On the Nash theory of gravity with matter contents

Phongpichit Chanhuie∗
College of Graduate Studies, Walailak University,
Thasala, Nakhon Si Thammarat, 80160, Thailand

Davood Momeni† and Mudhahir Al Ajmi‡
Department of Physics, College of Science, Sultan Qaboos University,
P.O. Box 36, Al-Khodh 123, Muscat, Sultanate of Oman

(Dated: September 11, 2018)

In this work, we generalize the original Nash theory of gravity by adding the matter fields. In other words, we specify a proper form of the field equations on more general footings for space with matter contents. We first derive the equations of motion in the flat FLRW spacetime and examine the behaviors of the solutions by invoking specific forms of the Hubble parameter. We also classify the physical behaviors of the solutions by employing the stability analysis. We check the consistency of the model by considering particular cosmological parameters. We use observational data from Supernovae Ia (SNeIa) and baryonic acoustic oscillations (BAO) to constrain the parameters of the model. Interestingly, we discover the best fit of our model with $\Omega_{m0} = 0.25$ and $H_0 = 0.4$ for $\tau_0 = 0.3$ and $\rho_0 = 0.5$ assuming the Hubble parameter $H \propto \rho^{-2}$.

Keywords: Nash gravitational theory, matter field contents, Om analysis

I. INTRODUCTION

Several cosmological observations convince us that the observable universe is undergoing a phase of accelerated expansion. The idea of a short period of extremely rapid expansion when the universe was very young is widely motivated. This behavior is impressively described based on the theory of cosmic inflation. The idea was initiated and primarily developed by many cosmologists; see for instance Refs.[1]. Inflation is an inevitable ingredient when describing the very early evolution period of the universe. The reason is that it does not only solve most of the puzzles that plague the standard Big Bang theory, but its prediction is also simultaneously consistent with the observations. Moreover, the discovery of the accelerating expansion of the Universe through observations of distant supernovae intimates the current expansion of the universe [2]. In addition, various cosmological observations

∗Electronic address: channuie@gmail.com
†Electronic address: davood@squ.edu.om
‡Electronic address: mudhahir@squ.edu.om
concede such an expansion. These include cosmic microwave background (CMB) radiation [3], large scale structure [4], baryon acoustic oscillations (BAO) [5] as well as weak lensing [6].

Regarding the late-time cosmic expansion, there are several possible explanations, to date. Intuitively, the first one is the introduction of the dark energy component in the universe [7]. Unfortunately, the dark energy sector of the universe still remains an open question. In addition, the second scenario is to interpret this unknown phenomenon by utilizing a purely geometrical picture. The latter is well known as the modified theory of gravity. Interestingly, modified theories of gravity have received more attraction lately due to numerous motivations ranging from high-energy physics, cosmology and astrophysics [8].

The $f(R)$ gravity constitutes one of the simple versions of such modification. Several version of $f(R)$ gravity have been proposed and investigated so far; for example Refs.[9]. Here the Lagrangian density is an arbitrary function of the scalar curvature, $R$. For more details, see comprehensive reviewed articles on $f(R)$ theories [10]. Note that the first model that can describe cosmic inflation was proposed by Starobinsky [11]. Notice that the modified $f(R)$ gravity gives good explanation for the cosmic acceleration without invoking the dark energy component implied from the cosmological data. Among numerous alternatives, Nash developed a modified theory of gravity for empty space by employing higher-derivatives instead of the usual Einstein-Hilbert action [12]. There are few number of papers about this interesting theory which was rarely investigated in literatures [13]-[15].

In this work, we anticipate to generalize the Nash theory by adding the matter fields in the original action. We specify a proper form of the field equations on more general footings for space with matter contents. In Sec.(II), we generalize the Nash theory of gravity by including the matter field in the original action. In addition, we derive the equations of motion in the flat FLRW spacetime and examine the behaviors of the solutions by invoking specific forms of the Hubble parameter. We also classify the physical behaviors of the solutions by employing the stability analysis. In Sec.(III), we check the consistency of the model by considering cosmological parameters, e.g., the Hubble parameter $H$, deceleration parameter $q$, and $Om(z)$ parameter. In Sec.(IV), we further use observational data from Supernovae Ia (SNeIa) and baryonic acoustic oscillations (BAO) to constrain the parameters of the model. Finally, we conclude our findings in the last section.

II. NASH GRAVITY WITH MATTER CONTENTS

J. Nash has actively worked in different fields of applied and pure Mathematics. Moreover, his contributions to physics have been recognized from differential geometry, for example the problem of the compactification in Riemanninan geometry as well as the non Riemannian one. In addition, he has worked on the gauge theories for gravity to figure out alternative forms of the Einstein general
relativity (GR) as well as to study their implications on gravitational wave physics. Likewise, he has also proposed nonlinear theories where the action is composed of second order curvature scalar sectors as well as some topological terms very similar to the Gauss-Bonnet theory. As an alternative theory of gravity, he developed a theory of modified gravity [12] by employing higher-derivatives and without the usual Einstein-Hilbert action. However, originally this theory aimed at solving field equations for empty space. A significant feature of his theory is that the scalar curvature term \( R \) in four dimensional Riemanninan manifold of spacetime satisfies wave equation. If one pass to the linear regime, it is possible to probably observe gravitational waves in a natural form by adopting a suitable gauge frame.

We in the present work are going to generalize his work by adding the matter fields in the original Nash theory. Our modified action takes the following form:

\[
S = \int d^4x \mathcal{L}_{\text{Nash}} + \int d^4x \mathcal{L}_{\text{matter}} = \frac{1}{2\kappa^2} \int \left( 2R^{\mu\nu} R_{\mu\nu} - R^2 \right) \sqrt{-g} \, d^4x + S_{\text{matter}},
\]

where \( R_{\mu\nu} \) and \( R \) are the Ricci tensor and Ricci (curvature) scalar, respectively, while \( g \) is the determinant of the background metric tensor, \( g_{\mu\nu} \) and \( \kappa^2 \equiv 8\pi G \). Here we have added the matter field sector, \( S_{\text{matter}} \), to the original Nash action. It is remarked that the general class of Lagrangians including the one without matter written above has been considered to be of interest in attempting to develop theories of quantum gravity. Using the above action, gravitational field equations are directly derived by taking into account the metric \( g_{\mu\nu} \) as a dynamical field to yield

\[
\Box G_{\mu\nu} + G_{\alpha\beta} \left( 2R^\alpha_{\mu\nu}R^\beta_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\alpha\beta} \right) = \kappa^2 T_{\mu\nu},
\]

where \( \Box = (\sqrt{-g})^{-1} \partial_{\mu}(\sqrt{-g} \partial^\mu) \) is the d’Alembertian operator, \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) and \( T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \).

If we consider the trace of the equation of motion (EoM) given in Eq. (2), we obtain:

\[
\Box R - 2G_{\alpha\beta} R^\alpha_{\mu\nu} R^\beta_{\mu\nu} + 2G_{\alpha\beta}R^{\alpha\beta} + \kappa^2 T = 0,
\]

where \( T \equiv T^\mu_\mu \) is the trace of the energy momentum tensor for matter action \( S_{\text{matter}} \). We assume that there exist a classical Einstein metric satisfying the vacuum field equation \( G_{\mu\nu} = 0 \). Using Eq.(3) we conclude that it solves Nash gravity as well. Consequently any vacuum Einstein metric is also a solution to Nash gravity. In this sense any asymptotic flat rotating metric (Kerr metric) is also solution to the Nash theory. In case of the cosmological constant it needs a sign reversion in the original action such that \( 2R^\alpha_{\mu\nu} R^\beta_{\mu\nu} \rightarrow -2R^\alpha_{\mu\nu} R^\beta_{\mu\nu} \) [14].

Nevertheless, it should be noted that this theory contains higher-order time derivative in the equations of motion. In a quantum version, higher-derivative interactions result (ghost) fields with negative norm yielding negative probabilities and possibly a breakdown in unitarity. In some cases, the higher-order time derivative in the equations of motion can be reduced to the second order in which such
stability can be solved, see e.g., [16, 17]. However, in the present work, investigating classical aspects of Nash’s theory with matter content in the cosmological background is our main objective.

A. A flat FLRW geometry

The geometry compatible with the homogeneous and isotropic universe is the Friedmann-Lemaitre-Robertson-Walker (FLRW) space-time with the corresponding line element:

\[ ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right), \]

where we have expressed the spatial section in terms of spherical coordinates, \((r, \theta, \phi)\). The constant \(k\) encodes the curvature of the space-time, with \(k = 0\) corresponding to flat (Euclidean) spatial sections, and \(k = \pm 1\) corresponding to positive and negative curvatures, respectively. Here we define the Hubble parameter as \(H \equiv \frac{d \ln a}{dt}\). The energy-momentum tensor of the matter contents is given in the following form:

\[ T_{\mu\nu} = (\sum_i \rho_i + \sum_i p_i) u_\mu u_\nu - (\sum_i p_i) g_{\mu\nu}, \]

where \(i = \{\text{dark energy, dark matter, radiation}\} := \{\text{de, dm, r}\}\). Here if we define a new parameter such that \(\zeta = H^{1/2}\), we obtain equations of motion (EoMs) in the flat FLRW space-time:

\[ \frac{d}{dt}(\dot{\zeta} + \zeta^3) = \kappa^2 \sum_i \rho_i, \]

\[ \zeta \frac{d^2}{dt^2}(\dot{\zeta} + \zeta^3) + \frac{3}{2} \frac{d}{dt}(\dot{\zeta} + \zeta^3)^2 = -\kappa^2 (\sum_i \rho_i + \sum_i p_i). \]

Notice that the solutions of Eqs.(6) and (7) are investigated in Ref.[13] for empty space. Moreover, by substituting Eq.(6) into Eq.(7), we find the following equation:

\[ \zeta \frac{d}{dt} \sum_i \rho_i + 3\kappa^2 \sum_i \sum_j \rho_i(t) \int t \rho_j(t') dt' + \sum_i (\rho_i + p_i) = 0, \]

which is the so-called "continuity equation". Note that in spite of the continuity equation in GR, this is a nonlocal equation in which the integration of \(\rho\) is primarily required in order to quantify \(\rho\). In the section below, we examine in more details this equation and aim to transform it into an ordinary differential equation rather than the integro-differential equation.

B. The collision term and effective energy density action

Notice in Eq.(8) that there is a term in which \(\rho_i\) couples with \(\rho_j\), i.e., \(\sum_i \sum_j \rho_i(t) \int t \rho_j(t') dt'\). It will drastically influence dynamics in cosmological behaviors. We use a new re-parametrized time
coordinate $\tau \equiv \int_{t}^{t'} H^{-1/2}(t') dt'$ to rewrite the continuity equation to yield

$$\frac{d}{d\tau} \sum_i \rho_i + 3\kappa^2 \sum_{i,j} \rho_i(\tau) \int_{\tau}^{\infty} \rho_j(\tau') \sqrt{H(\tau')} d\tau' + \sum_i (\rho_i + p_i) = 0. \quad (9)$$

Supposing $p_{\text{tot}} = p(\rho_{\text{tot}})$, we can rewrite the above continuity equation in terms of the total energy density to obtain:

$$\frac{d}{d\tau} \rho_{\text{tot}} + 3\kappa^2 \rho_{\text{tot}}(\tau) \int_{\tau}^{\infty} \rho_{\text{tot}}(\tau') \sqrt{H(\tau')} d\tau' + (\rho_{\text{tot}} + p(\rho_{\text{tot}})) = 0. \quad (10)$$

In the equivalent form ($\rho_{\text{tot}} \equiv \rho, p(\rho) \equiv f(\rho)$), it turns out to yield the following differential equation:

$$\frac{d}{d\tau} \left[ \frac{d}{d\tau} \ln \rho + \frac{f(\rho)}{\rho} \right] + 3\kappa^2 \rho \sqrt{H(\tau)} = 0. \quad (11)$$

Notice that Eq.(11) is a nonlinear second order differential equation for the energy density $\rho$ in which the solution depends on the time evolution of the cosmological background depending on $H(\tau)$. There is no any simple way to figure out the energy density profile except a reconstruction scheme. Here we will assume a form for the Hubble parameter allowing to integrate Eq.(11) analytically. Let us compare new continuity equation given in Eq.(11) with the one in Einstein relativity such that

$$\frac{d}{d\tau} \ln \rho + 3H^{3/2} \left(1 + \frac{f(\rho)}{\rho} \right) = 0. \quad (12)$$

Note that we use the same fluid model as $p = f(\rho)$ in the Einstein gravity to compare with our Nash case. It is worth mentioning here that the continuity equation Eq.(11) is a second-other differential equation instead of the first order one. The reason is that Nash gravity is constructed from the scalar curvature squared terms instead of the Ricci scalar. As a result, the continuity equation resulting from the Bianchi identity applied to the left-hand side of the modified Einstein field equation leads to a second order differential equation rather than first order one.

It is intuitive to derive an effective action for the matter content density in which the density function $\rho$ governs the EoM given in Eq.(11). Referring to the Euler-Lagrange equation, we conclude from Eq.(11) that there should be an effective conjugate momentum $p_\rho$ corresponding to the density function $\rho$ which is derived from an effective Lagrangian $L_\rho$ such that

$$p_\rho \equiv \frac{\partial L_\rho}{\partial \dot{\rho}} = \frac{d}{d\tau} \ln \rho + \frac{f(\rho)}{\rho}. \quad (13)$$

A possible potential function can be deduced to write

$$V(\rho) \equiv -3\kappa^2 \int_{\rho}^{\infty} \rho' \sqrt{H(\rho')} d\rho'. \quad (14)$$

Note that in order to explicitly quantify the potential it is needed to specify the functional form of the Hubble parameter and energy density. Hence, we propose the following irregular effective action of the
field $\rho$,

$$S_{\text{eff}}[\rho(\tau), \dot{\rho}(\tau)] = \int d\tau \rho \left( \frac{\dot{\rho} + f(\rho)}{\rho} - V(\rho) \right).$$  \hspace{1cm} (15)

Note that varying the above effective action with respect to the energy density $\rho$ gives us the irregular continuity equation present in Eq.(11). We expect that there are some physically important results related to the effective action given above. However, we will leave these interesting for our future work.

C. De Sitter like expansion and energy density profile

It is illustrative to show how the solution looks like when the Universe is dominated by barotropic fluid $p = w\rho$ and when $H(\tau) \approx H_0$ de sitter like epoch. In this general case, we discover the following solution for vacuum energy density from Eq. (11),

$$\rho(\tau) = \frac{\rho_0}{6\kappa^2 \sqrt{H_0}} \cosh^{-2} \left( \frac{\sqrt{\rho_0}}{2} (\tau - \tau_0) \right).$$  \hspace{1cm} (16)

Here $\rho_0, \tau_0, H_0$ are integration constants and we set $H_0 = 1$. In Fig.(1), we plot this barotropic energy density for different values of the present energy density, $\rho_0$. Note that in late time when $\tau \to \infty$ we find $\rho(\tau) \to 0$ which means that the Universe has an empty energy density. Remarkably, we observe that the total energy density both in early and late time Universe vanishes and the whole cosmological background undergoes a de Sitter expansion from vacuum to a future vacuum cosmological constant dominated epoch.

Although the total energy density vanishes at early (and late) times, there is no need to have a pure vacuum. Let us clarify this very carefully. At very early time, the universe has radiation as dominant energy density which in a classical Einstein-Hilbert action is governed by $\rho_r \sim a(t)^{-4}$. In our cosmological model, we need to assume that there are other types of matter contents. For example, we have a type of primary tachyonic field $\psi$ with energy density given by $\rho_\psi \sim \rho_r$ in which this equal sized energy content is balanced with radiation field maintaining the vanishing total energy density at time $\tau \to -\infty$. However, this possible form of energy content violated the null energy condition. We can suppose that it can be generated by a wormhole source at early universe. As a result we can predict the existence of a primary wormhole in a similar form as primary black holes. Unfortunately we are not able to probe the form of this type of the tachyonic matter. Since the total pressure is governed by a barotropic equation of state, we can just conclude that the tachyonic matter is also a barotropic exotic fluid.

---

1 This form of effective action is selected among a list of possibilities and there are more non-canonical forms for the effective action in holonomic (explicitly time independent) gauge. There are other possible action forms where the action can be constructed from time dependent * Lennard-Jones potential * terms like $V[\rho(\tau), \dot{\rho}(\tau); \tau]$. 
FIG. 1: The plot shows the time evolution of the energy density $\rho(\tau)$ with three different values of $\rho_0$. Notice that at late time the energy energy tends to zero. The acceptable scenario here is a viable energy content matter in the Universe initiated from a low dense matter at very early epochs and then the density slowly increases to a peak of energy density where the universe reaches a turning point with the maximum energy content. Note that under this circumstances the system passed an unstable phase transition point. The final state of the Universe is dominated by an energy density of order of a non fine-tuned cosmological constant $\Lambda_{\text{non-Fin}}$. The reason for absence a fine tuning cosmological constant is the potentially renormalized form of the Nash gravity as an alternative model for quantum gravity.

D. Universe filled with barotropic fluid with energy density $H \propto \rho^{-2}$

The energy density continuity equation given in Eq.(11) is nonlinear and highly coupled to the Hubble parameter $H$. To make it integrable, we should not only know the initial energy density profile at very early times, but also we need to know how Hubble parameter evolves with respect to the density. An easy case and physically remarkable situation is when we suppose that $H = \frac{H_0}{\rho^2}$. Using this assumption we can integrate Eq.(11) and find $^2$:

$$\rho(\tau) = \rho_0 \exp \left( \frac{\tau}{\tau_0} - \frac{3\kappa^2 \sqrt{H_0} \tau^2}{2} \right).$$

(17)

This is a Gaussian density profile plotted in Fig.(3). We observe that for very early times, when $\tau \to -\infty$, $\rho(\tau) \to 0$ as well as late time when $\tau \to \infty$. Interestingly, we also observe a maximum of energy density at a certain time. Note that this form of energy density has a same physics as the one obtained in Eq.(16). Although in this model the universe didn’t evolve exponentially, and the total

$^2$ Note that here $H_0$ is an arbitrary constant and it may related to the initial Hubble parameter $H(\tau = 0)$ as well as initial energy density $\rho(\tau = 0)$. 
Hubble scale factor is not de Sitter. Still the Universe will start from a vacuum at very early times and comes to an end to vacuum.

Notice that in this scenario the evolution form of the universe is Gaussian, and only at an instant of time namely $\tau_m$, it reaches the maximum of energy density and hence the minimum of the Hubble scale. This time scale corresponds to an unstable point in the cosmic time, implying that the energy density reaches its maximum as an unstable transition point. Then the system suddenly undergoes a very sharp phase transition from maximum energy content to the vacuum. In our new scenario, the Universe starts from de Sitter (supporting the idea of de Sitter inflationary era), followed by the matter rich content epoch as an unstable time, and then transits to the late time cosmological constant dominated epoch.

$$
\tau_m = \left( 3\kappa^2 \sqrt{H_0 \tau_0} \right)^{-1}, \quad \rho_m = \rho_0 \exp \left( \frac{1}{6 \sqrt{H_0 \tau_0^2} \kappa^2} \right). \quad (18)
$$

FIG. 2: The plot shows the time evolution of the energy density $\rho(\tau)$ with three different values of $\rho_0$. Notice that at late time the energy energy tends to zero. In this scenario the universe initiated from an almost de-Sitter epoch, where the dominated energy density was very low and after a time interval, the universe reaches the maximum amount of the energy at a turning point, where the phase changed from deceleration to the acceleration. At the late time the dominant energy content is cosmological constant and the energy density decreases. Note that in this scenario the cosmological constant is not fine tuned. There is an effective cosmological constant which mimics the background cosmological evolution. An existence of a non fine tuned unique cosmological constant is due to the renormalizable quantum version of Nash gravity and the appearance of higher order curvature terms.

Now we can find Hubble parameter in terms of the $\tau$ as cosmic time:

$$
H(\tau) = \frac{H_0}{\rho_0} \exp \left( -\frac{2\tau}{\tau_0} + 3\kappa^2 \sqrt{H_0 \tau_0^2} \right). \quad (19)
$$
Note that we will examine this model with observational data in section (IV). For this purpose, it is adequate to rewrite it in terms of the redshift
\( 1 + z = \frac{1}{a(\tau)} \) (we suppose that \( a(\tau = 0) = a_0 \equiv 1 \) for simplicity), where \( a(\tau) = \exp\left\{ \int_0^\tau H(\tau')d\tau' \right\} \). We know that

\[
\int \exp(-ax + bx^2)dx = \frac{\sqrt{\pi}H_0 e^{-a^2/4b} \text{erfi}\left(\frac{2bx-a}{2\sqrt{b}}\right)}{2\sqrt{bp_0^2}},
\]

is the "imaginary error function" defined by \( \text{erfi}(y) = -i \text{erf}(iy) \). Using this formula we obtain:

\[
a(\tau) = \exp\left( \frac{\sqrt{\pi}H_0 e^{-a^2/4b} \text{erfi}\left(\frac{2bx-a}{2\sqrt{b}}\right)}{2\sqrt{bp_0^2}} \right), \quad a \equiv \frac{2}{\tau_0}, \quad b \equiv 3\kappa^2\sqrt{H_0}.
\]

Consequently, we can write the redshift \( z \) in terms of time \( \tau \) as:

\[
1 + z = \exp\left( -\frac{\sqrt{\pi}H_0 e^{-a^2/4b} \text{erfi}\left(\frac{2bx-a}{2\sqrt{b}}\right)}{2\sqrt{bp_0^2}} \right).
\]

Using the above expression, we can solve it to obtain \( \tau \) written in terms of the redshift \( z \):

\[
\tau = \frac{\left( a + 2\sqrt{b} \right) \pi H_0}{8b \left( \sqrt{bp_0^2} e^{a^2/4b} \log(z + 1) \right)^2} \text{ _1F_1 } \left( \frac{1}{2}, \frac{3}{2}, \left( \frac{2\sqrt{bp_0^2} e^{a^2/4b} \rho_0^2 \log(z + 1)}{H_0 \sqrt{\pi}} \right)^2 \right).
\]

Here we have used the relation \( \text{erfi}(z) = -i \text{erf}(iz) \) and \( \text{erf}(z) = (2z/\sqrt{\pi})\text{ _1F_1 }[1/2, 3/2, -z^2] \) with \( \text{ _1F_1 }[1/2, 3/2, -z^2] \) the confluent hypergeometric function of the first kind. Using \( \tau \) given in Eq.(23) we can express the Hubble parameter in terms of the redshift, \( H = H(z) \). Therefore, we write

\[
H(z) = \frac{H_0}{\rho_0} \exp\left( -\frac{2\tau(z)}{\tau_0} + 3\kappa^2\sqrt{H_0\tau(z)} \right).
\]
FIG. 3: The plots show the Hubble parameter $H(\tau)$ given in Eq. (24) versus redshift $z$ explicitly expressed in Eq. (23). (left $\rho_0 = 0.5$) and (right $\rho_0 = 1.0$). In very early time universe when $z >> 1$, the parameter is an increasing function vs $\tau$ and near the big bang time it tends to a constant, in support of the idea of inflationary universe. At the present time near $z \sim 0$, the Hubble becomes very large and the phases is completely divided specially when $z < 0$ and the universe undergoes a deceleration phase. As a result our phenomenological model predicts a type of phase transition from deceleration to acceleration.

E. Exact Nash cosmology for almost uniform total energy density

Let us first start by considering Eq. (6) and figure out its solution. Suppose that an arbitrary form of total energy density is given by $\rho_{\text{tot}}(t) \equiv \sum_i \rho_i$. Using Eq. (6) we obtain:

$$\dot{\zeta} + \zeta^3 - \kappa^2 \int_t \rho_{\text{tot}}(t')dt' = 0.$$

(25)

Using a standard method, we can simply solve the above nonlinear equation by changing a variable such that $\zeta \to \psi = \zeta^{-2}$, and the exact solution for the above matter contents when $\int_t \rho_{\text{tot}}(t')dt' \approx \text{constant} = \rho_0 > 0$ takes the form:

$$\frac{1}{6} \log \left( \zeta(t)^2 + \rho_0^{\frac{1}{3}} \zeta(t) + \rho_0^{\frac{2}{3}} \right) + \frac{\sqrt{3}}{3} \arctan \left( \frac{2\zeta(t) + \rho_0^{\frac{1}{3}}}{\sqrt{3} \rho_0^{\frac{2}{3}}} \right) - \frac{1}{3} \log \left( \zeta(t) - \rho_0^{\frac{1}{3}} \right) = \rho_0^{\frac{2}{3}} (t - t_0).$$

(26)

Apart from the condition $\int_t \rho_{\text{tot}}(t')dt' \approx \text{constant} = \rho_0 > 0$, we can not figure out any exact solution of the above nonlinear differential equation.
The time evolution of the solution $\zeta(t)$ given in Eq. (26) is illustrated in Fig. (4). It can be attributed to the behavior of the Hubble parameter since $\zeta = H^{1/2}$. Notice that the Hubble parameters are decreasing functions and become constant in the present epoch in all cases. This is an exact solution for the cosmological EoM and as a dominant solution in the early epoch can be used to build up a quasi stable de Sitter solution.

F. A barotropic fluid solution in Nash cosmological models

In the previous subsection, we can only obtain the exact solution for $\zeta$ in the case of uniform total matter density. However, it is also possible to figure out exact solutions if we use the scale factor representation of this equation. Specially using the reconstruction technique we first suppose that $\rho_{\text{tot}} = \tilde{\rho}_0 a^{-3(1+w)}$ with $\tilde{\rho}_0 = \rho(a_0 = 1)$. This is a standard form of the energy density written in terms of the scale factor. Here $w$ is the equation of state parameter (EoS). In the barotropic fluid manner, the EoS is parametrized by $p = w \rho$ responsible for a time evolution. Therefore, the EoM in Eq. (6) can be recast as follows:

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = 2\tilde{\rho}_0 a^{-3(w+\frac{1}{2})} a^{1/2}. \tag{27}$$

The above equation can be analytically solved to obtain

$$\bar{\omega}(t - t_0) = a^{5/2} \left(\frac{c_1}{a^{3/2}} + \frac{3\tilde{\rho}_0 a^{1/2 - 3w}}{2 - 3w}\right) \, 2F_1(1, a_1; 1 + b \mid c), \tag{28}$$

where $c_1$ is a constant and $2F_1(1, a_1; b \mid c)$ is the hyper-geometric function with $a_1 = (8 - 3w)/(6 - 9w)$, $b = 2/(2 - 3w)$ and $c = (3a^{2-3w}\tilde{\rho}_0)/(3w - 2)c_1$. Furthermore, $t_0$ denotes the initial time, $\bar{\omega}$ is an integration constant which can be interpreted as a frequency of oscillation.
G. Perturbation analysis of energy density via Hartman-Grobman & Lyapunov linearization

In any linear system, we classify physical behaviors of any fixed point by using the eigenvalues of the matrix constructed from a \( n \)-dimensional differential equation. However, the situation has dramatically changed when working in the nonlinear one. In the latter, the behavior of the system is more difficult to handle. Fortunately, we can transform the nonlinear differential system to a linear one. In doing so, we find the Jacobian matrix, \( J \), corresponding to the system and evaluate it at the fixed point. As a result, regarding the Hartman-Grobman theorem, we obtain a linear system with a characteristic coefficient matrix. Let’s consider a differential equation (DE): \( \dot{X} = \frac{dX}{dN} = f(X) \) defined on \( \mathbb{R}^n \), where \( N \) plays the role of time and \( X \) is a vector field. If \( \bar{a} \) is an equilibrium point \( f(\bar{a}) = 0 \), the linear approximation of \( f(X) \) at \( \bar{a} \) yields

\[
f(\bar{X}) \approx \mathbb{D}f(\bar{a})(\bar{X} - \bar{a}),
\]

where

\[
(J)_{ij} \equiv \mathbb{D}f(\bar{a}) = \left( \frac{df_i}{dX_j} \right)_{\bar{X}=\bar{a}},
\]

is the derivative (Jacobian) matrix of \( f \). Therefore, with the given \( \bar{X}' = f(\bar{X}) \), we associate the linear DE

\[
\bar{U}' = \mathbb{D}f(\bar{a})\bar{U},
\]

where \( \bar{U} = \bar{X} - \bar{a} \), called the linearization of the DE at the equilibrium point \( \bar{a} \). It is worth noting that the solutions of Eq.(31) will approximate the solutions of the nonlinear DE in a neighborhood of the equilibrium point \( \bar{a} \) provided that the equilibrium point is hyperbolic. This means that all eigenvalues \( (\lambda_i) \) of \( \mathbb{D}f(\bar{a}) \) have non-zero real part, \( \Re(\lambda_i) \neq 0 \).

Now we are going to quantify the stability of the fixed points obtained from Eq.(11) by using the above linearization. Consider the EoM for the total energy density given in Eq.(11). We then transform it to the second-order differential equation to obtain

\[
\ddot{\rho} - \frac{\dot{\rho}^2}{\rho} - \left( \frac{f(\rho)}{\rho} - f'(\rho) \right) \dot{\rho} + 3\kappa^2 \rho^2 \sqrt{H(\tau)} = 0,
\]

where the dot denote derivatives with respect to \( \tau \). Specifically, the above DE is nonlinear. In order to figure out its solutions, we use the following change of the variables:

\[
\dot{X} = X \left( \frac{f(\rho)}{\rho} - f'(\rho) \right) - 3\kappa^2 \rho^2 \sqrt{H(\tau)} + \frac{X^2}{\rho},
\]

\[
\dot{\rho} = X
\]

\[
(33) \quad (34)
\]
Here $f'(\rho) = \frac{df(\rho)}{d\rho}$. This in a non-autonomous dynamical system describing a continuous-time nonlinear density $\rho$. In the language of the dynamical systems, $\rho(\tau)$ is the state of the system and we can treat the Hubble $H(\tau)$ as the control input. Note that the left hand side of the Eq. (34) is a Lipschitz or continuously differentiable nonlinear function. A standard way to study the time evolution of the density function is to investigate the trajectory $\phi_t(\rho_0)$ where $\rho_0 \equiv \rho(\tau = 0)$ is the initial density profile.

Using the Hartman-Grobman linearization theorem, we find that the fixed points for the system are located at $P = (X_c = 0, \rho_c = \epsilon(\rightarrow 0))$, and the corresponding Jacobian matrix reads

$$\begin{pmatrix} \frac{f(\epsilon)}{\epsilon} - f'(\epsilon) & -6\epsilon\kappa^2\sqrt{H(\tau_0)} \\ 1 & 0 \end{pmatrix}$$

(35)

To ensure stable solutions, it is enough to set all eigenvalues of the Jacobian matrix so that $\lambda_i$ satisfies $\text{Re}(\lambda_i) \neq 0$. Therefore in our case, we find

$$\lambda_{\pm} = \frac{-\epsilon f'(\epsilon) + f(\epsilon) \pm \sqrt{(\epsilon f'(\epsilon) - f(\epsilon))^2 - 24\kappa^2\epsilon^3\sqrt{H(\tau_0)}}}{2\epsilon}.$$  

(36)

Note that when $\epsilon \to 0$ we have (if $f(0) > 0$):

$$\lambda_{\pm} = \frac{\epsilon^2}{6|f(0)|} \left(36\kappa^2\sqrt{H(\tau_0)} + f(0)f'''(0) \pm |f(0)|f''(0)\right) \to 0.$$  

(37)

So the system is unstable under density perturbations.

Moreover, there is the following important alternative theorem to study stability of the above non-autonomous system: Lyapunov theorem. Regarding the Lyapunov theorem for nonautonomous dynamical systems [18], we see that the system of Eqs.(33,34) is global and over the entire connected domain $D$, uniformly over the entire time interval $[\tau_0, \infty)$, and asymptotically stable about its equilibrium $P = (X_c, \rho_c)$, if there exist a Lyapunov function $V(X, \rho, \tau) : D \times [\tau_0, \infty) \to \mathcal{R}$ and three functions $\alpha, \beta, \gamma$ satisfying the following conditions:

- a: $V(X_c, \rho_c, \tau_0) = 0$,
- b: $V(X \neq X_c, \rho \neq \rho_c, \tau > \tau_0) > 0$,
- c: $\alpha\sqrt{X^2 + \rho^2} \leq V(X, \rho, \tau) \leq \beta\sqrt{X^2 + \rho^2}$, and
- d: $\frac{d}{d\tau}V(X, \rho, \tau \geq \tau_0) \leq -\gamma\sqrt{X^2 + \rho^2} < 0$.

Now we need to quantify a suitable form for $V$. Note that from the above conditions, we find

$$\frac{d}{d\tau}V(X, \rho, \tau \geq \tau_0) = X \frac{\partial V}{\partial \rho} + \frac{\partial V}{\partial X} \left(X\left(\frac{f(\rho)}{\rho} - f'(\rho)\right) - 3\kappa^2 \rho^2 \sqrt{H(\tau)} + \frac{X^2}{\rho}\right).$$

(38)
A suitable Lyapunov function for a set of parameters \( \{\alpha, \beta, \gamma\} \) is given as the following:

\[
V(X, \rho, \tau) = V_0 \left[ \exp \left( -\alpha X^2 - \beta \rho^2 - \gamma (\tau - \tau_0)^2 \sqrt{X^2 + \rho^2} \right) - 1 \right].
\] (39)

Notice that it violates condition (d) at point \( P = (X_c = 0, \rho_c = \epsilon \rightarrow 0) \). As a result the density equation is globally, uniformly and asymptotically unstable. This instability ensures the cosmological phase transitions during cosmic epochs.

**H. Poincare portrait for \((\rho, \dot{\rho})\) and energy conditions**

In this subsection, we aim to quantify the phase of \((\rho, \dot{\rho})\) and examine the relation between continuity equation and energy conditions. We also explain the possible phase of the matter in Nash gravitational theory. In so doing, we first explain the main idea of the continuity equation which obeys Eq.(12). Having assumed that the matter is isotropic, it is worth noting that the energy-momentum tensor components, energy density \( \rho \) and pressure \( p \) should satisfy the following well known energy conditions [19] summarized in Table (I):

| Energy condition                  | parametrization               |
|----------------------------------|-------------------------------|
| null energy condition (NEC)      | \( \rho + p \geq 0 \)        |
| weak energy condition (WEC)      | \( \rho \geq 0, \rho + p \geq 0 \) |
| dominant energy condition (DEC)  | \( \rho \geq 0, \rho \pm p \geq 0 \) |
| strong energy condition (SEC)    | \( \rho + p \geq 0, \rho + 3p \geq 0 \) |

The above energy conditions are defined as a set of inequalities from a purely geometrical point of view [20] and are widely used in many different types of modified gravity theories, e.g., \( f(R) \) theory [21], \( f(T) \) theory [22], Gauss-Bonet theory [23] and other interesting models. Note here that the violation of the NEC violates all other energy conditions. Regarding the continuity equation in GR, if the WEC holds, then the only possibility is that \( H < 0 \) meaning that the cosmological background is in the deceleration phase. Furthermore, if we have NEC, then for an accelerating universe it implies that \( dp/d\tau < 0 \). In addition, if DEC holds, then the only possibility is to have a decelerating behavior. In the traversable wormhole scenarios, the NEC is violated. If we require a wormhole to be located in the accelerating universe, we need to have \( dp/d\tau < 0 \). In our Nash gravity case, note that if \( dp/d\tau < 0 \), from Eq.(11) and if \( \rho > 0 \), necessarily we will have

\[
\frac{d^2}{d\tau^2} \rho < 0.
\] (40)
The above inequality with the condition \( \rho > 0 \) implies that \( \frac{d}{d\tau}\rho < \dot{\rho}(\tau_0) \). Consider the phase portraits of \( \rho - \dot{\rho} \) displayed in Fig.(5). In our case, we need to consider \( \dot{\rho}(\tau_0) > 0 \) to have a compact phase space. From the Figs.(5), we observe that there are some regions where \( \frac{d}{d\tau}\rho < \dot{\rho}(\tau_0) \) and \( \rho > 0 \). Notice that shaded regions above those of \( \rho(0.01) = \dot{\rho}(0.01) = 0.5 \) display \( \rho(\tau_0) < \dot{\rho}(\tau_0) \). It demonstrates that the violation of the NEC is possible. Since this violation is detected for different types of the EoS \( w \), for a barotropic fluid, we claim that a class of the traversable wormholes probably exist in the primordial epochs of our Nash-modified cosmological model. Bear in mind that the existence of such wormholes were predicted before in Ref.[24]. However, our Poincaré phase portrait demonstrates a different point of view.

![Phase portraits for different EoS parameters](image)

**FIG. 5:** The plots show the phase portrait for pair \( \rho - \dot{\rho} \) based on the generalized continuity equation given in Eq.(11). From energy conditions point of view we conclude that there are regions of the cosmological epochs where the NEC is violated and they significantly indicate the existence of primordial transversal wormholes. Note that shaded regions above those of \( \rho(0.01) = \dot{\rho}(0.01) = 0.5 \) display \( \rho(\tau_0) < \dot{\rho}(\tau_0) \).

### III. OM DIAGNOSTIC ANALYSIS

Commonly the cosmological parameters like the Hubble parameter \( H \), deceleration parameter \( q \), and the equation of state (EoS) parameter \( \omega \) are importance on checking the consistency of a particular model. However, \( H \) and \( q \) are not adequate in differentiating among dark energy models. This is so since any dark energy models can generate a positive Hubble parameter and a negative deceleration
parameter, i.e., $H > 0$ and $q < 0$, for the present cosmological epoch. Therefore, higher-order time derivatives of the scale factor $a(t)$ is required to analyse the dark energy models [25, 26].

Let’s consider the first parameter, called the deceleration parameter $q(t)$, which is defined in terms of the Hubble parameter via:

$$q(t) \equiv - \left(1 + \frac{\dot{H}}{H}\right) \rightarrow q(\tau) \equiv - \left(1 + H^{-3/2} \frac{dH}{d\tau}\right),$$

(41)

where in order to transform $q(t)$ into $q(\tau)$, we have used $dH/dt = H^{-1/2} dH/d\tau$. Regarding our solution given in Eq.(19), we obtain from the above relation:

$$q(\tau) = -1 + \left(\frac{2 - 3\sqrt{\beta_0^2 \tau_0}}{\rho_0 H_0^1/2}\right) e^{-\frac{3\sqrt{\beta_0^2 \tau_0}}{2} \tau + \frac{\tau_0}{\tau_0}}. $$

(42)

FIG. 6: The plots show the evolution of the deceleration parameter $q(\tau)$ for different values of $\rho_0$ and $\tau_0$. We used on the left panel $\rho_0 = 0.5$ and on the right panel $\rho_0 = 1.0$. We used $\kappa = 1$ and $H_0 = 1$ for the plots. The left panel using $\rho_0 = 0.5$ displays an accelerating expansion for all time with $\tau_0 = 1.0$ and $5.0$. However, we have another case of phase transition from an acceleration to a deceleration with $\tau_0 = 0.5$. The right panel utilizing $\rho_0 = 1.0$ displays an accelerating expansion for all time with $\tau_0 = 1.0$ and $5.0$. However, we have another case of a deceleration with $\tau_0 = 0.5$.

The behavior of the deceleration parameter $q(\tau)$ can be clearly displayed in Fig.(6). Notice that the behavior of the deceleration parameter at very early time depends on $t_0$. Specifically, if we choose $t_0 < 2$ we find the positivity of the deceleration parameter at very early time. However, we clearly discover that the deceleration parameter is always negative at late time.

Other physical quantities are the statefinder parameters $\{r, s\}$ defined in terms of the Hubble parameters as follow:

$$r = 1 + 3 \frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3},$$

(43)

$$s = - \frac{3HH + \dot{H}}{3H \left(2\dot{H} + 3H^2\right)},$$

(44)
In terms of $\tau$, they become
\begin{align}
    r &= 1 + \frac{3\partial_\tau H}{H^{5/2}} + \frac{\partial_\tau H^{-1/2} \partial_\tau H}{H^{7/2}}, \\
    s &= -\frac{\partial_\tau H}{(2\partial_\tau H + 3H^{5/2})} - \frac{\partial_\tau H^{-1/2} \partial_\tau H}{3H(2\partial_\tau H + 3H^{5/2})}.
\end{align}

Substituting the Hubble parameter Eq.(19) into the above expressions, we find
\begin{align}
    r &= \frac{6M(\tau)^{9/2}\rho_0^3\tau_0(3\tau_0 - 2) + M(\tau)^3\rho_0^6(2 - 3\tau_0)^2 + 2\tau_0^2}{2\tau_0^2}, \\
    s &= -\frac{(3\tau_0 - 2)\left(6N(\tau)^{3/2}\tau_0 + 3\tau_0 - 2\right)}{6N(\tau)^{3/2}\tau_0 \left(3\sqrt{N(\tau)}\tau_0 + 6\tau_0 - 4\right)},
\end{align}

where we have used $\kappa = 1$ and $H_0 = 1$ and defined new parameters $M(\tau)$ and $N(\tau)$ as the following
\begin{align}
    M(\tau) &= e^{3\tau - \frac{2\tau_0}{\tau_0}}, \\
    N(\tau) &= M(\tau)/\rho_0^2.
\end{align}

We can write $\tau = \tau(s)$ by solving Eq.(48) and substituting $t(r)$ back into Eq.(47). Therefore we can obtain $r$ in terms of $s$, i.e., $r = r(s)$. Note that the statefinder parameters $\{r, s\} = \{1, 0\}$ represents the point where the flat $\Lambda$CDM model exists in the $r - s$ plane [27]. Therefore the departure of dark energy models from this fixed point can be used to obtain the distance of these models from the flat $\Lambda$CDM model. This allows us to display the statefinder parameters as shown in Fig.(7). Here we found two solutions in which the first one displays the existence of the $\Lambda$CDM model.

![FIG. 7: The plot shows the statefinder parameters in the $r - s$ plane. We notice that our model develops $(r, s) = (1, 0)$ for $\tau_0 = 2/3$ which displays the point where the flat $\Lambda$CDM model exists. Here we used $\rho_0 = 1.$](image)

Notice that above physical quantities, e.g. the statefinder parameters, are parametrized by higher-order time derivatives of the scale factor. However, another diagnostic parameter depends only on the first order temporal derivative of the scale factor can also be used to constrain the model. This is
so-called the $\Om$ analysis \cite{28}. This parameter only involves to the Hubble parameter. This has also been applied to some interesting models, e.g. Galileons models \cite{29}. The $\Om$ analysis is defined in terms of the Hubble parameter via

$$Om(z) = \left[ \frac{H(z)}{H_0} \right]^2 - 1 \frac{1}{(1 + z)^3 - 1},$$  \hspace{1cm} (50)

where $Om(z)$ is a function depending only on the redshift, $z$. Moreover, this can be parametrized further when having a constant equation of state (EoS) parameter $\omega$, and in this case we can write

$$Om(z) = \Omega_{m0} + (1 - \Omega_{m0}) \frac{(1 + z)^{3(1+\omega)} - 1}{(1 + z)^3 - 1}.$$  \hspace{1cm} (51)

It is worth noting that we have different values of $Om(z) = \Omega_{m0}$ for the $\Lambda$CDM model, quintessence, and phantom cosmological models.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{The plots show the relation between $Om(z)$ and $z$ given by Eq.(50). Here we have used $\rho_0 = 0.5$ and $\rho_0 = 1.0$ for the left panel and right panel, respectively.}
\end{figure}

The redshift dependence of the $Om(z)$ parameter is displayed in Fig.(8). The plots show the relation between $Om(z)$ and $z$ given by Eq.(50). Here we have used $\rho_0 = 0.5$ and $\rho_0 = 1.0$ for the left panel and right panel, respectively.
FIG. 9: We compare the Hubble parameter of Eq.(42) with that of Eq.(43). We find that they are in good agreement between $0.1 \leq z \leq 0.4$ for the left panel, while between $0.1 \leq z \leq 0.6$ for the right panel.

From the left-panel of Fig.(9), we discover that our model displays an equation of state $w = -9$ which corresponds to the hypothetical phantom energy. Here we used for Eq.(42) $\tau_0 = 1.0$ and $\rho_0 = 0.70$ and for Eq.(43) $\Omega_m = 0.3$, $\Omega_{r,m} = 10^{-4}$ and $w = -9$. In addition, on the right-panel of Fig.(9), we consider an equation of state $w = -6$ which also corresponds to the hypothetical phantom energy. Here we used for Eq.(42) $\tau_0 = 1.0$ and $\rho_0 = 0.55$ and for Eq.(43) $\Omega_m = 0.3$, $\Omega_{r,m} = 10^{-4}$ and $w = -6$.

IV. OBSERVATIONAL CONSTRAINTS

In this section, we use observational data from Supernovae Ia (SNeIa) and baryonic acoustic oscillations (BAO) to constrain parameters of the models. Consider the total $\chi^2$ for joint data set. Here we employ

$$
\chi^2_{\text{tot}} = \chi^2_{\text{SNeIa}} + \chi^2_{\text{BAO}},
$$

where the $\chi^2$ for each set of data have to be evaluated. In order to determine the parameter, we need the luminosity distance $D_L(z)$ parameter. Such parameter is defined in terms of the Hubble parameter and the redshift as:

$$
D_L(z) = (1 + z) \int_0^z \frac{H_0 dz'}{H(z')}.
$$

We consider the distance modulus $\mu$ defined by

$$
\mu = m - M = 5 \log D_L + \mu_0,
$$

where $m$ and $M$ are defined as the apparent and absolute magnitudes of the Supernovae. Here $\mu_0 = 5 \log \left( \frac{H_0^{-1}}{\text{Mpc}} \right) + 25$ is a nuisance parameter which will be marginalized. What we are going to do is to
estimate the luminosity and distance modulus for our theoretical model given in Eq.(19). Plugging the Hubble parameter given in Eq.(19) into (53) and performing integration, we obtain

$$D_L(z) = \rho_0^2 (1 + z) \int_0^z \exp \left( -\frac{2\tau(z')}{\tau_0} + 3\kappa^2 \sqrt{H_0 \tau(z')^2} \right) \, dz'$$

where $\tau(z)$ is given in Eq.(23). Consequently the $\mu$th takes the from

$$\mu_{th} = 5 \log \left[ (1 + z) \right] + 5 \log \left[ \int_0^z \exp \left( -\frac{2\tau(z')}{\tau_0} + 3\kappa^2 \sqrt{H_0 \tau(z')^2} \right) \, dz' \right] + \mu_0 + 10 \log \rho_0. \quad (56)$$

Note that the observed range of the redshift $z \subset [0.1, 0.74]$. In this case, it is possible to estimate the integral in Eq.(56) by using "the method of steepest descent or stationary-phase method or saddle-point method". Using this approach, the integral yields

$$I(z) \equiv \int_0^z \exp \left( -\frac{2\tau(z')}{\tau_0} + 3\kappa^2 \sqrt{H_0 \tau(z')^2} \right) \, dz' \approx \tau_0 \left[ 1 - \exp \left( -\frac{2\tau(z_{Max})}{\tau_0} \right) \right], \quad (57)$$

with $z_{Max} < 1$. Therefore, the approximated $\mu_{th}$ takes the form

$$\mu_{th} = m - M = 5 \log \left[ (1 + z) \right] + 5 \log \left[ 1 - \exp \left( -\frac{2\tau(z_{Max})}{\tau_0} \right) \right] + \mu_0 + 5 \log \rho_0^2 \tau_0. \quad (58)$$

Fig.(10) shows the approximation of the $\mu_{th}$ parameter estimated in Eq.(58). Here we have used $z_{Max} = 0.74$.

![Fig. 10: The plots show theoretical predictions for the $\mu_{th}$ estimated in Eq.(58). At very "present" time epochs, when $z \sim 0$, the theory results $\mu_{th} \rightarrow 0$. There are two regimes; the upper line displays an monotonically increasing value of $\mu_{th}$ (a.k.a. a nonlinear behavior), while the bottom line shows a linear regime. In addition, we observe that a lower initial density provides a higher $\mu_{th}$.](attachment:fig10.png)

From the observational point of view, we consider the corresponding $\chi^2_{SNeIa}$ for this data set,

$$\chi^2_{SNeIa}(\mu_0, \theta) = \sum_{i=1}^{580} \frac{[\mu_{th}(z_i, \mu_0, \theta) - \mu_{obs}(z_i)]^2}{\sigma_{\mu}(z_i)^2}, \quad (59)$$
where $\mu_{\text{obs}}$, $\mu_{\text{th}}$ and $\sigma_{\mu}$ denote the observed distance modulus, the theoretical distance modulus and the uncertainty in the distance modulus, respectively. Note that the parameters in the cosmological models are parametrized by $\theta$. In addition, Eq.\((59)\) can be rewritten in a more compact form as

$$\chi^2_{\text{SNBA}}(\theta) = A(\theta) - \frac{B(\theta)^2}{C(\theta)},$$

where

$$A(\theta) = \sum_{i=1}^{580} \frac{[\mu_{\text{th}}(z_i, \mu_0 = 0, \theta) - \mu_{\text{obs}}(z_i)]^2}{\sigma_{\mu}(z_i)^2},$$

$$B(\theta) = \sum_{i=1}^{580} \frac{\mu_{\text{th}}(z_i, \mu_0 = 0, \theta) - \mu_{\text{obs}}(z_i)}{\sigma_{\mu}(z_i)^2},$$

$$C(\theta) = \sum_{i=1}^{580} \frac{1}{\sigma_{\mu}(z_i)^2}.$$  \(\text{(60)}\)

**TABLE II:** The table shows the values of $\frac{d_A(z_\star)}{D_V(z_{\text{BAO}})}$ for distinct values of $z_{\text{BAO}}$ [30].

| $z_{\text{BAO}}$ | 0.106  | 0.20  | 0.35  | 0.44  | 0.60  | 0.73  |
|------------------|-------|-------|-------|-------|-------|-------|
| $\frac{d_A(z_\star)}{D_V(z_{\text{BAO}})}$ | 30.95 ± 1.46 | 17.55 ± 0.60 | 10.11 ± 0.37 | 8.44 ± 0.67 | 6.69 ± 0.33 | 5.45 ± 0.31 |

Using BAO data of $\frac{d_A(z_\star)}{D_V(z_{\text{BAO}})}$ displayed in Table (II), we find $z_\star \approx 1091$ as the decoupling time. The co-moving angular-diameter is given by $d_A(z) = \int_0^z H(z')^{-1}dz'$ and $D_V(z) = \sqrt[d_A(z)]{z/H(z)}$ is the dilation scale. In addition, using this data set, the $\chi^2_{\text{BAO}}$ is defined as

$$\chi^2_{\text{BAO}} = X^T C^{-1} X,$$

where $C^{-1}$ is the inverse covariance matrix. Here $X$ is the column vector given by

$$X = \begin{pmatrix}
\frac{d_A(z_\star)}{D_V(0.106)} & -30.95 \\
\frac{d_A(z_\star)}{D_V(0.20)} & -17.55 \\
\frac{d_A(z_\star)}{D_V(0.35)} & -10.11 \\
\frac{d_A(z_\star)}{D_V(0.44)} & -8.44 \\
\frac{d_A(z_\star)}{D_V(0.60)} & -6.69 \\
\frac{d_A(z_\star)}{D_V(0.73)} & -5.45
\end{pmatrix}.$$  \(\text{(65)}\)

Note that BAO exist in the decoupling redshift such that $z = 1.090$. The relevant scaling is the following quantity

$$A = \sqrt{\Omega_m} \left( \frac{1}{z_b} \int_{z_b}^z \frac{dz}{H(z)} \right)^{2/3}.$$  \(\text{(66)}\)
Regarding the WiggleZ-data [31], we obtain $A = 0.474 \pm 0.034, 0.442 \pm 0.020$ and $0.424 \pm 0.021$ at the redshifts $z_b = 0.44, 0.60$ and $0.73$, respectively. We compare our results with observational data. Fortunately, the SNeIa data allows us to constrain the following observational parameters for our models. With the Hubble parameter $H \propto \rho^{-2}$, we discover the best fit of our model summarized in Table (III) with $\Omega_{m0} = 0.25$ and $H_0 = 0.4$ for $\tau_0 = 0.3$ and $\rho_0 = 0.5$ which are in a good agreement with the observational data [32].

TABLE III: The table shows the best fit of the physical parameters for our present model. In our case, we assume that the universe is parametrized by the Hubble parameter of the form $H \propto \rho^{-2}$.

| Parameters | $\tau_0$ | $\rho_0$ | $H_0$ | $\Omega_{m0}$ |
|-----------|----------|----------|-------|---------------|
| Best fitted value | $0.3^{+0.35}_{-0.15}$ | $0.5^{+0.35}_{-0.4}$ | $0.4^{+0.35}_{-0.1}$ | $0.25^{+1.3}_{-2.7}$ |

V. CONCLUSION

Nash have proposed a new theory of gravity alternative to Einstein’s general theory of relativity. The formulation allows to obtain field equations for empty space, but did not include a description of matter field. In this work, we have generalized the original Nash theory by adding the matter fields in the original action. We have specified a proper form of the field equations on more general footings for space with matter contents. We have derived the equations of motion in the flat FLRW spacetime and examined the behaviors of the solutions by invoking specific forms of the Hubble parameter.

We have also classified the physical behaviors of the solutions by employing the stability analysis. We have checked the consistency of the model by considering cosmological parameters, e.g., the Hubble parameter $H$, deceleration parameter $q$, and $\Omega_{m}(z)$ parameter. We have further used observational data from Supernovae Ia (SNeIa) and baryonic acoustic oscillations (BAO) to constrain the parameters of the model.

With the Hubble parameter $H \propto \rho^{-2}$, we have discovered the best fit of our model with $\Omega_{m0} = 0.25$ and $H_0 = 0.4$ for $\tau_0 = 0.3$ and $\rho_0 = 0.5$. However, particular extensions of the present work are still possible. As well known, the dynamical systems analysis for analyzing the qualitative properties of cosmological models has proven to be very useful. It has been successfully used to study and to understand a number of cosmological models, e.g. the standard GR cosmology [33] and the scalar-tensor theories of gravity [34].

Moreover, the cosmological dynamics of the modified $f(R)$ gravity was extensively studied in Refs.[35–37] by employing the dynamical system analysis in both homogeneous & isotropic universe (a.k.a. the flat FLRW model). In addition, there were some interesting investigations regarding the
less anisotropic counterpart (Bianchi types and the others); see Refs.[38]-[41] for example. Interestingly, some possible scenarios will be worth investigating, e.g. the origin and evolution of primordial magnetic field in the Universe; see for example [42]. The extension of the present work can be in principle implemented by following those references on employing the dynamical system analysis.

Acknowledgment

PC is financially supported by the Institute for the Promotion of Teaching Science and Technology (IPST) under the project of the “Research Fund for DPST Graduate with First Placement”, under Grant No. 033/2557. This work is partially supported by Thailand Center of Excellence in Physics (ThEP).

[1] Phys. Lett. B 91, 99 (1980)
[2] S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999)
A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998)
[3] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 594, A13 (2016); P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 594, A20 (2016); P. A. R. Ade et al. [BICEP2 Collaboration], Phys. Rev. Lett. 112, 241101 (2014); P. A. R. Ade et al. [BICEP2 and Planck Collaborations], Phys. Rev. Lett. 114, 101301 (2015); P. A. R. Ade et al. [BICEP2 and Keck Array Collaborations], Phys. Rev. Lett. 116, 031302 (2016); E. Komatsu et al. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011); G. Hinshaw et al. [WMAP Collaboration], Astrophys. J. Suppl. 208, 19 (2013)
[4] M. Tegmark et al. [SDSS Collaboration], Phys. Rev. D 69, 103501 (2004); U. Seljak et al. [SDSS Collaboration], Phys. Rev. D 71, 103515 (2005)
[5] D. J. Eisenstein et al. [SDSS Collaboration], Astrophys. J. 633, 560 (2005)
[6] B. Jain and A. Taylor, Phys. Rev. Lett. 91, 141302 (2003)
[7] K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, Astrophys. Space Sci. 342, 155 (2012)
[8] S. Nojiri and S. D. Odintsov, eConf C 0602061, 06 (2006) Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007); Nojiri, S. D. Odintsov, Phys. Rept. 505 (2011) 59; S. Nojiri and S. D. Odintsov, Phys. Rept. 505, 59 (2011); M. Sami and R. Myrzakulov, Int. J. Mod. Phys. D 25 (2016) no.12, 1630031; M. Sami, Curr. Sci. 97 (2009) 887; M. Sami, arXiv:0901.0756 [hep-th]; M. Sami, Lect. Notes Phys. 720 (2007) 219; E. J. Copeland, M. Sami and S. Tsujikawa,Int. J. Mod. Phys. D 15 (2006) 1753; M. Sami and N. Dadhich, TSPU Bulletin 44N7 (2004) 25
[9] H. A. Buchdahl, Mon. Not. Roy. Astron. Soc. 150, 1 (1970); S. Capozziello, Int. J. Mod. Phys. D 11, 483 (2002); S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D 70, 043528 (2004)
[10] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010); A. De Felice and S. Tsujikawa, Living Rev. Rel. 13, 3 (2010)
[11] A. A. Starobinsky, Phys. Lett. 91B (1980) 99.
[12] Lecture by John F. Nash Jr. “An Interesting Equation.” http://sites.stat.psu.edu/babu/nash/intereq.pdf
[13] M. T. Aadne and Ø. G. Grøn, Universe 3 (2017) no.1, 10
[14] K. Lake, arXiv:1703.02653 [gr-qc]
[15] P. Channuie, D. Momeni and M. A. Ajmi, Eur. Phys. J. C 78, no. 7, 588 (2018)
[16] C. Lin, S. Mukohyama, R. Namba and R. Saitou, JCAP 1410, no. 10, 071 (2014)
[17] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, Phys. Rev. Lett. 114 (2015) no.21, 211101
[18] A. M. Lyapunov, The General Problem of Stability of Motion (100th Anniversary), Taylor & Francis, London, 1992
[19] S. W. Hawking, G.F.R. Ellis. The Large Scale Structure of Space-Time, (Cambridge University Press, 1973).
[20] M. Visser and C. Barcelo, gr-qc/0001099
[21] J. Wang, Y. B. Wu, Y. X. Guo, W. Q. Yang and L. Wang, Phys. Lett. B 689, 133 (2010)
[22] M. Jamil, D. Momeni and R. Myrzakulov, Gen. Rel. Grav. 45, 263 (2013)
[23] N. M. Garcia, T. Harko, F. S. N. Lobo and J. P. Mimiso, Phys. Rev. D 83, 104032 (2011)
[24] S. Nojiri, O. Obregon, S. D. Odintsov and K. E. Osetrin, Phys. Lett. B 458, 19 (1999)
[25] V. Sahni, T. D. Saini, A. A. Starobinsky and U. Alam, Soviet Journal of Experimental and Theoretical Physics Letters, 77,(2003) 201
[26] U. Alam, V. Sahni, T. D. Saini and A. A. Starobinsky, Mon. Not. R. Astron. Soc., 344,(2003) 1057
[27] Z. G. Huang, X. M. Song, H. Q. Lu and W. Fang W, Astrophys. Space Sci. 315,(2008) 175
[28] V. Sahni, A. Shafieloo and A. A. Starobinsky, Phys. Rev. D 78,(2008) 103502
[29] M. Jamil, D. Momeni and R. Myrzakulov, Eur. Phys. J. C 73, (2013) 2347
[30] O. Farooq and B. Ratra, Astrophys. J. 766, (2013) L7
[31] C. Blake et all, arXiv:1108.2635
[32] D. J. Eisenstein, et al., Astrophys. J. 633, 560 (2005); Y. Wang and P. Mukherjee, Astrophys. J. 650, 1 (2006); J. R. Bond, G. Efstathiou and M. Tegmark, Mon. Not. Roy. Astron. Soc. 291, L33 (1997)
[33] G. Leon and C. R. Fadragas, arXiv:1412.5701 [gr-qc]; Wainwright, G.F.R. Ellis. 1997. Dynamical System in Cosmology, Cambridge Univ. Press
[34] S. Carloni, S. Capozziello, J. A. Leach and P. K. S. Dunsby, Class. Quant. Grav. 25, 035008 (2008)
[35] S. Carloni, P. K. S. Dunsby, S. Capozziello and A. Troisi, Class. Quant. Grav. 22, 4839 (2005)
[36] N. Goheer, J. Larena and P. K. S. Dunsby, Phys. Rev. D 80, 061301 (2009)
[37] M. Abdelwahab, S. Carloni and P. K. S. Dunsby, Class. Quant. Grav. 25, 135002 (2008)
[38] J. A. Leach, S. Carloni and P. K. S. Dunsby, Class. Quant. Grav. 23, 4915 (2006)
[39] A. Chatrabhuti, V. Yingcharoenrat and P. Channuie, Phys. Rev. D 93, no. 4, 043515 (2016)
[40] D. Momeni, Int. J. Theor. Phys. 50, 1493 (2011)
[41] X. Liu, P. Channuie and D. Samart, Phys. Dark Univ. 17, 52 (2017)
[42] D. Grasso and H. R. Rubinstein, Phys. Rept. 348, 163 (2001)