Extracting the Chern number from the dynamics of a Fermi gas: Implementing a quantum Hall bar for cold atoms

Alexandre Dauphin\textsuperscript{1,2} and Nathan Goldman\textsuperscript{1,1}

\textsuperscript{1}Center for Nonlinear Phenomena and Complex Systems - Université Libre de Bruxelles, 231, Campus Plaine, B-1050 Brussels, Belgium
\textsuperscript{2}Departamento de Física Teórica I, Universidad Complutense, 28040 Madrid, Spain

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We propose a scheme to measure the quantized Hall conductivity of an ultracold Fermi gas initially prepared in a topological (Chern) insulating phase, and driven by a constant force. We show that the time evolution of the center of mass, after releasing the cloud, provides a direct and clear signature of the topologically invariant Chern number. We discuss the validity of this scheme, highlighting the importance of driving the system with a sufficiently strong force to displace the cloud over measurable distances while avoiding band-mixing effects.

The unusual shapes of the driven atomic cloud are qualitatively discussed in terms of a semi-classical approach.

Today, the manifestation of topology in physical systems is no longer restricted to the realm of solid-state setups, where the quantum Hall (QH) phases and topological insulators were initially observed and created\textsuperscript{[1-2]}. Indeed, the ingredients responsible for these topological phases, such as large magnetic fields or spin-orbit couplings, have recently been engineered in cold-atom setups\textsuperscript{[3-8]}. With such quantum simulators of topological matter in hand, topological phases can be explored from a different perspective, exploiting the unique probing and addressing techniques proper to cold-atom setups\textsuperscript{[9,10]}.

Topological quantum phases are characterized by two fundamental properties\textsuperscript{[11,12]}: (a) a topological invariant $\nu$ associated with a bulk gap, which remains constant as long as the gap remains open, and (b) robust edge states whose energies are located within the topological bulk gap. A first manifestation of topology was discovered in the quantum Hall effect\textsuperscript{[11,12]}, where the Hall conductivity is exactly equal to the quantized Hall conductivity quantum $\sigma_{\nu}$, i.e., $\sigma_H = \nu \sigma_0$. In solid materials subjected to large magnetic fields, the quantized Hall conductivity is measured through the transport equation $j = eE$, where $E = E_y L_y$ is an electric field and where a non-zero transverse conductivity $\sigma_{xy}$ signals the Hall current $j_x$ generated by the magnetic field\textsuperscript{[13]}. Engineering the analogue of a quantum Hall experiment with cold atoms subjected to synthetic magnetic fields\textsuperscript{[4]} would thus require to drive the system along a given direction, and to measure the Hall current in the transverse direction. For the analogy to be complete, reservoirs should be connected to the cold-atom systems, in order to inject and retrieve the driven particles. Although mesoscopic conduction properties have been demonstrated in an ultracold Fermi gas “connected” to two reservoirs\textsuperscript{[14]}, such a scheme would add a considerable complexity to the demanding setup that generates the synthetic magnetic field. To overcome this technical issue, strategies have been proposed to evaluate the topological invariant $\nu$ by other means, exploiting available imaging and driving techniques. These proposals rely on hybrid time-of-flight imaging\textsuperscript{[15]}, Bloch oscillations\textsuperscript{[16]}, the measure of the Zak’s phase\textsuperscript{[17,18]}, and state-dependent momentum density imaging\textsuperscript{[19,20]} (see also Refs.\textsuperscript{[21,22]}). Besides, the response of a bosonic cloud to modulations of the external confining potential has already revealed Hall-like behaviors in the presence of a synthetic magnetic field\textsuperscript{[23]} (see also Ref.\textsuperscript{[24]}). Alternatively, the topological edge states that are expected to be present when $\nu \neq 0$ could also be visualized through in-situ imaging\textsuperscript{[25,26]} and light scattering\textsuperscript{[27,29]}.

Here, we introduce a simple scheme to measure the quantized Hall conductivity of a Fermi gas trapped in a 2D optical lattice and driven along $y$ by a constant external force (e.g. a synthetic “electric” field $E = E_y L_y$ created by accelerating the lattice\textsuperscript{[30]}). Our method is based on the possibility to initially prepare the atomic gas in a Chern insulating phase, and to image the time evolution of its center-of-mass $x(t)$ after suddenly releasing the confining potential, see Fig.\textsuperscript{[1]} (a). This scheme, which constitutes a cold-atom analogue of quantum Hall experiments, leads to a satisfactory measure of the Chern number under two conditions: (1) the “electric” field $E_y$ should be strong enough to generate a measurable displacement after a realistic experimental time $t^*$, namely $x(t^*) \sim 10a$, where $a$ is the lattice spacing; (2) the strength of the driving field should be small compared to the topological bulk gap $E_y \rho a \ll \Delta$, to avoid band-mixing processes during the evolution\textsuperscript{[31]}. Under those assumptions, which are addressed in detail below, our method is valid for any 2D cold-atom system exhibiting the (anomalous) quantum Hall\textsuperscript{[28,29,32,33]} or quantum spin Hall effects\textsuperscript{[34,35]}.

To illustrate the method, we consider a non-interacting Fermi gas trapped in a 2D brick-wall optical lattice\textsuperscript{[15,18,38]} with laser-induced nearest-neighbor (NN) hopping of amplitude $J_2 < J$. Such a system can be realized by trapping atoms in two internal states $|g\rangle$ and $|e\rangle$ using state-dependent optical lattices and by inducing the NN hopping through laser-coupling methods\textsuperscript{[5,19,20,30,39]}. The second-quantized Hamiltonian is taken to be\textsuperscript{[19,20]}

$$\hat{H} = -J \sum_{\langle i,j \rangle} e^{i p \cdot (r_i - r_j) / 2} \hat{c}_i \hat{c}_j - J_2 \sum_{\langle k,l \rangle} \hat{c}^\dagger_k \hat{c}_l,$$

where $p$ is the recoil momentum associated with the laser coupling\textsuperscript{[30]}, and where $\hat{c}^\dagger_k$ creates an atom at lattice site $x_k$. 

\[\text{(1)}\]
This system realizes a generalization of the two-band Hal- 
dane model [40]: for certain values of the recoil momentum 
and ratio \( j_2/J \), a topological bulk gap opens with Chern 
number \( \nu = \pm 1 \) [19 20], leading to anomalous QH phases 
with quantized Hall conductivity \( \sigma_H = \pm 1 \) in units of the conductivity quantum [40]. The bulk energy gap can be as 
large as \( \Delta \sim 2J \), as shown in Fig. 1 (b) for \( j_2 = 0.3J \) 
and \( \bm{p} = (0, 4\pi/3a) \). Below, we describe a scheme to mea-
Sure the Chern number \( \nu = \pm 1 \), assuming that the atomic 
gas is initially prepared in such a QH phase. Other cold-atom 
setups leading to topological bands with large gaps \( \Delta \sim J \) 
could be equally considered.

Let us first assume that a single atom is confined in an 
optical lattice of size \( L \times L \), where \( L \) is the number of 
unit cells along each direction. In the presence of a syn-
thetic electric field directed along the \( y \) direction, \( H_{\text{electric}} = 
-\varepsilon_0 \sum_j E_y c_j^\dagger c_j \), the velocity in a state \( |u(-), k\rangle \) of the lowest 
band \( E_-(k) \) with quasi-momentum \( k \) is given by [41]

\[
\begin{align*}
  v_x(k) &= v_{\text{band}}^x + v_F^x = \frac{\partial E_-(k)}{\hbar \partial k_x} - \frac{i}{\hbar} E_y \mathcal{F}_{xy}^-(k), \\
  v_y(k) &= v_{\text{band}}^y = \frac{\partial E_-(k)}{\hbar \partial k_y},
\end{align*}
\]

(2)

where \( \mathcal{F}_{xy}^-(k) = \langle \partial_{k_x} u | \partial_{k_y} u \rangle - \langle \partial_{k_y} u | \partial_{k_x} u \rangle \) is the Berry’s 
curvature associated with the state \( |u(-), k\rangle \). Completely fill-
ing the lowest band \( E_-(k) \) and taking the limit \( L \to \infty \) yields 
the standard relation for the current density in the QH phase

\[
  j_x = -(\nu/\hbar) E_y, \quad j_y = 0,
\]

(3)

where \( \nu = (i/2\pi) \int_{\text{BZ}} \mathcal{F}_{xy}^- dk \) is the Chern number of the 
lowest band [12], and where BZ denotes the first Brillouin 
zone. In order to avoid measuring the current that flows 
through the lattice, which would require connecting reservoirs 
to the system, we propose to follow an alternative strategy. 
We initially confine the system in a region of size \( L_0 < L \) using 
a confining potential \( V_{\text{conf}} \), and we set the Fermi energy \( E_F \) 
inside the topological bulk gap, hence filling the lowest band 
\( E_-(k) \) completely. At time \( t = 0 \), we suddenly remove the 
confining walls \( V_{\text{conf}} \) and add the force along the \( y \) direction.

After the quench, all the initial states project onto the eigen-
states of the final Hamiltonian \( H_{\text{tot}} = H + H_{\text{electric}} - V_{\text{conf}} \), 
uniformly populating the lowest energy band \( E_-(k) \). We note 
that edge states lying within the bulk gap partially project unto 
states of the highest band \( E_+(k) \), but we verified that this ef-
fet is negligible [45]. Taking into account the anomalous ve-
locity associated with all the occupied states, and neglecting 
any contribution of states lying in the highest band \( E_+(k) \), 
one finds that the center-of-mass \( x(t) \) of the cloud follows the 
equations of motion

\[
x(t) = -\left( a^2 t E_y / \pi \hbar \right) \nu_{\text{approx}}, \quad y(t) = 0,
\]

(4)

where \( \nu_{\text{approx}} \) is a discretized expression of the Chern number 
that converges towards \( \nu \) as the initial system size \( L_0 \to \infty \), 
and where we used the fact that a unit cell of the brick-wall 
array has a volume \( 2a^2 \). The expression (4), which predicts a 
constant velocity \( \dot{x}(t) \), is only valid at sufficiently long times 
after the quench \( t \gg 0 \). Importantly, the initial filling of the 
lowest band cancels the undesired contribution of the band 
velocity \( \nu_{\text{band}} \). This constitutes a significant advantage with 
respect to proposals based on bosonic wave packets, where 
this effect has to be annihilated by other means in view of 
measuring the Chern number [16].

We now simulate such a protocol and discuss the regimes 
in which a stable Chern number \( \nu_{\text{approx}} \approx \nu \) can indeed be 
measured through in-situ imaging of the cloud. In order to 
minimize the effects due to band-mixing processes (e.g. 
Landau-Zener transitions [16]), we set the model parameters 
to the values \( j_2 = 0.3J \) and \( \bm{p} = (0, 4\pi/3a) \) that maxi-

\[
\text{fize the spectral gap} \Delta = 2J, \text{see Fig. 1 (b). We initially}
\]

configure the system with a perfectly sharp circular potential 
\( V_{\text{conf}}(r) = J (r/r_0)^\gamma \), with \( \gamma = \infty \), which can now be 
created in experiments [42 44]: smooth confinements are dis-

\[
\text{cussed in [45]. In this configuration, we set the Fermi energy}
\]

\( E_F \geq \max(E_-(k)) \) so as to fill the lowest band while 
limiting the population of edge states [45]. At time \( t = 0 \), we 
suddenly remove the confinement \( V_{\text{conf}} \) and act on the sys-

\[
\text{tem with a reasonably weak force} \ E = 0.2J/a1_y. \text{Figures 2}
\]

(a)-(c) show the time evolution of the particle density \( \rho(x, t) \), 
demonstrating a clear drift of the cloud along the transverse 
direction \( x \) only. For \( E_y = 0.2J/a \), this displacement is given by 
\( |x(t^*)| = 8a1_x \) after a typical time \( t^* = 40\pi \hbar /J \), which 
could be detected using imaging techniques with high (single-
\[
\text{site) resolution. Figure 2 (d) shows the displacement} \ x(t)
\]

along the \( x \) direction as a function of time, for different val-
ues of the force \( E = E_y 1_y. \) For \( E_y = 0.2J/a \), the system

\[
\]
is driven in the linear-response regime, and the center of mass follows the constant motion described by Eq. (4). A linear regression applied to the data \( x(t) \) yields a remarkably precise value for the measured Chern number \( \nu_{\text{approx}} \approx 1.00 \). Increasing the force \( E_y \) allows to significantly enlarge the displacement \( x(t) \), which is desirable to improve the detection under realistic experimental times. However, it is also crucial to avoid non-linear and band-mixing effects in order to measure the Chern number adequately through Eq. (4). For \( E_y = 0.4J/a \), the measured Chern number is still satisfactory \( \nu_{\text{approx}} \approx 0.90 \) while the displacement at time \( t^* = 40\pi \hbar / J \) is \( |x(t^*)| = 14a \). For \( E_y = 0.8J/a \), a clear Hall drift is still observed, however, the dramatically non-quantized Hall conductivity \( \nu_{\text{approx}} \approx 0.63 \) deduced from this data signals the breakdown of the single-populated-band approximation due to Landau-Zener transitions. From Fig. 2 we conclude that a moderate field \( E_y \approx \pm 0.3J/a \) constitutes a good compromise, allowing one to measure a quantized Chern number \( \nu_{\text{approx}} \approx \pm 1 \) through a center-of-mass displacement of about ten lattice sites. Let us point out that the flatness of the lowest band in Fig. 1(b) does not influence our result, as the displacement \( x(t) \) only relies on the Chern number of the lowest band, and not on the band velocity \( v_{\text{band}} = (1/\hbar) \partial k E_{\nu}(k) \).

Having described the motion of a driven QH atomic cloud initially prepared by setting the Fermi energy within the topological bulk gap, it is now interesting to study the stability of our method under variations of the initial atomic filling factor. This effect is investigated in Figs. 3(a)–(b), which compares the time evolution of the center-of-mass \( x(t) \) for different fillings \( n_F = 1/4, 1/2, 3/4 \). The half-filling case \( n_F = 1/2 \), which has already been discussed above, shows the constant Hall drift \( x(t) \) dictated by Eq. (4) and the immobility along the driven direction \( y(t) \approx 0 \). When \( n_F = 1/4 \), the first band is only partially filled and the system behaves as a metal: the longitudinal conductivity is non-zero and a clear motion along the transverse direction is observed, see Fig. 3(b). Interestingly, the Hall motion along the transverse direction \( x \) is characterized by an almost constant velocity, which when fitted with the filled-band expression Eq. (4) yields an approximatively quantized value for the Hall conductivity \( \nu_{\text{approx}} \approx 1.03 \). This result can be understood by realizing that the evolving occupied states contribute significantly to the total Berry’s velocity \( \sum_k v_F(k) \propto \nu \) while their contribution to the band velocity \( \sum_k v_{\text{band}} \approx 0 \) is vanishing by symmetry, see Fig. 3. When \( n_F = 3/4 \), the upper band \( E_{\nu_F}(k) \) is partially filled and the system has a conducting behavior along the \( y \) direction, similarly to the metallic phase \( n_F = 1/4 \). We note that the contribution of high-energy states strongly affects the Hall motion along the \( x \) direction, which when fitted with Eq. (4) naturally yields a non-quantized Hall conductivity, \( \nu_{\text{approx}} \approx 0.3 \): the contribution of the Berry’s curvature \( \chi_{xy} = -F_{xy} \) associated with the upper band spoils the evaluation of the Chern number. We have verified that the Chern number \( \nu_{\text{approx}} \approx \nu \) remains robust for small filling variations around the QH phase, \( n_F \approx 1/2 \) [43].

The Chern number characterizes the topological class of the quantum system [11, 12], and thus, it distinguishes between a trivial insulating phase (\( \nu = 0 \)) and a topological (Chern) insulating phase (\( \nu \neq 0 \)). To further evaluate the efficiency of our method, we propose to compare the time evolution of the center of mass \( x(t) \) discussed above, with a system configuration corresponding to a trivial topological order \( \nu = 0 \). To do so, we introduce a staggered potential \( H_{\text{stag}} = \lambda_{\text{stag}} \sum_j (-1)^j c_j^\dagger c_j \), which adds an onsite energy \( \pm \lambda_{\text{stag}} \) alternatively along both spatial directions. This perturbation opens a bulk gap leading to a trivial insulating phase with \( \nu = 0 \) [19, 20, 40]. We show in Figs. 3(c)–(d) the center-of-mass motion for this trivial configuration \( J_2 = 0, \lambda_{\text{stag}} \neq 0 \), considering the different filling factors \( n_F = 1/4, 1/2, 3/4 \). Here, the potential strength \( \lambda_{\text{stag}} = J_2 \) is chosen such that the width of the bulk gap \( \Delta = 2J \) is the same as for the topological situation shown in Figs. 2(a)–(b). At half filling, the system remains immune to the external force \( E = E_y 1_y \), i.e. \( x(t) \approx 0 \), in agreement with the behavior expected for an insulator. Fitting the data with Eq. (4) yields \( \nu_{\text{approx}} = 0 \) with less than 1% error. In the metallic phases, \( n_F = 1/4 \) and \( n_F = 3/4 \), the cloud performs Bloch oscillations along the driving direction \( y \), while no Hall transport is observed along the transverse direction, i.e. \( x(t) \approx 0 \). It is instructive to study the case when both competing effects \( J_2, \lambda_{\text{stag}} \) are present, which can potentially give rise to either
a trivial or a non-trivial topological phase \[19\] \[20\] \[40\]. We have verified that our method still allows to precisely measure the Chern number \(\nu_{\text{approx}} \simeq 0, \pm 1\) in this situation, hence revealing the topological order of the atomic system. When \(J_2 = 0.3J\) and \(\lambda_{\text{stag}} = 0.3J\), the system is in a topological phase characterized by a gap width \(\Delta = 1.9J\) and a Chern number \(\nu = 1\). The Chern number evaluated from the displacement \(x(t) = 40\pi h/J\), using a force \(E_y = 0.2J/a\), has been found to be \(\nu_{\text{approx}} = 1.00\) with less than 1% error. Besides, when \(J_2 = 0.3J\) and \(\lambda_{\text{stag}} = 2.3J\), the system is in an insulating state with the same gap width \(\Delta = 1.9J\) but with a vanishing Chern number; the measured \(\nu_{\text{approx}} = 0.00\) has been found with the same precision.

In our scheme, the dynamics of the atomic cloud is characterized by two different effects: (a) the center-of-mass displacement discussed above, which is associated with a quantized conductivity \(\nu_{\text{approx}} \simeq \nu\); and (b) the dynamical deformations of the cloud. The latter effect arises as a complicated interplay between the band structure and the external force applied to the system. These deformations can be qualitatively described considering a semi-classical picture \[16\] \[31\], in which the dynamics of the atomic cloud is decomposed into time-evolving wave packets \(\psi(x(t), k_n(t))\), localized around the center of mass \(x(t)\) and the many quasi-momenta \(k_n(t)\) \(\in\) BZ. When a force is applied along the \(y\) direction, the quasi-momentum of a single wave packet initially localized around \(k_0\) evolves according to \(k_x = k_x^0\) and \(k_y = k_y^0 + E_y/\hbar t\). The real space evolution of each wave packet is dictated by \(\dot{x}(t) = \nu(k)\), where the velocity \(\nu = \nu_{\text{band}} + \nu_x\) is given by Eq. \[4\] and represented in Fig. \[4\].

We now discuss the global movement of the atomic cloud by solving these semi-classical equations independently, for a few chosen values of \(k_0\) that capture the essential diffusion effects. We point out that the motion along the \(y\) direction is entirely determined by the band velocity \(\nu_{\text{band}}(k)\) (see Figs. \[4\] (a),\(c\)), which simply leads to Bloch oscillations: after a full period \(T = 2\pi h/E_y\), all the wave packets return to their initial position \(y(T) = 0\) and \(k(T) = k_0\), see Fig. \[5\]. The motion taking place along the transverse direction \(x\) is more exotic, as it is influenced by the Berry’s velocity \(\nu_{\text{Berry}}\). First, the motion of a wave packet initially at \(k_0 = 0\) is characterized by a finite Berry’s velocity and a zero band velocity \(\nu_{\text{band}} = 0\), see Fig. \[4\]. In the topological case \(\nu \neq 0\), the Berry’s velocity is always negative, which leads to a net drift along \(x\). This analysis can be readily extended to wave packets initially centered around other momenta \(k_0 \neq 0 \in\) BZ, whose transverse drift are affected by the band velocity \(\nu_{\text{band}} \neq 0\). By symmetry, the initial conditions shown in Fig. \[4\] corresponding to \(k_0 < 0\) and \(k_0 > 0\), will evolve with the same Berry’s velocity but opposite band velocity. As can be deduced from the patterns in Fig. \[4\] these two wave packets will undergo a net drift along opposite directions after each period \(T\). These opposite drifts, which are found to be qualitatively similar for arbitrary initial conditions, describe the progressive broadening of the cloud along the \(x\) direction. Combining this broadening effect along \(x\) together with the Bloch oscillations along \(y\) leads to the unusual shapes of the cloud at arbitrary times \(t \neq T \times \text{integer}\) (see also \[45\]).

As a final remark, we emphasize that our Chern number measurement does not rely on the methods used to generate the topological band structure, and thus, it can be applied to any 2D cold-atom setup characterized by non-trivial Chern numbers. Furthermore, this scheme could be applied to distinguish between different Chern insulators with \(\nu \neq \pm 1 \in\) \(\mathbb{Z}\). Finally, our method could be extended to the case of \(Z_2\) topological phases \[34\] \[35\], where the spin Chern number could be deduced by subtracting the center-of-masse displacements associated with the two spin species, \(\nu_{\text{spin}} \propto \mathbf{r}_\uparrow - \mathbf{r}_\downarrow\).

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Figure 4. The band $v_{b}$ and Berry’s $v_{F}$ velocities in units of $v_{a} = J a / h$ for a constant driving force of strength $E_{0} = 0.2 J / a$. (a)-(b) the non-trivial Chern insulating phase with parameters $J_{2} = 0.3 J, \lambda_{stag} = 0$; (c) the trivial insulating phase with parameters $J_{2} = 0, \lambda_{stag} = 1.0 J$. The green dots represent the momenta $k^{0}$ used to describe the dynamics of the atomic cloud in Fig. 5.

Figure 5. Time-evolving spatial density $\rho(x, t)$ for a driving force $E = (0.2 J / a) \mathbf{1}_{y}$. (a)-(c) the non-trivial Chern insulating phase with parameters $J_{2} = 0.3 J, \lambda_{stag} = 0$; (d)-(f) the trivial insulating phase with parameters $J_{2} = 0, \lambda_{stag} = 1.0 J$. The blue plain circle represents the semi-classical trajectories of the wave-packets with initial momenta $k^{0}$ shown in Fig. 4. The time is expressed in units of the period $T = 10 \pi a / J$.

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Appendix A: Effects of the populated edge states on the dynamics

Appendix B: Bloch oscillations and dynamics at arbitrary times: flat bands vs dispersive bands

Appendix C: The Chern number measurement using smooth confinements

Appendix D: Releasing the confinement along the transverse direction only

Supplementary Material

In the main text, we considered a Fermi gas initially confined by an infinitely abrupt potential \( V_{\text{conf}}(r) = (r/r_0)^\infty \) and prepared in a quantum Hall (QH) phase. The cold-atom system is characterized by the band structure illustrated in Fig. 6(a) and the Fermi energy is set within the bulk gap denoted \( \Delta \). In this configuration, the lowest band \( E_-(k) \) with Chern number \( \nu = 1 \) is totally filled. At time \( t = 0 \), the confinement is suddenly released \( V_{\text{conf}}(r) = 0 \) and a force is added along the \( y \) direction, \( E = E_y 1_y \). Assuming that all the initially populated states project unto states of the lowest energy band \( E_-(k) \), which is a valid hypothesis as far as the bulk states are concerned (see below), we obtained the equations of motion for the center of mass,

\[
x(t) = -(a^2 t^2 E_y / \pi \hbar) \nu_{\text{approx}}, \quad y(t) = 0,
\]

where \( \nu_{\text{approx}} \approx \nu \), see main text. Clearly, these equations neglect the fact that edge states, whose energies are located within the bulk gap, are initially populated. Indeed, the edge states that are spatially localized in the vicinity of the confining radius \( r \approx r_0 \), will potentially project on the (many) bulk states associated with the two bulk bands \( E_{\pm}(k) \). Since the Chern numbers of the two bands are opposite \( \nu(+) = -\nu(-) \), the contribution of the initially populated edge states to the dynamics can potentially perturb the Chern number measurement described in the main text. It is the scope of this Appendix to show to what extend their contribution can indeed be neglected.

The two-band spectrum \( E_{\pm}(k) \) shown in Fig. 6(a), which has been obtained by considering periodic boundary conditions, does not take into account the edge states that are present in the experimental setup: the non-zero Chern number \( \nu = 1 \) guarantees the presence of edge states that are spatially localized at \( r = r_0 \), and whose energies are located within the bulk gap. These edge states are visible in the spectrum \( E_{\alpha} \) represented in Fig. 6(b), which has been obtained for a trapped system with circular confining potential \( V_{\text{conf}}(r) = (r/r_0)^\infty \) and \( r_0 = 25a \).

Figure 6. (a) Bulk band structure \( E_{\pm}(k) \) in the case \( J_2 = 0.3J \) and \( p = (0, 4\pi/3a) \), see main text. The bulk energy gap is \( \Delta = 2J \); for all configurations presented in this Appendix \( \Delta = 2J \). (b) Discrete energy spectrum for the same system configuration, in the presence of an infinitely abrupt circular potential \( V_{\text{conf}}(r) \) with radius \( r_0 = 25a \). The three Fermi energies \( E_F^{(1,2,3)} \) are considered in this Appendix in order to study the effects of the edge states on the dynamics. The label \( \alpha \) classifies the energies in increasing order, \( E_\alpha < E_{\alpha+1} \).

We now consider three different values for the Fermi energy: \( E_F^{(1)} \) is set right above the lowest bulk band \( E_-(k) \), \( E_F^{(2)} = 0 \) is located well inside the bulk gap, and \( E_F^{(3)} \) is set right below the highest band \( E_+(k) \). Note that the value \( E_F^{(1)} = -1.3J \) corresponds to the situation considered in the main text. When suddenly releasing the confinement \( V_{\text{conf}}(r) \), a bulk state \( \chi_\alpha \) with energy \( E_\alpha \in E_-(k) \) will project on bulk states \( \phi_\lambda \) with energies \( \epsilon_\lambda = E_\alpha \), as shown in Fig. 7(a). On the contrary, an edge state...
with energy $E_\alpha \in \Delta$ will project on bulk states with energies $\epsilon_\lambda \in E_-(k)$ and also on bulk states with energies $\epsilon_\lambda \in E_+(k)$, as shown in Fig. 7(b). As a corollary, the population of states lying in the highest band $E_+(k)$ and taking part in the dynamics is reduced by setting the Fermi energy close to the band edge $E_F \approx E_F^{(1)}$, while this undesired population is increased for higher Fermi energies $E_F = E_F^{(2,3)} > E_F^{(1)}$.

Figure 7. (a) Projection of a bulk state $|\chi_\alpha\rangle$ of the confined system ($V_{\text{conf}} \neq 0$), with energy $E_\alpha = -1.54J$, onto the states $|\phi_\lambda\rangle$ of the unconfined system ($V_{\text{conf}} = 0$). (b) Projection of an edge state $|\chi_\alpha\rangle$ of the confined system with energy $E_\alpha = -0.97J$: after releasing the trap, the edge states project onto bulk states associated with the two bulk bands $E_\pm(k)$.

The states populations $P_\lambda = \sum_{E_\alpha < E_F} |\langle \chi_\alpha | \phi_\lambda \rangle|^2$ after the quench are represented in Fig. 8 for $E_F = E_F^{(1,2,3)}$. Here, $\chi_\alpha$ [resp. $\phi_\lambda$] denotes the eigenstate with energy $E_\alpha$ [resp. $\epsilon_\lambda$] before [resp. after] the quench. From Fig. 8 we deduce that the population of the highest band $E_+(k)$ is highly limited, even in the extreme case where all the edge states are initially filled, i.e. when $E_F = E_F^{(3)}$.

Figure 8. Population of the states $\phi_\lambda$ with energy $\epsilon_\lambda$ after releasing the confinement $V_{\text{conf}}$, for the different values of the Fermi energy $E_F = E_F^{(1,2,3)}$ shown in Fig. 6. In all figures, the green shaded region corresponds to the energy bulk gap $\Delta$. The partial filling of the highest band $E_+(k)$ is highlighted for all cases in small boxes.

We now illustrate how the population of the highest band modifies the dynamics of the cloud, and thus how it affects the Chern number measurement. The time-evolving density is shown in Fig. 9 for the three different values of the Fermi energy $E_F = E_F^{(1,2,3)}$ discussed above. By increasing the contrast of the corresponding density plots, we observe the appearance of a few particles that move to the right, i.e. in the direction opposite to the overall Hall drift. These few states, whose population increases with the Fermi energy, are associated with the highest band $E_+(k)$ and they have an opposite Berry’s velocity $v_F^{(+)} = -v_F^{(-)}$. We find that these few counter-propagating states only slightly affect the center-of-mass displacement: the Chern numbers evaluated from the dynamics are $\nu_{\text{approx}} = 1.00$ ($E_F = E_F^{(1)}$), $\nu_{\text{approx}} = 0.98$ ($E_F = E_F^{(2)}$) and $\nu_{\text{approx}} = 0.96$ ($E_F = E_F^{(3)}$). These results highlight the robustness of our scheme against variations of the atomic filling factor.
Figure 9. Spatial density $\rho(x,t)$ at time $t = 40\pi\hbar/J$ for the three Fermi energies $E_F = E_F^{(1,2,3)}$ presented in Fig. 6(b). The dashed blue circle represents the atomic cloud at $t = 0$, while the blue [resp. red] dots show the position of the center of mass at time $t = 0$ [resp. $t = 40\pi/J$]. The non-zero projection onto the upper band $E_+^{(+)}(k)$ gives rise to a small counter-propagating motion, due to the opposite value of the Berry’s curvature $F_{xy}^+ = -F_{xy}^-$, see also main text.

Appendix B: Bloch oscillations and dynamics at arbitrary times: flat bands vs dispersive bands

In this Appendix, we discuss the dynamics of the cloud at arbitrary times, so as to further reveal the interplay between the Hall drift taking place perpendicularly to the force $E = E_y 1_y$ – due to the Berry’s velocity $v_F^x(k)$ – and the Bloch oscillations stemming from the band velocity $v_{\text{band}} = (1/\hbar)\partial_k E(k)$, see main text.

We first consider the case where the filled energy band is dispersive, in which case the contribution from the band velocity is large. To study such a situation, we start with the band structure depicted in Fig. 6(a) and reverse the sign of the hopping amplitude $J_2 = 0.3J \rightarrow J_2 = -0.3J$ so as to interchange (and reverse) the upper and lower bands $E_+(k) \leftrightarrow E_-(k)$. Setting the Fermi energy in the gap, we now fill the dispersive band $E_+^{(+)}(k)$, instead of the nearly flat band $E_-^{(-)}(k)$. Note that the gap size $\Delta = 2J$ is the same as for the situation encountered in the main text ($J_2 = 0.3J$). The time-evolving density is shown in Fig. 10, where large Bloch oscillations are observed between the periods $t = \text{integer} \times T$, where $T = 2\pi\hbar/aE_y = 10\pi\hbar/J$ is the time after which a full cycle is performed in the Brillouin zone (see main text). Note that these Bloch oscillations take place along both spatial directions, leading to a large broadening of the cloud at arbitrary times (see for example $t = 6\pi\hbar/J$ in Fig. 10). At $t = \text{integer} \times T$, the contribution of the band velocity vanishes, and the Hall drift is clearly visualized (see for example $t = 40\pi\hbar/J$ in Fig. 10). Note that the band $E_+(k)$ is associated with the Chern number $\nu^{(+)} = -1$ (in contrast with $\nu^{(-)} = +1$ for $E_-(k)$), which leads to a transverse displacement towards the right. The dispersive motion of the cloud can be analyzed through a semi-classical treatment, as already discussed in the main text.

We emphasize that, in general, the topological bulk bands produced in cold-atom systems will be dispersive. Consequently, the behavior presented in Fig. 10 showing a center-of-mass motion accompanied with Bloch oscillations, should correspond to the typical dynamics that will be observed in such experiments.

To be complete, we show in Fig. 11 the full dynamics in the case of the nearly flat band configuration obtained by setting $J_2 = 0.3J$. In this case, the band velocity associated with the filled band $E_-(k)$ is small, and thus, the Bloch oscillations only take place on the scale of a few lattice sites: the flat-band configuration reveals the Hall drift in a clear manner at arbitrary times.
Figure 10. Time-evolving density $\rho(x, t)$ for a cloud initially trapped by an infinitely abrupt potential $V_{\text{conf}}(r) = (r/r_0)^\infty$ with $r_0 = 20a$. The system parameters are $J = -0.3J, \lambda_{\text{stag}} = 0$, so that the system is initially prepared in a QH phase associated with the filled (dispersive) band $E_+(k)$. The blue dashed circle and blue dot highlight the initial condition. The force applied after releasing the cloud is $E = (0, 0.2J/a)$.

Figure 11. Time-evolving density $\rho(x, t)$ for a cloud initially trapped by an infinitely abrupt potential $V_{\text{conf}}(r) = (r/r_0)^\infty$ with $r_0 = 25a$. The system parameters are $J = 0.3J, \lambda_{\text{stag}} = 0$, so that the system is initially prepared in a QH phase associated with the filled (nearly flat) band $E_-(k)$. The blue dashed circle and blue dot highlight the initial condition. The force applied after releasing the cloud is $E = (0, 0.2J/a)$. 
Appendix C: The Chern number measurement using smooth confinements

In the main text, we considered that the atomic cloud was initially trapped by an infinitely abrupt circular confinement, which was then suddenly removed at time $t = 0$ when the force $E = E_{y}1_{y}$ was applied. In this Appendix, we now study the time-evolved density $\rho(x, t)$ in the situation where the initial confinement is chosen to be smooth, which is generally the case in most experiments. We performed numerical simulations for the following cases (setting the Fermi energy at the value $E_{F} = E_{F}^{(1)} = -1.3J$):

- A system initially confined by an abrupt potential $V_{\text{conf}}(r) = 0.8J(r/r_{0})^{10}$, see Fig. 12.
- A system initially confined by a quartic potential $V_{\text{conf}}(r) = 0.8J(r/r_{0})^{4}$, see Fig. 13.
- A system initially confined by a harmonic potential $V_{\text{conf}}(r) = 0.8J(r/r_{0})^{2}$, see Fig. 14.
- A system initially confined by a harmonic potential $V_{\text{conf}}(r) = 0.8J(r/r_{0})^{2}$, which is then suddenly released in a larger harmonic potential $V_{\text{conf}}(r) = 0.8J(r/R_{0})^{2}$ with $R_{0} \gg r_{0}$, see Fig. 15.

In all these situations, the atomic cloud shows a clear Hall drift along the $x$ direction, while the center of mass remains nearly immobile along the driven direction, $|y(t)| < a$. The Chern numbers deduced from Eq. (4) remain remarkably close to the quantized value $\nu_{\text{approx}} \simeq \nu = 1$, as indicated in all Figs. 12-15. These numerical investigations demonstrate the applicability and robustness of our method in various confinement schemes.

Figure 12. Time-evolving density $\rho(x, t)$ for a cloud initially trapped by a sharp potential $V_{\text{conf}}(r) = 0.8J(r/r_{0})^{10}$ with $r_{0} = 25a$. The system parameters are $J_{2} = 0.3J, \lambda_{\text{stag}} = 0$, so that the system is initially prepared in a QH phase. The blue dashed circle and blue dot highlight the initial condition. The force applied after releasing the cloud is $E = (0, 0.2J/a)$. Figures (d)-(e) show the time evolution of the center of mass $x(t)$. The Chern number deduced from Fig. (d) is indicated.
Figure 13. Time-evolving density $\rho(x, t)$ for a cloud initially trapped by a quartic potential $V_{\text{conf}}(r) = 0.8 J(r/r_0)^4$ with $r_0 = 25a$. The system parameters are $J_2 = 0.3J, \lambda_{\text{stag}} = 0$, so that the system is initially prepared in a QH phase. The blue dashed circle and blue dot highlight the initial condition. The force applied after releasing the cloud is $E = (0, 0.2J/a)$. Figures (d)-(e) show the time evolution of the center of mass $x(t)$. The Chern number deduced from Fig. (d) is indicated.

Figure 14. Time-evolving density $\rho(x, t)$ for a cloud initially trapped by a harmonic potential $V_{\text{conf}}(r) = 0.8 J(r/r_0)^2$ with $r_0 = 25a$. The system parameters are $J_2 = 0.3J, \lambda_{\text{stag}} = 0$, so that the system is initially prepared in a QH phase. The blue dashed circle and blue dot highlight the initial condition. The force applied after releasing the cloud is $E = (0, 0.2J/a)$. Figures (d)-(e) show the time evolution of the center of mass $x(t)$. The Chern number deduced from Fig. (d) is indicated.
Figure 15. Time-evolving density $\rho(x,t)$ for a cloud initially trapped by a harmonic potential $V_{\text{conf}}(r) = 0.8J(r/r_0)^2$ with $r_0 = 25a$. At time $t = 0$, the system is released in a weaker harmonic potential $V_{\text{conf}}(r) = 0.8J(r/R_0)^2$, with $R_0 = 50a$. The system parameters are $J_z = 0.3J, \lambda_{\text{stag}} = 0$, so that the system is initially prepared in a QH phase. The blue dashed circle and blue dot highlight the initial condition. The force applied after releasing the cloud is $E = (0, 0.2J/a)$. Figures (d)-(e) show the time evolution of the center of mass $x(t)$. The Chern number deduced from Fig. (d) is indicated.

Appendix D: Releasing the confinement along the transverse direction only

The topological order associated with the QH phase is captured by the Chern number $\nu$, which was shown to be deduced from the transverse motion of the center of mass $x(t)$ (the force being applied along the $y$ direction). Since no relevant information is contained in the longitudinal displacement $y(t)$, which might potentially perform Bloch oscillations, we may simplify the measurement scheme by simply releasing the cloud along the $x$ direction only. This possibility is investigated in this Appendix for two situations:

- A system initially confined by a harmonic potential $V_{\text{conf}}(r) = 0.8J(r/r_0)^2$ and released along the $x$ direction only: at time $t = 0$ the cloud is confined by the anisotropic potential $V_{\text{conf}}^{(t)}(r) = 0.8J(y/r_0)^2$, while the force is applied along the $y$ direction; see Fig. 16.

- A system initially confined by a harmonic potential $V_{\text{conf}}(r) = 0.8J(r/r_0)^2$ and only partially released along the $x$ direction: at time $t = 0$ the cloud is confined by the anisotropic potential $V_{\text{conf}}^{(t)}(r) = 0.8J[(x/R_0)^2 + (y/r_0)^2]$ with $R_0 > r_0$, while the force is applied along the $y$ direction; see Fig. 17.

The Chern number deduced from these two anisotropic schemes remains close to the quantized value $\nu_{\text{approx}} \simeq \nu = 1$, indicating the validity of our method in these situations.
Figure 16. Time-evolving density $\rho(x, t)$ for a cloud initially trapped by a harmonic potential $V_{\text{conf}}(r) = 0.8J(r/r_0)^2$ with $r_0 = 25a$. At time $t = 0$, the system is released along the direction $x$ (transverse to the force $E$), while it is still trapped along the $y$ direction by an anisotropic harmonic potential $V_{\text{conf}}(t)(r) = 0.8J(y/r_0)^2$. The system parameters are $J_2 = 0.3J, \lambda_{\text{stag}} = 0$, so that the system is initially prepared in a QH phase. The blue dashed circle and blue dot highlight the initial condition. The force applied after releasing the cloud is $E = (0, 0.2J/a)$. Figures (d)-(e) show the time evolution of the center of mass $x(t)$. The Chern number deduced from Fig. (d) is indicated.

Figure 17. Time-evolving density $\rho(x, t)$ for a cloud initially trapped by a harmonic potential $V_{\text{conf}}(r) = 0.8J(r/r_0)^2$ with $r_0 = 25a$. At time $t = 0$, the system is partially released along the direction $x$ (transverse to the force $E$), meaning that the cloud is suddenly confined by an anisotropic harmonic potential $V_{\text{conf}}(t)(r) = 0.8J[(x/R_0)^2 + (y/r_0)^2]$ with $R_0 = 50a$. The system parameters are $J_2 = 0.3J, \lambda_{\text{stag}} = 0$, so that the system is initially prepared in a QH phase. The blue dashed circle and blue dot highlight the initial condition. The force applied after releasing the cloud is $E = (0, 0.2J/a)$. Figures (d)-(e) show the time evolution of the center of mass $x(t)$. The Chern number deduced from Fig. (d) is indicated.