Global Existence of the Compressible Euler Equations with Time-dependent Damping and Sign-changing State Equation

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Abstract

In mathematical physics, the pressure function is determined by the equation of state. There are two existing state equations: the state equation ($S_+$) for polytropic gas with adiabatic index greater than 1 and the state equation ($S_-$) for generalized Chaplygin gas in cosmology. In this paper, we provide some mathematical foundations for the sign-changing pressure ($S_*$) by establishing a global existence result for the one-dimensional Euler equations with time-dependent damping. It is found that the newly introduced sign-changing pressure shares many mathematical properties such as reduction to a symmetric hyperbolic system and finite propagation speed property with the classical positive pressure $S_+$ and the negative pressure $S_-$. Moreover, $S_+$ and $S_-$ are unified and generalized in our proposition which includes a global existence result for $S_+$, $S_-$ and $S_*$.

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1 Introduction

The $N$-dimensional compressible isentropic Euler equations with time-dependent damping can be expressed as

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\rho [u_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] + \nabla p + s(t) \rho \mathbf{u} &= 0,
\end{align*}
\]

where $\rho = \rho(t, x) : [0, \infty) \times \mathbb{R}^N \to [0, \infty)$, $\mathbf{u}(t, x) : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^N$ and $p$ are the density, the velocity, and the pressure function respectively. $s(t) > 0$ is a time-dependent damping term.

The first equation in (1) is derived from the mass conservation law while the second equation in (1) is a result of the momentum conservation law.

The pressure function $p = p(\rho)$ is given by the barotropic equation of state (the term barotropic means that the pressure $p$ can be expressed as a function of the density $\rho$). For classical Euler equations, the state equation is given by the $\gamma$-law:

\[
p = \frac{1}{\gamma} \rho^\gamma, \quad \gamma \geq 1.
\]

The classical Euler equations are one of the most fundamental equations in fluid dynamics. Many interesting fluid dynamic phenomena can be described by the classical Euler equations [9, 10].

When the state equation is given by

\[
p = -\frac{1}{\gamma} \rho^{-\gamma}, \quad 0 < \gamma \leq 1,
\]

system (1) is called the Euler equations for generalized Chaplygin gas (GCG).

The GCG model was introduced in [15] in 2012 as a unification of dark matter and dark energy in the latest model of cosmology.

Note that the pressures given by (2) and (3) for non-vacuum initial data are always positive and negative respectively and these two state equations dominate the mathematics and physics communities in the existing literature.
In this article, we propose the third and remaining possibility: a state equation for which the pressure changes signs. More precisely, we consider

\[ p = K_1 \ln \rho, \]

where \( K_1 \) is a key positive constant which cannot be reduced to 1 by a linear change of variables in (1). It is clear that \( p \) will be positive, zero and negative when \( \rho \) is greater than 1, equal to 1 and less than 1 respectively.

First, (4) satisfies the fundamental thermodynamic assumption that the sound speed \( \sqrt{p'\left(\rho\right)} \) is well-defined:

\[ p'\left(\rho\right) = \frac{K_1}{\rho} > 0 \]

for non-vacuum initial data.

Second, system (1) with pressure (4) shares several common mathematical properties with system (1) equipped with positive pressure (2) and negative pressure (3). To be specific, the logarithmic pressure (4) has the same asymptotic behaviours as (2) when \( \rho \) tends to infinity and (3) when \( \rho \) tends to zero. Moreover, system (1) with the logarithmic pressure (4) can be transformed to a symmetric hyperbolic system (15). It follows that finite propagation speed property for that system holds. Furthermore, it is shown in Proposition 2 that damping can prevent blowup for system (1) with the logarithmic pressure (4) and small initial data.

Third, logarithmic interaction terms like \( \log|x - y| \) are common in the interaction energy for planar electrostatics and random matrices. Expressions such as \( \rho \log \rho \) are common in information theory and the thermodynamic entropy. In some literature on the semi-geostrophic equations in meteorology [4], convexity conditions are imposed on the modified pressure. It would be significant to address which differential equations have long-term solutions and which blow up when the pressure is logarithmic, the marginal case for convexity. The case of pressure \( p = K_1 \ln \rho \) is therefore worth investigating.
2 Main Result

For linear systems, one can apply vector space analysis, by which methods in 1-dimensional case can usually be extended to general $N$-dimensional case. However, for non-linear systems, in particular, the Euler equations (1), the spatial dimension plays a key role in proving results such as global existence and finite-time blowup. For example, the singularity formation problem of the $n$-dimensional Naiver-stokes equations was solved for $n = 1$ and 2 while for $n = 3$, it is still one of the Millennium Prize Problems.

For system (1) with $s(t) = 0$ and state equation (2), the first general breakdown result for the compressible Euler equations in three spatial dimensions was achieved by Sideris [14] for adiabatic index $\gamma$ greater than 1. In [7], the author established the finite propagation speed property (F.P.S.P.) by which the followers are able to apply the integration techniques to study the blowup phenomena of system (1) when $s(t) = 0$. While the global existence result for system (1) with $s(t) = 0$ is still open, readers may refer to [1, 5, 11, 20, 7, 8, 16, 17, 19] for the blowup results of system (1) without damping.

Subsequently, Sideris et al. [15] showed system (1) with constant damping $s(t) = s > 0$ and state equation (2) still enjoys the finite propagation speed property for $N = 3$. Moreover, the authors in [15] proved the global existence result under the condition that the initial energy functional is small enough. In [3], the author extended the F.P.S.P. for system (1) with constant damping $s(t) = s > 0$ and state equation (2) from the 3-dimensional case to the general $N$-dimensional case. Consequently, Blowup results of the $N$-dimensional compressible Euler equations with constant damping $s(t) = s > 0$ are obtained.

For system (1) with $s(t) = 0$ and state equation (3), the author in [2] adopted the approach in [8] to study the blowup properties. It turned out that with different analysis of systems of ordinary differential equations, the the author applied the technique of phase diagram to obtain the long-time behaviours and blowup results of system (1) with state equation (3).

For $s(t) = \frac{\mu}{(1+t)}$, the author in [13, 12] proved the blowup and global existence results for system (1) with state equation (2) and $\lambda = 1$ in the 1-dimensional case while the authors in [6]
showed the blowup results for $\lambda \geq 1$ and global existence results for $0 \leq \lambda \leq 1$ for the same system with the same state equation in the 2 and 3-dimensional cases.

In this article, we consider system (1) with time-dependent damping

$$s(t) = \frac{\mu}{1 + t}$$

and state equations

$$p = \frac{K_1}{K_0 + 1} \rho^{K_0 + 1}, \quad \text{and}$$

$$p = K_1 \ln \rho,$$

where $K_0 \neq 0, -1$ and $K_1 > 0$ are given constants.

Without loss of generality, one may assume $K_1 = 1$ in (7) but $K_1$ is a key constant in (8). We unify the two cases by considering a general positive $K_1$.

As $K_0 + 1$ can be positive and less than 1, (7) is a generalization of (2) when $K_0 + 1 > 0$ in the isentropic case. Similarly, as $K_0 + 1$ can be less than $-1$, (7) is also a generalization of (3) when $K_0 + 1 < 0$. In conclusion, (7) is a unification and generalization of the state equations (2) and (3).

For $N = 1$, (1) becomes

$$\begin{cases} 
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2)_x + p_x + \frac{\mu}{1 + t} \rho u = 0 
\end{cases}$$

with the initial data

$$\begin{cases} 
(\rho(0, x), u(0, x)) = (\bar{\rho} + \varepsilon \rho_0(x), \varepsilon u_0(x)), \\
\text{supp}(\rho_0, u_0) \subseteq \{ x : |x| \leq R \}, 
\end{cases}$$

for some positive constants $\bar{\rho}$ and $R$. Here $\rho_0$ and $u_0 \in H^m(\mathbb{R})$, the Sobolev space with its norm

$$\|f\|_m := \sum_{k=0}^{m} \|\partial_x^k f\|_{L^p}.$$ 

**Remark 1** Without loss of generality, one may set $\bar{\rho} = 1$.

We are ready to present the main result, whose proof is given in the Section 3.
Proposition 2  Suppose \((\rho_0, u_0) \in H^m(\mathbb{R}), m \geq 3\) and \(\mu > 2\). Then, there exists a unique global classical solution \((\rho(x, t), u(x, t))\) of system (9) with state equations (7) or (8). Moreover, the following estimate holds.

\[
E^2_m(t) + L_m(t) \leq CE^2_m(0),
\]

for some constant \(C > 0\). Here, \(E_m\) and \(L_m\) are defined in (21) and (22) respectively.

3  Proof of Proposition 2

Set

\[
v := \frac{2}{K_0} \left( \frac{\sqrt{p'(\rho)}}{\rho} - \sigma \right),
\]

where

\[
\sigma := \sqrt{K_1}
\]

and \(K_0 \neq 0, -1\) and \(K_0 = -1\) correspond to the state equations (7) and (8) respectively.

Hence, (9)\(_1\) and (9)\(_2\) are transformed to

\[
\begin{aligned}
v_t + \sigma u_x &= -u v_x - K_0^2 v u_x, \\
u_t + \sigma v_x + \frac{\mu}{1 + t} u &= -u u_x - K_0^2 v v_x,
\end{aligned}
\]

with initial data

\[
(v(0, x), u(0, x)) = \varepsilon(v_0(x), u_0(x)).
\]

It follows that the system has the finite propagation speed property for such general form of pressure, (7) and (8).

From (15)\(_1\) and (15)\(_2\), one has

\[
v_{tt} - \sigma v_{xx} + \mu \frac{u}{1 + t} v_t = Q(v, u),
\]
Global Existence and Blowup

where $Q = Q_1 + Q_2 + Q_3$ with

\begin{align*}
Q_1 &= \frac{\mu}{1 + t} \left( -uv_x - \frac{K_0}{2}vu_x \right), \quad (18) \\
Q_2 &= -\partial_t \left( uv_x + \frac{K_0}{2}vu_x \right), \quad (19) \\
Q_3 &= \sigma \partial_x \left( uu_x + \frac{K_0}{2}vv_x \right). \quad (20)
\end{align*}

Define

\begin{align*}
E_m(T) := \sup_{0 < t < T} \left\{ \| (1 + t)v_t \|_{m-1}^2 + \| (1 + t)v_x \|_{m-1}^2 + \| (1 + t)u_x \|_{m-1}^2 + \|v\|_2^2 + \|u\|_2^2 \right\}^{1/2} \quad (21)
\end{align*}

and

\begin{align*}
L_m(t) := \int_0^t \left\{ (1 + \tau) \left( \| v\|_{m-1}^2 + \|v_x\|_{m-1}^2 + \|u_x\|_{m-1}^2 \right) + \frac{\|v\|_2^2}{(1 + \tau)} \right\} d\tau. \quad (22)
\end{align*}

Suppose

\begin{align*}
E_m(T) \leq M\varepsilon. \quad (23)
\end{align*}

If one can show that

\begin{align*}
E_m(T) \leq \frac{1}{2} M\varepsilon, \quad (24)
\end{align*}

the the result of global existence follows from the local existence of system (15). Here, $M \geq 1$ is independent of $\varepsilon$ and we may assume $\varepsilon$ is small and less than or equal to 1.

We divide the proof into 7 steps.

**Step 1.** Multiply \((1 + t)^2v_t\), we have

\begin{align*}
\frac{1}{2} \partial_t \left[ (1 + t)^2v_t^2 \right] + (\mu - 1)(1 + t)v_t^2 - \sigma^2(1 + t)^2v_t v_{xx} = (1 + t)^2v_t Q. \quad (25)
\end{align*}

Integrate \((25)\) over \([0, t] \times \mathbb{R}\), apply integration by parts and the F.P.S.P., we have

\begin{align*}
\frac{1}{2} \| (1 + t)v_t \|^2 + \frac{1}{2} \sigma^2 \| (1 + t)v_x \|^2 - \sigma^2 \int_0^t (1 + \tau)\| v_x \|^2 d\tau + (\mu - 1) \int_0^t (1 + \tau)\| v_r \|^2 d\tau \quad (26)
\end{align*}

\begin{align*}
= \frac{1}{2} \| v_t(0) \|^2 + \frac{1}{2} \sigma^2 \| v_x(0) \|^2 + \int_0^t \int_\mathbb{R} (1 + \tau)^2 v_x Q dx \, d\tau. \quad (27)
\end{align*}

**Step 2.** Multiply \((1 + t)v\) by \((1 + t)v\). Then, we have

\begin{align*}
\partial_t [(1 + t)vv] + \frac{\mu - 1}{2} \partial_t (v^2) + \sigma^2(1 + t)vv_{xx} - (1 + t)(v_t)^2 = (1 + t)vQ. \quad (28)
\end{align*}
Integrate (28) over $[0, t] \times \mathbb{R}$, apply integration by parts and the F.P.S.P., we have

$$
\int_{\mathbb{R}} (1 + t)v v_t dx + \frac{\mu - 1}{2} \|v\|^2 + \sigma^2 \int_{\mathbb{R}} (1 + \tau)\|v_x\|^2 d\tau - \int_{0}^{t} (1 + \tau)\|v_r\|^2 d\tau
$$

(29)

$$
= \int_{\mathbb{R}} (v v_t)(0) dx + \frac{\mu - 1}{2} \|v(0)\|^2 + \int_{0}^{t} \int_{\mathbb{R}} (1 + \tau)v Q dx d\tau.
$$

(30)

The first two terms of (27) and the second term of (30) are easy to handle as their sum is less than or equal to $CE_1^2(0)$, for some positive $C$. By the AM-GM inequality, the first term of (30) is less than or equal to $CE_1^2(0)$, for some positive $C$. In what follows, $C$, which may change from line to line, will denote a positive constant depending on $\mu$, $K_0$, $K_1$ and $\sigma$ only.

On the other hand, the first two terms of (26) and the second term of (29) should be kept as they appear in $E_2^1(T)$. Also, the fourth term of (26) and the fourth term of (29) should be kept as they appear in $L_1(t)$.

Now, we want to keep the term $\int_{0}^{t} (1 + \tau)\|v_x\|^2 d\tau$ as it appears in $L_1(t)$. Thus, it remains to cope with the first term of (29).

By AM-GM inequality,

$$
v(1 + t)v_t \geq -\frac{b}{4}v^2 - \frac{1}{b}(1 + t)^2(v_t)^2
$$

(31)

for any positive $b$. Thus,

$$
\text{the first term of (29)} \geq -\frac{b}{4}\|v\|^2 - \frac{1}{b}\|(1 + t)v_t\|^2.
$$

(32)

Thus, the first term of (32) can be absorbed by the second term of (29) if $b$ is small enough; and the second term of (32) can be absorbed by the first term of (26). However, we wish to keep the term $\int_{0}^{t} (1 + \tau)\|v_x\|^2 d\tau$. Thus, assuming $\mu > 2$, we multiply (26) by a constant $a$ and add the result to (29). In this way, one should carefully choose $a$ and $b$ to keep all the coefficients positive.

After examination,

$$
a = \frac{\mu}{2(\mu - 1)} \text{ and } b = \frac{8(\mu - 1)}{\mu - 2}
$$

(33)

will work. Thus, after $a \times (26) + (29)$ and using (32), one obtains

$$
\|(1 + t)v_t\|^2 + \|(1 + t)v_x\|^2 + \|v\|^2 + \int_{0}^{t} (1 + \tau)\|v_x\|^2 d\tau + \int_{0}^{t} (1 + \tau)\|v_r\|^2 d\tau
$$

(34)

$$
\leq CE_1^2(0) + C \left[ \int_{0}^{t} \int_{\mathbb{R}} a(1 + \tau)^2 v_x Q dx d\tau + \int_{0}^{t} \int_{\mathbb{R}} (1 + \tau)v Q dx d\tau \right],
$$

(35)
where $C$ is a positive constant independent of $M$ and $\varepsilon$.

**Step 3.** Multiply (15) by $u$, take integration over $[0, t] \times \mathbb{R}$, use integration by parts, one has

$$
\frac{1}{2} \|u\|^2 + \mu \int_0^t \frac{\|u\|^2}{1 + \tau} d\tau - \sigma \int_0^t \int_{\mathbb{R}} vu_x dx d\tau
= \frac{1}{2} \|u(0)\|^2 - \int_0^t \int_{\mathbb{R}} (u^2 u_x + \frac{K_0}{2} uv_x) dx d\tau.
$$

(36)

To handle the third term of (36), we use (15) and integration by parts to obtain

the third term of (36) = $\frac{1}{2} \|v\|^2 - \frac{1}{2} \|v(0)\|^2 + \int_0^t \int_{\mathbb{R}} uv_x dx d\tau$.

(37)

Thus, one has

$$
\|u\|^2 + \|v\|^2 + \int_0^t \frac{\|u\|^2}{1 + \tau} d\tau
\leq C \left[ E_1^2(0) + \int_0^t \int_{\mathbb{R}} \left[ -v_x^2 - (1 + K_0^2) vu_x - u^2 u_x \right] dx d\tau \right],
$$

(39)

for some positive constant $C$.

**Step 4.** Differentiate (15) with respect to $x$, multiply it by $(1 + t)^2 u_x$, integrate it over $[0, t] \times \mathbb{R}$ and apply integration by parts. One has

$$
\frac{1}{2} \|(1 + t) u_x\|^2 + (\mu - 1) \int_0^t (1 + \tau) \|u_x\|^2 d\tau - \sigma \int_0^t \int_{\mathbb{R}} (1 + \tau)^2 u_x dx d\tau
= \frac{1}{2} \|u_x(0)\|^2 + \int_0^t \int_{\mathbb{R}} (1 + \tau)^2 u_x \partial_x (-u u_x - \frac{K_0}{2} v_x) dx d\tau.
$$

(41)

By (15)1,

the third term of (H1) = $\int_0^t \int_{\mathbb{R}} (1 + \tau)^2 v_x \left[ v_{x\tau} + \partial_x (uv_x + \frac{K_0}{2} vu_x) \right] dx d\tau$

$$
= \frac{1}{2} \|(1 + t) v_x\|^2 - \frac{1}{2} \|v_x(0)\|^2 - \int_0^t (1 + \tau) \|v_x\|^2 d\tau + \int_0^t \int_{\mathbb{R}} (1 + \tau)^2 v_x \partial_x (uv_x + \frac{K_0}{2} vu_x) dx d\tau.
$$

(44)

Hence, (H1) and (H2) become

$$
\frac{1}{2} \|(1 + t) u_x\|^2 + \frac{1}{2} \|(1 + t) v_x\|^2 + (\mu - 1) \int_0^t (1 + \tau) \|u_x\|^2 d\tau - \int_0^t (1 + \tau) \|v_x\|^2 d\tau
\leq C E_1^2(0) + \int_0^t \int_{\mathbb{R}} (1 + \tau)^2 u_x \partial_x (-u u_x - \frac{K_0}{2} v_x) dx d\tau + \int_0^t \int_{\mathbb{R}} (1 + \tau)^2 v_x \partial_x (-u v_x - \frac{K_0}{2} v v_x) dx d\tau.
$$

(45)
Note that the negative term (the fourth term) of (45) can be absorbed by the forth term of (34).

Now we add the following.

\[ 2 \times (34) \text{ in Step 2} + (39) \text{ in Step 3} + (45) \text{ in Step 4} \quad (47) \]

to have

\[ E_1^2(T) + L_1(t) \leq CE_1^2(0) + C \left( |I_1| + |J_2| + |I_3| + |J_1| + |J_2| + |W_1| + |W_2| + |W_3| + |W_4| \right), \quad (48) \]

where

\[ I_1 = \int_0^t \int_R (1 + \tau)v_x \left( vu_x + \frac{K_0}{2} vu_x \right) dxd\tau, \quad (49) \]
\[ I_2 = \int_0^t \int_R v \left( vu_x + \frac{K_0}{2} vu_x \right) dxd\tau, \quad (50) \]
\[ I_3 = \int_0^t \int_R \left[ \left( 1 + \frac{K_0}{2} \right) u^2 v_x + uu_x \right] dxd\tau, \quad (51) \]
\[ J_1 = \int_0^t \int_R (1 + \tau)v \partial_x \left( vu_x + \frac{K_0}{2} vu_x \right) dxd\tau, \quad (52) \]
\[ J_2 = \int_0^t \int_R (1 + \tau)v \partial_x \left( uu_x + \frac{K_0}{2} vu_x \right) dxd\tau, \quad (53) \]
\[ W_1 = \int_0^t \int_R (1 + \tau)^2 v_x \partial_x \left( vu_x + \frac{K_0}{2} vu_x \right) dxd\tau, \quad (54) \]
\[ W_2 = \int_0^t \int_R (1 + \tau)^2 v \partial_x \left( vu_x + \frac{K_0}{2} vu_x \right) dxd\tau, \quad (55) \]
\[ W_3 = \int_0^t \int_R (1 + \tau)^2 u_x \partial_x \left( uu_x + \frac{K_0}{2} vu_x \right) dxd\tau, \quad (56) \]
\[ W_4 = \int_0^t \int_R (1 + \tau)^2 v_x \partial_x \left( uu_x + \frac{K_0}{2} vu_x \right) dxd\tau. \quad (57) \]

**Step 5.** In this step, we will handle \( I_k \) and \( J_k \).

First, by Sobolev’s inequality and (23), we have

\[ \sup_{t,x} \{ |u|, |v|, |u_x|, |v_x|, v_x|, (1 + t)|v|, (1 + t)|v_x|, (1 + t)|u_x| \} \leq CM \varepsilon. \quad (58) \]

For \( I_1 \), note that by (58) and AM-GM inequality, one has

\[ |v_x vu_x| + \frac{K_0}{2} v_x vu_x \leq CM \varepsilon (|v_x vu_x| + |v_x u_x|) \leq CM \varepsilon \left( |v_x|^2 + |v|^2 + |u_x|^2 \right). \quad (59) \]
Hence,

$$|I_1| \leq CM\varepsilon L_1(t). \quad (61)$$

For $I_2$, by integration by parts, one has

$$I_2 = \int_0^t \int_{\mathbb{R}} (1 - K_0) uv v_x dx d\tau. \quad (62)$$

As

$$|uv v_x| \leq CM\varepsilon |uv x| \leq CM\varepsilon \left( \frac{u^2}{1 + \tau} + (1 + \tau)(v_x)^2 \right), \quad (65)$$

one has

$$|I_2| \leq CM\varepsilon L_1(t). \quad (66)$$

Similarly, one also has

$$|I_3| \leq CM\varepsilon L_1(t). \quad (67)$$

For $J_1$, by integration by parts,

$$J_1 = \int_0^t \int_{\mathbb{R}} (1 + \tau) \left[ -v_x v_{\tau} u + \left( -1 + \frac{K_0}{2} \right) v v_{\tau} u_x + (1 - K_0) v v_{\tau} u_{\tau} \right] dx d\tau. \quad (68)$$

Hence,

$$|J_1| \leq CM\varepsilon L_1(t) + CM\varepsilon \int_0^t \int_{\mathbb{R}} (1 + \tau)|u_{\tau}|^2 dx d\tau. \quad (69)$$

Note that by (55) and (58),

$$|u_{\tau}| \leq CM \left( |v_x| + |u_x| + \frac{|u|}{1 + \tau} \right). \quad (70)$$

By AM-GM inequality,

$$|u_{\tau}|^2 \leq CM^2 \left[ |v_x|^2 + |u_x|^2 + \frac{|u|^2}{(1 + \tau)^2} \right]. \quad (71)$$
Hence,

\[ |J_1| \leq CM\varepsilon L_1(t) + CM^3\varepsilon L_1(t) \]  
\[ \leq CM^3\varepsilon L_1(t). \]  

(72) \hspace{1cm} (73)

For \( J_2 \), by integration by parts,

\[ J_2 = \int_0^t \int_R (1 + \tau) \left[ -uu_xv_x - \frac{K_0}{2} v(v_x)^2 \right] dxd\tau. \]  

(74)

Thus, one has

\[ |J_2| \leq CM^3\varepsilon L_1(t). \]  

(75)

**Step 6.** In this step, we shall handle \( W_k \).

First, by integration by parts, one has

\[ W_1 = \int_0^t \int_R (1 + \tau)^2 \left[ \left( \frac{K_0}{2} - \frac{1}{2} \right) u_x (v_x)^2 + u_x v_x v_x \right] dxd\tau \]  
\[ + \frac{K_0}{2} \int_0^t \int_R (1 + \tau)^2 vv_x u_x dxd\tau, \]  

(76) \hspace{1cm} (77)

\[ W_2 = \int_0^t \int_R (1 + \tau)^2 \left[ v_x (u_x)^2 - \frac{K_0}{2} vv_x v_x \right] dxd\tau \]  
\[ + \frac{1}{2} \int_0^t \int_R (1 + \tau)^2 uu_x v_x dxd\tau, \]  

(78) \hspace{1cm} (79)

\[ W_3 = \int_0^t \int_R (1 + \tau)^2 \left[ \frac{1}{2} u_x (u_x)^2 \right] dxd\tau \]  
\[ - \frac{K_0}{2} \int_0^t \int_R (1 + \tau)^2 vv_x u_x dxd\tau, \]  

(80) \hspace{1cm} (81)

\[ W_4 = \int_0^t \int_R (1 + \tau)^2 \left[ \left( \frac{K_0}{2} + \frac{1}{2} \right) u_x (v_x)^2 \right] dxd\tau \]  
\[ + \frac{K_0}{2} \int_0^t \int_R (1 + \tau)^2 vv_x u_x dxd\tau. \]  

(82) \hspace{1cm} (83)

Second, the right hand sides of (76), (78), (80) and (82) are easy to handle as they only contain at most first order derivatives. The exception is the term that contains \( v_x v_x \) in (78), which will be handled as follows.

Note that

\[ vv_x v_x = \frac{1}{2} v^2 \partial_x \left[ (v_x)^2 \right]. \]  

(84)
Hence, the second order derivative will be reduced to first order derivative when integration by parts is applied. More precisely,

\[-\frac{K_0}{2} \int_0^t \int_R (1 + \tau)^2 vv_x v_{x\tau} dxd\tau = -\frac{K_0}{2} \int_R v(1+t)^2(v_x)^2 dx + \frac{K_0}{2} \int_R v(0)(v_x(0))^2 dx \]

\[+ \frac{K_0}{2} \int_0^t \int_R (1 + \tau)(v_x)^2 [(1 + \tau)v_{\tau} + 2v] dxd\tau \]

\[\leq CM\varepsilon \|(1 + t)v_x\|^2 + CM\varepsilon E_1^2(0) + CM\varepsilon L_1(t). \] (87)

Hence,

Right hand side of (78) \leq CM\varepsilon \|(1 + t)v_x\|^2 + CM\varepsilon E_1^2(0) + CM\varepsilon L_1(t). \] (88)

Similarly, we have

Right hand side of (79) \leq CM\varepsilon L_1(t), \] (89)

Right hand side of (80) \leq CM\varepsilon L_1(t), \] (90)

Right hand side of (82) \leq CM\varepsilon L_1(t). \] (91)

It remains to handle (77), (79), (81) and (83). The method is similar to that one in (84). To be more specific, whenever \(v_x v_{xx}, v_{\tau} v_{\tau\tau}, v_x v_{\tau\tau}, v_x v_{x\tau}, u_x u_{xx}, u_{\tau} u_{\tau\tau}, u_{\tau} u_{x\tau}\) or \(u_x u_{x\tau}\) occur, one may apply integration by parts to reduce them to first order derivatives. Note that by (13), one has

\[u_{x\tau} = \frac{-uv_{x\tau} - uv_{\tau} - u_{\tau} v_x}{\sigma + \frac{K_0}{2} v} + \frac{K_0}{2} \frac{(v_\tau)^2 + uv_x v_{\tau}}{(\sigma + \frac{K_0}{2} v)^2} \] (92)

and

\[u_{xx} = \frac{-uv_{xx} - uv_{x\tau} - u_{x} v_x}{\sigma + \frac{K_0}{2} v} + \frac{K_0}{2} \frac{u(v_x)^2 + v_x v_{\tau}}{(\sigma + \frac{K_0}{2} v)^2}. \] (93)

To handle the denominator \((\sigma + \frac{K_0}{2} v)\), note that

\[\sigma + \frac{K_0}{2} v \geq \sigma(1 - C_1M\varepsilon) > 0 \] (94)
for some positive constant $C_1$. Hence,

\[
(77) \leq \frac{CM\varepsilon}{1-C_1M\varepsilon} \|v_t\|^2 + \frac{CM\varepsilon}{1-C_1M\varepsilon} E^2_1(0) + \frac{CM^3\varepsilon}{(1-C_1M\varepsilon)^2} L_1(t) \tag{95}
\]

\[
(78) \leq \frac{CM^2\varepsilon^2}{1-C_1M\varepsilon} \|v_x\|^2 + \frac{CM^2\varepsilon^2}{1-C_1M\varepsilon} E^2_1(0) + \frac{CM^3\varepsilon}{(1-C_1M\varepsilon)^2} L_1(t) \tag{96}
\]

\[
(81) \leq \frac{CM\varepsilon}{1-C_1M\varepsilon} \|v_x\|^2 + \frac{CM\varepsilon}{1-C_1M\varepsilon} E^2_1(0) + \frac{CM^3\varepsilon}{(1-C_1M\varepsilon)^2} L_1(t) \tag{97}
\]

\[
(83) \leq \frac{CM\varepsilon}{1-C_1M\varepsilon} \|v_x\|^2 + \frac{CM\varepsilon}{1-C_1M\varepsilon} E^2_1(0) + \frac{CM^3\varepsilon}{(1-C_1M\varepsilon)^2} L_1(t). \tag{98}
\]

**Final Step.** Now, (48) becomes

\[
E^2_2(t) + L_1(t) \leq \frac{C(1-C_1M\varepsilon + M^2\varepsilon)}{1-C_1M\varepsilon} E^2_1(0) + \frac{CM^3\varepsilon}{(1-C_1M\varepsilon)^2} E^2_1(t) + \frac{CM^3\varepsilon}{(1-C_1M\varepsilon)^2} L_1(t). \tag{99}
\]

In general, one has

\[
E^2_m(t) + L_m(t) \leq \frac{C(1-C_1M\varepsilon + M^2\varepsilon)}{1-C_1M\varepsilon} E^2_m(0) + \frac{CM^3\varepsilon}{(1-C_1M\varepsilon)^2} E^2_m(t) + \frac{CM^3\varepsilon}{(1-C_1M\varepsilon)^2} L_m(t). \tag{100}
\]

Hence,

\[
E^2_m(t) + L_m(t) \leq C_2 E^2_m(0), \tag{101}
\]

where

\[
C_2 = \frac{C(1-C_1M\varepsilon)(1-C_1M\varepsilon + M^2\varepsilon)}{(1-C_1M\varepsilon)^2 - CM^3\varepsilon}. \tag{102}
\]

There exists an $M_0 \geq 1$ such that for any given $M \geq M_0$, one may choose an $\varepsilon_0 < \frac{1}{C_1M}$ such that $C_2 \leq C_1\varepsilon_0 + 1$. Given $E^2_m(0) \leq C_3\varepsilon^2$, set $\frac{M^2}{4} := \max\{\frac{M^2}{4}, (C_1\varepsilon + 1)C_3\}$, then

\[
E^2_m(t) \leq \frac{1}{4} M^2\varepsilon^2. \tag{103}
\]

The proof is completed.

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