Localization Regions of Local Observables

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Localization Regions of Local Observables

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Abstract
Exploiting the properties of the Jost-Lehmann-Dyson representation, it is shown that a nonempty smallest localization region can be associated with every local observable in the sense of Araki, Haag, and Kastler that is not a multiple of the identity. But given such a localization prescription, the question whether local observables with spacelike separated localization regions commute turns out to be nontrivial despite the locality of the net. Necessary and sufficient conditions for this version of locality to hold are given.

1 Introduction
The algebraic approach to relativistic quantum physics founded by Araki, Haag, and Kastler aims at joining the structures familiar from nonrelativistic quantum mechanics to those of special relativity. The input of the theory is a net $\mathcal{A}$ of local observables which associates with every bounded open region $\mathcal{O}$ in the Minkowski spacetime $\mathbb{R}^{1+3}$ a unital C*-algebra $\mathcal{A}(\mathcal{O})$ of bounded operators in a Hilbert space $\mathcal{H}$ in such a way that set-theoretic inclusions are preserved, i.e., such that $\mathcal{O} \subset \mathcal{P} \subset \mathbb{R}^{1+3}$ implies $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{P})$ (isotony), and such that the elements of algebras associated with spacelike separated regions commute (locality). The elements of $\mathcal{A}(\mathcal{O})$ are interpreted as the observables measurable within the region $\mathcal{O}$, and it is one aim of algebraic quantum field theory to recover the field-theoretic structures familiar from the Wightman setting and the Standard Model – including (unobservable) fields with Fermi- and other statistics – from this input of locally measurable quantities [15].

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The main question to be investigated below is whether and how one can associate a smallest localization region with a given *single* local observable. Such localization regions have been of interest for the analysis of the Unruh effect and a closely related special form of PCT-symmetry [21, 22, 29]. There has been wide interest in these phenomena and their consequences during the last decades, see [10, 22] and references given there for details.

This article is structured as follows: Section 2.1 discusses the notation, concepts, and basic assumptions that play a role in this paper. The significance of the main assumptions is discussed briefly, and some elementary concepts and facts concerning the causal structure of Minkowski space will be recalled. Section 2.2 comments on the Reeh-Schlieder property, and Section 2.3 briefly discusses duality, another notion that will be used later on. Section 2.4 collects and completes the tools that will be used in what follows. Parts of the material presented in the introductory sections may be well-known to experts, but have been included nevertheless for the reader’s convenience. Section 3 contains the main results of the paper. In Section 3.1 a well-known theorem due to Landau is discussed and generalized: it states that the algebras of any two double cones with disjoint closures only have in common the complex multiples of the identity operator. Using the techniques introduced in the first sections, it is shown that this result can be generalized to the *empty-intersection theorem*: instead of considering two double cones, one can consider one double cone and any finite number of wedge regions; if the closures of the regions under consideration have an empty common intersection, then the corresponding algebras have in common exactly the c-numbers.

In Section 3.2, finally, it is discussed how one can use the empty-intersection theorem in order to associate a nonempty *localization region* with a given *single* local observable that is not a multiple of the identity. But even though locality of the net is assumed from the outset, the question whether local observables with spacelike separated localization regions commute turns out to be nontrivial. A necessary and sufficient criterion for the locality of such a localization prescription is provided by the *nonempty-intersection theorem*: the criterion requires that, given any finite family of wedges, all local observables contained in the algebras associated with all these wedges are contained in the algebra associated with any neighbourhood of the intersection of the wedges as well. It is shown that an additional additivity assumption, which is typically fulfilled by nets arising from Wightman fields, implies this criterion, so it should be quite difficult to find nonpathological counterexamples. On the other hand it is illustrated why the nonempty-intersection criterion, though looking quite natural, is far from self-evident. It does not
follow from the locality property of the net.

In the Conclusion, some related results are discussed briefly.

2 Preliminaries

2.1 Notation and assumptions

In what follows, $\mathcal{H}$ will be a (not necessarily separable) Hilbert space, and $\mathcal{A}$ will be a net of observables as defined above, i.e., a net which satisfies locality and isotomy. The union of all algebras $\mathcal{A}(\mathcal{O})$ associated with bounded open regions $\mathcal{O} \subset \mathbb{R}^{1+s}$ is an involutive algebra $\mathcal{A}_{\text{loc}}$ called the algebra of local observables. The following (standard) assumptions on $\mathcal{A}$ will be made:

(A) **Translation covariance.** $\mathcal{A}$ will be assumed to be covariant under a strongly continuous unitary representation $U$ of the group $(\mathbb{R}^{1+s}, +)$ of spacetime translations, i.e.,

$$U(a)\mathcal{A}(\mathcal{O})U(-a) = \mathcal{A}(\mathcal{O} + a)$$

for every bounded region $\mathcal{O}$ and every $a \in \mathbb{R}^{1+s}$.

(B) **Spectrum condition.** The spectrum of the four-momentum operator generating $U$ is contained in the closure of the forward light cone.

(C) **Existence and uniqueness of the vacuum:** The space of $U$-invariant vectors in $\mathcal{H}$ is one-dimensional. $\Omega$ will denote an arbitrary, but fixed unit vector $\Omega$ in this space, the vacuum vector. $\Omega$ is cyclic with respect to the algebra $\mathcal{A}_{\text{loc}}$, i.e., $\mathcal{A}_{\text{loc}}\Omega = \mathcal{H}$.

Throughout Section 3 an additional assumption will be made:

(D) **Reeh-Schlieder property:** For every nonempty bounded open region $\mathcal{O}$, the space $\mathcal{A}(\mathcal{O})\Omega$ is dense in $\mathcal{H}$.

Assumptions (A) and (B) make sure that the system described by the net has a well-defined four-momentum whose spectrum ensures energetic stability of the system.

Given Assumptions (A) and (B), Assumption (C) holds if and only if $\Omega$ induces a unique and pure vacuum state, which, in turn, holds if and only

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1 In this article we refer to arbitrary subsets of $\mathbb{R}^{1+s}$ as 'regions'.

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if the algebra $A_{\text{loc}}$ is irreducible. A sufficient condition for this uniqueness is that the bicommutant $A_{\text{loc}}''$ of $A_{\text{loc}}$ is a factor (Thm. III.3.2.6 in [15]). But as soon as $\mathcal{H}$ is separable, this implies that the uniqueness part of Assumption (C) does not mean any loss of generality: every von Neumann algebra in a separable Hilbert space admits a direct-integral decomposition into factors, and since the unitaries representing the translations commute with the elements of the center of $A_{\text{loc}}''$, one can conclude that almost all factors of the central decomposition inherit Properties (A) and (B) (cf. also the remarks in [15], Sect. III.3.2, and references therein).

Independent from the separability of $\mathcal{H}$, Assumption (C) is fulfilled as soon as all charges are fixed in the vacuum sector (cf. [15] for details).

The Reeh-Schlieder property, Assumption (D), holds for all Wightman fields [25]. Some more remarks concerning this technical, but standard assumption follow in Section 2.2.

In order to define a net $\mathcal{A}$, it is sufficient to associate algebras with the elements of some topological base and to define for any open set $M \subset \mathbb{R}^{1+s}$ the $C^*$-algebra $\mathcal{A}(M)$ to be the algebra generated by the algebras associated with elements of the base. A convenient base of $\mathbb{R}^{1+s}$ is the class $\mathcal{K}$ of all double cones, i.e., all regions of the form $(a+V_+) \cap (b-V_+)$, $a, b \in \mathbb{R}^{1+s}$. If one is given some general net $\mathcal{A}$ not necessarily defined this way, one can compare this to the net $\mathcal{A}^\mathcal{K}$ generated by the algebras associated with double cones. Due to isotony, the algebras $A_{\text{loc}}$ and $A_{\text{loc}}^\mathcal{K}$ of local observables associated with, respectively, the nets $\mathcal{A}$ and $\mathcal{A}^\mathcal{K}$ coincide, so for an analysis of special representations of these algebras there may be no loss of generality if one just considers $\mathcal{A}^\mathcal{K}$. This is the case in the analysis of superselection sectors in algebraic quantum field theory (cf. [15] and references given there), and this is why it has become common to assume that the local algebras are generated by the algebras associated with double cones. But since it is the localization behaviour which will be of interest in what follows, it will not be assumed that local algebras are generated by the ‘double cone algebras’ contained in them. On the other hand, the localization prescriptions discussed below are so coarse that they coincide for $\mathcal{A}$ and $\mathcal{A}^\mathcal{K}$. So in any analysis which does not rely on any finer localization properties of the observables than the ones discussed below, it can be assumed without loss of generality that the net be generated by its double cone algebras. But this is a result of what follows rather than an a priori assumption.

The class $\mathcal{W}$ of wedges consists of the region $W_1 := \{x \in \mathbb{R}^{1+s} : x_1 > |x_0|\}$ and its images under Poincaré transforms. If $M$ is a region in $\mathbb{R}^{1+s}$, one denotes by $M^\mathcal{C}$ the causal complement or spacelike complement, which is the region consisting of all points that are spacelike with respect to all points of
The spacelike complement of the spacelike complement \((M^\circ)^\circ = M^{cc}\) \(\subset M\) is called the causal completion of \(M\), and \(M\) is called causally complete if \(M = M^{cc}\). It is convenient to denote the interior of \(M^\circ\) by \(M^\circ_0\).

\(\mathcal{K}\) and \(\mathcal{W}\) are subclasses of the class \(\mathcal{C}\) of convex, causally complete and open proper subsets of \(\mathbb{R}^{1+2}\). The wedges in \(\mathcal{W}\) are maximal elements of \(\mathcal{C}\) in the sense that for every wedge \(W \in \mathcal{W}\), every element \(R \in \mathcal{C}\) with \(R \supset W\) is a wedge. Every element \(R\) of \(\mathcal{C}\) is an intersection of wedges (cf. [28], Thm. 3.2)\(^2\). The class of all wedges which contain a region \(R\) will be denoted by \(\mathcal{W}_R\).

In general, the causal complement of a region in \(\mathcal{C}\) is not convex. \(\mathcal{C}'\) will denote the class of open regions which are interiors of causal complements of regions in \(\mathcal{C}\). Every region \(R\) in \(\mathcal{C}'\) is a union of wedges ([28], Thm. 3.2); \(\mathcal{W}^R\) will denote the class of all wedges that are subsets of \(R\). Note that \(\mathcal{W} = \mathcal{C} \cap \mathcal{C}'\). If \(O\) is an open convex region and if \(P\) is a convex region that is spacelike separated from \(O\), there is a wedge \(X \in \mathcal{W}\) such that \(O \subset X\) and \(P \subset W^c\) (cf. [28], Prop. 3.1).

\(\mathcal{B}\) will denote the bounded elements of the class \(\mathcal{C}\). Clearly, the double cones are in \(\mathcal{B}\). Every element of \(\mathcal{B}\) is contained in some double cone, and it is precisely the intersection of all such double cones ([28], Prop. 3.8). The class of all double cones which contain a region \(O\) will be denoted by \(\mathcal{K}_O\), and the class of all double cones contained in an arbitrary region \(R\) will be called \(\mathcal{K}^R\).

Occasionally, terminology borrowed from PDEs and General Relativity will be used (timelike curves, Cauchy surfaces, etc.). These notions will not be defined in detail, but will be used as in [17].

### 2.2 Some remarks on the Reeh-Schlieder property

The Reeh-Schlieder property requires that the vacuum vector is cyclic with respect to all algebras \(A(O)\) associated with nonempty bounded open regions \(O\). If, instead, one looks at algebras \(A(R)\) associated with regions containing an open cone, cyclicity of the vacuum with respect to these algebras is a well-known consequence of Assumptions (A) through (C) (Actually, the uniqueness assumption in (C) is not even needed). We include a proof for the convenience of the reader.

\(^2\)For the proof of this statement it is essential that \(\mathcal{C}\) consists of open regions (the statement also implies to regions with a nonempty interior). As a counterexample, consider the lightlike half plane \(R := \{x \in \mathbb{R}^{1+2} : x_1 = x_0, x_2 > 0, \ldots, x_n > 0\}\). One checks that this region is causally complete and convex, while it is not an intersection of wedges.
2.2.1 Lemma

Let $A$ be a local net of local observables satisfying Conditions (A) through (C) above. If $R$ contains some nonempty open cone, then $\Omega$ is cyclic with respect to $A(R)$, i.e., $A(R)\Omega = \mathcal{H}$.

Proof. Let $\psi \in \mathcal{H}$ be orthogonal to $A(R)\Omega$, choose any $O \in \mathcal{K}$ and any $A \in A(O)$. Since $R$ contains a nonempty open cone, there is an $a \in \mathbb{R}^{1+\epsilon}$ such that $\overline{O} + a \subset R$, and the function $x \mapsto f(x) := \langle \psi, U(x)A\Omega \rangle$ vanishes in an open neighbourhood of $a$.

The spectrum condition implies that $f$ has a continuous extension to the closure of the forward tube $\mathbb{R}^{1+\epsilon} + i\overline{\nabla}_+$, that is analytic in the interior of this tube. But since $f$ vanishes in a neighbourhood of $a$, i.e., in an open subset of the boundary of the forward tube, it follows from the edge-of-the-wedge theorem that $f \equiv 0$.

Since this holds for all $A \in A_{bc}$, it follows that $\psi$ is orthogonal to $A_{bc}\Omega = \mathcal{H}$ by the cyclicity of $\Omega$, i.e., $\psi = 0$. This proves the lemma.

\[ \square \]

In particular, $\Omega$ is cyclic with respect to $A(O')$ for every bounded region $O$, which immediately implies that it is separating with respect to $A(O)$, i.e., $A\Omega = 0$ implies $A = 0$ for every $A \in A(O)$.

The Reeh-Schlieder property is a strengthening of what the above lemma derives from Assumptions (A) through (C). Given these assumptions, it can be derived from weak additivity ([7], cf. also Thm. 7.3.37 in [2]); one can also prove it by just mimicking the proof of the preceding lemma: if $O$ is any double cone, then

\[
\left( \bigcup_{a \in \mathbb{R}^{1+\epsilon}} A(O' + a) \right)'' = A_{bc}''.
\]

That, conversely, weak additivity can be derived from the Reeh-Schlieder property in the present setting, is not really new (cf., e.g., Lemma 2.6 in [29]), but for the reader’s convenience we include the following lemma.

2.2.2 Lemma

Let $A$ be a local net of local observables satisfying Conditions (A) through (D) above, let $O \subset \mathbb{R}^{1+\epsilon}$ be a bounded open region, and
let $a \in \mathbb{R}^{1+2}$ be some timelike vector. Then
\[
C_{\mathcal{O},a} := \left( \bigcup_{t \in \mathbb{R}} \mathcal{A}(\mathcal{O} + ta) \right)' = \mathcal{B}(\mathcal{H}).
\]

**Proof.** For any $a$ and $\mathcal{O}$ as above, let $A$ be any local observable commuting with all elements of $C_{\mathcal{O},a}$, and pick a $B \in \mathcal{A}(\mathcal{O})$. Define $f_+(t) := \langle \Omega, A^* U(ta) B \Omega \rangle$ and $f_-(t) := \langle \Omega, B U(-ta) A^* \Omega \rangle$. By the spectral theorem and the spectrum condition, the Fourier transforms of these functions are (not necessarily positive, but bounded) measures one of which has its support in the closed positive half axis, while the other one has its support in the closed negative half axis. Since $f_+$ and $f_-$ coincide by construction, it follows that the Fourier transform of $f_+$ (and of $f_-)$ is a measure with support $\{0\}$, i.e., some multiple of the Dirac measure, so that $f_+$ is a constant function. Using this, the spectral theorem, and uniqueness of the vacuum, one concludes
\[
\langle A \Omega, B \Omega \rangle = f_+(0) = f_+(t) = \langle \Omega, A \Omega \rangle \langle \Omega, B \Omega \rangle =: \pi \langle \Omega, B \Omega \rangle = \langle \alpha \Omega, B \Omega \rangle.
\]
Since by the Reeh-Schlieder property, $B \Omega$ runs through a dense set, one concludes $A \Omega = \alpha \Omega$, and since $A$ is a local observable, one obtains $A = \alpha \text{id}$ since $\Omega$ is separating with respect to $\mathcal{A}(\mathcal{O})$. This proves the lemma. \hfill \Box

### 2.3 Duality, PCT-symmetry, and Borchers classes

Given the net of observables $\mathcal{A}$, the **dual net** $\mathcal{A}^d$ is defined by $(\mathcal{A}^d(\mathcal{O})) := \mathcal{A}(\mathcal{O})'$. By locality of $\mathcal{A}$, one has $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}^d(\mathcal{O})$ for each $\mathcal{O} \in \mathcal{B}$. But in general, the dual net itself does not need to satisfy locality.

If it does satisfy locality, it coincides with its own dual net. It then follows that every other local net $(\mathcal{B}(\mathcal{O}))_{\mathcal{O} \in \mathcal{B}}$ of observables which is local with respect to $\mathcal{A}^d$ is a subnet of $\mathcal{A}^d$. Given locality of $\mathcal{A}^d$, any two subnets of $\mathcal{A}^d$ are local with respect to one another.

For the Wightman framework it has been shown by Borchers that mutually local fields have the same PCT-operator and the same scattering matrix [6]. Borchers also showed that mutual locality between irreducible local Wightman fields with PCT-symmetry is not only a reflexive and symmetric, but even a transitive relation, so that these fields form equivalence classes, called **Borchers classes**. In the algebraic setting the property corresponding to this behaviour is locality of the dual net; for a local net with
this property, the dual net contains all nets which are local with respect to $\mathcal{A}$, i.e., all nets in the ‘Borchers class’ of $\mathcal{A}$.

A local net of observables is said to satisfy **essential duality** if its dual net satisfies locality; it is said to satisfy **Haag duality** if $\mathcal{A} = \mathcal{A}^d$ in the sense that $\mathcal{A}(\mathcal{O})^d = \mathcal{A}(\mathcal{O}^d)$ for all $\mathcal{O} \in \mathcal{B}$. Clearly, if a net satisfies essential duality, its dual net satisfies Haag duality. In general, the distinction between a net of observables and its dual net is more than a technicality: Doplicher and Roberts have shown that representation associated with localized charges give rise to a net of field operators in an enlarged Hilbert space. The field system possesses a compact group of global gauge symmetries, and the observables are precisely the field operators which are invariant under the adjoint action of this group, while the vacuum Hilbert space consists of the vectors in the field Hilbert space which are invariant under these global gauge symmetries. But it can happen that the dual net contains field operators which break this symmetry; in this case, Haag duality has to be violated as a consequence. But essential duality can hold nevertheless, as examples show [26, 11].

$\mathcal{A}$ is said to satisfy **wedge duality** if the isotonous family $(\mathcal{A}(W'))'_{W \in W}$ satisfies locality, which is equivalent to $\mathcal{A}(W')'' = \mathcal{A}(W')'$. It follows from the Bisognano-Wichmann results quoted before [3, 4] that all nets arising from finite-component Wightman fields satisfy wedge duality.

One checks that wedge duality implies essential duality, since for any two spacelike separated regions in $\mathcal{B}$, one can find a wedge which contains one of the two, whereas its spacelike complement contains the other one (cf. Sect. 2.1 and Lemma 3.2.2 below).

### 2.4 Commutator functions and wave equation techniques

It is a classical result of the Wightman approach to quantum field theory that one can reconstruct a Wightman field from its vacuum expectation values [27, 18]. The following lemma shows how one can reconstruct commutation relations of a net of observables from the behaviour of its vacuum expectation values. Since these have some convenient properties, this will facilitate the subsequent investigations. $\mathcal{A}$ will be a local net of local observables satisfying the above Assumptions (A) through (C).

#### 2.4.1 Lemma

*For an arbitrary double cone $\mathcal{O} \in \mathcal{K}$, let $A$ be an element of $\mathcal{A}(\mathcal{O}^\prime)^\prime$.***
(i) If a region $R \subset \mathbb{R}^{1+s}$ contains some open cone and has the property that $\langle \Omega, AB\Omega \rangle = \langle \Omega, BA\Omega \rangle$ for all $B \in \mathcal{A}(R)$, then $A \in \mathcal{A}(R)'$.

(ii) Assume that $\mathcal{A}$ has the Reeh-Schlieder property, and suppose there is a double cone $P \in \mathcal{K}$ with the property that $\langle \Omega, AB\Omega \rangle = \langle \Omega, BA\Omega \rangle$ for all $B \in \mathcal{A}(P)$. If there is a double cone $Q \subset P$ with the property that $A \in \mathcal{A}(Q)'$, then $A \in \mathcal{A}(P)'$.

(iii) Assume that $\mathcal{A}$ exhibits the Reeh-Schlieder property, and suppose there is a double cone $P \in \mathcal{K}$ with the property that $\langle \Omega, AB\Omega \rangle = \langle \Omega, BA\Omega \rangle$ for all $B \in \mathcal{A}(P+a), a \in \mathbb{R}^{1+s}$. Then $A$ is a multiple of the identity.

Proof. (i): If $S$ is an open cone contained in $R$, there is a translation $a \in \mathbb{R}^{1+s}$ such that $S + a \subset R \cap \mathcal{O}'$. Choose $C$ and $D$ in $\mathcal{A}(S + a)$ and $B$ in $\mathcal{A}(R)$. Since $A \in \mathcal{A}(\mathcal{O}')'$, the operators $A$ and $C^*$ commute:

$$\langle C\Omega, ABD\Omega \rangle = \langle \Omega, C^*ABD\Omega \rangle = \langle \Omega, AC^*BD\Omega \rangle.$$ 

Since $C^*BD$ is in $\mathcal{A}(R)$, the assumption implies

$$\langle \Omega, AC^*BD\Omega \rangle = \langle \Omega, C^*BDA\Omega \rangle,$$

and since $D$ and $A$, in turn, commute because of $A \in \mathcal{A}(\mathcal{O}')'$, one concludes

$$\langle C\Omega, ABD\Omega \rangle = \langle \Omega, C^*BDA\Omega \rangle = \langle \Omega, BAD\Omega \rangle.$$ 

But since $C$ and $D$ are arbitrary elements of $\mathcal{A}(S+a)$, and since $\Omega$ is cyclic with respect to this algebra, it follows that $AB = BA$; since $B \in \mathcal{A}(R)$ was arbitrary, one obtains $A \in \mathcal{A}(R)'$, which is (i).

(ii) Choose $C$ and $D$ in $\mathcal{A}(Q)$ and $B$ in $\mathcal{A}(P)$. Since $A$ has been assumed to be in $\mathcal{A}(Q)'$, it commutes with $C^*$, so

$$\langle C\Omega, ABD\Omega \rangle = \langle \Omega, C^*ABD\Omega \rangle = \langle \Omega, AC^*BD\Omega \rangle.$$ 

Since $C^*BD$ is in $\mathcal{A}(P)$, the assumption implies

$$\langle \Omega, AC^*BD\Omega \rangle = \langle \Omega, C^*BDA\Omega \rangle,$$

and since $D$ and $A$ commute by the assumption that $A \in \mathcal{A}(Q)'$, one concludes

$$\langle C\Omega, ABD\Omega \rangle = \langle \Omega, C^*BDA\Omega \rangle = \langle C\Omega, BAD\Omega \rangle.$$
But since $C$ and $D$ are arbitrary elements of $\mathcal{A}(Q)$, and since by the Reeh-Schlieder property, $\Omega$ is cyclic with respect to this algebra, it follows that $AB = BA$; since $B \in \mathcal{A}(P)$ was arbitrary, one obtains $A \in \mathcal{A}(P)'$, which is (ii).

(iii) There is a translation $a \in \mathbb{R}^{1+3}$ such that $P + a \subset O'$, so that $A \in \mathcal{A}(O')' \subset \mathcal{A}(P + a)'$. Now choose a $b \in \mathbb{R}^{1+3}$ such that $P + b$ intersects $P + a$, and let $Q$ be a double cone contained in $(P + b) \cap (P + a)$. Isotony implies that $A \in \mathcal{A}(Q)'$. Since by assumption, $\langle \Omega, AB\Omega \rangle = \langle \Omega, BA\Omega \rangle$ for all $B \in \mathcal{A}(P + b)$, (ii) implies that $A \in \mathcal{A}(P + b)'$. Now one can iterate this procedure: choose an arbitrary $c \in \mathbb{R}^{1+3}$ such that $(P + c) \cap (P + b)$ is nonempty, choose a new double cone $Q$ in this intersection, and conclude from (ii) that $A \in \mathcal{A}(P + c)'$. Note that only the double cone $P + a$ chosen in the first step needs to be spacelike separated from $O$, since each step uses the result of the preceding one, so one finds that for every $a \in \mathbb{R}^{1+3}$, one proves that $A \in \mathcal{A}(P + a)'$ with a finite number of steps. The statement now follows from weak additivity, which follows from the Reeh-Schlieder property (see above), and from irreducibility.

\[\square\]

Given any two local observables $A, B \in \mathcal{A}_{\text{loc}}$, the commutator function $f_{A,B}$ will henceforth be defined by

\[\mathbb{R}^{1+3} \ni x \mapsto \langle \Omega, [A, U(x)BU(-x)]\Omega \rangle =: f_{A,B}(x).\]

Due to Lemma 2.4.1, the analysis of the support of this function yields information on the structure of the net. Crucial for this analysis is the fact that $f_{A,B}$ is a boundary value of a solution of the wave equation, and a well-known lemma due to Asgeirsson concerning such solutions (cf., e.g., [5], Sect. 4.4.D (p. 183 ff), or [13]) immediately implies the following lemma, which, for this reason, will be referred to as Asgeirsson’s Lemma. Another important consequence of the ‘wave nature’ of the function $f_{A,B}$ is a theorem due to Jost, Lehmann and Dyson [19, 14], which will also be recalled for the reader’s convenience.

2.4.2 Lemma (Asgeirsson)

If the commutator function $f_{A,B}$ and all its partial derivatives are zero along a timelike curve segment $\gamma$, $f_{A,B}$ vanishes in the entire double cone $\gamma^{\pm\varepsilon}$. 

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Proof. The Fourier transform of the operator valued function \( R^{1+s} \ni x \mapsto U(x) \) is the spectral measure of the four-momentum operator. It follows that the Fourier transform \( \hat{f}_{A,B} \) of the function \( f_{A,B} \) is a finite (not necessarily positive) measure, and by the spectrum condition, one has \( \text{supp} \hat{f}_{A,B} \subset \nabla \). It follows that the function

\[
F(x, \sigma) := (2\pi)^{-\frac{1+s}{2}} \int \cos(\sigma \sqrt{k^2}) e^{ikx} d\hat{f}_{A,B}(k)
\]

is a continuous function with \( F(x, 0) = f_{A,B}(x) \) for all \( x \in R^{1+s} \). This \( F \) is a solution of the \( 1+(s+1) \)-dimensional wave equation. This implies the statement by Asgeirsson’s result for solutions of the wave equation, see the references quoted above.

\[\square\]

Evidently, the assumption of the lemma is satisfied as soon as \( f_{A,B} \) vanishes in some open neighbourhood of \( \gamma \). In the proof of Theorem 2.4.5 below, however, the function \( F \) defined in the proof is analysed, and the information one has about \( f_{A,B} \) from locality merely implies that \( F \) vanishes in a null set of \( R^{1+(s+1)} \). In this case one makes use of the fact that \( F \) has been constructed in such a way that all its partial derivatives, including the one in the \( \sigma \)-direction, are zero at all points of this null set; one may then use the above lemma to show that the region where \( F \) vanishes also extends into the \( \sigma \)-direction.

2.4.3 Definition

Let \( R \) be a region in Minkowski space.

(i) \( R \) will be called **Asgeirsson complete** if for every timelike curve segment \( \gamma \subset R \), the double cone \( \gamma^\pm \) is a subset of \( R \) as well. The smallest Asgeirsson complete extension of \( R \) will be called the **Asgeirsson hull** of \( R \).

(ii) \( R \) will be called **timelike convex** if it contains as subsets all double cones with tips in \( R \), i.e., if \( (R + V_+) \cap (R - V_+) \subset R \).

(iii) \( R \) will be called a **Jost-Lehmann-Dyson region** if it is timelike convex and if every inextendible timelike curve in \( R^{1+s} \) intersects \( R \cup R^c \).

Timelike convex regions contain all timelike path segments connecting two points in the region, so the terminology is in harmony with other notions
of convexity. In [21] the term ‘double cone complete’ was used instead of ‘timelike convex’, but the latter term was also used in [28] (Par. IV) and will be used in what follows to facilitate reading. The following lemma collects some relations between these notions most of which will be used below.

2.4.4 Lemma

(i) Every causally complete region is timelike convex.

(ii) Every timelike convex region is Asgeirsson complete.

(iii) Every timelike convex and bounded open region is a Jost-Lehmann-Dyson region.

(iv) The causal complement of a Jost-Lehmann-Dyson region is a Jost-Lehmann-Dyson region.

(v) Let \( R \) and \( S \) be timelike convex regions, and assume that there exists a Cauchy surface \( T \) which is a subset of both \( R \) and \( S \). Then the region \( R \cup S \) is timelike convex (and, like \( R \) and \( S \), trivially, a Jost-Lehmann-Dyson region).

(vi) Let \((R_\rho)_\rho \geq 0\) be an increasing family of Jost-Lehmann-Dyson regions. Then \( R := \bigcup_\rho R_\rho \) is a Jost-Lehmann-Dyson region.

Before proving the lemma, we give some counterexamples to strengthened statements or converse implications. An example of a timelike convex region (and Jost-Lehmann-Dyson region) that is not causally complete (cf. (i)) is the time slice region \( \{ x \in \mathbb{R}^{1+1} : 0 \leq x_0 \leq 1 \} \). An example of an Asgeirsson complete region that is not timelike convex (cf. (ii)) is the union of two disjoint double cones at a timelike distance; this shows that the classes of timelike convex regions and of Jost-Lehmann-Dyson regions, respectively, are not stable under taking unions, so Statement (v) is far from tautological. The same holds for the class of Asgeirsson complete regions: consider the regions \( R_+ := \{ x \in \mathbb{R}^{1+1} : \rho x_1 < x_0 < \rho x_1 + 1 \} \) and \( R_- := \{ x \in \mathbb{R}^{1+1} - \rho x_1 < x_0 < 1 - \rho x_1 \} \) for some \( \rho \) with \( 0 < \rho \leq 1 \). One easily checks that both regions are Asgeirsson complete, while their union is not: its Asgeirsson hull is \( \mathbb{R}^{1+1} \). If \( \rho < 1 \), the two regions are even Jost-Lehmann-Dyson regions, while their union evidently is not (cf. (v) and (vi)).

An example of a timelike convex region which is neither causally complete nor a Jost-Lehmann-Dyson region (cf. (iii)) is the region

\[
R := \{ x \in \mathbb{R}^{1+1} : 1 < x^2 < 2, x_0 > 0 \},
\]
since there are timelike curves which do not intersect $R$, e.g., the curve $R \ni t \mapsto (\sinh t, \cosh t, 0, \ldots, 0)$.

**Proof of Lemma 2.4.4.** (i) Let $R$ be a causally complete region, and pick two points $x \in R$ and $y \in R \cap (x + V_+)$. The causal completion $(x, y)^\circ$ of the set $(x, y)$ is the closure of the double cone $(x + V_+) \cap (y - V_+)$, and since $(x, y) \subset R$ implies $(x, y)^\circ \subset R^\circ = R$, this immediately implies (i).

Statement (ii) immediately follows from the definition.

(iii) Let $R$ be timelike convex, bounded and open. Since $R$ is open, a point $x \in \mathbb{R}^{1+3}$ is not contained in the spacelike complement $R^c$ if and only if it is timelike with respect to some point of $R$, i.e., $\mathbb{R}^{1+3} \setminus R^c = R + V$, where $V$ is the open light cone. Now let $\gamma$ be an inextendible timelike curve that does not intersect $R \cap R^c$. Since $\gamma$ does not intersect $R^c$, it has to stay within the region $R + V$. But since $\gamma$ is an inextendible timelike curve, while $R$ is bounded, $\gamma$ cannot stay in the future $R + V_+$ or the past $R - V_+$ of $R$, i.e., it has to pass from $R - V_+$ to $R + V_+$. Since both these regions are open, while $\gamma$ is continuous, it follows that it has to hit the region $(R + V_+) \cap (R - V_+)$. But this region coincides with $R$ since $R$ is timelike convex and open, so $\gamma$ hits $R$, which is a contradiction and proves (iii).

(iv) The causal complement of any region is causally complete, by (i), this enhances timelike convexity of $R^c$. The condition that $R \cup R^c$ is intersected by every inextendible timelike curve implies that $R^c \cup R^\circ$ ($= R^c \cup R$) is intersected by every such curve. This proves (iv).

(v) Let $x$ and $y$ be points in $R \cup S$, and let $\gamma$ be any inextendible timelike curve hitting both $x$ and $y$, and let $z$ be the unique point where $\gamma$ hits $T$. Since $R$ and $S$ are timelike convex, and since $z \in T \subset R \cap S$, the closed double cones with the tips $z$ and $x$ and the tips $z$ and $y$, respectively, are subsets in $R \cup S$. If with respect to the time ordering along $\gamma$, $z$ is earlier or later than both $x$ and $y$, it follows that the double cone with tips $x$ and $y$ is contained in $R \cup S$ as well. If $z$ is between $x$ and $y$, then, as before, we can conclude that the segments of $\gamma$ between $z$ and $x$ and between $z$ and $y$ is a subset of $R \cup S$, and since $z \in T \subset R \cap S$, it follows that all of the segments of $\gamma$ joining $x$ to $y$ is a subset of $R \cup S$. Since $\gamma$ can be any inextendible timelike curve hitting $x$ and $y$, one obtains that all timelike curve segments joining $x$ and $y$ are contained in $R \cup S$, so the double cone with tips $x$ and $y$ is contained in $R \cup S$, which completes the proof of (v).

(vi) Let $x$ and $y$ be two points in $R$ at a timelike distance. There are $\rho_x > 0$ and $\rho_y > 0$ such that $x \in R_{\rho_x}$ and $y \in R_{\rho_y}$, so it follows that both $x$ and $y$ are elements of $R_{\max\{\rho_x, \rho_y\}}$. Since this region is timelike convex, it
follows that the double cone with tips $x$ and $y$ is in $R$, proving that $R$ is timelike convex.

It remains to be shown that every inextendible timelike curve intersects $R \cup R'$. Let $\gamma$ be an inextendible timelike curve that does not intersect $R$. Since all $R_{\rho}$ are Jost-Lehmann-Dyson regions, it follows that $\gamma$ has to intersect every $R'_{\rho}$, so it has to intersect the region $\bigcap_{\rho>0} R'_{\rho} = R'$. This completes the proof.

The above statements and proofs can be extended in a straightforward manner to any globally hyperbolic spacetime. For further results of the above kind, see [28]. The useful property of Jost-Lehmann-Dyson regions (which is the reason to call them so) is established by the following theorem.

**2.4.5 Theorem (Jost, Lehmann, Dyson)**

Let $A$ and $B$ be local observables, and assume that the commutator function $f_{AB}$ vanishes in a Jost-Lehmann-Dyson region $R$. Then the support of $f_{AB}$ is contained not only in the complement of $R$, but even in the (in general, smaller) union of all admissible mass hyperboloids of $R$, i.e., the mass hyperboloids

$$H_{a,\sigma} := \{ x \in \mathbb{R}^{1+s} : (x-a)^2 = \sigma^2 \}, \quad a \in \mathbb{R}^{1+s}, \sigma \in \mathbb{R},$$

which do not intersect $R$.

**Sketch of proof.** Define $F$ as in the proof of Lemma 2.4.2. Since $F$ is a solution of the wave equation, it is well-known that for every Cauchy surface $\zeta$ in $\mathbb{R}^{1+(s+1)}$, there exists a distribution $F_\zeta$ with support in $\zeta$ such that $F = F_\zeta * D_{1+(s+1)}$, where $D_{1+(s+1)}$ denotes a fundamental solution of the $1+(s+1)$-dimensional wave equation (see, e.g., [5], pp. 175-184). The support of $D_{1+(s+1)}$ is contained in the closed light cone $V$ of $\mathbb{R}^{1+(s+1)}$.

Since $R$ is a Jost-Lehmann-Dyson region in $\mathbb{R}^{1+s}$, its $1+(s+1)$-dimensional Asgeirsson hull $\hat{R}$ is easily seen to be a Jost-Lehmann-Dyson region in $\mathbb{R}^{1+(s+1)}$. Provided this region is 'well-behaved', there is a Cauchy surface $\zeta$ in $\hat{R} \cup \hat{R}$. This Cauchy surface has the property that for every point $z \in \zeta$, either both the forward and the backward part of $V + z$ or neither of them intersects $R$.

---

3 This notation is consistent since $\hat{V}$ is, indeed, the $1+(s+1)$-dimensional Asgeirsson hull of $V$. Note that $\hat{\hat{V}} \neq \hat{V}$.
The former case occurs if and only if \( z \in \zeta \cap \hat{R} \). The latter case occurs if and only if \( z \in \zeta \cap \hat{R}^c \), the Asgeirsson hull \( \hat{R} \) of \( R \) and the spacelike complement being taken in the spacetime \( \mathbb{R}^{1+(n+1)} \). But since all partial derivatives of \( F \) can be checked to vanish in all points in \( R \), one obtains from Lemma 2.4.2 that \( F \) vanishes in \( \hat{R} \), the support of \( F \) contains only points of the second kind, i.e., it is contained in \( \hat{R}^c \cap \zeta \). This implies that the support of \( F \) is contained in \((\hat{R} \cap \zeta) + \overline{V}\).

Since \( f_{A,B} \) is a boundary value of \( F \) and since the intersection of \( V + c \) with \( \mathbb{R}^{1+\ell} \) is the convex hull of a shifted mass hyperboloid, the support of the boundary value \( f_{A,B} \) of the function \( F \) is contained in the union of admissible mass hyperboloids, as stated.

We conclude this section with another lemma to be used below that concerns the geometry of Minkowski space.

2.5 Lemma

\[ \text{Let } P \in K \text{ be a double cone.} \]

(i) If \( O \) is a double cone, so is \((O + P)^{cc}\).

(ii) If \( W \) is a wedge, so is \((W + P)^{cc}\).

Proof. Denote by \( a_O \) and \( a_P \) the lower tips, and by \( b_O \) and \( b_P \) the upper tips of \( O \) and \( P \), respectively. Let \( x = a_O + \xi \) and \( y = a_P + \eta \) be points in \( O \) and \( P \), respectively. Then \( x + y = a_O + a_P + \xi + \eta \), and since \( \xi \) and \( \eta \) are elements of \( V_+ \), so is \( \xi + \eta \), so \( x + y \) is contained in \( a_O + a_P + V_+ \). In the same way one proves that \( x + y \in b_O + b_P - V_+ \), so one has

\[ O + P \subset (a_O + a_P + V_+) \cap (b_O + b_P - V_+). \]

Since the right hand side is a double cone and, hence, causally complete, it follows that \((O + P)^{cc}\) is a subset of this double cone as well. On the other hand it is straightforward to check that the tips \( a_O + a_P \) and \( b_O + b_P \) of this double cone and the straight line joining them are contained in \( O + P \), whence the converse inclusion follows as well, so the proof of (i) is complete.

The region \( W + P \) is a union of wedges that are images of \( W \) under translations. Consequently, \((W + P)^c\) is the intersection of the corresponding translates of \( W^c \) under translations. But this intersection is the closure of a wedge, so it follows that the causal complement of this region, \((W + P)^{cc}\), is a wedge. This proves (ii).
3 Results

By definition, a local net associates algebras with regions. In the sequel it will be discussed how to associate a localization region with a given algebra and even with a single local observable. The analysis is based on a theorem due to Landau [23]. In order to localize single observables, a new generalization of Landau’s theorem will be used. It will be stated and proved below.

This section is structured as follows: in Section 3.1, the theorem of Landau and its consequences for the localization of algebras will be discussed, and the mentioned generalization will be proved. This generalization will be the basis for the analysis of localization regions for single local observables, which is presented in Section 3.2.

In what follows, Assumptions (1) through (4) will be made without further mentioning.

3.1 Landau’s theorem and the empty-intersection theorem

Using the wave equation techniques discussed in the preceding section, Landau [23] proved the following:

3.1.1 Theorem (Landau)

If the closures of two double cones $\mathcal{O}$ and $\mathcal{P}$ are disjoint, then

$$\mathcal{A}(\mathcal{O}')' \cap \mathcal{A}(\mathcal{P}')' = \text{Cid}_\mathcal{H}.$$ 

This already implies that for an $\mathcal{O}$ satisfying the assumptions of the corollary, the region

$$L(\mathcal{A}(\mathcal{O}')) := \bigcup \{P \in \mathcal{K} : \mathcal{A}(P) \subset \mathcal{A}(\mathcal{O}')'\},$$

which will be called the localization region of the algebra $\mathcal{A}(\mathcal{O}')'$, coincides with $\mathcal{O}$ (cf. [1]):

3.1.2 Corollary

Let $\mathcal{O} \subset \mathbb{R}^{1+1}$ be a bounded, causally complete and convex open region.

(i) For every open region $M \subset \mathbb{R}^{1+1}$, one has $\mathcal{A}(M) \subset \mathcal{A}(\mathcal{O}')'$ if and only if $M \subset \mathcal{O}$.

(ii) $L(\mathcal{A}(\mathcal{O})) = \mathcal{O}$. 

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Proof. By isotony and locality, the condition in statement (i) is sufficient. To prove that it is necessary, assume \( M \not\subset \mathcal{O} \). Then, since \( \mathcal{K} \) is a topological base and since the region \( M \setminus \overline{\mathcal{O}} \) has a nonempty interior, \( M \setminus \overline{\mathcal{O}} \) contains a double cone \( P \in \mathcal{K} \) whose closure is disjoint from \( \overline{\mathcal{O}} \). Since \( \overline{\mathcal{O}} \) is an intersection of closures of wedges, it follows from this that a wedge \( W \) can be found whose closure is disjoint from \( \overline{P} \) and contains \( \overline{\mathcal{O}} \). Since \( \overline{P} \) is compact, the distance between \( \overline{P} \) and \( \overline{W} \) is \( > 0 \), so eventually shifting it a little bit, one can choose \( W \) in such a way that \( \overline{W} \) is a subset not only of \( \overline{W} \), but also of \( W \) itself.

By Proposition 3.8 (b) in [28], one can now conclude that there is a double cone \( Q \) with \( Q \subset W \) and \( Q \supset \mathcal{O} \) (note that \( \mathcal{O} \) itself does not need to be a double cone). Landau’s theorem now implies that \( \mathcal{A}(P) \cap \mathcal{A}(Q')' = \text{Cid}_H \). It follows from the Reeh-Schlieder property that \( \mathcal{A}(P) \not\subset \text{Cid}_H \), so \( \mathcal{A}(P) \not\subset \mathcal{A}(Q')' \). Since \( \mathcal{A}(P) \subset \mathcal{A}(M) \) follows from isotony, \( \mathcal{A}(M) \) cannot be a subset of \( \mathcal{A}(Q')' \), and since \( \mathcal{O} \subset Q \), it cannot be a subset of \( \mathcal{A}(O')' \). This proves (i) and, trivially, implies (ii).

\[ \square \]

The proof of Corollary 3.1.2 can be made shorter as soon as one knows that Landau’s theorem still works if one of the two double cones is replaced by a wedge. That this, indeed, is possible, has been shown in the context of the proof of the P,CT-part of the first uniqueness theorem for modular symmetries (Theorem 2.1 in [20]).

3.1.3 Theorem

*If the closures of a double cone \( \mathcal{O} \) and a wedge \( W \) are disjoint, then*

\[ \mathcal{A}(O')' \cap \mathcal{A}(W')' = \text{Cid}_H. \]

Using this generalized version of Landau’s theorem, one concludes that in Lemma 3.1.2, the assumption that \( \mathcal{O} \) is bounded may be omitted:

3.1.4 Corollary

*Let \( R \subset \mathbb{R}^{1+} \) be a causally complete convex open region.*

(i) *For every open region \( M \subset \mathbb{R}^{1+} \), one has \( \mathcal{A}(M) \subset \mathcal{A}(R')' \) if and only if \( M \subset R \).*

(ii) *\( L(\mathcal{A}(R)) = R \).*

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Proof. By isotony and locality, the condition is sufficient. To prove that it is necessary, assume $M \not\subset R$. Then, since $\mathcal{K}$ is a topological base and since the region $M \setminus \mathcal{F}$ has a nonempty interior, $M \setminus \mathcal{F}$ contains a double cone $O \in \mathcal{K}$ whose closure is disjoint from $\overline{R}$. As in the proof of Corollary 3.1.2, it follows that a wedge $W$ can be found whose closure is disjoint from $\overline{O}$ and whose interior contains $\overline{R}$. Landau’s theorem now implies that $\mathcal{A}(O) \cap \mathcal{A}(W')' = \text{Cid}_H$. It follows from the Reeh-Schlieder property that $\mathcal{A}(O) \not\subset \text{Cid}_H$, so $\mathcal{A}(O) \not\subset \mathcal{A}(W')'$. Since $\mathcal{A}(O) \subset \mathcal{A}(M)$ follows from isotony, $\mathcal{A}(M)$ cannot be a subset of $\mathcal{A}(W')'$, and since $R \subset W$, it cannot be a subset of $\mathcal{A}(R')'$, proving both statements.

In order to investigate the localization behaviour of a single local observable, a further generalization of Landau’s theorem will be used. It is the main result of this section. It is a generalization of Theorem 2.1 in [20]. $\mathcal{A}^d_{\text{loc}}$ will denote the algebra of local observables of the dual net $\mathcal{A}^d$.

3.1.5 Theorem (empty-intersection theorem)

Let $(W_i)_{1 \leq i \leq n}$ be a family of $n$ wedges in $\mathcal{W}$. If $\bigcap_{i} W_i = \emptyset$, then

$$\mathcal{A}^d_{\text{loc}} \cap \bigcap_{i} \mathcal{A}(W_i)' = \text{Cid}_H.$$ 

Proof. Choose an $A \in \mathcal{A}^d_{\text{loc}} \cap \bigcap_{i} \mathcal{A}(W_i)'$, and let $O$ be a double cone with $A \in \mathcal{A}(O)'$.

Since the wedges $W_i$ have empty common intersection, so do the compact regions $\overline{O} \cap W_i$. But if a finite family of compact regions have empty common intersection, there is an $\varepsilon > 0$ such that the family of $\varepsilon$-neighbourhoods of the regions still have empty common intersection. The proof of this is an elementary induction proof: any two disjoint compact regions have a positive distance, which implies the statement for two regions. Now assume the statement to hold for any family of $n$ compact sets, and let $C_1, \ldots, C_{n+1}$ be a family of $n+1$ regions. If $n$ of these regions already have empty common intersection, there is nothing more to prove. So consider the case that the set $\Gamma := \bigcap_{i=1}^{n+1} C_i$ is nonempty. This region is compact and, as shown, has a finite distance $\delta$ from $C_{n+1}$, so the $\delta/3$-neighbourhoods of the two regions still have a finite distance. But the $\delta/3$-neighbourhood of $\Gamma$ is the intersection of the $\delta/3$-neighbourhoods of $C_1, \ldots, C_n$, so the statement follows for $\varepsilon = \delta/3$. 

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It follows from this that there is a double cone \( P \) which is so small that the wedge \( W'_\nu := (W'_\nu - P)' = (W'_\nu - P)^\infty \), \( \nu \leq n \), and the double cone \( \tilde{O} := (O - P)' = (O - P)^\infty \) (cf. Lemma 2.5 above to see that these regions are a wedge and a double cone, respectively) still have empty common intersection. Choose any \( B \in \mathcal{A}(P) \). By locality, the commutator function \( f_{A,B} \) vanishes in the region \( R := \tilde{O}' \cup \bigcup_\nu W'_\nu \).

There is no admissible mass hyperboloid for this region. To see this, note that if a (shifted) mass hyperboloid is disjoint from a union of wedges, so is the unique shift \( x + V \), \( x \in \mathbb{R}^{1+s} \), of the closure of the full light cone which contains the hyperboloid. Now choose \( x \in \mathbb{R}^{1+s} \) such that \( x + V \) is disjoint from all \( W'_\nu \), \( \nu \leq n \), and from \( \tilde{O}' \), which is a union of wedges, too. This is equivalent to \( \{x\}' \supset \tilde{O}' \cup \bigcup_\nu W'_\nu \), i.e.,

\[
x \in \tilde{O}'' \cap \bigcap_\nu W''_\nu = \tilde{O} \cap \bigcap_\nu W'_\nu = \emptyset.
\]

Hence there is no admissible mass hyperboloid for \( R \).

If \( R \) is a Jost-Lehmann-Dyson region, it follows from Theorem 2.4.5 that \( f_{A,B}(x) \) vanishes for all \( x \in \mathbb{R}^{1+s} \) and all \( B \in \mathcal{A}(P) \), so using part (iii) of Lemma 2.4.1, one concludes that \( A \in \text{Cid}_\mathcal{H} \), and the proof is complete.

But since \( R \) does not need to be a Jost-Lehmann-Dyson region, Asgerson's lemma will be used to show that the function \( f_{A,B} \) vanishes in a larger region \( N \supseteq R \) which is a Jost-Lehmann-Dyson region. Since there is no admissible hyperboloid for \( R \), there is, a fortiori, no admissible hyperboloid for \( N \), so the proof will be complete as soon as \( N \) has been shown to exhibit the stated properties.

To this end, choose coordinates such that \( \tilde{O} \) is the double cone

\[
((-\rho_0, 0, 0, 0) + V_+) \cap ((\rho_0, 0, 0, 0) - V_+),
\]
where \( \rho_0 > 0 \) is the radius of the double cone \( \tilde{O} \). Let \( Z_\rho = \{x = (x_0, \vec{x}) \in \mathbb{R}^{1+s} : ||\vec{x}|| = \rho\} \) be the boundary of the cylinder of radius \( \rho \) around the time axis in \( \mathbb{R}^{1+s} \), and define

\[
\begin{align*}
R_{\rho,0} &:= \tilde{O}' \cap Z_\rho, \\
R_{\rho,\nu} &:= \tilde{W}'_\nu \cap Z_\rho, \quad \nu \leq n;
\end{align*}
\]
All these regions are bounded subsets of \( \mathbb{R}^{1+s} \). Due to our choice of coordinates, the region \( R_{\rho,0} \) is a strip:

\[
R_{\rho,0} = \{x \in Z_\rho : |x_0| \leq \rho - \rho_0\}
\]
(which is empty if \( \rho < \rho_0 \)).

For \( 1 \leq \nu \leq n \), the wedge \( \tilde{W}_\nu \) is timelike convex in \( \mathbb{R}^{1+n} \), so the region \( R_{\rho,\nu} \) is timelike convex with respect to the inherited spacetime structure of \( Z_\rho \). We now show that there is a \( \rho_\nu > 0 \) such that the union \( R_{\rho,0} \cup R_{\rho,\nu} \) is timelike convex as well for all \( \rho > \rho_\nu \). To this end, let \( C \) be a spacelike hypersurface in \( \tilde{W}_\nu \cup \tilde{W}_\nu' \). As a spacelike surface, it is a subset of \( \tilde{O}' \) up to a compact set. For \( \rho \) so large that this compact set is enclosed by \( Z_\rho \) one finds that \( C \cap Z_\rho \) is a subset of \( R_{\rho,0} \). Since \( C \cap Z_\rho \) is a Cauchy surface in the spacetime \( Z_\rho \), it follows that \( R_{\rho,\nu} \cup C \) and \( R_{\rho,0} \) are timelike convex regions in the spacetime \( Z_\rho \) whose intersection contains a Cauchy surface, so part (v) of Lemma 2.4.4 implies that \( R_{\rho,0} \cup R_{\rho,\nu} \) is timelike convex. This proves that \( \rho_\nu \) with the stated properties exists for \( 1 \leq \nu \leq n \).

Now choose \( \rho > \hat{\rho} := \max_{\nu} \rho_\nu \), and apply Lemma 2.4.4 (v) another \( n-1 \) times to conclude that the region

\[
R_\rho := R \cap Z_\rho = \bigcup_{0 \leq \nu \leq n} R_{\rho,\nu}
\]

is timelike convex in \( Z_\rho \). Since the \( \mathbb{R}^{1+n} \)-Asgeirsson hull \( \hat{R}_\rho \) is open, bounded, and timelike convex, it is a Jost-Lehmann-Dyson region by Lemma 2.4.4 (iii).

On the other hand, the part of \( \tilde{O}' \) and \( \tilde{W}_\nu' \), respectively, which is enclosed by \( Z_\rho \) is a subset of the \( \mathbb{R}^{1+n} \)-Asgeirsson hull \( \hat{R}_{\rho,\nu} \) of \( R_{\rho,\nu} \). It follows that

\[
R \subset N := \bigcup_{\rho > \hat{\rho}} \bigcup_{\nu \leq n} \hat{R}_{\rho,\nu} = \bigcup_{\rho > \hat{\rho}} \hat{R}_\rho,
\]

and by Asgeirsson's lemma, \( f_{A,B} \) vanishes in \( N \). Since the Jost-Lehmann-Dyson region \( \hat{R}_\rho \) increases with \( \rho \), it follows from Lemma 2.4.4 (vi) that \( N \) is a Jost-Lehmann-Dyson region. This is what remained to be shown, so the proof is complete.

\[\square\]

Actually, the following, slightly stronger version has been established by the preceding proof:

\[3.1.6 \text{ Corollary}\]

Let \( W_1, \ldots, W_n \) be wedges in \( W \), and let \( O \in \mathcal{K} \) be a double cone. If \( \overline{O} \cap \bigcap_{1 \leq \nu \leq n} \overline{W}_\nu = \emptyset \), then

\[
A(O) \cap \bigcap_{\nu} A(W_\nu')' = \text{Cid}.
\]
3.2 The localization region of a single local observable and the nonempty-intersection theorem

Theorem 3.1.5 prepares for the definition of a localization region for local observables. The existence of a nonempty localization region for every local observable is established by the following proposition.

3.2.1 Proposition

Let \( \mathcal{X} \) be any of the classes \( \mathcal{K}, \mathcal{B}, \mathcal{W}, \) and \( \mathcal{C} \). For every \( A \in \mathcal{A}_{\text{loc}} \), which is not a multiple of the identity, the localization regions

\[
L^X(A) := \bigcap \{ \overline{O} : O \in \mathcal{X} : A \in \mathcal{A}(O)' \}
\]

\[
L_d^X(A) := \bigcap \{ \overline{O} : O \in \mathcal{X} : A \in \mathcal{A}(O)' \}
\]

are nonempty, causally complete, convex, and compact sets. Between them, one has the following equalities and inclusions:

\[
\begin{align*}
L^K(A) &= L^B(A) \cup L_d^K(A) = L_d^K(A) \\
L^C(A) &= L^W(A) \cup L_d^C(A) = L_d^C(A)
\end{align*}
\]

Proof. We start with the proof of the equalities and inclusions. The equalities immediately follow from the definitions, since on the one hand, \( \mathcal{K} \subseteq \mathcal{B} \) and \( \mathcal{W} \subseteq \mathcal{C} \), while on the other hand, every region in \( \mathcal{B} \) is an intersection of double cones in \( \mathcal{K} \) and every region in \( \mathcal{C} \) is an intersection of wedges in \( \mathcal{W} \) (see Section 2.1). The inclusions in the upper and the lower row of the diagram immediately follow from locality. The inclusions in the two columns follow from the fact that every double cone is an intersection of wedges and that, by isotony, an observable contained in the algebra associated with a given double cone is contained in all algebras associated with wedges containing this double cone.

By these inclusions, it is sufficient to prove that \( L_d^W(A) \) is nonempty if \( A \not\in \mathcal{C}_{\text{id}} \). It already follows from Theorem 3.1.5 that the intersection of the closures of any finite family of wedges whose algebras contain \( A \) is nonempty. But the family of all wedges whose algebras contain \( A \) is never finite.

Since \( A \) is a local observable, there is a double cone \( O \) with \( A \in \mathcal{A}(O) \), and it follows from isotony, locality, and the above inclusions that \( L_d^W(A) \subseteq \overline{O} \). But this implies that

\[
L_d^W(A) = \bigcap \{ \overline{O} \cap W : W \in \mathcal{W}, A \in \mathcal{A}(W)' \}.
\]
which is an intersection of subsets of the compact set $\mathcal{O}$. But if for a class of closed subsets of a compact space, every finite subclass has a nonempty intersection, it follows from the Heine-Borel property that the whole class has a nonempty intersection. Now Corollary 3.1.6 implies the statement. □

In the sequel the maps $A_{\text{loc}} \ni A \mapsto L^X(A)$ and $A_{\text{loc}} \ni A \mapsto L^Y_d(A)$ will be referred to as localization prescriptions. Two problems arise if one wants to interpret the above definitions:

**Problem 1:** There are several of them, and others may easily be defined. One may ask whether there is one 'physical' localization prescription or whether several distinct localization prescriptions play different roles.

**Problem 2:** None of the above localization prescriptions is known to satisfy locality in the sense that two local observables commute if their localization regions are spacelike with respect to each other.

The localization prescription $L^W_d$ is the one which — compared with the other prescriptions suggested above — associates the smallest localization region with a local observable. Evidently, this is a first partial answer to Problem 1. But from a physical viewpoint, it is not necessarily the strongest localization prescription which can be regarded as the 'best' one, but one may prefer to look for the strongest localization prescription which satisfies locality (if such a prescription exists). It might happen that the localization prescriptions $L^W$ and $L^K_d$ both satisfy locality, while $L^W_d$ does not\(^4\). Since there is, in general, no inclusion relation between $L^W$ and $L^K_d$, it might occur in this case that there are two distinct ‘strongest’ localization prescriptions satisfying locality.

Clearly, the localization prescriptions $L^K$ and $L^K_d$ coincide if the net satisfies Haag duality, and the prescriptions $L^W$ and $L^W_d$ coincide if the net satisfies wedge duality. Furthermore, wedge duality also makes $L^W_d$ coincide with $L^K_d$ by the following lemma (cf. also [9], Lemma 4.1).

\(^4\)On the other hand it cannot happen that $L^K$ violates locality if the weaker localization prescription

$$A_{\text{loc}} \ni A \mapsto \bigcap \{ \mathcal{O} : \mathcal{O} \in \mathcal{K}, \ A \in \mathcal{A}(\mathcal{O}) \},$$

(where the local $C^\ast$-algebras themselves are tested instead of their weak closures) does satisfy locality. This is why this type of localization prescription is not discussed at this stage.
3.2.2 Lemma

Assume the net $A$ to satisfy wedge duality. For every region $R \in \mathcal{C}$, one has

$$A(R')' = \bigcap_{W \in \mathcal{W}_R} A(W)' =: \mathcal{M}(R),$$

and the net $\mathcal{M}$ satisfies locality.

Proof. We first show that the net $(A(R')')_{R \in \mathcal{C}}$ satisfies locality. This immediately follows from the fact remarked above that if $R$ and $S$ are spacelike separated regions in $\mathcal{C}$, there is a wedge $W \in \mathcal{W}$ with $R \cap W$ and $S \cap W'$. For such a constellation one has

$$A(R')' \subset A(W)' = A(W)' \subset A(S')'',$$

which is the stated locality for the net $(A(R')')_{R \in \mathcal{C}}$.

One proves in the same way that the net $\mathcal{M}$ satisfies locality with respect to $A$, i.e., $\mathcal{M}(R) \subset A(R')'$ for all $R \in \mathcal{C}$. On the other hand,

$$A(R')' \subset \bigcap_{W \in \mathcal{W}_R} A(W)' = \bigcap_{W \in \mathcal{W}_R} A(W)' = \mathcal{M}(R) \quad \text{for all } R \in \mathcal{C},$$

and this completes the proof. \hfill \square

In the following lemma and in the discussion of the second uniqueness theorem for modular symmetries, wedge duality will be assumed. Henceforth, the localization region $L^W(A) = L^W(A) = L^W_\perp(A)$ will simply be denoted by $L(A)$ for each $A \in \mathcal{A}_{\text{loc}}$.

3.2.3 Theorem (nonempty-intersection theorem)

Assume $A$ to satisfy wedge duality.

The localization prescription $\mathcal{A}_{\text{loc}} \ni A \mapsto L(A)$ satisfies locality if and only if for every finite family $W_1, \ldots, W_n$ of wedges and for every causally complete and convex region $R \in \mathcal{C}$ with $\bigcap_{\nu} \overline{W}_\nu \subset R$, one has

$$\mathcal{A}_{\text{loc}} \cap \bigcap_{1 \leq \nu \leq n} A(W'_\nu)'' \subset A(R')'.$$
Proof. To prove that the condition is sufficient, let \( \partial B_\varepsilon(L(A)) \) be the boundary of the open \( \varepsilon \)-neighbourhood \( B_\varepsilon(L(A)) \) of \( L(A) \) for \( \varepsilon > 0 \), and define

\[
W_A := \{ W \in W : \exists X \in W : A \in A(X)'' , \overline{X} \subseteq W \}.
\]

A class of closed subsets of the compact space \( \partial B_\varepsilon(L(A)) \) is defined by

\[
\mathcal{X} := \{ \partial B_\varepsilon(L(A)) \cap \overline{W} : W \in W_A \}.
\]

\( \mathcal{X} \) has empty intersection, and by the Heine-Borel property, there is a finite subclass of \( \mathcal{X} \) whose intersection is still empty, i.e., there are wedges \( W_1, \ldots, W_n \in W_A \) such that

\[
\partial B_\varepsilon(L(A)) \cap \bigcap_{\nu} W_\nu = \emptyset.
\]

Due to the convexity of \( L(A) \) and of wedges it follows that the region

\[
R := \bigcap_{\nu} W_\nu
\]

is a subset of \( B_\varepsilon(L(A)) \), and that \( R \in B \). By the definition of the class \( W_A \), there are wedges \( X_1, \ldots, X_n \in W_A \) such that \( \overline{X_\nu} \subseteq W_\nu \) for \( 1 \leq \nu \leq n \).

Since \( R \in B \subseteq \mathcal{C} \), one now obtains from the condition that

\[
A \in A_{\text{loc}} \cap \bigcap_{\nu} A(X_\nu)'' \subseteq A(R)' \subseteq A(B_\varepsilon(L(A))')',
\]

as stated. This holds for each \( \varepsilon > 0 \), and evidently, the same reasoning proves that \( B \in A(B_\varepsilon(L(B))')(\cdot)' \).

Since \( L(A) \) and \( L(B) \) are compact, convex, and spacelike separated, the euclidean distance between these regions is positive, and one can choose \( \varepsilon > 0 \) so small that \( B_\varepsilon(L(A)) \) and \( B_\varepsilon(L(B)) \) still are spacelike separated. As remarked in Section 2.1, there is a wedge \( X \) such that \( B_\varepsilon(L(A)) \subseteq X \) and \( B_\varepsilon(L(B)) \subseteq X' \). Using wedge duality and Lemma 3.2.2, one concludes

\[
A \in A(B_\varepsilon(L(A))')' \subseteq A(X'') \subseteq A(X)_0 \subseteq A(X)' = A(X)'',
\]

and

\[
B \in A(B_\varepsilon(L(B))'(\cdot'))' \subseteq A(X'(\cdot'))'' = A(X)' \subseteq A(X)'',
\]

so \( AB = BA \), proving that the condition is sufficient.

To prove that the condition is necessary, let \( W_1, \ldots, W_n \) be a family of wedges, and choose an \( R \in \mathcal{C} \) with \( \bigcap_{\nu} W_\nu \subseteq R \). Whenever \( A \in A_{\text{loc}} \cap A_\varepsilon(L(A))'' \),
\[ \bigcap_{\nu} A(W_{\nu})'' \text{ and } B \in A_{\text{loc}} \cap A(X)'' \text{ for any } X \in W'', \text{ locality of } L \text{ implies that } AB = BA, \text{ and one concludes that} \]
\[ A \in \bigcap_{X \in W''} (A_{\text{loc}} \cap A(X)'')' = \bigcap_{X \in W''} A(X)' = \bigcap_{X \in W''} A(X)'' = A(X)'' = A(R)'', \]

where Lemma 3.2.2 has been used in the last step.

\[ \square \]

### 3.2.4 Proposition

Assume \( A \) to satisfy wedge duality, and suppose that the dual net of \( A \) satisfies **strong additivity for wedges**, i.e., for every wedge \( W \in W \) and every double cone \( P \) with \( W \subset W + P \), one has

\[ A(W)'' \subset \left( \bigcup_{a \in W} A(a + P')' \right)'' . \]

Then the localization prescription \( L \) satisfies locality on \( A_{\text{loc}} \). 

**Proof.** Let \( A \) and \( B \) be local observables with spacelike separated localization regions. There is a wedge \( W \) such that \( L(A) \subset W \) and \( L(B) \subset W' \). So as soon as one proves that this implies \( A \in A(W)'' \) and \( B \in A(W')'' \), wedge duality implies the statement.

To this end, we consider any \( A \in A_{\text{loc}} \) and any wedge \( W \) whose closure is spacelike separated from \( L(A) \), and show that \( A \in A(W)' \). This follows from wedge additivity as soon as one has found a double cone \( P \) with the property that \( W \subset W + P \) and that \( f_{A, B} \) vanishes in \( W \) for all \( B \in A(P')' \).

So fix an \( \varepsilon > 0 \) such that the \( \varepsilon \)-neighbourhood \( B_\varepsilon(L(A)) \) of \( L(A) \) is still spacelike separated from \( \overline{W} \). As in the proof of Theorem 3.2.3, we choose a finite number of wedges \( X_1, \ldots, X_n \) in the class \( W_A \) such that

\[ \bigcap_{\nu} X_{\nu} \subset B_\varepsilon(L(A)). \]

Now define

\[ P := ((-\rho, 0, 0) + V_+) \cap ((\rho, 0, 0) - V_+) \]

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for some $\rho > 0$, and note that $W \subset W + P$. Fixing $\rho > 0$ sufficiently small, one can make sure that

$$W \subset \left( \bigcap_{\nu} (X_\nu - P) \right)' .$$

Choosing any $B \in A(P')'$, one obtains from wedge duality that the commutator function $f_{A,B}$ defined above vanishes in the region

$$R := \bigcup_{\nu} (X_\nu - P)' ,$$

which is a union of wedges.

As in the proof of Theorem 3.1.5, $f_{A,B}$ can be shown to vanish in a larger region $N \supset R$ which is a Jost-Lehmann-Dyson region. This can be shown by mimicking the corresponding part of that proof, as it does not depend on the assumption that the intersection of the closed wedges under consideration is empty. So one can keep $A$, $B$, and the double cone $P$, choose some double cone $O$ with $A \in A(O)$, replace $X_1, \ldots, X_n$ by $W_1, \ldots, W_n$, and proceed like above to construct $N$.

A mass hyperboloid $H$ is admissible with respect to $N$ only if it is admissible with respect to $R$, and as $R$ is a union of closed wedges, this is the case only if the whole unique shift of the open light cone which contains $H$ is disjoint from $R$. But by Theorem 2.4.5, this implies that in particular, $f_{A,B}$ vanishes in the region $W$, completing the proof.

$$\square$$

Strong additivity for wedges is a technical property shared by all theories which arise from Wightman fields. The assumption of strong additivity for wedges has been used extensively by Thomas and Wichmann in [29]. The authors have obtained results in the spirit of Proposition 3.2.4 and Theorem 3.1.5, but their results do not imply ours.

To illustrate a situation where the condition of Theorem 3.2.3 is violated, consider the wedge $X := W_1 + (0, 1, 0, 0)$ and its images $Y$ and $Z$ under rotations around the 3-axis by $120^\circ$ and $240^\circ$, respectively. Assume a local observable $A$ to be contained in $A(X)'$ and in $A(Y)'$, while another local observable $B$ is contained in $A(Y)''$ and $A(Z)''$. In this case, the localization regions $L(A)$ and $L(B)$ are spacelike with respect to one another, but there is no reason why $A$ and $B$ should commute, since not any two of the three wedges are spacelike separated.
To avoid these problems, one could just assume that \( A(O) \cap A(P) = A(O \cap P) \) for all \( O, P \in \mathcal{C} \). Clearly, this would imply the condition in Theorem 3.2.3. It is common in algebraic quantum field theory to consider the algebra \( A(O) \) as the algebra of all observables that can be measured in \( O \). If an observable can be measured in \( O \) and in \( P \), one could ask whether it should be measurable in \( O \cap P \) as well. But there is no reason why this should be the case. Considering generalized free fields, Landau has given examples of local nets which do not satisfy Haag duality, while they do satisfy essential duality [24] (and they satisfy wedge duality as well as weak additivity). The examples yield local nets that satisfy the assumptions of Theorem 3.2.3, while, in general, \( A(O) \cap A(P) \neq A(O \cap P) \) for \( O, P \in \mathcal{C} \).

To illustrate the geometrical trick of Landau’s example, start from some local net \( B \) of observables in \( 1+(s+1) \) dimensions, and with every double cone \( O = (a + V_+) \cap (b + V_-) \) in \( \mathbb{R}^{1+1} \), associate the algebra

\[
B_0(O) := B((a + \hat{V}_+) \cap (b + \hat{V}_-)) \equiv B(\hat{O}),
\]

where, as before, \( \hat{V}_+ \) and \( \hat{V}_- \) denote the \( 1+(s+1) \)-dimensional forward and backward light cone, respectively.

One easily checks that \( B_0(O) \cap B_0(P) \) might not coincide with \( B_0(O \cap P) \), since the intersection of the \( 1+(s+1) \)-dimensional Asgeirsson hulls of \( O \) and \( P \) differs from the \( 1+(s+1) \)-dimensional Asgeirsson hull of the intersection \( O \cap P \), i.e., \( \hat{O} \cap \hat{P} \neq \hat{O} \cap \hat{P} \). Indeed, Landau has given examples for theories where the corresponding algebras differ. In particular, they differ if the ‘large’ net \( B \) has the intersection property, i.e., if \( B(O) \cap B(P) = B(O \cap P) \) for all \( O, P \in B \). This shows that the intersection property cannot be a general property of all local nets of observables.

On the other hand, Landau has shown that all his \( B_0 \) satisfy essential duality. This implies that for every \( O \in \mathcal{K} \), one has \( B_0(O')' = B(\mathcal{O}')' \), where \( \mathcal{O} \) denotes the region \( \bigcup_{s \in \mathbb{R}} O + R e_{s+1} \), i.e., the double cone \( O \) smeared out in the \( s+1 \)st spacelike direction. In Landau’s examples, this yields the net associated with some generalized free field, and this net is known to satisfy all the conditions of Theorem 3.2.3 and Proposition 3.2.4.

### 4 Conclusion

Generalizing Landau’s result that the algebras associated with two strictly disjoint double cones have a trivial intersection, the empty-intersection theorem makes it possible to associate a nonempty causally complete, convex and compact localization region with every single local operator of a local
There are several ways how to define such a localization prescription, and from the outset it is not clear that the various prescriptions satisfy locality in the sense that observables with spacelike separated localization regions commute. As a necessary sufficient condition for this, the *nonempty-intersection theorem* establishes a special intersection property, and sufficient for this property is the additional condition of weak additivity for wedges, a property typically shared by models arising from Wightman fields.

It should be emphasized that, while localization prescriptions satisfying locality might be considered convenient, there would be nothing really wrong with a theory with nonlocal localization prescriptions. As the nonempty-intersection theorem has shown, such a theory typically would be one where $A(O) \cap A(P) \neq A(O \cap P)$. Now recall that for every region $O$, the algebra $A(O)$ is usually ’defined’ to contain all observables measurable by a laboratory placed in the region $O$. If an observable is measurable in a lab placed in the region $O$ as well as in a lab placed in the region $P$, there is no physical principle that makes sure that it is also measurable in the region $O \cap P$. Local observables that require a minimum size of a lab and a minimum time of measurement would not at all be in conflict with the principles of relativistic quantum physics, while they would violate the intersection property under discussion. What is more, they appear to be a rather typical phenomenon, as a look at the large particle accelerators shows. Locality of the above localization prescriptions can be convenient for the use of these localization prescriptions, but it is far from essential in order to avoid conflicts with relativity.

The question what the intersection of two algebras of local observables contains has arisen earlier, as, e.g., the remarks in Section III.4.2 of Haag’s monograph [15] show. Haag’s ‘Tentative Postulate’ that the map $O \mapsto A(O)$ be a homomorphism from the orthocomplemented lattice of all causally complete regions (which, in general, are neither bounded nor convex) of Minkowski space into the orthocomplemented lattice of von Neumann algebras on a Hilbert space is far from proved as it stands (cf. also Haag’s heuristic remarks which illustrate the physical limits of the postulate). But if a net satisfies wedge duality and strong additivity for wedges, the above results, indeed, imply parts of Haag’s conjecture: for arbitrary finite families of wedges, one obtains relations in the spirit of (III.4.7) through (III.4.11) in [15] for the dual net.

The results of this article have been used for the analysis of the Unruh effect and related symmetries of quantum fields [21, 22]. Proceeding, so to speak, in the converse direction, Thomas and Wichmann have investigated the implications that the symmetries providing the Unruh effect exert on the
localization behaviour of local observables. Assuming the theory to exhibit the Unruh-effect, and assuming strong additivity for wedges and an intersection property\(^5\), they found that the localization region of an observable \(A\) with respect to a minimal Poincaré covariant local net generated by \(A\) is the smallest region \(O_A\) in \(B\) with the property that for every \((\alpha, \Lambda) \in \mathcal{P}_+^1\), one has \(\alpha + \Lambda O_A \subseteq O'_{\alpha}\) if and only if \([A, U(\alpha, \Lambda) A U(\alpha, \Lambda)^+] = 0\), which is unique up to a translation [29]. This definition of a localization region no longer refers to any other operators of the net; it shows that a localization region of a local observable can be defined so that it is a property of the local operator itself without referring to any other local operators. While this interesting conclusion has been derived from the Unruh effect and other assumptions of relevance in the above discussion, all these assumptions have been avoided above since they are a goal rather than a starting point of the above analysis. In this sense, the results of Thomas and Wichmann are complementary to the above results.

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\(^5\) This additional assumption makes the definition of a nonempty localization region straightforward, but as it is a nonstandard assumption, it has been avoided above.
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