2D-gravity and the Hamilton-Jacobi formalism

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Abstract

Hamilton-Jacobi formalism is used to study 2D-gravity and its SL(2, R) hidden symmetry. If the contribution of the surface term is considered the obtained results coincide with those given by the Dirac and Faddeev-Jackiw approaches.

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1 Introduction

Two dimensional gravity models were subjected to an intense investigation during last years due to its relation with string theory and its potential applications to a better understanding of classical and quantum properties of gravity models \[1, 2, 3, 4, 5, 6, 7, 8, 9\]. Polyakov \[2\] proved that an induced 2D-gravity has a hidden $SL(2,R)$ symmetry. This symmetry was analysed in the context of the canonical and extended Hamiltonian formalisms \[10, 11, 12, 13\] as well as in the context of improper gauge transformations \[14\]. Despite of a huge number of publications on the above topics still a final conclusion and consensus on the main problems is missing.

Recently, Hamilton-Jacobi (HJ) formalism was investigated in the context of strings and p-Branes \[15\], strongly coupled gravitational systems \[16\], nonholonomic-constrained systems with second-class constraints and Proca’s model \[17\] and the quantum HJ formulation was obtained from the equivalence principle \[18\].

A new method of quantization of the system with constraints based on the Carathéodory’s equivalent Lagrangians method \[19\] was initiated in \[20, 21\] and developed in \[22, 23, 24, 25, 26\]. The connection between the integrability conditions and Dirac’s consistency conditions \[27\] were established \[28\]. The importance of the surface terms \[29\] was analyzed and the relation between Batalin-Fradkin-Tyutin \[30\] and HJ formalism \[25\] was traced. The advantage of HJ formalism is that we have no difference between first and second class constraints and we do not need gauge-fixing term because the gauge variables are separated in the processes of constructing an integrable system of total differential equations. The action of the formalism is suitable for the path integral quantization method of the constrained systems. However the role of the surface term for HJ formalism requires a deep analysis especially when we are dealing with gravity.

For these reasons the application of HJ formalism to 2D-gravity and the investigation of its $SL(2,R)$ hidden symmetry is an interesting issue.

The plan of the paper is as follows:

In sec. 2 HJ formalism is presented. 2D-gravity is analyzed using HJ formulation in sec.3 In sec. 4 conclusions are given.

2 Hamilton-Jacobi formalism

Let us assume that the Lagrangian $L$ is singular and the Hessian matrix has the rank $n-r$. The ”Hamiltonians” to start with are

$$H'_\alpha = H_\alpha(t_\beta,q_\alpha,p_\alpha) + p_\alpha, \quad (1)$$

where $\alpha, \beta = n - r + 1, \cdots, n, a = 1, \cdots n - r$. The usual Hamiltonian $H_0$ is defined as

$$H_0 = -L(t, q_\nu, \dot{q}_\alpha = w_\alpha) + p_\alpha w_\alpha + \dot{q}_\mu p_\mu \mid_{p_\nu = -H_\nu}, \nu = 0, n - r + 1, \cdots, n. \quad (2)$$
which is independent of \( \dot{q}_\mu \). Here \( \dot{q}_a = \frac{dq_a}{d\tau} \), where \( \tau \) is a parameter. The equations of motion are obtained as total differential equations in many variables as follows

\[
dq_a = \frac{\delta H'}{\delta p_a} dt_\alpha, dp_a = -\frac{\delta H'}{\delta q_a} dt_\alpha,
\]

\[
\begin{align*}
dp_\mu &= -\frac{\delta H'}{\delta t_\mu} dt_\alpha, & \mu = 1, \cdots, r, \\
dz &= (-H_\alpha + p_a \frac{\delta H'}{\delta p_a}) dt_\alpha, & (3)
\end{align*}
\]

where \( z = S(t_\alpha, q_a) \) and \( \frac{\delta H'}{\delta x} \) represents the variation of \( H'_\alpha \) with respect to \( x \). The variations of constraints will produce a complete set of "Hamiltonians" and some of them are not in the form given in (1) or they are divergence as in the Proca’s case. In order to maintain the physical significance of multi HJ formulation we are forced to modify the "Hamiltonians" by reducing or extending the initial phase space. We can adapt the above formalism to 2D-gravity considering the components of the metric as fields and taking into account that the canonical Hamiltonian is zero due to the reparametrization invariance.

### 3 Hamilton-Jacobi formulation of 2D-gravity

The action proposed by Polyakov for 2D-gravity is [2]

\[
S = -\frac{1}{2} \int \sqrt{-g} [g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \alpha R \phi] d\tau d\sigma.
\]

(5)

After some integrations by parts this action gives expression for the Lagrangian density as

\[
L = \frac{1}{2\sqrt{-g}} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{\alpha}{2\sqrt{-g}} (\epsilon^{\mu\nu\alpha} \partial_\mu \phi)(\epsilon^{\nu\beta\gamma} g_{\mu\nu}) - \frac{\alpha \phi}{8\sqrt{-g}} g^{\alpha\beta} \epsilon^{\rho\sigma\mu\nu} (\partial_\mu g_{\alpha\rho})(\partial_\nu g_{\beta\sigma}),
\]

(6)

where the determinant \( g \) is given by

\[
g = g_{00}g_{11} - g_{01}^2.
\]

(7)

Here \( g_{00}, g_{01} \) and \( g_{11} \) represent the independent fields.

In order to simplify the form of the Hamiltonian we add to (4) a surface term [10]. The new Lagrangian density has the form

\[
\tilde{L} = L + \partial_\mu (\frac{\phi}{2\sqrt{-g} g_{11}} \epsilon^{\mu\nu} \epsilon^{\alpha\beta} g_{1\alpha} \partial_\nu g_{1\beta})
\]

(8)
or explicitly

\[
\tilde{\bar{L}} = \frac{1}{2\sqrt{-g}} ((-g_{11}\dot{\phi}^2 + 2g_{01}\dot{\phi}\phi' - g_{00}\phi'^2) + \alpha \left( g_{11}\dot{\phi} - 2g_{01}\dot{\phi} + g_{00}'\phi' \right) + \frac{\alpha g_{01}}{g_{11}} (g_{11}'\dot{\phi} - g_{11}\phi')).
\] (9)

The overdots and primes denote time and space derivatives, respectively. The above expression leads us to the canonical momenta corresponding to \(\phi\) and \(g_{11}\) as

\[
\pi_{\phi} = \frac{(g_{01}\phi' - g_{11}\dot{\phi})}{\sqrt{-g}} + \frac{\alpha}{2\sqrt{-g}} (g_{11} - 2g_{01} + \frac{g_{01}g_{11}'}{g_{11}}),
\] (10)

\[
\pi^{11} = \frac{\alpha}{2\sqrt{-g}} (\dot{\phi} - \frac{g_{01}}{g_{11}} \phi').
\] (11)

The other momenta \(\pi_{00}, p_{01}\) are zero.

The canonical Hamiltonian is expressed as

\[
H_c = \int dx (-\sqrt{-g} \Phi_1 + \frac{g_{01}}{g_{11}} \Phi_2),
\] (12)

where \(\Phi_1\) and \(\Phi_2\) are

\[
\Phi_1 = \frac{1}{2} \left( \phi'^2 - 4\left( \frac{g_{11}\pi^{11}}{\alpha^2} \right)^2 - 4\alpha \left( \frac{g_{11}\pi^{11}}{\alpha g_{11}'\phi'} - 2\alpha \phi'' \right) \right),
\] (13)

\[
\Phi_2 = \pi_{\phi}\phi' - 2g_{11}\pi^{11'} - \pi^{11} g_{11}'.
\] (14)

Thus, the ”Hamiltonians” which are the basic components of the method, are expressed as

\[
H'_0 = p_0 + H_c, H'_1 = \pi_{00}, H'_2 = p_{01}.
\] (15)

Here \(g_{00}\) and \(g_{01}\) are gauge variables. Consistency conditions require the variations of \(H'_0, H'_1, H'_2\). These variations lead us to new constraints

\[
H'_3 = \Phi_1, H'_4 = \Phi_2.
\] (16)

Further variations of \(H'_3\) and \(H'_4\) do not give independent constraints. The above results are in agreement with those obtained by Dirac’s canonical formalism [10].

3.1 \(SL(2, R)\) symmetry in the light-cone gauge

To exhibit the \(SL(2, R)\) symmetry we introduce the light cone variables \(x^\pm\) which are given as \(x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^1)\). Besides, choosing the metric tensor as
\[
g_{\mu\nu} = \begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--} \end{pmatrix} \\
= \begin{pmatrix} h & -1 \\ -1 & 0 \end{pmatrix},
\]
(17)

where \( h \) is a field, the Lagrangian density becomes
\[
L = \partial_+ \phi \partial_- \phi + \frac{1}{2} h (\partial_- \phi)^2 - \frac{\alpha}{2} \partial_- \phi \partial_- h,
\]
(18)

where \( \partial_{\pm} = \frac{1}{\sqrt{2}} (\partial_0 \pm \partial_1) \). Treating \( x^+ \) as time, the momenta corresponding to \( \phi \) and \( h \) are defined as
\[
P = \partial_- \phi, \quad \Pi = 0.
\]
(19)

This formulation gives the canonical Hamiltonian density \( H_c = P \partial_+ \phi + \Pi h - L \) as
\[
H_c = \partial_+ \phi (P - \partial_- \phi) + \Pi h - \frac{1}{2} h (\partial_- \phi)^2 + \frac{\alpha}{2} \partial_- \phi \partial_- h
\]
(20)

Thus, the basic "Hamiltonians" to start with are
\[
H_0' = p_0 - h \partial_- \phi \partial_- \phi + \alpha \partial_- \phi \partial_- h, \\
H_1' = P - \partial_- \phi, H_2' = \Pi
\]
(21)

Again, the consistency conditions give a new constraint \( H_3' \) as
\[
H_3' = (\partial_- \phi)^2 + \alpha \partial_-^2 \phi
\]
(22)

The next step is to analyze the variation of \( H_3' \). From (21) we conclude that
\[
dP = (h \partial_-^2 \phi - \frac{\alpha}{2} \partial_-^2 h) d\tau
\]
(23)

Imposing \( dH_1' = 0 \) and using (23) we found the following constraint
\[
H_4' = \partial_-^3 h.
\]
(24)

The "Hamiltonians" satisfy the following algebra:
\[
\{ H_2'(x^-), H_4'(x^-) \} = \{ H_4'(x^-), H_2'(x^-) \} = \partial_-^3 \delta(x^- - x'^-),
\]
\[
\{ H_1'(x^-), H_1'(x^-) \} = -2\partial_- \delta(x^- - x'^-),
\]
\[
\{ H_1'(x^-), H_3'(x^-) \} = (2\partial_- \phi \partial_- - \alpha \partial_-^2) \delta(x^- - x'^-),
\]
\[
\{ H_3'(x^-), H_1'(x^-) \} = (2\partial_- \phi \partial_- + \alpha \partial_-^2) \delta(x^- - x'^-).
\]
(25)
We would like to mention that in \cite{12} the same constraints were obtained by using Faddeev-Jackiw formalism \cite{31}. Algebra (25) describes a system which is not integrable. Hence we would like to transform it to an integrable one. Solving (24) we obtain
\[ h(x^-, x^+) = B_1(x^+) + 2x^-B_2(x^+) + (x^-)^2B_3(x^+), \tag{26} \]
where \( B_1(x^+), B_2(x^+), B_3(x^+) \) are arbitrary functions of \( x^+ \).

On the other hand since \( H_c \) and \( H'_3 \) are the "Hamiltonian" densities we observed that they are proportional up to a surface term. Taking into account this result and using (26) we obtain
\[
\begin{align*}
H'_0 &= p_0 + B_1(x^+)\left(\partial_-\phi\right)^2 + B_3(x^+)\left[(x^-\partial_-\phi)^2 + \alpha\phi - \alpha\partial_-\phi x^-\right] \\
&+ 2B_2(x^+)\frac{1}{2}\left[2x^-\left(\partial_-\phi\right)^2 - \alpha\partial_-\phi\right]. 
\end{align*}
\tag{27}
\]
which becomes
\[
\begin{align*}
H'_0 &= p_0 + B_1(x^+)\left(P\partial_-\phi\right) + +2B_2(x^+)\frac{1}{2}\left[2x^-\partial_-\phi P - \alpha P\right] \\
&+ B_3(x^+)\left[(x^-)^2\partial_-\phi P + \alpha\phi - \alpha P x^-\right] 
\end{align*}
\tag{28}
\]
with the use of the constraints \( H'_1 = 0 \). This expression makes it possible to define quantities
\[
\begin{align*}
L^1(x^+) &= \int P\partial_-\phi dx^- ,
L^2(x^+) &= \frac{1}{2}\int \left[2x^-P\partial_-\phi - \alpha P\right] dx^- ,
L^3(x^+) &= \int \left[(x^-)^2P\partial_-\phi + \alpha\phi - \alpha P x^-\right] dx^- . 
\end{align*}
\tag{29}
\]
which fulfill the \( SL(2, R) \) algebra on the surface of constraint \( \int Pdx^- = 0 \).

This result coincides with that obtained by using Dirac’s canonical formalism, in \cite{10}

\section{4 Conclusions}

The surface term plays an important role for HJ formalism of constrained systems. This role becomes crucial when a Hamiltonian represents a total divergence. In this paper we applied the HJ formalism to the 2D-gravity model. The analysis of 2D-gravity in the light-cone coordinates is quite unusual for HJ formalism. The starting point was a theory with two "Hamiltonians" and two gauge variables. Imposing the integrability conditions we found the same constraints as obtained by using Dirac and Faddeev-Jackiw formalisms. \( SL(2,R) \) symmetry arose in the HJ formalism after we obtained the reduced phase space and eliminated the surface terms in order to find an integrable system of total differentiable equations.
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