UNIFORM STATIONARY PHASE ESTIMATE WITH LIMITED SMOOTHNESS

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Abstract. In this paper, we consider the uniform estimate for the oscillatory integral with stationary phase, which was previously studied by Alazard–Burq–Zuily. We significantly reduce the order of required regularity condition on the phase and amplitude functions for the uniform estimate. We also study estimates for the oscillatory integrals of which phase and amplitude functions depend on the oscillation parameter. The novelty of this article lies in the use of the wave packet decomposition, which transforms the decay estimate for the oscillatory integral to the disjointness property of the supports of wave packets. The latter is geometric in its nature and less sensitive to the smoothness of the phase and amplitude functions.

1. Introduction

Let \( \psi \in C_0^\infty(\mathbb{R}^d) \) and \( \Phi \) be smooth on the support of \( \psi \). For \( \lambda \geq 1 \), we set

\[
I(\lambda, \Phi, \psi) = \int e^{i\lambda \Phi(x)} \psi(x) \, dx.
\]

If \( \Phi \) is nondegenerate, that is to say, the hessian matrix of \( \Phi \) is nonsingular on the support of \( \psi \), the stationary phase method gives

\[
|I(\lambda, \Phi, \psi)| \leq B \lambda^{-\frac{d}{2}}
\]

with a constant \( B \) depending on \( \Phi \). (See, for example, Hörmander [6] or Stein [9].) The stationary phase method is an important tool, which has a wide range of applications.

There are various occasions where we need the estimate (1.2) with \( B \) independent of the phase \( \Phi \) and \( \psi \) while they are normalized in a suitable way (see [2, 4, 7]). Besides, a uniform bound on the stationary phase estimate makes it possible to recover the optimal bound for degenerate phase if a suitable decomposition away from the degeneracy is available (see, for example, [3]). In one dimension, thanks to van der Corput’s Lemma it is relatively easy to get the uniform estimate as long as we have a lower bound on the second derivative \( |\Phi''| \) (see [9]). However, in higher dimensions, precise control of the behavior, with respect to the phase and symbol, of the constant \( B \) becomes far less trivial.

Uniform stationary phase estimate. We set \( B_r = \{ x \in \mathbb{R}^d : |x| < r \} \). Without loss of generality, we may assume that \( \text{supp} \, \psi \subset B_1 \) and \( \Phi \) is smooth on \( B_1 \). For \( k \in \mathbb{N}_0 \) and \( 0 < \alpha < 1 \), let us denote by \( C^{k,\alpha} \) the Hölder space. For simplicity we denote \( C^{k+\alpha} = C^{k,\alpha} \). And define \( \| \phi \|_{C^r(B_1)} = \| \phi \|_{C^r(B_1)} + \sum_{|\alpha| = |r|} |D^\alpha \phi|_{C^{0, r-|\alpha|}(B_1)} \)
where $|f|_{C^{0,a}} = \sup_{(x,y) \in \mathbb{B}_1 \times \mathbb{B}_1; x \neq y} |f(x) - f(y)|/|x - y|^{\beta}$ for $0 < \beta < 1$. For $r \geq 2$, we define

$$P_r := \sum_{|\beta| = 2} \|D^\beta \Phi\|_{C^{r-2}(\mathbb{B}_1)}, \quad A_r := \|\psi\|_{C^r(\mathbb{B}_1)},$$

which respectively control the phase and the amplitude functions. Note that $P_r$ is determined by derivatives of degree bigger than or equal to 2. We also set

$$L^*(\Phi) = \inf_{x \in \mathbb{B}_1} |(\det H\Phi(x))|.$$

The following is due to Alazard–Burq–Zuily [1].

**Theorem 1.1** ([1, Theorem 3]). Let $\Phi \in C^{d+2}(\mathbb{B}_1)$ and $\psi \in C^{d+1}_c(\mathbb{B}_{1-\epsilon_0})$ for some small constant $\epsilon_0 > 0$. Suppose that $\mathbb{B}_1 \ni x \mapsto \nabla\Phi(x)$ is injective and $L^*(\Phi) > 0$, then we have

$$(1.3) \quad \lambda^{\frac{d}{2}} |I(\lambda, \Phi, \psi)| \leq \frac{C}{L^*(\Phi)} (1 + P_{d+2}) A_{d+1}$$

with $C$ depending only on $d$ and $\epsilon_0$.

The regularity assumption in Theorem 1.1 seems to be rather too strong and unlikely to be optimal. In this note, we attempt to relax the regularity condition for the uniform stationary phase estimate. However, even for the phase without a stationary point, i.e., $\nabla\Phi \neq 0$ on the support of $\psi$, in order to get the decay of order $\lambda^{-\frac{d}{2}}$ (1.2), it is necessary to carry out integration by parts at least $d/2$ times, so we need to assume that $\Phi, \psi \in C^k$, $k \geq d/2$. In this regard, it is natural to expect that a threshold should be bigger than $d/2$ for the uniform stationary phase estimate to hold.

The following is our first result which significantly improves the regularity requirement in Theorem 1.1. It is remarkable that this brings the order of regularity close to the threshold $d/2$.

**Theorem 1.2.** Let $n_0 > (d + 2)/2$. Let $\Phi \in C^{n_0}(\mathbb{B}_1)$ and $\psi \in C^{(d/2)+1}_c(\mathbb{B}_{1-\epsilon_0})$ for some small constant $\epsilon_0 > 0$. Suppose that $\mathbb{B}_1 \ni x \mapsto \nabla\Phi(x)$ is injective and $L^*(\Phi) > 0$, then there is a constant $C$, depending only on $n_0, \epsilon_0, d$, such that

$$(1.4) \quad \lambda^{\frac{d}{2}} |I(\lambda, \Phi, \psi)| \leq \frac{C}{L^*(\Phi)} (1 + P_{n_0}) A_{\lfloor\frac{d}{2}\rfloor+1}.$$

The approach in [1] relies on integration by parts with respect to the operator $L = |\nabla\Phi|^{-2} \nabla \Phi \cdot \nabla$ to control a part of the integral $I(\lambda, \Phi, \psi)$ which is away from the critical point. This requires extra order of differentiability for phase and amplitude functions. To avoid this, we take a different approach based on the wave packet decomposition which has been widely used in the study of the restriction problem. The localization in the phase space gives a significant advantage in controlling the phase function under weaker regularity assumptions. The primary advantage of wave packet decomposition is to reduce the estimate for the oscillatory integral to a problem of estimating a geometric quantity, more precisely, the degree of overlap between the supports of the wave packets (e.g., Section 2.1).

**Remark 1.** If we remove the injectivity assumption in Theorem 1.2 using [1] Lemma 7, one can easily show that

$$\lambda^{\frac{d}{2}} |I(\lambda, \Phi, \psi)| \leq \frac{C}{(L^*(\Phi))^{d+1}} (1 + P_{n_0}^{2d+2}) A_{\lfloor\frac{d}{2}\rfloor+1}.$$
Oscillatory integral depending on $\lambda$. Now we consider the oscillatory integrals of which phase and amplitude functions are no longer independent of the oscillatory parameter $\lambda$. In such cases, the oscillatory integral estimate becomes more involved than the case where the phase and amplitude are independent of $\lambda$. In practice, those kinds of oscillatory integrals typically appear as results of applying cutoff functions depending on $\lambda$.

Let us set
\[ \tilde{I}(\lambda) := I(\lambda, \Phi_\lambda, \psi_\lambda). \]

Here, we use the notation $\Phi_\lambda$ and $\psi_\lambda$ to indicate that the phase and amplitude functions depend on $\lambda$. We consider $\tilde{I}(\lambda)$ under the following conditions:
\begin{align}
|D^\gamma_2 \psi_\lambda(x)| &\leq \tilde{A} \lambda^{\vert \gamma \vert \beta}, \quad 0 \leq \vert \gamma \vert \leq d + 1, \\
|D^\gamma_2 \Phi_\lambda(x)| &\leq \tilde{P} \lambda^{\beta(\vert \gamma \vert - 2)}, \quad 2 \leq \vert \gamma \vert \leq d + 1,
\end{align}

for $x \in \mathbb{B}_1$ and some positive constants $\tilde{A}$ and $\tilde{P}$. A weaker decay in $\lambda$ is naturally expected by the presence of extra power of $\lambda$ in (1.5) and (1.6). However, for $0 < \beta \leq 1/2$, it is easy to show the best possible decay $\tilde{I}(\lambda) = O(\lambda^{-d/2})$, for example, by adapting the argument used to prove Theorem 1.2 (see Section 2.3). If $\beta$ in (1.5) and (1.6) is bigger than $1/2$, then the bound $O(\lambda^{-d/2})$ is no longer available (see Remark 2 below). Taky [8] showed that $\tilde{I}(\lambda) \leq CL^*(\Phi_\lambda)^{-1} \lambda^{-d(1-\beta)}$ under a restrictive assumption on the phase. More precisely, it was assumed in [8] that there exist $\mu \geq \lambda^{-1+\beta}$ and $C_\gamma > 0$ such that all eigenvalues of $H\Phi_\lambda(x_c)$ are in $[\mu, 2\mu]$ and $|D^\gamma_2 \Phi_\lambda(x)| \leq C_\gamma \mu$ for all multi-indices $\gamma$. Here, $x_c$ denotes the critical point of $\Phi$.

In what follows we show that a similar result remains valid in a more general and natural setting.

**Theorem 1.3.** Let $1/2 \leq \beta < 1$ and $0 < \epsilon_0 \ll 1$. Let $\Phi_\lambda \in C^{d+1}(\mathbb{B}_1)$ and $\psi_\lambda \in C_0^{d+1}(\mathbb{B}_{1-\epsilon_0})$ satisfy (1.5) and (1.6). Suppose that $\mathbb{B}_1 \ni x \mapsto \nabla_\ell \Phi_\lambda(x)$ is injective and $L^*(\Phi_\lambda) > 0$. Then there exists a constant $C$ such that
\begin{equation}
|\tilde{I}(\lambda)| \leq \frac{C}{L^*(\Phi_\lambda)(1 + \tilde{P}^{d\beta})} \tilde{A} \lambda^{-d(1-\beta)}.\end{equation}

**Remark 2.** When $1/2 \leq \beta \leq 1$, the decay rate $d(1-\beta)$ in (1.7) is optimal. To see this, set $\Phi_\lambda(x) = |x|^2$ and $\alpha = 2\beta - 1$. We consider
\[ \psi_\lambda(x) = \sum_{\nu \in \mathbb{Z}^{d+1} \cap \mathbb{B}_{1/2}} \phi(2\lambda^{\alpha}(\lambda^{(1-\alpha)/2}x - \nu)) e^{-i\lambda^{\alpha}|\nu|^2}, \]
where $\phi \in C_0^\infty(\mathbb{B}_1)$ is nonnegative and $\psi \geq 1$ on $\mathbb{B}_{1/2}$. Then, it is easy to see that $\psi$ satisfies (1.5) with $\beta = (\alpha + 1)/2$. Set
\[ I_{\lambda, \nu}(\lambda) = \int e^{i\lambda^{\alpha}(|\lambda^{(1-\alpha)/2}x|^2 - |\nu|^2)} \phi(2\lambda^{\alpha}(\lambda^{(1-\alpha)/2}x - \nu)) dx. \]

Hence, we have $\tilde{I}(\lambda) = \sum_{\nu} I_{\lambda, \nu}(\lambda)$, and $I_{\lambda, \nu}(\lambda) = \lambda^{-d(1+\alpha)/2} \int e^{i\lambda^{\alpha}(2\nu + \lambda^{-\alpha}x)} \phi(2x) dx$ changing variables. Since $|x|, |\nu| \leq 1/2$, $Re(I_{\lambda, \nu}(\lambda)) \geq C^{-1} \lambda^{-d(1+\alpha)/2}$ for some constant $C > 0$. This gives $|\tilde{I}(\lambda)| \geq \lambda^{-d(1-\beta)}$ as desired.

**Remark 3.** If $0 < \beta < 1/2$, we can get the same result under a weaker regularity assumption in a similar manner as in Theorem 1.2. Precisely, we only need (1.5) for $0 \leq \vert \gamma \vert \leq [d/(2 - 2\beta)] + 1$, and (1.6) for $2 \leq \vert \gamma \vert \leq [(d - 2)/(2 - 2\beta)] + 3$. This can
be shown by modifying the proof of Theorem 1.2. Instead of providing the detail we leave its proof to the interested reader.

**Notation.** For positive $A, B$ numbers, we say $A \lesssim B$ if there exists $C$ depending only on $d, \epsilon_0, n_0, \beta$ such that $A \leq CB$.

2. Uniform Stationary Phase Estimates

2.1. Wave packet decomposition. We begin with providing an alternative proof of the result in [1], making use of the wave packet decomposition. In fact, we show

\[(2.1)\]

\[\lambda^{\frac{d}{2}} |I(\lambda, \Phi, \psi)| \lesssim L^* (\Phi)^{-1} (1 + P_{d+1}^{d+1}) A_{d+1}.\]

Note that $I(\lambda, \Phi, \psi) = I(t, \lambda^{-1} \Phi, \psi)$. Taking $t = 1 + P_{d+1}$, we can recover the result of Alazard–Burq–Zuily, that is to say, Theorem 1.1. More precisely, we get a slightly stronger result since we need only to control derivatives of the phase and amplitude up to $(d + 1)$-th order.

Let $\phi$ be a smooth nonnegative function such that $\sum_{k \in \mathbb{Z}^d} \phi(x - k) = 1$ and $\text{supp} \phi \subset \mathbb{B}_1$. For $\nu \in \lambda^{-1/2} \mathbb{Z}^d$, we set

\[I(\lambda, \nu) = \int e^{i \lambda \Phi(x)} \psi(x) \phi(\lambda^{1/2} (x - \nu)) dx.\]

Note that $I(\lambda, \nu) \neq 0$ only if $\nu \in \text{supp} \psi + O(\lambda^{-1/2})$. Since $\text{supp} \psi \subset \mathbb{B}_{1-\epsilon_0}^\beta$, for $\lambda \geq 10/\epsilon_0^2$ we decompose

\[(2.2)\]

\[I(\lambda, \Phi, \psi) = \sum_{\nu \in L_\lambda} I(\lambda, \nu),\]

where $L_\lambda = \lambda^{-1/2} \mathbb{Z}^d \cap \mathbb{B}_1$. For $\nu \in L_\lambda$, we set

\[
\Phi^{\lambda, \nu}(x) = \Phi(\lambda^{-1} x + \nu) - \Phi(\nu) - \lambda^{-1} \nabla \Phi(\nu) \cdot x, \\
\chi^{\lambda, \nu}(x) = e^{i \lambda \Phi^{\lambda, \nu}(x)} \psi(\lambda^{-1} x + \nu) \tilde{\phi}(x),
\]

where $\tilde{\phi} \in C_0^\infty$ is fixed smooth and $\text{supp} \tilde{\phi} \subset \mathbb{B}_1$. Changing variables $x \to \lambda^{-1/2} x + \nu$, we have

\[I(\lambda, \nu) = \lambda^{-\frac{d}{2}} e^{i \lambda \Phi(\nu)} \int e^{i \lambda \frac{1}{2} \nabla \Phi(\nu) \cdot x} \chi^{\lambda, \nu}(x) dx.\]

By the mean value theorem we have

\[(2.4)\]

\[|\nabla (\lambda \Phi^{\lambda, \nu})(x)| \lesssim P_2, \quad x \in \text{supp} \chi^{\lambda, \nu}.
\]

A routine computation shows

\[|\nabla (\lambda \Phi^{\lambda, \nu})(x)| \lesssim P_2, \quad x \in \text{supp} \chi^{\lambda, \nu}.
\]

\[|\nabla (\lambda \Phi^{\lambda, \nu})(x)| \lesssim P_2, \quad x \in \text{supp} \chi^{\lambda, \nu}.
\]

\[(2.6)\]

\[|\nabla (\lambda \Phi^{\lambda, \nu})(x)| \lesssim P_2, \quad x \in \text{supp} \chi^{\lambda, \nu}.
\]

for $|\gamma| \geq 2$ and $0 < \beta < 1$. If $\Phi$ and $\psi \in C^{d+1}$, by (2.4) and (2.5) we have

\[|\chi^{\lambda, \nu}|_{C^{d+1}} \lesssim (1 + P_{d+1}^{d+1}) A_{d+1}.
\]

By integration by parts $(d + 1)$-times, one can easily see that $|I(\lambda, \nu)| \lesssim (1 + P_{d+1}^{d+1}) A_{d+1} (1 + \lambda^{\frac{1}{2}} |\nabla \Phi(\nu)|)^{-d-1}$. Hence,

\[\lambda^{\frac{d}{2}} |I(\lambda, \Phi, \psi)| \lesssim (1 + P_{d+1}^{d+1}) A_{d+1} \sum_{\nu \in L_\lambda} (1 + \lambda^{\frac{1}{2}} |\nabla \Phi(\nu)|)^{-d-1}.
\]
It is now sufficient for (2.1) to show the following lemma, which is in a slightly more general form than we need it here.

**Lemma 2.1.** Let \( k \in \mathbb{R}^d \) and \( \lambda \geq 10/\epsilon_0^2 \). Suppose that \( \mathbb{B}_1 \ni x \mapsto \nabla \Phi(x) \) is injective and \( L^*(\Phi) > 0 \). Then, for a constant \( C \) we have

\[
\sum_{\nu \in \mathcal{L}_\lambda} (1 + |k + \lambda^\frac{1}{2} \nabla \Phi(\nu)|)^{-d-1} \leq C L^*(\Phi)^{-1} (1 + P_2^d).
\]

**Proof of Lemma 2.1.** We split the sum \( \sum_{\nu \in \mathcal{L}_\lambda} (1 + |k + \lambda^\frac{1}{2} \nabla \Phi(\nu)|)^{-d-1} \) into two parts \( I \) and \( II \) which are given by

\[
I = \sum_{\nu \in \mathcal{L}_\lambda: |k + \lambda^\frac{1}{2} \nabla \Phi(\nu)| \geq \sqrt{d} P_2} (1 + |k + \lambda^\frac{1}{2} \nabla \Phi(\nu)|)^{-d-1},
\]

\[
II = \sum_{\nu \in \mathcal{L}_\lambda: |k + \lambda^\frac{1}{2} \nabla \Phi(\nu)| < \sqrt{d} P_2} (1 + |k + \lambda^\frac{1}{2} \nabla \Phi(\nu)|)^{-d-1}.
\]

If \( |k + \lambda^\frac{1}{2} \nabla \Phi(\nu)| \geq \sqrt{d} P_2 \), by the mean value inequality we have \( |k + \lambda^\frac{1}{2} \nabla \Phi(\nu)| \geq 2^{-1} |k + \lambda^\frac{1}{2} \nabla \Phi(x)| \) for \( x \in Q_\nu := \nu + \lambda^{-\frac{1}{2}} [-1/2, 1/2]^d \). Thus,

\[
(1 + |k + \lambda^\frac{1}{2} \nabla \Phi(\nu)|)^{-d-1} \leq C \lambda^\frac{d}{2} \int_{Q_\nu} (1 + |k + \lambda^\frac{1}{2} \nabla \Phi(x)|)^{-d-1} dx.
\]

Consequently, we have

\[
I \lesssim C_d \lambda^\frac{d}{2} \int_{\mathbb{B}_1} (1 + |k + \lambda^\frac{1}{2} \nabla \Phi(x)|)^{-d-1} dx
\]

for some constant \( C_d \). Since \( \mathbb{B}_1 \ni x \mapsto \nabla \Phi(x) \) is injective and since \( L^*(\Phi) > 0 \), changing variables \( x \mapsto (\nabla \Phi)^{-1}(x) \), we get

\[
I \lesssim \int_{\nabla \Phi(\mathbb{B}_1)} (1 + |k + y|)^{-d-1} |\det H\Phi((\nabla \Phi)^{-1}(y))|^{-1} dy \lesssim L^*(\Phi)^{-1}.
\]

Now we consider the sum \( II \). Using \( (1 + |k + \lambda^\frac{1}{2} \nabla \Phi(\nu)|)^{-d-1} \leq 1 \) and following the same argument, we see \( II \) is bounded above by

\[
\lambda^\frac{d}{2} \int_{x \in \mathbb{B}_1: |k + \lambda^\frac{1}{2} \nabla \Phi(x)| \leq 2\sqrt{d} P_2} dx \lesssim \int_{y \in \nabla \Phi(\mathbb{B}_1): |k + y| \leq 2\sqrt{d} P_2} |\det H\Phi((\nabla \Phi)^{-1}(y))|^{-1} dy.
\]

The last is clearly bounded by \( \lesssim L^*(\Phi)^{-1} P_2^d \). \( \square \)

### 2.2. Estimate with reduced regularity.

In the above we have observed that the wave packet decomposition provides an easy way to prove a uniform stationary phase estimate (1.3). We elaborate the argument to lower required order of regularity. To improve the preliminary result, we expand \( \chi^{\lambda,\nu} \) into Fourier series which gives rise to another summation. In order to guarantee absolute summability of the series we need to show the Fourier coefficients of \( \chi^{\lambda,\nu} \) decay fast enough, i.e., \( (\chi^{\lambda,\nu})^\wedge(k) = O(|k|^{-d-\epsilon}) \) for some \( \epsilon > 0 \). However, in view of the typical (integration by parts) argument, we need to differentiate more than \( d \)-times, hence we are forced to assume differentiability of order higher than \( d \). To get over this, we examine \( \chi^{\lambda,\nu} \) carefully.

Recalling that \( \chi^{\lambda,\nu} \) is supported in \( \mathbb{B}_2 \), for \( k \in \mathbb{N}_0^d \) we set

\[
C_k^{\lambda,\nu} = (2\pi)^{-d} \int e^{-ik \cdot x} \chi^{\lambda,\nu}(x) dx.
\]
Proof of Lemma 2.3.

With the implicit constant independent of (2.10) the main contribution comes from $|k| \lesssim \lambda^{1/2}$. Under such a condition, we have the following:

**Lemma 2.2.** Let $n_0 > (d + 2)/2$. Let $\Phi \in C^{n_0}(B_1)$ and $\psi \in C^{[d/2]+1}(B_1 - e_0)$. If $1 \leq |k| \leq 5 \max\{p_2, 1\} \lambda^{\frac{d}{2}}$, then there is a constant $C > 0$, independent of $\lambda, \nu$, such that

$$|C_{k}^{\lambda, \nu}| \leq C A_{[\frac{d}{2}]+1} (1 + P_{n_0}^{d+2}) |k|^{-(d+\epsilon)}$$

for some $\epsilon > 0$.

Compared with the typical argument using integration by parts, the desired decay of Fourier coefficients is obtained under a weaker regularity assumption. The key observation is that differentiations on $\chi^{\lambda, \nu}$ produce extra factors of $\lambda^{-\frac{1}{2}}$ and these factors can be converted to get additional decay because $|k| \lesssim \lambda^{\frac{1}{2}}$.

In order to exploit the extra regularity given by Hölder continuity, we make use of the following lemma which allows us to deal with functions of Hölder continuity of order $\alpha$ as if they were $\alpha$ times differentiable.

**Lemma 2.3.** Let $n$ be an nonnegative integer and $0 \leq \beta < 1$. If $f \in C^{n+\beta}(B_2)$ and $g \in C^{n+1+\beta}(B_2)$, then we have

$$\left| \int f(\lambda^{-p}x) g(x) e^{-ik \cdot x} dx \right| \lesssim \frac{(\lambda^{-p\beta} + |k|^{-1})}{|k|^{n+\beta}} \|f\|_{C^{n+\beta}(B_2)} \|g\|_{C^{n+1+\beta}(B_2)},$$

with the implicit constant independent of $p \geq 0$, $\lambda > 1$.

**Proof of Lemma 2.3.** We begin with recalling the well-known estimate

$$\left| \int h(x) e^{-ik \cdot x} dx \right| \lesssim \|h\|_{C^{\alpha}(B_2)} |k|^{-\beta}$$

for $0 \leq \beta < 1$ whenever $h \in C^{\beta}_{c}(B_2)$. After integration by parts $n$-times using

$$D_{(k)} := \frac{k}{|k|} \cdot \nabla,$$

we need only to show that (2.10) with $n = 0$. Using $2e^{-ik \cdot x} = e^{-ik \cdot x} - e^{-ik \cdot (x + \frac{\pi k}{|k|})}$ and change of variables $x \to x + \frac{\pi k}{|k|^2}$, we see that $\int f(\lambda^{-p}x) g(x) e^{-ik \cdot x} dx$ equals

$$\frac{1}{2} \int \left( f(\lambda^{-p}x) g(x) - f(\lambda^{-p}(x - \frac{\pi k}{|k|^2})) g(x - \frac{\pi k}{|k|^2}) \right) e^{-ik \cdot x} dx.$$

We break $\int f(\lambda^{-p}x) g(x) e^{-ik \cdot x} dx = I + II$, where

$$I = \int \left( f(\lambda^{-p}x) - f(\lambda^{-p}(x - \frac{\pi k}{|k|^2})) \right) g(x) e^{-ik \cdot x} dx,$$

$$II = \int f(\lambda^{-p}(x - \frac{\pi k}{|k|^2})) \left( g(x) - g(x - \frac{\pi k}{|k|^2}) \right) e^{-ik \cdot x} dx.$$
Since $f \in C^\beta$, it is clear that $|I| \leq C\|f\|_{C^\beta}\|g\|_{C^\sigma} \lambda^{-\beta}|k|^{-\beta}$. For $II$, we write
\[ g(x) - g(x - \frac{\pi k}{|k|^2}) = \int_0^{\frac{\pi k}{|k|^2}} k \cdot \nabla g(x - \tau \frac{k}{|k|}) d\tau. \]
Thus,
\[ II = \int_0^{\frac{\pi k}{|k|^2}} f(\lambda^{-\beta} \cdot \tau \frac{k}{|k|}) \frac{k}{|k|} \cdot \nabla g(x - \tau \frac{k}{|k|}) e^{-ik\cdot x} d\tau. \]

Since $f$ and $\nabla g \in C^\beta$, $\|f(\lambda^{-\beta}(\cdot - \frac{k}{|k|^2}) + \tau \frac{k}{|k|})\|_{C^\beta} \lesssim \|f\|_{C^\beta}\|g\|_{C^{1+\beta}}$. So, using (2.11), the inner integral is bounded by $|k|^{-\beta}\|f\|_{C^\beta}\|g\|_{C^{1+\beta}}$ and, hence, we see that $|II| \lesssim |k|^{-1-\beta}\|f\|_{C^\beta}\|g\|_{C^{1+\beta}}$.

We prove Lemma 2.2 by making use of Lemma 2.3. To show (2.9), we exploit the factor $\lambda^{-1/2}$ which is generated by differentiation. Since $n_0 > (d+2)/2$, we may assume
\[ (2.12) \quad n_0 = \frac{d+2}{2} + \alpha \]
for some $0 < \alpha < 1/2$.

**Proof of Lemma 2.2.** We write
\[ (D_k)^{\frac{d}{2}+1} \chi^{\lambda,\nu}(x) = \sum_{m_1+m_2+m_3=\lfloor \frac{d}{2} \rfloor+1} f_{m_1}(x)g_{m_2}(x)h_{m_3}(x), \]
where
\[ f_{m}(x) = D_{m}(\phi_k)(x), \]
\[ g_{m}(x) = D_{m}(\psi(\lambda^{-1/2} x + \nu)), \]
\[ h_{m}(x) = D_{m}(e^{i\lambda^2 x}). \]

Applying integration by parts $[d/2] + 1$ times, we have
\[ (2.13) \quad |C_k^{\lambda,\nu}| \lesssim |k|^{-\lfloor \frac{d}{2} \rfloor-1} \sum_{m_1+m_2+m_3=\lfloor \frac{d}{2} \rfloor+1} \left| \int e^{-ik\cdot x} f_{m_1}(x)g_{m_2}(x)h_{m_3}(x) dx \right|. \]

In order to show (2.9), we further apply integration by parts. However, we need to examine those functions to ensure uniform bounds. The factors $f_{m}$ and $g_{m}$ are easier to handle. Note that $\|f_{m}\|_{C^{d+1}} \lesssim 1$ and
\[ (2.14) \quad \|g_{m}(\lambda^{\frac{d}{2}})\|_{C^{\lfloor \frac{d}{2} \rfloor+1-m}} \lesssim A_{\lfloor \frac{d}{2} \rfloor+1} \lambda^{-\frac{d}{2}} \]
for any $0 \leq m \leq [d/2] + 1$. We first consider the case $m_3 = 0$ and claim
\[ (2.15) \quad \left| \int e^{-ik\cdot x} f_{m_1}g_{m_2}h_{0} dx \right| \lesssim A_{\lfloor \frac{d}{2} \rfloor+1}(1 + P_{n_0}^{d+2-m_2}) \lambda^{-\frac{d}{2}} |k|^{-\lfloor \frac{d}{2} \rfloor-1+m_2}. \]

Being combined with (2.13), this shows the contributions of the terms with $m_3 = 0$ are $O(|k|^{-d-1/2})$ since $|k| \leq 5 \max\{P_2, 1\} \lambda^{1/2}$. To show (2.15), from (2.4),(2.6) we note $|D_{m_3}^{(h)_{0}}| \lesssim (1 + P_{n_0}^{d+2-m_2})$ if $\gamma \leq [d/2] + 1 - m_2$. Thus, via integration by parts $[d/2] + 1 - m_2$ times, using (2.11) we get (2.15).

For the rest of the proof, we assume $m_3 \geq 1$. For $m \geq 1$, we write $h_m$ as
\[ h_{m}(x) = \sum_{n=1}^{m} \left( \sum_{i_1+\cdots+i_n=m, i_1 \leq i_2 \leq \cdots \leq i_n} C_{n,i_1,\ldots,i_n} h_{m,n}^{i_1,\ldots,i_n}(x) \right). \]
where \( C_{n,1,\ldots,i_n} \) is a constant and

\[
h_{i_1,\ldots,i_n}^{m_1,\ldots,m_n}(x) = \prod_{l=1}^{n}(D_{(k)}^iD_{(k)}^j\Phi(x))e^{iA\lambda^{\nu}(x)}.
\]

Note that \( h_{i_1,\ldots,i_n}^{m_1,\ldots,m_n} \in C^{n_0-i_1} \). Recalling (2.12), for (2.9) we only need to show

\[
|\int e^{-ikx}f_{m_1}g_{m_2}h_{i_1,\ldots,i_n}^{m_3,\ldots,m_n}dx| \lesssim A_{2j+1}(1 + P_{n_0}^{d+3-j})\lambda^{-\frac{d-2}{2}}
\]

for each fixed \( m_1, m_2, m_3, n \), and \( i_1, \ldots, i_n \) which satisfies \( i_1 + \ldots + i_3 = m_3 \) and \( 1 \leq i_1 \leq \ldots \leq i_3 \). Combining this with (2.13) immediately shows (2.9).

In order to show (2.16), we write

\[
f_{m_1}(x)g_{m_2}(x)h_{i_1,\ldots,i_n}(x) = F(\lambda^{\frac{1}{2}}x)G(x),
\]

where

\[
F(x) = g_{m_2}(\lambda^{\frac{1}{2}}x)\prod_{i_1 \geq 2}(D_{(k)}^i\Phi(x) + \nu),
\]

\[
G(x) = f_{m_1}(x)\prod_{i_1 = 1}(D_{(k)}^i\Phi(x))e^{iA\lambda^{\nu}(x)}.
\]

By Lemma 2.3, it is enough for (2.16) to prove

\[
\|F\|_{C^{n_0-j}}\|G\|_{C^{n_0-j+1}} \lesssim A_{2j+1}(1 + P_{n_0}^{d+3-j})\lambda^{-\frac{d-2}{2}}
\]

for some integer \( j \in [2, n_0] \). Indeed, by Lemma 2.3 and the assumption \( |k| \leq 5 \max\{p_2, 1\}\lambda^\frac{1}{2} \) we obtain (2.16).

It remains to show (2.17). Since \( \Phi \in C^{n_0}(\mathbb{B}_1) \) and \( \psi \in C_c^{d/2+1}(\mathbb{B}_1-n_0) \), we note \( F \in C_c^{d/2+1-m_2}(\mathbb{B}_1(0)) \cap C_c^{n_0-i_1}(\mathbb{B}_1(-\nu)) \) and \( G \in C^{n_0-1}(\mathbb{B}_2) \) where \( \mathbb{B}_1(-\nu) = \{ x \in \mathbb{R}^d : |x + \nu| < 1 \} \). We show (2.17) by considering the three cases

\[
i_1, m_2 \leq 1, \quad m_2 \geq \max\{i_1, 2\}, \quad i_1 > \max\{m_2, 1\},
\]

separately. First, if \( i_1, m_2 \leq 1 \), then \( F \in C^{n_0-2}(\mathbb{B}_2) \). Using (2.14) and (2.4)–(2.6), we see that (2.17) holds with \( j = 2 \). Secondly, if \( m_2 \geq \max\{i_1, 2\} \), then \( F \in C^{d+1-m_2}(\mathbb{B}_2) \subset C^{n_0-m_2}(\mathbb{B}_2) \). Similarly, by (2.14) and (2.4)–(2.6), it follows that (2.17) holds with \( j = m_2 + 1 \). Lastly, if \( i_1 > \max\{m_2, 1\} \), then \( F \in C^{n_0-i_1}(\mathbb{B}_2) \). Since \( F \) includes \( \lambda^{-\frac{d-2}{2}} \) as one of its factors, using (2.14) and (2.4)–(2.6), we see that (2.17) holds with \( j = i_1 \).

We are now ready to prove Theorem 1.2

**Proof of Theorem 1.2.** We may assume that \( \lambda \geq 10/\epsilon_0^2 \). We first consider the case \( |\nabla \Phi(0)| \geq 2 \max\{p_2, 1\} \). In this case, we have \( |\eta \cdot \nabla \Phi(x)| \geq \max\{p_2, 1\} \) for \( |x| \leq 1 \) where \( \eta = |\nabla \Phi(0)|^{-1} \nabla \Phi(0) \). Using the operator \( |\eta \cdot \nabla \Phi(0)|^{-1}(\eta \cdot \nabla \), via integration by parts \((d/2+1)\)–times we get

\[
|I(\lambda, \Phi, \psi)| \lesssim (\max\{p_2, 1\})^{\lambda^{-\frac{d}{2}}-1} A_{\frac{d}{2}+1}(1 + P_{[d/2]+1}^{d+3}).
\]

Thus, we may now assume \(|\nabla \Phi(0)| < 2 \max\{p_2, 1\} \). Recalling (2.8), we split the sum over \( k \) into two parts to get

\[
|\lambda^{\frac{d}{2}} I(\lambda, \Phi, \psi)| \leq \sum_{\nu \in \mathbb{Z}_\lambda} (A(\nu) + B(\nu)),
\]

where \( A(\nu) \) and \( B(\nu) \) are defined in (2.9).
where
\[
A(\nu) := \sum_{|k| \geq 5 \max\{\mathcal{P}_2, 1\} \lambda^2} |C_k^{\lambda, \nu}||\hat{\phi}(\lambda^{\frac{1}{2}} \nabla \Phi(\nu) + k)|, \\
B(\nu) := \sum_{|k| < 5 \max\{\mathcal{P}_2, 1\} \lambda^2} |C_k^{\lambda, \nu}||\hat{\phi}(\lambda^{\frac{1}{2}} \nabla \Phi(\nu) + k)|.
\]

We first deal with \(A(\nu)\). Since \(|\nabla \Phi(0)| < 2 \max\{\mathcal{P}_2, 1\}\), we have \(|\lambda^{\frac{1}{2}} \nabla \Phi(\nu)| \leq 4 \max\{\mathcal{P}_2, 1\} \lambda \frac{1}{2}\) for \(\nu \in \mathcal{L}_\lambda\). Since \(|k| \geq 5 \max\{\mathcal{P}_2, 1\} \lambda \frac{1}{2}\), we have \(|\lambda^{\frac{1}{2}} \nabla \Phi(\nu) + k| \sim |k|\).

By the rapid decay of \(\hat{\phi}\) it follows that
\[
|\hat{\phi}(\lambda^{\frac{1}{2}} \nabla \Phi(\nu) + k)| \leq (1 + |\lambda^{\frac{1}{2}} \nabla \Phi(\nu) + k|)^{-d-1}|k|^{-d-1}
\]
for \(|k| \geq 5 \max\{\mathcal{P}_2, 1\} \lambda \frac{1}{2}\). Since \(|C_k^{\lambda, \nu}| \leq \int |\psi(\lambda^{-\frac{1}{2}} x + \nu) \hat{\phi}(x)|\,dx \lesssim A_0\), we get
\[
A(\nu) \lesssim A_0 \sum_{|k| \geq 5 \max\{\mathcal{P}_2, 1\} \lambda^2} |k|^{-d-1}(1 + |\lambda^{\frac{1}{2}} \nabla \Phi(\nu) + k|)^{-d-1}.
\]

Thus, Lemma 2.1 yields
\[
(2.18) \quad \sum_{\nu \in \mathcal{L}_\lambda} A(\nu) \lesssim L^*(\Phi)^{-1} A_0(1 + \mathcal{P}_2^d).
\]

We now consider \(\sum_{\nu} B(\nu)\). Since \(|k| \leq 5 \lambda \frac{1}{2} \max\{\mathcal{P}_2, 1\}\), by (2.9) we have
\[
B(\nu) \lesssim A_{1/2} + 1(1 + \mathcal{P}_{n_0}^{d+2}) \sum_{|k| \leq 5 \max\{\mathcal{P}_2, 1\} \lambda^{\frac{1}{2}}} |k|^{-d-1}|k|^{-d-1}(1 + |\lambda^{\frac{1}{2}} \nabla \Phi(\nu) + k|)^{-d-1}.
\]

The rest of the argument is the same as before. In fact, using Lemma 2.1 we get
\[
\sum_{\nu \in \mathcal{L}_\lambda} B(\nu) \lesssim L^*(\Phi)^{-1} A_{1/2} + 1(1 + \mathcal{P}_{n_0}^{d+2})(1 + \mathcal{P}_2^d).
\]

Combining this and (2.18), we obtain
\[
\lambda^{\frac{d}{2}} |\hat{I}(\lambda, \Phi, \psi)| \lesssim L^*(\Phi)^{-1} A_{1/2} + 1(1 + \mathcal{P}_{n_0}^{2d+2}).
\]

Finally, since \(I(\lambda, \Phi, \psi) = I(\lambda, t^{-1} \Phi, \psi)\), taking \(t = 1 + \mathcal{P}_{n_0}\) we obtain (1.4). \(\square\)

### 2.3 Phase and amplitude depending on the oscillatory parameter \(\lambda\).

From the previous argument we can expect that, as long as the regularity assumption is high enough, a similar result holds even if \(\psi_\lambda(x)\) has some bad behavior in \(\lambda\).

Similarly as before, let us set
\[
\hat{\Phi}^{\lambda, \nu}_{\lambda} := (\Phi^{\lambda, \nu})^{\lambda, \nu}, \\
\hat{\chi}^{\lambda, \nu}_{\lambda}(x) := e^{i \lambda^{\frac{1}{2}} \Phi^{\lambda, \nu}_{\lambda}(x)} \psi_\lambda(\lambda^{\frac{1}{2}} x + \nu) \hat{\phi}(x).
\]

By the same decomposition as in Section 2.1 with \(\hat{\Phi}_\lambda, \psi_\lambda\), we have an analogue of (2.8):
\[
(2.19) \quad \lambda^{\frac{d}{2}} |\hat{\Phi}(\lambda)| \leq \sum_{k \in \mathbb{Z}^d} \sum_{\nu \in \mathcal{L}_\lambda} |(\hat{\chi}^{\lambda, \nu}_{\lambda})^{\lambda, \nu}(k)||\hat{\phi}(\lambda^{\frac{1}{2}} \nabla \Phi(\nu) + k)|.
\]

For the case \(0 \leq \beta \leq 1/2\), the optimal decay estimate \(\lambda^{-d/2}\) is relatively easy to show. In fact, the estimate follows once we have the desired estimate
\[
(2.20) \quad \|\hat{\chi}^{\lambda, \nu}_{\lambda}\|_{C_2 + 1} \lesssim (1 + \hat{\Phi}^{d+1}) \tilde{A}
\]
for $0 \leq \beta \leq 1/2$. From (1.5) it is clear that
\begin{equation}
|D^\gamma_x (\psi \lambda (x/2 + \nu))| \leq \tilde{A} \lambda^{-(\beta - \frac{d}{2})}, \quad 0 \leq |\gamma| \leq d + 1.
\end{equation}
From (1.6), we also have $|\nabla_x \Phi_\lambda^{\lambda, \nu}(x)| \leq \tilde{P}$ and
\begin{equation}
|D^\gamma_x \Phi_\lambda^{\lambda, \nu}(x)| \leq \tilde{P} \lambda^{-(\beta - \frac{d}{2})}, \quad 2 \leq |\gamma| \leq d + 1.
\end{equation}
Combining those estimates, one can easily obtain (2.20). By (2.20) it follows that
\begin{equation}
|D^\gamma_x e^{i \lambda (x/2 + \nu)}| \leq (1 + \tilde{P}^{d+1}) \lambda^{-(\beta - \frac{d}{2})} \lambda^{-(\beta - \frac{d}{2})}, \quad 0 \leq |\gamma| \leq d + 1.
\end{equation}
Note that the power of $\lambda$ is positive. Combining this and (2.21), by integration by parts $d + 1$ times we get
\begin{equation}
|\lambda^{\gamma} e^{i \lambda (x/2 + \nu)}(x)| \leq (1 + \tilde{P}^{d+1}) \lambda^{(\beta - \frac{d}{2})(d+1)} |\gamma|^{d-1}.
\end{equation}
(2.22) and the trivial estimate $|\lambda^{\gamma} e^{i \lambda (x/2 + \nu)}(x)| \leq \tilde{A}$ give
\begin{align*}
\sum_{k \in \mathbb{Z}^d} |\lambda^{\gamma} e^{i \lambda (x/2 + \nu)}(x)| &\leq \sum_{k \leq \lambda^{\frac{d}{2}}} \tilde{A} + \sum_{|k| \geq \lambda^{\beta - \frac{d}{2}}} \tilde{A} \lambda^{(\beta - \frac{d}{2})(d+1)} |k|^{-d-1} \\
&\leq \tilde{A} \lambda^{d(\beta - \frac{d}{2})} + (1 + \tilde{P}^{d+1}) \tilde{A} \lambda^{d(\beta - \frac{d}{2})}.
\end{align*}
Thus, we obtain $\lambda^{\frac{d}{2}} |\tilde{I}(\lambda)| \leq L^* (\Phi_\lambda)^{-1} (1 + \tilde{P}^{2d+1}) \tilde{A}$. Recalling $I(\lambda, \Phi_\lambda, \psi_\lambda) = I(t\lambda, t^{-1} \Phi_\lambda, \psi_\lambda)$ and taking $t = 1 + \tilde{P}$, we see that (1.7) holds.

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