STANDARD MODULE CONJECTURE

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Abstract. Let $G$ be a quasi-split $p$-adic group. Under the assumption that the local coefficients $C_\psi$ defined with respect to $\psi$-generic tempered representations of standard Levi subgroups of $G$ are regular in the negative Weyl chamber, we show that the standard module conjecture is true, which means that the Langlands quotient of a standard module is generic if and only if the standard module is irreducible.

Let $F$ be a non archimedean local field of characteristic 0. Let $G$ be the group of points of a quasi-split connected reductive $F$-group. Fix a $F$-Borel subgroup $B = TU$ of $G$ and a maximal $F$-split torus $T_0$ in $T$. If $M$ is any semi-standard $F$-Levi subgroup of $G$, a standard parabolic subgroup of $M$ will be a $F$-parabolic subgroup of $M$ which contains $B \cap M$.

Denote by $W$ the Weyl group of $G$ defined with respect to $T_0$ and by $w_0^F$ the longest element in $W$. After changing the splitting in $U$, for any generic representation $\pi$ of $G$, one can always find a non degenerate character $\psi$ of $U$, which is compatible with $w_0^F$, such that $\pi$ is $\psi$-generic [Sh2, section 3]. For any semi-standard Levi-subgroup $M$ of $G$, we will still denote by $\psi$ the restriction of $\psi$ to $M \cap U$. It is compatible with $w_0^M$. If we write in the sequel that a representation of a $F$-semi-standard Levi subgroup of $G$ is $\psi$-generic, then we always mean that $\psi$ is a non degenerate character of $U$ with the above properties.

Let $P = MU$ be a standard parabolic subgroup of $G$ and $T_M$ the maximal split torus in the center of $M$. We will write $a^*_M$ for the dual of the real Lie-algebra of $T_M$ and $a^{*_+}_M$ for the positive Weyl chamber in $a^*_M$ defined with respect to $P$. There is a canonical map $H_M : M \to a^*_M$, such that $|\chi(m)|_F = q^{-\langle \chi,H_M(m) \rangle}$ for every $F$-rational character $\chi \in a^*_M$ of $M$. (We remark that this is not the classical definition of $H_M$.) If $\pi$ is a smooth representation of $M$ and $\nu \in a^{*_+}_M$, we denote $\pi_{\nu}$ the smooth representation of $M$ defined by $\pi_{\nu}(m) = q^{-\langle \nu,H_M(m) \rangle}\pi(m)$. The symbol $i_P^G$ will denote the functor of parabolic induction normalized such that it sends unitary representations to unitary representations, $G$ acting on its space by right translations.

Let $\tau$ be a generic irreducible tempered representation of $M$ and $\nu \in a^{*_+}_M$. Then the induced representation $i_P^G \tau_{\nu}$ has a unique irreducible quotient $J(\tau, \nu)$, the so-called Langlands quotient.
The aim of our paper is to prove the standard module conjecture [CSh], which states that
\[ J(\tau, \nu) \text{ is generic, if and only if } i_G^{\tau, \nu} \text{ is irreducible.} \]

We achieve this aim under the assumption that the local coefficients \( C_{\psi} \) defined with respect to \( \psi \)-generic tempered representations of standard Levi subgroups of \( G \) are regular in the negative Weyl chamber. This property of the local coefficients \( C_{\psi} \) would be a consequence of Shahidi’s tempered \( L \)-function conjecture [Sh2, 7.1], which is now known in most cases [K]. Nevertheless, the result that we actually need may be weaker (in particular, we do not need to consider each component \( r_i \) of the adjoint representation \( r \) separately). So it may be possible to show it independently of the tempered \( L \)-function conjecture (see the remark in 1.6).

Our conditional proof of the standard module conjecture follows the method developed in [M], [M1], but using the description of the supercuspidal support of a discrete series representation of \( G \) given in [H1].

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1. Let \( P = MU \) be a standard \( F \)-parabolic subgroup of \( G \) and \((\pi, V)\) an irreducible \( \psi \)-generic admissible representation of \( M \). The parabolic subgroup of \( G \) which is opposite to \( P \) will be denoted \( \overline{P} = M\overline{U} \). The set of reduced roots of \( T_M \) in \( \text{Lie}(U) \) will be denoted \( \Sigma(P) \). We will use a superscript \( G \) to underline that the corresponding object is defined relative to \( G \).

1.1 For all \( \nu \) in an open subset of \( a_M^* \) we have an intertwining operator \( J_{\overline{P}, P}(\pi_\nu) : i_G^{\overline{P}, \nu} \pi_\nu \to i_G^{P, \nu} \pi_\nu \). For \( \nu \) in \( (a_M^*)^+ \) far away from the walls, it is defined by a convergent integral
\[
(J_{\overline{P}, P}(\pi_\nu)v)(g) = \int_{\overline{P}} v(ug) du.
\]
It is meromorphic in \( \nu \) and the map \( J_{P, \overline{P}} J_{\overline{P}, P} \) is scalar. Its inverse equals Harish-Chandra’s \( \mu \)-function up to a constant and will be denoted \( \mu(\pi, \nu) \).

1.2 Put \( \tilde{w} = u_0^G u_0^M \). Then \( \tilde{w} \overline{P} \tilde{w}^{-1} \) is a standard parabolic subgroup of \( G \). For any \( \nu \in a_M^* \) there is a Whittaker functional \( \lambda_P(\nu, \pi, \psi) \) on \( i_P^G V \). It is a linear functional on \( i_P^G V \), which is holomorphic in \( \nu \), such that for all \( v \in i_P^G V \) and all \( u \in U \) one has \( \lambda_P(\nu, \pi, \psi)((i_P^G \pi_\nu)(u)v) = \overline{\psi(u)} \lambda_P(\nu, \pi, \psi)(v) \). Remark that by Rodier’s theorem [R], \( i_P^G \pi_\nu \) has a unique \( \psi \)-generic irreducible sub-quotient.

Fix a representative \( w \) of \( \tilde{w} \) in \( K \). Let \( t(w) \) be the map \( i_P^G V \to i_P^G \pi_\nu V \), which sends \( v \) to \( v(w^{-1}) \). There is a complex number \( C_{\psi}(\nu, \pi, w) \) such that \( \lambda(\nu, \pi, \psi) = \overline{\psi(u)} \lambda_P(\nu, \pi, \psi)(v) \).
$C_\psi(\nu, \pi, w) \lambda(w\nu, w\pi, \psi) t(w) J_{|P|}(\pi_\nu)$. The function $a_\psi^* \rightarrow \mathbb{C}$, $\nu \mapsto C_\psi(\nu, \pi, w)$ is meromorphic.

The local coefficient $C_\psi$ satisfies the equality $C_\psi(\cdot, \pi, w) C_\psi(w(\cdot), w\pi, w^{-1}) = \mu(\pi, \nu)$ [Sh1].

1.3 We will use the following criterion which follows easily from the definitions and Rodier’s theorem [R]:

**Proposition:** If $(\pi, V)$ is an irreducible tempered representation of $M$ and $\nu \in a_M^+$, then the Langlands quotient of the induced representation $i_G^M \pi_\nu$ is $\psi$-generic if and only if $\pi$ is $\psi$-generic and $C_\psi(\cdot, \pi, w)$ is regular in $\nu$.

1.4 For $\alpha \in \Sigma(P)$, put $w_\alpha = w_0^M w_0^M$. With this notation, one has the following version of the multiplicative formula for the local coefficient $C_\psi(\cdot, \pi, \psi)$ [Sh1, proposition 3.2.1]:

**Proposition:** Let $Q = NV$ be a standard parabolic subgroup of $G$, $N \subseteq M$, and $\tau$ an irreducible generic representation of $N$, such that $\pi$ is a sub-representation of $i_M^Q \pi_\nu \cap M \tau$. Then one has

$$C_G^\psi(\cdot, \pi, w) = \prod_{\alpha \in \Sigma(Q) - \Sigma(Q \cap M)} C_N^\psi(\cdot, \tau, w_\alpha).$$

**Proof:** It follows from [Sh1, proposition 3.2.1] that

$$C_G^\psi(\cdot, \pi, w) = \prod_{\alpha \in \Sigma(P)} C_M^\psi(\cdot, \tau, w_\alpha).$$

Now fix $\alpha \in \Sigma(P)$. As $J_{P \cap M_n}^M(\pi_\nu)$ (resp. $\lambda^M(\nu, \pi, \psi)$) equals the restriction of $J_{Q \cap M_n}^M(\tau_\nu)$ (resp. $\lambda^M(\nu, \tau, \psi)$) to the space of $i_{P \cap M_n}^M \pi_\nu \subseteq i_{Q \cap M_n}^M \tau_\nu$, it follows that

$$C_M^\psi(\nu, \pi, w_\alpha) = C_N^\psi(\nu, \tau, w_\alpha).$$

Applying the above product formula to the expression on the right, one gets the required identity.

1.5 Recall the tempered $L$-function conjecture [Sh, 7.1]: if $\sigma$ is an irreducible tempered representation of $M$ and $\widetilde{M}$ is a maximal Levi-subgroup of $G_1$, then for every component $r_i$ of the adjoint representation $r$ the $L$-function $L(s, \sigma, r_{G_1, i})$ is holomorphic for $\Re(s) > 0$. 

**Proposition:** Let \( \pi \) be an irreducible generic tempered representation of \( M \). Assume that the tempered \( L \)-function holds for \( \pi \) relative to any \( M_\alpha, \alpha \in \Sigma(P) \).

Then \( C_\psi(w(\cdot), w\pi, w^{-1}) \) is regular in \((a_M^*)^\pm\).

**Proof:** Let \( \lambda \in (a_M^*)^\pm \). Denote by \( s_\alpha \bar{\alpha}, s_\alpha \in \mathbb{R} \), the orthogonal projection of \( \lambda \) on \( a_M^\alpha \ast \). Then \( s_\alpha > 0 \). By proposition 1.4 applied to \( \tau = \pi \),

\[
C_\psi(w(\lambda), w\pi, w^{-1}) = \prod_{\alpha \in \Sigma(P)} C_{\psi_M}^{M_\alpha}(-s_\alpha \bar{\alpha}, w\pi, w_{\alpha}^{-1}).
\]

By [Sh, 3.11, 7.8.1 and 7.3],

\[
C_{\psi_M}^{M_\alpha}(s \bar{\alpha}, w\pi, w_{\alpha}) = * \prod_i L(1 - is, w\pi, r_{M_\alpha} s, \sigma, r_{M_\alpha} s),
\]

where \(*\) denotes a monomial in \( q^{\pm s} \).

Now, by assumption, \( L(\cdot, w\pi, r_{M_\alpha} s) \) is regular in \( 1 + is_\alpha \). As \( 1/L(is_\alpha, w\pi, r_{M_\alpha} s) \) is polynomial, this proves the proposition.

**Remark:** In fact, what is really needed to prove the above proposition is a result that may be weaker than the tempered \( L \)-function conjecture: suppose for simplicity that \( \pi \) is square integrable and choose a standard parabolic subgroup \( Q = NV \) of \( G \) and a unitary supercuspidal representation \( \sigma \) of \( N \), \( N \subseteq M \), and \( \nu \in a_N^\alpha \), such that \( \pi \) is a sub-representation of \( i_{Q \cap M} M_\sigma \). Then, by 1.4,

\[
C_{\psi}^{G}(s \bar{\alpha}, \pi, w) = \prod_{\alpha \in \Sigma(Q) - \Sigma(Q \cap M)} C_{\psi}^{M_\alpha}(s \bar{\alpha} + \nu, \sigma, w_{\alpha}).
\]

This is, up to a meromorphic function on the real axes, equal to

\[
\prod_{\alpha \in \Sigma(Q) - \Sigma(Q \cap M)} \frac{L(1 - i_\alpha s, \sigma, r_{M_\alpha} s)}{L(i_\alpha s, \sigma, r_{M_\alpha} s)}
\]

where \( i_\alpha \in \{1, 2\} \) and \( i_\alpha = i_\beta \) if \( \alpha \) and \( \beta \) are conjugated.

Now let \( \Sigma_\sigma(N) \) be the set of reduced roots \( \alpha \) of \( T_N \) in \( Lie(V) \), such that Harish-Chandra’s \( \mu \)-function \( \mu_{N_\alpha} \) defined with respect to \( N_\alpha \) and \( \sigma \) has a pole. The set of these roots forms a root system [Si, 3.5]. Denote by \( \Sigma_{\sigma}(Q) \) the subset of those roots, which are positive for \( Q \) and by \( \Sigma_{\sigma}(Q \cap M) \) the one of those roots in \( \Sigma_{\sigma}(Q) \), which belong to \( M \). Then the above product equals up to a holomorphic function

\[
\prod_{\alpha \in \Sigma_{\sigma}(Q) - \Sigma_{\sigma}(Q \cap M)} \frac{1 - q^{-i_\alpha s + (\alpha, \nu)}}{1 - q^{-1 + i_\alpha s + (\alpha, \nu)}}.
\]
So, what we have to know, is that this meromorphic function is holomorphic for \( s < 0 \). The fact that \( \sigma_\nu \) lies in the supercuspidal support of a discrete series means by the main result of [H1], that \( \sigma_\nu \) is a pole of order \( rk_{ss}(M) - rk_{ss}(N) \) of Harish-Chandra’s \( \mu \)-function \( \mu^M \). This can be translated somehow to the assertion that \( \nu \) corresponds to a distinguished unipotent orbit [H2]. So, what remains, is a purely combinatorial problem in the theory of rootal hyperplane configurations, which can be stated independently of representation theory, although its validity may depend on the fact that the labels \( i_{\alpha} \) and \( \nu \) come from a “generic setting”. As, in particular, one does not need here to consider each component \( r_i \) of the adjoint representation \( r \) separately, this result may be weaker than Shahidi’s tempered \( L \)-function conjecture.

2. In this section we make the following assumption on \( G \) (see the remark in 1.5 for what we actually need):

\((TL)\) If \( M \) is a semi-standard Levi subgroup of \( G \) and if \( \pi \) is an irreducible generic tempered representation of \( M \) then \( L(s, \pi, r_i) \) is regular for \( \Re(s) > 0 \) for every \( i \).

We give a proof of the following lemma only for completeness:

2.1. Lemma: Let \( P = MU \) be a \( F \)-standard parabolic subgroup of \( G \) and \( \sigma \) an irreducible supercuspidal representation of \( M \). If the induced representation \( i_G^P \sigma \) has a sub-quotient, which lies in the discrete series of \( G \), then any tempered sub-quotient of \( i_G^P \sigma \) lies in the discrete series of \( G \).

**Proof:** If \( i_G^P \sigma \) has an irreducible sub-quotient, which is square-integrable, then by the main result of [H], \( \sigma \) is a pole of order \( rk_{ss}(G) - rk_{ss}(M) \) of \( \mu \) and \( \sigma|_{A_M} \) is unitary. It follows that the central character of \( i_G^P \sigma \) is unitary, too, which implies that the central character of any irreducible sub-quotient of \( i_G^P \sigma \) is unitary. In particular, any essentially tempered irreducible sub-quotient of \( i_G^P \sigma \) is tempered.

So, if \( \tau \) is an irreducible tempered sub-quotient of \( i_G^P \sigma \), then there is a \( F \)-parabolic subgroup \( P' = M'U' \) of \( G \) and a square-integrable representation \( \sigma' \) of \( M' \), such that \( \tau \) is a sub-representation of \( i_{G'}^{P'} \sigma' \). The supercuspidal support of \( \sigma' \) and the \( W \)-orbit of \( \sigma \) share a common element \( \sigma_0 \).

By the invariance of Harish-Chandra’s \( \mu \)-function, \( \sigma_0 \) is still a pole of \( \mu \) equal to the order given above. As this order is maximal and the central character of \( \sigma' \) must be unitary, this implies that \( M' \) must be equal to \( G \), and consequently \( \tau = \sigma' \) is square-integrable.

2.2 Theorem: Let \( G \) be a group that satisfies property (TL). Let \( P = MU \) be a
If the induced representation $i_P^G \sigma$ has a sub-quotient, which lies in the discrete series of $G$ (resp. is tempered), then any irreducible $\psi$-generic sub-quotient of $i_P^G \sigma$ lies in the discrete series of $G$ (resp. is tempered).

Proof: First assume that $i_P^G \sigma$ has a sub-quotient, which lies in the discrete series of $G$. Let $(\pi, V)$ be an irreducible, admissible $\psi$-generic representation of $G$, which is a sub-quotient of $i_P^G \sigma$. By the Langlands quotient theorem, there is a standard parabolic subgroup $P_1 = M_1 U_1$ of $G$, an irreducible tempered representation $\tau$ of $M_1$ and $\nu \in (a_M^* J)^+$, such that $\pi$ is the unique irreducible quotient of $i_P^G \nu$.

As any representation in the supercuspidal support of $\tau$, any such representation must be conjugated to $\sigma$. So, after conjugation by an element of $G$, we can assume that $M \subseteq M_1$ and that $\tau_\nu$ is a sub-representation of $i_{P_1 \cap M_1} M \sigma$.

We will actually show that $P_1 = G$, which means that $\pi$ is tempered and by 2.1 in fact square-integrable.

Following 1.3, $\tau$ must be $\psi$-generic and it is enough to show that $C_\psi(\cdot, \tau, w)$ has a pole in $\nu$, if $P_1 \neq G$.

For this we will use the assumption that $i_P^G \sigma$ has an irreducible sub-quotient which is square-integrable. By the main result of [H] this implies that $\mu$ has a pole of order equal to $rk_{ss} G - rk_{ss} M$ in $\sigma$. Remark that $\mu^{M_1}$ can have at most a pole of order $rk_{ss} M_1 - rk_{ss} M$ in $\sigma$. The order of the pole of $\mu$ in $\tau_\nu$ is equal to the one of $\mu/\mu^{M_1}$ in $\sigma$. It follows that the order of this pole must be $> 0$, if $P_1 \neq G$. As $\mu(\tau, \cdot) = C_\psi(\cdot, \tau, w) C_\psi(w(\cdot), w \tau, w^{-1})$, it follows that either $C_\psi(\cdot, \tau, w)$ or $C_\psi(w(\cdot), w \tau, w^{-1})$ must have a pole in $\nu$. As $\nu \in (a_M^* J)^+$, it follows from 1.5 that $C_\psi(w(\cdot), w \tau, w^{-1})$ cannot have a pole in $\nu$. So $C_\psi(\cdot, \tau, w)$ does. This gives us the desired contradiction.

Now assume that $i_P^G \sigma$ only has a tempered sub-quotient $\tau$. Then there is a standard parabolic subgroup $P_1 = M_1 U_1$ of $G$ and a discrete series representation $\pi_1$ of $M_1$, such that $\tau$ is a sub-representation of $i_{P_1}^G \pi_1$. As the supercuspidal support of $\pi_1$ is contained in the $G$-conjugacy class of $\sigma$, it follows that there is a standard Levi subgroup $M' \geq M_1$, such that $i_{P_1 \cap M_1} M' \sigma$ has a discrete series sub-quotient.

By what we have just shown, there exists a unique $\psi$-generic subquotient $\pi'$ of $i_{P_1 \cap M_1} M' \sigma$, which lies in the discrete series.

As $i_P^G \sigma$ and $i_P^G \pi'$ have each one a unique irreducible $\psi$-generic sub-quotient and any sub-quotient of $i_P^G \pi'$ is a sub-quotient of $i_P^G \sigma$, these irreducible $\psi$-generic sub-quotients must be equal and therefore tempered. \qed

2.3 Theorem: Let $G$ be a group that satisfies property (TL). Let $P = MU$ be a $F$-standard Levi subgroup of $G$, $\tau$ an irreducible tempered generic representation of $M$ and $\nu \in a_M^{*+}$. 

Then the Langlands quotient $J(\tau, \nu)$ is generic, if and only if $i^G_{P'} \tau_\nu$ is irreducible.

Proof: As $i^G_{P'} \tau_\nu$ always has a generic sub-quotient, one direction is trivial. So, assume $i^G_{P'} \tau_\nu$ is reducible. We will show that $\pi = J(\nu, \tau)$ is not $\psi$-generic for any $\psi$.

We can consider (and will) $\nu_\pi := \nu$ as an element of $a^*_T$. We denote by $<$ the partial order on $a^*_T$ explained in ([BW], Chapter XI, 2.1) (for our purpose it is not important to write it explicitly).

Let $\pi'$ be an irreducible sub-quotient of $i^G_{P'} \tau_\nu$, which is not isomorphic to $\pi$. Let $P' = M'U'$ be a $F$-standard parabolic subgroup, $\tau'$ an irreducible tempered representation of $M'$ and $\nu' \in a^*_M$, such that $\pi' = J(\nu', \tau')$. Let $\nu_{\pi'} := \nu'$. Then [BW, XI, Lemma 2.13]

\begin{equation}
\nu_{\pi'} < \nu_{\pi}.
\end{equation}

Choose an $F$-standard parabolic subgroup $P_1 = M_1U_1$, $M_1 \subseteq M$, with an irreducible $\psi$-generic supercuspidal representation $\sigma$ of $M_1$, such that $\tau$ is a sub-quotient of $i^M_{P_1 \cap M} \sigma$.

Then $\sigma_\nu$ lies as well in the supercuspidal support of $\pi$ as in the supercuspidal support of $\pi'$. It lies also in the $G$-conjugacy class of the supercuspidal support of $\tau_\nu'$ and $\tau_\nu$. Let $\pi_\psi$ be the unique $\psi$-generic irreducible sub-quotient of $i^G_{P_1} \sigma_\nu$. By 2.2, the unique $\psi$-generic irreducible sub-quotient $\tau''$ of $i^{M'}_{P_1 \cap M'} \sigma$ is tempered. The induced representation $i^G_{P_1} \tau''_\nu$ admits a unique $\psi$-generic irreducible sub-quotient, which is equal to the unique $\psi$-generic sub-quotient of $i^G_{P_1} \sigma_\nu$. Let $\pi'' = J(\nu', \tau'')$ be the Langlands quotient of $i^G_{P_1} \tau''_\nu$. Since (2.1) implies $\nu_{\pi''} = \nu_{\pi'} < \nu_{\pi}$, $\pi$ cannot be a sub-quotient of $i^G_{P_1} \tau''_\nu$ by [BW, XI, Lemma 2.13]. Therefore, $\pi$ is not $\psi$-generic. 

\begin{thebibliography}{99}

[BW] A. Borel and N. Wallach, Continuous Cohomology, discrete subgroups and representations of reductive groups, Princeton University Press, Princeton, 1980.

[CSh] W. Casselman and F. Shahidi, On irreducibility of standard modules for generic representations, Ann. Sci. École Norm. Sup. 31 (1998), 561–589.

[H1] V. Heiermann, Décomposition spectrale d’un groupe réductif p-adique, J. Inst. Math. Jussieu 3 (2004), 327–395.

[H2] V. Heiermann, Orbites unipotents et pôles d’ordre maximal de la fonction $\mu$ de Harish-Chandra, to appear in Canad. J. Math.,

[K] H. Kim, On Local $L$-Functions and Normalized Intertwining Operators, Canad. J. Math. 57 (2005), 535–597.
\end{thebibliography}
Some results on square integrable representations; Irreducibility of standard representations, Intern. Math. Research Notices 41 (1998), 705–726.

G. Muić, A proof of Casselman-Shahidi’s Conjecture for quasi-split classical groups, Canad. Math. Bull. 43 (2000), 90–99.

F. Rodier, Whittaker models for admissible representations, Proc. Sympos. Pure Math. AMS 26 (1973), 425–430.

F. Shahidi, On certain L-functions, Amer. J. Math. 103 (1981), 297–356.

F. Shahidi, A proof of Langlands’ conjecture on Plancherel measures; complementary series for p-adic groups, Ann. Math. 132 (1990), 273–330.

A. Silberger, Discrete Series and classification of p-adic groups I, Amer. J. Mathematics 103 (1981), 1241–1321.

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