ON GEOMETRIC BOTT-CHERN FORMALITY AND DEFORMATIONS

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ABSTRACT. A notion of geometric formality in the context of Bott-Chern and Aeppli cohomologies on a complex manifold is discussed. In particular, by using Aeppli-Bott-Chern-Massey triple products, it is proved that geometric Aeppli-Bott-Chern formality is not stable under small deformations of the complex structure.

INTRODUCTION

On a complex manifold one can consider two different kinds of invariants: the topological ones of the underline manifold and the complex ones. Among the first ones a fundamental role is played by de Rham cohomology, among the second ones we recall the Dolbeault, Bott-Chern and Aeppli cohomologies; where, Bott-Chern and Aeppli cohomologies of a complex manifold $X$ are, respectively, defined as

$$H_{BC}^{\bullet,\bullet}(X) := \frac{\text{Ker} \partial \cap \text{Ker} \overline{\partial}}{\text{Im} \partial \overline{\partial}}, \quad H_{A}^{\bullet,\bullet}(X) := \frac{\text{Ker} \partial \overline{\partial}}{\text{Im} \partial + \text{Im} \overline{\partial}}.$$ 

Since all the cohomologies just mentioned coincide on a compact Kähler manifold, more precisely $\partial \overline{\partial}$-lemma holds on $X$ (this is in particular true on a Kähler manifold) if and only if the maps induced by identity in the diagram

$$
\begin{array}{ccc}
H_{BC}^{\bullet,\bullet}(X) & \xrightarrow{\text{id}} & H_{BC}^{\bullet,\bullet}(X) \\
H_{\partial}^{\bullet,\bullet}(X) & & H_{\partial}^{\bullet,\bullet}(X) \\
H_{dR}^{\bullet,\bullet}(X, \mathbb{C}) & & H_{A}^{\bullet,\bullet}(X) \\
H_{A}^{\bullet,\bullet}(X) & \xleftarrow{\text{id}} & H_{A}^{\bullet,\bullet}(X)
\end{array}
$$

are all isomorphisms, then Bott-Chern and Aeppli cohomologies could provide more informations on the complex structure when $X$ does not admit a Kähler metric.

The theory of formality, developed by Sullivan, concerns with differential-graded-algebras, namely graded algebras endowed with a derivation with square equal to 0. An immediate example is given by the space of differential (resp. complex) forms on a differentiable manifold.
(resp. complex) manifold together with the exterior derivative. It is proven in [6] that compact complex manifolds satisfying $\partial\bar{\partial}$-lemma are formal in the sense of Sullivan.

On the other side, on a complex manifold $X$ the double complex of bigraded forms $(\Lambda^{\bullet,\bullet} X, \partial, \bar{\partial})$ is naturally defined; then, one could ask whether a notion of formality could be defined in case of bidifferential-bigraded-algebras. In this context Neisendorfer and Taylor developed a formality theory for the Dolbeault complex on complex manifolds (see [10]). In particular, we are interested in a formality notion for Bott-Chern-cohomology.

Inspired by Kotschick [8], D. Angella and the second author in [3], define a compact complex manifold $X$ being geometrically-H$_{BC}$-formal if there exists a Hermitian metric $g$ on $X$ such that the space of $\Delta_{BC}$-harmonic forms (in the sense of Schweitzer [13]) has a structure of algebra. Moreover, an obstruction to the existence of such a metric on $X$ is provided by Aeppli-Bott-Chern-Massey triple products (see Theorem 2.2).

In this note we are interested in studying the relationship of this new notion with the complex structure, in particular we discuss the behaviour of geometric-H$_{BC}$-formality under small deformations of the complex structure (see [14] for similar results for Dolbeault formality).

Indeed, considering compact complex surfaces diffeomorphic to solvmanifolds the property considered is open, however, more in general, we prove the following

**Theorem 1 (see Theorem 3.1 and Corollary 3.2).** The property of geometric-H$_{BC}$-formality is not stable under small deformations of the complex structure.

A key tool in the proof of Theorem 1 is Theorem 2.2. First of all we construct a complex curve $J_t$ of complex structures on $X = S^3 \times S^3$ such that $J_0$ is the the geometrically-H$_{BC}$-formal Calabi-Eckmann complex structure on $X$; then, by computing the Bott-Chern cohomology of $X_t = (S^3 \times S^3, J_t)$ for small $t$, we exhibit a non-trivial Aeppli-Bott-Chern Massey triple product on $X_t$, for $t \neq 0$. Furthermore, we show that the non holomorphically parallelizable Nakamura manifold has no geometrically H$_{BC}$-formal metric (see Example 2.3).

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1. **Bott-Chern cohomology and Aeppli-Bott-Chern geometrical formality**

Let $X$ be a compact complex manifold of complex dimension $n$. We will denote by $A^{p,q}(X)$ the space of complex $(p,q)$-forms on $X$. The Bott-Chern and Aeppli cohomology groups of $X$ are defined respectively as (see [1] and [4])

$$ H^*_{BC}(X) = \frac{\ker \partial \cap \ker \bar{\partial}}{\im \partial \bar{\partial}}, \quad H^*_A(X) = \frac{\ker \partial \bar{\partial}}{\im \partial + \im \bar{\partial}}. $$

Let $g$ be a Hermitian metric on $X$ and $*: A^{p,q}(X) \to A^{n-p,n-q}(X)$ be the complex Hodge operator associated with $g$. Let $\Delta_{BC}$ and $\Delta_A$ be the 4-th order elliptic self-adjoint differential operators defined respectively as

$$ \Delta_{BC}^g := (\partial \bar{\partial})^* (\partial \bar{\partial}) + (\partial \bar{\partial})^* (\partial \bar{\partial}) + (\partial \bar{\partial})^* (\partial \bar{\partial})^* + (\partial \bar{\partial})^* (\partial \bar{\partial}) + \partial^* \partial + \partial^* \partial $$

and

$$ \Delta_A^g := \partial \partial^* + \overline{\partial} \partial^* + (\partial \bar{\partial})^* (\partial \bar{\partial}) + (\partial \bar{\partial}) (\partial \bar{\partial})^* + (\partial \bar{\partial})^* (\partial \bar{\partial}) + (\partial \bar{\partial}) (\partial \bar{\partial})^*.$$
Then, accordingly to [3], it turns out that $H^{\bullet, \bullet}_{BC}(X) \simeq \ker \tilde{\Delta}_B$, and $H^{\bullet, \bullet}_{A}(X) \simeq \ker \tilde{\Delta}_A$, so that $H^{\bullet, \bullet}_{BC}(X)$ and $H^{\bullet, \bullet}_{A}(X)$ are finite dimensional complex vector spaces. Denoting by $\alpha$ a $(p,q)$-form on $X$, note that

$$\alpha \in \ker \tilde{\Delta}_B^q \iff \begin{cases} \partial \alpha = 0, \\
\bar{\partial} \alpha = 0, \\
\bar{\partial} \bar{\partial} + \alpha = 0,
\end{cases} \iff \ast \alpha \in \ker \tilde{\Delta}_A^q.$$ 

Therefore, $\ast$ induces an isomorphism between $H^p,q_{BC}(X)$ and $H^{n-p,n-q}(X)$. Furthermore, the wedge product induces a structure of algebra on $\bigoplus_{p,q} H^p,q_{BC}(X)$ and a structure of $\bigoplus_{p,q} H^p,q_{BC}(X)$-module on $\bigoplus_{p,q} H^p,q_{BC}(X)$ (see [13, Lemme 2.5]).

Since in general the wedge product of harmonic forms may be not a harmonic form, the following definition makes sense (see [8] for Riemannian metrics)

**Definition 1.1.** A Hermitian metric $g$ on $X$ is said to be geometrically-$H_{BC}$-formal if $\ker \tilde{\Delta}_B^q$ is an algebra. Similarly, a compact complex manifold $X$ is said to be geometrically-$H_{BC}$-formal if there exists a geometrically-$H_{BC}$-formal Hermitian metric on $X$.

2. Aeppli-Bott-Chern-Massey triple products

Let $X$ be a compact complex manifold and denote by $(\mathcal{A}^{\bullet, \bullet}(X), \partial, \bar{\partial})$ the bi-differential bi-graded algebra of $(p,q)$-forms on $X$. As we have already noted in section I on a compact complex manifold $X$, the Bott-Chern cohomology has a structure of algebra, instead, the Aeppli cohomology has a structure of $H^{\bullet, \bullet}_{BC}(X)$-module. This motivates the following (see [3])

**Definition 2.1.** Take

$$\alpha_{12} = [\alpha_{12}] \in H^p,q_{BC}(X), \quad \alpha_{23} = [\alpha_{23}] \in H^{r,s}_{BC}(X), \quad \alpha_{34} = [\alpha_{34}] \in H^{u,v}_{BC}(X),$$

such that $\alpha_{12} \cup \alpha_{23} = 0$ in $H^{p+q+r+s}_{BC}(X)$ and $\alpha_{23} \cup \alpha_{34} = 0$ in $H^{r+s+u+v}_{BC}(X)$: let

$$(-1)^{p+q} \alpha_{12} \wedge \alpha_{23} = \bar{\partial} \bar{\partial} \alpha_{13} \quad \text{and} \quad (-1)^{r+s} \alpha_{23} \wedge \alpha_{34} = \partial \partial \alpha_{24}.$$ 

The Aeppli-Bott-Chern-Massey product is defined as

$$\alpha_{1234} := \langle \alpha_{12}, \alpha_{23}, \alpha_{34} \rangle_{ABC} := \left[ (-1)^{p+q} \alpha_{12} \wedge \alpha_{24} - (-1)^{r+s} \alpha_{13} \wedge \alpha_{34} \right] \in$$

$$H^{p+r+u-1,q+s+v-1}_{BC}(X) \cup H^{p+u-1,q+s+v-1}_{A}(X) + H^{p+q+r+s-1}_{A}(X) \cup H^{u,v}_{BC}(X).$$

Similarly to the real case, Aeppli-Bott-Chern-Massey triple products provide an obstruction to geometric-$H_{BC}$-formality (see also [3]).

**Theorem 2.2.** Let $X$ be a compact complex manifold. If $X$ is geometrically-$H_{BC}$-formal then the Aeppli-Bott-Chern-Massey triple products are trivial.

**Proof.** Fix a Hermitian metric $g$ on $X$ such that $\ker \tilde{\Delta}_B^q$ has a structure of algebra. Take

$$\alpha_{12} = [\alpha_{12}] \in H^p,q_{BC}(X), \quad \alpha_{23} = [\alpha_{23}] \in H^{r,s}_{BC}(X), \quad \alpha_{34} = [\alpha_{34}] \in H^{u,v}_{BC}(X),$$

such that $\alpha_{12} \cup \alpha_{23} = 0$ in $H^{p+q+r+s}_{BC}(X)$ and $\alpha_{23} \cup \alpha_{34} = 0$ in $H^{r+s+u+v}_{BC}(X)$, with $\alpha_{12}, \alpha_{23}, \alpha_{34}$ harmonic representatives in the respective classes. Then $\alpha_{12} \wedge \alpha_{23}$ and $\alpha_{23} \wedge \alpha_{34}$ are
harmonic forms with respect to the Laplacian $\tilde{\Delta}^g_{BC}$. Hence, with the notation introduced, $\alpha_{13} = 0$ and $\alpha_{24} = 0$. Therefore, by definition, $\langle a_{12}, a_{23}, a_{34} \rangle_{ABC} = 0$. \hfill \Box

**Example 2.3.** Let $G = \mathbb{C} \ltimes \mathbb{C}^2$, where $\varphi : \mathbb{C} \to \text{GL}(2, \mathbb{C})$ is defined as

$$\varphi(x + iy) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}.$$  

Then for some $a \in \mathbb{R}$ the matrix $\begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}$ is conjugated to a matrix in $\text{SL}(2, \mathbb{Z})$. Then $\Gamma := (a\mathbb{Z} + i2\pi\mathbb{Z}) \ltimes \varphi \Gamma''$, with $\Gamma''$ lattice in $\mathbb{C}^2$, is a lattice in $G$ (see [2] and [9]). Denoting with $(z_1, z_2, z_3)$ global coordinates on $G$, the following forms

$$\psi_1 = dz_1, \quad \psi_2 = e^{-\frac{1}{2}(z_1 + \overline{z}_1)}dz_2, \quad \psi_3 = e^{\frac{1}{2}(z_1 + \overline{z}_1)}dz_3$$

are $\Gamma$-invariant. A direct computation shows that

$$\partial \psi_1 = 0 \quad \partial \psi_2 = -\frac{1}{2} \psi_2^{12} \quad \partial \psi_3 = \frac{1}{2} \psi_3^{13}$$

$$\overline{\partial} \psi_1 = 0 \quad \overline{\partial} \psi_2 = \frac{1}{2} \psi_2^{12} \quad \overline{\partial} \psi_3 = -\frac{1}{2} \psi_3^{13},$$

where $\psi^A \overline{\psi}^B = \psi^A \wedge \overline{\psi}^B$ and so on. Therefore $\{\psi_1, \psi_2, \psi_3\}$ give rise to complex $(1,0)$-forms on the compact manifold $X = \Gamma \backslash G$. We will show that $X$ is not geometrically-$H_{BC}$-formal. Let

$$a_{12} = [e^{\frac{1}{2}(z_1 - \overline{z}_1)} \psi^{17}] \in H^{1,1}_{BC}(X), \quad a_{23} = [e^{\frac{1}{2}(z_1 - \overline{z}_1)} \psi^{17}] \in H^{0,2}_{BC}(X),$$

$$a_{34} = [e^{-\frac{1}{2}(z_1 - \overline{z}_1)} \psi^{13}] \in H^{1,1}_{BC}(X).$$

Then,

$$e^{z_1 - \overline{z}_1} \psi^{123} = \overline{\partial}(-e^{z_1 - \overline{z}_1} \psi^{23}).$$

Therefore,

$$\langle a_{12}, a_{23}, a_{34} \rangle_{ABC} = \left[ e^{\frac{1}{2}(z_1 - \overline{z}_1)} \psi^{123} \right] \in \frac{H^{1,3}_{A}(X)}{H^{1,1}_{BC}(X) \cup H^{0,2}_{A}(X)}.$$  

According to the cohomology computations in [2] table 4], it follows that $e^{\frac{1}{2}(z_1 - \overline{z}_1)} \psi^{123} \in \text{Ker} \tilde{\Delta}^g_A$. Furthermore, a direct computation shows that $[e^{\frac{1}{2}(z_1 - \overline{z}_1)} \psi^{123}] \notin H^{1,1}_{BC}(X) \cup H^{0,2}_{A}(X)$, so that $\langle a_{12}, a_{23}, a_{34} \rangle_{ABC}$ is a non-trivial Aeppli-Bott-Chern Massey product.

3. Instability of Bott-Chern geometrical formality

In this section, starting with a geometrical-$H_{BC}$-formal compact complex manifold, we will construct a complex deformation which is no more geometrically-$H_{BC}$-formal.

Let $X = S^3 \times S^3$ and $S^3 \simeq \text{SU}(2)$ be the Lie group of special unitary $2 \times 2$ matrices and denote by $\text{su}(2)$ the Lie algebra of $\text{SU}(2)$. Denote by $\{e_1, e_2, e_3\}$, $\{f_1, f_2, f_3\}$ a basis of the first copy of $\text{su}(2)$, respectively of the second copy of $\text{su}(2)$ and by $\{e^1, e^2, e^3\}$, $\{f^1, f^2, f^3\}$ the corresponding dual co-frames. Then we have the following commutation rules:

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = 2e_1,$$
and the corresponding Cartan structure equations

\[
\begin{align*}
de^1 &= -2e^2 \wedge e^3 \\
de^2 &= 2e^1 \wedge e^3 \\
de^3 &= -2e^1 \wedge e^2 \\
df^1 &= -2f^2 \wedge f^3 \\
df^2 &= 2f^1 \wedge f^3 \\
df^3 &= -2f^1 \wedge f^2 
\end{align*}
\]

Define a complex structure \( J \) on \( X \) by setting

\[
J e_1 = e_2, \quad J f_1 = f_2, \quad J e_3 = f_3.
\]

Note that \( J \) is a Calabi-Eckmann structure or its conjugate on \( S^3 \times S^3 \) (see \cite{5} and \cite{11}).

We have the following

**Theorem 3.1.** Let \( X = S^3 \times S^3 \) be endowed with the complex structure \( J \). Then \( X \) is geometrically-BC-formal and there exists a small deformation \( \{ X_t \} \) of \( X \) such that \( X_t \) is not geometrically-BC-formal for \( t \neq 0 \).

**Proof.** For the sake of the completeness we will recall the proof of geometric \( H_{BC} \)-formality of \( X \) (see \cite{3}). According to the previous notation, a complex co-frame of \((1,0)\)-forms for \( J \) is given by

\[
\begin{align*}
\varphi^1 &= e^1 + ie^2 \\
\varphi^2 &= f^1 + if^2 \\
\varphi^3 &= e^3 + if^3 
\end{align*}
\]

Therefore the complex structure equations are given by

\[
\begin{align*}
d\varphi^1 &= i\varphi^1 \wedge \varphi^3 + i\varphi^1 \wedge \overline{\varphi^3} \\
d\varphi^2 &= \varphi^2 \wedge \varphi^3 - \varphi^2 \wedge \overline{\varphi^3} \\
d\varphi^3 &= -i\varphi^1 \wedge \overline{\varphi^1} + \varphi^2 \wedge \overline{\varphi^2} 
\end{align*}
\]

in particular,

\[
\begin{align*}
\partial \varphi^1 &= i\varphi^1 \wedge \varphi^3 \\
\partial \varphi^2 &= \varphi^2 \wedge \varphi^3 \\
\partial \varphi^3 &= 0 \\
\overline{\partial} \varphi^1 &= i\varphi^1 \wedge \overline{\varphi^3} \\
\overline{\partial} \varphi^2 &= -\varphi^2 \wedge \overline{\varphi^3} \\
\overline{\partial} \varphi^3 &= -i\varphi^1 \wedge \overline{\varphi^1} + \varphi^2 \wedge \overline{\varphi^2} 
\end{align*}
\]

Now fix the Hermitian metric whose associated fundamental form is

\[
\omega := i \sum_{j=1}^{3} \varphi^j \wedge \overline{\varphi^j}.
\]

As a matter of notation, from now on we shorten, e.g., \( \varphi^1^\top := \varphi^1 \wedge \overline{\varphi^1} \).

As regards the Bott-Chern cohomology, thanks to \cite[Theorem 1.3]{2} we have that the sub-complex

\[
\iota: \wedge \langle \varphi^1, \varphi^2, \varphi^3, \varphi^1, \varphi^2, \varphi^3 \rangle \hookrightarrow \mathcal{A}^{\bullet \bullet}(X)
\]
is such that $H_{BC}(\iota)$ is an isomorphism, hence, by explicit computations, we get

\begin{align*}
H_{BC}^{0,0}(X) &= \mathbb{C} \langle [1] \rangle, \\
H_{BC}^{1,1}(X) &= \mathbb{C} \langle [\varphi^1_1], [\varphi^2_2] \rangle, \\
H_{BC}^{2,1}(X) &= \mathbb{C} \langle [\varphi^{2T} + i\varphi^{1T}] \rangle, \\
H_{BC}^{1,2}(X) &= \mathbb{C} \langle [\varphi^{2T} - i\varphi^{1T}] \rangle, \\
H_{BC}^{2,2}(X) &= \mathbb{C} \langle [\varphi^{12T}] \rangle, \\
H_{BC}^{3,2}(X) &= \mathbb{C} \langle [\varphi^{123T}] \rangle, \\
H_{BC}^{2,3}(X) &= \mathbb{C} \langle [\varphi^{1213}] \rangle, \\
H_{BC}^{3,3}(X) &= \mathbb{C} \langle [\varphi^{1231}] \rangle.
\end{align*}

The other Bott-Chern cohomology groups are trivial.

Notice that the Hermitian metric $g$ associated to $\omega$ is geometrically-$H_{BC}$-formal, hence $S^3 \times S^3$ is geometrically-$H_{BC}$-formal. Now our purpose is to prove that geometrical-$H_{BC}$-formality is not stable under small deformations of the complex structure. In order to get this result, let $J_t$ be the almost complex structure on $X$ defined as

\begin{equation*}
\begin{cases}
\varphi^1_t := \varphi^1 \\
\varphi^2_t := \varphi^2 \\
\varphi^3_t := \varphi^3 - t\varphi^3
\end{cases}
\end{equation*}

then

\begin{equation*}
\begin{cases}
d\varphi^1_t &= \frac{i(t+1)}{1-|t|^2} \varphi^3_t + \frac{i(t+1)}{1-|t|^2} \varphi^3_t \\
d\varphi^2_t &= \frac{1-t}{1-|t|^2} \varphi^3_t + \frac{t-1}{1-|t|^2} \varphi^3_t \\
d\varphi^3_t &= (t-i)\varphi^1_t + (t+1)\varphi^2_t
\end{cases}
\end{equation*}

and consequently $J_t$ is integrable. Set $X_t = (X, J_t)$ and $g_t$ the Hermitian metric whose fundamental form is $\omega_t = \frac{i}{2} \sum \varphi^j_t \wedge \overline{\varphi^j_t}$. By applying again [2, Theorem 1.3], we compute the Bott-Chern cohomology of $X_t$.
if \(|t|^2 + \Re e t - \Im m t \neq 0\) we get

\[
\begin{align*}
H_{BC}^{0,0}(X_t) &= \mathbb{C}\langle [1] \rangle, \\
H_{BC}^{1,1}(X_t) &= \mathbb{C}\langle [\varphi_i^T, \varphi_i^T] \rangle, \\
H_{BC}^{2,1}(X_t) &= \mathbb{C}\langle [\varphi_{i+1}^T - i - t \varphi_i^T + t + 1 \varphi_i^T] \rangle, \\
H_{BC}^{1,2}(X_t) &= \mathbb{C}\langle [\varphi_{i-1}^T + i - t \varphi_i^T + t + 1 \varphi_i^T] \rangle, \\
H_{BC}^{2,2}(X_t) &= \mathbb{C}\langle [\varphi_{i+1}^{123}] \rangle, \\
H_{BC}^{2,3}(X_t) &= \mathbb{C}\langle [\varphi_{i+1}^{123}] \rangle, \\
H_{BC}^{3,2}(X_t) &= \mathbb{C}\langle [\varphi_{i+1}^{123}] \rangle,
\end{align*}
\]

where the other groups are trivial, in particular \(H_{BC}^{2,2}(X,J_t)\) vanishes.

Now we will show that there are no geometric-H_{BC}-formal Hermitian metric on \(X_t\). To this purpose we are going to exhibit a non-trivial ABC-Massey triple product.

Setting

\[
\begin{align*}
\alpha_{12} &= [\varphi_{i+1}^T] \in H_{BC}^{1,1}(X_t), \\
\alpha_{23} &= [\varphi_{i+2}^T] \in H_{BC}^{1,1}(X_t), \\
\alpha_{34} &= [\varphi_{i+3}^T] \in H_{BC}^{1,1}(X_t),
\end{align*}
\]

we get \(\alpha_{12} \cup \alpha_{23} = \alpha_{23} \cup \alpha_{34} = 0\). Indeed

\[
\frac{\partial}{\partial \varphi_{i+3}^T} = [(t-i)(t+1) + (t+1)(t+i)] \varphi_{i+1}^{123} =: A_t \varphi_{i+1}^{123},
\]

so, in the hypothesis that \(|t|^2 + \Re e t - \Im m t \neq 0\), we can take as representatives

\[
\alpha_{13} = -\frac{1}{A_t} \varphi_{i+3}^{T}, \quad \alpha_{24} = 0.
\]

Thus the corresponding ABC-Massey product is

\[
\langle \alpha_{12}, \alpha_{23}, \alpha_{34} \rangle_{ABC} = \left[ -\frac{1}{A_t} \varphi_{i+3}^{2323} \right] \in \frac{H_{BC}^{2,2}}{H_{BC}^{1,1} \cup H_A^{1,1} \cup H_{BC}^{1,1}}(X_t).
\]

It is easy to check that this ABC-Massey triple product is not zero, in fact \(\left[ -\frac{1}{A_t} \varphi_{i+3}^{2323} \right] \neq 0\) in \(H_A^{2,2}(X_t)\) since \(-\frac{1}{A_t} \varphi_{i+3}^{2323}\) is \(\bar{\Delta}^2_A\)-harmonic in \(X_t\). Furthermore \(H_A^{1,1}(X_t) = \{0\}\), since \(H_{BC}^{2,2}(X_t) = \{0\}\).

This concludes the proof. 

As a consequence, we obtain the following

**Corollary 3.2.** The property of geometric-H_{BC}-formality is not stable under small deformations of the complex structure.
Remark 3.3. Recall that a Hermitian metric $g$ on a complex manifold $X$ of dimension $n$ is said to be strong Kähler with torsion (shortly SKT), respectively Gauduchon, if its fundamental form $\omega$ satisfies $\partial\bar{\partial}\omega = 0$, respectively $\partial\bar{\partial}^{n-1}\omega = 0$.

A direct computation shows that the Hermitian metric $\omega := \frac{i}{2} \sum_{j=1}^{3} \varphi_j \wedge \bar{\varphi}_j$ defined on $S^3 \times S^3$ is SKT and Gauduchon. As regard the deformation previously considered, there are two different situations for $X_t$ in a small neighborhood of $t = 0$. If $|t|^2 + \Re t - \Im t \neq 0$ the Hermitian metric $\omega_t := \frac{i}{2} \sum \varphi_t^j \wedge \bar{\varphi}_t^j$ is not SKT, and, more precisely, $X_t$ does not admit such a metric.

Indeed, let
$$\partial\bar{\partial}\varphi_t^{33} = -2 \left(|t|^2 + \Re t - \Im t\right) \varphi_t^{122} =: U_t;$$

then, for $|t|^2 + \Re t - \Im t > 0$, the $(2, 2)$-form $U_t$ gives rise to a $\partial\bar{\partial}$-exact $(1, 1)$-positive non-zero current on $X_t$. Then, in view of the characterisation Theorem of the existence of SKT metrics in terms of currents (see [7] and [12]), it follows that $X_t$ has no SKT metrics for $|t|^2 + \Re t - \Im t > 0$.

Otherwise, if $|t|^2 + \Re t - \Im t = 0$, a straightforward computation shows that the Hermitian metric $\omega_t$ is SKT.

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