The connection problem associated with a Selberg type integral and the \(q\)-Racah polynomials

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Abstract

The connection problem associated with a Selberg type integral is solved. The connection coefficients are given in terms of the \(q\)-Racah polynomials. As an application of the explicit expression of the connection coefficients, examples of the monodromy-invariant Hermitian form of non-diagonal type are presented. It is noteworthy that such Hermitian forms are intimately related with the correlation functions of non-diagonal type in \(sl_2\)-conformal field theory.

Key Words and Phrases. A Selberg type integral, connection problem, connection coefficient, \(q\)-Racah polynomial, twisted homology, monodromy-invariant Hermitian form, conformal field theory, correlation functions of non-diagonal type.

Introduction

A Selberg type integral

\[
\int_{\gamma} \prod_{1 \leq i < j \leq m} (t_j - t_i)^g \prod_{1 \leq i \leq m} (t_i - z_j)^{\lambda_j} dt_1 \cdots dt_m, \tag{0.1}
\]

where \(g\) and \(\lambda_j\) are complex numbers and \(\gamma\) is a suitable cycle, is a natural generalization of the Gauss hypergeometric function and the Selberg integral. It is used to express conformal blocks in conformal field theory \cite{7} \cite{8} \cite{9} \cite{25} \cite{30} \cite{33} and to represent the hypergeometric function associated with a root system due to Heckman and Opdam \cite{12} \cite{20}. The integral (0.1) can be thought of as an element of the pairing between the de Rham cohomology group and the twisted homology group (the homology group with coefficients in local system). Such a viewpoint to the integral representation of special functions was introduced by Aomoto around 1970, and have been developed after the name of the \emph{twisted de Rham theory} \cite{1} \cite{2} \cite{3}.
The main purpose of this article is to solve the connection problem associated with a special case of (0.1):

\[ \int \gamma \prod_{1 \leq i < j \leq m} (t_j - t_i)^9 \prod_{1 \leq i \leq m} t_i^a (1 - t_i)^b (t_i - z)^c \ dt_1 \cdots dt_m, \]  

(0.2)

which satisfies an ordinary differential equation of order \( m + 1 \) with three regular singular points 0, 1 and \( \infty \). The connection problem we mean here is to give linear relations between the fundamental sets of solutions around the singularities, moreover, to write down such coefficients explicitly.

Generally speaking, the connection problem is important to know the global property of the solution space of a given differential equation, but only rare cases are known to be solved.

In the case of the Gauss hypergeometric function, Kummer discovered the relations between the fundamental sets of solutions around three singularities in 1836 [19] (Afterwards, it was found in his nachlass that Gauss had also discovered such relations in 1812). In the case of the generalized hypergeometric function \( _nF_{n-1} \) for \( n \geq 3 \), Thomae obtained the coefficients between the solutions around 0 and those around \( \infty \) in 1870 (See also [35] [29] and [22]). These are all the cases of regular singular type in which the connection problem is solved.

Contents in this article is the following. The sets of solutions around 0 and 1 are given in Proposition 2.1, and those around 0 and \( \infty \) in Proposition 2.6. The connection formula which connects the solutions in Proposition 2.1 is given in Theorem 2.3-4, and the formula which connects the solutions in Proposition 2.6 is given in Theorem 2.7-8. In particular, the connection coefficients in Theorem 2.4 and Theorem 2.8 are represented by the \( q \)-Racah polynomials. The \( q \)-Racah polynomials are essentially the same as the \( q \)-6j symbol and are known to be ingredients to construct some kinds of link invariants including the Jones polynomial [14]. It is also noteworthy that the formulas in Theorem 2.3-4 correspond to the braiding matrices and the formulas in Theorem 2.7-8 to the fusion matrices in the context of conformal field theory.

The connection formulas in Section 2 are actually derived in Section 3 in a unified form; this section is separated into two parts. The first part is devoted to the manipulation of twisted cycles to obtain the connection formula in Proposition 3.3. We note that Theorem 2.3 and Theorem 2.7 are two special cases of Proposition 3.3. Next, the second part is devoted to the change of the expression of Proposition 3.3 into several forms by means of the transformation formulas of the basic hypergeometric series. The expression in Proposition 3.4 is in terms of \( 4\phi_7 \), the expressions in Proposition 3.5 are in terms of \( 4\phi_3 \), and the expression in Proposition 3.5 is in terms of the \( q \)-Racah polynomial.

Finally, in Section 4, examples of the monodromy-invariant Hermitian form of non-diagonal type are presented in Theorem 4.3-4. These are intimately related with the correlation functions of non-diagonal type in conformal field theory classified by Kato [13] and Cappelli-Itzykson-Zuber [5].

It is also noteworthy that the intersection number of twisted cycles associated
with the function
\[ u(t) = \prod_{1 \leq i < j \leq m} (t_j - t_i)^g \prod_{1 \leq i \leq m} (t_i - z_j)^{-g/2} \]

is used to construct the Jones polynomial of link invariant \([23]\). On the other hand, \(q\)-6j symbols, or the \(q\)-Racah polynomials are used in \([14]\) to construct the Jones polynomial. To clarify the linkage of the \(q\)-Racah polynomials with the intersection number of twisted cycles is our future problem.

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1 Preliminaries

1.1 Twisted homology groups

Let \( u(t) = \prod_i f_i(t)^{\alpha_i} \) be a multivalued function on \( T \subset \mathbb{C}^m \), where \( \alpha_i \in \mathbb{C} \) and \( T \) is the complement of the singular locus \( \bigcup_i \{ t = (t_1, \ldots, t_m) \in \mathbb{C}^m \ | \ f_i(t) = 0 \} \) in \( \mathbb{C}^m \). Let \( \mathcal{L} \) be the local system (locally constant sheaf) defined by \( u \) the sheaf consisting of the local solutions of \( dL = L\omega \) for \( \omega = du(t)/u(t) \).

Let \( H_m(T, \mathcal{L}) \) be the \( m \)-th homology group with coefficients in \( \mathcal{L} \), \( H^m_m(T, \mathcal{L}) \) the \( m \)-th locally finite homology group with coefficients in \( \mathcal{L} \). Elements of these twisted homology groups, called twisted cycles or loaded cycles, are represented by \( \partial \)-closed twisted (finite or locally finite) chains

\[
C = \sum_\rho a_\rho \rho \otimes v_\rho, \quad (a_\rho \in \mathbb{C}),
\]

where each \( \rho \) is an \( m \)-simplex and \( v_\rho \) a section of \( \mathcal{L} \) on \( \rho \). The boundary operator \( \partial \) is defined to be a \( \mathbb{C} \)-linear mapping satisfying

\[
\partial(\rho \otimes v) = \sum_{i=0}^m (-1)^i \rho^i \otimes v|_{\rho^i},
\]

where \( \rho \) is an \( m \)-simplex, \( \rho^i \) denotes the \( i \)-th face of \( \rho \), and \( v|_{\rho^i} \) is the restriction of \( v \) on \( \rho^i \).

If each factor \( f_i(t) \) of \( u(t) \) is defined over \( \mathbb{R} \), and \( D \) is a domain of the real manifold \( T_R \) (the real locus of \( T \)), then it is convenient to load \( D \) with a section

\[
u_D(t) = \prod_i (\epsilon_i f_i(t))^{\alpha_i}
\]

of \( \mathcal{L} \) on \( D \), and to make a loaded cycle \( D \otimes u_D(t) \), where \( \epsilon_i = \pm \) is so determined that \( \epsilon_i f_i(t) \) is positive on \( D \), and the argument of \( \epsilon_i f_i(t) \) is assigned to be zero. This choice of a section is said to be standard.

In this paper, we adopt mainly the standard loading. Thus, we frequently omit the assignment of loading and denote just the topological cycles for simplicity. For example, in case \( T = \mathbb{C}\backslash\{0,1\} \) and \( u(t) = t^\alpha (1-t)^\beta \), we denote by \( (0,1) \) to express \( (0,1) \otimes u(t) \), and \( (1,\infty) \) for \( (1,\infty) \otimes t^\alpha (1-t)^\beta \).

Under some genericity condition on the exponents \( \alpha_i \), we have the isomorphism, called the regularization,

\[
\operatorname{reg} : H_m(T, \mathcal{L}) \rightarrow H_m(T, \mathcal{L}),
\]

which is the inverse of the natural map \( \iota : H_m(T, \mathcal{L}) \rightarrow H^m_m(T, \mathcal{L}) \).

For example, in case \( T = \mathbb{C}\backslash\{0,1\} \) and \( u(t) = t^\alpha (1-t)^\beta \), where \( \alpha, \beta, \alpha + \beta \in \mathbb{R}\backslash\mathbb{Z} \), a regularization (regularized cycle) \( \operatorname{reg} C \in H_1(T, \mathcal{L}) \) of \( C = (0,1) \in H_1^m(T, \mathcal{L}) \) can be given by

\[
\operatorname{reg} C = \left\{ \frac{1}{d_\alpha} S(\epsilon; 0) + \frac{1}{d_\epsilon} S(1 - \epsilon; 1) - \frac{1}{d_\beta} S(1 - \epsilon; 1) \right\} \otimes u(t).
\]
Here $d_a = e(2a) - 1$ with $e(A) = \exp(\pi \sqrt{-1}A)$, $\epsilon$ is a small positive number, the symbol $S(a; z)$ stands for the positively oriented circle centered at the point $z$ with starting and ending at the point $a$, and the argument of each factor of $u(t)$ on $S(\epsilon; 0)$ or $S(1 - \epsilon; 1)$ is defined so that $\arg t$ takes values from 0 to $2\pi$ on $S(\epsilon; 0)$, and $\arg(1 - t)$ from 0 to $2\pi$.

We refer the reader to [15] for the construction of regularized cycles in higher dimensional cases.

The intersection form

$$\bullet : H^H_{m}(T, L) \times H^H_{m}(T, L) \longrightarrow \mathbb{C}$$

is the Hermitian form defined by

$$(C, C') \mapsto C \bullet C' = \sum_{\rho, \sigma} a_{\rho} \overline{a_{\sigma}} \sum_{t \in \rho \cap \sigma} I_{x}(\rho, \sigma) u_{\rho}(t) \overline{u_{\sigma}(t)}/|u|^{2}$$

for $C, C' \in H^H_{m}(T, L)$, if $\text{reg} C$ and $C'$ are represented by

$$\text{reg} C = \sum_{\rho} a_{\rho} \rho \otimes v_{\rho}, \quad C' = \sum_{\sigma} a'_{\sigma} \sigma \otimes v'_{\sigma},$$

where $a_{\rho}, a'_{\sigma} \in \mathbb{C}$, each $\rho$ or $\sigma$ is an $m$-simplex, $v_{\rho}$ or $v'_{\sigma}$ a section of $L$ on $\rho$ or $\sigma$, $\overline{\cdot}$ the complex conjugation, and $I_{x}(\rho, \sigma)$ the topological intersection number of $\rho$ and $\sigma$ at $x$. The value $C \bullet C'$ of the intersection form for $C, C' \in H^H_{m}(T, L)$ is called the intersection number of $C$ and $C'$.

For example, if the local system $L$ is defined by

$$u(t) = \prod_{i=1}^{m} t_{i}^{\alpha} (1 - t_{i})^{\beta} \prod_{1 \leq i < j \leq m} (t_{j} - t_{i})^{2\gamma},$$

where $\alpha, \beta, \gamma \in \mathbb{R} \setminus \mathbb{Z}$ with some genericity condition on $\alpha, \beta, \gamma$, and

$$C := \sum_{\sigma \in S_{m}} C_{\sigma},$$

where $C_{\sigma} = D_{\sigma} \otimes u_{D_{\sigma}}(t)$ and $D_{\sigma}$ is a bounded domain

$$\{ (t_{1}, \ldots, t_{m}) \in \mathbb{R}^{m} \mid 0 < t_{\sigma(1)} < \cdots < t_{\sigma(m)} < 1 \}$$

with the standard orientation, then we have the self-intersection number

$$J_{m}(\alpha, \beta, \gamma) := C^{2} = C \bullet C$$

$$= m! \left( \frac{\sqrt{-1}}{2} \right)^{m} \prod_{j=1}^{m} \frac{s(\alpha + \beta + (m + j - 2)\gamma)s(\gamma)s(\alpha + (j - 1)\gamma)s(\beta + (j - 1)\gamma)s(\gamma)}{s(\alpha + (j - 1)\gamma)s(\beta + (j - 1)\gamma)s(\gamma)}.$$

(1.1)

where $s(A) = \sin(\pi A)$. We refer the reader to [17] [18] [24] [25] [26] for more details of the intersection numbers of twisted cycles.
1.2 A Selberg type integral

Let $\mathcal{L}_z$ be the local system determined by a function

$$u(t) = \prod_{1 \leq i < j \leq m} (t_j - t_i)^g \prod_{1 \leq i \leq m} t_i^g (1 - t_i)^b (t_i - z)^c$$

on the domain

$$T_z = \{ t = (t_1, \ldots, t_m) \in \mathbb{C}^m \mid t_i \neq t_j \ (i \neq j), \ t_i \neq 0, 1, z \}.$$ 

for $z \in \mathbb{C} \backslash \{0, 1\}$. Let $H^H_m(T_z, \mathcal{L}_z)_{\mathfrak{S}_m}$ and $H_m(T_z, \mathcal{L}_z)_{\mathfrak{S}_m}$ stand for the anti-symmetric part of $H^H_m(T_z, \mathcal{L}_z)$ and $H_m(T_z, \mathcal{L}_z)$ with respect to the action of the symmetric group $\mathfrak{S}_m$ on the coordinate $t = (t_1, \ldots, t_m)$ of $T_z$.

Under the genericity condition on the exponents is

$$0 \leq a, b, c \leq m$$

and the natural map $\nu: H_m(T_z, \mathcal{L}_z)_{\mathfrak{S}_m} \to \dim H^H_m(T_z, \mathcal{L}_z)_{\mathfrak{S}_m}$ is an isomorphism (See [3][6][16][24]). Here the genericity condition on the exponents is that none of the following is an integer:

$$ia + \left( \frac{i}{2} \right) g, \ ib + \left( \frac{i}{2} \right) g, \ ic + \left( \frac{i}{2} \right) g, \ i\lambda_\infty + \left( \frac{i}{2} \right) g, \ \left( \frac{1}{2} \right) g, \ \ (1 \leq i \leq m)$$

where

$$\lambda_\infty = -a - b - c - (m - 1)g \ \text{and} \ \left( \frac{1}{2} \right) = 0.$$ 

In this paper, the genericity condition is assumed and the inverse map

$$\text{reg} : H^H_m(T_z, \mathcal{L}_z)_{\mathfrak{S}_m} \to H_m(T_z, \mathcal{L}_z)_{\mathfrak{S}_m}$$

is freely used.

It is noteworthy that (1.2), or more directly

$$\dim H^H_m(T_z, \mathcal{L}_z)_{\mathfrak{S}_m} = m + 1,$$

where $H^m(T_z, \mathcal{L}_z)_{\mathfrak{S}_m}$ is the anti-symmetric part of the twisted de Rham cohomology $H^m(T_z, \mathcal{L}_z)$ ( $\mathcal{L}_z$ is the sheaf of the local solutions of $dL = -\omega L, \omega = du(t)/u(t)$) guarantees the existence of the ordinary differential equation of order $m + 1$ which is satisfied by the Selberg type integral

$$\int \prod_{1 \leq i < j \leq m} (t_j - t_i)^g \prod_{1 \leq i \leq m} t_i^g (1 - t_i)^b (t_i - z)^c \ dt_1 \cdots dt_m,$$  \hspace{1cm} (1.3)
where $\gamma$ is a suitable cycle. Indeed, we have the following differential equation of the first order with matrix coefficients: (20): For $0 \leq i \leq m$, set

$$\tilde{\varphi}_i = \sum_{\sigma \in S_m} \left\{ \prod_{1 \leq s \leq i} t_{\sigma(s)}^{-1} \prod_{i < s \leq m} (t_{\sigma(s)} - 1)^{-1} \right\},$$

which corresponds to an element of the basis of the twisted de Rham cohomology $H^m(T, \mathcal{L}^\vee)_{S_m}$. For the fixed cycle $\gamma$, set

$$\langle \varphi \rangle = \int_{\gamma} \varphi(t) \, dt_1 \cdots dt_m.$$

Then we have

$$\frac{d}{dz} \langle \tilde{\varphi}_0 \rangle = \frac{m}{z-1} \left\{ (b + c + (m - 1) \frac{g}{2}) \langle \tilde{\varphi}_0 \rangle + a \langle \tilde{\varphi}_1 \rangle \right\},$$

$$\frac{d}{dz} \langle \tilde{\varphi}_m \rangle = \frac{m}{z} \left\{ (a + c + (m - 1) \frac{g}{2}) \langle \tilde{\varphi}_m \rangle + b \langle \tilde{\varphi}_{m-1} \rangle \right\}$$

and

$$\frac{d}{dz} \langle \tilde{\varphi}_i \rangle = \frac{i}{z} \left\{ (a + c + (i - 1) \frac{g}{2}) \langle \tilde{\varphi}_i \rangle + (b + (m - i) \frac{g}{2}) \langle \tilde{\varphi}_{i-1} \rangle \right\}$$

$$+ \frac{m - i}{z - 1} \left\{ (b + c + (m - i - 1) \frac{g}{2}) \langle \tilde{\varphi}_i \rangle + (a + i \frac{g}{2}) \langle \tilde{\varphi}_{i+1} \rangle \right\},$$

for $1 < i < m$. This system of equations induces the the scalar-valued differential equation satisfied by (1,3), the order of which is $m + 1$ and the characteristic exponents $e_j^{(0)}, e_j^{(1)}, e_j^{(\infty)}$ of which at the singularities $0, 1, \infty$ are given by

$$e_j^{(0)} = (a + c + 1)j + \binom{j}{2}g,$$

$$e_j^{(1)} = (b + c + 1)j + \binom{j}{2}g,$$

$$e_j^{(\infty)} = -(a + b + 1)j - cm - \binom{j}{2} + j(m - j)g$$

for $0 \leq j \leq m$.

When $m = 1$, the differential equation satisfied by (1,3) is

$$z(z - 1)I'' + \{a + c - (a + b + 2c)z\}I' + c(a + b + c + 1)I = 0,$$

(1.4)

which is nothing but the hypergeometric differential equation.
When \( m = 2 \), it is
\[
z^2(z - 1)^2 I''' + (K_1 z + K_2(z - 1)) z(z - 1) I''
+ (L_1 z^2 + L_2(z - 1)^2 + L_3 z(z - 1)) I' + (M_1 z + M_2(z - 1)) I = 0 \tag{1.5}
\]
with
\[
K_1 = -g - 3b - 3c, \quad K_2 = -g - 3a - 3c,
L_1 = (b + c)(2b + 2c + g + 1), \quad L_2 = (a + c)(2a + 2c + g + 1),
L_3 = (b + c)(2a + 2c + g + 1) + (a + b + c) + (3c + g)(a + b + c + g + 1),
M_1 = -c(2b + 2c + g + 1)(2a + 2b + 2c + g + 2),
M_2 = -c(2a + 2c + g + 1)(2a + 2b + 2c + g + 2),
\]
which was first derived by Dotsenko-Fateev \[8\].

In more general \( m \) case, such an explicit expression is not known.

Let \( V = \sum_{1 \leq i \leq l} C_i \subset H^m_{\text{lf}}(T_z, \mathcal{L}_z) \otimes \mathbb{C} \) be an invariant subspace under the action of the fundamental group \( \pi_1(z, \mathbb{C}\setminus\{0,1\}) \). If the loaded cycle \( C_i \) is expressed as \( \sum_{\rho} a_{i\rho} \rho \otimes u_{\rho}(t) \), where each \( \rho \) is an \( m \)-simplex in \( T_z \), and \( u_{\rho}(t) \) a section of \( \mathcal{L}_z \) on \( \rho \), then we define a function \( I_i(z) \) by the integral
\[
\sum_{\rho} a_{i\rho} \int_{\rho} u_{\rho}(t) dt_1 \cdots dt_m.
\]
Let
\[
I_h = (C_i \cdot C_j)_{1 \leq i,j \leq l}
\]
be the intersection matrix. Then, the Hermitian form
\[
F(z, \overline{z}) = \sum_{1 \leq i,j \leq l} (I_h^{-1})_{ij} \overline{I_i(z)I_j(z)} \tag{1.6}
\]
is invariant under the action of \( \pi_1(z, \mathbb{C}\setminus\{0,1\}) \).

For a while, we assume that \( z \) is real and \( 0 < z < 1 \). Let us define \( C_k \) to be
\[
C_k = \sum_{\sigma \in \mathcal{S}_m} \sigma \left( \Delta_k(t) \otimes u_{\Delta_k(t)} \right), \quad 0 \leq k \leq m \tag{1.7}
\]
where
\[
\Delta_k(t) = \{ t \mid 0 < t_1 < \cdots < t_k < z, 1 < t_{k+1} < \cdots < t_m \}.
\]
Then $C_0, \ldots, C_m$ form a basis of $H^k_m(T_z, \mathcal{L}_z)^{-m}$. Note that $C_k$ is equal to the domain

$$
\binom{m}{k} \sum_{\sigma \in S_k} \{ (t_1, \ldots, t_k) \mid 0 < t_{\sigma(1)} < \cdots < t_{\sigma(k)} < z \} \times \sum_{\sigma \in S_{m-k}} \{ (t_{k+1}, \ldots, t_m) \mid 1 < t_{\sigma(k+1)} < \cdots < t_{\sigma(m)} < \infty \}
$$

standardly loaded with $u(t)$. Hence, (1.1) implies that

$$
C_k \cdot C_k = \binom{m}{k} J_k(a, c, g/2) J_{m-k}(b, -a - b - c - (m-1)g, g/2).
$$

(1.8)

Moreover, since $C_0, \ldots, C_m$ are mutually disjoint, if we set

$$
I_k(z) = \langle C_k, dt_1 \cdots dt_m \rangle = m! \int_{\Delta_k(t)} u_{\Delta_k} dt_1 \cdots dt_m,
$$

we obtain the monodromy-invariant Hermitian form

$$
F(z, \bar{z}) = \frac{1}{m!} \left( \frac{2}{\sqrt{-1}} \right)^m \sum_{k=0}^{m} \prod_{j=1}^{k} \frac{s \left( a + (j-1)\frac{g}{2} \right) s \left( c + (j-1)\frac{g}{2} \right) s \left( \frac{j\frac{g}{2}}{2} \right)}{s \left( a + c + (k + j - 2)\frac{g}{2} \right) s \left( \frac{k\frac{g}{2}}{2} \right)}
$$

$$
\cdot \prod_{j=1}^{m-k} \frac{s \left( -a - b - c - (m-1)g + (j-1)\frac{g}{2} \right) s \left( b + (j-1)\frac{g}{2} \right) s \left( \frac{j\frac{g}{2}}{2} \right)}{s \left( -a - c - (m-1)g + (m-k+j-2)\frac{g}{2} \right) s \left( \frac{k\frac{g}{2}}{2} \right)} \times |I_k(z)|^2.
$$

(1.9)

We refer the reader to [26] for more details of (1.9).

### 1.3 $q$-Racah polynomials

Let $\varphi_{m-1}$ be the basic hypergeometric series

$$
\varphi_{m-1} \left( \begin{array}{c}
a_1, \ldots, a_m \\
b_1, \ldots, b_{m-1}
\end{array} ; q, z \right) = \sum_{n \geq 0} \frac{\left( a_1, \ldots, a_m ; q \right)_n}{\left( b_1, \ldots, b_{m-1}, q ; q \right)_n} z^n,
$$

where

$$
\left( a_1, \ldots, a_m ; q \right)_n = (a_1 ; q)_n \cdots (a_m ; q)_n
$$

and

$$
(a ; q)_n = \prod_{0 \leq i \leq n-1} (1 - a q^i).
$$
The $q$-Racah polynomials $W_n(x; a, b, c, N; q)$ are defined by

$$W_n(x; q) = W_n(x; a, b, c, N; q) = \varphi_3 \left( q^{-n}, abq^{n+1}, q^{-x}, cq^{-N} \atop aq, q^{-N}, bcq ; q, q \right)$$

for $n = 0, 1, \ldots, N$, which is of degree $n$ in the variable $\mu(x) = q^{-x} + cq^{-N}$. Their orthogonality relation is

$$\sum_{x=0}^{N} \rho(x; q) W_m(x; q) W_n(x; q) = \frac{\delta_{m,n}}{h_n(q)},$$

where

$$\rho(x; q) = \rho(x; a, b, c, N; q) = \frac{(1 - cq^{2x-N}) (cq^{-N}, q^{-N}, aq, bcq ; q)_x (abq)^{-x}}{(1 - cq^{-N}) (ca^{-1}q^{-N}, b^{-1}q^{-N}, q, cq ; q)_x (abq)^{-x}}$$

and

$$h_n(q) = h_n(a, b, c, N ; q) = \frac{(bcq, aq/c ; q)_n (1 - abq^{2n+1}) (aq, abq, bcq, q^{-N} ; q)_n (q^N/c)^n}{(abq^2, 1/c ; q)_n (1 - abq) (q, bq, aq/c, abq^{N+2} ; q)_n (q^N/c)^n}.$$ 

We refer the reader to [10] for more detail on the $q$-Racah polynomials.

It is noteworthy that the $q$-Racah polynomial $W_n(x; ; q)$ is essentially the same as the $q$-6$j$ symbol [14]; actually we have

$$\begin{pmatrix} a, b, c \\ d, e, f \end{pmatrix}_q = \{ \rho(x; q)h_n(q) \}^{1/2} W_n(x; \alpha, \beta, \gamma, N; q),$$

where

$$n = a + b - e, \quad x = c + d - e, \quad N = a + b + c + d + 1,$$

$$\alpha = q^{-a-d+e+f}, \quad \beta = q^{-a-b-c+d-1}, \quad \gamma = q^{a+e+f-d+1}.$$ 

The orthogonality relation satisfied by these $q$-6$j$ symbols is

$$\sum_{0 \leq j \leq N} \begin{pmatrix} j_2, j_1, j \\ j_3, j_5, j_4 \end{pmatrix}_q \begin{pmatrix} j_3, j_1, j_6 \\ j_2, j_5, j \end{pmatrix}_q = \delta_{j_4,j_6}.$$ 

The $q$-6$j$ symbols are used to construct the invariants of links related with the quantum group $U_q(sl_2)$ [14].
2 Connection formulas

Let $L_z$ be the local system determined by the function
\[ u(t) = \prod_{1 \leq i < j \leq m} (t_j - t_i)^a \prod_{1 \leq i \leq m} (1 - t_i)^b (t_i - z)^c \]
on the domain
\[ T_z = \{ t = (t_1, \ldots, t_m) \in \mathbb{C}^m \mid t_i \neq t_j \ (i \neq j), \ t_i \neq 0, 1, z \} , \]
where $z \in \mathbb{C}\{0, 1\}$.

2.1 The solutions around 0 in terms of those around 1

In this subsection, we give two formulas that connect the fundamental set of solutions around 0 with that around 1.

For convenience to our purpose, we fix a complex variable $z$ to be real with $0 < z < 1$ and assign the domains $D_{0,j,0,m-j}(t)$ and $D_{m-j,0,j,0}(t)$ for $0 \leq j \leq m$ of the real manifold $T_R$ by
\[ D_{0,j,0,m-j}(t) = \{ (t_1, \ldots, t_m) \mid 0 < t_1 < \cdots < t_j < z, \ 1 < t_{j+1} < \cdots < t_m \} , \]
\[ D_{m-j,0,j,0}(t) = \{ (t_1, \ldots, t_m) \mid t_1 < \cdots < t_{m-j} < 0, \ z < t_{m-j+1} < \cdots < t_m < 1 \} , \]
where each orientation is fixed to be natural one induced from $T_R$. Moreover, we define the loaded cycle $C_{0,j,0,m-j}$ and $C_{m-j,0,j,0}$ to be
\[ C_{0,j,0,m-j} = \sum_{\sigma \in S_m} \sigma \{ D_{0,j,0,m-j}(t) \otimes u_{D_{0,j,0,m-j}(t)} \} \]
and
\[ C_{m-j,0,j,0} = \sum_{\sigma \in S_m} \sigma \{ D_{m-j,0,j,0}(t) \otimes u_{D_{m-j,0,j,0}(t)} \} , \]
where the action of $\sigma \in S_m$ is on the coordinates $t_1, \ldots, t_m$ of $\mathbb{C}^m$. The families of cycles \( \{ C_{0,j,0,m-j} \mid 0 \leq j \leq m \} \) and \( \{ C_{m-j,0,j,0} \mid 0 \leq j \leq m \} \) give fundamental sets of solutions around 0 and that around 1, respectively. Indeed, the integrals
\[ I_j(a, b, c; g; z) = \langle \text{reg} \ C_{0,j,0,m-j}, dt_1 \ldots dt_m \rangle \]
\[ = m! \int_{D_{0,j,0,m-j}(t)} u_{D_{0,j,0,m-j}(t)} dt_1 \cdots dt_m \]
and
\[ J_j(a, b, c; g; z) = \langle \text{reg} \ C_{m-j,0,j,0}, dt_1 \ldots dt_m \rangle \]
\[ = m! \int_{D_{m-j,0,j,0}(t)} u_{D_{m-j,0,j,0}(t)} dt_1 \cdots dt_m \]
Proposition 2.1. (1) For $0 \leq j \leq m$, we have

\[ I_j(a, b, c; g; z) = m! S_j(a + 1, b + 1, g/2) S_{m-j}(-a - b - c - (m - 1)g - 1, b + 1, g/2) \]
\[ \times z^{(a+c+1)j + (\frac{j}{2})g} (1 + O(z)) \quad (z \to 0). \]

(2) For $0 \leq j \leq m$, we have

\[ J_j(a, b, c; g; z) = m! S_j(b + 1, c + 1, g/2) S_{m-j}(-a - b - c - (m - 1)g - 1, a + 1, g/2) \]
\[ \times (1 - z)^{(b+c+1)j + (\frac{j}{2})g} (1 + O(1 - z)) \quad (z \to 1). \]

Here the arguments of $z$ of $z^{(a+c+1)j + (\frac{j}{2})g}$ and $1 - z$ of $(1 - z)^{(b+c+1)j + (\frac{j}{2})g}$ are fixed to be zero on $0 < z < 1$, and $S_m(\alpha, \beta, \gamma)$ denotes the Selberg integral [31]:

\[ S_m(\alpha, \beta, \gamma) = \int_{0<t_1<\cdots<t_m<1} \prod_{i=1}^m t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1 \leq i < j \leq m} (t_j - t_i)^2 dt_1 \cdots dt_m \]
\[ = \frac{1}{m!} \prod_{j=1}^m \frac{\Gamma(\alpha + (j - 1)\gamma)\Gamma(\beta + (j - 1)\gamma)\Gamma(j\gamma + 1)}{\Gamma(\alpha + \beta + (m + j - 2)\gamma)\Gamma(\gamma + 1)}. \quad (2.1) \]

Proof. (1) The change of the integration variables such as $t_i \mapsto zt_i \ (1 \leq i \leq j)$
and \( t_i \mapsto 1/t_i \ (j < i \leq m) \) leads to

\[
\frac{1}{m!} I_j(a, b, c; g; z) = z^{(a+c+1)j+\left(\frac{c}{2}\right)} \int \prod_{1 \leq i_1 < i_2 \leq j} (t_{i_2} - t_{i_1})^g \prod_{1 \leq i \leq j} t_i^a (1 - t_i)^c (1 - zt_i)^b \]
\[
\times \prod_{j < i_1 < i_2 \leq m} (t_{i_1} - t_{i_2})^g \prod_{j < i \leq m} t_i^{-a-b-c-(m-1)g-2} (1 - zt_i)^c (1 - t_i)^b \]
\[
\times \prod_{1 \leq i_1 \leq j \atop j \leq i_2 \leq m} (1 - zt_{i_1} t_{i_2})^g \ dt_1 \cdots dt_m,
\]

where the domain of integration is

\[
0 < t_1 < \cdots < t_j < 1, \quad 0 < t_m < \cdots < t_{j+1} < 1
\]

with the standard orientation. This implies the required result by using the binomial theorem

\[
(1 - zt_{i_1} t_{i_2})^g = \sum_{n \geq 0} \frac{(-g)_n}{n!} (zt_{i_1} t_{i_2})^n.
\]

(2) The change of the integration variables such as \( t_i \mapsto 1/(1 - t_i) \ (1 \leq i \leq m - j) \) and \( t_i \mapsto (1 - t_i)/(1 - z) \ (m - j < i \leq m) \) leads to

\[
\frac{1}{m!} I_j(a, b, c; g; z) = (1 - z)^{(b+c+1)j+\left(\frac{c}{2}\right)} g \]
\[
\times \int \prod_{1 \leq i_1 < i_2 \leq m-j} (t_{i_2} - t_{i_1})^g \prod_{1 \leq i \leq m-j} t_i^{a-b-c-(m-1)g-2} (1 - t_i)^a (1 - (1 - z) t_i)^c \]
\[
\times \prod_{m-j < i_1 < i_2 \leq m} (t_{i_2} - t_{i_1})^g \prod_{m-j+1 \leq i \leq m} t_i^b (1 - (1 - z) t_i)^a (1 - t_i)^c \]
\[
\times \prod_{1 \leq i_1 \leq m-j \atop m-j < i_2 \leq m} (1 - (1 - z) t_{i_1} t_{i_2})^g \ dt_1 \cdots dt_m,
\]

where the domain of integration is

\[
0 < t_{m-j} < \cdots < t_1 < 1 - z, \quad 0 < t_m < \cdots < t_{m-j+1} < 1
\]

with the standard orientation. This implies the required result. \( \square \)
In case of $m = 1$, i.e. Gauss hypergeometric functions, Proposition 2.1 corresponds to

\begin{align*}
I_0(a, b, c; z) &= B(b + 1, -a - b - c - 1) \ _2F_1\left(\begin{array}{c}
-c, -a - b - c - 1 \\
-a - c
\end{array} \ ; z \right), \\
I_1(a, b, c; z) &= B(a + 1, c + 1) \ z^{a+c+1} \ _2F_1\left(\begin{array}{c}
-b, a + 1 \\
-a - c
\end{array} \ ; z \right)
\end{align*}
(2.2)

and

\begin{align*}
J_0(a, b, c; z) &= B(a + 1, -a - b - c - 1) \ _2F_1\left(\begin{array}{c}
-c, -a - b - c - 1 \\
-b - c
\end{array} \ ; 1 - z \right), \\
J_1(a, b, c; z) &= B(b + 1, c + 1) \ (1 - z)^{b+c+1} \ _2F_1\left(\begin{array}{c}
-a, b + 1 \\
-b - c
\end{array} \ ; 1 - z \right),
\end{align*}
(2.3)

where $B(a, b)$ denotes the beta function

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

In this case, the linear relations between $\{I_0, I_1\}$ and $\{J_0, J_1\}$ are given by

\begin{align*}
I_0(a, b, c; z) &= \frac{s(a)}{s(b + c)} I_0(a, b, c; z) + \frac{-s(c)}{s(b + c)} J_1(a, b, c; z), \\
I_1(a, b, c; z) &= \frac{-s(a + b + c)}{s(b + c)} I_0(a, b, c; z) + \frac{-s(b)}{s(b + c)} J_1(a, b, c; z)
\end{align*}

and

\begin{align*}
J_0(a, b, c; z) &= \frac{s(b)}{s(a + c)} I_0(a, b, c; z) + \frac{-s(c)}{s(a + c)} I_1(a, b, c; z), \\
J_1(a, b, c; z) &= \frac{-s(a + b + c)}{s(a + c)} I_0(a, b, c; z) + \frac{-s(a)}{s(a + c)} I_1(a, b, c; z),
\end{align*}
where \( s(A) = \sin(\pi A) \).

These are called the connection formulas between the fundamental set of solutions around \( 0 \) and that around \( 1 \) in the case of the Gauss hypergeometric functions.

In general \( m \) case, if we define \( p_{ij}^{(0,1)} = p_{ij}^{(0,1)}(a,b,c;g) \) for \( 0 \leq i,j \leq m \), called the connection coefficients, by

\[
I_i(a,b,c;g;z) = \sum_{0 \leq j \leq m} p_{ij}^{(0,1)}(a,b,c;g) J_j(a,b,c;g;z),
\]

we have the following:

**Theorem 2.2.** For \( 0 \leq i,j \leq m \), we have

\[
p_{ij}^{(0,1)}(a,b,c;g) = (-)^j \sum_{0 \leq k \leq m-i} (-)^k \prod_{r=1}^{m-i-k} s(a + \frac{i+r-1}{2} g) s(b + c + (k + \frac{i+r-1}{2} g)) \prod_{r=1}^{k} s(c + \frac{i+r-1}{2} g) \\
\times \prod_{r=1}^{i-l} s(a + b + c + \frac{m+i+k-r-1}{2} g) s(\frac{m-i-k+r}{2} g) \prod_{r=1}^{l} s(b + \frac{k+r-1}{2} g) s(\frac{k+r}{2} g),
\]

and

\[
p_{ij}^{(0,1)}(a,b,c;g) = (-)^j \sum_{0 \leq k \leq i} (-)^k \prod_{r=1}^{i-k} s(a + b + c + \frac{m+i-r-1}{2} g) s\left(\frac{m-i-k+r}{2} g\right) \prod_{r=1}^{k} s\left(\frac{b + m-i-r-1}{2} g\right) \\
\times \prod_{r=1}^{m-i-l} s\left(\frac{a + \frac{i-k+r-1}{2} g}{2}\right) s\left(\frac{i-k+r}{2} g\right) \prod_{r=1}^{l} s\left(\frac{c + \frac{k+r-1}{2} g}{2}\right) s\left(\frac{k+r}{2} g\right),
\]\n
where \( s(A) = \sin(\pi A) \).

**Proof.** Set \( \lambda_1 = a, \lambda_2 = c, \lambda_3 = b \) and \( z_1 = 0, z_2 = z, z_3 = 1 \) in Proposition 3.3 in the next section. Then we obtain the required result. \( \square \)

When \( m = 2 \), Theorem 2.2 implies that
\[ P = \left( p_{ij}^{(0,1)} \right)_{0 \leq i, j \leq 2} \]

\[
P^{(0,1)}_{11} = -\frac{s(b)s(a + \frac{1}{2}g)}{s(b+c)s(b+c + \frac{1}{2}g)} + \frac{s(a+b+c+g)s(c + \frac{1}{2}g)}{s(b+c+g)s(b+c + \frac{1}{2}g)} - \frac{s(c)s(a + b + c + \frac{1}{2}g)}{s(b+c)s(b+c + \frac{1}{2}g)} - \frac{s(a)s(b + \frac{1}{2}g)}{s(b+c+g)s(b+c + \frac{1}{2}g)}
\]

where

The first expression of \( p_{11}^{(0,1)} \) is implied by (2.6), and the second one is by (2.7). The second expression coincides with (5.11) of [8].

We have many expressions of the connection coefficient \( p_{ij}^{(0,1)} (a, b, c; g) \) other than those in Theorem 2.2. As an example of them, we give the following, which is essentially given by the q-Racah polynomial.

**Theorem 2.3.** For \( 0 \leq i, j \leq m \), we have

\[
p_{ij}^{(0,1)} (a, b, c; g) = (-1)^{i+j} \prod_{r=1}^{i} \frac{s(\frac{m-r+1}{2}g)}{s(\frac{1}{2}g)} \times \frac{s(b+c+(j-\frac{1}{2})g) \prod_{r=1}^{i} s(a+b+c+\frac{m+r-2}{2}g) \prod_{r=1}^{m} s(a+r-1\frac{1}{2}g)}{\prod_{r=1}^{i} s(b+c + \frac{r+1}{2}g) \prod_{r=1}^{m} s(a + r-1\frac{1}{2}g)} \times \sum_{l \geq 0} \prod_{r=1}^{l} \frac{s(-j+r-1\frac{1}{2}g)s(-j+r-2\frac{1}{2}g)s(a+c + \frac{1+r-2}{2}g)s(a+b+c + \frac{m+r-2}{2}g)s(\frac{1}{2}g)}{s(-j+r-1\frac{1}{2}g)s(a+b+c + \frac{m+r-2}{2}g)s(\frac{1}{2}g)} ,
\]

or, equivalently,
\[
p_{ij}^{(0,1)}(a, b, c; g) = \frac{1 - e(2(b + c))q^{2j-1}}{1 - e(2(b + c))q^{j-1}} \frac{(e(2(b + c)), e(2c); q)_j}{(e(2(b + c))q^m, e(-2a)q^{1-m}; q)_j (e(2(b + c)); q)_m} \\
\times e((-m - j)a + (m - i)b + (m - i - j)c)q^i \\
\times 4\varphi_3 \left( \begin{array}{c} q^{-i}, e(2(a + c))q^{i-1}, q^{-j}, e(2(c + b))q^{j-1} \\
q(c), q^{-m}, e(2(a + b + c))q^{m-1} \end{array} ; q, q \right).
\]

Here

\[
4\varphi_3 \left( \begin{array}{c} q^{-i}, e(2(a + c))q^{i-1}, q^{-j}, e(2(c + b))q^{j-1} \\
q(c), q^{-m}, e(2(a + b + c))q^{m-1} \end{array} ; q, q \right)
\]
can be considered as

\[
W_j(j; e(2c)q^{-1}, e(2a)q^{-1}, e(2(b + c))q^{m-1}, m; q)
\]
or

\[
W_j(i; e(2c)q^{-1}, e(2b)q^{-1}, e(2(a + c))q^{m-1}, m; q),
\]
where \(W_n(x; a', b', c', N; q)\) denotes the q-Racah polynomial defined by (1.6).

**Proof.** In Proposition 3.6 in the next section, set \(\lambda_1 = a, \lambda_2 = c, \lambda_3 = b\) and \(z_1 = 0, z_2 = z, z_3 = 1\). Then we obtain the required result. □

On the other hand, it is easily seen from (2.2-5) that \(J_j(a, b, c; g; z) = I_j(b, a, c; g; 1 - z)\) for \(j = 0, 1\). In general \(m\) case, we have the following.

**Proposition 2.4.** For \(0 \leq j \leq m\),

\[
J_j(a, b, c; g; z) = I_j(b, a, c; g; 1 - z).
\]

**Proof.** The change of the integration variables \(t_i \leftrightarrow 1 - t_i\) (\(1 \leq i \leq m\)) in

\[
J_j(a, b, c; g; z) = m! \int_{D_{m-j,0,0,0}(t)} u_{D_{m-j,0,0,0}}(t) \ dt_1 \cdots \ dt_m
\]
leads to \(I_j(b, a, c; g; 1 - z)\). □

Therefore, combining Theorem 2.3 and Proposition 2.4, we reach the following.
Corollary 2.5. For $0 \leq i, j \leq m$, we have

$$\delta_{ij} = h_i(e(2c)q^{-1}, e(2a)q^{-1}, e(2(b+c))q^{m-1}, m; q)$$

$$\times \sum_{0 \leq x \leq m} \rho(x; a', b', c', m; q) W_i(x; a', b', c', m; q) W_j(x; a', b', c', m; q),$$

where

$$a' = e(2c)q^{-1}, \quad b' = e(2a)q^{-1}, \quad c' = e(2(b+c))q^{m-1}.$$

Proof. By Proposition 2.4, we have

$$I_i(a, b, c; g; z) = \sum_{x=0}^{m} p_{ix}^{(0,1)}(a, b, c; g) J_x(a, b, c; g; z)$$

$$= \sum_{x=0}^{m} p_{ix}^{(0,1)}(a, b, c; g) I_x(b, a, c; 1-z)$$

$$= \sum_{x=0}^{m} p_{ix}^{(0,1)}(a, b, c; g) \sum_{j=0}^{m} p_{xj}^{(0,1)}(b, a, c; g) I_j(a, b, c; g; z),$$

hence

$$\sum_{x=0}^{m} p_{ix}^{(0,1)}(a, b, c; g) p_{xj}^{(0,1)}(b, a, c; g) = \delta_{ij}.$$
\[ \times 4 \varphi_3 \left( q^{-i}, e(2(a + c))q^{i-1}, q^{-x}, e(2(c + b))q^{x-1} \right. \\
\left. e(2c), q^{-m}, e(2(a + b + c))q^{m-1} ; q, q \right) \]
\[ \times 4 \varphi_3 \left( q^{-x}, e(2(b + c))q^{x-1}, q^{-j}, e(2(c + a))q^{j-1} \right. \\
\left. e(2c), q^{-m}, e(2(a + b + c))q^{m-1} ; q, q \right). \]

Therefore we obtain the required result. \( \square \)

2.2 The solutions around 0 in terms of those around \( \infty \)

In this subsection, we give formulas that connect the fundamental set of solutions around 0 with the set of solutions around \( \infty \).

For convenience to our purpose, we fix a complex variable \( z \) to be real such that \( z < 0 \) and assign the names \( D_{0,j,0,m-j}(t) \) and \( D_{m-j,0,0,j}(t) \) for \( 0 \leq j \leq m \) to the domains of the real manifold \( T_R \) by

\[
D_{0,j,0,m-j}(t) = \{ (t_1, \ldots, t_m) | z < t_1 < \cdots < t_j < 0, 1 < t_{j+1} < \cdots < t_m \},
\]

\[
D_{m-j,0,0,j}(t) = \{ (t_1, \ldots, t_m) | t_1 < \cdots < t_{m-j} < z, 0 < t_{m-j+1} < \cdots < t_m < 1 \},
\]

where each orientation is natural one. Correspondingly, we define the loaded cycles \( C_{0,j,0,m-j} \) and \( C_{m-j,0,0,j} \) to be

\[
C_{0,j,0,m-j} = \sum_{\sigma \in S_m} \sigma \{ D_{0,j,0,m-j}(t) \otimes u_{D_{0,j,0,m-j}}(t) \}
\]

and

\[
C_{m-j,0,0,j} = \sum_{\sigma \in S_m} \sigma \{ D_{m-j,0,0,j}(t) \otimes u_{D_{m-j,0,0,j}}(t) \}.
\]

Then the integrals

\[
I_j(a, b, c; g; z) = \langle \text{reg} C_{0,j,0,m-j}, dt_1 \cdots dt_m \rangle
\]

\[
= m! \int_{D_{0,j,0,m-j}(t)} u_{D_{0,j,0,m-j}}(t) dt_1 \cdots dt_m
\]

and

\[
K_j(a, b, c; g; z) = \langle \text{reg} C_{m-j,0,0,j}, dt_1 \cdots dt_m \rangle
\]

\[
= m! \int_{D_{m-j,0,0,j}(t)} u_{D_{m-j,0,0,j}}(t) dt_1 \cdots dt_m
\]

give fundamental set of solutions around 0 and that around \( \infty \), respectively. Indeed, we have the following.

**Proposition 2.6.** (1) For \( 0 \leq j \leq m \), we have
\[ I_j(a, b, c; g; z) = m! \, S_j(c + 1, a + 1, g/2) \, S_{m-j}(-a - b - c - (m - 1)g - 1, b + 1, g/2) \]
\[ \times (-z)^{(a+c+1)j + \frac{j}{2}} \left(1 + O(z)\right) \quad (z \to 0). \]

(2) For \(0 \leq j \leq m\), we have
\[ K_j(a, b, c; g; z) = m! \, S_j(-a - b - c - (m - 1)g - 1, b + 1, g/2) \]
\[ \times \left( -z^{-1} \right)^{-(a+b+1)j - mc - \left\{ \frac{j}{2} + j(m-j) \right\}} \left(1 + O(z^{-1})\right) \quad (z \to \infty) \]

Here the arguments of \(-z\) of \((-z)^{(a+c+1)j + \frac{j}{2}}\) and \(-z^{-1}\) of \((-z^{-1})^{-(a+b+1)j - mc - \left\{ \frac{j}{2} + j(m-j) \right\}}\) are fixed to be zero on \(z < 0\), and \(S(\alpha, \beta, \gamma)\) is the Selberg integral (2.1).

**Proof.** (1) The change of the integration variables \(t_i \mapsto zt_i\) for \(1 \leq i \leq j\) and \(t_i \mapsto t_i^{-1}\) for \(j < i \leq m\) and the binomial theorem imply the result.

(2) The change of the integration variables \(t_i \mapsto z^{-1}t_i\) for \(1 \leq i \leq j\) (\(t_i\) for \(j < i \leq m\) is fixed) and the binomial theorem imply the result. \(\square\)

When \(m = 1\), Proposition 2.6 corresponds to
\[ I_0(a, b, c; z) = B(b + 1, -a - b - c - 1) \, {}_2F_1\left( -c, -a - b - c - 1 \atop -a - c \right) ; \quad (2.8) \]
\[ I_1(a, b, c; z) = B(a + 1, c + 1) \, (-z)^{a+c+1} \, {}_2F_1\left( -b, a + 1 \atop a + c + 2 \right) ; \quad (2.9) \]

and
\[ K_0(a, b, c; z) = B(c + 1, -a - b - c - 1) \]
\[ \times \left( \frac{-1}{z} \right)^{-a-b-c-1} \, {}_2F_1\left( -b, -a - b - c - 1 \atop -a - b \right) ; \quad (2.10) \]
\[ K_1(a, b, c; z) = B(a + 1, b + 1) \]
\[ \times \left( \frac{-1}{z} \right)^{-c} \, {}_2F_1\left( -c, a + 1 \atop a + b + 2 \right) ; \quad (2.11) \]
where $B(a, b)$ denotes the beta function.

The linear relations between $\{I_0, I_1\}$ and $\{K_0, K_1\}$ are expressed by

$$I_0(a, b, c; z) = \frac{s(c)}{s(a+b)} K_0(a, b, c; z) + \frac{-s(a)}{s(a+b)} K_1(a, b, c; z),$$

$$I_1(a, b, c; z) = -\frac{s(a+b+c)}{s(a+b)} K_0(a, b, c; z) + \frac{-s(b)}{s(a+b)} K_1(a, b, c; z)$$

and

$$K_0(a, b, c; z) = \frac{s(b)}{s(a+c)} I_0(a, b, c; z) + \frac{-s(a)}{s(a+c)} I_1(a, b, c; z),$$

$$K_1(a, b, c; z) = -\frac{s(a+b+c)}{s(a+c)} I_0(a, b, c; z) + \frac{-s(c)}{s(a+c)} I_1(a, b, c; z).$$

These are the connection formulas between the solutions around 0 and those around $\infty$.

In general $m$, if we define $p_{ij}^{(0,\infty)}(a, b, c; g)$ for $0 \leq i, j \leq m$ by

$$I_i(a, b, c; g; z) = \sum_{0 \leq j \leq m} p_{ij}^{(0,\infty)}(a, b, c; g) K_j(a, b, c; g; z),$$

we have the following two expressions.

**Theorem 2.7.** For $0 \leq i, j \leq m$, we have

$$p_{ij}^{(0,\infty)}(a, b, c; g)$$

$$= (-)^i \sum_{0 \leq k \leq m-i \leq -k} \prod_{r=1}^{m-i-k} \frac{s(c+i+r-1)g}{s(a+b+(k+i+r-1)g)} \prod_{r=1}^{k} \frac{s(a+i+r-1)g}{s(a+b+(k+i+r-1)g)}$$

$$\times \prod_{r=1}^{i-l} \frac{s(a+b+c+(\frac{m+i+k-r-1}{2})g) s(\frac{m-i-k+r}{2}g)}{s(a+b+(\frac{m+i+k-r-1}{2})g) s(\frac{m-i-k+r}{2}g)} \prod_{r=1}^{l} \frac{s(b+k+r-1)g s(\frac{k+r}{2}g)}{s(a+b+(\frac{j-r}{2})g) s(\frac{j+r}{2}g)}$$

(2.12)
Theorem 2.8. For or, equivalently, where \( s(A) = \sin(\pi A) \).

**Proof.** Set \( \lambda_1 = c, \lambda_2 = a, \lambda_3 = b \) and \( z_1 = z, z_2 = 0, z_3 = 1 \) in Proposition 3.3. Or change \( a \) and \( c \) in Theorem 2.2. Then we reach the required result. \( \square \)

Similar to Theorem 2.3, we also have the following.

**Theorem 2.8.** For \( 0 \leq i, j \leq m \), we have

\[
 p^{(0, \infty)}_{ij}(a, b, c; g) = (-1)^{i+j} \sum_{r=0}^{i} \frac{s(m+r\frac{g}{2})}{s(\frac{g}{2})} \times \frac{s(b+a+(j-\frac{1}{2})g)\prod_{r=1}^{i} s(a+b+c+m\frac{r-2}{2}g) \prod_{r=1}^{j} s(c+r\frac{g}{2}) \prod_{r=1}^{m} s(b+a+c+m\frac{r-1}{2}g)}{\prod_{r=1}^{m+1} s(b+a+c+m\frac{r-1}{2}g) \prod_{r=1}^{i} s(a+b+c+m\frac{r-2}{2}g) \prod_{r=1}^{j} s(c+r\frac{g}{2})} \times \sum_{l \geq 0} \prod_{r=1}^{l} \frac{s(-l\frac{r-2}{2}g)s(-l\frac{r-1}{2}g)s(b+a+i\frac{r-2}{2}g)s(a+c+i\frac{r-2}{2}g)}{s(-m-r\frac{2}{2}g)s(-l\frac{r-1}{2}g)s(a+i\frac{r-1}{2}g)s(a+b+c+m\frac{r-2}{2}g)}.
\]

or, equivalently,

\[
 p^{(0, \infty)}_{ij}(a, b, c; g) = 1 - \frac{e(2(a+b))q^{2j-1}}{1 - e(2(a+b))q^{2j-1}} \frac{(e(2a); q)_j}{(e(2a+b); q)_j} \frac{(e(2c); q)_m}{(e(2a+b); q)_m} \times c((m-i-j)a+(-m-j)c) q^i \times 4\varphi_3 \left( q^{-1}, e(2(a+c))q^{-1}, q^{-j}, e(2(a+b))q^{-1}; q, q \right).
\]
Here
\[ 4\varphi_3 \left( \begin{array}{c}
q^{-i}, e(2(a+c))q^{i-1}, q^{-j}, e(2(a+b))q^{j-1} \\
e(2a), q^{-m}, e(2(a+b+c))q^{m-1}
\end{array} ; q, q \right) \]
can be considered as
\[ W_i(j; e(2a)q^{-1}, e(2c)q^{-1}, e(2(a+b))q^{-m}, m; q) \]
or
\[ W_j(i; e(2a)q^{-1}, e(2b)q^{-1}, e(2(a+c))q^{-m}, m; q), \]
where \( W_n(x; a', b', c', N; q) \) denotes the \( q \)-Racah polynomial defined by (1.6).

**Proof.** In Proposition 3.6, set \( \lambda_1 = c, \lambda_2 = a, \lambda_3 = b \) and \( z_1 = z, z_2 = 0, z_3 = 1 \). Then we reach the required result. \( \square \)

On the other hand, it is seen from (2.8-11) that
\[ K_j(a, b, c; z) = (-z)^{a+b+c+1} I_j(a, c, b; z^{-1}) \]
for \( j = 0, 1 \). More generally, we have the following.

**Proposition 2.9.** For \( 0 \leq j \leq m \),
\[ K_j(a, b, c; g; z) = (-z)^{(a+b+c+1)m+(\varpi g)} I_j(a, c, b; g; z^{-1}). \]

**Proof.** The change of integration variables in \( K_j(a, b, c; g; z) \) such as \( t_i \mapsto zt_i \) \( (1 \leq i \leq m) \) implies the required result. \( \square \)

## 3 Derivation of the connection coefficients

In this section, let \( \mathcal{L} \) be the local system determined by the function
\[ u(t) = \prod_{1 \leq i < j \leq m} (t_j - t_i)^g \prod_{1 \leq j \leq 3} (t_i - z_j)^{\lambda_j} \]
on the domain
\[ T = \{ t = (t_1, \ldots, t_m) \in \mathbb{C}^m \mid t_i \neq t_j (i \neq j), t_i \neq z_1, z_2, z_3 \}, \]
where \( z_1, z_2, z_3 \) are fixed to be real and \( z_1 < z_2 < z_3 \).

Each connection formula in the previous section is obtained as a special case of the formula in this section. We obtain the formulas in §2.1, if we set \( z_1 = 0, z_2 = z, z_3 = 1 \) and \( \lambda_1 = a, \lambda_2 = c, \lambda_3 = b \). Similarly, we obtain the formulas in §2.2, if we set \( z_1 = z, z_2 = 0, z_3 = 1 \) and \( \lambda_1 = c, \lambda_2 = a, \lambda_3 = b \).
Set the loaded cycle
\[ C_{i_1,j_1,i_2,j_2} = \sum_{\sigma \in S_m} \sigma \{ D_{i_1,j_1,i_2,j_2}(t) \otimes u_{D_{i_1,j_1,i_2,j_2}}(t) \}, \]
for \( i_1, j_1, i_2, j_2 \in \mathbb{Z}_{\geq 0} \) with \( i_1 + j_1 + i_2 + j_2 = m \), where \( D_{i_1,j_1,i_2,j_2}(t) \) is the domain of \( T^R \) defined by the inequalities
\[
\begin{align*}
t_1 &< t_2 < \cdots < t_{i_1} < z_1, \\
z_1 &< t_{i_1+1} < t_{i_1+2} < \cdots < t_{i_1+j_1} < z_2, \\
z_2 &< t_{i_1+j_1+1} < t_{i_1+j_1+2} < \cdots < t_{i_1+j_1+i_2} < z_3, \\
z_3 &< t_{i_1+j_1+i_1+1} < t_{i_1+j_1+i_1+2} < \cdots < t_{i_1+j_1+i_1+j_2} 
\end{align*}
\]
with the standard orientation.

Define the symbols \( e(A) \), \( s(A) \), \( [n]_q \), \( \langle A \rangle_n \) and \( \lambda_{ijk\cdots l} \) to be
\[
\begin{align*}
e(A) &= e^{\pi \sqrt{-1} A}, & s(A) &= \sin \pi A, \\
[n]_q &= 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}, \\
\langle A \rangle_n &= A [n]_q - A^{-1} [n]_{q^{-1}}, \\
\lambda_{ijk\cdots l} &= \lambda_i + \lambda_j + \lambda_k + \cdots + \lambda_l 
\end{align*}
\]
for brevity. Note that
\[
\langle A \rangle_n = \langle A q^{n/2} \rangle_1 \langle q^{n/2} \rangle_1.
\]
Hence, when \( q = e(g) \), we have
\[
\langle e(\lambda) \rangle_n = 2 \sqrt{-1} s \left( \lambda + \frac{n - 1}{2} g \right) \frac{s(\frac{1}{2} g)}{s(\frac{1}{2} g)}.
\]
In what follows, we fix \( q \) to be \( e(g) \).

### 3.1 Connection coefficients

**Lemma 3.1.** (1) For integers \( i_1, i_2, j_1 \geq 0, j_2 \geq 1 \), we have
\[
\begin{align*}
C_{i_1,j_1,i_2,j_2} &= \frac{\langle e(\lambda_1) q^{\frac{i_1}{2}} \rangle_{i_1+1}}{\langle e(\lambda_{23}) q^{i_2+\frac{j_2}{2}} \rangle_{j_2}} C_{i_1+1,j_1,i_2,j_2-1} \\
&\quad - \frac{\langle e(\lambda_2) q^{\frac{j_2}{2}} \rangle_{j_2+1}}{\langle e(\lambda_{23}) q^{i_2+\frac{j_2}{2}} \rangle_{j_2}} C_{i_1,j_1,i_2+1,j_2-1}.
\end{align*}
\]
(2) For integers \( i_1, i_2, j_2 \geq 0, j_1 \geq 1 \), we have
\[
C_{i_1,j_1,i_2,j_2} = -\frac{\langle e(\lambda_{123})q^{i_2+j_1+j_2} \rangle_{i_1+1} \langle e(\lambda_{23})q^{i_2+j_2} \rangle_{j_1} C_{i_1+1,j_1-1,i_2,j_2}}{\langle e(\lambda_{23})q^{i_2} \rangle_{j_1} C_{i_1,j_1-1,i_2+1,j_2}}.
\]

Proof. (1) Fix a point \((t_1, \ldots, t_{m-1})\) of \(D_{i_1,j_1,i_2,j_2-1}(t_1, \ldots, t_{m-1})\), where \(i_1 + j_1 + i_2 + j_2 = m - 1\). Then it is seen that a trivial loop with clockwise direction in the lower half plane of the \(t_m\)-plane is homologous to
\[
\sum_{s=1}^{i_1+1} q^{s-1} D_{i_1+1,j_1,i_2,j_2-1}(t_1, \ldots, t_{s-1}, t_m, t_s, \ldots, t_{m-1})
\]
\[
+ \langle e(\lambda_1) \rangle \sum_{s=i_1+1}^{i_1+j_1+1} q^{s-1} D_{i_1,j_1+1,i_2,j_2-1}(t_1, \ldots, t_{s-1}, t_m, t_s, \ldots, t_{m-1})
\]
\[
+ \langle e(\lambda_{12}) \rangle \sum_{s=i_1+j_1}^{i_1+j_1+i_2+1} q^{s-1} D_{i_1,j_1,i_2+1,j_2-1}(t_1, \ldots, t_{s-1}, t_m, t_s, \ldots, t_{m-1})
\]
\[
+ \langle e(\lambda_{123}) \rangle \sum_{s=i_1+j_1+i_2+1}^{m} q^{s-1} D_{i_1,j_1,i_2,j_2}(t_1, \ldots, t_{s-1}, t_m, t_s, \ldots, t_{m-1})
\]
and a trivial loop with counterclockwise direction in the upper half plane of the \(t_m\)-plane is homologous to
\[
\sum_{s=1}^{i_1+1} q^{-s+1} D_{i_1+1,j_1,i_2,j_2-1}(t_1, \ldots, t_{s-1}, t_m, t_s, \ldots, t_{m-1})
\]
\[
+ \langle e(-\lambda_1) \rangle \sum_{s=i_1+1}^{i_1+j_1+1} q^{-s+1} D_{i_1,j_1+1,i_2,j_2-1}(t_1, \ldots, t_{s-1}, t_m, t_s, \ldots, t_{m-1})
\]
\[
+ \langle e(-\lambda_{12}) \rangle \sum_{s=i_1+j_1}^{i_1+j_1+i_2+1} q^{-s+1} D_{i_1,j_1,i_2+1,j_2-1}(t_1, \ldots, t_{s-1}, t_m, t_s, \ldots, t_{m-1})
\]
\[
+ \langle e(-\lambda_{123}) \rangle \sum_{s=i_1+j_1+i_2+1}^{m} q^{-s+1} D_{i_1,j_1,i_2,j_2}(t_1, \ldots, t_{s-1}, t_m, t_s, \ldots, t_{m-1}).
\]
It implies
\[
\sum_{s=1}^{i_1+1} q^{s-1} C_{i_1+1, i_2, j_2-1} + e(\lambda_1) \sum_{s=1}^{i_1+j_1+1} q^{s-1} C_{i_1, i_2, j_2-1} + e(\lambda_2) \sum_{s=1}^{i_1+j_1+1} q^{s-1} C_{i_1, i_2+1, j_2-1} + e(\lambda_2) \sum_{s=1}^{i_1+j_1+1} q^{s-1} C_{i_1, i_2, j_2-1} = 0,
\]
and
\[
\sum_{s=1}^{i_1+1} q^{-s+1} C_{i_1+1, i_2, j_2-1} + e(-\lambda_1) \sum_{s=1}^{i_1+j_1+1} q^{-s+1} C_{i_1, i_2, j_2-1} + e(-\lambda_2) \sum_{s=1}^{i_1+j_1+1} q^{-s+1} C_{i_1, i_2, j_2-1} + e(-\lambda_2) \sum_{s=1}^{i_1+j_1+1} q^{-s+1} C_{i_1, i_2, j_2-1} = 0.
\]
thus,
\[
[i_1 + 1]_q C_{i_1+1, i_2, j_2-1} + e(\lambda_1) [i_1 + 1]_q C_{i_1+1, i_2, j_2-1} + e(\lambda_2) [i_2 + 1]_q C_{i_1, i_2+1, j_2-1} + e(\lambda_2) [i_2 + 1]_q C_{i_1, i_2, j_2-1} = 0
\]
and
\[
[i_1 + 1]_q^{-1} C_{i_1+1, i_2, j_2-1} + e(-\lambda_1) [i_1 - 1]_q^{-1} C_{i_1+1, i_2, j_2-1} + e(-\lambda_2) [i_2 + 1]_q^{-1} C_{i_1, i_2+1, j_2-1} + e(-\lambda_2) [i_2 + 1]_q^{-1} C_{i_1, i_2, j_2-1} = 0
\]
in the sense of twisted homology.

Therefore, by eliminating the second terms of (3.3) and (3.4), we have
\[
\left( e(\lambda_1) q^i [j_1 + 1]_q - e(-\lambda_1) q^{-i} [j_1 + 1]_q^{-1} \right) C_{i_1+1, i_2, j_2-1} + \left( e(\lambda_2) q^i [j_1 + 1]_q - e(-\lambda_2) q^{-i} [j_1 + 1]_q^{-1} \right) C_{i_1, i_2+1, j_2-1} + \left( e(\lambda_2) q^i [j_2]_q - e(-\lambda_2) q^{-i} [j_2]_q^{-1} \right) C_{i_1, i_2, j_2-1} = 0,
\]
\[
\left( e(\lambda_1) q^{-i} [j_1 + 1]_q - e(-\lambda_1) q^i [j_1 + 1]_q^{-1} \right) C_{i_1+1, i_2, j_2-1} + \left( e(\lambda_2) q^{-i} [j_1 + 1]_q - e(-\lambda_2) q^i [j_1 + 1]_q^{-1} \right) C_{i_1, i_2+1, j_2-1} + \left( e(-\lambda_2) q^{-i} [j_2]_q - e(\lambda_2) q^i [j_2]_q^{-1} \right) C_{i_1, i_2, j_2-1} = 0.
\]
which is simplified to
\[
\langle e(\lambda_1)q^{\frac{j_2}{2}} \rangle_{i_1+1} C_{i_1+1,j_1,i_2,j_2-1} - \langle e(\lambda_2)q^{\frac{j_2}{2}} \rangle_{i_2+1} C_{i_1,j_1,i_2+1,j_2-1}
- \langle e(\lambda_{23})q^{\frac{j_2}{2}+\frac{1}{2}} \rangle_{j_2} C_{i_1,j_1,i_2,j_2} = 0.
\]
This is the required equality.

(2) Similarly, we have
\[
[i_1 + 1]_q C_{i_1+1,j_1-1,i_2,j_2} + e(\lambda_1)q^{i_1} [j_1]_q C_{i_1,j_1,i_2,j_2}
+ e(\lambda_{12})q^{i_1+j_1-1} [j_2 + 1]_q C_{i_1,j_1-1,i_2+1,j_2}
+ e(\lambda_{123})q^{i_1+j_1+i_2-1} [j_2 + 1]_q C_{i_1,j_1-1,i_2,j_2} = 0
\]
and
\[
[i_1 + 1]_q^{-1} C_{i_1+1,j_1-1,i_2,j_2-1} + e(-\lambda_1)q^{-i_1} [j_1]_q^{-1} C_{i_1,j_1,i_2,j_2}
+ e(-\lambda_{12})q^{-i_1-j_1-1} [j_2 + 1]_q^{-1} C_{i_1,j_1-1,i_2+1,j_2}
+ e(-\lambda_{123})q^{-i_1-j_1-i_2-1} [j_2 + 1]_q^{-1} C_{i_1,j_1-1,i_2,j_2+1} = 0,
\]
which imply, by eliminating the last terms,
\[
\langle e(\lambda_{123})q^{j_1 + j_2 + \frac{j_2}{2} - 1} \rangle_{i_1+1} C_{i_1+1,j_1-1,i_2,j_2}
+ \langle e(\lambda_{23})q^{j_2 + \frac{j_2}{2}} \rangle_{i_2} C_{i_1,j_1,i_2,j_2}
+ \langle e(\lambda_3)q^{\frac{j_2}{2}} \rangle_{i_2+1} C_{i_1,j_1-1,i_2+1,j_2} = 0.
\]
It completes the proof of Lemma 3.1. □

**Lemma 3.2.** For nonnegative integers \(i_1, j_1, i_2, j_2\), we have the following.

(1)
\[
C_{i_1,j_1,i_2,j_2} = \sum_{k=0}^{j_2} (-1)^k \left\{ \prod_{r=1}^{j_2-k} \frac{e(\lambda_1)q^{\frac{j_2}{2}}}{e(\lambda_{23})q^{\frac{j_2}{2}+\frac{1}{2}+k-r}} \right\}
\times \prod_{r=1}^{k} \left\{ \frac{e(\lambda_2)q^{\frac{j_2}{2}+r}}{e(\lambda_{23})q^{\frac{j_2}{2}+k-r}} \right\}
C_{i_1+j_2-k,j_1,i_2+k,0}. \tag{3.5}
\]
\[ C_{i_1,j_1,i_2,j_2} = (-1)^n \sum_{k=0}^{j_1} \left\{ \prod_{r=1}^{j_1-k} \frac{\langle e(\lambda_{123})q^{i_2+\frac{r}{2}+j_1-r} \rangle_{i_1+r}}{\langle e(\lambda_{23})q^{i_2+\frac{r}{2}+k} \rangle_r} \right. \\
\times \left. \prod_{r=1}^{k} \frac{\langle e(\lambda_3)q^{j_2+\frac{r}{2}} \rangle_{i_2+r}}{\langle e(\lambda_{23})q^{i_2+\frac{r}{2}+k-r} \rangle_r} \right\} C_{i_1+j_1-k,0,i_2+k,j_2}. \quad (3.6) \]

**Proof.** (1) We prove it by induction on \( j_2 \). The equality (3.5) in case \( j_2 = 1 \) is equal to (3.1) in case \( j_2 = 1 \). Hence the equality (3.5) in case \( j_2 = 1 \) is true. Next we assume the equality (3.5) for a fixed \( j_2 \). It follows from (3.1) by the change of \( j_2 \) into \( j_2 + 1 \) that

\[ C_{i_1,j_1,i_2,j_2+1} = \frac{\langle e(\lambda_1)q^{\frac{j_2}{2}} \rangle_{i_1+1}}{\langle e(\lambda_{23})q^{i_2+\frac{j_2}{2}} \rangle_{j_2+1}} C_{i_1+1,j_1,i_2,j_2} \]

\[ - \frac{\langle e(\lambda_2)q^{\frac{j_2}{2}} \rangle_{i_2+1}}{\langle e(\lambda_{23})q^{i_2+\frac{j_2}{2}} \rangle_{j_2+1}} C_{i_1,j_1,i_2+1,j_2}. \quad (3.7) \]

Substitute (3.5) with the change of \( i_1 \) into \( i_1 + 1 \) into (3.7), and substitute (3.5) with the change of \( i_2 \) into \( i_2 + 1 \) into (3.7). Then we have

\[ C_{i_1,j_1,i_2,j_2+1} = \frac{\langle e(\lambda_1)q^{\frac{j_2}{2}} \rangle_{i_1+1}}{\langle e(\lambda_{23})q^{i_2+\frac{j_2}{2}} \rangle_{j_2+1}} \left\{ \prod_{r=1}^{j_2} \frac{\langle e(\lambda_1)q^{\frac{r}{2}} \rangle_{i_1+1+r}}{\langle e(\lambda_{23})q^{i_2+\frac{r}{2}+k} \rangle_r} \right\} C_{i_1+j_2+1,j_1,i_2,0} \]

\[ + \sum_{k=1}^{j_2} (-1)^{j_2-k} \prod_{r=1}^{j_2-k} \frac{\langle e(\lambda_1)q^{\frac{r}{2}} \rangle_{i_1+1+r}}{\langle e(\lambda_{23})q^{i_2+\frac{r}{2}+k} \rangle_r} \prod_{r=1}^{k} \frac{\langle e(\lambda_2)q^{\frac{r}{2}} \rangle_{i_2+r}}{\langle e(\lambda_{23})q^{i_2+\frac{r}{2}+k-r} \rangle_r} C_{i_1+j_2+1-k,j_1,i_2+k,0} \]

\[ - \frac{\langle e(\lambda_2)q^{\frac{j_2}{2}} \rangle_{i_2+1}}{\langle e(\lambda_{23})q^{i_2+\frac{j_2}{2}} \rangle_{j_2+1}} \left\{ \sum_{k=1}^{j_2} (-1)^{j_2-k-1} \prod_{r=1}^{j_2-k-1} \frac{\langle e(\lambda_1)q^{\frac{r}{2}} \rangle_{i_1+1+r}}{\langle e(\lambda_{23})q^{i_2+\frac{r}{2}+k} \rangle_r} \prod_{r=1}^{k-1} \frac{\langle e(\lambda_2)q^{\frac{r}{2}} \rangle_{i_2+1+r}}{\langle e(\lambda_{23})q^{i_2+\frac{r}{2}+k-r} \rangle_r} \right. \]

\[ \times C_{i_1+j_2-k+1,j_1,i_2+k,0} + (-1)^{j_2-1} \prod_{r=1}^{j_2} \frac{\langle e(\lambda_2)q^{\frac{r}{2}} \rangle_{i_2+1+r}}{\langle e(\lambda_{23})q^{i_2+\frac{r}{2}+j_2+1-r} \rangle_r} C_{i_1,j_1,i_2+j_2+1,0} \left. \right\} \]

\[ = \sum_{k=0}^{j_2+1} (-1)^k \prod_{r=1}^{j_2+1-k} \frac{\langle e(\lambda_1)q^{\frac{r}{2}} \rangle_{i_1+r}}{\langle e(\lambda_{23})q^{i_2+\frac{r}{2}+k} \rangle_r} \prod_{r=1}^{k} \frac{\langle e(\lambda_2)q^{\frac{r}{2}} \rangle_{i_2+r}}{\langle e(\lambda_{23})q^{i_2+\frac{r}{2}+k-r} \rangle_r} \]

\[ \times C_{i_1+j_2+1-k,j_1,i_2+k,0}. \quad (3.8) \]

Here the last equality follows from

\[ \langle e(\lambda_{23})q^{i_2+\frac{j_2}{2}} \rangle_k + \langle e(\lambda_{23})q^{i_2+\frac{j_2}{2}+k} \rangle_{j_2-k+1} = \langle e(\lambda_{23})q^{i_2+\frac{j_2}{2}} \rangle_{j_2+1}. \]
The right most of (3.8) is the right of (3.5) with the change of \( j_2 \mapsto j_2 + 1 \). Thus we have proved the required equality.

(2) It is similarly proved by induction on \( j_1 \). The equality (3.6) in case \( j_1 = 1 \) is equal to (3.2) in case \( j_1 = 1 \). Hence the equality (3.6) holds true in case \( j_1 = 1 \). Next we assume the equality (3.6) for a fixed \( j_1 \). It follows from (3.2) by the change of \( j_1 \mapsto j_1 + 1 \) that

\[
C_{i_1,j_1+1,i_2,j_2} = -\frac{\langle e(\lambda_{123})q^{i_2+j_1+1}\rangle_{i_1+1}}{\langle e(\lambda_{23})q^{i_2+1}\rangle_{j_1+1}} C_{i_1+1,j_1,i_2,j_2} \]

\[
-\frac{\langle e(\lambda_2)q^{i_2+1}\rangle_{i_2+1}}{\langle e(\lambda_{23})q^{i_2+1}\rangle_{j_1+1}} C_{i_1,j_1+1,i_2,j_2}. \tag{3.9}
\]

Substitute (3.6) with the change of \( i_1 \mapsto i_1 + 1 \) into (3.9), and substitute (3.6) with the change of \( i_2 \mapsto i_2 + 1 \) into (3.9). Then we have

\[
C_{i_1,j_1+1,i_2,j_2} = (-)^{j_1+1} \frac{\langle e(\lambda_{123})q^{i_2+j_1+1}\rangle_{i_1+1}}{\langle e(\lambda_{23})q^{i_2+1}\rangle_{j_1+1}} \\
\times \left\{ \prod_{r=1}^{j_1} \frac{\langle e(\lambda_{123})q^{i_2+1+j_1-1}\rangle_{i_1+1+r}}{\langle e(\lambda_{23})q^{i_2+1}\rangle_{i_1+1+r}} C_{i_1+1,j_1+1,i_2,j_2} \right. \\
+ \sum_{k=1}^{j_1} \prod_{r=1}^{j_1-k} \frac{\langle e(\lambda_{123})q^{i_2+1+j_1-1}\rangle_{i_1+1+r}}{\langle e(\lambda_{23})q^{i_2+1}\rangle_{i_1+1+r}} \times \\
\left. \prod_{r=1}^{k} \frac{\langle e(\lambda_2)q^{i_2+1}\rangle_{i_2+r}}{\langle e(\lambda_{23})q^{i_2+1+k}\rangle_{i_1+1+r}} C_{i_1,j_1+1,i_2+1,k,j_2} \right\} \\
+ (-)^{j_1+1} \frac{\langle e(\lambda_2)q^{i_2+1}\rangle_{i_2+1}}{\langle e(\lambda_{23})q^{i_2+1}\rangle_{j_1+1}} \\
\times \left\{ \sum_{k=1}^{j_1} \prod_{r=1}^{j_1-k} \frac{\langle e(\lambda_{123})q^{i_2+1+j_1-1}\rangle_{i_1+r}}{\langle e(\lambda_{23})q^{i_2+1}\rangle_{i_1+r}} \times \\
\prod_{r=1}^{k} \frac{\langle e(\lambda_2)q^{i_2+1}\rangle_{i_2+r}}{\langle e(\lambda_{23})q^{i_2+1+k}\rangle_{i_1+r}} C_{i_1,j_1+1+k,i_2,j_2} \right. \\
+ \prod_{r=1}^{j_1} \frac{\langle e(\lambda_{123})q^{i_2+1}\rangle_{i_2+1+r}}{\langle e(\lambda_{23})q^{i_2+1}\rangle_{i_1+1+r}} C_{i_1,j_1+1,i_2,j_2} \right\} \\
= (-1)^{j_1+1} \sum_{k=0}^{j_1+1} \prod_{r=1}^{j_1+1-k} \frac{\langle e(\lambda_{123})q^{i_2+1+j_1-1}\rangle_{i_1+r}}{\langle e(\lambda_{23})q^{i_2+1}\rangle_{j_1+1+r}} \times \\
\prod_{r=1}^{j_1+1-k} \frac{\langle e(\lambda_2)q^{i_2+1}\rangle_{i_2+r}}{\langle e(\lambda_{23})q^{i_2+1+k}\rangle_{i_1+1+r}} C_{i_1,j_1+1,i_2+j_2}. \tag{3.10}
\]
\( \times \prod_{r=1}^{k} \frac{\langle e(\lambda_3)q^{i_2}\rangle_{r^2}}{\langle e(\lambda_23)q^{i_2+k-r}\rangle_{r}} C_{i_3+j_3+1-k,0,i_2+k,j_2}. \) \hspace{1cm} (3.10)

The right most of (3.10) is the right of (3.6) with the change of \( j_1 \mapsto j_1 + 1 \). It competes the proof. \( \Box \)

By using the equalities in Lemma 3.2, we obtain the expressions of the connection coefficients \( p_{ij} \) defined by

\[ C_{0,i,0,m-i} = \sum_{j=0}^{m} p_{ij} C_{m-j,0,j,0} \] \hspace{1cm} (3.11)

for \( 0 \leq i \leq m \).

**Proposition 3.3.** For \( 0 \leq i, j \leq m \), we have the following.

(1)

\[ p_{ij} = (-)^i \sum_{0 \leq k \leq m-i \atop 0 \leq l \leq i \atop k+l=j} (-)^k \prod_{r=1}^{m-i-k} \frac{s(\lambda_1 + i+r-1\frac{g}{2})}{s(\lambda_23 + (k + i+r-1\frac{g}{2}) g)} \prod_{r=1}^{k} \frac{s(\lambda_2 + i+r-1\frac{g}{2})}{s(\lambda_23 + (k + i+r-1\frac{g}{2}) g)} \times \prod_{r=1}^{i-l} \frac{s(\lambda_1 + m+i+k-r\frac{g}{2})s(m-i-k+r\frac{g}{2})}{s(\lambda_23 + (j + \frac{r-1}{2}) g)s(\frac{r}{2} g)} \prod_{r=1}^{l} \frac{s(\lambda_3 + k+r-1\frac{g}{2})s(k+r\frac{g}{2})}{s(\lambda_23 + (j - \frac{r+1}{2}) g)s(\frac{r}{2} g)}. \] \hspace{1cm} (3.12)

(2)

\[ p_{ij} = (-)^i \sum_{0 \leq k \leq i \atop 0 \leq l \leq m-i \atop k+l=j} (-)^k \prod_{r=1}^{i-k} \frac{s(\lambda_1 + m+i-r\frac{g}{2})}{s(\lambda_23 + (k + m-i-r\frac{g}{2}) g)} \prod_{r=1}^{k} \frac{s(\lambda_2 + m-i-r\frac{g}{2})}{s(\lambda_23 + (k + m-i-r\frac{g}{2}) g)} \times \prod_{r=1}^{m-i-l} \frac{s(\lambda_1 + i-k+r\frac{g}{2})s(i-k+r\frac{g}{2})}{s(\lambda_23 + (j - \frac{r}{2}) g)s(\frac{r}{2} g)} \prod_{r=1}^{l} \frac{s(\lambda_3 + k+r\frac{g}{2})s(k+r\frac{g}{2})}{s(\lambda_23 + (j + \frac{r+1}{2}) g)s(\frac{r}{2} g)}. \] \hspace{1cm} (3.13)

**Proof.** (1) When \( i_1 = i_2 = 0 \), \( j_1 = i \), \( j_2 = m - i \), (1) of Lemma 3.2 implies

\[ C_{0,i,0,m-i} = \sum_{k=0}^{m-i} (-)^k \left\{ \prod_{r=1}^{m-i-k} \frac{\langle e(\lambda_1)q^{\frac{i}{2}}\rangle_{r}}{\langle e(\lambda_23)q^{\frac{i}{2}+k}\rangle_{r}} \prod_{r=1}^{k} \frac{\langle e(\lambda_2)q^{\frac{i}{2}}\rangle_{r}}{\langle e(\lambda_23)q^{\frac{i}{2}+k-r}\rangle_{r}} \right\} C_{m-i-k,i,k,0}. \] \hspace{1cm} (3.14)
When $i_1 = m - i - k$, $i_2 = k$, $j_1 = i$, $j_2 = 0$, (2) of Lemma 3.2 implies

$$C_{m-i-k, i, k, 0}$$

$$= (-)^i \sum_{l=0}^{i-l} \left\{ \prod_{r=1}^{i-l} \frac{\langle e(\lambda_{123})q^{k+i-r} \rangle_{m-i-k+r}}{\langle e(\lambda_{23})q^l \rangle_r} \prod_{r=1}^{l} \frac{\langle e(\lambda_3)q^{k+r} \rangle_r}{\langle e(\lambda_{23})q^{l-r} \rangle_r} \right\} C_{m-k, l, 0, k+l, 0}. \quad (3.15)$$

Substituting (3.14) into (3.15) leads to the relation

$$p_{ij} = (-)^i \sum_{0 \leq k \leq m-i \atop 0 \leq l \leq i \atop k+l=j} (-)^k \prod_{r=1}^{m-i-k} \frac{\langle e(\lambda_1)q^r \rangle_{m-i-k+r}}{\langle e(\lambda_{23})q^l \rangle_r} \prod_{r=1}^{k} \frac{\langle e(\lambda_2)q^{k+r} \rangle_r}{\langle e(\lambda_{23})q^{k+r-k} \rangle_r}$$

$$\times \prod_{r=1}^{i-l} \frac{\langle e(\lambda_{123})q^{k+i-r} \rangle_{m-i-k+r}}{\langle e(\lambda_{23})q^l \rangle_r} \prod_{r=1}^{l} \frac{\langle e(\lambda_3)q^{k+r} \rangle_r}{\langle e(\lambda_{23})q^{l-r} \rangle_r}$$

$$= (-)^i \sum_{0 \leq k \leq m-i \atop 0 \leq l \leq i \atop k+l=j} (-)^k \prod_{r=1}^{m-i-k} \frac{\langle e(\lambda_1)q^{r \cdot \frac{1}{123}} \rangle_{1} \langle q^{m-i-k+r} \rangle_{1}}{\langle e(\lambda_{23})q^l \rangle_{1} \langle q^{l-r} \rangle_{1}} \prod_{r=1}^{k} \frac{\langle e(\lambda_2)q^{k+r} \rangle_{1} \langle q^{k+r-k} \rangle_{1}}{\langle e(\lambda_{23})q^{k+r-k} \rangle_{1} \langle q^{k+r} \rangle_{1}}$$

$$\times \prod_{r=1}^{i-l} \frac{\langle e(\lambda_{123})q^{k+i-r} \rangle_{1} \langle q^{m-i-k+r} \rangle_{1}}{\langle e(\lambda_{23})q^l \rangle_{1} \langle q^{l-r} \rangle_{1}} \prod_{r=1}^{l} \frac{\langle e(\lambda_3)q^{k+r} \rangle_{1} \langle q^{k+r-k} \rangle_{1}}{\langle e(\lambda_{23})q^{k+r-k} \rangle_{1} \langle q^{k+r} \rangle_{1}}. \quad (3.16)$$

which implies the required result (3.12).

(2) When $i_1 = i_2 = 0$, $j_1 = i$, $j_2 = m - i$, (2) of Lemma 3.2 implies

$$C_{0, i, 0, m-i}$$

$$= (-)^i \sum_{k=0}^{i-l} \left\{ \prod_{r=1}^{i} \frac{\langle e(\lambda_{123})q^{k+i-r} \rangle_{r}}{\langle e(\lambda_{23})q^{k} \rangle_{r}} \prod_{r=1}^{k} \frac{\langle e(\lambda_3)q^{k+r} \rangle_{r}}{\langle e(\lambda_{23})q^{k+r-k} \rangle_{r}} \right\} C_{i-k, 0, k, m-i}. \quad (3.17)$$

When $i_1 = i - k$, $i_2 = k$, $j_1 = 0$, $j_2 = m - i$, (1) of Lemma 3.2 implies
\[ C_{i-k,0,k,m-i} \]

\[ = \sum_{l=0}^{m-i} \left\{ \prod_{r=1}^{m-i-l} \frac{\langle e(\lambda_1) \rangle_{i-k+r}}{\langle e(\lambda_2) \rangle_{l+r}} \prod_{r=1}^{l} \frac{\langle e(\lambda_2) \rangle_{k+r}}{\langle e(\lambda_2) \rangle_{l+r}} \right\} C_{m-k-l,0,k+l,0}. \tag{3.18} \]

Substituting (3.18) into (3.17) leads to the relation

\[
\begin{align*}
p_{ij} &= (-)^i \sum_{0 \leq k \leq i} (-)^{i-k} \prod_{r=1}^{i-k} \frac{\langle e(\lambda_1) \rangle_{i-k+r}}{\langle e(\lambda_2) \rangle_{l+r}} \prod_{r=1}^{l} \frac{\langle e(\lambda_2) \rangle_{k+r}}{\langle e(\lambda_2) \rangle_{l+r}} \\
& \quad \times \prod_{r=1}^{m-i-l} \frac{\langle e(\lambda_1) \rangle_{i-k+r}}{\langle e(\lambda_2) \rangle_{l+r}} \prod_{r=1}^{l} \frac{\langle e(\lambda_2) \rangle_{k+r}}{\langle e(\lambda_2) \rangle_{l+r}} \\
& = (-)^i \sum_{0 \leq k \leq i} (-)^{i-k} \prod_{r=1}^{i-k} \frac{\langle e(\lambda_1) \rangle_{m+i-r}}{\langle e(\lambda_2) \rangle_{m+i-r}} \prod_{r=1}^{l} \frac{\langle e(\lambda_2) \rangle_{m+k-r}}{\langle e(\lambda_2) \rangle_{m+k-r}} \\
& \quad \times \prod_{r=1}^{m-i-l} \frac{\langle e(\lambda_1) \rangle_{i-k+r}}{\langle e(\lambda_2) \rangle_{l+r}} \prod_{r=1}^{l} \frac{\langle e(\lambda_2) \rangle_{k+r}}{\langle e(\lambda_2) \rangle_{l+r}} \tag{3.19}
\end{align*}
\]

which implies (3.13).

It completes the proof. \[ \square \]

In the next subsection, we shall give an expression of \( p_{ij} \) in terms of the basic hypergeometric polynomial \( 8\phi_7 \) and that in terms of \( 4\phi_3 \).

### 3.2 Connection coefficients in terms of the basic hypergeometric polynomials

**Proposition 3.4.** (1) For \( 0 \leq i + j \leq m \), we have
\[ p_{ij} = (-1)^{i+j} \prod_{r=1}^{m-i-j} \frac{\langle e(\lambda_1)q^{i+j-1} \rangle_1}{\langle e(\lambda_2)q^{i+j-r} \rangle_1} \prod_{r=1}^{j} \frac{\langle e(\lambda_2)q^{i+j-1} \rangle_1}{\langle e(\lambda_2)q^{j-r} \rangle_1} \times \prod_{r=1}^{i} \frac{\langle e(\lambda_1)q^{m-i+j-r} \rangle_1}{\langle e(\lambda_2)q^{m-i+j-r} \rangle_1} \times \varphi_7 \left( e(-\lambda_2)q^{-\frac{i+j}{2}}m, -e(-\lambda_2)q^{-\frac{i+j}{2}}m, e(2\lambda_1)q^{i+j}, e(-2\lambda_2)q^{m-i+j}, e(-2\lambda_2)q^{2-i-j}, e(-2\lambda_2)q^{2-i-j}, e(-2\lambda_2)q^{2-i-j}, e(-2\lambda_2)q^{2-i-j} \right). \]

(2) For \( 2m \geq i + j \geq m \), we have

\[ p_{ij} = (-1)^{m} \prod_{r=1}^{m-i} \frac{\langle e(\lambda_2)q^{i+j-1} \rangle_1}{\langle e(\lambda_2)q^{m-i+j-r} \rangle_1} \prod_{r=1}^{m-j} \frac{\langle e(\lambda_2)q^{m-i+j-r} \rangle_1}{\langle e(\lambda_2)q^{m-i+j-r} \rangle_1} \times \prod_{r=1}^{i+j-m} \frac{\langle e(\lambda_1)q^{m-i+j-r} \rangle_1}{\langle e(\lambda_2)q^{m-i+j-r} \rangle_1} \left( q^{\frac{i+j}{2}} \right)_1 \times \varphi_7 \left( e(-\lambda_2)q^{i+j-1}m, -e(-\lambda_2)q^{i+j-1}m, e(-2\lambda_2)q^{i+j-2m}, e(-2\lambda_2)q^{i+j-2m}, e(-2\lambda_2)q^{i+j-2m}, e(-2\lambda_2)q^{i+j-2m}, e(-2\lambda_2)q^{i+j-2m}, e(-2\lambda_2)q^{i+j-2m} \right). \]

**Proof.** (1) When \( 0 \leq i + j \leq m \), (1) of Proposition 3.3 shows

\[ p_{ij} = (-1)^{i+j} \sum_{0 \leq l \leq \min\{i, j\}} (-1)^{l} \prod_{r=1}^{m-i-j+l} \frac{\langle e(\lambda_1)q^{l+r-1} \rangle_1}{\langle e(\lambda_2)q^{l+i+j+r} \rangle_1} \prod_{r=1}^{j-l} \frac{\langle e(\lambda_2)q^{l+r-1} \rangle_1}{\langle e(\lambda_2)q^{l+i+j+r} \rangle_1}. \]
\[ \prod_{r=1}^{i-l} \frac{e(\lambda_{123})q^{\frac{m-i-j-r-1}{2}}}{e(\lambda_{23})q^{j+r-\frac{i}{2}}} \frac{1}{q^{\frac{r}{2}}} \prod_{r=1}^{j-l} \frac{e(\lambda_{2})q^{\frac{i-r-1}{2}}}{e(\lambda_{23})q^{j-r-\frac{i+1}{2}}} \frac{1}{q^{\frac{r}{2}}} \prod_{r=1}^{l} \frac{e(\lambda_{3})q^{\frac{i-r-1}{2}}}{e(\lambda_{23})q^{j-r-\frac{i+2}{2}}} \frac{1}{q^{\frac{r}{2}}} \]

(3.20)

It is seen that

\[ \prod_{r=1}^{m-i-j+l} \frac{e(\lambda_{1})q^{\frac{i+r-1}{2}}}{e(\lambda_{23})q^{j+\frac{i+1-r}{2}}} \frac{1}{q^{\frac{r}{2}}} \prod_{r=1}^{j-l} \frac{e(\lambda_{2})q^{\frac{i-r-1}{2}}}{e(\lambda_{23})q^{j-r-\frac{i+1}{2}}} \frac{1}{q^{\frac{r}{2}}} \prod_{r=1}^{l} \frac{e(\lambda_{3})q^{\frac{i-r-1}{2}}}{e(\lambda_{23})q^{j-r-\frac{i+2}{2}}} \frac{1}{q^{\frac{r}{2}}} \]

(3.21)

and

\[ \prod_{r=1}^{i-l} \frac{e(\lambda_{123})q^{\frac{m-i-j-r-1}{2}}}{e(\lambda_{23})q^{j+i-\frac{r}{2}}} \frac{1}{q^{\frac{r}{2}}} \prod_{r=1}^{j-l} \frac{e(\lambda_{2})q^{\frac{i-r-1}{2}}}{e(\lambda_{23})q^{j-r-\frac{i+1}{2}}} \frac{1}{q^{\frac{r}{2}}} \prod_{r=1}^{l} \frac{e(\lambda_{3})q^{\frac{i-r-1}{2}}}{e(\lambda_{23})q^{j-r-\frac{i+2}{2}}} \frac{1}{q^{\frac{r}{2}}} \]

(3.22)
and

\[
\prod_{r=1}^{l} \left( \frac{e(\lambda_3)q^{\frac{j-r-1}{2}}}{e(\lambda_3)q^{\frac{j-i}{2}}} \right)_1 \frac{q^{\frac{j-r-1}{2}}}{(q^2)_1} = \prod_{r=1}^{l} \left( \frac{e(\lambda_3)q^{\frac{j-m-1}{2}}}{e(\lambda_3)q^{\frac{j-i}{2}}} \right)_1 \frac{q^{\frac{j-m-1}{2}}}{(q^2)_1}
\]

\[
= (-)^{i+j} \prod_{r=1}^{l} \left( \frac{e(-\lambda_3)q^{\frac{r-j}{2}}}{e(-\lambda_3)q^{\frac{r-i}{2}}} \right)_1 \frac{q^{\frac{r-j}{2}}}{(q^2)_1} (3.23)
\]

Therefore, substituting (3.21-23) into (3.20) implies

\[
p_{ij} = (-)^{i+j} \prod_{r=1}^{m-i-j} \left( \frac{e(\lambda_1)q^{\frac{m-j-i}{2}}}{e(\lambda_1)q^{\frac{m-j-r}{2}}} \right)_1 \frac{q^{\frac{m-j-i}{2}}}{(q^2)_1} \prod_{r=1}^{j} \left( \frac{e(\lambda_2)q^{\frac{m+j-1}{2}}}{e(\lambda_2)q^{\frac{m+j-i}{2}}} \right)_1 \frac{q^{\frac{m+j-1}{2}}}{(q^2)_1}
\]

\[
\times \prod_{r=1}^{i} \left( \frac{e(\lambda_1)q^{\frac{m+j+i-r-2}{2}}}{e(\lambda_1)q^{\frac{m+j+i-2}{2}}} \right)_1 \frac{q^{\frac{m+j+i-r-2}{2}}}{(q^2)_1}
\]

\[
\times \sum_{0 \leq t \leq \min(i,j)} \prod_{r=1}^{l} \left( \frac{e(-\lambda_3)q^{\frac{j-i+t}{2}}}{e(-\lambda_3)q^{\frac{j-i+r}{2}}} \right)_1 \frac{q^{\frac{j-i+t}{2}}}{(q^2)_1} \prod_{r=1}^{j} \left( \frac{e(-\lambda_3)q^{\frac{r-j-i+t}{2}}}{e(-\lambda_3)q^{\frac{r-j-i+r}{2}}} \right)_1 \frac{q^{\frac{r-j-i+t}{2}}}{(q^2)_1}
\]

\[
\times \prod_{r=1}^{j} \left( \frac{e(-\lambda_3)q^{\frac{j-i-t}{2}}}{e(-\lambda_3)q^{\frac{j-i+r}{2}}} \right)_1 \frac{q^{\frac{j-i-t}{2}}}{(q^2)_1} \prod_{r=1}^{j} \left( \frac{e(-\lambda_3)q^{\frac{i+j-1-t}{2}}}{e(-\lambda_3)q^{\frac{i+j-1+r}{2}}} \right)_1 \frac{q^{\frac{i+j-1-t}{2}}}{(q^2)_1}
\]

\[
\times \prod_{r=1}^{i} \left( \frac{e(-\lambda_3)q^{\frac{m+j+i-2}{2}}}{e(-\lambda_3)q^{\frac{m+j+i-r-2}{2}}} \right)_1 \frac{q^{\frac{m+j+i-2}{2}}}{(q^2)_1}
\]

\[
\times \frac{\varphi(1+6 \varphi)}{e(-\lambda_3)q^{\frac{j-i+1}{2}}, -e(-\lambda_3)q^{\frac{j-i-1}{2}}, e(2\lambda_1)q^{m-j}, e(-\lambda_3)q^{\frac{j-i+1}{2}}, -e(-\lambda_3)q^{\frac{j-i-1}{2}}, e(-2\lambda_2)q^{i-j}, e(-2\lambda_3)q^{1-m-j}, e(-2\lambda_3)q^{2j-1-i}, q^{-i}, e(-2\lambda_3)q^{1-j}, q^{-j}, e(-2\lambda_2)q^{2-i-j}, e(-2\lambda_3)q^{2-m-i-j}, q^{m-i-j+1}, e(-2\lambda_3)q^{2-3j}}{q, e(-2\lambda_2)q^{2-i}}
\]

\[
(3.24)
\]
Here we have used the identities

\[
\prod_{r=1}^l \frac{\langle Aq^\frac{r}{2} \rangle_1}{\langle Bq^\frac{r}{2} \rangle_1} = \left( \frac{B}{A} \right)^l \prod_{r=1}^l \frac{1 - A^2 q^r}{1 - B^2 q^r} = \left( \frac{B}{A} \right)^l \frac{\langle A^2 q; q \rangle_l}{\langle B^2 q; q \rangle_l} \tag{3.25}
\]

and

\[
\prod_{r=1}^l \frac{\langle Aq^r \rangle_1}{\langle Bq^r \rangle_1} = \left( \frac{B}{A} \right)^l \frac{\langle A^2 q^2; q^2 \rangle_l}{\langle B^2 q^2; q^2 \rangle_l} = \left( \frac{B}{A} \right)^l \frac{\langle A q; q \rangle_l (-A q; q)_l}{\langle B q; q \rangle_l (-B q; q)_l}. \tag{3.26}
\]

(2) When \(2m \geq i+j \geq m\), (1) of Proposition 3.3 with the change of the running index \(k \mapsto m-i-k\) shows that

\[
p_{ij} = (-)^m \sum_{0 \leq k \leq \min\{m-i,m-j\}} (-)^k \prod_{r=1}^k \frac{\langle e(\lambda_1)q^{\frac{i+r-1}{2}} \rangle_1}{\langle e(\lambda_2)q^{m-k+\frac{i+r-1}{2}} \rangle_1} \prod_{r=1}^{m-i-k} \frac{\langle e(\lambda_2)q^{\frac{i+r-1}{2}} \rangle_1}{\langle e(\lambda_3)q^{m-k-\frac{i+r-1}{2}} \rangle_1} \times \prod_{r=1}^{m-j-k} \frac{\langle e(\lambda_{123})q^{m-\frac{i+j+r}{2}} \rangle_1}{\langle e(\lambda_{23})q^{m-\frac{i+j+r}{2}} \rangle_1} \times \prod_{r=1}^{i+j-m+k} \frac{\langle e(\lambda_{123})q^{\frac{i-j+r}{2}} \rangle_1}{\langle e(\lambda_{23})q^{\frac{i-j+r}{2}} \rangle_1} \times \prod_{r=1}^{i+j-m+k} \frac{\langle e(\lambda_{123})q^{\frac{i-j+r}{2}} \rangle_1}{\langle e(\lambda_{23})q^{\frac{i-j+r}{2}} \rangle_1} \tag{3.27}
\]

It is seen that

\[
\prod_{r=1}^k \frac{\langle e(\lambda_1)q^{\frac{i+r-1}{2}} \rangle_1}{\langle e(\lambda_3)q^{m-k+\frac{i+r-1}{2}} \rangle_1} \prod_{r=1}^{m-i-k} \frac{\langle e(\lambda_2)q^{\frac{i+r-1}{2}} \rangle_1}{\langle e(\lambda_3)q^{m-k-\frac{i+r-1}{2}} \rangle_1} = \frac{\langle e(\lambda_{123})q^{m-k-\frac{i+r}{2}} \rangle_1}{\langle e(\lambda_{123})q^{m-k+\frac{i+r}{2}} \rangle_1} \prod_{r=1}^{m-1} \frac{\langle e(\lambda_{23})q^{m-\frac{i+r}{2}} \rangle_1}{\langle e(\lambda_{123})q^{m-\frac{i+r}{2}} \rangle_1}
\]

\[
= \prod_{r=1}^k \frac{\langle e(\lambda_{23})q^{m-k-\frac{i+r}{2}} \rangle_1}{\langle e(\lambda_{23})q^{m-k+\frac{i+r}{2}} \rangle_1} \prod_{r=1}^{m-i} \frac{\langle e(\lambda_{23})q^{m-\frac{i+r}{2}} \rangle_1}{\langle e(\lambda_{23})q^{m-\frac{i+r}{2}} \rangle_1}
\]

\[
= (-)^k \prod_{r=1}^k \frac{\langle e(-\lambda_{23})q^{-m+\frac{i+r}{2}} \rangle_1}{\langle e(-\lambda_{23})q^{-m+\frac{i+r}{2}} \rangle_1} \prod_{r=1}^{m-i} \frac{\langle e(\lambda_{2})q^{\frac{i+r}{2}} \rangle_1}{\langle e(\lambda_{2})q^{\frac{i+r}{2}} \rangle_1} \tag{3.28}
\]

and

36
\[
\prod_{r=1}^{m-j-k} \left( e(\lambda_{123})q^{m-j-k-r+1} \right) \left( q^{r} \right)_{1} / \left( e(\lambda_{23})q^{j+r-1} \right)_{1} \left( q^{n} \right)_{1} = \prod_{r=1}^{m-j} \left( e(\lambda_{123})q^{m-j-r+2} \right)_{1} / \left( e(\lambda_{23})q^{j+r-2} \right)_{1} \prod_{r=1}^{k} \left( e(\lambda_{123})q^{m-j-r+1} \right) \left( q^{n} \right)_{1} \left( e(\lambda_{23})q^{j+r-1} \right)_{1} = (-k)^{m-j} \prod_{r=1}^{k} \left( e(\lambda_{123})q^{m-j-r+1} \right) \left( q^{n} \right)_{1} / \left( e(\lambda_{23})q^{j+r-1} \right)_{1} (3.29)
\]

and

\[
\prod_{r=1}^{i+j-m+k} \left( e(\lambda_{3})q^{m-i-k+r-1} \right) \left( q^{m} \right)_{1} / \left( e(\lambda_{23})q^{j+m-1} \right)_{1} = \prod_{r=1}^{k} \left( e(\lambda_{3})q^{m-j-r+1} \right) \left( q^{m} \right)_{1} / \left( e(\lambda_{23})q^{j+m-1} \right)_{1} \prod_{r=1}^{i+j-m} \left( e(\lambda_{3})q^{m-j-r+1} \right) \left( q^{m} \right)_{1} / \left( e(\lambda_{23})q^{j+m-1} \right)_{1} = (-k)^{i+j-m} \prod_{r=1}^{k} \left( e(\lambda_{3})q^{m-j-r+1} \right) \left( q^{m} \right)_{1} / \left( e(\lambda_{23})q^{j+m-1} \right)_{1} (3.30)
\]

Therefore, substituting (3.28-30) into (3.27) shows

\[
p_{ij} = (-m)^{m-j} \prod_{r=1}^{m-j} \left( e(\lambda_{2})q^{m-j} \right)_{1} / \left( e(\lambda_{23})q^{m-j-r} \right)_{1} \prod_{r=1}^{m-j} \left( e(\lambda_{123})q^{m-j-r+1} \right) \left( q^{r} \right)_{1} / \left( e(\lambda_{23})q^{j+r-1} \right)_{1} \prod_{r=1}^{i+j-m} \left( e(\lambda_{3})q^{m-j-r+1} \right) \left( q^{m} \right)_{1} / \left( e(\lambda_{23})q^{j+m-1} \right)_{1} \sum_{0 \leq b \leq \min(m-i,m-j)} \prod_{r=1}^{k} \left( e(-\lambda_{123})q^{i+m-r} \right)_{1} \left( e(-\lambda_{2})q^{i+m-r+1} \right)_{1} \left( e(\lambda_{1})q^{i+m-r+1} \right)_{1} \prod_{r=1}^{k} \left( e(-\lambda_{23})q^{i+m-r} \right)_{1} \left( e(-\lambda_{2})q^{i+m-r+1} \right)_{1} \left( e(\lambda_{123})q^{i+m-r+1} \right)_{1} \left( q^{i+m-r+1} \right)_{1} / \left( e(\lambda_{23})q^{i+m-r+1} \right)_{1} \left( q^{i+m-r+1} \right)_{1} = (-m)^{m-j} \prod_{r=1}^{m-j} \left( e(\lambda_{2})q^{i+m-r} \right)_{1} / \left( e(\lambda_{23})q^{i+m-r} \right)_{1} \prod_{r=1}^{m-j} \left( e(\lambda_{123})q^{i+m-r+2} \right)_{1} / \left( e(\lambda_{23})q^{i+m-r+1} \right)_{1}
\]

37
Here we have used the identity (3.25) and (3.26). □

(1)

\[
p_4 \times 4 \times 4 \times \langle p \rangle \langle p \rangle = (\langle p \rangle)_{1,1} (\langle p \rangle)_{1,1}
\]

\[
(\langle p \rangle)_{1,1} (\langle p \rangle)_{1,1}
\]

\[
\times s \varphi \left( e(-\lambda_{23})q^{i+j-m}, e(-\lambda_{23})q^{i+j-m}, e(-2\lambda_{23})q^{1+i-2m},
\right.
\]

\[
\left. e(-2\lambda_{23})q^{i+j-m}, e(2\lambda_1)q^i, e(-2\lambda_{23})q^{1+i-m}, q^j, q^i, q^j
\right)
\]

\[
; q, e(-2\lambda_{12})q^{2-i} \right).
\]

(3.31)

Here we have used the identity (3.25) and (3.26).

In the next step, we rewrite the connection coefficients \( p_{ij} \) in terms of \( 4 \varphi_3 \).

**Proposition 3.5.** (1) In case \( 0 \leq i + j \leq m \), we have

\[
p_{ij} = (-)^{i+j} \prod_{r=1}^{i} \frac{\langle q^{m-j-r} \rangle_1}{\langle q^2 \rangle_1}
\]

\[
\times \frac{\langle e(\lambda_{23})q^{i+j-r} \rangle_1 \prod_{r=1}^{i} \langle e(\lambda_{23})q^{m+j-r} \rangle_1 \prod_{r=1}^{j} \langle e(\lambda_2)q^{i+r} \rangle_1 \prod_{r=1}^{m-i} \langle e(\lambda_1)q^{i+r} \rangle_1}{\prod_{r=1}^{m+1} \langle e(\lambda_{23})q^{i+j-r} \rangle_1}
\]

\[
\times 4 \varphi_3 \left( q^{-j}, q^{-i}, e(-2\lambda_{23})q^{1-j-m}, e(-2\lambda_{12})q^{1-i-m}, q^m, q^j
\right)
\]

\[
; q, q \right).
\]

(3.32)

\[
= (-)^{i+j} \prod_{r=1}^{i} \frac{\langle q^{m-j-r} \rangle_1}{\langle q^2 \rangle_1}
\]

\[
\times \frac{\langle e(\lambda_{23})q^{i+j-r} \rangle_1 \prod_{r=1}^{i} \langle e(\lambda_{23})q^{m+j-r} \rangle_1 \prod_{r=1}^{j} \langle e(\lambda_2)q^{i+r} \rangle_1 \prod_{r=1}^{m-i} \langle e(\lambda_1)q^{i+r} \rangle_1}{\prod_{r=1}^{m+1} \langle e(\lambda_{23})q^{i+j-r} \rangle_1}
\]

\[
\times 4 \varphi_3 \left( q^{-j}, q^{-i}, e(2\lambda_{23})q^{i-j}, e(2\lambda_{12})q^{i-1}, q^m, e(2\lambda_2), e(2\lambda_{123})q^{m-1}
\right)
\]

\[
; q, q \right).
\]

(3.33)

(2) In case \( m \leq i + j \leq 2m \), we have

38
\[ p_{ij} = (-)^m \prod_{r=1}^{i+j} \left( q^{m-\frac{1}{2}r} \right) _1 \]

\[ \times \left( e(\lambda_2)q^{i+j-\frac{1}{2}} \right)_1 \prod_{r=1}^{i+j-m} \left( e(\lambda_1)q^{m-\frac{1}{2}r-1} \right)_1 \prod_{r=1}^{m-i-1} \left( e(\lambda_2)q^{\frac{1}{2}r-1} \right)_1 \]

\[ \times 4\varphi_3 \left( e(-2\lambda_1)q^{1-i-m}, e(-2\lambda_2)q^{1-j-m}, q^{j-m}, q^{-m} ; q, q \right) \]

\[ = (-)^{i+j} \prod_{r=1}^{i+j} \left( q^{\frac{m-i+j-r}{2}} \right) _1 \]

\[ \times \left( e(\lambda_2)q^{i+j-\frac{1}{2}} \right)_1 \prod_{r=1}^{i+j} \left( e(\lambda_1)q^{m+r-2} \right)_1 \prod_{r=1}^{m-i-1} \left( e(\lambda_2)q^{\frac{1}{2}r-1} \right)_1 \]

\[ \times 4\varphi_3 \left( q^{-i}, e(2\lambda_1)q^{-1}, q^{-j}, e(2\lambda_2)q^{-1} q^{-m}, e(2\lambda_2)q^{m-1} q^{-1} ; q, q \right). \]

**Proof.** (1) Watson’s transformation formula for a terminating very-well poised \( \varphi_7 \) series [10] is

\[ \varphi_7 \left( \frac{a, qa^{1/2}, -qa^{1/2}, b, c, d, e, q^{-n}}{a^{1/2}, -a^{1/2}, qa/b, qa/c, qa/d, qa/e, qa^{n+1}} ; q, q^{a^{2}q^{n+2}/bcde} \right) \]

\[ = \frac{(aq, qa/de; q)_n (aq/bc, d, e, q^{-n})}{(aq/d, qa/e; q)_n} 4\varphi_3 \left( \frac{aq/\bar{bc}, d, e, q^{-n}}{aq/b, qa/c, deq^{-n}/a} ; q, q \right). \]  

(3.36)

The substitution

\[ n = i, \quad a = e(-2\lambda_2)q^{-2j+1-i}, \quad b = e(2\lambda_1)q^{m-j}, \]

\[ c = e(-2\lambda_3)q^{1-i}, \quad d = e(-2\lambda_3)q^{1-m-j}, \quad e = q^{-j} \]

into Watson’s formula (3.36) leads to

\[ \varphi_7 \left( \frac{e(-\lambda_2)q^{-\frac{1}{2}j+\frac{1}{2}}}{e(-\lambda_2)q^{1-\frac{1}{2}j+\frac{1}{2}}}, \quad \frac{e(-\lambda_2)q^{-\frac{1}{2}j+\frac{1}{2}}}{e(-\lambda_2)q^{1-\frac{1}{2}j+\frac{1}{2}}}, \quad \frac{e(\lambda_1)q^{m-j}}{e(-\lambda_2)q^{1-\frac{1}{2}j+\frac{1}{2}}}, \quad \frac{e(-\lambda_2)q^{-1-\frac{1}{2}j}}{e(-\lambda_2)q^{1-\frac{1}{2}j+\frac{1}{2}}}, \quad e(-2\lambda_2)q^{1-i-j}, \quad e(-2\lambda_2)q^{1-2j} \right) \]

\[ e(-2\lambda_2)q^{1-m-j}, \quad e(-2\lambda_2)q^{1-2j+1-i}, \quad q^{-i}, \quad e(-2\lambda_3)q^{1-i-j}, \quad q^{-j} \]

\[ e(-2\lambda_2)q^{2-i-j}, \quad e(-2\lambda_2)q^{2-1-2m-i-j}, \quad q^{m-i-j+1}, \quad e(-2\lambda_2)q^{2-2j} \]
Hence (3.37) and (2) of Proposition 3.4 implies (3.32).

into the Sears’ formula (3.38) implies

which is equal to

which is equal to

Hence (3.37) and (2) of Proposition 3.4 implies (3.32).

Sears’ transformation formula for a terminating series \([10]\) is

The substitution of

into the Sears’ formula (3.38) implies

\[
\begin{align*}
4\varphi_3 \left( \frac{q^{-n}, a_1, a_2, a_3}{b_1, b_2, b_3} ; q, q \right) &= \frac{(b_2/a_1, b_3/a_1; q)_n}{(b_2, b_3; q)_n} a_n^q 4\varphi_3 \left( \frac{q^{-n}, a_1, b_1/a_2, b_1/a_3}{b_1, a_1 q^{-n}/b_2, a_1 q^{-n}/b_3} ; q, q \right). \tag{3.38}
\end{align*}
\]
Hence, combining (3.37), (3.39) with the expression (1) of Proposition 3.4 implies (3.33).

(2) The substitution

\[ n = m - i, \quad a = e(-2\lambda_{23})q^{-2m+1+i}, \quad b = e(2\lambda_1)q^i, \]
\[ c = e(-2\lambda_3)q^{1+i-m}, \quad d = e(-2\lambda_{23})q^{1-m-j}, \quad e = q^{i-m} \]

into Watson’s formula (3.36) implies

\[
8\varphi_7 \left( \frac{e(-2\lambda_{23})q^{1+i-2m}, e(-\lambda_{23})q^{\frac{i+3}{2}-m}, -e(-\lambda_{23})q^{\frac{i+1}{2}-m},}{e(-\lambda_{23})q^{rac{i+1}{2}-m}, -e(-\lambda_{23})q^{\frac{i+1}{2}-m}, e(-2\lambda_{123})q^{2-2m},}{e(2\lambda_1)q^i, e(-2\lambda_3)q^{1+i-m}, e(-2\lambda_{23})q^{1-m-j}, q^{i-m}, q^{i-m}}{e(-2\lambda_{12})q^{1-m}, q^{1+i-j-m}, e(-2\lambda_{23})q^{2-m+i-j}, e(-2\lambda_{23})q^{2-m}} \right) \]
\[
\times 4\varphi_3 \left( \frac{e(-2\lambda_{12})q^{1-1-i-m}, e(-2\lambda_{23})q^{1-j-m}, q^{i-m}, q^{j-m}}{e(-2\lambda_{123})q^{2-2m}, e(-2\lambda_2)q^{1-m}, q^{m}} \right) \]
\[ = \prod_{r=1}^{m-i} \frac{(e(\lambda_{23})q^{\frac{m+r+2}{2}})_{1}(q^{\frac{i+3}{4}})_{1}}{(e(\lambda_{23})q^{\frac{m+r+2}{2}})_{1}(q^{\frac{i+1}{4}+m-r})_{1}} \]
\[ \times 4\varphi_3 \left( \frac{e(-2\lambda_{12})q^{1-i-m}, e(-2\lambda_{23})q^{1-j-m}, q^{i-m}, q^{j-m}}{e(-2\lambda_{123})q^{2-2m}, e(-2\lambda_2)q^{1-m}, q^{m}} \right) \]  

Hence (3.40) and (2) of Proposition 3.4 implies (3.34).

The substitution

\[ n = m - j, \quad a_1 = e(-2\lambda_{12})q^{1-i-m}, \quad a_2 = q^{i-m}, \quad a_3 = e(-2\lambda_{23})q^{1-m-j} \]
\[ b_1 = q^{-m}, \quad b_2 = e(-2\lambda_{123})q^{2-2m}, \quad b_3 = e(-2\lambda_2)q^{1-m} \]

into Sears’s formula (3.38) implies
Hence, combining (3.40), (3.41), (3.42) with the expression (2) of Proposition 3.4 implies (3.35). It completes the proof.

The substitution

\[ n = i, \quad a_1 = e(2\lambda_{23})q^{j-1}, \quad a_2 = q^{j-m}, \quad a_3 = e(-2\lambda_{12})q^{1-i-m} \]
\[ b_1 = q^{-m}, \quad b_2 = e(2\lambda_{3})q^{j-i}, \quad b_3 = e(-2\lambda_{1})q^{1-m+j-i} \]

into Sears’s formula (3.38) implies

\[
4\varphi_3 \left( q^{j-m}, e(-2\lambda_{12})q^{1-i-m}, q^{i-m}, e(-2\lambda_{23})q^{1-j-m}; q, q \right)
\]
\[
= \frac{(e(-2\lambda_{3})q^{-m+i}, e(2\lambda_{1})q^j; q)_{m-j}}{(e(-2\lambda_{12})q^{2-2m}, e(-2\lambda_{2})q^{1-m}; q)_{m-j}} \cdot e(-2\lambda_{12})q^{1-i-m})^{m-j}
\]
\[
\times 4\varphi_3 \left( q^{j-m}, e(-2\lambda_{12})q^{1-i-m}, q^{i-j}, e(2\lambda_{23})q^{j-1} \right)
\]
\[
= \left( \frac{e(\lambda_{3})q^{-i+j}}{e(\lambda_{23})q^{-i+j}} \right)_{i} \cdot e(-2\lambda_{1})q^{1-m+j-i} \cdot e(2\lambda_{23})q^{j-1)_{i}}
\]
\[
\times 4\varphi_3 \left( q^{j-m}, e(2\lambda_{23})q^{j-1}, q^{j-i}, e(-2\lambda_{1})q^{1-i-m}; q, q \right)
\]
\[ \times 4\varphi_3 \left( q^{j-m}, e(2\lambda_{23})q^{j-1}, q^{j-i}, e(-2\lambda_{1})q^{1-i-m}; q, q \right) \, . \quad (3.41) \]

Hence, combining (3.40), (3.41), (3.42) with the expression (2) of Proposition 3.4 implies (3.35). It completes the proof. \( \square \)
The equality
\[ \frac{\prod_{r=1}^{m-i} \langle e(\lambda_1)q^{\frac{i+r-1}{2}} \rangle_1}{\prod_{r=1}^{i} \langle e(\lambda_1)q^{\frac{m-r-1}{2}} \rangle_1} = \left\{ \begin{array}{ll}
\prod_{r=1}^{m-i-j} \langle e(\lambda_1)q^{\frac{i+r-1}{2}} \rangle_1 & \text{for } i + j \leq m, \\
1 & \text{for } i + j \geq m
\end{array} \right. \]

rewrites (3.33) and (3.35) into the following.

**Proposition 3.6.** For \(0 \leq i, j \leq m\), we have

\[
p_{ij} = (-)^{i+j} \prod_{r=1}^{i} \frac{\langle q^{\frac{m-r}{2}} \rangle_1}{\langle q^2 \rangle_1} \times \frac{\pi_{i,j}^{(1)}}{\pi_{i-j}^{(1)}} \pi_{i,j}^{(2)} \pi_{i-j}^{(2)} \times \frac{4\varphi_3 \left( q^{-i}, e(2\lambda_2)q^{i-1}, q^{-j}, e(2\lambda_3)q^{j-1} \middle| e(2\lambda_2), q^{-m}, e(2\lambda_{123})q^{m-1} \right)}{1 - e(2\lambda_{123})q^{2i-1} \langle e(2\lambda_{123})q^{m-1}; q \rangle_i} \times \frac{\langle q^{-m}; q \rangle_i}{\langle q; q \rangle_i} e((-m-j)\lambda_1 + (m-i-j)\lambda_2 + (m-i)\lambda_3)q^i.
\]

\[
\times 4\varphi_3 \left( q^{-i}, e(2\lambda_2)q^{i-1}, q^{-j}, e(2\lambda_3)q^{j-1} \middle| e(2\lambda_2), q^{-m}, e(2\lambda_{123})q^{m-1} \right).
\]

### 4 Invariant Hermitian form of non-diagonal type

A Hermitian form which is invariant with respect to the monodromy group is called the monodromy-invariant Hermitian form or, simply, the invariant Hermitian form. One of the method for constructing an invariant Hermitian form is that by means of the intersection number of twisted cycles, as is explained in Subsection 1.2.

The invariant Hermitian form

\[ F(z, \overline{z}) = \sum_{i,j} a_{ij} I_i(z) \overline{I_j(z)}, \]

where \(\{I_i(z)\}\) is a fundamental set of solutions around a singularity, is called diagonal if \(a_{ij} = c_i \delta_{ij}\), where \(c_i\) is a multiplicative constant, and is called non-diagonal if not the case. The example displayed in (1.9) is an invariant Hermitian form of diagonal type.
In this section, as an application of our connection formulas, we provide some examples of the invariant Hermitian form of nondiagonal type. To obtain them, we consider the special case of the exponents $a, b, c, g$: for $\rho \in \mathbb{Z}_{\geq 0}$,

$$m = 2\rho, \quad a = b = c = -\frac{\rho}{2\rho + 1}, \quad g = \frac{1}{2\rho + 1}. \quad (4.1)$$

In this case, the characteristic exponents at $0, 1$ and $\infty$ are

$$e_j^{(0)} = e_j^{(1)} = \frac{j(j + 1)}{2(2\rho + 1)}, \quad e_j^{(\infty)} = \frac{2m\rho + (1 - 2m)j - j^2}{2(2\rho + 1)} \quad (4.2)$$

for $0 \leq j \leq m$, and it is seen that some of two differences of them are integer; indeed, $e_{2\rho-i}^{(0)} - e_i^{(0)} = \rho - i$ for $0 \leq i \leq m$. As a result of such degeneration of characteristic exponents, in the solution space of the differential equation of rank $m + 1$ a submodule emerges. This submodule induces the required Hermitian form, and the connection matrix in Theorem 2.2 is used to find it.

In what follows, under the condition (4.1), $T_z$ and $L_z$ are taken as in Section 2, hence

$$q = e(g) = e\left(\frac{1}{2\rho + 1}\right), \quad q^{2\rho+1} = -1 \quad \text{and} \quad e(a) = e(b) = e(c) = q^{-\rho}.$$

For a related work in conformal field theory, we refer the reader to [13] and [5], where the $\hat{sl}_2$ modular-invariant partition functions are classified; our functions below correspond to the $D_{2\rho+1}$ type invariants in the ADE-classification.

In case $\rho = 1$, (4.2) means

$$e_0^{(0)} = e_0^{(1)} = 0, \quad e_1^{(0)} = e_1^{(1)} = 1/3, \quad e_2^{(0)} = e_2^{(1)} = 1,$$

and Theorem 2.2 implies

$$\begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} s(1/6) & -1 & s(1/6) \\ -s(1/6) & 0 & s(1/6) \\ s(1/6) & 1 & s(1/6) \end{bmatrix} \begin{bmatrix} J_0 \\ J_1 \\ J_2 \end{bmatrix} = \begin{bmatrix} 2s(1/6) & 2s(1/6) \\ -s(1/6) & s(1/6) \end{bmatrix} \begin{bmatrix} J_0 \\ J_2 \end{bmatrix},$$

where $s(A) = \sin(\pi A)$. Thus, it is easily seen that

$$\begin{bmatrix} I_0 + I_2 \\ I_1 \end{bmatrix} = \begin{bmatrix} 2s(1/6) & 2s(1/6) \\ -s(1/6) & s(1/6) \end{bmatrix} \begin{bmatrix} J_0 \\ J_2 \end{bmatrix},$$

which leads to the submodule generated by $I_0 + I_2$ and $I_1$. Moreover, it is seen that $I_0 + I_2 = 2s(1/6)(J_0 + J_2)$, which leads to the submodule of rank one generated by $I_0 + I_2$. 44
In case \( \rho = 2 \), (4.2) means

\[
e_0^{(0)} = e_0^{(1)} = 0, \quad e_1^{(0)} = e_1^{(1)} = 1/5, \quad e_2^{(0)} = e_2^{(1)} = 3/5,
\]

\[
e_3^{(0)} = e_3^{(1)} = 6/5, \quad e_4^{(0)} = e_4^{(1)} = 2,
\]

and Theorem 2.2 implies

\[
\begin{bmatrix}
I_0 \\
\vdots \\
I_4
\end{bmatrix} =
\frac{\sqrt{5} - 1}{4} \begin{bmatrix} 1 & 0 & 0 & \sqrt{5} - 1 \\
1 & 0 & 0 & -\frac{1}{2} \\
0 & 1 - \sqrt{5} & 0 & \frac{\sqrt{5} - 1}{2} \\
0 & 0 & 1 & \frac{\sqrt{5} - 1}{4}
\end{bmatrix}
\begin{bmatrix}
J_0 \\
\vdots \\
J_4
\end{bmatrix},
\]

This induces

\[
\begin{bmatrix}
I_0 + I_4 \\
I_1 + I_3 \\
I_2
\end{bmatrix} =
\frac{\sqrt{5} - 1}{2} \begin{bmatrix} 2 & 0 & 0 & \sqrt{5} - 1 \\
1 - \sqrt{5} & 0 & \sqrt{5} - 1 & 0 \\
\sqrt{5} - 1 & 1 - \sqrt{5} & \sqrt{5} - 1 & 0
\end{bmatrix}
\begin{bmatrix}
J_0 \\
J_2 \\
J_4
\end{bmatrix},
\]

which leads to the submodule of rank 3 generated by \( I_0 + I_4, I_1 + I_3, I_2 \), and

\[
\begin{bmatrix}
I_0 + I_4 \\
I_2
\end{bmatrix} =
\frac{\sqrt{5} - 1}{4} \begin{bmatrix} 2 & 0 & 0 & \sqrt{5} - 1 \\
\sqrt{5} - 1 & 1 - \sqrt{5} & 0 & \sqrt{5} - 1
\end{bmatrix}
\begin{bmatrix}
J_0 + J_4 \\
J_2
\end{bmatrix},
\]

which leads to the submodule of rank 2 generated by \( I_0 + I_4, I_2 \).

Generally, we have the following.

**Proposition 4.1.** For \( 0 \leq i \leq \rho \), we have

\[
\begin{cases}
I_1 + I_{2^\rho - i} &= \sum_{0 \leq j \leq \rho} 2p_{i, 2j} J_{2j} \quad (0 \leq i < \rho), \\
I_\rho &= \sum_{0 \leq j \leq \rho} 2p_{\rho, 2j} J_{2j}.
\end{cases}
\]
Proof. It follows from (2) of Lemma 4.5 below. □

**Proposition 4.2.** If $\rho$ is even, we have

$$\begin{cases}
I_{2i} + I_{2\rho-2i} = \sum_{0 \leq j < \rho} 2p_{2i,2j}(J_{2j} + J_{2\rho-2j}) + 2p_{2i,\rho}J_{\rho} & (0 \leq i \leq \frac{\rho}{2} - 1), \\
I_{\rho} = \sum_{0 \leq j \leq \rho} 2p_{\rho,2j}(J_{2j} + J_{2\rho-2j}) + 2p_{\rho,\rho}J_{\rho}.
\end{cases}$$

If $\rho$ is odd, we have

$$I_{2i} + I_{2\rho-2i} = \sum_{0 \leq j \leq \rho} 2p_{2i,2j}(J_{2j} + J_{2\rho-2j}) & (0 \leq i \leq \frac{\rho - 1}{2}).$$

**Proof.** It follows from (1) with (2) of Lemma 4.5 below. □

Therefore, we reach the following:

**Theorem 4.3.** The Hermitian form given by

$$F(z, \overline{z}) = \sum_{0 \leq i \leq \rho-1} \frac{1}{(C_i + C_{2\rho-i})^2} |I_i + I_{2\rho-i}|^2 + \frac{1}{C_{2\rho}^2} |I_{\rho}|^2$$

is the monodromy-invariant Hermitian form. Here

$$(C_i + C_{2\rho-i})^2 = (C_i + C_{2\rho-i}) \cdot (C_i + C_{2\rho-i})$$

$$= C_i \cdot C_i + C_{2\rho-i} \cdot C_{2\rho-i} = 2C_i \cdot C_i = 2C_i^2$$

for $0 \leq i \leq \rho - 1$, and

$$C_i^2 = m! \left( \frac{\sqrt{-1}}{2} \right)^m \prod_{j=1}^{i} s \left( \frac{-2i+j+2}{2(2i+1)} \right) s \left( \frac{1}{2(2i+1)} \right) 2^{2i-2} \prod_{j=1}^{\rho-i} s \left( \frac{-2\rho+j-2}{2(2\rho+1)} \right) s \left( \frac{1}{2(2\rho+1)} \right)$$

for $0 \leq i \leq \rho$.

**Theorem 4.4.** The Hermitian form given by

$$F(z, \overline{z}) = \begin{cases}
\sum_{i=0}^{\frac{\rho-1}{2}} \frac{1}{(C_{2i} + C_{2\rho-2i})^2} |I_{2i} + I_{2\rho-2i}|^2 + \frac{1}{C_{\rho}^2} |I_{\rho}|^2, & (\rho : \text{even}) \\
\sum_{i=0}^{\frac{\rho-1}{2}} \frac{1}{(C_{2i} + C_{2\rho-2i})^2} |I_{2i} + I_{2\rho-2i}|^2, & (\rho : \text{odd})
\end{cases}$$

is the monodromy-invariant Hermitian form. Here

$$(C_{2i} + C_{2\rho-2i})^2 = 2C_{2i}^2$$

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for $0 \leq i < \rho/2$, and

$$C_i^2 = m! \left( \frac{\sqrt{-1}}{2} \right) \prod_{j=1}^{m} s \left( \frac{-2\rho+i-j-2}{2(2\rho+1)} \right) \prod_{j=1}^{2p-i} s \left( \frac{1}{2(2\rho+1)} \right) \prod_{j=1}^{2p-i} s \left( \frac{-2\rho+i-j-2}{2(2\rho+1)} \right)$$

for $0 \leq i \leq \rho$.

**Lemma 4.5.** We have

(1) $p_{ij} = (-)^i p_{i,m-j}$, \hspace{1cm} (2) $p_{ij} = (-)^j p_{m-i,j}$.

**Proof.** (1) Under the condition (4.1), the expression (2.6) reduces to

$$p_{ij} = (-)^i \sum_{0 \leq k \leq m-i \atop 0 \leq l \leq i \atop k+l=m-j} (-)^k \prod_{r=1}^{m-i-k} \left( \frac{q^{-\rho + \frac{i+r-1}{2}}}{q^{-2\rho+k+\frac{i+r-1}{2}}} \right) \prod_{r=1}^{k} \left( \frac{q^{-\rho + \frac{i+r-1}{2}}}{q^{-2\rho+k+\frac{i+r-1}{2}}} \right)$$

$$\times \prod_{r=1}^{l} \left( \frac{q^{-\rho + \frac{k+r-1}{2}}}{q^{-2\rho+j+\frac{k+r-1}{2}}} \right) \prod_{r=1}^{\rho + \frac{j+r-1}{2}} \left( q^{\frac{k+r-1}{2}} \right)$$

Hence we have

$$p_{i,m-j} = (-)^i \sum_{0 \leq k \leq m-i \atop 0 \leq l \leq i \atop k+l=m-j} (-)^k \prod_{r=1}^{m-i-k} \left( \frac{q^{-\rho + \frac{i+r-1}{2}}}{q^{-2\rho+k+\frac{i+r-1}{2}}} \right) \prod_{r=1}^{k} \left( \frac{q^{-\rho + \frac{i+r-1}{2}}}{q^{-2\rho+k+\frac{i+r-1}{2}}} \right)$$

$$\times \prod_{r=1}^{l} \left( \frac{q^{-\rho + \frac{k+r-1}{2}}}{q^{-2\rho+j+\frac{k+r-1}{2}}} \right) \prod_{r=1}^{\rho + \frac{j+r-1}{2}} \left( q^{\frac{k+r-1}{2}} \right)$$

$$\sum_{0 \leq k \leq m-i \atop 0 \leq l \leq i \atop k+l=m-j} (-)^k \prod_{r=1}^{m-i-k} \left( \frac{q^{-\rho + \frac{i+r-1}{2}}}{q^{-2\rho+k+\frac{i+r-1}{2}}} \right) \prod_{r=1}^{k} \left( \frac{q^{-\rho + \frac{i+r-1}{2}}}{q^{-2\rho+k+\frac{i+r-1}{2}}} \right)$$

$$\times \prod_{r=1}^{l} \left( \frac{q^{-\rho + \frac{k+r-1}{2}}}{q^{-2\rho+j+\frac{k+r-1}{2}}} \right) \prod_{r=1}^{\rho + \frac{j+r-1}{2}} \left( q^{\frac{k+r-1}{2}} \right)$$

$$\sum_{0 \leq k \leq m-i \atop 0 \leq l \leq i \atop k+l=m-j} (-)^k \prod_{r=1}^{m-i-k} \left( \frac{q^{-\rho + \frac{i+r-1}{2}}}{q^{-2\rho+k+\frac{i+r-1}{2}}} \right) \prod_{r=1}^{k} \left( \frac{q^{-\rho + \frac{i+r-1}{2}}}{q^{-2\rho+k+\frac{i+r-1}{2}}} \right)$$
Under the condition (4.1), the expression (2.7) reduces to

\[ \prod_{r=1}^{l} \frac{q^{\rho - k - l + 1}}{q^{j} \cdot \frac{j}{2}} \prod_{r=1}^{l} \frac{q^{\rho + k + l - 1}}{q^{j} \cdot \frac{j}{2}}. \tag{4.4} \]

Here the second equality follows from the change of the running indices \( k \rightarrow m - i - k \) and \( l \rightarrow i - l \).

Comparing (4.3) with (4.4) leads to

\[ p_{ij} = (-1)^{i} p_{i, m-j}, \]

because

\[ (q^{\alpha})_{1} = (q^{\beta})_{1}, \text{ if } \alpha + \beta = \pm (2\rho + 1). \tag{4.5} \]

(2) Under the condition (4.1), the expression (2.7) reduces to

\[ p_{ij} = (-1)^{i} \sum_{0 \leq k \leq l} \sum_{0 \leq l \leq m-i} \left( -1 \right)^{l-k} \prod_{r=1}^{i-k} \frac{q^{j - 2\rho + l - r - 1}}{q^{j} \cdot \frac{j}{2}} \prod_{r=1}^{k} \frac{q^{j - 2\rho + k - r - 1}}{q^{j} \cdot \frac{j}{2}} \prod_{r=1}^{l} \frac{q^{j - 2\rho + k - l - r - 1}}{q^{j} \cdot \frac{j}{2}}. \tag{4.6} \]

Hence we have

\[ p_{m-i,j} = (-1)^{i+j} \sum_{0 \leq k \leq m-i} \sum_{0 \leq l \leq i} \left( -1 \right)^{k} \prod_{r=1}^{m-i-k} \frac{q^{j - 2\rho + l - r - 1}}{q^{j} \cdot \frac{j}{2}} \prod_{r=1}^{k} \frac{q^{j - 2\rho + k - r - 1}}{q^{j} \cdot \frac{j}{2}} \prod_{r=1}^{l} \frac{q^{j - 2\rho + k - l - r - 1}}{q^{j} \cdot \frac{j}{2}} \prod_{r=1}^{i-l} \frac{q^{j - 2\rho + l - r - 1}}{q^{j} \cdot \frac{j}{2}} \prod_{r=1}^{l} \frac{q^{j - 2\rho + k - l - r - 1}}{q^{j} \cdot \frac{j}{2}}. \tag{4.7} \]

Therefore, by comparing (4.6) and (4.7) with noting (4.5), it is seen that

\[ p_{m-i,j} = (-1)^{j} p_{i,j}. \]

It completes the proof. \( \square \)
5 Appendix

Under the condition (4.1), Theorem 2.2 implies that, for $0 \leq j \leq m = 2\rho$,

$$p_{0j} = (-)^j s \left( \frac{2j + 1}{2(2\rho + 1)} \right), \quad p_{2\rho, j} = s \left( \frac{2j + 1}{2(2\rho + 1)} \right),$$

$$p_{1j} = (-)^{j+1} s \left( \frac{1}{2(2\rho + 1)} \right) s \left( \frac{2j + 1}{2(2\rho + 1)} \right), \quad p_{2\rho - 1, j} = -s \left( \frac{1}{2(2\rho + 1)} \right) s \left( \frac{2j + 1}{2(2\rho + 1)} \right),$$

$$p_{2, j} = (-)^j s \left( \frac{2j + 1}{2(2\rho + 1)} \right) \left\{ 1 - s \left( \frac{\rho}{2(2\rho + 1)} \right) s \left( \frac{2j}{2(2\rho + 1)} \right) s \left( \frac{3}{2(2\rho + 1)} \right) s \left( \frac{j+1}{2(2\rho + 1)} \right) s \left( \frac{j+2}{2(2\rho + 1)} \right) \right\}.$$

$$p_{m-2, j} = s \left( \frac{2j + 1}{2(2\rho + 1)} \right) \left\{ 1 - s \left( \frac{\rho}{2(2\rho + 1)} \right) s \left( \frac{2j}{2(2\rho + 1)} \right) s \left( \frac{3}{2(2\rho + 1)} \right) s \left( \frac{j+1}{2(2\rho + 1)} \right) s \left( \frac{j+2}{2(2\rho + 1)} \right) \right\}.$$

and, for $0 \leq i \leq m = 2\rho$,

$$p_{i, 0} = (-)^i s \left( \frac{1}{2(2\rho + 1)} \right), \quad p_{i, 2\rho} = s \left( \frac{1}{2(2\rho + 1)} \right),$$

$$p_{i, 1} = (-)^{i+1} s \left( \frac{3}{2(2\rho + 1)} \right) c \left( \frac{2i+1}{2(2\rho + 1)} \right), \quad p_{i, 2\rho - 1} = -s \left( \frac{3}{2(2\rho + 1)} \right) c \left( \frac{2i+1}{2(2\rho + 1)} \right).$$
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