq \frac{t}{2} - \frac{(\epsilon z_2 - \rho_0^2)}{2(t - 2z_2)}.$$  

(34)

In variables $z_1, z_2$ the exponential factor is:

$$E = \exp[ -2\int_0^{z_1} \sigma_t(\tau)d\tau - \frac{t - 2z_1}{t - z_1 - z_2} \int_{z_1}^{z_2} \sigma_t(\tau)d\tau],$$

(35)

The expression (33) is the main result of the article: it represents, in the assumptions made here, the two-fold contribution into a response of the LIDAR probing.

6 Conclusion

The expansion we use converges because the number of particles conservation law in elastic collisions. Estimation of the convergence speed we delivered for a homogeneous layer thickness $d$. In this case the expansion by scattering multiplicity is the power series by optic thickness $\sigma d$.

Consideration of the three-fold contribution into a response, obtained by analogous scheme in the case of $\sigma_{max} d < 1$ allows to establish conditions in which one can restrict himself by two-terms of the expansion. In this case, in principle, the inverse problem of the function $\sigma$ reconstruction may be posed.

The polarisation account may be held by the scheme we use and do not introduce principal complications.

A diffusion of the mean phase and interference also may be included in this apparatus and do not contradict simplifications procedure.

The theory development include more detailed modeling of the scattering processes and geometry conditions.

By the way the condition $\sigma_{max} d < 1$ is not necessary in a case of nonspherical indicatrix which is met in real LIDAR problems. The case of strongly nonspherical indicatrix, perhaps, can be considered on a way of unification of our results with ones of [5].

In this work we present relatively simple method for It means that we can control evaluation process.

References

[1] A.M. Kolchuzhkin, V.V. Uchaikin. Introduction into the Theory of Particle Penetration through a Matter, Moscow, Atomizdat (1978) (in Russian).

V.V. Uchaikin, A.A. Lagutin, The Stochastic Importance. Moscow, Energoatomizdat (1993) (in Russian). V.V. Uchaikin, V.V. Ryzhov, The Stochastic Theory of High Energy Particle Transport, Novosibirsk, Nauka, Siberian Branch (1988) (in Russian).
[2] A. Buzdin, S. Leble. Boltzmann Equation Solution in LIDAR Problem Theses Conference Lidar Zonding, Tomsk (1978) Theses Conference Lidar Zonding. Tomsk: 174–176, 1978.

[3] A. Buzdin, S. Leble. Lidar Problem Solution in Double-Scattering Approximation VINITI N2536-80dep p 126. 1980.

[4] A. Buzdin, S. Leble Boltzmann Equation Solution in LIDAR Problem Theses Conference Lidar Zonding, Tomsk (1978) Theses Conference Lidar Zonding. Tomsk: 174–176, 1978.

[5] B. V. Kaul, I. V. Samochvalov. Equation of LIDAR probing of atmosphere with two-fold scattering account. Izv. VUZov, Fizika, 159, 109, 1975. I. V. Samokhvalov, ”Double-scattering approximation of lidar equation for inhomogeneous atmosphere,” Opt. Lett. 4, 12–14 (1979).

[6] M.V. Kazarnovskii, V. E. Pafomov, Solution of the Fokker-Plank equation for a layered medium. ZhETF, 74, 846-848, 1978