THE ALEXANDER POLYNOMIAL AS AN INTERSECTION OF TWO CYCLES IN A SYMMETRIC POWER

NIKITA KALININ

Abstract. We consider a braid $\beta$ which acts on a punctured plane. Then we construct a local system on this plane and find a homology cycle $D$ in its symmetric power such that $D \cdot \beta(D)$ coincides with the Alexander polynomial of the plate closure of $\beta$.

Contents
1. Introduction 2
2. Local system of coefficients 3
3. Action of the braid group and homology 4
4. Definition of the invariant via external power 4
5. Examples 4
6. Action of pure braid group generators 5
7. Auxiliary facts 8
8. A proof of Theorem 1 10
9. Skein relation 11
10. Reformulation in the terms of $Symm_{n-1}(D')$ 13
11. Remarks 13
Appendix A. Symmetric power 15
Appendix B. Intersections with coefficients in the local system 15
Appendix C. Determinant of a pseudostochastic matrix 15
Appendix D. Action of $B_{n,n}$ on the total space of $\Theta$ 15
References 16

Date: February 28, 2014.
1. Introduction

Results obtained here.

We present a construction of the Alexander polynomial for a knot $K \subset S^3$; it uses a local system which imitate the action on $\mathbb{Z}[t, t^{-1}]$ on universal abel covering of $S^3 \setminus K$, and this construction is similar to a Floer homology construction: we intersect two manifolds in a symmetric power of a surface. We also obtain a higher Alexander polynomials. In article [S. Bigelow, 2001] (based on R. Lawrence, 1993) the Jones polynomial of a knot is presented as a intersection of two homology classes in a covering over symmetric power of punctured disk; this article is founded on similar ideas. Lawrence’s approach might lead to a construction of higher Jones polynomials, which are currently unknown; that was one of the main motivations for this work.

Acknowledgements.

This work appeared as my graduation thesis in Sankt-Petersburg State University, 2010. I thank my adviser Oleg Yanovich Viro who gave me the problem discussed here and who conducted and encouraged me through university. I thank A. Akopyan, M. Karev, S. Podkorytov, L. Positselsky and participants of the topology seminar for discussions, suggestions and simplifications. Recently S. Bigelow told me that the presented construction is not new and it is definitely known for experts. But since it is hard to find it written explicitly somewhere I translated the article in English. After that, Vincent Florens communicated to me that the same construction of the Alexander polynomial is contained in their paper [S. Bigelow, A. Cattabriga, V. Florens, 2012] (chapter 3), which was firstly written in 2008. Hence this paper may serve mostly as an example of concrete computations.

Short history of Alexander polynomial.

Alexander constructed by a diagram of a knot $K$ the polynomial $\Delta_K(t)$ which doesn’t change (modulo multiplication on $\pm t^k$) with Reidemeister moves, and, therefore, depends only on the isotopic class of $K$ (J. Alexander, 1928). Alexander’s construction works as follows: with each connected component of the complement for the knot projection we associate a variable, with each crossing in the projection we associate an equation on variables corresponding to the components touching this crossing. These equations are linear and contain a formal parameter $t$. This gives us an $n \times (n + 2)$ matrix $M$ consisting of all these equations; here $n$ is the number of crossings on the knot diagram. It happens that if we remove any two columns from $M$ then the determinant $\Delta_K(t)$ of the rest is an invariant (modulo multiplication on $\pm t^k$) with respect to the Reidemeister moves. Afterward, this invariant was called Alexander polynomial.

Each knot $K \subset S^3$ is bounded by an compact oriented surface $S$, which is called Seifert surface of $K$. The genus $g(K)$ of $K$ is, by definition, the minimal possible genus for a Seibert surface spanning $K$. H. Seifert (H. Seifert, 1934) found that $2g(K)$ is bounded from below by degree of Alexander polynomial, this required a new way to calculate the latter. Seifert considered the infinite cyclic covering over $S^3 \setminus K$; it can be obtained if we cut $S^3$ along Seibert surface of $K$ and then glue together a countable number of such a pieces along their boundaries. The cohomology group $H^1$ of the obtained space is a module over $\mathbb{Z}[t, t^{-1}]$, and the determinant of its representation matrix is the Alexander polynomial.

The equivalent way to obtain this determinant is to take loops presenting a basis of the first homology group of the Seibert surface, slightly push them from this surface to one side and then compute linked numbers between loops on the surface and moved ones. Resulting matrix $A$ of linking numbers is called the Seifert matrix. Now we can compute the Alexander polynomial in the following way: $\Delta_K(t) = \text{det}(tA - A^T)$ where $A^T$ is transpose to $A$.

After the work of Alexander J. Alexander, 1928 R. Fox considered a copresentation of the knot group $\pi_1(S^3 \setminus K)$, and introduced non-commutative differential calculus (R. Fox, 1961), which also permits to compute $\Delta_K(t)$. Detailed exposition of this approach and information about higher Alexander polynomials can be found in the book R. Crowell, R. Fox, 1963.
J. Conway ruled out problems with sign in the polynomial and defined it axiomatically via skein-relations [J.Conway, 1967].

There is a number of constructions of the Alexander polynomial via state sums derived from physic models. The first such a construction appeared in the article [L.Kauffman, 1983]. A survey of this topic and other connections with physics are given in [M.Khovanov, 2006].

Some properties of Alexander polynomial.

Alexander polynomial has many beautiful properties, for example, if a knot $K$ is slice then $\Delta_K(t) = f(t)f(t^{-1})$ where $f \in \mathbb{Z}[t] ([R.Fox,J.Milnor, 1966])$, more, the degree of Alexander polynomial is no more than $2g(K)$ ([H.Seifert, 1934], see also [W.Lickorish, 1997]).

Recently it has been found an extraordinary connection between the Alexander polynomial and Seiberg-Witten invariants of smooth four-dimensional manifolds ([R.Fintushel, R.Stern, 1996], Seiberg-Witten invariant specializes on recognition of different smooth structures on the same topological manifold). It happens that if we cut out a neighborhood of a torus from a four-dimensional manifold and glue $S^1 \times (S^3 \setminus K)$ in the obtained free space, then, with some additional assumptions, the obtained manifold will be homeomorphic to the one we start with and Seiberg-Witten invariant will change by multiplication on $\Delta_K(t)$.

2. LOCAL SYSTEM OF COEFFICIENTS

Let $D \subset \mathbb{R}^2$ be a unit disk on the plane, and let $p_1, p_2, \ldots, p_{2n} \in D$ be a collection of marked points which lay on the $x$-axis in index increasing order: $-1 < p_1 < p_2 < \ldots < p_{2n} < 1$. Let us color points with odd indices in blue, and with even indices in red. Denote $D' = \mathbb{R}^2 \setminus \bigcup p_i$, we shall call $D'$ punctured disk, though $D'$ is only homotopically equivalent to it.

In the following text we treat $t$ as a formal variable. Let us consider a local system $\Theta$ on $D'$ with fiber $\mathbb{Z}[\lfloor \frac{1}{2}, t^{-\frac{1}{2}} \rfloor]$ : a small counterclockwise rotation around a point $p_i$ corresponds to multiplication by $t^{-i\frac{1}{2}+1}$ in the fiber.

**Proposition 2.1.** Let us choose the following trivialization of $\Theta$ on $x$-axis: on $(-\infty, 0), (+\infty, 0) \cap D'$ it will be the product $((-\infty, 0), (+\infty, 0)) \cap D' \times \mathbb{Z}[\lfloor \frac{1}{2}, t^{-\frac{1}{2}} \rfloor]$. Further, for points $(x, 0), (y, 0)$ such that $x < p_i < y$ and $[x, y]$ contains no marked points except $p_i$, the path from $(x, 0)$ to $(y, 0)$ by semicircle in the bottom (upper) half-plane gives multiplication by $t^{(-i)\frac{1}{2}+1}$ (by $t^{(-i)\frac{1}{2}}$, respectively) in the fiber. Such a semicircles are called basic semi-circles.

So, the local system is fixed on the $x$-axe and we will not need its concretization on the rest of the plane. Notice that continuation of $\Theta$ from $x$-axe to the plane is homotopically unique because the bottom and the upper half-planes are simply-connected.

**Remark 2.2.** The expression ”total space of $\Theta$” means total space of the local system $\Theta$. The fibers of $\Theta$ are equipped with discrete topology, therefore the total space of $\Theta$ is a covering.

In $D'$ we consider two types of circles; $s'_i (i = 1..n)$ has as diameter an interval, slightly bigger than interval $[p_{2i-1}, p_{2i}]$, and therefore $s'_i$ contains no marked points except $p_{2i-1}, p_{2i}$; and $d'_i (i = 1..n-1)$ has as a diameter an interval slightly bigger than $[p_{2i}, p_{2i+1}]$, and hence $d'_i$ contains no marked points except $p_{2i}, p_{2i+1}$. We orient all these circles counterclockwise.

It is clear that we can lift $s'_i, d'_i$ into the total space of the local system such a way that the points of liftings over $x$-axis have coordinate 1 in the fiber of local system. Denote these liftings by $s_i$ and $d_i$ correspondingly.
Proposition 2.3. The first homology group (with coefficients in local system $\Theta$) $H_1(D'; \Theta)$ of punctured disk is freely generated (as a module over $\mathbb{Z}[t^{1/2}, t^{-1/2}]$) by the set $\{s_i (i = 1..n), d_i (i = 1..n - 1)\}$.

Proof. Let $H_{k>1}(D'; \Theta) = 0$ follows from the fact that $D'$ is homotopically equivalent to a bouquet of $2n$ circles. By $C_1, C_2, \ldots C_{2n}$ we denote the 1-cells of the bouquet, and by $x$ we denote its 0-cell. Then we orient all $C_i$ counterclockwise and lift them in the total space of $\Theta$: all liftings $C_i'$ will start from a point $x$ with coordinate 1 in the fiber over $x$. By the construction we have $s_i = C_{2i-1}' + t^{1/2} C_{2i}'$ and $d_i = C_{2i-1}' + t^{-1/2} C_{2i}'$. Now we consider any exact 1-chain $a = \sum a_i C_i$. Exactness means that $\sum a_{2i-1}(t^{1/2} - 1) + \sum a_{2i}(t^{-1/2} - 1) = 0$. Now we can subtract the exact chain $a_1 s_1$ from this expression and this kills $C_1$ in $a$. Then we subtract $d_1, s_2, d_2, s_3, \ldots$ with some coefficients. Thus we proved that all such expressions are generated by $s_i, d_i$. 

Remark 2.4. One can easily show that $H_0(D'; \Theta) = \mathbb{Z}[t^{1/2}, t^{-1/2}]/(t^{1/2} - 1) = \mathbb{Z}$.

Let $pr : \Theta \to D'$ be a projection, and we consider a loop $l$ in the total space of local system $\Theta$.

Proposition 2.5. Knowing the point $x \in l$ in some fiber and $pr(l)$ in the plane we can uniquely determine $l$.

This follows from the homotopy lifting property. 

3. Action of the braid group and homology

Let $B_{n,n}$ be the colored braid group with $2n$ strings, and $n$ strings are colored in blue and other $n$ strings are colored in red. In this article we consider only braids with such a coloring. Consider the group $Aut(D')$ of autodiffeomorphisms of $D$ which send marked point to a marked point preserving colors. In other words, $B_{n,n}$ is $Aut(D')$ modulo homotopy equivalence. Let us read braids from bottom to up. Below we explain how $B_{n,n}$ acts on $\Theta$.

Remark 3.1. How to understand figures. It is convenient to depict (up to homotopy) a loop $l$ in the total space $\Theta$ using the following convention: take the projection of $l$ in $D'$, deform it homotopically into basic semicircles (see Proposition 2.4) and draw the result. Therefore we already depicted $pr(l)$; its place in the total space of $\Theta$ is encoded by coordinates of $l$ in the fibers over intersection of $pr(l)$ with $x$-axis, see Fig 1.

Proposition 3.2. Action of the group $B_{n,n}$ can be naturally (see Appendix A) propagated on $\Theta$.

Let $\beta \in B_{n,n}$ be a colored braid, $l$ be a loop in the total space of $\Theta$. We know the action of $\beta$ on $D'$, therefore $pr(\beta l) = \beta \cdot pr(l)$. To completely determine $\beta l$ we need only one point in $\beta l$. To do this note that we can choose a concrete representative of $\beta$ which don’t act outside of a small neighborhood of the interval $[p_1, p_{2n}]$. The we anchor $l$, that means that in homotopy class of $l$ we chose a loop which passes through $(-1,0)$. Now we know that $\beta((-1,0)) = (-1,0)$ therefore the coordinate in fiber of $l$ and $\beta(l)$ are the same (see Fig 2), and then we use lifting homotopy property according to Proposition 4.
THE ALEXANDER POLYNOMIAL AS AN INTERSECTION OF TWO CYCLES IN A SYMMETRIC POWER

Figure 2. An anchored cycle.

1

Figure 3. Plat closure of a braid colored in two colors

Remark 3.3. In the following definition the class $r_i$ represents the intersection index with interval $[p_{2i-1}, p_{2i}]$, taken with respect to local system, see Appendix [B].

Definition 3.4. By definition by $r_i \in H^1(D'; \Theta^*)$ we denote the cohomological class, such that:

$\begin{align*}
r_i \times s_j &= 0 \quad (j = 1..n), r_i \times d_{i-1} = -1, r_i \times d_i = 1, r_i \times d_j = 0 \quad (j \neq i - 1, i)
\end{align*}$

Proposition 3.5. A class $\beta s_i$ is a linear combination of $s_j, d_j$ with coefficients in $\mathbb{Z} \left[ t^\frac{k}{2}, t^{-\frac{k}{2}} \right]$, therefore $r_k \times \beta s_i$ is an element of $\mathbb{Z} \left[ t^\frac{k}{2}, t^{-\frac{k}{2}} \right]$.

Proof. Let us anchor the all generators of the homology group, see Fig. 1. Consider a loop $\beta s_i$. At point $-1$ the value of $\beta s_i$ is 1, therefore using the description of local system (Proposition 2.1) the value of $\beta s_i$ in fiber at all the intersections of $\beta s_i$ with $x$-axis will be powers of $t^\frac{1}{2}$.

Examples can be found on page 6, the other proof of this proposition will be clear when we shall consider the action of braid group generators on $s_i, d_i$ in explicit way, see page 10.

4. Definition of the invariant via external power

Consider $n - 1$-th external power of (co)homologies of the punctured disc. We recall that $s_i$ morally is a small circle containing $[p_{2i-1}, p_{2i}]$ and $r_i$ morally is $[p_{2i-1}, p_{2i}]$.

Define the polynomial

$V_\beta = (n - 1)! (r_1 \wedge r_2 \wedge \cdots \wedge r_{n-1}) \times (\beta s_1 \wedge \cdots \wedge \beta s_{n-1}) =$

$= \det(r_i(\beta s_j)) = \sum_{\sigma \in S_{n-1}} \varepsilon(\sigma) \prod_{i=1}^{n-1} r_i \times \beta s_{\sigma(i)}.$

Take an arbitrary oriented link $L$. It can be presented as a plat closure of a braid $\beta \in B_{n,n}$, in such a way that blue strings (growing from the points with odd indices) will be oriented from the bottom to the top of this braid and red ones – oppositely: draw the link in a space and then push all local minimum to bottom and stretch local maximum to up.

Put, by definition $V_L = V_\beta \cdot e(\beta) \cdot (-1)^n$, where $e(\beta)$ is a coefficient of the type $(-1)^k t^l$, it is defined on the page 10.

The main results of this paper are

Theorem 1. The polynomial $V_L$ is well-defined, i.e. it does not depend on presentation of $L$ as a plat closure of some braid $\beta$.

Theorem 2. The $V_L$ is the Alexander polynomial of link $L$ in Conway normalization (we will prove it by skein relations).
Remark 6.1. We shall add one more cycle to our collections $s_i, d_i$. Let us take a circle with diameter slightly bigger then $[p_1, p_{2n}]$ and push it upwards such that the area bounded by deformed circle $d_n$ contains no marked points except $p_{2n}, p_1$ (see Fig. 5). Then in the group $H_1(D'; \Theta)$ we have

$$\sum_{i=1..n} d_i = t^{-\frac{1}{2}} \sum_{i=1..n} s_i$$

On Figs. 4, 5 the borrom part of $\sum_{i=1..n} s_i$ is depicted. If we draw $\sum_{i=1..n-1} d_i$ we will see the same set of curves but the intersection points will be multiplied par $t^{-\frac{1}{2}}$. \(\square\)

On can imagine that the $p_1, \ldots, p_{2n}$ are arranged on a circle and $p_{2n}$ is automatically sits near $p_1$ and all $d_i(i=1..n)$ can be defined in one way.

The generators of colored braid group are the following:

\[\begin{cases} 
  a) \quad M_i = \sigma_{2i-1}^{-1} \sigma_{2i} \sigma_{2i-1} = \sigma_{2i} \sigma_{2i-1} \sigma_{2i} \\
  b) \quad N_i = \sigma_2 \sigma_{2i+1} \sigma_{2i} = \sigma_{2i+1} \sigma_2 \sigma_{2i+1} \\
  c) \quad P_i = \sigma_{2i}^2 \\
  d) \quad Q_i = \sigma_{2i-1}^{-1}
\end{cases}\]
There are relations on them:

\[(M_i P_i^{-1} N_i)Q_i (M_i P_i^{-1} N_i)^{-1} = Q_{i+1}, (N_i Q_i^{-1} M_i + 1)P_i (N_i Q_i^{-1} M_i + 1)^{-1} = P_{i+1}\]

**Proposition 6.2.** Let us find the action of the colored braid group generators on the generators of \(H_1(D'; \Theta)\).

a) \(M_i\): See Fig. 6 It is clear that \(s_i \rightarrow t^{-\frac{1}{2}}d_i\); breaking \(\beta s_{i+1}\) into basic semicircles, as recommended in Remark 3.1 about pictures, we verify the equality \(s_{i+1} \rightarrow s_i + s_{i+1} - t^{-\frac{1}{2}}d_i\). The second row on the Figure the action of \(M_i\) on \(d_{i-1}, d_i\) is depicted.

\[M_i = \sigma_{2i-1} \sigma_{2i} \sigma_{2i-1} \begin{cases}
  d_{i-1} \rightarrow d_{i-1} + d_i - t^{-\frac{1}{2}}s_i \\
  s_i \rightarrow t^{-\frac{1}{2}}d_i \\
  d_i \rightarrow t^{-\frac{1}{2}}s_i \\
  s_{i+1} \rightarrow s_i + s_{i+1} - t^{-\frac{1}{2}}d_i
\end{cases}\]

b) \(N_i\): The action of the following generator is given from the previous one by substitutions \(s_i \rightarrow d_i, d_{i-1} \rightarrow s_i, s_{i+1} \rightarrow d_{i+1}, d_i \rightarrow s_{i+1}, t^{-\frac{1}{2}} \rightarrow t^{\frac{1}{2}}\).

\[N_i = \sigma_{2i} \sigma_{2i+1} \sigma_{2i} \begin{cases}
  s_i \rightarrow s_i + s_{i+1} - t^{\frac{1}{2}}d_i \\
  d_i \rightarrow t^{\frac{1}{2}}s_{i+1} \\
  s_{i+1} \rightarrow t^{\frac{1}{2}}d_i \\
  d_{i+1} \rightarrow d_i + d_{i+1} - t^{\frac{1}{2}}s_{i+1}
\end{cases}\]

c) \(P_i\): see Fig. 7

\[P_i = \sigma_{2i}^2 \begin{cases}
  s_i \rightarrow s_i + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})d_i \\
  d_i \rightarrow d_i \\
  s_{i+1} \rightarrow s_{i+1} - (t^{-\frac{1}{2}} - t^{\frac{1}{2}})d_i
\end{cases}\]
7. Auxiliary facts

Let us write $\beta$ as a matrix $B$, which acts on a linear space $W$ with basis $s_i, d_i$ ($i = 1..n$), odd rows encode the action of $\beta$ on $s_i$.

**Proposition 7.1.** $V_\beta$ is the determinant of the submatrix of $B$, which consists of $n - 1$ odd rows and $n - 1$ even columns.

Indeed, in the definition of $V_\beta$ we see $|\{r_i\}| = n - 1, |\{d_i\}| = n - 1$, therefore they must be paired. $r_i \times d_j = 0 (1 \leq j \leq n - 1, j \neq 1)$, therefore $r_1$ corresponds to $d_1$ (i.e. the first even column). Hence $r_2$ corresponds to $d_2$, because $d_3$ is already in a pair, etc. Now it is easy to see that the definition of $V_\beta$ is exactly the definition of the considered minor.

**Theorem 3.** $V_L$ does not depend on a collection of $n - 1$ odd rows (there are $n$ of them) which we take in submatrix. The first minors of $B$ will differ only by sign.

The proof consists of two lemmas:

**Lemma 7.2.** A submatrix $B'$ which consists of all odd rows and even columns of $B$ satisfies the following criterion: The sum of elements in each row is zero. The sum of elements in each column is zero. Let us call such matrices pseudostochastic.

Proof. It follows from the previous section that $\sum s_i$ is invariant under the action of colored braid group. Therefore $r_k \times \sum i \beta s_i = r_k \times \sum i s_i = 0$ and the fact about columns is proved.

Let us prove the condition about rows. For each $k$ we have $\sum r_i \times s_k = 0, \sum r_i \times d_k = 0$. Therefore $\sum r_i = 0$ in $H^1(D'; \Theta^*)$ and hence $\sum r_i \times \beta s_k = 0$. \(\square\)

**Lemma 7.3.** All the first minors of a pseudostochastic matrix $B$ are equal modulo sign change.

Proof. It follows from the definition of a pseudostochastic matrix that rows of $B$ (as vectors) lie in the hyperplane $\sum x_i = 0$ and their center of mass is 0. A first minor of $B$ is the oriented volume of a simplex consisting of $n - 1$ vectors and 0, projected on a coordinate hyperplane. All such volumes are equal because 0 is the center of mass on these $n$ points, all angles between coordinate hyperplanes and the hyperplane $\sum x_i = 0$ are also equal that finishes this proof. \(\square\)

**Corollary 7.4.** It follows from this theorem that $V_L$ also doesn’t depend on a particular collection of columns (i.e. $r_i$) the number of them must be $n - 1$ and that is all.

**Corollary 7.5.** a) Let $\eta_s, \eta_d: W \to W$ be the braids (presented as linear maps) that

$$\eta_s(s_i) = s_i (i = 1..n), \eta_d(d_i) = d_i + s s_{i,1} (i = 1..n), \eta_d(s_i) = s s_{i,2} (i = 1..n)$$

where $s s_{i,*}$ is any vector depending only on $s_j (j = 1..n)$. Then $V_\beta = V_{\eta_s, \beta} = V_{\beta \eta_d}$.

b) **Important Corollary** Let us consider more general situation. By $W_s$ we denote the subspace of $W$ generated by $s_1, \ldots s_n$; by $W_d$ we denote the subspace generated by $d_1, \ldots, d_n$. Suppose that $W_s$ and $W_d$ are invariant under the action of $\eta_s$, and $W_s$ is invariant under the action of $\eta_d$. Then

$$V_{\eta_s, \beta} = \det(\eta_s|_{W_s}) \cdot V_\beta, V_{\beta \eta_d} = \det(\eta_d|_{W_d}) \cdot V_\beta$$

\(^1\) Other proof can be found on page 15.
Lemma 8.4. Let \( \beta, \beta_2 \) hanging a circle is adding to a braid

Definition 8.3. \( r \)

Definition 8.5. \( \beta \)

Elements of \( V \)

Lemma 8.6. \( (J. \text{Birman, Theorem 1', [J.Birman, 1976]}) \). Two oriented braids \( \beta_1 \in B_{2n_1}, \beta_2 \in B_{2n_2} \) have the same plat closures if and only if after adding a number of trivial circle components \( \beta_1 \to \beta'_1 \in B_{2n}, \beta_2 \to \beta'_2 \in B_{2n} \) braids \( \beta'_1, \beta'_2 \) will lie in the same coset in \( K_{2n} \), i.e.

\[ \beta'_1 = g \beta'_2 h, \beta \in B_{2n}, g, h \in K_{2n} \]

Lemma 8.4. Let \( \beta \) be a colored braid, \( \beta' \) is obtained from \( \beta \) by hanging a circle. Let \( L, L' \) be the plat closures of \( \beta, \beta' \) correspondingly. Therefore \( V_L = V_L \cdot (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \)

Proof. \( V_\beta = (n-1)! \langle r_1 \wedge r_2 \wedge \cdots \wedge r_{n-1} \rangle \times (\beta s_1 \wedge \cdots \wedge \beta s_{n-1} \rangle \), in order to obtain \( V_{\beta'} \) we add \( s_{n+1} \) and \( r_{n+1} \). Therefore \( r_{n+1} \times \sigma^2_{2n} (s_{n+1}) = r_{n+1} \times (s_{n+1} + (t^\frac{1}{2} - t^{-\frac{1}{2}})d_n) = - (t^\frac{1}{2} - t^{-\frac{1}{2}}), r_{i=1..n-1} \times \sigma^2_{2n} (s_{n+1}) = 0 \), and the number of components is increased by 1 and is compensated by \( (-1)^n \) in the definition of \( V_L \).

Denote \( \sigma^2_{2i} \sigma_2 = A, \sigma_{2i-1} \sigma_{2i+1} \sigma_2 = B_i \).

Definition 8.5. Let \( R_n \) be the subgroup generated by the following elements:

\[
\begin{align*}
(a) & \quad \sigma^2_{2i-1}, \\
(b) & \quad \sigma_1 A \sigma_1, \\
(c) & \quad A, \\
(d) & \quad B_i, \\
(e) & \quad \sigma_{2i-1} B_i \sigma_{2i+1}, \sigma_{2i+1} B_i \sigma_{2i-1}
\end{align*}
\]

Lemma 8.6. Elements of \( R_n \) respect colors and orientation of plat closure, adding them to the bottom or top of \( \beta \) doesn’t change \( V_L \) (we call the new braid as \( \beta' \)).

In the following proof we use explicit formulas for action (Proposition D) and important Corollary

7.5

a) \( \sigma^2_{2i-1} \), applied in the bottom doesn’t change \( \beta s_i \) at all, and if we apply it in the very top it adds to \( d_i, d_{i-1} \) some amount of \( s_i \), this have no influence on \( V_\beta \).

b) \( \sigma_1 A \sigma_1 = \sigma_1 \sigma_2 \sigma_1 \cdot \sigma_1 \sigma_2 \sigma_1 \) explicit formulas are the following:

\[
\sigma_1 A \sigma_1 \begin{cases}
  d_0 \to d_0 + (1 - t^{-1})d_1 \\
  s_1 \to t^{-1} s_1 \\
  d_1 \to t^{-1} d_1 \\
  s_2 \to s_2 + s_1 (1 - t^{-1})
\end{cases}
\]
The matrix of action on $d_0, d_1$ (in which we are interested when we add $\sigma_1 A \sigma_1$ to the very top of the braid) is the same as the matrix of action on $s_2, s_1$ which corresponds to adding to the bottom.

\[
\begin{pmatrix}
1 & 1 - t^{-1} \\
0 & t^{-1}
\end{pmatrix}
\]

Matrix determinant equals $t^{-1}$, therefore $V_\beta = V_\beta \cdot t^{-1}$.

c) $A = \sigma_2 \sigma_1^2 \sigma_2 = \sigma_1 \sigma_2 \sigma_1 \cdot \sigma_2^2 \cdot \sigma_1 \sigma_2 \sigma_1$.

\[
A = \begin{cases}
    d_0 \to d_0 + (1 - t^{-1})d_1 + (t^{-\frac{3}{2}} - t^{-\frac{1}{2}})s_1 \\
    s_1 \to t^{-1}s_1 \\
    d_1 \to t^{-1}d_1 - (t^{-\frac{3}{2}} - t^{-\frac{1}{2}})s_1 \\
    s_2 \to s_2 + (1 - t^{-1})s_1
\end{cases}
\]

Analogous to b) we apply important Corollary \[\text{Corollary} \]

d) $B_i = \sigma_2 \sigma_2 i \sigma_2 i + 1 \sigma_2 i = \sigma_2 i - 1 \sigma_2 i \sigma_2 i - 1 \cdot \sigma_2^2 i \cdot \sigma_2 i + 1 \sigma_2 i$.

\[
B_i = \begin{cases}
    d_{i-1} \to d_{i-1} + d_i - t^{-\frac{1}{2}}s_i \\
    s_i \to s_{i+1} \\
    d_i \to t^\frac{1}{2}s_i + d_i + t^\frac{1}{2}s_{i+1} \\
    s_{i+1} \to s_i \\
    d_{i+1} \to d_i + d_{i+1} - t^\frac{1}{2}s_{i+1}
\end{cases}
\]

If we add this transformation to the very bottom of the braid such that $s_i$ switch with $s_{i+1}$ then this influences $V_\beta$ by multiplication on $-1$. Consider the case when we insert it in the top of the braid, let us look on the corresponding matrix; how does it act on vectors $d_{i-1}, d_i, d_{i+1}$.

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\]

Its determinant equals -1. Therefore $V_\beta' = -V_\beta$.

e) The both elements have the same action modulo important Corollary \[\text{Corollary} \]

we write all for only one of them:

\[
\sigma_2 i - 1 B_i \sigma_2 i + 1 = \begin{cases}
    d_{i-1} \to d_{i-1} - t^{-\frac{1}{2}}s_i + d_i \\
    s_i \to s_{i+1} \\
    d_i \to t^\frac{1}{2}s_i + d_i + t^\frac{1}{2}s_{i+1} - d_i \\
    s_{i+1} \to s_i \\
    d_{i+1} \to d_i + d_{i+1} - t^\frac{1}{2}s_{i+1}
\end{cases}
\]

Then the same as in d).

Now we define $e(\beta)$ in such a way that it kills the coefficients appeared in the above considerations. So, if we switch two blue strings we multiply by $t^\frac{1}{2}$, if we switch two red strings we multiply by $-t^{-\frac{1}{2}}$. In the cases b), c) we get $t^\frac{1}{2} \cdot t^\frac{1}{2} = t$, and for d), e) we get $t^\frac{1}{2} \cdot (-t^{-\frac{1}{2}}) = -1$; hanging of a circle does nothing.

Now we give a formal definition of $e(\beta)$: let $\beta_1 \in B_n$ be a braid consisting of only blue strings of $\beta$, and $\beta_2 \in B_n$ of only red ones. Consider the homomorphism $a: B_n \to \mathbb{Z}$, which sends all standard generators to 1. Then $e(\beta) = (t^\frac{1}{2})^{a(\beta_1)} \cdot (-t^{-\frac{1}{2}})^{a(\beta_2)}$.

\[\Box\]

Lemma 8.7. The subset of $K_n$, consisting of elements preserving coloring, is $R_n$. 

Proof. Let \( g \in K_n \) preserve coloring. We present \( g \) as a collection of exemplars \( A, B \) glued in a line with posers \( \sigma_i^k \).

Further we nip off generators (mentioned in the previous lemma) of \( R_n \) from the left of \( g \) and we use that \( \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| > 1 \). Always the following is true: \( g, h \) is glued from \( A, B \) using powers of \( \sigma_{2k-1} \). Killed all \( A, B \), we get \( \prod_i \sigma_{2k_i-1} \), and since \( g \) preserves colorings of braids we get 1 in the braid group.

**Lemma 8.8.** Let \( \beta' = g\beta h, \beta, \beta' \in B_{n,n}, g, h \in K_{2n} \), and plat closures of \( \beta \) \& \( \beta' \) are the same. Therefore \( g, h \in R_n \), and, consequently, \( e(\beta)V_\beta = e(\beta')V_{\beta'} \).

Proof. Consider cosets of \( K_n \) by subgroup which preserve coloring and orientation of plat closure, i.e. by \( R_n \).

Note that \( \sigma_1 \) switches points in pairs and other elements map pairs to pairs. Therefore coloring could break only of in some pairs the points are not in the right order. Now it is clear that cosets \( K_n \) by \( R_n \) corresponds to products of \( \sigma_{2k_i-1} \) which switch points in pairs.

Therefore by nipping of \( R_n \) on the right and on the left from \( g\beta h \), we get \( \prod_i \sigma_{2k_i-1} \beta \prod_j \sigma_{2l_j-1} \).

Now the key point: the link \( \prod_i \sigma_{2k_i-1} \beta \prod_j \sigma_{2l_j-1} \) differs from the link \( \beta \) by orientation of some components; we remind that strings from \( p_{2i+1} \) are oriented from bottom to top, and appositively for strings from \( p_{2i} \). Since we are looking for an invariant for oriented links we demand that transformation do not change the orientation. Therefore we get \( \beta \), and lemma is proved.

**Theorem 4.** \( V_L \) does not depend on \( \beta \), and depend only on \( L \).

Proof. Suppose \( L \) is presented as the plat closure of braids \( \beta_1, \beta_2 \). Using result of Birman we hang on \( \beta_1, \beta_2 \) appropriate (but equal) number of circles we get braids \( \beta_1', \beta_2' \in B_{n,n} \) which lie in the same coset of \( K_{2n} \). The above consideration assert that if \( \beta_1', \beta_2' \) are colored braids with equal plat closures then \( \beta_1', \beta_2' \in B_{n,n} \) lie in the same coset by subgroup \( R_n \), and \( V_L \) depend only on such a coset by Lemma [8.8].

**9. Skein relation**

Consider a crossing between two strings of the same color, in the bottom of the braid. In order to axiomatically define an Conway-type invariant it is enough to prove skein relation only for such a crossing.

**Theorem 9.1.** We prove a direct analog of the relation \( V_{L_+} - V_{L_-} = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{L_0} \), namely \( e(\beta_+)V_{\beta_+} - e(\beta_-)V_{\beta_-} = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})e(\beta_0)V_{\beta_0} \), where

\[
\begin{align*}
a) \quad & \beta_+ = \sigma_2 \sigma_2^{-1} \sigma_2 \beta, \quad \beta_- = \sigma_2 \sigma_2^{-1} \sigma_2 \beta, \quad \beta_0 = \sigma_2^2 \beta \\
b) \quad & \beta_+ = \sigma_2 \sigma_2^{-1} \sigma_2 \beta, \quad \beta_- = \sigma_2 \sigma_2^{-1} \sigma_2 \beta, \quad \beta_0 = \sigma_2^2 \beta
\end{align*}
\]

where \( \beta \) – is an arbitrary considered braid.

On Fig.9 the bottom part of a braid is depicted: a crossing and its resolutions; these two strings are of the same color therefore the resolution knows about their orientation.
Remark 9.1. Let \( \beta, \eta \) be two braids, hence

\[
V_{\eta\beta} = \sum_{\sigma \in S_{n-1}} e(\sigma) \prod_{i=1}^{n-1} r_i \times \beta(\eta s_{\sigma(i)})
\]

According to this remark in order to compute \( V_{\beta_+} \), we should replace \( \ldots \land \beta s_i \land \beta s_{i+1} \ldots \) with \( \ldots \land \beta(t^\frac{1}{2}d_i - (t - 1)s_i) \land \beta(s_i + s_{i+1} - t^\frac{1}{2}d_i + (t - 1)s_i) \land \ldots \) in the definition of \( V_\beta \). We are interested only in the right hand side of definition, i.e. in \( \beta s_1 \land \ldots \land \beta s_{n-1} \). Since \( \beta \) is a linear map, we have, making abuse in notation (we don’t write \( \beta \) everywhere!), we look on difference between \( \ldots \land s_i \land s_{i+1} \land \ldots \) and \( \ldots \land (t^\frac{1}{2}d_i - (t - 1)s_i) \land (s_i + s_{i+1} - t^\frac{1}{2}d_i + (t - 1)s_i) \land \ldots \)

Notice that \( e(\beta_+) = e(\beta)^{-\frac{1}{2}}, e(\beta_-) = e(\beta)^{\frac{1}{2}}, e(\beta_0) = e(\beta) \).

Therefore \( e(\beta_+)V_{\beta_+} - e(\beta_-)V_{\beta_-} = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})e(\beta_0)V_{\beta_0} \) becomes, after cancellation

\[
t^{-\frac{1}{2}}(((1-t)s_i + t^\frac{1}{2}d_i) \land (s_i + t^\frac{1}{2}d_i)) - t^\frac{1}{2}((t^{-\frac{1}{2}}d_i \land (s_i + s_{i+1} - t^{-\frac{1}{2}}d_i)) =
\]

\[
= (t^{-\frac{1}{2}} - t^{\frac{1}{2}})s_i \land s_{i+1}
\]

by direct calculation.

In the case a), analogously we get the following identity which is also true:

\[
(-t^\frac{1}{2})((s_i + t^{-1}s_{i+1} - t^{-\frac{1}{2}}(t - 1 + t^{-1})d_i) \land ((1 - t^{-1})s_{i+1} + t^{-\frac{1}{2}}(t - 1 + t^{-1})d_i)) -
\]

\[
-(t^{-\frac{1}{2}})((s_i - t^\frac{1}{2}d_i + s_{i+1}) \land t^\frac{1}{2}d_i) =
\]

\[
= (t^{-\frac{1}{2}} - t^{\frac{1}{2}})((s_i + (t^{-\frac{1}{2}} - t^\frac{1}{2})d_i) \land (s_{i+1} - (t^{-\frac{1}{2}} - t^\frac{1}{2})d_i)) \square
\]
10. Reformulation in the terms of $\text{Symm}_{n-1}(D')$

Let $\text{Symm}_k(M)$ be the space of all unordered tuples of $k$ points (points may coincide) in topological space $M$.

$$\text{Symm}_k(M) = M^k/S_k, M^k = \frac{M \times M \times \ldots \times M}{k \text{ summands}}$$

$S_k$ acts on the product by permutations.

Notice that a local system $\Phi$ on $M$ with fiber $C$ canonically lifts to $\text{Symm}_k(M)$. $\Phi$ is given, up to homotopy, by a homomorphism $l: \pi_1(M) \to \text{Aut}(C)$ from fundamental group of $M$ to the group of automorphisms of the fiber; since $\text{Aut}(C)$ is abelian, it is enough to fix a homomorphism $l: H_1(M; \mathbb{Z}) \to \text{Aut}(C)$; indeed, inclusion $M \to M^k$, as a first term, induces the map $M \to \text{Symm}_k M$, the latter gives the homomorphism $f: H_1(M; \mathbb{Z}) \to H_1(\text{Symm}_k M; \mathbb{Z})$, which is, in fact, an isomorphism. (For a proof use the fact that $\text{Aut}(C)$ is abelian, it is enough to fix a homomorphism $l: H_1(\text{Symm}_k M; \mathbb{Z}) \to \mathbb{Z}$, see Appendix C.). Put, by definition, $l(f(s)) = l(s), s \in H_1(M; \mathbb{Z})$. By $\Phi$ we denote the obtained local system on $\text{Symm}_k(M)$.

Returning to our question, we consider $\text{Symm}_{n-1}(D')$ and lift on it the local system $\Theta$. Homology with coefficients in the local system $\Theta$ of the punctured disc $D'$ concentrated in the dimension 1, therefore

$$H_*(\text{Symm}_{n-1}(D'); \tilde{\Theta}) = \bigwedge H_1(D'; \Theta)$$

as for all spaces with cohomology in the odd dimensions; in the right side we have external product of external powers of odd-dimensional (co)homology of considered space.

Thus, $V_\beta$ is the value of an element $r \in H^{n-1}(\text{Symm}(D'); \tilde{\Theta}^*)$ on certain element $S \in H_{n-1}(\text{Symm}_{n-1}(D'); \tilde{\Theta})$.

An equivalent definition is the following: replace $r$ by duality on a relative homology class $R \in H_{n-1}(\text{Symm}_{n-1}(D'); \tilde{\Theta})$ because $\dim \text{Symm}_{n-1}(D') = 2n - 2$; therefore $V_\beta$ is the intersection of these classes, i.e. $< R, S >$.

Now we can give more geometric definition of Alexander polynomial:

**Definition 10.1.** Consider two submanifolds in $\text{Symm}_{n-1}(D')$: the first, $S$, each its point is a set of $n - 1$ points in $D'$, one point in each $s_1, \ldots, s_{n-1}$, and the second $R$, each its point is a set of $n - 1$ points in $D'$, each one from each interval $[p_1, p_2], \ldots, [p_{2i-3}, p_{2i-2}]$. Let $\beta$ be a braid, plat closure is oriented link $L$. Action of $\beta$ lifts from $D'$ to $\text{Symm}_{n-1}(D')$ in the same way as we lift local system.

Now, $\Delta_L(t) = < R, \beta S >$.

11. Remarks

1. The construction of Bigelow [S.Bigelow, 2001] is done on language of coverings, but can easily be reinterpreted in terms of a local system on $D'$ where counterclockwise moving around a marked point gives a multiplication by $t$ in fiber, the we lift this local system to symmetric power and moving around the diagonal gives multiplication by $-t^{-1}$. Then John's polynomial is obtained via intersection of two cycles in this symmetric power.

2. Higher Alexander polynomials can be obtained in the similar way. We look at the changing of $B'$ during transformations of a braid. It is easy to see that the ideals generated by determinants of all submatrices of given size are also invariant under braid transformations.
In any case it is enough to prove that $B'$ is a copresentation matrix of Alexander’s module. See below a sketch:

a) $d_i (i = 1..n - 1)$ in the top of a braid are generators of homologies of infinite cyclic cover over complement of a knot.

b) $s_i$, and, therefore $\beta s_i$ equal to 0 in the homologies of cyclic covering. That means that the row corresponding to $s_i$ in $B'$ (i.e. the vector $W_d \cap \beta s_i$) is a relation for $d_j$.

c) now we prove that that is all relations: consider a linear combination $d = \sum c_j d_j, c_j \in \mathbb{Z}[\frac{1}{2}, t^{-\frac{1}{2}}]$, suppose it is 0 in homologies of the cyclic covering. Homologous of $d$ to zero implies that it is possible to find a surface $S$ with boundary $d$; this surface does not intersect the knot. Stretched closer to the knot, we see that $S$ has a number of "hats" in the top and in the bottom, each of those is the circle $s_i$ spanned by a disk. That means that $d \sum s_i + \sum \beta (s_i)$.

Hats in the top means that we need to add to $d$ a linear combination of $s_j$, lying higher than braid, hats in the bottom correspond to a linear combination of $s_i$ below the braid. Therefore we have $\beta (\sum c_j s_j) = d + \sum c_j s_j$.

With 1) in combination it possibly gives the higher polynomial of John’s.

3. Following Important Corollary 7.5 we consider the vector space $V$ generated by $s_i, d_i$. We can give the following definitions of $V_\beta$: let us chose some $n - 1$-dimensional subspace of $V$ with basis $f_1, \ldots, f_{n-1}$ and the complimentary subspace with basis $g_1, \ldots, g_{n-1}$. Hence $V_\beta = c(n - 1)! (f_1^* \wedge f_2^* \wedge \ldots \wedge f_{n-1}^*) \times (g_1 \wedge g_2 \wedge \ldots \wedge g_{n-1})$ where $c$ is a some constant.
Appendix A. Symmetric power

Let us find the homology groups $H_*(Symm_k(M))$ with coefficients in the local system $\Theta$. In follows from Künneth formula that $H_*(M^k) = \bigotimes H_*(M)$. Therefore $H_*(Symm_k(M); \Phi^*)$ is the factor by the action $S_k$ on this tensor product. We note that $a, b \in H_*(M)$, $\dim a = l, \dim b = m, a \otimes b = (-1)^{lm}b \otimes a$.

Therefore if all cohomology classes live in odd dimensions that $ab = -ba$ for any classes.

Hence we have $H_*(Symm_k(M); \Phi) = \bigwedge H_*(M; \Phi)$. We forgot about $H_0$ because we consider only top cohomology groups.

Appendix B. Intersections with coefficients in the local system

Firstly, let us note that we can think about any element of $H_*(D', \Theta)$ as about a chain living in the total covering space over $D'$ with the fiber $\mathbb{Z}[t^\frac{1}{2}, t^{-\frac{1}{2}}]$.

Suppose we have a cycle $a$ in $H_1(D', \Theta)$. Denote by $t^\frac{1}{2}r_i$ the lifting of the interval $r_i = [p_{2i-1}, p_{2i}]$ in the total space of $\Theta$, the lifted points have coordinate $t^\frac{1}{2}$ in the fibers. Now define $a \cdot r_i = \sum_{k \in \mathbb{Z}} < a \cdot t^\frac{1}{2}r_i > t^\frac{1}{2}$ where $\langle ., . \rangle$ is the usual intersection product of two homology cycles in topological space.

Furthermore, $R$ (defined earlier) is a cube, therefore we suppose that all its points have coordinate 1 in fibers. We say that $t^lS$ is the image of $S$ by monodromy action which corresponds to multiplication by $t^l$. Let us intersect $R, t^lS$ as usual manifolds, its intersection consists of a number of points, that is, integer number.

Definition B.1. $< R, S > = \sum_{k \in \mathbb{Z}} < R, t^\frac{k}{2}S > t^\frac{k}{2}$.

Appendix C. Determinant of a pseudostochastic matrix

Lemma C.1. All first minors of a pseudostochastic matrix $A$ are equal modulo sign.

Proof. Consider the matrix $B$ which is complimentary to $A$ Elements of $B$ are first minors of $A$. We have

$$AB = \text{det}(A)I = 0$$

all vectors-rows of $A$ lie in the hyperplane $\sum_{i=1..n} x_i = 0$, therefore are generated by vectors $e_{i+1} - e_i$, where $e_i$ are basis vectors. From the facts that $(e_{i+1} - e_i)B = 0$ we conclude that columns of $B$ are the same; it follows from $BA = 0$ that the same is true for rows. \qed

Appendix D. Action of $B_{n,n}$ on the total space of $\Theta$

An action of a group $G$ on a topological space $X$ is given by an homomorphism $a : G \to \text{Homeo}(X)$ from group $G$ to the automorphism group of the space $X$.

Coloured braid group acts on $D$, mapping marked points to marked points preserving colors; in other words we have an homomorphism $a : B_{n,n} \to \pi_0(\text{Homeo}(D'))$ where we put by definition $\text{Homeo}(D') = \text{Homeo}(D, \bigcup_{i=1..n} p_{2i+1}, \bigcup_{i=1..n} p_{2i})$ to be the group of autohomeomorphisms of $D$ mapping marked points to marked points and preserving pairity of their indices.

Let $\text{Homeo}(D', \Theta)$ be the group of autohomeomorphisms of the total space of $\Theta$, we require that elements of this group preserve fibers of $\Theta$, and after factorization by fibers they become elements of $\text{Homeo}(D')$.

Proposition D.1. Projection $\text{Homeo}(D', \Theta) \to \text{Homeo}(D')$ is an isomorphism. \qed

Therefore we have an isomorphism between $\pi_0(\text{Homeo}(D'))$ and $\pi_0(\text{Homeo}(D', \Theta))$. Hence we have a homomorphism $a : B_{n,n} \to \pi_0(\text{Homeo}(D', \Theta))$ which gives us an action of $B_{n,n}$ on $H_*(D'; \Theta)$.
References

[J. Alexander, 1928] J.W. Alexander, Topological Invariants of knots and links Trans. Amer. Math. Soc. 30 (1928) 275–306

[J. Birman, 1976] Joan S. Birman, On the stable equivalence of plat representations of knots and links, Canad. J. Math. 28 (1976), no. 2, 264–290

[S. Bigelow, 2001] Stephen Bigelow, A homological definition of the Jones polynomial, Geometry and Topology Monographs Volume 4: Invariants of knots and 3-manifolds (Kyoto 2001) Pages 29–41

[S. Bigelow, A. Cattabriga, V. Florens, 2012] S. Bigelow, A. Cattabriga, V. Florens Alexander representation of tangles, arXiv:1203.4590.

[J. Conway, 1967] J. H. Conway, An enumeration of knots and links and some of their algebraic properties, In: Computational Problems in Abstract Algebra, Proc. Conf. Oxford (1967) (edited by J. Leech), pp. 329–358; New York: Pergamon Press. MR 41:2661

[R. Crowell, R. Fox, 1963] R. H. Crowell and R. H. Fox, Introduction to Knot Theory, New York: Ginn and Co. (1963), or: Grad. Texts Math. 57, Berlin-Heidelberg-New York: Springer Verlag (1977). MR 26:4348; MR 56:3829

[R. Fintushel, R. Stern, 1996] Ronald Fintushel, Ronald J. Stern, Knots, Links, and 4-Manifolds (1996), arXiv:dg-ga/9612014v2

[A. Floer, 1988] A. Floer, Morse theory for Lagrangian intersections. J. Differential Geometry, 28:513–547, 1988.

[R. Fox, 1961] R. H. Fox., A quick trip through knot theory, In Topology of Three Manifolds - Proceedings of 1961 Topology Institute at Univ. of Georgia, edited by M. K. Fort, pp. 120–167. Englewood Cliffs, N. J. : Prentice-Hall. MR 25:3522

[R. Fox, J. Milnor, 1966] R. H. Fox and J. Milnor, Singularities of 2-spheres in 4-space and cobordism of knots. Osaka J. Math., Vol. 3 (1966), pp. 257–267. MR 35:2273

[M. Freedman, F. Quinn, 1990] Michael H. Freedman and Frank Quinn, Topology of 4-manifolds, Princeton Mathematical Series, vol 39, Princeton University Press, Princeton, NJ, 1990.

[L. Kauffman, 1983] Louis H. Kauffman, Formal Knot Theory, Mathematical Notes No. 30 (1983), Princeton University Press. MR 85b:57006

[L. Kauffman, 2001] Louis H. Kauffman, Knots and Physics (Series on Knots and Everything, Vol. 1), World Scientific Publishing Company (2001)

[M. Khovanov, 2006] Mikhail Khovanov, Link homology and categorification, Proceedings of the ICM-2006, Madrid, vol.2 989–999, arXiv:math.QA/0605339.

[R. Lawrence, 1993] Ruth J. Lawrence, A functorial approach to the one-variable Jones polynomial, Journal of Differential Geometry, 37 (1993) 689–710.

[W. Lickorish, 1997] W.B. Raymond Lickorish, An Introduction to knot theory. Graduate Texts in Mathematics, 175. Springer-Verlag, New York, (1997)

[P. Ozsváth, Z. Szabó, 2004] P. Ozsváth, Z. Szabó, Holomorphic disks and knot invariants. Adv. Math. 186 (1)(2004), 58–116.

[J. Rasmussen, 2003] J. Rasmussen, Floer homology and knot complements. PhD thesis, Harvard University, 2003; math.GT/0306378.

[H. Seifert, 1934] Herbert Seifert, Über das geschlecht von knoten, Math. Ann., Vol. 110, (1934), pp. 571–592.

E-mail address: Nikita.Kalinin@unige.ch, nikananspb@gmail.com