AREA LITTLEWOOD-PALEY FUNCTIONS ASSOCIATED WITH
HERMITE AND LAGUERRE OPERATORS

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Abstract. In this paper we study $L^p$-boundedness properties for area Littlewood-Paley functions associated with heat semigroups for Hermite and Laguerre operators.

1. Introduction.

We denote by $T = \{T_t\}_{t>0}$ a semigroup of linear and bounded operators on $L^p(\Omega,d\mu)$, $1 \leq p < \infty$, where $(\Omega, d\mu)$ is a measure space. Suppose that $\rho$ is a metric on $\Omega$ and that, for every $f \in L^p(\Omega,d\mu)$, the mapping

$$M_f : (0,\infty) \rightarrow L^p(\Omega,d\mu)$$

$$ t \rightarrow M_f(t) = T_t(f),$$

is a.e pointwise differentiable. For every $q > 1$, the area g-function $g_T^q(f)$ of $f \in L^p(\Omega,d\mu)$, $1 \leq p < \infty$, is defined by

$$g_T^q(f)(x) = \left\{ \int_{\Gamma(x)} \left| s \frac{\partial}{\partial s} T_s(f)(y) \right|_2^q \frac{dt \, dy}{t^2} \right\}^{1/q},$$

where $\Gamma(x) = \{(y,t) \in \Omega \times (0,\infty) : \rho(x,y) < t\}$, $x \in \Omega$. This area g-function can be seen as an extension of the Lusin area integral function. As it is wellknown Lusin area integral is related to the nontangential boundary behaviour of analytic and harmonic functions in the unit disc (see, for instance, the celebrated papers [7], [31] and [47]). $L^p$-boundedness properties of the (sublinear) operator $g_T^q$ (and some extensions of this one) when $T$ represents Poisson or heat semigroups associated with the Euclidean Laplacian operator and in other settings have been studied by several authors ([1], [2], [6], [9], [10], [20], [22], [27], [29], [37], [44], [48], amongst

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In this paper we prove $L^p$-boundedness properties for the function $g_T^q$, $q > 1$, when $T$ is the heat semigroup associated with Hermite and Laguerre operators.

For every $n \in \mathbb{N}$, we denote by $h_n$ the Hermite function defined by

$$h_n(x) = \frac{1}{\sqrt{\sqrt{\pi} 2^n n!}} H_n(x) e^{-x^2/2}, \quad x \in \mathbb{R},$$

where $H_n$ represents the Hermite polynomial of degree $n$. The sequence $\{h_n\}_{n \in \mathbb{N}}$ is complete and orthonormal in $L^2(\mathbb{R})$. Moreover, one has

$$\mathcal{H} h_n = \left( n + \frac{1}{2} \right) h_n, \quad n \in \mathbb{N},$$

where $\mathcal{H} = \frac{1}{2}(-\Delta + |x|^2)$ is the harmonic oscillator, also called, Hermite operator. This operator $\mathcal{H}$ is positive and symmetric in $L^2(\mathbb{R})$ on the domain $C_c^\infty(\mathbb{R})$, the space of the $C^\infty$-functions on $\mathbb{R}$ which have compact support.

The heat diffusion semigroup $\mathcal{W} = \{ W_t \}_{t > 0}$ generated by $L$ is given by

$$W_t f = \sum_{n=0}^{\infty} c_n(f) e^{-(n+1/2)t} h_n, \quad f \in L^2(\mathbb{R}),$$

being, for every $n \in \mathbb{N}$,

$$c_n(f) = \int_{-\infty}^{+\infty} h_n(x) f(x) \, dx.$$  

By using the Mehler formula ([55, p. 380]) we can write, for $f \in L^2(\mathbb{R})$,

$$W_t f(x) = \int_{-\infty}^{+\infty} W_t(x, y) f(y) \, dy, \quad x \in \mathbb{R},$$

where, for each $x, y \in \mathbb{R}$ and $t > 0$,

$$W_t(x, y) = \sum_{n=0}^{\infty} e^{-(n+1/2)t} h_n(x) h_n(y) = \frac{1}{\sqrt{\pi}} \left( \frac{e^{-t}}{1 - e^{-2t}} \right)^{1/2} e^{-\frac{1}{2(1-e^{-2t})} \frac{(x-e^{-tx})^2 + (y-e^{-ty})^2}{2(1-e^{-2t})}}.$$  

The study of harmonic analysis operators (such as maximal operators, Riesz transforms, g-functions,...) in the Hermite polynomial setting was begun by Muckenhoupt ([34] and [36]). We can remark also in this context the results of Sjögren ([45] and [46]), Pérez and Soria ([43]), García-Cuerva, Mauceri, Sjögren and Torrea ([18], [19]), García-Cuerva, Mauceri, Meda, Sjögren and Torrea ([17]), Fabes, Gutiérrrez and Scotto ([16]), Urbina ([57]) and Harboure, Torrea and Viviani ([25]), amongst others. Harmonic analysis associated with the Hermite operator $\mathcal{H}$ has been developed in the last years. Stempak and Torrea studied
maximal operators associated with \( W = \{W_t\}_{t>0} \), Riesz transforms and certain Littlewood-Paley g-functions in [51], [52] and [54]. Here we consider, for every \( q > 1 \), the area g-function \( g_q^W \) associated with the heat semigroup \( W = \{W_t\}_{t>0} \), defined by

\[
g^q_W(f)(x) = \left\{ \int_{\Gamma(x)} \left| s \frac{\partial}{\partial s} W_s(f)(y) \right|_{s=t^2}^q \frac{dtdy}{t^2} \right\}^{1/q}, \quad x \in \mathbb{R},
\]

where \( \Gamma(x) = \{(y,t) \in \mathbb{R} \times (0, \infty) : |x-y| < t\} \), for every \( x \in \mathbb{R} \). We establish the following \( L^p \)-boundedness properties for this g-function.

**Proposition 1.2.** Let \( q \geq 2 \). Then \( g^q_W \) defines a bounded operator from \( L^p(\mathbb{R}) \) into itself, for every \( 1 < p < \infty \), and from \( L^1(\mathbb{R}) \) into \( L^{1,\infty}(\mathbb{R}) \).

We remark that [28, Theorem 4.8] can not be used to establish the \( L^p \)-boundedness of the operator defined by \( g^2_W \) because

\[
\int_\mathbb{R} \frac{\partial}{\partial s} W_s(x,y) |_{s=t^2} dy \neq 0, \quad x \in \mathbb{R} \text{ and } t > 0,
\]

(see [53, Proposition 5.1]).

The Laguerre differential operator \( L_\alpha, \alpha > -1/2 \), can be written by

\[
L_\alpha = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}(\alpha^2 - \frac{1}{4}) \right).
\]

This operator is positive and symmetric in the domain \( C_c^\infty(0, \infty) \) with respect to \( L^2(0, \infty) \). Here \( C_c^\infty(0, \infty) \) denotes the space of \( C^\infty \)-functions that have compact support on \( (0, \infty) \). For every \( n \in \mathbb{N} \), one has

\[
L_\alpha \varphi_n^\alpha = (2n + \alpha + 1) \varphi_n^\alpha,
\]

where

\[
\varphi_n^\alpha(x) = \left( \frac{2\Gamma(n+1)}{\Gamma(n+1+\alpha)} \right)^{1/2} e^{-x^2/2} x^{\alpha+\frac{1}{2}} L_n^\alpha(x^2), \quad x \in (0, \infty),
\]

and \( L_n^\alpha \) is the Laguerre polynomial of type \( \alpha \) ([55, p. 100] and [56, p. 7]). The heat diffusion semigroup \( W^\alpha = \{W_t^\alpha\}_{t>0} \) generated by \( L_\alpha \) is defined by

\[
W_t^\alpha(f) = \sum_{n=0}^{\infty} \epsilon_n^\alpha(f) e^{-(2n+\alpha+1)t} \varphi_n^\alpha, \quad f \in L^2(0, \infty),
\]

being

\[
\epsilon_n^\alpha(f) = \int_0^\infty \varphi_n^\alpha(x)f(x) \, dx, \quad n \in \mathbb{N}.
\]
According to the Mehler formula for Laguerre polynomials ([56, (1.1.47)]) we can write, for 
\( f \in L^2(0, \infty) \),
\[
W_\alpha^\alpha(f)(x) = \int_0^\infty W_\alpha^\alpha(x, y) f(y) \, dy, \quad x \in (0, \infty),
\]
where, for each \( x, y, t \in (0, \infty) \),
\[
W_\alpha^\alpha(x, y) = \left( \frac{2 e^{-t}}{1 - e^{-2t}} \right)^{1/2} \left( \frac{2 x y e^{-t}}{1 - e^{-2t}} \right)^{1/2} I_\alpha \left( \frac{2 x y e^{-t}}{1 - e^{-2t}} \right) e^{-\frac{1}{2} (x^2 + y^2) \frac{1 + e^{-2t}}{1 - e^{-2t}}}. \]
Here \( I_\alpha \) denotes the modified Bessel function of the first kind and order \( \alpha \).

Muckenhoupt ([35]) investigated harmonic analysis associated with Laguerre polynomials \( \{L_\alpha^n\}_{n \in \mathbb{N}} \). More recently, harmonic analysis operators in the \( L_\alpha \)-setting have been studied by several authors. Stempak [50], Macías, Segovia and Torrea [32] and Chicco Ruiz and Harboure [8] studied maximal operator for the heat semigroup \( \mathbb{W}^\alpha = \{W_\alpha^t\}_{t \geq 0} \). Riesz transforms associated with Laguerre functions was investigated by Harboure, Torrea and Viviani ([26]) and Harboure, Segovia, Torrea and Viviani ([24]). Also, the papers of Nowak ([38] and [39]) and Nowak and Stempak ([40] and [41]) are remarkable. In [23], Gutiérrez, Incognito and Torrea investigated Riesz transforms and certain Littlewood-Paley functions in the Laguerre context by exploiting a relation between \( n \)-dimensional Hermite polynomials and Laguerre polynomials when \( \alpha = \frac{n-1}{2} \). This idea was also used in [21] to study higher order Riesz transform associated with Laguerre polynomials. In [4], Betancor, Fariña, Rodríguez-Mesa, Sanabria and Torrea developed a new procedure to analyze operators in the Laguerre setting. The operator under consideration is decomposed into a local part and into a global part. The boundedness properties of the local operator are deduced from the boundedness properties of the corresponding operator in the Hermite setting. This transference procedure works for every value of \( \alpha \) and it uses properties of operators in the one dimension Hermite context in contrast with the method employed in [21] and [23]. In this paper we apply the procedure introduced in [4] to establish \( L^p \)-boundedness properties for the area \( g \)-functions \( g_\mathbb{W}^q, q > 1 \), associated with the heat Laguerre semigroup \( \mathbb{W}^\alpha = \{W_\alpha^t\}_{t \geq 0} \), and defined by
\[
g_\mathbb{W}^q(f)(x) = \left\{ \int_{\Gamma_+(x)} \left| \left( s \frac{\partial}{\partial s} W_\alpha^s(f)(y) \right) \right|_{s=t^2} \frac{q \, dt \, dy}{t^2} \right\}^{1/q}, \quad x \in (0, \infty),
\]
where \( \Gamma_+(x) = \{(y, t) \in (0, \infty) \times (0, \infty) : |y - x| < t\} \), for every \( x \in (0, \infty) \). We transfer the result obtained in Proposition 1.2 from \( g_\mathbb{W}^q \) to \( g_\mathbb{W}^q \), and we get the following.
Proposition 1.3. Let \( \alpha > -1/2 \) and \( q \geq 2 \). Then \( g^{\alpha}_{wq} \) defines a bounded operator from \( L^p(0, \infty) \) into itself, for every \( 1 < p < \infty \), and from \( L^1(0, \infty) \) into \( L^{1, \infty}(0, \infty) \).

We complete this result with the next reverse \( L^p \)-boundedness property for \( g^{\alpha}_{wq} \), \( 1 < q \leq 2 \).

Proposition 1.4. Let \( \alpha > -1/2 \) and \( 1 < q \leq 2 \). For every \( 1 < p < \infty \), there exists \( C > 0 \) for which

\[
\|f\|_{L^p(0, \infty)} \leq C \|g^{\alpha}_{wq}(f)\|_{L^p(0, \infty)}, \quad f \in L^p(0, \infty).
\]

By combining Propositions 1.3 and 1.4 we can obtain the following.

Corollary 1.5. Let \( \alpha > -1/2 \) and \( 1 < p < \infty \). Then, there exists \( C > 0 \) such that

\[
\frac{1}{C} \|f\|_{L^p(0, \infty)} \leq \|g^{\alpha}_{wq}(f)\|_{L^p(0, \infty)} \leq C \|f\|_{L^p(0, \infty)}, \quad f \in L^p(0, \infty).
\]

Hardy spaces associated with Schrödinger operators were studied by Dziubański and Zienkiewicz ([13], [14] and [15]). The Hermite operator is a special case of the operators considered by these authors. The Hardy space \( H^1_H(\mathbb{R}) \) in the Hermite setting consists of all those functions \( f \in L^1(\mathbb{R}) \) such that \( \sup_{t > 0} |W_t(f)| \in L^1(\mathbb{R}) \). The norm \( \cdot \| \cdot \|_{H^1_H(\mathbb{R})} \) in \( H^1_H(\mathbb{R}) \) is defined by

\[
\|f\|_{H^1_H(\mathbb{R})} = \left\| \sup_{t > 0} |W_t(f)| \right\|_{L^1(\mathbb{R})}, \quad f \in H^1_H(\mathbb{R}).
\]

Hardy spaces in the Laguerre context have been investigated by Dziubański ([11] and [12]). A function \( f \in L^1(0, \infty) \) is in the Hardy space \( H^1_{L_\alpha}(0, \infty) \) when \( \sup_{t > 0} |W^\alpha_t(f)| \in L^1(0, \infty) \). The norm \( \| \cdot \|_{H^1_{L_\alpha}(0, \infty)} \) is given by

\[
\|f\|_{H^1_{L_\alpha}(0, \infty)} = \left\| \sup_{t > 0} |W^\alpha_t(f)| \right\|_{L^1(0, \infty)}, \quad f \in H^1_{L_\alpha}(0, \infty).
\]

Recently, Betancor, Dziubański and Garrigós ([3]) have established a useful connection between the spaces \( H^1_H(\mathbb{R}) \) and \( H^1_{L_\alpha}(0, \infty) \). Let \( f \in L^1(0, \infty) \). We denote by \( f_\ominus \) the odd extension of \( f \) to \( \mathbb{R} \). Then, \( f \in H^1_{L_\alpha}(0, \infty) \) if, and only if, \( f_\ominus \in H^1_H(\mathbb{R}) \). Moreover, the quantities \( \|f\|_{H^1_{L_\alpha}(0, \infty)} \) and \( \|f\|_{H^1_H(\mathbb{R})} \) are equivalent.

It is known that a function \( f \in L^1(\mathbb{R}) \) is in \( H^1_H(\mathbb{R}) \) when and only when \( g^{\alpha}_{wq}(f) \in L^1(\mathbb{R}) \) ([49, Proposition 4, p. 124]). In the following we establish the corresponding result in the Laguerre setting.
Proposition 1.6. Let $\alpha > -1/2$. Then, if $f \in L^1(0, \infty)$, $f \in H_{L,\alpha}^1(0, \infty)$ if and only if,
\[ g_{W^{\alpha}}^2(f) \in L^1(0, \infty). \] Moreover, there exists $C > 0$ such that
\[ \frac{1}{C} \| f \|_{H_{L,\alpha}^1(0, \infty)} \leq \| f \|_{L^1(0, \infty)} + \| g_{W^{\alpha}}^2(f) \|_{L^1(0, \infty)} \leq C \| f \|_{H_{L,\alpha}^1(0, \infty)}, \quad f \in H_{L,\alpha}^1(0, \infty). \]

In [58] Xu studied the Littlewood-Paley theory for functions with values in Banach spaces. He characterized the uniform convexity and smoothness of the underlying Banach spaces by the validity in the vector valued setting of the $L^p$-inequalities for the Lusin area integral associated with the Poisson semigroup on the unit disc on the complex plane. The procedure developed in this paper allows us to obtain the results in Propositions 1.2, 1.3 and 1.4 when the heat semigroup is replaced by the Poisson semigroup in the Hermite and Laguerre setting. We can also obtain new characterizations of the uniform convexity and smoothness of a Banach space in terms of the $L^p$-inequalities of the area Littlewood-Paley functions for the Poisson semigroup in the Laguerre setting by using [33, Theorem 2.1] (see also [5]).

The next useful properties for the Bessel functions $I_\alpha$, $\alpha > -1/2$, can be found in [30, Ch. 5].

(I1) $I_\alpha(z) \sim z^\alpha$, as $z \to 0^+$.  

(I2) 
\[
\sqrt{z} I_\alpha(z) = \frac{1}{\sqrt{2\pi}} e^z \left( \sum_{k=0}^{n} (-1)^k [\alpha, k](2z)^{-k} + O((z^{-n-1}) \right), \quad \text{as } z \to \infty,
\]
where $[\alpha, 0] = 1$, and 
\[
[\alpha, k] = \frac{(4\alpha^2 - 1)(4\alpha^2 - 3\alpha^2)\cdots (4\alpha^2 - (2k-1)^2)}{2^{2k} \Gamma(k+1)}, \quad k = 1, 2, \ldots.
\]

(I3) $\frac{d}{dz}(z^{-\alpha}I_\alpha(z)) = z^{-\alpha}I_{\alpha+1}(z), \quad z \in (0, \infty).$

We also state here the following elementary properties which will be often used along the paper.

(P1) For every $x, y \in \mathbb{R}$ and $s > 0$, one has

(a) $(x - e^{-s}y)^2 + (y e^{-s} x)^2 = 2(x - y)^2 e^{-s} + (x^2 + y^2)(1 - e^{-s})^2$.

(b) $(x - e^{-s}y)^2 + (y e^{-s} x)^2 = (x - y)^2 (1 + e^{-2s}) + 2xy(1 - e^{-s})^2 = (x^2 + y^2)(1 + e^{-2s}) - 4xy e^{-s}$.

(c) $(x - e^{-s}y)^2 + (y - e^{-s} x)^2 \geq \frac{(x-y)^2}{2}.$

(P2) $\frac{2}{u} \leq \frac{1+e^{-u}}{1-e^{-u}} \leq \frac{2}{1-e^{-u}}, \quad u > 0.$

(P3) For each $a > 0$ there exists $c > 0$ such that $ue^{-au} \leq c(1 - e^{-u}), \quad u > 0.$

(P4) For every $a > 0$ and $b \geq 0$, it can be found $c > 0$ for which $u^b e^{-au} \leq c, \quad u > 0.$

Throughout this paper by $C$ we always denote a positive constant that can change from a line to another one.
2. Proof of Proposition 1.2

Let us fix \( q \geq 2 \). We will use the vector-valued Calderón-Zygmund theory to see the \( L^p \)-boundedness properties of the \( g \)-function \( g^q_W \).

We first observe that we can write, for every \( f \in L^q(\mathbb{R}) \),
\[
g^q_W(f)(x) = \left\| \left( s \frac{\partial}{\partial s} W_s(f + y) \right)_{s=t^2} \right\|_{L^q(\Gamma(0), \frac{dtdu}{t^2})}, \quad x \in \mathbb{R}.
\]

Moreover, if \( f \in L^q(\mathbb{R}) \) then
\[
\frac{\partial}{\partial s} W_s(f)(x) = \int_{-\infty}^{+\infty} \frac{\partial}{\partial s} W_s(x, y) f(y) dy, \quad x \in \mathbb{R} \text{ and } s > 0.
\]

Indeed, let \( f \in L^q(\mathbb{R}) \). By using \((1.1)\) we can write, for every \( x, y \in \mathbb{R} \) and \( s > 0 \),
\[
\frac{\partial}{\partial s} W_s(x, y) = -\frac{1}{2\sqrt{\pi}} e^{-\frac{(y-x)^2}{4(1-e^{-2s})}} \frac{e^{-s/2}}{(1-e^{-2s})^{3/2}}
\]
\[
\times \left\{ 1 + e^{-2s} + 2e^{-s}(y(x - e^{-s}y) + x(y - e^{-s}x)) - 2e^{-2s} \left( x - e^{-s}y \right)^2 + \frac{(y-e^{-s}x)^2}{1-e^{-2s}} \right\}.
\]

By taking into account \((P1)(a)\) and \((P4)\), a straightforward manipulation leads to
\[
\left| \frac{\partial}{\partial s} W_s(x, y) \right| \leq C e^{-\frac{(x-y)^2}{32s}} \frac{e^{-s/2}}{(1-e^{-2s})^{3/2}} \left( 1 + (|x| + |y|) e^{-s} (1 - e^{-2s})^{1/2} \right)
\]
\[
\leq C e^{-\frac{(x-y)^2}{32s}} \frac{e^{-s/2}}{(1-e^{-2s})^{3/2}} \quad x, y \in \mathbb{R} \text{ and } s > 0.
\]

Moreover, if \( s_0 > 0 \) is fixed, by \((P1)(c)\) and \((P2)\) we obtain, for each \( x, y \in \mathbb{R} \),
\[
\left| \frac{\partial}{\partial s} W_s(x, y) \right| \leq C e^{-\frac{(x-y)^2}{32s_0}} \frac{e^{-s/2}}{(1-e^{-2s})^{3/2}} \leq C e^{-\frac{(x-y)^2}{64s_0}} \frac{e^{-s_0/4}}{(1-e^{-s_0})^{3/2}}, \quad \frac{s_0}{2} < s < 2s_0.
\]

Since \( f \in L^q(\mathbb{R}) \), by the Hölder inequality one gets
\[
\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{64s_0}} \frac{e^{-s_0/4}}{(1-e^{-s_0})^{3/2}} |f(y)| dy < \infty, \quad x \in \mathbb{R}.
\]
Then, the mean value theorem and dominated convergence theorem lead to (2.1).

Let us define the operator \( T \) on \( L^q(\mathbb{R}) \) as follows
\[
[Tf(x)](y, t) = \int_{-\infty}^{+\infty} K(x, z)(y, t) f(z) dz, \quad x, y \in \mathbb{R} \text{ and } t > 0,
\]
where \( K(x, z), x, z \in \mathbb{R}, \) denotes the function defined on \( \Gamma(0) \) by
\[
K(x, z)(y, t) = \left( s \frac{\partial}{\partial s} W_s(x + y, z) \right)_{|s=t^2}.
\]
Note that for every $f \in L^q(\mathbb{R})$, $g^q_{W}(f)(x) = ||Tf(x)||_{L^q(\Gamma(0), \frac{dt}{t^2})}$, $x \in \mathbb{R}$.

Our objective then is to prove the following assertions:

(i) $T$ is a bounded operator from $L^q(\mathbb{R})$ into $L^q_{L^q(\Gamma(0), \frac{dt}{t^2})}(\mathbb{R})$, or equivalently, $g^q_{W}$ is bounded from $L^q(\mathbb{R})$ into itself.

(ii) For every $x, z \in \mathbb{R}, x \neq z$,

\[ ||K(x, z)||_{L^q(\Gamma(0), \frac{dt}{t^2})} \leq \frac{C}{|x - z|}, \text{ and} \]

\[ \left\| \frac{\partial}{\partial x} K(x, z) \right\|_{L^q(\Gamma(0), \frac{dt}{t^2})} + \left\| \frac{\partial}{\partial z} K(x, z) \right\|_{L^q(\Gamma(0), \frac{dt}{t^2})} \leq \frac{C}{|x - z|^2}. \]

(iii) For every $f \in L^q(\mathbb{R})$,

\[ Tf(x) = \int_{-\infty}^{+\infty} K(x, y)f(y)dy, \quad x \notin \text{supp } f. \]

Then the vector valued Calderón-Zygmund theory allows us to obtain the desired result.

Let us establish (i). Consider, for each $r > 1$, the operator $g_r$, given by

\[ g_r(f)(x) = \left\{ \int_{0}^{\infty} \left| \frac{\partial}{\partial t} W_t(f)(x) \right|^r \frac{dt}{t} \right\}^{1/r}, \quad x \in \mathbb{R}. \]

$L^p$-boundedness properties for $g_2$ were analyzed in [56] and also in [52]. In [52, Theorem 2.2] it was established that $f \rightarrow g_2(f)$ is bounded from $L^p(\mathbb{R})$ into itself, for every $1 < p < \infty$. Thus, in particular $g_2$ is a bounded operator from $L^q(\mathbb{R})$ into itself.

We can see that $g_q$ is also a bounded operator from $L^q(\mathbb{R})$ into itself. In effect, since $q \geq 2$ one has

\[ ||g_q(f)||_{L^q(\mathbb{R})}^2 \leq \int_{-\infty}^{+\infty} \int_{0}^{\infty} \left| \frac{\partial}{\partial t} W_t(f)(x) \right|^2 \frac{dt}{t} \left( \sup_{t > 0} \left| \frac{\partial}{\partial t} W_t(f)(x) \right| \right)^{q-2} dx. \]

Also, from (2.3) and using (P1)(c) and (P3) we obtain

\[ \left| \frac{t}{\partial t} W_t(f)(x) \right| \leq C \int_{-\infty}^{+\infty} |f(y)| \left( \frac{t^{3/2}e^{-t/2}}{(1 - e^{-2t})^{3/2}} \right) dy \leq CW_*(|f|(x), \quad x \in \mathbb{R}, \]

where $W_*$ represents the maximal operator associated to the heat semigroup for the Laplacian operator, that is,

\[ W_*(f)(x) = \sup_{t > 0} \left| \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} f(y) e^{-\frac{(x-y)^2}{4t}} dy \right|. \]
Then, by the Hölder inequality and the fact that $g_2$ and $W_s$ are bounded operators from $L^q(\mathbb{R})$ into itself, we get

$$
\|g_q(f)\|^q_{L^q(\mathbb{R})} \leq C \int_{-\infty}^{+\infty} (g_2(f)(x))^2 (W_s(|f|)(x))^{q-2} dx
\leq C \left( \int_{-\infty}^{+\infty} |g_2(f)(x)|^q dx \right)^{2/q} \left( \int_{-\infty}^{+\infty} (W_s(|f|)(x)) q dx \right)^{(q-2)/q}
= C \|g_2(f)\|^2_{L^q(\mathbb{R})} \|W_s(|f|)\|^2_{L^q(\mathbb{R})} \leq C \|f\|^q_{L^q(\mathbb{R})}.
$$

Finally, we observe that

$$
\|g_2^q(f)\|^q_{L^q(\mathbb{R})} = \int_0^{+\infty} \int_{-\infty}^{+\infty} \left| \left( s \frac{\partial}{\partial s} W_s(f)(y) \right)_{s=t^2} \right|^q \frac{dxdy}{t^2}
= 2 \int_0^{+\infty} \int_{-\infty}^{+\infty} \left| \left( u \frac{\partial}{\partial u} W_u(f)(y) \right)_{u=t^2} \right|^q \frac{dy}{u} \frac{dt}{t}.
$$

By making the change of variables $u = t^2$, one gets

$$
\|g_2^q(f)\|^q_{L^q(\mathbb{R})} = \int_0^{+\infty} \int_{-\infty}^{+\infty} \left| u \frac{\partial}{\partial u} W_u(f)(y) \right|^q \frac{dy}{u} \frac{dt}{t} = \|g_q(f)\|^q_{L^q(\mathbb{R})}.
$$

Thus, (i) is proved.

We now establish (ii). First, let us prove that

$$(2.4) \quad \|K(x, z)\|_{L^q(\Gamma \cap \Omega_0, \frac{dxdy}{t^2})} = \left\{ \int_{\Gamma(x)} \left| \left( s \frac{\partial}{\partial s} W_s(y, z) \right)_{s=t^2} \right|^q \frac{dxdy}{t^2} \right\}^{1/q} \leq \frac{C}{|x-z|}, \quad x, z \in \mathbb{R}, \ x \neq z.
$$

From (2.3) and taking into account (P1)(c), (P3) and (P4) we can write

$$
\int_{\Gamma(x)} \left| s \frac{\partial}{\partial s} W_s(y, z) \right|_{s=t^2} \frac{dxdy}{t^2} \leq C \int_{-\infty}^{+\infty} \int_{|x-y|}^{+\infty} t^{q-2} e^{-\frac{2(x-y)^2}{32x^2}} \frac{e^{-\frac{y^2}{2}}}{(1-e^{-2t^2})^\frac{3}{2}} dt dy
\leq C \int_{-\infty}^{+\infty} \int_{|x-y|}^{+\infty} (t + |z-y|)^{q+1} dy \leq C \int_{-\infty}^{+\infty} \frac{1}{(|x-y| + |z-y|)^{q+1}} dy
\leq C \left( \frac{1}{|z-x|^{q+1}} + \int_{\mathbb{R}\setminus I_{x,z}} \frac{1}{|2y-x-z|^{q+1}} dy \right) \leq \frac{C}{|x-z|^q}, \quad x, z \in \mathbb{R}, \ x \neq z.
$$

Here $I_{x,z}$ represents the interval $I_{x,z} = (\min\{x, z\}, \max\{x, z\})$. Thus (2.4) is established.
We now see that
\begin{equation}
\left\| \frac{\partial}{\partial x} K(x, z) \right\|_{L^q(\Gamma(0), \frac{dt \, dy}{t^2})} + \left\| \frac{\partial}{\partial z} K(x, z) \right\|_{L^q(\Gamma(0), \frac{dt \, dy}{t^2})} \leq \frac{C}{|x - z|^2}, \quad x, z \in \mathbb{R}, \ x \neq z.
\end{equation}

We will show that, when \( x, z \in \mathbb{R}, x \neq z \),
\begin{equation}
\left\| \frac{\partial}{\partial x} K(x, z) \right\|_{L^q(\Gamma(0), \frac{dt \, dy}{t^2})} = \left\{ \int_{\Gamma(x)} \left| \left( s \frac{\partial^2}{\partial y \partial s} W_s(y, z) \right) \right|_{s=\tau^2} \frac{q \, dt \, dy}{t^2} \right\}^{1/q} \leq \frac{C}{|x - z|^2}.
\end{equation}
The analogous property for \( \left\| \frac{\partial}{\partial z} K(x, z) \right\|_{L^q(\Gamma(0), \frac{dt \, dy}{t^2})} \) can be established in a similar way.

From (2.2) we have that, for every \( y, z \in \mathbb{R} \) and \( s > 0 \),
\[
\frac{\partial^2}{\partial y \partial s} W_s(y, z) = -\frac{(y - e^{-s}z) - (z - e^{-s}y)e^{-s}}{1 - e^{-2s}} \frac{\partial}{\partial s} W_s(y, z)
\]
\[
- \frac{2}{\sqrt{\pi}} e^{\frac{(y - e^{-s}z)^2 + (z - e^{-s}y)^2}{2(1 - e^{-2s})}} e^{-s/2} \left( e^{-s}(z - e^{-s}y) - \frac{e^{-2s}y - e^{-s}z - e^{-s}(z - e^{-s}y)}{1 - e^{-2s}} \right).
\]

Hence, by using (2.3) we obtain
\begin{equation}
\left\| \frac{\partial^2}{\partial y \partial s} W_s(y, z) \right\| \leq C e^{-\frac{(y - e^{-s}z)^2 + (z - e^{-s}y)^2}{16(1 - e^{-2s})}} e^{-s/2} \frac{(1 - e^{-2s})^2}{(1 - e^{-2s})^2}, \quad y, z \in \mathbb{R}, \ s > 0,
\end{equation}
and by proceeding as in the proof of (2.4) we conclude (2.6).

To finish we prove (iii). Let \( f \in L^q(\mathbb{R}) \) and let \( K = \text{supp} \ f \). According to (ii) and the Minkowski inequality we can write
\[
\left\| \int_{-\infty}^{+\infty} K(x, z) f(z)dz \right\|_{L^q(\Gamma(0), \frac{dt \, dy}{t^2})} \leq \int_{-\infty}^{+\infty} \| K(x, z) \|_{L^q(\Gamma(0), \frac{dt \, dy}{t^2})} \| f(z) \|dz
\]
\[
\leq C \int_{-\infty}^{+\infty} \frac{1}{|x - z|} \| f(z) \|dz \leq C \left( \int_{\mathcal{K}} \frac{1}{|x - z|^q} dz \right)^{1/q} \| f \|_{L^q(\mathbb{R})} < \infty,
\]
for every \( x \notin \mathcal{K} \). Assume that \( g \in L^{q'} \left( \Gamma(0), \frac{dt \, dy}{t^2} \right) \). We have
\[
\int_{\Gamma(0)} g(y, t) \left( \int_{-\infty}^{+\infty} K(x, z) f(z)dz \right) (y, t) \frac{dt \, dy}{t^2} = \int_{-\infty}^{+\infty} f(z) \int_{\Gamma(0)} K(x, z) (y, t)g(y, t) \frac{dt \, dy}{t^2} dz
\]
\[
= \int_{\Gamma(0)} g(y, t) \int_{-\infty}^{+\infty} K(x, z) (y, t) f(z)dz \frac{dt \, dy}{t^2}, \quad x \notin \mathcal{K}.
\]
These equalities are justified by taking into account that
\[
\int_{\Gamma(0)} |g(y, t)| \left( \int_{-\infty}^{+\infty} |K(x, z) (y, t)| |f(z)|dz \right) \frac{dt \, dy}{t^2}
\]
\[
\leq ||g||_{L^{q'}(\Gamma(0), \frac{dt \, dy}{t^2})} ||f||_{L^q(\mathbb{R})} \left( \int_{\mathcal{K}} \frac{1}{|x - z|^q} dz \right)^{1/q'} < \infty, \quad x \notin \mathcal{K}.
\]
Hence

\[ Tf(x) = \int_{-\infty}^{+\infty} K(x, z)f(z)dz, \quad x \notin \mathcal{K}. \]

3. Proof of Proposition 1.3

In this section we exploit the arguments developed in [4] where the basic idea is to compare in some region the \( q \)-functions \( g_Q^q \), and \( g_{\tilde{Q}}^q \) and then transfer \( L^p \)-boundedness properties from \( g_Q^q \) to \( g_{\tilde{Q}}^q \). We first observe that if \( f_o \) denotes the odd extension to \( \mathbb{R} \) of a suitable function \( f \) defined on \( (0, \infty) \), we can write

\[ W_s(f_o)(x) = \int_0^\infty (W_s(x, y) - W_s(x, -y))f(y)dy, \quad x \in \mathbb{R} \text{ and } s > 0. \]  

Then \( W_s(f_o) \) is odd and \( g_{\tilde{Q}}^q(f_o) \) is even. Thus we get

\[
\begin{align*}
g_{\tilde{Q}}^q(f_o)(x) &= g_{\tilde{Q}}^q(f_o)(|x|) = \left\{ \int_{\Gamma(|x|)} \left| s \frac{\partial}{\partial s} W_s(f_o)(y)_{|s=t^2} \right|^q \frac{dt \, dy}{t^2} \right\}^{1/q} \\
&\leq \left\{ 2 \int_{\Gamma_+(|x|)} \left| s \frac{\partial}{\partial s} W_s(f_o)(y)_{|s=t^2} \right|^q \frac{dt \, dy}{t^2} \right\}^{1/q}, \quad x \in \mathbb{R}.
\end{align*}
\]

Moreover, it is clear that

\[
\left\{ \int_{\Gamma_+(|x|)} \left| s \frac{\partial}{\partial s} W_s(f_o)(y)_{|s=t^2} \right|^q \frac{dt \, dy}{t^2} \right\}^{1/q} \leq g_{\tilde{Q}}^q(f_o)(x), \quad x \in \mathbb{R}.
\]

Hence, according to Proposition 1.2, the operator \( g_{\tilde{Q}}^q \) defined by

\[ g_{\tilde{Q}}^q(f)(x) = \left\{ \int_{\Gamma_+(|x|)} \left| s \frac{\partial}{\partial s} W_s(f_o)(y)_{|s=t^2} \right|^q \frac{dt \, dy}{t^2} \right\}^{1/q}, \quad x \in (0, \infty), \]

is bounded from \( L^p(0, \infty) \) into itself, for every \( 1 < p < \infty \), and from \( L^1(0, \infty) \) into \( L^1(0, \infty) \).

Then, the proof of Proposition 1.3 will be finished when we establish the \( L^p \)-boundedness properties for the operator \( D_{q,\alpha} = g_{\tilde{Q}}^q - g_{\tilde{Q}}^{q+} \). Let us describe the main steps we follow.

By taking into account (3.1) and using triangular inequality in \( L^q(\Gamma(0), \frac{dt \, dy}{t^2}) \) we write

\[
\begin{align*}
|D_{q,\alpha}(f)(x)| &\leq \left\{ \int_{\Gamma_+(|x|)} \left| \int_0^\infty \left[ s \frac{\partial}{\partial s} (W_s^\alpha(y, z) - W_s(y, z)) \right]_{|s=t^2} f(z)dz \right|^q \frac{dt \, dy}{t^2} \right\}^{1/q} \\
&+ \left\{ \int_{\Gamma_+(|x|)} \left| \int_0^\infty \left[ s \frac{\partial}{\partial s} W_s(y, -z) \right]_{|s=t^2} f(z)dz \right|^q \frac{dt \, dy}{t^2} \right\}^{1/q}, \quad x \in (0, \infty).
\end{align*}
\]
We split the first integral in three parts and, by using Minkowsky inequality, we get

\[
|D_{q,\alpha}(f)(x)| \leq \left\{ \int_{\Gamma_+ (x)} \left( \int_0^x + \int_0^{2x} + \int_0^\infty \right) \left[ s \frac{\partial}{\partial s} (W_s^\alpha(y, z) - W_s(y, z)) \right] \left| f(z) \right| dz \right\}^{1/q} \left\{ \frac{q \, dtdy}{t^2} \right\}
\]

\[
+ \left\{ \int_{\Gamma_+ (x)} \left[ \int_0^\infty \left( s \frac{\partial}{\partial s} W_s(y, z) \right) \left| f(z) \right| dz \right] \right\}^{1/q} \left\{ \frac{q \, dtdy}{t^2} \right\}
\]

\[
\leq \left( \int_{(0, \infty) \setminus (\frac{x}{2}, 2x)} \left[ \int_{\Gamma_+ (x)} \left( s \frac{\partial}{\partial s} W_s^\alpha(y, z) \right) \left| f(z) \right| dz \right] \right\}^{1/q} \left\{ \frac{q \, dtdy}{t^2} \right\}
\]

\[
+ \left\{ \int_{\Gamma_+ (x)} \left[ \int_0^{\infty} \left( s \frac{\partial}{\partial s} W_s(y, z) \right) \left| f(z) \right| dz \right] \right\}^{1/q} \left\{ \frac{q \, dtdy}{t^2} \right\}
\]

\[
+ \left\{ \int_0^{\infty} \left[ \int_{\Gamma_+ (x)} \left( s \frac{\partial}{\partial s} (W_s^\alpha(y, z) - W_s(y, z)) \right) \left| f(z) \right| dz \right] \right\}^{1/q} \left\{ \frac{q \, dtdy}{t^2} \right\}
\]

\[
= A_1(|f|)(x) + A_2(|f|)(x) + A_3(|f|)(x), \quad x \in (0, \infty).
\]

Suitable estimates for the kernels of the operators $A_1$ and $A_2$ allow us to mayore $A_1$ and $A_2$ by certain Hardy type operators. Then the boundedness $L^p$-properties of $A_1$ and $A_2$ are obtained from the corresponding ones for Hardy type operators (see for instance, [59, p. 20]).

To prove that $A_3$ is bounded from $L^p(0, \infty)$ into itself, for each $1 \leq p \leq \infty$, we will find a nonnegative function $H(x, z)$, $0 < \frac{x}{2} < z < 2x < \infty$, verifying that

\[
\left\{ \int_{\Gamma_+ (x)} \left[ \left( s \frac{\partial}{\partial s} W_s^\alpha(y, z) - W_s(y, z) \right) \right] \left| f(z) \right| dz \right\}^{1/q} \left\{ \frac{q \, dtdy}{t^2} \right\} \leq H(x, z),
\]

and such that the operator $\mathfrak{B}$ defined by

\[
\mathfrak{B}(f)(x) = \int_{\frac{x}{2}}^{2x} H(x, z) f(z) dz
\]

is bounded from $L^p(0, \infty)$ into itself, for every $1 \leq p \leq \infty$.

Next we study the operators that we have defined above.
First, we obtain some estimations which will be used later. According to (I3) we can write, for every $s, y, z \in (0, \infty)$,

\[
\frac{\partial}{\partial s} W_\alpha^s(y, z) = \frac{\partial}{\partial s} \left[ 2^{\alpha+1} (yz)^{\alpha+1/2} \left( e^{-s} \right)^{\alpha+1} e^{-\frac{1}{2}(y^2+z^2) \frac{1+c-2s}{1-e^{-2s}}} \left( \frac{2yze^{-s}}{1-e^{-2s}} \right)^{-\alpha} \left( \frac{2yze^{-s}}{1-e^{-2s}} \right) \right]
\]

\[
\begin{align*}
(3.3) &= 2^{\alpha+1} (yz)^{\alpha+1/2} e^{-\frac{1}{2}(y^2+z^2) \frac{1+c-2s}{1-e^{-2s}}} \left[ - (\alpha+1) 1 + e^{-2s} \right] + 2(y^2+z^2) (e^{-s})^2 \\
&\quad \times \left( \frac{e^{-s}}{1-e^{-2s}} \right)^{\alpha+1} \left( \frac{2yze^{-s}}{1-e^{-2s}} \right)^{-\alpha} \left( \frac{2yze^{-s}}{1-e^{-2s}} \right) \\
&\quad \times \left( \frac{e^{-s}}{1-e^{-2s}} \right)^{\alpha+1} \left( \frac{2yze^{-s}}{1-e^{-2s}} \right)^{-\alpha} \left( \frac{2yze^{-s}}{1-e^{-2s}} \right) \\
&\quad \times \left( \frac{e^{-s}}{1-e^{-2s}} \right)^{\alpha+1} \left( \frac{2yze^{-s}}{1-e^{-2s}} \right)^{-\alpha} \left( \frac{2yze^{-s}}{1-e^{-2s}} \right)
\end{align*}
\]

From (I1) and (P2) one has, for every $s, y, z \in (0, \infty)$ such that $\frac{e^{-s}yz}{1-e^{-2s}} \leq 1$,

\[
(3.4) \left| \frac{\partial}{\partial s} W_\alpha^s(y, z) \right| \leq C (yz)^{\alpha+1/2} e^{-\frac{1}{2}(y^2+z^2) \frac{1+c-2s}{1-e^{-2s}}} \\
\times \left\{ \left[ \frac{1}{1-e^{-2s}} + (y^2+z^2) \left( \frac{1}{1-e^{-2s}} \right)^2 \right] \left( \frac{e^{-s}}{1-e^{-2s}} \right)^{\alpha+1} \\
+ \ (yz)^2 \left( \frac{e^{-s}}{1-e^{-2s}} \right)^{\alpha+3} \left( \frac{1}{1-e^{-2s}} \right) \right\} \\
\leq C (yz)^{\alpha+1/2} e^{-\frac{y^2+z^2}{8s} \frac{e^{-s}}{1-e^{-2s}}} \ e^{-\frac{(\alpha+1)s}{2(1-e^{-2s})}}
\]

Moreover, by using (I2) for $n = 0$, it follows that, for every $s, y, z \in (0, \infty)$ such that $\frac{e^{-s}yz}{1-e^{-2s}} \geq 1$,

\[
(3.5) \left| \frac{\partial}{\partial s} W_\alpha^s(y, z) \right| \leq C (yz)^{\alpha+1/2} e^{-\frac{1}{2}(y^2+z^2) \frac{1+c-2s}{1-e^{-2s}}+2yz e^{-s}} \\
\times \left\{ (yz)^{-\alpha-1/2} \left( \frac{e^{-s}}{1-e^{-2s}} \right)^{1/2} \left[ \frac{1}{1-e^{-2s}} + (y^2+z^2) \left( \frac{e^{-s}}{1-e^{-2s}} \right)^2 \right] \\
+ \ (yz)^{-\alpha+1/2} \left( \frac{e^{-s}}{1-e^{-2s}} \right)^{3/2} \left( \frac{1}{1-e^{-2s}} \right) \right\}
\]
\[
\frac{\partial}{\partial s} W_s^\alpha(y, z) = -\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(y^2+z^2)\frac{1+e^{-2s}}{1-e^{-2s}} + 2yz e^{-s}} e^{-s/2} 
\]
\[
\times \left\{ \left[ (\alpha + 1)(1 + e^{-2s}) - \frac{[\alpha + 1, 1]}{2} (1 + e^{-2s})(1 - e^{-2s}) \frac{y^2 + z^2}{4yze^{-s}} \right] + O\left( \frac{1 - e^{-2s}}{yze^{-s}} \right) \right\} 
\]
\[
= \frac{\partial}{\partial s} W_s(y, z) - \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(y^2+z^2)\frac{1+e^{-2s}}{1-e^{-2s}} + 2yz e^{-s}} e^{-s/2} 
\]
\[
\times \left\{ \left[ \frac{\alpha + 1}{2} (1 + e^{-2s}) - \frac{[\alpha + 1, 1]}{2} (1 + e^{-2s})(1 - e^{-2s}) \frac{y^2 + z^2}{4yze^{-s}} \right] + O\left( \frac{1 - e^{-2s}}{yze^{-s}} \right) \right\} 
\]
\[
= \frac{\partial}{\partial s} W_s(y, z) - \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(y^2+z^2)\frac{1+e^{-2s}}{1-e^{-2s}} + 2yz e^{-s}} e^{-s/2} 
\]
\[
\times \left\{ \frac{[\alpha + 1]}{2} (1 - e^{-s})^2 + O\left( \frac{(y - z)^2 e^{-s}}{yz} \right) + O\left( \frac{1 - e^{-2s}}{yze^{-s}} \right) \right\}. 
\]

Then, by using again (P1)(b) and (P2) we get, for every \( s, y, z \in (0, \infty) \) such that \( \frac{e^{-s}yz}{1-e^{-2s}} \geq 1 \),
\[
\left| \frac{\partial}{\partial s} W_s^\alpha(y, z) - \frac{\partial}{\partial s} W_s(y, z) \right| 
\]
\[
\leq C e^{\frac{-(y^2+z^2)(1+e^{-2s})}{2(1-e^{-2s})} - \frac{yz(1-e^{-s})}{2}} \left\{ (1 - e^{-s})^{1/2} + \frac{(y - z)^2 e^{-3s/2}}{yz(1-e^{-2s})^{3/2}} + \frac{e^{s/2}}{yz(1-e^{-s})^{1/2}} \right\} 
\]
\[
\leq C e^{\frac{-(y^2+z^2)}{2s}} \frac{e^{s/2}}{yz(1-e^{-s})^{1/2}}. 
\]

We now study operators \( A_1 \) and \( A_2 \). Let \( M \) and \( N \) be the functions defined by
\[
M(x, z) = \left\{ \int_{\Gamma_s(x)} \left| s \frac{\partial}{\partial s} W_s(y, z) \right|_{s=t^2} q \frac{dt}{t^2} \right\}^{1/q}, \quad x, z \in (0, \infty). 
\]
and
\[ N(x, z) = \left\{ \int_{\Gamma_+(x) \cap L(z)} \left| \left( s \frac{\partial}{\partial s} W_s^\alpha(y, z) \right) \right|_{s=t^2} \left| \frac{q \, dt \, dy}{t^2} \right| \right\}^{1/q}, \quad x, z \in (0, \infty), \]

By using (2.4) one gets
\[ |M(x, z)| \leq \frac{C}{|x - z|} \leq C \begin{cases} \frac{1}{x}, & 0 < z < x/2, \\ \frac{1}{z}, & 0 < 2x < z, \end{cases} \]
and that,
\[ |M(x, -z)| \leq \frac{C}{x + z}, \quad x, z \in (0, \infty). \]

We also claim that
\[ |N(x, z)| \leq C \begin{cases} \frac{1}{x}, & 0 < z < x/2, \\ \frac{1}{z}, & 0 < 2x < z. \end{cases} \]

Thus, we can write
\[ A_1(|f|)(x) + A_2(|f|)(x) \leq C \left( \frac{1}{x} \int_0^x |f(z)| \, dz + \int_x^\infty \frac{|f(z)|}{z} \, dz \right), \quad x \in (0, \infty). \]

Wellknown properties of Hardy operators ([59, p. 20]) allow us to establish that \( A_1 \) and \( A_2 \) are bounded operators from \( L^p(0, \infty) \) into itself, for every \( 1 < p < \infty \), and from \( L^1(0, \infty) \) into \( L^{1,\infty}(0, \infty) \).

Let us establish (3.7). Denote by \( L(z) \) and \( R(z) \), \( z \in (0, \infty) \), the sets
\[ L(z) = \left\{ (y, t) \in (0, \infty) \times (0, \infty) : \frac{yze^{-t^2}}{1 - e^{-2t^2}} \leq 1 \right\}, \]
and
\[ R(z) = \left\{ (y, t) \in (0, \infty) \times (0, \infty) : \frac{yze^{-t^2}}{1 - e^{-2t^2}} \geq 1 \right\}. \]

We have that \( N(x, z) \leq N_1(x, z) + N_2(x, z) \), \( x, z \in (0, \infty) \), where
\[ N_1(x, z) = \left\{ \int_{\Gamma_+(x) \cap L(z)} \left| \left( s \frac{\partial}{\partial s} W_s^\alpha(y, z) \right) \right|_{s=t^2} \left| \frac{q \, dt \, dy}{t^2} \right| \right\}^{1/q}, \quad x, z \in (0, \infty), \]
and
\[ N_2(x, z) = \left\{ \int_{\Gamma_+(x) \cap R(z)} \left| \left( s \frac{\partial}{\partial s} W_s^\alpha(y, z) \right) \right|_{s=t^2} \left| \frac{q \, dt \, dy}{t^2} \right| \right\}^{1/q}, \quad x, z \in (0, \infty). \]
From (3.4) and using (P3) and (P4) one gets
\[
N_1(x, z) \leq C z^{\alpha+1/2} \left\{ \int_0^\infty \int_{|x-y|}^\infty y^{\alpha+1/2} e^{-\frac{q(y-z)^2}{4t^2}} \frac{e^{-qt^2/2} t^{2q-2}}{(1 - e^{-2t^2})^{q(\alpha+1)}} dt dy \right\}^{1/q}
\]
\[
\leq C z^{\alpha+1/2} \left\{ \int_0^\infty \int_{|x-y|}^\infty y^{\alpha+1/2} \frac{1}{(t^2 + y^2 + z^2)^{q(\alpha+1)+1}} \left( \frac{t^2 e^{-\frac{\alpha+1}{2} t^2}}{1 - e^{-2t^2}} \right)^{q(\alpha+2)} dt dy \right\}^{1/q}
\]
\[
\leq C z^{\alpha+1/2} \left\{ \left( \int_0^x + \int_x^\infty \right) \frac{1}{((x-y) + y + z)^{q(\alpha+3/2)+1}} dy \right\}^{1/q}
\]
\[
\leq C \frac{z^{\alpha+1/2}}{(x+z)^{\alpha+3/2}} \leq C \begin{cases} 
  \frac{x^{\alpha+1/2}}{x^{\alpha+3/2}}, & 0 < z < x, \\
  \frac{1}{z}, & 0 < x < z.
\end{cases}
\]
(3.10)

On the other hand, let \( M_{x,z} = \max\{x, z\} \) and \( m_{x,z} = \min\{x, z\} \). By taking into account properties (P1)(c), (P2), (P3) and (P4) we obtain, from (3.5)
\[
N_2(x, z) \leq C \left\{ \int_0^\infty \int_{|x-y|}^\infty e^{-\frac{q(y-z)^2}{4t^2}} \frac{e^{-3qt^2/2} t^{2q-2}}{(1 - e^{-2t^2})^{5q/2}} (y^2 + z^2)^q dt dy \right\}^{1/q}
\]
\[
\leq C \left\{ \int_0^\infty \int_{|x-y|}^\infty \frac{(y^2 + z^2)^q}{(t^2 + |y-z|^2)^{3q/2+1}} dt dy \right\}^{1/q} \leq C \left\{ \int_0^\infty \frac{(y+z)^2}{(|x-y| + |y-z|)^{3q+1}} dy \right\}^{1/q}
\]
\[
\leq C \left( \frac{M^2_{x,z}}{(x-z)^3} \int_{m_{x,z}}^{m_{x,z}} \frac{1}{(x-z-2y)^{3q+1}} dy \right)^{1/q} + M^2_{x,z} \left( \int_{m_{x,z}}^{M_{x,z}} \frac{1}{|x-z|^3} dy \right)^{1/q}
\]
\[
\leq C M^2_{x,z} \left( \int_{m_{x,z}}^{M_{x,z}} \frac{1}{(2y-x-z)^{3q+1}} dy \right)^{1/q}
\]
\[
(3.11) \leq C \frac{M^2_{x,z}}{|x-z|^3} \leq C \begin{cases} 
  \frac{1}{x}, & 0 < z < \frac{x}{2}, \\
  \frac{1}{z}, & 0 < 2x < z.
\end{cases}
\]

In the fourth inequality we have used that, for each \( x, z \in (0, \infty) \), \( h_{x,z}(y) = \frac{y+z}{2y-x-z} \), is a decreasing function on \((0, \infty)\). Hence, \( h_{x,z}(y) \leq h_{x,z}(M_{x,z}) \leq 2 \frac{M_{x,z}}{|x-z|} \), when \( y \geq M_{x,z} \).

Estimations (3.10) and (3.11) lead to (3.7).
Finally we study the operator $A_3$. We need to estimate the function

$$G(x, z) = \left\{ \int_{\Gamma_+(x)} \left[ \frac{\partial}{\partial s} W^\alpha_s(y, z) - W_s(y, z) \right]_{s=t^2} |dtdy|^{\frac{1}{q}} \right\}^{1/q} , \quad 0 < \frac{x}{2} < z < 2x.$$  

By using again the sets $L(z)$ and $R(z)$, $z \in (0, \infty)$ (see (3.8) and (3.9)) we write

$$G(x, z) \leq C \left[ \int_{\Gamma_+(x) \cap L(z)} \left[ \frac{s}{s} W^\alpha_s(y, z) \right]_{s=t^2} |dtdy|^{\frac{1}{q}} \right]^{1/q} + \int_{\Gamma_+(x) \cap L(z)} \left[ \frac{\partial}{\partial s} W_s(y, z) \right]_{s=t^2} |dtdy|^{\frac{1}{q}} \right]^{1/q} + \int_{\Gamma_+(x) \cap R(z)} \left[ \frac{\partial}{\partial s} (W^\alpha_s(y, z) - W_s(y, z)) \right]_{s=t^2} |dtdy|^{\frac{1}{q}} \right]^{1/q}$$  

$$= C[N_1(x, z) + G_2(x, z) + G_3(x, z)], \quad 0 < \frac{x}{2} < z < 2x.$$  

From (3.10) we have that

$$(3.12) \quad N_1(x, z) \leq \frac{C}{z}, \quad 0 < \frac{x}{2} < z < 2x.$$  

On the other hand, by (2.3) and (P2) it follows that, when $(y, s) \in L(z)$,

$$(3.13) \quad \left| \frac{\partial}{\partial s} W_s(y, z) \right| \leq Ce^{-(y^2+z^2)} \frac{1+e^{-2s}}{e^{-s/2}} \frac{e^{-s/2}}{(1-e^{-2s})^{3/2}} \leq Ce^{-\frac{y^2+z^2}{s}} \frac{e^{-s/2}}{(1-e^{-2s})^{3/2}}.$$  

We note that the right side of (3.13) coincides with the right hand side of (3.4) when $\alpha = -1/2$. Hence by proceeding as in the proof of (3.10) we conclude that

$$(3.14) \quad G_2(x, z) \leq \frac{C}{z}, \quad 0 < \frac{x}{2} < z < 2x.$$  

Finally, considering (3.6) and again (P3) and (P4), we get, when $0 < \frac{x}{2} < z < 2x$,

\[
G_3(x, z) \leq C \left\{ \int_{\Gamma_+} \left( \frac{e^{t^2/2}e^{-t^2/2}}{zy(1-e^{-t^2})^{1/2}} \right)^q t^{2q-2} dt \right\}^{1/q} 
\]

\[
\leq C \left\{ \int_0^{\frac{z}{2}} \int_{|x-y|}^{\infty} e^{-\frac{(y-z)^2}{2t^2}} \left( \frac{e^{-t^2/2}}{(1-e^{-t^2})^{3/2}} \right)^q t^{2q-2} dt \right\}^{1/q} 
\]

\[
+ \int_{\frac{z}{2}}^{\infty} \int_{|x-y|}^{\infty} e^{-\frac{(y-z)^2}{2t^2}} \left( \frac{e^{-t^2/4}}{z(1-e^{-t^2})^{5/4}} \right)^q t^{2q-2} dt \right\}^{1/q} 
\]

\[
\leq C \left\{ \int_0^{\frac{z}{2}} \int_{|x-y|}^{\infty} \frac{1}{(t^2 + (y-z)^2)^{q/2+1}} dt \right\}^{1/q} 
\]

\[
\leq C \left\{ \int_0^{\frac{z}{2}} \frac{1}{(x-z)^2} \frac{1}{(x+y-z)^{q+1}} dy + \int_0^{\frac{z}{2}} \frac{1}{|x-y|} \frac{1}{(|y-z|+|x-y|)^{q/2+1}} dy \right\}^{1/q} 
\]

\[
\leq C \left\{ \frac{1}{z^{q/2}} + \frac{1}{z^{q/2}|x-z|^q} \right\}^{1/q}. 
\]

The second term in the last inequality can be obtained in the same way as in the proof of (2.4).

Then, we have

\[
G_3(x, z) \leq C \left( 1 + \sqrt{z} \right), \quad 0 < \frac{x}{2} < z < 2x.
\]

and by (3.12), (3.14) and (3.15) we conclude that

\[
G(x, z) \leq C \left( 1 + \sqrt{z} \right), \quad 0 < \frac{x}{2} < z < 2x.
\]

The operator $\mathfrak{B}$ defined by

\[
\mathfrak{B}(f)(x) = \int_\frac{z}{2}^{2x} \frac{1}{z} \left( 1 + \sqrt{\frac{z}{|z-x|}} \right) f(z) dz, \quad x \in (0, \infty),
\]

is bounded from $L^p(0, \infty)$ into itself for every $1 \leq p \leq \infty$. Indeed, note that

\[
\int_\frac{z}{2}^{2x} \frac{1}{z} \left( 1 + \sqrt{\frac{z}{|z-x|}} \right) \frac{1}{u} \left( 1 + \sqrt{\frac{u}{|1-u|}} \right) du > 0, \quad x \in (0, \infty).
\]
Jensen’s inequality allows us to show the boundedness on $L^p(0, \infty)$, $1 \leq p \leq \infty$, of the operator $B$. Hence the operator $A_3$ is bounded from $L^p(0, \infty)$ into itself, for every $1 \leq p \leq \infty$. The proof is thus completed.

4. PROOF OF PROPOSITION 1.4

In order to prove Proposition 1.4 we first establish the following property.

**Proposition 4.1.** Let $1 < p < \infty$. For every $f \in L^p(0, \infty)$ and $h \in L^{p'}(0, \infty)$, one has

$$
\int_0^\infty f(x)h(x)dx = 8 \int_0^\infty \int_{\Gamma_+(x)} \frac{s}{\partial s} W_s^\alpha(f)(y)|_{s=t^2} \frac{\partial}{\partial s} W_s^\alpha(h)(y)|_{s=t^2} \frac{dt dy}{t|J_y(y)|} dx.
$$

where $J(y) = \{x \in (0, \infty) : |x - y| < t\}$, $y, t > 0$. Here $p'$ denotes, as usual, the exponent conjugate of $p$.

**Proof.** Let us consider the bilinear mappings $T_1$ and $T_2$ defined on $L^p(0, \infty) \times L^{p'}(0, \infty)$ by

$$
T_1(f, h) = \int_0^\infty f(x)h(x)dx,
$$

and

$$
T_2(f, h) = 8 \int_0^\infty \int_{\Gamma_+(x)} \frac{s}{\partial s} W_s^\alpha(f)(y)|_{s=t^2} \frac{\partial}{\partial s} W_s^\alpha(h)(y)|_{s=t^2} \frac{dt dy}{t|J_y(y)|} dx.
$$

It is clear, by using the H"older inequality, that $T_1$ is a continuous bilinear functional. Also, since $t|J_y(y)| \geq t^2$, $t, y > 0$, the H"older inequality and Proposition 1.3 lead to

$$
|T_2(f, g)| \leq 8 \int_0^\infty \left( \int_{\Gamma_+(x)} \left| \frac{s}{\partial s} W_s^\alpha(f)(y)|_{s=t^2} \right|^2 \frac{dt dy}{t|J_y(y)|} \right)^{1/2}
$$

$$
\times \left( \int_{\Gamma_+(x)} \left| \frac{s}{\partial s} W_s^\alpha(h)(y)|_{s=t^2} \right|^2 \frac{dt dy}{t|J_y(y)|} \right)^{1/2} dx
$$

$$
\leq 8 \int_0^\infty g_{\alpha\alpha}^2(f)(x)g_{\alpha\alpha}^2(h)(x) dx
$$

$$
\leq 8 \|g_{\alpha\alpha}^2(f)\|_{L^r(0, \infty)}\|g_{\alpha\alpha}^2(h)\|_{L^{r'}(0, \infty)} \leq C\|f\|_{L^p(0, \infty)}\|h\|_{L^{p'}(0, \infty)}.
$$

Then, by taking into account that span $\{\varphi_n^\alpha\}_{n=1}^\infty$ is dense in $L^r(0, \infty)$, $1 < r < \infty$ (see [38, Lemma 4.3]), we will finish the proof when we show that $T_1(f, h) = T_2(f, h)$, for $f, g \in \text{span} \{\varphi_n^\alpha\}_{n=1}^\infty$. 

Let \( f = \sum_{n=0}^{\infty} a_n \varphi_n^\alpha \) and \( h = \sum_{n=0}^{\infty} b_n \varphi_n^\alpha \), where \( a_n, b_n \in \mathbb{R}, n \in \mathbb{N}, \) and \( a_n, b_n \neq 0 \), only for a finite number of \( n \). We can write

\[
T_2(f, h) = 8 \sum_{n,m=0}^{\infty} a_n b_m (2n + \alpha + 1)(2m + \alpha + 1) \int_0^\infty \int_{\Gamma_n(x)} t^3 e^{-2(n+m+\alpha+1)t^2} \varphi_n^\alpha(y) \varphi_m^\alpha(y) \frac{dy}{|J_t(y)|} dx
\]

\[
= 8 \sum_{n,m=0}^{\infty} a_n b_m (2n + \alpha + 1)(2m + \alpha + 1) \int_0^\infty t^3 e^{-2(n+m+\alpha+1)t^2} \int_0^\infty \frac{\varphi_n^\alpha(y) \varphi_m^\alpha(y)}{|J_t(y)|} \int_{J_t(y)} dx dy dt
\]

\[
= 8 \sum_{n=0}^{\infty} a_n b_n (2n + \alpha + 1)^2 \int_0^\infty t^3 e^{-2(2n+\alpha+1)t^2} dt
\]

\[
= \sum_{n=0}^{\infty} a_n b_n = T_1(f, h).
\]

Thus the proof is finished. \( \square \)

Let \( 1 < p < \infty \) and \( f \in L^p(0, \infty) \). By using duality and Proposition 4.1 we have

\[
\|f\|_{L^p(0,\infty)} = \sup_{h \in L^{p'}(0,\infty), \|h\|_{L^{p'}(0,\infty)} \leq 1} \left| \int_0^\infty f(x) h(x) dx \right|
\]

\[
= 8 \sup_{h \in L^{p'}(0,\infty), \|h\|_{L^{p'}(0,\infty)} \leq 1} \left| \int_0^\infty \int_{\Gamma_n(x)} \left[ \frac{\partial}{\partial s} W_s^\alpha f(s, y) \right]_{|s=0} \left[ \frac{\partial}{\partial s} W_s^\alpha h(s, y) \right]_{|s=0} \frac{dy}{|J_s(y)|} dx \right|
\]

By taking into account that \( t|J_s(y)| \geq t^2, y, t > 0 \), the Hölder inequality leads to

\[
\|f\|_{L^p(0,\infty)} \leq 8 \sup_{h \in L^{p'}(0,\infty), \|h\|_{L^{p'}(0,\infty)} \leq 1} \int_0^\infty g^{q}_{W_s}(f(x)) g^{q'}_{W_s}(h(x)) dx
\]

\[
\leq 8 \sup_{h \in L^{p'}(0,\infty), \|h\|_{L^{p'}(0,\infty)} \leq 1} \|g^{q}_{W_s}(f)\|_{L^p(0,\infty)} \|g^{q'}_{W_s}(h)\|_{L^{p'}(0,\infty)}
\]

Since \( q' \geq 2 \), Proposition 1.3 allows us finally to conclude that

\[
\|f\|_{L^p(0,\infty)} \leq C \|g^{q}_{W_s}(f)\|_{L^p(0,\infty)}.
\]

5. PROOF OF PROPOSITION 1.6

Let \( f \in L^1(0, \infty) \). We denote by \( f_o \) the odd extension of \( f \) to \( \mathbb{R} \). According to [3, Remark 2.4], \( f \in H^1_{\mathcal{L}_0}(0, \infty) \) if, and only if, \( f_o \in H^1_{\mathcal{H}}(\mathbb{R}) \). Moreover, \( f_o \in H^1_{\mathcal{H}}(\mathbb{R}) \) when, and only when, \( g^{2}_{W_s}(f_o) \in L^1(\mathbb{R}) \). Our objective is to see that \( g^{2}_{W_s}(f_o) \in L^1(\mathbb{R}) \) if, and only if, \( g^{2}_{W_s^o}(f) \), defined as in (3.2) with \( q = 2 \), belongs to \( L^1(0, \infty) \).
By using (3.1) and the Minkowski inequality, we get
\[
\left| \mathcal{G}^2_{f+}(f)(x) - \left\{ \int_{\Gamma_+} \left( \frac{s}{\partial s} \int_{\frac{t}{2}}^{2x} W_s(y, z) f(z) dz \right) \left| \frac{dt}{t^2} \right| \right\} \right|_{s=\frac{t}{2}}^{1/2}
\]
\[
\leq \int_{(0, \infty) \setminus (\frac{4x}{3}, 2x)} \left\{ \int_{\Gamma_+} \left( \frac{s}{\partial s} (W_s(y, z) - W_s(y, -z)) \right) \left| \frac{dt}{t^2} \right| \right\}^{1/2} |f(z)| dz
\]
\[
- \int_{\frac{2x}{3}}^{2x} \left\{ \int_{\Gamma_+} \left( \frac{s}{\partial s} W_s(y, z) \right) \left| \frac{dt}{t^2} \right| \right\}^{1/2} |f(z)| dz
\]
(5.1)
\[
= G_1(|f|)(x) - G_2(|f|)(x), \quad x \in (0, \infty).
\]

By taking into account (2.4) with \( q = 2 \) we obtain
\[
G_2(|f|)(x) \leq C \int_{\frac{x}{2}}^{2x} \frac{|f(z)|}{x + z} \, dz, \quad x \in (0, \infty),
\]
and, consequently, \( G_2 \) defines a bounded operator from \( L^1(0, \infty) \) into itself. On the other hand, according again to (2.4) with \( q = 2 \), we have that
\[
\left\{ \int_{\Gamma_+} \left( \frac{s}{\partial s} (W_s(y, z) - W_s(y, -z)) \right) \left| \frac{dt}{t^2} \right| \right\}^{1/2} \leq \frac{C}{z}, \quad 0 < 2x < z,
\]
and by (2.6), with \( q = 2 \), we get
\[
\left\{ \int_{\Gamma_+} \left( \frac{s}{\partial s} (W_s(y, z) - W_s(y, -z)) \right) \left| \frac{dt}{t^2} \right| \right\}^{1/2}
\]
\[
= \left\{ \int_{\Gamma_+} \left( s \int_{-z}^{z} \frac{\partial^2}{\partial u \partial s} W_s(y, u) du \right) \left| \frac{dt}{t^2} \right| \right\}^{1/2}
\]
\[
\leq \int_{-z}^{z} \left\{ \int_{\Gamma_+} \left( \frac{s}{\partial u \partial s} W_s(y, u) du \right) \left| \frac{dt}{t^2} \right| \right\}^{1/2} du
\]
\[
\leq C \int_{-z}^{z} \frac{1}{(x-u)^2} du \leq C \frac{z}{x^2}, \quad 0 < z < \frac{x}{2}.
\]

Then,
\[
G_1(|f|)(x) \leq C \left( \int_{2x}^{\infty} \frac{|f(z)|}{z} \, dz + \frac{1}{x^2} \int_{0}^{\frac{x}{2}} z |f(z)| \, dz \right), \quad x \in (0, \infty),
\]
and thus, $G_1$ is a bounded operator from $L^1(0, \infty)$ into itself.

From (5.1) we deduce that $g_{W^2}^{2,\ast}(f) \in L^1(0, \infty)$ if and only if

$$\left\{ \int_{\Gamma_+(x)} \left| \left( s \frac{\partial}{\partial s} \int_{\frac{t}{s}}^{2x} W_s(y, z) f(z) dz \right) \right|_{s=t^2}^2 \frac{dt dy}{t^2} \right\}^{1/2} \in L^1(0, \infty).$$

Note now that

$$\left| g_{W^2}^2(f)(x) - \left\{ \int_{\Gamma_+(x)} \left| \left( s \frac{\partial}{\partial s} \int_{\frac{t}{s}}^{2x} W_s(y, z) f(z) dz \right) \right|_{s=t^2}^2 \frac{dt dy}{t^2} \right\} \right|$$

$$\leq \int_{(0, \infty) \setminus \left( \frac{t}{s}, 2x \right)} \left\{ \int_{\Gamma_+(x)} \left| \left( s \frac{\partial}{\partial s} W_s^\alpha(y, z) \right) \right|_{s=t^2}^2 \frac{dt dy}{t^2} \right\}^{1/2} |f(z)| dz$$

$$+ \int_{\frac{t}{s}}^{2x} \left\{ \int_{\Gamma_+(x)} \left| \left( s \frac{\partial}{\partial s} (W_s^\alpha(y, z) - W_s(y, z)) \right) \right|_{s=t^2}^2 \frac{dt dy}{t^2} \right\}^{1/2} |f(z)| dz$$

(5.2)

$$= \Lambda_1(|f|)(x) + \Lambda_2(|f|)(x), \quad x \in (0, \infty).$$

As it was showed in the proof of Proposition 1.3, the operator $\Lambda_2$ is bounded from $L^1(0, \infty)$ into itself. (Note that $\Lambda_2(|f|) = A_3(|f|)$, with $q = 2$). On the other hand, from (3.7) for $q = 2$, we have that

$$\left\{ \int_{\Gamma_+(x)} \left| \left( s \frac{\partial}{\partial s} W_s^\alpha(y, z) \right) \right|_{s=t^2}^2 \frac{dt dy}{t^2} \right\}^{1/2} \leq \frac{C}{z}, \quad 0 < 2x < z.$$

Moreover, let $L(z)$ and $R(z)$, $z \in (0, \infty)$, be as in (3.8) and (3.9), respectively, and consider

$$N_1(x, z) = \left\{ \int_{\Gamma_+(x) \cap L(z)} \left| \left( s \frac{\partial}{\partial s} W_s^\alpha(y, z) \right) \right|_{s=t^2}^2 \frac{dt dy}{t^2} \right\}^{1/2}, \quad x, z \in (0, \infty),$$

and

$$N_2(x, z) = \left\{ \int_{\Gamma_+(x) \cap R(z)} \left| \left( s \frac{\partial}{\partial s} W_s^\alpha(y, z) \right) \right|_{s=t^2}^2 \frac{dt dy}{t^2} \right\}^{1/2}, \quad x, z \in (0, \infty).$$

By (3.10), we get

$$N_1(x, z) \leq C \frac{z^{\alpha+1/2}}{x^{\alpha+3/2}}, \quad 0 < z < \frac{x}{2}.$$

(5.4)
Also, by proceeding as in the proof of (3.11), we can obtain

\[ N_2(x, z) \leq C \left\{ \int_{\Gamma_+(x) \cap R(z)} e^{-\frac{(y-z)^2}{2t^2}} \frac{e^{-3t^2 l^2}}{(1-e^{-2t^2})^5} (y^2 + z^2)^2 dtdy \right\}^{1/2} \]

\[ \leq C \left\{ \int_0^\infty \int_{|x-y|}^\infty e^{-\frac{(y-z)^2}{2t^2}} \frac{e^{-3t^2 l^2}}{(1-e^{-2t^2})^5} \left( \frac{yz}{(1-e^{-2t^2})^2} \right)^{2a+1} (y^2 + z^2)^2 dtdy \right\}^{1/2} \]

\[ \leq C z^{a+1/2} \left\{ \int_0^\infty \int_{|x-y|}^\infty e^{-\frac{(y-z)^2}{2t^2}} \frac{e^{-3t^2 l^2}}{(1-e^{-2t^2})^5} (y + z)^4 y^{2a+1} dtdy \right\}^{1/2} \]

\[ \leq C z^{a+1/2} \left\{ \int_0^\infty \int_{|x-y|}^\infty \frac{(y + z)^4 y^{2a+1}}{(t^2 + |y-z|^2)^{2a+5}} dtdy \right\}^{1/2} \]

\[ \leq C \frac{z^{a+1/2}}{|x-z|^{2a+4}} \leq C \frac{z^{a+1/2}}{x^{a+3/2}}, \quad 0 < z < \frac{x}{2}. \]

From (5.3), (5.4) and (5.5) we deduce that

\[ \Lambda_1(||f||)(x) \leq C \left( \int_{2x}^\infty \frac{|f(z)|}{z} dz + \frac{1}{x^{a+3/2}} \int_0^{x/2} z^{a+1/2} |f(z)| dz \right), \quad x \in (0, \infty). \]

Hence, \( \Lambda_1 \) defines a bounded operator from \( L^1(0, \infty) \) into itself.

We infer from (5.2) that \( g_{W_\alpha}^2(f) \in L^1(0, \infty) \) if and only if

\[ \left\{ \int_{\Gamma_+(x)} \left| \left( \frac{\partial}{\partial s} \int_{s=0}^{2x} W_s(y, z) f(z) dz \right) \right|^2 \frac{dtdy}{t^2} \right\}^{1/2} \in L^1(0, \infty). \]

Thus, we conclude that \( f \in H_{L_\alpha}^1(0, \infty) \) if and only if \( g_{W_\alpha}^2(f) \in L^1(0, \infty) \). Moreover, the above estimations prove that the quantities \( ||f||_{H_{L_\alpha}^1(0, \infty)} \) and \( ||f||_{L^1(0, \infty)} + ||g_{W_\alpha}^2(f)||_{L^1(0, \infty)} \) are equivalent.

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