On the local systems Hamiltonian in the weakly nonlocal Poisson brackets.

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Abstract

We study in this work the important class of nonlocal Poisson Brackets (PB) which we call weakly nonlocal. They appeared recently in some investigations in the Soliton Theory. However there was no theory of such brackets except very special first order case. Even in this case the theory was not developed enough. In particular, we introduce the Physical forms and find Casimirs, Momentum and Canonical forms for the most important Hydrodynamic type PB of that kind and their dependence on the boundary conditions.

Introduction

The fundamental idea of the local field-theoretical Poisson Brackets (PB) on the spaces of fields started to circulate widely in the community of theoretical and mathematical physicists in the second half of the 70s as a by-product

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of the KdV theory. It is interesting that in 1940 L.D.Landau wrote the right formulas for the local Poisson Brackets of fields in hydrodynamics but he called them "Quantum Commutators" saying nothing about Poisson Brackets. He realized very soon that he did not need this type of quantization of hydrodynamics for the description of superfluid $He^4$, and this subject was forgotten until the late 70s.

In pure mathematics the idea of symplectic structure has been considered since the 60s as a most fundamental extension of classical Hamiltonian formalism. This point of view is certainly true for the geometry of finite-dimensional manifolds. However, people studying PDEs coming from the problems of physics found out soon that the Poisson Structures are more fundamental because they (not the symplectic structures) are local in most important cases. Already the first (Gardner-Zakharov-Faddeev's) Poisson structure for KdV (1971) is a nonstandard local PB; however, it was originally described in the less convenient terminology as a "nonlocal symplectic structure" [1], or even in the standard trigonometric canonical coordinates for periodic boundary conditions [2]. Hamiltonian formalism became very important in the development of KdV theory: for example, it played an extremely important role in Novikov's approach to the solution of the periodic problem through stationary higher KdVs starting from the work [3] in 1974–see also the survey article [4].

The second (Lenard-Magri) local Poisson Structure for KdV and the idea of $\lambda$-pencils of compatible PB structures were discovered in 1977 (see in the book [9]). After that a huge number of nontrivial local PB appeared in the problems of mathematical physics especially for the different classes of Integrable Systems. Another fundamental class of local "hydrodynamic type" PB was discovered in 1983 (see [11] and Section 3) describing the Hamiltonian formalism of the first order quasilinear (i.e. "hydrodynamic type") systems in terms of Riemannian Geometry. Some specific nonlocal extension of this class started in the works [30, 31] plays an important role in the theory of hydrodynamic type PD systems. We call such PB a Weakly Nonlocal Hydrodynamic Type PB–see Section 3. The main part of this work is dedicated to the study of such brackets. In particular, in the present work we found all sets of Casimirs and canonical forms for them; we clarified their dependence on the boundary conditions, which is very important in the nonlocal case (see sections 4 and 5).

As it was observed already in the late 70s, the compatible pair of two
Poisson Structures leads to the "Recursion Operator" (see Section 2 below). This recursion operator produces an infinite series of nonlocal Poisson Structures: therefore, the KdV equation, for example, is the Hamiltonian system relative to the infinite number of nonlocal Poisson Structures with different Kruskal integrals as the Hamiltonians. The structure of these nonlocalities was not clarified until the 90s. In the case of KdV the right formula for them was obtained in the work [8] in 1993. The case of NLS is more complicated: there is only one local PB for it, all others are nonlocal. In the present work we clarified the structure of their nonlocality for NLS (see Section 2). As a by-product of this result we introduce a general notion of the Weakly Nonlocal Poisson Bracket. This notion is very natural for the (1+1)-systems. Such brackets produce local PD systems for the broad classes of local Hamiltonians. They appear in many integrable systems but can describe also a lot of non-integrable local perturbations. As it was found recently by Maltsev, this class of Poisson Brackets is closed under the operation of the "Whitham Averaging" (it is a nonlinear analog of the WKB approximation; people frequently call it a "method of the slow modulations of parameters"). This method is based on the proper family of quasiperiodic solutions (invariant tori):

The slow modulation of parameters always leads to the hydrodynamic type system Hamiltonian in the weakly nonlocal hydrodynamic type PB if you started from the local evolution system generated by the local Hamiltonian in any weakly nonlocal PB (see [18, 19]). For the local case this theory was developed by Dubrovin and Novikov in 1983 [11] (see also the survey article [12]). However, in 1992 a gap was found in the general proof of the Jacoby Identity for the hydrodynamic type PB constructed in these works (see [14]). This gap was fulfilled by Maltsev in 1998: full proof can be found in the work [16].

Our results presented here can be divided into two parts:

Part I (see the Section 2) explains how Weakly Nonlocal Poisson Brackets and Symplectic Structures appear from the theory of the famous completely integrable soliton systems like KdV and NLS. It turns out that all Higher Poisson Brackets (known since the late 70s) are weakly nonlocal for $n \geq 0$. For the case of KdV this result follows from the work [8], for NLS it is completely new. We prove also that all Higher Symplectic Structures are weakly nonlocal for $n \leq 0$. This result is new for the both cases.

Part II (see the Sections 3–5) is dedicated to the Weakly Nonlocal Poisson
Brackets of Hydrodynamic Type associated with Riemannian Geometry. The most important results known before were obtained by Ferapontov, Mokhov and Pavlov (see [30], [34], [36]). We formulate their results below. In this work we found the Canonical forms, Casimirs and the Hamiltonians for the Structure Flows (proving therefore that they are the Hamiltonian Systems). It turns out that these important quantities depend on the boundary conditions. We prove also that the Symplectic Structure is weakly nonlocal for all such Poisson Brackets.

1 Local and Weakly Nonlocal Poisson and Symplectic Structures.

We are going to consider only one-dimensional (i.e. 1+1) systems, so we define the Poisson Structures either on the spaces of loops $L(\mathcal{M}^N)$ containing the mappings $f : S^1 \to \mathcal{M}^N$ of the circle (periodic boundary conditions)

or on the spaces $L(\mathcal{M}^N, y)$ of mappings of the line $R \to \mathcal{M}^N$ constant at infinity, into some manifold $\mathcal{M}^N$ with local coordinates $\varphi^1, \ldots \varphi^n$ where $\varphi(y) = 0$. Therefore we think about these mappings as vector-functions (fields) $\varphi(x) = (\varphi^1(x), \ldots \varphi^n(x))$. In the nonlocal case we consider only the boundary conditions constant at infinity, i.e. our mapping should be such that for $x \to \pm \infty$ we have $f(x) \to y \in \mathcal{M}^N$ and $\varphi(x) \to 0$. Here $y$ is some point of the manifold $\mathcal{M}^N$. We require also that all derivatives of the map $f$ tend to zero at infinity.

The local field-theoretical PB can be completely defined by the finite order differential Hamiltonian operator of the form:

$$J^{ij} = \sum_{L \geq k \geq 0} B^{ij}_k(x, \varphi(x), \varphi_x(x), \ldots) \partial_x^k$$

We call this PB translation invariant if the Hamiltonian operator does not depend on $x$. The Hamiltonian system generated by the Hamiltonian functional $I\{\varphi\}$ has the form

$$\varphi^i_t = J^{ij}_{ij} \frac{\delta I\{\varphi\}}{\delta \varphi^j(x)}$$
The Poisson bracket of two local functionals is defined by the formula

\[ \{I_1, I_2\} = \int \frac{\delta I_1}{\delta \varphi^i(x)} J^{ij} \frac{\delta I_2}{\delta \varphi^j(x)} \, dx \]

In the field theory people prefer to write the Poisson Brackets for fields in the local form convenient for calculations

\[ \{\varphi^i(x), \varphi^j(y)\} = J^{ij} \delta(x - y) \]

These formulas define a skew-symmetric bilinear operation satisfying to the so-called Leibnitz Identity \( \{fg, h\} = f \{g, h\} + g \{f, h\} \). It should satisfy also to the Jacoby Identity for 3 functionals:

\[ \{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0 \]

This requirement is very strong. It leads to specific, very serious restrictions on the class of admissible Hamiltonian operators.

We define the weakly non-local Poisson brackets (PB) through the corresponding class of Hamiltonian operators:

\[ J^{ij} = \sum_{k \geq 0} B^{ij}_k (\varphi, \varphi_x, \ldots) \partial_x^k + \sum_{k,l \geq 0} e_{kl} S_i^j (\varphi, \varphi_x, \ldots) \partial^{-1} S_j^i (\varphi, \varphi_x, \ldots) \quad (1.1) \]

where \( \partial_x \equiv d/dx \) and \( \partial^{-1} \) is defined here as a skew-symmetric operator on the line

\[ \partial^{-1} = \frac{1}{2} \int_{-\infty}^x dx - \frac{1}{2} \int_x^{+\infty} dx \quad (1.2) \]

on the space of rapidly decreasing at \( \pm \infty \) vector-functions. The constants \( e_{kl} = e_{lk} \) give a quadratic form in the linear space generated by the (linearly independent) flows \( S_{(k)}(\varphi, \varphi_x, \ldots) \). We say that the bracket is written in the Reduced Form if \( e_{kl} = e_k \delta_{kl} \) where \( e_k = \pm 1 \).

Let us point out that we define only translation invariant weakly nonlocal Hamiltonian operators. It is no problem to extend this definition to the coefficients dependent on \( x \), but such brackets will not be considered in this work.
In the same way we define a class of weakly-nonlocal Symplectic Structures. We call the symplectic structure weakly-nonlocal if the operator $J^{-1}$ (if it exists) has a weakly-nonlocal form, i.e.

$$(J^{-1})_{ij} = \sum_{k=0}^{N} C_{(k)ij}(\varphi, \varphi_x, \ldots) \partial^k + \sum_{k,l \geq 0} d_{kl} Q_{(k)ij}(\varphi, \varphi_x, \ldots) \partial^{-1} Q_{(l)ij}(\varphi, \varphi_x, \ldots)$$

(1.3)

As far as we know, the first example of Poisson bracket written in the literature precisely in this form was the Sokolov bracket (\textsuperscript{[5]})

$$\{ \varphi(x), \varphi(y) \} = \varphi_x \nu(x - y) \varphi_y$$

(1.4)

designed to prove that the so-called Krichever-Novikov equation is Hamiltonian:

$$\varphi_t = \varphi_{xxx} - \frac{3}{2} \varphi_{xx}^2 + \frac{h(\varphi)}{\varphi_x} = \varphi_x \partial^{-1} \varphi_x \frac{\delta H}{\delta \varphi_x}$$

where $h(\varphi) = c_3 \varphi^3 + c_2 \varphi^2 + c_1 \varphi + c_0$ and

$$H = \int \left( \frac{1}{2} \varphi_{xx}^2 + \frac{1}{3} \frac{h(\varphi)}{\varphi_x^2} \right) dx$$

For the Sokolov bracket the corresponding symplectic structure is local:

$$J^{-1} = \frac{1}{\varphi_x} \partial \frac{1}{\varphi_x}$$

This equation appeared originally in work \textsuperscript{[3]} describing the "rank 2" solutions of the KP system. In pure algebra it describes the deformations of the commuting genus 1 pairs OD operators of the rank 2 whose classification was obtained in this work. As it was found later, this equation is a unique third order in $x$ completely integrable evolution equation which cannot be reduced to KdV by Miura type transformations.

Let us mention that the local symplectic structures was considered by I.Dorfman and O.I.Mokhov (see Review \textsuperscript{[7]}).

For the Weakly Nonlocal Poisson Brackets of basic fields we use the same definition as above. They have the following form:
\{\varphi^i(x), \varphi^j(y)\} = J^i_j \delta(x-y) = \sum_{k \geq 0} B^i_j (\varphi, \varphi_x, \ldots) \delta^{(k)}(x-y) +
+ \sum_{k,l} e_{kl} S^i_{(k)} (\varphi, \varphi_x, \ldots) \nu(x-y) S^j_{(l)} (\varphi, \varphi_y, \ldots)
(1.5)

where $e_{kl} = e_{lk}$ is a constant symmetric matrix, $\delta^{(k)}(x-y) \equiv d^k/dx^k \delta(x-y)$, 
$\nu(x-y) = 1/2 \text{sgn}(x-y)$ and both sums contain the finite number of terms.
All functions involved depend on the finite numbers of derivatives of $\varphi$ with respect to $x$. The bracket is written in the Reduced form if $e_{kl} = e_k \delta_{kl}$ and $e_k = \pm 1$.

We include in the definition the following requirements: all flows with right-hand parts equal to $S^i_{(k)}, k = 1, 2, \ldots$ form a linearly independent set; the flows

$\varphi^i_t = S^i_{(k)} (\varphi, \varphi_x, \ldots)$
(1.6)

commute with each other. We call them the **Structure Flows** for a given weakly-nonlocal Hamiltonian structure. The structure flows should preserve the Poisson structure (1.5). Both requirements were proved in the works \cite{18, 19} as a corollary from the definition of the weakly nonlocal PB given above, but here we simply include them in the definition.

The brackets (1.5) were already used in fact in the recent works \cite{18} and \cite{19} where the nonlocal Hamiltonian version of the Whitham averaging method was considered for the local systems (PDEs) with the full necessary set of local commuting integrals. The Poisson Brackets used in these works are in fact weakly nonlocal.

We call the weakly nonlocal Poisson Bracket **fundamental** if it contains only one flow of the form $S^i_0 = \varphi^i_x$ in the non-local part (i.e. its nonlocal part exactly coincides with the Sokolov PB written above).

In this case **every local translation invariant Hamiltonian**

$$H = \int h(\varphi(x), \varphi_x(x), \ldots, \varphi_{x\ldots x}(x)) dx$$

generates the local (PDE) system because:

$$\varphi^i_x \frac{\delta H}{\delta \varphi^i(x)} = \partial_x Q(\varphi, \varphi_x, \ldots)$$
for some function $Q$.

For the more general weakly nonlocal Poisson brackets defined above this property is valid only for the special Hamiltonians: it was pointed out by E.V.Ferapontov (33), that for the nonlocal brackets of Hydrodynamic Type

**local Hamiltonian generates local system if and only if it is a conservative quantity for the flows $\varphi^i = S^i_{(k)}$ in the nonlocal part of the bracket.**

This statement is valid for any weakly nonlocal bracket, and the proof is the same.

Let us mention also that it is easy to check by direct calculation that for any closed 2-form (1.3) the corresponding forms $Q_{(k)i}(\varphi, \varphi_x, \ldots)$ should be closed 1-forms in the functional space $\varphi(x)$. In the weakly-nonlocal symplectic structures given by the integrable systems they usually appear as the Euler-Lagrange derivatives of local Hamiltonian functionals of the corresponding hierarchy (see below).

## 2 Weakly nonlocal Poisson and Symplectic Structures and famous Integrable Systems.

Let us mention here that in the late 70s many people considered an infinite series of the nonlocal Poisson Brackets for the famous integrable systems like KdV, NLS, etc., on the spaces of rapidly decreasing functions. They obtained these brackets following the Lenard-Magri scheme starting with the initial pair of local brackets for KdV (for the NLS system the second bracket is already nonlocal indeed, it is weakly nonlocal in our sense). For the KdV - equation: $\varphi_t = 6\varphi \varphi_x - \varphi_{xxx}$ we have $J_0 = \partial$ (Gardner - Zakharov - Faddeev bracket) and $J_1 = -\partial^3 + 2(\varphi \partial + \partial \varphi)$ (Lenard-Magri bracket)

The recursion operator $R = -\partial^2 + 4\varphi + 2\varphi_x \partial^{-1}$ such that $RJ_0 = J_1$, generates the next bracket with the Hamiltonian operator $R^2 J_0 = J_2$:

$$J_2 = \partial^5 - 8\varphi \partial^3 - 12\varphi_x \partial^2 - 8\varphi_{xx} \partial + 16\varphi^2 \partial - 2\varphi_{xxx} + 16\varphi_x - 4\varphi_x \partial^{-1}\varphi_x$$

In work (8) all nonlocal parts of higher PBs were calculated. These authors never defined any specific class of Poisson Brackets, but using their
results we are easily coming to the statement:

All higher brackets for KdV given by the formula $J_n = R^n J_0$, $n \geq 0$ are weakly nonlocal. The corresponding flows $S^{(k)}(\varphi, \varphi_x, \ldots)$ are exactly the Higher KdV systems and the exact formula for $J^n$ can be written in the form:

$$J^n = (\text{local part}) - \sum_{k=1}^{n-1} S^{(k)}(\varphi, \varphi_x, \ldots) \partial^{-1} S^{(n-k-1)}(\varphi, \varphi_x, \ldots)$$

where $S^{(1)}(\varphi, \varphi_x, \ldots) = 2\varphi_x$ and

$$S^{(k)}(\varphi, \varphi_x, \ldots) \equiv R S^{(k-1)}(\varphi, \varphi_x, \ldots)$$

are higher KdV flows.

The similar weakly-nonlocal expression for the positive powers of the recursion operator for KdV was also considered in [8]. Let us represent here the corresponding result

$$R^n = (\text{local part}) + \sum_{k=1}^{n} S^{(k)}(\varphi, \varphi_x, \ldots) \partial^{-1} \frac{\delta H^{(n-k)}}{\delta \varphi(x)} , \quad n \geq 0$$

where $S^{(k)} = \partial_x \frac{\delta H^{(k)}}{\delta \varphi(x)}$, $H^{(0)} = \int \varphi dx$ and

$$\frac{\delta H^{(k)}}{\delta \varphi(x)} \equiv \frac{\delta H^{(k-1)}}{\delta \varphi(x)} R$$

are Euler-Lagrange derivatives of higher Hamiltonian functions for KdV hierarchy. Let us mention also that in our notations $R$ acts from the left on the vectors and from the right on the 1-forms in the functional space.

Using these results we prove here the following

**Proposition.**

All symplectic structures for KdV $\Omega_{-n} = (J_{-n})^{-1}$ are weakly nonlocal for $n \geq 0$. The 1-forms $Q^{(k)}(\varphi, \varphi_x, \ldots)$ in the nonlocal part are the Euler-Lagrange derivatives of higher Hamiltonian functions $H^{(n)}$ and the formula for $\Omega_{-n}$ can be written as

$$\Omega_{-n} = (\text{local part}) + \sum_{k=0}^{n} \frac{\delta H^{(k)}}{\delta \varphi(x)} \partial^{-1} \frac{\delta H^{(n-k)}}{\delta \varphi(x)}$$

Proof. We have $\Omega_0 = \partial^{-1}$ and by the definition:
\[ \Omega_{-n} = \partial^{-1} R^n = \partial^{-1} (\text{local part}) + \sum_{k=1}^n \partial^{-1} S_{(k)}(\varphi, \varphi_x, \ldots) \partial^{-1} \frac{\delta H_{(n-k)}}{\delta \varphi(x)} = \]

\[ = \partial^{-1} (\text{local part}) + \sum_{k=1}^n \partial^{-1} \left( \frac{\delta H_{(k)}}{\delta \varphi(x)} \right) \partial^{-1} \frac{\delta H_{(n-k)}}{\delta \varphi(x)} = \]

\[ = \partial^{-1} (\text{local part}) + \sum_{k=1}^n \partial^{-1} \left( \partial \frac{\delta H_{(k)}}{\delta \varphi(x)} - \frac{\delta H_{(k)}}{\delta \varphi(x)} \partial \right) \partial^{-1} \frac{\delta H_{(n-k)}}{\delta \varphi(x)} = \]

\[ = \frac{\delta H_{(0)}}{\delta \varphi(x)} \partial^{-1} \frac{\delta H_{(0)}}{\delta \varphi(x)} (\text{local part}) + \]

\[ + \sum_{k=1}^n \frac{\delta H_{(k)}}{\delta \varphi(x)} \partial^{-1} \frac{\delta H_{(n-k)}}{\delta \varphi(x)} - \sum_{k=1}^n \frac{\delta H_{(k)}}{\delta \varphi(x)} \partial^{-1} \frac{\delta H_{(k)}}{\delta \varphi(x)} \frac{\delta H_{(n-k)}}{\delta \varphi(x)} \]

We can also write that

\[ \frac{\delta H_{(0)}}{\delta \varphi(x)} (\text{local part}) = \zeta(x) + \partial (\text{local part})' \]

where \( \zeta(x) \) is a result of the action of the local part of \( R^n \) on \( \delta H_{(0)}/\delta \varphi(x) \) (from the right) and \( (\text{local part})' \) is a differential operator. So we have

\[ \Omega_{-n} = (\text{local part})' + \sum_{k=1}^n \frac{\delta H_{(k)}}{\delta \varphi(x)} \partial^{-1} \frac{\delta H_{(n-k)}}{\delta \varphi(x)} + \]

\[ + \frac{\delta H_{(0)}}{\delta \varphi(x)} \partial^{-1} \left[ \zeta(x) - \sum_{k=1}^n \frac{\delta H_{(k)}}{\delta \varphi(x)} \frac{\delta H_{(n-k)}}{\delta \varphi(x)} \right] \]

where the expression in the brackets is equal to

\[ \frac{\delta H_{(0)}}{\delta \varphi(x)} R^n \equiv \frac{\delta H_{(n)}}{\delta \varphi(x)} \]

So we obtain the statement of the Proposition.

It seems that this property is very general for the brackets given by the recursion operators for integrable systems. We prove here the similar fact for the NLS-equation:
\[ i\psi_t = -\psi_{xx} + 2\kappa|\psi|^2\psi \] where \( \psi \) is a complex function. We have

\[ \{\psi(x), \bar{\psi}(y)\}_0 = i\delta(x-y) \] (2.1)

This is the first Hamiltonian structure. It corresponds to the operator

\[ J_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \]

There is an infinite number of Hamiltonian structures connected with this one through the recursion operator ([9], p. 218). The bracket \( \{\ldots,\ldots\}_1 \) corresponds to the Hamiltonian operator

\[ J_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} - 2\kappa \begin{pmatrix} -\psi\partial^{-1}\psi & \psi\partial^{-1}\bar{\psi} \\ \bar{\psi}\partial^{-1}\psi & -\bar{\psi}\partial^{-1}\bar{\psi} \end{pmatrix} \]

which has the form (1.1) with only one flow in the nonlocal part

\[ \left( \begin{array}{c} \psi \\ \bar{\psi} \end{array} \right)_t = \sqrt{2\kappa} \left( \begin{array}{c} i\psi \\ -i\bar{\psi} \end{array} \right) \] (2.2)

**Proposition.** All the brackets \( \{\ldots,\ldots\}_n, n \geq 0 \) given by the recursion \( R^nJ_0 \) have the form (1.1) with the flows from NLS-hierarchy in the nonlocal parts. All the symplectic structures \( \Omega_{-n} = (J_{-n})^{-1} = \Omega_0R^n \) have the form (1.3) where the forms \( Q_{(k)}(\psi, \bar{\psi}, \ldots) \) are the Euler-Lagrange derivatives of the higher Hamiltonian functionals of the NLS-hierarchy. The corresponding formulas for \( J_n, \Omega_{-n} \) and \( R^n (n \geq 0) \) can be written as

\[ J_n = \text{(local part)} - \sum_{k=1}^{n} S_{(k-1)}(\psi, \bar{\psi}, \ldots)\partial^{-1}S_{(n-k)}(\psi, \bar{\psi}, \ldots) \]

\[ R^n = \text{(local part)} + \sum_{k=1}^{n} S_{(k-1)}(\psi, \bar{\psi}, \ldots)\partial^{-1}\frac{\delta H_{(n-k)}}{\delta(\psi, \bar{\psi})(x)} \]

\[ \Omega_{-n} = \text{(local part)} + \sum_{k=1}^{n} \frac{\delta H_{(k-1)}}{\delta(\psi, \bar{\psi})(x)}\partial^{-1}\frac{\delta H_{(n-k)}}{\delta(\psi, \bar{\psi})(x)} \]

where
\[ S(k) \equiv J_0 \frac{\delta H(k)}{\delta(\psi, \bar{\psi})}(x), \quad H(0) = \sqrt{2\kappa} \int \psi \bar{\psi} dx, \quad \text{and} \quad \frac{\delta H(k)}{\delta(\psi, \bar{\psi})(x)} = R \frac{\delta H(k-1)}{\delta(\psi, \bar{\psi})(x)} \]

for any \( k \geq 1 \).

Proof. We use the induction. The recursion operator \( R = J_1 J_0^{-1} \) takes here the form:

\[
R = \begin{pmatrix} -i\partial & 0 \\ 0 & i\partial \end{pmatrix} + 2\kappa \begin{pmatrix} i\psi \partial^{-1} \bar{\psi} & i\psi \partial^{-1} \psi \\ -i\bar{\psi} \partial^{-1} \psi & -i\bar{\psi} \partial^{-1} \bar{\psi} \end{pmatrix}
\]

Suppose now that \( J_n \) has the required form:

\[
J_n = \sum_{k \geq 0} B(n)k(\psi, \bar{\psi}, \ldots) \partial^k - \sum_{k=1}^n J_0 \frac{\delta H(k-1)}{\delta(\psi, \bar{\psi})(x)} \partial^{-1} J_0 \frac{\delta H(n-k)}{\delta(\psi, \bar{\psi})(x)}
\]

(2.3)

where \( B(n)k \) are \( 2 \times 2 \) matrices and \( H(k) \) are the higher NLS Hamiltonians by the induction assumption. We note that \( R \) can be written in the form:

\[
R = \begin{pmatrix} -i\partial & 0 \\ 0 & i\partial \end{pmatrix} + \sqrt{2\kappa} \begin{pmatrix} i\psi \partial^{-1} \delta N/\delta \psi(x) & i\psi \partial^{-1} \delta N/\delta \bar{\psi}(x) \\ -i\bar{\psi} \partial^{-1} \delta N/\delta \psi(x) & -i\bar{\psi} \partial^{-1} \delta N/\delta \bar{\psi}(x) \end{pmatrix}
\]

Here \( N = \sqrt{2\kappa} \int \psi(x) \bar{\psi}(x) dx \) is the first Hamiltonian in the NLS hierarchy, generating the flow (2.2) with respect to the bracket (2.1). Since it commutes with any \( H(k) \) with respect to the bracket (2.1), we have

\[
\left( \frac{\delta N}{\delta \psi(x)} \frac{\delta N}{\delta \bar{\psi}(x)} \right) J_0 \left( \frac{\delta H(k-1)}{\delta \psi(x)} \frac{\delta H(k-1)}{\delta \bar{\psi}(x)} \right) \equiv \left( Q_{(k-1)} \right)_x = \partial Q_{(k-1)} - Q_{(k-1)} \partial
\]

for some functions \( Q_{(k-1)}(\psi, \bar{\psi}, \ldots) \). We have

\[
RJ_n = (\text{local terms}) - \sum_{k=1}^n \left( \begin{array}{cc} -i\partial & 0 \\ 0 & i\partial \end{array} \right) J_0 \frac{\delta H(k-1)}{\delta(\psi, \bar{\psi})(x)} \partial^{-1} J_0 \frac{\delta H(n-k)}{\delta(\psi, \bar{\psi})(x)} +
\]

\[+ \sqrt{2\kappa} \begin{pmatrix} i\psi \partial^{-1} \delta N/\delta \psi(x) & i\psi \partial^{-1} \delta N/\delta \bar{\psi}(x) \\ -i\bar{\psi} \partial^{-1} \delta N/\delta \psi(x) & -i\bar{\psi} \partial^{-1} \delta N/\delta \bar{\psi}(x) \end{pmatrix} \sum_{k \geq 0} B(n)k(\psi, \bar{\psi}, \ldots) \partial^k -
\]
\[-\sqrt{2}\kappa \sum_{k=1}^{n} \left( \begin{array}{cc} i\psi & -i\bar{\psi} \\ -i\bar{\psi} & i\psi \end{array} \right) \partial^{-1} \left( \partial Q_{(k-1)} - Q_{(k-1)} \partial \right) \partial^{-1} J_{0} \frac{\delta H_{(n-k)}}{\delta(\psi, \bar{\psi})(x)} \]

We can write the relations:

\(-i\partial \begin{array}{cc} 0 & 0 \\ 0 & i\partial \end{array} J_{0} \frac{\delta H_{(k-1)}}{\delta(\psi, \bar{\psi})(x)} = \left( \begin{array}{cc} 0 & 0 \\ -i & i \end{array} \right) \left[ \left( J_{0} \frac{\delta H_{(k-1)}}{\delta(\psi, \bar{\psi})(x)} \right)_{x} + J_{0} \frac{\delta H_{(k-1)}}{\delta(\psi, \bar{\psi})(x)} \partial \right] \]

and

\( \left( \frac{\delta N}{\delta \psi(x)} \frac{\delta N}{\delta \bar{\psi}(x)} \right) \sum_{k\geq 0} B_{(n)k}(\psi, \bar{\psi}, \ldots) \partial^{k} = (\zeta(x) \bar{\zeta}(x)) + \partial(\text{local terms}) \)

where \((\zeta(x) \bar{\zeta}(x))\) is a result of the action of the local part of \(J_{n}\) (from the right) on \((\delta N/\delta \psi(x) \delta N/\delta \bar{\psi}(x))\). So the nonlocal part of \(RJ_{n}\) has a form

\[-\sum_{k=1}^{n} \left[ \left( \begin{array}{cc} 0 & 0 \\ -i & i \end{array} \right) J_{0} \frac{\delta H_{(k-1)}}{\delta(\psi, \bar{\psi})(x)} \right] + \sqrt{2}\kappa \left( \begin{array}{cc} i\psi & -i\bar{\psi} \\ -i\bar{\psi} & i\psi \end{array} \right) Q_{(k-1)} \partial^{-1} J_{0} \frac{\delta H_{(n-k)}}{\delta(\psi, \bar{\psi})(x)} \]

\[+ \sqrt{2}\kappa \left( \begin{array}{cc} i\psi & -i\bar{\psi} \\ -i\bar{\psi} & i\psi \end{array} \right) \partial^{-1} \left[ (\zeta(x) \bar{\zeta}(x)) + \sqrt{2}\kappa \sum_{k=1}^{n} Q_{(k-1)} J_{0} \frac{\delta H_{(n-k)}}{\delta(\psi, \bar{\psi})(x)} \right] \]

Using the relations:

\(-i\partial \begin{array}{cc} 0 & 0 \\ 0 & i\partial \end{array} J_{0} \frac{\delta H_{(k-1)}}{\delta(\psi, \bar{\psi})(x)} + \sqrt{2}\kappa \left( \begin{array}{cc} i\psi & -i\bar{\psi} \\ -i\bar{\psi} & i\psi \end{array} \right) Q_{(k-1)} = RS_{(k-1)} = S_{(k)} \)

and

\( (\zeta(x) \bar{\zeta}(x)) + \sum_{k=1}^{n} Q_{(k-1)} J_{0} \frac{\delta H_{(n-k)}}{\delta(\psi, \bar{\psi})(x)} = \frac{\delta N}{\delta(\psi, \bar{\psi})(x)} J_{n} = -S_{(n)} \)

we obtain the required formula for \(J_{n+1}\).
Now using the expressions \( R^n = J_n \Omega_0 \) and \( \Omega_{-n} = \Omega_0 R^n = \Omega_0 J_n \Omega_0 \) where

\[
\Omega_0 = \begin{pmatrix}
0 & i \\
-i & 0
\end{pmatrix}
\]

it is easy to obtain the corresponding formulas for \( R^n \) and \( \Omega_{-n} \) for \( n \geq 1 \).

Proposition is proved.

3  Poisson brackets of Hydrodynamic Type.

The well-known Whitham averaging procedure (or the nonlinear WKB method) we consider for the local (PDE) evolution system only. It is based on the family of invariant tori (i.e. exact solutions quasiperiodic in \( x, t \)) dependent on some parameters \( U \). We assume that a family of the local integrals is given such that the value of parameters \( U \) can be chosen as the average values of the densities of integrals along the invariant tori. For the Hamiltonian systems we assume that these integrals are commuting (we call it ”Liouville Property”), and we have an invariant, completely integrable finite-dimensional subsystem or family of them.

This procedure leads to first order homogeneous quasilinear systems (Hydrodynamic type system) useful in many cases for asymptotic studies:

\[
U_T^\nu = V_\nu^\nu(U)U_X^\mu
\]

As it was established in 1983 (see \[1\]), they are Hamiltonian in the so-called Hydrodynamic Type Poisson Brackets (HTPB) or Dubrovin-Novikov (DN)–brackets in the local case (i.e. the original system was Hamiltonian in the local PB)

\[
\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(U)\delta'(X - Y) + b_\lambda^\nu(U)U_X^\lambda \delta(X - Y)
\]

introduced in \[1\] (see also \[2, \[3\] for more complete information). (There was a gap in the general proof of the Jacoby Identity for the ”averaged” Poisson Bracket constructed in 1983; this gap was finally fulfilled in work \[14\]).

\[1\] It is worth to mention that in the variables \( r = \sqrt{\psi \bar{\psi}}, \theta = -i(\psi_x / \psi - \bar{\psi}_x / \bar{\psi}) \) (i.e. \( \psi = r \exp(i \int \theta dx) \)) the NLS-equation has in fact three \((J_0, J_1, J_2)\) local Hamiltonian structures. We don’t know any place in the literature where this fact was clearly mentioned.
Let us remind here the general properties of the brackets (3.1) - (see [11]- [13]).

Any Hydrodynamic type Hamiltonian \( H = \int h(U(x)) \, dx \) (i.e. its density does not depend on derivatives) generates a H.T.System with Hydrodynamic type (or DN) P.B.. Consider the H.T. bracket (3.1) such that \( \text{det} \, g^{\nu \mu} \neq 0 \).

It follows from the Leibnitz property that the first coefficient (\( \delta \)-prime term) transforms under the "pointwise" change of coordinates \( u(w) \) as a Riemannian metric with upper indices, and the second coefficient (\( \delta \)-term) transforms as a set of Christoffel symbols (=connection) with two upper indices. Indices should be raised up with the same metric. It is skew-symmetric if and only if the tensor \( g^{\nu \mu} \) is symmetric (i.e. defines a pseudo-Riemannian metric), and the connection \( \Gamma_{\nu}^{\mu \lambda} = -g_{\mu \tau}b_{\lambda}^\tau \) is compatible with this metric: \( \nabla_\lambda g_{\mu \nu} \equiv 0 \).

The bracket (3.1) satisfies the Jacobi Identity if and only if connection \( \Gamma_{\nu}^{\mu \lambda} \) is symmetric, and the metric is flat. Therefore the signature of metric is a complete local invariant under the change of coordinates \( u(w) \).

Three types of coordinates for this kind of P.B. play an important role in applications:

1. The Canonical (flat) coordinates \( n^\mu \) where the connection coefficients are equal to zero. The integrals \( \int n^\nu(x) \, dx \) are Casimirs for this P.B..

2. The so-called "Physical Coordinates" \( U^\nu \) obtained by the averaging of densities of local commuting integrals. We say that the coordinates are "Liouville" or Physical for H.T.P.B. if it has the form:

\[
\{U^\nu(X), U^\mu(Y)\} = (\gamma^{\nu \mu}(U) + \gamma^{\mu \nu}(U)) \delta'(X - Y) + \frac{\partial \gamma^{\nu \mu}}{\partial U^\lambda} U^\lambda \delta(X - Y)
\]

for some functions \( \gamma^{\nu \mu}(U) \).

Any coordinates such that integrals of them define the commuting flows, are physical in that sense.

In particular, any coordinates such that PB is linear, are physical.

The general local Poisson Brackets of any order linearly dependent on the fields were studied in works [20, 21]. The especially interesting class of linear Hydrodynamic Type P.B. was studied in works [22]- [28]. It leads to the beautiful algebraic and differentially geometrical theory of the local translational invariant first order Lie algebras, Frobenius type algebras and their non-associative analogs ("Novikov Algebras"), and super-analogs of that theory.
3. The diagonal form where our metric and H.T. system both are diagonal: the coordinates are orthogonal for the metric, and the velocity tensor $V_{\mu}^{\nu}(U)$ is diagonal (in classical terminology the coordinates $U$ are the "Riemann Invariants" for our system). This form has the following fundamental property: all diagonal H.T. Systems Hamiltonian in such PB, are Completely Integrable. It was conjectured by S.P. Novikov and proved by S.P. Tsarev in his PhD thesis in 1985 (see [29]).

Let us point out that the beautiful Tsarev integration procedure based on the Riemannian metric turns out to be more broad than the class of H.T. Systems Hamiltonian corresponding to the local H.T.P.B.. It integrates also the systems called "semihamiltonian". The Riemannian metric is non-flat in that case. Probably, all semihamiltonian systems are in fact Hamiltonian corresponding to some weakly nonlocal H.T.P.B. with (maybe) an infinite number of terms in the nonlocal tail. Some investigation of this problem can be found in [34], [35] but this problem is still open.

The first weakly nonlocal fundamental PB of hydrodynamic type was found in [30], the more general class - in [31]- [34]. The weakly nonlocal PB of the types different from the hydrodynamic one, never have been studied to our knowledge. Let us describe the Mokhov-Ferapontov fundamental weakly nonlocal Poisson bracket of hydrodynamic type (MF-bracket, see [30]):

$$\{U^\nu(X), U^\mu(Y)\} = g^{\mu\nu}(U)\delta'(X - Y) + b^{\mu\nu}_\lambda(U)U^\lambda_X \delta(X - Y) + cU^\nu_Y \nu(X - Y)U^\mu_X$$

(3.2)

and more general Ferapontov weakly nonlocal brackets of hydrodynamic type (F-bracket [31]):

$$\{U^\nu(X), U^\mu(Y)\} = g^{\mu\nu}(U)\delta'(X - Y) + b^{\mu\nu}_\lambda(U)U^\lambda_X \delta(X - Y) +$$

$$+ \sum_{k=1}^{g} e_k w^{\mu}_{(k)\lambda}(U)U^\lambda_X \nu(X - Y) w^{\nu}_{(k)\delta}(U)U^\delta_Y$$

(3.3)

$e_k = \pm 1$.

Consider the fundamental MF-bracket (3.2) with non-degenerate metric tensor $g^{\mu\nu}(U)$. It is skew-symmetric if and only if the tensor $g^{\mu\nu}$ is symmetric, and the connection
\[ \Gamma^\nu_{\mu\lambda} = -g_{\mu\tau} b^\tau_{\lambda} \]

is compatible with this metric: \( \nabla_\lambda g_{\mu\nu} \equiv 0 \).

The bracket (3.2) satisfies Jacobi Identity if and only if its connection \( \Gamma^\nu_{\mu\lambda} \)

is symmetric (i.e. torsion tensor is equal to zero) and has constant curvature equal to \( c \), i.e.

\[ R^\nu_{\mu\lambda} = c \left( \delta^\nu_{\mu} \delta^\lambda_{\tau} - \delta^\nu_{\tau} \delta^\lambda_{\mu} \right) \]  (3.4)

Consider the F-bracket (3.3) with \( \text{det} \ g^{\nu\mu} \neq 0 \).

The bracket (3.3) is skew-symmetric if and only if tensor \( g^{\nu\mu} \) is symmetric and connection is compatible with this metric as above for the local case.

The bracket (3.3) satisfies Jacobi Identity if and only if its connection \( \Gamma^\nu_{\mu\lambda} \)

is symmetric, the metric \( g^{\nu\mu} \) (with lower indices) and tensors \( w^\nu_{(k)\mu} \) satisfy the equations:

\[ g_{\nu\tau} w^\tau_{(k)\mu} = g_{\mu\tau} w^\tau_{(k)\nu}, \quad \nabla_\nu w^\mu_{(k)\lambda} = \nabla_\lambda w^\mu_{(k)\nu} \]  (3.5)

\[ R^\nu_{\mu\lambda} = \sum_{k=1}^g e_k \left( w^\nu_{(k)\mu} w^\tau_{(k)\lambda} - w^\tau_{(k)\mu} w^\nu_{(k)\lambda} \right) \]  (3.6)

Moreover, this set is commutative \( [w_k, w_{k'}] = 0 \).

It was pointed out by E.V.Ferapontov that the equations written above are the Gauss-Codazzi equations for the submanifolds \( M^N \) with flat normal connection in the Pseudo-Euclidean space \( E^{N+g} \). Here \( g_{\nu\mu} \) is the first quadratic form of \( M^N \), and \( w_{(k)} \) are the Weingarten operators corresponding to the field of pairwise orthogonal unit normals \( \vec{n}_k \), see [31]-[34]. Moreover, it was proved by E.V.Ferapontov that these brackets can be obtained as a result of Dirac restriction of the local DN-bracket

\[ \{ N^I(X), N^J(Y) \} = \epsilon^I \delta^{IJ} \delta'(X - Y), \quad I, J = 1, \ldots, N + g, \quad \epsilon^I = \pm 1 \]

in \( E^{N+g} \) to the submanifold \( M^N \) (see [32], [34]).

Let us note that for the brackets (3.3) corresponding to the submanifolds with flat normal connection and holonomic net of the lines of curvature, the
commutativity of the flows \( w^{\nu}_{(k)\mu} u^\mu_X \) was proved in \([31]\)) for this particular case. Concerning their Hamiltonian structure, it was suggested in work \([31]\) to consider them as "being generated by formal Hamiltonian functions like \( H = \int 1 dx \). This statement makes no sense in any symplectic geometry where the Poisson Bracket is well-defined. Besides that, we demonstrate below the local Hamiltonians generating these flows.

Let us introduce the Physical or Liouville coordinates for the weakly nonlocal Poisson brackets of hydrodynamic type: this form is given by the formulas

\[
\{ U^\nu(X), U^\mu(Y) \} = \left( \gamma^{\nu\mu}(X) + \gamma^{\mu\nu}(X) - \sum_{k=1}^{g} e_k f^\nu_{(k)} f^\mu_{(k)} \right) \delta(X - Y) + \\
+ \left( \frac{\partial \gamma^{\nu\mu}}{\partial U^\lambda} U^\lambda_X - \sum_{k=1}^{g} e_k (f^\nu_{(k)} X f^\mu_{(k)}) \delta(X - Y) + \sum_{k=1}^{g} e_k (f^\nu_{(k)} X \nu(X - Y) (f^\mu_{(k)} Y) \right) \\
\text{(3.7)}
\]

for some functions \( \gamma^{\nu\mu}(U) \) and \( f^\nu_{(k)}(U) \). Like in the local case, we have the following property:

**Lemma 1.** The bracket \( \{3.3\} \) of F-type (i.e. weakly nonlocal H.T.P.B.) has Physical form in the coordinates \( U^\mu \) if and only if the integrals \( J^\nu = \int U^\nu(X) dX \) generate the set of local commuting flows according to the bracket \( \{3.3\} \).

Proof. Suppose all functionals \( J^\nu \) satisfy the conditions above. Then we have for any \( \nu \) and \( k \)

\[
w^{\nu}_{(k)\tau}(U) U^\tau_X \equiv f^\nu_{(k)X}
\]

for some functions \( f^\nu_{(k)}(U) \) as it follows from the condition of locality of the flow corresponding to \( J^\nu \). From the commutativity of the set of functionals \( \{ J^\nu \} \) we then have

\[
b^\nu_{\lambda} U^\lambda_X + \sum_{k \geq 0} e_k (f^\nu_{(k)} X f^\mu_{(k)}) \equiv \frac{d}{dx} \gamma^{\nu\mu}
\]
for some $\gamma^\nu\mu(U)$. Again, from the skew-symmetry property of the bracket we have

$$\frac{d}{dx}g^\nu\mu \equiv b^\nu\mu U^\lambda_X + b^\mu\nu U^\lambda_X = \frac{d}{dx} \left( \gamma^\nu\mu + \gamma^\mu\nu - \sum_{k\geq 0} e_k f^\nu_k f^\mu_k \right)$$

and it’s clear that we can choose $\gamma^\nu\mu$ in such a way that

$$g^\nu\mu = \gamma^\nu\mu + \gamma^\mu\nu - \sum_{k\geq 0} e_k f^\nu_k f^\mu_k$$

It is easy to obtain from (3.7) that functionals $J^\nu$ generate the local flows according to this bracket and commute with each other.

Lemma is proved.

Let us also point out here that the first non-local bracket for the averaged NLS-equation was constructed by M.V. Pavlov ([37]) from a nice differential-geometrical consideration. In [38] and [39] the construction of non-local brackets for the averaged KdV equation using its local bi-Hamiltonian structure was considered.

4 Canonical forms and Casimirs for Fundamental Poisson bracket.

The structure of the MF fundamental bracket written in the densities of Casimirs was first found by M.V. Pavlov in Theorem 6 of work [36] who never published any proof later. As far as we know, he never paid any attention to the dependence on the boundary conditions and linear Poisson $\lambda$-pencils in this structure.

**Definition.** Let MF type bracket (3.2) be given with non-degenerate metric tensor $g^{\nu\mu}(U)$. We say that it is written in the canonical form if:

$$\{n^\nu(X), n^\mu(Y)\} =$$

$$= (e^\nu \delta^\nu\mu - cn^\nu n^\mu) \delta(X-Y) - cn^\nu X n^\mu \delta(X-Y) + cn^\nu X n^\mu X n^\nu Y$$

where $e^\nu$ are equal to $\pm 1$, and the term $e^\nu \delta^\nu\mu$ has the same signature as metric tensor $g^{\nu\mu}(U)$. 


Let us formulate here the following Theorem:

**Theorem 1.** Every fundamental MF-bracket (3.2) with non-degenerate metric tensor \( g^{\nu\mu}(U) \) of the constant curvature \( c \) can be written locally in the canonical form (4.1) for any point \( U_0 \) after some change of coordinates \( n^{\nu} = n^{\nu}(U^1, \ldots, U^N) \), such that \( n^{\nu}(U_0) \equiv 0 \).

The bracket (4.1) represents the **Linear Poisson pencil** given by the compatible Poisson brackets

\[
\{ n^{\nu}(X), n^{\mu}(Y) \}_0 = \epsilon^{\nu} \delta^{\mu} \delta'(X - Y)
\]

and

\[
\{ n^{\nu}(X), n^{\mu}(Y) \}_1 =
\]

\[
= -n^{\nu}(X)n^{\mu}(X)\delta'(X - Y) - n^{\nu}_X n^{\mu}(X)\delta(X - Y) + n^{\nu}_X \nu(X - Y)n^{\mu}_Y
\]

where curvature \( c \) is the parameter of a pencil.

We postpone the proof of Theorem 1 for the general case of Ferapontov brackets (see Theorem 3). Here we just check that the expression (4.1) really defines the MF-bracket with constant curvature \( c \). Here we have:

\[
g^{\nu\mu} = \epsilon^{\nu} \delta^{\nu\mu} - cn^{\nu} n^{\mu}
\]

where \( \epsilon^{\nu} = \pm 1 \), and

\[
b^{\nu\mu}_\lambda = -c\delta^{\nu\mu} n^{\mu}
\]

The symmetry of \( g^{\nu\mu} \) is clear; the compatibility of the connection with metric can be written in upper indices as an equation

\[
\frac{\partial g^{\nu\mu}}{\partial n^{\lambda}} = b^{\nu\mu}_\lambda + b^{\mu\nu}_\lambda
\]

(see [12]), which is also clear in our case. So we should check symmetry of the connection and its curvature. The symmetry of connection in upper indices is equivalent to the equation:

\[
b^{\nu\mu}_\alpha g^{\alpha\lambda} - b^{\lambda\mu} g^{\alpha\nu} = 0
\]
which in our case is

\[-c\delta^\nu_\alpha n^\mu \left( e^\alpha \delta^\nu_\lambda - cn^\alpha n^\lambda \right) + c\delta^\lambda_\alpha n^\mu \left( e^\alpha \delta^\nu_\nu - cn^\alpha n^\nu \right) =
\]

\[= -cn^\mu e^\nu \delta^\nu_\lambda + c^2 n^\mu n^\nu n^\lambda + cn^\mu \epsilon^\lambda \delta^\lambda_\nu - c^2 n^\mu n^\lambda n^\nu \equiv 0\]

For the curvature we use the formula

\[-g^\nu_\alpha g^{\mu_\beta} R^\tau_\beta\lambda\alpha = \left( b^{\mu_\tau}_{\lambda,\alpha} - b^{\mu_\tau}_{\alpha,\lambda} \right) g^{\nu_\nu} + b^{\nu_\mu} b^{\lambda_\tau} - b^{\nu_\tau} b^{\lambda_\mu} =
\]

\[= (-c\delta^\mu_\lambda \delta^\tau_\alpha + c\delta^\mu_\alpha \delta^\tau_\lambda) g^{\nu_\nu} + c^2 \delta^\nu_\mu n^\mu n^\nu - c^2 \delta^\nu_\tau n^\tau \delta^\mu_\lambda n^\mu \equiv
\]

\[\equiv -c \left( \delta^\mu_\lambda \delta^\tau_\alpha - \delta^\tau_\lambda \delta^\mu_\alpha \right) g^{\nu_\nu}\]

so that finally we have

\[R^{\mu_\tau}_{\lambda\alpha} = g^{\mu_\beta} R^{\tau}_{\beta\lambda\alpha} = c \left( \delta^\mu_\lambda \delta^\tau_\alpha - \delta^\tau_\lambda \delta^\mu_\alpha \right)\]

which corresponds to the constant curvature $c$.

**Remark.** The metric $g^{\nu_\mu}$ in (4.1) is not well-defined for all values of $n^\nu$ but only for some small enough domain near the zero point. So our theorem claims in fact that for the space of constant curvature $\mathcal{M}^N$ (and given signature of the metric) there exists a standard Canonical form for the corresponding bracket (3.2) on the small neighborhood of the constant point in the ”loop space”, i.e space of mappings $R^1 \rightarrow \mathcal{M}^N$, such that $n(X) \rightarrow y$ at $X \rightarrow \pm \infty$ for some point $y \in \mathcal{M}^N$. We shall clarify in the next work the canonical forms globally.

In work [36] an implicit expression for the density of the momentum functional for the bracket (4.1) in terms of annihilators was also derived. \footnote{As it will be clear from our later considerations we can not actually speak about the Casimirs and momentum functional until we fix the boundary conditions at infinity. As we shall show in the general case it’s better to speak about the invariant set of $N + g$ canonical functionals playing the role of annihilators or some ”canonical Hamiltonian functions” depending on the boundary conditions. We postpone these considerations to the case of the general Ferapontov bracket and assume here the rapidly decreasing boundary conditions at infinity.} We formulate here this result in the explicit form.
Lemma 2. On the space of rapidly decreasing functions $n^\nu(X)$ the functionals
\[ N^\nu = \int n^\nu dX \] (4.2)
are the annihilators of the bracket (4.1). The functional
\[ P = \frac{1}{c} \int \left( 1 - \sqrt{1 - c \sum_{\nu=1}^{N} e^\nu n^\nu(X)n^\nu(X)} \right) dX \] (4.3)
is the momentum generating shifts along the coordinate $X$.

This lemma follows from a simple calculation. We just point out here that the constant in the density $P(n)$ of the functional $P$ is chosen in such a way that $P(X) \to 0$ at $X \to \pm \infty$ on the space of rapidly decreasing at $X \to \pm \infty$ functions $n^\nu(X)$. The expression (4.3) becomes the ordinary momentum operator
\[ \frac{1}{2} \int \sum_{\nu=1}^{N} e^\nu n^\nu(X)n^\nu(X) dX \]
for the corresponding DN bracket as $c \to 0$. Our definition of the operator $\partial^{-1}$ leads to the identity:
\[ \sum_{\nu} \partial^{-1} n_{X}^\nu \frac{\partial P(n)}{\partial n^\mu} \equiv P(n) \]

It is interesting that the functional $P$ is uniquely defined only on the part of the phase space where
\[ \sum_{\nu=1}^{N} e^\nu n^\nu(X)n^\nu(X) < 1/c \]
for any $X$. If it is not so, we have nonsmooth functions and the problem of choosing the sign. Let us postpone these questions.

The bracket (4.1) is written also in the Physical form with $\gamma^{\nu\mu}(n) = 1/2 \ e^\nu \delta^{\nu\mu}$, $e = \text{sgn} \ c$ and $f^\nu(n) = \sqrt{|c|} \ n^\nu$.

Lemma 3. Suppose we have a Poisson bracket (3.7) such that
\[ \gamma^{\nu\mu}(n) = A^{\nu\mu} + B^{\nu\mu}_\lambda n^\lambda \]

for some constants \( A^{\nu\mu} \) and \( B^{\nu\mu}_\lambda \), \( c = \text{sgn } c \) and \( f^\nu(n) = \sqrt{|c|} n^\nu \). Then \( B^{\nu\mu}_\lambda \equiv \delta^\nu_\lambda b^\mu \) for some constants \( b^\mu \); therefore the linear part can be removed from the bracket by the simple shifts of coordinates \( n^\nu = \tilde{n}^\nu + b^\nu/c \).

**Proof.** We have in our case
\[ g^{\nu\mu}(n) = A^{\nu\mu} + A^{\nu\mu} + B^{\nu\mu}_\lambda n^\lambda + B^{\nu\mu}_\lambda n^\lambda - cn^\nu n^\mu \]
\[ b^{\nu\mu}_\lambda = B^{\nu\mu}_\lambda - c\delta^{\nu\lambda} n^\mu \]

So from the symmetry of the connection
\[ b^{\nu\mu}_\alpha g^{\alpha\lambda} - b^{\lambda\mu}_\alpha g^{\alpha\nu} = 0 \]
we can obtain for the quadratic part in the variables \( n \):
\[ B^{\nu\mu}_\alpha n^\alpha n^\lambda - B^{\lambda\mu}_\beta n^\alpha n^\nu \equiv 0 \]
for \( \nu \neq \lambda \). So for \( \alpha \neq \nu \) or \( \beta \neq \lambda \) we obtain
\[ B^{\nu\mu}_\alpha \equiv 0 , \quad B^{\lambda\mu}_\beta \equiv 0 \]
and for \( \alpha = \nu \) and \( \beta = \lambda \)
\[ B^{\nu\mu}_\nu = B^{\lambda\mu}_\lambda \]
for any \( \nu \neq \lambda \). So we have \( B^{\nu\mu}_\lambda \equiv \delta^\nu_\lambda b^\mu \) for some \( b^\mu \).

Lemma is proved.

It is clear also that any change of the term \( \epsilon^\nu \delta^{\nu\mu} \) by a constant symmetric matrix \( A^{\nu\mu} \) leads to the Poisson pencil; however, in the degenerate case we do not claim that any bracket (3.2) of the MF type can be presented in that form after some coordinate transformation.

The MF-brackets can be obtained as the averaged ones according to the recent works [18]–[19] from the original weakly nonlocal fundamental brackets with terms like
\[ \varphi^i_x \nu(x - y) \varphi^j_y \]
in the non-local part.

5 General F-brackets.
Riemannian Geometry.

Let us consider more general F-brackets.

**Definition.** We say that the F-bracket is written in the Canonical form if

\[
\{ n^\nu(X), n^\mu(Y) \} = \left( \epsilon^{\nu} \delta^{\mu\mu} - \sum_{k=0}^{g} e_k f^\nu_{(k)}(n) f^\mu_{(k)}(n) \right) \delta'(X - Y) -
\]

\[
- \sum_{k=0}^{g} e_k \left( f^\nu_{(k)}(n) \right)_X \left( f^\mu_{(k)}(n) \right)_Y \delta(X - Y) +
\]

\[
+ \sum_{k=0}^{g} e_k \left( f^\nu_{(k)}(n) \right)_X \nu(X - Y) \left( f^\mu_{(k)}(n) \right)_Y
\]

(5.1)

with non-degenerate metric and some functions \( f^\nu_{(k)}(n) \) such that \( f^\nu_{(k)}(0) \equiv 0 \), \( e_k = \pm 1 \).

**Theorem 2.** The expression (5.1) with non-degenerate metric and linearly independent set of functions \( f^\nu_{(k)}(n) \) defines a Poisson bracket if and only if:

1) The flows

\[ n^\nu_{Tk} = \left( f^\nu_{(k)}(n) \right)_X \]

commute with each other, i.e. \( d/dT_k'(f^\nu_{(k')}(n)) = d/dT_k(f^\nu_{(k')}(n)) \) or

\[
\frac{\partial f^\nu_{(k)}}{\partial n^\lambda} \frac{\partial f^\lambda_{(k')}}{\partial n^\mu} - \frac{\partial f^\nu_{(k')}}{\partial n^\lambda} \frac{\partial f^\lambda_{(k)}}{\partial n^\mu} \equiv 0
\]

for any \( k, k' \).

The functions \( f^\nu_{(k)}(n) \) are such that 2) and 3) are true:
2) \[
\frac{\partial f^\nu}{\partial n^\lambda} \left( e^\lambda \delta^\nu - \sum_{s=0}^g e_s f^\lambda_{(s)} f^\nu_{(s)} \right) = \frac{\partial f^\mu}{\partial n^\lambda} \left( e^\lambda \delta^\nu - \sum_{s=0}^g e_s f^\lambda_{(s)} f^\nu_{(s)} \right)
\]

3) \[
\left( e^\lambda \delta^\nu - \sum_{s=0}^g e_s f^\lambda_{(s)} f^\nu_{(s)} \right) \left( \sum_{n=0}^g e_n \frac{\partial f^\mu_{(n)}}{\partial n^\beta} f^\nu_{(n)} \right) \frac{\partial f^\beta}{\partial n^\alpha} = \\
\left( e^\mu \delta^\alpha - \sum_{s=0}^g e_s f^\mu_{(s)} f^\alpha_{(s)} \right) \left( \sum_{n=0}^g e_n \frac{\partial f^\lambda_{(n)}}{\partial n^\beta} f^\nu_{(n)} \right) \frac{\partial f^\beta}{\partial n^\alpha}
\]

Proof. As was shown in [19], the commutativity of the flows \((f^\nu_{(k)})_X\) is the necessary requirement under the conditions of the Theorem.

The condition (2) is exactly the identity
\[ g_{\nu\alpha} w^{\alpha}_{(k)\mu} = g_{\mu\alpha} w^{\alpha}_{(k)\nu} \]
written in the upper indices for our case.

By direct calculation (like in Theorem 1) it’s not hard to show that in our case the conditions
\[ R^\nu_{\mu\lambda} = \sum_{k=1}^g e_k \left( w^\nu_{(k)\mu} w^\tau_{(k)\lambda} - w^\tau_{(k)\mu} w^\nu_{(k)\lambda} \right) \]
and the symmetry of connection are corollaries of (1) and (2), respectively.

The condition (3) is equivalent to the equality
\[ \nabla_\mu w^\nu_{(k)\lambda} = \nabla_\lambda w^\nu_{(k)\mu} \]

So from Ferapontov’s results we obtain the statement of our Theorem.

Let us mention also that the commutativity of affinors \([w_{(k)}, w_{(k')}]) = 0\) follows from the commutativity of the flows \(w^\nu_{(k)\mu} n^\mu_X\).

Theorem is proved.

Let the F-bracket (3.3) be given with the non-degenerate tensor \(g^\nu\mu(U)\). Fix some point \(U_0 = (U^1_0, \ldots, U^N_0)\). By the local Canonical Form of F-bracket corresponding to the point \(U_0\) we call the bracket (3.1) where \(n^\nu = n^\nu(U)\),
\( n^\nu(U_0) \equiv 0; \) the term \( \epsilon^\nu \delta^\mu \) has the same signature as \( g^{\nu\mu}(U) \) and \( f^\nu_{(k)}(U_0) \equiv 0 \) at the point \( U_0 \).

**Theorem 3.**

I) Every F-bracket (3.3) with the non-degenerate metric tensor \( g^{\nu\mu}(U) \) can be locally written in the Canonical form (5.1) after some coordinate transformation \( n^\nu = n^\nu(U) \). Moreover, for any given point \( U_0 \) it’s possible to choose the coordinates \( n^\nu(U) \) in such a way that \( n^\nu(U_0) \equiv 0, f^\nu_{(k)}(U_0) \equiv 0 \).

II) The integrals

\[
N^\nu = \int n^\nu(X) dX
\]

are annihilators of the bracket (5.1) on the domain in the space of rapidly decreasing functions \( n^\nu(X) \) bounded by the small enough constant;

III) The flows

\[
n^\nu_{tk} = \frac{d}{dX} f^\nu_{(k)}(n)
\]

are generated by the local Hamiltonians

\[
H_k = \int h_k(n) dX
\]
on the same phase space. The functions \( n^\nu(U), h_k(n(U)) \) can be represented as linear combinations of coordinates \( V^I \) in the pseudo-Euclidean space \( R^{N+g} \) for the local representation of our manifold as a submanifold \( M^N \subset R^{N+g} \) with flat normal connection.

Proof.

As follows from the results of Ferapontov (34) the relations (3.3)-(3.4) are the Gauss-Codazzi equations; so every space with non-degenerate metric \( g^{\nu\mu}(U) \) and the relations (3.3)-(3.4) for the operators \( w^\nu_{(k)\mu}(U) \) can be represented as a submanifold \( M^N \) with flat normal connection in the pseudo-Euclidean space \( R^{N+g} \) such that:

the restriction of the constant metric on \( M^N \) is equal to \( g^{\nu\mu}(U) \);
the restriction of this metric on the orthogonal subspace has at every point the signature corresponding to the signature of the nonlocal part of the bracket (3.3);
$w_{(k)\mu}(U)$ are the Weingarten operators corresponding to $g$ parallel normal vector fields.

Moreover, the bracket (3.3) can be obtained as a Dirac restriction of the constant bracket

\[ \{V^I(X), V^J(Y)\} = \epsilon_I \delta^{IJ} \delta'(X - Y) \quad (5.3) \]

where $I, J = 1, \ldots, N + g$, $\epsilon_I = \pm 1$.

Let us point out also that the Dirac restriction of (5.3) to any submanifold $M$ (not necessarily with flat normal connection) has the similar structure as (3.3) with $\partial^{-1}$ replaced by $\nabla^{-1}_\perp$ where $\nabla_\perp$ is the normal connection corresponding to $M$ (Ferapontov [34]).

We can assume that the coordinates $V^I$ are such that:
1) $V^I \equiv 0$ at the point $U_0$ of submanifold $M$;
2) The first $N$ coordinate lines are tangent to $M$ at the point $U_0$; the remaining $g$ coordinates are orthogonal to $M$.

It follows that the first $N$ quantities $\epsilon_I$ in this new system of coordinates correspond to the signature of the metric $g_{\nu\mu}(U)$, and the last $g$ coordinates correspond to the nonlocal part of (3.3).

We take $n^\nu(U) = V^\nu$, $\nu = 1, \ldots, N$ as the local coordinates on the manifold $M$ in the domain around of $U_0$ ($n^\nu(U_0) \equiv 0$); for the last of the quantities $V^I$ at the points of $M$ we have

\[ V^{N+k} = v_k(n) \]

for some functions $v_k(n)$ such that $v_k(U_0) = 0$, $\partial v_k/\partial n^\nu(U_0) = 0$.

Make now the Dirac restriction of the bracket (5.3) on $M$ in the coordinates $n^\nu$, $V^I$. We know that it should coincide with our initial PB of the F type.

According to the Dirac procedure, for any functional $F(n)$ we should find such linear combination

\[ \int m^s(X) g_s(X) dX \]

of the constraints:

\[ g_k(X) = V^{N+k}(X) - v_k(n(X)) \quad k = 1, \ldots, g \]

that the functional
\[ F(n) + \int m^s(X) g_s(X) dX \]

leaves \( \mathcal{M}^N \) invariant; i.e.

\[
\{ g_k(X), F(n) + \int m^s(Y) g_s(Y) dY \} =
\]

\[
= - \sum_{\nu=1}^{N} e_{\nu} \frac{\partial v_k}{\partial n^\nu} \frac{d}{dX} \delta F + e_{N+k} \frac{d}{dX} m^k(X) +
\]

\[
+ \sum_{\nu=1}^{N} e_{\nu} \frac{\partial v_k}{\partial n^\nu} \left( \frac{d}{dX} m^s(X) + \left( \frac{\partial v_s}{\partial n^\nu} \right)_X m^s(X) \right) \equiv 0 \quad (5.4)
\]

The restricted bracket \( \{ \ldots, \ldots \}^* \) on \( \mathcal{M}^N \) we define as

\[
\{ n^\nu(X), F(n) \}^* = \{ n^\nu(X), F(n) + \int m^s(Y) g_s(Y) dY \}
\]

The relations (5.4) for \( m^s(X) \) can be written in the form

\[
- \sum_{\nu=1}^{N} e_{\nu} \frac{\partial v_k}{\partial n^\nu} \frac{d}{dX} \delta F + \frac{d}{dX} \left( e_{N+k} \delta^{ks} + \sum_{\nu=1}^{N} e_{\nu} \frac{\partial v_k}{\partial n^\nu} \frac{\partial v_s}{\partial n^\nu} \right) m^s(X) -
\]

\[
- \sum_{\nu=1}^{N} e_{\nu} \left( \frac{\partial v_k}{\partial n^\nu} \right)_X \frac{\partial v_s}{\partial n^\nu} m^s(X) \equiv 0
\]

We note now that the vectors \( t_{\nu}(n) \) tangent to \( \mathcal{M}^N \) can be written as

\[
t_{\nu}(n) = (0, \ldots, 1, \ldots, 0, \frac{\partial v_1}{\partial n^\nu}, \ldots, \frac{\partial v_N}{\partial n^\nu})^t \quad (5.5)
\]

where 1 stays at the position \( \nu \). The vectors \( q_k(n) \) orthogonal to \( \mathcal{M}^N \) can be chosen as

\[
q_k(n) = (e_1 \frac{\partial v_k}{\partial n^1}, \ldots, e_N \frac{\partial v_k}{\partial n^N}, 0, \ldots, -e_{N+k}, \ldots, 0)^t \quad (5.6)
\]

Therefore the quantities

\[
e_{N+k} \delta^{ks} + \sum_{\nu=1}^{N} e_{\nu} \frac{\partial v_k}{\partial n^\nu} \frac{\partial v_s}{\partial n^\nu} = (q_k(n), q_s(n)) = G_{ks}(n) \quad (5.7)
\]

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are equal to the pairwise scalar products of the normal vectors \( q_k \).

Let us introduce the inverse matrix \( G^{ks}(n) \) such that

\[
G^{ks}(n)(q_s(n), q_n(n)) = \delta^k_n
\]

\( k, n = 1, \ldots, g. \)

It can be easily shown that the quantities

\[
\Omega^k_{n, \mu} = - \sum_{\nu=1}^{N} e_\nu \frac{\partial^2 v_n}{\partial n^\nu \partial n^{\mu}} G^{sk} = -G^{ks}(q_s, \frac{\partial q_n}{\partial n^\mu})
\]

(5.8)

are the connection coefficients of the normal connection for \( \mathcal{M}^N \) following from the pseudo-Euclidean structure in \( \mathbb{R}^{N+g} \) written in the basis \( \{q_s\} \) in the normal bundle.

So we can write the relations (5.4) as

\[
\nabla_\perp [G_{kp}m^p(X)] - \sum_{\nu=1}^{N} e_\nu \frac{\partial v_s}{\partial n^\nu} \left( \frac{\delta F}{\delta n^\nu(X)} \right) X \equiv 0
\]

where

\[
\nabla_\perp = \delta^k_s \frac{d}{dX} \pm \Omega^s_{k, \mu} n^\mu_X
\]

(5.9)

is the covariant derivative with respect to \( X \) applied to the lower(upper) indices in the normal bundle.

From this we get

\[
m^k(X) = G^{ks} \nabla_\perp^{-1} \sum_{\nu=1}^{N} e_\nu \frac{\partial v_s}{\partial n^\nu} \left( \frac{\delta F}{\delta n^\nu(X)} \right) X
\]

We define the operator \( \nabla_\perp^{-1} \) in a "skew-symmetric" way as \( \partial^{-1} \) above, i.e. \( \nabla_\perp^{-1} \kappa_s \), (where \( \kappa_s(X) \to 0 \) at \( X \to \pm\infty \)) is the sum of the solutions to the equation

\[
\frac{d}{dX} \tau_s + \Omega^t_{s, \mu} n^\mu_X \tau_t = \kappa_s
\]

equal to 0 at \( -\infty \) and \( +\infty \) respectively and divided by 2.

Besides that, we consider here the index \( k \) in the formula
\[
\sum_{\nu=1}^{N} e_{\nu} \frac{\partial v_{k}}{\partial n^{\nu}} \left( \frac{\delta F}{\delta n^{\nu}(X)} \right) \equiv q_{k} \left( \frac{\delta F}{\delta n^{I}(X)} \right)_{X} \tag{5.10}
\]

as a lower index in the basis \{q_{k}\}.

So, for the bracket restricted to \(M^{N}\) we can write

\[
\{n^{\nu}(X), F(n)\}^{*} = e_{\nu} \frac{d}{dX} \delta F^{\nu}(X) - e_{\nu} \left( \frac{\partial v_{s}}{\partial n^{\nu}} G^{sk} \nabla_{\perp}^{-1} \sum_{\mu=1}^{N} e_{\mu} \frac{\partial v_{k}}{\partial n^{\mu}} \left( \frac{\delta F}{\delta n^{\mu}(X)} \right)_{X} \right)
\]

After some elementary calculations (using the expressions for operators
\(d/dX\) and \(\nabla_{\perp}^{-1} d/dX\) following from the form (5.9) and the relations (5.7)
and (5.8)) we can write this bracket in the form:

\[
\{n^{\nu}(X), F(n)\}^{*} = e_{\nu} \frac{d}{dX} \delta F^{\nu}(X) - \sum_{\mu=1}^{N} e_{\mu} \frac{\partial v_{k}}{\partial n^{\mu}} \frac{\delta F}{\delta n^{\mu}(X)} + \sum_{\mu=1}^{N} \left( \nabla_{\perp} G^{sk} e_{\nu} \frac{\partial v_{k}}{\partial n^{\nu}} \right) \nabla_{\perp}^{-1} \left( \nabla_{\perp} e_{\mu} \frac{\partial v_{k}}{\partial n^{\mu}} \right) \frac{\delta F}{\delta n^{\mu}(X)} \tag{5.11}
\]

Here differential operators \(\nabla_{\perp}\) act only on the functions staying within the same brackets (\ldots). It is necessary to differ their actions on upper and lower indices. So the expressions \((\nabla_{\perp} G^{sk} e_{\nu} \partial v_{k}/\partial n^{\nu})\) and \((\nabla_{\perp} e_{\mu} \partial v_{s}/\partial n^{\mu})\) are just \((\nabla_{\perp} G^{sk} q_{k})^{\nu}\) and \((\nabla_{\perp} q_{s})^{\mu}\) where we take \(\nu, \mu = 1, \ldots, N\) for their components.

Formula (5.11) gives us the restriction of the bracket (5.3) to any submanifold \(M^{N} \subset R^{N+g}\) in the coordinates \(n^{\nu}, V^{I}\). Suppose now that \(M^{N}\) has a flat normal connection. This means that locally there exists such nondegenerate matrix \(S^{k}_{n}(n)\) that the normal connection

\[
\nabla_{\perp} = \left( S^{-1} \right)_{n} \frac{d}{dX} S^{k}_{s}
\]

- for the action on the covectors \(\kappa_{k}\) and

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\[ \nabla_{\perp} = S_n^s \frac{d}{dX} (S^{-1})_k^s \]  
(5.13)

for the action on the vectors \( e^k \) in the normal bundle.

We can take \( S_n^k(U_0) = I \) and introduce the basis of parallel vector fields

\[ \mathbf{N}_n(n) = S_n^k(n) \mathbf{q}_k(n) \]

where \( \mathbf{q}_k(n) \) were defined in (5.6).

Since our normal connection preserves the scalar product in \( \mathbb{R}^{N+g} \) we conclude that the pairwise scalar products of \( \mathbf{N}_k(n) \) are constant at any \( n \); i.e.

\[ (\mathbf{N}_k, \mathbf{N}_n) = e_{N+k} \delta_{kn} \]

Therefore we have for the matrix \( G^{kn}(n) \):

\[ G^{kn}(n) = \sum_{s=1}^{g} e_{N+s} S_n^k(n) S_s^n(n) \]  
(5.14)

Substituting (5.12), (5.13), (5.14) and the expression corresponding to \( \nabla^{-1}_{\perp} \) for the covectors in (5.11), we obtain after the simple calculation:

\[
\{n^\nu(X), F(n)\}^* = e_\nu \frac{d}{dX} \frac{\delta F}{\delta n^\nu(X)} - \\
- \sum_{\mu=1}^{N} \sum_{q=1}^{g} e_{N+q} \left( e_\nu \frac{\partial v_s}{\partial n^\nu} S_q^s \right) \left( e_\mu \frac{\partial v_k}{\partial n^\mu} S_q^k \right) \frac{d}{dX} \frac{\delta F}{\delta n^\nu(X)} + \\
- \sum_{\mu=1}^{N} \sum_{q=1}^{g} e_{N+q} \left( \frac{d}{dX} e_\nu \frac{\partial v_s}{\partial n^\nu} S_q^s \right) \left( e_\mu \frac{\partial v_k}{\partial n^\mu} S_q^k \right) \frac{\delta F}{\delta n^\mu(X)} \\
+ \sum_{\mu=1}^{N} \sum_{q=1}^{g} e_{N+q} \left( \frac{d}{dX} e_\nu \frac{\partial v_s}{\partial n^\nu} S_q^s \right) \left( \frac{d}{dX} \right)^{-1} \left( \frac{d}{dX} e_\mu \frac{\partial v_k}{\partial n^\mu} S_q^k \right) \frac{\delta F}{\delta n^\mu(X)}
\]  
(5.15)

(The operator \( (d/dX)^{-1} \) is defined as above in a "skew-symmetric" way.)

So if we put:

\[ f^\nu_{(k)}(n) \equiv e_\nu \frac{\partial v_s}{\partial n^\nu} S_q^s(n) = N^\nu_k(n) \]
for $\nu = 1, \ldots, N$ (summation with respect to $s$) we just obtain the expression corresponding to the bracket (5.1) where $f^\nu_{(k)}(U_0) = 0$. It is not hard to see that the flows $N^\nu_{(k),X}$ correspond to the Weingarten operators, the expressions corresponding to $\delta'$ and $\delta$ terms are equal to the restricted metric with upper indices and to the corresponding connection respectively in accordance with Ferapontov theorems. So we proved the part (I) of the Theorem.

Since $f^\nu_{(k)}(X) \to 0$ at $X \to \pm \infty$ on the space of rapidly decreasing functions $n^\nu(X)$, we write $(d/dX)^{-1} f^\nu_{(k),X} \equiv f^\nu_{(k)}$. It is easy to see that the functionals

$$\int n^\nu(X) dX$$

are annihilators of the bracket (5.1) on the space of rapidly decreasing functions.

Consider now the functionals:

$$H_n = \int v_n(n) dX$$

(5.16)

We have

$$\{n^\nu(X), H_n\}^* = e_\nu \frac{d}{dX} \frac{\partial v_n}{\partial n^\nu} -$$

$$- \sum_{\mu=1}^{N} \sum_{q=1}^{g} e_{N+q} \left( e_\nu \frac{\partial v_s}{\partial n^\nu} S^s_q \right) \left( e_\mu \frac{\partial v_k}{\partial n^\mu} S^k_q \right) \frac{d}{dX} \frac{\partial v_n}{\partial n^\mu}$$

$$- \sum_{\mu=1}^{N} \sum_{q=1}^{g} e_{N+q} \left( \frac{d}{dX} e_\nu \frac{\partial v_s}{\partial n^\nu} S^s_q \right) \left( e_\mu \frac{\partial v_k}{\partial n^\mu} S^k_q \right) \frac{d}{dX} \frac{\partial v_n}{\partial n^\mu} +$$

$$+ \sum_{\mu=1}^{N} \sum_{q=1}^{g} e_{N+q} \left( \frac{d}{dX} e_\nu \frac{\partial v_s}{\partial n^\nu} S^s_q \right) \left( \frac{d}{dX} \right)^{-1} \left( \frac{d}{dX} e_\mu \frac{\partial v_k}{\partial n^\mu} S^k_q \right) \frac{\partial v_n}{\partial n^\mu}$$

We can write

$$\sum_{\mu=1}^{N} \left( e_\mu \frac{\partial v_k}{\partial n^\mu} S^k_q \right) \frac{\partial v_n}{\partial n^\mu} = (N_q, q_n) + q^{N+n} = (N_q, q_n) - e_{N+n} S^q_n$$

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\[
\sum_{\mu=1}^{N} \left( e_{\mu} \frac{\partial v_{n}}{\partial n^{\mu}} S_{q}^{k} \right) \frac{d}{dX} \frac{\partial v_{n}}{\partial n^{\mu}} = (N_{q}, \frac{d}{dX} q_{n})
\]

\[
\sum_{\mu=1}^{N} \left( e_{\mu} \frac{\partial v_{k}}{\partial n^{\mu}} S_{q}^{k} \right) X \frac{\partial v_{n}}{\partial n^{\mu}} = \left( \frac{d}{dX} N_{q}, q_{n} \right) - e_{N+n} \left( S_{q}^{n} \right)_{X} = (\nabla_{\perp} N_{q}, q_{n}) - e_{N+n} \left( S_{q}^{n} \right)_{X}
\]

since \( \nabla_{\perp} N_{q} \) is tangent to \( M^{N} \).

So we have

\[
\{ n^{\nu}(X), H_{n} \}^{*} = \frac{d}{dX} \left( q_{n}^{\nu} - \sum_{q=1}^{g} e_{N+q} N_{q}^{\nu}(N_{q}, q_{n}) \right) +
\]

\[
+ \sum_{q=1}^{g} \left( \frac{d}{dX} e_{\nu} \frac{\partial v_{s}}{\partial n^{\nu}} S_{q}^{s} \right) \left( s_{n}^{q} - \left( \frac{d}{dX} \right)^{-1} \left( S_{q}^{n} \right)_{X} \right)
\]

for \( \nu = 1, \ldots, N \). As we know, any \( q_{n} \) is orthogonal to \( M^{N} \) and \( \{ N_{q} \} \) give the pseudo-orthonormal basis in the normal bundle. Therefore the first term in this expression is zero. As for the last term, we know that \( S_{q}^{n}(X) \to \delta_{q}^{n} \) at \( X \to \pm \infty \). So we have

\[
\left( \frac{d}{dX} \right)^{-1} \left( S_{q}^{n} \right)_{X} = S_{q}^{n}(X) - \frac{S_{q}^{n}(+\infty) + S_{q}^{n}(-\infty)}{2} = S_{q}^{n}(X) - \delta_{q}^{n}
\]

and

\[
\{ n^{\nu}(X), H_{n} \}^{*} = \frac{d}{dX} \left( e_{\nu} \frac{\partial v_{s}}{\partial n^{\nu}} S_{n}^{s} \right) = \frac{d}{dX} f_{(n)}^{\nu}
\]

on the space of rapidly decreasing functions \( n^{\nu}(X) \). So we proved the parts (II) and (III) of the Theorem.

Theorem is proved.

It is easy to see that the nonlocal tail of the MF bracket (3.2) has the same form in any coordinate system \( \tilde{U}^{\nu} = \tilde{U}^{\nu}(U) \). So if we require \( f^{\nu}(0) \equiv 0 \) in
the case of the MF bracket, we obtain \( f^\nu(n) \equiv n^\nu \) here. By direct calculation we obtain the following:

**Lemma 4.** For the variables

\[
v^\nu(X) = \partial^{-1} n^\nu(X)
\]  

(5.17)

with the nonlocal operator \( \partial^{-1} \) defined in (1.2) we have for the Poisson brackets (5.1)

\[
\{v^\nu(X), v^\mu(Y)\} =
\]

\[
= -\epsilon^\nu \delta^\mu \nu(X - Y) + \sum_{k=1}^{g} e_k f^\nu_{(k)}(v_X) \nu(X - Y) f^\mu_{(k)}(v_Y) =
\]

\[
= -\sum_{\lambda=1}^{N} \epsilon^\lambda \delta^\mu \nu(X - Y) \delta^\nu_{\lambda} + \sum_{k=1}^{g} e_k f^\nu_{(k)}(v_X) \nu(X - Y) f^\mu_{(k)}(v_Y)
\]  

(5.18)

So, the local and nonlocal parts of (5.1) can be unified after the non-local transformation (5.17). The flows

\[
v^\nu_{\mu} = \delta^\nu_{\mu}
\]

and

\[
v^\nu_{T_k} = f^\nu_{(k)}(v_X)
\]

commute with each other. They preserve the bracket (5.18) as follows from the general theorem.

For the MF-bracket we have in this case

\[
\{v^\nu(X), v^\mu(Y)\} =
\]

\[
= -\epsilon^\nu \delta^\mu \nu(X - Y) + c v^\nu_X \nu(X - Y) v^\mu_Y
\]
We would like to emphasize here that the annihilators (Casimirs) of the bracket (3.3) and the Hamiltonians for the flows \( w^{\nu}_{(k)\mu} U^\nu_X \) strongly depend on the boundary conditions for the functions \( U^\nu(X) \). We can see that it is not possible to divide a set of \( N + g \) "Canonical forms" (the restrictions of Euclidean coordinates to \( \mathcal{M}^N \)) to the Casimirs of the bracket and Hamiltonians for the flows from nonlocal tail until we fix a point \( y \in \mathcal{M}^N \) and define the corresponding loop space

\[
L(\mathcal{M}, y) = \{ \gamma : R^1 \to \mathcal{M}^N : \gamma(-\infty) = \gamma(+\infty) = y \in \mathcal{M}^N \}
\]
corresponding to any point \( y \in \mathcal{M}^N \). In this case any global function \( h(U) \) on \( \mathcal{M}^N \) such that \( h(y) = 0 \) gives a "hydrodynamic type Hamiltonian"

\[
H = \int h(\gamma(X)) dX
\]
on the loop space \( L(\mathcal{M}, y) \) and the Hamiltonian flow on \( L(\mathcal{M}, y) \) defined in the invariant way on \( \mathcal{M}^N \).

Remark.
If we consider the Poisson bracket \{\ldots, \ldots\}_g \) on the loop space \( L(\mathcal{M}, y) \) for the Dirac restriction (5.11) of \( \varepsilon_I \delta^{IJ} d/dX \) on the general space \( \mathcal{M}^N \) in \( R^{N+g} \) it is possible to introduce the relations (5.12)-(5.13) with some matrix \( S^k_n[\gamma](X) \) along every curve \( \gamma \in L(\mathcal{M}, y) \). The relations (5.15) can be also formally written on the space \( L(\mathcal{M}, y) \), but in this case the expressions \( S^k_n[\gamma](X) \) are nonlocal functionals. They are defined on the loop spaces \( \gamma \in L(\mathcal{M}, y) \) and depend on the whole curve \( \gamma \). However, in this case it can be shown by the same way that the integrals (5.2) are still the local Casimirs for the bracket (5.11); the functionals (5.16) generate the nonlocal flows

\[
n_{TN}^{\nu} = \frac{1}{2} \sum_{q=1}^{g} \frac{d}{dX} \left( e_\nu \frac{\partial v_s}{\partial n^\nu} S^s_q[\gamma](X) \right) \left( S^n_q(-\infty) + S^n_q(+\infty)[\gamma] \right)
\]
on \( L(\mathcal{M}, y) \).

Now we prove that the general F-bracket with nondegenerate \( g^{\nu\mu}(U) \) has exactly \( N \) Casimirs on the space \( L(\mathcal{M}^N, y) \) with fixed point \( y \in \mathcal{M}^N \).

**Theorem 4.** Suppose we have a manifold \( \mathcal{M}^N \) with non-degenerate \( g^{\nu\mu}(U) \) and the relations (3.3)-(3.4) for the affinors set \( w^{\nu}_{(k)\mu} \). Then:
For any point \( y \in \mathcal{M}^N \) there exist locally exactly \( N + g \) linearly independent functions \( V^I_y(U) \), \( I = 1, \ldots, N + g \) such that the functionals
\[
H^I = \int V^I_y(U) dX
\]
generate the flows proportional to \( w^\nu_{(k)\mu}(U)U^\mu_X \) on the space \( L(\mathcal{M}, y) \) of loops close enough to \( y \). The functions \( V^I_y(U) \) can be chosen in such a way that for close enough \( y, y' \)
\[
V^I_y(U) = A^I_{IJ}(y, y') V^J_{y'}(U) + B^I(y, y')
\]
with some constants \( B^I(y, y') \) and the matrix \( A^I_{IJ}(y, y') \) orthogonal with respect to diagonal metric
\[
g_{IJ} = \epsilon_I \delta_{IJ}, \quad I = 1, \ldots, N + g
\]
where \( \epsilon_I = \epsilon_I, \quad I = 1, \ldots, N, \quad \epsilon_I = \epsilon_{I-N}, \quad I = N + 1, \ldots, N + g \).

Proof.

We should prove that the linear space of functions \( F_y(n) \) such that:
\[
F_y(0) \equiv 0, \quad \frac{\partial F_y}{\partial n^\mu} f^\mu_{(k)X} \equiv G_{y(k)X}
\]
for some \( G_{y(k)}(U), \quad G_{y(k)}(y) \equiv 0 \) and
\[
\frac{d}{dX} \left( \epsilon^\nu \frac{\partial F_y}{\partial n^\nu} - \sum_{k=1}^g \epsilon_k \left( f^\nu_{(k)} f^\mu_{(k)} \frac{\partial F_y}{\partial n^\mu} - f^\nu_{(k)} G_{y(k)} \right) \right) = \sum_{k=1}^g \alpha_k f^\nu_{(k)X}
\]
(\( \alpha_1, \ldots, \alpha_g \) is at most \( N + g \) - dimensional \( \epsilon^\nu = \pm 1 \)). We have
\[
\epsilon^\nu \frac{\partial F_y}{\partial n^\nu} - \sum_{k=1}^g \left( f^\nu_{(k)} f^\mu_{(k)} \frac{\partial F_y}{\partial n^\mu} - f^\nu_{(k)} G_{y(k)} \right) = \sum_{k=1}^g \alpha_k f^\nu_{(k)} + \beta^\nu
\]
for some constants \( \beta^1, \ldots, \beta^N \), and
\[
\sum_{k=1}^g f^\mu_{(k)} \frac{\partial F_y}{\partial n^\mu} \left( \delta^{sk} - \epsilon_k \sum_{\nu=1}^N \epsilon^\nu f^\nu_{(s)} f^\nu_{(k)} \right) + \sum_{k=1}^g \epsilon_k \left( \sum_{\nu=1}^N \epsilon^\nu f^\nu_{(s)} f^\nu_{(k)} G_{y(k)} \right) = \]

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\[ = \sum_{k=1}^{g} \alpha_k \sum_{\nu=1}^{N} \epsilon^\nu f^{\nu}_{(s)} f^{\nu}_{(k)} + \sum_{\nu=1}^{N} \epsilon^\nu f^{\nu}_{(s)} \beta^\nu \]

For the points close enough to \( y \) the matrix

\[ \delta^{sk} - \epsilon_k \sum_{\nu=1}^{N} \epsilon^\nu f^{\nu}_{(s)} f^{\nu}_{(k)} \]

is non-degenerate, and we can locally express the value \( f^\mu(k) \partial F_y/\partial n^\mu \) as a function of \( G_{y(s)}^{\nu}(s), f^{\nu}_{(s)}, \alpha_k, \beta^\nu \) at every point. After that we obtain the equations:

\[ \frac{\partial F_y}{\partial n^\nu} = \epsilon^\nu \Lambda^\nu(G_{y(s)}^{\nu}(s), f^{\mu}_{(s)}, \alpha_k, \beta^\nu) \quad (5.22) \]

and the linear (non-homogeneous) equations

\[ \frac{\partial G_{y(k)}}{\partial n^\nu} = \sum_{\mu=1}^{N} \epsilon^\mu \frac{\partial f^\mu_{(k)}}{\partial n^\nu} \Lambda^\mu(G_{y(s)}^{\nu}(s), f^{\mu}_{(s)}, \alpha_k, \beta^\nu) \]

for every derivative of \( G_{y(k)} \) with the normalizing conditions \( G_{y(k)}(y) = 0 \) which give us the unique \( G_{y(k)}(U) \) at every \( \{\alpha_k, \beta^\nu\} \). So, from (5.22) we obtain that the family \( F_y(U) \) is at most \( N + g \) parametric.

The functions constructed in the proof of Theorem 3 then give us \( N \) linearly independent densities of annihilators of the bracket \( n^\nu(U) \) and \( g \) functions \( v_k(n) \) (generating linearly independent flows \( e_k f^{\nu}_{(k)} X \)) which satisfy all the conditions of the Theorem.

Theorem is proved.

The construction of the Dirac restriction for the F-brackets leads also to the following statement:

**Theorem 5.**

The symplectic form for any F-bracket with nondegenerate tensor \( g^{\nu\mu}(U) \) is weakly nonlocal and can be written in the form

\[ \Omega_{\nu\mu}(X, Y) = \sum_{I=1}^{N+g} \epsilon_I \frac{\partial V_I}{\partial U^\nu}(X) \nu(X - Y) \frac{\partial V_I}{\partial U^\mu}(Y) \quad (5.23) \]
where \( V_i(U) \) are the restrictions of the Euclidean coordinates to \( M^N \) (canonical forms) and \( \epsilon_I = \pm 1 \) according to the signature of the flat metric in \( R^{N+g} \).

Proof. For the proof of the Theorem we just note that the symplectic form of the F-bracket should coincide with the restriction of the symplectic form for the corresponding DN-bracket in \( R^{N+g} \). The symplectic form for the DN-bracket is given by

\[
\Omega_{IJ}(X,Y) = \epsilon_I \delta_{IJ} \nu(X - Y)
\]

and it is easy to see that its restriction to \( M^N \) is given by the formula (5.23).

Theorem is proved.

At the end we give some classification for the special class of brackets (5.1) which represent the multi-parametric Poisson pencils.

**Theorem 6.** I. The expression

\[
\{ n^\nu(X), n^\mu(Y) \} = \left( \epsilon^\nu \delta^\nu\mu - \sum_{k=0}^g \alpha_k f^\nu_{(k)}(n) f^\mu_{(k)}(n) \right) \delta'(X - Y) -
\]

\[
- \sum_{k=0}^g \alpha_k \left( f^\nu_{(k)}(n) \right)_X f^\mu_{(k)}(n) \delta(X - Y) + \sum_{k=0}^g \alpha_k \left( f^\nu_{(k)}(n) \right)_X \nu(X - Y) \left( f^\mu_{(k)}(n) \right)_Y
\]

with a linearly independent set of \( f^\nu_{(k)}(n) \) defines a Poisson bracket at any \((\alpha_1, \ldots, \alpha_N)\) if and only if:

1) The flows

\[
n^\nu_{T_k} = \left( f^\nu_{(k)} \right)_X
\]

are Hamiltonian with respect to local Poisson bracket

\[
\{ n^\nu(X), n^\mu(Y) \}_0 = \epsilon^\nu \delta^\nu\mu \delta'(X - Y)
\]

\(^3\)The symplectic form for the case of the MF-bracket was considered also by M.V.Pavlov (private communication).
with some local Hamiltonian functions $H_k$, i.e. there exist such functions $h_k(n)$ that

$$f_{(k)}^\nu(n) \equiv e^\nu \frac{\partial h_k}{\partial n^\nu}$$

2) The Hamiltonians

$$H_k = \int h_k(X) dX$$

commute with each other with respect to the bracket $\{\ldots,\ldots\}_0$ and moreover:

$$\{h_k, H_{k'}\}_0 = \tau_{kk'}(h_k) h_{kX} = (W_{kk'}(h_k))_X$$

for some $\tau(h_k)$ and $W(h_k)$.

II. If the conditions of (I) take place (i.e. we have a Poisson pencil) then the functionals $H_k$ generate local Hamiltonian flows and commute with each other with respect to the full bracket (5.24). Besides that, the maximal functionally independent subset of $h_k(n)$ defines at any $(\alpha_1, \ldots, \alpha_g)$ a closed sub-bracket on the space of corresponding $h_k(X)$.

**Proof.**

First of all we note that our bracket at any $(\alpha^1, \ldots, \alpha^g)$ can be written in the Canonical form (5.1) after the linear changes of the functions $f_{(k)}^\nu(n)$. So we can use here the results of Theorem 2 replacing the quantities $e_s$ by $\alpha_s$ in every case. After that from the constant part of the condition (2) of Theorem 2 we get the equation

$$e^\mu \frac{\partial f_{(k)}^\nu}{\partial n^\mu} = e^\nu \frac{\partial f_{(k)}^\mu}{\partial n^\nu}$$

So locally we have:

$$f_{(k)}^\nu \equiv e^\nu \frac{\partial h_k}{\partial n^\nu} \quad (5.25)$$

for some $h_k(n)$. Therefore we are coming to the first statement of our Theorem. The commutativity of the functionals $H_k = \int h_k(X) dX$ with respect to the bracket $\{\ldots,\ldots\}_0$ now follows from condition (1) of Theorem 2.
Now from the linear term (with respect to $\alpha$) of the condition (2) of Theorem 2 we have:

$$e^\nu e^\mu \sum_\lambda e^\lambda \frac{\partial^2 h_k}{\partial n^\lambda \partial n^\nu} \frac{\partial h_s}{\partial n^\lambda} \frac{\partial h_s}{\partial n^\mu} = e^\nu e^\mu \sum_\lambda e^\lambda \frac{\partial^2 h_k}{\partial n^\lambda \partial n^\nu} \frac{\partial h_s}{\partial n^\lambda} \frac{\partial h_s}{\partial n^\mu}$$

for any $k$, $s$, $\nu$, $\mu$. This fact means that the rows (with respect to $\nu$) $\frac{\partial h_s}{\partial n^\nu}$ and $\sum_\lambda e^\lambda \frac{\partial^2 h_k}{\partial n^\lambda \partial n^\nu} \frac{\partial h_s}{\partial n^\nu}$ are linearly dependent for any $k$ and $s$, i.e.

$$\sum_\lambda e^\lambda \frac{\partial^2 h_k}{\partial n^\lambda \partial n^\nu} \frac{\partial h_s}{\partial n^\nu} = \tau_{sk}(n) \frac{\partial h_s}{\partial n^\nu}$$ (5.26)

After multiplying (5.26) by $n^\nu_X$ and the summation with respect to $\nu$, we obtain the following equality

$$\frac{dh_s}{dT_k} = \tau_{sk}(n)h_s X$$

This expression should be equal to some $(W_{sk}(n))_X$ because of the commutativity of $H_s$ and $H_k$ with respect to the bracket $\{.,.,.\}_0$. So we conclude that $W_{sk}(n)$ can be expressed in terms of the values of $h_s(n)$:

$$\tau_{sk}(n) \equiv \tau_{sk}(h_s)$$ (5.27)

and $W_{sk}(\xi) = \int \tau_{sk}(\xi)d\xi$.

Therefore the requirements of our Theorem are equivalent to the requirements corresponding to the analogs of (1) and (2) of Theorem 2 for all $(\alpha_1, \ldots, \alpha_g)$. It is not hard to check by direct calculation that the analog of the condition (3) of Theorem 2 for any $(\alpha_1, \ldots, \alpha_g)$ follows also from (5.23), (5.26) and (5.27). So we proved part (I) of the Theorem.

To prove part (II) we note that all $H_s$ generate the local flows according to the full bracket (5.24) since they are the integrals of the flows $w_{(k)\mu} n^\nu_X$ from the non-local part of a bracket according to the commutativity of the $\{H_k\}$ with respect to the bracket $\{.,.,.\}_0$. To prove the rest of the Theorem we mention that we deduce from (5.26) and (5.27) the relation

$$\sum_\nu \frac{\partial h_k}{\partial n^\nu} e^\nu \left( \frac{\partial h_s}{\partial n^\nu} \right)_X = (W_{sk}(h_k))_X$$

We can obviously choose $W_{sk}(h_k)$ such that
\[
\sum_{\nu} \frac{\partial h_k}{\partial n^\nu} \epsilon^\nu \frac{\partial h_s}{\partial n^\nu} = W_{sk}(h_k) + W_{ks}(h_s)
\]

So for the bracket \{h_s(X), h_t(Y)\} after the simple calculation we can write the following equalities

\[
\{h_s(X), h_t(Y)\} = (W_{st}(h_s) + W_{ts}(h_t)) \delta'(X - Y) -
\]

\[
- \left( \sum_{k=1}^{9} \alpha_k (W_{sk}(h_s) + W_{ks}(h_k)) (W_{kt}(h_k) + W_{tk}(h_t)) \right) \delta'(X - Y) +
\]

\[
+ (W_{st}(h_s))_X \delta(X - Y) - \sum_{k=1}^{9} \alpha_k (W_{sk}(h_s)W_{kt}(h_k))_X \delta(X - Y) -
\]

\[
- \sum_{k=1}^{9} \alpha_k (W_{ks}(h_k)W_{kt}(h_k))_X (W_{sk}(h_s)_Y + (W_{ts}(h_s))_X W_{tk}(h_t)) \delta(X - Y) +
\]

\[
+ \sum_{k=1}^{9} \alpha_k (W_{sk}(h_s))_X \epsilon^\nu(X - Y) (W_{tk}(h_t))_Y
\]

(5.28)

where all \(W_{kk'}\) are expressed in terms of the corresponding \(h_k\) and

\[
\{h_s(X), H_t\} = (W_{st}(h_s))_X - \sum_{k=1}^{9} \alpha_k (W_{sk}(h_s)W_{kt}(h_k) - Z_{kst}(h_k))_X
\]

where \(Z_{kst}(\xi) = \int W_{ks}(\xi) W'_t(\xi) d\xi\).

So we conclude that any \(H_s\) and \(H_t\) commute with each other with respect to the full pencil (5.24), and the maximal functionally independent subset of \(\{h_k\}\) form a sub-bracket (5.28) at any \((\alpha_1, \ldots, \alpha_g)\).

Theorem is proved.

**Simple example.** Let us consider the bracket

\[
\{\epsilon^\nu(X), \epsilon^\mu(Y)\}_0 = \epsilon^\nu \delta^\mu \delta'(X - Y)
\]

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It’s easy to show that any set of functions $h_k(R^2)$ where $R^2 = \sum_\nu \epsilon^\nu n^\nu n^\nu$ satisfy the requirements of Theorem 3 as the Hamiltonian functions for the flows $(f^\nu_{(k)})_X$. So, any linearly independent set of such functions gives us the Poisson pencil $(5.24)$ according to the relations

$$f^\nu_{(k)}(n) = \epsilon^\nu \frac{\partial h_k}{\partial n^\nu} = 2 h'_k(R^2)n^\nu$$

In particular, if we take only one function $h(R^2) = R^2/2$ we obtain the Canonical form of the MF bracket corresponding to the constant curvature $\alpha$.

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