FIGURATE NUMBERS AND SUMS OF POWERS OF INTEGERS

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ABSTRACT. Recently, Litvinov and Marko conjectured that, for all positive integers \( n \) and \( p \), the \( p \)th power of \( n \) admits the representation \( n^p = \sum_{\ell=0}^{p-1} (-1)^\ell \frac{F_{n-p}^p}{F_{n-\ell}^p} \), where \( F_{n-p}^p \) is the \( n \)th hyper-tetrahedron number of dimension \( p-\ell \) and \( c_{p,\ell} \) denotes the number of \((p-\ell)\)-dimensional facets of the \( p \)-dimensional simplex \( x_{\sigma_1} \geq x_{\sigma_2} \geq \cdots \geq x_{\sigma_p} \) (where \( \sigma \) is a permutation of \( \{1,2,\ldots,p\} \)) formed by cutting the \( p \)-dimensional cube \( 0 \leq x_1, x_2, \ldots, x_p \leq n-1 \). In this note we show that this conjecture is true for every natural number \( p \) if, and only if, \( c_{p,\ell} = (p-\ell)!S(p,p-\ell) \), where \( S(p,p-\ell) \) are the Stirling numbers of the second kind. Furthermore, we provide several equivalent formulas expressing the sum of powers \( \sum_{i=1}^n i^p \), \( p = 1,2,\ldots \), as a linear combination of figurate numbers.

1. Introduction

Let \( F_n^k = \binom{n+k-1}{k} \) be the \( n \)th hyper-tetrahedron number of dimension \( k \). In particular, \( F_n^2 \) is the \( n \)th triangular number \( T_n = \frac{1}{2}n(n+1) \) and \( F_n^3 \) is the \( n \)th tetrahedral number \( T_n = \frac{1}{2}n(n+1)(n+2) \) (see, for instance, [1, Chapter 2]). Recently, Litvinov and Marko conjectured (see [2, Conjecture 16]) that, for all positive integers \( n \) and \( p \), the \( p \)th power of \( n \) can be put as

\[
n^p = \sum_{\ell=0}^{p-1} (-1)^\ell \frac{F_{n-p}^p}{F_{n-\ell}^p},
\]

for certain positive integer coefficients \( c_{p,0}, c_{p,1}, \ldots, c_{p,p-1} \) (note the corrected factor \((-1)^\ell\) in Equation (1) instead of the original one \((-1)^p \) appearing in [2]). Specifically, \( c_{p,\ell} \) is the number of \((p-\ell)\)-dimensional simplices defined by \( 0 \leq x_1, x_2, \ldots, x_p \leq n-1 \) in conjunction with the conditions

\[
x_{\sigma_1} L_{1} x_{\sigma_2} L_{2} \ldots L_{p-1} x_{\sigma_p},
\]

where exactly \( \ell \) symbols \( L_i \) are "\( = \)", the remaining \( p-\ell-1 \) symbols \( L_i \) are "\( \geq \)”, and where \( \sigma \) is a permutation of \( \{1,2,\ldots,p\} \). As indicated in [2], every such simplex is then a \((p-\ell)\)-dimensional facet of the \( p \)-dimensional simplex \( x_{\sigma_1} \geq x_{\sigma_2} \geq \cdots \geq x_{\sigma_p} \) formed by cutting the \( p \)-dimensional cube \( 0 \leq x_1, x_2, \ldots, x_p \leq n-1 \).

An alternative characterization of the coefficients \( c_{p,\ell} \) can be made in terms of \( m \)-tuples \( (k_1, k_2, \ldots, k_m) \) of nonnegative integers with content \( \ell = \sum_{i=1}^m k_i \) and support \( s \) (the latter being defined as the number of indices \( i \) such that \( k_i > 0 \)). As shown in [2, Proposition 14], the coefficients \( c_{p,\ell} \) are given by

\[
c_{p,\ell} = \sum_{(k_1, k_2, \ldots, k_m)} \frac{p!}{(k_1+1)! (k_2+1)! \cdots (k_m+1)!},
\]

where the sum runs over all \( m \)-tuples of nonnegative integers having the content \( \ell \) and the support \( s = m + \ell + 1 - p \) and such that \( k_j > 0 \) implies that \( k_{j+1} = 0 \) for every \( j < m \).
Comparing the formulas for $\Sigma_n^p = \sum_{r=1}^n r^p$. Since $F_n^k = \sum_{i=1}^n F_i^{k-1}$, representation (1) for $n^p$ immediately implies that

$$\Sigma_n^p = \sum_{i=1}^{p} (-1)^{i-1} c_{p,i-1} F_n^{p-i+2},$$

expressing $\Sigma_n^p$ as a linear combination of figurate numbers. In what follows, we refer to either Equation (1) or (3) (with the coefficients $c_{p,i}$ being given by Equation (2)) as the LM conjecture. The crucial point we want to remark here is that $\Sigma_n^p$ can, in fact, be expressed in the polynomial form (see, for example, [3, Equation (7.5)], [4], and [5, Section 4])

$$\Sigma_n^p = \sum_{i=1}^{p} (-1)^{p-i} S(p, i) \binom{n+i}{i+1},$$

where $S(p, i)$ are the Stirling numbers of the second kind. Correspondingly, formula (4) can be written in terms of the figurate numbers as follows:

$$\Sigma_n^p = \sum_{i=1}^{p} (-1)^{i-1}(p-i+1)! S(p, p-i+1) F_n^{p-i+2}.$$  

For example, letting $p = 5$ in Equation (5) yields

$$\Sigma_5^5 = 120F_n^6 - 240F_n^5 + 150F_n^4 - 30F_n^3 + F_n^2.$$ 

Comparing the formulas for $\Sigma_n^p$ in Equations (3) and (5), and noting that the polynomials representing the figurate numbers $F_n^k$ are linearly independent, it is clear that if formula (3) for $\Sigma_n^p$ is true then necessarily the coefficients $c_{p,\ell}$ in Equation (2) should be of the form $c_{p,\ell} = (p-\ell)! S(p, p-\ell)$ for $\ell = 0, 1, \ldots, p-1$. Conversely, if $c_{p,\ell} = (p-\ell)! S(p, p-\ell)$ for $\ell = 0, 1, \ldots, p-1$, then formula (3) for $\Sigma_n^p$ is true by virtue of Equation (5). This can be summarized as

$$\text{LM conjecture } \iff c_{p,\ell} = (p-\ell)! S(p, p-\ell), \text{ for } \ell = 0, 1, \ldots, p-1.$$  

## 2. Matrix formulation

One can also arrive at the equivalence in Equation (6) by considering the transition matrix connecting the bases $\{n, n^2, \ldots, n^p\}$ and $\{F_n^1, F_n^2, \ldots, F_n^p\}$. For $k \geq 0$, the figurate numbers $F_n^k$ can be expanded in the basis $\{n, n^2, \ldots, n^k\}$ as

$$F_n^k = \binom{n+k-1}{k} = \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} = \frac{1}{k!} \sum_{r=1}^{k} s(k, r)n^r,$$
where the numbers \( s(k, r) \) are (unsigned) Stirling numbers of the first kind. For example, we have that

\[
\begin{align*}
F_n^1 &= n, \\
F_n^2 &= \frac{1}{2} n^2 + \frac{1}{2} n, \\
F_n^3 &= \frac{1}{6} n^3 + \frac{1}{2} n^2 + \frac{1}{3} n, \\
F_n^4 &= \frac{1}{24} n^4 + \frac{1}{4} n^3 + \frac{11}{24} n^2 + \frac{1}{4} n, \\
F_n^5 &= \frac{1}{120} n^5 + \frac{1}{12} n^4 + \frac{7}{24} n^3 + \frac{5}{12} n^2 + \frac{1}{5} n,
\end{align*}
\]

or, in matrix form,

\[
\begin{pmatrix}
F_1^1 \\
F_2^2 \\
F_3^3 \\
F_4^4 \\
F_5^5
\end{pmatrix} = A_5
\begin{pmatrix}
n \\
n^2 \\
n^3 \\
n^4 \\
n^5
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{2} & 1 & 0 & 0 \\
\frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 0 \\
\frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1
\end{pmatrix}
\begin{pmatrix}
n \\
n^2 \\
n^3 \\
n^4 \\
n^5
\end{pmatrix},
\]

where, following [2], we call the matrices \( A_p \) (for any \( p \geq 1 \)) Fermat matrices. By inverting the matrix \( A_5 \) we get the transition matrix from the basis \( \{ F_n^1, F_n^2, F_n^3, F_n^4, F_n^5 \} \) to \( \{ n, n^2, n^3, n^4, n^5 \} \), namely,

\[
\begin{pmatrix}
n \\
n^2 \\
n^3 \\
n^4 \\
n^5
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 \\
1 & -6 & 6 & 0 & 0 \\
-1 & 14 & -36 & 24 & 0 \\
1 & -30 & 150 & -240 & 120
\end{pmatrix}
\begin{pmatrix}
F_1^1 \\
F_2^2 \\
F_3^3 \\
F_4^4 \\
F_5^5
\end{pmatrix}.
\]

Moreover, recalling that \( F_n^k = \sum_{i=1}^{n} F_i^{k-1} \), we can express the last matrix equation in the equivalent way

\[
\begin{pmatrix}
\Sigma_n^1 \\
\Sigma_n^2 \\
\Sigma_n^3 \\
\Sigma_n^4 \\
\Sigma_n^5
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 \\
1 & -6 & 6 & 0 & 0 \\
-1 & 14 & -36 & 24 & 0 \\
1 & -30 & 150 & -240 & 120
\end{pmatrix}
\begin{pmatrix}
F_n^2 \\
F_n^3 \\
F_n^4 \\
F_n^5
\end{pmatrix}.
\]

Let \( A_p \) be a Fermat matrix, and denote the entries of \( A_p^{-1} \) as \( a_{k,j}^p \), \( k, j = 1, 2, \ldots, p \). Thus, for arbitrary \( p \geq 1 \), we have

\[
\begin{pmatrix}
\Sigma_n^1 \\
\Sigma_n^2 \\
\vdots \\
\Sigma_n^p
\end{pmatrix} =
\begin{pmatrix}
a_{1,1}^p & a_{1,2}^p & \cdots & a_{1,p}^p \\
a_{2,1}^p & a_{2,2}^p & \cdots & a_{2,p}^p \\
\vdots & \vdots & \ddots & \vdots \\
a_{p,1}^p & a_{p,2}^p & \cdots & a_{p,p}^p
\end{pmatrix}
\begin{pmatrix}
F_n^2 \\
F_n^3 \\
\vdots \\
F_n^p
\end{pmatrix},
\]

from which it follows that

\[
\Sigma_n^p = \sum_{i=1}^{p} a_{p,i}^p F_n^{i+1} = \sum_{i=1}^{p} a_{p,p-i+1}^i F_n^{i+2}.
\]
Comparing Equation (8) to Equation (3), and noting that the polynomials $F_n^2$, $F_n^3$, ..., $F_n^{p+1}$ are linearly independent, it is concluded that the LM conjecture is equivalent to (cf. [2])

$$a'_p,i = (-1)^{p-i}c_{p,p-i}, \quad i = 1, 2, \ldots, p.$$  \hspace{1cm} (9)

Next, we show that

$$a'_{k,j} = (-1)^{k-j}j!S(k,j), \quad k, j = 1, 2, \ldots, p.$$  \hspace{1cm} (10)

To prove this, let us denote the entries of the Fermat matrix $A_p$ as $a_{k,j}$, $k, j = 1, 2, \ldots, p$.

Then, from Equation (7), it is clear that

$$a_{k,j} = \frac{s(k,j)}{k!}, \quad k, j = 1, 2, \ldots, p.$$  \hspace{1cm} (11)

Now, consider the product matrices $M_p = A_pA_p^{-1}$ and $N_p = A_p^{-1}A_p$, where $A_p^{-1}$ and $A_p$ have elements given in Equations (10) and (11), respectively. It is readily seen that the matrix elements $M_{k,j}$ and $N_{k,j}$ of $M_p$ and $N_p$ are

$$M_{k,j} = \frac{j!}{k!} \sum_{r=1}^{p} (-1)^{r-j}s(k,r)S(r,j),$$

$$N_{k,j} = \sum_{r=1}^{p} (-1)^{k-r}S(k,r)s(r,j),$$

for $1 \leq k, j \leq p$. Therefore, invoking the well-known identities (see, for instance, [6, Theorem 6.24])

$$\sum_{r=0}^{k} (-1)^{r}s(k,r)S(r,j) = (-1)^{k}\delta_{k,j},$$

$$\sum_{r=0}^{k} (-1)^{r}S(k,r)s(r,j) = (-1)^{k}\delta_{k,j},$$

and taking into account that $s(k,0) = S(k,0) = 0$ for $k \geq 1$, and that $s(k,j) = S(k,j) = 0$ for $k < j$, we obtain $M_{k,j} = (-1)^{k-j}j!/k!\delta_{k,j}$ and $N_{k,j} = (-1)^{2k}\delta_{k,j}$, and thus both $M_p$ and $N_p$ turn out to be the identity matrix $I_p$.

Note that, for the case in which $k = p$, Equation (10) reads (after renaming the index $j$ as $i$) $a'_{p,i} = (-1)^{p-i}i!S(p,i)$. Hence, from Equation (9), we conclude that the LM conjecture is true if, and only if, $c_{p,p-i} = i!S(p,i)$ or, equivalently, $c_{p,\ell} = (p-\ell)!S(p,p-\ell)$, for $\ell = 0, 1, \ldots, p-1$, thus recovering the statement in Equation (6).

Table 1 displays the first few rows of the triangular array for the numbers $c_{p,\ell} = (p-\ell)!S(p,p-\ell)$, where $\ell = 0, 1, \ldots, p-1$. It is worth pointing out that, starting from the well-known recurrence relation for the Stirling numbers of the second kind, namely, $S(k,j) = jS(k-1,j) + S(k-1, j-1)$ (with initial conditions $S(0,0) = 1$ and $S(0,j) = S(j,0) = 0$ for $j > 0$), one can derive the following recurrence relation which is fulfilled by the numbers $c_{p,\ell}$:

$$c_{p,\ell} = \begin{cases} p!, & \text{if } \ell = 0, \\ (p-\ell)[c_{p-1,\ell} + c_{p-1,\ell-1}], & \text{if } 0 < \ell < p-1, \\ 1, & \text{if } \ell = p-1. \end{cases}$$  \hspace{1cm} (12)

Of course, the entries in Table 1 may be computed using the recursive formula (12). For example, $c_{9,3}$ is determined by the values of $c_{8,3}$ and $c_{8,2}$ as follows: $c_{9,3} = 6(c_{8,3} + c_{8,2}) =$
6(126000 + 191520) = 1905120. Moreover, the alternating sum of the entries in the $p$th row in Table 1 is given by $\sum_{\ell=0}^{p-1} (-1)^\ell c_{p,\ell} = 1$ for all $p \geq 1$. This quickly follows from Equation (5) by noting that $\Sigma_p^\ell = 1$ for all $p \geq 1$.

Let us also observe that, since the recursive formula (12) completely defines the numbers $c_{p,\ell} = (p-\ell)!S(p, p-\ell)$, to prove the LM conjecture it suffices to show that the coefficients $c_{p,\ell}$ in Equation (2) satisfy such recurrence relation.

### 3. Concluding remarks

In addition to the formula in Equation (5), there are several alternative formulas expressing $\Sigma_n^p$ as a linear combination of the figurate numbers $F_{n,k}^p = \binom{n+k-1}{k}$. For example, the following two well-known polynomial formulas for $\Sigma_n^p$ (see, for instance, [5, 7, 8]):

$$
\Sigma_n^p = \sum_{j=1}^{p} j!S(p, j) \binom{n+1}{j+1},
$$

(13)

and

$$
\Sigma_n^p = \sum_{j=1}^{p} \left\langle \frac{p}{j} \right\rangle \binom{n+j}{p+1},
$$

(14)

can equivalently be written in terms of $F_{n,k}^p$ as

$$
\Sigma_n^p = \sum_{j=1}^{p} j!S(p, j) F_{n-j+1}^{j+1},
$$

(15)

and

$$
\Sigma_n^p = \sum_{j=1}^{p} \left\langle \frac{p}{j} \right\rangle F_{n+p-j}^{p+1},
$$

(16)

respectively, where $\left\langle \frac{p}{j} \right\rangle$ are the Eulerian numbers, with the initial values $\left\langle \frac{p}{1} \right\rangle = 1$ for all $p \geq 1$. Note that Equation (16) only involves figurate numbers of dimension $p+1$. Along with the above two formulas in Equations (13) and (14), we may quote another, not so well-known formula for $\Sigma_n^p$ which is a variant of that in Equation (13), namely (see [9,
Equation (9)] and [10])

\[
\sum_{p}^{p+1}(j-1)!S(p+1,j)\binom{n}{j} = \sum_{j=1}^{p+1}(j-1)!S(p+1,j)F_{n-j+1}^{2j-1}.
\]

As an example, for \( p = 8 \), from Equations (5), (15), (16), and (17), we obtain the equivalent polynomial representations

\[
\Sigma_{n}^{8} = 40320F_{n}^{9} - 141120F_{n}^{8} + 191520F_{n}^{7} - 126000F_{n}^{6} + 40824F_{n}^{5} - 5796F_{n}^{4} + 254F_{n}^{3} - F_{n}^{2}
\]

\[
= 40320F_{n-2}^{9} + 141120F_{n-6}^{8} + 191520F_{n-5}^{7} - 126000F_{n-4}^{6} + 40824F_{n-3}^{5} - 5796F_{n-2}^{4} + 254F_{n-1}^{3} + F_{n}^{2}
\]

\[
= F_{n}^{9} + 247F_{n-1}^{9} + 4293F_{n-2}^{9} + 15619F_{n-3}^{9} + 247F_{n-5}^{9} + F_{n-7}^{9}
\]

\[
= 40320F_{n-8}^{9} - 181440F_{n-7}^{8} + 332640F_{n-6}^{7} + 317520F_{n-5}^{6} + 166824F_{n-4}^{5} + 46620F_{n-3}^{4} + 6050F_{n-2}^{3} + 255F_{n-1}^{2} + F_{n}^{1}.
\]

For completeness, let us finally mention that, as is well known, the power sums \( \Sigma_{n}^{p} \) can be expressed as polynomials in the triangular numbers \( T_{n} \) (the so-called Faulhaber polynomials [11]) as follows:

\[
\Sigma_{n}^{2k} = \sum_{j=0}^{k}b_{k,j}T_{n}^{j} + \sum_{j=0}^{k}c_{k,j}T_{n}^{j},
\]

where \( b_{k,j} \) and \( c_{k,j} \) are numerical coefficients for \( j = 0, 1, \ldots, k \) and \( k \geq 1 \).

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