Schwinger-Dyson Study for Walking/Conformal Dynamics with IR Cutoffs

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Abstract

Motivated by recent progress on many flavor QCD on a lattice, we investigate conformal/walking dynamics by using Schwinger-Dyson (SD) equation within an improved ladder approximation for two-loop running coupling. By numerically solving the SD equation, we obtain a pole mass $m_p$, pion decay constant $f_\pi$, and investigate the chiral symmetry breaking and mass anomalous dimension $\gamma_m$ in the presence of IR cutoffs $\Lambda_{IR}$. We find that the chiral symmetry breaking is suppressed if IR cutoff $\Lambda_{IR}$ becomes larger than the critical value near the dynamical mass ($\Lambda_{IR} \approx m_D$). In the conformal phase the $\gamma_m$ is strongly suppressed by IR cutoffs for $\Lambda_{IR} \approx m_p$. We, then, obtain finite size hyperscaling (FSS) relation by adapting a linearized approximation for the SD equation, and examine the $\gamma_m$. The results offer valuable insight and suggestion for analyses in lattice gauge theories.

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I. INTRODUCTION

When a fermion multiplicity (flavor) exceeds a critical value \( N_f = N_f^* \) in non-Abelian gauge theories, a quantum phase transition from a chiral broken phase to the conformal window is anticipated to take place. In the conformal window, strong coupling gauge theories are asymptotically-free in ultra violet (UV) region, while dominated by a non-trivial infrared fixed point (IRFP) in infrared (IR) region. The IRFP-conformal dynamics can be probed via a response to a small fermion mass \( (m_0) \) perturbation, which gives rise to a bound state with a mass gap \( (M_H) \) satisfying the conformal hyperscaling relation with a mass anomalous dimension \( \gamma_m \). In the chiral broken phase near to the conformal window \( N_f \lesssim N_f^* \), it is advocated that the system shows an approximate conformality (walking) with a “would-be” \( \gamma_m \), which is of great interest in physics beyond the Standard Model \cite{1}. In recent lattice studies, the precise determination of \( \gamma_m \) in many flavor QCD is reported, while there exist tensions among the values \cite{2}. The discrepancy presumably originates to scale violations by UV/IR cutoffs (lattice spacing and finite box size) as well as a probe fermion mass. In this proceedings, we focus on the IR cutoff effects by using Schwinger-Dyson (SD) equation.

II. SCHWINGER-DYSON EQUATION ANALYSIS

We consider the SD equation for the fermion propagator \( iS_F(q) = \left[ p - \Sigma(p^2) \right]^{-1} \) within improved ladder approximation in the Landau gauge. After integrating over angular degrees of freedom in a momentum space, the SD equation is expressed as

\[
\Sigma(p^2) = m_0 + \int_{\Lambda_{\text{IR}}}^{\Lambda_{\text{UV}}} dq^2 \frac{3 C_2[F] \alpha(p^2 + q^2)}{4\pi} \left[ \frac{q^2 \theta_{p^2-q^2} + \theta_{q^2-p^2}}{q^2 + \Sigma^2(q^2)} \right],
\]

where \( m_0, \theta_{p^2}, \alpha(\mu) = g^2(\mu)/(4\pi^2), \) and \( C_2[F] \) represent a bare fermion mass, step function, running coupling constant in the two-loop perturbation theory, and quadratic Casimir operator, respectively. The UV/IR cutoffs \( \Lambda_{\text{UV/IR}} \) are introduced in the momentum space integral. For a given fermion mass \( m_0 \) and in the presence of \( \Lambda_{\text{UV/IR}} \), we numerically solve the SD equation \cite{11} and evaluate the physical pole mass \( m_p \) and pion decay constant \( f_\pi \) as

\[
m_p \equiv \Sigma(p^2 = m_p), \quad f_\pi^2 = \frac{N_c}{4\pi^2} \int_{\Lambda_{\text{UV}}}^{\Lambda_{\text{IR}}} dz \frac{1 - \frac{1}{3} \frac{d}{dz} \Sigma^2(z)}{(z + \Sigma^2(z))^2}.
\]

We take account of full momentum dependences of the two-loop running coupling without recourse to the usual step-function type approximation for the coupling. We then perform the hyperscaling fit analyses for the obtained numerical data similarly to analyses in lattice gauge theories. Moreover,
FIG. 1: Left: The running coupling for the various fermion multiplicity $9 \leq N_f \leq 16$ in the color $SU(N_c = 3)$ gauge theory. Middle and Right: Pole masses $m_p$ as a function of a bare fermion mass $m_f$ for various IR cutoffs ($\Lambda_{IR}/\Lambda \in [10^{-5}, 3.0]$, blue-thin-solid lines) with the UV cutoff $\Lambda_{UV}/\Lambda = 10$. in the chiral broken phase: (Middle panel) $N_f = 9$, (Right) $N_f = 11$.

we derive the SD-based finite-size hyperscaling formula (FSS), which allows us to handle the IR cutoff artifacts, even in the case of $\Lambda_{IR} \lesssim m_p, f_\pi$. These are the advantages to the previous work [3].

III. CHIRAL BROKEN AND CONFORMAL PHASES WITH IR CUTOFF

First, we investigate the chiral broken phase in the presence of IR cutoffs. The middle and right panels of Fig. 1 show the pole mass $m_p$ against $m_0$ for $N_f = 9$ and 11, respectively. Each blue-thin-solid line represents a result with a different IR cutoff $\Lambda_{IR}/\Lambda$.

For a small $\Lambda_{IR}/\Lambda$, the pole masses remain finite at small $m_0$, indicating the spontaneous chiral symmetry breaking. We find a critical value of the IR cutoff above which the dynamical fermion mass $m_D$ is strongly suppressed as shown in the figures:

$$\Lambda_{IR}^c = m_p(\Lambda_{IR} = 0, m_0 = 0) =: m_D.$$  \hspace{1cm} (3)

Thus, a Fake Conformality results from a large $\Lambda_{IR}$. As a system becomes closer to the conformal window ($N_f = 9 \rightarrow 11$), the fake conformality appears for a smaller $\Lambda_{IR}$. This gives a caveat for lattice works studying the border region of chiral broken and conformal phases.

Next, we investigate a conformal phase with IR cutoffs by using the SD results for $N_f = 12$. The mass anomalous dimension $\gamma_m$ is obtained by the scaling property of $m_p (f_\pi)$ against $m_0$. The left panel of Fig. 2 shows a logarithm plot of the pole mass $m_p$ as a function of fermion bare masses

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1 The scale $\Lambda$ denotes the intrinsic scale which is renormalization invariant and associated with the onset of non-perturbative dynamics [4].
$m_0$ for various IR cutoffs $\Lambda_{IR}$. For the smallest IR cutoff $\Lambda_{IR} = 10^{-8}$ (red-squares), the data points align on a straight line, therefore, the hyperscaling relation $m_p = Cm_0^{1/(1+\gamma_m)}$ is satisfied. For the fit range of $m_0 \in [0.001, 0.1]$ (non-shaded region in the figure), we obtain $\gamma_m = 0.56$. While, for larger IR cutoffs, the data points are on a polygonal line with a bend around $\Lambda_{IR} \approx m_p$. As seen in the figure, the fit range $m_0 \in [0.001, 0.1]$ starts affected by the bend down with increasing $\Lambda_{IR}$. As a result, the fit with the ansatz $m_p = Cm_0^{1/(1+\gamma_m)}$ results in a $\gamma_m$ strongly suppressed, as shown in the middle panel of Fig. 2. When the suppression sets in, the $\gamma_m$ evaluated by $m_p$ (red-squares) and $f_\pi$ (blue-circles) get impaired, and thus the universality of $\gamma_m$ does not hold any more.

**IV. FINITE SIZE HYPERSCALING (FSS) AND MASS ANOMALOUS DIMENSION**

We shall now derive a SD-based FSS which allows us to handle the IR cutoff artifacts explained above. To this end, we further approximate the SD equation \( \text{(1)} \) with a linearized fermion propagator and $\alpha(\mu) \to \alpha_s \theta_{\Lambda_{IR} - \mu}$ where $\alpha_s$ denotes the running coupling at IRFP, and obtain the analytic expression of the renormalization factor $Z_{M}^{-1} := m_0/m_p$. The $Z_{M}$ is a function of dimensionless variables, $\hat{m}_p := m_p/\Lambda$, $\hat{m}_0 := m_0/\Lambda$, $\hat{\Lambda} := \Lambda_{IR}/\Lambda$, and $\gamma = 1 - \sqrt{1 - \alpha_s/\alpha_{cr}}$, and the analytic expression allows us to derive the SD-based FSS formula \( \text{(2)} \):

\[
\frac{\hat{m}_p}{\hat{\Lambda}_{IR}} = C_X \cdot X, \quad X \equiv \hat{m}_0^{1/(1+\gamma_m)}/\hat{\Lambda}_{IR}, \quad \Lambda_{IR} \ll m_p \ll \Lambda, \quad \text{(4)}
\]

\[
\frac{\hat{m}_p}{\hat{\Lambda}_{IR}} = C_Y \cdot Y, \quad Y \equiv \hat{m}_0/\hat{\Lambda}_{IR}^{1+\gamma_m}, \quad m_p \ll \Lambda_{IR} \ll \Lambda, \quad \text{(5)}
\]

The equation \( \text{(4)} \) is same as what dictated by the renormalization group equation \( \text{(3)} \). Remarkably, the SD predicts another FSS formula \( \text{(5)} \) which is characterized by the scaling variable $Y$ rather than $X$ for larger $\Lambda_{IR}$.

We apply the SD-based FSS \( \text{(4)} \) and \( \text{(5)} \) for the data obtained by solving Eq. \( \text{(1)} \) numerically. Right panel of Fig. 2 shows the results. The various symbols (colors) show data with different IR cutoffs. For the case of $\Lambda_{IR} \ll m_p \ll \Lambda$, we adopt the ansatz \( \text{(4)} \). The fit works well (blue-solid line) and we obtain $\gamma_m \simeq 0.60 =: \gamma_1$ which is fairly consistent to $\gamma_m = 0.56$ obtained in the previous section for the smallest IR cutoff $\hat{\Lambda}_{IR} = 10^{-8}$. We find the alignment of the data points, indicating the universal nature of $\gamma_m = 0.60$. For the case of $m_p \ll \Lambda_{IR} \ll \Lambda$, we adopt the ansatz \( \text{(5)} \). The fit works well (black-solid line) and gives $\gamma_m \simeq 0.56 =: \gamma_2$, which is somewhat smaller than $\gamma_1 = 0.60$ but the strong suppression has disappeared. Thus, the right FSS formula in the right place recovers the approximate universality $\gamma_1 \simeq \gamma_2$, or equivalently, the approximate data alignments in the whole mass region including both $\Lambda_{IR} \ll m_p \ll \Lambda$ and $m_p \ll \Lambda_{IR} \ll \Lambda$. Then,
two scaling variables, $X(\gamma_1)$ and $Y(\gamma_2)$, are responsible for two slopes of the alignments. In lattice studies, indeed, the formula (4) is widely used in lattice study, but the formula (5) is rarely used [6]. However, these result are of the case without paying attention to the scope of application, and opposed to each other [2].

FIG. 2: Left: The pole mass $\hat{m}_p = m_p/\Lambda$ vs the fermion mass $\hat{m}_0 = m_0/\Lambda$ for various IR cutoffs $\hat{\Lambda}_{IR} = \Lambda_{IR}/\Lambda$ in $N_f = 12$. Middle: The mass anomalous dimension $\gamma_m$ obtained by fitting the non-shaded region data of the left panel with the conformal ansatz. Right: The pole mass $\hat{m}_p$ vs $X = \hat{m}_0^{1/(1+\gamma_m)}/\hat{\Lambda}_{IR}|_{\gamma_m=0.60}$ in the conformal phase $N_f = 12$. The data upper and lower than the shaded region are used for the fits, giving the blue- and black-solid lines.

Acknowledgments

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