A bending theory of thermoelastic diffusion plates based on Green-Naghdi theory

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Abstract

This article is concerned with bending plate theory for thermoelastic diffusion materials under Green-Naghdi theory. First, we present the basic equations which characterize the bending of thin thermoelastic diffusion plates for type II and III models. The theory allows for the effect of transverse shear deformation without any shear correction factor, and permits the propagation of waves at a finite speed without energy dissipation for type II model and with energy dissipation for type III model. By the semigroup theory of linear operators, we prove the well-posedness of the both models and the asymptotic behavior of the solutions of type III model. For unbounded plate of type III model, we prove that a measure associated with the thermodynamic process decays faster than an exponential of a polynomial of second degree. Finally, we investigate the impossibility of the localization in time of solutions. The main idea to prove this result is to show the uniqueness of solutions for the backward in-time problem.

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1. Introduction

Elastic plates play an important role in mechanical structures since they can support loads far in excess of their own weight. In addition, due to their geometric characteristics, thin plates can be studied mathematically by means of two-dimensional models instead of the full and much more complex equations of three-dimensional elasticity. The first plate-bending model was proposed by Kirchhoff (1850). Making a number of simplifying hypotheses, he arrived at the conclusion that, in terms of Cartesian coordinates $(x_1, x_2, x_3)$ with $(x_1, x_2)$ in the middle plane of the plate, the displacement field should be of the form $(x_3 u_1, x_3 u_2, u_3)$, where the functions $u_i = u_i(x_1, x_2)$, $i = 1, 2, 3$, satisfy $u_1 = -u_3, \quad u_2 = -u_3, \quad \Delta u_3 = q/D$, $q$ being the total load, $D$ the modulus of rigidity of the material and $\Delta$ the two-dimensional Laplacian. It is clear that the nonhomogeneous biharmonic equation for $u_3$ cannot take more than two boundary conditions.

Since there are numerous cases where the transverse shear force is not negligible and each of the three moments must be given on the contour, the need arose for more refined models with a more sophisticated mathematical content. One such model was proposed by Reissner (1947), who started with the stress tensor, postulating a linear dependence on $x_3$ for the components $t_{\alpha\beta}$, $\alpha, \beta \in \{1, 2\}$ and a certain type of parabolic dependence on $x_3$ for the components $t_{\alpha 3}$, $\alpha = 1, 2$. This model accepts three independent boundary conditions, but does not yield the explicit expressions of the displacements. Another model, proposed by...
Mindlin (1951), is based on Kirchhoff’s kinematic assumption on the displacement field but has no tie between its first two components and the third one, as above; however, it makes use of a correction factor in the constitutive equations, which interferes with its mathematical rigor. Again, this model accepts three boundary conditions. It should be pointed out that all Reissner and Mindlin type models also account for the transverse shear force in the plate.

On the other hand, the temperature plays a significant role in the process of bending thermoelastic plates. The deformation of such plates is of interest in a wide variety of practical problems, from microchip production to aerospace industry. Relevant theoretical developments on the subject were made by Green and Naghdi (1993, 1995). They developed three models for generalized thermoelasticity of homogeneous isotropic materials which are labeled as model I, II and III. This theory is developed in a rational way to produce a fully consistent theory that is capable of incorporating thermal pulse transmission in a very logical manner. When the theory of type I is linearized, the parabolic equation of the heat conduction arises. Type II theory predicts a finite speed for heat propagation and involves no energy dissipation, now referred to as thermoelasticity without energy dissipation. Type III theory permits propagation of thermal signals at both finite and infinite speeds.

We may think that the classical theory of thermoelastic bending plates is a good model to explain the thermal conduction in different structures. However, the research conducted in the development of high technologies after the second world war, confirmed that the field of diffusion in solids cannot be ignored. Thus, the obvious question is, what happens when the diffusion effect is considered with the thermal effect in the theory of bending elastic plate. Diffusion can be defined as the random walk of a set of particles from regions of high concentration to regions of lower concentration. Thermodiffusion in an elastic solid is due to coupling of the fields of strain, temperature and mass diffusion. The processes of heat and mass diffusion play important roles in many engineering applications, such as satellites problems, returning space vehicles and aircraft landing on water or land. There is now a great deal of interest in the process of diffusion in the manufacturing of integrated circuits, integrated resistors, semiconductor substrates and MOS transistors. Oil companies are also interested to this phenomenon to improve the conditions of oil extractions.

Recently, Aouadi et al. (2014) used the results of Green and Naghdi on thermo-mechanics of continua to derive a nonlinear theory of thermoelastic diffusion materials based on Green and Naghdi theory. In the present paper we use the results of Aouadi et al. (2014) to derive a bending theory of thermoelastic diffusion thin plates of Mindlin type in the context of the Green and Naghdi theory (models II and III). The theory allows for the effect of transverse shear deformation as in the Mindlin-Timoshenko model of plates (see Lagnese and Lions (1989)), but we do not introduce any shear correction factor.

A linear theory of thermoelastic plates with voids was investigated by Birsan (2003) under the classical theory of thermoelasticity based on Fourier’s law. Iesan and Quintanilla (2005) presented the basic equations which characterize the bending of thin microstretch elastic plates under Fourier’s law. Leseduarte and Quintanilla (2006) derived a bending theory of thermoelastic plates in the context of Green and Naghdi’s theory (model III). In (Ghiba, 2013a,b), Ghiba studies the temporal and the spatial behaviour of the solution of the bending plates of Mindlin type thermoelastic with voids.

The organization of this paper is as follows. In Section 2 we describe the theory established by Aouadi et al. (2014) to obtain, in section 3, the bending theory of thermoelastic diffusion plates based on Green-Naghdj theory of type II and III, which admits the possibility of "second sound". With the help of the semigroup theory of linear operators we investigate the well-posedness and the asymptotic behavior of solutions to the proposed model in sections 4 and 5. In section 6 we introduce a weighted surface measure associated with the dynamic process at issue and then we establish a second-order differential inequality whose integration gives a good information upon the spatial behavior. Finally, in Section 7, we study the problem of localization in time of the solutions. For this end, we use the uniqueness property of the backward in time problem.

It is worth noting that we focus on the analysis of the qualitative properties of solutions of type III problem. However, some particular aspects of the type II problem are also pointed out.
2. Basic Equations and Preliminaries

We refer the motion of the continuum to a fixed system of rectangular Cartesian axes $Ox_i$ ($i = 1, 2, 3$). We shall employ the usual summation and differentiation conventions. Latin subscripts are understood to range over the integers $\{1, 2, 3\}$ summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding material Cartesian coordinate. We consider a body that at time $t_0$ occupies a bounded regular region $V$ of the Euclidean three-dimensional space and is bounded by the surface $\partial V$. We deal with functions of position and time which have as their domain of definition the Cartesian product $V \times [0, \infty)$, where $\bar{V}$ is the closure of $V$. Letters in boldface stand for tensors of order $p \geq 1$ and, if $\mathbf{v}$ has order $p$, we write $v_{i_1, i_2, \ldots, i_p}$ in the Cartesian coordinate frame. In what follows, we use a superposed dot to denote material time differentiation.

On the basis of the theory established by Aouadi et al. (2014), the behavior of thermoelastic diffusion bodies under Green-Naghdi theory is governed by the following equations:

(i) the motion equation
\[ t_{ij,j} + f_i^* = \rho \ddot{u}_i, \]  
(ii) the energy equation
\[ \rho \dot{S} = -\Phi_i,i + s^*, \]  
(iii) the equation of conservation of mass diffusion
\[ \dot{C} = -\eta_i,i + e^*. \]

If the material is isotropic, then the constitutive equations become
\[ t_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} - d_1 T \delta_{ij} - d_2 \psi \delta_{ij}, \]
\[ \rho S = d_1 \epsilon_{kk} + \epsilon T + \kappa \psi, \]
\[ C = d_2 \epsilon_{kk} + \kappa T + r \psi, \]
\[ \Phi_i = -(k_1 \nu + h_1 \gamma + k_2 T + h_2 \psi)_{,i}, \]
\[ \eta_i = -(h_1 \gamma + h_1 \nu + h_2 T + h_2 \psi)_{,i}, \]
\[ q_i = T_0 \Phi_i = -T_0 (k_1 \nu + h_1 \gamma + k_2 T + h_2 \psi)_{,i}. \]

In these equations we have used the following notations: $t_{ij}$ is the stress tensor, $f_i^*$ is the body force per unit of initial volume, $\rho$ is the reference mass density, $T$ is the absolute temperature, $\mathbf{u} = (u_i)$ is the displacement vector, $\Phi_i$ is the entropy flux vector, $q_i$ is the heat flux vector, $\eta_i$ denotes the flow of the diffusing mass vector, $s^*$ is the external rate of supply of entropy per unit mass, $\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$ is the deformation tensor, $S$ is the entropy per unit mass, $\psi$ is the chemical potential per unit mass, $C$ is the concentration of diffusive material in the elastic body, $\nu$ is the thermal displacement whose derivative coincides with the absolute temperature, $\psi$ is the chemical potential displacement whose derivative coincides with the chemical potential, i.e.,
\[ \nu = \int_{t_0}^t \epsilon_T dt, \quad \gamma = \int_{t_0}^t \psi dt. \]

These scalars, on the macroscopic scale, are regarded, respectively, as representing some "mean" thermal and chemical potential displacements magnitudes on the molecular scale. The constants $\lambda$ and $\mu$ are elastic coefficients and $d_1$ and $d_2$ are the coefficients of thermal and mass diffusion expansions, respectively. The constants $\kappa$ and $r$ are measures of thermodiffusion and diffusive effects, respectively. $k_1$ and $h_1$ are coefficients of thermal and diffusion conductivity, respectively, while $h_2$ is a measure of thermodiffusion gradient displacement. $k_2$, $h_2$ and $h_3$ are coefficients characterizing the type III model.

Remark that the evolutive equations for the thermoelastic diffusion theory of type II (without energy dissipation) can be deduced from the above equations by taking $k_2 = h_2 = h_3 = 0$.

All the above coefficients are constitutive constants and satisfy the following conditions:
We say that the system of body loads
\[ \delta = cr - \kappa^2 > 0, \]
which implies that
\[ \delta \theta^2 + 2\kappa \theta P + r P^2 > 0. \]

(ii) For type III model, the coefficients \( k_2, h_2 \) and \( h_2 \) should satisfy
\[ k_2 h_2 - h_2^2 > 0, \]
which implies that
\[ k_2 \theta^2 + 2 h_2 \theta P + h_2 P^2 > 0 \]
to ensure the non-negativeness of the internal rate of production of entropy (see Aouadi et al. (2014)).

(iii) For both type II and III models, the necessary and sufficient conditions that the internal energy density be positive are
\[ \mu > 0, \quad \lambda + \mu > 0, \quad k_1 h_1 - h_1^2 > 0. \]

The components of surface traction \( t_i \), the internal flux of entropy per unit mass \( \Phi \) and the diffusion flux \( \eta \) at regular points of \( \partial V \), are given by
\[ t_i = t_{ij} n_j, \quad \Phi = \Phi_i n_i, \quad \eta = \eta_i n_i, \]
respectively. We denote by \( n_j \) the outward unit normal of \( \partial V \). We assume that \((f'_i, s^*, c^*)\) are continuous on \( V \times T \), where \( T \) is a temporal interval. To the system of field equation we must add boundary conditions and initial conditions.

3. Bending thermoelastic diffusion plates

In what follows we assume that the region \( V \) is the interior of a right cylinder of length \( 2h \) with open cross-section \( \Gamma \) and smooth lateral boundary \( \Pi \). Let \( \Gamma \) be the boundary of \( \Sigma \). The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that the plane \( x_1Ox_2 \) is the middle plane. Thus, we have
\[ V = \{ x : (x_1, x_2) \in \Sigma, \quad -h < x_3 < h \}, \quad \Pi = \{ x : (x_1, x_2) \in \Gamma, \quad -h < x_3 < h \}. \]
An admissible process \( p = [u_\alpha, v, T, \gamma, \psi] \) is a state of bending on \( V \times T \) provided
\[
\begin{align*}
& u_\alpha(x_1, x_2, x_3, t) = -u_\alpha(x_1, x_2, -x_3, t), \quad u_3(x_1, x_2, x_3, t) = u_3(x_1, x_2, -x_3, t), \\
& \nu(x_1, x_2, x_3, t) = -\nu(x_1, x_2, -x_3, t), \quad \gamma(x_1, x_2, x_3, t) = -\gamma(x_1, x_2, -x_3, t), \quad (x_1, x_2, x_3, t) \in V \times T.
\end{align*}
\]
Here and in what follows the Greek subscripts are confined to the range 1, 2. In view of Eqs. (1), (4) we find that
\[
\begin{align*}
&t_{\alpha\beta}(x_1, x_2, x_3, t) = -t_{\alpha\beta}(x_1, x_2, -x_3, t), \quad t_{33}(x_1, x_2, x_3, t) = -t_{33}(x_1, x_2, -x_3, t), \\
&t_{30}(x_1, x_2, x_3, t) = t_{30}(x_1, x_2, -x_3, t), \quad \Phi_3(x_1, x_2, x_3, t) = -\Phi_3(x_1, x_2, -x_3, t), \\
&S(x_1, x_2, x_3, t) = -S(x_1, x_2, -x_3, t), \quad \Phi_\alpha(x_1, x_2, x_3, t) = -\Phi_\alpha(x_1, x_2, -x_3, t), \\
&\Phi_3(x_1, x_2, x_3, t) = -\Phi_3(x_1, x_2, -x_3, t), \quad q_\alpha(x_1, x_2, x_3, t) = -q_\alpha(x_1, x_2, -x_3, t), \\
&q_3(x_1, x_2, x_3, t) = q_3(x_1, x_2, -x_3, t), \quad C(x_1, x_2, x_3, t) = -C(x_1, x_2, -x_3, t), \\
&\eta_\alpha(x_1, x_2, x_3, t) = -\eta_\alpha(x_1, x_2, -x_3, t), \quad \eta_3(x_1, x_2, x_3, t) = \eta_3(x_1, x_2, -x_3, t).
\end{align*}
\]
We say that the system of body loads \((f'_i, s^*, c^*)\) is compatible with a state of bending if
\[
\begin{align*}
f'_\alpha(x_1, x_2, x_3, t) &= -f'_\alpha(x_1, x_2, -x_3, t), \quad f'_3(x_1, x_2, x_3, t) = f'_3(x_1, x_2, -x_3, t), \\
s^*(x_1, x_2, x_3, t) &= -s^*(x_1, x_2, -x_3, t), \quad c^*(x_1, x_2, x_3, t) = -c^*(x_1, x_2, -x_3, t).
\end{align*}
\]
In what follows we assume that the body loads satisfy the restrictions (11). From Eqs. (10), (11) we get
\[
\int_{-h}^{h} t_{αβ} dx_3 = 0, \quad \int_{-h}^{h} t_{33} dx_3 = 0, \quad \int_{-h}^{h} S dx_3 = 0, \\
\int_{-h}^{h} Φ_α dx_3 = 0, \quad \int_{-h}^{h} q_α dx_3 = 0, \quad \int_{-h}^{h} C dx_3 = 0, \\
\int_{-h}^{h} η_α dx_3 = 0, \quad \int_{-h}^{h} f_α^* dx_3 = 0, \quad \int_{-h}^{h} s^* dx_3 = 0, \quad \int_{-h}^{h} c^* dx_3 = 0.
\] (12)

We derive a theory of thin plates of uniform thickness assuming that the fields \(u_1, \nu \) and \(γ \) do not vary violently with respect to \(x_3\). We denote
\[
N_α = \frac{1}{2h} \int_{-h}^{h} t_{αβ} dx_3.
\] (13)

We assume that the functions \(t_1, q \) and \(η \) are prescribed on the surfaces \(x_3 = \pm h\). We integrate (1) with respect to \(x_3\) between the limits \(-h\) and \(h\). According to Eqs. (9) and (13), we obtain
\[
N_{α,α} + f = ρ\ddot{v}, \quad \text{on} \ Σ \times \mathcal{T},
\]
where
\[
w = \frac{1}{2h} \int_{-h}^{h} u_3 dx_3, \quad f = \frac{1}{h} t_{33}(x_1, x_2, h, t) + \frac{1}{2h} \int_{-h}^{h} f_α^* dx_3.
\] (15)

If we multiply (1) by \(x_3\) and integrate from \(x_3 = -h\) to \(x_3 = h\), then we obtain
\[
M_{βα,β} - 2hN_α + f_α = ρI\ddot{v}_α, \quad \text{on} \ Σ \times \mathcal{T},
\] (16)

where
\[
M_{αβ} = \int_{-h}^{h} x_3 t_{αβ} dx_3, \quad I\ddot{v}_α = \int_{-h}^{h} x_3 u_α dx_3, \quad I = \frac{2}{3}h^3,
\]
\[
f_α = 2ht_{3α}(x_1, x_2, h, t) + \int_{-h}^{h} x_3 f_α^* dx_3.
\] (17)

Now multiply (2) and (3) by \(x_3\) and integrate from \(x_3 = -h\) to \(x_3 = h\). In view of (9)-(11) we obtain, respectively
\[
ρ\ddot{σ} = -Ψ_{α,α} + 2hR + W,
\] (18)
\[
\ddot{χ} = -Ω_{α,α} + 2hM + V,
\] (19)

where
\[
σ = \int_{-h}^{h} x_3 S dx_3, \quad Ψ_α = \int_{-h}^{h} x_3 Φ_α dx_3, \quad R = \frac{1}{2h} \int_{-h}^{h} Φ_3 dx_3,
\]
\[
W = -2hΦ_3(x_1, x_2, h, t) + \int_{-h}^{h} x_3 s^* dx_3,
\]
\[
χ = \int_{-h}^{h} x_3 C dx_3, \quad Ω_α = \int_{-h}^{h} x_3 η_α dx_3, \quad M = \frac{1}{2h} \int_{-h}^{h} η_3 dx_3,
\]
\[
V = -2hη_3(x_1, x_2, h, t) + \int_{-h}^{h} x_3 c^* dx_3.
\] (20)

The functions \(f, f_α, W\) and \(V\) are prescribed.
We restrict our attention to the state of bending characterized by

\[ u_\alpha = x_3 v_\alpha (x_1, x_2, t), \quad u_3 = w(x_1, x_2, t), \]
\[ \nu = x_3 \tau(x_1, x_2, t), \quad T = x_3 \theta(x_1, x_2, t), \]
\[ \gamma = x_3 \varphi(x_1, x_2, t), \quad \psi = x_3 P(x_1, x_2, t), \quad \text{on } \Sigma \times T. \] (21)

In view of (21) we have

\[ \epsilon_{\alpha\beta} = x_3 \epsilon_{\alpha\beta}, \quad 2\epsilon_{\alpha 3} = \gamma_\alpha, \quad \epsilon_{33} = 0, \]
\[ \epsilon_{\alpha\beta} = \frac{1}{2} (v_{\alpha, \beta} + v_{\beta, \alpha}), \quad \gamma_\alpha = v_\alpha + w_\alpha = 2\epsilon_{3\alpha} = 2\epsilon_{\alpha 3}. \] (22)

The quantities \( \gamma_\alpha \) represent the angles of rotation of the cross sections \( x_\alpha = \text{const} \) about the middle surface.

It follows from (4), (21) and (22) that

\[ \tau_{\alpha\beta} = x_3 (\lambda \epsilon_{rr} \delta_{\alpha\beta} + 2\mu \epsilon_{\alpha\beta} - d_1 \theta \delta_{\alpha\beta} - d_2 P \delta_{\alpha\beta}), \]
\[ \tau_{33} = x_3 (\lambda \epsilon_{rr} - d_1 \theta - d_2 P), \]
\[ \rho S = x_3 (d_1 \epsilon_{rr} + c \theta + \kappa P), \]
\[ C = x_3 (d_2 \epsilon_{rr} + \kappa \theta + rP), \]
\[ \Phi_\alpha = -x_3 (k_1 \tau + h_1 \varphi + k_2 \theta + h_2 P)_\alpha, \]
\[ \Phi_3 = -(k_1 \tau + h_1 \varphi + k_2 \theta + h_2 P), \]
\[ \eta_\alpha = -x_3 (h_1 \varphi + h_1 \tau + h_2 \theta + h_2 P)_\alpha, \]
\[ \eta_3 = -(h_1 \varphi + h_1 \tau + h_2 \theta + h_2 P), \]
\[ q_\alpha = -x_3 T_0 (k_1 \tau + h_1 \varphi + k_2 \theta + h_2 P)_\alpha, \]
\[ q_3 = -T_0 (k_1 \tau + h_1 \varphi + k_2 \theta + h_2 P). \] (23)

It follows from Eqs. (13), (17), (20) and (23) that

\[ M_{\alpha\beta} = I (\lambda \epsilon_{rr} \delta_{\alpha\beta} + 2\mu \epsilon_{\alpha\beta} - d_1 \theta \delta_{\alpha\beta} - d_2 P \delta_{\alpha\beta}), \]
\[ N_\alpha = \mu \gamma_\alpha, \]
\[ \rho \sigma = I (d_1 \epsilon_{rr} + c \theta + \kappa P), \]
\[ \Psi_\alpha = -I (k_1 \tau + h_1 \varphi + k_2 \theta + h_2 P)_\alpha, \]
\[ R = -(k_1 \tau + h_1 \varphi + k_2 \theta + h_2 P), \]
\[ \chi = I (d_2 \epsilon_{rr} + \kappa \theta + rP), \]
\[ \Omega_\alpha = -I (h_1 \varphi + h_1 \tau + h_2 \theta + h_2 P)_\alpha, \]
\[ M = -(h_1 \varphi + h_1 \tau + h_2 \theta + h_2 P). \] (24)

The equations of thermoelastic diffusion bending plates consist of the equations of motion (14) and (16), the entropy equation (18), the mass diffusion equation (19), the constitutive equations (24) and the geometrical equations (22). These equations can be expressed in terms of the functions \( v_\alpha, w, \tau, \theta, \varphi \) and \( P \). Thus, we obtain the evolutive equations of type III (with energy dissipation)

\[ \rho I \ddot{v}_\alpha = I (\mu \Delta v_\alpha + (\lambda + \mu) \psi_{\beta, \alpha} - d_1 \theta_{,\alpha} - d_2 P_{,\alpha}) - 2\hbar \rho (v_\alpha + w_\alpha) + f_\alpha, \]
\[ \rho \ddot{w} = \mu \Delta w + \mu w_{\alpha, \alpha} + f, \]
\[ c I \dddot{\tau} + \kappa I \dddot{\varphi} = I (k_1 \Delta \tau + h_1 \Delta \varphi + k_2 \Delta \theta + h_2 \Delta P) - I \dot{d}_1 \dot{v}_{\alpha, \alpha} - 2h (k_1 \tau + h_1 \varphi + k_2 \theta + h_2 P) + W, \]
\[ \kappa I \dddot{\tau} + r I \dddot{\varphi} = I (h_1 \Delta \varphi + h_1 \Delta \tau + h_2 \Delta \theta + h_2 \Delta P) - I \dot{d}_2 \dot{v}_{\alpha, \alpha} - 2h (h_1 \varphi + h_1 \tau + h_2 \theta + h_2 P) + V. \] (25)
Remark that the evolutive equations of the thermoelastic diffusion bending plates of type II (without energy dissipation) can be deduced from (25) by taking \( k_2 = h_2 = h_2 = 0 \).

Summarizing, the following initial boundary value problems are to be solved:

(i) Type II problem: Find \((v_\alpha, z_\alpha, w, y, \tau, \theta, \varphi, P)\) solution of (25) (with \( k_2 = h_2 = h_2 = 0 \)) subject to the initial conditions

\[
\begin{align*}
v_\alpha(x_1, x_2, 0) &= v^0_\alpha(x_1, x_2), & v_\alpha(x_1, x_2, 0) &= z^0_\alpha(x_1, x_2), \\
w(x_1, x_2, 0) &= w^0(x_1, x_2), & \dot{w}(x_1, x_2, 0) &= \gamma^0_\alpha(x_1, x_2), \\
\tau(x_1, x_2, 0) &= \tau^0(x_1, x_2), & \theta(x_1, x_2, 0) &= \theta^0(x_1, x_2), \\
\varphi(x_1, x_2, 0) &= \varphi^0(x_1, x_2), & P(x_1, x_2, 0) &= \Phi^0(x_1, x_2),
\end{align*}
\]

where \( v^0_\alpha, z^0_\alpha, w^0, \gamma^0, \theta^0, \varphi^0 \) and \( \Phi^0 \) are prescribed functions.

We consider the boundary conditions

\[
M_\beta \alpha n_\beta = M_\alpha, \quad N_\alpha n_\alpha = \bar{N}, \quad \Psi_\alpha n_\alpha = \bar{\Psi}, \quad \Omega_\alpha n_\alpha = \bar{\Omega}, \quad \text{on} \; \Gamma \times T,
\]

where the given functions \( M_\alpha, \bar{N}, \bar{\Psi} \) and \( \bar{\Omega} \) are piecewise regular and continuous in time.

(ii) Type III problem: Find \((v_\alpha, z_\alpha, w, y, \tau, \theta, \varphi, P)\) solution of (25) subject to the initial conditions (26) and the boundary conditions (27).

Some qualitative properties of the solutions of type III problem are studied in the following. However, some particular aspects of the type II problem are also pointed out.

4. Well-posedness

We shall use the results of the semigroup of linear operators theory to prove the existence, uniqueness and continuous dependence from the initial values and the external loads of the solution for the system (25). Seeking for simplicity, we will restrict ourselves to homogeneous boundary conditions

\[
v_\alpha = 0, \quad w = 0, \quad \tau = 0, \quad \varphi = 0 \quad \text{on} \; \Gamma \times T.
\]

In the rest of the paper we assume:

(i) Relations (5)-(7) are satisfied;

(ii) The positive definiteness of the internal energy density, i.e., there exists a positive constant \( c_0 \) such that

\[
I(\lambda \varepsilon_r \varepsilon_{\gamma} + 2\mu \varepsilon_{\alpha \beta} \varepsilon_{\alpha \beta}) + 2h \mu \gamma_\alpha \gamma_\alpha + I(k_1 \tau_\alpha \tau_\alpha + h_1 \varphi_\alpha \varphi_\alpha) + 2h_1 \tau_\alpha \varphi_\alpha + 2h_1 \tau \varphi
\]

\[
\geq c_0 (\varepsilon_{\alpha \beta} \varepsilon_{\alpha \beta} + \gamma_\alpha \gamma_\alpha + \tau_\alpha \tau_\alpha + \varphi_\alpha \varphi_\alpha + \tau^2 + \varphi^2)
\]

for any \( \varepsilon_{\alpha \beta}, \gamma_\alpha, \tau_\alpha, \varphi_\alpha \).

We now wish to transform the boundary-initial-value problem defined by system (25), the initial conditions (26) and the boundary conditions (28) to an abstract problem on a suitable Hilbert space. In what follows we use the notation \( z = \mathbf{v}, \; y = w, \; \theta = \dot{\tau}, \; P = \dot{\varphi} \). Let

\[
\mathcal{H} = \left\{(v, z, w, y, \tau, \theta, \varphi, P); \; v_\alpha, w, \tau, \varphi \in W_0^{1,2}(\Sigma); \; z_\alpha, y, \theta, P \in L^2(\Sigma)\right\},
\]

where \( W_0^{1,2}(\Sigma) \) and \( L^2(\Sigma) \) are the familiar Sobolev spaces.
We consider the following operators

\[ A_\alpha v = \rho^{-1}[\mu \Delta v_\alpha + (\lambda + \mu) v_{\beta,\beta\alpha}] - 2h \rho^{-1} I^{-1} \mu v_\alpha, \quad B_\alpha w = -2\rho^{-1} I^{-1} h w_\alpha, \]

\[ C_\alpha \theta = -\rho^{-1} d_1 \theta_\alpha, \quad D_\alpha P = -\rho^{-1} d_2 P_\alpha, \quad G v = \rho^{-1} \mu v_{\alpha,\alpha}, \quad H w = \rho^{-1} \mu D w, \]

\[ J z = -\delta^{-1}(rd_1 - k_d^2) z_{\alpha,\alpha}, \quad K \tau = -\delta^{-1}(rk_1 - k_h^2) \Delta \tau - 2h(I_\alpha \delta)^{-1}(rk_1 - k_h^1) \tau, \]

\[ L \theta = -\delta^{-1}(rk_2 - k_h^2) \Delta \theta - 2h(I_\alpha \delta)^{-1}(rk_2 - k_h^2) \theta, \quad M \varphi = -\delta^{-1}(rh_1 - k_h^1) \Delta \varphi - 2h(I_\alpha \delta)^{-1}(rh_1 - k_h^1) \varphi, \]

\[ N P = -\delta^{-1}(rh_2 - k_h^2) \Delta P - 2h(I_\alpha \delta)^{-1}(rh_2 - k_h^2) P, \quad L z = -\delta^{-1}(cd_2 - k_d^1) z_{\alpha,\alpha}, \]

\[ U \tau = -\delta^{-1}(ch_1 - k_h^1) \Delta \tau - 2h(I_\alpha \delta)^{-1}(ch_1 - k_h^1) \tau, \quad V \varphi = -\delta^{-1}(ch_2 - k_h^2) \Delta \varphi - 2h(I_\alpha \delta)^{-1}(ch_2 - k_h^2) \varphi, \]

\[ W \varphi = -\delta^{-1}(ch_1 - k_h^1) \Delta \varphi - 2h(I_\alpha \delta)^{-1}(ch_1 - k_h^1) \varphi, \]

\[ X \varphi = -\delta^{-1}(ch_2 - k_h^2) \Delta \varphi - 2h(I_\alpha \delta)^{-1}(ch_2 - k_h^2) P, \]

where \( \delta \) is given by (5), \( A = (A_i) \), \( B = (B_i) \), \( C = (C_i) \) and \( D = (D_i) \). Now consider the matrix operator \( \mathcal{A} \) on \( \mathcal{H} \) defined by

\[
\mathcal{A} = \begin{pmatrix}
0 & \text{Id} & 0 & 0 & 0 & 0 & 0 & 0 \\
A & 0 & B & 0 & 0 & C & 0 & D \\
0 & 0 & 0 & Id & 0 & 0 & 0 & 0 \\
G & 0 & \mathcal{H} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Id & 0 & 0 \\
0 & J & 0 & 0 & K & \mathcal{L} & \mathcal{M} & \mathcal{N} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Id \\
0 & L & 0 & 0 & \partial \mathcal{U} & \mathcal{V} & \mathcal{W} & \mathcal{X}
\end{pmatrix}
\]

with the domain

\[
\mathcal{D} = \mathcal{D}(\mathcal{A}) = (W_{0,2}^1 \cap W_{2,2}^1) \times W_{0,2}^1 \times (W_{0,2}^1 \cap W_{2,2}^1) \times W_{0,2}^1 \times (W_{0,2}^1 \cap W_{2,2}^1) \times (W_{0,2}^1 \cap W_{2,2}^1)
\]

\times (W_{0,2}^1 \cap W_{2,2}^1) \times (W_{0,2}^1 \cap W_{2,2}^1),
\]

where \( \text{Id} \) and \( Id \) are the identity operators in the respective spaces. We note that the domain \( \mathcal{D} \) is dense in \( \mathcal{H} \).

We introduce the inner product in \( \mathcal{H} \) defined by

\[
((v, w, \tau, \varphi, P), (v^*, w^*, \tau^*, \varphi^*, P^*)) = \frac{1}{2} \int_{\Sigma} \left( \rho I z_\alpha z_\alpha^* + 2h \rho g y^* + c I \theta^* + \kappa I (\theta^* P + \theta P^*) + r I P \varphi^* + \mathcal{W}[(v, w, \tau, \varphi), (v^*, w^*, \tau^*, \varphi^*)] \right) da,
\]

where

\[
\mathcal{W}[(v, w, \tau, \varphi), (v^*, w^*, \tau^*, \varphi^*)] = \int \left( \lambda \varepsilon_{\alpha,\alpha} \varepsilon_{\beta,\beta}^* + 2\mu \varepsilon_{\alpha,\beta} \varepsilon_{\alpha,\beta}^* + k_1 \tau_{\alpha} \tau_{\alpha}^* + h_1 \varphi_{\alpha} \varphi_{\alpha}^* + h_1 \tau_{\alpha} \varphi_{\alpha} \varphi_{\alpha}^* + k_1 \tau_{\alpha} \varphi_{\alpha} \varphi_{\alpha}^* + h_1 \tau_{\alpha} \varphi_{\alpha} \varphi_{\alpha}^* \right)
\]

\[ + 2h \left( \mu \gamma_{\alpha} \gamma_{\alpha}^* + k_1 \tau_{\alpha} \tau_{\alpha}^* + h_1 \varphi_{\alpha} \varphi_{\alpha}^* + h_1 \tau_{\alpha} \varphi_{\alpha} \varphi_{\alpha}^* \right). \]

In the above relations we have used the notation

\[
\varepsilon_{\alpha,\beta}^* = \frac{1}{2}(v_{\beta,\alpha} + v_{\alpha,\beta}), \quad \gamma_{\alpha}^* = v_{\alpha}^* + w_{\alpha}^* = 2\varepsilon_{\alpha,\alpha}^* = 2\varepsilon_{\alpha,\alpha}.
\]

If we recall the assumptions (29) and the first Korn inequality, we conclude that the norm induced in \( \mathcal{H} \) through the product (32) is equivalent to the usual one in \( \mathcal{H} \).

The boundary initial value problem (25), (26), (28) can be transformed into the following equation in the Hilbert space \( \mathcal{H} \),

\[
\frac{dU(t)}{dt} = \mathcal{A}U(t) + \mathcal{F}(t), \quad U(0) = U_0,
\]

(33)
where

\[ U = (v, z, w, y, \tau, \theta, \varphi, P), \quad U_0 = (v^0, z^0, w^0, y^0, \tau^0, \varphi^0, P^0), \]

\[ \mathcal{F} = (0, (\rho t)^{-1} f, 0, 0, (I\delta)^{-1} (rW - \kappa V), 0, (I\delta)^{-1} (cV - \kappa W)). \]

Now, we use the theory of semigroups of linear operators to obtain the existence of solutions for the Eq. (33).

**Lemma 1.** The operator \( \mathcal{A} \) satisfies the inequality \(< \mathcal{A} U, U > > 0 \) for type III model and the equality \(< \mathcal{A} U, U > = 0 \) for type II model, for every \( U \in \mathcal{D}(\mathcal{A}) \).

**Proof.** Let \( U = (v, z, w, y, \tau, \theta, \varphi, P) \in \mathcal{D}(\mathcal{A}) \). Using the divergence theorem and the boundary conditions, we have

\[
< \mathcal{A} U, U > = -\int_\Omega \left( z_{\alpha, \beta} M_{\beta \alpha} + 2h z_{\alpha} N_{\alpha} + 2hz_{\alpha} N_{\alpha} + I(k\delta, \alpha, \theta, \alpha + h2\delta, \alpha P, \alpha + h2P^2, \alpha, \alpha) \right) \] 
\[ + 2h(k2\theta^2 + 2h\theta P + h2P^2) + \int_0^1 \left( I(k2\theta^2 + 2h\theta P + h2P^2) + \int_0^1 \left( 0 \right) \right) da \] 
\[ = -\int_\Omega \left( I(k2\theta^2 + 2h\theta P + h2P^2) + \int_0^1 \left( 0 \right) \right) da. \]

In the context of type III model and from the assumption (6) we have \(< \mathcal{A} U, U > > 0 \), which means dissipation of the energy, i.e.,

\[ \frac{dE_0(t)}{dt} = -\int_\Omega \left( I(k2\theta^2 + 2h\theta P + h2P^2) + \int_0^1 \left( 0 \right) \right) da < 0, \]

where

\[ E_0(t) = \frac{1}{2} \int_\Omega \left( \rho I z_{\alpha} z_{\alpha} + 2h \rho y + c1 \theta^2 + 2k1 \theta P + r1P^2 + 2 \gamma \right) da, \]

and

\[ 2 \gamma = I \left( \lambda e_{\alpha, \beta} e_{\beta \alpha} + 2\mu e_{\alpha, \beta} e_{\beta \alpha} + k1 \tau, \alpha \tau, \alpha + h1 \tau, \alpha \tau, \alpha + h1 \tau, \alpha \tau, \alpha \right) + 2h \left( \mu \gamma, \alpha + k1 \tau^2 + 2h1 \tau \varphi + h1 \varphi^2 \right). \] (35)

In the context of type II model \((k_2 = h_2 = h_2 = 0)\), we have \(< \mathcal{A} U, U > = 0 \), which means conservation of the energy i.e., \( \frac{dE_0(t)}{dt} = 0 \).

It is worth remarking that this quantity is also conserved even if we do not impose conditions (i) – (ii).

**Lemma 2.** The operator \( \mathcal{A} \) has the property that

\[ \text{Range}(\mathcal{I} - \mathcal{A}) = \mathcal{H}, \]

where \( \mathcal{I} \) is the identity operator in \( \mathcal{H} \).

**Proof:** Let \( U^* = (v^*, z^*, w^*, y^*, \tau^*, \theta^*, \varphi^*, P^*) \in \mathcal{H} \). We must prove that

\[ U - \mathcal{A} U = U^* \]

has a solution \( U = (v, z, w, y, \tau, \theta, \varphi, P) \) in \( \mathcal{D} \). This equation leads to the system

\[
\begin{align*}
    v^* &= v - z, & w^* &= w - y, \\
    \tau^* &= \tau - \theta, & \varphi^* &= \varphi - P, \\
    z^* &= z - (Av + Bw + C\theta + DP), & y^* &= y - (Gv + Hw), \\
    \theta^* &= \theta - (Jz + K\tau + L\theta + MP + NP), & \varphi^* &= \varphi - (Lz + Ut + V\theta + WP + XP).
\end{align*}
\] (36)
Substituting the four first equations in the others, we obtain
\[
\begin{align*}
z^* + v^* - C\tau^* - D\psi^* &= v - (Av + Bw + C\tau + D\psi), \\
y^* + z^* &= w - (Gv + Hw), \\
\theta^* + \tau^* - L\tau^* - N\psi^* &= \tau - (Jv + (K + L)\tau + (M + N)\psi), \\
P^* + \psi^* - V\tau^* - X\psi^* &= \psi - (Lv + (U + V)\tau + (W + X)\psi).
\end{align*}
\] (37)

To solve this system, we introduce the following bilinear form on \(W_0^{1,2}\),
\[
\mathcal{B}(v, w, \tau, \psi, (\hat{v}, \hat{w}, \hat{\tau}, \hat{\psi})) = (v^{(1)}, w^{(1)}, \tau^{(1)}, \psi^{(1)}), (\rho I\hat{v}, 2h\rho\hat{w}, cI\hat{\tau}, rI\hat{\psi}) >
\] (38)
where
\[
\begin{align*}
v^{(1)} &= v - (Av + Bw + C\tau + D\psi), \\
w^{(1)} &= w - (Gv + Hw), \\
\tau^{(1)} &= \tau - (Jv + (K + L)\tau + (M + N)\psi), \\
\psi^{(1)} &= \psi - (Lv + (U + V)\tau + (W + X)\psi).
\end{align*}
\]

A direct calculation shows that \(\mathcal{B}\) is bounded in each variable. Using the divergence theorem, we have
\[
\mathcal{B}(v, w, \tau, \psi, (\hat{v}, \hat{w}, \hat{\tau}, \hat{\psi})) = \int \sum \left( \rho I v_{\alpha}\nu_{\alpha} + 2h\rho w^2 + c\theta^2 + 2\kappa I\theta P + r I P^2 + \mathcal{W}(v, w, \tau, \phi) \right) da
\]
\[
+ \int \left( I(k_2\theta, \theta, \alpha + 2h_2\theta, P, \alpha + h_2 P, \alpha) + 2h(k_2\theta^2 + 2h_2\theta P - h_2 P^2) \right) da.
\]

In view of our assumptions on the constitutive coefficients, we see that \(\mathcal{B}\) is coercive on \(W^{-1,2}\). On the other hand, it is easy to see that the vector
\[
(z^* + v^* - C\tau^* - D\psi^*, y^* + z^*, \theta^* + \tau^* - L\tau^* - N\psi^*, P^* + \psi^* - V\tau^* - X\psi^*)
\]
lies in \(W^{-1,2}\). Hence the Lax-Milgram theorem implies the existence of \((v, w, \tau, \phi) \in W_0^{1,2}\) which solves the system (37). Thus, the system (36) has also a solution. \(\square\)

The previous lemmas lead to next theorem.

**Theorem 1.** The operator \(\mathcal{A}\) generates a semigroup of contraction in \(\mathcal{H}\).

**Proof.** The proof follows from Lumer-Phillips corollary to the Hille-Yosida theorem (Pazzy, 1983). \(\square\)

It is worth remarking that this theorem implies that the dynamical system generated by the equations of bending thermoelastic diffusion plates of type III (or type II) is stable in the sense of Lyapunov.

**Theorem 2.** Assume that \(f, f, f, W, V \in C^1([0, \infty), L^2)\) and \(U_0\) is in the domain of the operator \(\mathcal{A}\). Then, there exists a unique solution \(U(t) \in C^1([0, \infty), \mathcal{H}) \cap C^0([0, \infty), \mathcal{D})\) to the problem (33).

Since the solutions are defined by means of a semigroup of contraction, we have the estimate
\[
\|U(t)\| \leq \|U_0\|_{\mathcal{H}} + \int_0^t \left( \|f_0(s)\|_{L^2} + \|f(s)\|_{L^2} + \|W(s)\|_{L^2} + \|V(s)\|_{L^2} \right) ds
\]
which proves the continuous dependence of the solutions upon initial data and body loads. Thus, under assumptions (i) – (ii) the problem of bending thermoelastic diffusion plates of type III (or type II) is well posed.
5. Asymptotic behavior of solutions

In this section we study the asymptotic behavior of solutions, whose existence has been proved previously, in the homogenous case \( f_n = 0, f = 0, W = 0, V = 0 \). In particular we are interested in the relation between dissipation effects and time decay of solutions. Therefore, we will continue to assume that the assumptions (i) – (ii) considered in the previous section hold. However it is worth noting that the results for this section only hold for type III theory.

To this end, we recall that for a semigroup of contraction, the precompact orbits tend to the \( \omega \)-limit sets if its generator \( \mathfrak{A} \) has only the fixed point \( 0 \) and the structure of the \( \omega \)-limit sets is determined by the eigenvectors of eigenvalue \( i\lambda \) (where \( \lambda \) is a real number) in the closed subspace

\[
\mathcal{L} = \ll \{ U \in \mathcal{H}; \langle \mathfrak{A} U, U \rangle = 0 \} \gg,
\]

where \( \ll \mathcal{C} \gg \) denotes the closed vectorial subspace generated by the set \( \mathcal{C} \).

From the assumptions (i)-(ii) it is easy to check that \( \mathfrak{A}^{-1}(0) = 0 \), while the precompactness of the orbits starting in \( \mathcal{D} \) is a consequence of the following Lemma (Pazy, 1983).

Lemma 3. The operator \((\mathcal{I} - \mathfrak{A})^{-1}\) is compact.

Proof. Let \((v_n, z_n, \tilde{w}_n, y_n, \tau_n, \theta_n, \tilde{\varphi}_n, \tilde{P}_n)\) be a bounded sequence in \( \mathcal{H} \) and let \((v_n, z_n, w_n, y_n, \tau_n, \theta_n, \varphi_n, P_n)\) be the sequence of the respective solutions of the system \((36)\). We have

\[
\Lambda [\Gamma_n, \Gamma_n] = \ll \Sigma_n, \Upsilon_n \gg,
\]

where

\[
\Gamma_n = (v_n, w_n, \tau_n, \varphi_n), \quad \Sigma_n = (v_n^{(1)}, w_n^{(1)}, \tau_n^{(1)}, \varphi_n^{(1)}), \quad \Upsilon_n = (\rho I v_n, 2h p \tilde{w}_n, c l \tau_n, r I \tilde{\varphi}_n)
\]

and

\[
\begin{align*}
v_n^{(1)} &= v_n - (Av_n + Bw_n + C\tau_n + D\varphi_n), \\
w_n^{(1)} &= w_n - (Gv_n + Hw_n), \\
\tau_n^{(1)} &= \tau_n - (Jv_n + (K + \mathcal{L})\tau_n + (M + N)\varphi_n), \\
\varphi_n^{(1)} &= \varphi_n - \big( L v_n + (R + V)\tau_n + (W + X)\varphi_n \big).
\end{align*}
\]

In view of the definition of \((v_n^{(1)}, w_n^{(1)}, \tau_n^{(1)}, \varphi_n^{(1)})\), it follows that it is a bounded sequence in \( L^2(\Sigma) \) and then the sequence \((v_n, w_n, \tau_n, \varphi_n)\) is a bounded sequence in \( W^{1,2}_0(\Sigma) \). The theorem of Rellich-Kondrasov (Ciarlet, 1988) implies that there is exists a subsequence converging in \( L^2(\Sigma) \). In a similar way

\[
z_{n_j} = v_{n_j} - \tilde{v}_{n_j}, \quad y_{n_j} = w_{n_j} - \tilde{w}_{n_j}, \quad \theta_{n_j} = \tau_{n_j} - \tilde{\tau}_{n_j}, \quad P_{n_j} = \varphi_{n_j} - \tilde{\varphi}_{n_j}
\]

has a sub-sequence converging in \( L^2(\Sigma) \). Thus we conclude the existence of a sub-sequence

\[
(v_{n_{j_k}}, z_{n_{j_k}}, w_{n_{j_k}}, y_{n_{j_k}}, \tau_{n_{j_k}}, \theta_{n_{j_k}}, \varphi_{n_{j_k}}, P_{n_{j_k}})
\]

which converges in \( \mathcal{H} \).

Now, we can state a theorem on the asymptotic behavior of solutions

Theorem 3. Let \( U_0 = (v^0, z^0, w^0, y^0, \tau^0, \theta^0, \varphi^0, P^0) \in \mathcal{D}(\mathfrak{A}) \) and \( U(t) \) be the solution of the boundary-initial-value problem \((33)\) with \( F = 0 \). Then

\[
\tau(t), \varphi(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \text{in} \quad W^{1,2}_0(\Sigma) \quad \text{and} \quad \theta(t), P(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \text{in} \quad L^2(\Sigma).
\]

Moreover

\[
v(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \text{in} \quad W^{1,2}_0(\Sigma) \quad \text{and} \quad z(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \text{in} \quad L^2(\Sigma)
\]

\[
w(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \text{in} \quad W^{1,2}_0(\Sigma) \quad \text{and} \quad y(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \text{in} \quad L^2(\Sigma).
\]
whenever the system
\[
\begin{align*}
\mathbf{Av} + \mathbf{Bw} + \lambda^2 \mathbf{v} &= 0 \text{ in } \Sigma \\
\mathbf{Gv} + \mathbf{Hw} + \lambda^2 \mathbf{w} &= 0 \text{ in } \Sigma \\
\mathbf{Jz} &= 0 \text{ in } \Sigma \\
\mathbf{Lz} &= 0 \text{ in } \Sigma \\
\mathbf{v} &= 0 \text{ on } \Gamma \\
\mathbf{w} &= 0 \text{ on } \Gamma
\end{align*}
\]
has only the null solution.

**Proof.** To prove the theorem we have to study the structure of the \( \omega \)-limit set. Thus, we must study the equation
\[
\mathbf{a}(U) = i\lambda U
\]
for some real number \( \lambda \), where \( U \in \mathcal{D}(\mathbf{a}) \) and \( \mathbf{a} = \mathbf{a}(\mathcal{L}) \) is the generator of a group on \( \mathcal{L} \). If \( U \in \mathcal{L} \) then \( < \mathbf{a}(U), U > = 0 \). Under the condition (6), it follows that \( \theta = P = 0 \) and then \( \tau = \varphi = 0 \). Thus, the asymptotic behavior of the temperature and the chemical potential is proved.

Now, Eq. (42) can be rewritten as
\[
\begin{pmatrix}
0 & \mathbf{Id} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{A} & 0 & \mathbf{B} & 0 & 0 & \mathbf{C} & 0 & \mathbf{D} \\
0 & 0 & 0 & \mathbf{Id} & 0 & 0 & 0 & 0 \\
\mathbf{G} & 0 & \mathbf{H} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{Id} & 0 & 0 & 0 \\
0 & \mathbf{J} & 0 & 0 & \mathbf{K} & \mathbf{L} & \mathbf{M} & \mathbf{N} \\
0 & 0 & 0 & 0 & 0 & \mathbf{Id} & 0 & 0 \\
0 & \mathbf{L} & 0 & 0 & \mathbf{U} & \mathbf{V} & \mathbf{W} & \mathbf{X}
\end{pmatrix}
\begin{pmatrix}
\mathbf{v} \\
\mathbf{z} \\
\mathbf{w} \\
\mathbf{y} \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
= i\lambda
\begin{pmatrix}
\mathbf{v} \\
\mathbf{z} \\
\mathbf{w} \\
\mathbf{y} \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]
Introducing the first equation into the others we obtain the system (41).

If the system (41) has only the trivial solution, we obtain \( \omega \)-limit \( (U_0) = 0 \) and \( \mathbf{v}(t) \to 0 \) in \( \mathbf{W}^{1,2}_0(\Sigma) \), \( \mathbf{z}(t) \to 0 \) in \( \mathbf{L}^2(\Sigma) \), \( \mathbf{w}(t) \to 0 \) in \( \mathbf{L}^2(\Sigma) \) and \( \mathbf{y}(t) \to 0 \) in \( \mathbf{L}^2(\Sigma) \) when \( t \to \infty \).

6. The spatial decay estimates

Let us consider an unbounded body, that is we assume that \( \Sigma \) is an unbounded regular region. For the plate whose middle surface has a specific shape, we prove that a measure associated with the thermodynamic process decays faster than an exponential of a polynomial of second degree.

Following Chirita and Ciarletta (1999), we consider a given time \( \mathcal{T} \in (0, \infty) \). We denote by \( \mathcal{D}_\mathcal{T} \) the support of the initial and boundary data, the body force, the heat supply and the diffusion supply on the time interval \( [0, \mathcal{T}] \), that is, the set of all \( \mathbf{x} \in \Sigma \) such that

(i) if \( \mathbf{x} \in \Sigma \), then
\[
\begin{align*}
v^0_\alpha(\mathbf{x}) &\neq 0 \text{ or } w^0_\alpha(\mathbf{x}) \neq 0 \text{ or } y^0_\alpha(\mathbf{x}) \neq 0 \text{ or } \tau^0(\mathbf{x}) \neq 0 \text{ or } \theta^0(\mathbf{x}) \neq 0 \\
or \varphi^0(\mathbf{x}) \neq 0 \text{ or } P^0(\mathbf{x}) \neq 0 \\
or \mathcal{f}_\alpha(\mathbf{x}, t) \neq 0 \text{ or } \mathcal{f}(\mathbf{x}, t) \neq 0 \text{ or } W(\mathbf{x}, t) \neq 0 \text{ or } V(\mathbf{x}, t) \neq 0 \text{ for some } t \in [0, \mathcal{T}],
\end{align*}
\]

(ii) if \( \mathbf{x} \in \partial \Sigma \), we have
\[
v_\alpha(\mathbf{x}, t) \neq 0 \text{ or } w(\mathbf{x}, t) \neq 0 \text{ or } \tau(\mathbf{x}, t) \neq 0 \text{ or } \theta(\mathbf{x}, t) \neq 0 \text{ or } \varphi(\mathbf{x}, t) \neq 0 \text{ or } P(\mathbf{x}, t) \neq 0 \text{ for some } t \in [0, \mathcal{T}].
\]

Following Horgan et al. (1984), we assume that the support \( \mathcal{D}_\mathcal{T} \) of the initial and boundary data is enclosed in the half-space \( x_2 < 0 \). We introduce the notation \( S_2 \) for the open cross-section of \( \Sigma \) for which \( x_2 = z, z \geq 0 \) and whose unit normal vector is \( (0, 1) \). We assume that the unbounded set \( \Sigma \) is so that \( S_2 \) is bounded for all finite \( z \in [0, \infty) \). We denote by \( \Sigma_z \) that portion of \( \Sigma \) for which \( x_2 > z \).
We introduce the following function,
\[
J(z, t) = -\int_0^t \int_{S_z} \left( M_{2z}(s) \tilde{v}_\alpha(s) + 2hN_2(s) \tilde{w}(s) - \Psi_2(s) \theta(s) - \Omega_2(s) P(s) \right) dx_1 ds.
\]  
(44)

As we want to obtain an upper estimate of the spatial decay of the solutions, it is natural to assume that
\[
\lim_{z \to \infty} J(z, t) = 0
\]  
(45)

for every finite time. Using the constitutive equations, the divergence theorem and having in mind the definition of \( S_z \) and \( \Sigma_z \), we deduce that the function \( J(z, t) \) can be written in the form
\[
J(z, t) = \int_{S_z} \left( \mathcal{K}(t) + \mathcal{V}(t) + \mathcal{Y}(t) \right) da + \int_0^t \int_{S_z} D(s) da ds,
\]  
(46)

where
\[
\mathcal{K}(t) = \frac{1}{2} \rho \left( \dot{v}_\alpha(t) \dot{v}_\alpha(t) + 2h \dot{w}^2(t) \right),
\]
\[
D(t) = I(k_2 \theta_\alpha \theta_\alpha + 2h \theta_\alpha P_\alpha + h_2 P_\alpha P_\alpha) + 2h(k_2 \theta^2 + 2h \theta P + h_2 P^2),
\]  
(47)

\[
\mathcal{V}(t) = \frac{1}{2} I(c \theta^2 + 2k \theta P + r P^2)
\]

and \( \mathcal{Y} \) is defined by (35). Further, we introduce the function
\[
E(z, t) = \int_z^\infty J(r, t) dr.
\]  
(48)

Lemma 4. There exists a positive constant \( \zeta \) depending on \( k_2 \) and \( h_2 \) such that
\[
\int_{S_z} I(k_2 \theta^2 + h_2 P^2) dx_1 \leq \zeta \frac{\partial^2 E}{\partial z^2}.
\]  
(49)

Proof. It is easy to see that
\[
\frac{\partial E}{\partial z} = -J(z, t)
\]  
(50)

and
\[
\frac{\partial^2 E}{\partial z^2}(z, t) = \int_{S_z} \left( \mathcal{K}(t) + \mathcal{V}(t) + \mathcal{Y}(t) \right) dx_1 + \int_0^t \int_{S_z} D(s) dx_1 ds.
\]  
(51)

On the other hand, we have
\[
k_2 \theta^2 + h_2 P^2 \leq \alpha (\theta^2 + P^2),
\]
with \( \alpha = \max\{k_2, h_2\} > 0 \). Moreover, being \( c \theta^2 + 2k \theta P + r P^2 \) a positive definite quadratic form (see (5)), denoting with \( 0 < k_{\min} \leq k_{\max} \) the smallest and the largest eigenvalues, we have
\[
k_{\min} (\theta^2 + P^2) \leq c \theta^2 + 2k \theta P + r P^2 \leq k_{\max} (\theta^2 + P^2).
\]

Then if we set \( \zeta = \alpha/k_{\min} \), we get
\[
k_2 \theta^2 + h_2 P^2 \leq \zeta k_{\min} (\theta^2 + P^2) \leq \zeta (c \theta^2 + 2k \theta P + r P^2),
\]
so that
\[
\frac{1}{2} \int_{S_z} I \left( k_2 \theta^2 + h_2 P^2 \right) dx_1 \leq \frac{1}{2} \zeta \int_{S_z} I \left( c \theta^2 + 2k \theta P + r P^2 \right) dx_1 = \zeta \int_{S_z} \mathcal{V}(t) dx_1
\]
\[
\leq \zeta \left( \int_{S_z} \left( \mathcal{K}(t) + \mathcal{V}(t) + \mathcal{Y}(t) \right) dx_1 + \int_0^t \int_{S_z} D(s) dx_1 ds \right) = \zeta \frac{\partial^2 E}{\partial z^2}(z, t).
\]

Now, we can state the main result of this section.
Theorem 4. There exists a positive constant $\xi$ depending on the constitutive constants such that the function $E$ decays exponentially in terms of the square of the distance $z$ from the support $\mathcal{D}_x$ of the external given data when $z > \xi t$.

Proof. From the constitutive equations, and the relations (44) and (48), we have
\[
\frac{\partial E}{\partial t}(z, t) = -\int_{\mathcal{D}_z} \left( M_{2a}(t)\dot{\nu}_a(t) + 2hN_2(t)\dot{\nu}(t) - \Psi_2(t)\theta(t) - \Omega_2(t)P(t) \right) da.
\]
(52)
It is worth remarking that
\[
\int_{\mathcal{D}_z} \Psi_2(t)\theta(t) da = -\int_{\mathcal{D}_z} I(k_1\tau_2 + h_1\dot{\nu}_2 + k_2\dot{\theta}_2 + h_2P_2)\theta da
\]
\[= -\int_{\mathcal{D}_z} I(k_1\tau_2 + h_1\dot{\nu}_2 + h_2P_2)\theta da - \frac{1}{2} \int_{\mathcal{D}_z} I\kappa_2\theta^2 dx_1.
\]
(53)
and
\[
\int_{\mathcal{D}_z} \Omega_2(t)P(t) da = -\int_{\mathcal{D}_z} I(h_1\dot{\nu}_2 + h_1\tau_2 + h_2\dot{\theta}_2 + h_2P_2)P da
\]
\[= -\int_{\mathcal{D}_z} I(h_1\dot{\nu}_2 + h_1\tau_2 + h_2P_2)P da - \frac{1}{2} \int_{\mathcal{D}_z} I\kappa_2P^2 dx_1.
\]
(54)
Thus we have
\[
\frac{\partial E}{\partial t}(z, t) = -\int_{\mathcal{D}_z} \left( M_{2a}(t)\dot{\nu}_a(t) + 2hN_2(t)\dot{\nu}(t) + I(k_1\tau_2 + h_1\dot{\nu}_2 + h_2P_2)\theta + I(h_1\dot{\nu}_2 + h_1\tau_2 + h_2P_2)P \right) da
\]
\[+ \frac{1}{2} \int_{\mathcal{D}_z} I\kappa_2\theta^2 + h_2P^2 dx_1.
\]
(55)
Our next step is to estimate the time derivative of $E$ in terms of the first two spatial derivatives of $E$. We have
\[
\left| \int_{\mathcal{D}_z} \left( M_{2a}(t)\dot{\nu}_a(t) + 2hN_2(t)\dot{\nu}(t) + I(k_1\tau_2 + h_1\dot{\nu}_2 + h_2P_2)\theta + I(h_1\dot{\nu}_2 + h_1\tau_2 + h_2P_2)P \right) da \right| \leq \xi J(z, t) = -\xi \frac{\partial E}{\partial z}
\]
where $\xi$ can be easily calculated in terms of the constitutive constants and parameters. This estimate together with (49) allow us to obtain the inequality
\[
\frac{\partial E}{\partial t} \leq \xi \frac{\partial^2 E}{\partial z^2} - \xi \frac{\partial E}{\partial z}.
\]
Following Horgan et al. (1984), such inequality can be further treated by the Comparison Principle (Tikhonov and Samarskii 1964). On this basis we can conclude that
\[
E(z, t) \leq \left( \max_{s \in [0, t]} E(0, s) \right) e^{\frac{\xi t}{\sqrt{\pi}}} H(z, t)
\]
where
\[
H(z, t) = \frac{1}{2\sqrt{\pi} J(t)} \int_0^t z s^{-3/2} e^{-\frac{s^2}{4z^2}} e^{rac{s^2}{4z^2}} ds.
\]
Using the estimate discussed by Pompei and Scalia (2002), we have
\[
H(z, t) \leq \frac{2z}{z^2 - \xi^2 t^2} e^{\frac{\xi t}{\sqrt{\pi}}(\xi^2 t + \frac{1}{t})}
\]
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for $z > \xi t$ and $t > 0$. Thus, we find the estimate

$$E(z, t) \leq 2 \max_{s \in [0, t]} E(0, s) \frac{z \sqrt{\xi}}{z^2 - \xi^2} t e^{-\frac{t^2}{2} e^{\frac{z^2}{2} - \xi^2}}$$

for $z > \xi t$ and $t > 0$.

**Remark 1.** Note that this result is not valid in the case of type II theory because of $k_2 = h_2 = 0$ gives $\zeta = 0$.

### 7. Impossibility of localization in time

In previous sections we have proved that the solutions of type III theory are stable in the sense of Lyapunov, decay asymptotically and spatially. A natural question is to ask if the decay is fast enough to guarantee that the solution vanishes in a finite time. In fact, when the dissipation mechanism in a system is sufficiently strong, the localization of solutions in the time variable can hold. This means that the decay of the solutions that sufficiency fast to guarantee that they vanish after a finite time.

In the context of Green-Naghdi thermoelasticity of type III, Quintanilla (2007) has shown that the thermal dissipation is not strong enough to obtain the localization in time of the solutions. In this section we assume the quadratic form (29) positive definite and prove that the further dissipation effects due to diffusion is not sufficiently strong to guarantee that the thermomechanical deformations vanish after a finite interval of time. This means that, in absence of sources, the only solution for the evolutive problem that vanishes after a finite time is the null solution, that is the following theorem holds.

**Theorem 5.** Let $(v, w, \tau, \wp)$ be a solution of the system (25), (26) and (28) which vanishes after a finite time $t_0$. Then $(v, w, \tau, \wp) \equiv (0, 0, 0, 0)$ for every $t \geq 0$.

In order to prove this theorem, generalizing the technique used in Quintanilla (2007), we show the uniqueness of solutions for the related backward in time problem. Theses problems are relevant from the mechanical point of view when we want to have some information about what happened in the past by means of the information that we have at this moment.

For our model, the system of equations which govern the backward in time problem is given by

$$
\begin{align*}
\rho I \ddot{v}_a &= I (\mu \Delta v_a + (\lambda + \mu)\varepsilon_{\beta,\beta v_a} + d_1 \tau, v_a + d_2 \phi, v_a) - 2h \mu (v_a + w, v_a), \\
\rho \dot{w} &= \mu \Delta w + \mu v_{\alpha, v_a}, \\
cI \ddot{\tau} + kI \ddot{\wp} &= I (k_1 \Delta \tau + h_1 \Delta \wp - k_2 \Delta \phi - h_2 \Delta \wp) + I d_1 \dot{v}_{a, v_a} - 2h (k_1 \tau + h_1 \wp - k_2 \phi - h_2 \wp), \\
kI \ddot{\tau} + cI \ddot{\wp} &= I (h_1 \Delta \wp + h_1 \Delta \tau - h_2 \Delta \phi - h_2 \Delta \wp) + I d_2 \dot{v}_{a, a} - 2h (h_1 \wp + h_1 \tau - h_2 \phi - h_2 \wp).
\end{align*}
$$

**Proposition 1 (Uniqueness).** Let $(v, w, \tau, \wp)$ be a solution of the system (56), (28) with null initial data and sources. Then $(v, w, \tau, \wp) = (0, 0, 0, 0)$ for every $t \geq 0$.

**Proof.** Let us introduce the following functionals

$$
\begin{align*}
E_1(t) &= \frac{1}{2} \int_{\Sigma} \left( \rho I \dot{v}_a \dot{v}_a + 2h \rho \dot{w}^2 + cI \ddot{\tau}^2 + 2kI \ddot{\wp} + rI \phi^2 + 2 \phi' \right) da, \\
E_2(t) &= \frac{1}{2} \int_{\Sigma} \left( \rho I \dot{v}_a \dot{v}_a + 2h \rho \dot{w}^2 - cI \ddot{\tau}^2 - 2kI \ddot{\wp} - rI \phi^2 + 2 \phi' \right) da, \\
E_3(t) &= \int_{\Sigma} I \left( \rho v_{\alpha, v_a} + 2h \rho \dot{w} - c\dot{\tau} - m \ddot{\wp} - \kappa (\tau \phi' + \phi) \\
&+ \frac{1}{2} (k_2 \tau, v_{\alpha, v_a} + h_2 \phi, v_{\alpha, v_a}) + \frac{h}{I} (k_2 \tau^2 + 2h_2 \tau \wp + h_2 \wp^2) \right) da,
\end{align*}
$$

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where $\mathcal{V}$ is defined by (35) and

$$2\mathcal{V} = I\left(\lambda \varepsilon_{\alpha \alpha} \varepsilon_{\beta \beta} + 2 \mu \varepsilon_{\alpha \beta} \varepsilon_{\alpha \beta} - k_1 \tau^2_{\alpha} - 2h_1 \tau_{\alpha} \phi_{\alpha} - h_1 \phi^2_{\alpha}\right) + 2h \left(\mu \gamma_{\alpha \gamma} - k_1 \tau^2 - 2h_1 \phi \rho + h_1 \phi^2\right). \quad (58)$$

We compute now their time derivatives. By multiplying the first equation of (56) by $\dot{w}$, the second one by $\dot{\tau}$ and the fourth one by $\dot{\phi}$, we get

$$\dot{E}_1(t) = \int_{\Sigma} I \left(k_2 \dot{\tau}_{\alpha} \dot{\tau}_{\alpha} + 2h_2 \dot{\phi}_{\alpha} \dot{\phi}_{\alpha} + 2h \dot{h}\left(k_2 \dot{\tau}^2 + 2h_2 \dot{\phi}^2 + h_2 \phi^2\right)\right) da.$$

On the other hand, if we multiply equation of (56) by $\dot{\phi}_\alpha$, the second one by $\dot{w}$, the third one by $-\dot{\tau}$ and the fourth one by $-\dot{\phi}$, we obtain

$$\dot{E}_2(t) = -\int_{\Sigma} I \left(2d_1 \dot{\tau}_{\alpha} \dot{\tau}_{\alpha} + 2d_2 \dot{\phi}_{\alpha} \dot{\phi}_{\alpha} + k_2 \dot{\tau}_{\alpha} \dot{\phi}_{\alpha} + 2h_2 \dot{\phi}_{\alpha} \dot{\phi}_{\alpha} + 2h \dot{h}\left(k_2 \dot{\tau}^2 + 2h_2 \dot{\phi}^2 + h_2 \phi^2\right)\right) da.$$

Finally, if we multiply the first equation of (56) by $v_{\alpha}$, the second one by $w$, the third one by $-\tau$ and the fourth one by $-\phi$, we get

$$\dot{E}_3(t) = -\int_{\Sigma} I \left(d_1 \dot{\tau}_{\alpha} v_{\alpha} - d_1 \tau_{\alpha} \dot{v}_{\alpha} + d_2 \dot{\phi}_{\alpha} v_{\alpha} - d_1 \phi_{\alpha} \dot{v}_{\alpha} - 2h \dot{h}\left(k_2 \dot{\tau}^2 + 2h_2 \dot{\phi}^2 + h_2 \phi^2\right)\right) da. \quad (59)$$

Moreover, a well-known identity for type III thermoelasticity (see Eqn. (3.9) in Quintanilla and Straughan (2000)) for our model becomes

$$\int_{\Sigma} I \left(c_{\xi}^2 + 2\kappa \dot{\phi} + m \phi^2\right) da = \int_{\Sigma} \left(\rho I \dot{v}_{\alpha} \dot{v}_{\alpha} + 2h \rho \dot{w}^2 - 2\mathcal{V}\right) da.$$

Then we have

$$E_2(t) = \int_{\Sigma} 2\mathcal{V} da$$

and

$$\dot{E}_3(t) = -\int_{\Sigma} I \left(d_1 \dot{\tau}_{\alpha} v_{\alpha} - d_1 \tau_{\alpha} \dot{v}_{\alpha} + d_2 \dot{\phi}_{\alpha} v_{\alpha} - d_1 \phi_{\alpha} \dot{v}_{\alpha}\right) da.$$

We consider the function

$$\mathcal{E}(t) = \int_0^t \left(\epsilon E_1(s) + E_2(s) + \lambda E_3(s)\right) ds$$

$$= \frac{1}{2} \int_0^t \int_{\Sigma} I \left(\epsilon \rho \dot{v}_{\alpha} \dot{v}_{\alpha} + 2h \rho \dot{w}^2 + \epsilon \dot{\tau}_{\alpha}^2 + 2\kappa \dot{\phi} + \mu_{\phi} \phi^2 + (\epsilon + 2)\left[\lambda \varepsilon_{\alpha \alpha} \varepsilon_{\beta \beta} + 2 \mu \varepsilon_{\alpha \beta} \varepsilon_{\alpha \beta} + 2h \mu \gamma_{\alpha \gamma}\right]\right) da ds$$

$$+ \frac{1}{2} \int_0^t \int_{\Sigma} 2h \left(\lambda k_2 + (\epsilon - 2) k_1\right) \tau_{\alpha}^2 + 2\left(\lambda h_2 + (\epsilon - 2) h_1\right) \phi_{\alpha}^2 da ds$$

$$+ \frac{1}{2} \int_0^t \int_{\Sigma} 2h \left(\lambda k_2 + (\epsilon - 2) k_1\right) \tau_{\alpha}^2 + 2\left(\lambda h_2 + (\epsilon - 2) h_1\right) \phi_{\alpha}^2 da ds$$

$$+ \frac{1}{2} \int_0^t \int_{\Sigma} \left(\rho \dot{v}_{\alpha} \dot{v}_{\alpha} + 2h \rho \dot{w}^2 - c_{\xi} \tau - m \phi \phi - \kappa \left(\dot{\tau} + \dot{\phi}\right)\right) da ds$$

where $\epsilon$ and $\lambda$ are positive suitable constants such that the quadratic forms

$$\int_{\Sigma} I \left(\lambda k_2 + (\epsilon - 2) k_1\right) \tau_{\alpha}^2 + 2\left(\lambda h_2 + (\epsilon - 2) h_1\right) \phi_{\alpha}^2 da$$

are negative definite.
and

$$\int_\Sigma I \left( \bar{\lambda} k_2 + (\epsilon - 2) k_1 \right) \tau^2 + 2[\bar{\lambda} h_2 + (\epsilon - 2) \tilde{h}_1] \tau \varphi + [\bar{\lambda} h_2 + (\epsilon - 2) h_1] \varphi^2 \right) da$$

are positive definite.

By using the null initial data hypothesis and the Poincaré inequality we have

$$\hat{\lambda} \int_0^t \int_\Sigma I \left( \rho \dot{v}_a \dot{v}_a + 2 \frac{h}{\tau} \rho \dot{w} - c \dot{\tau} - m \dot{\varphi} - \kappa (\tau \dot{\varphi} + \dot{\tau}) \right) da$$

$$\leq \frac{\epsilon}{4} \int_0^t \int_\Sigma I \left( \rho \dot{v}_a^2 + 2 \frac{h}{\tau} \rho \dot{w}^2 + c \dot{\tau}^2 + 2 \kappa \dot{\tau} \dot{\varphi} + m \dot{\varphi}^2 \right) dv ds$$

for any \( t \leq t_0 \), where \( t_0 \) is a positive time which depends on \( \hat{\lambda}, \epsilon \) and the constitutive coefficients. Therefore \( \mathcal{E}(t) \) is a positive definite quadratic form for \( 0 \leq t \leq t_0 \), in particular

$$\mathcal{E}(t) \geq \frac{1}{4} \int_0^t \int_\Sigma I \left( \rho \dot{v}_a \dot{v}_a + 2 \frac{h}{\tau} \rho \dot{w}^2 + c \dot{\tau}^2 + 2 \kappa \dot{\tau} \dot{\varphi} + m \dot{\varphi}^2 + (\epsilon + 2) \left[ \lambda \varepsilon_{a \beta} \varepsilon_{b \beta} + 2 \mu \varepsilon_{a \beta} \varepsilon_{a \beta} + 2 h \mu \gamma_{a \gamma} \right] \right) dv ds$$

$$+ \frac{1}{4} \int_0^t \int_\Sigma I \left( \bar{\lambda} k_2 + (\epsilon - 2) k_1 \right) \tau^2 + 2 [\bar{\lambda} h_2 + (\epsilon - 2) \tilde{h}_1] \tau \varphi + [\bar{\lambda} h_2 + (\epsilon - 2) h_1] \varphi^2 \right) da$$

$$+ \frac{1}{4} \int_0^t \int_\Sigma I \left( \bar{\lambda} k_2 + (\epsilon - 2) k_1 \right) \tau^2 + 2 [\bar{\lambda} h_2 + (\epsilon - 2) \tilde{h}_1] \tau \varphi + [\bar{\lambda} h_2 + (\epsilon - 2) h_1] \varphi^2 \right) da$$

(60)

Moreover, recalling the null initial data assumption, we have

$$\mathcal{E}(t) = (\epsilon - 1) \int_0^t \int_\Sigma I \left( k_2 \dot{x}_a \dot{x}_a + 2 h_2 \dot{x}_a \dot{\varphi}_a + h_2 \dot{\varphi}_a \dot{\varphi}_a + 2 \frac{h}{\tau} (k_2 \dot{x}^2 + 2 h_2 \dot{x} \dot{P} + h_2 \dot{\varphi}^2) \right) d\text{ads} - \int_0^t \int_\Sigma \mathcal{M} d\text{ads}$$

where

$$\mathcal{M} = I \left( 2 d_1 \dot{x}_a \dot{x}_a + 2 d_2 \dot{\varphi}_a \dot{x}_a + \beta \left[ d_1 \dot{x}_a \dot{\varphi}_a - d_1 \dot{\varphi}_a \dot{x}_a + d_2 \dot{\varphi}_a \dot{\varphi}_a - d_2 \dot{x}_a \dot{x}_a \right] \right).$$

Choosing \( 0 < \epsilon < 1 \) and using the inequality of arithmetic and geometric means, we have

$$\left| \int_0^t \int_\Sigma \mathcal{M} d\text{ads} \right| \leq (1 - \epsilon) \int_0^t \int_\Sigma I \left( k_2 \dot{x}_a \dot{x}_a + 2 h_2 \dot{x}_a \dot{\varphi}_a + h_2 \dot{\varphi}_a \dot{\varphi}_a + 2 \frac{h}{\tau} (k_2 \dot{x}^2 + 2 h_2 \dot{x} \dot{P} + h_2 \dot{\varphi}^2) \right) d\text{ads}$$

$$+ K_1 \int_0^t \int_\Sigma I \left( \rho \dot{v}_a \dot{v}_a + 2 h \rho \dot{w} \dot{w} \right) d\text{ads} + K_2 \int_0^t \int_\Sigma I \left( c \dot{\tau}^2 + 2 \kappa \dot{\tau} \dot{\varphi} + m \dot{\varphi}^2 \right) d\text{ads}$$

$$+ K_3 \int_0^t \int_\Sigma I \left( \lambda \varepsilon_{a \beta} \varepsilon_{b \beta} + 2 \mu \varepsilon_{a \beta} \varepsilon_{a \beta} + 2 h \mu \gamma_{a \gamma} \right) d\text{ads}$$

$$+ \frac{1}{2} \int_0^t \int_\Sigma I \left( [\bar{\lambda} k_2 + (\epsilon - 2) k_1] \tau^2 + 2 [\bar{\lambda} h_2 + (\epsilon - 2) \tilde{h}_1] \tau \varphi + [\bar{\lambda} h_2 + (\epsilon - 2) h_1] \varphi^2 \right) d\text{ads}$$

$$+ \frac{1}{2} \int_0^t \int_\Sigma I \left( [\bar{\lambda} k_2 + (\epsilon - 2) k_1] \tau^2 + 2 [\bar{\lambda} h_2 + (\epsilon - 2) \tilde{h}_1] \tau \varphi + [\bar{\lambda} h_2 + (\epsilon - 2) h_1] \varphi^2 \right) d\text{ads}$$

where the positive constants \( K_i \) can be calculated by standard methods, so that

$$\mathcal{E}(t) \leq K \int_0^t \int_\Sigma I \left( \rho \dot{v}_a \dot{v}_a + 2 h \rho \dot{w} \dot{w} + c \dot{\tau}^2 + 2 \kappa \dot{\tau} \dot{\varphi} + m \dot{\varphi}^2 + (\epsilon + 2) \left[ \lambda \varepsilon_{a \beta} \varepsilon_{b \beta} + 2 \mu \varepsilon_{a \beta} \varepsilon_{a \beta} + 2 h \mu \gamma_{a \gamma} \right] \right) d\text{ads}$$

$$+ K \int_0^t \int_\Sigma I \left( [\bar{\lambda} k_2 + (\epsilon - 2) k_1] \tau^2 + 2 [\bar{\lambda} h_2 + (\epsilon - 2) \tilde{h}_1] \tau \varphi + [\bar{\lambda} h_2 + (\epsilon - 2) h_1] \varphi^2 \right) d\text{ads}$$

$$+ K \int_0^t \int_\Sigma 2 h \left( [\bar{\lambda} k_2 + (\epsilon - 2) k_1] \tau^2 + 2 [\bar{\lambda} h_2 + (\epsilon - 2) \tilde{h}_1] \tau \varphi + [\bar{\lambda} h_2 + (\epsilon - 2) h_1] \varphi^2 \right) d\text{ads}$$

(61)
with $K = \max\{\frac{1}{2}, K_1, K_2, K_3\}$. Inequalities (60) and (61) yield

$$\dot{E}(t) \leq 4K E(t), \quad 0 \leq t \leq t_0.$$ 

This inequality and the null initial data imply $E(t) \equiv 0$ if $0 \leq t \leq t_0$. Reiterating this argument on each subinterval $[(n-1)t_0, nt_0]$ we obtain $E(t) \equiv 0$ for $t \geq 0$.

If we take into account the definition of $E(t)$, the uniqueness result is proved. □

8. Conclusions

The results established in this paper can be summarized as follows:

(i) We have derived the linear theory of bending plate for thermoelastic diffusion materials of type II and III in the frame of Green-Naghdi theory. We have shown that the type II theory is conservative and the solutions cannot decay with respect to time. It is well known that, in general, the solutions of type III decay with respect to time.

(ii) We have proved that the problem of bending plate for thermoelastic diffusion materials of type III (or type II) is well posed. This result proves that in the motion following any sufficiently small change in the external system, the solution of the initial-boundary value problem is everywhere arbitrary small in magnitude.

(iii) We have have shown the asymptotic behaviour of solutions of type III model.

(iv) We have have shown that the spatial decay for the solutions corresponding to unbounded plates, for a fixed time, at large distance to the support $\mathcal{D}_\mathcal{F}$ of the given data is dominated by term $e^{-\frac{\zeta}{T}t}$, where $\zeta$ depends only on the thermal and the diffusion coefficients characterizing the type III model. We can conclude that at large distance from the support of the external given data, the spatial decay is influenced only by the thermal and diffusion effects arising only in type III model (see Remark 1).

(v) We have derived the uniqueness of solutions of type III model for the backward in time problem. Thus, it says the impossibility of localization in time of the solutions. From a thermomechanical point of view, this result says that combination of thermal and diffusion dissipations in an elastic bending plate is not sufficiently strong to guarantee that the thermomechanical deformations vanish after a finite interval of time. Results of this kind are a good complement to the ones we have obtained in Passarella et al. (2013); Quintanilla (2007).

We believe that by adapting the same analysis, we can prove the impossibility of localization of solutions in the case of exterior domains, even when the solutions can be unbounded, whether the spatial variable goes to infinity. This proof is omitted for the sake of brevity.

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