Solving Totally Unimodular LPs with the Shadow Vertex Algorithm

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Abstract

We show that the shadow vertex simplex algorithm can be used to solve linear programs in strongly polynomial time with respect to the number $n$ of variables, the number $m$ of constraints, and $1/\delta$, where $\delta$ is a parameter that measures the flatness of the vertices of the polyhedron. This extends our recent result that the shadow vertex algorithm finds paths of polynomial length (w.r.t. $n$, $m$, and $1/\delta$) between two given vertices of a polyhedron [4].

Our result also complements a recent result due to Eisenbrand and Vempala [6] who have shown that a certain version of the random edge pivot rule solves linear programs with a running time that is strongly polynomial in the number of variables $n$ and $1/\delta$, but independent of the number $m$ of constraints. Even though the running time of our algorithm depends on $m$, it is significantly faster for the important special case of totally unimodular linear programs, for which $1/\delta \leq n$ and which have only $O(n^2)$ constraints.

1 Introduction

The shadow vertex algorithm is a well-known pivoting rule for the simplex method that has gained attention in recent years because it was shown to have polynomial running time in the model of smoothed analysis [9]. Recently we have observed that it can also be used to find short paths between given vertices of a polyhedron [4]. Here short means that the path length is $O\left(\frac{mn^2}{\delta^2}\right)$, where $n$ denotes the number of variables, $m$ denotes the number of constraints, and $\delta$ is a parameter of the polyhedron that we will define shortly.

Our result left open the question whether or not it is also possible to solve linear programs in polynomial time with respect to $n$, $m$, and $1/\delta$ by the shadow vertex simplex algorithm. In this article we resolve this question and introduce a variant of the shadow vertex simplex algorithm that solves linear programs in strongly polynomial time with respect to these parameters.

For a given matrix $A = [a_1, \ldots, a_m]^T \in \mathbb{R}^{m \times n}$ and vectors $b \in \mathbb{R}^m$ and $c_0 \in \mathbb{R}^n$ our goal is to solve the linear program $\max \{c_0^T x \mid Ax \leq b\}$. We assume without loss of generality that $\|c_0\| = 1$ and $\|a_i\| = 1$ for every row $a_i$ of the constraint matrix.

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Definition 1. The matrix $A$ satisfies the $\delta$-distance property if the following condition holds: For any $I \subseteq \{1, \ldots, m\}$ and any $j \in \{1, \ldots, m\}$, if $a_j \notin \text{span}\{a_i | i \in I\}$ then $\text{dist}(a_j, \text{span}\{a_i | i \in I\}) \geq \delta$. In other words, if $a_j$ does not lie in the subspace spanned by the $a_i$, $i \in I$, then its distance to this subspace is at least $\delta$.

We present a variant of the shadow vertex simplex algorithm that solves linear programs in strongly polynomial time with respect to $n$, $m$, and $1/\delta$, where $\delta$ denotes the largest $\delta'$ for which the constraint matrix of the linear program satisfies the $\delta'$-distance property. (In the following theorems, we assume $m \geq n$. If this is not the case, we use the method from Section 2.1 to add irrelevant constraints so that $A$ has rank $n$. Hence, for instances that have fewer constraints than variables, the parameter $m$ should be replaced by $n$ in all bounds.)

Theorem 2. There exists a randomized variant of the shadow vertex simplex algorithm (described in Section 2) that solves linear programs with $n$ variables and $m$ constraints satisfying the $\delta$-distance property using $O\left(\frac{mn^3}{\delta^2} \cdot \log\left(\frac{1}{\delta}\right)\right)$ pivots in expectation if a basic feasible solution is given. A basic feasible solution can be found using $O\left(\frac{m^5}{\delta^2} \cdot \log\left(\frac{1}{\delta}\right)\right)$ pivots in expectation.

We stress that the algorithm can be implemented without knowing the parameter $\delta$. From the theorem it follows that the running time of the algorithm is strongly polynomial with respect to the number $n$ of variables, the number $m$ of constraints, and $1/\delta$ because every pivot can be performed in time $O(mn)$ in the arithmetic model of computation (see Section 2.4).

Let $A \in \mathbb{Z}^{m \times n}$ be an integer matrix and let $A' \in \mathbb{R}^{m \times n}$ be the matrix that arises from $A$ by scaling each row such that its norm equals 1. If $\Delta$ denotes an upper bound for the absolute value of any sub-determinant of $A$, then $A'$ satisfies the $\delta$-distance property for $\delta = 1/(\Delta^2n)$ [4]. For such matrices $A$ Phase 1 of the simplex method can be implemented more efficiently and we obtain the following result.

Theorem 3. For integer matrices $A \in \mathbb{Z}^{m \times n}$, there exists a randomized variant of the shadow vertex simplex algorithm (described in Section 2) that solves linear programs with $n$ variables and $m$ constraints using $O(mn^3\Delta^4 \log(\Delta + 1))$ pivots in expectation if a basic feasible solution is given, where $\Delta$ denotes an upper bound for the absolute value of any sub-determinant of $A$. A basic feasible solution can be found using $O(m^8\Delta^4 \log(\Delta + 1))$ pivots in expectation.

Theorem 3 implies in particular that totally unimodular linear programs can be solved by our algorithm with $O(mn^5)$ pivots in expectation if a basic feasible solution is given and with $O(m^8)$ pivots in expectation otherwise.

Besides totally unimodular matrices there are also other classes of matrices for which $1/\delta$ is polynomially bounded in $n$. Eisenbrand and Vempala [6] observed, for example, that $\delta = \Omega(1/\sqrt{n})$ for edge-node incidence matrices of undirected graphs with $n$ vertices. One can also argue that $\delta$ can be interpreted as a condition number of the matrix $A$ in the following

\footnote{By strongly polynomial with respect to $n$, $m$, and $1/\delta$ we mean that the number of steps in the arithmetic model of computation is bounded polynomially in $n$, $m$, and $1/\delta$ and the size of the numbers occurring during the algorithm is polynomially bounded in the encoding size of the input.}
sense: If $1/\delta$ is large then there must be an $(n \times n)$-submatrix of $A$ of rank $n$ that is almost singular.

1.1 Related Work

**Shadow vertex simplex algorithm** We will briefly explain the geometric intuition behind the shadow vertex simplex algorithm. For a complete and more formal description, we refer the reader to [2] or [9]. Let us consider the linear program $\max \{c^T x \mid Ax \leq b\}$ and let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ denote the polyhedron of feasible solutions. Assume that an initial vertex $x_1$ of $P$ is known and assume, for the sake of simplicity, that there is a unique optimal vertex $x^*$ of $P$ that maximizes the objective function $c^T x$. The shadow vertex pivot rule first computes a vector $w \in \mathbb{R}^n$ such that the vertex $x_1$ minimizes the objective function $w^T x$ subject to $x \in P$. Again for the sake of simplicity, let us assume that the vectors $c_0$ and $w$ are linearly independent.

In the second step, the polyhedron $P$ is projected onto the plane spanned by the vectors $c_0$ and $w$. The resulting projection is a (possibly open) polygon $P'$ and one can show that the projections of both the initial vertex $x_1$ and the optimal vertex $x^*$ are vertices of this polygon. Additionally, every edge between two vertices $x$ and $y$ of $P'$ corresponds to an edge of $P$ between two vertices that are projected onto $x$ and $y$, respectively. Due to these properties a path from the projection of $x_1$ to the projection of $x^*$ along the edges of $P'$ corresponds to a path from $x_1$ to $x^*$ along the edges of $P$.

This way, the problem of finding a path from $x_1$ to $x^*$ on the polyhedron $P$ is reduced to finding a path between two vertices of a polygon. There are at most two such paths and the shadow vertex pivot rule chooses the one along which the objective $c_0^T x$ improves.

**Finding short paths** In [4] we considered the problem of finding a short path between two given vertices $x_1$ and $x_2$ of the polyhedron $P$ along the edges of $P$. Our algorithm is the following variant of the shadow vertex algorithm: Choose two vectors $w_1, w_2 \in \mathbb{R}^n$ such that $x_1$ uniquely minimizes $w_1^T x$ subject to $x \in P$ and $x_2$ uniquely maximizes $w_2^T x$ subject to $x \in P$. Then project the polyhedron $P$ onto the plane spanned by $w_1$ and $w_2$ in order to obtain a polygon $P'$. Let us call the projection $\pi$. By the same arguments as above, it follows that $\pi(x_1)$ and $\pi(x_2)$ are vertices of $P'$ and that a path from $\pi(x_1)$ to $\pi(x_2)$ along the edges of $P'$ can be translated into a path from $x_1$ to $x_2$ along the edges of $P$. Hence, it suffices to compute such a path to solve the problem. Again computing such a path is easy because $P'$ is a two-dimensional polygon.

The vectors $w_1$ and $w_2$ are not uniquely determined, but they can be chosen from cones that are determined by the vertices $x_1$ and $x_2$ and the polyhedron $P$. We proved in [4] that the expected path length is $O(mn^2/\delta^2)$ if $w_1$ and $w_2$ are chosen randomly from these cones. For totally unimodular matrices this implies that the diameter of the polyhedron is bounded by $O(mn^4)$, which improved a previous result by Dyer and Frieze [5] who showed that for this special case paths of length $O(m^3 n^{16} \log(mn))$ can be computed efficiently.

Additionally, Bonifas et al. [1] proved that in a polyhedron defined by an integer matrix $A$ between any pair of vertices there exists a path of length $O(\Delta^2 n^4 \log(n\Delta))$ where $\Delta$ is the largest absolute value of any sub-determinant of $A$. For the special case that $A$ is a totally unimodular matrix, this bound simplifies to $O(n^4 \log n)$. Their proof is
non-constructive, however.

**Geometric random edge** Eisenbrand and Vempala [6] have presented an algorithm that solves a linear program $\max\{c_0^T x | Ax \leq b\}$ in strongly polynomial time with respect to the parameters $n$ and $1/\delta$. Remarkably the running time of their algorithm does not depend on the number $m$ of constraints. Their algorithm is based on a variant of the random edge pivoting rule. The algorithm performs a random walk on the vertices of the polyhedron whose transition probabilities are chosen such that it quickly attains a distribution close to its stationary distribution.

In the stationary distribution the random walk is likely at a vertex $x_c$ that optimizes an objective function $c^T x$ with $\|c_0 - c\| < \frac{\delta \cdot 2^n}{2^n}$. The $\delta$-distance property guarantees that $x_c$ and the optimal vertex $x^*$ with respect to the objective function $c_0^T x$ lie on a common facet. This facet is then identified and the algorithm is run again in one dimension lower. This is repeated at most $n$ times until all facets of the optimal vertex $x^*$ are identified.

Let us point out that the number of pivots of our algorithm depends on the number $m$ of constraints. However, Heller showed that for the important special case of totally unimodular linear programs $m = O(n^2)$ [8]. Using this observation we also obtain a bound that depends polynomially only on $n$ for totally unimodular matrices.

**Combinatorial linear programs** Éva Tardos has proved in 1986 that combinatorial linear programs can be solved in strongly polynomial time [10]. Here combinatorial means that $A$ is an integer matrix whose largest entry is polynomially bounded in $n$. Her result implies in particular that totally unimodular linear programs can be solved in strongly polynomial time, which is also implied by Theorem [3]. However, the proof and the techniques used to prove Theorem [3] are completely different from those in [10].

### 1.2 Our Contribution

We replace the random walk in the algorithm of Eisenbrand and Vempala by the shadow vertex algorithm. Given a vertex $x_0$ of the polyhedron $P$ we choose an objective function $w^T x$ for which $x_0$ is an optimal solution. As in [4] we choose $w$ uniformly at random from the cone determined by $x_0$. Then we randomly perturb each coefficient in the given objective function $c_0^T x$ by a small amount. We denote by $c^T x$ the perturbed objective function. As in [4] we prove that the projection of the polyhedron $P$ onto the plane spanned by $w$ and $c$ has $O(\frac{mn^2}{\delta^2})$ edges in expectation. If the perturbation is so small that $\|c_0 - c\| < \frac{\delta \cdot 2^n}{2^n}$, then the shadow vertex algorithm yields with $O(\frac{mn^2}{\delta^2})$ pivots a solution that has a common facet with the optimal solution $x^*$. We follow the same approach as Eisenbrand and Vempala and identify the facets of $x^*$ one by one with at most $n$ calls of the shadow vertex algorithm.

The analysis in [4] exploits that the two objective functions possess the same type of randomness (both are chosen uniformly at random from some cones). This is not the case.
anymore because every component of $c$ is chosen independently uniformly at random from some interval. This changes the analysis significantly and introduces technical difficulties that we address in this article.

The problem when running the simplex method is that a feasible solution needs to be given upfront. Usually, such a solution is determined in Phase 1 by solving a modified linear program with a constraint matrix $A'$ for which a feasible solution is known and whose optimal solution is feasible for the linear program one actually wants to solve. There are several common constructions for this modified linear program, it is, however, not clear how the parameter $\delta$ is affected by modifying the linear program. To solve this problem, Eisenbrand and Vempala [6] have suggested a method for Phase 1 for which the modified constraint matrix $A'$ satisfies the $\delta$-distance property for the same $\delta$ as the matrix $A$. However, their method is very different from usual textbook methods and needs to solve $m$ different linear programs to find an initial feasible solution for the given linear program. We show that also one of the usual textbook methods can be applied. We argue that $\frac{1}{\delta}$ increases by a factor of at most $\sqrt{m}$ and that $\Delta$, the absolute value of any sub-determinant of $A$, does not change at all in case one considers integer matrices. In this construction, the number of variables increases from $n$ to $n + m$.

1.3 Outline and Notation

In the following we assume that we are given a linear program $\max \{c_0^T x \mid Ax \leq b\}$ with vectors $b \in \mathbb{R}^m$ and $c_0 \in \mathbb{R}^n$ and a matrix $A = [a_1, \ldots, a_m]^T \in \mathbb{R}^{m \times n}$. Moreover, we assume that $\|c_0\| = \|a_i\| = 1$ for all $i \in [m]$, where $[m] := \{1, \ldots, m\}$ and $\| \cdot \|$ denotes the Euclidean norm. This entails no loss of generality since any linear program can be brought into this form by scaling the objective function and the constraints appropriately. For a vector $x \in \mathbb{R}^n \setminus \{0^n\}$ we denote by $N(x) = \frac{1}{\|x\|} \cdot x$ the normalization of vector $x$.

For a vertex $v$ of the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ we call the set of row indices $B_v = \{i \in \{1, \ldots, m\} \mid a_i \cdot v = b_i\}$ basis of $v$. Then the normal cone $C_v$ of $v$ is given by the set

$$C_v = \left\{ \sum_{i \in B_v} \lambda_i a_i \mid \lambda_i \geq 0 \right\}.$$

We will describe our algorithm in Section 2.3 where we assume that the linear program in non-degenerate, that $A$ has full rank $n$, and that the polyhedron $P$ is bounded. We have already described in Section 3 of [4] that the linear program can be made non-degenerate by slightly perturbing the vector $b$. This does not affect the parameter $\delta$ because $\delta$ depends only on the matrix $A$. In Appendix D we discuss why we can assume that $A$ has full rank and why $P$ is bounded. There are, of course, textbook methods to transform a linear program into this form. However, we need to be careful that this transformation does not change $\delta$.

In Section 3 we analyze our algorithm and prove Theorem 2. In Section 4 we discuss how Phase 1 of the simplex method can be implemented and in Appendix A we give an alternative definition of $\delta$ and discuss some properties of this parameter.
2 Algorithm

Given a linear program \( \max \{ c_0^T x \mid Ax \leq b \} \) and a basic feasible solution \( x_0 \), our algorithm randomly perturbs each coefficient of the vector \( c_0 \) by at most \( \frac{1}{\phi} \) for some parameter \( \phi \) to be determined later. Let us call the resulting vector \( c \). The next step is then to use the shadow vertex algorithm to compute a path from \( x_0 \) to a vertex \( x_c \) which maximizes the function \( c^T x \) for \( x \in P \). For \( \phi > \frac{2n^{3/2}}{\delta} \) one can argue that the solution \( x \) has a facet in common with the optimal solution \( x^* \) of the given linear program with objective function \( c_0^T x \). Then the algorithm is run again on this facet one dimension lower until all facets that define \( x^* \) are identified.

This section is organized as follows. In Section 2.1 we repeat a construction from [6] to project a facet of the polyhedron \( P \) into the space \( \mathbb{R}^{n-1} \) without changing the parameter \( \delta \). This is crucial for being able to identify the facets that define \( x^* \) one after another. In Section 2.2 we also repeat an argument from [6] that shows how a common facet of \( x_c \) and \( x^* \) can be identified if \( x_c \) is given. Section 2.3 presents the shadow vertex algorithm, the main building block of our algorithm. Finally in Section 2.4 we discuss the running time of a single pivot step of the shadow vertex algorithm.

2.1 Reducing the Dimension

Assume that we have identified an element \( a_i, i \in [m] \), of the optimal basis \( x^* \) (i.e., \( a_i x^* = b_i \)). In [6] it is described how to reduce in this case the dimension of the linear program by one without changing the parameter \( \delta \). We repeat the details. Without loss of generality we may assume that \( a_1 \) is an element of the optimal basis. Let \( Q \in \mathbb{R}^{n \times n} \) be an orthogonal matrix that rotates \( a_1 \) into the first unit vector \( e_1 \). Then the following linear programs are equivalent:

\[
\max \{ c_0^T x \mid x \in \mathbb{R}^n, Ax \leq b \}
\]

and

\[
\max \{ c_0^T Qx \mid x \in \mathbb{R}^n, AQx \leq b \}.
\]

In the latter linear program the first constraint is of the form \( x_1 \leq b_1 \). We set this constraint to equality and subtract this equation from the other constraints (i.e., we project each row into the orthogonal complement of \( e_1 \)). Thus, we end up with a linear program of dimension \( n-1 \). Lemma 25 shows that the \( \delta \)-distance does not change under multiplication with an orthogonal matrix. Furthermore, Lemma 3 of [6] ensures that the \( \delta \)-distance property is not destroyed by the projection onto the orthogonal complement.

2.2 Identifying an Element of the Optimal Basis

In this section we repeat how an element of the optimal basis can be identified if an optimal solution \( x_c \) for an objective function \( c^T x \) with \( \|c_0 - c\| < \delta/(2n) \) is given (see also [6]).

**Lemma 4** (Lemma 2 of [6]). Let \( B \subseteq \{1, \ldots, m\} \) be the optimal basis of the linear program (1) and let \( B' \) be an optimal basis of the linear program (1) with \( c_0 \) being replaced
by \( c \), where \( \|c_0 - c\| < \delta/(2n) \) holds. Consider the conic combination
\[
c = \sum_{j \in B'} \mu_j a_j.
\]
For \( k \in B' \setminus B \), one has \( \|c_0 - c\| \geq \delta \cdot \mu_k \).

The following corollary whose proof can also be found in [3] gives a constructive way to identify an element of the optimal basis.

**Corollary 5.** Let \( c \in \mathbb{R}^n \) be such that \( \|c_0 - c\| < \delta/(2n) \) and let \( \mu_j \), \( B \), and \( B' \) be defined as in Lemma 4. There exists at least one coefficient \( \mu_k \) with \( \mu_k > 1/n \cdot (1 - \delta/(2 \cdot n)) \) and any \( k \) with this property is an element of the optimal basis \( B \) (assuming \( \|c_0\| = 1 \)).

The corollary implies that given a solution \( x_c \) that is optimal for an objective function \( c^T x \) with \( \|c_0 - c\| < \delta/(2n) \), one can identify an element of the optimal basis by solving the system of linear equations
\[
[a'_1, \ldots, a'_n] \cdot \mu = c,
\]
where the \( a'_i \) denote the constraints that are tight in \( x_c \).

### 2.3 The Shadow Vertex Method

In this section we assume that we are given a linear program of the form \( \max \{ c_0^T x \mid x \in P \} \), where \( P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \) is a bounded polyhedron (i.e., a polytope), and a basic feasible solution \( x_0 \in P \). We assume \( \|c_0\| = \|a_i\| = 1 \) for all rows \( a_i \) of \( A \). Furthermore, we assume that the linear program is non-degenerate.

Due to the assumption \( \|c_0\| = 1 \) it holds \( c_0 \in [-1, 1]^n \). Our algorithm slightly perturbs the given objective function \( c_0^T x \) at random. For each component \( c_0 \) of \( c_0 \) it chooses an arbitrary interval \( I_i \subseteq [-1, 1] \) of length \( 1/\phi \) with \( (c_0)_i \in I_i \), where \( \phi \) denotes a parameter that will be given to the algorithm. Then a random vector \( c \in [-1, 1]^n \) is drawn as follows: Each component \( c_i \) of \( c \) is chosen independently uniformly at random from the interval \( I_i \). We denote the resulting random vector by \( \text{pert}(c_0, \phi) \). Note that we can bound the norm of the difference \( \|c_0 - c\| \) between the vectors \( c_0 \) and \( c \) from above by \( \sqrt{n} \).

The shadow vertex algorithm is given as Algorithm 1. It is assumed that \( \phi \) is given to the algorithm as a parameter. We will discuss later how we can run the algorithm without knowing this parameter. Let us remark that the Steps 5 and 6 in Algorithm 1 are actually not executed separately. Instead of computing the whole projection \( P' \) in advance, the edges of \( P' \) are computed on the fly one after another.

Note that
\[
\|w\| \leq \sum_{k=1}^n \lambda_k \cdot \|u_k\| \leq \sum_{k=1}^n \lambda_k \leq n,
\]
where the second inequality follows because all rows of \( A \) are assumed to have norm 1.

The Shadow Vertex Algorithm yields a path from the vertex \( x_0 \) to a vertex \( x_c \) that is optimal for the linear program \( \max \{ c^T x \mid x \in P \} \) where \( P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \). The following theorem (whose proof can be found in Section 3) bounds the expected length of this path, i.e., the number of pivots.
Algorithm 1: Shadow Vertex Algorithm

1: Generate a random perturbation $c = \text{pert}(c_0, \phi)$ of $c_0$.
2: Determine $n$ linearly independent rows $u_k^T$ of $A$ for which $u_k^T x_0 = b_k$.
3: Draw a vector $\lambda \in (0, 1]^n$ uniformly at random.
4: Set $w = -[u_1, \ldots, u_n] \cdot \lambda$.
5: Use the function $\pi : x \mapsto (c^T x, w^T x)$ to project $P$ onto the Euclidean plane and obtain the shadow vertex polygon $P' = \pi(P)$.
6: Walk from $\pi(x_0)$ along the edges of $P'$ in increasing direction of the first coordinate until a rightmost vertex $\tilde{x}_c$ of $P'$ is found.
7: Output the vertex $x_c$ of $P$ that is projected onto $\tilde{x}_c$.

Theorem 6. For any $\phi \geq \sqrt{n}$ the expected number of edges on the path output by Algorithm 1 is $O\left(\frac{mn^2}{\phi^2} + \frac{m\sqrt{n}}{\phi}\right)$.

Since $\|c_0 - c\| \leq \frac{\sqrt{n}}{\phi}$ choosing $\phi > 2n^{3/2}/\delta$ suffices to ensure $\|c_0 - c\| < \frac{\delta}{2n}$. Hence, for such a choice of $\phi$, by Corollary 5, the vertex $x_c$ has a facet in common with the optimal solution of the linear program of the form $\max\{c_0^T x \mid x \in P\}$ and we can reduce the dimension of the linear program as discussed in Section 2.1. This step is repeated at most $n$ times. It is important that we can start each repetition with a known feasible solution because the transformation in Section 2.1 maps the optimal solution of the linear program of repetition $i$ onto a feasible solution with which repetition $i + 1$ can be initialized. Together with Theorem 6 this implies that an optimal solution of the linear program 1 can be found by performing in expectation $O\left(\frac{mn^3}{\phi} + \frac{mn^{3/2}\phi}{\delta}\right)$ pivots if a basic feasible solution $x_0$ and the right choice of $\phi$ are given. We will refer to this algorithm as repeated shadow vertex algorithm.

Since $\delta$ is not known to the algorithm, the right choice for $\phi$ cannot easily be computed. Instead we will try values for $\phi$ until an optimal solution is found. For $i \in \mathbb{N}$ let $\phi_i = 2n^{3/2}/\delta$. First we run the repeated shadow vertex algorithm with $\phi = \phi_0$ and check whether the returned solution is an optimal solution for the linear program $\max\{c_0^T x \mid x \in P\}$. If this is not the case, we run the repeated shadow vertex algorithm with $\phi = \phi_1$, and so on. We continue until an optimal solution is found. For $\phi = \phi_i$ with $i^* = \lceil \log_2 \left(\frac{1}{\delta}\right) \rceil + 2$ this is the case because $\phi_i > 2n^{3/2}/\delta$.

Since $\phi_i \leq \frac{8n^{3/2}}{\delta}$, in accordance with Theorem 6 each of the at most $i^* = O(\log(1/\delta))$ calls of the repeated shadow vertex algorithm uses in expectation

$$O\left(\frac{mn^3}{\delta^2} + \frac{mn^{3/2}\phi_i}{\delta}\right) = O\left(\frac{mn^3}{\delta^2}\right).$$

pivots. Together this proves the first part of Theorem 2. The second part follows with Lemma 22 which states that Phase 1 can be realized with increasing $1/\delta$ by at most $\sqrt{m}$ and increasing the number of variables from $n$ to $n + m \leq 2m$. This implies that the expected number of pivots of each call of the repeated shadow vertex algorithm in Phase 1 is $O(m(n + m)^{3/2}/(\delta^2)) = O(m^3/\delta^2)$. Since $1/\delta$ can increase by a factor of $\sqrt{m}$, the argument above yields that we need to run the repeated shadow vertex algorithm at
most \( i^* = O(\log(\sqrt{m}/\delta)) \) times in Phase 1 to find a basic feasible solution. By setting \( \phi_i = 2^{i}(n + m)^{3/2} \) instead of \( \phi_i = 2^{i}(n + m)^{3/2} \) this number can be reduced to \( i^* = O(\log(1/\delta)) \) again.

Theorem 3 follows from Theorem 2 using the following fact from [4]: Let \( A \in \mathbb{Z}^{m \times n} \) be an integer matrix and let \( A' \in \mathbb{R}^{m \times n} \) be the matrix that arises from \( A \) by scaling each row such that its norm equals 1. If \( \Delta \) denotes an upper bound for the absolute value of any sub-determinant of \( A \), then \( A' \) satisfies the \( \delta \)-distance property for \( \delta = 1/(\Delta^2 n) \). Additionally Lemma 2 states that Phase 1 can be realized without increasing \( \Delta \) but with increasing the number of variables from \( n \) to \( n + m \leq 2m \). Substituting \( 1/\delta = \Delta^2 n \) in Theorem 2 almost yields Theorem 3 except for a factor \( O(\log(\Delta^2 n)) \) instead of \( O(\log(\Delta + 1)) \). This factor results from the number \( i^* \) of calls of the repeated shadow vertex algorithm. The desired factor of \( O(\log(\Delta + 1)) \) can be achieved by setting \( \phi_i = 2^{i}n^{5/2} \) if a basic feasible solution is known and \( \phi_i = 2^i(n + m)^{5/2} \) in Phase 1.

### 2.4 Running Time

So far we have only discussed the number of pivots. Let us now calculate the actual running time of our algorithm. For an initial basic feasible solution \( x_0 \) the repeated shadow vertex algorithm repeats the following three steps until an optimal solution is found. Initially let \( P' = P \).

**Step 1:** Run the shadow vertex algorithm for the linear program \( \max\{c^T x \mid x \in P'\} \), where \( c = \text{pert}(c_0, \phi) \). We will denote this linear program by \( LP' \).

**Step 2:** Let \( x_c \) denote the returned vertex in Step 1, which is optimal for the objective function \( c^T x \). Identify an element \( a'_i \) of \( x_c \) that is in common with the optimal basis.

**Step 3:** Calculate an orthogonal matrix \( Q \in \mathbb{R}^{n \times n} \) that rotates \( a'_i \) into the first unit vector \( e_1 \) as described in Section 2.1 and set \( LP'' \) to the projection of the current \( LP' \) onto the orthogonal complement. Let \( P' \) denote the polyhedron of feasible solutions of \( LP'' \).

First note that the three steps are repeated at most \( n \) times during the algorithm. In Step 1 the shadow vertex algorithm is run once. Step 1 to Step 4 of Algorithm 1 can be performed in time \( O(m) \) as we assumed \( P \) to be non-degenerate (this implies \( P' \) to be non-degenerate in each further step). Step 5 and Step 6 can be implemented with strongly polynomial running time in a tableau form, described in [2]. The tableau can be set up in time \( O((m - d)d^3) = O(mn^3) \) where \( d \) is the dimension of \( P' \). By Theorem 1 of [2] we can identify for a vertex on a path the row which leaves the basis and the row which is added to the basis in order to move to the next vertex in time \( O(m) \) using the tableau. After that, the tableau has to be updated. This can be done in \( O((m - d)d) = O(mn) \) steps. Using this and Theorem 6 we can compute the path from \( x_0 \) to \( x_1 \) in expected time \( O(mn^3 + mn \cdot (m^2n^3 + m^3n^3/\delta)) \) for \( \phi \leq 8n^{3/2}/\delta \), as discussed above, yields a running time of \( O(m^2n^3) \).

Once we have calculated the basis of \( x_0 \) we can easily compute the element \( a_i \) of the basis that is also an element of the optimal basis. Assume the rows \( a'_1, \ldots, a'_n \) are the
Theorem 7. The repeated shadow vertex algorithm has a running time of $O\left(\frac{m^2n^4}{\delta^2}\right)$.

The entries of both $c$ and $\lambda$ in Algorithm 1 are continuous random variables. In practice it is, however, more realistic to assume that we can draw a finite number of random bits. In Appendix E we will show that our algorithm only needs to draw $\text{poly}(\log m, n, \log(1/\delta))$ random bits in order to obtain the expected running time stated in Theorem 2 if $\delta$ (or a good lower bound for it) is known. However, if the parameter $\delta$ is not known upfront and only discrete random variables with a finite precision can be drawn, we have to modify the shadow vertex algorithm. This will give us an additional factor of $O(n)$ in the expected running time.

3 Analysis of the Shadow Vertex Algorithm

For given linear functions $L_1: \mathbb{R}^n \rightarrow \mathbb{R}$ and $L_2: \mathbb{R}^n \rightarrow \mathbb{R}$ we denote by $\pi = \pi_{L_1,L_2}$ the function $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^2$, given by $\pi(x) = (L_1(x), L_2(x))$. Note that $n$-dimensional vectors can be treated as linear functions. By $P' = P'_{L_1,L_2}$ we denote the projection $\pi(P)$ of the polytope $P$ onto the Euclidean plane, and by $R = R_{L_1,L_2}$ we denote the path from the bottommost vertex of $P'$ to the rightmost vertex of $P'$ along the edges of the lower envelope of $P'$.

Our goal is to bound the expected number of edges of the path $R = R_{c,w}$, which is random since $c$ and $w$ are random. Each edge of $R$ corresponds to a slope in $(0, \infty)$. These slopes are pairwise distinct with probability one (see Lemma 5). Hence, the number of edges of $R$ equals the number of distinct slopes of $R$.

Definition 8. For a real $\varepsilon > 0$ let $\mathcal{F}_\varepsilon$ denote the event that there are three pairwise distinct vertices $z_1, z_2, z_3$ of $P$ such that $z_1$ and $z_3$ are neighbors of $z_2$ and such that

$$\left|\frac{w^T \cdot (z_2 - z_1)}{c^T \cdot (z_2 - z_1)} - \frac{w^T \cdot (z_3 - z_2)}{c^T \cdot (z_3 - z_2)}\right| \leq \varepsilon.$$ 

Note that if event $\mathcal{F}_\varepsilon$ does not occur, then all slopes of $R$ differ by more than $\varepsilon$. Particularly, all slopes are pairwise distinct. First of all we show that event $\mathcal{F}_\varepsilon$ is very unlikely to occur if $\varepsilon$ is chosen sufficiently small. The proof of the following lemma is almost identical to the corresponding proof in [4] except that we need to adapt it to the
different random model of $c$. The proof as well as the proofs of some other lemmas that are almost identical to their counterparts in [4] can be found in Appendix C for the sake of completeness. Proofs that are completely identical to [4] are omitted.

**Lemma 9.** The probability of event $F_\varepsilon$ tends to 0 for $\varepsilon \to 0$.

Let $p$ be a vertex of $R$, but not the bottommost vertex $\pi(x_0)$. We call the slope $s$ of the edge incident to $p$ to the left of $p$ the slope of $p$. As a convention, we set the slope of $\pi(x_0)$ to 0 which is smaller than the slope of any other vertex $p$ of $R$.

Let $t \geq 0$ be an arbitrary real, let $p^*$ be the rightmost vertex of $R$ whose slope is at most $t$, and let $\hat{p}$ be the right neighbor of $p^*$, i.e., $\hat{p}$ is the leftmost vertex of $R$ whose slope exceeds $t$ (see Figure 1). Let $x^*$ and $\hat{x}$ be the neighboring vertices of $P$ with $\pi(x^*) = p^*$ and $\pi(\hat{x}) = \hat{p}$. Now let $i = i(x^*, \hat{x}) \in [m]$ be the index for which $a_i^T x^* = b_i$ and for which $\hat{x}$ is the (unique) neighbor $x$ of $x^*$ for which $a_i^T x < b_i$. This index is unique due to the non-degeneracy of the polytope $P$. For an arbitrary real $\gamma \geq 0$ we consider the vector $\tilde{w} := w - \gamma \cdot a_i$.

**Lemma 10 (Lemma 9 of [4]).** Let $\tilde{\pi} = \pi_{c, \tilde{w}}$ and let $\tilde{R} = R_{c, \tilde{w}}$ be the path from $\tilde{\pi}(x_0)$ to the rightmost vertex $\tilde{p}_r$ of the projection $\tilde{\pi}(P)$ of polytope $P$. Furthermore, let $\hat{p}^*$ be the rightmost vertex of $\tilde{R}$ whose slope does not exceed $t$. Then $\tilde{p}^* = \tilde{\pi}(x^*)$.

Let us reformulate the statement of Lemma 10 as follows: The vertex $\tilde{p}^*$ is defined for the path $\tilde{R}$ of polygon $\tilde{\pi}(R)$ with the same rules as used to define the vertex $p^*$ of the original path $R$ of polygon $\pi(P)$. Even though $\tilde{R}$ and $\tilde{R}$ can be very different in shape, both vertices, $p^*$ and $\tilde{p}^*$, correspond to the same solution $x^*$ in the polytope $P$, that is, $p^* = \pi(x^*)$ and $\tilde{p}^* = \tilde{\pi}(x^*)$.

**Lemma 10** holds for any vector $\tilde{w}$ on the ray $\tilde{r} = \{ w - \gamma \cdot a_i \mid \gamma \geq 0 \}$. As $\|w\| \leq n$ (see Section 2.3), we have $w \in [-n, n]^n$. Hence, ray $\tilde{r}$ intersects the boundary of $[-n, n]^n$ in a unique point $z$. We choose $\tilde{w} = \tilde{w}(w, i) := z$ and obtain the following result.
**Corollary 11.** Let \( \tilde{\pi} = \pi_{c, \hat{\omega}(w,i)} \) and let \( \tilde{p}^* \) be the rightmost vertex of path \( \tilde{R} = R_{c, \hat{\omega}(w,i)} \) whose slope does not exceed \( t \). Then \( \tilde{p}^* = \tilde{\pi}(x^*) \).

Note that Corollary 11 only holds for the right choice of index \( i = i(x^*, \hat{x}) \). However, the vector \( \hat{\omega}(w,i) \) can be defined for any vector \( w \in [-n,n]^n \) and any index \( i \in [m] \). In the remainder, index \( i \) is an arbitrary index from \([m]\).

We can now define the following event that is parameterized in \( i, t, \) and a real \( \varepsilon > 0 \) and that depends on \( c \) and \( w \).

**Definition 12.** For an index \( i \in [m] \) and a real \( t \geq 0 \) let \( \hat{p}^* \) be the rightmost vertex of \( \hat{R} = R_{c, \hat{\omega}(w,i)} \) whose slope does not exceed \( t \) and let \( y^* \) be the corresponding vertex of \( P \).

For a real \( \varepsilon > 0 \) we denote by \( E_{i,t,\varepsilon} \) the event that the conditions

1. \( a_i^T y^* = b_i \) and
2. \( \frac{w^T (\hat{y} - y^*)}{c^T (\hat{y} - y^*)} \in (t, t + \varepsilon] \), where \( \hat{y} \) is the neighbor \( y \) of \( y^* \) for which \( a_i^T y < b_i \),

are met. Note that the vertex \( \hat{y} \) always exists and that it is unique since the polytope \( P \) is non-degenerate.

Let us remark that the vertices \( y^* \) and \( \hat{y} \), which depend on the index \( i \), equal \( x^* \) and \( \hat{x} \) if we choose \( i = i(x^*, \hat{x}) \). For other choices of \( i \), this is, in general, not the case.

Observe that all possible realizations of \( w \) from the line \( L := \{ w + x \cdot a_i \mid x \in \mathbb{R} \} \) are mapped to the same vector \( \hat{\omega}(w,i) \). Consequently, if \( c \) is fixed and if we only consider realizations of \( \lambda \) for which \( w \in L \), then vertex \( \hat{p}^* \) and, hence, vertex \( y^* \) from Definition 12 are already determined. However, since \( w \) is not completely specified, we have some randomness left for event \( E_{i,t,\varepsilon} \) to occur. This allows us to bound the probability of event \( E_{i,t,\varepsilon} \) from above (see proof of Lemma 14). The next lemma shows why this probability matters.

**Lemma 13** (Lemma 12 from [4]). For any \( t \geq 0 \) and \( \varepsilon > 0 \) let \( A_{t,\varepsilon} \) denote the event that the path \( R = R_{c,w} \) has a slope in \((t, t + \varepsilon]\). Then, \( A_{t,\varepsilon} \subseteq \bigcup_{i=1}^{n} E_{i,t,\varepsilon} \).

With Lemma 13 we can now bound the probability of event \( A_{t,\varepsilon} \). The proof of the next lemma is almost identical to the proof of Lemma 13 from [4]. We include it in the appendix for the sake of completeness. The only differences to Lemma 13 from [4] are that we can now use the stronger upper bound \( ||c|| \leq 2 \) instead of \( ||c|| \leq n \) and that we have more carefully analyzed the case of large \( t \).

**Lemma 14.** For any \( \phi \geq \sqrt{n} \), any \( t \geq 0 \), and any \( \varepsilon > 0 \) the probability of event \( A_{t,\varepsilon} \) is bounded by

\[
\Pr[A_{t,\varepsilon}] \leq \frac{2mn^2 \varepsilon}{\max \{ \frac{2}{\phi^2}, t \} \cdot \delta^2} \leq \frac{4mn \varepsilon}{\delta^2}.
\]

**Lemma 15.** For any interval \( I \) let \( X_I \) denote the number of slopes of \( R = R_{c,w} \) that lie in the interval \( I \). Then, for any \( \phi \geq \sqrt{n} \),

\[
\mathbb{E}[X_{(0,n]}] \leq \frac{4mn^2}{\delta^2}.
\]
Lemma 17. For any $\phi \geq \sqrt{n}$, any $t \geq 0$, and any $\varepsilon > 0$ the probability of event $A_{t,\varepsilon}^{-1}$ is bounded by

$$\Pr \left[ A_{t,\varepsilon}^{-1} \right] \leq \frac{2mn^{3/2}\varepsilon\phi}{\max \left\{ 1, \frac{m}{2} \right\} \cdot \delta} \leq \frac{2mn^{3/2}\varepsilon\phi}{\delta}.$$
Proof. Due to Lemma 13 (to be precise, due to its canonical adaption to the events with superscript $-1$) it suffices to show that
\[
\Pr \left[ E_{i,t,\varepsilon}^{-1} \right] \leq \frac{1}{m} \cdot \frac{2mn^{3/2} \varepsilon \phi}{\max \left\{ 1, \frac{nt}{T} \right\} \cdot \delta} = \frac{2n^{3/2} \varepsilon \phi}{\max \left\{ 1, \frac{nt}{T} \right\} \cdot \delta}
\]
for any index $i \in [m]$.

We apply the principle of deferred decisions and assume that vector $w$ is already fixed. Now we extend the normalized vector $a_i$ to an orthonormal basis $(q_1, \ldots, q_{n-1}, a_i)$ of $\mathbb{R}^n$ and consider the random vector $(Y_1, \ldots, Y_{n-1}, Z)^T = Q^T c$ given by the matrix vector product of the transpose of the orthogonal matrix $Q = [q_1, \ldots, q_{n-1}, a_i]$ and the vector $c = (c_1, \ldots, c_n)^T$. For fixed values $y_1, \ldots, y_{n-1}$ let us consider all realizations of $c$ such that $(Y_1, \ldots, Y_{n-1}) = (y_1, \ldots, y_{n-1})$. Then, $c$ is fixed up to the ray
\[
c(Z) = Q \cdot (y_1, \ldots, y_{n-1}, Z)^T = \sum_{j=1}^{n-1} y_j \cdot q_j + Z \cdot a_i = v + Z \cdot a_i
\]
for $v = \sum_{j=1}^{n-1} y_j \cdot q_j$. All realizations of $c(Z)$ that are under consideration are mapped to the same value $\hat{c}$ by the function $c \mapsto \hat{c}(c, i)$, i.e., $\hat{c}(c(Z), i) = \hat{c}$ for any possible realization of $Z$. In other words, if $c = c(Z)$ is specified up to this ray, then the path $R_{\hat{c}(c, i), w}$ and, hence, the vectors $y^*$ and $\hat{y}$ from the definition of event $E_{i,t,\varepsilon}^{-1}$ are already determined.

Let us only consider the case that the first condition of event $E_{i,t,\varepsilon}^{-1}$ is fulfilled. Otherwise, event $E_{i,t,\varepsilon}$ cannot occur. Thus, event $E_{i,t,\varepsilon}^{-1}$ occurs iff
\[
(t, t + \varepsilon) \ni \frac{c^T \cdot (\hat{y} - y^*)}{w^T \cdot (\hat{y} - y^*)} = \frac{v^T \cdot (\hat{y} - y^*)}{w^T \cdot (\hat{y} - y^*)} + Z \cdot \frac{a_i^T \cdot (\hat{y} - y^*)}{w^T \cdot (\hat{y} - y^*)}.
\]
The next step in this proof will be to show that the inequality $|\beta| \geq \max \left\{ 1, \sqrt{n} \cdot t \right\} \cdot \frac{\delta}{n}$ is necessary for event $E_{i,t,\varepsilon}^{-1}$ to happen. For the sake of simplicity let us assume that $\|\hat{y} - y^*\| = 1$ since $\beta$ is invariant under scaling. If event $E_{i,t,\varepsilon}^{-1}$ occurs, then $a_i^T y^* = b_i$, $\hat{y}$ is a neighbor of $y^*$, and $a_i^T \hat{y} \neq b_i$. That is, by Lemma 25 Claim 3 we obtain $|a_i^T \cdot (\hat{y} - y^*)| \geq \delta \cdot \|\hat{y} - y^*\| = \delta$ and, hence,
\[
|\beta| = \left| \frac{a_i^T \cdot (\hat{y} - y^*)}{w^T \cdot (\hat{y} - y^*)} \right| \geq \frac{\delta}{|w^T \cdot (\hat{y} - y^*)|}.
\]
On the one hand we have $|w^T \cdot (\hat{y} - y^*)| \leq \|w\| \cdot \|\hat{y} - y^*\| \leq \left( \sum_{i=1}^{n} \|u_i\| \right) \cdot 1 \leq n$. On the other hand, due to $\frac{c^T \cdot (\hat{y} - y^*)}{t} \geq t$ we have
\[
|w^T \cdot (\hat{y} - y^*)| \leq \frac{|c^T \cdot (\hat{y} - y^*)|}{t} \leq \frac{\|c\| \cdot \|\hat{y} - y^*\|}{t} \leq \frac{\left( 1 + \frac{\sqrt{n}}{\tau} \right)}{t} \leq \frac{2}{\tau},
\]
where the third inequality is due to the choice of $c$ as perturbation of the unit vector $c_0$ and the fourth inequality is due to the assumption $\phi \geq \sqrt{n}$. Consequently,
\[
|\beta| \geq \frac{\delta}{\min \left\{ n, \frac{2}{\tau} \right\}} = \max \left\{ 1, \frac{nt}{T} \right\} \cdot \frac{\delta}{n}.
\]
Summarizing the previous observations we can state that if event \( E_{i,t,\varepsilon}^{-1} \) occurs, then \( |\beta| \geq \max \{ 1, \frac{m}{T} \} \cdot \frac{\delta}{n} \) and \( \alpha + Z \cdot \beta \in (t, t + \varepsilon] \). Hence, 
\[
Z \cdot \beta \in (t, t + \varepsilon] - \alpha,
\]
i.e., \( Z \) falls into an interval \( I(y_1, \ldots, y_{n-1}) \) of length at most \( \varepsilon/(\max \{ 1, \frac{m}{T} \} \cdot \delta/n) = n\varepsilon/(\max \{ 1, \frac{m}{T} \} \cdot \delta) \) that only depends on the realizations \( y_1, \ldots, y_{n-1} \) of \( Y_1, \ldots, Y_{n-1} \). Let \( B_{i,t,\varepsilon}^{-1} \) denote the event that \( Z \) falls into the interval \( I(Y_1, \ldots, Y_{n-1}) \). We showed that \( E_{i,t,\varepsilon}^{-1} \subseteq B_{i,t,\varepsilon}^{-1} \). Consequently, 
\[
\Pr \left[ E_{i,t,\varepsilon}^{-1} \right] \leq \Pr \left[ B_{i,t,\varepsilon}^{-1} \right] \leq \frac{2\sqrt{n\varepsilon \phi}}{\max \{ 1, \frac{m}{T} \} \cdot \delta} \leq \frac{2n^{3/2}\varepsilon \phi}{\max \{ 1, \frac{m}{T} \} \cdot \delta},
\]
where the second inequality is due to Theorem 26 for the orthogonal matrix \( Q \).

**Lemma 18.** For any interval \( I \) let \( X_I^{-1} \) denote the number of slopes of \( R_{w,c} \) that lie in the interval \( I \). Then 
\[
\mathbb{E} \left[ X_{I (0,1/n)}^{-1} \right] \leq \frac{2m\sqrt{n\phi}}{\delta}.
\]

**Proof.** As in the proof of Lemma 15 we define for \( t \in \mathbb{R} \) and \( \varepsilon > 0 \) the random variable \( Z_{t,\varepsilon}^{-1} \) that indicates whether \( R_{w,c} \) has a slope in the interval \((t, t + \varepsilon]\) or not. For any integer \( k \geq 1 \) we obtain 
\[
\mathbb{E} \left[ X_{(0,1/n)}^{-1} \right] \leq \sum_{i=0}^{k-1} \mathbb{E} \left[ Z_{i,1/n}^{-1} \right] + \Pr \left[ F_{1/n}^{-1} \right] \cdot m^n
\]
\[
= \sum_{i=0}^{k-1} \Pr \left[ A_{i,1/n}^{-1} \right] + \Pr \left[ F_{1/n}^{-1} \right] \cdot m^n
\]
\[
\leq \sum_{i=0}^{k-1} \frac{2mn^{3/2}\phi}{kn\delta} + \Pr \left[ F_{1/n}^{-1} \right] \cdot m^n = \frac{2m\sqrt{n\phi}}{\delta} + \Pr \left[ F_{1/n}^{-1} \right] \cdot m^n.
\]
The second inequality stems from Lemma 17. Now the lemma follows because the bound holds for any integer \( k \geq 1 \) and \( \Pr \left[ F_{\varepsilon}^{-1} \right] \to 0 \) for \( \varepsilon \to 0 \) in accordance with Lemma 16.

The following corollary directly implies Theorem 6.

**Corollary 19.** The expected number of slopes of \( R = R_{c,w} \) is
\[
\mathbb{E} \left[ X_{(0,\infty)} \right] = \frac{4mn^2}{\delta^2} + \frac{2m\sqrt{n\phi}}{\delta}.
\]

**Proof.** We divide the interval \((0, \infty)\) into the subintervals \((0, n]\) and \((n, \infty)\). Using Lemma 15, Lemma 18, and linearity of expectation we obtain 
\[
\mathbb{E} \left[ X_{(0,\infty)} \right] = \mathbb{E} \left[ X_{(0,n]} \right] + \mathbb{E} \left[ X_{(n,\infty]} \right] = \mathbb{E} \left[ X_{(0,n]} \right] + \mathbb{E} \left[ X_{(0,1/n)}^{-1} \right]
\]
\[
\leq \frac{4mn^2}{\delta^2} + \frac{2m\sqrt{n\phi}}{\delta}.
\]
In the second step we have exploited that by definition \( X_{(a,b)} = X_{(1/b,1/a)}^{-1} \) for any interval \((a,b)\).
4 Finding a Basic Feasible Solution

In this section we discuss how Phase 1 can be realized. In general there are, of course, several known textbook methods how Phase 1 can be implemented. However, for our purposes it is crucial that the parameter $\delta$ (or $\Delta$) is not too small (or too large) for the linear program that needs to be solved in Phase 1. Ideally we would like it to be identical with the parameter $\delta$ (or $\Delta$) of the matrix $A$ of the original linear program. Eisenbrand and Vempala have addressed this problem and have presented a method to implement Phase 1. Their method is, however, very different from usual textbook methods and needs to solve $m$ different linear programs to find an initial feasible solution for the given linear program.

In this section we will argue that also one of the usual textbook methods can be applied. We argue that $1/\delta$ increases by a factor of at most $\sqrt{m}$ and that $\Delta$ does not change at all in case one considers integer matrices (in particular, for totally unimodular matrices).

Let $m$ and $n$ be arbitrary positive integers, let $A \in \mathbb{R}^{m \times n}$ be an arbitrary matrix without zero-rows, and let $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ be arbitrary vectors. For finding a basic feasible solution of the linear program

\[
\begin{align*}
\text{(LP)} & \quad \max c^T x \\
& \text{s.t. } Ax \leq b
\end{align*}
\]

if one exists, or detecting that none exists, otherwise, we can solve the following linear program:

\[
\begin{align*}
\text{(LP')} & \quad \min \sum_{i=1}^{m} y_i \\
& \text{s.t. } Ax - y \leq b \\
& y \geq 0
\end{align*}
\]

In the remainder of this section let us assume that matrix $A$ has full column rank, that is, $\text{rank}(A) = n$. Otherwise, we can transform the linear program (LP) as stated in Section D.1 before considering (LP’). Furthermore, let us assume that the matrix $\bar{A}$, formed by the first $n$ rows of matrix $A$, is invertible. This entails no loss of generality as this can always be achieved by permuting the rows of matrix $A$.

Let $\bar{b}$ denote the vector given by the first $n$ entries of vector $b$ and let $\bar{x}$ denote the vector for which $\bar{A}\bar{x} = \bar{b}$. The vector $(x', y') = (\bar{x}, \max\{A\bar{x} - b, 0\})$ is a feasible solution of (LP’), where the maximum is meant component-wise and $0$ denotes the $m$-dimensional null vector. This is true because $Ax' - y' \leq \bar{A}\bar{x} - (A\bar{x} - b) = b$ and $y' \geq 0$. Moreover, $(x', y')$ is a basic solution: By the choice of $\bar{x}$ the first $n$ inequalities of $Ax - y \leq b$ are tight as well as the first $n$ non-negativity constraints. For each $k > m$ the $k^{th}$ inequality of $Ax - y \leq b$ or the $k^{th}$ non-negativity constraint is tight. Hence, the number of tight constraints is at least $2n + (m - n) = m + n$, which equals the number of variables of (LP’).

Finally, we observe that a vector $(x, 0)$ is a basic feasible solution of (LP’) if and only if $x$ is a basic feasible solution of (LP). Consequently, by solving the linear program (LP’) we obtain a basic feasible solution of the linear program (LP) (if the optimal value is 0)
or we detect that (LP) is infeasible (if the optimal value is larger than 0). The linear program (LP') can be solved as described in Section 2.3. However, the running time is now expressed in the parameters $m' = 2m$, $n' = m + n$ and $\delta(B)$ (or $\Delta(B)$) of the matrix

$$B = \begin{bmatrix} A & -\mathbb{I}_m \\ \mathbb{O}_{m \times n} & -\mathbb{I}_m \end{bmatrix} \in \mathbb{R}^{(m+m) \times (n+m)}.$$  

Before analyzing the parameters $\delta(B)$ and $\Delta(B)$, let us show that matrix $B$ has full column rank.

**Lemma 20.** The rank of matrix $B$ is $m + n$.

**Proof.** Recall that we assumed that the matrix $\bar{A}$ given by the first $n$ rows of matrix $A$ is invertible. Now consider the first $n$ rows and the last $m$ rows of matrix $B$. These rows form a submatrix $\bar{B}$ of $B$ of the form

$$\bar{B} = \begin{bmatrix} \bar{A} & C \\ \mathbb{O}_{m \times n} & -\mathbb{I}_m \end{bmatrix}$$

for $C = [-\mathbb{I}_{n \times n}, \mathbb{O}_{n \times (m-n)}]$. As $\bar{B}$ is a $2 \times 2$-block-triangular matrix, we obtain $\det(\bar{B}) = \det(\bar{A}) \cdot \det(-\mathbb{I}_n) \neq 0$, that is, the first $n$ rows and the last $m$ rows of matrix $B$ are linearly independent. Hence, $\text{rank}(B) = m + n$. 

The remainder of this section is devoted to the analysis of $\delta(B)$ and $\Delta(B)$, respectively.

## 4.1 A Lower Bound for $\delta(B)$

Before we derive a bound for the value $\delta(B)$, let us give a characterization of $\delta(M)$ for a matrix $M$ with full column rank.

**Lemma 21.** Let $M \in \mathbb{R}^{m \times n}$ be a matrix with rank $n$. Then

$$\frac{1}{\delta(M)} = \max_{k \in [n]} \max \left\{ \|z\| \mid r_1^T, \ldots, r_n^T \text{ linear independent rows of } M \text{ and } [\mathcal{N}(r_1), \ldots, \mathcal{N}(r_n)]^T \cdot z = e_k \right\},$$

where $e_k$ denotes the $k^{th}$ unit vector.

**Proof.** The correctness of the above statement follows from

$$\frac{1}{\delta(M)} = \max \left\{ \frac{1}{\delta(r_1, \ldots, r_n)} \mid r_1^T, \ldots, r_n^T \text{ lin. indep. rows of } M \right\} = \max \left\{ \delta(\mathcal{N}(r_1), \ldots, \mathcal{N}(r_n)) \mid r_1^T, \ldots, r_n^T \text{ lin. indep. rows of } M \right\} = \max \left\{ \max_{k \in [n]} \|v_k\| \mid r_1^T, \ldots, r_n^T \text{ lin. indep. rows of } M \text{ and } [v_1, \ldots, v_n]^{-1} = [\mathcal{N}(r_1), \ldots, \mathcal{N}(r_n)]^T \right\}.$$ 

The first equation is due to the definition of $\delta$, the second equation holds as $\delta$ is invariant under scaling of rows, and the third equation is due to Claim 1 of Lemma 25. The vector $v_k$ from the last line is exactly the vector $z$ for which $[\mathcal{N}(r_1), \ldots, \mathcal{N}(r_n)]^T \cdot z = e_k$. This finishes the proof.
For the following lemma let us without loss of generality assume that the rows of matrix $A$ are normalized. This does neither change the rank of $A$ nor the value $\delta(A)$.

**Lemma 22.** Let $A$ and $B$ be matrices of the form described above. Then

$$\frac{1}{\delta(B)} \leq 2\sqrt{m - n + 1} \cdot \frac{1}{\delta(A)}.$$

**Proof.** In accordance with Lemma 21, it suffices to show that for any $m + n$ linearly independent rows $r_1^T, \ldots, r_{m+n}^T$ of $B$ and any $k = 1, \ldots, m + n$ the inequality

$$\|z\| \leq 2\sqrt{m - n + 1} \cdot \frac{1}{\delta(A)}$$

holds, where $z$ is the vector for which $[N(r_1), \ldots, N(r_{m+n})]^T \cdot z = e_k$.

Let $r_1^T, \ldots, r_{m+n}^T$ be arbitrary $m + n$ linearly independent rows of $B$ and let $k \in [m + n]$ be an arbitrary integer. We consider the equation $\hat{B} \cdot z = e_k$, where $\hat{B} = [N(r_1), \ldots, N(r_{m+n})]^T$. Each row $r_i$ is of either one of the two following types: Type 1 rows correspond to a row from $A$ and for these we have $\|r_i\| = 2$ as the rows of $A$ are normalized. Type 2 rows correspond to a non-negativity constraint of a variable $y_i$. For these we have $\|r_i\| = 1$. Observe that each row has exactly one ‘‘$-1$’’-entry within the last $m$ columns.

We categorize type 1 and type 2 rows further depending on the other selected rows:

Type 1a rows are type 1 rows for which a type 2 row exists among the rows $r_1, \ldots, r_{m+n}$ which has its ‘‘$-1$’’-entry in the same column. This type 2 row is then classified as a type 2a row. The remaining type 1 and type 2 rows are classified as type 1b and type 2b rows, respectively. Observe that we can permute the rows of matrix $\hat{B}$ arbitrarily as we show the claim for all unit vectors $e_k$. Furthermore, we can permute the columns of $\hat{B}$ arbitrarily because this only permutes the rows of the solution vector $z$. This does not influence its norm. Hence, without loss of generality, matrix $\hat{B}$ contains normalizations of type 1a, of type 2a, of type 1b, and of type 2b rows in this order and the normalizations of the type 2a rows are ordered the same way as the normalizations of their corresponding type 1a rows.

Let $m_1$, $m_2$, and $m_3$ denote the number of type 1a, type 1b, and type 2b rows, respectively. Observe that the number of type 2a rows is also $m_1$. As matrix $\hat{B}$ is invertible, each column contains at least one non-zero entry. Hence, we can permute the columns of $\hat{B}$ such that $\hat{B}$ is of the form

$$\hat{B} = \begin{bmatrix}
\frac{1}{2} A_1 & -\frac{1}{2} I_{m_1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\frac{1}{2} I_{m_1} & \mathbf{0} & \mathbf{0} \\
\frac{1}{2} A_2 & \mathbf{0} & -\frac{1}{2} I_{m_2} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -I_{m_3}
\end{bmatrix} \in \mathbb{R}^{(m+n) \times (m+n)},$$

where $A_1$ and $A_2$ are $m_1 \times n$- and $m_2 \times n$-submatrices of $A$, respectively. The number of rows of $\hat{B}$ is $2m_1 + m_2 + m_3 = m + n$, whereas the number of columns of $\hat{B}$ is $n + m_1 + m_2 + m_3 = m + n$. This implies $m_1 = n$ and $m_2 \leq m - n$. Particularly, $A_1$ is a square
matrix. As matrix $\hat{B}$ is a $2 \times 2$-block-triangular matrix and the top left and the bottom right block are $2 \times 2$-block-triangular matrices as well, we obtain

$$\det(\hat{B}) = \det\left(\frac{1}{2}A_1\right) \cdot (-1)^{m_1} \cdot \left(-\frac{1}{2}\right)^{m_2} \cdot (-1)^{m_3} = \pm \det(A_1) \cdot \frac{1}{2^{n+m_2}}.$$  

Due to the linear independence of the rows $r_1^T, \ldots, r_{m+n}^T$ we have $\det(\hat{B}) \neq 0$. Consequently, $\det(A_1) \neq 0$, that is, matrix $A_1$ is invertible.

We partition vector $z$ and vector $e_k$ into four components $z_1, \ldots, z_4$ and $e_k^{(1)}, \ldots, e_k^{(4)}$, respectively, and rewrite the system $\hat{B} \cdot z = e_k$ of linear equations as follows:

$$\frac{1}{2}A_1z_1 - \frac{1}{2}z_2 = e_k^{(1)}$$
$$-z_2 = e_k^{(2)}$$
$$\frac{1}{2}A_2z_1 - \frac{1}{2}z_3 = e_k^{(3)}$$
$$-z_4 = e_k^{(4)}$$

Now we distinguish between four pairwise distinct cases $e_k^{(i)} \neq 0$ for $i = 1, \ldots, 4$. In any case recall that the rows of $A_1$ and $A_2$ are rows of $A$, which are normalized. Furthermore, recall that the rows of $A_1$ are linearly independent.

- **Case 1**: $e_k^{(1)} \neq 0$. In this case we obtain $z_2 = 0$ and $z_4 = 0$. This implies $z_1 = 2\hat{z}$, where $\hat{z}$ is the solution of the equation $A_1\hat{z} = e_k^{(1)} + \frac{1}{2} \cdot 0 = e_k^{(1)}$. As the rows of matrix $A_1$ are normalized, Lemma 21 yields $\|\hat{z}\| \leq 1/\delta(A)$ and, hence, $\|z_1\| \leq 2/\delta(A)$. Next, we obtain $z_3 = A_2z_1 - 2 \cdot e_k^{(3)} = A_2z_1 - 0 = A_2z_1$. Each entry of $z_3$ is a dot product of a (normalized) row from $A$ and $z_1$. Hence, the absolute value of each entry is bounded by $\|z_1\| \leq 2/\delta(A)$. This yields the inequality

$$\|z\| = \sqrt{\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \|z_4\|^2} \leq \sqrt{(1 + m_2) \cdot (2/\delta(A))^2} \leq 2\sqrt{m-n+1}/\delta(A).$$

For the last inequality we used the fact that $m_2 \leq m - n$.

- **Case 2**: $e_k^{(2)} \neq 0$. Here we obtain $z_2 = -e_k^{(2)}$, $z_4 = 0$, and $A_1z_1 = 2 \cdot e_k^{(1)} + z_2 = 2 \cdot 0 - e_k^{(2)} = -e_k^{(2)}$, that is, $z_1 = -\hat{z}$, where $\hat{z}$ is the solution of the equation $A_1\hat{z} = e_k^{(2)}$. Analogously as in Case 1, we obtain $\|\hat{z}\| \leq 1/\delta(A)$ and, hence, $\|z_1\| \leq 1/\delta(A)$. Moreover, we obtain $z_3 = A_2z_1 - 2 \cdot e_k^{(3)} = A_2z_1 - 0 = A_2z_1$, that is, the absolute value of each entry of $z_3$ is bounded by $\|z_1\| \leq 1/\delta(A)$. Consequently,

$$\|z\| \leq \sqrt{1 + (1 + m_2) \cdot (1/\delta(A))^2} \leq \frac{\sqrt{m-n+2}}{\delta(A)} \leq \frac{2\sqrt{m-n+1}}{\delta(A)}.$$  

For the second inequality we used $m_2 \leq m - n$ and $\delta(A) \leq 1$ by definition of $\delta(A)$. In the last inequality we used the fact that $m - n + 1 \geq 1$ and $\sqrt{x+1} \leq 2\sqrt{x}$ for all $x \geq 1/3$.  

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Case 3: $e_k^{(3)} \neq 0$. In this case we obtain $z_2 = 0$, $z_4 = 0$, and hence, $z_1 = 0$. This yields $z_3 = -2 \cdot e_k^{(3)}$ and 

$$
\|z\| = \|z_3\| = 2 \leq \frac{2\sqrt{m-n+1}}{\delta(A)},
$$

where we again used $\delta(A) \leq 1$.

Case 4: $e_k^{(4)} \neq 0$. Here we obtain $z_2 = 0$, $z_4 = -e_k^{(4)}$, and hence, $z_1 = 0$ and $z_3 = 0$. Consequently, we get 

$$
\|z\| = \|z_4\| = 1 \leq \frac{2\sqrt{m-n+1}}{\delta(A)},
$$

which completes this case distinction.

As we have seen, in any case the inequality $\|z\| \leq 2\sqrt{m-n+1}/\delta(A)$ holds, which finishes the proof.

4.2 An Upper Bound for $\Delta(B)$

Although parameter $\Delta(B)$ can be defined for arbitrary real-valued matrices, its meaning is limited to integer matrices when considering our analysis of the expected running time of the shadow vertex method. Hence, in this section we only deal with the case that matrix $A$ is integral. Unlike in Section 4.1, we do not normalize the rows of matrix $A$ before considering the linear program (LP'). As a consequence, matrix $B$ is also integral.

The following lemma establishes a connection between $\Delta(A)$ and $\Delta(B)$.

Lemma 23. Let $A$ and $B$ be of the form described above. Then $\Delta(B) = \Delta(A)$.

Proof. It is clear that $\Delta(B) \geq \Delta(A)$ as matrix $B$ contains matrix $A$ as a submatrix. Thus, we can concentrate on proving that $\Delta(B) \leq \Delta(A)$. For this, consider an arbitrary $k \times k$-submatrix $\hat{B}$ of $B$. Matrix $\hat{B}$ is of the form

$$
\hat{B} = \begin{bmatrix}
A' & -I_1 \\
0_{k_1 \times (k-k_2)} & -I_2
\end{bmatrix},
$$

where $A'$ is a $(k-k_1) \times (k-k_2)$-submatrix of $A$ and $I_1$ and $I_2$ are $(k-k_1) \times k_2$- and $k_1 \times k_2$-submatrices of $I_m$, respectively. Our goal is to show that $|\det(\hat{B})| \leq \Delta(A)$. By analogy with the proof of Lemma 22 we partition the rows of $\hat{B}$ into classes. A row of $\hat{B}$ is of type 1 if it contains a row from $A'$. Otherwise, it is of type 2. Consequently, there are $k - k_1$ type 1 and $k_1$ type 2 rows.

These type 1 and type 2 rows are further categorized into three subtypes depending on the “$-1$”-entry (if exists) within the last $k_2$ columns. Type 1 and type 2 rows that only have zeros in the last $k_2$ entries are classified as type 1c and type 2c rows, respectively. The remaining type 1 and type 2 rows have exactly one “$-1$”-entry within the last $k_2$ columns. These are partitioned into subclasses as follows: If there are a type 1 row and a type 2 row that have their “$-1$”-entry in the same column, then these rows are classified as type 1a and type 2a, respectively. The type 1 and type 2 rows that are neither type 1a nor type 1c nor type 2a nor type 2c are referred to as type 1b and type 2b rows, respectively.
Note that type 2c rows only contain zeros. If matrix $\hat{B}$ contains such a row, then $|\det(\hat{B})| = 0 \leq \Delta(A)$. Hence, in the remainder we only consider the case that matrix $\hat{B}$ does not contain type 2c rows. With the same argument we can assume, without loss of generality, that matrix $\hat{B}$ does not contain a column with only zeros. As permuting the rows and columns of matrix $\hat{B}$ does not change the absolute value of its determinant, we can assume that $\hat{B}$ contains type 1a, type 1c, type 2a, type 1b, and type 2b rows in this order and that the type 2a rows are ordered the same ways as their corresponding type 1a rows. Furthermore, we can permute the columns of $\hat{B}$ such that it has the following form:

$$\hat{B} = \begin{bmatrix}
A_1 & -\mathbb{I} & \mathbb{0} & \mathbb{0} \\
A_2 & \mathbb{0} & \mathbb{0} & \mathbb{0} \\
\mathbb{0} & -\mathbb{I} & \mathbb{0} & \mathbb{0} \\
A_3 & \mathbb{0} & -\mathbb{I} & \mathbb{0} \\
\mathbb{0} & \mathbb{0} & \mathbb{0} & -\mathbb{I}
\end{bmatrix},$$

where $A_1$, $A_2$, and $A_3$ are submatrices of $A'$ and, hence, of $A$. Iteratively decomposing matrix $\hat{B}$ into blocks and exploiting the block-triangular form of the matrices obtained in each step yields

$$|\det(\hat{B})| = \det \left( \begin{bmatrix} A_1 & -\mathbb{I} \\ A_2 & \mathbb{0} \end{bmatrix} \right) \cdot |\det \left( \begin{bmatrix} -\mathbb{I} & \mathbb{0} \\ \mathbb{0} & -\mathbb{I} \end{bmatrix} \right)| = \det \left( \begin{bmatrix} A_1 & -\mathbb{I} \\ A_2 & \mathbb{0} \end{bmatrix} \right) \cdot |\mathbb{0} - \mathbb{I}| = \det \left( \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \right).$$

The absolute value of the latter determinant is bounded from above by $\Delta(A)$. This completes the proof.

5 Conclusions

We have shown that the shadow vertex algorithm can be used to solve linear programs possessing the $\delta$-distance property in strongly polynomial time with respect to $n$, $m$, and $1/\delta$. The bound we obtained in Theorem 2 depends quadratically on $1/\delta$. Roughly speaking, one term $1/\delta$ is due to the fact that the smaller $\delta$ the less random is the objective function $w^T x$. This term could in fact be replaced by $1/\delta(B)$ where $B$ is the matrix that contains only the rows that are tight for $x$. The other term $1/\delta$ is due to our application of the principle of deferred decisions in the proof of Lemma 14. The smaller $\delta$ the less random is $w(Z)$.

For packing linear programs, in which all coefficients of $A$ and $b$ are non-negative and one has $x \geq 0$ as additional constraint, it is, for example, clear that $x = 0^n$ is a basic feasible solution. That is, one does not need to run Phase 1. Furthermore as in this solution without loss of generality exactly the constraints $x \geq 0$ are tight, $\delta(B) = 1$ and one occurrence of $1/\delta$ in Theorem 2 can be removed.
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Appendix

In Appendix [A] we give an equivalent definition of $\delta$ and state some important properties that are used later. Appendix [B] contains some theorems from probability theory that will be used in Appendix [C] which contains the omitted proofs from Section [6]. In Appendix [D] we argue how to cope with unbounded linear programs and linear programs without full column rank. We conclude with Appendix [E] in which we analyze the number of random bits necessary to run the shadow vertex method.

A The Parameter $\delta$

In [4] we introduced the parameter $\delta$ only for $m \times n$-matrices $A$ with rank $n$. This was the only interesting case for the type of problem considered there. In this paper we cannot assume the constraint matrix to have full column rank. Hence, in Definition [1] we extended the definition of $\delta$ to arbitrary matrices (as Eisenbrand and Vempala [6]). We will now give a definition of $\delta$ that is equivalent to Definition [1] and allows to prove some important properties of $\delta$.

**Definition 24.**

1. Let $z_1, \ldots, z_k \in \mathbb{R}^n$ be $k \geq 2$ linearly independent vectors and let $\varphi \in (0, \frac{\pi}{2}]$ be the angle between $z_k$ and span${\{z_1, \ldots, z_{k-1}\}}$. By $\hat{\delta}(\{z_1, \ldots, z_{k-1}\}, z_k) = \sin \varphi$ we denote the sine of $\varphi$. Moreover, we set $\delta(z_1, \ldots, z_k) = \min_{\ell \in [k]} \hat{\delta}(\{z_i \mid i \in [k] \setminus \{\ell\}\}, z_\ell)$.

2. Given a matrix $A = [a_1, \ldots, a_m]^T \in \mathbb{R}^{m \times n}$ with rank $r = \text{rank}(A) \geq 2$, we set $\delta(A) = \min \{\delta(a_{i_1}, \ldots, a_{i_r}) \mid a_{i_1}, \ldots, a_{i_r} \text{ linearly independent}\}$.

Note that for the angle $\varphi$ in Definition [24] we obtain the equation

$$\varphi = \min \{\angle(z_k, z) \mid z \in \text{span}\{z_1, \ldots, z_{k-1}\}\}.$$

Furthermore, the minimum is attained for the orthogonal projection of the vector $z_k$ onto span${\{z_1, \ldots, z_{k-1}\}}$ when we use the convention $\angle(x, 0) := \frac{\pi}{2}$ for any vector $x \in \mathbb{R}^n$. For this reason the sine is given by the length of the orthogonal projection divided by $\|z_k\|$. In the case where $\|z_k\|$ has length 1 this equals the length of the orthogonal projection and thus the $\delta$-distance of $z_k$ to span${\{z_1, \ldots, z_{k-1}\}}$ as defined in Definition [1].

**Lemma 25** (Lemma 5 of [4]). Let $z_1, \ldots, z_n \in \mathbb{R}^n$ be linearly independent vectors of length 1, let $A \in \mathbb{R}^{m \times n}$ be a matrix with rank$(A) = n$, and let $\delta := \delta(A)$. Then the following properties hold:

1. If $M$ is the inverse of $[z_1, \ldots, z_n]^T$, then

$$\delta(z_1, \ldots, z_n) = \frac{1}{\max_{k \in [n]} \|m_k\|} \leq \frac{\sqrt{n}}{\max_{k \in [n]} \|M_k\|},$$

where $[m_1, \ldots, m_n] = M$ and $[M_1, \ldots, M_n] = M^T$.
2. If \( Q \in \mathbb{R}^{n \times n} \) is an orthogonal matrix, then \( \delta(Qz_1, \ldots, Qz_n) = \delta(z_1, \ldots, z_n) \).

3. Let \( y_1 \) and \( y_2 \) be two neighboring vertices of \( P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \) and let \( a_i^T \) be a row of \( A \). If \( a_i^T \cdot (y_2 - y_1) \neq 0 \), then \( |a_i^T \cdot (y_2 - y_1)| \geq \delta \cdot \|y_2 - y_1\| \).

4. If \( A \) is an integral matrix, then \( \frac{1}{\delta} \leq n\Delta_1\Delta_{n-1} \leq n\Delta^2 \), where \( \Delta, \Delta_1, \) and \( \Delta_{n-1} \) are the largest absolute values of any sub-determinant of \( A \) of arbitrary size, of size 1, and of size \( n-1 \), respectively.

### B Some Probability Theory

In this section we state and formulate the corollary about linear combinations of random variables used in Section [3]. This theorem follows from Theorem 3.3 of [3] which we will recite here in a simplified variant.

**Theorem 26** (cf. Theorem 3.3 of [3]). Let \( \varepsilon > 0 \) and \( \phi \geq 1 \) be reals, let \( I_1, \ldots, I_n \subseteq [-1, 1] \) be intervals of length \( 1/\phi \), and let \( X_1, \ldots, X_n \) be independent random variables such that \( X_k \) is uniformly distributed on \( I_k \) for \( k = 1, \ldots, n \). Moreover, let \( A \in \mathbb{R}^{n \times n} \) be an invertible matrix, let \( (Y_1, \ldots, Y_{n-1})^T = A \cdot (X_1, \ldots, X_n)^T \) be the linear combinations of \( X_1, \ldots, X_n \) given by \( A \), and let \( I : \mathbb{R}^{n-1} \to \{ [x, x+\varepsilon] \mid x \in \mathbb{R} \} \) be a function mapping a tuple \( (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1} \) to an interval \( I(y_1, \ldots, y_{n-1}) \) of length \( \varepsilon \). Then the probability that \( Z \) falls into the interval \( I(Y_1, \ldots, Y_{n-1}) \) can be bounded by

\[
\Pr[Z \in I(Y_1, \ldots, Y_{n-1})] \leq 2\varepsilon \phi \sum_{i=1}^{n} \frac{|\det A_n|}{|\det A|},
\]

where \( A_{n,i} \) is the \((n-1) \times (n-1)\)-submatrix of \( A \) obtained from \( A \) by removing row \( n \) and column \( i \).

Now we can state

**Corollary 27.** Let \( \varepsilon, \phi, X_1, \ldots, X_n, A, Y_1, \ldots, Y_{n-1}, Z, \) and \( I \) be as in Theorem 26. Then the probability that \( Z \) falls into the interval \( I(Y_1, \ldots, Y_{n-1}) \) can be bounded by

\[
\Pr[Z \in I(Y_1, \ldots, Y_{n-1})] \leq \frac{2n\varepsilon\phi}{\delta(a_1, \ldots, a_n) \cdot \min_{k \in [n]} \|a_k\|},
\]

where \( a_1, \ldots, a_n \) denote the columns of matrix \( A \). Furthermore, if \( A \) is orthogonal, then even the stronger bound

\[
\Pr[Z \in I(Y_1, \ldots, Y_{n-1})] \leq 2\sqrt{n}\varepsilon\phi
\]

holds.

**Proof.** In accordance with Theorem 26 it suffices to bound the sum \( \sum_{i=1}^{n} \frac{|\det(A_{n,i})|}{|\det(A)|} \) from above. For this, consider the equation \( Ax = e_n \), where \( e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n \) denotes the \( n^{th} \) unit vector. Following Cramer’s rule and Laplace’s formula, we obtain

\[
|x_i| = \frac{|\det([a_1, \ldots, a_{i-1}, e_n, a_{i+1}, \ldots, a_n])|}{|\det(A)|} = \frac{|\det(A_{n,i})|}{|\det(A)|},
\]
Hence, applying Theorem 26 yields
\[
\Pr \left[ Z \in I(\mathbf{Y}_1, \ldots, \mathbf{Y}_{n-1}) \right] \leq 2\varepsilon \phi \cdot \sum_{i=1}^{n} |x_i| = 2\varepsilon \phi \cdot \|x\|_1 \leq 2\sqrt{n} \varepsilon \phi \cdot \|x\|.
\]

Recall, that by \(\|x\|\) we refer to the Euclidean norm \(\|x\|_2\) of \(x\). The claim for orthogonal matrices \(A\) follows immediately since \(\|x\| = \|A^{-1}e_n\| = \|e_n\| = 1\) because \(A^{-1} = A^T\) is orthogonal as well.

For the general case we consider the equation \(\hat{A} \hat{x} = e_n\), where \(\hat{A} = [N(a_1), \ldots, N(a_n)]\) consists of the normalized columns of matrix \(A\). Vector \(\hat{x} = \hat{A}^{-1}e_n\) is the \(n^{th}\) column of the matrix \(\hat{A}^{-1}\). Thus, we obtain
\[
\|\hat{x}\| \leq \max_{r \text{ column of } \hat{A}^{-1}} \|r\| \leq \frac{\sqrt{n}}{\delta(a_1, \ldots, a_n)},
\]
where second inequality is due to Claim 1 of Lemma 25. Due to \(A = \hat{A} \cdot \text{diag}([\|a_1\|, \ldots, \|a_n\|])\), we have
\[
x = A^{-1}e_n = \text{diag} \left( \frac{1}{\|a_1\|}, \ldots, \frac{1}{\|a_n\|} \right) \cdot \hat{A}^{-1}e_n = \text{diag} \left( \frac{1}{\|a_1\|}, \ldots, \frac{1}{\|a_n\|} \right) \cdot \hat{x}.
\]
Consequently, \(\|x\| \leq \|\hat{x}\|/\min_{k \in [n]} \|a_k\|\) and, thus,
\[
\Pr \left[ Z \in I(\mathbf{Y}_1, \ldots, \mathbf{Y}_{n-1}) \right] \leq 2\sqrt{n} \varepsilon \phi \cdot \frac{\|\hat{x}\|}{\min_{k \in [n]} \|a_k\|} \leq \frac{2n \varepsilon \phi}{\delta(a_1, \ldots, a_n) \cdot \min_{k \in [n]} \|a_k\|}.
\]

\[\square\]

\section{C Proofs from Section 3}

In this section we give the omitted proofs from Section 3. These are merely contained for the sake of completeness because they are very similar to the corresponding proofs in 4.

\subsection*{C.1 Proof of Lemma 9}

\textbf{Lemma 28.} The probability that there are two neighboring vertices \(z_1, z_2\) of \(P\) such that \(|c^T \cdot (z_2 - z_1)| \leq \varepsilon \cdot \|z_2 - z_1\|\) is bounded from above by \(2m^n \varepsilon \phi\).

\textbf{Proof.} Let \(z_1\) and \(z_2\) be arbitrary points in \(\mathbb{R}^n\), let \(u = z_2 - z_1\), and let \(A_e\) denote the event that \(|c^T \cdot u| \leq \varepsilon \cdot \|u\|\). As this inequality is invariant under scaling, we can assume that \(\|u\| = 1\). Hence, there exists an index \(i\) for which \(|u_i| \geq 1/\sqrt{n} \geq 1/n\). We apply the principle of deferred decisions and assume that the coefficients \(c_j\) for \(j \neq i\) are already fixed arbitrarily. Then event \(A_e\) occurs if and only if \(c_i \cdot u_i \in [-\varepsilon, \varepsilon] - \sum_{j \neq i} c_j u_j\). Hence, for event \(A_e\) to occur the random coefficient \(c_i\) must fall into an interval of length \(2\varepsilon/|u_i| \leq 2n\varepsilon\). The probability for this is bounded from above by \(2n\varepsilon\).

As we have to consider at most \(\binom{m}{n-1}\) \(\leq m^n\) pairs of neighbors \((z_1, z_2)\), a union bound yields the additional factor of \(m^n\). \[\square\]
Proof of Lemma 2. Let $z_1, z_2, z_3$ be pairwise distinct vertices of $P$ such that $z_1$ and $z_3$ are neighbors of $z_2$ and let $\Delta_z := z_2 - z_1$ and $\Delta'_z := z_3 - z_2$. We assume that $\|\Delta_z\| = \|\Delta'_z\| = 1$. This entails no loss of generality as the fractions in Definition 3 are invariant under scaling. Let $i_1, \ldots, i_{n-1} \in [m]$ be the $n-1$ indices for which $a_{ik}^T z_1 = b_{ik} = a_{ik}^T z_2$. For the case of notation let us assume that $i_k = k$. The rows $a_1, \ldots, a_{n-1}$ are linearly independent because $P$ is non-degenerate. Since $z_1, z_2, z_3$ are distinct vertices of $P$ and since $z_1$ and $z_3$ are neighbors of $z_2$, there is exactly one index $\ell$ for which $a_\ell^T z_3 < b_\ell$, i.e., $a_\ell^T \Delta'_z \neq 0$. Otherwise, $z_1, z_2, z_3$ would be collinear which would contradict the fact that they are pairwise distinct vertices of $P$. Without loss of generality assume that $\ell = n-1$. Since $a_k^T \Delta_z = 0$ for each $k \in [n-1]$, the vectors $a_1, \ldots, a_{n-1}, \Delta_z$ are linearly independent.

We apply the principle of deferred decisions and assume that $c$ is already fixed. Thus, $c^T \Delta_z$ and $c^T \Delta'_z$ are fixed as well. Moreover, we assume that $c^T \Delta_z \neq 0$ and $c^T \Delta'_z \neq 0$ since this happens almost surely due to Lemma 28. Now consider the matrix $M = [a_1, \ldots, a_{n-2}, \Delta_z, a_{n-1}]$ and the random vector $(Y_1, \ldots, Y_{n-1}, Z)^T = M^{-1} \cdot w = -M^{-1} \cdot [u_1, \ldots, u_n] \cdot \lambda$. For fixed values $y_1, \ldots, y_{n-1}$ let us consider all realizations of $\lambda$ for which $(Y_1, \ldots, Y_{n-1}) = (y_1, \ldots, y_{n-1})$. Then

$$\begin{align*}
w^T \Delta_z &= (M \cdot (y_1, \ldots, y_{n-1}, Z)^T)^T \Delta_z \\
&= \sum_{k=1}^{n-2} y_k \cdot a_k^T \Delta_z + y_{n-1} \cdot \Delta_z^T \Delta_z + Z \cdot a_{n-1}^T \Delta_z \\
&= y_{n-1},
\end{align*}$$

i.e., the value of $w^T \Delta_z$ does not depend on the outcome of $Z$ since $\Delta_z$ is orthogonal to all $a_k$. For $\Delta'_z$ we obtain

$$\begin{align*}
w^T \Delta'_z &= (M \cdot (y_1, \ldots, y_{n-1}, Z)^T)^T \Delta'_z \\
&= \sum_{k=1}^{n-2} y_k \cdot a_k^T \Delta'_z + y_{n-1} \cdot \Delta_z^T \Delta'_z + Z \cdot a_{n-1}^T \Delta'_z \\
&= y_{n-1} \cdot \Delta_z^T \Delta'_z + Z \cdot a_{n-1}^T \Delta'_z
\end{align*}$$

as $\Delta'_z$ is orthogonal to all $a_k$ except for $k = \ell = n-1$. The chain of equivalences

$$\left| \frac{w^T \Delta_z}{c^T \Delta_z} - \frac{w^T \Delta'_z}{c^T \Delta'_z} \right| \leq \varepsilon$$

$$\iff \frac{w^T \Delta'_z}{c^T \Delta'_z} \in [-\varepsilon, \varepsilon] + \frac{w^T \Delta_z}{c^T \Delta_z}$$

$$\iff \frac{w^T \Delta'_z}{c^T \Delta'_z} \in [-\varepsilon \cdot |c^T \Delta'_z|, \varepsilon \cdot |c^T \Delta'_z|] + \frac{w^T \Delta_z}{c^T \Delta_z} \cdot c^T \Delta'_z$$

$$\iff Z \cdot a_{n-1}^T \Delta'_z \in [-\varepsilon \cdot |c^T \Delta'_z|, \varepsilon \cdot |c^T \Delta'_z|] + \frac{w^T \Delta_z}{c^T \Delta_z} \cdot c^T \Delta'_z - y_{n-1} \cdot \Delta_z^T \Delta'_z$$

implies, that for event $F_\varepsilon$ to occur $Z$ must fall into an interval $I = I(y_1, \ldots, y_{n-1})$ of length $2\varepsilon \cdot |c^T \Delta'_z|/|a_{n-1}^T \Delta'_z|$. The probability for this to happen is bounded from above
Proof of Lemma 10. We consider a linear auxiliary function \( \tilde{w} : \mathbb{R}^n \to \mathbb{R} \), given by \( \tilde{w}(x) := \tilde{w}^T x + \gamma \cdot b_i \). The paths \( \tilde{R} = R_{e,\tilde{w}} \) and \( \tilde{R} \) are identical except for a shift by \( \gamma \cdot b_i \) in the second coordinate because for \( \tilde{\pi} = \pi_{e,\tilde{w}} \) we obtain

\[
\tilde{\pi}(x) = (e^T x, \tilde{w}^T x + \gamma \cdot b_i) = (e^T x, \tilde{w}^T x) + (0, \gamma \cdot b_i) = \tilde{\pi}(x) + (0, \gamma \cdot b_i)
\]

for all \( x \in \mathbb{R}^n \). Consequently, the slopes of \( \tilde{R} \) and \( \tilde{R} \) are exactly the same (see Figure 2a).

Let \( x \in P \) be an arbitrary point from the polytope \( P \). Then, \( \tilde{w}^T x = w^T x - \gamma \cdot a_i^T x \geq w^T x - \gamma \cdot b_i \). The inequality is due to \( \gamma \geq 0 \) and \( a_i^T x \leq b_i \) for all \( x \in P \). Equality holds, among others, for \( x = x^* \) due to the choice of \( a_i \). Hence, for all points \( x \in P \) the two-dimensional points \( \pi(x) \) and \( \tilde{\pi}(x) \) agree in the first coordinate while the second coordinate of \( \pi(x) \) is at most the second coordinate of \( \tilde{\pi}(x) \) as \( \tilde{w}(x) = \tilde{w}^T x + \gamma \cdot b_i \geq w^T x \). Additionally, we have \( \pi(x^*) = \tilde{\pi}(x^*) \). Thus, path \( \tilde{R} \) is above path \( \tilde{R} \) but they have point \( p^* = \pi(x^*) \) in common. Hence, the slope of \( \tilde{R} \) to the left (right) of \( p^* \) is at most (at least)
the slope of $R$ to the left (right) of $p^*$ which is at most (greater than) $t$ (see Figure 2b). Consequently, $p^*$ is the rightmost vertex of $R$ whose slope does not exceed $t$. Since $R$ and $\tilde{R}$ are identical up to a shift of $(0, \gamma \cdot b_i)$, $\tilde{\pi}(x^*)$ is the rightmost vertex of $\tilde{R}$ whose slope does not exceed $t$, i.e., $\tilde{\pi}(x^*) = \tilde{p}^*$.

\[ \square \]

C.3 Proof of Lemma 14

Proof of Lemma 14. Due to Lemma 13 it suffices to show that

\[
\Pr[E_{i,t,\varepsilon}] \leq \frac{1}{m} \cdot \frac{2mn^2\varepsilon}{\max\left\{\frac{n}{2}, t\right\} \cdot \delta^2} = \frac{2n^2\varepsilon}{\max\left\{\frac{n}{2}, t\right\} \cdot \delta^2}
\]

for any index $i \in [m]$.

We apply the principle of deferred decisions and assume that vector $c$ is already fixed. Now we extend the normalized vector $a_i$ to an orthonormal basis $\{q_1, \ldots, q_n-1, a_i\}$ of $\mathbb{R}^n$ and consider the random vector $(Y_1, \ldots, Y_{n-1}, Z)^\top = Q^\top w$ given by the matrix vector product of the transpose of the orthogonal matrix $Q = [q_1, \ldots, q_{n-1}, a_i]$ and the vector $w = -[u_1, \ldots, u_n] \cdot \lambda$. For fixed values $y_1, \ldots, y_{n-1}$ let us consider all realizations of $\lambda$ such that $(Y_1, \ldots, Y_{n-1}) = (y_1, \ldots, y_{n-1})$. Then, $w$ is fixed up to the ray

\[
w(Z) = Q \cdot (y_1, \ldots, y_{n-1}, Z)^\top = \sum_{j=1}^{n-1} y_j \cdot q_j + Z \cdot a_i = v + Z \cdot a_i
\]

for $v = \sum_{j=1}^{n-1} y_j \cdot q_j$. All realizations of $w(Z)$ that are under consideration are mapped to the same value $\tilde{w}$ by the function $w \mapsto \tilde{w}(w, i)$, i.e., $\tilde{w}(w(Z), i) = \tilde{w}$ for any possible realization of $Z$. In other words, if $w = w(Z)$ is specified up to this ray, then the path $R_{c, \tilde{w}(w, i)}$ and, hence, the vectors $y^*$ and $\tilde{y}$ from the definition of event $E_{i,t,\varepsilon}$, are already determined.

Let us only consider the case that the first condition of event $E_{i,t,\varepsilon}$ is fulfilled. Otherwise, event $E_{i,t,\varepsilon}$ cannot occur. Thus, event $E_{i,t,\varepsilon}$ occurs iff

\[
(t, t + \varepsilon) \ni \frac{w^T \cdot (\tilde{y} - y^*)}{c^T \cdot (\tilde{y} - y^*)} = \frac{v^T \cdot (\tilde{y} - y^*)}{c^T \cdot (\tilde{y} - y^*)} + Z \cdot \frac{a_i^T \cdot (\tilde{y} - y^*)}{c^T \cdot (\tilde{y} - y^*)} = \alpha \geq \beta
\]

The next step in this proof will be to show that the inequality $|\beta| \geq \max\{\frac{\gamma}{2}, t\} \cdot \frac{\delta}{n}$ is necessary for event $E_{i,t,\varepsilon}$ to happen. For the sake of simplicity let us assume that $\|\tilde{y} - y^*\| = 1$ since $\beta$ is invariant under scaling. If event $E_{i,t,\varepsilon}$ occurs, then $a_i^T y^* = b_i$, $\tilde{y}$ is a neighbor of $y^*$, and $a_i^T \tilde{y} \neq b_i$. That is, by Lemma 25 Claim 3 we obtain $|a_i^T \cdot (\tilde{y} - y^*)| \geq \delta \cdot \|\tilde{y} - y^*\| = \delta$ and, hence,

\[
|\beta| = \left| \frac{a_i^T \cdot (\tilde{y} - y^*)}{c^T \cdot (\tilde{y} - y^*)} \right| \geq \frac{\delta}{|c^T \cdot (\tilde{y} - y^*)|}.
\]

On the one hand we have $|c^T \cdot (\tilde{y} - y^*)| \leq \|c\| \cdot \|\tilde{y} - y^*\| \leq (1 + \frac{\sqrt{n}}{\rho}) \cdot 1 \leq 2$, where the second inequality is due to the choice of $c$ as perturbation of the unit vector $c_0$ and the
third inequality is due to the assumption $\phi \geq \sqrt{n}$. On the other hand, due to $\frac{w^T \cdot (\hat{y} - y^*)}{c^T \cdot (\hat{y} - y^*)} \geq t$ we have

$$|c^T \cdot (\hat{y} - y^*)| \leq \frac{|w^T \cdot (\hat{y} - y^*)|}{c^T \cdot (\hat{y} - y^*)} \leq \frac{n}{t} \cdot \frac{\|w\|}{t} \cdot \frac{\|\hat{y} - y^*\|}{t} \leq \frac{n}{t}.$$  

Consequently,

$$|\beta| \geq \frac{\delta}{\min \left\{ \frac{2}{n}, \frac{n}{t} \right\}} = \max \left\{ \frac{n}{2}, t \right\} \cdot \frac{\delta}{n}.$$

Summarizing the previous observations we can state that if event $E_{i,t,\varepsilon}$ occurs, then $|\beta| \geq \max \left\{ \frac{n}{2}, t \right\} \cdot \frac{\delta}{n}$ and $\alpha + Z \cdot \beta \in \left( t, t + \varepsilon \right]$. Hence,

$$Z \cdot \beta \in (t, t + \varepsilon] - \alpha,$$

i.e., $Z$ falls into an interval $I(y_1, \ldots, y_{n-1})$ of length at most $\varepsilon/(\max \left\{ \frac{n}{2}, t \right\} \cdot \delta/n) = n\varepsilon/(\max \left\{ \frac{n}{2}, t \right\} \cdot \delta)$ that only depends on the realizations $y_1, \ldots, y_{n-1}$ of $Y_1, \ldots, Y_{n-1}$.

Let $B_{i,t,\varepsilon}$ denote the event that $Z$ falls into the interval $I(Y_1, \ldots, Y_{n-1})$. We showed that $E_{i,t,\varepsilon} \subseteq B_{i,t,\varepsilon}$. Consequently,

$$\Pr[E_{i,t,\varepsilon}] \leq \Pr[B_{i,t,\varepsilon}] \leq \frac{2n \cdot \max \left\{ \frac{n}{2}, t \right\} \cdot \delta}{\delta(Q^Tu_1, \ldots, Q^Tu_n)} \leq \frac{2n^2 \varepsilon}{\max \left\{ \frac{n}{2}, t \right\} \cdot \delta^2},$$

where the second inequality is due to Corollary 27 (applied with $\phi = 1$): By definition, we have

$$(Y_1, \ldots, Y_{n-1}, Z)^T = Q^Tw = Q^T \cdot [u_1, \ldots, u_n] \cdot \lambda = [-Q^Tu_1, \ldots, -Q^Tu_n] \cdot \lambda.$$

The third inequality stems from the fact that $\delta(-Q^Tu_1, \ldots, -Q^Tu_n) = \delta(u_1, \ldots, u_n) \geq \delta$, where the equality is due to the orthogonality of $-Q$ (Claim 2 of Lemma 25).

D Justification of Assumptions

We assumed the matrix $A \in \mathbb{R}^{m \times n}$ to have full column rank and we assumed the polyhedron $\{ x \in \mathbb{R}^n \mid Ax \leq b \}$ to be bounded. In this section we show that this entails no loss of generality by giving transformations of arbitrary linear programs into linear programs with full column rank whose polyhedra of feasible solutions are bounded.

D.1 Raising the Rank of Matrix A

For the algorithm we have assumed that the matrix $A$ determining the polyhedron $P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$ has full column rank. In this section we provide a solution if this condition is not met. For this, we describe the transformation of $A$ into a matrix $A'$ with full column rank by adding new linearly independent rows (we will ensure that the $\delta$-distance property respectively the value of $\Delta$ is not violated by the transformation of $A$ into $A'$).
D.1.1 Transformation with respect to $\delta$

Assume that we have an arbitrary matrix $A = [a_1, \ldots, a_m]^T \in \mathbb{R}^{m \times n}$ with rank $r = \text{rank}(A) < n$. This implies that the polyhedron $P = \{ x \mid Ax \leq b \}$ has no vertices. Let $c \in \mathbb{R}^n$ be an arbitrary vector. Then the linear program max $\{ c^T x \mid Ax \leq b \}$ has either no solution (this is true if $P$ is empty or $c^T x$ is unbounded) or infinitely many solutions. We distinguish two different cases.

**Case 1:** $c \in \text{span}\{a_1, \ldots, a_m\}$

Let $\text{span}\{a_1, \ldots, a_m\}^\perp$ denote the orthogonal complement of $\text{span}\{a_1, \ldots, a_m\}$. Furthermore let $o_1, \ldots, o_{n-r}$ be an orthonormal basis of $\text{span}\{a_1, \ldots, a_m\}^\perp$. Then the set of solutions $L = \{ \arg \max c^T x \mid Ax \leq b \}$ equals the set

$$L = \{ v + \arg \max c^T x \mid Ax \leq b, \hat{A}x = \emptyset, v \in \text{span}\{a_1, \ldots, a_m\}^\perp \} ,$$

where $\hat{A} = [a_1, \ldots, o_{n-r}]$. Thus we can add rows $[o_1, -o_1, \ldots, o_{n-r}, -o_{n-r}]$ and extend the vector $b$ by zero entries and calculate the set of solutions (note that the $\delta$-distance-property does not change under this extension of $A$ by Lemma 29). This equals the case where $n - r$ basis variables are known and we can proceed as in Section 2.1 by reducing the polyhedron to dimension $r$.

**Case 2:** $c \notin \text{span}\{a_1, \ldots, a_m\}$

We maintain the notation from above. Then we have a linear combination

$$c = \sum_{i=1}^{r} \ell_i \cdot a_i + \sum_{i=1}^{n-r} \ell_i \cdot o_i ,$$

where $\ell_k \neq 0$ for at least one $k \in [n-r]$. Without loss of generality we may assume that $\ell_k > 0$. But $x$ is not bounded for direction $o_k$ by $A$ and thus $o_k$’s coefficient in the linear combination of $x$ may be chosen arbitrarily large. Thus max $\{ c^T x \mid Ax \leq b \}$ is unbounded.

Finally we prove that adding rows from the orthogonal complement of $\{a_1, \ldots, a_m\}$ to $A$ does not change the $\delta$-distance property.

**Lemma 29.** Let $A = [a_1, \ldots, a_m]^T \in \mathbb{R}^{m \times n}$ be an arbitrary matrix of rank $r \leq n - 1$ and $\|a_i\| = 1$ for $i \in [m]$. Let $v \in \mathbb{R}^n$ be a vector such that $\|v\| = 1$ and $\langle v, a_i \rangle = 0$ for $i \in [m]$. Then $\text{rank}(A') = r + 1$ and furthermore $\delta(A') = \delta(A)$ where $A' = [a_1, \ldots, a_m, v]^T \in \mathbb{R}^{m \times n}$ is defined by adding the new row $v^T$ to matrix $A$.

**Proof.** First choose $r$ linearly independent rows of $A$. Without loss of generality we may assume $a_1, \ldots, a_r$. To calculate $\delta(\{a_1, \ldots, a_{r-1}\}, a_r)$ we choose a vertex $x \in \text{span}\{a_1, \ldots, a_r\}$ with $x \cdot a_i = 0$ for $i \in [r-1]$ and $x \cdot a_r = 1$. Let $\alpha$ be the angle between $a_r$ and $x$. Then $\delta(\{a_1, \ldots, a_{r-1}\}, a_r) = \sin(\pi/2 - \alpha) = \cos(\alpha) = \frac{\|a_r\|}{\|x\|} = \frac{1}{\|x\|}$. Moreover let $v, o_2, \ldots, o_{n-r}$ be an orthonormal basis of the orthogonal complement $\text{span}\{a_1, \ldots, a_r\}^\perp$. Then $x \cdot v = 0$ and $x \cdot o_{i+1} = 0$ for $i \in [n-r-1]$ because of $x \in \text{span}\{a_1, \ldots, a_r\}$. Thus, $x$ is the unique solution of the system of linear equations

$$[a_1, \ldots, a_r, v, o_2, \ldots, o_{n-r}]^T x = e_r ,$$

where $e_r \in \mathbb{R}^n$ denotes the $r^{th}$ canonical unit vector.
In accordance with Definition 24 and Lemma 25, we obtain $\delta(A)$ by choosing a solution with minimum norm over all such systems of linear equations and all vectors $e_i \in \mathbb{R}^n$ with $i \in [r]$. Consider now the matrix $A'$ which is obtained by adding the row $v$ to $A$. To calculate $\delta(A')$ we have to calculate the minimal norm $\frac{1}{\|x\|}$ of the set of solutions of the systems of linear equations of the form

$$[a_1', \ldots, a_r', v, o_2, \ldots, o_{n-r}]^T x = e_i,$$

with $i \in [r + 1]$. In the case where $i \leq r$ the set of systems of linear equations equals the set of systems of linear equation from the case where we calculate $\delta(A)$. Thus the minimum norm does not change and we obtain $\delta(A') = \delta(A)$.

In the case where $i = r + 1$ the solution of the systems of linear equations is given by $x = v$ and we obtain $1/\|x\| = 1$. But this is the maximum norm, which can be reached by a solution $x$ and thus the minimum norm does not change at all which completes the proof.

### D.1.2 Transformation with respect to $\Delta$

If we want to ensure that the value $\Delta(A)$ does not change under the tranformation (which means $\Delta(A') = \Delta(A)$) we have to consider a slight modification of the above transformation. Especially, we will add vectors $e_i$ for $i \in [n]$ which are part of the canonical basis of $\mathbb{R}^n$ such that $e_i \notin \text{span}\{a_1, \ldots, a_m\}$.

Again, we know that in Case 1 the polyhedron is either empty or has infinitely many solutions. Thus, if we find a solution

$$x' \in \{\arg\max c^T x \mid Ax \leq b, \hat{A}x = \emptyset\},$$

where $\hat{A} = [e_{i_1}, \ldots, e_{i_{n-r}}]$ we already know that $x'$ also maximizes $c$ with respect to $P$. Furthermore if we are in Case 2 which means $c \notin \text{span}\{a_1, \ldots, a_m\}$ then the function $c^T x$ is unbounded for elements $x \in P$. It remains to show that $\Delta$ does not change by adding rows $e_i$ to $A$.

**Lemma 30.** Let $A = [a_1, \ldots, a_m]^T \in \mathbb{Z}^{m \times n}$ be an arbitrary matrix of rank $r \leq n - 1$. Let $e_r \in \mathbb{R}^n$ be a vector part of the canonical basis of $\mathbb{R}^n$ such that $e_r \notin \text{span}\{a_1, \ldots, a_m\}$. Then $\text{rank}(A') = r + 1$ and furthermore $\Delta(A') = \Delta(A)$ where $A' = [a_1, \ldots, a_m, e_r]^T$ is defined by adding the new row $e_r^T$ to matrix $A$.

**Proof.** Let $B$ be a submatrix of $A'$. Then either $B$ contains no entries from row $e_r^T$ (which means $\det(B) \leq \Delta(A)$) or one row of $B$ is a subvector $e'$ of $e_i^T$. We distinguish two different cases:

**Case 1:** $e' = 0$. Then $B$ has a zero row and thus $\det(B) = 0 \leq \Delta(A)$.

**Case 2:** $e' \neq 0$. In this case $B$ has a row $e'^T$, which is element of the canonical basis. Then $B$ has the form

$$\begin{pmatrix} \hat{A} \\ e' \end{pmatrix}.$$

Using the Laplace expansion, the absolute value of the determinant of $B$ is at most the absolute value of a determinant of a submatrix of $\hat{A}$ which is a submatrix of $A$. We obtain $\det(B) \leq \Delta(A)$. This concludes the proof. \qed
D.2 Translation into a Bounded Polyhedron

For the algorithm we have assumed that the polyhedron \( P = \{ x \in \mathbb{R} \mid Ax \leq b \} \) is bounded. This may be done because in the case where \( P \) is unbounded we transform \( P \) into a polyhedron \( P' \) and run the algorithm for \( P' \). If the optimum solution is unique and not a vertex of \( P \), then we assert that the linear program \( \max \{ c^T x \mid Ax \leq b \} \) is unbounded. To transform \( P \) we use the construction applied in \( \mathbb{Q} \). First we choose \( n \) linearly independent rows of \( A \). Without loss of generality we may assume the rows are given by \( a_1, \ldots, a_n \).

If we find a ball \( B(0) \) with radius \( r \) which contains all vertices of \( P \), then we define a parallelpiped

\[
Z = \{ x \in \mathbb{R}^n \mid -r \leq a_i \cdot x \leq r, i \in [n] \},
\]

which contains all vertices of \( P \) and does not violate the \( \delta \)-distance property since it is defined by rows of \( A \). Finally, set \( P' = P \cap Z \) and start the algorithm on polytope \( P' \). Note that \( P' \) has \( \delta \)-distance since the set of rows of \( A \) did not change during the transformation.

To construct a ball with the desired properties we have to assume \( A \in \mathbb{Q}^{m \times n} \) such as \( b \in \mathbb{Q}^m \), which means no loss of generality for the implementation of the algorithm. By a slight generalization of Lemma 3.1.33 of \( [7] \) all vertices of the polyhedron \( P \) are contained in a ball \( B(0) \) with radius \( r = \sqrt{m} \cdot 2^{\text{enc}(A,b) - n^2} \cdot \text{lcm}(A)^n \) if \( A \in \mathbb{Q} \), where the function \( \text{enc} \) returns the encoding length and the function \( \text{lcm}(M) \) for a rational matrix \( M \in \mathbb{Q}^{m \times n} \) returns the least common multiple of the denominators of the entries of \( M \).

As a convention, the denominator of 0 is defined as 1.

**Lemma 31.** If \( P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \) for \( A \in \mathbb{Q}^{m \times n} \) and \( b \in \mathbb{Q}^m \), then all vertices of \( P \) are contained in a ball \( B \) around 0 with radius \( r = \sqrt{m} \cdot 2^{\text{enc}(A,b) - n^2} \cdot \text{lcm}(A)^n \).

**Proof.** We have to calculate an upper bound for the length of all vertices. Thus, for each submatrix \( B \) of \( A \) of rank \( n \) and the corresponding subvector \( b' \) of \( b \) we have to bound the length of the solution \( x \) of \( Bx = b' \). Applying Cramer’s rule, the components \( x_i \) of \( x \) are given by

\[
x_i = \frac{\det(B_i)}{\det(B)},
\]

where \( B_i \) equals \( B \) after replacing the \( i \)-th column of \( B \) by \( b' \). We obtain \( \text{lcm}(B)^n \cdot \det(B) = \det(\text{lcm}(B) \cdot B) \geq 1 \) since \( \text{lcm}(B) \cdot B \) is integral and non-singular. All together we obtain

\[
x_i = \frac{\det(B_i)}{\det(B)} \leq \det(B_i) \cdot \text{lcm}(B)^n \leq 2^{\text{enc}(B_i) - n^2} \cdot \text{lcm}(B)^n \leq 2^{\text{enc}(B,b') - n^2} \cdot \text{lcm}(B)^n.
\]

Thus, choosing \( r = \sqrt{n} \cdot 2^{\text{enc}(A,b) - n^2} \cdot \text{lcm}(A)^n \) the ball \( B(0) \) with radius \( r \) contains all vertices of \( P \).

\[\square\]

E An Upper Bound on the Number of Random Bits

For our analysis we assumed that we can draw continuous random variables. In practice it is, however, more realistic to assume that we can draw a finite number of random bits.
In this section we will show that our algorithm only needs to draw \( \text{poly}(\log m, n, \log(1/\delta)) \) bits in order to obtain the expected running time stated in Theorem 2. However, if the parameter \( \delta \) is not known to our algorithm, we have to modify the shadow vertex algorithm. This will give us an additional factor of \( O(n) \) in the expected running time.

Let us assume that we want to approximate a uniform random draw \( X \) from the interval \([0, 1]\) with \( k \) random bits \( Y_1, \ldots, Y_k \in \{0, 1\} \). (A draw from an arbitrary interval \([a, b]\) can be simulated by drawing a random variable from \([0, 1]\) and then applying the affine linear function \( x \mapsto a + (b - a) \cdot x \).) We consider the random variable \( Z = \sum_{\ell=1}^k Y_{\ell} \cdot 2^{-\ell} \).

We observe that the random variable \( Z \) has the same distribution as the random variable \( g(X) \), where \( g(x) = \lfloor x \cdot 2^k \rfloor / 2^k \). Note that \( |g(X) - X| \leq 2^{-k} \). Hence, instead of considering discrete variables and going through the whole analysis again, we will argue that, with high probability, the number of slopes of the shadow vertex polygon does not change if each random variable is perturbed by not more than a sufficiently small \( \varepsilon \). If we have proven such a statement, this implies that we can approximate our continuous uniform random draws as discussed above by using \( O(\log(1/\varepsilon)) \) bits for each draw. Recall that our algorithm draws two random vectors \( \lambda \in (0, 1]^n \) and \( c \in [-1, 1]^n \) that we have to deal with in this section.

For a vector \( x \in \mathbb{R}^n \) and a real \( \varepsilon > 0 \) let \( U_\varepsilon(x) \subseteq [-1, 1]^n \) denote the set of vectors \( x' \in [-1, 1]^n \) for which \( \|x' - x\|_\infty \leq \varepsilon \), that is, \( x' \) and \( x \) differ in each component by at most \( \varepsilon \). In the remainder let us only consider values \( \varepsilon \in (0, 1] \).

Whenever a vector \( c \in [-1, 1]^n \) and a vector \( \hat{c} \in U_\varepsilon(c) \) are defined, then by \( \Delta_c \) we refer to the difference \( \Delta_c := \hat{c} - c \). Observe that \( \|\Delta_c\| \leq \sqrt{n}\varepsilon \). The same holds for the vectors \( \lambda \in (0, 1]^n \), \( \hat{\lambda} \in U_\varepsilon(\lambda) \), and \( \Delta_\lambda := \hat{\lambda} - \lambda \). When the vectors \( \lambda \) and \( \hat{\lambda} \) are defined, then the vectors \( w \) and \( \hat{w} \) are defined as \( w := -[u_1, \ldots, u_n] \cdot \lambda \) and \( \hat{w} := -[u_1, \ldots, u_n] \cdot \hat{\lambda} \) (cf. Algorithm 1). Furthermore, the vector \( \Delta_w \) is defined as \( \Delta_w := \hat{w} - w \). Note that \( \|w\| = \|[u_1, \ldots, u_n] \cdot \lambda\| \leq \sum_{\ell=1}^n \|u_\ell\| \leq \lambda \) as the rows \( u_1^T, \ldots, u_n^T \) of matrix \( A \) are normalized. Similarly, \( \|\hat{w}\| \leq \lambda \) and \( \|\Delta_w\| \leq n\varepsilon \). We will frequently make use of these inequalities without discussing their correctness again.

If \( P \) denotes the non-degenerate bounded polyhedron \( \{x \in \mathbb{R}^n \mid Ax \leq b\} \), then we denote by \( V_k(P) \) the set of all \( k \)-tuples \( (z_1, \ldots, z_k) \) of pairwise distinct vertices \( z_1, \ldots, z_k \) of \( P \) such that for any \( i = 1, \ldots, k - 1 \) the vertices \( z_i \) and \( z_{i+1} \) are neighbors, that is, they share exactly \( n - 1 \) tight constraints. In other words, \( V_k(P) \) contains the set of all simple paths of length \( k - 1 \) of the edge graph of \( P \). Note that \( |V_k(P)| \leq \binom{m}{n} \cdot n^{k-1} \leq m^n n^{k-2} \). For our analysis only \( V_2(P) \) and \( V_3(P) \) are relevant.

The following lemma is an adaption of Lemma 28 for our needs in this section and follows from Lemma 28.

**Lemma 32.** The probability that there exist a pair \( (z_1, z_2) \in V_2(P) \) and a vector \( \hat{c} \in U_\varepsilon(c) \) for which \( \hat{c}^T \cdot (z_2 - z_1) = 0 \) is bounded from above by \( 2m^n n^{3/2}\varepsilon \).

**Proof.** Let \( c \in [-1, 1]^n \) be a vector such that there exists a vector \( \hat{c} \in U_\varepsilon(c) \) for which \( \hat{c}^T \cdot (z_2 - z_1) = 0 \) for an appropriate pair \( (z_1, z_2) \in V_2(P) \). Then
\[
\|c^T \cdot (z_2 - z_1)\| = |\hat{c}^T \cdot (z_2 - z_1) - \Delta_c^T \cdot (z_2 - z_1)|
\leq \|\Delta_c\| \cdot \|z_2 - z_1\|
\leq \sqrt{n}\varepsilon \cdot \|z_2 - z_1\|.
\]

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In accordance with Lemma 28, the probability of this event is bounded from above by $2m^m n^{3/2} \varepsilon \phi$.

A similar statement as Lemma 32 can be made for the objective $w$. However, for our purpose we need a slightly stronger statement.

**Lemma 33.** The probability that there exist a pair $(z_1, z_2) \in V_2(P)$ and a vector $\hat{\lambda} \in U_\varepsilon(\lambda)$ for which $|\hat{w}^T \cdot (z_2 - z_1)| \leq n \varepsilon^{1/3} \cdot \|z_2 - z_1\|$, where $\hat{w} = [-u_1, \ldots, u_n] \cdot \hat{\lambda}$ (cf. Algorithm 1), is bounded from above by $4m^m n^{2\varepsilon^{1/3}/\delta}$.

**Proof.** Fix a pair $(z_1, z_2) \in V_2(P)$ and let $\Delta_z := z_2 - z_1$. Without loss of generality let us assume that $\|\Delta_z\| = 1$. The event $\hat{w}^T \Delta_z \in [-n \varepsilon^{1/3}, n \varepsilon^{1/3}]$ is equivalent to

$$w^T \Delta_z \in [-n \varepsilon^{1/3}, n \varepsilon^{1/3}] - \Delta_w^T \Delta_z.$$  

This interval is a subinterval of $[-2n \varepsilon^{1/3}, 2n \varepsilon^{1/3}]$ as

$$|\Delta_w^T \Delta_z| \leq \|\Delta_w\| \cdot \|\Delta_z\| \leq n \varepsilon \cdot 1 \leq n \varepsilon^{1/3}$$

when recalling that $\varepsilon \leq 1$. Since

$$w^T \Delta_z \in [-2n \varepsilon^{1/3}, 2n \varepsilon^{1/3}] \iff (U\lambda)^T \Delta_z \in [-2n \varepsilon^{1/3}, 2n \varepsilon^{1/3}] \iff \lambda^T y \in [-2n \varepsilon^{1/3}, 2n \varepsilon^{1/3}]$$

for $U = [u_1, \ldots, u_n]$ and $y = U^T \Delta_z$, in the next part of this proof we will derive a lower bound for $\|y\|$. Particularly, we will show that $\|y\| \geq \delta/\sqrt{n}$.

Let $M := [m_1, \ldots, m_n] := (U^T)^{-1}$. Due to $\Delta_z = My$, we obtain $1 = \|\Delta_z\| \leq \|M\| \cdot \|y\|$, which implies $\|y\| \geq 1/\|M\|$. In accordance with Lemma 25 Claim 1 we obtain

$$\max_{k \in [n]} \|m_k\| = \frac{1}{\delta(u_1, \ldots, u_n)} \leq \frac{1}{\delta}.$$

Consequently,

$$\|Mx\| \leq \sum_{k=1}^n \|m_k\| \cdot |x_k| \leq \sum_{k=1}^n \frac{1}{\delta} \cdot |x_k| = \frac{\|x\|}{\delta} \leq \frac{\sqrt{n} \cdot \|x\|}{\delta}$$

for any vector $x \neq 0$, i.e., $\|M\| = \sup_{x \neq 0} \|Mx\|/\|x\| \leq \sqrt{n}/\delta$. Summarizing the previous observations, we obtain $\|y\| \geq 1/\|M\| \geq \delta/\sqrt{n}$.

For the last part of the proof we observe that there exists an index $i \in [n]$ such that $|y_i| \geq \delta/n$. We apply the principle of deferred decisions and assume that all coefficients $\lambda_j$ for $j \neq i$ are fixed arbitrarily. By the chain of equivalences

$$\lambda^T y \in [-2n \varepsilon^{1/3}, 2n \varepsilon^{1/3}]$$

$$\iff \sum_{k=1}^n \frac{\lambda_k \cdot y_k}{y_i} \in \left[-\frac{2n \varepsilon^{1/3}}{|y_i|}, \frac{2n \varepsilon^{1/3}}{|y_i|}\right]$$

$$\iff \lambda_i \in \left[-\frac{2n \varepsilon^{1/3}}{|y_i|}, \frac{2n \varepsilon^{1/3}}{|y_i|}\right] - \sum_{k \neq i} \frac{\lambda_k \cdot y_k}{y_i}$$

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we see that the event $\lambda^T y \in [-2n\varepsilon^{1/3}, 2n\varepsilon^{1/3}]$ occurs if and only if the coefficient $\lambda_i$, which we did not fix, falls into a certain fixed interval of length $4n\varepsilon^{1/3}/|y_i|$. The probability for this to happen is at most $4n\varepsilon^{1/3}/|y_i| \leq 4n^2\varepsilon^{1/3}/\delta$. The claim follows by applying a union bound over all pairs $(z_1, z_2) \in V_2(P)$, which gives us the additional factor of $m^n$. \square

The next observation characterizes the situation when the projections of two linearly independent vectors in $\mathbb{R}^n$ are projected onto two linearly dependent vectors in $\mathbb{R}^2$ by the function $x \mapsto (\hat{c}^T x, \hat{w}^T x)$.

**Observation 34.** Let $(z_1, z_2, z_3) \in V_3(P)$, let $\Delta_1 := z_2 - z_1$ and $\Delta_2 := z_3 - z_2$, and let $\hat{c}, \hat{w} \in \mathbb{R}^n$ be vectors for which $\hat{w}^T \Delta_1 \neq 0$, $\hat{w}^T \Delta_2 \neq 0$, and

\[
\frac{\hat{w}^T \Delta_1}{\hat{c}^T \Delta_1} = \frac{\hat{w}^T \Delta_2}{\hat{c}^T \Delta_2}.
\]

Then $\hat{c}^T x = 0$ for $x := \Delta_1 - \mu \cdot \Delta_2$, where $\mu = \hat{w}^T \Delta_1 / \hat{w}^T \Delta_2$.

Note that, by the definition of $x$, the equation $\hat{c}^T x = 0$ trivially holds. For the equation $\hat{c}^T x = 0$ we require that the projections of $\Delta_1$ and $\Delta_2$ are linearly dependent as it is assumed in Observation 34. Furthermore, let us remark that in the formulation above we allow $\hat{c}^T \Delta_1 = 0$ or $\hat{c}^T \Delta_2 = 0$ using the convention $x/0 = +\infty$ for $x > 0$ and $x/0 = -\infty$ for $x < 0$.

**Proof.** The claim follows from

\[
\hat{c}^T x = \hat{c}^T \Delta_1 - \mu \cdot \hat{c}^T \Delta_2 = \hat{c}^T \Delta_2 \cdot \frac{\hat{w}^T \Delta_1}{\hat{w}^T \Delta_2} - \mu \cdot \hat{c}^T \Delta_2 = 0.
\]

We are now able to prove an analog of Lemma 35.

**Lemma 35.** The probability that there exist a triple $(z_1, z_2, z_3) \in V_3(P)$ and vectors $\hat{\lambda} \in U_\varepsilon(\lambda)$ and $\hat{c} \in U_\varepsilon(c)$ for which

\[
\frac{\hat{w}^T \Delta_1}{\hat{c}^T \Delta_1} = \frac{\hat{w}^T \Delta_2}{\hat{c}^T \Delta_2},
\]

where $\Delta_1 := z_2 - z_1$, $\Delta_2 := z_3 - z_2$, and $\hat{w} = -[u_1, \ldots, u_n] \cdot \hat{\lambda}$, is bounded from above by $12m^2n^2\varepsilon^{1/3}/\delta$.

**Proof.** Let us introduce the following events:

- With event $A$ we refer to the event stated in Lemma 35.
- Event $B$ occurs if there exist a pair $(z_1, z_2) \in V_2(P)$ and a vector $\hat{\lambda} \in U_\varepsilon(\lambda)$ such that $|\hat{w}^T \cdot (z_2 - z_1)| \leq n\varepsilon^{1/3} \cdot |z_2 - z_1|$ (cf. Lemma 33).
- Event $C$ occurs if there is a triple $(z_1, z_2, z_3) \in V_3(P)$ such that $|c^T x| \leq (4\sqrt{n}\varepsilon^{1/3}/\delta) \cdot |x|$, where $x = x(u, z_1, z_2, z_3) := \Delta_1 - \mu \cdot \Delta_2$ for $\Delta_1 := z_2 - z_1$, $\Delta_2 := z_3 - z_2$, and $\mu = w^T \Delta_1 / w^T \Delta_2$ if $w^T \Delta_2 \neq 0$ and $\mu = 0$ otherwise (cf. Observation 34).
In the first part of the proof we will show that \( A \subseteq B \cup C \). For this, it suffices to show that \( A \setminus B \subseteq C \). Let us consider realizations \( w \in (0, 1]^n \) and \( c \in [-1, 1]^n \) for which event \( A \) occurs, but not event \( B \). Let \((z_1, z_2, z_3) \in V_3(P)\), \( \lambda \in U_{c}(\lambda) \), and \( \hat{c} \in U_{c}(c) \) be the vectors mentioned in the definition of event \( A \). Our goal is to show that \(|c^T x| \leq (4\sqrt{n} \varepsilon^{1/3} / \delta) \cdot ||x||\) for \( x = x(w, z_1, z_2, z_3) \). As event \( B \) does not occur, we know that

\[
|w^T \Delta_1| \geq n \varepsilon^{1/3} \cdot \|\Delta_1\|, \quad |\hat{w}^T \Delta_1| \geq n \varepsilon^{1/3} \cdot \|\Delta_2\|, \\
|w^T \Delta_2| \geq n \varepsilon^{1/3} \cdot \|\Delta_2\|, \quad \text{and} \quad |\hat{w}^T \Delta_2| \geq n \varepsilon^{1/3} \cdot \|\Delta_2\|.
\]

Furthermore, note that

\[
|\hat{w}^T \Delta_1 - w^T \Delta_1| \leq \|\Delta_w\| \cdot \|\Delta_1\| \leq n \varepsilon \cdot \|\Delta_1\|
\]

and, similarly,

\[
|\hat{w}^T \Delta_2 - w^T \Delta_2| \leq n \varepsilon \cdot \|\Delta_2\|.
\]

Therefore,

\[
|\hat{w}^T \Delta_1 - w^T \Delta_1| \leq n \varepsilon \cdot \|\Delta_1\| \leq \varepsilon^{2/3} \cdot \|\Delta_1\| \quad \text{and} \quad |\hat{w}^T \Delta_2 - w^T \Delta_2| \leq n \varepsilon \cdot \|\Delta_2\| \leq \varepsilon^{2/3} \cdot |\hat{w}^T \Delta_2|,
\]

and, consequently

\[
\frac{|\hat{w}^T \Delta_1|}{|\hat{w}^T \Delta_2|} \leq \frac{(1 + \varepsilon^{2/3})}{\frac{1 - e^{2/3}}{1 + \varepsilon^{2/3}}} \cdot \frac{|w^T \Delta_1|}{|w^T \Delta_2|} = \frac{1 + \varepsilon^{2/3}}{1 - e^{2/3}} \cdot \frac{|w^T \Delta_1|}{|w^T \Delta_2|} \leq (1 + 3 \varepsilon^{2/3}) \cdot \frac{|w^T \Delta_1|}{|w^T \Delta_2|} \quad \text{and} \quad \frac{|\hat{w}^T \Delta_1|}{|\hat{w}^T \Delta_2|} \geq \frac{(1 - \varepsilon^{2/3})}{\frac{1 - e^{2/3}}{1 + \varepsilon^{2/3}}} \cdot \frac{|w^T \Delta_1|}{|w^T \Delta_2|} = \frac{1 - \varepsilon^{2/3}}{1 - e^{2/3}} \cdot \frac{|w^T \Delta_1|}{|w^T \Delta_2|} \geq (1 - 3 \varepsilon^{2/3}) \cdot \frac{|w^T \Delta_1|}{|w^T \Delta_2|}.
\]

Here we again used \( \varepsilon \leq 1 \). Observe that both, \( w^T \Delta_1 \) and \( \hat{w}^T \Delta_1 \), as well as \( \hat{w}^T \Delta_2 \) and \( w^T \Delta_2 \), have the same sign, since their absolute values are larger than \( n \varepsilon^{1/3} \cdot \|\Delta_1\| \) and \( n \varepsilon^{1/3} \cdot \|\Delta_2\| \), but their difference is at most \( n \varepsilon \cdot \|\Delta_1\| \) and \( n \varepsilon \|\Delta_2\| \), respectively. Hence,

\[
\frac{|\hat{w}^T \Delta_1 - w^T \Delta_1|}{|w^T \Delta_2 - \hat{w}^T \Delta_2|} = \frac{|\hat{w}^T \Delta_1|}{|\hat{w}^T \Delta_2|} - \frac{|w^T \Delta_1|}{|w^T \Delta_2|} \leq 3 \varepsilon^{2/3} \cdot \frac{|w^T \Delta_1|}{|w^T \Delta_2|}.
\]

As event \( A \) occurs, but not event \( B \), Observation \[34\] yields \( \hat{c}^T x(\hat{w}, z_1, z_2, z_3) = 0 \). With
the previous inequality we obtain
\[
|\hat{c}^T x(w, z_1, z_2, z_3)| = \left| \hat{c}^T \cdot (x(w, z_1, z_2, z_3) - x(\hat{w}, z_1, z_2, z_3)) \right| \\
\leq \|\hat{c}\| \cdot \|x(w, z_1, z_2, z_3) - x(\hat{w}, z_1, z_2, z_3)\| \\
= \|\hat{c}\| \cdot \left| \frac{w_i \Delta_1}{w_i \Delta_2} - \frac{\hat{w}_i \Delta_1}{\hat{w}_i \Delta_2} \right| \cdot \|\Delta_2\| \\
\leq \sqrt{n} \cdot 3 \varepsilon^{2/3} \cdot \left| \frac{w_i \Delta_1}{w_i \Delta_2} \right| \cdot \|\Delta_2\| \\
\leq \sqrt{n} \cdot 3 \varepsilon^{2/3} \cdot \frac{\|w\| \cdot \|\Delta_1\|}{n \varepsilon^{1/3} \cdot \|\Delta_2\|} \cdot \|\Delta_2\| \\
\leq \sqrt{n} \cdot 3 \varepsilon^{2/3} \cdot \frac{n \cdot \|\Delta_1\|}{n \varepsilon^{1/3} \cdot \|\Delta_2\|} \cdot \|\Delta_2\| \\
= 3 \sqrt{n} \varepsilon^{1/3} \cdot \|\Delta_1\|.
\]

In the remainder of this proof, with \( x \) we refer to the vector \( x(w, z_1, z_2, z_3) \) (and not to, e.g., \( x(\hat{w}, z_1, z_2, z_3) \)). Now we show that \( \|x\| \geq \delta \cdot \|\Delta_1\| \). For this, let \( a_i^T \) be a row of matrix \( A \) for which \( a_i^T z_1 < b_i \), but \( a_i^T z_2 = a_i^T z_3 = b_i \), i.e., the \( i \)th constraint is tight for \( z_2 \) and \( z_3 \), but not for \( z_1 \). Such a constraint exists as \( z_1 \) and \( z_3 \) are distinct neighbors of \( z_2 \). Consequently, \( a_i^T \Delta_1 > 0 \) and \( a_i^T \Delta_2 = 0 \). Hence,
\[
|a_i^T x| = |a_i^T \cdot (\Delta_1 - \mu \cdot \Delta_2)| = |a_i^T \cdot \Delta_1| \geq \delta \cdot \|\Delta_1\|,
\]
where the last inequality is due to Lemma 25. Claim 3. As \( \|a_i\| = 1 \), we obtain
\[
\|x\| \geq \frac{|a_i^T x|}{\|a_i\|} = |a_i^T x| \geq \delta \cdot \|\Delta_1\|.
\]

Summarizing the previous observations yields
\[
|\hat{c}^T x| \leq 3 \sqrt{n} \varepsilon^{1/3} \cdot \|\Delta_1\| \leq \frac{3 \sqrt{n} \varepsilon^{1/3}}{\delta} \cdot \|x\|.
\]

Now that we have bounded \( |\hat{c}^T x| \) from above, we easily get an upper bound for \( |c^T x| \). Since
\[
|c^T x - \hat{c}^T x| \leq \|\Delta_c\| \cdot \|x\| \leq \sqrt{n} \varepsilon \cdot \|x\|
\]
we obtain
\[
|c^T x| \leq |\hat{c}^T x| + |c^T x - \hat{c}^T x| \leq \frac{3 \sqrt{n} \varepsilon^{1/3}}{\delta} \cdot \|x\| + \sqrt{n} \varepsilon \cdot \|x\| \leq \frac{4 \sqrt{n} \varepsilon^{1/3}}{\delta} \cdot \|x\|
\]
and \( c^T x \) is event \( \mathcal{C} \) occurs.

In the second part of the proof we show that \( \Pr[A] \leq 8m^2 n^2 \varepsilon^{1/3} \phi / \delta \). Due to \( A \subseteq B \cup C \), \( \phi \geq 1 \), and Lemma 33 it then follows that
\[
\Pr[A] \leq 4m^2 n^2 \varepsilon^{1/3} / \delta + 8m^2 n^2 \varepsilon^{1/3} \phi / \delta \leq 12m^2 n^2 \varepsilon^{1/3} \phi / \delta.
\]
Let $(z_1, z_2, z_3) \in V_3(P)$ be a triple of vertices of $P$. We apply the principle of deferred decisions twice: First, we assume that $\lambda$ has already been fixed arbitrarily. Hence, the vector $x = x(w, z_1, z_2, z_3) \neq 0$ is also fixed. Let $z = (1/\|x\|) \cdot x$ be the normalization of $x$. As $|c^T x| \leq (4\sqrt{\kappa \varepsilon^{1/3}}/\delta) \cdot \|x\|$ holds if and only if $|c^T z| \leq 4\sqrt{\kappa \varepsilon^{1/3}}/\delta$, we will analyze the probability of the latter event.

There exists an index $i$ such that $|z_i| \geq 1/\sqrt{n}$. Now we again apply the principle of deferred decisions assuming that all coefficients $c_j$ for $j \neq i$ are fixed arbitrarily. Then

$$|c^T z| \leq 4\sqrt{\kappa \varepsilon^{1/3}}/\delta \iff \sum_{j=1}^{n} c_j \cdot \frac{z_j}{z_i} \in \left[ -\frac{4\sqrt{\kappa \varepsilon^{1/3}}}{\delta \cdot |z_i|}, \frac{4\sqrt{\kappa \varepsilon^{1/3}}}{\delta \cdot |z_i|} \right]$$

$$\iff c_i \in \left[ -\frac{4\sqrt{\kappa \varepsilon^{1/3}}}{\delta \cdot |z_i|}, \frac{4\sqrt{\kappa \varepsilon^{1/3}}}{\delta \cdot |z_i|} \right] - \sum_{j \neq i} c_j \cdot \frac{z_j}{z_i}.$$

Hence, the random coefficient $c_i$ must fall into a fixed interval of length $8\sqrt{\kappa \varepsilon^{1/3}}/(\delta \cdot |z_i|)$. The probability for this to happen is at most

$$\frac{8\sqrt{\kappa \varepsilon^{1/3}}}{\delta \cdot |z_i|} \cdot \phi \leq \frac{8\sqrt{\kappa \varepsilon^{1/3}}}{\delta \cdot 1/\sqrt{n}} \cdot \phi = \frac{8n\varepsilon^{1/3}\phi}{\delta}.$$  

A union bound over all triples $(z_1, z_2, z_3) \in V_3(P)$ gives the additional factor of $V_3(P) \leq m^n n$.

**Lemma 36.** Let us consider the shadow vertex algorithm given as Algorithm 1 for $\phi \geq \sqrt{n}$. If we replace the draw of each continuous random variable by the draw of at least

$$B(m, n, \phi, \delta) := [6n \log_2 m + 6 \log_2 n + 3 \log_2 \phi + 3 \log_2 (1/\delta) + 12]$$

random bits as described earlier in this section, then the expected number of pivots is $O\left(\frac{mn^2}{\delta^2} + \frac{m\sqrt{\kappa \phi}}{\delta}\right)$.

**Proof.** As discussed in the beginning of this section, instead of drawing $k$ random bits to simulate a uniform random draw from an interval $[a, b]$, we can draw a uniform random variable $X$ from $[0, 1)$ and apply the function $g(X) = h((X \cdot 2^k)/2^k)$ for $h(x) = a + (b - a) \cdot x$ to obtain a discrete random variable with the same distribution. Observe, that $|X - g(X)| \leq (b - a)/2^k$. In the shadow vertex algorithm all intervals are of length 1 or of length $1/\phi \leq 1$. Hence, $|X - g(X)| \leq 2^{-k}$. As we use $k \geq B(m, n, \phi, \delta)$ bits for each draw, we obtain $g(X) \in U_\varepsilon(X)$ for

$$\varepsilon = 2^{-B(m, n, \phi, \delta)} \leq \frac{\delta^3}{212m^6 n^6\phi^3} = \left(\frac{\delta}{16m^2 n^2 \phi}\right)^3.$$

Now let $c$ and $\lambda$ denote the continuous random vectors and let $\bar{c} \in U_\varepsilon(c)$ and $\bar{\lambda} \in U_\varepsilon(\lambda)$ denote the discrete random vectors obtained from $c$ and $\lambda$ as described above. Furthermore, let $w = [u_1, \ldots, u_n] \cdot \lambda$ and $\bar{w} = [u_1, \ldots, u_n] \cdot \bar{\lambda}$. We introduce the event $D$ which occurs if one of the following holds:

1. There exists a pair $(z_1, z_2) \in V_3(P)$ such that $c^T z_1$ and $c^T z_2$ are not in the same relation as $\bar{c}^T z_1$ and $\bar{c}^T z_2$ or $c^T z_1 = c^T z_2$ or $c^T z_1 = c^T z_2$.  

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2. There exists a triple \((z_1, z_2, z_3) \in V_3(P)\) such that \(w^{T}(z_2 - z_1)\) and \(w^{T}(z_3 - z_2)\) are not in the same relation as \(\bar{w}^{T}(z_2 - z_1)\) and \(\bar{w}^{T}(z_3 - z_2)\). Here, \(a\) and \(b\) being in the same relation as \(\bar{a}\) and \(\bar{b}\) means that \(\text{sgn}(a - b) = \text{sgn}(\bar{a} - \bar{b})\), where \(\text{sgn}(x) = -1\) for \(x < 0\), \(\text{sgn}(x) = 0\) for \(x = 0\), and \(\text{sgn}(x) = +1\) for \(x > 0\).

Let \(X\) and \(\bar{X}\) denote the number of pivots of the shadow vertex algorithm with continuous random vectors \(c\) and \(\lambda\) and with discrete random vectors \(\bar{c}\) and \(\bar{\lambda}\), respectively. We will first argue that \(X = \bar{X}\) if event \(D\) does not occur. In both cases, we start in the same vertex \(x_0\). In each vertex \(x\), the algorithm chooses among the neighbors of \(x\) with a larger \(c\)-value (or \(\bar{c}\)-value, respectively) the neighbor \(z\) with the smallest slope \(w^{T}(z-x)\) (or \(\bar{w}^{T}(z-x)\), respectively). If event \(D\) does not occur, then in both cases the same neighbors of \(x\) are considered and, additionally, the order of their slopes is the same. Hence, in both cases the same sequence of vertices is considered.

Now let \(Y\) be the random variable that takes the value \(m^n\) if event \(D\) occurs and the value \(0\) otherwise. Clearly, \(\bar{X} \leq X + Y\) and, thus,

\[
E[\bar{X}] \leq E[X] + E[Y] \leq O \left( \frac{m n^2}{\delta^2} + \frac{m \sqrt{\eta \phi}}{\delta} \right) + m^n \cdot \Pr[D],
\]

where the last inequality stems from Theorem \(6\). In the remainder of this work we show that the probability \(\Pr[D]\) of event \(D\) is bounded from above by \(1/m^n\). For this, let us assume that the first part of the definition of event \(D\) is fulfilled for a pair \((z_1, z_2) \in V_2(P)\). If \(c^T z_1\) and \(c^T z_2\) are not in the same relation as \(c^T \bar{z}_1\) and \(c^T \bar{z}_2\), then there exists a \(\mu \in [0, 1]\) such that

\[
\mu \cdot (c^T z_1 - c^T \bar{z}_2) + (1 - \mu) \cdot (\bar{c}^T z_1 - \bar{c}^T \bar{z}_2) = 0.
\]

If we consider the vector \(\hat{c} := \mu \cdot c + (1 - \mu) \cdot \bar{c} \in U_c(c)\), then we obtain

\[
\hat{c}^T \cdot (z_2 - z_1) = \mu \cdot c^T \cdot (z_2 - z_1) + (1 - \mu) \cdot \bar{c}^T \cdot (z_2 - z_1) = 0.
\]

Hence, the event described in Lemma \(32\) occurs. This event also occurs if \(c^T z_1 = c^T \bar{z}_2\) or \(\bar{c}^T z_1 = \bar{c}^T \bar{z}_2\).

Let us now assume that the second part of the definition of event \(D\) is fulfilled for a triple \((z_1, z_2, z_3) \in V_3(P)\), but not the first one, and let us consider the function \(f : [0, 1] \rightarrow \mathbb{R}\), defined by

\[
f(\mu) = \frac{(\mu \cdot w + (1 - \mu) \cdot \bar{w})^T \cdot (z_2 - z_1)}{\mu \cdot c + (1 - \mu) \cdot \bar{c}} - \frac{(\mu \cdot w + (1 - \mu) \cdot \bar{w})^T \cdot (z_3 - z_2)}{\mu \cdot c + (1 - \mu) \cdot \bar{c}}.
\]

The denominators of both fractions are linear in \(\mu\) and, since the first part of the definition of event \(D\) does not hold, the signs for \(\mu = 0\) and \(\mu = 1\) are the same and different from \(0\). Hence, both denominators are different from \(0\) for all \(\mu \in [0, 1]\). Consequently, function \(f\) is continuous (on \([0, 1]\)). As we have

\[
f(0) = \frac{\bar{w}^T \cdot (z_2 - z_1)}{\bar{c}^T \cdot (z_2 - z_1)} - \frac{\bar{w}^T \cdot (z_3 - z_2)}{\bar{c}^T \cdot (z_3 - z_2)}.
\]
and
\[ f(1) = \frac{w^T \cdot (z_2 - z_1)}{c^T \cdot (z_2 - z_1)} - \frac{w^T \cdot (z_3 - z_2)}{c^T \cdot (z_3 - z_2)} \]
and these differences have different signs as the second part of the definition of event \( D \) is fulfilled, there must be a value \( \mu \in [0, 1] \) for which \( f(\mu) = 0 \). This implies
\[ \frac{\hat{w}^T \cdot (z_2 - z_1)}{c^T \cdot (z_2 - z_1)} = \frac{\hat{w}^T \cdot (z_3 - z_2)}{c^T \cdot (z_3 - z_2)} \]
for \( \hat{c} := \mu \cdot c + (1 - \mu) \cdot \tilde{c} \in U_\varepsilon(c) \), \( \hat{\lambda} := \mu \cdot \lambda + (1 - \mu) \cdot \tilde{\lambda} \in U_\varepsilon(\lambda) \), and \( \hat{w} := -[u_1, \ldots, u_n] \cdot \tilde{\lambda} = \mu \cdot w + (1 - \mu) \cdot \tilde{w} \). Thus, the event described in Lemma 35 occurs.

By applying Lemma 32 and Lemma 35 we obtain
\[ \Pr[D] \leq 2m^2n^{3/2} \varepsilon \phi + \frac{12m^2n^2 \varepsilon^{1/3} \phi}{\delta} \leq \frac{4m^2n^2 \varepsilon^{1/3} \phi}{\delta} + \frac{12m^2n^2 \varepsilon^{1/3} \phi}{\delta} = \frac{16m^2n^2 \phi}{\delta} \cdot \varepsilon^{1/3} \leq \frac{1}{m^n}. \]
This completes the proof.

Lemma 36 states that if we draw \( 2n \cdot B(m, n, \phi, \delta) \) random bits for the \( 2n \) components of \( c \) and \( \lambda \), then the expected number of pivots does not increase significantly. We consider now the case that the parameter \( \delta \) is not known (and also no good lower bound). We will use the fraction \( \hat{\delta} = \hat{\delta}(n, \phi) := 2n^{3/2} / \phi \) as an estimate for \( \delta \). For the case \( \phi > 2n^{3/2} / \delta \), in which the repeated shadow vertex algorithm is guaranteed to yield the optimal solution, this is a valid lower bound for \( \delta \). For the case \( \phi < 2n^{3/2} / \delta \) this estimate is too large and we would draw too few random bits, leading to a (for our analysis) unpredictable running time behavior of the shadow vertex method. To solve this problem, we stop the shadow vertex method after at most \( 8n \cdot p(m, n, \phi, \hat{\delta}(n, \phi)) \) pivots, where \( p(m, n, \phi, \delta) = O\left(\frac{mn^2}{\delta^2} + \frac{m^2n^2}{\delta^4} \right) \) is the upper bound for the expected number of pivots stated in Lemma 36. When the shadow vertex method stops, we assume that the current choice of \( \phi \) is too small (although this does not have to be the case) and restart the repeated shadow vertex algorithm with \( 2\phi \). Recall that this is the same doubling strategy that is applied when the repeated shadow vertex algorithm yields a non-optimal solution for the original linear program. We call this algorithm the shadow vertex algorithm with random bits.

**Theorem 37.** The shadow vertex algorithm with random bits solves linear programs with \( n \) variables and \( m \) constraints satisfying the \( \delta \)-distance property using \( O\left(\frac{mn^2}{\delta^2} \cdot \log \left(\frac{1}{\delta}\right)\right) \) pivots in expectation if a feasible solution is given.

Note that, in analogy, all other results stated in Theorem 2 and Theorem 3 also hold for the shadow vertex algorithm with random bits with an additional \( O(n) \)-factor (or \( O(m) \)-factor when no feasible solution is given).

**Proof.** Let us assume that the shadow vertex algorithm with random bits does not find the optimal solution before the first iteration \( i^* \) for which \( \phi_{i^*} > 2n^{3/2} / \delta \). For iterations \( i \geq i^* \) we know that the shadow vertex algorithm will return the optimal solution (or detect, that the linear program is unbounded) if it is not stopped because the number of
 pivots exceeds $8n \cdot p(m, n, \phi_i, \hat{\delta}(n, \phi_i))$. Due to Markov’s inequality, the probability of the latter event is bounded from above by $1/8n$ (for each facet of the optimal solution) because $p(m, n, \phi_i, \hat{\delta}(n, \phi_i)) \geq p(m, n, \phi_i, \delta)$ due to $\hat{\delta}(n, \phi_i) \leq \delta$ and $p(m, n, \phi_i, \delta)$ is an upper bound for the expected number of pivots. As $n$ facets have to be identified in iteration $i$, the probability that the shadow vertex method stops because of too many pivots is bounded by

$$\sum_{i=i^*}^{\infty} \left( \frac{1}{8} \right)^{i-i^*} \cdot \frac{7}{8} \cdot n \cdot 8n \cdot p(m, n, \phi_i, \hat{\delta}(n, \phi_i))$$

$$= 7n^2 \sum_{i=i^*}^{\infty} \frac{1}{8^{i-i^*}} \cdot p \left( m, n, \phi_i, \frac{2n^{3/2}}{\phi_i} \right)$$

$$= O \left( 8^{i^*-n^2} \cdot \sum_{i=i^*}^{\infty} \frac{1}{8^i} \cdot \frac{m\sqrt{n}\phi_i}{2n^{3/2}\phi_i} \right) = O \left( 8^{i^*-n} \cdot \sum_{i=i^*}^{\infty} \frac{1}{8^i} \cdot m\phi_i^2 \right)$$

$$= O \left( 8^{i^*-n} \cdot \sum_{i=i^*}^{\infty} \frac{1}{8^i} \cdot (2n^{3/2})^2 \right) = O \left( 8^{i^*-n} \cdot \sum_{i=i^*}^{\infty} \frac{1}{2^i} \cdot mn^3 \right)$$

$$= O(4^{i^*} mn^4) = O \left( \frac{mn^4}{\delta} \right) .$$

Some equations require further explanation. The factor $n \cdot 8n \cdot p(m, n, \phi_i, \hat{\delta}(n, \phi_i))$ stems from the fact that we have to identify $n$ facets, and for each we stop after at most $8n \cdot p(m, n, \phi_i, \hat{\delta}(n, \phi_i))$ pivots. The second equation is in accordance with Lemma 36 which states that $p(m, n, \phi, \delta) = O \left( \frac{mn^2}{\delta^2} + \frac{m\sqrt{n}\phi}{\delta} \right)$. As the term $mn^2/\delta^2$ is dominated by the term $m\sqrt{n}\phi/\delta$ when $\phi \geq n^{3/2}/\delta$, it can be omitted in the $O$-notation for such values. Above we only consider iterations $i \geq i^*$, i.e., $\phi_i \geq \phi_{i^*} > 2n^{3/2}/\delta$. The last equation is due to the fact that

$$2^{i^*-1}n^{3/2} = \phi_{i^*-1} \leq \frac{2n^{3/2}}{\delta},$$

i.e., $2^{i^*-1} \leq 4/\delta$ and, hence, $4^{i^*} = O(1/\delta^2)$.

To finish the proof, we observe that the iterations $i = 1, \ldots, i^*$ require at most

$$\sum_{i=1}^{i^*-1} n \cdot 8n \cdot p(m, n, \phi_i, \hat{\delta}(n, \phi)) = \sum_{i=1}^{i^*-1} n \cdot 8n \cdot p \left( m, n, \phi_i, \frac{2n^{3/2}}{\phi_i} \right)$$

$$= O \left( \sum_{i=1}^{i^*-1} n^2 \cdot \frac{mn^2}{\delta^2} \right) = O \left( i^* \cdot \frac{mn^4}{\delta^2} \right) = O \left( \log \left( \frac{1}{\delta} \right) \cdot \frac{mn^4}{\delta^2} \right)$$

 pivots in expectation. The second equation stems from Lemma 36 which states that $p(m, n, \phi, \delta) = O \left( \frac{mn^2}{\delta^2} + \frac{m\sqrt{n}\phi}{\delta} \right)$. The second term in the sum can be omitted if $\phi = O(n^{3/2}/\delta)$, which is the case for $\phi_1, \ldots, \phi_{i^*-1}$. Finally, $i^*$ is the smallest integer $i$ for which $2^{i^*}n^{3/2} > 2n^{3/2}/\delta$. Hence, $i^* = O(\log(1/\delta))$. 
