SECTORIAL EXTENSIONS FOR SOME ROUMIEU ULTRAHOLOMORPHIC CLASSES DEFINED BY WEIGHT FUNCTIONS

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Abstract. We prove several extension theorems for Roumieu ultraholomorphic classes of functions in sectors of the Riemann surface of the logarithm which are defined by means of a weight function or weight matrix. Our main aim is to transfer the results of V. Thilliez from the weight sequence case to these different, or more general, frameworks. As a byproduct, we obtain an extension in a mixed weight-sequence setting in which assumptions on the sequence are minimal.

1. Introduction

The main aim of this paper is to prove the surjectivity of the Borel map (via the existence of right inverses for this map) in ultraholomorphic classes of functions in unbounded sectors defined by means of weight functions or weight matrices, so generalizing to this framework previous results available only in the ultradifferentiable setting. Let us start by reviewing such results and motivating our approach.

Ultradifferentiable classes of smooth functions in sets of \( \mathbb{R}^n \), defined by suitably restricting the growth of their derivatives, have been extensively studied since the beginning of the 20th century. In particular, the study of the injectivity and surjectivity of, or the existence of right inverses for, the Borel map (respectively, the Whitney map), sending a function in this class to the family of its derivatives at a given point (resp., at every point in a given closed subset of \( \mathbb{R}^n \)), has attracted much attention. In case the restriction of growth is specified in terms of a sequence of positive real numbers, the corresponding classes are named after Denjoy and Carleman, who characterized the injectivity of the Borel map back in 1923. The surjectivity of, and the existence of right inverses for, the Borel map was solved 1988 by H.-J. Petzsche [23], and the Whitney extension result was treated by J. Chaumat and A. M. Chollet [17]. From the seminal work of R. W. Braun, R. Meise and B. A. Taylor [6], who modified the original approach of A. Beurling, it is also standard to consider classes in which the growth control is made by a weight function, whose properties allow one to conveniently apply Fourier analysis in this setting thanks to suitable Paley-Wiener-like results. The study of the surjectivity of the Borel and Whitney maps and their right inverses in this situation was done in the 1980’s and 1990’s by several authors, we mention J. Bonet, R. W. Braun, J. Bruna, M. Langenbruch, R. Meise and B. A. Taylor (see [22, 3] and the references therein). A last step in this context has been recently taken by A. Rainer and G. Schindl [33, 26], who considered classes defined by weight matrices, what strictly includes both the Denjoy-Carleman and the Braun-Meise-Taylor approaches, and also obtained results in the same line [27].
However, the study of similar problems for classes of holomorphic functions is much more recent, and it has been motivated by the increasing interest on asymptotic expansions, a theory put forward by H. Poincaré at the end of the 19th century. In order to give a full analytical meaning to the formal power series solutions of meromorphic linear systems of ordinary differential equations at an irregular singular point in the complex domain, in the 1980’s J. P. Ramis, B. Malgrange, Y. Sibuya and W. Balser, just to name a few, refined this concept by considering Gevrey asymptotic expansions of order $\alpha > 1$: Its existence for a function $f$, holomorphic in a sector $S$ (with vertex at 0) of the Riemann surface of the logarithm, amounts to the estimations $|f^{(n)}(z)| \leq CA_n(n!)^\alpha$ in proper subsectors of $S$, for suitable $C, A > 0$. This fact makes evident the close link between ultradifferentiable classes and those similarly introduced for holomorphic functions defined in sectors, which are called ultraholomorphic classes. In asymptotic theory it is also important to decide about the injectivity or surjectivity of the Borel map, sending a function to the sequence of its derivatives at the vertex (defined by an obvious limiting process). While the injectivity for Gevrey classes was already studied by Watson and Nevanlinna in the 1920’s, the surjectivity result, known as Borel-Ritt-Gevrey theorem, is due to B. Malgrange (see [29, 28]), and V. Thilliez [36] obtained right inverses for the Borel map. Corresponding results for Gevrey functions in several variables were obtained by Y. Haraoka [10] and the second author [30]. For general Denjoy-Carleman ultraholomorphic classes in unbounded sectors, in which the sequence $(n!)^\alpha_n$ is replaced by a general sequence $M = (M_n)_n$ subject to standard assumptions, the first results on the surjectivity of the Borel map and the existence of right inverses were obtained in 2000 by J. Schmets and M. Valdivia [35], and these were improved in some respects by V. Thilliez [37]. In this last paper, a growth index $\gamma(M)$ associated with the sequence $M$ plays a crucial role, limiting from above the opening of the sector for which extension operators exist. Finally, in case the sequence $M$ admits a proximate order definitive results for injectivity and surjectivity were obtained by the second author in [31], and a forthcoming paper [12] will completely solve the injectivity problem for general logarithmically convex sequences, and it will provide significantly improved information for the surjectivity as long as strongly regular sequences are considered. However, no attempt has been made so far to study these problems for ultraholomorphic classes defined by weight functions or matrices, and our present paper is a first step in this direction.

The main ingredient for our construction of extension operators is the use of a truncated Laplace-like integral transform whose kernel is obtained from optimal flat functions, i.e., functions which are not only flat (in the sense that they have a null asymptotic expansion, and so an exponential decrease in terms of the sequence $M$) but admit also exponential estimates from below. This technique rests on the fundamental idea of B. Malgrange, and it has already been fruitful in an alternative proof by A. Lastra, S. Malek and the second author [17] of the extension results of V. Thilliez [37], and also in [31]. While the construction of sectorially (optimal) flat functions is an adaptation of the ideas by V. Thilliez, we will not use any Whitney-type extension result from the ultradifferentiable setting: A suitable integral kernel is defined from the flat functions available, and its moments are proved to be estimated from above and below by sequences belonging to the weight matrix defining the ultraholomorphic class (see Proposition 6.3). The opening of the sectors for which the construction is possible is again controlled by a new growth index $\gamma(\omega)$, associated in this case with the defining weight function $\omega$, and which allows one to turn qualitative properties of $\omega$ into quantitative ones (see in this respect the Lemmas 4.2 and 4.3). For a detailed information about this and other indices for $\omega$, and their relation to the indices $\gamma(M)$ of Thilliez or $\omega(M)$ (introduced in [31]), we refer to a paper in preparation [11]. The main result, Theorem 6.4, states the surjectivity of the Borel map in ultraholomorphic classes, associated with a weight matrix which
is, in turn, obtained from a suitable weight function $\tau$ with $\gamma(\tau) > 0$, and in sectors of opening smaller than $\pi \gamma(\tau)$. Observe that, as a byproduct of the existence of optimal flat functions, we deduce that the Borel map is not injective for these classes in such narrow sectors.

A last paragraph in the paper is devoted to the implications of our main result when Denjoy-Carleman ultraholomorphic classes are considered. If the weight sequence $M$ is strongly regular we recover the result of Thilliez, but if we drop the moderate growth condition we are able to prove an extension result in a mixed setting, meaning that the weight sequence defining the class of sequences we depart from has to be changed into a precise, larger (nonequivalent one) weight sequence defining the ultraholomorphic class where the interpolating function dwells. It is worthy to emphasize that there do exist sequences which do not satisfy any of the standard growth properties assumed in previous extension results, as illustrated by Example 6.8.

The paper is organized as follows. Section 2 contains all the preliminary, mostly well-known, information concerning weight sequences, weight functions and weight matrices, and it introduces the ultraholomorphic classes we will consider, among which those associated with weight functions or matrices are new in the literature. It ends with Lemma 2.7, which will be important for rephrasing flatness in our ultraholomorphic classes by means of some standard auxiliary functions. In Section 3 we recall some basic facts about Legendre (also called Young) conjugates and, thanks to them and after several auxiliary important results, we prove in Theorem 3.8 that, under suitable hypotheses, the ultraholomorphic class associated with a weight matrix may be represented as that associated with a weight function. The information about Thilliez’s growth index for a weight sequence, and about a new growth index for weight functions, is described in Section 4. After a characterization of flat functions (Lemma 5.4), the construction of optimal flat functions is the aim of Section 5. Finally, Section 6 is devoted to the main result, Theorem 6.4, its rephrasing Corollary 6.6 in terms of classes defined by weight functions, and a closing subsection about a mixed setting extension procedure for classes defined by weight sequences.

### 2. Basic definitions

This section is devoted to fixing some notations, introducing the main properties of weight sequences, functions or matrices which we will deal with, and defining the ultraholomorphic classes of Roumieu type under consideration.

We denote by $\mathcal{H}$ the class of holomorphic functions. We will write $\mathbb{N}_> = \{1, 2, \ldots\}$ and $\mathbb{N} = \mathbb{N}_0 \cup \{0\}$, moreover we put $\mathbb{R}_> := \{x \in \mathbb{R} : x > 0\}$, i.e. the set of all positive real numbers.

#### 2.1. Weight sequences.

A sequence $M = (M_k)_k \in \mathbb{R}^\mathbb{N}_>$ is called a weight sequence. We define also $m = (m_k)_k$ by

$$m_k := \frac{M_k}{k!}, \quad k \in \mathbb{N},$$

and $\mu = (\mu_k)_k$ by

$$\mu_0 := 1; \quad \mu_k := \frac{M_k}{M_{k-1}}, \quad k \in \mathbb{N}_>.$$

$M$ is called normalized if $1 = M_0 \leq M_1$ (this condition may always be assumed without loss of generality).

We list now some interesting and standard properties for weight sequences:

1. $M$ is log-convex, if

$$\text{l.c.} :\iff \forall j \in \mathbb{N}_> : M_j^2 \leq M_{j-1}M_{j+1}$$
and strongly log-convex, if
\[ (\text{slc}) : \forall j \in \mathbb{N}_0 : m_j^2 \leq m_{j-1}m_{j+1}. \]

We recall that for every weight sequence \( M = (M_k)_k \in \mathbb{R}_{>0}^N \) one has
\[ \liminf_{k \to \infty} \mu_k \leq \liminf_{k \to \infty} (M_k)^{1/k} \leq \limsup_{k \to \infty} (M_k)^{1/k} \leq \limsup_{k \to \infty} \mu_k. \]

If \( M \) is log-convex and normalized, then \( M, ((M_k)^{1/k})_{k \in \mathbb{N}} \) and \( (\mu_k)_{k \in \mathbb{N}} \) are nondecreasing, and so \( \lim_{k \to \infty} (M_k)^{1/k} = +\infty \) if, and only if, \( \lim_{k \to \infty} \mu_k = +\infty \). Moreover, \( M_j M_k \leq M_{j+k} \) holds for all \( j, k \in \mathbb{N} \), e.g. see [37] Remark 2.0.3, Lemmata 2.0.4, 2.0.6].

(2) \( M \) has moderate growth if
\[ (\text{mg}) : \exists C \geq 1 \forall j, k \in \mathbb{N} : M_{j+k} \leq C^{j/k} M_j M_k. \]

Note that, by elementary estimates, \( M \) has \((\text{mg})\) if, and only if, \( m \) has \((\text{mg})\).

(3) \( M \) has \((\gamma_1)\) if
\[ (\gamma_1) : \sup_{p \in \mathbb{N}_{>0}} \mu_p \sum_{k \geq p} \frac{1}{\mu_k} < +\infty. \]

In the literature \((\gamma_1)\) is also called "strong non-quasianalyticity condition".
A sequence \( M \) is called strongly regular (see [37]) if it satisfies \((\text{slc})\), \((\text{mg})\), and \((\gamma_1)\).

At this point we want to make the reader aware that in [37] a slightly different notation and terminology is used, due to the fact that the main role in the statements there is assigned to the sequence which here is denoted by \( m \), and not to the sequence denoted here by \( M \).

We write \( M \preceq N \) if and only if \( M_p \leq N_p \) holds for all \( p \in \mathbb{N} \) and define
\[ M \preceq N :\iff \exists C \geq 1 \forall p \in \mathbb{N} : M_p \leq C^p N_p \iff \sup_{p \in \mathbb{N}_{>0}} \left( \frac{M_p}{N_p} \right)^{1/p} < +\infty. \]

\( M \) and \( N \) are called equivalent if
\[ M \simeq N :\iff M \preceq N \text{ and } N \preceq M. \]

Moreover, if we write \( \nu_0 := 1, \nu_p := N_p/N_{p-1}, \) \( p \in \mathbb{N}_{>0} \), we introduce the stronger relation
\[ M \preceq N :\iff \exists C \geq 1 \forall p \in \mathbb{N} : \mu_p \leq C\nu_p \iff \sup_{p \in \mathbb{N}} \frac{\mu_p}{\nu_p} < +\infty \]
and call them strongly equivalent if
\[ M \simeq N :\iff M \preceq N \text{ and } N \preceq M. \]

If we write \( n = (n_k)_k \) for \( n_k := \frac{N_k}{k!}, \) \( k \in \mathbb{N} \), then it is clear that \( M \preceq N \) if, and only if, \( n \preceq \gamma \), and that \( M \preceq N \) if, and only if, \( n \preceq \gamma \).

Define the set
\[ \mathcal{L}C := \{ M \in \mathbb{R}_{>0}^N : M \text{ normalized, log-convex, } \lim_{k \to \infty} (M_k)^{1/k} = +\infty \}. \]

We warn the reader that in previous works by the authors \([13, 14]\) the condition \( \lim_{k \to \infty} (M_k)^{1/k} = +\infty \) was equivalently expressed (see above) as \( \lim_{k \to \infty} \mu_k = +\infty \).

The Gevrey sequence of order \( s \geq 1 \) will be denoted by \( G^s := (p^s)_p \), for \( s > 1 \) it satisfies all properties listed above.
2.2. Weight functions \( \omega \) in the sense of Braun-Meise-Taylor. A function \( \omega : [0, \infty) \to [0, \infty) \) is called a weight function if it is continuous, nondecreasing, \( \omega(0) = 0 \) and \( \lim_{x \to \infty} \omega(x) = +\infty. \) In case we also have \( \omega(x) = 0 \) for all \( x \in [0, 1], \) we say \( \omega \) is a normalized weight.

Moreover we consider the following conditions:

1. \( \omega(2t) = O(\omega(t)) \) as \( t \to +\infty. \)
2. \( \log(t) = o(\omega(t)) \) as \( t \to +\infty \) (\( \iff \lim_{t \to +\infty} \frac{t}{\omega(t)} = 0. \))
3. \( \omega \) is the function \( \varphi \), given by \( \varphi(t) = \omega(e^t), \) is a convex function on \( \mathbb{R}. \)
4. \( \text{The condition } \omega_\varphi : \mathbb{R} \to \mathbb{R}, \text{ given by } \varphi_\omega(t) = \omega(e^t), \text{ is a convex function on } \mathbb{R}. \)
5. \( \exists \) \( H \geq 1 \forall t \geq 0 : 2 \omega(t) \leq \omega(Ht) + H. \)
6. \( C > 0 : \forall y > 0 : \int_1^\infty \frac{\omega(y)}{t^2} dt \leq C \omega(y) + C. \)

We mention that this list of properties is extracted from a larger one used already in [33], what explains the lack of \( (\omega_2) \), irrelevant in this paper.

An interesting example is the weight function \( \sigma_s(t) := \max\{0, \log(t)^s\}, s > 1, \) which satisfies all listed properties except \( (\omega_4) \). It is well-known that the weight \( t \mapsto t^{1/s} \) yields the Gevrey class \( G^s \) of index \( s > 1, \) it satisfies all listed properties (except normalization).

For a normalized weight \( \omega \) satisfying \( (\omega_3) \) we define the Legendre-Fenchel-Young-conjugate

\[
\varphi_\omega^*(x) := \sup\{xy - \varphi_\omega(y) : y \in \mathbb{R}\} = \sup\{xy - \varphi_\omega(y) : y \geq 0\}, \quad x \geq 0,
\]

with the following properties, e.g. see [6] Remark 1.3, Lemma 1.5: It is nonnegative, convex and nondecreasing, \( \varphi_\omega^*(0) = 0, \) the map \( x \mapsto \frac{\varphi_\omega^*(x)}{x} \) is nondecreasing in \([0, +\infty)\) and \( \lim_{x \to +\infty} \frac{\varphi_\omega^*(x)}{x} = \infty. \)

Moreover, \( \omega \) has also \( (\omega_4) \) if, and only if, \( \varphi_\omega^* = \varphi_\omega, \) and then the map \( x \mapsto \frac{\varphi_\omega^*(x)}{x} \) is also nondecreasing in \([0, +\infty). \)

Remark 2.1. It is interesting to note, as it was done in [33] p. 15, that condition \( (\omega_4) \) appearing in [6], was necessary in order to show that certain classes of compactly supported functions defined by decay properties of their Fourier transform in terms of a weight function \( \omega \) could be alternatively represented as those consisting of functions whose derivatives’ growth may be controlled by the Legendre-Fenchel-Young-conjugate of \( \omega. \) Since we will work in a different framework, we will only assume this condition whenever the equality \( \varphi_\omega^* = \varphi_\omega \) is needed in our arguments.

Given a weight function \( \omega \) and \( s > 0, \) we define a new weight function \( \omega^s \) by

\[
\omega^s(t) := \omega(t^s), \quad t \geq 0.
\]

If \( \omega \) satisfies any of the properties \( (\omega_1) \) \( (\omega_3) \) \( (\omega_4) \) or \( (\omega_6) \) then the same holds for \( \omega^s, \) but \( (\omega_5) \) or \( (\omega_{snq}) \) might not be preserved. Indeed, this last fact motivates the introduction of the index \( \gamma(\omega) \) in this paper, see Subsection 4.2.

Let \( \sigma, \tau \) be weight functions, we write

\[
\sigma \leq \tau : \iff \tau(t) = O(\sigma(t)) \text{ as } t \to +\infty
\]

and call them equivalent if

\[
\sigma \sim \tau : \iff \sigma \preceq \tau \text{ and } \tau \preceq \sigma.
\]

We recall [21] Proposition 1.3, where \( (\omega_{snq}) \) was characterized, and [21] Corollary 1.4:

**Proposition 2.2.** Let \( \omega : [0, +\infty) \to [0, +\infty) \) be a weight function. The following are equivalent::

(i) \( \lim_{t \to 0} \limsup_{t \to +\infty} \frac{\omega(t)}{\omega(2t)} = 0, \)
(ii) \( \exists K > 1 \text{ such that } \limsup_{t \to +\infty} \frac{\omega(Kt)}{\omega(t)} < K, \)
(iii) \( \omega \) satisfies \( (\omega_{snq}) \)
Moreover, we put
\[ \kappa(t) := \int_1^\infty \frac{\omega(tu)}{u^2} du = t \int_t^\infty \frac{\omega(u)}{u^2} du, \quad \forall \ t > 0 \quad \kappa(0) = 0. \]

Consequently, \( \omega \) has also \([\omega_1] \) and \([\omega_5] \). If \( \omega \) satisfies one of the equivalent conditions above, then there exists some \( 0 < \alpha < 1 \) such that \( \omega(t) = O(t^\alpha) \) as \( t \to \infty \).

It is well-known that each of the properties \([\omega_1] \) \([\omega_3] \) or \([\omega_4] \) can be transferred from \( \omega \) to \( \kappa \), see e.g. \([3, \text{Remark 3.2}] \).

Note that concavity of a weight function \( \omega \) implies sub-additivity (i.e. \( \omega(s + t) \leq \omega(s) + \omega(t) \) for every \( s, t \geq 0 \); the proof needs the fact that \( \omega(0) = 0 \), and this in turn yields \([\omega_2] \).

Finally, for a weight function \( \omega \) it will be useful to consider the function \( \omega^* \) given by \( \omega^*(t) := \omega(1/t), \ t > 0 \).

2.3. Weight matrices. For the following definitions and conditions see also \([26, \text{Section 4}] \).

Let \( \mathcal{I} = \mathbb{R}_{>0} \) denote the index set, a \textit{weight matrix} \( \mathcal{M} \) associated to \( \mathcal{I} \) is a (one parameter) family of weight sequences \( \mathcal{M} := \{ M^x, y \} : x \in \mathcal{I}, \ y \in \mathcal{I} \), such that

\[ (\mathcal{M}) : \forall x \in \mathcal{I} : M^x \text{ is normalized, nondecreasing, } M^x \leq M^y \text{ for } x \leq y. \]

We call a weight matrix \( \mathcal{M} \) \textit{standard log-convex}, if

\[ (\mathcal{M}_{sc}) : (\mathcal{M}) \text{ and } \forall x \in \mathcal{I} : M^x \in \mathcal{L} \]

Moreover, we put \( m^x_p := \frac{M^x_p}{M^x_0} \) for \( p \in \mathbb{N}, \) and \( \mu^x_p := \frac{M^x_p}{M^x_{p-1}} \) for \( p \in \mathbb{N}_{>0}, \mu^x_0 := 1. \)

A matrix is called \textit{constant} if \( \mathcal{M} = \{ M \} \) or more generally if \( M^x \equiv M^y \) for all \( x, y \in \mathcal{I} \).

We are going to consider the following properties for \( \mathcal{M} \):

\( (\mathcal{M}_{mg}) \) \quad \forall x \in \mathcal{I} \exists C > 0 \exists y \in \mathcal{I} \forall j, k \in \mathbb{N} : M^x_{j+k} \leq C^{j+k} M^y_j M^x_k. \)

\( (\mathcal{M}_{L}) \) \quad \forall C > 0 \forall x \in \mathcal{I} \exists D > 0 \exists y \in \mathcal{I} \forall k \in \mathbb{N} : C^k M^y_k \leq DM^x_k. \)

Let \( \mathcal{M} = \{ M^x : x \in \mathcal{I} \} \) and \( \mathcal{N} = \{ N^y : x \in \mathcal{J} \} \) be\([\mathcal{M}]\) \([\mathcal{N}]\) define

\[ \mathcal{M}\{\leq\}\mathcal{N} : \forall x \in \mathcal{I} \exists y \in \mathcal{J} : M^x \leq N^y, \]

and \textit{equivalence of matrices},

\[ \mathcal{M}\{\approx\}\mathcal{N} : \mathcal{M}\{\leq\}\mathcal{N} \text{ and } \mathcal{N}\{\leq\}\mathcal{M}. \]

2.4. Weight matrices obtained from weight functions. We summarize some facts which are shown in \([26, \text{Section 5}] \) and will be needed.

(i) A central new idea was that to each normalized weight function \( \omega \) that has \([\omega_3] \) we can associate a \([\mathcal{M}_{sc}] \) weight matrix \( \Omega := \{ W^j = (W^j_l)_{j \in \mathbb{N}} : l > 0 \} \) by

\[ W^j_l := \exp \left( \frac{1}{l} \omega^*_N(lj) \right), \]

which moreover satisfies \([\mathcal{M}_{mg}] \) more precisely

\[ \forall \ l > 0 \forall j, k \in \mathbb{N} : \ W_{j+k}^j \leq W_{j}^{2j} W_{k}^{2k}. \]
where $f$ and put $\Lambda$.

Similarly as for the ultradifferentiable case, we now define ultraholomorphic classes associated with $\omega$.

A normalized weight function $\omega$ satisfies $(\equiv)$ if, and only if, some/each $W^l$ satisfies $(\equiv)$ and this amounts to the fact that $W^s \equiv W^s$ for each $l, s > 0$. Consequently, $(\equiv)$ is characterizing the situation when $\Omega$ is constant, i.e. all the weight sequences it consists of are equivalent to each other.

2.5. Classes of ultraholomorphic functions of Roumieu type. For the following definitions, notation and more details we refer to [31, Section 2]. Let $\mathcal{R}$ be the Riemann surface of the logarithm. We wish to work in general unbounded sectors in $\mathbb{R}$ with vertex at 0, but all our results will be unchanged under rotation, so we will only consider sectors bisected by direction 0: For $\gamma > 0$ we set

$$S_\gamma := \{ z \in \mathcal{R} : |\arg(z)| < \frac{\gamma \pi}{2} \},$$

i.e. the unbounded sector of opening $\gamma \pi$, bisected by direction 0.

Let $M$ be a weight sequence, $S \subseteq \mathcal{R}$ an (unbounded) sector and $h > 0$. We define

$$\mathcal{A}_{M,h}(S) := \{ f \in \mathcal{H}(S) : \| f \|_{M,h} := \sup_{z \in S, p \in \mathbb{N}} \frac{|f^{(p)}(z)|}{h^p M_p} < +\infty \}.$$

$(\mathcal{A}_{M,h}(S), \| \cdot \|_{M,h})$ is a Banach space and we put

$$\mathcal{A}(M)(S) := \bigcup_{h > 0} \mathcal{A}_{M,h}(S).$$

$\mathcal{A}(M)(S)$ is called the Denjoy-Carleman ultraholomorphic class (of Roumieu type) associated with $M$ in the sector $S$ (it is a $(LB)$ space). Analogously, we introduce the space of complex sequences

$$\Lambda_{M,h} := \{ a = (a_p)_p \in \mathbb{C}^\mathbb{N} : |a|_{M,h} := \sup_{p \in \mathbb{N}} \frac{|a_p|}{h^p M_p} < +\infty \}$$

and put $\Lambda(M) := \bigcup_{h > 0} \Lambda_{M,h}$. The (asymptotic) Borel map $B$ is given by

$$B : \mathcal{A}(M)(S) \to \Lambda(M), \quad f \mapsto (f^{(p)}(0))_{p \in \mathbb{N}},$$

where $f^{(p)}(0) := \lim_{z \in S, z \to 0} f^{(p)}(z)$.

Similarly as for the ultradifferentiable case, we now define ultraholomorphic classes associated with a normalized weight function $\omega$ satisfying $(\equiv)$. Given an unbounded sector $S$, and for every $l > 0$, we first define

$$\mathcal{A}_{\omega,l}(S) := \{ f \in \mathcal{H}(S) : \| f \|_{\omega,l} := \sup_{z \in S, p \in \mathbb{N}} \frac{|f^{(p)}(z)|}{(\omega,\omega)(l)^p} \exp(\varphi^*_\omega(l^p)) < +\infty \}.$$

Remark 2.3. As can be seen in [33, Lemma 5.1.3], if $\omega$ is a normalized weight function satisfying $(\omega_3)$ and $(\omega_4)$ then $\omega$ satisfies $(\omega_6)$ if, and only if, some/each $W^l$ satisfies $(\equiv)$ and this amounts to the fact that $W^s \equiv W^s$ for each $l, s > 0$. Consequently, $(\equiv)$ is characterizing the situation when $\Omega$ is constant, i.e. all the weight sequences it consists of are equivalent to each other.
(\mathcal{A}_{\omega,l}(S), \| \cdot \|_{\omega,l}) is a Banach space and we put
\[
\mathcal{A}_{\omega}(S) := \bigcup_{l>0} \mathcal{A}_{\omega,l}(S).
\]
\(\mathcal{A}_{\omega}(S)\) is called the Denjoy-Carleman ultraholomorphic class (of Roumieu type) associated with \(\omega\) in the sector \(S\) (it is a \((LB)\) space). Correspondingly, we introduce the space of complex sequences
\[
\Lambda_{\omega,l} := \{ a = (a_p)_p \in \mathbb{C}^\mathbb{N} : |a|_{\omega,l} := \sup_{p \in \mathbb{N}} \frac{|a_p|}{\exp(l \varphi_\omega(p))} < +\infty \}
\]
and put \(\Lambda_{\omega} := \bigcup_{l>0} \Lambda_{\omega,l}\). So in this case we get the Borel map \(B : \mathcal{A}_{\omega}(S) \rightarrow \Lambda_{\omega}\).

Finally, we recall that ultradifferentiable function classes \(\mathcal{E}_{(M)}\), of Roumieu type and defined by a weight matrix \(M\), were introduced in \([33]\), see also \([26, 4.2]\). Similarly, given a weight matrix \(M = \{ M^x : x \in \mathbb{R}^n_0 \}\) and a sector \(S\) we may define ultraholomorphic classes \(\mathcal{A}_{(M)}(S)\) of Roumieu type as
\[
\mathcal{A}_{(M)}(S) := \bigcup_{x \in \mathbb{R}^n_0} \mathcal{A}_{(M^x)}(S),
\]
and accordingly, \(\Lambda_{(M)} := \bigcup_{x \in \mathbb{R}^n_0} \Lambda_{(M^x)}\).

As said before in Subsection 2.2, if \(\omega\) is a normalized weight function with \(\omega_1\) and \(\omega_3\) the weight matrix \(\Omega := \{ W^l = (W^l_j)_{j \in \mathbb{N}} : l > 0 \}\), given by \(W^l_j := \exp(\frac{1}{l} \varphi_\omega(j))\) satisfies \((\mathcal{M}_{\omega^3})\) (see 2.3) and \((\mathcal{M}_{\omega^3})\) (see 2.3), and moreover
\[
\mathcal{A}_{(\omega)}(S) = \mathcal{A}_{(\Omega)}(S)
\]
holds as locally convex vector spaces (this equality is an easy consequence of the way the seminorms are defined in these spaces and of property \((\mathcal{M}_{\omega^3})\)). As one also has \(\Lambda_{(\omega)} = \Lambda_{(\Omega)}\), the Borel map \(B\) makes sense in these last classes, \(B : \mathcal{A}_{(\Omega)}(S) \rightarrow \Lambda_{(\Omega)}\).

In any of the considered ultraholomorphic classes, an element \(f\) is said to be \textit{flat} if \(f^{(p)}(0) = 0\) for every \(p \in \mathbb{N}\), that is, \(B(f)\) is the null sequence.

2.6. Functions \(\omega_M\) and \(h_M\). Let \(M \in \mathbb{R}^n_0\) \((M_0 = 1)\), then the \textit{associated function} \(\omega_M : [0, \infty) \rightarrow \mathbb{R} \cup \{ +\infty \}\) is defined by
\[
\omega_M(t) := \sup_{p \in \mathbb{N}} \log \left( \frac{t^p}{M^p} \right) \quad \text{for } t > 0, \quad \omega_M(0) := 0.
\]

For an abstract introduction of the associated function we refer to \([19]\) Chapitre 1, see also \([16]\) Definition 3.1. If \(\lim_{p \rightarrow \infty} (M^p)_+^{1/p} > 0\), then \(\omega_M(t) = 0\) for sufficiently small \(t\), since \(\log \left( \frac{t^p}{M^p} \right) < 0 \Leftrightarrow t < (M^p)^{1/p}\) holds for all \(p \in \mathbb{N}_0\).

A basic assumption is \(\lim_{p \rightarrow \infty} (M^p)_+^{1/p} = +\infty\), which implies that \(\omega_M(t) < +\infty\) for any \(t > 0\), and so \(\omega_M\) is a weight function. If moreover \(M\) is normalized, then \(\omega_M\) also is.

According to the definition given in 2.2, for any \(t, s > 0\) we get
\[
(\omega_M)^s(t) = \omega_M(t^s) = \sup_{p \in \mathbb{N}} \log \left( \frac{t^p}{M^p} \right) = \sup_{p \in \mathbb{N}} \left( \frac{t^p}{(M^p)^{1/s}} \right)^s = s \omega_{M^{1/s}}(t),
\]
where \(M^{1/s} := ((M^p)_+^{1/p})_{p \in \mathbb{N}}\).

We summarize some more well-known facts for this function:

**Lemma 2.4.** Let \(M \in \mathcal{LC}\)

(i) \(\omega_M\) is a normalized weight function satisfying \([\omega_1]\) and \([\omega_3]\)
(ii) $\lim_{p \to \infty} (m_p)^{1/p} = +\infty$ implies $[\omega_5]$ for $\omega_M$.

(iii) $M$ has $[\omega_4]$ if and only if $\omega_M$ has $[\omega_5]$.

(iv) If $M$ satisfies $[\gamma_1]$ then $\omega_M$ has $[\omega_{nq}]$.

So for any strongly regular weight sequence $M$ the weight function $\omega_M$ satisfies $[\omega_3]$ $\omega_4]$ and $[\omega_{nq}]$.

Proof. (i) See [10] Definition 3.1.

(ii) That $\lim (m_p)^{1/p} = +\infty$ implies $[\omega_5]$ for $\omega_M$ follows along the same lines as a similar argument in [4] Lemma 12 (iv) $\Rightarrow$ (v).

(iii) See [16] Proposition 3.6.

(iv) It follows from [16, Proposition 4.4].

$\square$

Lemma 2.5. Let $\omega$ be a normalized weight satisfying $[\omega_3]$ and $\Omega = \{W^x = (W^x_p)_{p \in \mathbb{N}} : x > 0\}$ be its associated weight matrix, where $W^x_p = \exp \left( \frac{1}{x} \varphi^*_\omega(xp) \right)$. Then, we get

$$\forall \ x > 0 \ \forall \ t \geq 0: \ x \omega_{W^x}(t) \leq \omega(t).$$

If $\omega$ satisfies moreover $[\omega_4]$, then $\omega_{W^x}$ for each $x > 0$, more precisely we get

$$\forall \ x > 0 \ \exists \ C_x > 0 \ \forall \ t \geq 0: \ x \omega_{W^x}(t) \leq \omega(t) + 2 \omega_{W^x}(t) + C_x.$$ 

Proof. For these estimates we recall [20] Lemma 5.7, respectively [33] Theorem 4.0.3, Lemma 5.1.3: Given the normalized weight $\omega$ satisfying $[\omega_3]$ and the sequence $W^1 = \{\exp (\varphi^*_\omega(xp))\}_{p \in \mathbb{N}}$, there exists some $c > 0$ such that for all $t \geq 0$ we get $\omega_{W^1}(t) \leq \omega(t) \leq 2 \omega_{W^1}(t) + c$, where the first inequality does not need $[\omega_4]$ while the second does. Now, for any $x > 0$ we will apply the previous statement to the weight $\tau_x(t) := \omega(t)/x$, which has the same properties assumed for $\omega$. We need to compute, for $p \in \mathbb{N}$, the value

$$\varphi^*_\tau_x (p) = \sup_{y \geq 0} \{py - \varphi^*_\tau_x (y)\} = \sup_{y \geq 0} \{py - \tau_x(e^y)\} = \sup_{y \geq 0} \left\{py - \frac{1}{x} \omega(e^y)\right\}$$

$$= \sup_{y \geq 0} \left\{py - \frac{1}{x} \varphi_\omega (y)\right\} = \frac{1}{x} \sup_{y \geq 0} \{(xp) - \varphi_\omega (y)\} = \frac{1}{x} \varphi^*_\omega(xp).$$

So, it turns out that, in the same way that $W^1$ was the sequence corresponding to $\omega$ in our previous statement, the sequence $W^x$ is the one corresponding to $\tau_x$, and so there exists some $c_x > 0$ such that for all $t \geq 0$ we get $\omega_{W^x}(t) \leq \tau_x(t) \leq 2 \omega_{W^x}(t) + c_x$, under the same conditions as before. The conclusion is immediate by the definition of $\tau_x$.

$\square$

Another important function will be introduced now: Let $M \in \mathbb{R}_0^\mathbb{N}$ ($M_0 = 1$) and put

$$h_M(t) := \inf_{k \in \mathbb{N}} M_k t^k.$$

The functions $h_M$ and $\omega_M$ are related by

$$h_M(t) = \exp (-\omega_M(1/t)), \quad t > 0,$$

since $\log(h_M(t)) = \inf_{k \in \mathbb{N}} \log (t^k M_k) = -\sup_{k \in \mathbb{N}} - \log (t^k M_k) = -\omega_M(1/t)$ (e.g. see also [7, p. 11]).

By definition we immediately get:

Lemma 2.6. Let $M, N \in \mathbb{R}_0^\mathbb{N}$ be given, then

(i) The function $h_M(t)$ is nondecreasing.
(ii) If $M$ is normalized, then $h_M(t) \leq 1$ for all $t > 0$; if moreover $M \in \mathcal{LC}$ then $h_M(t) = 1$ for all $t$ sufficiently large and $\lim_{t \to 0} h_M(t) = 0$.

(iii) $M \leq N$ implies $h_M \leq h_N$, more generally $M \leq N$ implies that $h_M(t) \leq h_N(\lambda t)$ holds for some $\lambda \geq 1$ and all $t > 0$.

From Lemma 2.5 and the equality (2.8) we immediately deduce the following result. Here, we write as before $\omega^t(t) = \omega(1/t)$ for a given weight function $\omega$.

**Lemma 2.7.** Let $\omega$ be a normalized weight function satisfying $\langle \omega_3 \rangle$ then we get

\[ \forall x > 0 \forall t \geq 0 : \exp(-\omega^t(t)) \leq (h_{W^t}(t))^x. \]

If moreover $\omega$ has $\langle \omega_4 \rangle$, then

\[ \forall x > 0 \exists C_x > 0 \forall t \geq 0 : \exp(-C_x(h_{W^t}(t))^{2x} \leq \exp(-\omega^t(t)). \]

3. Legendre conjugates. From weight matrices to weight functions

3.1. Legendre conjugates of a weight $\omega$. For any $M \in \mathcal{LC}$ with $\lim_{p \to \infty} (m_p)^{1/p} = +\infty$ there exists a connection between $\omega_m$ and a different type of a conjugate for $\omega_M$, as considered in [24, Definition 1.4] and [3], see Lemma 3.1 below for more details. This conjugate must not be mixed with $\varphi^\omega_m$ as considered in (2.1). We warn the reader that the terminology differs from one author to another: In [1] the conjugates are named after Legendre, in [24] after Young.

Let $\omega$ be a weight function, then for any $s \geq 0$ we define

\[ \omega^*(s) := \sup_{t \geq 0} \{ \omega(t) - st \}. \]

$\omega^*$ is the upper Legendre conjugate (or upper Legendre envelope) of $\omega$.

We summarize some basic properties, see also [24, Remark 1.5].

(i) By definition, $\omega^*(0) = +\infty$. If $\omega$ has in addition $\langle \omega_5 \rangle$ then $\omega^*(s) < +\infty$ for all $s > 0$.

Indeed, we have that for any $s > 0$ (however small) there exists some $C_s > 0$ (large enough) such that for all $t \geq 0$ we get $\omega(t) \leq st + C_s$, and so $\omega^*(s) \leq C_s$. In this case, the function $\omega^*: (0, +\infty) \to [0, +\infty)$ is nonincreasing, continuous and convex, and $\lim_{s \to 0} \omega^*(s) = +\infty$, $\lim_{s \to \infty} \omega^*(s) = 0$.

(ii) So, whenever the weight function $\omega$ has $\langle \omega_5 \rangle$ the function $\tilde{\omega}: [0, +\infty) \to [0, +\infty)$ given by

\[ \tilde{\omega}(t) := (\omega^*)^x(t) = \omega^* \left( \frac{1}{t} \right), \quad t > 0; \quad \tilde{\omega}(0) := 0, \]

is a (nonnormalized) weight function.

We introduce now a new conjugate. For any $h: (0, +\infty) \to [0, +\infty)$ which is nonincreasing and such that $\lim_{s \to 0} h(s) = +\infty$, we can define the so-called lower Legendre conjugate (or envelope) $h_*: [0, +\infty) \to [0, +\infty)$ of $h$ by

\[ h_* : t \geq 0. \]

$h_*$ is clearly nondecreasing, continuous and concave, and $\lim_{t \to \infty} h_*(t) = \infty$, see [16] (8), p. 156]. Moreover, if $\lim_{s \to \infty} h(s) = 0$ then $h_*(0) = 0$, and so $h_*$ is a weight function.

In our work this second conjugate will be mainly applied to the case $h(t) := \omega^t(t) = \omega(1/t)$, where $\omega$ is a weight function, so that $(\omega^*)_*$ is again a weight function; in particular, we will frequently find the case $h(t) = \omega_M^t(t) = \omega_M(1/t)$ for $M \in \mathbb{R}^+_0$ with $\lim_{p \to \infty} (m_p)^{1/p} = +\infty$. 

\[ \omega(x) := \sup_{t \geq 0} \{ \omega(t) - st \}. \]

\[ \omega^*(s) := \sup_{t \geq 0} \{ \omega(t) - st \}. \]

\[ \forall x > 0 \forall t \geq 0 : \exp(-\omega^t(t)) \leq (h_{W^t}(t))^x. \]

\[ \forall x > 0 \exists C_x > 0 \forall t \geq 0 : \exp(-C_x(h_{W^t}(t))^{2x} \leq \exp(-\omega^t(t)). \]
In [24] Proposition 1.6 it was shown that for any \( \omega : [0, +\infty) \to [0, +\infty) \) concave and nondecreasing we get
\[
\forall t > 0 : \quad \omega(t) = \inf_{s \geq 0} \{ \omega^*(s) + st \} = (\omega^*)_*(t).
\]

In case \( \omega \) is a weight function satisfying \([\omega_5]\) \((\omega^*)_* \) is a weight function and it is indeed the least concave majorant of \( \omega \) (in the sense that, if \( \tau : [0, +\infty) \to [0, +\infty) \) is concave and \( \omega \leq \tau \), then \((\omega^*)_* \leq \tau \), see [3].

We prove now several properties for \( \omega^* \) which will be needed below. For this we use \( 0^0 := 1 \) and recall the following consequence of Stirling’s formula:
\[
(3.1) \quad \forall n \in \mathbb{N} : \quad \left( \frac{n}{e} \right)^n \leq n! \leq n^n.
\]

Lemma 3.1. (i) Let \( \sigma \) and \( \tau \) be two weight functions with \([\omega_5]\) and suppose there exist \( A, B > 0 \) such that
\[
\forall t > 0 : \quad \tau(t) \leq A\sigma(t) + B.
\]

Then
\[
\forall s > 0 : \quad \tau^*(s) \leq A\sigma^*(\frac{s}{A}) + B.
\]

Consequently \( \sigma \) \( \tau \) implies
\[
\exists C \geq 1 \forall s > 0 : \quad -C + C^{-1}\sigma^*(Cs) \leq \tau^*(s) \leq C\sigma^*(\frac{s}{C}) + C.
\]

(ii) Let \( M \in \mathbb{R}_+^n \) such that \( \lim_{p \to \infty}(p^{1-b}m_p)^{1/p} = +\infty \) for some \( b > 0 \). Then for each \( 0 < a \leq b \) the mappings \( s \mapsto ((\omega_M)^a)^*(s) \) and \( s \mapsto \omega_{M/G^a}(1/s^a) \), with \( G^a_p := p^{1/a} \), are equivalent in the previous sense, more precisely
\[
(3.2) \quad \forall s > 0 : \quad ((\omega_M)^a)^*(s) \leq \omega_{M/G^a}(\frac{s^a}{a^a}) \leq ((\omega_M)^a)^*(\frac{s}{a}).
\]

Proof. (i) Let \( s > 0 \), then
\[
\tau^*(s) = \sup_{t \geq 0} \{ \tau(t) - st \} \leq \sup_{t \geq 0} \{ A\sigma(t) + B - st \} = A\sup_{t \geq 0} \{ \sigma(t) - (sA^{-1})t \} + B = A\sigma^*(s/A) + B,
\]
see also [24] Remark 1.7.

(ii) For any \( a \in (0, b] \) we have \( \lim_{p \to \infty}(M_p/(p^a))^{1/p} = \lim_{p \to \infty}(p^{1-b-a}m_p)^{1/p} = +\infty \) if, and only if, \( \lim_{p \to \infty}(M_p^a/p^a)^{1/p} = \lim_{p \to \infty}(p^{1-b-a}m_p)^{1/(ap)} = +\infty \), hence we may apply Lemma 3.3 (ii) to deduce that \( \omega_{M^a} \) satisfies \([\omega_5]\). Consequently, as indicated in the study of the properties of the upper Legendre conjugate, the function \((\omega_{M^a})^* \) is well-defined from \((0, \infty) \) to \((0, \infty) \), and by [24] coincides with \((a(\omega_M)^a)^* \).

We follow now the proof [9] Lemma 5.7.8, where only the case \( a = 1 \) was treated. Let \( s > 0 \), then
\[
(a^{-1}\omega_{M^a})^*(s) = (\omega_M^a)^*(s) := \sup_{t \geq 0} \{ \omega_M(t^a) - st \} = \sup_{t \geq 0} \left\{ \sup_{p \in \mathbb{N}} \log \left( \frac{t^ap}{M_p} \right) - st \right\}.
\]

For \( s > 0 \) and \( p \in \mathbb{N} \) fixed we consider \( f_{s, p} : (0, +\infty) \to \mathbb{R} \) defined by
\[
f_{s, p}(t) := \log \left( \frac{t^ap}{M_p} \right) - st = ap \log(t) - \log(M_p) - st, \quad p \in \mathbb{N}_{>0}; \quad f_{s, 0}(t) = -st.
\]
Hence, \( \sup_{t \geq 0} f_{s,t} = 0 \) for any \( s > 0 \). Let \( p \geq 1 \), then \( f_{s,p}(t) = ap \frac{1}{e^t} - s = 0 \iff t = \frac{ap}{s} \) and \( f_{s,p} \) attains its maximum at this point. Hence we get

\[
f_{s,p}(ap/s) = ap \log(ap/s) - \frac{ap}{s} s - \log(M_p) = \log \left( \frac{a^p e^{ap}}{e^{ap} M_p} \right) - \log(ap) = \log \left( \frac{a^p}{e^{ap} M_p} \right),
\]

which holds also for \( p = 0 \) by \( 0^0 = 1 \). Thus we have shown

\[
\forall s > 0 : \quad (\omega_M)^* (s) = \sup_{p \in \mathbb{N}} \log \left( \frac{a^p}{e^{ap} M_p} \right).
\]

The left hand side of (3.3) gives \( \frac{p^p}{(e^p)^p M_p} \leq \frac{p^p}{M_p} \) for all \( p \in \mathbb{N} \), i.e. the left hand side of (3.4). By the right hand side of (3.3) we get \( \log \left( \frac{p^p}{M_p} \right) \leq \log \left( \frac{p^p}{M_p} \right) \) for all \( p \in \mathbb{N} \) and so the right hand side of (3.2).

Combining the previous lemma with results from [26, Section 5] we get the following consequences, which have already appeared, in a weaker form, in [25].

**Corollary 3.2.** Let \( \omega \) be a normalized weight with \([\omega_2]([\omega_3]) \) and \([\omega_5] \) let \( \Omega = \{W^x = (W_p^x)_{p \in \mathbb{N}} : x > 0 \} \) be its associated weight matrix, and put \( w^x = (W_p^x/p!p!p!) \in \mathbb{N} \), \( x > 0 \). Then,

\[
\forall x > 0 \exists C_x \geq 1 \forall s > 0 : \quad x\omega_w \left( \frac{s}{x} \right) \leq \omega^* (s) \leq 2x\omega_w \left( \frac{2x}{s} \right) + C_x
\]

and

\[
\forall x > 0 \exists C_x \geq 1 \forall s > 0 : \quad x\omega_w \left( \frac{x}{es} \right) \leq \omega^* (s) \leq 2x\omega_w \left( \frac{2x}{s} \right) + C_x,
\]

or equivalently,

\[
\forall x > 0 \exists C_x \geq 1 \forall s > 0 : \quad h_w \left( \frac{es}{x} \right) \cdot x \geq \exp(-\omega^* (s)) \geq \exp(-C_x)h_w \left( \frac{s}{2x} \right)^{2x},
\]

where, for all the inequalities on the left to hold, it is not necessary to impose \([\omega_4]\).

**Proof.** To prove (3.3) we apply (2.9), the stability of \([\omega_3]\) under equivalence, and Lemma (3.1) ii). For (3.4) we depart from (3.3) and recall (see Subsection 2.4) that \([\omega_5]\) implies \( \lim_{p \to \infty} (w^p_x)^{1/p} = +\infty \) for every \( x > 0 \), so we may apply also (ii) in the previous result for \( a = 1 \). The last inequalities are just a re-writing of (3.4) thanks to the very definition (2.8).

**3.2. From weight matrices to weight functions defining the same ultraholomorphic classes.** We will now show that, starting with a good matrix \( \Omega \) associated to a weight function with some standard properties, we can describe the matrix space associated to \( \hat{\Omega} \), consisting of all sequences from \( \Omega \) and multiplying each of them by a “factorial term”, by a (single) Braun-Meise-Taylor weight function.

**Remark 3.3.** Let \( M, N \in \mathbb{LC} \) and let \( \omega_M \) have \([\omega_1]\). Then \( M \subset \mathbb{V} \) implies \( \omega_M \subset \omega_N \) as follows: First \( M \subset \mathbb{V} \) implies \( \omega_M (A^{-1} t) \leq \omega_N (t) \leq \omega_M (At) \) for all \( t \geq 0 \) and some \( A \geq 1 \), hence by iterating \([\omega_1]\) there exists some \( B \geq 1 \) such that for all \( t \) sufficiently large:

\[
B^{-1} \omega_M (t) \leq \omega_M (A^{-1} t) \leq \omega_N (t) \leq \omega_M (At) \leq B \omega_M (t).
\]
Moreover we recall (see Lemma 3.1(ii) with \( a = 1 \)) that for \( M \in \mathbb{R}_{>0}^{\mathbb{N}} \) with \( \lim_{p \to \infty} (m_p)^{1/p} = +\infty \) one has
\[
\forall \ s > 0 : \quad \omega_M^*(s) \leq \omega_M \left( \frac{1}{s} \right) \leq \omega_M^* \left( \frac{s}{r} \right).
\]

**Lemma 3.4.**

(i) Let \( M \in \mathbb{R}_{>0}^{\mathbb{N}} \) with \( \lim_{p \to \infty} (m_p)^{1/p} = +\infty \) be given and define the sequence
\[
N_p := \sup_{t > 0} \exp(\omega_m^*(t)),
\]
where \( \omega_m^*(t) = \omega_m(1/t) \). Then we get \( N \leq (m_p)_p \) (here \( (m_p)_p \) stands for the log-convex regularization of the sequence \( (m_p)_p \), see for example [16] (3.2)), and so \( N \) is log-convex and equivalent to a strongly log-convex sequence.

(ii) Let \( Q \in \mathbb{R}_{>0}^{\mathbb{N}} \) such that \( Q \succeq M \) and \( m \in \mathbb{LC} \). Then \( \omega_Q \) is equivalent to a concave function, more precisely we get
\[
\forall \ x \geq \mu_1 : \quad \omega_M(x) \leq (\omega_m^*)_*(x) \leq 1 + \omega_M(\varepsilon x),
\]
and since \( (\omega_m^*)_* \) is concave, we have \( \omega_M(\varepsilon x) \).

**Proof.**

(i) For all \( p \in \mathbb{N} \) we get
\[
\sup_{t > 0} \exp(\omega_m^*(t)) = \exp \left( \sup_{t > 0} \{ p \log(t) - \omega_m^*(t) \} \right)
= \exp \left( \sup_{t > 0} \{ p \log(t) - \inf_{s > 0} \{ \omega_m(1/s) + st \} \} \right)
= \exp \left( \sup_{t > 0} \{ p \log(t) - \omega_m(1/s) - st \} \right).
\]
Let \( p \in \mathbb{N} \) and \( s > 0 \) be fixed and put
\[
f_{p,s}(t) := p \log(t) - \omega_m(1/s) - st, \quad f_{0,s}(t) = -\omega_m(1/s) - st.
\]
Clearly, \( \sup_{t > 0} f_{0,s} = -\omega_m(1/s) \). For all \( p \geq 1 \) we get
\[
f_{p,s}^*(t) = \frac{t}{s} - s = 0 \Leftrightarrow t = \frac{s}{s},
\]
the point where \( f_{p,s} \) attains its maximum. Hence
\[
f_{p,s}(p/s) = p \log(p/s) - \omega_m(1/s) - p = \log \left( \frac{p^p}{(es)^p} \right) - \omega_m(1/s),
\]
which holds also for the case \( p = 0 \) by \( 0^0 := 1 \). Thus for any \( p \in \mathbb{N} \),
\[
\sup_{t > 0} \exp(\omega_m^*(t)) = \exp \left( \sup_{s > 0} \{ \log \left( \frac{p^p}{(es)^p} \right) - \omega_m(1/s) \} \right)
= \frac{p^p}{e^p} \sup_{s > 0} \frac{1}{sp \exp(\omega_m(1/s))} = \frac{p^p}{e^p} \sup_{s > 0} \frac{s^p}{\exp(\omega_m(s))} = \frac{p^p}{e^p} m_p^*,
\]
where in the last step we have applied [16] Proposition 3.2).

(ii) We follow and recall the arguments of [15] p. 233). On the one hand we get by (3.5) for any \( x \geq 0 \):
\[
(\omega_m^*)_*(x) = \inf_{y > 0} \{ \omega_m(1/y) + xy \} \geq \inf_{y > 0} \{ \omega_M^*(y) + xy \} = \inf_{y > 0} \{ \sup_{u > 0} (\omega_M(u) - uy) + xy \}
= \inf_{y > 0} \sup_{u > 0} (\omega_M(u) + y(x - u)) \geq \omega_M(x).
\]
For this estimate the log-convexity of $m$ was not used. On the other hand, first we get
\[
\exp(-\langle \omega^\epsilon_m \rangle_\ast(x)) = \exp\left(-\inf_{y > 0} \left\{ \omega_m(1/y) + xy \right\} \right) = \exp\left(\sup_{y > 0} \{-xy - \omega_m(1/y)\} \right) \\
= \sup_{y > 0} \{\exp(-xy)\exp(-\omega_m(1/y))\} = \sup_{y > 0} \{\exp(-xy)h_m(y)\}.
\]
For convenience, we write $\mu_0^\ast := 1$, $\mu_n^\ast := \mu_{n/n}$, $n \in \mathbb{N}$. Let now $x \in [\mu_n, \mu_{n+1})$, $n \geq 1$, and put $y_0 := \max\{\frac{n}{x}, \frac{1}{\mu_{n+1}}\}$. Hence $n \cdot \frac{1}{\mu_{n+1}} < \frac{n}{x} \leq n \cdot \frac{1}{\mu_n}$ and by the strong log-convexity $\frac{1}{\mu_{n+1}} \leq n \frac{1}{\mu_n} = \frac{1}{\mu_n}$. So $\frac{1}{\mu_{n+1}} \leq y_0 \leq \frac{1}{\mu_n}$ and we estimate as follows:
\[
\exp(-\langle \omega^\epsilon_m \rangle_\ast(x)) = \sup_{y > 0} \{\exp(-xy)h_m(y)\} \geq \exp(-xy_0)h_m(y_0) = m_0 y_0^\epsilon \exp(-xy_0),
\]
where the last equality holds by the choice of $y_0$ as explained above. Finally, we have to show that $e \exp(\omega_M(ex)) = e \sup_{x \in \mathbb{R}^n} \left\{ \frac{e^{x_1}}{M_1} \geq \frac{1}{m_0 y_0} \exp(xy_0) \right\}$. And this holds true since, on the one hand, $\frac{1}{y_0} \leq \frac{n}{\mu_n} \leq \frac{n}{\mu_{n+1}}$, and on the other hand, $\exp(x(\mu_1/x)) = \exp(n)$ and $\exp(x(\mu_{n+1}/x)) = \exp(n+1)$. Together proves $e \exp(xy_0) \leq \exp(n+1)$.

So far we have shown $(\omega_m^\ast)$ has $(\omega_1)$ by concavity, and so we deduce that $\omega_M \leq \omega_m^\ast$ and that also $\omega_M$ has $(\omega_1)$. Finally, since $Q \in \mathcal{M}$, Remark 3.3 yields $\omega_M \leq \omega_m^\ast$ and we are done.

Using this result we can prove the following Corollaries:

**Corollary 3.5.** Let $M \in \mathbb{R}_{>0}^N$ with $\lim_{p \to \infty} (m_p)^{1/p} = +\infty$. Then the functions $\omega_L$, $L_p := p!m_p^{lc}$, and $(\omega_m^\ast)$, are equivalent with respect to $\leq$.

**Proof.** Recall that, by the very definition of $(\omega_m^\ast)$, we have $(\omega_m^\ast) = (\omega^{lc, m^*})$, and so, by (ii) in Lemma 3.2 applied for $Q = M = L$, i.e. with $m^{lc}$ instead of $m$, we get $(\omega_m^\ast) \leq L$.

Until the end of this section, we assume that $\tau$ is a normalized weight function with $(\omega_1)$, $(\omega_2)$ and $(\omega_3)$.

Denote by $\mathcal{T} := (T^x : x > 0)$ the associated weight matrix, i.e. $T^x_p := \exp\left(\frac{p}{p!} \tau^\ast(x)\right)$, and write also $\widetilde{T} := \{\widetilde{T}^x : x > 0\}$, defined by $\widetilde{T}_p^x := p!T^x_p$ for each $x > 0$ and $p \in \mathbb{N}$.

**Lemma 3.6.** For all $x, y > 0$ we get $(\omega_T^\ast) \leq (\omega_{\widetilde{T}}^\ast)$.

**Proof.** Let $x, y > 0$ be arbitrary but fixed. By [23] Lemma 5.7] we obtain
\[
(\omega_T^\ast)_\ast(s) = \inf_{u > 0} \left\{ \omega_T^\ast \left( \frac{1}{u} \right) + us \right\} \leq \inf_{u > 0} \left\{ C \omega_T^\ast \left( \frac{1}{u} \right) + us \right\} + C
\]
\[
= C \inf_{u > 0} \left\{ \omega_T^\ast \left( \frac{1}{u} \right) + \frac{u}{C} \right\} + C = C(\omega_T^\ast)_\ast(s/C) + C.
\]
Taking into account that each $(\omega_T^\ast)_\ast$ has $(\omega_1)$ (by concavity), we have shown $(\omega_T^\ast) \leq (\omega_{\widetilde{T}}^\ast)$, for all $x, y > 0$.

Combining the previous results we have shown so far:
Corollary 3.7. The associated functions $\omega^*_{T_x}$ and $\omega^*_{T_y}$ are all equivalent with respect to $\sim$, more precisely $\omega^*_{T_x} \sim (\omega^*_{T_y})^*$, holds for all $x, y > 0$.

Proof. Let $x > 0$, arbitrary but fixed, then take $m = T^x(= m^x)$ in Corollary 3.5 to show that $\omega^*_{T_x} \sim (\omega^*_{T_y})^*$, what leads to the conclusion by Lemma 3.6. □

Finally we can prove the following:

Theorem 3.8. For the considered weight $\tau$, the following identities hold as locally convex vector spaces for all sector $S$ and for all $x > 0$:

$$A_{(T^x)}(S) = A_{(\omega^*_{T_x})}(S).$$

So, $A_{(T^x)}(S)$ coincides with the space $A_{(\omega)}(S)$ associated with a normalized weight $\omega$ satisfying \([\omega_1]\) and \([\omega_2]\).

Proof. We do not wish to include here the details, since there is no significant difference with those carefully presented in [34] for a similar result in the ultradifferentiable case. The main idea behind the proof of this statement is that the ultraholomorphic classes considered here, associated either to a weight function or to a weight matrix, are introduced in exactly the same way as in the ultradifferentiable case, what lets us apply similar arguments as those developed in [33, 26, 34] as long as only the structural properties of the spaces are concerned.

Note that $T$ satisfies $[M_{(mg^x)}]$ and $[M_{(L^x)}]$ since it is associated to the weight $\tau$ (see [26, Section 5]). Both properties are also true immediately for the weight matrix $T$, and clearly each $T^x \in L^x$. Finally, by Corollary 3.7, we have every ingredient to mimic the proof of [34, Corollary 3.17] in order to obtain the result. □

Remark 3.9. Observe that we also have $A_{(T^x)}(S) = A_{(\omega^*_{T_x})}(S)$ for any $x > 0$, but the weight function $(\omega^*_{T_x})^*$ is concave and so it cannot satisfy the normalization condition; moreover, property \([\omega_2]\) is also not clear for this weight.

4. Growth indices

4.1. The growth index $\gamma(M)$ introduced by V. Thilliez. We revisit the definition of the growth index $\gamma(M)$ introduced in [37, Section 1.3]. This is necessary if we pretend to explain the result about the mixed setting as a complement to the extension results by V. Thilliez.

Let $\gamma \in \mathbb{R}$ be given, then $M \in \mathbb{R}^N_{>0}$ satisfies $(P_\gamma)$ (see [37, Definition 1.3.1] where only $\gamma > 0$ was considered), if

\[
\exists \nu = (\nu_p)_p \exists a \geq 1 \forall p \in \mathbb{N}: \quad a^{-1} \nu_p \leq \nu_p \leq a \mu_p
\]

and such that

\[
p \mapsto \frac{\nu_p}{p^\gamma} \quad \text{is nondecreasing.}
\]

\((4.1)\) is precisely $M(\geq N)$ for $N = (N_p)_p$ given by $N_0 := 1$, $N_p = \prod_{j=1}^p \nu_j$, $p \in \mathbb{N}_{>0}$. $(4.2)$ is equivalent to the fact that $(\frac{N_p}{(p!)^\gamma})_{p \in \mathbb{N}}$ is log-convex. The growth index of $M$, introduced in [37, Definition 1.3.5], is defined by

$$\gamma(M) := \sup \{ \gamma \in \mathbb{R} : (P_\gamma) \text{ is satisfied} \}.$$
Let \( \omega, \sigma \) be weight functions, and consider the following lemma.

**Lemma 4.2.** Let \( \omega \) be a weight function. Then, \( \gamma(\omega) > 0 \) if, and only if, \( \omega \) has \( (\omega_1) \).

Note that, while \( (\omega_1) \) is a qualitative property of \( \omega \), the condition \( \gamma(\omega) > 0 \) is quantitative in the sense that the value of the index, as it will be shown in the next sections, provides an upper bound (except for the factor \( \pi \)) for the opening of the sectors in which extension results will be available for ultraholomorphic classes associated with \( \omega \).
For a thorough study of the index $\gamma(\omega)$, its relationship with different properties for $\omega$ and the link with Thilliez’s index $\gamma(M)$, we refer also to [11]. In this work we will need the following result, whose proof is included for completeness.

**Lemma 4.3.** $\omega$ satisfies $[\omega_{\text{sq}}]$ if and only if $\gamma(\omega) > 1$.

**Proof.** If $\gamma(\omega) > 1$, then $\omega$ has $[\omega_{\text{sq}}]$ since $(P_{\omega,1})$ holds true (see Proposition 2.2). On the other hand let $\omega$ be given with $[\omega_{\text{sq}}]$. Then, as already shown in [21, Corollary 1.4] there exists some $K > 1$, $0 < \alpha < 1$ and $m \geq 0$ such that $\frac{\omega(Kt)}{\omega(t)} \leq K^\alpha$ for all $t \geq K^m$. Take some $\beta$ such that $\alpha < \beta < 1$ and so $\limsup_{t \to \infty} \frac{\omega(Kt)}{\omega(t)} \leq K^\alpha < K^\beta$ is valid, what proves $(P_{\omega,\beta-1})$ for any $\alpha < \beta < 1$.

**Remark 4.4.** We also mention without proof (see [11]) that for a sequence $L \in LC$ one always has $\gamma(\omega_L) \geq \gamma(L)$. This fact will be useful for an extension result in a mixed setting that will be described in the last section of this paper.

**Remark 4.5.** In the situation described in Theorem 3.8 we know that $\omega \ll \omega^\tau$. Moreover, the inequalities (3.5) imply, due to the concavity of $(\omega^\tau)_*$, that $\omega^\tau \ll (\omega^\tau)_*$. Finally, since $\tau \ll \tau^*$ (see (3.7)), it is easy to deduce that $(\omega^\tau)_* \ll (\tau^*)_*$. Altogether, we have proved that $\omega \ll (\tau^*)_*$, and so $\gamma(\omega) = \gamma((\tau^*)_*)$. As it can be seen in [11], for any weight function $\tau$ we always have that $\gamma((\tau^*)_*) \geq 1 + \gamma(\tau)$, and we may conclude that $\gamma(\omega) \geq 1 + \gamma(\tau) > 1$. Then, by Lemma 4.3 we see that the weight function $\omega$ obtained in Theorem 3.8 also has $[\omega_{\text{sq}}]$. We may also mention that, in the particular case that $\tau^*$ is convex, it turns out that $\gamma(\omega) = \gamma((\tau^*)_*) = 1 + \gamma(\tau)$.

5. **Existence of sectorially flat functions**

5.1. **Construction of outer functions.** The aim of this paragraph is to obtain holomorphic functions in the right half-plane whose growth is accurately controlled by a given weight function. The next result transfers $[\omega_{\text{sq}}]$ for a weight function $\tau$ into a property for $\tau^t$, where $\tau^t(t) = \tau(1/t)$. Compare this with [37, Lemma 2.1.1].

**Lemma 5.1.** Let $\tau$ be a weight function. Then, one has $\gamma(\tau) > 1$ if, and only if,

$$\exists C \geq 1 \forall y > 0 : \int_0^1 -\tau^t(ty)dt \geq -C(\tau^t(y) + 1).$$

**Proof.** First, by Lemma 4.3 we have $\gamma(\tau) > 1$ if, and only if, the weight $\tau$ has $[\omega_{\text{sq}}]$. In this condition we change $y \mapsto y^{-1}$, $t \mapsto t^{-1}$ and it is then equivalent to

$$\exists C \geq 1 \forall y > 0 : \int_0^1 \tau \left( \frac{1}{ty} \right) dt \leq C \tau \left( \frac{1}{y} \right) + C$$

(observe that, by putting $s := t^{-1}$, we get $\int_1^\infty \tau(ty)^{-1} dt = \int_0^1 \frac{\tau((s^{-1})y)}{s^{-1}} ds = \int_0^1 \tau(s^{-1}y) ds$. Now we multiply (5.1) by $-1$ and recall that $\tau^t(t) = \tau(1/t)$.

Moreover, we get the analogous result to [37, Lemma 2.1.2].

**Lemma 5.2.** Let $\tau$ be given as in Lemma 5.1 then

$$\int_{-\infty}^{+\infty} -\tau^t(\lvert t \rvert) \frac{dt}{1 + t^2} > -\infty.$$
Proof. Since by assumption Proposition 2.2 and Corollary 1.4 can be applied to \( \tau \), there exists some \( 0 < \alpha < 1 \) and \( C \geq 1 \) such that \( \tau(t) \leq Ct^\alpha + C \) for all \( t > 0 \). Hence by definition and multiplying this by \(-1\) we get \( -\tau'(t) \geq -C(t^{-\alpha} + 1) \) for all \( t > 0 \), from where the conclusion easily follows. \( \square \)

In the next step we transfer Lemma 2.1.3 to the weight function case.

Lemma 5.3. Let \( \tau \) be a weight function with \( \gamma(\tau) > 1 \). Then for all \( a > 0 \) there exists a function \( F_a \) which is holomorphic on the right half-plane \( H_1 := \{ w \in \mathbb{C} : \Re(w) > 0 \} \) and constants \( A, B \geq 1 \) (large) depending only on \( \tau \) such that

\[
\forall w \in H_1 : \quad B^{-a} \exp(-2a\tau'(B^{-1}\Re(w))) \leq |F_a(w)| \leq \exp(-\frac{a}{2}\tau'(A|w|)).
\]

Proof. We are following the idea of the proof of Lemma 2.1.3. For \( w \in H_1 \) put

\[
F_a(w) := \exp\left( \frac{1}{\pi} \int_{-\infty}^{\infty} -\alpha \tau'(|t|) \frac{tw - 1}{it - w} dt \right);
\]

Lemma 5.2 implies immediately that \( F_a \) is a holomorphic function in \( H_1 \). Since \( F_a(w) = (F_1(w))^a \), we need only consider in the proof \( a = 1 \) and put for simplicity \( F := F_1 \).

For \( w \in H_1 \) write \( \log(|F(w)|) = \frac{1}{\pi} \int_{\mathbb{R}} -\tau'(|t|) \frac{u}{(t-v)^2 + u^2} dt = -\frac{1}{\pi} f \ast g_u(v), \)

where \( f(t) := \tau'(|t|), \; g_u(t) := u/(t^2 + u^2). \; f \) and \( g_u \) are symmetrically nonincreasing functions, hence the convolution too. This means that \( (f \ast g_u)(x) \leq (f \ast g_u)(y) \leq (f \ast g_u)(0) \) for \( |x| \geq |y| \geq 0 \).

Consequently, the minimum for \( w \mapsto \log(|F(w)|) \) is attained for \( v = 0 \), so on the positive real axis and we have for all \( w \in H_1 \):

\[
\log(|F(w)|) \geq \log(|F(u)|) = \log(|F(\Re(w))|), \quad \log(|F(u)|) = \frac{1}{\pi} \int_{\mathbb{R}} -\tau'(|t|) \frac{u}{t^2 + u^2} dt = -\frac{1}{\pi} f \ast g_u(0).
\]

First we concentrate on the left hand side in (5.2). Consider \( K > 0 \) (small) and get

\[
\pi \log(|F(u)|) = \int_{\mathbb{R}} -\tau'(|t|) \frac{u}{t^2 + u^2} dt = \int_{\{|t| \geq Ku\}} -\tau'(|t|) \frac{u}{t^2 + u^2} dt + \int_{\{|t| \leq Ku\}} -\tau'(|t|) \frac{u}{t^2 + u^2} dt.
\]

For the first integral we estimate by

\[
\int_{\{|t| \geq Ku\}} -\tau'(|t|) \frac{u}{t^2 + u^2} dt \geq -\tau'(Ku) \int_{\{|t| \geq Ku\}} \frac{u}{t^2 + u^2} dt = -\tau'(Ku)(\pi - 2 \arctan(K)),
\]

since \( t \mapsto -\tau'(t) \) is nondecreasing.

For the second integral we get

\[
\int_{\{|t| \leq Ku\}} -\tau'(|t|) \frac{u}{t^2 + u^2} dt = \int_{\{|s| \leq K\}} -\tau'(Ku|s|) \frac{K}{K^2s^2 + 1} ds = K \int_{\{|s| \leq 1\}} -\tau'(Ku|s|) ds,
\]

since \( -\tau'(Ku|s|) \leq 0 \) holds for any \( K, u > 0 \) and \( |s| \leq 1 \). Let \( C \geq 1 \) be the constant appearing in Lemma 5.4 then

\[
K \int_{\{|s| \leq 1\}} -\tau'(Ku|s|) ds \geq 2KC(-\tau'(Ku) - 1).
\]
Thus we get for any \( u > 0 \):
\[
\pi \log(\|F(u)\|) \geq (\pi - 2\arctan(K) + 2KC)(-\tau^t(Ku)) \geq 2KC \geq (\pi + 3(C - 1)K)(-\tau^t(Ku)) - 2KC
\]
\[
\geq \pi(1 + \frac{3}{\pi}(C - 1)K)(-\tau^t(Ku)) - 2KC,
\]
since for all \( K > 0 \) chosen sufficiently small enough but arbitrarily \( \pi - 2\arctan(K) + 2KC \) behaves like \( \pi + 2(C - 1)K + O(K^2) \leq \pi + 3(C - 1)K \). Equivalently we have
\[
\forall \ u > 0: \quad |F(u)| \geq \exp(-2KC)(\exp(-\tau^t(Ku)))^{1 + \frac{3(C - 1)K}{\pi}}.
\]
If \( K > 0 \) is chosen small enough to have \( 1 + \frac{2}{\pi}(C - 1)K \leq 2 \Leftrightarrow K \leq \frac{\pi}{3(C - 1)} \), then since \( \exp(-\tau^t(t)) \leq 1 \) for any \( t > 0 \) we get
\[
\exp(-2KC)(\exp(-\tau^t(Ku)))^{1 + \frac{3(C - 1)K}{\pi}} \geq \exp(-2KC)(\exp(-\tau^t(Ku)))^2
\]
\[
= \exp(-2KC)\exp(-2\tau^t(Ku)).
\]
So the left hand side of (5.2) is shown (\(-\tau^t \) is nondecreasing).

For the right hand side assume that \( K > 0 \) is chosen arbitrarily (large), then
\[
|F(w)| \leq \exp \left( \frac{1}{\pi} \int_{|t|:|t-v| \leq K_u} -\tau^t(|t|) \frac{u}{(t-v)^2 + u^2} dt + \frac{1}{\pi} \int_{|t|:|t-v| \geq K_u} -\tau^t(|t|) \frac{u}{(t-v)^2 + u^2} dt \right).
\]
We estimate the second integral by 0 (since the integrand is negative). For the first one, since we have \( |t| \leq |t-v| + |v| \leq K_u + |w| \leq K|w| + |w| = (1 + K)|w| \) and \(-\tau^t \) is nondecreasing, we get for any \( w \in H_1: \)
\[
|F(w)| \leq \exp \left( \frac{1}{\pi}(-\tau^t((1 + K)|w|)) \int_{|t|:|t-v| \leq K_u} -\tau^t(|t|) \frac{u}{(t-v)^2 + u^2} dt \right).
\]
Since the last integral is equal to \( 2\arctan(K) \) we summarize:
\[
\forall \ w \in H_1: \quad |F(w)| \leq (\exp(-\tau^t((1 + K)|w|)))^{2\arctan(K)/\pi} \leq (\exp(-\tau^t((1 + K)|w|)))^{1 - 2/(\pi K)}
\]
holds, where \( K > 0 \) is chosen sufficiently large to guarantee \( \frac{2\arctan(K)}{\pi} \geq 1 - \frac{2}{\pi K} \). If \( K \) is chosen large enough to have \( 1 - \frac{2}{\pi K} \geq \frac{1}{2} \Leftrightarrow K \geq \frac{\pi}{2} \), then we get
\[
\forall \ w \in H_1: \quad |F(w)| \leq (\exp(-\tau^t(((1 + K)|w|)))^{1/2}
\]
which concludes the proof. \( \square \)

5.2. Construction of sectorially flat functions. Given a sequence \( M \in \mathbb{C} \) it is easy to express flatness in the classes \( A_{(1)}(S) \) by means of the associated functions \( \omega_M \) or \( h_M \). Indeed, as in the classical Gevrey case, flat functions are characterized as those exponentially decreasing in a precise sense. The proof of the following result is a straightforward adaptation of the arguments in [MS Proposition 4].

Lemma 5.4. Let \( S \) be an unbounded sector.

(i) Let \( M \in \mathcal{L}C \) be given such that \( \lim_{p \to \infty} m_p^{1/p} = \infty \). Then,

\[ (i.1) \quad \text{If } f \in A_{(1)}(S) \text{ is flat}, \]
\[ (5.3) \quad \exists C > 0 \exists k > 0: \forall z \in S, \quad |f(z)| \leq \chi_m(k|z|) = C \exp(-\omega_m(1/(k|z|))). \]

(ii) Conversely, if \( f \) is a holomorphic function in \( S \) verifying (5.3), then for every unbounded and proper subsector \( T \) of \( S \) one has \( f \in A_{(1)}(T) \) and \( f \) is flat.
(ii) Let $\mathcal{M} = \{M^x : x \in \mathbb{R}_{>0}\}$ be a standard log-convex weight matrix with $\lim_{k \to \infty} (m_k^x)^{1/k} = +\infty$ for every $x > 0$. Then,

(ii.1) If $f \in \mathcal{A}_{(\mathcal{M})}(S)$ is flat,

\begin{equation}
\exists C > 0 \exists k > 0 \exists x > 0 : \forall z \in S, \ |f(z)| \leq C h_m^x(\lambda |z|) = C \exp(-\omega_m^x(1/(k|x|))).
\end{equation}

(ii.2) Conversely, if $f$ is a holomorphic function in $S$ verifying \((5.4)\), then for every unbounded and proper subsector $T$ of $S$ one has $f \in \mathcal{A}_{(\mathcal{M})}(T)$ and $f$ is flat.

(iii) Let $\omega$ be a normalized weight function with $[\omega_1]$ and $[\omega_3]$, and $\Omega = \{W^x = (W_j^x)_{j \in \mathbb{N}} : x > 0\}$ the associated weight matrix, for which (as indicated in Subsection \(2.2\)) we have $A_{(\omega)}(S) = A_{(\Omega)}(S)$. Suppose moreover that $\lim_{p \to \infty} (w^x_p)^{1/p} = +\infty$ for every $x > 0$. Then,

(iii.1) If $f \in A_{(\omega)}(S)$ is flat,

\begin{equation}
\exists C > 0 \exists x > 0 : \forall z \in S, \ |f(z)| \leq C h_m^x(\Omega |z|) = C \exp(-\omega_m^x(1/|z|)).
\end{equation}

(iii.2) Conversely, if $f$ is a holomorphic function in $S$ verifying \((5.5)\), then for every unbounded and proper subsector $T$ of $S$ one has $f \in \mathcal{A}_{(\omega)}(T)$ and $f$ is flat.

**Proof.** We only prove (i), since the rest of items may be obtained similarly. If $f \in \mathcal{A}_{(\mathcal{M})}(S)$, there exists $k > 0$ such that $f \in \mathcal{A}_{M,k}(S)$, and so for every $z \in S$ and $p \in \mathbb{N}$ we have

\begin{equation}
|f^{(p)}(z)| \leq \|f\|_{M,k} k^p M_p.
\end{equation}

Now, by Taylor’s formula we may write, for any $\lambda \in (0,1),

\begin{equation*}
f(z) - \sum_{j=0}^{p-1} \frac{f^{(j)}(\lambda z)}{j!} (z - \lambda z)^j = \frac{z^p}{(p-1)!} \int_{\lambda}^{1} (1 - t)^{p-1} f^{(p)}(tz) dt,
\end{equation*}

and taking limits as $\lambda \to 0$ we deduce, since $f$ is flat, that

\begin{equation*}
f(z) = \frac{z^p}{(p-1)!} \int_{0}^{1} (1 - t)^{p-1} f^{(p)}(tz) dt.
\end{equation*}

So, we may apply \((5.6)\) in order to see that for every $p \in \mathbb{N}$ we have

\[ |f^{(p)}(z)| \leq \frac{|z|^p}{p!} \sup_{w \in S} |f^{(p)}(w)| \leq \|f\|_{M,k} k^p |z|^p M_p, \]

what implies \((5.3)\) by definition of $h_m$ and \((2.8)\).

Conversely, suppose $f$ satisfies \((5.3)\). Given a proper and unbounded subsector $T$ of $S$, there exists $\varepsilon > 0$ such that for every $z \in T$, the disc $D(z, \varepsilon |z|)$ is contained in $S$, and by Cauchy’s formula we have, for every $p \in \mathbb{N}$,

\begin{equation*}
f^{(p)}(z) = \frac{p!}{2\pi i} \int_{|w-z|=\varepsilon |z|} \frac{f(w)}{(w-z)^{p+1}} dw, \quad p \in \mathbb{N}.
\end{equation*}

So, we easily estimate

\[ |f^{(p)}(z)| \leq \frac{p!}{\varepsilon |z|} \max_{|w-z|=\varepsilon |z|} |f(w)| \leq \frac{C p!}{\varepsilon |z|^p} h_m(k(1 + \varepsilon)|z|). \]

Since $h_m(t) = \inf_{n \in \mathbb{N}} m_n t^n$, on the one hand we have that

\[ |f^{(p)}(z)| \leq \frac{C p!}{\varepsilon |z|^p} m_p(k(1 + \varepsilon)|z|)^p = C \left( \frac{k(1 + \varepsilon)}{\varepsilon} \right)^p M_p, \]
from where $f \in \mathcal{A}_{M,k(1+\varepsilon)/\varepsilon}(T) \subset \mathcal{A}_M(T)$, and on the other hand we deduce

$$|f^{(p)}(z)| \leq \frac{Cp}{(\varepsilon|z|)^p}m_{p+1}(k(1+\varepsilon)|z|)^{p+1} = C\frac{(k(1+\varepsilon))^{p+1}M_{p+1}}{\varepsilon^{p+1}}|z|,$$

what immediately implies that $f^{(p)}(0) = \lim_{z \to 0} z^p f^{(p)}(z) = 0$ for every $p \in \mathbb{N}$, and $f$ is flat. □

**Remark 5.5.** The condition $\lim_{p \to \infty} m_{1/p} = \infty$ is not necessary for item (i) to hold. However, note that whenever $\lim_{p \to \infty} m_{1/p} < \infty$ the statement is trivial, since $h_m$ identically vanishes in an interval with 0 as its left-end point, and we immediately deduce that the only flat function in the class is the null function. Similar observations can be made for the other two items.

**Remark 5.6.** Suppose given a normalized weight function $\omega$ with $[\omega_1],[\omega_3]$ and $[\omega_5]$ and let $\Omega = \{W^x = (W^x_p)_{p \in \mathbb{N}} : x > 0\}$ be its associated weight matrix. According to the information in Lemma 5.3(iii), one may characterize flatness in the ultraholomorphic class $\mathcal{A}_{\{\omega\}}(S)$ (which coincides with $\mathcal{A}_{\{\Omega\}}(S)$) in terms of exponential decrease of the type $\exp(-\omega(x)(1/|z|))$ for some $x > 0$. If $\omega$ has moreover $[\omega_1],[\omega_3]$ and $[\omega_5]$ we see that this could be also expressed by exponential decrease of the type $\exp(-C\omega(x)(D|z|))$ for suitable $C,D > 0$.

Using the results from the previous sections the aim is now to transfer Theorem 2.3.1] to the weight function setting. Although the considered weights $\tau$ will satisfy $[\omega_1]$ we use the equivalent condition $\gamma(\tau) > 0$ (see Lemma 4.2), as this quantity will essentially indicate the opening of the sectors where our constructions will be valid.

**Theorem 5.7.** Let $\tau$ be a weight function with $\gamma(\tau) > 0$. Then for any $0 < \gamma < \gamma(\tau)$ there exist constants $K_1,K_2,K_3 > 0$ depending only on $\tau$ and $\gamma$ such that for all $a > 0$ there exists a function $G_a$ holomorphic in $S_\gamma$ and satisfying

$$\forall \xi \in S_\gamma : \quad K_1^{-a} \exp\left(-2a\tau^\ast(K_2|\xi|)\right) \leq |G_a(\xi)| \leq \exp\left(-\frac{a}{2}\tau^\ast(K_3|\xi|)\right).$$

Moreover, if $\tau$ is normalized and satisfies $[\omega_3]$ and $T = \{T^x = (T^x_p)_{p \in \mathbb{N}} : x > 0\}$ is its associated weight matrix, then $G_a$ is a flat function in $\mathcal{A}_{\{T\}}(S_\gamma)$, where $T$ is the standard log-convex weight matrix consisting of the sequences $\tilde{T}^x = (p!T^x_p)_{p \in \mathbb{N}}, x > 0$. Finally, if we also assume that $\tau$ satisfies $[\omega_3]$ then there exist $x > 0$ and $K_4 > 0$, both depending on $a$, such that

$$\forall \xi \in S_\gamma : |G_a(\xi)| \geq K_4 h_{T^x}(K_2|\xi|).$$

We remark that $[\omega_3]$ tells us that $G_a$ is indeed an optimal flat function, in the sense that its size is controlled by the functions $h_{T^x}$ not only from above, as needed for flatness, but also from below.

**Proof.** Let $a > 0$ be arbitrary. Take $s, \delta > 0$ such that $\gamma < \delta < \gamma(\tau)$, $s \delta < 1 < s\gamma(\tau)$. By (4.3) we get $s\gamma(\tau) > 1$ and $\gamma(\tau) > 1$, hence $\tau^\ast(t) = \tau(t^{1/s}) = \tau(t)^{1/s}(t)$ satisfies $[\omega_{\text{func}}]$. So we can use Lemma 5.3 for the weight $\tau^\ast$ instead of $\tau$ and we obtain a function $F_a$ satisfying $\mathcal{A}_{\{\omega_3\}}(S_\gamma)$ with $\tau^\ast$ replaced by $(\tau^\ast)^{1/s}$. Then put

$$G_a(\xi) = F_a(\xi^{1/s}) \quad \xi \in S_\delta.$$

Note that, as $s \delta < 1$, the ramification $\xi \mapsto \xi^s$ maps holomorphically $S_\delta$ into $S_{s\delta} \subseteq S_1 = H_1$, and so $G_a$ is well-defined.

We show that the restriction of $G_a$ to $S_\gamma \subseteq S_\delta$ satisfies the desired properties by proving that $5.7$ holds indeed on the whole $S_\delta$. 
First we consider the lower estimate. Let $\xi \in S_{\delta}$ be given, then $\Re(\xi^*) \geq \cos(s\delta\pi/2)|\xi|^s$ (since $s\delta\pi/2 < \pi/2$). If $B \geq 1$ denotes the constant coming from the left hand side in (5.2) applied for the weight $\tau^{1/s}$, then
\[
|G_a(\xi)| = |F_a(\xi^*)| \geq B^{-a} \exp(-2\alpha(\tau^*)^{1/s}(B^{-1} (\Re(\xi^*))))) \\
\geq B^{-a} \exp(-2\alpha(\tau^*)^{1/s}(B^{-1} \cos(s\delta\pi/2)|\xi|^s)) \\
= B^{-a} \exp(-2\alpha(\tau^*)^{1/s}((B_1|\xi|)^s)) = B^{-a} \exp(-2\alpha(\tau^*)^{1/s}(B_1|\xi|)) ,
\]
where we have put $B_1 := (B^{-1} \cos(s\delta\pi/2))^{1/s}$.

Now we consider the right hand side in (5.7) and proceed as before. Let $A$ be the constant coming from the right hand side of (5.2) applied to $\tau^{1/s}$, so
\[
|G_a(\xi)| = |F_a(\xi^*)| \leq \exp(-\frac{a}{2}((\tau^*)^{1/s}(A|\xi|)^s)) = \exp(-\frac{a}{2}((\tau^*)^{1/s}(A^{1/s}|\xi|)^s)) ,
\]
and (5.7) has been proved for every $\xi \in S_{\delta}$.

Assume now that $\tau$ satisfies also $\omega_3$. First put in the estimate above $A_1 := A^{1/s}$. By using (2.9) for any $y > 0$ we get
\[
\exp(-\frac{a}{2}((A^{1/s}|\xi|)^y) \leq h_{T^*(A_1|\xi|)}^{y/2}.
\]
Hence taking $y := 2\alpha^{-1}$ proves that
\[
\forall \xi \in S_{\delta} : |G_a(\xi)| \leq h_{T^{2/\alpha}}(A_1|\xi|),
\]
and it suffices to take into account Lemma 5.4.(ii.2) in order to deduce that $G_a$ belongs to $A_{(T^{-1}(S_{\gamma})}$ and it is flat.

Finally, if $\tau$ satisfies moreover $\omega_4$ we may apply (2.10) for any $x > 0$ and deduce that
\[
\exp(-2\alpha(\tau^*)^{1/s}(B_1|\xi|)) \geq \exp(-2\alpha C_x h_{T^*(B_1|\xi|)}^{4x\alpha}) .
\]

Now we take $x := 1/(4\alpha)$ and prove that
\[
\forall \xi \in S_{\delta} : |G_a(\xi)| \geq B^{-a} \exp(-2\alpha C_x h_{T^{1/(4\alpha)}}(B_1|\xi|) ,
\]
as desired.

\[\square\]

6. Right inverses for the asymptotic Borel map in ultraholomorphic classes in sectors

The aim of this section is to obtain an extension result in the ultraholomorphic classes considered. The existence of the flat functions $G_a$ obtained in Theorem 5.7 will be the main ingredient in the proof, which will follow the same technique as in previous works of A. Lastra, S. Malek and the second author [17, 18]. Although for this construction the weight function $\tau$ needs not be normalized, we are interested in working with the weight matrix associated with it, which will be standard log-convex if we ask for normalization and $\omega_3$ to hold. Moreover, since the condition $\gamma(\tau) > 0$ is also necessary and this amounts to $\omega_1$, we will have the warranty that the ultraholomorphic spaces associated to the weight function and its corresponding weight matrix coincide, see the comments preceding 2.6.
Note that any weight function may be substituted by a normalized equivalent one, and equivalence preserves the properties \([\omega_3]\) and \(\gamma(\tau) > 0\), so it is no restriction to ask for normalization from the very beginning.

The next lemma provides us with suitable kernels entering the formal and analytic, truncated Laplace-like transforms we will need in our main statement.

**Lemma 6.1.** Let \(\tau\) be a normalized weight function with \(\gamma(\tau) > 0\) which satisfies \([\omega_3]\), let \(T = \{T^x = (T^x_p)_{p \in \mathbb{N}} : x > 0\}\) be its associated weight matrix, let \(0 < \gamma < \gamma(\tau)\), and for \(a > 0\) let \(G_a\) be the function constructed in Theorem [7.4]. Let us define the function \(e_a : S_\gamma \to \mathbb{C}\) by

\[
e_a(z) := zG_a(1/z), \quad z \in S_\gamma.
\]

The function \(e_a\) enjoys the following properties:

(i) \(z^{-1}e_a(z)\) is uniformly integrable at the origin, it is to say, for any \(t_0 > 0\) we have

\[
\sup_{|\sigma| < \gamma \pi / 2} \int_0^{t_0} t^{-1} |e_a(te^{i\sigma})| dt < \infty.
\]

(ii) There exist constants \(K > 0\), independent from \(a\), and \(C > 0\), depending on \(a\), such that

\[
|e_a(z)| \leq Ch_{T^{4/a}} \left( K \frac{1}{|z|} \right), \quad z \in S_\gamma.
\]

(iii) For \(\xi \in \mathbb{R}\), \(\xi > 0\), the values of \(e_a(\xi)\) are positive real.

**Proof.** (i) Let \(t_0 > 0\) and \(\sigma \in \mathbb{R}\) with \(|\sigma| < \frac{\gamma \pi}{2}\). From \([5.7]\) we deduce that there exists \(K_3 > 0\) such that

\[
\int_0^{t_0} \frac{|e_a(te^{i\sigma})|}{t} dt \leq \int_0^{t_0} \exp\left(-\frac{a}{2} \tau^t(K_3/t)\right) dt \leq t_0.
\]

(ii) For the second part, we may apply \([5.9]\) and write

\[
|e_a(z)| = |z||G_a(1/z)| \leq |z|h_{T^{2/a}}(A_1/|z|)
\]

for every \(z \in S_\gamma\), where \(A_1\) does not depend on \(a\).

We recall that from \([2.2]\) we know that \(2\omega_{T^2}(t) \leq \omega_{T^2}(t)\) for every \(x > 0\) and \(t \geq 0\), and so \(h_{T^2}(t) \leq h_{T^{2}}(t)^2\). Hence, combining this fact with the very definition of \(h_{T^{2}}\), we get

\[
|e_a(z)| \leq |z|h_{T^{2/a}}(A_1/|z|)^2 \leq |z|A_1T_1^{4/a}h_{T^{4/a}}(A_1/|z|) < A_1T_1^{4/a}h_{T^{4/a}}(A_1/|z|),
\]

as desired.

(iii) Finally, if \(\xi > 0\) then \(e_a(\xi) = \xi G_a(1/\xi)\). From the integral expression for \(G_a\), it is immediate to check that the imaginary part of the integrand is an odd function, so the imaginary part of \(G_a(1/\xi)\) is 0, while the real part is positive. \(\square\)

**Definition 6.2.** We define the moment function associated to the function \(e_a\) (introduced in the previous Lemma) as

\[
m_a(\lambda) := \int_0^\infty t^{\lambda-1}e_a(t) dt = \int_0^\infty t^\lambda G_a(1/t) dt.
\]

From Lemma [6.4] and the definition of \(h_{T^x}\) we see that for every \(p \in \mathbb{N}\),

\[
|e_a(z)| \leq C \frac{K_{T^p}^{4/a}}{|z|^p}, \quad z \in S_\gamma.
\]
So, we easily deduce that the function $m_a$ is well defined and continuous in $\{\Re(\lambda) \geq 0\}$, and holomorphic in $\{\Re(\lambda) > 0\}$. Moreover, $m_a(\xi)$ is positive for every $\xi \geq 0$, and the sequence $(m_a(p))_{p \in \mathbb{N}}$ is called the sequence of moments of $e_a$. The next result is similar to Proposition 3.6 in [17]. The fact that such estimates could also be obtained in the present situation became clear thanks to the arguments by O. Blasco in [2].

**Proposition 6.3.** Let $\tau$ be a normalized weight function with $\gamma(\tau) > 0$ which satisfies $[\omega_2]$; let $\mathcal{T} = \{T^x = (T^x_p)_{p \in \mathbb{N}} : x > 0\}$ be its associated weight matrix, and for $0 < \gamma < \gamma(\tau)$ and $a > 0$ let $G_a, e_a, m_a$ be the functions previously constructed. Then, there exist constants $C_1, C_2 > 0$, both depending on $a$, such that for every $p \in \mathbb{N}$ one has

\[
(6.2) \quad C_1 \left( \frac{K_2}{2} \right)^p T_p^{1/(2a)} \leq m_a(p) \leq C_2 K_3^{p} T_p^{4/a},
\]

where $K_2$ and $K_3$ are the constants, not depending on $a$, appearing in Theorem 5.7.

**Proof.** Let $p \in \mathbb{N}_0$. By the second inequality in (5.7), we have for every $s > 0$ that

\[
m_a(p) = \int_0^{\infty} t^p G_a(1/t) dt \leq \int_0^{s} t^p dt + \int_s^{\infty} \frac{1}{t^2} \exp \left( (p + 2) \log(t) - \frac{a}{2} \tau \left( \frac{t}{K_3} \right) \right) dt.
\]

Now, observe that

\[
\sup_{t > s} \left( (p + 2) \log(t) - \frac{a}{2} \tau \left( \frac{t}{K_3} \right) \right) = (p + 2) \log(K_3) + \sup_{u > s/K_3} \left( (p + 2) \log(u) - \frac{a}{2} \tau(u) \right)
\]

\[
\leq (p + 2) \log(K_3) + \sup_{u > 0} \left( (p + 2) \log(u) - \frac{a}{2} \tau(u) \right)
\]

\[
= (p + 2) \log(K_3) + \frac{a}{2} \sup_{v \in \mathbb{R}} \left( \frac{2(p + 2)}{a} \log(v) - \tau(e^v) \right)
\]

\[
= (p + 2) \log(K_3) + \frac{a}{2} \varphi^*_\tau \left( \frac{2(p + 2)}{a} \right).
\]

Hence, we deduce that

\[
m_a(p) \leq \frac{s^{p+1}}{p+1} + K_3^{p+2} \exp \left( \frac{a}{2} \varphi^*_\tau \left( \frac{2(p + 2)}{a} \right) \right) \frac{1}{s}.
\]

Since this is valid for any $s > 0$, we compute the infimum of such bounds as $s$ runs in $(0, \infty)$, whose value is

\[
\frac{p+2}{p+1} K_3^{p+1} \exp \left( \frac{a}{2} \varphi^*_\tau \left( \frac{2(p + 2)}{a} \right) \right),
\]

and obtain that

\[
m_a(p) \leq 2 K_3^{p+1} \exp \left( \frac{a}{2} \varphi^*_\tau \left( \frac{2(p + 2)}{a} \right) \right) = 2 K_3^{p+1} T_{p+2}^{2/a} \leq (2 K_3 T_2^{4/a}) K_3^{p} T_p^{4/a},
\]

where we have made use of (2.3).

For the second part of the estimates we use the first inequality in (6.7) and the fact that $\tau$ is nondecreasing in order to write, for every $s > 0$,

\[
m_a(p) \geq \int_0^{s} t^p G_a(1/t) dt \geq K_1^{-a} \int_0^{s} t^p \exp \left( -2a \tau \left( \frac{t}{K_2} \right) \right) dt \geq K_1^{-a} s^{p+1} \exp \left( -2a \tau \left( \frac{s}{K_2} \right) \right).
\]
We compute now the supremum of such bounds and, in a similar way, we deduce that
\[ m_a(p) \geq \frac{K_1^{-a}}{p+1} \exp \sup_{s>0} ((p+1) \log(s) - 2a\tau(\frac{s}{K_2})) = \frac{K_1^{-a}}{p+1} K_2^{p+1} \exp(2a\varphi_2^*(\frac{p+1}{2a})) \]
\[ = \frac{K_1^{-a}}{p+1} K_2^{p+1} T_{p+1}^{1/(2a)} \geq (K_1^{-a} K_2) \left( \frac{K_2}{2} \right)^p T_{p+1}^{1/(2a)}, \]
as desired.

The proof of the incoming result rests on a constructive procedure which combines a formal Borel transform and a truncated Laplace transform, like the original one in the Gevrey case (see [40], [30, Theorem 4.1]). The main tool needed is a suitable kernel, namely the function \( e_a \) obtained in Lemma 6.1, in terms of which both aforementioned transforms are explicitly given. This generalizes the classical situation, where the role of \( e_a \) is played by the exponential function, whose moment function is precisely the Euler Gamma function.

**Theorem 6.4.** Let \( \tau \) be a normalized weight function with \( \gamma(\tau) > 0 \) which satisfies \( [\omega_3] \), let \( 0 < \gamma < \gamma(\tau) \), let \( T = \{ T^x = (T^x_p)_{p \in \mathbb{N}} : x > 0 \} \) be its associated weight matrix, and consider the weight matrix \( \widehat{T} = \{ \hat{T}^x : x > 0 \} \) where \( \hat{T}^x = (p!T^x_p)_{p \in \mathbb{N}} \). Then, there exists a constant \( k_0 > 0 \) such that for every \( x > 0 \) and every \( h > 0 \), one can construct a linear and continuous map
\[ \lambda \in \Lambda_{\hat{T}^x,h} \mapsto f_\lambda \in \mathcal{A}_{\hat{T}^x,k_0h}(S_\gamma) \]
such that for every \( \lambda \) one has \( B(f_{\lambda}) = \lambda \).

In particular, the Borel map \( B : \mathcal{A}_{\{ \hat{T} \}}(S_\gamma) \to \Lambda_{\{ \hat{T} \}} \) is surjective.

**Proof.** Fix \( \delta > 0 \) such that \( \gamma < \delta < \gamma(\tau) \). Given \( \lambda = (\lambda_p)_{p \in \mathbb{N}} \in \Lambda_{\hat{T}^x,h} \), we have
\[ |\lambda_p| \leq ||\lambda||_{\hat{T}^x,h} h^p p! T^x_p, \quad p \in \mathbb{N}. \]

We choose \( a = 1/(2x) \), and consider the function \( G_a \), defined in \( S_\delta \), obtained in Theorem 5.7 for such value of \( a \), and the corresponding functions \( e_a \) and \( m_a \) previously defined. Next, we consider the formal power series
\[ \hat{f}_\lambda := \sum_{p=0}^{\infty} \frac{\lambda_p}{p!} z^p \]
and its formal (Borel-like) transform
\[ \mathcal{B}_a \hat{f}_\lambda := \sum_{p=0}^{\infty} \frac{\lambda_p}{p! m_a(p)} z^p. \]

By the choice of \( a \), (6.3) and the first part of the inequalities in (6.2), we deduce that
\[ \left| \frac{\lambda_p}{p! m_a(p)} \right| \leq \frac{||\lambda||_{\hat{T}^x,h} h^p p! T^x_p}{C_1(K_2/2)^p p! T^x_p} = \frac{||\lambda||_{\hat{T}^x,h} (2h)^p}{C_1 K_2^p}, \]
and so the series \( \mathcal{B}_a \hat{f}_\lambda \) converges in the disc of center 0 and radius \( K_2/(2h) \) (not depending on \( \lambda \)), where it defines a holomorphic function \( g_\lambda \). We set \( R_0 := K_2/(4h) \), and define
\[ f_\lambda(z) := \int_0^{R_0} e_a \left( \frac{u}{z} \right) g_\lambda(u) \frac{du}{u}, \quad z \in S_\delta. \]

By virtue of Leibniz’s theorem on analyticity of parametric integrals, \( f_\lambda \) is holomorphic in \( S_\delta \).
Our next aim is to obtain suitable estimates for the difference between \( f \) and the partial sums of the series \( \tilde{f}_\lambda \).

Let \( N \in \mathbb{N}_0 \) and \( z \in S_\delta \). We have

\[
\begin{align*}
f_\lambda(z) - \sum_{p=0}^{N-1} \lambda_p \frac{z^p}{p!} &= f_\lambda(z) - \sum_{p=0}^{N-1} \frac{\lambda_p}{m_a(p)} z^p \\
&= \int_0^{R_0} e_a \left( \frac{u}{z} \right) \sum_{p=0}^{\infty} \frac{\lambda_p}{m_a(p)} \frac{u^p}{p!} \frac{du}{u} - \sum_{p=0}^{N-1} \frac{\lambda_p}{m_a(p)} \int_0^\infty u^{p-1} e_a(u) \frac{du}{p!}.
\end{align*}
\]

In the second integral we make the change of variable \( v = zu \), which results in a rotation of the line of integration. By the estimate (6.1), one may use Cauchy’s residue theorem in order to obtain that

\[
z^p \int_0^{\infty} u^{p-1} e_a(u) \frac{du}{u} = \int_0^{\infty} v^p e_a \left( \frac{v}{z} \right) \frac{dv}{v},
\]

which allows us to write the preceding difference as

\[
\int_0^{R_0} e_a \left( \frac{u}{z} \right) \sum_{p=0}^{\infty} \frac{\lambda_p}{m_a(p)} \frac{u^p}{p!} \frac{du}{u} - \sum_{p=0}^{N-1} \frac{\lambda_p}{m_a(p)} \int_0^{\infty} u^{p} e_a \left( \frac{u}{z} \right) \frac{du}{u}.
\]

Then, we have

\[
(6.5) \quad \left| f_\lambda(z) - \sum_{p=0}^{N-1} \frac{\lambda_p z^p}{p!} \right| \leq |f_1(z)| + |f_2(z)|,
\]

where

\[
f_1(z) = \int_0^{R_0} e_a \left( \frac{u}{z} \right) \sum_{p=0}^{\infty} \frac{\lambda_p}{m_a(p)} \frac{u^p}{p!} \frac{du}{u}, \quad f_2(z) = \int_0^{\infty} e_a \left( \frac{u}{z} \right) \sum_{p=0}^{N-1} \frac{\lambda_p}{m_a(p)} \frac{u^p}{p!} \frac{du}{u}.
\]

From (6.4) we deduce that

\[
|f_1(z)| \leq \frac{\|f\|_{\mathcal{T}_x,h}}{C_1} \int_0^{R_0} e_a \left( \frac{u}{z} \right) \sum_{p=0}^{\infty} \frac{(2h/u)^p}{K_2^p} \frac{du}{u} = \frac{\|\lambda\|_{\mathcal{T}_x,h}}{C_1} \left( \frac{2h}{K_2} \right)^N \int_0^{R_0} e_a \left( \frac{u}{z} \right) \frac{u^N}{1 - \frac{2hu}{K_2}} \frac{du}{u}
\]

\[
(6.6) \quad \leq 2\frac{\|\lambda\|_{\mathcal{T}_x,h}}{C_1} \left( \frac{2h}{K_2} \right)^N \int_0^{R_0} e_a \left( \frac{u}{z} \right) u^{N-1} \frac{du}{u},
\]

where in the last step we have used that \( 0 < u < R_0 = K_2/(4h) \) we have \( 1 - 2hu/K_2 > 1/2 \). In order to estimate \( f_2(z) \), observe that for \( u \geq R_0 \) and \( 0 \leq p \leq N - 1 \) we always have \( u^p \leq R_0^p u^N / R_0^N \), and so, using again (6.4) and the value of \( R_0 \), we may write

\[
\sum_{p=0}^{N-1} \frac{\lambda_p u^p}{p! m_a(p)} \leq \frac{\|\lambda\|_{\mathcal{T}_x,h}}{C_1} \sum_{p=0}^{N-1} \frac{(2h/K_2)^p}{R_0^p} \frac{R_0^p}{R_0^N} \leq 2\frac{\|\lambda\|_{\mathcal{T}_x,h}}{C_1} \left( \frac{4h}{K_2} \right)^N u^N.
\]

Then, we deduce that

\[
(6.7) \quad |f_2(z)| \leq 2\frac{\|\lambda\|_{\mathcal{T}_x,h}}{C_1} \left( \frac{4h}{K_2} \right)^N \int_0^{\infty} e_a \left( \frac{u}{z} \right) \frac{u^{N-1} du}{u}.
\]
In order to conclude, it suffices then to obtain estimates for \( \int_0^\infty |e_a(u/z)|u^{N-1}du \). For this, note first that, by the estimates in (5.7),
\[
\int_0^\infty \left| e_a \left( \frac{u}{z} \right) \right| u^{N-1}du = \int_0^\infty \frac{u}{|z|} \left| G_a \left( \frac{z}{u} \right) \right| u^{N-1}du \leq \int_0^\infty u^N \exp \left( -\frac{a}{2} \tau \left( \frac{u}{K_3|z|} \right) \right) du = |z|^N \int_0^\infty t^N \exp \left( -\frac{a}{2} \tau \left( \frac{t}{K_3} \right) \right) dt.
\]
Now, we can follow the first part of the proof of Proposition 6.3 to obtain that
\[
\int_0^\infty \left| e_a \left( \frac{u}{z} \right) \right| u^{N-1}du \leq C_2K_3^{-N}\tau^4/|z|^N = C_2K_3^{-N}\tau^8/|z|^N.
\]
Gathering (6.5), (6.6), (6.7) and (6.8), we get
\[
\left| f_\lambda(z) - \sum_{p=0}^{N-1} \lambda_p z^p \right| \leq \frac{2C_2\|\lambda\|_{\mathfrak{f}_{\tau, h}}(4hK_3)^N}{C_1} T^{8\tau/|z|^N}.
\]
A straightforward application of Cauchy’s integral formula for the derivatives (as in the proof of Lemma 5.1) shows that there exists a constant \( r \), depending only on \( \gamma \) and \( \delta \), such that whenever \( z \) is restricted to belong to \( S_\gamma \), one has that for every \( p \in \mathbb{N} \),
\[
|f^{(p)}(z)| \leq \frac{2C_2\|\lambda\|_{\mathfrak{f}_{\tau, h}}(4hK_3)^N}{C_1} p!T^{8\tau/|z|^N}.
\]
So, putting \( k_0 := \frac{4K_3}{K_2} \) (independent of \( x \) and \( h \)), we see that \( f_\lambda \in \mathcal{A}_{\mathfrak{f}_{\tau, h}, k_0} (S_\gamma) \) and \( \|f_\lambda\|_{\mathfrak{f}_{\tau, h}, k_0} \leq \frac{2C_2\|\lambda\|_{\mathfrak{f}_{\tau, h}}(4hK_3)^N}{C_1} T^{8\tau/|z|^N} \). Since the map sending \( \lambda \) to \( f_\lambda \) is clearly linear, this last inequality implies that the map is also continuous from \( \Lambda_{\mathfrak{f}_{\tau, h}} \) into \( \mathcal{A}_{\mathfrak{f}_{\tau, h}, k_0} (S_\gamma) \). Finally, from (6.9) one may easily deduce that \( B(f_\lambda) = \lambda \), and we conclude.

Remark 6.5. Indeed, the estimates in (6.9) show precisely that the function \( f_\lambda \) admits the series \( \hat{f}_\lambda \) as its uniform asymptotic expansion in the sector \( S_\delta \), with constraints given mainly in terms of the sequence \( T^{8\tau/|z|^N} \). The link between the classes of functions admitting such an expansion and the ultraholomorphic classes studied in this paper is extremely strong, as it can be seen in [31].

We may infer also the existence of extension operators in the classes associated to the weight functions corresponding to the weight matrices \( \hat{T} \).

Corollary 6.6. Let \( \tau \) be a normalized weight function with \( \gamma(\tau) > 0 \) which satisfies (\( \omega_3 \)) and (\( \omega_4 \)), let \( \gamma \), \( \mathcal{T} \) and \( \hat{T} \) be as in the previous Theorem, and let \( \omega \) be the weight function given in Theorem 7.8 in such a way that \( \mathcal{A}_{\hat{T}}(S_\gamma) = \mathcal{A}_{\omega}(S_\gamma) \). Then, for every \( l > 0 \) there exists \( l_1 > 0 \) such that there exists a linear and continuous map
\[
\lambda \in \Lambda_{\omega,l} \mapsto f_\lambda \in \mathcal{A}_{\omega,l_1}(S_\gamma)
\]
such that for every \( \lambda \) one has \( B(f_\lambda) = \lambda \).

Proof. Let \( \Omega := \{W_x : x > 0\} \) be the weight matrix associated with the weight function \( \omega \), i.e. \( W_p := \exp \{a\varphi(x)p\} \), and \( \hat{T} := \{\hat{T}_p : x > 0\} \), where \( \hat{T}_p := p!T^p \) for each \( x > 0 \) and \( p \in \mathbb{N} \). We may apply (2.3) in order to deduce that \( \mathcal{A}_{\hat{T}}(S_\gamma) = \mathcal{A}_{\Omega}(S_\gamma) \). It turns out that, independently and by similar arguments, related only to the way the classes are defined, the same equality will hold for the corresponding ultradifferentiable spaces, introduced in [33, Chapter 7] (see also [26].
Now, Theorem 4.6 in [29] states that this equality in the ultradifferentiable case amounts to the equivalence of the corresponding weight matrices, in the sense that

\begin{align}
(6.10) & \forall x > 0 \exists y > 0 \exists C > 0 : \forall p \in \mathbb{N} \quad W_p^x \leq C^p p! T_p^x, \\
(6.11) & \forall y > 0 \exists x > 0 \exists D > 0 : \forall p \in \mathbb{N} \quad p! T_p^y \leq D^p W_p^x.
\end{align}

We fix \( l > 0 \). By (6.10), there exist \( x > 0 \) and \( C_1 > 0 \) such that for every \( p \in \mathbb{N} \) one has \( W_p^l \leq C_1^p p! T_p^l \). So, given \( \lambda \in \Lambda_{\omega,l} \), for every \( p \in \mathbb{N} \) we have

\[ |\lambda_p| \leq \|\lambda\|_{\omega,l} W_p^l \leq \|\lambda\|_{\omega,l} C_1^p p! T_p^l, \]

what implies that \( \lambda \in \Lambda_{\hat{T}^x, C_1} \) and \( \|\lambda\|_{\hat{T}^x, C_1} \leq \|\lambda\|_{\omega,l} \). Now, consider the function \( f_{\lambda} \) given by the previous theorem, which belongs to \( \mathcal{A}_{\hat{T}^x, k_0 C_1} (S_\gamma) \), depends on \( \lambda \) in a linear continuous way (so, there exists \( A > 0 \) with \( \|f_{\lambda}\|_{\hat{T}^x, k_0 C_1} \leq A \|\lambda\|_{\hat{T}^x, C_1} \)), and is such that \( B(f_{\lambda}) = \lambda \). By (6.11) there exists \( l_0 > 0 \) and \( C_2 > 0 \) (independent of \( \lambda \)) such that for every \( p \in \mathbb{N} \), \( \hat{T}_p^x \leq C_2^p W_p^{l_0} \), and by property (\( M_{(L)} \)) for \( \Omega \), there exist \( l_1 > 0 \) and \( D > 0 \) such that \( (k_0 C_1 C_2)^p W_p^{l_1} \leq D W_p^{l_1} \). Then, we obtain that for every \( p \in \mathbb{N} \) and every \( z \in S_\gamma \),

\[ |f^{(p)}(z)| \leq \|f\|_{\hat{T}^x, k_0 C_1} (k_0 C_1)^p \hat{T}_p^x \leq \|f\|_{\hat{T}^x, k_0 C_1} (k_0 C_1 C_2)^p W_p^{l_1} \leq D \|\lambda\|_{\hat{T}^x, C_1} W_p^{l_1} \leq AD \|\lambda\|_{\omega,l} W_p^{l_1} . \]

This means that \( f \in \mathcal{A}_{\omega,l} (S_\gamma) \) and, moreover, \( |f_{\lambda}|_{\omega,l} \leq AD \|\lambda\|_{\omega,l} \), so that the map is linear and continuous between the corresponding spaces.

\[ \square \]

6.1. Application to a mixed setting. As commented in the introduction, the known extension results for Denjoy-Carleman ultraholomorphic classes of Roumieu type in unbounded sectors by V. Thilliez [37] or J. Schmets and M. Valdivia [35] impose growth conditions on the weight sequence defining the classes, namely moderate growth in the first case, and (\( \beta_2 \)) condition (see (6.12)) in the second one. The aim in this last paragraph is to indicate how our previous results may be used in order to obtain extension results in a mixed setting under minimal assumptions on the sequence. We will discuss two situations:

(a) Let us consider a weight sequence \( \hat{M} \) which is \( [\omega_c] \) and has \( [\gamma_1] \). As a consequence of the results by H.-J. Petzsche (see [29]), \( \hat{M} \) may be substituted by a strongly equivalent sequence \( \hat{L} \) which is \( [\omega_c] \) and also has \( [\gamma_1] \). We write \( \hat{L} = (n! L_n)_{n \in \mathbb{N}} \), in such a way that

\[ L := (L_n)_{n \in \mathbb{N}} \in [\omega_c] \text{ and } \gamma(L) = \gamma(\hat{L}) - 1 = \gamma(\hat{M}) - 1 > 0 \]

(see Subsection 4.2). Consider now the associated weight function for \( L \), \( \tau := \omega_L \), which is a normalized weight function with \( [\omega_3] \) and satisfies \( \gamma(\tau) \geq \gamma(L) > 0 \) (see Remark 4.4). Let \( \mathcal{T} = \{ T^x = (T_p^x)_{p \in \mathbb{N}} : x > 0 \} \) be its associated weight matrix, and consider the weight matrix \( \hat{\mathcal{T}} = \{ T^x : x > 0 \} \) where \( T^x = (p! T_p^x)_{p \in \mathbb{N}} \). It turns out that \( T^1 = L \), and so \( \hat{T}^1 = \hat{L} \).

Hence, by Theorem 4.6 for any \( 0 < \gamma < \gamma(\tau) \) there exists \( k_0 > 0 \) such that for every \( h > 0 \), one can construct a linear and continuous map

\[ \lambda \in \Lambda_{\hat{T}^x, h} \equiv \Lambda_{\hat{M}, h} \mapsto f_{\lambda} \in \mathcal{A}_{\hat{T}^x, k_0 h} (S_\gamma) \]

such that for every \( \lambda \) one has \( B(f_{\lambda}) = \lambda \).

In particular, the Borel map \( \hat{B} : \mathcal{A}_{\hat{T}^x} (S_\gamma) \to \Lambda_{\hat{M}} \) is surjective.
(b) Suppose now that \( \hat{M} \) is \([\text{slo}]\) and \( m \in \mathcal{L} \mathcal{C} \) but \( \hat{M} \) does not have \([\gamma_1]\). According to the results in Subsection 2.1.1 one has \( \gamma(\hat{M}) = 1 \) and \( \gamma(m) = 0 \). Nevertheless, we know that for the weight function \( \tau := \omega_m \) we have \( \gamma(\tau) \geq \gamma(m) = 0 \), and it could perfectly be the case that \( \gamma(\tau) > 0 \) (an example of this situation is presented in [11]). So, it would be again possible to apply the previous procedure and obtain an extension operator in a mixed setting.

Remark 6.7. It is interesting to note that, in case the previously considered weight sequence \( \hat{M} \) has \([\text{le}]\) \([\gamma_1]\) and \([\text{mg}]\) we recover exactly the extension result of V. Thilliez [37, Theorem 3.2.1]. To see this, first note that \([\text{mg}]\) is stable under (weak or strong) equivalence of sequences, so \( \hat{L} \) will also have \([\text{mg}]\) and the same will hold for the sequence \( L \), as indicated in Subsection 2.1. So, by Lemma 2.3(iii) the weight function \( \tau \) has \([\omega_0]\) and Remark 2.3 implies that the matrix \( T \), and consequently also the matrix \( \hat{T} \), is constant, in the sense that all the weight sequences appearing in it are equivalent to each other. This means then that \( \hat{M} \) is equivalent to \( \hat{T}^8 \), and so the previous extension operator can be seen as

\[
\lambda \in \Lambda_{\hat{M}, h} \mapsto f_\lambda \in \mathcal{A}_{\hat{M}, k_1 h}(S_\gamma)
\]

for some suitable \( k_1 > 0 \). This is precisely the form of the extensions provided in Theorem 3.2.1 in [37].

The next example shows that there do exist sequences for which previously known extension results by V. Thilliez or J. Schmets and M. Valdivia cannot be applied.

Example 6.8. We first recall that condition \([\gamma_1]\) for a log-convex weight sequence \( M \) is equivalent to the following condition (see [23, Proposition 1.1]):

\[
(\beta_1) \iff \exists k \in \mathbb{N}_{>1} : \liminf_{p \to \infty} \frac{\mu_{kp}}{\mu_p} > k.
\]

Also in [23] the following condition was introduced:

\[
(\beta_2) \iff \forall \varepsilon > 0 \exists k \in \mathbb{N}_{>1} : \limsup_{p \to \infty} \left( \frac{M_{kp}}{M_p} \right)^{\frac{1}{p(k-1)}} \frac{1}{\mu_{kp}} \leq \varepsilon.
\]

In [35, Lemma 2.2 (b)] it was pointed out that, by Stirling’s formula, \( M \) has \([\beta_2]\) if, and only if, \( m \) has \([\beta_2]\).

We show now that there exist sequences \( M \in \mathbb{R}^N_{>0} \) which satisfy \( m \in \mathcal{L} \mathcal{C} \) and such that:

(i) \([\beta_2]\) and \([\beta_1]\) are violated.

(ii) \([\beta_2]\) and \([\text{mg}]\) are violated, but nevertheless \([\beta_1]\) holds.

(i) We define \( m := (m_p)_p \) by putting \( m_p := q^f(p) \), where \( q \geq \exp(1) \) and \( f : [0, +\infty) \to [0, +\infty) \) is a convex function with \( f(0) = 0 \) and \( \lim_{p \to \infty} f(p) = +\infty \) defined as follows:

Let \( (a_j)_{j \geq 1} \) be an increasing sequence in \( \mathbb{N}_{>0} \) with \( a_{j+1} \geq a_j \cdot j \) for all \( j \in \mathbb{N}_{>0} \). Furthermore denote by \( G^s = (G^s_p)_p \) the Gevrey-sequence, i.e. \( G^s_p = p^s \), \( s > 1 \). Consider now the set of points

\[
\mathcal{P} := \{(a_j, j \log(a_j!)) \}.
\]

For \( j \geq 1 \) let \( L_j \) be the line connecting the points \( (a_j, j \log(a_j!)) \) and \( (a_{j+1}, (j + 1) \log(a_{j+1}!)) \) with slope \( l_j := \frac{(j+1) \log(a_{j+1}!) - j \log(a_j!)}{a_{j+1} - a_j} \). For \( j = 0 \) let \( L_0 \) be the line connecting \((0,0)\) with the point \((a_1, \log(a_1!))\).

By the log-convexity of \( G^1 \), the points on the line \( L_0 \) lie above each point on \( \{(p, \log(p!)) : 0 < p < a_1\} \). By choosing \((a_j)\) increasing fast enough we can achieve that \((l_j)_j\) is increasing:
For this note that $l_j \geq \tilde{l}_j$, where $\tilde{l}_j$ is the slope of the straight line $L_j$ connecting the points $(a_j, j \log(a_j))$ and $(a_{j+1}, j \log(a_{j+1}))$. By convexity and the properties of the Gevrey-sequences $\tilde{l}_j$ is tending to infinity as $a_{j+1} \to \infty$, and so one recursively choose the $a_j$ in such a way that $l_{j+1} \geq \tilde{l}_{j+1} \geq l_j$.

We put $f(p)$ equal to the height of the segment $L_j$ at the point $p$ for all $p \in \mathbb{N}$ with $a_j \leq p \leq a_{j+1}$. Since $(l_j)_{j}$ is increasing, $m$ is log-convex (and so $\tilde{M}$ is strongly log-convex). Moreover by construction and convexity $f(p) \geq j \log(p!)$ for all $p \geq a_j$, which proves $m_p \geq G_l^p$ for all $p \geq a_j$ (and arbitrary $j \in \mathbb{N}_{>0}$). Since

$$\forall h > 0 \exists C \geq 1 \forall p \in \mathbb{N} : G^s_p \leq C h^p G^p_{\tilde{l}+1},$$

we get that

$$\forall h > 0 \exists C \geq 1 \forall p \in \mathbb{N} : G^s_p \leq C h^p m_p$$

for any $s > 1$. So the sequence $m$ is “beyond all Gevrey sequences”, and this fact excludes $\text{(mg)}$ (see [20]) for $m$ and, equivalently, for $M$.

Let us see that condition $\text{(β_2)}$ does not hold: The expression in this condition gives $q^{S(k,p)}$ with $S(k,p) := \frac{1}{p(k-1)}(f(kp) - f(p)) - (f(kp) - f(kp - 1))$. So, in fact $S(k,p)$ is measuring the difference of two different slopes of $\{(p, f(p)) : p \in \mathbb{N}\}$. For any $k \in \mathbb{N}$ and for all $j \geq k$ we get $ka_j \leq ja_j \leq a_{j+1}$, so $S(k,a_j) = 0$, what implies that

$$\limsup_{p \to \infty} q^{S(k,p)} \geq \limsup_{j \to \infty} q^{S(k,a_j)} = q^0 = 1,$$

what excludes $\text{(β_2)}$. Analogously we see that $\text{(β_1)}$ does not hold either: for any $p$ such that $a_j \leq p - 1 < p \leq a_{j+1}$ we have that $m_p / m_{p-1} = e^{h_j}$, and this implies that for any given $k \in \mathbb{N}$, and whenever $j \geq k$, we have $ka_j \leq ja_j \leq a_{j+1}$ and

$$\frac{\mu_{ka_j}}{\mu_{a_j}} = \frac{ka_j m_{ka_j}}{a_j m_{a_j - 1}} \leq k,$$

whence $\liminf_{p \to \infty} \frac{\mu_p}{\mu_p} = k$. So, $\gamma(m) = 0$ follows in this case.

It is worthy to comment that another example in this situation is the sequence $\tilde{M}$ mentioned in the previous item (b) before Remark 6.7, which is included in $\text{(slc)}$. $\tilde{M}$ is $\text{(slc)}$ and does not have $\text{(γ_1)}$ so that it does not have $\text{(β_2)}$ either (see [25] p. 223). Moreover, $\tilde{M}$ does not have $\text{(mg)}$.

(ii) However, from the previously constructed sequence $M$ (or the sequence $\tilde{M}$ in (b)) it is possible to get a sequence with $\text{(β_1)}$ and without $\text{(β_2)}$ and $\text{(mg)}$.

For this we point out the following: Let $\tilde{M}$ be log-convex, then $\liminf_{p \to \infty} \frac{\mu_p}{\mu_p} \geq 1$ holds. Then, the sequence $P := (p!^2 M_p)_p$ always satisfies $\text{(β_1)}$. Note $\pi_p = p^2 \mu_p$, hence $\liminf_{p \to \infty} \frac{\pi_p}{\pi_p} = k^2 \cdot \liminf_{p \to \infty} \frac{\mu_p}{\mu_p} \geq k^2 > k$. On the other hand, if $M$ is the sequence in (i), $M$ does not satisfy $\text{(β_2)}$ either $\text{(mg)}$, and the same is true for $P$ since, as already commented, these two properties are stable under multiplication by the factorials.

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