MULTI-WINDOW GABOR FRAMES IN AMALGAM SPACES

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ABSTRACT. We show that multi-window Gabor frames with windows in the Wiener algebra \( W(L^{\infty}, \ell^1) \) are Banach frames for all Wiener amalgam spaces. As a byproduct of our results we positively answer an open question that was posed by [Krishtal and Okoudjou, Invertibility of the Gabor frame operator on the Wiener amalgam space, J. Approx. Theory, 153(2), 2008] and concerns the continuity of the canonical dual of a Gabor frame with a continuous generator in the Wiener algebra. The proofs are based on a recent version of Wiener’s \( 1/f \) lemma.

1. INTRODUCTION

A Gabor system is a collection of functions on \( \mathbb{R}^d \)
\[ \mathcal{G}(g, \Lambda) = \{ \pi(\lambda) g \mid \lambda \in \Lambda \}, \]
where \( \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \) is a lattice, \( g \in L^p(\mathbb{R}^d), 1 \leq p \leq \infty \), and the time-frequency shifts of \( g \) are given by
\[ \pi(\lambda)g(y) = \pi(x, \omega)g(y) = e^{2\pi i \omega \cdot y}g(y - x), \quad y \in \mathbb{R}^d. \]
More generally, a multi-window Gabor system is a collection of functions on \( \mathbb{R}^d \)
\[ \mathcal{G}(\Lambda^1, \ldots, \Lambda^n, g^1, \ldots, g^n) = \{ \pi(\lambda^i) g^i \mid \lambda^i \in \Lambda^i, 1 \leq i \leq n \}, \]
where \( \Lambda^1, \ldots, \Lambda^n \subseteq \mathbb{R}^{2d} \) are lattices \( \Lambda^i = \alpha_i \mathbb{Z}^d \times \beta_i \mathbb{Z}^d \) and \( g^1, \ldots, g^n \in L^p(\mathbb{R}^d) \).
Such a system is called a Gabor frame if it satisfies,
\[ A\|f\|_2^2 \leq \sum_{i=1}^n \sum_{\lambda^i \in \Lambda^i} \left| \langle f, \pi(\lambda^i) g^i \rangle \right|^2 \leq B\|f\|_2^2, \quad f \in L^2(\mathbb{R}^d), \]
for some constants \( A, B > 0 \). These multi-window Gabor frames are related to the notion of superframes considered in \[2, 25\]. The condition in (1) can be suitably modified to hold for \( f \in L^p(\mathbb{R}^d) \) for some \( p \in [1, \infty] \). Such a system is then called a Gabor \( p \)-frame.

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For the standard case \( p = 2 \), the Gabor frame operator \( S_G : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \), defined by

\[
S_G(f) := \sum_{i=1}^{n} \sum_{\lambda^i \in \Lambda^i} \langle f, \pi(\lambda^i)g^i \rangle \pi(\lambda^i)g^i, \quad f \in L^2(\mathbb{R}^d),
\]

is bounded, positive definite and invertible. Moreover, every function \( f \in L^2(\mathbb{R}^d) \) admits the following two expansions:

\[
f = \sum_{i=1}^{n} \sum_{\lambda^i \in \Lambda^i} \langle f, \pi(\lambda^i)g^i \rangle S_G^{-1} \left( \pi(\lambda^i)g^i \right)
\]

\[
= \sum_{i=1}^{n} \sum_{\lambda^i \in \Lambda^i} \langle f, S_G^{-1} \left( \pi(\lambda^i)g^i \right) \rangle \pi(\lambda^i)g^i.
\]

Under additional conditions on the generators \( g^i \), the Gabor frame operator extends continuously from the Schwartz class \( \mathcal{S} \) not just to \( L^2(\mathbb{R}^d) \) but to a whole host of Banach spaces (cf. Section 2.2 and references therein); we use the same notation, \( S_G \), to denote all those extensions. Under more stringent conditions \( S_G \) remains invertible on these Banach spaces and the expansions in (2) also extend. In the latter case, a \( p \)-frame is usually called a Banach frame. When \( n = 1 \), for example, if \( g = g^1 \) belongs to the Feichtinger algebra then a Gabor frame for \( L^2(\mathbb{R}^d) \) is also a Banach frame for all the modulation spaces (see [15] and [19, Chapters 11–12]). Extension of (2) to \( L^p \) spaces was settled in [18, 21, 31] under suitable condition on the generator \( g \). In addition, when \( g \) belongs to the Wiener algebra, it was shown in [22] that the corresponding Gabor frame is a Banach frame for a class of Wiener amalgam spaces defined below. We also point out that a special case of the results obtained in [22] was considered in [3] and related results appear in [16].

In all these cases, the extension of Gabor frames from a pure \( L^2 \)-theory to a class of Banach spaces relies on the generator \( g \) of the frame and its canonical dual \( S_G^{-1} g \) having the same time-frequency properties. In a sense, one usually has to prove that if a Gabor frame for \( L^2 \) has a generator in a class, then its canonical dual belongs to the same class. Equivalently, one must show that the frame operator is continuously invertible on the class under consideration. This is exactly a Wiener-type lemma. Recall that the Wiener \( 1/f \) lemma asserts that if \( f \) is a periodic function with absolutely convergent Fourier series and if \( f \) never vanishes, then \( 1/f \) also has an absolutely convergent Fourier series. We refer to [11, 14, 15, 16, 17, 18, 19, 20, 21, 24, 26, 27] for some of the relevant extensions.

Generally, in the case of Wiener amalgam spaces or the \( L^p \) spaces, it is known that Gabor series converge only conditionally and the norm of a function \( f \) is not determined by the size of its Gabor coefficients. This is essentially related to the behavior of Fourier series on \( L^p([0,1]^d) \) spaces. We
refer to [32] for related results about the pointwise convergence of Gabor expansions.

One of our goals in this paper is to extend the $L^2$-theory of multi-window Gabor frames to the setting of amalgam spaces. In order to do this, we must understand the behavior of the canonical dual frame in this context. Indeed, in this multi-window setting the canonical dual frame is no longer of Gabor type, rather, it consists of atoms. Nonetheless, we prove that if a multi-window Gabor system is generated by windows in the Wiener algebra has a frame operator bounded below in some $L^p(\mathbb{R}^d)$, $p \in [1, \infty)$, then it is, in fact, a Banach frame for all reasonable Wiener amalgam spaces, and its canonical dual atoms belong to the same space. Moreover, if the windows are also assumed to be continuous on $\mathbb{R}^d$, then the above result holds for $p = \infty$ as well, and the corresponding dual atoms are also continuous. We point out that even when $n = 1$ and $p = 2$ this result is new. The question about continuity of the dual atoms was posed by [28].

The proofs of some of the above results rely mainly on the almost periodic non-commutative Wiener’s lemma developed in [4]. Other tools are furnished by the standard results in the spectral theory of linear operators.

Our paper is organized as follows. In Section 2 we define the Wiener amalgam spaces and recall their characterization via Gabor frames. In Section 3 we present the main technical result of this paper: a spectral invariance theorem for a sub-algebra of weighted-shift operators in $B(L^p(\mathbb{R}^d))$. In Section 4 we use the result of the previous section to extend the theory of multi-window Gabor frames to the class of Wiener amalgam spaces. In particular, this last section contains a Wiener-type lemma for multi-window Gabor frames.

2. Amalgam spaces and Gabor expansions

Before introducing the Wiener amalgam spaces, we first set the notation that will be used throughout the paper.

Given $x, \omega \in \mathbb{R}^d$, the translation and modulation operators act on a function $f : \mathbb{R}^d \to \mathbb{C}$ by

$$T_x f(y) := f(y - x), \quad M_\omega f(y) := e^{2\pi i \omega \cdot y} f(y),$$

where $\omega \cdot y$ is the usual dot product. The time-frequency shift associated with the point $\lambda = (x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$ is the operator $\pi(\lambda) = \pi(x, \omega) := M_\omega T_x$.

Given two non-negative functions $f, g$, we write $f \preceq g$ if $f \leq C g$, for some constant $C > 0$. If $E$ is a Banach space, we denote by $B(E)$ the Banach algebra of all bounded linear operators on $E$.

We use the following normalization of the Fourier transform of a function $f : \mathbb{R}^d \to \mathbb{C}$:

$$\hat{f}(\omega) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \omega \cdot x} dx.$$
2.1. Definition and properties of the amalgam spaces. A function \( w : \mathbb{R}^d \to (0, +\infty) \) is called a weight if it is continuous and symmetric (i.e. \( w(x) = w(-x) \)). A weight \( w \) is submultiplicative if
\[
w(x + y) \leq w(x)w(y), \quad x, y \in \mathbb{R}^d.
\]
Prototypical examples are given by the polynomial weights \( w(x) = (1 + |x|^s)^t \), which are submultiplicative if \( s \geq 0 \). The main results in this article require to consider an extra condition on the weights. A weight \( w \) is called admissible if \( w(0) = 1 \), it is submultiplicative and satisfies the Gelfand-Raikov-Shilov condition,
\[
\lim_{k \to \infty} w(kx)^{1/k} = 1, \quad x \in \mathbb{R}^d.
\]
Note that this condition, together with the submultiplicativity, implies that \( w(x) \geq 1, x \in \mathbb{R}^d \).

Given a submultiplicative weight \( w \), a second weight \( v : \mathbb{R}^d \to (0, +\infty) \) is called \( w \)-moderate if there exists a constant \( C_v > 0 \) such that,
\[
v(x + y) \leq C_v w(x)v(y), \quad x, y \in \mathbb{R}^d.
\]
For polynomial weights \( v(x) = (1 + |x|)^t, w(x) = (1 + |x|^s)^t \), \( v \) is \( w \)-moderate if \( |t| \leq s \). If \( v \) is \( w \)-moderate, it follows from (3) and the symmetry of \( w \) that \( 1/v \) is also \( w \)-moderate (with the same constant).

Let \( w \) be a submultiplicative weight and let \( v \) be \( w \)-moderate. This will be the standard assumption in this article. We will keep the weight \( w \) fixed and consider classes of function spaces related to various weights \( v \). For \( 1 \leq p, q \leq +\infty \), we define the Wiener amalgam space \( W(L^p, L^q_v) \) as the class of all measurable functions \( f : \mathbb{R}^d \to \mathbb{C} \) such that,
\[
\|f\|_{W(L^p, L^q_v)} := \left( \sum_{k \in \mathbb{Z}^d} \|f\|_{L^p([0,1]^d+k)}^q v(k)^q \right)^{1/q} < \infty,
\]
with the usual modifications when \( q = +\infty \). As with Lebesgue spaces, we identify two functions if they coincide almost everywhere. For a study of this class of spaces in a much broader context see [13, 14, 17]. We only point out that, as a consequence of the assumptions on the weights \( v \) and \( w \), it can be shown that the partition \( \{[0,1]^d+k : k \in \mathbb{Z}^d \} \) in (4) can be replaced by more general coverings yielding an equivalent norm.

Weighted amalgam spaces are solid. This means that if \( f \in W(L^p, L^q_v) \) and \( m \in L^\infty(\mathbb{R}^d) \), then \( mf \in W(L^p, L^q_v) \) and
\[
\|mf\|_{W(L^p, L^q_v)} \leq \|m\|_{L^\infty(\mathbb{R}^d)} \|f\|_{W(L^p, L^q_v)}.
\]
In addition, using the fact that \( v \) is \( w \)-moderate, it follows that \( W(L^p, L^q_v) \) is closed under translations and
\[
\|T_x f\|_{W(L^p, L^q_v)} \leq C_v w(x) \|f\|_{W(L^p, L^q_v)},
\]
where \( C_v \) is the constant in (3).
The K"othe-dual of $W(L^p, L^q_v)$ is the space of all measurable functions $g : \mathbb{R}^d \to \mathbb{C}$ such that $g \cdot W(L^p, L^q_v) \subseteq L^1(\mathbb{R}^d)$. It is equal to $W(L^{p'}, L^{q'}_{1/v})$, where $1/p + 1/p' = 1/q + 1/q' = 1$ for all $1 \leq p, q \leq \infty$. In particular, the pairing

$$\langle \cdot, \cdot \rangle : W(L^p, L^q_v) \times W(L^{p'}, L^{q'}_{1/v}) \to \mathbb{C}, \quad \langle f, g \rangle = \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx,$$

is bounded. The linear functionals arising from integration against functions in $W(L^{p'}, L^{q'}_{1/v})$ determine a topology in $W(L^p, L^q_v)$ that will be denoted by $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$.

2.2. Gabor expansions on amalgam spaces. We now recall the theory of Gabor expansions on Wiener amalgam spaces as developed in [16, 18, 21, 22]. Let $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$ be a (separable) lattice which will be used to index time-frequency shifts. For convenience we assume that $\alpha, \beta > 0$. We point out that the theory depends heavily on the assumption that $\Lambda$ is a separable lattice $\alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$.

We first recall the definition of the family of sequence spaces corresponding to amalgam spaces via Gabor frames. For a weight $v$ and $1 \leq p, q \leq +\infty$ we define the sequence space $S^p,q_v(\Lambda)$ in the following way. We let $\mathcal{F}L^p([0, 1/\beta]^d)$ stand for the image of $L^p([0, 1/\beta]^d)$ under the discrete Fourier transform. More precisely, a sequence $c \equiv \{ c_j \mid j \in \beta \mathbb{Z}^d \} \subseteq \mathbb{C}$ belongs to $\mathcal{F}L^p([0, 1/\beta]^d)$ if there exists a (unique) function $f \in L^p([0, 1/\beta]^d)$ such that,

$$c_j = \hat{f}(j) = \beta^d \int_{[0,1/\beta]^d} f(x) e^{-2\pi ij x} dx, \quad j \in \beta \mathbb{Z}^d.$$  

The space $\mathcal{F}L^p([0, 1/\beta]^d)$ is given the norm $\|c\|_{\mathcal{F}L^p([0, 1/\beta]^d)} := \|f\|_{L^p([0, 1/\beta]^d)}$.

We now let $S^p,q_v(\Lambda)$ be the set of all sequences $c \equiv \{ c_{\lambda} \mid \lambda \in \Lambda \} \subseteq \mathbb{C}$ such that, for each $k \in \alpha \mathbb{Z}^d$, the sequence $(c_{k,j})_{j \in \beta \mathbb{Z}^d}$ belongs to $\mathcal{F}L^p([0, 1/\beta]^d)$ and

$$\|c\|_{S^p,q_v(\Lambda)} := \left( \sum_{k \in \alpha \mathbb{Z}^d} \left\| (c_{k,j})_{j \in \beta \mathbb{Z}^d} \right\|_{\mathcal{F}L^p([0,1/\beta]^d)}^q \right)^{1/q} < +\infty,$$

with the usual modifications when $q = \infty$. When $1 < p < +\infty$ this is simply,

$$\|c\|_{S^p,q_v(\Lambda)} := \left( \sum_{k \in \alpha \mathbb{Z}^d} \left( \sum_{j \in \beta \mathbb{Z}^d} c_{k,j} e^{2\pi i j x} \right)^q \right)^{1/q} < +\infty,$$

and the usual modifications hold for $q = \infty$.

The following Theorem from [22] introduces the analysis and synthesis operators, clarifies their precise meaning and gives their mapping properties.
Theorem 1. [22, Theorem 3.2]. Let $w$ be a submultiplicative weight, $v$ a $w$-moderate weight, $g \in W(L^{\infty}, L^1_w)$, and $1 \leq p, q \leq +\infty$. Then the following properties hold.

(a) The analysis (coefficient) operator,
$$C_{g,\Lambda} : W(L^p, L^q_v) \to S^{p,q}_v(\Lambda), \quad C_{g,\Lambda}(f) := (\langle f, \pi(\lambda)g \rangle)_{\lambda \in \Lambda},$$
is bounded with a bound that only depends on $\alpha, \beta, \|g\|_{W(L^{\infty}, L^1_w)}$, and the constant $C_v$ in [3].

(b) Let $c \in S^{p,q}_v(\Lambda)$ and $m_k \in L^p([0, 1/\beta]^d)$ be the unique functions such that $\hat{m}_k(j) = c_k,j$. Then the series
$$R_{g,\Lambda}(c) := \sum_{k \in \alpha \mathbb{Z}^d} m_k T_k g,$$
converges unconditionally in the $\sigma(W(L^p, L^q_v), W(L^p, L^q_v))$-topology and, moreover, unconditionally in the norm topology of $W(L^p, L^q_v)$ if $p, q < \infty$.

(c) The synthesis (reconstruction) operator $R_{g,\Lambda} : S^{p,q}_v(\Lambda) \to W(L^p, L^q_v)$ is bounded with a bound that depends only on $\alpha, \beta, \|g\|_{W(L^{\infty}, L^1_w)}$, and the constant $C_v$ in [3].

The definition of the operator $R_{g,\Lambda}$ is rather abstract. As shown in [16], the convergence can be made explicit by means of a summability method.

For $g \in W(L^{\infty}, L^1_w)$, a sequence $c \in S^{p,q}_v(\Lambda)$, and $N, M \geq 0$ let us consider the partial sums
$$R_{N,M}(c)(x) := \sum_{|k|_\infty \leq \alpha N} \sum_{|j|_\infty \leq \beta M} c_k,j e^{2\pi ij x} g(x - k).$$

In the conditions "$|k|_\infty \leq N, |j|_\infty \leq M$" above we consider elements $(k, j) \in \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$; it is important that we use the max norm. We also consider the regularized partial sums,
$$\sigma_{N,M}(c)(x) := \sum_{|k|_\infty \leq \alpha N} \sum_{|j|_\infty \leq \beta M} r_{j,M} c_k,j e^{2\pi ij x} g(x - k),$$
where the regularizing weights are given by,
$$r_{j,M} := \prod_{h=1}^d \left(1 - \frac{|j_h|}{\beta(M + 1)}\right).$$

We then have the following convergence result [16, 22].

Theorem 2. Let $w$ be a submultiplicative weight, $v$ a $w$-moderate weight, $g \in W(L^{\infty}, L^1_w)$, and $1 \leq p, q \leq +\infty$. Then the following properties hold.

(a) If $1 < p < \infty$ and $q < \infty$, then
$$R_{N,M}(c) \to R_{g,\Lambda}(c), \quad \text{as } N, M \to \infty,$$
in the norm of $W(L^p, L^q_v)$. 


(b) For each \( c \in S_{b}^{+,q}(\Lambda) \),
\[
\sigma_{N,M}(c) \to R_{g,\Lambda}(c), \quad \text{as } N, M \to \infty,
\]
in the \( \sigma\left(W(L^{p}, L_{b}^{q}), W(L^{r'}, L_{1/(1/v)}^{q'})\right) \)-topology and also in the norm of \( W(L^{p}, L_{b}^{q}) \) if \( p, q < +\infty \).

**Remark 1.** A more refined convergence statement, with more general summability methods, can be found in [16]. We will only need the norm and weak convergence of Gabor expansions but we point out that the problem of pointwise summability has also been extensively studied [16] [18] [21] [22] [32].

**Proof.** Part (a) is proved in [22, Proposition 4.6].

The case \( p < +\infty \) of (b) is proved in [16, Theorem 4], where only unweighted amalgam spaces are considered, but the same proof extends with trivial modifications to the weighted case. The case \( p = +\infty \) is also treated in [16] but in a different direction, establishing norm convergence under additional assumptions on the sequence \( c \). Hence, we need to sketch a proof of (b) when \( p = +\infty \).

Let \( c \in S_{b}^{+\infty,q}(\Lambda) \). According to Theorem 11, \( R_{g,\Lambda}(c) \) is given by
\[
R_{g,\Lambda}(c) = \sum_{k \in \alpha \mathbb{Z}^{d}} m_{k} T_{k} g,
\]
where \( m_{k} \in L^{\infty}([0, 1/\beta)^{d}) \) and \( \|c\|_{S_{b}^{+\infty,q}} = \|(\|m_{k}\|_{\infty})_{k}\|_{\ell_{q}^{\prime}} \). Let us also write
\[
\sigma_{N,M}(c) := \sum_{|k|_{\infty} \leq \alpha N} \sigma_{M}(m_{k}) T_{k} g,
\]
where \( \sigma_{M}(m_{k})(x) = \sum_{|j|_{\infty} \leq \beta M} r_{j,M} c_{k,j} e^{2\pi i j x} \) are the Fejér means of \( m_{k} \).

Let \( f \in W(L^{1}, L_{1/(1/v)}^{q'}) \), we must show that \( \langle R_{g,\Lambda}(c) - \sigma_{N,M}(c), f \rangle \to 0 \). Let us write \( f = \sum_{j \in \alpha \mathbb{Z}^{d}} f_{j} \) and \( g = \sum_{j \in \alpha \mathbb{Z}^{d}} g_{j} \), where \( f_{j} = f \chi_{[0,\alpha)^{d}+j} \) and \( g_{j} = g \chi_{[0,\alpha)^{d}+j} \). Hence, for any subset \( L \subseteq \alpha \mathbb{Z}^{d} \),
\[
\left\langle \sum_{k \in L} m_{k} T_{k} g, f \right\rangle = \sum_{k \in L} \sum_{j' \in \alpha \mathbb{Z}^{d}} \langle m_{k} T_{k} g_{j}, f_{j'} \rangle
= \sum_{k \in L} \sum_{j \in \alpha \mathbb{Z}^{d}} \langle m_{k} T_{k} g_{j}, f_{j+k} \rangle.
\]

In addition,
\[
\sum_{j \in \alpha \mathbb{Z}^{d}} \sum_{k \in L} |\langle m_{k} T_{k} g_{j}, f_{j+k} \rangle|
\lesssim \sum_{k \in L} \sum_{j \in \alpha \mathbb{Z}^{d}} \|g_{j}\|_{\infty} w(j) \|m_{k}\|_{\infty} w(k) \|f_{j+k}\|_{1}(1/v)(j+k)
\lesssim \|g\|_{W(L^{\infty}, L_{1/(1/v)}^{q'})} \|c\|_{S_{b}^{+\infty,q}} \|f\|_{W(L^{1}, L_{1/(1/v)}^{q'})}.
\]
Hence, given \( \varepsilon > 0 \), there exists \( N_0 > 0 \) and a finite set \( J \subseteq \alpha \mathbb{Z}^d \) such that

\[
(11) \quad |\langle R_{g,\Lambda}(c), f \rangle - \sum_{|k| \leq \alpha N_0} \sum_{j \in J} \langle m_k T_k g, f \rangle| < \varepsilon.
\]

Similarly, using the fact \( \|\sigma_M(m_k)\|_\infty \lesssim \|m_k\|_\infty \) independently of \( M \), it follows that \( N_0 \) can be chosen so that, in addition,

\[
(12) \quad |\sigma_{N,M}(c) - \sum_{|k| \leq \alpha N_0} \sum_{j \in J} \langle \sigma_M(m_k) T_k g, f \rangle| < \varepsilon,
\]

for all \( M \geq 0 \) and \( N \geq N_0 \). Finally, for each \( j, k \),

\[
\langle (m_k - \sigma_M(m_k)) T_k g_j, f_{j+k} \rangle = \langle m_k - \sigma_M(m_k), T_k g_j f_{j+k} \rangle \to 0,
\]
as \( M \to +\infty \) because \( T_k g_j f_{j+k} \in L^1 \) and the Fejér means of an \( L^\infty \) function converge in the weak*-topology. This, together with the estimates in (11) and (12), yields the desired conclusion. \( \square \)

We now present a representation of Gabor frame operators that will be essential for the results to come. For proofs see [31] or [22] Theorem 4.2 and Lemma 5.2 for the weighted version.

**Theorem 3.** Let \( w \) be a submultiplicative weight, \( v \) a \( w \)-moderate weight, \( g, h \in W(L^\infty, L^1_w) \) and \( 1 \leq p, q \leq +\infty \). Then the operator \( R_{h,\Lambda}C_{g,\Lambda} : W(L^p, L^q_v) \to W(L^p, L^q_v) \) can be written as

\[
(13) \quad R_{h,\Lambda}C_{g,\Lambda} f = \beta^{-d} \sum_{j \in \mathbb{Z}^d} G_j T_{\beta} f,
\]

where,

\[
(14) \quad G_j(x) := \sum_{k \in \mathbb{Z}^d} \frac{g(x-j/\beta - \alpha k)}{h(x-\alpha k)}, \quad x \in \mathbb{R}^d.
\]

In addition, the functions \( G_j : \mathbb{R}^d \to \mathbb{C} \) satisfy

\[
(15) \quad \sum_{j \in \mathbb{Z}^d} \|G_j\|_{\ell^\infty} \|w(j/\beta)\| \lesssim \|g\|_{W(L^\infty, L^1_v)} \|h\|_{W(L^\infty, L^1_w)} < +\infty.
\]

As a consequence, the series in (13) converges absolutely in the norm of \( W(L^p, L^q_v) \).

### 3. The algebra of \( L^\infty \)-weighted shifts

**3.1. \( L^\infty \)-weighted shifts.** Guided by (13), we will now introduce a Banach *-algebra of operators on function spaces that will be the key technical object of the article. For an admissible weight \( w \) we let \( \mathcal{A}_w \) be the set of all families \( \mathcal{M} = (m_x)_{x \in \mathbb{R}^d} \in \ell^\infty_w(\mathbb{R}^d, L^\infty(\mathbb{R}^d)) \) with the standard Banach space norm

\[
(16) \quad \|\mathcal{M}\|_{\mathcal{A}_w} = \sum_{x \in \mathbb{R}^d} \|m_x\|_{L^\infty} < +\infty.
\]
The algebra structure and the involution on $A_w$, however, will be non-standard. They will come from the identification of $A_w$ with the class of operators on function spaces of the form

$$f \mapsto \sum_{x \in \mathbb{R}^d} m_x f(\cdot - x). \tag{17}$$

Observe that due to (16) the family $M = (m_x)_{x \in \mathbb{R}^d}$ has countable support and also that the operator in (17) is well-defined and bounded on all $L^p(\mathbb{R}^d)$, $p \in [1, \infty]$ (recall that the admissibility of $w$ implies that $w \geq 1$).

With a slight abuse of notation, given a function $m \in L^\infty(\mathbb{R}^d)$ we also denote by $m$ the multiplication operator $f \mapsto mf$. It is then convenient to write $M \in A_w$ as

$$M = \sum_{x \in \mathbb{R}^d} m_x T_x, \quad (m_x)_{x \in \mathbb{R}^d} \in \ell_1^w(\mathbb{R}^d, L^\infty(\mathbb{R}^d)),$$

and endow $A_w$ with the product and involution inherited from $B(L^2(\mathbb{R}^d))$.

More precisely, the product on $A_w$ is given by

$$\left( \sum_x m_x T_x \right) \left( \sum_x n_x T_x \right) = \sum_x \left( \sum_y m_y n_{x-y} (\cdot - y) \right) T_x,$$

and the involution – by

$$\left( \sum_x m_x T_x \right)^* = \sum_x m_x (\cdot + x) T_{-x} = \sum_x m_{-x} (\cdot - x) T_x.$$

It is straightforward to verify that with this structure $A_w$ is, indeed, a Banach $*$-algebra which embeds continuously into $B(L^2(\mathbb{R}^d))$. We shall establish a number of other continuity properties of the operators defined by families in $A_w$ in Proposition 1 below. These will be useful in dealing with Gabor expansions on amalgam spaces.

Before that, we mention that the identification of families in $A_w$ and operators on $B(L^p(\mathbb{R}^d))$ given by the operator in (17) is one-to-one; this follows from the characterization of $A_w$ in the following subsection and can easily be proved directly. Because of this we shall no longer distinguish between the families in $A_w$ and operators generated by them. We will write $A_w \subset B(L^p(\mathbb{R}^d))$ if we need to highlight that we treat members of $A_w$ as operators on $L^p(\mathbb{R}^d)$. We also point out that for $m \in L^\infty(\mathbb{R}^d)$ and $x, w \in \mathbb{R}^d$

$$M_\omega m T_x M_{-\omega} = e^{2\pi i \omega \cdot x} m T_x. \tag{18}$$

**Proposition 1.** Let $1 \leq p, q \leq +\infty$ and let $v$ be a $w$-moderate weight. Then the following statements hold.

(a) $A_w \hookrightarrow B(W(L^p, L^q_v))$. More precisely, every $M = \sum_x m_x T_x \in A_w$ defines a bounded operator on $W(L^p, L^q_v)$ given by the formula

$$M(f) := \sum_x m_x f(\cdot - x).$$
The series defining $\mathcal{M} : W(L^p, L^q) \to W(L^p, L^q)$ converges absolutely in the norm of $W(L^p, L^q)$ and $\|\mathcal{M}\|_{B(W(L^p, L^q))} \leq C_{\nu}\|\mathcal{M}\|_{A_w}$, where $C_{\nu}$ is the constant in $[8]$.

(b) For every $\mathcal{M} \in A_w$, $f \in W(L^p, L^q)$ and $g \in W(L^p, L^q)$,

$$\langle \mathcal{M}(f), g \rangle = \langle f, \mathcal{M}^*(g) \rangle.$$  

(c) For every $\mathcal{M} \in A_w$, the operator $\mathcal{M} : W(L^p, L^q) \to W(L^p, L^q)$ is continuous in the $\sigma(W(L^p, L^q))$-topology.

Proof. Part (a) follows immediately from (5) and (6). Part (b) follows from the fact the involution in $A_w$ coincides with taking adjoint. The interchange of summation and integration is justified by the absolute convergence in part (a). Part (c) follows immediately from (b).

3.2. Spectral invariance. In this section we shall exhibit the main technical result of the article. We remark that similar and more general results appear in $[8, 9, 29]$. We, however, feel obliged to present a proof here because the rest of our paper is based on this result. The key ingredient in the proof is the identification of the algebra $A_w$ with a class of almost periodic elements associated with a certain group representation. We give a brief account of the theory as required for our purposes. For a more general presentation see $[4]$ and references therein.

For $y \in \mathbb{R}^d$ and $\mathcal{M} \in B(L^p(\mathbb{R}^d))$, $p \in [1, \infty]$, let $\rho(y)\mathcal{M} := M_y \mathcal{M} M_{-y}$. Explicitly,

$$\rho(y)\mathcal{M} f(x) = e^{2\pi i y \cdot x}(\mathcal{M} g)(x), \quad g(x) = e^{-2\pi i y \cdot x} f(x)$$

The map $\rho : \mathbb{R}^d \to B(B(L^p(\mathbb{R}^d)))$ defines an isometric representation of $\mathbb{R}^d$ on the algebra $B(L^p(\mathbb{R}^d))$. This means that $\rho$ is a representation of $\mathbb{R}^d$ on the Banach space $B(L^p(\mathbb{R}^d))$ and, in addition, for each $y \in \mathbb{R}^d$, $\rho(y)$ is an algebra automorphism and an isometry.

A continuous map $Y : \mathbb{R}^d \to B(L^p(\mathbb{R}^d))$ is almost-periodic in the sense of Bohr if for every $\varepsilon > 0$ there is a compact $K = K_{\varepsilon} \subset \mathbb{R}^d$ such that for all $x \in \mathbb{R}^d$

$$(x + K) \cap \{y \in \mathbb{R}^d \mid \|Y(g + y) - Y(g)\| < \varepsilon, \forall g \in \mathbb{R}^d\} \neq \emptyset$$

Then $Y$ extends uniquely to a continuous map of the Bohr compactification $\hat{\mathbb{R}}^d$ of $\mathbb{R}^d$, also denoted by $Y$. Thus, now $Y : \hat{\mathbb{R}}^d \to B(L^p(\mathbb{R}^d))$, where $\hat{\mathbb{R}}^d$ represents the topological dual group (i.e. the group of characters) of $\mathbb{R}^d$ when $\mathbb{R}^d$ is endowed with the discrete topology. The normalized Haar measure on $\hat{\mathbb{R}}^d$ is denoted by $\hat{\mu}(dy)$.

For each $\mathcal{M} \in B(L^p(\mathbb{R}^d))$, we consider the map,

$$\hat{\mathcal{M}} : \mathbb{R}^d \to B(L^p(\mathbb{R}^d)), \quad \hat{\mathcal{M}}(y) := \rho(y)\mathcal{M} = M_y \mathcal{M} M_{-y}.$$  

An operator $\mathcal{M} \in B(L^p(\mathbb{R}^d))$ is said to be $\rho$-almost periodic if the map $\hat{\mathcal{M}}$ is continuous and almost-periodic in the sense of Bohr. For every $\rho$-almost
periodic operator $\mathcal{M}$, the function $\hat{\mathcal{M}}$ admits a $B(L^p(\mathbb{R}^d))$-valued Fourier series,
\begin{equation}
\hat{\mathcal{M}}(y) \sim \sum_{x \in \mathbb{R}^d} e^{2\pi i y \cdot x} C_x(\mathcal{M}), \quad (y \in \mathbb{R}^d).
\end{equation}

The coefficients $C_x(\mathcal{M}) \in B(L^p(\mathbb{R}^d))$ in (20) are uniquely determined by $\mathcal{M}$ via
\begin{equation}
C_x(\mathcal{M}) = \int_{\mathbb{R}^d} e^{-2\pi i y \cdot x} \hat{\mathcal{M}}(y) \, d\mu(y) = \lim_{T \to \infty} \frac{1}{2T} \int_{[-T,T]^d} e^{-2\pi i y \cdot x} \hat{\mathcal{M}}(y) \, dy
\end{equation}
and, therefore, satisfy
\begin{equation}
\rho(y) C_x(\mathcal{M}) = e^{2\pi i y \cdot x} C_x(\mathcal{M}).
\end{equation}

Hence, they are eigenvectors of $\rho$ (see [4] for details).

Within the class of $\rho$-almost periodic operators we consider $\text{AP}^p_w(\rho)$, the subclass of those operators for which the Fourier series in (20) is $w$-summable, where $w$ is an admissible weight. More precisely, a $\rho$-almost periodic operator $\mathcal{M}$ belongs to $\text{AP}^p_w(\rho)$ if its Fourier coefficients with respect to $\rho$ satisfy
\begin{equation}
\|\mathcal{M}\|_{\text{AP}^p_w(\rho)} := \sum_{x \in \mathbb{R}^d} \|C_x(\mathcal{M})\|_{B(L^p(\mathbb{R}^d))} w(x) < +\infty.
\end{equation}

By the submultiplicativity of $w$ we know that $w \geq 1$, so for operators in $\text{AP}^p_w(\rho)$ the series in (20) converges absolutely in the norm of $B(L^p(\mathbb{R}^d))$ to $\hat{\mathcal{M}}(y)$:
\begin{equation}
\hat{\mathcal{M}}(y) = \sum_{x \in \mathbb{R}^d} e^{2\pi i y \cdot x} C_x(\mathcal{M}), \quad y \in \mathbb{R}^d,
\end{equation}
where each $C_x \in B(L^p(\mathbb{R}^d))$ satisfies (21) and, hence, (22). In particular, for $y = 0$, it follows that each $\mathcal{M} \in \text{AP}^p_w(\rho)$ can be written as
\begin{equation}
\mathcal{M} = \sum_{x \in \mathbb{R}^d} C_x(\mathcal{M}).
\end{equation}

Conversely, if $\mathcal{M}$ is given by (25), with the coefficients $C_x$ satisfying (23) and (22), it follows from the theory of almost-periodic series that $\mathcal{M} \in \text{AP}^p_w(\rho)$ and $C_x$ satisfy (21).

Theorem 3.2 from [4] establishes the spectral invariance of $\text{AP}^p_w(\rho) \hookrightarrow B(L^p(\mathbb{R}^d))$, $p \in [1, \infty]$ (the result there applies to a more general context). Our goal here is to establish connection between $\mathcal{A}_w$ and $\text{AP}^p_w(\rho)$ and prove a spectral invariance result for $\mathcal{A}_w$.

To achieve this goal we first characterize the eigenvectors $C_x$ of the representation $\rho$. 
Lemma 1. For any $1 \leq p \leq \infty$ and any $m \in L^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, $C_x = mT_x$ is an eigenvector of $\rho : \mathbb{R}^d \to B(L^p(\mathbb{R}^d))$. For $1 \leq p < \infty$ these are the only eigenvectors.

Proof. If $C_x = mT_x$, then, according to (18), it satisfies (23).

The converse works only for $1 \leq p < \infty$. Suppose that $C_x \in B(L^p(\mathbb{R}^d))$ satisfies (23). Using (18) once again we have,

$$\rho(y)(C_xT_{-x}) = e^{2\pi iy \cdot x}C_x e^{-2\pi iy \cdot x}T_{-x} = C_xT_{-x}.$$ 

It follows that $C_xT_{-x}$ commutes with every modulation $M_y$. Hence, $C_xT_{-x}$ must be a multiplication operator $m$, so $C_x = mT_x$. □

For $p = \infty$ there are eigenvectors of $\rho$ which are not of the form $mT_x$. An example of such an eigenvector is given in [29, Section 5.1.11]. Hence, one would need additional conditions to conclude that $C_x = mT_x$ for some $m \in L^\infty(\mathbb{R}^d)$.

From the discussion above, $AP_0^w(\rho)$ consists of all the operators $M = \sum_{x \in \mathbb{R}^d} C_x$, with $C_x$ satisfying (23) and (22). In addition, by the previous lemma, for $1 \leq p < \infty$ an operator $C_x$ satisfies (22) if and only if it is of the form $C_x = mT_x$, for some function $m \in L^\infty(\mathbb{R}^d)$. In this case, $\|C_x\|_{B(L^2(\mathbb{R}^d))} = \|m\|_\infty$ and, thus, (23) reduces to (16). Hence we obtained

Proposition 2. For $p \in [1, \infty)$ the class $A_w \subset B(L^p(\mathbb{R}^d))$ coincides with $AP_0^w(\rho)$, the class of $\rho$-almost periodic elements, having $w$-summable Fourier coefficients.

For $p = \infty$, the two classes are different. Nevertheless, the results we have obtained so far are sufficient to prove our main technical result.

Theorem 4. Let $w$ be an admissible weight. Then, the embedding $A_w \hookrightarrow B(L^p(\mathbb{R}^d))$, $p \in [1, \infty)$ is spectral. In other words, if $M \in A_w$ defines an invertible operator $\sum_{x \in \mathbb{R}^d} m_xT_x \in B(L^p(\mathbb{R}^d))$ for some $p \in [1, \infty]$, then $M^{-1} \in A_w$.

Proof. For $1 \leq p < \infty$ the result follows from Proposition 2 and [4] Theorem 3.2]. This last result states that $AP_0^w(\rho)$ is spectral.

For $p = \infty$ we follow a different path. Given an operator

$$M = \sum_{x \in \mathbb{R}^d} m_xT_x \in A_w \subset B(L^\infty(\mathbb{R}^d))$$

with $\sum_{x \in \mathbb{R}^d} w(x)\|m_x\|_{L^\infty(\mathbb{R}^d)} < \infty$, we consider the operator

$$N = \sum_{x \in \mathbb{R}^d} T_{x}(m_{-x})T_{x} = \sum_{x \in \mathbb{R}^d} m_{-x}(-x)T_{x} \in A_w \subset B(L^1(\mathbb{R}^d)),$$

which is well defined since $\|T_{x}(m_{-x})\|_{L^\infty(\mathbb{R}^d)} = \|m_{-x}\|_{L^\infty(\mathbb{R}^d)}$. By direct computation, the transpose (Banach adjoint) of $N : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is precisely $M : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$. Thus, $M = N'$ and by [30] Theorem 3,
Chapter 20] it follows that \( N \) is invertible when \( M \) is invertible. Now, by spectrality of \( A_w \) in \( B(L^1(\mathbb{R}^d)) \) (as obtained earlier) and [30, Theorem 8(ii), Chapter 15], we obtain that \( M^{-1} = (N^{-1})' \in A_w \), that is \( M^{-1} = \sum_{x \in \mathbb{R}^d} n_x T_x \) for some bounded functions \( n_x \) such that \( \sum_{x \in \mathbb{R}^d} w(x)\|n_x\|_{L^\infty(\mathbb{R}^d)} < \infty \). □

Remark 2. In [28] two of us used a special case of Theorem 4 for \( \rho \)-periodic (rather than \( \rho \)-almost periodic) operators in \( B(L^2(\mathbb{R}^d)) \). In [28, Example 2.1], however, we neglected to mention this restriction and erroneously implied that all of the operators in \( B(L^2(\mathbb{R}^d)) \) were \( \rho \)-periodic.

We end this section by defining \( A_{w,c} \), an important closed Banach *-subalgebra of \( A_w \), and providing its intrinsic characterization. We let

\[
A_{w,c} = \left\{ M = \sum_{x \in \mathbb{R}^d} m_x T_x \in A_w \mid \text{all } m_x \text{ are continuous, } x \in \mathbb{R}^d \right\}.
\]

Theorem 5. Let \( w \) be an admissible weight. Then, the embedding \( A_{w,c} \hookrightarrow B(L^p(\mathbb{R}^d)), p \in [1, \infty] \) is spectral.

Remark 3. In general Banach algebras the spectrum of an element is not invariant when passing to a closed subalgebra. Usually additional hypotheses are required to obtain that the inverse remains in the subalgebra. Here the situation is different because \( A_w \) can be realized as an algebra of bounded operators on a Hilbert space. The statement follows from a lemma due to Hulanicki as stated, for instance, in [23, Lemma 3]. We will give a direct proof adapted to our setting.

Proof. We need to prove that if \( M \in A_{w,c} \) defines a bounded invertible operator in \( B(L^p(\mathbb{R}^d)) \) for some \( 1 \leq p \leq \infty \), then \( M^{-1} \in A_{w,c} \). Since \( A_w \) is spectral in \( B(L^p(\mathbb{R}^d)) \) by Theorem 4 it follows that \( M^{-1} \in A_w \) and \( M \) defines also a bounded invertible operator on \( B(L^2(\mathbb{R}^d)) \). In particular, we can write

\[
M^{-1} = (M^*M)^{-1}M^*
\]

where \( M^* \) is the adjoint of \( M \). Due to invertibility, it follows that \( A \leq M^*M \leq B \) in the sense of quadratic forms, for some positive real numbers \( 0 < A \leq B < \infty \). Then one can check straightforwardly that

\[
(M^*M)^{-1} = \frac{2}{A+B} \left[ I - \left( I - \frac{2}{A+B}M^*M \right) \right]^{-1} = \frac{2}{A+B} \sum_{n \geq 0} \left( I - \frac{2}{A+B}M^*M \right)^n,
\]

where the series converges in \( B(L^2(\mathbb{R}^d)) \) norm. Using the equality of the spectral radii, we see that the series converges in \( A_w \) norm as well. Since
each term is in $A_{w,c}$ and $A_{w,c}$ is closed, it follows that the limit belongs to $A_{w,c}$. Thus, $M^{-1} \in A_{w,c}$ as desired.

We remark that in the proof of Theorem 7 below we use a shorter argument which could also be applied to see that $(M^*M)^{-1} \in A_w$. 

Next we identify operators $M \in AP^\infty_w(\rho)$ which are actually in $A_{w,c}$. To state these necessary and sufficient conditions we need the following definition.

**Definition 1.** A bounded sequence of functions $\{f_n : n \in \mathbb{N}\} \subseteq L^1(\mathbb{R}^d)$ is called a bounded approximate identity, or, simply, b. a. i. if the following conditions hold:

(i) $\hat{f}_n(0) = 1$ for all $n \in \mathbb{N}$;
(ii) $\text{supp} \hat{f}_n$ is compact;
(iii) $\lim_{n \to \infty} \|f_n * f - f\|_{L^1(\mathbb{R}^d)} = 0$ for all $f \in L^1(\mathbb{R}^d)$.

Existence of such b. a. i. in $L^1(\mathbb{R}^d)$ is a well-known fact. Given $f \in L^1(\mathbb{R}^d)$, we denote by $T(f)$ the convolution operator with kernel $f$, which is bounded on all $L^p(\mathbb{R}^d)$, $p \in [1, \infty]$.

**Theorem 6.** Assume $M \in AP^\infty_w(\rho)$ and the following two conditions hold for some b. a. i. $(f_n)$ in $L^1(\mathbb{R}^d)$:

$$\lim_{n \to \infty} \|(\hat{f}_n \mathcal{M} - \mathcal{M} \hat{f}_n)h\|_\infty = 0 \quad \text{for all } h \in L^\infty(\mathbb{R}^d); \tag{27}$$

$$\lim_{n \to \infty} \|T(f_n)M - M T(f_n)h_0\|_\infty = 0 \quad \text{for all } h_0 \in L^\infty_0(\mathbb{R}^d), \tag{28}$$

where $L^\infty_0$ is the set of all functions in $L^\infty$ that vanish at infinity, i.e. $L^\infty_0 = \{f \in L^\infty(\mathbb{R}^d) \mid \forall \varepsilon > 0 \exists K = K_\varepsilon \text{ compact s.t. } |f(x)| < \varepsilon, \forall x \notin K\}$.

Then $M \in A_{w,c} \subset B(L^\infty(\mathbb{R}^d))$.

Conversely, if $M \in A_{w,c}$ then conditions (27) and (28) are satisfied.

**Proof.** Observe that by (21) the Fourier coefficients $C_\rho$ of $M$ and the operators $C_x T_{-x}$ satisfy conditions (27) and (28) whenever the operator $M$ does. Hence, we only need to establish that every $M \in B(L^\infty(\mathbb{R}^d))$ that satisfies (27), (28) and commutes with all modulations is, in fact, an operator of multiplication by a bounded continuous function $m$ (cf. the proof of Lemma 1). Obviously, this function $m \in L^\infty(\mathbb{R}^d)$ has to be given by $m = M e$, $e \equiv 1$, and we shall see below that under our assumptions this is indeed a continuous function.

Let $h_0 \in L^\infty_0$. Firstly, observe that (27) implies

$$M h_0 = \lim_{n \to \infty} M \hat{f}_n h_0 = \lim_{n \to \infty} \hat{f}_n M h_0 \in L^\infty_0. \tag{29}$$

Secondly, we show that $M \phi_0 h_0 = \phi_0 M h_0$ for any function $\phi_0 \in C_0$ — the space of continuous functions vanishing at infinity. Clearly, it is enough to consider $\phi_0$ in the dense subset of $C_0$ consisting of those functions with compact support and Fourier transform in $L^1$. Since $h_0 \in L^\infty_0$, the functions
\( s \mapsto M_s h_0, s \mapsto M_s M h_0 : \mathbb{R}^d \to L^\infty \) are continuous (the second function is continuous because of (29)). Integrating the equalities \( \hat{\phi}_0(s) M_s M h_0 = \hat{\phi}_0(s) M M_s h_0, s \in \mathbb{R}^d \), we obtain
\[
\mathcal{M} \phi_0 h_0 = \int_{\mathbb{R}^d} \hat{\phi}_0(s) M M_s h_0 ds = \int_{\mathbb{R}^d} \hat{\phi}_0(s) M_s M h_0 ds = \phi_0 M h_0.
\]
As a consequence, we deduce that \( \mathcal{M} \phi_0 = m \phi_0 \) for any \( \phi_0 \in C_0 \).

Next, we use (28) to show that \( \mathcal{M} \phi_0 \in C_0 \) for any \( \phi_0 \in C_0 \) and, hence, \( m \in C \). This follows from
\[
\lim_{n \to \infty} T(f_n) \mathcal{M} \phi_0 = \lim_{n \to \infty} \mathcal{M} T(f_n) \phi_0 = \mathcal{M} \phi_0, \quad \phi_0 \in C_0.
\]

Now, we can use (28) to extend the desired equality to \( L^\infty_0 \). This follows from
\[
\lim_{n \to \infty} T(f_n)(\mathcal{M} - m) h_0 = \lim_{n \to \infty} (\mathcal{M} - m) T(f_n) h_0 = 0, \quad h_0 \in L^\infty_0.
\]
Indeed, the above equalities imply \( T(f)(\mathcal{M} - m) h_0 = 0 \) for all \( f \in L^1 \), \( h_0 \in L^\infty_0 \), and, hence, \( \mathcal{M} h_0 = mh_0 \) for all \( h_0 \in L^\infty_0 \).

Finally, we use (27) in a similar way to extend the equality to all of \( L^\infty(\mathbb{R}^d) \). We have
\[
\lim_{n \to \infty} \hat{f}_n (\mathcal{M} - m) h = \lim_{n \to \infty} (\mathcal{M} - m) \hat{f}_n h = 0, \quad h \in L^\infty(\mathbb{R}^d),
\]
and we conclude \( \mathcal{M} = m \in B(L^\infty(\mathbb{R}^d)) \).

The converse part of the theorem follows by direct computations. \( \square \)

**Remark 4.** We remark that \( \mathcal{M} = \sum_x m_x T_x \in \mathcal{A}_w \subset B(L^\infty(\mathbb{R}^d)) \) always satisfies (27). The operator satisfies (28) if and only if all functions \( m_x \) are continuous. We refer to [10] Section 5 for other conditions of this kind.

### 3.3. Corollaries of spectral invariance.

Let us denote by \( \sigma_p(\mathcal{M}) \) and \( \sigma_{A_w}(\mathcal{M}) \) the spectra of the operator \( \mathcal{M} \in \mathcal{A}_w \) in the algebras \( B(L^p(\mathbb{R}^d)) \), \( p \in [1, \infty] \), and \( \mathcal{A}_w \), respectively.

**Corollary 1.** Consider \( \mathcal{M} = \sum_x m_x T_x \in \mathcal{A}_w \). Then \( \sigma_p(\mathcal{M}) = \sigma_{A_w}(\mathcal{M}) \) for all \( p \in [1, \infty] \). If, in addition, all \( \mathcal{M} \in \mathcal{A}_{w,c} \) then \( \sigma_p(\mathcal{M}) = \sigma_{A_w}(\mathcal{M}) = \sigma_{A_{w,c}}(\mathcal{M}) \).

We conclude the section with the following very important result.

**Theorem 7.** Assume that \( \mathcal{M} \in \mathcal{A}_w \) satisfies \( \mathcal{M}^* = \mathcal{M} = \sum_x m_x T_x \) and \( A_r \| f \|_r \leq \| M f \|_r \) for some \( A_r > 0 \) and all \( f \in L^r(\mathbb{R}^d) \) for some \( r \in [1, \infty] \). Then \( \mathcal{M}^{-1} \in \mathcal{A}_w \).

Moreover, suppose that \( E \subseteq W(L^p, L^q) \), \( 1 \leq p, q \leq +\infty \), is a closed subspace (in the norm of \( W(L^p, L^q) \)) such that \( \mathcal{M} E \subseteq E \). Then \( \mathcal{M}^{-1} E \subseteq E \) and, as a consequence, \( \mathcal{M} E = E \).

**Proof.** From Corollary 1 we deduce that \( \sigma_{A_w}(\mathcal{M}) = \sigma_r(\mathcal{M}) = \sigma_2(\mathcal{M}) \subset \mathbb{R} \) since \( \mathcal{M} \in B(L^2(\mathbb{R}^d)) \) is self-adjoint. Recall that in Banach algebras every boundary point of the spectrum belongs to the approximative spectrum.
The boundedness below condition, however, implies that 0 does not belong to the approximative spectrum of $\mathcal{M} \in B(L^r(\mathbb{R}^d))$. Hence, $0 \notin \sigma_r(\mathcal{M})$ and, by Theorem 4, $\mathcal{M}^{-1} \in \mathcal{A}_w$.

To prove the second part, let $\mathcal{A}_w(E)$ be the subalgebra of $\mathcal{A}_w$ formed by all those operators $S$ such that $\sigma E \subseteq E$. Since $E$ is closed in $W(L^p, L_q^d)$ and $\mathcal{A}_w \hookrightarrow B(W(L^p, L_q^d))$ by Proposition 1, it follows that $\mathcal{A}_w(E)$ is a closed subalgebra of $\mathcal{A}_w$ (we do not claim that it is closed under the involution). From the first part of the proof it follows that the set $\mathcal{C} \setminus \sigma_{\mathcal{A}_w}(\mathcal{M})$ is connected. Consequently, (see for example [12] Theorem VII 5.4), $\sigma_{\mathcal{A}_w(E)}(\mathcal{M}) = \sigma_{\mathcal{A}_w}(\mathcal{M})$. Finally, $0 \notin \sigma_{\mathcal{A}_w}(\mathcal{M}) = \sigma_{\mathcal{A}_w(E)}(\mathcal{M})$ which proves that $\mathcal{M}^{-1} \in \mathcal{A}_w(E)$, as desired.

4. Dual Gabor Frames on Amalgam Spaces

4.1. Multi-window Gabor frames. Let $\Lambda = \Lambda^1 \times \ldots \times \Lambda^n$ be the Cartesian product of separable lattices $\Lambda^i = \alpha_i \mathbb{Z}^d \times \beta_i \mathbb{Z}^d$ and let $g^1, \ldots, g^n \in W(L^{\infty}, L_w^d)$. We consider the (multi-window) Gabor system

$$G = \{ g_{\lambda}^i := \pi(\lambda^i)g^i \mid \lambda^i \in \Lambda^i, 1 \leq i \leq n \}.$$ 

We consider the system $G$ as an indexed set, hence $G$ might contain repeated elements. The frame operator of the system $G$ is given by,

$$S_G = S_{g^1, \Lambda^1} + \ldots + S_{g^n, \Lambda^n},$$

where $S_{g^i, \Lambda^i} = R_{g^i, \Lambda^i}C_{g^i, \Lambda^i}$ (see Section 2.2). For $1 \leq p, q \leq +\infty$ and a $w$-moderate weight $v$, we define the space $S^{p,q}_v(\Lambda) := S^{p,q}_v(\Lambda^1) \times \ldots \times S^{p,q}_v(\Lambda^n)$ endowed with the norm,

$$\|c = (c^1, \ldots, c^n)\|_{S^{p,q}_v(\Lambda)} := \sum_{i=1}^{n} \|c^i\|_{S^{p,q}_v(\Lambda^i)}.$$ 

The analysis map is $W(L^p, L_q^d) \ni f \mapsto C_G(f) := (C_{g^i, \Lambda^i}(f))_{1 \leq i \leq n} \in S^{p,q}_v(\Lambda)$, while the synthesis map is $S^{p,q}_v \ni c \mapsto R_G(c) := \sum_{i=1}^{n} R_{g^i, \Lambda^i}(c^i) \in W(L^p, L_q^d)$. With these definitions, the boundedness results in Theorem 4 extend immediately to the multi-window case. The frame expansions are however more complicated since the dual system of a frame of the form of $G$ may not be a multi-window Gabor frame. We now investigate this matter.

4.2. Invertibility of the frame operator and expansions.

**Theorem 8.** Let $w$ be an admissible weight, $g^1, \ldots, g^n \in W(L^{\infty}, L_w^d)$ and $\Lambda = \Lambda^1 \times \ldots \times \Lambda^n$, with $\Lambda^i = \alpha_i \mathbb{Z}^d \times \beta_i \mathbb{Z}^d$ separable lattices. Suppose that the Gabor system

$$G = \{ g_{\lambda}^i := \pi(\lambda^i)g^i \mid \lambda^i \in \Lambda^i, 1 \leq i \leq n \},$$

is such that its frame operator $S_G$ is bounded below in some $L^r(\mathbb{R}^d)$ for some $r \in [1, \infty]$, i.e.

$$A_r \|f\|_r \leq \|S_G f\|_r, \quad A_r > 0, \quad \text{for all } f \in L^r(\mathbb{R}^d).$$
Theorem 2 implies that for all $1 \leq p, q \leq +\infty$ and every $v$-moderate weight $v$. Moreover, the inverse operator $S_g^{-1} : W(L^p, L^q_v) \to W(L^p, L^q_v)$ is continuous both in $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$ and the norm topologies.

Proof. For each $1 \leq i \leq n$, the frame operator $S_{g^i, \Lambda^i} = R_{g^i, \Lambda^i} C_{g^i, \Lambda^i}$ belongs to the algebra $A_w$ as a consequence of the Walnut representation in Theorem 3. Hence, $S_g = S_{g^1, \Lambda^1} + \ldots + S_{g^n, \Lambda^n} \in A_w$. Since $S_g$ is bounded below in $L^p(\mathbb{R}^d)$, Theorem 7 implies that $S_g^{-1} \in A_w$. The conclusion now follows from Proposition 1.

We now derive the corresponding Gabor expansions.

**Theorem 9.** Under the conditions of Theorem 8, define the dual atoms by $g_{\Lambda^i}^i := S_g^{-1}(g_{\Lambda^i})$. Let $1 \leq p, q \leq +\infty$ and $v$ be a $w$-moderate weight. Then the following expansions hold.

(a) For every $f \in W(L^p, L^q_v)$,

$$f = \lim_{N,M \to \infty} \sum_{i=1}^{n} \sum_{|k|_\infty \leq N} \sum_{|j|_\infty \leq M} r_{\beta, j, M} \left< f, g_{\Lambda^i}^i \right> g_{\Lambda^i}^i,$$

where the regularizing weights $r_{\beta, j, M}$ are given in (7) and the series converge in the $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$-topology. For $p, q < +\infty$ the series also converge in the norm of $W(L^p, L^q_v)$.

(b) If $1 < p < +\infty$ and $q < +\infty$, for every $f \in W(L^p, L^q_v)$,

$$f = \lim_{N,M \to \infty} \sum_{i=1}^{n} \sum_{|k|_\infty \leq N} \sum_{|j|_\infty \leq M} \left< f, g_{\Lambda^i}^i \right> g_{\Lambda^i}^i,$$

where the series converge in the in the norm of $W(L^p, L^q_v)$.

**Remark 5.** A more refined convergence statement, including more sophisticated summability methods can be obtained using the results in [16].

Proof. Theorem 2 implies that for all $f \in W(L^p, L^q_v)$,

$$S_g(f) = \lim_{N,M \to \infty} \sum_{i=1}^{n} \sum_{|k|_\infty \leq N} \sum_{|j|_\infty \leq M} r_{\beta, j, M} \left< f, g_{\Lambda^i}^i \right> g_{\Lambda^i}^i,$$

with the kind of convergence required in (a). Since $S_g^{-1} \in A_w$, Proposition 1 implies that $S_g^{-1} : W(L^p, L^q_v) \to W(L^p, L^q_v)$ is continuous both in the norm and $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$-topology. Consequently, we can apply $S_g^{-1}$
to both sides of (30) to obtain the first expansion in (a). The second one follows by applying (30) to the function $S_g^{-1}(f)$ and using Proposition 1 to get,

$$\langle S_g^{-1}(f), g_{\lambda_i}^i \rangle = \langle f, S_g^{-1}(g_{\lambda_i}^i) \rangle = \langle f, \tilde{g}_{\lambda_i}^i \rangle.$$ 

The statement in (b) follows similarly, this time using the corresponding statement in Theorem 2. □

4.3. Continuity of dual generators. We now apply Theorem 7 to Gabor expansions.

**Theorem 10.** In the conditions of Theorem 8, let $1 \leq p, q \leq +\infty$ and let $v$ be a $w$-moderate weight. Let $E \subseteq W(L_p^1, L_q^w)$ be a closed subspace (in the norm of $W(L_p^1, L_q^w)$) such that $S_g E \subseteq E$. Suppose that the atoms $g^1, \ldots, g^n \in E$. Then the dual atoms, $\tilde{g}_{\lambda_i}^i = S_g^{-1}(g_{\lambda_i}^i) \in E$.

**Proof.** As seen in the proof of Theorem 8, $S_g \in A_w$. Hence, the conclusion follows from Theorem 7. □

As an application of Theorem 10 we obtain the following corollary, which was one of our main motivations. The case $n = 1$ was an open problem in [28].

**Corollary 2.** In the conditions of Theorem 8, if all the atoms $g^1, \ldots, g^n$ are continuous functions, so are all the dual atoms $\tilde{g}_{\lambda_i}^i = S_g^{-1}(g_{\lambda_i}^i)$.

**Proof.** We apply Theorem 10 to the subspace $W(C_0, L^1_w)$ formed by the functions of $W(L^\infty, L^1_w)$ that are continuous. To this end we need to observe that $S_g W(C_0, L^1_w) \subseteq W(C_0, L^1_w)$. Since $S_g = S_{g^1, \Lambda^1} + \ldots + S_{g^n, \Lambda^n}$, it suffices to show that each $S_{g^i, \Lambda^i}$ maps $W(C_0, L^1_w)$ into $W(C_0, L^1_w)$.

Let $f \in W(C_0, L^1_w)$. The Walnut representation of $S_{g^i, \Lambda^i}$ in Theorem 3 gives $S_{g^i, \Lambda^i}(f) = \beta_i^{-d} \sum_j G_j^i T_j f$ with absolute convergence in the norm of $W(L^\infty, L^1_w)$. Hence it suffices to observe that each of the functions $G_j^i$ is continuous. According to Theorem 3 these are given by

$$G_j^i(x) := \sum_{k \in \mathbb{Z}^d} g^i(x - j/\beta_i - \alpha_i k)g^i(x - \alpha_i k).$$ 

Since the function $g^i$ is continuous it suffices to note that in the last series the convergence is locally uniform. This is an easy consequence of the fact that $\|g^i\|_W(L^\infty, L^1_w) < +\infty$. □

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