A Generalized Variational Principle for Gaussian Random Fields

Flora Koukiou
Laboratoire de physique théorique et modélisation, Université de Cergy-Pontoise, 2, av. A. Chauvin, 95302 Cergy-Pontoise, France
E-mail: flora.koukiou@u-cergy.fr

Abstract.
The freezing property for a family of models involving Gaussian Random Fields is studied in a unified manner and it is related to the behaviour of the entropy of the Gibbs measure in terms of the variational principle. The analysis includes the Gaussian Multiplicative Chaos, Random Energy Model, Spin Glass Mean-Field, and the Random Polymers models.

1. Introduction and Motivation
The freezing phenomenon, expected to occur in a wide class of log-correlated Gaussian fields and Branching Brownian Motion has been recently (see [1] for instance) the subject of many studies. Initially defined for free Gaussian fields, it was extensively studied by physicists in case of random polymers on trees. In physics literature freezing is related to the behaviour of the free energy at low temperatures. The freezing phenomenon can be characterized either as the fact that the free energy becomes linear for temperatures beyond some "freezing temperature", or as the concentration of the Gibbs measure on the extreme values of fields.

In the following, we study the freezing property in relation with the entropy; in particular, for a class of Gaussian random models we give a generalized variational principle allowing to show that the freezing property is equivalent to the vanishing of the entropy of the corresponding Gibbs measure. We expect that the same is valid for the log-correlated Gaussian fields and this will be given in a future publication.

2. Random Gaussian Models
In order to have an unified description of the models to be considered, we first introduce a generic Gaussian model (GGM) as follows. Consider the dyadic tree defined from the alphabet: \( A = \{a, b\} \). The words (or paths) of length \( n \) define the natural configuration space denoted by \( A^n \), e.g. \( a = abaa \) with \( n = |a| = 4 \). To each configuration \( \alpha \in A^n \) of the GGM, we associate the random hamiltonian \( H_n(\alpha, \omega) \) whose explicit form is depending on specific model.

For \( \beta = \frac{1}{T} > 0 \) we define the partition function

\[
Z_n(\beta, \omega) = \sum_{\alpha} \frac{e^{-\beta H_n(\alpha, \omega)}}{e^{\frac{\beta n}{2}}} := \sum_{\alpha} W_n^\beta(\alpha, \omega).
\]
When the randomness $J$ is fixed, the corresponding conditional Gibbs probability measure is denoted by

$$\mu_{n,\beta,\omega}(\alpha) = \frac{W_\beta(\alpha,\omega)}{Z_\beta(\beta,\omega)}.$$ 

Moreover, if $E_J$ denotes the expectation with respect to the randomness $J$, the annealed free energy is given by

$$\overline{g}_n(\beta,\omega) = \frac{1}{n} \log E_J Z_n(\beta,\omega).$$ 

We recall moreover the definitions of the quenched free energy

$$g_n(\beta,\omega) = \frac{1}{n} \log Z_n(\beta,\omega),$$

and the entropy of $\mu_{n,\beta,\omega}(\alpha)$,

$$S(\mu_{n,\beta,\omega}) = - \sum_\alpha \mu_{n,\beta,\omega}(\alpha) \log \mu_{n,\beta,\omega}(\alpha).$$

It is well known that, $\forall \beta > 0$, the $n \to \infty$ (thermodynamic) limits exist $a.s.$ and are non random

$$g_\infty(\beta) = \lim_{n \to \infty} g_n(\beta,\omega)$$

$$S(\mu_\beta) := \lim_{n \to \infty} \frac{1}{n} S(\mu_{n,\beta,\omega}).$$

From the variational principle one has also that

$$g_\infty(\beta) = S(\mu_\beta) + \lim_{n \to \infty} \frac{1}{n} \sum_\alpha \mu_{n,\beta,\omega}(\alpha) \log \frac{W_\beta(\alpha,\omega)}{\mu_{n,\beta,\omega}(\alpha)}.$$ 

The reader can now easily verify that the following models are included in the GGM:

2.1. The Random Energy Model

This model is indeed defined [?] by associating to each vertex of the $n^{th}$ level of the tree the i.i.d. centered Gaussian weights “energies” $(J_\alpha(\omega))$ of variance $n/2$ indexed by the nodes. The Hamiltonian is given by

$$H_n(\alpha,\omega) = \sum_{\alpha \in A^n} J_\alpha(\omega),$$

and one can easily check that $EZ_n(\beta) = 2^n e^{n(\frac{\beta^2}{2} - \frac{\beta}{2})}$. 

---

Figure 1: The tree corresponding to the Generalized Gaussian Model.
2.2. Random Polymers on Trees and Multiplicative Chaos

At each configuration $\alpha \in \mathcal{A}^n$, we associate a family of i.i.d. Gaussian random “weights” $(J_{\text{parent}, \text{child}})$, indexed by the edges,

$$H_n(\alpha, \omega) = \sum_{a \in \mathcal{A}^n} \sum_{i=0}^{n} J_{a_i, a_{i+1}}(\omega).$$

2.3. Mean-Field Spin Glass Model

The Sherrington-Kirkpatrick (SK) mean-field model is defined in a complete graph $[?, ?, ?, ?, ?]$. For this, we consider the family of i.i.d. centered Gaussian random interactions $J_{ij}$ indexed by pairs of levels

$$H_n(\alpha, \omega) = \frac{1}{\sqrt{n-1}} \sum_{1 \leq i < j \leq n} J_{ij}(\omega) \sigma(a_i) \sigma(a_j) \text{ (where } \sigma(a) = -1, \sigma(b) = 1),$$

and it is very simple to show that $E_J Z_n(\beta, J) = 2^n e^{\frac{\beta^2}{4} - \frac{\beta}{2}} n$.

These models are studied extensively and in different levels of rigor. There is a rich physics and, more recently, mathematical literature and studies of the phase transitions and properties of the Gibbs measures $[?, ?, ?, ?, ?, ?]$. Here we only focus on the relationship of the freezing property with the entropy of the Gibbs measure and show that vanishing of the entropy defines freezing as well.

All models covered by the GGM undergo a phase transition at a critical temperature $\beta_c$ such that the high temperature region is characterized by

$$\beta_c = \sup\{\beta : g_\infty(\beta) = g_\infty(\beta)\}.$$ 

Hence, at low temperature, from the concavity of logarithm, we always have

$$g_\infty(\beta) < \overline{g}(\beta), \text{ for } \beta > \beta_c.$$ 

We remark moreover that, for $\beta \leq 1$, all previously defined models have similar thermodynamic behaviour and as we can see in the next section they have the same entropy $S(\mu_\beta)$.

3. Entropy versus Freezing

Returning to the GMM model, we define moreover the freezing temperature $\beta_f$ by

$$\beta_f = \min\{\beta \geq \beta_c : S(\mu_\beta) = 0\}.$$ 

One can verify that $\beta_c = \beta_f$ for the REM, Random Polymers on Trees and Multiplicative Chaos models. In the mean-field spin glass (SK) model, the freezing temperature is bigger than the critical temperature. In particular, the SK model undergoes a phase transition at $\beta = 1$ but its entropy remains positive for small values of $\beta > 1$.

In the following we shall prove the

**Theorem 3.1.** Almost surely, the freezing temperature for the Generic Gaussian Model occurs in the interval $[\beta_c, \beta_*]$ where $\beta_* = 4 \log 4 = 2.77258 \cdots$. Moreover, the specific entropy $S(\mu_{\beta_*})$ of the Gibbs measure vanishes.

$$S(\mu_{\beta_*}) := \lim_{n \to \infty} \frac{1}{n} S(\mu_{n, \beta_*}) = - \lim_{n \to \infty} \frac{1}{n} \sum_{\sigma} \mu_{n, \beta_*}(\sigma) \log \mu_{n, \beta_*}(\sigma) = 0.$$
Thus, for the Gaussian models we have considered, the value of $\beta_*$ gives the maximum inverse temperature of positive entropy. As a consequence, the freezing phenomenon is common for all models covered by the GGM in the region $\beta \geq \beta_*$ and coincides with the fact that the entropy of the Gibbs measures vanishes. Since the GGM includes the SK model our result provides with the low temperature region where the freezing property is valid for the SK model as well.

For the proof of the theorem we first make the following remark concerning the inverse temperature $\beta_*$: the particular value of $\beta_*$ is defined by the intersection of the graph of $g_\infty(\beta)$ and the straight line $\frac{\beta}{\beta_1} g_\infty(\beta_1)$. As one can see in figure 2, the annealed free energy $g_\infty(\beta) = \ln 2 + \frac{\beta^2}{4} - \frac{\beta}{2}$ is plotted as a function of $\beta$ and the straight line is defined by $\frac{\beta}{\beta_1} g_\infty(\beta_1) \equiv \beta g_\infty(\beta_1)$. The two graphs intersect at $\beta_1 \equiv 1$ and $\beta_* \equiv 4 \ln 2 = 2.77258 \ldots$. We can now readily check that the annealed limit, at $\beta = \beta_*$, is simply related to the limit at $\beta_1$ by

$$g_\infty(\beta_*) = \frac{\beta_*}{\beta_1} g_\infty(\beta_1).$$

Figure 2: The value $\beta_* = 4 \ln 2$, is given by the intersection of the graph of the annealed free energy $g_\infty(\beta)$ with the straight line $\beta g_\infty(\beta_1)$.

In figure 2, the graph of the convex function $g_\infty(\beta)$ is plotted and it is easy to check that the minimum is reached at $\beta = 1$. The same holds for the minimum of the limit $g_\infty(\beta)$ because $g_\infty(\beta_1) = g_\infty(\beta_1 - 0) = g_\infty(\beta_1 - 0) = 0$ and the convexity of $g_\infty$.

Introducing now the relative entropy entropy density $s(\lambda|\lambda')$ of the probability measure $\lambda$ w.r.t. the probability measure $\lambda'$, one can show easily that for all $\beta \geq \beta_1$, the limit $g_\infty(\beta)$ is related to the relative entropy by

$$g_\infty(\beta) = g_\infty(\beta_1) + s(\mu_\beta|\mu_{\beta_1}).$$

At the particular value $\beta = \beta_*$, one can show by a simple geometrical argument that

$$s(\mu_\beta|\mu_{\beta_*}) \geq (\beta_* - 2) g_\infty(\beta_1).$$

Using now the facts that at each point of the graph of a convex function there is a tangent line which never goes above the graph and the positivity of the entropy, we obtain, for $\beta_1 \leq \beta \leq \beta_*$, the convexity boundary of the limit $g_\infty(\beta)$ (represented by dashed lines $AC$ and $OC'$ in the figure).

Figure 2: The graph of the convex function $g_\infty(\beta)$ is plotted and it is easy to check that the minimum is reached at $\beta = 1$. The same holds for the minimum of the limit $g_\infty(\beta)$ because $g_\infty(\beta_1) = g_\infty(\beta_1 - 0) = g_\infty(\beta_1 - 0) = 0$ and the convexity of $g_\infty$.

Introducing now the relative entropy entropy density $s(\lambda|\lambda')$ of the probability measure $\lambda$ w.r.t. the probability measure $\lambda'$, one can show easily that for all $\beta \geq \beta_1$, the limit $g_\infty(\beta)$ is related to the relative entropy by

$$g_\infty(\beta) = g_\infty(\beta_1) + s(\mu_\beta|\mu_{\beta_1}).$$

At the particular value $\beta = \beta_*$, one can show by a simple geometrical argument that

$$s(\mu_\beta|\mu_{\beta_*}) \geq (\beta_* - 2) g_\infty(\beta_1).$$

Using now the facts that at each point of the graph of a convex function there is a tangent line which never goes above the graph and the positivity of the entropy, we obtain, for $\beta_1 \leq \beta \leq \beta_*$, the convexity boundary of the limit $g_\infty(\beta)$ (represented by dashed lines $AC$ and $OC'$ in the figure).
In order to show that the entropy $S(\mu_\beta)$ of the Gibbs measure vanishes at $\beta_*$, we assume that the relative entropy reaches its lower bound $s(\mu_\beta | \mu_{\beta_\ast})$. It follows, by the variational principle, that the entropy indeed vanishes. We conclude by remarking that, from convexity, if the relative entropy is bigger than its lower bound, the entropy of the Gibbs measure had already vanished at a value of $\beta$ smaller than $\beta_*$. 

4. Conclusion

In this note we showed the relationship between the entropy of Gibbs measure and the freezing property for a class of Gaussian Free Models. Another interesting question concerns the Hausdorff dimension of the support of the Gibbs measure. One can show using the theorem of this note that this dimension vanishes at $\beta_*$. We expect the same relationship for log-correlated Gaussian Fields and Brownian Branching Motion.

References

[1] M. Aizenman, J. L. Lebowitz, and D. Ruelle. Some rigorous results on the Sherrington-Kirkpatrick spin glass model. Comm. Math. Phys., 112(1):3–20, 1987.

[2] P. Collet and F. Koukiou. Large deviations for multiplicative chaos. Comm. Math. Phys., 147(2):329–342, 1992.

[3] Bernard Derrida. Random-energy model: an exactly solvable model of disordered systems. Phys. Rev. B (3), 24(5):2613–2626, 1981.

[4] Francesco Guerra and Fabio Lucio Toninelli. The thermodynamic limit in mean field spin glass models. Comm. Math. Phys., 230(1):71–79, 2002.

[5] F. Koukiou. The low-temperature free energy of the Sherrington-Kirkpatrick spin glass model. Europhys. Lett., 33(2):95–98, 1996.

[6] F. Koukiou. Low temperature behaviour of the entropy for mean-field spin-glass models, 2016. Submitted for publication to Reviews of Mathematical Physics.

[7] G. Parisi. A sequence of approximated solutions to the S-K model for spin glasses. J. Phys. A, 13(3):L115–L121, 1980.

[8] L. A. Pastur and M. V. Shcherbina. Absence of self-averaging of the order parameter in the Sherrington-Kirkpatrick model. J. Statist. Phys., 62(1-2):1–19, 1991.

[9] David Sherrington and Scott Kirkpatrick. Solvable model of a spin-glass. Phys. Rev. Lett., 35:1792–1796, Dec 1975.

[10] Eliran Subag and Ofer Zeitouni. Freezing and decorated Poisson point processes. Comm. Math. Phys., 337(1):55–92, 2015.

[11] Michel Talagrand. The Parisi formula. Ann. of Math. (2), 163(1):221–263, 2006.