UNRAMIFIED COHOMOLOGY, $\mathbb{A}^1$-CONNECTEDNESS, AND CHEVALLEY-WARNING PROBLEM IN GROTHENDIECK RING

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Abstract. We study the Chevalley-Warning problem in the Grothendieck ring $K_0(Var/k)$. We show that the $\mathbb{A}^1$-homotopy theory yields well defined invariants on $K_0(Var/k)/L$, in particular the Brauer group is such an invariant. We use this to give a concrete counterexample to the Chevalley-Warning conjecture over a $C_1$-field [BS11]. This also gives a negative answer to the question in [Bil11, Ques. 3.8].

Résulté : Cohomologie non ramifiée, $\mathbb{A}^1$-connexité et le problème de Chevalley-Warning dans l’anneau de Grothendieck. Nous étudions le problème de Chevalley-Warning dans l’anneau de Grothendieck $K_0(Var/k)$. Nous montrons que la théorie $\mathbb{A}^1$-homotopie fournit des invariants sur $K_0(Var/k)/L$. En particulier le groupe de Brauer est un tel invariant. Nous utilisons cela pour donner un contre-exemple concret à la conjecture de Chevalley-Warning sur un corps $C_1$ [BS11]. Cela donne aussi une réponse négative à la question dans [Bil11, Ques. 3.8].

1. Introduction

Let $k$ be a field and $Var/k$ be the category of varieties over $k$. We denote by $K_0(Var/k)$ the Grothendieck ring of varieties over $k$. Over a finite field $k = \mathbb{F}_q$, the Chevalley-Warning theorem (cf. [Ax64]) states that a projective hypersurface $X \subset \mathbb{P}^n$ of degree $d \leq n$ satisfies the congruence formula

$$|X(\mathbb{F}_q)| \equiv 1 \mod q.$$  

The counting point $X \mapsto |X(\mathbb{F}_q)|$ gives rise to a ring homomorphism

$$|-| : K_0(Var/\mathbb{F}_q) \to \mathbb{Z},$$

from which one may reformulate the congruence formula as $|[X]| \equiv 1 \mod |L|$, where we denote by $L$ the class of the affine line $[\mathbb{A}^1]$ in $K_0(Var/\mathbb{F}_q)$. The geometric Chevalley-Warning problem for smooth projective hypersurfaces concerns with the following question:

**Question 1.1.** Let $k$ be a field and $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $\leq n$ such that $X(k) \neq \emptyset$. Whether is it true that $[X] \equiv 1 \mod L$ in $K_0(Var/k)$, where $1 = [\text{Spec} k]$?

In [BS11 3.3], F. Brown and O. Schnetz conjectured that the question is always true for $C_1$-fields. The question over an arbitrary field $k$ is due to H. Esnault in general for the relationship between rational points and the Grothendieck ring $K_0(Var/k)$ (cf. [Bil11 Ques. 3.8]).
Theorem 1.2. Let $X$ be a smooth projective geometrically integral variety over a field $k$ of characteristic 0. If $[X] \equiv 1 \mod L$, then $Br(X) \cong Br(k)$.

The proof of 1.2 is simple. By Kollár-Larsen-Lunts theorem (cf. [Ko05], [LL03], one has $[X] \equiv 1 \mod L$ iff $X$ is stably $k$-rational. The fact that the Brauer group $Br(X)$ is a birational invariant is due to Grothendieck [Gro68, Cor. 7.3, p. 138]. Moreover, one has $Br(\mathbb{P}^n_X) \cong Br(X)$, because $Br(X)$ can be identified with the unramified Brauer group $Br_{nr}(k(X))$ from the exact sequence (cf. [CT95], (3.9))

$$0 \to Br(X) \to Br(k(X)) \to \bigoplus_{x \in X^{(1)}} H^1_{et}(k(x), \mathbb{Q}/\mathbb{Z})$$

and the later group $Br_{nr}(k(X))$ gives us a stably birational invariance [CTO89]. So the theorem follows, since $Br(\mathbb{P}^n_X) \cong Br(k)$. In fact, theorem 1.2 is a special case of a more general invariant coming from strictly $\mathbb{A}^1$-invariant sheaves (see Theorem 1.3 below). However, it is enough to produce a counter-example to the geometric Chevalley-Warning conjecture over non-algebraically closed $C_1$-fields.

Corollary 1.3. Let $k$ be a non-algebraically closed field of char($k$) $\neq 3$ and assume $k^x \backslash (k^x)^3$ is not empty. Let $X$ be a smooth cubic surface given by the equation

$$x_0^3 + x_1^3 + x_2^3 + ax_3^3 = 0,$$

where $a \notin (k^x)^3$. Then $Br(X)/Br(k)$ is non-trivial. In particular, if $k$ is a non-algebraically closed $C_1$-field of characteristic 0 with $k^x \backslash (k^x)^3 \neq \emptyset$, then $[X]$ is not $\equiv 1 \mod L$.

Proof. Obviously $X(k) \neq \emptyset$. If $k$ is a non-algebraically closed field with char($k$) $\neq 3$ containing a primitive cubic root of unity, then for the smooth cubic surface as above one has $Br(X)/Br(k) = \mathbb{Z}/3 \oplus \mathbb{Z}/3$ (cf. [Man86], Ex. 45.3 for number fields and [CTS87], 2.5.1 in general). If $k$ has no primitive cubic roots of unity, the quotient $Br(X)/Br(k)$ is still non-trivial and it is described in [CTW11], Prop. 2.1]. This gives a negative answer to the question 1.1 as desired. 

Now let $k$ be an arbitrary field and let $Ho_{A^1}(k)$ be the $A^1$-homotopy category constructed in [MV01]. For a space $\mathcal{X} \in \Delta^{op}Sh_{Nis}(Sm/k)$ let $\pi^A_0(\mathcal{X})$ be the sheaf associated to the presheaf

$$U \mapsto [U, \mathcal{X}]_{A^1} \overset{def}{=} \text{Hom}_{Ho_{A^1}(k)}(U, \mathcal{X}),$$

for $U \in Sm/k$. We say $\mathcal{X}$ is $A^1$-connected, if the canonical map $\mathcal{X} \to \text{Spec } k$ induces an isomorphism of sheaves $\pi^A_0(\mathcal{X}) \cong \pi^A_0(\text{Spec } k) = \text{Spec } k$, [AM11]. Let $D_{A^1}(k)$ denote the $A^1$-derived category introduced by F. Morel (see e.g. [Mor12], §5.2). Let us denote by $Ab_{A^1}^k$ the category of strictly $A^1$-invariant sheaves (cf. [Mor12], Def. 7, page 8 or [AM11], Def. 4.3.1), it is known that $D_{A^1}(k)$ has a homological $t$-structure and one can identify $Ab_{A^1}^k$ with the heart of this $t$-structure [Mor05], Lem. 6.2.11. Thus $Ab_{A^1}^k$ is an abelian category.
by [BBD82, Thm. 1.3.6]. For a strictly $\mathbb{A}^1$-invariant sheaf $M$ and an irreducible smooth $k$-scheme $X$ we write $M^{nr}(X)$ for the group of unramified elements ([A11, Def. 4.1]). Now in the context of $\mathbb{A}^1$-derived category one can prove

**Theorem 1.4.** Let $k$ be a field of characteristic $0$. If $X, Y$ are two irreducible smooth projective $k$-varieties, such that $[X] = [Y]$ in $K_0(Var/k)/\mathbb{L}$, then $M(X) \cong M(Y)$ for any strictly $\mathbb{A}^1$-invariant sheaf $M \in \mathcal{A}^b_{k,1}$, i.e. $M$ yields a well-defined invariant on $K_0(Var/k)/\mathbb{L}$. In particular, if $X$ is an integral smooth projective $k$-variety, whose class in $K_0(Var/k)$ satisfies $[X] \equiv 1 \mod \mathbb{L}$, then $X$ is $\mathbb{A}^1$-connected, hence for any strictly $\mathbb{A}^1$-invariant sheaf $M \in \mathcal{A}^b_{k,1}$ the canonical map $M(k) \to M^{nr}(X)$ is then a bijection, where $M^{nr}(X)$ denotes the group of unramified elements.

**Remark 1.5.** Theorem 1.4 is just a simple application of [A11, Thm. 3.9]. Our example 1.3 shows that this smooth cubic surface is $\mathbb{A}^1$-disconnected over non-algebraically closed fields, while [AM11, Cor. 2.4.7] asserts that a smooth proper surface over an algebraically closed field of characteristic $0$ is $\mathbb{A}^1$-connected if and only if it is rational.

2. **Proof of 1.4**

By Kollár-Larsen-Lunts theorem (cf. [K05], [LL03]), one has an isomorphism

$$K_0(Var/k)/\mathbb{L} \to \mathbb{Z}[SB],$$

where the right hand side denotes the free abelian group generated over the set of stably birational equivalences of smooth projective varieties. So if $[X] = [Y]$ in $K_0(Var/k)/\mathbb{L}$, then $X$ is stably birational to $Y$. We have then $\mathcal{H}^{\mathbb{A}^1}_0(X) \cong \mathcal{H}^{\mathbb{A}^1}_0(Y)$ by [A11, Thm. 3.9]. By representing theorem [A11, Lem. 3.3], which asserts that

$$H^{0}_{Nis}(X, M) = \text{Hom}_{\mathcal{A}^b_{k,1}}(\mathcal{H}^{\mathbb{A}^1}_0(X), M),$$

one obtains $M(X) \cong M(Y)$. Remark that one has $M(X) = M^{nr}(X)$, if $X$ is an irreducible smooth $k$-scheme ([A11, Lem. 4.2]). Now if $X$ is an integral smooth projective $k$-variety with $[X] \equiv 1 \mod \mathbb{L}$ in $K_0(Var/k)$, then $X$ is stably $k$-rational. From [CTS07, Prop. 1.4] one knows that $X$ is then retract $k$-rational in sense of Saltman. By [AM11, Thm. 2.3.6] $X$ is $\mathbb{A}^1$-chain connected, hence $\mathbb{A}^1$-connected by [Mor05, Lem. 6.1.3]. Thus the theorem is proved and we see also immediately that 1.2 is a special case of 1.4 by [AM11, Prop. 4.3.8].

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