Extremal metrics on Hartogs domains

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Abstract
An $n$-dimensional Hartogs domain $D_F$ with strongly pseudoconvex boundary can be equipped with a natural Kähler metric $g_F$. In this paper we prove that if $g_F$ is an extremal Kähler metric then $(D_F, g_F)$ is biholomorphically isometric to the $n$-dimensional complex hyperbolic space.

Keywords: Kähler metrics; Hartogs domain; extremal metrics; generalized curvatures; canonical metrics.

Subj.Class: 53C55, 32Q15, 32T15.

1 Introduction and statements of the main results

The study of the existence and uniqueness of a preferred Kähler metric on a given complex manifold $M$ is a very interesting and important area of research, both from the mathematical and from the physical point of view. Many definitions of canonical metrics (Einstein, constant scalar curvature, extremal, balanced and so on) have been given both in the compact and in the noncompact case (see e.g. [2], [12] and [20]). In the noncompact case many important questions are still open. For example Yau raised the question on the classification of Bergman Einstein metrics on strongly pseudoconvex domains and S.Y. Cheng conjectured that if the Bergman metric on a strongly pseudoconvex domain is Einstein, then the domain is biholomorphic to the ball (see [10]).

In this paper we are interested in extremal Kähler metrics on noncompact manifolds. Extremal metrics were introduced and christened by Calabi [4] in the compact case as the solution for the variational problem in a Kähler class defined by the square integral of the scalar curvature. Therefore they are a generalization of constant scalar curvature metrics. Calabi himself constructs some compact manifolds with an extremal metric which cannot admit a metric with constant scalar curvature.

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Only recently extremal Kähler metrics were rediscovered by several mathematicians due to their link with the stability of complex vector bundles (see e.g. [3], [7], [11], [15] and [18]).

Obviously extremal metrics cannot be defined in the noncompact case as the solutions of a variational problem involving some integral on the manifold. Nevertheless, in the compact case, one can give an alternative definition of these metrics using local coordinates (see (23) below) which makes sense also in the noncompact case. In this case, the existence and uniqueness of such metrics are far from being understood. For example, only recently [5] (see also [6]), it has been shown the existence of a nontrivial (namely with nonconstant scalar curvature) extremal and complete Kähler metric in a complex one-dimensional manifold.

Our main result is the following theorem which deals with extremal Kähler metrics on a particular class of strongly pseudoconvex domains, the so called Hartogs domains (see the next section for their definition and main properties).

**Theorem 1.1** Let \((D_F, g_F)\) be an \(n\)-dimensional strongly pseudoconvex Hartogs domain. Assume that \(g_F\) is an extremal Kähler metric. Then \((D_F, g_F)\) is biholomorphically isometric to the \(n\)-dimensional complex hyperbolic space \((\mathbb{C}^n, g_{hyp})\), where \(\mathbb{C}^n\) is the unit ball in \(\mathbb{C}^n\) and \(g_{hyp}\) denotes the hyperbolic metric.

Two remarks are in order (compare with Cheng’s conjecture above). First, it is worth pointing out that, in contrast to the Bergman metric, \(g_F\) is defined also if the domain \(D_F\) is unbounded. Secondly, the extremality assumption in Theorem 1.1 is weaker than Einstein’s condition (actually it is even weaker than the constancy of the scalar curvature).

The paper is organized as follows. In the next section we recall the definition of Hartogs domain \((D_F, g_F)\) and we analyze the relation between the pseudoconvexity of \(D_F\) and the Kähler condition of \(g_F\). We also compute its Ricci and scalar curvatures. The last section is dedicated to the proof of Theorem 1.1.

## 2 Strongly pseudoconvex Hartogs domains

Let \(x_0 \in \mathbb{R}^+ \cup \{+\infty\}\) and let \(F : [0, x_0) \rightarrow (0, +\infty)\) be a decreasing continuous function, smooth on \((0, x_0)\). The Hartogs domain \(D_F \subset \mathbb{C}^n\) associated
to the function $F$ is defined by

$$D_F = \{(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n \mid |z_0|^2 < x_0, \ |z_1|^2 + \cdots + |z_{n-1}|^2 < F(|z_0|^2)\}.$$  

One can prove that the assumption of strongly pseudoconvexity of $D_F$ is equivalent (see Proposition 2.1 below) to the fact that the natural $(1,1)$-form on $D_F$ given by

$$\omega_F = \frac{i}{2} \partial \bar{\partial} \log \frac{1}{F(|z_0|^2) - |z_1|^2 - \cdots - |z_{n-1}|^2}$$  \hspace{1cm} (1)$$

is a Kähler form on $D_F$. The Kähler metric $g_F$ associated to the Kähler form $\omega_F$ is the metric we will be dealing with in the present paper. (Observe that for $F(x) = 1-x, 0 \leq x < 1$, $D_F$ equals the $n$-dimensional complex hyperbolic space $\mathbb{C}H^n$ and $g_F$ is the hyperbolic metric). In the 2-dimensional case this metric has been considered in [9] and [17] in the framework of quantization of Kähler manifolds. In [16], the first author studied the Kähler immersions of $(D_F, g_F)$ into finite or infinite dimensional complex space forms and [8] is concerned with the existence of global symplectic coordinates on $(D_F, \omega_F)$.

**Proposition 2.1** Let $D_F$ be a Hartogs domain in $\mathbb{C}^n$. Then the following conditions are equivalent:

(i) the $(1,1)$-form $\omega_F$ given by (1) is a Kähler form;

(ii) the function $-\frac{xF'(x)}{F(x)}$ is strictly increasing, namely $-(\frac{xF'(x)}{F(x)})' > 0$ for every $x \in [0, x_0]$;

(iii) the boundary of $D_F$ is strongly pseudoconvex at all $z = (z_0, z_1, \ldots, z_{n-1})$ with $|z_0|^2 < x_0$;

(iv) $D_F$ is strongly pseudoconvex.

**Proof:** (i) $\Leftrightarrow$ (ii) Set

$$A = F(|z_0|^2) - |z_1|^2 - \cdots - |z_{n-1}|^2.$$  \hspace{1cm} (2)$$

Then $\omega_F$ is a Kähler form if and only if the real-valued function $\Phi = -\log A$ is strictly plurisubharmonic, i.e. the matrix $g_{\alpha \bar{\beta}} = \left(\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta}\right)$, $\alpha, \beta = 0, \ldots, n - 1$ is positive definite, where

$$\omega_F = \frac{i}{2} \sum_{\alpha,\beta=0}^{n-1} g_{\alpha \bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta.$$  \hspace{1cm} (3)$$
A straightforward computation gives

\[ \frac{\partial^2 \Phi}{\partial z_0 \partial \bar{z}_\beta} = \frac{F'(|z_0|^2)\bar{z}_0 z_\beta}{A^2}, \quad \beta = 1, \ldots, n-1 \]

and

\[ \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{\delta_{\alpha\beta} A + \bar{z}_\alpha z_\beta}{A^2}, \quad \alpha, \beta = 1, \ldots, n-1. \]

Then, by setting

\[ C = F'^2(|z_0|^2)|z_0|^2 - (F''(|z_0|^2)|z_0|^2 + F'(|z_0|^2))A, \]

one sees that the matrix \( h = (g_{\alpha\bar{\beta}}) = (\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta})_{\alpha,\bar{\beta}=0,\ldots,n-1} \) is given by:

\[ h = \frac{1}{A^2} \begin{pmatrix}
C & -F'z_0\bar{z}_1 & \cdots & -F'z_0\bar{z}_n & \cdots & -F'z_0\bar{z}_{n-1} \\
-F'z_0\bar{z}_1 & A + |z_1|^2 & \cdots & \bar{z}_1z_\alpha & \cdots & \bar{z}_1z_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-F'z_0\bar{z}_n & \bar{z}_n z_\alpha & \cdots & A + |z_\alpha|^2 & \cdots & \bar{z}_n z_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-F'z_0\bar{z}_{n-1} & \bar{z}_{n-1} z_\alpha & \cdots & z_\alpha \bar{z}_{n-1} & \cdots & A + |z_{n-1}|^2
\end{pmatrix}. \quad (5) \]

First notice that the \((n-1) \times (n-1)\) matrix obtained by deleting the first row and the first column of \( h \) is positive definite. Indeed it is not hard to see that, for all \( 1 \leq \alpha \leq n-1, \)

\[
\det \begin{pmatrix}
A + |z_\alpha|^2 & \bar{z}_\alpha z_{\alpha+1} & \cdots & \bar{z}_\alpha z_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{z}_{n-1} z_\alpha & \bar{z}_{n-1} \bar{z}_{\alpha+1} & \cdots & A + |z_{n-1}|^2
\end{pmatrix} = A^{n-\alpha} + A^{n-\alpha-1}(|z_\alpha|^2 + \cdots + |z_{n-1}|^2) > 0. \quad (6)
\]

On the other hand, by the Laplace expansion along the first row, we get

\[
\det(h) = \frac{C}{A^{2n}}[A^{n-1} + A^{n-2}(|z_1|^2 + \cdots + |z_{n-1}|^2)] +
\]
\[
\frac{F'z_0 \bar{z}_1}{A^{2n}} \det \begin{pmatrix}
-F'z_0 \bar{z}_1 & z_2 \bar{z}_1 & \cdots & z_{n-1} \bar{z}_1 \\
-F'z_0 \bar{z}_2 & A + |z_2|^2 & \cdots & z_{n-1} \bar{z}_2 \\
\vdots & \vdots & \ddots & \vdots \\
-F'z_0 \bar{z}_{n-1} & z_2 \bar{z}_{n-1} & \cdots & A + |z_{n-1}|^2
\end{pmatrix} + \cdots +
\]

\[
+(-1)^n \frac{F'z_0 \bar{z}_{n-1}}{A^{2n}} \det \begin{pmatrix}
-F'z_0 \bar{z}_1 & A + |z_1|^2 & \cdots & z_{n-2} \bar{z}_1 \\
-F'z_0 \bar{z}_2 & z_1 \bar{z}_2 & \cdots & z_{n-2} \bar{z}_2 \\
\vdots & \vdots & \ddots & \vdots \\
-F'z_0 \bar{z}_{n-1} & z_1 \bar{z}_{n-1} & \cdots & A + |z_{n-2}|^2
\end{pmatrix}
= \frac{C}{A^2} [A^{n-1} + A^{n-2}(|z_1|^2 + \cdots + |z_{n-1}|^2)] +
\]

\[
+ \frac{F'^2|z_0|^2|z_1|^2}{A^{2n}} \det \begin{pmatrix}
-1 & z_2 & \cdots & z_{n-1} \\
-\bar{z}_2 & A + |z_2|^2 & \cdots & z_{n-1} \bar{z}_2 \\
\vdots & \vdots & \ddots & \vdots \\
-\bar{z}_{n-1} & z_2 \bar{z}_{n-1} & \cdots & A + |z_{n-1}|^2
\end{pmatrix} + \cdots +
\]

\[
+(-1)^n \frac{F'^2|z_0|^2|z_{n-1}|^2}{A^{2n}} \det \begin{pmatrix}
-\bar{z}_1 & A + |z_1|^2 & \cdots & z_{n-2} \bar{z}_1 \\
-\bar{z}_2 & z_1 \bar{z}_2 & \cdots & z_{n-2} \bar{z}_2 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & z_1 & \cdots & z_{n-2}
\end{pmatrix}
= \frac{1}{A^{n+2}} [CA + (C - F'^2|z_0|^2)(|z_1|^2 + \cdots + |z_{n-1}|^2)].
\]

By substituting (2) and (4) into this last equality one gets

\[
\det(h) = -\frac{F'^2}{A^{n+1}} \left( \frac{xF'}{F} \right)' \bigg|_{x=|z_0|^2}.
\]  

(7)

Hence, by (6) and (7), the matrix \( \frac{\partial^2 \phi}{\partial z_0 \partial \bar{z}_0} \) is positive definite if and only if \( \left( \frac{xF'}{F} \right)' < 0 \).

Before proving equivalence (ii) \( \iff \) (iii) we briefly recall some facts on complex domains (see e.g. [13]). Let \( \Omega \subseteq \mathbb{C}^n \) be any complex domain of
$\mathbb{C}^n$ with smooth boundary $\partial\Omega$, and let $z \in \partial\Omega$. Assume that there exists a
smooth function $\rho : \mathbb{C}^n \to \mathbb{R}$ (called defining function for $\Omega$ at $z$) satisfying
the following: for some neighbourhood $U$ of $z$, $\rho < 0$ on $U \cap \Omega$, $\rho > 0$ on
$U \setminus \Omega$ and $\rho = 0$ on $U \cap \partial\Omega$; $\text{grad} \rho \neq 0$ on $\partial\Omega$. In this case $\partial\Omega$ is said to be
strongly pseudoconvex at $z$ if the Levi form

$$L(\rho, z)(X) = \sum_{\alpha, \beta=0}^{n-1} \frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z}_\beta}(z) X_\alpha \bar{X}_\beta$$

is positive definite on

$$S_\rho = \{(X_0, \ldots, X_{n-1}) \in \mathbb{C}^n \mid \sum_{\alpha=0}^{n-1} \frac{\partial \rho}{\partial z_\alpha}(z) X_\alpha = 0\}$$

(it is easily seen that this definition does not depend on the particular defining function $\rho$).

$(ii) \iff (iii)$ Let now $\Omega = D_F$ and let us fix $z = (z_0, z_1, \ldots, z_{n-1}) \in \partial D_F
with |z_0|^2 < x_0$. Then, $|z_1|^2 + \cdots + |z_{n-1}|^2 = F(|z_0|^2)$. In this case

$$\rho(z_0, z_1, \ldots, z_{n-1}) = |z_1|^2 + \cdots + |z_{n-1}|^2 - F(|z_0|^2)$$

is a (globally) defining function for $D_F$ at $z$, the Levi form for $D_F$ reads as

$$L(\rho, z)(X) = |X_1|^2 + \cdots + |X_{n-1}|^2 - (F' + F''|z_0|^2)|X_0|^2$$

and

$$S_\rho = \{(X_0, X_1, \ldots, X_{n-1}) \in \mathbb{C}^n \mid - F'z_0 X_0 + \bar{z}_1 X_1 + \cdots + \bar{z}_{n-1} X_{n-1} = 0\}. \quad (9)$$

We distinguish two cases: $z_0 = 0$ and $z_0 \neq 0$. At $z_0 = 0$ the Levi form
reads as

$$L(\rho, z)(X) = |X_1|^2 + \cdots + |X_{n-1}|^2 - F'(0)|X_0|^2$$

which is strictly positive for any non-zero vector $(X_0, X_1, \ldots, X_{n-1})$ (not
necessarily in $S_\rho$) because $F$ is assumed to be decreasing.

If $z_0 \neq 0$ by (9) we obtain $X_0 = \frac{z_1 X_1 + \cdots + \bar{z}_{n-1} X_{n-1}}{F'|z_0}$ which, substituted in
(8), gives:

$$L(X, z) = |X_1|^2 + \cdots + |X_{n-1}|^2 - \frac{F' + F''|z_0|^2}{F''|z_0|^2}|\bar{z}_1 X_1 + \cdots + \bar{z}_{n-1} X_{n-1}|^2. \quad (10)$$
Therefore we are reduced to show that:

\[(xF'/F)' < 0 \text{ for } x \in (0, x_0) \text{ if and only if } L(X, z) \text{ is strictly positive for every } (X_1, \ldots, X_{n-1}) \neq (0, \ldots, 0) \text{ and every } (z_0, z_1, \ldots, z_{n-1}) \in \partial D_F, 0 < |z_0|^2 < x_0.\]

If \((xF'/F)' < 0\) then \((F' + xF'')F < xF'^2\) and, since \(F(|z_0|^2) = |z_1|^2 + \cdots + |z_{n-1}|^2\), we get:

\[L(X, z) > |X_1|^2 + \cdots + |X_{n-1}|^2 - \frac{1}{F(|z_0|^2)}|\bar{z}_1 X_1 + \cdots + \bar{z}_{n-1} X_{n-1}|^2 = \]

\[= \frac{(|X_1|^2 + \cdots + |X_{n-1}|^2)(|z_1|^2 + \cdots + |z_{n-1}|^2) - |\bar{z}_1 X_1 + \cdots + \bar{z}_{n-1} X_{n-1}|^2}{|z_1|^2 + \cdots + |z_{n-1}|^2}\]

and the conclusion follows by the Cauchy-Schwarz inequality.

Conversely, assume that \(L(X, z)\) is strictly positive for every \((X_1, \ldots, X_{n-1}) \neq (0, \ldots, 0)\) and each \(z = (z_0, z_1, \ldots, z_{n-1})\) such that \(F(|z_0|^2) = |z_1|^2 + \cdots + |z_{n-1}|^2\). By inserting \((X_1, \ldots, X_{n-1}) = (z_1, \ldots, z_{n-1})\) in (10) we get

\[L(z, z) = F(|z_0|^2) \left(1 - \frac{F' + F''|z_0|^2}{F'^2|z_0|^2} F(|z_0|^2)\right) > 0\]

which implies \((xF'/F)' < 0\).

Finally, the proof of the equivalence \((ii) \Leftrightarrow (iv)\) is completely analogous to that given in [9] (Proposition 3.4 and Proposition 3.6) for the 2-dimensional case, to which the reader is referred.

\[\square\]

**Remark 2.2** Notice that the previous proposition is a generalization of Proposition 3.6 in [9] proved there for the 2-dimensional case.

Recall (see e.g. [14]) that the Ricci curvature and the scalar curvature of a Kähler metric \(g\) on an \(n\)-dimensional complex manifold \((M, g)\) are given respectively by

\[\text{Ric}_{\alpha\beta} = -\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \left(\log \det(h)\right), \quad \alpha, \beta = 0, \ldots, n - 1 \quad (11)\]

and

\[\text{scal}_g = \sum_{\alpha, \beta = 0}^{n-1} g^{\beta\bar{\gamma}} \text{Ric}_{\alpha\bar{\gamma}}, \quad (12)\]
where $g^{\bar{\alpha}\beta}$ are the entries of the inverse of $(g_{\alpha\bar{\beta}})$, namely $\sum_{\alpha=0}^{n-1} g^{\bar{\alpha}\beta} g_{\alpha\bar{\gamma}} = \delta_{\beta\gamma}$.

When $(M, g) = (D_F, g_F)$, using (5) it is not hard to check the validity of the following equalities.

$$g^{00} = \frac{A}{B} F,$$ (13)

$$g^{\beta 0} = \frac{A}{B} F' z_0 \bar{z}_{\beta}, \quad \beta = 1, \ldots, n - 1,$$ (14)

$$g^{\bar{\alpha}\bar{\beta}} = \frac{A}{B} (F' + F''|z_0|^2) z_{\alpha} \bar{z}_{\beta}, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \ldots, n - 1,$$ (15)

$$g^{\bar{\alpha}\beta} = \frac{A}{B} [B + (F' + F''|z_0|^2)|\bar{z}_{\beta}|^2], \quad \beta = 1, \ldots, n - 1,$$ (16)

where

$$B = B(|z_0|^2) = F'^2|z_0|^2 - F(F' + F''|z_0|^2).$$

Now, set

$$L(x) = \frac{d}{dx} [x \frac{d}{dx} \log(x F'^2 - F(F' + F''x))].$$

A straightforward computation using (7) and (11) gives:

$$\text{Ric}_{00} = -L(|z_0|^2) - (n + 1) g_{00},$$ (17)

$$\text{Ric}_{\alpha\beta} = -(n + 1) g_{\alpha\beta}, \quad \alpha > 0.$$ (18)

Then, by (12), the scalar curvature of the metric $g_F$ equals

$$\text{scal}_{g_F} = -L(|z_0|^2) g^{00} - (n + 1) \sum_{\alpha, \beta=0 \atop \alpha \neq \beta} g_{\alpha\bar{\beta}} g^{\bar{\alpha}\beta} = -L(|z_0|^2) g^{00} - n(n + 1),$$

which by (13) reads as

$$\text{scal}_{g_F} = -\frac{A}{B} FL - n(n + 1).$$ (19)
3 Proof of the main result

In order to prove Theorem 1.1, we need Lemma 3.1 below, interesting on its own sake, which is a generalization of a result proved by the first author for 2-dimensional Hartogs domains (see Theorem 4.8 in [17]).

We first recall the definition of generalized scalar curvatures. Given a Kähler metric $g$ on an $n$-dimensional complex manifold $M$, its generalized scalar curvatures are the $n$ smooth functions $\rho_0, \ldots, \rho_{n-1}$ on $M$ satisfying the following equation:

$$\frac{\det(g_{\alpha\bar{\beta}} + tRic_{\alpha\bar{\beta}})}{\det(g_{\alpha\bar{\beta}})} = 1 + \sum_{k=0}^{n-1} \rho_k t^{k+1},$$

where $g_{\alpha\bar{\beta}}$ are the entries of the metric in local coordinates. Observe that for $k = 0$ we recover the value of the scalar curvature, namely

$$\rho_0 = \text{scal}_g.$$  

The introduction and the study of these curvatures (in the compact case) is due to K. Ogiue [19] to whom the reader is referred for further results. In particular, in a joint paper with B.Y. Chen [1], he studies the constancy of one of the generalized scalar curvatures. Their main result is that, under suitable cohomological conditions, the constancy of one of the $\rho_k$'s, $k = 0, \ldots, n-1$, implies that the metric $g$ is Einstein.

Lemma 3.1 Let $(D_F, g_F)$ be an $n$-dimensional Hartogs domain. Assume that one of its generalized scalar curvatures is constant. Then $(D_F, g_F)$ is biholomorphically isometric to the $n$-dimensional hyperbolic space.

Proof: By (17), (18) we get

$$\frac{\det(g_{\alpha\bar{\beta}} + tRic_{\alpha\bar{\beta}})}{\det(g_{\alpha\bar{\beta}})} = (1 - (n + 1)t)^n - tL(1 - (n + 1)t)^{n-1} \frac{AF}{B}.$$ 

So the generalized curvatures of $(D_F, g_F)$ are given by

$$\rho_k = (n+1)^k(-1)^{k+1}\binom{n-1}{k} \left[ \frac{n(n+1)}{k+1} + \frac{AF}{B} \right], \quad k = 0, \ldots, n-1$$
Notice that, for $k = 0$, we get $\rho_0 = -\frac{AFL}{B} - n(n + 1) = \text{scal}_g$, (compare with (19)) in accordance with (21).

Thus, $\rho_k$ is constant for some (equivalently, for any) $k = 0, \ldots, n - 1$ if and only if $\frac{AFL}{B}$ is constant. Since $A = F(|z_0|^2) - |z_1|^2 - \cdots - |z_{n-1}|^2$ depends on $z_1, \ldots, z_{n-1}$ while $\frac{LF}{B}$ depends only on $z_0$, this implies that $L = 0$, i.e.

$$
\frac{d}{dx} \left[ x \frac{d}{dx} \log(xF'^2 - F(F' + F''x)) \right]_{x = |z_0|^2} \equiv 0.
$$

Now, we continue as in the proof of Theorem 4.8 in [17] and conclude that $F(x) = c_1 - c_2x$, $x = |z_0|^2$, with $c_1, c_2 > 0$, which implies that $D_F$ is biholomorphically isometric to the hyperbolic space $\mathbb{C}H^n$ via the map

$$
\phi : D_F \to \mathbb{C}H^n, \ (z_0, z_1, \ldots, z_{n-1}) \mapsto \left( \frac{z_0}{\sqrt{c_1/c_2}}, \frac{z_1}{\sqrt{c_1}}, \ldots, \frac{z_{n-1}}{\sqrt{c_1}} \right).
$$

\[ \square \]

**Proof of Theorem 1.1** The system of PDE’s which has to be satisfied by an extremal Kähler metric is the following (see [4]):

$$
\frac{\partial}{\partial z_\gamma} \left( \sum_{\beta=0}^{n-1} g^{\beta\bar{\alpha}} \frac{\partial \text{scal}_g}{\partial z_\beta} \right) = 0, \quad (23)
$$

for every $\alpha, \gamma = 0, \ldots, n - 1$ (indeed, this is equivalent to the requirement that the (1,0)-part of the Hamiltonian vector field associated to the scalar curvature is holomorphic).

In order to use equations (23) for $(D_F, g_F)$ we write the entries $g^{\beta\bar{\alpha}}$ by separating the terms depending only on $z_0$ from the other terms. More precisely, (13), (14), (15) and (16) can be written as follows.

$$
g^{00} = P_{00} + Q_{00}(|z_1|^2 + \cdots + |z_{n-1}|^2),
$$

$$
g^{0\alpha} = \bar{z}_0 z_\alpha \left[ P_{0\alpha} + Q_{0\alpha}(|z_1|^2 + \cdots + |z_{n-1}|^2) \right], \quad \alpha = 1, \ldots, n - 1,
$$

$$
g^{\alpha\bar{\alpha}} = F + P_{a\bar{a}} |z_\alpha|^2 - (1 + Q_{a\bar{a}} |z_\alpha|^2) \sum_{k \neq \alpha} |z_k|^2 - R_{a\bar{a}} |z_\alpha|^4, \quad \alpha = 1, \ldots, n - 1,
$$

10
\[ g^{\alpha \bar{\beta}} = \bar{z}_\beta z_\alpha [P_{ab} + Q_{ab}(|z_1|^2 + \cdots + |z_{n-1}|^2)], \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \ldots, n - 1, \]

where

\[
\begin{align*}
P_{00} &= \frac{F^2}{B}, & Q_{00} &= -\frac{F}{B}, \\
P_{0\alpha} &= \frac{F'F}{B}, & Q_{0\alpha} &= -\frac{F'}{B},
\end{align*}
\]

\[
\begin{align*}
P_{aa} &= \frac{F(F' + F''|z_0|^2)}{B} - 1, & Q_{aa} &= R_{aa} = \frac{F' + F''|z_0|^2}{B}, \\
P_{ab} &= \frac{F(F' + F''|z_0|^2)}{B}, & Q_{ab} &= -\frac{F' + F''|z_0|^2}{B}
\end{align*}
\]

are all functions depending only on \(|z_0|^2|\).

We also have (cfr. (19))

\[ \text{scal}_{g_F} = -n(n + 1) + G(F - |z_1|^2 - \cdots - |z_{n-1}|^2) \quad (24) \]

where

\[ G = G(|z_0|^2) = -\frac{L(|z_0|^2)F(|z_0|^2)}{B(|z_0|^2)}. \]

Assume that \(g_F\) is an extremal metric, namely equation (23) is satisfied. We are going to show that \(\text{scal}_{g_F}\) is constant and hence by Lemma 3.1 \((D_F, g_F)\) is biholomorphically isometric to \((\mathbb{C}H^n, g_{hyp})\). In order to do that, fix \(i \geq 1\) and let us consider equation (23) when \(g = g_F\) for \(\alpha = 0, \gamma = i\). We have

\[
\frac{\partial \text{scal}_{g_F}}{\partial \bar{z}_0} = G'z_0(F - |z_1|^2 - \cdots - |z_{n-1}|^2) + z_0GF'
\]

\[
\frac{\partial \text{scal}_{g_F}}{\partial z_i} = -Gz_i.
\]

So, equation (23) gives

\[
\frac{\partial}{\partial z_i} \left\{ P_{00} + Q_{00} \sum_{k=1}^{n-1} |z_k|^2 \right\} \left[ G'z_0(F - \sum_{k=1}^{n-1} |z_k|^2) + z_0GF' \right] = -
\]

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\[-z_0 G \left[ P_{0\alpha} + Q_{0\alpha} \sum_{k=1}^{n-1} |z_k|^2 \right] \sum_{k=1}^{n-1} |z_k|^2 \right] = 0,\]

namely

\[Q_{00} \bar{z}_i \left[ G' z_0( F - \sum_{k=1}^{n-1} |z_k|^2 ) + z_0 GF' \right] - G' z_0 \bar{z}_i \left[ P_{0\alpha} + Q_{0\alpha} \sum_{k=1}^{n-1} |z_k|^2 \right] -
\]

\[-z_0 GQ_{0a} \bar{z}_i \sum_{k=1}^{n-1} |z_k|^2 - z_0 z_i G \left[ P_{0\alpha} + Q_{0\alpha} \sum_{k=1}^{n-1} |z_k|^2 \right] = 0\]

Deriving again with respect to \( \bar{z}_i \), we get

\[-2Q_{00} G' z_0 z_i^2 - 2GQ_{0a} z_i^2 z_i = 0.\]

Let us assume \( z_0 \bar{z}_i \neq 0 \). This implies \( Q_{00} G' + GQ_{0a} = 0 \), i.e. \( GF' + FG' = 0 \)
or, equivalently, \( G = \frac{c}{F} \) for some constant \( c \in \mathbb{R} \). The proof of Theorem 1.1
will be completed by showing that \( c = 0 \). In fact, in this case \( G = 0 \) on the open and dense subset of \( D_F \) consisting of those points such that \( z_0 \bar{z}_i \neq 0 \)
and therefore, by (24), \( \text{scal}_F \) is constant on \( D_F \). In order to prove that \( c = 0 \), let us now consider equation (23) for \( \alpha = i, \gamma = i \).

\[
\frac{\partial}{\partial \bar{z}_i} \left\{ \bar{z}_0 z_i \left[ G' z_0 ( F - \sum_{k=1}^{n-1} |z_k|^2 ) + GF' z_0 \right] \left[ P_{0\alpha} + Q_{0\alpha} \sum_{k=1}^{n-1} |z_k|^2 \right] -
\right.
\]

\[-G z_i \left[ F + P_{aa} |z_i|^2 - (1 + Q_{aa} |z_i|^2) \sum_{k \neq 0, i} |z_k|^2 \right] -
\]

\[-G z_i \sum_{k \neq 0, i} |z_k|^2 \left[ P_{ab} + Q_{ab} \sum_{k=1}^{n-1} |z_k|^2 \right] \right\} = 0.\]

This implies

\[-G' |z_0|^2 z_i^2 \left[ P_{0\alpha} + Q_{0\alpha} \sum_{k=1}^{n-1} |z_k|^2 \right] + \bar{z}_0 z_i^2 Q_{0\alpha} \left[ G' z_0 ( F - \sum_{k=1}^{n-1} |z_k|^2 ) + GF' z_0 \right] -
\]

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\[-P_{aa}Gz_i^2 + Gz_i^2 Q_{aa} \sum_{k \neq 0, i} |z_k|^2 + 2Gz_i^2 \bar{z}_i R_{aa} - Gz_i^2 Q_{ab} \sum_{k \neq 0, i} |z_k|^2.\]

If we divide by $z_i^2$ (we are assuming $z_i \neq 0$) and derive again the above expression with respect to $\bar{z}_i$ we get

\[-G'|z_0|^2 Q_{0a} + GR_{aa} = 0.\]

By the definitions made at page 11 this is equivalent to

\[
\frac{G'F'|z_0|^2 + G(F' + F''|z_0|^2)}{B} = 0,
\]

i.e. $(GF'x)' = 0, x = |z_0|^2$. Substituting $G = \frac{c}{F'}$ in this equality we get $c(F'x)' = 0$. Since $(F'x)' < 0$ (by (ii) in Proposition 2.1) $c$ is forced to be zero, and this concludes the proof.

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