PRIMARILY QUASILOCAL FIELDS AND 1-DIMENSIONAL
ABSTRACT LOCAL CLASS FIELD THEORY

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Abstract. Let $E$ be a field satisfying the following conditions: (i) the $p$-component of the Brauer group $\text{Br}(E)$ is nontrivial whenever $p$ is a prime number for which $E$ is properly included in its maximal $p$-extension; (ii) the relative Brauer group $\text{Br}(L/E)$ equals the maximal subgroup of $\text{Br}(E)$ of exponent $p$, for every cyclic extension $L/E$ of degree $p$. The paper proves that finite abelian extensions of $E$ are uniquely determined by their norm groups and related essentially as in the classical local class field theory. This includes analogues to the fundamental correspondence, the local reciprocity law and the local Hasse symbol.

1. Introduction and statements of the main results

This paper is concerned with finite abelian extensions of primarily quasi-local (abbr, PQL) fields, and can be viewed as a continuation of [9], I. When $E$ is a strictly PQL-field, it shows that these extensions and their norm groups are related as in the fundamental correspondence of the classical local class field theory (see [20], page 101). The paper proves that they are subject to an exact analogue to the local reciprocity law (formulated, e.g., in [45], Ch. 6, Theorem 8, and [20], Theorem 7.1), and to a partial analogue to the local Hasse symbol (as characterized in [25], Ch. 2, see also [55], and [20], Theorems 6.9 and 6.10) of form determined by invariants of the Brauer group $\text{Br}(E)$. It takes a step towards characterizing the fields whose finite abelian extensions have the above properties. When $E$ belongs to some special classes of traditional interest, the present research enables one to achieve this aim and to find a fairly complete description of the norm groups of arbitrary finite separable extensions of $E$ (see Section 3, [8, 12] and the references in [13], Remark 3.9).

The basic notation, terminology and conventions kept in this paper are standard and virtually the same as in [9], I, and [13]. Throughout, $\mathbb{P}$ denotes the set of prime numbers and every algebra $A$ is understood to be associative with a unit lying in the considered subalgebras of $A$. Simple algebras are supposed to be finite-dimensional over their centres, Brauer and value groups are written additively, Galois groups are viewed as profinite with respect to the Krull topology, and the considered profinite group homomorphisms are continuous. For each field $E$, $E^*$ denotes its multiplicative group, $E_{\text{sep}}$ a separable closure of $E$, $\mathcal{G}_E = \mathcal{G}(E_{\text{sep}}/E)$ is the absolute Galois group of

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$E$, $s(E)$ stands for the class of central simple $E$-algebras, $[B]$ denotes the similarity class of any $B \in s(E)$, $d(E)$ is the subclass of division algebras $D \in s(E)$, and $P(E) = \{p \in \mathbb{P} : E(p) \neq E\}$, where $E(p)$ is the maximal $p$-extension of $E$ in $E_{\text{sep}}$. As usual, $E$ is said to be formally real, if $-1$ is not presentable over $E$ as a finite sum of squares; $E$ is called nonreal, otherwise. We say that $E$ is Pythagorean, if it is formally real and the set $E^{*2} = \{\lambda^2 : \lambda \in E^*\}$ is additively closed. For any field extension $F/E$, $\rho_{E/F}$ denotes the scalar extension map of $Br(E)$ into $Br(F)$, $Br(F/E)$ the relative Brauer group of $F/E$, and $I(F/E)$ is the set of intermediate fields of $F/E$. When $F/E$ is finite and separable, $\text{Cor}_{F/E}$ stands for the corestriction map of $Br(E)$ into $Br(F)$ (see [50]). We say that $E$ is $p$-quasilocal, for some $p \in \mathbb{P}$, if it satisfies one of the following conditions: (i) every cyclic degree $p$ extension of $E$ is embeddable as a subalgebra in each $\Delta \in d(E)$ of (Schur) index $p$; (ii) the $p$-component $Br(E)_p$ of $Br(E)$ is trivial or $p \notin P(E)$. The field $E$ is called PQL, if it is $p$-quasilocal for every $p \in \mathbb{P}$; when this holds, $E$ is said to be strictly PQL in case $Br(E)_p \neq \{0\}, p \in P(E)$. We say that $E$ is quasilocal, if its finite extensions are PQL.

Local fields and $p$-adically closed fields are strictly quasilocal (abbr, SQL), i.e. their finite extensions are strictly PQL (cf. [47], Ch. XIII, Sect. 3, [42], Theorem 3.1 and Lemma 2.9, and [7], Sect. 3). The strictly PQL-property has been fully characterized in the following two classes: (i) algebraic extensions of global fields [12]; (ii) Henselian discrete valued fields [7], Sect. 2. These facts extend the arithmetic basis of this research which is further motivated by results of [9], I, on Brauer groups of PQL-fields and on absolute Galois groups of quasilocal fields. For reasons clarified in the sequel, our approach to the main topic of this paper is purely algebraic. Our starting point is the fact that residue fields of Henselian valued stable fields are PQL in the case of totally indivisible value groups, in the sense of (2.1) (i). Specifically, a Henselian discrete valued field $(\hat{K}, v)$ with a perfect residue field $\hat{K}$ is stable if and only if $\hat{K}$ is stable and PQL [44], Proposition 2 (cf. also [9], I, and [14], Proposition 2.3). These and other related results show, with their proofs, that PQL-fields partly resemble local fields in a number of respects (see Propositions 2.2-2.3, [9], I, Lemma 4.3 and [9], II, Lemma 2.3). For example, by [9], I, Lemma 4.3, if $\hat{K}$ is $p$-quasilocal and $\hat{L}_1$ and $\hat{L}_2$ are different extensions of $\hat{K}$ in $\hat{K}(p)$ of degree $p$, then the inner product $N(\hat{L}_1/\hat{K})N(\hat{L}_2/\hat{K})$ of the norm groups $N(\hat{L}_i/\hat{K}), i = 2$, is equal to $\hat{K}_*$. This result and its key role in the proof of [9], I, Theorem 4.1, attract interest in the study of the PQL-property along the lines of the classical local class field theory, with the notion of a local field extended as follows:

**Definition 1.** Let $E$ be a field, $\text{Nr}(E)$ the set of norm groups of $E$, and $\Omega(E)$ the set of finite abelian extensions of $E$ in $E_{\text{sep}}$. We say that $E$ admits 1-dimensional local class field theory (abbr, LCFT), if the natural mapping of $\Omega(E)$ into $\text{Nr}(E)$ (by the rule $M \rightarrow N(M/E), M \in \Omega(E)$) is injective and the following condition holds, for each $M_1, M_2 \in \Omega(E)$:

$$(1.1) \text{ The norm group (over } E\text{) of the compositum } M_1M_2 \text{ equals the intersection } N(M_1/E) \cap N(M_2/E), \text{ and } N(M_1 \cap M_2/E) = N(M_1/E)N(M_2/E).$$
We say that $E$ is a field with 1-dimensional local $p$-class field theory (abbr, local $p$-CFT), for a given $p \in \mathbb{P}$, if the fields from the set $\Omega_p(E) = \{L \subseteq E(p) : L \subseteq E(p)\}$ are uniquely determined by their norm groups and satisfy condition (1.1). When this is the case and $p \in P(E)$, we have $Br(E)_p \neq \{0\}$ (see Proposition 3.3). Observe that $E$ admits LCFT if and only if it admits local $p$-CFT, for every $p \in P(E)$. This follows from Lemma 2.1 and shows that PQL-fields with LCFT are strictly PQL.

The main purpose of this paper is to shed light on the place of strictly PQL-fields in LCFT by proving the following:

**Theorem 1.1.** Strictly PQL-fields admit LCFT. Conversely, a field $E$ admitting such a theory is strictly PQL, if each $D \in d(E)$ of prime exponent $p$ is similar to a tensor product of cyclic division $E$-algebras of index $p$.

**Theorem 1.2.** Let $E$ and $M$ be fields, such that $E$ is strictly PQL, $P(E) \neq \phi$ and $M \in \Omega(E)$. For each $p \in P(E)$, let $\Phi_p(E) = \{b_p \in Br(E) : pb_p = 0\}$, $I_p$ be a basis and $d(p)$ the dimension of $\Phi_p(E)$ as a vector space over the field $\mathbb{F}_p$ with $p$ elements, $G(M/E)_p$ the Sylow $p$-subgroup of the Galois group $G(M/E)$, and $G(M/E)_{p^{(d(p)}}$ the direct product, indexed by $I_p$, of isomorphic copies of $G(M/E)_p$. Then the direct product $G(M/E)^{Br(E)} = \prod_{p \in P(E)} G(M/E)_{p^{(d(p)}}$ and the quotient group $E^*/N(M/E)$ are isomorphic.

Before stating our third main result, recall that a field $F$ is Pythagorean with $F(2) = F(\sqrt{-1})$ if and only if it is formally real and 2-quasilocal [9], I, Lemma 3.5. Note also that, by [56], Theorem 2, if $p \in P(F)$, then $F(p)$ contains as a subfield a $\mathbb{Z}_p$-extension $U_p$ of $F$ (i.e. $U_p/F$ is Galois with $G(U_p/F)$ isomorphic to the additive group $\mathbb{Z}_p$ of $p$-adic integers) unless $p = 2$ and $F$ is Pythagorean. We retain notation as in Theorem 1.2.

**Theorem 1.3.** Let $E$ be a strictly PQL-field, such that $P(E) \neq \phi$, and let $E_\infty \subseteq E_{sep}$ be the compositum of fields $E_p$, $p \in P(E)$, where $E_p/E$ is a $\mathbb{Z}_p$-extension, if $p > 2$ or $E$ is nonreal, and $E_2 = E(2)$ when $E$ is formally real. Then there exists a set $H_E = \{(M/E) : E^* \to G(M/E)^{Br(E)}, M \in \Omega(E)\}$ of surjective group homomorphisms with the following properties:

(i) The kernel of $(M/E)$ coincides with $N(M/E)$, for each $M \in \Omega(E)$;

(ii) If $M \in \Omega(E)$ and $K$ is an intermediate field of $M/E$, then $(K/E)$ equals the composition $\pi_{M/K} \circ (M/E)$, where $\pi_{M/K} : G(M/E)^{Br(E)} \to G(K/E)^{Br(E)}$ is the homomorphism mapping the $i_p$-th component of $G(M/E)_{p^{(d(p)}}$ on the $i_p$-th component of $G(K/E)_{p^{(d(p)}}$ as the natural projection $G(M/E)_{p^{(d(p)}} \to G(K/E)_{p^{(d(p)}}$, for each pair $(p, i_p) \in P(E) \times I_p$;

(iii) The set $H_E$ is uniquely determined by the mappings $(M/E)$, where $\Gamma$ runs through the set of finite extensions of $E$ in $E_\infty$ of primary degrees.

Theorems 1.1 and 1.2 show the strong influence of $Br(E)$ on a number of algebraic properties of a PQL-field $E$. They are obtained from similar results on finite abelian $p$-extensions of $p$-quasilocal fields, stated as Theorem 3.1. This approach uses several properties of $p$-quasilocal fields without
generally valid analogues for PQL-fields (see Propositions 2.2 and 2.3, [9], I, Corollary 8.5, and [13], Proposition 6.3). As shown in [13, 14] and [9], II, Sect. 3, it enables one to describe the isomorphism classes of Brauer groups of major types of PQL-fields, and of the reduced parts of Brauer groups of equicharacteristic Henselian valued absolutely stable fields with totally indivisible value groups. Thus it turns out that usually powerful methods of valuation theory are virtually inapplicable to many PQL and most presently known quasilocal fields (see (2.4) and (2.5) (iii), and compare (2.1) with (2.2) and Remark 2.4). At the same time, the proofs in [13] show at crucial points that the study of the PQL-property can effectively rely on constructive methods based on properties (established in the 1990’s, see e.g. [17], Proposition 2.6, and [46]) of relative Brauer groups of extensions obtained as transfers of function fields of Brauer-Severi and other varieties. This makes it possible to apply Theorem 3.1 and other results about $p$-quasilocal fields to Brauer groups of arbitrary fields, and so leads to a better understanding of the relations between Galois groups and norm groups of finite Galois extensions of quasilocal fields (see Remark 4.3 and [13]).

When $E$ is a local field, the former assertion of Theorem 1.1 yields the fundamental correspondence of the classical local class field theory. If $E$ is merely strictly PQL with $d(p) = 1$, for all $p \in P(E)$ (i.e. with $\text{Br}(E)$ embeddable in $\mathbb{Q}/\mathbb{Z}$, the quotient group of the additive group of rational numbers by the subgroup of integers), then Theorem 1.2 states that $E^*/N(M/E) \cong G(M/E)$. This holds, for instance, in the following cases:

(1.2) (i) $E$ is an algebraic strictly PQL-extension of a global field $E_0$; when $\text{Br}(E)_p \neq \{0\}$, by [12], Theorem 2.1, $p \in P(E)$ and $\text{Br}(E)_p$ is isomorphic to the quasicyclic $p$-group $\mathbb{Z}(p^{\infty})$ unless $p = 2$ and $E$ is formally real (big families of such $E$ can be constructed by applying [12], Theorem 2.2).

(ii) The triple $\text{AT}(E) = (G_E; \{G_F; F \in \Sigma\}, E_{\text{sep}}^*)$, $\Sigma$ being the set of finite extensions of $E$ in $E_{\text{sep}}$, is an Artin-Tate class formation (see [3], Ch. 14).

It is known that (1.2) (ii) holds, if $E$ is $p$-adically closed or has a Henselian discrete valuation with a quasifinite residue field. Note further that the statement of Theorem 1.2 coincides in case (1.2) (ii) with the local reciprocity law of the Artin-Tate abstract class field theory [3], Ch. 14, Sect. 5. As to Theorem 1.3, it can be viewed as a partial analogue to the local Hasse symbol (compare with Theorem 3.2, Remark 7.1 and [55], Theorem 3). The question of whether fields $E$ with LCFT are strictly PQL is open. By Theorem 1.1, its answer depends on the solution to one of the leading unsolved problems in Brauer group theory (see Remark 3.4, [35], Sect. 16, and [34], Sect. 5). This allows us to prove in Section 3 that SQL-fields are those whose finite extensions admit LCFT. Note also that when $p\text{Br}(E)$ is finite, for each $p \in P(E)$, the answer is positive if and only if $E^*/N(M/E) \cong G(M/E)^{\text{Br}(E)}$, $M \in \Omega(E)$ (apply Theorem 1.2 and [41], Sect. 15.1, Proposition b).

The paper is organized as follows: Section 2 includes generalities about PQL-fields. The former and the latter assertions of Theorem 1.1 are proved in Sections 4 and 3, respectively. Our main results on local $p$-CFT are stated in Section 3. Theorems 1.2, 1.3 and these results are proved in Sections 4, 6 and 7. Section 5 contains an interpretation of a part of local $p$-CFT
in terms of Galois cohomology. It generalizes [31], Theorem 1, as well as known relations between local fields and Demushkin groups (cf. [48], Ch. II, Theorem 4), and the sufficiency part of the main results of [36, 37].

2. Preliminaries on PQL-fields

The present research is based on the possibility to reduce the study of norm groups of finite abelian extensions to the special case of $p$-extensions. The reduction is obtained by applying the following lemma (which can be deduced from Galois theory (see e.g., [32], Ch. VIII) and [9], II, Lemma 2.2).

**Lemma 2.1.** Let $E$ be a field, $M \in \Omega(E)$, $M \neq E$, $\Pi$ the set of prime divisors of $[M : E]$, and $M_p = M \cap E(p)$, for each $p \in \Pi$. Then $M$ coincides with the compositum of the fields $M_p$: $p \in \Pi, N(M/E) = \cap_{p \in \Pi} N(M_p/E)$ and $E'/N(M/E)$ is isomorphic to the direct product $\prod_{p \in \Pi} E'/N(M_p/E)$.

The main results of [9], I, used in this paper can be stated as follows:

**Proposition 2.2.** Let $E$ be a $p$-quasilocal field with $Br(E)_p \neq \{0\}$, for some $p \in P(E)$. Suppose further that $R$ is a finite extension of $E$ in $E(p)$ and $D \in d(E)$ is an algebra of $p$-primary dimension. Then:

(i) $R$ is $p$-quasilocal and $D/E$ is a cyclic of exponent $\exp(D) = \text{ind}(D)$;

(ii) $Br(R)_p$ is a divisible group unless $p = 2$, $R = E$ and $E$ is formally real; when $E$ is formally real, $E(2) = E(\sqrt{-1})$ and $Br(E)_2$ is of order 2;

(iii) $P_{E/R}$ maps $Br(E)_p$ surjectively on $Br(R)_p$ and $Cor_{R/E}$ maps $Br(R)_p$ injectively in $Br(E)_p$: every $E$-automorphism $\psi$ of the field $R$ is extendable to a ring automorphism on each $D_R \in s(R)$ of $p$-primary index;

(iv) $R$ embeds in $D$ as an $E$-subalgebra if and only if the degree $[R : E]$ divides $\text{ind}(D)$; $R$ is a splitting field of $D$ if and only if $\text{ind}(D) \mid [R : E]$.

Our next result gives an equivalent form of Proposition 2.2 (iv), for PQL-fields, and shows its optimality in the class of algebraic strictly PQL-extensions of the field $\mathbb{Q}$ of rational numbers. We refer the reader to [9], II, for a proof of this result, which demonstrates the applicability of the arithmetic method of constructing such extensions, based on [12], Theorem 2.2.

**Proposition 2.3.** (i) Let $E$ be a PQL-field, $M/E$ a finite Galois extension and $R$ an intermediate field of $M/E$. If $G(M/E)$ is nilpotent, then $R$ embeds as an $E$-subalgebra in each $E \in d(E)$ of index divisible by $[R : E]$.

(ii) For each nonnilpotent finite group $G$, there exists a strictly PQL-field $\Psi = \Psi(G)$, such that $E/Q$ is an algebraic extension, $Br(\Psi) \cong \mathbb{Q}/\mathbb{Z}$ and there is a Galois extension $\Psi'$ of $\Psi$ with $G(\Psi'/\Psi) \cong G$, which does not embed as a $\Psi$-subalgebra in any $\Delta \in d(\Psi)$ of index $\text{ind}(\Delta) = [\Psi': \Psi]$.

The main results of [9], I, and [5], Sect. 3, show that $Br(E)$ and $G_E$ are the main algebraic structures associated with any quasilocal field $E$. Therefore, we would like to point out that $G_E$ is prosolvable and $Br(E)$ is embeddable in $\mathbb{Q}/\mathbb{Z}$ in the following two cases:
(2.1) (i) $E$ is an algebraic extension of a quasilocal field $K$ with a Henselian valuation $v$, such that $v(K)$ is totally indivisible (i.e. $v(K) \neq pv(K)$, for every $p \in \mathbb{P}$); then $\text{Br}(E)$ is divisible with $\text{Br}(E)_{p'} = \{0\}$, $p' \notin \mathbb{P}$. Concerning $\mathcal{G}_E$ and $\text{Br}(E)$, see [4], I, Lemma 1.2 and Proposition 3.1, and [14], Corollary 5.3 as well as [9], I, (1.3), respectively.

(ii) $E$ is formally real and quasilocal; then $\text{Br}(E)$ is of order 2 and $\mathcal{G}_E$ has every $p$-component $T_p$ of index 2, which is either procyclic or 2-generated as a topological group (see [5], Sect. 3, and [14], Proposition 3.1).

Condition (2.1) (i) holds, when $(K, v)$ is Henselian discrete valued quasilocal or $E$ is an algebraic quasilocal nonreal extension of a global field, such that $\text{Br}(E) \neq \{0\}$ (cf. [12], Sect. 3). The existence of a PQL-field $F$ of essentially nonarithmetic nature, i.e. with $\text{Br}(F)$ not embeddable in $\mathbb{Q}/\mathbb{Z}$, has been established in [46], Sect. 3, by observing that every abelian torsion group $A$ embeds in $\text{Br}(F(A))$, for a suitably chosen strictly PQL-field $F(A)$. This result and Proposition 2.2 (ii) are complemented by the following statement which describes, in conjunction with [19], Theorem 23.1, the isomorphism classes of Brauer groups of nonreal PQL-fields (see [13], Theorem 1.2 (i)-(ii), and for the formally real case, [13], Proposition 6.4). This statement also shows that the absolute Galois groups of nonreal SQL-fields need not be prosolvable and may have a very complex structure:

(2.2) An abelian torsion group $T$ is isomorphic to $\text{Br}(E)$, for some nonreal PQL-field $E = E(T)$ if and only if $T$ is divisible. If $T$ is divisible, $S$ is a set of finite groups, $E_0$ is an arbitrary field, and $T_0$ is a subgroup of $\text{Br}(E_0)$ embeddable in $T$, then $E$ can be chosen so as to satisfy the following:

(i) $E$ is quasilocal, $P(E) = \mathbb{P}$ and $\rho_{E/L}$ surjective, for any finite extension $L/E$ (apply also the Albert-Hochschild theorem (cf. [48], Ch. II, 2.2)).

(ii) $E/E_0$ is an extension, such that $E_0$ is algebraically closed in $E$, $T_0 \cap \text{Br}(E/E_0) = \{0\}$ and every $G \in S$ is realizable as a Galois group over $E$.

When the set $pT = \{t \in T: pt = 0\}$ is infinite, for each $p \in \mathbb{P}$, (2.2) implies the following result (combined with [14], Remark 6.6, it points up the need for the restrictions on the ground fields considered in [27] and [43]):

(2.3) In order that $A, B \in d(E)$ have a common set of splitting fields among the intermediate fields of $E_{\text{sep}}/E$ it is necessary and sufficient that $\text{ind}(A) = \text{ind}(B)$ [9], I, Corollary 8.5. For each $n \in \mathbb{N}$, $d(E)$ contains infinitely many nonisomorphic $E$-algebras of index $n$.

Statements (2.1) and (2.2) can be supplemented by the following partial conversion of the Koenigsmann-Neukirch theorem [26], Theorem B:

(2.4) If $(E, v)$ is a Henselian quasilocal field, such that all finite groups are isomorphic to subquotients of $\mathcal{G}_E$ (by open normal subgroups), then $v(E)$ is divisible and every $D \in d(E)$ is inertial relative to $v$ [14], Proposition 6.3.

Remark 2.4. Let $T$ be a divisible abelian torsion group and $\text{Supp}(T)$ the set of those $p \in \mathbb{P}$ for which the $p$-component $T_p$ of $T$ is nontrivial. Fix a field $E$ with $\text{Br}(E) \cong T$ as in (2.2) (i) and denote by $E_{\text{sol}}$ the maximal Galois extension of $E$ in $E_{\text{sep}}$ with a prosolvable Galois group. Then $E_{\text{sol}}/E$ possesses an intermediate field $E'$ that is strictly PQL with $\text{Br}(E') \cong T$ (take
as $E'$, e.g., the fixed field of some Hall pro-Supp($T$)-subgroup of $G(E_{sol}/E)$). Note also that if Supp($T$) = $\mathbb{P}$, then $E$ is SQL.

Remark 2.4 and statements (1.2) (i) and (2.1) ÷ (2.4) draw interest in the following open questions:

(ii) Let $E$ be an SQL-field with $G_E$ pro-solvable. Find whether the Sylow pro-$p$-subgroups of $G_E$ are of rank $r_p \leq 2$, for all $p \in \mathbb{P}$, with at most 2 exceptions.

Remainder 2.5. It is known (see [39]) that if $E$ is a field, then the triple AT($E$) defined in (1.2) (ii) is an Artin-Tate class formation if and only if $E$ is SQL, Br($E$) is embeddable in $\mathbb{Q}/\mathbb{Z}$, and $\rho_{E/R}$ is surjective, for every finite extension $R$ of $E$ in $E_{sep}$. Observing that (2.2) provides access to the richest presently known sources of such fields, we add to the examples given after the statement of (1.2) that AT($E$) is Artin-Tate when $E$ is an algebraic SQL-extension of a global field (see [39] and [12], Sect. 3).

3. Statements of the main results of local $p$-class field theory

Our main results on local $p$-CFT are stated as the following two theorems:

Theorem 3.1. Let $E$ be a $p$-quasilocal field with Br($E$)$_p \neq \{0\}$, for some $p \in P(E)$, and let $\Omega_p(E)$ and $d(p)$ be as in the Introduction. Then $E$ admits local $p$-CFT and, for each $M \in \Omega_p(E)$, $E^*/N(M/E)$ is isomorphic to the group $\mathcal{G}(M/E)^{d(p)}$ defined in Theorem 1.2.

Theorem 3.1 plays a major role in the proof of the embeddability of Br($E$) into $\mathbb{Q}/\mathbb{Z}$ in case (2.1) (i), which in turn enables one to characterize the quasilocal property in the class of Henselian valued fields with totally indivisible value groups (see [14], Sect. 6, and [7], II). Note also that Theorem 3.1 and Lemma 2.1 imply Theorem 1.2 and the former assertion of Theorem 1.1. In addition, Theorem 3.1 contains exact analogues to the fundamental correspondence and the local reciprocity law. Similarly, our next result can be viewed as such an analogue to Hasse’s symbol for local fields (see [25], Ch. 2, as well as Remark 7.1 and Corollary 7.3 below).
Theorem 3.2. Under the hypotheses of Theorem 3.1, let \( E_\infty \) be a \( \mathbb{Z}_p \)-extension of \( E \) in \( E(\mathcal{P}) \), and for any \( n \in \mathbb{N} \), let \( \Gamma_n \) be the intermediate field of \( E_\infty /E \) of degree \([\Gamma_n : E] = p^n\). Then there exist sets \( H_p(E') = \{(M',/E') : E' \to \mathcal{G}((M'//E')^{(p)}, M' \in \Omega_p(E')), E' \in \Omega_p(E)\}, \) of surjective group homomorphisms satisfying the following:

(i) The kernel of \((M'//E')\) is equal to \( N(M'//E')\), for each \( E' \in \Omega_p(E)\), \( M' \in \Omega_p(E')\);

(ii) If \( M \in \Omega_p(E) \) and \( K \) is an intermediate field of \( M/E \), then \((K/E) = \mathcal{G}(M/E)^{(p)} \to \mathcal{G}(K/E)^{(p)}\) is the homomorphism acting componentwise as the natural mapping of \( \mathcal{G}(M/E) \) on \( \mathcal{G}(K/E)\);

(iii) In the setting of (ii), \((\lambda, M/K) = (N_{E}^{K}(\lambda), M/E)\), for each \( \lambda \in K^*\);

(iv) The maps \((\Gamma_n/E)\), \( n \in \mathbb{N} \), determine the sets \( H_p(E'), E' \in \Omega_p(E)\).

The rest of this Section concerns the open question of whether fields with LCFT are strictly PQL. Lemma 2.1 and first result in this direction imply the latter assertion of Theorem 1.1.

Proposition 3.3. Let \( E \) be a field admitting local \( p \)-CFT, for some \( p \in P(E) \), and let \( L \) be a degree \( p \) extension of \( E \) in \( E(\mathcal{P}) \). Then \( \text{Br}(L/E) \neq \{0\} \) and \( \text{Br}(L/E) \) does not depend on the choice of \( L \). Furthermore, if each \( D \in d(E) \) of exponent \( p \) is similar to tensor products of cyclic division \( E \)-algebras of index \( p \), then \( E \) is \( p \)-quasilocal.

Proof. Our assumptions ensure that \( L/E \) is cyclic, whence \( \text{Br}(L/E) \cong E^*/N(L/E) \) (cf. [32], Ch. I, Sect. 6, and [41], Sect. 15.1, Proposition b). As \( E \) admits local \( p \)-CFT and \( L \neq E \), this yields \( N(L/E) \neq N(E/E) = E^* \) and \( \text{Br}(L/E) \neq \{0\} \). In order to complete our proof, it suffices to show that \( L \) embeds as an \( E \)-subalgebra in each cyclic division \( E \)-algebra of index \( p \). If \( \mathcal{G}(E(p)/E) \) is procyclic, this is evident, so we assume that \( \Omega_p(E) \) contains a field \( F \neq L \), such that \([F:E] = p\). It follows from Galois theory that \( LF \in \Omega(E), [LF:E] = p^2 \) and \( \mathcal{G}(LF/E) \) is noncyclic. Therefore, \( LF/E \) possesses \( p + 1 \) intermediate fields of degree \( p \) over \( E \). Let \( F' \) be such a field different from \( L \) and \( F \). Then \( N(F/E)N(F'/E) = E^* \) and \( LF' = LF \). Moreover, the norm maps \( N_{E}^L \) and \( N_{E}^{F'} \) are induced by \( N_{L}^{LF} \), so it turns out that \( E^* \subseteq N(LF/L) \). Hence, by [41], Sect. 15.1, Proposition b, \( L \) embeds over \( E \) in each \( \Delta \in d(E) \) split by \( F \), which proves Proposition 3.3. \( \square \)

Remark 3.4. Let \( E \) be a field and \( p \in P(E) \).

(i) If \( \mathcal{G}(E(p)/E) \cong \mathbb{Z}_p \) and \( L \) is the unique degree \( p \) extension of \( E \) in \( E(\mathcal{P}) \), then \( E \) admits local \( p \)-CFT if and only if \( \text{Br}(L/E) \neq \{0\} \) (apply Proposition 3.3 and [41], Sect. 15.1, Corollary b).

(ii) When \( \text{Br}(E)_p \neq \{0\} \), the concluding condition of Proposition 3.3 is satisfied in the following cases: (a) \( E \) is an algebraic extension of a global or local field \( E_0 \); (b) \( E \) contains a primitive \( p \)-th root of unity or char\( (E) = p \); (c) \( p = 3, 5 \). In cases (b) and (c), this follows from the Merkur’ev-Suslin theorem [35], (16.1), [2], Ch. VII, Theorem 28, [34], Sect. 4, Corollary, and [33]. In case (a), by class field theory, every \( A_0 \in s(E_0) \) is cyclic (cf. [3], Ch. 10, Theorem 5), which implies the same, for all \( A \in s(E) \).
(iii) It is not known whether $E$ is strictly PQL, if it admits LCFT and has a Henselian discrete valuation (see [7], Sect. 2).

**Corollary 3.5.** A finite purely inseparable extension $K$ of a field $E$ of characteristic $p > 0$ admits local $p$-CFT if and only if so does $E$.

**Proof.** By [9], I, Proposition 4.4, $E$ is $p$-quasilocal if and only if $K$ is $p$-quasilocal, and by the Albert-Hochschild theorem, $p_{E/K}$ is surjective. Since the exponent of $Br(K/E)$ divides $[K : E]$ (see [41], Sects. 13.4 and 14.4), and by Witt’s theorem (cf. [16], page 110), $Br(E)_p$ and $Br(K)_p$ are divisible, this implies that $Br(E)_p \neq \{0\} \iff Br(K)_p \neq \{0\}$, so Corollary 3.5 follows from Theorem 3.1 and Remark 3.4 (ii). □

The concluding result of this Section clarifies the role of SQL-fields in LCFT. In view of (2.2), Remark 2.5 and the main results of [13], it also determines the place in the study of strictly PQL-fields of the Neukirch-Perlis variant of the theory [39], built upon (1.2) (ii).

**Proposition 3.6.** A field $E$ is SQL if and only if its finite extensions admit LCFT. When occurs, $Br(E_p) \neq \{0\}$, provided that $E_p$ is the fixed field of a Sylow pro-$p$-subgroup of $G_E$, where $p \in \mathbb{P}$ is chosen so that $G_E$ is of cohomological $p$-dimension $cd_p(G_E) \neq 0$.

**Proof.** The left-to-right implication is contained in Theorem 1.1, so we prove the converse one and the nontriviality of $Br(E_p)_p$, for any $p \in \mathbb{P}$ satisfying $cd_p(G_E) \neq 0$. Suppose that every finite extension $L$ of $E$ admits LCFT, and $B_p$ is the extension of $E$ generated by the $p$-th roots of unity in $E_{sep}$. It is known (cf. [32], Ch. VIII, Sect. 3) that $B_p \subseteq E_p$, i.e. $E_p$ contains a primitive $p$-th root of unity unless $p = \text{char}(E)$. Also, Proposition 3.3 and Remark 3.4 (ii) indicate that $L$ is $p$-quasilocal, provided that $B_p \subseteq L$. As $p \vdash [L' : E]$, for any finite extension $L'$ of $E$ in $E_p$, this enables one to obtain from general properties of scalar extension maps and Schur indices (cf. [41], Sect. 13.4, and [9], I, (1.3)) that $E_p$ is $p$-quasilocal. Moreover, it follows from [9], I, Lemma 8.3, and the choice of $p$ that $E$ is quasilocal. Since, by Proposition 3.3, PQL-fields with LCFT are strictly PQL, the obtained result shows that $E$ is SQL. Note further that the inequality $cd_p(G_E) \neq 0$ ensures the existence of finite Galois extensions of $E$ in $E_{sep}$ of degrees divisible by $p$. Therefore, by Sylow’s theorems and Galois theory, there is a finite extension $L_0$ of $E$ in $E_p$, such that $p \in P(L_0)$. As in the proof of the statement that $E_p$ is $p$-quasilocal, these observations imply $p \in P(E_p)$ and $Br(E_p) \neq \{0\}$. □

Let us note that the abstract approach to LCFT dates back to the early 1950’s (cf. [3], Sect. 14). It fits the character of the related class formation theory, see [23], and accounts for the fact that the Neukirch-Perlis variant of LCFT goes substantially beyond the limits of (2.1). The fact itself is established by summing up Proposition 3.6, Remark 2.5, statements (2.1), (2.2) and (2.4), and the results on (2.5) (ii) mentioned in Section 2.
4. Proof of Theorem 3.1

Let $E$ be a $p$-quasilocal field, for a given $p \in P(E)$. By Proposition 2.2 (iv), then $\text{Br}(M/E) = \{b \in \text{Br}(E): [M:E]b = 0\}$, for every cyclic $p$-extension $M/E$. Hence, by the structure of divisible abelian torsion groups, and by the fact that $\text{Br}(M/E) \cong E^*/N(M/E)$ (cf. [19], Theorem 23.1, and [41], Sect. 15.1, Proposition b), $E^*/N(M/E) \cong \mathcal{G}(M/E)^{(p)}$. To prove the obtained isomorphism, for any $M \in \Omega_p(E)$, we need the following lemmas.

**Lemma 4.1.** Let $E$ be a $p$-quasilocal field, for some $p \in P(E)$, and let $E_1, \ldots, E_t$ be cyclic extensions of $E$ in $E(p)$, for a given integer $t \geq 2$. Assume that the composite $E'$ of the fields $E_j$: $j = 1, \ldots, t$, satisfies the equality $[E': E] = \prod_{j=1}^t [E_j: E]$. Then $N(E'/E) = \bigcap_{j=1}^t N(E_j/E)$.

**Proof.** The inclusion $N(E'/E) \subseteq \bigcap_{j=1}^t N(E_j/E)$ follows from the transitivity of norm maps in towers of finite separable extensions (cf. [32], Ch. VIII, Sect. 5). Conversely, let $c \in \bigcap_{j=1}^t N(E_j/E)$ and $\beta \in E_1^*$ be of norm $N_{E_1}^E(\beta) = c$. The equality $[E': E] = \prod_{j=1}^t [E_j: E]$, Galois theory and [6], Lemma 4.2, imply $[E': E_1] = \prod_{i=2}^t ([E_1E_i]: E_1)$ and $\beta \in \bigcap_{i=2}^t N(E_1E_i/E_1)$. This proves Lemma 4.1 in the case where $t = 2$. Since $(E_1E_i)/E_1$ is cyclic, for each $i \geq 2$, and by Proposition 2.2 (i), $E_1$ is $p$-quasilocal, the obtained result makes it easy to complete our proof by induction on $t$. □

**Lemma 4.2.** Assume that $E$, $F$, and $L$ are fields, such that $E$ is $p$-quasilocal, $L \in \Omega_p(E)$, $E \subseteq F \subseteq L$ and $F/E$ is cyclic. Then $\psi(\alpha)\alpha^{-1} \in N(L/F)$, for each $\alpha \in F^*$ and $\psi \in \mathcal{G}(F/E)$.

**Proof.** As $F$ is $p$-quasilocal and $L \in \Omega_p(F)$, whence $\mathcal{G}(L/F)$ decomposes into a direct product of cyclic groups, Galois theory and Lemma 4.1 allow one to consider only the case in which $L/F$ is cyclic. Let $\psi'$ be an automorphism of $L$ extending $\psi$. Fix a generator $\sigma$ of $\mathcal{G}(L/F)$, denote by $A_\alpha$ the cyclic $F$-algebra $(L/F, \sigma, \alpha)$, for an arbitrary $\alpha \in F^*$, put $m = [L:F]$, and take an invertible element $\eta \in A_\alpha$ so that $\eta^m = \alpha$ and $\eta\lambda\eta^{-1} = \sigma(\lambda)$, for every $\lambda \in L$. Then Proposition 2.2 (iii) and the Skolem-Noether theorem (cf. [41], Sect. 12.6) imply that $A_\alpha$ has a ring automorphism $\tilde{\psi}$, such that $\tilde{\psi}(\lambda) = \psi'(\lambda)$, for any $\lambda \in L$. Since $\sigma\psi' = \psi'\sigma$, this ensures that the element $\eta^{-1}\psi(\eta) := \mu$ lies in the centralizer of $L$ in $A_\alpha$. Thereby, we have $\mu \in L^*$ and $\psi(\eta)^m = \psi(\alpha) = \eta^m$ $N_L^F(\mu) = \alpha N_L^F(\mu)$, which proves Lemma 4.2. □

Assuming that $E$ is a $p$-quasilocal field, $M_u \in \Omega_p(E)$, $[M_u: E] = p^{\mu_u}$: $u = 1, 2$, and putting $M' = M_1M_2$, we prove the following assertions:

(4.1) (i) If $N(M_1/E) = N(M_2/E)$ and $M_1 \subseteq M_2$, then $M_1 = M_2$;

(ii) If $M_1 \cap M_2 = E$, then $N(M_1/E) \cap N(M_2/E) = N(M'/E)$, $N(M_1/E)N(M_2/E) = E^*$, each $\sigma \in \mathcal{G}(M_2/E)$ has a unique prolongation $\sigma' \in \mathcal{G}(M'/M_1)$, and the mapping of $\mathcal{G}(M_2/E)$ on $\mathcal{G}(M'/M_1)$ by the rule $\sigma \mapsto \sigma'$ is an isomorphism;

(iii) Under the hypotheses of (ii), if $M_2/E$ is cyclic and $\mathcal{G}(M_2/E) = \langle \sigma \rangle$, then $\mathcal{G}(M'/M_1) = \langle \sigma' \rangle$ and $\text{Cor}_{M_1/E}$ maps $\text{Br}(M'/M_1)$ into $\text{Br}(M_2/E)$.
by the formula \( [(M'/M_1, \sigma', \theta)] \to [(M_2/E, \sigma, N_{E}^{M_1}(\theta))] \), \( \theta \in M_1^* \); when \( \text{Br}(E)_p \) is divisible, \( \text{Cor}_{M_1/E} \) induces isomorphisms \( \text{Br}(M_1)_p \cong \text{Br}(E)_p \) and \( \text{Br}(M'/M_1) \cong \text{Br}(M_2/E) \).

It is sufficient to consider the special case where \( M_u \neq E \): \( u = 1, 2, \) and \( M_1 \neq M_2 \). In view of Proposition 2.2 (ii), then \( \text{Br}(E)_p \) is divisible. At the same time, Galois theory and the assumptions on \( M_1/E \) imply the existence of a cyclic degree \( p \) extension \( M_0 \) of \( E \) in \( M_1 \). Suppose that \( M_1 \subseteq M_2 \) and \( d \in N(M_u/E) \): \( u = 1, 2 \), fix a generator \( \psi \) of \( \mathcal{G}(M_0/E) \) and elements \( \eta_j \in M_j^*, j = 1, 2 \), so that \( N_{E}^{M_1}(\eta_1) = N_{E}^{M_2}(\eta_2) = d. \) It follows from Hilbert's Theorem 90 that \( N_{M_0}^{M_1}(\eta_1) = N_{M_0}^{M_2}(\eta_2)\psi(\beta)\beta^{-1} \), for some \( \beta \in M_0^* \). Hence, by Lemma 4.2, \( N(M_1/E) = N(M_2/E) \) if and only if \( N(M_1/M_0) = N(M_2/M_0) \). Observing also that \( M_0 \) is \( p \)-quasilocal and \( [M_1: M_0] = p^{u+1} \), and proceeding by induction on \( u \), one proves (4.1) (i).

Assume now that \( M_1 \cap M_2 = E \). Then the Galois-theoretic parts of (4.1) (ii) and (iii) are contained in [32], Ch. VIII, Theorem 4. The rest of the former assertion of (4.1) (iii) is implied by Proposition 2.2 (iii), the basic restriction-corestriction (abbr, RC) formula (cf. [50], Theorem 2.5) and the lemmas in [6], Sect. 4. The equality \( N(M'/E) = N(M_1/E) \cap N(M_2/E) \) follows from the presentability of \( M_1 \) and \( M_2 \) as composites of cyclic extensions of \( E \) satisfying the conditions of Lemma 4.1. It remains for us to show that \( N(M_1/E)N(M_2/E) = E^* \) and to prove the latter part of (4.1) (iii). Suppose first that \( M_2/E \) is cyclic and the automorphisms \( \sigma, \sigma' \) are determined as in (4.1) (iii), fix an element \( c \in E^* \) out of \( N(M_2/E) \), and put \( A_c = (M_2/E, \sigma, c) \). It is clear from [41], Sect. 15.1, Proposition b, and the choice of \( c \) that \( \text{ind}(A_c) | p^{u+2} \) and \( \text{ind}(A_c) > 1. \) As \( \text{Br}(E)_p \) is divisible, there exists \( \Delta_c \in d(E) \), such that \( p^{u+1}[\Delta_c] = [A_c] \). In addition, it follows from Proposition 2.2 (i) that \( \text{ind}(\Delta_c) = p^{u+1} \cdot \text{ind}(A_c) \). Observing also that \( [M': E] = p^\mu \) and \( \text{ind}(\Delta_c) | p^{\mu} \), where \( \mu = \mu_1 + \mu_2 \), one obtains from Proposition 2.2 (iv) that \( [\Delta_c] \in \text{Br}(M'/E) \). Since \( \Delta_c \otimes_E M' \cong (\Delta_c \otimes_E M_1) \otimes_{M_1} M' \) as \( M' \)-algebras (cf. [41], Sect. 9.4, Corollary a), this means that \( [\Delta_c \otimes_E M_1] \in \text{Br}(M'/M_1) \), or equivalently, that \( [\Delta_c \otimes_E M_1] = [(M'/M_1, \sigma', \alpha)] \), for some \( \alpha \in M_1^* \). Therefore, by Proposition 2.2 (iii), the RC-formula and the former statement of (4.1) (iii), \( [A_c] = [(M_2/E, \sigma, N_{E}^{M_1}(\alpha))] \). As \( [A_c : E] = [(M_2/E, \sigma, N_{E}^{M_1}(\alpha)): E] = [M_2 : E]^2 \), this proves that \( A_c \cong (M_2/E, \sigma, N_{E}^{M_1}(\alpha)) \) over \( E \). Hence, by [41], Sect. 15.1, Proposition b, \( cN_{E}^{M_1}(\alpha)^{-1} \in N(M_2/E) \), which yields \( N(M_1/E)N(M_2/E) = E^* \) in case \( M_2/E \) is cyclic. Our argument, combined with Proposition 2.2 (ii)-(iii) and the RC-formula, also proves the latter part of (4.1) (iii). Henceforth, we assume that \( M_2/E \) is noncyclic, i.e. \( \mathcal{G}(M_2/E) \) is an abelian \( p \)-group of rank \( r \geq 2 \). Then it follows from Galois theory and the structure of finite abelian groups that there exist cyclic extensions \( F_1 \) and \( F_2 \) of \( E \) in \( M_2 \) such that \( F_1 \cap F_2 = E \) and the \( p \)-groups \( \mathcal{G}(F_2/F_1) \) and \( \mathcal{G}(M_2/F_2) \) are of rank \( r - 1 \). As \( M_1 \cap M_2 = E \), one also sees that \( (M_1F_u) \cap M_2 = F_u : \ u = 1, 2 \). Taking now into account that \( F_1 \) and \( F_2 \) are \( p \)-quasilocal fields, and arguing by induction on \( r \), one concludes that it suffices to deduce the equality \( N(M_1/E)N(M_2/E) = E^* \) under the extra hypothesis that \( N(M_1F_u/F_u)N(M_2/F_u) = F_u^* : \ u = 1, 2 \). Then, by norm transitivity in
towers of finite extensions, $N(F_1/E)N(F_2/E) \subseteq N(M_1/E)N(M_2/E)$, and since $N(F_1/E)N(F_2/E) = E^*$, this completes the proof of (4.1).

**Remark 4.3.** Let $E$ be a field and $M_1, M_2$ be finite extensions of $E$ in $E_{\text{sep}}$, such that $M_2/E$ is cyclic and $M_1 \cap M_2 = E$. It is known that then the former statement of (4.1) (iii) remains valid. The assertions on $G(M_2/E)$ and $G(M'/M_1)$ follow from [32], Ch. VIII, Theorem 4, and the formula for the action of $\text{Cor}_{M_1/E}$ on $\text{Br}(M'/M_1)$ can be proved by a group-cohomological technique (cf. [54], Proposition 4.3.7). It is therefore worth noting that (4.1) (iii) plays a role in the proof not only of Theorems 3.1 and 3.2 but also of Proposition 2.2 (i) (see [9], I, Sect. 7). This, combined with [13], Theorem 1.2 (i)-(ii), enables one to find alternative field-theoretic proofs of the formula in (4.1) (iii) and of other results on $\text{Cor}_{F/E}$, for any finite extension $F$ of $E$ in $E_{\text{sep}}$ (by reduction to the setting of (2.2) (i), see [10]).

We are now in a position to prove Theorem 3.1 (and thereby, Theorem 1.2 and the former part of Theorem 1.1 as well). Assuming as above that $M_u \in \Omega_p(E)$: $u = 1, 2$, and $M_1M_2 = M'$, put $L' = M_1 \cap M_2$. We first show that $N(L'/E) = N(M_1/E)N(M_2/E)$ and $N(M'/E) = N(M_1/E) \cap N(M_2/E)$. In view of (4.1) (ii), it suffices to consider the case where $L' \neq E$. As $L'$ is $p$-quasilocal, (4.1) (ii) yields $L^* = N(M_1/L')N(M_2/L')$, so the equality $N(L'/E) = N(M_1/E)N(M_2/E)$ is obtained from norm transitivity in the towers $E \subseteq L' \subseteq M_u$: $u = 1, 2$. Suppose further that $[L': E] = p^m$ and fix a degree $p$ extension $L$ of $E$ in $L'$. As in the proof of (4.1) (i), it is seen that an element $\lambda \in L^*$ lies in $N(M_i/L)$, for some $i \in \{1, 2\}$, if and only if $N^*_E(\lambda) \in N(M_i/L)$. Since $[L': L] = p^m-1$ and $L$ is $p$-quasilocal, this makes it easy to prove inductively that $N(M'/E) = N(M_1/E) \cap N(M_2/E)$. It follows from this result and (4.1) (i) that the natural mapping $\Omega_p(E) \to \text{Br}(E)$ is injective. Thus the statement that $E$ admits local $p$-CFT is proved, which allows us to deduce Theorem 1.1 from Lemma 2.1 and Proposition 3.3.

To finish the proof of Theorem 3.1 (and Theorem 1.2) we show that $E^*/N(M/E) \cong G(M/E)^{d(p)}$, provided that $M \in \Omega_p(E)$ and $G(M/E)$ is noncyclic. Then $G(M/E)$ is an abelian $p$-group of rank $r(M) \geq 2$, so it follows from Galois theory that $M/E$ has intermediate fields $\Phi_1$ and $\Phi_2$, such that $\Phi_1 \Phi_2 = M$, $\Phi_1 \cap \Phi_2 = E$ and $\Phi_2/E$ is cyclic. This means that $G(M/E)$ is isomorphic to the direct product $G(\Phi_1/E) \times G(\Phi_2/E)$, and $G(\Phi_1/E)$ is of rank $r(M)-1$ as a $p$-group. Since $N(\Phi_1/E)N(\Phi_2/E) = E^*$ and $N(\Phi_1/E) \cap N(\Phi_2/E) = N(M/E)$, the natural diagonal embedding of $E^*$ into $E^* \times E^*$ induces a group isomorphism $E^*/N(M/E) \cong E^*/N(\Phi_1/E) \times E^*/N(\Phi_2/E)$. Now our proof is easily completed proceeding by induction on $r(M)$.

**Remark 4.4.** Given a PQI-field $E$, denote by $c(M)$ the intersection of fields $N \in \Omega(E)$ with $N(N(E) = N(M/E)$, for each $M \in \Omega(E)$. It follows from Theorem 3.1, Lemma 2.1 and [9], I, Lemma 4.2 (ii), that the natural map $\nu$ of $\Omega(E)$ on the set $N_{ab}(E) = \{N(M/E): M \in \Omega(E)\}$ satisfies (1.1). Therefore, $\text{Br}(E)p \neq \{0\}$ and $p \nmid [M: c(M)]$ whenever $M \in \Omega(E)$ and $p \in \mathbb{P}$ divides $[c(M): E]$. Note also that $N(\Phi_1/E)N(\Phi_2/E) = E^*$ and $N(\Phi_1/E) \neq N(M/E)$ in case $M_0 \in \Omega(E)$, $M_0 \subseteq c(M)$ and $M_0 \neq c(M)$. Since the set $\text{Cl}(E) = \{c(\Lambda): \Lambda \in \Omega(E)\}$ is closed under taking subextensions of $E$ and
finite compositums, whence \( \nu \) induces a bijection of \( \text{Cl}(E) \) on \( N_{ab}(E) \), these observations allow us to view \( c(M) \) as a class field of \( N(M/E) \).

5. Galois cohomological interpretation of Theorem 3.1

In this Section we consider some Galois cohomological aspects of the problem of characterizing fields with LCFT. Let \( P \) an infinite pro-\( p \)-group, \( \text{cd}(P) \) the cohomological dimension of \( P \) and \( \mathbb{F}_p \) a field with \( p \) elements, for some \( p \in \mathbb{P} \). We say that \( P \) is a \( p \)-group of Demushkin type, if the (continuous) cohomology group homomorphism \( \varphi_P: H^1(P, \mathbb{F}_p) \rightarrow H^2(P, \mathbb{F}_p) \) mapping each \( g \in H^1(P, \mathbb{F}_p) \) into the cup-product \( h \cup g \) is surjective, for every \( h \in H^1(P, \mathbb{F}_p) \setminus \{0\} \). We call a degree of \( P \) the dimension of \( H^2(P, \mathbb{F}_p) \) as an \( \mathbb{F}_p \)-vector space. The defined groups and local \( p \)-CFT are related as follows:

Proposition 5.1. Let \( E \) be a nonreal field containing a primitive \( p \)-th root of unity, for some \( p \in P(E) \). Then the following conditions are equivalent:

(i) \( E \) admits local \( p \)-CFT;
(ii) \( G(E(p)/E) \) is a \( p \)-group of Demushkin type of degree \( d \geq 1 \);
(iii) \( E \) is \( p \)-quasilocal with \( \text{Br}(E)_p \neq \{0\} \);
(iv) \( \text{cd}(G(E(p)/E)) = 2 \) and \( \text{Br}(E') \) is a trivial module over the integral group ring \( \mathbb{Z}G(E'/E) \), for every degree \( p \) extension \( E' \) of \( E \) in \( E(p) \).

Proof. The equivalence (i)\(\Leftrightarrow\)(iii) is implied by Proposition 3.3, Remark 3.4 (ii) and Theorem 3.1. Note also that \( \text{Br}(E)_p = \{0\} \) if and only if \( G(E(p)/E) \) is a free pro-\( p \)-group, i.e. a \( p \)-group of Demushkin type of degree zero (cf. [52], Theorem 3.1, [53], page 725, and [48], Ch. I, 4.1 and 4.2). In particular, this holds when \( \text{Br}(E(p)/E) \) is of rank 1 as a pro-\( p \)-group, since then \( G(E(p)/E) \cong \mathbb{Z}_p \) (see, e.g., [9], I, Remark 3.4 (ii)). These observations, [9], I, Lemma 3.8, and the end of Proposition 2.2 (ii) prove that (ii)\(\Leftrightarrow\)(iii). It remains to be seen that (iii)\(\Leftrightarrow\)(iv). As \( E \) is nonreal and \( p \in P(E) \), one obtains from Galois theory and [56], Theorem 2, that \( G(E(p)/E) \) possesses a closed normal subgroup \( H \), such that \( G(E(p)/E)/H \cong \mathbb{Z}_p \). Since \( \text{Br}(E)_p \neq \{0\} \), this means that \( H \neq \{1\} \), so it follows from [9], I, Proposition 4.6 (ii), and Galois cohomology (see [48], Ch. I, 4.2 and Proposition 15) that \( \text{cd}(H) = 1 \) and \( \text{cd}(G(E(p)/E)) = 2 \). Hence, by Teichmüller’s theorem (cf. [16], Ch. 9, Theorem 4) and Proposition 2.2 (iii), (iii)\(\Rightarrow\)(iv). To prove that (iv)\(\Rightarrow\)(iii) we need the following results (see [35], (11.5), and [24], Proposition 3.26):

(5.1) If \( F \) is a field containing a primitive \( p \)-th root of unity, and \( M \) is a finite Galois extension of \( F \) in \( F(p) \), then there exists an isomorphism \( \kappa_M: H^2(G(F(p)/M, \mathbb{F}_p) \cong \text{Br}(M) \) as \( \mathbb{Z}[G(M/F)] \)-modules, and the compositions \( \text{cor}_{M/F} \circ \kappa_M \) and \( \kappa_F \circ \text{cor}_{M/F} \) coincide, where \( \text{cor}_{M/F} \) is the corestriction map \( H^2(G(F(p)/M, \mathbb{F}_p) \rightarrow H^2(G(F(p)/F, \mathbb{F}_p)) \). Hence, \( \text{cor}_{M/F} \) is surjective if and only if so is \( \text{cor}_{M/F} \).

It suffices to show that \( \text{Br}(E'/E) = \text{Br}(E) \), for an arbitrary degree \( p \) extension \( E' \) of \( E \) in \( E(p) \). The equality \( \text{cd}(G(E(p)/E)) = 2 \) ensures that \( \text{Br}(E)_p \neq \{0\} \) and it follows from (5.1) and [40], Proposition 3.3.8, that
Corollary 5.2. Let $E$ be an SQL-field, such that $cd_p(G_E) \neq 0$, for a given $p \in \mathbb{P}$. Then the Sylow pro-$p$-subgroups of $G_E$ are pro-$p$-groups of Demushkin type of degree $d_p \geq 1$, unless $p = \text{char}(E)$ or $E$ is formally real and $p = 2$.

Proof. One may consider only the special case of $E_{\text{sep}} = E(p)$ (see the proof of Proposition 3.6). Then $\text{Br}(E)_p \neq \{0\}$ and our conclusion follows from Propositions 5.1, 2.2 (ii) and [48], Ch. II, Proposition 3. □

It is easily seen that the degrees of $p$-groups of Demushkin type are bounded by their ranks. The following conversion of this fact can be deduced from [35], (11.5), Proposition 5.1, [9], I, Theorem 8.1, and the sufficiency part of (2.2) (by specifying the cardinalities of the fields $E_0$ and $E$ in (2.2) (ii), see [13], Remark 5.4, for more details):

(5.2) For any system $d \geq N_0$ and $d_p \leq d$: $p \in \mathbb{P}$, of cardinal numbers, there is a field $E$ containing a primitive $p$-th root of unity, for every $p \in \mathbb{P}$, and such that $G(E(p)/E)$ and the Sylow pro-$p$-subgroups of $G_E$ are of rank $d$, Demushkin type and degree $d_p$.

By a Demushkin pro-$p$-group, we mean a $p$-group of Demushkin type of degree 1. Demushkin pro-$p$-groups of rank $r(P) \leq N_0$ have been classified by Demushkin, Labute and Serre (cf. [29, 30] and further references there). When $r(P) = N_0$, by [36] and [37], $P$ has $s$-invariant zero if and only if $P \cong G(F(p)/F)$, for some field $F$. Hence, by applying Lemma 3.5 of [7] (in a modified form adjusted to the case singled out by [37], Proposition 3.1 (iii)), and arguing as in the proof of (5.2), one supplements it as follows:
For each sequence $G_p: \ p \in \mathbb{P}$, of Demushkin pro-$p$-groups of rank $\aleph_0$ and $s$-invariant zero, there exists a field $E$ such that $G(E(p)/E)$ and the Sylow pro-$p$-subgroups of $G_E$ are isomorphic to $G_p$ when $p$ ranges over $\mathbb{P}$.

Next we show that the inequality $d \geq \aleph_0$ in (5.2) is essential. This result is a special case of [28], Corollary 5 (see also [31], Sects. 1 and 2). For convenience of the reader, we present it here with a short proof based on Theorem 3.1 and Proposition 5.1.

**Corollary 5.3.** Let $E$ be a field containing a primitive $p$-th root of unity, for some $p \in \mathbb{P}(E)$, and with $G(E(p)/E)$ a $p$-group of Demushkin type of finite rank $r(p)$. Then $G(E(p)/E)$ is a Demushkin group or a free pro-$p$-group.

**Proof.** In view of [53], Lemma 7, and Proposition 5.1 (with its proof), one may consider only the case where $r(p) \geq 3$ and $Br(E)_p \neq \{0\}$. Take a field $E' \in \Omega(E)$ so that $G(E'/E)$ has exponent $p$ and order $p^{r(p)-1}$. By Theorem 3.1 and [35], (11.5), then $E'^p \subseteq N(E'/E)$ and $N(E'/E)$ is a subgroup of $E^{*}$ of index $p^{(r(p)-1)d_p}$, so we have $(r(p)-1)d_p \leq r(p)$, where $d_p$ is the degree of $G(E(p)/E)$. In our case, this implies $d_p = 1$, which proves Corollary 5.3. □

**Remark 5.4.** Statement (5.1) and Corollary 5.3 show that the equivalence (ii)$\leftrightarrow$(iv) in Proposition 5.1 generalizes the concluding assertion of [31], Theorem 1 (independently of the elementary type conjecture formulated in [31]).

Corollary 5.3 gives us the possibility to determine the structure of the continuous character group $C(E(p)/E)$ of $G(E(p)/E)$, for a $p$-quasilocal nonreal field $E$ containing a primitive $p$-th root of unity. Recall that $C(E(p)/E)$ is an abelian torsion $p$-group, whence, it decomposes into the direct sum $D(E(p)/E) \oplus R(E(p)/E)$, where $D(E(p)/E)$ is the maximal divisible subgroup of $C(E(p)/E)$ and $R(E(p)/E)$ is a (reduced) subgroup of $C(E(p)/E)$ isomorphic to $C(E(p)/E)/D(E(p)/E)$ (see [22], Ch. 7, Sect. 5, and [19], Theorem 24.5). With this notation, our next result can be stated as follows:

**Proposition 5.5.** Let $E$ be a $p$-quasilocal nonreal field, $\mu_p(E)$ the group of roots of unity in $E$ of $p$-primary degrees, and $r_p(E)$ the rank of $G(E(p)/E)$ as a pro-$p$-group. Suppose that $\mu_p(E) \neq \{1\}$ and $\varepsilon_p \in E$ is a primitive $p$-th root of unity. Then:

(a) $C(E(p)/E) = D(E(p)/E)$ if and only if $\mu_p(E)$ is infinite or $Br(E)_p = \{0\}$; when $Br(E)_p \neq \{0\}$ and $\mu_p(E)$ is finite of order $p^\nu$, the group $R(E(p)/E)$ is isomorphic to the maximal subgroup of $Br(E)$ of period $p^\nu$;

(b) $Br(E)_p$ is embeddable as a subgroup of $D(E(p)/E)$.

**Proof.** (a): It follows from Kummer theory that $C(E(p)/E) = D(E(p)/E)$, provided that $\mu_p(E)$ is infinite. We show that the same equality holds in the case of $Br(E)_p = \{0\}$. Then it follows from [41], Sect. 15.1, Proposition b, that $\varepsilon_p$ lies in the norm group $N(L'/E)$, for every cyclic extension $L'$ of $E$ in $E(p)$; hence, by Albert’s height theorem (cf. [1], Ch. IX, Sect. 6, and [17], Sect. 2), there is a cyclic extension $L'_1$ of $E$ in $E(p)$, such that $L' \subseteq I(L'_1/E)$ and $[L'_1: L'] = p$. This implies $L' \subseteq I(L_1/E)$, for some $\mathbb{Z}_p$-extension $L_1$ of $E$ in $E(p)$, and so proves that $C(E(p)/E) = D(E(p)/E)$. It remains to consider
the case where \( Br(E)_p \neq \{0\} \) and \( \mu_p(E) \) has finite order \( p' \). As \( \mu_p(E) \neq \{1\} \), then we have \( E(p) \neq E \), so it follows from [56], Theorem 2, and the condition that \( E \) is a nonreal field that \( E(p) \) contains as a subfield a \( \mathbb{Z}_p \)-extension of \( E \). In view of Galois theory, this means that \( C(E(p)/E) \) possesses a quasicyclic \( p \)-subgroup, which proves that \( D(E(p)/E) \neq \{0\} \). Note further that \( Br(E)_p \) is a divisible group, since \( E \) is nonreal and \( p \)-quasilocality (cf. [9], I, Theorem 3.1). Therefore, by [19], Theorem 23.1, \( Br(E)_p \) decomposes into the direct sum \( \mathbb{Z}(p^\infty)^d(p) \) of isomorphic copies of the quasicyclic \( p \)-group \( \mathbb{Z}(p^\infty) \), indexed by a set \( I \) of cardinality \( d(p) \) equal to the dimension of \( Br(E)_p \approx H^2(G(E(p)/E), \mathbb{F}_p) \) as an \( \mathbb{F}_p \)-vector space. Moreover, it becomes clear that, for each \( m \in \mathbb{N} \), the maximal subgroup of \( Br(E)_p \) of period \( p^m \) decomposes into a direct sum of cyclic groups of order \( p^m \), indexed by \( I \). Thus the latter conclusion of Proposition 5.5 (a) is equivalent to the former part of the following assertions:

(5.4) (a) \( C(E(p)/E) \) and the direct sum \( D(E(p)/E) \oplus \mu_p(E)^d(p) \) are isomorphic, where \( \mu_p(E)^d(p) \) is a direct sum of isomorphic copies of \( \mu_p(E) \), indexed by a set of cardinality \( d(p) \) (for a proof, see [4], II, Lemma 2.3).

(b) A cyclic extension \( M \) of \( E \) in \( E(p) \) is a subfield of a \( \mathbb{Z}_p \)-extension of \( E \) in \( E(p) \) if and only if there is \( M' \in I(E(p)/M) \), such that \( M'/E \) is cyclic and \( [M': M] = p^\nu \); this is the case if and only if \( \mu_p(E) \subset N(M/E) \).

The former part of (5.4) (b) is implied by (5.4) (a) and Galois theory, and the latter one follows from Albert’s height theorem.

(b): Proposition 5.5 (a) allows us to consider only the case where \( Br(E)_p \neq \{0\} \) and \( \mu_p(E) \) has order \( p' \), for some \( \nu \in \mathbb{N} \). Then it follows from (5.4) (a) and the nontriviality of \( D(E(p)/E) \) that \( r_p(E) \geq 2 \). Using the observations preceding the statement of (5.4), one also sees that it is sufficient to prove the embeddability of \( \rho \Br(E) \) in \( D(E(p)/E) \). Let now \( \delta \rho \) be a primitive \( p' \)-th root of unity, and \( M_\lambda \) be an extension of \( E \) generated by a \( p \)-th root \( \eta_\lambda \in E(p) \) of an element \( \lambda \in E^* \setminus E^{*p} \). Then \( M_\lambda/E \) is cyclic, \([M_\lambda: E] = p\) and \( G(M_\lambda/E) \) contains a generator \( \sigma_\lambda \), such that the cyclic \( E \)-algebra \((M_\lambda/E, \sigma_\lambda, \delta_\lambda)\) is isomorphic to the symbol \( E \)-algebra \( A_{sp}(\lambda, \delta_\lambda; E) \). It is well-known that \( A_{sp}(\lambda, \delta_\lambda; E) \) and \( A_{sp}(\delta_\lambda, \lambda; E) \) are inversely-isomorphic \( E \)-algebras. This, combined with [41], Sect. 15.1, Proposition b, implies \( \delta_\mu \in N(M_{\lambda}/E) \) if and only if \( \lambda \in N(M_{\lambda}/E) \). Hence, by (5.4) (b), the divisibility of \( Br(E)_p \), and Theorem 23.1 of [19], Proposition 5.5 (b) and the latter assertion of Proposition 5.5 (a) are equivalent to the statement that \( \rho \Br(E) \) embeds as a subgroup of \( N(M_{\delta}/E)/E^{*p} \). Since \( N(M_{\delta}/E)/E^{*p} \) is an abelian group of period \( p \), this amounts to proving that \( \rho \Br(E) \) is its homomorphic image. We show that \( \rho \Br(E) \) is a homomorphic image of \( N(M_{\mu}/E)/E^{*p} \), for an arbitrary element \( \mu \in E^* \setminus E^{*p} \). Fix \( \mu' \in E^* \setminus E^{*p} \) so that \( M_{\mu'} \neq M_{\mu} \). Then \( E^*/N(M_{\mu}/E) \cong \rho \Br(E) \), by [41], Sect. 15.1, Proposition b (and the \( p \)-quasilocality of \( E \)), and \( N(M_{\mu}/E)N(M_{\mu'}/E) = E^* \), by [9], I, Lemma 4.3. Since, by Theorem 3.1, \( N(M_{\mu}/E) \cap N(M_{\mu'}/E) = N(M_{\mu}M_{\mu'}/E) \), this yields \( E^*/N(M_{\mu}/E) \cong N(M_{\mu}/E)/N(M_{\mu}M_{\mu'}/E), E^{*p} \leq N(M_{\mu}M_{\mu'}/E) \) and \( N(M_{\mu}/E)/N(M_{\mu}M_{\mu'}/E) \cong (N(M_{\mu}/E)/E^{*p})/(N(M_{\mu}M_{\mu'}/E)/E^{*p}); \) in particular, \( \rho \Br(E) \) is a homomorphic image of \( N(M_{\mu}/E)/E^{*p} \), which completes the proof of Proposition 5.5 (b). \( \square \)
The concluding results of this Section present applications of Proposition 5.5 and Corollary 5.3 to the study of $p$-primary index-exponent $K$-pairs, for a Henselian field $(K, v)$ with a $p$-quasilocal field $\hat{K}$, for some $p \in \mathbb{P}$. Recall first that, for any field $E$ and each $D \in d(E)$, $\exp(D)$ divides $\text{ind}(D)$ and is divisible by any $p \in \mathbb{P}$ dividing $\text{ind}(D)$; in addition, $(\text{ind}(D), \exp(D))$ is obtained as a componentwise product of index-exponent $l$-pairs, where $l$ runs across the set of prime divisors of $\text{ind}(D)$ (see [41], Sect. 14.4). Index-exponent relations of algebras from $d(E)$ depend essentially on specific properties of $E$, and their description reduces to the special case of algebras $D_p \in d(E)$ of $p$-primary dimensions, for an arbitrary $p \in \mathbb{P}$. The study of index-exponent $p$-primary $E$-pairs relies on the knowledge of the Brauer $p$-dimension $\text{Brd}_p(E)$, defined as the least integer $b(p) \geq 0$, for which $\text{ind}(D_p)$ divides $\exp(D_p)^{\hat{b}(p)}$ whenever $D_p \in d(E)$ and $[D_p] \in \text{Br}(E)_p$; when no such $b(p)$ exists, we say that $\text{Brd}_p(E)$ is infinite. Note that $(p^k, p^n)$: $k, n \in \mathbb{N}, k \geq n$, are index-exponent $L$-pairs whenever $(L, \lambda)$ is a Henselian field, such that the quotient group $\frac{\lambda(L)}{p\lambda(L)}$ of the value group $\lambda(L)$ is infinite, $\hat{L}$ is a nonreal field and $\mu_p(\hat{L}) \neq \{1\}$. This is demonstrated by the proof of [11], Corollary 4.5, which also shows that the result does not change, if the condition on $\mu_p(\hat{L})$ is replaced by the one that the rank $r_p(\hat{L})$ of $\mathcal{G}(\hat{L}(p)/\hat{L})$ is infinite. Therefore, we restrict our considerations to the case where $\hat{K}$ is a $p$-quasilocal nonreal field, $\mu_p(\hat{K}) \neq \{1\}$ and $v(K)/pv(K)$ is a nontrivial finite group. Then $\text{Brd}_p(K)$ is determined by [15], Theorem 4.1, as follows:

(5.5) If $v(K)/pv(K)$ has order $p^\tau(p)$, then $\text{Brd}_p(K) = [(m_p(\hat{K}) + \tau(p))/2]$, where $m_p(\hat{K}) = \min\{r_p(\hat{K}), \tau(p)\}$.

Relying on Proposition 5.5, we first show that if $r_p(\hat{K}) = \infty$, then index-exponent $p$-primary $K$-pairs are fully determined by $\text{Brd}_p(K)$.

**Corollary 5.6.** Let $(K, v)$ be a Henselian field with $\hat{K}$ satisfying the conditions of Proposition 5.5, for some $p \in \mathbb{P}$. Assume also that $r_p(\hat{K}) < \infty$, $v(K) \neq pv(K)$ and $v(K)/pv(K)$ is finite of order $p^\tau(p)$. Then:

(a) $\hat{K}$ has Galois extensions $U_n, U'_n$ in $E(p)$, $n \in \mathbb{N}$, such that $[U_n: E] = p$, $[U_1 \ldots U_n: E] = p^n$, $\mathcal{G}(U'_n/E) \cong \mathbb{Z}_p$ and $U_n \in I(U'_n/E)$, for each $n$;

(b) $(p^k, p^n)$: $k, n \in \mathbb{N}, k \leq n \tau(p)$, are all nontrivial $p$-primary index-exponent pairs over $K$.

**Proof.** (a): In view of (5.4) (b) and Galois theory, it suffices to prove that the maximal subgroup $pD(\hat{K}(p)/\hat{K})$ of $D(\hat{K}(p)/\hat{K})$ of period $p$ is infinite. The infinity of $pD(\hat{K}(p)/\hat{K})$ follows from Proposition 5.5 (b), if $p\text{Br}(\hat{K})$ is infinite. Note also that $r_p(\hat{K}) = \infty$ if and only if $\hat{K}^*/\hat{K}^{*p}$ is infinite (cf. [48], Ch. I, 4.1), which holds if and only if $C(\hat{K}(p)/\hat{K})$ contains infinitely many elements of order $p$. It is therefore clear from (5.4) (a) that if $p\text{Br}(\hat{K})$ is finite, then $pD(\hat{K}(p)/\hat{K})$ is infinite, as required.

(b): It follows from Corollary 5.6 (a) and Galois theory that, for each finite abelian $p$-group $G$, there exists a Galois extension $U_G$ of $K$ in $K_\text{ur}$ with $\mathcal{G}(U_G/K) \cong G$. When the rank of $G$ is $\leq \tau(p)$, one obtains from
Corollary 5.7. Assume that \((K, v)\) is a Henselian field, such that \(v(K)/pv(K)\) is finite of order \(p^\tau(p) > 1\), and \(\hat{K}\) is \(p\)-quasilocal with \(\mu_p(\hat{K}) \neq \{1\}\), \(r_p(\hat{K}) < \infty\), and \(C(\hat{K}(p)/\hat{K}) = D(\hat{K}(p)/\hat{K})\), for some \(p \in \mathbb{P}\). Then non-trivial \(p\)-primary index-exponent \(\hat{K}\)-pairs are described by as follows:

(a) \((p^k, p^n)\): \(k, n \in \mathbb{N}, n \leq k \leq \text{Br}_d p(K)\), if \(\mu_p(\hat{K})\) is infinite;

(b) \((p^k, p^n)\): \(k, n \in \mathbb{N}, n \leq k \leq n\text{Br}_d p(K)\), if \(\mu_p(\hat{K})\) is finite of order \(p^\nu\), and \(m_p(\hat{K})\) is defined as in (5.5).

Proof. The description of index-exponent \(p\)-primary \(K\)-pairs is obtained in the case of Corollary 5.7 (b) by the method of proving [15], Lemma 5.1 (replacing \(\min\{(r_p(\hat{K}) - 1, \tau(p)\}\) by \(m_p(\hat{K})\)). Henceforth, we assume that \(\mu_p(\hat{K})\) is infinite. Note that, for any finite abelian \(p\)-group \(G\) of rank \(\leq m_p(\hat{K})\), there exist a Galois extension \(U_G\) of \(K\) in \(K_{ur}\), and an algebra \(N_G \in d(K)\), such that \(\mathcal{G}(U_G/K) \cong G\), \(N_G/K\) is NSR and \(U_G\) is \(K\)-isomorphic to a maximal subfield of \(N_G\). As in the proof of Corollary 5.6 (b), one obtains further that there exist NSR-algebras \(N_{k,n} \in d(K): k, n \in \mathbb{N}, n \leq k \leq m_p(\hat{K})n\), with \(\text{ind}(N_{k,n}) = p^k\) and \(\exp(N_{k,n}) = p^n\), for each admissible pair \(k, n\). Since \(\text{Br}_d p(K) = [(m_p(\hat{K}) + \tau(p))/2]\), this fact proves Corollary 5.7 in case \(r_p(\hat{K}) \geq \tau(p) - 1\). Suppose now that \(\tau(p) - m_p(\hat{K}) \geq 2\), put \(\hat{m} = \text{Br}_d p(K) = [(m_p(\hat{K}) + \tau(p))/2]\), fix a divisible hull \(v(N_G)\) of \(v(N_G)\), and take a finite abelian \(p\)-group \(H\) of rank \(r(H) \leq [(\tau(p) - m_p(\hat{K}))/2]\).

Using [38], Theorem 1, and the natural bijection between \(I(Y/K)\) and the set of subgroups of \(v(Y)/v(K)\), for any totally ramified finite abelian \(p\)-extension \(Y/K\) (cf. [45], Ch. 3, Theorem 2), one obtains that there is \(T_H \in d(K)\) with the following properties: \(T_H/K\) is totally ramified, \(v(T_H) \subset v(N_G)\) and \(v(T_H)/v(K)\) is isomorphic to \(H \otimes H; v(N_G) \cap v(T_H) = v(K)\) and \(T_H \otimes_K K_{ur} \in d(K_{ur}); T_H\) is a tensor product of \(r(H)\) cyclic totally ramified \(K\)-algebras; \(\text{ind}(T_H)\) and \(\exp(T_H)\) equal the order and the period of \(H/v(K)\), respectively (see, e.g., [21], for more details). Note also that, by [38], Theorem 1, \(N_G \otimes_K T_G \in d(K)\). These observations indicate that, for any \(n \in \mathbb{N}\), there exist \(\Delta_n \in d(K)\) and \(T_{n,\rho} \in d(K): \rho = 1, \ldots, n\theta\), where \(\theta = [(\tau(p) - m_p(\hat{K}))/2]\), satisfying the following conditions:

5.6 (a) \(\Delta_n/K\) is NSR, \(\exp(\Delta_n) = p^n\) and \(\text{ind}(\Delta_n) = p^{n\hat{m}}\);

(b) For each index \(\rho\), \(T_{n,\rho}/K\) is totally ramified, \(\Delta_n \otimes_K T_{n,\rho} \in d(K)\), \(\exp(\Delta_n \otimes_K T_{n,\rho}) = p^n\), and \(\text{ind}(T_{n,\rho}) = p^\rho\).
Statements (5.6) (b) show that \( \text{ind}(\Delta_n \otimes_K T_{n,\rho}) = p^{n+\rho}, \rho = 1, \ldots, \theta, \) which completes the proof of Corollary 5.7. \( \square \)

Summing-up Corollaries 5.6, 5.7 and [15], Lemma 5.1, and combining the latter with Corollary 5.3, one one fully describes \( p \)-primary index-exponent \( K \)-pairs, for a Henselian field \( (K, v) \), such that \( v(K) \neq pv(K) \) and \( \hat{K} \) is nonreal and \( p \)-quasilocal with \( \mu_p(\hat{K}) \neq \{1\} \). We refer the reader to [15], Corollary 2.2 and Remark 4.2, for an analogous description in the case where \( (K, v) \) is Henselian, \( p = 2 \), and \( \hat{K} \) is formally real \( 2 \)-quasilocal.

6. Preparation for the proof of Theorems 3.2 and 1.3

Let \( E \) be a field, \( M \in \Omega(E) \), \( \Pi \) the set of prime divisors of \( [M: E] \), and \( M_p = M \cap E(p) \), for each \( p \in \Pi \). When \( M \neq E \), the homomorphism group \( \text{Hom}(E^*, G(M/E)) \) is isomorphic to the direct group product \( \prod_{p \in \Pi} \text{Hom}(E^*, G(M_p/E)) \), so Theorem 1.3 can be deduced from Theorem 3.2, Lemma 2.1 and the primary tensor decomposition of cyclic \( E \)-algebras (cf. \[41\], Sect. 15.3). The results of this Section serve as a basis for the proof of Theorem 3.2, presented in Section 7. Our starting point are the following two lemmas.

**Lemma 6.1.** Let \( E \) and \( M \) be fields, such that \( M \in \Omega_p(E) \), for some \( p \in P(E) \). Suppose that \( G(M/E) \) has rank \( t \geq 1 \) as a \( p \)-group, and \( F \) is an intermediate field of \( M/E \) of degree \( [F : E] = p \). Then there exist cyclic extensions \( E_1, \ldots, E_t \) of \( E \) in \( M \) with \( E_1 \ldots E_t = M, \prod_{i=1}^t [E_i : E] = [M : E] \) and \( \prod_{i=1}^t [(FE_i) : F] = [M : F] \).

**Proof.** In view of Galois theory, this is equivalent to the following statement:

(6.1) Let \( G \) be a finite abelian \( p \)-group of rank \( t \geq 1 \) and \( H \) a maximal subgroup of \( G \). Then these exist cyclic subgroups \( G_1, \ldots, G_t \) of \( G \), such that the (inner) products of \( G_i \): \( i = 1, \ldots, t \), and \( H \cap G_i \): \( i = 1, \ldots, t \), are direct and equal to \( G \) and \( H \), respectively.

To prove (6.1) take a cyclic subgroup \( G_1 \leq G \) of maximal order so that the order of the group \( H_1 = H \cap G_1 \) equals the exponent of \( H \). The choice of \( G_1 \) ensures that \( G_1G_0 = G \) whenever \( G_0 \) is maximal among the subgroups of \( G \), which trivially intersect \( G_1 \) (cf. \[19\], Sects. 15 and 27). This implies the existence of subgroups \( H_1' \leq H \) and \( G_1' \leq G \), such that \( H_1 \cap H_1' = G_1 \cap G_1' = \{1\}, H_1' \leq G_1' \) and the products \( H_1H_1' \) and \( G_1G_1' \) are equal to \( H \) and \( G \), respectively. Therefore, (6.1) can be proved by induction on \( t \). \( \square \)

**Lemma 6.2.** Assume that \( E, M, p \) and \( t \) satisfy the conditions of Lemma 6.1, and let \( F \) be a maximal subfield of \( M \) including \( E \). Then there exist cyclic extensions \( E_1, \ldots, E_t \) of \( E \) in \( M \), such that \( \prod_{i=1}^t [E_i : E] = [M : E], \prod_{i=1}^t [(E_i \cap F) : E] = [F : E], \) and the composites \( E_1 \ldots E_t \) and \( (E_i \cap F) \ldots (E_t \cap F) \) are equal to \( M \) and \( F \), respectively.

**Proof.** This is equivalent to the following statement:

(6.2) Let \( G \) be a finite abelian \( p \)-group of rank \( t \geq 1 \), \( H \) a subgroup of \( G \) of order \( p \), and \( \pi \) the natural homomorphism of \( G \) on \( G/H \). Then \( G \) contains
elements $g_1, \ldots, g_t$, for which the products of the cyclic groups $\langle g_1 \rangle, \ldots, \langle g_t \rangle$ and $\langle \pi(g_1) \rangle, \ldots, \langle \pi(g_t) \rangle$ are direct and equal to $G$ and $G/H$, respectively.

For the proof of (6.2), consider a cyclic subgroup $C_1 \leq G$ of maximal order, and such that $C_1 \cap H$ is of minimal possible order. Then one can find $C'_1 \leq G$ so that $C_1 \cap C'_1 = \{1\}$, $C_1C'_1 = G$ and $H \subseteq C_1 \cup C'_1$. This implies $G/H$ is isomorphic to the direct product $C_1H/H \times C'_1H/H$, which allows one to prove (6.2) by induction on $t$.

Let now $G$ be a finite abelian $p$-group of rank $t \geq 2$. A subset $g = \{g_1, \ldots, g_t\}$ of $G$ is called an ordered basis of $G$, if $g$ is a basis of $G$ (i.e. the group product $\langle g_1 \rangle \ldots \langle g_t \rangle$ is direct and equal to $G$) and the orders of the elements of $g$ satisfy the inequalities $o(g_1) \geq \cdots \geq o(g_t)$. It is known that the automorphism group $\text{Aut}(G)$ acts transitively on the set $\text{Ob}(G)$ of ordered bases of $G$. Arguing by induction on $t$, and using the fact that cyclic subgroups of $G$ of order $o(g_1)$ are direct summands in $G$ (see [19], Sect. 27), one obtains that $\text{Aut}(G)$ has the following system of generators:

$$\text{Aut}(G) = \langle d(k, m; h), t(i, j, s; h) : h \in \text{Ob}(G) \rangle,$$

where $i, j, k, s$ and $m$ are integers with $1 \leq k, i, j \leq t$, $i \neq j$, $1 < m < o(h_k)$, $1 \leq s < o(h_i)$, $\gcd(m, o(h_k)) = 1$ and $\max\{1, o(h_i)/o(h_j)\}$, and $d(k, m; h)$ are defined by the data $d(k, m; h) = h^{m^k}$, $d(k, m; h)(h_k') = h_k'$, $k' \neq k$, and $t(i, j, s; h)(h_j) = h_j h_i^s$, $t(i, j, s; h) = h_j$: $j' \neq j$.

This allows us to prove the following two lemmas without serious technical difficulties.

**Lemma 6.3.** Let $E$, $M$, $p$ and $t$ satisfy the conditions of Lemma 6.1, and let $E$ be a $p$-quasilocal field. Suppose that $B$ a cyclic subgroup of $\text{Br}(E)_p$ of order $o(B)$ divisible by $[M : E]$, $b$ is a generator of $B$ and $B(M/E)$ is the group of those $\beta \in E^*$, for which $[(L/E, \tau, \beta)] \in B$ whenever $L/E$ is a cyclic extension, $L \subseteq M$ and $(\tau) = \mathcal{G}(L/E)$. Assume also that $E_1, \ldots, E_t$ are cyclic extensions of $E$ in $M$, such that $E_1 \ldots E_t = M$ and $[E_j : E] = \prod_{i=1}^t [E_j : E]$, and for each index $j \leq t$, let $E_j$ be the compositum of the fields $E_i$: $i \neq j$, $\tau_j$ a generator of $\mathcal{G}(E_j/E)$, and $\sigma_j$ the unique $E_j$-automorphism of $M$ extending $\tau_j$. Then there exist a group homomorphism $\omega_{M/E,b} : B(M/E) \to \mathcal{G}(M/E)$, and elements $c_1, \ldots, c_t$ of $E^*$ with the following properties:

(i) $c_j \in N(E_j/E)$ and $[(E_j/E, \tau_j, c_j)] = [o(B)/[E_j : E]], b$, for each index $j$; the co-set $c_j N(M/E)$ is uniquely determined by $b$ and $\tau_j$;

(ii) $\omega_{M/E,b}$ is the unique homomorphism of $B(M/E)$ on $\mathcal{G}(M/E)$ mapping $c_j$ into $\sigma_j$: $j = 1, \ldots, t$, and with a kernel equal to $N(M/E)$;

(iii) $\omega_{M/E,b}$ does not depend on the choice of the $t$-tuples $(E_1, \ldots, E_t)$ and $(\tau_1, \ldots, \tau_t)$; it induces a group isomorphism $B(M/E)/N(M/E) \cong \mathcal{G}(M/E)$.

**Proof.** If $t = 1$, our assertions can be deduced from Theorem 3.1 and the general theory of cyclic algebras (see [41], Sect. 15.1). Henceforth, we assume that $t \geq 2$. Using consecutively Galois theory and Theorem 3.1, one obtains that $N(E_j/E)N(E_j'/E) = E^* : j = 1, \ldots, t$, which implies the existence of elements $c_1, \ldots, c_t$ of $E^*$ with the properties required by Lemma 6.3 (i). Denote by $T(M/E)$ the subgroup of $E^*$ generated by
As $E$ aims at proving that $G$ and proceeding by induction on $\omega$ and Ker($\omega_{M/E,b}$) = $N(M/E)$. The mapping $\omega_{M/E,b}$ is surjective, so it induces canonically an isomorphism of $T(M/E)/N(M/E)$ on $G(M/E)$. We aim at proving that $T(M/E) = B(M/E)$. Assuming that $[M:E] = p^m$, and proceeding by induction on $m$, one obtains that this can be deduced from Lemma 6.2 and [41], Sect. 15.1, Corollary b, if $\omega_{M/E,b}$ has the property claimed by the former part of Lemma 6.3 (iii). In order to establish this property (the crucial point in our proof), consider another basis $\sigma'_1, \ldots, \sigma'_t$ of $G(M/E)$, and for each $j \in \{1, \ldots, t\}$, let $H_j$ be the subgroup of $G(M/E)$ generated by the elements $\sigma'_i$: $i \neq j$, $F_j$ the fixed field of $H_j$, $\tau'_j$ the $E$-automorphism of $F_j$ induced by $\sigma'_j$, and $F'_j$ the compositum of all $F_i$ with $i \neq j$. It follows from Galois theory that $F_1 \cdots F_t = M,$ $\prod_{i=1}^{t}[F_i: E] = [M:E]$, $F_j/E$ is a cyclic extension, $G(F_j/E) = \langle \tau'_j \rangle$, and $\sigma'_j$ is the unique $F'_j$-automorphism of $M$ extending $\tau'_j$ ($j = 1, \ldots, t$). Since $\omega_{M/E,b}$ is surjective, it maps some elements $c'_1, \ldots, c'_t$ of $T(M/E)$ into $\sigma'_1, \ldots, \sigma'_t$, respectively. Applying Lemma 6.2 and [41], Sect. 15.1, Corollary b, one concludes that the proof of Lemma 6.3 will be complete, if we show that $c'_j \in N(F'_j/E)$ and $[(F'_j/E, \tau'_j, c'_j)] = [o(B)/[F_j:E]], b$, for every index $j$.

It is clearly sufficient to consider the special case in which $\{\sigma_1, \ldots, \sigma_t\}$ and $\{\sigma'_1, \ldots, \sigma'_t\}$ are ordered bases of $G(M/E)$ (which implies $[E_\rho:E] = [F_\rho:E]$, $\rho = 1, \ldots, t$). In view of [41], Sect. 15.1, Corollary a, and (6.3), one may assume in addition that $\sigma'_j = \sigma_j \sigma'_s$, $c'_j = c_j c'_s$, $\sigma'_u = \sigma_u$ and $c'_u = c_u$: $u \neq j$, for some pair $(i, j)$ of different indices, and some integer $s$ satisfying the inequalities $1 \leq s < o(\sigma_i)$ and divisible by $\max\{1, o(\sigma_1)/o(\sigma_j)\}$. Then it follows from Galois theory that $F_i F_j = E_i E_j$, $F_w = E_w$, $\tau'_w = \tau_w$: $w \neq i$, and $F'_u = E'_u$: $u \neq j$. One also sees that if $t \geq 3$ and $y \in \{\{1, \ldots, t\} \setminus \{i, j\}\}$, then $c'_y \in N(E_y/E)$. Since the cyclic $E$-algebras $(E_j/E, \tau_j, c_j)$ and $(E_j/E, \tau'_j, c'_j)$ are isomorphic (see [41], Sect. 15.1, Proposition b), and by Lemma 4.1, $N(F'_j/E) = \bigcap_{u \neq j} N(F_u/E)$, this reduces the proof of Lemma 6.3 (ii) to the one of the assertions that $c'_j \in N(F'_i/E)$ and there is an $E$-isomorphism $(E_i/E, \tau_i, c_i) \cong (F_i/E, \tau'_i, c'_i)$. We first show that $(E_i/E, \tau_i, c_i) \cong (F_i/E, \tau'_i, c'_i)$. The assumptions on $E_i$, $F_i$ and $E_j = F_j$ guarantee that the field $E_i E_j = F_i E_j$ is isomorphic as an $E$-algebra to $E_i \otimes_E E_j$ and $F_i \otimes_E E_j$ (cf. [41], Sect. 14.7, Lemma b). It is therefore easy to see from the general properties of tensor products (cf. [41], Sect. 9.2, Proposition c) that $(E_i/E, \tau_i, c_i) \otimes_E (E_j/E, \tau_j, c_j) \cong (F_i/E, \tau'_i, c'_i) \otimes_E (E_j/E, \tau_j, c_j c'_s)$ as $E$-algebras, where $s' = s$, $o(\sigma_j)/o(\sigma_i)$. Since $s' \in \mathbb{Z}$ and $c_i \in N(E_j/E)$, there exists an $E$-isomorphism $(E_j/E, \tau_j, c_j) \cong (E_j/E, \tau_j, c_j c'_s)$. The obtained results prove that $(E_i/E, \tau_i, c_i)$ and $(F_i/E, \tau'_i, c'_i)$ are similar over $E$. As $[E_i:E] = [F_i:E]$, we also have $[E_i/E, \tau_i, c_i] = [F_i/E, \tau'_i, c'_i]$, and $[E_i:E, \tau_i, c_i] = [E_i:E]^2$, so it turns out that $(E_i/E, \tau_i, c_i) \cong (F_i/E, \tau'_i, c'_i)$, as claimed.

It remains to be seen that $c'_j \in N(F_i/E)$. Suppose first that $F_i \cap E_j = E$, take elements $\beta_i \in E_i$, $\beta_j \in E_j$, $\delta \in Br(E)$ and an algebra $D \in d(E)$ so that $N_{E_i}^E(\beta_i) = c_i$, $N_{E_j}^E(\beta_j) = c_j$, $([E_i:E],[E_j:E])\delta = b$, and $o(B)\delta = [D]$,
and denote by $\varphi_i$ and $\varphi_j$ the automorphisms of $E_i E_j$ induced by $\sigma_i$ and $\sigma_j$, respectively. It is easily verified that $[E_j : E] [D] = [(E_j / E, \tau_i, c_i)]$ and $\mathcal{G}(E_i E_j / E_j) = (\varphi_i)$. This, combined with the fact that $E_i E_j = F_i E_j$, $E_i \cap E_j \neq E$ and $\varphi_i$ extends $\tau_i$, enables one to deduce from (4.1) (iii), Proposition 2.2 and the RC-formula that $D \otimes E E_j$ is similar to the cyclic $E_j$-algebra $(E_i E_j / E_j, \varphi_i, \beta_i)$. Since $[E_i : E] [D] = [(E_j / E, \tau_j, c_j)]$, $\mathcal{G}(E_i E_j / E_j) = (\varphi_j)$, and $\varphi_j$ is a prolongation of $\tau_j$, it is analogously proved that $[D \otimes E E_j] = [(E_i E_j / E_i, \varphi_j, \beta_j)]$ in $Br(E_i)$. At the same time, it is clear from Galois theory and the condition $F_i \cap E_i = E$ that $\mathcal{G}(E_i E_j / E) = (\varphi_j, \varphi_i$ (i.e. g.c.d.($s, p) = 1$), $[E_i : E] \leq [E_j : E]$, and $E_i F_i = E_i \Phi$, where $\Phi$ is the extension of $E$ in $E_j$ of degree $[\Phi : E] = [E_i : E]$. Note also that $\varphi_j$ extends $\tau_i^{-s}$, since $\sigma_j(\lambda) = \lambda$, for every $\lambda \in F_i$. Observing now that $[(E_i / E, \tau_i, c_i)] = \omega_i [(E_j / E, \tau_j, c_j)]$ and $[(E_i F_i / E_i, \psi_j, c_i)] = \omega_i [(E_i E_j / E_i, \varphi_j, c_i)]$, where $\omega_i = [E_i : E] / [E_i : E] = (E_i E_j, E_i F_i)$ and $\psi_j$ is the automorphism of $E_i F_i$ induced by $\varphi_j$ (cf. [41], Sect. 15.1, Corollary b), one obtains from (4.1) (iii) that $(E_i / E, \tau_i, c_i) \cong (\Phi / E, \psi_j, c_j) \cong (F_i / E, \tau_i^{-s}, c_j)$ over $E$ ($\psi_j$ being the automorphism of $\Phi$ induced by $\varphi_j$). Since $(E_i / E, \tau_i, c_i) \cong (F_i / E, \tau_i^{-s}, c_j)$, these results indicate that $(F_i / E, \tau_i^{-s}, c_j) \cong (F_i / E, \tau_i^{-s}, c_i)$, which means that $c_i \in N(F_i / E)$.

Let now $E_i \cap F_i = \bar{E}_i [\bar{E}_i : E] = \mu > 1$, $s / \mu = s$ and $E_i E_j = E_{i,j}$. Then $E_{i,j} \in \Omega(\bar{E})$ and $\mathcal{G}(E_{i,j} / \bar{E}) = \mathcal{G}(E_i / E_i) \mathcal{G}(E_j / F_i)$, which implies $s \in E$ and g.c.d.($s, p) = 1$. For each index $u \neq i$, take $\gamma_u \in E'_u$ so that $N_{E_u}^{E_u}(\gamma_u) = c_u$, and put $\bar{c}_u = N_{E_u}^{E_u}(\gamma_u)$, $\bar{E}_u = E_u \bar{E}$ and $\bar{F}_u = F_u \bar{E}$. Since $E_i \cap E_u = E_u$ follows from (4.1) (ii) that $E_i \cap \bar{E}_u = \bar{E}_i [\bar{E}_u : E] = [E_u : E]$, $\tau_u$ uniquely extends to a $\bar{E}$-automorphism $\tilde{\tau}_u$ of $\bar{E}_u$, and $\mathcal{G}(\bar{E}_u / \bar{E}) = (\tilde{\tau}_u)$. Note also that $N_{E_u}^{E_u}(\theta_u) = N_{E_u}^{E_u}(\theta_u)$, $\theta_u \in E_u$, whence $c_i \in N(\bar{E}_u / \bar{E})$. Fix elements $b_u \in Br(E)$ and $\Delta_u \in d(E)$ so that $\mu b_u = b$ and $[\Delta_u] = o(B) / [E_u : E] | b_u$, and put $\bar{b} = p_{E / \bar{E}}(b_u)$. It is clear from the double centralizer theorem (cf. [41], Sects. 12.7 and 13.3) that $[(E_i / E, \tau_i, c_i) \otimes_{E} \bar{E}] = [(E_i / \bar{E}, \tau_i^{\mu}, c_i)]$ in $Br(\bar{E})$. Applying Proposition 2.2, (4.1) (iii) and the RC-formula, one obtains consecutively that $o(B)$ equals the order of $\bar{b}$ in $Br(E)$, $[(E_i / E, \tau_i, c_i)] = o(B) / [E_i : \bar{E}] | b_u$, $[(E_i / \bar{E}, \tau_i^{\mu}, c_i)] = [E_i / \bar{E}] | b_u$, and for each $u \neq i$, $[E_u / E, \tau_u, c_u] = \mu [\Delta_u]$ and $[(\bar{E}_u / \bar{E}, \tilde{\tau}_u, c_u)] = [\Delta_u \otimes_{E} \bar{E}] = [o(B) / [\bar{E}_u : \bar{E}] | b_u]$. Consider now $M / E$, $\tilde{s}$, $\tilde{b}$, the fields $E_i, F_i, \bar{E}_i, \bar{F}_i$, $u \neq i$, and the algebras $(E_i / \bar{E}, \tau_i^{\mu}, c_i)$, $(E_i / \bar{E}, \tau_j, c_j)$, instead of $M / E$, $s$, $b$, $E_u$, $F_u$: $u' = 1, \ldots, t$, and $(E_i / E, \tau_i, c_i)$, $(F_i / E, \tau_i^{\mu}, c_i)$, $(E_j / E, \tau_j, c_j)$, respectively. As in the proof of our assertion in case g.c.d.($s, p) = 1$, one concludes that $\tilde{c}_j c_i^{\mu} \in N(F_i / E)$.

Since $N_{E_u}^{E_u}(\tilde{c}_j c_i^{\mu}) = c_j$, this yields $c_j \in N(F_i / E)$ (and $T(M / E) = B(M / E)$), which completes the proof of Lemma 6.3.

\[\square\]

**Lemma 6.4.** In the setting of Lemma 6.3, let $F$ be an extension of $E$ in $M$, $\pi_M / E$ the natural projection of $G(M / E)$ on $G(F / E)$, $\rho_E F / (b) = b'$, $B' = (b')$, and $B_k = (b_k)$, where $b_k = \kappa b$, for some $\kappa \in \mathbb{N}$ dividing $o(B) / [M : E]$. Then $[M : E] = o(B_k)$, $[M : F] = o(B')$, and the mappings $\omega_{M / E, b}$, $\omega_{M / E, b_k}$, $\omega_{F / E, b}$ and $\omega_{F / E, b_k}$, determined as required by Lemma 6.3, are related as follows:

(i) $\omega_{M / E, b} = \omega_{M / E, b_k}$ and $\omega_{F / E, b} = \pi_M / F \cdot \omega_{M / E, b}$;
(ii) \( \omega_{M/F,M}(\gamma) = \omega_{M/E,M}(N_E^F(\gamma)), \) for every \( \gamma \in F^\ast \).

Proof. Clearly, \( o(B_n) = o(B)/\kappa \), and Proposition 2.2 implies that \( o(B') = o(B)/[F : E] \), so the assertions that \([M : E] \mid o(B_n), [M : F] \mid o(B') \) and \( \omega_{M/E,b} = \omega_{M/E,b} \) are obvious. Note also that if \( t = 1 \), then the remaining statements of the lemma can be deduced from the general properties of cyclic algebras. Suppose further that \( t \geq 2 \). First we prove that \( \omega_{F/E,b} = \pi_{M/F} \circ \omega_{M/E,b} \) in case \( F \) is a maximal subfield of \( M \). Let \( E_1, \ldots, E_t \) be extensions of \( E \) in \( M \) with the properties described by Lemma 6.2. Then there exists an index \( j \), such that \( E_j \cap F \) is a maximal subfield of \( E_j \) and \( E_i \subseteq F \), for any other index \( i \). This allows us to prove our assertion by applying Lemma 6.3 and [41], Sect. 15.1, Corollary b.

We turn to the proof of Lemma 6.4 (ii) in the special case of \([F : E] = p\). It is briefly presented, since we argue along the same lines as the concluding part of the proof of Lemma 6.3. Take \( E_1, E_1', \ldots, E_t, E_t', \tau_1, \sigma_1, \ldots, \tau_t, \sigma_t \) and \( c_1, \ldots, c_t \) in accordance with Lemmas 6.1 and 6.3. Then \( F \subseteq E_j \) and \( E_i \cap F = E : u \neq j \), for some index \( j \). This implies \( F \subseteq E_0 \) and \( \tau_u \) uniquely extends to an \( F \)-automorphism \( \tilde{\tau}_u \) of the field \( E_0 F := \tilde{E}_u \), when \( u \) runs across the set \( W_j = \{1, \ldots, t\} \setminus \{j\} \). Choose elements \( \alpha_1 \in E_1', \ldots, \alpha_t \in E_t' \) so that \( N_E^{E_0}(\alpha_k) = c_k : k = 1, \ldots, t \), and put \( \tilde{E}_j = E_j, \tilde{c}_j = c_j, \tilde{\tau}_j = \tau_j^p, \tilde{\sigma}_j = \sigma_j^p \), and \( \tilde{\sigma}_u = N_{E_0}^{E_0}(\alpha_u), \tilde{\tau}_u = \sigma_u : u \in W_j \). It is easily verified that \( N_E^{E_0}(\tilde{c}_j) = \tilde{c}_j^p \) and \( N_E^{E_0}(\tilde{c}_u) = c_u, u \in W_j \). Therefore, it follows from (4.1) (iii), [41], Sect. 14.7, Lemma a, and the equality \( o(B') = o(B)/p \) that \( M/F, b' \), and \( \tilde{\tau}_p, \tilde{\sigma}_p, \tilde{\rho} \), where \( \rho \) runs through \( W_j \) or \( \{1, \ldots, t\} \) depending on whether or not \( F \) is related as in Lemma 6.3. Since the RC-formula and [41], Sect. 15.1, Proposition b, yield \( \text{Cor}_{F/E}(\{E_j/F, \tau_j^p, c_j\}) = \{E_j/F, \tau_j, c_j^p\} \), this proves Lemma 6.4 (ii) when \([F : E] = p\).

Let finally \( F \) be any proper extension of \( E \) in \( M \) different from \( M \). By Galois theory and the structure of finite abelian groups, then \( M \) possesses subfields \( F_0 \) and \( M_0 \), such that \( E \subseteq F_0 \subseteq F \subseteq M_0 \) and \([F_0 : E] = [M : M_0] = p\).

In view of the transitivity of norm mappings, canonical projections of Galois groups and scalar extension maps of Brauer groups in towers of intermediate fields of \( M/E \) (cf. [41], Sect. 9.4, Corollary a), the considered special cases of Lemma 6.4 enable one to complete inductively its proof. □

7. Existence and Form of Hasse Symbols

The main purpose of this Section is to prove Theorem 3.2. Fix an \( \mathbb{F}_p \)-basis \( I_p \) of \( p \cdot \text{Br}(E) \) and a generator \( \varphi_{\infty} \) of \( \mathcal{G}(E_{\infty}/E) \) as a topological group. Take a subset \( \Delta_p(E) \) of \( \text{Br}(E) \) as a subset of \( \mathbb{F}_p \)-vector space, and \( \varphi_{\infty}(E) \) as a topological group. Take a subset \( \Delta_p(E) = \{b_{i,n}(p) : i \in I_p, n \in \mathbb{N}\} \) of \( \text{Br}(E) \) such that \( \{b_{i,n}(p) : i \in I_p\} \) is a basis of \( p \cdot \text{Br}(E) \) as an \( \mathbb{F}_p \)-vector space, and \( \varphi_{\infty}(E) \) as a topological group. For each \( E' \in \Omega_p(E) \), let \( \rho_{E'/E}(b_{i,n}(p)) = b_{i,n}(E') : (i, n) \in I_p \times \mathbb{N}, E'^{\infty} = E\infty \cdot E', and [E' \cap E_{\infty}) : E] = m(E'). As \( \mathcal{G}(E_{\infty}/E) \cong \mathbb{Z}_p \cong H \) whenever \( H \) is an open subgroup of \( \mathbb{Z}_p \) (see [48], Ch. 1, 1.5), it follows from Galois theory that \( E'_{\infty}/E' \) is a \( \mathbb{Z}_p \)-extension and \( \varphi_{\infty}^{m(E')} \) uniquely extends to an \( E' \)-automorphism \( \varphi_{\infty}(E') \) of \( E'_{\infty} \); one also sees that \( \varphi_{\infty}(E') \)
topologically generates $G(E'_{\infty}/E')$. For each $n \in \mathbb{N}$ and $i \in I_p$, denote by $\varphi_n(E')$ the automorphism of $\Gamma_nE'$ induced by $\varphi_{\infty}(E')$, and by $g_{i,n}(E')$ the element of $G(\Gamma_nE'/E')^{d(p)}$ with components $g_{i,n}(E')_i = \varphi_n(E')_i$, and $g_{i,n}(E')_{i'} = 1$: $i' \in I_p \setminus \{i\}$. By Proposition 2.2 and [41], Sect. 15.1, Proposition a, $E'$ has a subset $C_p(E') = \{c_{i,n}(E') : i \in I_p, n \in \mathbb{N}\}$, such that $[(\Gamma_nE'/E', \varphi_n(E'), c_{i,n}(E'))] = b_{i,n}(E')$, for each $(i, n) \in I_p \times \mathbb{N}$. Observe that, for any $n \in \mathbb{N}$, there is a unique surjective homomorphism $(\cdot, \Gamma_nE'/E') : E'^* \to G(\Gamma_nE'/E')^{d(p)}$, whose kernel is $N(\Gamma_nE'/E')$, and which maps $c_{i,n}(E')$ into $g_{i,n}(E')$, when $i \in I_p$. Fix some $M' \in \Omega_p(E')$ and $\mu \in \mathbb{Z}$ so that $p^\mu \geq [M' : E]$, and for each $i \in I_p$, let $B(M'/E')_i = \{\beta_i \in E'^*: [(L'/E', \sigma'_i, \beta_i)] \in b_{i,\mu}(E')\}$, for every cyclic extension $L'$ of $E'$ in $M'$, where $\sigma'$ is a generator of $G(L'/E')$. It follows from Proposition 2.2 (ii)-(iii), the definition of $\Lambda_p(E)$ and Lemma 6.3 that the groups $B(M'/E')_i := B(M'/E')_i/N(M'/E')$, $i \in I_p$, have the following property:

$$(7.1) \quad \overline{B}(M'/E') \cong G(M'/E'), \quad \text{for each index } i, \text{ and the inner product of } \overline{B}(M'/E')_i : i \in I_p, \text{ is direct and coincides with } E'^*/N(M'/E').$$

Therefore, there exists a unique homomorphism $(\cdot, M'/E')$ of $E'^*$ into $G(M'/E')^{d(p)}$, mapping $B(M'/E')_i$, into the $i$-th component of $G(M'/E')^{d(p)}$ by the formula $\bar{\beta}_i \mapsto \omega_{M'/E', b_{i,\mu}(E')}(\bar{\beta}_i)$, for each $i \in I_p$, where $\omega_{M'/E', b_{i,\mu}(E')}$ is defined as in Lemma 6.3. Hence, $Ker(\cdot, M'/E') = N(M'/E')$, and by Lemma 6.4, the sets $H(E') = \{(M', E') : M' \in \Omega_p(E')\}, E' \in \Omega_p(E)$, consist of surjections related as required by Theorem 3.2 (ii)÷(iii).

Suppose now that $\Theta_p(E') = \{\theta(M'/E') : E'^* \to G(M'/E')^{d(p)}, M' \in \Omega_p(E')\}, E' \in \Omega_p(E)$, is a system of surjective homomorphisms with the same kernels and relations, and such that $\theta(\Gamma_n/E) = (\cdot, \Gamma_n/E)$, for every $n \in \mathbb{N}$. Then it follows from Proposition 2.2, the RC-formula and (4.1) (iii) that $\theta(\Gamma_nE'/E') = (\cdot, \Gamma_nE'/E')$, for each pair $(E', n) \in \Omega_p(E') \times \mathbb{N}$. We show that $\Theta_p(E') = H_p(E')$, for any $E' \in \Omega_p(E)$. This is obvious, if $E_\infty = E(p)$, so we assume further that $E(p) \neq E_\infty$. In view of Proposition 2.2 (i) and the already established part of Theorem 3.2, it is sufficient to prove that $\theta(M/E) = (\cdot, M/E)$, for an arbitrary fixed field $M \in \Omega_p(E)$. Note first that the composition $ME_\infty$ possesses a subfield $M_0 \in \Omega_p(E)$, such that $M_0 \cap E_\infty = E$ and $M_0E_\infty = ME_\infty$. This follows from Galois theory and the projectivity of $\mathbb{Z}_p$ as a profinite group (cf. [48], Ch. I, 5.9).

In particular, $M \subseteq M_0\Gamma_n$, for every sufficiently large index $n$. We also have $\theta(M/E) = \pi_{M_0\Gamma_n/m} \circ \theta(M_0\Gamma_n/E)$, which allows us to consider only the special case of $M = M_0\Gamma_n$ and $M_0 \neq E$, where $\kappa$ is chosen so that $[M_0 : E] | p^\kappa$. Let now $t$ be the rank of $G(M/E)$, and $E_1, \ldots, E_t$ be cyclic extensions of $E$ in $M$, such that $\prod_{i=1}^t [E_i : E] = [M : E]$, $E_1 = \Gamma_n$ and $E_2 \ldots E_t = M_0$. Take $E'_1, \ldots, E'_t$ as in Lemma 6.3, denote for convenience by $\tau_1$ the automorphism $\varphi_n(E)$ of $E_1$, and let $\tau_u$ be a generator of $G(E_u/E)$. For every $u \in \{2, \ldots, t\}$. Fix an index $x \in I_p$, put $b_{x,u}(p) = b_x$, and identifying $G(M/E)$ with the $x$-th component of $G(M/E)^{d(p)}$, consider elements $x_1, \ldots, x_t$ of $E^*$ determined so that $\theta(M/E)(x_u)$ equals the $E'_u$-automorphism $\sigma_u$ of $M$ extending $\tau_u$, for each positive integer $u \leq t$. The assumptions on $\Theta_p(E') : E' \subseteq \Omega_p(E)$, imply that $x_u \in N(E_u/E') : u = 1, \ldots, t$, and $[(E_1/E, \tau_1, x_1)] = b_x$. We show as in the proof of Lemma 6.3 (iii) that $[(E_u/E, \tau_u, x_u)] = (p^u/[E_u : E])b_x$, for
each \( u \). Fix an index \( u \geq 2 \), put \( \omega_u = p^n/[E_u : E] \), \( \sigma'_1 = \sigma_1 \sigma_u \), \( x'_1 = x_1 x_u \), and denote by \( F_u \) the fixed field of the subgroup of \( G(M/E) \) generated by \( \sigma'_1 \) and \( \sigma_u' : \ u' \notin \{1, u\} \). It is easily verified that \( F_1 \ldots F_l = M \) and \( \prod_{u=1}^l [F_u : E] = [M : E] \), where \( F_u = E_{u'} : \ u' \neq u \). As \( \ker(\theta(F_u/E)) = N(F_u/E), \theta(M/E)(x'_1) = \sigma'_1 \) and \( \sigma'_1 \in G(M/F_u), \) the equality \( \theta(F/E) = \pi_{M/E} \circ \theta(M/E) \) ensures that \( x'_1 \in N(F_u/E) \). Observe also that \( E_1 E_u = E_1 F_u \) and \( E_u \cap F_u = E \). This implies that \( (E_1/E, \tau_1, x_1) \otimes_E (E_u/E, \tau_u, x_u) \) is \( E \)-isomorphic to \( (E_1/E, \tau'_1, x_1 x_u) \otimes_E (F_u/E, f_u, x_u) \), where \( \tau'_1 \in G(E_1/E) \) and \( f_u \in G(F_u/E) \) are induced by \( \sigma'_1 \) and \( \sigma_u \), respectively. Since \( \sigma_u \in G(M/E) \) and \( x_u \in N(E_1/E) \), it thereby turns out that \( (E_u/E, \tau_u, x_u) \cong (F_u/E, f_u^{-1}, x_1) \cong (F_u/E, f_1, x_1) \) over \( E \). \( f_1 \) being the automorphism of \( F_u \) induced by \( \sigma_1 \). Furthermore, if \( \Phi_u \) is the extension of \( E \) in \( E_1 \) of degree \( [E_u : E] \), then \( E_u \Phi_u = E_u F_u \) and, because \( x_1 \in N(E_u/E) \), it follows that \( [(E_u/E, \tau_u, x_u)] = [(F_u/E, f_1, x_1)] = \omega_u[(E_1/E, \tau_1, x_1)] = \omega_u b_x \), as claimed. The obtained result indicates that \( \theta(M/E)(\alpha_x) = (\alpha_x, M/E), \) for each \( \alpha_x \in B(M/E) \). As \( x \) is an arbitrary element of \( I_p \), this enables one to complete the proof of Theorem 3.2 by applying (7.1).

Remark 7.1. Theorem 1.1 and [7], Theorem 2.1, show that if \( (E, v) \) is a Henselian discrete valued strictly \( \mu \)-field, then \( \widehat{\mathbb{E}}(p)/\widehat{\mathbb{E}} \) is a \( \mathbb{Z} \)-extension, for each \( p \in P(E) \). Therefore, one can take as \( E_\infty \) the compositum of all \( I \in \Omega(E) \) that are inertial over \( E \). Note also that if \( E \) is SQL, then \( \text{Br}(E) \) is isomorphic to the direct sum \( \oplus_{p \in P(E)} \mathbb{Z}(p^\infty) \) (by (2.1) (i)), and for each \( p \in P(E) \), the set \( \Lambda_p(E) \) can be chosen so that one may put \( C_p(E) = \{ c_n(p) = \pi : \ n \in \mathbb{N} \} \), where \( \pi \) is a uniformizer of \( (E, v) \). When \( E \) is a local field, and for each \( (p, n) \in P(E) \times \mathbb{N}, \varphi_p(p) \) is the Frobenius automorphism of the inertial extension of \( E \) in \( E_{\text{sep}} \) of degree \( p^n \), the sets \( H_p(E), p \in P(E) \), from the proof of Theorem 3.2, define the Hasse (the norm residue) symbol, in the sense of [25], and give rise to the Artin map (cf. [20], Ch. 6).

Corollary 7.2. In the setting of (2.2) (i), let \( M \in \Omega(E) \) and \( F \) be a finite extension of \( E \) in \( E_{\text{sep}} \). Then \( N(MF/F) = \{ \lambda \in F^* : \ N_E^F(\lambda) \in N(M/E) \} \).

Proof. It suffices to prove that \( N(MF/F) \) includes the preimage of \( N(M/E) \) in \( F^* \) under \( N_E^F \). In view of Theorem 3.1, Lemma 2.1, [9], I, Lemma 4.2 (ii), and of the \( \text{PQL}\)-property of \( F \), one may consider only the case where \( M/F_0 \) is cyclic, for \( F_0 = M \cap F \). Put \( \mu_0 = N_E^F(\mu) \), for each \( \mu \in F^* \). Theorem 3.2 (iii), [9], I, Lemma 4.2 (ii), and norm transitivity in towers of intermediate fields of \( MF/E \) imply that if \( N_E^F(\mu) \in N(M/E) \), then \( \mu_0 \in N(M/F_0) \). At the same time, it follows from Proposition 2.2 (ii), [9], I, Corollary 8.5, the RC-formula and the assumptions on \( E \) that \( \text{Cor}_{F/F_0} \) is injective. Therefore, one deduces from the former part of (4.1) (iii) (in the general form pointed out in Remark 4.3) that \( \mu \in N(M(F/F)) \), which proves our assertion.

Corollary 7.2 generalizes [20], Theorem 7.6. Applying the RC-formula, Propositions 2.2 (ii) and 2.3 (i), as well as Lemma 2.1, norm and corestriction transitivity, and already used known relations between norms and cyclic algebras, one obtains by the method of proving Theorem 3.2 the existence of an exact analogue to the local Hasse symbol in the following situation:
Corollary 7.3. Assume that $E$ is a nonreal field, such that every $L \in \Omega(E)$ is strictly PQL with $p_{E/L}$ surjective. Then the maps $\text{Cor}_{L/E} : L \in \Omega(E)$, are bijective, and there are sets $H(E') = \{(M'/E') : E'^* \to G(M'/E')^{\text{Br}(E')}, M' \in \Omega(E')\}$, $E' \in \Omega(E)$, of group homomorphisms satisfying the following:

(i) $(M'/E')$ is surjective and its kernel equals $N(M'/E')$, for each $E' \in \Omega(E)$, $M' \in \Omega(E')$;

(ii) $H(E')$ has the properties required by Theorem 1.3 (ii), for every $E' \in \Omega(E)$; furthermore, if $M \in \Omega(E)$ and $K$ is an intermediate field of $M/E$, then $(\lambda, M/K) = (N^{K}(\lambda), M/E)$, for any $\lambda \in K^{*}$;

(iii) The sets $H(E')$, $E' \in \Omega(E)$, are determined by the mappings $(\cdot, \Gamma/E)$, when $\Gamma$ ranges over finite extensions of $E$ in $E_{\infty}$ of primary degrees.

Corollary 7.3 has a partial analogue for a formally real strictly PQL-field $E$ and the sets $\Omega(E') = \{M' \in \Omega(E') : 2 \uparrow [M'/E']\}$, $E' \in \Omega(E)$. Specifically, statements (i)–(iii) hold when every $E' \in \Omega(E)$ is $p$-quasilocal with $\text{Br}(E') = \mathbb{P}$ included in the image of $p_{E/E'}$, for each $p \in \mathbb{P} \setminus \{2\}$; also, in this case, $\text{Cor}_{E'/E}$ induces isomorphisms $\text{Br}(E') = \text{Br}(E)p$, $p > 2$. This is proved in the same way as Corollary 7.3. On the other hand, it follows from the Artin-Schreier theory that if $E' \in \Omega(E)$ and $E' \not= E$, then $E'$ is formally real and the action of $G(E'/E)$ induces $[E': E]$ orderings on $E'$. Therefore, $E'$ is not 2-quasilocal, so it cannot admit LCFT.

Remark 7.4. Under the hypotheses of Theorem 1.2, let $E_{ab}$ be the compositum of all $M \in \Omega(E)$, $E_{ab}(p) = E_{ab} \cap E(p)$, for each $p \in P(E)$, and $N_{1}(E) = \cap_{M \in \Omega(E)} N(M/E)$. Suppose that $d(p) \in \mathbb{N}$, for all $p \in P(E)$, and $\text{Pr}_{c}(E^{*})$ is the profinite completion of $E^{*}/N_{1}(E)$ (concerning its existence, see [24], Sect. 1.2). Using Theorems 1.1, 1.3 and Galois theory, and arguing as in the proof of implication (iii) $\Rightarrow$ (i) of [24], Proposition 1.14, one obtains that $\text{Pr}_{c}(E^{*})$ is isomorphic to the topological group product $\prod_{p \in P(E)} G(E_{ab}(p)/E)^{d(p)}$, and so generalizes a part of [20], Proposition 6.3.

Note finally that every abelian torsion group $T$ admissible by Proposition 2.2 (ii) is isomorphic to $\text{Br}(E(T))$, for some strictly PQL-field $E(T)$ satisfying the conditions of Corollary 7.3 or its analogue in the formally real case. This follows from [13], Corollary 6.6 and (7.2) (when $T$ is divisible - from Remark 2.4 as well). Hence, all forms of the local reciprocity law and Hasse symbol admissible by Theorems 1.2, 1.3 and Corollary 7.3 can be realized.

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