The Gauss maps of Demoulin surfaces with conformal coordinates

In Memory of Professor Zhengguo Bai (1916–2015)

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Abstract

Demoulin surfaces in the real projective 3-space are investigated. Our result enables us to establish a generalized Weierstrass type representation for definite Demoulin surfaces by virtue of primitive maps into a certain semi-Riemannian 6-symmetric space.

Keywords

Demoulin surface, Wilczynski frame, Gauss map

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1 Introduction

Professor Zhengguo Bai has done great contributions in projective differential geometry. For example, he solved the so-called Fubini’s problem [22] (see also [29]).

Projective differential geometry of surfaces is a treasure box of infinite-dimensional integrable systems. For example, harmonic maps of Riemann surfaces into the complex projective space CP^n (the CP^n-sigma models in particle physics) are typical examples of 2-dimensional integrable systems. One of the key clue of the study of harmonic maps into the complex projective space is the use of harmonic sequences introduced by Chern and Wolfson [7]. It should be emphasized that the basic idea of the harmonic sequence goes back to the Laplace sequence in classical projective differential geometry (see [3]).

From the modern point of view, the Laplace sequence produces the 2-dimensional Toda field equation of type A∞ (see [9, 10, 26]). In particular, the periodic Laplace sequence produces the 2-dimensional periodic Toda field equations. For example, the Laplace sequences of period 2 produce the sinh-Gordon equation. The Tâteica equation is obtained as the Laplace sequence of period 3, and it is a structure equation of affine spheres [13]. The Laplace sequences of period 4 were studied by Su [27, 28], Hu [15] gave a Darboux matrix, i.e., the simple type dressing for such a sequence.

This article addresses the Laplace sequences of period 6. The Toda field equation derived from those sequences is a structure equation of Demoulin surfaces in the real projective 3-space RP^3 [11].
Godeaux gave a method for studying projective surfaces through their Plüker images in real projective 5-space $\mathbb{RP}^5$. His method relies on the consideration of the Laplace sequence associated with the Plücker image, called the Godeaux sequence. For a characterization of Demoulin surfaces in terms of Godeaux sequences, see [26, Subsection 4.8]. Pa [23] studied Godeaux sequences of quadrics.

In [16], Kobayashi considered two Gauss maps of surfaces in $\mathbb{RP}^3$ with indefinite projective metrics and characterized projective minimal surfaces and Demoulin surfaces in terms of harmonicity of the Gauss maps. In this paper, we consider those surfaces with positive definite projective metrics. This paper is organized as follows: After preparing prerequisite knowledge on projective surface theory in Sections 2–4, we parametrize the space of all the conformal 2-spheres in $\mathbb{RP}^3$ in Section 5. We will show that the space of all the conformal 2-spheres is realized as a semi-Riemannian symmetric space. The conformal first-order Gauss map. In addition, definite Demoulin surfaces and definite projective minimal surfaces are characterized projective minimal surfaces and Demoulin surfaces in terms of harmonicities of the Gauss maps introduced in this paper take the values in this symmetric space. In Section 6, we introduce the first-order Gauss map for a surface in $\mathbb{RP}^3$ as a congruence of conformal 2-spheres which has the first-order contact to the surface. Definite Demoulin surfaces are characterized as the surfaces with the conformal first-order Gauss map. In addition, definite Demoulin surfaces and definite projective minimal coincidence surfaces are characterized by the harmonicity of the first-order Gauss map. In the final section, we will show that every definite Demoulin surface can be constructed by a primitive map into a certain semi-Riemannian 6-symmetric space fibered over the semi-Riemannian symmetric space of all the conformal 2-spheres.

Throughout this paper, we use the following abbreviation:

$\text{diag}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \\ & & & a_n \end{pmatrix}$, \quad \text{offdiag}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} & & & \\ & a_1 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}.$

## 2 Projective surface theory

Let $f : M \to \mathbb{RP}^3$ be an immersed surface in the real projective 3-space $\mathbb{RP}^3$. Take a simply connected region $\mathbb{D} \subset M$ and a homogeneous coordinate vector field $f = (f^0, f^1, f^2, f^3) : \mathbb{D} \to \mathbb{R}^4 \setminus \{0\}$. Let $D$ be the natural affine connection on $\mathbb{R}^4$ and $\Omega$ be a volume element so that $D\Omega = 0$. Thus $(\mathbb{R}^4, D, \Omega)$ is an equiaffine 4-space. One can take a vector field $\xi$ transversal to both $f$ and the radial vector field $\zeta = \sum_{i=0}^{3} x^i \partial / \partial x^i$. Then $\xi$ induces an affine connection $\nabla$ on $\mathbb{D}$ and symmetric tensor fields $h$ and $T$ via the Gauss formula:

$$D_X f_Y = f_\ast (\nabla_X Y) + h(X, Y)\xi + T(X, Y)\zeta, \quad X, Y \in \Gamma(T\mathbb{D}).$$

Moreover, we have the following Weingarten formula:

$$D_X \xi = -f_\ast(SX) + \tau(X)\xi + \rho(X)\zeta.$$

The triplet $(\mathbb{D}, f, \xi)$ is a centroaffine surface (of codimension 2) in $\mathbb{R}^4$ in the sense of [20,21]. We introduce an area element $\vartheta$ on $\mathbb{D}$ by $\vartheta(X, Y) = \Omega(f_\ast X, f_\ast Y, f_\ast \xi, f_\ast \zeta)$. The cubic form $C$ is defined by $C = \nabla h + \tau \otimes h$.

The non-degeneracy of $h$ is independent of the choice of $\xi$. In addition, the conformal class $[h]$ of $h$ is independent of $\xi$. Thus the property "$h$ is positive definite" is well defined for $f$. Throughout this article, we assume that $h$ is positive definite.

When we take $\xi$ so that $\tau = 0$, then $(\mathbb{D}, f, \xi)$ is said to be equiaffine. An equiaffine centroaffine immersion $f$ is said to be Blaschke if $\vartheta$ coincides with the area element of the metric $h$.

On the other hand, Nomizu and Sasaki [20] showed that there exits a transversal vector field $\xi$ such that

$$\text{tr}_h T + \text{tr} S = 0. \quad (2.1)$$
Such a vector field is called a pre-normalized transversal vector field. In particular, the pre-normalized transversal vector field $\xi$ such that $(D, f, \xi)$ is a Blaschke immersion is unique up to sign. In such a choice, the pair surface $(f, \xi)$ is called a pre-normalized Blaschke immersion.

Let us take another homogeneous coordinate vector field $\tilde{f} = \phi f$. Here, $\phi$ is a smooth (non-zero) function. Then the connection $\nabla$ induced from $\tilde{f}$ is projectively equivalent to $\nabla$. The equiaffine property is preserved under the change $f$ by $\phi f$.

Let us denote by $\nabla^h$ the Levi-Civita connection of $h$. Then the scalar field $J = h(K, K)/2$ is called the Fubini-Pick invariant of $f$. Here, $K = \nabla - \nabla^h$. The Riemannian metric $Jh$ is projectively invariant and called the projective metric of $f$. Although $C$ itself is not projectively invariant, its conformal class is projectively invariant (see [19]).

When $(f, \xi)$ is pre-normalized Blaschke, the projective metric is given by $h(\nabla h, \nabla h)h/8$. For more details on centroaffine immersions and projective immersions, we refer to [20, 21].

### 3 Wilczynski frames

Let $f : M \to \mathbb{RP}^3$ be an immersed surface with a positive definite projective metric. We regard $M$ as a Riemann surface with respect to the conformal structure $[Jh]$ determined by the projective metric $Jh$.

We take a simply connected complex coordinate region $D$ with the coordinate $z = x + iy$ on $D$ and a lift $f = (f^0, f^1, f^2, f^3) : D \to \mathbb{R}^4 \setminus \{0\}$. Then the canonical system of $f$ is given by

$$f_{zz} = b\bar{f}_z + pf, \quad f_{\bar{z}z} = \bar{b}f_z + \bar{p}f$$

(3.1)

for some smooth functions $b$ and $p$ (see [26, p. 121]). Note that the subscripts $z$ and $\bar{z}$ denote the partial derivatives of $z$ and $\bar{z}$, respectively:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right).$$

Assume that $f^0 \neq 0$. Then $f$ is given by the inhomogeneous coordinate $\tilde{f} = (f^1, f^2, f^3)/f^0$. The canonical system is rewritten as

$$f_{zz} = b\bar{f}_z - 2(\log f^0)f_z, \quad f_{\bar{z}z} = \bar{b}f_z - 2(\log f^0)f_{\bar{z}}.$$

(3.2)

The integrability condition of the canonical system is (see [26, Subsection 2.3])

$$p_z = b\bar{b}z + \frac{1}{2}b\bar{b} - \frac{1}{2}b\bar{b},$$

$$\text{Im} (b_{zz\bar{z}} - b\bar{b}_{z\bar{z}} - 2b\bar{b}_{z\bar{z}} - 2b_{z\bar{z}} \bar{b}_{z\bar{z}} - 4b\bar{b}_{z\bar{z}}) = 0.$$

The Fubini-Pick invariant is given by $J = 8|b|^2$ and hence the projective metric is $8|b|^2 dzd\bar{z}$. The cubic form of $f$ is given by $C = -2(b dz^3 + \bar{b}d\bar{z}^3)$ (see [25, p. 54, Definition, Subsection 4.8]). Note that when $f$ is pre-normalized Blaschke, then the projective metric is expressed as $2|b|^2 dzd\bar{z}$.

Hereafter we assume that $b \neq 0$. Note that when $C = 0$, $f$ is a part of a quadratic surface [24, 31] (see also [25, Theorem 4.4]).

The Wilczynski frame $F$ of $f$ is defined by

$$F = (f, f_1, f_2, \eta),$$

where

$$f_1 := f_z - \frac{\bar{b}}{2b} f, \quad f_2 := f_{\bar{z}} - \frac{b}{2\bar{b}} f, \quad \eta = f_{zz} - \bar{b}f_z f_{\bar{z}} - \bar{b}f_z f_{\bar{z}} + \left(\frac{|b|^2}{4|b|^2} - \frac{|b|^2}{2}\right)f.$$

Then a straightforward computation shows that the Wilczynski frame $F$ satisfies the following equations:

$$F_z = FU \quad \text{and} \quad F_{\bar{z}} = FV,$$

(3.3)
where

\[
U = \begin{pmatrix}
\bar{b}_z/(2\bar{b}) & P & k & b\bar{P} \\
1 & -\bar{b}_z/(2\bar{b}) & 0 & k \\
0 & b & \bar{b}_z/(2\bar{b}) & P \\
0 & 0 & 1 & -\bar{b}_z/(2\bar{b})
\end{pmatrix},
\quad
V = \begin{pmatrix}
b_z/(2b) & \bar{k} & \bar{P} & \bar{b}P \\
0 & b_z/(2b) & \bar{b} & \bar{P} \\
1 & 0 & -b_z/(2b) & \bar{k} \\
0 & 1 & 0 & -b_z/(2b)
\end{pmatrix}.
\]

Here, we introduced functions \( k \) and \( P \) as follows:

\[
k = \frac{|b|^2 - (\log b)_{zz}}{2},
\]

\[
P = p + \frac{b_z}{2} - \frac{b_{zz}}{2b} + \frac{\bar{b}_z^2}{4\bar{b}^2}.
\]

The compatibility conditions of (3.3) are

\[
\bar{P}_z = k\bar{z} + k\frac{b_z}{\bar{b}},
\]

\[
\text{Im}(\bar{b}P_z + 2\bar{b}zP) = 0.
\]

These equations are nothing but the \textit{projective Gauss-Codazzi equations} of a surface \( f \). One can see that \( Pdz^2 \) and \( 2b^2 Pdz^4 \) are globally defined on \( M \) and projectively invariant \[14\].

Since both \( U \) and \( V \) are trace free, the Wilczynski frame \( F \) takes values in \( \text{SL}_4 \mathbb{C} \) up to the initial condition. Moreover, if we choose some base point \( z_0 \in \mathbb{D} \) and \( F(z_0) = \text{id} \), then the frame \( F \) takes values in \( \text{SL}_4 \mathbb{R} \) by conjugation of a simple complex matrix

\[
\text{Ad}(L)F \in \text{SL}_4 \mathbb{R}, \quad L = \frac{1}{\sqrt{2}}\begin{pmatrix}
\sqrt{2} & 0 & 0 & 0 \\
0 & \sqrt{-1} & -\sqrt{-1} & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & \sqrt{2}
\end{pmatrix}.
\]

4 \hspace{1em} \textbf{Projective minimal surfaces and definite Demoulin surfaces}

A surface \( f : M \to \mathbb{RP}^3 \) with a positive definite projective metric is said to be a \textit{projective minimal surface} if it is a critical point of the area functional of the projective metric (called the \textit{projective area functional}). Then the projective minimality can be computed as in \[30\]:

\[
\bar{b}P_z + 2b_zP = 0,
\]

where the function \( P \) is defined in (3.5). It should be remarked that the projective minimality (4.1) implies the second equation (3.7) of the projective Gauss-Codazzi equations. There is a particular class of projective minimal surfaces with positive definite projective metrics.

A surface with a positive definite projective metric is said to be a \textit{definite Demoulin surface} if it satisfies \( P = 0 \). The Demoulin property originates from Demoulin transformations of surfaces in \( \mathbb{RP}^3 \). For more details, we refer to \[26\].

5 \hspace{1em} \textbf{The Plücker quadric and the space of conformal spheres}

5.1 \hspace{1em} \textbf{The Plücker quadric}

Take a volume element \( \Omega \) on \( \mathbb{R}^4 \) parallel with respect to the natural affine connection \( D \). Then we can introduce a scalar product \( \langle \cdot, \cdot \rangle \) on \( \wedge^2 \mathbb{R}^4 \) by

\[
\langle \alpha, \beta \rangle = \Omega(\alpha \wedge \beta), \quad \alpha, \beta \in \wedge^2 \mathbb{R}^4.
\]
One can check that $\langle \cdot, \cdot \rangle$ is of signature $(3,3)$. In fact, let $\{e_0, e_1, e_2, e_3\}$ be the natural basis of $\mathbb{R}^4$. Denote by $\{e^0, e^1, e^2, e^3\}$ the dual basis of $\{e_0, e_1, e_2, e_3\}$. Then with respect to the volume element $\Omega = e^0 \wedge e^1 \wedge e^2 \wedge e^3$ and the basis $\{e_0 \wedge e_1, e_0 \wedge e_2, e_0 \wedge e_3, e_1 \wedge e_2, e_3 \wedge e_1, e_2 \wedge e_3\}$ of $\wedge^2 \mathbb{R}^4$, the scalar product $\langle \cdot, \cdot \rangle$ is determined by the matrix offdiag$(1,1,1,1,1)$. The special linear group $\text{SL}_4\mathbb{R}$ acts on $\wedge^2 \mathbb{R}^4$ via the action

$$\text{SL}_4\mathbb{R} \times \wedge^2 \mathbb{R}^4 \to \wedge^2 \mathbb{R}^4,$$

$$(g, v \wedge w) \mapsto gv \wedge gw.$$ 

One can see that this action is isometric with respect to $\langle \cdot, \cdot \rangle$. This fact implies the Lie group isomorphism $\text{PSL}_4\mathbb{R} \cong \text{SO}^+_{3,3}$. Here, $\text{SO}^+_{3,3}$ denotes the identity component of the semi-orthogonal group $\text{O}_{3,3}$.

Next, we consider the Plücker embedding of the Grassmannian manifold $\text{Gr}_2(\mathbb{R}^4)$ of 2-planes in $\mathbb{R}^4$ into the projective 5-space $\mathbb{RP}^5 = \mathbb{P}(\wedge^2 \mathbb{R}^4)$. The Plücker coordinates of the 2-plane spanned by $(a^0, a^1, a^2, a^3)$ and $(b^0, b^1, b^2, b^3)$ are $[p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}]$, where

$$p_{ij} = \det \begin{pmatrix} a^i & b^j \\ a^j & b^i \end{pmatrix}. \quad (5.1)$$

The Plücker coordinates $[p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}]$ of $a \wedge b$ satisfy the quadratic Plücker relation

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0. \quad (5.2)$$

Thus the Plücker image of $\text{Gr}_2(\mathbb{R}^4)$ is a projective variety (called the Plücker quadric) of $\mathbb{RP}^5$ determined by the equation $(5.2)$. Moreover, the Plücker relation means that $(p_{01}, p_{02}, p_{03}, p_{23}, p_{31}, p_{12})$ is null with respect to $\langle \cdot, \cdot \rangle$. Namely, the Plücker image of $\text{Gr}_2(\mathbb{R}^4)$ is the projective light cone $\mathbb{P}(\mathcal{L})$ of $\wedge^2 \mathbb{R}^4 = \mathbb{R}^{3,3}$.

Now let us consider a line $\ell$ in $\mathbb{RP}^3$ connecting two points $a = [a^0 : a^1 : a^2 : a^3]$ and $b = [b^0 : b^1 : b^2 : b^3]$. The Plücker image of $\ell$ in $\mathbb{RP}^5$ is $\mathbb{P}(\wedge^2 \mathbb{R}^4)$ is

$$a \wedge b = [p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}]$$

with $(5.1)$. Hence the space $\mathcal{P}$ of lines in $\mathbb{RP}^3$ is identified with the Plücker quadric. This identification is called the Klein correspondence.

**Remark 5.1.** The conformal compactification of semi-Euclidean 4-space $\mathbb{R}^{2,2}$ of neutral signature is obtained as the projective light cone $\mathbb{P}(\mathcal{L}) \subset \mathbb{RP}^5$ equipped with the conformal structure induced from $\mathbb{R}^{3,3}$. The action of $\text{PSL}_4\mathbb{R} \cong \text{SO}^+_{3,3}$ on $\mathbb{P}(\mathcal{L})$ is conformal. One can see that the Plücker quadric $\mathcal{P} = \mathbb{P}(\mathcal{L})$ is isomorphic to $\text{Gr}_2(\mathbb{R}^4) \cong (S^2 \times S^2)/\mathbb{Z}_2$ (equipped with the standard conformal structure of neutral signature) as a conformal manifold. Note that on $\mathbb{P}(\mathcal{L})$, there exits a complex structure compatible with the standard neutral metric. The standard neutral metric is neutral Kähler with respect to the complex structure. In particular, the Kähler form is regarded as a standard symplectic form on $\mathbb{P}(\mathcal{L})$. For more information on conformal geometry of $\mathbb{P}(\mathcal{L})$, see [18].

### 5.2 The space of conformal spheres

A quadric in $\mathbb{RP}^3$ is a surface of the form $\{v \in \mathbb{RP}^3 \mid q(v, v) = 0\}$, where $q$ is a scalar product of $\mathbb{R}^4$. For our purpose we choose a Lorentzian scalar product $q = \langle \cdot, \cdot \rangle$ on $\mathbb{R}^4$ and regard it as a Minkowski 4-space $\mathbb{R}^{1,3}$. Then the quadric is nothing but the conformal 2-sphere (Riemann sphere) in $\mathbb{RP}^3$. The space of conformal 2-spheres in $\mathbb{RP}^3$ is parametrized as the space $Q$ of $4 \times 4$ symmetric matrices with determinant one and signature $(1,3)$. In fact, the conformal 2-sphere is given by the Lorentzian scalar product $q(u, v) = uQv^T$ with $Q \in Q$.

The special linear group $\text{SL}_4\mathbb{R}$ acts transitively on $Q$ via the action $(g, Q) \mapsto gQg^T$ with $g \in \text{SL}_4\mathbb{R}$ and $Q \in Q$. The stabilizer at

$$\hat{J}_q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (5.3)$$
is given by $\tilde{K}_1 = \{ a \in \SL_4 \mathbb{R} \mid a \hat{J}_1 a^T = \hat{J}_1 \}$, which is isomorphic to the identity component $SO^{+}_{1,3}$ of the semi-orthogonal group $O^{+}_{1,3}$ of signature $(1, 3)$. Thus $Q$ is isomorphic to the homogeneous space $\SL_4 \mathbb{R}/SO^{+}_{3,3} \cong SO^{+}_{3,3}/SO^{+}_{1,3}$.

We introduce a scalar product $\langle \cdot, \cdot \rangle$ at $Q \in \mathcal{Q}$ by

$$
\langle X, Y \rangle_Q = \text{Tr} \left( Q^{-1} X Q^{-1} Y \right), \quad X, Y \in T_Q \mathcal{Q}.
$$

Note that at the origin of $SO^{+}_{3,3}/SO^{+}_{1,3}$, $8\langle \cdot, \cdot \rangle$ is the Killing form of $\mathfrak{sl}_4 \mathbb{R}$. This scalar product is invariant under the action of $\SL_4 \mathbb{R}$. In fact,

$$
\langle gXg^T, gYg^T \rangle_g = \text{Tr} \left( (gQg^T)^{-1} gXg^T (gQg^T)^{-1} gYg^T \right) = \langle X, Y \rangle_Q.
$$

Thus $Q = \SL_4 \mathbb{R}/\tilde{K}_1$ is a semi-Riemannian symmetric space corresponding to the outer involution

$$
\hat{\tau}_1(X) = \hat{J}_1(X^T)^{-1} \hat{J}_1.
$$

**Remark 5.2.** The space of lines in the Plücker quadric $\mathcal{P}$ is identified with the Grassmannian manifold of all the null 2-planes in $\mathbb{R}^{3,3}$:

$$
\mathcal{Z} = \{ W \in \text{Gr}_2(\mathbb{R}^{3,3}) \mid W \text{ is a null 2-plane in } \mathbb{R}^{3,3} \} \cong SO^{+}_{3,3}/SO^{+}_{2,2}.
$$

For surfaces in $\mathbb{R}^{3,3}$ with indefinite projective metrics, two kinds of Gauss maps are considered in our previous work [16]. Those Gauss maps take the value in the space of quadrics determined by scalar products of signature $(2, 2)$ of $\mathbb{R}^4$. The space of all the quadrics derived from such scalar products is identified with the semi-Riemannian symmetric space $SO^{+}_{3,3}/SO^{+}_{2,2}$.

## 6 Demoulin surfaces and the first-order Gauss maps

In this section, we define the first-order Gauss map for a surface in $\mathbb{R}^{3,3}$.

### 6.1 The first-order Gauss map

Let $f : M \to \mathbb{R}^{3,3}$ be a surface and $F$ the corresponding Wilczynski frame defined in (3.3) with a base point $z_0 \in \mathbb{D}$ and $F(z_0) = \text{id}$. Let $L$ be the matrix defined in (3.8) and $\hat{F}$ be the $\SL_4 \mathbb{R}$ matrix such that

$$
\Ad(L)F = \hat{F}.
$$

We now define the first-order Gauss map $g_1$ as follows:

$$
g_1 = \hat{F} \hat{J}_1 \hat{F}^T = \Ad(L)(F \hat{J}_1 F^T), \quad (6.1)
$$

where the matrix $\hat{J}_1$ is the one given by (5.3) and $J_1 = \text{offdiag}(1, 1, 1, 1)$. Note that $\Ad(L)J_1 = \hat{J}_1$.

Therefore the map $g_1$ takes values in the space of conformal 2-spheres:

$$
g_1 : M \to \mathcal{Q} \cong \SL_4 \mathbb{R}/\tilde{K}_1 = \SL_4 \mathbb{R}/SO^{+}_{1,3}.
$$

This map $g_1$ is known to be a quadric which has the first-order contact to the surface and it does not have the second-order contact (see [17, Section 22]).

We now characterize the Demoulin surface by the first-order Gauss map.

**Proposition 6.1.** The first-order Gauss map $g_1$ of a surface $f$ in $\mathbb{R}^{3,3}$ with a positive definite projective metric is conformal if and only if $f$ is a definite Demoulin surface.

**Proof.** A direct computation shows that

$$
\partial_z g_1 = 2(\mathcal{L}F) \begin{pmatrix}
 b \bar{P} & k & P & 0 \\
 k & 0 & 0 & 1 \\
P & 0 & b & 0 \\
 0 & 1 & 0 & 0
\end{pmatrix} (\mathcal{L}F)^T, \quad \partial_{\bar{z}} g_1 = 2(\mathcal{L}F) \begin{pmatrix}
 b \bar{P} & \bar{k} & 0 & 0 \\
 \bar{P} & b & 0 & 0 \\
 \bar{k} & 0 & 0 & 1 \\
 0 & 0 & 1 & 0
\end{pmatrix} (\mathcal{L}F)^T.
$$
Thus
\[ \langle \partial_z g_1, \partial_{\bar{z}} g_1 \rangle = 16P, \quad \langle \partial_z g_1, \partial_{\bar{z}} g_1 \rangle = 16\bar{P} \quad \text{and} \quad \langle \partial_z g_1, \partial_{\bar{z}} g_1 \rangle = \langle \partial_z g_1, \partial_{\bar{z}} g_1 \rangle = 8(k + \bar{k}) + 4|b|^2. \]
Since the coordinates \((z, \bar{z})\) are null for the conformal structure induced by \(f\), the first-order Gauss map \(g_1\) is conformal if and only if \(P = 0\).

6.2 Demoilin surfaces and projective minimal coincidence surfaces

We set
\[ G = \text{Ad}(L^{-1})\text{SL}_4\mathbb{R} = \{ L^{-1} X L \mid X \in \text{SL}_4\mathbb{R} \} \subset \text{SL}_4\mathbb{C}, \]
where \(L\) is defined in (3.8). The closed subgroup \(G\) is a real form of \(\text{SL}_4\mathbb{C}\) which is isomorphic to \(\text{SL}_4\mathbb{R}\). The space \(Q\) of conformal 2-spheres is isomorphic to \(G/K_1\), where \(K_1\) is
\[ K_1 = \{ a \in G \mid a J_1 a^T = J_1 \}. \]
Let \(\tau_1\) be the outer involution on the \(G\) associated with \(G/K_1\) given by
\[ \tau_1(a) = J_1(a^T)^{-1} J_1, \quad a \in G. \]
By abuse of notation, we denote the differential of \(\tau_1\) by the same letter \(\tau_1\):
\[ \tau_1(X) = -J_1 X^T J_1, \quad X \in \mathfrak{g}. \quad (6.3) \]
Let us consider the eigenspace decomposition of \(\mathfrak{g}\) with respect to \(\tau_1\), i.e., \(\mathfrak{g} = \mathfrak{t}_1 \oplus \mathfrak{p}_1\), where \(\mathfrak{t}_1\) is the \((+1)\)-eigenspace and \(\mathfrak{p}_1\) is the \((-1)\)-eigenspace as follows:
\[ \mathfrak{t}_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & 0 & -a_{13} \\ a_{31} & 0 & -a_{22} & -a_{12} \\ 0 & -a_{31} & -a_{21} & -a_{11} \end{pmatrix} \right\} \subset \mathfrak{g}, \quad \mathfrak{p}_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & -a_{11} & a_{23} & a_{13} \\ a_{31} & a_{32} & -a_{11} & a_{12} \\ a_{41} & a_{31} & a_{21} & a_{11} \end{pmatrix} \right\} \subset \mathfrak{g}. \]
We decompose the Maurer-Cartan form according to this decomposition \(\alpha = F^{-1}dF = Ud\bar{z} + Vd\bar{z}\) along
the Lie algebra decomposition \(\mathfrak{g} = \mathfrak{t}_1 \oplus \mathfrak{p}_1\). First, we decompose \(U\) and \(V\) as
\[ U = U_{\mathfrak{t}_1} + U_{\mathfrak{p}_1}, \quad V = V_{\mathfrak{t}_1} + V_{\mathfrak{p}_1}, \quad U_{\mathfrak{t}_1}, V_{\mathfrak{t}_1} \in \mathfrak{t}_1, \quad U_{\mathfrak{p}_1}, V_{\mathfrak{p}_1} \in \mathfrak{p}_1. \]
Next, set \(a_{\mathfrak{t}_1} = U_{\mathfrak{t}_1}dz + V_{\mathfrak{t}_1}d\bar{z}\) and \(a_{\mathfrak{p}_1} = U_{\mathfrak{p}_1}dz + V_{\mathfrak{p}_1}d\bar{z}\). Then we obtain the expression
\[ a = a_{\mathfrak{t}_1} + a_{\mathfrak{p}_1} = U_{\mathfrak{t}_1}dz + V_{\mathfrak{t}_1}d\bar{z} + U_{\mathfrak{p}_1}dz + V_{\mathfrak{p}_1}d\bar{z}, \]
where \(U = U_{\mathfrak{t}_1} + U_{\mathfrak{p}_1}\) and \(V = V_{\mathfrak{t}_1} + V_{\mathfrak{p}_1}\). Let us insert the spectral parameter \(\lambda \in S^1\) into \(U\) and \(V\) as follows:
\[ U^\lambda = U_{\mathfrak{t}_1} + \lambda^{-1} U_{\mathfrak{p}_1} \quad \text{and} \quad V^\lambda = V_{\mathfrak{t}_1} + \lambda V_{\mathfrak{p}_1}. \]
Then an \(S^1\)-family of 1-forms \(a_\lambda\) is defined as follows:
\[ a^\lambda = a_{\mathfrak{t}_1} + \lambda^{-1} a_{\mathfrak{p}_1} = U^\lambda dz + V^\lambda d\bar{z}. \]
By using the matrices \(U^\lambda\) and \(V^\lambda\), they are explicitly given as follows:
\[ U^\lambda = \begin{pmatrix} \frac{b_{\bar{z}}}{(2\bar{b})} & \lambda^{-1} P & \lambda^{-1} k & \lambda^{-1} b\bar{P} \\ \lambda^{-1} & -\frac{\bar{b}_{\bar{z}}}{(2\bar{b})} & 0 & \lambda^{-1} k \\ 0 & \lambda^{-1} b & -\frac{\bar{b}_{\bar{z}}}{(2\bar{b})} & \lambda^{-1} P \\ 0 & 0 & \lambda^{-1} & -\frac{b_{\bar{z}}}{(2b)} \end{pmatrix}, \quad V^\lambda = \begin{pmatrix} \frac{b_{\bar{z}}}{(2\bar{b})} & \lambda \bar{b} & \lambda \bar{P} & \lambda \bar{b} \bar{P} \\ 0 & \frac{\bar{b}_{\bar{z}}}{(2\bar{b})} & \lambda \bar{b} & \lambda \bar{P} \\ \lambda & 0 & -\frac{b_{\bar{z}}}{(2b)} & \lambda \bar{k} \\ 0 & \lambda & 0 & -\frac{b_{\bar{z}}}{(2b)} \end{pmatrix}. \]
After these preparations, we obtain the following theorem.
Theorem 6.2.  Let $f$ be a surface in $\mathbb{RP}^3$ with a positive definite projective metric and $g_1$ be the first-order Gauss map defined in (6.2). Moreover, let $\{\alpha^\lambda\}_{\lambda \in S^1}$ be a family of 1-forms defined in (6.4). Then the following three properties are mutually equivalent:

1) The surface $f$ is a definite Demoulin surface or a projective minimal coincidence surface.

2) The first-order Gauss map $g_1$ is a harmonic map into $\mathbb{Q}$.

3) $\{d + \alpha^\lambda\}_{\lambda \in S^1}$ is a family of flat connections on $\mathbb{D} \times G$.

Proof. Let us first compute the flatness

$$d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0, \quad \lambda \in S^1$$

for the connection $d + \alpha^\lambda$ on $\mathbb{D} \times G$. A straightforward computation shows that $d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0$ holds for all $\lambda \in S^1$ if and only if

$$P_x = 0, \quad k_x + k \frac{b_x}{b} = 0, \quad \bar{b}P_x + 2\bar{b}_xP = 0. \quad (6.5)$$

One can see that this system implies the projective Gauss-Codazzi equations (3.6)–(3.7). In particular, the third equation is nothing but the projective minimality equation (4.1).

Every definite Demoulin surface clearly satisfies the above flatness condition (zero curvature equations) since $P = 0$.

Assume that $P \neq 0$. The first equation of (6.5) means that $Pdz^2$ is a holomorphic differential. From the third equation together with the holomorphicity of $P$, one can deduce that $(\log \bar{b})z$ is holomorphic. Hence $(\log \bar{b}_x)_z = 0$. Via the holomorphic coordinate change of $z$ preserving the form of the canonical system, we can assume that $b = \bar{b}$, i.e., $b$ is real\(^1\). Then (3.4) implies that $2k = b^2$. By using the second equation of (6.5), $b$ is constant and $k$ is a real constant. By using the third equation again, we get $P$ is constant. This implies that $P = p$ is a non-zero constant. After this reparametrization, the canonical system is rewritten as

$$f_zz = bf_z + pf, \quad f_{zz} = bf_z + pf.$$

A surface satisfying the above equation is a special case of the coincidence surface \([26, \text{Example 2.19}]. In fact, it is easy to see that the surface is a projective minimal coincidence surface. Thus the equivalence of the claims (1) and (3) follows.

The equivalence of the claims (2) and (3) follows from Proposition A.4, since the $S^1$-family of 1-forms $\alpha^\lambda$ is given by the involution $\tau_1$ and it defines the semi-Riemannian symmetric space $\mathbb{Q} = \text{SL}_4\mathbb{R}/K_1$. \(\square\)

Corollary 6.3. Retain the assumptions in Theorem 6.2, and the following are equivalent:

1) The surface $f$ is a definite Demoulin surface.

2) The first-order Gauss map $g_1$ is a conformal harmonic map into $\mathbb{Q}$.

Proof. From Proposition 6.1, it is easy to see that the first-order Gauss map is conformal if and only if it satisfies that $P = 0$, i.e., the surface is a definite Demoulin surface. Moreover, from Theorem 6.2 the Gauss map of the Demoulin surface is harmonic. \(\square\)

This corollary implies that if $f$ is a definite Demoulin surface or a projective minimal coincidence surface, then there exists an $S^1$-parameter family of smooth maps $F_\lambda : \mathbb{D} \times S^1 \to G$ which is a solution to

$$(F_\lambda)^{-1}dF_\lambda = \alpha^\lambda$$

under the initial condition $F_\lambda (z_0) = \text{id}$. One can see that $F_\lambda$ is regarded as a smooth map of $\mathbb{D}$ into the following twisted loop group:

$$\Lambda G_{\tau_1} = \{g : S^1 \to G \mid \tau_1 g(\lambda) = g(-\lambda)\}$$

\(^1\) The transformation rule of $b$ under the conformal change of the coordinates $w(z)$ is given by $\bar{b} = (\bar{w}_z/w_z^2)b$, and thus $\bar{b} = \bar{b}$ can be achieved by a suitable choice of the function $w(z)$ under the condition $(\log b/\bar{b})_z = 0$ (see [14, Section 3]).
of $G$. The $\Lambda G_{\tau_1}$-valued map $F_\lambda$ is referred to as the extended Wilczynski frame of a definite Demoulin surface.

Precisely speaking, the extended Wilczynski frame $F_\lambda$ is not the Wilczynski frame of a Demoulin surface or a projective minimal coincidence surface except for $\lambda = 1$. By conjugating $F_\lambda$ by $DF_\lambda D^{-1}$ with $D = \text{diag}(1, \lambda, \lambda^{-1}, 1)$, the frames $DF_\lambda D^{-1}$ give a family of Wilczynski frames for Demoulin surfaces or projective minimal coincidence surfaces. The corresponding Demoulin surfaces or projective minimal coincidence surfaces have the same projective metric $8|b|^2dzd\bar{z}$ but the different conformal classes of cubic forms $\lambda^{-3}b dz^3$. Moreover, the function $P$ changes to $\lambda^{-2}P$.

### 7 Primitive lifts

We now show that the extended Wilczynski frame for a Demoulin surface has an additional order three cyclic symmetry. Let $\sigma$ be an order three automorphism on the complexification $\text{SL}_4\mathbb{C}$ of $G$ as follows:

$$\sigma X = \text{Ad}(E)X, \quad X \in \text{SL}_4\mathbb{C},$$

where $E = \text{diag}(1, \epsilon^2, \epsilon, 1)$ with $\epsilon = e^{2\pi\sqrt{-1}/3}$. It should be emphasized that $\sigma$ preserves the real form $G$. Thus $\sigma$ is regarded as an automorphism of $G$.

Next, one can check that $F_\lambda$ satisfies the symmetry $\sigma(F_\lambda) = F_\epsilon\lambda$, since $U^\lambda$ and $V^\lambda$ satisfy the same symmetry. It is also easy to see that $\tau_1$ and $\sigma$ commute, and $\kappa = \tau_1 \circ \sigma$ defines an automorphism of order six. We obtain a regular semi-Riemannian 6-symmetric space $G/K$ (see Appendix A.1), where

$$K = \{\text{diag}(k_1, k_2, k_2^{-1}, k_1^{-1}) \mid k_1 \in \mathbb{R}^+, k_2 \in \mathbb{S}^1\} \cong \text{SO}_{1,1} \times \text{SO}_2. \quad (7.1)$$

Note that $G/K$ is identified with $\{gJg^T \mid g \in G\}$, where $J = EJ_1$. There is a homogeneous projection

$$\pi : G/K \rightarrow G/K_1, \quad gK \mapsto gK_1.$$

The extended Wilczynski frame $F_\lambda$ satisfies the symmetry

$$\kappa(F_\lambda) = F_{-\epsilon\lambda}.$$ 

Note that $-\epsilon$ is the 6-th root of unity. From the above argument, it is easy to see that the extended Wilczynski frame $F_\lambda = F(\lambda)$ for a Demoulin surface is an element of the twisted loop group of $G$:

$$\Lambda G_\kappa = \{g : \mathbb{S}^1 \rightarrow G \mid \kappa g(\lambda) = g(-\epsilon\lambda)\}.$$

**Theorem 7.1.** The first-order Gauss map of a Demoulin surface, which is a conformal harmonic map into $Q = G/K_1$, can be obtained by the homogeneous projection of a primitive map into the regular semi-Riemannian 6-symmetric space

$$G/K \cong \text{SL}_4\mathbb{R}/\text{SO}_{1,1} \times \text{SO}_2.$$

**Proof.** The 0-th-eigenspace $g^C_0$ and $\pm 1$st-eigenspaces $g^C_{\pm 1}$ of the derivative of the order six automorphism $\kappa = \tau_1 \circ \sigma$ are described as follows:

$$g^C_0 = \{\text{diag}(a_{11}, a_{22}, -a_{22}, -a_{11}) \mid a_{11} \in \mathbb{R}, a_{22} \in \mathbb{C}\},$$

and

$$g^C_{-1} = \left\{ \begin{pmatrix} 0 & 0 & a_{13} & 0 \\ a_{21} & 0 & 0 & a_{13} \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{21} & 0 \end{pmatrix} \mid a_{ij} \in \mathbb{C} \right\}, \quad g^C_1 = \left\{ \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ a_{31} & 0 & 0 & a_{12} \\ 0 & a_{31} & 0 & 0 \end{pmatrix} \mid a_{ij} \in \mathbb{C} \right\}.$$
From the matrices $U^\lambda$ and $V^\lambda$ in (6.4) with $P = 0$, we see that the condition in Definition A.1 of the primitive map is satisfied. The stabilizer of $\kappa$ is the closed subgroup $K$ given by (7.1). Therefore there is a primitive map $g = FJF^TJ = EJ_1$ into the 6-symmetric space $G/K$ such that $\pi \circ g = \text{Ad}(L^{-1})g_1$. Since

$$\text{Ad}(L^{-1}) : \text{SL}_4 \mathbb{R}/\hat{K}_1 \to G/K_1$$

is an isometry, $g_1 = \text{Ad}(L)(\pi \circ g)$ is harmonic. \hfill \qed

This theorem enables us to establish a generalized Weierstrass type representation for definite Demoulin surfaces by virtue of primitive maps into the semi-Riemannian 6-symmetric space $G/K$ (see [12]).

**Remark 7.2.** The corresponding result theorem for indefinite Demoulin surfaces was obtained by Kobayashi in the preprint version of [16].

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Appendix A Primitive harmonic maps

Appendix A.1 Homogeneous geometry

Let $G$ be a semi-simple real Lie group with the automorphism $\tau$ of order $k \geq 2$. We consider a reductive homogeneous space $G/K$ equipped with a $G$-invariant semi-Riemannian metric satisfying the following three conditions:

- The closed subgroup $H$ satisfies $G^\tau_0 \subset K \subset G^\tau$. Here, $G^\tau$ is the Lie subgroup of all the fixed points of $\tau$ and $G^\tau_0$ is the identity component of it.
- The $G$-invariant semi-Riemannian metric is derived from (a constant multiple of) the Killing form of $G$.
- The Lie algebra $\mathfrak{k}$ of $K$ is non-degenerate with respect to the induced scalar product.

The resulting homogeneous semi-Riemannian space $G/K$ is called a regular semi-Riemannian $k$-symmetric space. Note that a regular semi-Riemannian 2-symmetric space is just a semi-Riemannian symmetric space. Since $\mathfrak{k}$ is non-degenerate, the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ is non-degenerate and can be identified with the tangent space of $G/K$ at the origin $o = K$. The Lie algebra $\mathfrak{g}$ is decomposed into the direct sum

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of linear subspaces.

We denote the induced Lie algebra automorphism of $\mathfrak{g}$ by the same letter $\tau$. Now we have the eigenspace decomposition of the complexified Lie algebra $\mathfrak{g}^C$, i.e.,

$$\mathfrak{g}^C = \sum_{j \in \mathbb{Z}_k} \mathfrak{g}^C_j,$$

where $\mathfrak{g}^C_j$ is the eigenspace of $\tau$ with the eigenvalue $\omega^j$. Here, $\omega$ is the (primitive) $k$-th root of unity. In particular, $\mathfrak{g}^C_0 = \mathfrak{t}^C$ and $\mathfrak{g}^C_{-1} = \mathfrak{g}^C_{1}$. Let us define a subbundle $[\mathfrak{g}^C_j]$ of $G/K \times \mathfrak{g}$ by

$$[\mathfrak{g}^C_j]_{\mathfrak{g}^C_0} = \text{Ad}(g)[\mathfrak{g}^C_j].$$

Then the complexified tangent bundle $T^C G/K$ is expressed as

$$T^C G/K = \sum_{j \in \mathbb{Z}_k, j \neq 0} [\mathfrak{g}^C_j].$$

Appendix A.2 Primitive maps

A smooth map $\psi : \Sigma \rightarrow N$ of a Riemann surface $\Sigma$ into a semi-Riemannian manifold $N$ is said to be a harmonic map if its tension field $\text{tr}(\nabla d\psi)$ vanishes.

For smooth maps into regular semi-Riemannian $k$-symmetric spaces with $k > 2$, the notion of the primitive map was introduced by Burstall and Pedit [6] (see also Bolton et al. [2]).

**Definition A.1.** Let $\psi : \Sigma \rightarrow G/K$ be a smooth map of a Riemann surface $\Sigma$ into a regular semi-Riemannian $k$-symmetric space with $k > 2$. Then $\psi$ is said to be a primitive map if $d\psi(T'\Sigma) \subset [\mathfrak{g}^C_{-1}]$. Here, $T'\Sigma$ denotes the $(1, 0)$-tangent bundle of $\Sigma$. 
Black [1] showed that primitive maps are \textit{equi-harmonic}, i.e., harmonic with respect to suitable invariant metrics on $G/K$ (see also [6]). In addition, primitive maps well behave with respect to homogeneous projections [6, Theorem 3.7].

\textbf{Theorem A.2.} Let $H$ be a closed subgroup of $G$ satisfying
- $K \subset H$;
- the Lie algebra $\mathfrak{h}$ of $H$ is non-degenerate;
- the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is reductive and stable under $\tau$, where $\mathfrak{q}$ is the orthogonal complement of $\mathfrak{h}$.

Denote by $\pi_H : G/K \to G/H$ the homogeneous projection. Then for any primitive map $\psi$, $\pi_H \circ \psi$ is a harmonic map into $G/H$.

Note that when $k = 2$, $[\mathfrak{g}_{-1}^C] = T^C G/K$ and the primitivity condition is vacuous. On the other hand when $k > 2$, every primitive map is harmonic with respect to the Killing metric. To provide a unified description, we recall the following terminology from [5].

\textbf{Definition A.3.} A smooth map $\psi : \Sigma \to G/K$ into a regular semi-Riemannian $k$-symmetric space is said to be a \textit{primitive harmonic map} if it is primitive for $k > 2$ and harmonic if $k = 2$.

Now let $\psi : \mathbb{D} \to G/H$ be a smooth map from a simply connected Riemann surface $\mathbb{D}$ into a regular semi-Riemannian $k$-symmetric space $G/K$ with $k \geq 2$. Take a frame $\Psi : \mathbb{D} \to G$ of $\psi$ and put $\alpha := \Psi^{-1} d\Psi$.

Then we have the identity (the Maurer-Cartan equation)

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$ 

Decompose $\alpha$ along the Lie algebra decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as

$$\alpha = \alpha_\mathfrak{k} + \alpha_\mathfrak{p}, \quad \alpha_\mathfrak{k} \in \mathfrak{k}, \quad \alpha_\mathfrak{p} \in \mathfrak{p}. $$

We decompose $\alpha_\mathfrak{p}$ with respect to the conformal structure of $\mathbb{D}$ as

$$\alpha_\mathfrak{p} = \alpha_\mathfrak{p}' + \alpha_\mathfrak{p}''.$$ 

Here, $\alpha_\mathfrak{p}'$ and $\alpha_\mathfrak{p}''$ are the $(1, 0)$- and $(0, 1)$-parts of $\alpha_\mathfrak{p}$, respectively. Since $G$ is a real Lie group, $\alpha_\mathfrak{p}''$ is the conjugate of $\alpha_\mathfrak{p}'$.

Now let us assume that $\psi$ is a primitive harmonic map. Then $\alpha_\mathfrak{p}'$ is $[\mathfrak{g}_{-1}^C]$-valued and $\alpha_\mathfrak{p}''$ is $[\mathfrak{g}_{+1}^C]$-valued, respectively. Hence the decomposition of $\alpha$ is rewritten as

$$\alpha = \alpha_{-1}' + \alpha_0 + \alpha_1'.$$

Now let us introduce a spectral parameter $\lambda \in \mathbb{S}^1$ into $\alpha$ as

$$\alpha^\lambda := \alpha_0 + \lambda^{-1} \alpha_{-1}' + \lambda \alpha_1''.$$ 

We arrive at the \textit{zero curvature representation} of primitive harmonic maps.

\textbf{Proposition A.4.} Let $\mathbb{D}$ be a connected open subset of $\mathbb{C}$. Let $\psi : \mathbb{D} \to G/K$ be a primitive harmonic map. Then the loop of connections $\omega + \alpha^\lambda$ is flat for all $\lambda$, i.e.,

$$d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0$$

for all $\lambda$.

Conversely assume that $\mathbb{D}$ is simply connected. Let $\alpha^\lambda = \alpha_0 + \lambda^{-1} \alpha_{-1}' + \lambda \alpha_1''$ be an $\mathbb{S}^1$-family of $\mathfrak{g}$-valued one-forms which satisfies

$$d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0$$

for all $\lambda \in \mathbb{S}^1$. Then there exists a one-parameter family of maps $\Psi^\lambda : \mathbb{D} \to G$ such that $\Psi^\lambda_{-1} d\Psi^\lambda = \alpha^\lambda$, and $\psi^\lambda = \Psi^\lambda_{-1} d\Psi^\lambda \mod K : \mathbb{D} \to G/K$ is a primitive harmonic map for all $\lambda$. 
Appendix B  Projective minimal surfaces and the conformal Gauss maps

Appendix B.1  The conformal Gauss map

Let \( f : M \to \mathbb{R}^3 \) be a surface with Wilczynski frame \( F \) as in Subsection 6.1. We define a map \( g_2 \) by

\[
g_2 = \hat{F} J_2 \hat{F}^T = -\text{Ad}(L)(F J_2 F^T), \quad \hat{J}_2 = -L J_2 L^T,
\]

where \( J_2 = \text{offdiag}(1, -1, -1, -1) \) (see [25, Subsection 4.1]). Analogously to the first-order Gauss map \( g_1 \), \( g_2 \) takes the value in the space \( Q \) of conformal 2-spheres in \( \mathbb{R}^3 \). More precisely, since the matrix \( \hat{J}_2 \) is of signature \((1, 3)\), it is a point of \( Q \). Thus \( Q \) is realized as a homogeneous space \( \text{SL}_4 \mathbb{R}/\hat{K}_2 \), where \( \hat{K}_2 \) is the stabilizer at \( \hat{J}_2 \) explicitly given by \( \hat{K}_2 = \{ X \in \text{SL}_4 \mathbb{R} \mid X J_2 X^T = J_2 \} \), which is also isomorphic to \( \text{SO}^+_{1,3} \). Thus the map \( g_2 \) takes the value in \( \text{SL}_4 \mathbb{R}/\hat{K}_2 \):

\[
g_2 : M \to Q \cong \text{SL}_4 \mathbb{R}/\hat{K}_2 = \text{SL}_4 \mathbb{R}/\text{SO}^+_{1,3}.
\]

This map \( g_2 \) is known to be a Lie quadric which has the second-order contact to the surface (see [17, Section 18]). The map \( g_2 \) has been called the conformal Gauss map for a surface \( f \) in \( \mathbb{R}^3 \) (see [4,30]). In [19], the conformal Gauss map \( g_2 \) was called the projective Gauss map. In the classical literature, \( g_2 \) was called the congruence of Lie quadrics.

**Proposition B.1** (See [4, Theorem 3]). The conformal Gauss map \( g_2 \) is a conformal map.

**Proof.** As in the proof of Proposition 6.1, a direct computation shows that

\[
\partial_z g_2 = -2(\mathbf{L}F) \text{diag}(b \hat{P}, 0, -b, 0)(\mathbf{L}F)^T \quad \text{and} \quad \partial_{\bar{z}} g_2 = -2(\mathbf{L}F) \text{diag}(\bar{b} \hat{P}, -\bar{b}, 0, 0)(\mathbf{L}F)^T.
\]

Thus

\[
\langle \partial_z g_2, \partial_{\bar{z}} g_2 \rangle = \langle \partial_{\bar{z}} g_2, \partial_z g_2 \rangle = 0 \quad \text{and} \quad \langle \partial_z g_2, \partial_{\bar{z}} g_2 \rangle = 4|b|^2 \neq 0.
\]

Since the coordinates \((z, \bar{z})\) are null for the conformal structure induced by \( f \), the conformal Gauss map \( g_2 \) is conformal. \( \square \)

**Remark B.2.** The Hodge star operator \( \ast \) on \( \wedge^2 \mathbb{R}^{1,3} \) is introduced by

\[
\langle a, b \rangle = \Omega(a \wedge \ast b).
\]

Since \( \langle \cdot, \cdot \rangle \) is Lorentzian, \( \ast \) satisfies \( \ast^2 = -1 \). Thus the complexification \( (\wedge^2 \mathbb{R}^{1,3})^C \cong \wedge^2 \mathbb{C}^{1,3} \) has the eigenspace decomposition

\[
(\wedge^2 \mathbb{R}^{4})^C = S \oplus \bar{S},
\]

where \( S \) is the \( \sqrt{-1} \)-eigenspace of \( \ast \). In this way, a quadric \( \mathcal{Q} \in Q \) corresponds to a complex linear subspace \( S \) of \( (\wedge^2 \mathbb{R}^{4})^C \). The correspondence \( Q \mapsto S \) defines a smooth bijection from the space \( Q \) of conformal 2-spheres in \( \mathbb{R}^3 \) to the space

\[
\mathcal{G}^{3,3}_{2,0} = \{ S \subset (\mathbb{R}^{3,3})^C \mid S \cap S^\perp = \{ 0 \}, \bar{S} = S^\perp \}.
\]

Under this identification, \( g_2 \) is regarded as a smooth map into \( \mathcal{G}^{3,3}_{2,0} \) in [4, p. 183] and [8, p. 30].

Appendix B.2  Projective minimal surfaces and the conformal Gauss maps

The space \( Q \) of conformal 2-spheres in \( \mathbb{R}^3 \) is isomorphic to the semi-Riemannian symmetric space \( G/K_2 \), where

\[
K_2 = \{ a \in G \mid a J_2 a^T = J_2 \}.
\]

Let \( \tau_2 \) be the outer involution on \( G \) associated with the symmetric space \( G/K_2 \) defined by

\[
\tau_2(a) = J_2(a^T)^{-1} J_2,
\]
where \( a \in G \). By abuse of notation, we denote the differential of \( \tau_2 \) by the same letter \( \tau_2 \) which is an outer involution on \( g \):

\[
\tau_2(X) = -J_2 X^T J_2,
\]

where \( X \in g \). Let us consider the eigenspace decomposition of \( g \) with respect to \( \tau_2 \), i.e., \( g = \mathfrak{f}_2 \oplus \mathfrak{p}_2 \), where \( \mathfrak{f}_2 \) is the \((+1)\)-eigenspace and \( \mathfrak{p}_2 \) is the \((-1)\)-eigenspace as follows:

\[
\begin{aligned}
\mathfrak{f}_2 &= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\
0 & a_{21} & a_{22} & 0 \\
0 & 0 & -a_{32} & a_{12} \\
0 & a_{31} & a_{21} & -a_{11} \end{pmatrix} \in g \right\}, \\
\mathfrak{p}_2 &= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\
0 & a_{21} & -a_{11} & a_{23} & -a_{13} \\
0 & a_{31} & a_{32} & -a_{11} & -a_{12} \\
0 & a_{41} & a_{31} & a_{21} & a_{11} \end{pmatrix} \in g \right\}.
\end{aligned}
\]

According to this decomposition \( g = \mathfrak{f}_2 \oplus \mathfrak{p}_2 \), the Maurer-Cartan form \( \alpha = F^{-1} dF = U dz + V d\bar{z} \) can be decomposed into

\[
\alpha = \alpha_{\mathfrak{f}_2} + \alpha_{\mathfrak{p}_2} = U_{\mathfrak{f}_2} dz + V_{\mathfrak{f}_2} d\bar{z} + U_{\mathfrak{p}_2} dz + V_{\mathfrak{p}_2} d\bar{z},
\]

where \( U = U_{\mathfrak{f}_2} + U_{\mathfrak{p}_2} \) and \( V = V_{\mathfrak{f}_2} + V_{\mathfrak{p}_2} \). Let us insert the parameter \( \lambda \in \mathbb{S}^1 \) into \( U \) and \( V \) in a manner similar to that in Subsection 6.2:

\[
U^\lambda = U_{\mathfrak{f}_2} + \lambda^{-1} U_{\mathfrak{p}_2} \quad \text{and} \quad V^\lambda = V_{\mathfrak{f}_2} + \lambda V_{\mathfrak{p}_2}.
\]

Then a family of 1-forms \( \alpha^\lambda \) is defined as follows:

\[
\alpha^\lambda = \alpha_{\mathfrak{f}_2} + \lambda^{-1} \alpha'_{\mathfrak{p}_2} + \lambda \alpha''_{\mathfrak{p}_2} = U^\lambda dz + V^\lambda d\bar{z}.
\]

In fact, the matrices \( U^\lambda \) and \( V^\lambda \) are explicitly given as follows:

\[
U^\lambda = \begin{pmatrix}
\frac{b_{25}}{25} & P & k & \lambda^{-1} b P \\
1 & \frac{-b_{25}}{25} & 0 & k \\
0 & \lambda^{-1} b & \frac{b_{25}}{25} & P \\
0 & 0 & 1 & \frac{-b_{25}}{25}
\end{pmatrix}, \quad V^\lambda = \begin{pmatrix}
\frac{b_{25}}{25} & \bar{b} & \bar{P} & \lambda \bar{b} P \\
0 & \frac{b_{25}}{25} & \lambda \bar{b} & \bar{P} \\
1 & 0 & \frac{-b_{26}}{25} & \bar{k} \\
0 & 1 & 0 & \frac{-b_{25}}{25}
\end{pmatrix}.
\]

Then the projective minimal surface can be characterized by the harmonicity of the conformal Gauss map (see [30] and [4, Theorem 7]), and by a family of flat connections.

**Theorem B.3** (See [4, 30]). Let \( f \) be a surface in \( \mathbb{R}P^3 \) and \( g_2 \) be the conformal Gauss map defined in (B.1). Moreover, let \( \{\alpha^\lambda\}_{\lambda \in \mathbb{S}^1} \) be a family of 1-forms defined in (B.3). Then the following are mutually equivalent:

1. The surface \( f \) is a projective minimal surface.
2. The conformal Gauss map \( g_2 \) is a conformal harmonic map into \( Q \).
3. \( \{\alpha^\lambda\}_{\lambda \in \mathbb{S}^1} \) is a family of flat connections on \( \mathbb{D} \times G \).

**Proof.** Let us compute the flatness conditions of \( d + \alpha^\lambda \), i.e., the Maurer-Cartan equation \( d\alpha^\lambda + \frac{1}{2} [\alpha^\lambda \wedge \alpha^\lambda] = 0 \). It is easy to see that except for the \((1, 4)\)-entry, the Maurer-Cartan equation is equivalent to (3.6). Moreover, the \( \lambda^{-1}\)-term and the \( \lambda \)-term of the \((1, 4)\)-entry are equivalent to the first equation and the second equation in (4.1), respectively. Thus the equivalence of (1) and (3) follows.

The equivalence of (2) and (3) follows from Proposition A.4, since the family of 1-forms \( \alpha^\lambda \) is given by the involution \( \tau_2 \) and it defines the semi-Riemannian symmetric space \( Q = \text{SL}_4 \mathbb{R} / K_2 \).

**Remark B.4.** The above theorem implies that if \( f \) is a projective minimal surface, then there exists a family of projective minimal surfaces \( f^\lambda \ (\lambda \in \mathbb{S}^1) \) such that \( f^\lambda|_{\lambda = 1} = f \). Projective minimal surfaces of the family have the same projective metric \( 8|b|^2 \, dz d\bar{z} \) but the different conformal classes of the cubic forms \( \lambda^{-1} b \, dz^3 \).