ON THE D-AFFINITY OF QUADRICS IN POSITIVE CHARACTERISTIC

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Abstract. In this paper we consider sheaves of differential operators on quadrics of low dimension in positive characteristic. We prove a vanishing theorem for the first term in the $p$-filtration of these sheaves. This vanishing is a necessary condition for the D-affinity of these quadrics.

1. Introduction

Let $X$ be a smooth proper algebraic variety over an algebraically closed field $k$ of arbitrary characteristic, $\mathcal{O}_X$ the structural sheaf of $X$, and $\mathcal{D}_X$ the sheaf of differential operators on $X$. Denote $\mathcal{M}(\mathcal{D}_X)$ the category of (left) $\mathcal{D}_X$-modules that are coherent over $\mathcal{D}_X$. The variety $X$ is said to be D-affine if the two following conditions hold: (i) for any $\mathcal{F} \in \mathcal{M}(\mathcal{D}_X)$ one has $H^k(X, \mathcal{F}) = 0$ for $k > 0$, and (ii) the natural morphism $\mathcal{D}_X \otimes_{\Gamma(\mathcal{D}_X)} \Gamma(\mathcal{M}) \to \mathcal{M}$ is surjective.

Beilinson and Bernstein proved ([3]) that homogeneous spaces of semisimple algebraic groups are D-affine if the base field $k$ has characteristic zero. On the other hand, as was shown by Kashiwara et Lauritzen in [7], homogeneous spaces over fields of positive characteristic can fail to be D-affine. In this paper we study quadrics of dimension less or equal to 4 in positive characteristic (these are homogeneous spaces of the orthogonal group). We prove a necessary condition for these quadrics to be D-affine. This theorem, which is the main result of the paper, is similar to that of Andersen and Kaneda ([1]), where they treated the case of the flag variety of the group in type $B_2$.

Throughout we fix an algebraically closed field $k$ of characteristic $p$. For a smooth variety $X$ over $k$ denote $F: X \to X$ the absolute Frobenius morphism. Note that since $X$ is smooth, the sheaf $F^*\mathcal{O}_X$ is locally free. Recall that the sheaf of differential operators $\mathcal{D}_X$ on $X$ admits the $p$-filtration defined as follows. For $r \geq 1$ denote $\mathcal{D}_r$ the endomorphism bundle $\text{End}_{\mathcal{O}_X}(F^r\mathcal{O}_X)$ (here $F^r = F \circ \cdots \circ F$ is the $r$-iteration of $F$). Then $\mathcal{D}_X = \bigcup \mathcal{D}_r$. Recall that $X$ is said to be Frobenius split if the sheaf $\mathcal{O}_X$ is a direct summand in $F_*\mathcal{O}_X$. Homogeneous spaces of semisimple algebraic groups are Frobenius split ([9]). For a Frobenius split variety $X$ and for all $i \in \mathbb{N}$ the following property holds (Proposition, Sec. 1, [1]):

\[(1.1) \quad H^i(X, \mathcal{D}_X) = 0 \iff H^i(X, \mathcal{D}_r) = 0 \text{ for any } r \in \mathbb{N}.\]

Here is the main result of the paper (cf. the main theorem of [1]):

**Theorem 1.1.** Let $Q_n$ be a quadric of dimension $n \leq 4$. Then $H^i(Q_n, \mathcal{D}_1) = 0$ for $i > 0$. 

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2. Preliminaries

Let $X$ be a smooth variety over $k$, and $\text{Coh}(X)$ the category of coherent sheaves on $X$. The direct image functor $F_*$ has a right adjoint functor $F^!$ in $\text{Coh}(X)$ ([6]). The duality theory for the finite flat morphism $F$ yields (loc.cit.):

**Lemma 2.1.** The functor $F^!$ is isomorphic to

\[ F^!(?) = F^*(?) \otimes \omega_X^{1-p}, \tag{2.1} \]

where $\omega_X$ is the canonical invertible sheaf on $X$.

For any $i \geq 0$ one has an isomorphism, the sheaf $F_*\mathcal{O}_X$ being locally free:

\[ H^i(X, \text{End}_{\mathcal{O}_X}(F_*\mathcal{O}_X)) = \text{Ext}^i(F_*\mathcal{O}_X, F_*\mathcal{O}_X). \tag{2.2} \]

From Lemma 2.1 we obtain:

\[ \text{Ext}^i(X, \mathcal{O}_X) = \text{Ext}^i(F_*\mathcal{O}_X, F_*\mathcal{O}_X) = H^i(X, F^!F_*\mathcal{O}_X \otimes \omega_X^{1-p}). \tag{2.3} \]

Consider the fibered square:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi_2} & X \\
\downarrow{\pi_1} & & \downarrow{F} \\
X & \xleftarrow{F} & X
\end{array}
\]

Let $i: \Delta \hookrightarrow X \times X$ be the diagonal embedding, and $\tilde{i}$ the embedding $\tilde{X} \hookrightarrow X \times X$ obtained from the above fibered square.

**Lemma 2.2.** One has an isomorphism of sheaves:

\[ \tilde{i}_*\mathcal{O}_{\tilde{X}} = (F \times F)^*(i_*\mathcal{O}_\Delta). \tag{2.4} \]

Here $F \times F$ is the Frobenius morphism on $X \times X$. The lemma is equivalent to saying that the fibered product $\tilde{X}$ is isomorphic to the Frobenius neighbourhood of the diagonal $\Delta \subset X \times X$.

**Proof.** Follows from the definition of fibered product. \qed

**Lemma 2.3.** There is an isomorphism of cohomology groups:

\[ H^i(X, F^*F_*\mathcal{O}_X) \otimes \omega_X^{1-p}) = H^i(X \times X, (F \times F)^*(i_*\mathcal{O}_\Delta) \otimes (\omega_X^{1-p} \boxtimes \mathcal{O}_X)). \tag{2.5} \]

**Proof.** Recall that the sign $\boxtimes$ in the right hand side of (2.5) denotes the external tensor product. Applying the flat base change to the above fibered square, we get an isomorphism of functors, the morphism $F$ being flat:

\[ F^*F_* = \pi_1^* \pi_2^*. \tag{2.6} \]
Note that all the functors $F_*$, $F^*$, $\pi_1*$, and $\pi_2^*$ are exact, the morphism $F$ being affine. The isomorphism (2.6) implies an isomorphism of cohomology groups

\[(2.7) \quad H^i(X, F^*F_*(O_X) \otimes \omega^{1-p}_X) = H^i(X, \pi_1^*\pi_2^*(O_X) \otimes \omega^{1-p}_X).\]

By the projection formula the right-hand side group in (2.7) is isomorphic to $H^i(\tilde{X}, \pi_2^*O_X \otimes \pi_1^*\omega^{1-p}_X)$. Let $p_1$ and $p_2$ be the projections of $X \times X$ onto the first and the second component respectively. One has $\pi_1 = p_1 \circ i$ and $\pi_2 = p_2 \circ i$. Hence an isomorphism of sheaves

\[(2.8) \quad \pi_2^*O_X \otimes \pi_1^*\omega^{1-p}_X = \tilde{i}^*(p_2^*O_X \otimes p_1^*\omega^{1-p}_X) = \tilde{i}^*(\omega^{1-p}_X \otimes O_X).\]

From (2.8) and the projection formula one obtains

\[(2.9) \quad H^i(\tilde{X}, \pi_2^*O_X \otimes \pi_1^*\omega^{1-p}_X) = H^i(\tilde{X}, \tilde{i}^*(\omega^{1-p}_X \otimes O_X)) = H^i(X \times X, \tilde{i}_*O_X \otimes (\omega^{1-p}_X \otimes O_X)).\]

Using Lemma 2.2 we get the statement. \qed

**Corollary 2.1.** One has as well an isomorphism:

\[(2.10) \quad H^i(X, F^*F_*(O_X) \otimes \omega^{1-p}_X) = H^i(X \times X, (F \times F)^*(i_*O_{\Delta}) \otimes (O_X \otimes \omega^{1-p}_X)).\]

Finally, we need a well-known lemma (e.g., SGA3):

**Lemma 2.4.** If a sheaf $F$ on a variety $X$ is quasi-isomorphic to a bounded complex $F^\bullet$ then $H^i(X, F) = 0$ provided that $H^p(X, F^q) = 0$ for all $p + q = i$.

### 3. Vanishing

In this section we prove Theorem 1.1. Let $Q_n$ be a smooth quadric of dimension $n \leq 4$. Note that $Q_1$ is isomorphic to $\mathbb{P}^1$, and $Q_2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. For projective spaces in positive characteristic the D-affinity was proved in [4] by B. Haastert. We need, therefore, to consider the case $n = 3$ and $n = 4$. We treat the case of three-dimensional quadrics, the four-dimensional case being similar. Let $G$ be a simply connected simple algebraic group over $k$ of type $B_2$, $B$ a Borel subgroup of $G$, and $P \subset G$ a parabolic subgroup such that $G/P$ is isomorphic to a quadric $Q_3$. Denote $\pi: G/B \to G/P$ the projection. There exists a line bundle $L$ over $G/B$ such that $R^0\pi_*L$ is a rank two vector bundle over $Q_3$, the spinor bundle. Denote $U$ the dual bundle to $R^0\pi_*L$. There is a short exact sequence:

\[(3.1) \quad 0 \to U \to V \otimes O_{Q_3} \to U^* \to 0,\]

where $V$ is a symplectic $k$-vector space of dimension 4, i.e. a space equipped with a non-degenerate skew form $\omega \in \bigwedge^2 V^*$.

**Lemma 3.1.** Let $Q_3$ be a smooth quadric of dimension 3, and $\iota: \Delta \subset Q_3 \times Q_3$ the diagonal embedding. Then the following complex is exact:

\[(3.2) \quad 0 \to U \otimes U(-2) \to \Psi_2 \otimes O_{Q_3}(-2) \to \Psi_1 \otimes O_{Q_3}(-1) \to O_{Q_3} \otimes O_{Q_3} \otimes i_*O_{\Delta} \to 0.\]

Here $\Psi_i$ for $i = 1, 2$ are some vector bundles on $Q_3$, which have right resolutions.
Let us show that $H^k$ truncated complex $\sigma$

Here categorical construction of a resolution of the diagonal (loc.cit.) that can be performed over fields of arbitrary characteristic. The bundles $U, \Psi_2, \Psi_1, O_{Q_3}$ are terms of the so-called right dual collection. Otherwise, one can suitably modify Kapranov’s argument so that it will hold over fields of positive characteristic.

\[ \text{Proof.} \] One can show, without recurrence to Kapranov’s theorem for quadrics ([5], Theorem 4.10), that the collection of bundles $U(-2), O_{Q_3}(-2), O_{Q_3}(-1), O_{Q_3}$ is a complete exceptional collection ([10]) in $D^b(Q_3)$, the bounded derived category of coherent sheaves on $Q_3$, and then use a purely categorical construction of a resolution of the diagonal (loc.cit.) that can be performed over fields of arbitrary characteristic.

Consider now the bundle $U \otimes (\otimes \omega_{Q_3})$. We thus need to show that $H^i(F_*, O_{Q_3}) = 0$ for $i > 0$. Recall that $\omega_{Q_3} = O_{Q_3}(-3)$. For a line bundle $L$ one has $F^* L = L^p$. Taking the pull-back under $(F \times F)^*$ of the resolution (3.2), we get a complex of coherent sheaves in degrees $-3, \ldots, 0$:

\[ 0 \rightarrow \Psi_1 \otimes \otimes_k O_{Q_3} \rightarrow \cdots \rightarrow B_1 \otimes_k O_{Q_3}(i-1) \rightarrow O_{Q_3}(i) \rightarrow 0, \]

and $B_i$ are $k$-vector spaces.

\[ \text{Proof.} \] By Corollary 2.1 we need to show that $H^i(Q_3 \times Q_3, (F \times F)^*(i_* \omega_\Delta) \otimes (O_{Q_3} \otimes \omega_{Q_3}^{-p})) = 0$ for $i > 0$. Recall that $\omega_{Q_3} = O_{Q_3}(-3)$. For a line bundle $\omega$ one has $F^* \omega = \omega^p$. Taking the pull-back under $(F \times F)^*$ of the resolution (3.2), we get a complex of coherent sheaves in degrees $-3, \ldots, 0$:

\[ 0 \rightarrow F^* U \otimes F^* U(-2) \rightarrow F^* \Psi_2 \otimes \otimes_k O_{Q_3}(-2p) \rightarrow F^* \Psi_1 \otimes \otimes_k O_{Q_3}(-p) \rightarrow \otimes_k O_{Q_3} \rightarrow 0. \]

Denote $C^\bullet$ the complex (3.4), and let $\tilde{C}^\bullet$ be the tensor product of $C^\bullet$ with the invertible sheaf $O_{Q_3} \otimes \omega_{Q_3}^{-p}$. Then $\tilde{C}^\bullet$ is quasiisomorphic to the sheaf $(F \times F)^*(i_* \omega_\Delta) \otimes (O_{Q_3} \otimes \omega_{Q_3}^{-p})$. We thus have to compute the hypercohomology of $\tilde{C}^\bullet$. There is a distinguished triangle in $D^b(Q_3)$:

\[ \cdots \rightarrow F^* U \otimes (F^* U \otimes \omega_{Q_3})[3] \rightarrow \tilde{C}^\bullet \rightarrow \sigma_{\geq -2} (\tilde{C}^\bullet) \rightarrow [1] \rightarrow \cdots. \]

Here $\sigma_{\geq -2}$ is the stupid truncation, and $[1]$ is a shift functor in $D^b(Q_3)$. Let us first look at the truncated complex $\sigma_{\geq -2} (\tilde{C}^\bullet)$, which is quasiisomorphic to

\[ 0 \rightarrow F^* \Psi_2 \otimes \otimes_k O_{Q_3}(p-3) \rightarrow F^* \Psi_1 \otimes \otimes_k O_{Q_3}(2p-3) \rightarrow \otimes_k O_{Q_3}(3p-3) \rightarrow 0. \]

Let us show that $H^k(Q_3, F^* \Psi_i) = 0$ for $k > i$ and $i = 1, 2$. Indeed, the sheaves $\Psi_1$ and $\Psi_2$ have resolutions as in (3.3). By the Kempf vanishing theorem ([8]), effective line bundles on homogeneous spaces have no higher cohomology. The terms of resolutions (3.3) for $n = 3$ and $i = 1, 2$ consist of direct sums of effective line bundles and of direct sums of the sheaf $O_{Q_3}$. Positivity of a line bundle is preserved under the Frobenius pullback, hence the terms of the resolutions of sheaves $F^* \Psi_i$ have only zero cohomology. Applying Lemma 2.1, we obtain the above vanishing. Further, line bundles that occur in the second argument of the terms of the complex (3.6) are effective for $p > 3$. For $p = 2$ and $p = 3$ the line bundle occurring in the leftmost term of (3.6) is isomorphic to $O_{Q_3}$ and $O_{Q_3}(-1)$, respectively. For all $p$ these line bundles have no higher cohomology. Using again Lemma 2.1, we get $H^i(\sigma_{\geq -2} (\tilde{C}^\bullet)) = 0$ for $i > 0$.

Consider now the bundle $F^* U \otimes (F^* U \otimes \omega_{Q_3})$. The proof of Theorem 3.1 will be completed if we show that $H^i(Q_3 \times Q_3, F^* U \otimes (F^* U \otimes \omega_{Q_3})) = 0$ for $i \neq 3$. Indeed, taking cohomology of the triangle (3.5), we see that $H^i(\tilde{C}^\bullet) = 0$ for $i > 0$, q.e.d.
Lemma 3.2. One has $H^i(Q_3 \times Q_3, F^* U \boxtimes (F^* U^* \otimes \omega_{Q_3})) = 0$ for $i \neq 3$.

Proof. By the Serre duality one has $H^k(Q_3, F^* U) = H^{3-k}(Q_3, F^* U^* \otimes \omega_{Q_3})$, for $0 \leq k \leq 3$. Using the Künneth formula, we see that it is sufficient to show that $H^k(Q_3, F^* U)$ is non-zero for just one value of $k$. In fact, $H^k(Q_3, F^* U) = 0$ if $k \neq 2$. To prove this, apply the Frobenius pullback $F^*$ to (3.1). We get

$$0 \to F^* U \to F^* V \otimes O_{Q_3} \to F^* U^* \to 0$$

We need a particular case of the following theorem due to Carter and Lusztig ([2], Theorem 6.2):

Theorem 3.2. Let $k$ be a field of characteristic $p$ and let $E$ be a vector space of dimension $r$ over $k$. Then there exists an exact sequence of $GL_r(k)$-modules

$$0 \to F^* E \to S^p(E) \to \Sigma^p(E) \to \cdots \to \Sigma^\lambda(E) \to 0$$

where $\lambda = (p - \min (p-1, r-1), 1, 1, \ldots)$. Here $\Sigma^\lambda$ are Schur functors. In particular, if a partition $\lambda$ is equal to $(p, 0, \ldots)$ then $\Sigma^\lambda$ is equal to $p$-th symmetric power functor $S^p$. This construction globalizes to produce a resolution of the Frobenius pull-back of a vector bundle ([2]).

Theorem 3.2 applied to the bundle $U^*$, furnishes a short exact sequence, the bundle $U^*$ having rank 2:

$$0 \to F^* U^* \to S^p U^* \to S^{p-2} U^* \otimes O_{Q_3}(1) \to 0$$

The vector bundles $S^p U^*$ and $S^{p-2} U^* \otimes O_{Q_3}(1)$ are pushforwards onto $Q_3$ of effective line bundles $L_{x_1}$ and $L_{x_1}$ over $G/B$ that correspond to the weights $x_1 = (p, 0)$ and $x_2 = (p-1, 1)$, respectively. Moreover, the restrictions of $L_{x_1}$ and $L_{x_1}$ to the fibers of $\pi$ have positive degrees. Considering the Leray spectral sequence for the morphism $\pi$ and using the Kempf vanishing theorem, we obtain that the bundles $S^p U^*$ and $S^{p-2} U^* \otimes O_{Q_3}(1)$ both have no higher cohomology. By Lemma 2.3, one then has $H^i(Q_3, F^* U^*) = 0$ for $i > 1$. Moreover, there is an isomorphism $H^0(Q_3, F^* U^*) = V$. Considering the long exact cohomology sequence associated to (3.7), we get the statement of Lemma 3.2. Hence, it remains to prove the isomorphism $H^0(Q_3, F^* U^*) = V$. Indeed, taking cohomology of (3.9), we get a short exact sequence:

$$0 \to H^0(Q_3, F^* U^*) \to H^0(Q_3, S^p U^*) \to H^0(Q_3, S^{p-2} U^* \otimes O_{Q_3}(1))$$

Using again the Kempf vanishing theorem, we obtain $H^0(Q_3, S^p U^*) = S^p V^*$ and $H^0(Q_3, S^{p-2} U^* \otimes O_{Q_3}(1)) = \Sigma^{(p-1,1)} V^*$. On the other hand, the resolution (3.8), applied to the vector space $V^*$, furnishes a short exact sequence:

$$0 \to F^* V^* \to S^p V^* \to \Sigma^{(p-1,1)} V^*$$

Comparing (3.10) and (3.11), and taking into account the above isomorphisms, we get the isomorphism $H^0(Q_3, F^* U^*) = V$. Finally, one has an isomorphism $V = V^*$ since $V$ is symplectic. This implies Lemma 3.2. \qed
The proof of Theorem 1.1 in the case of four-dimensional quadrics is essentially the same, the only difference being that there are two spinor bundles in this case.

**Remark 3.1.** A more general result, as well as applications to derived categories of coherent sheaves, are discussed in a forthcoming paper ([11]).

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