QUANTUM FAMILIES OF MAPS AND QUANTUM SEMIGROUPS ON FINITE QUANTUM SPACES

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Abstract. Quantum families of maps between quantum spaces are defined and studied. We prove that quantum semigroup (and sometimes quantum group) structures arise naturally on such objects out of more fundamental properties. As particular cases we study quantum semigroups of maps preserving a fixed state and quantum commutants of given quantum families of maps.

1. Introduction

Let $X$ be a set. Then the set $\text{Map}(X)$ of all maps $X \to X$ is a semigroup. Of course, the set of all maps fixing a given point of $X$ is a subsemigroup of $\text{Map}(X)$. So is the set of all maps leaving invariant a given measure on $X$ or commuting with a fixed family of maps $X \to X$. These statements border triviality. The situation does not change substantially if we introduce a topology on $X$ and require that all maps be continuous.

Our aim in this paper is to investigate the non commutative analogs of the above mentioned phenomena. More precisely we will recall the definition of a quantum family of maps between quantum spaces ([9]) and we will show that, just as in the commutative case mentioned above, quantum semigroup structures appear naturally on many such objects.

In [8] S. Wang investigated quantum automorphism groups of finite quantum spaces. He searched for universal objects in the category of quantum transformation groups of a given finite quantum space. He also mentioned quantum semigroups in [5, Remark (3), page 208]. We show that the quantum semigroup structure on these objects is there as a consequence of a more fundamental structure these objects possess. This makes them also easier to define.

Wang proved that for non classical finite quantum spaces the quantum automorphism group does not exist and turned to study the group preserving a fixed state. We will take a more general approach and define the quantum family of all maps preserving a given state. We will take a more general approach and define the quantum family of all maps preserving a given state. Again the quantum semigroup structure (and in some cases quantum group structure — cf. Section 7) appears from a more fundamental property of these objects.

We will give one more example of a similar situation, where a quantum subsemigroup of a given quantum semigroup is defined without reference to its semigroup structure, by constructing the quantum commutant of a given quantum family of maps. Again, the emergence of a quantum semigroup structure will be a consequence of a more fundamental property of the quantum commutant.

Let us now briefly describe the contents of the paper. Section 2 is a short summary of the standard language of non commutative topology. In particular we shall recall and discuss the definition of a quantum space. In Section 3 we shall define the concept of a quantum family of maps from one quantum space to another. This notion was introduced already in [9], where quantum spaces were called “pseudospaces”. We shall define what the quantum space of all maps from one quantum space to another is and prove its existence in a special (yet interesting) case. Then we shall define the crucial notion of composition of quantum families of maps.

The quantum space of all maps from a given finite quantum space to itself carries a natural structure of a compact quantum semigroup with unit. This is the content of Section 4. We shall
study the properties of this quantum semigroup and its action on the finite quantum space, like ergodicity.

The next two sections are devoted to natural constructions of quantum subsemigroups of the quantum semigroup defined in Section 4. First, in Section 5, we define and study the quantum family of all maps preserving a fixed state. This family is naturally endowed with a compact quantum semigroup structure. The existence of this structure follows from simple considerations concerning composition of quantum families (as defined in Section 3). Then, in Section 6, we construct the quantum commutant of a given quantum family of maps. This construction bears many similarities to the one presented in Section 5. It is based on the notion of *commuting quantum families of maps* which is briefly investigated at the beginning of the section. The quantum commutant has a natural structure of a quantum semigroup.

The constructions presented in Sections 5 and 6 clarify the mechanism of obtaining quantum semigroup structure. This has never been investigated before. In the last section we address the question when quantum group structures appear and when they should be expected and show how S. Wang’s quantum automorphism groups of finite spaces ([8]) fit into the framework developed in Section 5.

2. Quantum spaces

Let \( \mathcal{C} \) be the category of \( C^* \)-algebras described and studied in [9] and [12]. The objects of \( \mathcal{C} \) are all \( C^* \)-algebras and for any two objects \( A \) and \( B \) of \( \mathcal{C} \) the set \( \text{Mor}(A, B) \) consists of all non degenerate *-homomorphisms from \( A \) to \( M(B) \). The category \( \Omega \) of quantum spaces is by definition the dual category of \( \mathcal{C} \). By definition the class of objects of \( \Omega \) is the same as the class of objects of \( \mathcal{C} \). Nevertheless for any \( C^* \)-algebra \( A \) we shall write \( QS(A) \) for \( A \) regarded as an object of \( \Omega \).

From the point of non commutative geometry (topology) it is natural to work with objects of \( \Omega \). On the other hand all the tools at our disposal are from the world of \( C^* \)-algebras. We shall try to introduce a compromise between the two conventions by declaring that the phrase “let \( QS(A) \) be a quantum space” be taken to mean “let \( A \) be a \( C^* \)-algebra”. Moreover we shall use interchangeably the notation \( \Phi \in \text{Mor}(A, B) \) and \( \Phi : QS(B) \to QS(A) \).

The category of locally compact topological spaces with continuous maps is a full subcategory of \( \Omega \). A quantum space \( QS(A) \) is a locally compact space if and only if \( A \) is unital. If \( A \) is finite dimensional then \( QS(A) \) is said to be a compact quantum space. A more controversial idea to call a quantum space \( QS(A) \) a finite dimensional if \( A \) is finitely generated was proposed in [9].

An interesting step towards a better understanding of the category \( \Omega \) was taken in [10] and [12] (see also [4]). The results of these papers show that any (separable) \( C^* \)-algebra is of the form \( C_\infty(\mathcal{X}) \), where \( \mathcal{X} \) is a certain \( W^* \)-category and \( C_\infty(\cdot) \) has a whole new meaning (which reduces to the old one for commutative \( C^* \)-algebras). This means that, despite technical complications, it is possible to realize quantum spaces as concrete mathematical objects.

3. Quantum families of maps

In this section we shall introduce the objects of our study. These will be quantum spaces of maps or quantum families of maps. The latter concept is a generalization of a classical notion of a continuous family of maps between locally compact spaces labeled by some other locally compact space.

This is based on the fact that for topological spaces \( X, Y \) and \( Z \) such that \( Z \) is Hausdorff and \( X \) is locally compact (Hausdorff) we have

\[
C(Z \times X, Y) \approx C(Z, C(X, Y)),
\]
where “≈” means homeomorphism and all spaces are taken with compact-open topology \([3]\). Thus a continuous family of maps \(X \to Y\) labeled by \(Z\) can be represented by a continuous map \(Z \times X \to Y\).

**Definition 3.1.** Let \(\mathcal{QS}(A), \mathcal{QS}(B)\) and \(\mathcal{QS}(C)\) be quantum spaces.

1. A quantum family of maps \(\mathcal{QS}(C) \to \mathcal{QS}(B)\) labeled by \(\mathcal{QS}(A)\) is an element \(\Psi \in \text{Mor}(B, C \otimes A)\).
2. We say that \(\Phi \in \text{Mor}(B, C \otimes A)\) is the quantum family of all maps \(\mathcal{QS}(C) \to \mathcal{QS}(B)\) if for any quantum space \(\mathcal{QS}(D)\) and any quantum family \(\Psi\) of maps \(\mathcal{QS}(C) \to \mathcal{QS}(B)\) labeled by \(\mathcal{QS}(D)\) there exists a unique \(\Lambda \in \text{Mor}(A, D)\) such that the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\Phi} & C \otimes A \\
\downarrow & & \downarrow \\
B & \xrightarrow{\Psi} & C \otimes D \\
\end{array}
\]

is commutative. In this case we say that \(\mathcal{QS}(A)\) is the quantum space of all maps \(\mathcal{QS}(C) \to \mathcal{QS}(B)\).

3. In the special case when \(B = C\), we say that a quantum family \(\Psi \in \text{Mor}(B, B \otimes A)\) is trivial if \(\Psi(b) = b \otimes I\) for all \(b \in B\).

The property of \((A, \Phi)\) described in (2) will be referred to as the universal property of \((A, \Phi)\).

**Remark 3.2.**

1. Let \(\Psi \in \text{Mor}(B, C \otimes A)\) be a quantum family of maps \(\mathcal{QS}(C) \to \mathcal{QS}(B)\) labeled by \(\mathcal{QS}(A)\). Assume that all three spaces are classical, i.e. \(A, B\) and \(C\) are commutative. Then \(\mathcal{QS}(A)\) is a classical locally compact space labeling a family of elements of \(C(\mathcal{QS}(C), \mathcal{QS}(B))\) and the map from \(\mathcal{QS}(A)\) to \(C(\mathcal{QS}(C), \mathcal{QS}(B))\) is continuous for the compact-open topology.
2. It is clear that given two quantum spaces \(\mathcal{QS}(B)\) and \(\mathcal{QS}(C)\), the quantum space of all maps \(\mathcal{QS}(C) \to \mathcal{QS}(B)\) might not exist in the category \(\mathcal{QS}\). This can happen even for classical spaces. Non existence of the quantum space of all maps should be understood as meaning that this object is not locally compact in the compact open-topology, rather than that it does not exist at all.
3. If the quantum space \(\mathcal{QS}(A)\) of all maps \(\mathcal{QS}(C) \to \mathcal{QS}(B)\) exists and \(\Phi\) is the quantum family of all maps \(\mathcal{QS}(C) \to \mathcal{QS}(B)\) then the pair \((A, \Phi)\) is unique in the sense that if \((A', \Phi')\) is another such pair then there exists an isomorphism \(\Lambda \in \text{Mor}(A, A')\) such that \((\text{id} \otimes \Lambda) \circ \Phi = \Phi'\).

It was stated already in [9] that the quantum space of all maps from a finite quantum space (described by a finite dimensional \(C^*\)-algebra) to a compact finite dimensional one (corresponding to a unital finitely generated \(C^*\)-algebra) always exists:

**Theorem 3.3.** Let \(\mathcal{QS}(B)\) and \(\mathcal{QS}(C)\) be quantum spaces. Assume that \(C\) is finite dimensional and \(B\) is finitely generated and unital. Then

1. the quantum space \(\mathcal{QS}(A)\) of all maps \(\mathcal{QS}(C) \to \mathcal{QS}(B)\) exists.
2. The \(C^*\)-algebra \(A\) is unital and generated by \(\{\omega \otimes \text{id}(b) \mid b \in B, \omega \in C^*\}\), where \(\Phi \in \text{Mor}(B, C \otimes A)\) is the quantum family of all maps \(\mathcal{QS}(C) \to \mathcal{QS}(B)\).

**Proof.** Let \(x_1, \ldots, x_N\) be generators of \(B\). Since any element of \(B\) is a linear combinations of four unitaries, we can assume that \(x_1, \ldots, x_N\) are unitary. Let \(\{R_t \mid t \in T\}\) be the complete list of relations between \(x_1, \ldots, x_N\), so that

\[
\langle x_1, \ldots, x_N \mid R_t(x_1, \ldots, x_N) = 0, t \in T \rangle
\]

is a presentation of \(B\). In particular some of the relations \(\{R_t \mid t \in T\}\) say that each \(x_p\) is unitary.

The \(C^*\)-algebra \(C\) can be written as a finite direct sum of full matrix algebras:

\[
C = \bigoplus_{k=1}^K M_{n_k}.
\]
Let $\mathcal{A}$ be the $*$-algebra generated by elements
\[\{y^k_{p,r,s}| p \in \{1, \ldots, N\}, k \in \{1, \ldots, K\}, r, s \in \{1, \ldots, n_k\}\}\]
with the relations
\[\mathcal{R}_t \left( \begin{pmatrix} Y^1_1 & \cdots & Y^1_K \\ \vdots & \ddots & \vdots \\ Y^K_1 & \cdots & Y^K_K \end{pmatrix} \right) = 0, \quad (t \in \mathcal{T}),\]
where $Y^k_p$ is the matrix $(y^k_{p,r,s})_{r,s=1,\ldots,n_k}$.

The relation saying that $x_p$ is unitary guarantees that
\[\sum_{q=1}^{n_k} (y^k_{p,q,r})^* (y^k_{p,q,r}) = \delta_{r,s} I_A.\]

In particular, for any Hilbert space representation $\pi$ of $\mathcal{A}$ the norm $\|\pi(x^k_{p,r,s})\| \leq 1$.

This implies that for any $c \in \mathcal{A}$ the quantity
\[\sup_{\pi} \|\pi(c)\|\] (where the supremum is taken over all Hilbert space representations of $\mathcal{A}$) is finite. Now standard procedure leads to construction of the universal enveloping $C^*$-algebra $A$ of $\mathcal{A}$.

Let us define a map $\Phi : B \to C \otimes A$ as sending $x_p$ to the image in $C \otimes A$ of the matrix
\[\begin{pmatrix} Y^1_1 & \cdots & Y^1_K \\ \vdots & \ddots & \vdots \\ Y^K_1 & \cdots & Y^K_K \end{pmatrix} \in C \otimes \mathcal{A}.\]

After a moment of reflection, we see that $(A, \Phi)$ has the required universal property, so that $\mathcal{Q}\mathcal{S}(A)$ is the space of all maps $\mathcal{Q}\mathcal{S}(C) \to \mathcal{Q}\mathcal{S}(B)$ and $\Phi \in \text{Mor}(B, C \otimes A)$ is the quantum family of all these maps.

The second part of Statement (2) follows from the obvious observation that
\[\{(\omega \otimes \text{id})\Phi(b)| b \in B, \omega \in C^*\}\]
contains the images of the generators of $\mathcal{A}$ in $A$. Alternatively, one can prove this using the uniqueness of $(A, \Phi)$. \hfill \Box

Remark 3.4.

(1) In case when $C$ is the algebra $M_n$ of $n \times n$ complex matrices, the $C^*$-algebra $A$ defined in Theorem 3.3 coincides with $W_n(M_n)$, where $W_n$ is the left adjoint of the functor $D \mapsto M_n(D)$ (cf. [6] Page 174). Note, however that our morphism sets are different that those used in [6].

(2) It was noticed in [9] that the quantum space of maps between quantum spaces might be very interesting even if the two quantum spaces are finite classical spaces. For example if we take $B = C = \mathbb{C}^2$ then $\mathcal{Q}\mathcal{S}(B) = \mathcal{Q}\mathcal{S}(C)$ is the two-point space. The family of all maps $\mathcal{Q}\mathcal{S}(C) \to \mathcal{Q}\mathcal{S}(B)$ has four elements. However the corresponding quantum family is infinite in the sense that the corresponding $C^*$-algebra is infinite dimensional. It is amusing to check that in this case it is isomorphic to the $C^*$-algebra of all continuous functions from an interval to $M_2$ whose values at the endpoints are diagonal ([5, 2, Section 2,β]).

We shall now introduce the notion of composition of quantum families of maps. Let $A_1, A_2, B, C$ and $D$ be $C^*$-algebras and let
\[\Psi_1 \in \text{Mor}(C, D \otimes A_1), \quad \Psi_2 \in \text{Mor}(B, C \otimes A_2)\]
be quantum families of maps $\mathcal{Q}\mathcal{S}(D) \to \mathcal{Q}\mathcal{S}(C)$ and $\mathcal{Q}\mathcal{S}(C) \to \mathcal{Q}\mathcal{S}(B)$ labeled by $\mathcal{Q}\mathcal{S}(A_1)$ and $\mathcal{Q}\mathcal{S}(A_2)$ respectively. We define the quantum family $\Psi_1 \Delta \Psi_2$ of maps $\mathcal{Q}\mathcal{S}(D) \to \mathcal{Q}\mathcal{S}(B)$ labeled by $\mathcal{Q}\mathcal{S}(A_1 \otimes A_2)$ by
\[\Psi_1 \Delta \Psi_2 = (\Psi_1 \otimes \text{id}) \circ \Psi_2.\]
This quantum family of maps will be called the \textit{composition} of the families \(\Psi_1\) and \(\Psi_2\). We shall also refer to the operation taking \(\Psi_1\) and \(\Psi_2\) to \(\Psi_1 \circ \Psi_2\) as the \textit{operation of composition} of quantum families of maps. In case the families are classical, i.e. the \(\mathcal{C}^*\)-algebras \(A_1\) and \(A_2\) are commutative, the family \(\Psi_1 \circ \Psi_2\) is a classical family consisting of all compositions of members of \(\Psi_1\) and \(\Psi_2\).

The crucial property of composition of quantum families of maps is that it is associative:

\textbf{Proposition 3.5.} Let \(A_1, A_2, A_3, B_1, B_2, C\) and \(D\) be \(\mathcal{C}^*\)-algebras and let

\[
\begin{align*}
\Psi_1 &\in \text{Mor}(B_1, C \otimes A_1), \\
\Psi_2 &\in \text{Mor}(B_2, B_1 \otimes A_2), \\
\Psi_3 &\in \text{Mor}(D, B_2 \otimes A_3)
\end{align*}
\]

be quantum families of maps. Then

\[
\Psi_1 \circ (\Psi_2 \circ \Psi_3) = (\Psi_1 \circ \Psi_2) \circ \Psi_3.
\]

\textit{Proof.} This is a simple computation:

\[
\begin{align*}
\Psi_1 \circ (\Psi_2 \circ \Psi_3) &= (\Psi_1 \otimes \text{id}) \circ (\Psi_2 \circ \Psi_3) \\
&= (\Psi_1 \otimes \text{id} \otimes \text{id}) \circ (\Psi_2 \otimes \text{id}) \circ \Psi_3 \\
&= ([\Psi_1 \otimes \text{id}] \circ \Psi_2] \otimes \text{id}) \circ \Psi_3 \\
&= ([\Psi_1 \circ \Psi_2] \otimes \text{id}) \circ \Psi_3 \\
&= (\Psi_1 \circ \Psi_2) \circ \Psi_3.
\end{align*}
\]

\(\square\)

\section*{4. Quantum semigroup structure}

In this section we shall analyze the structure of the quantum space of all maps from a finite quantum space to itself. Thus let \(M\) be a finite dimensional \(\mathcal{C}^*\)-algebra. Then Theorem 3.3 guarantees that there exists the quantum space of all maps \(\mathcal{Q}(M) \to \mathcal{Q}(M)\). Let us denote the corresponding \(\mathcal{C}^*\)-algebra by \(A\) and let \(\Phi \in \text{Mor}(M, M \otimes A)\) be the quantum family of all maps \(\mathcal{Q}(M) \to \mathcal{Q}(M)\). This notation will be kept throughout this section.

The universal property of \((A, \Phi)\) will provide us with rich structure on \(A\) or, more appropriately, on \(\mathcal{Q}(A)\).

\textbf{Theorem 4.1.}

\begin{enumerate}
\item There exists a unique morphism \(\Delta \in \text{Mor}(A, A \otimes A)\) such that

\[
(\Phi \otimes \text{id}) \circ \Phi = (\text{id} \otimes \Delta) \circ \Phi. \tag{4.1}
\]

\item The morphism \(\Delta\) satisfies

\[
(\text{id} \otimes \Delta) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta. \tag{4.2}
\]

\item There exists a unique character \(\epsilon\) of \(A\) such that

\[
(\text{id} \otimes \epsilon) \circ \Phi = \text{id}. \tag{4.3}
\]

\item The character \(\epsilon\) satisfies

\[
(\text{id} \otimes \epsilon) \circ \Delta = (\epsilon \otimes \text{id}) \circ \Delta = \text{id}. \tag{4.4}
\]
\end{enumerate}

\textit{Proof.} Let us consider the quantum family \(\Phi \circ \Phi\) of maps \(\mathcal{Q}(M) \to \mathcal{Q}(M)\). It is labeled by \(\mathcal{Q}(A \otimes A)\) and the universal property of \((A, \Phi)\) implies that there exists a unique \(\Delta \in \text{Mor}(A, A \otimes A)\) such that

\[
(\text{id} \otimes \Delta) \circ \Phi = \Phi \circ \Phi.
\]

This is precisely (4.1).
To prove (4.2) we use (4.1) to compute $(\Phi \otimes \text{id} \otimes \text{id}) \circ (\Phi \otimes \text{id}) \circ \Phi$ in two ways:

$$(\Phi \otimes \text{id} \otimes \text{id}) \circ (\Phi \otimes \text{id}) \circ \Phi = (\Phi \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes \Delta) \circ (\Phi \otimes \text{id}) \circ \Phi$$

$$= (\text{id} \otimes \text{id} \otimes \Delta) \circ (\Phi \otimes \text{id}) \circ \Phi$$

and

$$(\Phi \otimes \text{id} \otimes \text{id}) \circ (\Phi \otimes \text{id}) \circ \Phi = \left(\left[(\Phi \otimes \text{id}) \circ \Phi\right] \otimes \text{id}\right) \circ \Phi$$

$$= \left(\left[(\text{id} \otimes \Delta) \circ \Phi\right] \otimes \text{id}\right) \circ \Phi$$

$$= (\text{id} \otimes \Delta \otimes \text{id}) \circ (\Phi \otimes \text{id}) \circ \Phi$$

$$= (\text{id} \otimes \Delta \otimes \text{id}) \circ (\text{id} \otimes \Delta) \circ \Phi.$$

Let $\omega$ be a functional on $M$. Applying $(\omega \otimes \text{id} \otimes \text{id})$ to both sides of the equation

$$(\text{id} \otimes \text{id} \otimes \Delta) \circ (\text{id} \otimes \Delta) \circ \Phi = (\text{id} \otimes \text{id} \otimes \Delta) \circ (\text{id} \otimes \Delta) \circ \Phi$$

we obtain

$$\left[(\text{id} \otimes \Delta) \circ \Delta\right] (\omega \otimes \text{id}) \Phi(m)) = \left[(\Delta \otimes \text{id}) \circ \Delta\right] ((\omega \otimes \text{id}) \Phi(m)),$$

for any $m \in M$. Thus formula (4.2) follows from Proposition 3.3(2).

Statement (3) follows again from the universal property of $(A, \Phi)$ (Definition 3.1.2) with $B = C = M$. More precisely we take $D = C$ and canonically identify $M$ with $M \otimes D$. Then we take $\Psi$ to be the identity morphism $M \to M \otimes D$. The universal property of $(A, \Phi)$ guarantees that there exists a unique $\epsilon \in \text{Mor}(A, D)$ satisfying (4.3).

The proof of Statement (4) is similar to that of Statement (2). Using (4.1) and (4.3) we arrive at

$$\left(\text{id} \otimes \left[(\epsilon \otimes \text{id}) \circ \Delta\right]\right) \circ \Phi = \Phi = \left(\text{id} \otimes \left[(\text{id} \otimes \epsilon) \circ \Delta\right]\right) \circ \Phi.$$

Applying $(\omega \otimes \text{id} \otimes \text{id})$ to both sides, we obtain

$$\left[(\epsilon \otimes \text{id}) \circ \Delta\right] ((\omega \otimes \text{id}) \Phi(m)) = (\omega \otimes \text{id}) \Phi(m) = \left[(\text{id} \otimes \epsilon) \circ \Delta\right] ((\omega \otimes \text{id}) \Phi(m))$$

for any $m \in M$ which proves (4.4). $\square$

The morphisms $\Delta \in \text{Mor}(A, A \otimes A)$ and $\epsilon \in \text{Mor}(A, C)$ are called the comultiplication and the counit of $A$ respectively. They endow $A$ with the structure of a quantum semigroup with unit as defined below.

**Definition 4.2.**

1. A pair $(B, \Delta_B)$ consisting of a $C^*$-algebra and a morphism $\Delta_B \in \text{Mor}(B, B \otimes B)$ is called a quantum semigroup if $\Delta_B$ is coassociative, i.e.

$$(\Delta_B \otimes \text{id}) \circ \Delta_B = (\text{id} \otimes \Delta_B) \circ \Delta_B.$$

2. A quantum semigroup $(B, \Delta_B)$ has a unit if $B$ admits a character $\epsilon_B$ satisfying

$$(\epsilon_B \otimes \text{id}) \circ \Delta_B = \text{id} = (\text{id} \otimes \epsilon_B) \circ \Delta_B.$$

3. Let $(B, \Delta_B)$ and $(C, \Delta_C)$ be quantum semigroups. An element $\Lambda \in \text{Mor}(B, C)$ is a morphism of quantum semigroups (or a quantum semigroup morphism) if it satisfies

$$(\Lambda \otimes \Lambda) \circ \Delta_B = \Delta_C \circ \Lambda.$$

4. An action of a quantum semigroup $(B, \Delta_B)$ on a quantum space $\mathcal{Q}(C)$ is a morphism $\Psi \in \text{Mor}(C, C \otimes B)$ satisfying

$$(\Psi \otimes \text{id}) \circ \Psi = (\text{id} \otimes \Delta_B) \circ \Psi.$$

We shall denote the quantum semigroup $(A, \Delta)$ constructed in Theorem 4.1.1 by $\mathcal{Q}(\text{Map}(\mathcal{Q}(M)))$. The quantum family $\Phi \in \text{Mor}(M, M \otimes A)$ of all maps $\mathcal{Q}(M) \to \mathcal{Q}(M)$ is then an action of $\mathcal{Q}(\text{Map}(\mathcal{Q}(M)))$ on the quantum space $\mathcal{Q}(M)$. Since $A$ is a unital $C^*$-algebra (Theorem 3.3.2), the quantum semigroup $\mathcal{Q}(\text{Map}(\mathcal{Q}(M)))$ is compact.
Remark 4.3.

1. The coassociativity of $\Delta$ as derived in the proof of Theorem 4.1 is, in fact, a consequence of the associativity of composition of quantum families of maps (Proposition 3.5).

2. The semigroup structure on $\Omega\text{-Map}(\Omega S(M))$ and the operation of composition of quantum families of maps are related in a very natural way. Let $B$ and $C$ be $C^*$-algebras and let $\Psi_B \in \text{Mor}(M, M \otimes B)$, $\Psi_C \in \text{Mor}(M, M \otimes C)$ be quantum families of maps $\Omega S(M) \to \Omega S(M)$ labeled by $\Omega S(B)$ and $\Omega S(C)$ respectively. Finally let $\Lambda_B \in \text{Mor}(A, B)$ and $\Lambda_C \in \text{Mor}(A, C)$ be the unique morphisms satisfying $(id \otimes \Lambda_B) \circ \Phi = \Psi_B$, $(id \otimes \Lambda_C) \circ \Phi = \Psi_C$. Then

$$\Psi_B \Delta \Psi_C = (id \otimes \Lambda_B \otimes \Lambda_C) \circ (id \otimes \Delta) \circ \Phi.$$

Clearly the morphisms $\Lambda_B$ and $\Lambda_C$ describe the “inclusions” of the two considered families into the quantum family of all maps, and their composition is the composition in the semigroup of all maps.

The coassociativity of $\Delta$ means, in particular that the operation of convolution product of continuous functionals on $A$, defined by

$$\phi \ast \psi = (\phi \otimes \psi) \circ \Delta$$

for $\phi, \psi \in A^*$, is associative. Note that the counit $\epsilon$ is, by Theorem 4.1(4), a neutral element for the convolution product. The notion of convolution product enters the formulation of the next theorem.

Also in the next theorem we shall use the concept of natural topology on the set of morphisms between $C^*$-algebras defined as follows: let $B$ and $C$ be $C^*$-algebras. The natural topology on $\text{Mor}(B, C)$ is the weakest topology such that for any $b \in B$ the maps $\text{Mor}(B, C) \ni \Psi \mapsto \Psi(b) \in M(C)$ is strictly continuous ([12, p. 491]). If $B = C_\infty(Y)$ and $C = C_\infty(X)$ are commutative, the set $\text{Mor}(B, C)$ can be identified with $C(X, Y)$ and the natural topology is the compact-open topology. In the relevant case, when $B = C = M$ is a finite dimensional $C^*$-algebra, this topology is the one inherited by $\text{Mor}(M, M)$ form the space of all linear maps $M \to M$ which is a finite dimensional vector space.

Theorem 4.4.

1. There exists a canonical bijection $\Theta$ between the space of all characters of $A$ and the set $\text{Mor}(M, M)$ given by

2. The operation of convolution endows the set of all characters of $A$ with the structure of a unital semigroup and $\Theta$ is an isomorphism of semigroups.

3. $\Theta$ is a homeomorphism for the weak* topology on the space of characters of $A$ and the natural topology on $\text{Mor}(M, M)$.

4. $\Theta$ maps the set of all convolution invertible characters onto the group of all automorphisms of $M$ and is an isomorphism of topological groups.

Proof. Statement (1) follows from the universal property of $(A, \Phi)$: any morphism $\Lambda \in \text{Mor}(M, M)$ gives a character $\lambda$ on $A$ such that $(id \otimes \lambda) \circ \Phi = \Lambda$. Conversely, any character $\lambda$ on $A$ defines $\Lambda = (id \otimes \lambda) \circ \Phi \in \text{Mor}(M, M)$. These correspondences are inverse to one another by the universal property of $(A, \Phi)$. 
The fact that $\Theta$ is a semigroup isomorphism follows from inspection of the following commutative diagram:

![Diagram](image)

We obtain the equality $(\id \otimes [\lambda * \mu]) \circ \Phi = \Theta(\lambda) \circ \Theta(\mu)$, so that

$$\Theta(\lambda * \mu) = \Theta(\lambda) \circ \Theta(\mu).$$

To prove Statement (3) it is enough to show that $\Theta$ is continuous for the weak* topology on $A$ and the natural topology on $\Mor(M,M)$. The fact that $\Theta$ is a homeomorphism will follow, because both spaces are compact and $\Theta$ is a bijection.

Take $m \in M$ and let $(\lambda_n)$ be a net of characters of $A$, weak* convergent to $\lambda$ (which must then be a character of $A$). It is clear that $(\id \otimes \lambda_n)\Phi(m)$ is norm convergent to $(\id \otimes \lambda)\Phi(m)$ ($M$ is finite dimensional). This means that $(\Theta(\lambda_n))(m)$ depends continuously on $\alpha$. Since $m$ is arbitrary, this means that $\Theta(\lambda_n)$ varies continuously for the natural topology on $\Mor(M,M)$.

Statement (3) is a consequence of (2) and (3).

The next statement says, in particular, that the quantum semigroup $\mathcal{Q}\text{-Map}(\mathcal{Q}S(M))$ is not too small, because it always contains the semigroup $\Mor(M,M)$. Containment must be understood in the sense of noncommutative geometry.

**Corollary 4.5.** The Gelfand transform of $A$ is an epimorphism onto $C(\Mor(M,M))$ and is a morphism of quantum semigroups.

An interesting question is whether the quantum semigroup $\mathcal{Q}\text{-Map}(\mathcal{Q}S(M))$ is significantly bigger than the semigroup $\Mor(M,M)$. A partial answer to this question is given by the following proposition.

**Proposition 4.6.** The action $\Phi$ of $\mathcal{Q}\text{-Map}(\mathcal{Q}S(M))$ on $\mathcal{Q}S(M)$ is ergodic: for any $m \in M$, the condition that $\Phi(m) = m \otimes I$ implies $m \in \mathcal{C}I$.

**Proof.** Consider the quantum family of maps $\mathcal{Q}S(M) \to \mathcal{Q}S(M)$ labeled by $\mathcal{Q}S(M)$ given by $\Psi \in \Mor(M,M \otimes M)$,

$$\Psi(m) = I \otimes m.$$

By the universal property of $(A, \Phi)$ there is an element $\Lambda \in \Mor(A,M)$ such that $I \otimes m = (\id \otimes \Lambda)\Phi(m)$. Now if $\Phi(m) = m \otimes I$ then $I \otimes m = (\id \otimes \Lambda)(m \otimes I) = m \otimes I$ and it follows that $m \in \mathcal{C}I$.

Note that the semigroup $\Mor(M,M)$ need not be ergodic in the sense that there can be non-trivial elements of $M$ fixed under every morphism.

**Proposition 4.7.** Let $B$ be a $C^*$-algebra and let $\Psi \in \Mor(M, M \otimes B)$ be a quantum family of maps $\mathcal{Q}S(M) \to \mathcal{Q}S(M)$ labeled by $\mathcal{Q}S(B)$. Assume that there exists a morphism $\Delta_B \in \Mor(B, B \otimes B)$ such that

$$(\id \otimes \Delta_B) \circ \Psi = (\Psi \otimes \id) \circ \Psi$$

and let $\Lambda \in \Mor(A,B)$ be the unique morphism such that $(\id \otimes \Lambda) \circ \Phi = \Psi$. Then $\Lambda$ satisfies

$$(\Lambda \otimes \Lambda) \circ \Delta = \Delta_B \circ \Lambda.$$
Proof. Looking at the commutative diagram

\[
\begin{array}{c}
M \xrightarrow{\Phi} M \otimes A \xrightarrow{\text{id} \otimes \Delta} M \otimes A \otimes A \\
\downarrow \Phi \quad \downarrow \Phi \otimes \text{id} \quad \downarrow \text{id} \otimes \Lambda \otimes \Lambda \\
M \otimes B \xrightarrow{\Psi \otimes \text{id}} M \otimes B \otimes B
\end{array}
\]

we find that \((\text{id} \otimes \Lambda \otimes \Lambda) \circ (\text{id} \otimes \Delta) \circ \Phi = (\Psi \otimes \text{id}) \circ (\text{id} \otimes \Lambda) \circ \Phi\). But \((\text{id} \otimes \Lambda) \circ \Phi = \Psi\) and using this fact twice we obtain

\[
(\text{id} \otimes \Lambda \otimes \Lambda) \circ (\text{id} \otimes \Delta) \circ \Phi = (\Psi \otimes \text{id}) \circ (\text{id} \otimes \Lambda) \circ \Phi = (\Psi \otimes \text{id}) \circ \Psi = (\text{id} \otimes \Delta_B) \circ \Psi = (\text{id} \otimes \Delta_B) \circ (\text{id} \otimes \Lambda) \circ \Phi.
\]

Take \(\omega \in M^*\) and let us apply \(\omega \otimes \text{id} \otimes \text{id}\) to both sides of the above equality. We find that for any \(m \in M\)

\[
(\Delta_B \circ \Lambda)(\omega \otimes \text{id})(\Phi(m)) = ((\Lambda \otimes \Lambda) \circ \Delta)(\omega \otimes \text{id})(\Phi(m))
\]

and our result follows from Theorem 3.3(2). □

5. Invariant states

We shall retain the notation introduced in Section 4. Thus \(M\) is a finite dimensional \(C^*\)-algebra and \(\mathcal{Q}\text{-Map}((\mathcal{Q}S(M))) = (A, \Delta)\) is the quantum semigroup of all maps \(\mathcal{Q}S(M) \to \mathcal{Q}S(M)\). The action of \(\mathcal{Q}\text{-Map}((\mathcal{Q}S(M)))\) on \(\mathcal{Q}S(M)\) will, as before, be denoted by \(\Phi\).

Definition 5.1. Let \(\Psi \in \text{Mor}(M, M \otimes B)\) be a quantum family of maps \(\mathcal{Q}S(M) \to \mathcal{Q}S(M)\) labeled by \(\mathcal{Q}S(B)\) and let \(\omega\) be a state on \(M\). We say that \(\omega\) is invariant for \(\Psi\) if

\[
(\omega \otimes \text{id})\Psi(m) = \omega(m)I
\]

for all \(m \in M\). We also say that the quantum family of maps \(\Psi\) preserves the state \(\omega\).

In the next proposition we shall use the notion of composition of quantum families of maps defined in Section 3.

Proposition 5.2. Let \(B\) and \(C\) be \(C^*\)-algebras and let \(\Psi_B \in \text{Mor}(M, M \otimes B), \Psi_C \in \text{Mor}(M, M \otimes C)\) be quantum families of maps \(\mathcal{Q}S(M) \to \mathcal{Q}S(M)\) labeled by \(\mathcal{Q}S(B)\) and \(\mathcal{Q}S(C)\) respectively. Let \(\omega\) be a state on \(M\) which is invariant for both \(\Psi_B\) and \(\Psi_C\). Then \(\omega\) is invariant for the composition \(\Psi_B \Delta \Psi_C\).

Proof. The proof of this result is purely computational. To make the computations more transparent let us denote them maps

\[
\begin{align*}
C \ni z & \mapsto zI \in B, \\
C \ni z & \mapsto zI \in C, \\
C \ni z & \mapsto zI \in B \otimes C,
\end{align*}
\]

by \(\eta_B\), \(\eta_C\) and \(\eta_{B \otimes C}\) respectively. Then we have

\[
(\omega \otimes \text{id}) \circ \Psi_B = \eta_B \circ \omega, \quad (\omega \otimes \text{id}) \circ \Psi_C = \eta_C \circ \omega
\]

and \(\eta_B \otimes \eta_C = \eta_{B \otimes C}\) (with the identification \(C \otimes C = C\)).
Now
\[
(\omega \otimes \text{id}) \circ (\Psi_B \Delta \Psi_C) = (\omega \otimes \text{id}) \circ (\Psi_B \otimes \text{id}) \circ \Psi_C
= \left(\left(\left(\omega \otimes \text{id}\right) \circ \Psi_B\right) \otimes \text{id}\right) \circ \Psi_C
= \left(\left(\eta_B \circ \omega\right) \otimes \text{id}\right) \circ \Psi_C
= (\eta_B \otimes \text{id}) \circ (\omega \otimes \text{id}) \circ \Psi_C
= (\eta_B \otimes \text{id}) \circ \eta_C \circ \omega
= (\eta_B \otimes \eta_B) \circ \omega = \eta_{B \otimes C} \circ \omega,
\]
so that for any \( m \in M \) we have
\[
(\omega \otimes \text{id})(\Psi_B \Delta \Psi_C)(m) = \omega(m)I. \tag{5.1}
\]
One can ask if there are any states on \( M \) preserved by the action of \( \mathcal{Q}\text{-Map}(\mathcal{Q}\mathcal{S}(M)) \), i.e. if there is a state \( \omega \) on \( M \) such that for any \( m \in M \) we have
\[
(\omega \otimes \text{id})\Phi(m) = \omega(m)I. \tag{5.1}
\]
It turns out that, unless \( M \) is one dimensional, there are no such states. This result is proved in the same way as Proposition 4.6.

**Proposition 5.3.** Let \( \omega \) be a state on \( M \). Assume that \( \omega \) is invariant under the action of \( \mathcal{Q}\text{-Map}(\mathcal{Q}\mathcal{S}(M)) \). Then \( M \) is one dimensional.

**Proof.** Let \( \Lambda \in \text{Mor}(A,M) \) be such that for any \( m \in M \) we have \((\text{id} \otimes \Lambda)\Phi(m) = I \otimes m \) (see proof of Proposition 4.6 for the existence of \( \Lambda \)). Applying \( \Lambda \) to both sides of (5.1) we obtain
\[
\Lambda\left((\omega \otimes \text{id})\Phi(m)\right) = \omega(m)\Lambda(I).
\]
The right hand side of this equality is \( \omega(m)I \), while the left hand side is
\[
(\omega \otimes \text{id})(\text{id} \otimes \Lambda)\Phi(m) = (\omega \otimes \text{id})(I \otimes m) = m,
\]
so that \( m = \omega(m)I \) for any \( m \in M \).

In the next theorem we shall describe the quantum subsemigroups of \( \mathcal{Q}\text{-Map}(\mathcal{Q}\mathcal{S}(M)) \) preserving a given state \( \omega \) on \( M \). The strategy is to define the smallest ideal that must be included in the kernel of any morphism from \( A \) to any \( C^* \)-algebra such that the resulting quantum family of maps \( \mathcal{Q}\mathcal{S}(M) \rightarrow \mathcal{Q}\mathcal{S}(M) \) preserves \( \omega \). Then the quotient of \( A \) by this ideal gives a quantum family of maps \( \mathcal{Q}\mathcal{S}(M) \rightarrow \mathcal{Q}\mathcal{S}(M) \) which is universal for all quantum families preserving \( \omega \). This universality will give a comultiplication on the quotient \( C^* \)-algebra.

**Theorem 5.4.** Let \( \omega \) be a state on \( M \) and let \( J \) be the ideal generated by the set
\[
\{(\omega \otimes \text{id})\Phi(m) - \omega(m)I \mid m \in M\}.
\]
Let \( \hat{A} \) be the quotient \( A/J \), let \( \pi : A \rightarrow \hat{A} \) be the canonical epimorphism and let \( \hat{\Phi} = (\text{id} \otimes \pi) \circ \Phi \).

Then
(1) the state \( \omega \) is invariant for the quantum family \( \hat{\Phi} \in \text{Mor}(\hat{A},\hat{A}) \) of maps \( \mathcal{Q}\mathcal{S}(M) \rightarrow \mathcal{Q}\mathcal{S}(M) \) labeled by \( \mathcal{Q}\mathcal{S}(\hat{A}) \).
(2) For any \( C^* \)-algebra \( B \) and any quantum family \( \Psi \in \text{Mor}(\hat{A},\hat{A} \otimes B) \) of maps \( \mathcal{Q}\mathcal{S}(M) \rightarrow \mathcal{Q}\mathcal{S}(M) \) labeled by \( \mathcal{Q}\mathcal{S}(B) \) such that \( \omega \) is invariant for \( \Psi \) there exists a unique morphism \( \Lambda \in \text{Mor}(\hat{A},\hat{A} \otimes B) \) such that \( \Psi = (\text{id} \otimes \Lambda) \circ \hat{\Phi} \).
(3) There exists a unique \( \Delta \in \text{Mor}(\hat{A},\hat{A} \otimes \hat{A}) \) such that
\[
(\hat{\Phi} \otimes \text{id}) \circ \hat{\Phi} = (\text{id} \otimes \Delta) \circ \hat{\Phi}. \tag{5.2}
\]
(4) The morphism \( \Delta \) is coassociative and \( (\hat{A},\Delta) \) is a compact quantum semigroup with unit. \( \hat{\Phi} \) is an action of \( (\hat{A},\Delta) \) on \( \mathcal{Q}\mathcal{S}(M) \).
(5) \( \pi \) is a quantum semigroup morphism.
(6) For any quantum semigroup \( (B,\Delta_B) \) acting on \( \mathcal{Q}\mathcal{S}(M) \) with action \( \Phi_B \in \text{Mor}(\hat{A},\hat{A} \otimes B) \) preserving \( \omega \), the unique morphism \( \Lambda \in \text{Mor}(\hat{A},\hat{A} \otimes B) \) such that \( \Phi_B = (\text{id} \otimes \Lambda) \circ \hat{\Phi} \) is a quantum semigroup morphism.
Proof. Statement (1) is almost obvious. Since \( \pi \) sends each element of the form
\[
(\omega \otimes \text{id})\Phi(m) - \omega(m)I
\]
to 0, we see that we have
\[
(\omega \otimes \text{id})\Phi(m) = \omega(m)I
\]
for all \( m \in M \).

Let \( B \) be a \( C^* \)-algebra and let \( \Psi \in \text{Mor}(\mathcal{M}, M \otimes B) \) be a quantum family of maps \( \mathcal{Q}(\mathcal{M}) \rightarrow \mathcal{Q}(\mathcal{M}) \) preserving \( \omega \). Then there is a unique map \( \Lambda_0 \in \text{Mor}(A, B) \) such that \( (\text{id} \otimes \Lambda_0) \circ \Phi = \Psi \). Moreover all elements of the form (5.3) are mapped to 0 by \( \Lambda_0 \). Therefore \( J \subset \ker \Lambda_0 \). This guarantees that there is a unique \( \Lambda \in \text{Mor}(\bar{A}, B) \) such that \( \Lambda \circ \pi = 0 \). This means that
\[
(id \otimes \Lambda) \circ \Phi = (id \otimes \Lambda) \circ (id \otimes \pi) \circ \Phi = (id \otimes \Lambda_0) \circ \Phi = \Psi
\]
and Statement (2) is proven.

To prove statement (3) note that by Proposition 4.7 the composition \( \Phi \circ \tilde{\Phi} \) is a quantum family of maps \( \mathcal{Q}(\mathcal{M}) \rightarrow \mathcal{Q}(\mathcal{M}) \) labeled by \( \mathcal{Q}(\bar{A} \otimes \bar{A}) \) for which \( \omega \) is invariant. Therefore there exists a unique \( \Delta \in \text{Mor}(\bar{A} \otimes \bar{A}, \bar{A}) \) such that
\[
(id \otimes \Delta) \circ \tilde{\Phi} = \tilde{\Phi} \circ \tilde{\Phi}.
\]
By definition of \( \tilde{\Phi} \) we have
\[
(id \otimes \Delta) \circ \tilde{\Phi} = (\tilde{\Phi} \otimes \text{id}) \circ \tilde{\Phi}.
\]
In fact (5.2) is equivalent to (5.4), so that \( \Delta \) is unique.

The proof that \( \bar{A}, \Delta \) is a quantum semigroup with unit can be copied verbatim from that of Theorem 4.4 and supplying \( \Phi \)'s and \( \Delta \)'s with dots. The necessary facts for this are:
- the uniqueness of \( \Lambda \) in Statement (2),
- \( \bar{A} \) is generated by \( \{ (\eta \otimes \text{id})\Phi(m) | m \in M, \eta \in M^* \} \) (this is clearly seen from the analogous property for \( A \) and the fact that \( \pi \) is an epimorphism).

The fact that \( \tilde{\Phi} \) is an action is obvious in view of (5.2).

Statement (5) follows from Proposition 4.7. The proof of (6) is analogous to the proof of Proposition 4.7 Again we need to use uniqueness of \( \Lambda \) stated in (2). \( \square \)

The quantum semigroup \( \bar{A}, \Delta \) constructed in Theorem 5.4 will be denoted by the symbol \( \mathcal{Q}\text{-Map}^\omega(\mathcal{Q}(\mathcal{M})) \). This is the quantum semigroup of all maps \( \mathcal{Q}(\mathcal{M}) \rightarrow \mathcal{Q}(\mathcal{M}) \) preserving \( \omega \).

Remark 5.5.

1. In Theorem 5.4 we obtained the comultiplication on \( \bar{A} \) through its universal property. However it is possible to show directly that
\[
\Delta(J) \subset J \otimes A + A \otimes J,
\]
which gives a unique \( \Delta \) on \( \bar{A} \) such that \( \pi \) is a quantum semigroup morphism. This can be seen for example by choosing a basis \( (m_k)_{k=1,...,N} \) of \( M \) and denoting by \( (a_{kl})_{k,l=1,...,N} \) the elements of \( A \) such that \( \Phi(m_l) = \sum_{k=1}^N m_k \otimes a_{kl} \). Then the ideal \( J \) is generated by the elements
\[
X_l = \sum_{k=1}^N \omega(m_k)a_{kl} - \omega(m_l)I. \text{ One finds that } \Delta(X_l) = \sum_{p=1}^N X_p \otimes a_{pl} + I \otimes X_l \text{ and this suffices for (5.5).}
\]
2. Since \( J \subset \ker \epsilon \), the counit of \( \bar{A}, \Delta \) composed with \( \pi \) is \( \epsilon \).
3. From the fact that \( \pi \) is an epimorphism and Theorem 3.3 it follows that the set
\[
\{(\eta \otimes \text{id})\Phi(m) | m \in M, \eta \in M^* \}
\]
generates \( \bar{A} \) as a \( C^* \)-algebra.
4. The construction of \( \mathcal{Q}\text{-Map}^\omega(\mathcal{Q}(\mathcal{M})) \) can be performed for any \( \omega \in M^* \), not necessarily a state. Theorem 5.4 remains true in that situation.
6. Quantum Commutants

In this section we shall use the notation introduced in Section 4. Let us consider two C*-algebras $B$ and $C$ and two quantum families

$$\Psi_B \in \text{Mor}(M, M \otimes B), \quad \Psi_C \in \text{Mor}(M, M \otimes C)$$

of maps $\mathcal{Q}S(M) \to \mathcal{Q}S(M)$ labeled by $\mathcal{Q}S(B)$ and $\mathcal{Q}S(C)$ respectively. We shall say that $\Psi_B$ commutes with $\Psi_C$ if

$$(\text{id} \otimes \sigma_{B,C})(\Psi_B \triangle \Psi_C)(m) = (\Psi_C \triangle \Psi_B)(m), \quad (6.1)$$

where $\sigma_{B,C} \in \text{Mor}(B \otimes C, C \otimes B)$ is the flip.

This is a straightforward generalization of the notion of commutation of classical families of maps. Note that $\Psi_B$ commutes with $\Psi_C$ if and only if $\Psi_C$ commutes with $\Psi_B$.

The most basic properties of this notion will be analyzed in the next proposition. In its formulation we use the concept of a trivial quantum family of maps defined in Definition 3.1 (3).

**Proposition 6.1.** Let $B$ be a C*-algebra and let $\Psi_B \in \text{Mor}(M, M \otimes B)$ be a quantum family of maps $\mathcal{Q}S(M) \to \mathcal{Q}S(M)$ labeled by $\mathcal{Q}S(B)$. Then

1. if $\Psi_B$ is trivial then any quantum family of maps $\mathcal{Q}S(M) \to \mathcal{Q}S(M)$ commutes $\Psi_B$.
2. If $\Psi_B$ commutes with $\Phi \in \text{Mor}(M, M \otimes A)$ then $\Psi_B$ is trivial.

**Proof.** Ad (1). Let $C$ be a C*-algebra and let $\Psi_C \in \text{Mor}(M, M \otimes C)$ be a quantum family of maps $\mathcal{Q}S(M) \to \mathcal{Q}S(M)$. Since $\Psi_B$ is assumed to be trivial, we have

$$(\text{id} \otimes \sigma_{C,B})(\Psi_C \triangle \Psi_B)(m) = (\Psi_C \triangle \Psi_B)(m)$$

for any $m \in M$.

Ad (2). Let $\Lambda \in \text{Mor}(A, M)$ be the unique morphism such that $(\text{id} \otimes \Lambda)(m) = I \otimes m$ for all $m \in M$ (see proofs of Propositions 4.6 and 5.3). Let us apply $(\text{id} \otimes \Lambda \otimes \text{id})$ to both sides of

$$(\text{id} \otimes \sigma_{B,A})(\Psi_B \triangle \Phi)(m) = (\Phi \triangle \Psi_B)(m)$$

we obtain $I \otimes m \otimes I = I \otimes \Psi_B(m)$. Since $m$ is arbitrary, $\Psi_B$ must be trivial. \qed

If two quantum families of maps $\mathcal{Q}S(M) \to \mathcal{Q}S(M)$ commute with a third one, then so does their composition.

**Proposition 6.2.** Let $B$, $B'$ and $C$ be C*-algebras and let

$$\Psi_B \in \text{Mor}(M, M \otimes B), \quad \Psi_{B'} \in \text{Mor}(M, M \otimes B'), \quad \Psi_C \in \text{Mor}(M, M \otimes C)$$

be quantum families of maps $\mathcal{Q}S(M) \to \mathcal{Q}S(M)$ labeled by $\mathcal{Q}S(B)$, $\mathcal{Q}S(B')$ and $\mathcal{Q}S(C)$ respectively. Assume that $\Psi_B$ commutes with $\Psi_C$ and that $\Psi_{B'}$ commutes with $\Psi_C$. Then $\Psi_B \triangle \Psi_{B'}$ commutes with $\Psi_C$.

**Proof.** In the following computation we shall use the fact that

$$\sigma_{B \otimes B',C} = (\sigma_{B,C} \otimes \text{id}) \circ (\text{id} \otimes \sigma_{B',C}).$$
Now
\[(id \otimes \sigma_{B \otimes B', C}) \circ ((\Psi_B \Delta \Psi_{B'}) \Delta \Psi_C) = (id \otimes \sigma_{B \otimes B', C}) \circ ((\Psi_B \otimes id) \otimes \Psi_{B'} \otimes \Psi_C)
= (id \otimes \sigma_{B \otimes B', C}) \circ ((\Psi_B \otimes id) \otimes (\Psi_{B'} \otimes id) \otimes \Psi_C)
= (id \otimes \sigma_{B, C \otimes id}) \circ (\Psi_B \otimes id) \otimes (\Psi_{B'} \otimes id) \otimes (\Psi_{B'} \Delta \Psi_C)
= (id \otimes \sigma_{B, C \otimes id}) \circ (\Psi_B \otimes id) \otimes (\Psi_{B'} \otimes id) \otimes (\Psi_{C} \Delta \Psi_{B'})
= (id \otimes \sigma_{B, C \otimes id}) \circ (\Psi_B \otimes id) \otimes (\Psi_{C} \otimes id) \otimes \Psi_{B'}
= ([id \otimes \sigma_{B, C}] \circ (\Psi_{B} \otimes id) \otimes \Psi_{C}] \otimes id) \otimes \Psi_{B'}
= ([id \otimes \sigma_{B, C}] \circ (\Psi_B \Delta \Psi_C)] \otimes id) \otimes \Psi_{B'}
= ([\Psi_C \Delta \Psi_{B}] \otimes id) \otimes \Psi_{B'}
= (\Psi_{C} \Delta \Psi_{B}) \Delta \Psi_{B'} = \Psi_{C} \Delta (\Psi_{B} \Delta \Psi_{B'})
\]
means that $\Psi_B \Delta \Psi_{B'}$ commutes with $\Psi_C$.

\[\Box\]

Remark 6.3. The notion of commuting quantum families of maps can be generalized further by introducing braiding. The point is that instead of the flip automorphism, say, $\Sigma_{B,C}$ (see e.g. [5, Section 9.2]). Propositions 6.1 and 6.2 remain valid, but commutation fails to be a symmetric relation unless $\Sigma_{B,C} = \Sigma_{C,B}$

Equipped with the conclusion of Proposition 6.2 we can now easily proceed to construct the quantum family of all maps $\Omega S(M) \to \Omega S(M)$ commuting with a given quantum family. Then we find that it has a structure of a quantum semigroup. The strategy is very similar to the one used in construction of $\Omega$-Map" of $\Omega S(M)$ given in Section 5

Theorem 6.4. Let $B$ be a $C^*$-algebra and let $\Psi_B \in Mor(M, M \otimes B)$ be a quantum family of maps $\Omega S(M) \to \Omega S(M)$ labeled by $\Omega S(B)$. Let $K$ be the ideal of $A$ generated by
\[\{(\omega \otimes id \otimes \eta)(\Delta \Psi_B)(m) - (\omega \otimes \eta \otimes id)(\Psi_B \Delta \Phi)(m) | m \in M, \omega \in M^*, \eta \in B^*\}\]

Let $\bar{A}$ be the quotient $A/K$, let $\rho : A \to \bar{A}$ be the canonical epimorphism and let $\bar{\Phi} = (id \otimes \rho) \circ \Phi$. Then

1. the quantum family $\bar{\Phi} \in Mor(\bar{A}, \bar{A} \otimes \bar{A})$ of maps $\Omega S(\bar{A}) \to \Omega S(\bar{A})$ labeled by $\Omega S(\bar{A})$
2. commutes with the quantum family $\Psi_B$.

2. For any $C^*$-algebra $C$ and any quantum family $\Psi_C \in Mor(M, M \otimes C)$ of maps $\Omega S(M) \to \Omega S(M)$ which commutes with $\Psi_B$ there exists a unique $\Lambda \in Mor(\bar{A}, C)$ such that $\Psi_C = (id \otimes \Lambda) \circ \bar{\Phi}$.

3. There exists a unique $\bar{\Delta} \in Mor(\bar{A}, \bar{A} \otimes \bar{A})$ such that
\[(\bar{\Phi} \otimes id) \circ \bar{\Phi} = (id \otimes \bar{\Delta}) \circ \bar{\Phi}.
\]

4. The morphism $\bar{\Delta}$ is coassociative and $(\bar{A}, \bar{\Delta})$ is a compact quantum semigroup with unit. $\bar{\Phi}$ is an action of $(\bar{A}, \bar{\Delta})$ on $\Omega S(\bar{A})$.

5. $\rho$ is a quantum semigroup morphism.

6. For any quantum semigroup $(C, \Delta_C)$ acting on $\Omega S(M)$ with action $\Phi_C \in Mor(M, M \otimes C)$ commuting with $\Psi_B$, the unique morphism $\Lambda \in Mor(\bar{A}, C)$ such that $\Phi_C = (id \otimes \Lambda) \circ \bar{\Phi}$ is a quantum semigroup morphism.

In the proof we will only indicate the main steps. The details are practically identical to those in the proofs of Theorems 4.1 and 5.3 as well as Proposition 4.4.

Proof of Theorem 6.4. Clearly $\bar{\Phi} \in Mor(M, M \otimes \bar{A})$ is a quantum family of maps commuting with $\Psi_B$ and it is universal in the sense of Statement (2). Therefore, by Proposition 6.2 there exists a
unique morphism $\Delta \in \text{Mor}(\bar{A}, \bar{A} \otimes \bar{A})$ such that

$$(\text{id} \otimes \Delta) \circ \Phi = \Phi \Delta = (\Phi \otimes \text{id}) \circ \Phi.$$  

The coassociativity of $\Delta$ follows exactly as in the proof of Theorem 4.1(2). We use the fact that

$$\{((\eta \otimes \text{id})\Phi(m)) \mid m \in M, \eta \in M^*\}$$

generates $\bar{A}$ ($\rho$ is an epimorphism). Thus $(\bar{A}, \bar{\Delta})$ is a compact quantum group with unit and $\Phi$ is its action on $\mathcal{Q}(M)$ which, by construction, commutes with the quantum family of maps $\Psi_B$.

Statement (1) follows from Proposition 4.7 and (6) can be proved along the same lines as that proposition.

The quantum semigroup $(\bar{A}, \bar{\Delta})$ constructed in Theorem 5.4 will be denoted by the symbol $\mathcal{Q}(\text{Map}_{\Psi_B}(\mathcal{Q}(M)))$.

7. Remarks on quantum groups inside quantum semigroups

Let $M$ be, as in previous sections, a finite dimensional $C^*$-algebra. We have constructed the compact quantum semigroup $\mathcal{Q}(\text{Map}(\mathcal{Q}(M)))$ and have shown how to find quantum sub-semigroups $\mathcal{Q}(\text{Map}^{\omega}(\mathcal{Q}(M)))$ for a state $\omega$ and $\mathcal{Q}(\text{Map}_{\Psi_B}(\mathcal{Q}(M)))$ for a given quantum family $\Psi_B \in \text{Mor}(M, M \otimes B)$ of maps $\mathcal{Q}(M) \rightarrow \mathcal{Q}(M)$ labeled by $\mathcal{Q}(B)$. An interesting question arises: when are the quantum semigroups $\mathcal{Q}(\text{Map}^{\omega}(\mathcal{Q}(M)))$ or $\mathcal{Q}(\text{Map}_{\Psi_B}(\mathcal{Q}(M)))$ compact quantum groups?

Let us note that the quantum groups of automorphisms of finite spaces considered by S. Wang in [8] are exactly the semigroups $\mathcal{Q}(\text{Map}^{\omega}(\mathcal{Q}(M)))$ for $\omega$ the canonical trace (see paragraph following the proof of Theorem 7.3). Considering these examples one is quickly led to a conjecture that in general, $\mathcal{Q}(\text{Map}^{\omega}(\mathcal{Q}(M)))$ is a compact quantum group if $\omega$ is faithful. Similarly one expects a compact quantum group structure on $\mathcal{Q}(\text{Map}_{\Psi_B}(\mathcal{Q}(M)))$ for an ergodic family $\Psi_B$.

We will give here a short argument that a compact quantum semigroup acting on $\mathcal{Q}(M)$, preserving a faithful state $\omega$ and satisfying a condition corresponding to (2) form Theorem 4.3 (cf. Remark 5.3), has cancellation from the left (see e.g. [13] Remark 3]). Moreover, one natural condition (cf. Remark 5.3) immediately guarantees that it also has cancellation from the right and is thus a compact quantum group. We will first concentrate on the case when $\omega$ is a trace. This is the case encountered in the construction of S. Wang’s quantum automorphism groups of finite spaces ([8]) which we will present within the framework of quantum families of maps. Then we will present the result for a general faithful state.

Before we procede let us recall that if $(B, \Delta_B)$ is a compact quantum semigroup then an n-dimensional representation of $(B, \Delta_B)$ is an $n \times n$-matrix $V = (v_{k,l})_{k,l=1,\ldots,n}$ of elements of $B$ such that

$$\Delta_B(v_{k,l}) = \sum_{r=1}^n v_{k,r} \otimes v_{r,l}. \quad (7.1)$$

We can also consider $V$ as an element in $M_n \otimes B$ (where $M_n$ is the $n \times n$-matrix algebra).

If $V \in M_n \otimes B$ and $W \in M_k \otimes B$ are representations of $(B, \Delta_B)$ of dimensions $n$ and $k$ respectively, then the tensor product $V \otimes W$ of $V$ and $W$ is the $nk$-dimensional representation

$$V \otimes W = V_{13}W_{23} \in M_{nk} \otimes B.$$  

The matrix elements of $V \otimes W$ are all possible products of a matrix element of $V$ and a matrix element of $W$. Note also that if $V$ and $W$ satisfy $V^*V = I$ and $W^*W = I$ then $(V \otimes W)^* (V \otimes W) = I$. Similarly $VV^* = I$ and $WW^* = I$ imply $(V \otimes W)(V \otimes W)^* = I$.

Our argument will on purpose be made in terms of matrices over $C^*$-algebras instead of adjointable maps of Hilbert $C^*$-modules. We hope that this will make it more transparent to a reader not familiar with these concepts.

**Lemma 7.1.** Let $(B, \Delta_B)$ be a quantum semigroup. Let $V = (v_{k,l})_{k,l=1,\ldots,n} \in M_n \otimes B$ be an n-dimensional representation of $(B, \Delta_B)$ such that $V^*V = I$. Then for any $c \in B$ and any $k,l \in \{1,\ldots,n\}$ the element $c \otimes v_{k,l}$ belongs to the linear span of the set $\{(a \otimes I) \Delta_B(b) \mid a, b \in B\}$. 

Proof. Equation (7.1) can be rewritten in matrix form as

\[
\begin{pmatrix}
\Delta_B(v_{1,1}) & \cdots & \Delta_B(v_{1,n}) \\
\vdots & \ddots & \vdots \\
\Delta_B(v_{n,1}) & \cdots & \Delta_B(v_{n,n})
\end{pmatrix}
= \begin{pmatrix}
v_{1,1} \otimes I & \cdots & v_{1,n} \otimes I \\
\vdots & \ddots & \vdots \\
v_{n,1} \otimes I & \cdots & v_{n,n} \otimes I
\end{pmatrix}
\begin{pmatrix}
I \otimes v_{1,1} & \cdots & I \otimes v_{1,n} \\
\vdots & \ddots & \vdots \\
I \otimes v_{n,1} & \cdots & I \otimes v_{n,n}
\end{pmatrix}
\]

Multiplying this equality from the left by

\[
\begin{pmatrix}
v_{1,1} \otimes I & \cdots & v_{n,1} \otimes I \\
\vdots & \ddots & \vdots \\
v_{1,n} \otimes I & \cdots & v_{n,n} \otimes I
\end{pmatrix}
\]

we get all elements of the form \( I \otimes v_{k,l} \) as linear combinations of elements of \((B \otimes I)\Delta_B(B)\).

In the same way we can prove the following lemma:

**Lemma 7.2.** Let \( (B, \Delta_B) \) be a quantum semigroup. Let \( V = (v_{k,l})_{k,l=1,...,n} \in M_n \otimes B \) be an \( n \)-dimensional representation of \((B, \Delta_B)\) such that \( VV^* = I \). Then for any \( c \in B \) and any \( k,l \in \{1,...,n\} \) the element \( v_{k,l} \otimes c \) belongs to the linear span of the set \( \{ \Delta_B(a)(I \otimes b) \mid a, b \in B \} \).

**Theorem 7.3.** Assume that \( \omega \) is a faithful trace and let \((B, \Delta_B)\) be a compact quantum semigroup acting on \( QS(M) \) with action \( \Phi_B \) and that

\[
\{(\eta \otimes \text{id})\Phi_B(m) \mid m \in M, \eta \in M^*\}
\]

generates \( B \) as a C*-algebra. Assume further that \( \omega \) is invariant for \( \Phi_B \). Then \((B, \Delta_B)\) has cancellation from the left, i.e., \( \{(a \otimes I)\Delta_B(b) \mid a, b \in B\} \) is linearly dense in \( B \otimes B \).

If, moreover, the set

\[
\{\Phi_B(m)(I \otimes a) \mid m \in M, a \in B\}
\]

is linearly dense in \( M \otimes B \) then \((B, \Delta_B)\) has cancellation from the right, i.e., the linear span of \( \{\Delta_B(a)(I \otimes b) \mid a, b \in B\} \) is dense in \( B \otimes B \).

**Proof.** Let \((m_i)_{i=1,...,n}\) be a basis of \( M \) which is orthonormal for the scalar product given by \( \omega \). Let \( \tilde{a} = (a_{k,l})_{k,l=1,...,n} \) be the matrix of elements of \( B \) such that

\[
\Phi_B(m_i) = \sum_{k=1}^n m_k \otimes a_{k,i}.
\]

Then \( \tilde{a} \) is an \( n \)-dimensional representation of \((B, \Delta_B)\).

The invariance of \( \omega \) for \( \Phi_B \) implies that \( \tilde{a} \) is an isometry:

\[
\delta_{l,t}I = \omega(m^*_l m_t)I = (\omega \otimes \text{id})\Phi_B(m^*_l m_t) = (\omega \otimes \text{id})\left(\Phi_B(m_l)^*\Phi_B(m_t)\right)
\]

\[
= (\omega \otimes \text{id})\left(\sum_k m^*_k \otimes a_{k,l}^*\right)\left(\sum_s m_s \otimes a_{s,t}\right)
\]

\[
= (\omega \otimes \text{id})\left(\sum_{k,s} m^*_k m_s \otimes a_{k,l}^* a_{s,t}\right)
\]

\[
= \sum_{k,s} \omega(m^*_k m_s)a_{k,l}^* a_{s,t} = \sum_{k,s} \delta_{k,s}a_{k,l}^* a_{s,t}
\]

\[
= \sum_k a_{k,l}^* a_{k,t} = [\tilde{a}^* \tilde{a}]_{l,t}
\]

The fact that \( \omega \) is a trace means that \((m^*_i)_{i=1,...,n}\) is also an orthonormal basis of \( M \) for the scalar product given by \( \omega \). Therefore the matrix \( \tilde{b} \) with \( a_{k,l}^* \) as the \((k, l)\)-entry, is also an \( n \)-dimensional representation of \((B, \Delta_B)\) and an isometry.
Let \( V \) be a representation of \((B, \Delta_B)\) constructed as a finite tensor product of representations \( \tilde{a} \) and \( \tilde{b} \) (in any order). Then \( V \) satisfies the conditions in Lemma 7.1 and so, for any matrix element \( x \) of \( V \) and any \( c \in B \), the element \( c \otimes x \) belongs to \((B \otimes I)\Delta_B(B)\).

Clearly any monomial in matrix elements of \( \tilde{a} \) and \( \tilde{b} \) is a matrix element of \( V \) of considered form. Our assumption that \( B \) is generated by (7.2) can be rephrased as saying that the span of all monomials in matrix elements of \( \tilde{a} \) and \( \tilde{b} \) is dense in \( B \). This ends the proof that \((B, \Delta_B)\) has cancellation from the left.

Now let us note that the closure of the span of (7.3) is the image of \( \tilde{a} \) considered as a map of \( M \otimes B \sim = B^N \) into itself. Thus from the linear density of (7.3) in \( M \otimes B \) it follows that \( \tilde{a} \) is unitary and we might as well apply our reasoning to \( \tilde{a}^* \). Using Lemma 7.2 we arrive at the conclusion that \((B, \Delta_B)\) as cancellation from the right. \( \square \)

**Remark 7.4.** Condition that (7.2) generated \( B \) of Theorem 7.3 is there to ensure that the action of \((B, \Delta_B)\) on \( QS(M) \) is in some sense faithful. One can imagine a large quantum semigroup acting via a quotient group and (7.2) excludes such a situation. The condition of density of 7.3 has been used before in connection with quantum group actions. It has been introduced by P. Podleś in his thesis [7, Definicja 2.2].

Let us now see how the quantum permutation groups of S. Wang ([8]) fit into the framework described above and in Section 5. In [8] S. Wang described quantum automorphism groups of finite spaces. For each natural \( n \) Wang considered the finite space \( X_n = \{1, \ldots, n\} = QS(C^n) \) and defined the quantum group \( \text{Aut}(X_n) = (A, \Delta) \) where \( A \) is the \( C^* \)-algebra generated by elements \( \{a_{ij} | i, j = 1, \ldots, n\} \) (7.6) with relations

\[
\begin{align*}
(a_{ij})^2 &= a_{ij}, & i, j = 1, \ldots, n, \\
(a_{ij})^* &= a_{ij}, & i, j = 1, \ldots, n, \\
\sum_{j=1}^{n} a_{ij} &= I, & i = 1, \ldots, n, \\
\sum_{i=1}^{n} a_{ij} &= I, & j = 1, \ldots, n.
\end{align*}
\]

(7.7a) \hspace{1cm} (7.7b) \hspace{1cm} (7.7c) \hspace{1cm} (7.7d)

It turns out that

\[ \text{Aut}(X_n) = Q\text{-Map}^\psi(X_n), \]

where \( \psi \) is the uniformly distributed probability measure on \( X_n \). Indeed, if we let \( \tilde{A} \) be the \( C^* \)-algebra generated by (7.6) with relations (7.7a), (7.7b) and (7.7c) then \( QS(\tilde{A}) \) is the quantum space of all maps \( X_n \to X_n \). The comultiplication

\[ \tilde{\Delta} : a_{ij} \mapsto \sum_{k=1}^{n} a_{ik} \otimes a_{kj} \]

coincides with the one described in Theorem 6.11. In other words

\[ (A, \tilde{\Delta}) = Q\text{-Map}(X_n). \]

Now the relation (7.7d) defines the ideal \( J \) related to \( \psi \) as in Theorem 5.4 and the comultiplication \( \Delta \) on \( A \) is given by the same formula as \( \tilde{\Delta} \). All this shows that (7.8) holds.

The action \( \Phi \) of \((A, \Delta)\) on \( X_n \) is given by

\[ \Phi(e_j) = \sum_{i=1}^{n} e_i \otimes a_{ij}, \]

where \( \{e_1, \ldots, e_n\} \) is the standard basis of \( \mathbb{C}^n = C(X_n) \). Now we will show that \( Q\text{-Map}^\psi(X_n) \) is in fact a compact quantum group and not merely a quantum semigroup. First let us note that
the density condition (7.9) is satisfied by Remark 5.3 [3] (with $M = \mathbb{C}^n$). Moreover linear density of (7.3) in $\mathbb{C}^n \otimes A$ also holds. This follows from the fact that we have
generates $B$ as a $C^*$-algebra and the set
is linearly dense in $M \otimes B$ then $(B, \Delta_B)$ is a compact quantum group.

The proof is almost the same as that of Theorem 7.3 with the difference that the basis $(m_i^*)_{l=1,\ldots,n}$ is not orthonormal. However the matrix $\tilde{a}$ with matrix element $(a_{i,j})_{i,j=1,\ldots,n}$ defined by (7.4) still is an isometry. The density condition (7.9) guarantees that $\tilde{a}$ is unitary. Let $\overline{a}$ be the matrix with elements $(a_{i,j})_{i,j=1,\ldots,n}$. Then $\overline{a}$ is not isometric. Still, by elementary linear algebra, there exists an invertible map $\sigma : M \rightarrow M$ such that

$$\omega(yx) = \omega(y \sigma(x))$$

for all $x, y \in M$. Let $S = (s_{i,j})_{i,j=1,\ldots,n}$ be the matrix of $\sigma$ in the basis $(m_i)_{i=1,\ldots,n}$.

$$\omega(m_i m_j^*) = \omega(m_j^* \sigma(m_i)) = \sum_{p=1}^n s_{i,p} \omega(m_p m_j^*) = s_{i,j}.$$ 

Therefore the computation (7.5) shows that

$$s_{i,j} I = \omega(m_i m_j^*) I = \sum_{p,q=1}^n \omega(m_p m_q^*) a_{p,i} a_{q,j}^* = \sum_{p,q=1}^n a_{p,i} s_{p,q} a_{q,j}^* = [\overline{a} \overline{(S \otimes I) a}]_{i,j}$$

1 Any family $\{p_1, \ldots, p_n\}$ of projections such that $\sum_{k=1}^n p_k = I$ must satisfy $p_k p_l = \delta_{k,l} p_k$ for $k, l = 1, \ldots, n$. 

Therefore $m \otimes a \in$ belongs to the span of (7.4) for any $m \in \mathbb{C}^n$ and any $a \in A$.

All this shows that the assumptions of Theorem 7.3 are satisfied and $(A, \Delta) = \text{Q-Map}^\psi(X_n)$ is a compact quantum group. Let us emphasize that this has already been established in [8]. The point of the above argument is that the existence of a quantum group structure on this quantum space is a consequence of its more fundamental properties, namely that $\text{QS}(A)$ is the universal quantum family of maps $X_n \rightarrow X_n$ preserving the measure $\psi$.

The statement of Theorem 7.3 gives explicit relationship between cancellation laws and density conditions. However the conclusion that $(B, \Delta_B)$ is a compact quantum group does not require that $\omega$ be a trace. In fact we have

**Theorem 7.5.** Let $(B, \Delta_B)$ be a quantum semigroup and let $\Phi_B \in \text{Mor}(M, M \otimes B)$ be an action of $(B, \Delta_B)$ on a finite dimensional $C^*$-algebra $M$. Let $\omega$ be an invariant faithful state on $M$. Then if

$$\{ (\eta \otimes \text{id}) \Phi_B(m) \big| m \in M, \eta \in M^* \}$$

generates $B$ as a $C^*$-algebra and the set

$$\{ \Phi_B(m)(I \otimes a) \big| m \in M, a \in B \}$$

(7.9)

is linearly dense in $M \otimes B$ then $(B, \Delta_B)$ is a compact quantum group.
or in other words
\[
S \otimes I = \overline{\overline{a}} (S \otimes I) \overline{\overline{a}}.
\] (7.10)

Multiplying (7.10) from the left by \(S^{-1} \otimes I\) we see that \(\overline{\overline{a}}\) is left invertible. However, as a map \(B^n \to B^n\) the matrix \(\overline{\overline{a}}\) has dense range because of the density condition (7.9). It follows that \(\overline{\overline{a}}\) is invertible, and thus so is
\[
\overline{\overline{a}}^\top = \overline{\overline{a}}.
\]

Therefore \(B\) is a \(C^*\)-algebra generated by elements of a unitary matrix \(\overline{\overline{a}} = (a_{ij})_{i,j=1,...,n}\) with a morphism \(\Delta_B \in \text{Mor}(B,B \otimes B)\) satisfying
\[
\Delta_B(a_{ij}) = \sum_{p=1}^n a_{ip} \otimes a_{pj}
\]
(this follows from (7.4) and the fact that \(\Phi_B\) is an action) and such that \(\overline{\overline{a}}^\top\) is invertible. By the results of [11] \((B,\Delta_B)\) is a compact quantum group.

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