Geometry of halo and Lissajous orbits in the circular restricted three-body problem with drag forces

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ABSTRACT
In this paper, we determine the effect of radiation pressure, Poynting-Robertson drag and solar wind drag on the Sun-(Earth-Moon) restricted three body problem. Here, we take the larger body of the Sun as a larger primary, and Earth+Moon as a smaller primary. With the help of the perturbation technique, we find the Lagrangian points, and see that the collinear points deviate from the axis joining the primaries, whereas the triangular points remain unchanged in their configuration. We also find that Lagrangian points move towards the Sun when radiation pressure increases. We have also analysed the stability of the triangular equilibrium points and have found that they are unstable because of the drag forces. Moreover, we have computed the halo orbits in the third-order approximation using Lindstedt-Poincaré method and have found the effect of the drag forces. According to this prevalence, the Sun-(Earth-Moon) model is used to design the trajectory for spacecraft traveling under the drag forces.

Key words: celestial mechanics solar wind planets and satellites: dynamical evolution and stability.

1 INTRODUCTION
During the last few years, many researchers have studied the effect of drag forces because of the significant role they have in the dynamical system. For example, Murray (1994) studied the dynamical effects of drag force in the circular restricted three body problem and found the approximate location and stability properties of the Lagrangian points. Liou et al. (1995) examined the effects of radiation pressure, Poynting Robertson (P-R) drag and solar wind drag on dust grains trapped in mean motion resonance with the Sun-Jupiter restricted three-body problem. Ishwar & Kushvah (2006) studied the linear stability of triangular equilibrium points in the generalized photogravitational restricted three body problem with P-R drag and found that the triangular equilibrium points are unstable. Also, Kushvah (2008) determined the effect of radiation pressure on the equilibrium points in the generalized photogravitational restricted three body problem, and noticed that the collinear points deviate from the axis joining the two primaries, whereas the triangular points are not symmetric because of the presence of radiation pressure. Moreover, Kumari & Kushvah (2013) studied the motion of the infinitesimal mass in the restricted four body problem with solar wind drag and found the range of radiation factor of the equilibrium points.

We know that the Lagrangian points are important for mission design and transfer of trajectories. A number of missions have been successfully operated in the vicinity of the Sun-Earth and Earth-Moon collinear Lagrangian points. In this regard, the International Sun-Earth Explorer (ISEE), program was established as a joint project of the National Aeronautics and Space Administration (NASA) and the European Space Agency (ESA). ISEE-3 was launched into a halo orbit around the Sun-Earth L1 point in 1978, allowing it to collect data on solar wind conditions upstream from the Earth. Farquhar et al. (1977) had designed the ISEE-3 scientific satellite in the vicinity of the Sun-Earth interior Lagrangian point to continuously monitor the space between the Sun and the Earth. WIND was launched on 1994 November 1 and was positioned in a sunward orbit. The Solar and Heliospheric Observatory (SOHO) project was launched in 1995 December to study the internal structure of the Sun. The Advanced Composition Explorer (ACE) was launched in 1997 and orbits the L1 Lagrangian point, which is a point of the Sun-Earth gravitational equilibrium about 1.5 million km from the Earth and 148.5 million km from the Sun. ARTEMIS was the first spacecraft to be in the vicinity of Earth-Moon Lagrangian point. In 2010 August, the ARTEMIS P1 spacecraft entered an orbit near the Earth-Moon L2 point for approximately 131 d, before transferring to an L1 quasi-halo orbit where it remained for an additional 85 d. On 2011 July 17, the ARTEMIS P2 spacecraft was successfully inserted into the Earth-Moon Lagrangian point orbit with an arrival near the L2 point in 2010 October.
point orbit design features quasi-halo orbits demonstrated recently by [Folta et al. 2013].

To our knowledge, there have been numerous published papers that have extensively covered topics related to halo orbits, Lagrangian point satellite operations in general, and their applications within the Earth-Moon and the Sun-Earth system. For more information on halo orbits, we refer to the three-dimensional periodic halo orbits near the collinear Lagrangian points in the restricted three-body problem obtained by [Howell 1984]. She obtained orbits that increase in size when increasing the mass parameter $\mu$. [Clarke 2003] discussed a discovery mission concept that utilize occultations from a lunar halo orbit by the Moon to enable detection of terrestrial planets. [Breakwell & Brown 1973] has computed halo orbits around the Earth-Moon $L_2$ point. [Calleja et al. 2012] computed the unstable manifolds of selected vertical and halo orbits, which in several cases have led to the detection of heteroclinic connections from such a periodic orbit to invariant tori. Other authors have carried out similar work [Di Giamberardino & Monaco 1992; Farquhar et al. 2001; Kim & Hall 2001; Junge et al. 2002; Kolemen et al. 2003; Hill & Born 2008]. However, they ignored the drag force, although, Eapen & Sharma (2014) discussed a discovery mission concept that utilize occultations from a lunar halo orbit by the Moon to enable detection of terrestrial planets. [Breakwell & Brown 1973] has computed halo orbits around the Earth-Moon $L_2$ point. [Calleja et al. 2012] computed the unstable manifolds of selected vertical and halo orbits, which in several cases have led to the detection of heteroclinic connections from such a periodic orbit to invariant tori. Other authors have carried out similar work [Di Giamberardino & Monaco 1992; Farquhar et al. 2001; Kim & Hall 2001; Junge et al. 2002; Kolemen et al. 2003; Hill & Born 2008]. However, they ignored the drag force, although, Eapen & Sharma (2014) studied the halo orbits at the Sun-Mars $L_1$ Lagrangian point in the photogravitational restricted three-body problem and have found that as the radiation pressure increases, the transition from Mars-centric to heliocentric path is delayed.

In this paper, we study the effect of radiation pressure, P-R drag, and solar wind drag on the Lagrangian points and use the Lindstedt-Poincaré method to compute halo orbits in vicinity of the $L_1$ point of the Sun-Earth-Moon system. This paper is organized as follows. In Section 2, we recall some well-known facts about the circular restricted three-body problem with drag forces (i.e., its equations of motion, equilibrium points, and stability). In Section 3, we describe the motion near the Lagrangian point $L_1$, and use the Lindstedt-Poincaré method to compute the halo orbits. In Section 4, we discuss our results. Finally, in Appendix A, we provide all the coefficients.

## 2 MATHEMATICAL FORMULATION OF THE PROBLEM AND EQUATIONS OF MOTION

We formulate the Sun-Earth-Moon system with the radiation pressure, P-R drag and solar wind drag [Parker 1965] and proceed with the problem as follows [Liu et al. 1993]. The equations of motion in the rotating reference frame are

\[ \ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} + (1 + sw)F_x, \]
\[ \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} + (1 + sw)F_y, \]
\[ \ddot{z} = \frac{\partial U}{\partial z} + (1 + sw)F_z, \]

where

\[ U = \frac{(1 - \beta)(1 - \mu)}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}(x^2 + y^2), \]

and drag force components are

\[ F_x = -\frac{\beta(1 - \mu)}{cr_1^2} \left\{ \left[ (x + \mu)\dot{x} + (y - \mu)y + z\dot{z} \right] (x + \mu) + (\dot{x} - y) \right\}, \]
\[ F_y = -\frac{\beta(1 - \mu)}{cr_1^2} \left\{ \left[ (x + \mu)\dot{y} + (y - \mu)y + z\dot{z} \right] y + (\dot{y} + x) \right\}, \]
\[ F_z = -\frac{\beta(1 - \mu)}{cr_1^2} \left\{ \left[ (x + \mu)\dot{z} + (y - \mu)y + z\dot{z} \right] z + \dot{z} \right\}. \]

It is supposed that $m_1$ is the mass of the Sun, and $m_2$ is the mass of the Earth plus Moon, and hence the mass parameter $\mu = m_2/(m_1 + m_2)$. However, the definition of mass parameter $\mu$ is different from that of [Liu et al. 1993]. They have used $\mu_1$ and $\mu_2$ to represent the masses of the Sun and Jupiter, respectively, whereas in the problem the masses are represented by $m_1$ and $m_2$. Here, $\beta$ is the ratio of radiation pressure force to the solar gravitation force, $sw$ is the ratio of solar wind drag to P-R drag [Barnes et al. 1974; Gustafson 1994], and $c$ is the unitless speed of light. The unit of mass is taken in such a way that $G(m_1 + m_2) = 1$; the unit of distance is taken as the distance of the center of mass (of the Earth-Moon) to the Sun, whereas the unit of time is taken to be the time period of the rotating frame.

In order to calculate the Lagrangian equilibrium points, we solve equations (1)-(5) with the condition that all derivatives are zero, and we obtain

\[ x = \frac{(1 - \beta)(1 - \mu)}{r_1^2} (x + \mu) - \frac{\mu}{r_2^2} (x + \mu - 1), \]
\[ + \left(1 + sw\right)\frac{\beta(1 - \mu)}{r_1^2} \left[ \frac{\mu y(x + \mu)}{r_2^2} + \frac{y}{c} \right] = 0, \]
\[ y = \frac{1}{\frac{1 - \beta(1 - \mu)}{r_1^2} - \frac{\mu}{r_2^2} + \left(1 + sw\right)\beta(1 - \mu)} \times \left[ \frac{\mu y^2}{x} - \frac{x}{c} \right] = 0, \]
\[ z = \frac{1 - \beta(1 - \mu)}{r_1^2} \frac{\mu}{r_2^2} - \frac{(1 + sw)\beta(1 - \mu)\mu y}{cr_1^2} = 0. \]

From equation (10), we obtain two possible solutions, either $z = 0$ or $z \neq 0$. If $z \neq 0$ then

\[ \frac{(1 - \beta)(1 - \mu)}{r_1^2} + \frac{\mu}{r_2^2} = \frac{(1 + sw)\beta(1 - \mu)\mu y}{cr_1^2}. \]

From equation (11), with $\beta = 0$, (i.e., no radiation pressure is taken into account), we obtain

\[ \frac{1 - \mu}{r_1^2} = -\frac{\mu}{r_2^2}. \]

From the right-hand side of equation (12), because $\mu > 0$, and also $r_2$ is the distance of the infinitesimal body from the second primary, we find that

\[ \frac{1 - \mu}{r_1^2} = -\frac{\mu}{r_2^2} < 0. \]

This gives $\mu > 1$, which is never possible. Therefore, at $\beta = 0$, there are no equilibrium points outside the $xy$-plane. Again, with $z \neq 0$ and if $0 < \beta < 1$, then it is obvious
from equation (11) that the only possible solution is to have \( y > 0 \). From equation (9) and (11), we obtain

\[
y - \frac{(1 + sw)\beta(1 - \mu)x}{cr^2} = 0.
\] (14)

With the condition that \( y > 0 \), equation (14) gives one possible solution \( x > 0 \), and therefore \( x + \mu > 0 \). Now, we divide both sides of equation (8) by \( x + \mu \), and using equation (11), we obtain

\[
\frac{x}{x + \mu} + \frac{\mu}{r^2(x + \mu)} + \frac{(1 + sw)\beta(1 - \mu)y}{cr^2(x + \mu)} = 0.
\] (15)

For \( x > 0 \) and \( y > 0 \), the left-hand side of equation (15) has a non-zero quantity. Therefore, \( z \) must be zero and have no equilibrium points outside the \( xy \)-plane.

Simmons et al. (1983) have shown that out-of-plane equilibrium points occur if \( (1 - \beta_1)/(1 - \beta_2) < 0 \), where \( \beta_{1,2} \) are the ratio of the magnitudes of radiation to the gravitational forces from \( m_{1,2} \). However, in the present Sun-Earth-Moon system, only the Sun is a radiating body. Therefore, \( \beta_2 = 0 \) and \( \beta_1 = \beta > 1 \), which is not possible. Consequently, out-of-plane equilibrium points do not exist in the present model.

Thus, for \( z = 0 \), we solve equations (8) and (9) using Taylor series expansions in \( r_1 \) and \( r_2 \) and some approximations, as in Murray & Dermott (1999). The general solutions are given by

\[
x^* = x_0 + \Delta x, \quad y^* = y_0 + \Delta y.
\] (16)

Here, \( \Delta x \) and \( \Delta y \) are the small quantities that are introduced for drag forces and \( (x_0, y_0) \) is a solution of the equations when there is no drag force. The corresponding distances of the infinitesimal body to the masses \( m_1 \) and \( m_2 \) are given by

\[
r_1^* = \sqrt{(x^* + \mu)^2 + y^2 + z^2},
\] (17)

and

\[
r_2^* = \sqrt{(x^* + \mu - 1)^2 + y^2 + z^2}.
\] (18)

We use the Taylor series expansions around \( (x_0, y_0, z_0) \) and neglect second- and higher-order terms in \( \Delta x \) and \( \Delta y \). Then, we solve the simultaneous equations in \( \Delta x \) and \( \Delta y \). We obtain following expressions of \( \Delta x \) and \( \Delta y \) for all the Lagrangian points. We obtain following expressions of \( \Delta x \) and \( \Delta y \) for all the Lagrangian points with a fixed value of \( sw = 0.35 \) (Gustafson 1994):

for \( L_1 \),

\[
\Delta x = 0.494997 - \frac{2.19061}{4.42551 - \beta},
\]

\[
\Delta y = \frac{0.470726(-3.74923 	imes 10^{-9} + 4.23594 \times 10^{-10}\beta)}{(-4.42551 + \beta)(-2.97008 + \beta)},
\]

for \( L_2 \),

\[
\Delta x = \frac{0.505037(-1.23265 \times 10^{-15} + \beta)}{-4.57611 + \beta},
\]

\[
\Delta y = \frac{0.530994(-3.57762 \times 10^{-9} + 3.90903 \times 10^{-10}\beta)\beta}{(-4.57611 + \beta)(-3.03032 + \beta)},
\]

for \( L_3 \),

\[
\Delta x = \frac{0.53\beta}{1.5 - \beta},
\]

\[
\Delta y = \frac{0.500004(1.22068 \times 10^{-9} - 4.06893 \times 10^{-10}\beta)\beta}{(-1.5 + \beta)(3.34357 \times 10^{-6} + \beta)},
\]

for \( L_{4,5} \),

\[
\Delta x = \frac{0.25\beta(0.0000126643 + \beta)(3.00001 + \beta)}{(-1.5 + \beta)(5.83558 \times 10^{-6} + \beta)(3.00001 + \beta)},
\]

\[
\Delta y = \frac{0.500002(3.08248 \times 10^{-6} + 0.866021\beta)\beta}{(-1.5 + \beta)(5.83558 \times 10^{-6} + \beta)}.
\] (19)

(20)

The effect of \( \beta \) in \( \Delta x \) on the \( L_1 \) and \( L_2 \) points is shown in the top panel of Fig. 1 whereas the bottom panel is for the \( L_3 \) point. For the \( L_1 \) and \( L_2 \) points, the effect of \( \beta \) is

\[
\Delta x = \frac{0.25\beta(0.0000126643 + \beta)(3.00001 + \beta)}{(-1.5 + \beta)(5.83558 \times 10^{-6} + \beta)(3.00001 + \beta)}.
\] (19)

\[
\Delta y = \frac{0.500002(3.08248 \times 10^{-6} + 0.866021\beta)\beta}{(-1.5 + \beta)(5.83558 \times 10^{-6} + \beta)}.
\] (20)
approximately equal and $\Delta x$ is increasing negatively (i.e. both the $L_1$ and $L_2$ points tend towards the Sun with increasing radiation pressure). For the $L_3$ point, $\Delta x$ increases positively when $\beta$ increases, and therefore the Lagrangian point $L_3$ also tends towards the Sun. $\Delta y$ corresponds to the case when $L_{1,2,3}$ increases negatively, for an increasing value of $\beta$, which is shown in Fig. 3. Therefore, when the radiation pressure increases, the collinear points perturb from their collinearity and tend towards the radiating body of the Sun.

The different Lagrangian points $L_i$, $(i = 1, 2, 3)$ at different values of $\beta$ are presented in Table 1. The effect of radiation pressure in $\Delta x$ of $L_{4,5}$ is the same, as shown in Fig. 3 while Fig. 4 shows that there is a symmetrical change in $\Delta y$ of $L_4$ and $L_5$. These equilibrium points also tend towards the Sun symmetrically when $\beta$ increases. The different triangular points $L_{4,5}$ at different values of $\beta$ are shown in Table 2.

### 2.1 Stability of the equilibrium points

The location of equilibrium points do not affect our knowledge of their stability. We now consider the stability property by using the standard technique of linearizing the perturbation equations in the vicinity of an equilibrium point. Our approach is based on that of Schuerman (1980) and Murray (1994).

Consider a small displacement from the equilibrium position $(x^*, y^*)$, and let the solution for the subsequent motion be of the form $x = x^* + X$, $y = y^* + Y$, where

$$ X = X_0 e^{\lambda t}, \quad Y = Y_0 e^{\lambda t}, $$

and $X_0$, $Y_0$ and $\lambda$ are constants. Using these substitutions in equations (8) and (9), and using Taylor series expansion, the following simultaneous linear equations in $X$ and $Y$ are given as

$$
X \left[ \lambda^2 + \left( \frac{(1 - \beta)(1 - \mu)}{r_1^3} \right) \left( \frac{3(x^* + \mu)^2}{r_1^2} \right) + \frac{\mu}{r_1^2} \right] + \left( \frac{\partial F_e}{\partial x} \right)_* \right] \right] \right] + Y \left[ -2\lambda - \left( \frac{3(1 - \beta)(1 - \mu)\gamma^*(x^* + \mu)}{r_1^3} \right) \right] - \left( \frac{\partial F_e}{\partial y} \right)_* = 0,
$$

and

$$
X \left[ 2\lambda - \left( \frac{3(1 - \beta)(1 - \mu)\gamma^*(x^* + \mu)}{r_1^3} \right) - \left( \frac{3\mu y^*(x^* + \mu - 1)}{r_1^3} \right) \right] - \left( \frac{\partial F_e}{\partial x} \right)_* \right] + Y \left[ \lambda^2 \right] + \left( \frac{\partial F_e}{\partial y} \right)_* \right] = 0.
$$

where $(),$ denotes the evaluation of a partial derivative at the displaced equilibrium point. We can now rewrite equations (22) and (23) as

$$
X[\lambda^2 + a^* - d^* - 1 - (1 + sw)(\lambda K_{x,x} + K_{x,y})] + Y[-2\lambda - c^* - (1 + sw)(\lambda K_{x,y} + K_{y,y})] = 0,
$$

and

$$
X[2\lambda - c^* - (1 + sw)(\lambda K_{y,x} + K_{y,y})] + Y[\lambda^2 + a^* - b^* - 1 - (1 + sw)(\lambda K_{y,y} + K_{x,y})] = 0,
$$

where the constants are

$$
a^* = \left( \frac{(1 - \beta)(1 - \mu)}{r_1^3} \right) + \frac{\mu}{r_1^2},
$$

$$
b^* = 3 \left( \frac{(1 - \beta)(1 - \mu)}{r_1^3} \right) + \frac{\mu}{r_1^2} \left( \frac{3\mu y^2}{r_1^2} \right),
$$

$$
c^* = 3 \left( \frac{(1 - \beta)(1 - \mu)(x^* + \mu)}{r_1^3} \right) + \frac{\mu(x^* + \mu - 1)}{r_1^2},
$$

$$
d^* = 3 \left( \frac{(1 - \beta)(1 - \mu)(x^* + \mu)^2}{r_1^3} \right) + \frac{\mu(x^* + \mu - 1)^2}{r_1^2},
$$

and

$$
K_{x,x} = \left( \frac{\partial F_e}{\partial x} \right)_*, \quad K_{x,y} = \left( \frac{\partial F_e}{\partial y} \right)_*, \quad K_{y,x} = \left( \frac{\partial F_e}{\partial x} \right)_*, \quad K_{y,y} = \left( \frac{\partial F_e}{\partial y} \right)_*.
$$

The condition for determinant of the linear equations defined by equations (24) and (25) needs to be zero. Neglecting the terms of $O(K^2)$, we obtain the characteristic equation
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Table 1. Lagrangian equilibrium points $L_i$ ($i = 1, 2, 3$) with $s_w = 0.35$.

| $\beta$ | $L_1$ | $L_2$ | $L_3$ |
|---------|-------|-------|-------|
| 0.0     | $(0.989991, 0)$ | $(1.01007, 0)$ | $(-1.00000326495, 0)$ |
| 0.1     | $(0.978547, -1.40554 \times 10^{-11})$ | $(0.998787, -1.43251 \times 10^{-11})$ | $(-0.964289, -4.21415 \times 10^{-10})$ |
| 0.2     | $(0.966652, -2.94742 \times 10^{-11})$ | $(0.986989, -3.0005 \times 10^{-11})$ | $(-0.923908, -4.3819 \times 10^{-10})$ |
| 0.3     | $(0.953996, -4.63459 \times 10^{-11})$ | $(0.974638, -4.72137 \times 10^{-11})$ | $(-0.875903, -4.57754 \times 10^{-10})$ |
| 0.4     | $(0.940805, -6.51505 \times 10^{-11})$ | $(0.961696, -6.6153 \times 10^{-11})$ | $(-0.818185, -4.80874 \times 10^{-10})$ |
| 0.5     | $(0.926942, -8.58655 \times 10^{-11})$ | $(0.948119, -8.70629 \times 10^{-11})$ | $(-0.750004, -5.08618 \times 10^{-10})$ |
| 0.6     | $(0.912355, -1.08874 \times 10^{-10})$ | $(0.93386, -1.10221 \times 10^{-10})$ | $(-0.666671, -5.42526 \times 10^{-10})$ |
| 0.7     | $(0.896984, -1.34524 \times 10^{-10})$ | $(0.918864, -1.35961 \times 10^{-10})$ | $(-0.562504, -5.84912 \times 10^{-10})$ |
| 0.8     | $(0.880766, -1.63235 \times 10^{-10})$ | $(0.903074, -1.64679 \times 10^{-10})$ | $(-0.428576, -6.39407 \times 10^{-10})$ |
| 0.9     | $(0.863627, -1.95512 \times 10^{-10})$ | $(0.886425, -1.96851 \times 10^{-10})$ | $(-0.250006, -7.12067 \times 10^{-10})$ |
| 1.0     | $(0.845488, -2.31971 \times 10^{-10})$ | $(0.868845, -2.33054 \times 10^{-10})$ | $(-7.51792 \times 10^{-6}, -8.13791 \times 10^{-10})$ |

Table 2. Triangular equilibrium points with $s_w = 0.35$

| $\beta$ | $L_{(4, 5)}$ |
|---------|-------------|
| 0.0     | $(0.499997, \pm 0.866025)$ |
| 0.1     | $(0.482139, \pm 0.835960)$ |
| 0.2     | $(0.461534, \pm 0.799408)$ |
| 0.3     | $(0.437496, \pm 0.757773)$ |
| 0.4     | $(0.409086, \pm 0.708567)$ |
| 0.5     | $(0.374995, \pm 0.64952)$ |
| 0.6     | $(0.333329, \pm 0.577351)$ |
| 0.7     | $(0.281245, \pm 0.487141)$ |
| 0.8     | $(0.214281, \pm 0.371155)$ |
| 0.9     | $(0.124995, \pm 0.216599)$ |
| 1.0     | $(-5.81166 \times 10^{-6}, \pm 3.08134 \times 10^{-6})$ |

\[ \lambda^4 + a_3 \lambda^3 + (a_{20} + a_2) \lambda^2 + a_1 \lambda + (a_{00} + a_0) = 0, \tag{34} \]

where the approximate expressions for constants $a_{00}$ and $a_{20}$, and the drag force terms $a_i$ ($i = 0, 1, 2, 3$) are given in Appendix A.

If we take $K = 0$, then the characteristic equation reduces to the following form

\[ \lambda^4 + a_{20} \lambda^2 + a_{00} = 0, \tag{35} \]

which gives the classical solutions. The four roots in the classical problem occur as real and pure imaginary pair, in the case of each $L_i$, ($i = 1, 2, 3$), whereas two pure imaginary pairs occur in the case of $L_4$ and $L_5$. The stability of the given equilibrium point depends on the sign and value of the roots of the characteristic equation. If any of the roots has a positive real part, the motion is unstable to small displacements with exponential growth, whereas in classical case, the $L_4$ and $L_5$ points are linearly stable to small displacements in the system.

If we restrict our analysis to $L_4$ and $L_5$, the four roots of the characteristic equation, without drag forces, are given as

\[ \lambda_n = \pm Z_i, \quad (n = 1, 2, 3, 4) \tag{36} \]

where

\[ Z = \sqrt{a_{20} \pm \sqrt{a_{20}^2 - 4a_{00}}}. \tag{37} \]

In the presence of drag forces, we assume that the roots are in the following form

\[ \lambda = \eta Z \pm i(1 + \gamma) Z, \tag{38} \]

that is,

\[ \lambda_{1,2} = \eta Z + (1 + \gamma) Z_i, \tag{39} \]

\[ \lambda_{3,4} = \eta Z - (1 + \gamma) Z_i, \tag{40} \]

where $\gamma$ and $\eta$ are small real quantities. With the help of equations (34), (39) and (40), neglecting the product of $\gamma$ and then $\eta$ with $a_i$, and solving the real and imaginary parts, we obtain

\[ \eta = \frac{(a_3 Z^2 - a_1)}{2Z(-2Z^2 + a_{20})}, \tag{41} \]

and

\[ \gamma = \frac{(a_{00} + a_0) - (a_{20} + a_2) Z^2 + Z^4}{2Z^2(a_{20} - 2Z^2)}. \tag{42} \]

In all the cases, the real part of at least one characteristic root is positive. Therefore, the equilibrium point is saddle point.

3 COMPUTATION OF HALO ORBIT

In order to discuss the motion near the Lagrangian point of the system, we choose a coordinate system centred at the Lagrangian point $L_1$ in the rotating reference frame. The equations of motion of an infinitesimal body are obtained by translating the origin to the location of $L_1$. The referred
translation is given as
\[ x = X - 1 + \mu + \gamma, \quad y = Y - l, \quad \text{and} \quad z = Z. \] (43)
In this new coordinate system, the variables \( x, y \) and \( z \) are scaled. The distances between \( L_1 \) and the smaller primary are \( \gamma \) in the \( x \)-axis and \( l \) in the \( y \)-axis. Using these facts in equations (1)-(3), we obtain the equations of motion with the help of Legendre polynomials for expanding the nonlinear terms and considering the linear parts, and we can write
\[ \ddot{x} - 2\dot{y} - x\nu_{10} + y\nu_{11} = 0, \] (44)
\[ \dot{y} + 2\dot{x} - y\nu_{21} - x\nu_{20} = 0, \] (45)
\[ \ddot{z} + \nu_* z = 0. \] (46)
All the coefficients are given in Appendix A. Here, the \( z \)-axis solution is simple harmonic, because \( \nu_* > 0 \). However, the motion in the \( xy \)-plane is coupled. Solving equations (44) and (45), the characteristic equation has two real and two complex roots. The complex roots are not pure imaginary but the real parts of the complex roots are very small with respect to the age of solar system. Therefore, we neglect them and the roots are \( \pm \alpha \) and \( \pm i\lambda \), where
\[ \alpha = \pm \sqrt{\frac{-4 + \nu_{21} + \nu_{10} + \sqrt{(4 - \nu_{21} - \nu_{10})^2 - 4\zeta_1}}{2}}, \] (47)
and
\[ \lambda = \pm \sqrt{\frac{4 - (\nu_{21} + \nu_{10}) + \sqrt{(4 - \nu_{21} - \nu_{10})^2 - 4\zeta_1}}{2}}. \] (48)
All the coefficients are given in Appendix A. Because the two real roots are opposite in sign, arbitrarily chosen initial conditions give rise to unbounded solutions as time increases. If, however, the initial conditions are restricted and only a non-divergent mode is allowed, the \( xy \)-solution will be bounded. In this case, the linearized equations have solutions of the form
\[ x(t) = -A_x \cos(\lambda t + \phi), \] (49)
\[ y(t) = \kappa A_x \{(2\lambda \sin(\lambda t + \phi) - \nu_{11} \cos(\lambda t + \phi)} \}, \] (50)
\[ z(t) = A_z \sin(\sqrt{\nu_*} t + \psi), \] (51)
with
\[ \kappa = \frac{\lambda^2 + \nu_{10}}{4\lambda^2 + \nu_{11}}. \] (52)
The in-plane and out-of-plane frequencies are not equal, they are incommensurable. Then the linearized motion produces the Lissajous-type trajectories for the Sun-(Earth-Moon) system around \( L_1 \). When the radiation pressure increases, the phase difference of the trajectories decrease. When there is no drag force (i.e. \( \beta = 0 \)), the trajectory of Lissajous orbit completes one period approximately at \( t = 3.0 \), but when radiation pressure force increases (i.e. \( \beta = 0.2 \)), it does not complete its period at that time, as shown in Fig. 5. Therefore, the period increases with an increase in the value of \( \beta \). Also, because of the increasing value of \( \beta \), the trajectories shrink in its amplitude. Figs. 7, 8 and 9 show the projections in the \( xy \)-, \( xz \)- and \( yz \)-plane, respectively, with the different values of \( \beta \). Here, black, blue and red coloured orbits are shown with \( \beta = 0, 0.05 \) and 0.1, respectively. Clearly, the orbits shrink when \( \beta \) increases.

3.1 Periodic orbits using the Lindstedt-Poincaré method

The equations of motions can be written using the Legendre polynomials \( P_n \). A third-order approximation was used in the circular restricted three-body without any drag force by Richardson (1980). Here, we include P-R drag and solar wind drag, and then find the third-order approximation as described by Thurman & Worfolk (1990):
\[ \ddot{x} - \ddot{y} = x\nu_{10} - y\nu_{11} + x^2\nu_{12} + y^2\nu_{13} + z^2\nu_{14} - x^2y\nu_{15} + x^3\nu_{16} + y^3\nu_{17} + xy^2\nu_{18} + xz^2\nu_{19} + x^2y\nu_{100} + y^2z\nu_{191} + O(4), \] (53)
\[ \ddot{y} + 2\ddot{x} = x\nu_{21} + y\nu_{22} + z^2\nu_{23} + x^2\nu_{24} + x^2y\nu_{25} + x^3\nu_{26} + y^3\nu_{27} + xy^2\nu_{28} + xz^2\nu_{290} + x^2y\nu_{291} + y^2z\nu_{292} + O(4), \] (54)
\[ \ddot{z} + \nu_* z = xz\nu_{31} + yz\nu_{32} + x^2z\nu_{33} + y^2z\nu_{34} + z^3\nu_{35} + O(4). \] (55)
Geometry of halo and Lissajous orbits in the CRTBP

All the coefficients are given in Appendix A.

3.2 Correction term

For the construction of higher-order applications, we linearize equation (55) and introduce a correction term

$$\Delta = \lambda^2 - \nu_*, \quad (56)$$

on the right-hand side. The new third-order $z$ equation becomes,

$$\ddot{z} + \lambda^2 z = xz\nu_{30} + yz\nu_{31} + x^2 z\nu_{32} + xy\nu_{33} + y^2 \nu_{34} + z^2 \nu_{35} + \Delta z + O(4). \quad (57)$$

Richardson (1980) developed a third-order periodic solution using the Lindstedt-Poincaré type of successive approximations. We follow their work with the P-R drag and solar wind drag, by removing secular terms. A new independent variable $\tau$ and a frequency connection $\omega$ are introduced via, $\tau = \omega t$. Then, the equations of motion at the second degree including the P-R drag and solar wind drag are given as

$$\omega^2 x'' - 2\omega y' = x\nu_{10} - y\nu_{11} + x^2 \nu_{12} + y^2 \nu_{13} + z^2 \nu_{14} - xy\nu_{15} + x^3 \nu_{16} + y^3 \nu_{17} + x^2 y^2 \nu_{18} + xz^2 \nu_{19} + x^2 y^2 \nu_{190} + yz^2 \nu_{191} + \nu^* + O(4), \quad (58)$$
In order to avoid secular solutions, we need to constraints on the constants $A_x$, $A_z$, $\phi$ and $\psi$, but for now they are arbitrary.

### 3.4 Second-order equations

The order of $O(\epsilon^2)$ equations depend on the first order solutions for $x_1, y_1, z_1$. Collecting only non-secular terms, we obtain

\[ x''_1 - 2y'_1 - \nu_{10}x_2 + \nu_{11}y_2 = \alpha_1 \cos 2\tau_1 - \alpha_2 \cos 2\tau_2 - \alpha_3 \sin 2\tau_1 + \alpha_4, \]

\[ y''_2 + 2x'_2 - \nu_{21}y_2 - \nu_{20}x_2 = -\delta_1 \sin 2\tau_1 + \delta_2 \cos 2\tau_1 - \delta_3 \cos 2\tau_2 + \delta_4, \]

where all the coefficients are given in Appendix A.

We remove all the secular terms by setting $\omega_1 = 0$. Thus, we find the following solutions,

\[ x_2 = \rho_{10} + \rho_{11} \cos 2\tau_1 + \rho_{12} \cos 2\tau_2 + \rho_{13} \sin 2\tau_2 + \rho_{14} \sin 2\tau_1, \]

\[ y_2 = \rho_{20} + \rho_{21} \sin 2\tau_1 + \rho_{22} \sin 2\tau_2 - \rho_{23} \cos 2\tau_2 - \rho_{24} \cos 2\tau_1, \]

\[ z_2 = \rho_{30} \sin(\tau_1 + \tau_2) + \rho_{31} \sin(\tau_2 - \tau_1) + \rho_{32} \cos(\tau_2 - \tau_1) + \rho_{33} \cos(\tau_1 + \tau_2), \]

where $\tau_1 = \lambda \tau + \phi$, and $\tau_2 = \lambda \tau + \psi$.

All the coefficients are given in Appendix A.

### 3.5 Third-order equations

The $O(\epsilon^3)$ equations are obtained by setting $\omega_1 = 0$ and substituting in the solutions for $x_1, y_1, z_1, x_2, y_2$, and $z_2$. Thus we obtain

\[ x'''_1 = -2y''_1 - \nu_{10}x_2 + \nu_{11}y_2 = [\alpha_{11} + 2\omega_2 \lambda^2 A_x (2\kappa - 1)] \cos \tau_1 + [\alpha_{12} + 2\kappa \lambda A_x \nu_{11} \omega_2] \sin \tau_1 + \alpha_{13} \cos 3\tau_1 + \alpha_{14} \cos(\tau_1 + 2\tau_2) + \alpha_{15} \cos(\tau_1 - 2\tau_2) + \alpha_{16} \sin(2\tau_2 + \tau_1) + \alpha_{17} \sin(2\tau_2 - \tau_1) + \alpha_{18} \sin 3\tau_1, \]

\[ y'''_2 + 2x''_2 - \nu_{21}y_2 - \nu_{20}x_2 = [\alpha_{21} - 2\kappa \lambda^2 A_x \nu_{21} \omega_2] \cos \tau_1 + [\alpha_{22} + 2\lambda A_x \nu_{21} \omega_2 (2\kappa \lambda^2 - 1)] \sin \tau_1 + \alpha_{23} \cos 3\tau_1 + \alpha_{24} \cos(\tau_1 + 2\tau_2) + \alpha_{25} \cos(2\tau_2 - \tau_1) + \alpha_{26} \sin(2\tau_2 + \tau_1) + \alpha_{27} \sin(2\tau_2 - \tau_1) + \alpha_{28} \sin 3\tau_1. \]
where the expressions of the coefficients are given in Appendix A. There are secular terms. We start by examining the secular terms in the $z_3$ equation, by simply setting a value for the frequency correction $\omega_2$. To remove the secular terms $\alpha_{33} \sin(\tau_2 - \tau_1)$ and $\alpha_{37} \cos(\tau_2 - \tau_1)$, we need the coefficients of these terms to be zero. Thus
\[ \phi = \psi + \frac{\pi}{2}, \quad \text{where} \quad n = 0, 1, 2, 3. \]

The solution will be bounded if
\[ \alpha_{31} + A_x \left[ 2 \omega \lambda^2 + \frac{A}{\epsilon^2} \right] \sin \tau_2 + \alpha_{32} \sin(2\tau_1 - \tau_2) + \alpha_{33} \sin(\tau_2 - 2\tau_1) + \alpha_{34} \sin \tau_2 + \alpha_{35} \cos \tau_2 + \alpha_{36} \cos 3\tau_2 + \alpha_{37} \cos(2\tau_2 - \tau_1) + \alpha_{38} \cos(2\tau_1 + \tau_2), \quad (80) \]

where $\zeta = (-1)^n$. This phase constraint affects the $x_3 - y_3$ equations; now each contains a secular term. The requirement of another constraint is from the simultaneous equations \(85\) and \(86\):
\[ -(\nu_{11} + \lambda^2)[\alpha_{11} + 2 \omega \lambda^2 A_x (2\kappa - 1)] + 2\lambda[\alpha_{22} + 2 \omega \lambda^2 A_x (2\kappa - 1)] + \nu_{11} \alpha_{22} - 2 \omega \lambda^2 A_x \nu_{11} \omega_2) + \zeta[-\alpha_{15} (\lambda^2 + \nu_{21}) + \lambda(\alpha_{27} + \alpha_{11} \alpha_{25})] = 0, \quad (81) \]
and
\[ -(\nu_{22} + \lambda^2)[(\alpha_{12} + 2 \omega \lambda^2 A_x \nu_{12} \omega_2) - 2 \lambda(\alpha_{21} - 2 \omega \lambda^2 A_x \nu_{11} \omega_2) + \nu_{11} \alpha_{22} + 2 \omega \lambda^2 A_x (2\kappa - 1)] + \zeta[-\alpha_{17} (\lambda^2 + \nu_{21}) + \lambda(\alpha_{25} + \nu_{11} \alpha_{27})] = 0. \quad (82) \]

From these equations we find
\[ \omega_2 = \left[ \lambda^2 \alpha_{11} + \lambda^2 \alpha_{15} - 2 \lambda \alpha_{22} - \lambda \alpha_{27} + \alpha_{11} \nu_{11} - \alpha_{11} \nu_{11} - \zeta(\alpha_{25} \nu_{21} - \alpha_{25} \nu_{21}) \right] \left[ 2 \lambda^2 A_x \left( 2 \nu_{21} + 1 \right) + \nu_{11} - 2 \nu_{11} - \kappa \nu_{11} - 2 \right]^{-1}. \quad (83) \]

The amplitude relation is obtained by putting the value of equation \(85\) into equation \(84\). Thus, we can satisfy this constraint by letting one amplitude be determined by the other.

Different amplitude $A_x$ at $A_x = 206000/(1.496 \times 10^8)$ with $sw = 0.35$ at different $\beta$ and $n$ are shown in Table \(8\) and \(9\). From these tables, we see that the amplitudes contract, whereas $\omega_2$ decreases negatively as $\beta$ increases. Using these constraints, the third-order equations are reduced to
\[ x_3'' - 2y_3' - \nu_{10} x_3 + \nu_{11} y_3 = s_{11} \cos \tau_1 + s_{12} \sin \tau_1 + (\alpha_{13} + \zeta \alpha_{14}) \cos 3\tau_1 + (\alpha_{14} + \zeta \alpha_{15}) \sin 3\tau_1, \quad (86) \]
\[ y_3'' + 2x_3' - \nu_{20} x_3 - \nu_{21} y_3 = s_{21} \cos \tau_1 + s_{22} \sin \tau_1 + (\alpha_{23} + \zeta \alpha_{24}) \cos 3\tau_1 + (\alpha_{24} + \zeta \alpha_{25}) \sin 3\tau_1, \quad (87) \]

\[ z_3'' + \lambda^2 z_3 = \zeta(\alpha_{32} \sin 3\tau_1 + \alpha_{38} \cos 3\tau_1) + \alpha_{34} \left\{ \begin{array}{l} \left( -1 \right)^n \sin 3\tau_1, \quad n = 0, 2, \\ \left( -1 \right)^{n+1} \cos 3\tau_1, \quad n = 1, 3 \end{array} \right. \]
\[ + \alpha_{36} \left\{ \begin{array}{l} \left( -1 \right)^n \cos 3\tau_1, \quad n = 0, 2, \\ \left( -1 \right)^{n+1} \sin 3\tau_1, \quad n = 1, 3. \end{array} \right. \quad (88) \]

The solutions are
\[ x_3 = \sigma_{10} \cos \tau_1 + \sigma_{11} \sin \tau_1 + \sigma_{12} \cos 3\tau_1 + \sigma_{13} \sin 3\tau_1, \quad (89) \]
\[ y_3 = \sigma_{20} \cos \tau_1 + \sigma_{21} \sin \tau_1 + \sigma_{22} \cos 3\tau_1 + \sigma_{23} \sin 3\tau_1, \quad (90) \]
\[ z_3 = \frac{1}{8 \lambda^2} (\zeta(\alpha_{32} \sin 3\tau_1 + \alpha_{38} \cos 3\tau_1)) + \left( \begin{array}{l} \left( -1 \right)^n \sigma_{34} \sin 3\tau_1 + \alpha_{36} \cos 3\tau_1, n = 0, 2, \\ \left( -1 \right)^{n+1} \alpha_{34} \cos 3\tau_1 - \alpha_{36} \sin 3\tau_1, n = 1, 3. \end{array} \right. \quad (91) \]

where the expressions of the coefficients are given in Appendix A.

### 3.6 The final approximation

Using all the first-, second- and third-order approximate solutions with the mapping $A_x \rightarrow A_x/\epsilon$ and $A_z \rightarrow A_z/\epsilon$, which remove $\epsilon$ from all the equations, we obtain
\[ x(\tau) = \rho_{10} + (A_x + \sigma_{10}) \cos \tau_1 + \sigma_{11} \sin \tau_1 + \left( \rho_{11} + \zeta \rho_{12} \right) \cos 2\tau_1 + \left( \rho_{14} + \zeta \rho_{13} \right) \sin 2\tau_1 + \sigma_{12} \cos 3\tau_1 + \sigma_{13} \sin 3\tau_1, \quad (92) \]
\[ y(\tau) = \rho_{20} + (2 \lambda \kappa A_x + \sigma_{20}) \cos \tau_1 + \left( -\kappa A_x \nu_{11} + \sigma_{20} \right) \cos \tau_1 + \left( \rho_{21} + \zeta \rho_{22} \right) \sin 2\tau_1 - \left( \rho_{24} + \zeta \rho_{23} \right) \cos 2\tau_1 + \sigma_{22} \cos 3\tau_1 + \sigma_{23} \sin 3\tau_1, \quad (93) \]
and
\[ z(\tau) = A_z \left( \begin{array}{l} \left( -1 \right)^{n+1} (A_z \sin \tau_1 + \rho_{30} \sin 2\tau_1), n = 0, 2, \\ \left( -1 \right)^n (A_z \cos \tau_1 + \rho_{30} \cos 2\tau_1), n = 1, 3 \end{array} \right. + \rho_{31} \left( \begin{array}{l} \left( -1 \right)^n \sin 3\tau_1, n = 0, 2, \\ \left( -1 \right)^{n+1} \cos 3\tau_1, n = 1, 3 \end{array} \right. \frac{1}{8 \lambda^2} (\zeta(\alpha_{32} \sin 3\tau_1 + \alpha_{38} \cos 3\tau_1)) + \left( \begin{array}{l} \left( -1 \right)^n \sigma_{34} \sin 3\tau_1 + \alpha_{36} \cos 3\tau_1, n = 0, 2, \\ \left( -1 \right)^{n+1} \alpha_{34} \cos 3\tau_1 - \alpha_{36} \sin 3\tau_1, n = 1, 3. \end{array} \right. \quad (94) \]

The expressions of the coefficients are given in Appendix A.

For the Sun-(Earth-Moon) $L_1$, the difference of both frequencies of the $z-$plane and $xy$-plane is quite small. The value of $A_x$ and $A_z$ are found using the relation of equations \(89\) and \(91\).

In the final approximation, the halo orbits at $n = 0$ and $2$ with $sw = 0.35$ are shown in Figs. \(10\) \(12\) and \(14\). For $n = 1$ and $3$, the halo orbits are depicted in Figs. \(16\) \(18\) and \(20\). The amplitude and frequency $\omega_2$ are both equal for each
Table 3. At $\beta = 0.13$

| $n$ | $A_z$ | $\omega_2$ |
|-----|-------|------------|
| 0   | $\pm 0.4292186$ | $-24.7175$ |
| 1   | $0.1120075$    | $-4.66144$ |
| 2   | $0.4292186$    | $-24.7175$ |
| 3   | $0.1120075$    | $-4.66144$ |

Table 4. At $\beta = 0.15$

| $n$ | $A_z$ | $\omega_2$ |
|-----|-------|------------|
| 0   | $\pm 0.219213958$ | $-5.14832$ |
| 1   | $0.165566928$    | $-3.32111$ |
| 2   | $0.219213958$    | $-5.14832$ |
| 3   | $0.165566928$    | $-3.32111$ |

Figure 10. Halo orbit at $n = 0$, and $n = 2$ at $\beta = 0.13$

Figure 12. Halo orbit at $n = 0$, and $n = 2$ at $\beta = 0.15$

Figure 11. Halo orbit at $n = 1$, and $n = 3$ at $\beta = 0.13$

Figure 13. Halo orbit at $n = 1$, and $n = 3$ at $\beta = 0.15$

4 CONCLUSIONS

In this paper, we have studied the circular restricted three-body problem of the Sun-Earth-Moon system by assuming the effect of radiation pressure, P-R drag and solar wind drag. We have found that the collinear Lagrangian points deviate from their axis joining the primaries, whereas the triangular points remain unchanged in their configuration. However, all points lie in a plane. If we increase the value of $\beta$, with a fixed value of $sw = 0.35$, the Lagrangian points $L_1$, $L_2$ and $L_3$ tend towards the radiating body (the Sun), whereas $L_{4,5}$ have symmetrical changes with the increasing case, $n = 0$, 2 and $n = 1$, 3. When $\beta$ increases, then the trajectory shrinks.

In the previous subsections, we have computed the first, second, third and final approximations for the halo orbits and we have seen the effect of radiation pressure, with P-R drag and solar wind drag. In the absence of drag forces, the results agree with those of the classical case (Thurman & Worfolk, 1996).
Table 5. At $\beta = 0.18$

| $n$ | $A_x$ | $\omega_x$ |
|-----|-------|------------|
| 0   | $\pm 0.09654386$ | -0.718982 |
| 1   | $\pm 0.05566177$ | -0.691666 |
| 2   | $\pm 0.09654386$ | -0.718982 |
| 3   | $\pm 0.05566177$ | -0.691666 |

Figure 14. Halo orbit at $n = 0$, and $n = 2$ at $\beta = 0.18$

Figure 15. Halo orbit at $n = 1$, and $n = 3$ at $\beta = 0.18$

value of $\beta$. We have examined the linear stability of the equilibrium points with the help of characteristic roots. It is observed that the Lagrangian points are unstable because of the drag forces. Further, we have computed the orbit around the $L_1$ point and have seen that when radiation pressure increases, the phase difference of the trajectory decreases. If there is no drag force (i.e., $\beta = 0$), then the trajectory of the Lissajous orbit completes one period approximately at $t = 3$.0, whereas if $\beta$ increases with $sw = 0.35$, it does not complete its period at the same time. Also, because of the increasing value of $\beta$, the trajectories shrink in its amplitude. In this study, we have used the Lindstedt-Poincaré method to compute the halo orbits in the third-order approximation with the radiation pressure, P-R drag and solar wind drag. In this analysis, we have fixed the value of $sw = 0.35$, which is the ratio of of solar wind drag to P-R drag, and we have varied the value of $\beta$, (i.e. the ratio of radiation pressure force to solar gravitation force). This model can be used to compute the higher-order approximation (four, fifth, etc.) expressions for halo orbits. Moreover, stable unstable manifolds of the halo orbits, and trajectory transfer would be interesting topics of future research with the similar dissipative forces.

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$\nu_{11} = \frac{3l}{D_f} \left[ \frac{c_2(\gamma - 1) + d_2\gamma}{D_f^2} + \frac{(1 + sw)\beta(1 - \mu)}{D_f} \left[ \mu(\gamma - 1) - \frac{1}{D_f^2} \right] \right]$ (103)

$\nu_{12} = \frac{15}{2} \left[ \frac{c_2(\gamma - 1)^3}{D_f^2} + \frac{d_2\gamma^2}{D_f^2} \right] - \frac{3}{2} \left[ \frac{c_2(\gamma - 1)^3}{D_f^2} + \frac{d_2\gamma^2}{D_f^2} \right] - \frac{9}{2} \left[ \frac{c_2(\gamma - 1)}{D_f} + \frac{d_2\gamma}{D_f} + \frac{(1 + sw)\beta(1 - \mu)}{c} \right] \frac{6\mu(\gamma - 1)l + 4l(\gamma - 1)}{D_f^2} \frac{l}{D_f} \frac{12\mu(\gamma - 1)^3}{D_f^8}$ (104)

$\nu_{13} = \frac{3}{2} \left[ \frac{c_2(\gamma - 1)}{D_f} + \frac{d_2\gamma}{D_f} \right] + \left[ \frac{4\mu(\gamma - 1)l}{D_f} \right] - \frac{2l}{D_f}$ (105)

$\nu_{14} = \frac{3}{2} \left[ \frac{c_2(\gamma - 1)}{D_f} + \frac{d_2\gamma}{D_f} \right] + \frac{(1 + sw)\beta(1 - \mu)}{c} \frac{2\mu(\gamma - 1)}{D_f^2} \frac{l}{D_f} \frac{1}{D_f}$ (106)

$\nu_{15} = \frac{15}{2} \left[ \frac{c_2(\gamma - 1)^2}{D_f^2} + \frac{d_2\gamma^2}{D_f^2} \right] - \frac{3}{2} \left[ \frac{c_2(\gamma - 1)^2}{D_f^2} + \frac{d_2\gamma^2}{D_f^2} \right] - \frac{9}{2} \left[ \frac{c_2(\gamma - 1)}{D_f} + \frac{d_2\gamma}{D_f} + \frac{(1 + sw)\beta(1 - \mu)}{c} \right] \frac{4\mu(\gamma - 1)^2}{D_f^2} \frac{\mu + 2\gamma(\gamma - 1)}{D_f}$ (107)

$\nu_{16} = -\frac{15}{2} \left[ \frac{c_2(\gamma - 1)^2}{D_f^2} + \frac{d_2\gamma^2}{D_f^2} \right] + \frac{35}{2} \left[ \frac{c_2(\gamma - 1)^4 + d_2(\gamma - 1)^2}{D_f^2} \right] + \frac{3}{2} (c_4 + d_4) + \frac{(1 + sw)\beta(1 - \mu)}{c} \frac{24\mu(\gamma - 1)^2 + 8\gamma(\gamma - 1)^3}{D_f^2}$ (108)

$\nu_{17} = \frac{15}{2} \left[ \frac{c_2(\gamma - 1)^2}{D_f^2} + \frac{d_2\gamma^2}{D_f^2} \right] + \frac{35}{2} \left[ \frac{c_2(\gamma - 1)^4}{D_f^2} + \frac{d_2(\gamma - 1)^2}{D_f^2} \right] + \frac{3}{2} (c_4 + d_4) + \frac{(1 + sw)\beta(1 - \mu)}{c} \frac{28((\gamma - 1)^4 + \mu)}{D_f^{10}} - \frac{2\mu + 4(\gamma - 1)}{D_f^{10}}$ (109)

$\nu_{18} = -\frac{15}{2} \left[ \frac{c_2(\gamma - 1)^2 + d_2\gamma^2}{D_f^2} \right] + \frac{3}{2} (c_4 + d_4) + \frac{3\mu + 2\gamma(\gamma - 1)}{D_f^2}$ (110)

$\nu_{19} = \frac{3}{2} \left[ \frac{c_2(\gamma - 1)^2 + d_2\gamma^2}{D_f^2} \right] + \frac{3}{2} (c_4 + d_4) + \frac{(1 + sw)\beta(1 - \mu)}{c} \frac{-6\mu - 12\gamma(\gamma - 1)}{D_f^2} + \frac{(24\mu - 12\mu)(\gamma - 1)^2}{D_f^2}$ (111)
\[\nu^* = 1 - \mu - \gamma + \frac{c_1(\gamma - 1)}{D_1} + \frac{d_3 \gamma}{D_2} + \frac{(1 + sw)\beta(1 - \mu)}{c} \left[ \frac{l}{D_1^2} - \frac{\mu(\gamma - 1)}{D_1^2} \right], \quad (112)\]
\[\nu_{20} = -3l \left[ \frac{c_2(\gamma - 1)}{D_1^3} + \frac{d_2 \gamma}{D_2^2} \right] - \frac{(1 + sw)\beta(1 - \mu)}{c} \left[ \frac{2(\gamma - 1)(1 - \mu - \gamma)}{D_1^3} + \frac{1}{D_1^3} \right], \quad (113)\]
\[\nu_{21} = 1 - c_2 - d_2 - \frac{2(1 + sw)\beta(1 - \mu)(\gamma - 1)l}{cD_1^4}, \quad (114)\]
\[\nu_{22} = -\frac{15}{2} \left[ \frac{c_3(\gamma - 1)^2}{D_1^3} + \frac{d_3 \gamma^2}{D_2^2} \right] + \frac{3l}{2} \left[ \frac{c_3}{D_1} + \frac{d_3}{D_2} \right] + \frac{(1 + sw)\beta(1 - \mu)}{c} \left[ \frac{3(1 - \gamma) - \mu - \gamma}{D_1^2} - \frac{4(\gamma - 1)^2(1 - \mu - \gamma)}{D_1^2} \right], \quad (115)\]
\[\nu_{23} = \frac{9l}{2} \left[ \frac{c_3}{D_1} + \frac{d_3}{D_2} \right] - \frac{(1 + sw)\beta(1 - \mu)}{c} \left[ \frac{\gamma - 1}{D_1^3} \right], \quad (116)\]
\[\nu_{24} = \frac{3l}{2} \left[ \frac{c_3}{D_1} + \frac{d_3}{D_2} \right] + \frac{(1 + sw)\beta(1 - \mu)(1 - \mu - \gamma)}{cD_1^4}, \quad (117)\]
\[\nu_{25} = -3 \left[ \frac{c_3(\gamma - 1)}{D_1} + \frac{d_3 \gamma}{D_2} \right] + \frac{(1 + sw)\beta(1 - \mu)}{c} \times \left[ \frac{8(\gamma - 1)^2l(\gamma - 1)(\gamma - 1)l + 2}{D_1^2} \right], \quad (118)\]
\[\nu_{26} = -\frac{35l}{2} \left[ \frac{c_4(\gamma - 1)^3}{D_1^3} + \frac{d_4 \gamma^3}{D_2^2} \right] + \frac{15l}{2} \left[ \frac{c_4(\gamma - 1)}{D_1^3} + \frac{d_4 \gamma}{D_2^2} \right] + \frac{(1 + sw)\beta(1 - \mu)}{c} \times \left[ \frac{1}{D_1^3} - \frac{8(\gamma - 1)^2 + 4\mu(\gamma - 1)}{D_1^3} \right], \quad (124)\]
\[\nu_{27} = \frac{3}{2} \left[ c_4 + d_4 \right] + \frac{(1 + sw)\beta(1 - \mu)4l(\gamma - 1 - \mu)}{cD_1^4}, \quad (125)\]
\[\nu_+ = c_2 + d_2 - \frac{(1 + sw)\beta(1 - \mu)\mu l}{cD_1^4}, \quad (126)\]
\[\nu_{29} = \frac{45l}{2} \left[ \frac{c_4(\gamma - 1)}{D_1^3} + \frac{d_4 \gamma}{D_2^2} \right] + \frac{(1 + sw)\beta(1 - \mu)}{c} \left[ \frac{1}{D_1^3} - \frac{4(\gamma - 1)^2}{D_1^3} \right], \quad (127)\]
\[\nu_{290} = \frac{15l}{2} \left[ \frac{c_4(\gamma - 1)}{D_1^3} + \frac{d_4 \gamma}{D_2^2} \right] + \frac{(1 + sw)\beta(1 - \mu)}{c} \left[ \frac{1}{D_1^3} + \frac{4(\gamma - 1)(1 - \mu - \gamma)}{D_1^3} \right], \quad (128)\]
\[\nu_{291} = -\frac{15}{2} \left[ \frac{c_4(\gamma - 1)^2}{D_1^3} + \frac{d_4 \gamma^2}{D_2^2} \right] + \frac{3}{2} \left[ c_4 + d_4 \right] + \frac{(1 + sw)\beta(1 - \mu)}{c} \times \left[ \frac{12l(\gamma - 1)}{D_1^3} - \frac{24(\gamma - 1)^2(\mu + 1)}{D_1^3} \right], \quad (129)\]
\[\nu_{292} = \frac{3}{2} \left[ c_4 + d_4 \right] + \frac{(1 + sw)\beta(1 - \mu)}{c} \left[ \frac{4l(\gamma - 1)}{D_1^3} \right], \quad (130)\]
\[\nu^{**} = l \left[ 1 - \left( \frac{c_4}{D_1} + \frac{d_4}{D_2} \right) \right] + \frac{(1 + sw)\beta(1 - \mu)(\gamma - 1 + \mu)}{cD_1^4}, \quad (131)\]
\[\nu_{32} = -\frac{15}{2} \left[ \frac{c_4(\gamma - 1)^2}{D_1^3} + \frac{d_4 \gamma^2}{D_2^2} \right] + \frac{3}{2} \left[ c_4 + d_4 \right] + \frac{(1 + sw)\beta(1 - \mu)\mu l}{cD_1^4} \left[ \frac{12l(\gamma - 1)}{D_1^3} - \frac{2}{D_1^3} \right], \quad (132)\]
\[\nu_{33} = \frac{15l}{2} \left[ \frac{c_4(\gamma - 1)}{D_1^3} + \frac{d_4 \gamma}{D_2^2} \right] + \frac{4(1 + sw)\beta(1 - \mu)\mu l}{cD_1^4} \left[ \frac{12l(\gamma - 1)}{D_1^3} + \frac{2}{D_1^3} \right], \quad (133)\]
\[\nu_{34} = \frac{3}{2} \left[ c_4 + d_4 \right] - \frac{6(1 + sw)\beta(1 - \mu)\mu l}{cD_1^4}, \quad (134)\]
\[\nu_{35} = \frac{3}{2} \left[ c_4 + d_4 \right] - \frac{2(1 + sw)\beta(1 - \mu)\mu l}{cD_1^4}, \quad (135)\]
\[\rho_{10} = -\frac{A_{14}}{\nu_{10}^* \nu_{20}}, \quad (136)\]
\[ \rho_{11} = \frac{-A_{11}B_{11} + 4\lambda A_{13}(\nu_{11} - \nu_{20})}{B_{11}^2 + 16\lambda^2(\nu_{11} - \nu_{20})^2}, \quad (137) \]
\[ c_n = \frac{(1 - \beta)(1 - \mu)}{D_n^{n+1}}, \quad d_n = \frac{\mu}{D_2^{n+1}}, \quad (159) \]
\[ \rho_{12} = \frac{B_{11}A_{12}}{B_{11}^2 + 16\lambda^2(\nu_{11} - \nu_{20})^2}, \quad (138) \]
\[ D_1 = \sqrt{(\gamma - 1)^2 + F^2}, \quad D_2 = \sqrt{\gamma^2 + F^2}, \quad (160) \]
\[ \rho_{13} = \frac{B_{11}A_{13} - 4\lambda A_{12}\lambda(\nu_{11} - \nu_{20})}{B_{11}^2 + 16\lambda^2(\nu_{11} - \nu_{20})^2}, \quad (139) \]
\[ \beta_{11} = -s_{11}\lambda^2 + \nu_{21}s_{11} + 2s_{22}\lambda - \nu_{11}s_{21}, \quad (161) \]
\[ \rho_{14} = \frac{4\lambda A_{11}(\nu_{11} - \nu_{20})}{B_{11}^2 + 16\lambda^2(\nu_{11} - \nu_{20})^2}, \quad (140) \]
\[ \beta_{12} = -s_{12}\lambda^2 - \nu_{21}s_{12} - 2s_{22}\lambda - \nu_{11}s_{22}, \quad (162) \]
\[ \rho_{20} = \frac{-\delta_1 + \nu_{21}\rho_{10}}{\nu_{21}}, \quad (141) \]
\[ \beta_{13} = -9\lambda^2(\alpha_{13} + \zeta\alpha_{14}) - \nu_{21}(\alpha_{13} + \zeta\alpha_{14}) + 3\lambda(\alpha_{24} + \zeta\alpha_{26}) - \nu_{11}(\alpha_{23} + \zeta\alpha_{24}), \quad (163) \]
\[ \rho_{21} = \frac{4\lambda\rho_{11} - \nu_{21}\rho_{14} - \delta_1}{-4\lambda^2 - \nu_{21}}, \quad (142) \]
\[ \beta_{14} = -9\lambda^2(\alpha_{18} + \zeta\alpha_{16}) - \nu_{21}(\alpha_{18} + \zeta\alpha_{16}) - 3\lambda(\alpha_{24} + \zeta\alpha_{26}) - \nu_{11}(\alpha_{23} + \zeta\alpha_{24}), \quad (164) \]
\[ \rho_{22} = \frac{4\lambda\rho_{12} - \nu_{21}\rho_{13}}{-4\lambda^2 - \nu_{21}}, \quad (143) \]
\[ \beta_{21} = \lambda^4 - \lambda^2(4 - \nu_{21} - \nu_{10}) + \nu_{10}\nu_{21} + \nu_{11}\nu_{20}, \quad (165) \]
\[ \rho_{23} = \frac{4\lambda\rho_{13} + \nu_{21}\rho_{12} + \delta_3}{-4\lambda^2 - \nu_{21}}, \quad (144) \]
\[ \beta_{22} = 2\lambda(\nu_{21} + \nu_{20}), \quad \beta_{22} = 6\lambda(\nu_{11} + \nu_{20}), \quad (166) \]
\[ \rho_{24} = \frac{4\lambda + \nu_{21}\rho_{11} - \delta_2}{-4\lambda^2 - \nu_{21}}, \quad (145) \]
\[ \beta_{23} = 81\lambda^4 - 9\lambda^2(4 - \nu_{21} - \nu_{10}) + \nu_{10}\nu_{21} + \nu_{11}\nu_{20}, \quad (167) \]
\[ \rho_{30} = \frac{h_1}{-3\lambda^2}, \quad \rho_{31} = \frac{h_1}{\lambda^2}, \quad \rho_{32} = \frac{h_2}{\lambda^2}, \quad (146) \]
\[ \sigma_{20} = \frac{-2\sigma_{11}\lambda + \nu_{20}\sigma_{10} + s_{21}}{-\lambda^2 - \nu_{21}}, \quad (168) \]
\[ \rho_{33} = \frac{h_2}{-3\lambda^2}, \quad (147) \]
\[ \sigma_{21} = \frac{2\sigma_{10}\lambda + \nu_{20}\sigma_{11} + s_{22}}{-\lambda^2 - \nu_{21}}, \quad (169) \]
\[ A_{11} = 4\lambda^2\alpha_1 + \alpha_1\nu_{21} + 4\lambda\delta_1 - \delta_2\nu_{11}, \quad (148) \]
\[ \sigma_{22} = \frac{-6\sigma_{13}\lambda + \nu_{20}\sigma_{12} + \alpha_{23} + \zeta\alpha_{24}}{-9\lambda^2 - \nu_{21}}, \quad (170) \]
\[ A_{12} = 4\lambda^2\alpha_2 + \alpha_2
u_{21} + \delta_3\nu_{11}, \quad (149) \]
\[ \alpha_1 = \frac{\nu_{21}A_2^2 + \nu_{21}\kappa^2A_2^2\nu_{10}^2}{2} - 2\nu_{23}\kappa^2A_2^2\lambda^2, \quad (171) \]
\[ A_{13} = 4\lambda^2\alpha_3 + \nu_{21}\alpha_3 - 4\delta_2\lambda + \delta_1\nu_{11}, \quad (150) \]
\[ \alpha_2 = \frac{\nu_{11}A_2^2}{2}, \quad \alpha_3 = 2\nu_{13}\kappa^2A_2^2\nu_{10}, \quad (172) \]
\[ A_{14} = \alpha_2\nu_{21} + \nu_{11}\delta_4, \quad (151) \]
\[ \alpha_4 = \frac{\nu_{12}A_2^2 + \nu_{13}\kappa^2A_2^2\nu_{10}^2 + \nu_{14}A_2^2}{2} + 2\nu_{23}\kappa^2A_2^2\lambda^2, \quad (173) \]
\[ B_{11} = 16\lambda^4 - 4\lambda^2(4 - \nu_{21} + \nu_{10}) + (\nu_{10}\nu_{21} - \nu_{20}\nu_{11}), \quad (152) \]
\[ \delta_1 = \nu_{25}\kappa^2A_2^2\lambda - 2\nu_{10}\nu_{23}\kappa^2A_2^2, \quad (174) \]
\[ s_{11} = \alpha_{11} + 2\nu_2\nu_{1}^2A_4(2\kappa - 1), \quad (153) \]
\[ \delta_2 = \frac{\nu_{22}A_2^2 - (\nu_{25} + \nu_{23}\kappa)\kappa^2A_2^2\nu_{10}}{2} - \lambda^2\nu_{23}\kappa^2A_2^2, \quad (175) \]
\[ s_{12} = \alpha_{12} + 2\nu_2A_2\nu_{1}^2\omega_2, \quad (154) \]
\[ \delta_3 = \frac{\nu_{24}A_2^2}{2}, \quad (176) \]
\[ \sigma_{10} = \frac{\beta_{11}\beta_{21} + \beta_{12}\beta_{22}}{\beta_{21}^2 + \beta_{22}^2}, \quad (155) \]
\[ \delta_4 = \frac{(\nu_{22} - \nu_{25}\nu_{10} + \nu_{23}\kappa^2\nu_{10}^2 + 2\lambda_2\kappa^2\nu_{23}^2)A_2^2 + \nu_{24}A_2^2}{2}, \quad (177) \]
\[ \sigma_{11} = \frac{\beta_{12}\beta_{21} - \beta_{11}\beta_{22}}{\beta_{21}^2 + \beta_{22}^2}, \quad (156) \]
\[ h_1 = \frac{(\nu_{20} + \nu_{21})A_4A_5}{2}, \quad (178) \]
\[ \sigma_{12} = \frac{\beta_{13}\beta_{21} + \beta_{14}\beta_{22}}{\beta_{21}^2 + \beta_{22}^2}, \quad (157) \]
\[ h_2 = \lambda\nu_{21}\kappa A_4A_5, \quad (179) \]
\[ \sigma_{13} = \frac{\beta_{14}\beta_{31} - \beta_{13}\beta_{31}\beta_{22}}{\beta_{21}^2 + \beta_{22}^2}, \quad (158) \]