Robustness of a bisimulation-type faster-than preorder

Katrin Iltgen, Walter Vogler
Inst. f. Informatik, Universität Augsburg
\{katrin.iltgen, walter.vogler\}@informatik.uni-augsburg.de

TACS is an extension of CCS where upper time bounds for delays can be specified. Lütgen and Vogler defined three variants of bismulation-type faster-than relations and showed that they all three lead to the same preorder, demonstrating the robustness of their approach. In the present paper, the operational semantics of TACS is extended; it is shown that two of the variants still give the same preorder as before, underlining robustness. An explanation is given why this result fails for the third variant. It is also shown that another variant, which mixes old and new operational semantics, can lead to smaller relations that prove the same preorder.

1 Introduction

To evaluate or compare the worst-case efficiency of asynchronous systems, it is adequate to introduce upper time bounds for their actions; see e.g. [8] for distributed algorithms, or [10] for Petri nets: since components can still be arbitrary fast, their relative speeds are indeterminate, i.e. one is truly dealing with asynchronous systems. In other words, everything that can happen when disregarding time can still happen in time zero; at the same time, Zeno behaviour is not really relevant when studying worst-case efficiency. In order to introduce upper time bounds, Milner’s CCS [9] is extended by a clock prefix $\sigma$ in [7] to obtain the process algebra TACS (Timed Asynchronous Communicating Systems), where $\sigma$ represents a potential delay of one time unit. Since time is treated as discrete, this means that e.g. process $\sigma.a.P$ can either delay action $a$ (performing a unit time step, also denoted by $\sigma$, leading to process $a.P$) or skip the clock prefix and perform $a$ immediately; thus, possibly using repeated clock prefixes, upper time bounds for communications can be specified. An elegant type of (so-called naive) faster-than relations is introduced that corresponds to bisimulation on actions (related processes must have the same functionality) and simulation on time steps (if the faster process performs a time step, i.e. the user has to wait, then the slower process must follow suit).

Furthermore, two variants of the faster-than relations (called delayed and indexed) are developed that are more complicated, but possibly more intuitive. As a validation of the first variant, it is shown that all three variants lead to the same faster-than preorder, demonstrating the robustness of the approach. Furthermore, the coarsest precongruence contained in this faster-than preorder is characterised, introducing a small modification regarding urgent actions, and these ideas are translated to a weak setting, where $\tau$-actions are invisible.

In the present paper, we give new results to emphasise robustness further. For this, we extend the operational semantics of TACS: while originally clock prefixes could only be skipped when performing an action, we now also allow this when performing a time step. Upper time bounds correspond to the idea that there are unobservable activities on a lower level of abstraction that can lead to varying delays; skipping a clock prefix during a time step means that, during this time step, it becomes clear that these activities will not lead to the maximal possible delay in this run. For example, the process $P \equiv \sigma.\sigma.\sigma.a.0$ may now skip one $\sigma$-prefix when performing a time step and behave like $\sigma.a.0$ afterwards. This new behaviour seems more realistic; observe that it can become visible with a progress bar. It should be noted...
that, with this extension, time determinism does not hold anymore; as a sort of compensation, we gain transitivity of the $\sigma$-transition relation.

In our new setting, we look at textually the same three variants from [7] mentioned above, and show: two of the three variants lead to the same faster-than preorder on TACS processes as the one in [7]. This robustness result does not hold for the third variant, which is an amortised bisimulation; see [6] for a very similar idea. Our counter-example reveals that the idea behind amortised bisimulation relies essentially on time determinism, so we cannot really expect a coincidence result in this case. Robustness against small variations is often regarded as a quality criterion, thus all in all our results further demonstrate the good quality of the TACS-approach.

We also show that analogous modifications as in [7] lead to the same coarsest precongruence for our new operational semantics; the same is true for the weak setting of [7], see [5]. But the bisimulation-like strong faster-than relations that serve as witness for the faster-than precongruence are different for the old and the new operational semantics: in some cases the first, in other cases the second type of relation can be smaller. As a final contribution, we define a third type of so-called strong combined-faster-than relations that mixes the old and the new operational semantics and serves to demonstrate the same precongruence; this type of relations includes the other types, and sometimes allows a smaller relation than any of the other two.

This paper is organised as follows. The next section presents the syntax as well as the old and the new operational semantics of the process algebra TACS. Moreover, we get familiar with the nature of the new $\sigma$-transitions and prove their transitivity. In Section 3, we compare the original time steps to those newly introduced here; the key concept is a syntactic faster-than relation from [7]. Subsequently, we demonstrate the robustness of the naive faster-than preorder against the transition extension in Section 4. Section 5 proves the robustness of the delayed faster-than preorder, while Section 6 demonstrates the defect of the extended indexed faster-than preorder. In Section 7, we establish the robustness of the precongruence; we further introduce the strong combined-faster-than relations and give two example processes where such a relation can be much smaller than a relation of one of the other two types. Finally, we draw a short conclusion in Section 8. The preliminary version [5] of this paper contains a number of proof details that are omitted here.

2 TACS

In this section, we introduce TACS [7] as an extension of CCS by the clock prefix $\sigma$, representing a delay of up to one unit of time. We define the operational semantics of TACS, including our new extension, and we give some first results.

Let $\Lambda$ be a countable set of action names or ports $a, b, c$; $\overline{\Lambda} = \{a | a \in \Lambda\}$ is the set of complementary action names $\overline{a}, \overline{b}, \overline{c}$, and $\mathcal{A} = \Lambda \cup \overline{\Lambda} \cup \{\tau\}$ is the set of all actions $\alpha, \beta, \gamma$, including the internal action $\tau$. As usual, $\overline{\overline{a}} = a$ for all $a \in \Lambda$, and an action $a$ will communicate with its complement $\overline{a}$ to produce the internal action $\tau$.

A TACS term is defined as follows, where the operators have the usual meaning:

$$P ::= 0 \mid x \mid \alpha.P \mid \sigma.P \mid P + P \mid P|P \mid P \setminus L \mid P[f] \mid \mu x.P$$

where $x$ is a variable taken from a countably infinite set $\mathcal{V}$ of variables, $L \subseteq \mathcal{A} \setminus \{\tau\}$ is a finite restriction set, and $f : \mathcal{A} \rightarrow \mathcal{A}$ is a finite relabelling. A finite relabelling satisfies the properties $f(\tau) = \tau$, $f(\overline{a}) = \overline{f(a)}$, and $|\{\alpha | f(\alpha) \neq \alpha\}| < \infty$. The set of all terms is abbreviated by $\widehat{\mathcal{P}}$ and, for convenience, we define
\( \overline{L} = \{ \overline{a} \mid a \in L \} \). We use the standard definitions for free and bound variables (where \( \mu x \) binds \( x \)), and open and closed terms. \( P[Q/x] \) stands for the term that results when substituting every free occurrence of \( x \) in \( P \) by \( Q \). A variable is called guarded in a term if each occurrence of the variable is in the scope of an action prefix. We require for terms of the form \( \mu x. P \) that \( x \) is guarded in \( P \). Closed, guarded terms are referred to as processes, with the set of all processes written as \( \mathcal{P} \), and syntactic equality is denoted by \( \equiv \).

| Table 1: Urgent action sets |
|--------------------------------|
| \( U(\sigma. P) = \text{df} \emptyset \) | \( U(0) = U(x) = \text{df} \emptyset \) | \( U(P \setminus L) = \text{df} U(P) \setminus (L \cup L) \) |
| \( U(\alpha. P) = \text{df} \{ \alpha \} \) | \( U(P + Q) = \text{df} U(P) \cup U(Q) \) | \( U(P(f)) = \{ f(\alpha) \mid \alpha \in U(P) \} \) |
| \( U(\mu x. P) = \text{df} U(P) \cup U(Q) \cup \{ \tau \mid U(P) \cap U(Q) \neq \emptyset \} \) |

| Table 2: Operational semantics for TACS (action transitions) |
|-------------------------------------------------------------|
| Act \( \alpha. P \xrightarrow{\sigma} P \) | Pre \( P \xrightarrow{\sigma. P} P' \) | Rec \( P \xrightarrow{\mu x. P} P' | [\mu x. P/x] \) |
| Sum1 \( P \xrightarrow{\sigma} P' \) | Com1 \( P \xrightarrow{\sigma} P' \) | Com3 \( P \xrightarrow{\sigma} P' \) |
| \( P + Q \xrightarrow{\sigma} P' \) | \( P/Q \xrightarrow{\sigma} P'|Q \) | \( P|Q \xrightarrow{\tau} P'|Q' \) |
| Rel \( P \xrightarrow{\sigma} P' \) | Res \( P \xrightarrow{\sigma} P' \) | \( P \xrightarrow{\sigma} P' \) |
| \( P[f] \xrightarrow{f(\alpha)} P'[f] \) | \( P \setminus L \xrightarrow{\sigma} P'|L \) | \( \alpha \notin L \cup L \) |

As a basis for the operational semantics, we first define the set \( U(P) \) of urgent actions of \( P \), i.e. those initial actions that cannot be delayed because of a \( \sigma \)-prefix, see Rule (Pre). On the basis of this set, we define the action transitions with the SOS-rules in Table 2; there are symmetric rules (Sum2) and (Com2) for (Sum1) and (Com1). These are not influenced by our extension, and are standard except for Rule (Pre), explained in the introduction.

The time steps, i.e. the \( \sigma \)-transitions, are defined in Table 3. We write \( \xrightarrow{\sigma} \) for the type-1 time steps according to [7] – they do not use Rule (tNew) – and \( \xrightarrow{\sigma} \) for our extended setting, using all rules. Observe that \( a.P \) can perform a time step, since it might have to wait for a synchronisation partner that can delay \( a \) or is unable to perform it. Furthermore, special care is taken in Rule (tCom) such that all in all, a time step is possible iff there is no urgent \( \tau \) (Maximal Progress Assumption). E.g. \( a.0 | \overline{a}.0 \) cannot perform a time step, while \( a.0 | \sigma.\overline{a}.0 \) can.

We will write \( \xrightarrow{\sigma}^+ \) (and \( \xrightarrow{\sigma}^* \)) for the transitive (and the reflexive-transitive resp.) closure of \( \xrightarrow{\sigma} \), for \( i \in \{1, 2\} \), and similarly for other relations.

Clearly, type-1 time steps like \( \sigma.\sigma.\sigma.a.0 \xrightarrow{\sigma} \sigma.\sigma.a.0 \) are also of type 2, but there are additional type-2 time steps as e.g. \( \sigma.\sigma.\sigma.a.0 \xrightarrow{\sigma} \sigma.a.0 \), which corresponds to a sequence of two type-1 time steps. But things are not always that easy: e.g. \( \sigma.\sigma.\sigma.a.0 \mid \sigma.\sigma.a.0 \xrightarrow{\sigma} a.0 \mid \sigma.a.0 \) reaches a process
Lemma 2.1 Let $P, P', Q \in \mathcal{P}$ and $\gamma \in \mathcal{A} \cup \{\sigma\}$.

1. $P \xrightarrow{\gamma} P'$ implies $P[\mu y. Q/y] \xrightarrow{\gamma} P'[\mu y. Q/y]$.
2. $y$ guarded in $P$ and $P[\mu y. Q/y] \xrightarrow{\gamma} P'$ implies $\exists P'' \in \mathcal{P}. P \xrightarrow{\gamma} P''$ and $P' \equiv P''[\mu y. Q/y]$.

Now we are able to prove our first result that $\xrightarrow{\sigma}$ is transitive. In its proof and also in the future, we will use the following preservation of guardedness under a time step (which is not hard to see): $P \xrightarrow{\sigma} P'$ for some $i \in \{1, 2\}$ and $x$ guarded in $P$ implies that $x$ is also guarded in $P'$.

Proposition 2.2 Let $P, P', P'' \in \mathcal{P}$.

$P \xrightarrow{\sigma} P'$ and $P'' \equiv P'[\mu y. Q/y]$ imply $P \xrightarrow{\sigma} P''[\mu y. Q/y]$.

Proof: This proposition can be proved by induction on the structure of $P$. The two more interesting cases are:

1. Let $P \equiv \sigma.P_1$. If $\sigma.P_1 \xrightarrow{\sigma} P_1$ by (tPre) and $P_1 \xrightarrow{\sigma} P''$, we can infer $\sigma.P_1 \xrightarrow{\sigma} P''$ by (tNew).

2. Let $P \equiv \mu x.P_1$. Consider the case $\mu x.P_1 \xrightarrow{\sigma} P_1[\mu x.P_1/x]$ due to $P_1 \xrightarrow{\sigma} P_1'$ by (tRec) and $P_1'[\mu x.P_1/x] \xrightarrow{\sigma} P''$. By our requirements for $\mu x.P_1$, $x$ is guarded in $P_1$ and, hence, also in $P_1'$. Using Lemma 2.1(2), we obtain $P_1' \xrightarrow{\sigma} P_1''$ and $P'' \equiv P''[\mu x.P_1/x]$ for some $P_1'' \in \mathcal{P}$, since $x$ is guarded in $P_1'$. By induction, we infer $P_1 \xrightarrow{\sigma} P_1''$ from $P_1 \xrightarrow{\sigma} P_1' \xrightarrow{\sigma} P_1''$, and conclude $\mu x.P_1 \xrightarrow{\sigma} P_1[\mu x.P_1/x] \equiv P''$ by (tRec).
3 Relating $\sigma_2$ and $\sigma_1$

The key to describing how $\sigma_2$ can be matched by $\sigma_1$ and to the new robustness results lies in the syntactic faster-than relation $\succeq$ from [7] and its transitive closure.

**Definition 3.1** The relation $\succeq \subseteq \widehat{\mathcal{P}} \times \widehat{\mathcal{P}}$ is defined as the smallest relation satisfying the following properties, for all $P, P', Q, Q' \in \widehat{\mathcal{P}}$.

Always:
1. $P \succeq P$
2. $P \succeq \sigma.P$
3. $P' \succeq P, Q' \succeq Q$
4. $P' \succeq P \land L$
5. $P' \succeq P \land P$
6. $P' \succeq P \land P$
7. $P' \succeq P \land P$

If $P' \succeq P, x$ guarded in $P$:
8. $P' \succeq P, x$ guarded in $P$

Observe that the syntactic relation is defined for arbitrary open terms. We note some technical results, partly taken from [7]:

**Lemma 3.2** Let $P, P', Q \in \widehat{\mathcal{P}}$ such that $P' \succeq P$, and let $y \in \mathcal{Y}$.

1. [7, Lemma 7(1)] Then $y$ is guarded in $P$ if and only if $y$ is guarded in $P'$.
2. [7, Lemma 7(2)] $P'[Q/y] \succeq P'[Q/y]$.
3. [7, Lemma 8(2)] $\mathcal{W}(P') \supseteq \mathcal{W}(P)$.

The following lemma establishes several properties of $\succeq$ similar to those of the syntactic relation in Def. 3.1:

**Lemma 3.3** Let $P, P', Q, Q' \in \widehat{\mathcal{P}}$.

1. $P' \succeq P, Q' \succeq Q$ then:
2. $P' \succeq P, Q' \succeq Q$
3. $P' \succeq P \land L$
4. $P' \succeq P \land L$
5. $P' \succeq P \land P$
6. $P' \succeq P \land P$
7. $P' \succeq P \land P$

**Proof:** In the proof of (1) and (2), one has to deal with the case that $P' \succeq P$ and $Q' \succeq Q$ hold because of $\succeq$-chains of different length: one simply extends the shorter chain using Def. 3.1(1). Parts (3) and (4) are easier.

For (5), take $n \geq 1$ and $P_0, \ldots, P_n \in \widehat{\mathcal{P}}$ with $P' \equiv P_0 \equiv P_1 \equiv \cdots \equiv P_n \equiv P$. Since $x$ is guarded in $P$, we infer $P_{n-1}[\mu x. P_n/x] \succeq \mu x. P_n$ from $P_{n-1} \succeq P_n$ by using Def. 3.1(7). Further, we obtain $P_{n-1}[\mu x. P_n/x] \succeq P_1[\mu x. P_n/x]$, due to $P_1 \equiv P_1$ for $1 \leq i \leq n-1$, by Lemma 3.2(2) and are done.

With a time step, a process should turn into a process that should be faster with everything it does; we can now prove this in terms of $\succeq$ – the main achievement here lies in the treatment of recursion. The other important point about $\succeq$ is that it describes a faster-than relationship in the sense of the semantic definitions in the next section; this holds by Proposition 4.2 for one variant, and Theorems 4.3 and 5.3 transfer this to other variants.

**Lemma 3.4** Let $P, P' \in \widehat{\mathcal{P}}$.

1. [7, Prop. 9(1)] $P \rightarrow_{\sigma_1} P'$ implies $P' \succeq P$, for all terms $P, P' \in \widehat{\mathcal{P}}$. 

(2) $P \xrightarrow{\sigma_2} P'$ implies $P' \geq^+ P$, for all terms $P, P' \in \mathcal{P}$.

**Proof:** We prove Part (2) by induction on the inference of $P \xrightarrow{\sigma_2} P'$.

$tNil$ $P \equiv 0 \equiv P'$. Since $\geq \geq^+$, $0 \geq 0$ holds by using Def. 3.1(1).

tAct $P \equiv a.P'' \equiv P'$. Since $\geq \geq^+$, $a.P'' \geq^+ a.P''$ holds by using Def. 3.1(1).

tPre $P \equiv \sigma.P'$. Since $\geq \geq^+$, $P' \geq^+ \sigma.P'$ holds by using Def. 3.1(2).

tNew $P \equiv \sigma.P_1$. Let $\sigma.P_1 \xrightarrow{\sigma_2} P'$ due to $P_1 \xrightarrow{\sigma_2} P'$. The latter implies $P' \geq^+ P_1$ by induction, and with $P_1 \geq^+ \sigma.P_1$ by Def. 3.1(2), we conclude $P' \geq^+ \sigma.P_1$.

tRec $P \equiv \mu.x.P_1$ and $P' \equiv P_2[\mu.x.P_1/x]$.

Let $\mu.x.P_1 \xrightarrow{\sigma_2} P_2[\mu.x.P_1/x]$ due to $P_1 \xrightarrow{\sigma_2} P_2$. By induction, the latter implies $P_2 \geq^+ P_1$. Since $x$ is guarded in $P_1$, we can infer $P_2[\mu.x.P_1/x] \geq^+ \mu.x.P_1$ from $P_2 \geq^+ P_1$ by Lemma 3.3(5).

tSum $P \equiv P_1 + Q_1$ and $P' \equiv P_2 + Q_2$.

Since $P \xrightarrow{\sigma_2} P'$, we have $P_1 \xrightarrow{\sigma_2} P_2$ and $Q_1 \xrightarrow{\sigma_2} Q_2$. $P_2 \geq^+ P_1$ and $Q_2 \geq^+ Q_1$ follows by induction hypothesis and $P_2 + Q_2 \geq^+ P_1 + Q_1$ results by application of Lemma 3.3(2).

tCom The treatment of this case is analogous to case tSum and uses Lemma 3.3(1).

tRel This case follows in analogy to case tRes, using Lemma 3.3(4). □

Now we will relate the result of a type-2 time step with the only result of a type-1 time step. Consider e.g. $P =_{df} \sigma.\sigma.\sigma.0 | \sigma.\nu.0 | \sigma.a.0 \xrightarrow{\sigma_2} a.0 | \sigma.\nu.0 | a.0 = P'$, while the only enabled type-1 time step leads to $\sigma.\sigma.a.0 | \sigma.\nu.0 | a.0 = P''$; observe $P'' \xrightarrow{\tau} \sigma$ as $\tau \in \mathcal{U}(P'')$. We see that $P'$ results from $P''$ by removing some leading $\sigma$-prefixes, so they should be related by $\geq^+$.

**Proposition 3.5** Let $P, P' \in \mathcal{P}$. $P \xrightarrow{\sigma_2} P'$ implies $\exists P'' \in \mathcal{P}. P \xrightarrow{\sigma_1} P''$ and $P' \geq^+ P''$.

**Proof:** The proof is an induction on the inference of $P \xrightarrow{\sigma_2} P'$. Most cases are straightforward, often using Lemma 3.3, we only show two:

$tNew$ $P \equiv \sigma.P''$. Observe that $\sigma.P'' \xrightarrow{\sigma_2} P''$, and let $\sigma.P'' \xrightarrow{\sigma_2} P'$ due to $P'' \xrightarrow{\sigma_2} P'$ by (tNew). The latter implies $P' \geq^+ P''$ by Lemma 3.4(2).

tRec $P \equiv \mu.x.P_1$. Let $\mu.x.P_1 \xrightarrow{\sigma_2} P'_1[\mu.x.P_1/x]$ due to $P_1 \xrightarrow{\sigma_2} P'_1$ by (tRec).

By the induction hypothesis, there exists $P'_2$ such that $P_1 \xrightarrow{\sigma_1} P'_2$ and $P'_2 \geq^+ P'_1$.

Thus, we may infer $\mu.x.P_1 \xrightarrow{\sigma_1} P'_2[\mu.x.P_1/x]$ by (tRec). Applying Lemma 3.2(2) to the $\geq$-chain behind $P'_1 \geq^+ P'_2$, we obtain $P'_1[\mu.x.P_1/x] \geq^+ P'_2[\mu.x.P_1/x]$. □

Generalising a result from [5], but with the same proof, one can show for any process $P$, where any occurrence of parallel composition or choice is ‘guarded’ that $P \xrightarrow{\sigma_2} P'$ implies $P \xrightarrow{\sigma_1} P'$.
4 The naive faster-than preorders

In [7], the naive faster-than preorder is introduced as an elegant and concise candidate for a faster-than preorder: the faster and the slower process are simply linked by a relation that is a simulation for time steps and a strong bisimulation for actions. The definition of the 1-naive faster-than preorder is adopted from [7] and extended to a second variant by considering our new type-2 transitions:

**Definition 4.1** For \( i \in \{1, 2\} \), a relation \( \simeq \subseteq \mathcal{P} \times \mathcal{P} \) is an \( i \)-naive faster-than relation if the following conditions hold for all \( (P, Q) \in \simeq \) and \( \alpha \in \mathcal{A} \).

1. \( P \xrightarrow{\alpha} P' \) implies \( \exists Q'. Q \xrightarrow{\alpha} Q' \) and \( (P', Q') \in \simeq \).
2. \( Q \xrightarrow{\alpha} Q' \) implies \( \exists P'. P \xrightarrow{\alpha} P' \) and \( (P', Q') \in \simeq \).
3. \( P \xrightarrow{\sigma_1} P' \) implies \( \exists Q'. Q \xrightarrow{\sigma_1} Q' \) and \( (P', Q') \in \simeq \).

We write \( P \simeq_{\sim_{i, -\text{nv}}} Q \) if \( (P, Q) \in \simeq \) for some i-naive faster-than relation \( \simeq \), and call \( \simeq_{i, -\text{nv}} \) i-naive faster-than preorder.

Part (1) and (2) require that the faster and the slower process are functionally equivalent, in other words, \( \simeq \) is a strong bisimulation. The i-naive faster-than relation refines strong bisimulation since additionally any time step of the faster process (corresponding to a delay of its user) must be simulated by the slower one. Extra time steps \( Q \) might perform are not considered (cf. Sec. 5) intuitively because they do not change the functional behaviour of \( Q \). (Formally, \( Q \xrightarrow{\sigma_2} Q' \) implies \( Q' \succeq^+ Q \) by Lemma 3.4, thus \( Q' \) and \( Q \) are strongly similar by Proposition 4.2 below.) Note that faster-than holds for equally fast processes, i.e. it is not strict.

As usual, one can show that \( \simeq_{-\text{nv}} \) is an i-naive faster-than relation and a preorder; in particular, the composition of two such relations is again one. The i-naive faster-than preorder is defined on processes; it can be extended to open terms as usual by considering all possible substitutions with processes.

In the sequel, we will prove that the faster-than preorders \( \simeq_{1, -\text{nv}} \) and \( \simeq_{2, -\text{nv}} \) coincide. For this, we take from [7] that the syntactic \( \succeq \) satisfies the definition of a 1-naive faster-than relation and show the same for \( \succeq^+ \). The same results for the 2-naive case can be useful, but we are able to avoid their tedious proofs.

**Proposition 4.2** (1) The relation \( \succeq \) satisfies the defining clauses of a 1-naive faster-than relation, also on open terms; hence, \( \succeq \) restricted to processes is a 1-naive faster-than relation and \( \succeq_{\mathcal{P} \times \mathcal{P}} = df \succeq \cap (\mathcal{P} \times \mathcal{P}) \subseteq \simeq_{1, -\text{nv}} \).

2. The same holds for \( \succeq^+ \).

**Proof:** To obtain Part (2) from Part (1), observe that \( \succeq^+ \) equals \( \succeq_0 \cup \succeq_1 \cup \ldots \) and the closure of 1-naive faster-than relations under union and composition.

**Theorem 4.3 (Coincidence I)** The preorders \( \simeq_{1, -\text{nv}} \) and \( \simeq_{2, -\text{nv}} \) coincide.

**Proof:** There are two places, where we insert some text in square brackets. With this text, this is essentially the proof of Theorem 7.3 here, the text should be ignored.

To prove \( \simeq_{1, -\text{nv}} \subseteq \simeq_{2, -\text{nv}} \), we show that \( \simeq_{1, -\text{nv}} \) satisfies the definition of a 2-naive faster-than relation; clearly, we have only to consider Part (3). Hence, consider some arbitrary processes \( P \) and \( Q \) such that \( P \simeq_{1, -\text{nv}} Q \). If \( P \xrightarrow{\sigma_2} P' \) for some process \( P' \), then \( P \xrightarrow{\sigma_1} P'' \) for some process \( P'' \) satisfying \( P'' \succeq^+ P'' \) by Proposition 3.5. By definition of \( \simeq_{1, -\text{nv}} \), \( \forall (Q) \subseteq \forall (P) \) and there exists some \( Q'' \) with \( Q \xrightarrow{\sigma_1} Q'' \) and
Let $P^\pi \succeq_{1-mv} Q^\pi$; hence, also $Q \overset{\sigma_2}{\rightarrow} Q''$. $P' \succeq^+ P''$ implies $P' \succeq_{1-mv} P''$ by Proposition 4.2(2), and we are done by transitivity of $\succeq_{1-mv}$.

For the inverse inclusion $\succeq_{2-mv} \subseteq \succeq_{1-mv}$, consider $P$ and $Q$ such that $P \succeq_{2-mv} Q$. If $P \overset{\sigma_1}{\rightarrow} P'$ for some process $P'$, then $P \overset{\sigma_2}{\rightarrow} P'$ by definition of $\succeq_{2-mv}$. By Proposition 3.5, there exists some $Q''$ with $Q \overset{\sigma_1}{\rightarrow} Q''$ and $Q' \succeq^+ Q''$. Hence, $Q' \succeq_{2-mv} Q''$ follows from $\succeq^+ \subseteq \succeq_{1-mv} \subseteq \succeq_{2-mv}$, and we are done by transitivity of $\succeq_{2-mv}$.

5 The delayed-faster-than-preorders

In the course of design choices in [7], an alternative candidate for a faster-than relation on processes is the delayed-faster-than preorder. Since the slower process might take more time, this preorder allows the slower process additional time steps when matching an action or time step. We define the i-delayed faster-than preorder, where case $i = 1$ is adopted from [7].

Before giving the definition, we present a technical lemma, highlighting an important property of the transitive closure of the syntactic relation. The result is intuitively convincing because, if the faster process skips time steps by performing a ‘real’ type-2 time step, it can only become even faster than the slower process $Q$.

**Lemma 5.1** Let $P, P' \in \mathcal{P}$ such that $P \succeq^+ P'$. Then $P \overset{\sigma_2}{\rightarrow} P$ implies $\exists P', P' \overset{\sigma_1}{\rightarrow} P$ and $P \succeq^+ P'$.

**Proof:** $P \overset{\sigma_2}{\rightarrow} P$ implies $P \overset{\sigma_1}{\rightarrow} P'$ and $P \succeq^+ P'$ by using Proposition 3.5. Then we get $P' \overset{\sigma_1}{\rightarrow} P'$ and $P \succeq^+ P'$ by Proposition 4.2(2) and are done by the obvious transitivity of $\succeq^+$. □

**Definition 5.2** For $i \in \{1, 2\}$, a relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is an i-delayed faster-than relation if the following conditions hold for all $(P, Q) \in \mathcal{R}$ and $\alpha \in \mathcal{A}$.

1. $P \overset{\alpha}{\rightarrow} P'$ implies $\exists Q'. Q \overset{\sigma_1}{\rightarrow} Q'$ and $(P', Q') \in \mathcal{R}$.
2. $Q \overset{\alpha}{\rightarrow} Q'$ implies $\exists P'. P \overset{\alpha}{\rightarrow} P'$ and $(P', Q') \in \mathcal{R}$.
3. $P \overset{\sigma_1}{\rightarrow} P'$ implies $\exists Q'. Q \overset{\sigma_1}{\rightarrow} Q'$ and $(P', Q') \in \mathcal{R}$.

We write $P \succeq_{i-dly} Q$ if $(P, Q) \in \mathcal{R}$ for some i-delayed faster-than relation $\mathcal{R}$ and call $\succeq_{i-dly}$ i-delayed faster-than preorder. □

As usual, $\succeq_{i-dly}$ is the largest delayed faster-than relation; somewhat similar to weak bisimulation, one can also show that it is a preorder.

It is shown in [7] that $\succeq_{1-mv}$ and $\succeq_{1-dly}$ coincide. To complete the picture, we will show that $\succeq_{1-dly}$ and $\succeq_{2-dly}$ coincide and get:

**Theorem 5.3** The preorders $\succeq_{1-mv}$, $\succeq_{2-mv}$, $\succeq_{1-dly}$ and $\succeq_{2-dly}$ coincide.

**Proof:** We must only show the last coincidence; Case (2) of 5.2 is obvious. As above, we consider some $P$ and $Q$ with $P \succeq_{1-dly} Q$.

1. If $P \overset{\alpha}{\rightarrow} P'$, then $Q \overset{\sigma_1}{\rightarrow} Q'$ and hence $Q \overset{\sigma_2}{\rightarrow} Q'$ such that $P' \succeq_{1-dly} Q'$ by definition of $\succeq_{1-dly}$.
(3) If $P \stackrel{\sigma}{\rightarrow} P'$ for some $P'$, we have $P \stackrel{\sigma}{\rightarrow} P''$ for some $P''$ such that $P' \geq^+ P''$ by Proposition 3.5. Thus, $Q \stackrel{\sigma}{\rightarrow} Q''$ and, hence, $Q \stackrel{\sigma}{\rightarrow} Q''$ for some $Q''$ with $P'' \geq_1 Q''$ by the definition of $\geq_1$. Further, $\geq^+ \subseteq \geq_1$ by Proposition 4.2(2) and [7], hence $P' \geq^+ P''$ implies $P' \geq_1 Q''$; we are done by transitivity of $\geq_1$.

For the reverse inclusion, we consider $(P, Q) \in \geq_2$. For a smooth presentation, we start with the simulation of a time step.

(3) If $P \stackrel{\sigma}{\rightarrow} P'$, then $P \stackrel{\sigma}{\rightarrow} P'$ as well. Then, we get some $Q'$ with $P' \geq_2 Q'$ and $Q \stackrel{\sigma}{\rightarrow} Q'$, i.e. $Q \stackrel{\sigma}{\rightarrow} Q'$ by transitivity. This gives us $Q \stackrel{\sigma}{\rightarrow} Q''$ with $Q' \geq^+ Q''$ by Proposition 3.5. Since $\geq_1$ and $\geq_1$ coincide and are included in $\geq_2$, we get $Q' \geq_2 Q''$ with Proposition 4.2(2), and are done by transitivity of $\geq_2$.

(1) For this part, cf. the figure below: if $P \sigma P'$, then $Q_0 \equiv Q_0 \sigma_2 \cdots \sigma_2 Q_n \sigma_{n+1} \sigma_2 \cdots \sigma_2 Q_m \equiv Q'$ for some $Q'$ and some $n$ and $m$ with $0 \leq n < m$ such that $P' \geq_2 Q'$ by definition of $\geq_2$. Considering that $Q_0 \geq^+ Q_0$ by definition, we obtain $Q_0 \sigma_{n+1} \sigma_1 \sigma_{n} \sigma \cdots \sigma_1 Q'_m$ for some $Q'_m$ with $Q_m \equiv Q'_m$ from Lemma 5.1 and Proposition 4.2(2). As above, we derive $Q_m \geq_2 Q'$ and are done by transitivity of $\geq_2$.

Figure 1:

Clearly, any 1-naive faster-than relation is also a 1-delayed one, as well as any 2-naive faster-than relation a 2-delayed one. But the sets of delayed relations of type 1, type 2 resp. are incomparable, and this also holds for the naive relations. We come back to this issue in Sec. [7] in a slightly different setting.

6 Indexed faster-than preorder

The second variant of a faster-than preorder in [7] is the indexed faster-than preorder, formalizing the idea of an account for time steps for the faster process. If a time step of the slower process is not (or cannot be) simulated immediately by the faster process, then this time step is credited and might be withdrawn if the process performs this time step later on. Obviously, the account balance may never be
negative. This much more complicated variant answers the question how time steps of the slower process are matched.

**Definition 6.1** For \( i \in \{1,2\} \), a family \(( R_j )_{j \in \mathbb{N}}\) of relations over \( \mathcal{P} \), indexed by natural numbers (including 0), is a family of \( i \)-indexed faster-than relations if, for all \( j \in \mathbb{N} \), \( (P,Q) \in R_j \) and \( \alpha, \beta \in \mathcal{A} \):

1. \( P \overset{\alpha}{\rightarrow} P' \) implies \( \exists Q'. P \overset{\beta}{\rightarrow} Q' \) and \( (P',Q') \in R_j \).
2. \( Q \overset{\beta}{\rightarrow} Q' \) implies \( \exists P'. P \overset{\alpha}{\rightarrow} P' \) and \( (P',Q') \in R_j \).
3. \( P \overset{\alpha}{\rightarrow} P' \) implies (a) \( \exists Q', Q \overset{\alpha}{\rightarrow} Q' \) and \( (P',Q') \in R_j \), or
   (b) \( j > 0 \) and \( (P',Q') \in R_{j-1} \).
4. \( Q \overset{\beta}{\rightarrow} Q' \) implies (a) \( \exists P', P \overset{\beta}{\rightarrow} P' \) and \( (P',Q') \in R_j \), or
   (b) \( (P,Q') \in R_{j+1} \).

We write \( P \preceq_{i,j} Q \) if \( (P,Q) \in R_j \) for some family of \( i \)-indexed faster-than relations \(( R_j )_{j \in \mathbb{N}} \) and call \( \preceq_{i,j} \) \( i,j \)-indexed faster-than preorder.

The latter notion is an abuse, since these relations are not really preorders. \( [7] \) proves that \( \preceq_{1,0} \) coincides with the 1-naive faster-than preorder. We would have expected to be able to prove the same coincidence result in the new setting. Unfortunately, this has turned out to be wrong, which can be explained by the absence of time determinism. According to their definitions, \( \preceq_{2,0} \subseteq \preceq_{2,0} \) obviously holds. However, the reverse inclusion fails; consider the following counterexample:

Let \( P =_{df} \tau.0 | \sigma.\tau.0 \) and \( Q =_{df} \sigma.\tau.0 | \sigma.\tau.0 \). Clearly, \( P \) is faster than \( Q \) in the sense of a naive faster-than preorder. In the sequel, we will try to build a family of relations \(( R_j )_{j \in \mathbb{N}} \) such that \( (P,Q) \in \preceq_{2,0} \) holds. Hence, we put \( (P,Q) \) into \( R_0 \) and first consider the 'real' type-2 time step \( Q \overset{\tau}{\rightarrow} \tau.0 | \tau.0 \); since \( P \) is not able to match this behaviour, as a time step is preempted by an urgent \( \tau \), we are forced to credit this time step and hence obtain:

\[
R_0 =_{df} \{ (\tau.0 | \sigma.\sigma.\tau.0, \sigma.\tau.0 | \sigma.\sigma.\tau.0), \ldots \} \\
R_1 =_{df} \{ (\tau.0 | \sigma.\sigma.\tau.0, 0 | \tau.0), \ldots \}
\]

Now we consider \( \tau.0 | \sigma.\sigma.\tau.0 \overset{\tau}{\rightarrow} 0 | \sigma.\sigma.\tau.0 \) of \( P \), which '\( Q \)' can either mimic with \( \tau.0 | \tau.0 \overset{\tau}{\rightarrow} 0 | \tau.0 \) or \( \tau.0 | \tau.0 \overset{\tau}{\rightarrow} \tau.0 \). Since the resulting processes have the same functional and waiting behaviour, we may consider any of them. Thus, we so far know:

\[
R_0 =_{df} \{ (\tau.0 | \sigma.\sigma.\tau.0, \sigma.\tau.0 | \sigma.\sigma.\tau.0), \ldots \} \\
R_1 =_{df} \{ (\tau.0 | \sigma.\sigma.\tau.0, 0 | \tau.0), (0 | \sigma.\sigma.\tau.0, 0 | \tau.0), \ldots \}
\]

Finally, \( 0 | \sigma.\sigma.\tau.0 \) may perform \( 0 | \sigma.\sigma.\tau.0 \overset{\sigma}{\rightarrow} 0 | \sigma.\tau.0 \). As this time step cannot be simulated by \( Q \), we have to withdraw the credited time step and put \( (0 | \sigma.\tau.0, 0 | \tau.0) \) in the relation \( R_0 \). This leads to a contradiction.

Summarizing, the problem lies in the fact that the slower process \( Q \) performs a 'real' type-2 time step and skips a \( \sigma \)-prefix, but only one time step is credited for the faster process \( P \). The amortisation mechanism does not ensure that \( P \) can match this 'real' type-2 time step properly later; it instead wastes its credit on a type-1 time step. This cannot happen in a sensible setting with time determinism. We leave the repair of this defect by altering the definition of the indexed faster-than preorder for future work.
7 Strong faster-than precongruence

A shortcoming of the 1-naive faster-than preorder is that it is not compositional wrt. |. As an example consider the processes \( P = \sigma.a.0 \) and \( Q = a.0 \) for which \( P \lessdot_{i-m} Q \) holds: the time step \( \sigma.a.0 \lessdot_i a.0 \) of \( P \) is matched by the time step \( a.0 \lessdot_i a.0 \), i.e. \( Q \) idles. Yet, if we compose both processes in parallel with \( R = \sigma_0.a.0 \), then we observe that \( P | R \lessdot_{i-m} Q | R \) does not hold as the clock transition \( P | R \sigma_i a.0 | R \sigma_i a.0 \) cannot be matched due to an urgent \( \tau \).

In any case, it is intuitively suggestive to exclude pairs \( (\sigma.P, P) \) from a faster-than preorder. In \cite{Tight}, the 1-naive faster-than preorder is modified taking urgent sets into account: if \( Q \) cannot be matched due to an urgent \( \tau \) \( \sigma \)-performs a time step and \( a \in \U(Q) \), then a context process does not really have to wait for synchronisation on \( a \); thus, when matching a time step the faster process must have a larger urgent set. The resulting preorder turns out to be the largest precongruence contained in \( \lesssim_{i-m} \); it is called strong faster-than precongruence. Again, we adopt the definition in our setting.

**Definition 7.1** For \( i \in \{1,2\} \), a relation \( \lesssim_i \subseteq \mathcal{P} \times \mathcal{P} \) is a strong \( i \)-faster-than relation if the following conditions hold for all \( (P, Q) \in \lesssim_i \) and \( \alpha \in \mathcal{A} \).

1. \( P \xrightarrow{\alpha} P' \) implies \( \exists Q'. Q \xrightarrow{\alpha} Q' \) and \( (P', Q') \in \lesssim_i \).
2. \( Q \xrightarrow{\alpha} Q' \) implies \( \exists P'. P \xrightarrow{\alpha} P' \) and \( (P', Q') \in \lesssim_i \).
3. \( P \xrightarrow{\sigma_i} P' \) implies \( \U(Q) \subseteq \U(P) \) and \( \exists Q'. Q \xrightarrow{\alpha_i} Q' \) and \( (P', Q') \in \lesssim_i \).

We write \( P \lesssim_i Q \) if \( (P, Q) \in \lesssim_i \) for some strong \( i \)-faster-than relation \( \lesssim_i \) and call \( \lesssim_i \) strong \( i \)-faster-than precongruence.

Clearly, \( \lesssim_i \) is contained in \( \lesssim_{i-m} \) for \( i \in \{1,2\} \). As usual, it is easy to prove that \( \lesssim_i \) is the largest strong \( i \)-faster-than relation and a preorder. Since the \( \lesssim_{i-m} \) coincide, they contain the same largest precongruence. Hence, the expected coincidence result also shows the most important fact that \( \lesssim_2 \) is this largest precongruence. We first present the following result.

**Proposition 7.2** The relation \( \gtrsim \) satisfies the defining clauses of a strong 1-faster-than relation, also on open terms; hence, \( \gtrsim \) restricted to processes is a strong 1-faster-than relation and \( \gtrsim = df \gtrsim \cap (\mathcal{P} \times \mathcal{P}) \subseteq \lesssim_1 \). The same holds for \( \lesssim_+ \).

The first statement is given in \cite{Tight}; the second can be shown as we have done above for Proposition 4.2.

Now the following coincidence result can be proven as indicated in the proof of Theorem 4.3.

**Theorem 7.3** The preorders \( \lesssim_1 \) and \( \lesssim_2 \) coincide.

Although \( \lesssim_1 \) and \( \lesssim_2 \) coincide, this does not hold for the strong 1- and 2-faster-than relations. In fact, in some cases one can find a smaller 2-faster-than relation because \( Q \) has more possibilities to match a clock transition. In other cases, one can find a smaller 1-faster-than relation because \( P \) can reach fewer processes by type-1 time steps; see \cite{Tight} for examples.

Another contribution of this paper is that we show how to combine these two advantages in a mixed setting. The new strong combined faster-than or strong c-faster-than precongruence matches type-1 time steps with type-2 time steps. (The same idea is also studied on the naive level in \cite{Tight}.)

**Definition 7.4** A relation \( \lesssim \subseteq \mathcal{P} \times \mathcal{P} \) is a strong c-faster-than relation if the following conditions hold for all \( (P, Q) \in \lesssim \) and \( \alpha \in \mathcal{A} \).
Proof: First, take processes $P$ and $Q$ such that $P \mathrel{\sim}_c Q$; we only have to consider the matching of time steps. If $P \mathrel{\alpha} P'$ for some process $P'$, then $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ and $Q \mathrel{\sigma} Q'$ for some $Q'$ satisfying $P' \mathrel{\sim}_c Q'$ by definition of $\mathrel{\sim}_c$. We are done since we also have $Q \mathrel{\sigma} Q'$.

For the reverse inclusion $\mathrel{\sim}_c \subseteq \mathrel{\sim}_1$, define the relation $\mathcal{R}$ by $(P, Q) \in \mathcal{R}$ if and only if $\exists R \in \mathcal{P}, P \mathrel{\sim}_c R \mathrel{\geq}^+ Q$ for $P, Q \in \mathcal{P}$. This relation contains $\mathrel{\sim}_c$ since $\geq^+$ is reflexive; hence, it suffices to check that $\mathcal{R}$ is a strong 1-faster-than relation; consider $P \mathrel{\sim}_c R \mathrel{\geq}^+ Q$.

1. If $P \mathrel{\alpha} P'$ for some $P'$, the definition of $\mathrel{\sim}_c$ shows $R \mathrel{\alpha} R'$ for some process $R'$ with $P' \mathrel{\sim}_c R'$.

Since $\geq^+$ is a strong 1-faster-than relation by Proposition 7.2, this implies $Q \mathrel{\alpha} Q'$ for some $Q'$ such that $R' \mathrel{\geq}^+ Q'$.

2. The case $Q \mathrel{\alpha} Q'$ for some $Q'$ is analogous to Part (1).

3. If $P \mathrel{\sigma} P'$ for some $P'$, then $\mathcal{U}(R) \subseteq \mathcal{U}(P)$ and $R \mathrel{\sigma} R'$ for some process $R'$ with $P' \mathrel{\sim}_c R'$ by definition of $\mathrel{\sim}_c$. Due to $R \mathrel{\sigma} R'$, we infer $Q \mathrel{\sigma} Q'$ for some $Q'$ satisfying $R' \mathrel{\geq}^+ Q'$ from Lemma 5.1. Moreover, we know $\mathcal{U}(Q) \subseteq \mathcal{U}(R)$ due to $R \mathrel{\geq}^+ Q$ by successive application of Lemma 3,2,5, implying $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$. □

The next result shows that strong c-faster-than relations indeed provide more possibilities to prove the strong faster-than precongruence.

Proposition 7.6

(1) Every strong 1-faster-than relation $\mathcal{R}$ is a strong c-faster-than relation.

(2) Every strong 2-faster-than-relation $\mathcal{R}$ is a strong c-faster-than relation.

Proof: Again, we only have to look at the matching of time steps. First, take a strong 1-faster-than relation $\mathcal{R}$ and some $(P, Q) \in \mathcal{R}$. If $P \mathrel{\sigma} P'$, then $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ and $Q \mathrel{\sigma} Q'$ for some process $Q'$ with $(P', Q') \in \mathcal{R}$ by definition of $\mathcal{R}$, then, also $Q \mathrel{\sigma} Q'$.

Now let $\mathcal{R}$ be a strong 2-faster-than relation and $(P, Q) \in \mathcal{R}$. If $P \mathrel{\sigma} P'$ for some process $P'$, then also $P \mathrel{\sigma} P'$. Hence, we obtain $\mathcal{U}(Q) \subseteq \mathcal{U}(P)$ and $Q \mathrel{\sigma} Q'$ for some $Q'$ such that $(P', Q') \in \mathcal{R}$ by the definition of $\mathcal{R}$. □

We remark that one could also look at relations where a type-2 time step is matched by a type-1 time step; these lead to the same strong faster-than precongruence and comprise all relations that are strong 1- and 2-faster-than relations.
We conclude by giving processes $P$ and $Q$, where the least strong $c$-faster-than relation proving $P \preceq_c Q$ is much smaller than any relation of type 1 or 2. In the following, we write $\sigma^n$ for a sequence of $n$ $\sigma$-prefixes. Let $P =_{df} (\mu.x.a.x) | (\mu.x.b.x) | ((\mu.x.\sigma^n.a.x) | (\mu.x.\sigma^n.c.x) \setminus \{c\}$ and $Q =_{df} (\mu.x.\sigma^n.a.x) | (\mu.x.\sigma^n.b.x) | ((\mu.x.\sigma^n.c.x) | (\mu.x.\sigma^n.d.x) \setminus \{c\}$ for some large $n$. For a smooth presentation, we code $Q$ and its resulting processes as a subset of $\{ijkl \mid 0 \leq i, j, k, l \leq n\}$, where $i$, $j$, $k$ and $l$ are the numbers of leading $\sigma$-prefixes for the four components. Analogously, we code the $P$-states as elements of $\{ijkl \mid i, j \in \{0, n\}, 0 \leq k, l \leq n\}$ where $k$ and $l$ refer to the third and fourth component; $i = n$ encodes $\mu.x.a.x$ (the original component as in all other cases) and $i = 0$ encodes $\mu.x.a.x$, resulting from a time step; similarly for $j$.

We first show that the following is a strong $c$-faster-than relation: $\{(ijkl, i j k k) \mid i, j \in \{0, n\}, 0 \leq k \leq n\}$, hence we obviously have $4(n + 1)$ pairs. Observe that, in each pair, the urgent set of the second process is contained in that of the first process, since $a$ and $b$ are always urgent on the $P$-side and $k$ is equal on both sides. Each time step on the $P$-side is of the form $ijkl \xrightarrow{a} 00(k - 1)(k - 1)$ and is matched by the ‘same’ time step of type-2 on the $Q$-side. For the actions, observe that $a, b$ and $\tau$ set the resp. $i$, $j$ or $k$ to $n$ on both sides.

Next we show that a strong 2-faster-than relation for $(P, Q)$ must contain pairs for $4n^2 + 4$ different $P$-states: $P$ is encoded as $nnnn$, and there is a $\xrightarrow{\sigma_2}$ transition to each $00kl$ with $0 \leq k, l \leq n - 1$. For each of these $n^2$ processes we have three additional ones, since action $a$ and/or action $b$ sets the first and/or second component to $n$. Furthermore, $0000$ can perform $\tau$ to become $00nn$, and also for this process we have three additional ones. Since $P$ can reach at least $4n^2 + 4$ processes, a strong 2-faster-than relation must contain at least $4n^2 + 4$ pairs. Further factors $n$ are possible by replacing the $c$- and $\tau$-components by more components that communicate in a ring: e.g. $((\mu.x.\sigma^n.c_1.\tau_1.x) | (\mu.x.\sigma^n.c_2.x) | (\mu.x.\sigma^n.\tau_2.c_3.x) | (\mu.x.\sigma^n.\tau_3.c_4.x)) \setminus \{1, \ldots, c_4\}$.

To present elements of a strong 1-faster-than relation for $(P, Q)$, we introduce the following notation: we let $[i]_n$ be $n$ if $i = n$, and $0$ otherwise; furthermore, we write $\text{pair}(ijkl)$ to denote the pair $([i]_n [j]_n kk, i j k k)$ for $0 \leq i, j, k \leq n$. We show now that the strong 1-faster-than relation must include all these pairs, hence we have at least $(n + 1)^3$ pairs. By way of example, assume that $j = i - 5, k = i - 2$. Starting from $\text{pair}(nnn)$, three time steps give $\text{pair}((n - 3)(n - 3)(n - 3))$, $\tau$ plus two time steps lead to $\text{pair}((n - 5)(n - 5)(n - 2))$ and action $a$ gives $\text{pair}(n(n - 5)(n - 2))$. Now we can reach the desired pair by $n - i$ time steps.

Further factors $n + 1$ can be obtained by adding further components $\mu.x.a'.x$ to $P$ and $\mu.x.\sigma^n.a'.x$ to $Q$; with each component, the factor 4 in the size of the $c$-relation only grows by a factor 2.

### 8 Conclusion and future work

In [7], the process algebra TACS and three types of faster-than relations were introduced, and it was shown that the three types lead to the same faster-than preorder. Here, we have extended the clock transitions of TACS processes by new time steps and studied, in this new setting, the three types of relations. With the exception of the indexed faster-than preorder, we have proved that the new definitions lead to the same preorder as in [7]. We have also obtained a coincidence result for the strong faster-than precongruence of [7], and developed a new type of faster-than relation that combines old and new operational semantics; this combined variant leads to the same precongruence and allows smaller relations for proving that the precongruence holds.

In [5], it is also shown that (at least on the naive level) faster-than relations up to $\preceq_{i-mv}$ can be employed, which can also lead to smaller relations as usual. Furthermore, coincidence also holds for weak
variants of the faster-than preorder in the old and the new operational semantics.

Summarizing, we have given a number of new results that further support the robustness of the approach in [7] for comparing the worst-case efficiency of asynchronous processes.

Although in [8] hundreds of pages are devoted to a setting with upper time bounds only, this is rarely treated in process algebra. Typically, e.g. in [4], σ represents a definite time step; then replacing a component by one with fewer time steps might change the overall system behaviour drastically instead of improving efficiency. The efficiency preorder studied in [1] counts τ’s; since parallel τ’s count twice, while parallel σ’s are counted only once, this does not compare time but some other cost, as is done in [6]. There is one other process algebra with upper time bounds, which has been studied in a testing scenario and related to fairness, see e.g. [2, 3].

As future work, we intend to repair the defect of the indexed-faster-than relations in our new setting by altering its definition. This approach is relevant since the indexed-faster-than preorder is quite a convincing candidate for a faster-than relation on processes. Another open issue is to consider combined definitions of the delayed and the weak variants. Moreover, one should study to what extent such a combined weak precongruence can be based on small relations. Finally, to reinforce robustness of the approach further, there is at least one additional intuitive variation of the operational semantics we will consider.

References

[1] S. Arun-Kumar & M.C.B. Hennessy (1992): An Efficiency Preorder for Processes. Acta Inform. 29(8), pp. 737–760.
[2] F. Corradini, M.R. Di Berardini & W. Vogler (2009): Liveness of a Mutex Algorithm in a Fair Process Algebra. Acta Informatica 46, pp. 209–235.
[3] F. Corradini & W. Vogler (2007): Performance of pipelined asynchronous systems. J. Logic and Algebraic Programming 70, pp. 201–221.
[4] M.C.B. Hennessy & T. Regan (1995): A Process Algebra for Timed Systems. Inform. and Comp. 117(2), pp. 221–239.
[5] K. Iltgen (2009): Robustness of a bismulation-type faster-than relation. Technical Report 2009-08, Universität Augsburg, Germany.
[6] A. Kiehn & S. Arun-Kumar (2005): Amortised Bisimulations. In: FORTE 2005, LNCS 3731. Springer-Verlag, pp. 185–204.
[7] G. Lüttgen & W. Vogler (2004): Bisimulation on speed: worst-case efficiency. Information and Computation 191(2), pp. 105–144.
[8] N. Lynch (1996): Distributed Algorithms. Morgan Kaufmann Publishers.
[9] R. Milner (1989): Communication and Concurrency. Prentice Hall.
[10] W. Vogler (2002): Efficiency of Asynchronous Systems, Read Arcs, and the MUTEX-problem. TCS 275(1–2), pp. 589–631.