Brieskorn module and Center conditions: pull-back of differential equations in projective space

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Abstract
The moduli space of algebraic foliations on \( \mathbb{P}^2 \) of a fixed degree and with a center singularity has many irreducible components. We find a basis of the Brieskorn module defined for a rational function and prove that pull-back foliations forms an irreducible component of the moduli space. The main tools are Picard-Lefschetz theory of a rational function in two variables, period integrals and Brieskorn module.

0 Introduction
A holomorphic foliation \( \mathcal{F}(\alpha) \) in \( \mathbb{P}^2 \) is defined by a 1-form \( \alpha = AdX + BdY + CdZ \), where \( A, B \) and \( C \) are homogeneous polynomials of degree \( d+1 \) satisfying the identity 
\[
\alpha(\nu) = XA + YB + ZC = 0 \text{ for } \nu = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z},
\]
the Euler vector field. For generic \( A, B \) and \( C \) the degree of the foliation \( \mathcal{F}(\alpha) \) is defined as \( d = \text{deg}(A) - 1 \) (see [LS] and [CL]). The space of algebraic foliations
\[
\mathcal{F}(\alpha), \quad \alpha \in \Omega^1_{d+1},
\]
where
\[
(0.1) \quad \Omega^1_{d+1} := \{ \alpha = AdX + BdY + CdZ \mid A, B, C \in \mathbb{C}[X, Y, Z]_{d+1}^h, \alpha(\nu) \equiv 0 \},
\]
is the projectivization of the vector space \( \Omega^1_{d+1} \) and is denoted by \( \mathcal{F}(2, d) \). From here on we use the notation \( \mathbb{C}[x, y, z]_{D_0}^h \) to denote the ring of homogeneous polynomials of degree \( D_0 \). The space \( \mathcal{F}(2, d) \) is a rational variety \( \mathbb{C}^N \) for some \( N \), the space of coefficients of polynomials \( A, B, C \). A singularity of \( \mathcal{F}(\alpha) \) is a common zero of \( A, B \) and \( C \). We denote the set of singularities of \( \mathcal{F}(\alpha) \) by \( \text{Sing}(\mathcal{F}(\alpha)) \). For an isolated singularity \( p \in \text{Sing}(\mathcal{F}(\alpha)) \), if there is a holomorphic coordinate system \((\bar{x}, \bar{y})\) in a neighborhood of the point \( p \) with \( \bar{x}(p) = 0, \bar{y}(p) = 0 \) such that in this coordinate system
\[
\alpha \wedge d(\bar{x}^2 + \bar{y}^2) = 0,
\]
holds then the point \( p \) is called a center singularity. The closure of the set of algebraic foliations of fixed degree \( d \) with at least one center in \( \mathcal{F}(2, d) \), which is denoted by \( \mathcal{M}(2, d) \), is an algebraic subset of \( \mathcal{F}(2, d) \) (see for instance, [Mo] and [LS]). Identifying irreducible components of \( \mathcal{M}(2, d) \) is the center condition problem in the context of polynomial differential equations on the real plane. The complete classification of irreducible components of \( \mathcal{M}(2, 2) \) is done by H. Dulac in [D] (see also [CL, p.601]). For the classification of polynomial 1-forms of degree 3 the reader can consult Zoladek’s articles [Z1, Z2]. This

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classical classification gives applications on the number of limit cycles in the context of polynomial differential equations on the real plane.

Consider the morphism

\begin{equation}
F : \mathbb{P}^2 \to \mathbb{P}^2 \\
[x, y, z] \to [R, S, T]
\end{equation}

where \(R, S, T \in \mathbb{C}[x, y, z]^h\).

Let \(P(2, a, s)\) be the set of foliation

\begin{equation}
\mathcal{F}(F^*(\alpha)) \quad \text{where} \quad \alpha \in \Omega^1_{a+1}.
\end{equation}

Let us denote by \(D\) the zero locus of the determinant of Jacobian matrix \(J_F\) of \(F\) i.e.

\begin{equation}
J_F(x, y, z) = \begin{pmatrix}
R_x & R_y & R_z \\
S_x & S_y & S_z \\
T_x & T_y & T_z
\end{pmatrix}
\end{equation}

\begin{equation}
D = \{[x : y : z] | \det(J_F(x, y, z)) = 0\} \subset \mathbb{P}^2.
\end{equation}

**Definition 0.1.** For a generic morphism \(F\) and foliation \(\mathcal{F}\), there exists local chart \((\phi, U)\) (resp. \((\psi, V)\)) of the critical point \(p \in D\) (resp. \(F(p)\)) such that \(F|U = (x^2, y)\) and the leaves of \(\mathcal{F}|(\psi, V)\) are given by \(X + Y^2 = t\). Therefore, \(F^*(\mathcal{F})\) with local expression \(x^2 + y^2 = t\) has a center singularity. We call this singularity of the foliation \(F^*(\mathcal{F})\) tangency center singularity.

We define the space \(P(2, a, s)\) of pull-back foliations \(\mathcal{F}(\omega)\) defined by

\begin{equation}
\omega = F^*(A)dR + F^*(B)dS + F^*(C)dT,
\end{equation}

where

\(R, S, T \in \mathbb{C}[x, y, z]^h\), \(A, B, C \in \mathbb{C}[X, Y, Z]^h\),

\(RF^*(A) + SF^*(B) + TF^*(C) = 0\).

**Theorem 0.1.** The space \(P(2, a, s)\) constitutes an irreducible component of \(M(2, d)\) for \(d = s(a + 2) - 2, s \geq 2\).

This article treats the question of the vector tangent to \(M(2, d)\) at \(F(F^*(\frac{d}{f}))\) for a rational \(f\), that has been studied for the case of pull-back of polynomial differential equations in \([Za]\). A sketch of contents of the article is as follows. In section 1 we prepare notations and notions that will be used in the course of exposition. Since the calculation of the tangent space of \(M(2, d)\) at a generic \(F_0 \in M(2, d)\) requires more information, we adopt a strategy to choose a special point \(F\) in the intersection of \(P(2, a, s)\) with the space of projective logarithmic foliations \([1, 5]\). Let \(F_{\epsilon}\) be a deformation of \(F\) such that for any \(\epsilon, F_{\epsilon}\) has a center singularity closed to a tangency center singularity of \(F\). We assume that \(\mathcal{F}\) (resp. \(F_{\epsilon}\)) is defined by \(F^*(\alpha_0)\) (resp. \(\omega_{\epsilon} = F^*(\alpha_0) + \epsilon^1\omega_1 + \cdots\)) for \(\alpha(\alpha_0) = F(\frac{d}{f})\). By using the results concerning vanishing cycles from section 2 and relatively exact 1-forms from section 3 we establish that there are 1-forms \(\tilde{\alpha}\) and \(\tilde{\omega}_{\epsilon}\) such that \(\omega_1\) be expressed as \(\omega_1 = F^*(\tilde{\alpha}) + \tilde{\omega}_{\epsilon}\). In section 4 by using the first order Melnikov function and period integral we calculate an explicit form of the tangent vector. In section 5 we find a basis for the Brieskorn module \(H_f\) with \(f = \frac{d}{Q}\) for \(P, Q\) generic polynomials and \(p, q\) co-prime. This is used to prove that \(F^*(B)\), \(B\) is a basis of \(H_f\), is extendable to a basis of \(H_{F^*(f)}\). This result furnishes us with the proof of Theorem 5.1 that is necessary to establish Corollary 3.2 that in its turn plays an essential rôle during the proof of the key Lemma 4.1.
1 Pull-back of projective foliations

Let $F$ be the morphism defined in (0.2). If the coefficients of the pull-back $F^*(\alpha)$ of $\alpha = AdX + BdY + CdZ$ for $A, B, C \in \mathbb{C}[X, Y, Z]_{a+1}$ do not have a common factor then the degree of the foliation $F^*(F(\alpha))$ is $s(a + 2) - 2$. An equivalent statement to the Theorem 0.1 is the following Remark

Theorem 1.1. Let $a, s$ be two natural numbers and $s \geq 2$ and $F = F(\omega) \in \mathcal{F}(2, a)$ and $F = (R, S, T) : \mathbb{P}^2 \to \mathbb{P}^2$, $R, S, T \in \mathbb{C}[x, y, z]_h$. Let $F^*(\omega) = \omega_0$ we consider the deformation $F_\epsilon$ that is generated by $F_\epsilon = \omega_0 + (\omega_1 + \cdots + \omega_d)$, $d = s(a + 2) - 2$ of the foliation $F(\omega)$. Let $p \in D$ be one of the center singularities of $F^*(F(\omega))$ which is obtained by taking the pull back. For a generic choice of $\omega$ and $F$, if the deformed foliation $F_\epsilon$ for all small $\epsilon$ has center singularity near $p$ then $F_\epsilon$ is also a pull back foliation. More precisely, there is a foliation $F_\epsilon \in F(2, a)$ and a morphism map $F_\epsilon = (R_\epsilon, S_\epsilon, T_\epsilon) : \mathbb{P}^2 \to \mathbb{P}^2$, $R_\epsilon, S_\epsilon, T_\epsilon \in \mathbb{C}[x, y, z]_s$ such that $F^*_\epsilon(F_\epsilon) = F_\epsilon$.

By taking the coefficient of polynomials as coordinates of the map from the space of projective foliations to projective space we can see that $\mathcal{P}(2, a, s)$ is an irreducible algebraic subset of $\mathcal{M}(2, d)$. We take a point $\mathcal{F}(F^*(\alpha))$ in $\mathcal{P}(2, a, s)$ for a 1-form $\alpha$ like in (0.1), make a deformation $F_\epsilon \in \mathcal{P}(2, a, s)$ and calculate the tangent vector space of $\mathcal{P}(2, a, s)$ at $\mathcal{F}(F^*(\alpha))$:

$$F^*_\epsilon(\alpha_\epsilon) = F^*_\epsilon(\alpha + \epsilon \alpha_1) + O(\epsilon^2)$$

(1.1)

for $\alpha_\epsilon = \alpha + \sum_{j \geq 1} \epsilon^j \alpha_j$ with $\alpha_j \in \Omega^1_{\leq d+1}$, $\forall j \geq 1$. The pull-back by a morphism $F_\epsilon = F + \epsilon F_1 + O(\epsilon^2)$ with $F = (R, S, T)$, $F_1 = (R_1, S_1, T_1)$ of the form (1.1)

$$F^*_\epsilon(\alpha + \epsilon \alpha_1 + \cdots) = F^*(\alpha) + \epsilon \omega_W + O(\epsilon^2).$$

(1.2)

Here the 1-form $\omega_W$ has the following form:

$$\omega_W = F^*(A)dR_1 + F^*(B)dS_1 + F^*(C)dT_1$$

(1.3)

$$+ (R_1.F^*(A_X) + S_1.F^*(A_Y) + T_1.F^*(A_Z)) dR$$

$$+ (R_1.F^*(B_X) + S_1.F^*(B_Y) + T_1.F^*(Q_Z)) dS$$

$$+ (R_1.F^*(C_X) + S_1.F^*(C_Y) + T_1.F^*(C_Z)) dT + F^*(\alpha_1)$$

Consider a foliation in $\mathbb{P}^2$ with the first integral

$$f : \mathbb{P}^2 \setminus \{(P = 0) \cap \{Q = 0\}\} \to \mathbb{C}$$

(1.4)

$$f(X, Y, Z) = \frac{P(X, Y, Z)^q}{Q(X, Y, Z)^p}$$

with $deg(P) = \frac{p}{q}$, $g.c.d(p, q) = 1$. We denote by

$$\mathcal{I}(deg(P) - 1, deg(Q) - 1)$$

(1.5)

the space of projective foliations $F(\alpha_0)$ with the first integrals of type (1.4).
In [Mo1, Theorem 5.1] it is shown that the space of foliations with a first integral of type (1.4) is an irreducible component of $\mathcal{M}(2, \text{deg}(P) + \text{deg}(Q) - 2)$ for $\text{deg}(P) + \text{deg}(Q) - 2 \geq 2$. This is a generalization of the result [I] established for the polynomial first integral to the case of foliations in $\mathbb{P}^2$ with a generic rational first integral of type (1.4).

**Remark 1.1.** By calculating the tangent vector $\omega_W$ at the point $F_0 = F^*(qQdP - pPdQ)$ we have $A = qQP_x - pPQ_x$, $B = qQP_y - pPQ_y$, $C = qQP_z - pPQ_z$, and then

\[
\omega_W = \omega_{pl} + F^*(\alpha_1),
\]

where

\[
\omega_{pl} = qF^*(Q)dP - pPdF^*(Q) + qQ_1dF^*(P) - pF^*(P)dQ_1,
\]

\[
P_1 = R_1F^*(P_x) + S_1F^*(P_y) + T_1F^*(P_z),
\]

\[
Q_1 = R_1F^*(P_x) + S_1F^*(P_y) + T_1F^*(P_z).
\]

Let $\mathcal{X}(a, s)$ be the irreducible component of $\mathcal{M}(2, d)$ containing $\mathcal{P}(2, a, s)$. To calculate tangent cone of $\mathcal{X}(a, s)$ at a special point in $\mathcal{I}(ms - 1, ns - 1) \cap \mathcal{P}(2, a, s)$. Let $F$ be a generic morphism of $\mathbb{P}^2$ into itself such that each component of $F$ and $f$ be a rational function $f = \frac{P}{Q}$ where $P, Q$ are two homogeneous polynomials degrees $m, n$ with

\[
a = n + m - 2
\]

i.e. $mq = np$ with the condition (2.1).

**Theorem 1.2.** Tangent cone of $\mathcal{X}(a, s)$ at the point $F_0 := F(F^*(df))$ is equal to tangent cone of $\mathcal{P}(2, a, s)$ at this point

\[
TC_{F_0}\mathcal{X}(a, s) = TC_{F_0}\mathcal{P}(2, a, s).
\]

Let us consider a deformation $F(\omega_\epsilon) \in \mathcal{X}(a, s)$ defined by the following 1-form:

\[
\omega_\epsilon = F^*(\alpha_0) + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots, \quad \text{deg}(\omega_j) \leq d + 1 \forall j
\]

where

\[
\frac{\alpha_0}{PQ} = \frac{QdP - PdQ}{PQ} = \frac{df}{f}.
\]

![Figure 1: Tangent Cone](image)
The hypothesis \( \mathcal{F}_0 \in \mathcal{X}(a, s) \) implies that \( \mathcal{F}(\omega) \) always has a tangency center singularity (Definition 0.1) near the tangency center singularity \( p \) of \( \mathcal{F}(\omega_0) \) defined by \( \omega_0 = F^*(\alpha_0) \). We call it also tangential critical point of the rational mapping \( F^*(f) : \mathbb{P}^2 \to \mathbb{P}^2 \) in view of the circumstance that requires analysis of vanishing cycles and their monodromy associated to \( F^*(f) \) in [2].

Let us consider the affine coordinate \( E \), and let \( \delta_t \) be a continuous family of vanishing cycles in \( (F^*(f))^{-1}(t) \) around tangent critical point \( p \) and \( \Sigma \equiv \mathbb{C} \) be a transverse section of \( \mathcal{F} \) at some points of \( \delta_t \). We write Taylor expansion of the deformed holonomy \( h_t(t) \)

\[
h_t(t) - t = M_1(t)\epsilon + M_2(t)\epsilon^2 + \cdots + M_i(t)\epsilon^i + \cdots
\]

where \( M_i(t) \) is \( i-\)th Melnikov function of the deformation (see [F] Theorem 1.1, [Mo1] Section 3]). If \( \Sigma \) is parametrized by the image of \( F^*(f) \) i.e. \( t = F^*(f)(\sigma_0) \) for \( \sigma_0 \in \Sigma \) then

\[
M_1(t) = -\int_{\delta_t} \frac{F^*(f)\omega_1}{F^*(PQ)} = -t \int_{\delta_t} \frac{\omega_1}{F^*(PQ)}.
\]

for \( \delta_t \in H_1((F^*(f))^{-1}(t), \mathbb{Z}) \), the vanishing cycle associated to the tangent critical point \( p \in C_{F^*(f)} \). In fact \( M_j(t), \forall j \geq 1 \) also vanishes for all \( t \) near to zero as \( F^*(\mathcal{F}(\omega_\eta)) \) has a tangency center singularity \( \forall \eta \in [0, 2\epsilon] \).

The condition \( M_k(t) = 0 \) plays a central role in the proof of the following main theorem. See Lemma 4.1 Proposition 4.1.

2 Monodromy action on tangency cycles

In this section we formulate Theorem 2.1 that establishes a relation between vanishing cycles associated to \( f \) and \( F^*(f) \). Let \( \mathcal{F} \) be a foliation in \( \mathbb{P}^2 \) with the first integral \( f = \frac{P}{Q} \) like in (1.4). In addition to (1.4) we impose on \( \mathcal{F} \) satisfies the following conditions:

**Conditions 2.1.** (1) The curves \( V(P) \subset \mathbb{P}^2 \) and \( V(Q) \subset \mathbb{P}^2 \) are smooth and intersect each other transversally, and also each one has transversal intersection with line at infinity. We denote the set \( V(P) \cap V(Q) \) by \( R \). (2) All critical points of \( f = \frac{P}{Q} \) are distinct and belong to the affine plane \( \mathbb{C}^2 \). (3) \( \deg(P) \geq \deg(Q) \geq 2 \), i.e. \( m \geq n \geq 2 \).

Let us denote by

\[
M_f : \pi_1(\mathbb{C} \setminus C_f, B) \longrightarrow GL(H_1(f^{-1}(B), \mathbb{Z}), \mathbb{Z})
\]

the monodromy representation of the fundamental group of the complement to the critical value set \( C_f \) of \( f \) acting on the first homology group of a smooth generic fiber \( H_1(f^{-1}(B), \mathbb{Z}) \).

In a similar manner we consider the following monodromy representation for the first homology group of a smooth generic fiber \( H_1((F^*(f))^{-1}(b), \mathbb{Z}) \):

\[
M_{F^*(f)} : \pi_1(\mathbb{C} \setminus C_{F^*(f)}, b) \longrightarrow GL(H_1((F^*(f))^{-1}(b), \mathbb{Z}), \mathbb{Z}).
\]

By virtue of the assumption made on the critical points of \( f \) and the genericity of \( F \) the monodromy representations \( M_f \) and \( M_{F^*(f)} \) can be realized with integer coefficients. Here \( C_{F^*(f)} \) denotes the critical value set of \( F^*(f) \).

Let us call vanishing cycle \( \delta_t \) around a tangent critical point tangency vanishing cycle (see (2.3) below).
Theorem 2.1. The morphism

\[(2.1)\quad F_\ast : H_1((F^\ast(f))^{-1}(b),\mathbb{Z}) \to H_1(f^{-1}(b),\mathbb{Z})\]

is surjective and $\text{Ker}(F_\ast)$ is generated by the result of the monodromy group action $M_{F^\ast(f)}(\pi_1(\mathbb{C} \setminus C_{F^\ast(f)}, b))$ on a tangency cycle $\delta_i$ around a tangent critical point.

**Proof.** For $D$ defined in (0.5) the morphism $F_1 : \mathbb{C}^2 \setminus D \to \mathbb{C}^2 \setminus F(D)$ is a covering map. Inverse image of each vanishing cycle $\Delta \in H_1(f^{-1}(b),\mathbb{Z})$ via $F^{-1}$ contains $s^2$ disjoints vanishing cycles. Moreover, $H_1((F^\ast(f))^{-1}(b),\mathbb{Z})$ contains a subgroup consisting of $s^2$ copies of a group isomorphic to $H_1(f^{-1}(B),\mathbb{Z})$.

Let us take a pull-back vanishing cycle $\delta_1$ such that $F_\ast(\delta_1) = \Delta$. According to [La (7.3.5)], [Mo5 Theorem 2.3] the action of the monodromy group $M_{F^\ast(f)}(\pi_1(\mathbb{C} \setminus C_{F^\ast(f)}, b))$ on the vanishing cycle $\delta_1$ generates a subgroup of $H_1((F^\ast(f))^{-1}(b),\mathbb{Z})$ which is isomorphic to $H_1(f^{-1}(B),\mathbb{Z})$.

It is well-known that Dynkin diagram of $F^\ast(f)$ is connected, see for instance [AGV]. If we remove the vertices which correspond to vanishing cycles around the tangent critical points in $D$ (we call them *tangency vertices*) the Dynkin diagram becomes $s^2$ disjoint graphs $P_i$, $1 \leq i \leq s^2$ each of which is isomorphic to the Dynkin diagram of $f$. See [Za Figure 9].

There exists a local chart around the tangent critical point $p \in D$ such that in this chart $F = (x^2, y)$. Therefore the graph $P_i$ is connected to a graph $P_j$ by tangency vertices. Indeed, a tangency vertex corresponding to a tangency cycle around a tangent critical point $p \in D$ is connected to some $P_i$ or to another vertex corresponding to a vanishing cycle around $p \in D$. Therefore we conclude that the monodromy group $M_{F^\ast(f)}(\pi_1(\mathbb{C} \setminus C_{F^\ast(f)}, b))$ actions on the tangency cycle $\delta_i$ generate cycles of the following two types. The difference between two vanishing cycles associated to different critical values $c_i \neq c_j$ of $F^\ast(f)$ satisfying $F(c_i) = F(c_j) = c$, $c \in C_{F^\ast(f)}$.

\[(2.2)\quad \delta^i - \delta^j, \quad 1 \leq i, j \leq s^2, \]

and vanishing cycles that are associated to $p \in D$,

\[(2.3)\quad \delta_p, \quad p \in D.\]

In [Za Theorem 4.9, Figure 9] cycles (2.3) are divided into *tangency* and *exceptional vanishing cycles*.

We denote by $H$ the free Abelian group generated by cycles (2.2), (2.3) that is a subgroup of $\ker(F_\ast)$.

The space of cycles of type (2.2) has rank $(s^2 - 1)\mu_f$ for $\mu_f = \text{rank } H_1(f^{-1}(b),\mathbb{Z})$. More precisely we calculate $\mu_f = (m + n - 1) - mn$ as $|V(P) \cap V(Q)| = mn$ (see Proposition 3.1). In a similar manner the equality $\mu = \text{rank } H_1((F^\ast(f))^{-1}(b),\mathbb{Z}) = (s(m + n) - 1)^2 - s^2mn$ holds.

Thus the rank of the space spanned by (2.2) and (2.3) is equal to

\[(2.4)\quad (s^2 - 1)\mu_f + \rho_D\]

for $\rho_D : \text{the rank of cycles (2.3)}$.

Here we remark that

\[(2.5)\quad \mu = \text{rank } H_1((F^\ast(f))^{-1}(b),\mathbb{Z}) = s^2\mu_f + \rho_D.\]
From the definition of the morphism $F_*$ we see that $\text{rank}(\ker(F_*)) = \mu - \mu_f$.

The combination of (2.4), (2.5) shows the equality $\text{rank}(H) = \text{rank}(\ker(F_*))$ thus together with $H \subseteq \ker(F_*)$ we conclude $H = \ker(F_*)$.

3 Relatively Exact 1-forms

A foliation $\mathcal{F} = F(\omega)$ defined by a holomorphic 1-form $\omega$ is called integrable if there exists a meromorphic function $f$ on $\mathbb{P}^2$ such that $df \wedge \omega|_{\mathcal{F}} = 0$. In this case the meromorphic function $f$ is said to be the first integral of $\mathcal{F}$.

The concept of relatively exact forms has been investigated by many authors, e.g. [Mu], [Mo1, Section 4].

Definition 3.1. A meromorphic 1-form $\omega$ on $\mathbb{P}^2$ is called relatively exact modulo a foliation $\mathcal{F}$ in $\mathbb{P}^2$ if the restriction of $\omega$ to each leaf $L$ of $\mathcal{F}$ is exact, i.e. there is a meromorphic function $g$ on $L$ such that $\omega|_L = dg$.

Let us call $f$-fiber the set $\{u \in \mathbb{P}^2 \setminus \mathcal{R} : f(u) = t\}$ defined for some $t \in \mathbb{C}$.

Proposition 3.1. [Mu, §2] A 1-form $\omega$ is relatively exact modulo the foliation $F(df)$ with rational first integral $f$ if and only if

$$\int_{\delta} \omega = 0$$

for every closed curve $\delta$ in a $f$-fiber.

Proof. Let $L$ be a line in $\mathbb{P}^2$ which is not $F$-invariant and does not pass through the point in $\mathcal{R}$ as in Conditions 2.1. For any point $u \in U := \mathbb{P}^2 \setminus \mathcal{R}$ let

$$f^{-1}(f(u)) \cap L = \{p_1, p_2, \ldots, p_r\}$$

where the intersection multiplicity of $p_i$ might be greater than 1.

Define

$$g : \mathbb{P}^2 \setminus \mathcal{R} \to \mathbb{C}$$

(3.1)

$$g(u) = \frac{1}{r} \left( \sum_{i=1}^{r} \int_{p_i} p_i \omega \right)$$

where $\int_{p_i}$ is an integral over a path in $f^{-1}(f(u))$ which connects $u$ to $p_i$. The function $g$ is well-defined and does not depend on the choice of the paths connecting $u$ to $p_i$ because $\int_{\delta} \omega = 0$ on any close curve $\delta$ in each level set $f^{-1}(f(u))$. Furthermore it is clear that a monodromy action on the line $L$ leaves the set $L \cap f^{-1}(f(u))$ invariant as it induces merely a permutation among its points. Thus we conclude that $g(u)$ is a meromorphic function on $\mathbb{P}^2 \setminus (V(P) \cup V(Q))$ in taking Levi extension theorem and Hartogs theorem into account.

A function $f$ is called non-composite if every generic $f$-fiber is irreducible. It is easy to see that $f$ is non-composite if and only if $f$ can not be factored as a composite

$$\mathbb{P}^2 \xrightarrow{f'} \mathbb{C} \xrightarrow{i} \bar{\mathbb{C}}$$

(3.2)
where $i$ is a non-constant holomorphic map. In fact, if the composite factorization like\footnote{3.2} does not take place then the generic $f$-fiber cannot be reducible. From \footnote{3.2} the reducibility of generic $f$-fiber follows.

Let $f = \frac{P}{Q}$ be a rational function satisfying Conditions \footnote{2.1} and suppose that for every $t \in \mathbb{C}$ the fiber $f^{-1}(t)$ is connected. Let $\mathcal{F}(\omega_0)$ be a foliation on $\mathbb{P}^2$ with the non-composite first integral $f$ not satisfying \footnote{3.2} for

\begin{equation}
\omega_0 = \frac{df}{f}.
\end{equation}

Let $\omega$ be a rational 1-form with the pole divisor

\begin{equation}
\check{D} = n_1 D_1 + n_2 D_2,
\end{equation}

where $D_1 := V(P), D_2 := V(Q)$.

**Theorem 3.1.** Every relatively exact rational 1-form $\omega$ modulo $\mathcal{F}(\omega_0)$ with pole divisor $\check{D}$ has the form

\begin{equation}
\omega = dg + T \omega_0
\end{equation}

where $g$ and $T$ are rational functions with the pole divisor $\check{D}$.

The following is a modification of \footnote{Mo1, Theorem 4.1} adapted to our situation.

**Proof.** The function $g$ in \footnote{3.1} is a holomorphic function in $\mathbb{P}^2 \setminus (D_1 \cup D_2)$. For a point $u \in U \setminus (D_1 \cup D_2)$, by the hypothesis $q, p$ are the multiplicities of $f$ along $D_1, D_2$, respectively. The function $f^{\frac{n_1}{q}}$ is an univalent function in a small neighborhood of the path connecting $u$ to $p_i$ and we have

\begin{equation}
\int_u^{p_i} \omega = f^{\left(-\frac{n_1}{q}\right)} \int_u^{p_i} f^{\left(\frac{n_1}{q}\right)} \omega
\end{equation}

$f^{\frac{n_1}{q}} \omega$ is a holomorphic 1-form along $U \cap D_1$ therefore the above integral has poles of order at most $n_1$ along $D_1$. By using the chart around infinity and applying the above argument to $D_2$ once again, one can check that each component integral in \footnote{3.1} has poles of order at most $n_2$ along $D_2$. The equalities

\begin{equation}
dg \land \omega_0 = \omega \land \omega_0 \Rightarrow (\omega - dg) \land \omega_0 = 0
\end{equation}

imply that there is a rational function $T$ with pole divisor $\check{D}$ such that $\omega = dg + T \omega_0$. \qed

**Corollary 3.1.** Suppose that $\omega$ is a polynomial homogeneous 1-form on $\mathbb{P}^2$ with $\deg(\mathcal{F}(\omega)) = \deg(\mathcal{F}(\omega_0))$ and $\frac{\omega}{PQ}$ is relatively exact modulo $\mathcal{F}(\omega_0)$. Then there are polynomials $(P_1, Q_1) \in \mathcal{P}_{ms} \times \mathcal{P}_{ns}$ such that $\omega$ has the form

\begin{equation}
\omega = qF^*(Q)dP_1 - pP_1dF^*(Q) + qQ_1dF^*(P) - pF^*(P)dQ_1.
\end{equation}

**Proof.** By the Theorem \footnote{3.1} there are polynomials $B, A$ of degree at most $(m + n)s$ such that

\begin{equation}
\omega = \frac{QP.dB - Bd(PQ) - A(qQdP - pPdQ)}{PQ}
\end{equation}

where $i$ is a non-constant holomorphic map. In fact, if the composite factorization like\footnote{3.2} does not take place then the generic $f$-fiber cannot be reducible. From \footnote{3.2} the reducibility of generic $f$-fiber follows.
This implies that

\[ P|B + qA, \quad Q|B - pA \Rightarrow \]

\[ B + qA = (p + q)PQ_1, \quad B - pA = (p + q)QP_1 \Rightarrow \]

\[ B = pPQ_1 + qQP_1, \quad A = -QF_1 + P_1 \]

where \( P_1, Q_1 \) are two polynomials of respective degrees at most \( ms, ns \). Substituting these in (3.7) we get the result.

**Corollary 3.2.** The morphism \( F^* : H_f \to H_{F^*(f)} \) is injective and the image of the basis of \( H_f \) is can be extended to a basis of \( H_{F^*(f)} \).

**Proof.** We consider the pull-back by \( F \) of the projectivized 1-form \( \tilde{\omega} \) for \( \omega \in H_f \). We restrict \( F^*(\tilde{\omega}) \) on \( \mathbb{P}^2 = \{ z = 1 \} \) in \( H_{F^*(f)} \). It is well known that \( F^* \) is injective (see [H Proposition 1.1]). For each element \( \omega_j \in \mathbb{P}^2 \) of the basis \( H_f \), \( \tilde{\omega}_j = \tilde{m}_j(X,Y,Z)\tilde{\eta} \) is a polynomial 1-form on \( \mathbb{P}^2 \) and \( F^*(\tilde{\omega}_j) = \tilde{m}_j(R,S,T)F^*(\tilde{\eta}) \) where the form \( \eta = axdy - bydx \) such that \( d\eta = dx \wedge dy \). According to Theorem 5.1, \( F^*(\tilde{\eta}) \) can be written as

\[ F^*(\tilde{\eta}) = \sum_{\ell=1}^{\mu} g_{\ell}(F^*(f))\omega_\ell, \]

where \( \omega_\ell \) is a basis of \( H_{F^*(f)} \). The inequality (5.6) entails

\[ deg(g_{\ell}) \leq deg(F^*(\tilde{\eta})) + ns - deg(Z(\omega_\ell)) - 1 \]

We recall here that \( deg(F^*(\tilde{\eta})) \leq s \) and \( m \geq n \geq 2 \) by Condition 2.1(3).

This means that the polynomial \( g_{\ell} \) is in fact a constant for every \( \ell \). Therefore the 1-form

\[ F^*(\tilde{\omega}_j) = \tilde{m}_j(R,S,T)F^*(\tilde{\eta}) \]

is free of \( F^*(f) \). This terminates the proof.

**4 Tangent vector**

We begin our discussion on the tangent vector of \( M(2,d) \) \( (d = s(a + 2) - 2) \) at the point \( F(F^*(a_0)) \in I(ms - 1, ns - 1) \cap P(2, a, s) \), for \( a_0 \) defined in (1.9). First of all, we show the following lemma on a decomposition (4.1) valid for \( \omega_1 \) used to define the first Melnikov function (1.10).

In the sequel we shall use the notation \( \tilde{\omega} \) defined on \( \mathbb{P}^2 \) for the projectivization of a form \( \omega \) given on an affine variety.

**Lemma 4.1.** For \( \omega_1 \) in (1.10) we find a homogeneous 1-form \( \tilde{\alpha} \) on \( \mathbb{P}^2 \) of degree \( a + 1 = m + n - 1 \) and two homogeneous polynomials \( P_1, Q_1 \) of respective degrees \( ms, ns \) such that

\[ \omega_1 = F^*(\tilde{\alpha}) + \tilde{\omega}_e, \]

where

\[ \tilde{\omega}_e = qF^*(Q)dP_1 - pP_1dF^*(Q) + qQ_1dF^*(P) - pF^*(P)dQ_1 \]

that is the projectivization of \( \omega_e \).
Proof. In the affine coordinate the polynomial map $F$ introduces a morphism $F^*: H_f \rightarrow H_{F^*(f)}$, between two $\mathbb{C}[\tau]$-module $H_f$ and $\mathbb{C}[t]$-module $H_{F^*(f)}$. The linear map

$$H_1(f^{-1}(b), \mathbb{Z}) \rightarrow \mathbb{C} \text{ given by } \Delta \rightarrow \int_{F^{-1}_*(\Delta)} \frac{\omega_1}{F^*(PQ)}$$

is well-defined because $\int_{\delta} \frac{\omega_1}{F^*(PQ)} = 0$, $\forall \delta \in \ker(F_*)$ by virtue of Theorem 2.1.

By the duality between de Rham cohomology and singular homology there is a $C^\infty$ differential form $h_\alpha$ in regular fiber $f^{-1}(b)$ such that

$$\int_{F^{-1}_*(\Delta)} \frac{\omega_1}{F^*(PQ)} = \int_\Delta \frac{\alpha_b}{PQ}.\]$$

According to Atiyah-Hodge theorem (See [AHL Theorem 4], [M-VL Chapter 4]) $\alpha_b$ can be taken holomorphic thus polynomial. The analytic continuation of $\alpha_b$ with respect to the parameter $b \in \mathbb{C} \setminus C_f$ gives rise to a holomorphic global section $\alpha$ of cohomology bundle of $f$. Thus in the affine coordinate $\mathbb{C}^2 \subset \mathbb{P}^2$ we have the following decomposition in $H_f$

$$\alpha = \sum_{\ell=1}^{\mu_f} h_\ell(f)[\eta_\ell]$$

where $h_\ell(\tau)$ is holomorphic in $\tau \in \mathbb{C} \setminus C_f$.

The coefficients $h_\ell(\tau)$ in (4.3) are rational functions in $\tau$ because of the following relation

$$\begin{bmatrix} h_1(\tau) \\ \vdots \\ h_{\mu_f}(\tau) \end{bmatrix} = \left[ \int_{\delta_k \eta_\ell}^{-1} \right]_{\mu_f \times \mu_f} \begin{bmatrix} \int_{\delta_1 \alpha} \\ \vdots \\ \int_{\delta_{\mu_f} \alpha} \end{bmatrix}$$

All the elements of the matrices in the right side of the equality have finite growth at critical values. This is an analogy of the argument used to show (5.11) with the aid of Cramer’s rule.

Pull-back of forms $\eta_\ell$, $\forall \ell$ are independent in $H_f$ under the map $F^*$ and can be extended to a basis for $H_{F^*(f)}$ in view of Corollary 3.2.

There is a polynomial $K(\tau) \in \mathbb{C}[\tau]$ such that $K(f)[\alpha]$ be a holomorphic form. We can write $K(f)[\alpha] = \sum_\ell h_\ell(f)[\eta_\ell]$ then $F^*(K)\omega_1 - F^*(K,\alpha) = 0$ in $H_{F^*(f)}$.

Now we shall show the Claim: $\omega_1 - F^*(\alpha) = 0$ in $H_{F^*(f)}$. The set $\{ F^*(\eta_\ell) \}_{\ell=1}^{\mu_f}$ can be extended to a basis of $H_{F^*(f)}$ by Corollary 3.2. So we have in $H_{F^*(f)}$

$$F^*(K)\omega_1 = \sum_{\ell=1}^{\mu_f} F^*(K), h_\ell(f)[\eta_\ell] + \sum_{\sigma=\mu_f+1}^{\mu} F^*(K)a_\sigma \tilde{\eta}_\sigma$$

Here $\{ \tilde{\eta}_\sigma \}_{\sigma=\mu_f+1}^{\mu}$ is a basis of $H_{F^*(f)}$ alien to $F^*(H_f)$.

Since each element of $H_{F^*(f)}$ can be uniquely written as a linear combination of the elements in this basis we get the vanishing coefficients $a_\sigma = 0$ for all $\sigma$. In other words, in view of (4.3), we have $F^*(K), h_\ell = F^*(h_\ell)$ hence $K|h_\ell$. This means that $\omega_1 - F^*(\alpha) = 0$ in $H_{F^*(f)}$. To find the degree of $\alpha$, let

$$\omega_1 = \sum_\ell h_\ell(F(f))[\eta_\ell] = \sum_\ell h_\ell(F(f)) \sum_{\beta} g_{\beta}(F(f)) \eta_\beta = \sum_\eta \sum_{l}(l)(g_{\beta} h_\ell)[\eta_\beta].$$
by applying Theorem \[5.1\] we conclude that \( \deg(h_i) = 0 \) so \( \deg(\tilde{\alpha}) = a + 1 \). This is nothing but the Claim in question.

Thanks to the Claim, we have

\[
\int_{\delta} \frac{\omega_1 - F^*(\alpha)}{F^*(PQ)} = 0, \quad \forall \delta \in H_1((F^*(f))^{-1}(b), \mathbb{Z}),
\]

which implies that the integrand rational form of \((4.5)\) is a relatively exact 1-form modulo the foliation \( F^*(\omega_0) \) for \((4.3)\). By Corollary \[3.1\] there is a 1-form \( \tilde{\omega} \) of the form \((3.6)\) such that \( \omega_1 = F^*(\alpha) + \tilde{\omega} \). In fact there are polynomials \( P_1 \) and \( Q_1 \) with degree \( ms \) and \( ns \) respectively such that \( \tilde{\omega} = qF^*(Q)dP_1 - pP_1dF^*(Q) - qQ_1dF^*(P) + pF^*(P)dQ_1 \).

We know that there are rational function \( \tilde{h}_1 \) and a 1-form \( \beta_1 \) on \( \mathbb{P}^2 \) such that

\[
\tilde{\omega} = \frac{\tilde{\omega}}{F^*(PQ)} = \frac{\beta_1 + \tilde{h}_1F^*(\alpha_0)}{F^*(PQ)}.
\]

**Proposition 4.1.** The 1-form \( \tilde{\omega} \) in the equality \((4.1)\) is of the form

\[
\tilde{\omega} := qF^*(Q)dP_1 - pP_1dF^*(Q) + qQ_1dF^*(P) - pF^*(P)dQ_1
\]

with \( P_1 = q < F_1, F^*(\text{grad}P) >, Q_1 = p < F_1, F^*(\text{grad}Q) > \) for a vector

\[
F_1 = (R_1, S_1, T_1),
\]

defined by some homogeneous polynomials \( R_1, S_1, T_1 \in \mathbb{C}[x, y, z]_3^b \).

**Proof.** First of all we introduce the following polynomials \( \{\lambda_j(X, Y, Z)\}_{j=1}^3 \) defined by the relation

\[
\alpha_0 = qQdP - pPdQ = \lambda_1dX + \lambda_2dY + \lambda_3dz.
\]

Secondly, we define polynomials \( \{\rho_j(x, y, z)\}_{j=1}^3 \) by means of \((0.4), (4.9)\),

\[
(F^*(\lambda_1), F^*(\lambda_2), F^*(\lambda_3)).J_F(x, y, z) = (\rho_1, \rho_2, \rho_3).
\]

In other words, \( F^*(\alpha_0) = \rho_1dx + \rho_2dy + \rho_3dz \).

Now we pass to the investigation of polynomials \( P_1, Q_1 \) present in \((4.2)\). By multiply the relation \((4.6)\) with \( F^*(\alpha_0) \) we get the following equality:

\[
(-Q_1F^*(P) + P_1F^*(Q))F^*(dP \wedge dQ) = (pq)^{-1}(\beta_1 - qF^*(Q)dP_1 + pF^*(P)dQ_1) \wedge F^*(\alpha_0).
\]

Now let us consider the ideals in \( \mathbb{C}[x, y, z] \)

\[
I_1 = < F^*(\lambda_1), F^*(\lambda_2), F^*(\lambda_3) >, \quad I = < \rho_1, \rho_2, \rho_3 >
\]

for polynomials from \((4.9), (4.10)\). We also consider the ideal \( J = J_F(j, \ell)_{1 \leq j, \ell \leq 3} \subset \mathbb{C}[x, y, z] \) generated by \( 2 \times 2 \) minors of \( J_F(x, y, z) \) \((0.4)\).

Since \( V(J) \cap V(I_1) = \emptyset \) we see that \( I_1 + J = \mathbb{C}[x, y, z] \) and \( I_1 \cap J = I_1.J \).
The equality (4.11) entails that \((-Q_1 F^*(P) + P_1 F^*(Q)) \in J \subseteq I \subseteq I_1\). This means that 
\((-Q_1 F^*(P) + P_1 F^*(Q)) \in I_1 \cap J = I_1, J\) thus 
\((-Q_1 F^*(P) + P_1 F^*(Q)) \in I_1\). In other words, there exist polynomials \(R_1, S_1, T_1\) of degree \(s\) such that

\[
Q_1 F^*(P) + P_1 F^*(Q) = R_1 F^*(\lambda_1) + S_1 F^*(\lambda_2) + T_1 F^*(\lambda_3).
\]

Since \(P, Q\) are co-prime we have the required expressions for \(P_1, Q_1\) (4.7):

\[
P_1 = q(R_1 F^*(P_X) + S_1 F^*(P_Y) + T_1 F^*(P_Z)),
\]
\[
Q_1 = p(R_1 F^*(Q_X) + S_1 F^*(Q_Y) + T_1 F^*(Q_Z)).
\]

\[\square\]

**Proof. of Theorem 1.2:** We know that the foliation \(F_0\) can be generated by both 1-form \(F^*(\alpha_0)\) or \(F^*(Q_0dP_0 - pP_0dQ_0)\), where \(P_0 = qP, \ Q_0 = pQ\) and deformation of the foliation \(F_0\) in the space of logarithmic foliation \(I(ms - 1, ns - 1)\) given by 1-form

\[
\omega^i = qQ_0dP_0 - pP_0dQ_0 + \epsilon[q(pQ_1)dP_0 - pP_0d(pQ_1) + qQ_0d(qP_1) - p(qP_1)dQ_0]
\]
\[
+ p\epsilon^2(qQ_1dP_1 - pP_1dQ_1) + \cdots
\]

which is come from \(d\left(\frac{(pP_1 + qQ_1)^{p}}{Q^q + Q_1}\right)\) or the 1-form

\[
\omega^i = qQ_0dP_0 - pP_0dQ_0 + \epsilon[qQ_1dP_0 - pP_0dQ_1 + qQdP_1 - pP_1dQ] + \epsilon^2(qQ_1dP_1 - pP_1dQ_1) + \cdots
\]

Which coming from \(d\left(\frac{(pP_1 + qQ_1)^{p}}{Q^q + Q_1}\right)\). Therefore, coefficient of \(\epsilon\) in deformation (1.28) of the the foliation \(F_0 = F(F^*(Q_0dP_0 - pP_0dQ_0))\) is given by \(\omega_1 = F^*(\tilde{\alpha}) + \tilde{\omega}_e\) where \(\tilde{\omega}_e\) has the form of the \(\epsilon\)’s coefficient of \(\omega^i\) in (4.13) which is equal to the \(\epsilon\)’s coefficient of \(\omega^i\) in (4.14). This implies that \(\omega_1 = \omega_W\). \(\square\)

## 5 Brieskorn/Petrov module

In this section we establish a new formulation of results concerning the Brieskorn/Petrov module defined for a rational function of type (1.4). This generalization furnishes us with the proof of Theorem 5.1 that is necessary to establish Corollary 3.2 that in its turn plays an essential rôle during the proof of the key Lemma 4.1.

Let us consider a rational function \(f : \mathbb{P}^2 \setminus \mathcal{R} \to \mathbb{C}\) as in (1.4) satisfying Conditions 2.1. Further in this section, we regard \(f = \frac{p}{q}\) with \(P, Q \in \mathbb{C}[X, Y]\) defined on \(\mathbb{C}^2\). In other words in the sequel \(P = P(X, Y, 1), Q = Q(X, Y, 1)\) in terms of polynomials in (1.4). This reduction is possible due to the fact that \(f\) is transversal to \(Z = 0\). We recall that \(D := f^{-1}(\infty) = V(Q)\) is smooth due to the Condition 2.1 (1). Let \(\Omega^i(\ast_D)\) be the set of rational \(i\)-forms on \(\mathbb{P}^2\) with poles of arbitrary order along \(D\). Let \(t\) be an affine coordinate of \(\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}\). The set \(\Omega^i(\ast_D)\) can be regarded as a \(\mathbb{C}[t]\)-module according to the following identification:

\[p(t) \omega = p(f) \omega, \quad \omega \in \Omega^i(\ast_D)\]
for \( p(t) \in \mathbb{C}[t] \). Any \( i \)-form \( \omega \in \Omega^i(\ast D) \) can be considered as a polynomial \( i \)-form in three variables \( X, Y, \zeta \) with \( d\zeta = -\zeta^2 dQ \). The \( \mathbb{C}[t] \)-module of relative rational 2-forms with poles of arbitrary order is defined as follows:

\[
\Omega^2_{\mathbb{P}^2/\mathbb{P}^1}(\ast D) = \frac{\Omega^2(\ast D)}{df \wedge \Omega^1(\ast D)}.
\]

One can find a \( \mathbb{C}[t] \)-module injective and surjective homomorphism from \( \Omega^2_{\mathbb{P}^2/\mathbb{P}^1}(\ast D)|_{\mathbb{C}^2} \) to the following \( \mathbb{C}[t] \)-module with quotient ring structure

\[
\mathcal{M}(\ast D) := \frac{\mathbb{C}[X, Y, \zeta]}{I}
\]

defined for the ideal

\[
I = \langle \zeta.Q - 1, qP_X - p\zeta.Q.X.P, qP_Y - p\zeta.Q.Y.P \rangle.
\]

We remark here that the element \( Q \) is invertible in the quotient ring \( \mathcal{M}(\ast D) \).

**Proposition 5.1.** \( \Omega^2_{\mathbb{P}^2/\mathbb{P}^1}(\ast D) \) has a structure of vector space of dimension \( \mu_f = (n + m - 1)^2 - nm \) where \( \mu_f \) is the global Milnor number of \( f \).

**Proof.** According to [13], [Mo3] Corollary 1.1, \( \Omega^2_{\mathbb{P}^2/\mathbb{P}^1}(\ast D) \) is a vector space with dimension \( \mu_f : \) the sum of local Milnor numbers of \( f \). We remark here that the cardinality of the set \( \mathcal{R} = V(P) \cap V(Q) \) is equal to \( nm \).

We can prove the Proposition 5.1 by the aid of the Gröbner basis. In fact by using the graded lex order on \( \mathbb{C}[X, Y, \zeta] \) one can find a Gröbner basis \( \bar{I} \) for the ideal \( I \) (5.2) and then by considering the leading part of \( \bar{I} \) the basis of \( \mathcal{M}(\ast D) \) is obtained. Indeed this basis depends on \( \zeta \) but by changing the basis we can write \( \zeta \) as a polynomial in variables \( X, Y \) because \( \zeta \) is an invertible.

By the Condition 2.1 (1) we have

\[
\mathcal{M}(\ast D) \cong \sum_{j=1}^{\mu_f} \mathbb{C}m_j(X, Y)
\]

for \( m_j(X, Y) \in \mathbb{C}[X, Y] \).

We see that the concrete calculation of the monomials \( m_j(X, Y) \) in (5.3) can be done with the following example that essentially covers all necessary cases (1.4) under the Conditions 2.1. Namely these conditions mean that the Newton polyhedron of the polynomial \( P(X, Y) \) (resp. \( Q(X, Y) \)) is a triangle with vertices \( \{ (0, 0), (m, 0), (0, m) \} \) (resp. \( \{ (0, 0), (n, 0), (0, n) \} \)) with non-degenerate condition on the edge \( s(m, 0) + (1-s)(0, m), s \in [0, 1] \) (resp. \( s(n, 0) + (1-s)(0, n), s \in [0, 1] \)).

Consider two generic polynomials of respective degrees \( n, m \) of the form \( P = p_1X^m + p_2Y^m \) and \( Q = q_1X^n + q_2Y^n \) where \( P, Q \) are co-prime. The module \( \mathcal{M}(\ast D) \) is generated by

\[
X^iY^\ell, \quad s.t \quad X^{m-1}Y^{n-1} \mid X^iY^\ell, \quad 0 \leq i, \ell \leq (m + n - 2),
\]

Thus we get monomials \( m_j(X, Y) \) for \( j = 1, \ldots, \mu_f = (n + m - 1)^2 - nm \).

See Figure 2.
Now let us define the relative cohomology associated to $f$ that is endowed with $\mathbb{C}[t]$-
module structure

$$H_f := \frac{\Omega^1(*)}{df \wedge \Omega^0(*) + d\Omega^0(*)},$$

where $t = f$ in the affine coordinate $\mathbb{C}$. This module is called Brieskorn $\mathbb{C}[t]$-module (or
Petrov module according to [G1]). It is worthy noticing that the denominator of (5.4)
represents the space of relatively exact rational 1-forms modulo $\mathcal{F}(\omega_0)$ for (3.3).

We define the degree of the zero divisor $Z(\eta)$ of a rational form

$$\eta = \frac{P^1_\eta(X,Y)dX + P^2_\eta(X,Y)dY}{Q^\ell}, \ell \geq 0$$

for $Q \nmid P^1_\eta, P^2_\eta$ as follows

$$\deg Z(\eta) = \max\{\deg P^1_\eta(X,Y), \deg P^2_\eta(X,Y)\}$$

Now we state the following (see [Mo6, Theorem 10.12.1]):

**Theorem 5.1.** The $\mathbb{C}[t]$-module $H_f$ is free and finitely generated by 1-forms $\alpha_j$, $j = 1, \ldots, \mu_f$ and each 1-form $\alpha_j$ can be defined by the condition

$$d\alpha_j = m_j(x,y)dx \wedge dy,$$

where $m_j(x,y)$ is an element of the monomial basis (5.3) for $\mathcal{M}(*D)$. Furthermore, a
rational 1-form $\alpha$ in $\Omega^1(*)$ can be written as follows

$$\alpha = \sum_{j=1}^{\mu} C_j(f)\alpha_j + df \wedge \zeta_1 + d\zeta_2,$$

where $\zeta_1, \zeta_2 \in \Omega^0(*)$ and $C_j$ is a polynomial with degree that admits an evaluation,

$$\deg(C_j) \leq \frac{\deg(Z(\alpha_j)) + \deg(Q) - \deg(Z(\alpha_j)) - 1}{q\deg(P)} = \frac{\deg(Z(\alpha)) + n - \deg(Z(\alpha_j)) - 1}{mq}.$$

Here $Z(\eta)$ is zero divisor of $\eta$.  

![Figure 2: $(i, j) \leftrightarrow X^i Y^j$](image)
In order to give a proof of this theorem we introduce period integrals associated to the rational function \( f \) as in \([1.4]\).

Let \( \delta_i(t) \), \( i = 1, \ldots, \mu_f \) be a continuous family of vanishing cycles around the critical points of \( f \) which form a basis of \( H_1(f^{-1}(t), \mathbb{Z}) \) for any regular value \( t \in \mathbb{C} \). Suppose that the 1-forms \( \alpha_1, \ldots, \alpha_{\mu_f} \in H_f \) such that \( d\alpha_j = m_j dx \wedge dy \) where \( m_j \) is an element of the basis \([5.3]\) for \( \mathcal{M}(\ast D) \). With these homology and cohomology bases one can associate the period matrix

\[
(5.7) \quad Y(t) := \begin{pmatrix}
\int_{\delta_1(t)} \alpha_1 & \cdots & \int_{\delta_{\mu_f}(t)} \alpha_1 \\
\vdots & \ddots & \vdots \\
\int_{\delta_1(t)} \alpha_{\mu_f} & \cdots & \int_{\delta_{\mu_f}(t)} \alpha_{\mu_f}
\end{pmatrix}
\]

which is an analytic multi-valued matrix-function ramified over the critical values \( C_f \) of \( f \). Also, \( W(t) := \det(Y(t)) \) is called Wronskian function.

**Proposition 5.2.** For \( f \) under the Conditions \([2.7]\), the determinant \( W(t) \) of any period matrix is a polynomial in \( t \) with zeros at \( C_f \). This fact holds regardless of the choice of the forms which constitute a basis of \( H_f \).

**Proof.** From the Picard-Lefschetz theorem we see that the monodromy action around each \( t = t_i \in C_f \) induces a monodromy transformation \( T_i Y(t) \) on \( Y(t) \) \([5.7]\) and \( \det(T_i Y(t)) = \det(Y(t)) \). This means that the determinant \( W(t) \) is a single-valued function on \( \mathbb{C} \). As \( t \) tends to infinity the integrals occurring in the entries of \( Y(t) \) grow no faster than a polynomial in \(|t|\) in any sector. This implies that \( W(t) \) is a polynomial. When \( t \) tends to a point \( c \in C_f \), at least one of vanishing cycles \( \delta_i(t) \) vanishes, hence one row of \( Y(t) \) is zero and the determinant of \( W(t) \) tends to zero. \( \square \)

Let us define the function

\[
(5.8) \quad \Delta(t) = (t - t_1)^{\mu_1} (t - t_2)^{\mu_2} \cdots (t - t_s)^{\mu_s}
\]

where \( \mu_i \) is the summation of local Milnor numbers of critical points located on the fiber \( f^{-1}(t_i) \) and \( \sum_{i=1}^s \mu_i = \mu \). When the \( f \) is a polynomial \( \Delta \) is called determinant function of \( f \) (see e.g. \([G1]\)).

**Lemma 5.1.** There exists a non-zero constant \( c \) such that \( W(t) = c \Delta(t) \).

**Proof.** If \( t \) tends to the critical value \( t_i \) then the vanishing cycles \( \delta_i(t) \) correspond to \( t_i \) tend to points therefore \( \int_{\delta_i(t)} \alpha_j \) tends to zero. This implies that the matrix \( Y(t) \) at point \( t_i \) is not of full rank, in other words \( t_i \) is a root of Wronskian function \( W \). If all critical values \( C_f \) of \( f \) are distinct then we \( W(t) = c \cdot \Delta(t) \) as the cardinality of \( C_f \) is \( \mu_f \).

If not, consider \( s \) deformation

\[
f_{A,t}(X,Y) = \frac{(P + a_1 X + b_1 Y)^q}{(Q + a_2 X + b_2 Y)^p} - t
\]

where \( A = (a_1, b_1, a_2, b_2) \). There is an open subset \( U \) of \((A, t) \in \mathbb{C}^5\) such that the function \( f_{A,t} \) has distinct critical values and all the critical points belongs to affine coordinate \( \mathbb{C}^2 \).

Let \( \Sigma_{A,t} = \{(A, t) \mid D(A, t) = 0\} \). As the Milnor number of \( f_{a,t} \) is equal to the Milnor number \( f \) then \( D(A, t) \) is also has degree \( \mu \) therefore if \( A = 0 \) then \( D(0, t) = \Delta(t) \). Let
\{\delta_j(t, A)\}_{j=1}^{\mu_f} be a continuous family of cycles which forms a basis of \(H_1(f^{-1}_A(0), \mathbb{Z})\) for any \((A, t) \notin \Sigma_{A,t}\). For the polynomial \(f_{A,t}\) consider the Wronskian function

\[
\tilde{W}(A, t) = det \left( \int_{\delta_i(t, A)} \alpha_j \right).
\]

This function is polynomial in \((A, t)\) and vanishes along \(\Sigma_{A,t}\), this implies that \(\tilde{W}(A, t) = C_A \cdot D(A, t)\). The polynomial \(C_A\) does not depend on \(t\) because \(deg(\tilde{W}(A, t))\) respect to \(t\) is equal to \(deg(D(A, t)) = \mu_f\). To finish the proof it is enough to recall that \(\tilde{W}(0, t) = W(t)\).

**Proof of Theorem 5.1**

First of all we see that \(\{\alpha_j\}_{j=1}^{\mu_f}\) are linearly independent in \(H_f\). The form \(\frac{d\alpha_j}{df}\) of \(\alpha_j\) coincides with the Gel'fand-Leray form of

\[
m_j \frac{dx \wedge dy}{df}.
\]

According to in [ZO Theorem 5.25, b] we have

\[
(5.9) \quad det \left( \int_{\delta_i(t)} \frac{d\alpha_j}{df} \right) = det \left( \frac{d}{dt} \int_{\delta_i(t)} \alpha_j \right) = c = \text{const} \neq 0.
\]

This means the required linear independence. We remark also that this relation can be deduced from a refinement of [T Theorem 2.7] established for the Gauss-Manin system satisfied by period integrals \(\int_{\delta_i(t)} \alpha_j\).

Let us consider a system of equations for a column vector of unknown functions \(C(t) = (C_1(t), \cdots, C_{\mu_f}(t))\)

\[
(5.10) \quad Y(t)C(t) = A_\delta(t)
\]

with \(Y(t)\) (5.7) and a \(\mu_f\) column vector:

\[
A_\delta(t) = \begin{pmatrix}
\int_{\delta_1(t)} \alpha \\
\vdots \\
\int_{\delta_{\mu_f}(t)} \alpha
\end{pmatrix}.
\]

Cramer’s rule solves (5.10) with solutions

\[
C_j(t) = det Y_{A,j}(t)/det Y(t) \quad j = 1, \cdots, \mu_f
\]

where \(Y_{A,j}(t)\) is a \(\mu_f \times \mu_f\) matrix obtained by replacing the \(j\)–th column of \(Y(t)\) with \(A_\delta(t)\). The function \(det Y_{A,j}(t)\) has at most polynomial growth as \(|t| \to \infty\) or as \(t\) approaches to a zero of \(\Delta(t)\). By Lemma 5.1 \(\forall j, C_j(t) \in \mathbb{C}[t]\) as \(Y_{A,j}(t)\) vanishes at zeros of \(\Delta(t)\).

In other words, for a fixed \(i\) we have

\[
\int_{\delta_i(t)} \alpha = \sum_{j=1}^{\mu_f} C_j(t) \int_{\delta_i(t)} \alpha_j.
\]
This implies that

\[ (5.11) \quad \alpha = \sum_{j=1}^{\mu} C_j(f) \alpha_j \]

in the \( \mathbb{C}[t] \)-module \( H_f \). Assume that \( \alpha = Q^{-l}(P_1^1 dx + P_2^2 dy) \) has poles of order \( l \) along \( V(Q) \) therefore \( d\alpha = hQ^{-l-1} dx \wedge dy \) where \( h \) is a polynomial of degree \( \text{deg}(Z(\alpha)) + n - 1 \). The derivation of each term of the right side of (5.11) becomes

\[
\begin{align*}
    d(C_j(f)\alpha_j) &= C_j'(f)f \frac{(qQdP - pPdQ) \wedge \alpha_j}{PQ} + C_j(f)d\alpha_j.
\end{align*}
\]

Each term of the above expression has degree \( (\text{deg}(C_j) - 1)mq + (mq - 1) + (n + m - 1) + \text{deg}Z(\alpha_j) \) and \( mq.\text{deg}(C_j) + \text{deg}Z(\alpha_j) - 1 \) respectively. We remark here that

\[
\text{deg}Z((qQdP - pPdQ) \wedge (xdy - ydx)) = n + m - 1.
\]

By Condition (2.1) (3) \( m \geq n \geq 2 \) and we conclude that

\[ (5.12) \quad \text{deg}(Z(\alpha)) + n - 1 \geq mq.\text{deg}(C_j) + \text{deg}(Z(\alpha_j)). \]

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