A periodic review integrated inventory model with controllable safety stock and setup cost under service level constraint and distribution-free demand

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Abstract. In this study, we consider a stochastic integrated manufacturer-retailer inventory model with service level constraint. The model analyzed in this article considers the situation in which the vendor and the buyer establish a long-term contract and strategic partnership to jointly determine the best strategy. The lead time and setup cost are assumed can be controlled by an additional crashing cost and an investment, respectively. It is assumed that shortages are allowed and partially backlogged on the buyer’s side, and that the protection interval (i.e., review period plus lead time) demand distribution is unknown but has given finite first and second moments. The objective is to apply the minmax distribution free approach to simultaneously optimize the review period, the lead time, the setup cost, the safety factor, and the number of deliveries in order to minimize the joint total expected annual cost. The service level constraint guarantees that the service level requirement can be satisfied at the worst case. By constructing Lagrange function, the analysis regarding the solution procedure is conducted, and a solution algorithm is then developed. Moreover, a numerical example and sensitivity analysis are given to illustrate the proposed model and to provide some observations and managerial implications.

1. Introduction

In today’s global market, the cooperation between the vendor and the buyer is an important way to gain competitive advantages as it lowers supply chain cost. It involves commitment to long-term coordination, information sharing, joint problem solving and shared benefits. This regenerated interest is encouraged by the growing focus on supply chain management. Many firms are recognizing that inventories across the entire supply chain can be more efficiently organized through better integration of the supply chain. Several policies are used to align the activities of the components of a supply chain to establish better performance of supply chain in terms of response time, cost, customer service, and timely supply. Supply chain collaboration is concerned with the development and implementation
of such strategies. Several researchers have shown that the vendor and the buyer can increase their mutual benefit or obtain their minimal total cost through strategic collaboration with each other. Goyal [7] was one of the first authors to develop a single-vendor single-buyer integrated inventory model. The joint vendor-buyer optimization was later reinforced by Banerje [3] and Goyal [8]. Pan and Yang [23] then generalized Goyal’s [8] model and presented an integrated inventory model by considering lead time as a decision variable, and obtained a shorter lead time and lower joint total expected cost. Other related studies of the integrated inventory system include Lin [16], Shah [26], and Jauhari et al. [10]. A review of related literature on coordination inventory models is given in Glock [6].

In the collaborative inventory management system, the continuous review integrated inventory model continuous to be widely discussed. However, it is noticed that the inventory literature discussing the integration between the vendor and the buyer on the periodic review inventory model is quite sparse. The periodic review inventory model can frequently be found in organizing inventory cases such as smaller retail stores, grocery stores, and drugstores. There are papers related to the periodic review inventory model studies. Pal and Chandra [22] investigated a periodic review inventory model with stock dependent demand and permissible delay in payment. Lin [18] explored a stochastic periodic review inventory model with back-order price discounts and ordering cost dependent on lead time. Soni and Joshi [27] developed a periodic review inventory model by considering lead-time and the backorder rate as control variables in fuzzy stochastic environment. Akyurt [1] studied the single step single-item periodic review inventory models under Poisson demand distribution. Chand et al. [16] examined a periodic review inventory model where the buyer has access to a mix of two delivery modes for the quantity it orders in a period; these include an emergency delivery mode that delivers instantaneously and a regular delivery mode that delivers within the length of the review period. However, all of the above periodic review inventory model focused on determining optimal policy for the buyer only. These models neglect the opportunity that buyer and vendor can cooperate with each other to obtain a better integrated strategy. We also notice that the traditional periodic review inventory models assumed that all products are perfect and protection interval demand is normally distributed. It is common in all industries that a certain percent of produced/ordered products are non-conforming (imperfect) quality, which may be resulted from an imperfect production process, insufficient process control, inadequate work instruction, wrongly planned maintenance, damage in transit, careless handling, or others negative factors. These defective items will impact the on-hand inventory level, the number of shortages and the frequency of orders in the inventory system. Hence, it is important to study the production/order policy in which an arrival lot contains defective items. Recently, some research (see, for instance, [12], [14], [17], [20], [30],) has been done in continuous review multi-stage lot sizing decisions for imperfect production processes. In addition, the information about the probability distribution of the lead time demand is often quite limited. There are many related studies such as Lin [17], Moon et al. [21], Sarkar et al. [25], Annadurai [2], etc.

Most of the integrated inventory models are generally developed without any constraint. Nevertheless, in real situations, this kind of inventory system rarely occurs. Therefore, many companies deal with several constraints such as service level and capital investment. While allowing backlogging, shortages may turn into possible loss of future sales because of an unacceptable service level. Hence, the maximal allowable level of shortages per cycle is usually set as an operating constraint of the business in order to achieve optimal service level while deriving the optimal lot-size decision. In addition, in many cases the shortage cost includes intangible components such as potential delays to other parts of the system and loss of goodwill. A common substitute for a shortage cost is a service level which is measured as proportion of demands satisfied directly from stock. Therefore, several scholars (see, [2] [11], [12], [14], [15], [21], [25]) have incorporated a service level constraint into their inventory models. Consequently, the shortage level per cycle is bounded. Furthermore, safety factor (safety stock) is one of the important factors which influences the service level. However, in most periodic review inventory models, safety stock was not taken into account, and they merely focused on the relationship between periodic review and lead time. In other words, they neglected the
possible impact of the safety stock on the economic ordering strategy. Such a phenomenon is usually not perfect in a real inventory situation. Hence, we consider the controllable safety stock in the proposed model in this paper.

Furthermore, setup cost is one of the important parts of the inventory organization. Setup cost is related with expenses incurred in surroundings up a machine, assembly line, or work center, to switch from one production career to the subsequently ([29]). The basic integrated inventory model is generally based on the assumption of fixed setup cost. By using investment, the setup cost can be reduced. That is, setup cost is assumed to be variable rather than constant. The initial investment may be high but expected total cost will be reduced in each stage by using the investment. The setup cost reduction is one of the important production activities in an integrated inventory control. The concept of setup cost reduction has been widely applied to the production system. The consequences of set-up cost reduction are significantly increasing production efficiency and decreasing inventory. In practical situations, setup cost can be reduced through worker training, procedural changes and specialized equipment acquisition. Porteus [24] investigated the impact of investing in reducing setup cost by considering the discounted model. Further, Kurdhi et al. [13], Lo [19], Tahami et al. [28] and Vijayashree and Uthayakumar [29], addressed the joint optimization cost of the vendor and the buyer. They showed that by investing in set-up cost reduction, a significant saving in joint total cost can be achieved. Moreover, we set the normal setup cost without any investment as the upper limit of the setup cost level, which implies that if the optimal set-up cost obtained does not satisfy the restriction on the cost, then no set-up cost reduction investment is made. This is to avoid losses caused by the investment.

Based on the above discussions, we present the periodic review integrated vendor-buyer inventory model with controllable lead time and safety stock, imperfect production process, investment for setup cost reduction, partial backordering, and stochastic demand under service level constraint. We assume that the probability density function of demand during protection period (i.e. review period plus lead time) is unknown with given mean and variance. A minimax distribution free procedure and Lagrange multiplier are applied in order to minimize the joint expected total cost of the vendor and the buyer by simultaneously optimizing the review period, safety factor, lead time, setup cost, and the number of shipments in one production cycle. An algorithm procedure is developed to determine the optimal solutions. Later, a numerical example is provided to illustrate the proposed model. Finally, the concluding remark is made. The rest of this paper is organized as follows: The notations and assumptions used in this paper are introduced in Section 2. In Section 3, we formulate the stochastic integrated periodic review inventory model containing single vendor and single buyer subject to a service level constraint in the partial backorder case. The minimax distribution free procedure, the Lagrangian multiplier technique and a detailed solution procedure to solve the proposed model are presented in Section 4. In Section 5, a numerical example and discussion of the results are provided. Finally, some conclusions and suggestions for some future research are given in Section 6.

2. Notations and Assumptions

The following notations and assumptions are used in developing mathematical model.

2.1. Notations

Variables:
- $T$ : buyer’s review period length
- $R$ : buyer’s target level
- $L$ : buyer’s lead time
- $S$ : vendor’s setup cost per setup
- $m$ : number of shipments from vendor to buyer per production run, a positive integer

Parameters:
- $P$ : vendor’s production rate per unit time
\( \nu \) : vendor’s production cost per unit
\( h_s \) : vendor’s average inventory holding cost per unit per unit time
\( D \) : buyer’s average demand per unit time
\( A \) : buyer’s ordering cost per order
\( \omega \) : vendor’s treatment cost of defective items
\( \gamma \) : percentage of defective items in each deliver, \( 0 \leq \gamma < 1 \)
\( x \) : buyer’s inspecting rate
\( s \) : buyer’s unit inspecting cost
\( p \) : unit price charged by the vendor to the buyer, \( p > \nu \)
\( h_{r1} \) : buyer’s inventory holding cost per non-defective items per unit per unit time
\( h_{r2} \) : buyer’s inventory holding cost per defective items per unit per unit time,
\( h_{r1} \geq h_{r2} > h_s \)
\( \beta \) : fraction of the shortage that will be backordered, i.e., backorder ratio, \( 0 \leq \beta < 1 \)
\( \beta_0 \) : upper bound of the backorder ratio
\( \pi_x \) : backorder price discount offered by the buyer per unit
\( \pi_0 \) : Marginal profit (i.e. cost of lost demand) per unit
\( X \) : The protection interval, \( T + L \), demand which has a normal probability density function
\( (p.d.f.) f_X \) with finite mean \( D(T + L) \) and standard deviation \( \sigma \sqrt{T+L} \), where \( \sigma \)
denotes the standard deviation of the demand per unit time
\( E(\cdot) \) : Mathematical expectation
\( y^+ \) : Maximum value of \( y \) and 0, i.e., \( y^+ = \text{Max}\{y, 0\} \).

### 2.2. Assumptions

1. There is a single vendor and a single buyer for a single product in this model, and the inventory system deals with only one type item
2. When buyer orders quantity \( DT \), vendor manufactures \( mDT \) at one set-up, with a finite production rate \( P \), and \( (1 - \gamma)P > D \). Each batch is dispatched to the buyer in \( m \) equally-sized shipments, where \( m \) is a positive integer.
3. The inventory level is reviewed every \( T \) units of time. A sufficient quantity is ordered up to the target level \( R \), and the ordering quantity is received after \( L \) units of time.
4. The lead time consists of \( n \) mutually independent components. The \( i \)-th component has a minimum duration \( a_i \) and normal duration \( b_i \), and a crashing cost per unit time \( c_i \), where \( c_1 \leq c_2 \leq \cdots \leq c_n \). \( L_i \) the length of lead time with components \( 1,2,...,i \) crashed to their minimum duration;
\[
L_i = \sum_{j=1}^{i} b_j - \sum_{j=1}^{i-1} (b_j - a_j), \quad i = 1, 2, ..., n.
\]
Also, let \( L_0 = \sum_{j=1}^{n} b_j \), \( C(L) \) the lead time crashing cost per cycle;
\[
C(L) = c_l(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j),
\]
for a given \( L \in [L_i, L_{i-1}] \).
5. The length of the lead time \( L \) does not exceed an inventory cycle time \( T \) so that there is never more than a single order outstanding in any cycle. That is, \( L \leq T \).
6. The target level \( R \) is expected demand during the protection interval + safety stock (SS), and \( SS = k \times (\text{standard deviation of protection interval demand}) \), i.e., \( R = D(T + L) + k \sigma \sqrt{T + L} \), where \( k \) is the safety factor.
7. The buyer makes decisions in order to obtain profits. Therefore, if the price discount, \( \pi_x \), is greater than the marginal profit, \( \pi_0 \), then the buyer may decide against offering the price discount. The choice of investing in reducing the setup cost for the buyer is available.
8. The investment required to reduce the ordering cost from original level \( S_0 \) to a target level \( S \) is denoted by \( I(S) \), where \( I(S) \) is a convex and strictly decreasing function.
3. Model Formulation

3.1. Buyer’s expected total cost per unit time

The unit price charged by the vendor to the buyer is \( p \). Hence, the purchase cost per unit time is \( pD \). It is assumed that an arriving order items may contain some defective items with defective rate, \( \gamma \). Upon arrival of an order, all items are inspected by the buyer with inspected rate, \( x \). The defective items in each lot will be returned to the vendor when the next lot is supplied. The unit inspecting cost for items is \( s \), so the buyer’s inspecting cost per unit time is \( sD \). Further, the buyer will have two extra costs: non-defective items holding cost and defective items holding cost. The number of non-defective items is \((1 - \gamma)DT\) and the buyer’s holding cost of non-defective items per unit time is \( h_{r_1} \). Hence, the buyer’s holding cost of non-defective items per unit time is

\[
\frac{h_{r_1}}{T} \left[ \frac{DT^2}{2} + \frac{\gamma(DT)^2}{2x} \right] + h_{r_1} \left[ R - D(T + L) + (1 - \beta)E[(X - R)^+] \right].
\]

In addition, the number of defective items in each shipment is \( \gamma DT \), inspection period time is \( \frac{DT}{x} \), and the buyer’s holding cost of defective items per unit time is \( h_{r_2} \). Hence, the buyer’s holding cost of defective items per unit time is

\[
\frac{h_{r_2}}{T} \left[ \gamma DT^2 - \frac{\gamma (DT)^2}{2x} \right].
\]

Therefore, the buyer’s holding cost of non-defective and defective items per unit time is

\[
\frac{h_{r_1}}{T} \left[ \frac{DT^2}{2} + \frac{\gamma(DT)^2}{2x} \right] + h_{r_1} \left[ R - D(T + L) + (1 - \beta)E[(X - R)^+] \right] + \frac{h_{r_2}}{T} \left[ \gamma DT^2 - \frac{\gamma (DT)^2}{2x} \right]
\]

which can be simplified as

\[
DT \left[ \frac{h_{r_1}}{2} + h_{r_2} \gamma + \frac{yD}{2x} (h_{r_1} - h_{r_2}) \right] + h_{r_1} \left[ R - D(T + L) + (1 - \beta)E[(X - R)^+] \right] + \frac{r \beta}{T} [r \beta + \pi_0 (1 - \beta)] E[(X - R)^+].
\]

Thus, the buyer’s expected total cost per unit time is

\[
TC_b(T, R, L) = (p + s)D + \frac{A + C(L)}{T} + DT \left[ \frac{h_{r_1}}{2} + h_{r_2} \gamma + \frac{yD}{2x} (h_{r_1} - h_{r_2}) \right] + h_{r_1} \left[ R - D(T + L) + (1 - \beta)E[(X - R)^+] \right] + \frac{r \beta}{T} [r \beta + \pi_0 (1 - \beta)] E[(X - R)^+].\tag{1}
\]

3.2. Vendor’s expected total cost per unit time

For each production run, the setup cost per unit time is \( S/mT \). When the vendor produces the first \( DT \) units, he/she will deliver them to the buyer, after that the vendor will make the deliver on the average every \( T \) unit of time until the inventory level falls to zero. Hence, the average inventory per unit time can be calculated as follows:

\[
\left( mDT \left[ \frac{DT}{P} + (m - 1)T \right] - \frac{mDT)^2}{2P} - \frac{D^2T^2}{D} [1 + 2 + \cdots + (m - 1)] \right) \cdot \frac{D}{mDT} = \frac{DT}{P} \left[ \frac{D}{P} (2 - m) + (m - 1) \right].
\]

Hence, the vendor’s average holding cost per unit time is \( h_s \cdot \frac{DT}{2} \left[ \frac{D}{P} (2 - m) + (m - 1) \right] \), and the number of defective items is \( \gamma DT \), the length of inventory cycle is \( T \), the treatment cost of defective items is \( \omega \). Hence, the treatment cost of defective items per unit time is \( \omega \gamma D \). Furthermore, the capital investment, \( I(S) \), is a logarithmic function of the setup cost \( S \). That is,

\[
I(S) = \frac{1}{\tau} \ln \left( \frac{S_0}{S} \right),
\]

where \( 0 < S \leq S_0 \) and \( \tau \) is the percentage decrease in \( S \) per dollar increase in \( I(S) \). The function is consistent with the Japanese experience as informed in Hall [9] and has been used in literature (see, [13], [19], [28], [29]). Moreover, to reduce the ordering cost, the retailer invests capital \( I(S) \) to
improve and avoid too highly ordering cost. This capital investment \( I(S) \) is the onetime investment cost whose benefits will intend indefinitely into the future. Therefore, the investment cost per unit time is \( \delta I(S) \), where \( \delta \) is fractional opportunity cost of capital per unit time. Thus, the vendor’s expected total cost per unit time is

\[
TC_v(T, S, m) = \text{production cost} + \text{setup cost} + \text{holding cost} + \text{inspecting cost} + \text{treatment defective cost} + \text{Investment in setup cost reduction} \]

\[
= Dv + \frac{S}{mT} + h_s \frac{D}{p} (2 - m) + (m - 1) + \omega \gamma D + \frac{\delta}{\tau} \ln \left( \frac{S_0}{S} \right). \tag{2}
\]

### 3.3. The joint expected total cost per unit time

In this subsection, we consider the situation where vendor and buyer coordinate their production and inventory strategies with each other to determine the best policy for their periodic review integrated supply chain system. Therefore, the joint expected total cost per unit time is given by:

\[
JT\text{C}(T, R, S, L, m) = TC_v(T, R, L) + TC_v(T, S, m)
\]

\[
= (p + s + v + \omega \gamma)D + \frac{A+C(L)}{T} + DT \left[ \frac{h_{r_1}}{2} + h_{r_2} \gamma + \frac{pD}{2x} (h_{r_1} - h_{r_2}) \right]
\]

\[
+h_{r_1} [R - D(T + L) + (1 - \beta)E[(X - R)^+]] + \frac{S}{mT} + \frac{\delta}{\tau} \ln \left( \frac{S_0}{S} \right)
\]

\[
+ \frac{1}{T} (\pi \beta + \pi_0 (1 - \beta)) E[(X - R)^+] + h_s \frac{D}{2} \frac{pD}{p} (2 - m) + (m - 1). \tag{3}
\]

For a given safety factor which satisfies the probability that lead time demand at the buyer exceeds buyer’s target level, the actual proportion of demands not met from stock should not exceed the desired value of \( \alpha \). Hence, the service level constraint can be established as:

\[
\text{Expected demand shortages at the end of the cycle} \leq \alpha.
\]

Symbolically, it can be expressed by

\[
\frac{E[(X-R)^+]}{D(T+L)} \leq \alpha.
\]

Moreover, we note that if the optimal set-up cost obtained does not satisfy the restriction on \( S \), then no set-up cost reduction investment is made. For this special case, the optimal set-up cost is the initial set-up cost. Therefore, the problem in this research can be represented as:

\[
\text{Min } JT\text{C}(T, R, S, L, m) = (p + s + v + \omega \gamma)D + \frac{A+C(L)}{T} + DT \left[ \frac{h_{r_1}}{2} + h_{r_2} \gamma + \frac{pD}{2x} (h_{r_1} - h_{r_2}) \right]
\]

\[
+h_{r_1} [R - D(T + L) + (1 - \beta)E[(X - R)^+]] + \frac{S}{mT} + \frac{\delta}{\tau} \ln \left( \frac{S_0}{S} \right)
\]

\[
+ h_s \frac{D}{2} \frac{pD}{p} (2 - m) + (m - 1), \tag{4}
\]

subject to \( \frac{E[(X-R)^+]}{D(T+L)} \leq \alpha, 0 < S \leq S_0 \).

In many practical situations, the distributional information of the protection interval demand is often quite limited. Hence, in this section, we relax the assumption about the normal distribution of the protection interval demand and only assume that the protection interval demand \( X \) has given finite first two moments; i.e., the \( p.d.f. f_X \) belongs to the class \( \Phi \) of \( p.d.f.'s \) with finite mean \( D(T + L) \) and standard deviation \( \sigma \sqrt{T + L} \). Since the probability distribution of \( X \) is unknown, we can not find the exact value of \( E[(X - R)^+] \). We propose to apply the minimax distribution free procedure for our problem. The minimax distribution free approach for this problem is to find the “most unfavorable” \( p.d.f. f_X \) in \( \Phi \) for each \( (T, R, S, L, m) \) and then minimize the total expected annual cost over \( (T, R, S, L, m) \); more exactly, we need to solve

\[
\text{Min}_{T,R,S,L,m} \text{Max}_{f \in \Phi} JT\text{C}(T, R, S, L, m)
\]

subject to \( \frac{E[(X-R)^+]}{D(T+L)} \leq \alpha. \tag{5}
\]

For this purpose, we need the following proposition which was asserted by Gallego and Moon [5].
**Proposition 1.** For any $f_x \in F$, 
\[
E[(X-r)^+] \leq \frac{1}{2} \left\{ \sqrt{\sigma^2(T+L)} + [R - D(T + L)]^2 - [R - D(T + L)] \right\}
\]
\[
= \frac{1}{2} \sigma \sqrt{T + L} \left( \sqrt{1 + k^2} - k \right).
\]
Moreover the upper bound (6) is tight.

Based on the results of (6) and $R = D(T + L) + k\sigma \sqrt{T + L}$, the safety factor $k$ can be viewed as a decision variable instead of $R$, and thus model (5) is reduced to 
\[
\begin{align*}
\text{Min JTC}(T, k, S, L, m) &= (p + s + v + \omega y)D + \frac{A + C(L)}{T} + DT \left[ \frac{h_{r1} + h_{r2}}{2} + \alpha \frac{\sqrt{\sigma^2(T+L)}}{T} \right] \\
&\quad + h_{r1} \sigma \sqrt{T + L} \left[ k + \frac{1}{2} (1 - \beta) \left( \sqrt{1 + k^2} - k \right) \right] + \frac{s}{mT} + \frac{\delta}{T} \ln \left( \frac{S_0}{S} \right) \\
&\quad + h_{s} \frac{DT}{2} \left[ \frac{D}{P} (2 - m) + (m - 1) \right].
\end{align*}
\]
subject to $\sigma \sqrt{T + L} \left( \sqrt{1 + k^2} - k \right) \leq 2\alpha D(T + L)$, $0 < S \leq S_0$.

**4. Solution Technique**

Our problem is to find the optimal values of $T, k, S, L,$ and $m$ such that $JTC(T, k, S, L, m)$ has a minimum value. To solve the nonlinear programming problem in (7), we provisionally ignore the setup cost constraint $0 < S \leq S_0$ and add a nonnegative slack variable $S_0^2 \geq 0$ to convert the service level constraint $\sigma \sqrt{T + L} \left( \sqrt{1 + k^2} - k \right) \leq 2\alpha D(T + L)$ to equality. One has 
\[
\begin{align*}
\text{Min JTC}(T, k, S, L, m, S_0) &= (p + s + v + \omega y)D + \frac{A + C(L)}{T} + DT \left[ \frac{h_{r1} + h_{r2}}{2} + \alpha \frac{\sqrt{\sigma^2(T+L)}}{T} \right] \\
&\quad + h_{r1} \sigma \sqrt{T + L} \left[ k + \frac{1}{2} (1 - \beta) \left( \sqrt{1 + k^2} - k \right) \right] + \frac{s}{mT} + \frac{\delta}{T} \ln \left( \frac{S_0}{S} \right) \\
&\quad + h_{s} \frac{DT}{2} \left[ \frac{D}{P} (2 - m) + (m - 1) \right].
\end{align*}
\]
subject to $\sigma \sqrt{T + L} \left( \sqrt{1 + k^2} - k \right) + S_0^2 = 2\alpha D(T + L)$.

The Lagrangian function of (8) is given by 
\[
\begin{align*}
JTC(T, k, S, L, m, \lambda, S_0) &= (p + s + v + \omega y)D + \frac{A + C(L)}{T} + DT \left[ \frac{h_{r1} + h_{r2}}{2} + \alpha \frac{\sqrt{\sigma^2(T+L)}}{T} \right] \\
&\quad + h_{r1} \sigma \sqrt{T + L} \left[ k + \frac{1}{2} (1 - \beta) \left( \sqrt{1 + k^2} - k \right) \right] + \frac{s}{mT} + \frac{\delta}{T} \ln \left( \frac{S_0}{S} \right) \\
&\quad + h_{s} \frac{DT}{2} \left[ \frac{D}{P} (2 - m) + (m - 1) \right] + \lambda \left[ \sigma \sqrt{T + L} \left( \sqrt{1 + k^2} - k \right) + S_0^2 - 2\alpha D(T + L) \right],
\end{align*}
\]
where $\lambda$ is a Lagrange multiplier. And, for any given $(T, k, S, L, m, \lambda, S_0)$, $JTC(T, k, S, L, m, \lambda, S_0)$ is a concave function in $L \in [L_i, L_{i-1}]$, because 
\[
\frac{\partial^2 JTC(T, k, S, L, m, \lambda, S_0)}{\partial L^2} = -\frac{\lambda \sigma (\sqrt{1 + k^2} - k)}{4(T + L)^2} - \frac{\sigma h_{r1}}{4(T + L)^2} \left[ k + \frac{1}{2} (1 - \beta) \left( \sqrt{1 + k^2} - k \right) \right] < 0.
\]
Therefore, for fixed $(T, R, S, m, \lambda, S_0)$, the minimum joint expected total cost will occur at the end points of the interval $[L_i, L_{i-1}]$. On the other hand, we can further prove that, for any given $L \in [L_i, L_{i-1}]$ and $m$, by the Kuhn-Tucker necessary conditions for minimization problem, we can obtain the slack variable $S_0^2 = 0$ (hence, the service level constraint is active when the optimal solution is obtained). Therefore for fixed $L \in [L_i, L_{i-1}]$ and $m$, the minimum value of Eq. (9) (in which the variable $S_0 = 0$) will occur at the point $(T, k, S, \lambda)$ which satisfies 
\[
0 = \frac{\partial JTC(T, k, S, L, m, \lambda)}{\partial T} = -\frac{A + C(L)}{T^2} - \frac{S}{mT^2} - 2D \alpha \lambda + \frac{s}{mT^2} + \frac{\sigma h_{r1}}{2} \left[ k + \frac{1}{2} (1 - \beta) \left( \sqrt{1 + k^2} - k \right) \right] \\
+ D \left[ \frac{h_{r1}}{2} + h_{r2}y + \frac{\gamma D}{2x} (h_{r1} - h_{r2}) \right] + h_{s} \frac{DT}{2} \left[ \frac{D}{P} (2 - m) + (m - 1) \right].
\]
\[
0 = \frac{\partial J(T, k, s, l, m, \lambda)}{\partial k} = \lambda \sqrt{T + L} \left( \frac{k}{\sqrt{k^2 + 1}} - 1 \right) + h_{r_1} \sqrt{T + L} \left[ 1 + \frac{1}{2} (1 - \beta) \left( \frac{k}{\sqrt{k^2 + 1}} - 1 \right) \right],
\]
\[
0 = \frac{\partial J(T, k, s, l, m, \lambda)}{\partial s} = \frac{1}{m_T} - \frac{\delta}{T^2},
\]
\[
0 = \frac{\partial J(T, k, s, l, m, \lambda)}{\partial \lambda} = \sigma \sqrt{T + L} (\sqrt{1 + k^2} - k) - 2\alpha D(T + L).
\]
Simplifying these equations leads to
\[
\left( A^4 C(L) \right) - D \left[ \frac{h_{r_1}}{2} + h_{r_2} x + \frac{y D}{2x} (h_{r_1} - h_{r_2}) \right] - h_{s} \frac{D}{p} (2 - m) + (m - 1) + \frac{s}{m_T} + 2\alpha D \lambda \sqrt{T + L} = \frac{h_{r_1} \sigma k}{2} + \sigma \left( \sqrt{1 + k^2} - k \right) h_{r_1} (1 - \beta) + 2\lambda,
\]
\[
\lambda = h_{r_1} \left[ \frac{\sqrt{1 + k^2} - k}{2} \left( 1 - \beta \right) \right],
\]
\[
S = \frac{m \delta r}{T},
\]
\[
\sqrt{1 + k^2} - k = \frac{2\alpha D}{\sigma} \sqrt{T + L}.
\]
By substituting (11), (12) and (13) into (10), we have
\[
\left( A^4 C(L) \right) - D \left[ \frac{h_{r_1}}{2} + h_{r_2} x + \frac{y D}{2x} (h_{r_1} - h_{r_2}) \right] - h_{s} \frac{D}{p} (2 - m) + (m - 1) + \frac{s}{m_T} + 2\alpha D h_{r_1} \left[ \frac{4a^2 D^2 (L + T) + \sigma^2}{8a D (L + T)} - \frac{1}{2} (1 - \beta) \right] \sqrt{T + L} = \frac{h_{r_1} \sigma^2 - 4\alpha D^2 (L + T) + \sigma^2}{8a D (L + T)} + \frac{\alpha D}{2} \sqrt{T + L} \left[ h_{r_1} (1 - \beta) + 2 \left[ \frac{4a^2 D^2 (L + T) + \sigma^2}{8a D (L + T)} - \frac{1}{2} (1 - \beta) \right] \right].
\]
Further, for fixed \( L \in [L_i, L_{i-1}] \), and the active constraint \( \sigma \sqrt{T + L} (\sqrt{1 + k^2} - k) = 2\alpha D (T + L) \), if we take the second derivatives of \( JTC(T, k, S, L, m) \) with respect to \( \lambda \), we have \( \frac{\delta^2 JTC(T, k, S, L, m)}{m^2} = \frac{2s}{m^2} > 0 \).
Therefore, for fixed \( T, k, S, L, m \), \( JTC(T, k, S, L, m) \) is a convex function of \( m \). Since the number of shipments per batch production run, \( m \), is an integer, consequently the optimal value of \( m \) (denoted by \( m^* \)) can be obtained when
\[
JTC(T_m^*, k_m^*, S_m^*, L_m^*, m^*) \leq JTC(T_m^{* - 1}, k_m^{* - 1}, S_m^{* - 1}, L_m^{* - 1}, m^* - 1)
\]
and
\[
JTC(T_m^*, k_m^*, S_m^*, L_m^*, m^*) \leq JTC(T_m^{* + 1}, k_m^{* + 1}, S_m^{* + 1}, L_m^{* + 1}, m^* + 1).
\]
The following algorithm can be utilized to find the optimal \( (T, k, S, L, m) \).

**Algorithm 1.**

**Step 1.** Set \( m = 1 \).

**Step 2.** For each \( L_i, i = 0, 1, \ldots, n \), we use Eq. (7) to evaluate \( T_i \), and then use Eq. (12) and (13) to compute \( S_i \) and \( k_i \), respectively.

**Step 3.** Compare \( S_i \) and \( S_0 \).

(i) If \( S_i \leq S_0 \), then \( S_i \) is feasible and we denote the solution found in Step 1 for given \( L_i \) and \( m \) by \( (T_i^*, k_i^*, S_i^*) \).

(ii) If \( S_i > S_0 \), then \( S_i \) is not feasible and for given \( L_i \) and \( m \), take \( S_i^* = S_0 \) and the corresponding value of \( T_i^* \) and \( k_i^* \) can be obtained from (12) and (13).

**Step 4.** Utilize (7) to compute the corresponding joint expected total cost \( JTC(T_i^*, k_i^*, S_i^*, L_i, m) \) for \( i = 0, 1, 2, \ldots, n \).

**Step 5.** Find \( \min_{i=0,1,2,\ldots,n} JTC(T_i^*, k_i^*, S_i^*, L_i, m) \).

If \( JTC(T_m^*, k_m^*, S_m^*, L_m^*, m) = \min_{i=0,1,2,\ldots,n} JTC(T_i^*, k_i^*, S_i^*, L_i, m) \), then \( (T_m^*, k_m^*, S_m^*, L_m^*, m) \) is the optimal solution for the fixed \( m \).

**Step 6.** Set \( m = m + 1 \), and repeat Step 2 to Step 5 to get \( JTC(T_m^*, k_m^*, S_m^*, L_m^*, m) \).
Step 7. If $JT(C^*T_m^*, k_m^*, S_m^*, L_m^*, m) \leq JT(C^*T_{m-1}^*, k_{m-1}^*, S_{m-1}^*, L_{m-1}^*, m - 1)$ then go to Step 6, otherwise go to Step 8.

Step 8. Set $JT(C^*(T^*, k^*, S^*, L^*, m^*)) = JT(C^*(T_{m-1}^*, k_{m-1}^*, S_{m-1}^*, L_{m-1}^*, m - 1)$, then $JT(C^*(T^*, k^*, S^*, L^*, m^*))$ is the minimum joint expected total cost and $(T^*, k^*, S^*, L^*, m^*)$ is the optimal solution. Hence the optimal target level is $R^* = D(T^* + L^*) + k\sigma\sqrt{T^* + L^*}$.

5. Numerical Example

We consider a stochastic periodic review integrated inventory system with the following data: $D = 600$ units per year, $A = $200 per order, $\beta = 0.8$, $\sigma = 15$ units per year, $P = 1200$ units per year, $S_0 = $1000 per setup, $v = $60, $h_S = $20 per unit per year, $h_r = $30 per unit per year, $h_{r_2} = $25 per unit per year, $x = 1500$, $s = $0.5 per unit, $\omega = $30 per unit, and $\gamma = 0.005$, $\delta = 0.1$, $1/T = 2800$, where $1$ year = 52 weeks. We assume that the worst service levels are $1 - \alpha = 98\%$, $98.5\%$, $99\%$, and $99.5\%$, i.e., the maximum allowable proportions of demands which are not met from stock are $\alpha = 2\%, 1.5\%$, and $0.5\%$. Besides, it is assumed that the lead time has three components with data shown in Table 1.

Table 1. Lead time and crashing cost data

| Lead time component $i$ | Normal duration $b_i$ (days) | Minimum duration $a_i$ (days) | Unit crashing cost $c_i$ ($/day$) |
|------------------------|-----------------------------|-------------------------------|-------------------------------|
| 1                      | 20                          | 6                             | 0.4                           |
| 2                      | 20                          | 6                             | 1.2                           |
| 3                      | 16                          | 9                             | 5.0                           |

Algorithm 1 procedure is applied to yield the results for various service level $(1 - \alpha)$ values as shown in Table 2. The optimal policy for each $(1 - \alpha)$ value can be determined by comparing $JT(C^*(T_m^*, k_m^*, S_m^*, L_m^*, m))$, $m = 1, 2, ..., $ and the results are summarized in Table 3. For example, in the case of service level $(1 - \alpha) = 99.5\%$, applying the above procedure yields $T^* = 8.30$ weeks, $L^* = 8$ weeks, $k^* = 2.12$ (i.e. $R^* = 205.96$ units), $S^* = $319.39 per setup and $m^* = 2$. The joint expected total cost of integrated model is $81731$. Moreover, as the value of $(1 - \alpha)$ increases, the integrated policy not only achieves the required service level but also has the higher period review, safety factor (target level), setup cost and joint expected total cost. Besides, the values of lead time and number of deliveries are not influenced by changes in the value of service level.

Table 2. The results for various service level using the solution procedures ($T$ and $L$ in weeks)

| Service level $(1 - \alpha)$ | $m$ | $L_m^*$ | $T_m^*$ | $k_m^*$ | $S_m^*$ | $JT(C^*(T_m^*, k_m^*, S_m^*, L_m^*, m))$ |
|-----------------------------|-----|---------|---------|---------|---------|--------------------------------------|
| 98.0%                       | 1   | 8       | 9.50    | 0.07    | 182.87  | 81435*                              |
|                             | 2   | 8       | 8.18    | 0.11    | 314.71  | 81240*                              |
|                             | 3   | 8       | 7.26    | 0.14    | 418.90  | 81272*                              |
| 98.5%                       | 1   | 8       | 9.49    | 0.37    | 182.46  | 81506*                              |
|                             | 2   | 8       | 8.17    | 0.41    | 314.24  | 81310*                              |
|                             | 3   | 8       | 7.25    | 0.44    | 418.46  | 81340*                              |
| 99.0%                       | 1   | 8       | 9.50    | 0.85    | 182.75  | 81624*                              |
|                             | 2   | 8       | 8.18    | 0.90    | 314.77  | 81426*                              |
|                             | 3   | 8       | 7.27    | 0.94    | 419.17  | 81455*                              |
| 99.5%                       | 1   | 8       | 9.66    | 2.03    | 185.85  | 81931*                              |
|                             | 2   | 8       | 8.30    | 2.12    | 319.39  | 81731*                              |
|                             | 3   | 8       | 7.36    | 2.19    | 424.58  | 81759*                              |

Table 3. Summary of the optimal integrated policy for various service level

| Service level $1 - \alpha$ | $m^*$ | $L^*$ | $T^*$ | $k^*(R^*)$ | $S^*$ | $JT(C^*(T^*, k^*, S^*, L^*, m^*))$ |
|-----------------------------|-------|------|-------|------------|-------|----------------------------------|
| 98.0%                       | 2     | 8    | 8.18  | 0.11 (187.67) | 314.71 | 81240*                           |
| 98.5%                       | 2     | 8    | 8.17  | 0.41 (190.03) | 314.24 | 81310*                           |
6. Conclusions
In this paper, we propose an integrated vendor-buyer inventory model with the consideration of defective items, controllable lead time and safety factor, investment for setup cost reduction, partial backlogging, stochastic demand with unknown distribution, and there is an upper bound on the actual proportion of demands not met from stock (service level constraint). Under these conditions, we formulated the mathematical problem as a nonlinear programming model with a constraint and recommended minimax distribution-free procedure and Lagrange multiplier method to solve it. The proposed algorithm procedure is developed to determine the optimal number of period review, safety factor, lead time, setup cost and the number of shipments. The results of the numerical example show that the higher service level will produce positive influences on customer loyalty and brand which are crucial to building competitive advantage in market.

Possible extensions for future research could be by considering learning in production. In this case, the vendor experiences learning in the production process while some of the units are defective. The effect of learning in buyer’s inspection errors on supply chain cost also can be investigated. Furthermore, the other possibility is considering fuzzy demand, inflation and the time value of money, and delay-in-payments concepts in this model.

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