No-signaling bounds for quantum cloning and metrology

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The impossibility of superluminal communication is a fundamental principle of physics. Here we use this principle to derive limitations on several fundamental tasks in quantum information processing and quantum metrology. In particular, we derive tight no-signaling bounds for probabilistic cloning and super-replication that coincide with the corresponding optimal achievable fidelities and rates known. In the context of quantum metrology, we derive the Heisenberg limit from the no-signaling principle for certain scenarios including reference frame alignment and maximum likelihood state estimation.

Nothing can travel faster than the speed of light. This is one of the pillars of modern physics and an explicit element of Einstein’s theory of relativity. Any violation of this principle would lead to problems with local causality giving rise to logical contradictions. This principle not only applies to matter, but also to information, rendering superluminal communication impossible. Whilst not explicitly contained in the postulates of quantum mechanics all attempts to construct or observe violations of this principle have failed, leading us to believe that this is indeed a basic ingredient of our description of nature. In fact, modifications of quantum mechanics, e.g., by allowing non-linear dynamics, would lead to signaling and a violation of this fundamental principle [13]. It is therefore natural to assume that no-signaling holds and try to deduce what follows under such an assumption.

Indeed, the no-signaling principle has been used to derive bounds and limitations on several physical processes and tasks. These include the observation that a perfect quantum copying machine would allow for superluminal communication [15–17], limitations on universal quantum 1 → 2 cloning [7–8], a security proof for quantum communication [9], optimal state discrimination [10], and bounds on the success probability of port-based teleportation [11]. However in a more general setting, where no quantum state structure is assumed, the no-signaling principle is not restrictive enough to recover the true physics. For instance, no-signaling allows for stronger non-local correlations than possible within quantum mechanics [12]. Hence, attempts have been made to further supplement the no-signaling principle in order to retrieve quantum mechanics [13–16].

Here we assume the Hilbert state structure of states and show how the no-signaling principle directly leads to tight bounds on different fundamental tasks in quantum information processing and quantum metrology. We apply the no-signaling principle to derive bounds on phase covariant quantum cloning [17–18], replication of unitary operations [19–20], and quantum metrology—more specifically phase reference alignment [21–22]. Not only does the no-signalling principle allow us to prove ultimate limits on these fundamentally important tasks, it also allows us to demonstrate the optimality of known protocols and shed light on the recently discovered possibility of probabilistic super-replication of states [18] and operations [20].

We start by showing how the impossibility of faster-than-light communication between Alice and Bob can be used to provide upper bounds on Bob’s ability to perform certain tasks, even if Bob has access to supra-quantum resources. Using this approach we derive a no-signaling bound on the global fidelity of $N \rightarrow M$ probabilistic phase-covariant cloning. Our derivation is constructive and we provide the optimal deterministic quantum protocol that achieves the bound [18]. In similar fashion, we derive a no-signaling bound on the replication of unitary operations which is tight in the large $M$ limit.

Finally, we derive the Heisenberg limit of quantum metrology solely from the no-signaling principle. We find a tight no-signaling bound on the maximal likelihood and a bound on the fidelity of reference frame alignment for phase. The latter bound is not tight but exhibits the proper scaling. This discrepancy arises from the fact that we consider a specific communication scenario to derive the no-signaling bound. We supplement this approach by a general argument, extending that of [9], showing that the no-signaling condition ultimately leads to the Born rule. Therefore optimal quantum protocols are at the edge of no-signaling. Furthermore, we derive a bound for Bayesian phase estimation for a general prior where the optimal quantum strategy is not known.

RESULTS

No-signaling

In this section we describe the operational setting underpinning all three tasks we consider (cloning, replication of unitaries, and metrology), as well as the no-signalling condition. All three tasks we consider can be described in the following operationally generic setting: a party, Bob, possess an $N$-qubit state, $|\Phi^N\rangle$, and receives the action of a unitary operator $U^N_\theta$ such that

$$|\Phi^N_\theta\rangle = U^N_\theta |\Phi^N\rangle,$$  

(1)
where \( U_\theta = e^{i\theta H} \) with \( H \) an arbitrary Hamiltonian acting on 2-level systems (qubits) with spectral radius \( \sigma(H) \), \( \theta \) uniformly chosen from \((0, 2\pi \sigma(H)]\). Bob has to process \( U_\theta^{\otimes N} \) for some quantum information task in an optimal way. Throughout this paper we assume the state structure of quantum mechanics, i.e. that states are described by vectors in a Hilbert space, however make no assumptions on Bob’s processing capabilities. In fact, we allow Bob processing power that is supra-quantum, i.e., not restricted to quantum physics. What Bob has to output varies depending on which task he performs. For example, if the required task is the cloning of the state \(|\theta\rangle\), then Bob has to output an \( M\)-qubit state, \( \rho_M^\theta \), that is a close approximation to \(|\theta\rangle^{\otimes M}\). If the required task is the replication of the unitary operator \( U_\theta \), then Bob has to output a quantum channel acting on the Hilbert space of \( M \) qubits that is a close approximation to \( U_\theta^{\otimes M} \). Finally, if the required task is to estimate the parameter \( \theta \), then Bob must output a probability distribution corresponding to his updated knowledge about parameter \( \theta \). We denote the outcome of Bob’s processing, be it a quantum state, channel, or probability distribution, by \( \mathcal{O}(\theta) \).

To incorporate the no-signaling condition we consider the above task in a communication scenario. In particular, we consider that Bob holds one part of a suitably chosen entangled state, \( |\Psi\rangle_{AB} \), which he shares with Alice. The state \( |\Psi\rangle_{AB} \) can always be chosen such that Bob receives the action of \( U_\theta^{\otimes N} \) on an arbitrary input state. This is achieved by Alice first performing \((U_\theta^{\otimes N} \otimes I)|\Psi\rangle_{AB}\) followed by a suitable measurement (see Fig. 1 and methods). Denoting by \( \mathcal{P}(\theta|k) \) the outcome of Bob’s processing, given Alice’s measurement outcome was \( k \), the no-signaling condition implies that (see methods section)

\[
\mathcal{O}(\theta) = \sum_k p_k \mathcal{P}(\theta|k)
\]

is independent of the setting \( \theta \) chosen by Alice, i.e., \( \mathcal{O}(\theta) = \mathcal{O} \). Eq. (2) says that Alice’s choice of \( \theta \) cannot be detected by Bob [7], and thus Alice cannot use the entanglement present in \(|\Psi\rangle_{AB}\) to superluminally communicate any information to Bob.

We stress that we make no assumptions about the processing Bob uses to obtain \( \mathcal{P}(\theta|k) \). Indeed the processing can even be stronger than what is allowed by quantum mechanics. Our only requirement is that Bob’s output is valid for the task at hand, i.e., an \( M \)-qubit quantum state in the case of cloning, a quantum channel in the case of unitary replication, and a classical probability distribution in the case of phase reference alignment.

**Probabilistic phase covariant cloning**

We first apply the no-signaling condition to the case of phase-covariant quantum cloning (PCC). The latter task involves cloning an unknown state from the set \( \{ |\psi(\theta)\rangle = U(\theta) |\psi\rangle \} \) [17-18]. We focus on PCC of equatorial states, \( |\psi(\theta)\rangle = 1/\sqrt{2}(|0\rangle + e^{i\theta} |1\rangle) \), which play a crucial role in proving the security of quantum key distribution [10]. Specifically, we provide a bound for the optimal PCC of \( N \) qubits into \( M > N \) qubits and show that this bound is achievable by a deterministic quantum mechanical strategy. The latter strategy involves the use of a suitable \( N \)-partite entangled input state on which \( U_\theta^{\otimes N} \) is applied. We then show how our deterministic strategy is equivalent to the probabilistic PCC of [13], by introducing a suitable filter operation that maps \( |\psi(\theta)\rangle^{\otimes N} \) to the suitable \( N\)-partite entangled state.

A **deterministic**, phase-covariant quantum cloning machine is some transformation, \( C \), whose input is \( N \) copies of an unknown equatorial qubit state \( |\psi(\theta)\rangle \), that outputs an \( M \)-qubit state \( \rho^M(\theta) = C \langle |\psi(\theta)\rangle |\psi(\theta)\rangle^{\otimes N} \). Optimal deterministic cloning machines, be it state-dependent [17-23] or state-independent [24-27], have been constructed and tight bounds, for the case of \( 1 \rightarrow 2 \) cloning, based on the no-signaling condition have been derived [17, 18]. A probabilistic cloning machine is more powerful in that it allows for a much higher number of copies at the cost of succeeding only some of the time. Indeed, probabilistic PCC when successful can output up to \( N^2 \) faithful copies of \( |\psi(\theta)\rangle \). However, the probability of success is exponentially small [18].

In order to quantify the success of the cloning procedure we use the **mean global fidelity**

\[
F_{CN \rightarrow M} = \int \frac{d\theta}{2\pi} F(\rho^M_\theta, \langle |\psi(\theta)\rangle |\psi(\theta)\rangle^{\otimes M}),
\]

where \( F(\rho^M_\theta, \langle |\psi(\theta)\rangle |\psi(\theta)\rangle^{\otimes M}) = \text{Tr} \{ \rho^M_\theta \langle |\psi(\theta)\rangle |\psi(\theta)\rangle^{\otimes M} \} \) is the fidelity between the the output of the cloner and \( M \) perfect copies of the input state [28]. If the input state to the probabilistic PCC machine is remotely prepared by Alice, as explained in the methods section, then the no-signaling condition on the output of Bob’s probabilistic PCC procedure has to be independent of \( \theta \), i.e.,

\[
\mathcal{O}_M^N = \mathcal{O}_M^N(\theta) = \frac{1}{N+1} \sum_{k \geq 0} \rho^M_{\theta + \frac{k}{N+1}}.
\]
It remains to determine the optimal global mean fidelity (Eq. (3)) allowed by the no-signaling constraint (Eq. (4)). Using the symmetries of the output state in Eq. (1) and the concavity and joint concavity of the fidelity [29], it can be shown (see methods) that the global mean fidelity is upper bounded by

$$F^{\mathcal{N} \rightarrow M} \leq \frac{1}{2M} \sum_{\lambda = 0}^{N} \left( \frac{M}{M+N} + \lambda \right),$$

where \([\cdot]\) denotes the floor function. Assuming that \(M \gg N\), and performing the Gaussian approximation to the binomial distribution the mean global fidelity is

$$F^{\mathcal{N} \rightarrow M} \leq \text{erf} \left( \frac{\sqrt{2}(N+1)}{\sqrt{\pi M}} \right).$$

We note that as long as \(M = \mathcal{O}(N^2)\) the cloning fidelity approaches unity in the limit \(N \rightarrow \infty\). Indeed, one can make an even stronger claim. Any replication procedure that respects the no-signaling condition and produces a number of replicas \(M = \mathcal{O}(N^{2+c})\) does so with a cloning fidelity that tends to zero exponentially fast.

We now show how one can achieve the no-signaling bound of Eq. (5) using a deterministic quantum mechanical strategy. Instead of \(N\)-copies of \(|\psi(\theta)\rangle\), suppose Bob prepares the entangled state

$$|\Phi^N\rangle \propto \sum_{\lambda = 0}^{N} \sqrt{\frac{M}{M+N} + \lambda} |N,\lambda\rangle,$$

where \(|N,\lambda\rangle\) are the symmetric states of \(N\) qubits with \(\lambda\) excitations. Bob now applies the cloning map \(\mathcal{C}: |N,\lambda\rangle \mapsto |M, (\frac{M-N}{2} + \lambda)\rangle\), that maps totally symmetric \(N\)-qubit states to totally symmetric \(M\)-qubit states. This strategy achieves the bound of Eq. (5) as the latter is valid for all input states.

The bound of Eq. (5) is the ultimate bound that can be achieved even by a probabilistic strategy. Indeed, the best probabilistic quantum mechanical PCC attains precisely the no-signaling bound of Eq. (5) [18]. In fact there is an easy way to see how the probabilistic strategy of [18] and the deterministic strategy described above are related. Starting from \(N\) copies of the state \(|\psi(\theta)\rangle\), the probabilistic PCC of [18] has Bob first apply the probabilistic filter that projects onto the state \(|\Phi^N\rangle\) of Eq. (7) and succeeds with probability \(p_{\text{succ}} = |\langle \Phi^N | + \rangle^{\otimes N}|^2\). As such a filter commutes with the unitary \(U_\theta^{\otimes N}\) it can be seen as part of the overall state preparation. The advantage, then, of probabilistic PCC can be simply understood as a passage from the standard quantum limit in quantum metrology, achieved for separable input states, to the Heisenberg limit achieved by entangled input states. Notice that no probabilistic advantage exists for the case of \(1 \rightarrow M\) cloning. For the latter, the fidelity of Eq. (5) takes the simple form \(F^{\mathcal{1} \rightarrow M} = \frac{1}{2M} \left( \frac{M+1}{M+2} \right)\) for \(M\) odd and \(F^{\mathcal{1} \rightarrow M} = \frac{1}{2M} \left( \frac{M+1}{M+2} \right)\) for \(M\) even and is known to be achievable by a deterministic strategy [17].

**FIG. 2.** Equivalence between probabilistic PCC and deterministic PCC for arbitrary states. The filter in the probabilistic cloning can be viewed as part of a probabilistic preparation of a general state from a separable \(N\)-qubit state. Allowing for arbitrary input states makes the preparation process deterministic.

**Replication of unitaries**

We now consider the task where Bob has to output an approximation, \(V_\theta\), of \(U_\theta^{\otimes M}\) having received only \(N\) uses of the black box implementing the unitary transformation \(U_\theta^{\otimes 1}\) [19] [20]. The figure of merit that one uses is the global Jamiołkowski fidelity (process fidelity) [30],

$$F(U_\theta^{\otimes M}, V_\theta) = \langle \psi_{V_\theta^{\otimes M}} | \rho_{\psi_\theta} | \psi_{U_\theta^{\otimes M}} \rangle,$$

averaged over all \(\theta\), where \(|\psi_{V_\theta^{\otimes M}}\rangle = (I \otimes U_\theta^{\otimes M}) |\Phi^+\rangle\) and \(\rho_{\psi_\theta} = (I \otimes V_\theta) |\Phi^+\rangle \langle \Phi^+ |\), with \(|\Phi^+\rangle = 1/\sqrt{2} \sum_{\tilde{n}} |\tilde{n}\rangle |\tilde{n}\rangle\), where \(\tilde{n}\) are the \(M\)-qubit bit strings, are the corresponding Choi-Jamiołkowski states [31] for \(U_\theta^{\otimes M}\) and \(V_\theta\), respectively. It was shown in [20] that when \(M < N^2\) Bob can approximate \(U_\theta^{\otimes M}\) almost perfectly, i.e., with process fidelity approaching unity in the large \(N\) limit. We now show that the protocol in [20] saturates the no-signaling bound.

In order to apply the no-signaling condition for the case of unitary replication in an easy way we consider the following communication scenario. Alice prepares the Choi-Jamiołkowski state corresponding to \(U_\theta^{\otimes N} = |\psi_{U_\theta^{\otimes N}}\rangle\), at Bob’s side, which he can then use to probabilistically implement \(U_\theta^{\otimes N}\) on an arbitrary input state [32]. Consequently, the protocol for which we shall derive a no-signaling bound is inherently probabilistic. We note that a bound for a probabilistic protocol is automatically a bound for a deterministic protocol as well, as the former are less restrictive than the latter.

The no-signaling constraint for unitary replication takes the form

$$R_N^{\mathcal{N}} = \frac{1}{N+1} \sum_{k=0}^{N} \| \mathbb{I} \otimes V_{\theta^k} + \frac{1}{N+1} |\Phi^+\rangle \langle \Phi^+ | \|^2,$$

and is independent of \(\theta\). As the average process fidelity (Eq. (8)) is identical to the mean global fidelity used for
PCC (Eq. 4) the no-signaling bound for probabilistic replication of unitaries reads

\[ F(U_{\theta}^M, V_{\theta}) \leq \frac{1}{2\pi M} \sum_{\lambda=0}^{N} \left( \frac{M}{2N} + \lambda \right). \] (10)

This bound is achieved, in the limit of large M, by the deterministic strategy in [20]. This implies that probabilistic processes offer no advantage in this case. Thus, the optimal deterministic replication of unitary operations allowed by quantum mechanics is at the edge of no-signaling.

**Quantum Metrology**

We now apply the no-signaling condition to provide bounds for quantum metrology. The latter task involves the use of N systems, known as the probes, prepared in a suitable state \( |\psi\rangle \in H^\otimes N \), and subjected to a dynamical evolution described by a completely positive map, \( \mathcal{E}_\theta \), that imprints the value of \( \theta \) onto their state, i.e., \( \rho_0 = \mathcal{E}_\theta(|\psi\rangle\langle\psi|) \). Information about the value of \( \theta \) is retrieved by a suitable measurement of the N probes. The goal in quantum metrology is to choose the initial state \( |\psi\rangle \) and final measurement such that the value of \( \theta \) can be inferred as precisely as possible.

If the N quantum probes are prepared in a separable quantum state, i.e., \( |\psi\rangle = |\phi\rangle^\otimes N \), then the mean square error with which \( \theta \) can be estimated, optimizing over all allowable measurements, scales inversely proportional with N [33]. This limit is known as the **standard quantum limit**. If, however, the N probes are prepared in a suitably entangled state, then the mean square error with which \( \theta \) can be estimated scales inversely proportional with \( N^2 \) [33]. This limit is known as the **Heisenberg limit**. By allowing for a probabilistic strategy, the Heisenberg limit in precision can be obtained even with separable states [34, 35]. Recently, it was shown that both the standard and Heisenberg limits are related with the maximum replication rates corresponding to a deterministic and probabilistic PCC strategies respectively [13, 35].

We now show how the no-signaling condition implies that the ultimate bound in precision for metrology is the Heisenberg limit, even if supra-quantum processing is allowed. We shall consider a particular type of quantum metrology—phase alignment—where the relevant parameter to be estimated is the phase of a local oscillator, \( \theta \in (0, 2\pi] \), which is initially completely unknown [37]. This is an example of Bayesian quantum metrology, where the relevant parameter \( \theta \in (0, 2\pi] \) is known with probability \( p(\theta) = 1/2\pi \). We shall consider two different ways of quantifying the precision of estimation of \( \theta \): the maximum likelihood of a correct guess, \( \mu = p(\theta|\theta) \) [22], and the **fidelity** of alignment, given by the payoff function \( f = \cos^2 \left( \frac{\theta - \theta'}{2} \right) \) [21].

For the case of phase alignment the no-signaling condition (Eq. 2) takes the form (see methods)

\[ p(\theta') = \frac{1}{N + 1} \sum_{k=0}^{N} p \left( \theta' + \frac{2\pi k}{N + 1} \right) \] (11)

and is independent of \( \theta \) (the same holds for a measurement with discrete outcomes). Note that we make no assumptions on how Bob obtains the probability distribution of Eq. (11). In particular we do not restrict Bob’s processing to be quantum mechanical. We only require that the inputs and outputs to Bob’s processing apparatus be valid quantum states and probability distributions respectively.

For the case where the precision is quantified by the maximum likelihood the no-signaling bound (Eq. (11)) gives \( p(\theta|\theta) \leq (N + 1)p(\theta) \). If the estimate \( \theta' \) is unbiased, all outcomes are equally likely and the no-signaling bound takes the simple form \( p(\theta|\theta) \leq N + 1 \). This bound is known to be achievable using the state \( |\Phi^N\rangle = \frac{1}{\sqrt{N+1}} \sum_{n} |n\rangle \) [22]. We note that this is the same state as the one maximizing the cloning fidelity. This is not a coincidence as it is known that asymptotic cloning is equivalent to state estimation [33, 35].

For the case where the precision is quantified by the fidelity of alignment, for each choice of \( \theta, \theta' \) the fidelity must be properly weighted by the joint probability distribution, \( p(\theta|\theta)p(\theta) \). The average fidelity of alignment is thus

\[ \bar{f} = \int \frac{d\theta}{2\pi} \int d\theta' \cos^2 \left( \frac{\theta - \theta'}{2} \right) p(\theta'|\theta) \] (12)

The probability distribution that both maximizes the average fidelity and is compatible with no-signaling is (see methods)

\[ p(\theta'|\theta) = \begin{cases} \frac{N+1}{2\pi} & \text{if } |\theta' - \theta| \leq \frac{\pi}{N+1} \\ 0 & \text{otherwise} \end{cases} \] (13)

This distribution leads to \( \bar{f} \approx 1 - \frac{\pi^2}{12N^2} \) in the large N limit. The maximum average fidelity achievable by a quantum mechanical strategy is \( \bar{f} \approx 1 - \frac{\pi^2}{12} \) [21], strictly smaller than the bound achieved by no-signaling. Nevertheless, the no-signaling bound gives rise to the right scaling with respect to N.

Let us now consider a more general metrological scenario where Bob has some prior knowledge, \( p(\theta) \), of the parameter \( \theta \). Following [39] we consider the prior, \( p(\theta|t) = \frac{1}{2\pi} \left( 1 + 2 \sum_{n=1}^{\infty} \cos(n\theta) e^{-n^2t} \right) \), that arises from a diffusive evolution of \( p(\theta) = \delta(\theta) \). The mean fidelity (Eq. (12)) now reads

\[ \bar{f}_t = \int d\theta' \int d\theta p(\theta'|\theta)p(\theta|t) \cos^2 \left( \frac{\theta - \theta'}{2} \right) \] (14)

An efficient algorithm optimizing \( \bar{f}_t \) for moderate N was derived in [39]. However, the optimization becomes intractable, even numerically, when N increases. Indeed,
optimizing $f_t$ for large $N$ is in general a hard task. Nevertheless the no-signaling constraint allows us to derive an upper bound for $f_t$ for large enough $N$ (see methods)

$$f_t \leq 1 - \frac{\pi^2}{12N^2} \theta_4(0, e^{-t}), \quad (15)$$

where $\theta_4(0, e^{-t}) = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2 t}$ is the Jacobi theta function ranging from 0, when $p(\theta; 0) = \delta(\theta)$, to 1, when $p(\theta; \infty) = 1/2\pi$. Again we discover that the ultimate bound in precision scales inversely proportional to $N^2$.

**DISCUSSION**

**Tightness of bounds**

We have shown that the no-signaling condition can set upper bounds on several important quantum information tasks, such as cloning, unitary replication and metrology. In the case of PCC cloning and unitary replication, we have shown that the no-signaling bound coincides with the optimal quantum mechanical strategy, implying that quantum mechanical strategies for PCC cloning and unitary replication are at the edge of no-signaling. However, for the case of metrology, and in particular for the average fidelity of estimation, we see that there is a gap between the no signaling bound and the optimal quantum strategy. Could this gap be an indication of the existence of a supra-quantum strategy, compatible with no-signaling, that outperforms the best quantum mechanical strategy? The answer is no, as we now explain.

In deriving the no-signaling constraint of Eq. (2) we only considered one particular way for Alice and Bob to attempt for faster-than-light communication; using a suitably entangled state $|\Psi\rangle_{AB}$. This, in turn led to the sharp probability distribution of Eq. (15). Indeed, one can construct a rather contrived scenario to show that the probability distribution of Eq. (15) leads to signaling (see methods).

A tighter no-signaling bound can be obtained if we optimize over all possible no-signaling scenarios, i.e., over all possible bi-partite entangled states $|\Psi\rangle_{AB}$. In fact any ensemble $\{\rho_k, p_k\}$ corresponding to a density matrix, $\rho_B = \sum_k p_k \rho_k$, at Bob’s side can be remotely prepared by Alice, if they initially share a suitable entangled state $|\Psi\rangle_{AB}$ (that only depends on $\rho_B$) and Alice does an appropriate measurement [1, 40].

**Quantum Mechanics at the edge of no-signaling**

The above argument shows that the probability distribution of Eq. (15) is only valid for one possible no-signaling scenario, and that in order to obtain a tighter bound we should consider all possible states shared between Alice and Bob and all possible measurements at Alice’s side that steer Bob’s partial state into different ensembles leading to the same density matrix. Would such an optimization close the gap between our no-signaling bound and the optimal quantum strategy? Following [3], we now show that such an optimization is not even necessary, as the only processing compatible with no-signaling is given by the Born rule.

Indeed, any ensemble leading to the same density matrix for Bob can be remotely prepared by Alice [1, 40]. This together with the no-signaling condition implies the linearity of Bob’s processing, $\mathcal{P}$. The latter states that for any two ensembles $\{p_k, \rho_k\}$ and $\{q_k, \sigma_k\}$ corresponding to $\rho_B$ one has

$$\sum_k p_k \mathcal{P}(\rho_k) = \sum_k q_k \mathcal{P}(\sigma_k) \equiv \mathcal{P}(\rho_B). \quad (16)$$

This must be the case as Bob can not detect in which ensemble Alice decided to steer his state $\rho_B$. Thus, this shows that any dynamical evolution of quantum states that respects no-signaling is necessarily described by a completely positive (CP) map [3]. The results of [3] are concerned with situations where the outputs of Bob’s processing are quantum mechanical states. In this case [3] implies that the optimal probabilistic quantum mechanical strategies are at the edge of no-signaling.

However, in the case of quantum metrology the outputs are probability distributions. We remark than in [3] the validity of the Born rule was assumed, and used to derive the possibility for remote preparation of any ensemble. In this case supra-quantum metrology is ruled out. However, if we make no assumptions on how probabilities are assigned to measurement outcomes of quantum states but only take remote state preparation as experimental fact, one can show that the no-signaling constraint implies the Born rule (see methods). In the case where the dimensions of Bob’s system is larger than two this can be seen as a consequence of Gleason’s theorem [11]. As we show in the methods this extends also to the case of dimension two. In other words, any rule $\mathcal{P}$ to assign probabilities for possible measurement outcomes to quantum states, that is compatible with no-signaling, is necessarily given by the Born rule, i.e., $\mathcal{P}(\rho) = \text{tr}_E \mathcal{P}_E \rho$ for a certain POVM $\{E_k\}$. This shows that also in the case of quantum metrology the optimal probabilistic quantum mechanical strategies are at the edge of no-signaling.

**Probabilistic vs deterministic bounds**

Notice that all the no-signaling bounds derived here are concerned with probabilistic strategies. This is most transparent in our derivation of a no-signaling bound for unitary replication. In some cases, the optimal deterministic strategy coincides with the optimal probabilistic one
as is the case with the cloning of states and unitaries. In these protocols the filtering operation of the optimal probabilistic strategy can be moved to the preparation of an appropriate input state for the optimal deterministic strategy. This is the case in the protocols we consider here, for which the filter can be moved to the initial state preparation step.

METHODS

In this section we elaborate on the methods used to derive our results. In particular, we first show how the no-signaling condition can be incorporated in our operationally generic scenario. We then derive the explicit no-signaling bound for phase covariant cloning and quantum metrology for both uniform and arbitrary prior. For the latter we also derive the conditional probabilities, assigned by Bob’s processing $P$ that are compatible with the no-signaling condition. Finally, we show that ensemble steering and the no-signaling condition give rise to the Born rule for assigning probabilities to measurement outcomes.

The no-signaling condition

Recall that, whereas the transformations Bob can perform can be supra-quantum, we demand that the inputs and outputs of such a transformation be quantum mechanical. To that end, any $N$-qubit state held by Bob can be expressed as

$$|\Phi^N_\theta\rangle = \sum_{n=0}^N p_n |n\rangle_A |n\rangle_B,$$

where $n$ runs over all $N$-bit strings, and $|n\rangle_B$ is a superposition over all states with Hamming weight $|n| = n$.

The action of $U^N_\theta$ on an arbitrary state $|\Phi^N\rangle$ can be remotely prepared by Alice in the following way. Alice prepares the bi-partite entangled state

$$|\Psi\rangle_{AB} = \sum_{n=0}^N p_n |n\rangle_A |n\rangle_B,$$

where she keeps the $(N+1)$-level system and transmits the remaining $2^N$-dimensional system to Bob. She then applies a unitary transformation $U_\theta = e^{i\theta h}$ on her part of the state followed by a measurement in the Fourier basis $\{ |\psi_k\rangle \propto \sum_n e^{\frac{i\pi n^2}{N+1}} |n\rangle_A \}$ with $k = 0, \ldots, N$. If Alice obtains outcome $|\psi_k\rangle$ then Bob’s state becomes

$$|\Phi^N_{\theta + \frac{2\pi k}{N+1}}\rangle = U^\otimes N_{\theta + \frac{2\pi k}{N+1}} |\Phi^N\rangle. $$

As all outcomes, $|k\rangle$, are equally likely Bob ends up with a random state from the ensemble $\{ |\Phi^N_{\theta + \frac{2\pi k}{N+1}}\rangle, k \in \{0, \ldots, N\} \}$.

The no-signaling condition now requires that Bob, who does not know which unitary $U_\theta, \theta \in (0, 2\pi]$ was chosen by Alice, can not learn $\theta$ from the above ensemble. If this were not the case then Alice and Bob, who are spatially separated, can use the above construction to perform faster-than-light communication. In particular, the no-signaling condition implies that no matter what processing Bob does to his ensemble, be it quantum or otherwise, he cannot infer the value of $\theta$. Denoting the outcome of Bob’s processing by $P(\theta|k)$ the no-signaling condition requires that the mixture

$$O(\theta) = \frac{1}{N+1} \sum_{k=0}^N P(\theta|k)$$

is independent of $\theta$ chosen by Alice.

No-signaling bound on cloning

We now derive a bound on the optimal mean global fidelity of cloning based on so-signaling, and also construct the optimal deterministic strategy that achieves the bound.

Our goal is to bound the global mean fidelity

$$F^{C_{N-M}} = \int \frac{d\theta}{2\pi} F(\rho^M_\theta, (|\psi(\theta)\rangle\langle\psi(\theta)|)^\otimes M),$$

where $F(\rho^M_\theta, (|\psi(\theta)\rangle\langle\psi(\theta)|)^\otimes M) = Tr(\rho^M_\theta (|\psi(\theta)\rangle\langle\psi(\theta)|)^\otimes M)$. We will make use of the following three properties of the fidelity: (i) unitary invariance, $F(U\rho U^\dagger, U^\dagger \sigma U) = F(\rho, \sigma)$, (ii) concavity, $F(\sum p_i \rho_i + \sum p_i \sigma i) \geq \sum p_i F(\rho_i, \sigma_i)$, and (iii) joint concavity, $F(\sum p_i \rho_i, \sum p_i \sigma i) \geq \sum p_i F(\rho_i, \sigma_i)$.

Using the fact that $(2\pi)^{-1} \int_0^{2\pi} d\theta = \sum_{k=0}^N (2\pi)^{-1} \int_0^{2\pi} (N+1) d\theta$ and the joint concavity of the fidelity we arrive at

$$F^{C_{N-M}} \leq \int \frac{d\theta}{2\pi} F\left(\rho^M, \sum_{k=0}^N \frac{(U_{2\pi k}^{\otimes N} |\psi(\theta)\rangle\langle\psi(\theta)| U_{2\pi k}^{\dagger \otimes M})}{N+1}\right). $$




where $G_{N+1} \equiv \frac{1}{N+1} \sum_{k=0}^{N} U_{k}^{\otimes M} (\cdot) U_{k}^{\dagger \otimes M}$. Finally, using the concavity of the fidelity we arrive at

$$F^{C_{N-M}} \leq F \left( G_{\{1\}}^{\dagger} [\rho^{M}], G_{N+1} [\{+\}^{\otimes M}] \right),$$

(24)

where $G_{\{1\}}^{\dagger} [\cdot] \equiv \frac{1}{2\pi} \int_{0}^{2\pi} d\theta U_{\theta}^{\dagger \otimes M} (\cdot) U_{\theta}^{\otimes M}$. We note that $G_{\{1\}}^{\dagger} = G_{\{1\}}$.

The maps $G$ impose a block-diagonal structure on any density matrix on which they act, which makes computation of the right hand side of Eq. (24) simple. In particular

$$G_{N+1} [\{+\}^{\otimes M}] = \bigoplus_{\lambda=0}^{N} \left| \lambda \right\rangle \left\langle \lambda \right| \otimes M \lambda, \lambda + \alpha(N+1) \right|, \lambda = 0, \ldots, M$$

(25)

where $|\phi(\lambda)\rangle = \sum_{\alpha=0}^{M-1} \rho_{\lambda,\alpha} |M, \lambda + \alpha(N+1)\rangle$, and we note the set of all permutations symmetric states of $M$ qubits whose Hamming weight is $\lambda$ modulo $N$. Notice that the states $|\phi(\lambda)\rangle$ are unnormalized. The probability of projecting onto a sector with a given $\lambda$ is $p_{\lambda} = \frac{1}{2\pi} \sum_{\alpha=0}^{M-1} \left( \alpha \right|_{\lambda} M \lambda, \lambda + \alpha(N+1) \right|$, whereas the corresponding probability for $G_{\{1\}}[\rho^{M}]$ is $q_{\lambda} = \sum_{\alpha=0}^{M-1} \rho_{\lambda,\alpha} M \lambda$. For two density matrices that are block diagonal in the same basis the fidelity can be easily shown to satisfy $F(\bigoplus_{\lambda=0}^{N} \rho_{\lambda,\alpha} \bigoplus_{\lambda=0}^{N} \sigma_{\lambda,\beta}) = \sum_{\lambda=0}^{N} F(\rho_{\lambda,\alpha} \sigma_{\lambda,\beta})$. Using this, and a bit of algebra one can show that

$$F^{C_{N-M}} \leq \frac{1}{2M} \sum_{\lambda=0}^{N} \sum_{\alpha=0}^{M-1} \rho_{\lambda,\alpha}^{M} \left( \lambda + \alpha(N+1) \right).$$

(26)

In order to determine the best possible cloning procedure we need to first optimize $\rho_{\lambda,\alpha}^{M}$, within each sector $\lambda$, subject to the constraint $q_{\lambda} = \sum_{\alpha=0}^{M-1} \rho_{\lambda,\alpha}^{M}$, followed by optimizing $q_{\lambda}$ across each sector. The first optimization is evident; simply choose $\rho_{\lambda,\alpha_{\text{max}}}^{M} = q_{\lambda}$, where $\alpha_{\text{max}} \in \{0, \ldots, \left[ \frac{M-1}{N} \right] \}$ such that $\forall \beta \in \{0, \ldots, \left[ \frac{M-1}{N} \right] \}, \left( \lambda + \beta(N+1) \right) \leq \left( \lambda + \alpha_{\text{max}}(N+1) \right)$. We then have

$$F^{C_{N-M}} \leq \frac{1}{2M} \sum_{\lambda=0}^{N} q_{\lambda} \max_{\alpha \in \{0, \ldots, \left[ \frac{M-1}{N} \right] \}} \left( \lambda + \alpha(N+1) \right) \left( \lambda + \alpha(N+1) \right) \left( \lambda + \alpha(N+1) \right) \left( \lambda + \alpha(N+1) \right)$$

$$\leq \frac{1}{2M} \sum_{\lambda=0}^{N} \left( M \right|_{\lambda} + \lambda \right).$$

(27)

Finally, noting that the binomial distribution, $\{2^{-M} \left( \lambda_{\alpha} \right) \}$, can be approximated by a Gaussian, $N(\mu=M/2, \sigma=\sqrt{\lambda}/2)$, the sum in Eq. (27) can be approximated, for $M \gg N$, by

$$\int_{-\frac{N+1}{2\pi}}^{\frac{N+1}{2\pi}} N(\mu+\lambda, \sigma) d\lambda \approx \text{erf} \left( \frac{\sqrt{2}(N+1)}{\sqrt{\pi}M} \right).$$

(28)

No-signaling bounds for quantum metrology

In this section we derive the probability distribution $p(\theta | \theta)$ that optimizes the average fidelity of alignment and is compatible with the no-signaling condition. Recall that the latter states that the conditional probabilities

$$p(\theta | \theta) = \frac{1}{N+1} \sum_{k=0}^{N} p(\theta | \theta + \frac{2\pi k}{N+1})$$

(29)

are independent of $\theta$, i.e., $p(\theta | \theta) = p(\theta')$. Hence, we are free to distribute the probability $p(\theta')$ amongst $N+1$ terms subject to the constraint that $\int d\theta' p(\theta') = 1$.

Without loss of generality assume that $\theta \in \left( 0, \frac{2\pi}{N+1} \right)$. If this is not the case we can always relabel the measurement outcomes $k \in \{0, \ldots, N\}$ such that $\theta$ lies in $\left( 0, \frac{2\pi}{N+1} \right)$. As $\cos \left( \frac{\theta - \theta'}{2} \right)$ is largest when $\theta - \theta' = 0$ the average fidelity is optimized by setting $p(\theta | \theta + \frac{2\pi k}{N+1}) = 0$ for $k \neq 0$. As this is true for all randomly chosen $\theta$, and using the constraint $\int d\theta' p(\theta') = 1$, it follows that $p(\theta | \theta) = \frac{N+1}{2\pi} \left( \theta - \theta' \right)$ $\leq \frac{\pi}{N+1}$ and zero everywhere else.

We now derive the maximum average fidelity compatible with no-signaling. Recall that the average fidelity is given by Eq. (12). As the conditional probability distribution $p(\theta | \theta)$ above is also a function only of the difference between $\theta'$ and $\theta$ we may write the average fidelity as

$$\tilde{f} = 1 - \int_{-\frac{\pi}{2\pi}}^{\frac{\pi}{2\pi}} \frac{d\theta}{2\pi} \int_{-\frac{\pi}{2\pi}}^{\frac{\pi}{2\pi}} d\theta' p(\theta' - \theta) \sin^{2} \left( \frac{\theta - \theta'}{2} \right)$$

(30)

where we have used the identity $\cos^{2}(x) = 1 - \sin^{2}(x)$. As the integrand in Eq. (30) depends only on the difference between $\theta' - \theta$ we may define $\phi = \theta' - \theta$ and $d\phi = d\theta'$ so that

$$\tilde{f} = 1 - \int_{-\frac{\pi}{2\pi}}^{\frac{\pi}{2\pi}} d\phi \sin^{2} \left( \frac{\phi}{2} \right).$$

(31)

Substituting the no-signaling probability distribution of Eq. (13) in place of $p(\phi)$ in Eq. (31) one obtains

$$\tilde{f} = 1 - \frac{N+1}{2\pi} \int_{-\frac{\pi}{N+1}}^{\frac{\pi}{N+1}} \sin^{2} \left( \frac{\phi}{2} \right).$$

(32)
In the limit of large $N$ the limits of integration in Eq. (32) become narrower and we can use the small angle approximation to write $\sin(\phi/2) \approx \phi/2$. Substituting the latter into Eq. (32) and evaluating the integral one easily obtains $f = 1 - \frac{2m^2}{2N^2}$.

It is interesting to note that the optimal phase alignment protocol [21] directly translates into a measure and prepare strategy for probabilistic quantum cloning with a mean fidelity per copy $\text{Tr} (\rho^M (|\psi_i(\theta)\rangle\langle\psi_i(\theta)| \otimes \mathbb{I}^{\otimes (M-1)})$ equal to $\bar{f}$, where $M$ can be arbitrarily large. The best deterministic cloning protocol achieves a fidelity per copy, $f$, by $f = 1 - \frac{1}{N^2}$ in the large $N$ limit [17], whereas $\bar{f} = 1 - \frac{1}{N^2}$. This shows that probabilistic super-replication is also possible even if one considers the fidelity, rather than the mean fidelity, per copy. Explicit protocols, however, are not known.

**Metrology with general prior**

In this section we derive the optimal average fidelity when the prior information about the the parameter $\theta$ is given by

$$p(\theta; t) = \frac{1}{2\pi} (1 + 2 \sum_{n=1}^{\infty} \cos(n\theta)e^{-n^2t}), \quad (33)$$

corresponding to the dynamical diffusion over time of $p(\theta, 0) = \delta(\theta)$. In this case the mean fidelity reads

$$\bar{f}_t = 1 - \int d\theta' \int p(\theta'|\theta)p(\theta; t) \sin^2\left(\frac{\theta - \theta'}{2}\right) d\theta \quad (34)$$

Our goal is to minimize the integrand under the no-signaling constraint of Eq. (11). The latter demands that the conditional probabilities $p(\theta'|\theta)$ are independent of $t$ and are given by Eq. (29).

For a fixed value of $t$ the product

$$g(\theta, \theta', t) = p(\theta; t) \sin^2\left(\frac{\theta - \theta'}{2}\right) \quad (35)$$

in Eq. (34) obtains its minimum value when $\theta - \theta' = 0$. In addition $g(\theta, \theta', t)$ is monotonically increasing so long as the derivative of $g(\theta, \theta', t)$ around $\theta' = \theta$ is greater than zero. This is true so long as

$$\tan^2\left(\frac{\theta - \theta'}{2}\right) < \left(\frac{m}{M}\right)^2 \tan^2\left(\frac{\Delta}{2}\right), \quad (36)$$

where $m \equiv \min \rho p(\theta; t)$ and $M = \max \rho \delta(\theta, p(\theta; t))$. Outside the interval $[\theta - \Delta, \theta + \Delta]$ the function $g(\theta, \theta', t)$ is larger than $m \sin^2(\Delta/2)$. Therefore, $g(\theta, \theta', t)$ attains its global minimum in the finite interval satisfying the condition

$$\sin^2\left(\frac{\theta - \theta'}{2}\right) < m \sin^2\left(\frac{\Delta}{2}\right) = \frac{m^3}{M^2 + m^2}. \quad (37)$$

Now consider the narrowest probability distribution compatible with no-signaling given by $\frac{N+1}{2N}p(\theta')$, where $p(\theta')$ is the probability distribution given in Eq. (11), for $|\theta - \theta'| < \frac{\Delta}{\sqrt{2}}$ and zero elsewhere. For large enough $N$ this probability distribution is contained entirely in the interval $[\theta' - \Delta, \theta' + \Delta]$ where $g(\theta; \theta', t)$ attains its minimum and therefore minimizes the integrant of Eq. (34). Plugging this probability into Eq. (34) and using the condition $\int d\theta' p(\theta') = 1$ leads to

$$\bar{f}_t \approx 1 - \frac{\pi^2}{12N^2} \vartheta_4(0, e^{-t}), \quad (38)$$

where $\vartheta_4(0, e^{-t}) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2t} = m$ is the Jacobi theta function.

**Tightness of no-signaling bound for average fidelity of phase alignment**

In this section we illustrate that the probability distribution of Eq. (13), derived for a particular no-signaling scenario, can be used to perform faster-than-light communication. For simplicity we consider the qubit case ($N = 1$). Let Alice and Bob share the entangled state $|\Psi\rangle_{AB} = (\cos(\epsilon)|00\rangle + \sin(\epsilon)|11\rangle)$. Alice can choose to measure her system in either the computational basis $\{|0\}, |1\}$ or the x-basis $\{|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ steering Bob’s state into the ensembles $\mathcal{E}^{(1)} = \{\cos^2(\epsilon)|00\rangle|00\rangle, \sin^2(\epsilon)|11\rangle|11\rangle\}$ and $\mathcal{E}^{(2)} = \{|\pm\rangle|\pm\rangle\}$. As shown in Fig. 3

This construction obviously holds if all the states are rotated by the same angle. In particular, we can always set this angle such that the probability distribution in Eq. (13) yields $p(\theta'||0\rangle) = p(\theta'||\pm\rangle) = 0$ and $p(\theta'||1\rangle) = p(\theta'||\mp\rangle) = \frac{1}{2}$. In this case the two ensembles give a different probability to observe the outcome $\theta'$, $p(\theta'|\mathcal{E}^{(1)}) = \frac{\sin^2(\epsilon)}{\pi}$ and $p(\theta'|\mathcal{E}^{(2)}) = \frac{1}{2\pi}$. Hence, Bob can distinguish the two ensembles with non-zero probability an infer Alice’s choice of measurement instantaneously.

Let us now consider the general case. Any probability distribution $p(\theta)$ defines a continuous ensemble $\mathcal{E}^{(\rho)} = \{p(\theta)|\Phi^{(\rho)}_\theta \rangle\langle \Phi^{(\rho)}_\theta |\}$, where $|\Phi^{(\rho)}_\theta \rangle = \sum_{n=0}^{N} \psi_n e^{in\theta_n} |n\rangle$ are the $N$-qubit states of Eq. (1). Without loss of generality we consider $p(\theta)$ such that

$$p(\theta) = \frac{1}{2\pi} (1 + 2 \sum_{k=1}^{\infty} p_k \cos(k\theta)). \quad (39)$$

The density matrix for the ensemble $\mathcal{E}^{(\rho)}$ is given by $\rho = \sum_{n,m=0}^{N} \psi_n \psi_m^* |n\rangle\langle m| p_{n-m}$, in such a way that it only depends on the first $N$ coefficients, $p_k$, of the Fourier series in Eq. (39). For any two distributions $p_1(\theta)$ and $p_2(\theta)$ that are identical in the first $N$ components of the Fourier series the ensembles $\mathcal{E}^{(p_1)}$ and $\mathcal{E}^{(p_2)}$ give rise to
the same density matrix \( \rho \), and therefore can not be distinguished by Bob.

In particular, the ensemble given by the probability distributions \( p_1(\theta) = \frac{1}{2\pi} \) and \( p_2(\theta) = \frac{1}{2\pi}(1 + \cos(M\theta)) \), for \( M > N \), correspond to the same density matrix. However, with the outcome probability distribution of Eq. (13), Bob can distinguish the two probability distributions with non-zero probability as \( p(\theta'|\mathcal{E}(p_1)) - p(\theta'|\mathcal{E}(p_2)) = \frac{N+1}{2\pi} \int \frac{1}{2\pi} (p_2(\theta) - p_1(\theta)) d\theta = \sin\left(\frac{M\pi}{N+1}\right) \neq 0 \). Therefore, the probability distribution of Eq. (13) leads to signaling when Alice can chose to prepare \( \mathcal{E}(p_1) \) or \( \mathcal{E}(p_2) \).

More generally, the above argument implies that any outcome probability \( p(\theta'|\theta) \) compatible with no-signaling has to satisfy \( \int p(\theta'|\theta) \cos(M\theta) d\theta = 0 \) for \( M > N \), i.e. the Fourier components \( p_k \) of \( p(\theta'|\theta) \) are necessarily zero for \( k > N \). Therefore, for finite \( N \), probability distributions with sharp edges such as the one in Eq. (13) are ruled out.

![FIG. 3. The two ensemble decompositions of \( \rho_k \) for a qubit, leading to faster-than-light communication for the probability distribution Eq. (13) (represented by the red semi-circle).](image)

### The Born Ultimatum

In this section we show that the no-signaling condition and the ability to remotely prepare any ensemble corresponding to the same density matrix implies that the only possible way for Bob to assign outcome probabilities to quantum states is given by the Born rule. Notice that the converse holds trivially.

Let Alice and Bob share the maximally entangled state \( |\Psi\rangle_{AB} = \frac{1}{\sqrt{N+1}} \sum_{n=0}^N |n\rangle |n\rangle \). By a suitable choice of measurement basis Alice can prepare Bob’s system in any ensemble \( \{\frac{1}{\sqrt{N+1}} U |n\rangle |n\rangle U^\dagger\} \) composed of orthonormal states forming a basis. We view this property as an experimental fact. No-signaling imposes that all such ensembles can not be distinguished by Bob, i.e., the probability that Bob observes an outcome \( k \) satisfies

\[
\frac{1}{N+1} \sum_{n=0}^N p_k(U |n\rangle |n\rangle U^\dagger) = \text{cst}
\]

(40)

for all choice of orthonormal bases. By Gleason’s theorem if \( N > 2 \) the above equation implies that \( p_k \) is given by the Born rule [11]. In other words there is a positive operator \( E_k \) such that for any state \( \rho \) the conditional probability is given by \( p_k(\rho) = \text{Tr}(\rho E_k) \).

Gleason’s theorem does not hold for \( N = 2 \), because the restriction to ensembles composed of orthonormal basis states is not enough to single out the Born rule [42]. However imposing that all ensembles leading to the same density matrix are indistinguishable leads to the Born rule as we now show.

Let us denote by \( p_k(\rho) \) the probability to observe outcome \( k \) given the input state \( \rho \). The no-signaling condition together with the practical possibility to remotely prepare any ensemble \( \{\rho_k\} \) —corresponding to the density matrix \( \rho = \sum p_k \rho_k \)—at Bob’s side by an appropriate choice of measurement at Alice’s side [11] [40] implies that the probability assignment, \( p_k \), satisfies the linearity condition

\[
p_k(\rho) = \sum_{\ell} p_{\ell k}(\rho_{\ell}) = \sum_{\ell} p_{\ell k}(\rho_{\ell}).
\]

(41)

Let us define the following probabilities

\[
K_{11} = p_k(0|0\rangle) \quad K_{22} = p_k(1|1\rangle) \\
K^R_{12} = p_k(|+\rangle|+\rangle) - \frac{1}{2} K_{11} \quad K^R_{21} = \frac{1}{2} K_{22}
\]

\[
K^I_{12} = p_k(|i\rangle|-i\rangle) - \frac{1}{2} K_{11} \quad K^I_{21} = \frac{1}{2} K_{22},
\]

(42)

where \(|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)\) and \(|-i\rangle = \frac{1}{\sqrt{2}} (|0\rangle - i|1\rangle)\).

Assume that a state \( \rho = \left( \begin{array}{cc} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{array} \right) \) has \( \rho^R_{12} \) and \( \rho^I_{12} \) bigger then zero. Using \( |0\rangle|1\rangle + |1\rangle|0\rangle = 2 |+\rangle|+\rangle \) and \( |0\rangle|0\rangle - |1\rangle|1\rangle = 2 |-i\rangle|-i\rangle \) we may write

\[
\rho + (\rho^R_{12} + \rho^I_{12})(|0\rangle|0\rangle + |1\rangle|1\rangle) = \rho_{00} |0\rangle|0\rangle \\
\rho_{11} |1\rangle|1\rangle + 2 \rho^R_{12} |+\rangle|+\rangle + 2 \rho^I_{12} |-i\rangle|-i\rangle
\]

(43)

which involves only mixtures of physical states on both sides. Applying linearity and using Eq. (42) gives \( p_k(\rho) = \text{Tr}(\rho E_k) \), where \( E_k = \left( \begin{array}{cc} K_{11} & K_{12} \\ K^R_{12} & K^R_{21} \end{array} \right) = K^I_{21} \) is a Hermitian operator. A similar argument holds when \( \rho^R_{12} \) and \( \rho^I_{12} \) are not lager than zero. The same construction can be applied for all possible measurement outcomes \( k \), and the non-negativity of each \( E_k \) is ensured by the non-negativity of the probability \( p_k(\rho) \) on all states \( \rho \). Finally the normalization condition, \( \sum_k E_k = 1 \), is ensured by the fact that the sum of all outcome probabilities is equal to one for all input states. This completes the proof.

Note that this proof can be extended for \( N > 2 \), however for these case it is a direct corollary of Gleason’s theorem.

### CONCLUSION

In this paper we derived no-signaling bounds for various quantum information processing tasks. These include...
phase covariant cloning of states and unitary operations, as well as quantum metrology. In the latter case we showed the validity of the Heisenberg limit purely from the no-signaling condition. In general, following [3], we have shown that the optimal probabilistic quantum mechanical strategy is at the edge of no-signaling also for the case of metrology. Furthermore, we have found that for some tasks, such as phase covariant cloning of states and unitaries, the optimal probabilistic and deterministic strategies coincide.

On the one hand it is clear that a bound for probabilistic strategies is also a bound for deterministic ones. However it might be possible to derive tighter no-signaling bounds for deterministic strategies. It is an interesting open question how to incorporate the requirement that the protocol be deterministic in a no-signaling scenario.

On the other hand, there are several tasks for which the optimal quantum strategy is not known. In such cases the techniques and methods we provide here can be particularly useful in deriving limitations to these tasks based on no-signaling. We have demonstrated one such example for Bayesian metrology for arbitrary prior.

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