GALOIS MODULE STRUCTURE OF GALOIS
COHOMOLOGY AND PARTIAL EULER-POINCARÉ
CHARACTERISTICS

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Abstract. Let $F$ be a field containing a primitive $p$th root of
unity, and let $U$ be an open normal subgroup of index $p$ of the
absolute Galois group $G_F$ of $F$. Using the Bloch-Kato Conjecture
we determine the structure of the cohomology group $H^n(U, \mathbb{F}_p)$
as an $\mathbb{F}_p[G_F/U]$-module for all $n \in \mathbb{N}$. Previously this structure
was known only for $n = 1$, and until recently the structure even
of $H^1(U, \mathbb{F}_p)$ was determined only for $F$ a local field, a case set-
tled by Borevič and Faddeev in the 1960s. For the case when the
maximal pro-$p$ quotient $T$ of $G_F$ is finitely generated, we apply
these results to study the partial Euler-Poincaré characteristics of
$\chi_n(N)$ of open subgroups $N$ of $T$. We show in particular that the
$n$th partial Euler-Poincaré characteristic $\chi_n(N)$ is determined by
only $\chi_n(T)$ and the conorm in $H^n(T, \mathbb{F}_p)$.

Let $F$ be a field containing a primitive $p$th root of unity $\xi_p$. Let $G_F$
be the absolute Galois group of $F$, $U$ an open normal subgroup of $G_F$
of index $p$, and $G = G_F/U$.

In the 1960s Z. I. Borevič and D. K. Faddeev classified the possible
$G$-module structures of the first cohomology groups $H^1(U, \mathbb{F}_p)$ in the
case $F$ a local field [Bf]. Recently this result was extended from local
fields to all fields $F$ as above [MS1], and in this more general context
the result has been further developed and applied in [MS2], [MS3],

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Security Agency grant MDA904-02-1-0061.
It is important in the study of Galois cohomology to extend these results to all cohomology groups $H^n(U, \mathbb{F}_p)$, $n \in \mathbb{N}$.

The celebrated 1982 paper of Merkurjev-Suslin [MeSu] achieved substantial progress in the investigation of Galois cohomology, and in hindsight we know that these results were already sufficient to extend the results of Borevič and Faddeev to $H^2(U, \mathbb{F}_p)$ for any field $F$ containing a primitive $p$th root of unity. Because of connections with Brauer groups, Galois embedding problems, and Galois pro-$p$-groups, a precise knowledge of the structure $H^2(U, \mathbb{F}_p)$ as an $\mathbb{F}_p[G]$-module has significant applications: see [LLMS2], for example, for consequences characterizing Galois Demuškin pro-$p$-groups.

The main conjectures in contemporary Galois cohomology include the Bloch-Kato Conjecture, the natural generalization to Milnor $K$-theory of Hilbert 90, and generalization of the exact sequences in Milnor $K$-theory contained in [MeSu] from the case $n = 2$ to any $n \in \mathbb{N}$. V. Voevodsky’s partially published proof of the Bloch-Kato Conjecture, based upon earlier work of A. S. Merkurjev, M. Rost, and A. A. Suslin, encompasses all of these. (See [R], [Su], [V1] and [V2].) In this paper we show that two results related to the Bloch-Kato Conjecture, contained in [V1] and [V2] and quoted precisely in section 2, are sufficient to determine the structure of all $\mathbb{F}_p[G]$-modules $H^n(U, \mathbb{F}_p)$ for all $n \in \mathbb{N}$. Moreover, this structure depends only on simple arithmetical invariants attached to the field extension $E/F$, where $E$ is the fixed field of $U$ in the separable closure $F_{\text{sep}}$ of $F$. In particular, we use Hilbert 90 for Milnor $K$-theory in an essential way to prove that no cyclic $\mathbb{F}_p[G]$-module of dimension $j$ with $2 < j < p$ can occur as a summand of $H^n(U, \mathbb{F}_p)$. Because we rely only on the main conjectures mentioned, our results could already have been obtained conditionally in the early 1980s.

We have already used the results on the $\mathbb{F}_p[G]$-structure of $H^n(U, \mathbb{F}_p)$ to obtain a generalization of Schreier’s formula to higher Galois cohomology groups [LLMS1], to obtain a new characterization of Demuškin groups [LLMS2], and to construct fields with free or trivial Galois cohomology modules [LMS]. Moreover, using ideas in this paper, we clarify when an analogue of Hilbert 90 is valid for Milnor $k$-theory and Galois cohomology [LMSS]. Our results here also lead to a characterization of all possible decompositions of $H^n(U, F_p)$ into a sum of indecomposable $\mathbb{F}_p[G]$-modules. In a special case, this has been achieved [LLMS2, Theorem 4]. An investigation of the moduli space of all decompositions of $H^n(U, \mathbb{F}_p)$ is a work in progress.
Here we use the results to produce remarkably simple formulas for the \( n \)th partial Euler-Poincaré characteristic of \( U \). These formulas provide a new tool for studying the structure of Sylow \( p \)-subgroups of absolute Galois groups, about which little is currently known. Recall Artin-Schreier’s important observation that the only finite subgroups of absolute Galois groups are the trivial subgroup and \( \mathbb{Z}/2\mathbb{Z} \), and Becker’s generalization of this result to the case of the Galois group of a maximal \( p \)-extension \([Be]\). These results lead naturally to the search for a classification of finitely generated subgroups of Sylow \( p \)-subgroups of absolute Galois groups. The question is settled for algebraic number fields and for fields of transcendence degree 1 over local fields \([E1]\), \([E2]\), \([E3]\). In the case of \( p = 2 \) for arbitrary fields \( F \) with \( |F^\times /F^\times 2| \leq 8 \) \([JW1]\) partial progress was made, but there are still open questions about possible Galois pro-2 groups with three generators. Thus, we know only very few types of such groups, and while the elementary type conjecture asserts that there are no others, there is not much evidence for the validity of the conjecture. (See section 7 for additional comments and references on this conjecture.) It is significant, therefore, that we are able to derive results on the good behavior of partial Euler-Poincaré characteristics—and its relation to cohomological dimension—without appealing to the conjecture.

In the Appendix \([BeLMS]\) written with Dave Benson, we use these results to provide examples of pro-\( p \)-groups which cannot be realized as absolute Galois groups.

1. Main Theorems

The main ingredient for our determination of the \( G \)-module structure of \( H^n(U, \mathbb{F}_p) \) is Milnor \( K \)-theory. (See \([Mi]\) and \([EV\) Chap. IX].) For \( i \geq 0 \), let \( K_iF \) denote the \( i \)th Milnor \( K \)-group of the field \( F \), with standard generators denoted by \( \{f_1, \ldots, f_i\}, f_1, \ldots, f_i \in F \setminus \{0\} \). For \( \alpha \in K_iF \), we denote by \( \bar{\alpha} \) the class of \( \alpha \) modulo \( p \), and we use the usual abbreviation \( k_nF \) for \( K_nF/pK_nF \). We denote by \( (f) \) the element in \( H^1(U, \mathbb{F}_p) \) corresponding to \( f \) in \( F \setminus \{0\} \) and by \( (f_1, \ldots, f_n) \) the cup product \( (f_1) \cdots (f_n) \) in \( H^n(U, \mathbb{F}_p) \).

Let \( E \) be the fixed field of \( U \) in the separable closure \( F_{\text{sep}} \) of \( F \). We write \( N_{E/F} \) for the norm map \( K_nE \rightarrow K_nF \), and we use the same notation for the induced map modulo \( p \). We denote by \( i_E \) the natural
homomorphism in the reverse direction. We also apply the same notation $N_{E/F}$ and $i_E$ for the corresponding homomorphisms between cohomology groups, although when convenient we replace these with the equivalent maps $\text{cor}_{E/F}$ and $\text{res}_{F,E}$. The image of an element $\alpha \in K_i F$ in $H^i(G_F, \mathbb{F}_p)$ we also denote by $\alpha$. We formulate our results in terms of Galois cohomology for intended applications, but we use Milnor $K$-theory in our proof.

Our decomposition depends on four arithmetic invariants $Y_1, Y_2, y, z$, which we define as follows. Fix $a \in F$ such that $E = F(\sqrt[p]{a})$, and let $\sigma \in G$ satisfy $\sqrt[p]{a}^\sigma - 1 = \xi_p$. First, for an element $\bar{\alpha}$ of $k_i F$, let

$$\text{ann}_{k_{n-1} F} \bar{\alpha} = \text{ann}(k_{n-1} F \xrightarrow{\bar{\alpha}} k_{n-1 + i} F)$$

denote the annihilator of the product with $\bar{\alpha}$. When the domain of $\bar{\alpha}$ is clear, we omit the subscript on the map and write simply $\text{ann} \bar{\alpha}$. We use analogous notation for the annihilator of $\alpha$ in $H^i(U, \mathbb{F}_p)$ when we work in Galois cohomology rather than in Milnor $K$-theory. Because we will often use the elements $\{a\}, \{\xi_p\}, \{a, a\}$, and $\{a, \xi_p\}$, we omit the bars for these elements. We also omit the bar in the element $\{\sqrt[p]{a}\} \in k_n E$.

Fix $n \in \mathbb{N}$ and $U$ an open normal subgroup of $G_F$ of index $p$, and write $E = \text{Fix}(U)$. Define invariants associated to $E/F$ and $n$ as follows:

$$d := \dim_{\mathbb{F}_p} k_n F / N_{E/F} k_n E, \quad e := \dim_{\mathbb{F}_p} N_{E/F} k_n E$$

$$Y_1 := \dim_{\mathbb{F}_p} \text{ann}_{k_{n-1} F} \{a, \xi_p\} / \text{ann}\{a\}, \quad Y_2 := \dim_{\mathbb{F}_p} k_{n-1} F / \text{ann}\{a, \xi_p\}$$

$$y := \begin{cases} 
\dim_{\mathbb{F}_p} (N_{E/F} k_n E) / \{a\} \cdot k_{n-1} F, & p > 2 \\
\dim_{\mathbb{F}_2} (N_{E/F} k_n E) / \{a\} \cdot \text{ann}_{k_{n-1} F} \{a, -1\}, & p = 2 
\end{cases}$$

$$z := \begin{cases} 
\dim_{\mathbb{F}_p} (k_n F) / \{\xi_p\} \cdot k_{n-1} F + N_{E/F} k_n E), & p > 2 \\
\dim_{\mathbb{F}_2} (k_n F) / \{a\} \cdot k_{n-1} F + N_{E/F} k_n E), & p = 2 
\end{cases}$$

Our main results are then the following.

**Theorem 1.** If $p > 2$ and $n \in \mathbb{N}$ then

$$H^n(U, \mathbb{F}_p) \simeq X_1 \oplus X_2 \oplus Y \oplus Z$$

where

1. $X_1$ is a trivial $\mathbb{F}_p[G]$-module of dimension $Y_1$ and

   $$X_1 \cap \text{res}_{F,E} H^n(G_F, \mathbb{F}_p) = \{0\}$$

2. $X_2$ is a direct sum of $Y_2$ cyclic $\mathbb{F}_p[G]$-modules of dimension 2
(3) $Y$ is a free $\mathbb{F}_p[G]$-module of rank $y$
(4) $Z$ is a trivial $\mathbb{F}_p[G]$-module of dimension $z$ and

$$Z \subset \text{res}_{F,E} H^n(G_F, \mathbb{F}_p).$$

Further we have

(5) $Y^G = \text{res}_{F,E} \text{cor}_{E/F} H^n(U, \mathbb{F}_p)$
(6) $\text{cor}_{E/F}: X_1 \oplus X_2 \rightarrow (a) \cdot H^{n-1}(G_F, \mathbb{F}_p)$ is surjective
(7) $\Upsilon_1 + \Upsilon_2 + y = e$
(8) $\Upsilon_2 + z = d$.

**Theorem 2.** If $p = 2$ and $n \in \mathbb{N}$ then

$$H^n(U, \mathbb{F}_2) \simeq X_1 \oplus Y \oplus Z$$

where

(1) $X_1$ is a trivial $\mathbb{F}_2[G]$-module of dimension $\Upsilon_1$ and

$$X_1 \cap \text{res}_{F,E} H^n(G_F, \mathbb{F}_2) = \{0\}$$
(2) $Y$ is a free $\mathbb{F}_2[G]$-module of rank $y$
(3) $Z$ is a trivial $\mathbb{F}_2[G]$-module of dimension $z$ and

$$Z \subset \text{res}_{F,E} H^n(G_F, \mathbb{F}_2).$$

Further we have

(4) $Y^G = \text{res}_{F,E} \text{cor}_{E/F} H^n(U, \mathbb{F}_2)$
(5) $\text{cor}_{E/F}: X_1 \rightarrow (a) \cdot \text{ann}(a, -1)$ is an isomorphism
(6) $\Upsilon_1 + y = e$
(7) $\Upsilon_2 + z = d$.

**Remark.** The case $n = 1$ in the two theorems recovers the results of [MS1].

The isomorphism classes of $\mathbb{F}_p[G]$-modules $X_1 \oplus Z$, $X_2$, and $Y$ are determined by $H^n(U, \mathbb{F}_p)$, but the decomposition of $H^n(U, \mathbb{F}_p)$ into summands $X_1$, $X_2$, $Y$, and $Z$ is not canonical. However, there is an equivalent, canonical version of our results, contained in the following statements. These statements describe $(k_n E)^G$ and the intersection of $(k_n E)^G$ with images of $k_n E$ under successive powers of the augmentation ideal of $\mathbb{F}_p[G]$. Here the augmentation ideal is generated by $\sigma - 1$, but the statements are independent of the choice of a generator $\sigma$ of $G$ as well as a primitive $p$th root of unity $\xi_p$. Assume first $p > 2$. (See Proposition 4 and Lemma 3
(1) for each $i \geq 3$,

$$(\sigma - 1)^{i-1}k_n E \cap (k_n E)^G = i_{E:F}k_n E = (\sigma - 1)^{p-1}k_n E.$$  

(2) $$(\sigma - 1)k_n E \cap (k_n E)^G = i_E(\{\xi_p\} \cdot k_{n-1} F) + i_{E:F}k_n E.$$

(3) $0 \to \text{ann}\{a\} \to k_{n-1} F \xrightarrow{[a]} k_n F \xrightarrow{i_E} (k_n E)^G \xrightarrow{N_{E/F}} \text{ann}\{a, \xi_p\} \to 0.$$

If $p = 2$ the exact sequence (3) is equivalent to Theorem 2. This equivalent, canonical reformulation of Theorems 1 and 2 follows from their proofs in sections 2, 3, and 4.

Now we turn to an application of the results above to Galois pro-$p$-groups. For a pro-$p$-group $T$ with finite cohomology groups $H^i(T, \mathbb{F}_p)$ for $0 \leq i \leq n$, the $n$th partial Euler-Poincaré characteristic $\chi_n(T)$ is defined as

$$\chi_n(T) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{F}_p} H^i(T, \mathbb{F}_p).$$

Generalizing a previous result of Šafarevič, Koch observed a very interesting criterion for cohomological dimension $\text{cd}(G)$ being $\leq n$, using the Euler-Poincaré characteristic of open subgroups of $T$. (See [Ko, Thm. 5.5].) For a strengthening of Koch’s result see [Sc].

Suppose now that $T = G_{F}(p)$ is the maximal pro-$p$-quotient of an absolute Galois group $G_F$ of a field $F$ containing a primitive $p$th root of unity, and suppose $T$ has finite rank. In this paper we show that for $N$ an open subgroup of $T$ of index $p$, the single condition that $\chi_n(N) = p\chi_n(T)$ is equivalent to the condition that $\text{cor} : H^n(N, \mathbb{F}_p) \to H^n(T, \mathbb{F}_p)$ is surjective. Using this result, we strengthen Koch’s criterion in our case, showing that if $\chi_n(N) = p\chi_n(T)$ for all open subgroups $N$ of index $p$, then $\text{cd}(T) \leq n$.

The essential part of section 7 is a formula for $\chi_n(N)$, where $N$ is an open subgroup of $T$ of index $p$. This formula depends only on $p$, $\chi_n(T)$, and the conorm in degree $n$. The connection with module structure takes particular notice of the maximal free submodules in the cohomology groups; we show that $\chi_n(N)$ differs from $p\chi_n(T)$ by a certain multiple of the degree $n$ conorm, and if $p > 2$ then $\chi_n(N)$ is determined by the maximal free submodules in the cohomology groups together with this conorm. To see the role of the free submodules more clearly when the dimensions are finite, we set $F^i(N)$ to be a maximal
free $\mathbb{F}_p[T/N]$-submodule of $H^i(N, \mathbb{F}_p)$ and define the $n$th partial free Euler-Poincaré characteristic $\chi^\delta_n(N)$ by

$$
\chi^\delta_n(N) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{F}_p} F^i(N).
$$

(Observe that $\chi^\delta_n(N)$ is independent of the choice of $F^i(N)$ for each $i$.) Our theorem is as follows.

**Theorem 3.** Let $p$ be a prime and $F$ a field containing a primitive $p$th root of unity. Suppose $T = \text{Gal}(F(p)/F)$ is a pro-$p$-group of finite rank, $N$ is a subgroup of $T$ of index $p$, and $n \in \mathbb{N}$.

(a) We have

$$
p\chi_n(T) - \chi_n(N) = (-1)^n(p - 1) \dim_{\mathbb{F}_p} \frac{H^n(T, \mathbb{F}_p)}{\text{cor} H^n(N, \mathbb{F}_p)}.
$$

(b) Now additionally assume either that $p > 2$ or that $p = 2$ and $-1 \in N_{E/F}(E)$, where $E = \text{Fix}(N)$. Then

$$
\chi_n(N) = \chi^\delta_n(N) + (-1)^n \dim_{\mathbb{F}_p} \frac{H^n(T, \mathbb{F}_p)}{\text{cor} H^n(N, \mathbb{F}_p)}.
$$

Under the initial hypotheses of Theorem 3 we deduce the following corollaries.

**Corollary 1.** The following are equivalent.

1. $\chi_n(N) = p\chi_n(T)$
2. $\text{cor} H^n(N, \mathbb{F}_p) = H^n(T, \mathbb{F}_p)$

Under the hypothesis either that $p > 2$ or that $p = 2$ and $-1 \in N_{E/F}(E)$, then these are additionally equivalent to

3. $\chi_n(N) = \chi^\delta_n(N)$.

**Corollary 2.** Let $n \in \mathbb{N}$. The following are equivalent:

1. $\chi_n(N) = p\chi_n(T)$ for all open subgroups $N$ of $T$ of index $p$
2. $\text{cd}(T) \leq n$
3. $\chi_n(V) = p\chi(U)$ for all open subgroups $U$ of $T$ and all open subgroups $V$ of $U$ of index $p$ in $U$. 
2. Bloch-Kato and Milnor $K$-theory

Our proofs rely on the following two results in Voevodsky’s work on the Bloch-Kato Conjecture. Because we apply Voevodsky’s results in the case when the base field contains a primitive $p$th root of unity we shall formulate Voevodsky’s results restricted to this case. The first is the Bloch-Kato Conjecture itself:

**Theorem 4** ([V1, Lemma 6.11 and §7] and [V2, §6 and Thm. 7.1]).

(1) Let $F$ be a field containing a primitive $p$th root of unity and $m \in \mathbb{N}$. Then the norm residue homomorphism

$$k_m F \rightarrow H^m(G_F, \mu_p)$$

is an isomorphism.

(2) For any cyclic extension $E/F$ of degree $p$, the sequence

$$K_{m-1} E \xrightarrow{\sigma^{-1}} K_m E \xrightarrow{N_{E/F}} K_m F$$

is exact.

The second result establishes an exact sequence connecting $k_m F$ and $k_m E$ for consecutive $m$. (We translate the statement of the original result to $K$-theory using the previous theorem.) In the following result $a$ is chosen to satisfy $E = F(\sqrt[p]{a})$.

**Theorem 5** ([V1, Def. 5.1 and Prop. 5.2]). Let $F$ be a field containing a primitive $p$th root of unity with no extensions of degree prime to $p$. Then for any cyclic extension $E/F$ of degree $p$ and $m \in \mathbb{N}$, the sequence

$$k_{m-1} E \xrightarrow{N_{E/F}} k_m F \xrightarrow{(a) -} k_m F \xrightarrow{i_E} k_m E$$

is exact.

We observe that we may remove the hypothesis that the field $F$ has no extensions of degree prime to $p$. We give a proof of the following theorem in section 5.

**Theorem 6** (Modification of Theorem 5). Let $F$ be a field containing a primitive $p$th root of unity. Then for any cyclic extension $E/F$ of degree $p$ and $m \in \mathbb{N}$ the sequence

$$k_{m-1} E \xrightarrow{N_{E/F}} k_m F \xrightarrow{(a) -} k_m F \xrightarrow{i_E} k_m E$$

is exact.
We fix \( n \in \mathbb{N} \) and the cyclic extension \( E = F(\sqrt[n]{a}) \), and we write \( k_{n-1}F = \text{ann}\{a\} \oplus V \oplus W \), where \( \text{ann}\{a, \xi_p\} = \text{ann}\{a\} \oplus V \). Observe that \( \Upsilon_1 = \dim_F V \) and \( \Upsilon_2 = \dim_F W \). We denote by \( i_E : K_nF \rightarrow K_nE \) the map induced by the inclusion of \( F \) in \( E \). In what follows we will frequently refer to the element \( \sqrt[n]{a} \), and so we abbreviate it by \( A \). We recall that if \( p = 2 \) then \( \{a, \xi_p\} = \{a, -1\} = \{a, a\} \in k_2F \), while if \( p > 2 \) then \( \{a, a\} = 0 \in k_2F \). Finally, we will use the projection formula for taking the norms of standard generators of \( K_iF \) (see [FW, p. 81]).

**Lemma 1.** We have a vector space isomorphism

\[
V \oplus W \xrightarrow{\{a\} -} \{a\} \cdot k_{n-1}F
\]

and, if \( p > 2 \), the compositum of the maps \( \{\xi_p\} - \) and \( i_E \)

\[
W \xrightarrow{\{\xi_p\} -} \{\xi_p\} \cdot W \xrightarrow{i_E} i_E(\{\xi_p\} \cdot W)
\]

is a vector space isomorphism as well.

**Proof.** The first isomorphism follows from the fact that \( V \oplus W \) is a complement in \( k_{n-1}F \) of the kernel of multiplication by \( \{a\} \). For the second, assume \( p > 2 \). Suppose that \( \bar{w} \in W \) and \( \bar{\alpha} = \{\xi_p\} \cdot \bar{w} \in \ker i_E \). Then by Theorem 6, \( \bar{\alpha} = \{a\} \cdot \bar{c} \) for \( c \in K_{n-1}F \). Since \( \{a, a\} = 0 \) we see that \( \{a\} \cdot \bar{\alpha} = 0 \). But then \( \bar{w} \in \text{ann}\{a, \xi_p\} \) and so \( \bar{w} = 0 \). \( \square \)

For \( \gamma \in K_nE \), let \( l(\gamma) \) denote the dimension of the cyclic \( F_p[G] \)-submodule \( \langle \gamma \rangle \) of \( k_nE \) generated by \( \gamma \). Then we have

\[
(\sigma - 1)^{l(\gamma)}-1 \langle \gamma \rangle = \langle \gamma \rangle^G \neq 0 \quad \text{and} \quad (\sigma - 1)^{l(\gamma)} \langle \gamma \rangle = 0.
\]

We denote by \( N \) the map \( (\sigma - 1)^{p-1} \) on \( k_nE \). Because \( (\sigma - 1)^{p-1} = 1 + \sigma + \cdots + \sigma^{p-1} \) in \( F_p[G] \), we may use \( i_E N_{E/F} \) and \( N \) interchangeably on \( k_nE \).

**Lemma 2.** Let \( p > 2 \). Suppose \( \gamma \in K_nE \) with \( l = l(\gamma) \geq 2 \). Then if \( l \geq 3 \),

\[
(\sigma - 1)^{l-1} \bar{\gamma} \in i_E N_{E/F}k_nE,
\]

and if \( l = 2 \)

\[
(\sigma - 1) \bar{\gamma} \in i_E(\{\xi_p\} \cdot k_{n-1}F) + i_E N_{E/F}k_nE.
\]
Proof. If $l = p$, then $(\sigma - 1)^{p-1}k_nE = i_E N_{E/F}k_nE$ shows the result in this case.

Suppose $l < p$. Then $\bar{\gamma} \in \ker(\sigma - 1)^{p-1}$ and so $i_E N_{E/F}(\bar{\gamma}) = 0$. By Theorem 6 there exists $b \in K_{n-1}F$ such that $N_{E/F}\bar{\gamma} = \{a\} \cdot \bar{b}$. Equivalently $N_{E/F}\gamma = \{a\} \cdot b + pf$ for some $f \in K_nF$. By the projection formula (see [FW, p. 81]),

$$N_{E/F}(\gamma - \{A\} \cdot i_E(b) - i_E(f)) = 0.$$

Then by Theorem 4 there exists $\omega \in K_nE$ such that

$$(\sigma - 1)\omega = \gamma - (\{A\} \cdot i_E(b) + i_E(f))$$

and hence, since $l \geq 2$,

$$(\sigma - 1)^{l-1}\bar{\gamma} = (\sigma - 1)^l\bar{\omega} + (\sigma - 1)^{l-2}i_E(\{\xi_p\} \cdot b).$$

If $l(\gamma) \geq 3$ we deduce

$$(\sigma - 1)^{l-1}\bar{\gamma} = (\sigma - 1)^l\bar{\omega}$$

where $l(\omega) = l + 1$. Set $\bar{\gamma}_{l+1} = \omega$ and repeat the argument. We obtain $\gamma_k \in K_nE$ of lengths $l < k \leq p$ with

$$(\sigma - 1)^{l-1}\bar{\gamma} = (\sigma - 1)^{k-1}\bar{\gamma}_k.$$

Take $\bar{\alpha} = \bar{\gamma}_p$ to obtain

$$(\sigma - 1)^{l-1}\bar{\gamma} \in i_E N_{E/F}k_nE,$$

as required.

If $l(\gamma) = 2$ we have that

$$(\sigma - 1)\bar{\gamma} = (\sigma - 1)^2\bar{\omega} + i_E(\{\xi_p\} \cdot \bar{b})$$

for some $\omega \in K_nE$ and some $b \in K_{n-1}F$. We see that $l(\omega) \leq 3$. If $l(\omega) < 3$ then

$$(\sigma - 1)\bar{\gamma} \in i_E(\{\xi_p\} \cdot k_{n-1}F),$$

while if $l(\omega) = 3$ then by the previous case we see that $(\sigma - 1)^2\bar{\omega} \in i_E N_{E/F}k_nE$, so the result holds in either case. □

Proposition 1. If $p > 2$, then

$$(\sigma - 1)k_nE \cap (k_nE)^G = i_E(\{\xi_p\} \cdot k_{n-1}F) + i_E N_{E/F}k_nE$$

and for $3 \leq i \leq p$,

$$(\sigma - 1)^{i-1}k_nE \cap (k_nE)^G = i_E N_{E/F}k_nE.$$

If $p = 2$ then

$$(\sigma - 1)k_nE \cap (k_nE)^G = i_E N_{E/F}k_nE.$$
Proof. We have that for $2 \leq i \leq p$,
\[ i_E N_{E/F} k_n E = (\sigma - 1)^{p-1} k_n E \cap (k_n E)^G \subset (\sigma - 1)^i k_n E \cap (k_n E)^G. \]
The statement for $p = 2$ follows immediately. Now assume that $p > 2$. Note also that
\[ i_E(\{\xi_p\} \cdot k_{n-1} F) = (\sigma - 1)(\{\xi\} \cdot k_{n-1} F) \subset (\sigma - 1)k_n E \cap (k_n E)^G. \]
Thus $i_E(\{\xi_p\} \cdot k_{n-1} F) + i_E N_{E/F} k_n E \in (\sigma - 1)k_n E \cap (k_n E)^G$ and the reverse inclusion follows from Lemma 2.

Now we shall prove that if $3 \leq i \leq p$, then
\[ (\sigma - 1)^{i-1} k_n E \cap (k_n E)^G = i_E N_{E/F} k_n E. \]
Because $i_E N_{E/F} k_n E = (\sigma - 1)^{p-1} k_n E$ we see that
\[ i_E N_{E/F} k_n E \subset (\sigma - 1)^{i-1} k_n E \cap (k_n E)^G \]
for each $1 \leq i \leq p$. In order to establish the reverse inclusion when $3 \leq i \leq p$, consider $(\sigma - 1)^{i-1} \gamma \in (\sigma - 1)^{i-1} k_n E \cap (k_n E)^G$. We may assume without loss of generality that $l(\gamma) = i$, and then since $p > 2$, we have $(\sigma - 1)^{i-1} \gamma \in i_E N_{E/F} k_n E$ by Lemma 2 as desired. \[ \square \]

In the following lemma we elongate the exact sequence of Theorem 6.

**Lemma 3.** The following sequence is exact:
\[ 0 \to \text{ann}\{a\} \to k_{n-1} F \xrightarrow{\{a\} -} k_n F \xrightarrow{i_E} (k_n E)^G \xrightarrow{N_{E/F}} \{a\} \cdot \text{ann}\{a, \xi_p\} \to 0. \]
Here the map $\text{ann}\{a\} \to k_{n-1} F$ is the natural inclusion.

**Proof.** We show first that $N_{E/F}((k_n E)^G) \subset \{a\} \cdot \text{ann}\{a, \xi_p\}$. Let $\bar{a} \in (k_n E)^G$ and $\beta = N_{E/F} \alpha$. Since $i_E(N_{E/F} \bar{a}) = (\sigma - 1)^{p-1} \bar{a} = 0$ we have that $\bar{b} = N_{E/F} \bar{a} = \{a\} \cdot \bar{b}$ for some $b \in K_{n-1} F$ by Theorem 6.

Suppose $p = 2$. Since $\bar{b}$ is in the image of $N_{E/F}$, we have by Theorem 6 that $\{a\} \cdot \bar{b} = \{a, a\} \cdot \bar{b} = 0$. Since $\{a, a\} = \{a, -1\}$, we have $\bar{b} \in \text{ann}\{a, -1\}$.

Now suppose that $p > 2$. Write $\beta = \{a\} \cdot b + pf$ for some $f \in K_n F$. Then by the projection formula (see [FW], p. 81),
\[ N_{E/F}(\alpha - (\{A\} \cdot i_E(b) + i_E(f))) = 0. \]

By Theorem 6 there exists $\omega \in K_n E$ such that
\[ (\sigma - 1)\omega = \alpha - (\{A\} \cdot i_E(b) - i_E(f)) \]
and hence \((\sigma - 1)^2 \bar{\omega} = \{\xi_p\} \cdot i_E(\bar{b})\).

If \((\sigma - 1)^2 \bar{\omega} = 0\) then by Theorem 6, \(\{\xi_p\} \cdot \bar{b} = \{a\} \cdot \bar{h}\) for some \(h \in K_{n-1}F\). Because \(\{a, a\} = 0\), the right-hand side of the preceding equation is annihilated by \(\{a\}\). Therefore \(\bar{b} \in \text{ann}\{a, \xi_p\}\).

If \((\sigma - 1)^2 \bar{\omega} \neq 0\) then \(l(\omega) = 3\) and Lemma 2 shows that
\[
i_E(\{\xi_p\} \cdot \bar{b}) = i_E(N_{E/F}(\lambda))
\]
for some \(\lambda \in K_nE\). By Theorem 6, we have \(\{\xi_p\} \cdot \bar{b} = N_{E/F}(\lambda) + \{a\} \cdot \bar{h}\) for some \(h \in K_{n-1}F\). Now by Theorem 6 and the fact that \(\{a, a\} = 0\), the right-hand side of the preceding equation is annihilated by \(\{a\}\). Then \(\bar{b} \in \text{ann}\{a, \xi_p\}\). Hence in all cases \(N_{E/F}\bar{\alpha} \in \{a\} \cdot \text{ann}\{a, \xi_p\}\).

Exactness at the first two terms is obvious, and exactness at the third term follows from Theorem 6.

For exactness at the fourth term, suppose \(\bar{\gamma} \in (k_nE)^G\) and \(N_{E/F}\bar{\gamma} = 0\).

Then \(N_{E/F}\bar{\gamma} = pf\) for \(f \in K_nF\). Let \(\beta = \gamma - i_E(f)\). Then \(N_{E/F}\beta = 0\) and by Theorem 6, there exists \(\alpha \in K_nE\) such that \((\sigma - 1)\alpha = \beta\). If \(p = 2\) then \(\bar{\beta} = i_E(N_{E/F}\bar{\alpha}) \in i_Ek_nF\) and we are done. Thus assume \(p > 2\).

Now if \((\sigma - 1)\bar{\alpha} = \bar{\beta} = 0\) we are done as then \(\bar{\gamma} = i_E(f)\). Hence assume \((\sigma - 1)\bar{\alpha} \neq 0\). Then \(l(\alpha) = 2\) and by Lemma 2, we see that
\[
\bar{\beta} = (\sigma - 1)\bar{\alpha} \in \{\xi_p\} \cdot i_Ek_{n-1}F + i_EN_{E/F}k_nF \subset i_Ek_nF
\]
and exactness at the fourth term is established.

Finally we show the exactness at the fifth term. Since
\[
\{a\} \cdot \text{ann}\{a, \xi_p\} = \{a\} \cdot V
\]
it is enough to show that each element \(\{a\} \cdot \bar{v}\) where \(\bar{v} \in V\) can be written as \(N_{E/F}\bar{\alpha}\) for some \(\bar{\alpha} \in (k_nE)^G\). Observe that
\[
(\sigma - 1)(\{A\} \cdot i_E\bar{v}) = \{\xi_p\} \cdot i_E(\bar{v}).
\]
Also we have
\[
N_{E/F}(\{A\} \cdot i_E(\bar{v})) = \begin{cases} \{a\} \cdot \bar{v} & \text{if } p > 2 \\ \{-a\} \cdot \bar{v} & \text{if } p = 2. \end{cases}
\]
Therefore it is enough to show that there exists an element \( \bar{\gamma} \in k_nE \) such that \((\sigma - 1)\bar{\gamma} = \{\xi_p\} \cdot i_E(\bar{v}) \) and

\[
N_{E/F}\bar{\gamma} = \begin{cases} 0 & \text{if } p > 2 \\ \{-1\} \cdot \bar{v} & \text{if } p = 2. \end{cases}
\]

Indeed then we can set \( \bar{\alpha} = \{A\} \cdot i_E(\bar{v}) - \bar{\gamma} \).

Because \( \bar{v} \in \text{ann}\{a, \xi_p\} \) we see that \( \{\xi_p\} \cdot i_E(\bar{v}) \in \text{ann}\{a\} \). By Theorem [1] there exists \( \bar{\beta} \in k_nE \) such that \( \{\xi_p\} \cdot \bar{v} = N_{E/F}\bar{\beta} \) and \( i_E(N_{E/F}\bar{\beta}) = (\sigma - 1)^{p-1}\bar{\beta} \).

Then setting \( \bar{\gamma} = (\sigma - 1)^{p-2}\bar{\beta} \) we obtain our required element. The proof of our lemma is now complete. \( \square \)

Next, we need a general lemma about \( \mathbb{F}_p[G] \)-modules. The straightforward proof of this lemma is omitted.

**Lemma 4.** Let \( M_i, i \in I \), be a family of \( \mathbb{F}_p[G] \)-modules contained in a common \( \mathbb{F}_p[G] \)-module \( N \). Suppose that the \( \mathbb{F}_p \)-vector subspace \( R \) of \( N \) generated by all \( M_i^G \) has the form \( R = \bigoplus_{i \in I} M_i^G \). Then the \( \mathbb{F}_p[G] \)-module \( M \) generated by \( M_i, i \in I \), has the form \( M = \bigoplus_{i \in I} M_i \).

Finally, we need a general structure proposition about \( \mathbb{F}_p[G] \)-modules that shows that the structure of an \( \mathbb{F}_p[G] \)-module \( X \) can be determined from the structure of \( X^G \).

**Proposition 2.** Let \( X \) be an \( \mathbb{F}_p[G] \)-module. Set \( L_p = (\sigma - 1)^{p-1}X \) and for \( 1 \leq i < p \), suppose that \( L_i \) is an \( \mathbb{F}_p \)-complement of \( (\sigma - 1)^i X \cap X^G \) in \( (\sigma - 1)^i - 1 X \cap X^G \).

Then there exist \( \mathbb{F}_p[G] \)-modules \( X_i, i = 1, \ldots, p \), such that

(1) \( X = \bigoplus_{i=1}^p X_i \),

(2) \( X_i^G = L_i \) for \( i = 1, \ldots, p \),

(3) each \( X_i \) is a direct sum of \( \text{dim}_{\mathbb{F}_p}(L_i) \) cyclic \( \mathbb{F}_p[G] \)-modules of length \( i \), and

(4) for each \( i = 1, \ldots, p \), there exists an \( \mathbb{F}_p \)-submodule \( Y_i \) of \( X_i \) with \( \text{dim}_{\mathbb{F}_p}(Y_i) = \text{dim}_{\mathbb{F}_p}(L_i) \) such that \( \mathbb{F}_p[G]Y_i = X_i \).

**Proof.** Since \( (\sigma - 1)^{p-1}X \subset X^G \), we see that \( L_p = (\sigma - 1)^{p-1}X \cap X^G \). By reverse induction on \( i = p, p - 1, \ldots, 1 \), we obtain

\[
(\sigma - 1)^i - 1 X \cap X^G = \bigoplus_{j=i}^p L_j.
\]
Observe that because $L_i$ is an $F_p$-complement of $(\sigma - 1)X \cap X^G$ in $(\sigma - 1)^{i-1}X \cap X^G$, for each $i = 1, \ldots, p$, there exists an $F_p$-submodule $Y_i$ of $X$ such that $(\sigma - 1)^{i-1}Y_i = 0$ and $(\sigma - 1)^{i-1} : Y_i \rightarrow L_i$ is an $F_p$-

isomorphism. Set $X_i$ to be the $F_p[G]$-submodule of $X$ generated by $Y_i$. By Exclusion Lemma 4, the sum of the $X_k$ is direct since the sum of the $X_k^G = L_k$ is direct.

We need only show that $X = \bigoplus_{k=1}^p X_k$. Let $J_k = \ker(\sigma - 1)^k \subset X$, $k = 1, \ldots, p$. Then we have the filtration $X^G = J_1 \subset J_2 \subset \cdots \subset J_p = X$. Let $x \in X$. Then $(\sigma - 1)^{p-1}x \in L_p$, and hence there exists $y_p \in Y_p$ with $(\sigma - 1)^{p-1}x = (\sigma - 1)^{p-1}y_p$. Therefore $x - y_p \in J_{p-1}$.

More generally we show for all $k \in \{2, 3, \ldots, p\}$ that if $x \in J_k$ then there exist $x_i \in X_i$, $k \leq i \leq p$, such that $x - \sum_{i=k}^p x_i \in J_{k-1}$. We have already shown our statement for $k = p$. Hence assume that $x \in J_k$, $k < p$. Then

\[(\sigma - 1)^{k-1}x \in (\sigma - 1)^{k-1}X \cap X^G = \bigoplus_{i=k}^p L_i.\]

Hence there exist $l_i \in L_i$, $i = k, \ldots, p$, such that

\[(\sigma - 1)^{k-1}x = \sum_{i=k}^p l_i,\]

and there exist $y_i \in Y_i$, $i = k, \ldots, p$, such that $(\sigma - 1)^{i-1}y_i = l_i$. Therefore

\[(\sigma - 1)^{k-1}x = (\sigma - 1)^{k-1} \left( \sum_{i=k}^p (\sigma - 1)^{i-k}y_i \right).\]

Setting $x_k = y_k \in X_k$, and $x_i = (\sigma - 1)^{i-k}y_i \in X_i$, $i > k$, we have

\[x - \sum_{i=k}^p x_i \in J_{k-1}\]

as required.

Therefore, using the fact that

\[J_1 = X^G = \bigoplus_{i=1}^p L_i = \bigoplus_{i=1}^p X_i^G \subset \bigoplus_{i=1}^p X_i,\]

as well as our statement above, we show by induction on $k$ that $J_k \subset \bigoplus_{i=1}^p X_i$ for all $k = 1, 2, \ldots, p$. In particular $J_p = X \subset \bigoplus_{i=1}^p X_i$, which completes our proof. □
4. Proofs of Theorems 1 and 2

We determine now the structure of $k_nE$ as an $\mathbb{F}_p[G]$-module. We do so by invoking Proposition 1 to determine $\mathbb{F}_p$-modules $L_i$, $i = 1, \ldots, p$, such that

$$L_p = (\sigma - 1)^{p-1}k_nE = (\sigma - 1)^{p-1}k_nE \cap (k_nE)^G,$$

and such that $L_i$ is an $\mathbb{F}_p$-complement of $(\sigma - 1)^i k_nE \cap (k_nE)^G$ in $(\sigma - 1)^{i-1}k_nE \cap (k_nE)^G$. In this way we may then apply Proposition 2.

**Proof of Theorem 1** Assume that $p > 2$, and set

$$L_1 = X_1 + Z, \quad L_2 = i_E(\{\xi_p\} \cdot W), \quad L_p = i_E N_{E/F}k_nE,$$

where we choose $\mathbb{F}_p$-complements $X_1$, $W$, and $Z$, as follows:

$$X_1 \oplus i_E k_nF = (k_nE)^G$$

$$W \oplus \text{ann}\{a, \xi_p\} = k_{n-1}F$$

$$Z \oplus (i_E(\{\xi_p\} \cdot k_{n-1}F) + i_E N_{E/F}k_nE) = i_E k_nF.$$

Observe that $X_1 \cap i_E k_nF = \{0\}$ and therefore $X_1$ satisfies the second part of condition (11) in Theorem 1. Similarly, the second part of condition (11) follows from the definition of $Z$.

Further set $L_i = 0$ if $i \neq 1, 2, p$. We claim that the $\mathbb{F}_p$-modules $L_i$ satisfy the hypotheses of Proposition 2.

First observe that $L_p = i_E N_{E/F}k_nE = (\sigma - 1)^{p-1}k_nE$ satisfies the hypotheses of Proposition 2 for $L_p$. Next, by Proposition 1 if $i \geq 3$ then

$$(\sigma - 1)^{i-1}k_nE \cap (k_nE)^G = i_E N_{E/F}k_nE = (\sigma - 1)^{p-1}k_nE,$$

so that the $L_i = 0$, $3 \leq i < p$, satisfy the hypotheses for $L_i$. Moreover, again by Proposition 1

$$(\sigma - 1)k_nE \cap (k_nE)^G = i_E(\{\xi_p\} \cdot k_{n-1}F) + i_E N_{E/F}k_nE.$$

We show next that

$$L_2 \oplus L_p = i_E(\{\xi_p\} \cdot k_{n-1}F) + i_E N_{E/F}k_nE.$$

First let $\bar{\gamma} \in i_E(\{\xi_p\} \cdot k_{n-1}F) \cap i_E N_{E/F}k_nE$. Then $\bar{\gamma} = \{\xi_p\} \cdot i_E(\bar{f}) = i_E N_{E/F}k_nE \cdot \bar{\alpha}$ for some $f \in K_{n-1}F$ and some $\alpha \in K_nE$. By Theorem 6, $\{\xi_p\} \cdot \bar{f} - N_{E/F}k_nE = \{a\} \cdot \bar{b}$ for some $b \in k_{n-1}F$. But then $\{a, \xi_p\} \cdot \bar{f} = 0$, since $N_{E/F}k_nE = \text{ann}\{a\}$ by Theorem 6 and $\{a, a\} = 0$. Therefore
\( \tilde{f} \in \text{ann}\{a, \xi_p\}. \) Conversely, if \( \tilde{f} \in \text{ann}\{\xi_p, a\} \) then \( \{\xi_p\} \cdot \tilde{f} \in N_{E/F}k_nE. \) Because \( W \) is an \( \mathbb{F}_p \)-complement of \( \text{ann}\{a, \xi_p\} \) in \( k_{n-1}F \), we obtain that

\[
i_E(\{\xi_p\} \cdot k_{n-1}F) + i_E N_{E/F}k_nE = i_E(\{\xi_p\} \cdot W) \oplus i_E N_{E/F}k_nE = L_2 \oplus L_p.
\]

Finally, by construction of \( X_1 \) and \( Z \), observe that

\[
(k_nE)^G = X_1 \oplus Z \oplus L_2 \oplus L_p.
\]

Therefore \( L_1 = X_1 \oplus Z \) is an \( \mathbb{F}_p \)-complement of \((\sigma - 1)k_nE \cap (k_nE)^G = L_2 \oplus L_p \) in \((k_nE)^G\), as desired.

Applying Proposition 2, we obtain \( \mathbb{F}_p[G] \)-modules \( X_2 \) and \( Y \) such that \((X_2)^G = L_2\), \( Y^G = L_p = i_E N_{E/F}k_nE \), \( X_2 \) is a direct sum of \( \dim_{\mathbb{F}_p} L_2 \) cyclic \( \mathbb{F}_p[G] \)-modules of length 2, and \( Y \) is a free \( \mathbb{F}_p[G] \)-module of rank \( \dim_{\mathbb{F}_p} L_p \). By construction \( X_1 \) and \( Z \) are trivial \( \mathbb{F}_p[G] \)-modules.

We turn to the calculation of the dimensions of \( X_1, Z, L_2, \) and \( L_p \). Note that for \( p > 2 \), \( \ker(i_E) \subset N_{E/F}k_nE \). Indeed, by Theorem 6, \( \ker(i_E) = \{a\} \cdot k_{n-1}F \) and \( N_{E/F}k_nE = \text{ann}\{a\} \), and from \( N_{E/F}\{A\} = \{a\} \) and the projection formula (see [FW, p. 81]) we obtain \( \ker(i_E) \subset N_{E/F}k_nE \). Then we calculate

\[
\dim_{\mathbb{F}_p} L_p = \dim_{\mathbb{F}_p} i_E N_{E/F}k_nE = \dim_{\mathbb{F}_p} N_{E/F}k_nE / \ker(i_E) \\
= \dim_{\mathbb{F}_p} N_{E/F}k_nE / \{a\} \cdot k_{n-1}F \\
y;
\]

\[
\dim_{\mathbb{F}_p} Z = \dim_{\mathbb{F}_p} i_E k_nF / i_E(\{\xi_p\} \cdot k_{n-1}F) + i_E N_{E/F}k_nE \\
= \dim_{\mathbb{F}_p} k_nF / (\{\xi_p\} \cdot k_{n-1}F + N_{E/F}k_nE) \\
z; \quad \text{and}
\]

\[
\dim_{\mathbb{F}_p} L_2 = \dim_{\mathbb{F}_p} i_E(\{\xi_p\} \cdot W) = \dim_{\mathbb{F}_p} W \\
= \dim_{\mathbb{F}_p} k_{n-1}F / \text{ann}\{a, \xi_p\} \\
= \Upsilon_2.
\]

(In the determination of \( \dim_{\mathbb{F}_p} L_2 \) we use both Lemma 10 and the definition of \( W \).)

As for \( X_1 \), by Lemma 8 we have \( \ker(N_{E/F}) \cap (k_nE)^G = i_E k_nF \) as well as \( N_{E/F}(k_nE)^G = \{a\} \cdot \text{ann}\{a, \xi_p\} \). Then \( X_1 \), as an \( \mathbb{F}_p \)-complement of \( i_E k_nF \) in \((k_nE)^G\), is mapped isomorphically under \( N_{E/F} \) onto \( \{a\} \).
ann\{a, \xi\} = \{a\} \cdot V. Therefore we calculate
\[\dim_{F_p} X_1 = \dim_{F_p} \{a\} \cdot \text{ann}\{a, \xi\} = \dim_{F_p} \text{ann}\{a, \xi\}/\text{ann}\{a\} = \Upsilon_1.\]

Next we claim that \(N_{E/F}(X_1 \oplus X_2) = \{a\} \cdot k_{n-1}F\). First observe that because \(i_E N_{E/F}(X_1 \oplus X_2) = (\sigma - 1)^{p-1}(X_1 \oplus X_2) = \{0\}\) we have
\(N_{E/F}(X_1 \oplus X_2) \subset \ker i_E = \{a\} \cdot k_{n-1}F.\)

Since \(\{a\} \cdot k_{n-1}F = \{a\} \cdot (V \oplus W)\), it suffices to show
\(\{a\} \cdot V \subset N_{E/F}(X_1 \oplus X_2)\) and \(\{a\} \cdot W \subset N_{E/F}(X_1 \oplus X_2).\)

By the above, \(N_{E/F}\) maps \(X_1\) isomorphically onto \(\{a\} \cdot V\), so we have the first inclusion. For the second, let \(\{a\} \cdot \bar{w} \in \{a\} \cdot W\). Then
\(N_{E/F}(\{A\} \cdot i_E(\bar{w})) = \{a\} \cdot \bar{w}\).

Observe that \(X_2 = F_p[G]Y_2\), where \(Y_2\) is an \(F_p\)-submodule of \(X_2\) mapped isomorphically under \((\sigma - 1)\) onto \(i_E(\{\xi\} \cdot W)\). Therefore there exists \(\gamma \in Y_2 \subset X_2\) such that
\((\sigma - 1)\gamma = (\sigma - 1)(\{A\} \cdot i_E \bar{w}).\)

Then \(\{A\} \cdot i_E \bar{w} - \gamma \in (k_nE)^G\). Now
\(N_{E/F}((k_nE)^G) = \{a\} \cdot \text{ann}\{a, \xi_p\} = \{a\} \cdot V = N_{E/F}(X_1)\).

Applying the projection formula, \(\{a\} \cdot \bar{w} - N_{E/F}(\gamma) \in N_{E/F}(X_1)\). Hence \(\{a\} \cdot W \subset N_{E/F}(X_1 \oplus X_2)\), as required.

Lastly, we need to show the relations between the dimensions of these modules. From
\[y = \dim_{F_p} N_{E/F}k_nE/\{a\} \cdot k_{n-1}F\]
and
\[\dim_{F_p}(\{a\} \cdot k_{n-1}F) = \dim_{F_p}(\{a\} \cdot (V \oplus W) = \dim_{F_p}(V \oplus W) = \Upsilon_1 + \Upsilon_2\]
(again using Lemma [1]), we obtain that
\[\Upsilon_1 + \Upsilon_2 + y = \dim_{F_p} N_{E/F}k_nE = e.\]

Furthermore, since
\[z = \dim_{F_p} \frac{i_E k_nF}{i_E(\{\xi\} \cdot k_{n-1}F) + i_E N_{E/F}k_nE} = \dim_{F_p} \frac{i_E k_nF}{i_E(\{\xi\} \cdot W) \oplus i_E N_{E/F}k_nE}\]
and \( \Upsilon_2 = \dim_{\mathbb{F}_p} i_E(\{\xi_p\} \cdot W) \), we use \( \ker(i_E) \subset N_{E/F}k_nE \) to deduce
\[
\Upsilon_2 + z = \dim_{\mathbb{F}_p} k_nF/N_{E/F}k_nE = d.
\]
The proof is complete. \( \square \)

**Proof of Theorem 2.** Assume \( p = 2 \), and set
\[
L_1 = X_1 + Z \\
L_2 = i_E N_{E/F}k_nE,
\]
where we choose \( \mathbb{F}_2 \)-complements \( X_1 \) and \( Z \), as follows:
\[
X_1 \oplus i_E k_nF = (k_nE)^G \\
Z \oplus i_E N_{E/F}k_nE = i_E k_nF.
\]
Observe that \( X_1 \cap i_E k_nE = \{0\} \) and therefore \( X_1 \) satisfies the second part of condition (1) of Theorem 2. Similarly, the second part of condition (3) follows from the definition of \( Z \). Note also that by construction the sum \( X_1 + Z \) is direct.

Since by Proposition 1
\[
(\sigma - 1)k_nE \cap (k_nE)^G = i_E N_{E/F}k_nE,
\]
we see that \( X_1 + Z \) is an \( \mathbb{F}_2 \)-complement of \( (\sigma - 1)k_nE \cap (k_nE)^G \) in \( (k_nE)^G \). Therefore \( L_1 \) and \( L_2 \) satisfy the hypotheses of Proposition 2 and there exist a free \( \mathbb{F}_2[G] \)-module \( Y \) with \( Y^G = i_E N_{E/F}k_nE \) and trivial \( \mathbb{F}_2[G] \)-modules \( X_1 \) and \( Z \) such that
\[
k_nE = X_1 \oplus Y \oplus Z.
\]

We next determine the rank of \( Y \) and the dimensions of \( X_1 \) and \( Z \). We claim first that
\[
\ker(i_E) \cap N_{E/F}k_nE = \{a\} \cdot \text{ann}\{a, -1\}.
\]
Indeed, by Theorem 3 \( \ker(i_E) = \{a\} \cdot k_{n-1}F \) and \( N_{E/F}k_nE = \text{ann}\{a\} \). Then, we deduce from \( \{a\} \cdot f \in \ker(i_E) \cap N_{E/F}k_nE \) and the fact that \( \{a, a\} = \{a, -1\} \) for \( p = 2 \) that
\[
\{a, -1\} \cdot \overline{f} = \{a, a\} \cdot \overline{f} = 0.
\]
Conversely, suppose \( \overline{f} \in \text{ann}\{a, -1\} \). Then because \( \{a, a\} = \{a, -1\} \) we have
\[
\{a\} \cdot \overline{f} \in \{a\} \cdot k_{n-1}F \cap \text{ann}\{a\} = \ker(i_E) \cap N_{E/F}k_nE.
\]
We then calculate
\[ \dim_{\mathbb{F}_2} L_2 = \dim_{\mathbb{F}_2} i_E N_{E/F} k_n E \]
\[ = \dim_{\mathbb{F}_2} N_{E/F} k_n E / \{a\} \cdot \operatorname{ann}\{a, -1\} = y \]
and
\[ \dim_{\mathbb{F}_2} Z = \dim_{\mathbb{F}_2} i_E k_n F / i_E N_{E/F} k_n F \]
\[ = \dim_{\mathbb{F}_2} k_n F / (N_{E/F} k_n E + \{a\} \cdot k_{n-1} F) = z. \]

As for \( X_1 \), by Lemma \[ 3 \] we observe that \( \ker(N_{E/F}) \cap (k_n E)^G = i_E k_n F \) and \( N_{E/F}((k_n E)^G) = \{a\} \cdot \operatorname{ann}\{a, -1\} \). Hence \( X_1 \), as an \( \mathbb{F}_2 \)-complement of \( i_E k_n F \) in \( (k_n E)^G \), is mapped isomorphically under \( N_{E/F} \) onto \( \{a\} \cdot \operatorname{ann}\{a, -1\} = \{a\} \cdot V \). Therefore
\[ \dim_{\mathbb{F}_2} X_1 = \dim_{\mathbb{F}_2} \{a\} \cdot \operatorname{ann}\{a, -1\} \]
\[ = \dim_{\mathbb{F}_2} \operatorname{ann}\{a, -1\} / \operatorname{ann}\{a\} = \Upsilon_1. \]

Now we establish the relations between these dimensions. Since
\[ y = \dim_{\mathbb{F}_2} N_{E/F} k_n E / \{a\} \cdot \operatorname{ann}\{a, -1\}, \]
we have
\[ \Upsilon_1 + y = \dim_{\mathbb{F}_2} N_{E/F} k_n E = e. \]
Finally, we calculate
\[ d = \dim_{\mathbb{F}_2} k_n F / N_{E/F} k_n E \]
\[ = \dim_{\mathbb{F}_2} k_n F / (N_{E/F} k_n E + \{a\} \cdot k_{n-1} F) + \]
\[ \dim_{\mathbb{F}_2} (N_{E/F} k_n E + \{a\} \cdot k_{n-1} F) / N_{E/F} k_n E \]
\[ = z + \dim_{\mathbb{F}_2} \{a\} \cdot k_{n-1} F / \{a\} \cdot \operatorname{ann}\{a, -1\} \]
\[ = z + \dim_{\mathbb{F}_2} k_{n-1} F / \operatorname{ann}\{a, -1\} = z + \Upsilon_2. \]

The proof is complete. \[ \square \]

5. Proof of Theorem \[ 5 \]

For the case \( p = 2 \) we have the long exact sequence of Galois cohomology groups due to Arason \[ [4] \text{ Satz 4.5} \]. Suppose then that \( p > 2 \), and assume first that \( F \) is perfect. Let \( S \) be any Sylow \( p \)-subgroup of \( G_F = \text{Gal}(F_{\text{sep}}/F) \), and set \( L \) to be the fixed field of \( S \). Because \( F \) is perfect, the separable closure \( F_{\text{sep}} \) is identical to the algebraic closure \( \bar{F} \), and hence each finite extension of \( L \) has degree a power of \( p \). In particular, all of the hypotheses of Theorem \[ 5 \] are valid for the field \( L \) in place of \( F \). Furthermore, \( ([L : F], p) = 1 \). (Here we use basic
properties of supernatural numbers and Sylow $p$-subgroups. See [Se, Chapter 1].) Therefore if $E = F(\sqrt[p]{a})$ is a cyclic extension of $F$ of degree $p$, so is $EL = L(\sqrt[p]{a})$ over $L$. By Theorem 5 we see that the sequence

$$k_{m-1}EL \xrightarrow{N_{EL/L}} k_{m-1}L \xrightarrow{\{a\}} k_{m-1}L \xrightarrow{i_{EL}} k_{m}EL$$

is exact for each $m \in \mathbb{N}$.

We claim that $i_L: k_{m}F \to k_{m}L$ is injective. Indeed, suppose that $i_L(\alpha) = 0$ for some $\alpha \in k_{m}F$. Then there exists a finite subextension $M/F$ of $L/F$ such that $i_M(\alpha) = 0$. Then

$0 = N_{M/F}(i_M(\alpha)) = [M : F]\alpha.$

(See [FV, Thm. XI.3.8].) Because $[M : F]$ is coprime with $p$, we see that $\alpha = 0$ and $i_L$ is injective as asserted. Similarly we have that $i_{EL}: k_{m}E \to k_{m}EL$ is injective.

We then have the following commutative diagram:

$$
\begin{array}{cccc}
k_{m-1}EL & \xrightarrow{N_{EL/L}} & k_{m-1}L & \xrightarrow{\{a\}} & k_{m-1}L & \xrightarrow{i_{EL}} & k_{m}EL \\
n_{EL/F} & & n_{L/F} & & n_{L/F} & & n_{EL/F} \\
k_{m-1}E & \xrightarrow{i_{EL}} & k_{m-1}E & \xrightarrow{i_E} & k_{m}E & & \\
\end{array}
$$

The fact that the first square is commutative follows from [BT, p. 383]. The commutativity of the remaining part of the diagram is clear. Because the vertical maps are injective, we see that the bottom row of the diagram is a complex: the composition of any two consecutive maps is the zero map. We now establish exactness at the second and third terms of the complex.

Let $\alpha \in k_{m-1}F$ such that $\{a\} \cdot \alpha = 0$. Then $\{a\} \cdot i_L(\alpha) = 0$ and therefore there exists an element $\beta \in k_{m-1}EL$ such that $N_{EL/L}(\beta) = i_L(\alpha)$. Let $M/F$ be a finite extension such that $\beta$ is defined over $EM$. Then $N_{EM/M}(\beta) = i_M(\alpha)$ (see [BT, p. 383]), and we have

$$N_{EM/F}(\beta) = N_{M/F}(N_{EM/M}(\beta)) = N_{M/F}(i_M(\alpha)) = [M : F]\alpha.$$

Thus $N_{E/F}(N_{EM/E}(\beta)) = N_{EM/F}(\beta) = [M : F]\alpha$.

Because $([M : F], p) = 1$ we see that $\alpha \in N_{E/F}(k_{m-1}E)$. Therefore we have established the exactness of our complex at $k_{m-1}F$.

Now assume that $\alpha \in k_{m}F$ such that $i_E(\alpha) = 0 \in k_{m}E$. Then arguing as above, we see that there exist a finite extension $M/F$ and $\beta \in k_{m-1}M$ such that $\{a\} \cdot \beta = i_M(\alpha) \in k_{m}M$. 
Applying $N_{M/F}$ and using the projection formula we see that
\[ \{a\} \cdot N_{M/F}(\beta) = N_{M/F}(i_M(\alpha)) = [M : F] \alpha. \]
Because $[M : F]$ is coprime with $p$, $\alpha \in \{a\} \cdot N_{M/F}(\gamma)$ for a suitable element $\gamma \in k_{m-1}M$. Hence we see that our complex is also exact at $k_mF$ and the full complex is exact.

Now if $F$ is not perfect, let $F_{pc}$ denote the perfect closure of $F$ (see, for instance, [Ka, pp. 69–70]). Since finite purely inseparable extensions of $F$ are of $q$th-power degree, where $q$ is the characteristic of $F$, and $q \neq p$ since $\xi_p \in F$, we obtain $([F_{pc} : F], p) = 1$. The argument above establishes the theorem for the perfect field $F_{pc}$, and a similar transfer argument descends from $F_{pc}$ to $F$.

\[\square\]

6. From $H^i(G_F, \mathbb{F}_p)$ to $H^i(G_F(p), \mathbb{F}_p)$

In the previous sections, we worked with cohomology groups of absolute Galois groups. In this section we observe that, assuming the Bloch-Kato Conjecture, we may replace these groups with cohomology groups of maximal pro-$p$-quotients of absolute Galois groups.

Let $F(p)$ be the compositum of all finite Galois $p$-power extensions of $F$ in a fixed separable closure $F_{sep}$ of $F$, $G_F = \text{Gal}(F_{sep}/F)$, and $T = \text{Gal}(F(p)/F) = G_F(p)$. Since $F$ contains $\xi_p$ we see that $F(p)$ is closed under taking $p$th roots and hence $H^1(G_F(p), \mathbb{F}_p) = \{0\}$. By the Bloch-Kato Conjecture (see Theorem 4) the subring of the cohomology ring $H^*(G_F(p), \mathbb{F}_p)$ consisting of elements of positive degree is generated by $H^1(G_F(p), \mathbb{F}_p)$. Hence for each $i \in \mathbb{N}$, $H^i(G_F(p), \mathbb{F}_p) = \{0\}$. Then, considering the Lyndon-Hochschild-Serre spectral sequence associated to the exact sequence

\[ 1 \to G_{F(p)} \to G_F \to T \to 1, \]

we have that $\inf: H^i(T, \mathbb{F}_p) \to H^i(G_F, \mathbb{F}_p)$ is an isomorphism for all $i \in \mathbb{N} \cup \{0\}$.

Recall that $E = F(\sqrt[p]{a})$, and let $N = \text{Gal}(F(p)/E)$. By the same argument, we have that $\inf: H^i(N, \mathbb{F}_p) \to H^i(G_E, \mathbb{F}_p)$ is an isomorphism. A straightforward argument with cochains shows that this isomorphism is $T/N$-equivariant.

We denote by $(a) \in H^1(T, \mathbb{F}_p)$ the element corresponding to $a$ and by $(a, \xi_p)$ the cup product $(a) \cdot (\xi_p)$ in $H^2(T, \mathbb{F}_p)$. From Theorems 4
and the above argument we see that the following sequence is exact for $i \geq 0$:

$$H^i(N, \mathbb{F}_p) \xrightarrow{\text{cor}} H^i(T, \mathbb{F}_p) \xrightarrow{-(a)} H^{i+1}(T, \mathbb{F}_p) \xrightarrow{\text{res}} H^{i+1}(N, \mathbb{F}_p).$$

Hence $\text{cor} H^i(N, \mathbb{F}_p) = \text{ann} H^i(T, \mathbb{F}_p)(a)$. Observe that, by naturality, the inflation map $\text{inf}$ induces the following isomorphisms:

$$\text{cor} H^i(N, \mathbb{F}_p) = \text{ann} H^i(T, \mathbb{F}_p)(a) \xrightarrow{\text{inf}} \text{cor} H^i(G, \mathbb{F}_p) = \text{ann} H^i(G, \mathbb{F}_p)(a),$$

$$\text{ann} H^i(T, \mathbb{F}_p)(a, \xi_p) \xrightarrow{\text{inf}} \text{ann} H^i(G, \mathbb{F}_p)(a, \xi_p).$$

From now on we shall freely use the isomorphisms related to the cohomology groups $H^i(T, \mathbb{F}_p)$, $H^i(G, \mathbb{F}_p)$ and the cohomology groups of their open subgroups observed in this section.

7. Partial Euler-Poincaré Characteristics

In this section we determine $\dim_{\mathbb{F}_p} H^i(N, \mathbb{F}_p)$ for all open subgroups of $T$. In the case when $T$ is finitely generated we use this result to calculate the $n$th partial Euler-Poincaré characteristic of $N$. We show that a very simple relation exists between the $n$th partial Euler-Poincaré characteristics of $N$ and $T$, and we then sharpen Koch’s well-known criterion for the cohomological dimension of $T$. These results help clarify the structure of finitely generated subgroups of Sylow $p$-subgroups of absolute Galois groups, a class about which we know comparatively little.

Recall that the elementary type conjecture asserts a classification of finitely generated subgroups of Sylow $p$-subgroups of absolute Galois groups. A version of this conjecture was first introduced in the context of Witt rings of quadratic forms by Marshall [M1] (see also [M2]), and Jacob and Ware then considered the conjecture in terms of Galois groups $G_F(2)$ [JW1], [JW2]. Efrat [E1] extended the conjecture to possible groups $G_F(p)$: each finitely generated group $G_F(p)$ may be constructed from groups $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}_p$, and $G_F(p)$ for $F$ a local field, using some specific semidirect products and free products in the category of pro-$p$-groups. (For a precise definition, see [E1] and [E3]. Another variant of the elementary type conjecture may be found in [Ku]. For a more recent investigation of the elementary type conjecture, see [En].)

Although the elementary type conjecture attempts a far-reaching generalization of the work of Artin-Schreier and Becker [E3], at the moment there is little evidence for the conjecture in such generality,
although the conjecture holds for algebraic number fields and fields of transcendence degree 1 over local fields \([E_1], [E_2], [E_3]\) and for \(G_F(2)\) when \(|F^\times/F^\times 2| \leq 8\) \([JW1]\). The significance of this work on the \(n\)th partial Euler-Poincaré characteristic of finitely generated subgroups and their connections with cohomological dimension lies in the fact that we do not use the elementary type conjecture to achieve the new results. As a consequence, the formulas provide a new tool for the investigation of the validity of the elementary type conjecture. (For further connections among the elementary type conjecture, Demuškin groups, and Galois modules, see \([LLMS2]\).)

For the remainder of the section we will use without mention the fact that \(\text{cor} \ H^i(N, \mathbb{F}_p) = \text{ann}_{H^i(T, \mathbb{F}_p)}(a)\), which follows from Theorem \(6\). Moreover, from this point on, we will write \(\text{ann}(a)\) for \(\text{ann}_{H^i(T, \mathbb{F}_p)}(a)\) and \(\text{ann}(a, \xi_p)\) for \(\text{ann}_{H^i(T, \mathbb{F}_p)}(a, \xi_p)\). For each \(i \geq 0\) we set

\[
\begin{align*}
a_i &= \dim_{\mathbb{F}_p} \text{ann}(a) \\
d_i &= \dim_{\mathbb{F}_p} H^i(T, \mathbb{F}_p) / \text{ann}(a) \\
h_i &= \dim_{\mathbb{F}_p} H^i(T, \mathbb{F}_p) = a_i + d_i.
\end{align*}
\]

**Proposition 3.** Let \(N\) be an open subgroup of \(T = \text{Gal}(F(p)/F)\) of index \(p\). (We do not assume that \(T\) has finite rank.)

(a) We have

\[
\dim_{\mathbb{F}_p} H^i(N, \mathbb{F}_p) = d_{i-1} + d_i + p \dim_{\mathbb{F}_p} \frac{\text{cor} \ H^i(N, \mathbb{F}_p)}{\text{ann}(a)}. \quad H^{i-1}(T, \mathbb{F}_p).
\]

(b) Assume further either that \(p > 2\) or that \(p = 2\) and \(a\) is a sum of two squares in \(F\). Then

\[
\dim_{\mathbb{F}_p} H^i(N, \mathbb{F}_p) = d_{i-1} + d_i + py.
\]

(c) Assume further either that \(p > 2\) and \(a_i\) is finite, that \(p = 2\), \(a\) is a sum of two squares in \(F\), and \(a_i\) is finite, or that \(p = 2\) and \(h_{i-1}\) and \(h_i\) are finite. Then

\[
\dim_{\mathbb{F}_p} H^i(N, \mathbb{F}_p) = d_{i-1} + d_i + p(a_i - d_{i-1}).
\]

**Proof.** Assume first that \(p > 2\). Observe that

\[
d_{i-1} = \dim_{\mathbb{F}_p} \frac{H^{i-1}(T, \mathbb{F}_p)}{\text{ann}(a, \xi_p)} + \dim_{\mathbb{F}_p} \frac{\text{ann}(a, \xi_p)}{\text{ann}(a)} = \gamma_2 + \gamma_1.
\]
By Theorem 1, we have $d_i = \Upsilon_2 + z$. Then from Theorem 1 we conclude

$$\dim_{\mathbb{F}_p} H^i(N, \mathbb{F}_p) = \Upsilon_1 + 2\Upsilon_2 + py + z = (\Upsilon_1 + \Upsilon_2) + (\Upsilon_2 + z) + py = d_{i-1} + d_i + py.$$ 

In the case when $a_i < \infty$ we deduce

$$\dim_{\mathbb{F}_p} H^i(N, \mathbb{F}_p) = d_{i-1} + d_i + p(a_i - d_{i-1}).$$

Now assume that $p = 2$ and that $a$ is a sum of two squares in $F$. In this case, using Theorem 2 we have:

$$\dim_{\mathbb{F}_2} H^i(N, \mathbb{F}_2) = \Upsilon_1 + 2y + z.$$ 

Because $a$ is a sum of two squares in $F$, we see that $-1 \in N_{E/F}(E^\times)$. Then $(a) \cdot (-1) = 0$ in $H^2(T, \mathbb{F}_2)$ and we obtain that $\text{ann}(a,-1) = H^{i-1}(T, \mathbb{F}_2)$ and $\Upsilon_1 = \dim_{\mathbb{F}_2} H^{i-1}(T, \mathbb{F}_2)/\text{ann}(a) = d_{i-1}$. Furthermore, in this case $(a) \cdot H^{i-1}(T, \mathbb{F}_2) \subset \text{cor} H^i(N, \mathbb{F}_2)$ and therefore

$$z = \dim_{\mathbb{F}_2} \frac{H^i(T, \mathbb{F}_2)}{\text{cor} H^i(N, \mathbb{F}_2)} = d_i.$$ 

Thus we recover the same formula as above:

$$\dim_{\mathbb{F}_2} H^i(N, \mathbb{F}_2) = d_{i-1} + d_i + 2y.$$ 

In the case when $a_i < \infty$ we deduce

$$\dim_{\mathbb{F}_2} H^i(N, \mathbb{F}_2) = d_{i-1} + d_i + 2(a_i - d_{i-1}).$$

Finally assume that $p = 2$ and that $a$ is not necessarily a sum of two squares in $F$. Then from Theorem 2

$$\dim_{\mathbb{F}_2} H^i(N, \mathbb{F}_2) = \Upsilon_1 + 2y + z$$

$$= \dim_{\mathbb{F}_2} \frac{\text{ann}(a,-1)}{\text{ann}(a)} + 2 \dim_{\mathbb{F}_2} \frac{\text{cor} H^i(N, \mathbb{F}_2)}{(a) \cdot \text{ann}(a,-1)}$$

$$+ \dim_{\mathbb{F}_2} \frac{H^i(T, \mathbb{F}_2)}{(a) \cdot H^{i-1}(T, \mathbb{F}_2) + \text{cor} H^i(N, \mathbb{F}_2)}.$$ (2)
Now assume that \( h_{i-1} \) and \( h_i \) are finite. Then equation (2) may be simplified, as follows. Observe that

\[
\dim_{\mathbb{F}_2} \frac{(a) \cdot H^{i-1}(T, \mathbb{F}_2) + \text{cor} H^i(N, \mathbb{F}_2)}{\text{cor} H^i(N, \mathbb{F}_2)} = \dim_{\mathbb{F}_2} \frac{(a) \cdot H^{i-1}(T, \mathbb{F}_2)}{(a) \cdot H^{i-1}(T, \mathbb{F}_2) \cap \text{cor} H^i(N, \mathbb{F}_2)}
\]

We claim that

\[(a) \cdot H^{i-1}(T, \mathbb{F}_2) \cap \text{cor} H^i(N, \mathbb{F}_2) = (a) \cdot \text{ann}(a, -1).\]

The right-hand side is contained in the left-hand side by Theorem 2(5). Now suppose that for some \( h \in H^{i-1}(T, \mathbb{F}_2) \), \( (a, -1) \cdot h = (a, a) \cdot h = 0 \) since \( \text{cor} H^i(N, \mathbb{F}_2) = \text{ann}(a) \) and, since \( p = 2 \), \( (a, a) = (a, -1) \). Then \( h \in \text{ann}(a, -1) \) and our claim is proved. Therefore

\[
\dim_{\mathbb{F}_2} \frac{(a) \cdot H^{i-1}(T, \mathbb{F}_2) + \text{cor} H^i(N, \mathbb{F}_2)}{\text{cor} H^i(N, \mathbb{F}_2)} = \dim_{\mathbb{F}_2} \frac{(a) \cdot H^{i-1}(T, \mathbb{F}_2)}{(a) \cdot \text{ann}(a, -1)} = \dim_{\mathbb{F}_2} \frac{H^{i-1}(T, \mathbb{F}_2)}{\text{ann}(a, -1)}.
\]

Now in equation (2) we add and subtract two copies of

\[
\dim_{\mathbb{F}_2} \frac{(a) \cdot H^{i-1}(T, \mathbb{F}_2)}{(a) \cdot \text{ann}(a, -1)} = \dim_{\mathbb{F}_2} \frac{H^{i-1}(T, \mathbb{F}_2)}{\text{ann}(a, -1)}.
\]

We then obtain

\[
\dim_{\mathbb{F}_2} H^i(N, \mathbb{F}_2) = \left( \dim_{\mathbb{F}_2} \frac{\text{ann}(a, -1)}{\text{ann}(a)} + \dim_{\mathbb{F}_2} \frac{H^{i-1}(T, \mathbb{F}_2)}{\text{ann}(a, -1)} \right) + \left( \dim_{\mathbb{F}_2} \frac{H^i(T, \mathbb{F}_2)}{\text{cor} H^i(N, \mathbb{F}_2)} \right) + 2 \left( \dim_{\mathbb{F}_2} \text{cor} H^i(N, \mathbb{F}_2) - \dim_{\mathbb{F}_2} (a) \cdot H^{i-1}(T, \mathbb{F}_2) \right) = d_{i-1} + d_i + 2(a_i - d_{i-1}).
\]

\[\square\]

**Remark.** We observe that if \( p = 2 \) and \( h_{i-1} \) and \( h_i \) are finite, the last formula of the proposition also follows from Arason’s long exact sequence [A, Satz 4.5].
8. Proofs of Theorem 3 and Corollary 2

Proof of Theorem 3 Let $T = G_F(p)$ be of finite rank and $N$ an open subgroup of index $p$. Hence the $n$th partial Euler-Poincaré characteristic $\chi_n(T)$ is well-defined. For $0 \leq i \leq n$, let $a_i$, $d_i$, and $h_i$ be defined as in section 7. Because $\dim F\times/F\times_p < \infty$ implies that $\dim F\times/E\times_p < \infty$ by Theorems 1 and 2 in the case $n = 1$, we see that $a_i$, $d_i$, $h_i$, and $\chi_i(N)$ are all well-defined for all $i \in \mathbb{N}$.

By Proposition 3 we calculate

$$\chi_n(N) = \sum_{i=0}^{n} (-1)^i \dim_{F_p} H^i(N, F_p)$$

$$= 1 + \sum_{i=1}^{n} (-1)^i (d_{i-1} + d_i + p(a_i - d_{i-1}))$$

$$= (-1)^n d_n + p \left(\sum_{i=1}^{n} (-1)^i a_i\right) - p \left(\sum_{i=1}^{n} (-1)^i d_{i-1}\right).$$

Since $h_i = a_i + d_i$, we have

$$p\chi_n(T) - \chi_n(N) = p \left( 1 + \sum_{i=1}^{n} (-1)^i (a_i + d_i)\right) + (-1)^{n+1} d_n$$

$$- p \left(\sum_{i=1}^{n} (-1)^i a_i\right) + p \left(\sum_{i=1}^{n} (-1)^i d_{i-1}\right)$$

$$= p(-1)^n d_n + (-1)^{n+1} d_n$$

$$= (-1)^n (p - 1) d_n.$$ 

We have therefore shown item (a).

Now if $p > 2$ then by Theorem 1(7) and the fact that $F^0(N) = \{0\}$ we have

$$\chi^3_n(N) = p \sum_{i=1}^{n} (-1)^i (a_i - d_{i-1}).$$

Moreover, if $p = 2$ and $-1 \in N_{E/F}(E)$, then $(a_i - 1) = 0$ in $H^2(T, F_2)$ and $\text{ann}_{H^{n-1}(T, F_2)}(a_i - 1) = H^{n-1}(T, F_2)$. By Theorem 2(6) we obtain that the formula for the rank of $Y$ has the same shape as the formula...
in the case \( p > 2 \):

\[
\text{rank}_{\mathbb{F}_2[G]} Y = \dim_{\mathbb{F}_2} \text{cor} H^n(N, \mathbb{F}_2)/(a) \cdot H^{n-1}(T, \mathbb{F}_2)
= a_n - \Upsilon_1
= a_n - d_{n-1}.
\]

Therefore in this case we have

\[
\chi_n^\delta(N) = 2 \sum_{i=1}^{n} (-1)^i(a_i - d_{i-1})
\]

as well. In either case \( p > 2 \) or \( p = 2 \) and \(-1 \in N_{E/F}(E)\), we then calculate

\[
\chi_n(N) = 1 + \sum_{i=1}^{n} (-1)^i(d_{i-1} + d_i + p(a_i - d_{i-1}))
= (-1)^n d_n + \chi_n^\delta(N),
\]

showing equation (3).

\( \square \)

Corollary 1 is an immediate consequence of Theorem 3.

**Proof of Corollary 2.**

(1) \( \Rightarrow \) (2): Assume that \( \chi_n(N) = p\chi_n(T) \) for all subgroups \( N \) of \( T \) of index \( p \). Then by Corollary 1 the corestriction map \( \text{cor}_n : H^n(N, \mathbb{F}_p) \to H^n(T, \mathbb{F}_p) \) is surjective for all such subgroups \( N \). By exact sequence (1) of section 6, \( \text{res}_{n+1} : H^{n+1}(T, \mathbb{F}_p) \to H^{n+1}(N, \mathbb{F}_p) \) is injective for all such subgroups \( N \).

By the Bloch-Kato Conjecture (see Theorem 4), \( H^{n+1}(T, \mathbb{F}_p) \) is generated by elements \((a_1) \cdot (a_2) \cdots (a_{n+1})\), \( a_i \in F^\times \), \( a_i \in H^1(T, \mathbb{F}_p) \). We claim that these elements are equal to zero. If \((a_1) = 0\), then \((a_1) \cdots (a_{n+1}) = 0\). If \((a_1) \neq 0\), then let \( E = F(\sqrt[p]{a_1}) \) be the cyclic extension of degree \( p \) and set \( N = \text{Gal}(F(p)/E) \). Then \((a_1) \cdots (a_{n+1})\) lies in the kernel of \( \text{res}_{n+1} : H^{n+1}(T, \mathbb{F}_p) \to H^{n+1}(N, \mathbb{F}_p) \) since \( \text{res}_{n+1} \) is injective, \((a_1) \cdots (a_{n+1}) = 0\). Thus \( H^{n+1}(T, \mathbb{F}_p) = \{0\} \), as required.

(2) \( \Rightarrow \) (3) follows from the fact that for each open subgroup \( U \) of \( T \), \( \text{cd}(U) = \text{cd}(T) \leq n \) (see [NSW Prop. 3.3.5]) and hence for each open subgroup \( V \) of \( U \) of index \( p \), \( \text{cor} : H^n(V, \mathbb{F}_p) \to H^n(U, \mathbb{F}_p) \) is surjective. (See [NSW Prop. 3.3.8].) Then by Corollary 1 we have \( \chi_n(V) = p\chi_n(U) \).

(3) \( \Rightarrow \) (1) follows by setting \( U = T \).
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