Sturmian words and the Stern sequence

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Abstract

Central, standard, and Christoffel words are three strongly inter-related classes of binary finite words which represent a finite counterpart of characteristic Sturmian words. A natural arithmetization of the theory is obtained by representing central and Christoffel words by irreducible fractions labeling respectively two binary trees, the Raney (or Calkin-Wilf) tree and the Stern-Brocot tree. The sequence of denominators of the fractions in Raney’s tree is the famous Stern di-atomic numerical sequence. An interpretation of the terms s(n) of Stern’s sequence as lengths of Christoffel words when n is odd, and as minimal periods of central words when n is even, allows one to interpret several results on Christoffel and central words in terms of Stern’s sequence and, conversely, to obtain a new insight in the combinatorics of Christoffel and central words by using properties of Stern’s sequence. One of our main results is a non-commutative version of the “alternating bit sets theorem” by Calkin and Wilf. We also study the length distribution of Christoffel words corresponding to nodes of equal height in the tree, obtaining some interesting bounds and inequalities.

Key words. Sturmian words, Central words, standard words, Christoffel words, Stern sequence, Raney tree.

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1 Introduction

Sturmian words are of great interest in combinatorics of infinite words since they are the most simple words which are not ultimately periodic. Since

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the seminal paper of 1940 by Marston Morse and Gustav A. Hedlund \cite{29}, there is a large literature on this subject (see, for instance, \cite{27} Chap. 2). Sturmian words can be defined in many different ways, of combinatorial or geometric nature.

In the theory a key role is played by characteristic (or standard) Sturmian words which can be generated in several different ways and, in particular, by a palindromization map $\psi$, introduced by the first author in \cite{11}, which maps injectively each finite word $v$ into a palindrome (cf. Section 2.1). The map $\psi$ can be naturally extended to infinite words. In such a case if $v$ is any infinite binary word in which all letters occur infinitely often, one generates all characteristic Sturmian words. An infinite word is Sturmian if it has the same set of finite factors of a characteristic Sturmian word. The set of all $\psi(v)$, with $v$ any finite word on the binary alphabet $\mathcal{A} = \{a, b\}$, coincides with the set of palindromic prefixes of all characteristic Sturmian words \cite{16, 11}.

The words $\psi(v)$, called central, may be also defined in a purely combinatorial way as the set of all words having two coprime periods periods $p$ and $q$ such that the length $|\psi(v)| = p + q - 2$. Central words $\psi(v)$ are strongly related \cite{11, 2} to proper finite standard words which may be defined as $\psi(v)xy$ with $x, y \in \mathcal{A}$ and to proper Christoffel words $a\psi(v)b$.

Central, standard, and Christoffel words are considered in Section 3. They represent a finite counterpart of characteristic Sturmian words of great interest since there exist several faithful representations of the preceding words by trees, binary matrices, and continued fractions \cite{2}. These representations give a natural arithmetization of the theory. Some new results are proved at the end of the section.

As regards trees, we mainly refer in Section 4 to the Raney tree. The tree is a complete binary tree rooted at the fraction $\frac{1}{1}$ and any rational number represented in a node as the irreducible fraction $\frac{p}{q}$ has two children representing the numbers $\frac{p}{p+q}$ and $\frac{p+q}{q}$. Every positive rational number appears exactly once in the tree. This tree is usually named in the literature the Calkin-Wilf tree after Neil Calkin and Herbert Wilf, who considered it in their 2000 paper \cite{6}. However, the tree was introduced earlier by Jean Berstel and the first author \cite{2} as Raney tree, since they drew some ideas from a paper by George N. Raney \cite{31}.

The fraction $Ra(w)$ in the node of Raney’s tree represented by the binary word $w$ is equal to the ratio $\frac{p}{q}$ of the periods of the central word $\psi(w)$, where $p$ (resp., $q$) is the minimal period of $\psi(w)$ if $w$ terminates with the letter $a$ (resp., $b$).

Another very important tree which can be considered as dual of Raney tree is the Stern-Brocot tree (see, for instance, \cite{28, 21}). One can prove
(cf. [2]) that the fraction $Sb(w)$ in the node $w$ of the Stern-Brocot tree is equal to the slope $\frac{|aw(b)|}{|aw(b)|}$ of the Christoffel word $av(w)b$. The duality is due to the fact that

$$Sb(w) = Ra(w^-).$$

The sequence formed by the denominators of the fractions labeling the Raney tree is the famous diatomic sequence introduced in 1858 by Moritz A. Stern [33]. There exists a large literature on this sequence, that we shall simply refer to as Stern’s sequence, since its terms admit interpretations in several parts of combinatorics and satisfy many surprising and beautiful properties (see, for instance, [25, 30, 6, 7, 10, 19, 34] and references therein).

In this paper we are mainly interested in the properties of Stern’s sequence which are related to combinatorics of Christoffel and central words. In Section 5, using some properties of Raney’s tree, we prove that there exists a basic correspondence (cf. Theorem 5.2) between the values of Stern’s sequence on odd integers and the lengths of Christoffel words as well as a correspondence between the values of the sequence on even integers and the minimal periods of central words. Thus there exists a strong relation between Sturmian words and Stern’s sequence which strangely, with the only exception of [19], has not been observed in the literature.

As a consequence of the previous correspondence several results on Stern’s sequence can be proved by using the theory of Sturmian words and, conversely, properties of Stern’s sequence can give a new insight in the combinatorics of Christoffel and central words.

In Section 6 we show that one can compute the terms of Stern’s sequence by continuants in two different ways. The first uses a result concerning the length of a Christoffel word $av(v)b$ and the minimal period of the central word $\psi(v)$ which can be expressed in terms of continuants operating on the integral representation of the directive word $v$. The second is of a more arithmetical nature and uses known results on Stern’s sequence.

Section 7 is devoted to a very interesting and unpublished theorem of Calkin and Wilf on Stern’s sequence [7, Theorem 5]. The Calkin-Wilf theorem states that $s(n)$ represents for each $n$ the number of “alternating bit sets” in $n$, i.e., the number of occurrences of subsequences (subwords) in the binary representation of $n$ belonging to the set $b(ab)^*$. We give two new proofs of the Calkin-Wilf theorem which are based on the combinatorics of Christoffel words. We also give a formula allowing to compute the length of the Christoffel word $av(v)b$ in terms of the number occurrences of subwords $u \in b(ab)^*$ in $bv$b. Moreover, the minimal period of $\psi(v)$ equals the number of occurrences $u \in b(ab)^*$ in $bv_+b$, where $v_+$ is the longest prefix of $v$ immediately followed by a letter different from the last letter of $v$. A further
formula shows that $|a\psi(v)b|_a$ is equal to the number of initial occurrences of subwords $u \in b(ab)^*$ in $bvb$.

The main result of the section is a theorem (cf. Theorem 7.4) showing the quite surprising result that for any $w \in A^*$, if we consider the reversed occurrences of words of the set $b(ab)^*$ as subwords in $bvb$, then sorting these in decreasing lexicographic order, and marking the reversed initial occurrences with $a$ and the reversed non-initial ones with $b$, one yields the standard word $\psi(w)ba$. This can be regarded as a non-commutative version of the Calkin-Wilf theorem.

In Section 8 we shall prove a formula (cf. Theorem 8.2) relating for each $w \in A^*$ the length of the Christoffel word $a\psi(w)b$ with the occurrences in $bwb$ of a certain kind of factors whose number is weighted by the lengths of Christoffel words associated to suitable directive words which are factors of $w$. The result is a consequence of an interesting theorem on Stern’s sequence due to Michael Coons and Jeffrey Shallit [10].

In Section 9 we study the distribution of the lengths of Christoffel words $a\psi(v)b$ of order $k$, i.e., the directive word $v$ has a fixed length $k$. Using a property of Stern’s sequence we show that the average value of the length is $2(3/2)^k$. Moreover, the maximal value given by $F_{k+1}$, where $(F_k)_{k \geq -1}$ is the Fibonacci numerical sequence, is reached if and only if $v$ is alternating, i.e., any letter in $v$ is immediately followed in $v$ by its complementary.

One of the main results of the section (cf. Theorem 9.5) is that if $v \in A^k$, with $k \geq 3$ is not alternating, then $|a\psi(v)b| \leq F_{k+1} - F_{k-4}$, where the upper bound is reached if and only if $v$ is an almost alternating word. From this some identities on Stern’s sequence are obtained. Moreover, the number of missing lengths for $k \geq 3$ has the lower bound $F_{k-4}$, so that it is exponentially increasing with $k$. Finally, we consider for each $k$ the maximal value $M_k$ of the number of Christoffel words of order $k$ having the same length. We prove that $M_k$ has a lower bound which is exponentially increasing with $k$.

## 2 Preliminaries and notation

Let $A$ be a finite non-empty alphabet and $A^*$ be the free monoid generated by $A$. The elements of $A$ are usually called letters and those of $A^*$ words. The identity element of $A^*$ is called empty word and denoted by $\varepsilon$. We shall set $A^+ = A^* \setminus \{\varepsilon\}$. A word $w \in A^+$ can be written uniquely as a sequence $w = w_1 w_2 \cdots w_n$, with $w_i \in A$, $i = 1, \ldots, n$. The integer $n$ is called the length of $w$ and is denoted by $|w|$. The length of $\varepsilon$ is conventionally 0.

Let $w \in A^*$. A word $v$ is a factor of $w$ if there exist words $r$ and $s$ such that $w = rva_s$; $v$ is a proper factor if $v \neq w$. If $r = \varepsilon$ (resp., $s = \varepsilon$), then $v$
is called a prefix (resp., a suffix) of \( w \). If \( w = r v s \), then \(|r| + 1 \) is called an occurrence of the factor \( v \) in \( w \). The number of all distinct occurrences of \( v \) in \( w \) is denoted by \(|w|_v\).

A word \( v = v_1 v_2 \cdots v_m \), \( v_i \in A \), \( i = 1, \ldots, m \), is a subword of \( w = w_1 w_2 \cdots w_n \) if there exists an \( m \)-tuple \((j_1, j_2, \ldots, j_m)\) such that

\[
1 \leq j_1 < j_2 < \cdots < j_m \leq n \quad \text{and} \quad v_h = w_{j_h}, \quad \text{for all} \quad h = 1, 2, \ldots, m.
\]

Any \( m \)-tuple \((j_1, j_2, \ldots, j_m)\) for which the previous condition is satisfied is called an occurrence of the subword \( v \) in \( w \). We shall represent such an occurrence also as a word \( j_1 j_2 \cdots j_m \) on the alphabet \( \{1, 2, \ldots, n\} \). An occurrence is said to be initial (resp., final) if \( j_1 = 1 \) (resp., \( j_m = n \)). A factor \( v \) of \( w \) is trivially a subword of \( w \), whereas the converse is not in general true. The number of all distinct occurrences of the subword \( v \) in \( w \) is usually denoted by \( \binom{w}{v} \) and called the binomial coefficient of \( w \) and \( v \) (see [26, Chap. 6]).

Let \( w = w_1 \cdots w_n \), \( w_i \in A \), \( 1 \leq i \leq n \). The reversal of \( w \) is the word \( w^\sim = w_n \cdots w_1 \). One defines also \( \varepsilon^\sim = \varepsilon \). A word is called palindrome if it is equal to its reversal. We let \( \text{PAL}(A) \), or simply PAL, denote the set of all palindromes on the alphabet \( A \).

Let \( p \) be a positive integer. A word \( w = w_1 \cdots w_n \), \( w_i \in A \), \( 1 \leq i \leq n \), has period \( p \) if the following condition is satisfied: for all integers \( i \) and \( j \) such that \( 1 \leq i, j \leq n \),

\[
\text{if } i \equiv j \pmod{p}, \text{ then } w_i = w_j.
\]

We let \( \pi(w) \) denote the minimal period of \( w \). In the sequel, we set \( \pi(\varepsilon) = 1 \). A word \( w \) is said to be constant if \( \pi(w) = 1 \), i.e., \( w = z^k \) with \( k \geq 0 \) and \( z \in A \).

An infinite word (from left to right) \( w \) is just an infinite sequence of letters:

\[
w = w_1 w_2 \cdots w_n \cdots \quad \text{where } w_i \in A, \quad \text{for all } i \geq 1.
\]

A (finite) factor of \( w \) is either the empty word or any sequence \( u = u_i \cdots u_j \) with \( i \leq j \), i.e., a finite block of consecutive letters in \( w \). If \( i = 1 \), then \( u \) is a prefix of \( w \); for any \( n \) we let \( w[n] \) denote its prefix of length \( n \), i.e., \( w[n] = w_1 \cdots w_n \). The set of all infinite words over \( A \) is denoted by \( A^\omega \). The set of all factors of a finite or infinite word \( w \) is denoted by \( \text{Fact } w \).

In the following we shall mainly concern with two-letter alphabets. We let \( \mathcal{A} \) denote the alphabet whose elements are the letters \( a \) and \( b \) that we shall identify respectively with the digits 0 and 1; moreover, we totally order \( \mathcal{A} \) by setting \( a < b \). We let \((\cdot)\) denote the automorphism of \( \mathcal{A}^\ast \) defined by \( \tilde{a} = b \) and \( \tilde{b} = a \). For each \( w \in \mathcal{A}^\ast \), the word \( \tilde{w} \) is called the complementary word, or simply the complement, of \( w \).
For each \( w \in A^* \) let \( \langle w \rangle_2 \), or simply \( \langle w \rangle \), denote the standard interpretation of \( w \) as an integer at base 2; conversely, for each integer \( n \geq 0 \), we let \([n]_2\) denote the expansion of \( n \) at base 2. For instance, \( \langle a \rangle = 0 \), \( \langle b \rangle = 0 \), \( \langle baaba \rangle = 18 \), and \([21]_2 = babab\).

2.1 The palindrome map

We introduce in \( A^* \) the operator \((+): A^* \rightarrow \text{PAL}\) which maps any word \( w \in A^* \) to the word \( w^{(+)} \) defined as the shortest palindrome having the prefix \( w \) (cf. \([11]\)). The palindrome \( w^{(+)} \) is called the right palindromic closure of \( w \). If \( Q \) is the longest palindromic suffix of \( w = uQ \), then one has \( w^{(+)} = uQu^{\sim} \).

Let us now define the map

\[
\psi: A^* \rightarrow \text{PAL},
\]

called palindrome map over \( A^* \), as follows: \( \psi(\varepsilon) = \varepsilon \) and for all \( u \in A^*, x \in A, \psi(ux) = (\psi(u)x)^{(+)} \).

For instance, if \( u = aba \), one has \( \psi(a) = a, \psi(ab) = (\psi(a)b)^{(+)} = aba \), and \( \psi(aba) = (abaa)^{(+)} = abaaba \).

The following proposition summarizes some simple but noteworthy properties of the palindrome map (cf., for instance, \([20, 11]\)):

**Proposition 2.1.** Let \( \psi \) be the palindrome map over \( A^* \). For \( u, v \in A^* \) the following hold:

- **P1.** If \( u \) is a prefix of \( v \), then \( \psi(u) \) is a palindromic prefix (and suffix) of \( \psi(v) \).
- **P2.** If \( p \) is a prefix of \( \psi(v) \), then \( p^{(+)} \) is a prefix of \( \psi(v) \).
- **P3.** Every palindromic prefix of \( \psi(v) \) is of the form \( \psi(u) \) for some prefix \( u \) of \( v \).
- **P4.** The palindrome map is injective.
- **P5.** \( |\psi(u^{\sim})| = |\psi(u)| \).
- **P6.** \( \psi(\overline{u}) = \overline{\psi(u)} \).

For any \( w \in \psi(A^*) \) the unique word \( u \) such that \( \psi(u) = w \) is called the directive word of \( w \).
For any $x \in A$ let $\mu_x$ denote the injective endomorphism of $A^*$

$$
\mu_x : A^* \to A^*
$$
defined by

$$
\mu_x(x) = x, \quad \mu_x(y) = xy, \text{ for } y \in A \setminus \{x\}.
$$

(1)

If $v = x_1x_2 \cdots x_n$, with $x_i \in A$, $i = 1, \ldots, n$, then we set:

$$
\mu_v = \mu_{x_1} \circ \cdots \circ \mu_{x_n};
$$
morover, if $v = \varepsilon$ then $\mu_v = \text{id}$.

The following interesting theorem, proved by Jacques Justin [23] in the case of an arbitrary alphabet, relates the palindromization map to morphisms $\mu_v$.

**Theorem 2.2.** For all $v, u \in A^*$,

$$
\psi(vu) = \mu_v(\psi(u))\psi(v).
$$

In particular, if $x \in A$, one has

$$
\psi(xu) = \mu_x(\psi(u))x \quad \text{and} \quad \psi(vx) = \mu_v(x)\psi(v).
$$

The palindromization map $\psi$ can be extended to $A^\omega$ as follows: let $w \in A^\omega$ be an infinite word

$$
w = w_1w_2 \cdots w_n \cdots, \quad w_i \in A, \quad i \geq 1.
$$

By property P1 of Proposition [23] for all $n$, $\psi(w_{[n]})$ is a prefix of $\psi(w_{[n+1]})$, so one can define the infinite word $\psi(w)$ as:

$$
\psi(w) = \lim_{n \to \infty} \psi(w_{[n]}).
$$

The extended map $\psi : A^\omega \to A^\omega$ is injective. The word $w$ is called the **directive word** of $\psi(w)$.

As proved in [11] an infinite word $s \in A^\omega$ is a **characteristic Sturmian word** if and only if $s = \psi(w)$ with $w \in A^\omega$ such that each letter $x \in A$ occurs infinitely often in $w$. An infinite word $s \in A^\omega$ is called **Sturmian** if there exists a characteristic Sturmian word $t$ such that $\text{Fact } s = \text{Fact } t$.

**Example 2.3.** If $w = (ab)^\omega$, then the characteristic Sturmian word $f = \psi((ab)^\omega)$ having the directive word $w$ is the famous **Fibonacci word**

$$
f = ababaabaabaab \cdots
$$

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1 If one extends the action of palindromization map to infinite words over arbitrary finite alphabets, one can generate a wider class of words, called **standard episturmian**, introduced in [20]. Some further extensions and generalizations of the palindromization map are in [14, 15, 24].

7
3 Central, standard, and Christoffel words

In the combinatorics of Sturmian words a key role is played by three classes of finite words called \textit{central}, \textit{standard}, and \textit{Christoffel words}. They are closely interrelated and satisfy remarkable structural properties.

A word \( w \) is called \textit{central} if \( w \) has two periods \( p \) and \( q \) such that \( \gcd(p, q) = 1 \) and \( |w| = p + q - 2 \). The set of central words, usually denoted by \( \text{PER} \), was introduced in \cite{16} where its main properties were studied; in particular, it has been proved that \( \text{PER} \) is equal to the set of the palindromic prefixes of all characteristic Sturmian words, i.e., \( \text{PER} = \psi(\mathcal{A}^*) \).

There exist several different characterizations of central words (see, for instance \cite{1} and the references therein). We recall here the following noteworthy structural characterization \cite{11, 8}:

\textbf{Proposition 3.1.} A word \( w \) is central if and only if it is constant or satisfies the equation

\[ w = w_1 ab w_2 = w_2 ba w_1 \]

with \( w_1, w_2 \in \mathcal{A}^* \). Moreover, in this latter case, \( w_1 \) and \( w_2 \) are uniquely determined central words, \( p = |w_1| + 2 \) and \( q = |w_2| + 2 \) are coprime periods of \( w \), and \( \min\{p, q\} \) is the minimal period of \( w \).

Another important family of finite words is the class of finite standard words. In fact, characteristic Sturmian words can be equivalently defined in the following way. Let \( c_1, \ldots, c_n, \ldots \) be any sequence of integers such that \( c_1 \geq 0 \) and \( c_i > 0 \) for \( i > 1 \). We define, inductively, the sequence of words \( (s_n)_{n \geq -1} \), where

\[ s_{-1} = b, \; s_0 = a, \; \text{and} \; s_n = s_{n-1}^{c_n} s_{n-2} \text{ for } n \geq 1. \]

Since for any \( n \geq 0 \), \( s_n \) is a proper prefix of \( s_{n+1} \), the sequence \( (s_n)_{n \geq -1} \) converges to a limit \( s \) which is a characteristic Sturmian word (cf. \cite{27}). Any characteristic Sturmian word is obtained in this way. The Fibonacci word is obtained when \( c_i = 1 \) for all \( i \geq 1 \).

We shall denote by \( \text{Stand} \) the set of all the words \( s_n, \; n \geq -1 \), of any sequence \( (s_n)_{n \geq -1} \). Any word of \( \text{Stand} \) is called \textit{finite standard word}, or simply \textit{standard word}. The following noteworthy relation exists \cite{16} between standard and central words:

\[ \text{Stand} = \mathcal{A} \cup \text{PER}\{ab, ba\}. \]

More precisely, the following holds (see, for instance \cite{12} Propositions 4.9 and 4.10):
Proposition 3.2. Any standard word different from a single letter can be uniquely expressed as $\mu_v(xy)$ with $\{x,y\} = \{a,b\}$ and $v \in A^*$. Moreover, one has

$$\mu_v(xy) = \psi(v)xy.$$  

Let us set for any $v \in A^*$ and $x \in A$,

$$p_x(v) = |\mu_v(x)|. \quad (2)$$

From Justin’s formula one derives (cf. [17, Proposition 3.6]) that $p_x(v)$ is the minimal period of $\psi(vx)$ and then a period of $\psi(v)$. Moreover, one has (cf. [18, Lemma 5.1])

$$p_x(v) = \pi(\psi(vx)) = \pi(\psi(v)x) \quad (3)$$

and $\gcd(p_x(v), p_y(v)) = 1$, so that

$$\pi(\psi(v)) = \min\{p_x(v), p_y(v)\}. \quad (4)$$

Since $|\mu_v(xy)| = |\mu_v(x)| + |\mu_v(y)|$, from Proposition 3.2 and (2) one has

$$|\psi(v)| = p_x(v) + p_y(v) - 2. \quad (5)$$

The following lemma is readily derived from (2),

Lemma 3.3. For $w \in A^*$ and $x, y \in A$ one has

$$p_x(wx) = p_x(w), \quad p_y(wx) = p_x(w) + p_y(w), \text{ for } y \in A \setminus \{x\}.$$  

Let us now consider the class CH of words, introduced in 1875 by Elwin B. Christoffel [9] (see also [3, 4]), usually called Christoffel words. Let $p$ and $q$ be positive relatively prime integers such that $n = p + q$. The Christoffel word $w$ of slope $\frac{p}{q}$ is defined as $w = x_1 \cdots x_n$ with

$$x_i = \begin{cases} 
    a, & \text{if } ip \mod n > (i - 1)p \mod n; \\
    b, & \text{if } ip \mod n < (i - 1)p \mod n. 
\end{cases}$$

for $i = 1, \ldots, n$ where $k \mod n$ denotes the remainder of the Euclidean division of $k$ by $n$. The term slope given to the irreducible fraction $\frac{p}{q}$ is due to the fact that, as one easily derives from the definition, $p = |w|_b$ and $q = |w|_a$. The words $a$ and $b$ are also Christoffel words with a respective slope $\frac{1}{0}$ and $\frac{1}{0}$. The Christoffel words of slope $\frac{p}{q}$ with $p$ and $q$ positive integers are called proper Christoffel words.
The following result [2], shows a basic relation existing between central and Christoffel words:

\[ \text{CH} = a \text{ PER } b \cup A. \]

Hence, any proper Christoffel word \( w \) can be uniquely represented as \( a\psi(v)b \) for a suitable \( v \in A^* \). We say that \( w \) is of order \( k \) if \( v \in A^k \).

Let \( < \) denote the lexicographic order of \( A^* \) and let Lynd be the set of Lyndon words (see, for instance, [26, Chap. 5]) of \( A^* \) and St be the set of (finite) factors of all Sturmian words. The following theorem summarizes some results on Christoffel words proved in [5, 2, 4, 18].

**Theorem 3.4.** Let \( w = a\psi(v)b \) with \( v \in A^* \) be a proper Christoffel word. Then the following hold:

1) \( \text{CH} = \text{St} \cap \text{Lynd} \), i.e., CH equals the set of all factors of Sturmian words which are Lyndon words.

2) There exist and are unique two Christoffel words \( w_1 \) and \( w_2 \) such that \( w = w_1w_2 \). Moreover, \( w_1 < w_2 \), and \((w_1, w_2)\) is the standard factorization of \( w \) in Lyndon words.

3) If \( w \) has the slope \( \eta(w) = \frac{p}{q} \), then \( |w_1| = p', |w_2| = q' \), where \( p' \) and \( q' \) are the respective multiplicative inverse of \( p \) and \( q \), mod \( |w| \). Moreover, \( p' = p_a(v) \), \( q' = p_b(v) \) and \( p = p_a(v^\sim) \), \( q = p_b(v^\sim) \).

**Example 3.5.** Let \( p = 4 \) and \( q = 7 \). The Christoffel construction is represented by the following diagram

\[
\begin{align*}
0 &\rightarrow 4 \rightarrow 8 \rightarrow b \rightarrow 1 \rightarrow 5 \rightarrow 9 \rightarrow b \rightarrow 2 \rightarrow 6 \rightarrow 10 \rightarrow b \rightarrow 3 \rightarrow a \rightarrow 7 \rightarrow b \rightarrow 0
\end{align*}
\]

so that the Christoffel word \( w \) having slope \( \frac{4}{7} \) is

\[ w = aabaabaabab = aub, \]

where \( u = abaabaaba = \psi(aba^2) \) is the central word of length 9 having the two coprime periods \( p_a(v) = 3 \) and \( p_b(v) = 8 \) with \( v = aba^2 \). The word \( w \) can be uniquely factorized as \( w = w_1w_2 \), where \( w_1 \) and \( w_2 \) are the Lyndon words \( w_1 = aab \) and \( w_2 = aabaabab \). One has \( w_1 < w_2 \) with \( |w_1| = 3 = p_a(v) \) and \( |w_2| = 8 = p_b(v) \). Moreover, \( w_2 \) is the proper suffix of \( w \) of maximal length which is a Lyndon word. Finally, \( \psi(v) = \psi(a^2ba) = aabaabaa, p_a(v^\sim) = 4 = |w|_b, p_b(v^\sim) = 7 = |w|_a, \) and \( |w|_{b p_a(v)} = 4 \cdot 3 = 12 \equiv |w|_a p_b(v) = 7 \cdot 8 = 56 \equiv 1 \mod 11. \)

For any word \( v \in A^+ \), we let \( v^F \) (resp. \( v^L \)) denote the first (resp., last) letter of \( v \).
Lemma 3.6. For any $v \in \mathcal{A}^+$, $\pi(\psi(v^\sim)) = |a\psi(v)b|_{v^F}$.

Proof. By [4] and item 3) of Theorem 3.4 one has

\[
\pi(\psi(v^\sim)) = \min\{p_a(v^\sim), p_b(v^\sim)\} = \min\{|a\psi(v)b|_{a}, |a\psi(v)b|_{b}\}.
\]

The result follows since for each $v \in \mathcal{A}^+$ one has $|\psi(v)|_{v^F} > |\psi(v)|_{v^F}$, as one easily derives using Proposition 3.1, observing that $(\psi(v))^F = v^F$ and by making induction on the lengths of central words.

Let $v$ be a non-empty word. We let $v^-$ (resp., $-v$) denote the word obtained from $v$ by deleting the last (resp., first) letter. If $v$ is not constant, we let $v_+$ (resp., $_+v$) denote the longest prefix (resp., suffix) of $v$ which is immediately followed (resp., preceded) by the complementary of the last (resp., first) letter of $v$. For instance, if $v = abbabab$, one has $v^- = ababa$, $v_+ = abbab$, $-v = bbabab$, and $_+v = babab.$

Proposition 3.7. If $v \in \mathcal{A}^*$ is not constant, then

\[
|a\psi(v)b| = |a\psi(v^-)b| + |a\psi(v_+)b| = |a\psi(-v)b| + |a\psi(_+v)b|.
\]

Moreover,

\[
|a\psi(v_+)b| = \pi(\psi(v)) \quad \text{and} \quad |a\psi(_+v)b| = |a\psi(v)b|_{v^F}.
\]

Proof. Let $x$ be the last letter of $v$. By Justin’s formula one has

\[
\psi(v) = \psi(v^-x) = \mu_{v^-}(x)\psi(v^-).
\]

Now if $y = \bar{x}$ one has $v^- \in (v_+)y\mathcal{A}^*$, and by Proposition 3.2, $\mu_{v^-}(x) = \mu_{v_+}(yx) = \psi(v_+)yx$. Thus

\[
\psi(v) = \psi(v_+)yx\psi(v^-) = \psi(v^-xy\psi(v_+)) \quad (6)
\]

and $|a\psi(v)b| = |a\psi(v^-)b| + |a\psi(v_+)b|$. By Proposition 3.1 $|a\psi(v^-)b| = p$ and $|a\psi(v_+)b| = q$ are two coprime periods such that $|\psi(v)| = p + q - 2$. Since $v_+$ is a proper prefix of $v^-$ one has $q < p$ and therefore $\pi(\psi(v)) = |a\psi(v_+)b|$.

By item P5 of Proposition 2.1 one has $|\psi(v)| = |\psi(v^\sim)|$, so that from the preceding result $|a\psi(v^-)b| = |a\psi((v^-)^-)b| + |a\psi((v^-)_+)b|$. As it is readily verified, $(v^-)^- = (v^-)\sim$ and $(v^-)_+ = (v^-)^\sim$, so that $|a\psi(v)b| = |a\psi(v^-)b| + |a\psi(_+v)b|$. Since $|a\psi(_+v)b| = |a\psi((v^-)_+)b| = \pi(\psi(v^-))$ the result follows from Lemma 3.6.
Corollary 3.8. For any non-constant \( v \in \mathcal{A}^* \), the standard factorization of \( a\psi(v)b \) in Lyndon words is
\[
(a\psi(v_+),b,a\psi(v^-)b) \text{ if } v^L = a \text{ and } (a\psi(v^-)b,a\psi(v_+)b) \text{ if } v^L = b.
\]
As a consequence for any \( v \in \mathcal{A}^+ \)
\[
\pi(\psi(v)) = p_{\cdot v}(v).
\]

Proof. From \((\text{10})\) if \( v \) terminates with \( a \) (i.e., \( x = a \)) one has \( a\psi(v)b = a\psi(v+)ba\psi(v^-)b \). On the contrary, if \( v \) terminates with \( b \) (i.e., \( x = b \)) one has \( a\psi(v)b = a\psi(v^-)bba\psi(v_+)b \). Since \( a\psi(v_+)b \) and \( a\psi(v^-)b \) are Christoffel words, the first result follows from item 2) of Theorem 3.4.

Let \( v \in \mathcal{A}^+ \). If \( v \) is constant, i.e., \( v = x^h \) with \( x \in \mathcal{A} \) and \( h \geq 1 \), then trivially \( p_{\cdot x}(x^h) = 1 = \pi(\psi(x^h)) \). If \( v \) is not constant, then by Proposition 3.7 one has \( \pi(\psi(v)) = |a\psi(v_+)b| \). From the result proved above and from item 3) of Theorem 3.4 it follows \( \pi(\psi(v)) = p_{\cdot v}(v) \).

Corollary 3.9. For any \( v \in \mathcal{A}^+ \) one has that \( |a\psi(va)b| \) is less (resp., greater) than \( |a\psi(vb)b| \) if and only if \( v^L = a \) (resp., \( v^L = b \)).

Proof. Let us first suppose that \( v \) is constant, i.e., \( v = a^n \) or \( v = b^n \), with \( n > 0 \). In this case the result is trivial since \( |a\psi(a^{n+1})b| = |a\psi(b^{n+1})b| = n+3 \) and \( |a\psi(a^n)b| = |a\psi(b^n)a| = 2n + 3 \). Let us then suppose that \( v \) is not constant. One has \( (va)^- = (vb)^- = v \). If \( v = ua \) with \( u \in \mathcal{A}^* \), then one has \( (vb)_+ = (uab)_+ = u \) and \( (va)_+ = (uaa)_+ = u_1 \), with \( u_1 \) a proper prefix of \( u \). By Proposition 3.7 one has:
\[
|a\psi(va)b| = |a\psi(v)b| + |a\psi(u_1)b| < |a\psi(v)b| + |a\psi(u)b| = |a\psi(vb)b|.
\]
In a similar way one proves that if \( v = ub \), one has \( |a\psi(va)b| > |a\psi(vb)b| \).

Proposition 3.10. For any word \( v = v_1 \cdots v_n \), with \( n > 0 \), \( v_i \in \mathcal{A} \), \( i = 1, \ldots, n \), one has
\[
|\psi(v)| = \sum_{i=1}^{n} \pi(\psi(v_1 \cdots v_i)) = \sum_{i=1}^{n} |a\psi(v_1 \cdots v_i)b|_{\psi_i}.
\]

Proof. The result is trivial if \( v \) is constant, i.e., \( v = x^n \) with \( x \in \mathcal{A} \). Indeed, \( |\psi(x^n)| = |x^n| = n \) and for all \( 1 \leq i \leq n \) one has \( \pi(\psi(x^i)) = 1 \) and \( |a\psi(x^i)b|_{\bar{x}} = |ax^ib|_{\bar{x}} = 1 \). Let us then suppose that \( v \) is not constant. The proof is obtained by induction on the length of \( v \). Let us prove the first equality. By Proposition 3.7 one has
\[
|a\psi(v)b| = |a\psi(v_+)b| + |a\psi(v^-)b| = \pi(\psi(v)) + |a\psi(v^-)b|,
\]
\[
|a\psi(v_+)b| = \sum_{i=1}^{n} \pi(\psi(v_1 \cdots v_i)) = \sum_{i=1}^{n} |a\psi(v_1 \cdots v_i)b|_{\psi_i}.
\]
so that by induction
\[ |\psi(v)| = \pi(\psi(v)) + \sum_{i=1}^{n-1} \pi(\psi(v_1 \cdots v_i)) = \sum_{i=1}^{n} \pi(\psi(v_1 \cdots v_i)). \]

Let us now prove the second equality. By Proposition 3.7 one has
\[ |a\psi(v)b| = |a\psi(v_1)v_1 + v_2 b| + |a\psi(v) - v)b|, \]
so that by induction
\[ |\psi(v)| = |a\psi(v)b|_{\bar{v}_1} + |\psi(\bar{v})| = |a\psi(v)b|_{\bar{v}_1} + \sum_{i=2}^{n} |a\psi(v_1 \cdots v_n)b|_{\bar{v}_1}. \]

which concludes the proof. \(\square\)

4 The Raney and the Stern-Brocot trees

Let us consider the complete binary tree. Trivially, each path from the root to a particular node can be represented by a word \(w \in A^\ast\). More precisely, if \(w = b_{h_0} a_{h_1} b_{h_2} \cdots a_{h_{n-1}} b_{h_n}\) with \(h_0, h_n \geq 0\) and \(h_i > 0, 1 \leq i \leq n - 1\), then the sequence of letters read from left to right gives the sequence of right and left moves in order to reach the node starting from the root. Since for every node there exists a unique path going from the root to the node, one has that the nodes are faithfully represented by the words over \(A\). Thus one can identify the nodes of the tree with the binary words of \(A^\ast\).

Let us now label each node of the tree with an irreducible fraction \(\frac{p}{q}\), where \(p\) and \(q\) are positive and relatively prime integers, in the following way. The root has the label \(\frac{1}{1}\). If a node has the label \(\frac{p}{q}\), then the left child has the label \(\frac{p}{p+q}\) and the right child has the label \(\frac{p+q}{q}\).

This tree was introduced by J. Berstel and the first author in [2] and called the Raney tree since it was implicitly contained in the work of Raney [31]. The Raney tree was reconsidered in [6] and is usually referred in the literature as the Calkin-Wilf tree.

As proved in [2] (see also [6]) all irreducible fractions can be faithfully represented by the Raney tree. We let \(Ra(w)\) denote the fraction labeling the node represented by the word \(w\).

Another famous labeling the complete binary tree by irreducible fractions is the Stern-Brocot tree (see, for instance, [28, 21]). The labeling is constructed as follows. The label \(\frac{p}{q}\) in a node is given by \(\frac{p' + p''}{q' + q''}\), where \(\frac{p'}{q'}\) is the nearest ancestor above and to the left and \(\frac{p''}{q''}\) is the nearest ancestor.
Figure 1: The Raney tree

above and to the right (in order to construct the tree one needs also two 
extra nodes labeled by $\frac{1}{0}$ and $\frac{0}{1}$). We let $Sb(w)$ denote the fraction labeling the node represented by the binary word $w$.

An important relation between the Raney and the Stern-Brocot tree is given by the following lemma\(^\text{2}\) (see, for instance, [2])

**Lemma 4.1.** For all $w \in A^*$, one has $Sb(w) = Ra(w^\sim)$.

Moreover, the following hold:

**Lemma 4.2.** For all $w \in A^*$,

$$Ra(\bar{w}) = \frac{1}{Ra(w)} \quad \text{and} \quad Sb(\bar{w}) = \frac{1}{Sb(w)}.$$ 

The Raney and Stern-Brocot numbers $Ra(w)$ and $Sb(w)$ are strictly related respectively to the ratio of periods of the central word $\psi(w)$ and to the slope of Cristoffel word $a\psi(w)b$ as follows [2]:

**Proposition 4.3.** Let $w$ be the directive word of the central word $\psi(w)$. Then

$$Ra(w) = \frac{p_a(w)}{p_b(w)}, \quad Sb(w) = \frac{|a\psi(w)b|_b}{|a\psi(w)b|_a}.$$ 

\(^2\)A suitable generalization of the Raney and of Stern-Brocot tree in the case of alphabets with more than two letters and of Lemma 4.1 is in [18].
By the preceding proposition and Lemma 4.1, one derives the following important duality property expressed in item 3) of Theorem 3.4

\[ |a\psi(w)b|_b = p_a(w^\sim), \quad |a\psi(w)b|_a = p_b(w^\sim). \]

By Lemma 3.3 one has for any \( w \in A^* \)

\[ Ra(wa) = \frac{p_a(w)}{p_a(w) + p_b(w)}, \quad Ra(wb) = \frac{p_a(w) + p_b(w)}{p_b(w)}. \] (7)

Finally, we mention that both the trees can be viewed as specializations of a tree formed by ordered pairs of binary words, called the Christoffel tree \cite{2,19}.

5 The Stern sequence

Let us enumerate the nodes of the complete binary tree as follows. The root \( \varepsilon \) is numbered by 2. If a node \( w \) has the number \( \nu(w) \), then the left child \( wa \) has the number \( \nu(wa) = 2\nu(w) - 1 \) and the right child \( wb \) has the number \( \nu(wb) = 2\nu(w) \). The numbering \( \nu \) is a bijection of \( A^* \) into the set \( \mathbb{N}_2 \) of all integers \( \geq 2 \) having for all \( w \in A^* \) and \( n > 1 \)

\[ \nu(w) = \langle bw \rangle + 1 \quad \text{and} \quad \nu^{-1}(n) = b^{-1}[n - 1]_2, \]

where \( b^{-1}[n - 1]_2 \) is the word obtained by cancelling the first digit in the binary expansion of \( n - 1 \). Let us set for \( n > 1 \)

\[ ra(n) = Ra(\nu^{-1}(n)), \]

and let \( s(n) \) denote the denominator of the fraction \( ra(n) \). By induction on \( n \) and by (7) one derives:

\[ ra(n) = \frac{s(n - 1)}{s(n)}. \] (8)

The sequence \( s(n) \) is the famous Stern sequence \cite{33} which can be inductively defined as: \( s(0) = 0, s(1) = 1 \) and for \( n \geq 1 \),

\[
\begin{cases}
  s(2n) = s(n) \\
  s(2n + 1) = s(n) + s(n + 1).
\end{cases}
\]

The first few terms of Stern’s sequence are (cf. sequence A2487 in \cite{32}):

0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, 3, 7, 4, 5, 1, ...
There exists a large literature on Stern’s sequence since it satisfies a great number of beautiful and surprising properties mainly for what concerns various combinatorial interpretations that can be associated to its terms. In this paper we are mainly interested in the combinatorial properties which are related to Christoffel and central words.

In the following for any \( w \in A^* \), we shall set \( \hat{s}(w) = s(\nu(w)) \).

From Lemma 4.2 one easily derives a well known identity on Stern’s sequence (see, for instance, [30])

**Lemma 5.1.** For any \( k \geq 0 \) and \( 1 \leq p \leq 2^k \),

\[
 s(2^k + p) = s(2^{k+1} - p).
\]

**Proof.** Let \( k \geq 0 \) and \( w \in A^k \). By [8] one has \( Ra(w) = \frac{s(\nu(w) - 1)}{s(\nu(w))} \). By Lemma 4.2 \( Ra(\tilde{w}) = \frac{1}{Ra(w)} \) so that

\[
 s(\nu(\tilde{w}) - 1) = s(\nu(w)).
\]

Setting \( \nu(w) = 2^k + p \), with \( 1 \leq p \leq 2^k \), one has \( \nu(\tilde{w}) = 2^{k+1} - p + 1 \) and the result follows. \( \square \)

Let us observe that there exists a basic correspondence between the values of Stern’s sequence on odd integers and the lengths of Christoffel words as well as a correspondence between the values of the sequence on even integers and the minimal periods of central words. More precisely the following hold:

**Theorem 5.2.** For any \( w \in A^* \) one has

\[
 \hat{s}(wa) = s(\langle bwb \rangle) = |a\psi(w)b|, \quad \hat{s}(wb) = s(\langle bwb \rangle + 1) = \pi(\psi(wb)).
\]

**Proof.** By (5) one has \( |a\psi(w)b| = 2 + |\psi(w)| = p_a(w) + p_b(w) \). Moreover,

\[
 \hat{s}(wa) = s(\nu(wa)) = s(2\nu(w) - 1) = s(\langle bwb \rangle).
\]

By (7) one has \( \hat{s}(wa) = p_a(w) + p_b(w) \), so that the first result follows. By (7) and (3) one has

\[
 \hat{s}(wb) = s(\nu(wb)) = s(2\nu(w)) = s(\langle bwb \rangle + 1) = s(\nu(w)) = p_b(w) = \pi(\psi(wb)),
\]

which proves the second assertion of the proposition. \( \square \)
As a consequence of the previous correspondence several results on Stern’s sequence can be proved by using the theory of Sturmian words and, conversely, properties of Stern’s sequence can give a new insight in the combinatorics of Christoffel and central words.

**Proposition 5.3.** For each $k \geq 3$ and $0 \leq p \leq 2^{k-3} - 1$ one has:

$$s(2^k + 8p + 1) < s(2^k + 8p + 3) \quad \text{and} \quad s(2^k + 8p + 5) > s(2^k + 8p + 7).$$

*Proof.* By Theorem 5.2 one has $|a\psi(va)b| = s((bvab))$ and $|a\psi(vb)b| = s((bbvb))$; moreover, $(bvbb) = (bvab) + 2$. If $v = ua$, with $u \in A^*$, one has $(bvab) = 1 + 8|u| + 2|u|^3$. Let us set $|u| + 3 = k$; one has $0 \leq |u| < 2^{k-3}$. Thus the first inequality follows from Corollary 3.9. If $v = ub$ one derives the second inequality in a similar way. \hfill \Box

Let us define a function $R : \mathbb{N} \to \mathbb{N}$ by $R(n) = \langle [n]_2^\sim \rangle$, i.e., $R(n)$ is the integer obtained by reversing the binary expansion of $n$. Note that, with the exception of $R(0) = 0$, $R(n)$ is always odd.

Let $n = \sum_{i=0}^\ell d_i 2^{\ell-i} > 0$, with $\ell = \lfloor \log_2 n \rfloor$ and $d_i \in \{0,1\}$ for all $i$, so that $d_i$ is the $(i + 1)$th binary digit of $n$. By definition, one derives $d_{i-i} = \lfloor n/2^i \rfloor - 2 \lfloor n/2^{i+1} \rfloor$ for $0 \leq i \leq \ell$, so that

$$R(n) = \sum_{i=0}^\ell \left( \left\lfloor \frac{n}{2^i} \right\rfloor - 2 \left\lfloor \frac{n}{2^{i+1}} \right\rfloor \right) 2^{\ell-i} = 2^\ell n - 3 \sum_{i=1}^\ell \left\lfloor \frac{n}{2^i} \right\rfloor 2^{\ell-i}.$$

The following known result (see, for instance, [34, 30]) can be simply proved using Proposition 2.1 and Theorem 5.2.

**Proposition 5.4.** For all $n \geq 0$, the identity $s(n) = s(R(n))$ holds.

*Proof.* The assertion is trivial if $n = 0$ or if $n$ is a power of 2, so we can write $[n]_2 = bwba^k$ for some $w \in A^*$ and $k \geq 0$. By Theorem 5.2 Proposition 2.1 and the definition of $s$, we have

$$s(R(n)) = s(\langle a^k bw^{-} b \rangle) = s(\langle bw^{-} b \rangle) = |a\psi(w^{-} b)|$$

$$= |a\psi(w) b| = s(\langle bw b \rangle) = s(2^k (bw b)) = s(\langle bwba^k \rangle) = s(n),$$

as desired. \hfill \Box

For each $n > 1$, let us set $L(n) = \lceil \log_2 n \rceil - 1$ and define for $1 \leq k \leq L(n)$,

$$\delta_k(n) = \left\lceil \frac{n - 2^L(n) - 1}{2^L(n) - k} \right\rceil - \left\lfloor \frac{n - 2^L(n) - 1}{2^L(n) - k + 1} \right\rfloor.$$
Proposition 5.5. For $n > 1$,

$$s(2n - 1) = 2 + \sum_{k=1}^{L(n)} s(2^{k-1} + \delta_k(n)).$$

Proof. For any $n > 1$ there exists a word $w$ such that $2n - 1 = \langle bwb \rangle$ and $|w| = m = L(n)$. If $n = 2$, then since $L(2) = 0$ the result is trivially verified. Let us then suppose $n > 2$. By Theorem 5.2 and Proposition 3.10 one has

$$s(\langle bwb \rangle) = |a\psi(w)b| = 2 + |\psi(w)| = 2 + \sum_{k=1}^{m} \pi(\psi(w_1 \cdot \cdot \cdot w_k)).$$

By Corollary 3.8, Proposition 4.3, and (8) one derives

$$\pi(\psi(w_1 \cdot \cdot \cdot w_k)) = \pi_w(k_{w_1 \cdot \cdot \cdot w_k}) = s(2 \cdot \langle w_1 \cdot \cdot \cdot w_k \rangle + \langle w \rangle).$$

Since

$$\langle w_1 \cdot \cdot \cdot w_k \rangle = \left\lfloor \frac{\langle w \rangle}{2^{m-k}} \right\rfloor$$

and

$$\langle w \rangle = \left\lfloor \frac{\langle w \rangle}{2^{m-k+1}} \right\rfloor - 2 \left\lfloor \frac{\langle w \rangle}{2^{m-k}} \right\rfloor,$$

one obtains, as $s(2x) = s(x)$, for $x \geq 0$,

$$\pi(\psi(w_1 \cdot \cdot \cdot w_k)) = s \left( 2^{k-1} + \left\lfloor \frac{\langle w \rangle}{2^{m-k+1}} \right\rfloor - \left\lfloor \frac{\langle w \rangle}{2^{m-k+1}} \right\rfloor \right).$$

Since $\langle w \rangle = n - 2^m - 1$ and $m = L(n)$, the result follows.

6 Stern’s sequence and continuants

Any word $v \in A^*$ can be uniquely represented as:

$$v = b^{a_0} a_1 b^{a_2} \cdot \cdot \cdot a_{n-1} b^{a_n},$$

where $n$ is an even integer, $a_i > 0$, $i = 1, \ldots, n - 1$, and $a_0 \geq 0$, $a_n \geq 0$. We call the list $(a_0, a_1, \ldots, a_n)$ the integral representation of the word $v$. If $a_n = 0$ the list $(a_0, a_1, \ldots, a_{n-1})$ is called the reduced integral representation of $v$.

We can identify the word $v$ with its integral representation and write $v \equiv (a_0, a_1, \ldots, a_n)$. One has

$$|v| = \sum_{i=0}^{n} a_i.$$
For instance, the words $v_1 = b^2aba^2$ and $v_2 = a^3bab^2$ have the integral representations $v_1 \equiv (2, 1, 1, 2, 0)$ and $v_2 \equiv (0, 3, 1, 1, 2)$. The empty word $\varepsilon$ has the integral representation $\varepsilon \equiv (0)$.

The following proposition (cf. [2]), called also mirror formula, permits to represent the Stern-Brocot and the Raney number of a word $v \in A^*$ in terms of continued fractions (see, for instance, [28, 21]) on the elements of the integral representation of $v$.

**Proposition 6.1.** Let $v \in A^*$ have the integral representation $(a_0, a_1, \ldots, a_n)$. If $n = 0$, $Sb(v) = Ra(v) = [a_0; 1] = [a_0 + 1]$. If $n > 0$, then

$$Sb(v) = [a_0; a_1, \ldots, a_{n-1}, a_n + 1], \quad Ra(v) = [a_n; a_{n-1}, \ldots, a_1, a_0 + 1].$$

Let $a_0, a_1, \ldots, a_n, \ldots$ be any sequence of numbers. We consider the continuant $K[a_0, \ldots, a_n]$ defined by recurrence as: $K[\ ] = 1, K[a_0] = a_0$, and for $n \geq 1$,

$$K[a_0, a_1, \ldots, a_n] = a_nK[a_0, a_1, \ldots, a_{n-1}] + K[a_0, a_1, \ldots, a_{n-2}]. \quad (9)$$

As it is readily verified, for any $n \geq 0$, $K[a_0, a_1, \ldots, a_n]$ is a multivariate polynomial in the variables $a_0, a_1, \ldots, a_n$ which is obtained by starting with the product $a_0a_1 \cdots a_n$ and then striking out adjacent pairs $a_ka_{k+1}$ in all possible ways. For instance, $K[a_0, a_1, a_2, a_3, a_4] = a_0a_1a_2a_3a_4 + a_2a_3a_4 + a_0a_3a_4 + a_0a_1a_4 + a_0a_1a_2 + a_0 + a_2 + a_4$.

We recall (cf. [21, 28]) that for every $n \geq 0$,

$$K[a_0, \ldots, a_n] = K[a_n, \ldots, a_0], \quad (10)$$

i.e., a continuant does not change its value by reversing the order of its elements; moreover, one has $K[1^n] = F_{n-1}$, where $1^n$ denotes the sequence of length $n$, $(1, 1, \ldots, 1)$ and $(F_n)_{n \geq 1}$ is the Fibonacci numerical sequence defined by $F_1 = F_0 = 1$, and for $n \geq 0$, $F_{n+1} = F_n + F_{n-1}$. From (9) one derives

$$K[a_0, \ldots, a_n, 1] = K[a_0, \ldots, a_{n-1}, a_n + 1]. \quad (11)$$

Any continued fraction can be expressed in terms of continuants as follows:

$$[a_0; a_1, \ldots, a_n] = \frac{K[a_0, a_1, \ldots, a_n]}{K[a_1, \ldots, a_n]}. \quad (12)$$

The following holds [13]:

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Theorem 6.2. Let \( w = a\psi(v)b \) be a proper Christoffel word and \((a_0, a_1, \ldots, a_n), \ n \geq 0,\) be the reduced integral representation of \( v. \) If \( n = 0, \) then \( |a\psi(v)b| = K[a_0 + 1, 1] = K[a_0 + 2] \) and \( \pi(\psi(v)) = K[1] = 1. \) If \( n > 0, \) then
\[
|a\psi(v)b| = K[a_0 + 1, a_1, \ldots, a_n - 1, a_n + 1]
\]
and
\[
\pi(\psi(v)) = K[a_0 + 1, a_1, \ldots, a_n - 2, a_n - 1].
\]

The following proposition shows that one can compute the values of Stern’s sequence by continuants (cf. [34]):

Proposition 6.3. If \( w \in A^* \) has the integral representation \( w \equiv (a_0, a_1, \ldots, a_n), \) then
\[
\hat{s}(w) = s(\nu(w)) = K[a_0 + 1, a_1, \ldots, a_n - 1].
\]

Proof. One has \( s(\nu(w)) = s(2\nu(w)) = \hat{s}(wb). \) By Proposition 5.2 one has \( \hat{s}(wb) = \pi(\psi(wb)). \) The word \( wb \) has the integral representation
\[
wb \equiv (a_0, a_1, \ldots, a_n + 1)
\]
which is reduced. By Theorem 6.2 the result follows. \( \square \)

Example 6.4. Let \( w = ab^2a. \) One has \( \nu(w) = 23 \) and the integral representation of \( w \) is \((0, 1, 2, 1, 0). \) One has \( s(23) = K[1, 1, 2, 1] = 7. \)

For any \( n > 0, \) let \( e(n) \) the exponent of the highest power of 2 dividing \( n. \) The sequence \( e = (e(n))_{n>0} \) (cf. the sequence A007814 in [32]) is
\[
e = 01020103010201010 \cdots
\]
It is noteworthy that the sequence \( e, \) called \( \omega-Rauzy \) or \( \omega-bonacci \) word in [22], can be expressed using the palindromization map \( \psi \) acting on the infinite word \( w = 0123456 \cdots \) on the alphabet \( \mathbb{N}, \) as \( e = \psi(0123\cdots). \) It is known [34] that for \( n > 0, \)
\[
\left\lfloor \frac{s(n - 1)}{s(n)} \right\rfloor = e(n).
\]

By using a result attributed to Moshe Newman (see, for instance, sequence A2487 in [32]) the following holds: for all \( n > 0, \)
\[
\frac{s(n)}{s(n + 1)} = \frac{1}{2e(n) + 1 - \frac{s(n - 1)}{s(n)}}. \tag{13}
\]
Let us now define for all \(n > 0\),
\[
\zeta(n) = (-1)^{n+1}(2e(n) + 1).
\]
The sequence \(\zeta = (\zeta_n)_{n>0}\), with \(\zeta_n = \zeta(n)\), is
\[
\zeta = 1(-3)1(-5)1(-3)1(-7)\ldots.
\]

**Proposition 6.5.** For all \(n > 0\),
\[
\frac{s(n)}{s(n+1)} = (-1)^{n+1}[0;\zeta_n,\ldots,\zeta_1].
\]

**Proof.** The proof is by induction on the integer \(n\). For \(n = 1\) one has
\[
\frac{s(1)}{s(2)} = \frac{1}{1} = 1.
\]
Suppose the formula true up to \(n-1\) and prove it for \(n\).
By (13) one has
\[
\frac{s(n)}{s(n+1)} = (-1)^{n+1} \frac{\zeta_n - (-1)^{n+1}s(n-1)}{s(n)}
\]
By induction
\[
\frac{s(n)}{s(n+1)} = \frac{(-1)^{n+1}}{\zeta_n + [0;\zeta_{n-1},\ldots,\zeta_1]} = (-1)^{n+1}[0;\zeta_n,\ldots,\zeta_1],
\]
which concludes the proof. \(\square\)

**Theorem 6.6.** For \(n > 1\) one has
\[
s(n) = (-1)^{\frac{n-1}{2}} K[\zeta_1,\ldots,\zeta_{n-1}].
\]

**Proof.** By (12) one has
\[
[0;\zeta_n,\ldots,\zeta_1] = \frac{K[0,\zeta_n,\ldots,\zeta_1]}{K[\zeta_n,\ldots,\zeta_1]}.
\]
By (2) and (10), \(K[0,\zeta_n,\ldots,\zeta_1] = K[\zeta_1,\ldots,\zeta_{n-1}]\), so that by the preceding proposition
\[
\frac{s(n)}{s(n+1)} = (-1)^{n+1} \frac{K[\zeta_1,\ldots,\zeta_{n-1}]}{K[\zeta_1,\ldots,\zeta_n]}.
\]
(14)
Since \(K[\zeta_1,\ldots,\zeta_{n-1}]\) and \(K[\zeta_1,\ldots,\zeta_n]\) are relatively prime, one has for all \(n > 1\), \(s(n) = |K[\zeta_1,\ldots,\zeta_{n-1}]|\). Moreover, from (14), \(K[\zeta_1] > 0\), \(K[\zeta_1,\zeta_2] < 0\), \(K[\zeta_1,\zeta_2,\zeta_3] < 0\), \(K[\zeta_1,\zeta_2,\zeta_3,\zeta_4] > 0\), etc. Hence, it follows \(K[\zeta_1,\ldots,\zeta_{n-1}] > 0\) if and only if \(\frac{n-1}{2}\) is even, which proves our result. \(\square\)

**Example 6.7.** One has \(s(4) = -K[\zeta_1,\zeta_2,\zeta_3] = -K[1,-3,1] = -(1(-3)1+1+1) = 1\), \(s(5) = K[1,-3,1,-5] = 1(-3)1(-5)+1(-3)+1(-5)+1(-5)+1 = 3\).
7 The Calkin-Wilf theorem

Let us recall the following important theorem on Stern’s sequence due to Calkin and Wilf [7, Theorem 5]. We shall give a new proof based on the combinatorics of Christoffel words. A second proof is a consequence of Theorem [7.4] (see Remark [7.6]).

Theorem 7.1. For each \( n \geq 0 \), the term \( s(n) \) of Stern’s sequence is equal to the number of occurrences of the subwords \( u \in b(ab)^* \) in the binary expansion of the integer \( n \).

Proof. We shall first consider the case when the integer \( n \) is odd. The result is trivial if \( n = 1 \). Let us then suppose \( n > 1 \). Letting \( bwb \) be the binary expansion of \( n \), by Theorem [5.2] one has \( s(n) = |a\psi(w)|b \). The proof is by induction on the length of the directive word \( w \). If \( w = z^p \) with \( z \in A \) and \( p \geq 0 \), then the number of occurrences of subwords \( u \in b(ab)^* \) in \( bz^p b \) is \( p + 2 = |a\psi(z^p)|b \), so in this case the result is trivially achieved. Let us then suppose that \( w \) is not constant. We can write \( w = x^hy(\_w) \) with \( x,y \in A \), \( x \neq y \), and \( h \geq 1 \). We have to consider two cases.

Case 1. The letter \( x \) is equal to \( a \). Thus \( bwb = ba^hb(\_w)b \). The number of non-initial occurrences of subwords \( u \in b(ab)^* \) in \( bwb \) is equal to the number of all occurrences of the subwords \( u \) in \( b(\_w)b \). By induction this number is equal to \( |a\psi(\_w)b| \).

The number of all occurrences of subwords \( u \) in \( ba^{h-1}b(\_w)b = b(\_w)b \) is by induction equal to \( |a\psi(\_w)b| \). This number is equal to the number of the initial occurrences of the subwords \( u \) in the word \( bwb \). Indeed, recall that an occurrence of a subword \( u \in b(ab)^* \) in \( v = bwb \) is a word \( j_1j_2\ldots j_m \) on the alphabet \( \{1,2,\ldots,k+2\} \), with \( m = |u| \) and \( k = |w| \), such that \( u_h = v_{j_h} \), \( h = 1,\ldots,m \). Any initial occurrence of \( u \) in \( bwb \) in which the symbol 2 does not appear (i.e., \( j_1 = 1, j_2 > 2 \)) is an initial occurrence of \( u \) in \( b(\_w)b \). Conversely, any initial occurrence of \( u \) in \( b(\_w)b \) is an initial occurrence of \( u \) in \( bwb \) in which the symbol 2 does not appear. Moreover, there exists a one-to-one correspondence between the initial occurrences of \( u \) in \( bwb \) beginning with 12 and the non-initial occurrences of \( u \) in \( b(\_w)b \).

Hence, the total number of occurrences of subwords \( u \in b(ab)^* \) in \( bwb \) is given by \( |a\psi(\_w)b| + |a\psi(\_w)b| \). By Proposition [3.7] this number is equal to \( |a\psi(w)b| \).

Case 2. The letter \( x \) is equal to \( b \). Thus \( bwb = b^{h+1}a(\_w)b \). The number of initial occurrences of subwords \( u \in b(ab)^* \) in \( bwb \) is equal to the number of all occurrences of the subwords \( u \) in \( b(\_w)b \). By induction this number is equal to \( |a\psi(\_w)b| \).
Any non-initial occurrence of a subword $u$ in $bwb$ is a word $j_1j_2\cdots j_m$ over the alphabet $\{1, 2, \ldots, k + 2\}$ with $j_1 \geq 2$, so that any such occurrence is an occurrence of $u$ in $b(-w)b$. Conversely, any occurrence of $u$ in $b(-w)b$ is a non-initial occurrence of $u$ in $bwb$. Hence, by induction, the number of all non-initial occurrences of subwords $u$ in $bwb$ is given by $|a\psi(-w)b|$ and the result is achieved also in this case by Proposition 3.7.

Let us now consider the case of $s(n)$ when $n$ is an even integer. The result is trivial if $n = 0$. Let us then suppose $n > 0$. We can write $n = 2^e(n)p$ where $e(n)$ is the highest integer such that $2^e(n)$ divides $n$. Thus $e(n) > 0$, $p$ is odd, and $s(n) = s(p)$. Since $p$ is odd, by the preceding result one has that $s(p)$ is equal to the number of occurrences of subwords $u \in b(ab)^*$ in the binary expansion $bwb$ of the integer $p$. Now $[bwb\sigma_0(n)]_2 = 2^{e(n)}[bwb]_2 = 2^{e(n)}p = n$. Since the number of occurrences of the subwords $u \in b(ab)^*$ in $bwb\sigma_0(n)$ is equal to the number of occurrences of the subwords $u$ in $bwb$ the result follows.

A consequence of Theorem 7.1 on Christoffel and central words is

**Proposition 7.2.** For each $v \in A^*$

$$|a\psi(v)b| = \sum_{u \in b(ab)^*} \binom{bwb}{u}.$$ 

If $v$ is not constant, then

$$\pi(\psi(v)) = \sum_{u \in b(ab)^*} \binom{bv+1}{u}.$$ 

**Proof.** For each $v$, one has by Proposition 5.2, $s(2\nu(v) - 1) = s(\langle bv \rangle) = |a\psi(v)b|$, so the first equality follows from Theorem 7.1. The second equality is derived from the second statement of Proposition 3.7. □

**Proposition 7.3.** For any $v \in A^*$ the number of initial occurrences of the subwords $u \in b(ab)^*$ in $bwb$ is given by $|a\psi(v)b|_a$.

**Proof.** We shall prove equivalently that the number of non-initial occurrences of the subwords $u \in b(ab)^*$ in $bwb$ is given by $|a\psi(v)b|_b$. This number equals the number of all occurrences of the subwords $u$ in $vb$.

The result is trivial if $v$ is constant. Let us then suppose that $v$ is not constant. We can write $v = x^ky(+v)$ with $k > 0$ and $x, y \in A$, $x \neq y$. We first suppose that $x = a$. By Proposition 7.2 one has

$$\sum_{u \in b(ab)^*} \binom{vb}{u} = \sum_{u \in b(ab)^*} \binom{a^kbv}{u} = \sum_{u \in b(ab)^*} \binom{bv}{u} = |a\psi(+v)b|.$$
Since by Proposition 3.7 one has $|a\psi_{\pm}v)b| = |a\psi(v)b|_b$, in this case the result is achieved. If $x = b$, then

$$\sum_{u \in b(ab)^+} \binom{vb}{u} = \sum_{u \in b(ab)^+} \binom{b^ka_{\pm}v)b}{u} = \sum_{u \in b(ab)^+} \binom{bbk^{-1}a_{\pm}v)b}{u} = |a\psi(-v)b|.$$  

By Proposition 3.7, one has $|a\psi(-v)b| = |a\psi(v)b| - |a\psi_{\pm}v)b| = |a\psi(v)b| - |a\psi(v)b|_a = |a\psi(v)b|_b$. From this the result follows.  

We have seen in Proposition 7.2 as a consequence of the Calkin-Wilf theorem, that the number of occurrences of words $u \in b(ab)^*$ as subwords of $bwb$ is equal to $|a\psi(w)b| = |\psi(w)ba|$. The following theorem shows that distinguishing between initial and non-initial occurrences and sorting them in a suitable way, one can construct the standard word $\psi(w)ba$. This result can be regarded as a non-commutative version of the Calkin-Wilf theorem.

**Theorem 7.4.** Let $w \in \mathcal{A}^*$, and consider the reversed occurrences of words of the set $b(ab)^*$ as subwords in $bwb$. Sorting these in decreasing lexicographic order, and marking the reversed initial occurrences with $a$ and the reversed non-initial ones with $b$, yields the standard word $\psi(w)ba$.

**Proof.** Let $w \in \mathcal{A}^n$ for some integer $n \geq 0$ and let $\mathcal{B} = \{1, 2, \ldots, n+2\}$ the $(n+2)$-letter alphabet totally ordered by the natural integer order $h < h+1$, $h = 1, 2, \ldots, n+1$. This order can be extended to the lexicographic order $\prec$ in $\mathcal{B}^*$. We say that $\psi(w)ba$ describes the reversed occurrences in $bwb$ of subwords in the set $b(ab)^*$ if it is generated by the sequence of markers associated with the sequence of the previous occurrences sorted in decreasing lexicographic order. In what follows for simplicity we shall use the term occurrence instead of reversed occurrence of words of the set $b(ab)^*$ as subwords in $bwb$. Hence, an occurrence is initial if it ends with 1.

If $w = a^n$, then clearly the desired sequence of occurrences in decreasing lexicographic order is $(n+2)(n+1)1 > (n+2)n1 > \cdots > (n+2)21 > (n+2) > 1$, which gives rise to the sequence of markers $a^nba = \psi(a^n)ba$. If $w = b^n$, the result is trivial.

Let us now suppose, by induction, that the result holds for $w \in \mathcal{A}^*$ and all shorter words, and prove it for both $wa$ and $wb$. In the case of $wa$, we can assume that $b$ occurs in $w$ and write $w = w'ba^k$ for some $w' \in \mathcal{A}^*$ and $k \geq 0$. Since $(wa)_+ = w'$ and $(wa)^- = w$, by (1) we have

$$\psi(wa)ba = \psi(w')bab\psi(w')ba = \psi(w')bab\psi(w'ba^k)ba. \quad (15)$$

Let $|bw'b| = h$, and observe that the occurrences in $bwab = bw'bba^kab$ containing the position $h + k + 1$ (i.e., the last $a$) are all greater (in lexicographic
order) than the ones not containing it; moreover, such occurrences cannot contain the preceding \(k\) letters \(a\), and have to start with \((h+k+2)(h+k+1)\). Hence, we can identify any such occurrence \((h+k+2)(h+k+1)\alpha, \alpha \in B^*\), with the occurrence \(\alpha\) in \(bw'b\), and identify any occurrence not containing \((h+k+1)\) as an occurrence in \(bw'ba'b\). These bijections preserve the lexicographic order and the property of being initial or not. By induction, the occurrences in \(bw'b\) are described by \(\psi(w')ba\) and the occurrences in \(bw'b\) by \(\psi(w)ba\). It follows by \(\text{(15)}\) that \(\psi(wa)ba\) describes the occurrences in \(bwab\) of subwords in the set \(b(ab)^*\).

Let us now consider \(wb\); in this case we can assume that \(a\) occurs in \(w\), and write \(w = w'ab^k\) for some \(k \geq 0\) and \(w' \in A^*\). Therefore, since \((wb)_+ = w'\) and \((wb)^- = w\), by \(\text{(6)}\) we have

\[
\psi(wb)ba = \psi(w)ba\psi(w')ba = \psi(w'ab^k)ba\psi(w')ba.
\]

Now, occurrences of words in \(b(ab)^*\) as subwords of \(bwbb = bw'ab^k+1\) can be divided in the following three classes, in decreasing lexicographic order:

1. occurrences containing a position greater than \(h+1\), where \(h = |bw'a|\),
2. occurrences containing positions \(h+1\) and \(h\) (the \(ab\) right after \(w'\)),
3. occurrences containing position \(h+1\) but not \(h\).

Members of the first class can never contain position \(h+1\), so that they can naturally be identified with occurrences in \(bw'ab^{k+1}\) (and so that the three classes are disjoint). Members of the second class can also be identified with occurrences in \(bw'ab^{k+1}\) after discarding positions \(h+1\) and \(h\). Under such correspondences (that do not alter the lexicographic order nor the property of being initial), it is easy to see that together, the first two classes make up all occurrences of words in \(b(ab)^*\) as subwords of \(bw'ab^{k+1} = bw'b\), so that by induction hypothesis they are described by \(\psi(w)ba\).

Occurrences in the third class can obviously be seen as occurrences in \(bw'b\), and are therefore described by \(\psi(w')ba\); the assertion then follows by \(\text{(15)}\).

**Example 7.5.** Let \(w = abbaa\), so that \(\psi(w) = ababaabababa\). The occurrences of words in \(b(ab)^*\) as subwords of \(bwbaab\) are:

- **Initial occurrences:** 1, 123, 12357, 12367, 124, 12457, 12467, 127, 157, 167.
- **Non-initial occurrences:** 3, 357, 367, 4, 457, 467, 7.

Sorting the reversed occurrences in decreasing lexicographic order, one has the standard word \(\psi(w)ba\) as shown by the following diagram:
Remark 7.6. Theorem 7.4 gives also a different proof of the Calkin-Wilf theorem. Indeed, from Theorem 7.4 one derives that for any \( w \in A^\ast \), the total number of occurrences of words in \( b(ab)^\ast \) as subwords of \( bwb \) is equal to \( |\psi(w)ba| = |a\psi(w)b| \).

8 The Coons-Shallit theorem

In this section we shall prove a formula relating for each \( w \in A^\ast \) the length of the Christoffel word \( a\psi(w)b \) with the occurrences in \( bwb \) of a certain kind of factors whose number is weighted by the lengths of Christoffel words associated to suitable directive words which are factors of \( w \). The result is a consequence of the following interesting theorem on Stern’s sequence due to Coons and Shallit [10].

For any \( n \geq 0 \) and \( w \in A^\ast \) let \( \alpha_w(n) \) simply denote the number of occurrences of \( w \) in the binary expansion of the integer \( n \), i.e., \( \alpha_w(n) = |[n]_2|_w \).

**Theorem 8.1.** For any \( n \geq 0 \) and \( w \in A^\ast \),

\[
s(n) = \alpha_b(n) + \sum_{u \in bA^\ast} s(\langle \bar{w} \rangle) \alpha_{wb}(n).
\]

Let us now define the two following sets of words \( \Gamma_1 = \{ u \in bA^\ast b \mid |u|_a = 1 \} \) and \( \Gamma_2 = \{ u \in bA^\ast b \mid |u|_a \geq 2 \} \). Moreover, to each word \( u \in \Gamma_2 \) we can associate the unique word \( \hat{u} \) such that \( u \in b^+a\hat{u}ab^+ \), i.e., \( \hat{u} \) is the unique factor of \( u \) between the first and the last occurrence of \( a \) in \( u \). The following holds:

**Theorem 8.2.** For any \( w \in A^\ast \) one has

\[
|a\psi(w)b| = |bwb|_b + \sum_{u \in \Gamma_1} |bwb|_u + \sum_{u \in \Gamma_2} |a\psi(\hat{u})b||bwb|_u.
\]

**Proof.** By Proposition 5.2, the Coons-Shallit theorem implies that for any \( w \in A^\ast \) we have

\[
|a\psi(w)b| = s(\langle bwb \rangle) = |bwb|_b + \sum_{u \in bA^\ast} s(\langle \bar{u} \rangle)|bwb|_{ub}.
\]

(17)
As \( s(\langle v \rangle) = s(2\langle v \rangle) = s(\langle va \rangle) \) for any \( v \in \mathcal{A}^* \), the last sum in (17) can be replaced by
\[
\sum_{u \in b\mathcal{A}^*b} s(\langle \bar{u} \rangle)|bwb|_u.
\]
Now, the set \( b\mathcal{A}^*b \) is clearly a disjoint union of \( b* \), \( \Gamma_1 \), and \( \Gamma_2 \). Since \( s(0) = 0 \), we only need to calculate that sum on \( \Gamma_1 \) and \( \Gamma_2 \).

For any \( u \in \Gamma_1 \), \( \langle \bar{u} \rangle \) is a power of 2, so that \( s(\langle \bar{u} \rangle) = 1 \) and
\[
\sum_{u \in \Gamma_1} s(\langle \bar{u} \rangle)|bwb|_u = \sum_{u \in \Gamma_1} |bwb|_u.
\]
If \( u \in \Gamma_2 \), then by the properties of \( s \) we have
\[
s(\langle \bar{u} \rangle) = s(\langle \bar{u}a \rangle) = s(\langle \bar{b} \rangle) = |a\psi(\bar{u})b|,
\]
where the last equality comes from Theorem 5.2 and the fact that \( |\psi(\bar{v})| = |\psi(v)| \) for any \( v \in \mathcal{A}^* \) (cf. item P6 of Proposition 2.1). Therefore, the assertion follows from (17).

**Example 8.3.** Let \( w = ababa \), so \( \psi(w) = abaabaababaababaaba \) and \( |a\psi(w)b| = 21 \). In \( bwb = bababab \) there is only one factor namely \( bab \) beginning and terminating with \( b \) and having only one occurrence of the letter \( a \). One has \( |bwb|_{bab} = 3 \). There are two factors \( u \) in \( bwb \) beginning and terminating with \( b \) such that \( |u|_b \geq 2 \). The first is \( u_1 = babab \) and occurs two times in \( bwb \) and the second \( u_2 = bababab \) occurring only once in \( bwb \). Moreover, \( \hat{u}_1 = b \) and \( \hat{u}_2 = bab \). Since \( |bwb|_{bab} = 4 \), \( |a\psi(b)b| = 3 \), and \( |a\psi(bab)b| = 8 \), one obtains by Theorem 8.2, \( |a\psi(w)b| = 4 + 3 + 6 + 8 = 21 \).

### 9 Length distribution of Christoffel words

We recall that a proper Christoffel word \( w \) is of order \( k \), \( k \geq 0 \), if \( w = a\psi(v)b \) with \( v \in \mathcal{A}^k \). In this section we are interested in the distribution of the lengths of Christoffel words of order \( k \).

By Theorem 5.2 one has that
\[
\{|a\psi(v)b| \mid v \in \mathcal{A}^k \} = \{s(2n - 1) \mid 2^k + 1 \leq n \leq 2^{k+1}\}.
\]

**Lemma 9.1.** For each \( k \geq 0 \),
\[
\sum_{v \in \mathcal{A}^k} |a\psi(v)b| = 2 \cdot 3^k.
\]
Proof. From (18) one has

\[ \sum_{v \in A^k} |a\psi(v)b| = \sum_{n=2^k+1}^{2^{k+1}} s(2n-1) = \sum_{n=2^k+1}^{2^{k+1}} s(n) + \sum_{n=2^k+1}^{2^{k+1}} s(n-1). \]

As is well known (see, for instance, [25]), \( \sum_{n=2^k+1}^{2^{k+1}} s(n) = 3^k \). Moreover, since \( s(2^k) = s(2^{k+1}) = s(1) = 1 \), one has

\[ \sum_{n=2^k+1}^{2^{k+1}} s(n-1) = \sum_{n=2^k+1}^{2^{k+1}} s(n) = 3^k. \]

From this the result follows. \( \square \)

Let us observe that from the preceding lemma one has that the average length of the Christoffel words of order \( k \) is \( 2(3/4)^{k} \).

We say that a word \( v \in A^k \) is alternating if for \( x, y \in A \) and \( x \neq y \), \( v = (xy)^{\frac{k}{2}} \) if \( k \) is even and \( v = (xy)^{\lfloor \frac{k}{2} \rfloor}x \) if \( k \) is odd, i.e., any letter in \( v \) is immediately followed by its complementary.

The following lemma, as regards the upper bound, was proved in [13] as an extremal property of the Fibonacci word. A different proof is obtained from (18) as a property of Stern’s sequence (see, for instance, [30]). As regards the lower bound the proof is trivial.

**Lemma 9.2.** For all \( v \in A^k \) one has

\[ k + 2 \leq |a\psi(v)b| \leq F_{k+1}, \]

where the lower bound is reached if and only if \( v \) is constant and the upper bound is reached if and only if \( v \) is alternating.

For each \( k \) let \( u_k \) be the alternating word of length \( k \) beginning with the letter \( a \). One has that

\[ \langle bu_{k-1}b \rangle = \frac{2^{k+2} - (-1)^{k+1}}{3}, \quad \langle b\bar{u}_{k-1}b \rangle = \frac{5 \cdot 2^k + (-1)^{k}}{3}. \]

Thus by Theorem 5.2 and the preceding lemma, one has (see, for instance, [30], Theorem 2.1)

\[ F_k = |a\psi(u_{k-1})b| = |a\psi(\bar{u}_{k-1})b| = s \left( \frac{2^{k+2} - (-1)^{k+1}}{3} \right) = s \left( \frac{5 \cdot 2^k + (-1)^k}{3} \right). \]

In the following for each word \( v \in A^k \) we let \( [v] \) denote the set \( [v] = \{ v, v^\sim, \bar{v}, \bar{v}^\sim \} \). From Proposition 2.1 all Christoffel words \( a\psi(z)b \) with a directive word \( z \in [v] \) have the same length.
Proposition 9.3. If \( v \in \mathcal{A}^k \) is not constant, then
\[
|a\psi(v)| \geq 2k + 1,
\]
where the lower bound is reached if and only if \( v \in [ab^{k-1}] \).

Proof. The proof is by induction on \( k \). The result is trivial if \( k = 2 \) and \( k = 3 \). Let \( v \in \mathcal{A}^k \), \( k > 3 \), be non-constant. We first suppose that \( v^- \) is not constant. We say that a word \( \bar{v} \in \mathcal{A}^k \) is almost alternating if \( \bar{v} \in \mathcal{A}^k \), but the second one, is immediately followed by its complementary. One has that \( v_k \in \mathcal{A}^k \) is almost alternating if \( \bar{v} \in \mathcal{A}^k \), \( \bar{v} \) is not constant, then
\[
|a\psi(v)| = |a\psi(v_+) + |a\psi(v_-)| \geq 2 + 2(k - 1) + 1 = 2k + 1.
\]

Suppose now that \( |a\psi(v)| = 2k + 1 \). From the preceding equation one has \( 2k + 1 \geq |a\psi(v_+)| + 2(k - 1) + 1 \), so that \( |a\psi(v_+)| \leq 2 \) that implies \( v_+ = \varepsilon \) and \( |a\psi(v_-)| = 2(k - 1) + 1 \). By induction \( v^- \in [ab^{k-2}] \). Since \( v_+ = \varepsilon \), one derives that either \( v = ab^{k-1} \) or \( v = ba^{k-1} \).

Let us now suppose that \( v^- \) is constant, i.e., \( v^- = x^{k-1} \) with \( x \in \mathcal{A} \). One has \( v = x^{k-1}y \) with \( y = \bar{x} \) and \( |a\psi(v)| = |ax^{k-1}yx^{k-1}b| = 2k + 1 \). It follows that in all cases the lower bound is reached if and only if \( v \in [ab^{k-1}] \). \( \square \)

Let us now introduce for each \( k \geq 3 \) the word \( v_k \) as follows:
\[
v_k = \begin{cases} 
ab^2(ab) \downarrow, & \text{if } k \text{ is odd;} \\
ab^2(ab) \downarrow \varepsilon_3, & \text{if } k \text{ is even.}
\end{cases}
\]

Note that each letter of \( v_k \) but the second one, is immediately followed by its complementary. One has that \( v_{k+1} = v_ka \) if \( k \) is odd and \( v_{k+1} = v_kb \) if \( k \) is even.

Lemma 9.4. For each \( k \geq 3 \),
\[
|a\psi(v_k)| = F_{k+1} - F_{k-4}.
\]

Proof. The proof is by induction on the value of \( k \). The result is true for \( k = 3 \) and \( k = 4 \). Indeed, \( |a\psi(ab^2)| = 7 = F_4 - F_1 \) and \( |a\psi(ab^2a)| = 12 = F_6 - F_0 \). Let us take \( k > 4 \). One has \( \psi(v_{k+1}) = \psi(v_kx) \) with \( x = a \) if \( k \) is odd and \( x = b \) if \( k \) is even. In both cases one has \( (v_{k+1})^- = v_k \) and \( (v_{k+1})^+ = v_{k-1} \), so by Proposition 3.7 and using the inductive hypothesis
\[
|a\psi(v_{k+1})| = |a\psi(v_k)| + |a\psi(v_{k-1})| = F_{k+1} - F_{k-4} + F_k - F_{k-5} = F_{k+2} - F_{k-3},
\]
which proves the assertion. \( \square \)

We say that a word \( v \in \mathcal{A}^k \) is almost alternating if \( v \in [v_k] \).
Theorem 9.5. Let \( k \geq 3 \). If \( v \in \mathcal{A}^k \) is not alternating, then
\[
|a\psi(v)b| \leq F_{k+1} - F_{k-4},
\]
where the upper bound is reached if and only if \( v \) is almost alternating.

Proof. The proof is by induction on the integer \( k \). If \( k = 3 \), then if \( v \neq xyx \) with \( \{x, y\} = \{a, b\} \) then \( |a\psi(v)b| \leq 7 = F_4 - F_1 \) and the upper bound is reached if and only if \( v \in \{ab^2, b^2a, ba^2, a^2b\} \). If \( k = 4 \) and \( v \) is not alternating, then \( |a\psi(v)b| \leq 12 = F_5 - F_2 \) and the maximal value is reached if and only if \( v \in \{ab^2a, ba^2b\} \).

Let us now consider the word \( vxy \) with \( x, y \in \mathcal{A} \) and \( |vxy| = k > 4 \) and first prove that if \( vxy \) is not alternating, then \( |a\psi(vxy)b| \leq F_{k+1} - F_{k-4} \). Let us first suppose that \( y = x \).

If \( v = x^{k-2} \), then \( |a\psi(vxx)b| = |ax^k|b| = k + 2 < F_{k+1} - F_{k-4} \) and we are done. If \( v \neq x^{k-2} \), then \( vxx \) is not constant, so that by Proposition 3.7 one has
\[
|a\psi(vxx)b| = |a\psi(vx)b| + |a\psi(v')b|,
\]
where \( v' = (vxx)_+ \) is a prefix of \( v \) of length \( < k - 2 \). By Lemma 9.2 one has \( |a\psi(vx)b| \leq F_k \) and \( |a\psi(v')b| \leq F_{k-2} \). Thus, since \( k > 4 \), one has
\[
|a\psi(vxx)b| \leq F_k + F_{k-2} = F_{k+1} - F_{k-3} < F_{k+1} - F_{k-4}.
\]

Let us now suppose \( x \neq y \). By Proposition 3.7 one has
\[
|a\psi(vxy)b| = |a\psi(vx)b| + |a\psi(v)b|.
\]
Since \( vxy \) is not alternating, so will be \( vx \). By induction \( |a\psi(vx)b| \leq F_k - F_{k-5} \).

If \( v \) is not alternating, then, by induction, \( |a\psi(v)b| \leq F_{k-1} - F_{k-6} \). Hence, \( |a\psi(vxy)b| \leq F_k - F_{k-5} + F_{k-1} - F_{k-6} = F_{k+1} - F_{k-4} \) and we are done.

If \( v \) is alternating, then as \( vx \) is not alternating, the only possibility is \( v = (yx)^{\frac{k-3}{2}} \) if \( k \) is even and \( v = (xy)^{\frac{k-3}{2}}x \) if \( k \) is odd. Hence, if \( k \) is even, \( vxy = y(xy)^{\frac{k-3}{2}}x^2y \) and if \( k \) is odd, \( vxy = (xy)^{\frac{k-3}{2}}x^2y \). In both the cases \( vxy \in [v_k] \). By Lemma 9.4 \( |a\psi(vxy)b| = F_{k+1} - F_{k-4} \). Thus the first assertion is proved.

In view of Lemma 9.4 it remains to prove that if \( |a\psi(vxy)b| = F_{k+1} - F_{k-4} \), then \( vxy \in [v_k] \). In this case, in view of (19), necessarily \( x \neq y \). If \( v \) is alternating, as we have previously seen, \( |a\psi(vxy)b| \) reaches its maximal value and \( vxy \in [v_k] \). If \( v \) is not alternating one has to require that both \( |a\psi(vx)b| \) and \( |a\psi(v)b| \) reach their maximal values. By induction this occurs if and only if \( vx \in [v_{k-1}] \) and \( v \in [v_{k-2}] \). If \( k \) is odd, the only possibility is \( v = xy^2(xy)^{\frac{k-3}{2}} \), so that \( vxy \in [v_k] \). If \( k \) is even, then necessarily \( v = yx^2(yx)^{\frac{k-3}{2}}y \), so that also in this case \( vxy \in [v_k] \). \( \square \)
One easily verifies that for each \( k \geq 3 \), \( \langle bv_k b \rangle = \frac{17 \cdot 2^{k-1} + (-1)^{k+1}}{3} \), \( \langle b\bar{v}_k b \rangle = \frac{19 \cdot 2^{k-1} + (-1)^k}{3} \), \( \langle b\bar{v}_k \rangle = \frac{7 \cdot 2^k (9 + (-1)^k)}{3} \), and \( \langle b\bar{v}_k^* \rangle = \frac{-7 \cdot 2^k (9 + (-1)^k)}{3} \).

By the preceding theorem and Lemma 9.2, one derives some identities on Stern’s sequence. For instance, for any \( k \geq 3 \)

\[
\begin{align*}
\frac{s \left( \frac{17 \cdot 2^{k-1} + (-1)^{k+1}}{3} \right)}{3} &= s \left( \frac{2^{k+3} + (-1)^{k+2}}{3} \right) - s \left( \frac{2^{k-2} + (-1)^{k-3}}{3} \right).
\end{align*}
\]

**Remark 9.6.** Let us observe that setting \( R_k = \frac{17 \cdot 2^{k-1} + (-1)^{k+1}}{3} \) for each \( k \geq 1 \), one has \( R_1 = 6 \) and \( R_{k+1} = 2R_k + (-1)^k \). Similarly, if one defines inductively the three sequences \( (S_k)_{k>0} \), \( (T_k)_{k>0} \), and \( (U_k)_{k>0} \) respectively as \( S_1 = 6, T_1 = 9, U_1 = 3 \) and for \( k \geq 1 \)

\[
\begin{align*}
S_{k+1} &= 2S_k + (-1)^{k+1}, \\
T_{k+1} &= T_k + (3 + (-1)^k)2^k, \\
U_{k+1} &= U_k + (3 + (-1)^{k+1})2^k,
\end{align*}
\]

one has that for \( k \geq 3 \), \( \langle b\bar{v}_k b \rangle = S_k, \langle b\bar{v}_k^* \rangle = T_k, \) and \( \langle b\bar{v}_k^* \rangle = U_k \).

For any \( k \geq 0 \) and \( n \geq 0 \) let \( C_k(n) \) denote the number of Christoffel words of length \( n \) and order \( k \). By [16, Lemma 5], and since the palindromization map is injective, one has

\[
\sum_{k \geq 0} C_k(n) = \phi(n) \quad \text{and} \quad \sum_{n \geq 0} C_k(n) = 2^k,
\]

(21)

where \( \phi \) is the Euler totient function. By Lemma 9.2 one has that \( C_k(n) = 0 \) for \( n < k + 2 \) and for \( n > F_{k+1} \). Moreover, from Proposition 9.3 and Theorem 9.5 one has that \( C_k(n) = 0 \) for \( k + 2 < n < 2k + 1 \) and for \( F_{k+1} - F_{k-4} < n < F_{k+1} \).

For each \( k \geq 0 \) we introduce the set of the missing lengths of order \( k \)

\[
ML_k = \{ n \mid k + 2 \leq n \leq F_{k+1} \text{ and } C_k(n) = 0 \}.
\]

The first values of \( \text{card}(ML_k) \), \( 0 \leq k \leq 20 \), are reported below

0, 0, 1, 2, 5, 11, 18, 29, 51, 74, 119, 195, 323, 498, 828, 1361, 2289, 3801, 6305, 10560.

By Proposition 9.3 and Theorem 9.5 one has that for \( k \geq 3 \)

\[
\text{card}(ML_k) \geq F_{k-4} + k - 3.
\]

Since for large \( k \), one has \( F_{k-4} \approx \frac{g^{k-2}}{\sqrt{5}} \), where \( g \) is the golden number \( g = \frac{1 + \sqrt{5}}{2} = 1.618 \cdots \), it follows that the lower bound to \( \text{card}(ML_k) \) is exponentially increasing with \( k \). Moreover, one easily derives that

\[
\liminf_{k \to \infty} \frac{\text{card}(ML_k)}{F_{k+1}} \geq \frac{1}{g^2}.
\]

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For each $k \geq 0$ let $L_k$ denote the set of all lengths of Christoffel words of order $k$.

The following lemma shows that for $k \geq 2$ there exist Christoffel words of order $k$ having lengths which are consecutive integers. Some examples are given in the following lemma.

**Lemma 9.7.** For $k \geq 2$ one has $3k - 2, 3k - 1 \in L_k$ and for $k \geq 3$, one has $5k - 8, 5k - 7 \in L_k$.

**Proof.** One easily verifies that for $k \geq 2$, one has $|a\psi(a^2b^{k-2})b| = 3k - 2$ and $|a\psi(aba^{k-2})b| = 3k - 1$. For $k \geq 3$, one has $|a\psi(ab^2a^{k-3})b| = 5k - 8$ and $|a\psi(abab^{k-3})b| = 5k - 7$.

For each $k \geq 1$, we set $M_k = \max\{C_k(n) | n \geq 0\}$.

The first values of $M_k$, $0 < k \leq 22$ and the values $n_k$ for which $C_k(n_k) = M_k$ are reported in the following table:

| $k$ | $M_k$ | $n_k$ | $k$ | $M_k$ | $n_k$ |
|-----|-------|-------|-----|-------|-------|
| 1   | 2     | 3     | 12  | 36    | 199, 283 |
| 2   | 2     | 4,5   | 13  | 48    | 449 |
| 3   | 4     | 7     | 14  | 64    | 433 |
| 4   | 4     | 9, 11 | 15  | 72    | 839 |
| 5   | 4     | 11,13,14,17,18,19 | 16  | 102   | 1433 |
| 6   | 8     | 23    | 17  | 124   | 1997 |
| 7   | 12    | 41    | 18  | 160   | 1987 |
| 8   | 12    | 43    | 19  | 212   | 3361 |
| 9   | 16    | 71,73,83 | 20  | 256   | 5557 |
| 10  | 24    | 113   | 21  | 332   | 8689 |
| 11  | 28    | 227   | 22  | 444   | 8507 |

**Proposition 9.8.** $\lim_{k \to \infty} M_k = \infty$.

**Proof.** By Lemma 9.2 and (21) one has

$$2^k = \sum_{n \geq 0} C_k(n) = \sum_{n \geq k+2} C_k(n) \leq M_k F_{k+1}.$$
By Binet’s formula of Fibonacci numbers one has that $F_{k+1} < g^{k+3}$. Hence, $M_k \geq \frac{1}{g^3} \left( \frac{2}{g} \right)^k$. From this the result follows.

From the proof of previous proposition one has that $M_k$ has a lower bound which is exponentially increasing with $k$, whose values are much less than those of $M_k$ given in the table above. It would be interesting to find tight lower and upper bounds for $M_k$ and possibly a formula to compute its values and also the values of the lengths of Christoffel words for which $C_k(n)$ is equal to $M_k$. Moreover, from the table one has that for all $0 < k < 22$, $M_k$ is non-decreasing with $k$ and $M_{k+1} \leq M_k + M_{k-1}$. We conjecture that this is true for all $k$.

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