PSEUDO-DIFFERENTIAL CALCULUS IN ANISOTROPIC GELFAND-SHILOV SETTING

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Abstract. We study some classes of pseudo-differential operators with symbols admitting anisotropic exponential growth at infinity and we prove mapping properties for these operators on Gelfand-Shilov spaces of type $\mathcal{S}$. Moreover, we deduce algebraic and certain invariance properties of these classes.

0. Introduction

Gelfand-Shilov spaces of type $\mathcal{S}$ have been introduced in the book [16] as an alternative functional setting to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of smooth and rapidly decreasing functions for Fourier analysis and for the study of partial differential equations. Namely, fixed $s > 0, \sigma > 0$, the space $\mathcal{S}_\sigma^s(\mathbb{R}^d)$ can be defined as the space of all functions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ satisfying an estimate of the form

$$\sup_{\alpha, \beta \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|x^\beta \partial^\alpha f(x)|}{h^{(|\alpha| + |\beta|)}} < \infty$$

for some constant $h > 0$, or the equivalent condition

$$\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|e^r |x|^{\frac{1}{s}} \partial^\alpha f(x)|}{h^{(\alpha)}} < \infty$$

for some constants $h, r > 0$. For $\sigma > 1$, $\mathcal{S}_\sigma^s(\mathbb{R}^d)$ represents a natural global counterpart of the Gevrey class $G^\sigma(\mathbb{R}^d)$ but, in addition, the condition (0.2) encodes a precise description of the behavior at infinity of $f$. Together with $\mathcal{S}_\sigma^s(\mathbb{R}^d)$ one can also consider the space $\Sigma_\sigma^s(\mathbb{R}^d)$, which has been defined in [25] by requiring (0.1) (respectively (0.2)) to hold for every $\varepsilon > 0$ (respectively for every $h, r > 0$). The duals of $\mathcal{S}_\sigma^s(\mathbb{R}^d)$ and $\Sigma_\sigma^s(\mathbb{R}^d)$ and further generalizations of these spaces have been then introduced in the spirit of Komatsu theory of ultradistributions, see [14, 25].

After their appearance, Gelfand-Shilov spaces have been recognized as a natural functional setting for pseudo-differential and Fourier integral operators, due to their nice behavior under Fourier transformation, and applied in the study of several classes of partial differential equations, see e.g. [1, 3–8].

According to the condition on the decay at infinity of the elements of $\mathcal{S}_\sigma^s(\mathbb{R}^d)$ and $\Sigma_\sigma^s(\mathbb{R}^d)$, we can define on these spaces pseudo-differential
operators with symbols $a(x, \xi)$ admitting an exponential growth at infinity. These operators are commonly known as 
operators of infinite order and they have been studied in \cite{2} in the analytic class and in \cite{12, 24, 34} in the Gevrey spaces where the symbol has an exponential growth only with respect to $\xi$ and applied to the Cauchy problem for hyperbolic and Schrödinger equations in Gevrey classes, see \cite{12, 13, 15, 23}. Parallel results have been obtained in Gelfand-Shilov spaces for symbols admitting exponential growth both in $x$ and $\xi$, see \cite{3, 4, 7, 8, 11, 27}.

We stress that the above results concern the non-quasi-analytic isotropic case $s = \sigma > 1$. In \cite{10} we considered the more general case $s = \sigma > 0$, which is interesting in particular in connection with Shubin-type pseudo-differential operators, cf. \cite{3, 9}. Although the extension of the complete calculus developed in \cite{3, 4} in this case is out of reach due to the lack of compactly supported functions in $S_\sigma^s(\mathbb{R}^d)$ and $\Sigma_\sigma^s(\mathbb{R}^d)$, nevertheless some interesting results can be achieved also in this case by using different tools than the usual micro-local techniques, namely a method based on the use of modulation spaces and of the short time Fourier transform.

In the present paper, we further generalize the results of \cite{10} to the case when $s > 0$ and $\sigma > 0$ may be different from each other. Thus the symbols we consider may have different rates of exponential growth and anisotropic Gevrey-type regularity in $x$ and $\xi$. More precisely, the symbols should obey conditions of the form

$$\sup_{\alpha, \beta \in \mathbb{N}^d} \sup_{x, \xi \in \mathbb{R}^d} \left| \frac{e^{-r(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{\sigma}})}}{h^{\alpha + \beta} \alpha! \beta! s} \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| < \infty$$

(0.3)

for suitable restrictions on the constants $h, r > 0$ (cf. (0.2)). We prove that if $h > 0$, and (0.3) holds true for every $r > 0$, then the pseudo-differential operator $\text{Op}(a)$ is continuous on $S_\sigma^s$ and on $(S_\sigma^s)'$. If instead $r > 0$, and (0.3) holds true for every $h > 0$, then we prove that $\text{Op}(a)$ is continuous on $\Sigma_\sigma^s$ and on $(\Sigma_\sigma^s)'$ (cf. Theorems 3.7 and 3.13). We also prove that pseudo-differential operators with symbols satisfying such conditions form algebras (cf. Theorems 3.16 and 3.17). Finally we show that our span of pseudo-differential operators is invariant under the choice of representation (cf. Theorem 3.9).

An important ingredient in the analysis which is used to reach these properties concerns characterizations of symbols above in terms of suitable estimates of their short-time Fourier transforms. Such characterizations are deduced in Section 2.

The paper is organized as follows. In Section 1, after recalling some basic properties of the spaces $S_\sigma^s(\mathbb{R}^d)$ and $\Sigma_\sigma^s(\mathbb{R}^d)$, we introduce several general symbol classes. In Section 2 we characterize these symbols in terms of the behavior of their short time Fourier transform. In Section 3 we deduce continuity on $S_\sigma^s(\mathbb{R}^d)$ and $\Sigma_\sigma^s(\mathbb{R}^d)$, composition and invariance properties for pseudo-differential operators in our classes.
1. Preliminaries

In this section we recall some basic facts, especially concerning Gelfand-Shilov spaces, the short-time Fourier transform and pseudo-differential operators.

We let \( \mathcal{S}(\mathbb{R}^d) \) be the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^d \) together with their derivatives, and by \( \mathcal{S}'(\mathbb{R}^d) \) the corresponding dual space of tempered distributions. Moreover \( \mathcal{M}(d, \mathbb{R}) \) will denote the vector space of real \( d \times d \) matrices.

1.1. Gelfand-Shilov spaces. We start by recalling some facts about Gelfand-Shilov spaces. Let \( 0 < h, s, \sigma \in \mathbb{R} \) be fixed. Then \( \mathcal{S}_{s,h}^\sigma(\mathbb{R}^d) \) is the Banach space of all \( f \in C^\infty(\mathbb{R}^d) \) such that

\[
\| f \|_{\mathcal{S}_{s,h}^\sigma} \equiv \sup_{\alpha, \beta \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|x^\alpha \partial^\beta f(x)|}{h^{\| \alpha \| + \| \beta \|}} < \infty, \tag{1.1}
\]

endowed with the norm \((1.1)\).

The Gelfand-Shilov spaces \( \mathcal{S}_s^\sigma(\mathbb{R}^d) \) and \( \Sigma_s^\sigma(\mathbb{R}^d) \) are defined as the inductive and projective limits respectively of \( \mathcal{S}_{s,h}^\sigma(\mathbb{R}^d) \). This implies that

\[
\mathcal{S}_s^\sigma(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}^\sigma(\mathbb{R}^d) \quad \text{and} \quad \Sigma_s^\sigma(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}^\sigma(\mathbb{R}^d), \tag{1.2}
\]

and that the topology for \( \mathcal{S}_s^\sigma(\mathbb{R}^d) \) is the strongest possible one such that the inclusion map from \( \mathcal{S}_{s,h}^\sigma(\mathbb{R}^d) \) to \( \mathcal{S}_s^\sigma(\mathbb{R}^d) \) is continuous, for every choice of \( h > 0 \). The space \( \Sigma_s^\sigma(\mathbb{R}^d) \) is a Fréchet space with seminorms \( \| \cdot \|_{\mathcal{S}_{s,h}^\sigma} \), \( h > 0 \). Moreover, \( \Sigma_s^\sigma(\mathbb{R}^d) \neq \{0\} \), if and only if \( s + \sigma \geq 1 \) and \( (s, \sigma) \neq (1, \frac{1}{2}) \), and \( \mathcal{S}_s^\sigma(\mathbb{R}^d) \neq \{0\} \), if and only if \( s + \sigma \geq 1 \).

The spaces \( \mathcal{S}_s^\sigma(\mathbb{R}^d) \) and \( \Sigma_s^\sigma(\mathbb{R}^d) \) can be characterized also in terms of the exponential decay of their elements, namely \( f \in \mathcal{S}_s^\sigma(\mathbb{R}^d) \) (respectively \( f \in \Sigma_s^\sigma(\mathbb{R}^d) \)) if and only if

\[
|\partial^\alpha f(x)| \lesssim e^{\| \alpha \| (s!)^\sigma} e^{-h|x|^{1/2}},
\]

for some \( h > 0, \varepsilon > 0 \) (respectively for every \( h > 0, \varepsilon > 0 \)). Moreover we recall that for \( s < 1 \) the elements of \( \mathcal{S}_s^\sigma(\mathbb{R}^d) \) admit entire extensions to \( \mathbb{C}^d \) satisfying suitable exponential bounds, cf. [16] for details.

The Gelfand-Shilov distribution spaces \( (\mathcal{S}_s^\sigma)'(\mathbb{R}^d) \) and \( (\Sigma_s^\sigma)'(\mathbb{R}^d) \) are the projective and inductive limit respectively of \( (\mathcal{S}_{s,h}^\sigma)'(\mathbb{R}^d) \). This means that

\[
(\mathcal{S}_s^\sigma)'(\mathbb{R}^d) = \bigcap_{h>0} (\mathcal{S}_{s,h}^\sigma)'(\mathbb{R}^d) \quad \text{and} \quad (\Sigma_s^\sigma)'(\mathbb{R}^d) = \bigcup_{h>0} (\mathcal{S}_{s,h}^\sigma)'(\mathbb{R}^d). \tag{1.2}'
\]

We remark that in [26] it is proved that \( (\mathcal{S}_s^\sigma)'(\mathbb{R}^d) \) is the dual of \( \mathcal{S}_{s,\sigma}(\mathbb{R}^d) \), and \( (\Sigma_s^\sigma)'(\mathbb{R}^d) \) is the dual of \( \Sigma_{s,\sigma}(\mathbb{R}^d) \) (also in topological sense).
For every $s, \sigma > 0$ we have

$$\Sigma_s^\sigma (\mathbb{R}^d) \hookrightarrow \mathcal{S}_s^\sigma (\mathbb{R}^d) \hookrightarrow \Sigma_{s+\varepsilon}^\sigma (\mathbb{R}^d) \hookrightarrow \mathcal{F}(\mathbb{R}^d)$$ (1.3)

for every $\varepsilon > 0$. If $s + \sigma \geq 1$, then the last two inclusions in (1.3) are dense, and if in addition $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, then the first inclusion in (1.3) is dense.

From these properties it follows that $\mathcal{F}'(\mathbb{R}^d) \hookrightarrow (\mathcal{S}_s^\sigma)'(\mathbb{R}^d)$ when $s + \sigma \geq 1$, and if in addition $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, then $(\mathcal{S}_s^\sigma)'(\mathbb{R}^d) \hookrightarrow (\Sigma_s^\sigma)'(\mathbb{R}^d)$.

The Gelfand-Shilov spaces possess several convenient mapping properties. For example they are invariant under translations, dilations, and to some extent tensor products and (partial) Fourier transformations.

The Fourier transform $\mathcal{F}$ is the linear and continuous map on $\mathcal{F}(\mathbb{R}^d)$, given by the formula

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-i(x,\xi)} \, dx$$

when $f \in \mathcal{F}(\mathbb{R}^d)$. Here $(\cdot, \cdot)$ denotes the usual scalar product on $\mathbb{R}^d$. The Fourier transform extends uniquely to homeomorphisms from $(\mathcal{S}_s^\sigma)'(\mathbb{R}^d)$ to $(\mathcal{S}_s^\sigma)'(\mathbb{R}^d)$, and from $(\Sigma_s^\sigma)'(\mathbb{R}^d)$ to $(\Sigma_s^\sigma)'(\mathbb{R}^d)$. Furthermore, it restricts to homeomorphisms from $\mathcal{S}_s^\sigma(\mathbb{R}^d)$ to $\mathcal{S}_s^\sigma(\mathbb{R}^d)$, and from $\Sigma_s^\sigma(\mathbb{R}^d)$ to $\Sigma_s^\sigma(\mathbb{R}^d)$.

Some considerations later on involve a broader family of Gelfand-Shilov spaces. More precisely, for $s_j, \sigma_j \in \mathbb{R}_+$, $j = 1, 2$, the Gelfand-Shilov spaces $\mathcal{S}_{s_1, s_2}(\mathbb{R}^{d_1+d_2})$ and $\Sigma_{s_1, s_2}(\mathbb{R}^{d_1+d_2})$ consist of all functions $F \in C^\infty(\mathbb{R}^{d_1+d_2})$ such that

$$|x_1^{\alpha_1} x_2^{\alpha_2} \frac{\partial_{x_1}^\beta_1 \partial_{x_2}^\beta_2}{\partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2}} F(x_1, x_2)| \lesssim h^{[\alpha_1 + \alpha_2 + \beta_1 + \beta_2]} \alpha_1! \alpha_2! \beta_1! \beta_2! s_1^\sigma s_2^\sigma (1.4)$$

for some $h > 0$ respective for every $h > 0$. The topologies, and the duals

$$(\mathcal{S}_{s_1, s_2}'(\mathbb{R}^{d_1+d_2})) \quad \text{and} \quad (\Sigma_{s_1, s_2}'(\mathbb{R}^{d_1+d_2}))$$

of

$$\mathcal{S}_{s_1, s_2}(\mathbb{R}^{d_1+d_2}) \quad \text{and} \quad \Sigma_{s_1, s_2}(\mathbb{R}^{d_1+d_2}),$$

respectively, and their topologies are defined in analogous ways as for the spaces $\mathcal{S}_s^\sigma(\mathbb{R}^d)$ and $\Sigma_s^\sigma(\mathbb{R}^d)$ above.

The following proposition explains mapping properties of partial Fourier transforms on Gelfand-Shilov spaces, and follows by similar arguments as in analogous situations in [13]. The proof is therefore omitted. Here, $\mathcal{F}_1 F$ and $\mathcal{F}_2 F$ are the partial Fourier transforms of $F(x_1, x_2)$ with respect to $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$, respectively.

**Proposition 1.1.** Let $s_j, \sigma_j > 0$, $j = 1, 2$. Then the following is true:
(1) the mappings \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) on \( \mathcal{I}(\mathbb{R}^{d_1+d_2}) \) restrict to homeomorphisms

\[
\mathcal{F}_1 : S^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) \to S^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2})
\]

and

\[
\mathcal{F}_2 : S^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) \to S^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2});
\]

(2) the mappings \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) on \( \mathcal{I}(\mathbb{R}^{d_1+d_2}) \) are uniquely extendable to homeomorphisms

\[
\mathcal{F}_1 : (S^{\sigma_1,\sigma_2}_{s_1,s_2})'(\mathbb{R}^{d_1+d_2}) \to (S^{\sigma_1,\sigma_2}_{s_1,s_2})'(\mathbb{R}^{d_1+d_2})
\]

and

\[
\mathcal{F}_2 : (S^{\sigma_1,\sigma_2}_{s_1,s_2})'(\mathbb{R}^{d_1+d_2}) \to (S^{\sigma_1,\sigma_2}_{s_1,s_2})'(\mathbb{R}^{d_1+d_2}).
\]

The same holds true if the \( S^{\sigma_1,\sigma_2}_{s_1,s_2} \)-spaces and their duals are replaced by corresponding \( \Sigma^{\sigma_1,\sigma_2}_{s_1,s_2} \)-spaces and their duals.

The next two results follow from [14]. The proofs are therefore omitted.

**Proposition 1.2.** Let \( s_j, \sigma_j > 0, j = 1, 2 \). Then the following conditions are equivalent.

1. \( F \in S^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) \) \( (F \in \Sigma^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) \)

2. for some \( h > 0 \) (for every \( h > 0 \)) it holds

\[
|F(x_1, x_2)| \lesssim e^{-h(|x_1|^\frac{1}{d_1}+|x_2|^\frac{1}{d_2})} \quad \text{and} \quad |\hat{F}(\xi_1, \xi_2)| \lesssim e^{-h(|\xi_1|^\frac{1}{d_1}+|\xi_2|^\frac{1}{d_2})}.
\]

We notice that if \( s_j + \sigma_j < 1 \) for some \( j = 1, 2 \), then \( S^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) \) and \( \Sigma^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) \) are equal to the trivial space \( \{0\} \). Likewise, if \( s_j = \sigma_j = \frac{1}{2} \) for some \( j = 1, 2 \), then \( \Sigma^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) = \{0\} \).

1.2. The short time Fourier transform and Gelfand-Shilov spaces.

We recall here some basic facts about the short-time Fourier transform and weights.

Let \( \phi \in S'(\mathbb{R}^d) \setminus \{0\} \) be fixed. Then the short-time Fourier transform of \( f \in (S'_s)'(\mathbb{R}^d) \) is given by

\[
(V_{\phi} f)(x, \xi) = (2\pi)^{-\frac{d}{2}} (f, \phi(\cdot - x) e^{i(\cdot, \xi)})_{L^2}.
\]

Here \( (\cdot, \cdot)_{L^2} \) is the unique extension of the \( L^2 \)-form on \( S'_s(\mathbb{R}^d) \) to a continuous sesqui-linear form on \( (S'_s)'(\mathbb{R}^d) \times S'_s(\mathbb{R}^d) \). In the case \( f \in L^p(\mathbb{R}^d) \), for some \( p \in [1, \infty] \), then \( V_{\phi} f \) is given by

\[
V_{\phi} f(x, \xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y) \overline{\phi(y - x)} e^{-i(y, \xi)} dy.
\]

The following characterizations of the \( S^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) \), \( \Sigma^{\sigma_1,\sigma_2}_{s_1,s_2}(\mathbb{R}^{d_1+d_2}) \) and their duals follow by similar arguments as in the proofs of Propositions 2.1 and 2.2 in [31]. The details are left for the reader.
Proposition 1.3. Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \geq 1$, $j = 1, 2$, $s_0 \leq s$ and $\sigma_0 \geq \sigma$. Also let $\phi \in \mathcal{S}_{s_1, s_2}^{\sigma_1, \sigma_2}(\mathbb{R}^{d_1+d_2}) \setminus 0$ and let $f$ be a Gelfand-Shilov distribution on $\mathbb{R}^{d_1+d_2}$. Then the following is true:

(1) $f \in \mathcal{S}_{s_1, s_2}^{\sigma_1, \sigma_2}(\mathbb{R}^{d_1+d_2})$, if and only if

$$|V_{\phi}f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^{-r(|x_1|^{\frac{1}{s_1}} + |x_2|^{\frac{1}{s_2}} + |\xi_1|^{\frac{1}{\sigma_1}} + |\xi_2|^{\frac{1}{\sigma_2}})},$$

(1.5) holds for some $r > 0$;

(2) if in addition $\phi \in \Sigma_{s_1, s_2}^{\sigma_1, \sigma_2}(\mathbb{R}^{d_1+d_2}) \setminus 0$, then $f \in \Sigma_{s_1, s_2}^{\sigma_1, \sigma_2}(\mathbb{R}^{d_1+d_2})$ if and only if

$$|V_{\phi}f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^{-r(|x_1|^{\frac{1}{s_1}} + |x_2|^{\frac{1}{s_2}} + |\xi_1|^{\frac{1}{\sigma_1}} + |\xi_2|^{\frac{1}{\sigma_2}})},$$

(1.6) holds for every $r > 0$.

A proof of Proposition 1.3 can be found in e.g. [20] (cf. [20, Theorem 2.7]). The corresponding result for Gelfand-Shilov distributions is the following improvement of [30, Theorem 2.5].

Proposition 1.4. Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \geq 1$, $j = 1, 2$, $s_0 \leq s$ and $t_0 \leq t$. Also let $\phi \in \mathcal{S}_{s_1, s_2}^{\sigma_1, \sigma_2}(\mathbb{R}^{d_1+d_2}) \setminus 0$ and let $f$ be a Gelfand-Shilov distribution on $\mathbb{R}^{d_1+d_2}$. Then the following is true:

(1) $f \in (\mathcal{S}_{s_1, s_2}^{\sigma_1, \sigma_2})(\mathbb{R}^{d_1+d_2})$, if and only if

$$|V_{\phi}f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^r(|x_1|^{\frac{1}{s_1}} + |x_2|^{\frac{1}{s_2}} + |\xi_1|^{\frac{1}{\sigma_1}} + |\xi_2|^{\frac{1}{\sigma_2}}),$$

(1.7) holds for every $r > 0$;

(2) if in addition $\phi \in \Sigma_{s_1, s_2}^{\sigma_1, \sigma_2}(\mathbb{R}^{d_1+d_2}) \setminus 0$, then $f \in (\Sigma_{s_1, s_2}^{\sigma_1, \sigma_2})(\mathbb{R}^{d_1+d_2})$, if and only if

$$|V_{\phi}f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^r(|x_1|^{\frac{1}{s_1}} + |x_2|^{\frac{1}{s_2}} + |\xi_1|^{\frac{1}{\sigma_1}} + |\xi_2|^{\frac{1}{\sigma_2}}),$$

(1.8) holds for some $r > 0$.

A function $\omega$ on $\mathbb{R}^d$ is called a weight or weight function, if $\omega, 1/\omega \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ are positive everywhere. It is often assumed that $\omega$ is $v$-moderate for some positive function $v$ on $\mathbb{R}^d$. This means that

$$\omega(x + y) \lesssim \omega(x)v(y), \quad x, y \in \mathbb{R}^d. \quad (1.9)$$

If $v$ is even and satisfies (1.9) with $\omega = v$, then $v$ is called submultiplicative.

For any $s > 0$, let $\mathcal{P}_s(\mathbb{R}^d)$ ($\mathcal{P}_s^0(\mathbb{R}^d)$) be the set of all weights $\omega$ on $\mathbb{R}^d$ such that

$$e^{-r|x|^{\frac{1}{s}}} \lesssim \omega(x) \lesssim e^{r|x|^{\frac{1}{s}}}$$

for some $r > 0$ (for every $r > 0$). In similar ways, if $s, \sigma > 0$, then $\mathcal{P}_{s, \sigma}(\mathbb{R}^{2d})$ ($\mathcal{P}_{s, \sigma}^0(\mathbb{R}^{2d})$) consists of all submultiplicative weight functions $\omega$ on $\mathbb{R}^{2d}$ such that

$$e^{-r(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{\sigma}})} \lesssim \omega(x, \xi) \lesssim e^{r(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{\sigma}})}.$$
for some \( r > 0 \) (for every \( r > 0 \)). In particular, if \( \omega \in \mathcal{P}_{s,\sigma}(\mathbb{R}^{2d}) \) \((\mathcal{P}_{s,\sigma}^{0}(\mathbb{R}^{2d}))\), then
\[
\omega(x + y, \xi + \eta) \lesssim \omega(x, \xi)e^{r(|y|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}})}, \quad x, y, \xi, \eta \in \mathbb{R}^{d},
\]
for some \( r > 0 \) (for every \( r > 0 \)).

1.3. Pseudo-differential operators. Let \( A \in \mathcal{M}(d, \mathbb{R}) \) and \( s \geq \frac{1}{2} \) be fixed, and let \( a \in \mathcal{S}_{s}(\mathbb{R}^{2d}) \). Then the pseudo-differential operator \( \text{Op}_{A}(a) \) with symbol \( a \) is the continuous operator on \( \mathcal{S}_{s}(\mathbb{R}^{d}) \), defined by the formula
\[
(\text{Op}_{A}(a)f)(x) = (2\pi)^{-d} \int \int a(x - A(x), \xi)f(y)e^{i(x-y, \xi)}\, dyd\xi.
\]
We set \( \text{Op}_{t}(a) = \text{Op}_{A}(a) \) when \( t \in \mathbb{R}, \ A = t \cdot I \) and \( I \) is the identity matrix, and notice that this definition agrees with the Shubin type pseudo-differential operators (cf. e. g. [29]).

If instead \( a \in (\mathcal{S}_{s}^{\sigma})(\mathbb{R}^{2d}) \), then \( \text{Op}_{A}(a) \) is defined to be the continuous operator from \( \mathcal{S}^{\sigma}_{s}(\mathbb{R}^{d}) \) to \( \mathcal{S}^{\sigma}_{s}(\mathbb{R}^{d}) \) with the kernel in \( (\mathcal{S}_{s}^{\sigma})(\mathbb{R}^{2d}) \), given by
\[
K_{a,A}(x, y) \equiv (\mathcal{F}^{-1}_{2}(a)(x - A(x) - y, x - y).
\]
It is easily seen that the latter definition agrees with \((1.11)\) when \( a \in L^{1}(\mathbb{R}^{2d}) \).

If \( t = \frac{1}{2} \), then \( \text{Op}_{t}(a) \) is equal to the Weyl operator \( \text{Op}^{w}_{a}(a) \) for \( a \). If instead \( t = 0 \), then the standard (Kohn-Nirenberg) representation \( a(x, D) \) is obtained.

1.4. Symbol classes. Next we introduce function spaces related to symbol classes of the pseudo-differential operators. These functions should obey various conditions of the form
\[
|\partial^{\alpha}_{x}\partial^{\beta}_{\xi}a(x, \xi)| \lesssim h^{[\alpha + \beta]^{\sigma}!\sigma^{!}\!\!\beta^{!}\!\!\omega(x, \xi)},
\]
for functions on the phase space \( \mathbb{R}^{2d} \). For this reason we consider semi-norms of the form
\[
\|a\|_{\Gamma_{s,\sigma,h}^{\alpha,\beta}(\omega)} \equiv \sup_{\alpha,\beta \in \mathbb{N}^{d}} \left( \sup_{x,\xi \in \mathbb{R}^{d}} \left( \frac{|\partial^{\alpha}_{x}\partial^{\beta}_{\xi}a(x, \xi)|}{h^{[\alpha + \beta]^{\sigma}!\sigma^{!}\!\!\beta^{!}\!\!\omega(x, \xi)}} \right) \right),
\]
indexed by \( h > 0 \).

**Definition 1.5.** Let \( s, \sigma \) and \( h \) be positive constants, let \( \omega \) be a weight on \( \mathbb{R}^{2d} \), and let \( \omega_{r}(x, \xi) \equiv e^{r(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{\sigma}})} \).

1. The set \( \Gamma_{s,\sigma,h}^{\alpha,\beta}(\omega)(\mathbb{R}^{2d}) \) consists of all \( a \in C^{\infty}(\mathbb{R}^{2d}) \) such that \( \|a\|_{\Gamma_{s,\sigma,h}^{\alpha,\beta}(\omega)} \) in \((1.13)\) is finite. The set \( \Gamma_{0}^{\sigma,\sigma,h}((\mathbb{R}^{2d}) \) consists of all \( a \in C^{\infty}(\mathbb{R}^{2d}) \) such that \( \|a\|_{\Gamma_{0}^{\sigma,\sigma,\omega}(\mathbb{R}^{2d})} \) is finite for every \( r > 0 \), and the topology
is the projective limit topology of $\Gamma_{\sigma,s;h}^{\omega}(R^d)$ with respect to $r > 0$;

(2) The sets $\Gamma_{\sigma,s}^0(\omega)(R^d)$ and $\Gamma_{\sigma,s;0}(\omega)(R^d)$ are given by

$$\Gamma_{\sigma,s}^0(\omega)(R^d) = \bigcup_{h > 0} \Gamma_{\sigma,s;h}^0(\omega)(R^d) \quad \text{and} \quad \Gamma_{\sigma,s;0}(\omega)(R^d) = \bigcap_{h > 0} \Gamma_{\sigma,s;h}^0(\omega)(R^d),$$

and their topologies are the inductive respective the projective topologies of $\Gamma_{\sigma,s;h}(\omega)(R^d)$ with respect to $h > 0$.

Furthermore we have the following classes.

**Definition 1.6.** For $s_j, \sigma_j \geq 0$, $j = 1, 2$, and $h, r > 0$ and $f \in C^\infty(R^{d_1 + d_2})$, let

$$\|f\|_{(h,r)} = \sup \left( \frac{|\partial_{x_1}^\alpha \partial_{x_2}^\beta f(x_1, x_2)|}{\sqrt{h_{\alpha_1 + \beta_2}^{\alpha_2} + r_{\alpha_2}^{\beta_2} |x_1|^2 + |x_2|^2}} \right), \quad (1.15)$$

where the supremum is taken over all $\alpha_1 \in \mathbb{N}^{d_1}$, $\alpha_2 \in \mathbb{N}^{d_2}$, $x_1 \in R^{d_1}$ and $x_2 \in R^{d_2}$.

(1) $\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2}(R^{d_1 + d_2})$ consists of all $f \in C^\infty(R^{d_1 + d_2})$ such that $\|f\|_{(h,r)}$ is finite for some $h, r > 0$;

(2) $\Gamma_{s_1,s_2;0}^{\sigma_1,\sigma_2}(R^{d_1 + d_2})$ consists of all $f \in C^\infty(R^{d_1 + d_2})$ such that for some $h > 0$, $\|f\|_{(h,r)}$ is finite for every $r > 0$;

(3) $\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2;0}(R^{d_1 + d_2})$ consists of all $f \in C^\infty(R^{d_1 + d_2})$ such that for some $r > 0$, $\|f\|_{(h,r)}$ is finite for every $h > 0$;

(4) $\Gamma_{s_1,s_2;0}^{\sigma_1,\sigma_2}(R^{d_1 + d_2})$ consists of all $f \in C^\infty(R^{d_1 + d_2})$ such that $\|f\|_{(h,r)}$ is finite for every $h, r > 0$.

In order to define suitable topologies of the spaces in Definition 1.6, let $\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2}(h,r)(R^{d_1 + d_2})$ be the set of $f \in C^\infty(R^{d_1 + d_2})$ such that $\|f\|_{(h,r)}$ is finite. Then $\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2}(h,r)(R^{d_1 + d_2})$ is a Banach space, and the sets in Definition 1.6 are given by

$$\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2}(R^{d_1 + d_2}) = \bigcup_{h,r > 0} (\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2}(h,r)(R^{d_1 + d_2}));$$

$$\Gamma_{s_1,s_2;0}^{\sigma_1,\sigma_2}(R^{d_1 + d_2}) = \bigcup_{h > 0} \left( \bigcap_{r > 0} (\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2}(h,r)(R^{d_1 + d_2})) \right);$$

$$\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2;0}(R^{d_1 + d_2}) = \bigcup_{r > 0} \left( \bigcap_{h > 0} (\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2}(h,r)(R^{d_1 + d_2})) \right)$$

and

$$\Gamma_{s_1,s_2;0}^{\sigma_1,\sigma_2}(R^{d_1 + d_2}) = \bigcap_{h,r > 0} (\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2}(h,r)(R^{d_1 + d_2}));$$
and we equip these spaces by suitable mixed inductive and projective limit topologies of \((\Gamma_{s_1,s_2}^{\sigma_1,\sigma_2})(h,r)(\mathbb{R}^{d_1+d_2})\).

In Appendix A we show some further continuity results of the symbol classes in Definition 1.6.

2. THE SHORT-TIME FOURIER TRANSFORM AND REGULARITY

In this section we deduce equivalences between conditions on the short-time Fourier transforms of functions or distributions and estimates on derivatives.

In what follows we let \(\kappa\) be defined as

\[
\kappa(r) = \begin{cases} 
1 & \text{when } r \leq 1, \\
2^{r-1} & \text{when } r > 1.
\end{cases}
\]  

(2.1)

In the sequel we shall frequently use the well known inequality

\[
|x + y|^{\frac{1}{s}} \leq \kappa(s^{-1})(|x|^{\frac{1}{s}} + |y|^{\frac{1}{s}}), \quad s > 0, \quad x, y \in \mathbb{R}^d.
\]

**Proposition 2.1.** Let \(s, \sigma > 0\) be such that \(s + \sigma \geq 1\) and \((s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})\), \(\phi \in \Sigma^s_\sigma(\mathbb{R}^d) \setminus 0\), \(r > 0\) and let \(f\) be a Gelfand-Shilov distribution on \(\mathbb{R}^d\). Then the following is true:

1. If \(f \in C^\infty(\mathbb{R}^d)\) and satisfies

\[
|\partial^\alpha f(x)| \lesssim h^{|\alpha|} \alpha! \sigma e^{r|x|^{\frac{1}{s}}},
\]

for every \(h > 0\) (resp. for some \(h > 0\)), then

\[
|V_\phi f(x, \xi)| \lesssim e^{c(s^{-1})|\xi|^{1-h} |\xi|^s},
\]

for every \(h > 0\) (resp. for some new \(h > 0\));

2. If

\[
|V_\phi f(x, \xi)| \lesssim e^{r|x|^{\frac{1}{s}} - h |\xi|^s},
\]

for every \(h > 0\) (resp. for some \(h > 0\)), then \(f \in C^\infty(\mathbb{R}^d)\) and satisfies

\[
|\partial^\alpha f(x)| \lesssim h^{|\alpha|} \alpha! \sigma e^{c(s^{-1})|x|^{\frac{1}{s}}},
\]

for every \(h > 0\) (resp. for some new \(h > 0\)).

**Proof.** We only prove the assertion when (2.2) or (2.4) are true for every \(h > 0\), leaving the straight-forward modifications of the other cases to the reader.

Assume that (2.2) holds. Then for every \(x \in \mathbb{R}^d\) the function

\[
y \mapsto F_x(y) \equiv f(y + x)\overline{\phi(y)}
\]

belongs to \(\Sigma^s_\sigma(\mathbb{R}^d)\), and

\[
|\partial^n F_x(y)| \lesssim h^{|\alpha|} \alpha! \sigma e^{c(s^{-1})|x|^{\frac{1}{s}}} e^{-ro|y|^{\frac{1}{s}}},
\]
for every $h, r_0 > 0$. In particular,
\[ |F_k(y)| \lesssim e^{\kappa (s^{-1}) |x|^\frac{1}{h}} e^{-r_0 |y|^\frac{1}{h}} \quad \text{and} \quad |\hat{F}_k(\xi)| \lesssim e^{\kappa (s^{-1}) |x|^\frac{1}{h}} e^{-r_0 |\xi|^\frac{1}{h}}, \tag{2.5} \]
for every $r_0 > 0$. Since $|V_\phi f(x, \xi)| = |\hat{F}_k(\xi)|$, the estimate \((2.3)\) follows from the second inequality in \((2.5)\), and \((1)\) follows.

Next we prove \((2)\). By the inversion formula we get
\[ f(x) = (2\pi)^{-\frac{d}{2}} \|\phi\|^{-\frac{1}{2}} \int_{\mathbb{R}^d} V_\phi f(y, \eta) \phi(x-y) e^{i(x,\eta)} \, dyd\eta. \tag{2.6} \]
Here we notice that
\[(x, y, \eta) \mapsto V_\phi f(y, \eta) \phi(x-y) e^{i(x,\eta)}\]
is smooth and
\[(y, \eta) \mapsto \eta^\alpha V_\phi f(y, \eta) \partial^\beta \phi(x-y) e^{i(x,\eta)}\]
is an integrable function for every $x$, $\alpha$ and $\beta$, giving that $f$ in \((2.6)\) is smooth.

By differentiation and the fact that $\phi \in \Sigma_s^\sigma$ we get
\[
|\partial^\alpha f(x)| \lesssim \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left| \int_{\mathbb{R}^d} \eta^\beta V_\phi f(y, \eta)(\partial^{\alpha-\beta} \phi)(x-y) e^{i(x,\eta)} \, dyd\eta \right|
\[
\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left| \int_{\mathbb{R}^d} \eta^\beta |\phi(y)^{\ast}| e^{-|y|^\frac{1}{h}} \phi(x-y) \right| |\phi(y)^{\ast}| \, dyd\eta
\[
\lesssim \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left| \int_{\mathbb{R}^d} \eta^\beta |\phi(y)^{\ast}| e^{-|y|^\frac{1}{h}} e^{-h_1 |x-y|^\frac{1}{h}} \, dyd\eta \right|
\]
for every $h_1 > 0$ and $h_2 > 0$. Since
\[
|\eta^\beta e^{-|y|^\frac{1}{h}}| \lesssim h_2^{|\beta|} |\beta|! e^{-\frac{h_1}{2} |y|^\frac{1}{h}}, \tag{2.7} \]
we get
\[
|\partial^\alpha f(x)|
\[
\lesssim h_2^{|\alpha|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\beta!(\alpha-\beta)!) \left| \int_{\mathbb{R}^d} e^{-\frac{h}{2} |y|^\frac{1}{h}} e^{r|y|^\frac{1}{h}} e^{-h_1 |x-y|^\frac{1}{h}} \, dyd\eta \right|
\[
\lesssim (2^{1-s} h_2)^{|\alpha|} |\alpha|! \int_{\mathbb{R}^n} e^{r|y|^\frac{1}{h}} e^{-h_1 |x-y|^\frac{1}{h}} \, dy. \tag{2.8} \]
Since $|y|^{\frac{1}{2}} \leq \kappa(s^{-1})(|x|^{\frac{1}{2}} + |y - x|^{\frac{1}{2}})$ and $h_1$ can be chosen arbitrarily large, it follows from the last estimate that

$$|\partial^\alpha f(x)| \lesssim (2h_2)^{|\alpha|}(\alpha!)^\kappa e^{\kappa(s^{-1})|x|^s},$$

for every $h_2 > 0$. This gives the result. \qed

By similar arguments we get the following result. The details are left for the reader.

**Proposition 2.1.** Let $s_1, \sigma_j > 0$ be such that $s_j + \sigma_j \geq 1$ and $(s_j, \sigma_j) \neq (\frac{1}{2}, \frac{1}{2})$, $j = 1, 2$, $\phi \in \Sigma_{s_1, s_2}^\sigma(\mathbb{R}^{d_1+d_2}) \setminus \{0\}$ and let $f$ be a Gelfand-Shilov distribution on $\mathbb{R}^{d_1+d_2}$. Then the following is true:

1. If $f \in C^\infty(\mathbb{R}^{d_1+d_2})$ and satisfies

$$|\partial^\alpha_1 \partial^\alpha_2 f(x_1, x_2)| \lesssim h^{|\alpha_1|+|\alpha_2|} \alpha_1! \alpha_2! \epsilon_{\alpha}(|x_1|^{\frac{1}{2}} + |x_2|^{\frac{1}{2}}),$$

for every $h > 0$ (resp. for some $h > 0$), then

$$|V_\phi f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^{\kappa(s^{-1})r|x_1|^{\frac{1}{2}} - h(|\xi_1|^{\frac{1}{2}} + |\xi_2|^{\frac{1}{2}})}$$

for every $h > 0$ (resp. for some new $h > 0$);

2. If

$$|V_\phi f(x_1, x_2, \xi_1, \xi_2)| \lesssim e^{r|x_1|^{\frac{1}{2}} - h(|\xi_1|^{\frac{1}{2}} + |\xi_2|^{\frac{1}{2}})},

$$

for every $h > 0$ (resp. for some new $h > 0$), then $f \in C^\infty(\mathbb{R}^{d_1+d_2})$

and satisfies

$$|\partial^\alpha_1 \partial^\alpha_2 f(x_1, x_2)| \lesssim h^{|\alpha_1|+|\alpha_2|} \alpha_1! \alpha_2! \epsilon_{\alpha}(|x_1|^{\frac{1}{2}} + |x_2|^{\frac{1}{2}}),$$

for every $h > 0$ (resp. for some new $h > 0$).

As a consequence of the previous result we get the following.

**Proposition 2.2.** Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \geq 1$ and $(s_j, \sigma_j) \neq (\frac{1}{2}, \frac{1}{2})$, $j = 1, 2$, $\phi \in \Sigma_{s_1, s_2}^\sigma(\mathbb{R}^{d_1+d_2}) \setminus \{0\}$ and let $f$ be a Gelfand-Shilov distribution on $\mathbb{R}^{d_1+d_2}$. Then the following is true:

1. there exist $h, r > 0$ such that \([2.4]^'\) holds, if and only if $f \in \Gamma_{s_1, s_2}(\mathbb{R}^{d_1+d_2})$;

2. there exists $r > 0$ such that \([2.4]^'\) holds for every $h > 0$, if and only if $f \in \Gamma_{s_1, s_2, 0}(\mathbb{R}^{d_1+d_2})$;

3. \([2.4]^'\) holds for every $h, r > 0$, if and only if $f \in \Gamma_{s_1, s_2, 0}(\mathbb{R}^{d_1+d_2})$.

By similar arguments that led to Proposition 2.2 we also get the following. The details are left for the reader.

**Proposition 2.3.** Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \geq 1$, $j = 1, 2$, $\phi \in \Sigma_{s_1, s_2}^\sigma(\mathbb{R}^{d_1+d_2}) \setminus \{0\}$ and let $f$ be a Gelfand-Shilov distribution on $\mathbb{R}^{d_1+d_2}$. Then there exists $h > 0$ such that \([2.4]^'\) holds for every $r > 0$, if and only if $f \in \Gamma_{s_1, s_2, 0}(\mathbb{R}^{d_1+d_2})$.\]
We also have the following version of Proposition 2.1, involving certain types of moderate weights.

**Proposition 2.4.** Let $s, \sigma > 0$ be such that $s + \sigma \geq 1$, $\phi \in \Sigma^s_s(R^{2d}) \setminus 0$ ($\phi \in \mathcal{S}^s_s(R^{2d}) \setminus 0$), $r > 0$, $\omega \in \mathcal{P}_{s,\sigma}(R^{2d})$ ($\omega \in \mathcal{P}_{s,\sigma}(R^{2d})$) and let $a$ be a Gelfand-Shilov distribution on $R^{2d}$. Then the following is true:

1. If $a \in C^\infty(R^{2d})$ and satisfies

   $$|\partial_\alpha^\beta \partial_\xi^\eta a(x, \xi)| \lesssim h^{1 + |r|} \alpha! \beta! e^{-r(|y|^{1/2} + |\xi|^{1/2})},$$

   then the fact that $\omega(X) \approx \omega(Y) e^{r_0(|y|^{1/2} + |\eta|^{1/2})}$ gives that $\{Y \mapsto F_X(Y) ; X \in R^{2d}\}$ is a bounded set of $\Sigma^s_s$.

2. If (2.10) holds for every $r > 0$ (for some $r > 0$), then $a \in C^\infty(R^{2d})$ and (2.9) holds for every $h > 0$ (for some $h > 0$).

**Proof.** We shall use similar arguments as in the proof of Proposition 2.1. Let $X = (x, \xi) \in R^{2d}$, $Y = (y, \eta) \in R^{2d}$, $Z = (z, \zeta) \in R^{2d}$ and let $\phi \in \mathcal{S}^s_s(R^{2d}) \setminus 0$. Suppose that $\omega \in \mathcal{P}_{s,\sigma}(R^{2d})$ and that (2.9) holds for all $h > 0$. If

$$F_X(Y) = \frac{a_k(Y + X) \phi(Y)}{\omega(X)}$$

then the fact that $\omega(X) \approx \omega(Y + X) e^{r_0(|y|^{1/2} + |\eta|^{1/2})}$ gives that

$$\{Y \mapsto F_X(Y) ; X \in R^{2d}\}$$

is a bounded set of $\Sigma^s_s$. Hence

$$|\partial_\alpha^\beta \partial_\xi^\eta F_X(y, \eta)| \lesssim h^{1 + |r|} \alpha! \beta! e^{-r(|y|^{1/2} + |\eta|^{1/2})},$$

for every $h, r > 0$. In particular,

$$|F_X(y, \eta)| \lesssim e^{-r(|y|^{1/2} + |\eta|^{1/2})}$$

(2.11)

and

$$|\langle \mathcal{F} F_X \rangle(\zeta, z)| \lesssim e^{-r(|z|^{1/2} + |\zeta|^{1/2})},$$

for every $r > 0$. Since

$$|V_\phi a(x, \xi, \eta, y)| = |\langle \mathcal{F} F_X \rangle(\eta, y) \omega(X)|,$$

it follows that (2.10) holds for all $r > 0$. This gives (1) in the case when $\omega \in \mathcal{P}_{s,\sigma}(R^{2d})$ and $\phi \in \mathcal{S}^s_s(R^{2d}) \setminus 0$. In the same way, (1) follows in the case when $\omega \in \mathcal{P}_{s,\sigma}(R^{2d})$ and $\phi \in \mathcal{S}^s_s(R^{2d}) \setminus 0$. The details are left for the reader.

Next we prove (2) in the case $\ldots$. Therefore, suppose (2.10) holds for all $r > 0$. Then $a$ is smooth in view of Proposition 2.1.

By differentiation, (2.6), the fact that

$$\omega(Z) \lesssim \omega(X) e^{r_0(|x|^{1/2} + |\xi - \zeta|^{1/2})},$$

for all $X, Z \in R^{2d}$ and $r > 0$, we have $\omega \in \mathcal{P}_{s,\sigma}(R^{2d})$ and $\phi \in \mathcal{S}^s_s(R^{2d}) \setminus 0$. Then $a$ is smooth in view of Proposition 2.1.
and the fact that \( \phi \in \Sigma_{\sigma} \) we get

\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \\
\lesssim \sum_{\gamma \leq \alpha} \left( \frac{\alpha}{\gamma} \right) \left( \frac{\beta}{\delta} \right) \int_{\mathbb{R}^{4d}} |\eta^\gamma y^\delta V_\phi a(z, \eta, \xi) (\partial_x^{\alpha-\gamma} \partial_\xi^{\beta-\delta} \phi)(X-Z)| \, dY dZ \\
\lesssim \sum_{\gamma \leq \alpha} \left( \frac{\alpha}{\gamma} \right) \left( \frac{\beta}{\delta} \right) h^{\alpha+\beta-\gamma-\delta} (\alpha-\gamma)! (\beta-\delta)! I_{\gamma,\delta}(X),
\]

where

\[
I_{\gamma,\delta}(X) = \int_{\mathbb{R}^{4d}} \omega(Z) |\eta^\gamma y^\delta e^{-r+ru_i(|x-z|^\frac{1}{s} + |y|^\frac{1}{s} + |\xi-\zeta|^\frac{1}{\sigma} + |\eta|^\frac{1}{\sigma})} \, dY dZ \\
\lesssim \omega(X) \int_{\mathbb{R}^{4d}} |\eta^\gamma y^\delta e^{-r(|z|^\frac{1}{s} + |y|^\frac{1}{s} + |\xi|^\frac{1}{\sigma} + |\eta|^\frac{1}{\sigma})} \, dY dZ \\
\lesssim h^{|\gamma+\delta|} \gamma! \delta! \omega(X) \int_{\mathbb{R}^{4d}} e^{-\frac{1}{2} (|z|^\frac{1}{s} + |y|^\frac{1}{s} + |\xi|^\frac{1}{\sigma} + |\eta|^\frac{1}{\sigma})} \, dY dZ \\
\lesssim h^{|\gamma+\delta|} \gamma! \delta! \omega(X)
\]

for every \( h, r > 0 \). Here the last inequality follows from (2.7). It follows that (2.9) holds for every \( h > 0 \) by using the estimates above and similar computations as in (2.8).

The remaining case follows by similar arguments and is left for the reader. \( \square \)

3. INVARIANCE, CONTINUITY AND COMPOSITION PROPERTIES FOR PSEUDO-DIFFERENTIAL OPERATORS

In this section we deduce invariance, continuity and composition properties for pseudo-differential operators with symbols in the classes considered in the previous sections. In the first part we show that for any such class \( S \), the set \( \text{Op}_A(S) \) of pseudo-differential operators is independent of the matrix \( A \). Thereafter we deduce that such operators are continuous on Gelfand-Shilov spaces and their duals. In the last part we deduce that these operator classes are closed under compositions.

3.1. Invariance properties. An important ingredient in these considerations concerns mapping properties for the operator \( e^{i(AD_c, D_x)} \). In fact we have the following.

**Theorem 3.1.** Let \( s, s_1, s_2, \sigma, \sigma_1, \sigma_2 > 0 \) be such that

\[
s + \sigma \geq 1, \quad s_1 + \sigma_1 \geq 1, \quad s_2 + \sigma_2 \geq 1, \quad s_2 \leq s_1 \quad \text{and} \quad \sigma_1 \leq \sigma_2,
\]

and let \( A \in \mathbf{M}(d, \mathbf{R}) \). Then the following is true:

...
(1) $e^{i(AD_\xi,D_\eta)}$ on $\mathcal{F}(\mathbb{R}^{2d})$ restricts to a homeomorphism on $\mathcal{S}_r^{\sigma_1,\sigma_2}(\mathbb{R}^{2d})$, and extends uniquely to a homeomorphism on $(\mathcal{S}_r^{\sigma_1,\sigma_2})'(\mathbb{R}^{2d})$;

(2) if in addition $(s_1,\sigma_1) \neq (\frac{s_1}{2}, \frac{1}{2})$ and $(s_2,\sigma_2) \neq (\frac{s_2}{2}, \frac{1}{2})$, then $e^{i(AD_\xi,D_\eta)}$ on $\mathcal{F}(\mathbb{R}^{2d})$ restricts to a homeomorphism on $\Sigma_r^{\sigma_1,\sigma_2}(\mathbb{R}^{2d})$, and extends uniquely to a homeomorphism on $(\Sigma_r^{\sigma_1,\sigma_2})'(\mathbb{R}^{2d})$;

(3) $e^{i(AD_\xi,D_\eta)}$ is a homeomorphism on $\Gamma^{\sigma,s}_{\sigma_0,0}(\mathbb{R}^{2d})$;

(4) if in addition $(s,\sigma) \neq (\frac{s}{2}, \frac{1}{2})$, then $e^{i(AD_\xi,D_\eta)}$ is a homeomorphism on $\Gamma^{\sigma,s,0}_{\sigma_0,0}(\mathbb{R}^{2d})$ and on $\Gamma^{\sigma,s,0}_{\sigma_0,0}(\mathbb{R}^{2d})$.

The assertion (1) in the previous theorem is proved in [10] and is essentially a special case of Theorem 32 in [33], whereas (2) can be found in [10, 11]. Thus we need to prove Theorem 3.1 (3) and (4), which are extensions of [10, Theorem 4.6 (3)].

\textbf{Proof.} We need to prove (3) and (4) and begin with (3). Let $\phi \in \mathcal{S}_{s,\sigma}(\mathbb{R}^{2d})$ and $\phi_A = e^{i(AD_\xi,D_\eta)}\phi$. Then $\phi_A \in \mathcal{S}_{s,\sigma}(\mathbb{R}^{2d})$, in view of (1), and

$$|(V_0 a(e^{i(AD_\xi,D_\eta)}a))(x, \xi, \eta, y)| = |(V_0 a)(x - Ay, \xi - A^*\eta, \eta, y)|$$

by straightforward computations. Then $a \in \Gamma^{s,\sigma,0}_{\sigma_0,0}(\mathbb{R}^{2d})$ is equivalent to that for some $h > 0$,

$$|V_0 a(x, \xi, \eta, y)| \lesssim e^{r(\frac{1}{2} + |\xi|^2)} - h(|\eta|^{\frac{1}{2}} + |y|^{\frac{1}{2}}),$$

holds for every $r > 0$, in view of Proposition 2.3. By (3.1) and (1) it follows by straightforward computation, that the latter condition is invariant under the mapping $e^{i(AD_\xi,D_\eta)}$, and (3) follows from these invariance properties. By similar arguments, taking $\phi \in \Sigma_r^{\sigma,s}(\mathbb{R}^{2d})$ and using (2) instead of (1), we deduce (4).

\textbf{Corollary 3.2.} Let $s, \sigma > 0$ be such that $s + \sigma \geq 1$ and $\sigma \leq s$. Then $e^{i(AD_\xi,D_\eta)}$ is a homeomorphism on $\mathcal{S}_s^{\sigma}(\mathbb{R}^{2d})$, $\Sigma_s^{\sigma}(\mathbb{R}^{2d})$, $(\mathcal{S}_s')'(\mathbb{R}^{2d})$ and on $(\Sigma_s')'(\mathbb{R}^{2d})$.

We also have the following extension of (3) and (4) in [10] Theorem 4.1.

\textbf{Theorem 3.3.} Let $\omega \in \mathcal{D}_{s,\sigma}(\mathbb{R}^{2d})$, $s, \sigma > 0$ be such that $s + \sigma \geq 1$. Then $a \in \Gamma^{s,\sigma,0}_{\omega}(\mathbb{R}^{2d})$ if and only if $e^{i(AD_\xi,D_\eta)a} \in \Gamma^{s,\sigma,0}_{\omega}(\mathbb{R}^{2d})$.

We need some preparation for the proof and start with the following proposition.

\textbf{Proposition 3.4.} Let $s, \sigma > 0$ be such that $s + \sigma \geq 1$ and $(s, \sigma) \neq (\frac{s}{2}, \frac{1}{2})$, $\phi \in \Sigma_r^{s,\sigma}(\mathbb{R}^{2d}) \setminus \{0\}$, $\omega \in \mathcal{D}_{s,\sigma}(\mathbb{R}^{2d})$ and let $a$ be a Gelfand-Shilov distribution on $\mathbb{R}^{2d}$. Then the following conditions are equivalent:

1. $a \in \Gamma^{s,\sigma,0}_{\omega}(\mathbb{R}^{2d})$;
Proof. Obviously, (2) implies (3). Assume now that (1) holds. Let
\[ \left| \partial_x \partial_\xi^\beta \left( e^{i \langle x, \xi \rangle} V_{\phi a}(x, \xi, \eta, y) \right) \right| \lesssim h^{\alpha + \beta} |\alpha|^\sigma |\beta|^s \omega(x, \xi) e^{- R \langle |y|^{\frac{1}{2}} + |\eta|^{\frac{1}{2}} \rangle}; \]

(3.2)

(3) for \( \alpha = \beta = 0 \), (3.2) holds for every \( h > 0, R > 0 \) and \( x, y, \xi, \eta \in \mathbb{R}^d \).

By straightforward application of Leibniz rule in combination with (1.10) we obtain
\[ |\partial_x \partial_\xi^\beta F_a(x, \xi, y, \eta)| \lesssim h^{\alpha + \beta} |\alpha|^\sigma |\beta|^s \omega(x, \xi) e^{- R \langle |y|^{\frac{1}{2}} + |\eta|^{\frac{1}{2}} \rangle} \]
for every \( h > 0 \) and \( R > 0 \). Hence, if
\[ G_{a,h,x,\xi}(y, \eta) = \frac{\partial_x \partial_\xi^\beta F_a(x, \xi, y, \eta)}{h^{\alpha + \beta} |\alpha|^\sigma |\beta|^s \omega(x, \xi)}, \]
then \( \{G_{a,h,x,\xi}\}_{x, \xi \in \mathbb{R}^d} \) is a bounded set in \( \Sigma^{\sigma, s}_{s, \sigma}(\mathbb{R}^{2d}) \) for every fixed \( h > 0 \).

Proposition 3.5. Let \( R > 0, q \in [1, \infty] \), \( s, \sigma > 0 \) be such that \( s + \sigma \geq 1 \) and \( (s, \sigma) \neq \left( \frac{1}{2}, \frac{1}{2} \right) \), \( \phi \in \Sigma^{\sigma, s}_{s, \sigma}(\mathbb{R}^{2d}) \setminus \emptyset \), \( \omega \in \mathcal{D}_{s, \sigma}(\mathbb{R}^{2d}) \), and let
\[ \omega_R(x, \xi, \eta, y) = \omega(x, \xi) e^{- R \langle |y|^{\frac{1}{2}} + |\eta|^{\frac{1}{2}} \rangle}. \]

Then
\[ \Gamma^{\sigma, s}_{(\omega)}(\mathbb{R}^{2d}) = \bigcap_{R > 0} \{ a \in (\Sigma^{\sigma, s}_{s, \sigma})'(\mathbb{R}^{2d}) : \| \omega_R^{-1} V_{\phi a} \|_{L^\infty, q} < \infty \}. \]

Proof. Let \( \phi_0 \in \Sigma^{\sigma, s}_{s, \sigma}(\mathbb{R}^{2d}) \setminus \emptyset \), \( a \in (\Sigma^{\sigma, s}_{s, \sigma})'(\mathbb{R}^{2d}) \), and set
\[ F_{0,a}(X, Y) = |(V_{\phi a})(x, \xi, \eta, y)|, \quad F_a(X, Y) = |(V_{\phi a})(x, \xi, \eta, y)| \]
and \( G(x, \xi, \eta, y) = |(V_{\phi a})(x, \xi, \eta, y)| \),
where \( X = (x, \xi) \) and \( Y = (y, \eta) \). Since \( V_{\phi a} \in \Sigma^{\sigma, s}_{s, \sigma}(\mathbb{R}^{4d}) \), we have
\[ 0 \leq G(x, \xi, \eta, y) \lesssim e^{- R \langle |x|^{\frac{1}{2}} + |\xi|^{\frac{1}{2}} + |\eta|^{\frac{1}{2}} + |y|^{\frac{1}{2}} \rangle} \quad \text{for every} \ R > 0. \]

(3.4)
By Lemma 11.3.3, we have $F_a \lesssim F_{0,a} * G$. We obtain

\[ (\omega_R^{-1} \cdot F_a)(X,Y) \lesssim \omega(X)^{-1} e^{R(|y|^{1/2} + |\eta|^{1/2})} \int_{\mathbb{R}^d} F_{0,a}(X - X_1, Y - Y_1) G(X_1, Y_1) \, dX_1 dY_1 \]

\[ \lesssim \int_{\mathbb{R}^d} (\omega_R^{-1} \cdot F_{0,a})(X - X_1, Y - Y_1) G_1(X_1, Y_1) \, dX_1 dY_1 \quad (3.5) \]

for some $G_1$ satisfying (3.3) in place of $G$ and some $c > 0$ independent of $R$. By applying the $L^\infty$-norm on the last inequality we get

\[ \|\omega_R^{-1} F_a\|_{L^\infty(\mathbb{R}^d)} \lesssim \sup_Y \left( \int_{\mathbb{R}^d} (\sup(\omega_R^{-1} \cdot F_{0,a})(\cdot, Y - Y_1)) G_1(X_1, Y_1) \, dX_1 dY_1 \right) \]

\[ \leq \sup_Y (\| (\omega_R^{-1} \cdot F_{0,a})(\cdot - (0, Y)) \|_{L^\infty,q}) \| G_1 \|_{L^{1,q}} \asymp \| \omega_R^{-1} \cdot F_{0,a} \|_{L^\infty,q} \cdot \]

We only consider the case $q < \infty$ when proving the opposite inequality. The case $q = \infty$ follows by similar arguments and is left for the reader.

By (3.5) we have

\[ \|\omega_R^{-1} F_a\|_{L^\infty(\mathbb{R}^d)} \lesssim \int_{\mathbb{R}^d} (\sup H(\cdot, Y)^q) \, dY, \]

where $H = K_1 * G$ and $K_j = \omega_{jR}^{-1} \cdot F_{0,a}$, $j \geq 1$.

By Minkowski’s inequality, letting $Y_1 = (y_1, \eta_1)$ as variables of integration, we get

\[ \sup_X H(X,Y) \]

\[ \lesssim \int_{\mathbb{R}^d} (\sup K_2(\cdot, Y - Y_1)) e^{-cR(|y - y_1|^{1/2} + |\eta - \eta_1|^{1/2})} G(X_1, Y_1) \, dX_1 dY_1 \]

\[ \lesssim \| K_2 \|_{L^\infty} \int_{\mathbb{R}^d} e^{-cR(|y - y_1|^{1/2} + |\eta - \eta_1|^{1/2})} G(X_1, Y_1) \, dX_1 dY_1. \]

By combining these estimates we get

\[ \|\omega_R^{-1} F_a\|_{L^\infty,q}^q \lesssim \| K_2 \|_{L^\infty}^q \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} e^{-cR(|y - y_1|^{1/2} + |\eta - \eta_1|^{1/2})} G(X_1, Y_1) \, dX_1 dY_1 \right)^q \, dY \]

\[ \asymp \| K_2 \|_{L^\infty}^q \]

That is,

\[ \|\omega_R^{-1} F_a\|_{L^\infty,q} \lesssim \|\omega_{2cR}^{-1} \cdot F_{0,a}\|_{L^\infty}, \]
and the result follows.

Proof of Theorem 3.3. The case $s = \sigma = \frac{1}{2}$ follows from [10, Theorem 4.1]. We may therefore assume that $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$. Let $\phi \in \Sigma_{s,\sigma}(\mathbb{R}^{2d})$ and $\phi_A = e^{i(AD_{\xi} - D_{\eta})} \phi$. Then $\phi_A \in \Sigma_{s,\sigma}(\mathbb{R}^{2d})$, in view of (2) in Theorem 3.1. Hence Proposition 3.5 gives

$$\omega_{A,R}(x, \xi, \eta, y) = \omega(x - Ay, \xi - A^*\eta)e^{-R||y||^{\frac{1}{2}} + ||\eta||^{\frac{1}{2}}},$$

By straight-forward applications of Parseval’s formula, we get

$$|(V_{\phi_A}(e^{i(AD_{\xi} - D_{\eta})} \phi))(x, \xi, \eta, y)| = |(V_{\phi}(x - Ay, \xi - A^*\eta, \eta, y)|$$

(cf. Proposition 1.7 in [29] and its proof). This gives

$$\|\omega_{0,R}^{-1}V_{\phi_A}\|_{L^{p,q}} = \|\omega_{A,R}^{-1}V_{\phi_A}(e^{i(AD_{\xi} - D_{\eta})} \phi)\|_{L^{p,q}}.$$ 

Hence Proposition 3.5 gives

$$a \in \Gamma^{s,\sigma,0}_{(\omega)}(\mathbb{R}^{2d}) \iff \|\omega_{0,R}^{-1}V_{\phi_A}\|_{L^{\infty}} < \infty \quad \text{for every } R > 0$$

$$\iff \|\omega_{0,R}^{-1}V_{\phi_A}(e^{i(AD_{\xi} - D_{\eta})} \phi)\|_{L^{\infty}} < \infty \quad \text{for every } R > 0$$

$$\iff \|\omega_{0,R}^{-1}V_{\phi_A}(e^{i(AD_{\xi} - D_{\eta})} \phi)\|_{L^{\infty}} < \infty \quad \text{for every } R > 0$$

$$\iff e^{i(AD_{\xi} - D_{\eta})} \phi \in \Gamma^{s,\sigma,0}_{(\omega)}(\mathbb{R}^{2d}),$$

and the result follows in this case. Here the third equivalence follows from the fact that

$$\omega_{0,R+c} \lesssim \omega_{0,R} \lesssim \omega_{0,R-c},$$

for some $c > 0$.

We note that if $A, B \in \mathbf{M}(d, \mathbb{R})$ and $a, b \in (\mathcal{S}_{s,\sigma}^{1,\sigma})(\mathbb{R}^{2d})$ or $a, b \in (\Sigma_{s,\sigma}^{1,\sigma})(\mathbb{R}^{2d})$, then the first part of the previous proof shows that

$$\text{Op}_A(a) = \text{Op}_B(b) \iff e^{i(AD_{\xi} - D_{\eta})} a = e^{i(BD_{\xi} - D_{\eta})} b. \quad (3.6)$$

The following result follows from Theorems 3.1 and 3.3. The details are left for the reader.

Theorem 3.6. Let $s, s_1, s_2, \sigma, \sigma_1, \sigma_2 > 0$ be such that

$s + \sigma \geq 1, \quad s_1 + \sigma_1 \geq 1, \quad s_2 + \sigma_2 \geq 1, \quad s_2 \leq s_1 \quad \text{and} \quad \sigma_1 \leq \sigma_2,$

$A, B \in \mathbf{M}(d, \mathbb{R}), \quad \omega \in \mathcal{P}_{s,\sigma}(\mathbb{R}^{2d})$, and let $a$ and $b$ be Gelfand-Shilov distributions such that $\text{Op}_A(a) = \text{Op}_B(b)$. Then the following is true:

1. $a \in \mathcal{S}_{s_1,\sigma_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{2d})$ if and only if $b \in \mathcal{S}_{s_1,\sigma_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{2d})$, and $a \in (\mathcal{S}_{s_1,\sigma_2}^{\sigma_1,\sigma_2})(\mathbb{R}^{2d})$

2. $a \in \Sigma_{s_1,\sigma_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{2d})$ if and only if $b \in \Sigma_{s_1,\sigma_2}^{\sigma_1,\sigma_2}(\mathbb{R}^{2d})$, and $a \in (\Sigma_{s_1,\sigma_2}^{\sigma_1,\sigma_2})(\mathbb{R}^{2d})$
(3) \( a \in \Gamma_{s,\sigma,0}^\sigma(R^{2d}) \) if and only if \( b \in \Gamma_{s,\sigma,0}^\sigma(R^{2d}) \). If in addition \((s, \sigma) \neq \left(\frac{1}{2}, \frac{1}{2}\right)\), then \( a \in \Gamma_{s,\sigma}^{s,0}(R^{2d}) \) if and only if \( b \in \Gamma_{s,\sigma}^{s,0}(R^{2d}) \), and \( a \in \Gamma_{s,\sigma,0}^\sigma(R^{2d}) \) if and only if \( b \in \Gamma_{s,\sigma,0}^\sigma(R^{2d}) \);

(4) \( a \in \Gamma_{(\omega)}^\sigma(R^{2d}) \) if and only if \( b \in \Gamma_{(\omega)}^\sigma(R^{2d}) \).

3.2. Continuity for pseudo-differential operators with symbols of infinite order on Gelfand-Shilov spaces of functions and distributions. Next we deduce continuity for pseudo-differential operators with symbols in the classes in Definitions 1.5 and 1.6. We begin with the case when the symbols belong to \( \Gamma_{s,\sigma}^{s,0}(R^{2d}) \).

Theorem 3.7. Let \( A \in \mathcal{M}(d, R) \), \( s, \sigma > 0 \) be such that \( s + \sigma \geq 1 \), and let \( a \in \Gamma_{s,\sigma,0}^\sigma(R^{2d}) \). Then \( \text{Op}_A(a) \) is continuous on \( S_\sigma^s(R^d) \) and on \( (S_\sigma^s)'(R^d) \).

For the proof we need the following result.

Lemma 3.8. Let \( s, \sigma > 0 \) be such that \( s + \sigma \geq 1 \), \( h_1 > 0 \), \( \Omega_1 \) be a bounded set in \( S_\sigma^s(R^d) \), and let \( h_2 \geq 2^{1+s}h_1 \) and \( h_3 \geq 2^{2+s}h_1 \).

Then

\[
\Omega_2 = \left\{ x \mapsto \frac{x^\gamma f(x)}{(2^{1+s}h_1)^{\lvert \gamma \lvert s}} ; f \in \Omega_1, \gamma \in \mathbb{N}^d \right\}
\]

is a bounded set in \( S_\sigma^s_{s,h_2}(R^d) \), and

\[
\Omega_3 = \left\{ x \mapsto \frac{D^\delta x^\gamma f(x)}{(2^{2+s}h_1)^{\lvert \gamma \lvert s}} ; f \in \Omega_1, \gamma, \delta \in \mathbb{N}^d \right\}
\]

is a bounded sets in \( S_\sigma^s_{s,h_3}(R^d) \).

Proof. Since \( \Omega_1 \) is a bounded set in \( S_\sigma^s_{s,h_1}(R^d) \), there are constants \( C > 0 \) such that

\[
|x^\alpha D^\beta f(x)| \leq C h_1^{\lvert \alpha + \beta \lvert s} \alpha! \beta! s, \quad \alpha, \beta \in \mathbb{N}^d,
\]

(3.7)

for every \( f \in \Omega_1 \). We shall prove that (3.7) is true for all \( f \in \Omega_2 \) for a new choice of \( C > 0 \), and \( h_2 \) in place of \( h_1 \).

Let \( f \in \Omega_2 \). Then

\[
f = \frac{x^\gamma f_0(x)}{(2^{1+s}h_1)^{\lvert \gamma \lvert s}}
\]
for some $f_0 \in \Omega_1$ and $\gamma \in \mathbb{N}^d$. Then
\[
|x^\alpha D^\beta f(x)| = \left| \frac{x^\alpha D^\beta (x^\gamma f_0)(x)}{(2^{1+s}h_1)^{\gamma!}} \right|
\leq \sum_{\gamma_0 \leq \gamma, \beta} \binom{\beta}{\gamma_0} (\gamma_0!)(\gamma_0 - \gamma)! \frac{|x^{\alpha + \gamma - \gamma_0} D^\beta f_0(x)|}{(2^{1+s}h_1)^{\gamma!}}.
\]
\[
\leq \sum_{\gamma_0 \leq \gamma, \beta} \binom{\beta}{\gamma_0} (\gamma_0!)(\gamma_0 - \gamma)! \frac{|h_1^{\alpha + \beta + \gamma - 2\gamma_0}(\alpha + \gamma - \gamma_0)^s(\beta - \gamma_0)!}{(2^{1+s}h_1)^{\gamma!}}.
\]
Since
\[
\sum_{\gamma_0 \leq \gamma, \beta} 1 \leq 2^{s|\beta|},
\]
we get
\[
|x^\alpha D^\beta f(x)| \leq C2^{s|\alpha|+2(1+s)|\beta|} h_1^{\alpha + \beta + \gamma} \alpha^s \beta^\sigma \lesssim Ch_2^{\alpha + \beta + \gamma} \alpha^s \beta^\sigma
\]
for some constant $C$ which is independent of $f$, and the assertion on $\Omega_2$ follows.

The same type of arguments shows that
\[
\left\{ x \mapsto \frac{D^\beta f(x)}{(2^{1+s}h_1)^{\beta!\sigma}} ; f \in \Omega_1, \delta \in \mathbb{N}^d \right\}
\]
is a bounded set in $S^\sigma_{s,h_2}(\mathbb{R}^d)$, and the boundedness of $\Omega_3$ in $S^\sigma_{s,h_3}(\mathbb{R}^d)$ follows by combining the boundedness of $\Omega_2$ and the boundedness of (3.8) in $S^\sigma_{s,h_2}(\mathbb{R}^d)$.

Lemma 3.9. Let $s, \tau > 0$, and set
\[
m_{s,\tau}(x) = \sum_{j=0}^{\infty} \frac{t^j}{(j!)^2} s \quad \text{and} \quad m_s(x) = m_s(\tau(x)^2) \quad t \geq 0, \ x \in \mathbb{R}^d
\]

Then
\[
C^{-1} e^{(2s-\epsilon)\frac{\tau^t(x)}{s}} \leq m_{s,\tau}(x) \leq C e^{(2s+\epsilon)\frac{\tau^t(x)}{s}}, \quad (3.9)
\]
for every $\varepsilon > 0$, and

$$\frac{x^\alpha}{m_{s,\tau}(x)} \lesssim h_0^{[\alpha]} \alpha! s e^{-r|\alpha|^s},$$  \hspace{1cm} (3.10)

for some positive constant $r$ which depends on $d$, $s$ and $\tau$ only.

The estimate (3.9) follows from [22], and (3.10) also follows from computations given in e.g. [10,22]. For sake of completeness we present a proof of (3.10).

**Proof.** We have

$$\frac{x^\alpha}{m_{s,\tau}(x)} \lesssim \prod_{j=1}^d g_{\alpha_j}(x_j),$$

where

$$g_k(t) = t^k e^{-2r_0 t^{1/s}}, \quad t \geq 0,$$

for some $r_0 > 0$ depending only on $d$, $s$ and $\tau$. Let

$$g_{0,k}(t) = C_k e^{-r_0 t}, \quad t \geq 0,$$

where

$$C_k = \sup_{t \geq 0} (t^k e^{-r_0 t}).$$

Then $g_k(t) \leq g_{0,k}(t^{1/s})$, and the result follows if we prove $C_k \lesssim h_0^k k!^s$.

By straight-forward computations, it follows that the maximum of $t^k e^{-r_0 t}$ is attained at $t = sk/r_0$, giving that

$$C_k = \left(\frac{s}{r_0 e}\right)^{sk} \left(k^k\right)^s \lesssim h_0^k k!^s, \quad h_0 = \left(\frac{s}{r_0}\right)^s,$$

where the last inequality follows from Stirling’s formula. This gives the result. \hfill $\square$

**Proof of Theorem 3.7.** By Theorem 3.1 it suffices to consider the case $A = 0$, that is for the operator

$$Op_0(a)f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} a(x,\xi) \hat{f}(\xi) e^{i(x,\xi)} \, d\xi.$$

Observe that

$$\frac{1}{m_{s,\tau}(x)} \sum_{j=0}^\infty \frac{\tau^j}{(j!)^{2s}} (1 - \Delta_\xi)^j e^{i(x,\xi)} = e^{i(x,\xi)}.$$
Let now \( h_1 > 0 \) and \( f \in \Omega \), where \( \Omega \) is a bounded subset of \( S^\sigma_{s,h_1}(\mathbb{R}^d) \). For fixed \( \alpha, \beta \in \mathbb{N}^d \) we get

\[
(2\pi)^\frac{d}{2} x^\alpha D^\beta_x (\text{Op}_0(a)f)(x) = x^\alpha \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \int_{\mathbb{R}^d} \xi^\gamma D^\beta_x a(x, \xi) \hat{f}(\xi) e^{i(x, \xi)} d\xi = \frac{x^\alpha}{m_s(x)} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) g_{r, \beta, \gamma}(x), \quad (3.11)
\]

\[
g_{r, \beta, \gamma}(x) = \sum_{j=0}^{\infty} \frac{\tau_j}{(j!)^{2s}} \int_{\mathbb{R}^d} (1 - \Delta \xi)^j \left( \xi^\gamma D^\beta_x a(x, \xi) \hat{f}(\xi) \right) e^{i(x, \xi)} d\xi.
\]

By Lemma 3.8 and the fact that \( (2j)! \leq 2^{|j|^2} \), it follows that for some \( h > 0 \),

\[
\Omega = \left\{ \xi \mapsto \frac{(1 - \Delta \xi)^j (\xi^\gamma D^\beta_x a(x, \xi) \hat{f}(\xi))}{h^{\beta + |j| + j \cdot (2s \gamma) |\beta\sigma| \epsilon r|x|^s}} \ ; \ j \geq 0, \ \beta, \gamma \in \mathbb{N}^d \right\}
\]

is bounded in \( S^\sigma_s(\mathbb{R}^d) \) for every \( r > 0 \). This implies that for some positive constants \( h \) and \( r_0 \) we get

\[
|(1 - \Delta \xi)^j (\xi^\gamma D^\beta_x a(x, \xi) \hat{f}(\xi))| \leq h^{\beta + |j| + j \cdot (2s \gamma) |\beta\sigma| \epsilon r|x|^s},
\]

for every \( r > 0 \). Hence,

\[
|g_{r, \beta, \gamma}(x)| \leq \sum_{j=0}^{\infty} \frac{\tau_j}{(j!)^{2s}} \int_{\mathbb{R}^d} e^{-\tau \xi^{\frac{1}{2}}} \int_{\mathbb{R}^d} \xi^{\beta - \gamma} e^{i|x|^{\frac{1}{2}}} d\xi 
\]

\[
\leq h^{\beta} |\beta\sigma| \epsilon r|x|^s \sum_{j=0}^{\infty} \tau h^j \leq h^{\beta} |\beta\sigma| \epsilon r|x|^s
\]

for every \( r > 0 \), provided \( \tau \) is chosen such that \( \tau h < 1 \).

By inserting this into (3.11) and using Lemma 3.9 we get for some \( h > 0 \) and some \( r_0 > 0 \) that

\[
|x^\alpha D^\beta_x (\text{Op}_0(a)f)(x)| \leq h^{\alpha + |\beta| + r_0} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) |g_{r, \beta, \gamma}(x)| 
\]

\[
\leq h^{\alpha + |\beta| + r_0} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \leq (2h)^{|\alpha + |\beta| + r_0} \alpha! \beta! \sigma,
\]

provided that \( r \) above is chosen to be smaller than \( r_0 \). Then the continuity of \( \text{Op}_A(a) \) on \( S^\sigma_s(\mathbb{R}^d) \) follows. The continuity of \( \text{Op}_A(a) \) on \( (S^\sigma_s)'(\mathbb{R}^d) \) now follows from the preceding continuity and duality. \( \square \)
Next we shall discuss corresponding continuity in the Beurling case. The main idea is to deduce such properties by suitable estimates on short-time Fourier transforms of involved functions and distributions. First we have the following relation between the short-time Fourier transforms of the symbols and kernels of a pseudo-differential operator.

**Lemma 3.10.** Let \( A \in \mathbf{M}(d, \mathbb{R}) \), \( s, \sigma > 0 \) be such that \( s + \sigma \geq 1 \), \( a \in \mathcal{S}^{s, s}_s(R^{2d}) \) \( (\sigma \in \mathcal{S}^{s, s}_s(R^{2d})) \), \( \phi \in \mathcal{S}^{s, s}(R^{2d}) \) \( (\phi \in \mathcal{S}^{s, s}(R^{2d})) \), and let

\[
K_{A,a}(x,y) = (2\pi)^{-\frac{d}{2}} (\mathcal{F}^{-1} a)(x - A(x-y), x-y)
\]

and

\[
\psi(x,y) = (2\pi)^{-\frac{d}{2}} (\mathcal{F}^{-1} \phi)(x - A(x-y), x-y)
\]

be the kernels of \( \text{Op}_A(a) \) and \( \text{Op}_A(\phi) \), respectively. Then

\[
(V_\psi K_{a,A})(x,y,\xi,\eta) = (2\pi)^{-d} e^{i(x-y,\eta - A^{*}(\xi + \eta))} (V_\phi a)(x - A(x-y), -\eta + A^{*}(\xi + \eta), \xi + \eta, y-x).
\]

(3.12)

The essential parts of (3.12) is presented in the proof of [32, Proposition 2.5]. In order to be self-contained we here present a short proof.

**Proof.** Let

\[
T_A(x, y) = x - A(x-y)
\]

and

\[
Q = Q(x, x_1, y, \xi, \xi_1, \eta) = \langle x - y, \xi_1 - T_A^{*}(-\eta, \xi) \rangle - \langle x_1, \xi + \eta \rangle.
\]

By formal computations, using Fourier’s inversion formula we get

\[
(V_\psi K_{a,A})(x,y,\xi,\eta)
\]

\[
= (2\pi)^{-3d} \int \int K_{a,A}(x_1, y_1) \overline{\psi(x_1 - x, y_1 - y)} e^{-i\langle x_1, \xi \rangle + \langle y_1, \eta \rangle} dx_1 dy_1
\]

\[
= (2\pi)^{-2d} \int a(x_1, \xi_1) \phi(x_1 - T_A(x, y), \xi_1 - T_A^{*}(-\eta, \xi)) e^{iQ(x,x_1,y,\xi,\xi_1,\eta)} dx_1 d\xi_1
\]

\[
= (2\pi)^{-d} e^{i(x-y,\eta - A^{*}(\xi + \eta))} (V_\phi a)(T_A(x, y), T_A^{*}(-\eta, \xi), \xi + \eta, y-x),
\]

where all integrals should be interpreted as suitable Fourier transforms. This gives the result. \(\square\)

Before continuing discussing continuity of pseudo-differential operators, we observe that the previous lemma in combination with Propositions [2.2 and 2.3] give the following.
Proposition 3.11. Let $A \in \mathcal{M}(d, \mathbb{R})$, $s, \sigma > 0$ be such that $s + \sigma \geq 1$ and $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, $\phi \in \Sigma^s_\sigma(\mathbb{R}^{2d}) \setminus 0$, $a$ be a Gelfand-Shilov distribution on $\mathbb{R}^{2d}$ and let $K_{a,A}$ be the kernel of $\text{Op}_A(a)$. Then the following conditions are equivalent:

1. $a \in \Gamma_{s,\sigma}^\sigma(\mathbb{R}^{2d})$ (resp. $a \in \Gamma_{s,\sigma}^{s,0}(\mathbb{R}^{2d})$);
2. for some $r > 0$,

$$|V_\phi K_{a,A}(x, y, \xi, \eta)| \lesssim e^{r(|x-A(x-y)|^2 + |\eta - A^*(\xi+\eta)|^2) - h(|(\xi+\eta)^\frac{1}{2} + |x-y|^2)}$$

holds for some $h > 0$ (for every $h > 0$).

By similar arguments we also get the following. The details are left for the reader.

Proposition 3.12. Let $A \in \mathcal{M}(d, \mathbb{R})$, $s, \sigma > 0$ be such that $s + \sigma \geq 1$, $\phi \in \Sigma^s_\sigma(\mathbb{R}^{2d}) \setminus 0$, $a$ be a Gelfand-Shilov distribution on $\mathbb{R}^{2d}$ and let $K_{a,A}$ be the kernel of $\text{Op}_A(a)$. Then the following conditions are equivalent:

1. $a \in \Gamma_{s,\sigma}^\sigma(\mathbb{R}^{2d})$ (resp. $a \in \Gamma_{s,\sigma}^{s,0}(\mathbb{R}^{2d})$);
2. for some $h > 0$ (for every $h > 0$),

$$|V_\phi K_{a,A}(x, y, \xi, \eta)| \lesssim e^{r(|x-A(x-y)|^2 + |\eta - A^*(\xi+\eta)|^2) - h(|(\xi+\eta)^\frac{1}{2} + |x-y|^2)}$$

holds for every $r > 0$.

Theorem 3.13. Let $A \in \mathcal{M}(d, \mathbb{R})$, $s, \sigma > 0$ be such that $s + \sigma \geq 1$ and $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, and let $a \in \Gamma_{s,\sigma}^{s,0}(\mathbb{R}^{2d})$. Then $\text{Op}_A(a)$ is continuous on $\Sigma^s_\sigma(\mathbb{R}^{2d})$, and is uniquely extendable to a continuous map on $(\Sigma^s_\sigma)'(\mathbb{R}^{2d})$.

Proof. By Theorem 3.1 we may assume that $A = 0$. Let

$$g(x) = \text{Op}_0(a)f(x) = (K_{0,a}(x, \cdot), \overline{J}) = (h_{a,x}, \overline{J}),$$

where $h_{a,x} = K_{0,a}(x, \cdot)$, and let $\phi_j \in \Sigma^s_\sigma(\mathbb{R}^d)$ be such that $\|\phi_j\|_{L^2} = 1$, $j = 1, 2$. By Moyal’s identity we get

$$g(x) = (h_{a,x}, \overline{J})_{L^2(\mathbb{R}^d)} = (V_{\phi_1} h_{a,x}, V_{\phi_1} \overline{J})_{L^2(\mathbb{R}^{2d})},$$

and applying the short-time Fourier transform on $g$ and using Fubini’s theorem on distributions we get

$$V_{\phi_2} g(x, \xi) = \langle J(x, \xi, \cdot), F \rangle,$$

where

$$F(y, \eta) = V_{\phi_1} f(y, -\eta), \quad J(x, \xi, y, \eta) = V_{\phi} K_{0,a}(x, y, \xi, \eta)$$

and $\phi = \phi_2 \otimes \phi_1$.

Now suppose that $r > 0$ is arbitrarily chosen. By Proposition 2.2 we get for some $c \in (0, 1)$ which depends on $s$ and $\sigma$ only, that for some
$r_0 > 0$ and with $r_1 = (r + r_0)/c$ that
\[
|J(x, \xi, y, \eta)| \lesssim e^{r_0(|x|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}})} e^{-r_1(|y-x|^{\frac{1}{2}}+|\xi+\eta|^{\frac{1}{2}})}
\]
\[
\lesssim e^{-(cr_1-r_0)|x|^{\frac{1}{2}}+cr_1|\xi|^{\frac{1}{2}}}e^{r_1|y|^{\frac{1}{2}}+(r_1+r_0)|\eta|^{\frac{1}{2}}}
\lesssim e^{-r(|x|^{\frac{1}{2}}+|\xi|^{\frac{1}{2}})}e^{r_2(|y|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}})},
\]
where $r_2$ only depends on $r$ and $r_0$.

Since $f \in \Sigma^s_\alpha(R^d)$ we have
\[
|F(x, \xi)| \lesssim \|f\|_{S^s_{\alpha,h}} e^{-(1+r_2)(|x|^{\frac{1}{2}}+|\xi|^{\frac{1}{2}})}
\]
for some $h > 0$ depending on $r_2$, and thereby by $r$ and $r_0$ only. This implies
\[
|V_{\phi_2}g(x, \xi)| = |\langle J(x, \xi, \cdot, F) \rangle|
\lesssim \|f\|_{S^s_{\alpha,h}} \left( \int e^{-(1+r_2)(|x|^{\frac{1}{2}}+|\xi|^{\frac{1}{2}})} d\eta \right) e^{-r(|x|^{\frac{1}{2}}+|\xi|^{\frac{1}{2}})}
\lesssim \|f\|_{S^s_{\alpha,h}} e^{-r(|x|^{\frac{1}{2}}+|\xi|^{\frac{1}{2}})} \quad (3.13)
\]
which shows that $g \in \Sigma^s_\alpha(R^d)$ in view of $[31]$, Proposition 2.1.

Since the topology of $\Sigma^s_\alpha(R^d)$ is given by the semi-norms
\[
g \mapsto \sup_{x, \xi \in \mathbb{R}^d} |V_{\phi_2}g(x, \xi)e^{r(|x|^{\frac{1}{2}}+|\xi|^{\frac{1}{2}})}|
\]
it follows from $(3.13)$ that $\text{Op}(a)$ is continuous on $\Sigma^s_\alpha(R^d)$.

By duality it follows that $\text{Op}(a)$ is uniquely extendable to a continuous map on $(\Sigma^s_\alpha)'(R^d)$. \hfill \Box

The following result follows by similar arguments as in the previous proof. The verifications are left for the reader.

**Theorem 3.14.** Let $A \in M(d, R)$, $s, \sigma > 0$ be such that $s + \sigma \geq 1$ and $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, and let $a \in \Gamma^s_{s,\sigma}(R^{2d})$. Then $\text{Op}_A(a)$ is continuous from $\Sigma^s_\alpha(R^d)$ to $S^s_\sigma(R^d)$, and from $(S^s_\sigma)'(R^d)$ to $(\Sigma^s_\alpha)'(R^d)$.

### 3.3. Compositions of pseudo-differential operators

Next we deduce algebraic properties of pseudo-differential operators considered in Theorems 3.10 and 3.14. We recall that for pseudo-differential operators with symbols in e.g. Hörmander classes, we have
\[
\text{Op}_0(a_1 \#_0 a_2) = \text{Op}_0(a_1) \circ \text{Op}_0(a_2),
\]
when
\[
a_1 \#_0 a_2(x, \xi) \equiv \left( e^{i(D_\xi, D_\eta)}(a_1(x, \xi) a_2(y, \eta)) \right)_{(y, \eta) = (x, \xi)}.
\]
More generally, if $A \in M(d, R)$ and $a_1 \#_A a_2$ is defined by
\[
a_1 \#_A a_2 \equiv e^{i(AD_\xi, D_\eta)} \left( (e^{-i(AD_\xi, D_\eta)} a_1) \#_0 (e^{-i(AD_\xi, D_\eta)} a_2) \right),
\]

(3.14)
for \(a_1\) and \(a_2\) belonging to certain Hörmander symbol classes, then it follows from the analysis in \[21\] that
\[
\text{Op}_A(a_1 \#_A a_2) = \text{Op}_A(a_1) \circ \text{Op}_A(a_2) \tag{3.15}
\]
for suitable \(a_1\) and \(a_2\).

We recall that the map \(a \mapsto K_{a,A}\) is a homeomorphism from \(S^{\sigma,s}_A(R^{2d})\) to \(S^0_s(R^{2d})\) and from \(\Sigma^{\sigma,s}_A(R^{2d})\) to \(\Sigma^0_s(R^{2d})\). It is also obvious that the map
\[
(K_1, K_2) \mapsto \left( (x, y) \mapsto (K_1 \circ K_2)(x, y) = \int_{R^d} K_1(x, z)K_2(z, y) \, dz \right)
\]
is sequentially continuous from \(S^0_s(R^{2d}) \times S^0_s(R^{2d})\) to \(S^0_s(R^{2d})\), and from \(\Sigma^0_s(R^{2d}) \times \Sigma^0_s(R^{2d})\) to \(\Sigma^0_s(R^{2d})\). Here we have identified operators with their kernels. Since
\[
(K_1 \circ K_2 \circ K_3)(x, y) = \langle K_2, T_{K_1,K_3}(x, y, \cdot) \rangle
\]
with
\[
T_{K_1,K_3}(x, y, z_1, z_2) = K_1(x, z_1)K_3(z_2, y)
\]
when \(K_j \in L^2(R^{2d}), j = 1, 2, 3\), and that
\[
(K_1, K_2, K_3) \mapsto \left( (x, y) \mapsto \langle K_2, T_{K_1,K_3}(x, y, \cdot) \rangle \right)
\]
is sequentially continuous from \(S^0_s(R^{2d}) \times (S^0_s)'(R^{2d}) \times S^0_s(R^{2d})\) to \(S^0_s(R^{2d})\), and from \(\Sigma^0_s(R^{2d}) \times (\Sigma^0_s)'(R^{2d}) \times \Sigma^0_s(R^{2d})\) to \(\Sigma^0_s(R^{2d})\), the following result follows from these continuity results and \[31]\.

**Proposition 3.15.** Let \(A \in M(d, R)\), and let \(s, \sigma > 0\) be such that \(s + \sigma \geq 1\). Then the following is true:

1. the map \((a_1, a_2) \mapsto a_1 \#_A a_2\) is continuous from \(S^{\sigma,s}_A(R^{2d}) \times S^{\sigma,s}_A(R^{2d})\) to \(S^{\sigma,s}_A(R^{2d})\);
2. the map \((a_1, a_2) \mapsto a_1 \#_A a_2\) is continuous from \(\Sigma^{\sigma,s}_A(R^{2d}) \times \Sigma^{\sigma,s}_A(R^{2d})\) to \(\Sigma^{\sigma,s}_A(R^{2d})\);
3. the map \((a_1, a_2, a_3) \mapsto a_1 \#_A a_2 \#_A a_3\) from \(S^{\sigma,s}_A(R^{2d}) \times S^{\sigma,s}_A(R^{2d}) \times S^{\sigma,s}_A(R^{2d})\) to \(S^{\sigma,s}_A(R^{2d})\) extends uniquely to a sequentially continuous and associative map from \(S^{\sigma,s}_A(R^{2d}) \times (S^{\sigma,s}_A)'(R^{2d}) \times S^{\sigma,s}_A(R^{2d})\) to \(S^{\sigma,s}_A(R^{2d})\);
4. the map \((a_1, a_2, a_3) \mapsto a_1 \#_A a_2 \#_A a_3\) from \(\Sigma^{\sigma,s}_A(R^{2d}) \times \Sigma^{\sigma,s}_A(R^{2d}) \times \Sigma^{\sigma,s}_A(R^{2d})\) to \(\Sigma^{\sigma,s}_A(R^{2d})\) extends uniquely to a sequentially continuous and associative map from \(\Sigma^{\sigma,s}_A(R^{2d}) \times (\Sigma^{\sigma,s}_A)'(R^{2d}) \times \Sigma^{\sigma,s}_A(R^{2d})\) to \(\Sigma^{\sigma,s}_A(R^{2d})\);

We have the following corresponding algebra result for \(\Gamma^{\sigma,s,0}_{\sigma,s}\) and related symbol classes.

**Theorem 3.16.** Let \(A \in M(d, R)\), and let \(s, \sigma > 0\) be such that \(s + \sigma \geq 1\). Then the following is true:
(1) the map (1) in Proposition 3.12 extends uniquely to a continuous map from \( \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \times \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \) to \( \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \), and from \( \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \times \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \) to \( \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \); (2) if in addition \( (s, \sigma) \neq (\frac{1}{2}, \frac{1}{2}) \), the map (2) in Proposition 3.12 extends uniquely to a continuous map from \( \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \times \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \) to \( \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \), and from \( \Gamma_{s,\sigma}(\mathbb{R}^{2d}) \times \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \) or from \( \Gamma_{s,\sigma}(\mathbb{R}^{2d}) \) to \( \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \).

Proof. We prove only the first assertion in (2). The other statements follow by similar arguments and are left for the reader.

By Theorem 3.6 it suffices to consider the case when \( A = 0 \). Let \( \phi_1, \phi_2, \phi_3 \in \Sigma_{s}^0(\mathbb{R}^d) \setminus 0 \), \( a_j \in \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \), \( j = 1, 2 \), and let \( K \) be the kernel of \( \text{Op}_0(a_1) \circ \text{Op}_0(a_2) \). By Proposition 3.11 we need to prove that for some \( r > 0 \),

\[
|V_{\phi_1 \otimes \phi_3} K(x, y, \xi, \eta)| \lesssim e^{r(|x|^\frac{4}{3} + |\eta|^{\frac{2}{3}} - h(|\xi + \eta|^{\frac{1}{3}} + |x - y|^{\frac{2}{3}}))} 
\]

(3.17) for every \( h > 0 \). Therefore, let \( h > 0 \) be arbitrarily chosen but fixed, and let \( K_j \) be the kernel of \( \text{Op}_0(a_j) \), \( j = 1, 2 \),

\[
F_1(x, y, \xi, \eta) = V_{\phi_1 \otimes \phi_2} K_1(x, y, \xi, \eta), \\
F_2(x, y, \xi, \eta) = V_{\phi_2 \otimes \phi_3} K_2(x, y, -\xi, \eta)
\]

and

\[
G(x, y, \xi, \eta) = V_{\phi_1 \otimes \phi_3} K(x, y, \xi, \eta).
\]

Then

\[
G(x, y, \xi, \eta) = \int_{\mathbb{R}^{2d}} F_1(x, z, \xi, \zeta) F_2(z, y, \zeta, \eta) \, dzd\zeta
\]

(3.18) by Moyal’s identity (cf. proof of Theorem 3.13). Since \( a_j \in \Gamma_{s,\sigma}^0(\mathbb{R}^{2d}) \) we have for some \( r > 0 \) that

\[
|F_1(x, y, \xi, \eta)| \lesssim e^{r(|x|^\frac{4}{3} + |\eta|^{\frac{2}{3}} - h_1(|\xi + \eta|^{\frac{1}{3}} + |x - y|^{\frac{2}{3}}))}
\]

and

\[
|F_2(x, y, \xi, \eta)| \lesssim e^{r(|x|^\frac{4}{3} + |\eta|^{\frac{2}{3}} - h_1(|\xi - \eta|^{\frac{1}{3}} + |x - y|^{\frac{2}{3}}))}
\]

for every \( h_1 > 0 \). By combining this with (3.18) we get for some \( r > 0 \),

\[
|G(x, y, \xi, \eta)| \lesssim \int_{\mathbb{R}^{2d}} e^{r_{h_1} h_2 (x, y; z, \xi, \eta; \zeta) + \psi_{h_1} h_2 (x, y; z, \xi, \eta; \zeta) dzd\zeta},
\]

(3.19) where \( h_2 \geq 2ch + cr \),

\[
\varphi_{h, h}(x, y, z, \xi, \eta, \zeta) = r(|x|^\frac{4}{3} + |\eta|^{\frac{2}{3}} - h(|\xi - \eta|^{\frac{1}{3}} + |y - z|^{\frac{2}{3}}))
\]
and
\[ \psi_{r,h}(x, y, z, \xi, \eta, \zeta) = r(|x|^{\frac{1}{r}} + |\eta|^{\frac{1}{r}}) - h(|\xi + \zeta|^{\frac{1}{r}} + |x - z|^{\frac{1}{r}}). \]
and \( c \geq 1 \) is chosen such that
\[ |x+y|^{\frac{1}{r}} \leq c(|x|^{\frac{1}{r}} + |y|^{\frac{1}{r}}) \quad \text{and} \quad |\xi + \eta|^{\frac{1}{r}} \leq c(|\xi|^{\frac{1}{r}} + |\eta|^{\frac{1}{r}}), \quad x, y, \xi, \eta \in \mathbb{R}^d. \]
Then
\[ \varphi_{r,h_2}(x, y, z, \xi, \eta, \zeta) \leq cr(|x|^{\frac{1}{r}} + |\eta|^{\frac{1}{r}}) - (h_2 - cr)(|\xi - \eta|^{\frac{1}{r}} + |y - z|^{\frac{1}{r}}) \]
\[ \leq cr(|x|^{\frac{1}{r}} + |\eta|^{\frac{1}{r}}) - 2ch(|\xi - \eta|^{\frac{1}{r}} + |y - z|^{\frac{1}{r}}) \]
and
\[ \psi_{r,h_2}(x, y, z, \xi, \eta, \zeta) \leq cr(|x|^{\frac{1}{r}} + |\eta|^{\frac{1}{r}}) - 2ch(|\xi + \zeta|^{\frac{1}{r}} + |x - z|^{\frac{1}{r}}). \]
This gives
\[ \varphi_{r,h_2}(x, y, z, \xi, \eta, \zeta) + \psi_{r,h_2}(x, y, z, \xi, \eta, \zeta) \leq 2cr(|x|^{\frac{1}{r}} + |\eta|^{\frac{1}{r}}) - 2ch(|\xi + \zeta|^{\frac{1}{r}} + |\xi - \eta|^{\frac{1}{r}} + |x - z|^{\frac{1}{r}} + |y - z|^{\frac{1}{r}}). \]
Since
\[ -2ch(|\xi + \zeta|^{\frac{1}{r}} + |\xi - \eta|^{\frac{1}{r}} + |x - z|^{\frac{1}{r}} + |y - z|^{\frac{1}{r}}) \]
\[ \leq -h(|\xi + \eta|^{\frac{1}{r}} + |x - y|^{\frac{1}{r}}) - ch(|\xi + \zeta|^{\frac{1}{r}} + |\xi - \eta|^{\frac{1}{r}} + |x - z|^{\frac{1}{r}} + |y - z|^{\frac{1}{r}}) \]
we get by combining these estimates with (3.19) that
\[ |G(x, y, \xi, \eta)| \lesssim \int_{\mathbb{R}^{2d}} e^{2cr(|x|^{\frac{1}{r}} + |\eta|^{\frac{1}{r}}) - h(|\xi + \eta|^{\frac{1}{r}} + |x - y|^{\frac{1}{r}})} dzd\zeta, \]
\[ \lesssim e^{2cr(|x|^{\frac{1}{r}} + |\eta|^{\frac{1}{r}}) - h(|\xi + \eta|^{\frac{1}{r}} + |x - y|^{\frac{1}{r}})}. \]
Since \( r > 0 \) is fixed and \( h > 0 \) can be chosen arbitrarily, the result follows.

**Theorem 3.17.** Let \( A \in M(d, \mathbb{R}) \), \( s, \sigma > 0 \) be such that \( s + \sigma \geq 1 \), and let \( \omega_j \in \mathcal{P}_{s,\sigma}(\mathbb{R}^d) \), \( j = 1, 2 \). Then the following is true:

1. the map \((a_1, a_2) \mapsto a_1 \#_A a_2 \) from \( \Sigma^{\sigma,s}_{s,\sigma}(\mathbb{R}^d) \times \Sigma^{\sigma,s}_{s,\sigma}(\mathbb{R}^d) \) to \( \Sigma^{\sigma,s}_{s,\sigma}(\mathbb{R}^d) \) is uniquely extendable to a continuous map from \( \Gamma^{\sigma,s,0}_{(\omega_1)}(\mathbb{R}^d) \times \Gamma^{\sigma,s,0}_{(\omega_2)}(\mathbb{R}^d) \) to \( \Gamma^{\sigma,s,0}_{(\omega_1 \omega_2)}(\mathbb{R}^d) \);

2. if in addition \( \omega_j \in \mathcal{P}^{0}_{s,\sigma}(\mathbb{R}^d) \), \( j = 1, 2 \), then the map \((a_1, a_2) \mapsto a_1 \#_A a_2 \) from \( S^{\sigma,s}_{s,\sigma}(\mathbb{R}^d) \times S^{\sigma,s}_{s,\sigma}(\mathbb{R}^d) \) to \( S^{\sigma,s}_{s,\sigma}(\mathbb{R}^d) \) is uniquely extendable to a continuous map from \( \Gamma^{\sigma,s}_{(\omega_1)}(\mathbb{R}^d) \times \Gamma^{\sigma,s}_{(\omega_2)}(\mathbb{R}^d) \) to \( \Gamma^{\sigma,s}_{(\omega_1 \omega_2)}(\mathbb{R}^d) \).
Proof. We may assume that \( A = 0 \) by Theorem 3.1. We only prove (2). The assertion (1) follows by similar arguments and is left for the reader.

Let

\[
F_{a_1,a_2}(x_1,x_2,\xi_1,\xi_2) = a_1(x_1,\xi_1)a_2(x_2,\xi_2)
\]

and

\[
\omega(x_1,x_2,\xi_1,\xi_2) = \omega_1(x_1,\xi_1)\omega_2(x_2,\xi_2).
\]

By the definitions it follows that the map \( T_1 \) which takes \((a_1,a_2)\) into \( F_{a_1,a_2} \) is continuous from \( \Gamma_{\omega_1}^{\sigma,s}(\mathbb{R}^{2d}) \times \Gamma_{\omega_2}^{\sigma,s}(\mathbb{R}^{2d}) \) to \( \Gamma_{\omega}^{\sigma,s}(\mathbb{R}^{4d}) \).

Theorem 3.3 declare that the map \( T_2 \) which takes \( F(x_1,x_2,\xi_1,\xi_2) \) to \( e^{i(D_{\xi_1}D_{\xi_2})}F(x_1,x_2,\xi_1,\xi_2) \) is continuous on \( \Gamma_{\omega}^{\sigma,s}(\mathbb{R}^{4d}) \). Hence, if \( T_3 \) is the trace operator which takes \( F(x_1,x_2,\xi_1,\xi_2) \) into \( F_0(x,\xi) \equiv F(x,x,\xi,\xi) \), Proposition A.2 shows that \( T \equiv T_3\circ T_2\circ T_1 \) is continuous from \( \Gamma_{\omega_1}^{\sigma,s}(\mathbb{R}^{2d}) \times \Gamma_{\omega_2}^{\sigma,s}(\mathbb{R}^{2d}) \) to \( \Gamma_{\omega}^{\sigma,s}(\mathbb{R}^{4d}) \).

By [21, Theorem 18.1.8] we have \( T(a_1,a_2) = a_1 \#_0 a_2 \) when \( a_1,a_2 \in \Sigma_{s,\sigma}(\mathbb{R}^{2d}) \). If instead \( a_k \in \Gamma_{\omega_k}^{\sigma,s}(\mathbb{R}^{2d}) \), \( k = 1,2 \), then we take \( T(a_1,a_2) \) as the definition of \( a_1 \#_0 a_2 \). By the continuity of \( T \) it follows that \( (a_1,a_2) \mapsto a_1 \#_0 a_2 \) is continuous from \( \Gamma_{\omega_1}^{\sigma,s}(\mathbb{R}^{2d}) \times \Gamma_{\omega_2}^{\sigma,s}(\mathbb{R}^{2d}) \) to \( \Gamma_{\omega_0}^{\sigma,s}(\mathbb{R}^{4d}) \).

Since \( \Gamma_{\omega_0}^{\sigma,s}(\mathbb{R}^{4d}) \subseteq \Gamma_{\omega_0}^{\sigma,s}(\mathbb{R}^{4d}) \), we get \( \text{Op}_0(a_1 \#_0 a_2) = \text{Op}_0(a_1) \circ \text{Op}_0(a_2) \) and that \( a_1 \#_0 a_2 \) is uniquely defined as an element in \( \Gamma_{\omega_0}^{\sigma,s}(\mathbb{R}^{2d}) \), in view of Theorem 3.11. Hence \( a_1 \#_0 a_2 \) is uniquely defined in \( \Gamma_{\omega_0}^{\sigma,s}(\mathbb{R}^{2d}) \), since all these symbol classes are subspaces of \( C^\infty(\mathbb{R}^{2d}) \). This gives the result. \( \square \)

APPENDIX A

In what follows we prove some auxiliary results on continuity of Gevrey symbol classes.

Proposition A.1. Let \( \sigma,s > 0, \omega \in \mathcal{S}_{s,\sigma}^0(\mathbb{R}^{2d}) \), \( a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \) and let \( a_\varepsilon = \phi(\varepsilon \cdot) a, \varepsilon > 0 \), where \( \phi \in \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \) satisfies \( \phi(0) = 1 \). Then the following is true:

1. \( a_\varepsilon \to a \) in \( \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \) as \( \varepsilon \to 0^+ \);

2. If in addition \( a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \), then \( a_\varepsilon \to a \) in \( \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \) as \( \varepsilon \to 0^+ \);

3. If in addition \( a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \) (\( a \in \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \)) and \( \phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \), then \( a_\varepsilon \to a \) in \( \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \) (in \( \Gamma_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \)) as \( \varepsilon \to 0^+ \).

Proof. We only prove (2). The other assertions follow by similar arguments and is left for the reader. Since \( \phi \in \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbb{R}^{2d}) \), there are constants \( C, r_0, h_0 > 0 \) which are independent of \( \delta > 0 \) such that

\[
|\partial^\alpha_x \partial^\beta_x \phi_\varepsilon(x,\xi)| \leq C h_0^{r_0} \varepsilon^{r_0} |\alpha + \beta|! \varepsilon^{-r \delta} (||x||^1 + ||\xi||^1).
\]

28
For convenience we also let

\[ \|a\|_{s,\sigma,h,r,\alpha,\beta} \equiv \sup_{x,\xi \in \mathbb{R}^d} \left( \frac{e^{-r(|x|^\frac{1}{2} + |\xi|^\frac{1}{2})} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)|}{\alpha! \beta! \gamma! h^{|\alpha|}} \right) \]

and

\[ \|a\|_{s,\sigma,h,r} \equiv \sup_{\alpha,\beta \in \mathbb{N}^d} \|a\|_{s,\sigma,h,r,\alpha,\beta}. \]

By Leibniz rule we get

\[ \partial_x^\alpha \partial_\xi^\beta a = \sum_{\gamma \leq \alpha} \sum_{\delta \leq \beta} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \partial_x^\gamma \partial_\xi^\delta \phi \cdot \partial_x^{\alpha-\gamma} \partial_\xi^{\beta-\delta} a. \]

Hence, if \( r > 0 \) is arbitrary and \( h \geq h_0 \) is chosen such that \( \|a\|_{s,\sigma,h,r} < \infty \), we get

\[ \|a - a\|_{s,\sigma,h,r,\alpha,\beta} \leq J_1(s, \sigma, h, r, \varepsilon) + J_2(s, \sigma, h, r, \varepsilon, \alpha, \beta), \]

where

\[ J_1(s, \sigma, h, r, \varepsilon) = \sup_{\alpha, \beta \in \mathbb{N}^d} \frac{\|e^{-r|\cdot|^2 - \phi(\varepsilon \cdot) \partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty}}}{\alpha! \beta! \gamma! h^{\alpha+\beta}}, \]

and

\[ J_2(s, \sigma, h, r, \varepsilon, \alpha, \beta) \]

\[ = (\alpha! \beta! \gamma! h^{\alpha+\beta})^{-1} \sum_{\gamma \leq \alpha} \sum_{\delta \leq \beta} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \|\partial_x^\gamma \partial_\xi^\delta \phi \|_{L^\infty} \|\partial_x^{\alpha-\gamma} \partial_\xi^{\beta-\delta} a\|_{L^\infty}, \]

where the last sum is taken over all \( \gamma \in \mathbb{N}^d \) and \( \delta \in \mathbb{N}^d \) such that \( \gamma \leq \alpha, \delta \leq \beta \) and \( (\gamma, \delta) \neq (0, 0) \).

Evidently, since \( a \in \Gamma_{s,\sigma,s}^\sigma(\mathbb{R}^{2d}) \), it follows by straightforward estimates that \( J_1(s, \sigma, \varepsilon_1, h, \delta) \to 0 \) as \( \delta \to 0^+ \). For \( J_2(s, \sigma, \varepsilon_1, \delta, \alpha) \) we have

\[ J_2(s, \sigma, \varepsilon_1, \delta, \alpha) \lesssim (\alpha! \varepsilon_1^{|\alpha|})^{-1} \sum_{0 \neq \gamma \leq \alpha} \binom{\alpha}{\gamma} \varepsilon_0^{|\gamma|} \varepsilon_1^{|\alpha-\gamma|} \delta^{|\gamma|} \delta^{|\alpha-\gamma|} = \delta^{|\gamma|} \frac{\varepsilon_0^{|\alpha|}}{\varepsilon_1^{|\alpha|}} \lesssim \delta^{|\gamma|}, \]

provided \( \varepsilon_1 \) is chosen larger than \( \varepsilon_0 + \varepsilon \). This gives the result. \( \square \)

The next result concerns mapping properties of \( \Gamma_{s,\sigma}^\sigma \) spaces under trace operators.
Proposition A.2. Let \( \omega \) be a weight on \( \mathbb{R}^{4d} \), \( \omega_0(x, \xi) = \omega(x, x, \xi, \xi) \) when \( x, \xi \in \mathbb{R}^d \), \( s, \sigma > 0 \) be such that \( s + \sigma \geq 1 \). Then the trace map which takes
\[
\mathbb{R}^{4d} \ni (x, y, \xi, \eta) \mapsto F(x, y, \xi, \eta)
\]
to
\[
\mathbb{R}^{2d} \ni (x, \xi) \mapsto F(x, x, \xi, \xi)
\]
is linear and continuous from \( \Gamma_{(\omega)}^{s, \sigma}((\mathbb{R}^{4d})) \) into \( \Gamma_{(\omega_0)}^{s, \sigma}((\mathbb{R}^{2d})) \). The same holds true with \( \Gamma_{(\omega)}^{s,0} \) and \( \Gamma_{(\omega_0)}^{s,0} \) in place of \( \Gamma_{(\omega)}^{s,\sigma} \) and \( \Gamma_{(\omega_0)}^{s,\sigma} \), respectively, at each occurrence.

Proposition A.2 follows by similar arguments as in the proof of Lemma 3.8, using the Leibniz type rule
\[
(\partial_1^{\alpha} \partial_2^{\beta} F(x, x, \xi, \xi)) = \sum_{\gamma \leq \alpha} \sum_{\delta \leq \beta} \binom{\alpha}{\gamma} \binom{\beta}{\delta} (\partial_1^{\alpha-\gamma} \partial_2^{\beta-\delta} \partial_3^{\gamma} \partial_4^{\delta} F)(x, x, \xi, \xi).
\]
The details are left for the reader.

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