Generic Feasibility of Perfect Reconstruction With Short FIR Filters in Multichannel Systems

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Abstract—We study the feasibility of short finite impulse response (FIR) synthesis for perfect reconstruction (PR) in generic FIR filter banks. Among all PR synthesis banks, we focus on the one with the minimum filter length. For filter banks with oversampling factors of at least two, we provide prescriptions for the shortest filter length of the synthesis bank that would guarantee PR almost surely. The prescribed length is as short or shorter than the analysis filters and has an approximate inverse relationship with the oversampling factor. Our results are in form of necessary and sufficient statements for PR feasibility that hold generically, hence only fail for elaborately designed nongeneric examples. We provide extensive numerical verification of the theoretical results and demonstrate that the gap between the derived filter length prescriptions and the true minimum is small. Finally, we demonstrate that when our PR conditions are not satisfied, the unavoidable distortion in the reconstruction is substantial. The results have potential applications in synthesis FB design problems, where the analysis bank is given, and for analysis of fundamental limitations in blind signals reconstruction from data collected by unknown subsampled multichannel systems.

Index Terms—Filter banks, finite impulse response, generic, minimum filter length, multichannel, multirate, oversampled, perfect reconstruction.

I. INTRODUCTION

A. Oversampled Filter Banks

FILTER banks with perfect reconstruction (PR), or near PR, are the most ubiquitous signal processing structure in multirate digital systems with applications in broad areas of signal, image, and video processing [1]–[3]. Fig. 1(a) shows a C-channel D-fold subsampled filter bank (FB). Here, a FB is considered to achieve PR if it reconstructs an exact though possibly delayed replica $\hat{x}[n - n_0]$ of the input $x[n]$, that is, $\hat{x}[n] = x[n - n_0]$ for some integer delay $n_0$. Filter banks can be categorized as: (i) critically sampled or maximally decimated, i.e., when the downsampling factor $D$ equals the number of channels $C$; or (ii) oversampled, when there are more channels than the downsampling factor, i.e., $C > D$.

For critically sampled FBs, the PR requirement is typically in conflict with other desirable design specifications. In the oversampled case, however, with a given set of analysis filters, an infinite number of PR synthesis filters exist [4]. The main advantage of oversampled FBs are the added degrees of design freedom gained from this redundancy, which have been exploited for the reduction of quantization noise in digital communication systems [5], [6], for improved precoding and robustness in data communication [7]–[9], and in image transmission [10] and image coding [11].

B. The Role of Filter Banks in Multichannel Systems

Multichannel data acquisition/sampling arises in various sensing, imaging, and data processing modalities including data communication/storage applications, remote sensing/imaging, and medical imaging systems such as magnetic resonance imaging (MRI) [12]. The continuous-time model for a C-channel sampling (data acquisition) and reconstruction system is illustrated in Fig. 1(b). The channel outputs are sampled prior to digital processing, say, with a uniform sampling rate $T$. The objective is to perfectly reconstruct the input signal from the sampled output signals $y_k[n] = y_k(nT)$, i.e., to design the signal reconstruction mechanism in Fig. 1(b), so that $\hat{x}(t) = x(t)$.

If the channel characteristics are known, the problem reduces to a well-studied problem in sampling theory, widely known as Papoulis generalized sampling [13], [14] and its generalizations to the oversampled case [14], [15] and to multidimensional (MD) sampling [16] of MD signals [17]. Examples of MD multichannel systems include sensor arrays with sampling in space and time, and multichannel (parallel) MRI [12] with spatio-temporal sampling [18], [19].

The analysis simplifies by a fully discrete-time formulation. Under mild conditions [20], we can convert the continuous-time channel model in Fig. 1(b) into the discrete-time model (FB structure) in Fig. 1(a), where $x[n]$ and $\hat{x}[n]$ represent samples of $x(t)$ and $\hat{x}(t)$ taken at a sufficiently high rate, and the D-fold subsampling models the sub-Nyquist sampling of the channels in Fig. 1(b). Although all channels in Fig. 1(a) have identical uniform subsampling, this setup is fairly general and subsumes periodic nonuniform subsampling [21]. It follows that FBs represent equivalent discrete-time models for a wide class of multichannel data acquisition/sampling systems.

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In most FB-related problems, it is assumed that one has the luxury to almost freely manipulate the analysis filters, and the goal is to design the entire FB with desirable properties such as robustness to noise or erasures, frequency selectivity, etc. However, considering FBs as models for multichannel systems, often little or no control can be exerted over the analysis filters since their characteristics are dictated by the underlying physics. This is the case in important sensing, imaging, and image processing applications. Examples of such “sensor filters” include spatial sensitivity of receivers in multichannel MRI [12], [22]–[25], or blurring kernels associated with remote imaging applications [26], [27]. Furthermore, since the sensor filters result from complex physical processes, a pathological case would be extremely unlikely; in other words, the corresponding analysis filters are “generic” (further described in Section II-A). Therefore, in this work, we focus on scenarios wherein the set of analysis filters, also called the analysis bank, is fixed and generic. Such a generic analysis covers most practical applications of multichannel sensing and imaging. It has the added advantage of revealing the properties that are inherited from the FB structure rather than specific values of the filter taps.

Another important aspect of FBs is the length of the filters in the synthesis bank: infinite or finite impulse response (IIR/FIR). Given the analysis bank, the PR synthesis bank is typically designed to equal (or closely approximate) the so-called para-pseudoinverse or dual-frame synthesis bank [4], [28], to minimize the reconstruction noise gain. Unless the analysis bank meets stringent condition [28], [29], which almost surely do not hold for generic FBs, the dual-frame synthesis is IIR. Even if the exact PR condition is relaxed to approximate PR, accurate approximations of the dual-frame synthesis bank requires long synthesis filters [30], [31]. An alternative to such approaches is to search for the “best” FIR synthesis bank that achieves PR—and simultaneously satisfies additional optimality criteria [32], [33]. In particular, short FIR PR synthesis banks have important advantages including the following:

— Short PR FB reconstruction is computationally very efficient, as compared to the IIR dual-frame synthesis bank or alternative non-FB reconstructions such as the least-squares solution.

— A short FIR synthesis bank implies a low-dimensional search space for the synthesis FB design algorithm, hence reducing the computational complexity and, more importantly, improving the optimized design. This is especially significant for practical design techniques that employ additional desired criteria (besides PR) in a non-convex optimization scheme (e.g., [33]).

— In certain applications, such as data communications or storage, the multichannel data is corrupted by “impulsive noise” [8], [34], which can also model channel erasures [9], [10], or dead pixels in a sensor. The noise-optimal dual-frame synthesis bank (generically IIR) or non-FB reconstruction using the maximum-likelihood (least squares) solution would corrupt the entire reconstructed signal. Instead, short FIR synthesis can provide “good” reconstruction in such scenarios, achieving PR everywhere except in the neighborhood of the noise spikes [35].

These advantages and the prevalence of the generic FB scenario in multichannel systems motivate the work in this paper.

D. Present Work

This work addresses the feasibility of short FIR synthesis for PR (allowing for reconstruction delay—delayed PR) in generic complex-valued FIR one-dimensional (1D) FBs. Specifically, we address the following two questions for a C-channel FIR generic analysis FB with D-fold subsampling and analysis filter length $m_A$:

$Q.1$ Feasibility of FIR Synthesis: What are the conditions for existence of an FIR synthesis bank that achieves PR or delayed PR?

$Q.2$ Feasibility of Short Synthesis and Effect of Oversampling: what is the minimum feasible filter length $m^D_A$ among all PR (or delayed PR) synthesis banks—and how does it depend on the oversampling factor $D$?

With $D = 1$, the FB is nonsubsampled [28] and the reconstruction problem is equivalent to multichannel deconvolution. For this case, Q.1 has been extensively studied in the theory of polynomials [36], [37]. Further, Q.2 has been addressed in previous work by Harikumar and Bresler in 1D (with $C \geq 2$) [38] and 2D (with $C \geq 3$) [35]. In the 1D case with generic channels, they provide prescriptions for deconvolver lengths that are both necessary and sufficient for PR [38].

For the subsampled ($D > 1$) case, although the conditions for PR for a given critically sampled and oversampled FB have been known for many years [1], [28], Q.1 has only been recently answered by Law et al. [39], in fact for the more general case

1A “snug” (almost tight) analysis frame is an exception [4], [31], which does not occur for generic FBs.
of MD signals. (We discuss their relevant result in Section V.) However, to the best of our knowledge, Q.2 is an open problem.

The significance of Q.2 is perhaps best appreciated in the context of blind signal PR, i.e., the perfect inversion of subsampled multichannel systems (i.e., the analysis FB) by identification of a PR synthesis bank without any prior knowledge of the channels. For this class of problems only the nonsubsampled case, i.e., the problem of blind multichannel deconvolution, is fully studied (see [40], [41], and references therein). In subsampled systems little rigorous analysis is available of both necessary and sufficient conditions for blind PR (for relevant work in 2D see [42]–[44] and references therein). In such problems, where an FIR synthesis bank is to be identified, the length (support size) of the synthesis filters is a fundamental issue as it dictates the dimensionality of the unknown parameter space. In fact, given limited available data, as the length allocated for the synthesis filters increases (more unknowns to solve for), the inverse problem of estimating the synthesis filters becomes progressively more difficult and/or ill-posed. (This is compounded by the concomitant increase in computational cost with more unknowns.) Therefore, it is important to constrain the FB length. On the other hand, reducing the dimensionality too much would make PR infeasible. Consequently, knowing where this phase transition between PR feasibility/infeasibility occurs becomes critical. Having the answer to Q.2, one would be able to: (i) find the minimally required dimensionality of the parameter space to enable PR; and (ii) analyze the fundamental tradeoffs, such as the oversampling factor needed to guarantee a “feasible” (low dimensional) search space.

In this paper, we address the above-raised two questions for generic FIR analysis banks that are at least twofold oversampled; this subsumes most practical cases of oversampled FBs. We show that with such oversampling factors, PR is almost surely feasible with a synthesis bank that consists of filters as short or shorter than the analysis filters. Furthermore, we show that the required length for the synthesis filters has an inverse relationship to the oversampling factor.

Our results indicate that satisfying the PR condition per se is quite easy—even a random choice will do—if the filter lengths satisfy certain conditions. Hence, we can guarantee feasibility of exact PR in the design of a FB by prescribing the analysis/synthesis filter lengths. This implies that the degrees of freedom in the design process can be mostly driven by other desired criteria, e.g., reconstruction noise gain [5], frequency selectivity, time/frequency localization, subband attenuation [45], or coding gain [46]—all while guaranteeing PR. For the problem of designing a PR synthesis bank given the analysis bank, our results provide an alternative to the expensive exhaustive search for the synthesis filter length [33].

The paper is organized as follows. Section II contains basic definitions and notations. In Sections III and IV, respectively, we present necessary and sufficient requirements on the minimal filter length for the PR synthesis bank. Section V uses these results to address Questions Q.1 and Q.2. In Section VI, we provide numerical verification of the theoretical results; further, we study the feasibility of near-PR using synthesis filter lengths below those prescribed by our propositions. Finally, Section VII summarizes the results and concludes the paper.

II. PRELIMINARIES

A. Notations and Setup

For $a \in \mathbb{R}$, $[a]$ (respectively, $[a]!$) denotes the smallest (respectively, largest) integer larger (respectively, smaller) than or equal to a. Column vectors and matrices are denoted by lowercase and uppercase letters, respectively. The elements of a vector $a$ are indexed as $a_i$, $0 \leq i \leq m_a - 1$; similarly, the elements of a matrix are denoted as $A_{jk}$, with upper-left element $A_{00}$. Signals and column vectors are used interchangeably. The length of a signal or vector $s$ is denoted by $m_s$. For a matrix $A$, its transpose and Hermitian are denoted by $A^T$ and $A^H$, respectively; $\mathcal{R}(A)$ is its range (column) space and $A^\dagger$ is its Moore-Penrose pseudoinverse. The notation vec$[A]$ denotes the vector obtained by concatenating columns of $A$ in lexicographical order. Similarly, concatenation of a sequence of vectors $\{a_i\}_{i=1}^N$ into a single vector is denoted by vec$[\{a_i\}_{i=1}^N]$.

The shifted unit pulse $\delta_{m}(N)$ is defined as the $(m + 1)$th column of the $N \times N$ identity matrix, $I_N$. In most cases, the $N$ argument in $\delta_{m}(N)$ can be inferred from the context and is dropped for notational brevity. The convolution of $s[n]$ and $h[n]$ is denoted by $(s * h)[n]$ and is equivalently written in vector form as $C_m\{h\} \cdot s$, where $C_m\{h\}$ is the matrix representation of the convolution operator, which is a Toeplitz matrix of size $(m_t + m_s - 1) \times m_s$. Finally, we define the “stack” of all convolution matrices corresponding to the analysis channels as follows:

$$C = \begin{bmatrix} C_{m_a}\{h_1\} & C_{m_a}\{h_2\} & \cdots & C_{m_a}\{h_C\} \end{bmatrix}_{m_a}$$ \hspace{1cm} (1)

Consider the standard $C$-channel filter bank (FB) structure with $D$-fold subsampling shown in Fig. 1(a). We focus on oversampled FBs, i.e., where the oversampling factor $\frac{m}{d} > 1$. The transfer functions of the analysis and synthesis filters are denoted by $H(z)$ and $V(z)$, respectively, and their corresponding impulse responses, assumed to be FIR, by $h[n]$ and $v[n]$. For brevity, we refer to the set of analysis (respectively, synthesis) filters as the analysis (respectively, synthesis) bank. It is assumed that the support of all filters in the analysis (respectively, synthesis) bank is the same and—without loss of generality—is right-sided (that is, the filters are causal). Consequently, all analysis (respectively, synthesis) filters have equal length, denoted by $m_h$ (respectively, $m_v$). For example, the support for $h[n]$ is $0 \leq n \leq m_h - 1$. We therefore have the following expressions for the $z$-transforms of the filters: $H(z) = \sum_{n=0}^{m_h-1} h[n] z^{-n}$ and $V(z) = \sum_{n=0}^{m_v-1} v[n] z^{-n}$.

B. Generic Properties and Key Result on Generic Full Rank

Most of the theoretical work here involves study of “generic” properties of vectors and matrices. We use the same definition for a property to hold “generically” as in previous works [35], [38]: If a property $P$ of a vector $a \in \mathbb{R}^n$ fails to hold only on a

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2This is not a limiting assumption as one can take the length of a set of filters $\{f_i[n]\}_{i=1}^n$ to be $m_f = \max_i m_{f_i}$.
closed set of measure zero that is nowhere dense\(^3\) in \(\mathbb{R}^n\), we say that the property holds for \textit{generic} \(a\), or equivalently, \(P\) holds \textit{generically}. As a result, \(P\) will hold with probability 1 (short form: “w.p.1”) when the elements of vector \(a\) are drawn independently from a probability distribution that admits a probability density function.\(^4\) Furthermore, property \(P\) is \textit{robust}, in the sense that \(P\) continues to hold for any sufficiently small perturbation of such a randomly generated \(a\). A more mathematically rigorous definition \(39\) can be found elsewhere.

Our main tool for the results on generic properties in this paper is the following result, which provides a test for a matrix function of a generic complex vector to have full rank.

\textit{Theorem 1 (Harikumar and Bresler \[38\]):} Let \(A(u)\) be an \(m \times n\) complex matrix function with elements \(A_{ij}(u)\) that are multivariate polynomials in the elements of \(u \in \mathbb{C}^k\). Then, \(A(u)\) has full column rank for almost all \(u \in \mathbb{C}^k\) if it has full column rank for at least one \(u \in \mathbb{C}^k\).

The main idea behind this theorem \(38\) is to establish a connection between generic full rank property of structured matrices and \textit{algebraic sets} \(49\) in the Euclidean space of variables.\(^5\) The result follows by noting that all algebraic sets have zero Lebesgue measure in the Euclidean space.

\(\text{C. Perfect Reconstruction in Polyphase Domain}\)

The \textit{polyphase decomposition} \(1, 2\) of the analysis filters \(H_i(z)\) is given by

\[
H_i(z) = \sum_{p=0}^{D-1} z^p H_{i,p}(z^D), \text{ where } \quad H_{i,p}(z) = \sum_{n=-\infty}^{\infty} h_i[nD-p]z^{-n}.
\]

(2)

is the \(p\)th polyphase component \((p = 0, \ldots, D - 1)\) of the \(i\)th analysis filter. The polyphase decomposition of the synthesis filters is similar, but with opposite signs for the index \(p\):

\[
V_i(z) = \sum_{p=0}^{D-1} z^{-p} V_{i,p}(z^D),
\]

\[
V_{i,p}(z) = \sum_{n=-\infty}^{\infty} v_i[nD+p]z^{-n}.
\]

(3)

There is a corresponding time-domain representation on all analysis and synthesis subchannels, for each analysis/synthesis channel, involving partitioning of the filters taps into \(D\) subsequences—based on congruency of their indices modulo \(D\).

Define the impulse response corresponding to \(H_{i,p}(z)\) as:

\(h_{i,p}[n] = h_i[nD-p]\), and denote its length by \(m_{i,p}\), which is equal to \(m_{i} = \left\lfloor \frac{m_i}{D} \right\rfloor \). Similarly, the length of the impulse response \(v_{i,p}[n] = v_i[nD+p]\) corresponding to \(V_{i,p}(z)\)

\(\text{is denoted by } m_{v,p} \text{ and is equal to } m_{v,p} = \left\lfloor \frac{m_v}{D} \right\rfloor \). The following properties, for \(\alpha = h \text{ and } \alpha = v\), are easy consequences of these definitions:

\[
(a) \sum_{p=0}^{D-1} m_{\alpha,p} = m_{\alpha}, \quad (b) \frac{m_{\alpha}}{D} \leq m_{\alpha,p} \leq \frac{m_{\alpha}}{D}
\]

\[
(c) m_{\alpha,p} = \left\lfloor \frac{m_{\alpha}}{D} \right\rfloor \text{ for some } 0 \leq p \leq D - 1.
\]

(4)

Based on the theory of filter banks \(1, 2\), the \textit{polyphase-domain} condition for \(PR\) with an output delay of \(m_0 = m_{0D}\) for all inputs \(x[n]\) is as follows:

\[
\left[ V_{1,p}(z), V_{2,p}(z), \ldots, V_{C,p}(z) \right] A(z) = z^{-m_0} \delta_p^T(D) \quad \forall z \in \mathbb{C}, \quad p = 0, \ldots, D - 1
\]

(5)

where the so-called \textit{analysis polyphase matrix} \(A(z)\), which is a \(C \times D\) (Laurent) polynomial matrix, has entries \(A_{ij}(z) = H_{i+j-1,j}(z)\), \(i = 0, \ldots, C - 1, j = 0, \ldots, D - 1\). Collecting all \(D\) polyphase conditions in (5) into a single equivalent \(PR\) condition yields

\[
\mathcal{R}(z) A(z) = z^{-m_0} I_D \quad \forall z \in \mathbb{C}
\]

(6)

where the \(D \times C\) matrix \(\mathcal{R}(z)\) with entries \(\mathcal{R}_{i,j}(z) = V_{j+1,i}(z)\), \(i = 0, \ldots, D - 1, j = 0, \ldots, C - 1\), is referred to as the \textit{synthesis polyphase matrix}. For the case of zero delay, \(6\) states that \(PR\) is achieved when the synthesis polyphase matrix is a left inverse of the analysis polyphase matrix, for all \(z \in \mathbb{C}\).

The sampling- (time-) domain counterpart of (5) can be written in the following form:

\[
\left[ C \{h_{0,0}\}_{i=1}^C \right] m_{v,p} \left[ \begin{array}{c} \mathcal{C} \{h_{i,1}\}_{i=1}^C \end{array} m_{v,p} \right] \quad \mathcal{C} \{v_{i,p}\}_{i=1}^C = \delta_{p,m_0D}
\]

(7)

where \(H_p\) is the sampling-domain analysis polyphase matrix and is of size \((m_0 + Dm_{v,p} - D) \times (Cm_{v,p})\).\(^6\) Assuming \(0 \leq m_0 \leq \left\lfloor \frac{m_v}{D} \right\rfloor + \left\lfloor \frac{m_h}{D} \right\rfloor = 2\), the right-hand side (RHS) is a shifted unit pulse with the amount of shift \(\kappa(p,m_{0D})\) given by

\[
\kappa(p,m_{0D}) = \begin{cases} m_0 & \text{if } p = 0, \\ m_0 + \sum_{k=0}^{p-1} (m_{h,k} + m_{v,p} - 1) & \text{if } p = 1, \ldots, (D - 1). \end{cases}
\]

(8)

The aforementioned PR conditions correspond to cases where the delay allowed in \(PR\) is a multiple of the subsampling factor, i.e., \(m_0 = m_{0D}\). Nevertheless, the \(PR\) condition in (6) can be

\(^3\)A set \(S\) is dense in \(\mathbb{R}\) if for all \(x \in \mathbb{R}\), any neighborhood of \(x\) contains at least one point from \(S\) \([47]\).

\(^4\)The probability distribution should be absolutely continuous with respect to Lebesgue measure.

\(^5\)A set is called an \textit{algebraic set} if it can be written as the set of common zeroes of a system of polynomials \(49\).

\(^6\)In (7), each of the matrices \(\mathcal{C} \{h_{i,k}\}_{i=1}^C \) for \(k = 0, \ldots, (D - 1)\) is of size \((m_{v,k} + m_{v,p} - 1) \times (Cm_{v,p})\). Therefore, using \(D-1\) \(m_{v,p} = m_h\), it is seen that the matrix \(H_p\) is of size \((m_h + Dm_{v,p} - D) \times (Cm_{v,p})\).

\(^7\)For (8) to hold we need \(0 \leq m_0 \leq m_{h,k} + m_{v,p} - 1\) for 0 \(\leq k \leq p \leq D - 1\), which implies the condition on \(m_0\) given above.
extended to the general delayed PR with a delay of \( r_0 = m_0D + r_0, 0 \leq r_0 \leq D - 1 \), as follows (cf. [1, Ch. 5.6]):

\[
\mathcal{R}(z)A(z) = z^{-m_0} \begin{bmatrix} 0 & z^{-1}I_0 \\ I_{D-r_0} & 0 \end{bmatrix}
\]

(9)

where, in the \( D \times D \) matrix on the RHS, 0 and \( \bar{0} \) denote zero matrices of appropriate size. The corresponding sampling-domain condition will differ from (7) only in the location of the 1 in the delay vector. In short, assuming\(^8\) \( 0 \leq \lceil \frac{m_0}{D} \rceil \leq \lceil \frac{m_0 + \ell}{D} \rceil + 1 \leq D - 2 \), the PR condition with \( r_0 = m_0D + r_0 \) delay is

\[
H_p y_{i\ell} \in \mathbb{C}^{1 \times 1} = \delta_{i\ell, r_0} \quad p = 0, \ldots, (D - 1)
\]

(10)

where \( \delta_{i\ell, r_0} \) is the inverse \( z \)-transform of the \( r_0 \)th row of the RHS of (9). The assumption on the range for \( r_0 \) is needed for (6) to be feasible; however, the present formulation can account for delays outside this range by proper zero-padding of analysis/synthesis impulse responses. The general closed-form expression for \( \delta_{i\ell, r_0} \), given in (8) for the special case of \( r_0 = m_0D \), is somewhat complicated and of no significance in this paper; hence, it is skipped here.

To illustrate the structure of the sampling-domain analysis polyphase matrix \( H_p \) given in (7), consider for example a \( C = 6 \) channel FB with \( D = 3 \), \( m_h = 7 \), and \( m_v = 6 \). The corresponding \( H_2 \) is shown in Fig. 2. In general, the structure of \( H_p \) consists of Toeplitz (rectangular) blocks of the form \( C_{m_0, \ell} \{ h_{i\ell, \ell} \} \) with \( C \) block-columns \((1 \leq c_0 \leq C)\) and \( D \) block-rows \((0 \leq \ell \leq D - 1)\). The zeros in the Toeplitz blocks are referred to as structural zeros—they are underlined to distinguish them from assigned zeros in our matrix constructions in the following sections. On occasion, we need to refer to an indeterminate zero, i.e., one that can be either assigned or structural, for which we use the notation \( \odot \).

When \( m_h < D \) or \( m_v < D \), the system of equations in (10) should be interpreted with some care: equations that correspond to \( m_{lp} \) = 0, and blocks in \( H_p \) corresponding to polyphase components in \( \{ h_i \} \) that do not exist, should all be removed. Because of these complications and considering that the case \( m_h < D \) is of limited practical interest, we assume \( m_h \geq D \) throughout.

Finally, given the analysis filters, the following sampling-domain necessary and sufficient condition for existence of a PR synthesis bank follows immediately from (10).

**Lemma 1:** For a \( C \)-channel FB with given FIR analysis filters \( \{ h_i[n] \}_{i=1}^C \), a set of length-\( m_v \) synthesis filters achieving PR with delay \( r_0 \) exists if: (i) \( 0 \leq \lceil \frac{m_0}{D} \rceil \leq \lceil \frac{m_0 + m_h}{D} \rceil + 1 \), and (ii) \( \delta_{i\ell, r_0} \in \mathbb{R} \), for all \( p = 0, \ldots, D - 1 \), and does not exist if either of the two conditions are not satisfied.

### III. Minimum Length for PR Synthesis Filters: Generic Sufficient Condition

In this section we aim to answer Question Q.2 raised in Section I. Specifically, for a length-\( m_h \) generic analysis FB with \( C \)-channels and \( D \)-fold subsampling, we propose a functional \( m_v^*(C, D, m_h) \): \( \mathbb{N} \to \mathbb{N} \) such that for all integers \( m_v \geq m_v^*(C, D, m_h) \) there exists a PR synthesis bank with filter length \( m_v \).

#### A. Statement of the Result

**Definition:** Denote by \( m_v^*(C, D, m_h) \) the minimal value of \( m_v \in \mathbb{N} \) that satisfies

\[
\frac{C}{D} \geq 1 + \frac{1}{D} \sum_{\ell=0}^{D-1} \left[ \frac{m_h}{\ell} - 1 \right]
\]

(11)

where \( m_{lp} \) = \( \lceil \frac{m_{lp}D}{D} \rceil \). We refer to \( m_v^* \) as the sufficient synthesis filter length—in short, the sufficient length.

It is easy to show that (11) is satisfied for any \( m_v \geq m_v^*(C, D, m_h) \). Therefore, the set of \( m_v \in \mathbb{N} \) satisfying (11) is a right-sided interval, i.e., all integers in: \( m_v^*(C, D, m_h), \infty \). The following lemma shows that for \( m_v^* \) to be finite, at least twofold oversampling is needed (assuming \( m_h \geq 2D \), which is the case in most practical scenarios). The proof is provided in [50].

**Lemma 2:** Suppose \( m_h \geq 2D \). Then, \( \frac{C}{D} < 2 \), (11) is not satisfied for any finite \( m_v \).

In the following proposition, the main result of this section, we consider generic FIR analysis banks that are at least twofold oversampled, which, as stated in Lemma 2, is a requirement for the sufficient-length defined by (11) to be feasible. We will show

\[
\begin{bmatrix}
    h_1[0] & 0 & h_2[0] & 0 & h_3[0] & 0 & h_4[0] & 0 & h_5[0] & 0 & h_6[0] & 0 \\
    h_1[1] & h_2[1] & h_3[1] & h_4[1] & h_5[1] & h_6[1] \\
    h_1[2] & 0 & h_2[2] & 0 & h_3[2] & 0 & h_4[2] & 0 & h_5[2] & 0 & h_6[2] \\
    h_1[3] & h_2[3] & h_3[3] & h_4[3] & h_5[3] & h_6[3] \\
    h_1[4] & 0 & h_2[4] & 0 & h_3[4] & 0 & h_4[4] & 0 & h_5[4] & 0 & h_6[4] \\
    h_1[5] & h_2[5] & h_3[5] & h_4[5] & h_5[5] & h_6[5] \\
    h_1[6] & 0 & h_2[6] & 0 & h_3[6] & 0 & h_4[6] & 0 & h_5[6] & 0 & h_6[6] \\
    h_1[7] & h_2[7] & h_3[7] & h_4[7] & h_5[7] & h_6[7] \\
    h_1[8] & 0 & h_2[8] & 0 & h_3[8] & 0 & h_4[8] & 0 & h_5[8] & 0 & h_6[8] \\
    h_1[9] & h_2[9] & h_3[9] & h_4[9] & h_5[9] & h_6[9] \\
    h_1[10] & 0 & h_2[10] & 0 & h_3[10] & 0 & h_4[10] & 0 & h_5[10] & 0 & h_6[10] \\
    h_1[11] & h_2[11] & h_3[11] & h_4[11] & h_5[11] & h_6[11] \\
    h_1[12] & 0 & h_2[12] & 0 & h_3[12] & 0 & h_4[12] & 0 & h_5[12] & 0 & h_6[12] \\
\end{bmatrix}
\]

Fig. 2. Structure of the sampling-domain analysis polyphase matrix \( H_2 \) (for \( p = 2 \)) of size \( 10 \times 12 \) corresponding to a 6-channel analysis FB with threefold subsampling (\( D = 3 \)) and a filter length of \( m_h = 7 \) (i.e., \( m_{hp} = 3 \), \( m_{hp} = m_{hp} = 2 \)). The synthesis filter length is \( m_v = 6 \) (i.e., \( m_{vp} = 2 \)). The structural zeros are underlined.
that PR is generically feasible with a synthesis bank that consists of filters with lengths \(m_v \geq m^S(C, D, m_b)\).

**Proposition 1 (Sufficient Length):** A \(D\)-fold subsampled \(C\)-channel length-\(m_b\) FIR analysis FB with \(\frac{C}{D} \geq 2\) is generically invertible by a length-\(m_v\) synthesis FB if \(m_v \geq m^S(C, D, m_b)\), i.e., the FB admits PR with any delay \(n_0\), \(0 \leq \lfloor \frac{n_0}{D} \rfloor \leq \lfloor \frac{m_b}{D} \rfloor + \lfloor \frac{n_0}{D} \rfloor = 2\).

**B. Proof of the Sufficient-Length Proposition**

We start by noting that, by Lemma 1, a sufficient condition for PR is that all \(H_p\), \(p = 0, \ldots, D - 1\), have full row rank. Hence, the following result implies the result of Proposition 1.

**Proposition 2:** For a \(D\)-fold subsampled \(C\)-channel length-\(m_b\), FIR analysis bank with \(\frac{C}{D} \geq 2\), the following property holds generically: the sampling-domain analysis polyphase matrix \(H_p\) corresponding to synthesis filter lengths \(m_v \geq m^S(C, D, m_b)\) has full row rank, for all \(p = 0, \ldots, D - 1\).

The rest of this section therefore addresses the proof of Proposition 2. Our main tool for proving this proposition is Theorem 1 in Section II-B. The proof has the following steps. We apply Theorem 1 with the variable vector defined as \(u = \text{vec}\{h_{11}, \ldots, h_{C}\}\) and \(A(u) = H_p^T\) (for each \(p = 0, \ldots, D - 1\)). It is easy to see that the entries of \(H_p^T\) are polynomials of order zero or one in \(u\). Therefore, to satisfy the condition in Theorem 1, we need to find one example of a set of analysis filters \(\{h_i\}_{i=1}^\infty\) for which \(H_p\) has full column rank for all \(p = 0, \ldots, D - 1\). In other words, we need to construct a particular matrix, with the same structure as \(H_p\), that has full row rank for all \(p\). Once this is done, the result of Proposition 2 follows by Theorem 1.

In the remainder of this section, we provide a constructive proof that a full rank matrix with the same structure as \(H_p\) exists. The following lemma provides the basic idea behind construction of such a matrix.

**Lemma 3:** Matrix \(A\) has full row rank if: (i) each column has at most one nonzero element; (ii) each row has at least one nonzero element.

To use Lemma 3, we construct an analysis polyphase matrix \(H_p\) (corresponding to a certain choice of the analysis bank) that possesses the two properties listed in the lemma. To this end, we introduce an explicit algorithm for construction of \(H_p\) and prove that the algorithm will successfully construct \(H_p\) with the two properties in Lemma 3 for all \(p = 0, \ldots, D - 1\). By the preceding arguments, this establishes Proposition 2, and in turn Proposition 1. Given the notions of blocks and block-columns described in Section II-C, let us consider the analysis polyphase matrix constructed by Algorithm 1 below. The algorithm sequentially assigns a single nonzero diagonal or sub-diagonal, called a \(1\)-diagonal, to each block-column; here, a row of \(H_p\) is called covered if at least one of its entries is assigned to be 1.

**Algorithm 1.** Matrix construction for proof of Proposition 2

(i) Initialize: \(H_p = 0\) by assigning 0 to all free entries
(ii) For the first block-column, assign 1’s to the top diagonal (corresponding to \(h_{1,0}(0) = 1\))
(iii) While the last row of \(H_p\) is not covered, assign the 1-diagonal in the \(k\)th block-column, \(k = 2, \ldots, C\), according to the following procedure:
   • Case 1: If no structural zero is present along the trajectory of the 1-diagonal extended from the previous block-column, assign the 1-diagonal (for the \(k\)th block-column) such that the one in the previous block-column is extended, as shown in Fig. 4(a).
   • Case 2: Otherwise, assign the 1-diagonal (for the \(k\)th block-column) such that it starts immediately above the structural zeros in its first column—two examples of this case are shown in Fig. 4(b).
(iv) If the last row of \(H_p\) (last row of \(C\)th block-column) is not covered declare Failure.
blocks; and (ii) structural zeros that limit the number of free (assignable) entries. The idea in Algorithm 1 is to assign a single 1-diagonal to each block-column, which corresponds to assigning only one nonzero tap to each analysis filter. It is easy to see that this strategy ensures that the constructed $H_p$ would enjoy Property (i) in Lemma 3—hence, all we need to be concerned with is Property (ii).

The algorithm starts by initializing the matrix to be all zeros and assigning 1’s to the top diagonal in the first block-column. Assume that we have assigned all of the entries in the first $k - 1$ block-columns, $k = 2, \ldots, C$. For the $k$th block-column, which corresponds to $h_k[n]$, there are two possible cases in terms of the structural zeros, as shown in Fig. 4(a) and (b). Panel (a) shows the simple case, i.e., when no structural zero are present along the trajectory of the 1-diagonal assigned for the $(k - 1)$th block-column. That is, we can simply extend the diagonal trace of 1’s by assigning the nonzero entry of the $k$th column block accordingly. In the example given in Fig. 3, this case applies to all but the 4th and 6th block-columns, where the structural zeros prevent the extension procedure just described. Two general instances for such nontrivial cases are shown in Fig. 4(b). To satisfy Property (ii) in Lemma 3, we have to assign at least one nonzero entry to each row. As described in Fig. 4(b), this is accomplished by assigning the 1-diagonal for the $k$th block-column to start at the last free entry in its first column, i.e., just above the structural zeros.

According to Algorithm 1, the rows in a block-row are progressively covered—i.e., in each step, the noncovered rows are at the bottom of the block-row. Hence, covering of a block-row implies that the last row is covered. Assume that the assigned nonzero elements used to cover these rows are located in column-blocks $k_1$ to $k_2 < C$. Then, the filter taps in block-column $(k_2 + 1)$ are assigned to cover the first $m_{\ell C P}$ rows of the next block-row without any overlap with those used to cover the previous block-row—in short, Algorithm 1 does not “revisit” any block-rows.9

Consequently, what remains to be shown (to establish Property (ii) in Lemma 3) is that there are enough block-columns (out of a total of $C$) to cover all rows of the constructed $H_p$ matrix. Fig. 4(c) shows a possible scenario where the last block-column fails to cover all rows (the bracket on the RHS shows the bottom-right edge of the matrix). We first show this for $p = p_0$ where, in accordance with Property (c) in (4), $p_0$ is such that $m_{C P} = \left\lceil \frac{m_p}{D} \right\rceil$. Let us recall the assumption in the statement of Proposition 2, $m_0 \geq m_0^2(C, D, m_p)$. As pointed out earlier, this means that $m_0$ satisfies (11), which can be equivalently written as follows:

$$\sum_{\ell=0}^{D-1} \left[ \frac{m_{\ell C P} + m_{\ell C P} - 1}{m_{C P}} \right] \leq C. \quad (12)$$

Now, consider the following lemma (proof provided in [50]).

**Lemma 4:** Using Algorithm 1, the number of consecutive block-columns required to cover the $m_{\ell C P} + m_{\ell C P} - 1$ rows of the $\ell$th block-row in $H_p$ ($0 \leq \ell \leq D - 1$) is equal to

$$\lceil \frac{(m_{C P} + m_{\ell C P} - 1)}{m_{C P}} \rceil.$$ 

By Lemma 4, (12) guarantees that the $C$ block-columns of $H_p$, suffice for the algorithm to cover all rows of $H_p$—i.e., avoid the case shown in Fig. 4(c). Finally, since by (4) we have $m_{\ell C P} \geq m_{\ell C P}$, (12) implies that the same argument holds for all $H_p$ with $0 \leq p \leq D - 1$. As a result, the condition in Line (iv) of Algorithm 1 is not met for any $p$; hence, the algorithm successfully finishes the construction of $H_p$ for all $p = 0, \ldots, D - 1$. This completes the proof. \hfill $\square$
IV. MINIMUM LENGTH FOR PR SYNTHESIS FILTERS: GENERIC NECESSARY CONDITION

In the previous section, we provided a “sufficient length” condition for PR synthesis banks. However, this only partially answers Question Q.2 (Section 1) as it raises the possibility that the prescribed minimum length $m_{\text{loc}}^{N}(C, D, m_{h})$ is too conservative. To refute this possibility, in this section, we propose a necessary condition counterpart to Proposition 1. It states that for synthesis filter lengths below a certain necessary length $m_{\text{loc}}^{N}(C, D, m_{h})$ (defined below) PR or delayed PR cannot generically be achieved. Subsequently, we exactly quantify the gap between the sufficient and the necessary lengths for each choice of $(C, D, m_{h})$.

In Section V, we provide numerical results demonstrating that the gap between these two lengths is indeed small (for moderately high oversampling factors).

A. Statement of the Result

First, let us define the counterpart to the sufficient length $m_{\text{loc}}^{S}(C, D, m_{h})$.

**Definition:** Denote by $m_{\text{loc}}^{N}(C, D, m_{h})$ the minimal value of $m_{\text{loc}} \in \mathbb{N}$ that satisfies:

$$\frac{C}{D} \geq 1 + \frac{1}{D} \sum_{j=0}^{D-1} \left\lfloor \frac{m_{\text{loc}} - 1}{m_{h}} \right\rfloor$$

(13)

where $m_{\text{loc}} = \left\lfloor \frac{m_{h} - 1}{D} \right\rfloor$. We refer to $m_{\text{loc}}^{N}$ as the necessary synthesis filter length—in short, the necessary length.

It is easy to show that all $m_{\text{loc}} \geq m_{\text{loc}}^{N}(C, D, m_{h})$ satisfy (13). Therefore, the set of $m_{\text{loc}} \in \mathbb{N}$ satisfying (13) is a right-sided interval, i.e., all integers in $[m_{\text{loc}}^{N}(C, D, m_{h}), \infty)$. Furthermore, the only difference between the definitions of the necessary and sufficient lengths (Section III) is a floor/ceiling operation in the summand. The following proposition, the main result of this section, provides a necessary condition counterpart to Proposition 1 for PR in generic FBs.

**Proposition 3 (Necessary Length):** A $D$-fold subsampled $C$-channel length-$m_{h}$ FIR analysis bank with $\frac{C}{D} \geq 2$ and $m_{h} > D$ is generically not invertible by a synthesis FB of length $m_{\text{loc}} < m_{\text{loc}}^{N}(C, D, m_{h})$, i.e., the FB does not admit PR with any integer delay $n_{0}$.

B. Proof of the Necessary-Length Proposition

The condition $m_{\text{loc}} < m_{\text{loc}}^{N}(C, D, m_{h})$ implies that (13) is not satisfied (since the set satisfying (13) is a right-sided interval as pointed out earlier). Here, we consider delays in the range $0 \leq [\frac{n_{0}}{D}] \leq \left\lfloor \frac{n_{0}}{D} \right\rfloor + \left\lfloor \frac{m_{h} - 1}{D} \right\rfloor - 2$, since for $n_{0}$ values outside this range PR is infeasible (Lemma 1). We show that a violation of (13) in turn implies that generically there exists an integer $p \in \{0, \ldots, D-1\}$ such that $\delta_{\epsilon(p,n_{0})} \notin \mathcal{R}(H_{p})$ for any $n_{0}$, which by Lemma 1 is equivalent to the proposition. To this end, we construct the augmented matrix $A_{p} = [H_{p} \, \delta_{\epsilon(p,n_{0})}]$, and will establish that $A_{p}$ is generically full column rank, for some $p$.

The proof has the following steps. We apply Theorem 1 with $A(x) = A_{p}$, and $x = \text{vec}(\{h_{1}, \ldots, h_{C}\})$. The entries of $A_{p}$ are polynomials of order zero or one in $x$. Hence, if we find a particular set of analysis filters $\{h_{i}\}_{i=1}^{C}$ such that the corresponding $A_{p}$ matrix has full column rank, then it will be generically full column rank, which in turn proves the proposition. Accomplishing this is equivalent to constructing a sampling-domain analysis polyphase matrix $H_{p}$ (for at least one $p$) such that: (a) $H_{p}$ has full column rank; (b) $\delta_{\epsilon(p,n_{0})}$ does not lie in the span of the columns of $H_{p}$.

In the remainder of this section, we provide a constructive proof that such an analysis polyphase matrix $H_{p}$ exists. We provide an explicit algorithm for construction of $H_{p}$ for a particular $p$, prove that this algorithm is guaranteed to terminate successfully, and that the resulting $H_{p}$ has the requisite properties. In what follows, the matrix construction corresponds to $p = p_{0}$, where $p_{0}$ was defined in Property (c) of (4). Let us consider the matrix $H_{p_{0}}$ constructed by Algorithm 2. Note that for the assignment in Line (ii) of the algorithm to be feasible, we need $m_{h0} > m_{\text{loc}}^{N}$, which is guaranteed by the following lemma.10

The proof is provided in [50].

10 For $h_{1,0}[m_{\text{loc}}+1] = 1$ to be feasible the length of $h_{1,0}$ should be at least $m_{\text{loc}}+1$. 

---

**Fig. 5.** (a,b) The two possible cases addressed in Algorithm 2 for allocation of the 1-diagonal for the $k$th block-column, separated by the solid vertical line from the $(k-1)$th block-column. (a) No structural zero is present along the trajectory of the 1-diagonal. (b) Structural zeros force breaking of the 1-diagonal: the 1-diagonal for the $k$th block-column is chosen such that it would start just below the structural zeros in its 1st column. (c) An example of the undesired case for the last block-column, wherein there are no free entries left below the structural zeros.
Algorithm 2. Matrix construction for proof of Proposition 3

(i) Initialize: $H_{p_0} = 0$ by assigning 0 to all free entries.
(ii) For the first block-column, select the 1-diagonal to be the $(m_{v_2p_0} + 1)$th diagonal in the first Toeplitz block (corresponding to $h_{1,0}[m_{v_2p_0}] = 1$).
(iii) For the $k$th block-column, $k = 2, \ldots, C$, select the 1-diagonal according to the following procedure:
- Case 1: If no structural zero is present along the trajectory of the 1-diagonal extended from the previous block-column, assign the 1-diagonal (for the $k$th block-column) such that the one in the previous block-column is extended, as shown in Fig. 5(a).
- Otherwise, —Case 2: Assign the 1-diagonal (for the $k$th block-column) such that it starts immediately below the structural zeros in its first column—an example of this case is shown in Fig. 5(b).
—Declare Failure if there are no free entries left below the structural zeros, e.g., as shown in Fig. 5(c).
(iv) Let $e^*$ be the block-column index of the nonzero entry in the row indexed by $k(p_0, n_0)$.
(v) If all entries of the row indexed by $k(p_0, n_0)$ are zero then Exit.
Else, in the $e^*$th block-column, assign the top diagonal to be nonzero (corresponding to $h_{e^*,0}[0] = 1$).

Lemma 5: With the assumptions in Proposition 3, we have $m_{H_{p_0}} > m_{v_2p_0}$.

Fig. 6 provides an example of the matrix $H_{p_0}$ with $p_0 = 2$ constructed by Algorithm 2 for a FB with $C = 6$ channels, $D = 3$ subsampling, and analysis filter length of $m_h = 13$. The reconstruction delay is $n_0 = 6$ (corresponding to $k(p_0, n_0) = 13$). In accordance with the assumption $m_c < m_v(C, D, m_h)$, we have $m_c = m_v(C, D, m_h) - 1 = 8$. The arrow next to $H_2$ indicates the row indexed by $k(p_0, n_0)$, which is the location of the 1 in $\delta_{k(p_0, n_0)}$. The corresponding analysis bank is: $h_1[r_1] = \delta_0(13), h_2[r_2] = \delta_2(13), h_3[r_3] = \delta_1(13), h_4[r_4] = \delta_7(13), h_5[r_5] = \delta_2(13), h_6[r_6] = \delta_0(13) + \delta_5(13)$.

The first task is to prove that the constructed $H_{p_0}$ has full column rank. This is accomplished by showing that it has the two properties given in the following lemma, which is equivalent to Lemma 3.

Lemma 6: Matrix $A$ has full column rank if: (i) each row has at most one nonzero element; (ii) each column has at least one nonzero element.

By construction, Algorithm 2: (a) allocates at most a single 1-diagonal to each Toeplitz block in $H_{p_0}$; (b) if the 1-diagonal for some block-column has a maximum row index of $r_c$, then the next block-column would not have any nonzero entries at or above the $r_c$th row. These two observations together imply Property (i) above.

Next, we prove that Property (ii) in Lemma 6 holds. We call a block-column covered (by Algorithm 2) if all of its $m_{v_2p_0}$ columns satisfy Property (ii). Owing to the Toeplitz structure of the blocks, any 1-diagonal assignment by the algorithm corresponds to covering of a block-column. However, it could be that, at a certain iteration of the loop in Line (iii) of the algorithm, there would be no free entries (that are below the structural zeros) left to assign. Fig. 5(c) shows an example of such a scenario for the last $(C/2)$th column-block. In the following we show that, given the assumption of the proof, i.e., $m_c < m_v(C, D, m_h)$, this latter scenario will never materialize. Considering the following lemma. (The proof is analogous to that of Lemma 4.)

Lemma 7: Using Algorithm 2, the number of consecutive block-columns in $H_{p_0}$ covered by the $m_{l_{12}} + m_{v_2p_0} - 1$
rows of the $\ell$th block-row ($0 \leq \ell \leq D - 1$) is equal to \( \left( \frac{m_{c,\ell} + m_{v,\ell}}{m_{v,\ell}} - 1 \right) \).

The opposite of (13), which is implied by the assumption \( m_v < m_v^C(C, D, m_h) \), can be rewritten as

\[
\frac{C}{D} < 1 + \frac{1}{D} \sum_{\ell = 0}^{D-1} \left( \frac{m_{h,\ell} - 1}{m_{v,\ell}} \right)
\Rightarrow C - D < \sum_{\ell = 0}^{D-1} \left( \frac{m_{h,\ell} - 1}{m_{v,\ell}} \right)
\Rightarrow D + \sum_{\ell = 0}^{D-1} \left( \frac{m_{h,\ell} - 1}{m_{v,\ell}} \right) \geq C + 1 \tag{14}
\]

where, in the last step, we assumed \( m_{v,\ell} = \left[ \frac{m_v}{D} \right] \). Now, summing up the number of consecutive block-columns that can be covered by all rows, we have

\[
\frac{m_{h,0} - 1}{m_{v,\ell}} + \sum_{\ell = 1}^{D-1} \left( \frac{m_{h,\ell} + m_{v,\ell} - 1}{m_{v,\ell}} \right) = -1
\]

\[
\left( D + \sum_{\ell = 0}^{D-1} \left( \frac{m_{h,\ell} - 1}{m_{v,\ell}} \right) \right) \geq -1 + C + 1 = C \tag{15}
\]

where we applied (14). The inequality in (15) implies that there are enough rows to cover all $C$ block-columns. Using a similar argument as the one at the end of Section III-B, it is easy to see that Algorithm 2 covers the block-columns progressively, i.e., once a block-column is covered it is not revisited. This in combination with (15) guarantees that the Failure condition in Line (iii) of the algorithm is never met; hence, the matrix construction finishes successfully—satisfying both properties in Lemma 6.

Finally, we show that $\delta_{k(p_0, n_0)}$ is linearly independent of all columns of the constructed $H_{p_0}$, which—together with the above—establishes that the constructed (augmented) matrix $A_{p_0} = [H_{p_0} \ \delta_{k(p_0, n_0)}]$ is full column rank. By inspection, it is clear that the top $m_{v,\ell}^0$ rows of the matrix constructed in Lines (i)–(iv) of Algorithm 2 are set to zero. Consider two cases based on whether $m_{v,\ell}^0$ is larger or smaller than $\kappa(p_0, n_0)$. If $\kappa(p_0, n_0) < m_{v,\ell}^0$, the condition in Line (v) of Algorithm 2 holds and no linear combination of columns of $H_{p_0}$ can produce $\delta_{k(p_0, n_0)}$. For the alternative case of $\kappa(p_0, n_0) \geq m_{v,\ell}^0$, the $1$ in $\delta_{k(p_0, n_0)}$ may share the same row index with at most one nonzero element in $H_{p_0}$—denote the corresponding column in $H_{p_0}$ by $e$.\(^{11}\) Line (v) of Algorithm 2 (after “Else, …”) ensures that $e$ will have another nonzero element in the top $m_{v,\ell}^0$ rows of the matrix, i.e., in the first row-block of $H_{p_0}$. This is demonstrated in Fig. 6 where $e$ is the eleventh column and $m_{v,\ell}^0 = 2$. Consequently, since $\kappa(p_0, n_0) \geq m_{v,\ell}^0$, column $e$ cannot belong to any linear combination producing $\delta_{k(p_0, n_0)}$, which in turn implies that $\delta_{k(p_0, n_0)}$ is linearly independent of all columns of $H_{p_0}$. This completes the proof. \( \square \)

V. TIGHTNESS, GAPS, AND CLOSED-FORM EXPRESSIONS

In the previous two sections, we provided generic necessary and sufficient requirements for the length of the PR synthesis bank. Here we show that the gap between the necessary length $m_v^N(C, D, m_h)$ and the sufficient length $m_v^S(C, D, m_h)$ is small. In addition, to answer the second part of Question Q.2 of Section I, we study the fundamental relation between the oversampling ratio, the analysis filter length, and the required synthesis length. This is facilitated by providing various bounds including closed-form upper and lower bounds for the sufficient and necessary synthesis filter lengths, respectively. The derived relations are verified and illustrated numerically in Section VI.

Assuming that the FB (Fig. 1) is oversampled, we start with a simple necessary condition, referred to as the counting condition, for all $H_p$, $p = 0,\ldots, D - 1$, to be full row rank.

Lemma 8 (Counting Length): Assuming $C > D$, a necessary condition for all $H_p$ ($p = 0,\ldots, D - 1$) to be full row rank is

\[
m_v \geq m_v^C(C, D, m_h) \triangleq D \left\lceil \frac{m_h - D - 1}{D} \right\rceil \tag{16}
\]

where the integer functional $m_v^C(C, D, m_h)$ is referred to as the counting length.

Proof of this lemma is provided in [50]. The counting length is simply derived by requiring the number of rows to be no more than the number of columns for all $H_p$. Note that, because the condition in Lemma 1 may be satisfied even when none of $\{H_p\}_{p=0}^{D-1}$ have full rank, condition (16) is neither necessary nor sufficient for existence of a length-$m_v$ PR synthesis bank. Remarkably, as described below (also revisited in Section VI), $m_v^C(C, D, m_h)$ is closely related to $m_v^S(C, D, m_h)$ and to $m_v^N(C, D, m_h)$, derived earlier.

Applying the basic inequality $\alpha \leq |\alpha| < \alpha + 1$, we have

\[
m_h - D \leq m_v^S(C, D, m_h) < D + \frac{m_h - D}{D} \tag{17}
\]

i.e., $m_v^C(C, D, m_h)$ is bounded from below and above by functions that each have an approximate inverse relationship to $C$, for a fixed $m_h$. Since the gap between the bounds in (17) is small (equal to $D$), the counting length $m_v^C(C, D, m_h)$ has an approximate inverse relationship to the oversampling factor $C$.

The following proposition, proved in [50], describes the relationship between the necessary, sufficient, and counting filter lengths; it further provides closed-form lower/upper bounds for $m_v^S$ and $m_v^N$.

Proposition 4: Assuming $\frac{C}{D} \geq 2$ and $m_h > D$, we have the following relations between the sufficient length $m_v^S(C, D, m_h)$, the necessary length $m_v^N(C, D, m_h)$, and the counting length $m_v^C(C, D, m_h)$:

\[
\max(D, m_v^L) \leq m_v^N \leq m_v^C \leq m_v^S \leq \min(m_h, m_v^U)
\]

where $m_v^L(C, D, m_h) \triangleq \left\lceil \frac{m_h - D}{D} \right\rceil$ and $m_v^U(C, D, m_h) \triangleq D + \left\lceil \frac{m_h - D}{D} \right\rceil$.\(^{11}\)

\(^{11}\)The case where no such element exists is trivial.
In order to draw conclusions from Proposition 4 in terms of the behavior of the various length, we first need to quantify the gaps between them—as given in the following corollary (proved in [50]).

**Corollary 1:** Assuming $D \geq 2$ and $\frac{C}{D} \geq 2$, the following bounds apply for the gaps between various length functionals:

\[
\Gamma_{\text{SN}}(C, D, m_h) \leq m_{v_c}(C, D, m_h) - m_{v_c}^N(C, D, m_h) < D + 1 + \frac{2m_h}{C} \left( \frac{C}{D} - 1 \right)
\]

\[
\Gamma_{\text{UIC}}(C, D, m_h) \leq m_{v_c}^U(C, D, m_h) - m_{v_c}^C(C, D, m_h) < D + 1 + \frac{(m_h - 1)(m_h - 2)}{C(D - 1)}
\]

\[
\Gamma_{\text{CLI}}(C, D, m_h) \leq m_{v_c}^C(C, D, m_h) - m_{v_c}(C, D, m_h) < D + 1 + \frac{m_h}{C(D - 1)}.
\]

For a generic $C$-channel $D$-fold subsampled FB, the true minimal PR synthesis filter length, i.e., where the “phase transition” between PR-infeasibility and PR-feasibility occurs, is denoted by $m_{v_c}^*(C, D, m_h)$. It follows that $m_{v_c}^N \leq m_{v_c}^* \leq m_{v_c}^S$. Although our results do not explicitly pinpoint the true minimal filter length $m_{v_c}^*$ (see Section VI for a conjecture that it coincides with $m_{v_c}^C$), we can exactly quantify the gap between $m_{v_c}^N$ and $m_{v_c}^S$ using their respective definitions. Here instead, combining Corollary 1 and Proposition 4, we study the qualitative behavior of the necessary and sufficient lengths and the gap between the two, which in turn enables us to address Question Q.2 raised in Section I regarding $m_{v_c}^*$.

- **The gap between $m_{v_c}^N$ and $m_{v_c}^S$ is small.** This is because, based on Corollary 1, $\Gamma_{\text{SN}}(C, D, m_h)$ drops roughly as $(\frac{C}{D})^{-2}$ and is small (relative to $m_h$) for moderately high oversampling factors. For example, with $D = 3$, for $\frac{C}{D} \geq 3$, the gap is smaller than $4 + 0.12 \times m_h$, and for $\frac{C}{D} \geq 4$ it is smaller than $4 + 0.18 \times m_h$. Further, in Section VI, we illustrate numerically that this gap is small.

- **$m_{v_c}^S$ has an approximately inverse relation to the oversampling factor.** Consider the set of inequalities in Proposition 4 bounding the sufficient length $m_{v_c}^S$ and $m_{v_c}^N$ and $m_{v_c}^S$. For a fixed analysis filter length $m_h = m_0^D$ and subsampling factor $D = D_h$, this relation shows that the integer function $m_{v_c}^S(C, D_h, m_0)$, defined on the integer line $C \in \mathbb{N}_i$, is “sandwiched” between two other integer functions both of which roughly drop as $\frac{C}{D}$ with increasing $C$. Moreover, the gap between $m_{v_c}^S$ and $m_{v_c}^C$, denoted as $\Gamma_{\text{UIC}}$ in Corollary 1, is small since (i) it drops as $(\frac{C}{D})^{-2}$ with increasing $C$; (ii) based on the proof of Proposition 4 [50] this gap is zero for all $C = kD_h$ with $k \geq 2$. Therefore, $m_{v_c}^S$ itself should behave similarly. The same argument can be repeated for a fixed $C$ and decreasing $D$. This shows that $m_{v_c}^S$ has an approximately inverse relation to $\frac{C}{D}$.

- **$m_{v_c}^*_{v_c}$ has an approximately inverse relation to the oversampling factor.** Similarly to the above, based on Proposition 4, the necessary length $m_{v_c}^N$ is sandwiched between $m_{v_c}^L$ and $m_{v_c}^C$. Moreover, the gap between $m_{v_c}^L$ and $m_{v_c}^C$, denoted $\Gamma_{\text{CLI}}(C, D, m_h)$ in Corollary 1, drops roughly as $(\frac{C}{D})^{-2}$ and is small. For example, for $\frac{C}{D} \geq 3$, the gap is smaller than $D + 1 + (\frac{m_h}{C})$. Specifically, for $m_h = 17$ and $D = 2$, $\Gamma_{\text{CLI}} \leq 5$. Hence, $m_{v_c}^*(C, D, m_h)$ itself has an approximate inverse relation to $\frac{C}{D}$.

- **$m_{v_c}^*_{v_c}$ has an approximate inverse relation to the oversampling factor.** Summarizing the aforementioned relations, the integer function $m_{v_c}^S$ and $m_{v_c}^N$ both behave (approximately) as $(\frac{C}{D})^{-2}$. Also, the gap in between them drops roughly as $(\frac{C}{D})^{-2}$ and is small (relative to $m_h$). This implies that $m_{v_c}^*$, which lies between $m_{v_c}^N$ and $m_{v_c}^C$, should itself have an approximate inverse relation with respect to the oversampling factor $\frac{C}{D}$. This answers the second part of Question Q.2 in Section I and is further demonstrated in Section VI.

The inequality $m_{v_c}^S \leq m_h$ stated in Proposition 4 implies that, with $\frac{C}{D} \geq 2$, the true minimal PR synthesis filter length $m_{v_c}^*(C, D, m_h)$ is generically less than the analysis filter length $m_h$. Note the lower bound $D$ for all lengths, which implies $m_{v_c}^*(C, D, m_h) \geq D$; this can be inferred from the FB structure (Panel (a) in Fig. 1), as a shorter synthesis filter would not be able to “fill in” the $D$-sample gap at the output of the up-samplers.

Finally, let us consider Question Q.1 raised in Section I. Based on Proposition 1 (with $m_0 = 0$), a length-$m_h$ PR synthesis bank exists generically for any $m_v \geq m_{v_c}^*(C, D, m_h)$, where, based on Proposition 4, $m_{v_c}^N(C, D, m_h) < m_h < \infty$. This implies feasibility of PR using FIR synthesis filters. The following corollary states this observation.

**Corollary 2:** For a $D$-fold subsampled $C$-channel FIR analysis bank with $\frac{C}{D} \geq 2$ and $m_h > D$, the following property holds generically: an FIR synthesis bank achieving PR with any $m_0$ delay $0 \leq \left\lfloor \frac{m_0}{D} \right\rfloor \leq \left\lfloor \frac{m_h}{2} \right\rfloor + 1$ exists.

A recent result due to Law et al. [39], specialized to a single-variate polynomial matrix, states that if $C \geq D + 1$, then a $C \times D$ single-variate polynomial matrix is generically (Laurent) polynomial left invertible. Applying this result to the analysis polyphase matrix $A(z)$, we can deduce that for generic oversampled FBs $A(z)$ has an FIR left inverse, which corresponds to the synthesis polyphase matrix $R(z)$ according to (6). Therefore, the result in [39] implies Corollary 2 and hence is stronger. However, as mentioned in Section I, Law et al. [39] do not address the filter support/length question (Q.2 in Section I), which is the focus of the present work.

**VI. NUMERICAL RESULTS: VERIFICATION OF THE PROPOSITIONS AND FURTHER OBSERVATIONS**

The first part of this section provides numerical verification of the results in Section V, as summarized in Proposition 4. In the second part, we provide Monte Carlo simulations results that provide a numerical verification of the Propositions 1 and 3 in Sections III and IV, respectively. A byproduct of the presented numerical results is a conjecture on the true minimal filter length for PR. In the last part of this section, we study the feasibility
Fig. 7 shows the necessary, sufficient, and counting filter lengths as functions of the number of channels $C$, for an analysis filter length of $m_h = 30$ and subsampling factors of: (a) $D = 1$; (b) $D = 2$; (c) $D = 3$; (d) $D = 4$. The upper bound $m_v^U$ is also overlaid on the plots and the dashed line in each panel marks the analysis filter length $m_h$. Therefore, it is easy to verify the upper bound $m_v^U$ is given in Proposition 4; (ii) the relationships between the different filter lengths, also given in Proposition 4.

Next, we move on to verification of the propositions. Fig. 9 shows Monte Carlo (M-C) simulation results for studying PR of imperfect (near perfect) reconstruction using synthesis filter lengths below those prescribed (for PR) by our propositions.
feasibility among randomly generated analysis FBs with threefold subsampling \((D = 3)\) and \(C = 1, \ldots, 24\) channels (horizontal axis). Each M-C run (from a total of 200) corresponded to generating an analysis FB comprising \(C\) real-valued analysis filters of length \(m_{b} = 30\), based on a uniform distribution with zero mean and variance of 100. The color-bar in the figure encodes the number of M-C runs (out of 200) for which a PR synthesis bank (allowing for multiple-of-3 delays) consisting of \(C\) filters with length \(m_{v}\) (vertical axis) exists. For each generated analysis FB, the computational process for determining numerically whether or not a PR synthesis FB exists involves checking the range condition in Lemma 1 for all feasible choices of \(n_{0}\).

The counting and sufficient lengths, \(m_{C}^{S}\) and \(m_{S}^{S}\), are overlaid on the graph as a function of \(C\). As is seen from the figure, for \(C \geq 2D = 6\), all synthesis filter length choices satisfying \(m_{v} \geq m_{S}^{S}\) resulted in 100% success in achieving PR. This verifies the claim of Proposition 1. The figure also shows that none of synthesis banks with filter lengths below the counting length \(m_{C}^{C}\) achieved PR (or delayed PR). Moreover, since the necessary length \(m_{S}^{S}\) bounds the counting length from below (Proposition 4), the M-C results verify that none of the synthesis lengths \(m_{v}\) that satisfied \(m_{v} < m_{S}^{S} \leq m_{C}^{C}\) resulted in PR—therefore, the minimal filter length that would allow for PR is at least as large as the derived necessary length. Consequently, the M-C results indirectly verify the claim of Proposition 3 as well.

The simulation results in Fig. 9 suggest that the counting length \(m_{C}^{C}(C, D, m_{b})\) is both sufficient and necessary for feasibility of PR with a length-\(m_{v}\) synthesis FB, i.e., the true minimum length \(m_{v}^{*}\), introduced in Section V, coincides with \(m_{C}^{C}\). This provides grounds for the following conjecture. (Note that, based on the bounds in Proposition 4, this conjecture is consistent with the requirement that the true minimum length should lie in between the necessary and sufficient lengths.)

**Conjecture:** For a \(D\)-fold subsampled FIR analysis bank \(\{h_{i}\}_{i=1}^{C}\) with \(m_{v} > D\), the following property holds generically: a length-\(m_{v}\) synthesis bank that achieves PR exists if and only if \(m_{v} \geq m_{C}^{C}(C, D, m_{b})\).

Regardless, as was pointed out in Section V and verified above, the gap between the counting length and our proven necessary and sufficient lengths is small, especially for moderately high oversampling.

In the last part of this section, we consider the feasibility of imperfect (near-perfect) reconstruction using synthesis filter lengths below those prescribed for PR by our theoretical results. To this end, assuming noise-free data, we compute the **reconstruction distortion** (error) resulting from FIR synthesis with insufficient synthesis filter length (i.e., below \(m_{C}^{C}\) based on the above conjecture). Consider an input signal \(x[n]\) that is a white random process with unit variance—and define the distortion metric to be the root-mean-square (RMS) value of reconstruction error at the FB output. Using fundamental relations for polyphase-domain analysis of FBs [1], [2] and the notation in Section II-C, it can be shown that this RMS error is

\[
\sqrt{\text{Var}(x - \hat{x})} = \left( \sum_{l=0}^{D-1} \left\| H_{M \times C} \left[ \{u_{lP}\}_{l=1}^{C} \right] - \delta_{w_0,P} \right\|_{2}^{2} \right)^{\frac{1}{2}}.
\]

(18)

It follows that, to minimize the RMS error we need to determine the least-squares (LS) solution to the synthesis FB by solving each of the \(D\) polyphase matrix equations in (10) using the pseudo-inverse operator \(H_{p}^{\dagger}\). Furthermore, it can be shown (following similar analysis for single channel interpolation [51]) that the LS solution also minimizes a second distortion metric, defined by \(\text{sup}_{n} |x[n] - \hat{x}[n]|\), i.e., the peak (\(\infty\)-norm) instantaneous output error, over all unit-energy (2-norm equal to 1) inputs—and that this second distortion metric coincides with the first, given by (18). In summary, the LS solution is optimal for the aforementioned two classes of input signals and distortion metrics, and is therefore used here as the method to compute synthesis FB given the analysis filters and the synthesis filter-length constraint. Furthermore, the two distortion metrics coincide, and their common expression in (18) is used here as the criterion to evaluate the impact of failing to satisfy the PR conditions.\(^{12}\)

Fig. 10(a) shows the above-defined reconstruction distortion (RMS error for random input, or worst-case peak error for unit energy input) as a function of \(m_{v}\), the synthesis filter length, for a randomly generated analysis FB with \(D = 3\) subsampling and \(m_{b} = 30\) analysis filter length. Each RMS value corresponds to the minimum of (18) over all feasible output delays \(n_{0}\) (Section II-C). It is seen from the figure that the distortion is zero for \(m_{v}\) values satisfying \(m_{v} \geq m_{C}^{C} = 21\), which is consistent with Fig. 9, Proposition 1, and the aforementioned conjecture, but it increases sharply and monotonically as \(m_{v}\) drops below \(m_{C}^{C}\). In fact, the RMS value jumps to 3.7% of the input variance (unity) when \(m_{v} = m_{C}^{C} - 1 = 20\) and is more than 30% with \(m_{v} = 17\).

Finally, to study the effect of different analysis FB realizations on the resulting distortion, we conducted the following M-C simulation: reconstruction distortions using a \(C\)-channel threefold subsampled FIR synthesis FB were computed as a function of \(C\) corresponding to 200 Monte Carlo runs and a synthesis bank of length \(m_{v} = [0.8 \times m_{C}^{C}(C, D, m_{b})]\).

\(^{12}\)If, however, the distortion metric is the 2-norm (rather than \(\infty\)-norm) of the worst-case error over all unit-energy inputs, then the solution involves \(H_{\infty}\)-norm optimization, which is outside the scope of the current analysis.
In this paper, we addressed the feasibility of PR using short FIR synthesis filters given an oversampled but otherwise general FIR analysis filter bank. We provided prescriptions for the shortest filter length of the synthesis bank that would guarantee PR. Specifically, we showed that for a length-$m_h$ FIR analysis FB with $C$ channels and $D$-fold subsampling that is at least twofold oversampled:

- PR is generically possible (with any delay $m_0$, $0 \leq \left\lceil \frac{m_0}{D} \right\rceil \leq \left\lfloor \frac{m_{hsp}}{D} \right\rfloor + \left\lfloor \frac{m_{hsp}}{D} \right\rfloor - 2$) using a FIR synthesis FB with filter length $m_v$ that is larger than the sufficient minimum length $m_v^S(C, D, m_h)$

$$m_v^S(C, D, m_h) = \min \left\{ m_v \in \mathbb{N} \mid \frac{C}{D} \geq 1 + \frac{1}{D} \sum_{l=0}^{D-1} \left\lceil \frac{m_{hsp} - 1 - l}{D} \right\rceil \right\}$$

where $m_{hsp} = \left\lfloor \frac{m_h}{D} \right\rfloor$.

- PR is generically impossible for any delay if $m_v$ is smaller than the necessary minimum length $m_v^N(C, D, m_h)$

$$m_v^N(C, D, m_h) = \min \left\{ m_v \in \mathbb{N} \mid \frac{C}{D} \geq 1 + \frac{1}{D} \sum_{l=0}^{D-1} \left\lceil \frac{m_{hsp} - 1 - l}{D} \right\rceil \right\}.$$

Furthermore, for oversampling factors of at least two, we showed that the sufficient minimum length is smaller than or equal to the length of the analysis filters, it decreases with increasing oversampling, and is close to the derived necessary length for moderately high oversampling factors. Finally, our numerical studies demonstrated that choosing filter lengths that are only slightly below the prescribed regime results in significant signal distortion. Our results are in the form of necessary and sufficient statements that hold generically, that is, they only fail for contrived examples and pathological cases of analysis filters. The results therefore also hold for specific classes of PR FBs, e.g., modulated FBs [1], as long as the analysis filters in the class are not selected as one of the pathological examples.

The results have potential applications in synthesis FB design problems where the analysis bank is given, e.g., in multichannel sensing/imaging systems where the channel characteristics are determined by the physics of the problem and cannot be fully manipulated. Practical instances of such systems include multichannel (parallel) MRI [12], [22] or blurring kernels associated with remote imaging applications [26], [27]. In particular, our recent work on reconstruction in parallel MR imaging [23], [24] extends the results in this work to multichannel perfect interpolation and shows that accurate multichannel interpolation (GRAPPA-type reconstruction [22]) is feasible with short FIR synthesis FBs. In addition, the presented work helps analyze and understand fundamental limitations in blind reconstruction of signals from data collected by unknown subsampled multichannel systems. A future area of research is to extend the results to multidimensional FBs, with applications to the problem of image superresolution.
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