The Hurwitz-Hopf map and harmonic wave functions for integer and half-integer angular momentum

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Abstract

Harmonic wave functions for integer and half-integer angular momentum are given in terms of the Euler angles (θ, φ, ψ) that define a rotation in SO(3), and the Euclidean norm r in \( \mathbb{R}^3 \), keeping the usual meaning of the spherical coordinates (r, θ, φ). They form a Hilbert (super-) space decomposed in the form \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \). Following a classical work by Schwinger, 2-dimensional harmonic oscillators are used to produce raising and lowering operators that change the total angular momentum eigenvalue of the wave functions in half units. The nature of the representation space \( \mathcal{H} \) is approached from the double covering group homomorphism \( SU(2) \to SO(3) \) and the topology involved is taken care of by using the Hurwitz-Hopf map \( H : \mathbb{R}^4 \to \mathbb{R}^3 \). It is shown how to reconsider \( H \) as a 2-to-1 group map, \( G_0 = \mathbb{R}^+ \times SU(2) \to \mathbb{R}^+ \times SO(3) \), translating it into an assignment \( (z_1, z_2) \mapsto (r, \theta, \phi, \psi) \) whose domain consists of pairs \( (z_1, z_2) \) of complex variables, under the appropriate identification of \( \mathbb{R}^4 \) with \( \mathbb{C}^2 \). It is shown how the Lie algebra of \( G_0 \) is coupled with two Heisenberg Lie algebras of 2-dimensional (Schwinger’s) harmonic oscillators generated by the operators \( \{z_1, z_2, \bar{z}_1, \bar{z}_2\} \) and their adjoints. The whole set of operators gets algebraically closed either into a 13-dimensional Lie algebra or into a 4(8)-dimensional Lie superalgebra. The wave functions in \( \mathcal{H} \) can be written in terms of polynomials in the complex coordinates \( z_1, z_2 \) and their complex conjugates \( \bar{z}_1, \bar{z}_2 \) and the representations are explicitly constructed via the various highest weight (or lowest weight) vector representations of \( G_0 \). Finally, a new nonrelativistic quantum (Schrödinger-like) equation for the hydrogen atom that takes into account the electron spin is introduced and expressed in terms of \( (r, \theta, \phi, \psi) \) and the time \( t \). The equation is susceptible to be solved exactly in terms of the harmonic wave functions hereby introduced.

1. Introduction and summary of results

Following Schwinger’s idea of connecting the ladder operators for the harmonic oscillator with those for the usual angular momentum, a realisation is constructed which, besides yielding the latter in terms of Euler’s angles, yields new operators which change the total angular momentum \( j \) by half steps: \( \hbar/2 \). This realisation thus gives, in a unified manner, the angular momentum eigenfunctions with integer and half-integer values. Schwinger’s original work goes back to 1965 (see [1]). Later on, after the concept of supersymmetry had been introduced, in 1979 he wrote (see [2]):

‘The recent recognition of the existence and possible utility of transformations between particles of different statistics [For a mathematically oriented review see [3–5]] had its origin in the properties of certain two-dimensional dual models, although the concept was clearly prefigured in the simple unification of all spins and statistics by means of

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Supersymmetry is a quest for a unified theory of elementary particles and their interactions [7] and, apart from being a promise for unified field theories, presents softer divergencies than the current models. Its natural language is that of Lie algebras, Lie groups, and their representations; the mathematical language describing symmetries, which plays a central role in modern mathematical physics [8]. So-called Lie superalgebras are a generalisation of Lie algebras, which include the notion of a super vector space; i.e., a vector space graded by an additional parity or degree structure, allowing for the interaction (via a Lie super bracket) of elements with equal or different parity: the Lie super bracket of two elements with the same parity leads to an even element, whereas that of two elements with different parity leads to an odd element (cf e.g. [3, 9]). While no supersymmetric particles have been discovered at the present energy scales, experimental observations in trapped ion simulators have been reported [10], and applications in various fields such as optics [11], quantum optics [12, 13], quantum chaos [14], quantum simulations [15], and others, have been devised.

The purpose of this work is to show how Schwinger’s ideas produce a Hilbert space (a super vector space, actually) of harmonic wave functions describing both, integer and half-integer angular momentum states from a unified mathematical point of view and having an easy algebraic and physical interpretation. In fact, the produced wave functions depend on the Euler angles \((\theta, \phi, \psi)\) that define a rotation in \(\mathbb{R}^3\), keeping the usual meaning of the spherical coordinates \((r, \theta, \phi)\). The harmonic wave functions obtained for integer angular momentum states can be expressed — up to normalization— in the form \(r^j Y_{jm}(\theta, \phi)\), with non-negative integral powers of \(r\) when the total angular momentum \(j = n\) takes values in \(\mathbb{N} \cup \{0\}\); in this case, the functions \(Y_{jm}(\theta, \phi)\) are the usual spherical harmonics defined on the 2-sphere embedded in \(\mathbb{R}^3\). On the other hand, the harmonic wave functions obtained for half-integer angular momentum states are expressed in the form \(r^j e^{i \psi / 2} y_{jm}(\theta, \phi)\) with \(j \in (2(\mathbb{N} \cup \{0\}) + 1)/2\) and the functions \(y_{jm}\) (like the \(Y_{jm}\)’s for non-negative integer values of \(j\)) belonging to the corresponding subspaces \((j, m)\) spanned by those harmonic functions \(y\) satisfying \(L_f = j f\) and \(L_z = m f\) with respect to the angular momentum operators described in equations (33) in section 7 and in proposition 8.2 in section 8 below. In both cases, \(m \in \{-j, -j + 1, \ldots, j - 1, j\}\). Convention: Here and in what follows we set \(h = 1\).

All the harmonic wave functions obtained for integer and half-integer angular momentum states can also be written in terms of two complex variables \((z_1, z_2)\) and their complex conjugates \((\bar{z}_1, \bar{z}_2)\), related to \((r, \theta, \phi, \psi)\) by,

\[
\begin{align*}
  z_1 &= \sqrt{r} \exp \left( i \frac{\psi + \phi}{2} \right) \cos \frac{\theta}{2}, \\
  z_2 &= \sqrt{r} \exp \left( i \frac{\psi - \phi}{2} \right) \sin \frac{\theta}{2}.
\end{align*}
\]

The nature of this correspondence is approached from the well-known double covering group homomorphism \(SU(2) \to SO(3)\). The topology involved is taken care of by using the Hurwitz-Hopf map \(H: \mathbb{R}^4 \to \mathbb{R}^3\) as in Hage-Hassan and Kibler [16]. The harmonic wave functions for integer and half-integer angular momentum states are then obtained from the corresponding representation theory of the groups \(SU(2)\) and \(SO(3)\). Under appropriate identifications, the domain \(\mathbb{R}^4\) of \(H\) can be viewed as Hamilton’s quaternions \(\mathbb{H}\) which are in turn identified with the domain \(\mathbb{C}^2\) of the complex variables \((z_1, z_2)\). The relationship between the Schwinger transformation and the Hurwitz-Hopf map from \(S^3\) to \(S^2\) is here given explicitly.

Our approach consists of changing the point of view of \(H\) and consider it as a 2-to-1 group map, \(G_0 = \mathbb{R}^+ \times SU(2) \to \mathbb{R}^+ \times SO(3)\), which translates into the correspondence \(\mathbb{C}^2 \ni (z_1, z_2) \mapsto (r, \theta, \phi, \psi) \in \mathbb{R}^+ \times SO(3)\), whenever \((z_1/\sqrt{r}, -z_2/\sqrt{r})\) \(\in SU(2)\), and

\[|z_1|^2 + |z_2|^2 = R^2 = r > 0.\]

In particular, \(z_1\) and \(z_2\) can be written in terms of \((\sqrt{r}, \theta/2, \phi/2, \psi/2)\) as in (1) above. The Lie algebra generators, \(\{L_x, L_y, L_z, L_0\}\) of \(g = \text{Lie}(G_0)\) find simple expressions in terms of the monomials \((z_1, z_2, \bar{z}_1, \bar{z}_2)\) and the partial derivatives with respect to these variables (see equation (33) in section 7 below). A direct computation shows that,

\[L_x^2 + L_y^2 + L_z^2 = L^2 + \frac{r}{4} \Delta,\]
where,
\[
\Delta = \left( \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} \right)
\]

is the Laplace operator in the domain \( \mathbb{R}^4 \cong \mathbb{C}^2 \). It turns out that the wave functions \( f \) produced through our approach satisfy \( \Delta f = 0 \).

Moreover, in terms of the complex variables \( (z_1, z_2, x_1, x_2) \), the wave functions hereby produced are obtained in one of two ways as follows: either, obtain first the highest weight vector \( | j \rangle = (z_1 z_2)^{\frac{1}{2}} \) (if \( j \in \mathbb{N} \cup \{0\} \)), or \( | j, -j \rangle = z_1 (z_1 z_2)^{\frac{1}{2}} \) (if \( j \in \mathbb{N} \cup \{0\} \)), and then move from it by successive application of the ladder operator \( L_+ = L_x - i L_y \), which preserves the eigenvalue \( j \) but lowers the eigenvalue \( m \) in one unit at each step, until reaching a non-zero scalar multiple of the lowest weight vector \( | j, -j \rangle \), characterized by \( L_- | j, -j \rangle = 0 \); or else, produce first the lowest weight vector \( | j, -j \rangle = (z_1 z_2)^{\frac{1}{2}} (z_1 z_2)^{\frac{1}{2}} \) (if \( j \in \mathbb{N} \cup \{0\} \)), or \( | j, -j \rangle = z_1 (z_1 z_2)^{\frac{1}{2}} \) (if \( j \in \mathbb{N} \cup \{0\} \)), and then move from it by successive application of the ladder operator \( L_- = L_x + i L_y \), which preserves the eigenvalue \( j \) but raises the eigenvalue \( m \) in one unit at each step, until reaching a non-zero scalar multiple of the highest weight vector \( | j, j \rangle \), characterized by \( L_- | j, j \rangle = 0 \).

The whole picture is completed by showing how the Lie algebra of \( G_0 \) is coupled with two Heisenberg Lie algebras of 2-dimensional harmonic oscillators. Following Schwinger, the 2-dimensional harmonic oscillators are used to produce raising and lowering operators that change the total angular momentum eigenvalue of the wave functions in half units. It is explained how the complete set of operators gets closed either into a 13-dimensional Lie algebra or into a \((4|8)\)-dimensional Lie superalgebra. We mention in passing that mathematically related work in a similar direction was done in the mid 70’s of the last century by M. Kashiwara and M. Vergne following ideas introduced by R. Howe (see [17] and [18]). It turns out that their approach is adapted to a joint action of a given symplectic and a given orthogonal group, which is a clear indicative of supersymmetry (see [3] and [17]).

The Hilbert space where our wave functions live can be naturally decomposed into a direct sum \( \mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_0 \), where \( \mathcal{H}_0 \) contains all the states having integer angular momentum (or space of bosons) and \( \mathcal{H}_0 \) contains all the states having half-integer angular momentum (or space of fermions) thus providing an explicit realisation of a superspace in which both, the Lie algebra and the Lie superalgebra are faithfully represented.

Finally, as an example of our approach, we write in section 14 a new nonrelativistic quantum (Schrödinger-like) equation in terms of the four dimensional spatial coordinates \((r, \theta, \phi, \psi)\) and the time \( t \), using the Hydrogen atom potential energy. As far as we know, this non-relativistic quantum equation is new indeed and it takes into account the electron spin. Besides, it can be solved exactly in terms of the new harmonic wave functions hereby introduced. For the new solutions with half-integer values of \( j \), we have provided the highest weight wave functions \( \psi_{j, m} \) in \( | j, m \rangle \) together with their energy eigenvalues \( E_{j, m} \). The solutions \( \psi_{j, m} \) are obtained, up to a constant, via \( (L_x)^k \psi_{j, m} \) for non-negative integer values of \( k \) satisfying \( k \leq 2j + 1 \).

N.B.- After submission of this manuscript, the work by V Fock [19] was brought to our attention. In this work Fock addresses the SO(4) symmetry of the Hydrogen atom Hamiltonian solving the Schrödinger equation in four dimensions, and making clear the degeneracy due to the quantum number \( j \) in the energy spectrum (apart from the ‘magnetic’ degeneracy coming from the quantum number \( m \)). Our results applied to the Schrödinger equation for the Coulomb potential in section 14 recover Fock’s findings, together with new results arising from the half-integer eigenvalues \( j \) and \( m \) of our approach, and their corresponding eigenfunctions.

### 2. Schwinger’s Heisenberg-Angular-Momentum Lie Algebra

Schwinger’s Lie algebra is defined in terms of the 2-dimensional quantum harmonic oscillator operators, \( a_x, a_y \), and their adjoints, \( a_x^+, a_y^+ \), satisfying the Heisenberg commutation relations,
\[
[a_x, a_x^+] = 1, \quad [a_y, a_y^+] = 1.
\]

Then define,
\[
L_+ = a_y^+ a_y, \quad L_- = a_y a_y^+, \quad L_z = \frac{1}{2}(a_x^+ a_x - a_y^+ a_y), \quad J = \frac{1}{2}(a_x^+ a_x + a_y^+ a_y).
\]
It follows that,
\[ [J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z, \]
\[ [J_z, J_z] = 0, \quad [J_z, J_z] = 0. \] (6)

It is not difficult to prove that
\[ [J_x, a_x] = -\frac{1}{2} a_x, \quad [J_x, a_+^x] = \frac{1}{2} a_x, \]
\[ [J_x, a_x] = -\frac{1}{2} a_x, \quad [J_x, a_+^x] = \frac{1}{2} a_x, \]
\[ [J_x, a_+] = -\frac{1}{2} a_+, \quad [J_x, a_+^+] = -\frac{1}{2} a_+, \] (7)

and also that
\[ [J_y, a_\pm] = \mp a_\pm, \quad [J_y, a_\pm^\pm] = \pm a_\pm, \]
\[ [J_y, a_\pm] = \mp a_\pm, \quad [J_y, a_\pm^\pm] = \pm a_\pm, \]
\[ [J_y, a_y] = -a_y, \quad [J_y, a_y^\pm] = 0. \] (8)

We shall give a Lie algebra, realised in terms of operators adapted to the geometry and the symmetry involved, that fully contains the subalgebra of the \( J \)'s and the \( a \)'s but, for the sake of completeness in the variables used to represent the operators, it must include an additional Heisenberg algebra.

3. The Hurwitz-Hopf map

From the point of view of Lie's theory, the Schwinger operators \( a_\pm \) and \( a^\pm_\pm \) transform representation spaces of the 3-dimensional rotation group \( \text{SO}(3) \) describing states having a non-negative integer total angular momentum \( j \) (or spin value \( s \); i.e., \( j \in \mathbb{N} \cup \{0\} \)), into representation spaces of its spin group or double covering group, \( \text{SU}(2) \); that is, into states whose total angular momentum is \( j \pm 1/2 \) (or whose spin value is \( s \pm 1/2 \)), and vice versa.

Even though the Lie groups \( \text{SU}(2) \) and \( \text{SO}(3) \) are locally isomorphic, they are topologically different. To unveil the spin representations of \( \text{SO}(3) \) one needs to consider the double covering map \( \text{SU}(2) \to \text{SO}(3) \). One way to take the topology involved into account is as in Hage-Hassan and Kibler (see [16]) by considering the Hurwitz-Hopf map, \( H: \mathbb{R}^4 \ni u \mapsto H(u) = x \in \mathbb{R}^3 \), given by,
\[
x_1 = H_1(u_1, u_2, u_3, u_4) = 2(u_1 u_3 - u_2 u_4), \]
\[
x_2 = H_2(u_1, u_2, u_3, u_4) = 2(u_1 u_3 + u_2 u_4), \]
\[
x_3 = H_3(u_1, u_2, u_3, u_4) = u_1^2 + u_2^2 - u_3^2 - u_4^2, \]

which has the property that,
\[
r = \sqrt{x_1^2 + x_2^2 + x_3^2} = u_1^2 + u_2^2 + u_3^2 + u_4^2 = R^2. \] (9)

Thus, \( H \) maps 3-spheres centered at the origin in \( \mathbb{R}^4 \) onto 2-spheres centered at the origin in \( \mathbb{R}^3 \). The Hurwitz-Hopf map can also be described as a map \( H: \mathbb{C}^2 \to \mathbb{R}^3 \), using the complex variables,
\[
z_1 = u_1 + i u_2 \quad \text{and} \quad z_2 = u_3 + i u_4, \] (11)

so that,
\[
x_1 = \bar{z}_1 \bar{z}_2 + \bar{z}_2 \bar{z}_1, \quad x_2 = -i(\bar{z}_1 \bar{z}_2 - \bar{z}_2 \bar{z}_1), \quad x_3 = z_1 \bar{z}_1 - z_2 \bar{z}_2, \] (12)

that is,
\[
x_1 = H_1(z_1, z_2) = 2 \text{Re}(z_1 \bar{z}_2), \]
\[
x_2 = H_2(z_1, z_2) = 2 \text{Im}(z_1 \bar{z}_2), \]
\[
x_3 = H_3(z_1, z_2) = |z_1|^2 - |z_2|^2. \] (13)

One possible symmetry group associated to the Hurwitz-Hopf map is the real Lie group,
\[
G = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \in \mathbb{C}^4, \quad \text{and} \quad R^2 = |z_1|^2 + |z_2|^2 > 0 \right\}, \] (14)

whose connected component to the identity \( G_0 \) is isomorphic to \( \mathbb{R}^+ \times \text{SU}(2) \), where \( \mathbb{R}^+ \) stands for the multiplicative group of positive real numbers, obtained when \( R = \sqrt{r} > 0 \). The Lie group \( \text{SU}(2) \) can be
identified with the 3-sphere \(|z_1|^2 + |z_2|^2 = 1\) of unit quaternions. It acts on 3-spheres in \(\mathbb{R}^4 \simeq \mathbb{H}\) via left (or right) multiplication in \(\mathbb{H}\). Thus, the Hurwitz-Hopf map transforms 3-spheres in \(\mathbb{R}^4\) into 2-spheres in \(\mathbb{R}^3\), and \(\mathbb{R}^+\) acts either in \(\mathbb{R}^4\) or in \(\mathbb{R}^3\) by changing the spheres radii.

4. The Lie group \(SU(2)\) and its Lie algebra \(\mathfrak{su}(2)\)

To fix the notation we recall that the special unitary group \(SU(2)\) is defined as,

\[
SU(2) = \{ A \in \text{Mat}_{2 \times 2}(\mathbb{C}) | A^\dagger A = I_2, \quad \text{det } A = 1 \},
\]

where \(A^\dagger = (\bar{A})^T\) stands for the conjugate transpose matrix of \(A\) and \(I_2\) is the \(2 \times 2\) unit matrix. Thus, \(A \in \text{Mat}_{2 \times 2}(\mathbb{C})\) belongs to \(SU(2)\) if and only if it is invertible and its inverse \(A^{-1}\) is equal to \(A^\dagger\). That is,

\[
(a \ b) (d \ c)^{-1} = \frac{1}{\text{det } A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger.
\]

Since \(\text{det } A = 1\) for any \(A \in SU(2)\), it follows that,

\[
d = \bar{a}, \quad b = -\bar{c}, \quad \text{and } \text{det } A = |a|^2 + |c|^2 = 1.
\]

Therefore,

\[
SU(2) \simeq \left\{ \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \in \text{Mat}_{2 \times 2}(\mathbb{C}) | |a|^2 + |c|^2 = 1 \right\},
\]

making clear the fact that \(SU(2)\) is actually diffeomorphic to the unit 3-sphere \(S^3\) in \(\mathbb{R}^4 \simeq \mathbb{C}^2\).

The Lie algebra \(\mathfrak{su}(2)\) of \(SU(2)\) is the real vector space,

\[
\mathfrak{su}(2) \simeq \{ X \in \text{Mat}_{2 \times 2}(\mathbb{C}) | X + X^\dagger = 0, \quad \text{and } \text{Tr } X = 0 \}.
\]

It is well known that \(\mathfrak{su}(2)\) is isomorphic to \(\mathbb{R}^3\) via,

\[
\mathfrak{su}(2) \ni X \iff X = i \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}, \quad \text{with } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.
\]

A convenient set of linearly independent generators for \(\mathfrak{su}(2)\) over the real field \(\mathbb{R}\) is, \([i\sigma_1, i\sigma_2, i\sigma_3]\), where, the \(\sigma_i\)'s are the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and,

\[
[i\sigma_1, i\sigma_2] = -2i\sigma_3, \quad [i\sigma_2, i\sigma_3] = -2i\sigma_1, \quad [i\sigma_3, i\sigma_1] = -2i\sigma_2.
\]

Clearly,

\[
\mathfrak{su}(2) \ni X \implies \text{det } X = x^2 + y^2 + z^2.
\]

5. The double covering map \(SU(2) \to SO(3)\)

For each \(A \in SU(2)\), the assignment \(X \mapsto A X A^\dagger\) defines a transformation \(\rho(A)\): \(\mathfrak{su}(2) \to \mathfrak{su}(2)\) which preserves the quadratic form, \(\text{det } X = x^2 + y^2 + z^2\), since \(A^\dagger A = I_2\) and \(\text{det}(A X A^\dagger) = \text{det}(A X A) = \text{det } X\).

Therefore, \(\rho(A) \in SO(3)\).

It can be proved (and it is well known) that \(\rho\) is a 2 to 1 map, and in fact, a local diffeomorphism. This is usually expressed by saying that \(\rho\) induces the exact sequence of groups,

\[
\{I_2\} \xrightarrow{\rho} \mathbb{Z}_2 \xrightarrow{\rho} SU(2) \to SO(3) \to \{I_1\},
\]

where, \(\mathbb{Z}_2 = \{ \pm I_2 \}\) as a subgroup of \(SU(2)\), and \(\rho\): \(SU(2) \to SO(3)\) is actually a group homomorphism, as it satisfies,

\[
\rho(A_1 A_2) = \rho(A_1) \rho(A_2), \quad \text{for any } A_1, A_2 \in SU(2),
\]

and it is straightforward to rewrite \(\rho(A_1)\) and \(\rho(A_2)\) as \(3 \times 3\) matrices with real entries acting (in that order) on column vectors \(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3\), whenever, \(X = i \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \in \mathfrak{su}(2)\).

**Proposition 5.1.** The parametrisation of a given rotation \(\rho(A) \in SO(3)\) that takes an initial \(x\)-\(y\)-\(z\) frame into a final \(X\)-\(Y\)-\(Z\) frame in \(\mathbb{R}^3\) in terms of the Euler angles \((\theta, \phi, \psi)\) is obtained from the double covering map

\(SU(2) \ni A \mapsto \rho(A) \in SO(3)\), by writing \(A \in SU(2)\) in the form,
For any $g = \left( \begin{array}{cc} \tilde{z}_1 & -\tilde{z}_2 \\ \tilde{z}_2 & \tilde{z}_1 \end{array} \right) \in G$, choose a positive real number $R = \sqrt{r} \in \mathbb{R}^+$, so as to have $g$ in the identity component $G_0$ of $G$, and write,

\begin{align}
A &= \left( e^{-i\varphi} 0 \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \begin{array}{cc} e^{-i\varphi} 0 \\ 0 e^{i\varphi} \end{array} \\ e^{i\varphi} 0 \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \begin{array}{cc} e^{i\varphi} 0 \\ 0 e^{-i\varphi} \end{array} \right) \\
&= \left( e^{-i\varphi} \cos \frac{\theta}{2} - e^{-i\varphi} \sin \frac{\theta}{2} \\ e^{i\varphi} \sin \frac{\theta}{2} e^{i\varphi} \cos \frac{\theta}{2} \right)
\end{align}

so that $\rho(A) = R_x(\psi) \circ R_y(\theta) \circ R_z(\phi) \in SO(3)$.

**Proof.** This is well-known and we shall briefly recall how the result follows from the following observations:

1. Take $A = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \in SU(2)$. Then,
   \[
   \rho(A)X = A \cdot X^t = \begin{pmatrix} z & e^{-2i\varphi}(x - iy) \\ z^{-1} & -z \end{pmatrix}
   \]
   corresponds to a counterclockwise rotation, $R_x(2\varphi)$, of the $x$-$y$ plane (performed around the $z$-axis in $\mathbb{R}^3$) by an angle of $2\varphi$.

2. Take $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in SU(2)$. Then,
   \[
   \rho(A)X = A \cdot X^t = \begin{pmatrix} z \cos 2\theta - x \sin 2\theta & z \sin 2\theta + x \cos 2\theta - iy \\ z \sin 2\theta + x \cos 2\theta + iy & -(z \cos 2\theta - x \sin 2\theta) \end{pmatrix}
   \]
   corresponds to a counterclockwise rotation, $R_y(2\theta)$, of the $z$-$x$ plane (performed around the $y$-axis in $\mathbb{R}^3$) by an angle of $2\theta$.

Now, in passing from a given $x$-$y$-$z$ (fixed) frame in $\mathbb{R}^3$ to a (moving) $X$-$Y$-$Z$ frame (assuming the most general situation in which the directions defined by $z$ and $Z$ are linearly independent), we fix first the orientation of the line of nodes $\ell$ defined by the intersection of the $x$-$y$ plane with the $X$-$Y$ plane, in such a way that the positive direction of $\ell$ coincides with the direction of $\hat{z} \times \hat{Z}$, where $\hat{z}$ and $\hat{Z}$ are unit vectors along the corresponding $z$ and $Z$ axes. Then, the three rotations defined by the Euler angles $(\theta, \phi, \psi)$ to go from the $x$-$y$-$z$ frame to the $X$-$Y$-$Z$ frame, must be performed as follows:

(1) First make a counterclockwise rotation by an angle $\phi$ around the $z$-axis, to take the $x$-axis into the positive direction of the line of nodes.

(2) Then make a counterclockwise rotation by an angle $\theta$ around the positive direction of the line of nodes, to make the $z$-axis coincide with the $Z$-axis.

(3) Finally, make a counterclockwise rotation by an angle $\psi$ around the $Z$-axis to make the positive direction of the line of nodes, coincide with the positive direction defined by the $X$-axis.

Observe that due to the symmetry under rotations in the $x$-$y$ plane, the positive direction of the line of nodes can be labeled either by $x'$ or by $y'$. All that matters is that the auxiliary $x'$-$y'$-$z$ frame must have the same orientation as the $x$-$y$-$z$ frame does. We shall adhere to the convention that the positive direction of the line of nodes is precisely the auxiliary $y'$-axis, so that the $x'$-axis will have the direction of $\hat{y}' \times \hat{z}$, where $\hat{y}'$ and $\hat{z}$ are unit vectors along the positive direction of the line of nodes and the $z$-axis, respectively. From these conventions, the result (26) in the statement is now evident. □

### 6. The Hurwitz-Hopf map in terms of the four real variables $(r, \theta, \phi, \psi)$

Consider the 4-dimensional real Lie group,

\[
G = \left\{ \begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix} \in \text{Mat}_{2 \times 2}(\mathbb{C}) \middle| |z_1|^2 + |z_2|^2 = R^2 > 0 \right\}.
\]

For any $g = \left( \begin{array}{cc} \tilde{z}_1 & -\tilde{z}_2 \\ \tilde{z}_2 & \tilde{z}_1 \end{array} \right) \in G$, choose a positive real number $R = \sqrt{r} \in \mathbb{R}^+$, so as to have $g$ in the identity component $G_0$ of $G$, and write,
Thus, using (26),

\[
\frac{1}{\sqrt{r}} \left( \frac{\bar{z}_1 - z_1}{z_2} \right) = \left( \begin{array}{c}
e^{-i(\frac{\pi}{4})} \cos \theta \frac{1}{2} - e^{-i(\frac{\pi}{4})} \sin \theta \frac{1}{2} \\
e^{i(\frac{\pi}{4})} \sin \theta \frac{1}{2} e^{i(\frac{\pi}{4})} \cos \theta \frac{1}{2}
\end{array} \right) \in SU(2),
\]

and,

\[
z_1 = \sqrt{r} e^{i(\frac{\pi}{4})} \cos \theta \frac{1}{2},
\]

\[
z_2 = \sqrt{r} e^{i(\frac{\pi}{4})} \sin \theta \frac{1}{2}.
\]

In particular, using these expressions for \(z_1\) and \(z_2\), the cartesian coordinates \(x_1, x_2, \) and \(x_3\) — given by the Hurwitz-Hopf map, \(x_i = H(z_1, z_2) (1 \leq i \leq 3)\) — can be written in terms of the spherical coordinates \(r, \theta, \phi\), in the usual way,

\[
x_1 = r \sin \theta \cos \phi, \quad x_2 = r \sin \theta \sin \phi, \quad x_3 = r \cos \theta.
\]

Also, \(H\) yields the following expressions for \(r\) and the Euler angles \((\theta, \phi, \psi)\) in terms of the complex variables \(z_1\) and \(z_2\):

\[
H^r = |z_1|^2 + |z_2|^2,
\]

\[
H^\theta = \cos^{-1} \left( \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2} \right),
\]

\[
H^\phi = \tan^{-1} \left( -i \frac{z_1 \bar{z}_2 - z_2 \bar{z}_1}{z_1 \bar{z}_2 + z_2 \bar{z}_1} \right),
\]

\[
H^\psi = \tan^{-1} \left( -i \frac{z_1 \bar{z}_2 - z_2 \bar{z}_1}{z_1 \bar{z}_2 + z_2 \bar{z}_1} \right),
\]

where, \(H^r\) stands for the pullback of \(H\), so that \(H^f\) means \(f \circ H\) for any smooth function \(f\). Thus, for example, \(H^r = r (H(z_1, z_2)) = |z_1|^2 + |z_2|^2\), etc. In particular, \(H\) itself gets described as a map changing two specific local coordinate charts; say, \((z_1, z_2) \mapsto (r, \theta, \phi, \psi)\), thus allowing a reinterpretation of the Hurwitz-Hopf map as a 2-to-1 differentiable group map,

\[
H: \mathbb{R}^+ \times SU(2) \to \mathbb{R}^+ \times SO(3)
\]

\[
(z_1, z_2) \mapsto (r, \theta, \phi, \psi), \quad |z_1|^2 + |z_2|^2 = r > 0.
\]

### 7. The Lie algebra \(\mathfrak{g}\) of the symmetry group \(G\) associated to the Hurwitz-Hopf map

Since \(G_0 \simeq \mathbb{R}^+ \times SU(2)\), it is immediate to see that the Lie algebra \(\mathfrak{g}\) of \(G\) is isomorphic to the real 4-dimensional space,

\[
\mathfrak{g} = \text{Span}_{\mathbb{R}} \{ I_2 \} \oplus \text{su}(2).
\]

Its complexification \(\mathfrak{g}_C\) is isomorphic to the Lie algebra \(\mathfrak{gl}_2(C)\) of complex 2 \times 2 matrices. Geometrically, we may realize the Lie algebras \(\mathfrak{g}\) and \(\mathfrak{g}_C\) in terms of real and complex 4-dimensional vector spaces of left \(G\)-invariant vector fields, respectively. A convenient set of linearly independent generators for either \(\mathfrak{g}\) or \(\mathfrak{g}_C\), is \([L, L_2, L_3, L_4, L_5]\), where,

\[
L = \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right),
\]

\[
L_2 = \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} \right),
\]

\[
L_+ = \left( z_1 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_1} \right), \quad \text{and},
\]

\[
L_- = \left( \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} \right).
\]
It is a straightforward matter to verify that \( L \) commutes with all the operators and that,
\[
[L_z, L_\pm] = \pm L_\pm, \quad \text{and} \quad [L_+, L_-] = 2L_z,
\]
thus proving that \( \text{Span}_\mathbb{C}\{L, L_x, L_y, L_z\} \) is isomorphic to the Lie algebra \( \mathfrak{gl}(2, \mathbb{C}) \). Clearly, the operators (34) may act on the space of complex polynomials in the variables \( \{z_1, z_2, \bar{z}_1, \bar{z}_2\} \) producing polynomials in the same variables.

8. Computations with the angular momentum complex operators

Take the Lie algebra generators \(\{L, L_x, L_y, L_z\}\) as given before, and define the operators \(L_x\) and \(L_y\) through,
\[
L_x = \frac{1}{2}(L_+ + L_-), \quad \text{and} \quad L_y = \frac{1}{2i}(L_+ - L_-).
\]
Therefore,
\[
L_x = \frac{1}{2}(L_+ + L_-), \quad \text{and} \quad L_y = \frac{1}{2i}(L_+ - L_-).
\]
It turns out that,
\[
[L_x, L_z] = i L_z, \quad [L_z, L_x] = i L_y, \quad [L_y, L_x] = i L_z.
\]
Clearly,
\[
L_x L_z + L_z L_x = L_x^2 + L_y^2 - i \left[L_x, L_y\right] = L_x^2 + L_y^2 + L_z,
\]
whence,
\[
L_x^2 + L_y^2 + L_z^2 = L_+ L_- + L_z^2 - L_z.
\]

Proposition 8.1. A direct computation shows that,
\[
L_x^2 + L_y^2 + L_z^2 = \mathbf{1} + L + (|z_1|^2 + |z_2|^2) \left( \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2} \right),
\]
where,
\[
\left( \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2} \right) = \Delta = \frac{1}{4} \left( \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2} \right),
\]
is the Laplace operator in \( \mathbb{R}^4 \).

Proof. This is a straightforward computation. One only has to make use of the explicit expressions for \(L_\pm\) and \(L_z\) given in (34) and apply the operator \(L_+ L_- + L_z^2 - L_z\) appearing on the right hand side of (40) to an arbitrary differentiable function \(f\) depending on the local coordinates \(\{z_1, z_2, \bar{z}_1, \bar{z}_2\}\). The calculation shows that \((L_+ L_- + L_z^2 - L_z)f\) is equal to \(\mathbf{1} + L + (|z_1|^2 + |z_2|^2) \Delta f\) where \(\Delta\) is the Laplace operator in the statement.

Proposition 8.2. Using the expressions \(H^*r, H^*\theta, H^*\phi, \text{ and } H^*\psi\) given in (31), together with the covering map (32) stating that
\[
\begin{align*}
z_1 &= \sqrt{r} \, e^{i\left(\frac{\theta}{2}\right)} \cos \frac{\theta}{2}, & \text{whenever } \left(\frac{z_1}{z_2}, \frac{\bar{z}_1}{\bar{z}_2}\right) \in \mathcal{G}_0 = \mathbb{R}^+ \times SU(2), \\
z_2 &= \sqrt{r} \, e^{i\left(\frac{\theta}{2}\right)} \sin \frac{\theta}{2},
\end{align*}
\]
then
\[
\begin{align*}
\frac{\partial}{\partial z_1} &= \sqrt{r} \, e^{-i\left(\frac{\theta}{2}\right)} \left( \cos \frac{\theta}{2} \frac{\partial}{\partial r} - \frac{1}{r} \sin \frac{\theta}{2} \frac{\partial}{\partial \theta} - \frac{i}{2r} \cos \frac{\theta}{2} \left( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi} \right) \right), \\
\frac{\partial}{\partial z_2} &= \sqrt{r} \, e^{-i\left(\frac{\theta}{2}\right)} \left( \sin \frac{\theta}{2} \frac{\partial}{\partial r} + \frac{1}{r} \cos \frac{\theta}{2} \frac{\partial}{\partial \theta} + \frac{i}{2r} \sin \frac{\theta}{2} \left( \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \psi} \right) \right), \\
\frac{\partial}{\partial \bar{z}_1} &= \sqrt{r} \, e^{i\left(\frac{\theta}{2}\right)} \left( \cos \frac{\theta}{2} \frac{\partial}{\partial r} - \frac{1}{r} \sin \frac{\theta}{2} \frac{\partial}{\partial \theta} + \frac{i}{2r} \cos \frac{\theta}{2} \left( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi} \right) \right), \\
\frac{\partial}{\partial \bar{z}_2} &= \sqrt{r} \, e^{i\left(\frac{\theta}{2}\right)} \left( \sin \frac{\theta}{2} \frac{\partial}{\partial r} + \frac{1}{r} \cos \frac{\theta}{2} \frac{\partial}{\partial \theta} - \frac{i}{2r} \sin \frac{\theta}{2} \left( \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \psi} \right) \right).
\end{align*}
\]
\[
\frac{\partial}{\partial z_2} = \sqrt{r} e^{i\left(\frac{\pi}{2}\right)} \left( \sin \frac{\theta}{2} \frac{\partial}{\partial r} + \frac{1}{r} \cos \frac{\theta}{2} \frac{\partial}{\partial \theta} - \frac{i}{2r \sin \frac{\theta}{2}} \left( \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \psi} \right) \right).
\]

Moreover,
\[
L = \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) = r \frac{\partial}{\partial r},
\]
\[
L_z = \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) = -i \frac{\partial}{\partial \phi},
\]
\[
L_+ = \left( z_1 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_1} \right) = e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \frac{\cos \theta}{\sin \theta} \left( \frac{\partial}{\partial \phi} - \frac{1}{\cos \theta} \frac{\partial}{\partial \psi} \right) \right),
\]
\[
L_- = \left( \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} \right) = -e^{-i\phi} \left( \frac{\partial}{\partial \theta} - i \frac{\cos \theta}{\sin \theta} \left( \frac{\partial}{\partial \phi} - \frac{1}{\cos \theta} \frac{\partial}{\partial \psi} \right) \right).
\]

**Proof.** The operators \(\partial/\partial z_1, \partial/\partial z_2, \partial/\partial \bar{z}_1\) and \(\partial/\partial \bar{z}_2\) can be written in terms of \(\partial/\partial r, \partial/\partial \theta, \partial/\partial \phi\) and \(\partial/\partial \psi\), by using the chain rule:
\[
\begin{align*}
\frac{\partial}{\partial z_1} &= \frac{\partial H^* r}{\partial z_1} \frac{\partial}{\partial r} + \frac{\partial H^* \theta}{\partial z_1} \frac{\partial}{\partial \theta} + \frac{\partial H^* \phi}{\partial z_1} \frac{\partial}{\partial \phi} + \frac{\partial H^* \psi}{\partial z_1} \frac{\partial}{\partial \psi}, \\
\frac{\partial}{\partial z_2} &= \frac{\partial H^* r}{\partial z_2} \frac{\partial}{\partial r} + \frac{\partial H^* \theta}{\partial z_2} \frac{\partial}{\partial \theta} + \frac{\partial H^* \phi}{\partial z_2} \frac{\partial}{\partial \phi} + \frac{\partial H^* \psi}{\partial z_2} \frac{\partial}{\partial \psi}, \\
\frac{\partial}{\partial \bar{z}_1} &= \frac{\partial H^* r}{\partial \bar{z}_1} \frac{\partial}{\partial r} + \frac{\partial H^* \theta}{\partial \bar{z}_1} \frac{\partial}{\partial \theta} + \frac{\partial H^* \phi}{\partial \bar{z}_1} \frac{\partial}{\partial \phi} + \frac{\partial H^* \psi}{\partial \bar{z}_1} \frac{\partial}{\partial \psi}, \\
\frac{\partial}{\partial \bar{z}_2} &= \frac{\partial H^* r}{\partial \bar{z}_2} \frac{\partial}{\partial r} + \frac{\partial H^* \theta}{\partial \bar{z}_2} \frac{\partial}{\partial \theta} + \frac{\partial H^* \phi}{\partial \bar{z}_2} \frac{\partial}{\partial \phi} + \frac{\partial H^* \psi}{\partial \bar{z}_2} \frac{\partial}{\partial \psi}.
\end{align*}
\]

The rest are just straightforward calculations of the partial derivatives of \(H^* r, H^* \theta, H^* \phi\) and \(H^* \psi\) with respect to \(z_1, z_2, \bar{z}_1\) and \(\bar{z}_2\) using the expressions in the statement. \(\square\)

**Corollary 8.3. Since**
\[
L_x = \frac{1}{2} (L_+ + L_-), \quad \text{and} \quad L_y = \frac{1}{2i} (L_+ - L_-),
\]

it follows from proposition 8.2 that,
\[
L_x = i \left( \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \frac{\cos \theta}{\sin \theta} \left( \frac{\partial}{\partial \phi} - \frac{1}{\cos \theta} \frac{\partial}{\partial \psi} \right) \right),
\]
and
\[
L_y = -i \left( \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \frac{\cos \theta}{\sin \theta} \left( \frac{\partial}{\partial \phi} - \frac{1}{\cos \theta} \frac{\partial}{\partial \psi} \right) \right). \square
\]

**Remark 8.4.** It is worthwhile to observe the form of the slightly simplified expressions just obtained for the angular momentum operators \(L_x\) and \(L_z\), as well as the expression that results for \(L_x^2 + L_y^2 + L_z^2\), namely,
\[
\begin{align*}
L_+ &= e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} - i \frac{1}{\sin \theta} \frac{\partial}{\partial \psi} \right), \\
L_- &= e^{-i\phi} \left( \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} + i \frac{1}{\sin \theta} \frac{\partial}{\partial \psi} \right), \\
L_z &= -i \frac{\partial}{\partial \phi}, \\
L_x^2 + L_y^2 + L_z^2 &= \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \csc^2 \theta \left( \frac{\partial^2}{\partial \phi^2} \right) \right) \\
&- \csc^2 \theta \left( \frac{\partial^2}{\partial \phi^2} - 2 \csc \theta \frac{\partial}{\partial \phi} \frac{\partial}{\partial \psi} \right). \quad (41)
\end{align*}
\]

which coincide with the usual ones, except for the terms depending on \(\partial/\partial \psi\). Observe that such terms vanish when acting on the usual spherical wave functions depending on \((r, \theta, \phi)\). Nevertheless, they are crucial when
acting on spherical harmonics half-integer spin wave functions (see sections 12-14 below). At any rate, it is important to remark that these new angular momentum operators satisfy the usual angular momentum commutation relations, as they are simply a particular realisation of the Lie algebra $su_2 \simeq so_3$.

9. The complete set of Schwinger’s operators

Following Schwinger’s works [1] and [2], introduce the following set of eight operators:

$$
\begin{align*}
\hat{L}_1^+ &= \frac{\partial}{\partial z_1}, & \hat{L}_1^- &= \frac{\partial}{\partial \bar{z}_1}, \\
\hat{L}_2^+ &= \frac{\partial}{\partial z_2}, & \hat{L}_2^- &= \frac{\partial}{\partial \bar{z}_2}.
\end{align*}
$$

We shall refer to them collectively as ‘the $j^{\pm\pm}_a$-operators’, or simply as ‘the $j$s’. Just like the operators from the set $\{L, L_1, L_2, L_3\} \simeq gl_2(\mathbb{C})$ — to which we shall refer as the ‘the $L$s’ — the $j$s also act on the space of polynomials in the variables $(z_1, z_2, \bar{z}_1, \bar{z}_2)$, to produce polynomials in the same variables.

**Proposition 9.1.** The commutators of the $L$s with the $j$s, produce $j^{\pm\pm}_a$-operators. In other words, under commutators between $L$s and $j$s, the $j$s form a representation space for the Lie algebra $gl_2(\mathbb{C})$ spanned by the $L$s. Moreover, the subspace generated by the $j$s gets decomposed into four copies of the 2-dimensional irreducible representation of $gl_2(\mathbb{C})$ according to:

1. On $\text{Span} \{j^+_{12}, j^-_{12}\}$

$$
\begin{align*}
L &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & L_z &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & L_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & L_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{align*}
$$

2. On $\text{Span} \{j^{+-}_{12}, j^{-+}_{12}\}$

$$
\begin{align*}
L &= -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & L_z &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & L_+ &= -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & L_- &= -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{align*}
$$

3. On $\text{Span} \{j^{++}_{12}, j^{-+}_{12}\}$

$$
\begin{align*}
L &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & L_z &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & L_+ &= -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & L_- &= -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{align*}
$$

4. On $\text{Span} \{j^{++}_{12}, j^{-+}_{12}\}$

$$
\begin{align*}
L &= -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & L_z &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & L_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & L_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{align*}
$$

**Proof.** This is a straightforward calculation using (34) and (43). The results can be just listed as follows:

$$
\begin{align*}
[L, j^{++}_1] &= \frac{1}{2} j^{++}_1, & [L, j^{+-}_1] &= \frac{1}{2} j^{+-}_1, & [L, j^{-+}_1] &= -\frac{1}{2} j^{-+}_1, & [L, j^{--}_1] &= -\frac{1}{2} j^{--}_1, \\
[L, j^{++}_2] &= \frac{1}{2} j^{++}_2, & [L, j^{+-}_2] &= \frac{1}{2} j^{+-}_2, & [L, j^{-+}_2] &= -\frac{1}{2} j^{-+}_2, & [L, j^{--}_2] &= -\frac{1}{2} j^{--}_2, \\
[L, j^{++}_3] &= \frac{1}{2} j^{++}_3, & [L, j^{+-}_3] &= \frac{1}{2} j^{+-}_3, & [L, j^{-+}_3] &= -\frac{1}{2} j^{-+}_3, & [L, j^{--}_3] &= -\frac{1}{2} j^{--}_3, \\
[L, j^{++}_4] &= \frac{1}{2} j^{++}_4, & [L, j^{+-}_4] &= \frac{1}{2} j^{+-}_4, & [L, j^{-+}_4] &= -\frac{1}{2} j^{-+}_4, & [L, j^{--}_4] &= -\frac{1}{2} j^{--}_4.
\end{align*}
$$

\[\square\]
We write
\[ W_1 = W_1^{+1/2} \oplus W_1^{-1/2} = \text{Span}\{j_1^{++}, j_1^{+-}\} \oplus \text{Span}\{j_1^{+-}, j_1^{--}\}, \]
\[ W_2 = W_2^{+1/2} \oplus W_2^{-1/2} = \text{Span}\{j_2^{++}, j_2^{+-}\} \oplus \text{Span}\{j_2^{+-}, j_2^{--}\}. \] (44)

According to proposition 9.1, the four 2-dimensional subspaces involved in these direct sums are 2-dimensional irreducible representations for \( \mathfrak{gl}_2(\mathbb{C}) \) on which the operator \( \text{Id} \) acts as the scalar \( j = \pm 1/2 \), as indicated by the superscript in the notation \( W_i^{\pm1/2} \) (1 \( \leq i \leq 2 \)). On each of these 2-dimensional subspaces, the operator \( \text{Id} \) acts diagonally in the given ordered bases as \( \text{diag}(1/2, -1/2) \). Both, \( W_1 \) and \( W_2 \) are symplectic vector spaces and the given decompositions \( W_1^{+1/2} \oplus W_1^{-1/2} \) correspond to the totally isotropic 2-dimensional subspaces defined by their corresponding symplectic forms with which one may define 5-dimensional Heisenberg Lie algebras as in \([1]\) and \([2]\) (see section 10 below).

**Definition 9.2.** Let \( \rho: \mathfrak{gl}_2(\mathbb{C}) \rightarrow \mathfrak{gl}(W_1 \oplus W_2) \) be the representation defined by the Lie brackets computed in proposition 9.1. That is, for any \( x \in \mathfrak{gl}_2(\mathbb{C}) \) and any \( w \in W_1 \oplus W_2 \), \( \rho(x)(w) = [x, w] \).

**Corollary 9.3.** One may deduce the following possible correspondences with the original Schwinger operators \((4)\) and \((5)\) given in section 2. Either,
\[ a_i^+ \leftrightarrow j_i^{++}, \quad a_i^- \leftrightarrow j_i^{+-}, \quad a_k \leftrightarrow j_k^{--}, \quad a_y \leftrightarrow j_y^{++}, \quad \text{or}, \]
\[ a_i^+ \leftrightarrow j_i^{++}, \quad a_i^- \leftrightarrow j_i^{+-}, \quad a_k \leftrightarrow j_k^{--}, \quad a_y \leftrightarrow j_y^{+-}. \] \(\square\)

### 10. The Lie algebra versus the Lie superalgebra approach

Assuming that the operators \( \text{L}_a \) and \( \text{j}_a^\star \) act on an appropriate Hilbert space of functions, \( \mathcal{H} \), one may consider the Lie algebra,
\[ \mathfrak{g} = \langle \text{Id} \rangle \oplus \mathfrak{gl}_1 \oplus W_1 \oplus W_2, \] (45)
where \( \langle \text{Id} \rangle \) stands for the one-dimensional subspace generated by the identity operator \( \text{Id} \): \( \mathcal{H} \rightarrow \mathcal{H} \). In this case, the subspaces,
\[ h_1 = W_2 \oplus \langle \text{Id} \rangle, \quad \text{and} \quad h_2 = W_2 \oplus \langle \text{Id} \rangle, \] (46)
close under the Lie bracket \( [\cdot, \cdot] \) which is the ordinary commutator of the input operators, and each \( h_a \) is a 5-dimensional Heisenberg Lie algebra. Moreover, \( [h_1, h_2] = 0 \), and \( h_1 \cap h_2 = \langle \text{Id} \rangle \). Observe that one requires \( W_1 \) and \( W_2 \) to be symplectic spaces in order to close the algebraic structure via Lie brackets (i.e., commutators) between \( j \)'s, giving thus rise to the two Heisenberg Lie algebras (46).

On the other hand, one may consider the Lie superalgebra,
\[ \mathfrak{g}/\langle \text{Id} \rangle \cong \mathfrak{g}' = \mathfrak{gl}_2 \oplus W_1 \oplus W_2, \] (47)
where the algebraic structure for pairs of elements in \( W_1 \oplus W_2 \) gets closed through a symmetric bilinear \( \mathfrak{gl}_2(\mathbb{C}) \)-valued map \( \Gamma: (W_1 \oplus W_2) \times (W_1 \oplus W_2) \rightarrow \mathfrak{gl}_2(\mathbb{C}) \), satisfying the equivariance condition,
\[ [x, \Gamma(v, w)] = \Gamma(\rho(x)(v), w) + \Gamma(v, \rho(x)(w)), \] (48)
for the representation \( \rho \) defined in 9.2 and for any \( x \in \mathfrak{gl}_2(\mathbb{C}) \), and any pair \( v \) and \( w \) in \( W_1 \oplus W_2 \). The underlying \( \mathbb{Z}_2 \)-graded vector space of this Lie superalgebra is,
\[ \mathfrak{g}' = \mathfrak{g}'_0 \oplus \mathfrak{g}'_1, \quad \text{where,} \quad \begin{cases} \mathfrak{g}'_0 = \mathfrak{gl}_2, \\ \mathfrak{g}'_1 = W_1 \oplus W_2, \end{cases} \] (49)
and the Lie superbracket is computed as,
\[ [x + v, y + w] = [x, y] + \rho(x)(w) - \rho(y)(v) + \Gamma(v, w), \] (50)
for any pair \( x \) and \( y \) in \( \mathfrak{gl}_2(\mathbb{C}) \) and any pair \( v \) and \( w \) in \( \mathfrak{g}'_1 = W_1 \oplus W_2 \) (see \([3]\) and \([20]\)). Moreover, since \( \mathfrak{g}'_1 \) may be decomposed in terms of various copies of the fundamental 2-dimensional irreducible representation of \( \mathfrak{g}_0 = \mathfrak{gl}_2 \), we may apply the results from \([20]\) to conclude the following:

**Proposition 10.1.** For the Lie superalgebra just described through (47)–(50), any symmetric bilinear \( \rho \)-equivariant \( \mathfrak{gl}_2(\mathbb{C}) \)-valued map \( \Gamma: (W_1 \oplus W_2) \times (W_1 \oplus W_2) \rightarrow \mathfrak{gl}_2(\mathbb{C}) \), must be identically zero.

**Remark 10.2.** For the Lie superalgebra (49) with Lie superbracket (50), the parity of all the the generators of \( W_1 \oplus W_2 \) (i.e., the \( j \)-operators (43)) is 1; that is, they are odd operators. Thus, in order to build up a Lie supergroup
whose Lie superalgebra is \((49)\), one would have to consider first the \(GL_2\)-homogeneous vector bundle defined over \(GL_2\) itself, whose typical fiber is the odd subspace \(W_n \oplus W_o\), and then look at the supermanifold based on \(GL_2\) whose defining supermanifold sheaf is the sheaf of sections of the exterior algebra \((GL_2\text{-homogeneous})\) bundle \(\wedge (W_n \oplus W_o)\). In particular, the Lie superbracket \(\Gamma(u, v)\) of any two odd generators is symmetric, and when viewed as sections of the corresponding supermanifold sheaf, they anti-commute (see \([21]\) or \([22]\)).

11. The representation space and the wave functions

For either the Lie algebra \(\mathfrak{g}\) or the Lie superalgebra \(\mathfrak{g}'\), we shall take the representation space to be the \(L^2\)-completion with respect to the hermitian product given by,

\[
\langle \psi_1, \psi_2 \rangle = \int_{\mathbb{R}^2} \psi_1(z_1, z_2) \overline{\psi_2(z_1, z_2)} e^{-\frac{1}{2}|z_1|^2 + |z_2|^2} \, dz_1 \, dz_2 \, d\bar{z}_1 \, d\bar{z}_2,
\]

in the vector space generated by the harmonic polynomials that result — up to normalization constants — from \((L_u)^n(z_1, z_2)^n\), together with those that result from \((L_u)^n(z_1, z_2)^n (0 \leq k \leq n \text{ and } n \in \{0\} \cup \mathbb{N})\). It will be clear from our work in sections 12 and 13 below, that these expressions are linear combinations of monomials of the form \(z_2^k z_2^{m-k}\) and \(\bar{z}_2^k \bar{z}_2^{m-k}\) \((0 \leq k \leq m, m = n \text{ or } m = n + 1, \text{ and } n \in \{0\} \cup \mathbb{N})\), and these are harmonic as they satisfy, \(\Delta \psi = 0\), where \(\Delta\) is the Laplace operator given in proposition 8.1.

12. Examples

Define \(|0, 0\rangle = 1\) and leave the normalization questions aside, for the moment. Define the \(j = 1/2\) kets by,

\[
|1/2, +1/2\rangle = j^{1+}|0, 0\rangle = z_1|0, 0\rangle = z_1,
|1/2, -1/2\rangle = j^{-1+}|0, 0\rangle = z_2|0, 0\rangle = z_2,
\]

to get the \(j = 1/2\) harmonic functions,

\[
f_{1/2, +1/2}(r, \theta, \phi, \psi) = \sqrt{\mathcal{R}} e^{j\psi} e^{j\phi} \cos \frac{\theta}{2},
\]

\[
f_{1/2, -1/2}(r, \theta, \phi, \psi) = \sqrt{\mathcal{R}} e^{j\psi} e^{j\phi} \sin \frac{\theta}{2},
\]

In terms of the complex coordinates \((z_1, z_2, \bar{z}_1, \bar{z}_2)\), we have,

\[
f_{1/2, +1/2}(z_1, z_2, \bar{z}_1, \bar{z}_2) = z_1,
f_{1/2, -1/2}(z_1, z_2, \bar{z}_1, \bar{z}_2) = z_2.
\]

Using the \(L\)’s as given in \((34)\), it follows from \((34)\) that,

\[
L_{1/2, +1/2} f_{1/2, +1/2} = \frac{1}{2} f_{1/2, +1/2}, \quad \text{and} \quad L_{1/2, -1/2} f_{1/2, -1/2} = \frac{1}{2} f_{1/2, -1/2},
\]

whereas,

\[
L_{1/2, +1/2} f_{1/2, +1/2} = \frac{1}{2} f_{1/2, +1/2}, \quad \text{and} \quad L_{1/2, -1/2} f_{1/2, -1/2} = \frac{1}{2} f_{1/2, -1/2}.
\]

On the other hand,

\[
L_1 z_1 = z_2, \quad \text{and} \quad L_1 z_2 = z_1,
\]

That is,

\[
L_1 f_{1/2, +1/2} = f_{1/2, +1/2}; \quad \text{i.e.,} \quad L_1 |1/2, +1/2\rangle = |1/2, -1/2\rangle,
L_1 f_{1/2, -1/2} = f_{1/2, -1/2}; \quad \text{i.e.,} \quad L_1 |1/2, -1/2\rangle = |1/2, +1/2\rangle.
\]

Observe that the pair \(|z_1\rangle = \begin{pmatrix} f_{1/2, +1/2} \\ f_{1/2, -1/2} \end{pmatrix}\) corresponds to the 2-dimensional \textit{spin representation}. If required, this spinor may be acted upon by the \textit{gauge transformation}, \(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto e^{-i\phi/2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\) to eliminate its dependence on \(\psi\).

\textbf{Notation 12.1.} Let \(|j, m\rangle\) stand for the subspace generated by the wave functions \(|j, m\rangle\) — also denoted by \(f_{j,m}\) — satisfying,

\[
L |j, m\rangle = j |j, m\rangle, \quad \text{and} \quad L_1 |j, m\rangle = m |j, m\rangle.
\]
In what follows, we shall use the well known relations
\[
L_+ | j, m \rangle = \sqrt{(j - m)(j + m + 1)} | j, m + 1 \rangle,
L_- | j, m \rangle = \sqrt{(j + m)(j - m + 1)} | j, m - 1 \rangle.
\] (60)

**Proposition 12.2.**
\[
\begin{align*}
& j^+_1(1/2, -1/2) \subset \langle 1, -1 \rangle, \quad \text{and} \quad j^+_2(1/2, +1/2) \subset \langle 1, +1 \rangle.
\end{align*}
\] (61)

**Proof.** Since \( \langle 1/2, -1/2 \rangle = \langle z_2 \rangle \), it follows that, \( j^+_1 z_2 = z_1 z_2 \). Therefore, using (34) one obtains that,
\[
L(z_1 z_2) = z_1 z_2, \quad \text{and} \quad L_2(z_1 z_2) = -z_1 z_2.
\]
Similarly, since \( \langle 1/2, +1/2 \rangle = \langle z_1 \rangle \), it follows that, \( j^+_2 z_1 = z_1 z_2 \), and,
\[
L(z_1 z_2) = z_1 z_2, \quad \text{and} \quad L_2(z_1 z_2) = z_1 z_2.
\]

**Proposition 12.3.**
\[
\begin{align*}
& |1, -1\rangle = j^+_1 z_2 = j^+_1 |1/2, -1/2\rangle = j^+_1 (j^+_2 |0, 0\rangle) = z_1 z_2, \\
& |1, +1\rangle = j^+_2 z_1 = j^+_2 |1/2, +1/2\rangle = j^+_1 (j^+_2 |0, 0\rangle) = z_1 z_2.
\end{align*}
\]

**Proof.** Straightforward computations using (34) show that,
\[
L(z_1 z_2) = z_1 z_2 \Rightarrow j = +1, \quad \text{and} \quad L_2(z_1 z_2) = z_1 z_2 \Rightarrow j = +1,
\]
\[
L_2(z_1 z_2) = -z_1 z_2 \Rightarrow m = -1, \quad \text{and} \quad L_2(z_1 z_2) = z_1 z_2 \Rightarrow m = +1.
\]

Straightforward computations also show that,
\[
\begin{align*}
L_+(z_1 z_2) &= |z_1|^2 - |z_2|^2, \quad L_+ | |z_1|^2 - |z_2|^2 \rangle = -2z_1 z_2, \quad L_+(-2z_1 z_2) = 0, \\
L_-(z_1 z_2) &= |z_1|^2 - |z_2|^2, \quad L_- | |z_1|^2 - |z_2|^2 \rangle = -2\bar{z}_1 \bar{z}_2, \quad L_-(-2\bar{z}_1 \bar{z}_2) = 0.
\end{align*}
\]

In particular,
\[
|z_1|^2 - |z_2|^2 = r \cos \theta = L_+ | 1, -1 \rangle = \sqrt{2} | 1, 0 \rangle
\]
\[
\Rightarrow | 1, 0 \rangle = \frac{r}{\sqrt{2}} \cos \theta,
\]
\[
-2z_1 \bar{z}_2 = -r e^{-i\phi} \sin \theta = \langle L_- |^2 | 1, -1 \rangle = \sqrt{2} L_+ | 1, 0 \rangle = 2 | 1, -1 \rangle
\]
\[
\Rightarrow | 1, -1 \rangle = -\frac{r}{2} e^{-i\phi} \sin \theta.
\]

Therefore, we may write,
\[
\begin{pmatrix}
\hat{f}_{1, 1}(r, \theta, \phi) \\
\hat{f}_{0, 0}(r, \theta, \phi) \\
\hat{f}_{-1, 1}(r, \theta, \phi)
\end{pmatrix} = \begin{pmatrix}
r/2 e^{i\phi} \sin \theta \\
r/\sqrt{2} \cos \theta \\
-r/2 e^{-i\phi} \sin \theta
\end{pmatrix}.
\] (62)

Observe that up to an overall scalar multiple of \( r \), the triple \( \begin{pmatrix} \hat{f}_{1, 1} \\ \hat{f}_{0, 0} \\ \hat{f}_{-1, 1} \end{pmatrix} \) of the spin-one representation yields the ordinary spherical harmonics \( \begin{pmatrix} Y_{1, 1}(\phi, \theta) \\ Y_{0, 0}(\phi, \theta) \\ Y_{-1, 1}(\phi, \theta) \end{pmatrix} \) as expected.

**Summary 12.4.** One may summarize the results obtained in passing from \( j = 1/2 \) to \( j = 1 \) by means of the following ‘starting’ diagram:
\[
\begin{align*}
\langle 1, -1 \rangle & \xrightarrow{L_+} \langle 1, 0 \rangle & \langle 1, 0 \rangle & \xrightarrow{L_-} \langle 1, +1 \rangle \\
\langle 1/2, -1/2 \rangle & \xrightarrow{\alpha} \langle 1/2, +1/2 \rangle & \langle 1/2, +1/2 \rangle & \xrightarrow{\beta} \langle 1/2, -1/2 \rangle \\
\langle 0, 0 \rangle & \xrightarrow{J^-_1} \langle 0, 0 \rangle & \langle 0, 0 \rangle & \xrightarrow{J^+_1} \langle 0, 0 \rangle
\end{align*}
\]

(63)

where, the maps \( \alpha \) and \( \beta \) can be determined using the commutation relations between the \( L \)'s and the \( j^{**} \)'s:

\[
\alpha = L_+ j^{**} - j^{**} L_-, \\
\beta = L_- j^{**} - j^{**} L_+.
\]

(64)

13. Generalisation

Proposition 13.1. For the \( j = n \) or the \( j = n + 1/2 \) \((n \in \mathbb{N} \cup \{0\})\) spin representation, the corresponding lowest weight and highest weight vectors, respectively characterised by \( -n = j^{**} L_j \), \( 0 \rangle = \langle z_1 z_2 \rangle^n \)

\[
\vert n, -n \rangle = (j^{**} L_j - j^{**} L)^n \langle 0, 0 \rangle = (z_1 z_2)^n
\]

\[
\vert n, +n \rangle = (j^{**} L_j + j^{**} L)^n \langle 0, 0 \rangle = (z_1 z_2)^n
\]

\[
\vert n + 1/2, -(n + 1/2) \rangle = j^{**} L_j (j^{**} L_j - j^{**} L)^n \langle 0, 0 \rangle = z_2 (z_1 z_2)^n
\]

\[
\vert n + 1/2, +(n + 1/2) \rangle = j^{**} L_j (j^{**} L_j + j^{**} L)^n \langle 0, 0 \rangle = z_2 (z_1 z_2)^n
\]

Proof. Straightforward computations using the \( L \)'s given in (34), show indeed that,

\[
L(z_1 z_2)^n = n(z_1 z_2)^n,
\]

\[
L(z_1 z_2)^n = n(z_1 z_2)^n
\]

and

\[
L(z_1 z_2)^n = n(z_1 z_2)^n,
\]

\[
L(z_1 z_2)^n = n(z_1 z_2)^n
\]

To complete the proof one needs to show that,

\[
L_-(z_1 z_2)^n = 0, \quad \text{and} \quad L_-(z_1 z_2)^n = 0,
\]

and that,

\[
L_+(z_1 z_2)^n = 0, \quad \text{and} \quad L_+(z_1 z_2)^n = 0.
\]

The general cases of these results, need induction together with the statements of lemma 13.2 and corollary 13.3 below.

Lemma 13.2.

\[
L_+(j^{**} L_j - j^{**} L_j) L_+ + (j^{**} L_j - j^{**} L_j - j^{**} L_j) = (j^{**} L_j - j^{**} L_j) L_+ + (|z_1|^2 - |z_2|^2) \text{Id}
\]

\[
L_-(j^{**} L_j + j^{**} L_j) L_- + (j^{**} L_j + j^{**} L_j + j^{**} L_j) = (j^{**} L_j + j^{**} L_j) L_- + (|z_1|^2 - |z_2|^2) \text{Id}
\]

Proof. This is a straightforward calculation using the expressions for the \( L \)'s in (34) and those for the \( j \)'s in (43), together with the commutation relations of proposition 9.1.
Corollary 13.3. Since \([L_+, j^+_2] = j^+_1\) and \([L_-, j^-_1] = j^-_2\), it follows that,
\[
L_+ j^+_2 = j^+_2 L_+ + j^+_1 L_+,
\]
implies that,
\[
L_+ j^+_2 (j^+_1 j^-_2) = j^+_2 (j^+_1 j^-_2) + (|z|^2 z)\text{id}.
\]

Proof. Indeed,
\[
L_+ j^+_2 (j^+_1 j^-_2) = \left( j^+_2 L_+ + j^+_1 L_+ \right) j^+_2 j^-_2 + (|z|^2 z) j^+_2 j^-_2
\]
\[
= \left( j^+_2 L_+ j^+_2 j^-_2 \right) L_+ + (|z|^2 z) j^+_2 j^-_2
\]
\[
= j^+_2 (j^+_1 j^-_2) L_+ + (|z|^2 z) j^+_2 j^-_2
\]
as claimed.

Corollary 13.4. For each non-negative integer \(n\), the components \(f\) of the \(2n + 1\)-tuples having \(j = n + 1/2\), can be expressed in the form,
\[
f(r, \theta, \phi, \psi) = r^{n+1/2} e^{i\psi/2} y(\theta, \phi), \quad \text{with} \quad y(\theta, \phi) \in \bigoplus_{m=-j}^{+j} (j, m),
\]
whereas the components \(f\) of the \(2n + 1\)-tuples having \(j = n\), can be expressed in the form,
\[
f(r, \theta, \phi) = r^n Y(\theta, \phi), \quad \text{with} \quad Y(\theta, \phi) \in \bigoplus_{m=-j}^{+j} (j, m).
\]

Corollary 13.5. The procedure that follows from the results above to produce the harmonic wave functions for integer and half-integer angular momentum states is this: Either,

1. Go from \((0, 0)\) to \((j, -j)\) using \((j^+ \downarrow)^n\) if \(j = n\),
   (or using \((j^- \uparrow)^n\) if \(j = n + 1/2\),
   and then move from \((j, -j)\) to \((j, +j)\) by applying successively \(L_+\); or,

2. Go from \((0, 0)\) to \((j, +j)\) using \((j^+ \uparrow)^n\) if \(j = n\),
   (or using \((j^- \downarrow)^n\) if \(j = n + 1/2\),
   and then move from \((j, +j)\) to \((j, -j)\) by applying successively \(L_-\).

In the first case, each application of \(L_+\) changes the eigenvalue \(m\) of \(L_z\) by adding 1 in each step, whereas in the second case, each application of \(L_-\) changes \(m\) by subtracting 1 in each step. In particular,

3. Application of the operators \(j^+\) and \(j^-\) to any of the above states raise the eigenvalue \(j\) by a 1/2 step, whereas the operators \(j^+\) and \(j^-\) lower \(j\) by a 1/2 step.

14. Nonrelativistic quantum equation for the Hydrogen atom including the electron spin

As an example we now write a new nonrelativistic quantum equation (NNRQE) for the Hydrogen atom that takes into account the electron spin. The equation can be solved exactly in terms of the new angular momentum operators depending on the Euler angle \(\psi\) and the new harmonic wave functions introduced before. The NNRQE is expressed in terms of four spatial variables plus time. The four spatial variables are the usual three-dimensional spherical coordinates \(r, \theta, \phi\), plus the third Euler angle \(\psi\).

For comparison purposes, we use here the standard units in terms of \(\hbar\). The NNRQE is then written in the usual fashion,
\[
H\Psi(r, \theta, \phi, \psi, t) = i\hbar \frac{\partial \Psi(r, \theta, \phi, \psi, t)}{\partial t},
\]
for the Hamiltonian operator \(H\) given by
\[
H = -\frac{\hbar^2}{2\mu} \nabla^2 + V(r, \theta, \phi, \psi, t).
\]

Here, \(\mu\) is the reduced mass of the proton-electron system. The only difference with the usual treatment is due to the presence of the new variable \(\psi\), which implies that the four-dimensional Laplacian.
\[ \nabla^2 r, \theta, \phi, \psi, \partial_r, \partial_\theta, \partial_\phi, \partial_\psi \) is related to the usual three-dimensional Laplacian \( \nabla^2 (r, \theta, \phi, \psi, \partial_r, \partial_\theta, \partial_\phi, \partial_\psi) \) by
\[
\nabla^2 = \nabla^2 \left[ \frac{1}{r^2} \left( \frac{\csc^2 \theta}{\partial \theta^2} - 2 \csc^2 \theta \cot \theta \frac{\partial}{\partial \phi} \right) \right],
\]
where the correction (in square brackets) arises from the difference between the angular momentum operators in four and three dimensions given in (42). Therefore, the Hamiltonian \( H \) written in full detail is
\[
H = -\frac{\hbar^2}{2 \mu r^2} \left( \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)
- \frac{\hbar^2}{2 \mu r^2} \left( \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} - 2 \csc \theta \cot \theta \frac{\partial}{\partial \phi} \right) + V(r, \theta, \phi, \psi, t).
\]
Consider the Coulomb potential for the Hydrogen atom,
\[
V(r, \theta, \phi, \psi, t) = -\frac{\alpha}{r}.
\]
Observe that the four-dimensional Hamiltonian \( H \) does not differ from the three dimensional one when acting on \( \psi \)-independent wavefunctions, so the usual solutions for integer momentum \( j \) (spinless electrons) are recovered. There are, however, new exact solutions for the case of half-integer \( j \) which solve exactly the NNRQE. The wavefunction,
\[
\Psi_{j\ell}(r, \theta, \phi, \psi, t) = C r^\ell e^{\frac{1}{2} \frac{E_{j\ell}}{\mu q^4}} \frac{\mu q^4}{(j+1)\hbar^4} (\sin \theta)^j \cos \frac{\theta}{2}
\]
with energy,
\[
E_{j\ell} = -\frac{\mu q^4}{2 \hbar^4 (j+1)^2}.
\]
solves (65) for the Coulomb potential for any half-integer \( j \) and \( m = j \). It is straightforward to check the validity of these solutions by following the standard techniques of separation of variables; one makes use of (2) in corollary 13.5 and a solution of the form \( r e^{\alpha r} \) for the radial equation. Finally, one may apply \( L_z \) to \( \Psi_{j\ell} \) repeatedly to get solutions \( \Psi_{j\ell} \) with the same energy \( E_{j\ell} = E_{j\ell} \). This degeneracy is due to the SO(4)-symmetry of the problem, as the NNRQE can be transformed into an equation in the cartesian coordinates \( u_1, u_2, u_3 \) and \( u_4 \) of \( \mathbb{R}^4 \) introduced in (9) and its corresponding four-dimensional Laplace operator. For the SO(4)-symmetry of the Hydrogen atom and the degeneration of its states see [23].

15. Concluding remarks

We have given a new interpretation of the classical Hurwitz-Hopf map \( H : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \), providing an assignment \( \mathbb{C}^2 \ni (z_1, z_2) \rightarrow (r, \theta, \phi, \psi) \) that realizes the double covering map of groups,
\[
G_0 = \mathbb{R}^+ \times SU(2) \rightarrow SU(4) \times SO(3),
\]
keeping the real 3-dimensional euclidean space interpretation of both, the radius \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \), and the Euler angles \( (\theta, \phi, \psi) \) that parametrize rotations in the group \( SO(3) \). The interpretation only requires that \( \left( \frac{z_1}{\sqrt{r}}, \frac{z_2}{\sqrt{r}} \right) \in SU(2) \), whenever \( r = |z_1|^2 + |z_2|^2 \). The Lie algebra generators, \( \{ L_x, L_y, L_z \} \) of \( \mathfrak{g} = \text{Lie}(G_0) \), written in terms of the complex variables \( z_1, z_2, \bar{z}_1, \bar{z}_2 \), satisfy,
\[
L_x^2 + L_y^2 + L_z^2 = L^2 = L + \frac{r}{4} \Delta,
\]
where \( \Delta \) is the Laplace operator in the domain \( \mathbb{R}^4 \approx \mathbb{C}^2 \). We use these facts to produce a Hilbert space \( \mathcal{H} \) of harmonic wave functions —depending on the four real variables \( (r, \theta, \phi, \psi) \)— with which one can describe either integer or half-integer angular momentum states. The angular momentum operators \( \{ L_x, L_y, L_z \} \) coincide with their usual expressions in the spherical coordinates \( (r, \theta, \phi) \) when they act on states having integer angular momentum, but they show an additional term with a derivative depending on the angle \( \psi \) when acting on half-integer angular momentum states. The total angular momentum operator \( L \) gets expressed as \( r \partial_r \). Finally, following Schwinger (see [1] and [2]), the 4-dimensional Lie algebra of \( G_0 \) is coupled with two Heisenberg Lie algebras of 2-dimensional harmonic oscillators generated by \( \{ z_1, z_2, \bar{z}_1, \bar{z}_2 \} \) and their adjoints, thus producing raising and lowering operators that change the total angular momentum in half-units. These two Heisenberg Lie algebras intersect at the identity operator \( 1_{d\mathcal{H}} \). All the operators close either into a 13-dimensional Lie algebra or,
leaving out only the operator $\text{Id}_{\mathcal{H}}$, into a $(4j)$-dimensional Lie superalgebra. Both algebraic structures can be faithfully represented in the Hilbert space $\mathcal{H}$ obtained by the \(L^2\)-completion of the span of the subspace of complex polynomials in the complex variables \((z_1, z_2, \bar{z}_1, \bar{z}_2)\) with the usual Hermitian product having the Hermite weight factor $\exp\left(-\frac{1}{2}(z_1^2 + z_2^2)\right)$. Moreover, $\mathcal{H}$ can be decomposed in the form $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_i$, where $\mathcal{H}_0$ contains all the states having integer angular momentum (or space of bosons) and $\mathcal{H}_i$ contains all the states having half-integer angular momentum (or space of fermions). The closed subspaces $\mathcal{H}_0$ and $\mathcal{H}_i$ are obtained by producing first the highest weight vector $|j, +j\rangle = (z_1 \bar{z}_2)^j$ (if $j$ is integer, with $j \geq 1$) or $|j, +j\rangle = z_1 (z_2 \bar{z}_2)^j$ (if $j$ is half-integer, with $j \geq 1/2$), and then move from it by successive application of the ladder operator $L_0$, which preserves the eigenvalue $j$ but lowers the eigenvalue $m$ in one unit at each step, until reaching the lowest weight vector $|j, -j\rangle$, characterized by $L_0 |j, -j\rangle = 0$, or by doing the analogue constructions producing first the lowest weight vectors $|j, -j\rangle$.

In summary, we have shown how to construct all of the eigenfunctions of the angular momentum operators with integer and half-integer eigenvalues $j$ and $m$, starting from the ground eigenstate $j = 0, m = 0$ by using the $j^+, j^{-}, j^-, j^-$ operators in conjunction with the ladder operators $L_{\pm}$. Note that one gets a vanishing result when applying the operators $j^+, j^-$, and $j^-$ to the ground eigenstate $j = 0, m = 0$. As it is crucial to recover the spherical harmonics for integer eigenvalues, we chose to work with positive eigenvalues $j$ only according to the usual treatment of angular momentum in Quantum Mechanics.

As an example of this approach, a new nonrelativistic quantum equation for the hydrogen atom that includes the electron spin is given, together with its eigensystem for half-integer angular momentum.

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Data availability statement

No new data were created or analysed in this study.

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