SUB-ADDITIVE ERGODIC THEOREMS FOR COUNTABLE
AMENABLE GROUPS

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Abstract. In this paper we generalize Kingman’s sub-additive ergodic theorem to a large class of infinite countable discrete amenable group actions.

1. Introduction

The study of ergodic theorems was started in 1931 by von Neumann and Birkhoff, growing from problems in statistical mechanics. Ergodic theory soon earned its own place as an important part of functional analysis and probability, and grew into the study of measure-preserving transformations of a measure space. In 1968 an important new impetus to this area was received from Kingman’s proof of the sub-additive ergodic theorem. This theorem opened up an impressive array of new applications [12, 13, 14]. Krengel [15] showed that Kingman’s theorem can be used to derive the multiplicative ergodic theorem of Oseledec [24], which is of considerable current interest in the study of differentiable dynamical systems. Today, there are many elegant proofs of the theorem [5, 11, 13, 20, 26, 27]. Among these, perhaps the shortest proof is that of Steele [27]: this relies neither on a maximal inequality nor on a combinatorial Riesz lemma. A lovely exposition of the whole theory is given in Krengel’s book [15].

A statement of Kingman’s sub-additive ergodic theorem is as follows:

Theorem 1.1. Let \( \vartheta \) be a measure preserving transformation over the Lebesgue space \( (Y, \mathcal{D}, \nu) \) and \( \{f_n : n \in \mathbb{N}\} \subseteq L^1(Y, \mathcal{D}, \nu) \) satisfy \( f_{n+m}(y) \leq f_n(y) + f_m(\vartheta^n y) \) for \( \nu \)-a.e. \( y \in Y \) and all \( n, m \in \mathbb{N} \). Then

\[
\lim_{n \to \infty} \frac{1}{n} f_n(y) = f(y) \geq -\infty
\]

for \( \nu \)-a.e. \( y \in Y \), where \( f \) is an invariant measurable function over \( (Y, \mathcal{D}, \nu) \).

If all the \( f_n \) are constant functions, equal to \( a_n \) (say), the theorem reduces to a well-known basic fact in analysis: if the sequence \( \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R} \) satisfies \( a_{n+m} \leq a_n + a_m \) for all \( n, m \in \mathbb{N} \), then

\[
\lim_{n \to \infty} \frac{a_n}{n} = \inf_{n \in \mathbb{N}} \frac{a_n}{n} \geq -\infty.
\]

It is an easy extension of Kingman’s theorem that if \( \inf_n \frac{\int f_n d\nu}{n} > -\infty \), then the convergence also holds in \( L^1 \).

In this paper, we shall discuss extensions of Kingman’s theorem to the class of countable discrete amenable groups.

The class of amenable groups includes all finite groups, solvable groups and compact groups, and actions of these groups on Lebesgue space are a natural extension of the \( \mathbb{Z} \)-actions considered in Kingman’s theorem: the foundations of the theory of
Amenable group actions were laid by Ornstein and Weiss in their pioneering paper [22].

Lindenstrauss [17] established the pointwise ergodic theorem for general locally compact amenable group actions along Følner sequences (with some natural conditions), which generalizes the of Birkhoff theorem from $\mathbb{Z}$-actions to general amenable group actions, see also Benji Weiss’ lovely survey article [32]. For other related work, see [1, 2, 3, 7, 8, 10, 21, 23, 25, 28, 29].

In contrast to the amenable group $\mathbb{Z}$, a general infinite countable discrete amenable group may have a complicated combinatorial structure, and our challenge is to consider the limiting behaviour of the Kingman type in this context.

In general, we define a subset $D = \{d_F : F \in \mathcal{F}\}$ of functions in $L^1(Y, \mathcal{D}, \nu)$, indexed by the family $\mathcal{F}$ of all non-empty finite subsets of $G$, to be $G$-invariant and sub-additive if $d_{Eg}(y) = d_E(gy)$ and $d_{E \cup F}(y) \leq d_E(y) + d_F(y)$ for $\nu$-a.e. $y \in Y$, any $g \in G$ and all disjoint $E, F \in \mathcal{F}$. Then a natural generalization of Kingman’s theorem is to ask whether, for an invariant sub-additive family, the limit

$$
\lim_{n \to \infty} \frac{1}{|F_n|} \int d_{F_n}(y) dy,
$$

exists for a Følner sequence $\{F_n : n \in \mathbb{N}\}$ of $G$, either pointwise almost everywhere, or, if $\liminf_{n \to \infty} \frac{1}{|F_n|} \int d_{F_n}(y) d\nu(y) > -\infty$, in $L^1$.

This theorem reduces to Kingman’s theorem for the Følner sequence $F_n = \{0, 1, \cdots, n-1\}$, although it is not a priori clear whether it holds for arbitrary good Følner sequences even in the integers. Our overall aim, for amenable groups, is to find conditions on the Følner sequence, and the group, under which this theorem holds.

The first step, motivated by [6] (in particular [6, Proposition 9.2 and Proposition 10.4] and their proof), is to replace the limit by the limit superior. Under some natural assumptions, we can prove versions of the Kingman theorem: precisely, if either the group is abelian or if the family is strongly sub-additive, we show the existence of the lim sup. Moreover, if the group has the property of self-similarity (see Section 5), then Kingman’s theorem can be generalized completely to an action over a Lebesgue space. We shall see in the last section of the paper that this class of infinite countable discrete amenable groups includes many interesting groups.

The paper is organized as follows. Sections 2 and 3 contain some preliminary remarks on infinite countable discrete amenable groups and pointwise ergodic theorems for their actions on Lebesgue space. Motivated by the results of [6], in Section 4 we analyze the limit superior behavior of a sub-additive family of integrable functions on an infinite countable discrete amenable group action, under some natural assumptions. In Section 5 we present our main results for some special infinite countable discrete amenable groups. Finally in Section 6 we give some direct applications of the results obtained in previous sections.

We believe that the theorems we establish will lead to further developments along the lines of Oseledec’ theorem and other applications.

2. Preliminaries

Throughout the current paper, we will assume that $G$ is an infinite countable discrete amenable group. We begin by recalling some basic properties of $G$. These and many further details can be found in [22].
Denote by $\mathcal{F}_G$ the set of all non-empty finite subsets of $G$. $G$ is called amenable, if for each $K \in \mathcal{F}_G$ and any $\delta > 0$ there exists $F \in \mathcal{F}_G$ such that

$$|F \Delta K F| < \delta |F|,$$

where $| \bullet |$ is counting measure on the set $\bullet$, $KF = \{kf : k \in K, f \in F\}$ and $F \Delta K F = (F \setminus KF) \cup (KF \setminus F)$. Let $K \in \mathcal{F}_G$ and $\delta > 0$. Set $K^{-1} = \{k^{-1} : k \in K\}$. $A \in \mathcal{F}_G$ is called $(K, \delta)$-invariant, if

$$|K^{-1}A \cap K^{-1}(G \setminus A)| < \delta |A|.$$ 

A sequence $\{F_n : n \in \mathbb{N}\}$ in $\mathcal{F}_G$ is called a Følner sequence, if for any $K \in \mathcal{F}_G$ and for any $\delta > 0$, $F_n$ is $(K, \delta)$-invariant whenever $n \in \mathbb{N}$ is sufficiently large, i.e.

$$(2.1) \quad \lim_{n \to \infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0$$

for each $g \in G$. It is not hard to see from this the usual asymptotic invariance property: $G$ is amenable if and only if $G$ has a Følner sequence $\{F_n\}_{n \in \mathbb{N}}$.

For example, for $G = \mathbb{Z}$, a Følner sequence is defined by $F_n = \{0, 1, \ldots, n-1\}$, or, indeed, $\{a_n, a_n + 1, \ldots, a_n + n - 1\}$ for any sequence $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{Z}$.

A measurable dynamical $G$-system (MDS) $(Y, D, \nu, G)$ is a Lebesgue space $(Y, D, \nu)$ and a group $G$ of invertible measure preserving transformations of $(Y, D, \nu)$ with $e_G$ acting as the identity transformation.

From now on $(Y, D, \nu, G)$ will denote an MDS.

Let $I$ be the sub-$\sigma$-algebra $\{D \in D : \nu(gD \Delta D) = 0 \text{ for each } g \in G\}$, and for each $f \in L^1(Y, D, \nu)$ denote by $\mathbb{E}(f|I)$ the conditional expectation of $f$ over $I$ with respect to $\nu$. If in addition, the MDS $(Y, D, \nu, G)$ is ergodic, (i.e. $\nu(D)$ is equal to either zero or one for all $D \in I$), then

$$\mathbb{E}(f|I)(y) = \int_Y f(y') d\nu(y')$$

for $\nu$-a.e. $y \in Y$. The measurable function $\mathbb{E}(f|I)$ is $G$-invariant and $\mathbb{E}(f|I) \in L^p(Y, D, \nu)$ if $f \in L^p(Y, D, \nu)$ for each $1 \leq p \leq \infty$.

A sequence $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{F}_G$ is said to be tempered if there exists $M > 0$ such that $| \bigcup_{k=1}^{n} E_{k+1}^- E_{n+1}^- | \leq M |E_{n+1}|$ for each $n \in \mathbb{N}$. It is easy to obtain a tempered sub-sequence from any given Følner sequence of the group $G$ (cf. \cite{17}).

Lindenstrauss (\cite{17} Theorem 1.2) and \cite{16} showed how to generalize the Birkhoff pointwise convergence theorem to an amenable group action as follows:

**Theorem 2.1.** Let $f \in L^p(Y, D, \nu)$ and $\{F_n : n \in \mathbb{N}\}$ be a tempered Følner sequence of $G$, where $1 \leq p < \infty$. Then

$$\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{F_n} f(gy) = \mathbb{E}(f|I)(y)$$

for $\nu$-a.e. $y \in Y$ and in the sense of $L^p$.

**Remark 2.2.** Lindenstrauss actually states the conclusion for pointwise convergence in the case of $p = 1$. It was remarked in \cite{10} that mean convergence can be easily proved by the methods for $\mathbb{Z}$-actions. Moreover, as shown by \cite{3}, the restriction on the Følner sequence is essential even for the special case of $G = \mathbb{Z}$.

Note further that \cite{32} Theorem 2.1 discusses mean convergence in the case of $p = 2$, and shows that this holds for any Følner sequence.
A function $f \in L^1(Y, \mathcal{D}, \nu)$ is non-negative if $f(y) \geq 0$ for $\nu$-a.e. $y \in Y$; and $G$-invariant if $f(gy) = f(y)$ for $\nu$-a.e. $y \in Y$ and all $g \in G$.

Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\}$ be a family of functions in $L^1(Y, \mathcal{D}, \nu)$. We make some natural assumptions on the family $\mathbf{D}$ which will enable us to state analogues of theorem 1.1.

We say that $\mathbf{D}$ is:

1. non-negative if: each element from $\mathbf{D}$ is non-negative;
2. $G$-invariant if: $d_{F_1}(y) = d_F(gy)$ for $\nu$-a.e. $y \in Y$ and all $F \in \mathcal{F}_G, g \in G$;
3. monotone if: $d_E(y) \leq d_F(y)$ for $\nu$-a.e. $y \in Y$ and all $\emptyset \neq E \subseteq F \in \mathcal{F}_G$;
4. sub-additive (sup-additive, respectively) if: $d_{E \cup F}(y) \leq d_{E}(y) + d_F(y)$ for $\nu$-a.e. $y \in Y$ and all disjoint $E, F \in \mathcal{F}_G$ ($-\mathbf{D}$ is sub-additive, respectively);
5. strongly sub-additive (strongly sup-additive, respectively) if $d_{E \cup F}(y) + d_{E \cup F}(y) \leq d_{E}(y) + d_F(y)$ for $\nu$-a.e. $y \in Y$ and all $E, F \in \mathcal{F}_G$; here by convention we set $d_{\emptyset}(y) = 0$ for each $y \in Y$ ($-\mathbf{D}$ is strongly sub-additive, respectively).

For example, for each $f \in L^1(Y, \mathcal{D}, \nu)$, it is easy to check that

$$\mathbf{D}^f = \{d_f^F(y) = \sum_{g \in F} f(gy) : F \in \mathcal{F}_G\}$$

is a strongly sub-additive $G$-invariant family in $L^1(Y, \mathcal{D}, \nu)$. Observe that in $L^1(Y, \mathcal{D}, \nu)$ not every strongly sub-additive $G$-invariant family is in this form, in fact, if $f \in L^1(Y, \mathcal{D}, \nu)$ then the following family is also strongly sub-additive and $G$-invariant:

$$\{d_F(y) = \sum_{g \in F} f(gy) - |F|^2 : F \in \mathcal{F}_G\} \subseteq L^1(Y, \mathcal{D}, \nu).$$

Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq L^1(Y, \mathcal{D}, \nu)$ be a sub-additive $G$-invariant family. We are interested in the convergence of

$$\lim_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y),$$

where $\{F_n : n \in \mathbb{N}\}$ is a Følner sequence of $G$.

These are the basic ingredients for our counterpart of Kingman’s sub-additive ergodic theorem for infinite countable discrete amenable group actions.

3. Preparations

In this section, we consider the asymptotic limiting behavior of (2.2) in the simplest case, when all functions $f_n$ are constant functions. We also present a version of the maximal inequality which will be used in later sections following the ideas of [15].

In contrast to a $\mathbb{Z}$-action, for a general infinite countable discrete amenable group action it is not clear whether the limit (2.2) exists, even if all the functions in $\mathbf{D}$ are constant functions. Thus, we will add some natural conditions.

Let $\emptyset \neq T \subseteq G$. We say that $T$ tiles $G$ if there exists $\emptyset \neq G_T \subseteq G$ such that \{\text{range} \in G \mid \text{range} \in G_T\} forms a partition of $G$, that is, $T_{c_1} \cap T_{c_2} = \emptyset$ if $c_1$ and $c_2$ are different elements from $G_T$ and $\bigcup_{c \in G_T} T_{c} = G$.

Denote by $\mathcal{T}_G$ the set of all non-empty finite subsets of $G$ which tile $G$. Observe that $\mathcal{T}_G \neq \emptyset$, as $\mathcal{T}_G \supseteq \{\{g\} : g \in G\}$.

Let $(Y, \mathcal{D}, \nu, G)$ be an MDS. We say that $G$ acts freely on $(Y, \mathcal{D}, \nu)$ if $\{y \in Y : gy = y\}$ has zero $\nu$-measure for any $g \in G \setminus \{e_G\}$. 
Tiling sets play a key role in establishing a version of Rokhlin's Lemma for infinite countable discrete amenable group actions (see 32 Theorem 3.3 and Proposition 3.6]). In particular, we have:

**Proposition 3.1.** Let $T \in \mathcal{F}_G$. Then $T \in \mathcal{T}_G$ if and only if, for every MDS $(Y, \mathcal{D}, \nu, G)$, where $G$ acts freely on $(Y, \mathcal{D}, \nu)$, for each $\epsilon > 0$ there exists $B \in \mathcal{D}$ such that the family \( \{tB : t \in T\} \) is disjoint and $\nu(\bigcup_{t \in T} tB) \geq 1 - \epsilon$.

The class of countable amenable groups admitting a tiling Følner sequence (i.e. a Følner sequence consisting of tiling subsets of the group) is large, and includes all countable amenable linear groups and all countable residually finite amenable groups [31]. Recall that a linear group is an abstract group which is isomorphic to a matrix group over a field $K$ (i.e. a group consisting of invertible matrices over some field $K$); a group is residually finite if the intersection of all its normal subgroups of finite index is trivial. Note that any finitely generated nilpotent group is residually finite. The question of whether every countable discrete amenable group admits a tiling Følner sequence remains open [22].

The following results are [6] Proposition 2.3 and Proposition 2.7. See also [18, Theorem 6.1], [32, Theorem 5.9] and [19, 30].

**Proposition 3.2.** Let $f : \mathcal{F}_G \to \mathbb{R}$ be a function. Assume that $f(Eg) = f(E)$ and $f(E \cup F) \leq f(E) + f(F)$ whenever $g \in G$ and $E, F \in \mathcal{F}_G$ satisfy $E \cap F = \emptyset$. Then for any tiling Følner sequence $\{F_n : n \in \mathbb{N}\}$ of $G$, the sequence $\{\frac{f(F_n)}{|F_n|} : n \in \mathbb{N}\}$ converges and the value of the limit is independent of the selection of the tiling Følner sequence $\{F_n : n \in \mathbb{N}\}$; in fact:

$$
\lim_{n \to \infty} \frac{f(F_n)}{|F_n|} = \inf_{F \in \mathcal{T}_G} \frac{f(F)}{|F|} \quad \text{(and so } \inf_{n \in \mathbb{N}} \frac{f(F_n)}{|F_n|}.)
$$

**Proposition 3.3.** Let $f : \mathcal{F}_G \to \mathbb{R}$ be a function. Assume that $f(Eg) = f(E)$ and $f(E \cap F) + f(E \cup F) \leq f(E) + f(F)$ whenever $g \in G$ and $E, F \in \mathcal{F}_G$ (here, we set $f(\emptyset) = 0$ by convention). Then for any Følner sequence $\{F_n : n \in \mathbb{N}\}$ of $G$, we have that the sequence $\{\frac{f(F_n)}{|F_n|} : n \in \mathbb{N}\}$ converges and the value of the limit is independent of the selection of the Følner sequence $\{F_n : n \in \mathbb{N}\}$; in fact:

$$
\lim_{n \to \infty} \frac{f(F_n)}{|F_n|} = \inf_{F \in \mathcal{T}_G} \frac{f(F)}{|F|} \quad \text{(and so } \inf_{n \in \mathbb{N}} \frac{f(F_n)}{|F_n|}.)
$$

The difference between Proposition 3.2 and Proposition 3.3 was shown in [6] Example 2.8 in the special case of $G = \mathbb{Z}$.

Let $D = \{d_F : F \in \mathcal{F}_G\} \subseteq L^1(Y, \mathcal{D}, \nu)$ be a sub-additive $G$-invariant family. By Proposition 3.2 the limit

$$
\lim_{n \to \infty} \frac{1}{|F_n|} \int_Y d_{F_n}(y) d\nu(y)
$$

exists for (and independent of) any tiling Følner sequence $\{F_n : n \in \mathbb{N}\}$ of $G$, and is equal to

$$
\inf_{n \in \mathbb{N}} \frac{1}{|F_n|} \int_Y d_{F_n}(y) d\nu(y) = \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \int_Y d_T(y) d\nu(y) < \infty.
$$

Now if $D$ is a strongly sub-additive $G$-invariant family, alternatively using Proposition 3.3 the limit (3.1) exists for (and independent of) any Følner sequence.
\( \{ F_n : n \in \mathbb{N} \} \) of \( G \), and is equal to
\[
\inf_{n \in \mathbb{N}} \frac{1}{|F_n|} \int_Y d_{F_n}(y) d\nu(y) = \inf_{F \in \mathcal{E}} \frac{1}{|F|} \int_Y d_F(y) d\nu(y) < \infty.
\]
Dually, if \( D \) is a sup-additive or strongly sup-additive \( G \)-invariant family, we can talk about the limit similarly. We will denote by \( \nu(D) \) these limits in the sequel. Remark that they need not to be a finite constant.

Recall that the sub-\( \sigma \)-algebra \( \mathcal{I} \) is introduced in previous section as \( \{ D \in \mathcal{D} : \nu(gD\Delta D) = 0 \text{ for each } g \in G \} \). Let \( \nu = \int_Y \nu_y d\nu(y) \) be the disintegration of \( \nu \) over \( \mathcal{I} \). Then \( (Y, \mathcal{D}, \nu_y, G) \) will be an ergodic MDS for \( \nu \)-a.e. \( y \in Y \). The disintegration is known as the ergodic decomposition of \( \nu \) (cf [9, Theorem 3.22]), which can be characterized as follows: for each \( f \in L^1(Y, \mathcal{D}, \nu) \), \( f \in L^1(Y, \mathcal{D}, \nu_y) \) for \( \nu \)-a.e. \( y \in Y \) and the function \( y \mapsto \int_Y f d\nu_y \) is in \( L^1(Y, \mathcal{I}, \nu) \). In fact, \( \int_Y f d\nu_y = \mathbb{E}(f|\mathcal{I})(y) \) for \( \nu \)-a.e. \( y \in Y \) and hence \( \int_Y (\int_Y f d\nu_y) d\nu(y) = \int_Y f d\nu \).

Thus, we have:

**Proposition 3.4.** Let \( D = \{ d_F : F \in \mathcal{F}_G \} \subseteq L^1(Y, \mathcal{D}, \nu) \) be a sub-additive (or strongly sub-additive, strongly sup-additive, and so on) \( G \)-invariant family. Assume that \( \nu = \int_Y \nu_y d\nu(y) \) is the ergodic decomposition of \( \nu \). Then

\[
(3.2) \quad \nu(D) = \int_Y \nu_y(D) d\nu(y).
\]

**Proof.** Dually, we only consider the case of \( D \) being sup-additive and \( \{ F_n : n \in \mathbb{N} \} \) a tiling Følner sequence. Let \( \nu = \int_Y \nu_y d\nu(y) \) be the disintegration of \( \nu \) over \( \mathcal{I} \). Then \( (Y, \mathcal{D}, \nu_y, G) \) will be an ergodic MDS for \( \nu \)-a.e. \( y \in Y \). The disintegration is known as the ergodic decomposition of \( \nu \) (cf [9, Theorem 3.22]), which can be characterized as follows: for each \( f \in L^1(Y, \mathcal{D}, \nu) \), \( f \in L^1(Y, \mathcal{D}, \nu_y) \) for \( \nu \)-a.e. \( y \in Y \) and the function \( y \mapsto \int_Y f d\nu_y \) is in \( L^1(Y, \mathcal{I}, \nu) \). In fact, \( \int_Y f d\nu_y = \mathbb{E}(f|\mathcal{I})(y) \) for \( \nu \)-a.e. \( y \in Y \) and hence \( \int_Y (\int_Y f d\nu_y) d\nu(y) = \int_Y f d\nu \).

Thus, we have:

**Proposition 3.4.** Let \( D = \{ d_F : F \in \mathcal{F}_G \} \subseteq L^1(Y, \mathcal{D}, \nu) \) be a sub-additive (or strongly sub-additive, strongly sup-additive, and so on) \( G \)-invariant family. Assume that \( \nu = \int_Y \nu_y d\nu(y) \) is the ergodic decomposition of \( \nu \). Then

\[
(3.2) \quad \nu(D) = \int_Y \nu_y(D) d\nu(y).
\]

**Proof.** Dually, we only consider the case of \( D \) being sup-additive and \( \{ F_n : n \in \mathbb{N} \} \) a tiling Følner sequence. And so \( \nu(D) > -\infty \). Observe that \( \int_Y d_F d\nu = \int_Y (\int_Y d_{F_n} d\nu_n) d\nu(y) \) for each \( F \in \mathcal{F}_G \), then: on one hand,

\[
\nu(D) = \sup_{T \in \mathcal{T}_G} \frac{1}{|T|} \int_Y d_T d\nu \leq \int_Y (\sup_{T \in \mathcal{T}_G} \frac{1}{|T|} \int_Y d_T d\nu_n) d\nu(y) = \int_Y \nu_y(D) d\nu(y);
\]
on the other hand, for

\[
d_{F_n}(y) = d_{F_n}(y) - \sum_{y \in F_n} d_{\{e_G\}}(gy) \geq 0,
\]

using Fatou’s Lemma one has
\[
\int_Y \nu_y(D) d\nu(y) - \int_Y (\int_Y d_{\{e_G\}} d\nu_y) d\nu(y)
\]
\[
= \int_Y (\lim_{n \to \infty} \frac{1}{|F_n|} \int_Y d_{F_n} d\nu_y) d\nu(y)
\]
\[
\leq \lim_{n \to \infty} \frac{1}{|F_n|} \int_Y (\int_Y d_{F_n} d\nu_y) d\nu(y)
\]
equivalently, \( \int_Y \nu_y(D) d\nu(y) \leq \nu(D) \). Summing up, we obtain \( \nu(D) \). □

In the remainder of the section, we present a version of the maximal inequality following the ideas of [15], which will be used in later sections.

A sequence \( \{ E_n : n \in \mathbb{N} \} \subseteq \mathcal{F}_G \) is called to satisfy Tempelman condition if there exists \( M > 0 \) such that \( \bigcup_{k=1}^n E_k^{-1} E_n \leq M |E_n| \) for each \( n \in \mathbb{N} \), which is a stronger property than a tempered sequence.

Note that not all infinite countable discrete amenable groups contain a Følner sequence satisfying Tempelman condition (e.g. the well-known “lamplighter group”), for details see [17].
The following result is a version of [15, Chapter 6, Theorem 4.2]: in fact, Krengel states it a version compatible with (3.5), while we will need a version in the style of equation (3.4). We present a proof of it here for completeness: the ideas are taken from [15].

Lemma 3.5. Assume that \( D = \{ d_F : F \in \mathcal{F}_G \} \subseteq L^1(Y, D, \nu) \) is a non-negative sup-additive \( G \)-invariant family and \( \{ F_n : n \in \mathbb{N} \} \) is a Følner sequence of \( G \) satisfying Tempelman condition with constant \( M > 0 \). Then

\[
\nu(Y_{\alpha,N}) \leq \frac{M}{\alpha} \cdot \inf_{n \in \mathbb{N}} \frac{1}{|F_n \cap \bigcap_{i=1}^{N} g^{-1} F_n|} \int_Y dF_n(y) d\nu(y) \tag{3.3}
\]

\[
\leq \frac{M}{\alpha} \cdot \liminf_{n \to \infty} \frac{1}{|F_n|} \int_Y dF_n(y) d\nu(y) \tag{3.4}
\]

\[
\leq \frac{M}{\alpha} \cdot \sup_{F \in \mathcal{F}_G} \frac{1}{|F|} \int_Y dF(y) d\nu(y) \tag{3.5}
\]

for each \( \alpha > 0 \) and any \( N \in \mathbb{N} \), where

\[
Y_{\alpha,N} = \{ y \in Y : \max_{k=1,\ldots,N} \frac{dF_k(y)}{|F_k|} > \alpha \}.
\]

Proof. In fact, the proof will be finished once we prove (3.3).

By convention we set \( d_0(y) = 0 \) for each \( y \in Y \) without any loss of generality.

Let \( y \in Y \) and \( n \in \mathbb{N} \) such that \( F_n^* = F_n \cap \bigcap_{i=1}^{N} g^{-1} F_n \neq \emptyset \). As \( \{ F_m : m \in \mathbb{N} \} \) is a Følner sequence of \( G \), \( F_m^* \neq \emptyset \) once \( m \in \mathbb{N} \) is large enough. Set

\[
C_1 = \{ g \in F_n^* : \frac{dF_1(gy)}{|F_1|} > \alpha \},
\]

and for \( j = 2, \ldots, N \), set

\[
C_j = \{ g \in F_n^* : \frac{dF_j(gy)}{|F_j|} > \alpha \} \setminus \bigcup_{i=1}^{j-1} C_i.
\]

Now let \( C_N \subseteq C_N \) be a maximal disjoint family in \( \{ F_{NC} : c \in C_N \} \). In particular,

\[
C_N \subseteq F_N^{-1} F_N C_N^\prime.
\]

Once the \( C_N' \subseteq C_N, \ldots, C_i' \subseteq C_i \) have been constructed for some \( i > 1 \), then let \( C_{i-1}' \subseteq C_{i-1} \) be a maximal disjoint family in \( \{ F_{i-1}c : c \in C_{i-1} \} \) which is also disjoint from the elements in \( \{ F_{ij}c_j : c_j \in C_{j}', j = i, \ldots, N \} \). In particular,

\[
C_{i-1} \subseteq F_{i-1}^{-1} \bigcup_{j=i}^{N} F_j C_{j}'.
\]

From the construction we have

\[
\{ g \in F_n^* : gy \in Y_{\alpha,N} \} = \bigcup_{i=1}^{N} C_i \subseteq \bigcup_{i=1}^{N} \bigcup_{j=1}^{i} F_{ij} C_{i}'.
\]
Moreover, as the sequence \( \{ F_n : n \in \mathbb{N} \} \) satisfies the Tempelman condition, one has

\[
\text{(3.6)} \quad |\{ g \in F_n^* : g y \in Y_{\alpha,N} \}| \leq \sum_{i=1}^{N} | \bigcup_{j=1}^{i} F_j^{-1} F_i | \cdot |C_i'| \leq M \sum_{i=1}^{N} |F_i| \cdot |C_i'|. 
\]

As \( D = \{ d_F : F \in \mathcal{F}_G \} \subseteq L^1(Y, \mathcal{D}, \nu) \) is a non-negative sup-additive (and so monotone) \( G \)-invariant family, by the construction of \( C_i, C_i', i = 1, \ldots, N \), one has

\[
d_{F_n}(y) \geq d \left( \bigcup_{i=1}^{N} F_i C_i'(y) \right) \geq \sum_{i=1}^{N} \sum_{g_i \in C_i'} d_{F_i g_i}(y) \geq \alpha \sum_{i=1}^{N} |F_i| \cdot |C_i'| 
\]

\[
\text{(3.7)} \quad \geq \frac{\alpha}{M} |\{ g \in F_n^* : g y \in Y_{\alpha,N} \}| \quad \text{(using (3.6))}
\]

for \( \nu \)-a.e. \( y \in Y \). Thus applying Fubini’s Theorem we obtain

\[
\frac{1}{|F_n^*|} \int_Y d_{F_n}(y) d\nu(y) \geq \frac{1}{|F_n^*|} \cdot \frac{\alpha}{M} \int_Y |\{ g \in F_n^* : g y \in Y_{\alpha,N} \}| d\nu(y) \quad \text{(using (3.7))}
\]

\[
= \frac{1}{|F_n^*|} \cdot \frac{\alpha}{M} \sum_{g \in F_n^*} \int_Y 1_{Y_{\alpha,N}}(gy) d\nu(y) = \frac{\alpha}{M} \cdot \nu(Y_{\alpha,N}),
\]

which implies (3.6). This finishes our proof. \( \square \)

**Remark 3.6.** Inspired by [17, Theorem 3.2] or [32, Section 7], it seems that it may be possible to strengthen Lemma 3.3 by replacing the assumption that the Følner sequence satisfies the Tempelman condition by the assumption that it is a tempered Følner sequence. We have not so far been able to prove this, however.

As a direct corollary of (3.4) and (3.5), we have:

**Corollary 3.7.** Assume that \( D = \{ d_F : F \in \mathcal{F}_G \} \subseteq L^1(Y, \mathcal{D}, \nu) \) is a non-negative sup-additive \( G \)-invariant family and \( \{ F_n : n \in \mathbb{N} \} \) is a Følner sequence of \( G \) satisfying Tempelman condition with constant \( M > 0 \). If either each \( F_n, n \in \mathbb{N} \) tiles \( G \), or the family \( D \) is strongly sup-additive, then, for any \( \alpha > 0 \),

\[
\nu(\{ y \in Y : \sup_{k \in \mathbb{N}} \frac{d_{F_k}(y)}{|F_k|} > \alpha \}) \leq \frac{M}{\alpha} \nu(D).
\]

4. **The superior limit behavior of the family**

Motivated by [9, Proposition 9.2 and Proposition 10.4] (and their proofs), in this section, we aim to study the superior limit behavior of (2.2).

The main results of this section are stated as follows.

**Theorem 4.1.** Let \( D = \{ d_F : F \in \mathcal{F}_G \} \subseteq L^1(Y, \mathcal{D}, \nu) \) be a sub-additive \( G \)-invariant family and \( \{ F_n : n \in \mathbb{N} \} \) a tempered tiling Følner sequence of \( G \). Assume that the group \( G \) is abelian. Then

\[
\text{(4.1)} \quad \limsup_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) = \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y)
\]

for \( \nu \)-a.e. \( y \in Y \), which is an invariant measurable function over \( (Y, \mathcal{D}, \nu) \), and

\[
\text{(4.2)} \quad \int_Y \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y) d\nu(y) = \nu(D).
\]
Theorem 4.2. Let $D = \{ d_F : F \in \mathcal{F}_G \} \subseteq L^1(Y, \mathcal{D}, \nu)$ be a strongly sub-additive $G$-invariant family and $\{ F_n : n \in \mathbb{N} \}$ a tempered Følner sequence of $G$. Then

$$\limsup_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) d\nu(y) = \inf_{F \in \mathcal{F}_G} \frac{1}{|F|} \mathbb{E}(d_F|\mathcal{I})(y) d\nu(y)$$

for $\nu$-a.e. $y \in Y$, which is an invariant measurable function over $(Y, \mathcal{D}, \nu)$, and

$$\int_Y \inf_{F \in \mathcal{F}_G} \frac{1}{|F|} \mathbb{E}(d_F|\mathcal{I})(y) d\nu(y) = \nu(D).$$

Before proceeding, we need the following easy observation.

Lemma 4.3. Let $\{ F_n : n \in \mathbb{N} \}$ be a tempered Følner sequence of $G$. Assume that $\emptyset \neq E_n \subseteq F_n$ for each $n \in \mathbb{N}$ satisfying $\lim_{n \to \infty} \frac{|E_n|}{|F_n|} = 1$. Then $\{ E_n : n \in \mathbb{N} \}$ is also a tempered Følner sequence of $G$.

Proof. Combined with the assumptions, the conclusion follows from the facts that $E_n \Delta gE_n \subseteq F_n \Delta gF_n \cup \{ g, e_G \} (F_n \setminus E_n)$ and

$$\bigcup_{i=1}^n E_i E_{i+1} \subseteq \bigcup_{i=1}^n F_i F_{i+1}$$

for each $n \in \mathbb{N}$ and $g \in G$, which are easy to check. \qed

The following result from [6, Lemma 10.6] is used in the proof of Theorem 4.1. Observe that whilst the whole discussion of [6, §10] is in the setting where $\{ F_n : n \in \mathbb{N} \}$ is an increasing tiling Følner sequence of $G$, the proof of [6, Lemma 10.6] uses only the fact that $\{ F_n : n \in \mathbb{N} \}$ is a Følner sequence of $G$.

Lemma 4.4. Let $D = \{ d_F : F \in \mathcal{F}_G \} \subseteq L^1(Y, \mathcal{D}, \nu)$ be a sub-additive $G$-invariant family, $\{ F_n : n \in \mathbb{N} \}$ a Følner sequence of $G$ and $T \in \mathcal{F}_G, \epsilon > 0$. Assume that $G$ is abelian and the family $-D$ is non-negative. Then, whenever $n \in \mathbb{N}$ is large enough, there exists $H_n \subseteq F_n$ such that $|F_n \setminus H_n| < 2\epsilon |F_n|$ and, for $\nu$-a.e. $y \in Y$,

$$d_{F_n}(y) \leq \frac{1}{|T|} \sum_{g \in H_n} d_T(gy).$$

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. As $D$ is a sub-additive $G$-invariant family, the function

$$\sum_{g \in F_n} d_{\{e_G\}}(gy) - d_{F_n}(y)$$

is non-negative for each $n \in \mathbb{N}$, and observe that the sequence $\{ F_n : n \in \mathbb{N} \}$ is tempered, by the Fatou Lemma, one has

$$\int_Y d_{\{e_G\}}(y) d\nu(y) - \nu(D) = \liminf_{n \to \infty} \int_Y \frac{1}{|F_n|} \left( \sum_{g \in F_n} d_{\{e_G\}}(gy) - d_{F_n}(y) \right) d\nu(y)$$

$$\geq \int_Y \liminf_{n \to \infty} \frac{1}{|F_n|} \left( \sum_{g \in F_n} d_{\{e_G\}}(gy) - d_{F_n}(y) \right) d\nu(y)$$

$$= \int_Y \mathbb{E}(d_{\{e_G\}}|\mathcal{I})(y) - \limsup_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) d\nu(y)$$

(using Theorem 2.3.1)

$$= \int_Y d_{\{e_G\}}(y) d\nu(y) - \int_Y \limsup_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) d\nu(y),$$
which implies
\begin{equation}
\int_Y \limsup_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) d\nu(y) \geq \nu(D).
\end{equation}

Similarly to the above, we may assume that the family \(-D\) is non-negative. Now applying Lemma 4.3 to \(D\) we obtain that, once \(T \in \mathcal{T}_G\) (fixed) and \(\epsilon > 0\), if \(n \in \mathbb{N}\) is large enough then there exists \(T_n \subseteq F_n\) such that \(|F_n \setminus T_n| \leq 2\epsilon|F_n|\) and, for \(\nu\)-a.e. \(y \in Y\),
\begin{equation}
d_{F_n}(y) \leq \frac{1}{|T|} \sum_{g \in T_n} d_T(gy).
\end{equation}

In fact, from this, without loss of generality, we may assume that \(\emptyset \neq T_n \subseteq F_n\) satisfies that \(\lim_{n \to \infty} \frac{|T_n|}{|F_n|} = 1\) and (4.6) holds for \(\nu\)-a.e. \(y \in Y\). Now applying Lemma 4.3 one sees that \(\{T_n : n \in \mathbb{N}\}\) is a tempered Følner sequence of \(G\), and so
\begin{equation}
\limsup_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) \leq \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y) \quad \text{(using Theorem 4.1)}
\end{equation}
for \(\nu\)-a.e. \(y \in Y\). Thus,
\begin{equation}
\limsup_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) \leq \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y)
\end{equation}
for \(\nu\)-a.e. \(y \in Y\). We should observe first (using Proposition 3.2) that:
\begin{equation}
\int_Y \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y) d\nu(y) \leq \inf_{T \in \mathcal{T}_G} \int_Y \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y) d\nu(y) = \nu(D).
\end{equation}

Combined with (4.5) and (4.6), we obtain a stronger version of (4.2):
\begin{equation}
\int_Y \limsup_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) d\nu(y) = \int_Y \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y) d\nu(y) = \nu(D).
\end{equation}

Obviously, the function \(\inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y)\) is measurable and \(G\)-invariant over \((Y, D, \nu)\). It remains to prove (4.1). Applying the ergodic decomposition, without loss of generality, we may assume that the MDS \((Y, D, \nu, G)\) is ergodic. By the above discussion, it is not hard to deduce that
\begin{equation}
\limsup_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) = \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y) = \nu(D)
\end{equation}
for \(\nu\)-a.e. \(y \in Y\), no matter \(\nu(D) = -\infty\) or \(> -\infty\). This finishes the proof. \(\square\)

The following results of [1] Lemma 2.4 and Lemma 9.3 are used in the proof of Theorem 4.2.

**Lemma 4.5.** Let \(T, E \in \mathcal{F}_G\). Then \(\sum_{t \in T} 1_{tE} = \sum_{g \in E} 1_{Tg}\).

**Lemma 4.6.** Let \(D = \{d_F : F \in \mathcal{F}_G\} \subseteq L^1(Y, D, \nu)\) be a strongly sub-additive family. If \(E, E_1, \ldots, E_n \in \mathcal{F}_G, n \in \mathbb{N}\) satisfy \(1_E = \sum_{i=1}^n a_i 1_{E_i}\), where all \(a_1, \ldots, a_n > 0\) are rational numbers, then \(d_E(y) \leq \sum_{i=1}^n a_i d_{E_i}(y)\) for \(\nu\)-a.e. \(y \in Y\).

Now we prove Theorem 4.2.
Proposition 5.1. Let \( \{F_n : n \in \mathbb{N}\} \) be a (tiling) Følner sequence of \( G \) and \( T_1, T_2 \in \mathcal{T}_G \) tile \( G \) self-similarly. If there exists \( T \in \mathcal{T}_G \), such that \( \{Tg : g \in G_{T_1}\} \) forms a partition of \( G_{T_1} \), then \( \{T \pi_{T_1}(F_n) : n \in \mathbb{N}\} \) is a (tiling) Følner sequence of \( G_{T_1} \).

Proof. The tiling property is easy to check once \( F_n, n \in \mathbb{N} \) has the tiling property.

Now we aim to prove its asymptotic invariance property. Let \( g \in G_{T_1} \). As \( T \in \mathcal{T}_G \), such that \( \{Tg : g \in G_{T_2}\} \) forms a partition of \( G_{T_2} \), then for each \( t \in T \) there exist \( g_t \in G_{T_2} \) and \( t_g \in T \) such that \( gt = t_g g_t \), moreover, if \( t_1 \) and \( t_2 \) are different elements from \( T \) then \( (t_1)_{g_t} \neq (t_2)_{g_t} \), otherwise \( g_{t_1} \neq g_{t_2} \) and

\[
\emptyset \neq g T g_t^{-1} \cap gT g_{t_2}^{-1} = g(T g_{t_1}^{-1} \cap T g_{t_2}^{-1}) \quad \text{and so} \quad T g_{t_1}^{-1} \cap T g_{t_2}^{-1} \neq \emptyset,
\]

5. Our main technical results

In this section, we aim to strengthen results in previous section for suitably well-behaved infinite countable discrete amenable groups.

First, we need introduce the following definition.

Let \( T \in \mathcal{T}_G \). We say that \( T \) tiles \( G \) self-similarly if by a suitable selection, \( G_T \) is a subgroup of \( G \) isomorphic to \( G \) via a group isomorphism \( \pi_T : G \to G_T \).

Then we have the following useful observation.

Proposition 5.1. Let \( \{F_n : n \in \mathbb{N}\} \) be a (tiling) Følner sequence of \( G \) and \( T_1, T_2 \in \mathcal{T}_G \) tile \( G \) self-similarly. If there exists \( T \in \mathcal{T}_{G_{T_1}} \), such that \( \{Tg : g \in G_{T_2}\} \) forms a partition of \( G_{T_1} \), then \( \{T \pi_{T_1}(F_n) : n \in \mathbb{N}\} \) is a (tiling) Følner sequence of \( G_{T_1} \).

Proof. The tiling property is easy to check once \( F_n, n \in \mathbb{N} \) has the tiling property.

Now we aim to prove its asymptotic invariance property. Let \( g \in G_{T_1} \). As \( T \in \mathcal{T}_{G_{T_1}} \), such that \( \{Tg : g \in G_{T_2}\} \) forms a partition of \( G_{T_1} \), then for each \( t \in T \) there exist \( g_t \in G_{T_2} \) and \( t_g \in T \) such that \( gt = t_g g_t \), moreover, if \( t_1 \) and \( t_2 \) are different elements from \( T \) then \( (t_1)_g \neq (t_2)_g \), otherwise \( g_{t_1} \neq g_{t_2} \) and

\[
\emptyset \neq g T g_{t_1}^{-1} \cap gT g_{t_2}^{-1} = g(T g_{t_1}^{-1} \cap T g_{t_2}^{-1}) \quad \text{and so} \quad T g_{t_1}^{-1} \cap T g_{t_2}^{-1} \neq \emptyset,
\]
a contradiction to the assumption that \{T_g : g \in G_T\} forms a partition of \(G_T\) and the selection of \(g_t, g_\nu \in G_{T_2}\). That is, there exists a bijection \(f : T \to T\) such that \(gt = f(t)g_t\) for some \(g_t \in G_{T_2}\).

Thus for each \(n \in \mathbb{N}\) one has
\[
T\pi_{T_2}(F_n)\Delta gT\pi_{T_2}(F_n) = \bigcup_{t \in T} t\pi_{T_2}(F_n)\Delta \bigcup_{t \in T} t g_{f^{-1}(t)}\pi_{T_2}(F_n) \\
\subseteq \bigcup_{t \in T} t(\pi_{T_2}(F_n)\Delta g_{f^{-1}(t)}\pi_{T_2}(F_n)).
\]

Observe that \(\{F_n : n \in \mathbb{N}\}\) is a Følner sequence of \(G\), and so \(\{\pi_{T_2}(F_n) : n \in \mathbb{N}\}\) is a Følner sequence of \(G_{T_2}\). Then
\[
\lim_{n \to \infty} \frac{|T\pi_{T_2}(F_n)\Delta gT\pi_{T_2}(F_n)|}{|T\pi_{T_2}(F_n)|} \leq \lim_{n \to \infty} \frac{1}{|T|} \sum_{t \in T} \frac{|\pi_{T_2}(F_n)\Delta g_{f^{-1}(t)}\pi_{T_2}(F_n)|}{|\pi_{T_2}(F_n)|} = 0.
\]

That is, \(\{T\pi_{T_2}(F_n) : n \in \mathbb{N}\}\) is also a Følner sequence of \(G_{T_1}\). \(\square\)

As a direct corollary, we have:

**Corollary 5.2.** Let \(\{F_n : n \in \mathbb{N}\}\) be a (tiling) Følner sequence of \(G\) and \(T \in T_\mathcal{G}\); tile \(G\) self-similarly. Then \(\{T\pi_T(F_n) : n \in \mathbb{N}\}\) is a (tiling) Følner sequence of \(G\).

Our main technical result is stated as follows.

**Theorem 5.3.** Assume that \(\mathcal{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq L^1(Y, \mathcal{D}, \nu)\) is a sub-additive \(G\)-invariant family satisfying \(\nu(\mathcal{D}) > -\infty\) and \(\{F_n : n \in \mathbb{N}\}\) is a Følner sequence of \(G\) consisting of self-similar tiling subsets. If, additionally,

(a) \(\{F_n : n \in \mathbb{N}\}\) satisfies the Tempelman condition with constant \(M > 0\) and
(b) there exists an infinite subset \(\mathcal{N} \subseteq \mathbb{N}\) such that, for each \(m \in \mathcal{N}, \) once \(p \in \mathbb{N}\) is large enough then there exist \(n_1, n_2 \in \mathbb{N}\) such that \(F_m\pi_{F_m}(F_{n_1}) \supseteq F_p \supseteq F_m\pi_{F_m}(F_{n_2})\) and \(\frac{|F_{n_1}| - |F_{n_2}|}{|F_p|}\) is small enough.

Then

1. There exists \(d \in L^1(Y, \mathcal{D}, \nu)\) such that
   \[
   \lim_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) = d(y)
   \]
   for \(\nu\)-a.e. \(y \in Y\) and \(\int Y d(y) d\nu(y) = \nu(\mathcal{D})\).
2. If the limit function \(d\) is \(G\)-invariant, then, for \(\nu\)-a.e. \(y \in Y\),
   \[
   d(y) = \inf_{T \in T_\mathcal{G}} \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y).
   \]
3. If, in addition, one of the following conditions holds:
   (i) \(G\) is abelian.
   (ii) The family \(\mathcal{D}\) is strongly sub-additive.
   (iii) There exist \(p, m \in \mathcal{N}\) large enough (in the sense that once \(N \in \mathbb{N}\) there exist such \(p, m \in \mathcal{N}\) satisfying \(p, m \geq N\) ) such that \(G_{F_m} G_{F_p} = G\).
   Then the limit function \(d\) is \(G\)-invariant.
4. If, in addition, there exist sequences \(\{r_1 < r_2 < \cdots\} \subseteq \mathcal{N}\) and \(\{t_n : n \in \mathbb{N}\}\) such that \(F_{r_{n+1}} = F_{r_n\pi_{F_{r_n}}(F_{t_n})}\) for each \(n \in \mathbb{N}\), then, in the sense of \(L^1\),
   \[
   \lim_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) = d(y).
   \]
Proof. We will follow the ideas of the proof of [15] Chapter 1, Theorem 5.3.

As the Følner sequence \( \{ F_n : n \in \mathbb{N} \} \) satisfies the Tempelman condition, using Theorem 2.1 it is equivalent to consider the family \( D' = \{ d'_F : F \in \mathcal{F}_G \} \) given by

\[
d'_F(y) = \sum_{g \in F} d_{\{e_G\}}(gy) - d_F(y) \ 	ext{for each} \ F \in \mathcal{F}_G.
\]

Then the family \( D' \) is non-negative, sup-additive and \( G \)-invariant and \( \nu(D') < \infty \).

For convenience, we may assume that the family \( D' \) satisfies the assumptions of non-negativity, sup-additivity and \( G \)-invariance for each point \( y \in Y \) without any ambiguity (for example, \( d_F(y) = d_F(gy) \) for each \( F \in \mathcal{F}_G \) and any \( y \in Y \)).

Set \( E = \{ y \in Y : \mathcal{T}(y) > d'(y) \} \), where

\[
\mathcal{T}(y) = \limsup_{n \to \infty} \frac{1}{|F_n|} d'_{F_n}(y)
\]

and

\[
d'(y) = \liminf_{n \to \infty} \frac{1}{|F_n|} d'_{F_n}(y) \geq 0.
\]

Observe that by Fatou’s Lemma one has

\[
\int_Y d'(y) d\nu(y) \leq \liminf_{n \to \infty} \frac{1}{|F_n|} \int_Y d'_{F_n}(y) d\nu(y) = \nu(D'),
\]

and so \( d'(y) \in L^1(Y, D, \nu) \), as \( \nu(D) \) is finite, equivalently, \( \nu(D') \) is finite.

Now for each \( m \in \mathbb{N} \) we introduce

\[
D'_m = \{ d'_{m,\pi F_m}(F)(y) = d'_{F_m,\pi F_m}(F)(y) - \sum_{g \in F} d'_{F_m}(\pi F_m(g)y) : F \in \mathcal{F}_G \}.
\]

Then the family \( D'_m \) is non-negative, sup-additive and \( G_{F_m} \)-invariant. Here, the \( G_{F_m} \)-invariance of \( D'_m \) means that, for all \( F \in \mathcal{F}_G, g \in G_{F_m} \) and \( y \in Y \),

\[
d'_{m,\pi F_m}(F)(y) = d'_{m,\pi F_m}(F)(gy).
\]

In the following, first we shall prove that

\[
\limsup_{n \to \infty} \frac{1}{|F_m||F_n|} d'_{m,\pi F_m}(F_n)(y) \geq \mathcal{T}(y)
\]

and

\[
\liminf_{n \to \infty} \frac{1}{|F_m||F_n|} d'_{m,\pi F_m}(F_n)(y) \leq d'(y).
\]

In fact, suppose that \( \{ k_1 < k_2 < \cdots \}, \{ p_1 < p_2 < \cdots \} \subseteq \mathbb{N} \) such that

\[
\mathcal{T}(y) = \lim_{n \to \infty} \frac{1}{|F_{k_n}|} d'_{F_{k_n}}(y) \text{ and } d'(y) = \lim_{n \to \infty} \frac{1}{|F_{p_n}|} d'_{F_{p_n}}(y).
\]

By assumption (b) for each \( n \in \mathbb{N} \) large enough we can select \( l_n, q_n \in \mathbb{N} \) such that

\[
F_{l_n} \supseteq F_{k_n} \text{ and } \lim_{n \to \infty} \frac{|F_{l_n}|}{|F_{k_n}|} = 1
\]

and

\[
F_{q_n} \supseteq F_{p_n} \text{ and } \lim_{n \to \infty} \frac{|F_{q_n}|}{|F_{p_n}|} = 1.
\]

As the family \( D' \) is non-negative and sup-additive (and hence monotone), one has:

\[
\limsup_{n \to \infty} \frac{1}{|F_m||F_n|} d'_{m,\pi F_m}(F_n)(y) \geq \limsup_{n \to \infty} \frac{1}{|F_m||F_{l_n}|} d'_{m,\pi F_m}(F_{l_n})(y)
\]

\[
\geq \limsup_{n \to \infty} \frac{1}{|F_{k_n}|} d'_{F_{k_n}}(y) \text{ (using (5.3))}
\]

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and
\[
\liminf_{n \to \infty} \frac{1}{|F_n||F_m|} d_{F_n \pi F_m}(F_n)(y) \leq \liminf_{n \to \infty} \frac{1}{|F_n||F_m|} d_{F_n \pi F_m}(F_n)(y) \\
\leq \liminf_{n \to \infty} \frac{1}{|F_n|} d_{F_{pn}}(y) \quad \text{(using (5.6))},
\]
which implies the direction “\(\leq\)” of (5.2) and (5.3). The other direction is obvious. Observe again that the family \(D'\) is non-negative and sup-additive. Hence from (5.2) and (5.3) we have, for \(\nu\)-a.e. \(y \in Y\),
\[
\overline{d}'(y) - \overline{d}(y) \\
\leq \limsup_{n \to \infty} \frac{1}{|F_m||F_n|} d_{F_m \pi F_m}(F_n)(y) - \liminf_{n \to \infty} \frac{1}{|F_m||F_n|} d_{F_m \pi F_m}(F_n)(y) \\
\leq \limsup_{n \to \infty} \frac{1}{|F_m||F_n|} d_{F_m \pi F_m}(F_n)(y) - \liminf_{n \to \infty} \frac{1}{|F_m||F_n|} \sum_{g \in F_n} d_{F_m}(\pi F_m(g))y \\
= \limsup_{n \to \infty} \frac{1}{|F_m||F_n|} \left[ d_{F_m \pi F_m}(F_n)(y) - \sum_{g \in F_n} d_{F_m}(\pi F_m(g))y \right] \\
\leq \sup_{n \in \mathbb{N}} \frac{1}{|F_m||F_n|} d_{F_m \pi F_m}(F_n)(y).
\] (5.6)
Moreover, by Theorem 2.1 the pointwise limit of the sequence
\[
\frac{1}{|F_m||F_n|} \sum_{g \in F_n} d_{F_m}(\pi F_m(g))y
\]
extists (denote it by \(d_m'\)), is \(G_{F_m}\)-invariant and is dominated by \(\frac{1}{|F_m||F_n|} d_{F_m \pi F_m}(F_n)(y)\). Combining (5.2) and (5.6) with (5.4) we also obtain, for \(\nu\)-a.e. \(y \in Y\),
\[
0 \leq \overline{d}'(y) - d_m'(y) \leq \sup_{n \in \mathbb{N}} \frac{1}{|F_m||F_n|} d_{m \pi F_m}(F_n)(y).
\] (5.8)
Applying Corollary 5.2 to \(D_m\) we obtain, for each \(\alpha > 0\),
\[
\nu\{y \in Y : \sup_{n \in \mathbb{N}} \frac{1}{|F_m||F_n|} d_{m \pi F_m}(F_n)(y) > \alpha\} \leq \frac{M}{\alpha} \frac{\nu(D_m)}{|F_m|}.
\] (5.9)
Step One: proof of (1)
In the following, first we will prove \(\nu(E) = 0\). Recall \(E = \{y \in Y : \overline{d}'(y) > \overline{d}(y)\}\).
Let \(\epsilon > 0\). Obviously, once \(m \in \mathcal{N}\) is sufficiently large then
\[
\frac{1}{|F_m|} \int_Y d_{F_m}(y) d\nu(y) > \nu(D') - \epsilon.
\] (5.10)
As \(\{F_n : n \in \mathbb{N}\}\) is a Følner sequence of \(G\) consisting of self-similar tiling subsets, by Corollary 5.2 the sequence \(\{F_m \pi F_m(F_n) : n \in \mathbb{N}\}\) is a tiling Følner sequence of
\( G \), and so by Proposition 3.2 one has

\[
\nu(D'_m) = \lim_{n \to \infty} \frac{1}{F_n} \int_Y [d'_{F_m \pi F_m} (F_n)](y) - \sum_{g \in F_n} d'_{F_m} (\pi F_m (g) y)]d\nu(y)
\]

\[
= \lim_{n \to \infty} \frac{1}{F_n} \int_Y d'_{F_m \pi F_m} (F_n)(y)d\nu(y) - \int_Y d'_{F_m} (y)d\nu(y)
\]

(5.11)

\[
= |F_m| \nu(D') - \int_Y d'_{F_m} (y)d\nu(y) \leq |F_m| \epsilon \text{ (using (5.10)).}
\]

In particular, combining this with (5.7) and (5.9) we obtain

\[
\nu(\{y \in Y : \tilde{d}'(y) - d'(y) > \alpha\})
\]

\[
\leq \nu(\{y \in Y : \sup_{n \in \mathbb{N}} 1/|F_n| |d'_{F_m \pi F_m} (F_n)(y)| > \alpha\}) \leq \frac{M \epsilon}{\alpha}.
\]

First letting \( \epsilon \to 0 \) and then letting \( \alpha \to 0 \) we obtain \( \nu(E) = 0 \) and so \( \tilde{d}'(y) = d'(y) \) (denoted by \( d'(y) \)) for \( \nu \)-a.e. \( y \in Y \).

For each \( m \in \mathcal{N} \) observe that by (5.8) we have

\[
\int_Y \tilde{d}'(y)d\nu(y) \geq \int_Y d'_m(y)d\nu(y)
\]

(5.13)

\[
= \frac{1}{|F_m|} \int_Y d'_{F_m} (y)d\nu(y) \text{ (using Theorem 2.1).}
\]

Letting \( m \to \infty \) we obtain \( \int_Y d'(y)d\nu(y) \geq \nu(D') \) and hence \( \int_Y d'(y)d\nu(y) = \nu(D') \) (using (5.11)), equivalently, \( \int_Y d(y)d\nu(y) = \nu(D) \), here, \( d \) is the pointwise limit function of the sequence \( \frac{1}{F_n} d_{F_n} (y) \).

**Step Two: proof of (2)**

Applying the ergodic decomposition, without loss of generality, we may assume that the MDS \((Y, D, \nu, G)\) is ergodic. If the limit function \( d \) is \( G \)-invariant, by (1) one has that \( d(y) = \nu(D) \) for \( \nu \)-a.e. \( y \in Y \). For each \( T \in \mathcal{T}_G \), observe that the measurable function \( E(d_T | I) \) is invariant over \((Y, D, \nu)\), and so \( E(d_T | I)(y) = \int_Y d_T(y)d\nu \) for \( \nu \)-a.e. \( y \in Y \). Thus, for \( \nu \)-a.e. \( y \in Y \),

\[
\inf_{T \in \mathcal{T}_G} \int_Y \frac{1}{|T|} E(d_T | I)(y) = \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \int_Y d_T(y)d\nu = \nu(D) = d(y) \text{ (using Proposition 3.2).}
\]

**Step Three: proof of (3)**

Now we prove the \( G \)-invariance of the limit function \( d' \) under the assumptions.

If either \( G \) is abelian or the family \( D \) is strongly sub-additive, this follows directly from Theorem 4.1 and Theorem 4.2 respectively. Now we assume that (iii) holds.

Let \( g \in G \). We aim to prove \( \nu(E_g) = 0 \), where \( E_g = \{y \in Y : d'(y) \neq d'(gy)\} \).

Let \( \epsilon > 0 \). By (iii), there exist \( p, m \in \mathcal{N} \) sufficiently large that \( G_{F_m} G_{F_p} = G \). Thus

\[
\frac{1}{|F_m|} \int_Y d'_{F_m} (y)d\nu(y) > \nu(D') - \epsilon,
\]

(5.14)

\[
\frac{1}{|F_p|} \int_Y d'_{F_p} (y)d\nu(y) > \nu(D') - \epsilon
\]

(5.15)
and there exist \( g_m \in G_{F_m} \) and \( g_p \in G_F \) such that \( q = g_m g_p \). Observe that \( d_m'(y) = d_m'(g_m y) \) and \( d_p'(y) = d_p'(g_p y) \) for \( \nu \)-a.e. \( y \in Y \), thus

\[
\nu(\{ y \in Y : |d'(y) - d'(gy)| > 4\alpha \}) = \nu(\{ y \in Y : |d'(y) - d'(g_m y)| > 4\alpha \}) \\
\leq \nu(\{ y \in Y : |d'(y) - d'(gy)| > \alpha \}) + \nu(\{ y \in Y : |d'(g_m y) - d'(gy)| > \alpha \}) + \nu(\{ y \in Y : |d'(y) - d'(g_m g_p y)| > \alpha \}) \\
< \frac{4M\epsilon}{\alpha} \quad (\text{similar to reasoning of (5.12), using (5.8), (5.14) and (5.15)})
\]

for any \( \alpha > 0 \). First letting \( \epsilon \to 0 \) and then letting \( \alpha \to 0 \) we obtain \( d'(y) = d'(gy) \) for \( \nu \)-a.e. \( y \in Y \). In other words, \( \nu(E_\beta) = 0 \).

**Step Four: proof of [4]**

First, we claim that, by assumption, the sequence of the functions \( \{ d_{F_n} : n \in \mathbb{N} \} \) increases to some \( d_\infty' \in L^1(Y, D, \nu) \).

Let \( n \in \mathbb{N} \) be fixed. As \( F_{n+1} = F_n \pi_{F_n}(F_n) \), in particular, \( \{ \pi_{F_n}(F_n) g : g \in G_{F_{n+1}} \} \) forms a partition of \( G_{F_n} \) and \( |F_{n+1}| = |F_n| \cdot |\pi_{F_n}(F_n)| \) and so (recall that the family \( D' \) is sup-additive)

\[
\frac{1}{|F_{n+1}|} \sum_{g \in F_m} d_{F_{n+1}}' (\pi_{F_{n+1}}(g)y) \\
\geq \frac{1}{|F_n| \cdot |\pi_{F_n}(F_n)| \cdot |F_m|} \sum_{g \in \pi_{F_n}(F_n) \pi_{F_{n+1}}(F_m)} d_{F_n}'(gy) \\
= \frac{1}{|F_n| \cdot |\pi_{F_n}(F_n) \pi_{F_{n+1}}(F_m)|} \sum_{g \in \pi_{F_n}(F_n) \pi_{F_{n+1}}(F_m)} d_{F_n}'(gy)
\]

(5.16)

for each \( m \in \mathbb{N} \). Observe that by assumptions, \( \{ \pi_{F_n}(F_n) \pi_{F_{n+1}}(F_m) : m \in \mathbb{N} \} \) is a Følner sequence of \( G_{F_n} \) (using Proposition 5.1), and so by Theorem 2.1 one has that (to obtain (5.18), we may take a sub-sequence of \( m \in \mathbb{N} \) if necessary; note that, as remarked by [17] Proposition 1.4) every Følner sequence of \( G \) contains a tempered sub-sequence) both

\[
\lim_{m \to \infty} \frac{1}{|F_{n+1}|} \sum_{g \in F_m} d_{F_{n+1}}' (\pi_{F_{n+1}}(g)y) = d_{F_{n+1}}'(y) \\
\]

(5.17)

and

\[
\lim_{m \to \infty} \frac{1}{|F_n| \cdot |\pi_{F_n}(F_n) \pi_{F_{n+1}}(F_m)|} \sum_{g \in \pi_{F_n}(F_n) \pi_{F_{n+1}}(F_m)} d_{F_n}'(gy) = d_{F_n}'(y) \\
\]

(5.18)

for \( \nu \)-a.e. \( y \in Y \) and in the sense of \( L^1 \). Combining (5.16) with (5.17) and (5.18) we obtain that the sequence of the functions \( \{ d_{F_n} : n \in \mathbb{N} \} \) increases. Let \( d_\infty' \) be the limit function (which is non-negative). Observe that, by Theorem 2.1

\[
\int_Y d_m'(y) d\nu(y) = \frac{1}{|F_m|} \int_Y d_{F_m}'(y) d\nu(y) \leq \nu(D')
\]
for each \( m \in \mathbb{N} \). As \( \nu(D) > -\infty \), equivalently, \( \nu(D') < \infty \), we obtain

\[
\int_Y \nu'(y) d\nu(y) \leq \nu(D') < \infty \quad \text{and hence} \quad \int_Y \nu'(y) d\nu(y) = \nu(D')
\]

in particular, \( \nu' \in L^1(Y, D, \nu) \). Since \( \int_Y \nu'(y) d\nu(y) = \nu(D') \), one has that \( \nu'(y) = d'_{\infty}(y) \) for \( \nu \)-a.e. \( y \in Y \).

To complete the proof, we only need to prove that

\[
\lim_{n \to \infty} \frac{1}{|F_n|} \nu'(y) = d'_{\infty}(y)
\]

in the sense of \( L^1 \).

Let \( \epsilon > 0 \). Obviously there exists \( n, m \in \mathbb{N} \) such that

\[
\int_Y |d'_{\infty}(y) - d'_{r_n}(y)| d\nu(y) < \epsilon \tag{5.19}
\]

and

\[
\left| \int_Y d'_{r_n}(y) d\nu(y) - \nu(D') \right| = \left| \frac{1}{|F_{r_n}|} \int_Y d'_{F_{r_n}}(y) d\nu(y) - \nu(D') \right| < \epsilon \tag{5.20}
\]

By our assumptions, once \( m \in \mathbb{N} \) is large enough, there exists \( s_m \in \mathbb{N} \) such that

\[
F_m \supseteq F_{r_n} \pi F_{r_n}(F_{s_m}) \tag{5.21}
\]

and

\[
\int_Y \left| \frac{1}{|F_{r_n}|} \sum_{g \in F_{s_m}} d'_{F_{r_n}}(\pi F_{r_n}(g)y) - d'_{r_n}(y) \right| d\nu(y) < \epsilon \tag{using (5.17)} \tag{5.22}
\]

Thus, once \( m \in \mathbb{N} \) is large enough we have

\[
0 \leq d'_{F_{r_n}}(y) - \sum_{g \in F_{s_m}} d'_{F_{r_n}}(\pi F_{r_n}(g)y) \tag{5.23}
\]

and

\[
\int_Y \left| d'_{F_{r_n}}(y) - \sum_{g \in F_{s_m}} d'_{F_{r_n}}(\pi F_{r_n}(g)y) \right| d\nu(y) \leq |F_m| \nu(D') - |F_{s_m}| \int_Y d'_{F_{r_n}}(y) d\nu(y) \\
\leq |F_m| \nu(D') - |F_{s_m}| \cdot |F_{r_n}| (\nu(D') - \epsilon) \text{ (using (5.20))} \\
= |F_m \setminus F_{r_n} \pi F_{r_n}(F_{s_m})| \nu(D') + |F_{s_m}| \cdot |F_{r_n}| \epsilon \tag{5.24}
\]
Letting
\begin{equation}
(5.25)
\end{equation}
\[\epsilon \rightarrow 0\] we obtain the conclusion. This finishes our proof. \(\square\)

In fact, even if \(\nu(D) = -\infty\) we have a similar result.

**Theorem 5.4.** Assume that \(D = \{d_F : F \in \mathcal{F}_G\} \subseteq L^1(Y, \mathcal{D}, \nu)\) is a sub-additive \(G\)-invariant family satisfying \(\nu(D) = -\infty\) and \(\{F_n : n \in \mathbb{N}\}\) is a Følner sequence of \(G\) consisting of self-similar tiling subsets satisfying the assumptions of (a) and (b) in Theorem 5.3. Then

1. There exists a measurable function \(d\) over \((Y, \mathcal{D}, \nu)\) such that
   \[
   \lim_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) = d(y)
   \]
   for \(\nu\)-a.e. \(y \in Y\) and \(\int_Y d(y) d\nu(y) = -\infty\).
2. If the limit function \(d\) is \(G\)-invariant, then, for \(\nu\)-a.e. \(y \in Y\),
   \[
   d(y) = \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \mathbb{E}(d_T|\mathcal{I})(y).
   \]
3. If, additionally, one of the following conditions holds:
   (i) \(G\) is abelian.
   (ii) The family \(D\) is strongly sub-additive.
   (iii) There exist \(p, m \in \mathbb{N}\) large enough such that \(G_{F_n}G_{F_p} = G\).
   Then the limit function \(d\) is \(G\)-invariant.

**Proof.** With the help of Theorem 5.3, the proof of the conclusion straightforward.

As in the proof of Theorem 5.3, we may assume that the family \(-D\) is non-negative. For each \(N \in \mathbb{N}\), we consider the family
\[
D^{(N)} = \{d_E^{(N)}(y) = \max\{-N|E|, d_E(y)\} : E \in \mathcal{F}_G\} \subseteq L^1(Y, \mathcal{D}, \nu).
\]
It is easy to check that \(D^{(N)}\) is a sub-additive \(G\)-invariant family satisfying \(\nu(D^{(N)}) \geq -N\). Thus we can apply Theorem 5.3 to the family \(D^{(N)}\) to see that there exists \(d^{(N)} \in L^1(Y, \mathcal{D}, \nu)\) such that
\[
\lim_{n \to \infty} \frac{1}{|F_n|} d_{F_n}^{(N)}(y) = d^{(N)}(y)
\]
for $\nu$-a.e. $y \in Y$ and $\int_Y d^{(N)}(y)\,d\nu(y) = \nu(D^{(N)})$. Clearly, the sequence of functions
$\{d^{(N)}(y) : N \in \mathbb{N}\}$ decreases and set $d$ to be the limit function of it. So, $d$ is a measurable function over $(Y, \mathcal{D}, \nu)$. Moreover,
\[
\int_Y d(y)\,d\nu(y) = \inf_{N \in \mathbb{N}} \int_Y d^{(N)}(y)\,d\nu(y) = \inf_{N \in \mathbb{N}} \inf_{n \in \mathbb{N}} \frac{1}{|F_n|} \int_Y d^{(N)}_x(y)\,d\nu(y)
= \inf_{n \in \mathbb{N}} \frac{1}{|F_n|} \int_Y d^{(N)}_n(y)\,d\nu(y)
= \inf_{n \in \mathbb{N}} \frac{1}{|F_n|} \int_Y d_n(y)\,d\nu(y) = \nu(D).
\]

Once the measurable function $d$ is $G$-invariant, as in the proof of Theorem 5.3 it is not hard to obtain that, for $\nu$-a.e. $y \in Y$, 
\[
d(y) = \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} \mathbb{E}(d_T|I)(y).
\]

We aim that, for $\nu$-a.e. $y \in Y$,
\[
\lim_{n \to \infty} \frac{1}{|F_n|} d_n(y) = d(y) = \inf_{N \in \mathbb{N}} \lim_{n \to \infty} \frac{1}{|F_n|} d^{(N)}_n(y).
\]

As for $\nu$-a.e. $y \in Y$ the limit $\lim_{n \to \infty} \frac{1}{|F_n|} d^{(N)}_n(y)$ exists for each $N \in \mathbb{N}$. Fix such a point. Thus if $d(y) > -\infty$, say $d(y) > -M$ for some $M \in \mathbb{N}$ then, once $n \in \mathbb{N}$ is large enough, then for any $N \geq M$, $d^{(N)}_n(y) > -|F_n|M$ (in particular, $d_n(y) = d^{(N)}_n(y)$) and so the limit of the sequence $\frac{1}{|F_n|} d^{(N)}_n(y)$ exists and equals $\lim_{n \to \infty} \frac{1}{|F_n|} d^{(N)}_n(y)$ (and hence equals $d(y)$). If $d(y) = -\infty$, observe that it is almost direct to check that $\limsup_{n \to \infty} \frac{1}{|F_n|} d_n(y) \leq d(y)$. Summing up, we obtain (5.26).

Finally, we are to prove the $G$-invariance of the limit function $d$ under the assumptions. If either $G$ is abelian or the family $D$ is strongly sub-additive, it follows directly from Theorem 4.1 and Theorem 4.2 respectively. Now we assume that (iii) holds. Then $d^{(N)}$ is $G$-invariant for each $N \in \mathbb{N}$, which implies immediately the $G$-invariance of $d$. This ends the proof. \hfill $\square$

**Remark 5.5.** The only place in the proofs of Theorem 5.3 and Theorem 5.4 where we used the assumption that $\{F_n : n \in \mathbb{N}\}$ satisfies the Tempelman condition, is in applying Corollary 5.7. Thus we can weaken the assumption to $\{F_n : n \in \mathbb{N}\}$ being tempered if Corollary 5.7 holds for a tempered Følner sequence of some particular group. Moreover, we can drop the assumption that $\{F_n : n \in \mathbb{N}\}$ is tempered if Theorem 2.7 and Corollary 3.7 both hold for any Følner sequence of a given group.

We also remark that, the family $D^{(N)}$ (in the proof of Theorem 5.3) need not to be strongly sub-additive even if $D$ is strongly sub-additive.

Except for assumption (ii), all other assumptions in Theorems depend only on the group $G$ and the Følner sequence $\{F_n : n \in \mathbb{N}\}$ of $G$ (independent of the considered family $D$). Thus, we end this section with the following remarks.

**Remark 5.6.** Let $N \in \mathbb{N}$. Assume that, for each $m = 1, \cdots , N$, $G_m$ is an infinite countable discrete amenable group with $\{F^{(m)}_n : n \in \mathbb{N}\}$ a Følner sequence satisfying the assumptions appearing in Theorems (with $M_m$ as its Tempelman condition constant). Then it is not hard to check that, $\bigotimes_{m=1}^N G_m$ (an infinite countable discrete
amenable group) and \( \{ F_{(n_1, \ldots, n_N)} : (n_1, \ldots, n_N) \in \bigotimes_{m=1}^N \mathbb{N} \} \) (a Følner sequence of \( \bigotimes_{m=1}^N G_m \)) also satisfy these assumptions, where, the subset \( F_{(n_1, \ldots, n_N)} \) is given by
\[
\bigotimes_{m=1}^N F_{n_m} \quad \text{for each} \quad (n_1, \ldots, n_N) \in \bigotimes_{m=1}^N \mathbb{N}.
\]

Observe that in \( \bigotimes_{m=1}^N \mathbb{N} \), \( (n_1, \ldots, n_N) \) tends to \( \infty \) just means that \( \min\{n_1, \ldots, n_N\} \) tends to \( \infty \) (if some \( n_i \), say \( n_1 \), is bounded, the problem is reduced to the case of \( N-1 \)). Moreover, \( \prod_{m=1}^N M_m \) will be the Tempelman condition constant for \( \{ F_{(n_1, \ldots, n_N)} : (n_1, \ldots, n_N) \in \bigotimes_{m=1}^N \mathbb{N} \} \).

**Remark 5.7.** For the case of \( N = \infty \) in Remark 5.6, if, additionally,
\[
M = \prod_{m \in \mathbb{N}} M_m < \infty,
\]
then we can carry out a similar discussion for \( \bigoplus_{m \in \mathbb{N}} G_m \) (also an infinite countable discrete amenable group) and \( \{ F_n : n \in \bigoplus_{m \in \mathbb{N}} \mathbb{N} \} \) (a Følner sequence of \( \bigoplus_{m \in \mathbb{N}} G_m \)), where
\[
\bigoplus_{m \in \mathbb{N}} G_m = \bigcup_{N' \in \mathbb{N}} \{(g, e_{G_{N'+1}}, e_{G_{N'+2}}, \ldots) : g \in \bigotimes_{m=1}^{N'} G_m\},
\]
\[
\bigoplus_{m \in \mathbb{N}} \mathbb{N} = \{(n_1, \ldots, n_m) \in \mathbb{N}^m : m \in \mathbb{N}\},
\]
and then for each \( n \in \bigoplus_{m \in \mathbb{N}} \mathbb{N} \), say \( n = (n_1, \ldots, n_m) \) for some \( m \in \mathbb{N} \),
\[
F_n = \{(g, e_{G_{m+1}}, e_{G_{m+2}}, \ldots) : g \in \bigotimes_{i=1}^m F_{n_i}^{(i)}\}.
\]

Observe that in \( \bigoplus_{m \in \mathbb{N}} \mathbb{N} \), \( (n_1, \ldots, n_m) \) tends to \( \infty \) just means that both \( m \) and \( \min\{n_1, \ldots, n_m\} \) tend to \( \infty \) (if \( m \) is bounded, it is reduced to the case of Remark 5.6). Moreover, \( M \) will be the Tempelman condition condition for \( \{ F_n : n \in \bigoplus_{m \in \mathbb{N}} \mathbb{N} \} \).

### 6. Direct applications of the main technical results

In this section, we aim to give some direct applications of the results obtained in previous sections. Although we only consider some special groups in this section, we believe that these results (and the ideas in proving them) will have wider applications.

#### 6.1. The case of \( G = \mathbb{Z}^m, m \in \mathbb{N} \)

As shown by Remark 5.6, the case of \( \mathbb{Z}^m, m \in \mathbb{N} \) is reduced to the case of \( \mathbb{Z} \). If \( G = \mathbb{Z} \) which is abelian, consider \( F_n = \{0, \ldots, n-1\} \) with \( G_{F_n} = n\mathbb{Z} \) for each \( n \in \mathbb{N} \), then 2 is its associated Tempelman condition constant and \( F_{m \pi F_m}(F_n) = \{0, 1, \ldots, mn - 1\} = F_{mn} \) for \( m, n \in \mathbb{N} \).

Thus, applying Theorem 5.3 and Theorem 5.4 we obtain:
Theorem 6.1. Let \( G = \mathbb{Z}^m, m \in \mathbb{N}, (Y,D,\nu,G) \) be an MDS and \( \{F_n : n \in \bigoplus \mathbb{N}\} \) the sequence introduced as in Remark 5.7. Assume that \( D = \{d_F : F \in \mathcal{F}(Y)\} \subseteq L^1(Y,D,\nu) \) is a sub-additive \( G \)-invariant family. Then, for \( \nu \)-a.e. \( y \in Y \),

\[
(6.1) \quad \lim_{n \to \infty} \frac{1}{|F_n|} d_{F_n}(y) = \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} E(d_T|T)(y),
\]

which is an invariant measurable function over \( (Y,D,\nu) \), and

\[
\int_Y \inf_{T \in \mathcal{T}_G} \frac{1}{|T|} E(d_T|T)(y) d\nu(y) = \nu(D).
\]

Moreover, if \( \nu(D) > -\infty \) then (6.1) also holds in the sense of \( L^1 \).

6.2. The case of \( G = \bigoplus K_n \) with each \( K_n \) a non-trivial finite group.

First, we consider the case where each \( K_n, n \in \mathbb{N} \) is equal to a fixed non-trivial finite abelian group \( K \) (a special case is \( \bigoplus \mathbb{Z}_p, p \in \mathbb{N} \setminus \{1\} \)), where \( \mathbb{Z}_p \) is the additive group \( \{0,1,\ldots,p-1\} \). Obviously, \( G \) is abelian. We consider \( F_n = \{(g,e_K,e_K,\cdots) : g \in \bigotimes K\} \) with \( G_{F_n} = \{(g_1,\cdots,g_n,g) : g_1 = \cdots = g_n = e_K, g \in \bigotimes K\} \) for each \( n \in \mathbb{N} \). Then 1 is its associated Tempelman condition constant and \( F_m \pi F_m(F_n) = F_{m+n} \) for \( m, n \in \mathbb{N} \). A result similar to Theorem 6.1 holds.

For the case of \( G = \bigoplus K_n \) where each \( K_n, n \in \mathbb{N} \) is a non-trivial finite abelian group: here \( K_n \) need not be a fixed. At first sight, it appears that we cannot apply our technical results directly. However, \( G \) is still an abelian group, and if we consider \( F_n = \{(g,e_{K_{i+1}},e_{K_{i+2}},\cdots) : g \in \bigotimes K_i\} \) with \( G_{F_n} = \{(e_{K_1},\cdots,e_{K_n},g) : g \in \bigotimes K_i\} \) for each \( n \in \mathbb{N} \), then 1 is also its associated Tempelman condition constant. Though in general \( G_{F_n} \) is not isomorphic to \( G \) via a group isomorphism, a rewriting of the argument of the previous section leads to the same conclusion as for the case of \( \bigoplus K \) with \( K \) a non-trivial finite abelian group.

Moreover, if we drop the assumption that each \( K_n, n \in \mathbb{N} \) is abelian, we can still obtain the result if the family is strongly sub-additive.

6.3. The case of \( G = \bigoplus \mathbb{Z} \).

For the abelian group \( \bigoplus \mathbb{Z} \), we consider the sequence \( \{F_n : n \in \bigoplus \mathbb{N}\} \), where, for \( n \in \bigoplus \mathbb{N}, \) say \( n = (n_1,\cdots,n_m) \) for some \( m \in \mathbb{N} \), the subset \( F_n \) is given as

\[
\{(g_1,\cdots,g_m,0,0,\cdots) : g_i \in \{0,1,\cdots,n_i-1\}, i = 1,\cdots,m\}.
\]

As in Remark 5.7, for \( \bigoplus \mathbb{N}, (n_1,\cdots,n_m) \) tends to \( \infty \) just means that both \( m \) and \( \min\{n_1,\cdots,n_m\} \) tend to \( \infty \). For each \( n \in \bigoplus \mathbb{N}, \) say \( n = (n_1,\cdots,n_m) \) for some
\( m \in \mathbb{N} \), obviously \( F_n \) tiles \( \bigoplus \mathbb{Z} \) self-similarly with

\[
G_F = \bigoplus_{i=1}^m n_i \mathbb{Z} \bigoplus \bigoplus \mathbb{Z},
\]

and so for \( n' \in \bigoplus \mathbb{N} \), say \( n' = (n_1', \ldots, n_m') \) for some \( m' \in \mathbb{N} \), if \( m' \geq m \) then

\[
F_{n^*} F_n (F_{n'}^*) = F_{n^*}, F_n F_{n'} = F_{n^*}, F_n \cup F_{n'} = F_{n^*}
\]

with

\[
n^* = (n_1 n_1', \ldots, n_m n_m', n_{m+1}', \ldots, n_{m'}'),
\]

\[
n^{**} = (n_1 + n_1' - 1, \ldots, n_m + n_m' - 1, n_{m+1}', \ldots, n_{m'}'),
\]

\[
n^{***} = (\max\{n_1, n_1'\}, \ldots, \max\{n_m, n_m'\}, n_{m+1}', \ldots, n_{m'}').
\]

We have similar formulas in the case of \( m' < m \).

It is easy to see that even the sequence \( \{F_n : n \in \bigoplus \mathbb{N}\} \) is not tempered, and so we cannot apply directly any of the results of the previous sections to \( \{F_n : n \in \bigoplus \mathbb{N}\} \).

Before proceeding, let us first recall some well-known results.

The first one can be found in any standard book about ergodic theory.

**Theorem 6.2.** Let \( \vartheta \) be an invertible measure-preserving transformation over a Lebesgue space \((Y, \mathcal{D}, \nu)\) and \( f \in L^p(Y, \mathcal{D}, \nu) \), where \( 1 \leq p < \infty \). Then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\vartheta^i y) = \mathbb{E}(f|\mathcal{I})(y)
\]

for \( \nu \)-a.e. \( y \in Y \) and in the sense of \( L^p \).

The second one is also standard: it is a variation of [15, Chapter 1, Theorem 5.2] (in fact, it is another version of the maximal ergodic inequality).

**Proposition 6.3.** Let \( \vartheta \) be an invertible measure-preserving transformation over a Lebesgue space \((Y, \mathcal{D}, \nu)\) and \( D = \{d_F : F \in \mathcal{F}_\mathbb{Z}\} \subseteq L^1(Y, \mathcal{D}, \nu) \) a non-negative sup-additive \( \vartheta \)-invariant family. Then, for each \( \alpha > 0 \),

\[
\nu\{y \in Y : \sup_{n \in \mathbb{N}} \frac{1}{n} d_{\{0, \ldots, n-1\}}(y) > \alpha\} \leq \frac{1}{\alpha} \nu(D).
\]

Now using Theorem 6.2 and Proposition 6.3 and applying the results in previous sections (especially, Remark 5.5) we obtain a result similar to Theorem 6.1 (as \( \bigoplus \mathbb{Z} \) is an abelian group).

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