THE TWISTED GROUP RING ISOMORPHISM PROBLEM
OVER FIELDS

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ABSTRACT. Similarly to how the classical group ring isomorphism problem
asks, for a commutative ring $R$, which information about a finite group $G$ is
encoded in the group ring $RG$, the twisted group ring isomorphism problem
asks which information about $G$ is encoded in all the twisted group rings of $G$
over $R$.

We investigate this problem over fields. We start with abelian groups and
show how the results depend on the characteristic of $R$. In order to deal
with non-abelian groups we construct a generalization of a Schur cover which
exists also when $R$ is not an algebraically closed field, but still linearizes all
projective representations of a group. We then show that groups from the
celebrated example of Everett Dade which have isomorphic group algebras
over any field can be distinguished by their twisted group algebras over finite
fields.

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1. INTRODUCTION

In [MS18] we proposed a twisted version of the celebrated group ring isomorphism
problem (GRIP), namely “the twisted group ring isomorphism problem” (TGRIP).

Recall that for a finite group $G$ and a commutative ring $R$, the group ring
isomorphism problem asks whether the ring structure of $RG$ determines $G$ up to
isomorphism. In other words, does, $RG \cong RH$ imply $G \cong H$ for groups $G$ and $H$?
Roughly speaking the twisted group ring isomorphism problem asks if for a
group $G$ and a commutative ring $R$, the ring structure of all the twisted group
rings of $G$ over $R$ determines the group $G$. The role twisted group rings of $G$
over $R$ play for the projective representation theory is in many ways the same
played by the group ring $RG$ for the representation theory of $G$ over $R$, as it was
shown in the ground laying work of I. Schur [Sch07]. In this sense the (TGRIP)
can also be understood as a question on how strongly the projective representation
theory of a group influences its structure. For results on the classical (GRIP) see
[RS87, Sch93, Her01] for the case $R = \mathbb{Z}$. Also questions on character degrees, as
addressed e.g. in [Ida76, Nav18], can be viewed as results for the case $R = \mathbb{C}$.

We denote by $R^*$ the unit group in a ring $R$. For a 2-cocycle $\alpha \in Z^2(G, R^*)$ the
twisted group ring $R^\alpha G$ of $G$ over $R$ with respect to $\alpha$ is the free $R$-module with
basis $\{u_g\}_{g \in G}$ where the multiplication on the basis is defined via

$$u_gu_h = \alpha(g, h)u_{gh} \text{ for all } g, h \in G.$$

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and any \( u_g \) commutes with the elements of \( R \). Notice that if we consider \( \alpha \) only as a function (not necessarily a 2-cocycle) from \( G \times G \) to \( R^* \), then \( R^\alpha G \) is associative if and only if \( \alpha \) is a 2-cocycle. The ring structure of \( R^\alpha G \) depends only on the cohomology class \( [\alpha] \in H^2(G, R^*) \) of \( \alpha \) and not on the particular 2-cocycle. Notice that if we consider \( \alpha \) only as a function (not necessarily a 2-cocycle) from \( G \times G \) to \( R^* \), then \( R^\alpha G \) is associative if and only if \( \alpha \) is a 2-cocycle. The ring structure of \( R^\alpha G \) depends only on the cohomology class \( [\alpha] \in H^2(G, R^*) \) of \( \alpha \) and not on the particular 2-cocycle. Notice that the ring \( R \) is central in the twisted group ring \( R^\alpha G \) and correspondingly the associated second cohomology group is with respect to a trivial action of \( G \) on \( R^* \). See [Kar85, Chapter 3] for details.

Let \( G \) and \( H \) be groups and let \( R \) be a commutative ring. We define an equivalence relation which corresponds to the regular (GRIP) by

\[
G \Delta_R H \text{ if and only if } RG \cong RH,
\]

and the twisted problem is defined using a refinement of this relation as follows.

**Definition 1.1.** Let \( R \) be a commutative ring and let \( G \) and \( H \) be finite groups. We say that \( G \sim_R H \) if there exists a group isomorphism

\[
\psi : H^2(G, R^*) \to H^2(H, R^*)
\]

such that for any \( [\alpha] \in H^2(G, R^*) \),

\[
R^\alpha G \cong R^{\psi(\alpha)} H.
\]

It is easy to see that \( \sim_R \) is indeed a refinement of \( \Delta_R \), cf. Corollary 2.4. The main problem we are interested in is the following.

**The twisted group ring isomorphism problem [TGRIP].** For a given commutative ring \( R \), determine the \( \sim_R \)-classes. Answer in particular, for which groups \( G \sim_R H \) implies \( G \cong H \).

In [MS18] we investigated (TGRIP) over the complex numbers and gave some results for families of groups, e.g. abelian groups, \( p \)-groups, groups of central type and groups of cardinality \( p^4 \) and \( p^2q^3 \) for \( p, q \) primes. In this paper we investigate (TGRIP) and related problems for fields other than \( \mathbb{C} \). In particular, our main motivation is to explore:

1. The differences between the (TGRIP) and the (GRIP).
2. The differences between the (TGRIP) over \( \mathbb{C} \) and the (TGRIP) over other fields.

For example we showed in [MS18] Lemma 1.2 that any abelian group is a \( \sim_{\mathbb{C}} \)-singleton which is clearly not true for \( \Delta_{\mathbb{C}} \). We show that over other fields \( F \), abelian groups are no longer necessarily \( \sim_F \)-singletons (see Example 3.1). This is particularly interesting since, when \( \text{char}(F) \) does not divide \( |G| \), i.e. the semi-simple case, \( G \Delta_F H \) implies \( G\Delta_{\mathbb{C}} H \), while we show that \( G \sim_F H \) not necessarily implies \( G \sim_{\mathbb{C}} H \). In this sense, \( \mathbb{C} \) is no longer “the worst” field in distinguishing between groups in the semi-simple case.

A main result is related to the so called Dade’s Example. In [Dad71] E. Dade gave a family of examples of non-isomorphic groups \( G \) and \( H \) of order \( p^3q^6 \) for \( p, q \) primes satisfying some arithmetic conditions, such that \( FG \cong FH \) for any field \( F \) while \( \mathbb{Z}G \not\cong \mathbb{Z}H \). Consequently, the ring structure of all the group rings of a group over all fields is not sufficient to determine the group up to isomorphism. We prove:

**Theorem 1.** Let \( G \) and \( H \) be the groups from Dade’s example of even order. Then there exists an infinite number of fields \( F \) such that \( G \not\cong_F H \).
Theorem 2. Let $G$ and $H$ be groups such that $G \sim_F H$ for all fields $F$. 

1. Is it true that $G$ and $H$ are necessarily isomorphic?
2. Find families of groups for which the answer to the question above is positive.

An example of such a family are the abelian groups. In fact, if two abelian groups $G$ and $H$ satisfy $CG \cong CH$ and $H^2(G, C^*) \cong H^2(H, C^*)$ then $G \cong H$ (see [MS18, Lemma 1.2]). Moreover, it is clear that the above two conditions, namely isomorphic group rings and isomorphic second cohomology groups, are necessary for groups in order to be in the twisted relation. In [MS18, Examples 3.2, 3.5] we proved that for non-abelian groups the combination of these two conditions is not sufficient even to imply that $G \sim_C H$. Here we prove the following

Theorem 2. Let $G$ and $H$ be finite abelian groups. Assume there exists a field $F$ of characteristic zero which satisfies $FG \cong FH$ and $H^2(G, t(F^*)) \cong H^2(G, t(F^*))$. Then $G$ and $H$ are isomorphic.

1. There exist non-isomorphic abelian groups $G$ and $H$ and a finite field $F$ such that $FG$ is semisimple and $G \sim_F H$. In particular, $\text{char}(F) \nmid |G|$ does not imply that $\sim_F$ is a refinement of $\sim_C$.
2. There exist abelian groups $G$ and $H$ and a finite field $F$ such that $FG \cong FH$ and $H^2(G, F^*) \cong H^2(H, F^*)$, but $G \not\sim_F H$.

The paper is organized as follows. Most of Section 2 is devoted to well-known definitions and tools related to twisted group rings and the second cohomology group of a finite group. However, we also prove in Proposition 2.5 an interesting
result about simple commutative components of twisted group rings. In Section 3 we deal with the twisted relation for abelian groups. In particular we prove Theorem 2. In Section 4 we introduce and construct the Yamazaki cover of a group which is a generalization of a Schur cover of a group which exists also when $F$ is not algebraically closed. Lastly, in Section 5 we prove Theorem 1 by constructing the Yamazaki covers for the groups from Dade’s example and then evaluating their Wedderburn decompositions.

2. Preliminaries

In this section we will recall some definitions and tools that will be useful later on. Recall that throughout this paper we will assume for a finite group $G$ and a field $F$ that $H^2(G, F^*) \cong H^2(G, t(F^*))$, although it is sometimes redundant.

Clearly two main objects that we need to understand in order to study the (TGRIP) are the ring structure of twisted group rings, and the structure of the second cohomology group of a finite group.

We use standard group theoretical notation. In particular we denote by $C_n$ a cyclic group of order $n$, by $\text{o}(g)$ the order of a group element $g$ in a group $G$, by $Z(G)$ the center and by $G'$ the commutator subgroup of $G$, by $\text{exp}(G)$ the exponent of $G$, by $GL(V)$ the general linear group acting on a vector space $V$ and by $PGL(V)$ the projective general linear group, i.e. $GL(V)/Z(GL(V))$. Moreover for an abelian group $G$ we denote by $\text{rk}(G)$ the rank of $G$, i.e. the minimal number of generators of $G$. We denote by $\mathbb{F}_q$ a finite field of order $q$.

2.1. Projective representations and twisted group rings. The theory presented here is standard and can be found e.g. in [Kar85, Chapter 3]. A projective representation of a group $G$ over a field $F$ is a map $\eta : G \to GL(V)$, where $V$ is an $F$-vector space, such that the composition of $\eta$ with the natural projection from $GL(V)$ to $PGL(V)$ is a group homomorphism. As in the ordinary case, two projective representations are equivalent if they differ by a basis change of $V$. A projective representation $\eta : G \to GL(V)$ is irreducible if $V$ admits no proper $G$-subspace. Two projective representations $\eta_1 : G \to GL(V_1)$ and $\eta_2 : G \to GL(V_2)$ are called projectively equivalent if there is a map $\mu : G \to F^*$ satisfying $\mu(1) = 1$ and a vector space isomorphism $f : V_1 \to V_2$ such that

$$\eta_1(g) = \mu(g)f^{-1}\eta_2(g)f$$

for every $g \in G$.

With the above notation, we can define $\alpha \in Z^2(G, F^*)$ by

$$\alpha(g_1, g_2) = \eta(g_1)\eta(g_2)\eta(g_1g_2)^{-1},$$

and refer to $\eta$ as an $\alpha$-representation of $G$. For a fixed 2-cocycle $\alpha$, the set of projective equivalence classes of irreducible $\alpha$-representations of $G$ is denoted by $\text{Irr}(G, \alpha)$. As in the ordinary case, there is a natural correspondence between projective representations of $G$ over $F$ with an associated 2-cocycle $[\alpha]$, and $F^\alpha G$-modules.

A projective representation $\eta : G \to GL(V)$ can be extended to a homomorphism of algebras

$$\tilde{\eta} : F^\alpha G \to \text{End}_F(V)$$

$$\sum_{g \in G} a_g u_g \mapsto \sum_{g \in G} a_g \eta(g).$$
For any ring $R$ and an irreducible $R$-module $M$, there is a surjective ring homomorphism $R \to \text{End}_D M$ for $D = \text{End}_G M$. A generalized Maschke’s theorem states that if $\text{char}(F) \nmid |G|$ then any twisted group algebra $F^\alpha G$ is semisimple. Therefore, with the above notations for any irreducible $\alpha$-representation $V$ of $G$, the ring $\text{End}_D V$ can be identified with one of the components of the Artin-Wedderburn decomposition of the semisimple algebra $F^\alpha G$. In other words, $F^\alpha G$ admits a decomposition

$$F^\alpha G = \bigoplus_{[W] \in \text{Irr}(G, \alpha)} \text{End}_{D_W}(W),$$

where $D_W = \text{End}_{F^\alpha G} W$. In particular, if $F$ is a finite field such that $\text{char}(F) \nmid |G|$ then

$$F^\alpha G = \bigoplus_{[W] \in \text{Irr}(G, \alpha)} \text{End}_{F_W}(W),$$

where here $F_W$ is a field extension of $F$ corresponding to $W$.

In some of our examples later on we will use the structure of the center of a twisted group algebra. Let $G$ be a finite group and let $\alpha \in Z^2(G, F^*)$. An element $g \in G$ is called $\alpha$-regular if $\alpha(g, h) = \alpha(h, g)$ for any $h \in G$ which commutes with $g$. Note that if $g$ is $\alpha$-regular and $\beta \in Z^2(G, F^*)$ such that $[\alpha] = [\beta]$ in $H^2(G, F^*)$ then $g$ is also $\beta$-regular. The following is well known (see e.g. [NVO88, Theorem 2.4]).

**Lemma 2.1.** Let $G$ be a finite group, let $\alpha \in Z^2(G, F^*)$, let $g \in G$ be an $\alpha$-regular element and let $T$ be a transversal of the centralizer of $g$ in $G$. Then

1. The element

$$S_g = \sum_{t \in T} u_t u_g u_t^{-1}$$

is a central element in $F^\alpha G$.

2. The elements $S_g$, where $g$ runs over all the $\alpha$-regular conjugacy classes in $G$, form an $F$-basis for the center of $F^\alpha G$.

### 2.2. The second cohomology group of a finite group.

The second cohomology group of a group $G$ over the complex numbers in denoted by $M(G)$ and is called the **Schur multiplier**. An important tool to understand $H^2(G, F^*)$ is the following exact sequence (see [Kar93, Theorem 11.5.2])

$$1 \to \text{Ext}(G, F^*) \to H^2(G, F^*) \to \text{Hom}(M(G), F^*) \to 1.$$ 

Moreover, this sequence splits (not canonically). Here, for abelian groups $G, A$

$$\text{Ext}(G, A) = \{[\alpha] \in H^2(G, A) \mid \alpha \text{ is symmetric}\},$$

where a cocycle $\alpha \in Z^2(G, A)$ is called symmetric if $\alpha(x, y) = \alpha(y, x)$ for all $x, y \in G$ (see [Kar85, Chapter 2, §1]). Notice that $\text{Ext}(G, A)$ corresponds to equivalence classes of abelian central extensions of a group $G$ by a group $A$. The map in (1) from $\text{Ext}(G, G', F^*)$ to $H^2(G, F^*)$ is the restriction of the inflation map hereby explained. Let $G$ be a finite group with normal subgroup $N$, let $A$ be an abelian group and let $\varphi : G \to G/N$ be the quotient map. Then, for any $\beta \in Z^2(G/N, A)$ we can define $\alpha \in Z^2(G, A)$ by

$$\alpha(x, y) = \beta(\varphi(x), \varphi(y)).$$
The map from \( Z^2(G/N, A) \) to \( Z^2(G, A) \) sending \( \beta \) to \( \alpha \) induces a map

\[
\text{inf} : H^2(G/N, A) \to H^2(G, A)
\]

which is called the inflation map. The map in (1) from \( \text{Ext}(G/G', F^*) \) to \( H^2(G, F^*) \) is the restriction to the subgroup \( \text{Ext}(G/G', F^*) \) of the inflation map from \( H^2(G/G', F^*) \) to \( H^2(G, F^*) \). In the sequel we will sometimes abuse notations and denote the image of this map in \( H^2(G, F^*) \) as \( \text{Ext}(G/G', F^*) \) and its complement in \( H^2(G, F^*) \) by \( \text{Hom}(M(G), F^*) \).

For the sake of completeness and for later use, before going forward with the description of the second cohomology group, we would like to introduce a third map which is associated to the second cohomology group. Let

\[
1 \to N \to H \xrightarrow{\alpha} G \to 1
\]

be a central extension, let \( \mu \) be a section of \( \alpha \) and define \( f \in Z^2(G, N) \) by \( f(x, y) = \mu(x)\mu(y)\mu(xy)^{-1} \). Then, for any abelian group \( A \) and any \( \chi \in \text{Hom}(N, A) \) we have \( \chi \circ f \in Z^2(G, A) \) and the cohomology class \([\chi \circ f]\) does not depend on the choice of \( \mu \).

**Definition 2.2.** With the above notation, the map \( \text{Tra} : \text{Hom}(N, A) \to H^2(G, A) \) defined by \( \chi \mapsto [\chi \circ f] \) is called the transgression map.

We like to point out that the three maps mentioned above, inflation, restriction and transgression, are connected to each other as demonstrated in the celebrated Hochschild and Serre exact sequence.

Now recall that (see e.g. [Kar85, Corollary 2.3.17]) for any natural numbers \( n_1, \ldots, n_r \)

\[
\text{Ext}(\Pi_{i=1}^{r} C_{n_i}, F^*) \cong \prod_{i=1}^{r} \text{Ext}(C_{n_i}, F^*).
\]

Therefore, in order to understand \( \text{Ext}(G/G', F^*) \) it is sufficient to understand the description of \( \text{Ext}(C_{n_i}, F^*) \cong H^2(C_{n_i}, F^*) \). This is well known (see e.g. [Kar85, Theorem 1.3.1]):

\[
\text{Ext}(C_{n_i}, F^*) \cong H^2(C_{n_i}, F^*) \cong F^*/(F^*)^n_i.
\]

Notice that by our assumption that always \( H^2(G, F^*) \cong H^2(G, t(F^*)) \), we deduce that \( H^2(C_{n_i}, F^*) \cong F^*/(F^*)^n_i \cong t(F^*)/(t(F^*))^n_i \). This is a finite cyclic group for any field \( F \) as any two elements \( a, b \in t(F^*) \) generate a finite, and hence cyclic, group and so also \( (a, b)/(a, b)^n \) is cyclic.

We will use the above to recall the known structure of the second cohomology group of abelian groups (see e.g. [Yam64B, Corollary in §2.2]).

Let \( G \) be an abelian group. Then \( G \) admits a decomposition

\[
G = C_{n_1} \times C_{n_2} \times \ldots \times C_{n_r} \cong \langle x_1 \rangle \times \langle x_2 \rangle \times \ldots \times \langle x_r \rangle
\]

such that \( n_i \) is a divisor of \( n_{i+1} \) for any \( 1 \leq i \leq r - 1 \). Clearly,

\[
\text{Ext}(G/G', F^*) \cong \prod_{i=1}^{r} F^*/(F^*)^{n_i}.
\]

We want to describe \( \text{Hom}(M(G), F^*) \). First notice, that if \( g \) and \( h \) are commuting elements in a group \( G \) with orders \( n \) and \( m \) correspondingly, then \([u_g, u_h] = \lambda \) in the twisted group algebra \( F^*G \), and \( \lambda \) is a root of unity dividing \( \gcd(m, n) \). This follows directly from the fact that for any \( x \in G \) the element \( u_x^{n} \) is central in
$F^\alpha G$ and therefore $[u_g^{\alpha(g)}, u_h] = \lambda^{\alpha(g)} = 1$. Now, for any natural numbers $n$ and $m$ denote by $d(m, n, F)$ the maximal order of a root of unity in $F$ which divides the greatest common divisor of $m$ and $n$. If $m$ is a divisor of $n$, we denote $d(m, F)$ by $d(m, F)$. By the above, for $G$ as in (4),

$$\text{Hom}(M(G), F^*) \cong \prod_{i=1}^{r-1} C_{d(n_i, F)}^{r-i},$$

generated by the tuple of functions

$$(\alpha_{ij})_{1 \leq i < j \leq r},$$

where $\alpha_{ij}(x_i, x_j)$ is a primitive $d(n_i, F)$-th root of unity and 1 elsewhere. From (1), (5) and (6), for $G$ as in (4) we have

$$H^2(G, F^*) \cong \left( \prod_{i=1}^{r} F^*/(F^*)^{n_i} \right) \times \left( \prod_{i=1}^{r-1} C_{d(n_i, F)}^{r-i} \right).$$

As a consequence of the above, over the complex numbers, non-isomorphic abelian groups of the same cardinality admit non-isomorphic cohomology groups (see [Sch07] or [Kar85, Corollary 2.3.16]).

2.3. Commutative components of twisted group rings. In this section we study twisted group rings admitting a commutative component in their Wedderburn decomposition. We start with a straightforward result.

**Lemma 2.3.** Let $G$ be a group, $R$ a commutative ring and let $\alpha \in Z^2(G, R^*)$. If there exists an $\alpha$-projective representation of dimension 1, then $\alpha$ is cohomologically trivial.

**Proof.** This is clear by the definition of co-boundary.

**Corollary 2.4.** Let $G$ and $H$ be groups, let $R$ be a commutative ring and let $\alpha \in Z^2(G, R^*)$. Then $R^\alpha G$ admits a 1-dimensional simple module if and only if $\alpha$ is cohomologically trivial. In particular, $\sim_R$ is a refinement of $\Delta_R$.

We wish to generalize this result to commutative components with dimension not necessarily 1 over fields.

**Proposition 2.5.** Let $G$ be a group, let $F$ be a field such that char$(F) \nmid |G|$ and let $[\alpha] \in H^2(G, F^*)$. Then $F^\alpha G$ admits a commutative simple component if and only if $[\alpha]$ is in the image of the inflation map from $\text{Ext}(G/G', F^*)$ to $H^2(G, F^*)$ as defined in Section 2.1.

**Proof.** Denote by $\bar{F}$ the algebraic closure of $F$. Consider the following commutative diagram related to the exact sequence in (1). Here the vertical maps are just obtained by understanding elements of $Z^2(G, F^*)$ as elements of $Z^2(G, \bar{F}^*)$.

$$
\begin{array}{ccc}
1 & \longrightarrow & \text{Ext}(G/G', F^*) \\
& & \text{inf} \\
& & H^2(G, F^*) \\
& & \longrightarrow \\
& & \text{Hom}(M(G), F^*) \\
& & \longrightarrow \\
& & 1
\end{array}
\quad \quad
\begin{array}{ccc}
1 & \longrightarrow & \text{Ext}(G/G', \bar{F}^*) \\
& & \text{inf} \\
& & H^2(G, \bar{F}^*) \\
& & \longrightarrow \\
& & \text{Hom}(M(G), \bar{F}^*) \\
& & \longrightarrow \\
& & 1
\end{array}
$$
Assume first that \([\alpha]\) is in the image of the inflation map from \(\text{Ext}(G/G', F^*)\) to \(H^2(G, F^*)\) and denote its (unique) pre image in \(\text{Ext}(G/G', F^*)\) by \([\beta]\). Then, since \(\text{Ext}(G/G', F^*)\) is trivial, \([\beta]\) is also trivial as an element of \(\text{Ext}(G/G', \tilde{F}^*)\) and therefore \(\gamma := \inf([\beta])\) is the trivial cohomology class in \(H^2(G, \tilde{F}^*)\). Hence \(\tilde{F}^*G \cong \tilde{F}G\) admits \(\tilde{F}\) as a simple component. Now, since \(\tilde{F}^*G \cong \tilde{F}^*G \otimes_F \tilde{F}\) we conclude that \(F^*G\) admits a commutative simple component.

Conversely, assume that \(F^*G\) admits a commutative simple component. Let \([\gamma]\) be the cohomology class in \(H^2(G, F^*)\) obtained from \([\alpha]\). Then, \(F^*G \otimes_F F \cong F^*G\) also admits a commutative simple component. However, since \(\tilde{F}\) is algebraically closed this component is \(\tilde{F}\) itself. Consequently, by Corollary 2.4 \([\gamma]\) is the trivial cohomology class. Clearly from the diagram above \([\alpha]\) is in the image of the inflation map from \(\text{Ext}(G/G', F^*)\) to \(H^2(G, F^*)\) \(\square\)

### 3. Abelian groups

The main result of this section is Theorem 2. The proof is done in three steps. In Theorem 3.3 we prove Theorem 2(1), Example 3.1 shows Theorem 2(2) and lastly, Proposition 3.5 gives Theorem 2(3).

In a way, the group ring isomorphism problem asks whether it is possible to distinguish groups by their group ring structure over a commutative ring \(R\). For this purpose it is clear that the ring of integers is “the best” ring since for any commutative ring \(R\) and finite groups \(G\) and \(H\) the isomorphism \(\mathbb{Z}G \cong \mathbb{Z}H\) implies that \(RG \cong RH\). Also, in a sense, in the semi-simple case, the field of complex numbers is “the worst” commutative domain in the sense that if \(F\) is a commutative domain, \(G\) and \(H\) are finite groups such that \(FG \cong FH\) is semi-simple then \(\mathbb{C}G \cong \mathbb{C}H\). This follows from the fact that if \(\tilde{F}\) denotes the algebraic closure of the quotient field of \(F\) then \(FG \cong \tilde{F} \otimes_F FG\) and the character theories over algebraically closed fields coincide in the semi-simple case [CR81 Corollary 18.11]. We don’t know yet, if \(\mathbb{Z}\) is also “best” in distinguishing groups in the twisted case, but it is clear that \(\mathbb{C}\) is no longer the “worst” in the semi-simple case.

**Example 3.1.** Let \(G = C_3 \times C_3\), let \(H = C_9\) and let \(F = \mathbb{F}_{17}\). Then, \(H^2(G, F^*)\) and \(H^2(H, F^*)\) are trivial and

\[FG \cong FH \cong F \oplus 4\mathbb{F}_{17^2}.\]

So \(G \sim_F H\).

It is clear that \(G \not\cong \mathbb{C}H\), since these groups admit non-isomorphic Schur multipliers by [7] (see also [MS18, Lemma 1.2]).

Notice, that for abelian groups \(G\) and \(H\), if \(\mathbb{C}G \cong \mathbb{C}H\) and \(M(G) \cong M(H)\) then \(G\) and \(H\) are isomorphic. By the above example, this is not true in general over other fields. However, due to our example it is natural to ask the following.

**Question 3.2.** Let \(G\) and \(H\) be finite abelian groups and let \(F\) be a field such that \(FG \cong FH\) and \(H^2(G, F^*) \cong H^2(H, F^*)\). Is it true that \(G \sim_F H\)?

It turns out that over fields of characteristics 0, the answer is yes, and even more the group algebra and the second cohomology group together determine the group up to isomorphism. In fact the situation here is similar to the complex case, however the proof is more evolved. We will use the following lemma.
Lemma 3.3. Let $G$ and $H$ be finite abelian $p$-groups for a prime $p$ such that $|G| = |H|$. Let $F$ be a field and let $p^m$ be the cardinality of the maximal $p$-subgroup of $F^*$ (here $m$ being infinity is allowed). If $H^2(G, F^*) \cong H^2(H, F^*)$ then the maximal subgroups of $G$ and $H$ of exponent dividing $p^m$ are isomorphic. In particular, for $m \geq 1$ the groups $G$ and $H$ have the same rank.

Proof. First, the lemma is clear for $m = 0$, that is if $F$ contains no primitive $p$-th roots of unity. Second, if $F^*$ admits a $p$-subgroup of infinite order then $\text{Ext}(G/G', F^*)$ and $\text{Ext}(H/H', F^*)$ are trivial and hence by (1)

$$M(G) \cong H^2(G, F^*) \cong H^2(H, F^*) \cong M(H).$$

Consequently by [MS18, Lemma 1.2] $G$ and $H$ are isomorphic. We are left with the case $m$ is some natural number. Assume

$$G = \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle \times \ldots \times \langle \sigma_k \rangle,$$

$$H = \langle \tau_1 \rangle \times \langle \tau_2 \rangle \times \ldots \times \langle \tau_r \rangle.$$

Now define for $1 \leq i \leq m$

$$b_i(G) = \left\{ \begin{array}{ll} |\{ j : \sigma_j = p^i \}|, & i < m. \\ |\{ j : \sigma_j \geq p^m \}|, & i = m. \end{array} \right. b_i(H) = \left\{ \begin{array}{ll} |\{ j : \tau_j = p^i \}|, & i < m. \\ |\{ j : \tau_j \geq p^m \}|, & i = m. \end{array} \right.$$

By (1) we have

$$H^2(G, F^*) \cong H^2(H, F^*) \cong a_mC_{p^m} \times a_{m-1}C_{p^{m-1}} \times \ldots \times a_1C_p$$

for some natural numbers $a_1, \ldots, a_m$. Also by (1) we can express the $a_i$ in terms of the $b_i$ such that $a_m$ only depends on $b_m$, $a_{m-1}$ only depends on $b_m$ and $b_{m-1}$ etc. Namely:

$$b_i(G) + b_i(H) + \sum_{j=i+1}^m b_j(G) = a_i = b_i(H) + \left( b_i(H) \right)^2 + b_i(H) \left( \sum_{j=i+1}^m b_j(H) \right).$$

This formula follows as, in the notation of (1), the first of the two products which appear as direct factors contributes $b_i(G)$ copies of $C_{p^i}$, the $b_i(G)$ cyclic groups of order exactly $C_{p^i}$ contribute $b_i(G)^2$, one for each choice of two such groups, and each cyclic group of order bigger than $p^i$ contributes $b_i(G)$ copies.

Consequently, $b_i(G) = b_i(H)$ for any $1 \leq i \leq m$ and the result follows. \hfill \Box

We are now ready do prove

Theorem 3.4. Let $F$ be a field of characteristic zero and let $G$ and $H$ be abelian groups, such that $FG \cong FH$ and $H^2(G, F^*) \cong H^2(H, F^*)$. Then $G$ and $H$ are isomorphic.

Proof. Clearly it is sufficient to prove the theorem for abelian $p$-groups for primes $p$. Set $e_G = \log_p(\exp(G))$ and $e_H = \log_p(\exp(H))$. Let $p^m$ be the cardinality of the maximal $p$-subgroup of $F^*$ (here $m$ being infinity is allowed). If $m \geq \max\{ e_G, e_H \}$ the result follows from Lemma 3.3. In the following argument we use the fact that the characteristic of $F$ is zero. Namely will use that if $\zeta$ is a primitive $p^m$-th root of unity with $n > m$ then $F[\zeta^n]$ contains no elements of order $p^{m+1}$. Assume $m < e_G$, then in the Artin-Wederbrun decomposition of $FG$ the maximal field extension appearing has a certain degree $\frac{p^m(p^n\phi)}{\phi(p^n\phi)}$, where $\phi$ denotes Euler’s totient function. Since $FG \cong FH$, the degree of the maximal field extension in the Artin-Wederbrun
decomposition of $FH$ is also $\frac{\varphi(p G)}{\varphi(p)} = \frac{\varphi(p H)}{\varphi(p)}$. Consequently, $e_G = e_H =: e$. Since $FG \cong FC_p \otimes F(G/C_p)$ and $FH \cong FC_p \otimes F(G/C_p)$ we conclude by induction that $FG \cong FH$.

However, it turns out that in general the answer to Question 3.2 is negative.

**Proposition 3.5.** Let $G \cong C_8 \times C_2$ and let $H \cong C_4 \times C_4$. Then

1. There exist fields $F$ such that $FG \cong FH$ and $H^2(G, F) \cong H^2(H, F)$. But
2. For any field $F$ the relation $G \cong F$ does not hold.

**Proof.** Let $\mathbb{F}_q$ be a finite field such that $q - 1$ is divisible by 2 but not divisible by 4, that is $\mathbb{F}_q$ contains roots of unity of order 2 but does not contain roots of unity of order 4. In this case it follows from (7) that

$$H^2(G, \mathbb{F}_q^*) \cong H^2(H, \mathbb{F}_q^*) \cong C_2 \times C_2 \times C_2.$$  

If additionally $q^2 - 1$ is divisible by 8, then

$$\mathbb{F}_q G \cong \mathbb{F}_q H \cong 4 \mathbb{F}_q \oplus \mathbb{F}_q^2.$$  

This concludes the first part of the proposition. We want to show that for $F$ any field, $G \not\cong F$. By Theorem 3.4 this is clear for fields of characteristic zero. Let $F$ be a field of positive characteristic, char($\mathbb{F}_q$) = $p$. If $p = 2$ then, since the modular isomorphism problem has a positive solution for abelian groups, $FG \not\cong FH$ and therefore, $G \not\cong H$ [Pas65] Corollary 5.

Consequently we may assume $p > 2$. Let

$$G \cong C_8 \times C_2 = \langle g_1 \rangle \times \langle g_2 \rangle, \quad H \cong C_4 \times C_4 = \langle h_1 \rangle \times \langle h_2 \rangle.$$  

In the following arguments about the center of twisted group algebras we use Lemma 2.1. If there exists a primitive 4-th root of unity $\zeta$ in $F$ then there exists a twisted group algebra over $H$ with a 1-dimensional center, determined by the relation $[u_{h_1}, u_{h_2}] = \zeta$. But all twisted group rings over $G$ admit a center of dimension at least 4 spanned by $u_{g_1}^2$. We are left with the case $p > 2$ and $p - 1$ is not divisible by 4. Consider $[\alpha] \in H^2(H, F^*)$ determined in $F^0H$ by

$$[\alpha] : \quad [u_{h_1}, u_{h_2}] = -1, \quad u_{h_1}^4 = u_{h_2}^4 = 1.$$  

Then the center of $F^0H$ is isomorphic to $F(C_2 \times C_2) \cong 4F$ and in particular, 4 is not a divisor of the order of any central element of $F^0H$. However, for any $[\beta] \in H^2(G, F^*)$ the element $u_{g_1}^2$ is central in $F^3G$ of order multiple of 4. This completes the proof. 

It is interesting to compare the situation in Proposition 3.5 to the following example.

**Example 3.6.** Let $G = C_{16} \times C_4$ and let $H = C_8 \times C_8$, then $G \not\cong H$.

**Proof.** Let $\mathbb{F}_31 = F$ and $\mathbb{F}_{31^2} = K$. Assume

$$G \cong C_{16} \times C_4 = \langle g_1 \rangle \times \langle g_2 \rangle, \quad H \cong C_8 \times C_8 = \langle h_1 \rangle \times \langle h_2 \rangle.$$  

Since, $F^*$ admits an element of order 2 but no elements of order 4, by (7)

$$H^2(G, F^*) \cong H^2(H, F^*) \cong C_2 \times C_2 \times C_2.$$  

In order to prove that $G \not\cong H$ we will need also to describe generators for the cohomology groups. For $H^2(G, F^*)$ we have the generators

$$[\alpha_1] : \quad [u_{g_1}, u_{g_2}] = -1, \quad u_{g_1}^{16} = 1, \quad u_{g_2}^4 = 1,$$  

For $H^2(H, F^*)$ we have the generators

\[ \begin{align*}
[\beta_1] & : [u_{h_1}, u_{h_2}] = -1, \quad u_{h_1}^8 = 1, \quad u_{h_2}^8 = 1, \\
[\beta_2] & : [u_{h_1}, u_{h_2}] = 1, \quad u_{h_1}^8 = -1, \quad u_{h_2}^8 = 1, \\
[\beta_3] & : [u_{h_1}, u_{h_2}] = 1, \quad u_{h_1}^8 = 1, \quad u_{h_2}^8 = -1.
\end{align*} \]

We claim that the isomorphism from $\psi : H^2(G, F^*) \to H^2(H, F^*)$ sending $[\alpha]$ to $[\beta]$ induces a ring isomorphism $F^\alpha G \cong F^\psi(\alpha) H$ for any $[\alpha] \in H^2(G, F^*)$. The group rings $FG$ and $FH$ are clearly isomorphic, namely to $4F \oplus 30K$. Now, let $[\alpha] \in H^2(G, F^*)$ and $[\beta] \in H^2(H, F^*)$ be non-trivial cohomology classes such that $F^\alpha G$ and $F^\beta H$ are commutative. By Lemma 2.3 the twisted group rings admit no 1-dimensional components (over $F$). And therefore, since $K^*$ admits elements of order 32 we conclude that

\[ F^\alpha G \cong F^\beta H \cong \bigoplus_{i=1}^{32} K. \]

A well known result says that for any group $G$, the order of a cohomology class $[\gamma] \in H^2(G, F^*)$ divides the dimension of each $\gamma$-projective representation of $G$ [Kar85, Proposition 6.2.6]. Therefore, by Proposition 2.3 for any $[\alpha] \in H^2(G, F^*)$ and $[\beta] \in H^2(H, F^*)$ such that $F^\alpha G$ and $F^\beta H$ are non-commutative, they are isomorphic to a direct sum of $2 \times 2$ matrix rings over $F$ and $K$. Therefore they are isomorphic if and only their center is isomorphic.

By Lemma 2.3 for any $[\alpha] \in H^2(G, F^*)$ and $[\beta] \in H^2(H, F^*)$ such that $F^\alpha G$ and $F^\beta H$ are non-commutative, the center of $F^\alpha G$ is generated (as an algebra) by $u_{g_1}^2, u_{g_2}^2$ and similarly the center of $F^\beta H$ is generated (as an algebra) by $u_{h_1}^2, u_{h_2}^2$. Again, by Lemma 2.3 if the restriction of $\alpha$ (similarly $\beta$) to the subgroup generated by $g_1, g_2$ (similarly $h_1, h_2$) is non-trivial then

\[ Z(F^\alpha G) \cong Z(F^\beta H) \cong 8K. \]

This holds for the cohomology classes

\[ [\alpha_1 \alpha_2], [\alpha_1 \alpha_3], [\alpha_1 \alpha_2 \alpha_3] \in H^2(G, F^*), \quad [\beta_1 \beta_2], [\beta_1 \beta_3], [\beta_1 \beta_2 \beta_3] \in H^2(H, F^*). \]

Finally,

\[ Z(F^{\alpha_1} G) \cong Z(F^{\beta_1} H) \cong 4F \oplus 2K. \]

This completes the proof. \qed

4. The Yamazaki cover

Let $p$ be prime, let $F$ be a field and let $\zeta$ be a primitive root of unity of order $p^k$ which is maximal in the sense that there are no primitive roots of unity in $F$ of order $p^{k+1}$. Then, by our assumption that $H^2(G, F^*) \cong H^2(G, t(F^*))$, we may always assume that for a cyclic group $C_{p^r}$ with generator $\sigma$, the group $H^2(C_{p^r}, F^*)$ is generated by a cohomology class which admits a 2-cocycle which is determined by $u^2 = \zeta$ (see e.g. [Yam64a, p.31]). Notice that this does not necessarily hold without our assumption on the field. For example $H^2(C_2, \mathbb{Q}^*)$ is an infinite group.

Let $G$ be a finite group and let $F$ be an algebraically closed field of characteristic 0. Then there exists a group $G^*$ with an abelian normal subgroup $A \cong H^2(G, F^*)$ such that

\[ 1 \to A \to G^* \to G \to 1 \]
is a stem-extension, i.e. \( A \leq Z(G^*) \cap (G^*)' \). This \( G^* \) is called a representation group of \( G \) or a Schur cover of \( G \). Clearly, \( |G^*| = |G||H^2(G, F^*)| \). In general the isomorphism type of a Schur cover is not unique, but each cover satisfies

\[
FG^* \cong \bigoplus_{[\alpha] \in H^2(G, F^*)} F\alpha G.
\]

See [Kar85] Chapter 3, §3 for the details.

Different variations and generalizations of representation groups have been studied, see e.g. [LT17, Sam15] for some of the most recent.

The following example demonstrates that over non-algebraically closed fields there is no Schur cover, and at the same time suggests how to find an analog, in a sense as in [8], in the non-algebraically closed case.

**Example 4.1.** Let \( G \cong C_{2^n} \) be generated by an element \( g \) and let \( F = \mathbb{F}_5 \). We can define a 2-cocycle \( \beta \in Z^2(G, F^*) \) by \( u_5^2 = \zeta \) where \( \zeta \) is of order 4. Notice that \( u_g \) is an element of order 8 in \( F\beta G \). It is clear that \( H^2(G, F^*) \cong C_2 \) and therefore, if \( G \) admits a Schur cover it is of order 4. However, \( F\mathbb{C}_4 \cong F(C_2 \times C_2) \cong 4F \) and in particular it does not contain elements of order 8. Consequently, [8] is not satisfied and there is no Schur cover for \( G \) over \( F \). However, it is not hard to check that

\[
\mathbb{F}_5C_8 \cong 4\mathbb{F}_5 \oplus 2\mathbb{F}_{25} = \left( \mathbb{F}_5C_2 \oplus \mathbb{F}_5^2C_2 \right) = 2 \left( \oplus_{[\alpha] \in H^2(\mathbb{C}_2, F^*)} \mathbb{F}_5^2C_2 \right).
\]

We wish to find a group \( G^* \) which will play a similar role of the Schur cover over non-algebraically closed fields in the sense that any twisted group ring over \( G \) will be a direct summand of the group ring over \( G^* \). Since the construction of this group is based on a proof of Yamazaki [Yam64a] we will give here the existence theorem with a sketch of the part of the proof which describes how to construct this object. Again, for a field \( F \) we will denote by \( t(F^*) \) the torsion part of \( F^* \).

**Theorem 4.2.** [Yam64a] (see also [Kar85] Theorem 3.3.2) Let \( G \) be a finite group and let \( F \) be a field such that \( H^2(G, F^*) = H^2(G, t(F^*)) \). There exists a finite central extension

\[
1 \to A \to G^* \to G \to 1,
\]

such that any projective representation of \( G \) is projectively equivalent to a linear representation of \( G^* \).

**Construction of \( G^* \).** First, we need to describe the group \( A \) in (9). Since \( H^2(G, F^*) \) is a finite abelian group we may write

\[
H^2(G, F^*) = \langle c_1 \rangle \times \langle c_2 \rangle \times \cdots \times \langle c_m \rangle.
\]

Construct a new group as follows. Choose in any cohomology class \( c_i \) a cocycle \( \alpha_i \) of order \( d_i \), let \( A_i \cong C_{d_i} \) and let

\[
A = A_1 \times A_2 \times \cdots \times A_m.
\]

Now, the group \( G^* \) will be determined by a cohomology class \( \beta \in H^2(G, A) \). This \( \beta \) can be considered as

\[
(\beta_1, \beta_2, \ldots, \beta_m) \in H^2(G, A_1) \times H^2(G, A_2) \times \cdots \times H^2(G, A_m),
\]

while the only restriction on \( \beta_i \) is that \( \tilde{\chi}_i(\beta_i) = c_i \) for the natural morphism \( \tilde{\chi}_i : H^2(G, A_i) \to H^2(G, F^*) \). \( \square \)
Definition 4.3. We will call the group \( G^* \) in Theorem 4.2 a Yamazaki cover and will denote a Yamazaki cover of a group \( G \) over a field \( F \) by \( Y_F(G) \).

If there is no proper quotient of \( G^* \) which is also a Yamazaki cover of \( G \) we call \( G^* \) a minimal Yamazaki cover.

The following remarks are in order.

Remark 4.4. With the notations above we have a surjective morphism \( \psi : A \to H^2(G, F^*) \). In fact this is the well-known transgression map \( \text{Hom}(A, F^*) \to H^2(G, F^*) \), cf. Definition 2.2 or [Kar85, Theorem 3.2.9].

Remark 4.5. Notice that with the above notations, \( A \) is not uniquely determined, and in fact even its cardinality is not uniquely determined, since there could be in \( c \) cocycles \( \alpha \) and \( \alpha' \) of distinct order. Furthermore, like in the situation with the classical Schur cover, for a fixed \( A \) different choices of \( \beta \) can lead to non-isomorphic Yamazaki covers.

Remark 4.6. The existence of \( Y_F(G) \) depends on the condition that \( H^2(G, F^*) = H^2(G, t(F^*)) \). This condition was also investigated by Yamazaki. He showed that \( H^2(G, F^*) = H^2(G, t(F^*)) \) if and only if \( F^* = (F^*)^{\text{exp}(G/F)} t(F^*) \) [Yam64a] (cf. also [Kar85, Corollary 3.3.4]). In particular over every finite field, the real and the complex numbers Yamazaki covers always exist.

The following is immediate now from Theorem 4.2 and the construction of the Yamazaki cover.

Corollary 4.7. Let \( Y_F(G) \) be a Yamazaki cover of a group \( G \) over a field \( F \) which corresponds to \([a]\). Then

\[
FY_F(G) \cong \frac{|A|}{|H^2(G, F^*)|} \oplus [\alpha] \in H^2(G, F^*) F^\alpha G.
\]

For given groups \( G \) and \( H \) there is a well-known group theoretical condition how to determine whether \( H \) is a Schur cover of \( G \), assuming we know the order of \( H^2(G, \mathbb{C}^*) \) [Kar85, Theorem 3.3.7]. For minimal Yamazaki covers we can provide a similar criterion which requires a few more things to check though. For an abelian group \( A \) and a prime \( p \) denote by \( A_p \) the Sylow \( p \)-subgroup of \( A \).

Theorem 4.8. Let \( 1 \to Z \to H \to G \to 1 \) be a central extension of a finite group \( G \) and \( F \) a field such that \( H^2(G, F^*) \cong H^2(G, t(F^*)) \). Assume that this extension satisfies the following:

- \( Z \cap H^1 \cong \text{Hom}(M(G), F^*) \).
- \( \text{rk}(G/G^*) = \text{rk}(H/H^1) \).
- For each prime \( p \) we have the following: If \( F^* \) contains a maximal finite \( p \)-subgroup and the order of this group is \( p^m \) then \( (Z/Z \cap H^1)_p \) is a direct product of \( \text{rk}((G/G^*)_p, F^*) \) cyclic \( p \)-groups of order \( p^m \).
- \( H^1 \cap Z \) has a complement in \( Z \), i.e. the short exact sequence \( 1 \to Z \cap H^1 \to Z \to Z/(Z \cap H^1) \to 1 \) is split.

Then \( H \) is a minimal Yamazaki cover of \( G \) over \( F \).

Proof. Short exact sequence illustrating the steps of the proof can be found in [10]. Note that by assumption the exponent of \( Z \) divides the exponent of \( F^* \), so \( Z \cong \text{Hom}(Z, F^*) \). We need to show that the transgression map (see Definition 2.2)

\[
\text{Tra} : \text{Hom}(Z, F^*) \to H^2(G, F^*)
\]

is surjective and moreover that this is not the case.
for any central extension \(1 \to \tilde{Z} \to H/\tilde{Z} \to G \to 1\) for \(\tilde{Z}\) a proper subgroup of \(Z\).

Let \(Z = (Z \cap H') \times C\) for a subgroup \(C\) of \(Z\) and identify \(C\) and \(Z/(Z \cap H')\). By our assumption that \(Z \cap H' \cong \text{Hom}(M(G), F^*)\) and \([\text{Kar85}]\) Lemma 11.5.1 it follows that the image of \(\text{Tra}|_{Z \cap H'}\) is isomorphic to \(\text{Hom}(M(G), F^*)\). Define \(H_0^2(G, F^*)\) as in \([\text{Kar85}]\) Definition before Theorem 2.2.9 to be the part of \(H^2(G, F^*)\) which corresponds to all central extensions \(1 \to A \to E \to G \to 1\) with the property that \(A' \cap E = 1\). Then \([\text{Kar85}][\text{Theorem 2.2.9}]\) implies that \(H_0^2(G, F^*)\) is exactly the image of \(\text{Ext}(G/G', F^*)\) under the inflation map. In particular the transgression map \(\text{Tra} : C \to H^2(G, \mathbb{F}^*)\) related to the short exact sequence

\[
1 \to Z/(Z \cap H') \to H/(Z \cap H') \to G \to 1
\]

has an image lying in \(H_0^2(G, F^*)\). It remains to show that this is indeed the whole image and that this is not the case for any group smaller than \(H/(Z \cap H')\). It is enough to show this for a non-trivial Sylow \(p\)-subgroup \(P\) of \(C\) for some fixed prime \(p\) with respect to the Sylow \(p\)-subgroup of \(H_0^2(G, (F^*)_p)\) as it follows for each Sylow subgroup of \(C\) in the same way.

It follows from our second and third assumptions that \(\text{rk}((H/H')_p) = \text{rk}((G/G')_p)\). Let \(P = \langle a_1 \rangle \times \langle a_2 \rangle \times \ldots \times \langle a_r \rangle\) for some \(a_1, \ldots, a_r\). Then each \(a_i\) has order \(p^n\) by assumption and \(r = \text{rk}(\text{Ext}((G/G')_p, (F^*)_p)) = \text{rk}(H_0^2(G,(F^*)_p))\). Fix some \(1 \leq i \leq r\). An abelian extension of \(G/G'\) by \((F^*)_p\) corresponding to \(a_i\) is not of the form \(1 \to (F^*)_p \to (F^*)_p \times G/G' \to G/G' \to 1\), as \(\text{rk}((H/H')_p) = \text{rk}((G/G')_p)\). So by \([\text{Kar85}][\text{Theorem 2.1.2 and Corollary 2.1.3}]\) the coclass \(\text{Tra}(a_i)\) is not a coboundary for any \(1 \leq i \leq r\). So \(\text{rk}(\text{Tra}(P)) = \text{rk}(H_0^2(G,(F^*)_p))\).

Assume that \(\text{Tra}(P)\) is a proper subgroup of \(H_0^2(G, (F^*)_p)\). Then there is an \(1 \leq i \leq r\) and a cocycle \(b \in Z^2(G, (F^*)_p)\) such that \(b^p = \text{Tra}(a_i)\). But then the \(b\) must have a value which is a \(p^{m+1}\)-th primitive root of unity in \((F^*)_p\), contradicting our choice of \(m\).

Lastly, the minimality of \(H\), follows from the fact that \(a_i\) corresponds to an element in \(\text{Ext}(G/G', (F_p)_p)\), that is an abelian extension with kernel \(C_{p^m}\) and hence \(a_i\) must have order at least \(p^n\).

\[
\begin{array}{cccccc}
1 & \to & Z & \to & H & \to & G & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Z/(Z \cap H') & \to & H/(Z \cap H') & \to & G & \to & 1 \\
(10) & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Z/(Z \cap H') & \to & H'/H' & \to & G/G' & \to & 1
\end{array}
\]

\[\square\]

**Example 4.9.** We provide an example for \(Y_{p_1}(D_8)\) where \(D_8\) denotes a dihedral group of order 8. We also give an example that the last condition in Theorem 4.8 is necessary. Let \(G = D_8\) and \(F = \mathbb{F}_3\).
We have $G/G' \cong C_2 \times C_2$, so $\text{Ext}(G/G', F^*) \cong C_2 \times C_2$. Moreover $M(G) \cong C_2$ [Kar85 Proposition 4.6.4]. A minimal Yamazaki cover of $G$ is given by

$$Y(G) = (\langle a \rangle \times \langle b \rangle) \times \langle c \rangle : \quad a^2 = 1, \ b^8 = 1, \ c^4 = 1, \ a^c = a, \ b^c = ab^3.$$ 

Then $Z(Y(G)) = \langle ab^2, c^2 \rangle$ is an elementary abelian group of order 8. Moreover $Y(G)' = \langle ab^2 \rangle$ is a cyclic group of order 4. Setting $Z = Z(Y(G))'$ we observe that all conditions from Theorem 4.8 are satisfied. Using the package Wedderga [BCHK+15] of the computer algebra system GAP [GAP16] we obtain moreover

$$FY(G) \cong 4F_3 \oplus 6F_9 \oplus 8M_2(F_3) \oplus 2M_2(F_9).$$

We now exhibit an example that the last condition in Theorem 4.8 is necessary. Set

$H = (\langle a \rangle \times \langle b \rangle) \times \langle c \rangle : \quad a^4 = 1, \ b^4 = 1, \ c^4 = 1, \ a^c = a^{-1}, \ b^c = ab.$

Then we have that $Z(H) = \langle ab^2, c^2 \rangle \cong C_4 \times C_2$ and $H' = \langle a \rangle \cong C_4$. Set $Z = Z(H)$. Then $H/Z \cong G, \ Z \cap H' = \langle a^2 \rangle \cong C_2$, $\text{rk}(Z/(Z \cap H')) = 2$, $Z/(Z \cap H') \cong C_4 \times C_2$ and $\text{rk}(H/H') = 2$. So the extension $1 \to Z \to H \to G \to 1$ satisfies all the conditions of Theorem 4.8 except the last one. But $H$ is not a Yamazaki cover of $G$ as its group algebra over $F$ is not isomorphic with the group algebra of $Y(G)$ given above. Indeed,

$$FH \cong 4F_3 \oplus 6F_9 \oplus 4M_2(F_3) \oplus 4M_2(F_9),$$

which again can be calculated using [BCHK+15].

5. The Dade example

In 1971 E. Dade, answering a question of R. Brauer [Bra63 Problem 2*], provided a family of examples of non-isomorphic finite groups $G$ and $H$ such that the group algebras of $G$ and $H$ are isomorphic over any field $F$. We will show that for a subclass of Dade’s examples there are fields $F$ such that $G \not\cong_F H$. Note that the groups of Dade are metabelian and hence have non-isomorphic group rings over the integers, a result due to Whitcomb already known at the time Dade solved Brauer’s problem [Whi68].

We will first describe the groups given by Dade. Let $p$ and $q$ be primes such that $q \equiv 1 \mod p^2$ and let $w$ be an integer such that $w \not\equiv 1 \mod q^2$, but $w^q \equiv 1 \mod q^2$. Let $Q_1$ and $Q_2$ be the following two non-abelian groups of order $q^3$.

$$Q_1 = (\langle \tau_1 \rangle \times \langle \sigma_1 \rangle) \rtimes \langle \rho_1 \rangle,$$

$$Q_2 = (\sigma_2) \times \langle \rho_2 \rangle.$$

$$\tau_1^q = \sigma_1^q = \rho_1^q = \sigma_2^q = \rho_2^q = 1, \ \sigma_1^q = \tau_2,$$

$$\tau_1^{\rho_1} = \tau_1, \ \sigma_1^{\sigma_1} = \tau_1 \sigma_1, \ \sigma_1^{\rho_1} = \tau_2 \sigma_2$$

So $Q_1$ and $Q_2$ are just the two non-abelian groups of order $q^3$ such that $Q_1$ has exponent $q$ (aka the Heisenberg group).

Let $\langle \pi_1 \rangle \cong C_{p^2}$, $\langle \pi_2 \rangle \cong C_p$ and for $i, j \in \{1, 2\}$ let

$$\rho_i = \rho_i, \ \sigma_i^\pi = \sigma_j^w, \ \tau_i^\pi = \tau_i^w.$$
Define two groups by
\[ G = (Q_1 \times \langle \pi_1 \rangle) \times (Q_2 \times \langle \pi_2 \rangle), \]
\[ H = (Q_1 \times \langle \pi_2 \rangle) \times (Q_2 \times \langle \pi_1 \rangle). \]
These are the groups constructed by Dade as a counterexample to Brauer’s question. Notice that \( G = G_1 \times G_2 \) and \( H = H_1 \times H_2 \) for
\[ G_1 = Q_1 \times \langle \pi_1 \rangle, \quad G_2 = Q_2 \times \langle \pi_2 \rangle, \quad H_1 = Q_1 \times \langle \pi_2 \rangle, \quad H_2 = Q_2 \times \langle \pi_1 \rangle. \]

5.1. The second cohomology groups of \( G \) and \( H \). In order to calculate the Schur multipliers of \( G \) and \( H \) we will use a result of Schur [Sch07] about the Schur multiplier of direct products of groups (see also [Kar85, Corollary 2.3.14]). Define the tensor product of two finite groups \( A \) and \( B \) by
\[ A \otimes B = A/A' \otimes B/B'. \]

**Theorem 5.1.** Let \( A \) and \( B \) be finite groups. Then
\[ M(A \times B) = M(A) \times M(B) \times (A \otimes B). \]

Notice that (slightly abusing notation)
\[ G_1' = \langle \tau_1 \rangle \times \langle \sigma_1 \rangle = H_1' \text{ and } G_2' = \langle \sigma_2 \rangle = H_2'. \]
Therefore
\[ G_1/G_1' \cong C_q \times C_p \cong H_2/H_2' \text{ and } G_2/G_2' \cong C_q \times C_p \cong H_1/H_1'. \]
Consequently
\[ G_1 \otimes G_2 \cong H_1 \otimes H_2 \cong C_q \times C_p. \]
We will use Theorem 5.1 to compute the Schur multipliers of \( G_1, G_2, H_1 \) and \( H_2 \).

**Lemma 5.2.** (see [Kar85] Corollary 2.2.6) Let \( N \) and \( T \) be subgroups of a group \( G \) of co-prime order and assume \( G = N \times T \). Then
\[ M(G) = M(T) \times M(N)^T. \]
Here \( M(N)^T \) are the elements in \( M(N) \) which are invariants under the \( T \)-action.

First, by [Kar85] Theorem 4.7.3, \( M(Q_1) \cong C_q \times C_q \) and \( Q_2 \) admits a trivial Schur multiplier. Therefore, since a Schur multiplier of a cyclic group is trivial, we get by Lemma 5.2 that
\[ M(G_2) = M(H_2) = 1. \]
We are left with the computation of \( M(G_1) \) and \( M(H_1) \). As written above \( M(Q_1) \cong C_q \times C_q \). In fact, \( M(Q_1) \) is generated by the cohomology classes \( \alpha \) and \( \beta \) which are determined by the following relations in the corresponding twisted group algebras \( \mathbb{C}^\alpha Q_1 \) and \( \mathbb{C}^\beta Q_1 \)
\[ \alpha : [u_r, u_\sigma] = \zeta, \quad [u_r, u_\rho] = 1, \]
\[ \beta : [u_r, u_\sigma] = 1, \quad [u_r, u_\rho] = \zeta. \]
Here \( \zeta \) denotes a primitive \( q \)-th roots of unity. Notice, that for \( i = 1, 2 \)
\[ [u_r^{\tau_i}, u_\rho^{\nu_i}] = [u_r^{\tau w_i}, u_\rho] = [u_r, u_\rho]^w. \]
Therefore, $\beta$ is not invariant under the action of $\langle \pi_i \rangle$ for $i = 1, 2$. We need to check whether $\alpha$ is invariant. It turns out that $\alpha$ is invariant if and only if $p = 2$. Indeed, in $\mathbb{C}^\alpha Q_1$

$$[u_{r \pi_i}, u_{\sigma \pi_i}] = [u_w^r, u_w^\sigma] = \zeta^{w^2}.$$ 

Therefore, $\alpha$ is invariant if and only if $w^2 \equiv 1 \mod q$ which happens if and only if $p = 2$ because $w^p \equiv 1 \mod q^2$. As a consequence of the above we obtain the following.

**Proposition 5.3.** With the above notations, if $p = 2$

$$M(G) \cong M(H) \cong C_q \times C_q \times C_p,$$

and for $p > 2$

$$M(G) \cong M(H) \cong C_q \times C_p.$$ 

We proceed to construct $H^2(G, F^*)$ using the exact sequence given in (1). Observe that

$$G/G' \cong C_q \times C_q \times C_2 \times C_4 = \langle G' \rho_1 \rangle \times \langle G' \rho_2 \rangle \times \langle G' \pi_1 \rangle \times \langle G' \pi_2 \rangle.$$ 

Therefore, by equations (2) and (3) we get

$$\text{Ext}(G/G', F^*) \cong C_q \times C_q \times C_p \times C_{p^2}.$$ 

**Corollary 5.4.** For $p = 2$ we have

$$H^2(G, F^*) \cong \text{Ext}(G/G', F^*) \times \text{Hom}(M(G), F^*) \cong (C_q \times C_q \times C_p \times C_{p^2}) \times (C_q \times C_q \times C_p).$$

and for $p > 2$ we get

$$H^2(G, F^*) \cong \text{Ext}(G/G', F^*) \times \text{Hom}(M(G), F^*) \cong (C_q \times C_q \times C_p \times C_{p^2}) \times (C_q \times C_p).$$

Notice that all the arguments above about $G$ are true also for $H$.

5.2. The Yamazaki covers of $G$ and $H$. From now on we will assume that $p = 2$ and $q$ is any prime satisfying the relations in Dade’s groups. Note that we can then assume w.l.o.g. $w = -1$. Moreover we assume that $F = \mathbb{F}_p$ is a finite field such that

- $r - 1$ is divisible by $q$ but not by $q^2$,
- $r - 1$ is divisible by $4$ but not by $8$ and
- $r^2 - 1$ is divisible by $8$ but not by $16$.

There exist infinitely many such fields, e.g. by Dirichlet’s theorem on primes in arithmetic progressions.

This allows us to give the Yamazaki covers of $G$ and $H$ using less notation, though it is not hard to give them also in case $p > 2$. But the difference observed between the Schur multipliers in Proposition 5.3 turns out to be crucial for our arguments, so we concentrate on this case. See Remark 5.8 about the case $p > 2$.

Let $\zeta$ be a primitive $q$-th and $\xi$ a primitive $4$-th root of unity in $F$. In order to construct the Yamazaki covers of $G$ and $H$ we will need to describe the group $A$ in the construction after Theorem 5.2 as computed in the previous subsection and in particular in Corollary 5.4. Let

$$H^2(G, F^*) = \text{Hom}(M(G), F^*) \times \text{Ext}(G/G', F^*)$$

$$= (\langle \alpha \rangle \times \langle \beta \rangle \times \langle \gamma \rangle) \times (\langle \kappa \rangle \times \langle \lambda \rangle \times \langle \mu \rangle \times \langle \nu \rangle),$$

where
\begin{itemize}
\item $\alpha$ is of order $q$, determined by $[u_{\rho_1}, u_{\rho_2}] = \zeta$.
\item $\beta$ is of order 2, determined by $[u_{\pi_1}, u_{\pi_2}] = -1$.
\item $\gamma$ is of order $q$, determined by $[u_{\nu_1}, u_{\nu_2}] = \zeta$.
\item $\kappa$ is of order $q$ determined by $u_{\rho_1}^q = \zeta$.
\item $\lambda$ is of order $q$ determined by $u_{\rho_2}^q = \zeta$.
\item $\mu$ is of order 4 determined by $u_{\pi_1}^4 = \xi$.
\item $\nu$ is of order 2 determined by $u_{\pi_2}^2 = \xi$.
\end{itemize}

Notice, that from the above the only cohomology class in which the order of the cocycle is bigger than the order of the cohomology class is for $\nu$. Here the order of $\nu$ is 2 and the order of the corresponding cocycle is 4. Therefore we may consider $\nu$ cocycle is bigger than the order of the cohomology class is for $\alpha$. We will generate a class $\beta \in H^2(G,A)$ in which the order of the cohomology class is for $\beta$. Here we will use the following relations $u_{\rho_1}^q = \zeta, \quad u_{\rho_2}^q = \zeta, \quad u_{\pi_1}^4 = \xi, \quad u_{\pi_2}^2 = \xi$.

Now in order to construct the Yamazaki cover we need to construct a cohomology class $\beta(G,A) \in H^2(G,A)$ which will correspond to the central extension $\mathfrak{g}$. Let $\{\tilde{g}\}_{g \in G}$ be a section of $G$ in $G^*$ corresponding to $\mathfrak{g}$. Then, abusing notation, $\tilde{\beta(G,A)}$ can be chosen to be the cohomology class determined by (compare with the classes given above)

$$[u_{\rho_1}, u_{\rho_2}] = \zeta, \quad [u_{\pi_1}, u_{\pi_2}] = -1, \quad [u_{\nu_1}, u_{\nu_2}] = \zeta,$$

$$u_{\rho_1}^q = \zeta, \quad u_{\rho_2}^q = \zeta, \quad u_{\pi_1}^4 = \xi, \quad u_{\pi_2}^2 = \xi.$$

This leads us also to the Yamazaki covers of $G$ and $H$ over $F$. Since from now on we will only work with these covers and their subgroups we will use the same notations for the elements as before in the “uncovered” groups. Here we will introduce cyclic subgroup $\langle x \rangle, \langle y \rangle$ and $\langle z \rangle$ corresponding to the cohomology classes $\alpha, \beta$ and $\gamma$ respectively. The orders of the other generators change according to the cohomology classes $\kappa, \lambda, \mu$ and $\nu$. We will construct both Yamazaki covers as the quotient of the same infinite group.

**Notation:** Let $Y$ be a group generated by elements $\sigma_1, \sigma_2, \rho_1, \rho_2, \tau_1, \pi_1, \pi_2, x, y$ and $z$ subject to the following relations:

$$\sigma_2^2 = x^q = \rho_2^q = y^q = \pi_2^q = \tau_1^q = \rho_1^q = \sigma_1^q = \pi_1^{16} = 1, \quad \sigma_2^2 =: \tau_2, \quad \sigma_2^2 = \sigma_2 \tau_2, \quad \rho_1^q = \tau_1 \rho_1, \quad \tau_1^q = z \tau_1, \quad \rho_2^q = x \rho_2.$$

Moreover we have $x, y, z \in Z(Y)$ and unless otherwise specified in the relations above for $g, h \in \{\sigma_1, \sigma_2, \rho_1, \rho_2, \tau_1\}$ we have $[g, h] = 1$ in $Y$.

**Lemma 5.5.** Let $Y$ be the group described above. Let $Y(G)$ be the quotient of $Y$ in which $\pi_i$ commutes with $\sigma_j, \rho_j, \tau_j$ for $i \neq j$ and which is additionally subject to the following relations

$$\sigma_1^{-1}, \quad \sigma_2^{-1}, \quad \pi_1^{-1}, \quad \pi_2^{-1} = y \pi_2.$$

Let $Y(H)$ be the quotient of $Y$ in which $\pi_i$ commutes with $\sigma_i, \rho_i, \tau_i$ for $i \in \{1, 2\}$ and where additionally we have the relations

$$\sigma_1^{-1}, \quad \sigma_2^{-1}, \quad \pi_1^{-1} = z \tau_1, \quad \sigma_1^{-1}, \quad \pi_2^{-1} = y \pi_1.$$

Then $Y(G)$ and $Y(H)$ are minimal Yamazaki covers of $G$ and $H$ respectively.
Remark: Using semi-direct products one can write:

\[ Y(G) = (((\sigma_2) \times (\langle x \rangle \times \langle \rho_2 \rangle)) \times ((\langle y \rangle \times \langle \pi_2 \rangle)) \times (((\langle z \rangle \times \langle \tau_1 \rangle \times \langle \rho_1 \rangle) \times (\sigma_1)) \times (\pi_1)), \]

\[ Y(H) = (((\sigma_2) \times (\langle x \rangle \times \langle \rho_2 \rangle)) \times ((\langle y \rangle \times \langle \pi_1 \rangle)) \times (((\langle z \rangle \times \langle \tau_1 \rangle \times \langle \rho_1 \rangle) \times (\sigma_1)) \times (\pi_2)). \]

Note that the only difference when writing this way is an interchange between \( \pi_1 \) and \( \pi_2 \).

Proof. We will use Theorem 4.8 and Corollary 5.4. In the notation of Theorem 4.8 we have

\[ Z = \langle x \rangle \times \langle y \rangle \times \langle z \rangle \times \langle \rho_1 \rangle \times \langle \rho_2 \rangle \times \langle \pi_1 \rangle \times \langle \pi_2 \rangle. \]

Moreover

\[ Y(G) = \langle x, y, z, \sigma_2, \tau_1, \sigma_1 \rangle. \]

So \( Y(G) \cap Z = \langle x \rangle \times \langle y \rangle \times \langle z \rangle \cong \text{Hom}(M(G), F^*) \). The other conditions are now easy to check.

The same statements hold for \( Y(H) \), even using formally the same elements. \( \square \)

5.3. Proof of Theorem 1. We keep the assumptions from the previous subsection and we will show that in this case \( G \nicht F H \). We will use the minimal Yamazaki covers \( Y(G) \) and \( Y(H) \) introduced in Lemma 5.5 and explicit elements will refer to these groups.

To show that \( G \) and \( H \) are not in relation over \( F \) we will work with Wedderburn decompositions of \( FY(G) \) and \( FY(H) \). The groups \( Y(G) \) and \( Y(H) \) are supersolvable as can be seen by their defining relations and hence both groups are monomial, i.e. each irreducible character of these groups is induced by a linear character of a subgroup. This holds over \( \mathbb{C} \) by [Isa76, Theorem 6.22] and over finite fields of characteristic not dividing \( |G| \) by [BdR07, Corollary 8].

Each Wedderburn component of \( FY(G) \) and \( FY(H) \) corresponds to a Wedderburn component of a twisted group algebra \( F^\psi G \) and \( F^\varphi H \) respectively. Let \( B \) be such a Wedderburn component. Then in fact we can easily determine \( \varphi \) from the character \( \chi \) corresponding to \( B \). Namely if we view \( \varphi \) as a product of powers of the generators \( \alpha, \gamma, \kappa, \lambda, \beta, \mu \) and \( \nu \), then we can read off \( \varphi \) from the powers of \( \zeta \), \( -1 \) and \( \xi \) appearing in the values of \( \varphi \) on \( x, z, \rho_1^0, \rho_2^0, y, \pi_1^0 \) and \( \pi_2^0 \) respectively. This follows from the natural correspondence between projective representations and 2-cocycles as explained in Section 2.4.

Denote by \( F_2 \) the field obtained from adjoining a primitive 8-th root of unity to \( F \) and by \( F_4 \) the field obtained from adjoining a primitive 16-th root of unity to \( F \). Note that these fields are different by our choice of \( F \).

We will show that there is a cohomology class \( \psi \) in \( H^2(G, F^*) \) such that every Wedderburn component of \( F^\psi G \) is a matrix ring over the field \( F_4 \), but there is no cohomology class \( \varphi \) in \( H^2(H, F^*) \) such that \( F^\varphi H \) is the direct sum of matrix rings over \( F_4 \). This will be proven in the next two lemmas and clearly imply \( G \nicht F H \).

Lemma 5.6. For \( \psi = \gamma \mu \) the Wedderburn decomposition of \( F^\psi G \) is a direct sum of matrix rings over \( F_4 \).

Proof. Both \( \gamma \) and \( \mu \) only influence the subgroup \( G_1 = Q_1 \times (\pi_1) \), in the sense that we can choose a cocycle \( \psi' \) representing \( \psi \) such that \( \psi'((g_1, g_2), (1, \tilde{g}_2)) = 1 \) for every \( g_1 \in G_1 \) and \( g_2, \tilde{g}_2 \in G_2 \). So \( k^\psi G = kG_2 \otimes k^\psi G_1 \), where \( \psi_1 \) denotes the
restriction of \( \psi \) to \( G_1 \). It is hence sufficient to show that \( k^{\psi_1}G_1 \) is a direct sum of matrix rings over \( F_1 \). A minimal Yamazaki cover of \( G_1 \) over \( F \) is given by

\[
Y(G_1) = (\langle z \rangle \times \langle \tau_1 \rangle \times \langle \rho_1 \rangle \times \langle \sigma_1 \rangle) \rtimes \langle \tau_1 \rangle
\]

where the orders of the generators and the relations between them are exactly as in \( Y(G) \).

The Wedderburn decompositions of \( FY(G_1) \) can also be computed in positive characteristic as described in \([BdR07]\). In particular each Wedderburn component corresponds to a pair \((S,T)\) of subgroups in \( Y(G_1) \) such that \( S \) has a linear character \( \chi \) with kernel \( T \) and the induction \( \text{ind}_S^Y(G_1)(\chi) \) of \( \chi \) to \( Y(G_1) \) is irreducible. Moreover assume that \( \text{ind}_S^Y(G_1)(\chi) \) corresponds to some Wedderburn component of \( F^{\psi_1}G_1 \), i.e. we have \( z, \pi_1^8 \not\in T \) and \( \rho_1^q \in T \). Our claim will follow once we show that \( S \) necessarily contains an element of order 16 or equivalently:

**Claim**: Every irreducible character of \( Y(G_1) \) whose kernel contains \( \rho_1 \), but not \( z \) and \( \pi_1^8 \), has odd degree.

The claim is true over \( F \) if and only if it is true over \( \mathbb{C} \). To make the calculations a bit easier we use the bar-notation to denote the natural projection modulo \( \langle \rho_1^q, \pi_1^8 \rangle \) and the reduction of \( Y(G_1) \) and set \( R = Y(G_1)/\langle \rho_1^q, \pi_1^8 \rangle \). We will prove that any irreducible character of \( R \) whose kernel does not contain \( z \) has odd degree which will imply the claim.

First of all observe that \( \langle \bar{z}, \bar{\tau}_1, \bar{\rho}_1 \rangle \) is an abelian normal subgroup of \( R \) of index \( 2q \) and so Ito’s Theorem \([ Isa76, \text{Theorem 6.15}]\) implies that the character degree of each irreducible character of \( R \) divides \( 2q \). So each irreducible character of odd degree has degree 1 or \( q \). Note that the number of characters of degree 1 of \( R \) equals \( |R/R'| = |R/\langle \bar{z}, \bar{\tau}_1, \bar{\sigma}_1 \rangle| = 2q \). By \([ Isa76, \text{Theorem 13.26}]\), a very special version of the McKay-conjecture, the number of irreducible characters of odd degree of \( R \) is the same as that of \( N_R(\langle \bar{\tau}_1 \rangle) \). Now \( N_R(\langle \bar{\tau}_1 \rangle) = \langle \bar{z}, \bar{\rho}_1, \bar{\tau}_1 \rangle \) is an abelian group of order \( 2q^2 \) and has \( 2q^2 \) irreducible characters of odd degree. Moreover \( R/\langle \bar{z} \rangle \) has also \( 2q \) characters of degree 1 and \( \frac{q(q-1)}{2} \) irreducible characters of degree 2 which are those having \( \bar{\tau}_1 \) in its kernel. This follows since

\[
R/\langle \bar{z}, \bar{\tau}_1 \rangle \cong \langle \bar{\rho}_1 \rangle \times (\langle \bar{\sigma}_1 \rangle \times \langle \bar{\pi}_1 \rangle) \cong C_q \times D_{2q},
\]

where \( D_{2q} \) denotes a dihedral group of order \( 2q \), and \( D_{2q} \) has exactly \( \frac{q-1}{2} \) irreducible characters of degree 2. Moreover the subgroup \( \langle \bar{\tau}_1, \bar{\rho}_1, \bar{\sigma}_1 \rangle \) of \( R/\langle \bar{z} \rangle \), which is an extraspecial \( q \)-group, has \( q - 1 \) irreducible characters of degree \( q \), see e.g. \([ Dor71, \text{Theorem 31.5}]\). The induction of each of these characters, which are all not real-valued, to \( R/\langle \bar{z} \rangle \) is irreducible, since it is real on the real conjugacy class of \( \bar{\tau}_1 \), and two of them induce the same character. So \( R/\langle \bar{z} \rangle \) has \( \frac{q-1}{2} \) irreducible characters of degree \( 2q \). Summing the squares of the degrees of the irreducible characters of \( R/\langle \bar{z} \rangle \) obtained so far we obtain

\[
2q \cdot 1^2 + \frac{q(q-1)}{2} \cdot 2^2 + \frac{q-1}{2} \cdot (2q)^2 = 2q^3.
\]

So there are no further irreducible characters of \( R/\langle \bar{z} \rangle \). In particular from all irreducible odd degree characters of \( R \) only the \( 2q \) linear characters of \( R \) have \( \bar{z} \) in its kernel. But since any other irreducible odd degree character has degree \( q \), there are \( 2q^2 \) such characters and since

\[
(2q^2 - 2q)q^2 = 2q^4 - 2q^3 = |R| - |R/\langle \bar{z} \rangle|.
\]
there are actually all irreducible characters of $R$ which do not contain $\bar{z}$ in its center. Hence the claim follows. This also finishes the proof of the lemma. \hfill \Box

**Lemma 5.7.** There is no $\varphi \in H^2(H, F^*)$ such that every direct summand of $F^*H$ is a matrix algebra over $F_1$.

**Proof.** Since all groups involved are monomial a Wedderburn component of $F^*H$ is determined by a pair $(S, T)$ of subgroups of $Y(H)$ which satisfy the following. $S$ has a linear character $\chi$ with kernel $T$ such that $\text{ind}_S^{Y(H)}(\chi)$ is irreducible and $\chi$ has values on $x, y, z, \rho_1^q, \rho_2^q, \pi_1^q, \pi_2^q$ which correspond to the powers of the natural generators $\alpha, \beta, \gamma, \kappa, \lambda, \mu$ and $\nu$ appearing in $\varphi$ respectively. The corresponding matrix algebra lies over $F_2$ if and only if $S$ contains an element of order 16 none of whose powers lies in $T$. So it is sufficient to show that for any $\varphi \in H^2(H, F^*)$ there is a corresponding pair $(S, T)$ such that $S$ contains no element of order 16. Instead of describing $\varphi$ we will distinguish the different $T$. For example the condition $x \in T$ means that in writing $\varphi$ in the natural generators the factor $\alpha$ does not appear. We will study some cases separately. Note that we can make assumptions only on $\langle x, y, z, \rho_1^q, \rho_2^q, \pi_1^q, \pi_2^q \rangle \cap T$, since this fixes which natural generators appear in $\varphi$.

The general goal in all cases will be to achieve $\sigma_2 \in S \setminus T$, because then an element of order 16 does not commute with $S/T$, so there can be no element of order 16 in $S$ which has no power in $T$. Set $Z = Z(Y(H)) = \langle x, \rho_2^q, y, \pi_2^q, z, \rho_1^q, \pi_1^q \rangle$.

**Case 1:** $x, z \in T$.

Let $S = \langle Z, \sigma_2, \rho_2, \tau_1, \rho_1, \sigma_1, \pi_2 \rangle$. Then $S' = \langle \sigma_2^q, z, \tau_1, \sigma_1 \rangle$ and let $T$ be a subgroup of $S$ containing $S'$ such that $S/T$ is cyclic and $T$ does not contain $\sigma_2$. Let $\chi$ be a linear character of $S$ with kernel $T$. Then $\chi' = \text{ind}_S^{Y(H)} \chi$ is of degree 2 and $\chi'(\sigma_2) = \chi(\sigma_2) + \chi(\sigma_2)^{-1} \neq 2$. Moreover $\chi'$ is irreducible, since otherwise it would decompose into two linear characters. But linear characters contain $\sigma_2$ in its kernel, since $\sigma_2 \in Y(H)'$, and then we would have $\chi'(\sigma_2) = 2$.

**Case 2:** $x \notin T$, $z \in T$.

Set $S = \langle Z, \sigma_2, \rho_2, \tau_1, \rho_1, \sigma_1, \pi_2 \rangle$. Then $S' = \langle \sigma_2^q, z, \tau_1, \sigma_1 \rangle$. Let $T$ again be a subgroup of $S$ containing $S'$ such that $S/T$ is cyclic, $\sigma_2 \notin T$ and let $\chi$ and $\chi'$ be defined similarly as in Case 1. Then $\chi'$ is a character of degree 10 such that $\chi'(\sigma_2) = 5(\chi(\sigma_2) + \chi(\sigma_2)^{-1})$. This means that the restriction of $\chi'$ to $\langle \sigma_2, \pi_1 \rangle$ decomposes into five 2-dimensional characters. So if $\chi'$ decomposes it decomposes into characters of even degree. But on the other hand its restriction to $\langle (x) \times \langle \rho_2 \rangle \rangle \times \langle \rho_1 \rangle$ has to decompose into characters of degree 5, since these are the only characters of this group not having $x$ in the kernel.

**Case 3:** $x \in T$, $z \notin T$.

Set $S = \langle Z, \sigma_2, \rho_2, \tau_1, \rho_1, \pi_2 \rangle$. Note that $\pi_1^q = z\pi_1^{-1} = \tau_1(z\pi_1^3)$. So we have $S' = \langle \sigma_2^q, x, z\pi_1^3 \rangle$. Again let $T$ be a normal subgroup of $S$ such that $S/T$ is cyclic, $\sigma_2 \notin T$ and let $\chi$ and $\chi'$ be defined as in the previous cases. If $\chi'$ decomposes then the summands have even degree, due to the value of $\chi'$ on $\sigma_2$ and its restriction to $\langle \sigma_2, \pi_1 \rangle$. But at the same time the degree of a summand would be divisible by 5, due to its character value 0 on $z$ and the character theory of the extraspecial $q$-group $\langle z, \tau_1, \sigma_1 \rangle$.

**Case 4:** $x, z \notin T$.

Set $S = \langle Z, \sigma_2, \tau_1, \rho_1, \pi_2 \rangle$. Then $S' = \langle z\pi_1^3 \rangle$. Let again $T$, $\chi$ and $\chi'$ have
analogues properties as before such that $\sigma_2^n \notin T$. Note that $\tau_1 \notin T$. By Frobenius reciprocity and Clifford theory we have, considering the scalar product of characters,

$$\langle \chi', \chi'' \rangle_{Y(H)} = \sum_{g \in Y(H)/S} \langle \chi, \chi'' \rangle_{S}.$$

Now a system of coset representatives of $Y(H)/S$ is given by

$$\{ \pi_i^1 \rho_j^2 \sigma_1^k \mid 0 \leq i \leq 1, \ 0 \leq j, k \leq q - 1 \}.$$

Set $a_{i,j,k} = \pi_i^1 \rho_j^2 \sigma_1^k$. Then $\sigma_2^{a_{i,j,k}} = \sigma^{(1+q)j}(-1)^i$ and $\tau_1^{a_{i,j,k}} = z^k \tau_1$. Since $\langle \tau_1, \sigma_2 \rangle \cap T = 1$ we hence have $\chi^{a_{i,j,k}} = \chi$ if and only if $i = j = k = 0$. So $\chi'$ is irreducible.

\[ \square \]

**Remark 5.8.** The calculations of the cohomology groups for the groups $G$ and $H$ from Dade’s example suggest that if the groups are of odd order then it is very well possible that $G \sim_F H$ over any field $F$. In the words of Passman the “surprise” in the proof of Dade is the fact that $FG \cong FH$ for fields of characteristic $q$ and that “this isomorphism is so easily proved” [Pass77] p. 664. This proof relies on the fact that setting $e = \frac{1}{p} \sum_{i=0}^{q-1} \pi_i^1$ in $FG$ and $FH$ respectively $FG$ and $FH$ are direct sums of algebras isomorphic to $eFG$ and $eFH$ respectively. As $eFG \cong F(G/\langle \pi_1^1 \rangle) \cong F(H/\langle \pi_1^1 \rangle) \cong eFH$ the isomorphism of $FG$ and $FH$ follows immediately.

It seems impossible to imitate this argument in the twisted case, since there is no natural idempotent in the twisted group ring of a cyclic group corresponding to $e$. For example $\mathbb{F}_5^0 C_4$ is a simple algebra isomorphic to $\mathbb{F}_5^0$ for $[a] \in H^2(G, \mathbb{F}_5^0)$ of order 4, so it has no quotients which “kill” exactly the cyclic group of order 2. This is a special instance of the fact that a twisted group ring of $G$ has no “obvious homomorphism” [Pass77] p. 14 to some twisted group ring of a given quotient of $G$. So though $G \sim_F H$ might still be true for any field $F$ the arguments to prove this would be different from the argument of Dade.

Also Yamazaki covers can not bring the whole solution as $H^2(G, \mathbb{F}^*)$ can be infinite, e.g. for $F = \mathbb{Q}$, and then no Yamazaki cover exists.

**Remark 5.9.** The probably most famous example obtained in the study of the classical group ring isomorphism problem is Hertweck’s counterexample to the integral isomorphism problem [Her01]. This counterexample consists of two non-isomorphic groups $G$ and $H$ of order $2^{21} \cdot 97^{28}$ such that $\mathbb{Z}G \cong \mathbb{Z}H$. It is not clear to us if there exists a ring $R$ such that $G \not\cong_R H$. But it clear that $RG \cong RH$ and $H^2(G, R^*) \cong H^2(H, R^*)$ for any commutative ring $R$. This follows from the fact that $RG \cong R \otimes \mathbb{Z}G$ and the functorial definition of group cohomology, $H^n(G, M) \cong \text{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, G)$ for any $G$-module $M$ and where $\text{Ext}$ denotes the Ext-functor. So $H^2(G, M)$ depends only on the group ring $\mathbb{Z}G$ and not $G$ itself. It would be very interesting to determine if $G \sim_R H$ indeed holds independently of $R$.

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