New series expansions of the Gauss hypergeometric function

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Abstract The Gauss hypergeometric function \( {}_2F_1(a, b, c; z) \) can be computed by using the power series in powers of \( z, z/(z - 1), 1 - z, 1/z, 1/(1 - z), \) and \((z - 1)/z\). With these expansions, \( {}_2F_1(a, b, c; z) \) is not completely computable for all complex values of \( z \). As pointed out in Gil et al. (2007, §2.3), the points \( z = e^{\pm i\pi/3} \) are always excluded from the domains of convergence of these expansions. Bühring (SIAM J Math Anal 18:884–889, 1987) has given a power series expansion that allows computation at and near these points. But, when \( b - a \) is an integer, the coefficients of that expansion become indeterminate and its computation requires a nontrivial limiting process. Moreover, the convergence becomes slower and slower in that case. In this paper, we obtain new expansions of the Gauss hypergeometric function in terms of rational functions of \( z \) for which the points \( z = e^{\pm i\pi/3} \) are well inside their domains of convergence. In addition, these expansions are well defined when \( b - a \) is an integer and no limits are needed in that case. Numerical computations show that these expansions converge faster than Bühring’s expansion for \( z \) in the neighborhood of the points \( e^{\pm i\pi/3} \), especially when \( b - a \) is close to an integer number.

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1 Introduction

The power series of the Gauss hypergeometric function\( _2F_1(a, b, c; z) \)

\[
_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,
\]

converges inside the unit disk. For numerical computations, we can use the right-hand side of (1) to compute \( _2F_1(a, b, c; z) \) only in the disk \(|z| \leq \rho < 1\), with \( \rho \) depending on numerical requirements, such as precision and efficiency. From [3, §§2.3.1 and 2.3.2] or [7, Eq. 15.2.1 and §§15.8(i) and 15.8(ii)], we see that the Gauss hypergeometric function \( _2F_1(a, b, c; z) \) may be written in terms of one or two other \( _2F_1 \) functions with any of the following arguments:

\[
\frac{1}{z}, \quad 1 - z, \quad \frac{1}{1 - z}, \quad \frac{z}{1 - z}, \quad \frac{z - 1}{z}.
\]

As explained in [3, §2.3.2], when these formulas are combined with the series expansion (1), we obtain a set of series expansions of \( _2F_1(a, b, c; z) \) in powers of some of the rational functions given in (2). The domains of convergence of the whole set of the expansions obtained in this way are the regions

\[
|z| \leq \rho < 1, \quad \left| \frac{1}{z} \right| \leq \rho < 1, \quad |1 - z| \leq \rho < 1,
\]

\[
\left| \frac{1}{1 - z} \right| \leq \rho < 1, \quad \left| \frac{z}{1 - z} \right| \leq \rho < 1, \quad \left| \frac{z - 1}{z} \right| \leq \rho < 1.
\]

These regions (interior or exterior of certain circles) do not cover the entire \( z \)-plane; the points \( z = e^{\pm i\pi/3} \), which are the intersection points of the circles \(|z| = 1\) and \(|z - 1| = 1\), are excluded for any value of \( \rho < 1 \) (see Fig. 1).

When \( \rho \to 1 \), the set of points of the \( z \)-plane excluded from the union of these regions shrinks to the exceptional points \( z = e^{\pm i\pi/3} \); in addition, the convergence of those expansions becomes slower and slower when \( z \to e^{\pm i\pi/3} \). To compute the Gauss hypergeometric function in a neighborhood of these points, other methods are indicated in [3], the most useful one being Bühring’s analytic continuation formula [1]. Bühring’s expansion reads as follows