SU(3) Maxwell equations and the classical chromodynamics

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Abstract

We study the equations of motion of the SU(3) Yang-Mills theory. Since the gluons, at scales of the order of 1 fm, can be considered as classical fields, we suppose that the gauge fields \( A^a_{\mu} \) of this theory are the gluonic fields and then it is possible to consider the Quantum Chromodynamics in a classical regime. For the case in which the condition \([A^a_{\mu}, A^b_{\rho}] = 0\) is satisfied, we show that the abelian equations of motion of the Classical Chromodynamics (CCD) have the same form as those of the classical electrodynamics without sources. Additionally, we obtain the non-abelian Maxwell equations for the CCD with sources. We observe that there exist electric and magnetic colour fields whose origin is not fermionic. We show as the gluons can be assumed as the sources of the electric and magnetic colour fields. We note that the gluons are the only responsible for the existence of a magnetic colour monopole in the CCD.

Keywords: SU(3) Yang-Mills equations of motion, Maxwell equations, Classical chromodynamics, Electric and magnetic colour fields, Magnetic colour monopole.

1 Introduction

Hadrons are composed by quarks which interact among them through the intermediate boson fields of the strong interaction (gluons). The strong interaction between quarks and gluons is described by the Quantum Chromodynamics (QCD), a non-abelian gauge theory based in the \( SU(3)_c \) colour gauge symmetry. The QCD can describe the observed phenomenology in the high-energy regime through the use of perturbation theory. However, for the low-energy regime, it is necessary to use non-perturbative methods because the running coupling constant is large. In this non-perturbative regime, in which the quarks are confined forming hadrons, several

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methods have been developed, as for instance, lattice theory, relativistic quarks models and effective theories. In the context of a particular class of relativistic quarks model \[1\] it is possible to describe the charmonium and bottomonium spectrum, in good agreement with the experimental data, by solving the Dirac equation in presence of SU(3) Yang-Mills fields representing gluonic fields. The obtaining solutions can model the quark confinement in a satisfactory way. At this respect, the results shown in \[1\] suggest that the mechanism of quark confinement should occur within the framework of QCD. Explicit calculations performed by Yu Goncharov \[2, 3\] show how the gluon concentration is huge at scales of the order of 1 fm. This fact suggests that the gluonic fields form a boson condensate and therefore, the gluons at large distances can be considered as classical fields \[5\].

The main goal of this paper is to study the equations of motion of the SU(3) Yang-Mills theory. For this theory, we suppose that the classical gauge fields \((A^a_\mu)\) represent the gluonic fields. At large distances, of the order of 1 fm, these fields are assumed as classical fields and then the theory is a classical version of the QCD. For the case in which the condition \([A^a_\mu, A^b_\rho] = 0\) is satisfied, we show that the abelian equations of motion of the Classical Chromodynamics (CCD) have the same form as those of the classical electrodynamics without sources. Additionally, if we consider the sources as having only one colour charge and assume that there only exist two diagonal gluon fields, we find that the equations of motion of the CCD also have the same form as those of the electrodynamics with sources. On the other hand, for the SU(3) Yang-Mills theory with sources, we obtain the non-abelian Maxwell equations. We observe that the divergence of the electric and magnetic colour fields are non vanishing. The latter implies that there exist electric and magnetic colour sources whose origin is not fermionic. The origin of these sources is related to the fact that the gluons have colour charge and therefore they can be assumed as the sources of the electric and magnetic colour fields. We note that the gluons are the only responsible for the existence of a magnetic colour monopole in the CCD.

2 Equations of motion of the classical chromodynamics

The equations of motion of the SU(3) Yang-Mills theory with quarks sources \(J^c_\nu\) are:

\[
\partial_\mu F_{b}^{\mu \nu} + g C^c_{ab} A^a_\mu F_\nu^{\mu} = g J^c_\nu = g \bar{\psi} \gamma^\nu \lambda_b \psi,
\]

being \(F_{b}^{\mu \nu}\) the non-abelian gauge field tensor, \(\lambda_b\) the Gell-Mann matrices, \(g\) the running coupling constant of the SU(3) group, \(C^c_{ab}\) the structure constants of the Lie algebra associated to this gauge group and \(\psi\) the quark field. The equations of motion given by \(1\) represents a system of non-lineal equations, whose solutions are supposed to contain at least the components of the SU(3) field which are Coulomb...
like or linear in the distance between quarks \((r)\). In this way, the solutions of (1) can model the quark confinement [1]. Since the Coulomb potential is solution of the equations of motion of the classic electrodynamics in a problem with fermionic sources, it is relevant to ask us under which conditions the motion equations of the chromodynamics have the same form of those of the electrodynamics. The main goal of this section is to answer this question starting from the SU(3) Yang-Mills motion equations given by (1).

Since the Lie algebra of the \(SU(3)\) gauge group is defined by \([\lambda_a, \lambda_b] = C^c_{ab} \lambda^c\) and rewriting the non-abelian gauge fields as \(A_\mu = A^a_\mu \lambda_a\), it is possible to write the non-abelian gauge field tensor as:

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\rho].
\]  

(2)

In the last expression, the colour index does not appear explicitly and each term is a matrix. For the case in which the quark sources vanish, i. e. for \(J^b_\nu = 0\), the SU(3) Yang-Mills equations of motion given by (1) can be written as

\[
\partial^\mu F_{\mu\nu} = -ig [A^\mu, F_{\mu\nu}].
\]  

(3)

Substituting (2) in (3), the equations of motion of the CCD without sources are

\[
\partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\rho]) = \partial^\mu F_{\mu\nu} = -ig [A^\mu, F_{\mu\nu}].
\]  

(4)

We observe that the system of equations given by (4) is clearly not linear, implying that the solutions cannot be obtained easily. A special case of (4) corresponds to the situation in which the abelian condition given by (5) is satisfied.

\[
[A_\mu, A_\rho] = 0
\]  

(5)

is satisfied. This condition, i.e. \([A^a_\mu \lambda_a, A^b_\nu \lambda_b] = 0\), can be satisfied in a non-trivial way if and only if one of the two following conditions is satisfied: i) If only the Gell-Mann matrices of the Cartan subalgebra, which is a maximal abelian of the \(SU(3)\) gauge group Lie algebra, appear in the system of equations of motion; ii) If \(A^a_\mu \lambda_a = m A^a_\mu \lambda_a\), being \(m \) a constant. The first condition means that there only exist two gluon fields in the system, precisely the associated with the \(\lambda_3\) and \(\lambda_8\) generators. These generators are shown in Appendix A. The second condition implies that each component of \(A_\mu\) is transmitted through the same gluonic configuration. This last possibility is not clear from a physical point of view.

Applying the condition (5) in (4), we obtain that the abelian equations of motion of the SU(3) Yang-Mills theory are

\[
\partial^\mu F'_{\mu\nu} = 0.
\]  

(6)
being $F'_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $A_\mu$ given by

$$A_\mu^a \lambda_a = \begin{pmatrix} A_\mu^3 + \frac{1}{\sqrt{3}} A_\mu^8 & A_\mu^4 - iA_\mu^5 & A_\mu^4 - iA_\mu^5 \\ A_\mu^4 + iA_\mu^5 & A_\mu^6 - \frac{1}{\sqrt{3}} A_\mu^8 & A_\mu^4 - iA_\mu^5 \\ A_\mu^4 + iA_\mu^5 & A_\mu^6 - \frac{2}{\sqrt{3}} A_\mu^8 & -A_\mu^3 - \frac{1}{\sqrt{3}} A_\mu^8 \end{pmatrix}. \quad (7)$$

We observe that there exist two differential equations for each one of the first seven fields and three for the eighth field. It is very easy to probe that the abelian equations of motion (6) can be written as:

$$\partial^\mu F'^{\mu a}_{\nu} = 0, \quad (8)$$

which means that there exists an equation similar to (6) for each field $A_\mu^a$. We note that the equations (8) have the same form as the electrodynamics ones. Under this similarity, the equations of motion of the CCD have the same behaviour as those of the classical electrodynamics. All the solutions of the equations (6) are also solutions of the equations (1) with $J^\nu_b = 0$.

The Yang-Mills equations given by (8) can be written in a similar form as the Maxwell equations of the electrodynamics without sources. These abelian Maxwell equations for the CCD are:

$$\vec{\nabla} \cdot \vec{E}_c = 0, \quad (9)$$
$$\vec{\nabla} \times \vec{E}_c = -\frac{\partial \vec{B}_c}{\partial t}, \quad (10)$$
$$\vec{\nabla} \cdot \vec{B}_c = 0, \quad (11)$$
$$\vec{\nabla} \times \vec{B}_c = \frac{1}{c^2} \frac{\partial \vec{E}_c}{\partial t}, \quad (12)$$

being $\vec{E}_c$ and $\vec{B}_c$ the electric and magnetic colour fields, respectively. Using the equations (9)-(12) is possible to predict the existence of CCD waves.

Now, we consider the equations of motion of the SU(3) Yang-Mills theory with sources, i. e. for the case $J \neq 0$. Following a procedure similar as the $J = 0$ case and demanding that the equations of motion satisfy the abelian condition $[A_\mu, A_\nu] = 0$, we can write the equations of motion (1) as

$$\partial_\mu F'^{\mu\nu} = \lambda^a \bar{\psi} \gamma^\nu \lambda_a \psi. \quad (13)$$

Because the quark field $\psi$ is a triplet in the colour space

$$\psi = \begin{pmatrix} \psi_{\text{Red}} \\ \psi_{\text{Blue}} \\ \psi_{\text{Green}} \end{pmatrix} = \begin{pmatrix} \bar{\psi}_R \\ \bar{\psi}_B \\ \bar{\psi}_G \end{pmatrix}, \quad (14)$$
then, the right side of (13) has the following form:

\[
\begin{pmatrix}
\frac{2}{3}(2\bar{\psi}_B\psi_B - \bar{\psi}_R\psi_R - \bar{\psi}_G\psi_G) \\
2\bar{\psi}_R\psi_B \\
2\bar{\psi}_G\psi_B
\end{pmatrix}
\begin{pmatrix}
\frac{2}{3}(\bar{\psi}_B\psi_B - 2\bar{\psi}_R\psi_R + \bar{\psi}_G\psi_G) \\
2\bar{\psi}_G\psi_R \\
2\bar{\psi}_R\psi_G
\end{pmatrix}
\begin{pmatrix}
2\bar{\psi}_B\psi_G \\
2\bar{\psi}_R\psi_G \\
2\bar{\psi}_G\psi_R
\end{pmatrix},
\]

where each element of the matrix has a $\gamma^\nu$ between the fields $\bar{\psi}\psi$. Since the left side of (13) is a system of equations, as is shown in (12), it is possible to consider the different situations in which the equation (13) is satisfied. Our interest is focused for the case in which the Yang-Mills equations is uncoupled respect to the components in the colour space of the quark fields. This case is considered if the abelian condition $[A_\mu, A_\rho] = 0$ is satisfied. For this abelian case, there are only two gluonic fields, the associated with the generators $\lambda_3$ and $\lambda_8$, different to zero and then it is necessary that two components in the colour space of the quark fields vanish independently. This means that for the CCD there only exists one colour charge, in a similar way as what happens for the electrodynamics in which there only exists one electric charge. This analysis leads to assume that any solution for the electrodynamics with sources is also a solution of the CCD. For instance, the Coulomb potential:

\[
A_t = \sum_{i=1}^{N} \frac{\alpha}{|\vec{r} - \vec{a}_i|},
\]

is solution of the SU(3)-Yang-Mills equations of motion given by (6). The latter assures that each fermionic source having colour charge contributes to the Coloumb potential. It is possible that there exist more potentials [4], but this result only assures that there exist at least one having the form of a Coloumb potential. This result is important because is a justification to take a Coloumb potential in the description of problems with three or more quarks.

## 3 Non-abelian Maxwell equations

The electric ($\vec{E}$) and magnetic ($\vec{B}$) fields, in the classical electrodynamics, are defined as the components of the electromagnetic field tensor ($F_{\mu\nu}$), in the following form:

\[
E_i := F_{0i},
\]

\[
B_n := -\frac{1}{2} \varepsilon_{nij} F_{ij},
\]

being $n, i, j = 1, 2, 3$. Starting from the Lagrangian of the electromagnetic field with sources is possible to obtain the Yang-Mills equations using the Euler-Lagrange equations. From these equations is possible to obtain the homogeneous Maxwell equations.
In an analogous way, we consider the non-abelian Maxwell equations for the CCD. For this case, the gauge field tensor is

\[ F_{\mu \nu}^a = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a + gC_{bc}^{a}A_{\mu}^bA_{\nu}^c, \]  

(19)

where the non-abelian gauge field is \( A_{\mu}^a = (A^0, -\vec{A}) \). The electric and magnetic colour fields, for the CCD, are defined respectively as:

\[ E_i^a := F_{0i}^a = -\partial_0 A_i^a - \partial_i A_0^a - gC_{bc}^{a}A_{0}^bA_{i}^c, \]  

(20)

\[ B_j^a := -\frac{1}{2} \varepsilon_{ijk} F_{ik}^a = -\frac{1}{2} \varepsilon_{ijk} \left(-\partial_i A_k^a + \partial_k A_i^a + gC_{bc}^{a}A_{i}^bA_{k}^c \right), \]  

(21)

being \( n, i, j = 1, 2, 3 \). In vectorial notation, the electric and magnetic colour fields are

\[ \vec{E}^a = -\partial_\mu \vec{A}^a - \vec{\nabla} A_0^a - gC_{bc}^{a}A_{0}^b\vec{A}^c, \]  

(22)

\[ \vec{B}^a = \vec{\nabla} \times \vec{A}^a - \frac{1}{2} gC_{bc}^{a} (\vec{A}^b \times \vec{A}^c). \]  

(23)

In contrast with the magnetic field of the electrodynamics, the magnetic colour field for the CCD is written as the sum of a rotor term and a non-rotor term.

Using the SU(3) Yang-Mills equations of motion (1), we obtain that the first Maxwell equation for the CCD with sources is given by:

\[ \partial^\mu E_\mu^a = -gC_{bc}^{a}A_{\mu}^bE_{\mu}^c + \rho^a, \]  

(24)

or in vectorial notation:

\[ \vec{\nabla} \cdot \vec{E}^a = -gC_{bc}^{a}A^b \cdot \vec{E}^c + \rho^a, \]  

(25)

where we have used the fact that the fermionic source can be written as \( J_{\mu}^a = (\rho^a, -\vec{J}^a) \). Using the following relation:

\[ B_j^a := -\frac{1}{2} \varepsilon_{ijk} F_{ik}^a, \]

\[ \varepsilon_{jqp} B_j^a := -\frac{1}{2} \varepsilon_{jqp} \varepsilon_{ijk} F_{ik}^a, \]

\[ \varepsilon_{jqp} B_j^a := -\frac{1}{2} (\delta_{pq} \delta_{qk} - \delta_{pq} \delta_{ik}) F_{ik}^a, \]

\[ \varepsilon_{jqp} B_j^a := -F_{pq}^a, \]  

(26)

then, it is possible to obtain the second Maxwell equation:

\[ \partial^\mu F_{\mu j}^a = -gC_{bc}^{a}A_{\mu}^bF_{\mu j}^c - J_{j}^a, \]

\[ \partial^0 E_j^a - \partial^\mu F_{\mu j}^a = -gC_{bc}^{a}A_{0}^bE_{j}^c + gC_{bc}^{a}A_{j}^bF_{ij}^c - J_{j}^a, \]

\[ \partial^0 E_j^a + \partial^i \varepsilon_{ijl} B_l^a = -gC_{bc}^{a}A_{0}^bE_{j}^c - gC_{bc}^{a}A_{j}^b \varepsilon_{ijl} B_l^c - J_{j}^a, \]  

(27)
that in vectorial notation can be written as
\[\vec{\nabla} \times \vec{B}^a - \partial_t \vec{E}^a = \vec{J}^a + gC_{bc}^a A_0^b \vec{E}^c - gC_{bc}^a \vec{A}^b \times \vec{B}^c.\] (28)

The other two Maxwell equations are obtained from the definitions of the electric (20) and magnetic (21) colour fields:
\[\vec{\nabla} \cdot \vec{B}^a = -\frac{1}{2} gC_{bc}^a \nabla \cdot (\vec{A}^b \times \vec{A}^c)\] (29)

and
\[\vec{\nabla} \times \vec{E}^a + \partial_t \vec{B}^a = \frac{1}{2} gC_{bc}^a \partial_t \left( \vec{A}^b \times \vec{A}^c \right) - gC_{bc}^a \left[ \vec{\nabla} \times (\vec{A}_0^b \vec{A}^c) \right].\] (30)

The Maxwell equations (25), (28), (29) and (30) can be extended directly for any $SU(N)$ gauge group. The interest here is for the $SU(3)$ gauge group. The $SU(3)$ Yang-Mills equations have solutions and these solutions are unique for the case in which $\vec{E}$ and $\vec{A}$ are independent fields and if there exist particular boundary conditions [4].

The wave equations for the CCD are given by (see Appendix B):
\[\Delta A^a_\nu = gC_{bc}^a A^b_\mu (\partial_\nu A^c_\mu - 2 \partial_\mu A^c_\nu - gC_{mn}^c A^m_\mu A^n_\nu).\] (31)

4 Conclusions

In the first part of this paper, we have found that the abelian equations of motion for the CCD without sources have the same form as those of electrodynamics without sources. These equations of motion have been obtained imposing that the $SU(3)$ Yang-Mills equations of motion satisfying the abelian condition given by $[A_\mu, A_\rho] = 0$. Starting of this result is possible to predict the existence of waves for the CCD. When considering sources, using the same abelian condition, it is possible to find Yang-Mills equations uncoupled to the components in the colour space of the quark fields. We have found, under the abelian condition, that there only exist one colour charge, reminding us the electrodynamics case where there only exists one electric charge. For this case, as maximum there are two gluonic fields as the mediators of the strong interaction. The equations of motion of the CCD with sources have the same form as those of the electrodynamics with sources, for each one of the two independent bosons. This last result assures that in a system with $N$ quarks, each quark contributes with a Coulomb potential. Surely in this problem there exist more potentials, but at least one of them is a Coulomb potential.

In the second part of the paper, we have obtained the equations of motion for the CCD analogous to the Maxwell equations for the electrodynamics. These non-abelian Maxwell equations have been obtained for a $SU(3)$ Yang-Mills theory, but
they are directly extended for a SU(N) Yang-Mills theory. We have found that the Maxwell equations does not only depend on $\vec{E}^a$ and $\vec{B}^a$ but also on $\vec{A}^a$ and $A^a_0$. From the divergences of $\vec{E}^a$ and $\vec{B}^a$, it is possible to conclude that there exist sources of electric and magnetic colour fields which are not fermions. It is possible to see as well that the bosonic field is charged and simultaneously is source of magnetic field, i.e. the gluonic fields have colour charge. Additionally, as the divergence of $\vec{B}^a$ non vanishing then there exist colour magnetic monopoles and the sources are not the quarks but the gluons.

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Appendix

Appendix A: Generators of $SU(3)$ group

The special unitary group in 3 dimensions $SU(3)$ has $3^2 - 1 = 8$ generators. These generators are labeled as $\lambda_1, \lambda_2, ..., \lambda_8$. The generators $\lambda_3$ and $\lambda_8$ are explicitly:

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Appendix B: Wave equations

Using the SU(3)-Yang-Mills equations [1] and the definition of the non-abelian gauge field tensor [19], it is possible to obtain:

$$\partial^\mu F^a_{\mu\nu} = -gC^a_{bc}A^b_\mu F^c_{\mu\nu},$$

$$\partial^\mu (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^a_{bc}A^b_\mu A^c_\nu) = -gC^a_{bc}A^b_\mu (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + gC^c_{mn}A^m_\mu A^n_\nu),$$

$$\Delta A^a_\nu - \partial_\nu \partial^\mu A^a_\mu + gC^a_{bc}\partial^\mu (A^b_\mu A^c_\nu) = -gC^a_{bc}A^b_\mu (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + gC^c_{mn}A^m_\mu A^n_\nu),$$

$$\Delta A^a_\nu - \partial_\nu \partial^\mu A^a_\mu + gC^a_{bc}\partial^\mu (A^b_\mu A^c_\nu) = -gC^a_{bc}A^b_\mu (2\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + gC^c_{mn}A^m_\mu A^n_\nu).$$

Using the gauge of Lorentz (fixing the gauge), we obtain a wave equation given by

$$\Delta A^a_\nu = gC^a_{bc}A^b_\mu (\partial_\nu A^c_\mu - 2\partial_\mu A^c_\nu - gC^c_{mn}A^m_\mu A^n_\nu).$$

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