Non-linear shallow water dynamics with odd viscosity

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(Dated: September 2, 2020)

In this letter, we derive the Korteweg-de Vries (KdV) equation corresponding to the surface dynamics of a shallow depth (h) two-dimensional fluid with odd viscosity ($\nu_o$) subject to gravity ($g$) in the long wavelength weakly nonlinear limit. In the long wavelength limit, the odd viscosity term plays the role of surface tension albeit with opposite signs for the right and left movers. We show that there exists two regimes with a sharp transition point within the applicability of the KdV dynamics, which we refer to as weak ($|\nu_o| < \sqrt{gh^3/6}$) and strong ($|\nu_o| > \sqrt{gh^3/6}$) parity-breaking regimes. While the 'weak' parity breaking regime results in minor qualitative differences in the soliton amplitude and velocity between the right and left movers, the 'strong' parity breaking regime on the contrary results in solitons of depression (negative amplitude) in one of the chiral sectors.

Introduction: Parity-breaking phenomena in two-dimensional fluids such as odd viscosity effects have been at the center of investigation in diverse platforms. For two dimensional isotropic and non-dissipative flows, odd viscosity is the only non-vanishing coefficient of the viscosity tensor and has a geometrical connection to the adiabatic curvature on the space of flat background metrics \cite{1,2}. Examples of quantum systems where odd viscous effects are important include electron fluids in mesoscopic systems \cite{3,5}, quantum Hall fluids \cite{6,20}, and chiral superfluids/superconductors \cite{27}.

In classical fluids, odd viscosity shows up in polyatomic gases \cite{28-31}, chiral active matter \cite{32-34}, vortex dynamics in 2D \cite{35,38} and chiral active fluids \cite{32,34}. For incompressible flows, it has been shown by one of the authors that odd viscosity effects are absent when the fluid is spread on the entire plane or confined in rigid domains with no-slip boundary conditions \cite{29}. In other words, the velocity profile is independent of the odd viscosity. Nevertheless, the signature of this parity-breaking coefficient is present in surface waves and in the interface between two fluids governed by kinematic and no-stress boundary conditions, which explicitly depends on the odd viscosity. The dynamical surface problem in the presence of odd viscosity results in an oscillating boundary layer where the vorticity is confined within some thickness of $\delta \propto \sqrt{g \eta}$, \cite{40,41} (where $\nu_e$ is the kinematic shear viscosity) for the dissipative case and $\delta \propto c_s^{-1}$ (where $c_s$ is the sound velocity) for the non-dissipative compressible case \cite{42}. In the limit of a very thin boundary layer, that is, $\nu_e \to 0$ or $c_s \to \infty$, both the fluid pressure at the edge and the surface vorticity diverge as $1/\sqrt{\nu_e}$ or $c_s$, but the quantity $\tilde{p} = p - \nu_e \rho \omega$ remains finite. We refer to $\tilde{p}$ as modified pressure, where $\nu_e$ is the odd viscosity and the variables $\rho$, $\omega$ and $\rho$ are the fluid pressure, vorticity and constant background density respectively. This cancelation of divergences allows us to write the dynamical surface problem with odd viscosity as an effective irrotational system where all the effects of odd viscosity and boundary layer can be absorbed into a modified pressure term at the edge.

In short, for an irrotational flow, that is, $\nu = \nabla \theta$, the effective boundary dynamics can be expressed as a Laplace equation for the velocity potential $\theta$ in the bulk and Bernoulli’s equation at the boundary with the modified pressure \cite{40}. The variational principle for this boundary dynamics was later formulated in terms of a geometric action which resulted in the odd viscosity induced effective pressure at the boundary \cite{41}. In the limit of infinitely deep fluid the weakly non-linear dynamics within a small angle approximation was shown to be governed by the novel Chiral-Burgers equation \cite{40}.

In this letter, we study the shallow depth limit of the weakly non-linear surface dynamics with odd viscosity and gravitational force (confining potential) (see the schematic in Fig. 1). We assume that the boundary layer is the shortest length scale and the effective dynamics is irrotational, using the hydrodynamic equations from \cite{11} as the starting point. We show that, for later times and long wavelengths, the weakly non-linear dynamics is given by the integrable Korteweg De Vries (KdV) equation with the kinematic odd viscosity $\nu_o$ entering the coefficient of the dispersive term,

$$\eta_{xx} \pm \frac{\eta_x}{\sqrt{gh^3}} + \frac{3}{2h} \eta \eta_{xx} + h^2 \left( \frac{1}{6} \pm \frac{\nu_o}{\sqrt{gh^3}} \right) \eta_{xxx} = 0. \quad (1)$$

Here, $\eta(x,t)$ is the boundary shape profile, $h$ is the average depth of the fluid and $g$ is the acceleration of gravity. The positive sign refers to right-moving solitons whereas the negative sign refers to the left-moving ones. The manifestation of odd viscosity in the above equation is similar to that of the surface tension although with different signs for the left and right movers.

The odd viscosity entering the KdV equation has major consequences due to its parity breaking effects. We show that there exists two regimes with a sharp transition point within the applicability of the KdV dynamics, which we refer to as weak ($|\nu_o| < \sqrt{gh^3/6}$) and strong ($|\nu_o| > \sqrt{gh^3/6}$) parity-breaking regimes. In the
KdV dynamics without odd viscosity. In contrast to the parity preserving case of shallow water parity breaking KdV dynamics discussed here is in stark higher order derivative terms become important. The ing solitons slightly differ in amplitude and speed. In exaggerated to highlight these features.

Equations (2) and (3) can be expressed in the form of an action principle, as shown in [41]. Following [41], the hydrodynamic action becomes

\[
S = -\int dt \int_{-h}^{\eta(t,x)} dy \left[ \partial_t + \frac{1}{2} \left( \partial_x^2 + \partial_y^2 \right) \right] - \int dt \int dx \left[ \frac{1}{2} \eta^2 - \nu_o \eta \tan^{-1}(\eta_x) \right],
\]

where \( \eta(t,x) \) is the top surface shape function. The domain of the fluid is bounded by a finite depth at the bottom. The bulk and boundary equations of motion are obtained by varying the above action for a finite depth fluid. The bulk equation for the irrotational system is simply the Laplace equation for the potential \( \Delta \theta = 0 \) defined in the domain \( -h < y < \eta(t,x) \). The boundary conditions at the top and bottom of the fluid domain can be written as,

\[
\theta_y = 0, \quad y = -h, \quad \eta_x + \nu_o \theta_x = \theta_y, \quad y = \eta(t,x), \quad \theta_t + \frac{\theta_x^2 + \theta_y^2}{2} + g\eta = \frac{2\nu_o}{\sqrt{1 + \eta_x^2}} \frac{\eta}{\sqrt{1 + \eta_x^2}}, \quad y = \eta(t,x).
\]

Eq. (7) is the kinematic boundary condition and Eq. (8) is the odd viscosity modified dynamic boundary condition.

**Linear waves in long wavelength limit:** Before we dive into the non-linear dynamics of the shallow fluid
regime, let us focus on the linearized free surface problem with finite depth. In this limit, we drop all the quadratic terms in Eqs. \[ \text{and evaluate the derivatives of } \theta \text{ at } y = 0. \] For the monochromatic surface profile \( \eta(x,t) = a \cos(kx - \Omega t) \) as an input, we find the velocity potential to be

\[
\theta(x,y,t) = \frac{a \Omega}{k \sinh(kh)} \cosh[k(y + h)] \sin(kx - \Omega t),
\]

with the surface dispersion relation given by

\[
\Omega = \tanh kh \left[ -\nu_o k^2 \pm \sqrt{\nu_o^2 k^4 + gk \coth(kh)} \right]. \tag{9}
\]

In the absence of gravity \((g = 0)\) and in the deep ocean limit \((h \to \infty)\), we have that \(\tanh kh \approx k/|k|\) and we recover the known odd viscosity dominated dispersion \(\Omega = \{0, -2\nu_o k |k|\}\). The weakly non-linear dynamics for this system was discussed in Ref. [10].

Shallow waves, on the other hand, arise when the fluid depth \(h\) is much smaller than the characteristic wavelengths of the system. In other words, they are characterized by \(kh \ll 1\). In this approximation, the leading terms in the dispersion \(\Omega\) are given by

\[
\Omega \approx \sqrt{\frac{g}{h}} [\pm kh - \left( \frac{\nu_o}{\sqrt{gh^3} + \frac{1}{6}} \right) (kh)^3]. \tag{10}
\]

The first term is the usual shallow water gravity wave dispersion, whereas the second one is the odd viscosity modified Korteweg de-Vries dispersive term. Prima facie it seems that the odd viscosity is qualitatively similar to the surface tension effect for the shallow water surface dispersion. However, the physical manifestation of odd viscosity is completely different, since the coefficient of the cubic term can develop a relative sign change between the left mover and right mover for \(\nu_o > \frac{1}{2} \sqrt{gh^3}\), what indicates a strong parity-breaking phenomena.

The shallow wave condition naturally introduces a power expansion in \(kh\). Formally, it is convenient to define an expansion parameter \(\varepsilon \ll 1\), such that \(kh = \sqrt{\varepsilon} \tilde{k}\) and \(\tilde{k}\) is a dimensionless wavenumber. Since an expansion in powers of \(kh\) can be translated into a derivative expansion for the fluid dynamics, this rescaling is equivalent to the redefinition \(x = \frac{\sqrt{\varepsilon}}{h} X\), where \(X\) is the dimensionless horizontal coordinate. In the same way, we can define \(y = hY\), with \(Y\) being the dimensionless vertical coordinate. In this counting scheme, we have that \(\partial_x \sim \mathcal{O}(\varepsilon^{1/2})\), whereas \(\partial_y \sim \mathcal{O}(1)\).

The wave dynamics dictated by Eq. \[(9)\] evolves according to two distinct time scales. The linear term scales with \(\sqrt{\varepsilon}\) and governs the splitting of an initial disturbance into right-moving and left-moving wavepackets. The first time scale, which we denote by \(T\), is the characteristic time in which left and right-movers are so far apart, we can study them separately. In other words, for sufficiently later times, the only role of \(T\) is to account for the boost of the center of mass and it only appears in the combination \(X - T\), for right-movers, or \(X + T\), for left-movers. On the other hand, the cubic term scales as \(\varepsilon^{3/2}\) and give rise to a dispersive group velocity of the boosted wavepacket. This effect becomes relevant at much later times, in comparison to \(T\), and introduce a second time scale, which governs the time evolution of the boosted wavepacket and we denote by \(\tau\). This means that both variables \(\theta\) and \(\eta\) evolve according to this double time scale, such that, the time derivative becomes

\[
\partial_t = \sqrt{\frac{g}{h}} \partial_T + \sqrt{\frac{g}{h^3}} \partial_{\tau}. \tag{11}
\]

In the following, we derive the full non-linear shallow water dynamics with odd viscosity and discuss how the parity breaking effects manifest in the non-linear dynamics. In particular we show that \(\nu_o = \frac{1}{2} \sqrt{gh^3}\) manifests as a critical point that separates two qualitatively different regimes of non-linear dynamics.

**Non-linear shallow depth waves**: Korteweg-de Vries (KdV) equation arises in the study of shallow water waves of long wavelengths and small amplitudes. In the following analysis, we show how the KdV equation corresponding to the shallow depth limit is modified by the presence of the odd viscosity term. The counting scheme for the KdV equation is chosen such that small amplitudes regime corresponds to \(\eta \sim \mathcal{O}(\varepsilon)\), that is, \(\eta\) is of the same order as \(\partial_x^2\). Thus, we can rescale the boundary shape in terms of \(h\) as \(\eta = \varepsilon h\).

KdV regime happens for sufficiently later times, so that right-moving and left-moving solutions are independent and well-separated. Here, we restrict ourselves to only right-moving propagation, since the analysis for the left-movers follows similarly. Hence, let us assume \(\theta\) and \(\eta\) of the form

\[
\theta(t,x,y) = \sqrt{\varepsilon h^3} \theta(\tau,\sigma,Y;\varepsilon), \tag{12}
\]

\[
\eta(t,x) = \varepsilon h \eta(\tau,\sigma;\varepsilon), \tag{13}
\]

with \(\sigma = X - T\). Under these conditions, the bulk equation of motion and the boundary condition at the flat bottom become

\[
\varepsilon \partial_{\sigma \sigma} + \partial_{YY} = 0, \quad -1 < Y < \varepsilon h, \tag{14}
\]

\[
\partial_{\sigma} = 0, \quad Y = -1. \tag{15}
\]

Let us denote \(\psi(\tau,\sigma,-1;\varepsilon)\) by \(\phi(\tau,\sigma;\varepsilon)\). This way, the solution of Eq. \[(13)\] with the condition \[(14)\] can be written as

\[
\partial(\tau,\sigma,Y;\varepsilon) = \sum_{n=0}^{\infty} \frac{(-\varepsilon)^n(1+Y)^{2n}}{(2n)!} \partial_{\sigma}^{2n} \phi. \tag{15}
\]

Plugging Eq. \[(15)\] into Eqs. \[(7)\] and neglecting terms of \(\mathcal{O}(\varepsilon^2)\) or higher, we obtain

\[
\phi_\sigma = \eta + \varepsilon \left( \phi_\tau + \frac{1}{2} \phi_{\sigma \sigma} + \frac{1}{2} \phi_{\sigma \sigma \sigma} + 2 \bar{\nu}_o \eta_{\sigma \sigma} \right), \tag{16}
\]

\[
\eta_{\eta} - \phi_{\sigma \sigma} = \varepsilon \left[ \frac{1}{2} \eta_{\sigma \sigma \sigma} - \psi_{\tau} + \frac{1}{2} \phi_{\sigma \sigma \sigma} + \partial_{\sigma}(\phi_{\sigma} \eta) \right]. \tag{17}
\]
Here, we denoted \( \bar{\nu}_o = \nu_o / \sqrt{gh^3} \). Eq. (16) allow us to perturbatively express \( \bar{\phi}_\sigma \) in terms of \( \eta \). Substituting this expression into Eq. (17), the leading order equation for the right-moving surface wave in the boosted reference frame becomes

\[
\eta_\tau + \frac{2}{3} \eta \eta_\sigma + \left( \frac{1}{6} + \bar{\nu}_o \right) \eta_{\sigma \sigma} = 0, \tag{18}
\]

which is nothing but the well-known KdV equation. In terms of the dimensionful variables \( t, x \) and \( \eta(t, x) \), it becomes Eq. (1).

As previously mentioned, we could repeat the same analysis for the left-moving solitons. For \( \xi = X + T \), we obtain

\[
\eta_\tau - \frac{3}{2} \eta \eta_\xi - \left( \frac{1}{6} - \bar{\nu}_o \right) \eta_{\xi \xi} = 0. \tag{19}
\]

Under reflection about \( y \)-axis (parity operation), \( \eta \rightarrow -\eta \), \( \xi \rightarrow -\sigma \) and Eq. (19) becomes

\[
\bar{\eta}_\tau + \frac{3}{2} \bar{\eta} \bar{\eta}_\sigma + \left( \frac{1}{6} - \bar{\nu}_o \right) \bar{\eta}_{\sigma \sigma} = 0. \tag{20}
\]

The odd viscosity term breaks parity symmetry of the problem [44], since the left-moving soliton under reflection about the \( y \)-axis do not behave like the right-moving soliton. The odd viscosity term entering the KdV equation is similar to the presence of the surface tension. Within this analogy of odd viscosity as surface tension, the left mover and right mover will have opposite signs of surface tension due to the parity breaking effects of odd viscosity. In other words, odd viscosity in the KdV regime acts as chirality dependent surface tension term.

**Soliton solution:** In the following we analyze the role of odd viscosity in the single soliton solution of Eqs. (18) and (19). Although multi-soliton solutions also show the same qualitative behavior, they are out of the scope of this letter. The single soliton solution corresponding to the left and right movers can be written as,

\[
\eta(t, x_\pm) = 8\bar{k}^2 \left( \frac{1}{6} \mp \bar{\nu}_o \right) \text{sech}^2 \left[ \bar{k} x_\pm \pm 4\bar{k}^3 \tau \left( \frac{1}{6} \mp \bar{\nu}_o \right) \right], \tag{21}
\]

where \( \bar{k} \) is the dimensionless wavenumber and we denoted \( x_+ = \xi \) and \( x_- = \sigma \) in order to shorten the notation. Moreover, \( \tau = 0 \) was chosen such that the soliton center of mass is at \( x_\pm = 0 \). Note that \( \left( \frac{1}{6} \mp \bar{\nu}_o \right) \) enters both the amplitude and the wave speed. Therefore, the odd viscosity modification to the KdV soliton dynamics can be separated into three regimes depending on the value of \( \left( \frac{1}{6} \mp \bar{\nu}_o \right) \).

- **Weak’ parity breaking regime** \( |\bar{\nu}_o| < \frac{1}{6} \): In this case, \( \left( \frac{1}{6} \mp \bar{\nu}_o \right) > 0 \), with left and right moving solitons only differ in the magnitude of the amplitude and velocity as shown in Fig. 2. We refer to this as weak parity breaking regime.

- **Strong’ parity breaking regime** \( |\bar{\nu}_o| > \frac{1}{6} \): In this case, \( \left( \frac{1}{6} \mp \bar{\nu}_o \right) \) have opposite signs. We call this strong parity breaking regime, because the difference between left and right moving solitons is visual, that is, one sector has positive amplitude, whereas the other corresponds to solitonic waves of depression or depletion as shown in Fig. 3.

**Critical’ dynamics** \( |\bar{\nu}_o| = \frac{1}{6} \): At this critical points, the dispersive term in one of the sectors vanish and we end up with the inviscid Burger’s equation for such sector as shown in Fig. 4. In fact, it is known that solutions of the inviscid Burger’s equation are subjected to a blow up time, in which the spatial derivative of \( \eta \) becomes infinite and higher order derivative terms become important.

In general, non-localized solutions for the KdV are given in terms of the Jacobi elliptic function, also known as cnoidal functions,

\[
\eta(t, x_\pm) = 8\bar{k}^2 \left( \frac{1}{6} \mp \bar{\nu}_o \right) \text{cn}^2 \left[ \bar{k} x_\pm \pm 4\bar{k}^3 \tau \left( \frac{1}{6} \mp \bar{\nu}_o \right) ; \kappa \right]. \tag{22}
\]

The parameter \( \kappa \in [0, 1] \) interpolates between the long linear waves for \( \kappa \rightarrow 0 \) (\( \text{cn} \rightarrow \cos \)) and the single soliton...
solution for $\kappa \to 1$ ($\text{cn} \to \text{sech}$).

**Discussion and Outlook:** In this letter, we derived parity broken generalization of the Korteweg de-Vries equation for shallow depth fluid with odd viscosity and gravity in the long wavelength weakly non-linear limit. The presence of odd viscosity manifests weak and strong parity breaking regimes in the two chiral sectors of the KdV dynamics. The odd viscosity term plays the role of surface tension albeit with opposite signs for the right and left movers. In future work, we aim to specialize this result to chiral active fluids, where odd viscous effects have been observed in free surface dynamics.

In order to make contact with experiments, we will numerically study the Cauchy initial value problem of an equation for shallow depth fluid with odd viscosity and parity broken generalization of the Korteweg de-Vries equation with conductivity.

**Acknowledgments.** We thank Alexander Abanov and Vincenzo Vitelli for helpful discussions and suggestions about to this project. This work is supported by NSF CAREER Grant No. DMR-1944967 (SG) and partly from PSC-CUNY Award. GM was supported by 21st century foundation startup award from CCNY.

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