Composition operators with surjective symbol and small approximation numbers

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Abstract. We give a new proof of the existence of a surjective symbol whose associated composition operator on \( H^2(\mathbb{D}) \) is in all Schatten classes, with the improvement that its approximation numbers can be, in some sense, arbitrarily small. We show, as an application, that, contrary to the 1-dimensional case, for \( N \geq 2 \), the behavior of the approximation numbers \( a_n = a_n(C_\varphi) \), or rather of \( \beta^-_N = \liminf_{n \to \infty} [a_n]^{1/n^1/N} \) or \( \beta^+_N = \limsup_{n \to \infty} [a_n]^{1/n^1/N} \), of composition operators on \( H^2(\mathbb{D}^N) \) cannot be determined by the image of the symbol.

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1 Introduction

We start by recalling some notations and facts.

Let \( \mathbb{D} \) be the open unit disk, \( H^2 \) the Hardy space on \( \mathbb{D} \), and \( \varphi : \mathbb{D} \to \mathbb{D} \) a non-constant analytic self-map. It is well-known ([14]) that \( \varphi \) induces a composition operator \( C_\varphi : H^2 \to H^2 \) by the formula:

\[
C_\varphi(f) = f \circ \varphi,
\]

and the connection between the "symbol" \( \varphi \) and the properties of the operator \( C_\varphi : H^2 \to H^2 \), in particular its compactness, can be further studied ([14]).

We also recall that the \( n \)th approximation number \( a_n(T) \), \( n = 1, 2, \ldots \), of an operator \( T : H_1 \to H_2 \), between Hilbert spaces \( H_1 \) and \( H_2 \), is defined as the distance of \( T \) to operators of rank \( < n \), for the operator-norm:

\[
a_n(T) = \inf_{\text{rank } R < n} \| T - R \|.
\]
The $p$-Schatten class $S_p(H_1, H_2)$, $p > 0$ consists of all $T: H_1 \to H_2$ such that $(a_n(T))_n \in \ell^p$. The approximation numbers have the ideal property:

$$a_n(ATB) \leq \|A\| a_n(T) \|B\|.$$ 

Let now, for $\xi \in \mathbb{T} = \partial \mathbb{D}$ and $h > 0$, the Carleson window $S(\xi, h)$ be defined as:

$$(1.2) \quad S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\}.$$ 

For a symbol $\varphi$, we define $m_{\varphi} = \varphi^*(m)$ where $m$ is the Haar measure of $\mathbb{T}$ and $\varphi^*: \mathbb{T} \to \overline{\mathbb{D}}$ the (almost everywhere defined) radial limit function associated with $\varphi$, namely:

$$\varphi^*(\xi) = \lim_{r \to 1^{-}} \varphi(r\xi).$$ 

Finally, we set for $h > 0$:

$$(1.3) \quad \rho_{\varphi}(h) = \sup_{\xi \in \mathbb{T}} m_{\varphi}[S(\xi, h)].$$ 

It is known ([14]) that $\rho_{\varphi}(h) = O(h)$ and ([12]) that $C_{\varphi}$ is compact if and only if $\rho_{\varphi}(h) = o(h)$ as $h \to 0$. Simpler criteria ([14]) exist when $\varphi$ is injective, or even $p$-valent, meaning that for any $w \in \mathbb{D}$, the equation $\varphi(z) = w$ has at most $p$ solutions.

A measure $\mu$ on $\mathbb{D}$ is called $\alpha$-Carleson, $\alpha \geq 1$, if $\sup_{|\xi| = 1} \mu[S(\xi, h)] = O(h^\alpha)$.

B. MacCluer and J. Shapiro showed in [13, Example 3.12] the following result, paradoxical at first glance.

**Theorem 1.1** (MacCluer-Shapiro). There exists a surjective and four-valent symbol $\varphi: \mathbb{D} \to \mathbb{D}$ such that the composition operator $C_{\varphi}: H^2 \to H^2$ is compact.

Observe that such a symbol $\varphi$ cannot be one-valent (injective), because it would be an automorphism of $\mathbb{D}$, and $C_{\varphi}$ would be invertible and therefore not compact. In [6, Theorem 4.1], we gave the following improved statement.

**Theorem 1.2.** For every non-decreasing function $\delta: (0, 1) \to (0, 1)$, there exists a two-valent symbol and nearly surjective (i.e. $\varphi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$) symbol $\varphi$, and $0 < h_0 < 1$, such that:

$$(1.4) \quad m(\{z \in \mathbb{T}; |\varphi^*(z)| \geq 1 - h\}) \leq \delta(h) \quad \text{for } 0 < h \leq h_0.$$ 

As a consequence, there exists a surjective and four-valent symbol $\psi: \mathbb{D} \to \mathbb{D}$ such that the composition operator $C_{\psi}: H^2 \to H^2$ is in every Schatten class $S_p(H^2)$, $p > 0$. 

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Our proof was rather technical and complicated, and based on arguments of barriers and harmonic measures.

The goal of this paper is to give a more precise statement of Theorem 1.2 in terms of approximation numbers \(a_n(C_\phi)\), and not only in terms of Schatten classes, and with a simpler proof. We then apply this result to show that for the polydisk \(\mathbb{D}^N, N \geq 2\), the nature (boundedness, compactness, asymptotic behavior of approximation numbers) of the composition operator cannot be determined by the geometry of the image \(\varphi(\mathbb{D}^N)\) of its symbol \(\varphi\). For certain asymptotic behavior of approximation numbers, this is contrary to the 1-dimensional case (see [10, Theorem 3.1 and Theorem 3.14]).

The notation \(A \lesssim B\) means that \(A \leq CB\) for some positive constant \(C\), and \(A \approx B\) that \(A \lesssim B\) and \(B \lesssim A\).

2 Background and preliminary results

We initiated the study of approximation numbers of composition operators on \(H^2\) in [8], and proved the following basic results:

**Theorem 2.1.** If \(\varphi\) is any symbol, then, for some \(\delta > 0\) and \(r > 0\), or \(a > 0\):

\[
a_n(C_\varphi) \geq \delta r^n = \delta e^{-an}.
\]

Moreover, as soon as \(\|\varphi\|_\infty = 1\), there exists some sequence \(\varepsilon_n\) tending to 0 such that:

\[
a_n(C_\varphi) \geq \delta e^{-n\varepsilon_n}.
\]

We also proved in [8, Theorem 5.1] that:

**Proposition 2.2.** For any symbol \(\varphi\), we have:

\[
a_n(C_\varphi) \lesssim \inf_{0 < h < 1} \left[ e^{-nh} + \sqrt{\rho_{\varphi}(h)} \frac{1}{h} \right].
\]

We also recall (see [8]) that, for \(\gamma > -1\), the weighted Bergman space \(\mathcal{B}_\gamma\) is the space of functions \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) such that:

\[
\|f\|_{\gamma}^2 := \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{\gamma+1}} < \infty.
\]

Equivalently, \(\mathcal{B}_\gamma\) is the space of analytic functions \(f : \mathbb{D} \to \mathbb{C}\) such that:

\[
\int_{\mathbb{D}} |f(z)|^2 (\gamma + 1)(1 - |z|^2)^\gamma dA(z) < \infty,
\]
where $dA$ is the normalized area measure on $\mathbb{D}$, and then:

\begin{equation}
\int_{\mathbb{D}} |f(z)|^2 (\gamma + 1)(1 - |z|^2)^\gamma dA(z) \approx \|f\|^2_\gamma.
\end{equation}

The case $\gamma = 0$ corresponds to the usual Bergman space $\mathcal{B}^2$, and the limiting case $\gamma = -1$ to the Hardy space $H^2$. We wish to note in passing (we will make use of that elsewhere) that the proof of Theorem 5.1 in [3] easily gives the following result.

**Proposition 2.3.** Let $\gamma > -1$ and $\varphi$ a symbol inducing a bounded composition operator $C_\varphi : \mathcal{B}_\gamma \to H^2$. Then:

\begin{equation}
\inf_{0 < h < 1} \left( (n + 1)^{(\gamma + 1)/2} e^{-nh} + \sup_{0 < t \leq h} \sqrt{\rho_\varphi(t) / t^{2+\gamma}} \right).
\end{equation}

**Proof.** Take $E = z^n \mathcal{B}_\gamma$; this is a subspace of $\mathcal{B}_\gamma$ of codimension $\leq n$. Let $f \in E$ with $\|f\|_\gamma = 1$. Writing $f = z^ng$ with $\|g\|^2_\gamma \leq (n + 1)^{\gamma + 1}$ and splitting the integral into two parts, we have, for $0 < h < 1$:

\begin{equation}
\|C_\varphi f\|^2_{H^2} = \int_{\mathbb{D}} |f|^2 dm_\varphi \leq (1 - h)^{2n} \int_{(1-h)\mathbb{D}} |g|^2 dm_\varphi + \int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 dm_\varphi.
\end{equation}

For the first integral, we have:

\begin{equation}
\int_{(1-h)\mathbb{D}} |g|^2 dm_\varphi \leq \int_{\mathbb{D}} |g|^2 dm_\varphi = \|C_\varphi g\|^2_{H^2} \leq \|C_\varphi\|^2_{\mathcal{B}_\gamma \to H^2} \|g\|^2_\gamma.
\end{equation}

For the second integral, we have:

\begin{equation}
\int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 dm_\varphi \leq \|J : \mathcal{B}_\gamma \to L^2(\mu_h)\|^2,
\end{equation}

where $\mu_h$ is the restriction of $m_\varphi$ to the annulus $\{z \in \mathbb{D} : 1 - h < |z| < 1\}$ and $J$ the canonical injection of $\mathcal{B}_\gamma$ into $L^2(\mu_h)$. Hence Stegenga’s version of the Carleson embedding theorem for $\mathcal{B}_\gamma$ ([16, Theorem 1.2]; see [1] for the unweighted case; see also [3, p. 62] or [17, p. 167]) gives us:

\begin{equation}
\int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 dm_\varphi \lesssim \sup_{0 < t \leq h} \frac{\rho_\varphi(t)}{t^{2+\gamma}}.
\end{equation}

Putting (2.4) and (2.5) together, that gives:

\begin{equation}
\|C_\varphi f\|_{H^2} \lesssim e^{-nh} (n + 1)^{(\gamma + 1)/2} + \sup_{0 < t \leq h} \sqrt{\rho_\varphi(t) / t^{2+\gamma}}.
\end{equation}
In other terms, using the Gelfand numbers $c_k$:
\[
c_{n+1}(C_\varphi : B_\gamma \to H^2) \lesssim (n + 1)^{(\gamma + 1)/2} \text{e}^{-nh} + \sup_{0 < t \leq h} \sqrt{\frac{\rho_\varphi(t)}{t^{2+\gamma}}},
\]
As $a_{n+1} = c_{n+1}$ and as we can ignore the difference between $a_n$ and $a_{n+1}$, that finishes the proof.

As an application, we mention the following result. We refer to [9, Section 4.1] for the definition of the cusp map, denoted $\chi$.

**Theorem 2.4.** Let $\chi : \mathbb{D} \to \mathbb{D}$ be the cusp map and $\Phi : \mathbb{D}^N \to \mathbb{D}^N$ the diagonal map defined by:
\[
\Phi(z_1, z_2, \ldots, z_N) = (\chi(z_1), \chi(z_1), \ldots, \chi(z_1)).
\]
Then, the composition operator $C_\Phi$ maps $H^2(\mathbb{D}^N)$ to itself and:
\[
a_n(C_\Phi) \lesssim e^{-d\sqrt{n}}
\]
where $d$ is a positive constant depending only on $N$.

**Remark.** We have to compare with [11, Theorem 6.2] where, for:
\[
\Psi(z_1, \ldots, z_N) = (\chi(z_1), \ldots, \chi(z_N)),
\]
it is shown that, for constants $b \geq a > 0$ depending only on $N$:
\[
e^{-b(n^{1/N}/\log n)} \lesssim a_n(C_\Psi) \lesssim e^{-a(n^{1/N}/\log n)}.
\]
Note also that for $N = 1$, the estimate of Theorem 2.4 is very crude.

**Proof of Theorem 2.4.** Take $\gamma = N - 2$. As in [11, Section 4], we have thanks to the Cauchy-Schwarz inequality, and the fact that $\sum_{|\alpha|=n} 1 \approx (n + 1)^{N-1}$, a factorization:
\[
C_\Phi = J\chi M,
\]
where $M : H^2(\mathbb{D}^N) \to B_\gamma$ is defined by $Mf = g$ with:
\[
g(z) = f(z, z, \ldots, z) = \sum_{n=0}^{\infty} \left( \sum_{|\alpha|=n} a_\alpha \right) z^n, \quad z \in \mathbb{D},
\]
for
\[
f(z_1, z_2, \ldots, z_N) = \sum_\alpha a_\alpha z_1^{a_1} \cdots z_N^{a_N},
\]
\[
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\]
and where \( J : H^2(\mathbb{D}) \to H^2(\mathbb{D}^N) \) is the canonical injection given by:

\[
(Jh)(z_1, z_2, \ldots, z_N) = h(z_1).
\]

This corresponds to a diagram:

\[
H^2(\mathbb{D}^N) \xrightarrow{M} \mathcal{B}_\gamma \xrightarrow{C_\chi} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^N),
\]

where \( C_\chi : \mathcal{B}_\gamma = \mathcal{B}_{N-2} \to H^2(\mathbb{D}) \) is a bounded operator. Indeed, we have the behavior (\[9, Lemma 4.2\]):

\[
|1 - \chi^*(e^{i\theta})| \approx \frac{1}{\log(1/|\theta|)},
\]

and this implies, with \( c \) an absolute constant:

\[
m_\chi[S(\xi, h)] \lesssim m_\chi[S(1, h)] = m(\{|\chi^*(e^{i\theta}) - 1| < h\}) \lesssim m(\{c/ \log(1/|\theta|) < h\}) \leq e^{-c/h};
\]

in particular \( \rho_\chi(h) \leq e^{-c/h} = O(h^N) \), so \( m_\chi \) is an \( N \)-Carleson measure and the Stengenga-Carleson theorem (\[16, Theorem 1.2\]) says that the operator \( C_\chi : \mathcal{B}_{N-2} \to H^2(\mathbb{D}) \) is bounded.

Now Proposition \[2.3\] with \( (2.11) \) give:

\[
a_n(C_\chi : \mathcal{B}_\gamma \to H^2) \lesssim \inf_{0 < h < 1} [(n + 1)(N-1)/2 e^{-nh} + e^{-c/h}h^{-N/2}].
\]

Adjusting \( h = 1/\sqrt{n} \), we get \( a_n(C_\chi : \mathcal{B}_\gamma \to H^2) \lesssim e^{-d\sqrt{n}} \) for some positive constant \( d \). Finally, the factorization \( C_\Phi = JC_\chi M \) and the ideal property of approximation numbers give the result.

In the case of lens maps, Proposition \[2.3\] gives very poor estimates. We avoid using this theorem in \[11, Section 4\], when \( N = 2 \), using the semi-group property of those lens maps. The same proof gives for arbitrary \( N \geq 2 \) the following result.

**Theorem 2.5.** Let \( \lambda_\theta \) the lens map with parameter \( \theta \), \( 0 < \theta < 1 \), and let \( \Phi : \mathbb{D}^N \to \mathbb{D}^N \) be the diagonal map defined by:

\[
\Phi(z_1, z_2, \ldots, z_N) = (\lambda_\theta(z_1), \lambda_\theta(z_1), \ldots, \lambda_\theta(z_1)).
\]

Then:

1) \( \text{if } \theta > 1/N, \text{ } C_\Phi \text{ is unbounded on } H^2(\mathbb{D}^N); \)
2) if $\theta = 1/N$, $C_\Phi$ is bounded and not compact on $H^2(\mathbb{D}^N)$;

3) if $\theta < 1/N$, $C_\Phi$ is compact on $H^2(\mathbb{D}^N)$ and moreover:

$$a_n(C_\Phi) \lesssim e^{-d\sqrt{n}}$$

for a constant $d > 0$ depending only on $\theta$ and $N$.

**Remark.** In [1, Theorem 6.1], it is shown that, for:

$$\Psi(z_1, \ldots, z_N) = \left(\lambda_\theta(z_1), \ldots, \lambda_\theta(z_N)\right),$$

we have, for constants $b \geq a > 0$, depending only on $\theta$ and $N$:

$$e^{-b n^{1/(2N)}} \lesssim a_n(C_\Psi) \lesssim e^{-a n^{1/(2N)}}.$$

**Proof of Theorem 2.5.** That had been proved, for $N = 2$ in [11, Theorem 4.2 and Theorem 4.4]. For convenience of the reader, we sketch the proof.

Assume first $\theta \leq 1/N$, and write $\lambda_\theta = \lambda_{N\theta} \circ \lambda_{1/N}$, where we set, for convenience, $\lambda_1(z) = z$, so $C_{\lambda_1} = \text{Id}$. As in the proof of Theorem 2.4 (see [11, Section 4]), we have a factorization:

$$C_\Phi = JC_{\lambda_{N\theta}}C_{\lambda_{1/N}}M,$$

where $M$ and $J$ are defined in (2.8) and (2.9).

This corresponds to a diagram (recall that $\gamma = N - 2$):

$$H^2(\mathbb{D}^N) \xrightarrow{M} B_\gamma \xrightarrow{C_{\lambda_{1/N}}} H^2(\mathbb{D}) \xrightarrow{C_{\lambda_{N\theta}}} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^N).$$

The second arrow is bounded, since we know ([7, Lemma 3.3]) that the pullback measure $m_{\lambda_{1/N}}$ is $N$-Carleson, so that $C_{\lambda_{1/N}}$ maps $B_{N-2}$ to $H^2(\mathbb{D})$ by the Stegenga-Carleson embedding theorem ([16, Theorem 1.2]).

For $\theta < 1/N$, we have $N\theta < 1$ and $C_{\lambda_{N\theta}}$ is compact and, for some constant $b = b(\theta)$, we have $a_n(C_{\lambda_{N\theta}}) \lesssim e^{-b\sqrt{n}}$ ([7, Theorem 2.1]). Hence $C_\Phi$ is compact and $a_n(C_\Phi) \lesssim e^{-b\sqrt{n}}$.

Now, for $\theta \geq 1/N$, we consider the reproducing kernels:

$$K_{a_1, \ldots, a_N}(z_1, \ldots, z_N) = \prod_{j=1}^N \frac{1}{1 - \overline{a_j}z_j}.$$

We have:

$$\|K_{a_1, \ldots, a_N}\|^2 = \prod_{j=1}^N \frac{1}{1 - |a_j|^2}.$$
and:
\[ C_\Phi^* (K_{a_1},...,a_N) = K_{\lambda_\theta(a_1),...,\lambda_\theta(a_1)}, \]
so:
\[ \left\| C_\Phi^* (K_{a_1},...,a_N) \right\|^2 = \left( \frac{1}{1 - |\lambda_\theta(a_1)|^2} \right)^N. \]
Since:
\[ 1 - |\lambda_\theta(a_1)|^2 \approx 1 - |\lambda_\theta(a_1)| \approx (1 - |a_1|)^\theta, \]
we see that \( \| C_\Phi^* (K_{a_1},...,a_N) \| / \| K_{a_1},...,a_N \| \) is not bounded for \( \theta > 1/N \), so \( C_\Phi \) is then not bounded; and it does not converge to 0 for \( \theta = 1/N \), so \( C_\Phi \) is then not compact.

3 Surjectivity

Let us come back to our surjectivity issues.

Let us first remark that Theorem 1.2 gives the following result.

**Theorem 3.1.** For every non-decreasing function \( \delta : (0, 1) \to (0, 1) \), there exists a surjective and four-valent symbol \( \psi \), and \( 0 < h_0 < 1 \), such that, for \( 0 < h \leq h_0 \):
\[
(3.1) \quad m(\{z \in \mathbb{T}; |\varphi^*(z)| \geq 1 - h\}) \leq \delta(h).
\]

**Proof.** Just observe that the passage from “\( \varphi \) two-valent and nearly surjective” to “\( \psi \) four-valent and surjective” is harmless: for this, consider the Blaschke product:
\[
B(z) = \left( \frac{z - a}{1 - az} \right)^2,
\]
where \( 0 < a < 1 \), and take \( \psi = B \circ \varphi \); we observe that \( B(\mathbb{D} \setminus \{0\}) = \mathbb{D} \) since \( a^2 = B(\frac{2a}{1+a^2}) \), and, for \( z \in \mathbb{D} \):
\[
1 - |B(z)| \geq \frac{1 - \frac{|z - a|}{1 - az}}{1 - |z|^2} = \frac{1 - a^2}{|1 - az|^2} \geq \frac{1 - a^2}{4},
\]
so that:
\[
m(|\psi^*| > 1 - h) = m(1 - |B \circ \varphi^*| < h) \leq m(1 - |\varphi^*| \leq \kappa_\alpha h),
\]
with \( \kappa_\alpha = 4/(1 - a^2) \). Hence, this map \( \psi \) is surjective, four-valent, and satisfies (5.1), as well, up to a change of \( \delta(h) \) to \( \delta(h/\kappa_\alpha) \) for \( \varphi \) at the beginning. \( \square \)
3.1 A more precise statement

Our new statement is as follows.

**Theorem 3.2.** For every positive sequence \((\varepsilon_n)_n\) with limit 0, there exists a surjective and four-valent symbol \(\varphi\) such that:

\[
a_n(C_{\varphi}) \lesssim e^{-n\varepsilon_n}.
\]

Consequently, there exists a surjective and four-valent symbol \(\varphi\) : \(\mathbb{D} \to \mathbb{D}\) such that the composition operator \(C_{\varphi} : H^2 \to H^2\) is in every Schatten class \(S_p(H^2), p > 0\).

**Proof.** Observe first that \(\|\varphi\|_\infty = 1\) when \(\varphi\) is surjective, so that, in view of Theorem 2.1, we cannot dispense with the numbers \(\varepsilon_n\), even if they can tend to 0 arbitrarily slowly.

Now, we can choose \(\delta : (0, 1) \to (0, 1)\) non-decreasing such that \(\delta(\varepsilon_n) \leq e^{-n\varepsilon_n}\) for all \(n\), and then, using Theorem 3.1, we get a surjective and four-valent symbol \(\varphi\), satisfying for all \(h\) small enough:

\[
\rho_{\varphi}(h) \leq h \delta^2(h).
\]

Proposition 2.2 gives:

\[
a_n(C_{\varphi}) \lesssim \inf_{0 < h < 1} \left[ e^{-nh} + \delta(h) \right].
\]

Adjusting \(h = \varepsilon_n\), we get \(a_n(C_{\varphi}) \lesssim e^{-n\varepsilon_n}\).

To get the second part of the theorem, just take \(\varepsilon_n = n^{-1/2}\).

3.2 A simplified proof of Theorem 1.2

We give here the announced simplified proof of Theorem 1.2. This proof is based on the following key lemma, in which \(\mathcal{H}(\mathbb{D})\) denotes the set of holomorphic functions on \(\mathbb{D}\).

**Lemma 3.3.** There exists a numerical constant \(C\) such that, if \(f \in \mathcal{H}(\mathbb{D})\) satisfies, for some \(\alpha \in \mathbb{R}\):

\[
\begin{align*}
\{ \text{Im} [f(0)] < \alpha \} \\
\{ f(\mathbb{D}) \subseteq \{ z \in \mathbb{C} ; 0 < \text{Re} z < \pi \} \cup \{ z \in \mathbb{C} ; \text{Im} z < \alpha \},
\end{align*}
\]

then:

\[
m(\{ \text{Im} f^* > y \}) \leq C e^{\alpha - y}, \quad \text{for } y \geq \alpha.
\]
We first show how this lemma allows us to conclude.

**Proof of Theorem 1.2.** Let \( g: (0, \infty) \to (0, \infty) \) be a continuous decreasing function such that:

\[
\lim_{t \to 0^+} g(t) = +\infty, \quad g(\pi) = \pi, \quad \lim_{t \to +\infty} g(t) = 0.
\]

Then let \( \Omega \) be the simply connected region defined by:

\[
\Omega = \{ x + iy ; \ x > 0, \ g(x) < y < g(x) + 4\pi \},
\]

and \( f: \mathbb{D} \to \Omega \) be a Riemann map such that \( f(0) = \pi + 3i\pi \). Observe that we can apply Lemma 3.3 to \( f \) with \( \alpha = 5\pi \) since \( \text{Im} f(0) = 3\pi \) and if \( f(z) = x + iy \) with \( x \geq \pi \); hence:

\[
\text{Im} f(z) = y < g(x) + 4\pi \leq g(\pi) + 4\pi = 5\pi.
\]

Finally, consider the symbol \( \varphi = e^{-f} \). It is nearly surjective: \( \varphi(\mathbb{D}) = \mathbb{D} \setminus \{0\} \), and two-valent, as easily checked.

For \( 0 < h \leq 1/2 \), we have for \( \xi \in \mathbb{T} \) and \( |\varphi^*(\xi)| > 1 - h \):

\[
e^{-2h} \leq 1 - h < |\varphi^*(\xi)| = \exp \left( -\mathfrak{R} e f^*(\xi) \right);
\]

hence \( \mathfrak{R} e f^*(\xi) < 2h \).

But if \( 2h > x = \mathfrak{R} e f^*(\xi), \) we have \( g(x) > g(2h) \). As \( f^*(\xi) = x + iy \in \overline{\Omega}, \) we get \( \text{Im} f^*(\xi) = y \geq g(x) > g(2h) \). Lemma 3.3 now gives:

\[
(3.2) \ m(\{ \xi ; \ |\varphi^*(\xi)| > 1 - h \}) \leq m(\{ \xi ; \ \text{Im} f^*(\xi) > g(2h) \}) \leq C e^{5\pi - g(2h)}.
\]

It is now enough to adjust \( g \) so as to have \( e^{g(t)} \geq C e^{5\pi / \delta(t/2)} \) for \( t \) small enough to get (1.4) from (3.2). \[\square\]

**Proof of Lemma 3.3.** We now prove Lemma 3.3. If \( e^{g-\alpha} < 2 \), there is nothing to prove, since then:

\[
m(\text{Im} f^* > y) \leq 1 \leq 2 e^{\alpha-y}.
\]

We can hence assume that \( e^{g-\alpha} \geq 2 \). First, we make a comment. If the Riemann mapping theorem is very general and flexible, it gives very few informations on the parametrization \( t \mapsto f^*(e^{it}) \) when \( f: \mathbb{D} \to \Omega \) is a conformal map, except in some specific cases (lens maps, cusps, etc.: see [9]). Here, the Kolmogorov weak type inequality provides a substitute. Write:

\[
f = u + iv
\]
and set:

\[ f_1 = -if + i\frac{\pi}{2} - \alpha = v - \alpha + i\left(\frac{\pi}{2} - u\right) \]

and:

\[ F_1 = 1 + e^{f_1} = (1 + e^{v-\alpha} \sin u) + ie^{v-\alpha} \cos u. \]

If \( v < \alpha \), then \( \Re F_1 > 1 - |\sin u| \geq 0 \). If \( v \geq \alpha \), then \( 0 < u < \pi \) and \( \Re F_1 \geq 1 \). Hence \( F_1 \) maps \( \mathbb{D} \) to the right half-plane \( \mathbb{C}_0 = \{z : \Re z > 0\} \).

Finally, let \( F = U + iV : \mathbb{D} \to \mathbb{C}_0 \) be defined by:

\[ F = F_1 - i\Im F_1(0), \]

so that \( V(0) = 0 \). By the Kolmogorov inequality for the conjugation map \( U \mapsto V \), and the harmonicity of \( U \), we have, for all \( \lambda > 0 \) (\( a \) designating an absolute constant):

\[ m(|F^*| > \lambda) \leq \frac{a}{\lambda} \|U^*\|_1 = \frac{a}{\lambda} \int_T U^* \, dm = \frac{a}{\lambda} U(0). \]  

Next, we claim that:

\[ |\Im F_1(0)| < 1 \quad \text{and} \quad U(0) < 2. \]  

Indeed, \( v(0) < \alpha \) by hypothesis, so that \( |\Im F_1(0)| = e^{v(0)-\alpha} |\cos u(0)| < 1 \), and \( U(0) = 1 + e^{v(0)-\alpha} \sin u(0) < 2 \). Suppose now that, for some \( y > \alpha \) and \( z \in \mathbb{D} \), we have \( v(z) > y \). Then, \( 0 < u(z) < \pi \) by our second assumption, and this implies \( \Re e^{f_1(z)} = e^{v(z)-\alpha} \sin u(z) > 0 \), so that, using \( |1 + w| \geq |w| \) if \( \Re w > 0 \) and (3.4), and remembering that \( e^{y-\alpha} \geq 2 \):

\[ |F(z)| = |1 + e^{f_1(z)} - i\Im F_1(0)| \geq |1 + e^{f_1(z)}| - 1 \]

\[ \geq |e^{f_1(z)}| - 1 = e^{v(z)-\alpha} - 1 > e^{y-\alpha} - 1 \geq \frac{1}{2} e^{y-\alpha}. \]

Taking radial limits and using (3.3) and (3.4), we get:

\[ m(\Im f^* > y) \leq m(|F^*| > e^{y-\alpha}/2) \leq 4a e^{\alpha-y}. \]

This ends the proof of Lemma 3.3 with \( C = \max(2, 4a) \).
4 Application to the multidimensional case

In this section, we apply Theorem 3.1 and Theorem 3.2 to show that, for $N \geq 2$, the image of the symbol cannot determine the behavior of the approximation numbers, or rather of $\beta_N(C_\varphi)$, of the associated composition operator $C_\varphi: H^2(\D^N) \to H^2(\D^N)$.

Recall that for an operator $T: H_1 \to H_2$, we set:

$$
\beta_N^-(T) = \liminf_{n \to \infty} [a_n(T)]^{1/n^{1/N}} \quad \text{and} \quad \beta_N^+(T) = \limsup_{n \to \infty} [a_n(T)]^{1/n^{1/N}},
$$

and write $\beta_N(T)$ when $\beta_N^-(T) = \beta_N^+(T)$.

**Theorem 4.1.** For $N \geq 2$, there exist pairs of symbols $\Phi_1, \Phi_2: \D^N \to \D^N$, such that $\Phi_1(\D^N) = \Phi_2(\D^N)$ and:

1) $C_{\Phi_1}$ is not bounded, but $C_{\Phi_2}$ is compact, and even $\beta_N(C_{\Phi_2}) = 0$;

2) $C_{\Phi_1}$ is bounded but not compact, so $\beta_N(C_{\Phi_1}) = 1$, and $C_{\Phi_2}$ is compact, with $\beta_N(C_{\Phi_2}) = 0$;

3) $C_{\Phi_1}$ is compact, with $\beta_N^-(C_{\Phi_1}) > 0$ and $\beta_N^+(C_{\Phi_1}) < 1$, and $C_{\Phi_2}$ is compact, with $\beta_N(C_{\Phi_2}) = 0$;

4) $C_{\Phi_1}$ is compact, with $\beta_N(C_{\Phi_1}) = 1$, and $C_{\Phi_2}$ is compact, but with $\beta_N(C_{\Phi_2}) = 0$.

**Proof.** Let $\sigma: \D \to \D$ be a surjective symbol such that $\rho_\sigma(h) \leq h^N e^{-2/h^2}$ given by Theorem 3.1. By Proposition 2.3 we have, with $\gamma = N - 2$:

$$
a_n(C_\sigma: B_\gamma \to H^2) \lesssim \inf_{0<h<1} (n^{(N-1)/2} e^{-nh} + e^{-1/h^2}),
$$

and, with $h = 1/n^{1/3}$, we get $a_n(C_\sigma: B_\gamma \to H^2) \lesssim e^{-d n^{2/3}}$.

We choose the exponent $2/3$ for fixing the ideas, but every exponent $\alpha > 1/2$, with $\alpha < 1$, (i.e. $a_n(C_\sigma: B_\gamma \to H^2) \lesssim e^{-d n^\alpha}$) would be suitable.

1) We take $\Phi_1(z_1, z_2, z_3, \ldots, z_N) = (z_1, z_1, \ldots, z_1)$. The composition operator $C_{\Phi_1}$ is not bounded because if $f_n(z_1, \ldots, z_N) = (\frac{n + 1}{2})^n$, then $\|f_n\|_2^2 = 4^{-n} \sum_{k=0}^n (\frac{n}{k})^2 = 4^{-n} (\frac{2^n}{n}) \approx 1/\sqrt{n}$, though $(C_{\Phi_1} f_n)(z_1, \ldots, z_N) = z_1^n$ and $\|C_{\Phi_1} f_n\|_2 = 1$.

We define $\Phi_2$ by:

$$
\Phi_2(z_1, z_2, \ldots, z_N) = (\sigma(z_1), \sigma(z_1), \ldots, \sigma(z_1)).
$$
Since $\sigma$ is surjective, we have $\Phi_2(D^N) = \Phi_1(D^N)$. Now, as in the proof of Theorem 2.4 we have $C_{\Phi_2} = JC_{\sigma}M$, so:

$$a_n(C_{\Phi_2}) \leq a_n(C_{\sigma}: B_{N-2} \to H^2) \lesssim e^{-dn^{2/3}},$$

by the ideal property. Hence $[a_n(C_{\Phi_2})]^{1/n^2/N} \lesssim e^{-dn^{2/3}}$ and therefore $\beta_N(C_{\Phi_2}) = 0$ since $2\frac{2}{3} - \frac{1}{N} > 0$.

2) We consider the lens map $\lambda = \lambda_{1/N}$ of parameter $1/N$. We define:

$$\begin{align*}
\Phi_1(z_1, \ldots, z_N) &= (\lambda(z_1), \lambda(z_1), \ldots, \lambda(z_1)) \\
\Phi_2(z_1, \ldots, z_N) &= (\lambda[\sigma(z_1)], \lambda[\sigma(z_1)], \ldots, \lambda[\sigma(z_1)]).
\end{align*}$$

Since $\sigma$ is surjective, we have $\Phi_1(D^N) = \Phi_2(D^N)$ and we saw in Theorem 2.5 that $C_{\Phi_1}$ is bounded but not compact.

On the other hand, we have the factorization $C_{\Phi_2} = JC_{\sigma}C_{\lambda}M$. Hence $C_{\Phi_2}$ is compact, and, as in 1), $\beta_N(C_{\Phi_2}) = 0$.

3) For this item, the map $\sigma$ does not suffice, and we will use another surjective symbol $s: D \to D$. By Theorem 3.1 there exists such a map $s$ with:

$$\rho_s(t) \leq t^2 e^{-2/t^2}$$

and

$$\rho_s(t) \leq t \delta^2(t)$$

for $t$ small enough, where $\delta: (0, 1) \to (0, 1)$ is a non-decreasing function such that $\delta(\varepsilon_n) \leq e^{-n\varepsilon_n}$ and:

$$\varepsilon_n = n^{-\frac{1}{4N-7}}.$$  

By the proof of Theorem 3.2 (4.3) implies that:

$$a_n(C_s) \leq e^{-n\varepsilon_n}.$$  

We also consider a lens map $\lambda = \lambda_{\theta}$, with parameter $\theta < 1/N$, and we set:

$$\begin{align*}
\Phi_1(z_1, \ldots, z_N) &= (\lambda(z_1), \lambda(z_1), \frac{z_3}{2}, \ldots, \frac{z_N}{2}) \\
\Phi_2(z_1, \ldots, z_N) &= (\lambda[s(z_1)], \lambda[s(z_1)], \frac{s(z_3)}{2}, \ldots, \frac{s(z_N)}{2}).
\end{align*}$$
Since \( s \) is surjective, we have \( \Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N) \).

a) Let us prove that \( \beta_N^+(C_{\Phi_1}) > 0 \) and \( \beta_N^-(C_{\Phi_1}) < 1 \). Note that:

\[
C_{\Phi_1} = C_u \otimes C_{v_3} \otimes \cdots \otimes C_{v_N},
\]

where \( u: \mathbb{D}^2 \to \mathbb{D}^2 \) is defined by \( u(z_1, z_2) = (\lambda(z_1), \lambda(z_1)) \) and \( v_j: \mathbb{D} \to \mathbb{D} \) is defined by \( v_j(z_j) = z_j/2 \). In fact, if \( f \in H^2(\mathbb{D}^2) \) and \( g_j \in H^2(\mathbb{D}) \), \( 3 \leq j \leq N \), we have:

\[
[C_{\Phi_1}(f \otimes g_3 \otimes \cdots \otimes g_N)](z_1, z_2, z_3, \ldots, z_N)
= (f \otimes g_3 \otimes \cdots \otimes g_N)(u(z_1, z_2), v_3(z_3), \ldots, v_N(z_N))
= f[\lambda(z_1), \lambda(z_1)] g_3[v_3(z_3)] \cdots g_N[v_N(z_N)]
= (C_u f)(z_1, z_2) (C_{v_3} g_3)(z_3) \cdots (C_{v_N} g_N)(z_N)
= [(C_u \otimes C_{v_3} \otimes \cdots \otimes C_{v_N})(f \otimes g_3 \otimes \cdots \otimes g_N)](z_1, z_2, z_3, \ldots, z_N),
\]

hence the result since \( H^2(\mathbb{D}^2) \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}) \) is dense in \( H^2(\mathbb{D}^N) \). That proves in particular that \( C_{\Phi_1} \) is compact since \( C_u \) and \( C_{v_3}, \ldots, C_{v_N} \) are (by Theorem 2.5 for \( C_u \)).

By the supermultiplicativity of singular numbers of tensor products (see [11] Lemma 3.2), it ensues that:

\[
a_n^N(C_{\Phi_1}) \geq a_n^2(C_u) \prod_{j=3}^{N} a_n(C_{v_j}) = a_n^2(C_u) \left( \frac{1}{2} \right)^{n(N-2)}.
\]

By [11] Remark at the end of Section 4, we have \( a_n^2(C_u) \gtrsim e^{-bn} \) for some positive constant \( b = b(\theta) \). Indeed, if \( J = J_2: H^2(\mathbb{D}) \to H^2(\mathbb{D}^2) \) is the canonical injection defined by \( (Jh)(z_1, z_2) = h(z_1) \) and \( Q: H^2(\mathbb{D}^2) \to H^2(\mathbb{D}) \) is defined by \( (Qf)(z_1) = f(z_1, 0) \), we have \( C_\lambda = QC_uJ \). Hence \( a_k(C_u) \gtrsim a_k(C_\lambda) \gtrsim e^{-bn/k} \).

Therefore we get:

\[
a_n^N(C_{\Phi_1}) \gtrsim e^{-cn}
\]

for some positive constant depending only on \( \theta \) and \( N \). It follows that \( \beta_N^-(C_{\Phi_1}) > 0 \).

To see that \( \beta_N^+(C_{\Phi_1}) < 1 \), we need the following lemma, whose proof is postponed.
Lemma 4.2. Let $S: H_1 \to H_1$ and $T: H_2 \to H_2$ be two operators between Hilbert spaces and $A, B$ a pair of positive numbers. Then, whenever:

$$a_{[n^A]}(S) \leq e^{-cn} \text{ and } a_{[n^B]}(T) \leq e^{-cn},$$

where $[.]$ stands for the integer part, we have, for some constant integer $M = M(A, B) > 0$:

$$a_{M[n^{A+B}]}(S \otimes T) \leq e^{-cn}.$$ 

Let $S = C_u$ and $T = C_{v_3} \otimes \cdots \otimes C_{v_N}$. For $c$ small enough, we have $a_{nN-2}(T) \leq C(1/2)^n \leq e^{-cn}$ and, using (2.13), $a_{n^2}(S) \leq e^{-dn} \leq e^{-cn}$. Hence, with $A = 2$, $B = N - 2$, Lemma 4.2 gives:

$$a_{Mn^N}(C_{\Phi_1}) \lesssim e^{-cn}.$$ 

Therefore $\beta^+_N(C_{\Phi_1}) \leq e^{-c/M^{1/N}} < 1$.

b) Define $\Psi: \mathbb{D}^N \to \mathbb{D}^N$ by:

$$\Psi(z_1, z_2, z_3, \ldots, z_N) = (s(z_1), s(z_1), s(z_3), \ldots, s(z_N)).$$

If $\tau_1: \mathbb{D}^2 \to \mathbb{D}^2$ is defined by $\tau_1(z_1, z_2) = (s(z_1), s(z_1))$ and the map $\tau_2: \mathbb{D}^{N-2} \to \mathbb{D}^{N-2}$ by $\tau_2(z_3, \ldots, z_N) = (s(z_3), \ldots, s(z_N))$, we have:

$$C_{\Psi} = C_{\tau_1} \otimes C_{\tau_2}.$$ 

As in the proof of Theorem 2.4, we have the factorization:

$$\tau_1: H^2(\mathbb{D}^2) \xrightarrow{M} \mathcal{B}_0 = \mathcal{B}^2 \xrightarrow{C_s} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^2).$$

Hence $a_n(C_{\tau_1}) \leq \|M\| \|J\| a_n(C_s: \mathcal{B}^2 \to H^2)$.

By Proposition 2.3 we have:

$$a_n(C_s: \mathcal{B}^2 \to H^2) \lesssim \inf_{0 < h < 1} \left( \sqrt{n} e^{-nh} + \sup_{0 < t \leq h} \sqrt{\frac{\rho_s(t)}{t^2}} \right);$$

so (4.2) implies that $a_n(C_s: \mathcal{B}^2 \to H^2) \lesssim \inf_{0 < h < 1} (\sqrt{n} e^{-nh} + e^{-1/h^2})$ and, taking $h = n^{-1/3}$, we get, with some $c$ small enough:

$$a_n(C_s: \mathcal{B}^2 \to H^2) \lesssim e^{-cn^{2/3}}.$$ 

It follows that $a_n(C_{\tau_1}) \lesssim e^{-cn^{2/3}}$ and hence:

$$a_{[n^{2/3}]}(C_{\tau_1}) \lesssim e^{-cn}.$$ 

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On the other hand, [1] Theorem 5.5] says that:

\[ a_n(C_{\tau_2}) \leq 2^{N-3} \| C_s \|^{N-2} \inf_{n_3 \cdots n_N \leq n} (a_{n_3}(C_s) + \cdots + a_{n_N}(C_s)). \]

Taking \( n_3 = \cdots = n_N = n^{\frac{1}{N-2}} \), we get, using (4.5):

\[ a_n(C_{\tau_2}) \leq K^N N \exp \left( -n^{\frac{1}{N-2}} \frac{\varepsilon}{n^{\frac{1}{N-2}}} \right). \]

Using (4.4), that gives:

\[ a_n(C_{\tau_2}) \lesssim \exp \left( -n^{\frac{1}{N-2}} (1 - \frac{1}{4N-7}) \right) = \exp \left( -n^{\frac{4}{4N-7}} \right), \]

or:

\[
(4.7) \quad a_{[n^{N-\frac{4}{2}}]}(C_{\tau_2}) \lesssim e^{-n} \leq e^{-cn}.
\]

Now, (4.6) and (4.7) allow to use Lemma 4.2 with \( A = \frac{3}{2} \) and \( B = N - \frac{7}{4} \), and we get:

\[ a_M[n^{N-\frac{2}{4}}](C_{\psi}) \lesssim e^{-cn}. \]

Equivalently:

\[ a_k(C_{\psi}) \lesssim \exp \left( -c' k^{\frac{4}{4N-7}} \right) \]

and:

\[
(a_k(C_{\psi}))^{1/k^{1/N}} \lesssim \exp \left( -c' k^{\frac{4}{4N-7} - \frac{1}{N}} \right) = \exp \left( -c' k^{\frac{1}{4N-3}} \right),
\]

which gives \( \beta_N(C_{\psi}) = 0 \).

To end the proof, it suffices to remark that \( C_{\Phi_2} = C_{\psi} \circ C_{\Phi_1} \), since \( \Phi_2 = \Phi_1 \circ \Psi \), and hence \( \beta_N^\Phi(C_{\Phi_2}) \leq \beta_N^\Psi(C_{\psi}) = 0 \), so \( \beta_N(C_{\Phi_2}) = 0 \).

4) We use a Shapiro-Taylor map. This one-parameter map \( \varsigma_\theta \), \( \theta > 0 \), was introduced by J. Shapiro and P. Taylor in 1973 ([15]) and was further studied, with a slightly different definition, in [5] Section 5]. J. Shapiro and P. Taylor proved that \( C_{\varsigma_\theta}: H^2 \rightarrow H^2 \) is always compact, but is Hilbert-Schmidt if and only if \( \theta > 2 \). Let us recall their definition.

For \( 0 < \varepsilon < 1 \), we set \( V_\varepsilon = \{ z \in \mathbb{C}; \Re z > 0\text{ and } |z| < \varepsilon \} \). For \( \varepsilon = \varepsilon_\theta > 0 \) small enough, one can define:

\[ f_\theta(z) = z(- \log z)^\theta, \]
for $z \in V_{\varepsilon}$, where $\log z$ will be the principal determination of the logarithm. Let now $g_\theta$ be the conformal mapping from $\mathbb{D}$ onto $V_{\varepsilon}$, which maps $T = \partial \mathbb{D}$ onto $\partial V_{\varepsilon}$, defined by $g_\theta(z) = \varepsilon \varphi_0(z)$, where $\varphi_0$ is given by:

$$\varphi_0(z) = \frac{(z - i) \left( \frac{i}{z - 1} \right)^{1/2} - i}{-i \left( \frac{i}{z - 1} \right)^{1/2} + 1}.$$ 

Then, we define:

$$\varsigma_\theta = \exp(-f_\theta \circ g_\theta).$$

We proved in [9, Section 4.2] (though it is not sharp) that:

\begin{equation}
(4.8) \quad a_n(C_{\varsigma_\theta}) \geq \frac{1}{n^{\theta/2}}.
\end{equation}

We define $\Phi_1 : \mathbb{D}^N \to \mathbb{D}^N$ as:

\begin{equation}
(4.9) \quad \Phi_1(z_1, z_2, \ldots, z_N) = (\varsigma_\theta(z_1), 0, \ldots, 0).
\end{equation}

If $J = J_N : H^2(\mathbb{D}) \to H^2(\mathbb{D}^N)$ is the canonical injection defined by $(Jh)(z_1, \ldots, z_N) = h(z_1)$ and $Q = Q_N : H^2(\mathbb{D}^N) \to H^2(\mathbb{D})$ is defined by $(Qf)(z_1) = f(z_1, 0, \ldots, 0)$, then $C_{\Phi_1} = JC_{\varsigma_\theta}Q$; hence $C_{\Phi_1}$ is compact. On the other hand, we also have $QC_{\Phi_1}J = C_{\varsigma_\theta}$, which implies that $a_n(C_{\Phi_1}) \geq a_n(C_{\varsigma_\theta}) \geq n^{-\theta/2}$. It follows that:

$$\beta_N(C_{\Phi_1}) \geq \lim_{n \to \infty} (n^{-\theta/2})^{1/n^1/N} = 1,$$

and hence $\beta_N(C_{\Phi_1}) = 1$.

Now, if:

$$\Phi_2(z_1, \ldots, z_N) = (\varsigma_\theta[\sigma(z_1)], 0, \ldots, 0),$$

since $\sigma$ is surjective, we have $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$. Moreover, we have $C_{\Phi_2} = JC_{\varsigma_\theta}\sigma Q = JC_{\sigma}C_{\varsigma}Q$, so $a_n(C_{\Phi_2}) \leq a_n(C_{\sigma})$. Since $\rho_\sigma(h) \leq h^{-N+1} e^{-2/h^2}$, Proposition [2.2] gives, with $h = 1/n^{1/3}$:

$$a_n(C_{\sigma}) \lesssim e^{-cn^{2/3}},$$

so $[a_n(C_{\Phi_2})]^{1/n^1/N} \lesssim \exp(-cn^{2/3} - \frac{1}{N})$ and $\beta_N(C_{\Phi_2}) = 0$. $\square$

**Proof of Lemma [4.2]** In [11], we observed that the singular numbers of $S \otimes T$ are the non-increasing rearrangement of the numbers $s_j t_k$, where $s_j$ and $t_k$ denote respectively the $j$-th and the $k$-th singular number of $S$ and $T$. We
can assume \( s_1 = t_1 = 1 \). Using this observation, we will majorize the number of pairs \( (j, k) \) such that \( s_jt_k > e^{-cn} \). Let \((j, k)\) be such a pair. Since \( s_j \leq s_1 = 1 \), we have \( t_k \geq e^{-cn} \) so that \( k \leq [n^B] \leq n^B \). Hence, for some \( 2 \leq l \leq n \), we have \((l-1)^B < k \leq l^B\). Then, due to the assumption on \( T \), \( t_k < e^{-c(l-1)} \) and \( s_j \geq e^{-cn}t_{k}^{-1} \geq e^{-c(n-l+1)} \), implying that \( j \lesssim (n-l+1)^A \), thanks to the assumption on \( S \). As a consequence, since the number of integers \( k \) such that \((l-1)^B < k \leq l^B\) is dominated by \( l^B - 1 \), the number \( \nu_n \) of pairs \((j, k)\) such that \( s_jt_k > e^{-cn} \) is dominated by:

\[
\sum_{l=1}^{n} (n-l+1)^A l^{B-1} \sim n^{A+B} \int_{0}^{1} t^{A}(1-t)^B \, dt ,
\]

by a Riemann sum argument. Next, let \( M \in \mathbb{N} \) big enough to have:

\[
\sum_{l=1}^{n} (n-l+1)^A l^{B-1} \leq Mn^{A+B} - 1 , \quad \text{for all } n.
\]

By definition, \( a_{M[n^{A+B}]}(S \otimes T) \leq a_{\nu_n+1}(S \otimes T) \leq e^{-cn} \), giving the result. \( \square \)

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