Lattice Polytopes of Degree 2

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Abstract. A theorem of Scott gives an upper bound for the normalized volume of lattice polygons with exactly $i > 0$ interior lattice points. We will show that the same bound is true for the normalized volume of lattice polytopes of degree 2 even in higher dimensions. In particular, there is only a finite number of quadratic polynomials with fixed leading coefficient being the $h^*$-polynomial of a lattice polytope.

1. Introduction

An $n$-dimensional lattice polytope $P \subset \mathbb{R}^n$ is the convex hull of a finite number of elements of $\mathbb{Z}^n$. In the following, we denote by $\text{Vol}(P) = n!\text{vol}(P)$ the normalized volume of $P$ and may call it the volume of $P$. By $\Pi^{(1)} := \Pi(P) \subset \mathbb{R}^{n+1}$, we denote the convex hull of $(P,0) \subset \mathbb{R}^{n+1}$ and $(0,\ldots,0,1) \in \mathbb{R}^{n+1}$, which we will call the standard pyramid over $P$. Recursively we define $\Pi^{(k)}(P) = \Pi\left(\Pi^{(k-1)}(P)\right)$ for all $k > 0$. $\Delta_n$ will denote the $n$-dimensional basic lattice simplex throughout, i.e. $\text{Vol}(\Delta_n) = 1$. If two lattice polytopes $P$ and $Q$ of the same dimension are equivalent via some affine unimodular transformation, we will write $P \equiv Q$. The $k$-fold of a polytope $P$ will be the convex hull of the $k$-fold vertices of $P$ for every $k \geq 0$.

Pick’s formula gives a relation between the normalized volume, the number of interior lattice points and the number of lattice points of a lattice polygon, i.e. of a two-dimensional lattice polytope: $\text{Vol}(P) = |P \cap \mathbb{Z}^2| + |P^o \cap \mathbb{Z}^2| - 2$. Here $P^o$ means the interior of the polytope $P$.

In 1976 Paul Scott [9] proved that the volume of a lattice polygon with exactly $i \geq 1$ interior lattice points is constrained by $i$:

**Theorem 1.1 (Scott).** Let $P \subset \mathbb{R}^2$ be a lattice polygon such that $|P^o \cap \mathbb{Z}^2| = i \geq 1$. If $P \cong 3\Delta_2$, then $\text{Vol}(P) = 9$ and $i = 1$. Otherwise the normalized volume is bounded by $\text{Vol}(P) \leq 4(i + 1)$. According to Pick’s formula, this implies $|P \cap \mathbb{Z}^2| \leq 3i + 6$ and $|P \cap \mathbb{Z}^2| \leq \frac{3}{2}\text{Vol}(P) + 3$.

Besides Scott’s proof, there are two proofs by Christian Haase and Joseph Schicho [5]. Another proof is given in [13].

Our aim is to generalize Scott’s theorem. Therefore we need to introduce another invariant, the degree of a lattice polytope:

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It is known from [4], [10] and [11] that $h^*_P(t) := (1-t)^{n+1} \sum_{k \geq 0} |kP \cap \mathbb{Z}^n| t^k \in \mathbb{Z}[t]$ is a polynomial of degree $d \leq n$. This number is described as the degree of $P$ and is the largest number $k \in \mathbb{N}$ such that there is an interior lattice point in $(n+1-k)P$ (cf. [2]). The leading coefficient of $h^*_P$ is the number of interior lattice points in $(n+1-d)P$ and the constant coefficient is $h^*_P(0) = 1$. Moreover the sum of all coefficients is the normalized volume of $P$ and all coefficients are non-negative integers by the non-negativity theorem of Richard P. Stanley [10].

It is easy to show that the $h^*$-polynomial of $P$ and $\Pi(P)$ are equal. So $P$ and $\Pi(P)$ have the same degree and the same normalized volume, which is the sum of all coefficients of the $h^*$-polynomial. Moreover

$$\left|\left((n+2-d)\Pi(P)\right)^\circ \cap \mathbb{Z}^{n+1}\right| = \left|\left((n+1-d)P\right)^\circ \cap \mathbb{Z}^n\right|.$$ 

Scott’s theorem shows that the normalized volume of a two-dimensional lattice polytope of degree 2 with exactly $i > 0$ interior lattice points is bounded by $4(i+1)$, except for one single polytope: $3\Delta_2$. We generalize this result to the case of $n$-dimensional lattice polytopes of degree 2.

**Theorem 1.2.** Let $P \subset \mathbb{R}^n$ be a $n$-dimensional lattice polytope of degree 2. If $P \cong \Pi^{(n-2)}(3\Delta_2)$, then $\text{Vol}(P) = 9$, $|P \cap \mathbb{Z}^n| = 8 + n$ and $\left|\left((n-1)P\right)^\circ \cap \mathbb{Z}^n\right| = 1$. Otherwise the following equivalent statements hold:

1. $\text{Vol}(P) \leq 4(i+1)$
2. $b \leq 3i + n + 4$
3. $b \leq \frac{3}{4} \text{Vol}(P) + n + 1$,

where $b := |P \cap \mathbb{Z}^n|$ and $i := \left|\left((n-1)P\right)^\circ \cap \mathbb{Z}^n\right| \geq 1$.

The following theorem of Victor Batyrev [1] motivates our estimation of the normalized volume of a lattice polytope of degree $d$:

**Theorem 1.3 (Batyrev).** Let $P \subset \mathbb{R}^n$ be an $n$-dimensional lattice polytope of degree $d$. If $n \geq 4d \left(2d + \text{Vol}(P) - 1\right)$, then $P$ is a standard pyramid over an $(n-1)$-dimensional lattice polytope.

There is a recent result by Benjamin Nill [7] which even strengthens this bound:

**Theorem 1.4 (Nill).** Let $P \subset \mathbb{R}^n$ be a $n$-dimensional lattice polytope of degree $d$. If $n \geq (\text{Vol}(P) - 1)(2d + 1)$, then $P$ is a standard pyramid over an $(n-1)$-dimensional lattice polytope.

Jeffrey C. Lagarias and Günter M. Ziegler showed in [6] that up to unimodular transformation there is only a finite number of $n$-dimensional lattice polytopes having a fixed volume. From Theorem 1.3 or Theorem 1.4 follows

**Corollary 1.5 (Batyrev).** For a family $\mathcal{F}$ of lattice polytopes of degree $d$, the following is equivalent:

1. $\mathcal{F}$ is finite modulo standard pyramids and affine unimodular transformation,
There is a constant $C_d > 0$ such that $\text{Vol}(P) \leq C_d$ for all $P \in \mathcal{F}$.

**Conjecture 1.6 (Batyrev).** Let $P$ be a lattice polytope of degree $d$ with exactly $i \geq 1$ interior lattice points in its $(\dim(P) + 1 - d)$-fold. Its normalized volume $\text{Vol}(P)$ can then be bounded by a constant $C_{d,i}$, only depending on $d$ and $i$. The finiteness of lattice polytopes of degree $d$ with this property up to standard pyramids and affine unimodular transformation follows from Theorem 1.3.

Theorem 1.2 proves Conjecture 1.6 in the case $d = 2$.

**Corollary 1.7.** Up to affine unimodular transformations and standard pyramids there is only a finite number of lattice polytopes of degree 2 having exactly $i \geq 1$ interior lattice points in their adequate multiple. This follows from Theorem 1.2 and Theorem 1.3.

**Corollary 1.8.** There is only a finite number of quadratic polynomials $h \in \mathbb{Z}[t]$ with leading coefficient $i \in \mathbb{N}$, such that $h$ is the $h^*$-polynomial of a lattice polytope. This follows from Theorem 1.2 and the fact that all coefficients of $h^*_P$ are positive integers summing up to $\text{Vol}(P)$.

In the remaining part of the paper we prove Theorem 1.2.

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**2. Preparations**

The formula of Pick can be easily generalized for higher dimensional polytopes of degree 2 using their $h^*$-polynomial. This shows that statements (1) – (3) in Theorem 1.2 are equivalent.

**Lemma 2.1.** An $n$-dimensional lattice polytope of degree 2 has normalized volume $\text{Vol}(P) = b + i - n$, where $b := |P \cap \mathbb{Z}^n|$ and $i := \left| \left( (n - 1)P \right)^\circ \cap \mathbb{Z}^n \right|$.

**Proof.** The normalized volume of $P$ can be computed by adding the coefficients of the $h^*$-polynomial of $P$. Consequently $\text{Vol}(P) = 1 + (b - n - 1) + i$. \qed

Let $s \subset P$ be a face of $P$. By $\text{st}(s) = \bigcup F$, we denote the star of $s$ in $P$, where the union is over all faces $F \subset P$ of $P$ containing $s$.

**Lemma 2.2.** Let $P$ be an $n$-dimensional lattice polytope of degree 2 and $s \subset P$ a face of $P$ having exactly $j > 0$ interior lattice points in its $(n-2)$-fold:

$$\left( (n - 2)s \right)^\circ \cap \mathbb{Z}^n = \{x_1, \ldots, x_j\}.$$  

Moreover, we suppose

$$z := \left| P \setminus \text{st}(s) \cap \mathbb{Z}^n \right| \geq 1.$$  

Then $0 < j + z - 1 \leq \left| \left( (n - 1)P \right)^\circ \cap \mathbb{Z}^n \right|$.
Remark 2.3. Let us first consider an easy case. If \( z = 1 \), i.e., \( P \setminus \text{st}(s) \cap \mathbb{Z}^n = \{ p \} \), then
\[
p + x = (n - 1) \left( \frac{n - 2}{n - 1} x + \frac{p}{n - 1} \right) \in (n - 1)P \cap \mathbb{Z}^n \quad \forall x \in (n - 2)s \cap \mathbb{Z}^n
\]
yield \( j > 0 \) distinct lattice points in \( (n - 1)P \). So \( 0 < j \leq \left| (n - 1)P \right| \cap \mathbb{Z}^n \) as claimed.

Proof. If \( l = 1 \), the claim is certainly correct. Hence let \( l \geq 2 \).
There is a lattice point \( z_i \in y_i \cap \left( D \setminus \left( \frac{x_i}{n} \right) \right) \cap \mathbb{Z}^{n+1} \). Define \( \pi_l := \text{conv}(s_l, z_i) \). Obviously \( \pi_l \cap s = s_l \). By induction, there are further pyramids \( \pi_1, \ldots, \pi_{l-1} \) satisfying \( \pi_k \cap s = s_k \) and \( \pi_k \cap \pi_{k'} \subset \{ z_1, \ldots, z_{k'} \} \subset D \cap \mathbb{Z}^{n+1} \forall k < k' < l \).

Assume \( \pi_l \cap \pi_k \not\subset \{ z_1, \ldots, z_l \} \), i.e. there exists a point \( q \in \pi_l \cap \pi_k \), \( q \not\in \{ z_1, \ldots, z_l \} \) and \( k < l \). Therefore \( y_k|\pi_k \geq 0 \), because \( y_k|s_k \geq 0 \) and \( y_k(z_i) = 0 \).
In particular, \( y_k(q) \geq 0 \), and \( y_l(q) \geq 0 \) as well. As \( q \in \pi_l \cap \text{conv}(s_l, z_i) \), there is a point \( p \in s_l \) and a number \( \lambda \in [0, 1] \) such that \( q = \lambda p + (1 - \lambda)z_i \). Therefore \( 0 \leq y_k(q) = y_k(p) + (1 - \lambda)y_k(z_i) \) with \( y_k(z_i) \leq 0 \) as \( z_i \in D \) and \( y_k|D \leq 0 \).

If \( y_k(p) > 0 \), then \( p \in s \cap \{ x \in \mathbb{R}^{n+1} : y_k(x) \geq 0 \} = s_k \cap \bigcup_{r \leq k} s_r \). But this is a contradiction to \( p \in s_l \). So
\[
0 \leq y_k(q) = \lambda y_k(p) + (1 - \lambda)y_k(z_i) \leq 0
\]
with equality only in the case of \( \lambda = 0 \) and \( y_k(z_i) = 0 \). Therefore the intersection of \( \pi_k \) and \( \pi_l \) is \( q = z_i \) or empty. This is a contradiction to \( \pi_l \cap \pi_k \not\subset \{ z_1, \ldots, z_l \} \) and so the claim is proven.

The pyramids \( \pi_1, \ldots, \pi_k \) intersect with \( D \) only in faces of \( D \).
To any \( k \in \{ 1, \ldots, K \} \) denote by \( a_k := \left| \left( (n - 2)s_k \right) \cap \mathbb{Z}^{n+1} \right| \) the number of interior lattice points of \( (n - 2)s_k \). By Remark 2.3, there are \( a_k \geq 0 \) interior lattice points of \( (n - 1)s \) in \( (n - 1)\pi_k \). By adding up the number of interior lattice points in \( (n - 1)\pi_1, \ldots, (n - 1)\pi_K \), we derive from the claim
\[
\left| \bigcup_{k=1}^K \left( (n - 1)\pi_k \right) \cap \mathbb{Z}^{n+1} \right| \geq \sum_{k=1}^K a_k = j - 1.
\]
Furthermore to every \( p \in D \setminus \left( \frac{x_i}{n} \right) \) we get a lattice point of \( \left( (n - 1)\left( D \setminus \left( \frac{x_i}{n} \right) \right) \right) \cap \mathbb{Z}^{n+1} \) in \( (n - 1)P \) in the following way:
\[
p + x_1 = (n - 1) \left( \frac{n - 2}{n - 1} x_1 + \frac{p}{n - 1} \right) \in (n - 1)P \cap \mathbb{Z}^{n+1}.
\]
Finally we get \( \left| (n - 1)P \right| \cap \mathbb{Z}^{n+1} \geq j - 1 + z \). □

3. The Proof of the Main Theorem
If \( n = 2 \), then Theorem 1.2 is equal to Scott’s Theorem 1.1. So let \( n > 2 \).
The monotonicity theorem of Stanley 12 says that the degree of every face of a polytope is not greater than the degree of the polytope itself. In particular this is true for every facet. So we will distinguish the two cases that there is a facet of \( P \) having degree 2 or there is not.
For the second case we need a result of Victor Batyrev and Benjamin Nill. They
proved in [2] that every \( n \)-dimensional lattice polytope of degree less than 2 either is equivalent to a pyramid over the exceptional lattice simplex \( 2\Delta_2 \) or it is a Lawrence polytope, i.e. a lattice polytope projecting along an edge onto an \((n-1)\)-dimensional basic simplex.

**Case 1:** There is a facet \( F \subset P \) of \( P \) having degree two, i.e.
\[
\left| \left( P \setminus \right) \cap \mathbb{Z}^n \right|^o = j \geq 1.
\]
Define \( z := |\left( P \setminus \cap \mathbb{Z}^n \right)|. \) From Lemma 2.2 we get \( z + j - 1 \leq i. \) Thus, by induction, we get, if \( F \notin \Pi^{(n-3)}(3\Delta_2), \)
\[
|P \cap \mathbb{Z}^n| = |F \cap \mathbb{Z}^n| + |(P \setminus F) \cap \mathbb{Z}^n| \leq 3j + n - 1 + 4 + z = 3j + z - 1 - 2z + 2 + n + 4 \leq 3i + n + 4.
\]
Otherwise \( F \equiv \Pi^{(n-3)}(3\Delta_2) \) and again by induction and Lemma 2.2
\[
|P \cap \mathbb{Z}^n| = (n - 1) + 8, \ z \leq i \text{ and so } |P \cap \mathbb{Z}^n| = n - 1 + 8 + z \leq i + 7 + n. \text{ This term is smaller than } 3i + n + 4 \text{ if } i \geq 2. \text{ If } i = 1 \text{ however, we get}
\]
\[
n + 8 \leq |P \cap \mathbb{Z}^n| = n + 7 + z \leq i + 7 + n = 8 + n,
\]
so \( |P \cap \mathbb{Z}^n| = 8 + n \) and \( \text{Vol}(P) = 9 \) by Lemma 2.1 In this case \( P \equiv \Pi^{(n-2)}(3\Delta_2) \) because \( \text{Vol}(F) = 9 \) and \( F \equiv \Pi^{(n-3)}(3\Delta_2). \)

**Case 2:** Every facet \( F \) of \( P \) has degree \( \text{deg}(F) \leq 1. \)

Let \( y \) be an edge of \( P \) having the maximal number of lattice points; its length will be denoted by \( h_1 \), i.e. \( h_1 = |y \cap \mathbb{Z}^n| - 1 \). Among all \( 2 \)-codimensional faces of \( P \) containing \( y \), \( s \) should be the face having the maximal number of lattice points. We will denote by \( F_1 \) and \( F_2 \) the two facets of \( P \) containing \( s \).

Again the monotonicity theorem of Stanley [12] implies \( \text{deg}(s) \leq \text{deg}(F_1) = 1. \)

Similarly to case 1, we will denote by \( z := |P \setminus \{F_1 \cup F_2\} \cap \mathbb{Z}^n| \) the number of lattice points of \( P \) not in \( F_1 \) and \( F_2 \).

By the result of Victor Batyrev and Benjamin Nill [2] we find that the facets \( F_1 \) and \( F_2 \) are either \((n-1)\)-dimensional Lawrence polytopes or pyramids over \( 2\Delta_2. \)

(A) \( F_1 \) and \( F_2 \) are Lawrence polytope with heights \( h_1^{(k)}, h_2^{(k)}, \ldots, h_{n-1}^{(k)} \forall k \in \{1, 2\} \), where we assume that \( h_1^{(1)} = h_2^{(1)} = h_1 \forall l \in \{1, \ldots, n - 2\}, \)
\[
s = \text{conv}(0, h_1e_1, e_i, e_l + h_1e_1 : 2 \leq l \leq n - 2),
\]
where \( \{e_1, \ldots, e_{n-2}, e_{n-1}^{(k)}\} \) should denote a lattice basis of \( \text{lin}(F_k) \cap \mathbb{Z}^n \) such that \( F_k = \text{conv}(s, e_{n-1}^{(k)}, e_{n-1}^{(k)} + h_1^{(k)} e_1) \) for \( k \in \{1, 2\} \). Since the degree of the Lawrence prism \( s \) is at most one, we obtain
\[
\left| \left( P \setminus s \right)^o \cap \mathbb{Z}^n \right| = \text{Vol}(s) - 1 = \left( \sum_{i=1}^{n-2} h_1 \right) - 1.
\]
We may assume \( z = |\left( P \setminus \right) \cap \mathbb{Z}^n| \neq 0 \) because otherwise \( P \) would be a prism over the face \( P \cap \{X_1 = 0\} \), which is an \((n-1)\)-dimensional lattice simplex of degree at most 1, whose only lattice points are vertices. By [2] this is a basic simplex and
hence $P$ is a Lawrence polytope. Consequently $\deg(P) < 2$, a contradiction.

We have to distinguish the following two cases:

(i) $\left(\left((n-2)s\right)^{\circ}\cap\mathbb{Z}^n\right) \cap \mathbb{Z}^n \geq 1$.

Because of Lemma 2.2, we get the estimation

$$z + \left(\sum_{i=1}^{n-2} h_i - 1\right) - 1 \leq i.$$ 

So we can bound the number of lattice points of $P$:

$$|P \cap \mathbb{Z}^n| = \left|\{(F_1 \cup F_2) \cap \mathbb{Z}^n\} + z = |s \cap \mathbb{Z}^n| + h_{n-1}^{(1)} + h_{n-1}^{(2)} + 1 + z\right.$$ 

$$= \sum_{i=1}^{n-2} h_i + (n-2) + h_{n-1}^{(1)} + h_{n-1}^{(2)} + 2 + z \leq i + n + 2h_1 + 2$$ 

$$h_1 \leq i + 1 \leq i + n + 2(i + 1) + 2 = 3i + n + 4.$$

(ii) $\left(\left((n-2)s\right)^{\circ}\cap\mathbb{Z}^n\right) = 0$.

In this case, $s$ has degree zero, so it is a basic simplex. Our assumption on $s$ implies that every lattice point of $P$ is a vertex. If $n = 3$, then Howe’s theorem \cite{8} yields that $P$ has at most 8 vertices, therefore $|P \cap \mathbb{Z}^n| \leq 8 < n + 4 + 3i$. So let $n \geq 4$.

In that case, since every 2-codimensional face is a simplex and every facet is a Lawrence prism, we see that $P$ is simplicial, i.e. every facet is a simplex. We may suppose that $P$ is not a simplex. Let $S$ be a subset of the vertices of $P$ such that the convex hull of $S$ is not a face of $P$. Then the sum over the vertices of $S$ is a lattice point in the interior of $|S| \cdot P$. Since the degree of $P$ is two, this implies $|S| \geq n - 1$. In other words, every subset of the vertices of $P$ that has cardinality at most $n - 2$ forms the vertex set of a face of $P$, i.e. $P$ is $(n - 2)$-neighbourly. As is known from \cite{3}, a polytope of dimension $n$ that is not a simplex is at most $\lfloor \frac{n}{2} \rfloor$-neighbourly. Therefore $n - 2 \leq \frac{3}{2}$. This shows $n = 4$.

Let $f_j \geq 0$ be the number of $j$-dimensional faces of $P$. Since $P$ is a 2-neighbourly simplicial 4-dimensional polytope we get $f_1 = \left(\frac{f_2}{2}\right)$ and $f_2 = 2f_3$. Since the Euler characteristic of the boundary of $P$ vanishes, i.e. $f_0 - f_1 + f_2 - f_3 = 0$, we deduce $f_3 = \frac{f_0(f_0-3)}{6}$. Let $D$ denote the set of subsets $\Delta$ of the vertices of $P$ such that $\Delta$ has cardinality three but $\Delta$ is not the vertex set of a face of $P$. Therefore, $|D| = \left(\frac{f_0}{2}\right) - f_2 = f_0 \left(\frac{(f_0-1)(f_0-2)}{6} - f_0 - 3\right)$. Since $|\{(e, \Delta) : e \text{ is an edge of } P, \Delta \in D, e \subset D\}| = 3|D|$, double counting yields that there exists an edge $e$ of $P$ that is contained in at least $\frac{3|D|}{f_1}$ many elements $\Delta \in D$. Therefore, any such $\Delta$ contains one vertex that is not in the star of $e$, and hence Lemma 2.2 yields

$$i \geq \frac{3|D|}{f_1} = f_0 - 2 - 6\frac{f_0 - 3}{f_0 - 1} \geq f_0 - 8.$$ 

Thus, $|P \cap \mathbb{Z}^n| = f_0 \leq 8 + i < n + 4 + 3i$.

(A') $F_1$, $F_2$ and $s$ have no common projection direction.

Without loss of generality let $F_1$ and $s$ have two different projection directions. If $s$ contains an edge of length at least 2, then this has to be a common projection
direction with $F_1$, because $s$ and $F_1$ are Lawrence prisms. But this is a contradiction. Hence, all lattice points in $s$ are vertices. In particular, $y$ has length one, so also all lattice points of $P$ are vertices.

Since any of the two different projection directions of the Lawrence prism $s$ maps a four-gon face onto the edge of an unimodular base simplex and two edges of the four-gon give the projection direction, we see that there is at most one four-gon face in $s$. Therefore, $s$ contains at most $(n-2)+2=n$ lattice points.

Since $F_k$ contains at most two vertices not in $s$ for $k \in \{1, 2\}$, we get $|(F_1 \cup F_2) \cap \mathbb{Z}^n| \leq n+4 < n+4+3i$. Therefore we may assume $z:=|P \setminus (F_1 \cup F_2) \cap \mathbb{Z}^n| \neq 0$.

If $\left((n-2)s\right)^\circ \cap \mathbb{Z}^n = 0$, then we will proceed exactly like in case (ii) from (A).

So let $j:=\left|\left((n-2)s\right)^\circ \cap \mathbb{Z}^n\right| \geq 1$.

Because of Lemma [2,2] we get the estimation $z+j-1 \leq i$, in particular $z \leq i$. Hence we can bound the number of lattice points of $P$:

$$|P \cap \mathbb{Z}^n| = |(F_1 \cup F_2) \cap \mathbb{Z}^n| + z \leq n + 4 + i < 3i + n + 4.$$  

(B) $F_1$ is a Lawrence polytope with the heights $h_1 \geq h_2 \geq \ldots \geq h_{n-1}$, $F_2 \cong \Pi^{(n-3)}(2\Delta_2)$.

Here

$s \cong \text{conv}(0, h_1 e_1, e_l, \ 2 \leq l \leq n-2)$

and $h_1 = 2$, $h_2 = \ldots = h_{n-2} = 0$, because $s$ is contained in the simplex $F_2$. If $z = |P \setminus (F_1 \cup F_2) \cap \mathbb{Z}^n| = 0$, then

$$|P \cap \mathbb{Z}^n| = |F_2 \cap \mathbb{Z}^n| + |F_1 \setminus F_2 \cap \mathbb{Z}^n| = 6 + (n-3) + h_{n-1} + 1 \leq 4 + n + 2 < 3i + n + 4.$$  

Otherwise if $z \geq 1$, we obtain just like in (A) $0 < z + (h_1 - 1) - 1 \leq i$. Therefore

$$|P \cap \mathbb{Z}^n| = |(F_1 \cup F_2) \cap \mathbb{Z}^n| + z = |s \cap \mathbb{Z}^n| + (h_{n-1} + 1) + 3 + z \leq h_{n-1} + (n-2) + (h_{n-1} + 1) + 3 + z \leq i + 4 + h_{n-1} + n \leq 3i + n + 4.$$  

(C) $F_1 \cong F_2 \cong \Pi^{(n-3)}(2\Delta_2)$.

Here either $s$ is a pyramid over $2\Delta_1$ or $s \cong \Pi^{(n-4)}(2\Delta_2)$. Again $h_1 = 2$.

If $z = |P \setminus (F_1 \cup F_2) \cap \mathbb{Z}^n| = 0$, then

$$|P \cap \mathbb{Z}^n| = |F_2 \cap \mathbb{Z}^n| + |F_1 \setminus F_2 \cap \mathbb{Z}^n| \leq 6 + (n-3) + 3 < 3i + n + 4.$$  

Otherwise if $z \geq 1$, we obtain $z \leq i$ because of $\left|(n-2)s\right)^\circ \cap \mathbb{Z}^n \geq 1$ and Lemma [2,2]. So as a result

$$|P \cap \mathbb{Z}^n| = |F_1 \cap \mathbb{Z}^n| + |F_2 \setminus F_1 \cap \mathbb{Z}^n| + z \leq (6 + n - 3) + 3 + z = n + z + 6 \leq n + i + 6 \leq n + 3i + 4.$$  

This completes the proof. \qedsymbol
Remark 3.1. In [11], Stanley shows that the coefficients of $h^*_P$, also appear in the polynomial $(1-t)^{n+1}\sum_{k\geq 0} |(kP)^{\circ}\cap \mathbb{Z}^n| t^k \in \mathbb{Z}[t]$. So we can also compute the coefficients of $h^*_P$ in a different way than in Lemma 2.1. Then it is easy to show that the bounds of Theorem 1.2 are also equivalent to the following estimations:

$$|(nP)^{\circ}\cap \mathbb{Z}^n| \leq (n+4)i + 3,$$

$$|2P\cap \mathbb{Z}^n| \leq (4+3n)(i+1) + \frac{n(n+3)}{2}.$$

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