Small Fluctuations in $\lambda\phi^{n+1}$ Theory in a Finite Domain: An Hirota’s Method Approach

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Abstract

We present a method to calculate small stationary fluctuations around static solutions describing bound states in a (1 + 1)-dimensional $\lambda\phi^{n+1}$ theory in a finite domain. We also calculate explicitly fluctuations for the $\lambda\phi^4$. These solutions are written in terms of Jacobi Elliptic functions and are obtained from both linear and nonlinear equations. For the linear case we get eigenvalues of a Lamé type Equation and the nonlinear one relies on Hirota’s Method.

1 Introduction

It is well known that, for some applications, quantum fields can be thought as classical fields upon which are added quantum corrections \cite{1}. In addition non-linear field theories are nowadays powerful tools to describe fundamental physical theories including cosmology, mainly for the inflationary model of the universe \cite{2,3}. On the other hand it is also well known that quantum systems when placed in finite domains (cavities) can present significant alterations on their behaviour. A step further is the study of interacting fields in finite domains \cite{4}. A modern paradigm is the famous Casimir Effect \cite{5,6}. In particular since these corrections are sensitive to boundary conditions an effect could be expected on bound states of the physical system under consideration.

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In a previous work Carrillo et al \cite{7} obtained a number of solutions of a \((1 + 1)\)-dimensional \(\lambda \phi^4\)-Theory in a finite domain in terms of Jacobi Elliptic functions. In this work we take a step further in the direction of a semi-classical approach of the theory by studying small fluctuations around one of the above mentioned solutions describing its bound states. We think this is enough to exemplify our approach to this problem. Nevertheless in this work we do not pursue on quantization. Below we show these fluctuations are described by a non-linear equation. In section 2 we present the linear case and in section 3 the non-linear case which is solved by Hirota’s method \cite{8}. In section 4 we present some conclusions and perspectives for future work.

We consider the Lagrangian density \(\mathcal{L}\) of a scalar field \(\phi\): 
\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \partial^{\mu} \phi + \frac{1}{2} M^2 \phi^2 - \frac{\lambda}{n + 1} \phi^{n+1},
\]
where \(\lambda\) is a coupling constant and \(\mu = x, t\).

From (1) we derive the following equation of motion,
\[
- \partial_{\mu} \partial^{\mu} \phi + M^2 \phi - \lambda \phi^n = 0.
\]

With the signature \((+ - - -)\), we can write the equation (2) in dimension \((1 + 1)\) as:
\[
\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} + M^2 \phi - \lambda \phi^n = 0.
\]

The problem now is to solve the equation (3) for \(\phi\) decomposed in a sum of a static solution \(\phi_0(x)\) and a small fluctuation \(\eta(x, t)\). In fact, in order to compare our results in linear case with the non-linear one, which will be presented in section 3 we can, without loss of generality, write
\[
\phi(x, t) = \phi_0(x) + \eta(x, t).
\]

Using (4) in (3) and taking the equations for \(\phi_0\) and \(\eta\) we find:
\[
\frac{d^2 \phi_0}{dx^2} + M^2 \phi_0 - \lambda \phi_0^n = 0,
\]
and
\[
\frac{\partial^2 \eta}{\partial x^2} - \frac{\partial^2 \eta}{\partial t^2} + (M^2 - \lambda n \phi_0^{n-1}) \eta - \lambda \sum_{k=2}^{n} \binom{n}{k} \phi_0^{n-k} \eta^k = 0.
\]

Solutions for the equation (5) are, in general, given by Abelian Theta Functions using the First Integral formalism \cite{9}. Now we want to solve the equation (6), and, for this we divide the analysis in two cases:
2 The Linear Case

Since \( \eta \) is a perturbation in the equation \((6)\), we assume that \( \eta(x, t) = \epsilon \xi(x, t) \)
where \( \epsilon \) is a small constant and then, the last term, which is of order \( O(\eta^2) \) can be
neglected. So the equation reduces to:

\[
\frac{\partial^2 \xi}{\partial x^2} - \frac{\partial^2 \xi}{\partial t^2} + \left(M^2 - \lambda n \phi_0^{-1}\right) \xi = 0. \tag{7}
\]

Here we are interested in stationary solutions of form:

\[
\xi(x, t) = e^{i\omega t} \psi(x). \tag{8}
\]

So, Eq.\((7)\) is reduced to equation for \( \psi \):

\[
\frac{d^2 \psi}{dx^2} + \left(M^2 + \omega^2 - \lambda n \phi_0^{-1}\right) \psi = 0. \tag{9}
\]

Now, we define a parameter \( E \) by equation:

\[
E^2 = M^2 + \omega^2, \tag{10}
\]

and then, the equation \((9)\) takes the form:

\[
\frac{d^2 \psi}{dx^2} + \left(\frac{E}{2} - \lambda n \phi_0^{-1}\right) \psi = 0. \tag{11}
\]

This equation can be interpreted as Schrödinger Equation for a potential of the form \( \phi_0^{-1} \). The problem now is, given a static solution \( \phi_0 \) of the equation \((5)\), find a
general solution of the equation \((11)\).

On the other hand, \( \lambda \phi^4 \)-Theory \((n = 3)\), the general static solutions of the classical
equation of motion to the field \( \phi \) are given by sn-type elliptic functions \[10\]

\[
\phi_0(x) = \pm \frac{M \sqrt{2c}}{\sqrt{\lambda \sqrt{1 + \sqrt{1 - 2c}}}} \text{sn} \left( \frac{M x}{\sqrt{2}} \sqrt{1 + \sqrt{1 - 2c}}, l \right), \tag{12}
\]

where \( c \) is a parameter belonging to interval \((0, \frac{1}{2}]\) and

\[
l = \frac{1}{-1 + \frac{1 + \sqrt{1 - 2c}}{c}}. \tag{13}
\]

From this relation, clearly \( l \in (0, 1] \).

Now, substituting \( \phi_0(x) \) given by Eq. \((12)\) in Eq. \((11)\) with \( n = 3 \) we get:

\[
\frac{d^2 \psi(\alpha)}{d\alpha^2} = \left(6 l \text{sn}^2(\alpha, l) - \frac{(1 + l)}{2} E \right) \psi(\alpha), \tag{14}
\]
where we have used also Eq. (10) and the change of variable
\[ \alpha = \frac{M x}{\sqrt{1 + l}}. \]

In the literature the general form of Eq. (14) is the Lamé differential equation, which is given by
\[ \frac{d^2 \Lambda(\alpha)}{d\alpha^2} - (m(m + 1) k^2 \text{sn}^2(\alpha, k) + C) \Lambda(\alpha) = 0, \quad (15) \]
where \( m \) is a positive real number, \( k^2 \) is the parameter of the Jacobian Elliptic Function \( \text{sn} \), and \( C \) is an arbitrary constant. Therefore, comparing (14) with (15) we obtain, \( m = 2 \) and \( C = -\frac{(1 + l)}{2} E. \)

The solutions of the Lamé equation have been studied in the literature \[12\]. With a few algebraic manipulations we get the following eigenfunctions and their respective eigenvalues:

1. \( \eta_l(x, t) = \epsilon \exp(i \sqrt{\frac{3}{1 + l}} Mt) \text{sn}(\frac{M x}{\sqrt{1 + l}}, l) \text{cn}(\frac{M x}{\sqrt{1 + l}}, l) \)
   for \( \omega^2_1 = \left( \frac{3}{1 + l} \right) M^2, \)
   (16)

2. \( \eta_l(x, t) = \epsilon \exp(i \sqrt{\frac{3l}{1 + l}} Mt) \text{sn}(\frac{M x}{\sqrt{1 + l}}, l) \text{dn}(\frac{M x}{\sqrt{1 + l}}, l) \)
   for \( \omega^2_2 = \left( \frac{3l}{1 + l} \right) M^2, \)
   (17)

3. \( \eta_l(x, t) = \epsilon \text{cn}(\frac{M x}{\sqrt{1 + l}}, l) \text{dn}(\frac{M x}{\sqrt{1 + l}}, l) \) for \( \omega^2_3 = 0, \)
   (18)

4. \( \eta_l(x, t) = \epsilon \exp(i \omega_4 t) \left( \text{sn}^2(\frac{M x}{\sqrt{1 + l}}, l) - \frac{1 + l + \sqrt{l^2 - l + 1}}{3l} \right) \)
   for \( \omega^2_4 = \left( \frac{1 + l - 2\sqrt{l^2 - l + 1}}{1 + l} \right) M^2, \)
   (19)

5. \( \eta_l(x, t) = \epsilon \exp(i \omega_5 t) \left( \text{sn}^2(\frac{M x}{\sqrt{1 + l}}, l) - \frac{1 + l - \sqrt{l^2 - l + 1}}{3l} \right) \)
   for \( \omega^2_5 = \left( \frac{1 + l + 2\sqrt{l^2 - l + 1}}{1 + l} \right) M^2. \)
   (20)

From the previous relations, it is worth to note that taking \( l = 1 \) in (18), (19) and (16), (17), we recover respectively, the eigenfunctions and energy levels of the ground-state and the first excited state (static case) of the Dashen-Hasslacher-Neveu (DHN model) \[13\]. For the fifth case, we observe that for \( l = 1, \omega^2_5 = 2M^2 \), which corresponds to the continuum part of the spectrum.
3 Bound States Fluctuations by Hirota’s Method

Now let us come back to the nonlinear model of the Equation (6). Again we will find the fluctuations \( \eta \), but now using a slight modification of the Hirota’s Method \([8]\). This is done as follows. Firstly we make a dependent variable transformation, namely,

\[
\eta(x, t) = \frac{\Phi_x(x, t)}{\Phi^{n-2}(x, t)}. \tag{21}
\]

In original Hirota’s Method, a \( \beta \) parameter enters the above equation as a multiplicative constant in order to simplify the differential equations obtained. In our method, this is not necessary, so the \( \beta \) parameter does not appear in the equation (21).

Thus, substituting (21) in (6), we get the following equation

\[
\Phi^2 \Phi_{xxx} + (6 - 3n)\Phi \Phi_x \Phi_{xx} + (n - 1)(n - 2)\Phi^3_x - (n - 1)(n - 2)\Phi_x \Phi_t^2 +
\]

\[
+ (n - 2)\Phi \Phi_x \Phi_{tt} + (2n - 4)\Phi \Phi_t \Phi_{xt} - \Phi^2 \Phi_{xxt} + M^2 \Phi^2 \Phi_x +
\]

\[
- \lambda \Phi^n \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \phi_0^{n-k} \Phi^k_x \Phi^{2k-nk} = 0. \tag{22}
\]

The Hirota’s Method prescribes that \( \Phi(x) \) can be expanded as a power series:

\[
\Phi(x, t) = 1 + \gamma f_1 + \gamma^2 f_2 + \ldots , \tag{23}
\]

where \( f_i(x, t) \) are unknown functions \( (i = 1, 2, 3, \ldots) \) and \( \gamma \) is an arbitrary constant \([8]\). Substituting (23) into (22) and equating the coefficients for every degree of \( \gamma \) to zero, we obtain an infinite system of differential equations (we show below only the first two):

\[
f_{1,xxx} - f_{1,xtt} + (M^2 - \lambda n\phi_0^{n-1}) f_{1,x} = 0, \tag{24}
\]

and

\[
f_{2,xxx} - f_{2,xtt} + (M^2 - \lambda n\phi_0^{n-1}) f_{2,x} =
\]

\[
= 2f_1 f_{1,xtt} - 2 (M^2 - \lambda n\phi_0^{n-1}) f_1 f_{1,x} +
\]

\[
+ ((3n - 6)f_{1,xx} - (n - 2)f_{1,tt}) f_{1,x} - (2n - 4)f_{1,t} f_{1,xt} - 2 f_{1,xxx} f_1
\]

\[- \lambda \frac{n(n - 1)}{2} \phi_0^{n-1} f_{1,x}^2. \tag{25}
\]
For now, we specialize for $\lambda \phi^4$ theory. In this case the above equation reduces to:

$$-\Phi_{xxt} \Phi^2 + 2 \Phi \Phi_{xt} \Phi_t + \Phi \Phi_{tt} \Phi_t - 2 \Phi_x \Phi_t^2 +$$

$$\Phi_{xxx} \Phi^2 - 3 \Phi \Phi_{xx} \Phi_x + (2 - \lambda) \Phi_x^3 + M^2 \Phi_x \Phi^2 - 3 \lambda \phi_0^2 \Phi_x \Phi^2 - 3 \lambda \phi_0 \Phi_x^2 \Phi = 0. \quad (26)$$

The equations for $f_1$ and $f_2$ are now:

$$- f_{1,xtt} + f_{1,xxx} + (M^2 - 3 \lambda \phi_0^2) f_{1,x} = 0, \quad (27)$$

$$f_{2,xxx} - f_{2,xtt} + (M^2 - 3 \lambda \phi_0^2) f_{2,x} = 2 f_{1,1,xtt} - 2 \left(M^2 - 3 \lambda \phi_0^2\right) f_{1,1,x} +$$

$$+ (3 f_{1,xx} - f_{1,tt}) f_{1,x} - 2 f_{1,t} f_{1,xt} - 2 f_{1,xxx} f_{1} - 3 \lambda \phi_0^2 f_{1,x}^2 \quad (28)$$

and so on for higher orders $O(\gamma^3)$.

Notice that the function $f_i$ will be determined by the previous functions $\{f_1, f_2, ... f_{i-1}\}$ only. So, we have an infinite set of recurrent equations. The next step is the usual one, as the first part of this work linear case). Thus, defining the new function

$$\xi(x,t) = f_{1,x}, \quad (29)$$

the Eq. (27) can be written as

$$- \xi_{tt} + \xi_{xx} + (M^2 - 3 \lambda \phi_0^2) \xi = 0. \quad (30)$$

Observe that the above equation is pretty the same as linearized equation (7). Since $\xi$ satisfies (7) we can then use the solutions ((16) – (20)) to obtain the functions $f_1$, which are given by:

(1) $f_1(x,t) = -\sqrt{1 + l} \frac{M}{M} \exp(i \sqrt{\frac{3}{1 + l}} M t) \text{dn} \left(\frac{M x}{\sqrt{1 + l}}, l\right)$, \quad for \quad $w_1^2 = \left(\frac{3}{1 + l}\right) M^2$. \quad (31)

(2) $f_1(x,t) = -\sqrt{1 + l} \frac{M}{M} \exp(i \sqrt{\frac{3l}{1 + l}} M t) \text{cn} \left(\frac{M x}{\sqrt{1 + l}}, l\right)$, \quad for \quad $w_2^2 = \left(\frac{3l}{1 + l}\right) M^2$. \quad (32)

(3) $f_1(x) = \sqrt{1 + l} \frac{M}{M} \text{sn} \left(\frac{M x}{\sqrt{1 + l}}, l\right)$ \quad for \quad $w_3^2 = 0$. \quad (33)

6
\[ f_1(x, t) = \exp\left( i \sqrt{\frac{1 + l - 2\sqrt{l^2 - l + 1}}{1 + l}} \cdot M t \right) \frac{M x}{\sqrt{1 + l}} - E(\text{am} \frac{M x}{\sqrt{1 + l}}, l) - \left( \frac{1 + l + \sqrt{l^2 - l + 1}}{3l} \right) x, \]

for \( w_4^2 = \left( \frac{1 + l - 2\sqrt{l^2 - l + 1}}{1 + l} \right) M^2. \) \( (34) \)

\[ f_2(x, t) = \exp\left( i \sqrt{\frac{1 + l + 2\sqrt{l^2 - l + 1}}{1 + l}} \cdot M t \right) \frac{M x}{\sqrt{1 + l}} - E(\text{am} \frac{M x}{\sqrt{1 + l}}, l) - \left( \frac{1 + l - \sqrt{l^2 - l + 1}}{3l} \right) x, \]

for \( w_5^2 = \left( \frac{1 + l + 2\sqrt{l^2 - l + 1}}{1 + l} \right) M^2, \) \( (35) \)

where, in all cases the constants of integration are taken as equal to zero.

The next step consists in the determination of the function \( f_2. \) In this case, using the Eq. (27) into Eq. (3) results the equation for \( f_2: \)

\[ f_{2,xxx} - f_{2,xtt} + (M^2 - 3\lambda \phi_0^2) f_{2,x} = (3f_{1,xx} - f_{1,tt}) f_{1,x} - 2f_{1,t}f_{1,xt} - 3\lambda \phi_0^2 f_{1,x}^2 \] \( (36) \)

One can see that substituting the functions \( f_1 \) given by (31) - (35) in above equation, we obtain inhomogeneous differential equations for the function \( f_2. \) Unfortunately due to the its great complexity, it was not possible to obtain the contribution of \( f_2 \) with this approach. Nevertheless we have obtained solutions for the nonlinear equation (6), considering only the contribution of the function \( f_1. \) On the other hand, \( \gamma \) is an arbitrary constant in the Hirota’s Method, since the series in equation (23) is just a formal series [8]. On the other hand, as mentioned above, in Eq. (4), the field \( \eta(x, t) \) is a small fluctuation on a static solution \( \phi_0(x). \) We have found solutions below whose amplitudes are of order \( \gamma \) (first approximation). So, for our application, \( \gamma \) is a small number controlling the fluctuation amplitude. Using (21), we obtain the corresponding fluctuations \( \eta, \) which are given by:

\[ (1) \ \eta(x, t) = \frac{\gamma \exp(i \sqrt{\frac{3}{1 + l}} M t) \sn(M x/\sqrt{1 + l}, l) \cn(M x/\sqrt{1 + l}, l)}{1 - \gamma \sqrt{\frac{3}{M} + l} \exp(i \sqrt{\frac{3}{1 + l}} M t) \dn(M x/\sqrt{1 + l}, l)}, \] \( (37) \)

\[ (2) \ \eta(x, t) = \frac{\gamma \exp(i \sqrt{\frac{3l}{1 + l}} M t) \sn(M x/\sqrt{1 + l}, l) \dn(M x/\sqrt{1 + l}, l)}{1 - \gamma \sqrt{\frac{3l}{M + l}} \exp(i \sqrt{\frac{3l}{1 + l}} M t) \cn(M x/\sqrt{1 + l}, l)}, \] \( (38) \)
\[ \eta_l(x) = \frac{\gamma \text{cn}(\frac{M\sqrt{1+l}}{\sqrt{1+l+1}}, l) \text{dn}(\frac{M\sqrt{1+l}}{\sqrt{1+l+1}}, l)}{1 - \gamma \frac{\sqrt{1+l}}{M} \text{sn}(\frac{M\sqrt{1+l}}{\sqrt{1+l+1}}, l)}, \quad (39) \]

\[ \eta_l(x,t) = \frac{\gamma \exp(i\sqrt{\frac{1+2\sqrt{1+l+1}}{1+l}} Mt) \left( \text{sn}^2\left(\frac{M\sqrt{1+l}}{\sqrt{1+l+1}}, l\right) - \frac{1+\sqrt{1+l+1}}{3l} \text{E}(am\frac{M\sqrt{1+l}}{\sqrt{1+l+1}}, l) \right)}{1 + \gamma \exp(i\sqrt{\frac{1+2\sqrt{1+l+1}}{1+l}} Mt) \left( \frac{x(2-l-v^2-l+1)}{3l} - \frac{\sqrt{1+l}}{M} \text{E}(am\frac{M\sqrt{1+l}}{\sqrt{1+l+1}}, l) \right)}, \quad (40) \]

\[ \eta_l(x,t) = \frac{\gamma \exp(i\sqrt{\frac{1+2\sqrt{1+l+1}}{1+l}} Mt) \left( \text{sn}^2\left(\frac{M\sqrt{1+l}}{\sqrt{1+l+1}}, l\right) - \frac{1+\sqrt{1+l+1}}{3l} \text{E}(am\frac{M\sqrt{1+l}}{\sqrt{1+l+1}}, l) \right)}{1 + \gamma \exp(i\sqrt{\frac{1+2\sqrt{1+l+1}}{1+l}} Mt) \left( \frac{x(2-l+\sqrt{1+l+1})}{3l} - \frac{\sqrt{1+l}}{M} \text{E}(am\frac{M\sqrt{1+l}}{\sqrt{1+l+1}}, l) \right)}, \quad (41) \]

Now observe that, at first approximation \((0 < \gamma << 1)\), the correspondent fluctuations of linear and non-linear cases coincides, which implies \(\gamma = \epsilon\) in equations \((37-41)\). This show that linear case is, in fact, an first order approximation of the non-linear one.

On the other hand, in [7] the authors showed that imposing Dirichlet boundary conditions on the field \(\phi\), given by \((12)\), confined to a box of size \(L\), it must satisfy the condition

\[ ML = 4\sqrt{1+l}K(l), \quad (42) \]

where \(4K(l)\) is a period of the Jacobi Elliptic Function \(\text{sn}(u, l)\) [15]. So, imposing the same boundary conditions for the fluctuations \(\eta\) (same kind of confinement), implies that the same \(l = l(L)\), obtained from above relation, must be used for \(\eta\). It is not difficult to see from our calculation, that, when \(\eta\) is confined, the only consistent linear and nonlinear fluctuations satisfying the condition \((12)\) are those given by \((16)\), \((17)\), \((37)\) and \((38)\), for they have the Jacobi Elliptic function \(\text{sn}(u, l)\), from which the relation \((12)\) comes from.

In the more general case, of the \(\lambda\phi^{n+1}\) theory, the first approximation for fluctuations can be calculated from Eq.\((21)\) and Eq.\((23)\). It is given by

\[ \eta(x,t) = \gamma \frac{f_{1,x}}{(1 + \gamma f_{1})^{n-2}}, \quad (43) \]

where the functions \(f_{1}\), obtained by Hirota's Method.

### 4 Conclusions

We developed an adaptation of the Hirota’s Method in order to calculate small fluctuations around static bound states in a \(\lambda\phi^{n+1}\)-theory. For the case \(n = 3\), we
calculate those fluctuations in a Finite Domain (interval) in (1+1)-dimensions in order to solve a non-linear equation of motion. We showed that those fluctuations are written in terms of Jacobi Elliptic functions. Imposing that fluctuations must satisfy the same boundary conditions of the unperturbed bound states we can select a subset of solutions. We must stress that, in the limit \( l = 1 \) ones get the results by Knyazev on fluctuations around a kink solution [14], for the fluctuations in the Eqs. (37) and (39). Even in this case (kink) we have obtained three new fluctuations, (38), (40) and (41). Dirichlet conditions imply the only possibility for fluctuation (40) is \( l = 1 \), so it does not exist in finite domains. The solution (41) there exists in any domain.

Acknowledgments

This work was partially supported by CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior).

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