Tolman-Bondi Space-Time in Brane World Scenario

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In the present work, inhomogeneous Tolman-Bondi type dust space-time is studied on the brane. There are two sets of solutions of the above model. The first solution represents either a collapsing model starting from an infinite volume at infinite past to the singularity or a model starting from a singularity and expanding for ever having a transition from decelerating phase to accelerating phase. The first solution shows that the end state of collapse may be black hole or a naked singularity depending signs of various parameters involved. The second solution represents a bouncing model where the bounce occurs at different comoving radii at different epochs.

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In the seminal papers [1, 2] the introduction of branes into cosmology offered a novel approach to the understanding of the evolution of our universe. It was proposed that our 4D universe is said to be a singular hypersurface, a 3-brane embedded in a (4 + d) dimensional space-time, which is called the bulk. Here in the bulk, we have considered no matter field. The matter field is all confined to the brane and the gravity propagates in the extra dimensions as well. The effective equations for gravity in four dimensions were obtained by Shiromizu et al [3] and discussed by Roy Maartens [4]. Here Israel’s boundary conditions were used along with $z^2$ symmetry for the five dimensional bulk space-time embedding our brane universe at a fixed point $y = y_0$. If we add a cosmological constant in the bulk, the solutions of the Einstein’s equations $G_{ab} = \kappa_5^2 T_{ab}$ can be obtained, where the universe starts with a non-conventional phase and then enters the standard cosmological phase [5 - 11]. Friedmann like equations are usually solved on the brane for matter localized on the brane [12]. The usual case in the brane model recollapses if it is embedded in an anti de-Sitter ($\Lambda_5 < 0$) five dimensional bulk, where the extra dimension is space-like. In a recent work of Ponce de Leon [13] the acceleration of the present universe is studied as a consequence of the time evolution of the vacuum energy in cosmological models based on brane world theories in 5 dimensions. The basic two assumptions in this paper are that the universe is spatially flat and the matter density decrease as an inverse power of the scale factor. If $\Lambda_5 = 0$ then the induced cosmological term in 4D is positive and the brane universe expands continuously in a power law time dependence. Again for $\Lambda_5 > 0$, this universe becomes dominated by a positive cosmological term $\Lambda_4$. The effect is an asymptotic de-Sitter expansion which occurs regardless of the signature of the extra dimension. On the other hand, with a single timelike extra dimension the cosmological brane world model experiences a natural bounce without ever-reaching the singular state [14].

In the present work, we have considered the Tolman-Bondi space-time on the brane. This is an inhomogeneous and anisotropic space-time and differs from the homogeneous Friedmann space-time so far mentioned above regarding their dynamical behaviour. It is perhaps for the first time that Tolman-Bondi type dust space-time is studied on the brane. We’ll consider only the spatially flat model ($f(r) = 0$), which is consistent with the present observational data. The projection of the bulk Weyl tensor on the brane is also assumed to be zero. There occur two different cases so long as we assume the anti de-Sitter bulk and the extra dimension to be space-like. In the case I, the brane world collapses from an infinite past to a singularity at a finite epoch or otherwise begins from the singularity and expands indefinitely with deceleration near the big bang and acceleration at the late stage. In case II, the brane world model has a lower bound and hence singularity-free, similar to the case mentioned above (Ref. [14]). Here we get a contracting model bouncing from a lower bound.
which however takes place at different shells with different comoving radii at different epochs. This is usual for an inhomogeneous space-time. It is to be noted further that in the Tolman-Bondi space-time on brane under consideration, there is no recollapsing model.

**Field equations in the Brane:**

The modified 4D Einstein’s equations on the brane embedded in the 5D bulk are usually presented in the form [1, 3]

\[ G_{\mu\nu} = -\Lambda_4 g_{\mu\nu} + \kappa_4^2 T_{\mu\nu} + \kappa_5^4 S_{\mu\nu} - E_{\mu\nu} \]  

(1)

where \( \Lambda_4 \) is the effective four dimensional cosmological constant, \( \kappa_5 \) and \( \kappa_4 \) are the coupling constants related to the gravitational constants in 5D bulk and 4D brane respectively. They are further related with the fundamental 5D Planck mass and the effective Planck mass on the brane. \( T_{\mu\nu} \) is the usual matter energy tensor on the brane. The two additional terms \( S_{\mu\nu} \) and \( E_{\mu\nu} \) on the r.h.s of (1) stand respectively for the local quadratic energy-momentum correction term and the free energy induced on the brane by the 5D Weyl tensor in the bulk. The local correction term arises from the extrinsic curvature of the brane. Their explicit expressions are given by

\[ S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu\alpha} T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} (3 T^{\alpha\beta} T_{\alpha\beta} - T^2) \]  

(2)

Finally, \( E_{\mu\nu} \) in (1) is the projection of the bulk Weyl tensor on the brane defined by

\[ E_{ab} = C_{abcd} n^c n^d \]  

(3)

Here \( C_{abcd} \) is the usual Weyl tensor in the bulk, \( n^a \) is the unit normal to the hypersurface \( y = \text{constant} \), representing the brane world, where \( y \) represents the extra fifth dimension. In the above the latin indices refer to the 5D bulk and greek indices correspond to the brane world. The four dimensional \( \Lambda_4 \) and 5D cosmological constant \( \Lambda_5 \) [2, 3] are related by the following relation

\[ \Lambda_4 = \frac{\kappa_5^2}{2} (\Lambda_5 + \frac{\kappa_5^2 \lambda^2}{6}) \]  

(4)

where \( \lambda \) is the so-called brane tension or the vacuum energy of the brane world. This brane tension is further related to the 4D gravitational constant by a relation of the form

\[ \kappa_4^2 = \frac{\kappa_5^2 \lambda}{6} \]  

(5)

In this work, we assume that our brane model embedded at \( y = \text{constant} \) in the conformally flat bulk (zero Weyl tensor) is in homogeneous and an isotropic. As a result the term \( E_{\mu\nu} \) is neglected and is justified in view of the late time dominance of the vacuum energy compared to the brane matter density \( \rho \) i.e., \( \lambda >> \rho \) [4] being compatible with observations. Using spherically symmetric form the metric is given by

\[ ds^2 = dt^2 - e^{-\alpha} dr^2 - R^2 d\Omega^2 \]  

(6)

where \( \alpha = \alpha(r, t) \) and \( R = R(r, t) \).

Now considering matter in the form of dust on brane the independent field equations are given by

\[ \frac{1}{R^2} - e^{-\alpha} \left( 2 \frac{R''}{R} + \frac{R'^2}{R^2} - \alpha' \frac{R'}{R} \right) + \frac{\dot{R}^2}{R^2} + \alpha \frac{\dot{R}}{R} = \Lambda + \rho + \frac{\rho^2}{2\lambda} \]  

(7)
\[
\frac{1}{R^2} - e^{-\alpha} \frac{R'^2}{R^2} + 2 \frac{\dot{R}}{R} + \frac{\ddot{R}^2}{R^2} = \Lambda - \frac{\rho^2}{2\lambda}
\]  
(8)

\[-e^{-\alpha} \left( \frac{R''}{R} - \frac{\alpha'}{2} \frac{R'}{R} \right) + \frac{\ddot{\alpha}}{2} + \frac{\dot{\alpha}^2}{4} + \frac{\dot{\alpha}}{2} \frac{\ddot{R}}{R} + \frac{\ddot{R}}{R} = \Lambda - \frac{\rho^2}{2\lambda}\]

(9)

and

\[2 \frac{\dot{R}'}{R} - \dot{\alpha} \frac{R'}{R} = 0\]

(10)

It follows immediately from the equation (10)

\[e^\alpha = \frac{R'^2}{1 + f(r)}\]

(11)

In general there may be three different choices \(f(r) = 0, f(r) > 0, f(r) < 0\). For simplicity \(f(r)\) is assumed to be zero (marginally bound case). In this particular case the field equations (7) to (9) simplify to

\[\frac{\ddot{R}^2}{R^2} + 2 \frac{\ddot{R}}{R} \frac{\dot{R}}{R'} = \Lambda + \rho + \frac{\rho^2}{2\lambda}\]

(12)

\[2 \frac{\dot{R}}{R} + \frac{\ddot{R}^2}{R^2} = \Lambda - \frac{\rho^2}{2\lambda}\]

(13)

\[\frac{\ddot{R}'}{R'} + \frac{\ddot{R}}{R} \frac{\dot{R}}{R'} + \frac{\ddot{R}}{R} = \Lambda - \frac{\rho^2}{2\lambda}\]

(14)

It is to be noted that the energy density enters the field equations quadratically in contrast with the usual general relativistic 4D equations.

Subtracting (13) from (14) we get

\[\frac{\ddot{R}'}{R'} + \frac{\ddot{R}}{R} \frac{\dot{R}}{R'} - \frac{\ddot{R}}{R} - \frac{\ddot{R}^2}{R^2} = 0,\]

(15)

which can be integrated twice to yield

\[\dot{R}^2 = \mu R^2 + \frac{D(r)}{R}\]

(16)

In (16), \(\mu\) is in general a function of time and \(D(r)\) is a function of radial co-ordinate alone. In order to integrate equation (16) we assume further \(\mu = \text{constant}\), which in fact has two groups of solutions. Also from the field equations (12) - (14) using (16), we finally get

\[\rho = 6\mu - 2\Lambda + \frac{D^2(r)}{R^2 R'}\]

(17)

and

\[\frac{\rho^2}{2\Lambda} = \Lambda - 3\mu\]

(18)
Solutions:

There are three possible types of solutions of the evolution equation (16) depending on the signs of the parameter \( \mu \) and the function \( D(r) \) namely, (i) \( \mu > 0 \), \( D(r) > 0 \), (ii) \( \mu > 0 \), \( D(r) < 0 \), (iii) \( \mu < 0 \), \( D(r) > 0 \). However, the third possibility will not be discussed as it gives oscillatory solution, not of much interest in the present context.

**Case I : \( \mu > 0 \), \( D(r) > 0 \):**

The solution for the scale factor \( R \) from equation (16) gives

\[
R = R_0(r) \sinh^{2/3} u
\]

where, \( R_0^3 = \frac{D(r)}{\rho} \), \( u = a(t_0(r) - t) \), \( a = \sqrt{\mu} \) and \( t_0(r) \) is an arbitrary function of the radial co-ordinate \( r \).

Differentiating \( R \) we get in turn

\[
\dot{R} = -\frac{2aR_0Cosh u}{3\sinh^{1/3} u} \quad \text{and} \quad \ddot{R} = \frac{2a^2R_0[Cosh 2u - 2]}{9\sinh^{4/3} u}
\]

Here there may be two situations. The first case corresponds to \( t < t_0 \). The collapse starts from the infinite past \((t \to -\infty)\) i.e., when \( R \to \infty \) and finally converges to the singularity \( R \to 0 \) at the time \( t_0(r) \). The rate of contraction at both ends is infinitely large. It is a collapsing model of the brane. One should note however that different shells collapse at different epochs \( t_0(r) \) because of inhomogeneity. The time of formation of central singularity is given by \[15 - 17\]

\[
t_s = t_i + \frac{1}{a} \lim_{r \to 0} \sinh^{-1} \left( \frac{\mu r^3}{D(r)} \right)^{1/2}
\]

where we have chosen \( R = r \) at \( t = t_i \) as an initial condition.

Now at the time of formation of apparent horizon \((t_{ah}(r))\) for a shell having comoving radial co-ordinate \( r \), we must have \( \dot{R}^2 = 1 \). Using dynamical equation (16) it results

\[
\mu R^3 - R + D(r) = 0.
\]

Thus the time difference between the formation of apparent horizon and the central singularity is \[15 - 17\]

\[
t_{ah}(r) - t_s = \frac{1}{a} \left[ \sinh^{-1} \left( \frac{r}{R_0} \right)^{3/2} - \sinh^{-1} \left( \frac{R_{ah}}{R_0} \right)^{3/2} \right] - \frac{1}{a} \lim_{r \to 0} \sinh^{-1} \left( \frac{\mu r^3}{D(r)} \right)^{1/2}
\]

where \( R_{ah} \) is the positive real root of the above cubic equation (22) in \( R \). Further, in order to have the collapsing process starts from a regular initial data one should have the density to be finite at \( r = 0 \) when \( t = t_i \). Thus near \( r = 0 \) the function \( D(r) \) should have the power series expansion

\[
D(r) = D_0r^3 + D_1r^4 + D_2r^5 + \ldots
\]

Then the above time difference simplifies to

\[
t_{ah}(r) - t_s = -\frac{D_1}{2D_0\sqrt{\mu} + D_0} r + \frac{(2\mu + 3D_0)D_2 - 4D_0D_2(\mu + D_0)}{8D_0^3(\mu + D_0)^{3/2}} r^2 + O(r^3)
\]

Thus we note that the sign of this time difference depends on various parameters involved. Therefore, the spherically symmetric space-time shows collapse to a black hole or a naked singularity near the centre.
Fig. 1 shows the behaviour of $\dot{R}$ of eq.(20) with time in case I for $\mu = 1$, $D_0 = 1$ and $t_0 = 1$. Depending on the nature and properties of certain parameters.

On the other hand, if $t \geq t_0$, $R(t)$ starts from the singularity $R = 0$ at $t = t_0$ and expands indefinitely. Although this shows a monotonic expansion $\ddot{R} < 0$ initially and subsequently changes sign (see figure 1) to $\ddot{R} > 0$ at the later stage. Thus there is a transition from deceleration to acceleration during the evolution of the brane model. This result seems to be interesting because the model is consistent with the possibility of the structure formation in the early stage ($\ddot{R} < 0$) and the accelerated expansion in the later stage getting support from the observational data.

**Case II : $\mu > 0$, $D(r) < 0$:**

In view of (16) this case is restricted by the condition that $\mu R^3 > |D(r)|$. The solutions are given by
\[ R = R_0(r) \cosh^{2/3} u \]  

(26)

Hence,

\[ \dot{R} = -\frac{2aR_0 \sinh u}{3 \cosh^{1/3} u} \quad \text{and} \quad \ddot{R} = \frac{2a^2 R_0 [\cosh 2u + 2]}{9 \cosh^{4/3} u} \]  

(27)

In this case, \( \dot{R} \) is never zero, which shows that the brane world model is singularity free. This conclusion follows immediately from the positive value of \( \ddot{R} \) throughout the evolution and hence there is a minimum only and no maximum at any stage. \( R \) has an infinite magnitude at \( t \to -\infty \), subsequently contracts to \( R = R_0 \) when \( t = t_0 \) and then again explodes to infinity at \( t \to \infty \) (see figure 2). So this particular brane world model shows bounce. However, it should be noted that due to inhomogeneity in the energy distribution the bounce occurs at different comoving radii at different epochs.

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