On the minimum of independent collecting processes via the Stirling numbers of the second kind

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Abstract

We consider the combinatorial problem where $p$ players aim to a complete set of $N$ different types of items (species) which are uniformly distributed. Let the random variables $T_{N(i)}$, $i = 1, 2, \ldots, p$ denoting the number of trials needed until all $N$ types are detected (at least once), respectively for each player. This paper studies the impact of the number $p$ in the asymptotics of the expectation, the second moment, and the variance of the random variable

$$M_{N(p)} := \bigwedge_{i=1}^{p} T_{N(i)}, \quad N \to \infty.$$ 

The main ingredient in the expression of these quantities are sums involving the Stirling numbers of the second kind; for which the asymptotics are explored. At the end of the paper we conjecture on a remarkable combinatorial identity, regarding alternating binomial sums. These sums have been studied (mainly) by P. Flajolet due to their applications to digital search trees and quadtrees.

Keywords. Species detection; Coupon collector’s problem; Stirling numbers of the second kind, digital search trees, quadtrees.

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1 Introduction

The coupon collector’s problem (CCP) in its classic form refers to a population whose members are of $N$ different species. For $1 \leq j \leq N$ we denote by $p_j$ the probability that a member of the population is of type $j$, where $p_j > 0$ and
\[ \sum_{j=1}^{N} p_j = 1. \] We refer to the \( p_j \)'s as the coupon probabilities. The members of the population are sampled independently with replacement (alternatively, the population is assumed very large) and their types are recorded.

Let \( T_{N(1)}, T_{N(2)}, \ldots, T_{N(p)} \), the random variables denoting the number of trials needed until all \( N \) types are detected (at least once) for each one of \( p \) independent collectors (here and in what follows \( p \) is a fixed positive integer). A main task of this note is to study the expectation of the random variable

\[ M_{N(p)} := \bigwedge_{i=1}^{p} T_{N(i)}, \quad N \to \infty, \]

i.e., the minimum of the random variables \( \{ T_{N(1)}, T_{N(2)}, \ldots, T_{N(p)} \} \), when the coupon probabilities are uniformly distributed, namely when \( p_j = 1/N, \ j = 1, 2, \ldots, N \). Notice that for \( p = 1 \) we have the classic version of the CCP. Since \( M_{N(p)} \) is a non negative random variable

\[
E \left[ M_{N(p)} \right] = \sum_{k=1}^{\infty} P \left\{ M_{N(p)} \geq k \right\}.
\] (1.1)

By independence we have

\[
P \left\{ M_{N(p)} \geq k \right\} = P \left\{ T_N \geq k \right\}^p, \quad k = 1, 2, \ldots, \]

where \( T_N \) is the random variable denoting the number of trials one collector needs until all \( N \) different types are detected. Let us consider the probability \( P \{ T_N \geq k \} \) for general values of \( p_j \). Clearly, \( k \geq N \). For each \( j \in \{1, \ldots, N\} \) it is convenient to introduce the event \( A^k_j \), that the type \( j \) is not detected until trial \( k \) (included). Then

\[
P \{ T_N \geq k \} = P \left( A^k_1 \cup \cdots \cup A^k_N \right).
\]

By invoking the inclusion-exclusion principle one gets (see, e.g. [2])

\[
P \{ T_N \geq k \} = \sum_{J \subseteq \{1, \ldots, N\}, J \neq \emptyset} (-1)^{|J| - 1} \left[ 1 - \left( \sum_{j \in J} p_j \right) \right]^{k-1},
\]

where the sum extends over all \( 2^N - 1 \) nonempty subsets \( J \) of \( \{1, \ldots, N\} \), while \( |J| \) denotes the cardinality of \( J \). Setting \( p_j = 1/N \) we get

\[
P \{ T_N \geq k \} = (-1)^{N-1} \sum_{n=0}^{N-1} (-1)^n \binom{N}{n} \left( \frac{n}{N} \right)^{k-1}.
\] (1.3)

Recall that the Stirling numbers of the second kind count the ways to partition a set of \( k \) labeled objects into \( N \) nonempty unlabeled subsets and they can be
calculated by the so-called Euler’s formula for Stirling numbers (see, e.g. [8], pp.118–119):

\[ S(k, N) = \frac{1}{N!} \sum_{n=0}^{N} (-1)^{N-n} \binom{N}{n} n^k. \]

By invoking the Stirling numbers of the second kind in (1.3) we have

\[ P\{T_N \geq k\} = 1 - S(k - 1, N) \frac{N!}{N^{k-1}}. \]  \hfill (1.4)

In view of (1.2) and (1.4), relation (1.1) yields the interesting formula

\[ E\left[M_N(p)\right] = \sum_{k=0}^{\infty} \left(1 - S(k, N) \frac{N!}{N^k}\right)^p, \]  \hfill (1.5)

where, of course, \( S(k, N) = 0 \) for \( k < N \). The rest of our analysis is, mainly, devoted to the asymptotics of \( E\left[M_N(p)\right] \) as \( N \to \infty \).

## 2 Asymptotic analysis

Fix a positive integer \( N \). Let us set

\[ c_N := \min_{j \in \mathbb{Z}^+} \left\{ \frac{N + j}{\ln(N + j)} > N \right\}. \]  \hfill (2.1)

In words \( c_N \) is the smallest positive integer such that the fraction above is greater than \( N \). We have

\[ E\left[M_N(p)\right] = \sum_{k=0}^{N+c_N-1} \left(1 - S(k, N) \frac{N!}{N^k}\right)^p + \sum_{k=N+c_N}^{\infty} \left(1 - S(k, N) \frac{N!}{N^k}\right)^p. \]  \hfill (2.2)

The idea behind (2.2) is that the asymptotic behavior of the Stirling numbers of the second kind is known thanks to Erdős and Szekeres when

\[ N < \frac{k}{\ln k} \]  \hfill (2.3)

and thanks to [9] otherwise\(^1\) (Notice that the inequality (2.3) may also be written in terms of the Lambert \( W \) function). From here and it what follows we will call the first of the sums of (2.2) as \( S_1(N) \) and the second one as \( S_2(N) \). We start with \( S_2(N) \). In case where (2.3) holds, we have (see [9], pp.164)

\[ S(k, N) = \frac{N^k}{N!} \exp \left( \left(\frac{k}{2N} - N\right) e^{-\frac{k}{N}} \right) (1 + o(1)). \]  \hfill (2.4)

\(^1\)For more results regarding the behavior of the Stirling numbers of the second kind, we refer the interested reader to [9], [7], and [10].
Using (2.4) in $S_2(N)$ (of (2.2)), and from the comparison of sums and integrals we get

$$S_2(N) = \int_{N+c_N}^{\infty} \left(1 - \exp \left[ \left( \frac{x}{2N} - N \right) e^{-\frac{x}{N}} \right] \right) (1 + o(1)) \frac{p}{dx} \left(1 + o(1)\right). \quad (2.5)$$

Let us consider the integral

$$J(N; p) := \int_{N+c_N}^{\infty} \left(1 - \exp \left[ \left( \frac{t}{2N} - N \right) e^{-\frac{t}{N}} \right] \right) x^{p-1} \ln x \, dx. \quad (2.6)$$

Changing the variables as $e^{\frac{N+x}{N} - \frac{t}{N}} = x$ and integrating by parts yields

$$J(N; p) = -e^{-\frac{N+c_N}{N}} pN \int_0^1 \left(N + \frac{1}{2} \ln x + c_N \frac{x}{2N} \right) e^{-\frac{N+c_N}{N}} \left(N + \frac{\ln x}{2} - \frac{N+c_N}{2N} \right) x$$

$$\times \left(1 - e^{-\frac{N+c_N}{N}} \left(N + \frac{\ln x}{2} - \frac{N+c_N}{2N} \right) x \right) \ln x \, dx.$$

Thanks to the binomial theorem we get ($p$ is always a positive integer)

$$J(N; p) = -e^{-\frac{N+c_N}{N}} pN \sum_{j=0}^{p-1} (-1)^j \left(\binom{p-1}{j} \int_0^1 \left[N \left(\ln x + \frac{1}{2} \left(\ln x \right)^2 + \frac{c_N \ln x}{2N} \right) \right)$$

$$\times \left(1 - (j+1) e^{-\frac{N+c_N}{N}} \frac{\ln x}{2} + (j+1) e^{-\frac{N+c_N}{N}} \frac{N + c_N}{2N} x + O(x \ln x)^2 \right)$$

$$\times e^{- (j+1) x N e^{-\frac{N+c_N}{N}}} \, dx,$$

where we have also used the Taylor expansion of the exponential around $x = 0$.

Set

$$I_1(N) := \int_0^1 \left(\ln x \right) e^{- (j+1) x N e^{-\frac{N+c_N}{N}}} \, dx.$$

Changing the variables as $(j+1) x Ne^{-\frac{N+c_N}{N}} = y$ yields

$$I_1(N) = \frac{e^{\frac{N+c_N}{N}}}{(j+1) N} \times \left[\int_0^\infty e^{-y \ln y} dy - \int_{(j+1) Ne^{-\frac{N+c_N}{N}}}^{\infty} e^{-y} \ln y \, dy \right.$$

$$\left. - \ln[(j+1) Ne^{-\frac{N+c_N}{N}}] \left(1 - e^{-N e^{-\frac{N+c_N}{N}}} \right) \right].$$

But

$$\int_0^\infty e^{-y \ln y} \, dy = -\gamma,$$
where \( \gamma = 0.5772... \) is the Euler–Mascheroni constant (see, e.g. [1]). Hence,

\[
I_1(N) = \frac{e^{N+c_N}}{(j+1)/N} \left[ - \ln N - \gamma - \ln (1+j) + 1 + \frac{c_N}{N} + O \left( e^{-Nc} \right) \ln N \right].
\]

Working in a similar way with the rest of the integrals appearing in (2.7) and finally invoking (2.8) in (2.7) we get

\[
J(N; p) = pN \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} \left[ \ln N + \gamma - 1 + \ln (1+j) \right] - \frac{c_N}{N} \ln (j+1)
+ O \left( \frac{(\ln N)^2}{N} \right).
\]

Next observe that

\[
\sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} \frac{1}{j+1} = \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} \left( \int_0^1 x^j dx \right) = \int_0^1 (1-x)^{p-1} dx = \frac{1}{p}.
\]

Relation (2.10) simplifies (2.9). On the other hand let us define the constant

\[
c_p := \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} \ln (1+j). \tag{2.11}
\]

For example if \( p = 2 \) (namely, the case we have two independent collectors), then \( c_2 = -\ln 2/2 \). If \( p = 3 \) we have \( c_3 = -\ln 2 + (\ln 3/3) \), if \( p = 5 \) then \( c_5 = -4 \ln 2 + 2 \ln 3 + (\ln 5/5) \), etc. It is worth mentioning that for large values of \( p \) sums of the type of (2.11) are of importance in applied Mathematics. We will come back to this after Theorem 2.2 below. Using (2.10) in (2.9), and by invoking (2.9) and (2.7) one has as \( N \to \infty \)

\[
S_2(N) = N \left[ \ln N + \left( \gamma + p c_p - 1 - \frac{c_N}{N} \right) + O \left( \frac{(\ln N)^2}{N} \right) \right].
\]

Last task of our analysis is \( S_1(N) \) of (2.2). We need the behaviour of the Stirling numbers of the second kind \( S(k, N) \) when

\[
N \geq \frac{k}{\ln k}. \tag{2.13}
\]
G. Louchard (see [6]), studied this behaviour in the large deviation region, namely when
\[ N = k - k^\alpha, \quad a > \frac{1}{2}, \] (2.14)
In particular, he proved that
\[ S(k, N) = \frac{1}{\sqrt{2\pi k\alpha/2}} \exp \left\{ k^\alpha \left[ (2 - \alpha) \ln k + (1 - \ln 2) + O \left( k^{\alpha-1} \right) \right] \right\} \left( 1 + O \left( k^{\alpha-1} \right) \right). \] (2.15)
Our case is covered by (2.14). By invoking (2.15) in \( S_1(N) \) of (2.2), and applying Stirling’s formula we get
\[ S_1(N) = \sum_{k=0}^{N+cN-1} (1 + o(1))^p = N + c_N (1 + o(1)), \quad N \to \infty. \] (2.16)
From (2.2), (2.12) and (2.16) we arrive to our first main result, which we state in the following

**Theorem 2.1** Consider the classical coupon collector’s problem and \( p \) independent collectors aiming to complete a set of \( N \) different types of coupons, which are uniformly distributed. Let \( T_{N(1)}, T_{N(2)}, \ldots, T_{N(p)} \) the random variables denoting the number of trials needed until all \( N \) types are detected for each one of the \( p \) collectors. If we set
\[ M_{N(p)} := \bigwedge_{i=1}^{p} T_{N(i)}, \]
then
\[ E \left[ M_{N(p)} \right] = \sum_{k=0}^{\infty} \left( 1 - S(k, N) \frac{N!}{N^k} \right)^p, \]
where \( S(k, N) \) are the Stirling numbers of the second kind. Moreover as \( N \to \infty \) we have
\[ E \left[ M_{N(p)} \right] = N \left[ \ln N + (\gamma + p c_p) + O \left( \frac{(\ln N)^2}{N} \right) \right], \]
where \( c_p \) is the constant given by the formula
\[ c_p := \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} \frac{\ln (1 + j)}{j+1}. \] (2.17)

### 2.1 Second moment and Variance of \( M_{N(p)} \)

Here we will briefly present the asymptotics of the second moment for the non negative random variable \( M_{N(p)} \) of (2.5). We have
\[ E \left[ M_{N(p)}^2 \right] = 2 \sum_{k=1}^{\infty} k P \{ M_{N(p)} \geq k \} - \sum_{k=1}^{\infty} P \{ M_{N(p)} \geq k \} \cdot \]
Following the steps of \( E \left[ M_N(p) \right] \) we get

\[
E \left[ M_N^2(p) \right] = 2 \sum_{k=0}^{N+c_N-1} k \left( 1 - S(k,N) \frac{N!}{N^k} \right)^p + 2 \sum_{k=N+c_N}^{\infty} k \left( 1 - S(k,N) \frac{N!}{N^k} \right)^p - E \left[ M_N(p) \right].
\]

(2.18)

Let \( S_3(N) \) and \( S_4(N) \) the sums appearing in (2.18) above. Using the same approximation for the Stirling numbers of the second kind, we see that the key is to obtain asymptotics (as \( N \to \infty \)) for the integral

\[
L(N;p) := 2 \int_{N+c_N}^{\infty} t \left( 1 - \exp \left[ \left( \frac{t}{2N} - \frac{N}{2} \right) \right] \right)^p dt,
\]

(2.19)

which is the analog of the integral \( J(N;p) \) of (2.6). It is now straightforward to get asymptotics for \( L(N;p) \)

\[
L(N;p) = pN^2 \sum_{j=0}^{p-1} \frac{(-1)^j}{j!(p-1)} \left( \ln N \right)^2 + 2 (\gamma + \ln (1+j)) \ln N + \frac{2\gamma (1+j) - 1 + \frac{\pi^2}{6} + \gamma^2}{1+j}
\]

\[
+ \left( \frac{\ln (1+j)}{j+1} + O \left( \frac{\ln N}{N} \right) \right)
\]

as \( N \to \infty \), which in turn provides asymptotics for \( S_3(N) \). On the other hand if we treat \( S_4(N) \) as we treated \( S_1(N) \) of (2.16), we have

\[
S_4(N) = 2 \sum_{k=0}^{N+c_N-1} k \left( 1 - S(k,N) \frac{N!}{N^k} \right)^p
\]

\[
= (N+c_N) (N+c_N-1) (1+o(1))^p = N^2 (1+o(1)), \quad N \to \infty
\]

and finally, arrive at the following

**Theorem 2.2** Let \( M_N(p) \) as defined in Theorem 2.1. Then

\[
E \left[ M_{N(p)}^2 \right] = 2 \sum_{k=0}^{\infty} k \left( 1 - S(k,N) \frac{N!}{N^k} \right)^p - E \left[ M_{N(p)} \right],
\]

where \( S(k,N) \) are the Stirling numbers of the second kind. In particular as \( N \to \infty \) we have

\[
E \left[ M_{N(p)}^2 \right] = N^2 \left[ (\ln N)^2 + 2 (\gamma + p \ln \frac{p}{6}) \ln N + \gamma^2 + \frac{\pi^2}{6} + 2pc\ln N \right]
\]

\[
+ p w_p + O \left( \frac{(\ln N)^3}{N} \right),
\]

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where \( c_p \) is the constant given in Theorem 2.1, and \( w_p \) is the constant which may be calculated explicitly by the formula

\[
w_p := \sum_{j=0}^{p-1} (-1)^j \left( \frac{p-1}{j} \right) \frac{(\ln (1+j))^2}{j+1}.
\] (2.20)

Moreover, for the variance of the random variable \( M_N \) we have as \( N \to \infty \)

\[
V \left[ M_{N(p)} \right] \sim \left( \frac{\pi^2}{6} + pw_p - p^2c_p^2 \right) N^2.
\] (2.21)

**Remark 2.3** From Theorems 2.1 and 2.2 we see that the leading term of the first and the second moment of the random variable \( M_{N(p)} \) is the same with the classic version and independent of \( p \), which first appears in the second term. However, \( p \) appears in the leading term of the variance. We remind the reader that

\[
E \left[ T_{N(1)} \right] = N \sum_{j=1}^{N} \frac{1}{j} = N \left( \ln N + \gamma + \frac{1}{2N} + O \left( \frac{1}{N^2} \right) \right)
\]

\[
E \left[ T_{N(1)}^2 \right] = N^2 \left[ \sum_{j=1}^{N} \frac{1}{j} \right] + N \sum_{j=1}^{N} \frac{1}{j^2}
\]

\[
= N^2 \left[ \ln^2 N + 2\gamma \ln N + \gamma^2 + \frac{\pi^2}{6} + O \left( \frac{\ln N}{N} \right) \right]
\]

\[
V \left[ T_{N(1)} \right] \sim \frac{\pi^2}{6} N^2, \quad N \to \infty,
\]

see, e.g., [2].

### 2.2 A few words for the case when \( p \) becomes infinitely large

The delicate problem of estimating asymptotically high order differences of some fixed numerical sequence \( \{f_k\} \)

\[
D_n [f] := \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k
\]

go back in mid 1960s to De Bruijn, Knuth, and Rice who showed their central role in the evaluation of data structures based on a binary representation of data. Applications of these binomial sums refer mainly to digital search trees (an alternative of Rice integrals, see [3]) and quadtrees (see, [4 – 5]).

Using the identity

\[
\frac{1}{j+1} \binom{n-1}{j} = \frac{1}{n} \binom{n}{j+1}
\]
the quantities $c_p$ and $w_p$ of (2.17) and (2.20) become

\[
c_p = -\frac{1}{p} \sum_{k=1}^{p} (-1)^k \binom{p}{k} \ln k, \quad w_p = -\frac{1}{p} \sum_{k=1}^{p} \binom{p}{k} (-1)^k (\ln k)^2
\]  

(2.22)

respectively. By exploiting the techniques presented by P. Flajolet and R. Sedgewick (see [4]) one has as $n \to \infty$:

\[
\sum_{k=1}^{n} \binom{n}{k} (-1)^k \ln k = \ln (\ln n) + \gamma + \frac{\gamma}{\ln n} + \frac{\gamma^2 + \pi^2}{2(\ln n)^2} + O\left(\frac{1}{(\ln n)^3}\right)
\]

\[
\sum_{k=1}^{n} \binom{n}{k} (-1)^k (\ln k)^2 = -\left(\ln (\ln n)\right)^2 - 2\gamma \ln (\ln n) + \frac{\pi^2}{6} - \gamma^2 - 2\gamma \frac{\ln (\ln n)}{\ln n}
\]

\[
+ \frac{\gamma^2 + \frac{\pi^2}{6}}{(\ln n)^2} \frac{\ln (\ln n)}{\ln n} + \frac{2\gamma^2}{(\ln n)^3} + O\left(\frac{\ln (\ln n)}{(\ln n)^4}\right).
\]

By invoking the above asymptotics in (2.22) we get the very interesting result

**Corollary 2.4**

\[
\lim_{p \to \infty} \left( p^2 c_p^2 - pw_p \right) = \frac{\pi^2}{6}.
\]  

(2.23)

It is remarkable that $\pi^2/6$ appears in this limit above, which is a difference of binomial alternating sums involving logarithms. Now Theorem 2.2. implies that as the number of the independent collectors goes to infinity the variance $V[M_{N(p)}]$, naturally, vanishes.

**Conjecture 2.5** The sequence

\[
a_p := \frac{\pi^2}{6} + pw_p - p^2 c_p^2
\]  

(2.24)

is decreasing in $p$.

**Remark 2.6** From the conjecture above one can prove the following remarkable identity

\[
\frac{\pi^2}{6} + pw_p - p^2 c_p^2 > 0, \quad p \in \mathbb{N}.
\]  

(2.25)

We continue with the following

**Examples.**

(i) The case $p = 1$, namely the case of the classic coupon collector’s problem.
Then \( c_1 = w_1 = 0 \) and Theorems 2.1–2.2 yield 

\[
E \left[ M_{N(1)} \right] = N \left[ \ln N + \gamma + O \left( \frac{(\ln N)^2}{N} \right) \right]
\]

\[
E \left[ M^2_{N(1)} \right] = N^2 \left[ (\ln N)^2 + 2 \gamma \ln N + \gamma^2 + \frac{\pi^2}{6} + O \left( \frac{(\ln N)^3}{N} \right) \right]
\]

\[
V \left[ M_{N(1)} \right] \sim \frac{\pi^2}{6} N^2,
\]

in accordance with Remark 2.3.

(ii) If \( p = 2 \), then we have two independent collectors. Hence, \( c_2 = -\ln 2/2 \) and \( w_2 = -(\ln 2)^2/2 \), and Theorems 2.1–2.2 immediately imply 

\[
E \left[ M_{N(2)} \right] = N \left[ \ln N + (\gamma - \ln 2) + O \left( \frac{(\ln N)^2}{N} \right) \right]
\]

\[
E \left[ M^2_{N(2)} \right] = N^2 \left[ (\ln N)^2 + 2 (\gamma - \ln 2) \ln N + \gamma^2 + \frac{\pi^2}{6} - 2 \gamma \ln 2 - (\ln 2)^2 + O \left( \frac{(\ln N)^3}{N} \right) \right]
\]

\[
V \left[ M_{N(2)} \right] \sim \left( \frac{\pi^2}{6} - 2 (\ln 2)^2 \right) N^2.
\]

Notice that the variance decreases when the collectors become two instead of one (as expected). We will give closure with the following

**Remark 2.7** Let us consider the case where \( p = 2 \). In view of (1.2) and (1.3) relation (1.1) yields

\[
E \left[ M_{N(2)} \right] = \sum_{k=1}^{\infty} \left[ \sum_{n=0}^{N-1} (-1)^n \left( \frac{N}{n} \right) \left( \frac{n}{N} \right)^{k-1} \right]^2
\]

\[
= \sum_{n=0}^{N-1} \left( \frac{N}{n} \right)^2 \left( \frac{n}{N} \right)^{2k-2} + 2 \sum_{0 \leq n_1, n_2 \leq N-1} (-1)^{n_1+n_2} \left( \frac{N}{n_1} \right) \left( \frac{n_1}{N} \right) \left( \frac{n_2}{N} \right) \left( \frac{n_1 n_2}{N^2} \right)^{k-1}
\]

\[
= \sum_{n=0}^{N-1} \left( \frac{N}{n} \right)^2 \frac{1}{1 - \left( \frac{n}{N} \right)^2} + 2 \sum_{0 \leq n_1, n_2 \leq N-1} (-1)^{n_1+n_2} \left( \frac{N}{n_1} \right) \left( \frac{n_1}{N} \right) \frac{1}{1 - \frac{n_1 n_2}{N^2}}
\]  

(2.26)

(2.27)

where the last equation follows by summing the geometric series of (2.26). An aside result of Theorem 2.1 is that it provides asymptotics for expressions similar to (2.26)–(2.27).
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