Fractal solutions of dispersive partial differential equations on the torus

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The Talbot Effect

Figure 1: An 1836 Optical experiment of Talbot.
Schrödinger’s equation

- Berry and Klein in 1996 used linear Schrödinger to model the Talbot effect
- Time is distance from the grating

\[ iq_t + q_{xx} = 0. \]
Consider the differential equation

\[
\begin{cases}
    iq_t + L_\omega q = 0, & t \in \mathbb{R}, \ x \in \mathbb{T} := \mathbb{R}/(2\pi \mathbb{Z}), \\
    q(0, \cdot) = g(\cdot) \in L^2(\mathbb{T}).
\end{cases}
\]  \tag{1}

where

\[\hat{L_\omega} q(n) = \omega(n) \hat{q}(n), \quad n \in \mathbb{Z}.\]

We say \(\omega\) is the dispersion relation.

**Key example (Schrödinger)**

\[\omega(n) = -n^2, \quad g(x) = \chi[0,\pi]\]
Other Examples

Airy (linear KdV): \( u_t + u_{xxx} = 0, \quad \omega(n) = n^3 \)

Fractional Schrödinger: \( iu_t + |\partial_x|^{\alpha}, \quad \omega(n) = |n|^\alpha \)

Boussinesq: \( \omega(n) = \sqrt{n^2 + n^4} \)

Gravity–Capillary wave: \( \omega(n) = \sqrt{(n + n^3 \tanh(n))} \)
Figure 2: Initial Data: $g(x) = \chi_{[0, \pi]}(x)$. 
Figure 3: Initial Data: $g(x) = \chi_{[0,\pi]}(x)$. 
Polynomial dispersion at rational times

Can prove at rational times $t = 2\pi \frac{p}{q}$, one has

$$q(t, x) = \frac{1}{q} \sum_{j=0}^{q-1} G_{p,q}(j) g(x - 2\pi \frac{j}{q}),$$

where

$$G_{p,q}(j) = \sum_{\ell=0}^{q-1} e(i \omega (\ell \frac{p}{q}) e(\ell \frac{j}{q}), \quad e(\theta) = e^{2\pi i \theta}.$$

Thus $q(t, x)$ is a finite $(2q)$ linear combination of characteristic functions of intervals of length $1/(2q)$. 
Irrational times

Figure 4: Initial Data: \( g(x) = \chi_{[0, \pi]}(x) \).
At a rational time $t = 2\pi^p\frac{p}{q}$, $q(t, x)$ is a linear combination of $\chi[\pi\frac{i}{q}, \pi\frac{j+1}{q}]$, for $0 \leq j \leq 2q - 1$.

- Coefficients are Gauss sums
- For $t \notin 2\pi\mathbb{Q}$, consider $2\pi\frac{p_n}{q_n} \rightarrow t$, where $q_n \rightarrow \infty$.
- Each $u(2\pi\frac{p_n}{q_n}, x)$ increases in complexity
Fractal behavior in a classical example

Riemann’s proposed continuous but nowhere differential function:

\[ \phi(t) = \sum_{n \neq 0} \frac{e^{itn^2}}{n^2}. \]

- We know now that \( \phi(t) \) is differentiable at certain \( t \in 2\pi \mathbb{Q} \)
- Obtained from integrating fundamental solution of Schrödinger along vertical line \( x = 0 \)
- Jaffard (1996) proved the multifractility of \( \phi \)
Recall that the fractal (also known as upper Minkowski or upper box–counting) dimension, $\dim(E)$, of a bounded set $E$ is given by

$$\limsup_{\epsilon \to 0} \frac{\log(N(E, \epsilon))}{\log(\frac{1}{\epsilon})},$$

where $N(E, \epsilon)$ is the minimum number of $\epsilon$–balls required to cover $E$. 

Fractal Dimension
Fractal Dimension
Fractal Dimension

In our case, $E$ will be the graph of an $f : [0, 2\pi] \rightarrow \mathbb{R}$ and so the dimension will lie between 1 and 2.

- If $f$ is differentiable, then the dimension is 1
- A space filling curve has dimension 2
Conjectures of Berry and coauthors

For step function initial data of the Schrödinger, Berry conjectured

- Fractal dimension \( \frac{3}{2} \) at irrational times
- There are space slices with fractal dimension \( \frac{7}{4} \)
- There are diagonal slices with fractal dimension \( \frac{5}{4} \)
- Same holds for non–linear perturbation
The General Question

Given the solution, \( q(t, x) \) of a dispersive PDE and a line \( \mathcal{L} \) in space–time, determine the fractal dimension of the real and imaginary parts of \( q_{\mathcal{L}}(t, x) \).

For a fixed \( t \) we denote

\[
D_t(\omega, g)
\]

the maximum fractal dimension of the real and imaginary parts of \( q(t, x) \) (horizontal line).
The solution to the dispersive PDE

\[
\begin{align*}
\left\{
\begin{array}{l}
iq_t + L_\omega q = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{T} := \mathbb{R}/(2\pi \mathbb{Z}), \\
q(0, \cdot) = g(\cdot) \in L^2(\mathbb{T})
\end{array}
\right.
\end{align*}
\]

is

\[
q(t, x) = \sum_{n \in \mathbb{Z}} \hat{g}(n) e^{i t \omega(n) + i n x}.
\]

In our running example,

\[
q(t, x) = \frac{1}{2} - \sum_{n \in \mathbb{Z}} \frac{1}{in} e^{i t n^2 + in x}.
\]

n \equiv 1 \mod 2
Oskolkov bounded variation initial data and polynomial dispersion, \( q(t, x) \) is continuous at irrational \( t \)

Rodnianski: Berry’s conjecture for linear Schrödinger, dimension \( = \frac{3}{2} \) (discrete Hilbert transform)

Chamizo and Cordoba, dimension of

\[
\sum_{n=1}^{\infty} \frac{e^{in^k}}{n^a}, \quad \frac{k + 1}{2} \leq a \leq k + \frac{1}{2},
\]

is exactly \( 2 - \frac{2a - 1}{2k} \).

Erdoğan and Tzirakis, smoothing estimates to non-linear equations

Chousionis, Erdoğan and Tzirakis addressed Berry’s conjecture on \(|e^{it\partial_{xx}}|^2\) and applied to the vortex filament equation.
Determining the fractal dimension

Spatial smoothness $\iff$ Frequency Decay

- The graph of a $C^\gamma(\mathbb{T})$ function has fractal dimension $\leq 2 - \gamma$
  
  $$|f(x) - f(y)| \lesssim |x - y|^{\gamma}$$

- If $f \in C^\gamma(\mathbb{T})$ then $|\hat{f}(n)| \lesssim |n|^{-\gamma}$

- Dispersive PDE preserve Sobolev norm

- Besov Spaces and Littlewood Paley projections
For \((t, x) \in \mathcal{L}\), we want to determine the fractal dimension of the real and imaginary parts of

\[ q(t, x) = \sum_{n \in \mathbb{Z}} \hat{g}(n) e^{it\omega(n) + inx}. \]

Using Littlewood–Paley and Besov theory, it is enough to estimate

\[ P_N(q(t, x)) = \sum_{n \sim N} \hat{g}(n) e^{it\omega(n) + inx}, \]

in \(L^p(\mathbb{T})\) for various \(p\), where \(p = \infty\) is “most wanted.”
Removing the weight

We have

\[ P_N(q(t, x)) = \sum_{n \sim N} \hat{g}(n) e^{it\omega(n) + inx}, \]

Assuming some smoothness condition on \( g \) (i.e. bounded variation, as in our key example) it is enough to study

\[ H_N(t, x) = \sum_{n \sim N} e^{it\omega(n) + inx}. \]
Theorem (\(L^\infty\) decay and fractal behavior)

Fix \(t\) and \(0 < \gamma \leq 1/2\). Let \(g \in BV(\mathbb{T})\) such that \(g \notin H^{1/2+}(\mathbb{T})\) and \(e^{itL_\omega}g\) is continuous. Suppose

\[
\|H_{N,w}(t, x)\|_{L^\infty} \lesssim N^{1-\gamma}, \quad N \in \mathbb{N}. \tag{3}
\]

Then \(1 + \gamma \leq D_t(\omega, g) \leq 2 - \gamma.\)
To use the Besov space theory, we need non–trivial estimates for
\[ \| H_N(t, x) \|_{L^\infty_x}, \quad H_N(t, x) = \sum_{n \sim N} e^{it\omega(n) + inx}, \quad (t, x) \in \mathcal{L}. \]

- Trivial bound $N$, tight when $t = x = 0$.
- Generically, expect $N^{1/2+}$, “square root cancellation” (i.e. Khintchine’s inequality, Rademacher functions, Central Limit Theorem, etc.)
Let $q_n$ be the sequence of numbers satisfying

$$|\frac{t}{2\pi} - \frac{p_n}{q_n}| \leq \frac{1}{q_n^2},$$

(4)

We say $t$ is Khinchin–Lévy if for all $\epsilon > 0$ and $n$ large depending on $\epsilon$, one has $q_{n+1} \leq q_n^{1+\epsilon}$.
Theorem (Weyl’s inequality)

Suppose $t$ satisfies the Khinchin–Lévy hypothesis. Then

$$\left| \sum_{N \leq n < 2N} e^{itn^d + ixn} \right| \lesssim N^{1 - 2^{1-d} +}.$$

- Even the $d = 3$ case is far from square–root cancellation: this is an old important problem in number theory
- Recent progress on Vinogradov’s mean value theorem, due to Bourgain, Demeter, and Guth, allows for improved bounds for $d \geq 7$
Theorem (Vinogradov’s mean value theorem)

We set

\[ J_{s,d}(N) := \int_{[0,1]^d} \left| \sum_{j=1}^{N} e(x_1 j + \ldots + x_d j^d) \right|^{2s} \, dx_1 \ldots dx_d \]

Then

\[ J_{s,d}(N) \lesssim N^{s+} + N^{\frac{2s-d(d+1)}{2}}. \]

- Suppose \(| \sum_{N \leq n < 2N} e(itn^d + ixn) |\) is large, then

\[ \sum_{N \leq n < 2N} e(t(n+q)^d + ix(n+q)) \sim \sum_{N \leq n \leq 2N} e(\alpha_1(q)n + \ldots + \alpha_d(q)n^d) \]

is large for many small \(q\)

- The points \((\alpha_1(q), \ldots, \alpha_d(q))\) are well–distributed in \([0, 1]^d\) which makes the \(L^{2s}\) norm large
We no longer have an optimal $L^\infty_x$ estimate.

Can still obtain conjectured lower bounds

**Theorem (Fractal behavior and $L^q$ decay)**

Fix $t$ and $0 < \gamma \leq 1/2$. Let $g \in BV(\mathbb{T})$ such that $g \notin H^{1/2+}(\mathbb{T})$ and $e^{it\omega}g$ is continuous. Suppose

$$\|H_{N,w}(t,x)\|_{L^q_x} \lesssim N^{1-\gamma+}, \quad N \in \mathbb{N}, \quad 2 < q < \infty. \quad (5)$$

Then

$$D_t(\omega, g) \geq 2 - \frac{1 - \gamma q'}{2 - q'}, \quad q' := \frac{q}{q - 1}.$$

In particular when $\gamma = 1/2$ (square root), $D_t(\omega, g) \geq \frac{3}{2}$. 
Oblique Lines for Schrödinger

- \( \mathcal{L} : t = c - \frac{k}{\ell} x \) where \( k \) and \( \ell \) are coprime
- We modify the Besov space theory, the Sobolev index changes and the fractal dimension increases to \( 7/4 \),

\[
\sum_{n \sim N} e^{i t n^2 + i n x} = \sum_{n \sim N} e^{i ((c - \frac{k}{\ell} x)n^2 + i n x} \approx P_N^2(q(t, x)),
\]

so the Archimedean size of the dispersion relation changes the relevant Besov space.
The lower bound

- We utilize an $L^4$ argument (Chamizo–Cordoba) which leads to an analog of the following

$$\#\{a, b, c, d \sim N : a^2 - b^2 = c^2 - d^2\}.$$  

- $(a - b)(a + b) = (c - d)(c + d)$, Divisor Bound

A sample calculation when $x = 0$ (vertical line):

$$\int \left| \sum_{n \sim N} e^{itn^2} \right|^4 dt = \int \sum_{n_1, n_2, n_3, n_4 \sim N} e^{it(n_1^2 + n_2^2 - n_3^2 - n_4^2)} dt.$$
Oblique Lines

- Our $L^\infty_x$ bound follows from our exponential sum estimate

$$\sup_x \left| \sum_{n \sim N} e^{inx + in^2(c-rx)} \right| \lesssim N^{4 \frac{4}{5} +}, \quad \text{a.e. } c \in \mathbb{R}, \ r \in \mathbb{Q}$$

- Have to get cancellation from both quadratic term and linear term

- Expect to be able to replace $4 \frac{4}{5}$ with $\frac{1}{2}$. 
Theorem (Erdo\u{g}an–S.)

For any \( g \in BV(\mathbb{T}) \), \( g \notin H^{1/2+} \) and for each \( t \neq 0 \), the linear fractional Schrödinger evolution, \( e^{it(-\Delta)^{3/4}}g \), satisfies

\[
1 + \frac{1}{4} \leq D_t(\omega, g) \leq 2 - \frac{1}{4}.
\]

Thus the rational/irrational dichotomy is not valid for fractional Schrödinger. Our methods indicate that there is some dependence on algebraic properties of \( t \), but it is not clear if this is just an artifact of our methods.

For instance, we showed \( 1 + \frac{3}{8} \leq D_t(\omega, g) \leq 2 - \frac{3}{8} \), for Khinchin–Lévy \( t \).

Techniques: A and B processes from exponential sums (Weyl Differencing and Poisson Summation/Stationary Phase)
KdV on oblique lines

Theorem (Erdoğan-S.)

Let $g$ be a non–constant, mean–zero, and real valued step function on the torus. Let $u(t, x)$ solve the Airy equation or the KdV equation with data $g$. Fix $k, \ell \in \mathbb{N}$ with $(k, \ell) = 1$. For $c \in \mathbb{R}$, let $F_c(x) = u(c - \frac{k}{\ell}x, x)$, $x \in [0, 2\pi\ell]$. Then for a.e. $c$, $F_c \in C^{1 - \frac{1}{27}}$ and the dimension of the graph of $F_c$ is in $\left[\frac{11}{6}, \frac{53}{27}\right]$.

- Our results extended to non–linear Schrödinger and KdV using the smoothing estimates of Erdoğan and Tzirakis: the solution to the non–linear PDE is the linear plus a “smooth perturbation”
Thank you!

Further Reading

- M. B. Erdoğan and N. Tzirakis, *Dispersive Partial Differential Equations: Wellposedness and Applications*, London Mathematical Society Student Texts 86, Cambridge University Press, 2016.

- G. Chen and P. J. Olver, *Numerical simulation of nonlinear dispersive quantization*, Discrete Contin. Dyn. Syst. 34 (2014), no. 3, 991–1008.

- B. Erdoğan, G. Shakan, *Fractal solutions of dispersive partial differential equations on the torus*, arXiv:1803.00674 (2018).

- These slides can be found on my website: gshakan.wordpress.com