A-statistical convergence of Mittag-Leffler operators

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1. INTRODUCTION

The function defined by [11]
\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (z \in \mathbb{C}; \Re(\alpha) > 0) \]
is known as the Mittag-Leffler function. The two-index Mittag-Leffler function is defined by [14]
\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z, \beta \in \mathbb{C}; \Re(\alpha) > 0). \]

Note that \( E_{\alpha,1}(z) = E_\alpha(z) \) and
\[ E_{1,1}(z) = e^z, \quad E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E_{1,m+1}(z) = \frac{e^z - \sum_{k=0}^{m-1} \frac{z^k}{k!}}{z^m}. \]

Moreover, for \(|z| < 2\pi\), we have
\[ \frac{1}{E_{1,2}(z)} = \sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{z^n}{n!}, \quad \frac{1}{E_{1,m+1}(z)} = \sum_{n=0}^{\infty} \frac{B_n^{(m)}}{n!} \frac{z^n}{n!}. \]
where the coefficients \((B_n)\) are the familiar Bernoulli numbers and \((B_n^{(m)})\) are the generalized Bernoulli numbers (see [2]).

Let \((b_n)\) be a sequence of positive real numbers and let \(\beta > 0\) be fixed. For all \(n \in \mathbb{N}\), we introduce the Mittag-Leffler operators by

\[
L_n^{(\beta)}(f; x) = \frac{1}{E_{1,\beta}(\frac{nx}{b_n})} \sum_{k=0}^{\infty} f \left( \frac{k}{n} b_n \right) \frac{(nx)^k}{b_n^k \Gamma(k + \beta)}, \tag{1.1}
\]

where \(f \in E := \left\{ f \in C[0, +\infty) : \lim_{x \to +\infty} \frac{f(x)}{1 + x^2} \text{ is finite} \right\}\) and \(C[0, +\infty)\) denotes the space of continuous functions defined on \([0, +\infty)\). Recall that the Banach lattice \(E\) is endowed with the norm

\[
\|f\|_* := \sup_{x \in [0, +\infty)} |f(x)| \left/ \left(1 + x^2\right)\right..
\]

It is obvious that the operators \(L_n^{(\beta)}(f; x)\) defined in (1.1) are linear and positive.

Note that for \(\beta = 1\), we have

\[
L_n^{(1)}(f; x) = e^{-nx/b_n} \sum_{k=0}^{\infty} f \left( \frac{k}{n} b_n \right) \frac{(nx)^k}{b_n^k k!} = S_n(f; x)
\]

where the operators \(S_n\) are the modified Szász-Mirakjan operators considered in [1].

By direct computations one can state the following lemma:

**Lemma 1.** Let \(\psi_x^2(t) = (t - x)^2\). Then, for each \(x \geq 0\) and \(n \in \mathbb{N}\), we have

(a) \(L_n^{(\beta)}(1; x) = 1,\)

(b) \(\left| L_n^{(\beta)}(t; x) - x \right| \leq \frac{|1 - \beta| b_n}{n},\)

(c) \(\left| L_n^{(\beta)}(t^2; x) - x^2 \right| \leq \frac{(2|1 - \beta| + 1) b_n}{n} x + \frac{\left(2(1 - \beta)^2 + |1 - \beta| + |1 - \beta||\beta - 2|\right) b_n^2}{n^2}\)

(d) \(L_n^{(\beta)}(\psi_x^2; x) \leq \frac{(4|1 - \beta| + 1) b_n}{n} x + \frac{\left(2(1 - \beta)^2 + |1 - \beta| + |1 - \beta||\beta - 2|\right) b_n^2}{n^2}.\)
Proof. Since
\[
\sum_{k=0}^{\infty} \frac{(nx)^k}{b_n^k \Gamma(k + \beta)} = E_{1, \beta} \left(\frac{nx}{b_n}\right),
\]
then \(L_n(\beta; 1; x) = 1\). Using the fact that \(\Gamma(k + \beta) = (k + \beta - 1) \Gamma(k + \beta - 1)\), we get
\[
L_n(\beta; t; x) = \frac{1}{E_{1, \beta} \left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \frac{k b_n}{n} \frac{(nx)^k}{b_n^k \Gamma(k + \beta)}
\]
\[
= \frac{1}{E_{1, \beta} \left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \frac{[(k + \beta - 1) + 1 - \beta] b_n}{n} \frac{(nx)^k}{b_n^k (k + \beta - 1) \Gamma(k + \beta - 1)}
\]
\[
= x + \frac{1}{E_{1, \beta} \left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \frac{1 - \beta}{n} \frac{b_n (nx)^k}{b_n^k \Gamma(k + \beta)}. \tag{1.2}
\]
Hence
\[
\left| L_n(\beta; t; x) - x \right| = \frac{|1 - \beta| b_n}{n} \frac{1}{E_{1, \beta} \left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \frac{(nx)^k}{b_n^k \Gamma(k + \beta)} \leq \frac{|1 - \beta| b_n}{n}.
\]
Similarly, by \(k(k - 1) = (k + \beta - 1)(k + \beta - 2) + 2(1 - \beta)k + (1 - \beta)(\beta - 2)\) and \(\Gamma(k + \beta) = (k + \beta - 1)(k + \beta - 2) \Gamma(k + \beta - 2)\), we get
\[
L_n(\beta; t^2; x) = \frac{1}{E_{1, \beta} \left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \left(\frac{k}{n}\right)^2 \frac{(nx)^k}{b_n^k \Gamma(k + \beta)}
\]
\[
= \frac{1}{E_{1, \beta} \left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \frac{(k(k - 1) + k) b_n^2}{n^2} \frac{(nx)^k}{b_n^k \Gamma(k + \beta)}
\]
\[
= \frac{1}{E_{1, \beta} \left(\frac{nx}{b_n}\right)} \sum_{k=2}^{\infty} \frac{k(k - 1) b_n^2}{n^2} \frac{(nx)^k}{b_n^k \Gamma(k + \beta)} + \frac{b_n L_n(\beta; t; x)}{n}
\]
\[
= \frac{1}{E_{1, \beta} \left(\frac{nx}{b_n}\right)} \sum_{k=2}^{\infty} \frac{(k + \beta - 1)(k + \beta - 2) b_n^2}{n^2} \frac{(nx)^k}{b_n^k (k + \beta - 1)(k + \beta - 2) \Gamma(k + \beta - 2)}
\]
\[
\times \frac{(nx)^k}{b_n^k (k + \beta - 1)(k + \beta - 2) \Gamma(k + \beta - 2)}
\]
\[ + \frac{1}{E_{1, \beta} \left( \frac{nx}{b_n} \right)} \sum_{k=2}^{\infty} \frac{2(1 - \beta) k b_n^2}{n^2} \frac{(nx)^k}{b_n^k \Gamma (k + \beta)} \]
\[ + \frac{(1 - \beta)(\beta - 2)b_n^2}{n^2 E_{1, \beta} \left( \frac{nx}{b_n} \right)} \sum_{k=2}^{\infty} \frac{(nx)^k}{b_n^k \Gamma (k + \beta)} + \frac{b_n L_n^\beta (t; x)}{n}. \]

Therefore
\[
\left| L_n^\beta (t^2; x) - x^2 \right| \leq \frac{2 |1 - \beta| b_n}{n E_{1, \beta} \left( \frac{nx}{b_n} \right)} \sum_{k=2}^{\infty} \frac{kb_n}{n} \frac{(nx)^k}{b_n^k \Gamma (k + \beta)}
\]
\[ + \frac{|1 - \beta| |\beta - 2| b_n^2}{n^2 E_{1, \beta} \left( \frac{nx}{b_n} \right)} \sum_{k=2}^{\infty} \frac{(nx)^k}{b_n^k \Gamma (k + \beta)} + \frac{|L_n^\beta (t; x)|}{n} \]
\[ \leq \frac{2 |1 - \beta| + 1}{n} b_n \left| L_n^\beta (t; x) \right| + \frac{|1 - \beta| |\beta - 2| b_n^2}{n^2}. \]

Using (1.2), we obtain
\[
\left| L_n^\beta (t^2; x) - x^2 \right| \leq \frac{2 |1 - \beta| + 1}{n} b_n x
\]
\[ + \frac{2 (1 - \beta)^2 + |1 - \beta| + |1 - \beta| |\beta - 2|}{n^2} b_n^2. \]

Finally,
\[
L_n^\beta (\psi^2; x)
\]
\[ \leq \frac{4 |1 - \beta| + 1}{n} b_n x + \frac{2 (1 - \beta)^2 + |1 - \beta| + |1 - \beta| |\beta - 2|}{n^2} b_n^2 \]
which completes the proof. \[ \square \]

We organize the paper as follows: In Section 2, we give the transformation properties of the operators $L_n^\beta$ and compute the rate of convergence by using the modulus of continuity. In Section 3, we prove an $A$-statistical Korovkin type approximation theorem.

2. Transformation properties and Rate of Convergence

We start with the following lemma, which proves that $L_n^\beta$ maps $E$ into itself.
Lemma 2. Let $\left(\frac{b_n}{n}\right)$ be a bounded sequence of positive numbers and $\beta > 0$ be fixed. Then there exists a constant $M(\beta)$ such that, for $w(x) = (1 + x^2)^{-1}$, we have
\[ w(x)L_n^{(\beta)}\left(\frac{1}{w} : x\right) \leq M(\beta) \]
holds for all $x \in [0, \infty)$ and $n \in \mathbb{N}$. Furthermore, for all $f \in E$, we have
\[ \left\| L_n^{(\beta)}(f) \right\|_* \leq M(\beta) \| f \|_* . \]

Proof. Using Lemma 1, we can write that
\[ w(x)L_n^{(\beta)}\left(\frac{1}{w} : x\right) = \frac{1}{1 + x^2} \left[ L_n^{(\beta)}(1 : x) + L_n^{(\beta)}(t^2 : x) \right] \]
\[ \leq \frac{1}{1 + x^2} \left[ 1 + x^2 + \frac{(2|1-\beta| + 1)b_n}{n} x \right. \]
\[ \left. \quad + \frac{(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2|)b_n^2}{n^2} \right] \]
\[ \leq M(\beta). \]

On the other hand
\[ w(x)\left| L_n^{(\beta)}(f : x) \right| = w(x)\left| L_n^{(\beta)}(w f : x) \right| \leq \| f \|_* \| w(x)L_n^{(\beta)}\left(\frac{1}{w} : x\right) \| \leq M(\beta) \| f \|_* . \]

Taking supremum over $x \in [0, \infty)$ in the above inequality, gives the result. \[ \square \]

Now, recall that the usual modulus of continuity of $f$ on the closed interval $[0, B]$ is defined by
\[ \omega_B(f, \delta) = \sup_{|t-x| \leq \delta, x, t \in [0, B]} |f(t) - f(x)|. \]

It is well known that, for a function $f \in E$, we have $\lim_{\delta \to \infty} \omega_B(f, \delta) = 0$.

The next theorem gives the rate of convergence of the operators $L_n^{(\beta)}(f : x)$ to $f(x)$, for all $f \in E$.

Theorem 1. Let $\beta > 0$ be fixed, $\left(\frac{b_n}{n}\right)$ be a bounded sequence of positive numbers, $f \in E$ and $\omega_{B+1}(f, \delta) (B > 0)$ be its modulus of continuity on the finite interval $[0, B+1] \subset [0, \infty)$. Then
\[ \left\| L_n^{(\beta)}(f : x) - f(x) \right\|_{C[0,B]} \leq M_f(\beta, B)\delta_n(\beta, B) + 2\omega_{B+1}(f, \delta_n^{1/2}(\beta, B)) \]
where $\delta_n(\beta, B) = N_f(\beta, B) \frac{b_n}{n} \left[ 1 + \frac{b_n}{n} \right],$

$$N_f(\beta, B) = \max \left\{ (4|1 - \beta| + 1) B, \left( 2 (1 - \beta)^2 + |1 - \beta| + |1 - \beta| |\beta - 2| \right) \right\}$$

and $M_f(\beta, B)$ is an absolute constant depending on $f, \beta$ and $B$.

**Proof.** Let $\beta > 0$ be fixed. For $x \in [0, B]$ and $t \leq B + 1$, we have the inequality

$$|f(t) - f(x)| \leq \omega_{B+1}(f, |t - x|) \leq \left( 1 + \frac{|t - x|}{\delta} \right) \omega_{B+1}(f, \delta) \quad (2.1)$$

where $\delta > 0$. On the other hand, for $x \in [0, B]$ and $t > B + 1$, using the fact that $t - x > 1$, we have

$$|f(t) - f(x)| \leq A_f(1 + x^2 + t^2) \leq A_f(2 + 3x^2 + 2(t - x)^2) \leq 6A_f(1 + B^2) (t - x)^2 \quad (2.2)$$

By (2.1) and (2.2), we get for all $x \in [0, B]$ and $t \geq 0$ that

$$|f(t) - f(x)| \leq 6A_f(1 + B^2) (t - x)^2 + \left( 1 + \frac{|t - x|}{\delta} \right) \omega_{B+1}(f, \delta).$$

Therefore

$$\left| L_n^{(\beta)}(f; x) - f(x) \right| \leq 6A_f(1 + B^2) L_n^{(\beta)}((t - x)^2; x) + \left( 1 + \frac{L_n^{(\beta)}(|t - x|; x)}{\delta} \right) \omega_{B+1}(f, \delta).$$

Applying Cauchy-Schwarz inequality andLemma 1, we get

$$\left| L_n^{(\beta)}(f; x) - f(x) \right| \leq 6A_f(1 + B^2) L_n^{(\beta)}(\psi_x^2; x) + \left( 1 + \frac{L_n^{(\beta)}(\psi_x^2; x)}{\delta} \right) \omega_{B+1}(f, \delta)$$

$$\leq 6A_f(1 + B^2) \times \left[ (4|1 - \beta| + 1) B \frac{b_n}{n} + \left( 2 (1 - \beta)^2 + |1 - \beta| + |1 - \beta| |\beta - 2| \right) \frac{b_n^2}{n^2} \right]$$

$$\times \left( 1 + \frac{L_n^{(\beta)}(\psi_x^2; x)}{\delta} \right) \omega_{B+1}(f, \delta) \leq M_f(\beta, B) \delta_n(\beta, B) + 2\omega_{B+1}(f, (\delta_n(\beta, B))^{1/2}).$$
where
\[ N_f(\beta, B) = \max \left\{ \left(4|1-\beta| + 1\right)B, \left(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2|\right) \right\}, \]
\[ M_f(\beta, B) = 6A_f \left(1 + B^2\right) \] and \[ \delta_n(\beta, B) = N_f(\beta, B) \frac{b_n}{n} \left[1 + \frac{b_n}{n}\right]. \]
Whence the result follows. □

3. \(A\)-STATISTICAL CONVERGENCE

Recently, \(A\)-statistical convergence of linear positive operators have been an active research area (see [3–5, 12]). We start to this section by recalling concepts of \(A\)-statistical convergence.

Let \(A = (a_{jk})\) be a non-negative regular summability matrix.

**Definition 1.** The \(A\)-density of a subset \(K\) of \(\mathbb{N}\) is given by
\[
\delta_A(K) = \lim_{n \to \infty} \frac{1}{n} \sum_{k \in K} a_{j,k}, \tag{3.1}
\]
provided that limit exists (see [7]).

**Definition 2.** A sequence \(x = (x_n)\) is said to be \(A\)-statistically convergent to \(l\) and denoted by \(\text{st}_A \lim x = l\) if for every \(\varepsilon > 0\), \(\delta_A \{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\} = 0\) (see [6, 13]).

Taking \(A = C_1\), the Cesaro matrix of order one in (3.1), \(A\)-statistical convergence reduces to statistical convergence [8, 10]. Taking \(A = I\), the identity matrix then \(A\)-statistical convergence reduces to ordinary convergence. Kolk [9] proved that in the case of \(\lim_n \max_n |a_{j,n}| = 0\), \(A\)-statistical convergence is stronger than ordinary convergence.

Now let \(A = (a_{jn})\) be a non-negative regular summability matrix. Assume that \((b_n)_{n \in \mathbb{N}}\) is a sequence in \([0, \infty)\) satisfying
\[
\text{st}_A \lim \frac{b_n}{n} = 0. \tag{3.2}
\]
Then we have
\[
\text{st}_A \lim \left(\frac{b_n}{n}\right)^2 = 0. \tag{3.3}
\]
Such a sequence \((b_n)_{n \in \mathbb{N}}\) satisfying (3.2), can be constructed as follows: Take \(A = C_1\), and define
\[
b_n := \begin{cases} n, & \text{if } n = m^2 \ (m \in \mathbb{N}) \\ n^{1/3}, & \text{otherwise}. \end{cases} \tag{3.4}
\]
Then clearly \(\text{st}_{C_1} \lim \frac{b_n}{n} = \text{st} \lim \frac{b_n}{n} = 0.\)
Theorem 2. Let $A = (a_{jk})$ be a non-negative regular summability matrix and $\beta > 0$ be fixed. If

$$st_A\lim_n \frac{b_n}{n} = 0$$

then

$$st_A\lim_n \left\| L_n^{(\beta)}(f; x) - f(x) \right\|_{C[0, B]} = 0$$

holds for every $f \in E$.

Proof. Given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. For fixed $\beta > 0$, define the following sets:

$$U := \{n : \delta_n(\beta, B) \geq r\},$$

$$U_1 := \left\{n : \frac{b_n}{n} \geq \frac{r - \varepsilon}{2N_f(\beta, B)} \right\},$$

$$U_2 := \left\{n : \left(\frac{b_n}{n}\right)^2 \geq \frac{r - \varepsilon}{2N_f(\beta, B)} \right\},$$

where $N_f(\beta, B)$ and $\delta_n(\beta, B)$ be the same as in Theorem 1. Then it is clear that $U \subseteq U_1 \cup U_2$, which gives

$$\sum_{k \in U} a_{jk} \leq \sum_{k \in U_1} a_{jk} + \sum_{k \in U_2} a_{jk}. \quad (3.5)$$

Letting $j \to \infty$ in (3.5) and using (3.2) and (3.3), we have $\lim_j \sum_{k \in U} a_{jk} = 0$. This proves that $st_A\lim_n \delta_n(\beta, B) = 0$ which also implies

$$st_A\lim_n \omega_B + 1(f, \delta_n^{1/2}(\beta, B)) = 0.$$ 

Using Theorem 1 we get the result. \qed

Remark that choosing the sequence $(b_n)_{n \in \mathbb{N}}$ as in (3.4), the statistical approximation results in Theorem 2 works, however its classical case does not work since $(\frac{b_n}{n})_{n \in \mathbb{N}}$ is not convergent in the ordinary sense.

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