Linking stability of nonlinear sampled-data systems and their continuous-time limits

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Abstract

Discrete-time (DT) models of sampled-data systems can be regarded as predictions of the future state value given current values for the state, input and sampling period. A DT model is exact if its prediction coincides with the true value and is otherwise approximate. Conditions under which Semiglobal Exponential Stability under nonuniform sampling (SES-VSR) is preserved among different (exact or approximate) DT models when fed back with the same sampling-period dependent control law exist. The current paper proves that the exact DT model is SES-VSR if and only if its corresponding continuous-time (CT) limit (for infinitesimally small sampling period) is Globally Exponentially Stable (GES), which is restrictive. The contribution of this note consists in extending and relaxing the assumptions of previous results, as follows: (i) only mild conditions are given in order to link stability properties between DT models fed back with different control laws, and (ii) the DT model properties making the CT limit Globally Asymptotically and Locally Exponentially Stable (GALES), instead of GES, are provided.

Keywords: sampled-data systems, nonlinear systems, nonuniform sampling, control redesign, DT models.

1. Introduction

Most control applications involve measuring available signals via a sampling mechanism, computing a control action based on signal samples, and applying this action usually via Zero-Order Hold (ZOH). The sampled-data controller can be designed directly in DT based on a model [Nesić et al., 1999, Nesić and Teel, 2004, Liu et al., 2008, Nesić et al., 2004, Ustüntürk, 2012, Ustüntürk and Kocaabdullah, 2013, Noroozi et al., 2018, Beikzadeh et al., 2018, Vallarella and Haimovich, 2018, 2019] or obtained by discretization of a CT controller that ignores the sampling. In this last case, the discretization may involve redesigning the CT controller for the sampled-data implementation [Nesić and Grüne, 2005, Monaco and Normand-Cyrot, 2007, Grüne et al., 2008].

It is known that controller emulation, i.e. applying the control action given directly by the CT controller, stabilizes the sampled-data system under sufficiently fast sampling [Nesić et al., 2009, Proskurnikov, 2020] and that redesign may lead to better performance [Nesić and Grüne, 2005, Postovan et al., 2008]. Existing results give conditions for the preservation of the stability—in a certain sense— of the CT closed-loop system when implemented in a sampled-data setting [Pepe and Fridman, 2017, Lin, 2020, Lin and Wei, 2018, Di Ferdinando and Pepe, 2019, Lin and Sun, 2021, Di Ferdinando et al., 2021a]. In particular, sampled-data stabilization under nonuniform sampling has lately been an active research topic [Li and Zhao, 2018, Hetel et al., 2017, Omiran et al., 2016].

We introduce a set of mild conditions (local Lipschitz continuity) on the plant and DT control laws that allow to establish links between closed-loop stability in a sampled-data setting (involving possibly sampling period-dependent control laws) and the stability of the CT system obtained as the limit as the sampling becomes infinitely frequent. First, we show that these conditions ensure that different DT models of the plant exhibit the same stability properties when fed back with different control laws. Then we show that these are the key conditions that link the Semiglobal (asymptotical) and Local Exponential stability for the exact DT model (the model which coincides with the sampled-data system at sampling instants) with GALES of its CT limit. These conditions are applicable also under aperiodic (i.e. nonuniform) sampling and may complement several of the aforementioned results on control redesign and DT design allowing, for example, to simplify the derivation of stability guarantees [Vallarella et al., 2021, Di Ferdinando et al., 2021b, Monaco and Normand-Cyrot, 2007, Grüne et al., 2008, Vallarella and Haimovich, 2018].

Notation: \( R, R_{\omega}, N \) and \( N_0 \) denote the sets of real, nonnegative real, natural and nonnegative integer numbers, respectively. We write \( \alpha \in \mathcal{K} \) if \( \alpha : R_{\omega} \to R_{\omega} \) is strictly increasing, continuous and \( \alpha(0) = 0 \). We write \( \alpha \in \mathcal{K}_{\infty} \) if \( \alpha \in \mathcal{K} \) and \( \alpha \) is unbounded. We write \( \beta \in \mathcal{KL} \) if \( \beta : R_{\omega} \times R_{\omega} \to R_{\omega} \), \( \beta(t, s) \in \mathcal{K} \) for all \( t \geq 0 \), and \( \beta(s, \cdot) \) is strictly decreasing asymptotically to 0 for every \( s \). We denote the Euclidean norm of a vector \( x \in R^n \) by \( \| x \| \) and an infinite sequence as \( \{ T_i \} := \{ T_i \}_{i=0}^{\infty} \). For any sequence \( \{ T_i \} \subset R_{\omega} \) we take the following convention: \( \sum_{i=0}^{\infty} \sigma_i = 0 \). Given a real number \( T > 0 \) we denote by \( \Phi(T) := \{ [T_i] : \{ T_i \} \) is such that \( T_i \in (0, T) \) for all \( i \in N_0 \} \) the set of all sequences of real numbers in the open interval \( (0, T) \).
2. Problem statement

The problem to be addressed is explained next.

2.1. Continuous-time system

Consider a nonlinear plant of the form

\[ \dot{x} = f(x, u), \]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \) are the state and control vectors, and assume that some control law \( u_c : \mathbb{R}^n \to \mathbb{R}^m \) renders

\[ \dot{x} = f(x, u_c(x)) =: h(x) \]

stable as per one of the following definitions.

Definition 2.1. The system (3) is said to be

i) Globally Asymptotically Stable (GAS) if there exists \( \beta \in \mathcal{KL} \) such that for any \( x_0 \in \mathbb{R}^n \) the solutions satisfy \( |x(t)| \leq \beta(|x_0|, t), \forall t \geq 0 \). If additionally \( \beta \in \mathcal{KL} \) can be chosen as \( \beta(r, t) := K e^{-\lambda t} \) with \( K \geq 1 \) and \( \lambda > 0 \) it is said to be Globally Exponentially Stable (GES).

ii) Locally Exponentially Stable (LES) if there exist \( K \geq 1 \) and \( R, \lambda > 0 \) such that for all \( |x_0| \leq R \) the solutions satisfy \( |x(t)| \leq K|x_0|e^{-\lambda t}, \forall t \geq 0 \).

iii) GALES if it is GAS and LES.

The function \( f \) in (1) and the control law \( u_c \) in (2) fulfill the following local Lipschitzian assumptions.

Assumption 2.2. \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) fulfills \( f(0, 0) = 0 \) and for every \( M, M_u \geq 0 \) there exists \( L = L(M, M_u) > 0 \) such that for all \( |x|, |y| \leq M \) and \( |u|, |v| \leq M_u \) we have \( |f(x, u) - f(y, v)| \leq L(|x - y| + |u - v|) \).

Assumption 2.3. \( u_c : \mathbb{R}^n \to \mathbb{R}^m \) fulfills \( u_c(0) = 0 \) and for every \( M \geq 0 \) there exists \( L = L(M) > 0 \) such that for all \( |x|, |y| \leq M \) we have \( |u_c(x) - u_c(y)| \leq L|x - y| \).

2.2. Sampled-data system and discrete-time models

Consider sampling instants \( t_k, k \in \mathbb{N}_0, t_0 = 0 \) and \( t_{k+1} = t_k + T_k \), where the sampling periods \( T_k \geq 0 \) may vary following any possible sequence \( \{T_k\}_{k=0}^{\infty} \). We refer to this situation as Varying Sampling Rate (VSR). We assume that the sampling period \( T_k \) becomes either known or determined at instant \( t_k \), so that this information may be used to perform the current control action:

\[ u_k = U(x_k, T_k). \]

This is always the case under periodic sampling (i.e. \( T_k \equiv T > 0 \)), where the control law is designed based on prior knowledge of the sampling period. The sampled-data system that arises from (1) in feedback with (3) under ZOH results

\[ \dot{x}(t) = f(x(t), U(x(t), T_k)) \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}_0. \]

Under ZOH, a DT model \( x_{k+1} = F^0(x_k, u_k, T_k) \) for this system can be thought of as an estimate of the value \( x_{k+1} = x(t_{k+1}) \), given \( x_k \) and \( u_k \) at the sampling instant \( t_k \). For nonlinear plants, the exact DT model, i.e. the model whose state coincides with that of the CT plant state at sampling instants, may be difficult or impossible to obtain given that it requires to solve the differential equations (1) in closed form. This model satisfies \( F^0(x, u, T) = x + \int_0^T f(F^0(x, u, s), u) ds \). Due to the unavailability of \( F^0 \), a suitable approach is to design the control law based on a sufficiently good approximate DT model of the plant, such as Runge-Kutta models [Stuart and Humphries, 1996]. The simplest of these models, the Euler model, is given by \( F^b(x, u, T) := x + T f(x, u) \). A DT model \( F^b \) fed back with a DT control of the form (4) will be denoted by \( F^b(x, u, T) := F^b(x, U(x, T), T) \). The sampling of (2) can even be interpreted as a closed-loop DT model \( H^b \) corresponding to zero DT control and which satisfies \( H^b(x, T) = x + \int_0^T h(H^b(x, s), s) ds \).

2.3. Consistency and stability of discrete-time models

Consistency properties of DT models are suitable mismatch bounds that allow to maintain specific stability properties. Equilibrium-Preserving Consistency (EPC) bounds the mismatch between DT model solutions in one sampling period and becomes equivalent to the REPC property in Vallarelli et al. (2021) when no errors affect the control input.

Definition 2.4. The DT model \( F^b \) is said to be Equilibrium-Preserving Consistent (EPC) with \( F^b \) if for each \( M \geq 0 \) there exist constants \( K := K(M) > 0, T^* := T^*(M) > 0 \) and a function \( p \in \mathcal{K}_\omega \) such that

\[ |\tilde{F}^b(x, T) - \tilde{F}^b(y, T)| \leq (1 + KT)|x - y| + Tp(T)|x_0| \]

for all \( |x|, |y| \leq M \) and \( T \in (0, T^*) \). The pair \((\tilde{F}^b, \tilde{F}^b)\) is said to be EPC if \( F^b \) is EPC with \( F^b \).

The following stability properties for DT models are suitable under uniform sampling (see also Vallarelli and Haimovich, 2019; Vallarelli et al., 2021).

Definition 2.5. The system \( x_{k+1} = F^0(x_k, T_k) \) is said to be

i) Semiglobally Practically Stable-Var (SPS-VSR) if there exists \( \beta \in \mathcal{KL} \) such that for every \( M \geq 0 \) and \( R > 0 \) there exists \( T^* := T^*(M, R) > 0 \) such that for all \( k \in \mathbb{N}_0, |T_k| \in \Phi(T^*) \) and \( |x_0| \leq M \) the solutions satisfy \( |x_k| \leq \beta(|x_0|, \sum_{i=0}^{k-1} T_i) + R \).

ii) Locally Exponentially Stable-Var (LES-VSR) if there exist \( K \geq 1 \) and \( R, T^*, A > 0 \) such that for all \( k \in \mathbb{N}_0, |T_k| \in \Phi(T^*) \) and \( |x_0| \leq R \) the solutions satisfy \( |x_k| \leq K|x_0|e^{-\lambda T_k^\beta} \).

iii) Semiglobally (asymptotically) and Locally Exponentially Stable (SLES-VSR) if it is SPS-VSR and LES-VSR.

iv) Semiglobally (asymptotically) Stable-Var (SS-VSR) if there exists \( \beta \in \mathcal{KL} \) such that for every \( M \geq 0 \) there exists \( T^* := T^*(M) > 0 \) such that for all \( k \in \mathbb{N}_0, |T_k| \in \Phi(T^*) \) and \( |x_0| \leq M \) the solutions satisfy \( |x_k| \leq \beta(|x_0|, \sum_{i=0}^{k-1} T_i) + R \). If additionally \( \beta \in \mathcal{KL} \) can be chosen as \( \beta(r, t) := K r e^{(-At)} \) with \( K \geq 1 \) and \( A > 0 \) it is said to be Semiglobally Exponentially Stable-Var (SES-VSR).
Remark 2.6. SLES-VSR implies SS-VSR, as expected (see Lemma 4.2 in Section 4.3). The converse does not hold: SS-VSR implies SPS-VSR but not necessarily LES-VSR because the bound may not be exponential.

DT models that are EPC necessarily share some stability properties. This enables to design a stabilizing control law based on an approximate model with the certainty that it will stabilize the exact model, provided the two models are EPC.

Theorem 2.7 (Proof in Section 4.1). Consider that $(F^a, F^b)$ is EPC. Then

i) $F^a$ is SPS-VSR $\Leftrightarrow$ $F^b$ is SPS-VSR.

ii) $F^a$ is LES-VSR $\Leftrightarrow$ $F^b$ is LES-VSR.

iii) $F^a$ is SLES-VSR $\Leftrightarrow$ $F^b$ is SLES-VSR.

The proof follows from the proofs of [Vallarella et al., 2021] and [Vallarella and Haimovich, 2019] using imposing that no errors affect the control input. [Vallarella et al., 2021] gives conditions for ensuring the required EPC property (REPC therein) between two closed-loop models that correspond to different open-loop models under the same control law, e.g. $(F^a_U, F^b_U)$.

3. Main results

In this section, we present reasonable sufficient conditions on the DT models and on the control laws under which the CT stabilization of the plant $(\bar{U}, \bar{x})$ by a control law $u_c(x)$ implies the sampled-data stabilization by sampling period-dependent control laws of the form $U(x, T)$ for sufficiently small sampling periods, and vice versa. To state the results we need the consistency and regularity conditions presented in Definitions 3.1-3.3.

Definition 3.1. The pair $(U, V)$ is said to be Semiglobally small-time convergent Consistent (StC) if for each $M \geq 0$ there exist a function $p \in \mathcal{K}_\infty$ and $T^* := T^*(M) > 0$ such that for all

$$|x| \leq M \text{ and } T \in [0,T^*)$$

we have

$$|U(x, T) - V(x, T)| \leq \rho(T)|x|.$$  \hspace{1cm} (6)

Definition 3.2. The function $U$ is said to be Semiglobally small-time Lipschitz (StL) if for each $M \geq 0$ there exist $K := K(M) > 0$, $T^* := T^*(M) > 0$ with $T^*(\cdot)$ nonincreasing such that for all $|x|, |y| \leq M$ and $T \in [0,T^*)$ we have $U(0, T) = 0$ and

$$|U(x, T) - U(y, T)| \leq K|x - y|.$$  \hspace{1cm} (7)

Definition 3.3. The DT model $F^a$ is said to be Semiglobally small-time Lipschitz Consistent (StLC) if for each $M, E \geq 0$ there exist $K := K(M, E) > 0$, $T^* := T^*(M, E) > 0$ such that for all $|x|, |y| \leq M$, $|u|, |v| \leq E$, and $T \in [0,T^*)$

$$|F^a(x, u, T) - F^a(y, v, T)| \leq (1 + KT)|x - y| + KT|u - v|.$$  \hspace{1cm} (8)

Theorem 3.4 gives conditions on the control laws to ensure that the closed-loop models arising from feeding back the same open-loop model with two different control laws are EPC.

Theorem 3.4 (Proof in Section 4.2). Suppose that i) $F^a$ is StLC, ii) $U$ is StL, iii) $(U, V)$ is StC. Then $(F^a_U, F^b_U)$ is EPC.

The combination of Theorem 2.7 and Theorem 3.4 provides conditions on the open-loop models and on the control laws so that a stability property ensured for one of the closed-loop models is satisfied also by the other one.

Theorem 3.5 (Proof in Section 4.3). Consider system $(\bar{U}, \bar{x})$, a DT control $U(x, T)$, and the CT control $u_c(x) := U(x, 0)$. Suppose that

i) Assumptions 2.2 and 2.3 hold.

ii) $U$ is StL.

iii) $(U_c, U)$ is StC, with $U_c(x, T) := u_c(x)$ for all $T$.

Then the CT closed-loop plant $(\bar{U})$ is

a) LES $\Leftrightarrow$ $F^a_U$ is LES-VSR.

b) GALES $\Leftrightarrow$ $F^b_U$ is SLES-VSR.

c) GES $\Leftrightarrow$ $F^b_U$ is SES-VSR.

From Theorem 3.5, an exact DT model is SES-VSR if and only if its corresponding CT limit is GES, which is restrictive. Under the same assumptions, Theorem 3.5 links the weaker (GA)LES property of the CT limit with (S)LES-VSR of the DT model. The combination of Theorems 2.7 and 3.5 provides conditions for ensuring the stability of the sampled-data system by means of an approximate DT model and its corresponding DT control. These results allow to establish different types of stability of the sampled-data system for a wide range of situations without necessarily obtaining a Lyapunov function for the DT-closed-loop plant, but by examining its continuous-time limit's stability properties. For an analysis of the intersample behaviour, see e.g. [Vallarella et al., 2021].

As a comment of the presented properties in Definitions 3.1-3.3, note that if the pair $(U, V)$ is StC the control laws $U$ and $V$ become increasingly similar as the sampling period is reduced, and converge to the same value for $T \to 0^+$. If a control law $U(x, T)$ is StC with some law $u_c(x, T) \equiv u_c(x)$, independent of the sampling period, then the equality $u_c(x) = U(x, 0)$ necessarily holds. The main difference between consistency properties EPC and StC lies in the product of an extra $T$. Note that for $x = y$ the EPC bound in (5) is $K|T|x$ while for StC in (6) is $\rho(T)|x|$. Similarly for StLC and StL, by taking $u = v$ the StLC bound in (7) results $(1 + KT)|x - y|$ which converges to $|x - y|$ when $T \to 0^+$. This differs from the StL bound in (7) where the function $K$ not necessarily fulfills $K(0) = 1$.

4. Proofs

4.1. Proof of Theorem 2.7.

i). According to [Vallarella and Haimovich, 2019] Theorem 3.1 the fact that $(F^a, F^b)$ is MSEC is a sufficient condition to ensure that, if the model $\bar{F}^a$ fulfills the Semiglobal
Practical Input-to-State Stability under nonuniform sampling property (SP-ISS-VSR) then \( \tilde{F}^a \) does it too, and viceversa. Note that the SP-ISS-VSR property in the absence of errors reduces to the present SPS-VSR property, i.e. taking \( e = 0 \) in [Vallarella and Haimovich 2019, Theorem 3.1]. Now we will prove that if the pair \((\bar{F}^a, \tilde{F}^a)\) is EPC then it also fulfills the MSEC property in absence of errors, and thus the present result holds. Suppose \( X \subset \mathbb{R}^n \) given and define \( M := \max\{|x| : x \in X\}\). Let the EPC definition generate \( K > 0, T^* > 0 \) and \( \rho \in \mathcal{K}_\infty \) such that
\[
|F^a(x^a, T) - F^a(x^b, T)| \leq (1 + KT)|x^a - x^b| + T\rho(T)\max\{|x^a|, |x^b|\}
\] (9)
for all \(|x^a|, |x^b| \leq M \) and \( T \in (0, T^*) \). Define \( \rho \in \mathcal{K} \) via \( \rho(s) := M\rho(s) \) for \( s \leq s^\star \) and arbitrary for \( s > s^\star \) and \( \sigma = K \) thus, condition (11) in [Vallarella and Haimovich 2019] holds and \((\bar{F}^a, \tilde{F}^a)\) fulfills the MSEC property in absence of errors.

\section{Proof of Theorem 3.3}

Consider \( M \geq 0 \) given and \(|x|, |y| \leq M \). Let the StC property of \((U, V)\) generate \( T^* := T^*(M) > 0 \) and \( \tilde{\rho} \in \mathcal{K}_\infty \) and the StL property of \( U \) generate \( K_U := K_U(M) > 0 \) and \( \tilde{\rho}^U := T^*(M) > 0 \). Thus \( |U(x, T)| \leq K_U M \) for all \(|x| \leq M \) and \( T \in (0, T^*) \). We can bound \(|V(x, T)| = |V(x, T) - U(x, T) + U(x, T)| \leq |V(x, T) - U(x, T)| + |U(x, T)| \leq \tilde{\rho}(T^*)M + K_U =: E \). Let the StLC property of \( F^a \) generate \( K(M, E) > 0 \) and \( T^* := T^*(M, E) > 0 \). Define \( \tilde{K} := K(1 + K_U) \) and \( \rho \in \mathcal{K}_\infty \) via \( \rho(s) := \tilde{K}\rho(s) \). Define \( T^* := \min(T^*, T^*, T^*) \) then, for all \(|x|, |y| \leq M \) and \( T \in [0, T^*) \) we have
\[
|F^a(x, U(x, T), T) - F^a(y, V(y, T), T)| \\
\leq (1 + KT)|x - y| + KT|U(x, T) - V(y, T)| \leq (1 + KT)|x - y| + KK_U|U(x, T) - U(y, T)| + U(y, T) - V(y, T) |y| \leq (1 + KT)|x - y| + KK_U|T| |x - y| + KT\tilde{\rho}(T)|y| \leq (1 + KT)|x - y| + KT\tilde{\rho}(T)\max\{|x|, |y|\}
\] (11)
(12)
In (11) and (12) we have used the facts that \( F^a \) is StLC, \( U \) is StL and \((U, V)\) is StC, respectively.

\section{Proof of Theorem 3.5}

To prove Theorem 3.5 we need Theorem 3.3 and the following Lemma 4.1 and Proposition 4.2. Lemma 4.1 establishes the relationship between CT stability properties of \((2)\) and the DT model \( H^c \), given by its samples.

Lemma 4.1. The CT closed-loop plant \((2)\) is i) \( GAS \Rightarrow H^c \) is SPS-VSR, ii) \( LES \Rightarrow H^c \) is LES-VSR, iii) \( GALES \Rightarrow H^c \) is SLS-VSR, iv) \( GES \Rightarrow H^c \) is SES-VSR.

Proposition 4.2 shows that under Assumptions \(2.2 \) and \( 2.3 \) the easily checkable sufficient conditions of StL for the control law \( U \) and StC for the pair \((U, E)\) ensure that the pair \((H^c, \tilde{F}^c_U)\) is EPC. This establish a correspondence between CT and sampled-data systems via Lemma 4.1.

Proposition 4.2. Under the assumptions of Theorem 3.3 the pair \((H^c, \tilde{F}^c_U)\) is EPC.

Since EPC is not ensured to link SS-VSR between DT models (recall Remark \(2.6 \) and Theorem \(2.7 \)), then the GAS property of \((2)\) is not necessarily equivalent to SS-VSR of \( \tilde{F}^c_U \). Next we use Theorem 4.3, Lemma 4.1 and the following Lemma 4.3 to prove Proposition 4.2. We define \( U_c(x, T) := u_c(x) \) for all \( T \) for notation consistency.

Lemma 4.3. The following implications hold

i) Assumptions \(2.2 \) and \( 2.3 \) \( \Rightarrow h \) is locally Lipschitz.

ii) \( h \) is locally Lipschitz \( \Rightarrow H^c \) is StLC. Moreover, for each \( M \geq 0 \) there exists \( \tilde{T} := \tilde{T}(M) \) such that for all \(|x| \leq M \) and \( T \in (0, \tilde{T}) \) we have \(|H^c(x, T)| \leq 2M\).

iii) \( h \) is locally Lipschitz \( \Rightarrow (H^c, \tilde{F}^c_U) \) is EPC.

iv) Assumption \(2.2 \) \( \Rightarrow F^c \) is StLC.
v) \text{Assumption 2.2 and } U \text{ is SlT}. \\
vii) \text{Condition of Theorem 3.4 hold, thus the result is immediate.}

vii) We will prove that Assumptions 2.1-2.3 (A2.1-2.3 in the following) and conditions i) and ii) of \cite{vall2021theory}. Theorem 3.9) hold. For sake of consistency with the notation of \cite{vall2021theory}, define \( U(x, e, T) := U(x, T) \) for each \( q \in \mathbb{N} \) and all \( e \in \mathbb{R}^q \).

A2.1: It is a direct consequence of Assumption 2.2.

A2.2: Consider \( M, C_0 \geq 0 \) given and \( |x| \leq M \) and \( |u| \leq C_0 \). From Assumption 2.2 define \( L := L(M, C_0) \) and function \( C_f(r, s) := L(r + s) \), then for all \( |x| \leq M \) and \( |u| \leq C_0 \), we have \( |f(x, u)| = |f(x, u) - f(0, 0)| \leq L(|x| + |u|) \leq L(M + C_0) := C_f(M, C_0) \).

A2.3: Consider \( M, E \geq 0 \) given. Define \( L := L(M) \) from Assumption 2.3 and \( K := K(M) \) from the fact that \( U = \text{StL} \). Define \( T_{E}(M, E) := \min \{1, T^{*(M)}\} \) and for all \( |x| \leq M, |e| \leq E \) and \( T \in (0, T_{E}) \), we have \( \|U(x, e, T)\| \leq T_{E}(M, E) \leq K(x) \leq K(M) := C_f(M, C_0) \).

Condition i): The result is immediate due to by assumptions we have \( f(0, U(0, e, T)) = f(0, U(0, T)) = f(0, 0) = 0 \).

Condition ii): From the last Assumption 2.3, and \( T \in (0, T_{E}) \) we have \( \|U(x, e, T)\| \leq C_0(M, E) \). Define \( L := L(M, C_0(M, E)) \). For all \( x_i^T, x_i^T \leq M, |e| \leq E \) and \( T \in (0, T_{E}) \) we have \( \|f(x_i^T, u_i^T, e, T)\| \leq L(x_i^T + |u_i^T|) = L(1 + K) \leq L(1 + K) |x_i^T - x_i^T| + L(|x_i^T| - |x_i^T|) \leq L(1 + K) \). Hence, by \cite{vall2021theory}, Theorem 3.9, \( F_{E}(x, U) \) is EPC. By assuming that no errors affect the control input, i.e. \( e = 0 \), we obtain that the REPC coincide with the EPC property and therefore \( F_{E}(x, U) \) is EPC. Given that \( F_E \) is a particular RK model, \( F_{E}(U), F_{E}(U) \) is EPC and the result follows.

\text{Proof of Proposition 4.4.2}: By the implications in Lemma 4.3 the pairs \( (H^E, F_{E}(U)), (F_{E}(U), F_{E}(U)) \) and \( (F_{E}(U), F_{E}(U)) \) are EPC. By the transitivity of the EPC property \( (H^E, F_{E}(U)) \) is EPC.

\text{Proof of Theorem 5.4.5: } Conditions of Proposition 4.4.2 imply that \( (H^E, F_{E}(U)) \) is EPC. Consider that \( F_{E}(U) \) fulfills the SES-VSR, LES-VSR or SLES-VSR properties. By Theorem 5.4.2 \( H^E \) or \( F_{E}(U) \) fulfills the same properties, respectively. By the implications of Lemma 5.4.1 the result follows.

4.4. Additional results

\text{Lemma 4.4.4, SLES-VSR }\Rightarrow\text{ SS-VSR.}

\text{Proof.} Let the LES-VSR property generate \( K \geq 1 \) and \( R, T^*, \lambda > 0 \). Thus for all \( k \in \mathbb{N}_0, \{T_k\} \in \Phi(T^{*}) \) and \( |x_0| \leq R \) we have \( |x_k| \leq K|x_0|e^{-\lambda \sum_{i=0}^{k-1} T_i} \). Analogously, by the SPS-VSR definition, given \( M > 0 \) there exists \( T^* := T^* := T^* (M, \frac{\sqrt{\lambda}}{\lambda}) > 0 \) such that for all \( k \in \mathbb{N}_0, \{T_k\} \in \Phi(T^{*}) \) and \( |x_0| \leq M \) we have \( |x_k| \leq K|x_0|e^{-\lambda \sum_{i=0}^{k-1} T_i} \). Define \( \tau(s) := \inf \{t \geq 0 : \beta(s, t) \leq \frac{\beta(x, 0)}{\lambda} \} \), \( \beta \in \mathcal{K}_\infty \).

\[ \beta(x, t) := \begin{cases} 2K\beta(s, t) & \text{if } 0 \leq t < \tau(s) \\ \min\{R, 2K\beta(s, 0)\}e^{\lambda t} & \text{if } \tau(s) \leq t \end{cases} \]
and $T^{\delta}(M) := \min \{ T^{\delta} \left( M, \frac{\beta}{\alpha} \right), T^{\delta}(M) \}$. Now we will prove that for all $k \in \mathbb{N}_0$, $T_i \in \Phi(T^{\delta}(M))$ and $|x_0| \leq M$ the solutions satisfy $|x|_k \leq \beta \left( x_0, \frac{1}{\alpha} \sum_{i=0}^{k-1} T_i \right)$.

Define $t_k := \sum_{i=0}^{k-1} T_i$ for the sake of notation. Suppose $|x_0| \leq M$ given and $T^{\delta} = T^{\delta}(M)$. Define $\alpha \in \mathcal{K}$ via $\alpha(s) := \beta(s, 0)$. We will consider two cases according to the $\tau(s)$ definition. First consider that $|x_0| \leq \alpha^{-1} \left( \frac{\beta}{M} \right)$ then $\beta(|x_0|, 0) \leq \frac{\beta}{M}$ and $\tau(|x_0|) = 0$. Thus, from (13) we have $2K\beta(|x_0|, 0) = \min(\mathbb{R}, 2K\beta(|x_0|, 0))$. By the SPS-VSR property, we have that $|x_0| \leq \beta(|x_0|, 0) + \frac{\beta}{M} \leq \frac{\beta}{M} \leq R$. Thus, the bound given by LES-ISR holds. According to (13) we have, for all $k \in \mathbb{N}_0$, $|x_k| \leq \alpha^{-1} \left( \frac{\beta}{M} \right)$ and $(T_i) \in \Phi(T^{\delta}(M))$, that $|x_k| \leq K|x_0|e^{-\frac{\beta}{M} \tau} \leq 2K\beta(|x_0|, 0)e^{-\frac{\beta}{M} \tau} \leq \beta(|x_0|, t_k)$, where we have used the fact that $s \leq \beta(s, 0)$ for all $s \geq 0$.

Now consider the second case, namely $\alpha^{-1} \left( \frac{\beta}{M} \right) < |x_0|$. From the $\tau(s)$ definition we have $|x_0|, t_0 > \frac{\beta}{M}$ for all $t \in [0, \tau(|x_0|))$ with $\tau(|x_0|) > 0$. Thus $R = \min(\mathbb{R}, 2K\beta(|x_0|, 0))$. By the SPS-VSR property and (13) we have for all $\alpha^{-1} \left( \frac{\beta}{M} \right) < |x_0| \leq M$, $(T_i) \in \Phi(T^{\delta}(M))$ and all $k \in \mathbb{N}_0$ that fulfill $t_k \in [0, \tau(|x_0|))$ that $|x_k| \leq \beta(|x_0|, t_k) + \frac{\beta}{M} \leq 2\beta(|x_0|, t_k) \leq 2K\beta(|x_0|, 0) \leq \beta(|x_0|, t_k)$.

On the other hand, for all $k \in \mathbb{N}_0$ that fulfill $\tau(|x_0|) \leq t_k$ we have $|x_k| \leq \beta(|x_0|, \tau(|x_0|)) + \frac{\beta}{M} \leq \frac{\beta}{M} \leq R$, thus the bound given by LES-ISR holds and therefore $|x_k| \leq K\frac{\beta}{M}e^{-\frac{\beta}{M}\tau} \leq R e^{-R\frac{\beta}{M}e^{-\frac{\beta}{M}\tau}} \leq \beta(|x_0|, t_k)$. Thus, $|x_k| \leq \beta(|x_0|, t_k)$ for all $k \in \mathbb{N}_0$, $(T_i) \in \Phi(T^{\delta}(M))$ and $|x_0| \leq M$ which concludes the proof.

Lemma 4.5. StC is an equivalence relation (reflexive, symmetric and transitive).

Proof. The proofs that StC is reflexive and symmetric are immediate. Now we prove that it is transitive. Consider $M \geq 0$ given. Suppose that the pairs $(U^a, U^b)$ and $(U^b, U^c)$ are StC and define $\rho_{ab} \in \mathcal{K}_{bc} \cdot T^{ab} := T^{\delta}(M)$ and $\rho_{bc} \in \mathcal{K}_{bc}$. $T^{bc} := T^{\delta}(M)$ for both pairs, respectively. Define $\rho \in \mathcal{K}_{bc}$ via $\rho(s) := \rho_{ab}(s) + \rho_{bc}(s)$ and $\tilde{T} := \min\{T^{ab}, T^{bc}\}$, then for all $|x| \leq M$ and $T \in (0, T)$ we have

$$
\begin{align*}
[U^a(x, T) - U^b(x, T)] & = \left[ U^b(x, T) - U^{bc}(x, T) + U^{bc}(x, T) - U^c(x, T) \right] \\
& \leq \left[ U^b(x, T) - U^{bc}(x, T) \right] + \left[ U^{bc}(x, T) - U^c(x, T) \right] \\
& \leq \rho_{ab}(T)|x| + \rho_{bc}(T)|x| = \rho(T)|x|.
\end{align*}
$$

5. Conclusions

We presented mild consistency and regularity conditions on the plant and control laws that allow to link the stability of a sampled-data system fed back with sampling-period-dependent control laws with the stability of the CT closed-loop plant obtained in the limit as the sampling period becomes infinitesimally small. The given results extend previous ones under milder assumptions.

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