Zeta-functions for germs of meromorphic functions and Newton diagrams

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Abstract
For a germ of a meromorphic function $f = \frac{P}{Q}$, we offer notions of the monodromy operators at zero and at infinity. If the holomorphic functions $P$ and $Q$ are non-degenerated with respect to their Newton diagrams, we give an analogue of the formula of Varchenko for the zeta-functions of these monodromy operators.

1 Germs of meromorphic functions

A polynomial $f$ of $(n + 1)$ complex variables of degree $d$ determines a meromorphic function $f$ on $\mathbb{C}P^{n+1}$. If one wants to understand the behaviour of $f$ at infinity, it is natural to analyze germs of the meromorphic function $f$ at points from the infinite hyperplane $\mathbb{C}P^{n}_\infty \subset \mathbb{C}P^{n+1}$. In local analytic coordinates $z_0, z_1, \ldots, z_n$, centred at a point $p \in \mathbb{C}P^n_\infty$ such that the infinite hyperplane $\mathbb{C}P^n_\infty$ is given by the equation \{ $z_0 = 0$ \}, the germ of the function $f$ has the form $f = \frac{P(z_0, \ldots, z_n)}{z_0^d}$. Let us consider germs of meromorphic functions of a general form.

DEFINITION 1 A germ of a meromorphic function on $(\mathbb{C}^{n+1}, 0)$ is a fraction $f = \frac{P}{Q}$, where $P$ and $Q$ are germs of holomorphic functions $(\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$. Two germs of meromorphic functions $f = \frac{P}{Q}$ and $f' = \frac{P'}{Q'}$ are equal if there exists a germ of a holomorphic function $U : (\mathbb{C}^{n+1}, 0) \rightarrow \mathbb{C}$ such that $U(0) \neq 0$, $P' = U \cdot P$ and $Q' = U \cdot Q$.

REMARKS. (1) For our convenience here we do not consider functions of the type $\frac{1}{Q(z)}$ or $\frac{P(z)}{1}$.

(2) According to the definition $\frac{x}{y} \neq \frac{x^2}{xy}$, but $\frac{x}{y} = \frac{x \exp(x)}{y \exp(x)}$.

Recently V.I. Arnold had obtained classifications of simple germs of meromorphic functions for certain equivalence relations.
In what follows we shall consistently use resolutions of germs of meromorphic functions.

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DEFINITION 2 A resolution of the germ $f$ is a modification of $(\mathbb{C}^{n+1}, 0)$ (i.e. a properanalytic map $\pi : \mathcal{X} \to \mathcal{U}$ of a smooth analytic manifold $\mathcal{X}$ onto a neighbourhood $\mathcal{U}$ of the origin in $\mathbb{C}^{n+1}$, which is an isomorphism outside of a proper analytic subspace in $\mathcal{U}$) such that the total transform $\pi^{-1}(H)$ of the hypersurface $H = \{ P = 0 \} \cup \{ Q = 0 \}$ is a normal crossing divisor at each point of $\mathcal{X}$.

The fact that the preimage $\pi^{-1}(H)$ is a divisor with normal crossings implies that in a neighbourhood of any point of it, there exists a local system of coordinates $y_0, y_1, \ldots, y_n$ such that the liftings $\tilde{P} = P \circ \pi$ and $\tilde{Q} = Q \circ \pi$ of the functions $P$ and $Q$ to the space $\mathcal{X}$ of the modification are equal to $u y_0^{k_0} y_1^{k_1} \cdots y_n^{k_n}$ and $v y_0^{l_0} y_1^{l_1} \cdots y_n^{l_n}$ respectively, where $u(0) \neq 0$ and $v(0) \neq 0$.

Let $B_{\varepsilon}$ be the closed ball of radius $\varepsilon$ with the centre at the origin in $\mathbb{C}^{n+1}$ and $\varepsilon$ be small enough such that (representatives of) the functions $P$ and $Q$ are defined in $B_{\varepsilon}$ and for any positive $\varepsilon' < \varepsilon$ the sphere $S_{\varepsilon'} = \partial B_{\varepsilon'}$ intersects the analytic spaces $\{ P = 0 \}$, $\{ Q = 0 \}$ and $\{ P = Q = 0 \}$ transversally (in the stratified sense). We choose $\delta$ small enough and take the ball $B_\delta \subset \mathbb{C}^2$ of radius $\delta$ centred at the origin.

DEFINITION 3 Let $c \in \mathbb{C}$ be such that $\| c \|$ is small enough, the $0$-Milnor fibre $\mathcal{M}^0_f$ of the germ $f$ is the set

$$\mathcal{M}^0_f = \{ z \in B_{\varepsilon} : (P(z), Q(z)) \in B_{\delta} \subset \mathbb{C}^2, f(z) = \frac{P(z)}{Q(z)} = c \}.$$

In the same way, for $c \in \mathbb{C}$ such that $\| c \|$ is large enough, the $\infty$-Milnor fibre $\mathcal{M}^\infty_f$ of the germ $f$ is the set

$$\mathcal{M}^\infty_f = \{ z \in B_{\varepsilon} : (P(z), Q(z)) \in B_{\delta} \subset \mathbb{C}^2, f(z) = \frac{P(z)}{Q(z)} = c \}.$$

LEMMA 1 The notion of the $0$- (respectively of the $\infty$-) Milnor fibre is well defined, i.e. for $\| c \|$ small enough: $\| c \| \ll \delta \ll \varepsilon$ (respectively for $\| c \|$ large enough: $\| c \|^{-1} \ll \delta \ll \varepsilon$) the differentiable type of $\mathcal{M}^0_f$ (respectively of $\mathcal{M}^\infty_f$) does not depend on $\varepsilon, \delta$ and $c$.

Proof. Let $\pi : \mathcal{X} \to \mathcal{U}$ be a resolution of the germ $f$ which is an isomorphism outside the hypersurface $H = \{ P = 0 \} \cup \{ Q = 0 \}$. Let $r : \mathbb{C}^{n+1} \to \mathbb{R}$ be the function $r(z) = \| z \|^2$, let $\tilde{r} = r \circ \pi : \mathcal{X} \to \mathbb{R}$ be the lifting of $r$ to the space $\mathcal{X}$ of the resolution. For $\varepsilon$ small enough, the hypersurface $\tilde{S}_{\varepsilon} = \{ \tilde{r} = \varepsilon^2 \}$ (the preimage of the sphere $S_{\varepsilon} \subset \mathbb{C}^{n+1}$) is transversal to all components of the total transform $\pi^{-1}(H)$. At each point of $\pi^{-1}(H)$ in a local coordinate system one has $P \circ \pi = uy_0^{k_0} \cdots y_n^{k_n}, Q \circ \pi = vy_0^{l_0} \cdots y_n^{l_n}$ with $u(0) \neq 0$ and $v(0) \neq 0$. Thus $f \circ \pi = wy_0^{m_0} \cdots y_n^{m_n}$ with $w(0) \neq 0$. The real hypersurface $\tilde{S}_{\varepsilon}$ is transversal to all coordinate subspaces (of different dimensions). It is not difficult to show that this implies transversality of $\tilde{S}_{\varepsilon}$ to the (complex) hypersurfaces $\{ wy_0^{m_0} \cdots y_n^{m_n} = c \}$ for $\| c \|$ small enough and for $\| c \|$ large enough. Now the proof follows from the standard arguments.
REMARKS. (1) The definition means that $\mathcal{M}_0^f$ or $\mathcal{M}_\infty^f$ is equal to

$$\{z \in B_\varepsilon : (P(z), Q(z)) \in B_\delta \subset \mathbb{C}^2, P(z) = c Q(z), \ P(z) \neq 0\}$$

and thus the Milnor fibres of the functions $\frac{P}{Q}$ and $\frac{R P}{R Q}$ with $R(0) = 0$ are, generally speaking, different.

(2) For $f = \frac{P}{Q}$, let $f^{-1} = \frac{Q}{P}$. It is not difficult to understand that $\mathcal{M}_0^{f^{-1}} = \mathcal{M}_\infty^f$ and $\mathcal{M}_\infty^{f^{-1}} = \mathcal{M}_0^f$. Just the same properties hold for the monodromy transformations and for the zeta-functions discussed below.

(3) It is possible (and sometimes more convenient) to define the Milnor fibres as follows:

$$\mathcal{M}_0^f = \{z \in B_\varepsilon : \|Q(z)\| \leq \delta, P(z) = c Q(z) \neq 0\}$$

with $\|c\| \ll \delta \ll \varepsilon$, and

$$\mathcal{M}_\infty^f = \{z \in B_\varepsilon : \|P(z)\| \leq \delta, P(z) = c Q(z) \neq 0\}$$

with $\|c\|^{-1} \ll \delta \ll \varepsilon$.

The meromorphic function $f$ determines a map from $B_\varepsilon \{P = Q = 0\}$ to the projective line $\mathbb{C}P^1 (z \mapsto (P(z) : Q(z)))$, which also will be denoted by $f$. Lemma 1 implies that this map is a locally trivial fibration in punctured neighbourhoods of the points $0 = (0 : 1)$ and $\infty = (1 : 0)$ of $\mathbb{C}P^1$.

DEFINITION 4 The 0-monodromy transformation $h_0^f$ (respectively the $\infty$-monodromy transformation $h_\infty^f$) of the germ $f$ is the monodromy transformation of the fibration $f$ over the loop $c \cdot \exp(2\pi i t), t \in [0, 1]$, with $\|c\|$ small enough (respectively large enough).

The 0- or $\infty$- monodromy operator is the action of the corresponding monodromy transformation in a homology group of the Milnor fibre. We are interested to apply the results for meromorphic functions to the problem of calculating the zeta-function of a polynomial at infinity. Thus we shall consider the zeta-functions $\zeta_0^f(t)$ and $\zeta_\infty^f(t)$ of the corresponding monodromy transformations:

$$\zeta_f^* = \prod_{q \geq 0} \{\det [id - t h_{f_{q+1}}(\mathcal{M}_q^f; \mathbb{C})]\}^{(-1)^q}$$

($\bullet = 0$ or $\infty$). This definition coincides with that in [2] and differs by minus sign in the exponent from that in [1].

2 Resolution of singularities and the formula of A’Campo for germs of meromorphic functions

Let $f = \frac{P}{Q}$ be a germ of a meromorphic function on $(\mathbb{C}^{n+1}, 0)$ and let $\pi : \mathcal{X} \to \mathcal{U}$ be a resolution of the germ $f$. The preimage $\mathcal{D} = \pi^{-1}(0)$ of the origin of $\mathbb{C}^{n+1}$, is a
normal crossing divisor. Let $S_{k,l}$ be the set of points of $D$ in a neighbourhood of which the functions $P \circ \pi$ and $Q \circ \pi$ in some local coordinates have the form $u y_0^k$ and $v y_0^l$ respectively ($u(0) \neq 0$, $v(0) \neq 0$). A slight modification of the arguments of A’Campo ([1]) permits to obtain the following version of his formula for the zeta-function of the monodromy of a meromorphic function.

THEOREM 1 Let the resolution $\pi : X \to U$ be an isomorphism outside the hypersurface $H = \{P = 0\} \cup \{Q = 0\}$. Then

$$\zeta_0^f(t) = \prod_{k \geq l} (1 - t^{k-l}) \chi(S_{k,l}),$$

$$\zeta_\infty^f(t) = \prod_{k < l} (1 - t^{l-k}) \chi(S_{k,l}).$$

REMARK. A resolution $\pi$ of the germ $f' = \frac{R_P}{R_Q}$ is at the same time a resolution of the germ $f = P Q$. Moreover the multiplicities of any component $C$ of the exceptional divisor in the zero divisors of the liftings $(RP) \circ \pi$ and $(RQ) \circ \pi$ of the germs $RP$ and $RQ$ are obtained from those of the germs $P$ and $Q$ by adding one and the same integer, the multiplicity $m = m(C)$ of $R$. Nevertheless the meromorphic functions $f$ and $f'$ can have different zeta-functions. The reason why formulae in the previous theorem give different results for $f$ and $f'$ consists in the fact that if an open part of the component $C$ lies in $S_{k,l}(f)$ then, generally speaking, its part which lies in $S_{k+m,l+m}(f')$ is smaller.

3 Zeta-functions of meromorphic functions via partial resolutions

Let $f = \frac{P}{Q}$ be a germ of a meromorphic function on $(\mathbb{C}^{n+1}, 0)$ and let $\pi : (X, D) \to (\mathbb{C}^{n+1}, 0)$ be an arbitrary modification of $(\mathbb{C}^{n+1}, 0)$, which is an isomorphism outside the hypersurface $H = \{P = 0\} \cup \{Q = 0\}$ (i.e. $\pi$ is not necessarily a resolution). Let $\varphi = f \circ \pi$ be the lifting of $f$ to the space of the modification, i.e. the meromorphic function $\frac{R_P}{Q \circ \pi}$. For a point $x \in \pi^{-1}(H)$, let $\zeta^0_{\varphi,x}(t)$ and $\zeta^\infty_{\varphi,x}(t)$ be the zeta-functions of the 0- and $\infty$-monodromies of the germ of the function $\varphi$ at $x$. Let $S = \{\Xi\}$ be a prestratification of $D = \pi^{-1}(0)$ (that is a partitioning into semi-analytic subspaces without any regularity conditions) such that, for each stratum $\Xi$ of $S$, the zeta-functions $\zeta^0_{\varphi,x}(t)$ and $\zeta^\infty_{\varphi,x}(t)$ do not depend on $x$, for $x \in \Xi$. We denote this zeta-functions by $\zeta^0_\Xi$ and by $\zeta^\infty_\Xi$ respectively. The same arguments which were used in [4] imply

THEOREM 2 For $\bullet = 0$ or $\infty$,

$$\zeta^\bullet_f(t) = \prod_{\Xi \in S} [\zeta^\bullet_\Xi(t)]^{\chi(\Xi)}.$$
4 Zeta-functions via Newton diagrams

For a germ $R = \sum a_k x^k : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ of a holomorphic function $(k = (k_0, k_1, \ldots, k_n), x^k = x_0^{k_0} x_1^{k_1} \cdots x_n^{k_n})$, its Newton diagram $\Gamma = \Gamma(R)$ is the union of the compact faces of the polytope $\Gamma_+ = \Gamma_+(R)$ = convex hull of $\bigcup_{k_i \neq 0} (k + \mathbb{R}_{n+1}^+) \subset \mathbb{R}_{n+1}^+$.

Let $f = \frac{P}{Q}$ be a germ of a meromorphic function on $(\mathbb{C}^{n+1}, 0)$ and let $\Gamma_1 = \Gamma(P)$ and $\Gamma_2 = \Gamma(Q)$ be the Newton diagrams of $P$ and $Q$. We call the pair $\Lambda = (\Gamma_1, \Gamma_2)$ of Newton diagrams $\Gamma_1$ and $\Gamma_2$ the Newight pair of $f$. We say that the germ of the meromorphic function $f$ is non-degenerated with respect to its Newton pair $\Lambda = (\Gamma_1, \Gamma_2)$ if the pair of germs $(P, Q)$ is non-degenerated with respect to the pair $\Lambda = (\Gamma_1, \Gamma_2)$ in the sense of [7] (which is an adaptation for germs of complete intersections of the definition of A.G. Khovanskii, [5]).

Let us define zeta-functions $\zeta^0_\Lambda(t)$ and $\zeta^\infty_\Lambda(t)$ for a Newton pair $\Lambda = (\Gamma_1, \Gamma_2)$. Let $1 \leq l \leq n + 1$ and let $\mathcal{I}$ be a subset of $\{0, 1, \ldots, n\}$ with the number of elements $|\mathcal{I}|$ equal to $l$. Let $L_{\mathcal{I}}$ be the coordinate subspace $L_{\mathcal{I}} = \{k \in \mathbb{R}_{n+1}^n : k_i = 0$ for $i \not\in \mathcal{I}\}$ and $\Gamma_{\mathcal{I}, \mathcal{I}} = \Gamma_1 \cap L_{\mathcal{I}} \subset L_{\mathcal{I}}$. Let $L_{\mathcal{I}}^*$ be the dual of $L_{\mathcal{I}}$ and $L_{\mathcal{I}}^+$ the positive orthant of it (the set of covectors which have positive values on $L_{\mathcal{I}}$). Let $\zeta_{\mathcal{I}}$ be a germ of a meromorphic function on $(\mathbb{C}^*)^n_{\mathcal{I}}$ such that $\dim(\Delta(a, \Gamma_1) + \Delta(a, \Gamma_2)) = l - 1$ (the Minkowski sum $\Delta_1 + \Delta_2$ of two polytopes $\Delta_1$ and $\Delta_2$ is the polytope $\{x = x_1 + x_2 : x_1 \in \Delta_1, x_2 \in \Delta_2\}$). There exists only a finite number of such covectors. For $a \in E_{\mathcal{I}}$, let $\Delta_1 = \Delta(a, \Gamma_1)$, $\Delta_2 = \Delta(a, \Gamma_2)$ and

$$V_a = \sum_{s=0}^{l-1} V_{l-1}(\underline{\Delta_1}, \ldots, \underline{\Delta_1}, \underline{\Delta_2}, \ldots, \underline{\Delta_2}),$$

where the definition of the (Minkowski) mixed volume $V(\Delta_1, \ldots, \Delta_m)$ can be found e.g. in [3] or [7]; $(l - 1)$-dimensional volume in a rational $(l - 1)$-dimensional affine subspace of $L_{\mathcal{I}}$ has to be normalized in such a way that the volume of the unit cube spanned by any integer basis of the corresponding linear subspace is equal to 1. Let us recall that $V_m(\underline{\Delta}, \ldots, \underline{\Delta})$ is simply the $m$-dimensional volume of $\Delta$. We have to assume that $m$ terms $V_0$(nothing) = 1, (this is necessary to define $V_a$ for $l = 1$). Let:

$$\zeta^0_{\mathcal{I}}(t) = \prod_{a \in E_{\mathcal{I}} : m(a, \Gamma_1) > m(a, \Gamma_2)} (1 - t^{m(a, \Gamma_1) - m(a, \Gamma_2)})^{(l-1)! V_a},$$

$$\zeta^\infty_{\mathcal{I}}(t) = \prod_{a \in E_{\mathcal{I}} : m(a, \Gamma_1) < m(a, \Gamma_2)} (1 - t^{m(a, \Gamma_2) - m(a, \Gamma_1)})^{(l-1)! V_a},$$

$$\zeta^*_{\mathcal{I}}(t) = \prod_{\mathcal{I} : \#(\mathcal{I}) = l} \zeta^\bullet_{\mathcal{I}}(t),$$

$$\zeta_\Lambda(t) = \prod_{l=1}^{n+1} (\zeta^*_{\mathcal{I}}(t))^{(-1)^{l-1}},$$

where $\zeta^\bullet_{\mathcal{I}}(t)$ and $\zeta^\bullet_{\mathcal{I}}(t)$ are the coordinate subspace $\zeta^\bullet_{\mathcal{I}}(t)$ and $\zeta^\bullet_{\mathcal{I}}(t)$.
where $\bullet = 0$ or $\infty$.

**Theorem 3** Let $f = \frac{P}{Q}$ be a germ of a meromorphic function on $(\mathbb{C}^{n+1}, 0)$ non-degenerated with respect to its Newton pair $\Lambda = (\Gamma_1, \Gamma_2)$. Then

$$\zeta^f_\Lambda(t) = \zeta^0_\Lambda(t) \quad \text{and} \quad \zeta^\infty_\Lambda(t) = \zeta^\infty_f(t).$$

**Proof.** Let $\Sigma$ be an unimodular simplicial subdivision of $\mathbb{R}^{n+1}$ which corresponds to the pair $(\Gamma_1, \Gamma_2)$ of Newton diagrams in the sense of [7] Section 4. This subdivision is consistent with each of the Newton diagrams $\Gamma_1$ and $\Gamma_2$ in the sense of [8].

Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{C}^{n+1}, 0)$ be the toroidal modification map corresponding to $\Sigma$. Since the pair $(P, Q)$ is non-degenerated with respect to $(\Gamma_1, \Gamma_2)$, $\pi$ is a resolution of the germ $f = \frac{P}{Q}$ (see [7]). We have the sets $S_{k,l} = S_k(P) \cap S_l(Q)$. The description of $S_k(P)$ (and of $S_l(Q)$) can be found in [8], Section 7. Each of them consists of open parts of certain complex tori of some dimensions.

Tori of dimension $n$ correspond to one-dimensional cone of $\Sigma$ which are positive (i.e., lie in $(\mathbb{R}^*)^{n+1}$). The multiplicity of $P \circ \pi$ (respectively of $Q \circ \pi$) at such a torus is equal to $m(a, \Gamma_1)$, (respectively to $m(a, \Gamma_2)$) for the primitive integer covector $a$ which spans the corresponding cone.

Tori of dimension $(l-1)$ correspond to positive simplicial $(n+2-l)$-dimensional cones of $\Sigma$ which have a cone of the form

$$\mathcal{G} = \{ a \in (\mathbb{R}^*)^{n+1} : a_j > 0 \text{ for } j \notin \mathcal{I}, \quad a_j = 0 \text{ for } j \in \mathcal{I} \}$$

with $\#(\mathcal{I}) = l$ (these cones are elements of $\Sigma$) as its face. Moreover these cones correspond to one-dimensional cones of a partitioning of $L_{\mathcal{I}}$ which is consistent with the Newton diagrams $\Gamma_{i,\mathcal{I}} = \Gamma_i \cap L_{\mathcal{I}} \subset L_{\mathcal{I}}$. The multiplicities of $P \circ \pi$ and $Q \circ \pi$ at such a torus again are equal to $m_{\mathcal{I}}(a, \Gamma_{1,\mathcal{I}})$ and $m_{\mathcal{I}}(a, \Gamma_{2,\mathcal{I}})$ for the primitive integer covector $a$ from the corresponding one-dimensional cone.

In order to apply Theorem 1 we have to calculate the Euler characteristic of the corresponding part of an $l-1$-dimensional torus $T$ : the complement to the intersection with the strict transform of the hypersurface $H = \{ P = 0 \} \cup \{ Q = 0 \}$. Let $A$ (respectively $B$) the intersection of the torus $T$ with the strict transform of the hypersurface $\{ P = 0 \}$ (respectively of $\{ Q = 0 \}$), let $\Delta_i := \Delta(a, \Gamma_{i,\mathcal{I}})$. From the results of Khovanskii ([6]) it follows that the Euler characteristic of $A$ (respectively of $B$) is equal to $(-1)^{l-1}(l-1)! \overline{V}_{l-1}(\Delta_1, \ldots, \Delta_1)$ (respectively to $(-1)^{l-1}(l-1)! \overline{V}_{l-1}(\Delta_2, \ldots, \Delta_2)$), the Euler characteristic of $A \cap B$ is equal to

$$(-1)^{l-1}(l-1)! [V_{l-1}(\Delta_1, \ldots, \Delta_1, \Delta_2) + V_{l-1}(\Delta_1, \ldots, \Delta_1, \Delta_2, \Delta_2) + \ldots + V_{l-1}(\Delta_1, \Delta_2, \ldots, \Delta_2)],$$

where

$$(-1)^{l-1}(l-1)! [V_{l-1}(\Delta_1, \ldots, \Delta_1, \Delta_2) + V_{l-1}(\Delta_1, \ldots, \Delta_1, \Delta_2, \Delta_2) + \ldots + V_{l-1}(\Delta_1, \Delta_2, \ldots, \Delta_2)].$$

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Thus the Euler characteristic of the complement of $A \cup B$ in the torus $T$ is equal to
\[
\chi(T) - \chi(A) - \chi(B) + \chi(A \cap B) =
\]
\[
(-1)^{l-1}(l-1)!\left[V_{l-1}(\Delta_1, \ldots, \Delta_1) + V_{l-1}(\Delta_2, \Delta_1, \Delta_2) + \ldots + V_{l-1}(\Delta_2, \ldots, \Delta_2)\right],
\]
which implies the statement. □

5 The Varchenko type formula for $f = \frac{P}{z_0^d}$

As we have mentioned at the beginning, in order to study the behaviour of polynomials at infinity, germs of meromorphic functions of the form $\frac{P(z_0, z_1, \ldots, z_n)}{z_0^d}$ have to be of interest. In this case the formulae for the zeta-functions $\zeta_0^I(t)$ and $\zeta_\infty^I(t)$ are considerably reduced. Thus let us reformulate the definition of these zeta-functions for the case when the Newton diagram $\Gamma_2$ consists of one point $(d, 0, \ldots, 0)$ (in terms of the Newton diagram $\Gamma := \Gamma_1$ of $P$). The description is as follows.

Let $1 \leq l \leq n+1$ and let $\mathcal{I}$ be a subset of $\{1, \ldots, n\}$ with the number of elements $\#\mathcal{I}$ equal to $l-1$. Let $\gamma_1^I, \ldots, \gamma_j^I(\mathcal{I})$ be all $(l-1)$-dimensional faces of $\Gamma_{\mathcal{I} \cup \{0\}}$ and $a_{\mathcal{I}, 1}, \ldots, a_{\mathcal{I}, j(\mathcal{I})}$ the corresponding primitive covectors (normal to $\gamma_1^I, \ldots, \gamma_j^I(\mathcal{I})$), $a_{\mathcal{I}, s}^0$ is the 0th coordinate of $a_{\mathcal{I}, s}$, let $m_s(\mathcal{I}) = (a_{\mathcal{I}, s}, k)$ for $k \in \gamma_s^I$. Then
\[
\zeta_0^{I \cup \{0\}}(t) = \prod_{1 \leq s \leq j(\mathcal{I}) : m_s(\mathcal{I}) > d a_{\mathcal{I}, s}^0} (1 - t^{m_s(\mathcal{I}) - d a_{\mathcal{I}, s}^0})^{(l-1)!V_{l-1}(\gamma_s^I)},
\]
\[
\zeta_\infty^{I \cup \{0\}}(t) = \prod_{1 \leq s \leq j(\mathcal{I}) : m_s(\mathcal{I}) < d a_{\mathcal{I}, s}^0} (1 - t^{d a_{\mathcal{I}, s}^0 - m_s(\mathcal{I})})^{(l-1)!V_{l-1}(\gamma_s^I)},
\]
\[
\zeta^I(t) = \prod_{\mathcal{I} \subset \{1, \ldots, n\} : \#\mathcal{I} = l-1} \zeta_0^{I \cup \{0\}}(t),
\]
\[
\zeta_\infty^I(t) = \prod_{l=1}^{n+1} (\zeta^I(t))^{(-1)^{l-1}}.
\]
($\bullet = 0$ or $\infty$) where $V_{l-1}(\gamma_s^I)$ is the (usual) $(l-1)$-dimensional volume of the face $\gamma_s^I$ (in the hyperplane spanned by it in $L_{\mathcal{I} \cup \{0\}}$).

6 Examples

Example 1. Let $f = \frac{x^3-xy}{y}$. The Milnor fibre $\mathcal{M}_f^0$ (respectively $\mathcal{M}_f^\infty$) is $\{(x, y) : \|(x, y)\| < \varepsilon, (x^3-xy, y) \in B_\delta, x^3-xy = cy\}$ \{$(0, 0)$\}, where $\|c\|$ is small (respectively large). From the equation $x^3-xy = cy$ one has $y = \frac{x^3}{x+c}$ and thus $\mathcal{M}_f^0$ is diffeomorphic to the disk $D$ in the $x$-plane with two points removed: $-c$ and the origin. In the same way $\mathcal{M}_f^\infty$ is
diffeomorphic to the punctured disk $\mathcal{D}^*$. It is not difficult to understand that the action of the monodromy transformation in the homology groups is trivial in both cases. Thus

$$\zeta_f^0(t) = (1 - t)^{-1} \quad \text{and} \quad \zeta_f^\infty(t) = 1.$$ 

Now let us calculate these zeta functions from their Newton diagrams, Fig 1.

![Figure 1](image)

We have $\zeta^*_f(t) = 1$ since each coordinate axis intersects only one Newton diagram. There is only one linear function (namely $a = k_x + 2 k_y$) such that $\dim \Delta(a, \Gamma_1) = 1$. The one-dimensional volume $V_1(\Delta(a, \Gamma_1))$ of $\Delta(a, \Gamma_1)$ is equal to 1 and $V_1(\Delta(a, \Gamma_2)) = 0$. We have $m(a, \Gamma_1) = 3$ and $m(a, \Gamma_2) = 2$. Thus $\zeta_2^0(t) = (1 - t)$, $\zeta_2^\infty(t) = 1$, $\zeta_{\Gamma_1, \Gamma_2}(t) = (1 - t)^{-1}$ and $\zeta_{\Gamma_1, \Gamma_2}(t) = 1$ which coincides with the formulae for $f$ written above.

**Example 2.** Let $P = x y z + x^p + y^q + z^r$ be a $T_{p,q,r}$ singularity, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and let $Q = x^d + y^d + z^d$ be a homogeneous polynomial of degree $d$. Suppose that $p > q > r > d > 3$ and that $p, q, r$ are pairwise prime. Let us compute the zeta-functions of $f = \frac{P}{Q}$ using Theorems 2 and 3.

(a) It is clear that $f$ is non-degenerated with respect to its Newton pair $\Lambda = (\Gamma_1, \Gamma_2)$. Thus

$$\zeta_f^*(t) = \zeta_\Lambda^*(t) = \zeta_f^*(\zeta_2^*)^{-1} \zeta_3^* \ (\bullet = 0 \text{ or } \infty).$$

One has $\zeta_f^\infty = \zeta_2^\infty = 1$ and the unique covector which is necessary for computing $\zeta_3^\infty$ is $a = (1, 1, 1)$. In this case $m(a, \Gamma_1) = 3$, $m(a, \Gamma_2) = d$, $\Delta(a, \Gamma_1) = \{(1, 1, 1)\}$ and $\Delta(a, \Gamma_2)$ is the simplex $\{k_x + k_y + k_z = d, k_x \geq 0, k_y \geq 0, k_z \geq 0\}$, its two-dimensional volume is equal to $\frac{d^2}{2}$. Thus $\zeta_f^\infty = (1 - t)^{d-3} d^2$.

We have

$$\zeta_1^0 = (1 - t^{p-d})(1 - t^{q-d})(1 - t^{r-d}),$$
$$\zeta_2^0 = (1 - t^{p-a-d})(1 - t^{q-p-d})(1 - t^{r-p-d})^2(1 - t^{q-d})^d.$$

To compute $\zeta_3^0$ one has to take into account both covectors $(rq - q - r, r, q)$, $(r, pr - p - r, p)$, and $(q, p, qp - p - q)$, corresponding to two-dimensional faces of $\Gamma_1$, and covectors $(1, r - 2, 1)$, $(r - 2, 1, 1)$, and $(q - 2, 1, 1)$, corresponding to pairs of the form (one-dimensional face of $\Gamma_1$, one-dimensional face of $\Gamma_2$). E.g., for $a = (1, r - 2, 1)$, $\Delta(a, \Gamma_1)$ (respectively $\Delta(a, \Gamma_2)$) is the segment between $m(0, 0, r)$ to $(1, 1, 1)$ (respectively between
Pay attention to the “absence of the symmetry”: last three covectors are not obtained from each other by permuting the coordinates and the numbers $p$, $q$, and $r$. This way

$$\zeta^0_\delta = (1 - t^{p-q-d})(1 - t^{q-p-d})(1 - t^{q-p}) (1 - t^{q-d})^2 (1 - t^{q-d})^d$$

and

$$\zeta^0_f = (1 - t^{p-d})(1 - t^{q-d})(1 - t^{r-d}).$$

(b) For computing the zeta-functions of $f$ with the help of Theorem 2, let $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^3, 0)$ be the blowing-up of the origin in $\mathbb{C}^3$ and let $\varphi$ be the lifting $f \circ \pi$ of $f$ to the space $\mathcal{X}$. The exceptional divisor $\mathcal{D}$ is the complex projective plane $\mathbb{CP}^2$. Let $H_1$ and $H_2$ be the strict transforms of the hypersurfaces $\{P = 0\}$ and $\{Q = 0\}$. $D_1$ is $\mathcal{D} \cap H_1$. The curve $D_1$ consists of three transversal lines $l_1, l_2, l_3$ and has three singular points $S_1 = l_2 \cap l_3 = (0, 0, 1)$, $S_2 = l_1 \cap l_3 = (0, 1, 0)$, and $S_3 = l_1 \cap l_3 = (1, 0, 0)$. The curve $D_2$ is a smooth curve of degree $d$, it intersects $D_1$ at $3d$ different points $\{P_1, \ldots, P_{3d}\}$. One has the following natural stratification of the exceptional divisor $\mathcal{D}$:

(i) 0-dimensional strata $\Lambda^0_i$ ($i = 1, 2, 3$), each consisting of one point $S_i$;

(ii) 0-dimensional strata $\Xi^0_i$ consisting of one point $P_i$ each ($i = 1, \ldots, 3d$);

(iii) 1-dimensional strata $\Xi^1_i = l_i \setminus \{D_2 \cup l_j \cup l_k\}$ ($i = 1, 2, 3$) and $\Xi^1_4 = D_2 \setminus D_1$;

(iv) 2-dimensional stratum $\Xi^2 = \mathcal{D} \setminus (D_1 \cup D_2)$.

It is not difficult to see that $\zeta^0_{\Xi^0_1} (t) = 1$, $\zeta^\infty_{\Xi^0_1} (t) = 1 - t^{d-3}$, for each stratum $\Xi$ from $\Xi^0_i$ ($1 \leq i \leq 3d$), $\Xi^1_i$ ($1 \leq i \leq 4$) one has $\zeta^\infty_{\Xi^i} (t) = 1$ ($\bullet = 0$ or $\infty$).

In what follows the exceptional divisor $\mathcal{D}$ has the local equation $u = 0$. At the point $S_1$ the lifting $\varphi$ of the function $f$ is of the form $\frac{u^2x_1y_1 + u^3x_2y_2 + y^q}{u^2x_1^2 + u^3y_1^2 + u^q}$. This germ has the same Newton pair as the germ $\frac{u^3y_2 + u^r}{u^d}$. Using theorem 3 one has $\zeta^0_{\Lambda^1_1} = 1$, $\zeta^\infty_{\Lambda^1_1} = 1 - t^{r-d}$.

At the point $S_2$ the function $\varphi$ has the form $\frac{u^3x_1z_1 + z^r + x^q}{u^2x_1^2 + u^3z_1^2 + u^q}$. It has the same Newton pair as $\frac{u^3x_1z_1 + z^r + u^q}{u^2x_1^2 + u^3z_1^2 + u^q}$. Using Theorem 3 one has $\zeta^\infty_{\Lambda^2_1} (t) = 1$, $\zeta^0_{\Lambda^2_1} (t) = 1 - t^{r-d}$. Just in the same way $\zeta^\infty_{\Lambda^2_2} (t) = 1$, $\zeta^0_{\Lambda^2_3} (t) = 1 - t^{p-d}$. Combining these computations together, one has the same results as above (without using a partial resolution).

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