Bifurcation and chaos of BNNT-reinforced piezoelectric plate under complex load

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ABSTRACT

By employing piezoelectric theory with thermal effects and von Kármán nonlinear plate theory, the constitutive equations of the boron nitride nanotube (BNNT)-reinforced piezoelectric plate under complex load are set up. The material constants are calculated by using the "XY" rectangle model. Referring to the Reissner variational principle, the nonlinear motion governing equations of the structure are derived and resolved by the fourth-order Runge–Kutta method. The numerical results show that decreasing voltage and temperature and increasing volume ratio can delay the chaotic or multiple periodic motions of BNNT-reinforced piezoelectric plates, thus improving the dynamic stability of the structure.

KEYWORDS: BNNT-reinforced piezoelectric plate, Reissner variational principle, bifurcation and chaos, Runge–Kutta method

1. INTRODUCTION

Boron nitride nanotubes (BNNTs) are structurally similar to carbon nanotubes (CNTs), and both have extraordinary mechanical properties. However, BNNTs have stronger piezoelectric characteristics and higher oxidation resistance. Furthermore, BNNTs also have stable semiconductor properties. Hence, BNNTs have attracted considerable attention and are extensively used as reinforcing fillers to improve the properties of materials. However, it is worth noting that the number of research works on BNNT-reinforced composite (BNNTRC) piezoelectric structures is limited and most of them only consider the static problem. Therefore, it is necessary to research the dynamic characteristics of the structure more deeply.

At present, the dynamic behavior of CNT-reinforced composites (CNTRCs) has been studied extensively. Gholami and Ansari [1] presented the effect of initial geometric defects on the nonlinear dynamical characteristics of functionally graded CNTRC rectangular plates. Using the first-order shear deformation theory of shells, Mohammadimehr and Rostami [2] examined the bending and free vibrations of a rotating sandwich cylindrical shell with nanocomposite core and piezoelectric layers under the action of thermal and magnetic fields. Civalek and Baltaciglu [3] presented the numerical solution and modeling of free vibrations of the annular sector plate of CNTRCs by the discrete singular convolution method. Nonlinear vibration of CNTRC plates exposed to the parametric and forced excitations was improved by Guo and Zhang [4]. Jorge and Rajamohan [5] investigated a 3D multiscale finite element model and exploited dynamic response of polymer material reinforced with CNTs. In view of Donald shell theory and the effective model of multiwalled CNTs, Wang et al. [6] provided an analytical method for studying the dynamic stability of composites reinforced with multiwalled CNTs. Moumen et al. [7] investigated polymer composite impact responses with random distributions of CNTs. Mohandes and Ghasemi [8] evaluated free vibration behavior of thin cylindrical shells made of single-walled CNT-reinforced fiber–metal composites. Fiorenzo and Fazzo [9] investigated stability and thermoelastic vibrations of CNTRC structures in thermal environments. Considering the agglomeration effect of CNTs, Kamarian et al. [10] discussed free vibration behavior of CNTRC conical shells.

In recent years, BNNTs have attracted an increasing attention of researchers because of their excellent mechanical properties. Based on Cooper–Naghdi theory, Mohammadimehr et al. [11] investigated the vibration of CNTRC/BNNTRC panels and double-bonded micro-sandwich cylindrical shells under various physical loads. Using the finite element method, Rouhi et al. [12] analyzed the elastic properties of concentric boron nitride and carbon multiwalled nanotubes. Through molecular dynamics simulation, Gong et al. [13] discussed the effect of the radial deformation of BNNTs on the application properties of BNNTs. Li et al. [14] performed molecular dynamics simulations to study the mechanical and thermal properties of CNTs and BNNTs. Wang et al. [15] used density functional theory to research the mechanical properties and failure process of boron nitride hybrid nanotubes filled with hexagonal or triangular graphene sheets under axial tensile load. By using the B3LYP/6-31G(d, p) density functional level of theory, the effects of the substitution of the boron or nitrite atom by carbon atom impurity and closing two ends of nanotube have been investigated by Yaghobi et al. [16]. Zeighampour and Beni [17] studied the axial buckling of BNNTs. Nikkar et al. [18] used finite element simulation methods to study the behavior of concentric multiwalled BNNTs and CNTs under compressive loads. Using molecular
dynamics simulation methods, Kundalwal and Choyal [19] predicted the piezoelectric coefficients of BNNTs containing vacancies under the action of Tersoff potential field. Chaudhuri et al. [20] studied a series of zigzag and armchair nanotubes of carbon, boron and nitrogen with various values of tube diameters by density functional theory. Rahmat et al. [21] investigated the quasi-static and dynamic properties of epoxy adhesives with and without BNNTs. Based on nonlocal theory, Khosravi et al. [22] considered the free and forced axial vibrations of sawtooth single-walled CNTs under two different linear and harmonic axial concentrated forces. Zeighmou et al. [23] studied the axial buckling of BNNTs under the action of surface and electric field. Wang et al. [24] used nonresonant Raman spectroscopy to study the stress transfer mechanism of hexagonal BNNTs in the polyvinyl alcohol matrix. Movlarooy and Minaie [25] investigated the electronic and structural properties of single-walled BNNTs with a diameter range of 4–22 Å. Machado et al. [26] reported on the structure and dynamics of BNNTs fired at a high speed at solid targets. Kalay et al. [27] first introduced the synthesis method and surface modification strategy of boron nitride, gradually eliminated the toxicity of boron nitride and summarized the application research. Casanova et al. [28] analyzed for the first time the preparation of a BNNT film capable of repelling nanoparticles smaller than the inner diameter.

In addition, some research works on BNNT-reinforced piezoelectric structures have also appeared. Mosalliaie Barzoki et al. [29] investigated torsional linear buckling of a polyvinylidene fluoride (PVDF) cylindrical shell reinforced with BNNTs, and concluded that the buckling strength is significantly improved through the use of hard core foam. By using most general strain gradient theories (MGSGTs) and sinusoidal shear deformation theory (SSDT), Mohammadimehr et al. [30] studied the buckling and free vibration analysis of microcomposite sandwich plates. An analysis on the chaos of agile beams subjected to the piezoelectric effect and temperature was given by Krysko et al. [31]. Mercan and Civalek [32] proposed a nonlocal continuum mechanical model of BNNTs and studied in detail the influence of some geometric parameters of BNNTs on the bucking performance. In combination with the Eshelby–Mori–Tanaka method and using the modified couple stress theory, Mohammadimehr et al. [33] studied the vibration and buckling of a double-layer nanocomposite piezoelectric plate reinforced by BNNTs under electro-thermo-mechanical loading on an elastic foundation. By SSDT and MGSGTs, Mohammadimehr et al. [34] studied the buckling and free vibrations of CNT-coupled microcomposite sandwich plates reinforced by CNTs and BNNTs. Ghorbanpour Arani et al. [35] analyzed the nonlinear vibration of PVDF by the double-walled BNNT-reinforced polymer piezoelectric rectangular microchannel plate under the electrical–thermal loads. Based on the effects of transverse shear deformation and rotary inertia, Yang and Zhang [36] investigated the nonlinear dynamic response of a piezoelectric plate reinforced with BNNTs under electro-thermo-mechanical loadings. Using the variational principle, a new Timoshenko beam model based on the modified gradient elasticity was developed by Zhao et al. [37]. Ma et al. [38] proposed an inverse research strategy toward the establishment of a contact force model for complex contacting surfaces by utilizing parameter identification methods. Jia et al. [39] studied the electrical and dielectric properties of BNNT-reinforced ceramic composites prepared by the polymer source ceramic process. So far, research on chaos and bifurcation of piezoelectric plates reinforced by BNNTs has not been reported in printed publication.

With this in mind, we plan to research the chaos and bifurcation of the BNNTRC piezoelectric plate subjected to the electro-thermo-mechanical loads. Employing the variational principle, equations of nonlinear motion of plate were gained. Through the introduction of additional state variable and the Galerkin method, the equations of nonlinear motion were converted to FNODE and were resolved employing the fourth-order Runge–Kutta approach. The obtained numerical results were graphically depicted, which showed the influences of volume fraction, temperature and voltage on bifurcation and chaos of the BNNTRC piezoelectric plate.

2. THEORETICAL FORMULATIONS

Figure 1 depicts a BNNTRC piezoelectric plate in a typical coordinate system \((x, y, z)\), \(b\) is the width of the plate, \(h\) is the thickness and \(\rho_0\) is the mass density. In addition, the plate is under uniform temperature rise \(\Delta T\), applied voltage \(V\) and transverse dynamic load \(q(x, y, t)\).

2.1. Strain–displacement relationships

Using von Kármán nonlinear plate theory, it is assumed that \(u_\parallel, v_\parallel, w\) and \(u, v, w\) denote the displacement components of the point along axes \(x, y, z\), and the displacement components of the corresponding point of mid-plane, respectively. Thus, we have

\[
\begin{align*}
\delta u(x, y, z, t) &= u(x, y, t) - zw_x(x, y, t), \\
\delta v(x, y, z, t) &= v(x, y, t) - zw_y(x, y, t), \\
\delta w(x, y, z, t) &= w(x, y, t).
\end{align*}
\]  

(1)

Among them, the subscript \((\_\parallel)\) indicates that the partial derivative is a variable coordinate.

The relationships of nonlinear strain–displacements for the piezoelectric plate are

\[
\begin{align*}
\varepsilon_x &= \varepsilon_x + 2K_x, \\
\varepsilon_y &= \varepsilon_y + 2K_y, \\
\varepsilon_{xy} &= \varepsilon_{xy} + 2K_{xy},
\end{align*}
\]  

(2)

where \(K_x, K_y, K_{xy}\) and \(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}\), respectively, represent curvature, torsion and strain components on the median plane, and

---

**Figure 1** Geometry of a piezoelectric plate reinforced with BNNTs.
Table 1 Mechanical, electrical and thermal properties of PVDF and BNNT.

| PVDF | BNNT |
|------|------|
| $C_{11} = 238.24 \text{ GPa}$, $C_{22} = 23.6 \text{ GPa}$, $C_{12} = 3.98 \text{ GPa}$, $C_{66} = 6.43 \text{ GPa}$ | $E = 1.8 \text{ TPa}$, $\nu = 0.34$, $\epsilon_{11} = 0.95 \text{ C/m}^2$ |
| $\epsilon_{ij} = -0.135 \text{ C/m}^2$, $\epsilon_{12} = -0.145 \text{ C/m}^2$, $\epsilon_{11}^* = 1.68 \times 10^{-8} \text{ F/m}$ | $\alpha_x = 7.1 \times 10^{-5} \text{ K}^{-1}$, $\alpha_y = 7.1 \times 10^{-5} \text{ K}^{-1}$ |
| $\alpha_x = 1.2 \times 10^{-6} \text{ K}^{-1}$, $\alpha_y = 0.6 \times 10^{-6} \text{ K}^{-1}$ | |

They are

$$
\varepsilon_x = u_x + \frac{1}{2} w_x^2, \quad \varepsilon_y = v_y + \frac{1}{2} w_y^2,
$$

$$
\varepsilon_{xy} = u_y + v_x + w_x w_y, \quad \kappa_x = -w_{xx},
$$

$$
\kappa_y = -w_{yy}, \quad \kappa_{xy} = -2w_{xy}.
$$

(3)

### 2.2. Constitutive equations

Under combined mechanical, electrical and thermal loads, the constitutive equations of the BNNTRC piezoelectric plate can be written as [29]

$$
\sigma_i = C_{ik} (\epsilon_k - \alpha_k \Delta T) - \epsilon_i^* E_j \quad (i, k = 1, 2, \ldots, 6),
$$

$$
D_i = \epsilon_{ik} (\epsilon_k - \alpha_k \Delta T) - \epsilon_i^* E_j \quad (i, j = 1, 2, 3),
$$

(4)

where $E_k$ ($k = x, y, z$), $\Delta T$ and $\alpha_k$ represent electric field, temperature rise and thermal expansion coefficient, respectively. $C_{ij}$ ($i, j = 1, \ldots, 6$), $\epsilon_{ij}$, and $\epsilon_{ij}^*$ are elastic constants, piezoelectric constant and dielectric constants, respectively. The magnitude of these constants depends on the structure of the material itself.

In general, a BNNT has two symmetrical forms: armchair and zigzag [40]. The armchair shows the electric dipole moment linearly coupled with torsion, while the zigzag shape shows longitudinal piezoelectric response. Because the chaos and bifurcation of composite piezoelectric plates are studied in this paper, the zigzag structure of the BNNT is chosen. The intelligent composite plate is composed of BNNTs and PVDF, which are used as reinforcement and matrix, respectively. The material constants of the plate $C_{ij}$ can be obtained by the "XY (or YX) rectangle model", which are given in Appendix A.

### 2.3. Governing equations

For the BNNT-reinforced piezoelectric plate, total potential energy $\Pi$ is given as

$$
\Pi = K - U + \Gamma,
$$

(5)

where $K$ represents the kinetic energy, $U$ represents the strain energy and $\Gamma$ represents the work done by the transverse dynamic load.

The expression of the strain energy is

$$
U = \frac{1}{2} \iint_V \sigma \varepsilon + \frac{1}{2} \iint_V E_i D_i \mathrm{d}V.
$$

(6)

Because of the zigzag structure for BNNTs employed here, we have $E_y = E_z = 0$. Suppose that the voltage applied to both sides of the plate along the coordinate $x$ is $V$, then the electric field component is

$$
E_x = V / a.
$$

(7)

The expression of the strain energy is

$$
K = \int_{-h}^{h/2} \int_{-h}^{h/2} \rho_0 \left[ \ddot{w}^2 + (\dot{u})^2 + (\dot{v})^2 \right] \mathrm{d}x \mathrm{d}y
$$

(8)

The work done by the transverse dynamic load $q(x, y, t)$ is

$$
\Gamma = \iiint_V q(x, y, t) w \mathrm{d}x \mathrm{d}y.
$$

(9)

Using the Reissner variational principle ($\delta \Pi = 0$), it can be deduced that the nonlinear governing equation of the BNNTRC piezoelectric plate is

$$
N_{xx} + N_{xy,y} = \rho_0 h u_{tt},
$$

$$
N_{xy,x} + N_{yy,y} = \rho_0 h v_{tt},
$$

$$
M_{xx} + 2M_{xy,y} + M_{yy,yy} + 2N_{yx} w_{xy} + N_{yy} w_{yy} + q = \rho_0 h w_{tt},
$$

(10)

where

$$
\mathbf{N} = [\mathbf{A}] \{\varepsilon\} - \{\mathbf{N}^T\} - \{\mathbf{N}^p\},
$$

$$
\mathbf{M} = [\mathbf{D}] \{\kappa\} - \{\mathbf{M}^T\} - \{\mathbf{M}^p\},
$$

(11)

in which

$$
\{\mathbf{N}^T, \mathbf{M}^T\} = \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} C_{ij} \{\alpha_k\} \{1, z\} \Delta T \mathrm{d}z,
$$

$$
\{\mathbf{N}^p, \mathbf{M}^p\} = \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \epsilon_{ij} \{E_k\} \{1, z\} \mathrm{d}z.
$$

(12)
Figure 3 Volume ratio effect on the bifurcation diagram for the BNNTRC piezoelectric plate ($V=0, \Delta T=0$): (a) $V_f=0.2$ and (b) $V_f=0.6$.

$A_{ij}$ and $D_{ij}$ in Eq. (11) are the stretching stiffness and bending stiffness, respectively, which are expressed as

$$(A_{ij}, D_{ij}) = \int_{-h/2}^{h/2} C_{ij}(1, z^2)dz \quad (i, j = 1, 2, 6). \quad (13)$$

Using the following dimensionless parameters:

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{b}, \quad \bar{U} = \frac{u}{a}, \quad \bar{V} = \frac{v}{b}, \quad \bar{W} = \frac{w}{h},$$

$$\bar{A}_{11} = \frac{A_{11}}{C_{22}h^4}, \quad \bar{A}_{12} = \frac{A_{12}}{C_{22}h^2}, \quad \bar{A}_{22} = \frac{A_{22}}{C_{22}h}, \quad \bar{A}_{66} = \frac{A_{66}}{C_{22}h^2},$$

$$\bar{D}_{11} = \frac{D_{11}}{C_{22}h^4}, \quad \bar{D}_{12} = \frac{D_{12}}{C_{22}h^2}, \quad \bar{D}_{22} = \frac{D_{22}}{C_{22}h^4},$$

$$\bar{D}_{66} = \frac{D_{66}}{C_{22}h^2}, \quad \lambda_1 = \frac{h}{a}, \quad \lambda_2 = \frac{h}{b}, \quad Q = \frac{q}{C_{22}},$$

$$\tau = t\sqrt{C_{22}/(\rho_0 h^2)}, \quad (14)$$

the nonlinear dynamic equations of the BNNTRC piezoelectric plate under electro-thermo-mechanical loads can be deduced as

$$\bar{A}_{11}(\bar{U}, \bar{V}, \bar{W}) + \lambda_1^2 \bar{W} \bar{V} + \lambda_2^2 \bar{W} \bar{U} + \lambda_3^2 \bar{W} \bar{W} = 0, \quad (15)$$

$$\bar{A}_{12}(\bar{U}, \bar{V}, \bar{W}) + \bar{A}_{22}(\bar{V}, \bar{U}, \bar{W}) = 0, \quad (16)$$

where $f(\xi)$ and $g(\eta)$ are defined as

$$f(\xi) = \frac{\lambda_2}{2} \bar{W}_\xi^2 - \phi_1, \quad g(\eta) = \frac{\lambda_1}{2} \bar{W}_\eta^2 - \phi_2, \quad \text{on} \quad \xi = 0 \quad \text{and} \quad \eta = 1,$$

and

$$\phi_1 = \lambda_2^2 \left( \frac{\epsilon_{12}V}{C_{22}} + \left( C_{12} \alpha_x + C_{12} \alpha_y \right) \Delta T \right),$$

$$\phi_2 = \lambda_1^2 \left( \frac{\epsilon_{11}V}{C_{11}} + \left( C_{11} \alpha_x + C_{11} \alpha_y \right) \Delta T \right).$$

The dynamic load was assumed to be

$$Q = F \sin \pi \xi \sin \pi \eta, \quad F = F_0 \sin \omega \tau.$$

In the above equation, $\omega$ and $F_0$ are the frequency and the dimensionless amplitude of dynamic loading, respectively.

After substituting Eq. (17) into Eq. (15), the first one of the resulting equations is multiplied with $\cos i \pi \xi \sin j \pi \eta$, the second one is multiplied with $\sin i \pi \xi \cos j \pi \eta$ and the third one is multiplied with $\sin i \pi \xi \sin j \pi \eta$. Then, the resulting equation is integrated in the range of 0 to 1 and truncated by first-order Galerkin. The nonlinear ordinary differential equation is given by the time functions $\bar{u}_{11}, \bar{v}_{11}$ and $\bar{w}_{11}$ (when $m=1, n=1$) as

$$L_{11} \ddot{u}_{11} + L_{12} \ddot{v}_{11} + L_{13} \ddot{w}_{11} + L_{14} = 0,$$

$$L_{21} \ddot{u}_{11} + L_{22} \ddot{v}_{11} + L_{23} \ddot{w}_{11} + L_{24} = 0,$$

$$L_{31} \ddot{w}_{11} + L_{32} \ddot{w}_{11} + L_{33} \ddot{w}_{11} + L_{34} \ddot{u}_{11} + \frac{1}{4} F = \frac{1}{4} \frac{d^2 \bar{w}_{11}}{dt^2}, \quad (18)$$

3. SOLUTION METHODOLOGY

To meet boundary conditions provided in Eq. (16), the formal solution of Eq. (15) is considered to be

$$\bar{W} = \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} \tilde{w}_{mn}(\tau) \sin m\pi \xi \sin n\pi \eta,$$

$$\bar{U} = -\frac{1}{\pi^2} f(\xi) \sin 2\pi \xi \nu + \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} \tilde{u}_{mn}(\tau) \cos m\pi \xi \sin n\pi \eta, \quad (17)$$

$$\bar{V} = -\frac{1}{\pi^2} g(\eta) \sin 2\pi \eta \nu + \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} \tilde{v}_{mn}(\tau) \sin m\pi \xi \cos n\pi \eta,$$

assuming simply supported conditions for four sides of the plate, the following dimensionless boundary conditions were obtained:

$$\xi = 0, \quad 1 : \bar{V} = 0, \quad \bar{W} = 0, \quad N_{\xi} = 0, \quad M_\xi = 0, \quad \eta = 0, \quad 1 : \bar{U} = 0, \quad \bar{W} = 0, \quad N_\eta = 0, \quad M_\eta = 0. \quad (16)$$
\section*{Figure 4} Comparison of nonlinear dynamic characteristics of the BNNTRC piezoelectric plate relative to different volume ratios ($F_0 = 671.8$): (a) phase-plane trajectory, $V_f = 0.2$; (b) phase-plane trajectory, $V_f = 0.6$; (c) Poincaré map, $V_f = 0.2$; (d) Poincaré map, $V_f = 0.6$; (e) time course curve, $V_f = 0.2$; and (f) time course curve, $V_f = 0.6$.

where $L_{ij}$ are given in Appendix B. 

\begin{align*}
\ddot{u}_{11} \text{ and } \dot{v}_{11} \text{ can be solved by the first two formulas of Eq. (18), that is} \\
\begin{bmatrix}
\ddot{u}_{11} \\
\dot{v}_{11}
\end{bmatrix} &= -\begin{bmatrix}
L_{11} & L_{12} \\
L_{21} & L_{22}
\end{bmatrix}^{-1} \begin{bmatrix}
L_{13}\dot{w}_{11}^2 + L_{14} \\
L_{23}\dot{w}_{11}^2 + L_{24}
\end{bmatrix},
\end{align*}

which is expressed as

\begin{align*}
\ddot{u}_{11} &= \varphi_u + \Gamma_u \dot{w}_{11}^2, \\
\dot{v}_{11} &= \varphi_v + \Gamma_v \dot{w}_{11}^2.
\end{align*}

Substituting Eq. (20) into the third equation of Eq. (18) and bringing linear damping term $\mu \ddot{w}_{11}$, the nonlinear governing equation only expressed by $\dot{w}_{11}$ is given as

\begin{equation}
\varphi_1 \ddot{w}_{11} + \varphi_2 \dot{w}_{11}^3 + \frac{1}{4} F - \mu \frac{d\dot{w}_{11}}{d\tau} = \frac{1}{4} \frac{d^2 \dot{w}_{11}}{d\tau^2},
\end{equation}

where

\begin{align*}
\varphi_1 &= L_{31} + L_{33} \varphi_u + L_{34} \varphi_v, \\
\varphi_2 &= L_{32} + L_{33} \Gamma_u + L_{34} \Gamma_v.
\end{align*}

By the introduction of state variables $y_1(\tau) = \dot{w}_{11}(\tau)$ and $y_2(\tau) = \frac{d\dot{w}_{11}}{d\tau}$, Eq. (21) is transformed to the first-order differential equations given below:

\begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= 4 (\varphi_1 y_1 + \varphi_2 y_1^3 - \mu y_2 + \frac{1}{4} F).
\end{align*}
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Figure 5 Comparison of nonlinear dynamic characteristics of the BNNTRC piezoelectric plate with different volume ratios ($V_f = 0.685$): (a) phase-plane trajectory, $V_f = 0.2$; (b) phase-plane trajectory, $V_f = 0.6$; (c) Poincaré map, $V_f = 0.2$; (d) Poincaré map, $V_f = 0.6$; (e) time course curve, $V_f = 0.2$; and (f) time course curve, $V_f = 0.6$.

Equation (22) is resolved using the fourth-order Runge–Kutta method. Then, phase plane trajectory, bifurcation diagram, time history curve and Poincaré map are achieved through uniting the numerical method used for nonlinear dynamics.

4. NUMERICAL RESULTS AND DISCUSSION

In order to verify the current analysis, first the dynamic response of the BNNT-enhanced piezoelectric plate under the combined action of electro-thermo-mechanical loadings is discussed. The material constants and geometrical parameters of BNNTs and PVDF are similar to those of [36]. Applied voltage $V = 25$, temperature $\Delta T = 50$ and volume ratio $V_f = 0.4$. Figure 2 shows

the time histories of the dimensionless center deflection $W_0$ and compares it with the results given by Yang and Zhang [36]. A good agreement with 8.9% relative error is observed.

In the following study, the chaos and bifurcation of BNNTRC piezoelectric plates under thermo-electro-mechanical action are calculated. The geometrical parameters of the piezoelectric plate are $a/h = 20$ and $b/h = 20$, and the boundary condition is simply supported on four sides. The external excitation frequency is $\omega = 0.5$. The material constants of BNNTs and PVDF are shown in Table 1.

Figure 3 shows the influence of BNNT volume ratio $(V_f)$ on the bifurcation diagram of BNNTRC piezoelectric plates. In this example, voltage and temperature are zero. From Fig. 3, it can
Figure 6 Bifurcation diagram of deflection versus load ($V = +250$, $\Delta T = 0$, $V_t = 0.2$): (a) bifurcation diagram and (b) partial enlarged detail.

Figure 7 Nonlinear dynamic characteristics of the BNNTRC piezoelectric plate ($V = +250$, $\Delta T = 0$, $V_t = 0.2$): (a) phase-plane trajectory ($F = 650$); (b) Poincaré map ($F = 650$); (c) phase-plane trajectory ($F = 656.2$); (d) Poincaré map ($F = 656.2$); (e) phase-plane trajectory ($F = 663.9$); and (f) Poincaré map ($F = 663.9$).
be seen that by increasing load, the system goes through one period motion, several period bifurcation motions, one period motion and double period bifurcation motion, and finally gets into a complex state containing multiple periodic motion, quasi-periodic motion or chaotic motion. However, BNNT volume fraction being different, the two kinds of situations create different critical loads corresponding to the system entering chaotic motion, and critical load increases when volume fraction $V_t$ increases.

Figure 4 exhibits the nonlinear dynamic characteristics of the BNNTRC piezoelectric plate relative to different volume ratios when the load $F_0 = 671.8$. It can be seen that the piezoelectric plate ($V_t = 0.2$) shows periodic motion bifurcation and the piezoelectric plate ($V_t = 0.6$) still maintains a stable one period motion.

Figure 5 also shows the nonlinear dynamics of the BNNTRC piezoelectric plate relative to different volume ratios when the load $F_0 = 685$. It is witnessed that plate structure with $V_t = 0.2$ shows chaotic motion while that with $V_t = 0.6$ still has one period motion. Consequently, the piezoelectric structure delays the multiple periodic or chaotic motions by the increase of BNNT volume fraction. In other words, increase of BNNT volume fraction stabilizes dynamic characteristics of structures, which is conducive to structure dynamic stability.

Figure 6 shows the bifurcation diagram for the BNNTRC piezoelectric plate when a positive voltage is applied to the system. The other parameters are same as those in Fig. 3a. Recuring to phase-plane trajectory and Poincaré map, the nonlinear dynamic characteristics of the system changing with load are discussed at length, as shown in Fig. 7. When the load is small, the system shows the periodic motion as shown in Fig. 7a and b. As the load increases to a certain value, the system shows the period bifurcation as shown in Fig. 7c and d. When $F_0 = 663.9$, the system first enters the chaotic motion as shown in Fig. 7e and f, and the motion of the system is unstable at this time. Furthermore, from Fig. 5 we notice that the critical load for the system to enter into chaotic motion for the first time is $F_0 = 685$ when the volume ratio $V_t = 0.2$. Therefore, applying a positive voltage to the system will lead to the emergence of multiple periodic or chaotic motions ahead of time. This shows that positive voltage is not conducive to the dynamic stability of the structure.

Figure 8 shows the bifurcation diagram for the BNNTRC piezoelectric plate when a negative voltage is applied to the system. The remaining parameters are also same as those in Fig. 3a. Recuring to phase-plane trajectory and Poincaré map, the nonlinear dynamic characteristics of the system changing with load are also discussed in detail, as shown in Fig. 9. From Fig. 9, we notice that the critical load for the system to enter into chaotic motion for the first time is $F_0 = 702.8$. Comparing to Fig. 5, we know that the critical load is $F_0 = 685$ when the volume ratio $V_t = 0.2$. Therefore, applying negative voltage to the system will postpone the chaotic motion of the piezoelectric plate. This indicates that the negative voltage can enhance stability of the system.

The influence of temperature on the bifurcation diagram of the BNNTRC piezoelectric plate is exhibited in Fig. 10. BNNT volume ratio is $V_t = 0.2$ V and the applied voltage $V$ is zero. In the figure, it can be shown that by increasing load, the system goes through one period motion, several period bifurcation motions, one period motion and several period bifurcation motions, and eventually a complicated chaotic motion is created. However, due to different temperatures, different critical loads are created in the system entering chaotic motion. Figures 11 and 12, respectively, show the phase-plane trajectory and the Poincaré map of the system when it first enters the chaotic state at two different temperatures. As can be seen from the figures, when the temperature $T = 100$, the critical load of the system entering chaotic motion is $F_0 = 664.4$ and when the temperature $T = 500$, the critical load is $F_0 = 658.352$. Consequently, increase of temperature will lead to the emergence of multiple periodic or chaotic motions ahead of time. This shows that temperature increase is not conducive to the dynamic stability of the structure.

5. CONCLUSIONS

In this paper, the chaos and bifurcation of the BNNT-reinforced piezoelectric plate under thermo-electro-mechanical loads are studied. The governing equations of the BNNT-reinforced piezoelectric plate are derived by the Reissner variational principle, and the chaos and bifurcation are studied by the Runge–Kutta method. The numerical results show that decreasing voltage and temperature and increasing volume ratio can delay...
Figure 9 Nonlinear dynamic characteristics of the BNNTRC piezoelectric plate ($V = -250$, $\Delta T = 0$, $V_f = 0.2$): (a) phase-plane trajectory ($F = 650$); (b) Poincaré map ($F = 650$); (c) phase-plane trajectory ($F = 686.54$); (d) Poincaré map ($F = 686.54$); (e) phase-plane trajectory ($F = 702.8$); and (f) Poincaré map ($F = 702.8$).

The chaotic or multiple periodic motions of BNNT-reinforced piezoelectric plates, thus improving the dynamic stability of the structure.

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**CONFLICT OF INTEREST**

The authors declare that they have no conflict of interest.
Figure 10 Effect of temperature on the bifurcation diagram of the piezoelectric plate reinforced with BNNTs: (a) $T = 100$ and (b) $T = 500$.

Figure 11 Phase-plane trajectories of the BNNTRC piezoelectric plate under different temperatures: (a) $T = 100$ and (b) $T = 500$.

Figure 12 Poincaré map of the BNNTRC piezoelectric plate under different temperatures: (a) $T = 100$ and (b) $T = 500$.

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APPENDIX A

\[ C_{11} = \frac{C_{11}^{m} + (1 - V)C_{11}^{m,31}}{C_{11}^{m}}, \]
\[ C_{12} = C_{11} \left[ \frac{V C_{11}^{m} + (1 - V)C_{11}^{m,31}}{C_{11}^{m}} \right], \]
\[ C_{13} = C_{11} \left[ \frac{V C_{11}^{m} + (1 - V)C_{11}^{m,31}}{C_{11}^{m}} \right], \]
\[ C_{22} = V C_{22} + (1 - V)C_{22}^{m} + \frac{C_{22}^{m}}{C_{11}^{m}} - \frac{V (C_{11}^{m})^2}{C_{11}^{m}} - \frac{(1 - V) (C_{11}^{m})^2}{C_{11}^{m}}, \]
\[ C_{44} = V C_{44} + (1 - V)C_{44}^{m}, \]
\[ C_{55} = \frac{2 + AC}{B + AC}, \]
\[ C_{66} = V C_{66} + (1 - V)C_{66}^{m}, \]
\[ e_{31} = C_{11} \left[ \frac{V C_{11}^{m} + (1 - V)C_{11}^{m,31}}{C_{11}^{m}} \right], \]
\[ e_{32} = V e_{32} + (1 - V)e_{32}^{m} + \frac{C_{12}e_{31}}{C_{11}} - \frac{V C_{12}^{m} e_{31}}{C_{11}} - \frac{(1 - V) C_{12}^{m,31}}{C_{11}}, \]
\[ e_{33} = V e_{33} + (1 - V)e_{33}^{m} + \frac{C_{13}e_{11}}{C_{11}} - \frac{V C_{13}^{m} e_{11}}{C_{11}} - \frac{(1 - V) C_{13}^{m,11}}{C_{11}}, \]
\[ e_{15} = \frac{B + AC}{2 + AC}, \]
\[ e_{15}^{e} = \frac{B + AC}{2 + AC}, \]
\[ e_{12}^{e} = V e_{12}^{e} + (1 - V)e_{12}^{m}, \]
\[ e_{33}^{e} = V e_{33}^{e} + (1 - V)e_{33}^{m} - \frac{\epsilon_{11}}{C_{11}} + \frac{V (\epsilon_{11})^2}{C_{11}} + \frac{(1 - V) (\epsilon_{11})^2}{C_{11}}. \]

where

\[ A = \frac{V C_{55}}{(e_{15}^{e})^2 + C_{55} e_{11}^{m}} + \frac{(1 - V) C_{55}^{m}}{(e_{15}^{e})^2 + C_{55} e_{11}^{m}}, \]
\[ B = \frac{V C_{11}^{m}}{(e_{15}^{e})^2 + C_{11} e_{11}^{m}} + \frac{(1 - V) C_{11}^{m,31}}{(e_{15}^{e})^2 + C_{11} e_{11}^{m}}, \]
\[ C = \frac{V C_{11}^{m}}{(e_{15}^{e})^2 + C_{11} e_{11}^{m}} + \frac{(1 - V) C_{11}^{m,11}}{(e_{15}^{e})^2 + C_{11} e_{11}^{m}}. \]

\[ L_{11} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{11} A_{11.\xi,\xi} \eta \right) A_{1} d\xi d\eta, \]
\[ L_{12} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{12} + \lambda_{21} A_{2,\eta,\eta} A_{1} \right) d\xi d\eta, \]
\[ L_{13} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{13} A_{1,3,\eta,\eta} \right) A_{1} d\xi d\eta + \int_{0}^{1} j_{0}^{1} \left( \lambda_{13} + \lambda_{11} A_{11,\eta,\eta} \right) A_{1} d\xi d\eta \]
\[ + \int_{0}^{1} \int_{0}^{1} \left( \lambda_{12} A_{11,3,\eta,\eta} \right) A_{12} d\xi d\eta + \int_{0}^{1} j_{0}^{1} \left( \lambda_{12} + \lambda_{13} A_{1,3,\eta,\eta} \right) A_{12} d\xi d\eta, \]
\[ L_{14} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{14} A_{14,\xi,\eta} A_{1} \right) A_{1} d\xi d\eta, \]
\[ L_{21} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{14} + \lambda_{11} A_{11,\eta,\eta} A_{2} \right) d\xi d\eta, \]
\[ L_{22} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{12} A_{12,\eta,\eta} + \lambda_{21} A_{2,\xi,\xi} \right) A_{2} d\xi d\eta, \]
\[ L_{23} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{13} A_{13,\xi,\eta,\eta} \right) A_{2} d\xi d\eta + \int_{0}^{1} j_{0}^{1} \left( \lambda_{13} + \lambda_{21} A_{2,\eta,\eta} \right) A_{2} d\xi d\eta \]
\[ + \int_{0}^{1} \int_{0}^{1} \left( \lambda_{22} A_{22,\eta,\eta} \right) A_{2} d\xi d\eta, \]
\[ L_{24} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{22} A_{22,\eta,\eta} A_{2} \right) d\xi d\eta, \]
\[ L_{31} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{31} A_{31,\xi,\eta} \right) A_{3} d\xi d\eta \]
\[ + \int_{0}^{1} j_{0}^{1} \left( \lambda_{31} A_{31,\xi,\eta} \right) A_{3} d\xi d\eta, \]
\[ L_{32} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{32} A_{32,\xi,\xi} \right) A_{3} d\xi d\eta \]
\[ + \int_{0}^{1} j_{0}^{1} \left( \lambda_{32} A_{32,\xi,\xi} \right) A_{3} d\xi d\eta, \]
\[ L_{33} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{31} A_{31,\xi,\eta} + \lambda_{32} A_{32,\xi,\xi} \right) A_{3} d\xi d\eta, \]
\[ L_{34} = \int_{0}^{1} \int_{0}^{1} \left( \lambda_{32} A_{32,\xi,\xi} + \lambda_{31} A_{31,\xi,\eta} \right) A_{3} d\xi d\eta, \]
\[ \Lambda_{1} = \cos \pi \xi \sin \pi \eta, \quad \Lambda_{2} = \sin \pi \xi \cos \pi \eta, \quad \Lambda_{3} = \sin \pi \xi \sin \pi \eta, \quad \Lambda_{4} = -\frac{\pi}{4} \sin \pi \xi \sin \pi \eta, \]
\[ \Lambda_{5} = -\frac{\pi}{4} \sin \pi \xi \sin \pi \eta. \]