On quantum analogue of instable oscillating states of an inverted oscillator in external poly-harmonic field

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Abstract. Nonstationary Schroedinger equation (NSE) is solved analytically and numerically to study a phenomenon of dynamical stabilization of the inverted oscillator driven by poly-harmonic in time and spatially uniform force with specially chosen phase shifts. It is shown that for Gaussian wave packet asymptotically fitting the initial condition (IC) it occurs temporary delay of the packet center about top of the parabolic potential for about 2 fundamental time periods followed by the center bifurcation.

1. Introduction
Inverted harmonic oscillator (IHO) driven by time periodic spatial uniform field is an open quantum model widely used for unstable objects such as a particle on excited energy level and subjected to coherent radiation before its decay or ionization or another destructive process [1]. Besides, the IHO itself is of great interest from many viewpoints of both physical and mathematical sciences. In fact, the model permits exact mathematical solution for certain initial condition (IC) of generalized Gaussian type even when mass of the particle and its “stiffness” depend on time [2]. It made possible to investigate the quantum analogue of classical phenomenon of dynamical stabilization of the particle in IHO potential additionally driven by sinusoidal force with a certain phase shift [3,4]. Coupling thus driven IHO with thermal bath to take into consideration an energy dissipation also preserves the stabilization phenomenon but makes it not so pronounced [5]. In addition to all above the IHO is also of interest from a viewpoint of quantum-classical correspondence [6] which requires delicate approach for starting wave functions or density matrix corresponding to the classical “ball on hill top” instable initial state. And in discrete configurational space the difference Schroedinger equation predicts classically forbidden localization of the wave aside potential barrier like “a cloud captured by hill” [7]. The present research is devoted to the stabilization effect when driving is a discrete sum of sinusoidal forces of various frequencies/amplitudes and in first turn a Fourier sum with subsequent aliquot frequencies.

2. Dynamical stabilization of classical IHO driven poly-harmonically
The classical dynamics of a conservative inverted oscillator driven in poly-harmonic manner at dimensionless frequencies $\Omega_\xi$, amplitudes $f_\xi$ and phases $\phi_\xi$ is described by following matrix equation in also dimensionless time $\tau$:

$$\frac{d}{d\tau}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \sum_\xi f_\xi \sin(\Omega_\xi \tau + \phi_\xi) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tau \geq 0 \tag{1}$$

With IC as $q(0)=q_0$ and $p(0)=p_0$ its general solution for the coordinate $q$ is
\[ q(\tau) = \frac{1}{2} \left( p_0 + q_0 + \sum_k f_k \sin(\phi_k + \Omega_k \cos(\phi_k)) + 1 + \Omega_k \right) e^\tau + \frac{1}{2} \left( q_0 - p_0 + \sum_k f_k \sin(\phi_k - \Omega_k \cos(\phi_k)) + 1 + \Omega_k \right) e^{-\tau} - \sum_k f_k \sin(\Omega_k \tau + \phi_k) \]

(2)

and it bounded if and only if ideally takes place following dynamical stabilization condition

\[ p_0 + q_0 + \sum f_k \sin(\phi_k + \arctg \Omega_k) = 0 \]

(3).

By given IC and amplitudes \( f_k \) and frequencies \( \Omega_k \) the condition (4) defines some hypersurface (HS) in \( N \)-dimensional product of semi-intervals \((-\pi, +\pi] \times (-\pi, +\pi] \times \ldots \times (-\pi, +\pi]\) of phase shifts \( \phi_k, k = 1, 2, \ldots, N \). Any set of phases taken from the HS provides an everlasting bounded solution for the coordinate \( q(\tau) \) and pulse \( p(\tau) \). An ultrathin subsurface \( \delta \)-layer with \( \delta \ll 1 \) adjacent to this hypersurface but except itself contains the set of phases \( (\phi_1, \phi_2, \ldots, \phi_N) \) providing limitedness during long but finite time while first summand in (2) stays negligible compared with the rest ones. So, the \( \delta \)-layer guarantees us sufficient long dynamical stabilization for the solution (3). The \( N \)-D-measure of the layer equals \( \mu(\delta) = S\delta \), \( S \) being HS ‘area’ and it is negligible compared with full measure \( 2\pi^N \) of the entire space \( \Sigma \) above. However, the share of \( \delta \)-layer grows with the dimensionality \( N \) and becomes significant for \( N \gg 1 \). So, the case of poly-harmonic driving has more chance to fit approximately constrain (3) for limitedness of the solution (3) and therefore the dynamical stabilization of IHO is more probable for multi-harmonic enforcing.

**Example.** In simple case of \( p_0 + q_0 = 0 \) and \( N = 2 \) frequencies \( \Omega_1 = 1, \Omega_2 = 2 \) and equal amplitudes \( f_1 = f_2 \) constrain permits an exact analytical description hypersurface inside of quadrat \((-\pi, +\pi] \times (-\pi, +\pi] \):

\[
\sin \left( \phi_1 + \frac{\pi}{4} \right) + \sin \left( \phi_2 + \arctg 2 \right) = 0 \iff \begin{cases} \phi_1 - \phi_2 = \arctg 2 + \frac{3\pi}{4}, & \phi_1 - \phi_2 = \arctg 2 - \frac{5\pi}{4}, \\ \phi_1 + \phi_2 = \arctg 2, & \phi_1 + \phi_2 = 7\pi - \arctg 2. \end{cases}
\]

The HS is equivalent to contour of another quadrat \( \sqrt{2}\pi \times \sqrt{2}\pi \) been rotated on \( \frac{\pi}{4} \) which \( \delta \)-layer has \( 2\text{D} \) measure of \( 4\sqrt{2}\pi \delta \) what consists \( \frac{4\sqrt{2}\pi \delta}{(2\pi)^2} = \frac{\sqrt{2}\pi \delta}{\pi} \)-share of whole measure of the phase shifts’ space (figure 1). This share is \( \sqrt{2} \) times greater than for simple sinusoidal driving with two stabilizing phase-points with the 1D-measure of the \( \delta \)-layer being \( 2\delta \) of whole \( 2\pi \). Therefore, we rely on suggestion that the higher the dimensionality \( N \) of ND-cube, the greater the volume adjacent to the hypersurface defining by condition (4) of dynamical stabilization. So, this phenomenon doesn’t seem as unlikely and exotically both for poly-harmonic driving and that with continuous set of frequencies satisfying integral analogue of (3) [3]. first paragraph after a heading is not indented.

In fact, for analogous case of three subsequent aliquot harmonics of 1, 2 and 3 and analogously equal amplitudes with the same null IC the constrain looks out as

\[
\sin \left( \phi_1 + \frac{\pi}{4} \right) + \sin \left( \phi_2 + \arctg 2 \right) + \sin \left( \phi_3 + \arctg 3 \right) = 0.
\]

The corresponding \( \delta \)-layer adjacent to hypersurface looks out more significant than that for 2D case (figure 2). (And the HS itself reminds open Fermi surface for cubic crystal built by Harrison method.)

The solutions with phase shifts in (1) taken exactly from the HS will be asymptotically sinusoidal at large time \( \tau \) and hence bounded. As for those taken from \( \delta \)-layer around the HS i.e. distanced from it by
an infinitesimal deviation the solution (3) loses its limitedness at relatively large time and bifurcates by outgoing to right or left infinity (figure 3). And the stabilization time is the longer, the smaller initial deviation of the phases taken from the δ-layer.

Figure 1. The HS of dynamical stabilization (line) and its δ-layer in 2D phase shifts’ space for double Ωs driving. The parts compose a square.

Figure 2. HS of dynamical stabilization for null IC, N=3 aliquot Ωs=1,2,3 in (1) and equal amplitudes.

3. Mathieu equation dynamics

Another classical case of possible dynamical stabilization refers to well-known Kapitza pendulum when driving term is coupling with negative potential as follows

\[ H(p,q) = \frac{1}{2} p^2 + \frac{1}{2} \left( 1 + h \cos \Omega \tau \right) q^2. \]

The dynamical equation, written in second form, becomes the Mathieu equation

\[ \frac{d^2q}{d\tau^2} + (-1 + h \cos \Omega \tau) q = 0, \]

or in canonical form:

\[ \frac{d^2q}{dz^2} + (\varepsilon - b \cos 2z) q = 0, \quad z = \frac{\Omega \tau}{2}, \quad \varepsilon = -\frac{4}{\Omega^2}, \quad b = \frac{4h}{\Omega^2} \] (4).

This equation has regions of stability in the space of parameters (Ω, h). The eq. (4) may be viewed as the Schrödinger equation for a particle in a periodic potential \( V(z) = -b \cos 2z \) within 1D crystal on energy level \( \varepsilon < 0 \). The condition of limitedness in time \( \tau \) can be interpreted then as the requirement that negative \( \varepsilon \) doesn’t lie in a bandgap (see e.g. [8]) where there is no solutions.
4. Quantum case
In dimensionless coordinate \( \xi \) and time \( \tau \) the NSE Schroedinger equation for evolving in time wave function (w.f.) \( \Psi(\xi, \tau) \) looks as

\[
\frac{i}{\hbar} \frac{\partial \Psi(\xi, \tau)}{\partial \tau} = -\frac{1}{2} \frac{\partial^2 \Psi(\xi, \tau)}{\partial \xi^2} - \frac{\xi^2}{2} \Psi(\xi, \tau) - i \sum_k f_k \sin(\Omega_k \tau + \phi_k) \Psi(\xi, \tau) \tag{5}
\]

To fit asymptotically quantum analogue of the IC \( q(0)+p(0)=0 \) [3], i.e. \( -i \frac{\partial}{\partial \xi} + \xi \Psi(\xi,0)=0 \) the generalized Gaussian function \( \Psi(\xi,0)=\left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^{\frac{1}{2}} \exp\left(-\frac{i\xi^2}{2} - \frac{\xi^2}{4\sigma_0^2}\right) \) had been chosen with extremely large initial wave packet half-width \( \sigma_0>>1 \).

The solution of eq. (5) was searched as

\[
\Psi(\xi,\tau)=\left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^{\frac{1}{2}} \exp\left(\alpha(\tau) + \beta(\tau)\xi - \left(\frac{i}{2} + \frac{1}{4\sigma_0^2} - \gamma(\tau)\right)\xi^2\right), \quad \alpha(0) = \beta(0) = \gamma(0) = 0 \tag{6}
\]

This reduced partial differential equation (5) to system of ordinary differential equations relatively the functions in (6). And its solution gave important values of an expectation of the dimensionless coordinate \( \langle \xi(\tau) \rangle = \xi_{av}(\tau) \) and mean spreading of this value \( \sigma(\tau) = \sqrt{\left\langle (\xi - \xi_{av}(\tau))^2 \right\rangle} \)

\[ \sigma_0 = 2^{12}, f_1 = 1; f_2 = 1.00000034997850000003 \ (red) \]
\[ f_2 = 1.00000034997865 \ (blue) \]
\[ \Omega_1 = 1, \Omega_2 = 2; \phi_1 = -\frac{1}{4} n = 0.00151900292394915, \]
\[ \phi_2 = -\arctan(2) - 0.0340478345094495 \]

Figure 3. Bifurcation of the trajectories with differing phase shifts on phase plane in case of \( N=2 \\) \( \Omega \)'s in (1). All trajectories start from one point (-1,1). Red line is a separatrix.

Figure 4. Wave packet center \( \xi_{av}(\tau) \) temporary stabilization and bifurcation at double harmonic driving with \( \Omega \_1=1 \) and \( \Omega \_2=2 \) and negligibly differing amplitudes \( f_2/f_1 \approx 3.5 \times 10^{-8} \).

The latter didn’t depend on driving terms in (5) and obeyed the same law as in [3] with initial collapsing of wave packet at times \( \tau \leq \tau_0 = 0.125\ln(16\sigma_0^4+1) \) from quite macroscopic starting values
of order $\sigma_0 = 2^{12} \cdot 2^{15}$ to minimal one of about 0.5 then followed by further irreversible expansion to infinity. Both the collapsing and unbounded expansion occur due to nearly exponential formulas, the former being descending the latter ascending (figures 4—6, green lines).

![Figure 5](image1.png)  
*Figure 5. Stabilization and bifurcation of the $\xi_{av}(\tau)$ for low $\Omega_1=1/16$ and high $\Omega_2=16$ freqs.*

![Figure 6](image2.png)  
*Figure 6. Temporary stabilization for “resonant” $\Omega_1=1$ and high $\Omega_2=16$ freqs.*

It was studied numerically only the case of two close to each other or strongly differing frequencies $\Omega_1$ and $\Omega_2$ and there were obtained dependences with temporary stabilization of the $\xi_{av}(\tau)$ near top of the potential during about 4$\pi$ time units (figures 4—6). It may be easily seen that $\xi_{av}(\tau)$ curves at times $\tau<4\pi$ include both $\Omega_1$ and $\Omega_2$ harmonics.

5. Discussion

So we see that temporary dynamical stabilization of wave packet center during of about two fundamental “periods” is retained for the case of poly-harmonic driving of quantum IHO. Moreover, it is more likely due to more significant weight of $\delta$-layer adjacent to HS (4) in entire space of possible harmonica’s phases. Close to it phenomena were predicted and observed for many quantum systems such as the stability to ionization of an atom under strong laser radiation [9] and others (see refs. in [3]).

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