Compressed Sparse Linear Regression

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Abstract

High-dimensional sparse linear regression is a basic problem in machine learning and statistics. Consider a linear model $y = X\theta^* + w$, where $y \in \mathbb{R}^n$ is the vector of observations, $X \in \mathbb{R}^{n \times d}$ is the covariate matrix with $i$th row representing the covariates for the $i$th observation, and $w \in \mathbb{R}^n$ is an unknown noise vector. In many applications, the linear regression model is high-dimensional in nature, meaning that the number of observations $n$ may be substantially smaller than the number of covariates $d$. In these cases, it is common to assume that $\theta^*$ is sparse, and the goal in sparse linear regression is to estimate this sparse $\theta^*$, given $(X, y)$.

In this paper, we study a variant of the traditional sparse linear regression problem where each of the $n$ covariate vectors in $\mathbb{R}^d$ are individually projected by a random linear transformation to $\mathbb{R}^m$ with $m \ll d$. Such transformations are commonly applied in practice for computational savings in resources such as storage space, transmission bandwidth, and processing time. Our main result shows that one can estimate $\theta^*$ with a low $\ell_2$-error, even with access to only these projected covariate vectors, under some mild assumptions on the problem instance. Our approach is based on solving a variant of the popular Lasso optimization problem. While the conditions (such as the restricted eigenvalue condition on $X$) for success of a Lasso formulation in estimating $\theta^*$ are well-understood, we investigate conditions under which this variant of Lasso estimates $\theta^*$. The main technical ingredient of our result, a bound on the restricted eigenvalue on certain projections of a deterministic matrix satisfying a stable rank condition, could be of interest beyond sparse regression.

As a simple consequence, our approach also provides a new way for estimating $\theta^*$ in the traditional sparse linear regression problem setting, which operates (even) under a weaker assumption on the design matrix than previously known, albeit achieving a weaker convergence bound.

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1 Introduction

Problems in high-dimensional statistical inference have attracted a great deal of attention in recent years. Many fields in modern science and engineering such as computational biology, medical imaging, and natural language processing regularly involve collecting datasets in which the dimension of the data exceeds the sample size. In this paper, we consider a prototypical problem in high-dimensional statistics, sparse linear regression.

Consider a linear model: \( y = X\theta^\star + w \), where \( y = (y_1, \ldots, y_n) \) is the vector of responses, \( X \in \mathbb{R}^{n \times d} \) is the covariate matrix (in which \( i \)th row \( x_i^\top \) represents the covariates (features) for the \( i \)th observation), and \( w \) is an unknown \( n \)-dimensional noise vector. The goal of linear regression, given \( (X, y) \), is to estimate the vector \( \theta^\star \), known as the regression vector. If the linear regression model is high-dimensional, which means that the number of observations \( n \) is substantially smaller than the number of covariates \( d \), the model is unidentifiable and it is not meaningful to estimate \( \theta^\star \in \mathbb{R}^d \). However, many machine learning and statistics applications exhibit special structure that can lead to an identifiable model. In particular, in many settings, the vector \( \theta^\star \) is sparse, which leads to a sparse linear regression problem. Given such a problem, the most direct approach would be to seek an exact sparse minimizer of the least-squares cost, \( \|y - X\theta\|^2 \), thereby obtaining an \( \ell_0 \)-based estimator. However, since this problem is non-convex, a standard approach is to replace the \( \ell_0 \)-constraint with its \( \ell_1 \)-norm, in either a constrained or penalized form, which leads to the “Lasso” (least absolute shrinkage and selection operator) formulation [18].

Random projections are a class of extremely popular technique for dimensionality reduction (compression), where the original high-dimensional data is projected onto a lower-dimensional subspace using some appropriately chosen random matrix. Random projection techniques, such as the Johnson-Lindenstrauss transform, are attractive for machine learning applications for several reasons: (i) they lead to substantial reduction in resources such as computation time, storage space, and transmission bandwidth, (ii) they are oblivious to the data set, meaning that the method does not require any prior knowledge of the data set as input, (iii) in a distributed data setting, they can be carried out locally by each party, independent of others, (iv) they are easy to implement and computationally inexpensive, and (v) they come with rigorous theoretical guarantees.

In this paper, we initiate the study of sparse linear regression in the compressed feature setting. A celebrated result in sparse linear regression is that, under a variety of mild assumptions on the instance, the \( \ell_2 \)-error of a Lasso estimate decays roughly at the rate \( \sqrt{k \log d/n} \), where \( k \) is the sparsity level of \( \theta^\star \) [21, 1, 11]. We ask: can we achieve a small \( \ell_2 \)-error bound, under some mild assumptions, when we have access to only to the compressed representation of the data? In this paper, we answer this question in affirmative by establishing both the sufficient conditions and the corresponding achievable error bound in this setting.

Our Model. Compressed sampling has been studied in the context of machine learning applications from two points of view. One idea is to use random projections to compress the dataset by combining input vectors using random projections [17, 23, 24]. This does not reduce the dimensionality of the data but rather generates a set of fewer datapoints (reduces \( n \)). Another idea is to project each input vector into a lower dimensional space (thereby reducing \( d \)), and then perform the learning with those compressed features. In the context of sparse linear regression, this would mean to estimate \( \theta^\star \) given \( (\Phi x_1, y_1), \ldots, (\Phi x_n, y_n) \), where \( \Phi \in \mathbb{R}^{m \times d} \) is a random projection matrix with \( m \ll d \). For sparse linear regression (when \( d \gg n \)), this form of feature compression has multiple advantages over compressing the number of observations. For example, consider a setting where we care about the cost of communicating the data to the server (e.g.,
remote devices communicating to the cloud). If $d$ is large then communicating $x_i \in \mathbb{R}^d$ is costly. A natural scheme here is that the server chooses and announces a single random projection matrix $\Phi$, and every input point $x_i$ can be compressed and sent as $\Phi x_i$ to the server.\footnote{Note that communicating $\Phi$ can be very efficient, e.g., by sending a seed to a pseudorandom generator.} Such a scheme can be applied locally (i.e., on each $x_i$, independent of the other), something that is not possible if the aim is to compress the number of observations. Additionally, for a fixed $m$, reducing the dimensionality leads to more storage space savings than the reducing the number of observations, as storing $n$ compressed features takes $\approx O(mn)$ space whereas storing the reduced observations takes $\approx O(md)$ space and $d \gg n$. In fact, in a high-dimensional setting, reducing the dimensionality seems intuitively the desirable way of achieving compression.

1.1 Our Contributions

We consider algorithms for linear regression that seek a sparse vector of regression coefficients. Our main result shows that, under a set of mild assumptions on the problem instance, we can estimate $\theta^*$ even with access to only the compressed features. To put our results into context, we start with some background discussion about sparse linear regression using Lasso.

Error Analysis of Lasso. In a traditional sparse linear regression problem, given $(X, y)$ that satisfies a linear system $y = X\theta^* + w$ where $\theta^*$ is $k$-sparse (i.e., has at most $k$ non-zero entries) in $\mathbb{R}^d$ and $w \in \mathbb{R}^n$ is the noise vector, the goal is to estimate $\theta^*$. Typically, $\theta^*$ is $k$-sparse for $k \ll d$. Throughout this paper, our main focus will be on the standard Gaussian model for sparse linear regression, in which the entries of the noise vector $w$ are i.i.d. subgaussian and the matrix $X$ is a deterministic matrix. For the purposes of this section, we make some simplifying assumptions and omit dependence on all but key variables.

A popular approach for solving a (traditional) sparse linear regression problem is the Lasso technique of $\ell_1$-penalized regression. Lasso minimizes the usual mean squared error loss penalized with (a multiple of) the $\ell_1$-norm of $\theta$:

$$
\theta^{\text{Lasso}} \in \argmin_{\theta \in \mathbb{R}^d} \frac{1}{n} \|y - X\theta\|^2 + \lambda \|\theta\|_1.
$$

(1)

The consistency properties of the Lasso are now well-understood under a variety of assumptions on the instance \cite{Bickel2009,Baron2016}. One of weakest known sufficient condition for the convergence of the Lasso estimator ($\theta^{\text{Lasso}}$) to $\theta^*$ is the restricted eigenvalue (RE) condition due to Bickel \textit{et al.} \cite{Bickel2009}\footnote{The RE condition is less severe than the Restricted Isometry Property (RIP) and other related conditions that can also be used for similar analyses \cite{Bickel2009}.}. Informally, the RE condition on $X$ lower bounds the quadratic form defined by $X$ over a subset of sparse vectors (formally defined in Definition 1). If $X$ satisfies the RE condition then it can be shown that with an appropriate choice of the regularization parameter $\lambda$, $\theta^{\text{Lasso}}$ satisfies the error bound: $\|\theta^{\text{Lasso}} - \theta^*\| = O(\sqrt{k \log d/n})$, with high probability over $w$. The above error decay rate is known to be minimax optimal, meaning that it cannot be substantially improved upon by any estimator \cite{Baron2016}.

Our Results and Techniques. In a compressed sparse linear regression setting, the goal is still to estimate $\theta^*$ where $y = X\theta^* + w$, but however we now have to do with just the compressed representation of $(x_1, y_1), \ldots, (x_n, y_n)$ (i.e., $(\Phi x_1, y_1), \ldots, (\Phi x_n, y_n)$), where $\Phi \in \mathbb{R}^{m \times d}$ is a random projection matrix.\footnote{Note that given $\Phi x_i$, it is not possible to accurately infer $x_i$ without some strong (sparsity-like) assumptions on $x_i$. More discussion on this is provided in Section 3.}

Since the $x_i$'s are not provided, directly applying an approach like Lasso is ruled out. Also since the $x_i$'s are not available, it is \textit{a priori} unclear whether a good reconstruction of $\theta^*$ is even possible. The aim of this paper is to resolve this question. For this, we consider a natural extension to the Lasso
formulation (1) that is based on using the projected covariate vectors:

\[ \theta^{\text{comp}} \in \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} (y_i - (\Phi x_i, \Phi \theta))^2 + \lambda \|\theta\|_1 \equiv \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \|y - X \Phi^\top \Phi \theta\|^2 + \lambda \|\theta\|_1. \]  

(2)

Our goal then is to show that the \( \ell_2 \)-error between \( \theta^{\text{comp}} \) and \( \theta^* \) is small under some reasonable assumptions on the instance.

Our main result (Theorem 3.4) shows if the stable rank of \( X \) is \( \Omega(m) \), then \( \theta^{\text{comp}} \) satisfies the error bound:

\[ \|\theta^{\text{comp}} - \theta^*\| = O \left( \frac{k^{5/2} \log^{3/2}(d)}{\|X\|_F} + \frac{k^{7/2} \log d}{\sqrt{d}} \right), \]

with high probability over \( \Phi, \omega \). Ignoring polylog factors, note that the second term \( k^{7/2}/\sqrt{d} \) is much smaller than \( k^{7/2}/\sqrt{n} \) as \( d \gg n \). Also, as we discuss in Section 3.2, for many interesting families of covariate vectors, \( \|X\|_F = \Omega(\sqrt{nd}) \). Therefore, in these cases, the error in estimation decays at a rate much greater than \( k^{7/2}/\sqrt{n} \).

Let us now talk about stable rank. Stable rank of a matrix \( X \) (denoted by \( \text{sr}(X) \)), defined as the squared ratio of Frobenius and spectral norms of \( X \), is a commonly used robust surrogate to usual matrix rank in linear algebra. It is natural to ask how stable rank relates to the commonly used RE condition in Lasso analysis. We provide a detailed comparison between these two matrix conditions in Appendix C. The picture that emerges is that stable rank is in fact a less restrictive\(^4\) condition to impose on the design matrix than RE. The stable rank is independent of the coordinate system, unlike RE which is tied to a concrete coordinate structure.

Our analysis follows the framework used in the traditional Lasso error analysis. For the purposes of the analysis, we consider a modified linear model: \( y = X \Phi^\top \Phi \theta^* + \tilde{\omega} \). The matrix of interest now becomes \( X \Phi^\top \Phi \), which we show satisfies a RE bound under the above stable rank condition on \( X \). The randomness of \( \Phi \) makes the required condition on \( X \) coordinate independent. To establish a RE bound, we need a lower bound on \( \|\Phi^\top \Phi \theta\| \) on all unit vectors \( \theta \) from a certain sparse set. The proof is challenging because applying standard concentration tools directly do not give strong enough probability estimates on this quantity for a fixed \( \theta \) to successfully apply an \( \varepsilon \)-net argument. To overcome this problem, we develop an orthogonal projection idea that allows us to decouple dependencies and reduce the problem to a state that is amenable to an application of an \( \varepsilon \)-net argument. Throughout the proof, we rely on the Hanson-Wright inequality and several of its consequences. With a RE bound on \( X \Phi^\top \Phi \), we investigate the setting of the regularization parameter \( \lambda \) that leads to a small \( \ell_2 \)-error between \( \theta^{\text{comp}} \) and \( \theta^* \).

**Consequences for Traditional Sparse Linear Regression.** Our results also trivially hold for the traditional sparse linear regression problem setting, as given \((x_1, y_1), \ldots, (x_n, y_n)\), the algorithm can pick \( \Phi \) and generate the input \((\Phi x_1, y_1), \ldots, (\Phi x_n, y_n)\) before using (2). While as discussed above this results in a weaker \( \ell_2 \)-error bound than using Lasso directly on \((x_1, y_1), \ldots, (x_n, y_n)\), nevertheless it does provide a new approach for the traditional sparse linear regression problem than operates (even) under a weaker assumption on \( X \) than any previous work in this area.

### 1.2 Related Work

**Lasso and Sparse Regression.** Sparsity is the most widely studied structure of data that also provides attractive statistical properties and computational advantages. There is an extensive literature on the topic of

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\(^4\)In that a RE bound implies a non-trivial stable rank bound, whereas other direction does not always hold.
sparse machine learning which have explored the close connections between it and areas such as compressed sensing, high-dimensional geometry, convex optimization, etc. (we refer the reader to books by Eldar et al. [3] and Rish et al. [13] for a detailed treatment). Lasso, is the most widely studied scheme for sparse linear regression. There has been a large and rapidly growing body of literature for Lasso and its variants which include theoretical explorations of its behavior and computationally efficient procedures for solving it. We refer the reader to the recent book by Hastie et al. [7] for a detailed survey about developments here. In this paper, we draw on the rich literature studying theoretical properties of Lasso for sparse linear regression.

A recent area of research is that of distributed (communication efficient) sparse linear regression, where the dataset is assumed to be distributed across multiple machines (see, e.g., [9] and references therein). We do not know of a direct connection between these works and our setting.

Zhou et al. [24] considered sparse linear regression in a setting where the covariate matrix $X$ is multiplied by a Gaussian random projection matrix to generate $m$ new datapoints in $d$-dimensions. They provide a convergence analysis of the Lasso estimator built from this compressed dataset. This setting is however different from ours, as we consider reducing the dimensionality of each covariate vector, which as we discussed earlier has advantages in the context of sparse linear regression.

**Compression on the Feature Space (Compressed Learning).** Our problem setting is related to the framework of compressed learning [3], where the goal is to “learn” directly from the compressed features. Compressed learning algorithms have been developed for a variety of common machine learning tasks such as ordinary least squares [10, 4, 8], classification [2], sparse subspace clustering [22], and robust PCA [5]. To the best of our knowledge ours is the first work dealing with the problem of sparse linear regression given only the projected data.

**Speeding up Regression using Random Projections.** There is a long line of work in using Johnson-Lindenstrauss style transforms for speeding up linear regression and its variants. For linear regression, the general idea is to consider the problem $\min_{\theta} \| R_y - RX \theta \|_2^2$ instead of the original least-squares problem, where $R$ is some appropriate choice of random matrix. Recent work in this space, have used structured random projections, such as those based on randomized Hadamard transform or Fourier transform, to generate a subsampled matrix, which is then used for estimating the regression coefficient $\theta$ (we refer the reader to the survey by Woodruff [23] for more details). An open question here is to extend the results in this paper to $\Phi$’s that come from structured random projections as it could lead to better computational efficiency.

## 2 Preliminaries

**Notation.** We denote $[n] = \{1, \ldots, n\}$. For a set $S \subseteq [d]$, $S^{\complement}$ denotes its complement set. Vectors are in column-wise fashion, denoted by boldface letters. For a vector $v$, $v^\top$ denotes its transpose, $\|v\|_p$ it’s $\ell_p$-norm, and $\text{supp}(v)$ its support. We use $e_j \in \mathbb{R}^d$ to denote the standard basis vector with $j$th entry set to 1. For a matrix $M$, $\|M\|$ denotes its spectral norm which equals its largest singular value, and $\|M\|_F$ its Frobenius norm. $I_d$ represents the $d \times d$ identity matrix. For a vector $x$ and set of indices $S$, let $x_S$ be the vector formed by the entries in $x$ whose indices are in $S$, and similarly, $X_S$ is the matrix formed by columns of $X$ whose indices are in $S$. The $d$-dimensional unit ball in $\ell_p$-norm centered at origin is denoted by $B_p^d$. The Euclidean sphere in $\mathbb{R}^d$ centered at origin is denoted by $S^{d-1}$.

We call a vector $a \in \mathbb{R}^d$, $k$-sparse, if it has at most $k$ non-zero entries. Denote by $\Sigma_k$ the set of all vectors $a \in B_2^d$ with support size at most $k$: $\Sigma_k = \{a \in B_2^d : |\text{supp}(a)| \leq k\}$.

Throughout this paper, we assume covariate-response pairs come from some domain $\mathcal{X} \times \mathcal{Y}$ where $\mathcal{X} \subseteq \mathbb{R}^d$ and $\mathcal{Y} \subseteq \mathbb{R}$.
In Appendix B, we also review a few additional concepts related to \( \varepsilon \)-nets, subgaussian random variables, and randomized dimensionality reduction techniques.

**Background on Lasso for Sparse Linear Regression.** Here we describe necessary background on how Lasso provides an estimate of sparse regression vector (we refer the reader to the book by Hastie et al. [7] for a detailed treatment on this topic).

A dominant goal\(^5\) in this line of work has been to establish conditions on the instance under which the \( \ell_2 \)-error on estimating \( \theta^* \) is well-controlled. For aiding this discussion, we would need few additional definitions. We also assume access to the original \( (x_i, y_i) \)'s.

For a set \( S \subset [d] \), let us define a cone set \( C(S) \) as:

\[
C(S) = \{ \theta \in \mathbb{R}^d : \| \theta_{S^c} \|_1 \leq 3\| \theta_S \|_1 \}.
\]

Restricted eigenvalue is a mild condition on the covariate matrix that is sufficient for estimating \( \theta^* \) in a noisy linear model setup.\(^6\)

**Definition 1 (Restricted Eigenvalue [1]).** A matrix \( X \in \mathbb{R}^{n \times d} \) satisfies the restricted eigenvalue (RE) condition with parameter \( \xi \) if,

\[
\inf_{S \subset [d], |S| = k, \theta \in C(S)} \frac{\|X\theta\|^2}{n} \geq \xi \|\theta\|^2.
\]

Restricted eigenvalue is in fact a special case of a general property of loss functions, known as the restricted strong convexity, which imposes a type of strong convexity condition for some subset of vectors [11].

We now state a well-known result in sparse linear regression that provides a bound on the Lasso error, based on the linear observation model \( y = X\theta^* + w \).

**Theorem 2.1 ([1, 11, 7]).** Let \( y = X\theta^* + w \) for a noise vector \( w \in \mathbb{R}^n \) and \( \theta^* \) is \( k \)-sparse. Let \( \lambda_n \geq 2\|X^\top w\|_\infty / n \). Suppose \( X \) satisfies the restricted eigenvalue condition with parameter \( \xi > 0 \), then any optimal minimizer, \( \hat{\theta} \in \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \|y - X\theta\|^2 + \lambda_n \|\theta\|_1 \), satisfies: \( \|\hat{\theta} - \theta^*\| \leq 3\sqrt{k}\lambda_n / \xi \).

**Remark 2.2.** [A Note on Assumptions] While the above RE condition is common for analyzing the \( \ell_2 \)-error of the Lasso estimator [11], stronger conditions are used for achieving the stronger guarantee of consistent support selection [21, 7]. These include mutual incoherence and minimum eigenvalue conditions on \( X \), and minimum signal value condition on \( \theta^* \). These conditions are known to be highly restrictive [19].

### 3 Sparse Linear Regression with Compressed Features

In this section, we consider the problem of sparse linear regression in a model where the algorithm only gets access to \( \Phi x_i \)'s and \( \Phi \), and not \( x_i \)'s. A first idea given only \( \Phi x_i \)'s will be to: (a) for all \( i \), construct \( \hat{x}_i \), an approximation to \( x_i \) from \( \Phi x_i \), (b) use the Lasso formulation (1) on \( (\hat{x}_i, y_i) \)'s. This idea, however, is problematic because good reconstruction of \( x_i \)'s from \( \Phi x_i \)'s will require (sparsity-like) assumptions on the structure of the \( x_i \)'s. Additionally, sparse linear regression analyses (such as for Lasso) require certain assumptions (such as RE) about the instance, which may not be satisfied by \( \hat{x}_i \)'s, even if the original \( x_i \)'s satisfy these assumptions.

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\(^5\)Other goals considered in the literature include establishing conditions for recovery of the support set of the unknown regression vector [7]. More on this in Remark 2.2.

\(^6\)Given that we observe only a noisy version of the product \( X\theta^* \), it is then difficult to distinguish \( \theta^* \) from other sparse vectors. Thus, it is natural to impose an RE condition if the goal is to produce an estimate \( \hat{\theta} \) such that \( \|\hat{\theta} - \theta^*\| \) is small.
Our idea for tackling the compressed sparse linear regression problem is based on using a variant of the Lasso formulation. Let $\Phi$ be an $m \times d$ random matrix with independent subgaussian entries. If the algorithm has only access to $(\Phi x_1, y_1), \ldots, (\Phi x_n, y_n)$ and $\Phi$, a natural extension to Lasso is:

$$\theta^{\text{comp}} \in \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n (y_i - (\Phi x_i, \Phi \theta))^2 + \lambda_n \|\theta\|_1 \equiv \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \|y - X\Phi^\top \Phi \theta\|_2^2 + \lambda_n \|\theta\|_1$$

Our goal is to establish a bound on the $\ell_2$-error between $\theta^{\text{comp}}$ and $\theta^*$ (Theorem 3.4). For this, we consider a modified linear model: $y = X\Phi^\top \Phi \theta^* + \tilde{w}$ (note that the true linear model is $y = X\theta^* + w$). In the following, we establish the conditions needed for invoking Theorem 2.1 on this modified linear model. The matrix of interest is now $X\Phi^\top \Phi$. We start off by establishing a RE bound on this matrix (Section 3.1).

In Section 3.2, we investigate the setting of the regularization parameter $\lambda_n$. Putting these pieces together in the framework of Theorem 2.1 bounds $\|\theta^{\text{comp}} - \theta^*\|_2$.

### 3.1 Restricted Eigenvalue Condition on $X\Phi^\top \Phi$

In this section, we show how a stable rank condition on $X$ translates into a RE bound on the matrix $X\Phi^\top \Phi$. We start with the definition of stable rank (denoted by $sr$) of a matrix $X$.

$$sr(X) = \|X\|_F^2 / \|X\|_2^2.$$  

Stable rank cannot exceed the usual rank. The stable rank is a more robust notion than the usual rank because it is largely unaffected by tiny singular values. Throughout this section, $C, C_1, c, c_1, \ldots$ denote positive constants which may depend on the subgaussian norm of the entries of the involved matrices.

For the proof, it will be convenient to work with a slightly modified (and a more general) definition of restricted eigenvalue that we state here.

**Definition 2.** Let $V$ be an $N \times M$ matrix, and let $k < M$, $\alpha > 0$. Define

$$\text{RE}(V, k, \alpha) = \inf \left\{ \frac{\|Vz\|}{\|z_J\|} : z_J \text{ is the coordinate projection of } z \text{ to } \mathbb{R}^d, \text{ and the infimum is taken over all sets } J \subset [M], |J| = k \text{ and all } z \in \mathbb{R}^m \setminus \{0\} \text{ satisfying} \|z_J\|_1 \leq \alpha \|z_J\|_1 \right\}.$$

Note that $\alpha = 3$ in Definition 1. Also given $\text{RE}(V, k, \alpha)$, we can get a lower bound on $\xi$ in Definition 1 as $\xi \geq \text{RE}(V, k, 3)^2 / k$.

Our primary result in this section is the following theorem which establishes a lower bound on $\text{RE}(X \Psi^\top \Psi, k, \alpha)$. The proof assumes a stable rank condition on $X$ that we define below. In Appendix C, we provide a detailed discussion about how stable rank compares with RE. The randomness of $\Phi$ makes the required condition on $X$ coordinate independent, unlike the RE condition which is tied to a concrete coordinate structure in $\mathbb{R}^d$.

**Theorem 3.1.** Let $m,n,d \in \mathbb{N}$, $m \leq n \leq d$, and let $X$ be a fixed $n \times d$ matrix satisfying

**Stable Rank Condition:** $2 \leq m \leq sr(X)/2$.

Let $\Psi = (\Psi_{ij})$ be an $m \times d$ random matrix with independent entries such that $\mathbb{E}[\Psi_{ij}] = 0$, $\mathbb{E}[\Psi_{ij}^2] = 1$, and $\|\Psi_{ij}\|_{\psi_2}$ is bounded. Let $p \in (0,1)$. Then for any $k \in \mathbb{N}$, $\alpha > 0$ such that

$$1 \leq \alpha \sqrt{k} \leq \sqrt{\frac{cm}{k \log d + \log(2/p)}}$$
the matrix $X\Psi^\top\Psi$ satisfies
\[
\text{RE}(X\Psi^\top\Psi, k, \alpha) \geq \frac{1}{32}\sqrt{m}\|X\|_F
\]
with probability at least $1 - p$.

**Corollary 3.2.** Let $X$ and $\Psi$ be matrices satisfying the conditions in Theorem 3.1 with
\[
1 \leq 3\sqrt{k} \leq \sqrt{\frac{cm}{k\log d + \log(2/\beta)}},
\]
Let $\Phi = \Psi/\sqrt{m}$. Then the matrix $X\Phi^\top\Phi$ satisfies:
\[
\inf_{S \subset [d], |S| = k, \theta \in \mathcal{C}(S)} \frac{\|X\Phi^\top\Phi\theta\|^2}{n} \geq \frac{\|X\|^2_F\|\theta\|^2}{1024nmk},
\]
with probability at least $1 - \beta$.

The complete proof of the above theorem is presented in Section 4. Here we provide a high-level description of the proof idea.

**Idea of the Proof of Theorem 3.1.** We now explain the idea behind the proof of the above theorem. Take any $J \subset [d], \ |J| = k$ and any $y \in S^{d-1}$ with supp$(y) \subseteq J$. We wish to show that with overwhelming probability, any $x \in \mathbb{R}^d$ with supp$(x) \subseteq J^\co$ and $\|x\|_1 \leq \alpha \|y\|_1 \leq \alpha \sqrt{k}$ satisfies
\[
\|X\Psi^\top\Psi(y + x)\| \geq r
\]
for some $r > 0$. If the probability estimate is strong enough, we would be able to run an $\varepsilon$-net argument over all such $y$ and take the union bound over all $J$ showing that $\text{RE}(X\Psi^\top\Psi, k, \alpha) \geq r/2$. The condition above requires checking infinitely many $x$. To make the problem tractable, let us introduce an orthogonal projection $Q : \mathbb{R}^n \to \mathbb{R}^n$ which we discuss more about later. Assume that $QX\Psi^\top\Psi y \neq 0$, and let $u$ be the unit vector in the direction of $QX\Psi^\top\Psi y$. Then
\[
\|X\Psi^\top\Psi(y + x)\| \geq \|QX\Psi^\top\Psi(y + x)\| \geq u^\top QX\Psi^\top\Psi(y + x)
\]
\[
= \|QX\Psi^\top\Psi y\| + u^\top QX\Psi^\top\Psi x
\]
The quantity above is affine in $x$, so it is minimized at one of the extreme points of the set \{ $x \in \mathbb{R}^d$ : supp$(x) \subseteq J^\co, \|x\|_1 \leq \alpha \sqrt{k}$ \}, i.e., at a vector $\pm \alpha \sqrt{k}e_j, j \in J^\co$. This observation allows us to pass from an infinite set of $x$’s to a finite set.

Next, we have to establish the concentration bounds on $\|QX\Psi^\top\Psi y\|$ and $u^\top QX\Psi^\top\Psi e_j$. Notice that $\Psi y$ and $\Psi e_j$ are independent centered (mean 0) subgaussian vectors with the unit variance of the coordinates. If these vectors were independent of the random matrix $\Psi^\top$ as well, we would have used the Hanson-Wright inequality to derive the necessary concentration. However, this is obviously not the case. At this moment, the projection $Q$ comes to the rescue. The idea is to carefully construct the projection to take care of the dependencies.
3.2 Bounding the $\ell_2$-error

In this section, we bound the $\ell_2$-error between $\theta^{\text{comp}}$ and $\theta^*$. We do so by using the RE bound established in Corollary 3.2 and some additional simple conditions needed for our analysis. We start with the definition of a well-behaved instance that precisely state these additional conditions.

**Definition 3** (Well-behaved Instance). An instance $(X, y, \theta^*)$, where $X \in \mathbb{R}^{n \times d}$ and $y \in \mathbb{R}^n$, and $\theta \in \mathbb{R}^d$, is $(k, \sigma)$-well behaved if there exists a $w \in \mathbb{R}^d$ such that $y = X\theta^* + w$ and:

1. Bounded estimator vector: $\theta^* \in \Sigma_k$ (i.e., $\theta^*$ is $k$-sparse and $\|\theta^*\| \leq 1$).
2. Noise condition: The entries of the noise vector $w = (w_1, \ldots, w_n)$ are independent centered subgaussians with $\|w_i\|_{\psi_2} \leq \sigma$ (Definition 5).

Note that these above assumptions are typical in the analysis of Lasso and related approaches to sparse linear regression (see, e.g., Hastie et al. [7]).

We now assume that $(X, y, \theta^*)$ is $(k, \sigma)$-well behaved. Again consider the modified linear model: $y = X\Phi^T\Phi\theta^* + \tilde{w}$. To establish the necessary bound on $\lambda_n$ for Theorem 2.1, we bound $\|(X\Phi^T\Phi)^\top \tilde{w}\|_\infty/n$. The proof of the following proposition is presented in Appendix D.

**Proposition 3.3.** Let $(X, y, \theta^*)$ be $(k, \sigma)$-well behaved. Let $\Psi = (\Psi_{ij})$ be an $m \times d$ random matrix with independent entries such that $\mathbb{E}[\Psi_{ij}] = 0$, $\mathbb{E}[\Psi_{ij}^2] = 1$, and $\|\Psi_{ij}\|_{\psi_2}$ is bounded. Let $m = \Theta(k^2 \log(d/\beta))$ and $\Phi = \Psi/\sqrt{m}$. Then with probability at least $1 - \beta$,

$$
\left(\frac{\|X\|_F^2 \log(d/\beta)}{n\sqrt{m}} + \frac{\|X\|_F^2}{m\sqrt{d}}\right)
$$

Our main result now follows by invoking the Theorem 2.1 on the modified linear model $y = X\Phi^T\Phi\theta^* + \tilde{w}$ with the results from Corollary 3.2 and Proposition 3.3.

**Theorem 3.4** (Main Theorem). Let $\Psi = (\Psi_{ij})$ be an $m \times d$ random matrix with independent entries such that $\mathbb{E}[\Psi_{ij}] = 0$, $\mathbb{E}[\Psi_{ij}^2] = 1$, and $\|\Psi_{ij}\|_{\psi_2}$ is bounded. Let $\Phi = \Psi/\sqrt{m}$. Let $(X, y, \theta^*)$ be $(k, \sigma)$-well behaved. Let $m = \Theta(k^2 \log(d/\beta))$ for $0 \leq \beta \leq 1/2$. If $X$ satisfies the stable rank condition: $\text{sr}(X) \geq 2m$, then any optimal minimizer,

$$
\theta^{\text{comp}} \in \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n (y_i - \langle \Phi x_i, \Phi \theta \rangle)^2 + \lambda_n \|\theta\|_1, \text{ with } \lambda_n = \Theta\left(\frac{\sigma\|X\|_F \log(d/\beta)}{n\sqrt{m}} + \frac{\|X\|_F^2}{m\sqrt{d}}\right),
$$

with probability at least $1 - \beta$ satisfies:

$$
\|\theta^{\text{comp}} - \theta^*\| = O\left(\frac{k^{3/2}\sqrt{m}\sigma \log(d/\beta)}{\|X\|_F} + \frac{k^{3/2}m}{\sqrt{d}}\right) = O\left(\frac{k^{5/2}\sigma \log^{3/2}(d/\beta)}{\|X\|_F} + \frac{k^{7/2} \log(d/\beta)}{\sqrt{d}}\right).
$$

**Discussion about Theorem 3.4.** In the first term of the error bound, note that $\|X\|_F$ is a function of both $n$ and $d$. As a point of comparison, for a very broad class of random matrices $X$, including ones with significant dependencies between the entries, with high probability, $\|X\|_F = \Omega(\sqrt{nd})$ [16]. In general, if $X$ satisfies the RE condition (Definition 1) with parameter $\xi$, then $\|X\|_F \geq \sqrt{\xi nd}$ as:

$$
\|X\|_F = \left(\sum_{j=1}^d \|Xe_j\|^2\right)^{1/2} \geq \sqrt{\xi nd}.
$$

\(^{7}\)To simplify presentation, we assume $\|\theta^*\| \leq 1$, but our results directly extend to any bound on $\|\theta^*\|$. 

9
The second term in the error bound is independent of \( n \), but since \( d \gg n \), it implies that \( k^{7/2} \sqrt{d} \ll k^{7/2} / \sqrt{n} \). Therefore, when \( \| X \|_F = \Omega(\sqrt{nd}) \), the estimation error decays at a rate much greater than \( k^{7/2} / \sqrt{n} \). In other words, the estimator \( \theta^{\text{comp}} \) is consistent when \( n = \omega(k^7) \).

We suspect that the dependence on the sparsity factor in bound of Theorem 3.4 could possibly be reduced with a tighter analysis of Theorem 3.1.

4 Restricted Eigenvalue Bound on \( X^T \Phi^T \Phi \): Proof of Theorem 3.1

In this section, we present the complete proof of Theorem 3.1. In Section 4.1, we use the Hanson-Wright theorem and its corollaries to get probabilistic estimates for norms of certain matrix products. In Section 4.2, we prove Theorem 3.1 for a fixed vector of a special form. We finish the proof in Section 4.3.

4.1 Hanson-Wright Preliminaries

We start by establishing probability estimates for the spectral and Frobenius norms for certain matrix products. The results in this section form the basic building blocks that are used throughout the proof. An important tool used here is the Hanson-Wright inequality and its several consequences. Hanson-Wright inequality establishes the concentration of a quadratic form of independent centered subgaussian random variables. An original (slightly weaker) version of this inequality was first proved in [6].

**Theorem 4.1** (Hanson-Wright Inequality [15]). Let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) be a random vector with independent components \( x_i \) which satisfy \( \mathbb{E}[x_i] = 0 \) and \( \| x_i \|_{\psi_2} \) is bounded. Let \( A \) be an \( n \times n \) matrix. Then, for every \( t \geq 0 \),

\[
\Pr \left[ \left\| x^T A x - \mathbb{E}[x^T A x] \right\| > t \right] \leq 2 \exp \left( -c \min \left( \frac{t^2}{\| A \|_F^2}, \frac{t}{\| A \|} \right) \right).
\]

Besides the theorem itself, we need several corollaries.

**Corollary 4.2** (Spectral Norm of the Product). Let \( B \) be a fixed \( n \times d \) matrix, and let \( G = (G_{ij}) \) be an \( m \times d \) random matrix with independent entries that satisfy: \( \mathbb{E}[G_{ij}] = 0 \), \( \mathbb{E}[G_{ij}^2] = 1 \), and \( G_{ij} \|_{\psi_2} \) is bounded. Then for any \( s, t \geq 1 \),

\[
\Pr \left[ \left\| B G^T \right\| > C (s \| B \|_F + t \sqrt{m} \| B \|) \right] \leq 2 \exp(-s^2 \mathbf{sr}(B) - t^2 m)
\]

and

\[
\Pr \left[ \left\| B G^T \right\| < \frac{1}{2} \| B \|_F \right] \leq 2 \exp(-c \mathbf{sr}(B)).
\]

Corollary 4.2 can be found in [15]. Assuming that \( m \leq \mathbf{sr}(B) \), we can rewrite the above inequalities as

\[
\Pr \left[ \frac{1}{2} \| B \|_F < \left\| B G^T \right\| < C \| B \|_F \right] \geq 1 - 2 \exp(-c \mathbf{sr}(B)). \tag{3}
\]

Applying this corollary in the case \( m = 1 \), we obtain a small ball probability estimate for the image of a subgaussian vector. The small ball probability bounds the probability \( \| B g \| \) is small for a fixed matrix \( B \) and a subgaussian vector \( g \).
Corollary 4.3 (Concentration for the Norm of a Vector). Let $B$ be a fixed $n \times d$ matrix, and let $g = (g_1, \ldots, g_d) \in \mathbb{R}^d$ be a random vector with independent entries that satisfy $\mathbb{E}[g_j] = 0$, $\mathbb{E}[g_j^2] = 1$, and $\|g_j\|_{\psi_2}$ is bounded. Then

$$\Pr \left[ \frac{1}{2} \|B\|_F < \|Bg\| < C \|B\|_F \right] \geq 1 - 2 \exp(-c \cdot sr(B)).$$

Using this inequality, we can easily derive a small ball probability estimate for the Frobenius norm.

Corollary 4.4 (Frobenius Norm of the Product). Let $B$ be a fixed $n \times d$ matrix, and let $G = (G_{ij})$ be an $m \times d$ random matrix with independent entries that satisfy: $\mathbb{E}[G_{ij}] = 0$, $\mathbb{E}[G_{ij}^2] = 1$, and $\|G_{ij}\|_{\psi_2}$ is bounded. Then

$$\Pr \left[ \frac{1}{\sqrt{m}} \|B\|_F < \left\|BG^\top\right\|_F < C \sqrt{m} \|B\|_F \right] \geq 1 - 2 \exp(-c \cdot sr(B)).$$

Proof. Denote the rows of $G$ by $\gamma_1, \ldots, \gamma_m$. Then,

$$\left\|BG^\top\right\|_F = \left( \sum_{j=1}^m \|B\gamma_j\|^2 \right)^{1/2}.$$

The right-hand side can be interpreted as the Euclidean norm of the image of the vector $\tilde{\gamma} \in \mathbb{R}^{dm}$ obtained by concatenation of the vectors $\gamma_1, \ldots, \gamma_m$ under the $nm \times dm$ block-diagonal matrix $\tilde{B} = \text{diag}(B, \ldots, B)$.

The result follows from the Corollary 4.3, since $\left\|\tilde{B}\right\|^2 = m \|B\|_F^2$ implying $\left\|\tilde{B}\right\|_F = \sqrt{m} \|B\|_F$. \hfill $\square$

We will need a similar estimate for the Frobenius norm of the triple product of the form $GHG^\top$, where $H$ is a positive semidefinite matrix. Let $\text{tr}(\cdot)$ denote the trace of a matrix.

Corollary 4.5 (Frobenius norm of the Triple Product). Let $H$ be a fixed $d \times d$ symmetric positive semidefinite matrix, and let $G = (G_{ij})$ be an $m \times d$ random matrix with independent entries that satisfy: $\mathbb{E}[G_{ij}] = 0$, $\mathbb{E}[G_{ij}^2] = 1$, and $\|G_{ij}\|_{\psi_2}$ is bounded. If $m \leq \text{tr}(H) / \|H\|$, then

$$\Pr \left[ \left\|GHG^\top\right\|_F \geq C \sqrt{m} \cdot \text{tr}(H) \right] \leq 4 \exp\left(-c \frac{\text{tr}(H)}{\|H\|}\right).$$

Proof. Let $H^{1/2}$ be the square root of the matrix $H$. Since $\text{tr}(H) = \|H^{1/2}\|_F^2$, the assumption of the corollary reads $m \leq \text{sr}(H^{1/2})$. By Corollary 4.3,

$$\Pr \left[ \left\|H^{1/2}G^\top\right\|_F \geq C \left\|H^{1/2}\right\|_F \right] \leq 2 \exp(-c \cdot \text{sr}(H^{1/2})).$$

Similarly, Corollary 4.4 implies

$$\text{Prob} \left[ \left\|H^{1/2}G^\top\right\|_F \geq C \sqrt{m} \left\|H^{1/2}\right\|_F \right] \leq 2 \exp(-c \cdot \text{sr}(H^{1/2})).$$

As $\|GHG^\top\|_F \leq \|H^{1/2}G^\top\|_F \cdot \|H^{1/2}G^\top\|$, we have

$$\Pr \left[ \left\|GHG^\top\right\|_F \geq C \sqrt{m} \cdot \left\|H^{1/2}\right\|_F^2 \right] \leq 4 \exp(-c \cdot \text{sr}(H^{1/2})) = 4 \exp\left(-c \frac{\text{tr}(H)}{\|H\|}\right),$$

which completes the proof. \hfill $\square$
4.2 Bounds for a Fixed Vector

In this section, our goal will be to investigate a special case of Theorem 3.1. In particular, we investigate the RE condition in Definition 2 when restricted to vectors of the kind \( z = e_j + x \) for a fixed \( j \) where \( j \notin \text{supp}(x) \) (Proposition 4.8). The proof is based on two technical lemmas that use careful conditioning arguments along with the probabilistic inequalities established in the previous section. We use \( \text{conv}() \) and \( \text{span}() \) to denote the convex hull and span of a set of vectors. We use \( \text{Ker}() \) to denote the kernel of a matrix.

The following lemma bounds the small ball probability of \( BG^\top g \), for a fixed matrix \( B \), random matrix \( G \), and a random vector \( g \).

**Lemma 4.6.** Let \( B \) be a fixed \( n \times d \) matrix, let \( G = (G_{ij}) \) be an \( m \times d \) random matrix with independent entries and let \( g = (g_1, \ldots, g_m) \in \mathbb{R}^m \) be a random vector with independent entries that satisfy: \( \mathbb{E}[G_{ij}] = \mathbb{E}[g_j] = 0 \), \( \mathbb{E}[G_{ij}^2] = \mathbb{E}[g_j^2] = 1 \), and \( \|G_{ij}\|_{\psi_2}, \|g_j\|_{\psi_2} \) are bounded. Then

\[
\Pr \left[ \|BG^\top g\| < \frac{1}{4} \sqrt{m} \|B\|_F \right] \leq 8 \left( \exp \left( -\frac{m}{2} \right) + \exp(-cm) \right).
\]

**Proof.** Conditioning on \( G \) and applying Corollary 4.3, we obtain

\[
\Pr \left[ \|BG^\top g\| \leq \frac{1}{2} \|BG^\top\|_F \mid G \right] \leq 2 \exp(-c \text{sr}(BG^\top)).
\]

Define the events \( \Omega_F \) and \( \Omega_{op} \) as in Corollary 4.5:

\[
\Omega_F = \left\{ G : \left\| BG^\top \right\|_F \geq \frac{1}{2} \sqrt{m} \|B\|_F \right\}
\]

\[
\Omega_{op} = \left\{ G : \left\| BG^\top \right\| \leq C(\|B\|_F + \sqrt{m} \|B\|) \right\}
\]

Let \( \Omega_F^C \) and \( \Omega_{op}^C \) denote the complement of these events respectively. Then by Corollaries 4.4 and 4.2,

\[
\Pr \left[ \|BG^\top g\| \leq \frac{1}{4} \sqrt{m} \|B\|_F \right]
\]

\[
\leq \Pr \left[ \|BG^\top g\| \leq \frac{1}{2} \|BG^\top\|_F \mid G \in \Omega_F \cap \Omega_{op} \right] + \Pr \left[ \Omega_F^C \right] + \Pr \left[ \Omega_{op}^C \right]
\]

\[
\leq 2 \exp \left( -c \frac{m \|B\|^2_F}{2 \|B\|^2_F + m \|B\|^2} \right) + 4 \exp(-c \text{sr}(B))
\]

\[
\leq 8 \left( \exp \left( -\frac{m}{2} \right) + \exp(-cm) \right).
\]

The following lemma provides a large deviation bound for a certain product form.

**Lemma 4.7.** Let \( B \) be a fixed \( n \times d \) matrix, let \( G = (G_{ij}) \) be an \( m \times d \) random matrix with independent entries and let \( g_1 = (g_{11}, \ldots, g_{1m}) \in \mathbb{R}^m \) and \( g_2 = (g_{21}, \ldots, g_{2m}) \in \mathbb{R}^m \) be random vectors with independent entries that satisfy: \( \mathbb{E}[G_{ij}] = \mathbb{E}[g_{ij}] = 0 \), \( \mathbb{E}[G_{ij}^2] = \mathbb{E}[g_{ij}^2] = 1 \), and \( \|G_{ij}\|_{\psi_2}, \|g_{ij}\|_{\psi_2} \) are all bounded for \( l \in \{1, 2\} \). Assume that \( m \leq \text{sr}(B) \). Then for any \( t \in \left[ 0, m \|B\|^2_F \right] \),

\[
\Pr \left[ \|g_1^\top GB^\top g_2\| \geq t \right] \leq 10 \exp \left( -c \frac{t^2}{m \|B\|^4_F} \right).
\]
Proof. Define the vector \( g \in \mathbb{R}^{2m} \) and the \( 2m \times 2m \) matrix \( \Gamma \) by
\[
g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & GB^T \\ GB^T & 0 \end{pmatrix}.
\]
Condition on \( G \). By Theorem 4.1, for any \( t \geq 0 \),
\[
\Pr \left[ |g^\top \Gamma g| > t \right] \leq 2 \exp \left[ -c \min \left( \frac{t^2}{\| \Gamma \|_F^2}, \frac{t}{\| \Gamma \|} \right) \right].
\]
Note that \( \| \Gamma \| = \| GB^T BG^\top \| = \| BG^\top \|^2 \). Let \( \Omega_{HS} \) and \( \Omega_{op} \) be the events defined by
\[
\Omega_{HS} = \{ G : \| GB^T BG^\top \|_F \leq C \left( m \| B^\top B \|_F + \sqrt{m} \cdot \text{tr}(B^\top B) \right) \}
\]
\[
\Omega_{op} = \{ G : \frac{1}{4} \| B \|_F^2 \leq \| GB^T BG^\top \| \leq C \| B \|_F^2 \}
\]
Again, let \( \Omega_{F}^c \) and \( \Omega_{op}^c \) denote the complement events. For any \( G \in \Omega_F \),
\[
\| \Gamma \|_F^2 \leq C m \cdot \text{tr}(B^\top B)^2 = C' m \| B \|_F^4.
\]
Notice that
\[
\frac{\text{tr}(B^\top B)}{\| B^\top B \|} = \text{sr}(B).
\]
Finally, combining this with Corollary 4.5, and (3), we obtain
\[
\Pr \left[ |g_1^\top GB^\top BG^\top g_2| \geq t \right]
\leq 2 \exp \left[ -c \min \left( \frac{t^2}{m \| B \|_F^4}, \frac{t}{\| B \|_F^2} \right) \right] + \Pr [\Omega_F^c] + \Pr [\Omega_{op}^c]
\leq 4 \exp \left( -c \frac{t^2}{m \| B \|_F^4} \right) + 6 \exp(-c \text{sr}(B))
\]
for any \( t \in \left[ 0, m \| B \|_F^2 \right] \). Since \( m \leq \text{sr}(B) \), the first term in the right hand side dominates the second one, and the proof is complete.

Using Lemmas 4.6 and 4.7, we are ready to prove the following proposition. The main idea here is to introduce an orthogonal projection matrix which lets us decouple various dependencies that appear across various quantities.

**Proposition 4.8.** Let \( R \) be a fixed \( n \times d \) matrix, and let \( G = (G_{i,j}) \) be an \( m \times d \) random matrix with independent entries that satisfy: \( \mathbb{E}[G_{i,j}] = 0, \mathbb{E}[G_{i,j}^2] = 1 \), and \( \| G_{i,j} \|_{\psi_2} \) is bounded. Assume that
\[
2 \leq m \leq \text{sr}(R)/2.
\]
Then for any \( s \geq 1 \),
\[
\Pr \left[ \exists x \in s \cdot \text{conv}(\pm e_2, \ldots, \pm e_d), \| RG^\top G'(e_1 + x) \| \leq \frac{1}{8} \sqrt{m} \| R \|_F \right] \leq 2d \exp \left( -c \frac{m}{s^2} \right).
\]
Proof: Let $P_1$ be the orthogonal projection in $\mathbb{R}^n$ with $\text{Ker}(P_1) = \text{span}(Re_1)$, where $\text{span}()$ denote the span. Assume that $P_1 RG^\top Ge_1 \neq 0$ and set

$$u = \frac{P_1 RG^\top Ge_1}{\| P_1 RG^\top Ge_1 \|}.$$

Then

$$\left\| RG^\top G(e_1 + x) \right\| \geq \left\| P_1 RG^\top G(e_1 + x) \right\| \geq \left\| P_1 RG^\top Ge_1 \right\| - u^\top P_1 RG^\top Gx. \quad (4)$$

The minimal value of this expression over $x \in s \cdot \text{conv}(\pm e_2, \ldots, \pm e_d)$ is attained at the extreme points of this set. Consider $x = se_2$ since all other extreme points are treated the same way. Since $sr(R) > 4$ and by the interlacing, we have

$$\| P_1 R \|_F^2 \geq \| R \|_F^2 - \| R \|_2^2 \geq \| R \|_F^2 / 2$$

and so, $sr(P_1 R) \geq (1/2) sr(R)$ (as $\| P_1 R \| = \| R \|$).

Denote by $g_1$ and $g_2$ the first and the second columns of $G$. We have introduced $P_1$ to ensure that the matrix $P_1 RG^\top$ is independent of $g_1$. This allows us to replace the vector $g_1$ by its copy independent of $G$. Hence, by Lemma 4.6,

$$\Pr \left[ \left\| P_1 RG^\top Ge_1 \right\| < \frac{1}{4} \sqrt{m} \| R \|_F \right] = \Pr \left[ \left\| P_1 RG^\top g_1 \right\| < \frac{1}{4} \sqrt{m} \| R \|_F \right] \leq 8 \left( \exp \left( -c \cdot sr(R) \right) + \exp(-cm) \right) \leq 2 \exp(-c'm),$$

where we used that $m \leq sr(R)$.

The estimate of the inner product is a little more complicated. Let $P_2$ be the orthogonal projection with $\text{Ker}(P_2) = \text{span}(Re_1, P_1 Re_2)$. Then we can write

$$P_1 RG^\top Ge_1 = P_2 RG^\top g_1 + P_1 Re_2 g_2^\top g_1$$

$$P_1 RG^\top Ge_2 = P_2 RG^\top g_2 + P_1 Re_2 g_2^\top g_2$$

and therefore,

$$(P_1 RG^\top Ge_1)^\top P_1 RG^\top Ge_2 = (P_2 RG^\top g_1)^\top P_2 RG^\top g_2 + (P_1 Re_2 g_2^\top g_1)^\top P_1 Re_2 g_2^\top g_2.$$

Note that $P_2 RG^\top$ is independent of $g_1$ and $g_2$. Similarly to (8), we have

$$\| P_2 R \|_F^2 \geq \| R \|_F^2 - 2 \| R \|_2^2 \geq \| R \|_F^2 / 2$$

and so, $sr(P_2 R) \geq (1/2) sr(R) \geq m$. This allows us to use Lemma 4.7 to estimate

$$\Pr \left[ \left| g_1^\top G(P_2 R)^\top P_2 RG^\top g_2 \right| \geq t \right] \leq 8 \exp \left( -c \frac{t^2}{m \| P_2 R \|_F^4} \right) \quad (6)$$

for any $t \in [0, m \| P_2 R \|_F^2]$.

The estimate for the last term is straightforward as $P_1 Re_2$ is deterministic. Since

$$\forall s \geq 0 \quad \Pr \left[ \left| g_2^\top g_1 \right| > Cs \right] \leq 2 \exp \left( -c \frac{s^2}{m} \right) + \exp(-m),$$

Thus, we conclude that

$$\forall s \geq 0 \quad \Pr \left[ \left| g_2^\top G e_2 \right| > Cs \right] \leq 2 \exp \left( -c \frac{s^2}{m} \right) + \exp(-m).$$
and
\[
\Pr \left[ \|g_2^T g_2\| > Cm \right] \leq \exp(-m),
\]
we obtain
\[
\Pr \left[ \|(P_1R_{e2}g_2)\| \geq sm \|P_1R_{e2}\|^2 \right] \leq 2 \exp\left( -\frac{c^2}{m} \right) + \exp(-m)
\]
or
\[
\Pr \left[ \|(P_1R_{e2}g_2)\|^2 \geq t \right] \leq 2 \exp\left( -\frac{c^2}{m^3 \|P_1R_{e2}\|^4} \frac{t^2}{m} \right) + \exp(-m)
\]
for all \( t \geq 0 \). Combining (6) and (7), we conclude that
\[
\Pr \left[ \|u^T P_1R_{e1} G e_1\| > \tau \right] \leq 2 \exp\left( -\frac{c}{m} \frac{t^2}{\|R\|_F^4} \right) + \exp(-cm)
\]
for any \( \tau \in [0, m \|P_2R\|_F^2] \). Here we used the inequality
\[
m \|P_1R_{e2}\|^2 \leq m \|R\|^2 \leq \|R\|_F^2,
\]
where the last one follows from the assumption \( m \leq s \text{rank}(R) \). Taking into account the result from (5), we see that
\[
\Pr \left[ \|u^T P_1R G e\| > \tau \right] \leq 2 \exp\left( -\frac{c}{m} \frac{\tau^2}{\|R\|_F^4} \right) + \exp(-cm),
\]
for all \( \tau \in [0, \frac{1}{s} \sqrt{m} \|R\|_F] \). After taking the union bound, we show that
\[
\Pr \left[ \exists j \geq 2, \|u^T P_1R G e_j\| > \tau \right] \leq 2d \exp\left( -\frac{c}{m} \frac{\tau^2}{\|R\|_F^4} \right) + \exp(-cm).
\]
Recall (4). Setting \( \tau = \frac{1}{ss} \sqrt{m} \|R\|_F \) with \( s \geq 1 \), and using together (5) and (8), we conclude that
\[
\Pr \left[ \exists x \in s \cdot \text{conv}(\pm e_2, \ldots, \pm e_d), \|RG^T G(e_1 + x)\| \leq \frac{1}{s} \sqrt{m} \|R\|_F \right] \leq 2d \exp\left( -c \frac{m}{s^2} \right),
\]
as the second term in the right-hand side gets absorbed in the first one. The proof of the proposition is complete.

4.3 Finishing the Proof of Theorem 3.1: Net Argument

The next theorem is the main technical step in proving Theorem 3.1. Invoking this theorem with appropriate parameters (that we explain later in this section) gives the proof of Theorem 3.1. The proof of the following theorem is based on generating an orthogonal matrix to reduce the general case to the special case discussed in Proposition 4.8, and then employing an \( \varepsilon \)-net argument.
Theorem 4.9. Let $X$ be a fixed $n \times d$ matrix satisfying,

$$2 \leq m \leq sr(X)/2.$$ 

Let $\Psi = (\Psi_{ij})$ be an $m \times d$ random matrix with independent entries such that $E[\Psi_{ij}] = 0$, $E[\Psi_{ij}^2] = 1$, and $\|\Psi_{ij}\|_{\psi_2}$ is bounded. Let $p \in (0, 1)$, and let $k \in \mathbb{N}$. Then for any $s$ such that

$$1 \leq s \leq \sqrt{\frac{cm}{k \log d + \log(2/p)}},$$

$$\Pr[\exists I \subset [d] \text{ with } |I| = k, \exists y \in \mathbb{S}^{d-1} \text{ with } \text{supp}(y) \subseteq I, \exists x \in s \cdot \text{conv}(\pm e_i, i \notin I),$$

$$\left\|X\Psi^\top(y + x)\right\| \leq \frac{1}{32} \sqrt{m \|X\|_F} \leq p.$$ 

Note that the condition $s \geq 1$ in the formulation of the theorem implicitly sets a lower bound on $p$ and an upper bound on $k$.

Proof. Fix the set $I$ with $|I| = k$. For instance, consider $I = [k] \subset [d]$. Fix also a point $y \in \mathbb{S}^{k-1}$. Define the subspace $E \subset \mathbb{R}^d$ as

$$E = \text{span}(y, e_j, j > k).$$

Note that the vectors $y$ and $e_j$, $j > k$ form an orthonormal basis of $E$. Let $P_E : \mathbb{R}^d \to E$ be matrix of the orthogonal projection onto $E$ with respect to this basis and the standard basis in $\mathbb{R}^d$. Then $P_E^\top$ is the matrix of the embedding of $E$ into $\mathbb{R}^d$.

Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection with $\text{Ker}(Q) =XE^\perp$, where $E^\perp$ represents the orthogonal complement of $E$. Then for any $z \in E$,

$$\left\|X\Psi^\top\Psi z\right\| \geq \left\|QX\Psi^\top\Psi z\right\|.$$ 

(9)

We can represent the restriction of the linear operator $QX\Psi^\top\Psi$ to $E$ as the following composition of linear operators:

$$E \xrightarrow{P_E^\top} \mathbb{R}^d \xrightarrow{\Psi^\top} \mathbb{R}^m \xrightarrow{\Psi} \mathbb{R}^d \xrightarrow{P_E} \mathbb{R}^d \xrightarrow{X} \mathbb{R}^n \xrightarrow{Q} \mathbb{R}^n.$$ 

Since $\|y\| = 1$ and $\text{supp}(y) \subseteq [k]$, the $m \times (d - k + 1)$ matrix $G = \Psi P_E^\top$ in the basis $\{y, e_j, j > k\}$ has centered subgaussian entries of unit variance. Denote $R = QX P_E^\top$. Then by the interlacing

$$\|X\|_F^2 \geq \|R\|_F^2 \geq \|X\|_F^2 - 2k \|X\|_F^2 \geq \frac{1}{2} \|X\|_F^2,$$

since by the assumptions on $k$ and $X$, $k \leq m/8 \leq sr(X)/16$. This implies

$$sr(R) \geq \frac{1}{2} sr(X) \geq m.$$ 

Applying Proposition 4.8 to the matrices $G, R$, with $y$ playing the role of $e_1$, and taking into account (9), we obtain

$$\Pr \left[ \exists x \in s \cdot \text{conv}(\pm e_j, j > k), \left\|X\Psi^\top(y + x)\right\| \leq \frac{1}{16} \sqrt{m \|X\|_F} \right] \leq 2d \exp \left( -\frac{cm}{s^2} \right)$$ 

16
for any $s \geq 1$.

In the rest of the proof, we employ the net argument. Since $\Psi$ is a subgaussian random matrix,

$$\left\| X\Psi^\top \Psi \right\| \leq \left\| X\Psi^\top \right\| \cdot \left\| \Psi \right\| \leq C'(\|X\|_F + \sqrt{m}\|X\|) \cdot C''(\sqrt{d} + \sqrt{m})$$

with probability at least $1 - \exp(-m)$, where we used Corollary 4.2. Let $\varepsilon > 0$ be a number to be chosen later, and (by Proposition B.1) let $\mathcal{N} \subset S^{k-1}$ be an $\varepsilon$-net of cardinality

$$|\mathcal{N}| \leq \left( \frac{3}{\varepsilon} \right)^k.$$

Assume that for any $y \in \mathcal{N}$, and for any $x \in s \cdot \text{conv}(\pm e_j, j > k)$,

$$\left\| X\Psi^\top \Psi(y + x) \right\| \geq \frac{1}{16} \sqrt{m}\|X\|_F.$$

Assume also that $\left\| X\Psi^\top \Psi \right\| \leq C\sqrt{d}\|X\|_F$. Let $z \in S^{k-1}$, and chose $y \in \mathcal{N}$ such that $\|z - y\| < \varepsilon$. Then setting $\varepsilon = c\sqrt{m/d}$ for an appropriately small constant $c > 0$, we obtain

$$\left\| X\Psi^\top \Psi(z + x) \right\| \geq \left\| X\Psi^\top \Psi(y + x) \right\| - \left\| X\Psi^\top \Psi \right\| \cdot \|z - y\| \geq \frac{1}{32} \sqrt{m}\|X\|_F.$$

Thus,

$$\Pr \left[ \exists y \in S^{k-1}, \exists x \in s \cdot \text{conv}(\pm e_i, i > k), \left\| X\Psi^\top \Psi(y + x) \right\| \leq \frac{1}{32} \sqrt{m}\|\Psi\|_F \right]$$

$$\leq |\mathcal{N}| \cdot 2d \exp \left( -\frac{m}{s^2} \right) + \exp(-m)$$

$$\leq 2 \exp \left( -\frac{m}{s^2} + k \log \left( \frac{C\sqrt{d}}{\sqrt{m}} \right) \right).$$

It remains to take the union bound over all possible supports of $y$. It yields,

$$\Pr[\exists I \subset [d] \text{ with } |I| = k, \exists y \in S^{d-1} \text{ with } \text{supp}(y) \subseteq I, \exists x \in s \cdot \text{conv}(\pm e_i, i \notin I),$$

$$\left\| X\Psi^\top \Psi(y + x) \right\| \leq \frac{1}{32} \sqrt{m}\|\Psi\|_F]$$

$$\leq \binom{d}{k} \cdot 2 \exp \left( -\frac{m}{s^2} + k \log \left( \frac{C\sqrt{d}}{\sqrt{m}} \right) \right)$$

$$\leq 2 \exp \left( -\frac{m}{s^2} + \frac{k}{2} \log \left( \frac{Cd^2}{mk} \right) \right).$$

The last quantity is smaller than $p$ provided that\(^8\)

$$1 \leq s \leq \sqrt{\frac{cm}{k \log d + \log(2/p)}}.$$

This completes the proof of the theorem. \(\square\)

\(^8\)Here we ignored smaller order terms assuming $d^2 \gg mk$. If this does not hold, one can obtain a slightly better estimate.
We now have all the ingredients to complete the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Assume that the complement of the event described in Theorem 4.9 occurs. Namely, assume that
\[
\forall I \subset [d] \text{ with } |I| = k, \forall y \in S^{d-1} \text{ with } \text{supp}(y) \subseteq I, \forall x \in s \cdot \text{conv}(\pm e_i, i \notin I)
\]
\[
\|X\Psi^\top \Psi(y + x)\| \geq \frac{1}{32} \sqrt{m} \|X\|_F.
\]
If \(s\) satisfies the condition of this theorem, then the event above occurs with probability at least \(1 - p\). Pick any \(I \subset [d] |I| = k\) and any \(z \in \mathbb{R}^d \setminus \{0\}\) with
\[
\|z_{I^{\infty}}\|_1 \leq \alpha \|z_I\|_1.
\]
Without loss of generality, we may assume that \(y = z_I \in S^{d-1}\). Then, \(\|y\|_1 \leq \sqrt{k}\), and so \(\|z_{I^{\infty}}\|_1 \leq \alpha \sqrt{k}\).

Theorem 3.1 now follows from Theorem 4.9 applied with \(s = \alpha \sqrt{k}\).

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A Background on Sparse Linear Regression

If the linear model $y = X\theta^* + w$, where $X \in \mathbb{R}^{n \times d}$ is high-dimensional in nature, meaning that the number of observations $n$ is substantially smaller than $d$, then it is easy to see that without further constraints on $\theta^*$, the statistical model $y = X\theta^* + w$ is not identifiable. This is because (even when $w = 0$), there are many vectors $\theta^*$ that are consistent with the observations $y$ and $X$. This identifiability concern may be eliminated by imposing some type of sparsity assumption on the regression vector $\theta^*$. Typically, $\theta^*$ is $k$-sparse for $k \ll d$. Under this assumption, the goal of sparse linear regression is to find a sparse $\theta$ with few nonzero entries such that $\langle x_i, \theta \rangle \approx y_i$ for “most” $(x_i, y_i)$ pairs. Disregarding computational cost, the
most direct approach to estimating a $k$-sparse $\theta$ in the linear regression model would be solving a quadratic optimization problem with an $\ell_0$-constraint:

$$\theta^\text{sparse} \in \arg\min_{\theta \in \Sigma_k} \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \mathbf{x}_i, \theta \rangle)^2.$$  (10)

**Lasso Regression.** Since (10) leads to a non-convex problem, a natural alternative is obtained by replacing the $\ell_0$-constraint with its tightest convex relaxation, the $\ell_1$-norm. This leads to the popular Lasso regression, defined as,

\[
\text{Lasso Regression (penalized form): } \hat{\theta}^{\text{Lasso}} \in \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \mathbf{x}_i, \theta \rangle)^2 + \lambda \|\theta\|_1,
\]

for some choice $\lambda > 0$.

The consistency properties of Lasso are now well-understood. Under a variety of mild assumptions on the instance, the Lasso estimator ($\hat{\theta}^{\text{Lasso}}$) is known to converge to the sparse $\theta^*$ in the $\ell_2$-norm. Under stronger assumptions (such as mutual incoherence, minimum eigenvalue, and minimum signal condition) on the instance, it is also known that $\hat{\theta}^{\text{Lasso}}$ will have the same support as $\theta^*$.

**B Additional Preliminaries**

**Background on $\varepsilon$-Nets.** Consider a subset $T$ of $\mathbb{R}^d$, and let $\varepsilon > 0$. A $\varepsilon$-net of $T$ is a subset $N \subseteq T$ such that for every $x \in T$, there exists a $y \in N$ such that $\|x - y\| \leq \varepsilon$.

**Proposition B.1 (Volumetric Estimate).** Let $T$ be a subset of $B^d_2$ and let $\varepsilon > 0$. Then there exists an $\varepsilon$-net $N$ of $T$ of cardinality at most $(1 + 2/\varepsilon)^d$. For any $\varepsilon \leq 1$, this can be simplified as $(1 + 2/\varepsilon)^d \leq (3/\varepsilon)^d$.

**Background on Subgaussian Random Variables.** Subgaussian random variables are a wide class of random variables, which contains in particular the standard normal, Bernoulli, and all bounded random variables.

**Definition 4 (Subgaussian Random Variable).** We call a random variable $x \in \mathbb{R}$ subgaussian if there exists a constant $C > 0$ if $\Pr[|x| > t] \leq 2 \exp(-t^2/C^2)$ for all $t > 0$.

**Definition 5 (Norm of a Subgaussian Random Variable).** The $\psi_2$-norm of a subgaussian random variable $x \in \mathbb{R}$, denoted by $\|x\|_{\psi_2}$ is: $\|x\|_{\psi_2} = \inf \{ t > 0 : \mathbb{E} [\exp(|x|^2/t^2)] \leq 2 \}$.

Note that the $\psi_2$ condition on a scalar random variable $x$ is equivalent to the subgaussian tail decay of $x$.

**Johnson-Lindenstrauss (JL) Transformations.** In the following few paragraphs, we will review some useful facts about randomized dimension reduction using Johnson-Lindenstrauss transformation. Johnson-Lindenstrauss (JL) transformation is a low-dimensional embedding which preserves, up to a small distortion, pairwise $\ell_2$-distances between vectors according to the JL lemma.

**Lemma B.2 (JL Lemma).** For any $0 < \gamma, \beta < 1/2$ and positive integer $d$, there exists a distribution $\mathcal{D}$ over $\mathbb{R}^{m \times d}$ for $m = O((1/\beta)/\gamma^2)$ such that for any $a \in \mathbb{R}^d$, $\Pr_{\Phi \sim \mathcal{D}}[\|\Phi a\|^2 - \|a\|^2 \geq \gamma \|a\|^2] \leq \beta$. 

20
The original proof of the JL lemma chose $\Phi$ as a scaled projection onto a random $m$-dimensional linear subspace, whereas subsequent works showed that the entries of $\Phi$ can be i.i.d. subgaussian random variables.

A simple consequence of the JL lemma is that, for any $a, b \in \mathbb{R}^d$, the inner-product between $a$ and $b$ is approximately preserved under these transformations, in that, if $m = \Omega(\log(1/\beta)/\gamma^2)$,

$$\Pr[|\langle \Phi a, \Phi b \rangle - \langle a, b \rangle| \geq \gamma \|a\| \|b\|] \leq \beta. \quad (11)$$

Using (11) and a net argument$^9$ over the set of sparse vectors gives the following standard fact.

**Proposition B.3.** Let $\Psi = (\Psi_{ij})$ be an $m \times d$ random matrix with independent entries such that $E[\Psi_{ij}] = 0$, $E[\Psi_{ij}^2] = 1$, and $\|\Psi_{ij}\|_2$ is bounded. Let $m = \Theta(k \log(d/\beta)/\gamma^2)$ and $\Phi = \Psi/\sqrt{m}$. Then for any fixed set of vectors $x_1, \ldots, x_n \in \mathbb{R}^d$, with $n \leq d$, we have,

$$\Pr[|\langle \Phi x_i, \Phi \theta \rangle - \langle x_i, \theta \rangle| \geq \gamma \|x_i\| \|	heta\| \text{ for all } i \in [n], \theta \in \Sigma_k] \leq \beta.$$

### C Comparison between Stable Rank and Restricted Eigenvalue Conditions

In this section, we investigate how stable rank relates to the restricted eigenvalue (RE) condition that is commonly used in the analysis of Lasso [7]. The picture that emerges is the following: stable rank is a less restrictive condition to impose on a design matrix than RE. We show this by establishing that a RE bound on a matrix implies a non-trivial$^{10}$ stable rank for that matrix, whereas other direction does not always hold.

We first look at the case, when we have a stable rank condition on $X$. The RE condition (and of course, RIP) governs the behavior of the matrix on all coordinate subspaces of a small dimension. In this sense, a bound on the stable rank on $X$ is much more relaxed. We now provide a simple pedagogical example to illustrate this fact. We rely on the fact that if $X e_j = 0$ for even one $j \in d$, then no RE condition holds. Consider, for example the $d \times n$ matrix

$$X = \begin{pmatrix} \mathbb{1}_{2m} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathbb{1}_{2m}$ is the identity $2m \times 2m$ matrix. Then, $sr(X) = 2m$, while the RE condition does not hold for $X$. This simple example illustrates that there exist families of matrices for which a stable rank condition (as required in Theorem 3.1) holds, but a RE condition is not satisfied.

To make the comparison in the other direction, we need an additional normalization of $X$, as $sr(X)$ is invariant under scaling, and $RE(X, k, \alpha)$ is degree 1 homogenous (in that scaling each element in $X$ by a factor $c$ changes $RE(X, k, \alpha)$ by $c$). Assume that $RE(X, k, \alpha) \geq r$ and define

$$\|X\|_{(k)} = \max_{J \subset [d]} \|X_J\| \leq R.$$

An upper bound on $\|X\|_{(k)}$ is usually applied together with a lower bound on $RE(X, k, \alpha) \geq r$ in derivation of the vector reconstruction conditions (see, e.g. [16]). These assumptions yield that

$$\|X\|_F = \left(\sum_{j=1}^d \|X e_j\|^2\right)^{1/2} \geq r \sqrt{d}.$$

---

$^9$Let $\mathcal{N}$ be an $\varepsilon$-net over $\Sigma_k$ (set of $k$-sparse vectors in $B^d_2$). Using Proposition B.1, $|\mathcal{N}| \leq \binom{d}{k} \cdot \left(\frac{2}{\varepsilon}\right)^k = O\left(\frac{2^k}{\varepsilon^k}\right).

$^{10}$A direct numerical extension is not possible as stable rank is invariant to matrix scaling, whereas RE is not.
Also, assume for simplicity that \( d = kL \) and decompose \([d] = \bigcup_{j=1}^{L} J_j \), where \( J_j \subset [d] \) are consecutive sets of \( k \) coordinates. Let \( y \in \mathbb{S}^{d-1} \). Then
\[
\|XY\| \leq \sum_{l=1}^{L} \|X_{J_l}\| \cdot \|y_{J_l}\| \leq \left( \sum_{l=1}^{L} \|X_{J_l}\|^2 \right)^{1/2} \left( \sum_{l=1}^{L} \|y_{J_l}\|^2 \right)^{1/2} \leq R\sqrt{L} = R\sqrt{\frac{d}{k}}.
\]
Therefore, \( \|X\| \leq R\sqrt{\frac{d}{k}} \) and so
\[
\operatorname{sr}(X) \geq \left( \frac{r}{R} \right)^2 k.
\]
This shows that a RE bound on \( X \) implies a non-trivial stable rank bound on \( X \).

Putting both these directions together implies that while a RE bound always translates into stable rank bound, the other direction does not always hold.

## D Proof of Proposition 3.3

The following Hoeffding bound will be useful in our analysis.

**Proposition D.1** (Hoeffding Bound). Suppose that the variables \( x_i, i = 1, \ldots, n \) are independent, and \( x_i \) has mean \( \mu_i \) and \( \|x_i\|_{\psi_2} \leq \sigma_i \). Then for all \( t \geq 0 \), we have
\[
\Pr \left[ \sum_{i=1}^{n} (x_i - \mu_i) \geq t \right] \leq \exp \left( \frac{-t^2}{2 \sum_{i=1}^{n} \sigma_i^2} \right).
\]

**Proposition D.2** (Proposition 3.3 Restated). Let \((X, y, \theta^*)\) be \((k, \sigma)\)-well behaved. Let \( \Psi = (\Psi_{ij}) \) be an \( m \times d \) random matrix with independent entries such that \( \mathbb{E}[\Psi_{ij}] = 0, \mathbb{E}[\Psi_{ij}^2] = 1 \), and \( \|\Psi_{ij}\|_{\psi_2} \) is bounded. Let \( m = \Theta(k \log(d/\beta)) \) and \( \Phi = \Psi / \sqrt{m} \). Then with probability at least \( 1 - \beta \),
\[
\left\| \frac{(X\Phi^\top \Phi)^\top \tilde{w}}{n} \right\|_{\infty} = O \left( \frac{\sigma \|X\|_F \log(d/\beta)}{n \sqrt{m}} + \frac{\|X\|_F^2}{n \sqrt{d}} \right).
\]

**Proof.** Let \( \mathbf{w} = (w_1, \ldots, w_n) \) and \( y = (y_1, \ldots, y_n) \). By Definition 3, \( w_i = y_i - \langle x_i, \theta^* \rangle \). Let \( \tilde{w} = y - X\Phi^\top \Phi \theta^* \). Therefore, by invoking Proposition B.3 with \( \gamma = O(1) \) provides that with probability at least \( 1 - \beta \),
\[
\left\| \frac{(X\Phi^\top \Phi)^\top \tilde{w}}{n} \right\|_{\infty} = \left\| \frac{1}{n} \sum_{i=1}^{n} \Phi^\top \Phi x_i (y_i - \langle \Phi x_i, \Phi \theta^* \rangle) \right\|_{\infty}
\leq \left\| \frac{1}{n} \sum_{i=1}^{n} \Phi^\top \Phi x_i (y_i - \langle x_i, \theta^* \rangle \pm \|x_i\| \|\theta^*\|) \right\|_{\infty}
= \left\| \frac{1}{n} \sum_{i=1}^{n} \Phi^\top \Phi x_i (y_i - \langle x_i, \theta^* \rangle \pm \|x_i\|) \right\|_{\infty}
= \left\| \frac{1}{n} \sum_{i=1}^{n} \Phi^\top \Phi x_i (w_i \pm \|x_i\|) \right\|_{\infty}
= \max_{j \in [d]} \left\{ \frac{1}{n} \langle \mathbf{e}_j, \tilde{w} \rangle \right\}
\]
where \( c_j = (c_{j1}, \ldots, c_{jn}) \) is the \( j \)th column in \( X \Phi^\top \Phi \), and \( \bar{w} = (\bar{w}_1, \ldots, \bar{w}_n) \) with \( \bar{w}_i = w_i \pm \| x_i \| \). Note that we used Proposition B.3 for the first inequality.

We now bound the term in the right-hand side of (12). For a fixed \( j \), using Proposition D.1 on the set of subgaussian variables \( c_{j1}, \bar{w}_1, \ldots, c_{jn}, \bar{w}_n \) gives that

\[
\Pr \left[ \sum_{i=1}^n (c_{ji} \bar{w}_i - \| x_i \|) \geq t \right] \leq \exp \left( \frac{-t^2}{2\sigma^2 \| c_j \|^2} \right).
\]

Taking a union bound over \( j \in [d] \),

\[
\Pr \left[ \max_{j \in [d]} \left\{ \sum_{i=1}^n (c_{ji} \bar{w}_i - \| x_i \|) \right\} \geq t \right] \leq d \exp \left( \frac{-t^2}{2\sigma^2 \| c_j \|^2} \right). \tag{13}
\]

We now investigate the norm of \( c_j \). Let \( \phi_j \) be the \( j \)th column in \( \Phi \). Now, the \( i \)th entry in \( c_j \in \mathbb{R}^n \), can be expressed as \( c_{ji} = \langle \Phi x_i, \phi_j \rangle \) i.e., the \((i, j)\)th entry of \( X \Phi^\top \Phi \) equals \( \langle \Phi x_i, \phi_j \rangle \). With the choice of \( m \), with probability at least \( 1 - \beta \), \( \| \Phi x_i \| = O(\| x_i \|) \) for all \( i \in [n] \), where the first expression follows from the norm preservation property of JL-style transform (Lemma B.2). Using definition of subgaussian random variables yields that with probability at least \( 1 - \beta \), for each \( j \in [d] \), \( \| c_j \| = O(\| X \|_F \sqrt{\log(d/\beta)/m}) \). Using this bound in (13) and setting \( t = O(\sigma \| X \|_F \log(d/\beta)/\sqrt{m}) \), and proper conditioning, gives that with probability at least \( 1 - \beta \),

\[
\max_{j \in [d]} \{ \langle c_j, \bar{w} \rangle \} = O \left( \frac{\sigma \| X \|_F \log(d/\beta)}{\sqrt{m}} + \sum_{i=1}^n \| x_i \| \right) = O \left( \frac{\sigma \| X \|_F \log(d/\beta)}{\sqrt{m}} + \frac{\| X \|_F^2}{\sqrt{d}} \right).
\]

Plugging this bound into (12) gives the claimed result. \( \square \)