MEAN CONVEX HULLS AND LEAST AREA DISKS SPANNING EXTREME CURVES

BARIS COSKUNUZER

ABSTRACT. We show that for any extreme curve in a 3-manifold $M$, there exist a canonical mean convex hull containing all least area disks spanning the curve. Similar result is true for asymptotic case in $\mathbb{H}^3$ such that for any asymptotic curve $\Gamma \subset S^2_\infty(\mathbb{H}^3)$, there is a canonical mean convex hull containing all minimal planes spanning $\Gamma$. Applying this to quasi-Fuchsian manifolds, we show that for any quasi-Fuchsian manifold, there exist a canonical mean convex core capturing all essential minimal surfaces. On the other hand, we also show that for a generic $C^3$-smooth curve in the boundary of $C^3$-smooth mean convex domain in $\mathbb{R}^3$, there exist a unique least area disk spanning the curve.

1. INTRODUCTION

We study the Plateau problem for extreme curves in a Riemannian 3-manifold. The existence of least area disks for any simple closed curve in a Riemannian manifold was proved by Morrey half a century ago. The regularity properties and number of solutions questions have been studied in the following decades. On the other hand, for extreme curves, this problem has many interesting properties. Meeks-Yau, Hass-Scott and Fanghua Lin have studied the Plateau problem in mean convex domains, and proved many important features of these least area disks, [MY1], [HS], [Li].

In this paper, we will concentrate on the same problem. We improve and reformulate some known results in a natural way, and try to clarify the picture with a simplified proof by using the techniques of [MY1] and [HS]. Moreover, we will extend these properties to asymptotic Plateau problem in $\mathbb{H}^3$. On the other hand, we prove a new generic uniqueness result for extreme curves in $\mathbb{R}^3$ by using some topological techniques and analytical results of [TT].

Now, we list the main results of the paper. First, we show existence of a canonical object called \textit{mean convex hull} for any extreme curve.
**Theorem 3.2.** Let $\Omega$ be a mean convex domain in a 3-manifold $M$ and $\Gamma \subset \partial \Omega$ be a simple closed curve. Then either there exist a unique least area disk $\Sigma$ in $\Omega$ with $\partial \Sigma = \Gamma$, or there exist a canonical mean convex hull $N$ in $\Omega$ such that $\partial N = \Sigma^+ \cup \Sigma^-$ where $\Sigma^\pm$ are uniquely defined extremal least area disks in $\Omega$ with $\partial \Sigma^\pm = \Gamma$. Moreover, all least area disks $\Sigma' \subset \Omega$ spanning $\Gamma$ are contained in $N$.

By extending the techniques of the above theorem to asymptotic Plateau problem in $H^3$, we got the following result.

**Theorem 4.5.** Let $\Gamma \subset S^2_\infty(H^3)$ be a simple closed curve. Then either there exist a unique minimal plane $\Sigma$ with $\partial \infty \Sigma = \Gamma$, or there exist a canonical mean convex hull $N \subset H^3$ such that all minimal planes $\Sigma'$ with $\partial \infty \Sigma' = \Gamma$ are contained in $N$.

By applying the above theorem to the limit set of a quasi-Fuchsian hyperbolic 3-manifold, we got the following corollary.

**Corollary 4.6.** Let $M$ be a quasi-Fuchsian hyperbolic 3-manifold. Then either there exist a unique minimal surface homotopy equivalent to $M$ or there exist a canonical mean convex core $N \subset M$ containing all minimal surfaces homotopy equivalent to $M$.

Finally, we have a generic uniqueness result for extreme curves in $\mathbb{R}^3$.

**Theorem 5.10.** Let $\Omega$ be a $C^3$-smooth mean convex domain in $\mathbb{R}^3$, and $A = \{ \alpha \in C^3(S^1, \partial \Omega) \mid \alpha \text{ embedding} \}$. Then there exist an open dense subset $A' \subset A$ in $C^3$ topology, such that for any $\Gamma \in A'$, there exist a unique least area disk with boundary $\Gamma$.

The organization of the paper is as follows. In Section 2, we will give basic definitions and results which will be used throughout the paper. In Section 3, canonical mean convex hull results will be proved for extreme curves. In Section 4, we will extend these to the asymptotic curves for $H^3$. In Section 5, we will prove a generic uniqueness result for the extreme curves in $\mathbb{R}^3$. Finally, we will have some remarks on the results in Section 6.

1.1. **Acknowledgements:** I would like to thank Bill Meeks, Yair Minsky, and Peter Li for very helpful conversations.
2. Preliminaries

In this section, we will overview the basic definitions and results which we use in the following sections.

**Definition 2.1.** A minimal disk (plane) is a disk (plane) such that the mean curvature is 0 at every point. A least area disk is a disk which has the smallest area among the disks with same boundary. A least area disk is minimal, but the converse is not true in general as minimal disks are just ”locally” area minimizing. A least area plane is a plane such that any subdisk in the plane is a least area disk.

**Definition 2.2.** Let $\Omega$ be a compact submanifold of a Riemannian 3-manifold $M$. Then $\Omega$ is a mean convex domain if the following conditions hold.

- $\partial \Omega$ is piecewise smooth.
- Each smooth subsurface of $\partial \Omega$ has nonnegative curvature with respect to inward normal.
- In the singular curves of $\partial \Omega$, the angle between the neighboring surfaces is less than $\pi$ (inward direction).

**Definition 2.3.** $\Gamma \subset M$ is an extreme curve if it is a curve in the boundary of a mean convex domain in $M$.

**Remark 2.1.** The above definition for extreme curves are different from the usual definition. We will abuse the name for our purposes. This definition is more general than the definition in the literature, which says an extreme curve is a curve in the boundary of a convex domain.

**Definition 2.4.** The sequence $\{\Sigma_i\}$ of embedded surfaces in a Riemannian manifold $M$ converges to the surface $\Sigma$ if

- $\Sigma$ contains all the limit points of the sequence $\{\Sigma_i\}$, i.e. $\Sigma = \{x = \lim_{i \to \infty} x_i \mid x_i \in S_i \text{ and } \{x_i\} \text{ convergent in } M\}$
- Given $x \in \Sigma$ and $x_i \in \Sigma_i$ as above. Then there exist embeddings $f : D^2 \to \Sigma$, and $f_i : D^2 \to \Sigma_i$ with $f(0) = x$, and $f_i(0) = x_i$ such that $f_i$ converges to $f$ in $C^\infty$ topology.

Now, we can state the main facts which we use in the first part.

**Lemma 2.1.** [MY1] Let $\Omega$ be a mean convex domain, and $\Gamma \subset \partial \Omega$ be a simple closed curve. Then there exist a least area disk $\Sigma \subset \Omega$ with $\partial \Sigma = \Gamma$. Moreover, all such disks are properly embedded in $\Omega$ and they are pairwise disjoint. Moreover, If $\Gamma_1, \Gamma_2 \subset \partial \Omega$ are disjoint simple closed curves, then the least area disks $\Sigma_1, \Sigma_2$ spanning $\Gamma_1, \Gamma_2$ are also disjoint.
Lemma 2.2. [H•S] Let $\Omega$ be a mean convex domain and let $\{\Sigma_i\}$ be a sequence of embedded least area disks in $\Omega$. Then there is a subsequence $\{\Sigma_j\}$ of $\{\Sigma_i\}$ such that $\Sigma_j \to \Sigma$, a countable collection of embedded least area disks in $\Omega$.

3. Mean Convex Hulls

In this section, we will show that for any simple closed extreme curve in a Riemannian 3-manifold, there exist a canonical neighborhood which is mean convex or there exist a unique least area disk spanning the curve. The idea is simple. Let $\Gamma \subset \partial \Omega$ be an extreme curve, and $\Gamma_i^\pm \subset \partial \Omega$ be two sequences converging to $\Gamma$ from different sides. Then the induced sequences of least area disks $\Sigma_i^\pm \subset \Omega$ with $\partial \Sigma_i^\pm = \Gamma_i^\pm$ limits to the two least area disks $\Sigma^+$ and $\Sigma^-$ with $\Sigma^\pm = \Gamma$. These least area disks will be a barrier for the other least area disks with same boundary, and they will define a canonical neighborhood $N$ with $\partial N = \Sigma^+ \cup \Gamma \cup \Sigma^-$, which we will call the mean convex hull of $\Gamma$.

The combination of the above two lemmas gives us the following lemma, which is the main tool of this part.

Lemma 3.1. Let $\Gamma \subset \partial \Omega$ be a simple closed extreme curve where $\Omega$ is the mean convex domain in a 3-manifold $M$. Then there are uniquely defined two canonical extremal least area disks (which might be same) $\Sigma^+$ and $\Sigma^-$ in $\Omega$ with $\partial \Sigma^\pm = \Gamma$, and they are limits of sequences of least area disks.

Proof: $\Gamma \subset \partial \Omega$ is a simple closed extreme curve, and $\Omega$ is the mean convex domain in a 3-manifold $M$. Take a small neighborhood $A$ of $\Gamma$ in $\partial \Omega$, which will be a thin annulus with $\Gamma$ is the core. $\Gamma$ separates the annulus $A$ into two parts, say $A^+$ and $A^-$ by giving a local orientation. Define a sequence of pairwise disjoint simple closed curves $\Gamma_i^+ \subset A^+ \subset \partial \Omega$ such that $\lim \Gamma_i^+ = \Gamma$. Now, by Lemma 2.1, for any curve $\Gamma_i^+$, there exist an embedded least area disk $\Sigma_i^+$ with $\partial \Sigma_i^+ = \Gamma_i^+$. This defines a sequence of least area disks $\{\Sigma_i^+\}$ in $\Omega$. By Lemma 2.2, there exist a subsequence $\{\Sigma_j^+\}$ converging to a countable collection of least area disks $\hat{\Sigma}^+$ with $\partial \hat{\Sigma}^+ = \Gamma$.

We claim that this collection $\hat{\Sigma}^+$ consists of only one least area disk. Assume that there are two disks in the collection say $\Sigma_a^+$ and $\Sigma_b^+$, and say $\Sigma_a^+$ is ”above” $\Sigma_b^+$. By Lemma 2.1, $\Sigma_a^+$ and $\Sigma_b^+$ are embedded and disjoint. They have same boundary $\Gamma \subset \Omega$. $\Sigma_b^+$ is also limit of the sequence $\{\Sigma_i^+\}$. But, since for any least area disk $\Sigma_i^+ \subset \Omega$, $\partial \Sigma_i^+ = \Gamma_i^+$ is disjoint from $\partial \Sigma_a^+ = \Gamma$, $\Sigma_i^+$ disjoint from $\Sigma_a^+$, by exchange roundoff trick. This means
\( \Sigma^+_i \) is a barrier between the sequence \( \{ \Sigma^+_i \} \) and \( \Sigma^+_b \), and so, \( \Sigma^+_b \) cannot be limit of this sequence. This is a contradiction. So \( \hat{\Sigma}^+ \) is just one least area disk, say \( \Sigma^+ \). Similarly, \( \hat{\Sigma}^- = \Sigma^- \).

Now, we claim these least area disks \( \Sigma^+ \) and \( \Sigma^- \) are canonical, depending only on \( \Gamma \) and \( \Omega \), and independent of the choice of the sequence \( \{ \Gamma_i \} \) and \( \{ \Sigma_i \} \). Assume that there exist another least area disk \( S^+ \) with \( \partial S^+ = \Gamma \) and \( S^+ \) is a limit of the sequence of least area disks \( S^+_i \) with \( \partial S^+_i = \gamma^+_i \subset A^+ \). By Lemma 2.1, \( \Sigma^+ \) and \( S^+ \) are disjoint. Then one of them is "above" the other one. If \( \Sigma^+ \) is above \( S^+ \), then \( \Sigma^+ \) between the sequence \( S^+_i \) and \( S^+ \). This is because, all \( S^+_i \) are disjoint and above \( S^+ \) as \( \partial S^+_i = \gamma^+_i \) are disjoint and "above" \( \Gamma \). Similarly, \( \Sigma^+ \) is below \( S^+_i \) for any \( i \), as \( \partial \Sigma^+ = \Gamma \) is below the curves \( \gamma^+_i \subset A^+ \). Now, since \( \Sigma^+ \) is between the sequence \( \{ S^+_i \} \) and its limit \( S^+ \), and \( S^+ \) and \( \Sigma^+ \) are disjoint, \( \Sigma^+ \) will be a barrier for the sequence \( \{ S^+_i \} \), and so it cannot limit on \( S^+ \). But, this is a contradiction. Similarly, \( \Sigma^+ \) cannot be below \( S^+ \), so they must be same. So, \( \Sigma^+ \) and \( \Sigma^- \) are canonical least area disks for \( \Gamma \).

**Theorem 3.2.** Let \( \Omega \) be a mean convex domain in a 3-manifold \( M \) and \( \Gamma \subset \partial \Omega \) be a simple closed curve. Then either there exist a unique least area disk \( \Sigma \) in \( \Omega \) with \( \partial \Sigma = \Gamma \), or there exist a canonical mean convex hull \( N \) in \( \Omega \) such that \( \partial N = \Sigma^+ \cup \Sigma^- \) where \( \Sigma^+ \) are uniquely defined extremal least area disks in \( \Omega \) with \( \partial \Sigma^+ = \Gamma \). Moreover, all least area disks \( \Sigma' \subset \Omega \) spanning \( \Gamma \) are contained in \( N \).

**Proof:** \( \Gamma \subset \partial \Omega \) is a simple closed extreme curve, and \( \Omega \) is the mean convex domain in a 3-manifold \( M \). Let \( \Sigma^+ \) and \( \Sigma^- \) be the extremal least area disks for \( \Gamma \) from Lemma 3.1. Let \( N \subset M \) be the region between \( \Sigma^+ \) and \( \Sigma^- \), i.e. \( \partial N = \Sigma^+ \cup \Sigma^- \). Assume \( \Sigma' \) is a least area disk with \( \partial \Sigma' = \Gamma \). We claim that \( \Sigma' \subset N \). Assume on the contrary. Since all least area disks are disjoint by Lemma 2.1, \( \Sigma' \cap \Sigma^+ = \emptyset \), which implies \( \Sigma' \cap N = \emptyset \).

Then either \( \Sigma' \) is "above" \( \Sigma^+ \) or "below" \( \Sigma^- \). If \( \Sigma' \) is "above" \( \Sigma^+ \), then since there is a sequence \( \{ \Sigma^+_i \} \) such that \( \Sigma^+_i \to \Sigma^+ \), for sufficiently large \( k \), \( \Sigma^+_k \cap \Sigma' \neq \emptyset \) as \( \partial \Sigma^+_k = \Gamma^+_k \) is above \( \partial \Sigma' = \Gamma \). But by the choice of the sequence, \( \Gamma^+_k \cap \Gamma = \emptyset \). So, by Lemma 2.1, the least area disks \( \Sigma^+_k \) and \( \Sigma' \) must be disjoint. This is a contradiction. Similarly, \( \Sigma' \) cannot be "below" \( \Sigma^- \). So \( \Sigma' \subset N \).

If \( \Sigma^+ = \Sigma^- \), then \( N = \Sigma^+ = \Sigma^- \). Since for any least area disk \( \Sigma' \subset \Omega \) with \( \partial \Sigma' = \Gamma \), \( \Sigma' \) is contained in \( N \), then \( \Sigma' = \Sigma^+ = \Sigma^- \). This means there exist a unique least area disk spanning \( \Gamma \). 

\( \square \)
Remark 3.1. In Lemma 3.1, and Theorem 3.2, the canonical least area disks, and mean convex hull for $\Gamma \subset \partial \Omega$ are depending also on the mean convex domain $\Omega$. This is because, even though $\Gamma$ is in the boundary of mean convex domain $\Omega$, it does not mean that any least area disk in $M$ spanning $\Gamma$ must be in $\Omega$.

We call the region $N$ assigned to $\Gamma$ as mean convex hull. The reason for this $N \subset \Omega$ is itself a mean convex domain. The importance of this object is that it is canonically defined and uniquely determined by $\Gamma$ and $\Omega$. One can think of this object as a pseudo-convex hull living in $\Omega$ and in the convex hull of $\Gamma$.

If you want the canonical least area disks $\Sigma^\pm$, and mean convex hull $N$ for $\Gamma$ to be independent from the mean convex domain $\Omega$, you need to make sure that all least area disks spanning $\Gamma$ are in the mean convex domain. One condition to guarantee that is that $\Gamma$ has convex hull property and the convex hull of $\Gamma$ is in $\Omega$. If $\Omega$ is convex domain, this is automatic. In other words, if $\Omega$ is convex domain (i.e. $\Gamma$ is an extreme curve in the usual sense), than the defined mean convex hull, and extremal least area disks are independent of $\Omega$. Moreover, all least area disks in $M$ spanning $\Gamma$ are in the mean convex hull, i.e. between the extremal least area disks.

Also, a similar result was obtained by Brian White by using geometric measure theory methods in [Wh].

4. Mean Convex Hulls in Hyperbolic Space

Now, we are going to generalize the above results to the least area planes in hyperbolic space. Our aim in this section to show the existence of canonical mean convex hulls for a simple closed curve in $S^2_{\infty}(\mathbb{H}^3)$. The technique is basically same. To prove this, we need analogies of the lemmas in Section 2.

Lemma 4.1. [An2] Let $\Gamma \subset S^2_{\infty}(\mathbb{H}^3)$ be a simple closed curve in the sphere at infinity of hyperbolic 3-space. Then there exist a properly embedded least area plane $\Sigma$ spanning $\Gamma$, i.e. $\partial_{\infty}\Sigma = \Gamma$.

Lemma 4.2. [Ga] Let $\{\Sigma_i\}$ be a sequence of least area planes in $\mathbb{H}^3$ with $\partial_{\infty}\Sigma_i = \Gamma_i \subset S^2_{\infty}(\mathbb{H}^3)$ simple closed curve for any $i$. If $\Gamma_i \to \Gamma$, then there exist a subsequence $\{\Sigma_j\}$ of $\{\Sigma_i\}$ such that $\Sigma_j \to \Sigma$ a collection of least area planes whose asymptotic boundaries are $\Gamma$. 
Lemma 4.3. Let $\Gamma_1$ and $\Gamma_2$ be two disjoint simple closed curves in $S^2_\infty(\mathbb{H}^3)$. Then, if $\Sigma_1$ and $\Sigma_2$ are least area planes with $\partial_\infty \Sigma_i = \Gamma_i$, then $\Sigma_1$ and $\Sigma_2$ are disjoint, too.

Proof: Assume that $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. Since asymptotic boundaries $\Gamma_1$, and $\Gamma_2$ are disjoint, the intersection cannot contain an infinite line. So, the intersection between $\Sigma_1$ and $\Sigma_2$ must contain a simple closed curve $\gamma$. Now, $\gamma$ bounds two least area disks $D_1$ and $D_2$ in $\mathbb{H}^3$, with $D_i \subset \Sigma_i$. Now, take a larger subdisk $E_1$ of $\Sigma_1$ containing $D_1$, i.e. $D_1 \subset E_1 \subset \Sigma_1$. By definition, $E_1$ is also a least area disk. Now, modify $E_1$ by swapping the disks $D_1$ and $D_2$. Then, we get a new disk $E'_1 = \{E_1 - D_1\} \cup D_2$. Now, $E_1$ and $E'_1$ have same area, but $E'_1$ have folding curve along $\gamma$. By smoothing out this curve as in [MY1], we get a disk with smaller area, which contradicts to $E_1$ being least area. Note that this technique is known as Meeks-Yau exchange roundoff trick.

Now, we will adapt Lemma 3.1 to our context.

Lemma 4.4. Let $\Gamma \subset S^2_\infty(\mathbb{H}^3)$ be a simple closed curve. Then there are two canonical extremal least area planes (which might be same) $\Sigma^+$ and $\Sigma^-$ in $\mathbb{H}^3$ with $\partial_\infty \Sigma^\pm = \Gamma$. Moreover, any least area plane $\Sigma$ with $\partial_\infty \Sigma = \Gamma$ is disjoint from $\Sigma^\pm$, and it is captured in the region bounded by $\Sigma^+$ and $\Sigma^-$ in $\mathbb{H}^3$.

Proof: Let $\Gamma \subset S^2_\infty(\mathbb{H}^3)$ be a simple closed curve. $\Gamma$ separates $S^2_\infty(\mathbb{H}^3)$ into two parts, say $D^+$ and $D^-$. Define sequences of pairwise disjoint simple closed curves $\{\Gamma^+_i\}$ and $\{\Gamma^-_i\}$ such that $\Gamma^+_i \subset D^+$, and $\Gamma^-_i \subset D^-$ for any $i$, and $\Gamma^+_i \to \Gamma$, and $\Gamma^-_i \to \Gamma$.

By Lemma 4.1, for any $\Gamma^+_i \subset S^2_\infty(\mathbb{H}^3)$, there exist a least area plane $\Sigma^+_i \subset \mathbb{H}^3$. This defines a sequence of least area planes $\{\Sigma^+_i\}$. Now, by using Lemma 4.2, we get a collection of least area planes $\hat{\Sigma}^+$ with $\partial_\infty \hat{\Sigma}^+ = \Gamma$, as $\partial_\infty \Sigma^+_i = \Gamma^+_i \to \Gamma$.

As in the proof of Lemma 3.1, we claim that the collection $\hat{\Sigma}^+$ consists of only one least area plane. But this time, we do not know that if two least area planes have same asymptotic boundary, then they are disjoint, as in the compact domain case (Lemma 2.1). We only know that if two least area planes have disjoint asymptotic boundary, then they are disjoint by Lemma 4.3.

Assume that there are two least area planes $\Sigma^+_a$ and $\Sigma^+_b$ in the collection $\hat{\Sigma}^+$. Since $\partial_\infty \Sigma^+_a = \partial_\infty \Sigma^+_b = \Gamma$, $\Sigma^+_a$ and $\Sigma^+_b$ might not be disjoint, but they
are disjoint from least area planes in the sequence, i.e. \( \Sigma_i^+ \cap \Sigma_{i,b}^+ = \emptyset \) for any \( i \), by Lemma 4.3.

If \( \Sigma_a^+ \) and \( \Sigma_b^+ \) are disjoint, we can use the argument in the proof Lemma 3.1, and conclude that \( \Sigma_a^+ \) is a barrier between the sequence \( \{ \Sigma_i^+ \} \) and \( \Sigma_b^+ \), and so, \( \Sigma_i^+ \) cannot be limit of this sequence.

If \( \Sigma_a^+ \) and \( \Sigma_b^+ \) are not disjoint, then they intersect each other, and in some region, \( \Sigma_a^+ \) is ”above” \( \Sigma_b^+ \). But since \( \Sigma_a^+ \) is the limit of the sequence \( \{ \Sigma_i^+ \} \), this would imply \( \Sigma_b^+ \) must intersect planes \( \Sigma_i^+ \) for sufficiently large \( i \). But, this contradicts \( \Sigma_b^+ \) is disjoint from \( \Sigma_i^+ \) for any \( i \), as they have disjoint asymptotic boundary. So, there exist unique least area plane \( \Sigma^+ \) in the collection \( \Sigma_i^+ \). Similarly, \( \Sigma^- = \Sigma^- \).

By using similar arguments to Lemma 3.1, one can conclude that these least area planes \( \Sigma_i^+ \), and \( \Sigma_i^- \) are canonical, and independent of the choice of the sequence \( \{ \Gamma_i^\pm \} \) and \( \{ \Sigma_i^\pm \} \).

Now, if we show the last statement of the theorem, then we are done. Let \( \Sigma' \) be any least area plane with \( \partial_\infty \Sigma' = \Gamma \). If \( \Sigma' \cap \Sigma^+ \neq \emptyset \), then some part of \( \Sigma' \) must be ”above” \( \Sigma^+ \). Since \( \Sigma^+ = \lim \Sigma_i^+ \), for sufficiently large \( i \), \( \Sigma' \cap \Sigma_i^+ \neq \emptyset \). But, \( \partial_\infty \Sigma_i^+ = \Gamma_i^+ \) is disjoint from \( \Gamma = \partial_\infty \Sigma' \). Then, by Lemma 4.3, \( \Sigma' \) must be disjoint from \( \Sigma_i^+ \), which is a contradiction.

Similarly, this is true for \( \Sigma^- \), too. Moreover, let \( N \subset \mathbb{H}^3 \) be the region between \( \Sigma^+ \) and \( \Sigma^- \), i.e. \( \partial N = \Sigma^+ \cup \Sigma^- \). Then, by construction, \( N \) is also a canonical region for \( \Gamma \), and for any least area plane \( \Sigma' \) with \( \partial_\infty \Sigma' = \Gamma \), \( \Sigma' \) is contained in the region \( N \), i.e. \( \Sigma \subset N \).

\[ \square \]

Remark 4.1. This lemma shows that for any given simple closed curve \( \Gamma \) in \( S^2_\infty(\mathbb{H}^3) \), we can get two canonical least area planes \( \Sigma^+_\Gamma \) (which might be same). Moreover, these least area planes are disjoint from any other least area plane with asymptotic boundary \( \Gamma \). Because of this reason, we call these canonical least area planes \( \Sigma^+_\Gamma \) as untouchable least area planes. Also, the region between them is a canonical mean convex hull of \( \Gamma \), which captures all least area planes in \( \mathbb{H}^3 \) spanning \( \Gamma \).

**Theorem 4.5.** Let \( \Gamma \subset S^2_\infty(\mathbb{H}^3) \) be a simple closed curve. Then either there exist a unique minimal plane \( \Sigma \) with \( \partial_\infty \Sigma = \Gamma \), or there exist a canonical mean convex hull \( N \subset \mathbb{H}^3 \) such that all minimal planes \( \Sigma' \) with \( \partial_\infty \Sigma' = \Gamma \) are contained in \( N \).

**Proof:** Let \( \Gamma \subset S^2_\infty(\mathbb{H}^3) \) be a simple closed curve and let \( X_\Gamma = \{ P \subset \mathbb{H}^3 \mid \partial_\infty P = \Gamma, P \text{ is a minimal plane} \} \). By [An1], we know that for any \( P \in X_\Gamma \), \( P \) is in the convex hull of its asymptotic boundary, i.e. \( P \subset CH(\Gamma) \).
Also, name the complement of $\Gamma$ in $S^2_\infty(\mathbb{H}^3)$ as $D^\pm$, i.e. $S^2_\infty(\mathbb{H}^3) - \Gamma = D^+ \cup D^-$. 

Now, for any $P_i \in X_\Gamma$, define domains $\Delta_i^\pm \subset \mathbb{H}^3$ such that $\Delta_i^+$ is the unbounded component of $\mathbb{H}^3 - P_i$ with asymptotic boundary $D^+ \subset S^2_\infty(\mathbb{H}^3)$. $\Delta^-$ is defined similarly. Now, let $\Delta^+ = \bigcap_i \Delta_i^+$, and $\Delta^-$ is defined similarly. Now, $\Delta^\pm$ are nonempty as all minimal planes stays in convex hull of $\Gamma$. Also, $\Delta^\pm$ are canonical and mean convex as boundaries coming from minimal planes.

Now, as in Lemma 4.4, we will define extremal least area planes in $\Delta^+$. Now, the crucial point here is that they are no more least area planes in $\mathbb{H}^3$, but least area planes in $\Delta^+$. As in the proof of Lemma 4.4, we will start with two sequences of simple closed curves converging to $\Gamma$ from different sides. Define sequences of pairwise disjoint simple closed curves $\{\Gamma_i^+\}$ and $\{\Gamma_i^-\}$ such that $\Gamma_i^+ \subset D^+$, and $\Gamma_i^- \subset D^-$ for any $i$, and $\Gamma_i^+ \to \Gamma$, and $\Gamma_i^- \to \Gamma$.

Since, $\Delta^+$ is mean convex domain, we can define a least area plane $\Sigma_i^+$ in $\Delta^+$ with asymptotic boundary $\Gamma_i^+$ as follows. Take a sequence of extremal simple closed curves in $\Delta^+$ such that these curves converge to $\Gamma_i^+$. Then by [MY2], there exist least area disks in the mean convex domain $\Delta^+$ spanning the extremal simple closed curves in the sequence. Then by taking a limit of the sequence of these least area disks, we will get a least area plane in $\Delta^+$ as in [Ga]. Note that this least area plane is not a least area plane of $\mathbb{H}^3$, but least area plane of $\Delta^+$.

Now, as in Lemma 4.4, we will take the limit of least area planes $\Sigma_i^+$, and we get uniquely defined limit least area plane $\Sigma^+$ in $\Delta^+$ by Lemma 4.4. Similarly, construct the least area plane $\Sigma^-$ in $\Delta^-$. Note that even though $\Sigma^\pm$ are canonical least area planes in $\Delta^\pm$, they might not have least area property in the whole $\mathbb{H}^3$.

Now, let $\Sigma^+$ and $\Sigma^-$ be the canonical planes for $\Gamma$ as above, and let $N \subset \mathbb{H}^3$ be the region between $\Sigma^+$ and $\Sigma^-$, i.e. $\partial N = \Sigma^+ \cup \Sigma^-$. We claim that for any minimal plane $P_i \in X_\Gamma$, $P_i \subset N$. Indeed, this is clear by construction. $\Sigma^+ \subset \Delta^+$ and by definition $\Delta^+ = \bigcap_i \Delta_i^+$. Since $P_i$ is "below" $\Delta^+$, then it is "below" $\Delta_i^+$, and so it is "below" $\Sigma^+$. Similarly $P_i$ is "above" $\Delta^-$ and so it is above $\Sigma^-$. This implies that $P_i \subset N$.

If $\Sigma^+ = \Sigma^-$, then $N = \Sigma^+ = \Sigma^-$. For any minimal plane with $\partial_{\infty} P = \Gamma$, $\Sigma$ is contained in $N$. So, $P = \Sigma^+ = \Sigma^-$. This means there exist a unique minimal plane $P \subset \mathbb{H}^3$ spanning $\Gamma \subset S^2_\infty(\mathbb{H}^3)$. Moreover, this unique minimal plane is also least area by existence of least area planes by [An2].
Remark 4.2. This result is also true for cocompact Gromov hyperbolic spaces. If $M$ is a compact Gromov hyperbolic 3-manifold, then the above statements are true for the universal cover $\widetilde{M}$. In other words, if you replace $S^2_\infty(\mathbb{H}^3)$ with $S^2_\infty(\widetilde{M})$ and $\mathbb{H}^3$ with $\widetilde{M}$ in the statement of Theorem 4.5, it is still true. This is because one can prove Lemma 4.4 by modifying the results in [Co1], and the proof of Theorem 4.5 simply goes through.

4.1. Quasi-Fuchsian Manifolds.

Definition 4.1. Let $M$ be a hyperbolic 3-manifold with $\pi_1(M) = \pi_1(S)$ for some compact oriented surface $S$. Then $M$ is called quasi-Fuchsian if the limit set (asymptotic limit of the orbit of a point in $\widetilde{M} = \mathbb{H}^3$ under covering transformations) is a simple closed curve in $S^2_\infty(\mathbb{H}^3)$.

Corollary 4.6. Let $M$ be a quasi-Fuchsian hyperbolic 3-manifold. Then either there exist a unique minimal surface homotopy equivalent to $M$ or there exist a canonical mean convex core $N \subset M$ containing all minimal surfaces homotopy equivalent to $M$.

Proof: We will use the notation of Theorem 4.5. Since $M$ is quasi-Fuchsian, its limit set is a simple closed curve, say $\Gamma \subset S^2_\infty(\mathbb{H}^3)$. By Theorem 4.5, we have a canonical mean convex hull $\mathcal{N} \subset \mathbb{H}^3$ with $\partial_{\infty}\mathcal{N} = \Gamma$. Since $\mathcal{N}$ is only depending on $\Gamma$, in order to get the canonical mean convex core, all we need to show that $\mathcal{N}$ is invariant under covering transformation.

So, if we can show that the planes $\partial\mathcal{N} = \Sigma^\pm$ are invariant under covering transformations, then we are done.

Let $G \simeq \pi_1(M)$ be the covering transformations of $M$. Then for any $\alpha \in G$, $\alpha$ induces a homeomorphism on $S^2_\infty(\mathbb{H}^3)$ and $\alpha(\Gamma) = \Gamma$. Since $\Sigma^+$ is the limit of the sequence $\{\Sigma^+\}$ with $\partial_{\infty}\Sigma^+ = \Gamma \subset D^+$, then $\alpha(\Sigma^+)$ is the limit of the sequence $\{\alpha(\Sigma^+)\}$. Since $M$ is orientable, $\alpha(D^\pm) = D^\pm$ and $\alpha(\Gamma^\pm) \subset D^\pm$. So $\{\Gamma^\pm\}$ is a sequence of pairwise disjoint simple closed curves in $D^\pm$, and $\alpha(\Gamma^\pm) \to \Gamma$. But the proof of Lemma 4.4 implies there is only one limit for such a sequence of least area planes, i.e. $\alpha(\Sigma^+) = \Sigma^+$. Similarly, $\alpha(\Sigma^-) = \Sigma^-$. So $\mathcal{N}$ is invariant under covering transformations.

So, under covering projection $\pi : \mathbb{H}^3 \to M$, $\pi(\mathcal{N}) = N \subset M$ defines the desired canonical mean convex core.

Now, we claim that if $S \subset M$ is a minimal surface homotopy equivalent to $M$, then $S$ is contained in our canonical mean convex core $N$. Since $S$ is homotopy equivalent to $M$, it is $\pi_1$-injective surface in $M$ and its universal
cover $\mathcal{F} \subset H^3$ is a minimal plane such that $\partial_\infty \mathcal{F} = \Gamma$. Then, Theorem 4.5 implies that $\mathcal{F} \subset \hat{N}$. So if we take the projection of both, we get $F \subset N$.

In the case, $\hat{N} = \Sigma^+ = \Sigma^-$, $N$ will be a least area surface, which is homotopy equivalent to $M$. Since any minimal surface homotopy equivalent to $M$ contained in $N$, this implies there exist a unique minimal surface homotopy equivalent to $M$. Moreover, this unique minimal surface is indeed least area by existence of least area planes by [An2].

**Remark 4.3.** The mean convex core lives in the convex core of $M$. This is because all minimal planes $\Sigma$ with $\partial_\infty \Sigma = \Gamma$, $\Sigma$ is contained in the convex hull of $\Gamma$, say $CH(\Gamma)$ (smallest convex subset of $H^3$ with asymptotic boundary $\Gamma \subset S^2_\infty(H^3)$). Then the mean convex hull is in the convex hull, $\hat{N} \subset CH(\Gamma)$, which implies mean convex core is in the convex core of $M$, since the convex core of $M$ is the projection of $CH(\Gamma)$.

## 5. Generic Uniqueness

In this section, we will give a generic uniqueness result for least area disks spanning an extreme curve. In other words, we will show that a generic curve on the boundary of a mean convex domain bounds a unique least area disk. Similar result for least area planes in $H^3$ has been proved in [Co3].

Let $\Omega$ be a $C^3$-smooth mean convex domain in $\mathbb{R}^3$. We need to define the following spaces.

- $A = \{ \alpha \in C^3(S^1, \partial \Omega) \mid \alpha \text{ embedding} \}$
- $D = \{ u \in C^3(S^1, S^1) \mid u \text{ diffeomorphism and satisfies three point condition, i.e. } u(e^{2k\pi i}) = e^{2k\pi i}, k = 1, 2, 3 \}$
- $M = \{ f : D^2 \to \mathbb{R}^3 \mid f(D^2) \text{ minimal and } f|_{\partial D^2} \in A \}$

Now, we will quote Tomi and Tromba’s results from [TT] on the structure of these spaces. They consider a minimal map $f : D^2 \to \mathbb{R}^3$, as conformal harmonic map, and realize the space of minimal maps as a subspace of space of harmonic maps. The minimal ones correspond to the conformal ones in this space. On the other hand, one can identify a harmonic map from a disk to $\mathbb{R}^3$ with its boundary parametrization, by unique extension property. So, we can think of the space of minimal maps $M$, as a subspace of harmonic maps or their boundary parametrizations, $A$ in above notation. If you don’t care about the parametrization but the image curve, you can augment the space of boundary parametrizations with a "reparametrization" factor $D$ to capture conformality.
So, they consider $A \times D$ such that $(\alpha, u) \in A \times D$ identified with $\tilde{\alpha} \circ u : D^2 \to \mathbb{R}^3$, the harmonic extension of $\alpha \circ u : S^1 \to \mathbb{R}^3$. Then, any minimal disk spanning $\alpha(S^1) \subset \partial \Omega$, will correspond to a point in the slice $\{ \alpha \} \times D$ identified with $\tilde{\alpha} \circ u : D^2 \to \mathbb{R}^3$, the harmonic extension of $\alpha \circ u : S^1 \to \mathbb{R}^3$. Then, any minimal disk spanning $\alpha(S^1) \subset \partial \Omega$, will correspond to a point in the slice $\{ \alpha \} \times D$, the space of minimal disks with boundary in $\partial \Omega$.

In [TT], Tomi and Tromba proved that the second component of the derivative of conformality operator, $D_u k : T_u D \to Z \subset C^2(S^1)$ is almost an isomorphism, which is a Fredholm map of index 0. Then by using basic linear algebra they show that the projection maps $\Pi_1 : A \times D \to A$ restricted to the $M = ker(k)$, i.e. $\Pi_1|_M : M \to A$ is Fredholm of index 0. This means the restriction map from minimal maps to their boundary parametrizations is almost an isomorphism.

Now, in our case, $A$ is different from Tomi and Tromba’s setting. But, if one look at the Tomi and Tromba’s proof, everything happens in the second component of $A \times D$, and they get the ”almost isomorphism” between $D$ and image of $k$. So, by simple modifications of these proofs would give the following desired result for our purposes. To see the modifications in detail, one can look at [Co2], and [Co3].

**Lemma 5.1.** [TT] Let $A$, $D$, and $M$ be as above, and $\Pi_1 : A \times D \to A$ be the projection map. Then the space of minimal maps $M$, is a submanifold of $A \times D$, and restriction of projection map to $M$, $\Pi_1|_M$ is Fredholm of index 0.

The second lemma which we use is the classical inverse function theorem for Banach manifolds, [La].

**Lemma 5.2. (Inverse Function Theorem)** Let $M$ and $N$ be Banach manifolds, and let $F : M \to N$ be a $C^p$ map. Let $x_0 \in M$ and $dF$ is isomorphism at $x_0$. Then $F$ is local $C^p$ diffeomorphism, i.e. there exist an open neighborhood of $U \subset M$ of $x_0$ and an open neighborhood $V \subset N$ of $F(x_0)$ such that $F|_U : U \to V$ is $C^p$ diffeomorphism.

The last ingredient is the generalization of Sard’s theorem to infinite dimensional spaces [Sm].

**Lemma 5.3. (Sard-Smale Theorem)** Let $F : X \to Y$ be a Fredholm map. Then the regular values of $F$ are almost all of $Y$, i.e except a set of first category.
From now on, we will call the regular values of the Fredholm map $\Pi_1|_M$ as generic curve. Now, we can establish the main analytical tool for our generic uniqueness result.

**Lemma 5.4.** Let $\alpha \in A$ be a generic curve. Then for any $\Sigma \in \Pi_1^{-1}(\alpha)$, there exist a neighborhood $U_\Sigma \subset M$ such that $\Pi_1|_{U_\Sigma}$ is a homeomorphism onto a neighborhood of $\alpha$ in $A$.

**Proof:** By Lemma 5.1, the map $\Pi_1|_M : M \to A$ is Fredholm of index 0. Let $\alpha \in A$ be a generic curve and $\Sigma \in \Pi_1^{-1}(\alpha) \subset M$. Since $\alpha$ is regular value, $D\Pi_1(\Sigma) : T_\Sigma M \to T_\alpha A$ is surjective, i.e. $\dim(\text{coker}(D\Pi_1)) = 0$. Moreover, we know that $\Pi_1$ is Fredholm of index 0. This implies $\dim(\ker(D\Pi_1)) = \dim(\text{coker}(D\Pi_1)) = 0$, and so $D\Pi_1$ is isomorphism at the point $\Sigma \in M$. By the Inverse Function Theorem, there exist a neighborhood of $\Sigma$ which $\Pi_1$ maps homeomorphically onto a neighborhood of $\alpha \in A$. \hfill $\square$

Now, our aim is to construct a foliated neighborhood for any least area disk spanning a given generic curve (regular value of the Fredholm map) in $A$. Moreover, we will show that the leaves of this foliation are embedded least area disks with pairwise disjoint boundary. By using this, we will show that uniqueness of the least area disk spanning the generic curve.

We will abuse the notation by using interchangeably the map $\Gamma : S^1 \to \partial \Omega$ with its image $\Gamma(S^1)$. Similarly same is true for $\Sigma : D^2 \to \Omega$ and its image $\Sigma(D^2)$.

Let $\Gamma_0 \in A$ be a generic curve, and let $\Sigma_0 \in \pi_1^{-1}(\Gamma_0) \subset M$ be a least area disk whose existence guaranteed by [MY1]. Then by Lemma 5.4, there is a neighborhood of $\Sigma_0 \in U \subset M$ homeomorphic to the neighborhood $\Gamma_0 \in V \subset A$.

Let $\Gamma : [-\epsilon, \epsilon] \to V$ be a path such that $\Gamma(0) = \Gamma_0$ and for any $t, t' \in [-\epsilon, \epsilon]$, $\Gamma_t \cap \Gamma_{t'} = \emptyset$. In other words, $\{\Gamma_t\}$ foliates a neighborhood of $\Gamma_0$ in $\partial \Omega$. Let $\Sigma_t \in U$ be the preimage of $\Gamma_t$ under the local homeomorphism.

**Lemma 5.5.** $\{\Sigma_t\}$ is a foliation of a neighborhood of $\Sigma_0$ in $\Omega$ by embedded least area disks.

**Proof:** We will prove the lemma in three steps.

**Claim 1:** For any $s \in [-\epsilon, \epsilon]$, $\Sigma_s$ is an embedded disk.

**Proof:** Since $\Sigma_0$ is a least area disk, by Lemma 2.1, $\Sigma_0$ is an embedded disk. Now, $\{\Sigma_t\}$ is continuous family of minimal disks. We cannot apply
the Lemma 2.1 to these disks, since the lemma is true for least area disks, while our disks are only minimal.

Let \( s_0 = \inf \{ s \in (0, \varepsilon] \mid \Sigma_s \text{ is not embedded} \} \). But, since \( \{\Sigma_t\} \) is continuous family of disks, and this can only happen when \( \Sigma_{s_0} \) has tangential self intersection (locally lying on on side). But this contradicts to maximum principle for minimal surfaces. So for all \( s \in [0, \varepsilon] \), \( \Sigma_s \) is embedded. Similarly, this is true for \( s \in [-\varepsilon, 0] \), and the result follows.

**Claim 2:** \( \{\Sigma_t\} \) is a foliation, i.e. for any \( t, t' \in [-\varepsilon, \varepsilon] \), \( \Sigma_t \cap \Sigma_{t'} = \emptyset \).

**Proof:** Assume on the contrary that there exist \( t_1 < t_2 \) such that \( \Sigma_{t_1} \cap \Sigma_{t_2} \neq \emptyset \). First, since the boundaries \( \Gamma_{t_1} \) and \( \Gamma_{t_2} \) are disjoint, the intersection cannot contain a line segment. So the intersection must be a collection of closed curves. We will show that in this situation, there must be a tangential intersection between two disks, and this will contradict to the maximum principle for minimal surfaces.

If \( \Sigma_{t_2} \) does not intersect all the minimal disks \( \Sigma_s \) for \( s \in [-\varepsilon, t_2] \), let \( s_0 = \sup \{ s \in [-\varepsilon, t_2] \mid \Sigma_{t_2} \cap \Sigma_s = \emptyset \} \). Then, since \( \{\Sigma_t\} \) is continuous family of minimal disks, it is clear that \( \Sigma_{t_2} \) must intersect \( \Sigma_{s_0} \) tangentially, and lie in one side of \( \Sigma_{s_0} \). But this contradicts to maximum principle for minimal surfaces.

So, let’s assume \( \Sigma_{t_2} \) intersects all minimal disks \( \Sigma_s \) for \( s \in [-\varepsilon, t_2] \). Let \( s_0 = \sup \{ s \in [-\varepsilon, +\varepsilon] \mid \Sigma_{-\varepsilon} \cap \Sigma_s = \emptyset \} \). If \( s_0 > -\varepsilon \), then this would imply a tangential intersection as above, which is a contradiction. Otherwise, the supremum is \( -\varepsilon \), which implies for any \( t > -\varepsilon \), \( \Sigma_t \) intersects \( \Sigma_{-\varepsilon} \). In particular, this implies \( \Sigma_0 \cap \Sigma_{-\varepsilon} \neq \emptyset \). Now, \( \Sigma_{-\varepsilon} \) separates \( \Omega \), and defines a mean convex domain. But, \( \Gamma_0 \) is in the boundary of this new mean convex domain, and the least area disk \( \Sigma_0 \) must be embedded inside of this mean convex domain by Lemma 2.1. So, \( \Sigma_0 \) cannot intersect \( \Sigma_{-\varepsilon} \). This is a contradiction, and the result follows.

**Claim 3:** For any \( s \in [-\varepsilon, \varepsilon] \), \( \Sigma_s \) is a least area disk.

**Proof:** Fix \( \Sigma_s \) for \( s \in (-\varepsilon, \varepsilon) \). Now, let \( [\Sigma_{-\varepsilon}, \Sigma_\varepsilon] \) be the region bounded by embedded disks \( \Sigma_{-\varepsilon} \) and \( \Sigma_\varepsilon \) in \( \Omega \). By above results, \( \Sigma_s \subset [\Sigma_{-\varepsilon}, \Sigma_\varepsilon] \). Since the boundaries are minimal disks, \( [\Sigma_{-\varepsilon}, \Sigma_\varepsilon] \) is a mean convex region. Let \( \gamma \subset \Sigma_s \) be a simple closed curve. By [MY1], there exist a least area embedded disk \( D \) spanning \( \gamma \) in the mean convex domain \( [\Sigma_{-\varepsilon}, \Sigma_\varepsilon] \). If \( D \) is not in \( \Sigma_s \), it must intersect other leaves nontrivially. Then \( \{\Sigma_t\} \cap D \) induce a singular 1-dimensional foliation \( F \) on \( D \). The singularities of the foliation
are isolated as \( \{ \Sigma_\epsilon \} \) and \( D \) are minimal disks. Since Euler characteristic of the disk is 1, by Poincare-Hopf index formula there must be a positive index singularity implying tangential (lying on one side) intersection of \( D \) with some leave \( \Sigma_s \). But this contradicts to maximum principle for minimal surfaces. Since \( \epsilon \) was chosen arbitrarily at the beginning, one can start with suitable \( \epsilon' > \epsilon \). The whole proof will go through, and this shows that for any \( s \in [-\epsilon, \epsilon] \), \( \Sigma_s \) is a least area disk.

Existence of such a foliated neighborhood for a least area disk, implies uniqueness.

**Lemma 5.6.** \( \Sigma_0 \) is the unique least area disk with boundary \( \Gamma_0 \).

**Proof:** Let \( \Sigma' \) be another least area disk with boundary \( \Gamma_0 \). If \( \Sigma_0 \neq \Sigma' \) then \( \Sigma' \) must intersect a leave in the foliated neighborhood of \( \Sigma_0 \), say \( \Sigma_s \). But, since \( \partial \Sigma_s = \Gamma_s \) is disjoint from \( \partial \Sigma' = \Gamma_0 \), Lemma 2.1 implies that the least area disks \( \Sigma_s \) and \( \Sigma' \) are disjoint. This is a contradiction.

So, we have proved the following theorem:

**Theorem 5.7.** Let \( \Gamma \in A \) be a generic curve as described above. Then there exist a unique least area disk \( \Sigma \subset \Omega \) with \( \partial \Sigma = \Gamma \).

**Remark 5.1.** This theorem does not say that there exist a unique minimal disk spanning a given generic curve. In the proof of Lemma 5.5, we essentially use the disk \( \Sigma_0 \) being least area.

So far we have proved the uniqueness of least area disks for a subset \( \hat{A} \subset A \), where \( A - \hat{A} \) is a set of first category. In the following subsection, we will show that this is true for a more general class of curves, i.e. an open dense subset of \( A \).

### 5.1. Open dense set of curves.

Now, we will show that any regular curve has an open neighborhood such that the uniqueness result holds for any curve in this neighborhood.

Let \( \Gamma_0 \in A \) be a regular curve, and let \( \Sigma_0 \in \pi^{-1}(\Gamma_0) \subset M \) be the unique least area disk spanning \( \Gamma_0 \). Let \( U \subset M \) be the neighborhood of \( \Sigma_0 \) homeomorphic to the neighborhood \( V \subset A \) of \( \Gamma_0 \) as above. We will show that \( \Gamma_0 \) has a smaller open neighborhood \( V' \subset V \) such that for any \( \Gamma \in V' \), there exist unique least area disk in \( \Omega \) with \( \partial \Sigma = \Gamma \).
First we will show that the curves disjoint from $\Gamma_0$ in the open neighborhood also bounds a unique least area disk in $\Omega$.

**Lemma 5.8.** Let $\beta \in V$ with $\beta \cap \Gamma_0 = \emptyset$. Then there exist a unique least area disk spanning $\beta$.

**Proof:** Since $\beta \in V$ is disjoint from $\Gamma_0$, we can find a path $\Gamma : (-\epsilon, \epsilon) \to V$, such that $\{\Gamma_t\}$ foliates a neighborhood of $\Gamma_0$ in $\partial \Omega$, and $\beta$ is one of the leaves, i.e. $\beta = \Gamma_s$ for some $s \in (-\epsilon, \epsilon)$. Then the proofs of the previous section implies that $\Sigma_\beta = \Sigma_s$ and $\{\Sigma_t\}$ also gives a foliation of a neighborhood of $\Sigma_\beta$ by least area disks. Then proof of Lemma 5.6 implies that $\Sigma_\beta$ is the unique least area disk spanning $\beta$. \hfill $\Box$

Now, if we can show same result for the curves in $V$ intersecting $\Gamma_0$, then we are done. Unfortunately, we cannot do that, but we will bypass this by going to a smaller neighborhood.

**Lemma 5.9.** There exist a neighborhood $V' \subset V$ of $\Gamma_0$ such that for any $\Gamma'_0 \in V'$, there exist a unique least area disk with boundary $\Gamma'_0$.

**Proof:** Let $V' \subset V$ be an open neighborhood containing $\Gamma_0$ such that there exist disjoint two curves $\beta_1, \beta_2 \in V$ with $\beta_1$ and $\beta_2$ are both disjoint from $\Gamma_0$ and $\Gamma'_0$, for any $\Gamma'_0 \in V'$. We also assume that if $B \subset \partial \Omega$ is the annulus bounded by $\beta_1$ and $\beta_2$, $\Gamma_0, \Gamma'_0$ are contained in $B$. To see the existence of such a neighborhood, one can fix two curves in $V$ disjoint from $\Gamma_0$, and lying in the opposite sides of $\Gamma_0$ in $\partial \Omega$. Then suitable complements of these two curves in $V$ will give us the desired neighborhood of $\Gamma_0$.

Now, fix $\Gamma'_0 \in V'$. By the assumption on $V'$, there are two curves $\beta_1, \beta_2$ disjoint from both $\Gamma_0, \Gamma'_0$ and bounding the annulus $B$ in $\partial \Omega$ such that $\Gamma_0, \Gamma'_0 \subset B \subset \partial \Omega$. Then, we can find two paths $\Gamma, \Gamma' : [-\epsilon, \epsilon] \to V$ with $\{\Gamma_t\}, \{\Gamma'_t\}$ foliates $B$ such that $\Gamma(\epsilon) = \Gamma'(\epsilon) = \beta_1, \Gamma(-\epsilon) = \Gamma'(-\epsilon) = \beta_2$, and $\Gamma(0) = \Gamma'_0, \Gamma'(0) = \Gamma'_0$.

By Lemma 5.5, we know that $\{\Gamma_t\}$ induces $\{\Sigma_t\}$ family of embedded least area disks spanning $\{\Gamma_t\}$. Moreover, these least area disks are unique with the given boundary, and leaves of the foliation in the neighborhood of $\Sigma_0$.

Now, consider the preimage of the path $\Gamma'$ under the homeomorphism $\pi_{1|U} : U \to V$. This will give us a path $\Sigma' \subset U \subset M$, which is a continuous family of minimal disks, say $\{\Sigma'_t\}$. We claim that this is also a family of embedded least area disks inducing a foliated neighborhood of $\Sigma'_0$. By previous paragraphs, we know that $\Sigma_\epsilon$ and the $\Sigma_{-\epsilon}$ are the unique least area
disks with boundary $\beta_1$ and $\beta_2$, respectively. This means $\Sigma'_\pm \epsilon = \Sigma_{\pm \epsilon}$. So, the family $\{\Sigma'_t\}$ has embedded least area disks $\Sigma'_\pm \epsilon$. Then by slight modification of the proof of Lemma 5.5 imply that $\{\Sigma'_t\}$ is a family of embedded least area disk inducing a foliation of a neighborhood of $\Sigma'_0$. By Lemma 5.6, $\Sigma'_0$ is the unique least area disk spanning $\Gamma'_0$.

So, we got the following theorem.

**Theorem 5.10.** Let $\Omega$ be a $C^3$-smooth mean convex domain in $\mathbb{R}^3$, and $A = \{\alpha \in C^3(S^1, \partial \Omega) \mid \alpha \text{ embedding}\}$. Then there exist an open dense subset $A' \subset A$ in $C^3$ topology, such that for any $\Gamma \in A'$, there exist a unique least area disk with boundary $\Gamma$.

**Proof:** The set of regular values of Fredholm map, say $\hat{A}$, is the whole set except a set of first category by Sard-Smale theorem. So, the regular curves are dense in $A$. By above lemmas, for any regular curve $\Gamma_0$, there exist an open neighborhood $V'_{\Gamma_0} \subset A$ which the uniqueness result holds. So, $A' = \bigcup_{\Gamma \in \hat{A}} V'_{\Gamma}$ is an open dense subset of $A$ with the desired properties.

6. **Concluding Remarks**

Different versions of the results in the first part of the paper has been proved by Meeks and Yau in [MY2] by using differential geometry techniques, by Brian White in [Wh] by using geometric measure theory methods, and by Fanghua Lin in [Li] by using global analysis methods. Here, we reformulate those results and extend it to more general class of mean convex domains. Our approach is topological, and seems more natural to the question. On the other hand, similar results for hyperbolicspace has been proved by Michael Anderson in [An2], by using geometric measure theory techniques.

In this paper, we are trying to promote the idea of mean convex hulls. These objects are naturally defined for any simple closed extreme curve, and any asymptotic curve in $S^2_\infty(\mathbb{H}^3)$. They are living in convex hulls of the curve, and have piecewise smooth boundary. As a corollary to the mean convex hulls in $\mathbb{H}^3$, we assign a mean convex core to any quasi-Fuchsian hyperbolic 3-manifold capturing any minimal surface homotopy equivalent to the manifold.

In the second part, we give a generic uniqueness result, mainly by adapting the techniques of [Co3]. There has been different types of generic uniqueness results for the curves in $\mathbb{R}^3$, see [Tr]. This is a new generic
uniqueness result, and says that if you have a mean convex domain with $C^3$-smooth boundary in $\mathbb{R}^3$, then simple closed curves in the boundary generically bounds a unique least area disk in $\mathbb{R}^3$.

REFERENCES

[An1] M. Anderson, *Complete minimal varieties in hyperbolic space*, Invent. Math. **69**, (1982) 477–494.

[An2] M. Anderson, *Complete minimal hypersurfaces in hyperbolic n-manifolds*, Comment. Math. Helv. **58**, (1983) 264–290.

[Co1] B. Coskunuzer, *Uniform 1-cochains and Genuine Laminations*, to appear in Topology.

[Co2] B. Coskunuzer, *Minimal Planes in Hyperbolic Space*, Comm. Anal. Geom. **12**, (2004) 821–836.

[Co3] B. Coskunuzer, *Generic Uniqueness of Least Area Planes in Hyperbolic Space*, eprint; math.GT/0408066

[Ga] D. Gabai, *On the geometric and topological rigidity of hyperbolic 3-manifolds*, J. Amer. Math. Soc. **10**, (1997) 37–74.

[HS] J. Hass and P. Scott, *The Existence of Least Area Surfaces in 3-manifolds*, Trans. AMS **310**, (1988) 87–114.

[La] S. Lang, *Real analysis*, Addison-Wesley, MA, (1983).

[Li] F.H. Lin, *Plateau’s problem for H-convex curves*, Manuscripta Math. **58**, (1987) 497–511.

[MY1] W. Meeks and S.T. Yau, *The classical Plateau problem and the topology of three manifolds*, Topology **21**, (1982) 409–442.

[MY2] W. Meeks and S.T. Yau, *The existence of embedded minimal surfaces and the problem of uniqueness*, Math. Z. **179**, (1982) 151–168.

[Sm] S. Smale, *An infinite dimensional version of Sard’s Theorem*, Amer. J. Math. **87**, (1965) 861–866.

[TT] F. Tomi and A.J. Tromba, *Extreme curves bound embedded minimal surfaces of the type of the disc*, Math. Z. **158**, (1978) 137–145.

[Tr] A. J. Tromba, *The set of curves of uniqueness for Plateau’s problem has a dense interior*, Geometry and topology, Lecture Notes in Math., Vol. 597, 696–706, Springer, Berlin, (1977).

[Wh] B. White, *On the topological type of minimal submanifolds*, Topology **31**, (1992) 445–448.

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN CT 06520

E-mail address: baris.coskunuzer@yale.edu