Consider a protocol in which Belinda seals a (classical) message. She gives the resulting sealed message to Charlie, who can either unseal and read the message or return it unopened to Belinda. If he returns it unopened, Belinda should be able to verify that Charlie neither read the message nor made a copy that would allow him to read it later. Such a protocol is impossible with classical cryptography: Charlie can copy a message and do anything he likes to that copy without damaging the original. With quantum cryptography, on the other hand, the no cloning theorem implies that Charlie cannot simply copy a message and unseal the copy.

In this paper, I prove that any conventional quantum cryptographic protocol can give at best a very weak security guarantee. However, quantum cryptography in conjunction with classical functions that can only be inverted by humans (i.e. CAPTCHAs) can potentially give exponential security.

I. INTRODUCTION

Imagine that you want to blackmail someone with incriminating documents. You are worried that your victim might kill you instead of paying up; for insurance, you give a copy of your incriminating documents to your attorney to be unsealed and published in the event of your untimely death. You meet with your victim and extort some quantum money.

Of course, your reputation as an honest blackmailer would be destroyed if the incriminating documents got out even though your victim paid up. To be safe, you ask your attorney for the documents back.

Is there any way your attorney can prove that he or she did not open and copy the documents before returning them?

With only classical techniques, the best you can do is to physically seal the documents before giving them to your attorney and check that the seal is intact when you get the documents back. If your attorney is good at covertly opening envelopes or if you want to send the documents to your attorney over the internet instead of in person, this is not good enough. Encrypting the documents would not help because your attorney needs to be able to read them if you die.

By the no-cloning theorem, it is conceivable that you could encode the documents into a sealed state and give a subsystem of that state to your attorney. He or she cannot directly copy the state, so perhaps any attempt to read the state would be detectable.

A quantum blackmail protocol has two players: Belinda the blackmailer and Charlie the co-conspiring attorney. Belinda has some secret message and, from that message, produces a state on two registers, $B$ and $C$. She gives register $C$ to Charlie. Charlie can do one of three things:

- He can unseal the state to learn the message.
- He can give the state in register $C$ back to Belinda and tell her that he did not unseal the message.
- He can cheat, trying to learn something about the message while lying to Belinda and saying he hasn’t.

If Charlie unseals the state, then the protocol is done; Belinda doesn’t need to do anything. If Charlie tells Belinda that he did not unseal the message, then Belinda will make a measurement to decide whether to believe him. If, in fact, Charlie did not cheat, then Belinda will believe him with probability $1 - \epsilon_c$, where $\epsilon_c$ is the protocol’s completeness error. If Charlie did cheat, then Belinda may or may not catch him and Charlie may learn something about the message.

In the straightforward case, the message being sealed is a single classical message, and Charlie must be able to decode the message correctly with some probability $p$ if he chooses to unseal the state. If he cheats, then Belinda should catch him with high probability. This turns out to be impossible to achieve with good security: in quantum computing, any measurement that has a nearly deterministic outcome can be performed with almost no damage to the state. If $p$ is large, Charlie can learn the message with minimal damage to his state and is unlikely to be caught. If $p$ is small, the protocol is not very useful, and Charlie’s chance of getting away with cheating does not decrease rapidly as $p$ decreases. In particular, Charlie can always recover the message with probability $p$, and Belinda will catch him with at most probability $\epsilon_c + p/(1-p)$ time. Charlie’s cheating strategy does not require difficult computation. The proof is given in Section II.
Nevertheless, there are at least two ways to achieve secure quantum blackmail protocols.

The simpler case is when Belinda has a collection of distinct pieces of classical information and unsealing the state only needs to reveal one of them to Charlie. This is realistic: if Belinda has several compromising pictures of her victim, then it may be sufficient for Charlie to have access to a single random picture as insurance. Belinda can put a random picture in the $C$ register and keep the purification of the state in the $B$ register; if Charlie copies the picture, he will know which picture it was and break the entanglement between the registers. Section III discusses the security properties of this protocol.

The more complicated case uses a new type of quantum computing resource: a computation that cannot be done on a quantum computer. The generic attack against sealed states relies on reversibly unsealing the state. If part of the unsealing process cannot be done on a quantum computer, then the attack may fail. Of course, any calculation that can be done on a classical computer can, in principle, be done on a quantum computer as well. Until someone invents perfect artificial intelligence, however, there will be calculations that can be done by humans but not by any computer. Since humans are not coherent quantum computers, we can use such a calculation to implement general-purpose sealed states. The idea is to encode a message as a superposition of many different images or sounds, any of which can be decoded by a human but not by a computer. Since humans cannot be run in reverse, showing the image or sound to a human will break the superposition. Section IV gives an example of this type of protocol.

II. A BOUND ON SOUNDNESS AND COMPLETENESS

Suppose that Belinda has a single classical message $m$. She prepares a sealed message in register $C$ and gives it to Charlie. Without loss of generality, we assume that Belinda keeps a purification of Charlie’s state in register $B$, so the full state is $|\psi_m\rangle_{BC}$. Charlie knows some efficient algorithm that takes register $C$ as input and outputs $m$ with probability $p$.

If Charlie returns register $C$ unmodified to Belinda, then Belinda will perform some measurement on registers $B$ and $C$ to determine whether Charlie cheated. If Charlie did not cheat, then Belinda will believe him with probability $1 - \epsilon_c$, where $\epsilon_c$ is the completeness error of the protocol.

Charlie can try to cheat. For example, he could perform some measurement on register $C$ to try to learn $m$ and give the state that remains in register $C$ back to Belinda. If he tells her that he did not look at the message, she will believe him with probability $1 - s$, where $s$ is the soundness of the protocol. In general, $s$ can depend on Charlie’s cheating strategy.

**Theorem. (No quantum blackmail)** Charlie has an efficient strategy that recovers $m$ with probability $p$ such that Belinda will catch him with probability $s \leq \epsilon_c + p\sqrt{1 - p} + (1 - p)$.

**Proof.** If Charlie were to open the message instead of cheating, he would perform some measurement. Without loss of generality, that measurement consists of a unitary operator $U$ followed by a projective measurement $\{P_i\}$, both on register $C$ [5]. The projective measurement has one outcome per possible message.

Charlie can cheat with a simple algorithm. Charlie acts only on register $C$, but we will keep track of the joint state of both registers. The initial state is $|\psi\rangle_{BC}$. Charlie applies $I \otimes U$ and then makes the projective measurement $\{I \otimes P_i\}$. With probability $p$, Charlie gets the outcome corresponding to $m$ and learns the state. With probability $1 - p$, Charlie gets a different outcome and does not learn the state. In either case, Charlie applies $I \otimes U^\dagger$ to the state that remains after the measurement and gives that state to Belinda.

Let $|\phi_i\rangle$ be the final state of registers $B$ and $C$ conditioned on Charlie obtaining outcome $i$. Suppose that outcome $i$ occurs with probability $q_i$, so that $p = q_m$. Marginalizing over the outcomes of Charlie’s measurement, Belinda receives the mixed state $\sigma = \sum_i q_i |\phi_i\rangle\langle\phi_i|$.
The trace distance between the initial (i.e. untampered-with) state and the state that Belinda tests is
\[
D(|\psi\rangle\langle\psi|, \sigma) \leq \sum_i q_i D(|\psi\rangle\langle\psi|, |\phi_i\rangle\langle\phi_i|) \text{ by convexity of the trace distance} \\
\leq q_m D(|\psi\rangle\langle\psi|, |\phi_m\rangle\langle\phi_m|) + \sum_{i \neq q_m} q_i \\
= p D(|\psi\rangle\langle\psi|, |\phi_m\rangle\langle\phi_m|) + (1 - p) \\
= p \sqrt{1 - |\langle\psi|\phi_m\rangle|^2} + (1 - p) \\
= p \sqrt{1 - p} + (1 - p)
\]

The trace distance is an upper bound on the difference between the probability that Belinda accepts $|\psi\rangle$ and the probability that Belinda accepts $|\phi\rangle$. So $s - \epsilon_c \leq p \sqrt{1 - p} + (1 - p)$. \(\square\)

This bound is rather weak. The derivation assumes that Belinda has access to Charlie’s entire final state, including any ancilla qubits; in practice, if Charlie uses ancillas, he would likely be better off zeroing them before returning them to Belinda.

It is also not obvious that even this bound is achievable in practice. A straightforward protocol using no ancillas comes up far short. Suppose that Belinda sends Charlie the $C$ register of
\[
|\psi_m\rangle_{BC} = \frac{1}{\sqrt{2}}(|0\rangle_B|0\rangle_C + |m\rangle_B|m\rangle_C),
\]
where $|0\rangle$ is an arbitrary nonsense state. If Charlie returns the state unopened, Belinda verifies it by projecting onto $|\psi_m\rangle$. This protocol is complete (i.e. $\epsilon_c = 0$). If Charlie measures his qubit in the computational basis, Belinda will detect his cheating with probability 50%, which is worse than the bound of 85%.

An improved protocol would replace $|0\rangle_B|0\rangle_C$ with a superposition of many garbage states. This would allow Belinda to detect Charlie’s cheating with probability near one if Charlie gets unlucky and fails to recover the message; that is, the second approximation above would be tight. This would improve the probability of detection to 75%.

Regardless of how close to the soundness bound a protocol of this type is, it will be subject to other types of attack. For example, if Charlie is worried that Belinda is blackmailing his friend, then he can verify that $m$ is not a compromising message about his friend. That is, Charlie can define a classical function $g(m') \in \{0, 1\}$ over the space of possible messages such that $g(m') = 1$ indicates that $m'$ is a picture of Charlie’s friend. Then Charlie can apply the message recovery protocol coherently and measure $g$ on the result. If the result is 1, then Belinda is probably blackmailing Charlie’s friend, and Belinda has a reasonable chance of detecting that Charlie cheated. If, on the other hand, the result was zero, then Charlie did not damage the state at all and will not be detected.

To work around these attacks, we need to look at a broader class of protocols.

III. MULTIPLE COMPROMISING PICTURES

If Belinda has more than one message and she is willing to give Charlie a sealed state that can only reveal one of them, then she can escape the no-quantum-blackmail bound. For example, suppose she has $n$ compromising pictures $m_1, \ldots, m_n$ of her victim, and she considers the threat of any one of them being published to be adequate to ensure her safety. Belinda can generate a superposition of pictures
\[
|\psi\rangle_{BC} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle|m_i\rangle
\]
and send register $C$ to Charlie. Of course, any tabloid would take great pains to authenticate a damaging photograph before publishing it, so we will assume that Charlie has no interest in cheating unless he can recover an entire intact photograph. If he can do this, then he knows which photograph Belinda sent him, and register $B$ must collapse to a single value of $i$. No matter what state Charlie sends Belinda, Belinda can detect that he cheated with probability at least $\frac{n-1}{n}$. As $n$ becomes large, this protocol becomes more secure.
This protocol is vulnerable to other attacks, though. Charlie could measure a property that all \( n \) pictures have in common without damaging the state at all. For example, he could use a quantum image recognition program to figure out who is in the pictures.

To prevent this type of attack and allow exponential security, we need a new type of resource.

IV. CAPTCHAS

The fundamental weakness of quantum cryptography that makes quantum blackmail difficult is that any computation can be performed coherently. If Charlie can calculate something based on a sealed state that Belinda gives him, then he can do the same calculation coherently, measure some property of the result, and undo the original calculation. If the unsealing process required that Charlie does something that cannot be done on a quantum computer, then we could block these attacks. If quantum computers ever become as powerful as classical computers, then the only things that quantum computers will not be able to do are things that no computer at all can do.

CAPTCHAs, a type of spam-preventing technology on the internet, are based on a computation that can only be reliably done by humans. A CAPTCHA is a “Completely Automated Public Turing test to tell Computers and Humans Apart” [1]. To use a CAPTCHA, a website generates a random word or number \( x \). The website then computes a picture, sound, or other message based on \( x \) that is meant to be decoded by a human. Any human should be able to find \( x \) by inspecting the web page, but no polynomial-time algorithm should be able to find \( x \) with non-negligible probability.

A CAPTCHA is defined as a communication protocol, possibly with multiple rounds. We need a function instead of a communication protocol, and we need the function to be secure against quantum adversaries. We therefore propose the following definition:

**Definition 1.** A human-invertible one-way function is a one-to-one function \( f : \{0,1\}^* \rightarrow I \) that maps strings of bits to a set \( I \) and has these properties:

- The function \( f \) can be evaluated in quantum polynomial time.
- Given \( f(x) \) for unknown \( x \), a human acts as an oracle that computes \( x \) in one query with negligible probability of error. This process is incoherent: \( f(x) \) leaks to the environment. Most likely, \( I \) will be a set of images, sounds, or other media that a human can look at.
- There is no polynomial-time quantum algorithm that can compute \( f^{-1}(z) \) with non-negligible probability of success. This must remain true even if the algorithm can query \( f^{-1} \) on other inputs in the range of \( f \).

Standard CAPTCHAs are randomized, but we will assume that \( f \) can derive any randomness it needs from its argument.

A human-invertible one-way function is very similar to a trapdoor one-way function; the difference is that instead of being invertible using a secret, it is invertible by asking a human to invert it. Without access to a human, it works just like a trapdoor one-way function with an unknown secret. We will therefore construct a cipher from \( f \) in much the same way that public-key ciphers are constructed using trapdoor functions. The standard construction is Bellare and Rogaway’s OAEP [2], and we will use a similar construction. OAEP is randomized, and we will replace the randomness with entanglement. To unseal a message, Charlie must show a particular value of \( f \) to a human, breaking the entanglement in the process.

Let \( n \) be the length of the message being sealed. Following OAEP, choose security parameters \( k \) and \( k_0 \), where \( n = k - k_0 \). Let \( G : \{0,1\}^{k_0} \rightarrow \{0,1\}^n \) be a pseudorandom generator and \( H : \{0,1\}^n \rightarrow \{0,1\}^{k_0} \) be an ideal hash function. Both \( G \) and \( H \) are publicly known. The sealed state encoding \( y \) is

\[
|\psi_y\rangle_{BC} = \frac{1}{\sqrt{2^{k_0}}} \sum_{r \in \{0,1\}^{k_0}} |r\rangle_B |E_r^{G,H}(y) \rangle_C,
\]

where

\[
E_r^{G,H}(y) = f(y \oplus G(r) \parallel r \oplus H(y \oplus G(r)))
\]

is the encrypted version of \( y \). The \( \parallel \) operator denotes concatenation of strings of bits, and \( \oplus \) is bitwise exclusive or.
To seal a message $y$, Belinda generates $|\psi_y\rangle_{BC}$ and gives Charlie the $C$ register. To unseal the message, Charlie measures register $C$ and shows it to a human. The human inverts $f$ to recover $y \oplus G(r) \parallel r \oplus H(y \oplus G(r))$ for some unknown value of $r$. Charlie then computes $H(y \oplus G(r))$ to recover $r$, $G(r)$, and $y$.

If Charlie does not unseal the message, he returns register $C$ to Belinda, who measures $|\psi_y\rangle\langle\psi_y|$ on the combined state in registers $B$ and $C$. If the outcome is 1, then she believes Charlie; if not, she accuses Charlie of cheating. This protocol is fully complete – if Charlie does not cheat, then Belinda will believe him with probability one.

If Charlie cheats, he can use an arbitrarily complicated strategy to select the values of $f$ that he asks a human to invert. Let $Q \subseteq I$ be the set of all $f$ values that he will ever ask a human to invert. Of course, Charlie can ask a human to evaluate $f^{-1}$ even after Belinda has made her measurement, so Belinda cannot possibly know the set $Q$. Nonetheless, Charlie is constrained to make a polynomial number of queries, so $|Q| = O(\text{poly}(k, k_0))$. Let

$$R = \{r : \mathcal{E}_{r,H}^{G,H}(y) \in Q\}$$

and define projectors

$$T = |\psi_y\rangle\langle\psi_y| = \frac{1}{2^{k_0}} \sum_{r \in \{0,1\}^{k_0}} |r\rangle_B |\mathcal{E}_{r,H}^{G,H}(y)\rangle_{CC} \langle\mathcal{E}_{r,H}^{G,H}(y)|_B \langle r|$$

$$U = \frac{1}{2^{k_0} - |R|} \sum_{r = \{0,1\}^{k_0} \setminus R} |r\rangle_B |\mathcal{E}_{r,H}^{G,H}(y)\rangle_{CC} \langle\mathcal{E}_{r,H}^{G,H}(y)|_B \langle r'|.$$  

Belinda’s measurement is $T$, the projector onto the uniform superposition of all $r$ values, each paired with its corresponding $\mathcal{E}_r$ value. The projector $U$ is almost the same; it projects onto the uniform superposition of $r$ values that are useless to Charlie – this is the set of values of $r$ for which Charlie will never ask a human to invert $f(\mathcal{E}_{r,H}^{G,H})$. If Belinda somehow measured $U$ and obtained the outcome 1, then Charlie could only attempt to decrypt $f(\mathcal{E}_{r,H}^{G,H})$ without asking a human to evaluate $f^{-1}(f(\mathcal{E}_{r,H}^{G,H}))$.

Belinda cannot measure $U$ because she does not know the set $R$, but she can measure $T$ instead. $T$ and $U$ are both rank 1 projectors, and they project onto states that differ negligibly from each other. If Belinda obtains the outcome 1 from $T$ and therefore believes Charlie, then either she got unlucky due to the difference between $T$ and $U$, which occurs with negligible probability, or Charlie’s queries are all useless. [6]

In the event that Charlie’s queries are useless, then recovering $x$ is mostly equivalent to breaking OAEP, which should be impossible as long as the human-invertible one-way function $f$ is secure. The standard OAEP security proof [2] fails in this context for two reasons: it assumes a classical adversary, and it assumes that the trapdoor function is a permutation; $f$ is merely one-way. The latter problem should be easily correctable, but the former will be more challenging. I am unaware of any meaningful security proofs of OAEP (or, for that matter, any other public-key cipher construction) against quantum adversaries.

This protocol is also somewhat resistant to attacks that break the human-invertible one-way function after Belinda makes her measurement – if Belinda’s test passes with non-negligible probability, then Charlie does not know anything that would let him reliably determine the value of $r$. It is hard to imagine that any quantum state he generated from register $C$ without being able to invert $f$ would let him later recover $y$ even with unlimited computational power.

If Belinda wants to seal a very long message, it could be more efficient to generate a short random key, encrypt the message against the key, and seal the key instead of the message.

V. OPEN QUESTIONS

These protocols require Belinda to store a quantum state that is entangled with the sealed state. It should be possible to eliminate the entanglement. One naive approach is to eliminate register $B$, making the sealed state

$$\frac{1}{\sqrt{2^{k_0}}} \sum_{r \in \{0,1\}^{k_0}} |\mathcal{E}_{r,H}^{G,H}(y)|.$$

This is neither practical nor secure: preparing this state is likely as hard as index erasure [3], and if Charlie unseals the message then he can recreate the original state with whatever algorithm Belinda used to prepare it in the first place. The difficulty in preparing the state can be resolved by giving Charlie both registers, that is

$$\frac{1}{\sqrt{2^{k_0}}} \sum_{r \in \{0,1\}^{k_0}} |r\rangle|\mathcal{E}_{r,H}^{G,H}(y)||.$$
This is insecure for the same reason. An improved version would be

\[ \frac{1}{\sqrt{2^{k_0}}} \sum_{r \in \{0,1\}^{k_0}} e^{i \phi(r)} |r\rangle |E_{G,H}^r(y)\rangle, \]

where the function \( \phi(r) \) is a secret known only to Belinda. This seems likely to be secure, although the security proof would be more complicated.

Regardless of whether Belinda needs to store a quantum state, all of these protocols involve giving Charlie a highly entangled state. A protocol in which Charlie’s state was a tensor product of a large number of low-dimension systems would be very interesting, since it could be built with much simpler quantum computing technology. Designing such a protocol that is secure even if Charlie can make entangling measurements may be difficult.

Putting aside quantum blackmail in particular, the technology for analysing the security of even classical cryptographic constructions against quantum attack is limited. There is extensive literature on the security of constructions as varied as the Luby-Rackoff block cipher, pseudorandom functions, public-key cryptosystems, and modern block cipher modes. Many of these are provably secure against classical attack assuming that some underlying primitive is secure. Little progress has been made in defining security against quantum attacks. Classical attacks can take many forms (e.g. known-plaintext attacks, adaptive chosen-ciphertext attacks, etc.); this taxonomy will need to be extended to meaningfully discuss security in a post-quantum world (e.g. what is a non-adaptive chosen-quantum-ciphertext attack, and when is it relevant?). Even less progress has been made in proving the security of cryptographic constructions against quantum attack; the best we can say is that no one has found generic attacks better than Grover’s algorithm.

Until the basic technology for analyzing the security of constructions like OAEP against quantum attacks is in place, it will be difficult to make rigorous statements about the security of even straightforward quantum cryptographic constructions like quantum blackmail.

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* Electronic address: andy@luto.us

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[5] If unsealing the state requires ancilla qubits, then those qubits can be treated as part of register \( C \). The final state of those qubits does not matter, since Belinda does not expect Charlie to preserve anything after unsealing the state.

[6] This is not the same thing as saying that if Belinda believes Charlie, then Charlie’s queries are useless with high probability. Charlie could, for example, unseal the message and give Belinda the classical state \( |E_{G,H}^r(y)\rangle \) in register \( C \). Belinda will believe him with negligible probability, but if she believes him then his queries are still useful with probability 1.