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TIAN’S INVARIANT OF THE GRASSMANN MANIFOLD

JULIEN GRIVAUX

Abstract. — We prove that Tian’s invariant on the complex Grassmann manifold $G_{p,q}(\mathbb{C})$ is equal to $1/(p+q)$. The method introduced here uses a Lie group of holomorphic isometries which operates transitively on the considered manifolds and a natural imbedding of $\mathbb{P}^1(\mathbb{C})^p$ in $G_{p,q}(\mathbb{C})$.

Résumé. — On prouve que l’invariant de Tian sur la grassmannienne $G_{p,q}(\mathbb{C})$ est $1/(p+q)$. La méthode présentée dans cet article utilise un groupe de Lie d’isométries holomorphes qui opère transitivement sur les variétés considérées ainsi qu’un plongement naturel de $\mathbb{P}^1(\mathbb{C})^p$ dans $G_{p,q}(\mathbb{C})$.

1. Introduction

On a complex manifold, an hermitian metric $h$ is characterized by the 1-1 symplectic form $\omega$ defined by $\omega = ig_{\lambda\mu} dz^\lambda \wedge dz^\mu$, where $g_{\lambda\mu} = h_{\lambda\mu}/2$. The metric is a Kähler metric if $\omega$ is closed, i.e. $d\omega = 0$; then $M$ is a Kähler manifold.

On a Kähler manifold, we can define the Ricci form by $R = i R_{\lambda\mu} dz^\lambda \wedge dz^\mu$, where $R_{\lambda\mu} = -\partial_{\lambda\mu} \log |g|$. A Kähler manifold is Einstein with factor $k$ if $R = k \omega$. For instance, choosing a local coordinate system $Z = (z_1, \ldots, z_m)$, the projective space $\mathbb{P}_m(\mathbb{C})$ with the Fubini-Study metric $\omega = i \partial \bar{\partial} \log (1 + ||Z||^2)$ is Einstein with factor $m+1$.

On a Kähler manifold $M$, the first Chern class $C^1(M)$ is the cohomology class of the Ricci tensor, that is the set of the forms $R + i \partial \bar{\partial} \varphi$, where $\varphi$ is $C^\infty$ on $M$. If there is a form in $C^1(M)$ which is positive (resp. negative, zero), then $C^1(M)$ is positive (resp. negative, zero). If a Kähler manifold is Einstein, then $C^1(M)$ and $k$ are both positive (resp. negative, zero). In the negative case, it was proved by Aubin ([Au1], see also [Au4]), that there exists a unique Einstein-Kähler metric (E.K. metric) on $M$. It is so for the zero case too ([Au1], [Ya]). The question for the positive case is still open: some manifolds, such as the complex projective space blown up at one point, do not admit an E.K. metric (for obstructions, see [Li] and [Fu]). Aubin [Au2] and Tian [Ti] have shown that for suitable values of holomorphic invariants of the metric, there exists an E.K. metric on $M$.

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For $\omega/2\pi$ in $C^1(M)$, Tian’s invariant $\alpha(M)$ is the supremum of the set of the real numbers $\alpha$ satisfying the following: there exists a constant $C$ such that the inequality $\int_M e^{-\alpha \varphi} \leq C$ holds for all the $C^\infty$ functions $\varphi$ with $\omega + i\partial\bar{\partial}\varphi > 0$ and $\sup \varphi \geq 0$, where $\omega = ig_{\lambda\mu} \, dz^\lambda \wedge d\bar{z}^\mu$ is the metric form. Such functions $\varphi$ are said $\omega$-admissible. In [Ti], Tian established that if $\alpha(M) > m/(m+1)$, $m$ being the dimension of $M$, there exists an E.K. metric on $M$. This condition is not necessary: it does not hold on the projective space, where Tian’s invariant is $1/(m+1)$.

In the same paper, Tian introduces a more restrictive invariant $\alpha_G(M)$, considering only the admissible functions $\varphi$ invariant by the action of a compact group $G$ of holomorphic isometries. The sufficient condition for the existence of an E.K. metric on $M$ remains $\alpha_G(M) > m/(m+1)$; it is more easily satisfied if the group $G$ is rich enough.

In many cases, the group $G$ is a non-discrete Lie group. The invariant $\alpha_G(M)$ can be computed using subharmonic functions methods and the maximum principle (for effective examples, see [Be1], [Be2], [Be-Ch1], [Be-Ch2], [Re]).

In this paper, we prove the following theorem:

**Theorem 1.1.** Tian’s invariant on $G_{p,q}(\mathbb{C})$ is given by $\alpha(G_{p,q}(\mathbb{C})) = 1/(p+q)$.

This generalizes the known result on $\mathbb{P}^m(\mathbb{C})$ ([Ti], see also [Au3]). Let us also mention that Tian’s invariant has been computed on $\mathbb{P}^m(\mathbb{C})$ blown up at one point and on certain Fermat hypersurfaces using Hörmander $L^2$ estimates for the $\bar{\partial}$-equation ([Ti]).

We first compute the volume element of the metric $\langle \cdot, \cdot \rangle_{G_{p,q}}$; then we will establish some general preliminary results concerning Tian’s invariant as well as imbeddings of $\{\mathbb{P}^1(\mathbb{C})\}^p$ in $G_{p,q}(\mathbb{C})$ which allow us to deduce $\alpha(G_{p,q}(\mathbb{C}))$ from $\alpha(\mathbb{P}^1(\mathbb{C}))$.

2. Basic properties of the Grassmann manifold

We propose here a short survey of the properties of Grassmann manifold (for more details, see [Ko-No]). We denote by $G_{p,q}(\mathbb{C})$ the set of the subspaces of dimension $p$ in $\mathbb{C}^{p+q}$; in particular, $G_{1,m}(\mathbb{C})$ is the complex projective space of dimension $m$. It is known (see [Au3]) that on $\mathbb{P}_m(\mathbb{C})$, the Fubini-Study metric is Einstein with factor $m+1$ and that Tian’s invariant is $1/(m+1)$. Now, let $M^*(p+q,p)$ be the set of the matrices of rank $p$ in $M_{p+q,p}(\mathbb{C})$. The group $Gl_p(\mathbb{C})$ acts by multiplication on the right on $M^*(p+q,p)$. More precisely $(M^*(p+q,p), \pi, G_{p,q}(\mathbb{C}))$ is a principal fiber bundle with group $Gl_p(\mathbb{C})$. The group $Gl_{p+q}(\mathbb{C})$ acts by multiplication on the left on $M^*(p+q,p)$ and induces an action on $G_{p,q}(\mathbb{C})$; so does the unitary group $U(p+q)$. These groups act transitively on $G_{p,q}(\mathbb{C})$, which shows that $G_{p,q}(\mathbb{C})$ is compact.
We denote by \( \mathcal{I} \) the set of all increasing-ordered subsets of \( p \) elements in \( \{1, \ldots, p + q\} \). Let \( P \) be an element of \( M^*(p + q, p) \), \( P = (p_{ij})_{1 \leq i \leq p + q} \). By Cauchy-Binet formula we get:

\[
\det(\mathcal{P}^T \mathcal{P}) = \sum_{I \in \mathcal{I}} |\det m_I(P)|^2,
\]

where \( m_I(P) \) is the matrix \( (p_{ij})_{1 \leq i \leq p + q} \). The form \( \omega \), where \( \omega = i \partial \bar{\partial} \log \det(\mathcal{P}^T \mathcal{P}) \), is invariant by the action of \( GL_p(\mathbb{C}) \) on \( M^*(p + q, p) \), and so it projects onto a form \( \mathcal{G}_{pq} \). The metric \( \mathcal{G}_{pq} \) is a Kähler metric form on \( G_{pq}(\mathbb{C}) \).

For \( p = 1 \), this metric on \( G_{1,m}(\mathbb{C}) \) is the Fubini-Study metric on the complex projective space. The action of the unitary group \( U(p + q) \) on \( G_{pq}(\mathbb{C}) \) preserves the metric \( \mathcal{G}_{pq} \) so that \( U(p + q) \) is a group of holomorphic isometries which operates transitively on \( G_{pq}(\mathbb{C}) \).

For \( I \) in \( \mathcal{I} \), let \( U_I \) be the set of the matrices \( P \) in \( M^*(p + q, p) \) such that \( \det(m_I(P)) \) is non-zero. Then \( \pi(U_I) \) is a coordinate open set on \( G_{pq}(\mathbb{C}) \), the matrix \( Z_I \) in \( M_{q,p}(\mathbb{C}) \) is the coordinate, the inverse of the chart \( \varphi_I \) sends \( M^*(p + q, p) \) onto \( \pi(U_I) \), and we have \( m_I(\varphi_I^{-1}(Z_I)) = I(p) \) where \( I(p) \) is the \( p \times p \) identity matrix, and \( m_I(\varphi_I^{-1}(Z_I)) = Z_I \).

**Lemma 2.1.** For \( I \) in \( \mathcal{I} \), let \( \lambda_I \) be the map from \( \pi(U_I) \) to \( \mathbb{R}_+ \) defined by

\[
\lambda_I(Z_I) = |\det(Id + t \bar{Z}_I Z_I)|^{-(p+q)}.
\]

Then \( (\lambda_I)_{I \in \mathcal{I}} \) are the components of a maximal differential form \( \eta \) on \( G_{pq}(\mathbb{C}) \), namely:

\[
\eta = \lambda_I(i/2)^{pq} (dZ \wedge d\bar{Z})_I.
\]

**Proof.** It suffices to show that the following transformation rule holds:

for every \( I, I' \) in \( \mathcal{I} \), \( \lambda_I(\varphi_I^{-1}(Z_I)) = \lambda_{I'}(\varphi_{I'}^{-1}(Z_{I'})) \)

Let \( P_I \) be the matrix \( \varphi_I^{-1}(Z_I) \). Then \( P_I \{m_I(P_I)\}^{-1} = P_{I'} \), so \( Z_I = m_{I'}(P_{I'}) \{m_I(P_I)\}^{-1} \). The differential of the map which sends \( Z_I \) on \( P_I \) is the map which sends \( H \) on \( \tilde{H} \), where \( m_{I'}(\tilde{H}) = H \) and \( m_I(\tilde{H}) = 0 \). The change of charts sending \( Z_I \) on \( Z_{I'} \), we obtain

\[
D Z_{I'}(H) = m_{I'}(\tilde{H}) \{m_I(P_I)\}^{-1} - m_{I'}(P_I) \{m_I(P_I)\}^{-1} m_{I'}(\tilde{H}) \{m_I(P_I)\}^{-1}
\]

\[
= \left(m_{I'}(\tilde{H}) - \gamma m_I(\tilde{H})\right) \alpha^{-1},
\]

where \( \alpha = m_{I'}(P_I) \), \( \beta = m_{I'}(P_I) \) and \( \gamma = \beta \alpha^{-1} \).

Let us define a map \( u \) from \( M_{q,p}(\mathbb{C}) \) to \( M_{q,p}(\mathbb{C}) \) by \( u(H) = m_{I'}(\tilde{H}) - \gamma m_I(\tilde{H}) \). We can choose \( I = \{q + 1, \ldots, q + p\} \) and \( \tilde{I} = \{1, \ldots, r\} \cup \{q + 1 + r, \ldots, q + p\} \), where \( 0 \leq r \leq \inf(p,q) \). We define the \( k \times l \) matrix \( E_{k \times l}^{(k \times l)} \) by \( (E_{k \times l}^{(k \times l)})_{\lambda \mu} = \delta_{\lambda i} \delta_{j \mu} \). We have

\[
m_{I'}(E_{i \times j}^{(p \times q)}) = E_{i \times j}^{(p \times q)} \quad \text{if } i \leq r, \quad \text{and } 0 \quad \text{if } i > r,
\]

and \( m_{I'}(E_{i \times j}^{(q \times p)}) = E_{i \times j}^{(q \times p)} \quad \text{if } i > r, \quad \text{and } 0 \quad \text{if } i \leq r \).

Hence

\[
\gamma m_{I'}(E_{i \times j}^{(p \times q)})_{\alpha \beta} = \gamma_{\alpha i} m_{I'}(E_{i \times j}^{(p \times q)})_{ij} \delta_{j \beta} = \gamma_{\alpha i} \delta_{j \beta} \quad \text{if } i \leq r, \quad \text{and } 0 \quad \text{elsewhere}.
\]
Now the map which sends $H$ to $\gamma_{m_{\tilde{I}}(\tilde{H})}$ can be restricted if $1 \leq j \leq p$ to the span $B_j$ of the $(E_{i,j})_{1 \leq i \leq q}$. The $r$ first columns of its matrix are those of $\gamma$, the others are 0. The map which sends $H$ to $\gamma_{m_{\tilde{I}}(\tilde{H})}$ maps also $B_j$ into itself. The right upper block of its matrix is $I^{(q-r)}$, the other elements are 0. This allows us to compute the matrix of the restriction of $u$ to $B_j$, whose determinant is $(-1)^{r \times (q-r)} \det(\gamma_{\tilde{I}})_{q-r+1 \leq i \leq q}$. So $\det u = (-1)^{p \times r \times (q-r)} \left[ \det(\gamma_{\tilde{I}})_{q-r+1 \leq i \leq q} \right]^p$. For $1 \leq i \leq q$, let $C_i$ be the span of the $(E_{i,j})_{1 \leq j \leq p}$. Each $C_i$ is stable by the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends $H$ to $H_{\alpha}^{-1}$. The matrix of the restriction is $\alpha^{-1}$, so the determinant of the map is $(\det \alpha)^{-q}$. Hence

$$\left| \det DZ_I(H) \right|^2 = \left| \det (\gamma_{\tilde{I}})_{q-r+1 \leq i \leq q} \right|^{2p} \times \left| \det \alpha \right|^{-2q}.$$

Let $A$ be the right $r \times r$ upper block of $\alpha$. The left $(p-r) \times (p-r)$ lower block of $\alpha$ is $I^{(p-r)}$ and the right $(p-r) \times r$ lower block is 0, so $\det \alpha = (-1)^{r(p-r)} \det A$. The left $r \times (p-r)$ lower block of $\beta$ is 0, the right $r \times r$ block is $I^{(r)}$ so that the left $r \times r$ lower block of $\gamma$ is $A^{-1}$.

From this we deduce $\left| \det DZ_I(H) \right|^2 = \left| \det \alpha \right|^{-2(p+q)}$. Since $P_I \alpha^{-1} = P_I$, we have

$$\lambda_I = \left| \det (P_I \tilde{P}_I) \right|^{-(p+q)} = \left| \det \alpha \right|^{2(p+q)} \lambda_I = \left| \det \frac{\partial Z_I}{\partial Z_I} \right|^{-2} \lambda_I.$$

\[
\square
\]

**Lemma 2.2.** The unitary group $U(p+q)$ preserves $\eta$.

**Proof.** We call $I$ the set $\{q+1, \ldots, q+p\}$. We define $P_I$ in $\pi(U_I)$ by $P_I = \varphi_I^{-1}(Z_I)$. Let $U$ be an element in $U(p+q)$ such that $m_I(U P_I)$ is invertible. Let $\tilde{P}_I = UP_I \left\{ m_I(U P_I) \right\}^{-1}$ and $\tilde{Z}_I = m_I(\tilde{P}_I)$. We have $\tilde{Z}_I = m_I(U) P_I \left\{ m_I(U) P_I \right\}^{-1}$. So

$$D\tilde{Z}_I(H) = m_I(U) \left[ \tilde{H} \left\{ m_I(U) P_I \right\}^{-1} - P_I \left\{ m_I(U) P_I \right\}^{-1} m_I(U) \tilde{H} \left\{ m_I(U) P_I \right\}^{-1} \right].$$

Thus $D\tilde{Z}_I(H) = X \tilde{H} \delta^{-1}$, where $\delta = m_I(U) P_I$ and $X = m_I(U) \left[ I^{(p+q)} - P_I \delta^{-1} m_I(U) \right]$. Let $X_1$ be the $q \times q$ matrix of the $q$ first columns of $X$. Then, $X \tilde{H} = X_1 H$ and we get $D\tilde{Z}_I(H) = X_1 H \delta^{-1}$. The determinant of the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends $H$ to $H \delta^{-1}$ is $(\det \delta)^{-q}$. The determinant of the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends $H$ to $X_1 H$ is $(\det X_1)^q$, so $\det D\tilde{Z}_I = (\det X_1)^p (\det \delta)^{-q}$. We divide $U$ into four blocks:

$$U = \begin{pmatrix} U_q & U_{q,p} \\ U_{p,q} & U_p \end{pmatrix}, \quad U_q \in M_q(\mathbb{C}), \ U_p \in M_p(\mathbb{C}), \ U_{p,q} \in M_{p,q}(\mathbb{C}), \ U_{q,p} \in M_{q,p}(\mathbb{C}).$$
Then $\delta = U_{p,q} Z_I + U_p$, so $X_1 = U_q - \left( U_q Z_I + U_{q,p} \right) \left( U_{p,q} Z_I + U_p \right)^{-1} U_{q,p}$. Let $Z$ in $M_{p+q,p+q}(\mathbb{C})$ be the matrix with blocks $Z_q = I^{(q)}$, $Z_{p,q} = 0$, $Z_{q,p} = Z_I$, $Z_p = I^{(p)}$, the notations being the same as above. Writing $\det U = \det(UZ)$ and using the column transformation $C_1 \leftarrow C_1 - C_2 \left( U_{p,q} Z_I + U_p \right)^{-1} U_{q,p}$ where $C_1$ is made of the first $q$ columns and $C_2$ of the remaining ones, we get

$$
\det U = \det \left[ U_q - \left( U_q Z_I + U_{q,p} \right) \left( U_{p,q} Z_I + U_p \right)^{-1} U_{q,p} \right] \times \det \left( U_{p,q} Z_I + U_p \right).
$$

Hence $|\det D\tilde{Z}_I|^2 = |\det \delta|^{-2(p+q)}$. We have $\tilde{P}_I = AP_I \delta^{-1}$, so

$$
\lambda_I = \det (i\tilde{P}_I \tilde{P}_I)^{-2} = \det (iP_I \tilde{P}_I)^{-2} \times |\det \delta|^{2(p+q)} = \lambda_I |\det D\tilde{Z}_I|^{-2},
$$

which proves the result. \qed

**Proposition 2.3.**

1. $dV \left( \mathcal{G}_{p,q} \right) = \eta$.

2. If $I \in \mathcal{I}$, $\left| \mathcal{G}_{p,q} \right|_I = \left\{ \det (I^{(p)} + iZ_I Z_I) \right\}^{-(p+q)}$.

3. $\mathcal{R} \left( \mathcal{G}_{p,q} \right) = (p + q) \mathcal{G}_{p,q}$. 

**Proof.**

1. Let $I \in \mathcal{I}$. It is easy to compute $\mathcal{G}_{p,q}$ at the point $Z_I = 0$: $\mathcal{G}_{p,q}(H,K) = \text{Tr}(H \overline{K})$. Then $dV(\mathcal{G}_{p,q})|_{Z_I=0} = \left( i/2 \right)^{pq} (dZ \wedge d\overline{Z})_I = \eta|_{Z_I=0}$. Since $dV(\mathcal{G}_{p,q})$ and $\eta$ are invariant by the transitive action of $U(p+q)$, we have $dV(\mathcal{G}_{p,q}) = \eta$.

2. Since $dV(\mathcal{G}_{p,q}) = \left| \mathcal{G}_{p,q} \right|_I \left( i/2 \right)^{pq} (dZ \wedge d\overline{Z})_I$, property 1 gives the result.

3. Remark that $\mathcal{G}_{p,q} = i \partial \overline{\partial} \log \left\{ \det (I^{(p)} + iZ_I \overline{Z}_I) \right\}$. Since $\mathcal{R}(\mathcal{G}_{p,q}) = -i \partial \overline{\partial} \log \left| \mathcal{G}_{p,q} \right|_I$, we obtain $\mathcal{R}(\mathcal{G}_{p,q}) = (p + q) \mathcal{G}_{p,q}$, which expresses that $\mathcal{G}_{p,q}$ is Einstein, with factor $p + q$. \qed

### 3. Some general results about Tian’s invariant

#### 3.1. Tian’s invariant with a normalization on a finite set.

If $X$ is a manifold, we will denote by $\mu_X$ a measure on $X$ compatible with the manifold structure.

**Theorem 3.1.** Let $M$ be a compact Kähler manifold. We suppose that there exists a compact Lie group $G$ of holomorphic isometries. Let $\Delta_n = \{ P_1, \ldots, P_n \}$ be a finite subset of $M$. Let $\alpha(\omega)$ (resp. $\alpha_{\Delta_n}(\omega)$) be the supremum of the set of the nonnegative real numbers $\alpha$ satisfying the condition: there exists a constant $C$ such that the inequality $\int_M e^{-\alpha} \varphi \leq C$ holds for all the $\omega$-admissible functions $\varphi$ with $\sup \varphi \geq 0$ (resp. with $\varphi(P_i) \geq 0$ for $1 \leq i \leq n$). Suppose in addition that the orbit of each $P_i$ under the action of $G$ has positive measure. Then $\alpha(\omega) = \alpha_{\Delta_n}(\omega)$. 


We first establish a few lemmas which will be useful for the proof.

**Lemma 3.2.** Let \((\varphi_n)_{n \geq 0}\) be a sequence of admissible functions with nonnegative maxima. Then there exists a subset \(\Omega\) of \(M\), with \(\mu_M(\Omega) = \mu_M(M)\), and a subsequence \(\varphi_{n_k}\) of \(\varphi_n\), such that for every \(p\) in \(\Omega\), the sequence \((\varphi_{n_k}(p))_{k \geq 0}\) has a finite lower bound (depending on \(p\)).

**Proof.** It is sufficient to assume that \(\varphi_n\) has null maxima. Let \(Q_n\) be a point such that \(\varphi_n(Q_n)\) vanishes. Green’s formula runs as follows:

\[
\varphi_n(Q_n) = \frac{1}{V} \int_M \varphi_n + \int_M G(Q, R) \Delta \varphi_n(R) dV(R),
\]

with \(G(Q, R) \geq 0\) and \(\int_M G(Q, R) dV(R) = C\), where \(C\) is a positive constant (see [Au4]). Since \(\varphi_n\) is admissible, \(\Delta \varphi_n\) is less than \(m\), \(m\) being the dimension of \(M\). Thus

\[
\int_M |\varphi_n| \leq C m V.
\]

Furthermore,

\[
\int_M \Delta \varphi_n = 0,
\]

and

\[
\int_M |\Delta \varphi_n| = 2 \int_{\{\Delta \varphi_n > 0\}} \Delta \varphi_n \leq 2mV.
\]

For every \(Q\) in \(M\), we have \(\nabla \varphi_n(Q) = \int_M \nabla_Q G(Q, R) \Delta \varphi_n(R) dV(R)\), so that

\[
\int_M |\nabla \varphi_n| \leq \int_M \left[ \int_M |\nabla_Q G(Q, R)| dV(Q) \right] |\Delta \varphi_n(R)| dV(R) \leq 2m \tilde{C} V,
\]

since \(\int_M |\nabla_Q G(Q, R)| dV(Q)\) is a continuous, hence a bounded function on \(M\). Thus \((\varphi_n)_{n \geq 0}\) is bounded in the Sobolev space \(H^{1,1}(M)\). By Kondrakov’s theorem, we can extract from \((\varphi_n)_{n \geq 0}\) a subsequence which converges in \(L^1(M)\), and after another extraction we can suppose that this sequence converges almost everywhere to a function \(\varphi\) of \(L^1(M)\). Since \(\varphi\) is finite almost everywhere, we get the result.

**Lemma 3.3.** Let \((\varphi_n)_{n \geq 0}\) be a sequence of admissible functions with nonnegative maxima and suppose that there exists a compact group \(G\) of holomorphic isometries of \(M\) such that the orbit of each \(P_i\) has positive measure. Let \(\Phi : G \to \mathbb{R} \cup \{-\infty\}\) be the map defined by

\[
\Phi(g) = \inf_{\Delta \varphi \geq 0} \inf (\varphi_k \circ g).
\]

Then there exists \(g\) in \(G\) such that \(\Phi(g)\) is finite.

**Proof.** Suppose that \(\Phi \equiv -\infty\). For \(i = 1, \ldots, n\), let \(A_i\) be the set of the \(g\) in \(G\) such that \(\inf_{k \geq 0} (\varphi_k \circ g)(P_i) = -\infty\). The sets \(A_i\) are measurable and \(\bigcup_{i=1}^n A_i = G\), so there exists \(i\) such that \(A_i\) has positive measure. From Lemma (3.2), \(A_i, P_i\) is a subset of \(\Omega^c\). Since \(\Omega\) and \(M\) have the same measure, the measure of \(A_i, P_i\) vanishes. Let \(u_i\) be the map from \(G\) to \(M\) which sends \(g\) to \(g(P_i)\). Then \(u_i\) has constant rank on \(G\). Indeed, \(u_i \circ L(g) = \sigma_g \circ u_i\), where \(L(g)\) is the left translation by \(g\) and \(\sigma_g\) the map from \(M\) to \(M\) which sends \(x\) to \(g.x\). Since \(G, P_i\) has positive measure, \(u_i\) is a submersion on \(G\), so that \(u_i(A_i)\) has positive measure. This is a contradiction since \(u_i(A_i) = A_i, P_i\).
We can now prove Theorem (3.1).

**Proof.** It is clear that \( \alpha(\omega) \leq \alpha_{\Delta_n}(\omega) \). Conversely, let \( \varepsilon > 0 \). There exists a sequence \((\varphi_n)_{n \geq 0}\) of admissible functions with positive maxima such that \( \int_M e^{-(\alpha(\omega)+\varepsilon)\varphi_n} \) goes to infinity as \( k \) goes to infinity. Replacing \( \varphi_n \) by \( \varphi_n - \sup \varphi_n \), we can take \( \sup \varphi_n = 0 \). First we apply Lemma (3.2). For the sake of simplicity, we take \( \varphi_{n_k} = \varphi_k \). From Lemma (3.3), there exists an element \( g \) in \( G \) such that \( \Phi(g) \) is finite; we define \( \Psi_k = \varphi_k \circ g - \Phi(g) \). Since \( g \) is an isometry, \( \Psi_k \) is \( \omega \)-admissible, and from the very definition of \( \Phi \), \( \Psi_k(P_k) \) is nonnegative. Furthermore, \( \int_M e^{-(\alpha(\omega)+\varepsilon)\varphi_k} = e^{(\alpha(\omega)+\varepsilon)\Phi(g)} \int_M e^{-(\alpha(\omega)+\varepsilon)\varphi_k} \). This proves that \( \int_M e^{-(\alpha(\omega)+\varepsilon)\varphi_k} \) goes to infinity as \( k \) goes to infinity. Then, \( \alpha_{\Delta_n}(\omega) \leq \alpha(\omega) + \varepsilon \). This inequality holds for every positive \( \varepsilon \), and so \( \alpha_{\Delta_n}(\omega) \leq \alpha(\omega) \). \( \square \)

3.2. Tian’s invariant on a product. For a Kähler form \( \omega \) on a compact Kähler manifold \( M \), \( \alpha(\omega) \) is defined as in Theorem (3.1).

**Proposition 3.4.** Let \((M_i)_{1 \leq i \leq n}\) be compact Kähler manifolds with metric forms \((\omega_i)_{1 \leq i \leq n}\). We endow the product \( M_1 \times \cdots \times M_n \) with the metric \( \omega_1 + \cdots + \omega_n \). Then \( \alpha(\omega_1 + \cdots + \omega_n) = \inf_{1 \leq i \leq n} \alpha(\omega_i) \).

**Proof.** It suffices to make the proof when \( n = 2 \), the general result will follow by induction.

1. Suppose that \( \alpha(\omega_1) \leq \alpha(\omega_2) \), and let \( \varepsilon > 0 \). There exists a sequence \((\varphi_n)_{n \geq 0}\) of \( \omega_1 \)-admissible functions on \( M_1 \) with positive maxima such that \( \int_{M_1} e^{-(\alpha(\omega_1)+\varepsilon)\varphi_n} \) goes to infinity when \( n \) goes to infinity. We define \( \psi_n \) on \( M_1 \times M_2 \) by \( \psi_n(m_1, m_2) = \varphi_n(m_1) \). Thus \( \psi_n \) is \((\omega_1 + \omega_2)\)-admissible on \( M_1 \times M_2 \), with positive maximum, and \( \int_{M_1 \times M_2} e^{-(\alpha(\omega_1)+\varepsilon)\psi_n} = V(M_2) \int_{M_1} e^{-(\alpha(\omega_1)+\varepsilon)\varphi_n} \), so that \( \int_{M_1 \times M_2} e^{-(\alpha(\omega_1)+\varepsilon)\psi_n} \) goes to infinity when \( n \) goes to infinity. We have therefore \( \alpha(\omega_1 + \omega_2) \leq \alpha(\omega_1) + \varepsilon \). This yields \( \alpha(\omega_1 + \omega_2) \leq \alpha(\omega_1) \).

2. Let us now prove the opposite inequality. Let \( \alpha \) be a real number such that \( \alpha < \inf(\alpha(\omega_1), \alpha(\omega_2)) \) and \( \varphi \) an \((\omega_1 + \omega_2)\)-admissible function on \( M_1 \times M_2 \). If \( m_2 \) is in \( M_2 \), the function which sends \( m_1 \) to \( \varphi(m_1, m_2) \) is \( \omega_1 \)-admissible. The same holds for \( M_1 \). Let \((u, v)\) in \( M_1 \times M_2 \) be such that \( \varphi(u, v) \geq 0 \). Then
\[
\int_{M_1 \times M_2} e^{-\alpha \varphi(m_1, m_2)} dV_1 dV_2 = \int_{M_1} e^{-\alpha \varphi(m_1, v)} \left( \int_{M_2} e^{-\alpha \varphi(m_1, m_2) - \varphi(m_1, v)} dV_2 \right) dV_1 \leq C_2 \int_{M_1} e^{-\alpha \varphi(m_1, v)} dV_1 \leq C_1 C_2.
\]
Thus, $\alpha \leq \alpha(\omega_1 \oplus \omega_2)$ and we get $\inf \{\alpha(\omega_1), \alpha(\omega_2)\} \leq \alpha(\omega_1 \oplus \omega_2)$. 

3.3. Tian’s invariant on $G_{p,q}(\mathbb{C})$. Since there is a natural duality isomorphism between $G_{p,q}(\mathbb{C})$ and $G_{q,p}(\mathbb{C})$, we can assume that $p \leq q$ without loss of generality.

3.3.1. Imbedding of $\{\mathbb{P}^1(\mathbb{C})\}^p$ into $G_{p,q}(\mathbb{C})$ when $p \leq q$. For $w \in \mathbb{C}^{p(q-1)}$, $w = \{w_{i,j}\}_{\begin{subarray}{c} 1 \leq i \leq q \\ 1 \leq j \leq p \\ i \neq j \end{subarray}}$, we define the map $\tilde{\rho}_w$ from $\{\mathbb{C}^2 \setminus (0,0)\}^p$ to $M_{p+q,p}(\mathbb{C})$ by

$$
\tilde{\rho}_w((\lambda_i, \mu_i)_{1 \leq i \leq p}) = \begin{cases} 
\lambda_i \delta_{ij} & \text{if } i \leq p \\
w_{i-p,j} \lambda_j & \text{if } i > p \text{ and } i \neq j + p \\
\mu_i & \text{if } i > p \text{ and } i = j + p
\end{cases}
$$

We make, for $p+1 \leq i \leq p+q$, the following row transformations: $L_i \leftarrow L_i - \sum_{\begin{subarray}{c} 1 \leq i \leq q \\ 1 \leq j \leq p \\ i \neq j \end{subarray}} w_{i-p,j} L_j$.

We get a matrix $(c_{ij})_{\begin{subarray}{c} 1 \leq i \leq p+q \\ 1 \leq j \leq p \end{subarray}}$ with $c_{ij} = \delta_{ij} \lambda_i$ if $1 \leq i \leq p$ and $c_{ij} = \delta_{i-p,j} \mu_j$ if $p+1 \leq i \leq p+q$, which has rank $p$. $\tilde{\rho}_w$ induces a map from $\{\mathbb{P}^1(\mathbb{C})\}^p$ into $G_{p,q}(\mathbb{C})$ as shown on the following diagram, where $\gamma$ is the projection of the principal fiber bundle $\{\mathbb{C}^2 \setminus (0,0)\}^p$ onto $\{\mathbb{P}^1(\mathbb{C})\}^p$. Remark that $\tilde{\rho}_w$ sends $[0,1] \times \cdots \times [0,1]$ onto $\pi(A)$, where $m_{(p+1,\ldots,2p)}(A) = I^{(p)}$ and $m_{(p+1,\ldots,2p)}(A) = 0^{(q \times p)}$.

$$
\begin{align*}
\{\mathbb{C}^2 \setminus (0,0)\}^p & \xrightarrow{\tilde{\rho}_w} M^*(p+q,p) \\
\gamma & \downarrow \\
\{\mathbb{P}^1(\mathbb{C})\}^p & \xrightarrow{\rho_w} G_{p,q}(\mathbb{C})
\end{align*}
$$

We have

$$
(\pi \circ \tilde{\rho}_w)^*(\mathcal{O}_{p,q}) = i \partial \overline{\partial} \log \left(\det(\tilde{\rho}_w \tilde{\rho}_w)\right)
= i \partial \overline{\partial} \log \left(\prod_{k=1}^{p} (|\lambda_k|^2 + |\mu_k|^2)\right)
= i \partial \overline{\partial} \log \Phi + \gamma^*(FS_1 \oplus \cdots \oplus FS_1),
$$

where $FS_1$ is the Fubini-Study metric on $\mathbb{P}^1(\mathbb{C})$. $\Phi$ is invariant by the action of the structural group $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$, so it induces a map $\Phi$ from $\{\mathbb{P}^1(\mathbb{C})\}^p$ into $\mathbb{C}$. Note that $\Phi([0,1] \times \cdots \times [0,1]) = 1$. Then $(\pi \circ \tilde{\rho}_w)^*(\mathcal{O}_{p,q}) = \pi^*(i \partial \overline{\partial} \log \Phi + FS_1 \oplus \cdots \oplus FS_1)$, so that $\rho_w^*(\mathcal{O}_{p,q}) = i \partial \overline{\partial} \log \Phi + FS_1 \oplus \cdots \oplus FS_1$. 


3.3.2. Lower bound of $\alpha(C_{p,q})$. For $I$ in $\mathcal{I}$, we define $P_I$ by $m_I(P_I) = I^{(p)}$ and $m_I^{-1}(P_I) = 0^{(q \times p)}$. If $n = \binom{p+q}{p}$, we set $\Delta_n = \{P_I\}_{I \in \mathcal{I}}$. Since $U(p+q)$ is a transitive group of holomorphic isometries of $G_{p,q}(\mathbb{C})$, we know from proposition (3.1), that $\alpha(C_{p,q}) = \alpha_\Delta(C_{p,q})$. We set $I = \{p+1, \ldots, 2p\}$. Let $\varphi$ be an admissible function on $G_{p,q}(\mathbb{C})$, nonnegative on $\Delta_n$. The last equality of the preceding section shows that the function $\varphi \circ \rho_w + \log \Phi$ is $(FS_1 \oplus \cdots \oplus FS_1)$-admissible for every $w$ in $\mathbb{C}^{p(q-1)}$. Furthermore, $(\varphi \circ \rho_w + \log \Phi)$ sends $[0,1] \times \cdots \times [0,1]$ to the nonnegative number $\varphi(P_I)$. It is known that $\alpha(FS_1) = 1$ (see [Au3]). Proposition (3.4) yields $\alpha(FS_1 \oplus \cdots \oplus FS_1) = 1$.

Let $\alpha$ be a real number such that $\alpha < 1$. There exists a constant $C$, independent of $\varphi$, such that

$$\int_{\{P^1(\mathbb{C})\}^p} e^{-\alpha \varphi \rho_w} \Phi^{-\alpha} \leq C.$$ We define the map $F_I$ from $\pi(U_I)$ to $\mathbb{R}_+$ by $F_I(Z_I) = \det(Id + Z_I Z_I^t)$. On $\{P^1(\mathbb{C})\}^p$, we work with the coordinates $\mu_1, \ldots, \mu_p$ in the chart $\lambda_1 = \cdots = \lambda_p = 1$. Thus

$$\Phi(\mu) = \frac{F_I \circ \rho_w(\mu)}{\prod_{k=1}^p (1 + |\mu_k|^2)}, \text{ so that } \int_{\mu \in \mathbb{C}^p} e^{-\alpha \rho_w(\mu)} \frac{dV_p(\mathbb{C}^p)}{\prod_{k=1}^p (1 + |\mu_k|^2)^{2-\alpha} (F_I \circ \rho_w(\mu))^\alpha} \leq C.$$ We have the inequality $\sum_{i=1}^q \sum_{j=1}^p |Z_{ij}|^2 \leq F_I(P_I)$. In particular, for every $k$ in $\{1, \ldots, p\}$,

$$1 + |\mu_k|^2 \leq F_I \circ \rho_w(\mu), \text{ and } f_I \circ \rho_w(\mu) \geq 1 + \sum_{1 \leq i \leq q \atop 1 \leq j \leq p \atop i \neq j} |w_{ij}|^2.$$ Thus, for $\kappa > 0$ and $w \in \mathbb{C}^{p(q-1)},$

$$\prod_{k=1}^p (1 + |\mu_k|^2)^{2-\alpha} \leq \frac{1}{(F_I \circ \rho_w(\mu))^{\kappa + p + q - \alpha}} \leq \left(1 + \sum_{1 \leq i \leq q \atop 1 \leq j \leq p} |w_{ij}|^2 \right)^\alpha = \left(1 + \|w\|^2 \right)^\kappa.$$
We have, according to Proposition (2.3),
\[
\int_{\pi(U_I)} e^{-\alpha \varphi} F_I^{\kappa} \leq \int_{w \in \mathbb{C}^{p(q-1)}} \int_{\mu \in \mathbb{C}^p} \frac{e^{-\alpha \varphi \rho_w(\mu)}}{\prod_{k=1}^p (1 + |\mu_k|^2)^{2-\alpha} (F_I \circ \rho_w(\mu))^\alpha} dV_\mu(\mathbb{C}^p) dV_w(\mathbb{C}^{p(q-1)})
\]

\[
= \int_{w \in \mathbb{C}^{p(q-1)}} \int_{\mu \in \mathbb{C}^p} \left( \frac{e^{-\alpha \varphi \rho_w(\mu)}}{\prod_{k=1}^p (1 + |\mu_k|^2)^{2-\alpha} (F_I \circ \rho_w(\mu))^\alpha} \right) dV_\mu(\mathbb{C}^p) dV_w(\mathbb{C}^{p(q-1)})
\]

\[
= \int_{w \in \mathbb{C}^{p(q-1)}} \left( \int_{\mu \in \mathbb{C}^p} \frac{e^{-\alpha \varphi \rho_w(\mu)}}{\prod_{k=1}^p (1 + |\mu_k|^2)^{2-\alpha} (F_I \circ \rho_w(\mu))^\alpha} \right) dV_w(\mathbb{C}^{p(q-1)}) \leq C \int_{w \in \mathbb{C}^{p(q-1)}} \frac{dV_w(\mathbb{C}^{p(q-1)})}{(1 + ||w||^2)^\kappa}
\]

Thus, we obtain that for all $I$ in $\mathcal{I}$, $\int_{\pi(U_I)} \frac{e^{-\alpha \varphi}}{F_I^{\kappa}} \leq C$, where $C$ is independent of $\varphi$.

Since $G_{p,q}(\mathbb{C})$ is compact, there exists a family $(V_I)_{I \in \mathcal{I}}$ of open sets of $G_{p,q}(\mathbb{C})$ such that $V_I$ is relatively compact in $\pi(U_I)$ for every $I \in \mathcal{I}$, and $\bigcup_{I \in \mathcal{I}} V_I = G_{p,q}(\mathbb{C})$. There exists $M > 0$ such that $F_I \leq M$ on $V_I$ for every $I \in \mathcal{I}$.

Thus
\[
\int_{G_{p,q}(\mathbb{C})} e^{-\alpha \varphi} \leq \sum_{I \in \mathcal{I}} \int_{V_I} e^{-\alpha \varphi} \leq \sum_{I \in \mathcal{I}} M^\kappa \int_{V_I} \frac{e^{-\alpha \varphi}}{F_I^{\kappa}} \leq M^\kappa \sum_{I \in \mathcal{I}} \frac{1}{\pi(U_I)} \frac{e^{-\alpha \varphi}}{F_I^{\kappa}} \leq C M^\kappa \left( \frac{p + q}{p} \right).
\]

We deduce that $\alpha(\mathcal{G}_{p,q}) \geq 1$.

3.3.3. Upper bound of $\alpha(\mathcal{G}_{p,q})$. We use here a method which can be found in [Re] for the complex projective space. Let $I$ in $\mathcal{I}$. We define $\tilde{K}$ from $M^*(p + q, p)$ to $\mathbb{P}^1(\mathbb{R})$ by the relation $\tilde{K}(M) = [\det m_1(H), \det M^*]$. $\tilde{K}$ is invariant by the action of the structural group $G_{p,q}(\mathbb{C})$, so it induces a $C^\infty$ map $K$ from $G_{p,q}(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{R})$. Remark that $\psi = \log K$ is a Kähler potential on $U_I$ for the metric $\mathcal{G}_{p,q}$.

Lemma 3.5. There exists a decreasing sequence $(\varphi_n)_{n \geq 0}$ of admissible functions with positive maxima which converges pointwise to $-\psi$ on $\pi(U_I)$.

Proof. We construct a decreasing sequence $(f_n)_{n \geq 0}$ of $C^\infty$ convex functions on $\mathbb{R}_+$ satisfying the conditions $1 + f'_n > 0$, $f_n(x) = -(1 - 1/n)x$ for $x$ in $[0, n]$ and $f_n(x) = -n$ for
\( x \geq 2n \). Let \( y \) be an element of \( \pi(U_1)^c \) and \( \Omega_n \) the set of the elements \( x \) in \( \pi(U_1) \) such that \( \psi(x) > 2n \). Since \( F_I(y) = [0, 1] \), there exists a neighborhood \( V \) of \( y \) such that the inequality \( z > e^{2n} \) holds for every point \([1, z]\) in \( F_I(V) \). Thus \( V \cap \pi(U_1) \) is included in \( \Omega_n \).

We have proved that \( W_n = \Omega_n \cup \pi(U_1)^c \), so that \( W_n \) is an open neighborhood of \( \pi(U_1)^c \).

We define \( \varphi_n \) by \( \varphi_n = f_n \circ \psi \) on \( \pi(U_1) \) and \( \varphi_n = -n \) on \( W_n \). Thus \( \varphi_n \) is well defined and \( \varphi_n(0) = 0 \). It remains to show that \( \varphi_n \) is admissible on \( \pi(U_1) \).

We have

\[
\left( G_{p,q} + i \partial \overline{\partial} \varphi_n \right)_{\lambda \mu} = \partial_{\lambda \mu} \psi + \partial_{\lambda} \left( f'_n \circ \psi \right) \partial_{\mu} \psi = \left( 1 + f'_n \circ \psi \right) \partial_{\lambda \mu} \psi + f''_n \circ \psi \partial_{\lambda} \psi \partial_{\mu} \psi.
\]

Hence the matrix of the metric \( G_{p,q} + i \partial \overline{\partial} \varphi_n \) is of the form \( A + T \) where \( A \) is positive definite and \( T \) has rank one and positive trace. So \( A + T \) is positive definite and we get the result.

\[\square\]

Lemma 3.6. Let \( n \) in \( \mathbb{N}^* \) and \( r \) a positive real number. Then

\[ \int_{||X|| \leq r} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2} = +\infty. \]

Proof. We can write

\[ \int_{||X|| \leq r} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2} = \sum_{k=0}^{\infty} \int_{r^{2k+1} \leq ||X|| \leq r^{2k}} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2}. \]

We put \( Y = 2^k X \), so

\[ \int_{r^{2k+1} \leq ||X|| \leq r^{2k}} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2} = \int_{1/2 \leq ||Y|| \leq 1} \frac{dV_Y(M_n(\mathbb{C}))}{|\det Y|^2}. \]

The terms in the series are strictly positive and independent of \( k \). The sum is therefore infinite. \[\square\]

We can now prove that \( \alpha(G_{p,q}) \) is upper bounded by 1. Suppose that \( \alpha(G_{p,q}) > 1 \). Then there exists a positive \( C \) such that for every integer \( n \), \( \int_{\pi(U_1)} e^{-\varphi_n} \leq C \). Using Lemma (3.5) and monotous convergence, \( \int_{\pi(U_1)} F_I \leq C \). Since \( \pi(U_1)^c \) has zero measure, \( \int_{G_{p,q}(\mathbb{C})} F_I \leq C \). Let \( \hat{I} \) in \( \mathcal{I} \) be such that \( I \cap \hat{I} = \emptyset \) (this is possible since \( p \leq q \)). We have \( P_I \{ m_I \{ P_I \}^{-1} = P_I \). Remark that \( m_I \{ P_I \} = m_I \{ Z_I \} \). Thus \( \det (Id + t Z_I \overline{Z_I}) = \det \{ P_I \overline{P_I} \} |\det m_I \{ Z_I \}|^{-2}. \)

For \( ||Z_I|| \leq r \), \( \det \{ P_I \overline{P_I} \} \leq M \), so that

\[ \int_{||Z_I|| \leq r} \frac{dV_{Z_I}(M_{q,p}(\mathbb{C}))}{|\det m_I \{ Z_I \}|^2} < +\infty. \]

Integrating over the remaining variables \( (Z_{ij})_{i \leq P_I \cap \hat{I}} \), yields
\[
\int_{\|Z\| \leq r} \frac{dV_Z(M_p(C))}{|\det Z|^2} < +\infty,
\]
which is in contradiction with the result of Lemma (3.6).

Thus we obtain \( \alpha(G_{p,q}) \leq 1 \).

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E-mail address: julien.grivaux@free.fr

Université Pierre et Marie Curie