Numerical Research of Bending Flexible Plates

Radek Gabbasov¹, Vladimir Filatov¹ and Ngoc Khoa Dao¹

¹National research Moscow State University of civil engineering, Yaroslavskoe Shosse 26, Moscow, 129337, Russian Federation
dnkhoa83@gmail.com

Abstract. The problem considered in the article belongs to the class of geometrically nonlinear problems. A numerical method basing on the use of difference equations of successive approximation method (MSA) is proposed to solve the problem. The results obtaining from example of this article are compared with the results received by A.S. Volmir for confirmation of efficiency of methodology.

1. Introduction

The high popularity of thin-walled structures in mechanical engineering and large-span structures in construction requires calculations to determine the stress-strain state of the above structures in geometrical non-linear formulation. The calculation of the above-mentioned structures taking into account large deflections, i.e. in geometrical non-linear formulation can be performed with use of method proposed in the following parts. In papers [1-8] calculation theories, analytical and numerical methods to solve geometrical non-linear problems of structural analysis are presented. The analyses of these problems using finite element method are illustrated in the articles [9-10]. In the articles [11-12], a method of successive approximation is proposed.

2. The difference form of the resolving equation in a geometrically non-linear formulation

The analysis of rectangular flexible plates, subjected to transverse load [1] is simplified into solving two non-linear partial differential equations of fourth order, formulated with respect to deflections w and the stress function Φ.

Based on the equations (1.60'), (1.61') and (1.62) of Volmir [1], the differential equations in following form are written:

\[
\frac{D}{H} \nabla^2 \nabla^2 w = \frac{\partial^2 \Phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 \Phi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{1}{H} q; \quad (1)
\]

\[
\frac{1}{E} \nabla^2 \nabla^2 \Phi = - \frac{1}{2} \left( \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right); \quad (2)
\]

In which, H – thickness of plate;
D – Cylindrical stiffness;
qu – Intensity of transverse load, distributed according to given law;
E – Modulus of elasticity;
x, y – coordinates.

Using (1) and (2), in [13] a calculation algorithm for plates in geometrical non-linear formulation based on generalized equations of finite difference method was established. In this article the same calculation procedure is applied using differential equations of successive approximation method [14].

Applying the dimensionless parameters:

\[ \xi = \frac{x}{a}; \eta = \frac{y}{a}; \nabla = a^2 \partial \nabla; q = \frac{q.a^3}{D}; \psi = \frac{\Phi H}{D}; w = \frac{w}{a}, \]

(3)

In which a – length of plate sides.

After transformations (1) and (2) we get:

\[ \nabla^2 \nabla^2 w = \frac{\partial^2 \psi}{\partial \xi^2} \cdot \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} \cdot \frac{\partial^2 w}{\partial \eta^2} - 2 \frac{\partial^2 \psi}{\partial \xi \partial \eta} \cdot \frac{\partial^2 w}{\partial \xi \partial \eta} + q; \]

(4)

\[ \nabla^2 \nabla^2 \psi = -k \left\{ \frac{\partial^2 w}{\partial \xi^2} \cdot \frac{\partial^2 w}{\partial \eta^2} - \frac{\partial^2 w}{\partial \xi \partial \eta} \cdot \frac{\partial^2 w}{\partial \xi \partial \eta} \right\}, \]

(5)

In which: \( k = \frac{EHa^2}{D} \).

Using following symbols:

\[ \nabla^2 w = -m; \frac{\partial^2 w}{\partial \xi^2} = l; \frac{\partial^2 w}{\partial \eta^2} = n; \frac{\partial^2 w}{\partial \xi \partial \eta} = t. \]

(6)

\[ \nabla^2 \psi = -f; \frac{\partial^2 \psi}{\partial \xi^2} = b; \frac{\partial^2 \psi}{\partial \eta^2} = c; \frac{\partial^2 \psi}{\partial \xi \partial \eta} = d. \]

(7)

In view of (6) and (7), equations (4) and (5) are written as follow:

\[ \nabla^2 m = -(cl + bn - 2dt) - q. \]

(8)

\[ \nabla^2 f = k(nl - t^2). \]

(9)

Then the algorithm simplifies into joint solution of following differential equations:

\[ \frac{\partial^2 m}{\partial \xi^2} + \frac{\partial^2 m}{\partial \eta^2} = -g. \]

(10)

\[ g = \lambda + q. \]

(11)

\[ \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = -m. \]

(12)
\[
\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = -f. \quad (13)
\]
\[
\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} = \alpha. \quad (14)
\]

In which
\[
\lambda = cl + bn - 2dt. \quad (15)
\]
\[
\alpha = k(nl - t^2). \quad (16)
\]

Solution of above-mentioned problem is illustrated in following scheme:

Figure 1. Block–scheme of calculation program of square plate with geometrical nonlinearity using difference equations of successive approximation method (MSA).

In which \( g \)– last value \( g \); \( g_{i-1} \)– penultimate.

The difference equations MSA in absence of all discontinuities according to \([14]\) are written to solve equations (10):

For interior points:
\[
m_{i-1,j-1} + 4m_{i-1,j} + m_{i-1,j+1} + 4m_{i,j-1} - 20m_{ij} + 4m_{i,j+1} + m_{i+1,j-1} + 4m_{i+1,j} + m_{i+1,j+1} = -6h^2 \cdot p_{ij}. \quad (17)
\]

Since the differential equations (12), (13) and (14) are the same type, the same difference pattern is used for their approximation in absence of all discontinuities of successive approximation method \([14]\):

For interior points:
\[
w_{i-1,j-1} + 4w_{i-1,j} + w_{i-1,j+1} + 4w_{i,j-1} - 20w_{ij} + 4w_{i,j+1} + w_{i+1,j-1} + 4w_{i+1,j} + w_{i+1,j+1} =
\]
\[
= \frac{h^2}{12} (m_{i-1,j-1} + 4m_{i-1,j} + m_{i-1,j+1} + 4m_{i,j-1} + 52m_{ij} + 4m_{i,j+1} + m_{i+1,j-1} + 4m_{i+1,j} + m_{i+1,j+1}). \quad (18)
\]

For edge points:
\[
4w_{i-1,j} + 2w_{i-1,j+1} - 12hw_{ij} + 20w_{ij} + 8w_{i+1,j+1} + 4w_{i+1,j} + 2w_{i+1,j+1} = \frac{h^2}{24} (5m_{i-1,j} + 8m_{i-1,j+1} +
\]
\[
- m_{i-1,j+2} + 74m_{ij} + 56m_{i+1,j+1} - 10m_{i+1,j+2} + 5m_{i+1,j+1} + 8m_{i+1,j+1} - m_{i-1,j+2}). \quad (19)
\]

If the parameters \( \overline{w} \) and \( \psi \) will be found, all other values corresponding to the solution [1] can be determined.
3. Example of calculation
The rectangular flexible plate fixed along nearly free edges, subjected to distributed load is analyzed as calculation example (figure 2).

Thickness of plate, sides $a = 10\ cm$, load $q = 0.5\ kg/cm^2$, $E = 0.75 \cdot 10^6\ kg/cm^2$ and $\mu = 0.316$.

Boundary conditions on the sides $\xi = 0$ and $\xi = 2h$ :

$\bar{w} = 0, w^{\xi} = 0, \frac{\partial^2 \psi}{\partial \eta^2} = 0, \frac{\partial^2 \psi}{\partial \xi \partial \eta} = 0$.

On the sides $\eta = 0$ and $\eta = 2h$ the same boundary conditions are obtained by changing $\xi$ into $\eta$.

Consider a square grid with a minimum number of divisions and the length of step $h=1/2$ (figure 2).

![Figure 2](image_url). Square plate, fixed along nearly free edges.

Determine derivatives (6) and (7) using successive approximation method and take into account boundary condition, it will be obtained:

$\lambda_{11} = \frac{1}{2} m_{11} ; f_{11}$. \hspace{1cm} (20)

$\alpha_{11} = k(n_{11} l_{11} - t_{11}^2) = \frac{k}{4} m_{11}^2$. \hspace{1cm} (21)

Writing the difference approximation (10), (12), (14), (13) according to the equations of successive approximation method [14] and taking into account the boundary conditions, it will be obtained:

$16m_{10} - 20m_{11} = 6h^2 g_{11}$. \hspace{1cm} (22)

$w_{11} = - \frac{h^2}{24} (10m_{10} + 7m_{11})$. \hspace{1cm} (23)

$-20w_{11} = - \frac{h^2}{12} (16m_{10} + 52m_{11})$. \hspace{1cm} (24)

From (22), (23) and (24) it will be found:

$- \bar{w}_{11} = \frac{17}{712} h^4 g_{11}; m_{10} = - \frac{61}{356} h^2 g_{11}; m_{11} = \frac{29}{178} h^2 g_{11}$. \hspace{1cm} (25)

$f_{11} = - \frac{13h^2}{60} \alpha_{11}$. \hspace{1cm} (26)
\[ \psi_{11} = \frac{13h^2}{60} f_{11}. \]  \hfill (27)

Taking into account (25), it will be obtained:

\[ g_{11} = \frac{712}{17h^4} w_{11}. \]  \hfill (28)

In accounting (20), (21), (25), (24) and (26):

\[ \lambda_{11} = -\frac{8,60457k}{h^4} w_{11}. \]  \hfill (29)

Using (28) and (29), it will be obtained:

\[ \frac{712}{17h^4} w_{11} = -\frac{8,60457k}{h^4} w_{11}. \]

In accounting \( q = \frac{q.a^3}{D} = \frac{0,5.10^3}{69,433} = 7,2 \) and \( k = \frac{EHa^2}{D} = \frac{0,75.10^6.0,1.10^2}{69,433} = 108817,8 \) it will be obtained:

\[ 936330,38 w_{11} + \frac{712}{17} w_{11} - 0,45 = 0 \Leftrightarrow w_{11} = 0,00598. \text{Then} \ W_{11} = 0,0598 \text{cm}. \]

Using block-scheme (figure 1), it will be received \( W_{11} = 0,06 \text{cm} \) after 16 cycles of iterations.

For square plate fixed along edges (the edges of plate move freely) in case \( \mu = 0,316 \), it will be determined \( \zeta = \frac{W}{H} \) from equations [1] (2.149):

\[ \frac{533\pi}{3200} \zeta^3 + \frac{2\pi^4}{3(1-\mu^2)} \zeta = q^*, \]

In which \( q^* = \frac{q.E(1-\mu^2)}{a^4} = \frac{0,5}{0,75.10^6} \left( \frac{10}{0,1} \right)^4(1-0,316^2) = 60,01. \)

\[ 16,192\zeta^3 + 71,997\zeta = 60,667; \zeta = 0,8075; \text{then} \ W_{\text{max}} = \zeta.H = 0,08075 \text{cm}. \]

The difference between the above mentioned numerical and analytical results is 25%. This is explained by the fact that our solutions were obtained with large sides of minimum possible calculation grid and division of each side into two sections (figure 2). This grid is especially chosen for methodological purposes to illustrate the algorithm. Table 1 shows the results obtained on several grids changed from one into another. They demonstrate the convergence of the numerical solution and high accuracy with the use of the proposed method. The calculation procedure is carried out using the Matlab program.
Figure 3. Diagrams of the changes of maximum deflection according to number of iteration cycle with steps a) $h = 1/4$; b) $h = 1/8$; c) $h = 1/16$; d) $h = 1/32$. 
Figure 4. Diagrams of the maximum deflection according to number of iteration cycle with steps 

a) $h = 1/4$; b) $h = 1/8$; c) $h = 1/16$; d) $h = 1/32$.

Table 1. Maximum deflection and comparison with the result of A.S. Volmir at different grid’s steps.

| $h$     | $h = 1/4$ | $h = 1/8$ | $h = 1/16$ | $h = 1/32$ |
|---------|-----------|-----------|-----------|-----------|
| Number of iteration cycle | 29        | 37        | 38        | 38        |
| Maximum deflection (cm)    | 0.075     | 0.0764    | 0.0778    | 0.0781    |
| $w_{\text{max}}$, compared with the result obtained by A.S. Volmir | 7.12%     | 5.39%     | 3.65%     | 3.28%     |

The table 1 shows that the solution obtained by our proposed method approaches to the solution of Volmir.

4. Conclusion

The results show the effectiveness of the application of the equations of successive approximation method to the analysis of plate in non-linear formulation using computer.

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