Identification of exponential trend models with fractional white noise

D V Ivanov¹, N V Chertykovtseva², A A Terekhova (Zharkova)³ and E A Andreeva³

¹Samara National Research University, Moskovskoe Shosse 34A, Samara, Russia, 443062
²Samara State University of transport, Svobody str. 2B, Samara, Russia, 443066
³Moscow State University of technologies and management, Zemlyanoj Val str. 73, Moscow, Russia, 109004

e-mail: dvi85@list.ru

Abstract. The paper suggests algorithms for identifying parameters of exponential trend models in the presence of fractional white noise. The paper considers three types of models that are solutions of a homogeneous linear differential equation of the second order. Identification of the solution of a differential equation makes it possible to increase accuracy by taking into account a priori information about the nature of the roots of the differential equation and initial conditions. However, identification of the solution is fraught with difficulties due to nonlinearity in the parameters of the obtained solutions. Two-step algorithms are proposed, allowing to determine the estimates of the parameters of the considered trend models. Test examples showed high accuracy of the estimates obtained using the developed algorithms.

1. Introduction

Long-memory models are recently finding increasing use in economic applications. Long memory, or long-term memory, is a property that describes the high-order correlation structure of time series [1,2]. If the series has a long memory, even observations that are widely spaced in time exhibit a relationship. One common long-memory model is the ARFIMA (autoregressive fractionally integrated moving average) model. A special case of the ARFIMA model is fractional white noise. References [3,4] address estimating the parameters of dynamical systems in the presence of fractional white noise. This paper considers algorithms for identifying exponential models with fractional white noise.

2. Problem statement

This paper addresses the problem of identifying trends as solutions to second-order linear homogeneous differential equations in the presence of fractional white noise. Estimating the parameters of solutions to differential equations is difficult because the solutions are nonlinear functions of the parameters. This paper deals with developing identification algorithms that allow the parameter estimates of trend models to be determined.

Consider a second-order linear homogeneous differential equation with constant coefficients:

\[ z'' + pz' + qz = 0, \]  

(1)
where \( z \) is the sought-for function, \( p \) and \( q \) are numbers.

Corresponding to equation (1) is the characteristic equation
\[
k^2 + pk + q = 0.
\]

Let us denote the roots of equation (2) by \( k_1 \) and \( k_2 \).

Depending on the nature of the roots, the solution to the differential equation can be described with three different functions:

1. If the roots of the characteristic equation are real and different, \( k_1 \neq k_2 \) and \( k_1, k_2 \in \mathbb{R} \), then the general solution to the linear homogeneous differential equation with constant coefficients (1) can be written as
\[
z = C_1 e^{k_1 t} + C_2 e^{k_2 t}, \quad C_1, C_2 \in \mathbb{R}.
\]

2. If the roots of the characteristic equation are real and \( k_1 = k_2 = k \in \mathbb{R} \), then the general solution to the linear homogeneous differential equation with constant coefficients (1) is the function
\[
z = C_1 e^{kt} + C_2 t e^{kt}, \quad C_1, C_2 \in \mathbb{R}.
\]

3. If the roots of the characteristic equation are conjugate complex, \( k_1 = b + iw \), \( k_2 = b - iw \), \( k_1, k_2 \in \mathbb{R} \), then the general solution to the linear homogeneous differential equation with constant coefficients (1) can be written as
\[
z = e^{bt} (C_1 \cos wt + C_2 \sin wt), \quad C_1, C_2 \in \mathbb{R}.
\]

In economic applications, the process usually has a stochastic component alongside a trend component. The stochastic component often fails to satisfy the Gauss–Markov conditions. This paper considers a generalization of white noise—fractional white noise, or \( 1/f \) noise.

Reference [6] shows that the fractional derivative is closely connected with the \( 1/f \) noise spectrum. A discrete analog of \( 1/f \) noise can be modeled by using a fractional-order difference. The process observed can be written as
\[
y_i = z_i + \Delta^\alpha \xi_i,
\]
where
\[
\Delta^\alpha \xi_i = \sum_{j=0}^{\infty} (-1)^j \left(\begin{array}{c}\alpha \\ j\end{array}\right) \xi_{i-j}, \quad \left(\begin{array}{c}\alpha \\ j\end{array}\right) = (-1)^{j+1} \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}, \quad -1/2 < \alpha < 1/2, \quad \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} dt.
\]

This paper deals with the identification of the trend parameters for the trend models (3), (4), and (5) using the discrete values of \( y_i \).

3. Identification algorithm

We will present two-step algorithms to estimate parameters for the trend models under study.

The first step consists in passing from the static trend model with nonlinear parameters to a dynamical model with linear parameters (autoregression). The parameters are identified based on the assumption that the model is a sum of the trend and fractional white noise. This makes it impossible to use the known methods for estimating autoregression parameters. The parameters of the dynamical model are estimable with the estimation algorithm proposed here, which is based on the minimization of a generalized Rayleigh ratio.

The second step consists in estimating the trend parameters on the basis of the estimates for autoregression parameters.

3.1. The case of real unequal roots. Let us write the solution to the difference equation (1) for discrete values:
\[
y_i = A_1 \exp(-b_1 \cdot T_i) + A_2 \exp(-b_2 \cdot T_i) + \Delta^\alpha \xi_i,
\]
where \( T_i = i \cdot \Delta t \) and \( \Delta t \) is a sampling interval.

This paper proposes a two-step algorithm that allows the parameters of equation (7) to be identified:

Step 1. Identifying the nonlinear parameters (\( b_1 \) and \( b_2 \)) by passing from equation (7) to a linear difference equation.
An efficient identification technique for short-range samples is expressing equation (1) as a linear difference equation based on Z-transform tools. Since expression (1) contains a constant coefficient, we will write a difference equation in relation to first-order differences for $k > 0$:

$$z_k = c_1 z_{k-1} + c_2 z_{k-2},$$

$$y_i = z_i + \Delta^n \xi_i,$$  \hspace{1cm} (8)

where $c_1 = \exp(-b_1 \cdot \Delta t) + \exp(-b_2 \cdot \Delta t)$ and $c_2 = \exp(-b_1 \cdot \Delta t) \cdot \exp(-b_2 \cdot \Delta t)$.

The forecast error for the model (8) can be written as

$$\varepsilon_k = \Delta^n \xi_k - c_1 \Delta^n \xi_{k-1} + c_2 \Delta^n \xi_{k-2}$$  \hspace{1cm} (9)

The use of the classical least-squares technique does not allow valid parameter estimates to be obtained for equation (4) because of the autocorrelation in the forecast error $\varepsilon_k$.

Let us find the expectation for the squared error:

$$E(\varepsilon_i^2) = \sigma^2 + \varepsilon_i^2 \cdot c \cdot h_{(1)}(1) \cdot h_{(2)}(2) \cdot \frac{1}{N} \sum_{j=0}^{N-j} \left( \frac{\alpha}{\sigma} \right)^j N - j.$$  \hspace{1cm} (10)

Already in the first identification step, the conditions to referring the time series under analysis to the model of the sum of two exponents (7) will be the inequality system

$$\begin{align*}
0 < c_1 < 2, \\
0 < c_2 < 0.25 \cdot c_1^2.
\end{align*}$$  \hspace{1cm} (11)

Conditions (11) can be considered an implicit regularization of expression (10).

Estimates for the coefficients of the vector $b = (b_1 \ b_2)^T$ are obtainable from

$$\hat{b}_1 = -\frac{\ln(\hat{\varepsilon}_i) - \sqrt{\hat{\varepsilon}_i^2 - 4\hat{\varepsilon}_2^2} + \ln(2)}{\Delta t},$$

$$\hat{b}_2 = -\frac{\ln(\hat{\varepsilon}_i) + \sqrt{\hat{\varepsilon}_i^2 - 4\hat{\varepsilon}_2^2} + \ln(2)}{\Delta t}.$$  \hspace{1cm} (12)

**Step 2.** Identifying the linear parameters in equation (7).

The vector of linear parameters is obtainable with the classical least-squares technique:

$$\hat{A} = (\Phi^T \Phi)^{-1} \Phi^T \Phi,$$  \hspace{1cm} (13)

where $\hat{A} = (\hat{\alpha}_1 \ \hat{\alpha}_2)^T$, $\Phi = \begin{pmatrix} \exp(-b_1 \cdot T) & \exp(-b_2 \cdot T) & \cdots & \exp(-b_1 \cdot T_n) & \exp(-b_2 \cdot T_n) \end{pmatrix}$.

It is noteworthy that the least-squares technique does not provide optimal accuracy where autocorrelated noise is present. The best unbiased estimates for these parameters are generalized least-squares estimates (GLS), or Markov estimates.

GLS estimate are obtainable as

$$\hat{A} = (\Phi^T D^{-1} \Phi)^{-1} \Phi^T D^{-1} \Phi,$$  \hspace{1cm} (14)

where $D = E \begin{pmatrix} \Delta^n \xi_{1} \ & \Delta^n \xi_{2}^T \\
\vdots & \vdots \\
\Delta^n \xi_{N} & \Delta^n \xi_{N}^T \end{pmatrix}$, $D_{mn} = E(\Delta^n \xi_m \Delta^n \xi_n) = \sigma_\xi^2 \frac{1}{\min(m,n)} \sum_{j=\min(m,n)}^{\min(m,n)} \left( \frac{\alpha}{\max(0, j-m-n)} \right)^j \frac{\max(m,n) - j - \max(m,n)}{\max(m,n)}$.
Generally, the use of the generalized least-squares technique is much more complicated because growing at every step, the general covariance matrix needs to be inverted. Reference [9] shows that procedures for finding GLSEs can be simplified to a certain degree for stationary processes. Procedures for Markov processes can be even simpler [10]. Reference [11] reports results that allow the accuracy of GLSEs to be assessed in comparison with the least-squares technique. It is shown that sometimes the use of the generalized least-squares technique improves accuracy by up to 10%.

3.2. The case of real unequal roots. Let us write the solution to the difference equation (1) for discrete values with fractional white noise:

\[ y_i = (A_i + A_i T_i) \exp(-h_i \cdot T_i) + \Delta^\alpha \xi_i, \]  

(15)

**Step 1.** Applying a Z-transform to equation (14), we obtain the equation system (8), where \( c_i = 2 \exp(-h_i \cdot \Delta t) \). Coefficient estimates can be found from relation (10).

The conditions for applying the model of the exponent times the linear argument form, that is, model (15) are the following relations

\[ \begin{cases} 0 < c_i < 2, \\ c_2 = 0.25 \cdot c_1^2. \end{cases} \]

The parameter \( h_1 \) of the model (15) is obtainable from the equation

\[ \hat{h}_1 = -\frac{\ln(\hat{c}_1/2)}{\Delta t}, \]  

(16)

**Step 2.** The vector of linear parameters can be found from equation (13) or (14), where \( \Phi \) is obtained from

\[ \Phi = \begin{bmatrix} \exp(-h_1 \cdot T_1) & T_1 \exp(-h_2 \cdot T_2) \\ \vdots & \vdots \\ \exp(-h_1 \cdot T_N) & T_N \exp(-h_2 \cdot T_N) \end{bmatrix}. \]

3.3. The case of real unequal roots. For the case of conjugate complex roots, the discrete values of the solution to equation (1) with fractional white noise can be written as

\[ y_i = A \exp(-h_i T_i) \cos(\omega T_i + \varphi) + \Delta^\alpha \xi_i. \]  

(17)

**Step 1.** The parameters are estimated similarly to the real-root cases. The autoregression coefficients depend on the parameters of equation (17) as follows:

\[ c_1 = 2 \exp(-h_1 \cdot \Delta t) \cos(\omega \cdot \Delta t), \quad c_1 = 2 \exp(-h_1 \cdot \Delta t). \]

For the autoregression coefficients, the condition \( 0.25c_1 < c_2^2 < 1 \) must be satisfied. Estimates for the parameters \( \hat{h}_1 \) and \( \hat{w} \) are obtainable from

\[ \hat{h}_1 = -\frac{\ln(\hat{c}_1/2)}{2 \Delta t}, \quad \hat{w} = \frac{1}{\Delta t} \arccos \left( \frac{c_1}{2 \sqrt{c_2}} \right). \]

**Step 2.** The vector of linear parameters can be found from equation (13) or (14), where \( \Phi \) is obtained from

\[ \Phi = \begin{bmatrix} \exp(-h_1 \cdot T_1) \sin(\omega \cdot T_1) & \exp(-h_1 \cdot T_1) \cos(\omega \cdot T_1) \\ \vdots & \vdots \\ \exp(-h_1 \cdot T_N) \sin(\omega \cdot T_N) & \exp(-h_1 \cdot T_N) \sin(\omega \cdot T_N) \end{bmatrix}, \]

then

\[ A = \sqrt{A_1^2 + A_2^2}, \quad \varphi = \arctg \left( \frac{A_1}{A_2} \right). \]
4. Test Example
The proposed identification algorithms (the nonlinear least-squares technique) were run in MATLAB and compared with the algorithms that estimate autoregression parameters in the first step of the least-squares technique [11].

We used the determination coefficient

\[ R^2 = 1 - \frac{\sum_{i=1}^{N} (y_i - \hat{y}_i)^2}{\sum_{i=1}^{N} (y_i - E[y_i])^2} \]

and the relative mean-square error of parameter estimation

\[ \delta_c = \frac{\sqrt{\|\hat{c} - c\|^2}}{\|c\|} \times 100\% \]

as quality indicators for the model.

The number of observations is \( N = 40 \), the sampling interval is \( \Delta t = 0.2 \).

4.1. The case of real unequal roots
The trend model is described by the equation

\[ y_i = 4 \exp(-0.5 \cdot T_i) + 2 \exp(-10 \cdot T_i) + \Delta^{0.3} \xi_i, \]  

Table 1 lists the results of identification of the model (18).

| True       | 1.4536 | 0.4966 | 0.5  | 3    | 0.983 |
| LS         | 0.5744 | -0.243 | 74.8 | 0.76 | 6.30  | 0.938 |
| GRQ        | 1.4083 | 0.4517 | 0.5  | 0.448| 3.52  | 0.987 |

4.2 The case of real roots
The trend model is described by the equation:

\[ y_i = (4T_i - 4) \exp(-0.5 \cdot T_i) + \Delta^{0.3} \xi_i, \]  

Table 2 lists the results of identification of the model (19).

| True       | 1.8097 | -0.8187 | 0.5  | 0.995 |
| LS         | 1.6566 | -0.6800 | 10.4 | 0.559 |
| GRQ        | 1.8119 | -0.8206 | 0.15 | 0.494 | 0.996 |

Figure 1. The results of the identification of the model (18).
4.3 The case of complex conjugate roots

The trend model is described by the equation:

\[ y_t = 4 \exp(-0.5T_i) \cos(2T_i + 0.5) + \Delta^{0.3} \xi, \]

Table 3 lists the results of identification of the model (20).

|       | \( c_1 \) | \( c_2 \) | \( \delta \), % | \( b_1 \) | \( b_2 \) | \( R^2 \) |
|-------|----------|----------|---------------|----------|----------|--------|
| True  | 1.6668   | -0.8187  | 0.5           | 2        |          | 0.972  |
| LS    | 1.2572   | -0.4996  | 27.9          | 1.73     | 2.38     | 0.768  |
| GRQ   | 1.6624   | -0.8268  | 0.5           | 0.475    | 2.09     | 0.973  |

Figure 2. The results of the identification of the model (19).

Figure 3. The results of the identification of the model (20).

5. Conclusion

This paper proposed algorithms for identifying solutions to second-order homogeneous linear differential equations in the presence of fractional white noise. We intend to further advance this research by developing and testing identification algorithms that do not require the knowledge of the parameter \( \alpha \) of fractional white noise (using genetic algorithms [12]).

6. References

[1] Granger C W J and Joyeux R 1980 An Introduction to Long Memory Time Series Models and Fractional Differencing *Journal of Time Series Analysis* **1**(1) 15-29

[2] Baillie R T 1996 Long Memory Processes and Fractional Integration in Econometrics *Journal of Econometrics* **73**(1) 5-59
[3] Das S, Kumar A, Pan I, Acharya A, Das Sh and Gupta 2011 Least square and instrumental variable system identification of AC servo position control system with fractional Gaussian noise Proc International Conference on Energy, Automation and Signal, ICEAS 545-550
[4] Ivanov D V and Ivanov A V 2017 Identification Fractional Linear Dynamic Systems with fractional errors-in-variables J. Phys.: Conf. Ser. 803(1)
[5] Refaat E A 2005 Lecture notes on Z-Transform (Morrisville NC: Lulu Press)
[6] Rekhviazhvili S Sh 2006 Simulation of Flicker Noise by Fractional Integro-Differentiation Technical Physics 51(6) 803-805
[7] Ivanov D V 2017 Determination of frequency in three-phase electric circuits with autocorrelated noise Russian Electrical Engineering 88(3) 123-126
[8] Aved'yav E D 1974 Recurrent method of least squares in the presence of correlated interferences Autom. Remote Control 36(5) 760-768
[9] Brinkulov U N 1991 Peculiarities of the use of the generalized least squares method in problems of the identification of Markov random processes Autom. Remote Control 52(1) 57-64
[10] Borodyuk V P and Kuznetsov V Ye 1974 Efficiency of a generalized method of least squares in identification problems Autom. Remote Control 35(7) 1053-1057
[11] Semenychev V K 2004 Identification of economic dynamics based on autoregression models (Samara: ANO Publishing House of SSC RAS)
[12] Ivanov D V, Engelgardt V V and Sandler I L 2018 Genetic Algorithm of Structural and Parametric Identification of Gegenbauer Autoregressive with Noise on Output Procedia Computer Science 131 619-625