Abstract

We study two related problems: the Maximum weight $m'$-edge cover (MWEC) problem and the Fixed cost minimum edge cover (FCEC) problem. In the MWEC problem, we are given an undirected simple graph $G = (V, E)$ with integral vertex weights. The goal is to select a set $U \subseteq V$ of maximum weight so that the number of edges with at least one endpoint in $U$ is at most $m'$. Goldschmidt and Hochbaum [7] show that the problem is NP-hard and they give a 3-approximation algorithm for the problem. We present an approximation algorithm that achieves a guarantee of 2, thereby improving the bound of 3 [7].

In the FCEC problem, we are given a vertex weighted graph, a bound $k$, and our goal is to find a subset of vertices $U$ of total weight at least $k$ such that the number of edges with at least one edges in in $U$ is minimized. A $2(1 + \epsilon)$-approximation for the problem follows from the work of Carnes and Shmoys [4]. We improve the approximation ratio by giving a 2-approximation algorithm for the problem. Can we get better results using methods based on linear programming? We take a first step and show that the natural LP for FCEC has an
integrality gap of $2 - o(1)$. We improve the NP-completeness result for MWEC \cite{GoldschmidtHochbaum} by showing that unless a well-known instance of the Dense $k$-subgraph admits a constant ratio, FCEC and MWEC do not admit a PTAS. Note that the best approximation guarantee known for this instance of Dense $k$-subgraph is $O(n^{2/9})$ \cite{Chuzhoy}. We show that for any constant $\rho > 1$, an approximation guarantee of $\rho$ for the FCEC problem implies a $\rho(1 + o(1))$ approximation for MWEC. Finally, we define the Degrees density augmentation problem which is the density version of the FCEC problem. In this problem we are given an undirected graph $G = (V, E)$ and a set $U \subseteq V$. The objective is to find a set $W$ so that $(e(W) + e(U, W))/\text{deg}(W)$ is maximum. This problem admits an LP-based exact solution \cite{Chuzhoy}. We give a combinatorial algorithm for this problem.

1 Introduction

We study two natural budgeted edge covering problems in undirected simple graphs with integral weights on vertices. The budget is given either as a bound on the number of edges to be covered or as a bound on the total weight of the vertices. We say that an edge $e$ is touched by a set of vertices $U$ or that $e$ touches the set of vertices $U$, if at least one of $e$’s endpoints is in $U$. Specifically, the problems that we study are as follows. The Maximum weight $m'$-edge cover (MWEC) problem that we study was first introduced by Goldschmidt and Hochbaum \cite{GoldschmidtHochbaum}. In this problem, we are given an undirected simple graph $G = (V, E)$ with integral vertex weights. The goal is to select a subset $U \subseteq V$ of maximum weight so that the number of edges touching $U$ is at most $m'$. This problem is motivated by application in loading of semi-conductor components to be assembled into products \cite{GoldschmidtHochbaum}.

We also study the closely related Fixed cost minimum edge cover (FCEC) problem in which given a graph $G = (V, E)$ with vertex weights and a number $W$, our goal is to find $U \subseteq V$ of weight at least $W$ such that the number of edges touching $U$ is minimized.

Finally, we study the Degrees density augmentation problem which is the density version of the FCEC problem. In the Degrees density augmentation problem, we are given an undirected graph graph $G = (V, E)$ and a set $U \subseteq V$ and our goal is to find a set $W$ with maximum augmenting density i.e., a set $W$ that maximizes $(e(W) + e(U, W))/\text{deg}(W)$. 
1.1 Related Work

Goldschmidt and Hochbaum [7] introduced the MWEC problem. They show that the problem is NP-complete and give algorithms that yield 2-approximate and 3-approximate algorithm for the unweighted and the weighted versions of the problem, respectively.

A class of related problems are the density problems – problems in which we are to find a subgraph and the objective function considers the ratio of the total number or weight of edges in the subgraph to the number of vertices in the subgraph. A well known problem in this class is the Dense \( k \)-subgraph problem (\( DkS \)) in which we want to find a subset of vertices \( U \) of size \( k \) such that the total number of edges in the subgraph induced by \( U \) is maximized. The best ratio known for the problem is \( n^{1/4+\epsilon} \) [5, 2], which is an improvement over the bound of \( O(n^{1/3-\epsilon}) \), for \( \epsilon \) close to 1/60 [4]. The Dense \( k \)-subgraph problem is APX-hard under the assumption that NP problems can not be solved in subexponential time [9]. Interestingly, if there is no bound on the the size of \( U \) then the problem can be solved in polynomial time [11, 6].

Consider an objective function in which we minimize \( \deg(U) \). One can associate a cost \( c_u = \deg(u) \) with each vertex \( u \) and a size \( s_u = w(u) \) for each vertex \( u \), and then the objective is just to minimize \( \deg(U) \) subject to \( \sum s_u x_u \geq k \). Carnes and Shmoys [4] give a \((1 + \epsilon)\)-approximation for the problem. Using this result and the observation that the objective function is at most a factor of 2 away from the objective function for the FCEC problem, a \( 2(1 + \epsilon) \)-approximation follows for the FCEC problem.

Variations of the Dense \( k \)-subgraph problem in which the size of \( U \) is at least \( k \) (\( Dalk \)) and the size of \( U \) is at most \( k \) (\( Damk \)) have been studied [1, 10]. In [1, 10], they give evidence that \( Damk \) is just as hard as \( DkS \). They also give 2-approximate solutions to the \( Dalk \) problem. In [10], they also consider the density versions of the problems in directed graphs. Gajewar and Sarma [8] consider a generalization in which we are give a partition of vertices \( U_1, U_2, \ldots, U_t \), and non-negative integers \( r_1, r_2, \ldots, r_t \). the goal is to find a densest subgraph such that partition \( U_i \) contributes at least \( r_i \) vertices to the densest subgraph. They give a 3-approximation for the problem, which was improved to 2 by Chakravarthy et al. [3], who also consider other generalizations. They also show using linear programming that the Degrees density augmentation problem can be solved optimally.

A problem parameterized by \( k \) is Fixed Parameter Tractable [12], if it admits an exact algorithm with running time of \( f(k) \cdot n^{O(1)} \). The function \( f \) can be exponential in \( k \) or larger. Proving that a problem is W[1]-hard (with respect to parameter \( k \)) is a strong indication that it has no FPT algorithm with parameter \( k \) (similar to NP-hardness implying the likelihood
of no polynomial time algorithm). The FCEC problem parameterized by $k$ is W[1] hard but admits a $f(k, \epsilon) \cdot n^{O(1)}$ time, $(1 + \epsilon)$-approximation, for any constant $\epsilon > 0$ [12]. This is in contrast to our result that shows that it is highly unlikely that FCEC admits a polynomial time approximation scheme (PTAS), if the running time is bounded by a polynomial in $k$.

1.2 Preliminaries

The input is an undirected simple graph $G = (V, E)$ and vertex weights are given by $w(\cdot)$. Let $n = |V|$ and $m = |E|$. For any subset $S \subseteq V$, let $\overline{S} = V \setminus S$. Let $e(P, Q)$ be the set of edges with one endpoint in $P$ and the other in $Q$. Let $\deg(S)$ denote the sum of degrees of all vertices in $S$, i.e., $\deg(S) = \sum_{v \in S} \deg(v)$. Let $\deg_H(v)$ denote the number of neighbors of $v$ among the vertices in $H$. Let $\deg_H(S)$ denote the quantity $\sum_{v \in S} \deg_H(v)$. We use $OPT$ to denote an optimal solution as well as the cost of an optimal solution. The meaning will be clear from the context in which it is used.

For set $U \subseteq V$, let $T(U)$ be the collection of all edges with at least one endpoint in $U$. Namely, is the set of edges touching $U$. We denote $t(U) = |T(U)|$. The set of edges with both endpoints in $U$, also called internal edges of $U$, is denoted by $E(U)$. We denote $e(U) = |E(U)|$. We denote by $e(X, Y)$ the number of edges with one endpoint in $X$ and one in $Y$. Let $e_U(X, Y)$ be the number of edges between $X \cap U$ and $Y \cap U$ in the graph $G(U)$ induced by $U$.

**Lemma 1.1** The FCEC problem admits a simple 2-approximate solution in case of uniform vertex weights.

Proof: Let $Z$ be the set of $k$ lowest degree vertices in $G$. The set $Z$ is a 2-approximate solution. Why? Let $b$ be the average degree of vertices in $Z$. Thus $t(Z) \leq bk$. The claim follows since $t(OPT) \geq \deg(OPT)/2 \geq bk/2$. □

From Lemma 1.1 if $\deg(OPT) \geq bk(1 + \epsilon)$, we obtain a $2/(1 + \epsilon) < 2$ approximation guarantee using the set $Z$ as the solution. Henceforth we assume that $\deg(OPT) < bk(1 + \epsilon)$

**Claim 1.2** For every set $U$, $t(U) = \deg(U) - e(U)$

Proof: Consider separately the edges $E(U, V \setminus U)$ and $E(U)$. Note that the edges $E(U, V \setminus U)$ are counted once in the sum of degrees, but edges in $E(U)$ are counted twice. Thus in order to get the number of edges touching $U$, we need to subtract $e(U)$ from $\deg(U)$. □
1.3 Our results

Our contributions in this paper are as follows.

• For the MWEC problem we give an algorithm that yields an approximation guarantee of 2, thereby improving the guarantee of 3 given by Goldschmidt and Hochbaum [7].

• We give a 2-approximate solution to the FCEC problem. This improves the $2(1 + \epsilon)$-ratio that follows from the work of Carnes and Shmoys [4].

• Can linear programming be used to improve the ratio of 2 for FCEC and MWEC problems? We take a first step and show that a natural LP for FCEC has an integrality gap of $2(1 - o(1))$, even for the unweighted case.

• We show that unless a well-known instance of the Dense $k$-subgraph admits a constant ratio, FCEC and MWEC do not admit a PTAS. Note that the best approximation guarantee known for this instance of Dense $k$-subgraph is $O(n^{2/9})$ [2]. This gives a stronger hardness result than the NP-completeness result known for MWEC [7].

• For any constant $\rho > 1$, we show that if FCEC admits a $\rho$-approximation algorithm then MWEC admits a $\rho(1 + o(1))$-approximation algorithm.

• We give a combinatorial algorithm that solves the Degrees density augmentation problem optimally.

2 A 2-approximation for Maximum Weight $m'$-Edge Cover

In this section we give a dynamic programming based solution for the MWEC problem. The idea of using dynamic programming in this context was first proposed by Goldschmidt and Hochbaum [7]. Recall that in the MWEC problem, we are given an undirected simple graph $G = (V, E)$ with integral vertex weights. The goal is to select a subset $U \subseteq V$ of maximum weight so that the number of edges touching $U$ is at most $m'$.

We will guess the following entities (by trying all possibilities) and for each guess, we use dynamic programming to solve the problem.

1. $H^* = \{v_h\}$, where $v_h$ is the heaviest vertex in an optimal solution.
2. $P_H = e(H^*, OPT \setminus H^*)$ – the number of neighbors of $v_h$ in the optimal solution. There are at most $n$ possibilities.

3. $D_H = \deg_H(OPT \setminus H^*)$: total degree of vertices in $OPT \setminus H^*$ in the graph induced by vertices in $V \setminus H^*$. There are at most $n^2$ possibilities.

We will try all combinations of the above entities. Since there are at most polynomial number of possibilities for each entity, we have at most polynomial number of possibilities in total. We define the following subproblems as part of our dynamic programming solution. Let $H$ be a guess for the singleton set $H^*$ that contains the heaviest vertex in an optimal solution. Let $\{v_1, v_2, \ldots, v_{n-1}\}$ be the vertices in $V \setminus H^*$ (recall $V \setminus H^* = \overline{H}$. Then, for any $H$, we solve the following subproblems.

$$A[H, i, P_H, D_H]$$ denote the maximum weighted subset $Q \subseteq \{v_1, v_2, \ldots, v_i\}$ such that $e(H, Q) \geq P_H$ and $\deg_H(Q) \leq D_H/2$.

Note that while the natural bound on $\deg_H(Q)$ is $D_H$, using such a bound will lead to an infeasible solution. For fixed parameters $H$, $P_H$, and $D_H$, we are interested in $A[H, n-1, P_H, D_H]$. We use the following recurrence as the basis for our dynamic programming solution: the value of $A[H, i, P_H, D_H] = -\infty$ in any of the following three cases – (i) $i = 0$ and $P_H > 0$, (ii) $i = 0$ and $D_H < 0$, and (iii) $D_H/2 > m' - e(H, \overline{H})$. When $i = 0$, $P_H \leq 0$ and $D_H \geq 0$, the value of $A[H, i, P_H, D_H] = 0$. Otherwise, we have

$$A[H, i, P_H, D_H] = \max\{A[H, i-1, P_H, D_H], w(v_i) + A[H, i-1, P'_H, D'_H]\}$$

where, $P'_H = P_H - \deg_H(v_i)$ and $D'_H = D_H - 2(\deg_H(v_i))$. Our solution is given by $\max_{H, P_H, D_H}\{w(H) + A[H, n-1, P_H, D_H]\}$.

**Analysis**

**Lemma 2.1** Our algorithm yields a feasible solution.

**Proof:** Let $H' \cup Q'$, where $Q' \subseteq V \setminus H'$, be the set of vertices returned by our solution. The number of edges with at least one endpoint in $H' \cup Q'$, is

$$= e(H', \overline{H'}) + e(Q', \overline{H'})$$

$$\leq e(H', \overline{H'}) + \deg_H(Q')$$

$$\leq e(H', \overline{H'}) + D_H/2$$

$$\leq e(H', \overline{H'}) + (m' - e(H', \overline{H'}))$$

(using the base case)

$$= m'$$

\qed
Lemma 2.2 The above algorithm results in a 2-approximate solution.

Proof: Recall that $H^*$ consists of the highest degree vertex in the optimal solution. Let $Q^*$ be the remaining vertices in the optimal solution. Consider the scenario when our algorithm makes the correct guess for $H^*$. Let $Q \subseteq H^*$ be the solution returned by the dynamic program in this setting. We know that

$$\deg_{H^*}(Q) \leq \frac{\deg_{H^*}(Q^*)}{2}$$

We now use ideas from [7] to show that $w(H^* \cup Q) \geq 2w(H^* \cup Q^*)$. Recall that $H' \cup Q'$ be the output of our algorithm. Since $w(H' \cup Q') \geq w(H^* \cup Q^*)$, it follows that our solution is a factor of at most 2 away from $OPT$.

Consider any arbitrary ordering of vertices $v_1, v_2, \ldots$ in $Q^*$. Note that the weight of each vertex in $Q^*$ is at most $w(H^*)$. Let $Q^*_r$ denote the the first $r$ vertices in the above ordering of vertices of $Q^*$. Let $p$ be the first index such that $\deg_{H^*}(Q^*_p) > \deg_{H^*}(Q^*)/2$. This implies the following – (i) $\deg_{H^*}(Q^*_{p-1}) \leq \frac{\deg_{H^*}(Q^*)}{2}$, and (ii) $\deg_{H^*}(Q^* \setminus Q^*_p) < \frac{\deg_{H^*}(Q^*)}{2}$. Note that both the sets $Q^*_{p-1}$ and $Q^* \setminus Q^*_p$ (neither set contains $v_p$) are feasible candidates for the set $Q$, the solution returned by our algorithm when the heaviest vertex set was chosen to be $H^*$. Since $w(Q) \geq w(Q^*_{p-1})$, $w(Q) \geq w(Q^* \setminus Q^*_p)$, and $w(v_p) \leq w(H^*)$, we have

$$w(OPT) \leq w(H^* \cup Q^*)$$

$$\leq w(H^*) + w(Q^*)$$

$$\leq w(H^*) + w(Q^*_{p-1}) + w(v_p) + w(Q^* \setminus Q^*_p)$$

$$\leq w(H^*) + w(Q) + w(H^*) + w(Q)$$

$$= 2w(H^* \cup Q)$$

$$\leq 2w(H' \cup Q')$$

\[ \square \]

3 A 2-approximation for Fixed Weight Minimum Edge Cover

Recall the FCEC problem: Given a graph $G = (V, E)$ with arbitrary vertex weights and a positive integer $W$, our objective is to choose a set $S \subseteq V$ of vertices of total weight at least $W$ such that the number of edges with at least one end point in $S$ is minimized.

We will solve the following related problem optimally and then show that an optimal solution to the problem is a 2-approximation to FCEC: we want
to find a subset \( S \) of vertices such that \( \text{deg}(S) \) is smallest and \( w(S) \) is at least \( W \).

We use the dynamic programming algorithm of the well-known Knapsack problem to find a solution to the above problem. For completeness, we restate the dynamic programming formulation below.

\[
P[i, D]: \text{maximum weight of set } Q \subseteq \{v_1, v_2, \ldots, v_i\} \text{ such that } \text{deg}(Q) \text{ is at most } D.
\]

Note that \( P[0, D] = 0 \), for all values of \( D \) is the base case. For all other case, we invoke the following recurrence.

\[
P[i, D] = \max\{P[i - 1, D], w(v_i) + P[i - 1, D - w(v_i)]\}
\]

After filling the table \( P \) using dynamic programming, we scan all entries of the form \( P[|V|, D] \) to find the smallest value of \( D \) for which \( P[|V|, D] \geq W \). Let \( S \) be the corresponding set.

**Lemma 3.1** The is a 2-approximate solution to the Fixed Cost Minimum Edge Cover Problem as follows.

\[
t(S) \leq \text{deg}(S) \leq \text{deg}(OPT) = 2(\text{deg}(OPT)/2) \leq 2OPT
\]

### 4 Integrality gap for Fixed Cost Minimum Edge Cover

Consider the following natural integer linear program for the problem

\[
\min \sum_{e} y_e
\]

subject to

\[
\sum_{v \in V} x_v \geq k,
\]

\[
y_e \geq x_u, \quad \forall e = (u, v)
\]

\[
y_e \geq x_v, \quad \forall e = (u, v)
\]

\[
x_v \in \{0, 1\}, \quad \forall v \in V
\]

\[
y_e \in \{0, 1\}, \quad \forall e \in E
\]

The LP relaxation can be obtained by relaxing the integrality constraints on \( x_v \) and \( y_e \) to \( x_v \geq 0, \forall v \in V \) and \( y_e \geq 0, \forall e \in E \).

**Theorem 4.1** The above LP has an integrality gap of \( 2(1 - o(1)) \).
Let \( k = \lfloor \sqrt{n} \rfloor \). Construct a graph \( G \) on \( n \) vertices as follows. For each pair of vertices, include an edge between the pair with a probability \( 1/\lfloor \sqrt{n} \rfloor \). For any vertex \( v \), \( E[\text{deg}(v)] = n(1/\lfloor \sqrt{n} \rfloor) \leq \lfloor \sqrt{n} \rfloor \). Using Chernoff bounds, for \( 0 < \delta < 1 \), we have

\[
\sqrt{n}(1 - o(1)) \leq \text{deg}(v) \leq \sqrt{n}(1 + o(1))
\]

Consider any subset \( Q \) of vertices in \( G \) such that \( |Q| = \lfloor \sqrt{n} \rfloor \). Then we have

\[
E[e(Q)] = \frac{1}{\lfloor \sqrt{n} \rfloor} \left( \frac{Q}{2} \right) = \frac{\lfloor \sqrt{n} \rfloor (\lfloor \sqrt{n} \rfloor - 1)}{2 \lfloor \sqrt{n} \rfloor} = \frac{\lfloor \sqrt{n} \rfloor - 1}{2}
\]

Thus, \( n \geq 4 \), we have \( \sqrt{n}/4 \leq E[e(Q)] < \sqrt{n}/2 \). We use the following Chernoff bound to obtain the probability that \( e(Q) \geq n^{1-\epsilon} \), for a constant \( \epsilon \).

\[
\Pr[e(Q) \geq (1 + \delta)E[e(Q)]] \leq \left( \frac{\exp(\delta)}{(1 + \delta)^{(1+\delta)}} \right)^{E[e(Q)]}
\]

In our case, \( 2n^{1/2-\epsilon} \leq 1 + \delta \leq 4n^{1/2-\epsilon} \), thus we get

\[
\Pr[e(Q) \geq n^{1-\epsilon}] \leq \left( \frac{\exp(4n^{1/2-\epsilon})}{(2n^{1/2-\epsilon})2n^{1/2-\epsilon}} \right)^{\sqrt{n}/4}
\]

Let \( f(n, \epsilon) = \left( \frac{\exp(n^{1/2-\epsilon})(2n^{1/2-\epsilon}-2)}{(2n^{1/2-\epsilon})^{n^{1/2-\epsilon}}} \right)^{\sqrt{n}} \). The number of sets of size \( \lfloor \sqrt{n} \rfloor \) is given by \( \left( \frac{n}{\sqrt{n}} \right) \leq (ne/\lfloor \sqrt{n} \rfloor)\sqrt{n} = (\lfloor \sqrt{n} \rfloor e)^{\sqrt{n}} \). The probability that there is no subset of size \( \lfloor \sqrt{n} \rfloor \) that has at least \( n^{1-\epsilon} \) edges is given by the union-bound as follows

\[
f(n, \epsilon) \left( \frac{n}{\sqrt{n}} \right) << 1
\]

The number of edges with at least one end point in \( Q \) is given by

\[
t(Q) = \text{deg}(Q) - e(Q) \\
\geq \lfloor \sqrt{n} \rfloor \cdot \sqrt{n}(1 - o(1)) - n^{1-\epsilon} \\
= n(1 - o(1))
\]

On the other hand, consider the fractional solution in which \( x_v = 1/\sqrt{n} \), for each \( v \) and \( y_e = 1/\sqrt{n} \), for each \( e \in E \). This LP solution is feasible and has a cost of \( |E|/\sqrt{n} \). The number of edges \( |E| = n\sqrt{n}/2(1 + o(1)) \). Thus the cost of the LP solution is at most \( n(1 + o(1))/2 \), which results in a gap of \( 2(1 - o(1)) \).
5 APX-hardness for unweighted Fixed Cost Minimum Edge Cover and Maximum Weight \(m'-\)Edge Cover

Let \(G\) be the input for the Dense \(k\)-subgraph problem and let \(OPT\) be an optimal subset of \(k\) vertices. To prove the hardness result we consider the following important instance \((G, k)\) of the Dense \(k\)-subgraph problem.

\(P_1.\) The \(k/2\) largest degree vertices, \(H\), in \(G\), have average degree \(d_H = \Theta(n^{1/3})\)

\(P_2.\) \(k = \Theta(n^{2/3})\), and

\(P_3.\) \(OPT\) has average degree \(d^* = \Theta(n^{1/3})\).

Feige et al. \cite{Feige} gave a relatively simple \(n^{1/3}\) ratio for the Dense \(k\)-subgraph problem. The ratio was improved to \(n^{1/3-1/60}\) by improving the ratio of \(n^{1/3}\) for two very specific instances. One of the instances was the important instance defined above. We now show that if FCEC admits a PTAS then this important instance for the Dense \(k\)-subgraph problem admits a constant factor approximation algorithm.

Consider the important instance and assume that \(e(H) = O(k)\). Note that a constant-factor approximation for the important instance implies a constant approximation for the case when \(e(H) = O(k)\). Clearly, removing \(H\) does not change the value of the optimum up to lower order terms. This modified instance has a maximum degree of \(O(n^{1/3})\) and it also satisfies properties \(P_2\) and \(P_3\) of the important instance. The best ratio, given the state-of-the-art, for the modified instance is \(\Theta(n^{2/9})\) (M. Charikar, Private Communication) and hence the following conjecture seems highly likely: The modified instance does not admit a constant approximation.

Claim 5.1 We can modify \(G\) into a graph \(G'\) for which the optimal solutions for the Dense \(k\)-subgraph problem and the FCEC are the same, and in addition, the value of the optimum solution for the Dense \(k\)-subgraph problem does not change.

Proof: Let the largest degree of \(G\) be \(\Delta = c_1 \cdot n^{1/3}\).

We show how to make the graph \(\Delta\) regular without changing the optimum value for the Dense \(k\)-subgraph instance. For every vertex \(v \in V\) add \(\Delta - \deg(v)\) vertices \(F_v\) and connect \(v\) to all the vertices of \(F_v\). The sets \(F_v\) for different vertices are disjoint. Let \(F = \bigcup_v F_v\). We now add a set of \(n^2\) disjoint edges
(no two edges share a vertex) $F'$ (and thus $2n^2$ new vertices). We then make $F \cup F'$ regular by adding a random $\Delta - 1$ regular graph on $F' \cup F$. Let $G'$ be the new graph.

Clearly, every vertex has degree $\Delta$ now. Indeed all vertices in $F$ and the $2n^2$ vertices that were added had degree exactly one before the random $\Delta - 1$ subgraph is added. Since $G'$ is regular, the sum of degrees in $G'$ for any $k$ vertices is the same. As $t(U) = \text{deg}(U) - e(U)$ the optimal solutions for FCEC and Dense $k$-subgraph are the same on regular graphs. Since the graph on $F \cup F'$ is basically a random graph with degrees $O(n^{1/3})$, and at least $n^2$ vertices, basic calculations show that for every $F'' \subseteq F \cup F'$, $e(F'') = O(|F''|)$. In addition, every vertex in $F' \cup F$ has degree at most 1 in $G$. Therefore any $F'' \subseteq F \cup F'$, can contribute at most $\text{deg}(F'') = O(|F''|)$ to the number of edges in a Dense $k$-subgraph solution. As $|F''| \leq k$, it follows that $F''$ can contribute $O(k)$ edges to the Dense $k$-subgraph solution. Observe that this number is negligible compared to the Dense $k$-subgraph in $G$. The number of edges in the Dense $k$-subgraph optimum in $G$ is $c'kn^{1/3}$, for some constant $c'$. Hence the value of the optimum solution does not change (up to lower order terms) by the addition of $F \cup F'$.

**Theorem 5.2** A PTAS for FCEC problem implies a constant factor approximation for the modified Dense $k$-subgraph instance.

**Proof:** Since $G'$ is a regular graph, the optimal FCEC solution is the same as the Dense $k$-subgraph optimum solution. In fact the number of touching edges is $\Delta k - c'kn^{1/3}$. Recall that $\Delta = c_1 \cdot n^{1/3}$. Thus the optimum is $c_1 kn^{1/3} - c'kn^{1/3}$.

By the value of $k$, the optimum value is $c_1 n - c' n$. If FCEC has a PTAS then there exists a $1 + c'/c_1$-approximation for the FCEC problem. Assuming this ratio, a set $U$ is output so that it is touched by at most $c_1 n - c' n + (c'/c_1)(c_1 n - c' n) = c_1 n - (c^2/c_1)n$ edges. This implies that $e(U) = (c^2/c_1)n$. Thus we find a subgraph with $k$ vertices and at least $c^2 n/c_1$ edges. Therefore the ratio obtained on the modified instance is $c'/((c^2/c_1)) = c_1/c'$, contradicting the conjecture that the modified instance does not admit a constant approximation.

### 5.1 APX-hardness for Maximum Weight $m'$-Edge Cover

We show that PTAS for (unweighted) MWEC implies a PTAS for (unweighted) FCEC on the modified instance. As we showed that this is not possible, MWEC is APX-hard as well.

Recall that the optimum for the modified instance had $c_1 n - c' n$ edges
and size k. Furthermore, the modified instance is Δ-regular.

Let \( OPT \) be the optimum solution for the FCEC instance. The number of edges touching \( OPT \) is at least: \( t(OPT) \geq k \cdot \Delta/2 \). We impose a bound of \( c_1 n - c' n \) on the number of edges to the hypothetical PTAS for the MWEC problem. A PTAS algorithm for MWEC will return a set \( S \) with size at least \( k/(1 + \epsilon) \), touched by at most \( c_1 n - c' n \) edges.

The amount of vertices still required to be added to transform \( S \) to a legal FCEC output is \( k - k/(1 + \epsilon) = \epsilon \cdot k/(1 + \epsilon) \). We can complete the set \( S \) to size \( k \) by any set \( S' \) of \( \epsilon \cdot k/(1 + \epsilon) \) vertices. In such case \( t(S') \leq \epsilon \cdot \Delta \cdot k/(1 + \epsilon) \). As we showed before that \( t(OPT) \geq k \cdot \Delta/2 \), it follows that \( t(S \cup S') \leq t(OPT) + 2\epsilon \cdot t(OPT) \).

Thus for getting a ratio of \( 1 + \delta \) for any constant \( \delta \) just set \( \epsilon = \delta/2 \). Therefore, the assumption that the MWEC problem admits a PTAS, implies that the FCEC problem admits a PTAS on the modified instance, which is highly unlikely, by Theorem 5.2.

6 An approximation for Fixed Cost Minimum Edge Cover implies the same approximation for Maximum Weight \( m' \)-Edge Cover

We first transform the input instance for the MWEC problem to one in which the optimum value of the objective function is at most \( n^5 \) by paying a very small penalty in the approximation ratio.

**Lemma 6.1** For the Maximum weight \( m' \)-subgraph problem, we can convert the input instance \( \langle G, w, m' \rangle \), with an optimal solution denoted by \( OPT \) into an instance \( \langle G', w', m' \rangle \), with optimal solution \( OPT'' \), such that \( OPT'' \leq n^5 \). Furthermore, if \( OPT'' \) is the total weight of the vertices in \( OPT'' \) under the weight function \( w \), then

\[
OPT'' \geq OPT(1 - 1/n)(1 - 1/n^2)
\]

**Proof:** Let \( v_1, v_2, \ldots, v_n \) be the vertices in \( G \) such that \( w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n) \). Let \( v_p \) be the last vertex in the ordering such that \( w(v_p) \geq w(v_1)/n^2 \). In other words, for each \( j, p < j \leq n \), \( w(v_1) > n^2 w(v_j) \). Let \( G' \) is the graph induced on vertices \( v_1, v_2, \ldots, v_p \). Let \( OPT_1 \) be the optimal solution for the instance \( \langle G', w, m' \rangle \). Note that \( OPT \) may choose some vertices from the set \( \{v_{p+1}, v_{p+1}, \ldots, v_n\} \). The error incurred in not considering these
vertices is at most \(n(w(v_1)/n^2) \leq \text{OPT}/n\). Thus we get

\[
\text{OPT}_1 \geq \text{OPT}(1 - 1/n)
\]

We now scale the weights of vertices in \(G'\) to create an instance \(\langle G', w', m' \rangle\), where

\[
w'(v_j) = \left\lfloor \left(\frac{w(v_j)}{w(v_p)}\right)n^2 \right\rfloor
\]

Let \(\text{OPT}''\) be an optimal solution to \(\langle G', w', m' \rangle\). Clearly, \(\text{OPT}'' \leq n^5\). Let \(\text{OPT}'\) be the cost of the solution \(\text{OPT}''\) under the weight function \(w\), i.e.,

\[
\text{OPT}' = \sum_{v \in \text{OPT}''} w(v).
\]

Thus we have

\[
\text{OPT}' \geq \text{OPT}_1 \left(1 - \frac{1}{n^2}\right) \geq \text{OPT} \left(1 - \frac{1}{n}\right) \left(1 - \frac{1}{n^2}\right)
\]

(1)

\[\square\]

**Theorem 6.2** For some constant \(\alpha\), an \(\alpha\) approximation guarantee for FCEC implies an \(\alpha(1 + o(1))\) approximation guarantee for MWEC.

**Proof:** Suppose that we have an \(\alpha > 1\) approximation algorithm for FCEC, for some constant \(\alpha\). Using Lemma 6.1, we transform the MWEC instance \((G, m')\) with an optimal weight \(W^*\) to an instance in which the optimum weight \(W^* \leq n^4\). This increase the approximation ratio by a factor of only \((1 + o(1))\). We now consider the modified instance \((G', m')\) as an input to FCEC. We guess the value of \(W^*\) by trying all possible integral values between 1 and \(n^4\). For each guess of \(W^*\), we apply the \(\alpha\)-approximation algorithm for FCEC to the new instance. When our guess \(W^*\) is correct and we apply the algorithm, we obtain a set \(U\) of vertices of cost at least \(W^*\) and that touch at most \(\alpha \cdot m'\) edges.

Create a new set \(B\) in which every vertex from \(U\) is chosen with a probability \(1/\alpha\). We say that an edge \(e\) is deleted if \(e \notin E(B)\). Let \(\tau\) be a constant.

We consider the following "bad" events: (i) \(w(B) \leq W^*/((1 + \tau)\alpha)\), (ii) \(t(B) > m'\).

We first bound the probability that \(w(B) \leq W^*/((1 + \tau)\alpha)\). The expected cost of \(B\) is \(w(U)/\alpha = W^*/\alpha\). Consider the expected cost of \(U \setminus B\). The expected cost is \(W^* - W^*/\alpha\). The event that \(w(B) \leq W^*/(\alpha(1 + \tau))\) is equivalent to the event \(w(U) - w(B) \geq W^* - W^*/(\alpha(1 + \tau)) = W^*(1 - 1/(\rho(1 + \tau)))\). By the Markov’s inequality, the last event has probability at most \((1 - 1/\alpha)/(1 - 1/(\alpha(1 + \tau))) = 1 - \tau/(\alpha + \alpha \cdot \tau - 1)\).

We now bound the probability of the second bad event. The expected number of edges in \(E(B)\) is at most \(m'(1 - (1 - 1/\alpha)^2)\). Note that the events
that edges are deleted are positively correlated because given that an edge $(v, u)$ is deleted, one of the possibilities that can cause this event, is that $v$ is deleted, and in that case all edges of $v$ are deleted with probability 1. Clearly, we can assume that $m' \geq c$ for any constant $c$. Otherwise, we can solve the MWEC problem in polynomial time by checking all subsets of edges. By the Chernoff bound, the probability that the number of edges is more than $m'$ is bounded by $\exp(-c\delta^2/2)$, for some $\delta < 1$. We can choose a large enough $c$ so that the above probability is at most $\tau/(2(\alpha + \alpha \cdot \tau - 1))$. This would mean that the sum of probabilities of bad events is strictly smaller than 1. This construction can be derandomized by the method of conditional expectations.

7 Exact algorithm for the Degrees Density Augmentation Problem

The Degrees density augmentation problem is as follows: Given a graph $G = (V, E)$ and a subset $U \subseteq V$, the objective is to find a subset $W \subseteq V \setminus U$ such that

$$\rho = \frac{e(W) + e(U, W)}{\deg(W)}$$

is maximized.

The Degrees density augmentation problem is related to the FCEC problem in the same way as the Densest subgraph problem is related to the Dense $k$-subgraph problem. A natural heuristic for the FCEC problem would be to iteratively find a set $W$ with good augmentation degrees density. A polynomial time exact solution for the problem using linear programming is given in [3]. Here we present a combinatorial algorithm.

We solve the Degrees density augmentation problem exactly by finding minimum $s$-$t$ cut in the flow network constructed as follows. Let $\overline{U}$ denote the set $V \setminus U$. In addition to the source $s$ and the sink $t$, the vertex set contains $V_{E'} \cup \overline{U}$, where $V_{E'} = \{v_e | e \in E \text{ and both end points of } e \text{ are in } \overline{U}\}$. There is an edge from $s$ to every vertex in $V_{E'} \cup \overline{U}$. If $a$ is a vertex in $V_{E'}$ then the capacity of the edge $(s, a)$ is 1, otherwise, the capacity of the edge is $\deg_U(a)$. For each vertex $v_e \in V_{E'}$, where $e = (p, q)$, there are edges $(v_e, p)$ and $(v_e, q)$. Each such edge has a large capacity of $M = \infty$ (any capacity of at least $n^5$ would work). Finally, each vertex $p \in \overline{U}$ is connected to $t$ and has a capacity of $\rho \cdot \deg(p)$. 

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### 7.1 Algorithm

For a particular value of $\rho$, let $W_s \subseteq U$ be the vertices that are on the $s(t)$ side of a minimum $s$-$t$ cut. Let $V_{E'}^s \subseteq V_{E'}(V_{\overline{E'}} \subseteq V_{E'})$ be the vertices in $V_{E'}$ that are on the $s(t)$ side of the minimum $s$-$t$ cut. We now state the algorithm.

1. Construct the flow network as shown above.

2. For each value of $\rho$, compute a minimum $s$-$t$ cut and find the resulting value of $e(W_s) + e(U,W_s) - \rho \deg(W_s)$. Find the largest value of $\rho$ for which the expression is at least 0.

3. Return $W_s$ corresponding to the largest value of $\rho$.

### 7.2 Analysis

**Lemma 7.1** Any minimum $s$-$t$ cut in the above flow network has capacity at most $2n^2$.

*Proof:* This follows because the $s$-$t$ cut $(s, V \setminus \{s\})$ has capacity at most $2n^2$. \qed

**Lemma 7.2** For any minimum $s$-$t$ cut $C$, $|V_{E'}^s| = e(W_s)$.

*Proof:* Note that it cannot be the case that $|V_{E'}^s| > e(W_s)$, as this will result in the capacity of the cut $C$ being at least $M$, which is not possible by Lemma 7.1. Note that any $s$-$t$ cut for which $|V_{E'}^s| < e(W_s)$ can be transformed into another $s$-$t$ cut of a lower capacity in which $|V_{E'}^s| = e(W_s)$ by moving vertices in $V_{E'}^s$ that correspond to edges in $W_s$ to $V_{\overline{E'}}$. Since edges from $s$ to vertices in $V_{E'}$ (vertices in $V_{\overline{E'}}$, in particular) have capacity of 1, the capacity of the cut reduces. The claim follows. \qed

**Lemma 7.3** The Degrees Density Augmentation problem admits a polynomial time exact solution.

*Proof:* We are interested in finding a non-empty set $W_s \subseteq \overline{U}$ such that

$$\frac{e(W_s) + e(U,W_s)}{\deg(W_s)}$$

is maximized. Note that there are at most $2n^4$ possible values of $\rho$ that our algorithm needs to try. Indeed, the numerator is an integer between 1 and $2n^2$ and the denominator is an integer between 1 and $n^2$.

Since minimum $s$-$t$ cut can be computed in polynomial time, our algorithm runs in polynomial time.

For any fixed guess for $\rho$, the capacity of the min $s$-$t$ cut is given by

$$\min_{W_s \subseteq \overline{U}} |V_{E'}^s| + d_{U}(W_t) + \rho \deg(W_s)$$

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\[
\begin{align*}
= & \min_{W_s \subseteq U} |V_{E'}| - |V_E^s| + deg_U(U) - deg_U(W_s) + \rho deg(W_s) \\
= & |V_{E'}| + deg_U(U) - \max_{W_s \subseteq U} |V_{E'}| + deg_U(W_s) - \rho deg(W_s) \\
= & |V_{E'}| + deg_U(V \setminus U) - \max_{W_s \subseteq U} e(W_s) + e(U, W_S) - \rho deg(W_s) (\text{using Lemma 7.2})
\end{align*}
\]

Our algorithm ensures that \( \rho deg(W_s) \geq e(W_s) + e(U, W_S) \), which eliminates the possibility of \( W_s = \emptyset \). Thus, finding the minimum \( s-t \) cut for a fixed \( \rho \) in the above flow network is equivalent to finding a set \( W_s \) with the largest degree density. Thus we have

\[
\frac{e(W_s) + e(U, W_S)}{deg(W_s)} \geq \rho
\]

Since our algorithm finds such \( W_s \) for each possible fraction that \( \rho \) can assume and returns the \( W_s \) with the highest degree density, our solution is optimal. \( \square \)

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