Inflation in $\tilde{\delta}$ Gravity.

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Abstract. $\tilde{\delta}$ Gravity is a gravitational fields model based on two symmetric tensors, $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, and new matter fields given by $\tilde{\phi}_I = \delta\phi_I$. This model has great properties at the quantum level. For example: It lives at one loop only; the classical equations of motion of the original fields are preserved and it is a finite quantum theory in the vacuum. Besides, massive particles follow an anomalous geodesic, while trajectories of massless particles are given by a null geodesic of the effective metric $g_{\mu\nu} = g_{\mu\nu} + \kappa^2\tilde{g}_{\mu\nu}$. In previous works, the accelerated expansion of the universe was explained without dark energy. Additionally, we saw that $\tilde{\delta}$ matter helps to explain dark matter. All these are motivations to study the effect of $\tilde{\delta}$ Gravity in Cosmic Inflation. For this, we introduce one traditional inflaton, $\phi$, plus his $\tilde{\delta}$ partner, $\tilde{\phi}$. Additionally, we have the contribution of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$.

1. Introduction
In the physics word, the Quantum Mechanics (QM) and General Relativity (GR) are the principal theories used to explain the universe. On one side, QM explain the sub-atomic word, and on the other side, GR describe the physics from scales larger than a millimeter to Cosmological scales [1]. Both theories are really successful, but when we try to mix them, some problems appear. That is because GR is non-renormalizable [2]. Moreover, recent discoveries in cosmology have revealed that most part of matter is in the form of unknown matter, dark matter [3], and that the dynamics of the expansion of the Universe is governed by a mysterious component that accelerates its expansion, the so called dark energy [4, 5]. This is the Dark Sector. Although GR is able to accommodate the Dark Sector, its interpretation in terms of fundamental theories of elementary particles is problematic [5]. Some candidates exist that could play the role of dark matter, but none have been detected yet. On the other side, dark energy can be explained if a small cosmological constant ($\Lambda$) is present. However $\Lambda$ is too small to be generated in quantum field theory (QFT) models, because $\Lambda$ is the vacuum energy, which is usually predicted to be very large [6]. For these reasons, there has been various proposals to explain the observed acceleration of the Universe.

In [7], a model of gravitation that is very similar to GR is presented, but works different at the quantum level. In that paper, we considered two different points. The first is that GR is finite on shell at one loop in vacuum [2], so renormalization is not necessary in this case. The second is a special kind of modification, called the $\tilde{\delta}$ gauge theories (DGT), originally presented in [8], where the main properties are: (a) A new kind of field $\tilde{\phi}_I$ is introduced, different from the original set $\phi_I$. (b) The classical equations of motion of $\tilde{\phi}_I$ are satisfied even in the full quantum theory. (c) The model lives at one loop. (d) The action is obtained through the extension of the original gauge symmetry of the model, introducing an extra symmetry that we call $\tilde{\delta}$ symmetry,
since it is formally obtained as the variation of the original symmetry. When we apply this
modification to GR, we obtain $\delta$ Gravity.

The original motivation was to develop the quantum properties of this model (See [7]) and
then we studied the classical properties of $\delta$ Gravity (See [9]), where we saw that: (a) it agrees
with GR, far from the sources. In particular, the causal structure of $\delta$ Gravity in vacuum is the
same as in general relativity. (b) The necessary quantity of dark matter could be considerably
less that we expected. (c) When we study the evolution of the Universe, it predicts an accelerated
expansion without a cosmological constant or additional scalar fields. The Universe ends in a
Big-Rip, similar to the scenario considered in [10]. (d) The scale factor agrees with the standard
cosmology at early times and shows acceleration only at late times. Therefore we expect that
primordial density perturbations should not have large corrections.

Now, Inflation is a brief period of exponential expansion before the Radiation era [11],
governed by a scalar field $\phi$ called Inflaton. Some inflationary parameters can be defined to
show the expansion behavior. The more important is $\epsilon$, that represent the acceleration of the
expansion. As long as $\epsilon \ll 1$, the expansion is exponential and it will be ended when $\epsilon = 1$.
In that moment, the Inflaton decay on matter and radiation in a process called Reheating [12].
In the same way, the quantum fluctuation of Inflaton produce density fluctuations and they
will give rise to the large structures of the universe, including dark matter distribution. The
anisotropy produced by these fluctuation is represented by the Power Spectrum of Inflation,
including information about the evolution of the universe produced by dark energy. Therefore,
it is useful to study the Dark Sector. Motivated by this, it is important to study the effect of $\delta$
Gravity in Cosmic Inflation, that is $\delta$ Inflation.

Unlike traditional Inflation, in $\delta$ Inflation the accelerated expansion could be affected by
$\delta$ Gravity, just like the expansion is accelerated in the present era. Besides, we will have an
additional inflationary field $\delta$ and new inflationary parameters. In the same way, $\delta$ Gravity will
affect the perturbative equation and then the Power Spectrum. With this, we can understand
much better how $\delta$ Gravity explain the Dark Sector and verify if it is phenomenologically viable.
In this work, we will present an introduction to $\delta$ Inflation, just like it was presented in XX
Simposio SOCHIFI 2016. This work is in progress and a complete article will be published soon.

2. Cosmology in $\delta$ Gravity.

From [9], we know that the action of $\delta$ Gravity is:

$$ S = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa} + L_M - \frac{\kappa_2}{2\kappa} \left( G^{\alpha\beta} - \kappa T^{\alpha\beta} \right) \delta g_{\alpha\beta} + \kappa_2 L_M \right), \quad (1) $$

where $L_M = L_M(\phi_I, \partial_m \phi_I)$ is the lagrangian of the matter fields $\phi_I$, $\delta g_{\mu\nu} = \hat{\delta} g_{\mu\nu}$, $T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} [\sqrt{-g} L_M]$ and $\hat{L}_M = \hat{\delta}_I \frac{\delta L_M}{\delta \phi_I} + (\partial_{\mu} \phi_I) \frac{\delta L_M}{\delta (\partial_{\mu} \phi_I)}$, where $\hat{\delta}_I = \delta \phi_I$ are the $\delta$ matter fields.

From this action, we can obtain the equations of motion of $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$. It is easy to see that the Einstein’s equations are still valid. Besides, the equation for $\hat{g}_{\mu\nu}$ is:

$$ G^{\mu\nu} = \kappa T^{\mu\nu} \wedge F^{(\mu\nu)(\alpha\beta)\rho\lambda} D_\rho D_\lambda \hat{g}_{\alpha\beta} + \frac{1}{2} g^{\mu\nu} R^{\alpha\beta} \hat{g}_{\alpha\beta} - \frac{1}{2} \hat{g}^{\mu\nu} R = \kappa \hat{T}^{\mu\nu}. \quad (2) $$

with $F^{(\mu\nu)(\alpha\beta)\rho\lambda} = P^{((\mu\nu)(\alpha\beta))\rho\lambda} g^{\rho\lambda} + P^{((\mu\nu)(\alpha\beta))} g^{\rho\lambda} - P^{((\mu\nu)(\alpha\beta))} g^{\rho\lambda} - P^{((\rho\lambda)(\alpha\beta))} g^{\mu\nu}$,

$$ P^{((\alpha\beta)(\mu\nu))} = \frac{1}{2} \left( g^{\alpha\beta} g^{\mu\nu} + g^{\alpha\nu} g^{\beta\mu} - g^{\alpha\mu} g^{\beta\nu} \right), $$

where $((\mu\nu))$ denotes that $\mu$ and $\nu$ are in a totally symmetric combination and $\hat{T}_{\mu\nu} = \delta T_{\mu\nu}$ is the $\delta$ energy momentum tensor, related to $\delta$ matter fields. An important fact to notice is that our equations are of second order in derivatives which is needed to preserve causality. Besides, we have two conservation rules:

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\[ D_{\nu} T^{\mu\nu} = 0 \quad \land \quad D_{\nu} \tilde{T}^{\mu\nu} = \frac{1}{2} T^{\alpha\beta} D^{\mu} \tilde{g}_{\alpha\beta} - \frac{1}{2} T^{\mu\beta} D_{\beta} \tilde{g}_{\alpha} + D_{\beta}(\tilde{g}_{\alpha}^{\beta} T^\alpha{}_{\mu}). \] (3)

Particularly, in the cosmological case, we have:

\[
\begin{align*}
g_{\mu\nu} dx^\mu dx^\nu & = -N^2(t) c^2 dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right) \quad (4) \\
g_{\mu\nu} dx^\mu dx^\nu & = -\tilde{F}_0(t) N^2(t) c^2 dt^2 + \tilde{F}_1(t) a^2(t) \left( dx^2 + dy^2 + dz^2 \right), \quad (5)
\end{align*}
\]

where \( N(t) \) and \( \tilde{F}_0(t) \) will be determined fixing the gauge. In \( \delta \) Gravity, the proper time is given by \( g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1 \), so we will use \( N(t) = 1 \) such that \( t \) is the proper time. In this case, we can use \( \tilde{F}_0(t) = 3\tilde{F}_1(t) \) (See Appendix B of [9]). We will impose the gauge below.

On the other side, in our model, the light (or any massless particle) move in a effective null geodesic given by \( g_{\mu\nu} + \kappa g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \). For this reason, the expansion of the universe is produced by an effective scale factor \( a_{eff}(t) = a(t)(\frac{1+\kappa \tilde{F}_1(t)}{1+\kappa \tilde{F}_0(t)}) \), such that \( N(t) dt = a_{eff}(t) dr \).

This means that all definitions must be modified such that \( a(t) \rightarrow a_{eff}(t) \). For example, the redshift and luminosity distance:

\[
1 + z(t_1) = \frac{a_{eff}(t_0)}{a_{eff}(t_1)} \quad \land \quad d_L = \frac{a_{eff}(t_0)}{a_{eff}(t_1)} \int_{t_1}^{t_0} N(t) dt \quad (6)
\]

With all these, we explained the accelerated expansion of the universe without dark energy in [9]. Now, in this paper, we will developed inflation with \( \delta \) Gravity, producing the expansion in the same way. But first, in the next chapter, we will analyze the solution for a perfect fluid using the Cosmological case.

3. Analysis of the Perfect Fluid.

In this chapter, as an example of how the expansion works in \( \delta \) Gravity, we will study the perfect fluid solution. The Energy-Momentum Tensors for a perfect fluid are \( T_{\mu\nu} = p g_{\mu\nu} + (\rho + p) U_\mu U_\nu \) and \( \tilde{T}_{\mu\nu} = \tilde{p} g_{\mu\nu} + \tilde{p} g_{\mu\nu} + (\tilde{\rho} + \tilde{p}) U_\mu U_\nu + (\rho + p) \left( \tilde{U}_\mu U_\nu + U_\mu \tilde{U}_\nu \right) \), where \( U_\mu = (1, 0, 0, 0) \) and \( \tilde{U}_\mu = \left( \frac{3}{2} \tilde{F}_1(t), 0, 0, 0 \right) \). Besides, \( \rho, p, \tilde{\rho} \) and \( \tilde{p} \) just depend on time in a Background level. With all these, the equations in (2) are reduced to:

\[
\begin{align*}
H^2(t) & = \frac{k}{3} \rho(t) \quad \land \quad \dot{H}(t) = -\frac{k}{2} (p(t) + \rho(t)) \quad (7) \\
3H(t) \left( \dot{\tilde{F}}_1(t) - 3H(t) \tilde{F}_1(t) \right) & = \kappa \tilde{\rho}(t) \quad \land \quad \ddot{\tilde{F}}_1(t) + 3\kappa \rho(t) \tilde{F}_1(t) = -\kappa \tilde{\rho}(t). \quad (8)
\end{align*}
\]

Additionally, when we have a fluid, equations of state are required to solve the system. In general, they are \( p(t) = \omega(t) \rho(t) \) and \( \tilde{p}(t) = \omega(t) \tilde{\rho}(t) + \tilde{\omega}(t) \rho(t) \), where \( \omega(t) \) is generally assumed to be constant and \( \tilde{\omega}(t) \) go to zero in that case. So, when we use these equations of state in (7-8), we obtain:

\[
\dot{H}(t) + \frac{3}{2} (\omega(t) + 1) H^2(t) = 0 \quad \land \quad \ddot{\tilde{F}}_1(t) + 3\omega(t)H(t) \tilde{F}_1(t) + 3\tilde{\omega}(t)H^2(t) = 0 \quad (9)
\]
In order to understand the behavior of these equations, we will solve the case where \( \omega(t) \) is constant and \( \bar{\omega}(t) = 0 \). That is:

\[
a(t) = \begin{cases} 
  a_0 \left( \frac{t}{t_0} \right)^{\frac{\epsilon}{3(1+\omega)}} & \text{, with: } \omega \neq -1 \quad \land \quad \bar{F}_1(t) = \begin{cases} 
    \left( \frac{a(t)}{a_0} \right)^{\frac{2}{3}(1-\omega)} + b_1 & \text{, with: } \omega \neq 1 \\
    b_2 \ln \left( \frac{a(t)}{a_0} \right) & \text{, with: } \omega = 1
  \end{cases}
\end{cases}
\]

where \( t_0, a_0, a_1, a_2, b_1 \) and \( b_2 \) are integration constants. From this solution and (7), we obtain the well-known result:

\[
H^2(X) = \begin{cases} 
  \frac{n_0}{3X^{3(1+\omega)}} & \text{, with: } \omega \neq -1 \\
  H^2_0 & \text{, with: } \omega = -1
\end{cases}
\land \quad t(X) = \begin{cases} 
  \frac{2X^{\frac{2}{3}(1+\omega)}}{\sqrt{3n_0}(1+\omega)} & \text{, with: } \omega \neq -1 \\
  \frac{\ln(X)}{H_0} & \text{, with: } \omega = -1,
\end{cases}
\]

where the new independent variable is now the normalized scale factor, \( X \equiv a(t)/a_0 \). (11) correspond to the usual GR solution, however we have new contribution given by \( \bar{F}_1(t) \) and \( \bar{\delta} \) fluid, \( \bar{\rho}(t) \). Additionally, we can rewrite (10) such that:

\[
\bar{F}_1(X) = \begin{cases} 
  b_0 X^{\frac{2}{3}(1-\omega)} + b_1 & \text{, with: } \omega \neq 1 \\
  b_2 \ln (X) + b_3 & \text{, with: } \omega = 1
\end{cases}
\]

Then, using the modified scale factor and equations (8,11-12), we obtain:

\[
X(X) = \frac{a_{eff}(t)}{a_0} = \begin{cases} 
  X \sqrt{c_1+2c_2-X^{3(1-\epsilon)}} & \text{, with: } \epsilon \neq 3 \\
  X \sqrt{d_1+2d_2-\ln(X)} & \text{, with: } \epsilon = 3
\end{cases}
\]

\[
\mathcal{H}(X) = \frac{\dot{a}_{eff}(t)}{a_{eff}(t)} = \begin{cases} 
  H(X) \left( 1 + \frac{c_2(3-\epsilon)X^{3(1-\epsilon)}}{c_1+2c_2-X^{3(1-\epsilon)}} \right) & \text{, with: } \epsilon \neq 3 \\
  H(X) \left( 1 + \frac{d_2}{d_1+2d_2-\ln(X)} \right) & \text{, with: } \epsilon = 3
\end{cases}
\]

\[
\epsilon_{eff}(X) = \begin{cases} 
  \epsilon - \frac{c_2(3-\epsilon)X^{3(1-\epsilon)}(-2c_1+(-1)c_2)X^{3(1-\epsilon)}+3c_1(c_1+2c_2)}{(X^{3(1-\epsilon)}-(2c_1+(-1)c_2)X^{3(1-\epsilon)}+c_1(c_1+2c_2))^2} & \text{, with: } \epsilon \neq 3 \\
  3 - \frac{d_2(3\ln^2(X)-2(1+3d_1+3d_2)\ln(X)+d_1(3d_1+6d_2+2)+5d_2)}{(\ln(X)-2(d_1+d_2)\ln(X)+d_1(d_1+2d_2)+d_2)^2} & \text{, with: } \epsilon = 3,
\end{cases}
\]

where we used \( \epsilon = \frac{3}{2} (1+\omega) \), \( c_1 = -\frac{1+3c_2b_1}{3\sigma_2b_0} \), \( c_2 = -\frac{1}{3\sigma_2b_0} \), \( d_1 = -\frac{1+3c_2b_1}{3\sigma_2b_2} \), \( d_2 = -\frac{1}{3\sigma_2b_2} \), \( X(X) = \) is the modified scale factor, \( \mathcal{H}(X) \) is the effective Hubble parameter and \( \epsilon_{eff}(X) \equiv -\frac{\mathcal{H}(X)}{H(X)} = \frac{3}{2} (1+\epsilon_{eff}(X)) \) is the effective inflationary parameter. This parameter is used in inflation to impose slow-roll conditions. For this reason, we will use it in this paper.

In conclusion, \( \epsilon_{eff}(X) \) represent the acceleration of the expansion in \( \bar{\delta} \) Gravity. When inflation is ended, \( \epsilon_{eff}(X) \) has to be closed to one. Unfortunately, that is impossible with a perfect fluid: As was developed in [9], the expansion is produced by a pole in the modified scale factor in (13), producing a Big-Rip. Additionally, the numerator of \( X(X) \) must be positive when
Figure 1. $\varepsilon_{\text{eff}}(X)$ vs $X$. Behavior of equation (15) for different values of the parameters. (a) We have two cases where $\varepsilon_{\text{eff}}(1) = 1$, so $c_2$ is given by (16). We can see that the accelerated expansion is produced very late and it is permanent. (b) We have two cases where $\varepsilon_{\text{eff}}(X \sim 0)$ is small. Clearly, the expansion is too strong. (c) We have two cases with $\varepsilon = 3$. In the first place, $\varepsilon_{\text{eff}}(1) = 1$, so $d_2$ is given by (17). Then, we have a case where $\varepsilon_{\text{eff}}(X \sim 0)$ is small. We obtain similar result to (a) and (b).

$X < X_{BR}$. So, defining the end of inflation when $X = 1$, then $c_1 > 1$, $c_2 > 0$, $d_1 > 0$ and $d_2 > 0$. These conditions are enough to obtain an accelerated expansion, however it is too strong. We can prove that, if $\varepsilon_{\text{eff}}(1) = 1$, then:

$$c_2 = \frac{(c_1 - 1)}{2} \left( (3 - \epsilon) \left( (5 - 2\epsilon) c_1 + \epsilon - 2 + \sqrt{(4\epsilon^2 - 24\epsilon + 29) c_1^2 + (4\epsilon + 2) c_1 - 4\epsilon + 5} \right) - 1 \right), \quad (16)$$

when $1 < \epsilon < 3$ and $1 < c_1 < \frac{2\epsilon^2 - 7\epsilon + 11 + (3 - \epsilon)\sqrt{4\epsilon^2 - 8\epsilon + 13}}{4(\epsilon - 1)}$, or:

$$d_2 = \frac{d_1 \left( 2 - d_1 - 8d_1^2 - \sqrt{4 - 4d_1^2 - 7d_1^4} \right)}{2(4d_1 + 3)(2d_1 - 1)}, \quad (17)$$

when $\epsilon = 3$ and $0 < d_1 < \frac{1}{2}$. For $0 \leq \epsilon \leq 1$, the condition is impossible to satisfy. Besides, we can prove, in all these cases, that $\varepsilon_{\text{eff}}(X < 1) > 1$ and $\varepsilon_{\text{eff}}(X > 1) < 1$. This means
that $\epsilon_{eff}(X)$ takes bigger values during "inflation" and the expansion is begun later. This result is a contradiction, so inflation does not work. On the other side, if we redefine $X$ such that $\epsilon_{eff}(X \sim 0) \sim 0$ (Exponential expansion) or $< 0$ (Phantom expansion), then we obtain that the accelerated expansion is perfectly produced, but inflation will never end. In fact, $\epsilon_{eff}(X_{BR}) = -2$. Few example are presented in Figure 1. In conclusion, inflation does not work with a single perfect fluid. A possible solution is to use two or more fluids, just like [9] where we explain the accelerated expansion of the universe with radiation ($\omega = \frac{1}{3}$) and non-relativistic matter ($\omega = 0$), so that the condition can be satisfied. However, the behavior of $\epsilon_{eff}(X)$ ($\omega_{eff}(X)$) for two fluids is really similar to one fluid, so explain inflation could be difficult (See Appendix A). A best option to explain inflation is a fluid with a dynamic $\omega(t)$ in (7-9), so that we have a new component represented by $\omega(t) \neq 0$. In the next chapter, we will focus on this option using a scalar field, the inflaton $\varphi$, to produce inflation.

4. $\delta$ Inflation.
In inflation, the fluid is usually represented by a single scalar field with a lagrangian $L_M = -\frac{1}{2}g_{\alpha\beta}(\partial_\alpha \varphi)(\partial_\beta \varphi) - V(\varphi)$, where $\varphi$ is the inflaton and $V(\varphi)$ is the potential. The modified Action is obtained using (1). So, with (4) and (5), the action in the background level is:

$$S^{(0)} = \int d^4x a^3 \left( \frac{3(1 + \kappa_2 \dot{F}_1)}{\kappa a} \frac{d}{dt} \left( \frac{\dot{a}}{N} \right) + \frac{3(2 + \kappa_2 \dot{F}_1 + \kappa_2 \dot{\phi}_0)}{2\kappa Na^2} \dot{\varphi}_0^2 + \frac{1}{2N} \left( 1 + \frac{\kappa_2}{2} \left( 3\dot{\phi}_0 - \dot{F}_0 \right) \right) \dot{\varphi}_0 \right)$$

where $V(\varphi, \tilde{\varphi}) = \left( 1 + \frac{\kappa_2}{2} \left( \dot{F}_0 + 3\dot{\phi}_0 \right) \right) V(\varphi_0) + \kappa_2 V_{,\varphi}(\varphi_0) \tilde{\varphi}_0$ is the effective potential, with $V_{,\varphi}(\varphi) = \frac{\partial V}{\partial \varphi}(\varphi)$. Additionally, we have another scalar field, $\tilde{\varphi}$, called $\tilde{\delta}$ inflaton. In a background level, they are $\varphi = \varphi_0(t)$ and $\tilde{\varphi} = \tilde{\varphi}_0(t)$. Now, calculating the equations of motions and fixing the gauge, we obtain:

$$3H^2 - \kappa \left( \frac{1}{2} \varphi_0^2 + V(\varphi_0) \right) = 0 \land \dot{H} + \frac{\kappa}{2} \dot{\varphi}_0^2 = 0$$

(19)

$$\ddot{F}_1 - 3H\dot{F}_1 + 2\kappa \dot{\varphi}_0 \dot{\varphi}_0 = 0 \land 3H\dot{F}_1 - \kappa \left( \dot{\varphi}_0 \dot{\varphi}_0 + 3\dot{F}_1 V(\varphi_0) + V_{,\varphi}(\varphi_0) \tilde{\varphi}_0 \right) = 0$$

(20)

$$\ddot{\varphi}_0 + 3H\dot{\varphi}_0 + V_{,\varphi}(\varphi_0) = 0 \land \ddot{\varphi}_0 + 3H\dot{\varphi}_0 + V_{,\varphi}(\varphi_0) \tilde{\varphi}_0 + 3\dot{F}_1 V_{,\varphi}(\varphi_0) = 0.$$ (21)

We can verify that (19-21) are respectively (7-9) when $\omega(t) = \frac{\varphi_0^2 - 2V(\varphi_0)}{\varphi_0^2 + 2V(\varphi_0)}$ and $\tilde{\omega}(t) = \frac{\varphi_0^2 - 2V(\varphi_0)}{\varphi_0^2 + 2V(\varphi_0)}$. In the next section, we will rewrite them using the field space formalism and we will define the effective inflationary parameters.

4.1. Fields Space.
In this formalism (For instance see [13]), we have to define a vector in the fields space given by

$$\phi_0^b = \left( \begin{array}{c} \varphi_0 \\ \tilde{\varphi}_0 \end{array} \right).$$

So, the background action (18) can be rewritten as:

$$S^{(0)} = \int d^4x a^3 N \left( L_{GR} + \frac{1}{2} \gamma_{ab} \dot{\phi}_0^a \dot{\phi}_0^b - V(\varphi_0) \right).$$ (22)
where \( L_{GR} \) is the Background of General Relativity lagrangian, \( V(\phi_0^a) \) and \( \gamma_{ab} \) are the effective potential and the metric in the fields space. After to fixing the gauge, they are given by \( V(\phi_0^a) = \left( 1 + 3\kappa_2 \tilde{F}_1 \right) V(\varphi_0) + \kappa_2 V_{\varphi}(\varphi_0) \tilde{\varphi}_0 \) and \( \gamma_{ab} = \left( \begin{array}{cc} 1 & \kappa_2 \\ \kappa_2 & 0 \end{array} \right) \). In this notation, the equations in (21) are reduced to:

\[
\ddot{\varphi}_0 + 3H \dot{\varphi}_0 + V_T = 0 \quad \land \quad \dot{T}^a - \frac{V_{\varphi}}{\phi_0} N^a = 0, \quad (23)
\]

with \( V^1 = V_{\varphi}(\varphi_0) \), \( V^2 = V_{\varphi \varphi}(\varphi_0) \tilde{\varphi}_0 + 3\tilde{F}_1 \tilde{V}_{\varphi}(\varphi_0) \), \( \dot{\varphi}_0 \equiv \sqrt{\gamma_{ab} \dot{\varphi}_0^a \dot{\varphi}_0^b} = \sqrt{\tilde{\varphi}_0^2 + 2\kappa_2 \tilde{\varphi}_0}, \)

\( T^a = \frac{\tilde{\varphi}_0}{\phi_0} \) and \( N_a = \sqrt{-\gamma} \dot{e}_{ab} T^b \). To obtain (23), we use that \( T_0 T^a = 1, N_0 N^a = -1 \) and \( T_0 N^a = 0 \). This means that \( T^a \) and \( N^a \) is a complete base in the field space of \( \varphi_0 \) and \( \tilde{\varphi}_0 \). At this point, we must say that our model is anomalous, because it has phantom fields. For this reason the determinant of the fields space metric is negative, specifically \( \gamma = -\kappa_2^2 \). For this reason, we have that \( V^a = V_T T^a - V_N N^a \) with \( V_T = T^a V_0 \) and \( V_N = N^a V_a \). Additionally, we have the condition \( \gamma_{ab} = \left( \begin{array}{cc} 1 & \kappa_2 \\ \kappa_2 & 0 \end{array} \right) = T_a T_b - N_a N_b \). To satisfy this, we need \( T^a = \left( \frac{T}{1 - \frac{T^2}{2\kappa_2 T}} \right) \)

and \( N^a = \left( -\frac{T}{1 + \frac{T^2}{2\kappa_2 T}} \right) \), with \( T + \frac{V_N}{\phi_0} T = 0 \).

In conclusion, \( T \) represent the direction of \( \phi^a \) in the field space in \( \tilde{\delta} \) Inflation, where \( \tilde{\varphi}_0 = T \dot{\varphi}_0 \) and \( \tilde{\dot{\varphi}}_0 = \frac{1 - T^2}{2\kappa_2 T} \phi_0 \). Now, we can define the inflationary parameters and simplify the equations of motion.

4.2. Inflationary Parameters and Background Equations.

Usually, slow-roll parameters are defined to study inflation. In this model, we can define similar parameters. In first place, we have the usual parameters: \( \epsilon \equiv -\frac{H}{\dot{H}} \) and \( \eta \equiv -\frac{\ddot{\phi}_0}{H \dot{\phi}_0} \). Then, using \( a(t) \propto e^N \), we have:

\[
\epsilon = \frac{\kappa_2 \ddot{\phi}_0}{2 T \dot{\phi}_0}, \quad \eta_T \equiv \frac{\eta a}{T} = 3 + \frac{V_T}{H^2 \phi_0^2} \quad \text{and} \quad \eta_N \equiv \frac{\eta a N}{H^2 \phi_0} = \frac{V_N}{H^2 \phi_0}. \quad (24)
\]

Besides, the equation of motion say \( H^2 = \frac{\kappa V(\varphi_0)}{3 - \frac{5}{2} \kappa_2 \phi_0^2} \). On the other side, we have a effective scale factor, given now by \( a_{eff}(t) \equiv a(t) \sqrt{1 + \kappa_2 \tilde{F}_1(t) \frac{1 + 3\kappa_2 \tilde{F}_1(t)}{1 + 3\kappa_2 \tilde{F}_1(t)}} \). As we said in Chapter 3, an accelerated growth can be produced by a pole in \( a_{eff}(t) \). In [9], we fixed the parameters to produce a pole to explain the accelerate expansion of the universe without dark energy. However, a Big-Rip is generated. For this reason, the dynamic of the scalar fields have to produce a \( \omega(t) \) such that the expansion slow down when inflation ends. To consider this effect in inflation, we need to define additional parameters, like the effective Hubble parameter and the effective expansion parameter:

\[
\mathcal{H} \equiv \frac{\dot{a}_{eff}(t)}{a_{eff}(t)} = H \left( 1 - \frac{\kappa_2 \tilde{F}_1'}{1 + \kappa_2 \tilde{F}_1} \right), \quad (25)
\]

\[
\epsilon_{eff} \equiv -\frac{\dot{\mathcal{H}}}{\mathcal{H}^2} = \frac{H^2}{\mathcal{H}^2} \left( \epsilon + \frac{\kappa_2 \left( 3 \tilde{F}_1 - 2 \epsilon \right)}{1 + \kappa_2 \tilde{F}_1} - \frac{2 \left( 2 + 3\kappa_2 \tilde{F}_1 \right) \kappa_2 \tilde{F}_1'}{1 + 3\kappa_2 \tilde{F}_1} \right), \quad (26)
\]
where \( \phi'(t) = \frac{d}{dN} \) and we use the background equations (19-21) and \( \dot{\epsilon} \equiv \dot{\varphi}_0 \ddot{\varphi}_0 = \frac{\kappa}{2\kappa_2} (1 - T^2) \phi_0^2 \). In GR, inflation needs \( \epsilon \ll 1 \), because it must obey \( V(\varphi_0(t)) \gg \dot{\varphi}_0^2(t) \) to obtain an exponential expansion. On the other side, in \( \bar{\delta} \) Gravity, the accelerated expansion could be produced by a pole in \( a_{eff}(t) \), such that \( \mathcal{H} \) is the effective expansion rate. For this reason, the condition to guarantee inflation in \( \bar{\delta} \) Gravity is \( \epsilon_{eff} \ll 1 \). On the other side, the background equations of motion, (19-21), can be reduced to:

\[
\dot{\Phi}_0'' + (3 - \epsilon) \left( \dot{\Phi}_0 + \frac{V_T}{\kappa V(\varphi_0)} \right) = 0 \quad (27)
\]

\[
T'' + \frac{V_N}{H^2 \Phi_0^2} T = 0 \quad (28)
\]

\[
\ddot{F}_1'' + (3 - \epsilon) \left( \ddot{F}_1 - 6\dot{F}_1 - \frac{2V_{,\varphi}(\varphi_0)}{V(\varphi_0)} \ddot{\varphi}_0 \right) = 0, \quad (29)
\]

where \( V_T = \frac{1+T^2}{2T} V_{,\varphi}(\varphi_0) + \kappa_2 T \left( 3V_{,\varphi}(\varphi_0) \ddot{F}_1 + V_{,\varphi\varphi}(\varphi_0) \dot{\varphi}_0 \right) \) and \( V_N = \frac{1-T^2}{2T} V_{,\varphi}(\varphi_0) - \kappa_2 T \left( 3V_{,\varphi}(\varphi_0) \ddot{F}_1 + V_{,\varphi\varphi}(\varphi_0) \dot{\varphi}_0 \right) \). From (25), we can see that the simplest case is \( \ddot{F}_1 \rightarrow cte \), because \( \mathcal{H} \rightarrow H \). In fact, the effective scale factor is \( a_{eff}(t) \propto a(t) \). This means that \( \epsilon_{eff} \rightarrow \epsilon \) and \( \bar{\epsilon} \rightarrow 0 \). Now, from equation (29), we have \( \ddot{\varphi}_0 \rightarrow -3\ddot{F}_1 \frac{V(\varphi_0)}{V_{,\varphi}(\varphi_0)} \), so \( \bar{\epsilon} \epsilon \left( \frac{V(\varphi_0) V_{,\varphi\varphi}(\varphi_0)}{V_{,\varphi}(\varphi_0)} - 1 \right) \rightarrow 0 \). This means that \( \epsilon \rightarrow 0 \), unless \( \ddot{F}_1 = 0 \) or \( V(\varphi) = V_0 e^{\lambda \varphi} \). Then, in any other case, if \( \bar{\epsilon} \ll 1 \), the inflation expansion is totally slow-rolling and the evolution is just like the usual inflation with GR. A similar situation is happened when \( \ddot{F}_1 \gg 1 \), such that \( a_{eff}(t) \rightarrow a(t) \sqrt{3} \).

In conclusion, if \( \ddot{F}_1 \rightarrow cte \) or \( \ddot{F}_1 \gg 1 \), the \( \bar{\delta} \) Gravity effect is negligible and the expansion is governed by the original inflaton \( \varphi_0 \). Otherwise \( \bar{\delta} \) Gravity is important for the expansion and we have to take it into account for a specifical model, given by \( V(\varphi_0) \). This work is in progress and we will publish an article very soon studying different models and we will compute the power spectrum.

5. Conclusions.

In this work, we have introduced basics concept and an initial analysis of \( \bar{\delta} \) Gravity in inflation. \( \bar{\delta} \) Gravity is a modified model with a new gravitational field \( g_{\mu\nu} \). A quantum field theory analysis of \( \bar{\delta} \) Gravity has been developed in [7], where we proved that it lives at just one loop. Besides, in a classical level, we saw that the necessary quantity of dark matter could be considerably less that we expected and it predicts an accelerated expansion without a cosmological constant or additional scalar fields [9]). Additionally, the scale factor agrees with the standard cosmology at early times and shows acceleration only at late times. Therefore we expect that primordial density perturbations should not have large corrections. All these are the motivation to study inflation with \( \bar{\delta} \) Gravity.

In first place, we studied the equation of motion for just one perfect fluid with equations of state given by \( p(t) = \omega \rho(t) \) and \( \bar{p}(t) = \omega \bar{\rho}(t) \). In that case, we conclude that an accelerated expansion is possible, but inflation never ends. Therefore, a perfect fluid with a \( \omega = cte \) can not explain inflation. A natural second option is to use two or more fluids. In [9] we used two fluid, radiation (\( \omega = \frac{1}{3} \)) and non-relativistic matter (\( \omega = 0 \)), to explain the accelerated expansion of the universe. However, the behavior of \( \omega_{eff}(X) \) for one and two fluid are very similar. For that reason, probably the best option is to use a fluid with a non-constant \( \omega(t) \). In that case, we have a new element given by a \( \bar{\omega}(t) \neq 0 \). For this, we used a scalar field to produce inflation.

In second place, we developed \( \bar{\delta} \) inflation. It is a model with a scalar field called inflaton and its \( \bar{\delta} \) partner, called \( \bar{\delta} \) inflaton. In traditional inflation with GR, the exponential expansion is
produced by the inflaton and its potential $V(\varphi)$, in a slow-roll approximation. In $\tilde{\delta}$ inflation, the expansion could be produced by a $\tilde{\delta}$ Gravity effect, given by the modified scale factor. To describe the dynamic of the expansion, we defined effective inflationary parameters. For example, if $\epsilon_{\text{eff}} \sim 0$ or $\epsilon_{\text{eff}} < 0$, the expansion is exponential or phantom-like respectively, and if $\epsilon_{\text{eff}} = 1$, inflation is finished. Then, $\epsilon_{\text{eff}}$ represent the effects in the expansion by inflatons and $\tilde{\delta}$ Gravity. In fact, the expansion could be accelerated and $\epsilon \sim 1$.

In resume, $\tilde{\delta}$ Gravity could be important for the expansion in inflation for a specific potential. In a future work, different potentials will be elaborated, including a calculation of the power spectrum to understand much better how $\tilde{\delta}$ Gravity explain the Dark Sector.

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Appendix A: Cosmological Case.
From [9], we know that the effective scale factor in the cosmological case is:

$$Y(Y) = \frac{1 - L_2 \sqrt{Y + C} + L_1 \sqrt{Y}}{1 - L_2 Y \sqrt{Y + C} + 3L_1 \sqrt{Y}} \left( \sqrt{\frac{Y}{C}} + 1 \ln \left( \sqrt{\frac{Y + 1}{C}} - 2 \right) \right),$$  \hspace{1cm} (A.1)

where $Y = \frac{a(t)}{a_0}$ is the usual scale factor normalized in the present and $L_1, L_2$ and $C = \frac{\Omega_{\Lambda}}{\Omega_M}$ are parameters in the model. The effective equation of state parameter is given by $\omega_{\text{eff}}(Y) = -1 - \frac{2YH(Y)\mathcal{H}(Y)}{3\mathcal{H}^2(Y)}$, with $\mathcal{H}(Y) = \frac{YH(Y)Y'(Y)}{3(Y)}$ and $(Y')' = \frac{dY'}{dY}$. The expression of this parameter is very complicated, however we can prove that:

$$\omega_{\text{eff}}(Y \to 0) = \frac{1}{3} + O(C)$$ \hspace{1cm} (A.2)

$$\omega_{\text{eff}}(Y \to 1) = -\frac{3L_2 \left( 16L_1^2 - 8L_1L_2 + 32L_1 - 5L_2 + 12 \right)}{(8L_1^2 - 8L_1L_2 + 2L_2^2 + 16L_1 - 5L_2 + 6)^2} + O(C)$$ \hspace{1cm} (A.3)

$$\omega_{\text{eff}}(Y \to Y_{\text{Rip}}) = -\frac{7}{3} + O(C),$$ \hspace{1cm} (A.4)

where $Y_{\text{Rip}} = \left( \frac{1 + 2L_1}{L_2} \right)^{\frac{2}{3}}$ and $C \ll 1$. Therefore, the effective equation of state evolve from a radiation fluid to a phantom-like fluid ($\omega_{\text{eff}} < -1$). In Figure A1, we can see the behavior of $\omega_{\text{eff}}(Y)$ using the parameters values obtained with the supernovae data [9]. They are $L_1 = 1.565$, $L_2 = 2.262$ and $C = 1.82 \times 10^{-4}$. In this case, we used two different fluids to explain the accelerated expansion of the universe, non-relativistic matter ($\omega = 0$) and radiation ($\omega = \frac{1}{3}$). With these parameters, we have that $\omega_{\text{eff}}(1) = -0.926.

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Figure A1. $\omega_{\text{eff}}(Y)$ vs $Y$ with non-relativistic matter ($\omega = 0$) and radiation ($\omega = \frac{1}{3}$), using the parameters values obtained in [9] to explain the accelerated expansion of the universe. (a) Behavior of $\omega_{\text{eff}}(Y)$ from the beginning of the universe to the Big-Rip. (b) Behavior of $\omega_{\text{eff}}(Y)$ in the early universe until the matter-radiation equality epoch, where $Y = C = 1.82 \times 10^{-4}$. (c) Behavior of $\omega_{\text{eff}}(Y)$ close to the moment where the effective fluid is non-relativistic ($\omega = 0$), like cold dark matter (CDM). Clearly, we can see how the effective equation of state evolve from a radiation fluid ($\omega_{\text{eff}} = \frac{1}{3}$) to a phantom-like fluid ($\omega_{\text{eff}} < -1$) close to Big-Rip epoch.

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