Generalized Quantum Dynamics as Pre-Quantum Mechanics

Stephen L. Adler
Institute for Advanced Study
Princeton, NJ 08540

Andrew C. Millard
Department of Physics, Jadwin Laboratory
Princeton University, Princeton, NJ 08544
ABSTRACT

We address the issue of when generalized quantum dynamics, which is a classical symplectic dynamics for noncommuting operator phase space variables based on a graded total trace Hamiltonian $H$, reduces to Heisenberg picture complex quantum mechanics. We begin by showing that when $H = \text{Tr}H$, with $H$ a Weyl ordered operator Hamiltonian, then the generalized quantum dynamics operator equations of motion agree with those obtained from $H$ in the Heisenberg picture by using canonical commutation relations. The remainder of the paper is devoted to a study of how an effective canonical algebra can arise, without this condition simply being imposed by fiat on the operator initial values. We first show that for any total trace Hamiltonian which involves no noncommutative constants, there is a conserved anti–self–adjoint operator $\tilde{C}$ with a structure which is closely related to the canonical commutator algebra. We study the canonical transformations of generalized quantum dynamics, and show that $\tilde{C}$ is a canonical invariant, as is the operator phase space volume element. The latter result is a generalization of Liouville’s theorem, and permits the application of statistical mechanical methods to determine the canonical ensemble governing the equilibrium distribution of operator initial values. We give arguments based on a Ward identity analogous to the equipartition theorem of classical statistical mechanics, suggesting that statistical ensemble averages of Weyl ordered polynomials in the operator phase space variables correspond to the Wightman functions of a unitary complex quantum mechanics, with a conserved operator Hamiltonian and with the standard canonical commutation relations obeyed by Weyl ordered operator strings. Thus there is a well–defined sense in which complex quantum field theory can emerge as a statistical approximation to an underlying generalized quantum dynamics.
1. Introduction and Brief Review of Generalized Quantum Dynamics

In several recent publications one of us (S.L.A.) formulated [1, 2] and with collaborators elaborated [3, 4] an operator dynamics, called generalized quantum dynamics, which gives a symplectic dynamics for general noncommutative degrees of freedom. This permits the direct derivation of equations of motion for field operators, dispensing with the conventional canonical procedure of “quantizing” a classical theory. Although we observed that generalized quantum dynamics in a complex Hilbert space is compatible with canonical quantization, the precise connection between the two formalisms was not established, and it is this issue which we address in the present paper. We will not in fact find it necessary to restrict ourselves to complex Hilbert space, and the derivations and conclusions given here apply (with some specific differences which we discuss) in quaternionic Hilbert space and real Hilbert space as well.

Generalized quantum dynamics can be given in either Lagrangian or Hamiltonian form, and for brevity we review only the Hamiltonian formalism, since this is what we will need. We shall assume an underlying Hilbert space which is the direct sum of bosonic and fermionic subspaces, and a grading operator \((-1)^F\) with eigenvalue 1(\(-1\)) for states in the bosonic (fermionic) subspace. For a general operator \(\mathcal{O}\), we define the graded trace operation \(\text{Tr}\mathcal{O}\) by

\[
\text{Tr}\mathcal{O} = \text{Re}\text{Tr}(-1)^F\mathcal{O} = \text{Re} \sum_n \langle n | (-1)^F \mathcal{O} | n \rangle = \text{Re} \sum_{n,B} \langle n | \mathcal{O} | n \rangle - \text{Re} \sum_{n,F} \langle n | \mathcal{O} | n \rangle,
\]

with the subscripts \(B, F\) on the sums indicating summations over bosonic and fermionic states, respectively. We call operators \(\text{bosonic}\) if they commute with \((-1)^F\) and \(\text{fermionic}\) if
they anticommute with \((-1)^F\). Given sufficient convergence, it is then easy to see that \(\text{Tr} \mathcal{O}\) vanishes if \(\mathcal{O}\) is fermionic, and that \(\text{Tr}\) obeys the cyclic property

\[
\text{Tr} \mathcal{O}_{(1)} \mathcal{O}_{(2)} = \pm \text{Tr} \mathcal{O}_{(2)} \mathcal{O}_{(1)} ,
\]

with the \(+(-)\) sign holding when \(\mathcal{O}_{(1)}\) and \(\mathcal{O}_{(2)}\) are both bosonic (fermionic).

Let now \(\{q_r(t)\}, \{p_r(t)\}, \ r = 1, ..., N\) be a set of operator phase space variables. For each \(r\), we assume that \(q_r\) and \(p_r\) are either both bosonic or both fermionic, but we make no a priori assumption about the commutativity of the phase space variables with one another. Letting \(A[\{q_r\}, \{p_r\}]\) be a polynomial (or Laurent expandable) operator function of the phase space variables, we define the real number–valued total trace functional \(A\) by

\[
A[\{q_r\}, \{p_r\}] = \text{Tr} A[\{q_r\}, \{p_r\}] .
\]

Although noncommutativity of the phase space variables prevents us from simply differentiating \(A\) with respect to them, we can use the cyclic property of \(\text{Tr}\) to define derivatives of \(A\) by forming \(\delta A\) and cyclically reordering all the operator variations \(\delta q_r, \delta p_r\) to the right, giving the fundamental definition

\[
\delta A = \text{Tr} \sum_r \left( \frac{\delta A}{\delta q_r} \delta q_r + \frac{\delta A}{\delta p_r} \delta p_r \right) ,
\]

in which \(\delta A/\delta q_r\) and \(\delta A/\delta p_r\) are themselves operators.

Let us now introduce an operator Hamiltonian \(H[\{q_r\}, \{p_r\}]\) and a corresponding total trace Hamiltonian \(H = \text{Tr} H\), which generates the dynamics of the phase space variables via the operator Hamilton equations

\[
\frac{\delta H}{\delta q_r} = -\dot{p}_r , \quad \frac{\delta H}{\delta p_r} = \epsilon_r \dot{q}_r ,
\]
with \( \epsilon_r = 1(-1) \) according to whether \( q_r \) and \( p_r \) are bosonic (fermionic), and with the dot denoting the time derivative. (As discussed in Refs. [1, 2], this Hamiltonian formulation can be derived from a total trace Lagrangian action principle, in strict analogy with standard derivations of classical mechanics.) If \( A[\{q_r\}, \{p_r\}, t] \) is an arbitrary total trace functional which can have an explicit time dependence, as well as an implicit time dependence through its dependence on the phase space variables, then a simple application of Eqs. (1–5) shows that the total time derivative of \( A \) is given by

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\}, \tag{6a}
\]

where we have denoted by \( \{A, B\} \) the generalized Poisson bracket defined by

\[
\{A, B\} = \text{Tr} \sum_{r=1}^{N} \epsilon_r \left( \frac{\delta A}{\delta q_r} \frac{\delta B}{\delta p_r} - \frac{\delta B}{\delta q_r} \frac{\delta A}{\delta p_r} \right). \tag{6b}
\]

Since the generalized Poisson bracket is antisymmetric in its arguments, by taking \( A \) to be the total trace Hamiltonian \( H \), which has no explicit time dependence, we learn from Eq. (6a) that \( H \) is a constant of the motion.

In addition to its antisymmetry, the generalized Poisson bracket can also be shown [3] to satisfy the Jacobi identity

\[
0 = \{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\}. \tag{7}
\]

As a consequence, the phase space flows in generalized quantum dynamics exhibit many features [4] analogous to those of ordinary classical mechanics. In order to exhibit the symplectic structure of the generalized Poisson bracket, we use the cyclic property of \( \text{Tr} \) to rewrite Eq. (6b) as

\[
\{A, B\} = \text{Tr} \left[ \sum_{r,B} \left( \frac{\delta A}{\delta q_r} \frac{\delta B}{\delta p_r} - \frac{\delta A}{\delta q_r} \frac{\delta B}{\delta p_r} \right) - \sum_{r,F} \left( \frac{\delta A}{\delta q_r} \frac{\delta B}{\delta p_r} + \frac{\delta A}{\delta q_r} \frac{\delta B}{\delta p_r} \right) \right]. \tag{8}
\]
with the subscripts $B, F$ on the sums respectively indicating summations over the bosonic and fermionic phase space variables. If we now introduce the notation $x_1 = q_1, x_2 = p_1, x_3 = q_2, x_4 = p_2, ..., x_{2N-1} = q_N, x_{2N} = p_N$ for the operator phase space variables, Eq. (8) can be compactly rewritten as

$$\{A, B\} = \text{Tr} \sum_{r,s=1}^{2N} \left( \frac{\delta A}{\delta x_r} \omega_{rs} \frac{\delta B}{\delta x_s} \right), \quad (9a)$$

and similarly, the operator Hamilton equations of Eq. (5) can be compactly rewritten as

$$\dot{x}_r = \sum_{s=1}^{2N} \omega_{rs} \frac{\delta H}{\delta x_s} . \quad (9b)$$

(Henceforth, we will not explicitly indicate the range of summation indices; the index $r$ on $q_r, p_r$ will be understood to have an upper summation limit of $N$, while the index $r$ on $x_r$ will be understood to have an upper summation limit of $2N$.) If for convenience we order the bosonic variables before the fermionic ones in the $2N$ dimensional phase space vector $x_r$, then the matrix $\omega_{rs}$ which appears in Eqs. (9a, b) is given by

$$\omega = \text{diag}(\Omega_B, ..., \Omega_B, \Omega_F, ..., \Omega_F) , \quad (10a)$$

with the $2 \times 2$ matrices $\Omega_{B,F}$ given by

$$\Omega_B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \Omega_F = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad (10b)$$

and $\omega$ obeys

$$\left(\omega^2\right)_{rs} = -\epsilon_r \delta_{rs} , \quad \omega^4 = 1 , \quad \omega_{sr} = -\epsilon_r \omega_{rs} = -\epsilon_s \omega_{rs} , \quad (10c)$$

$$\sum_r \omega_{rs} \omega_{rt} = \sum_r \omega_{sr} \omega_{tf} = \delta_{st} .$$

This concludes our review of generalized quantum dynamics; the reader interested in further details, including the Lagrangian formulation and some concrete examples of models constructed using the total trace formalism, should consult Refs. [1–4].
2. Weyl Ordered Hamiltonians

Let us now consider a special class of operator Hamiltonians called *Weyl ordered* Hamiltonians, in which the bosonic operators are all totally symmetrized with respect to one another and to the fermionic operators, and in which the fermionic operators are totally antisymmetrized with respect to one another. Clearly, the most general Weyl ordered Hamiltonian which is a polynomial in the operator phase space variables \( \{ x_r \} \) will be a sum of terms, which may be of different degrees, each obtained by Weyl ordering a distinct monomial in the phase space variables. The contribution of all such monomials of degree \( n \) may be simply represented by a generating function \( G_n \) constructed as follows. Let \( \sigma_r, \ r = 1, ..., 2N \) be a set of parameters which are real numbers when \( \epsilon_r = 1 \) and which are real Grassmann numbers, which anticommute with each other and with all of the fermionic phase space variables, when \( \epsilon_r = -1 \). Then if we form

\[
G_n = g^n, \quad g = \sum_s \sigma_s x_s, \quad (11a)
\]

the coefficient of each distinct monomial in the parameters \( \sigma_r \) will be a distinct Weyl ordered polynomial of degree \( n \) in the phase space variables \( \{ x_r \} \). Corresponding to the operator generating function \( G_n \), we define a total trace functional generating function

\[
G_n = \text{Tr} G_n, \quad (11b)
\]

where we specify the action of \( \text{Tr} \) on the Grassmann parameters in \( G_n \) by requiring that each Grassmann \( \sigma_r \) anticommutes with \( (-1)^F \). The part of \( G_n \) which is even in the Grassmann parameters is then a generating function for all nonvanishing total trace functionals that correspond to the bosonic Weyl ordered monomials generated by \( G_n \).

Let us now compare the generalized quantum dynamics equations of motion produced
by $G_n$ for general operators $\{x_r\}$, with the corresponding Heisenberg picture equations of motion produced by $G_n$ when the phase space variables $\{x_r\}$ are assumed to obey the canonical algebra of complex quantum mechanics. In our compact phase space notation, this algebra takes the form

$$x_r x_s - \epsilon_r x_s x_r = i \epsilon_r \omega_{rs} \ ,$$

$$[x_r, i] = 0 \ ,$$

where we adopt the convention that if only one of $x_r, x_s$ is bosonic, it is taken to be the operator $x_r$; alternatively, we can rewrite the first part of Eq. (12) with no restrictions on the indices $r, s$ by including a factor $\sigma_s$, giving

$$[x_r, \sigma_s x_s] = i \omega_{rs} \sigma_s \ .$$

Applying the equations of motion of Eq. (9b) with $G_n$ playing the role of the total trace Hamiltonian, we get

$$\dot{x}_r = \sum_s \omega_{rs} \delta G_n \delta x_s = \sum_s \omega_{rs} n g^{n-1} \sigma_s \ .$$

On the other hand, from the canonical algebra of Eq. (13) we find, for both bosonic and fermionic $x_r$, that

$$[x_r, g] = i \sum_s \omega_{rs} \sigma_s \ ,$$

which in turn implies that

$$[x_r, G_n] = n g^{n-1} i \sum_s \omega_{rs} \sigma_s \ .$$

But the Heisenberg picture equations of motion for the phase space variables, taking $G_n$ as the operator Hamiltonian, are

$$\dot{x}_r = i [G_n, x_r] \ ,$$

8
which substituting Eq. (15b) becomes
\[ \dot{x}_r = \sum_s \omega_{rs} n g^{n-1} \sigma_s, \]  
(16a)
in agreement with Eq. (14). We can now sum over all generating function contributions $G_n$ weighted by $c$–number coefficients to obtain a general Weyl ordered Hamiltonian $H$, which has a corresponding total trace Hamiltonian $H = \text{Tr} H$, which respectively generate the Heisenberg picture equation of motion
\[ \dot{x}_r = i[H, x_r] \]  
(16b)
and the corresponding generalized quantum dynamics equation of motion of Eq. (9b).

Thus, for Weyl ordered Hamiltonians formed with $c$–number coefficients, we conclude that the generalized quantum dynamics equations of motion generated by $H$ agree with the Heisenberg picture equations of motion generated by $H$, on an initial time slice on which the phase space variables are canonical, and that on this time slice
\[ [H, i] = 0. \]  
(16c)
But since Eq. (16c) guarantees that the Heisenberg picture equations of motion preserve the canonical algebra on the next time slice, integrating forward in time step by step then implies that generalized quantum dynamics agrees with Heisenberg picture dynamics at all subsequent times, and therefore defines a unitary dynamics. We have given the argument here in a form which applies in complex Hilbert space, where a $c$–number is a general complex number; in quaternionic Hilbert space $i$ in the above equations is actually an operator in the left quaternion algebra, denoted by $I$ in [1, 2], and the most general $c$–number is a real number; while in (even dimensional) real Hilbert space $i$ is represented by a $2 \times 2$ real matrix
i_2, which has the form of $-\Omega_B$ of Eq. (10b), and the most general c-number is again a real number.

When $H$ is not Weyl ordered, one can give explicit examples in which the generalized quantum dynamics equations of motion do not agree with those computed from Heisenberg picture quantum mechanics. With one pair of bosonic phase space variables $q, p$, for example, and the fifth degree Hamiltonian

$$H = \gamma_1 (p^3 q^2 + p^2 q^2 p + pq^2 p^2 + q^2 p^3 + qp^3 q)$$

$$+ \gamma_2 (p^2 qpq + pqpq + qppq^2 + qp^2 q + qp^2 qp) \quad ,$$

(17a)

explicit calculation gives

$$[iH, q] - \frac{\delta H}{\delta p} = 2(\gamma_1 - \gamma_2) \quad ,$$

(17b)

which vanishes only in the Weyl ordered case $\gamma_1 = \gamma_2$. The nonvanishing contribution to the right hand side of Eq. (17b) can be traced to the operator rearrangement

$$(q^2 p^2 - qp^2 q) + (p^2 q^2 - qp^2 q) = 2iqp - 2ipq = -2 \quad ,$$

(17c)

which involves two successive applications of the canonical commutator $[q, p] = i$. Taking the trace of the left and right hand sides of Eq. (17c) clearly leads to a contradiction if cyclic invariance of the trace is assumed. This example serves as a warning that, even though taking $\mathbf{Tr}$ of Eq. (12) does not directly lead to an inconsistency (because while the cyclic property of the trace implies that

$$\mathbf{Tr}(x_r x_s - \epsilon_r x_s x_r) = \mathbf{Tr}[x_r x_s - (\epsilon_r)^2 x_r x_s] = 0 \quad ,$$

(17d)

the inclusion of the real part $\text{Re}$ in the definition of Eq. (1) makes $\mathbf{Tr}i = 0$, the canonical algebra is in general inconsistent with the cyclic trace property. Hence we cannot simply
impose the canonical algebra by fiat as an initial condition in generalized quantum dynamics!

But we shall see that an effective canonical algebra can arise as an emergent property of ensemble averages in the statistical mechanics of generalized quantum dynamics.

To complete the analysis of this section, we note that Weyl ordering is not a necessary condition for the two forms of dynamics to agree, as can be seen by considering the fourth degree Hamiltonian

\[ H = \gamma_1(p^2q^2 + q^2p^2) + \gamma_2pq^2p + \gamma_3qp^2q + \gamma_4(pqpq + qpqp) , \] (17e)

for which one finds that the two forms of dynamics coincide for arbitrary values of the coefficients \(\gamma_1,...,4\) which multiply distinct self-adjoint combinations of operators. This example, and analogs with more than one degree of freedom, are relevant for the behavior of the operator gauge invariant extensions of standard gauge theories formulated in Refs. [1, 2], which we will study in detail elsewhere.

3. The Conserved Operator \( \tilde{C} \)

Let us now make a further application of the generating function \( G_n \) of Eq. (11b). Taking the operator derivative with respect to \( x_s \) and using the fact that \( \sigma_s \) commutes with \( g \), we have

\[ \frac{\delta G_n}{\delta x_s} = ng^{n-1} \sigma_s \]

\[ = \sigma_s ng^{n-1} . \] (18)

Multiplying the first equality in Eq. (18) by \( x_s \) from the right and summing over \( s \), we get

\[ \sum_s \frac{\delta G_n}{\delta x_s} x_s = ng^{n-1} \sum_s \sigma_s x_s = ng^n , \] (19a)

while multiplying the second equality in Eq. (18) by \( \epsilon_s x_s \) from the left and summing over \( s \),
we get
\[
\sum_s \epsilon_s x_s \frac{\delta G_n}{\delta x_s} = \left( \sum_s \epsilon_s x_s \sigma_s \right) n g^{n-1} = \left( \sum_s \sigma_s x_s \right) n g^{n-1} = n g^n . \tag{19b}
\]
Since the right-hand sides of Eqs. (19a, b) are the same, subtracting them gives
\[
\sum_s \left( \frac{\delta G_n}{\delta x_s} x_s - \epsilon_s x_s \frac{\delta G_n}{\delta x_s} \right) = 0 , \tag{19c}
\]
which when summed with \(c\)-number coefficients over all monomial contributions to the Weyl ordered total trace Hamiltonian \(H\) yields the important identity
\[
\sum_s \left( \frac{\delta H}{\delta x_s} x_s - \epsilon_s x_s \frac{\delta H}{\delta x_s} \right) = 0 . \tag{20}
\]

The identity of Eq. (20) is in fact more general than is suggested by the preceding derivation, and holds even if \(H = \text{Tr} H\) is not Weyl ordered, provided only that \(H\) is constructed from monomials formed from the \(\{x_r\}\) using only coefficients that commute with all bosonic operators in Hilbert space (that is, as discussed above, real coefficients in quaternionic and real Hilbert space and complex coefficients in complex Hilbert space; in addition to these \(c\)-numbers, the coefficients can also depend on the grading operator \((-1)^F\).) To see this, let us consider two distinct variations of \(H\) which we label \(\delta_1\) and \(\delta_2\), defined respectively by
\[
\delta_1 x_r = x_r \delta \Lambda ,
\]
\[
\delta_2 x_r = (\delta \Lambda) x_r ,
\]
with \(\delta \Lambda\) an arbitrary infinitesimal anti-self-adjoint bosonic operator variation. As long as the only noncommutativity in \(H\) arises from the phase space variables \(\{x_r\}\), which is the case when only \(c\)-number coefficients are employed in constructing \(H\), cyclic invariance of \(\text{Tr}\) implies that the two variations give the same result when applied to \(H\), since the term
$\mathcal{O}_L \delta \Lambda \mathcal{O}_R$ arising from $\delta_1$ acting on the right–most operator in $\mathcal{O}_L$ is identical to the term arising from $\delta_2$ acting on the left–most operator in $\mathcal{O}_R$. Applying Eq. (4) to Eq. (21) gives

$$0 = \text{Tr} \sum_s \frac{\delta H}{\delta x_s} [x_s \delta \Lambda - (\delta \Lambda) x_s]$$

$$= \text{Tr} \sum_s \left[ \frac{\delta H}{\delta x_s} x_s - \epsilon_s x_s \frac{\delta H}{\delta x_s} \right] \delta \Lambda \ ,$$

which since $\delta \Lambda$ is an arbitrary anti–self–adjoint bosonic operator implies the anti–self–adjoint operator relation

$$0 = \sum_s \left[ \frac{\delta H}{\delta x_s} x_s - \epsilon_s x_s \frac{\delta H}{\delta x_s} \right] ,$$

(giving the same identity as was obtained in the Weyl ordered case in Eq. (20).

We shall now rewrite the identity of Eq. (22b) in an alternative useful form. Let us define the operator $\tilde{C}$ by

$$\tilde{C} = \sum_{r,s} x_r \omega_{rs} x_s$$

$$= \sum_{r,B} [q_r, p_r] - \sum_{r,F} \{q_r, p_r\} ;$$

(23)

that is, $\tilde{C}$ is the difference between the sums of bosonic commutators and fermionic anticommutators. (The tilde is a reminder that $\tilde{C}$ is anti–self–adjoint, a point that will be discussed in detail shortly.) Differentiating the first line of Eq. (23) with respect to time, and using the Hamilton equations of Eq. (9b) and the properties of $\omega_{rs}$ summarized in Eq. (10c), we find

$$\dot{\tilde{C}} = \sum_{r,s} (\dot{x}_r \omega_{rs} x_s + x_r \omega_{rs} \dot{x}_s)$$

$$= \sum_{r,s,t} \left( \frac{\delta H}{\delta x_t} \omega_{rt} \omega_{rs} x_s + x_r \omega_{rs} \omega_{st} \frac{\delta H}{\delta x_t} \right)$$

$$= \sum_r \left( \frac{\delta H}{\delta x_r} x_r - \epsilon_r x_r \frac{\delta H}{\delta x_r} \right)$$

$$= 0 \ .$$
where in the final equality we have used the identity of Eq. (22b). Thus the operator $\tilde{C}$ is a constant of the motion for generalized quantum dynamics, as long as the total trace Hamiltonian $H$ is constructed from the operator phase space variables using only coefficients that commute with all bosonic phase space operators.

We will exploit the conservation of $\tilde{C}$ in the following two sections, but pause to make two remarks.

(1) We first comment on the adjointness properties of $\tilde{C}$. Taking the bosonic coordinates $\{q_r\}$ and conjugate momenta $\{p_r\}$ to be self–adjoint operators, the bosonic commutator terms in Eq. (23) defining $\tilde{C}$ are evidently anti–self–adjoint. (In Refs. [1, 2] the possibility of anti–self–adjoint bosonic $\{q_r\}$ and $\{p_r\}$ was considered, and this adjointness assignment also leads to an anti–self–adjoint $\tilde{C}$; however, because of the complex quantum mechanical structure of the final results of this paper we expect the anti–self–adjoint case to resemble the self–adjoint case, and do not consider it further.) In complex Hilbert space, where $i$ is a $c$–number, the usual fermionic Lagrangians lead to the identifications (see Sec. 13.6 of Ref. [2]) $q_r = \psi_r, \ p_r = i\psi^\dagger_r$, with $\psi_r$ a fermionic operator which is neither self–adjoint nor anti–self–adjoint. This gives $q^\dagger_r = -ip_r, \ p^\dagger_r = -iq_r$, as a consequence of which the fermionic anticommutator terms in Eq. (23) are also anti–self–adjoint. An analogous construction also applies in quaternionic Hilbert space, with the left algebra operator $I$ playing the role of $i$. One must now pay attention to factor ordering and use of a manifestly self–adjoint Lagrangian, giving for each fermionic degree of freedom a pair of phase space operators $q_{r1} = \psi_r, \ p_{r1} = \frac{1}{2}\psi^\dagger_r I, \ q_{r2} = \frac{1}{2}I\psi_r,$ and $p_{r2} = \psi^\dagger_r$, so that $p^\dagger_{r2} = q_{r1}, \ q^\dagger_{r2} = -p_{r1}$, which makes $\{q_{r1}, p_{r1}\} + \{q_{r2}, p_{r2}\}$ anti–self–adjoint. In quaternionic Hilbert space, the only way to construct a total trace Lagrangian for fermions without using an explicit imaginary unit is
to introduce the fermions in pairs, with the real matrix $-\Omega_B$ introduced above playing the role of the imaginary unit (see Sec. 13.7 of Ref. [2]). For each $r$ one then has $q_{r1}, p_{r1}, q_{r2}, p_{r2}$, with adjointness properties assigned according to $p_{r2}^\dagger = \pm q_{r1}, \ q_{r2}^\dagger = \mp p_{r1}$ (but now with no relation between $p_{r1}$ and $q_{r1}$), which again makes the two–term sum $\{q_{r1}, p_{r1}\} + \{q_{r2}, p_{r2}\}$ anti–self–adjoint. This fermionic construction also applies to real Hilbert space.

(2) Not surprisingly, the conservation of $\tilde{C}$ can be given a Noether’s theorem formulation in terms of the total trace Lagrangian $L = \text{Tr}L[\{q_r\}, \{\dot{q}_r\}]$ which corresponds to $H$. We discuss here the simplest case, in which $L$ is constructed from its operator arguments using only $c$–number coefficients, and involves no constraints. Cyclic invariance of $\text{Tr}$ then implies that $L$ is invariant under the operator variations $\delta q_r = \delta_2 q_r - \delta_1 q_r = [\delta \Lambda, q_r], \ r = 1, ..., N$ for a time–independent bosonic variation $\delta \Lambda$. The generalization of Noether’s theorem to total trace Lagrangians given in Sec. 13.5 of Ref. [2] then implies that there is a conserved charge $Q_{\Lambda}$ obeying $\dot{Q}_{\Lambda} = 0$ and given by

$$Q_{\Lambda} = \frac{\delta L}{\delta \Lambda} = \sum_{r,B} \left[ q_r, \frac{\delta L}{\delta \dot{q}_r} \right] - \sum_{r,F} \left\{ q_r, \frac{\delta L}{\delta \dot{q}_r} \right\}, \quad (25)$$

which on substituting $p_r = \frac{\delta L}{\delta \dot{q}_r}$ becomes identical to $\tilde{C}$. If the label $r$ is a composite label comprising a spatial coordinate $\vec{x}$ as well as a discrete field index $r$, then the Noether’s theorem argument implies that there is a current $J^\mu$ which obeys $\partial_\mu J^\mu = 0$ and is given by

$$J^\mu = \sum_{r,B} \left[ q_r, \frac{\delta L}{\delta \partial_\mu q_r} \right] - \sum_{r,F} \left\{ q_r, \frac{\delta L}{\delta \partial_\mu q_r} \right\}, \quad (26)$$

with $\tilde{C} = \int d^3x \ J^0$ the associated charge.
4. Canonical Transformations

We turn now to an analysis of the structure of canonical transformations and symmetry transformations in generalized quantum dynamics. Generalizing from the structure [5] of infinitesimal canonical transformations in classical mechanics, an infinitesimal canonical transformation in generalized quantum dynamics is defined by

\[ x'_r - x_r \equiv \delta x_r = \sum_s \omega_{rs} \frac{\delta G}{\delta x_s} . \]  

(27)

Here \( G = \text{Tr}G \), with \( G \) self-adjoint, is a total trace functional constructed from the phase space operators \( \{ x_r \} \) using arbitrary coefficients, which can be fixed operators as well as \( c\)-numbers. When Eq. (27) is restricted by requiring that the coefficients used to form \( G \) are composed of either \( c\)-numbers, Grassmann \( c\)-numbers, or the grading operator \((-1)^F\), the transformation will be termed an *intrinsic* canonical transformation, and when the further condition of a Weyl ordered \( G \) is imposed, the transformation will be termed a *Weyl ordered intrinsic* canonical transformation.

Letting \( A \equiv A[\{ x_r \}] \) be an arbitrary total trace functional, we find immediately that to first order under a canonical transformation,

\[ A' \equiv A[\{ x'_r \}] = A + \text{Tr} \sum_r \frac{\delta A}{\delta x_r} \delta x_r = A + \text{Tr} \sum_{r,s} \frac{\delta A}{\delta x_r} \omega_{rs} \frac{\delta G}{\delta x_s} = A + \{ A, G \} , \]  

(28a)

that is,

\[ \delta A \equiv A' - A = \{ A, G \} . \]  

(28b)

Comparing Eq. (27) with Eq. (9b), we see that when \( G \) is taken as \( Hdt \), with \( H \) the total
trace Hamiltonian and $dt$ an infinitesimal time step, then $\delta x_r = \dot{x}_r dt$ gives the small change in $x_r$ resulting from the dynamics of the system over that time step. From Eq. (28a), we see that when $A = \text{Tr} A$ with $A$ a Weyl ordered polynomial in the arguments $\{x_r\}$, the canonically transformed total trace functional $A + \{A, G\}$ obtained by applying a canonical transformation is again a Weyl ordered polynomial in the new arguments $\{x'_r\}$. However, since we shall see in Appendix F (where we discuss further details of canonical transformations) that the class of Weyl ordered total trace functionals is not closed under the generalized Poisson bracket operation, the transformed functional $A'$ is not in general a Weyl ordered polynomial in the original arguments $\{x_r\}$.

In Sec. 2 we saw that for Weyl ordered Hamiltonians, there is a close relationship between the time evolution under generalized quantum dynamics and the corresponding Heisenberg dynamics generated when the operator variables are assumed to obey canonical commutators. Generalizing the calculation of Eqs. (14–16b) to the case when $H$ is replaced by any Weyl ordered intrinsic canonical generator $G$ [the use now of Grassmann $c$–number coefficients in adding monomials causes no problems since $g$ in Eq. (11a) is bosonic], we see that in this case Eq. (27) can be represented over the canonical algebra by a commutator as well as by an operator derivative,

$$
\delta x_r = \sum_s \omega_{rs} \frac{\delta G}{\delta x_s} = i[G, x_r],
$$

(29)

with the first equality in Eq. (29) holding for arbitrary operator arguments, and the second equality holding only over the canonical algebra.

In the remainder of this section we establish two important invariances under canonical transformations in generalized quantum dynamics. We consider first the change in the
conserved operator $\tilde{C}$ under an intrinsic canonical transformation, giving by use of Eq. (27)

$$\delta \tilde{C} = \sum_{r,s} [\delta x_r \omega_{rs} x_s + x_r \omega_{rs} \delta x_s]$$

$$= \sum_{r,s,t} \left( \frac{\delta G}{\delta x_t} \omega_{rt} \omega_{rs} x_s + x_r \omega_{rs} \omega_{st} \frac{\delta G}{\delta x_t} \right)$$

$$= \sum_r \left( \frac{\delta G}{\delta x_r} x_r - \epsilon_r x_r \frac{\delta G}{\delta x_r} \right),$$

where we have again used the identities of Eq. (10c). We recognize that the right hand side of Eq. (30) has the same structure as was encountered in Sec. 3, with $G$ now playing the role of $H$. Therefore by the same cyclic invariance argument as was used previously in Eqs. (21–22), we conclude that as long as only $c$–numbers, Grassmann $c$–numbers, or the grading operator $(-1)^F$, all of which commute with an arbitrary bosonic $\delta\Lambda$, are used in constructing $G$, the right hand side of Eq. (30) vanishes and $\tilde{C}$ is intrinsic canonical invariant.

The second invariance concerns the phase space measure for the phase space operators $\{x_r\}$. Let us introduce a complete set of states $\{|n\rangle\}$ in the underlying Hilbert space, so that the phase space operators are completely characterized by their matrix elements $\langle m| x_r |n\rangle \equiv (x_r)_{mn}$, which have the following form in real, complex, and quaternionic Hilbert space:

In real Hilbert space : $(x_r)_{mn} = (x_r)_{mn}^0$

In complex Hilbert space : $(x_r)_{mn} = (x_r)_{mn}^0 + i(x_r)_{mn}^1$

In quaternionic Hilbert space : $(x_r)_{mn} = (x_r)_{mn}^0 + i(x_r)_{mn}^1 + j(x_r)_{mn}^2 + k(x_r)_{mn}^3$, with $(x_r)^A_{mn}$, $A = 0, 1, 2, 3$ real numbers. (Note that this is true for fermionic as well as bosonic operators; the matrix elements of fermionic operators are still real numbers, not real Grassmann numbers! Grassmann numbers are employed only as auxiliary quantities in forming Weyl ordered products and in grade–changing transformations.)
we ignore adjointness restrictions, the phase space measure is defined by

\[ d\mu = \prod_A d\mu^A, \]
\[ d\mu^A = \prod_{r,m,n} d(x_r)^A_{mn}; \]  

when adjointness restrictions are taken into account, certain factors in Eq. (32) become redundant and are omitted. Our strategy is to first ignore adjointness restrictions and to prove the canonical invariance of each individual factor \( d\mu^A \) in the first line of Eq. (32); in Appendix A we describe how the adjointness restrictions modify Eq. (32) and show that the proof remains valid when these modifications are taken into account.

Under the canonical transformation of Eq. (27), the matrix elements of the new variables \( x'_r \) are related to those of the original variables \( x_r \) by

\[ (x'_r)^A_{mn} = (x_r)^A_{mn} + \sum_s \omega_{rs} \left( \frac{\delta G}{\delta x^A_s} \right)^A_{mn}. \]  

Inserting a complete set of intermediate states into the fundamental definition

\[ \delta G = \text{Tr} \sum_s \frac{\delta G}{\delta x_s} \delta x_s, \]  

we get

\[ \delta G = \sum_{s,m,n,A} \epsilon_m \epsilon^A \left( \frac{\delta G}{\delta x^A_s} \right)_{mn} \left( \delta x^A_s \right)_{mn}, \]  

where \( \epsilon_m = 1(-1) \) according to whether the state \( |m\rangle \) is bosonic (fermionic), and where \( \epsilon^0 = 1 \) and \( \epsilon^A = -1, \quad A = 1, 2, 3. \) (In Refs. [1, 2] the factor of \( \epsilon^A \) was inadvertently omitted, but this does not affect the alternative proof of the Jacobi identity given there.) Thus, we see that

\[ \left( \frac{\delta G}{\delta x^A_s} \right)^A_{mn} = \epsilon_m \epsilon^A \frac{\partial G}{\partial (x^A_s)_{mn}}, \]  

19
allowing us to rewrite Eq. (33) in terms of ordinary partial derivatives of the total trace functional $G$,

$$
(x'_r)^A_{mn} = (x_r)^A_{mn} + \sum_s \omega_{rs} \epsilon_m \epsilon^n \frac{\partial G}{\partial (x_s)^A_{nm}} .
$$

(37)

Differentiating Eq. (37) with respect to $(x_r')^A_{m'n'}$, we get for the transformation matrix

$$
\frac{\partial (x'_r)^A_{mn}}{\partial (x_r')^A_{m'n'}} = \delta_{rr'} \delta_{mn'} \delta_{nn'} + \sum_s \omega_{rs} \epsilon_m \epsilon^n \frac{\partial^2 G}{\partial (x_s)^A_{nm} \partial (x_r')^A_{m'n'}} .
$$

(38)

Since for an infinitesimal matrix $\delta X$ we have $\det(1 + \delta X) \approx 1 + \text{Tr} \delta X$, we learn from Eq. (38) that the Jacobian of the transformation is

$$
J = 1 + \Sigma ,
$$

$$(39)$$

$$
\Sigma = \sum_{r,s,m,n} \omega_{rs} \epsilon_m \epsilon^n \frac{\partial^2 G}{\partial (x_s)^A_{nm} \partial (x_r')^A_{m'n'}} .
$$

(39)

Interchanging in $\Sigma$ the summation indices $r$ and $s$, and also interchanging the summation indices $m$ and $n$, we get

$$
\Sigma = \sum_{r,s,m,n} \omega_{sr} \epsilon_n \epsilon^m \frac{\partial^2 G}{\partial (x_s)^A_{nm} \partial (x_r')^A_{m'n'}} ,
$$

(40)

but now using $\omega_{sr} = -\epsilon_r \omega_{rs}$ together with the relation $\epsilon_r = \epsilon_m \epsilon_n$ [which expresses the fact that bosonic (fermionic) operators can only connect states of like (unlike) fermion number], we obtain

$$
\Sigma = -\sum_{r,s,m,n} \omega_{rs} \epsilon_m \epsilon^n \frac{\partial^2 G}{\partial (x_s)^A_{nm} \partial (x_r')^A_{m'n'}} .
$$

(41)

But since the order of second partial derivatives with respect to real matrix elements is immaterial, this is just the statement $\Sigma = -\Sigma$; hence $\Sigma$ vanishes and the Jacobian of the transformation is unity. Although we have ignored adjointness restrictions in this argument, as shown in Appendix A the conclusion is unaltered when these are taken into account.
To summarize, we have shown that the operator phase space integration measure $d\mu$ is invariant under canonical transformations. An important corollary of this result follows when $G$ is taken as the generator $dtH$ of an infinitesimal time translation, since we then learn that $d\mu$ is invariant under the dynamical evolution of the system, giving a generalized quantum dynamics analog of Liouville’s theorem of classical mechanics. Since no restrictions on the form of the generator $G$ were needed in the above argument for the invariance of $d\mu$, the argument applies even when $G$ is formed from the operator phase space variables using operator coefficients. Thus, the integration measure $d\mu$ is invariant under a unitary transformation on the basis of states in Hilbert space, the effect of which on the variables $\{x_r\}$ can be represented by Eq. (27) with $G = -\text{Tr} \sum_r [\tilde{G}, p_r] q_r$, with $\tilde{G}$ a fixed bosonic anti-self-adjoint operator. This transformation, however, is not an intrinsic canonical transformation and is only a covariance, rather than an invariance, of the operator $\tilde{C}$.

5. Equilibrium Ensemble of Operator Initial Values

The operator equations of motion of generalized quantum dynamics determine the time evolution of the operator coordinates and momenta at all times, given their values on an initial time slice. However, these initial values are themselves not determined. We shall now make the assumption that for a large enough system, the statistical distribution of initial values can be treated by the methods of statistical mechanics. Specifically, we shall assume that the a priori distribution of initial values is uniform over the operator phase space, so that the equilibrium distribution is determined solely by maximizing the combinatoric probability subject to the constraints imposed by the generic conservation laws. Liouville’s theorem implies that if the assumption of a uniform a priori probability distribution is made at one time, then it is valid at all later times, assuring the consistency of the concept of an
equilibrium ensemble. We do not propose to address the question of how the randomness in the initial value distribution arises: It could come from a random initial condition, an ordered initial condition followed by evolution under an ergodic dynamics, or some combination of the two.

More specifically, let \( d\mu = d\mu[x_r] \) denote the operator phase space measure discussed in detail in the preceding section. In what follows we shall not need the specific form of this measure, but only the properties that it obeys Liouville’s theorem, and that the measure is invariant under infinitesimal operator shifts \( \delta x_r \), that is

\[
d\mu[x_r + \delta x_r] = d\mu[x_r] \quad .
\]

(This property will not be used until Sec. 6, where we discuss the equipartition or Ward identities.) For a system in statistical equilibrium, there is an equilibrium distribution of operator initial values \( \rho[x_r] \), such that

\[
dP = d\mu[x_r]\rho[x_r]
\]

is the infinitesimal probability of finding the system in the operator phase space volume element \( d\mu \), with the total probability equal to unity,

\[
1 = \int dP = \int d\mu[x_r]\rho[x_r] \quad .
\]

The first task in a statistical mechanical analysis is to determine the equilibrium distribution \( \rho \).

Since equilibrium implies that \( \dot{\rho} = 0 \), the equilibrium distribution can only depend on conserved operators and total trace functionals. In the generic case for a Lorentz invariant system, the only conserved operator is \( \tilde{C} \) and the only conserved total trace functionals are
the total trace Hamiltonian $\mathbf{H} = p^0$, the total trace three momentum $\mathbf{p}$, and the total trace angular momentum $\mathbf{J}$. However, because the graded trace functionals are all indefinite in sign, standard statistical methods lead to a divergent partition function in the generic case. We shall therefore restrict our discussion to total trace Hamiltonians $\mathbf{H} = \text{Tr} H$ for which the generalized quantum dynamics equations of motion imply conservation of the ungraded trace functional $\hat{\mathbf{H}} \equiv \hat{\text{Tr}} H \equiv \text{ReTr} H$ as well as conservation of $\mathbf{H}$; we shall see in Sec. 7 below that a large class of models has this property. These models are characterized (see Appendices C and G) by having an additional conserved operator $\tilde{F}$, which when restricted to the canonical algebra corresponds to the operator for the conserved fermion number $F$.

In addition to the conservation of $\hat{\mathbf{H}}$, we shall assume the more restrictive condition that $\hat{\mathbf{H}}$ is bounded from below; conditions for achieving this are also discussed in Sec. 7. We shall also assume henceforth an ensemble which is translation invariant, rotation invariant, and Lorentz invariant. Since $\tilde{C}$ is invariant under intrinsic canonical transformations, it is Lorentz invariant, and so the equilibrium distribution depends on $\tilde{C}$; similarly, since $\tilde{F}$ is obtained from $\tilde{C}$ by a reordering of fermion factors it is Lorentz invariant as well, and so the equilibrium distribution also depends on $\tilde{F}$. In the ensemble rest frame the equilibrium distribution can also depend on $\hat{\mathbf{H}}$ and $\mathbf{H}$ (since these are the ungraded and graded mass functionals in the rest frame), giving the general equilibrium distribution

$$
\rho = \rho(\tilde{C}, \tilde{F}, \hat{\mathbf{H}}, \mathbf{H}) .
$$

(44a)

The analysis of the implications of this general equilibrium distribution entails considerable algebraic complexity, the full details of which will be published elsewhere. However, as summarized in Appendices C and G, the results are all qualitatively similar to those obtained
from the simpler equilibrium distribution

\[ \rho = \rho(\tilde{C}, \tilde{H}, H) \quad , \tag{44b} \]

in which the \( \tilde{F} \) dependence is dropped. We shall focus on this simplified case in the exposition that follows.

In addition to its dependence on the dynamical variables, \( \rho \) can also depend on constant parameter values, with the functional form of \( \rho \) and the values of the parameters together defining a statistical ensemble. Including an anti–self–adjoint operator parameter \( \tilde{\lambda} \) (which corresponds to the structure of \( \tilde{C} \)) and real number parameters \( \hat{\tau} \) and \( \tau \), which correspond to the structure of \( \hat{H} \) and \( H \), the general form of the equilibrium ensemble corresponding to Eq. (44b) is

\[ \rho = \rho(\tilde{C}, \tilde{\lambda}; \hat{H}, \hat{\tau}; H, \tau) \quad . \quad \tag{44c} \]

In the canonical ensemble, we shall see that the dependence on \( \tilde{C} \) and \( \tilde{\lambda} \) is only through the single real number \( \text{Tr} \tilde{\lambda} \tilde{C} \), and so specializing to this case, Eq. (44c) becomes

\[ \rho = \rho(\text{Tr} \tilde{\lambda} \tilde{C}; \hat{H}, \hat{\tau}; H, \tau) \quad . \quad \tag{44d} \]

We shall now show that some significant consequences follow from the general form of Eq. (44d), together with the fact that the real function \( \rho \) on the right hand side of Eq. (44d) is constructed from its real number arguments using only real number coefficients, and the assumption that \( \hat{H} \) and \( H \) are constructed from the operators \( \{ x_r \} \) using only \( c \)-number coefficients and the grading operator \( (-1)^F \). For a general operator \( \mathcal{O} \), let us define the ensemble average \( \langle \mathcal{O} \rangle_{AV} \) by

\[ \langle \mathcal{O} \rangle_{AV} = \frac{\int d\mu \rho \mathcal{O}}{\int d\mu \rho} \quad . \quad \tag{45a} \]
Then when $O$ is constructed from the $\{x_r\}$ using only $c$–numbers and $(-1)^F$ as coefficients, the ensemble average $\langle O \rangle_{AV}$ must have the form
\[ \langle O \rangle_{AV} = F_O(\tilde{\lambda}, (-1)^F) , \quad (45b) \]
with the function $F_O$ constructed from its arguments using only $c$–number coefficients (in which we include the $\hat{\tau}$ and $\tau$ dependence). This further implies that both the grading operator $(-1)^F$ and the ensemble parameter $\tilde{\lambda}$ commute with $\langle O \rangle_{AV}$,
\[ [\tilde{\lambda}, \langle O \rangle_{AV}] = [(-1)^F, \langle O \rangle_{AV}] = 0 . \quad (45c) \]

Let us now exploit the fact that the anti–self–adjoint operator $\tilde{\lambda}$ can always be diagonalized (or, in the case of an even dimensional real Hilbert space, reduced to $2 \times 2$ diagonal blocks), by a unitary transformation on the basis of states in Hilbert space, which we have seen is also an invariance of the integration measure $d\mu$. The functional relationship between $\tilde{\lambda}$ and $\langle \tilde{C} \rangle_{AV}$ then implies that $\langle \tilde{C} \rangle_{AV}$ is diagonal (or block diagonal) in this basis as well. As described more fully in Appendix B, this brings $\langle \tilde{C} \rangle_{AV}$ into the following canonical form in real (when even dimensional), complex, and quaternionic Hilbert space,
\[ \langle \tilde{C} \rangle_{AV} = i_{eff} D , \quad i_{eff} = -i_{eff}^\dagger , \quad i_{eff}^2 = -1 , \quad (46a) \]
\[ [i_{eff}, D] = 0 , \quad D \text{ real diagonal and nonnegative} . \]
Although the case of general $D$ is interesting, we shall restrict ourselves in this paper to the special case in which $D$ is a real constant times the unit operator; in other words, we are assuming that the ensemble does not favor any state in Hilbert space over any other. Benefiting from some prescience, we denote this real constant by $\hbar$, and so have
\[ \langle \tilde{C} \rangle_{AV} = i_{eff} \hbar , \quad (46b) \]
\[ \{i_{eff}, \langle \tilde{C} \rangle_{AV}\} = -2\hbar . \]
We turn now to the calculation of the functional form of $\rho$ in the canonical ensemble, which is the ensemble relevant for describing the behavior of a large system which is a subsystem of a still larger system. The form of $\rho$ is determined \cite{6, 7} by minimizing the negative of the entropy,

$$-S = \int d\mu \rho \log \rho ,$$  

subject to the constraints

$$\int d\mu \rho = 1 ,$$
$$\int d\mu \rho \tilde{C} = \langle \tilde{C} \rangle_{AV} ,$$
$$\int d\mu \rho \hat{\text{Tr}} H = \langle \hat{\text{Tr}} H \rangle_{AV} ,$$
$$\int d\mu \rho \text{Tr} H = \langle \text{Tr} H \rangle_{AV} .$$

The standard procedure is to impose the constraints with Lagrange multipliers by writing

$$\mathcal{F} = \int d\mu \rho \log \rho + \theta \int d\mu \rho + \int d\mu \rho \hat{\text{Tr}} \tilde{\lambda} \tilde{C} + \hat{\tau} \int d\mu \rho \hat{\text{Tr}} H + \tau \int d\mu \rho \text{Tr} H ,$$

and minimizing $\mathcal{F}$, treating all variations of $\rho$ as independent. (Note that it makes no difference whether the constraint for $\tilde{C}$ is introduced through a graded or an ungraded trace; the difference is a factor of $(-1)^F$ which can be absorbed into the definition of $\tilde{\lambda}$.) In order for $\mathcal{F}$ to have a minimum, it must be bounded below; we shall assume that this is the case for sufficiently large $\hat{\tau}$ (at a minimum, one needs $\hat{\tau} \geq |\tau|$, so that the coefficients of the trace of $H$ over the bosonic and fermionic subspaces, proportional respectively to $\hat{\tau} + \tau$ and $\hat{\tau} - \tau$, are both positive). Varying Eq. (48a) with respect to $\rho$ then gives

$$\rho = \exp(-1 - \theta - \hat{\text{Tr}} \tilde{\lambda} \tilde{C} - \hat{\tau} \hat{\text{Tr}} H - \tau \text{Tr} H) ,$$

$$\rho = \exp(-1 - \theta - \hat{\text{Tr}} \tilde{\lambda} \tilde{C} - \hat{\tau} \hat{\text{Tr}} H - \tau \text{Tr} H) ,$$

26
which on imposing the condition that \( \rho \) be normalized to unity gives finally

\[
\rho = Z^{-1} \exp(-\text{Tr} \tilde{\lambda} \tilde{C} - \hat{\tau} \hat{\text{Tr}} \hat{H} - \tau \text{Tr} H) ,
\]

\[
Z = \int d\mu \exp(-\text{Tr} \tilde{\lambda} \tilde{C} - \hat{\tau} \hat{\text{Tr}} \hat{H} - \tau \text{Tr} H) .
\]

From Eq. (48c) we can easily derive some elementary statistical properties of the equilibrium ensemble. For the entropy \( S \), we find

\[
S = -\int d\mu \log \rho = \log Z + \text{Tr} \tilde{\lambda} \langle \tilde{C} \rangle_{AV} + \hat{\tau} \langle \hat{\text{Tr}} \hat{H} \rangle_{AV} + \tau \langle \text{Tr} H \rangle_{AV} .
\]

(49a)

Since the ensemble averages which appear in Eq. (49a) are given by

\[
\langle \tilde{C} \rangle_{AV} = -\frac{\delta \log Z}{\delta \tilde{\lambda}} ,
\]

\[
\langle \hat{\text{Tr}} H \rangle_{AV} = -\frac{\partial \log Z}{\partial \hat{\tau}} ,
\]

\[
\langle \text{Tr} H \rangle_{AV} = -\frac{\partial \log Z}{\partial \tau} ,
\]

(49b)

Eq. (49a) takes the form

\[
S = \log Z - \text{Tr} \tilde{\lambda} \frac{\delta \log Z}{\delta \tilde{\lambda}} - \hat{\tau} \frac{\partial \log Z}{\partial \hat{\tau}} - \tau \frac{\partial \log Z}{\partial \tau} .
\]

(49c)

Thus the entropy is a thermodynamic quantity determined solely by the partition function. Taking second derivatives of the partition function, we can derive thermodynamic formulas for the averaged mean square fluctuations of the conserved quantities \( \tilde{C}, \hat{H} = \hat{\text{Tr}} H, \) and \( H = \text{Tr} H, \)

\[
\Delta_{\text{Tr} \tilde{P} \tilde{C}}^2 = \langle (\text{Tr} \hat{P} \tilde{C} - \langle \text{Tr} \hat{P} \tilde{C} \rangle_{AV})^2 \rangle_{AV} = \langle (\text{Tr} \hat{P} \tilde{C})^2 \rangle_{AV} - \langle \text{Tr} \hat{P} \tilde{C} \rangle_{AV}^2 = (\text{Tr} \hat{P} \frac{\delta}{\delta \tilde{\lambda}})^2 \log Z ,
\]

\[
\Delta_{\hat{H}}^2 = \langle (\hat{H} - \langle \hat{H} \rangle_{AV})^2 \rangle_{AV} = \langle \hat{H}^2 \rangle_{AV} - \langle \hat{H} \rangle_{AV}^2 = \frac{\partial^2 \log Z}{(\partial \hat{\tau})^2} ,
\]

\[
\Delta_{\hat{H}}^2 = \langle (H - \langle H \rangle_{AV})^2 \rangle_{AV} = \langle H^2 \rangle_{AV} - \langle H \rangle_{AV}^2 = \frac{\partial^2 \log Z}{(\partial \tau)^2} ,
\]

(49d)

with \( \hat{P} \) an arbitrary fixed anti-self-adjoint operator. Equations (49a–d) show that the entropy, the expectations of \( \tilde{C}, \hat{H}, \) and \( H, \) and the mean square fluctuations of the latter three
quantities, are all extensive quantities which grow linearly with the size \(N\) of the system. This implies that the ratio of the root mean square fluctuation to the mean for \(\tilde{C}, \hat{H},\) and \(H\) vanishes as \(N^{-1/2}\) in the limit \(N \to \infty\), and justifies using mean values in imposing the constraints in Eq. (48a).

In the Ward identity derivation of the following section, the distribution \(\rho\) enters under a phase space integral in two ways. There are a number of terms in which \(\rho\) appears simply as a weighting factor; these terms appear to be dominant and in them we assume that the distribution is sharp enough so that unvaried factors of the conserved extensive quantity \(\tilde{C}\) can be replaced by the ensemble average \(\langle \tilde{C} \rangle_{AV}\), an approximation that should become exact in the \(N \to \infty\) limit. In addition there is a term involving the variation \(\delta \rho\) of the equilibrium distribution, which gives corrections to the Ward identity that we presume to come from very high energy physics. The correction term is evaluated using the formula (the normalization factor \(Z\) is not varied in the following equations, since we are interested here only in variations for which \(\delta Z = 0\),

\[
\delta \rho = \rho \delta \log \rho = \rho \left( \text{Tr} \frac{\delta \log \rho}{\delta \tilde{C}} \delta \tilde{C} + \frac{\partial \log \rho}{\partial \hat{H}} \delta \hat{H} + \frac{\partial \log \rho}{\partial H} \delta H \right), \tag{50a}
\]

and from Eq. (48c) we find for the variations of \(\log \rho\),

\[
\frac{\delta \log \rho}{\delta \tilde{C}} = -\tilde{\lambda},
\]

\[
\frac{\partial \log \rho}{\partial \hat{H}} = -\hat{\tau}, \tag{50b}
\]

\[
\frac{\partial \log \rho}{\partial H} = -\tau.
\]

The variations \(\delta_x \tilde{C}, \delta_x \hat{H},\) and \(\delta_x H\) corresponding to the variation \(\delta x_s\) are computed in
Appendix C, with the results

\[ \delta x_s \hat{C} = \sum_r \omega_{rs} \left( x_r \delta x_s - \delta x_s x_r \right), \]

\[ \delta x_s \hat{H} = \delta x_s \text{Tr} H = \text{Tr} (-1)^F \sum_r \dot{x}_r \hat{\omega}_{rs} \delta x_s, \] (50c)

\[ \delta x_s H = \delta x_s \text{Tr} H = \text{Tr} \sum_r \dot{x}_r \omega_{rs} \delta x_s, \]

with \( \hat{\omega}_{rs} \) and \( \alpha_{ur} \equiv \sum_s \omega_{us} \hat{\omega}_{rs} \) given in Appendix C. In evaluating the correction term, we assume that unvaried factors of the conserved extensive quantities \( \hat{H} \) and \( H \) can also be replaced by their corresponding ensemble averages, again an approximation that should become exact in the \( N \to \infty \) limit.

As a final remark, in the derivation of the next section we shall follow the conventional practice of introducing, for each phase space operator, an operator source which can be varied and which is then set to zero after all variations have been performed. It is convenient to define the sources \( \rho_r \) so that they are all bosonic and self-adjoint. To couple such sources to the phase space operators, we employ the auxiliary real or Grassmann real parameters \( \sigma_r \) introduced in Sec. 2, and couple the source term as \( \text{Tr} \rho_r \sigma_r x_r \). When \( r \) is a bosonic index, the source \( \rho_r \) takes a distinct value for each \( r \), since \( \sigma_r x_r \) is already self-adjoint in this case.

When \( r \) is a fermionic index in complex quantum mechanics, the source \( \rho_r \) takes a distinct value only for each pair of fermionic phase space operators \( q_r, p_r \), so that each distinct \( \rho_r \) multiplies the combination \( \sigma_{q_r} q_r + \sigma_{p_r} p_r \), which (remembering that \( q_r^\dagger = -i p_r \)) is self-adjoint when the Grassmann parameters are taken to obey \( \sigma_{q_r}^\dagger = -i \sigma_{p_r} \). When \( r \) is a fermionic index in real or quaternionic quantum mechanics, the source \( \rho_r \) takes a distinct value only for each quartet of fermionic phase space operators \( q_{r1}, p_{r1}, q_{r2}, p_{r2} \), so that each distinct \( \rho_r \) multiplies the combination \( \sigma_{q_{r1}} q_{r1} + \sigma_{p_{r1}} p_{r1} + \sigma_{q_{r2}} q_{r2} + \sigma_{p_{r2}} p_{r2} \), which (remembering that \( q_{r2}^\dagger = \mp p_{r1}, \ p_{r2}^\dagger = \pm q_{r1} \)) is self-adjoint when the Grassmann parameters are taken to obey...
\( \sigma_{q,2}^\dagger = \pm \sigma_{q,1}^\dagger, \quad \sigma_{p,2}^\dagger = \mp \sigma_{p,1}. \) We shall follow the practice of writing the source term as
\[ \text{Tr} \sum_r \rho_r(\sigma_r x_r), \]
with the parentheses a reminder of this implicit grouping. With the sources included, the equilibrium distribution and partition function take the form
\[
\begin{align*}
\rho & = Z^{-1} \exp[-\text{Tr} \sum_r \rho_r(\sigma_r x_r)] \exp(-\text{Tr} \tilde{\lambda} \tilde{C} - \hat{\tau} \hat{H} - \tau H), \\
Z & = \int d\mu \exp[-\text{Tr} \sum_r \rho_r(\sigma_r x_r)] \exp(-\text{Tr} \tilde{\lambda} \tilde{C} - \hat{\tau} \hat{H} - \tau H). \quad (51a)
\end{align*}
\]
Continuing to use the expression \( \langle O \rangle_{AV} \) to denote the average of a general operator over the equilibrium distribution of Eq. (51a) which includes sources, the variations of \( \log Z \) with respect to its source arguments are related to the averages of the \( x_r \) by
\[
\langle (\sigma_r x_r) \rangle_{AV} = -\frac{\delta \log Z}{\delta \rho_r}. \quad (51b)
\]

6. Ward Identities, Unitarity, and the Canonical Algebra

In the previous sections we have seen that in generalized quantum dynamics there is a conserved operator \( \tilde{C} \), given by the sum of commutators for all of the bosonic degrees of freedom minus the corresponding sum of fermionic anticommutators, and that this operator plays a role in equilibrium statistical mechanics closely analogous to that played by the summed energy of independent degrees of freedom in classical statistical physics. This naturally suggests the idea that the canonical commutation relations of quantum mechanics may arise from a generalized quantum dynamics analog of the classical theorem of equipartition of energy. To pursue this thought, let us begin by reviewing a simple derivation [8] of the classical equipartition theorem. Let \( H(\{x_r\}) \) be the classical Hamiltonian as a function of classical phase space variables \( \{x_r\} \), and let \( d\mu(\{x_r\}) \) be the classical phase space integration.
measure. We consider the integral
\begin{align}
\int d\mu \frac{\partial [x_r \exp(-\beta H)]}{\partial x_s} \\
= \int d\mu \delta_{rs} \exp(-\beta H) \\
- \int d\mu x_r \frac{\partial [\beta H]}{\partial x_s} \exp(-\beta H) ,
\end{align}
(52a)
the left hand side of which is the integral of a total derivative and vanishes when the integrand
is sufficiently rapidly vanishing at infinity. Assuming this, we get
\[ \delta_{rs} = \frac{\int d\mu x_r \beta (\partial H/\partial x_s) \exp(-\beta H)}{\int d\mu \exp(-\beta H)} , \]  
(52b)
which is the classical theorem of equipartition of energy. The method of derivation is similar
to that used to derive Ward identities from functional integrals in quantum field theory (see, e.g. [9]),
and the equipartition theorem can be viewed as a Ward identity application in
classical statistical mechanics.

We proceed now to derive a Ward identity for the statistical ensemble of generalized
quantum dynamics. The derivation is based on Eq. (42), which asserts the invariance of the
operator phase space measure \( d\mu \) under finite operator shifts \( \delta x_r \), which can be arbitrary
apart from the obvious restriction that they must satisfy the same self–adjointness restric-
tions as the corresponding operators \( x_r \). We take account of the adjointness restrictions on
the variations by using the following Lemma, proved in Appendix D, which insures that when
we equate the variation of a total trace functional to zero, we do not inadvertently “deduce”
an operator relation which arises from the variation of an anti–self–adjoint operator, which
is identically zero when acted on by \( \text{Tr} \).

Lemma:

Let \( Y_1 \) and \( Y_2 \) be two self–adjoint bosonic or two anti–self–adjoint bosonic operators con-
structured from the phase space variables. Then in \(0 = \delta \text{Tr} Y_1 Y_2\), the self-adjointness restrictions on the variations can be ignored.

To derive the Ward identity, we consider the expression

\[
0 = \int d\mu \delta x_s \left[ \exp[-\text{Tr} \tilde{\lambda} \tilde{C} - \tilde{\tau} \tilde{H} - \tau H - \text{Tr} \sum_r \rho_r(\sigma_r x_r)] \text{Tr}\{\tilde{C}, i_{eff} \} V \right],
\]

(53a)

where the operator variation \(\delta x_s\) is defined to act on an arbitrary operator \(X[\{x_t\}]\) as

\[
\delta x_s X[\{x_t\}] = X[\{x_t, t \neq s; x_s + \delta x_s\}] - X[\{x_t\}],
\]

(53b)

and where in equating the shift to zero we are using the shift invariance of the measure and the assumption that the integrals are sufficiently convergent that contributions from infinity can be ignored. In Eq. (53a), the expression \(V\) denotes any self-adjoint polynomial in the variables \(\{(\sigma_s x_s)\}\) constructed using coefficients which are \(c\)-numbers apart from a possible dependence on the operators \((-1)^F\) and \(i_{eff}\). [Inclusion of the auxiliary factors \(\sigma_s\) in the combination \((\sigma_s x_s)\), which was defined in the discussion preceding Eq. (51a), is purely a matter of convenience; it permits working with bosonic quantities throughout, and also facilitates comparison with the form of the canonical algebra given in Eq. (13).] The traces in the exponent in Eq. (53a) and the trace involving \(V\) both have the form specified by the Lemma, so we can proceed with taking variations, giving [c.f. Eqs. (50a, b)]

\[
0 = \int d\mu \exp[-\text{Tr} \tilde{\lambda} \tilde{C} - \tilde{\tau} \tilde{H} - \tau H - \text{Tr} \sum_r \rho_r(\sigma_r x_r)]
\times \left[ [-\text{Tr} \tilde{\lambda} \delta x_s \tilde{C} - \tilde{\tau} \delta x_s \tilde{H} - \tau \delta x_s H - \text{Tr} \rho_s \sigma_s \delta x_s \text{Tr}\{\tilde{C}, i_{eff} \} V + \text{Tr}(\{\delta x_s \tilde{C}, i_{eff} \} V + \{\tilde{C}, i_{eff} \} \delta x_s V) \right].
\]

(53c)

We now make two assumptions: First, when the extensive quantity \(\tilde{C}\) appears in \textit{unvaried} form as a factor in an ensemble average over the equilibrium distribution, we assume
that it can be replaced by its ensemble average $\langle \tilde{C} \rangle_{AV}$ (but this is \textit{not}, of course, applied to the $\tilde{C}$ in the exponent of the equilibrium distribution). This assumption amounts to neglecting the fluctuations of $\tilde{C}$ when it appears as a factor in the integrand, and as we argued in Sec. 5, should be justified in the limit $N \to \infty$. Second, we assume the form of Eq. (46b) for $\langle \tilde{C} \rangle_{AV}$. Replacing unvaried factors of $\tilde{C}$ by their ensemble averages, substituting Eqs. (46b) and (50c), and making some cyclic permutations under the graded trace, Eq. (53c) becomes

$$0 = \int d\mu \exp[-\text{Tr} \hat{\lambda} \tilde{C} - \hat{\tau} \hat{H} - \tau H - \text{Tr} \sum_r \rho_r(x_r)]$$

$$\times \left[\left(-\text{Tr}[\hat{\lambda}, \sum_r \omega_{rs} x_r] \delta x_s - \hat{\tau} \text{Tr}(\tau F) \sum_r \hat{x}_r \omega_{rs} \delta x_s\right) (\tau - 2\hat{h})V + \text{Tr}\{i_{\text{eff}}, V\}, \sum_r \omega_{rs} x_r] \delta x_s - 2\hat{h} \text{Tr} \frac{V}{\delta x_s} \delta x_s\right],$$

where in the final line we have used the definition of the operator derivative of the total trace functional $V = \text{Tr} V$,

$$\text{Tr} \delta_{x_s} V = \text{Tr} \frac{\delta V}{\delta x_s} \delta x_s .$$

Before proceeding further, let us examine the structure of the first term on the right hand side of Eq. (54a), which contains the factor $[\hat{\lambda}, \sum_r \omega_{rs} x_r]$. After varying with respect to the sources and setting the sources to zero, this term is proportional to

$$\text{Tr}[\hat{\lambda}, \int d\mu \rho P(\{x_r\}) V] \delta x_s ,$$

with $\rho$ the zero source equilibrium distribution of Eq. (48c) and with $P$ the polynomial which results after variation with respect to the sources. Since $V$, by assumption, involves no operator coefficients other than $(-1)^F$ and $i_{\text{eff}}$, the reasoning of Eqs. (45a–c) implies
that the ensemble average in Eq. (54c) is a function $G(\tilde{\lambda}, (-1)^F)$, and so commutes with $\tilde{\lambda}$. Hence the first term on the right hand side of Eq. (54a) vanishes. Since each remaining term in Eq. (54a) is the graded trace of an operator times the variation $\delta x_s$, we can equate the total operator coefficient of this variation to zero. After multiplying through by $\frac{1}{2} \sum_s \omega_{us}$, and using Eq. (10c) and the definition of $\alpha_{ur}$ following Eq. (50c), this gives the operator Ward (or equipartition) identity

$$0 = \int d\mu \exp[-\text{Tr} \tilde{\lambda} \tilde{C} - \tau \tilde{H} - \tau H - \text{Tr} \sum_r \rho_r(\sigma_r x_r)]$$

$$\times \left[ [\tilde{\tau}(-1)^F \sum_r \alpha_{ur} \hat{x}_r + \tau \hat{x}_u + \sum_s \omega_{us} \sigma_s \rho_s] \hbar V \right. \right.$$

$$\left. \left. + \left[ \frac{1}{2} \{i_{eff}, V\}, x_u \right] - \hbar \sum_s \omega_{us} \frac{\delta V}{\delta x_s} \right] \right].$$

(55a)

Dividing by the partition function $Z$, Eq. (55a) can be rewritten in the compact form

$$0 = \langle [\tilde{\tau}(-1)^F \sum_r \alpha_{ur} \hat{x}_r + \tau \hat{x}_u + \sum_s \omega_{us} \sigma_s \rho_s] \hbar V + \left[ \frac{1}{2} \{i_{eff}, V\}, x_u \right] \rangle_{AV} - \hbar \sum_s \omega_{us} \frac{\delta V}{\delta x_s} \rangle_{AV},$$

(55b)

with $\langle \rangle_{AV}$ denoting the ensemble average with sources present, and with the understanding that after variation with respect to the sources, the sources are to be set equal to zero. A detailed discussion of how symmetrized polynomials in the phase space operators may be built up through source variation is given in Appendix E, and a discussion of a second Ward identity connected to the conserved operator $\tilde{F}$ is given in Appendix G. Although the explicit source term in Eq. (55b) proportional to $\sum_s \omega_{us} \sigma_s \rho_s$ does contribute when varied with respect to a source $\rho_v$ for which $\omega_{uv}$ is nonzero, the fact that $\omega_{uv}$ is antisymmetric in bosonic indices, and symmetric in fermionic indices, implies that this term drops out of Weyl ordered expressions in $x_u$ and the additional factors of $x_v, \ldots$ brought down by source variation, as is discussed in more detail in Appendix E. Hence we shall drop this term.
in the Ward identity applications that follow. A closely related remark is that although Eq. (55b) makes a statement of effective equality (when the $\hat{\tau}$ and $\tau$ terms are dropped) between the commutator expression in the next to last term and the total trace derivative in the final term, this does not contradict the counterexample of Eqs. (17a, b) above, since the Ward identity does not imply that the commutator can be evaluated in terms of the canonical algebra by using the Leibnitz product rule. Use of the Leibnitz product rule for $V$ is justified in Appendix E only when $V$ can be constructed as a Weyl ordered polynomial in the operator phase space variables $\{x_r\}$.

We proceed now to give three applications of Eq. (55b). As our first application we choose $V$ to be the operator Hamiltonian $H$, so that $V$ becomes the conserved quantity $H$. Substituting the generalized quantum dynamics equation of motion of Eq. (9b), the Ward identity becomes

$$0 = \langle [\hat{\tau}(-1)^F \sum_r \alpha_{ur} \dot{x}_r + \tau \dot{x}_u] hH + \left[ \frac{1}{2} \{i_{eff}, H \}, x_u \right] - \bar{h} \dot{x}_u \rangle_{AV}. \hspace{1cm} (56a)$$

We can simplify Eq. (56a) considerably by noting that since $H$ is a conserved extensive quantity, in the large $N$ limit we can approximate it by its ensemble average $\langle H \rangle_{AV}$; its coefficient in Eq. (56a) is then proportional to the partition function variation $\sum_s \omega_s \delta Z/\delta x_s = 0$. So dropping the $\hat{\tau}$ and $\tau$ terms, and multiplying Eq. (56a) by $-1$ we are then left with

$$0 = \langle h\dot{x}_u - \left[ \frac{1}{2} \{i_{eff}, H \}, x_u \right] \rangle_{AV}, \hspace{1cm} (56b)$$

where the ensemble used to form the average is understood to still contain nonzero sources.

It is convenient at this point to recall the properties of $i_{eff}$ given in Eq. (46a), and to make the definition

$$\tilde{H}_{eff} \equiv \frac{1}{2} \{i_{eff}, H \}, \quad \tilde{H}_{eff}^\dagger = -\tilde{H}_{eff}. \hspace{1cm} (57a)$$
with the tilde indicating that $\tilde{H}_{\text{eff}}$ is an anti–self–adjoint operator. We find from Eq. (57a) that

$$i_{\text{eff}}\tilde{H}_{\text{eff}} = \frac{1}{2}(-H + i_{\text{eff}}H_{\text{eff}}) = \tilde{H}_{\text{eff}}i_{\text{eff}}$$

(57b)
in other words, $i_{\text{eff}}$ and $\tilde{H}_{\text{eff}}$ commute. We can now write Eq. (56b) as

$$0 = \langle \hbar \dot{x}_u - [\tilde{H}_{\text{eff}}, x_u] \rangle_{AV}$$

(57c)

which is an effective Heisenberg picture equation of motion for $x_u$ in anti–self–adjoint generator form. Let us now vary with respect to the sources, leading (as described in Appendix E) to the replacement of Eq. (57c) by the expression

$$0 = \langle \hbar \dot{P}(\{x_r\}) - [\tilde{H}_{\text{eff}}, P(\{x_r\})] \rangle_{AV}$$

(57d)

with $P(\{x_r\})$ a Weyl ordered polynomial formed with coefficients which are $c$–numbers (apart from a possible dependence on $(-1)^F$ and $i_{\text{eff}}$). In particular, letting $P$ be the effective Hamiltonian $\tilde{H}_{\text{eff}}$, we find that

$$\frac{d}{dt}(\langle \tilde{H}_{\text{eff}} \rangle_{AV} = 0$$

(57e)

and so $\langle \tilde{H}_{\text{eff}} \rangle_{AV}$, still in the presence of sources, is a constant of the motion. Thus the ensemble averages of products of the coordinates have an effective unitary time development, involving an anti–self–adjoint effective time independent Hamiltonian.

This time development, however, cannot immediately be put into the standard Heisenberg picture form of Eq. (16b), which involves a self–adjoint Hamiltonian. We now show that we can extract from the $\{x_r\}$ a set of new operators $\{x_r^\text{eff}\}$, which do obey an effective dynamics of the standard Heisenberg form. We begin by introducing the self–adjoint
effective Hamiltonian $H_{\text{eff}}$ defined by

$$H_{\text{eff}} = -i_{\text{eff}} \tilde{H}_{\text{eff}} = \frac{1}{2}(H - i_{\text{eff}} H i_{\text{eff}}), \quad (58a)$$

which evidently also commutes with $i_{\text{eff}}$. In analogy with Eq. (58a), we further define

$$x_{r \text{ eff}} = \frac{1}{2}(x_{r} - i_{\text{eff}} x_{r} i_{\text{eff}}), \quad (58b)$$

which obeys

$$i_{\text{eff}} x_{r \text{ eff}} = \frac{1}{2}(i_{\text{eff}} x_{r} + x_{r} i_{\text{eff}}) = x_{r \text{ eff}} i_{\text{eff}}, \quad (58c)$$

and thus also commutes with $i_{\text{eff}}$. For any operators $x_{1}, x_{2}$ this definition evidently obeys

$$(x_{1} x_{2 \text{ eff}})_{\text{eff}} = (x_{1 \text{ eff}} x_{2})_{\text{eff}} = x_{1 \text{ eff}} x_{2 \text{ eff}}, \quad (58d)$$

and so taking the effective projection of the equation of motion of Eq. (57c) gives

$$0 = \langle \hbar \dot{x}_{u \text{ eff}} - [\tilde{H}_{\text{eff}}, x_{u \text{ eff}}] \rangle_{AV}, \quad (59a)$$

which by Eq. (58c) can now be written directly in terms of the self–adjoint effective Hamiltonian defined in Eq. (58a),

$$0 = \langle \hbar \dot{x}_{u \text{ eff}} - i_{\text{eff}} [H_{\text{eff}}, x_{u \text{ eff}}] \rangle_{AV}, \quad (59b)$$

and so has the standard Heisenberg picture form of complex quantum mechanics. In a similar fashion, by repeated applications of Eq. (58c) to polynomials of successively one higher degree, we can derive a self–adjoint analog of Eq. (57d),

$$0 = \langle \hbar \dot{P}([x_{r \text{ eff}}]) - i_{\text{eff}} [H_{\text{eff}}, P([x_{r \text{ eff}}])] \rangle_{AV}, \quad (59c)$$

Before proceeding to further Ward identity applications, we make two remarks. The first is that the method of projecting out effective operators which commute with $i_{\text{eff}}$ is
simply a complex analog of the method of extracting “formally real” components of operators in quaternionic quantum mechanics [2]. The second is that the operators $x_r$ and $x_{r \text{ eff}}$ always differ in the cases of real and quaternionic Hilbert spaces. Even in the case of a complex Hilbert space they differ when $i_{\text{eff}} \neq i$, as shown in Appendix B, while in Appendix G we show that the complex case with $i_{\text{eff}} = i$ is excluded. When $x_r$ and $x_{r \text{ eff}}$ differ, there is a “hidden sector” which cannot be attained by acting with arbitrary polynomials formed from the effective operators alone.

We turn now to the second Ward identity application, which we derive by taking $V$ in Eq. (55b) to be $V = (\sigma_t x_t)$. The parentheses here indicate summation over the one, two, or four $t$ values corresponding to a distinct source term $\rho_t$ in the equilibrium distribution, which according to the discussion preceding Eq. (51a) makes $V$ self-adjoint. Since the $\sigma$ parameters which appear in this sum are linearly independent, we can ignore the parentheses and simply substitute the single term $V = \sigma_t x_t$ into Eq. (55b), corresponding to $\delta V / \delta x_s = \sigma_t \delta_{ts}$, giving

$$0 = \langle [\hat{\tau} (-1)^F \sum_r \alpha_{ur} \dot{x}_r + \tau \dot{x}_u] \hbar \text{Tr} \sigma_t x_t + \left[ \frac{1}{2} (i_{\text{eff}} , \sigma_t x_t) , x_u \right] - \hbar \omega \sigma_t \rangle_{\text{AV}} \quad \text{(60a)}$$

Referring back to Eqs. (58a–d), we see that Eq. (60a) can be rewritten in terms of $x_{t \text{ eff}}$ as

$$0 = \langle [\hat{\tau} (-1)^F \sum_r \alpha_{ur} \dot{x}_r + \tau \dot{x}_u] \hbar \text{Tr} \sigma_t x_{t \text{ eff}} + \left[ i_{\text{eff}} \sigma_t x_{t \text{ eff}} , x_u \right] - \hbar \omega \sigma_t \rangle_{\text{AV}} \quad \text{(60b)}$$

Taking the effective projection of Eq. (60b) by using Eq. (58d), and then multiplying through by $i_{\text{eff}}$, we get finally

$$0 = \langle i_{\text{eff}} [\hat{\tau} (-1)^F \sum_r \alpha_{ur} \dot{x}_r + \tau \dot{x}_u] \hbar \text{Tr} \sigma_t x_{t \text{ eff}} + \left[ x_u , \sigma_t x_{t \text{ eff}} \right] - i_{\text{eff}} \hbar \omega \sigma_t \rangle_{\text{AV}} \quad \text{(60c)}$$

which coincides in form with the canonical algebra of Eq. (13) (where we had set $\hbar = 1$) when the dynamics dependent terms proportional to $\hat{\tau}$ and $\tau$ are dropped. Since these two
terms are proportional to time derivatives \( \dot{x}_{\text{eff}} \), their neglect should be justified in a regime characterized by energies much lower than the energy scale of the underlying dynamics. (Also, since these two terms are proportional to the graded trace \( \text{Tr}\sigma_t x_t \text{eff} \), they may be further suppressed by boson fermion cancellation.) By repeated applications of Eq. (58d) to the cases in which \( V \) is a successively one degree higher self–adjoint polynomial, one proves similarly from Eq. (55b) that after the \( \hat{\tau} \) and \( \tau \) terms are dropped,

\[
0 = \langle [x_u \text{eff}, V(\{x_r \text{eff}\})] - i_{\text{eff}} \hbar \sum_s \omega_{us} \frac{\delta V(\{x_r \text{eff}\})}{\delta x_s \text{eff}} \rangle_{AV},
\]

(60d)
corresponding in form (when \( \hbar = 1 \)) to the canonical algebra expression comprising the second and third terms of Eq. (29). Similarly, as discussed in Appendix E, by varying the sources in Eq. (60b) one can justify (when \( V \) is Weyl ordered) the evaluation of the commutator in Eq. (60d) in terms of the canonical algebra of Eq. (13) using the Leibnitz product rule.

As a final application of Eq. (55b), we examine its implications for canonical transformations. Replacing \( V \) now by \( G \), with \( G \) the generator of a Weyl ordered intrinsic canonical transformation, comparing with Eq. (27) and dividing by \( \hbar \), and dropping the \( \hat{\tau} \) and \( \tau \) terms, we see that Eq. (55b) takes the form

\[
0 = \langle \left[ \frac{1}{2} \hbar^{-1} \{i_{\text{eff}}, G\}, x_u \right] - \delta x_u \rangle_{AV} .
\]

(61a)

Defining the anti–self–adjoint generator \( \tilde{G}_{\text{eff}} \) by

\[
\tilde{G}_{\text{eff}} = \frac{1}{2} \{i_{\text{eff}}, G\} ,
\]

(61b)

Eq. (61a) can be rewritten in the form

\[
0 = \langle \delta x_u - \hbar^{-1} [\tilde{G}_{\text{eff}}, x_u] \rangle_{AV} ,
\]

(61c)
which after taking the effective projection gives

\[ 0 = \langle \delta x_{u \text{ eff}} - \hbar^{-1}[\tilde{G}_{\text{eff}}, x_{u \text{ eff}}] \rangle_{AV} , \quad (61d) \]

and corresponds in form (when \( \hbar = 1 \)) to the first and third terms of Eq. (29). Equations (61c, d), and their Weyl ordered polynomial generalizations

\[ 0 = \langle \delta P(\{x_r\}) - \hbar^{-1}[\tilde{G}_{\text{eff}}, P(\{x_r\})] \rangle_{AV} , \quad (61e) \]

which are analogous to Eqs. (57d) and (59c), indicate that there is a correspondence between Weyl ordered intrinsic canonical transformations in the underlying generalized quantum dynamics, and unitary transformations \( U_{\text{eff}} \) of the form

\[ U_{\text{eff}} = \exp(\hbar^{-1}\tilde{G}_{\text{eff}}) \quad (62) \]

acting on the variables \( \{x_{r \text{ eff}}\} \). The invariance of \( \iota_{\text{eff}} \) under unitary transformation by \( U_{\text{eff}} \) is an image, in the effective theory, of the invariance of \( \tilde{C} \) under intrinsic canonical transformations in the underlying generalized quantum dynamics.

To summarize, we have shown that there is a striking correspondence between the structure of the set of ensemble averages calculated in generalized quantum dynamics and the structure of canonical quantum field theory. To what specific field theoretic structures do these averages correspond? To answer this question we note that in the absence of sources, the averages of monomials constructed from the phase space operators all are functions, constructed with real number coefficients, solely of the operator \( \iota_{\text{eff}} \) and of the grading operator \( (-1)^F \). (This statement follows from an argument given at the end of Appendix B.) Since the Hamiltonian \( H \) is necessarily bosonic, physical processes cannot change the value of \( (-1)^F \), and so the \( (-1)^F = 1 \) and \( (-1)^F = -1 \) sectors of Hilbert space are superselection sectors, the states of which do not superimpose. Within each superselection sector, the
ensemble averages of monomials are simply linear combinations with real coefficients of the two operators $1$ and $i_{\text{eff}}$; this implies that they are to be identified with matrix elements, rather than operators, in an effective complex field theory in which $i_{\text{eff}}$ plays the role of the imaginary unit. Since for any self–adjoint operator $A$ formed from the phase space operators $\{x_r\}$, reality and positivity of the equilibrium phase space density $\rho$ imply that $\langle A \rangle_{AV}$ is real and $\langle A^\dagger A \rangle_{AV}$ is nonnegative, the ensemble averages must correspond to expectation values in some pure or mixed state in the effective field theory. But since the ensemble in which the averages are formed is Lorentz invariant (recall that in a general Lorentz frame $\hat{H}$ gets replaced by the invariant $[-(\hat{\text{Tr}} p_\mu)^2]^\frac{1}{2}$, and similarly for $H$), and since the vacuum is the only Lorentz invariant state in quantum field theory, it is then natural to identify the ensemble averages with vacuum expectation values in an effective quantum field theory. This leads us to conjecture that in the limit of infinitely many degrees of freedom and an infinite dimensional underlying Hilbert space, \emph{the ensemble averages of Weyl ordered polynomials formed from the canonical phase space operators in generalized quantum dynamics are isomorphic in structure to the vacuum expectation values of the corresponding Weyl ordered polynomials formed from the canonical operators in complex quantum field theory.}

An interesting feature that has emerged from our calculations is that an effective complex structure results irrespective of whether one starts from an underlying real, complex, or quaternionic Hilbert space. In particular, there now seems to be no reason to exclude the aesthetically appealing case of real Hilbert space, since we have given a natural and automatic mechanism for the complexification which is needed to describe the observed physical world. We also note that there is a natural connection between our calculations and matrix models containing $2N$ matrices, which correspond to our $2N$ phase space operators, acting on an $M$
dimensional space of states in the large $M$ limit (conventionally called the “large $N$” limit in the matrix model literature). When the term in the equilibrium ensemble containing $\tilde{\lambda}$ vanishes, the ensemble is invariant under unitary transformations, in which case the large $M$ limit is known to be described by the classical field theory [10, 11, 12] of a so-called “master field”. When the $\tilde{\lambda}$ term is nontrivial it breaks the unitary invariance, raising the interesting possibility that the master field in this case is a complex quantum field.

We make next some observations concerning the use of Weyl ordering in our analysis. It is clear from the discussion of Secs. 2–4 and Appendix E that the proposed isomorphism must fail for certain types of polynomials which are not Weyl ordered. There is evidently a subtlety in the non–Weyl ordered case, an understanding of which will require a more detailed investigation. Since the operator gauge invariant models discussed in Refs. [1, 2] do not have Weyl ordered Hamiltonians, it is important to extend our derivations so that this case (which involves fourth and lower degree polynomials in the phase space operators ) is explicitly included; this issue will be addressed in a separate publication.

Finally, we remark that in generalized quantum dynamics the concept of “temperature” is not defined, because the operator Hamiltonian $H$ is not a constant of the motion: only its graded and (in the class of models studied here) ungraded traces $\text{Tr} H$ and $\hat{\text{Tr}} H$ are conserved. However, in the effective complex theory the effective Hamiltonian is conserved, making possible further fine grained equilibria governed by $H_{\text{eff}}$, and thus leading to the emergence of the standard thermal ensemble $\exp(-\beta H_{\text{eff}})$ parameterized by the temperature $\beta^{-1}$.

7. Conditions for Convergence

The Ward identities derived in the preceding section are meaningful only when the
phase space integrals which appear in them are convergent. As we have already noted, the
generic conserved quantities $\tilde{C}$ and $\tilde{H}$ are both indefinite in sign, and so when standard
statistical mechanical methods are applied to them one finds a partition function which is
divergent. A necessary condition for convergence is that there be at least one constant of
motion which is bounded below, and an obvious candidate for this is the ungraded trace
$\hat{\text{Tr}}H$. A simple criterion can be given for constructing Lagrangians, in which the total trace
equations of motion obtained using the graded trace Lagrangian $\text{Tr}L$ imply conservation of
$\hat{\text{Tr}}H$ as well as conservation of $\text{Tr}H$. Let us consider Lagrangians in which the fermion fields
appear only through one of the standard bilinears of the form $\psi_l^\dagger \ldots \psi_r$; then when bosonic
variables to which the fermions couple are ordered to the outside of the fermion factors, as in
the vector coupling $\psi_l^\dagger \gamma^0 \gamma^\mu \ldots \psi_r B_\mu$ or the scalar coupling $\psi_l^\dagger \gamma^0 \ldots \psi_r B$, the ungraded trace and
graded trace equations of motion are the same and imply conservation of $\hat{\text{Tr}}H$. The reason
is that when the bosons are ordered to the outside of the fermions, only cyclic permutations
of bosonic factors are required to construct the equations of motion for the bosons and for
$\psi_r$, and so these are the same irrespective of whether or not a grading factor is included
inside the trace. The equation of motion for $\psi_l^\dagger$ can then be obtained as the adjoint of one
of the $\psi_r$ equations of motion, or if calculated directly by permuting the factor $\delta \psi_l^\dagger$ to the
right, the same grading factor appears in all terms and so drops out of the $\psi_l$ equation of
motion. Closer examination shows that in the models with a conserved $\hat{\text{Tr}}H$, the equations
of motion also imply conservation of an ungauged fermion number current $\tilde{F}^\mu$, with $\int d^3x \tilde{F}^0$
yielding the conserved charge $\tilde{F}$ discussed briefly in Sec. 5 and in Appendices C and G.

On the other hand, if the bosonic variables to which the fermions couple are ordered
to the inside of the fermion factors, as in the vector coupling $\psi_l^\dagger \gamma^0 \gamma^\mu B_\mu \psi_r$, then $\psi_r$ must be
cyclically permuted to the left in order to obtain the $B_\mu$ equation of motion, resulting in differing signs for the $B_\mu$ source terms in the graded and ungraded cases. These features of the generalized quantum dynamics equations of motion are readily verified by examination of the catalog of models given in Secs. 13.6–13.7 of [2]. In generalizing the Lagrangians for the various gauge theory and Higgs components of the standard model, and its grand unified extensions, so as to give generalized quantum dynamics Lagrangians, one is always free to adopt the ordering convention which leads to conservation of $\hat{\text{Tr}} H$. However, one can readily construct generalized quantum dynamics models in which $\hat{\text{Tr}} H$ is not conserved; an example is the maximally operator gauge invariant model constructed in [1, 2] using two fermion fields, in which the gauge potential $B'_\mu$ is ordered to the outside of the fermion fields and a second gauge potential $B_\mu$ is ordered to the inside, the difference in ordering being precisely what distinguishes the action of the two gaugings. In this case, one readily verifies that the source term for $B_\mu$ changes sign as one goes from the graded to the ungraded equations of motion, while that for $B'_\mu$ has the same sign in both cases.

In addition to the requirement that $\hat{\text{Tr}} H$ should be conserved, it is also necessary that $\hat{\tau} \hat{\text{Tr}} H$ should be bounded below and should dominate over the indefinite terms $\text{Tr} \tilde{\lambda} \tilde{C}$ and $\tau H$, in order for the partition function to converge. In general, $\hat{\text{Tr}} H$ contains three types of terms: bosonic kinetic energy terms, fermionic kinetic energy terms, and potential terms. The bosonic kinetic energy terms in standard field theory models are always positive, and we believe that in many models one will be able to establish that the potential is bounded below. However, because of negative energy states the fermionic kinetic energy terms are problematic. For a single Dirac fermion, before any reorderings of field operators are done,
the kinetic energy (including the mass term) has the form

\[ H_{\text{kin}} = \sum_{\vec{p},s} (\vec{p}^2 + m^2)^{1/2} (b^\dagger_{\vec{p},s} b_{\vec{p},s} - d^\dagger_{\vec{p},s} d_{\vec{p},s}) , \]  

(62a)

with \( m \) the fermion mass. Since Eq. (62a) is the difference of two positive semidefinite terms, \( \text{Tr} H_{\text{kin}} \) is unbounded below when the operators \( b_{\vec{p},s} \) and \( d_{\vec{p},s} \) are independent. However, for a self–conjugate Majorana fermion field one has \( d_{\vec{p},s} = b^T_{\vec{p},s} \), with \( T \) the operator transpose, and so in this case the trace of the second term in Eq. (62a) is

\[ \text{Tr} d_{\vec{p},s} d^\dagger_{\vec{p},s} = \text{Tr} b^T_{\vec{p},s} b^\dagger_{\vec{p},s} = \text{Tr} b^\dagger_{\vec{p},s} b_{\vec{p},s} , \]  

(62b)

and cancels the trace of the first term; thus the trace of the kinetic energy term vanishes and the negative energy catastrophe is avoided. Hence in order for a field theoretic model to be extendable to a generalized quantum dynamics model with a convergent partition function, one must be able to rewrite its fermion kinetic energy terms entirely in terms of Majorana fermions, which requires that every chiral fermion be accompanied by an opposite chirality partner. To complete the discussion of the fermion kinetic energy terms, let us examine the behavior of the graded trace of the fermion kinetic energy. For any two operators \( O_1 \) and \( O_2 \), irrespective of their grade, we have

\[ \text{Tr} O_1^T O_2^T = \text{Re} \sum_{m,n} \epsilon_m O_{1mn}^T O_{2nm} = \text{Re} \sum_{m,n} \epsilon_m O_{1nm} O_{2mn} \]

\( = \text{Re} \sum_{m,n} \epsilon_m O_{2mn} O_{1nm} = \text{Tr} O_2 O_1 \) ;  

(62c)

hence the rearrangement of Eq. (62b) is also valid inside \( \text{Tr} \), and so for Majorana fermions the graded trace of the kinetic energy term also vanishes. [The difference between Eq. (62c) and the cyclic identity of Eq. (2) is that to interchange the untransposed operators one must replace \( \epsilon_m \) by \( \epsilon_n \), and the product \( \epsilon_m \epsilon_n \) is just the grade of the operators \( O_{1,2} \), leading to the \( \pm \) sign in Eq. (2).]
Although a discussion of sufficient conditions for convergence of the partition function is beyond the scope of this paper, let us give one example which suggests that convergence can be attained. Let us work in complex Hilbert space, and consider special simple choices for the ensemble parameter $\tilde{\lambda}$. The simplest possibility, $\tilde{\lambda} = iR$, with $R$ a positive real number, is inadmissible because $i$ is a $c$-number and so $\text{Tr} \tilde{\lambda} \tilde{C}$ reduces to a multiple of $\text{Tr} \tilde{C}$, which is identically zero,

$$\text{Tr} \tilde{C} = \text{Tr} \sum_{r,s} x_r \omega_{rs} x_s = \text{Tr} \sum_{r,s} x_s \epsilon_{rs} \omega_{rs} x_r = -\text{Tr} \sum_{s,r} x_s \omega_{sr} x_r = 0. \quad (63a)$$

So let us consider the next simplest case, which is $\tilde{\lambda} = (-1)^F iR$, for which

$$\text{Tr} \tilde{\lambda} C = \text{Re} \ iR \text{Tr} \tilde{C}$$

$$= \text{Re} \ iR \text{Tr} \left( \sum_{r,B} [q_r, p_r] - \sum_{r,F} \{q_r, p_r\} \right) \quad (63b)$$

$$= - \text{Re} \ iR \text{Tr} \sum_{r,F} \{q_r, p_r\} = R \text{Tr} \sum_{r,F} \{q_r, q_r^\dagger\} ,$$

where in the final line we have assumed that all fermions have been constructed with the standard adjointness assignment $p_r = iq_r^\dagger$. Since both terms in the anticommutator on the right hand side of Eq. (63b) are positive semidefinite, the term $\text{Tr} \tilde{\lambda} \tilde{C}$ is bounded below in this case, and does not have to be dominated by the energy term $\hat{\tau} \text{Tr} H$. So a model in which the ungraded trace energy is conserved, bounded below, and becomes positive infinite on all paths approaching infinity in operator phase space (which our preliminary investigations suggest may be possible for gauge theories) then gives a convergent partition function. Although in this special case the $\tilde{\lambda}$ factor in the exponent has decoupled from the bosons, as long as the bosons and fermions interact the breaking of unitary transformation invariance implied by the presence of $\tilde{\lambda}$ is still felt by the bosons, and the model is satisfactory. For a noninteracting theory, of course, the fermionic and bosonic contributions to $\tilde{C}$ are separately conserved; both
must then be included in the statistical mechanical equilibrium distribution, and the choice \( \lambda = (-1)^F i R \) would then lead to classical mechanics, rather than quantum mechanics, for expectations defined within the bosonic ensemble.

8. Discussion

In the foregoing, we have given strong evidence that the canonical quantization rules, which are the basis of conventional quantum mechanics and quantum field theory, can arise as an emergent property in a generalized quantum operator dynamics when statistical mechanical methods are applied to the ensemble of operator initial values. In models where the index \( r \) is a composite index composed of a spatial coordinate \( \vec{x} \) as well as a discrete field index, the emergent canonical algebra implies locality, even though the underlying generalized quantum dynamics is highly nonlocal. To our knowledge, this is the first time that an embedding of quantum mechanics in a larger structure has been achieved that applies to local relativistic quantum field theories.

An interesting feature of the Ward identity applications of Sec. 6 is that the derivation of a unitary effective dynamics requires a less stringent approximation (the replacement of \( H \) by its ensemble average) than does the derivation of the canonical algebra (where a “low” frequency approximation is needed). Thus, there may be a high energy domain where the statistical mechanical analysis is still valid and takes the form of a nonlocal but unitary complex effective field dynamics, such as a string–type theory. This effective theory would still be only a statistical approximation to the fully nonlocal underlying generalized quantum dynamics, and its structure may well prove more intractable than that of the underlying theory.

An exciting aspect of our construction is that the principal features of quantum
mechanics basically become statements about the geometry of matrices. Thus, if in labeling
the rows and columns of matrix operators one orders all the bosonic states before all the
fermionic states, then the distinction between bosonic and fermionic operators is simply that
between matrices that are block diagonal, and ones that are block skew diagonal, respectively.
Furthermore, as we have seen in Sec. 6, the complex structure, canonical algebra, and
unitary dynamics of quantum mechanics are all reflections of the cyclic invariance of the
trace, which is both the origin of the conserved operator $\hat{C}$ and the basis for erecting a
generalized dynamics on noncommutative phase space. These observations suggest that the
distinction between matter degrees of freedom on the one hand, and gravitational or metric
degrees of freedom on the other, may be similarly rooted in some simple geometric property
of generalized quantum dynamics.

If our conjectured isomorphism can be proved, and if a generalized quantum dynam-
ics underlies the observed universe, there will be profound implications for some of the vexing
issues in conventional quantum mechanics. One of these issues is the quantum measurement
problem. In the underlying generalized dynamics there are no “dice”: the underlying dynam-
ics is a generalization of classical mechanics to noncommuting phase space operators and is
deterministic, although not in general unitary. However, the ability to follow this determin-
istic evolution in detail is lost at the level of the statistical ensemble average, where a unitary
conventional quantum mechanics emerges. In this picture, a calculation of corrections to the
ensemble average approximation should permit the resolution of the troubling “paradoxes” of
quantum measurement theory. A second issue where our picture has important ramifications
is the problem of infinities in quantum field theory. These divergences arise, fundamentally,
because of the singularity of the local canonical commutator/anticommutator structure of
field theory, which we have argued is an emergent property of ensemble averages in generalized quantum dynamics. The underlying dynamics is nonlocal and nonsingular, and should give finite answers to physical calculations.

There are clearly a number of important questions that must be addressed in future work. One of them is to give a detailed justification of the statistical mechanical aspects of our calculation, including a complete classification of Hamiltonians for which one can prove convergence of the partition function, a proof that a thermodynamic limit exists, and a justification of the assumption that each unvaried factor of $\tilde{C}$ (and of other conserved extensive quantities) in the Ward identity can be replaced by the corresponding ensemble average. The emergence of quantum mechanics from the Ward identity required the neglect of the $\hat{\tau}$ and $\tau$ terms, which may be reasonable only in models that have a very large ratio of the high mass scale characterizing pre-quantum mechanical physics to the low mass scale characterizing quantum physics, and possibly also a high degree of boson–fermion symmetry as well. Thus, finding a model in which neglect of the dynamics dependent terms can be justified may be tantamount to finding a model that solves the “hierarchy problem” of explaining the extraordinarily large ratio between the Planck mass scale and the standard model mass scale in particle physics. Since the entire generalized quantum dynamics formalism seems to naturally invite the incorporation of boson–fermion symmetries, such as generalized forms of supersymmetry, it will be important to analyze such symmetries. A possibly related question is to determine what happens when the numbers $N_B$, $N_F$ of bosonic and fermionic degrees of freedom are equal. This issue arises because if it were evaluated by restriction to the canonical algebra, the conserved operator $\tilde{C}$ would be equal (with $h=1$) to $i_{\text{eff}}(N_B - N_F)$, which vanishes when $N_B = N_F$. This suggests that the conditions for validity
of our analysis may be particularly delicate – and interesting – in models with a high degree of boson–fermion symmetry. Yet another question is what happens in constrained theories, such as the operator gauge invariant theories formulated in Refs. [1, 2]. In canonical gauges, where the constraints can be explicitly eliminated, our analysis should apply directly to the Hamiltonian restricted to the constraint surface, which involves only the physical degrees of freedom. However, there is likely to be an analog in generalized quantum dynamics of the methods used to treat constrained quantum field theories (such as Faddeev-Popov determinants and BRST invariance), which would permit working in non–canonical gauges as well, and it would be interesting, and possibly important, to find it. Beyond these basically technical questions is of course the larger issue of whether one can incorporate gravitation in a natural way, and whether one can find a compellingly beautiful total trace dynamics which gives rise to all of the observed forces and matter fields.

Acknowledgments

This work was supported in part by the Department of Energy under Grant #DE–FG02–90ER40542. One of the authors (S. L. A.) wishes to acknowledge the hospitality of the Aspen Center for Physics, and both the Department of Applied Mathematics and Theoretical Physics and Clare Hall at Cambridge University, where parts of this work were done. He also wishes to thank S. B. Treiman for suggesting the study of canonical transformations, F. J. Dyson for expressing strong reservations with approximations employed in the initial formulation of this work, Y. Suhov for several important conversations about statistical mechanics, and A. Kempf and G. Mangano for asking about corrections to the canonical algebra. The other author (A. C. M.) wishes to thank T. Tao for valuable discussions.
Appendix A. Inclusion of Adjointness Restrictions in the Argument for Invariance of the Phase Space Measure

Inspection of the argument of Eqs. (39–41) shows that the diagonal ($m = n$) and off–diagonal ($m \neq n$) terms in the sum $\Sigma$ vanish separately, and for each of these, the summed contribution from the canonical coordinate and momentum pair $q_r, p_r$ for each fixed $r$ also vanishes separately. This observation permits us to take the adjointness restrictions into account; in the following discussion we shall write $d\mu = d\mu_B d\mu_F$, with $d\mu_B$ and $d\mu_F$ respectively the bosonic and fermionic integration measures. There are three cases to be considered:

(1) For a bosonic pair of phase space variables $q_r, p_r$, the $x_r$ variables are independent but are both self–adjoint, and thus

\[(x_r)^A_{mn} = \epsilon^A (x_r)^A_{nm}.\]  

(A1a)

This means that the integration measure must be redefined to include all diagonal terms in $m, n$, but only the upper diagonal off–diagonal terms, so that the bosonic integration measure becomes

\[d\mu_B = \prod_A d\mu^A_B,\]

\[d\mu^A_B \equiv \prod_{r,m \leq n} d(x_r)^A_{mn}.\]  

(A1b)

The argument for the diagonal terms in this product proceeds just as did that for the diagonal terms in the unrestricted case, while the argument for the off–diagonal terms uses Eq. (A1a) in place of an interchange of the summation index pair $m, n$, together with the fact that for a boson $\epsilon_r = 1$, again leading to the conclusion that the diagonal and off–diagonal contributions to $\Sigma$ vanish independently.
(2) For a fermionic pair of phase space variables in complex Hilbert space constructed according to the recipe \( q_r = \psi_r, \ p_r = i\psi_r^\dagger = iq_r^\dagger \), the \( x_r \) variables are no longer independent. However, this construction implies that

\[
(q_r)_r^{1, mn} = (p_r)_r^{0, mn}, \quad (p_r)_r^{1, mn} = (q_r)_r^{0, mn},
\]

and thus, in a complex Hilbert space, the fermionic integration measure must be redefined as

\[
d\mu_F = d\mu_F^0, \quad d\mu_F^0 \equiv \prod_{r, m, n} d(x_r)_r^{0, mn}.
\]

Similarly, for the analogous fermionic construction \( p_{r1} = \frac{1}{2} \psi_{r1}^\dagger I, ... \) in a quaternionic Hilbert space, the fermionic integration measure must be redefined as

\[
d\mu_F = \prod_{A=0}^{2} d\mu_F^A, \quad d\mu_F^A \equiv \prod_{r, m, n} d(x_{r1})_r^{A, mn}.
\]

Since the argument for the unrestricted case worked for each \( A \) value separately, it still goes through as before.

(3) Finally, for a fermionic pair of phase space variables constructed using a real representation of the imaginary unit, with a pair of fermions for each \( r \) obeying \( q_{r2} = p_{r1}^\dagger, \ p_{r2} = -q_{r1}^\dagger \) but with no relation between \( p_{r1} \) and \( q_{r1} \), one simply omits the variables \( x_{r2} \) from \( d\mu_F \) and uses the unrestricted form of the fermionic measure for the variables \( x_{r1} \), and the invariance argument then proceeds just as before.
Appendix B. Canonical Form of $\langle \hat{C} \rangle_{AV}$ and Implications for the Structure of $x_r$ and $\tilde{\lambda}$

We give here the decomposition of the phase space operators $\{x_r\}$ with respect to the canonical form for $\langle \hat{C} \rangle_{AV}$ given in Eq. (46a) and the relationship this implies between $x_r$ and $x_r^{\text{eff}}$. We also discuss the canonical form for $\tilde{\lambda}$ which corresponds to that for $\langle \hat{C} \rangle_{AV}$.

1. **Real Hilbert space.** In real Hilbert space, the anti–self–adjoint operator $\langle \hat{C} \rangle_{AV}$ is skew symmetric, and when the Hilbert space is even dimensional can be brought by a real unitary (i.e., orthogonal) transformation to the canonical form $\langle \hat{C} \rangle_{AV} = i_2 \otimes C_d$, with $C_d$ a real diagonal matrix and with $i_2$ the $2 \times 2$ skew symmetric matrix

\[
i_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{B1}\]

(When the Hilbert space is odd dimensional the canonical form consists of a block of the form $i_2 \otimes C_d$, and one further element 0 on the principal diagonal which does not correspond to a symplectic structure.) The matrix $i_2$ spans a two dimensional real Hilbert subspace, and a complete set of operators [2] in this subspace can be taken as $1_2$, $i_2$, $W$ and $Wi_2$, with

\[
1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad Wi_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} , \quad \tag{B2}\]

from which one sees that $i_2$ and $W$ anticommute. The operator $x_r$ can now be expanded over the operator basis provided by Eqs. (B1, B2), with coefficients $x_{rA}$, $A = 0, 1, 2, 3$ which are still operators but which commute with $i_2$ and $W$,

\[
x_r = x_{r0} 1_2 + x_{r1} i_2 + x_{r2} W + x_{r3} Wi_2 . \tag{B3a}\]

It is convenient to rewrite this expansion in terms of “complex” components denoted by
\( x_{ra}, x_{rb} \) which both commute with \( i_2 \), according to

\[
x_r = x_{ra} + W x_{rb},
\]
\[
x_{ra} = x_{r0} 1_2 + x_{r1} i_2, \quad x_{rb} = x_{r2} + x_{r3} i_2.
\]  

(B3b)

Writing now \( C_d = \mathcal{E} D \), with \( D \) nonnegative and with \( \mathcal{E} \) a diagonal matrix with elements \( \pm 1 \), and writing \( i_{\text{eff}} = i_2 \mathcal{E} \) (with a direct product \( \otimes \) understood), we find

\[
x_{r \text{ eff}} = \frac{1}{2} \left[ x_{ra} + W x_{rb} - i_2 \mathcal{E} (x_{ra} + W x_{rb}) i_2 \mathcal{E} \right] = \frac{1}{2} \left[ x_{ra} + \mathcal{E} x_{ra} \mathcal{E} + W (x_{rb} - \mathcal{E} x_{rb} \mathcal{E}) \right],
\]  

(B4a)

which reduces in the special case when \( \mathcal{E} = 1 \) to

\[
x_{r \text{ eff}} = x_{ra}.
\]  

(B4b)

(2) **Complex Hilbert space.** In complex Hilbert space an anti–self–adjoint operator can always be written as the \( c \)-number \( i \) times a self–adjoint operator, and a self–adjoint operator can always be brought by a complex unitary transformation to real diagonal form. So we have the canonical form \( \langle \tilde{C} \rangle_{AV} = i C_d \); writing \( C_d = \mathcal{E} D \), with \( D \) nonnegative and with \( \mathcal{E} \) again a diagonal matrix with elements \( \pm 1 \), and writing \( i_{\text{eff}} = i \mathcal{E} \), we have

\[
x_{r \text{ eff}} = \frac{1}{2} (x_r - i \mathcal{E} x_r i \mathcal{E}) = \frac{1}{2} (x_r + \mathcal{E} x_r \mathcal{E}),
\]  

(B5a)

which reduces in the special case when \( \mathcal{E} = 1 \) to

\[
x_{r \text{ eff}} = x_r.
\]  

(B5b)

(3) **Quaternionic Hilbert space.** In quaternionic Hilbert space the spectral analysis for an anti–self–adjoint operator differs in a nontrivial way from that for a self–adjoint operator
(see [2] for a detailed discussion and references), and implies that by a quaternion unitary
transformation, \( \langle \tilde{C} \rangle_{AV} \) can be brought to the form \( ID \), with \( I \) and \( D \) commuting operators
of the form
\[
I = \sum_n |n\rangle i\langle n| , \\
D = \sum_n |n\rangle D_n \langle n| ,
\]
and with \( D_n \) real and nonnegative. We adjoin to this set of operators the additional two
operators \( J, K \), chosen to commute with \( D \) and to form a quaternion algebra with \( I \); one
possible (but not unique) choice for these operators is
\[
J = \sum_n |n\rangle j\langle n| , \\
K = \sum_n |n\rangle k\langle n| ,
\]
with \( i, j, k \) quaternion scalars. [The \( J, K \) of Eq. (B6b) commute with \( (-1)^F \). In a Hilbert
space with equal numbers of bosonic and fermionic states, so that they can be put into one
to one correspondence, it is easy to construct an alternative set of operators \( J, K \) which
anticommute with \( (-1)^F \).] Let us now expand the operator \( x_r \) over the basis \( 1, I, J, K \)
with formally real expansion coefficients \( x_{rA} \), \( A = 0, 1, 2, 3 \), which commute with the entire
\( I, J, K \) quaternion algebra (the theory of this is explained in detail in [2]), giving
\[
x_r = x_{r0} + x_{r1}I + x_{r2}J + x_{r3}K . \tag{B7a}
\]
It is convenient to rewrite Eq. (B7a) in terms of formally complex so-called symplectic
components \( x_{ra}, x_{r\beta} \) which both commute with \( I \), according to
\[
x_r = x_{ra} + Jx_{r\beta} , \\
x_{ra} = x_{r0} + x_{r1}I , \quad x_{r\beta} = x_{r2} - x_{r3}I . \tag{B7b}
\]
Writing now \( i_{eff} = I \), we get
\[
x_{r\ eff} = \frac{1}{2}[x_{ra} + Jx_{r\beta} - I(x_{ra} + Jx_{r\beta})I] = \frac{1}{2}[x_{ra}(1 + 1) + Jx_{r\beta}(1 - 1)] = x_{ra} . \tag{B8}
\]
Although a canonical form for $\tilde{\lambda}$ is not needed for the Ward identity derivation of Sec. 6, it is nonetheless instructive to examine the implications that the canonical form for $\langle \tilde{C} \rangle_{AV}$ has for $\tilde{\lambda}$. According to Eq. (45b), there is a function $F$ for which

$$\langle \tilde{C} \rangle_{AV} = F(\tilde{\lambda}, (-1)^F),$$

which implies the inverse functional relation

$$\tilde{\lambda} = G(\langle \tilde{C} \rangle_{AV}, (-1)^F),$$

$$= G_1(\langle \tilde{C} \rangle_{AV}, (-1)^F) + \langle \tilde{C} \rangle_{AV} G_2(\langle \tilde{C} \rangle_{AV}, (-1)^F),$$

where $G_1$ and $G_2$ are even functions of the argument $\langle \tilde{C} \rangle_{AV}$. When we specialize to the form $\langle \tilde{C} \rangle_{AV} = i_{eff} D$, with $D$ a real positive $c$–number, as assumed in Eq. (46a), the functions $G_{1,2}$ reduce to $c$–number functions of $(-1)^F$. Moreover, $G_1$ must be anti–self–adjoint, which implies that it must vanish in real and quaternionic Hilbert spaces, giving in these cases the canonical form

$$\tilde{\lambda} = i_{eff} D [G_0^0 + G_1^1 (-1)^F],$$

with $G_{0,1}^{0,1}$ real constants.

In complex Hilbert space, $i$ is an anti–self–adjoint $c$–number, so we get the canonical form

$$\tilde{\lambda} = i [G_0^0 + G_1^1 (-1)^F] + i_{eff} D [G_0^0 + G_1^1 (-1)^F],$$

with $G_{0,1}^{0,1}$ real constants. To achieve a further simplification in this case, we assume that $\hat{H}$ and $H$ can be expressed in terms of the phase space operators $\{x_r\}$ using only real number coefficients. This implies that $\langle \tilde{C} \rangle_{AV}$, and hence $i_{eff}$, can depend on $i$ only through $\tilde{\lambda}$, and conversely, that $\tilde{\lambda}$ can depend on $i$ only through $i_{eff}$; then the function $G_1$ must vanish, and
Eq. (B10b) reduces to the simpler form given in Eq. (B10a). More generally, this implies that in complex Hilbert space, ensemble averages of monomials constructed from the phase space variables can depend on $i$ only through $i_{\text{eff}}$, a result used at the end of Sec. 6.

**Appendix C. Evaluation of the $\delta x_s$ Variations of $\tilde{C}$, $H$, $\hat{H}$, and $\tilde{F}$**

Varying the definition

$$\tilde{C} = \sum_{r,s} x_r \omega_{rs} x_s \quad (C1a)$$

with respect to $x_s$, we get

$$\delta_{x_s} \tilde{C} = \sum_r (\delta x_s \omega_{sr} x_r + x_r \omega_{rs} \delta x_s) \quad , \quad (C1b)$$

which by Eq. (10c) becomes

$$\delta_{x_s} \tilde{C} = \sum_r \omega_{rs} (x_r \delta x_s - \delta x_s \epsilon_r x_r) \quad . \quad (C1c)$$

Varying $H$ with respect to $x_s$ and using the definition of operator derivative, we get

$$\delta_{x_s} H = \text{Tr} \frac{\delta H}{\delta x_s} \delta x_s \quad . \quad (C2a)$$

Applying Eq. (10c) to Eq. (9b) then gives

$$\delta_{x_s} H = \text{Tr} \sum_r \dot{x}_r \omega_{rs} \delta x_s \quad . \quad (C2b)$$

We turn our attention next to $\hat{H} = \hat{\text{Tr}} H = \text{ReTr} H$, which is naturally conserved under the trace dynamics generated by $\hat{L} = \hat{\text{Tr}} L = \text{ReTr} L$, where $L$ is the same self-adjoint operator Lagrangian as appears in the graded total trace Lagrangian $L = \text{Tr} L = \text{ReTr}(-1)^F L$. Since we are assuming that $\hat{H}$ is conserved under the equations of motion for
the graded trace dynamics, the Euler–Lagrange equations for \( \hat{L} \) and \( L \) must agree, up to an overall sign \( \eta_r = \pm 1 \) which can be chosen independently for each degree of freedom,

\[
\frac{\delta \hat{L}}{\delta q_r} = \hat{p}_r = \eta_r \frac{\delta L}{\delta \dot{q}_r} = \eta_r p_r \, , \\
\frac{\delta \hat{L}}{\delta \dot{q}_r} = \eta_r \frac{\delta L}{\delta q_r} .
\]

(C3)

In fact, for Lagrangians in which fermion time derivative terms all have the structure \( \psi^\dagger \psi \), no cyclic permutation is involved in varying with respect to \( \dot{\psi}_r \) since this already stands to the right, and so \( \eta_r = 1 \) for all \( r \) and \( \hat{p}_r = p_r \), a result that will be assumed henceforth.

Making a Legendre transformation from \( \hat{L} \) to \( \hat{H} \),

\[
\hat{H} = \hat{\text{Tr}} \sum_r p_r \dot{q}_r - \hat{L} \, , 
\]

(C4a)

we find

\[
\delta \hat{H} = \hat{\text{Tr}} \sum_r (\dot{q}_r \delta p_r - \dot{p}_r \delta q_r) .
\]

(C4b)

Thus the variation of \( \hat{H} \) is given by

\[
\delta x_s \hat{H} = \hat{\text{Tr}} \sum_r (\dot{q}_r \delta x_s p_r - \dot{p}_r \delta x_s q_r) = \text{Tr}(-1)^F \sum_r \hat{x}_r \hat{\omega}_{rs} \delta x_s ,
\]

(C5a)

with \( \hat{\omega} = \text{diag}(\Omega_B, \Omega_B, ..., \Omega_B) \) in the notation of Eqs. (10a, b). Forming the sum which is needed in the Ward identity derivation, we find

\[
\sum_s \omega_{us} \hat{\omega}_{rs} \equiv \alpha_{ur} \, ,
\]

(C5b)

\[
\alpha = \text{diag}(1_2, ..., 1_2, -W, ..., -W) \, ,
\]

with \( 1_2 \) (for the bosonic variables) and \( W \) (for the fermionic variables) as defined in Eq. (B2).

Carrying out an analog of the discussion of Sec. 3, now using the ungraded trace Hamiltonian \( \hat{H} \), we find a second conserved operator \( \hat{C} \) given by

\[
\hat{C} = \sum_{r,s} x_r \hat{\omega}_{rs} x_s = \sum_{r,B} [q_r, p_r] + \sum_{r,F} [q_r, p_r] .
\]

(C6a)
Defining the auxiliary conserved operator \( \tilde{F} = -\frac{1}{2}(\tilde{C} - \hat{\tilde{C}}) \), we have

\[
\tilde{F} = \sum_{r,F} q_r p_r ,
\]

which when restricted to the canonical algebra is (up to an additive constant) an anti–self–adjoint version of the fermion number operator. To take \( \tilde{F} \) into account, we include a term \( \text{Tr} \tilde{\kappa} \tilde{F} \), or equivalently, a term \( \hat{\text{Tr}} \hat{\tilde{\lambda}} \hat{\tilde{C}} \), in the exponent of the equilibrium distribution; the latter makes the algebraic calculations more symmetric and is thus more convenient. We now make the essential assumption that the ensemble averages of \( \tilde{F} \) and \( \hat{\tilde{C}} \) are functions solely of the ensemble average of \( \tilde{C} \) and of the grading operator \((-1)^F\), or equivalently, that the anti–self–adjoint operator ensemble parameters \( \tilde{\kappa} \) and \( \hat{\tilde{\lambda}} \) are functions solely of \( \hat{\lambda} \) and of \((-1)^F\), and thus commute with \( \hat{\lambda} \). The validity of this assumption is demonstrated in Appendix G, where we study implications of the full Ward identity structure. A straightforward calculation shows that

\[
\delta_{x_s} \hat{\text{Tr}} \hat{\tilde{\lambda}} \hat{\tilde{C}} = \text{Tr}(-1)^F[\hat{\tilde{\lambda}}, \sum_r \hat{\omega}_{rs} x_r] \delta x_s ,
\]

which has a similar commutator structure to the relation

\[
\delta_{x_s} \text{Tr} \hat{\lambda} \hat{\tilde{C}} = \text{Tr}[\hat{\lambda}, \sum_r \omega_{rs} x_r] \delta x_s
\]

deduced from Eq. (C1c). Consequently, the commutativity of the ensemble parameters allows us to use the argument of Eqs. (54a–c) to conclude that the variation of the \( \hat{\tilde{C}} \) term in the equilibrium distribution does not contribute to the Ward identity. The functional relation assumption also implies the continuing validity of the other parts of our analysis that depended on the structure of \( \hat{\lambda} \). Another interesting consequence of commutativity of the ensemble parameters is that it implies a vanishing generalized Poisson bracket of the
\( \tilde{C} \) term in the equilibrium distribution with the \( \hat{C} \) term, with the result that the four first integrals appearing in the equilibrium distribution all have vanishing generalized Poisson brackets with one another. Inclusion of the \( \tilde{C} \) term in the equilibrium distribution of course also implies that this term now appears, in a role analogous to that of the \( \tilde{C} \) term, in the thermodynamic expressions of Eqs. (49a–d).

Appendix D. Proof of the Lemma of Sec. 6

We restate and then prove the Lemma used in Sec. 6 to take account of adjointness restrictions on the variations.

Lemma:

Let \( Y_1 \) and \( Y_2 \) be two self–adjoint bosonic or two anti–self–adjoint bosonic operators constructed from the phase space variables. Then in \( 0 = \delta \text{Tr} Y_1 Y_2 \), the self–adjointness restrictions on the variations can be ignored.

Proof:

By the cyclic property of \( \text{Tr} \), we have

\[
\text{Tr} Y_1 Y_2 = \text{Tr} Y ,
\]

\[
Y \equiv \frac{1}{2} (Y_1 Y_2 + Y_2 Y_1) = Y^\dagger ,
\]

(D1)

and so it suffices to prove that in \( 0 = \delta \text{Tr} Y \) with manifestly self–adjoint \( Y \), the self–adjointness restrictions on the variations can be ignored.

We consider first the case of the variation of a bosonic variable \( x_r \), for which the self–adjointness of \( x_r \) implies that \( (\delta x_r)^\dagger = \delta x_r \). Self–adjointness of \( Y \) implies that for each term in \( Y \) of the form \( O_L x_r O_R \) there must be a corresponding term \( O_R^\dagger x_r O_L^\dagger \), with the grade \( \epsilon_L \) of \( O_L \) equal to the grade \( \epsilon_R \) of \( O_R \) for there to be a nonvanishing graded trace. The summed

---

60
contribution of the two terms when \( x_r \) is varied is

\[
\text{Tr} \epsilon_R(O_R O_L + O^\dagger_L O_R^\dagger) \delta x_r \ ,
\]

(D2)

and since the coefficient of \( \delta x_r \) is manifestly self–adjoint we are justified in equating the coefficient of \( \delta x_r \) to zero.

We consider next the variation of a fermionic variable \( x_r \), for which there is another fermionic variable \( x_s(r) \) for which \( x_r^\dagger = c_r x_s(r) \) and \( x_s^\dagger(r) = c_r x_r \), with \( c_r \) a \( c \)–number of unit magnitude with conjugate \( \overline{c_r} \), so that \( c_r \overline{c_r} = \overline{c_r} c_r = 1 \). (The methods for including fermions described in Sec. 3 take this form with either \( c_r = -i \) or \( c_r = \pm 1 \).) The corresponding variations must thus be related by the self–adjointness restriction \( \delta x_r^\dagger = c_r \delta x_s(r) \). Self–adjointness of \( Y \) now implies that for each term in \( Y \) of the form \( O_L x_r O_R \) there must be a corresponding term \( O^\dagger_R c_r x_s(r) O^\dagger_L \), with the grade \( \epsilon_L \) of \( O_L \) opposite to the grade \( \epsilon_R \) of \( O_R \) for there to be a nonvanishing graded trace. The summed contribution of the two terms when \( x_r \) is varied is then

\[
\text{Tr} (\epsilon_R O_R O_L \delta x_r + \epsilon_L O^\dagger_L O^\dagger_R c_r \delta x_s(r)) \ .
\]

(D3)

The self–adjointness restriction on the variations implies that the second term in Eq. (D3) is equal to

\[
\text{Tr} \epsilon_L O^\dagger_L O^\dagger_R \delta x_r^\dagger \ ,
\]

(D4)

which using the fact that \( \text{Tr} \) of any operator is equal to \( \text{Tr} \) of the adjoint of the same operator, is equal to

\[
\text{Tr} \epsilon_L \delta x_r O_R O_L = \text{Tr} (-\epsilon_L) O_R O_L \delta x_r = \text{Tr} \epsilon_R O_R O_L \delta x_r \ ,
\]

(D5)

which just doubles the contribution from the first term in Eq. (D3). Hence we get the correct
answer by equating the coefficients of $\delta x_r$ and $\delta x_s(r)$ independently to zero in Eq. (D3).

Appendix E. Use of the Sources to Generate the Polynomial

$P(\{x_r\})$ in Eqs. (57d) and (61e)

We show here that by variation of the sources in Eqs. (57c) and (61c), one obtains Eqs. (57d) and (61e), in which $\frac{\partial}{\partial t}$, $\delta$, $[\tilde{H}_{\text{eff}}, \ ]$, and $[\tilde{G}_{\text{eff}}, \ ]$ all act on the Weyl ordered polynomial $P(\{x_r\})$ by the Leibnitz product rule. Our argument also applies to Eq. (60b) (after dropping the $\hat{\tau}$ and $\tau$ terms), and shows that when $V$ in Eq. (60d) is Weyl ordered, the commutator appearing in Eq. (60d) can be evaluated in terms of the canonical algebra of Eq. (13) by the Leibnitz product rule. Representing Eq. (57c), Eq. (60b) (with the parentheses indicating implicit summation restored and $\sigma_t$ replaced by $\sigma'_t$, so that the terms inside the expectation read $[i_{\text{eff}}(\sigma'_t x_t x_{\text{eff}}), x_u] - \hbar \omega \sigma'_t$), and Eq. (61c), after multiplication through by the partition function $Z$, by the generic structure

$$0 = Z\langle D x_u \rangle_{AV}, \quad (E1a)$$

we wish to show that by varying the sources in Eq. (E1a) we can also derive

$$0 = Z\langle DP(\{x_r\}) \rangle_{AV}, \quad (E1b)$$

where $D$ acts on the Weyl ordered polynomial $P$ by the Leibnitz product rule

$$D(x_r x_s) = (D x_r)x_s + x_r(D x_s). \quad (E1c)$$

We begin by observing that Eq. (E1a) also implies that

$$0 = Z\langle D(\sigma_u x_u) \rangle_{AV}, \quad (E2a)$$
and we shall work with the self-adjoint bosonic variables \((\sigma_u x_u)\) henceforth. Multiplying Eq. (E2a) by \(\text{Tr}\delta \rho_u\), with \(\delta \rho_u\) an arbitrary self-adjoint bosonic operator, Eq. (E2a) becomes

\[
0 = Z \langle \text{Tr}\delta \rho_u \mathcal{D}(\sigma_u x_u) \rangle_{AV} , \tag{E2b}
\]

which we take as the starting point for our discussion. Writing \(P(\{x_r\}) = S[\{\sigma_r x_r\}]\), with \(S\) a totally symmetrized polynomial in its arguments, application of Eq. (E1c) gives

\[
\mathcal{D} S[\{\sigma_r x_r\}] = \sum_u S[\{\sigma_r x_r, r \neq u\}; \mathcal{D}(\sigma_u x_u)] , \tag{E2b}
\]

where the sum over \(u\) ranges over the indices of all variables \(x_r\) which appear as arguments of \(P\). Hence to derive Eq. (E1b) it suffices to derive

\[
0 = Z \langle \sum_u S[\{\sigma_r x_r, r \neq u\}; \mathcal{D}(\sigma_u x_u)] \rangle_{AV} . \tag{E3}
\]

We begin by varying Eq. (E2b) with respect to the source corresponding to each \(r \neq u\) (if a variable \(x_R\) appears multiple times, we perform multiple independent variations with respect to its source \(\rho_R\)), and after taking these variations, then summing over all choices of \(u\) from the among the indices appearing in \(P\). There are two types of source dependence which contribute: there are the source terms in the equilibrium distribution of Eq. (51a) that we use to form the ensemble averages, and also the explicit source term in the Ward identity that was suppressed in writing Eqs. (57c), (60b), (61c), and (E2b). From Eq. (55b) we see that the contribution to the summand of this latter term, after left multiplication by \(\text{Tr}\delta \rho_u \sigma_u\), is equal to

\[
Z \langle \text{Tr}\delta \rho_u \sum_s \omega_{us} \sigma_u \sigma_s \rho_s \hbar V \rangle_{AV} , \tag{E4a}
\]

with \(V\) appropriate to one of the applications discussed in the text. When Eq. (E4a) is
varied with respect to the source $\rho_v$, it contributes

$$Z \langle \text{Tr} \delta \rho_u \omega_{uv} \sigma_u \sigma_v \delta \rho_v \hat{h} V \rangle_{AV}, \quad (E4b)$$

which can be rewritten as

$$Z \langle \text{Tr} \delta \rho_u \delta \rho_v \hat{h} V \rangle_{AV} \omega_{uv} \sigma_u \sigma_v. \quad (E4c)$$

Now for each term with the form of Eq. (E4c) in the sum over $u$, symmetrization implies that there is a corresponding term with the roles of $u$ and $v$ interchanged, giving a total contribution of

$$Z \langle \text{Tr} \delta \rho_u \delta \rho_v \hat{h} V \rangle_{AV} (\omega_{uv} \sigma_u \sigma_v + \omega_{vu} \sigma_v \sigma_u). \quad (E4c)$$

But this vanishes because for bosonic $u,v$, the auxiliary quantities $\sigma_u$ and $\sigma_v$ commute and $\omega_{uv} + \omega_{vu} = 0$, while for fermionic $u,v$, we have $\omega_{uv} = \omega_{vu}$ and $\{\sigma_u, \sigma_v\} = 0$. Hence the explicit source term in the Ward identity makes no contribution to symmetrized expressions.

In the remaining terms in the sum over $u$, the variations $\delta \rho_u$ and $\delta \rho_v$ each appear in a separate graded trace. Thus, after implementing the cancellation of Eq. (E4c), it suffices to show that we can derive the generic term in the summand of Eq. (E3) by operations on a product of source variation factors, since once the generic term has the correct symmetrized polynomial form, all terms in the sum over $u$ are guaranteed to have this form.

At this point let us take advantage of the fact that the $\delta \rho$ are all arbitrary self–adjoint bosonic operators, permitting us to replace them by $(-1)^F \delta \rho$, with the $\delta \rho$ again arbitrary self–adjoint bosonic operators, thereby converting all graded traces involving the source variations to ungraded traces. We are thus left with the simpler problem of deriving

$$0 = Z \langle S[\{(\sigma_r x_r), r \neq u\}; D(\sigma_u x_u)] \rangle_{AV}, \quad (E5a)$$
given the identity

\[ 0 = Z \langle \prod_{r \neq u} [\text{ReTr}(\sigma_r x_r) \delta \rho_r] \text{ReTr} \delta \rho_u \mathcal{D}(\sigma_u x_u) \rangle_{AV} \quad . \]  

(E5b)

To do this, we exploit the arbitrariness of the variations \( \delta \rho \), as follows. Let \( \Lambda_A \) be a complete basis of trace normalized bosonic self–adjoint operators, that is, this basis is characterized by the properties that

\[ \text{Tr} \Lambda_A \Lambda_B = \delta_{AB} \quad , \]  

(E6a)

and that any bosonic self–adjoint operator \( \mathcal{O}_1 \) can be expanded in the form

\[ \mathcal{O}_1 = \sum_A \mathcal{O}_{1A} \Lambda_A \quad , \quad \mathcal{O}_{1A} = \text{ReTr} \mathcal{O}_1 \Lambda_A \quad , \]  

(E6b)

which implies the formula

\[ \text{ReTr} \mathcal{O}_1 \mathcal{O}_2 = \sum_A \text{ReTr} \mathcal{O}_1 \Lambda_A \text{ReTr} \mathcal{O}_2 \Lambda_A \quad . \]  

(E6c)

Now let us take \( \delta \rho_R = \{ \Lambda_A, \kappa \} \) and \( \delta \rho_S = \Lambda_A \) in Eq. (E5b), with \( \kappa \) an arbitrary bosonic self–adjoint operator, and sum over \( A \). By Eq. (E6c) this leads to the replacement of the product of factors \( \text{ReTr} \mathcal{O}_R \delta \rho_R \text{ReTr} \mathcal{O}_S \delta \rho_S \) with the single factor

\[ \text{ReTr} \{ \mathcal{O}_R, \kappa \} \mathcal{O}_S = \text{ReTr} \kappa \{ \mathcal{O}_R, \mathcal{O}_S \} \quad , \]  

(E7a)

which involves the symmetrized (or Jordan) product of the two operators \( \mathcal{O}_{R,S} \). Proceeding in this fashion, and using the freedom of the \( \kappa \) operators just as we use the freedom of the operators \( \delta \rho \), we can build up from Eq. (E5b) any trace identity of the form

\[ 0 = \text{ReTr} \kappa Z \langle \hat{S}[(\sigma_r x_r), r \neq u]; \mathcal{D}(\sigma_u x_u) \rangle_{AV} \quad , \]  

(E7b)

with \( \kappa \) an arbitrary bosonic self–adjoint operator, which further implies the operator identity

\[ 0 = Z \langle \hat{S}[(\sigma_r x_r), r \neq u]; \mathcal{D}(\sigma_u x_u) \rangle_{AV} \quad , \]  

(E7c)
in which $\hat{S}$ is any self-adjoint polynomial that can be constructed from its arguments by repeated applications of the symmetrized product. But repeated application of the identity

$$2S(x_1,\ldots,x_L) = \sum_w \{x_w,S(x_1,\ldots,(x_w),\ldots,x_L)\}, \quad (E8)$$

with $(x_w)$ indicating that $x_w$ is to be omitted from the argument list, shows that any totally symmetrized polynomial $S$ can be built up by repeated applications of the symmetrized product to its arguments; hence $\hat{S}$ in Eq. (E7c) can be taken to be $S$, completing the derivation of Eq. (E5a)

Although we have phrased this derivation entirely in terms of symmetrizing operations, it is likely that it can be significantly extended as follows. If we take $\delta \rho_R = [\Lambda_A, \tilde{\kappa}]$, with $\tilde{\kappa}$ now anti-self-adjoint, then Eq. (E7a) is replaced by

$$\text{ReTr}[\tilde{\kappa},\mathcal{O}_R]\mathcal{O}_S = \text{ReTr}\tilde{\kappa}[\mathcal{O}_R,\mathcal{O}_S], \quad (E9)$$

and thus from the source variations we can in fact build up polynomials which are anti-symmetrized in some variables. Moreover, the argument for the vanishing of the explicit source term contribution requires not total symmetrization, but only symmetrization in all arguments $x_u, x_v$ for which the symplectic structure $\omega_{uv}$ is nonzero. Thus, if we define a partially Weyl ordered polynomial to be a polynomial which is symmetrized with respect to all arguments $x_u, x_v$ for which $\omega_{uv} \neq 0$, then it appears likely that with careful attention to the combinatorics, one should be able to show that Eq. (E1a) implies the extension of Eq. (E1b) in which $P$ is any partially Weyl ordered polynomial in its arguments. The example of Eq. (17e) also indicates that there will be extensions of Eq. (E1b) to certain cases in which $P$ is a non-Weyl ordered polynomial of low degree. We expect these cases to play an important role in the study of operator gauge invariant theories, which we will take up
in detail elsewhere.

**Appendix F. Canonical and Symmetry Transformations**

We give here further details of the structure of canonical transformations in generalized quantum dynamics, with special emphasis on their role as symmetry transformations.

A particularly interesting class of canonical transformations are what we shall term linear symmetry transformations, defined as transformations of the type Eq. (27) generated by total trace functionals of the special form

\[ G_h = \text{Tr} G_h, \]

\[ G_h = \sum_{r,s} p_r h_{rs} q_s. \]  

(F1a)

These transformations linearly transform the canonical coordinates \( \{ q_r \} \) among themselves, with a corresponding transformation on the canonical momenta, but do not mix coordinates with momenta. When the indices \( r, s \) in Eq. (F1a) are both bosonic or both fermionic, the coefficients \( h_{rs} \) are taken to be ordinary \( c \)-numbers, while when one index is fermionic and one is bosonic, the coefficients \( h_{rs} \) are taken to be Grassmann \( c \)-numbers. Thus, the linear symmetry transformations include grade-changing transformations which mix the bosonic and fermionic coordinates. As a consequence of the grading structure of \( h \), we have

\[ \text{Tr} p_r h_{rs} q_s = \text{Tr}(\pm) h_{rs} p_r q_s, \]  

(F1b)

which together with the cyclic property of \( \text{Tr} \) implies that \( G_h \) is Weyl ordered. Thus, a linear symmetry transformation is also a Weyl ordered intrinsic canonical transformation. Under the generalized Poisson bracket operation, two linear symmetry transformations compose as
\{G_g, G_h\} = G_{[g,h]} \quad ,
\tag{F2a}

with \([g, h]\) the commutator

\[[g, h]_{rs} = \sum_t (g_{rt}h_{ts} - h_{rt}g_{ts}) \quad ,
\tag{F2b}

and hence linear symmetry transformations form a Lie commutator algebra under the generalized Poisson bracket. Clearly a linear rearrangement of the canonical coordinates among themselves, with \(c\)-number or Grassmann \(c\)-number coefficients, together with a corresponding linear transformation among the momenta, transforms a Weyl ordered polynomial into another such polynomial. Therefore the set of Weyl ordered total trace functionals, and thus of Weyl ordered intrinsic canonical transformations, is closed under the action of linear symmetry transformations. [We note in passing that in Refs. [1, 2] we also introduced linear symmetry transformations in which \(h\) is an arbitrary quaternionic (hence non–commutative) coefficient matrix; although the Lie property of Eqs. (F2a, b) holds for this generalization, most of the other properties of canonical transformations derived in Sec. 4 and this Appendix do not. For example, symmetry transformations based on quaternionic representations of compact Lie groups do not leave \(\tilde{C}\) invariant.] One can also define a generalization of linear symmetry transformations with generators which can mix coordinates and momenta according to

\[G_{h} = \text{Tr}G_{\hat{h}} \quad ,
\]

\[G_{\hat{h}} = \sum_{r,s} x_r \hat{h}_{rs} x_s \quad ,
\tag{F2c}

with the grading structure of \(\hat{h}_{rs}\) analogous to that for \(h_{rs}\), which implies that these transformations are also Weyl ordered. These transformations also form a Lie algebra under the
generalized Poisson bracket, with the structure
\[ \{ G_y, G_h \} = G_k \, , \]
\[ \hat{k}_{tu} = \sum_{r,s} (\hat{g}_{tr} + \epsilon_r \hat{g}_{rt}) \omega_{rs} (\hat{h}_{us} + \epsilon_s \hat{h}_{su}) \, ; \]
certain Bogoliubov transformations are of this more general type.

We shall now derive a second relation which is similar in structure to Eq. (29) of the text, and which describes the action of a linear symmetry generator \( G_h = \text{Tr} G_h \) on a Weyl ordered intrinsic canonical generator \( G = \text{Tr} G \), when the phase space operators \( \{ x_r \} \) are specialized to the canonical algebra,

\[ \{ G_h, G \} = -\text{Tr} i[G_h, G] \, . \] (F3)

Since by the Weyl ordering hypothesis \( G \) is symmetrized, we can represent it as a sum of monomial terms produced by generating functions with the form \( g^n \) of Eq. (11a), and it then suffices to prove the identity for only one such term. Writing \( g \) in the form
\[ g = \sum_r (\xi_r q_r + \eta_r p_r) \, , \] (F4)
use of the generalized Poisson bracket in the form given in Eq. (6b) gives
\[ \{ G_h, \text{Tr} g^n \} = n \text{Tr} g h g^{n-1} \, , \] (F5a)
with \( g_h \) defined by
\[ g_h = \sum_{r,s} (p_s \epsilon_r h_{sr} \eta_r - \xi_r h_{rs} q_s) \, . \] (F5b)
One can now check that over the canonical algebra one has
\[ i g_h = \left[ \sum_{r,s} p_r h_{rs} q_s, \sum_t (\xi_t q_t + \eta_t p_t) \right] \, . \] (F5c)
But the commutator $[-iG_h, g^n]$ reduces to the totally symmetrized product of $g_h$ with $g^{n-1}$, which is equal to the right hand side of Eq. (F5a) under the trace, completing the proof of Eq. (F3). Thus, there is an isomorphism between (a) the action of a linear symmetry transformation on an arbitrary Weyl ordered intrinsic canonical transformation, under the generalized Poisson bracket operation of generalized quantum dynamics, and (b) the corresponding behavior of the canonical algebra specializations of these transformations, under the usual commutator operation. This isomorphism extends to the more general Bogoliubov type transformation of Eq. (F2c), where we find

$$\{G_h, G\} = -\text{Tr}[G_h, G] = n\text{Tr}g_h g^{n-1} ,$$

$$g_h = \sum_{r,s,t} (\omega_{rt}\sigma_t \hat{h}_{rs} x_s + x_r \hat{h}_{rs} \omega_{st} \sigma_t) . \quad (F6)$$

The isomorphism does not extend, however, to the action of generic Weyl ordered intrinsic canonical transformations on one another. To see this, let us consider the case of two such canonical transformations with generators $G_1$ and $G_2$ which are generating functions for Weyl ordered monomials,

$$G_1 = \text{Tr}G_1, \quad G_2 = \text{Tr}G_2,$$

$$G_1 = g_1^{n_1}, \quad G_2 = g_2^{n_2} , \quad (F7a)$$

$$g_{1,2} = \sum_{r} \sigma_{1,2r} x_r .$$

Then from Eq. (9a) we find

$$\{G_1, G_2\} = \text{Tr} \sum_{r,s} n_1 g_1^{n_1-1} \sigma_{1r} \omega_{rs} n_2 g_2^{n_2-1} \sigma_{2s}$$

$$= C \text{Tr} g_1^{n_1-1} g_2^{n_2-1} , \quad (F7b)$$

$$C = n_1 n_2 \sum_{r,s} \sigma_{1r} \omega_{rs} \sigma_{2s} ,$$

which is clearly not Weyl ordered when $n_1$ and $n_2$ are both greater than 2. Thus the generalized Poisson bracket of the generators for two Weyl ordered intrinsic canonical trans-
formations is in general not a Weyl ordered canonical transformation. On the other hand, if we specialize to the canonical algebra and then evaluate the commutator of $G_1$ and $G_2$, we easily find using Eq. (13) that

$$[G_1, G_2] = iCS(g_1^{n_1-1} g_2^{n_2-1}) ,$$

with $S$ the polynomial formed from completely symmetrizing $n_1 - 1$ factors of $g_1$ with respect to $n_2 - 1$ factors of $g_2$, which is Weyl ordered. In other words, over the canonical algebra, the commutator of two Weyl ordered generators is Weyl ordered, but this does not correspond to the composition properties of Weyl ordered canonical generators under the generalized Poisson bracket operation.

In future work we plan to give a more detailed analysis of the Poincaré transformations in generalized quantum dynamics than was given in [1, 2], including a study of their relationship to the classification of canonical transformations given in Sec. 4 and here.

**Appendix G. The Full Ward Identity Structure**

We describe here the full Ward identity structure resulting when the presence of the additional conserved anti–self–adjoint operator $\hat{F}$ of Eq. (C6b), or equivalently $\hat{\hat{C}}$ of Eq. (C6a), is taken into account. Including the latter in the equilibrium distribution, Eq. (51a) becomes

$$\rho = Z^{-1} \exp[-\operatorname{Tr} \sum_r \rho_r(\sigma_r x_r)] \exp(-\hat{\operatorname{Tr}} \hat{\lambda} \hat{\hat{C}} - \operatorname{Tr} \hat{\lambda} \hat{\hat{C}} - \tau \hat{H} - \lambda \hat{H}) ,$$

$$Z = \int d\mu \exp[-\operatorname{Tr} \sum_r \rho_r(\sigma_r x_r)] \exp(-\hat{\operatorname{Tr}} \hat{\lambda} \hat{\hat{C}} - \operatorname{Tr} \hat{\lambda} \hat{\hat{C}} - \tau \hat{H} - \lambda \hat{H}) ;$$

ensemble averages denoted by the subscript “AV” will be understood henceforth to be taken in the distribution of Eq. (G1). Including the $\hat{C}$ term in the Ward identity of Eq. (55b), and
for the moment retaining all terms coming from the variation of the equilibrium distribution, we get

\[ 0 = \langle \left( \hat{\lambda}, x_u \right) + \left( -1 \right)^F \hat{\lambda}, \sum_r \alpha_{ur} r_r \rangle + \hat{\tau} \left( -1 \right)^F \sum_r \alpha_{ur} \dot{x}_r + \tau \dot{x}_u + \sum_s \omega_{us} \sigma_s \rho_s \rangle h V \]

\[ + \left[ i_{s} \left\{ i_{eff}, V \right\}, x_u \right] - h \sum_s \omega_{us} \delta Z / \delta x_s \rangle AV , \]

providing the full version of the Ward identity derived in Sec. 6. Let us now take \( V = H \), and make the approximation (expected to be valid in the large N limit) of replacing \( H \) by its ensemble expectation \( \langle H \rangle_{AV} \). The coefficient of this term in Eq. (G2) then becomes the proportional to the variation \( \sum_s \omega_{us} \delta Z / \delta x_s \) of the partition function of Eq. (G1), which is zero, and so Eq. (G2) reduces, after use of the equation of motion of Eq. (9b) and division by \( h \), to

\[ 0 = \langle \left[ \frac{1}{2} \left\{ i_{eff}, H \right\}, x_u \right] - \dot{x}_u \rangle_{AV} . \]  

This is the same relation as was obtained in Eq. (56b) of the text, but we have now shown that its derivation does not depend on the assumption that the ensemble parameters \( \hat{\lambda} \) and \( \tilde{\lambda} \) are functionally related.

Let us now derive a second Ward identity by using properties of the operator \( \hat{C} \). Since this is an anti–self–adjoint operator, its ensemble expectation can be written in a polar form analogous to Eq. (46a) for \( \langle \hat{C} \rangle_{AV} \),

\[ \langle \hat{C} \rangle_{AV} = i_{eff} \hat{D} , \quad \hat{e}_{eff} = -i_{eff} , \quad \hat{e}_{eff} = -1 , \]

\[ [i_{eff}, \hat{D}] = 0 , \quad \hat{D} \text{ self–adjoint and nonnegative}. \]  

The difference is that we do not now assume \( \hat{D} \) to be diagonal, but we shall assume it to be nonsingular, so that the inverse \( (\hat{D})^{-1} \) exists. Let us now derive a second Ward identity, by
considering the expression

\[ 0 = \int d\mu \delta x \left[ \rho \hat{\text{Tr}} \{ \hat{C}, (\hat{D})^{-1} \hat{i}_{\text{eff}} \} V \right], \quad (G5a) \]

with \( \rho \) the full equilibrium distribution of Eq. (G1). Proceeding as in Eqs. (53–55) of the text, with the one exception that we now multiply through by \( \frac{1}{2} \sum_s \hat{\omega}_us \), we end up with the following analog of Eq. (G2),

\[
0 = \langle \left[ \{ \hat{\lambda}, x_u \} + (-1)^F \{ \hat{\lambda}, \sum_r \alpha_{ur} x_r \} + \tau (-1)^F \sum_r \alpha_{ur} \hat{x}_r + \hat{\tau} \hat{x}_u + (-1)^F \sum_s \hat{\omega}_{us} \sigma_s \rho_s \right] \hat{V} \\
+ \frac{1}{2} \{(\hat{D})^{-1} \hat{i}_{\text{eff}}, H \}, x_u \rangle - \sum_s \hat{\omega}_us \hat{\delta} \hat{V} \hat{\delta}x_s \rangle_{AV}, \quad (G5b)
\]

where the hatted operator derivative \( \hat{\delta} \hat{V}/\hat{\delta}x_s \) is defined by

\[ \hat{\delta} \hat{V} = \hat{\text{Tr}} \sum_s \hat{\delta} \hat{V} \hat{\delta}x_s. \quad (G5c) \]

Let us now take \( V = H \) in Eq. (G5b), and replace the conserved extensive quantity \( \hat{H} \) by its ensemble average \( \langle \hat{H} \rangle_{AV} \). The coefficient multiplying this quantity in Eq. (G5b) now becomes proportional to the partition function variation \( \sum_s \hat{\omega}_{us} \hat{\delta}Z/\hat{\delta}x_s \), which again vanishes, and so Eq. (G5c) reduces in this special case to

\[ 0 = \langle \left[ \{ (\hat{D})^{-1} \hat{i}_{\text{eff}}, H \}, x_u \rangle - \sum_s \hat{\omega}_us \hat{\delta} \hat{H} \hat{\delta}x_s \rangle_{AV}. \quad (G6a) \]

But comparing the definition of Eq. (G5c) with Eq. (C4b), we see that the fact that the same equations of motion follow from the ungraded and graded trace variational principles can be expressed succinctly by the identity

\[ \hat{x}_r = \sum_s \omega_{rs} \frac{\delta H}{\delta x_s} = \sum_s \hat{\omega}_{rs} \frac{\hat{\delta} \hat{H}}{\hat{\delta}x_s}, \quad (G6b) \]
and so Eq. (G6a) simplifies further to

$$0 = \langle \left\{ \frac{1}{2} \{ (\hat{D})^{-1} \hat{i}_{eff}, H \}, x_u \} - \dot{x}_u \rangle_{AV} . \quad (G7a)$$

Comparing this expression for the ensemble average of $\dot{x}_u$ (still in the presence of sources!) with the similar expression obtained from the original Ward identity in Eq. (G3), we conclude that we must have the relation

$$\hat{h}^{-1} i_{eff} = (\hat{D})^{-1} \hat{i}_{eff} \quad (G7b)$$

which since $\hat{D}$ is nonnegative implies the further relations

$$\hat{D} = \hat{h} \quad \hat{i}_{eff} = \hat{i}_{eff} \quad (G7c)$$

justifying the functional relation assumption made in Appendix C. Substituting Eq. (G7b) into the second Ward identity of Eq. (G5b), and multiplying through by $\hat{h}$, we get

$$0 = \langle \left\{ \hat{\lambda}, x_u \right\} + (-1)^F [\hat{\lambda}, \sum_r \alpha_{ur} x_r] + \tau (-1)^F \sum_r \alpha_{ur} \dot{x}_r + \tau \dot{x}_u + (-1)^F \sum_s \hat{\omega}_{us} \sigma_s \rho_s) \hat{h} \hat{V} + \frac{1}{2} \{ i_{eff}, V \}, x_u \rangle - \hat{h} \sum_s \hat{\omega}_{us} \frac{\delta \hat{V}}{\delta x_s} \rangle_{AV} , \quad (G8)$$

giving the form of the second Ward identity analogous to Eq. (G2).

We must now perform an important consistency check. If instead of taking $V = H$ in the Ward identities of Eqs. (G2) and (G8), we take instead $V = (-1)^F H$, the effect is to simply interchange the roles of $H$ and $\dot{H}$ in the argument showing that the variation of the equilibrium distribution does not contribute, and so this argument remains valid. Hence we get two new relations, which we must check are not in contradiction with Eq. (G3). The two new relations are seen to be identical when one uses the identity

$$\sum_s \hat{\omega}_{rs} \frac{\delta \hat{H}}{\delta x_s} = \sum_s \omega_{rs} \frac{\delta \hat{H}}{\delta x_s} \quad (G9a)$$
which like Eq. (G6b) is a consequence of the fact that the same equations of motion follow from the graded and ungraded trace variational principles. When the index \( u \) is bosonic, the new relation is easily seen to be identical in form to Eq. (G3), and so is automatically consistent. When the index \( u \) is fermionic, we introduce the definition \( \tilde{H}_{\text{eff}} \) as in Eq. (57a), and rewrite Eq. (G3) in the form

\[
\langle [\tilde{H}_{\text{eff}}, x_u] \rangle_{AV} = \hbar \langle \sum_s \omega_{us} \frac{\delta H}{\delta x_s} \rangle_{AV} ;
\]

in this notation, the new relation takes the form

\[
\langle \{ \tilde{H}_{\text{eff}}, x_u \} \rangle_{AV} = \hbar \langle \sum_s \hat{\omega}_{us} \frac{\delta H}{\delta x_s} \rangle_{AV} ,
\]

which involves a commutator rather than an anticommutator on the left hand side. Treating separately the fermionic cases in which \( u = 2r - 1 \), \( x_u = q_r \) and \( u = 2r \), \( x_u = p_r \), and taking linear combinations of Eqs. (G9b) and (G9c) which eliminate the right hand side, we find that the new relation of Eq. (G9c) implies that

\[
\langle \tilde{H}_{\text{eff}} q_r \rangle_{AV} = 0 ,
\]

\[
\langle p_r \tilde{H}_{\text{eff}} \rangle_{AV} = 0 .
\]

These relations are compatible with what one expects for vacuum expectations in quantum field theory, when one defines the fermionic vacuum so that \( \langle 0 | p_r \text{eff} = q_r \text{eff} | 0 \rangle = 0 \); hence the new relation of Eq. (G9c) is consistent with the isomorphism conjectured in the text.

We also remark that Eq. (G7c) for the expectation of \( \tilde{F} \), which can be rewritten as

\[
\langle \sum_{r,F} q_r p_r \rangle_{AV} = 0 ,
\]

and Eq. (G10) do not contradict the fermionic canonical algebra of Eq. (60c), because in general

\[
q_r p_r \neq q_r \text{eff} p_r \text{eff} ;
\]
referring to the analysis of Appendix B we see that equality in Eq. (G11b) can hold only for the special case [see Eq. (B5b)] of complex Hilbert space with $\mathcal{E} = 1$, which is thus ruled out.

Having established the equality of the ensemble averages of $\tilde{C}$ and $\hat{C}$, and therefore a functional relationship between the corresponding ensemble parameters $\tilde{\lambda}$ and $\hat{\lambda}$, and $(-1)^F$, we can apply the argument of Eq. (54c) to conclude that the commutator terms involving $\tilde{\lambda}$ and $\hat{\lambda}$ drop out of the Ward identities. The Ward identities of Eqs. (G2) ad (G8) then simplify to

$$0 = \langle (\hat{\tau}(-1)^F \sum_r \alpha_{ur} \hat{x}_r + \hat{\tau} \hat{x}_u + \sum_s \omega_{us} \sigma_s \rho_s) \hbar V \rangle$$

$$+ \left[ \frac{1}{2} \{ i_{\text{eff}}, V \}, x_u \right] - \hbar \sum_s \omega_{us} \frac{\delta V}{\delta x_s} \rangle_{AV}, \quad (G12a)$$

and

$$0 = \langle (\tau(-1)^F \sum_r \alpha_{ur} \hat{x}_r + \hat{\tau} \hat{x}_u + (-1)^F \sum_s \hat{\omega}_{us} \sigma_s \rho_s) \hbar \hat{V} \rangle$$

$$+ \left[ \frac{1}{2} \{ i_{\text{eff}}, V \}, x_u \right] - \hbar \sum_s \hat{\omega}_{us} \frac{\delta \hat{V}}{\delta x_s} \rangle_{AV}, \quad (G12b)$$

with Eq. (G12a) the basis of further applications as discussed in the text. For any $V$ obeying the conditions

$$\sum_s \omega_{us} \frac{\delta V}{\delta x_s} = \sum_s \hat{\omega}_{us} \frac{\delta \hat{V}}{\delta x_s}, \quad (G13a)$$

an analysis paralleling the consistency check on the Hamiltonian given above shows that the Ward identities obtained from Eqs. (G12a, b) when the $\hat{\tau}$ and $\tau$ terms are neglected, and those similarly obtained when $V$ is replaced by $(-1)^F V$, are consistent as long as the conditions

$$\langle V_{\text{eff}} q_r \rangle_{AV} = 0, \quad (G13b)$$

$$\langle p_r V_{\text{eff}} \rangle_{AV} = 0, \quad (G13b)$$

76
with $\tilde{V}_{eff} = \frac{1}{2}\{i_{eff}, V\}$, are obeyed for fermionic $r$. These conditions are compatible with those of Eq. (G10), and have the same interpretation in terms of fermionic vacuum structure.
References

[1] S. L. Adler, Nucl. Phys. B 145 (1994) 195

[2] S. L. Adler, Quaternionic quantum mechanics and quantum fields, Secs. 13.5–13.7 and Appendix A (Oxford, New York, 1995)

[3] S. L. Adler, G. V. Bhanot and J. D. Weckel, J. Math. Phys. 35 (1994) 531

[4] S. L. Adler and Y.–S. Wu, Phys. Rev. D 49 (1994) 6705

[5] H. Goldstein, Classical Mechanics, pp. 260–261 (Addison–Wesley, Reading MA, 1950)

[6] D. ter Haar, Elements of Statistical Mechanics, Third ed., Sec. 5.13 (Butterworth Heinemann, Oxford and Boston, 1995)

[7] A. Sommerfeld, Thermodynamics and Statistical Mechanics, Secs. 28, 29, 36, and 40 (Academic Press, New York, 1956)

[8] F. Mohling, Statistical mechanics: methods and applications, pp. 270–272 (Halsted Press/John Wiley, New York, 1982)
[9] M. Kaku, Quantum Field Theory, pp. 407–410 (Oxford, New York, 1993)

[10] E. Witten, “The 1/N Expansion in Atomic and Particle Physics”, in Recent Developments in Gauge Theories, eds. G. ’tHooft et. al. (Plenum Press, New York and London, 1980)

[11] R. Gopakumar and D. J. Gross, “Mastering the Master Field”, Princeton preprint PUPT–1520, October, 1994

[12] I. Ya. Aref’eva and I. V. Volovich, “The Master Field for QCD and q–Deformed Quantum Field Theory”, Steklov preprint SMI–25–95, November, 1995