The nonleptonic charmless decays of $B_c$ meson

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Abstract

In this paper, with the framework of (p)NRQCD and SCET, the processes $B_c \rightarrow M_1M_2$ are investigated. Here $M_{1(2)}$ denotes the light charmless meson, such as $\pi$, $\rho$, $K$ or $K^*$. Based on the SCET power counting rules, the leading transition amplitudes are picked out, which include $A_{wA}^2$, $A_{wB}^2$, $A_{wC}^2$, $A_{wD}^2$ and $A_c^0$. From SCET, their factorization formulae are proven. Based on the obtained factorization formulae, in particular, the numerical calculation on $A_{wB}^2$ is performed.

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I. INTRODUCTION

Within the Standard Model, the $B_c$ meson is the only pseudo-scalar meson formed by two different heavy flavor quarks. Due to its mass being under the $BD$ threshold and the explicit flavors, $B_c$ meson decays weakly but behaves stably via the strong and electromagnetic interactions. Its weak decay modes are expected to be rich, because the $B_c$ meson contains two heavy quarks. Either can decay independent, or both of them annihilate to a virtual $W$ boson.

In the recent decades, the decays of $B_c$ meson have been widely studied. In this work, we lay stress on the two-body charmless processes $B_c \to M_1 M_2$. These charmless decays have particular features. First, they are not influenced by the penguin diagrams, which are expected to be sensitive to the new physics. Thus, they provide pure laboratories to examine the QCD effective methods. Second, they receive the contributions only from the annihilation amplitudes, which offer an ideal opportunity to study the annihilation effects singly.

In the paper [1], the nonleptonic charmless $B_c \to M_1 M_2$ decays have been calculated within the “QCD Factorization” approach (QCDF), while in Refs. [2, 3], these processes are calculated in the “perturbative QCD” (pQCD) scheme method. However, in this work, a sequence of effective field theories are employed to analysis the $B_c \to M_1 M_2$ transitions. Considering that the initial meson of the $B_c \to M_1 M_2$ transitions is $B_c$, which include two heavy quarks, we use the non-relativistic effective theory of QCD (NRQCD) [4, 5] to deal with them. Due to the relationship $M_{B_c} \gg M_{M_1} \sim M_{M_2}$, which makes that the final mesons are relativistically boosted and back-to-back move, we use soft collinear effective theory (SCET) [6–11] to describe these degrees of freedom (DOF). Under the SCET, it is convenient to explore the factorizations properties of the transition amplitudes.

This paper is organized as follows. In Sec. II we introduce the theoretical details. We classify the transition amplitudes and focus on leading contributions. Within the framework of SCET, we prove the factorization formulae. Within Sec. III according to the obtained factorization formulae, we calculate $A_{wB}^2$ and present the numerical results.
II. THEORETICAL DETAILS

In this section, we present the theoretical details. First of all, the general frameworks are shown and the transition amplitudes are classified into categories, $A_w$ and $A_c$. Next, we pick out the leading contributions of $A_w$ and $A_c$, respectively and prove the according factorization formulae.

A. Frameworks and Power Counting Rules

As to the $B_c \to M_1 M_2$ processes, there are three typical scales, $m_b$, $\sqrt{m_b \Lambda_H}$ and $\Lambda_H$. $\Lambda_H$ is the typical hadronic scale. Conventionally, $\Lambda_H \sim 500$ MeV [11].

In order to describe the DOFs at scales $\sim m_b$, we use the full QCD and low-energy effective Hamiltonian [12], which is

$$H_W = \frac{2G_F}{\sqrt{2}} \sum_{q=d,s} V_{ub} V_{us}^* (C_1 \bar{c}_L \gamma^\mu b_L \bar{q}_L \gamma_\mu u_L + C_2 \bar{c}_L \gamma^\mu b_L \bar{q}_L \gamma_\mu u_L \gamma^\mu b_L \bar{q}_L \gamma_\mu u_L \gamma^\mu b_L \bar{q}_L \gamma_\mu u_L \gamma^\mu b_L \bar{q}_L \gamma_\mu u_L) + h.c. \quad (1)$$

In Eq. (1), $G_F$ denotes the Fermi coupling constant and $V_{q_1 q_2}$s stand for the CKM matrix elements. $C_1(2)$ is the Wilson coefficients. $\mu$ represents the Lorentz index, while $\alpha(\beta)$ is the color index.

For investigating the $\sqrt{m_b \Lambda_H}$ fluctuations, we need to integrate out the hard modes $\sim m_b$, obtaining several transition currents $J^I$ and the intermediate effective theory SECT$_I$+NRQCD. Within SECT$_I$ [6–9], there are three kinds of DOFs [7]: 1) the $n$-collinear quarks $\xi_n^I$ and gluons $A_{n}^I$ with the momentum scaling $p_c = (n \cdot p_c, \bar{n} \cdot p_c, p_{c \perp}) \sim m_b(\lambda^2, 1, \lambda)$; 2) the $\bar{n}$-collinear quarks $\xi_{\bar{n}}^I$ and gluons $A_{\bar{n}}^I$ with the momenta $p_c \sim m_b(1, \lambda^2, \lambda)$; 3) the ultra-soft quarks $\xi_{us}^I$ and gluons $A_{us}^I$ with $p_{us} \sim m_b(\lambda^2, \lambda^2, \lambda^2)$. $\lambda = \sqrt{\Lambda_H/m_b}$ is the expansion parameter. The power counting rules for these SECT$_I$ fields [7] are summarized in Table. II.

Within NRQCD [4, 5], there are four typical fields [13]: 1) the Pauli spinor quark field $\psi(\chi)$ with momentum $p_{NR}^{\psi(\chi)} = (E, \vec{p}) \sim (|q_{Bc}|/M_{Bc}, \vec{q}_{Bc})$; 2) the potential gluon field $A_{p}^{NR}$ with momentum $p_p^{NR} \sim (|q_{Bc}|/M_{Bc}, \vec{q}_{Bc})$; 3) the soft gluon field $A_{s}^{NR}$ with momentum $p_s^{NR} \sim (|q_{Bc}|, \vec{q}_{Bc})$; 4) the ultra-soft gluon field $A_{us}^{NR}$ with momentum $p_{us}^{NR} \sim (|q_{Bc}|, \vec{q}_{Bc})$. $\vec{q}_{Bc}$ is the relative momentum between the quark and the anti-quark of the $B_c$ meson. According the recent analysis [14], we take $|q_{Bc}^2| \sim 1 \text{ GeV}^2$. Therefore, numerically, we have $\sqrt{(p_{NR}^{\psi(\chi)})^2} \sim \sqrt{(p_{NR}^{\psi(\chi)})^2} \sim$
As to the transition currents $J_s$, they fall into two categories: 1) the weak flavor transition currents $J_w^I$, which are induced by $H_W$; 2) the QCD currents $J_c^I$, which are caused by the pure QCD interactions and obtained by integrating out the hard ($\sim m_b$) QCD interactions. According to the number of $J_w^I$, it is convenient to classify the transition amplitudes into two types, $A_w$ which are induced by no $J_w^I$, and $A_c$ those are mediated by at least one $J_w^I$.

For describing the DOFs $\sim \Lambda_H$, the intermediate fluctuations $\sim \sqrt{m_b \Lambda_H}$ are integrated out. Then, the transition currents $J^I$ and final effective theory pNRQCD + SCET$_{II}$ are matched onto, corresponding to the $\Lambda_H$ momentum modes. In the framework of pNRQCD [5], the momentum modes $p_s^{NR}$ and $p_s^{NR}$ are integrated out, leaving only the ultra-soft gluon $A^{NR}_{us}$ and the Pauli spinor quark field $\psi(\chi)$. In SCET$_{II}$ [10, 11], similar to the case of SCET$_I$, there are also three typical momentum regions: 1) the $n$-collinear quarks $\xi^n$ and gluons $A^n$ with $p_c \sim m_b(\eta^2, 1, \eta)$; 2) the $\bar{n}$-collinear quarks $\xi^n$ and gluons $A^n_{\bar{n}}$ with $p_c \sim m_b(1, \eta^2, \eta)$; 3) the soft quarks $\xi^s$ and gluons $A^s$ with $p_s \sim m_b(\eta, \eta, \eta)$. Here $\eta = \lambda^2 = \Lambda_H/m_b$ is the expansion parameter. The field scalings for these SCET$_{II}$ fields are also listed in Table. 1.

| Fields | Field Scaling | Fields | Field Scaling |
|--------|---------------|--------|---------------|
| $\xi^n$ | $\lambda$ | $\xi^n$ | $\eta^3/2$ |
| $\bar{\xi}^n$ | $\lambda^3$ | $\bar{\xi}^n$ | $\eta$ |
| $(A^I, n, A^n, \bar{n}, A^n_{\perp})$ | $(\lambda^2, 1, \lambda)$ | $(A^I, n, A^n_{\perp}, \bar{n}, A^n_{\perp})$ | $(\eta^2, 1, \eta)$ |
| $(A^I, n, A^n, \bar{n}, A^n_{\perp})$ | $(1, \lambda^2, \lambda)$ | $(A^I, n, A^n_{\perp}, \bar{n}, A^n_{\perp})$ | $(1, \eta^2, \eta)$ |
| $A^I_{us}$ | $\lambda^2$ | $A^I_{us}$ | $\eta$ |

**B. The Leading contributions of $A_w$**

In this part, we pick out the leading contributions of $A_w$. At the scale $\sim \sqrt{m_b \Lambda_H}$, $A_w$ is induced by $J_w^I$ and the SCET$_I$ Lagrangian $\mathcal{L}_c$ and $\mathcal{L}_{us}$. Here we have $\mathcal{L}_c = \mathcal{L}_c^0 + \mathcal{L}_c^1 + \mathcal{L}_c^2 + \mathcal{L}_c^3 + \mathcal{L}_c^4 + \mathcal{L}_c^5 + \mathcal{L}_c^6 + \mathcal{L}_c^7 + \mathcal{L}_c^8 + \mathcal{L}_c^9 + \mathcal{L}_c^{10} + \mathcal{L}_c^{11} + \mathcal{L}_c^{12}$. The explicit forms of these SCET$_I$ Lagrangian can...
be found in Ref. \[11\]. The relevant \( J_w^I \)'s in this work are

\[
\begin{align*}
J_w^0 &= \int d\omega d\omega_1 \left[ C_{w}^{01}(\omega_2, \omega_1) \left( \chi_{\bar{c}}^{\dagger} \Gamma_{A}^{01} \psi_{b} \right) \left( \bar{q}_{\bar{h},\omega_2}^{\dagger} \Gamma_{B} \bar{q}_{h,\omega,4} \right) + C_{w}^{02} \left( \chi_{\bar{c}}^{\dagger} \Gamma_{A}^{02} \psi_{b,\alpha} \right) \left( \bar{q}_{\bar{h},\omega_2,\alpha}^{\dagger} \Gamma_{B} \bar{q}_{h,\omega,4,\beta} \right) \right], \\
J_w^1 &= \int d\omega d\omega_2 d\omega_3 \left[ C_{w}^{1}(\omega, \omega_2, \omega_3) \left( \chi_{\bar{c}}^{\dagger} \Gamma_{B} \psi_{b} \right) \left( \bar{q}_{\bar{h},\omega_2}^{\dagger} \Gamma_{B} \bar{q}_{h,\omega,3} \right), \\
J_w^{2A} &= \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \left[ C_{w}^{2A}(\omega_1, \omega_2, \omega_3, \omega_4) \left( \chi_{\bar{c}}^{\dagger} \Gamma_{B} \psi_{b} \right) \left( \bar{q}_{\bar{h},\omega_2}^{\dagger} \Gamma_{B} \bar{q}_{h,\omega,3} \right) \right. \\
& \quad \left. \left( \bar{q}_{\bar{h},\omega_2}^{\dagger} \Gamma_{B} \bar{q}_{h,\omega,4} \right) \right], \\
J_w^{2B} &= \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \left[ C_{w}^{2B}(\omega_1, \omega_2, \omega_3, \omega_4) \left( \chi_{\bar{c}}^{\dagger} \Gamma_{B} \psi_{b} \right) \left( \bar{q}_{\bar{h},\omega_2}^{\dagger} \Gamma_{B} \bar{q}_{h,\omega,3} \right) \right. \\
& \quad \left. \left( \bar{q}_{\bar{h},\omega_2}^{\dagger} \Gamma_{B} \bar{q}_{h,\omega,4} \right) \right],
\end{align*}
\]

(2)

where \( \chi_{\bar{c}} \) and \( \psi_{b} \) are the Pauli spinor fields corresponding to the \( \bar{c} \) and \( b \) quarks, respectively. \( q_{n,\omega} \)'s are defined as \( q_{n,\omega} \equiv \left[ \delta(\bar{n} \cdot \mathcal{P} - \omega) W_{\bar{n} L}^{1} \right] \) \[15\]. \( \mathcal{P} \) is the operator picking out the large label momenta. \( W_{\bar{n}} \) is the conventional Wilson line \( W_{\bar{n}[\bar{n} \cdot A_{L}]} \) after extracting the phase exponent \( e^{-\mathcal{P} \cdot x} \). \( \xi_{\bar{n}} \) is the \( n \)-collinear field in SCET, as introduced in Sec. \[IIA\].

In Eq. (2), \( B_{n,\omega}^{1} \) is also introduced, which is defined as \( B_{n,\omega}^{1} \equiv \left[ B_{n,\omega}^{1} \delta(\bar{n} \cdot \mathcal{P}^{1} - \omega) \right] \). Here we have \[16\]

\[
B_{n,\omega}^{1} = \frac{1}{g} \left[ \frac{1}{\bar{n} \cdot \mathcal{P}} W_{n}^{1}[i\bar{n} \cdot D_{n}, iD_{n}^{1\mu}] W_{n} \right],
\]

(3)

where \( i\bar{n} \cdot D_{n} = \bar{n} \cdot \mathcal{P} + g\bar{n} \cdot A_{L} \) and \( iD_{n}^{1\mu} = \mathcal{P}^{1\mu} + g A_{L}^{1\mu} \). Using the building operators \( q_{n,\omega} \) and \( B_{n,\omega} \) to construct the currents is quite convenient, because these building blocks are invariant under the collinear-gauge transformations \[9\].

Within SCET, the scaling of \( A_{w} \) can be expressed as \( \lambda^{N_{J} + N_{\bar{c}}} (N_{J}, N_{\bar{c}} \geq 0) \). \( \lambda^{N_{J}} \) is the power counting for \( J_w^I \)'s. For instance, \( \lambda^{1} \) corresponds to \( J_w^1 \). \( \lambda^{N_{c}} \) stands for the scaling caused by SCET Lagrangian. As an example, if we consider \( A_{w} \) is induced by the time-product \( T \left[ J_{w}^{0}, \mathcal{L}_{\xi}^{2}, \mathcal{L}_{\xi}^{1} \right] \), then we have \( N_{c} = 3 \).

If we integrating out the DOFs \( \sim \sqrt{m_{b} \Lambda_{H}} \), then the SCETII are matched onto. According to Ref. \[11\], within SCETII, the power counting for \( A_{w} \) is \( \eta^{(N_{J} + N_{\bar{c}})/2 + N_{uc}} (N_{uc} \geq 0) \). \( N_{uc} \) is caused by lowering the off-shellness of the un-contracted collinear fields.

In this way, the leading contributions of \( A_{w} \) in \( \eta \) can be picked out.

1. Case of \( N_{J} = 0, N_{\bar{c}} = 0 \). Here we show that this kind of amplitudes do not contribute to the \( B_{c} \rightarrow M_{1}M_{2} \) processes. As to \( B_{c} \rightarrow M_{1}M_{2} \) decays, the final mesons involve even \( n(\bar{n}) \)-collinear quarks. However, as shown in Eq. (2), there is odd \( n(\bar{n}) \)-collinear quark field. No matter how many \( \mathcal{L}_{\xi}^{0} \) and \( \mathcal{L}_{\xi}^{0} \)'s are contracted with \( J_{w}^{0} \), there are
still odd final \(n(\bar{n})\)-collinear quark fields. Therefore, the \(B_c \rightarrow M_1 M_2\) processes do not include this kind of amplitudes.

2. Case of \(N_J = 1, N_L = 0\). Although there are even \(n(\bar{n})\)-collinear quarks in \(J^1_w\), \(B_c \rightarrow M_1 M_2\) transition still receives no contributions from this case. This is because the \(n(\bar{n})\) DOF in \(J^1_w\) is color-octet. In the leading SCET\(_1\) Lagrangian, namely, \(N_L = 0\), the \(n\) collinear DOFs decouple from the \(\bar{n}\) and ultra-soft ones. Thus, the final \(n(\bar{n})\) fields are all generated originally from \(B_{n,\omega}^\perp\) in \(J^1_w\), which makes the final \(n(\bar{n})\) meson color-octet. So the \(B_c \rightarrow M_1 M_2\) decays do not contain the amplitudes for this case.

3. Case of \(N_J = 0, N_L = 1\). This case is similar to the \(N_J = 0, N_L = 0\) one, which also produces odd \(n(\bar{n})\)-collinear quarks. Thus, there is no overlapping amplitude for the \(B_c \rightarrow M_1 M_2\) transitions.

4. Case of \(N_J = 2, N_L = 0\). \(J^2_B\) will contribute to \(B_c \rightarrow M_1 M_2\) decays. \(J^2_A\) contributes only for the isosinglet final states, such as \(\eta, \eta'(958)\) mesons. Their typical diagrams are plotted in Figs. 1 (a,b).

5. Case of \(N_J = 0, N_L = 2\). In order to produce even \(n(\bar{n})\)-collinear quarks, only \(T[J^0_w, \mathcal{L}_{\xi_0 q}^1, \mathcal{L}_{\xi_0 q}^1]\) is possible. But in this time-product, the number of \(iD_{n(\bar{n})}^\perp\) is odd, which introduces extra suppressions from \(N_{uc}\). In this case, \(N_{uc} \geq 1\). Therefore, at the leading order in \(\eta\), the amplitudes for \(N_J = 0, N_L = 2\) do not contribute.

6. Case of \(N_J = 1, N_L = 1\). In this case, the product \(T[J^1_w, \mathcal{L}_{\xi q}^1]\) will not contribute, since it does not produce the even \(n(\bar{n})\)-collinear quarks. But the products \(T[J^1_w, \mathcal{L}_{\xi q}^1]\) and \(T[J^1_w, \mathcal{L}_{cg}^1]\) do. The examples of these two products are illustrated in Figs. 1 (c,d).

In summary, the operators \(J^2_A, J^2_B, T[J^1_w, \mathcal{L}_{\xi q}^1]\) and \(T[J^1_w, \mathcal{L}_{cg}^1]\) contribute to the \(B_c \rightarrow M_1 M_2\) processes in the leading order in \(\eta\). The according transition amplitudes are

\[
\begin{align*}
A_{\omega A}^2 &= \langle M_1 M_2 | J^2_A (0) | B_c^- \rangle, \\
A_{\omega B}^2 &= \langle M_1 M_2 | J^2_B (0) | B_c^- \rangle, \\
A_{\omega C}^2 &= \langle M_1 M_2 | \int dx \ T[J^1_w (0), \mathcal{L}_{\xi q}^1 (x)] | B_c^- \rangle, \\
A_{\omega D}^2 &= \langle M_1 M_2 | \int dx \ T[J^1_w (0), \mathcal{L}_{cg}^1 (x)] | B_c^- \rangle.
\end{align*}
\]
FIG. 1: Typical diagrams for $A_{wA}^2$, $A_{wB}^2$, $A_{wC}^2$ and $A_{wD}^2$. The solid lines stand for the initial $b(\bar{c})$ quarks, while the dash lines denote the final collinear quarks. A spring is the (ultra-)soft gluon, but the spring with a line though it represents the collinear gluon. Figs. (a,d) contribute only to the isosinglet final meson, such $\eta, \eta'$.  

Consider that in the leading SCET$_I$ Lagrangian the collinear fields decouple from the ultra-soft fields. Therefore, we have

\[
A_{wA}^2 = \int \omega_1 \omega_2 \omega_3 \omega_4 \ C_{wA}^{2A}(\omega_1, \omega_2, \omega_3, \omega_4) \ \langle 0|\chi \Gamma^{2A}_{\mu \nu} \psi_b B_{c}^{-}|M_1|\bar{q}_{\bar{n}, \omega_2} \Gamma_n q_{n, \omega_3}|0\rangle \\
\langle M_2|\text{Tr}[B_{n, \omega_1}^{\perp \mu} B_{n, \omega_4}^{\perp \nu}]|0\rangle,
\]

\[
A_{wB}^2 = \int \omega_1 \omega_2 \omega_3 \omega_4 \ C_{wB}^{2B}(\omega_1, \omega_2, \omega_3, \omega_4) \ \langle 0|\chi \Gamma^{2B}_{\mu \nu} \psi_b B_{c}^{-}|M_1|\bar{q}_{\bar{n}, \omega_2} \Gamma_n q_{n, \omega_3}|0\rangle \\
\langle M_2|\bar{q}_{\bar{n}, \omega_1}^{\prime} \Gamma_n q_{n, \omega_4}|0\rangle.
\] (5)

However, there are interactions between the collinear and ultra-soft fields in $\mathcal{L}^{1}_{\xi\xi}$ and $\mathcal{L}^{1}_{cg}$. Thus, we have

\[
A_{wC}^2 = \int d\omega d\omega_2 d\omega_3 C_w^{1B}(\omega, \omega_2, \omega_3) \ \langle M_2|T \left[ \left( \chi \Gamma^{1}_{\mu \nu} \psi_b B_{n, \omega}^{\perp \mu} \right) (0), \mathcal{L}^{1}_{\xi\xi}(x) \right] |B_{c}^{-}\rangle \langle M_1|\bar{q}_{\bar{n}, \omega_2} \Gamma_n q_{n, \omega_3}|0\rangle,
\]

\[
A_{wD}^2 = \int d\omega d\omega_2 d\omega_3 C_w^{1C}(\omega, \omega_2, \omega_3) \ \langle M_2|T \left[ \left( \chi \Gamma^{1}_{\mu \nu} \psi_b B_{n, \omega}^{\perp \mu} \right) (0), \mathcal{L}^{1}_{cg}(x) \right] |B_{c}^{-}\rangle \langle M_1|\bar{q}_{\bar{n}, \omega_2} \Gamma_n q_{n, \omega_3}|0\rangle.
\] (6)
The typical diagrams of $A_{wA}^2$, $A_{wB}^2$, $A_{wC}^2$ and $A_{wD}^2$ are illustrated in Fig. 1.

C. The analysis of $A_{wB}^2$

In the last subsection, we prove the factorization formulae of $A_{wA}^2$, $A_{wB}^2$, $A_{wC}^2$ and $A_{wD}^2$. Here we lay stress on the calculations of $A_{wB}^2$. The analysis of $A_{wB}^2$ can be performed in a similar manner. The estimations of $A_{wC}^2$ and $A_{wD}^2$ involve the non-factorizable matrix elements $\langle M_2 | T \left( \left( \chi c^\dagger \bar{\psi} B_{n,\omega^\prime}^{1+\mu} \right)(0), \mathcal{L}_1^\dagger(x) \right) | B^- \rangle$ and $\langle M_2 | T \left( \left( \chi c^\dagger \bar{\psi} B_{n,\omega^\prime}^{1+\mu} \right)(0), \mathcal{L}_c(x) \right) | B^- \rangle$. We expect them to be determined from the future experimental data or the non-perturbative method.

For the amplitude $A_{wB}^2$, as shown in Eq. (5), the hadronic matrix elements $\langle 0 | \chi c^\dagger \Gamma^{2B} \bar{\psi} B^- \rangle$, $\langle M_1 | q_0^\dagger \Gamma_n q_{n,\omega} | 0 \rangle$ and $\langle M_2 | q_0^\dagger \Gamma_n q_{n,\omega} | 0 \rangle$ are involved.

Considering that the $B_c$ meson is dominated by the $1S_0^1$ Fock state, the matrix $\Gamma^{2B}$ should be $I$ and the initial hadronic matrix element can be parameterized as

$$\langle 0 | \chi c^\dagger \bar{\psi} B^- \rangle = i f_{B_c} M_{B_c},$$

where $f_{B_c}$ is decay constant of the $B_c$ meson.

As to the final hadronic matrix elements, the matrices $\Gamma_n$ and $\Gamma_{\bar{n}}$ are involved. In general, they can be represented by the following basis

$$\{ I, \gamma_5, \gamma, \gamma_\perp, \gamma_{\perp}^\mu, \gamma_5 \gamma_5, \gamma_\perp \gamma_\perp, \gamma_\perp \gamma_{\perp}^\mu, (\gamma_\perp \gamma_\perp - 2) \}.$$

If the properties of the SCETI fields $\nu f^\dagger q_{n,\omega} = q_{n,\omega_i}$ and $\nu f q_{n,\omega} = q_{\bar{n},\omega_i}$ are considered, only the set of matrices $\Gamma_n = \hat{p} P_L$ and $\Gamma_{\bar{n}} = \hat{p} P_L$ contribute. (The constructions of these local six-quark operators are discussed detailedly in Ref. [17]. Here we directly use their results.)

Therefore, we have $\langle M_1 | q_0^\dagger \hat{p} P_L q_{n,\omega} | 0 \rangle$ and $\langle M_2 | q_0^\dagger \hat{p} P_L q_{n,\omega_i} | 0 \rangle$. According to Ref. [15], these two hadronic matrix elements are just the conventional light cone wave functions in the momentum space. Based on Ref. [18], we have

$$\langle P(p) | q_{n,\omega} \hat{p} P_L q_{n,\omega_i}' | 0 \rangle = -i f_{pP} \frac{p \cdot n}{2} \int_0^1 dx \delta(xp \cdot n - \omega_q) \delta((xp \cdot n + \omega_q') \phi_P,$$

$$\langle V(p) | q_{n,\omega} \hat{p} P_L q_{n,\omega_i}' | 0 \rangle = -i f_{pP} \frac{p \cdot n}{2} \int_0^1 dx \delta(xp \cdot n - \omega_q) \delta((xp \cdot n + \omega_q') \phi_V.$$
where $x = 1 - \bar{x}$. Usually, $\phi_{P(V)}$ can be expanded in the Gegenbauer polynomials

$$
\phi_{P(V)} = 6x(1-x) \left[ 1 + \sum_{n=1}^{\infty} a_n^{P(V)} C_{n}^{3/2}(2x-1) \right],
$$

(9)

where $a_n^{P(V)}$ are the Gegenbauer moments, which can be obtained from lattice simulations [20, 21]. $C_n^{3/2}(u)$s are the Gegenbauer polynomials. In our numerical calculations, we truncate this expansion at $n = 2$, using $C_1(u) = 3u$ and $C_2 = \frac{3}{2}(5u^2 - 1)$.

Plugging Eqs. (7-8) into Eq. (5), $A_{wB}^2$ can be re-written as

$$
A_{wB}^2 = \frac{f_{Bc} f_{M_1} f_{M_2}}{27} \int_{0}^{1} \int_{0}^{1} dxdy C_1^w(x,y) \phi_{M_1}(x) \phi_{M_2}(y).
$$

(10)

Matching at tree level, as shown in Fig. 2, we have

$$
C_1^w(x,y) = \frac{4\pi G_F \alpha_s(m_b) C_2 V_{cb} V^*_{uq}}{\sqrt{2} y(\bar{x}y - \alpha_1 \bar{x} - \alpha_1 y + i\epsilon)},
$$

(11)

where $\alpha_1 = m_b/M_{B_c}$. This result is in agreement with the one in Ref. [1]. If we take $\alpha_1 \to 1$, Eq. (11) also agree with the results in Refs. [17, 19].

D. The Leading contributions of $A_c$

In this part, we turn to analyzing the leading $A_c$ in $\eta$. $A_c$'s are induced by one $J_w$ and at least one $J_c$. From the SCET power counting rules, at the leading order in $\eta$, $A_c^0$ is mediated by $T[J_w^0, J_c^0]$. The expression of $J_w^0$ has been given in Eq. (2). For $J_c^0$, at the tree level
FIG. 3: Matching procedure for \( J^0_c \) in QCD (left diagram) and SCET (right diagram).

matching, as shown in Fig. 3, we have

\[
J^0_c = \int d\omega_1 d\omega_3 \left[ D_1 (\chi_{c}^{\dagger} \sigma_1^\mu \psi_c) (\bar{q}_{n,\omega_1} \gamma_{\perp \mu} q_{\bar{n},\omega_3}) + D_2 (\chi_{\bar{c}}^{\dagger} \sigma_1^\mu \psi_{\bar{c},c}) (\bar{q}_{n,\omega_1} \gamma_{\perp \mu} q_{\bar{n},\omega_3}) \right].
\]

(12)

Here \( D_1 = \frac{2\alpha_s(m_b)}{3\omega_1\omega_3} \) and \( D_2 = -\frac{2\alpha_s(m_b)}{\omega_1\omega_3} \).

The factorization properties of SCET yield that \( A^0_c \) can be re-written as

\[
A^0_c \propto \sum_{i,j} \int dz d\omega_1 d\omega_2 d\omega_3 d\omega_4 C^{0i}_{w} D_j e^{-i(\omega_1-\omega_3)z} \langle 0 | T \left\{ \left[ \chi_{c}^{\dagger} \Gamma^0_{A} \psi_b \right](0), \left[ \chi_{\bar{c}}^{\dagger} \sigma_1^\mu \psi_{\bar{c}} \right](z) \right\} | B^- \rangle

\langle M_1 | \bar{q}_{n,\omega_2} \Gamma_\alpha n q_{\bar{n},\omega_3} | 0 \rangle \langle M_2 | \bar{q}_{n,\omega_1} \Gamma_\beta n q_{\bar{n},\omega_4} | 0 \rangle.
\]

(13)

In Eq. (13), the color indices are implicit for readability. The example of this amplitude is illustrated in Fig. 4.

FIG. 4: Typical diagrams for \( A^0_c \).

Here we interpret the first hadronic term in Eq. (13) as the non-perturbative soft functions. This is because that the soft gluons may be exchanged between the produced \( c \) quark and the initial constituent \( b(\bar{c}) \) quark.

In order to see this, we approximatively consider \( \omega_2 = -\omega_3 = \omega_1 = -\omega_4 = M_{B_c}/2 \).
In this way, the $c$ quark produced by $J_w^0$ moves non-relativistically and is almost on-shell. When this $c$ quark and the initial constituent $\bar{c}$ quark are annihilated by $J_0^c$, it is observed that $(\bar{P}_c + P_c)^2 \sim M_{J/\psi}^2$. ($\bar{P}_c$ denotes the momentum of the propagated $c$ quark, while $P_c$ stands for the initial constituent $\bar{c}$ quark.) Therefore, it is reasonable to expect soft gluons exchanged between the propagated $c$ and the initial partons.

Actually, this situation is not unique in the analysis of SCET. In the $B \rightarrow M_1 M_2$ processes, there are long-distance charming penguins [22], in which soft gluons are also exchanged among the produced $c$ quarks, the spectator quark and the initial $b$ quark.

III. NUMERICAL RESULTS AND THE DISCUSSIONS

In this part, we present the numerical results and phenomenal analysis. In Sec. III A, the inputs in the calculations are introduced. Within Sec. III B, the numerical results are shown.

A. Inputs in calculations

The masses and lifetimes of the involved mesons are presented in Table. 1. The mass for $b$ quark is taken as $m_b = 4.8$ GeV [23], while the mass of $c$ quark is used as $m_c = 1.6$ GeV [23].

| Meson | $B_c$ | $\pi$ | $K$ | $\rho$ | $K^*$ |
|-------|-------|-------|-----|-------|-------|
| Mass  | 6.3 GeV | 0.14 GeV | 0.49 GeV | 0.77 GeV | 0.89 GeV |
| Lifetime | $0.51 \times 10^{-12}$s | |

In Eq. (1) and Eq. (11), $\alpha_s$ and the Wilson coefficients $C_1$ and $C_2$ are involved. Here we take $\alpha_s(m_b) = 0.22$, $C_1 = 1.078$ and $C_2 = -0.184$ [12].

In Eqs. (7-8), the decay constants $f_{B_c, P, V}$ and the Gegenbauer moments $a_{1,2}$ are involved. According to Ref. [24], we employ $f_{B_c} = 0.322$ GeV. The other inputs are summerized in Table. 2.
TABLE III: Decay constants and the Gegenbauer moments for the light mesons.

| Meson |  \( \pi \) |  \( K \) |  \( \rho \) |  \( K^* \) |
|-------|----------|----------|----------|----------|
| \( f_M \) | \( 0.130 \) GeV | \( 0.156 \) GeV | \( 0.208 \) GeV | \( 0.217 \) GeV |
| \( a_1 \) | 20, 21 | - | 0.0383 | \( a_1^{(21)} \) | - | 0.0716 |
| \( a_2 \) | 20, 21 | 0.136 | 0.175 | \( a_2^{(21)} \) | 0.204 | 0.145 |

B. Numerical results

Here we only show the numerical results of \( A_{wB}^2 \). \( A_{wA}^2 \) does not contribute to the open flavor final states, while the evaluations of \( A_{wC}^0 \), \( A_{wC}^2 \) and \( A_{wD}^2 \) involve the non-perturbative hadronic matrix elements. We leave the calculations on \( A_{wC}^0 \), \( A_{wC}^2 \) and \( A_{wD}^2 \) to the future work.

TABLE IV: Numerical results of \( A_{wB}^2 \) in \( 10^{-10} \) GeV.

| \( A_{wB}^2(B_c^- \rightarrow K^- K^0) \) | Results |
|---------------------------------|---------|
| \( A_{wB}^2(B_c^- \rightarrow K^*^- K^0) \) | -4.50 |
| \( A_{wB}^2(B_c^- \rightarrow K^- K^{*0}) \) | -5.94 |
| \( A_{wB}^2(B_c^- \rightarrow \pi^- \bar{K}^0) \) | -6.27 |
| \( A_{wB}^2(B_c^- \rightarrow \pi^- \bar{K}^{*0}) \) | -0.89 |

\( A_{wB}^2 \) can be obtained from Eqs. (10-11). The numerical results are listed in Table. IV. First, from Table. IV all of the \( A_{wB} \) results are real. This also happens in the local annihilation amplitudes of the \( B \rightarrow M_1 M_2 \) decays [17]. Second, one may note that \( A_{wB}^2(B_c^- \rightarrow K^- K^0) \) are comparable with the ones of the \( B_c^- \rightarrow K^*^- K^0 \) and \( B_c^- \rightarrow K^- K^{*0} \) processes, but much larger than the \( B_c^- \rightarrow \pi^- \bar{K}^0 \) and \( B_c^- \rightarrow \pi^- \bar{K}^{*0} \) cases. This is caused by the suppressed CKM matrix, namely, \( V_{us}/V_{ud} \sim \lambda = 0.22 \) [23]. Third, although our expression of \( A_{wB}^2 \) is formally identical to the one in Ref. [1], the results in Table. IV are different from them. In Ref. [1], the integration in Eq. (11) is done with expanding the parameter \( \alpha_1 \) and take the asymptotic wave functions. However, in this work, the calculations are performed without these approximations.
IV. CONCLUSION

In this paper, we investigate the $B_c \to M_1 M_2$ decays with the framework of (p)NRQCD+SCET. Our analysis shows that the leading amplitudes for $B_c \to M_1 M_2$ processes include $A_{wA}^2$, $A_{wB}^2$, $A_{wC}^2$, $A_{wD}^2$ and $A_c^0$.

As to $A_{wA}^2$ and $A_{wB}^2$, from the SCET properties, they can be factorized into the following form

$$H \otimes \Phi_{B_c} \otimes \Phi_{\bar{n}} \otimes \Phi_n.$$  \hspace{1cm} (14)

Here $H$ denotes the hard kernel, while $\Phi_{B_c}$ and $\Phi_{n(\bar{n})}$ stand for the initial and final wave functions, respectively. This factorization formulae is in agreement with the PQCD \cite{2, 3} and QCDF \cite{1} results. And our result on $A_{wB}^2$ is formally identical to Ref. \cite{1}.

But for $A_{wC}^2$, $A_{wD}^2$ and $A_c^0$, the situations are different. The amplitude $A_c^0$ includes the initial soft functions, while the ones $A_{wC}^2$ and $A_{wD}^2$ involve the lagrangian $L_{\xi\xi}^{1}$ and $L_{cg}^{1}$, where the collinear fields are tangled with ultra-soft gluons. Therefore, we expect the amplitudes $A_{wC}^2$, $A_{wD}^2$ and $A_c^0$ can not be expressed as the form in Eq. (14).

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Appendix A: Details on the $A_{wB}^2$ calculation

In this part, we introduce the details on the $A_{wB}^2$ calculation. From Eq. (10), it is observed that the numerator of the integrand is a polynomial of $x$ and $y$. Hence, we can expand $A_{wB}^2$ in terms of $I(m,n)$s, namely,

$$A_{wB}^2 = \sum_{m,n=0}^{\infty} B(m,n) I(m,n).$$

$B(m,n)$ is the according parameter, while the elemental integration $I(m,n)$ ($m,n \geq 0$) is defined as

$$I(m,n) = \int_0^1 dx dy \frac{x^m y^n}{xy - \alpha_1 x - \alpha_1 y + i\epsilon}.$$  \hspace{1cm} (A1)

From Eq. (A1), we see $I(m,n) = I(n,m)$. Hence, in the following paragraphs only $I(m,n)$ ($m \geq n \geq 0$) is introduced. The case for $n > m > 0$ can be obtained from the symmetries.

For the term $I(0,0)$, we have

$$I(0,0) = -\text{Li}_2\left(-\frac{(1-\alpha_1)^2}{-\alpha_1^2 + i\epsilon}\right) + \text{Li}_2\left(\frac{(1-\alpha_1)\alpha_1}{-\alpha_1^2 + i\epsilon}\right) + \text{Li}_2\left(-\frac{(1+\alpha_1)\alpha_1}{-\alpha_1^2 + i\epsilon}\right) - \text{Li}_2\left(-\frac{\alpha_1^2}{-\alpha_1^2 + i\epsilon}\right).$$ \hspace{1cm} (A2)

It seems that the analysis from the Landau equations \[25, 26\] implies the end-point singularities in $I(0,0)$. But a careful study shows that those singularities are not in the principal sheet. Hence, $I(0,0)$ is finite. Compared with other $I(m,n)$s, it is observed that $I(0,0)$ is the most singular term. Thus, all $I(m,n)$s are also finite. This conclusion agrees with Ref. [1].

For the term $I(m,0)$ ($m \geq 1$), we have

$$I(m,0) = \alpha^n I(0,0) + \int_{-\alpha_1}^{1-\alpha_1} du \sum_{i=0}^{n-1} \left(C_n^i u^{n-1-i} \alpha_1^i \right) \left[\log(u - \alpha_1 u - \alpha_1^2 + i\epsilon) - \log(-\alpha_1 u - \alpha_1^2 + i\epsilon)\right],$$ \hspace{1cm} (A3)

where $C_n^i$ is the binomial coefficient.

As to the term $I(m,n)$ ($m \geq n \geq 1$), it is

$$I(m,n) = \sum_{j=0}^{n} C_n^j \alpha_1^n I(n-j,m-n+j) + \int_0^1 dx dy \sum_{i=0}^{n-1} C_n^i y^{m-n} (\alpha_1 x + \alpha_1 y)^i (xy - \alpha_1 x - \alpha_1 y)^{n-1-i}.$$ \hspace{1cm} (A4)
We can evaluate this equation inductively, because the powers of $I(n-j, m-n+j)$s are no more than $m$. For instance, $I(1, 1) = 2\alpha_1 I(1, 0) + 1$, where $I(1, 0)$ can be computed from Eq. (A3).

Consequently, based on Eqs. (A2-A4), all of $I(m,n)$s can be evaluated. The use of these $I(m,n)$s are quite general. They can not only be employed to calculate Eq. (10), if we make proper replacements of $\alpha_1$, they are also useful in the calculations of Ref. [1].

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