The General $O(n)$ Quartic Matrix Model and its application to Counting Tangles and Links

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The counting of alternating tangles in terms of their crossing number, number of external legs and connected components is presented here in a unified framework using quantum field-theoretic methods applied to a matrix model of colored links. The overcounting related to topological equivalence of diagrams is removed by means of a renormalization scheme of the matrix model; the corresponding “renormalization equations” are derived. Some particular cases are studied in detail and solved exactly.
1. Introduction

The goal of this paper is to investigate a fairly general enumeration problem related to the theory of knots, links and tangles: we want to count objects which live in 3-dimensional space and are (loosely) made of a certain collection of “ropes”, some of which open (with fixed endpoints) and some closed on themselves, intertwined together in an alternating way. As usual in knot theory, these objects will be considered up to topological equivalence (deformation or ambient isotopy), and represented by their projections on the plane; we shall then classify them according to the (minimal) number of crossings, the number of connected components, and the way the various “external legs” connect to each other, see for example Fig. 1.

![Fig. 1: A diagram with 3 open and 1 closed line, intertwined together with 7 crossings.](image)

Without going into too much detail for now, we see that a convenient way to keep track of the number of connected components and of the connections of the external legs is to use colors, see Fig. 2.

![Fig. 2: Coloring the diagram of Fig. 1. Open lines have fixed colors (distinct from each other), whereas closed lines have arbitrary color.](image)
The colors allow us to distinguish the various external legs and add an extra power series variable in the theory (the number of colors \( n \)) to count separately objects with different numbers of connected components.

The idea to use colors was already suggested in [1,2] and present in the work [3]. At this stage it is natural to define a matrix model whose Feynman diagram expansion will produce such diagrams with \( n \) colors. Here we shall give a unified quantum field-theoretic treatment of this \( O(n) \)-invariant matrix model, which simplifies and generalizes the equations obtained in [3] (section 2 below). In particular, it gives a practical way to do the enumeration by computer; this procedure was recently used in the numerical work [4].

Even though the matrix model we propose is fairly natural, since as we shall see it is the most general quartic \( O(n) \)-invariant matrix model with a single trace in the action, it is in general unsolvable (or at least unsolved). It can be thought of as describing a statistical model on random dynamical lattices; more precisely, it is a model of fully packed loops drawn in \( n \) colors on random tetravalent planar diagrams with weights attached to vertices (intersections or tangencies of loops). Even the corresponding model on a regular (flat) square lattice is not fully understood. However it is tempting to speculate on its universality class; and that putting it on random lattices will correspond to the usual coupling of two-dimensional conformal field theory to gravity, which allows to predict the critical exponents of the theory based on the KPZ relation [5]. This in turn leads to various conjectures on the asymptotic number of large links and tangles, made in [2], which have been checked numerically in [4]. We shall not come back to these conjectures here, but instead produce exact analytic solutions of two particular cases of our matrix model (section 3 below): the classical case \( n = 1 \) (no colors), with some generalizations of the results of [1]; and the case \( n = -2 \), which is interesting because its asymptotic behavior cannot be obviously guessed by the universality arguments mentioned above.

2. General principle

We assume the reader familiar with the concept of links and tangles. Let us recall here that once projected on a plane, they give rise to planar diagrams with tetravalent vertices which must be “decorated” to distinguish under/over-crossings. Link diagrams are closed, whereas tangle diagrams have external legs. The diagrams are said to be alternating if one meets undercrossings and overcrossings alternatingly as one follows the various closed
loops of the diagram. The alternating property allows to ignore the decorations of the
vertices since they can be recovered from the diagram alone (up to a mirror symmetry for
the closed diagrams, see below).

2.1. Definition of the $O(n)$ matrix model

As in [2] and [3], we start with the following matrix integral over $N \times N$ hermitean
matrices

$$Z^{(N)}(n, g) = \int \prod_{a=1}^{n} dM_a e^{N \text{tr} \left( -\frac{1}{2} \sum_{a=1}^{n} M_a^2 + \frac{g}{4} \sum_{a,b=1}^{n} M_a M_b M_a M_b \right)}$$ (2.1)

where $n$ is (for now) a positive integer. The integral is normalized so that $Z^{(N)}(n, 0) = 1$. The partition function (2.1) displays a $O(n)$ symmetry where the $M_a$ form a vector of $O(n)$.

Expanding in power series in $g$ generates Feynman diagrams with double edges (“fat
graphs”) drawn in $n$ colors in such a way that colors cross each other at the vertices. By
taking the large $N$ limit one selects the planar diagrams,\footnote{Note that the next orders in the $1/N$ expansion of the free energy $\log Z^{(N)}$ would correspond to link diagrams drawn on thickened surfaces of higher genus, cf [6].} which are closely related to alternating link diagrams, cf Fig. 3.

![Fig. 3: A planar Feynman diagram of (2.1) and the corresponding alternating link diagram.](image)

More precisely, the large $N$ “free energy”

$$F(n, g) = \lim_{N \to \infty} \frac{\log Z^{(N)}(n, g)}{N^2}$$ (2.2)
is a double generating function of the number \( f_{k;p} \) of alternating link diagrams with \( k \) connected components and \( n \) crossings (weighted by the inverse of their symmetry factor, and with mirror images identified):

\[
F(n, g) = \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} f_{k;p} n^k g^p
\]  

(2.3)

Note that it is clearly possible to analytically continue \( Z^{(N)}(n, g) \) to arbitrary values of \( n \) (using, for example, a Hubbard–Stratonovitch transformation) so that Eq. (2.3) still holds. In particular, the counting of knot diagrams is given by \( F_{1;p} \) and can be obtained by formally taking the limit \( n \to 0 \), in the spirit of the replica method. Also, if \( n \) is an even negative integer one can write fermionic analogues of (2.1), see section 3.2, which display \( Sp(|n|) \) symmetry.

If one is interested in counting objects with a weight of 1, one cannot consider the free energy which corresponds to closed diagrams, but instead correlation functions of the model which generate diagrams with external legs: these are essentially tangle diagrams. Typically, we shall be interested in the two-point function

\[
G(n, g) \equiv \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr} M_a^2 \right\rangle
\]  

(2.4)

where the measure on the matrices \( M_a \) is given by Eq. (2.1) and \( a \) is any fixed index, which generates tangle diagrams with two external legs; and the connected four-point functions

\[
\Gamma_1(n, g) = \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr}(M_a M_b)^2 \right\rangle
\]  

(2.5.1)

\[
\Gamma_2(n, g) = \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr}(M_a^2 M_b^2) \right\rangle - G(n, g)^2
\]  

(2.5.2)

where \( a \) and \( b \) are two distinct indices, which generate tangle diagrams with four external legs of type 1 and 2 (see Fig. 4). Note that the freedom to replace a link diagram with its mirror image by inverting all under/over-crossings is, in the case of correlation functions, removed by fixing conventionally the first crossing encountered starting from a given external leg.

Let us briefly mention for now that the definition of \( G(n, g) \) again assumes \( n \) to be a positive integer, and has a natural continuation to any \( n \); however the definitions of \( \Gamma_i(n, g) \) are only meaningful for \( n \) integer greater or equal to 2, and there is a difficulty associated to this, which will be explained in section 2.3.
2.2. Renormalization of the $O(n)$ model

The model presented above is not sufficient to properly count colored tangles. Essentially, this comes from the fact that there is not a one-to-one correspondence between diagrams and the objects they are obtained from by projection. This generates a redundancy in the counting since to a given knot will correspond many (an infinity of) diagrams, each counted once. In the case of alternating diagrams one can distinguish two steps to remove this redundancy. First one must find a way to restrict ourselves to reduced diagrams which contain no irrelevant crossings (Fig. 5 a)); such diagrams will have minimum number of crossings. It turns out to be convenient to introduce at this point a closely related notion: a link is said to be prime if it cannot be decomposed into two pieces in the way depicted on Fig. 5 b). It is clear that at the level of diagrams, forbidding decompositions of the type of Fig. 5 b) automatically implies that the diagram is reduced; and we shall therefore restrict ourselves to prime links and tangles.

There may still be several reduced diagrams corresponding to the same link: according to the flyping conjecture, proved in [7], two such diagrams are related by a finite sequence of flypes, see Fig. 6.
To summarize, there are two problems: a) the diagrams generated by applying Feynman rules are not necessarily reduced or prime; b) several reduced diagrams may correspond to the same knot due to the flyping equivalence. A study of Figs. 5 and 6 shows that this “overcounting” is local in the diagrams in the sense that problem a) is related to the existence of sub-diagrams with 2 external legs, whereas problem b) is related to a certain class of sub-diagrams with 4 external legs. Clearly such graphs can be cancelled by the inclusion of appropriate counterterms in the action. We are therefore led to the conclusion that we must renormalize the quadratic and quartic interactions of (2.1). Now renormalization theory tell us that we should include in the action from the start every term compatible with the symmetries of the model, since they will be generated dynamically by the renormalization. In order to preserve connectedness we only look for terms of the form of a single trace. A key observation is that, while there is only one such quadratic $O(n)$-invariant term, there are two quartic $O(n)$-invariant terms, which leads to a generalized model with 3 coupling constants in the action (bare coupling constants):

$$Z^{(N)}(n, t, g_1, g_2) = \int \prod_{a=1}^{n} dM_a e^{N \text{ tr} \left( -\frac{t}{2} \sum_{a=1}^{n} M_a^2 + \frac{g_1}{4} \sum_{a,b=1}^{n} (M_a M_b)^2 + \frac{g_2}{2} \sum_{a,b=1}^{n} M_a^2 M_b^2 \right) }$$

(2.6)

The Feynman rules of this model now allow loops of different colors to “avoid” each other, which one can imagine as tangencies (Fig. 7).

![Fig. 7: Vertices of the generalized $O(n)$ matrix model.](image)

We define again the correlation functions $G(n, t, g_1, g_2)$ and $\Gamma_i(n, t, g_1, g_2)$ (Eqs. (2.4) and (2.5)), and want to extract from them the counting of colored alternating tangles with external legs.

The idea is to find the expressions of $t(g)$, $g_1(g)$ and $g_2(g)$ as a function of the renormalized coupling constant $g$, in such a way that the overcounting is suppressed and the correlation functions are generating series in $g$ of the number of colored tangles. At leading order, we shall have $t(g) = 1 + o(1)$, $g_1(g) = g + o(g)$ and $g_2(g) = o(g)$ so that we recover the original model (2.1). However there will be higher order corrections which correspond to the counterterms.
Let us consider $t(g)$ first. It is clear that one must remove all two-legged subdiagrams, that is impose

$$G(n, t, g_1, g_2) = 1 \quad (2.7)$$

\[G = \frac{1}{t - \Sigma} \quad (2.8)\]

where $\Sigma$ is the generating function of 1PI (one-particle irreducible, i.e. which cannot be made disconnected by removing one edge) two-legged diagrams, one finds equivalently that

$$t(g) = 1 + \Sigma(g) \quad (2.9)$$

i.e. the counterterms generated by $t(g)$ must cancel all 1PI two-legged subdiagrams. This is almost a tautology; notice however that one must not cancel all two-legged subdiagrams, since one-particle reducible diagrams would be subtracted multiple times.

![Fig. 8: Decomposition of the two-point function. Reexpanding in powers of $t - 1$ will cancel the powers of $\Sigma$ iff $t = 1 + \Sigma$.](image)

![Fig. 9: Breaking a flype into two elementary flypes.](image)

Next, we must consider the flyping equivalence. Again, it is important to notice that a flype can be made of several “elementary” flypes (Fig. 9), an elementary flype being by definition one that cannot be decomposed any more in this way. In the terminology of QFT, an elementary flype consists precisely of one simple vertex connected by two edges to a non-trivial H-2PI (two-particle irreducible in the horizontal channel) tangle diagram. Non-trivial means not reduced to a single vertex; H-2PI means that the tangle diagram cannot be cut into two pieces containing the left and right external legs respectively, by
removing two edges. We therefore need to introduce auxiliary generating functions $H'_1(g)$, $H'_2(g)$ and $V'_2(g)$ for non-trivial H-2PI tangles of type 1, of type 2 and of type 2 rotated by $\pi/2$ respectively. Only these must be included in the counterterms. It is now a simple matter to consider all possible insertions of elementary flypes as tangle sub-diagrams of a diagram; taking into account the two types of tangle sub-diagrams and the two channels (horizontal and vertical), we find (Fig. 10) that the renormalization of $g_1$ and $g_2$ is simply:

$$g_1(g) = g(1 - 2H'_2(g)) \quad (2.10.1)$$
$$g_2(g) = -g(H'_1(g) + V'_2(g)) \quad (2.10.2)$$

![Counterterms needed to cancel flypes.](image)

Fig. 10: Counterterms needed to cancel flypes.

All that is left is to find the expressions of the auxiliary generating functions in terms of known quantities. They are easily obtained by decomposing the four-point functions in the horizontal and vertical channels, and will not be rederived here (the reader is referred to e.g. [3] for details).

$$H'_2 \pm H'_1 = 1 - \frac{1}{(1 \mp g)(1 + T_2 \pm T_1)} \quad (2.11a)$$
$$H'_2 + nV'_2 + H'_1 = 1 - \frac{1}{(1 - g)(1 + (n + 1)T_2 + T_1)} \quad (2.11b)$$
2.3. Summary and discussion

We shall now summarize and rewrite more explicitly the formulae found previously, as well as discuss their implications.

Let us assume that for a certain \( n \), we have computed the free energy \( F(n, t, g_1, g_2) \).

What can we extract from the formulae of the previous paragraph, and how?

First, let us differentiate \( F \); we find

\[
G = -2 \frac{\partial F}{\partial t} \frac{1}{n}
\]

as well as two other quantities,

\[
F_1 = 4 \frac{\partial}{\partial g_1} \frac{F}{n} = \frac{1}{n} \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr} \sum_{a,b} (M_a M_b)^2 \right\rangle
\]

\[
F_2 = 2 \frac{\partial}{\partial g_2} \frac{F}{n} = \frac{1}{n} \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr} \sum_{a,b} M_a^2 M_b^2 \right\rangle
\]

According to the equations of motion, these three quantities are not independent:

\[
tG = 1 + g_1 F_1 + 2g_2 F_2
\]

Comparing (2.13) with the definition (2.5) of the \( \Gamma_i \), we see that there are two different choices of basis of the four-point functions;\(^2\) using \( O(n) \)-invariance of the measure it is easy to relate them:

\[
F_1 = n \Gamma_1 + 2(\Gamma_2 + G^2)
\]

\[
F_2 = \Gamma_1 + (n + 1)(\Gamma_2 + G^2)
\]

These relations also have a simple diagrammatic interpretation, which proves in particular that they are valid for any (complex) \( n \). One observes that relations (2.15) can be inverted to extract \( \Gamma_1 \) and \( \Gamma_2 \) only if \( n \neq 1, -2 \). These two cases will be the object of study of the next section, they are the first in the series of bosonic / fermionic matrix models; and as will be shown these are the values of \( n \) for which the model possesses only one quartic \( O(n) \)-invariant, contrary to the generic case. For now we simply observe that for \( n = 1, -2 \),

\(^2\) Using the \( \Gamma_i \) as the preferred basis is not only natural diagrammatically; it is also imposed by the structure of the equations such as (2.10) and (2.11).
\( F_1 = F_2 \) and therefore according to (2.13), the free energy \( F \) is a function of \( g_1 + 2g_2 \) only; while for \( n = -2 \), \( F_1 = -2F \) and \( F \) is a function of \( g_1 - g_2 \) only.

Once we have computed \( G, \Gamma_1 \) and \( \Gamma_2 \), we can slightly simplify the renormalization equations using the obvious scaling property:

\[
\begin{align*}
G(n, t, g_1, g_2) &= \frac{1}{t} G(n, 1, g_1/t^2, g_2/t^2) \\
\Gamma_i(n, t, g_1, g_2) &= \frac{1}{t^2} \Gamma_i(n, 1, g_1/t^2, g_2/t^2)
\end{align*}
\tag{2.16a}
\tag{2.16b}
\]

Combining this with Eq. (2.7) results in fixing \( t(g) \):

\[
t(g) = G(n, 1, g_1(g)/t(g)^2, g_2(g)/t(g)^2)
\tag{2.17}
\]

At this stage the three unknowns \( t(g), g_1(g), g_2(g) \) only appear through the combinations \( h_1(g) \equiv g_1(g)/t(g)^2, h_2(g) \equiv g_2(g)/t(g)^2 \); in particular we have the following expressions for the \( \Gamma_i \equiv \Gamma_i(n, t(g), g_1(g), g_2(g)) \):

\[
\Gamma_i = \frac{\Gamma_i(n, 1, h_1, h_2)}{G(n, 1, h_1, h_2)^2}
\tag{2.18}
\]

We only need to solve the two remaining renormalization equations (2.10), which we rewrite here:

\[
\begin{align*}
h_1(g) G(n, 1, h_1, h_2)^2 &= g(1 - 2H'_2(g)) \\
h_2(g) G(n, 1, h_1, h_2)^2 &= -g(H'_1(g) + V'_2(g))
\end{align*}
\tag{2.19.1}
\tag{2.19.2}
\]

where the auxiliary generating functions are still given in terms of the \( \Gamma_i \) by Eqs. (2.11).

Finally, solving Eqs. (2.19) gives access to the \( \Gamma_i \), which are the generating series of the numbers of prime alternating tangles of type \( i \). However, we can go further. By computing other correlation functions in the model and composing them with the solutions \( t(g), g_1(g), g_2(g) \) of the equations above, one can extract the generating functions of the number of alternating tangles with an arbitrary number of external legs. The correlation functions we consider are traces of non-commutative words in the \( M_\alpha \) of degree \( 2k \) (for \( 2k \) external legs). We usually restrict ourselves to connected correlation functions (free cumulants in the language of free probabilities), which exclude configurations in which some strings have no crossings with the other strings and can be pulled out altogether. This choice is only a matter of taste.
For example, there are five $O(n)$-invariants of degree 6, except, as before, for a finite set of values of $n$ for which there are fewer: only 4 for $n = -4$, 3 for $n = 2$, 2 for $n = -2$, 1 for $n = 1$. They are given by:

$$\Xi_1 = \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr}(M_a M_b M_c M_a M_b M_c) \right\rangle - \text{disc. terms} \quad (2.20.1)$$

$$\Xi_2 = \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr}(M_a M_b M_c M_a M_c M_b) \right\rangle - \text{disc. terms} \quad (2.20.2)$$

$$\Xi_3 = \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr}(M_a M_a M_b M_c M_b M_c) \right\rangle - \text{disc. terms} \quad (2.20.3)$$

$$\Xi_4 = \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr}(M_a M_b M_b M_a M_c M_c) \right\rangle - \text{disc. terms} \quad (2.20.4)$$

$$\Xi_5 = \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr}(M_a M_b M_a M_b M_c M_c) \right\rangle - \text{disc. terms} \quad (2.20.5)$$

$(a, b, c$ distinct) and give rise to the various six-legged diagrams depicted on Fig. 11.

![Fig. 11: The five types of tangles with 6 external legs.](image-url)
3. Application: two solvable cases

There are currently two values of \( n \) for which the corresponding matrix model has been exactly solved: \( n = 1 \) and \( n = 2 \). The case \( n = 1 \) is particularly important since it corresponds to counting all alternating tangles regardless of the number of connected components; we shall investigate it here in detail, generalizing known results [8,1].

The application of the \( O(n = 2) \) matrix model (also known as six-vertex model on dynamical random lattices) to knot theory has already been made in [3], using slightly different methods than in the present paper, and we shall not come back to it.

However, we have found earlier that aside from \( n = 1 \), there is another special value of \( n \), namely \(-2\), for which a simplification in the model occurs and we can expect some exact analytic results. We shall present below an analysis of this case.

3.1. The case \( n = 1 \): the usual tangles, and more

As an illustration of the general principle developed above, we present an elementary solution of the case \( n = 1 \), that is the counting of alternating tangles. Since there are no colors one cannot distinguish the way the various external legs are connected; the correlation functions available to us will be specified by the number of external legs only.

Note that this solution, which generalizes the original calculation of the number of prime alternating tangles with 4 external legs found in [8], is technically different from it.

We start by setting \( n = 1 \) in the definition of the partition function (Eq. (2.6)); we find:

\[
Z^{(N)}(t, g_0) = \int dM \, e^{N \text{tr} \left( -\frac{t}{2} M^2 + \frac{g_0}{4} M^4 \right)}
\]  

(3.1)

where \( g_0 \equiv g_1 + 2g_2 \). The fact that the partition function only depends on a particular combination of \( g_1 \) and \( g_2 \) is consistent with what was found in Section 2.3 and related to the existence of only one quartic \( O(n) \)-invariant for \( n = 1 \). The most general “planar” correlation functions of the model are of the form

\[
G_{2\ell}(t, g_0) \equiv \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr} M^{2\ell} \right\rangle
\]  

(3.2)

for which we introduce the generating function:

\[
\omega(\lambda) \equiv \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr} \frac{1}{\lambda - M} \right\rangle = \frac{1}{\lambda} + \sum_{\ell=1}^{\infty} G_{2\ell} \frac{1}{\lambda^{2\ell+1}}
\]  

(3.3)
and the corresponding connected correlation functions $G_{2\ell}^c$, whose generating function is the inverse function $\lambda(\omega)$:

$$\lambda(\omega) = \frac{1}{\omega} + \sum_{\ell=1}^{\infty} G_{2\ell}^c \omega^{2\ell-1}$$  \hspace{1cm} (3.4)

Among them we have the two-point function $G \equiv G_2^c = G_2$ and the connected four-point function $\Gamma \equiv G_4^c = G_4 - 2G_2^2$ which is nothing but the generating function of all tangles (regardless of type): $\Gamma = \Gamma_1 + 2\Gamma_2$. Since we do not have access to $\Gamma_1$ and $\Gamma_2$ separately, we need to recombine the equations of Section 2.2 so that only $\Gamma$ appears in them. Fortunately, this turns out to be possible; taking $(2.10.1)+2\times(2.10.2)$ results in

$$g_0(g) = g(1 - 2H(g))$$ \hspace{1cm} (3.5)

where $H(g) \equiv H_2^c(g) + H_1^c(g) + V_2^c(g)$ is the generating function of all H-2PI non-trivial tangles. Eq. (3.5) can of course be derived directly in a manner similar to Eqs. (2.10), by simply disregarding the types of the tangles i.e. how the outgoing strings are connected to each other inside the tangle.

Setting $n = 1$ in Eq. (2.11b), we also find that

$$H(g) = 1 - \frac{1}{(1 - g)(1 + \Gamma)}$$ \hspace{1cm} (3.6)

so that for $n = 1$ (and $n = 1$ only) we have a closed subset of equations.

We now turn to the solution of our matrix model. We do not repeat the calculation of the $G_{2\ell}$ here since it is a standard result of matrix models, see [9]. Starting from the following expression:

$$\omega(\lambda) = \frac{1}{2} \left( t\lambda - g_0\lambda^3 - (t - g_0\lambda^2 - g_0A)\sqrt{\lambda^2 - 2A} \right)$$ \hspace{1cm} (3.7)

with $A = \frac{1}{3} \frac{t - \sqrt{t^2 - 12g_0}}{g_0}$ solution of

$$3A^2g_0 - 2At + 4 = 0$$ \hspace{1cm} (3.8)

we find that

$$G_{2\ell} = A^\ell \frac{(2\ell - 1)!!}{(\ell + 2)!} \left( 2(\ell + 1) - \frac{\ell}{2} At \right)$$ \hspace{1cm} (3.9)

In particular $G = \frac{1}{6} A(4 - \frac{At}{2})$, and since $G = 1$ according to Eq. (2.7), we can express $t$ as a function of $A$:

$$t = \frac{4}{A^2}(2A - 3)$$ \hspace{1cm} (3.10)
Similarly, using the explicit expression of $\Gamma = G_4 - 2$, plugging it into Eqs. (3.5), (3.6), and using Eqs. (3.8), (3.10) to express $t$ and $g_0$ in terms of $A$ results in the following fifth degree equation for $A$:

$$32(1-g) - 64(1-g)A + 32(1-g)A^2 - 4(1+2g-g^2)A^3 + 6g(1-g)A^4 - g(1-g)A^5 = 0 \quad (3.11)$$

$A(g)$, specified by Eq. (3.11) and $A(g = 0) = 2$, is a well-defined analytic function of $g$ in a neighborhood of $g = 0$. The data of $A(g)$ is enough to recover all correlation functions since we have, combining Eqs. (3.9) and (3.10):

$$G_{2\ell} = 2A^{\ell-1}\frac{(2\ell - 1)!!}{(\ell + 2)!} (3\ell - (\ell - 1)A) \quad (3.12)$$

Similarly, one can extract the connected correlation functions, using the fact that $\lambda(\omega)$ satisfies a cubic equation (cf Eq. (3.7)); after a tedious calculation one finds

$$G^c_{2\ell} = \frac{c_\ell}{\ell!} (A - 2)^{\ell-1} (3\ell - 2 - (\ell - 1)A) \quad (3.13)$$

where $c_\ell$ is a constant (which already appeared in [9]):

$$c_{\ell+1} = \frac{1}{3\ell + 1} \sum_{\ell/2 \leq q \leq \ell} (-4)^{q-\ell} \frac{(\ell + q)!}{(2q - \ell)(\ell - q)!} \quad (3.14)$$

This concludes the calculation of the generating series of the number of tangles with any given number of external legs. In the appendix, the first few orders of $G_4^c$, $G_6^c$, $G_8^c$ are given.

Let us now discuss the asymptotic behavior of the coefficients of the various series for which we found an exact expression. All are simple polynomials in $A(g)$, so that we need to study the latter only. As can be easily checked, the singularity of $A(g)$ closest to the origin is the usual singularity of 2D pure gravity, which is given by $g_{0c} = 4/27$ and $t_c = 4/3$, so that $A_c = 3$, and, plugging into Eq. (3.11),

$$g_c = \frac{\sqrt{21001} - 101}{270} \quad (3.15)$$

We expand $A$ around $g \uparrow g_c$ and find

$$A = 3 - a (g_c - g)^{1/2} + b (g_c - g) + O((g_c - g)^{3/2}) \quad (3.16)$$
with \((a > 0)\)

\[
a^2 = \frac{2877137 + 7087\sqrt{21001}}{339696}
\]

\[
b = \frac{5(99397733 + 2127733\sqrt{21001})}{901510722}
\]

This provides the leading singular part of \(G^c_{2\ell}\):

\[
G^c_{2\ell} = \text{reg} + \frac{c_{\ell}}{(\ell - 2)!}a\left(b + \frac{1}{3}(\ell - 2)a^2\right)(g_c - g)^{3/2} + \cdots
\]

which finally yields the large order behavior of \(G^c_{2\ell}\): if \(G^c_{2\ell} = \sum_{p=1}^{\infty} \gamma_{2\ell;p} g^p\) then

\[
\gamma_{2\ell;p} \xrightarrow{p \to \infty} \frac{3}{4\sqrt{\pi}} \frac{c_{\ell}}{(\ell - 2)!}a\left(b + \frac{1}{3}(\ell - 2)a^2\right) p^{-5/2} g_c^{3/2-p}
\]

For \(\ell = 2\) this result coincides with the theorem 1 of [8]. Note that for any \(\ell\) the asymptotic behavior is the same up to a constant. One can of course send \(\ell\) and \(p\) to infinity in a correlated manner to obtain a non-trivial scaling limit (here, \(\ell \propto p^{1/2}\)); but the result is known to be universal and is therefore the usual scaling loop function of pure gravity, which will not be reproduced here.

### 3.2. The case \(n = -2\): a fermionic matrix model

For \(n\) negative even integer, it is natural, in the spirit of supersymmetry, to look for realizations of our model of colored links under the form of a fermionic matrix model with \(Sp(-n)\) symmetry. Let us show how this works in the simplest case, that is \(n = -2\).

Our fields will be a “complex fermionic matrix”, that is a matrix \(\Psi = (\Psi_{ij})\) where the \(\Psi_{ij}\) are independent Grassmann variables, and its formal adjoint \(\Psi^\dagger = (\bar{\Psi}_{ji})\), which together form the fundamental representation of \(Sp(2)\). We apply to them the usual rules of Berezin integration. We must next look for \(Sp(2)\)-invariant quadratic and quartic invariants of the form \(\text{tr} P(\Psi, \Psi^\dagger)\). Since the \(\Psi\) are non-commutative objects, one must consider arbitrary tensor products of the (dual of the) fundamental representation; however the trace property combined with the anti-commutativity of the matrix elements implies that an elementary circular permutation must have eigenvalue \(-1\) in this representation. Very explicitly, there are one quadratic invariant, \(\Psi \Psi^\dagger - \Psi^\dagger \Psi\), and two independent quartic invariants, say \(\Psi \Psi^\dagger \Psi \Psi^\dagger - \Psi^\dagger \Psi \Psi^\dagger \Psi - \Psi \Psi^\dagger \Psi^\dagger \Psi + \Psi^\dagger \Psi \Psi^\dagger \Psi\) and \(\Psi \Psi^\dagger \Psi^\dagger - \Psi^\dagger \Psi \Psi^\dagger -\)
However it is clear that the first quartic invariant is stable by circular permutation and therefore its trace is zero. We are thus left with the following expression:

$$Z^{(N)}(t, g_0) = \int d\Psi d\Psi^\dagger e^N \text{tr} \left(-t\Psi\Psi^\dagger + g_0\Psi\Psi^\dagger\Psi\Psi\right)$$

(3.20)

It is no surprise that the partition function only depends on one coupling constant $g_0$, since the analysis of section 2.3 has shown us that for $n = -2$ all large $N$ quantities depend only on the combination $g_1 - g_2$; and indeed, by direct inspection one can identify $g_0 = g_1 - g_2$.

As in the case $n = 1$, we have access to only one four-point fonction

$$\Gamma \equiv \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr} \Psi\Psi^\dagger\Psi^\dagger\Psi \right\rangle_c = \Gamma_1 - \Gamma_2$$

(3.21)

Again, a “miracle” happens in that a particular subset of the renormalization equations becomes closed for $n = -2$; namely, taking (2.10.1)–(2.10.2)

$$g_0(g) = g(1 - 2H'_2(g) + H'_1(g) + V'_2(g))$$

(3.22)

a combination of the H-2PI diagrams appears, which can be related to $\Gamma$ via the use of Eqs. (2.11):

$$V'_2 - 2H'_2 + H'_1 = -2 + \frac{3}{2(1 + g)(1 - \Gamma)} + \frac{1}{2(1 - g)(1 + \Gamma)}$$

(3.23)

We now briefly describe how to compute integral (3.20) in the large $N$ limit. We can set $t = 1$ without loss of generality, as explained in section 2.3. We perform the standard Hubbard–Stratonovitch transformation by introducing a hermitean matrix $A$:

$$Z^{(N)}(t, g_0) = \int d\Psi d\Psi^\dagger \int dA e^N \text{tr} \left(-\Psi\Psi^\dagger - \frac{1}{2}A^2 + \sqrt{-g_0}A(\Psi\Psi^\dagger + \Psi^\dagger\Psi)\right)$$

(3.24)

The gaussian integral over $\Psi$ and $\Psi^\dagger$ can then be performed:

$$Z^{(N)}(t, g_0) = \int dA \det(1 \otimes 1 + \sqrt{-g_0}(A \otimes 1 + 1 \otimes A)) e^{-\frac{N}{2} \text{tr}A^2}$$

(3.25)

We recognize at this point the usual $O(-2)$ fully packed (non-intersecting) loops model\(^3\) [10]. We perform the change of variables: $M = (A - a_0)^2$ with $a_0 = -\frac{1}{2\sqrt{-g_0}}$, which absorbs the determinant, resulting in

$$Z^{(N)}(t, g_0) = \int dM e^{-\frac{N}{2} \text{tr}(\sqrt{M} + a_0)^2}$$

(3.26)

\(^3\) Of course this might have been expected from the start, since for any $n$ the $O(n)$ model of fully packed non-intersecting loops corresponds to the particular case $g_1 = 0$ of our model, and therefore here $g_0 = -g_2$. 

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One can then diagonalize \( M \) and compute the integral over eigenvalues using standard large \( N \) saddle point techniques. The resolvent of \( M \) is given by a complete elliptic integral of the third kind; in particular,

\[
G - \frac{1}{4g_0} + 2 = \lim_{N \to \infty} \left\langle \frac{1}{N} \text{tr} M \right\rangle = \frac{1}{96\pi^4 g_0^3} K^3 ((8 - 8k^2 + 3k^4)K + 4(k^2 - 2)E) \tag{3.27}
\]

where \( K \) and \( E \) are complete elliptic integrals of the first and second kind with modulus \( k \), and the coupling constant \( g_0 \) is given by

\[
g_0 = -\frac{1}{8\pi^2} K((2 - k^2)K - 2E) \tag{3.28}
\]

(cf also [11] for a similar solution). Finally, inserting the expression of \( \Gamma = \frac{1}{2g_0} + 1 \) obtained from (3.27) into renormalization Eqs. (3.22)–(3.23) results in a \( g \)-dependent transcendental equation for the modulus \( k^2 \). This equation is too complicated to be solved exactly; however it can be easily solved numerically order by order. Finally, \( \Gamma \) is the desired generating function \( \Gamma_1 - \Gamma_2 \) of tangles; in the appendix we present the first few orders of the expansion of \( \Gamma(g) \).

We now turn to the asymptotic behavior of the coefficients of \( \Gamma(g) \). It is known that the \( O(-2) \) matrix model does not have any critical point of the usual form of 2D quantum gravity; for example in [11], the solution of the \( O(-2) \) model of dense loops, equivalent to ours, is studied in detail when the elliptic modulus \( k \) is in the range \([0,1]\), which corresponds to \( g_0 \in [-\infty,0] \) in our notations, and no singularity is found. This does not mean, of course, that \( \Gamma(g) \) has no singularity, since \( g_0 \) (and \( g \)) can move in the whole complex plane. Generically, these singularities are square root type singularities and given by the equation \( \frac{dg}{dg} = 0 \). It turns out that this equation has plenty of solutions, though one can unfortunately not write them down analytically. Numerically, one finds that the solutions with smallest modulus of \( g \) are:

\[
g_c \approx -0.239 \pm 0.135i \tag{3.29}
\]

They are pairs of complex conjugate solutions: this indicates oscillatory behavior of the series, which is due to the fact that \( n < 0 \). We therefore find that if \( \Gamma = \sum_{p=1}^{\infty} \gamma_p g^p \),

\[
\gamma_p \sim \text{Re}(\text{cst} p^{-3/2}g_c^{-p}) \tag{3.30}
\]

where \( \text{cst} \approx -0.237 \pm 0.090i \). It would be interesting to find a physical interpretation of this critical point à la Yang–Lee.
Finally, let us note that one could combine the results of \( n = +2 \) \([3]\) and of \( n = -2 \): this would give rise to a model of oriented tangles in which one counts separately tangles with odd and even numbers of connected components. Since the coefficients of \( \Gamma(g) \) (and presumably also of \( \Gamma_1(g) \) and \( \Gamma_2(g) \) separately) in the case \( n = -2 \) satisfy, according to Eq. (3.30), \( \gamma_p = O(3.64 \ldots p) \) whereas in the case \( n = +2 \) these coefficients are of the order \( 6.28 \ldots p \), we conclude that there are, up to exponentially small corrections, as many odd tangles as there are even tangles... 

Appendix A. Tables for \( n = 1 \) and \( n = -2 \) up to 32 crossings.
| $p$ | $G^4_c$ | $G^6_c$ | $G^8_c$ |
|-----|---------|---------|---------|
| 1   | 1       |         |         |
| 2   | 2       | 3       |         |
| 3   | 4       | 14      | 12      |
| 4   | 10      | 51      | 90      |
| 5   | 29      | 186     | 468     |
| 6   | 98      | 708     | 2196    |
| 7   | 372     | 2850    | 10044   |
| 8   | 1538    | 12099   | 46170   |
| 9   | 6755    | 53756   | 215832  |
| 10  | 30996   | 247911  | 1029564 |
| 11  | 146982  | 1178352 | 5010192 |
| 12  | 715120  | 5740224 | 24830640|
| 13  | 3552254 | 28535604| 125073288|
| 14  | 17951322| 144283404| 639037476|
| 15  | 92045058| 740126242| 3306068412|
| 16  | 477882876| 3843972303| 17292904722|
| 17  | 2508122859| 20180815236| 91335814848|
| 18  | 13289437362| 106957362161| 486589812240|
| 19  | 71010166670| 571643594646| 2612379495996|
| 20  | 382291606570| 3078146310603| 14122834373034|
| 21  | 2072025828101| 16686687494650| 76829648302716|
| 22  | 11298920776704| 91009054240656| 420345016423632|
| 23  | 61954857579594| 499101633250932| 2311716994208856|
| 24  | 341427364138880| 2750883342029780| 12773922263423472|
| 25  | 1890257328958788| 15231756014050908| 70893591427443456|
| 26  | 10509472317890690| 84695579659496748| 395034114129257304|
| 27  | 58659056351295672| 472782954018549456| 2209407034450182552|
| 28  | 328591560659948828| 2648662349568626736| 12399753592080373248|
| 29  | 1846850410940949702| 14888203427107319436| 69813861782757325992|
| 30  | 10412612510292744992| 83947527137925001240| 39424596054017041532|
| 31  | 58877494436409193754| 474714688448707647894| 223256841495863372020|
| 32  | 333824674188182988872| 2691749836124970938595| 12675855143073018219570|

**Tab. 1:** Table of the number of prime alternating tangles ($n = 1$) with 4, 6, 8 external legs.
| $p$ | $\Gamma$ |
|-----|---------|
| 1   | 1       |
| 2   | -1      |
| 3   | 1       |
| 4   | 1       |
| 5   | -7      |
| 6   | 23      |
| 7   | -51     |
| 8   | 50      |
| 9   | 212     |
| 10  | -1596   |
| 11  | 6492    |
| 12  | -19124  |
| 13  | 37094   |
| 14  | 1878    |
| 15  | -437322 |
| 16  | 2557800 |
| 17  | -10055712 |
| 18  | 29767944 |
| 19  | -58631365 |
| 20  | -4689017 |
| 21  | 740682974 |
| 22  | -4462194156 |
| 23  | 18243692937 |
| 24  | -57186253699 |
| 25  | 127394803329 |
| 26  | -81353773012 |
| 27  | -1062951245376 |
| 28  | 7538741871041 |
| 29  | -33359417764221 |
| 30  | 112902256367630 |
| 31  | -286176860146756 |
| 32  | 379259745656069 |

**Tab. 2:** Table of the coefficients of $\Gamma$ ($n = -2$ tangles with 4 external legs).
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