Optimal design problems in rough inhomogeneous media: existence theory

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OPTIMAL DESIGN PROBLEMS IN ROUGH INHOMOGENEOUS MEDIA: EXISTENCE THEORY

By Eduardo V. Teixeira

Abstract. This paper settles the existence question for a rather general class of convex optimal design problems with a volume constraint. In low dimensions, we prove existence of an optimal configuration for general convex minimization problems ruled by bounded measurable degenerate elliptic operators. Under a mild continuity assumption on the medium, the free boundary is proven to have an appropriate weak geometry and we establish existence of an optimal design in any dimension.

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1. Introduction. Well known for modeling important problems in applied mathematics, respected for the challenging mathematical questions they give rise to, and admired for their intrinsic beauty, optimization problems with volume constraints have received overwhelming attention in the past few decades. In general, the usual techniques of the Calculus of Variations are not sufficiently powerful, or even appropriate, to establish existence of optimal configurations for those classes of problems. This fact has inspired remarkable recent advances in a number of branches of applied analysis in an attempt to develop the right set of analytical and geometrical tools to study optimal design problems with volume constraints.

One of the fundamental motivations of this present work can be, in its most basic form, stated as follows: given an \( n \)-dimensional body and a fixed amount...
of insulating material, what is the best way of insulating it? Depending on the flexibility allowed, the mathematical set-up used to model this classic question can also be employed in the analysis of a variety of other problems in applied mathematics. In more precise mathematical terms, but still using the language of heat conduction, the above question takes the following form: let $D$ be a fixed Lipschitz bounded domain in $\mathbb{R}^n$ (the body to be insulated), $\varphi: \partial D \to \mathbb{R}$ be a prescribed positive function (the temperature distribution on $D$), and $\iota > 0$ be a given positive number (the amount of insulating material available). For each configuration $\Omega$ that surrounds $D$ and satisfies $\mathcal{L}^n(\Omega \setminus D) \leq \iota$, we compute the flux associated to it:

$$\Omega \mapsto \mathcal{J}(\Omega).$$

In general, $\mathcal{J}$ is related to a boundary integral involving a potential $u(\Omega)$, linked to $\Omega$ by a prescribed PDE. The optimal design problem is then to find

$$\text{Min} \{ \mathcal{J}(\Omega) \mid \Omega \subset D \text{ and } \mathcal{L}^n(\Omega \setminus D) \leq \iota \}. \quad (1.1)$$

Probably the first and still one of the most influential works in this line of research is the pioneering article of Aguilera, Alt and Caffarelli, [AAC86]. In this paper, the authors address the question of minimizing the Dirichlet integral when the volume of the zero set is prescribed. More precisely, they study the optimization problem

$$\text{Min} \left\{ \int_{\Omega} |\nabla u|^2 \, dX \mid u \in H^1(\Omega), \quad u = \varphi \geq 0 \text{ on } \partial \Omega \right\} \quad \text{and} \quad \mathcal{L}^n(\{u = 0\}) = \alpha \quad (1.2)$$

for a fixed $\alpha < \mathcal{L}^n(\Omega)$. In the case of an exterior domain, $\Omega = \mathbb{R}^n \setminus D$, problem (1.2) can be used to model a very simple, yet interesting optimal design problem with volume constraint as stated above. Namely, suppose $D$ is evenly heated. If one tries to minimize the heat flux given by $\int_{\partial \Omega} u \mu dH^{n-1}(X)$, where $u$ is the capacity potential associated to $\Omega$, and $\mu$ stands for the normal derivative of $u$ along $\Omega$, with $\mathcal{L}^n(\Omega \setminus D)$ prescribed, a simple application of Green’s identity reveals that the heat flux equals the Dirichlet integral, and therefore the problem becomes identical to (1.2). Fine regularity properties of the free boundary, $\partial \{u^* > 0\} \cap \Omega$, where $u^*$ is a minimizer of (1.2) rely on the powerful geometric-measure machinery developed by Alt and Caffarelli in [AC81]: the magnum opus of free boundary regularity theory for variational problems.

A significant generalization of problem (1.2) was carried out by Lederman in [Led96]. In this paper, the author studies the non-homogeneous minimization problem, that is, the Dirichlet integral is replaced by $\int |\nabla u|^2 dX - \int gu$, for a given $g$ bounded away from zero.
In an important paper, Ambrosio, Fonseca, Marcellini and Tartar, [AFMT99], address another major generalization of problem (1.2). Namely they establish the existence of a minimizer to the functional
\[ F := \int_\Omega W(Du) \, dx, \]
for \( W : \mathbb{R}^{d \times n} \to (0, \infty) \) \( C^1 \) and quasi-convex, with the multiple volume constraint \( L^n(\{u = z_i\}) = \alpha_i, 0 \leq i \leq k \). In a subsequence article, Tilli, in [Tilli99], showed, for \( W(\xi) := |\xi|^2 \), that in the case of just two level constraints, the minimizers are locally Lipschitz continuous.

Still assuming a constant temperature distribution, Oliveira and the author in [OT06] studied the optimization problem (1.1), governed by the \( p \)-Laplacian operator when the flux is given by \( J(u) := \int_{\partial\Omega} (u_\mu)^{p-1} \, d\mathcal{H}^{n-1}(X) \). This translates into the analysis of the minimization problem (1.2), for the \( p \)-Dirichlet integral, that is, \( W(\xi) = |\xi|^p \), for \( p > 1 \).

The first work to deal with optimal design problems with non-constant temperature distribution \( \varphi : \partial D \to (0, \infty) \) is [ACS87]. In this paper, the authors consider the linear functional: \( \mathcal{J}(\Omega) = \int_{\Omega \setminus D} \Delta u \, dX \), where \( u \) is the harmonic function in \( \Omega \setminus D \), taking boundary data \( \varphi \) on \( \partial D \) and zero on \( \partial\Omega \). Even for this simple functional, major difficulties arise. For instance, the free boundary condition, that is, the behavior of \( \nabla u^* \) along the free boundary, \( \partial\Omega^* \), is non-local and it required a new machinery to establish the appropriate geometric-measure properties of the free boundary necessary to perform suitable smooth perturbations. The latter is used in a crucial way to finally conclude the existence of an optimal design.

At least for smooth competing configurations, \( \Omega \), for the linear functional studied in [ACS87] we have
\[ \mathcal{J}(\Omega) := \int_{\Omega \setminus D} \Delta u \, dX = \int_{\partial\Omega} u_\nu \, d\mathcal{H}^{n-1}(X) = \int_{\partial D} u_\mu \, d\mathcal{H}^{n-1}(X). \]

This is a naïve, yet important observation, as the latter integral is taken over the fixed boundary. Therefore, at least in an intuitive perspective, a non-linear theory for this class of minimization problems should use \( \int_{\partial D} u_\mu \, d\mathcal{H}^{n-1}(X) \) as its linear pattern. From the applied viewpoint, if one allows a nonlinear flux, \( \mathcal{J} \) that might also depend upon the local structure of the boundary of the body \( D \), i.e.,
\[ \mathcal{J}(\Omega) := \int_{\partial D} \Gamma(X, u_\mu(X)) \, d\mathcal{H}^{n-1}(X) \]
the mathematical model (1.1) would address several other physical situations, such as: optimal configurations in electrostatics, problems in material science, flux dynamics, among many others. This nonlinear setting, however still only for problems governed by the Laplacian operator, has been studied by the author in [Teix05] and [Teix07].

In this present paper, we settle the existence theory for optimal design problem (1.1) with nonlinear functionals as in (1.3), when \( u(\Omega) \) is linked with \( \Omega \) by a rather general class of degenerate elliptic PDEs. In terms of applications, it greatly
extends the range of physical systems that can be modeled by this set-up. For instance, it opens the possibility of formulating classical optimal design problems with volume constraints within a Riemannian manifold. From the mathematical viewpoint, this project brings a number of new rather challenging difficulties in its analysis, and modern solutions to various issues commonly found in free boundary problems are developed throughout the paper. Free boundary regularity theory for uniform elliptic operators in divergence form with merely Hölder continuous coefficients has been recently developed in order to establish $C^{1,\gamma}$ smoothness of an optimal configuration, up to a possible negligible singular set, [Teix].

The article is organized as follows: in Section 2, we describe all the mathematical elements involved in the model and the optimization problem is accurately stated in that section. Still in Section 2, we introduce weak formulations of the optimal design problem (1.1) that are somewhat simpler to be tackled from the mathematical perspective. Basic properties of the functional to be minimized are established in Section 3. The first existence theorem for a weak formulation of the original optimization problem is delivered in Section 4. In Section 5, by letting the penalty term blow-up, we establish the existence of an optimal configuration to the optimal design problem with volume constraint (1.1) ruled by totally discontinuous degenerate elliptic operators. For that though, a technical restriction on the dimension is necessary. In Section 6, under $C^\epsilon$ regularity on the medium, a series of results concerning the weak geometric properties of the boundary of an optimal configuration to the weak formulation of the original problem (1.1) are achieved. These are used in Section 7 to ultimately derive existence of an optimal configuration in all dimensions.

We finish the introduction by setting-up the main notations adopted throughout the article:

- The dimension of the euclidian space the problem is modeled in will be denoted by $n$. $D$ will be a fixed bounded domain in $\mathbb{R}^n$, for $n \geq 2$ and $D^C$ will mean the complement of the set $D$. Equivalently, $D^C := \mathbb{R}^n \setminus D$. For a domain $\mathcal{O} \subset \mathbb{R}^n$, $\partial \mathcal{O}$ will represent the boundary of the domain $\mathcal{O}$. $\chi_S$ will stand for the characteristic function of the set $S$.

- The $n$-dimensional Lebesgue measure of a set $C$ will be denoted by $\mathcal{L}^n(C)$. $\mathcal{H}^{n-1}$ will stand for the $(n-1)$-Hausdorff measure.

- $\langle \cdot, \cdot \rangle$ will be the standard scalar product in $\mathbb{R}^n$. For a vector $\xi \in \mathbb{R}^n$, its euclidian norm will be denoted by $|\xi| := \sqrt{\langle \xi, \xi \rangle}$. $B_r(p)$ will be the open ball centered at $p$ with radius $r$. Furthermore, we shall denote $kB = kB_r(p) := B_{kr}(p)$, for any $k > 0$.

- $W^{1,p}(\mathcal{O})$ will denote the Sobolev space of $p$-integrable functions with distributional derivatives also in $L^p$.

- $A(X,\xi)$ will be a $p$-degenerate elliptic map (see below). $\mathcal{L}v := \text{div}(A(X, Dv))$ will be the operator in the distributional sense associated to $A$. The $A$-inward normal derivative of a function $u$ will be $\partial_Au := \langle A(X, Du), \mu \rangle$. See beginning of Section 2 for further details.
• Universal constants $C_1, C_2, \ldots$ will be constants depending on dimension and ellipticity. Any additional dependence will be emphasized.

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2. Mathematical set-up. Throughout the paper, $D$ denotes a fixed $C^{1,1}$ bounded domain in $\mathbb{R}^n$, $\varphi: \partial D \to \mathbb{R}$ is a prescribed positive function and $\iota > 0$ is a given positive number. Hereafter, $A$ will be a uniformly elliptic symmetric matrix with measurable coefficients. That is for positive constants $0 < \lambda \leq \Lambda$,

$$\lambda \text{Id} \leq A(X) \leq \Lambda \text{Id}.$$  

In this paper we shall consider degenerate elliptic problems of the form

$$F(X, \xi) = A(X)p^{-1}|\xi|^p, \quad A(X, \xi) = A(X)|\xi|^{p-2}\xi, \quad \text{for } p > 1.$$  

Our optimization problem is then formulated as follows: for each domain $\Omega \subset D$ satisfying

$$(2.1) \quad \mathcal{L}^n(\Omega \setminus D) \leq \iota,$$  

we consider the $A$-potential, $u = u(\Omega)$, with the prescribed boundary value $\varphi$ on the fixed boundary $\partial D$, associated to $\Omega$, i.e. the unique solution to

$$(2.2) \begin{cases} \Sigma u := \text{div}(A(X, Du)) = 0 \quad \text{in } \Omega \setminus D \\ u = \varphi \quad \text{on } \partial D \\ u = 0 \quad \text{on } \partial \Omega \end{cases}$$  

and compute

$$\mathcal{I}(\Omega) := \int_{\partial D} \Gamma(X, \partial_A u(X)) d\mathcal{H}^{n-1}(X) \quad \text{(the flux: quantity to be minimized).}$$  

Here $\Gamma: \partial D \times \mathbb{R} \to \mathbb{R}$ is a given function, whose properties will be described soon, and

$$(2.3) \quad \partial_A u(X) := \langle A(X, \nabla u(X)), \mu(X) \rangle$$
where $\mu$ denotes the inward normal vector defined $\mathcal{H}^{n-1}$ a.e. on $\partial D$. The optimal design problem we are interested in is the following:

(2.4) \[ \text{Minimize } \{ J(\Omega) \mid \Omega \supset D \text{ and } \mathcal{L}^n(\Omega \setminus D) \leq \iota \}. \]

The analytical (and naturally mild) properties assumed on the nonlinearity $\Gamma$ are:

1. For each $X \in \partial D$ fixed, $\Gamma(X, \cdot)$ is convex and increasing.
2. For each $t \in \mathbb{R}$ fixed, $\partial_t \Gamma(\cdot, t)$ is continuous.
3. If $\Gamma(X_0, t_0) = 0$ then $\Gamma(Y, t_0) = 0 \; \forall Y \in \partial D$; otherwise $\frac{\Gamma(Y, t)}{\Gamma(X, t)} \leq L$, for a universal constant $L > 0$.

Notice that from 1 the following coercivity condition holds:

(2.5) \[ \lim_{t \to +\infty} \int_{\partial D} \Gamma(X, t) d\mathcal{H}^{n-1}(X) = +\infty. \]

If $\psi$ is a positive continuous function defined on $\partial D$ and $\gamma$ is an increasing convex function, then

\[ \Gamma(X, t) = \psi(X)\gamma(t) \]

gives a typical nonlinearity that fulfills the above properties. As in the Calculus of Variations, $\Gamma$ is chosen based upon the particular problem we are trying to model and no relation whatsoever is imposed upon the nonlinearity $\Gamma$ and $A$.

We point out that our approach can be successfully employed to optimal design problems governed by a more general class of degenerate elliptic PDEs. Indeed, we could consider general variational kernels $F: D^C \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following structural assumptions:

(F1) For each $\xi \in \mathbb{R}^n$, the mapping $X \mapsto F(X, \xi)$ is measurable.
(F2) For a.e. $X \in D^C$, the mapping $\xi \mapsto F(X, \xi)$ is strictly convex and differentiable.
(F3) There exists constants $0 < \lambda \leq \Lambda < \infty$ and a $p > 1$, such that, for a.e. $X \in D^C$ and all $\xi \in \mathbb{R}^n$,
\[ \lambda |\xi|^p \leq F(X, \xi) \leq 2^{-p} \Lambda |\xi|^p. \]

(H4) $F(X, \alpha \xi) = |\alpha|^p F(X, \xi)$, for $\alpha \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$.

It is simple to verify that the above hypothesis on $F$ imply that

\[ A(X, \xi) = \nabla_\xi F(X, \xi). \]

is a measurable monotone $p$-degenerate elliptic map.
Let us point out that in fact the Dirichlet problem (2.2) has a unique solution. Such a solution is obtained by minimizing the variational functional \( E_F \) defined as

\[
E_F(v) := \int_{\Omega \setminus D} F(X, Dv) \, dX,
\]

among all functions \( v \in W^{1,p}(\Omega \setminus D) \), with \( v = \varphi \) on \( \partial D \) and \( v = 0 \) on \( \partial \Omega \).

Sometimes it is convenient to use the language of heat conduction theory to describe the elements involved in our analysis. Thus, \( D \) is the body to be insulated, \( \varphi \) represents the temperature distribution on \( \partial D \), \( \iota \) corresponds to the maximum amount of insulating material available, \( J \) plays the role of the (generalized) heat flux, which is the quantity to be minimized, and \( A \) determines the inhomogeneous and complexity features of the medium. However it is important to highlight that this model is widely applicable to several other situations beyond the bounds of the classical heat conduction theory and other interpretations of the model might provide different insights on what is reasonable to expect to hold.

It is noteworthy to point out that, since we are not forcing any regularity assumption on the medium \( A \), in principle just Hölder continuity estimates are available for an \( A \)-potential \( u = u(\Omega) \). This restriction appears even for linear uniform elliptic operators, e.g., DeGiorgi-Nash-Moser regularity theory. Harnack inequality for \( A \)-potentials was first proven by Serrin, [Serrin63], [Serrin64]. Hölder continuity is a consequence of Harnack inequality. The fact that, in principle, \( A \)-potentials \( u = u(\Omega) \) are merely locally Hölder continuous functions leads to technical issues in the definition of the \( A \)-normal derivative of \( u = u(\Omega) \), as defined in (2.3). Some of our primary results concerning geometric properties of the free boundary will not depend upon any smoothness condition on the medium. However, just to grapple with this inconsistence, we will assume throughout the paper that there exists a small \( 1 \gg \delta_0 > 0 \), such that

\[
\begin{align*}
(i) \quad & A \text{ is Hölder continuous in } D_{\delta_0} := \{ X \in \mathbb{R}^n \mid \text{dist}(X, \partial D) < \delta_0 \} \\
(ii) \quad & \varphi: \partial D \to \mathbb{R} \text{ is smooth: } C^{1,1} \text{ smoothness is sufficient.}
\end{align*}
\]

Once more we emphasize that for the first part of this project, condition (2.6) plays merely a technical role and, for sake of applications, it should not be seen as a constraint. For instance, there are genuine interests in the study of the behavior as \( p \to \infty \) of optimal design problems, see [RT09]. Section 5 of present paper guarantees that such analysis can be carried out in discontinuous media.

2.1. Penalty method and weak formulation. From the mathematical point of view, the minimization problem (2.4) carries too many difficulties to be approached directly. Instead, we will employ a fruitful penalty method in order to
formulate weak versions of problem (2.4). Such a technique has been successfully employed to study a variety of problems in applied mathematics.

The intuitive idea behind a penalization strategy is the following: suppose our problem has an “undesired” (from the mathematical perspective) constraint on the competing configurations (in our case a volume constraint). We then allow any configuration to compete; however we “charge a fee” for those configurations that do not obey the previously set constraint. We expect that, if the fee is too high, optimal configurations will indeed “prefer” to satisfy the original constraint.

Still in a philosophical perspective, one should expect an optimal configuration, \( \Omega^* \), of problem (2.4) to satisfy

\[
\mathcal{L}^n (\Omega^* \setminus D) = \iota.
\]

For that, think of \( \iota \) as the budget available and \( \Omega^* \) as the ultimate object to be built up. Mathematically this fact is indeed easily justified. For instance suppose for an optimal configuration \( \Omega^* \) we had

\[
\mathcal{L}^n (\Omega^* \setminus D) < \iota - \varepsilon,
\]

for some \( \varepsilon > 0 \). Let \( X_0 \in \partial \Omega^* \) be a free boundary point and \( \rho > 0 \) so that \( \omega_n \rho^n < \varepsilon \). Consider

\[
\tilde{\Omega} := \Omega^* \cup B_{\rho}(X_0).
\]

Thus, \( \tilde{\Omega} \) competes with \( \Omega^* \) in the minimization problem (2.4) and, because of maximum principle, \( u(\tilde{\Omega}) > u(\Omega^*) \). Taking into account that \( \Gamma \) is increasing and applying Hopf maximum principle on \( \partial D \), we would conclude

\[
\bar{\omega}(\Omega^*) > \bar{\omega}(\tilde{\Omega}),
\]

which contradicts the minimality property of \( \Omega^* \). Our conclusion is that in problem (2.4) we can regard the condition \( \mathcal{L}^n (\Omega^* \setminus D) \leq \iota \) as \( \mathcal{L}^n (\Omega^* \setminus D) = \iota \). For future reference, let us state this fact as a Lemma.

**Lemma 2.1.** Let \( \Omega^* \) be a minimizer of problem (2.4). Then \( \mathcal{L}^n (\Omega^* \setminus D) = \iota \).

Another general comment: we will always extend the \( A \)-potential \( u(\Omega) \) by zero outside \( \Omega \). Thus, in the distributional sense,

\[
(2.7) \quad \mathcal{L}[u(\Omega)] = 0, \text{ in } \Omega = \{ u(\Omega) > 0 \} \quad \text{and} \quad \mathcal{L}[u(\Omega)] \geq 0, \text{ in } \mathbb{R}^n \setminus D.
\]

Returning to the penalty technique issue: we shall borrow the simple, yet quite clever penalty term suggested in [Tilli99], that is, for each \( \lambda > 0 \), we will
consider the penalization term \( \varrho : \mathbb{R}_+ \to \mathbb{R}_+ \), defined by

\[
(2.8) \quad \varrho(t) := \lambda (t - \iota)^+.
\]

We then define the \( \lambda \)-perturbed functional, \( \hat{J}_\lambda \), to be

\[
(2.9) \quad \hat{J}_\lambda(\Omega) := \int_{\partial D} \Gamma(X, \partial A u(X)) d\mathcal{H}^{n-1}(X) + \varrho_\lambda \left( \mathcal{L}^n \left( \Omega \setminus D \right) \right).
\]

Once more, the idea is the following: we allow \( \hat{J}_\lambda \) to act on any configuration \( \Omega \supset D \) and, when \( \lambda \) is big enough, we hope that an optimal design \( \Omega^* \) for \( \hat{J}_\lambda \) will satisfy \( \mathcal{L}^n \left( \Omega^*_\lambda \setminus D \right) = \iota \), thus it will also be a minimizer for our original optimization problem with volume constraint. Our initial goal is then study existence and geometric properties of the penalized problem:

\[
(2.10) \quad (\mathcal{P}_\lambda) \quad \text{Minimize } \left\{ \hat{J}_\lambda(\Omega) \text{ among all sets } \Omega \supset D \right\}.
\]

However, even the penalty problem (2.10) is, in principle, too hard to be directly approached. Thus, for the time being, it will be more appropriate to initially deal with a weak formulation of problem (2.10), which we start describing now. Let \( \delta_0 \) be the technical number in (2.6). For each \( \delta \ll \delta_0 \), we us define the functional set

\[
(2.11) \quad \mathcal{V}(\delta) := \left\{ f \in W^{1,p}(D^C) \mid f = \varphi \text{ on } \partial D, \ f \geq 0, \ \mathcal{L} f \geq 0, \ \mathcal{L} f = 0 \text{ in } D_\delta \right\}.
\]

Then we define the sample functional set:

\[
(2.12) \quad \mathcal{V} := \bigcup_{\delta \downarrow 0} \mathcal{V}(\delta)
\]

and the weak formulation of problem (2.10) can then be stated as

\[
(2.13) \quad (\mathcal{P}_{\lambda}^{\text{weak}}) \quad \min_{f \in \mathcal{V}} \left\{ \int_{\partial D} \Gamma(X, \partial_A f(X)) d\mathcal{H}^{n-1}(X) + \varrho_\lambda \left( \mathcal{L}^n \left( \{ f > 0 \} \setminus D \right) \right) \right\}.
\]

3. Basic functional and analytic properties. In this section we establish all the basic and necessary properties of the mathematical elements of the problems we are interested in, namely, problems (2.4), (2.10) and (2.13); however we will mostly be concerned with the latter, as it the the weakest formulation among them all.

We start off this section with some fundamental measure theory facts surrounding problems (2.4), (2.10) and (2.13). Initially, we point out that for any
\( f \in \mathcal{V} \), as in (2.12), and for any \( g \in C_0^\infty(D^C) \), with \( g \geq 0 \), there holds

\[
\int_{D^C} g(X) d\Omega f := - \int_{D^C} \langle A(X, Df), Dg \rangle dX \geq 0.
\]

Thus, classical considerations together with Riesz-Markov Theorem assure that \( \Omega f \) defines a nonnegative Radon measure on \( D^C \).

Recall that for a fixed \( f \in \mathcal{V} \), there is an \( \eta = \eta(f) \ll \delta_0 \), such that \( \Omega f = 0 \) in \( D_\eta \). Let \( F := A(X, Df) \) and \( \mathcal{I}_\epsilon \), \( 2\epsilon < \eta \), be an approximation of identity defined on \( D_\epsilon \). One simply verify that

(3.1) \[
\text{div}(F \ast \mathcal{I}_\epsilon) = \mathcal{I}_\epsilon \ast \text{div}(F),
\]

where \( \ast \) denotes the usual convolution with vector fields and with Radon measures respectively. Furthermore,

(3.2) \[
\mathcal{I}_\epsilon \ast \text{div}(F) \rightharpoonup \text{div} F
\]

in the sense of Radon measures. Given a function \( g \in W^{1,p}(D^C) \), we can apply classical Gauss-Green Theorem to \( F \ast \mathcal{I}_\epsilon \) and obtain

(3.3) \[
\int_{D^C} g(X) \text{div}(F \ast \mathcal{I}_\epsilon) dX = \int_{D_{\eta'}} g(X) \text{div}(F \ast \mathcal{I}_\epsilon) dX
\]

\[
= - \int_{D^C} \langle F \ast \mathcal{I}_\epsilon, Dg \rangle dX
\]

\[
+ \int_{\partial D_{\eta'}} g(S) \langle F \ast \mathcal{I}_\epsilon, \nu \rangle dH^{n-1}(S),
\]

for any \( 0 < \eta' < \eta(f) \ll \delta_0 \). Now, since \( F \in L^{\frac{p}{p-1}} \), up to subsequence,

\[
F \ast \mathcal{I}_\epsilon(X) \rightarrow F(X), \quad \text{for a.e. } X \in D^C.
\]

In particular, for almost all \( 0 < \eta' < \eta(f) \),

(3.4) \[
F \ast \mathcal{I}_\epsilon(X) \rightarrow F(X) \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } S \in \partial D_{\eta'}.
\]

Letting \( \epsilon \rightarrow 0 \) in (3.4) and afterwards \( \eta' \rightarrow 0 \), we reach a generalized version of Gauss-Green Theorem for \( \Omega f \), with \( f \in \mathcal{V} \). We will state this result, and few other consequences, as a Lemma for future references.
**Lemma 3.1.** Let $f \in V$, defined in (2.12). Then $Lf$ defines a nonnegative Radon measure, $\mu_f$, in $D^C$. Furthermore, for any $\psi \in C(D^C) \cap W^{1,p}(D^C)$,

$$
\int_{B_r(Y)} \psi(X) d\mu_f(X) + \int_{B_r(Y)} \langle A(X,Df), D\psi(X) \rangle dX = \int_{\partial B_r(Y)} \psi(S) \cdot \partial_A f(S) d\mathcal{H}^{n-1}(S),
$$

for almost all $0 \leq r < \text{dist}(Y, \partial D)$. Also, if $\psi \in W^{1,p}(D^C)$, we have

$$
\int_{\mathbb{R}^n \setminus D} \psi(X) d\mu_f(X) + \int_{\mathbb{R}^n \setminus D} \langle A(X,Df), D\psi(X) \rangle dX = \int_{\partial D} \psi(S) \cdot \partial_A f(S) d\mathcal{H}^{n-1}(S).
$$

Therefore, if $\psi \in W^{1,p}_0(D^C)$, there holds

$$
\int_{D^C} \langle A(X,Df), D\psi(X) \rangle dX = \int_{D^C} \psi(X) d\mu_f(X).
$$

In addition,

$$
\mu_f(\mathbb{R}^n \setminus D) = \int_{\partial D} \partial_A f(S) d\mathcal{H}^{n-1}(S).
$$

For further generalizations of Gauss-Green Theorem for divergence-measure vector fields we refer the readers to [CF99] and [CF03]. Next Lemma provides a monotonicity feature for our variational kernel $F$ (see [DP05]).

**Lemma 3.2.** Let $O$ be a domain in $\mathbb{R}^n$ and $f \in W^{1,p}(O)$. There exists a constant $c = c(n, p, \lambda, \Lambda) > 0$, such that

$$
\int_O [F(X,Df) - F(X,Dh)] dX \geq c \begin{cases} 
\int_O |\nabla(f - h)|^p dX & \text{if } p \geq 2 \\
\alpha(f) \cdot \left[ \int_O |\nabla(f - h)|^p dX \right]^{2/p} & \text{if } 1 < p \leq 2.
\end{cases}
$$

where

$$
\alpha(f) := \left[ \int_O |\nabla f|^p dX \right]^{\frac{1-2/p}{2}}
$$

and $h$ is the $A$-harmonic function in $O$ that agrees with $f$ on $\partial O$.

**Proof.** For each $0 \leq \tau \leq 1$, let $\phi_\tau$ denote the linear interpolation between $f$ and $h$, i.e., $\phi_\tau := \tau f + (1 - \tau)h$. Fundamental Theorem of Calculus yields

$$
\int_D [F(X,Df) - F(X,Dh)] dX = \int_0^1 \frac{d}{d\tau} \left( \int_D F(X,D\phi_\tau) dX \right) d\tau.
$$
Passing the derivative through and using the fact that \( \text{div} (A(X, Dh)) \cdot (f - h) = 0 \) in \( O \), we find

\[
(3.9) \quad \int_0^1 \frac{d}{d\tau} \left( \int_O F(X, D\phi_\tau) dX \right) d\tau \\
= \int_0^1 d\tau \int_O (A(X, D\phi_\tau) - A(X, Dh)) D(f - h) dX \\
= \int_0^1 \frac{1}{\tau} d\tau \int_O (A(X, D\phi_\tau) - A(X, Dh)) D(\phi_\tau - h) dX,
\]

because \( \phi_\tau - h = \tau(f - h) \). Lemma now follows easily from monotonicity of \( A \), namely

\[
(3.10) \quad \langle A(X, \xi_1) - A(X, \xi_2), \xi_1 - \xi_2 \rangle \\
> c_{p, \lambda} \begin{cases} 
|\xi_1 - \xi_2|^p & \text{if } p \geq 2 \\
|\xi_1 - \xi_2|^2(|\xi_1| + |\xi_2|)^{p-2} & \text{if } 1 < p \leq 2.
\end{cases}
\]

Indeed, for \( p \geq 2 \), combining (3.8), (3.9) and taking into account (3.10) we reach

\[
\int_O [F(X, Df) - F(X, Dh)] dX \geq c_{p, \lambda} \int_0^1 \tau^{-1} d\tau \int_O |\nabla \phi_\tau - h|^p dX \\
= pc_{p, \lambda} \int_O |\nabla \phi_\tau - h|^p dX.
\]

Similarly we argue for the case \( 1 < p < 2 \). The Lemma is proven.

Our next Proposition provides an energy estimate for a minimizing sequences to our optimization problems. More precisely, we have:

**Proposition 3.3.** Let \( u_j \) be a minimizing sequence for the functional \( J_\lambda \). Then,

\[
\|\nabla u_j\|_{L^p(D^C)} \leq C,
\]

where \( C \) depends only on dimension, \( A, D, \varphi \) and \( \Gamma \).

**Proof.** Let \( h = h_p \) be the \( p \)-harmonic function in \( D^C \) that agrees with \( \varphi \) on \( \partial D^C \) and vanishes at infinity, that is the solution to

\[
(3.11) \quad \begin{cases} 
\Delta_p h = 0 \text{ in } D^C \\
h = \varphi \text{ on } \partial D^C \\
\lim_{|x| \to \infty} h = 0.
\end{cases}
\]

Existence of such a function is proven as follows: for \( R \gg 1 \), let \( h_R \) be the \( p \)-harmonic function in \( B_R \setminus D \), \( h_R = \varphi \) on \( \partial D \) and \( h_R = 0 \) on \( \partial B_R \). One verifies
that \( \{h_R\} \) as well as \( \{\|h_R\|_{W^{1,p}}\} \) are decreasing sequences. Thus the solution to (3.11) is obtained as \( h := \lim_{R \to \infty} h_R \).

Returning to the proof of Proposition 3.3: from the maximum principle, there holds

\[
0 \leq h \leq \sup_{\partial D} \varphi.
\]

For sake of notation convenience, let us denote \( \mathcal{L} u_j dX := \mu_{u_j} = \mu_j \), as in Lemma 3.1. We clearly have

\[
\int_{\mathbb{R}^n \setminus D} (h - u_j) d\mu_j = \int_{\mathbb{R}^n \setminus D} \langle A(X, Du_j), D(h - u_j) \rangle dX
\]

\[
= \int_{\mathbb{R}^n \setminus D} \langle A(X, Du_j), Dh(X) \rangle dX
\]

\[
- \int_{\mathbb{R}^n \setminus D} \langle A(X, Du_j), Du_j \rangle dX.
\]

From the degenerate ellipticity of \( A \), we can deduce from (3.12) that

\[
\lambda \int_{\mathbb{R}^n \setminus D} |Du_j(X)|^p dX \leq \left| \int_{\mathbb{R}^n \setminus D} (h - u_j) d\mu_j - \int_{\mathbb{R}^n \setminus D} \langle A(X, Du_j), Dh \rangle dX \right|
\]

\[
\leq \sup_{\partial D} \varphi \cdot \mu_j(\mathbb{R}^n \setminus D) + \Lambda \int_{\mathbb{R}^n \setminus D} |Du_j|^{p-1} |Dh| dX
\]

\[
\leq \sup_{\partial D} \varphi \cdot \mu_j(\mathbb{R}^n \setminus D)
\]

\[
+ \frac{\lambda}{2} \int_{\mathbb{R}^n \setminus D} |Du_j|^p dX + C_1 \int_{\mathbb{R}^n \setminus D} |Dh|^p dX.
\]

In the last step we have used Young’s inequality and \( C_1 = \epsilon^{-p/p} \) where \( \epsilon \) satisfies \( \epsilon^{p/(p-1)} = \rho \lambda / 2(p-1) \). In view of (3.7) and the estimate in (3.13), we reach the conclusion that there exists a constant \( C_2 \), depending only on \( \mathcal{A}, D \) and \( \varphi \), such that

\[
\|\nabla u_j\|_{L^p(D^c)}^p \leq C_2 \left( 1 + \frac{1}{2\alpha} \int_{\partial D} \partial D u_j(X) d\mathcal{H}^{n-1}(X) \right),
\]

where \( \alpha := \mathcal{H}^{n-1}(\partial D) \). From the monotonicity and convexity properties of the non-linearity \( \Gamma \), we derive, for each \( Y \in \partial D \) fixed, that

\[
2\Gamma \left( Y, \|\nabla u_j\|_{L^p(D^c)}^p \right) \leq C_3 + \Gamma \left( Y, \frac{1}{\alpha} \int_{\partial D} \partial D u_j(X) d\mathcal{H}^{n-1}(X) \right),
\]

where \( C_3 \) is a constant depending only on \( \mathcal{A}, D, \varphi \) and \( \Gamma \). Once more using the
convexity of \(\Gamma(Y, \cdot)\), it follows from Jensen’s inequality that

\[
2\Gamma\left(Y, \|\nabla u_j\|_{L^p(D_\delta)}^p\right) \leq C_3 + \frac{1}{\alpha} \int_{\partial D} \Gamma(Y, \partial A u_j(X)) \, d\mathcal{H}^{n-1}(X).
\]

Integrating inequality (3.15) with respect to \(Y\) over \(\partial D\) and taking into account property 3 of the non-linearity \(\Gamma\), we derive

\[
\int_{\partial D} \Gamma\left(Y, \|\nabla u_j\|_{L^p(D_\delta)}^p\right) \, d\mathcal{H}^{n-1}(Y) \\
\leq C_4 \left(1 + \int_{\partial D} \Gamma(X, \partial A u_j(X)) \, d\mathcal{H}^{n-1}(X)\right),
\]

where again \(C_4\) depends only upon \(A\), \(D\), \(\varphi\) and \(\Gamma\). Finally, (3.16) and the coercivity of the function

\[
t \mapsto \int_{\partial D} \Gamma(X, t) \, d\mathcal{H}^{n-1}(X),
\]

see (2.5), complete the proof of Proposition 3.3.

\[\square\]

In view of the energy estimate provided in Proposition 3.3, it becomes natural to investigate the behavior of \(\mathcal{J}_\lambda\) over weakly convergent sequences in \(W^{1,p}\).

**Proposition 3.4.** For \(\delta > 0\) fixed the functional set

\[
\mathcal{V}(\delta) := \{f \in W^{1,p}(D^C) \mid f = \varphi \text{ on } \partial D, \ f \geq 0, \ \mathcal{L} f \geq 0, \ \mathcal{L} f = 0 \text{ in } D_\delta\}
\]

is weakly closed in \(W^{1,p}(D^C)\).

**Proof.** The proof follows by slight modifications from the proof of Theorem 3.75 in [HKM]. We omit the details here. \[\square\]

We now prove our functional \(\mathcal{J}_\lambda\) is lower semicontinuous with respect to weak convergence in \(W^{1,p}\) when restricted to the appropriate functional set.

**Lemma 3.5.** Let \(f_j \in \mathcal{V}(\delta)\) be a sequence of functions satisfying that converges weakly to \(f\) in \(W^{1,p}(D^C)\). Then,

\[
\mathcal{J}(f) + \varrho_\lambda (|\{f > 0\}|) \leq \liminf_{j \to \infty} \left\{\mathcal{J}(f_j) + \varrho_\lambda (|\{f_j > 0\}|)\right\}.
\]

**Proof.** Initially, the volume penalty term of the functional \(\mathcal{J}_\lambda\) is indeed weak lower semicontinuous, since, up to a subsequence, \(f_j(X) \to f(X)\) for a.e. \(X \in D^C\).
Thus, by Fatou’s Lemma

\[ |\{ f > 0 \}| \leq \liminf_{j \to \infty} |\{ f_j > 0 \}|. \]

Since the penalty factor \( g_\lambda \) is non-decreasing and continuous, there holds

\[ g_\lambda (|\{ f > 0 \}|) \leq \liminf_{j \to \infty} g_\lambda (|\{ f_j > 0 \}|), \]

as desired. We now focus our attention on the functional

\[ \mathcal{J}(v) = \int_{\partial D} \Gamma(X, \partial_A v) d\mathcal{H}^{n-1}(X). \]

As in the Calculus of Variations, in order to establish the \( W^{1,p} \)-weak lower semi-continuity of \( \mathcal{J} \), we shall explore the convexity assumption on \( \Gamma(X, \cdot) \). Indeed, we start by analyzing functionals with piecewise linear potential, i.e., functionals with the particular profile:

\[ F_m(X, t) = \max_{1 \leq k \leq m} \{ B_k(X) t + C_k(X) \}, \quad B_k, C_k \in C(\partial D). \]

We then label, for each \( k = 1, 2, \ldots, m \), the sets

\[ \mathcal{D}_k(f) := \{ X \in \partial D \mid F_m(X, \partial_A f(X)) = B_k(X) \partial_A f(X) + C_k(X) \}. \]

Thus \( \partial D = \bigcup_{k=1}^m \mathcal{D}_k(f) \), and we may assume that \( \mathcal{D}_k(f) \cap \mathcal{D}_i(f) = \emptyset \), whenever \( k \neq i \). Also, recall that \( \mathcal{L} f_j \) and \( \mathcal{L} f \) define Radon measures in \( D^C \), and as proven in Proposition 3.4, there holds

\[ \mathcal{L} f_j \overset{\ast}{\rightharpoonup} \mathcal{L} f, \]

in the sense of Radon measures. Therefore, using a representation as in (3.5), we obtain that, for any continuous function \( \zeta \in C(\partial D) \),

\[ \int_{\partial D} \zeta(X) \partial_A f(X) d\mathcal{H}^{n-1}(X) \leq \liminf_{j \to \infty} \int_{\partial D} \zeta(X) \partial_A f_j(X) d\mathcal{H}^{n-1}(X). \]
With (3.19), we estimate

\[
\tilde{J}_m(f) = \sum_{k=1}^m \int_{D_k(f)} \left\{ B_k(X) \partial_A f + C_k(X) \right\} d\mathcal{H}^{n-1}(X)
\]

\[
\leq \liminf_{j \to \infty} \sum_{k=1}^m \int_{D_k(f_j)} \left\{ B_k(X) \partial_A f_j + C_k(X) \right\} d\mathcal{H}^{n-1}(X)
\]

\[
\leq \liminf_{j \to \infty} \tilde{J}_m(f_j),
\]

In other words, we have proven functionals as in (3.17) are \( W^{1,p} \)-weak lower semicontinuous. Finally, under the assumption that \( \Gamma(X, \cdot) \) is convex we know that for each \( X \in \partial D \) there exits a sequence of functions \( F_m(X, t) \) as in (3.18) such that, for any \( t \),

\[
(3.20) \quad \Gamma(X, t) = \lim_{m \to \infty} F_m(X, t).
\]

As a combination of (3.20) and the \( W^{1,p} \)-weak lower semicontinuity of each \( \tilde{J}_m \), the Lemma follows.

The results proven in Proposition 3.3 and in Lemma 3.5 are important pieces of information towards establishing the existence of an optimal shape for problem (2.10); however, at this precise stage, those are not enough. We would like to invite the readers to take a small pause in order to appreciate the intrinsic difficulty involved in proving the existence of a minimal configuration to the penalized problem (2.10).

Following the natural scheme, one considers a minimizing sequence, \( \Omega_j \), to the functional \( \mathcal{J}_\lambda \), i.e.,

\[
\mathcal{J}_\lambda(\Omega_j) \overset{j \to \infty}{\longrightarrow} \min_{\Omega \supset D} \mathcal{J}_\lambda.
\]

If \( u_j \) denotes the \( A \)-potential associated to the configuration \( \Omega_j \), it follows from Proposition 3.3 that, up to a subsequence, the \( u_j \) converge weakly and almost everywhere to a function \( u \in W^{1,p}(D^C) \) which is nonnegative. As a consequence of Proposition 3.4 we have that \( \mathcal{L} u \geq 0 \). Lemma 3.5 assures

\[
\int_{\partial D} \Gamma(X, \partial_A u) d\mathcal{H}^{n-1}(X) + \varrho_\lambda (|\{ u > 0 \}|) \leq \min_{\Omega \supset D} \mathcal{J}_\lambda.
\]

Therefore, a natural candidate for an optimal shape to problem (2.10) is

\[
\Omega := \{ X \in \mathbb{R} \setminus D \mid u(X) > 0 \}.
\]

However, with the information we have so far, it is not possible to guarantee that \( (\Omega, u) \) is an admissible pair, i.e., that \( u \) is the \( A \)-potential associated to \( \Omega \), or
equivalently that

\[ Lu = 0 \text{ in } \Omega. \]

In fact, it is not true, in general, that if an ordinary sequence of functions \( u_j \), satisfying \( \Delta u_j = 0 \) in \( \{ u_j > 0 \} \), converges weakly in \( H^1 \) to \( u \), then \( \Delta u = 0 \) in \( \{ u > 0 \} \). As a general comment, the above described difficulty is one of the features that makes problems with varying domains (free boundary problems) notably more delicate.

4. Existence of weak solutions. The scheme presented at the end of the previous section is not pointless: since our sequence is converging to a special configuration, namely a minimizer for the functional \( J_\lambda \), we should keep the hope that this strong additional ingredient will assure that in fact \( Lu = 0 \) in \( \{ u > 0 \} \).

In this section, we will carry this delicate analysis out, which will ultimately allow us to conclude problem (2.13) has always a minimizer. As we will see, even the weak formulation of the penalty version of our primary goal presents rather delicate mathematical issues. This is due in part to the adverse environment generated by the non-linear and degeneracy features of \( A \), and in part to the non-local structure of the problem. The latter makes local perturbations inefficient, and thus more creativity is needed to furnish appropriate competing configurations.

As for our first result towards the existence of a minimizer for problem (2.13), we will provide an a-priori estimate on the distance from the free boundary to the fixed boundary. This is an important supporting result as it allows to seek minimizers in a more suitable class of configurations.

However, in order to accomplish such a result, we initially need to study an auxiliary free boundary problem in the spirit of [AC81], which we present now.

**Theorem 4.1.** Let \( O \) be a domain in \( \mathbb{R}^n \) and \( \psi: O \to \mathbb{R} \) a nonnegative function. Let \( A \) be a \( p \)-degenerate elliptic map and assume \( A(\cdot, \xi) \in C^\epsilon \) for all \( \xi \in \mathbb{R}^n \). Then, for any constant \( \tau > 0 \), there exists a minimizer \( v = \nu_\tau \) to the problem

\[
\text{Minimize} \quad \left\{ E_\tau(f) := \int_O \left\{ (F(X, Df) + \tau \chi_{\{f > 0\}}) \right\} \, dX \mid f \in W^{1,p}(O), f|_{\partial O} = \psi \right\}.
\]

Furthermore, \( v \) is nonnegative, Lipschitz continuous and grows linearly away from the free boundary \( \partial \{ \nu_\tau > 0 \} \).

With the free boundary tools available nowadays, it is not hard to establish the existence as well as optimal regularity and non-degeneracy of a minimizer to the above problem. Basically there are two procedures that lead to these results: one can directly approach the minimization problem, by mixing the strategy as in [AC81] and [DP05]. Another charming and fruitful strategy is to employ a regularizing technique method, basically by mixing the estimates in [MT07] or...
Mathematically the latter is described as follows: choose your favorite nonnegative bounded real function \( \beta \), such that \( \text{supp} \beta = [0, 1] \) and, say, \( \int_0^1 \beta(\zeta) d\zeta = 1 \). For each \( \varepsilon > 0 \) define
\[
\beta_\varepsilon(t) := \frac{1}{\varepsilon} \beta \left( \frac{t}{\varepsilon} \right),
\]
and finally put \( B_\varepsilon(s) := \int_0^s \beta_\varepsilon(\zeta) d\zeta \). The \( \varepsilon \) regularizing problem then becomes
\[
(4.1) \quad \text{Minimize } \left\{ E_\varepsilon(f) := \int_O \left\{ \langle A(X, Df), Df \rangle + \tau B_\varepsilon(f) \right\} dX \mid f \in W^{1,p}(O), \quad f|_{\partial O} = \psi \right\},
\]
The existence of minimizers \( v^\varepsilon \) of (4.1) is standard. One then proves Lipschitz regularity and non-degeneracy for \( v^\varepsilon \), uniform in \( \varepsilon \). By letting \( \varepsilon \downarrow 0 \), up to a subsequence, \( v^\varepsilon \) will converge to a locally Lipschitz function \( v \) that is a minimizer of \( E_\tau \). We omit the details of the proof of Theorem 4.1.

**Proposition 4.2.** There exists a positive constant \( \gamma > 0 \), depending on dimension, \( \lambda, \partial D, \Gamma \) and \( \varphi \), but independent of \( \delta \), such that any minimizer \( u^\delta_\lambda \) of \( J_\lambda \) over \( V(\delta) \) satisfies
\[
D_\gamma := \left\{ X \in D^C \mid \text{dist}(X, \partial D) \leq \gamma \right\} \subset \{ u^\delta_\lambda > 0 \}.
\]

**Proof.** Initially, notice that the first part of Lemma 3.5 assures that \( V(\delta) \) is a weakly closed subset of \( W^{1,p}(D^C) \). Therefore, the lower weak semi-continuity property of \( J_\lambda \), proven in the second part of Lemma 3.5, guarantees that for each \( 0 < \delta \ll 1 \), there exists a function \( u^\delta_\lambda \in V(\delta) \), satisfying
\[
J_\lambda(u^\delta_\lambda) = \min_{V(\delta)} J_\lambda.
\]
Let \( P \in \partial D \) be fixed and \( B = B_\varepsilon(Y) \subset D \) satisfy
\[
\overline{B} \cap \partial D = \{ P \}.
\]
The existence of such a ball is guaranteed by the boundedness of the curvature of \( \partial D \); recall \( D \) is a \( C^{1,1} \) domain. By a compactness argument on \( \partial D \), we can select an \( r < 5\delta_0 \), where \( \delta_0 \) is the universal number from (2.6), such that the above holds for a.e. \( P \in \partial D \). In view of Theorem 4.1, there exists a minimizer,
\( v = v(\tau) \) to the variational free boundary problem:

\[
\text{Min} \left\{ \begin{array}{l}
E_\tau(f) := \int_{5B \setminus B} \left\{ (A(X, Df), Df) + \tau \chi_{\{f > 0\}} \right\} dX, \quad f \in W^{1,p}(5B \setminus B), \\
f\mid_{\partial SB} = 0, \quad \text{and} \quad f\mid_B = \inf_{\partial D} \varphi
\end{array} \right\}
\]

Here \( \tau > 0 \) is a constant to be chosen later. For future reference, let us label the following the sets

\[
\Theta := \{ X \in D^C \cap 5B \mid v(X) > u_\delta^\lambda(X) \} \quad \text{and} \quad \mathcal{O} := \{ X \in D^C \cap 5B \mid v(X) > 0 \}.
\]

It is important to keep in mind that, from the properties of \( v \), we can ensure that there exist constants \( \theta, \hat{\delta} > 0 \), depending only on \( A, \tau, \partial D \) and \( \inf \varphi \) such that

\[
|\mathcal{O} \cap D^C| > \theta(\tau), \quad \text{and} \quad \text{dist} \left( P, \left( \partial \mathcal{O} \cap D^C \right) \right) > \hat{\delta}(\tau).
\]

We now define the function \( m : 5B \setminus B \to \mathbb{R}_+ \) as

\[
m(X) := \begin{cases}
v(X) & \text{in} \quad (D \setminus B) \cap 5B \\
\min\{u_\delta^\lambda(X), v(X)\} & \text{in} \quad D^C \cap 5B,
\end{cases}
\]

Since \( m \) competes with \( v \) in the minimization problem (4.2), we have \( E_\tau(v) \leq E_\tau(m) \). Hence, the following inequality holds

\[
\int_{\Theta} \langle A(X, Du_\delta^\lambda), Du_\delta^\lambda \rangle dX - \int_{\Theta} \langle A(X, Dv), Dv \rangle dX \\
\geq \tau \left\{ \mathcal{L}^n(\mathcal{O}) - \mathcal{L}^n \left( \{ u_\delta^\lambda > 0 \} \cap \mathcal{O} \right) \right\}
\]

Our strategy now is to obtain a competing inequality to (4.4). To this end, let us consider the function \( \mathfrak{M} : D^C \to \mathbb{R}_+ \) defined as

\[
\mathfrak{M}(X) := \max\{v(X), u_\delta^\lambda(X)\},
\]

and compare it with \( u_\delta^\lambda \) in terms of the functional \( \mathfrak{I}_\lambda \). Notice that \( \mathfrak{M} \) as defined above is an element of \( \mathfrak{V}(\delta) \). Using the minimality feature of \( u_\delta^\lambda \), we obtain

\[
\rho_\lambda \left( \mathcal{L}^n(\mathcal{O}) + \mathcal{L}^n \left( \{ u_\delta^\lambda > 0 \} \right) - \mathcal{L}^n \left( \{ u_\delta^\lambda > 0 \} \cap \mathcal{O} \right) \right) \\
- \rho_\lambda \left( \mathcal{L}^n \left( \{ u_\delta^\lambda > 0 \} \right) \right) \\
\geq \int_{\partial D} \Gamma(X, \partial_A u_\delta^\lambda) - \Gamma(X, \partial_A \mathfrak{M}) d\mathcal{H}^{n-1}(X).
\]
From properties 1 and 2 of $\Gamma$, and the Lipschitz continuity of the penalty term $\varrho$, we conclude from (4.5) that there exists a small constant $\alpha_0 = \alpha_0(\partial D, \Gamma)$ such that

$$\frac{\lambda}{\alpha_0} \left( L^n(\mathcal{O}) - L^n \left( \{u^\delta_\lambda > 0\} \cap \mathcal{O} \right) \right) \geq \int_{\partial D} \left\{ \partial_A u^\delta_\lambda - \partial_A \mathcal{M} \right\} d\mathcal{H}^{n-1}(X).$$

Applying the Divergence Theorem (see the representation in (3.6)) and taking into account that $v(X)Lv(X) = 0$ a.e., we obtain

$$\int_{\partial D} \left\{ \partial_A u^\delta_\lambda - \partial_A \mathcal{M} \right\} d\mathcal{H}^{n-1}(X) \geq \frac{1}{\sup_{\partial D} \varphi} \int_{\Theta} \langle ADu^\delta_\lambda, Du^\delta_\lambda \rangle - \langle ADv, Dv \rangle dX.$$

As a combination of (4.4), (4.6) and (4.7) we deduce that

$$\sup_{\partial D} \varphi \cdot \frac{\lambda}{\alpha_0} \left[ L^n(\mathcal{O}) - L^n \left( \{u^\delta_\lambda > 0\} \cap \mathcal{O} \right) \right] \geq \tau \left[ L^n(\mathcal{O}) - L^n \left( \{u^\delta_\lambda > 0\} \cap \mathcal{O} \right) \right].$$

Thus, if $\tau$ is chosen big enough, depending only upon dimension, $A$, $\partial D$ and $\varphi$, there must be the case that $\mathcal{O} \subset \{u^\delta_\lambda > 0\}$.

This together with (4.3) ultimately finishes the proof of the Proposition. \qed

In order to advance in our analysis, we need another related free boundary problem: an $A$-obstacle type problem, which again, with the free boundary technology available, is easy accomplished and therefore we omit the details.

**Theorem 4.3.** Let $\mathcal{M}$ be a measurable set in $D^C$. There exists a unique function $b$, solution to the following obstacle-type problem:

$$\text{Min} \left\{ \int_{D^C} \langle A(X, Df), Df \rangle dX \mid f \in W^{1,p}(D^C) f = \varphi \text{ on } \partial D \text{ and } f \leq 0 \text{ in } \mathcal{M} \right\}.$$

Furthermore, $\sup \varphi \geq b \geq 0$, $\mathcal{L}b = 0$ in $\{b > 0\}$ and $\int b \mathcal{L}b dX = 0$.

We now can state and prove our main theorem concerning the existence of an optimal configuration to weak formulations of problem (2.4), namely problems $(\mathcal{P}^\text{weak}_\lambda)$ and $(\mathcal{P}_\lambda)$.

**Theorem 4.4.** There exists an optimal configuration $\Omega^\star_\lambda$ to problem (2.10) (the penalized problem $(\mathcal{P}_\lambda)$). Furthermore, for a universal modulus of continuity $\sigma$, the $A$-potential associated to $\Omega^\star_\lambda$, $u^\star_\lambda$, is $\sigma$-continuous in $D^C$ and $\|u^\star_\lambda\|_{C^\sigma} \lesssim K(\lambda, D, \varphi, \Gamma, A)$. 

Proof. Let $u^\delta_\lambda$ be a minimizer of $\mathcal{H}_\lambda$ over $\mathcal{V}$. As reported before, the existence $u^\delta_\lambda$ follows directly from Lemma 3.5, and the fact that $\mathcal{V}$ is weakly closed. From Theorem 4.2 we know that

$$D_\gamma \subset \{u^\delta_\lambda > 0\}, \quad \forall \delta > 0.$$ 

Let $B = B_r(x_0)$ be a fixed ball in $D^C$ and $b$ be the solution provided by Theorem 4.3 to

$$\min \left\{ \int_{D^C} \langle A(X,Df),Df \rangle dX \mid f \in W^{1,p}(D^C) \ f = \varphi \phantom{\text{on } \partial D} \text{and } f \leq 0 \text{ in } \{u^\delta_\lambda = 0\} \setminus B \right\}.$$ 

We also consider $h$ to be the $\mathcal{A}$-harmonic function in $B$ that agrees with $u^\delta_\lambda$ on $B^C$. It is standard to verify that

$$0 \leq u^\delta_\lambda \leq b \leq h \leq \sup \varphi.$$ 

As before (more precisely, as in the proof of Proposition 4.2), taking into account that $\int b \mathcal{L} b dX = 0$, we find

$$\int_{\partial D} \Gamma(X,\partial_A u^\delta_\lambda) - \Gamma(X,\partial_A b) \geq c_1 \left( \int_{D^C} \langle A(X,Du^\delta_\lambda),Du^\delta_\lambda \rangle dX - \int_{D^C} \langle A(X,Db),Db \rangle dX \right),$$

for a universal positive constant $c_1 > 0$. However, $h$ competes with $b$ in the obstacle problem (4.8), thus, (4.10) becomes

$$\int_{\partial D} \Gamma(X,\partial_A u^\delta_\lambda) - \Gamma(X,\partial_A b) \geq c_1 \left( \int_{D^C} \langle A(X,Du^\delta_\lambda),Du^\delta_\lambda \rangle dX - \int_{D^C} \langle A(X,Dh),Dh \rangle dX \right).$$

For the moment, let us assume $p \geq 2$. If we take into account Lemma 3.2, we can enhance the estimate from below in (4.11) as

$$\int_{\partial D} \Gamma(X,\partial_A u^\delta_\lambda) - \Gamma(X,\partial_A b) \geq c_2 \left( \int_{D^C} \left( \nabla \left( u^\delta_\lambda - h \right)(X) \right)^p dX \right),$$

for an appropriate positive but small constant $c_2$. Our next step is to compare $u^\delta_\lambda$.
and $b$ in terms of the functional $\mathcal{J}_\lambda$. By doing so, in view of (4.12), we obtain

$$\lambda \mathcal{L}^n \left( \{ X \in B_r(X_0) \mid u^\delta_\lambda(X) = 0 \} \right) \geq c_3 \left( \int_{\partial C} \left| \nabla \left( u^\delta_\lambda - h \right)(X) \right|^p \, dX \right),$$

for another constant $c_3 > 0$, depending on dimension, $A$, $\sup \phi$, and $\Gamma$. If $1 < p \leq 2$, we obtain

$$\lambda \mathcal{L}^n \left( \{ X \in B_r(X_0) \mid u^\delta_\lambda(X) = 0 \} \right)^{p/2} \cdot \left[ \int_{\partial C} \left| \nabla u^\delta_\lambda \right|^p \, dX \right]^{1 - \frac{p}{2}} \geq c_3 \left( \int_{\partial C} \left| \nabla \left( u^\delta_\lambda - h \right)(X) \right|^p \, dX \right).$$

In any case, our conclusion is that if $B_r(X_0) \subset D_\gamma$, then $\# \{ X \in B_r(X_0) \mid u^\delta_\lambda(X) = 0 \} = 0$ and consequently, from either (4.13) or (4.14), $u^\delta_\lambda$ is $A$-harmonic there. Of course $\mathcal{J}_\lambda(u^\delta_\lambda) \leq \mathcal{J}_\lambda(u^{\delta_2}_\lambda)$, provided $\delta_1 \leq \delta_2$. However, from the fact that $\mathcal{L} u^\delta_\lambda = 0$ in $D_\gamma$ we have a much stronger conclusion:

$$\mathcal{J}_\lambda(u^\delta_\lambda) = \mathcal{J}_\lambda(u^{\delta_2}_\lambda),$$

whenever $\delta_1, \delta_2 \leq \gamma$. We have proven the existence of a minimizer $u^*_\lambda$ to $(P\lambda^{\text{weak}})$, that is, problem (2.13).

Our next step is now to prove that $\Omega^* := \{ u^*_\lambda > 0 \}$ is a minimizer to problem (2.10). For that, we have to show

$$\mathcal{L} u^*_\lambda = 0 \text{ in } \Omega^*.$$

Well, but again it is a standard argument to show from either (4.13) or (4.14) that $u^*_\lambda$ belongs to an appropriate De Giorgi’s class (recall $u^*_\lambda$ is a sub-solution and $h$ is Hölder continuous by elliptic estimates). Therefore, there indeed exists a modulus of continuity $\omega (\omega(t) = |t|^\sigma$, for some $\sigma > 0$), such that

$$|u^*_\lambda(X) - u^*_\lambda(Y)| \leq C\lambda \sigma(|X - Y|).$$

In order to prove that $\mathcal{L} u = 0$ in $\{ u > 0 \}$, we argue as follows: let $X_0 \in \{ u > 0 \}$ be a generic point. By the continuity of $u$, there exists an $r_0 > 0$ such that $B_{r_0}(X_0) \subset \{ u > 0 \}$. Therefore, in view of (4.13) or (4.14), we conclude, as before that

$$u^*_\lambda = h \text{ in } B_{r_0}(X_0),$$

and the Theorem is finally proven. \(\square\)
5. Existence of optimal shape in low dimensions. In this section, upon a technical restriction on dimension, we will show that the original volume constrained problem (2.4) admits an optimal configuration. The theory that addresses the existence of an optimal design for problem (2.4) in any Euclidian space will be developed in section 7.

Our strategy is based on a limiting analysis on the penalized problem (2.10). Our first step towards implementing such an analysis is the following simple lemma.

**Lemma 5.1.** There exists a constant \( C > 0 \), depending on \( A, \Gamma, D \) and \( \varphi \), but independent of \( \lambda \), such that if \( u^*_\lambda \) is the \( A \)-potential associated to an optimal shape \( \Omega^*_\lambda \) of problem (2.10), then

\[
\int_{D^c} |\nabla u^*_\lambda(X)|^p dX < C.
\]

**Proof.** Let \( \mathcal{O} \) be your favorite smooth configuration surrounding \( D \) that satisfies

\[
\mathcal{L}^n (\mathcal{O} \setminus D) = \iota,
\]

and let \( \omega \) be its \( A \)-potential, i.e., the \( A \)-harmonic function in \( \mathcal{O} \setminus D \) taking \( \varphi \) and 0 as boundary data on \( \partial D \) and \( \partial \mathcal{O} \) respectively. By the minimality property of \( \Omega^*_\lambda \) we know

\[
\int_{\partial D} \Gamma(X, \partial_A u^*_\lambda(X)) d\mathcal{H}^{n-1}(X) \leq \mathcal{J}_\lambda(\Omega^*_\lambda) \leq \mathcal{J}_\lambda(\mathcal{O}) \leq \int_{\partial D} \Gamma(X, \partial_A \omega(X)) d\mathcal{H}^{n-1}(X) = \mathcal{C}_0,
\]

where \( \mathcal{C}_0 \) is universal, as it depends only on your choice for \( \mathcal{O} \). On the other hand, using the results and notations of Lemma 3.1, we have

\[
\int_{\mathbb{R}^n \setminus D} \langle A(X, Du^*_\lambda), Du^*_\lambda \rangle dX = \int_{\mathbb{R}^n \setminus D} u^*_\lambda(X) d\mu^*_\lambda(X) \leq \sup_{\partial D} \varphi \cdot \mu^*_\lambda(\mathbb{R}^n \setminus D) = \sup_{\partial D} \varphi \cdot \int_{\partial D} \rightda^*_\lambda d\mathcal{H}^{n-1}(S).
\]
From ellipticity and (5.2) we conclude

\[
\varepsilon_1 \int_{DC} |\nabla u^*_\lambda(X)|^p dX \leq \frac{1}{\mathcal{H}^{n-1}(\partial D)} \int_{\partial D} \partial A u^*_\lambda(S) d\mathcal{H}^{n-1}(S),
\]

where \( \varepsilon_1 \) is a positive number that depends on \( A, \varphi \) and \( D \). Now, for each \( Y \in \partial D \) fixed, we obtain from (5.3)

\[
\Gamma \left( Y, \varepsilon_1 \int_{DC} |\nabla u^*_\lambda(X)|^p dX \right) \\
\leq \frac{1}{\mathcal{H}^{n-1}(\partial D)} \int_{\partial D} \partial A u^*_\lambda(S) d\mathcal{H}^{n-1}(S)
\]

In the last inequality we have used Jensen’s Theorem. If we integrate (5.4) with respect to \( Y \) over \( \partial D \), we reach the following conclusion

\[
\int_{\partial D} \Gamma \left( Y, \varepsilon_1 \int_{DC} |\nabla u^*_\lambda(X)|^p dX \right) d\mathcal{H}^{n-1}(Y) \\
\leq \mathcal{C}_2 \int_{\partial D} \Gamma(X, \partial A u^*_\lambda) d\mathcal{H}^{n-1}(X),
\]

where \( \mathcal{C}_2 \) depends only on \( \partial D \) and the non-linearity \( \Gamma \). Finally, if we combine (5.1), (5.5) and (2.5), we deduce that there must exist a constant \( C > 0 \) depending only on \( A, \Gamma, D \) and \( \varphi \), such that

\[
\int_{DC} |\nabla u^*_\lambda(X)|^p dX \leq C,
\]

which is precisely the thesis of the lemma.

**Theorem 5.2.** Assume the dimension \( n \) is less than \( p \). Then there exists an optimal configuration \( \Omega^* \) to problem (2.4).

**Proof.** Because of Lemma 5.1, up to a subsequence, we can assume \( u_\lambda \) converges, as \( \lambda \to \infty \), weakly in \( W^{1,p}(DC) \) to a function \( u^* \). Furthermore, since we have assumed \( n < p \), it follows by the classical Sobolev Imbedding (see, for instance, \cite{Adams75}) that, passing to another subsequence if necessary, we can further assume that \( u_\lambda \) converges locally uniformly to \( u^* \) in \( \mathbb{R}^n \setminus D \) and thus, \( u^* \) is continuous in \( DC \). We claim that

\[
\mathcal{L}u^* = 0 \text{ in } \Omega^* := \{X \in DC \mid u^*(X) > 0\}.
\]

Indeed, let \( X_0 \in \Omega^* \) be an arbitrary point in the set of positivity of \( u^* \), say
$u^*(X_0) = \delta_0 > 0$. By continuity, there exists an $r_0 > 0$ such that

$$u^*(X) > \frac{\delta_0}{3} \text{ in } B_{r_0}(X_0).$$

Since $u^*_\lambda$ converges uniformly to $u^*$ in $B_{r_0}(X_0)$, there exists a $\lambda_0$ large enough, such that

$$u^*_\lambda(X) > \frac{\delta_0}{7} \text{ in } B_{r_0}(X_0), \quad \forall \lambda > \lambda_0.$$

However, we have proven that $\mathcal{L}u^*_\lambda = 0$ in $\{u^*_\lambda > 0\}$. Therefore, for $\lambda$ large enough, each $u^*_\lambda$ is $A$-harmonic in $B_{r_0}(X_0)$. Thus, as argued in the proof of Lemma 3.5, we in fact conclude $u^*$ is $A$-harmonic in its set of positivity and the first claim is proven. Notice furthermore that, in view of Proposition 4.2,

$$\text{dist}(\partial D, \partial \Omega^*) > \gamma,$$

for some $\gamma > 0$. From inequality (5.1), we have, in particular, that

$$\lambda (\mathcal{L}u^n (\Omega^*_\lambda \setminus D) - \iota)^+ \leq C_0,$$

for a universal constant $C_0$. Thus, using Fatou’s Lemma we see that

$$(\mathcal{L}u^n (\Omega^* \setminus D) - \iota)^+ \leq \liminf_{\lambda \to \infty} (\mathcal{L}u^n (\Omega^*_\lambda \setminus D) - \iota)^+ = 0.$$

That is, our candidate to an optimal design for problem (2.4), $\Omega^*$, does satisfy

$$\mathcal{L}u^n (\Omega^* \setminus D) \leq \iota,$$

so it competes in problem (2.4). Our final step is to show that in fact $\Omega^*$ is an optimal configuration for problem (2.4). For that, let $\mathcal{C}$ be any competing configuration for problem (2.4), i.e., $\mathcal{L}u^n (\mathcal{C} \setminus D) \leq \iota$, and $v$ its $A$-potential, that is, $v$ satisfies

$$\mathcal{L}v = 0 \text{ in } \mathcal{C} \setminus D, \quad v = \varphi \text{ on } \partial D, \quad v = 0 \text{ on } \partial \mathcal{C}.$$

In particular $\mathcal{C}$ competes with $u^*_\lambda$ in $(\mathfrak{P}_\lambda)$, problem (2.10); therefore,

$$\mathfrak{F}(\mathcal{C}) := \int_{\partial D} \Gamma (X, \partial_A v(X)) d\mathcal{H}^{n-1}(X)$$

$$= \mathfrak{F}_\lambda (\mathcal{C})$$

$$\geq \mathfrak{F}_\lambda (\Omega^*_\lambda)$$
because of the weak lower semi-continuity feature of $J$ proven in Lemma 3.5. Finally if we let $\lambda \to \infty$ in the above chain of inequalities, the Theorem is proven. \hfill \square

It is worth to point out that Theorem 5.2 gives the existence of an optimal configuration to problem (2.4) with no regularity whatsoever on the medium. That is, up to this point of the program, the operator $A$ has been a general bounded measurable degenerated elliptic map. However, it turns out that in order to advance on the study of existence of optimal shapes for problem (2.4), with no restriction on the dimension, some extra information is needed to perform appropriate perturbations on the optimal designs $\Omega^*_\lambda$. This will be the contents of the next two sections.

6. Continuous medium and fine weak geometric properties of the free boundary. In this section we will prove that the free boundary, $\partial \Omega_{\lambda}^*$, enjoys the appropriate weak geometry. This feature will allow us to produce geometric measure perturbations that will ultimately lead us to conclude that, if the penalty term $\lambda$ is too large, but still finite, then $\Omega^*_\lambda$, in fact, obey $\mathcal{L}^n(\Omega^*_\lambda \setminus D) \leq \iota$. The latter will be carried out in Section 7.

As highlighted in the last paragraph of the previous section, in order to accomplish a deeper understanding on the free boundary $\partial \Omega^*_\lambda$, we will need to enforce a mild continuity assumption on the medium. Thus, hereafter, unless otherwise stated, we shall assume that for some $\epsilon > 0$, the map

$$X \mapsto A(X, \xi) \in C^\epsilon(\mathbb{R}^n \setminus D), \quad \forall \xi \in \mathbb{R}^n \text{ and } \sup_{\xi \in \mathbb{R}^n} \|A(X, \xi)\|_{C^\epsilon} < K. \tag{6.1}$$

Mathematically condition (6.1) enables $C^{1,\alpha}$ elliptic estimates for solutions to

$$\mathcal{L}\psi = 0,$$

see, for instance, [Tolksdorf84], [Manfredi88] or [Giusti03], Chapter 8. At least equally important for our purposes is the fact that condition (6.1) unlocks the Hopf’s maximum principle for $A$-harmonic functions. For strong maximum principles see, for instance, [Vazquez84] or [Montenegro99].

As for our first theorem in this section, we will obtain optimal regularity for $A$-potentials $u^*_\lambda$ associated to optimal configurations $\Omega^*_\lambda$ of Problem (P$_\lambda$), that is, Problem (2.10). Notice that inside $\Omega^*_\lambda$, the function $u^*_\lambda$ satisfies $\mathcal{L}u^*_\lambda$; therefore, it is locally $C^{1,\alpha}$ smooth. However, from the Hopf’s maximum principle, $u^*_\lambda$
reaches the free boundary with a positive slope, thus \( \nabla u^\star \) jumps from a positive value to zero through the free boundary, \( \partial \Omega^\star \). The conclusion is that the optimal regularity we can hope for \( u^\star \) is Lipschitz continuity. This is the contents of the next theorem.

**Theorem 6.1.** Let \( \Omega^\star \) be an optimal configuration to Problem (2.10) and \( u^\star \) its \( \mathcal{A} \) potential. Then,

\[
\| \nabla u^\star \|_{L^\infty(\mathbb{R}^n \setminus D)} \leq C \lambda^{1/p},
\]

for a constant \( C \) that depends only on \( \mathcal{A}, \Gamma, \varphi \), and \( D \).

**Proof.** We will provide two proofs of this important theorem. The first one follows the glamorous approach suggested in [AC81]. Unfortunately, for non-local problems like ours, the efficiency of that method is restricted to the case \( p \geq 2 \) and a new and more modern argument is required to establish Lipschitz continuity for \( \mathcal{A} \)-potential associated to an optimal design with when \( 1 < p < 2 \). The second proof, inspired by arguments from Lemma 3.2 in [DP05], works for all \( p > 1 \).

1st Proof. The case \( p \geq 2 \). We shall initially obtain a competing estimate for Inequality (4.13), with \( u^\delta \lambda \) replaced by \( u^\star \lambda \). Enhancing the notation in the proof of Theorem 4.4, \( B = B_d(X_0) \) will be a ball centered at a point in \( \Omega^\star \), \( \text{dist}(X_0, \partial D) \gg d \geq \text{dist}(X_0, \partial \Omega^\star) \) and \( \mathfrak{h} \) the \( \mathcal{A} \)-harmonic function in \( B \) that agrees with \( u^\star \lambda \) on \( \partial B \). For any direction \( \nu \in S_{n-1} \), we define

\[
r^\nu := \min \left\{ r \mid \frac{1}{4} \leq r \leq 1 \text{ and } u^\star \lambda(X_0 + dr \nu) = 0 \right\}
\]

if such a set is nonempty; otherwise, we put \( r^\nu = 1 \). For almost every direction \( \nu \) the map \( r \mapsto u^\star \lambda(X_0 + dr \nu) \) is in \( W^{1, p}[\frac{1}{4}, 1] \). Thus, taking into account that \( u^\star \lambda(X_0 + dr \nu) = 0 \) whenever \( r^\nu < 1 \), we can compute,

\[
\mathfrak{h}(X_0 + dr \nu) = \int_{r^\nu}^{1} \frac{d}{dr}(u^\star \lambda - \mathfrak{h})(X_0 + dr \nu)dr \\
\leq d \cdot (1 - r^\nu)^{1/p'} \times \left[ \int_{r^\nu}^{1} |\nabla(u^\star \lambda - \mathfrak{h})(X_0 + r \nu)|^p dr \right]^{1/p},
\]

where, as usual, \( p' \) denotes the conjugate of \( p \), i.e., \( \frac{1}{p} + \frac{1}{p'} = 1 \). Now, by the Harnack Inequality, we know

\[
\inf_{B\setminus \mathfrak{h}} \mathfrak{h} \geq c_1 \mathfrak{h}(X_0),
\]

for a constant \( c_1 \) that depends only on \( \mathcal{A}, \Gamma, \varphi \), and \( D \).
for a constant $c_1 > 0$ that depends only on dimension and $\mathcal{A}$. Here $B_{\frac{7}{8}}$ stands for $B_{\frac{7}{8}}(X_0)$. Let us consider the universal barrier, $\mathcal{B}$, given by

$$
\begin{cases}
\text{div} (\mathcal{A}(X_0 + dX)D\mathcal{B}(X)) = 0 \text{ in } B_1(0) \setminus B_{\frac{7}{8}}(0) \\
\mathcal{B} = 0 \text{ on } \partial B_1(0) \\
\mathcal{B} = c_1 \text{ in } B_{\frac{7}{8}}(0),
\end{cases}
$$

where $c_1$ is the universal constant in (6.3). By the Hopf’s maximum principle, there exists a universal constant $c_2 > 0$, such that

$$
\mathcal{B}(X) \geq c_2 (1 - |X|).
$$

By the maximum principle and (6.5) we can write

$$
\mathcal{B}(X) \geq c_2 \mathcal{B}(X_0) \cdot (1 - |X|).
$$

Combining (6.2) and (6.6) we end up with

$$
d^p \cdot \left[ \int_{r_\nu}^{\frac{1}{r_\nu}} |\nabla (h - u^*_\lambda)(X_0 + r\nu)|^p dr \right] \geq c_3 h^p(X_0) \cdot (1 - r_\nu).
$$

Integrating (6.7) with respect to $\nu$ over $S^{n-1}$, taking into account the definition of $r_\nu$, we find

$$
\left( \frac{h(X_0)}{d} \right)^p \cdot \int_{B_d(X_0) \setminus B_{d/4}(X_0)} \chi \{ u^*_\lambda = 0 \} dX \leq C_4 \int_{B_d(X_0)} |\nabla (h - u^*_\lambda)(X)|^p dX.
$$

If we replace, in all of our arguments so far, $B_{d/4}(X_0)$ by $B_{d/4}(X)$, for any $X \in \partial B_{d/2}(X_0)$, we obtain

$$
\left( \frac{h(X_0)}{d} \right)^p \cdot \int_{B_d(X_0) \setminus B_{d/4}(X)} \chi \{ u^*_\lambda = 0 \} dX \leq C_4 \int_{B_d(X_0)} |\nabla (h - u^*_\lambda)(X)|^p dX, \quad \forall X \in \partial B_{d/2}(X_0).
$$

Integrating (6.9) with respect to $X$, we prove the following important estimate:

$$
\left( \frac{h(X_0)}{d} \right)^p \cdot |\{ X \in B_d(X_0) \mid u^*_\lambda(X) = 0 \}| \leq C_5 \int_{B_d(X_0)} |\nabla (h - u^*_\lambda)(X)|^p dX.
$$

Now we argue as follows: let $\rho := \text{dist}(X_0, \partial \Omega)$ and for each $0 < \delta \ll 1$, denote $h_\delta$ the $\mathcal{A}$-harmonic function in $B_{\rho + \delta}(X_0)$ that agrees with $u^*_\lambda$ on $\partial B_{\rho + \delta}(X_0)$.
Combining (4.13) and (6.10) together with standard elliptic estimate, we deduce

\begin{equation}
(u^*_\lambda(X_0) = h_\delta(X_0) + o(1) \\
\leq C \lambda^{1/p}(\rho + \delta) + o(1).
\end{equation}

Letting \( \delta \searrow 0 \) in (6.11) we finally conclude

\begin{equation}
u^*_\lambda(X_0) \leq C \text{dist}(X_0, \partial \Omega^*_\lambda),
\end{equation}

which clearly implies that \( u^*_\lambda \) is Lipschitz continuous up to the free boundary \( \partial \Omega^*_\lambda \) and \( \| \nabla u^*_\lambda \|_\infty \lesssim \lambda^{1/p} \).

**2nd Proof. The general case.** We may assume, with no loss of generality, that the origin is a free boundary point. From Proposition 4.2, there exists a positive constant \( \gamma > 0 \) such that \( B_\gamma \cap D = \emptyset \). We will show that \( u^*_\lambda \) is Lipschitz continuous within \( B_{\frac{1}{10} \gamma} \). To this end, let us assume, for purpose of contradiction, that there exists a sequence of points \( X_k \in B_{\frac{1}{10} \gamma} \), with

\begin{equation}
X_k \to \partial \Omega^*_\lambda, \quad \text{and} \quad \frac{u^*_\lambda(X_k)}{\text{dist}(X_k, \partial \Omega^*_\lambda)} \nearrow +\infty.
\end{equation}

Hereafter, let us denote

\[ d_k := \text{dist}(X_k, \partial \Omega^*_\lambda), \quad N_k := u(X_k). \]

For each \( k \) fixed, let \( Y_k \) be a point on \( \partial \Omega^*_\lambda \) that satisfies

\[ |Y_k - X_k| = \text{dist}(X_k, \partial \Omega^*_\lambda) = d_k. \]

By Harnack inequality, there exists a universal constant \( c > 0 \) independent of \( k \) such that

\begin{equation}
\sup_{B_{\frac{1}{10} \gamma}(Y_k)} u^*_\lambda \geq c N_k.
\end{equation}

In the sequel, we shall use the cut-off function \( Z \mapsto \text{dist}(Z, \partial B_{d_k}(Y_k)) \) to obtain a local control of \( u^*_\lambda \) near the free boundary. Let us initially consider the set \( A_k \subset B_{d_k}(Y_k) \) to be the points in \( B_{d_k}(Y_k) \) such that their distance to the free boundary \( \partial \Omega^*_\lambda \) is at most \( \frac{1}{3} \) of their distance to the boundary of \( B_{d_k}(Y_k) \), that is

\begin{equation}
A_k := \{ Z \in B_{d_k}(Y_k) \mid \text{dist}(Z, \partial \Omega^*_\lambda) \leq \frac{1}{3} \text{dist}(Z, \partial B_{d_k}(Y_k)) \}.
\end{equation}
If we define $M_k$ to be the maximum of $\text{dist}(Z, \partial B_{d_k}(Y_k))u_\lambda^*(Z)$ over $A_k$, we have

$$M_k := \sup_{Z \in A_k} \text{dist}(Z, \partial B_{d_k}(Y_k))u_\lambda^*(Z) = \text{dist}(Z_k, \partial B_{d_k}(Y_k))u_\lambda^*(Z_k) \geq \frac{3}{4}d_k \sup_{B_{\frac{1}{3}d_k}(Y_k)} u_\lambda^*,$$

for some $Z_k \in A_k$. The last estimate comes from the fact that $Y_k \in \partial \Omega_\lambda^*$, thus if $|Z - Y_k| \leq \frac{d_k}{4}$, then $\frac{1}{3} \text{dist}(Z, \partial B_{d_k}(Y_k)) \geq \frac{1}{3} \times \frac{3}{4}d_k \geq \text{dist}(Z, \partial \Omega_\lambda^*)$. In particular,

$$u_\lambda^*(Z_k) \geq \frac{d_k}{\text{dist}(Z_k, \partial B_{d_k}(Y_k))} \cdot \frac{3}{4} \sup_{B_{\frac{1}{3}d_k}(Y_k)} u_\lambda^* \geq \frac{3}{4} \sup_{B_{\frac{1}{3}d_k}(Y_k)} u_\lambda^*. \tag{6.16}$$

For each $k$, let $W_k \in \partial \Omega_\lambda^*$ be such that

$$\rho_k := |Z_k - W_k| = \text{dist}(Z_k, \partial \Omega_\lambda^*) \leq \frac{1}{3} \text{dist}(Z_k, \partial B_{d_k}(Y_k)) \leq \frac{1}{3}d_k. \tag{6.17}$$

From 6.17, we see that that

$$\rho_k \leq \frac{1}{3}(d_k - |Z_k - Y_k|) \leq \frac{1}{3}(d_k - \rho_k).$$

That is,

$$\rho_k \leq \frac{1}{4}d_k. \tag{6.18}$$

Combining 6.13), 6.16) and 6.18), we reach

$$\frac{N_k}{d_k} \leq C \frac{u_\lambda^*(Z_k)}{\rho_k}, \tag{6.19}$$

for a universal constant $C > 0$. Therefore, our proof will be complete if we show that there exists a universal constant $R > 0$, independent of $k$, such that

$$u_\lambda^*(Z_k) \leq R\rho_k. \tag{6.20}$$

The remaining part of the proof is devoted to verify 6.20). Given a point $\xi \in B_{2\rho_k}(W_k)$, we can estimate, in view of 6.17),

$$|\xi - Y_k| \leq |\xi - W_k| + |W_k - Z_k| + |Z_k - Y_k| \leq 2\rho_k + \rho_k + |Z_k - Y_k| \leq d_k,$$

because $Z_k \in A_k$, that is, $B_{2\rho_k}(W_k)$ is contained in $B_{d_k}(Y_k)$. For any $\zeta \in B_{\frac{1}{2}\rho_k}(W_k)$,
we have
\[ \text{dist} (\zeta, \partial \Omega_k^\star) \leq \frac{1}{2} \rho_k. \]

By triangular inequality and 6.17),
\[
\text{dist} (\zeta, \partial B_{\rho_k} (Y_k)) \geq \text{dist} (Z_k, \partial B_{\rho_k} (Y_k)) - |Z_k - \zeta| \\
\geq \text{dist} (Z_k, \partial B_{\rho_k} (Y_k)) - \frac{3}{2} \rho_k \\
\geq \frac{3}{2} \rho_k \\
\geq \frac{1}{2} \text{dist} (Z_k, \partial B_{\rho_k} (Y_k)).
\]

In particular we conclude that \( B_{\frac{3}{2} \rho_k} (W_k) \subset A_k \). It also follows from above estimate that, for any \( \zeta \in B_{\frac{3}{2} \rho_k} (W_k) \), there holds
\[
u^\star_k (Z_k) = \max_{Z \in A_k} \frac{\text{dist} (Z, \partial B_{\rho_k} (Y_k)) \nu^\star_k (Z)}{\text{dist} (Z_k, \partial B_{\rho_k} (Y_k))} \geq \frac{\text{dist} (\zeta, \partial B_{\rho_k} (Y_k)) \nu^\star_k (\zeta)}{\text{dist} (Z_k, \partial B_{\rho_k} (Y_k))} \geq \frac{1}{2} \nu^\star_k (\zeta).
\]

Taking the maximum over \( B_{\frac{3}{2} \rho_k} (W_k) \) in the above inequality we conclude
\[
(6.22) \quad \max_{B_{\frac{3}{2} \rho_k} (W_k)} \nu^\star_k \leq 2 \nu^\star_k (Z_k).
\]

Recall that \( \mathfrak{L} \nu^\star_k = 0 \) in its positive set. Since \( B_{\rho_k} (Z_k) \subset \{ \nu^\star_k > 0 \} \), by Harnack inequality, there exists a universal constant \( c' > 0 \), for which,
\[
(6.23) \quad \inf_{B_{\frac{3}{2} \rho_k} (Z_k)} \nu^\star_k \geq c' \nu^\star_k (Z_k).
\]

Therefore we have
\[
(6.24) \quad \sup_{B_{\frac{3}{2} \rho_k} (W_k)} \nu^\star_k \geq c' \nu^\star_k (Z_k).
\]

We shall use a blow-up technique. Let us define the re-scaled function \( \Psi_k : B_{\frac{3}{2} \rho_k} \rightarrow \mathbb{R} \) as
\[
(6.25) \quad \Psi_k (\xi) := \frac{1}{u^\star_k (Z_k)} u^\star_k \left( W_k + \frac{1}{2} \rho_k \xi \right).
\]
It follows from 6.22) and 6.24) that

\[ \max_{\overline{B}_1} \Psi_k \leq 2, \quad \max_{\overline{B}_{1/2}} \Psi_k \geq c', \quad \Psi_k(0) = 0. \]

Now, let $\mathfrak{h}$ be the $L$-harmonic function in $B_{1/2}^k(W_k)$ taking boundary data equal to $u_\lambda^*$. Let us also consider, $b_k$, the solution to the obstacle problem in Theorem 4.3 with $\mathcal{M} = \{u_\lambda^* = 0\} \setminus B_{1/2}^k(W_k)$. By comparing $u_\lambda^*$ and $b_k$ in terms of the optimal design problem (2.10) we deduce, as in the proof of Theorem 4.4, that

\[ \int_{B_{1/2}^k(W_k)} \langle A(X, Du_\lambda^*), Du_\lambda^* \rangle - \langle A(X, D\mathfrak{h}), D\mathfrak{h} \rangle dX \leq \lambda \left( \frac{1}{2} \rho_k \right)^n. \]

For each $k \geq 1$, consider the function $H_k: B_1 \rightarrow (0, 1)$ given by

\[ H_k(\xi) := \frac{1}{u_\lambda^*(Z_k)} \mathfrak{h} \left( W_k + \frac{1}{2} \rho_k \xi \right). \]

We know that $H_k$ is the unique minimizer of

\[ \mathcal{D}(v) := \int_{B_1} \langle A \left( W_k + \frac{1}{2} \rho_k X, Dv \right), Dv \rangle dX, \]

among functions $v \in W^{1,p}_0(B_1) + H_k$ and it satisfies

\[ \begin{cases} 
\text{div} \left( A \left( W_k + \frac{1}{2} \rho_k X, DH_k \right) \right) = 0 & \text{in } B_1 \\
H_k = \Psi_k & \text{on } \partial B_1.
\end{cases} \]

A direct computation reveals that

\[ \nabla \Psi_k(Z) = \frac{1}{2} \frac{\rho_k}{u_\lambda^*(Z_k)} \nabla u_\lambda^* \left( W_k + \frac{1}{2} \rho_k Z \right) \quad \text{and similarly} \]

\[ \nabla H_k(Z) = \frac{1}{2} \frac{\rho_k}{u_\lambda^*(Z_k)} \mathfrak{h} \left( W_k + \frac{1}{2} \rho_k Z \right). \]

If we apply the Change of Variables Theorem, taking into account (6.31), we reach

\[ \int_{B_{1/2}^k(W_k)} \langle A(X, Du_\lambda^*), Du_\lambda^* \rangle dX \]

\[ = \left\{ \frac{1}{2} \frac{\rho_k}{u_\lambda^*(Z_k)} \right\}^{-p} \left( \frac{1}{2} \rho_k \right)^n \int_{B_1} \langle A \left( W_k + \frac{1}{2} dk, D\Psi_k \right), D\Psi_k \rangle dX, \]
and the same holds when we replace \( u^*_k \) by \( h \) and \( \Psi_k \) by \( H_k \), that is

\[
\int_{B_1} \langle \mathcal{A}(X, D\mathcal{h}), D\mathcal{h} \rangle dX = \left\{ \frac{1}{2} \rho_k u^*_k(Z_k) \right\}^p \cdot \left( \frac{1}{2} \rho_k \right)^n \int_{B_1} \langle \mathcal{A} \left( X + \frac{1}{2} d_k X, DH_k \right), DH_k \rangle dX.
\]  

(6.33)

If no such an \( R > 0 \) exists in 6.20, in view of (6.27), 6.32 and 6.33, we would obtain

\[
\left( \int_{B_1} \langle \mathcal{A} \left( Y_k + \frac{1}{2} \rho_k X, D\Psi_k \right), D\Psi_k \rangle - \langle \mathcal{A} \left( Y_k + \frac{1}{2} \rho_k X, DH_k \right), DH_k \rangle dX \right) = o(1)
\]  

(6.34)

as \( k \to \infty \). Since \( 0 \leq \Psi_k \leq 2 \), in \( B_1 \), it follows by Morrey’s Theorem, as in the proof of Theorem 4.4, that \( \Psi_k \) and \( H_k \) are uniformly Hölder continuous in, say \( B_{7/8} \). By Ascoli’s Theorem, up to a subsequence, we have

\[
\Psi_k \to \Psi \quad \text{and} \quad H_k \to H,
\]  

(6.35)

uniformly in \( \overline{B}_{6/7} \) and weakly in \( W^{1,p} \). Also, up to subsequence, the sequence of free boundary points \( W_k \) converges, say \( W_k \to W_0 \in \partial \Omega^*_\lambda \). From (6.29) and (6.30), we conclude that

\[
\begin{cases}
\text{div} \left( \mathcal{A}(W_0, DH) \right) = 0 & \text{in } B_1 \\
H = U & \text{on } \partial B_1
\end{cases}
\]  

(6.36)

We can also assure that \( H \) is the unique minimizer of

\[
\mathcal{D}_0(\mathcal{v}) := \int_{B_1} \langle \mathcal{A}(Y_0, D\mathcal{v}), D\mathcal{v} \rangle dX,
\]  

(6.37)

among functions \( \mathcal{v} \in W^{1,p}_0(B_1) + H \). However, from (6.34) and Proposition 3.4, we obtain that

\[
\mathcal{D}_0(\mathcal{w}) = \mathcal{D}_0(\mathcal{h}).
\]  

(6.38)

Therefore, \( \Psi \equiv H \). In particular, \( \Psi \) solves the elliptic PDE

\[
\text{div} \left( \mathcal{A}(Y_0, D\Psi) \right) = 0 \text{ in } B_1.
\]  

(6.39)

On the other hand, since \( \Psi(0) = 0 \), by the strong maximum principle we would
conclude that $\Psi \equiv 0$, which contradicts the fact that
\[
\max_{B_{1/2}} \Psi \geq c' > 0,
\]
thus, 6.18) is verified and the general proof of Theorem 6.1 is finally concluded.

Our next step is to prove that $u_\lambda^*$ grows linearly away from $\partial \Omega_\lambda^*$. Notice that this is the largest admissible growth rate allowed by the Lipschitz regularity previously proven in Theorem 6.1. Here is the precise statement:

**THEOREM 6.2.** There exists a constant $c_\gamma > 0$, depending on dimension $A$, $D$, $\Gamma$ and $\phi$, such that
\[
\lambda^{-1/p} c_\gamma \cdot \text{dist} \left( X_0, \partial \Omega_\lambda^* \right) \leq u_\lambda^*(X_0),
\]
for any $X_0 \in \Omega_\lambda^*$.

**Proof.** Let us fix $X_0 \in \Omega_\lambda^*$ near the free boundary and label $d := \text{dist} \left( X_0, \partial \Omega_\lambda^* \right)$. From Theorem 4.3, there exists a unique solution, $\phi$, to the following obstacle problem

\[
(6.39) \quad \text{Min} \left\{ \int_{DC} \langle A(X, Df), Df \rangle dX \mid f \in W^{1,p}(D^C) f = \phi \right. \nonumber\]
\[
\text{on } \partial D \text{ and } f \leq 0 \text{ in } \{u_\lambda^* = 0\} \cup B_d(X_0).\nonumber
\]

Recall, in Theorem 4.4, we proved that $u_\lambda^*$ is also a minimizer for problem $(\mathcal{P}_\lambda^\text{weak})$, that is problem (2.13), and clearly $\phi$ competes with $u_\lambda^*$ in such a problem; therefore

\[
(6.40) \quad \int_{\partial D} \left( \Gamma(X, \partial A \phi) - \Gamma(X, \partial A u_\lambda^*) \right) d\mathcal{H}^{n-1}(X) \geq \lambda^{-1} c_n d^n,
\]
for a dimensional constant $c_n$. Since both $u_\lambda^*$ and $\phi$ are $A$-harmonic in $D_\gamma$, where $\gamma$ is the number in Proposition 4.2, for a constant $C_1 = C_1(\Gamma)$, we can estimate

\[
(6.41) \quad \int_{\partial D} \left( \Gamma(X, \partial A \phi) - \Gamma(X, \partial A u_\lambda^*) \right) d\mathcal{H}^{n-1}
\]
\[
\leq C_1 \int_{\partial D} \left( \partial A \phi - \partial A u_\lambda^* \right) d\mathcal{H}^{n-1}
\]
\[
\leq \frac{C_1}{\inf_{\partial D} \phi} \int \left( \langle A(X, D\phi), D\phi \rangle - \langle A(X, D u_\lambda^*), D u_\lambda^* \rangle \right) dX.
\]
Here we have used the measure representation provided by Lemma 3.1. Now let \( h \) satisfy

\[
\mathcal{L} h = 0 \text{ in } B_{\frac{3}{2}d}(X_0), \quad h = 0 \text{ in } B_{\frac{3}{2}d}(X_0), \quad \text{and} \quad h = 1 \text{ on } \partial B_{\frac{3}{2}d}(X_0).
\]

By the Harnack inequality, there exists a constant \( c_2 > 0 \), such that

\[
(6.42) \quad u_\lambda^*(X) \geq c_2 u_\lambda^*(X_0) h(X) \text{ in } B_{\frac{3}{2}d}(X_0).
\]

Consider the auxiliary function

\[
g(X) := \begin{cases} \min \{ u_\lambda^*(X), c_2 u_\lambda^*(X_0) h(X) \} & \text{in } B_{\frac{3}{2}d}(X_0) \\ u_\lambda^*(X) & \text{in } \Omega \setminus B_{\frac{3}{2}d}(X_0). \end{cases}
\]

Notice that \( g \) competes with \( \phi \) in the obstacle problem, thus, combining (6.40), (6.41) and replacing \( \phi \) by \( g \), we obtain

\[
(6.43) \quad \lambda^{-1} c_3 \leq \frac{1}{d^n} \int_{\Pi} \left( \langle A(X, Dg), Dg \rangle - \langle A(X, Du_\lambda^*), Du_\lambda^* \rangle \right) dX.
\]

Where set set of integration in the above estimate can be taken to be

\[
\Pi := \left\{ X \in B_{\frac{3}{2}d}(X_0) \setminus B_{\frac{3}{2}d}(X_0) \mid c_2 u_\lambda^*(X_0) h(X) \leq u_\lambda^*(X) \right\}.
\]

However, in this set, we can estimate

\[
(6.44) \quad \langle A(X, Dg), Dg(X) \rangle \leq \Lambda |Dg|^p \
\leq |Dh(X)|^p \cdot [c_2 u_\lambda^*(X_0)]^p \
\leq C \left[ \frac{c_2 u_\lambda^*(X_0)}{d} \right]^p.
\]

In the last inequality we have used the \( C^{1,\alpha} \) estimate for \( h \). Finally, a combination of (6.43) and (6.44) leads us to

\[
\lambda^{-1/p} \varepsilon d \leq u_\lambda^*(X_0),
\]

for a constant \( \varepsilon = \varepsilon(n, \mathcal{A}, D, \Gamma, \varphi) \), and the theorem is proven. \( \square \)

Sometimes it is convenient to express non-degeneracy in any ball centered at a point \( X_0 \in \Omega^\ast \). This is the contents of the next theorem.
THEOREM 6.3. Let $K$ be a compact set and $X_0 \in \bar{\Omega}_\lambda \cap K$. Then,

$$\sup_{B_r(X)} u_\lambda^* \geq cr,$$

for some constant $c > 0$ depending on dimension, $K$, $A$, $D$, $\Gamma$, $\varphi$ and $\lambda$.

Proof. The proof is basically the same of as the proof of Theorem 6.2. The only difference is that $u_\lambda^*$ is no longer $A$-harmonic near a free boundary point $X_0$, thus we replace the employment of Harnack inequality in (6.42) by:

$$v(X) := \sup_{B_r} u_\lambda^* \cdot h(X) \geq u_\lambda^*(X) \text{ on } \partial B_r,$$

where $h$ is the $A$-harmonic function in $B_r \setminus B_{r/2}$ taking boundary data 1 on $\partial B_r$ and 0 in $B_{r/2}$. We then define the auxiliary function $g(X) := \min\{u_\lambda^*(X), v(X)\}$. The proof now follows the same path as in the proof of Theorem 6.2. \qed

As usual, optimal regularity, Theorem 6.1 and non-degeneracy, Theorem 6.2 or Theorem 6.3, as you like, allow a deeper understanding on the geometric-measure properties of the free boundary. In the next Theorem we will show that the free boundary $\partial \Omega_\lambda^*$ has the appropriate weak geometry.

THEOREM 6.4. There exists a constant $0 < \varsigma < 1$, depending on dimension, $A$, $D$, $\Gamma$, $\varphi$, and $\lambda^{1/p}$, such that,

$$\varsigma \omega_n r^n \leq \mathcal{L}^n (B_r(Z) \cap \Omega_\lambda^*) \leq (1 - \varsigma) \omega_n r^n, \quad (6.45)$$

for any ball $B_r(Z)$ centered at a free boundary point $Z \in \partial \Omega_\lambda^*$. Furthermore, the optimal configuration $\Omega_\lambda^*$ is a set of locally finite perimeter and for positive constants $\varsigma$, $\overline{C}$, depending on $A$, $D$, $\Gamma$, $\varphi$, and $\lambda^{1/p}$, there holds

$$\varsigma r^{n-1} \leq \mathcal{H}^{n-1} (\partial \Omega_\lambda^* \cap B_r(Z)) \leq \overline{C} r^{n-1}, \quad (6.46)$$

for any ball $B_r(Z)$ centered at a free boundary point. In particular,

$$\mathcal{H}^{n-1} (\partial \Omega_\lambda^* \setminus \partial_{\text{red}} \Omega_\lambda^*) = 0.$$

Proof. The estimate by below in (6.45), that is, $\varsigma \omega_n r^n \leq \mathcal{L}^n (B_r(Z) \cap \Omega_\lambda^*)$, is an immediate consequence of Lipschitz regularity and strong non-degeneracy.

Let us focus our effort to prove the uniform density of the zero phase, $\mathbb{R}^n \setminus \Omega_\lambda^*$. Let us assume, for purpose of contradiction, the existence of a sequence of positive real numbers $r_j$ with $r_j \searrow 0$ as $j \to \infty$ and

$$\frac{\mathcal{L}^n (B_{r_j}(Z) \cap \{u_\lambda^* = 0\})}{r_j^n} = o(1). \quad (6.47)$$
We consider then the blow-up sequence $q_j : B_1 \to \mathbb{R}$, defined as

\[ q_j(Y) := \frac{1}{r_j} u_\lambda^*(Z + r_j Y). \]

Let $h_j$ be the solution to

\[ \begin{cases} \text{div} \left( A(Z + r_j X, D h_j) \right) = 0 \text{ in } B_1 \\ h_j = q_j \text{ on } \partial B_1. \end{cases} \]

A renormalization of (4.13), when $p \geq 2$ or (4.14) when $1 < p \leq 2$, under the assumption (6.47), reveals

\[ \int_{B_1} |\nabla (h_j - q_j) (Y)|^p dY = o(1). \]

By Lipschitz regularity of $u_\lambda^*$, and $C^{1,\alpha}$ elliptic estimate, up to a subsequence, we may assume

\[ q_j \to q_0 \quad \text{and} \quad h_j \to h_0 \]

uniformly in $B_{9/11}$. From (5.3) $h_0$ satisfies $\text{div} (A(Z, Dh_0(Y)) = 0$, and from (6.50) so does $q_0$, that is,

\[ \text{div} (A(Z, Dq_0)(Y)) = 0 \text{ in } B_{1/2}. \]

Since $q(0) = 0$, by the strong maximum principle, we conclude $q(0) \equiv 0$ in $B_{1/2}$. However, this is a contradiction to the nondegeneracy property guaranteed by Theorem 6.3.

We now turn our attention to (6.46). The estimate from above, that is $H^{n-1} (\partial \Omega^*_\lambda \cap B_r (Z)) \leq C r^{n-1}$ is a consequence of Lipschitz regularity of $u_\lambda^*$. In order to prove the estimate from below in (6.46), as before, let us assume, for the sake of contradiction, that there exists a sequence $r_j \searrow 0$ such that

\[ \frac{H^{n-1} (\partial \Omega^*_\lambda \cap B_{r_j} (Z))}{r_j^{n-1}} = o(1). \]

With the notation as in (6.48), let us define the sequence of nonnegative measures $\nu_j$, in $B_{2/3}$, as

\[ \nu_j := \text{div} \left( A(Z + r_j X, D q_j) \right) dX. \]

Via a compactness argument, we may assume, passing to a subsequence if necessary, that $\nu_j \overset{*}{\rightharpoonup} \nu_0$ in the sense of measures. However, condition (6.53) translates
Moreover, by Lipschitz regularity, nondegeneracy and uniform positive density of both phases, estimate (6.45), it is not hard to verify that

\[
\nu_j \rightharpoonup \nu := \text{div} \left( A(Z, Dq_0) \right) dX. \quad (6.56)
\]

Indeed, from (6.45), \( \mathcal{L}^n(\partial \{ q_0 > 0 \}) = 0 \), thus in order to justify (6.56), it is enough to attest such an identity holds true for balls entirely contained in \( \{ q_0 > 0 \} \) and in \( \{ q_0 = 0 \} \). If \( B \subset \{ q_0 > 0 \} \), then by elliptic estimate, \( q_j \) converges to \( q_0 \) in a \( C^{1,\alpha} \) fashion in \( B \). Thus clearly (6.56) is true. Now, if \( B \subset \{ q_0 = 0 \} \), then

\[
\left[ \text{div} \left( A(Z, Dq_0) \right) dX \right](B) = 0,
\]

so we have to show that \( \nu_j(B) \to 0 \) as \( j \to \infty \). This is a consequence of non-degeneracy. In fact, let \( \bar{B} \subset B \). If there were a subsequence, \( q_{j_k} \), with each \( q_{j_k} \not\equiv 0 \) in \( \bar{B} \), then by Theorem 6.3, there should exist points \( P_{k_j} \in \bar{B} \), such that \( q_{j_k}(P_{k_j}) \geq c > 0 \). Then, passing to another subsequence, \( P_{k_j} \to P \in \bar{B} \), and since \( q_{j_k} \) converges uniformly to \( q_0 \), we would reach the conclusion that \( q_0(P) > c \), which is not possible. In conclusion, if \( B_k \) is a nested sequence of balls, with \( B_k \not\subset B \), then, for some \( j_k \in \mathbb{N} \), \( q_j \equiv 0 \) in \( B_k \), for any \( j > j_k \). Therefore, \( \nu_j(B) \rightharpoonup 0 \) as desired.

Having verified (6.56), the observation in (6.55) tells us that

\[
\text{div} \left( A(Z, Dq_0) \right) = 0 \text{ in } B_{2/3},
\]

and as argued before, this leads us to a contradiction on the non-degeneracy feature of \( q_0 \) assured in Theorem 6.3.

An immediate, yet quite important consequence of Theorem 6.4 is a substantial enhancement of Lemma 3.1 for the measure \( \mathcal{L}u_\lambda^* \).

**Theorem 6.5.** There exists a Borel function \( Q_\lambda \), such that \( \mathcal{L}u_\lambda^* = Q_\lambda \lfloor \partial \Omega_\lambda^* \). That is,

\[
\int \text{div} \left( A(X, Du_\lambda^*) \right) \phi(X) dX = \int_{\partial \Omega_\lambda^*} Q_\lambda(S) \phi(S) d\mathcal{H}^{n-1}(S),
\]

for any \( \phi \in C_0^1(\mathbb{R}^n \setminus D) \). Moreover, \( Q_\lambda \) bounded away from zero and infinity, that
is for a positive constant $C = C(\lambda, n, A, D, \Gamma, \varphi)$, there holds

$$0 < C^{-1} \leq Q_{\lambda} \leq C < \infty.$$  

In particular, the free boundary is a set of finite perimeter.

As to provide some further insight, allow us to make some loose comments regarding the representation Theorem 6.5. The Borel function $Q_{\lambda}$ should be understood as a weak notion for $\partial A u_{\lambda}^*$ along the reduced free boundary $\partial_{\text{red}} \Omega_{\lambda}^*$. Indeed, in any $C^1$ piece of $\partial \Omega_{\lambda}^*$, there holds

$$Q_{\lambda}(S) = \langle A(S, Du_{\lambda}^*(S)), \nu(S) \rangle,$$

where $\nu$ is the unit inward normal vector to $\partial \Omega_{\lambda}^*$ at $S$. However, $\nu(S) = \frac{\nabla u_{\lambda}^*(S)}{||\nabla u_{\lambda}^*(S)||}$, thus, taking into account the scaling feature of $A$, property (c)(iv), from identity (6.57) we reach that

$$|\nabla u_{\lambda}^*(S)| = \frac{p^{-1/2} Q_{\lambda}(S)}{\sqrt{\langle A(S, Du_{\lambda}^*(S)), \nu(S) \rangle}}.$$

In a more rigorous way, expression (6.58) can be proven to hold in terms of an asymptotic approximation, that is, the following is true:

**Theorem 6.6.** Let $X_0 \in \partial_{\text{red}} \Omega_{\lambda}^*$. Then, for any $X \in \Omega_{\lambda}^*$ near $X_0$, we have

$$u_{\lambda}^*(X) = \theta_{\lambda}(X_0) \langle X - X_0, \nu(X_0) \rangle^+ + o(|X - X_0|),$$

where $\theta_{\lambda}(X_0) = \frac{p^{-1/2} Q_{\lambda}(X_0)}{\sqrt{\langle A(X_0, Du_{\lambda}^*(X_0)), \nu(X_0) \rangle}}$.

**Proof.** Indeed, consider a convergent blow-up sequence

$$q_r(Y) := \frac{1}{r} u_{\lambda}^*(X_0 + rY) \xrightarrow{r \searrow 0} q_0.$$

Easily, from standard geometric-measures arguments, combined with non-degeneracy and the convergence in (6.59), we see that

$$q_0 \equiv 0 \text{ in } \{X \in \mathbb{R}^n | \langle X, \nu(X_0) \rangle < 0\} \text{ and } \{q_0 > 0\} = \{X \in \mathbb{R}^n | \langle X, \nu(X_0) \rangle > 0\}.$$  

Moreover

$$\text{div} (A(X_0, Dq_0), Dq_0) = 0 \text{ in } \{q_0 > 0\}.$$
Notice that $\partial \{ q_0 > 0 \}$ is the hyperplane \( \{ X \in \mathbb{R}^n \; | \; \langle X, \nu(X_0) \rangle = 0 \} \): a smooth surface. One verifies from Theorem 6.5 that

\[
\text{div} (A(X_0, Dq_0), Dq_0) = Q_\lambda(X_0) \big| \{ X \in \mathbb{R}^n \; | \; \langle X, \nu(X_0) \rangle = 0 \},
\]

hence, reasoning as before, we reach the following conclusion

\[
\nabla q_0(Y) \cdot \nu(X_0) = \theta_\lambda(X_0), \quad \forall Y \in \{ \langle X, \nu(X_0) \rangle = 0 \}.
\]

Recall $q_0$ is Lipschitz continuous in the entire $\mathbb{R}^n$. Let $q^*_0$ be the odd reflection of $q_0$ with respect to the hyperplane \( \{ X \in \mathbb{R}^n \; | \; \langle X, \nu(X_0) \rangle = 0 \} \). It is easy to verify that $\| \nabla q_0^* \|_{L^\infty(\mathbb{R}^n)} = \| \nabla q_0 \|_{L^\infty(\mathbb{R}^n)} < C$ and that \( \text{div} (A(X_0, Dq^*_0), Dq^*_0) = 0 \) in all of $\mathbb{R}^n$. From the $C^1, \alpha$ regularity of $q^*_0$, we can employ the beautiful and recent blow-up argument from [KSZ] to conclude that $q^*_0$ is an affine function. Thus in view of (6.63), we obtain

\[
q_0(X) = \theta_\lambda(X_0) \langle X - X_0, \nu(X_0) \rangle^+,
\]

and the theorem is proven.

To finish this section, we mention that the fact that the free boundary is countably $n - 1$ rectifiable can actually be improved. Indeed the $\partial_{\text{red}} \Omega^*_\lambda$ admits a $C^1$-structure, when cover by compact sets, rather than open sets.

**Theorem 6.7.** There exists a collection of $C^1$ hypersurfaces \( \{ \mathcal{S}_j \}_{j \geq 1} \), and compact subsets $K_j \subset \mathcal{S}_j$, such that

\[
\mathcal{H}^{n-1} \left( \partial_{\text{red}} \Omega^*_\lambda \setminus \bigcup_{j \geq 1} K_j \right) = 0.
\]

Furthermore, if $X \in K_j$, the unit outward theoretical normal vector $-\nu(X)$ to $\partial_{\text{red}} \Omega^*_\lambda$ is normal to $\mathcal{S}_j$.

**Proof.** Let $B = B_r(X_0)$ be a generically ball centered at a point of the reduced free boundary. By the Lipschitz continuity of $u^*_\lambda$ and the ellipticity of $A$, we know there exists a constant $L$, such that

\[
\sup_B A(X, Du^*_\lambda) \leq \frac{L}{6}.
\]

Let $\mathcal{I}$ be your favorite nonnegative radially symmetric smooth function whose support is $B_1$. Normalize it so that $0 \leq \mathcal{I} \leq 1$; $\| \mathcal{I} \|_{L^1(B_1)} = 1$. Let $\mathcal{I}_\epsilon$ be the family of mollifiers induced by $\mathcal{I}$, that is, $\mathcal{I}_\epsilon(X) = \epsilon^{-n} \mathcal{I}(\epsilon^{-1} X)$. Also, select your favorite nonnegative function $\eta \in C_0^\infty(B)$, satisfying $\sup \eta = L^{-1}$. For sake of
notation convenience, let us call \( V(X) := A(X, Du^*_\lambda) \). If \( \varpi \) denotes the Radon measure \( D\chi_{\Omega^*_\lambda} \), we have, for \( \epsilon \ll 1 \),

\[
0 \geq \int_{\Omega^*_\lambda} \text{div} ((\eta V) * I_\epsilon) \, dX \\
= \int_{\Omega^*_\lambda} I_\epsilon * \text{div} (\eta V) \, dX \\
= \int_{\Omega^*_\lambda} \text{div} (\eta V) \, dX + o(1) \\
= \int_{\Omega^*_\lambda} V \cdot \nabla \eta dX + \int_{\partial_{\text{red}}\Omega^*_\lambda} Q_\lambda(S) \eta(S) d\mathcal{H}^{n-1}(S) + o(1).
\]

Letting \( \epsilon \to 0 \) in (6.65) and afterwards letting \( \eta \to L^{-1} \), we conclude there exists a constant \( c(\lambda, A, n, \Gamma, \phi) \), such that

\[
\varpi(B) \geq c \mathcal{H}^{n-1}(B \cap \partial_{\text{red}}\Omega^*_\lambda).
\]

In particular \( \mathcal{H}^{n-1}(\partial_{\text{red}}\Omega^*_\lambda) \) is absolutely continuous with respect to \( D\chi_{\Omega^*_\lambda} \). Now, arguing as in [DeGiorgi55] (see also [Giusti84] page 54 or [EG92] page 205) we prove the theorem.

**7. General existence theory in any dimension.** In Section 5, upon a restriction on the dimension, we have shown problem (2.4) has a minimal configuration. The strategy there was to let the penalizing parameter \( \lambda \) go to infinity and use appropriate estimates that becomes available under the constraint \( n < p \), due to the Sobolev Imbedding Theorem.

The goal of this section is to explore the geometric measure properties of the free boundary \( \partial\Omega^*_\lambda \), established in the previous section, to settle to existence of an optimal design for problem (2.4) in all dimensions. However, as the readers should expect, the analysis here is rather more delicate as we will not be able to pass the limit on the penalty parameter \( \lambda \). Instead, we will show that if we adjust the penalty term \( \varrho_\lambda \) properly, any optimal configuration, \( \Omega^* = \Omega^*_\lambda \), for problem (2.10) will obey

\[
\mathcal{L}^n(\Omega^* \setminus D) \leq \iota.
\]

Therefore, \( \Omega^* \) itself will be an optimal design for our primary optimization problem (2.4) and all the regularity features proven to hold for a solution to problem (2.10) will automatically extend to a solution to problem (2.4).
Before continuing, let us explain our strategy in a bit more technical terms. We will perform a small perturbation on an optimal configuration $\Omega^\star_\lambda$, around a point on the reduced free boundary: the portion of $\partial \Omega^\star_\lambda$ where we can replace classical differential geometry arguments by geometric-measures ones. We will not compute the Borel function $Q_\lambda$ of Theorem 6.5, as it is an extraordinary hard task: the free boundary condition for problem (2.10) is expected to be highly non-local. Instead, we will show that assuming $\mathcal{L}^n(\Omega^\star_\lambda \setminus D) > \nu$ enforces a universal bound to the penalty parameter $\lambda$.

With the strategy well understood, let us establish the first supporting result towards the main goal of this section.

**Lemma 7.1.** There exists a constant $M > 0$, depending on dimension, $D$, $\varphi$, $\Gamma$ and $A$, but independent of $\lambda$, such that

$$\inf_{\partial_{\text{red}} \Omega^\star_\lambda} Q_\lambda < M,$$

where $Q_\lambda$ is the Borel function in Theorem 6.5.

**Proof.** Indeed, in the lights of Lemma 5.1, there exists a constant $C$, independent of $\lambda$, such that $\|u^\star_\lambda\|_{W^{1,p}} \leq C$. Thus, from the Trace Theorem for Sobolev functions, we can write

$$\|u^\star_\lambda\|_{W^{1,p}} \cdot \left[ \mathcal{L}^n(\{u^\star_\lambda > 0\}) \right]^\frac{1}{p} \geq \int_{\partial D} \varphi(Z) d\mathcal{H}^{n-1}(Z).$$

The above estimate combined with the Isoperimetric Inequality assures the existence of a constant $c_1 > 0$, independent of $\lambda$, for which the following estimate holds

$$(7.1) \quad \mathcal{H}^{n-1}(\partial_{\text{red}} \Omega^\star_\lambda) \geq c_1.$$ 

From (7.1) and the representation in Theorem 6.5, we have

$$(7.2) \quad \int_{\partial D} \partial_A u^\star_\lambda(X) d\mathcal{H}^{n-1}(X) = \int_{\partial_{\text{red}} \Omega^\star_\lambda} Q_\lambda(X) d\mathcal{H}^{n-1}(X) \geq c_1 \inf_{\partial_{\text{red}} \Omega^\star_\lambda} Q_\lambda.$$

Now, in view of estimate (5.4), for each $Y \in \partial D$ fixed, we establish the following estimate

$$\int_{\partial D} \Gamma \left( Y, \xi_2 \cdot \left[ \inf_{\partial_{\text{red}} \Omega^\star_\lambda} Q_\lambda \right] \right) d\mathcal{H}^{n-1}(X) \leq \int_{\partial D} \Gamma \left( Y, \int_{\partial D} \partial_A u^\star_\lambda \right) d\mathcal{H}^{n-1}(X) \leq C_2.$$

Integrating the above estimate with respect to $Y$ over $\partial D$ and arguing as before, we conclude the proof of the lemma. \qed
We will next describe the mathematical setup for the suitable perturbation technique we shall employ on $\Omega_\star^\lambda$ near a point on the reduced free boundary. Initially, select and fix, throughout this section, a free boundary point $Z_0 \in \partial_{\text{red}} \Omega_\star^\lambda$, such that
\begin{equation}
Q_\lambda(Z_0) \leq 5 \inf_{\partial_{\text{red}} \Omega_\star^\lambda} Q_\lambda \leq M_1, \tag{7.3}
\end{equation}
where $M_1$ depends only on dimension, $A$, $D$, $\Gamma$ and $\varphi$, but it is independent of $\lambda$. The existence of such a point is guaranteed by Lemma 7.1.

Let $\psi: \mathbb{R} \to \mathbb{R}$ be your favorite nonnegative smooth function whose support equals $[0, 1]$. Normalize it so that
\begin{equation}
\int \psi(\tau) d\tau = 1.
\end{equation}

For a fixed positive, but small, real number $\alpha$, we define the inward perturbation map around $Z_0$ as
\begin{equation}
\Phi_r(X) := \begin{cases} X - \alpha r \psi \left( \frac{|X - Z_0|}{r} \right) \nu(Z_0) & X \in B_r(Z_0) \\ X & X \notin B_r(Z_0). \end{cases} \tag{7.4}
\end{equation}

Here, $\nu(Z_0)$ denotes the measure theoretical outward normal at $Z_0$. The idea now is to compare $\Omega_\star^\lambda$ with its inward perturbed configuration given by:
\begin{equation}
\Omega_r := \Phi_r(\Omega_\star^\lambda). \tag{7.5}
\end{equation}

For that, let us call $u_r$ the $A$-potential associated to $\Omega_r$, that is, $u_r$ is the solution to
\begin{equation}
\begin{cases} \Delta u_r = 0 & \text{in } \Omega_r \setminus D \\ u_r = \varphi & \text{on } \partial D \\ u_r = 0 & \text{on } \partial \Omega_r \setminus \partial D. \tag{7.6} \end{cases}
\end{equation}

Although it is possible to compare $u_r$ and $u$ directly, it turns out to the more convenient to use the auxiliary function, $v_r$, given implicitly by
\begin{equation}
v_r(\Phi_r(X)) = u_\lambda^*(X). \tag{7.7}
\end{equation}

Notice that $\{v_r > 0\}$ is not suitable for our minimization problem (2.10). It is also not efficient to compare it with $u_\lambda^*$ in terms of the minimization problem (2.13), since $\partial_A u_\lambda^* \equiv \partial_A v_r$. Our strategy is to compare $v_r$ with $u_\lambda^*$ and with $u_r$ separately and then combine these information using $v_r$ as a bridge from $u_r$ and $u_\lambda^*$.
The next two lemmas are from [OT06], Section 4, though in that paper the computations are carried out only for the $p$-Laplacian operator. Thus we decide to include in this present work “economic versions” of their proofs as a courtesy to the readers.

**Lemma 7.2.** With the notation previously set, we have

$$\mathcal{L}^n(\{u > 0\}) - \mathcal{L}^n(\{v_r > 0\}) = M_2 r^n + o(r^n),$$

for a universal constant $M_2 > 0$.

**Proof.** For sake of notation convenience, we will write $u$ for $u^*_\lambda$. For each $r > 0$ small, we consider the $r$-normalization of $u$ around $Z_0$, $u_r: B_1 \to \mathbb{R}$, defined as

$$u_r(Y) := \frac{1}{r} u(Z_0 + rY).$$

Since $Z_0 \in \partial_{\text{red}} \Omega^*_\lambda$,

$$B_1 \cap \{u_r > 0\} \xrightarrow{r \to 0} \{Y \in B_1 \mid \langle Y, \nu(Z_0) \rangle < 0\},$$

in the sense that the characteristic functions of the above sets in the LHS converge to the characteristic function of the set in the RHS in the $L^1_{\text{loc}}(\mathbb{R}^n)$ topology. One easily sees, by the Change of Variables Theorem, that

$$\frac{\mathcal{L}^n(B_r(Z_0) \cap \{v_r > 0\})}{r^n} = \frac{1}{r^n} \int_{B_r(Z_0) \cap \{v_r > 0\}} dX$$

$$= \int_{B_1 \cap \{v_r(Z_0 + rY) > 0\}} dY$$

$$= \int_{B_1 \cap \{u_r > 0\}} \det(D\Phi_r(Z_0 + rY)) dY$$

$$\xrightarrow{r \to 0} \int_{B_1 \cap \{Y, v_r(Z_0) < 0\}} 1 - \alpha \psi'(|Y|) \left\langle \frac{Y}{|Y|}, \nu(Z_0) \right\rangle dY,$$

It is important to highlight that for any unit vector $\nu \in \mathbb{S}^{n-1}$,

$$\int_{B_1 \cap \{Y, \nu < 0\}} \psi'(|Y|) \left\langle \frac{Y}{|Y|}, \nu \right\rangle dY \equiv M_2,$$

where $M_2$ is a constant that depends only on your choice for $\psi$. Similarly, one
finds that
\[
\mathcal{L}^n (B_r(Z_0) \cap \{ u^*_\lambda > 0 \}) \to 0 \quad \text{as} \quad r \to 0,
\]
\[
\int_{B_1 \cap \{ Y, v(Z_0) < 0 \}} dY.
\]
(7.11)

Combining (7.8), (7.9), (7.10) and (7.11), we conclude the lemma.

Our next lemma measures the differential on the \( A \)-Dirichlet integral passing from \( u^*_\lambda \) to \( u_r \).

**Lemma 7.3.** There exists a constant \( M_3 > 0 \) depends on dimension, \( D, \Gamma, \varphi \) and \( \psi \), but it is independent of \( \lambda \) such that

\[
\frac{1}{r^n} \int_{B_r(Z_0)} \{ \langle A(X, Du_r), Du_r \rangle - \langle A(X, Du^*_\lambda), Du^*_\lambda \rangle \} dX \leq \alpha M_3 + o(\alpha) + o(1).
\]

**Proof.** Again, for sake of notation convenience, we will write \( u \) for \( u^*_\lambda \). Yet for notation convenience, let us write, for any vector field \( \mathbf{V} \),

\[
\Theta(\mathbf{V})(X) := \langle A(X, \mathbf{V}), \mathbf{V} \rangle.
\]

Applying the Change of Variables Theorem twice and taking into account that \( P_r \) maps \( B_r(X) \) diffeomorphically onto itself, we can write

\[
\frac{1}{r^n} \int_{B_r(Z_0)} \Theta(Du_r)(X) \cdot \nabla u_r(X) \cdot \nu(Z_0) dX
\]

(7.12)

\[
= \frac{1}{r^n} \int_{B_r(Z_0)} \Theta \left( D\Phi_r^{-1}(X) \cdot \nabla \Phi_r^{-1}(X) \right) dX
\]

\[
= \frac{1}{r^n} \int_{B_r(Z_0)} \Theta \left( D\Phi_r(Y) \cdot \nabla u_r(Y) \right) \times |\det(D\Phi_r(Y))| dY
\]

\[
= \int_{B_1 \cap \{ u_r > 0 \}} \Theta \left( D\Phi_r(Z_0 + rZ) \cdot \nabla u_r(Z) \right)
\]

\[
\times |\det(D\Phi_r(Z_0 + rZ))| dZ.
\]

By an explicit computation it is easy to verify that

\[
D\Phi_r(Z_0 + rZ)^{-1} \cdot \nabla u_r(Z) = \nabla u_r(Z) + \alpha \frac{\psi'(|Z|)}{|Z|} \langle Z, \nabla v(Z) \rangle v(Z_0) + o(\alpha).
\]

(7.13)

Furthermore, we can compute explicitly that

\[
|\det(D\Phi_r(Z_0 + rZ))| = 1 - \alpha \frac{\psi'(|Z|)}{|Z|} \langle Z, v(Z_0) \rangle.
\]

(7.14)
A straight combination of (7.12), (7.13) and (7.14), reveals that

\[
\frac{1}{r^n} \int_{B_r(Z_0)} \Theta(Du_r)(X) - \Theta(Du)(X) dX \quad (7.15)
\]

\[
= -\alpha \int_{B_1 \cap \{u_r > 0\}} \Theta(Du_r(Z)) \frac{\psi'(|Z|)}{|Z|} \langle Z, \nu(Z_0) \rangle dZ + o(\alpha).
\]

It is simple to verify, from the Divergence Theorem, that

\[
\int_{B_1 \cap \{\langle X, \nu(Z_0) \rangle < 0\}} \psi(|Z|) d\mathcal{H}^{n-1}(Z) = I > 0,
\]

(7.16)

with the appropriate integral orientation. Furthermore, by the Lipschitz regularity of \( u \) and standard geometric-measure arguments we verify that

\[
\langle A(Z_0 + rY, \nabla u_r), \nabla u_r \rangle \to Q_{\lambda}(Z_0)\nu(Z_0)\chi_{B_1 \cap \{Y, \nu(X_i)\} < 0},
\]

(7.17)

in \( L^p(B_1) \). Thus, letting \( r \to 0 \) in (7.15), and taking into account (7.16) and estimate (7.3), we conclude the proof of the lemma.

We are ready to prove the existence of an optimal design for problem (2.4) in all dimensions.

**Theorem 7.4.** There exists a positive number \( \lambda_0 \), such that if \( \Omega^*_\lambda \) is an optimal configuration for problem (2.10) and \( \mathcal{L}^n(\Omega^*_\lambda \setminus D) > I \), then necessarily, \( \lambda < \lambda_0 \). In particular, there exists an optimal configuration for problem (2.4) and it enjoys all the weak geometric features derived in Section 6.

**Proof.** Throughout the proof we fix an optimal configuration \( \Omega^*_\lambda \) and assume

\[
\mathcal{L}^n(\Omega^*_\lambda \setminus D) > I.
\]

(7.18)

Initially we recall the variational characterization of the \( A \)-potential \( u_r \), namely

\[
\int \langle A(X, Du_r), Du_r \rangle dX \quad (7.19)
\]

\[
= \min \left\{ \int \langle A(X, Dv), Dv \rangle dX \mid v = \varphi \text{ on } \partial D \text{ and } v = 0 \text{ on } \partial \Omega_r \right\}.
\]

Now we compare \( \Omega^*_\lambda \) with \( \Omega_r \) in terms of the minimization problem (2.10).
From the minimality feature of the configuration $\Omega^\ast$, if $r$ is small enough as to $L^n(\Omega_r \setminus D) > \iota$, we have

\[(7.20)\]
\[
\lambda \left\{ L^n(\Omega^\ast_r \setminus D) - L^n(\Omega_r \setminus D) \right\} \\
\leq \int_{\partial D} \Gamma(X, \partial_A u_r) - \Gamma(X, \partial_A u^\ast) d\mathcal{H}^{n-1}(X).
\]

As argued before, we have the following estimate

\[(7.21)\]
\[
\int_{\partial D} \Gamma(X, \partial_A u_r) - \Gamma(X, \partial_A u^\ast) d\mathcal{H}^{n-1}(X) \\
\leq C(\partial D, \Gamma) \int_{\partial D} \{ \partial_A u_r - \partial_A u^\ast \} d\mathcal{H}^{n-1}(X) \\
\leq C(\partial D, \Gamma, \inf \varphi) \int \langle A(X, Du_r), Du_r \rangle \\
- \langle A(X, Du^\ast), Du^\ast \rangle dX.
\]

Now combining Lemmas 7.2 and 7.3 with (7.19), (7.20) and (7.21), we obtain

\[(7.22)\]
\[
\lambda \{ M_2 \alpha r^n + o(r^n) \} \leq C(\partial D, \Gamma, \inf \varphi) r^n \times [\alpha M_3 + o(\alpha) + o(1)].
\]

If we divide expression (7.22) by $r^n$, let $r \to 0$ and afterwards divide the result by $\alpha$ and let $\alpha \searrow 0$, we finally conclude the proof of the theorem. \qed

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