ABSTRACT. In this work we show that based on a conjecture for the pair correlation of integers representable as sums of two squares, which was first suggested by Connors and Keating and reformulated here, the second moment of the distribution of the number of representable integers in short intervals is consistent with a Poissonian distribution, where “short” means of length comparable to the mean spacing between sums of two squares. In addition we present a method for producing such conjectures through calculations in prime power residue rings and describe how these conjectures, as well as the above stated result, may by generalized to other binary quadratic forms. While producing these pair correlation conjectures we arrive at a surprising result regarding Mertens’ formula for primes in arithmetic progressions, and in order to test the validity of the conjectures, we present numerical computations which support our approach.

1. INTRODUCTION

Throughout this work $n$, $k$ and $h$ will denote positive integers, $p$ will denote prime numbers and for abbreviation reasons we use $a \equiv b \pmod{c}$ instead of $a \equiv b \pmod{c}$. In addition, we say $m_p(n) = k$ if $p^k \mid n$ but $p^{k+1} \nmid n$.

1.1. Background and motivation. When studying the distribution of a sequence of integers, for example the sequence of primes or of those representable as a sum of two squares, a natural first step would be to understand the mean density of such integers. For prime numbers this was achieved by Hadamard and de la Vallée-Poussin with their famous Prime Number Theorem, and for sums of squares by Landau [7]. In order to learn more about the distribution of such a set the next step would be to look at the $k$–point correlation, or in other words to find an expression for

$$
\frac{1}{n} \sum_{m=1}^{n} f(m + d_1) \cdots f(m + d_k)
$$

as $n \to \infty$, where $f$ is the characteristic function of the set at hand and $d_1, ..., d_k$ are distinct integers. These correlations give increasingly more precise data about the distribution, where the 2–point correlation provides the leading quantitative estimate of the fluctuations about the mean density of the sequence.
Regarding the sequence of primes, Hardy and Littlewood gave \[6\] the following \(k\)-tuple conjecture for the number \(\pi_d(n)\) of positive integers \(m \leq n\) for which all of \(m + d_1, \ldots, m + d_k\) are prime, \(d = (d_1, \ldots, d_k)\) and \(d_1, \ldots, d_k\) distinct integers. The conjecture is

\[
\pi_d(n) \sim \mathcal{S}_d \frac{n}{(\log n)^k}
\]
as \(n \to \infty\), provided \(\mathcal{S}_d \neq 0\), where the “singular series” \(\mathcal{S}_d\) is

\[
\mathcal{S}_d = \prod_p \frac{p^{k-1} (p - \nu_d(p))}{(p-1)^k}
\]
and \(\nu_d(p)\) stands for the number of residue classes modulo \(p\) occupied by \(d_1, \ldots, d_k\).

For \(k = 1\) this is exactly the Prime Number Theorem, and for \(k \geq 2\) it has not been proved for any \(d\).

**1.2. From a \(k\)-tuple conjecture to distribution in short intervals.**

We will follow Gallagher’s work \[5\] on primes in order to obtain the moments of the distribution of the number of integers representable as a sum of two squares in short intervals. Consider first the set of primes and the Prime Number Theorem, which states that as \(n \to \infty\)

\[
\pi(n) \sim \frac{n}{\log n}.
\]

This relation can be understood as the statement that the number of primes in an interval \((m, m + \alpha)\), averaged over \(m \leq n\), tends to the limit \(\lambda\), when \(n\) and \(\alpha\) tend to infinity in such a way that \(\alpha \sim \lambda \log n\) with \(\lambda\) a positive constant.

Gallagher studies the distribution of values of \(\pi(m + \alpha) - \pi(m)\) for \(m \leq n\) and \(\alpha \sim \lambda \log n\), and shows that, assuming the prime \(k\)-tuple conjecture of Hardy and Littlewood \[1.1\], it suffices that

\[
\sum_{1 \leq d_1, \ldots, d_k \leq H} \mathcal{S}_d \sim H^k
\]
as \(H \to \infty\) holds for all \(k \in \mathbb{N}\) in order to prove that all the moments of the distribution tend to moments of a Poisson distribution, and so the distribution tends to a Poisson distribution with parameter \(\lambda\) as \(n \to \infty\). This means that the distribution of primes in such intervals is similar to the distribution of a random set of integers with mean \(\lambda\), and so even though clearly primes are not distributed randomly, in the perspective of intervals such as those we deal with here they do. Gallagher has proved \[1.2\] in \[3\], and a simpler proof was presented by Ford \[4\]. We shall refer to this result as Gallagher’s Lemma.

Consider now the set of integers which are representable as a sum of two squares and Landau’s theorem, which states that \(B(n)\), the number of such
integers up to \( n \), is given asymptotically by

\[
B(n) \sim \beta \frac{n}{\sqrt{\log n}} + O\left( \frac{n}{\log 3 n} \right)
\]

as \( n \to \infty \), where \( \beta = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 (4)} (1 - p^{-2})^{-1/2} \) is the Landau-Ramanujan constant (see [12]).

This relation can be understood as the statement that the number of integers representable as a sum of two squares in an interval \((m, m + \alpha)\), averaged over \( m \leq n \), tends to the limit \( \lambda \), when \( n \) and \( \alpha \) tend to infinity in such a way that \( \alpha \sim \frac{\lambda}{\beta} \sqrt{\log n} \) with \( \lambda \) a positive constant.

We wish to study the distribution of values of \( B(m + \alpha) - B(m) \) for \( m \leq n \) and \( \alpha \sim \frac{\lambda}{\beta} \sqrt{\log n} \). In order to follow Gallagher’s method we need first a conjecture analogous to Hardy and Littlewood’s conjecture for sums of two squares, that is an asymptotic formula for the number \( B_d(n) \) of positive integers \( m \leq n \) for which all of \( m + d_1, \ldots, m + d_k \) can be represented as a sum of two squares, \( d = (d_1, \ldots, d_k) \) and \( d_1, \ldots, d_k \) distinct integers. The conjecture, analogous to (1.1), is that there exists a function \( \mathcal{T}_d \), the “singular series for our problem”, for which the limit

\[
B_d(n) \sim \mathcal{T}_d \frac{n}{(\sqrt{\log n})^k}
\]

holds. If this is so, then the function \( \mathcal{T}_d \) depends only on the differences between the \( d_1, \ldots, d_k \), in the sense that \( \mathcal{T}_d = \mathcal{T}_{d+1} \) where \( 1 = (1, \ldots, 1) \).

Assuming this conjecture, it is enough to show that the singular series \( \mathcal{T}_d \) has mean value \( \beta \), that is the limit

\[
\sum_{1 \leq d_1, \ldots, d_k \leq H} \mathcal{T}_d \sim (\beta H)^k
\]

as \( H \to \infty \) holds, for the moments to be Poisson with parameter \( \lambda \).

1.3. **Main result.** Connors and Keating conjectured in [11] that for \( k = 2 \) and \( h = |d_2 - d_1| \) we have

\[
B_h(n) \sim \mathcal{T}_h \frac{n}{(\sqrt{\log n})^2}
\]

as \( n \to \infty \), with the following “singular series”

\[
\mathcal{T}_h = 2W_2(h) \prod_{p \equiv 3 (4)} \prod_{p|h} \frac{1 - p^{-(m_p(h)+1)}}{1 - p^{-1}}
\]
where \( m_p(h) \) is the power to which the prime \( p \) is raised in the prime decomposition of \( h \) and

\[
W_2(h) = \begin{cases} 
\frac{1}{2^{m_2(h)+1}} & m_2(h) = 0 \\
\frac{1}{2^{m_2(h)+2}} & m_2(h) \geq 1 
\end{cases}
\]

**Theorem 1.** For \( k = 2 \) and \( h = |d_2 - d_1| \) the singular series \( T_h \) has mean value \( \beta \). More explicitly

\[
\sum_{1 \leq d_1 \neq d_2 \leq H} T_h = 2 \sum_{1 \leq h \leq H-1} (H-h) T_h = \beta^2 H^2 + O(\varepsilon(H^{1+\varepsilon}))
\]

as \( H \to \infty \), for all \( \varepsilon > 0 \).

Assuming the validity of Connors and Keating’s pair correlation conjecture this result implies that Gallagher’s Lemma for sums of two squares and \( k = 2 \) holds, or in other words we show that assuming the conjecture, the second moment of the distribution of values of \( B(m+\alpha) - B(m) \) for \( m \leq n \) and \( \alpha \sim \frac{\lambda}{\beta} \sqrt{\log n} \) is consistent with a Poissonian distribution with parameter \( \lambda \).

### 1.4. Mean density and pair correlation

We provide a new method of conjecturing estimates for the pair correlation function stated above, which goes through the mean density and pair correlation of elements representable as a sum of two squares in residue rings of the form \( \mathbb{Z}/p^k\mathbb{Z} \) for primes \( p \) and \( k \to \infty \). The naive expectation for the density of sums of two squares is

\[
(1.7) \quad \mathcal{M}(n) := \frac{1}{2} \prod_{\substack{p \equiv 3 (4) \\
p \leq n}} (1 + p^{-1})^{-1}
\]

where \( \mathcal{M}(n) \) is simply the product of the densities in the residue rings described above. We compare this expression with the leading term of the analytic result for the density of representable integers given by Landau

\[
(1.8) \quad \mathcal{L}(n) = \frac{\beta}{\sqrt{\log n}}
\]

and produce the precise ratio between the two and show that

\[
(1.9) \quad y := \lim_{n \to \infty} \frac{\mathcal{M}(n)}{\mathcal{L}(n)} = \frac{1}{2} \sqrt{\pi e^\gamma}
\]

where \( \gamma \) is Euler’s constant, using a version of Mertens’ formula in geometric progressions described in [8]. Comparing this to the case of the Prime Number Theorem and Mertens’ original formula

\[
\lim_{n \to \infty} \frac{\prod_{p \leq n} (1 - p^{-1})}{\pi(n)} = \frac{1}{e^\gamma}
\]

we see that as in the case of the primes we are off by a factor.
Next we derive (1.6) in similar methods to those used for the mean density (1.7). Denote by \( M^{(2)}(n, h) \) the product of densities of representable pairs \((a, a + h)\) in the rings \( \mathbb{Z}/p^k\mathbb{Z} \), \( k \to \infty \) where the product is over primes \( p \leq n \) (see Section 5 for the detailed definition). We then make the following conjecture, which is equivalent to that of Keating and Connors.

**Conjecture 2.** Let

\[
Y_h(n) := \frac{M^{(2)}(n, h)}{\frac{1}{n} \# \{m \leq n: m \text{ and } m + h \text{ are representable} \}}.
\]

Then \( Y_h(n) \) converges and the following limit holds for every \( h \in \mathbb{N} \)

\[
\lim_{n \to \infty} Y_h(n) = \frac{1}{4} \frac{\pi}{e^\gamma} =: y^2.
\]

Notice that according to our conjecture the ratio defined above converges to a universal constant which does not depend on the difference \( h \). In Section 5 we present numeric calculations to support this conjecture.

1.5. **Generalization to other binary quadratic forms.** Our methods allow us to expand our observation also to integers representable by other binary quadratic forms \( x^2 + dy^2 \) with \( d = 2, 3, 4, 7 \) in addition to \( d = 1 \), which are the sums of two squares. The reason we examine these values of \( d \) is that these are the convenient (idoneal) numbers such that the forms \( x^2 + dy^2 \) are of class number 1, see [2] and Definition 8. A surprising result is that the ratio between the product formulas \( M_d(n) \) we present and the analytic results using variations on Landau’s theorem \( L_d(n) \), for \( n \to \infty \), is in fact constant for the five different quadratic forms inspected and is again

\[
\lim_{n \to \infty} \frac{M_d(n)}{L_d(n)} = \frac{1}{2} \sqrt{\frac{\pi}{e^\gamma}}.
\]

We next produce conjectures analogous to (1.6) and therefore to (1.4) with \( k = 2 \) for integers representable by the forms at hand, and finally prove that assuming our conjectures Gallagher’s Lemma holds for \( k = 2 \).

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2. Distribution In Short Intervals - Gallagher’s Lemma

We define $B_h(n)$ to be the number of positive integers $m \leq n$ for which both $m$ and $m + h$ can be represented as a sum of two squares. It is conjectured that $B_h(n) \sim \mathcal{T}_h \frac{n}{\log n}$, where by the Connors and Keating conjecture

$$\mathcal{T}_h = 2W_2(h) \prod_{p \equiv 3 (4)} \frac{1 - p^{-(m_p(h)+1)}}{1 - p^{-1}}$$

and

$$W_2(h) = \begin{cases} 
1 & m_2(h) = 0 \\
\frac{2m_2(h)+1-3}{2m_2(h)+2} & m_2(h) \geq 1
\end{cases}$$

In this section we prove Theorem 1 that is we show that

$$\sum_{1 \leq d_1 \neq d_2 \leq H} \mathcal{S}_d = \sum_{1 \leq d_1 \neq d_2 \leq H} \mathcal{S}_{|d_2-d_1|} = 2 \sum_{1 \leq h \leq H-1} (H-h) \mathcal{T}_h = \beta^2 H^2 + O_\epsilon(H^{1+\epsilon})$$

for all $\epsilon > 0$.

Following Gallagher’s work for primes described in the introduction, this calculation will let us obtain the second moment for the distribution of representable integers in the short intervals described above.

2.1. Dirichlet series. Set

$$a(h) = 2\mathcal{T}_h = 4W_2(h) \prod_{p \equiv 3 (4)} \frac{1 - p^{-(m_p(h)+1)}}{1 - p^{-1}}.$$ 

Notice that $a(h)$ is multiplicative: obviously $a(1) = 1$ since 1 is odd and has no prime factors, and for $(m, n) = 1$ we have $a(mn) = a(m)a(n)$ because our function is composed of products depending only on the prime factorizations.

Computing $a(p^k)$ gives

$$a(p^k) = \begin{cases} 
1 & p \equiv 1(4) \\
2 - \frac{3}{p} & p = 2 \\
\frac{1-p^k+1}{1-p} & p \equiv 3(4)
\end{cases}.$$ 

We can thus write

$$D(s) = \sum_{h=1}^{\infty} a(h)h^{-s} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{a(p^k)}{p^{ks}}\right)$$

$$= \left(1 + \sum_{k=1}^{\infty} \frac{2 - \frac{3}{p}}{2^{ks}}\right) \prod_{p \equiv 1 (4)} \left(1 + \frac{p^{-s}}{1-p^{-s}}\right) \prod_{p \equiv 3 (4)} \left(1 + \sum_{k=1}^{\infty} \frac{1-p^k+1}{p^{ks}(1-p)}\right)$$

$$= R(s)P(s)Q(s)$$

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where

\[ R(s) = 1 + 2 \frac{2^{-s}}{1 - 2^{-s}} - 3 \frac{2^{-(s+1)}}{1 - 2^{-(s+1)}} \]

\[ P(s) = \prod_{p \equiv 1(4)} (1 - p^{-s})^{-1} \]

\[ Q(s) = \prod_{p \equiv 3(4)} \left( 1 + \frac{1}{1 - p^{-1}} \frac{p^{-s}}{1 - p^{-s}} - \frac{p^{-1}}{1 - p^{-1}} \frac{p^{-(s+1)}}{1 - p^{-(s+1)}} \right). \]

2.2. Comparison to Riemann’s \( \zeta \) function. Taking \( D(s) = \prod_p (1 - p^{-s})^{-1} \), we will now show that \( \frac{D(s)}{\zeta(s)} \) is analytic for \( \sigma > 0 \) where \( s = \sigma + it \), thus \( D(s) \) is analytic in that region with a simple pole at \( s = 1 \).

\[ \frac{D(s)}{\zeta(s)} = 1 + 2 \frac{2^{-s}}{1 - 2^{-s}} - 3 \frac{2^{-(s+1)}}{1 - 2^{-(s+1)}} \cdot \prod_{p \equiv 3(4)} \left( 1 + \frac{1}{1 - p^{-1}} \frac{p^{-s}}{1 - p^{-s}} - \frac{p^{-1}}{1 - p^{-1}} \frac{p^{-(s+1)}}{1 - p^{-(s+1)}} \right). \]

The first expression turns out to be

\[ \frac{R(s)}{(1 - 2^{-s})^{-1}} = 1 - 2^{-s} + 2 \frac{2^{-s} - 2^{-2s}}{1 - 2^{-s}} - 3 \frac{2^{-(s+1)} - 2^{-(2s+1)}}{1 - 2^{-(s+1)}} \]

which is clearly analytic in the desired region.

The second expression is

\[ \prod_{p \equiv 3(4)} \frac{Q(s)}{(1 - p^{-s})^{-1}} = \prod_{p \equiv 3(4)} \left( 1 - p^{-s} + \frac{p^{-s}}{1 - p^{-1}} - \frac{p^{-(s+2)} - p^{-(2s+2)}}{(1 - p^{-1}) (1 - p^{-(s+1)})} \right). \]

Notice that

\[ 1 - p^{-s} + \frac{p^{-s}}{1 - p^{-1}} - \frac{p^{-(s+2)} - p^{-(2s+2)}}{(1 - p^{-1}) (1 - p^{-(s+1)})} = 1 + O \left( \frac{1}{p^{s+1}} \right) \]

and so the product

\[ \prod_{p \equiv 3(4)} \frac{Q(s)}{(1 - p^{-s})^{-1}} = \prod_{p \equiv 3(4)} \left( 1 + O \left( \frac{1}{p^{s+1}} \right) \right) \]

converges in the desired region \( \sigma > 0 \) in which it is analytic, implying that \( \frac{D(s)}{\zeta(s)} \) is also analytic there.

Let \( A(s) \) be an analytic function in \( \sigma > 0 \) defined by \( D(s) = A(s) \zeta(s) \). Since \( \text{Res}_{s=1} \zeta(s) = 1 \), in order to compute \( \text{Res}_{s=1} D(s) \) we can simply compute \( A(1) \) and so

\[ \text{Res}_{s=1} D(s) = A(1) = \prod_{p \equiv 3(4)} \frac{1}{1 - \frac{1}{p^2}} = 2 \beta^2. \]
2.3. Proof of Theorem 1. We write \( D(s) = \zeta(s)A(s) \) where \( A(s) \) is absolutely convergent for \( \sigma > 0 \) and so is bounded. We need the following version of Perron’s formula (see for example [12]): If
\[
D(s) = \sum_{n=1}^{\infty} a(h)h^{-s}
\]
is absolutely convergent for \( \sigma > 1 \), then
\[
\sum_{1 \leq h \leq H} a(h)(H-h) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{D(s)}{s(s+1)} H^{s+1} ds.
\]

Applying Perron’s formula in our case, we are left with evaluating the contour integral. We want to shift the contour of integration to \( \sigma = \sigma_0 \), with \( 0 < \sigma_0 < 1 \), and so we need to bound \( D(s) \) in this region. First notice that \( |A(\sigma+it)| \leq C(\sigma) \) is bounded as it is given by an absolutely convergent product in \( \sigma > 0 \). In order to bound \( \zeta(s) \) we use the classical convexity bound (see [12, Chap. II.3])
\[
|\zeta(\sigma+it)| \ll \varepsilon (1 + |t|)^{1/2 - \sigma + \varepsilon}, \quad 0 \leq \sigma \leq 1, |t| > 1.
\]
for all \( \varepsilon > 0 \). Hence the integrand is bounded by
\[
(2.1) \quad \left| \frac{D(s)}{s(s+1)} H^{s+1} \right| \ll \sigma \left\{ \begin{array}{ll}
H^{\sigma+1}, & |t| \leq 1, 0 < \sigma < 1 \ (a) \\
H^{\sigma+1} |t|^{-2+1/2 - \sigma + \varepsilon}, & |t| > 1, 0 < \sigma \leq 1 \ (b)
\end{array} \right.
\]
and so by shifting contour using bound (b) and picking up a residue from the simple pole of \( \zeta(s) \) at \( s = 1 \) (recall that \( \text{Res}_{s=1} \zeta(s) = 1 \)) we have
\[
\sum_{1 \leq h \leq H} a(h)(H-h) = \frac{A(1)}{2} H^2 + \frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{D(s)}{s(s+1)} H^{s+1} ds.
\]

Applying the bounds (a) and (b) in (2.1) allows us to bound the integral by \( O(H^{\sigma_0+1}) \). In conclusion we find
\[
\sum_{1 \leq h \leq H} a(h)(H-h) = \beta^2 H^2 + O_\varepsilon(H^{1+\varepsilon})
\]
for all \( \varepsilon > 0 \). Therefore
\[
\sum_{1 \leq d \neq d \leq H} \mathcal{T}_d = \sum_{1 \leq h \leq H-1} (H-h)a(h) = \beta^2 H^2 + O_\varepsilon(H^{1+\varepsilon})
\]
which is effectively Gallagher’s Lemma for sums of two squares and \( k = 2 \).

3. Sums of Squares in Residue Rings

Following Keating and Connors we attempt to produce a 2–tuple conjecture using essentially heuristic methods and Landau’s theorem. The key step is to reduce our problem to prime power residue rings, a step which is made possible by Lemma 4 presented below.
3.1. Representable elements in residue rings.

**Proposition 3.** Denote by $Sq(p,k)$ the set of elements representable as a sum of two squares in $\mathbb{Z}/p^k\mathbb{Z}$

$$Sq(p,k) = \{a \in \mathbb{Z}/p^k\mathbb{Z} | a \text{ representable as a sum of two squares}\}.$$ 

The following holds:

(a) For $p \equiv 1 \pmod{4}$, $Sq(p,k) = \mathbb{Z}/p^k\mathbb{Z}$.
(b) For $p \equiv 3 \pmod{4}$, $Sq(p,k) = \{a \in \mathbb{Z}/p^k\mathbb{Z} | m_p(a) \text{ is even or } a = 0\}$.
(c) For $p = 2$, $Sq(2,k) = \{a \in \mathbb{Z}/2^k\mathbb{Z} | a = 2^j(1 + 4n), 0 \leq j \leq k - 1 \text{ or } a = 0\}$.

Detailed proofs for Proposition 3 and the other propositions presented in this section can be found in [11], and they can also be deduced from Lemma A.2 in [3].

**Lemma 4.** An integer is representable as a sum of two squares if and only if it is representable as a sum of two squares in $\mathbb{Z}/p^k\mathbb{Z}$ for every prime $p$ and integer $k \in \mathbb{N}$.

**Proof.** This is a corollary of Proposition 3 and of the famous classical result that an integer $n$ is as sum of two squares if and only if $m_p(a)$ is even for all $p \equiv 3 \pmod{4}$. Say $a = x^2 + y^2$, so obviously $a \equiv x^2 + y^2 \pmod{p^k}$. Conversely assume $a$ is not representable hence $m_p(a)$ is odd for some $p \equiv 3 \pmod{4}$ and so $a$ is not representable in $\mathbb{Z}/p^k\mathbb{Z}$ for $k \geq m_p(a)$. \[\square\]

Equipped with this lemma we shall examine $\mathbb{Z}/p^k\mathbb{Z}$ for all primes $p$ and $k \in \mathbb{N}$, and determine which are the representable elements in these residue rings. This will allow us to give an expression for the the density of representable elements, and then of representable pairs.

3.2. Mean density of representable elements in residue rings. We now wish to calculate the densities of representable elements in $\mathbb{Z}/p^k\mathbb{Z}$ for all primes, $k \to \infty$. The following propositions provide a method for deriving these limits, and present ideas which can be useful also for calculating correlations of higher degrees.

**Proposition 5.** Denote by $Med(p)$ the limit of the mean density of representable elements in $\mathbb{Z}/p^k\mathbb{Z}$ as $k \to \infty$

$$Med(p) = \lim_{k \to \infty} \frac{\#Sq(p,k)}{p^k}.$$ 

The following holds:

(a) For $p \equiv 1 \pmod{4}$, $Med(p) = 1$.
(b) For $p \equiv 3 \pmod{4}$, $Med(p) = (1 + p^{-1})^{-1}$.
(c) For $p = 2$, $Med(2) = \frac{1}{4}$. 

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4. Ratio Between the Product of Densities and Landau’s Result

4.1. Density of integers representable as a sum of two squares. We wish to calculate the mean density of integers representable as a sum of two squares, so following our approach we take the product of all the above densities for \( p \leq n \):

\[
\mathcal{M}(n) := \prod_{p \leq n} Med(p) = \frac{1}{2} \prod_{p \equiv 3 (4) \atop p \leq n} (1 + p^{-1})^{-1}.
\]

Even though we do not expect that this expression \( \mathcal{M}(n) \) will give us the correct asymptotics, we will show that as for the case of the primes this Mertens-type product provides the correct answer up to some constant, and this constant will show a universal property we will see in Section 6. The leading term in Landau’s analytic expression for the mean density of representable integers is

\[
\mathcal{L}(n) = \frac{\beta}{\sqrt{\log n}}.
\]

The events that an integer is representable in residue rings associated with different primes show some dependency, a dependency which gives rise to a term \( y(n) \). Taking this term into consideration we should have

\[
\mathcal{L}(n) \sim \frac{\mathcal{M}(n)}{y(n)}.
\]

4.2. The Ratio. Mertens’ original formula states that

\[
\prod_{p \leq n} (1 - p^{-1}) = \frac{e^{-\gamma}}{\log n} + O \left( \frac{1}{\log^2 n} \right)
\]

where \( \gamma \) denotes Euler’s constant.

For co-prime integers \( a, q \), Languasco and Zaccagnini show [8] a generalization of Mertens’ formula

\[
\lim_{n \to \infty} (\log n)^{1/\varphi(q)} \prod_{p \equiv a \atop p \leq n} (1 - p^{-1}) = \left[ e^{-\gamma} \prod_{p \atop p \leq n} (1 - p^{-1})^{a(p; a, q)} \right]^{1/\varphi(q)}
\]

where \( \varphi \) is Euler’s totient function, and \( a(p; a, q) \) is given by

\[
a(p; a, q) = \begin{cases} 
\varphi(q) - 1 & , p \equiv a \ (q) \\
-1 & , \text{otherwise}
\end{cases}
\]

**Theorem 6.** Let \( y(n) = \frac{\mathcal{M}(n)}{\mathcal{L}(n)} \) be the ratio between the product of densities in prime power residue rings and Landau’s leading term. Then \( y(n) \) converges
as $n$ tends to infinity and the limit is given by
\[
y := \lim_{n \to \infty} y(n) = \lim_{n \to \infty} \frac{M(n)}{\mathcal{L}(n)} = \frac{1}{2} \sqrt{\frac{\pi}{e^\gamma}}.
\]

Proof. Plugging $a = 3$, $q = 4$ in Mertens’ formula for primes in arithmetic progression we have
\[
\prod_{p \equiv 3 (4)} (1 - p^{-1}) \sim \frac{e^{-\gamma/2}}{\sqrt{\log n}} \prod_{p \equiv 3 (4)} \left(1 - \frac{1}{p-1}\right) \prod_{p \equiv 1 (4)} \left(1 - \frac{1}{p-1}\right)^{1/2}
\]
and since
\[
(1 + p^{-1})^{-1} = \frac{1 - p^{-1}}{1 - p^{-2}}
\]
we arrive at
\[
\prod_{p \equiv 3 (4)} (1 + p^{-1})^{-1} \sim \frac{2e^{-\gamma/2}}{\sqrt{\log n}} \prod_{p \equiv 3 (4)} (1 - p^{-1})^{-1/2} (1 + p^{-1})^{-1} \prod_{p \equiv 1 (4)} (1 - p^{-1})^{-1/2}.
\]

We are interested in the ratio
\[
\lim_{n \to \infty} \frac{M(n)}{\mathcal{L}(n)} = \lim_{n \to \infty} \frac{M(n)}{\beta/\sqrt{\log n}}
\]
with $\beta$ the Landau-Ramanujan constant given by
\[
\beta = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 (4)} (1 - p^{-2})^{-1/2}
\]
and so
\[
\lim_{n \to \infty} \frac{M(n)}{\mathcal{L}(n)} = \frac{1}{2} \cdot 2e^{-\gamma/2} \prod_{p \equiv 3 (4)} (1 + p^{-1})^{-1/2} \prod_{p \equiv 1 (4)} (1 - p^{-1})^{-1/2}.
\]

The two products are exactly $\sqrt{L(1)}$ which is calculated in [10], where $L(s)$ is the Dirichlet series for the non principal character modulo 4. Therefore
\[
y = \lim_{n \to \infty} \frac{M(n)}{\mathcal{L}(n)} = e^{-\gamma/2} \sqrt{\frac{\pi}{4}} = \frac{1}{2} \sqrt{\frac{\pi}{e^\gamma}}.
\]

This is quite an elegant result, which can be easily generalized using similar tools as will be done in section 6.
5. Representable Pairs In Residue Rings And Their Densities

We are now in a position to look at the distribution of representable pairs \( a, a + h \) in \( \mathbb{Z}/p^k\mathbb{Z} \). The following proposition states the densities of representable pairs, for a detailed proof see [11].

Proposition 7. Denote by \( \text{Med}^{(2)}(p, h) \) the limit of the mean density of representable pairs \( (a, a + h) \) in \( \mathbb{Z}/p^k\mathbb{Z} \) as \( k \to \infty \)

\[
\text{Med}^{(2)}(p, h) = \lim_{k \to \infty} \frac{\# \{a, a + h \in \text{Sq}(p, k)\}}{p^k}.
\]

The following holds:
(a) For \( p \equiv 1 (4) \), \( \text{Med}^{(2)}(p, h) = 1 \).
(b) For \( p \equiv 3 (4) \), \( \text{Med}^{(2)}(p, h) = \frac{1 - p^{-(m_2(h)+1)}}{1 + p^{-1}} \).
(c) For \( p = 2 \), \( \text{Med}^{(2)}(2, h) = W_2(h) = \left\{ \begin{array}{ll} 1 & m_2(h) = 0 \\ \frac{1 - 2^{m_2(h)-1}}{2^{m_2(h)+2}} & m_2(h) \geq 1 \end{array} \right. \).

We define \( \mathcal{M}^{(2)}(n, h) \) to be the product of the above densities

\[
\mathcal{M}^{(2)}(n, h) = \prod_{p \leq n} \text{Med}^{(2)}(p, h) = W_2(h) \prod_{p \equiv 3 (4)} \prod_{p \leq n} \frac{1 - p^{-(m_2(h)+1)}}{1 + p^{-1}}.
\]

And as in the introduction we denote

\[
B_h(n) = \# \{m \leq n \mid m, m + h \text{ are representable as a sum of two squares}\}
\]

and define

\[
Y_h(n) := \frac{\mathcal{M}^{(2)}(n, h)}{\frac{1}{n}B_h(n)}.
\]

The density of representable pairs is thus:

\[
\frac{1}{n}B_h(n) \sim \frac{1}{Y_h(n)} \cdot W_2(h) \prod_{p \equiv 3 (4)} \prod_{p \leq n} \frac{1 - p^{-(m_2(h)+1)}}{1 + p^{-1}}.
\]

We extract the asymptotic term depending on \( n \) from the above expression using the ratio computed in Section 4. Recall

\[
\mathcal{L}(n) = \beta \frac{\log n}{\sqrt{2 \log n}} = \frac{\prod_{p \equiv 3 (4)} (1 - p^{-2})^{-\frac{1}{2}}}{\sqrt{\log n}}
\]

and so from Landau’s Theorem together with the previous sections we have for all \( n \)

\[
1 = \left( \frac{\mathcal{L}(n)}{\mathcal{M}(n)/y(n)} \right)^2 = \frac{\prod_{p \equiv 3 (4)} (1 - p^{-2})^{-1}}{\log n} \prod_{p \leq n} (1 + p^{-1})^{-2}
\]
and so we write
\[ \frac{1}{n} B_h(n) \sim \frac{1}{Y_h(n)} \cdot W_2(h) \prod_{\substack{p \equiv 3 \pmod{4} \atop p \leq n}} \frac{1 - p^{-(m_p(h)+1)}}{1 + p^{-1}} \cdot \left( \frac{L(n)}{M(n)/y(n)} \right)^2. \]

Since
\[ \frac{(1 - p^{-(m_p(h)+1)})}{(1 + p^{-1})(1 + p^{-2})^{-1}} = \frac{1 - p^{-(m_p(h)+1)}}{1 - p^{-1}}, \]
we have
\[ (5.1) \quad \frac{1}{n} B_h(n) \sim \frac{1}{\log n} \cdot \left( \frac{y(n)^2}{Y_h(n)} \right) \cdot 2W_2(h) \prod_{\substack{p \equiv 3 \pmod{4} \atop p \leq n}} \frac{1 - p^{-(m_p(h)+1)}}{1 - p^{-1}}. \]

For \( p \) such that \( m_p(h) = 0 \) the product is 1, and since we are interested in \( n \to \infty \), we can assume \( n \geq h \) and so the product is over all \( p \equiv 3 \pmod{4} \) such that \( p \mid h \). The conjecture presented by Connors and Keating is thus equivalent to the conjecture that for all \( h \)
\[ \lim_{n \to \infty} \frac{y(n)^2}{Y_h(n)} = 1 \]
as \( n \to \infty \), which can be also stated as
\[ \lim_{n \to \infty} Y_h(n) = \frac{1}{4} \frac{\pi}{e} = y^2 \text{ for all } h \in \mathbb{N}, \]
a conjecture for which we present numerical computations in Section 7. Another interpretation of this conjecture would be that as in the case of the density of integers representable as a sum of two squares, the product expression \( M^2(n, h) \) gives the correct estimate up to a constant.

Assuming the validity of this conjecture the density of representable pairs is given by
\[ \frac{1}{n} B_h(n) = \frac{1}{\log n} \cdot 2W_2(h) \prod_{\substack{p \equiv 3 \pmod{4} \atop p \mid h}} \frac{1 - p^{-(m_p(h)+1)}}{1 - p^{-1}} \]
and so
\[ \mathcal{F}_h = 2W_2(h) \prod_{\substack{p \equiv 3 \pmod{4} \atop p \mid h}} \frac{1 - p^{-(m_p(h)+1)}}{1 - p^{-1}}. \]

6. Generalization to Other Binary Quadratic Forms

In this section we generalize our conjectures and results for additional binary quadratic forms.
6.1. Preliminaries. Let us look at the following family of positive definite binary quadratic forms

\[ q(d; x, y) = x^2 + dy^2. \]

**Definition 8.** We say that \( d \in \mathbb{N} \) is a convenient (idoneal) number if there is finite set of primes \( S \), an integer \( N \) and congruence classes \( c_1, \ldots, c_k \mod N \) such that for all primes \( p \notin S \)

\[ p = x^2 + dy^2 \iff p \equiv c_1, \ldots, c_k \mod N. \]

**Example 9.** For \( d = 1 \), \( S = \{2\} \), \( c_1 = 1 \) and \( N = 4 \) we have Fermat’s result for sums of two squares.

We focus here on convenient \( d \)'s such that the form \( x^2 + dy^2 \) is of class number 1 which are \( d = 1, 2, 3, 4, 7 \). In these cases one can fully determine if an integer \( n \) is representable by the form simply by making sure that the primes which are not representable appear with an even multiplicity in the integer’s prime factorization.

Again we are first interested in the mean density of representable integers, and we can calculate the densities in the residue rings in the exact same way that we did for \( d = 1 \) and thus generalize (4.1). In [9] Shanks produces Landau’s constants \( \beta_1, \beta_2, \beta_3, \beta_4, \beta_7 \) for which

\[ B(d, n) := \# \{ m \leq n \mid \text{mis of the form } x^2 + dy^2 \} \sim \beta_d \frac{n}{\sqrt{\log n}} \]

as \( n \to \infty \), and so we can again calculate the ratio between the product and the analytic expressions as was done in Section 4 for sums of squares.

First let us recall the following classical results (see [2]):

**Theorem 10.** An integer \( n \) is representable by the form \( x^2 + dy^2 \) if and only if:

- If \( d = 1 \), \( m_p(n) \) is even for all primes \( p \equiv 3 \mod 4 \).
- If \( d = 2 \), \( m_p(n) \) is even for all primes \( p \equiv 5, 7 \mod 8 \).
- If \( d = 3 \), \( m_p(n) \) is even for all primes \( p \equiv 2 \mod 3 \).
- If \( d = 4 \), \( m_p(n) \) is even for all primes \( p \equiv 3 \mod 4 \) and \( m_2(n) \neq 1 \).
- If \( d = 7 \), \( m_p(n) \) is even for all primes \( p \equiv 3, 5, 6 \mod 7 \) and \( m_2(n) \neq 1 \).

The conditions for representation by these forms bare obvious resemblance.

**Definition 11.** For convenience reasons we divide the primes into the following sets:

- Say \( p \in Q_d \) if \( p \) is a prime such that \( n = x^2 + dy^2 \Rightarrow m_p(n) \) is even. Notice that by Theorem 10 and by the Prime Number Theorem for Arithmetic Progressions this set consists of approximately half of the primes.
- Say \( p \in R_d \) if \( p \) is a prime such that \( p \notin Q_d \) and \( n = x^2 + dy^2 \Rightarrow m_p(n) \) has some constraint as described in Theorem 10 or if \( p \) is such that \( (p, N) \neq 1 \) where \( N \) is as described in Definition 8 that is \( N \) such
that $p = x^2 + dy^2 \iff p \equiv c_1, \ldots, c_k \mod N$. Notice that $R_d$ is a finite set.

- say $p \in P_d$ if $p$ is a prime such that $p \notin Q_d \cup R_d$, or more directly if $p$ is of the form $x^2 + dy^2$ and $(p, N) = 1$. Again this set consists of approximately half of the primes.

Example 12. For the case of sums of squares, that is $d = 1$, we write

$$Q_1 = \{p \text{ prime: } p \equiv 3 \mod 4\}, P_1 = \{p \text{ prime: } p \equiv 1 \mod 4\},$$
\[R_1 = \{2\} \]

For $d = 7$ we write

$$Q_7 = \{p \text{ prime: } p \equiv 3, 5, 6 \mod 7\}, P_7 = \{p \text{ prime: } p \equiv 1, 2, 4 \mod 7, p \neq 2\},$$
\[R_7 = \{2, 7\}\]

It is important to note that the reason we define the sets of primes $Q_d, R_d, P_d$ the way we do and not by the values of $\left(\frac{-d}{p}\right)$, which stands for the Legendre symbol, is that the Legendre symbol is only defined for odd primes $p$, while the prime $p = 2$ plays an important role in our computations. On the other hand it will be useful for us to notice that for $d = 1, 2, 3, 4, 7$ indeed

$$\left(\frac{-d}{p}\right) = 1 \iff p \in P_d$$

and

$$\left(\frac{-d}{p}\right) = -1 \iff p \in Q_d$$

unless $d = 3$, in which case $Q_3 = \{p : \left(\frac{-d}{p}\right) = -1\} \cup \{2\}$.

6.2. Ratio between the product density and Landau’s density. We continue by following the same methods established in Sections 3 and 4 for the definition of $\mathcal{M}(n)$ in order to define a product expression $\mathcal{M}_d(n)$ associated with the mean density of integers of the form $x^2 + dy^2$. Notice that for all the above $d$’s the condition for being representable by the form is over the primes in $Q_d$, plus some local conditions over the primes in $R_d$. Similarly to what we have done in the previous sections we define the naive expectation of the density of integers representable by the form $x^2 + dy^2$ as

$$\mathcal{M}_d(n) = \prod_{p \in R_d} w_d(p) \prod_{p \in Q_d, p \leq n} (1 + p^{-1})^{-1}$$

with $w_d(p)$ the mean density of representable element in $\mathbb{Z}/p^k\mathbb{Z}$, $k \to \infty$, for $p \in R_d$. The primes $p \in P_d$ do not participate here since similarly to the case of sums of two squares, the mean density of representable elements in $\mathbb{Z}/p^k\mathbb{Z}$, $k \to \infty$, is 1.
These products can be computed using Mertens’ formula for arithmetic progressions, as was done in the previous section for \( d = 1 \):

\[
\prod_{\substack{p \in \mathbb{Q}_d \\ p \leq n}} (1 + p^{-1})^{-1} \sim \frac{e^{-\gamma/2}}{\sqrt{\log n}} \prod_{p \in \mathbb{Q}_d} (1 - p^{-1})^{-\frac{1}{2}} (1 + p^{-1})^{-1} \prod_{p \in P_d \cup R_d} (1 - p^{-1})^{-\frac{1}{2}}.
\]

Again we are interested in the analogue of (1.10), that is in the ratio between these products and the leading term of the analytic expression given by the generalization of Landau’s theorem as shown in [9]:

\[(6.1) \quad L_d(n) = \frac{\beta_d}{\sqrt{\log n}}, \quad \beta_d = \delta_d \cdot g_d \cdot \left( \frac{L_d(1) \cdot 2 |d|}{\pi \varphi(2 |d|)} \right)^{\frac{1}{2}}\]

with \( \varphi \) the Euler totient function and

\[
g_d = \prod_{\left( \frac{d}{p} \right) = -1} (1 - p^{-2})^{-\frac{1}{2}}
\]

\[
L_d(s) = \sum_{\text{odd } n} \left( \frac{-d}{n} \right) n^{-s} = \prod_{\left( \frac{d}{p} \right) = 1} (1 - p^{-s})^{-1} \prod_{\left( \frac{d}{p} \right) = -1} (1 + p^{-s})^{-1}
\]

\[
\delta_d = \begin{cases} 
1 & , d = 1, 2 \\
\frac{2}{3} & , d = 3 \\
\frac{3}{4} & , d = 4, 7
\end{cases}
\]

Reformulating the products above we have

\[
g_d = \prod_{2 \neq p \in \mathbb{Q}_d} (1 - p^{-2})^{-\frac{1}{2}} = \gamma_d \prod_{p \in \mathbb{Q}_d} (1 - p^{-2})^{-\frac{1}{2}}
\]

where \( \gamma_d = \begin{cases} 
1 & , d = 1, 2, 4, 7 \\
\frac{\sqrt{3}}{2} & , d = 3
\end{cases} \), and

\[
\sqrt{L_d(1)} = \prod_{p \in P_d} (1 - p^{-1})^{-\frac{1}{2}} \prod_{2 \neq p \in \mathbb{Q}_d} (1 + p^{-1})^{-\frac{1}{2}}
\]

\[
= \prod_{p \in P_d} (1 - p^{-1})^{-\frac{1}{2}} \lambda_d \prod_{p \in \mathbb{Q}_d} (1 + p^{-1})^{-\frac{1}{2}}
\]
where $\lambda_d = \begin{cases} 
1 & , d = 1, 2, 4, 7 
\sqrt{\frac{3}{2}} & , d = 3 
\end{cases}$.

The ratio in question is therefore given by

\[
\lim_{n \to \infty} y_d(n) = \lim_{n \to \infty} \frac{\mathcal{M}_d(n)}{\mathcal{L}_d(n)} = \lim_{n \to \infty} \frac{\mathcal{M}_d(n)}{\beta_d / \sqrt{\log n}} = \prod_{p \in R_d} w_d(p) \left( \frac{1}{m} \cdot \sqrt{\frac{\pi}{e^\gamma}} \cdot \sqrt{\frac{\varphi(2d)}{2d^3}} \right) \frac{1}{\gamma_d \lambda_d} \prod_{p \in R_d} (1 - p^{-1})^{-\frac{1}{2}}.
\]

Recall $\frac{\varphi(n)}{n} = \prod_{p \mid n} (1 - p^{-1})$. For $d = 1, 2, 4, 7$ we have $p | 2d \iff p \in R_d$ and so the products cancel each other. For $d = 3$ we have $2 | 2d$ and $2 \notin R_3$, so we are left with the term $(1 - 2^{-1})^{\frac{1}{2}} = \frac{1}{\sqrt{2}}$. Since $\frac{1}{\sqrt{2 \sqrt{3 \sqrt{3}}} = \frac{2}{3}}$ we can write

\[
\lim_{n \to \infty} y_d(n) = \prod_{p \in R_d} w_d(p) \left( \frac{1}{m} \cdot \sqrt{\frac{\pi}{e^\gamma}} \cdot \delta_d \cdot s_d \right)
\]

where $s_d = \begin{cases} 
1 & , d = 1, 2, 4, 7 
\frac{2}{3} & , d = 3 
\end{cases}$.

Computing case by case we prove the following theorem:

**Theorem 13.** For $d = 1, 2, 3, 4, 7$ the ratio between the product of densities in the residue rings and Landau’s density of integers representable by the forms $x^2 + dy^2$ converges to $\frac{1}{2} \sqrt{\frac{\pi}{e^\gamma}}$ as $n \to \infty$, that is

\[
\lim_{n \to \infty} y_d(n) = \lim_{n \to \infty} \frac{\mathcal{M}_d(n)}{\mathcal{L}_d(n)} = \frac{1}{2} \sqrt{\frac{\pi}{e^\gamma}} = y.
\]

This is quite a surprising result, which makes the constant $y = \frac{1}{2} \sqrt{\frac{\pi}{e^\gamma}}$ somewhat universal as the ratio between the density of integers representable by the forms at hand and the naively constructed Mertens-type products we have presented.

### 6.3. Pair correlation conjecture.

We can now propose a conjecture for the pair correlation function for the forms $x^2 + dy^2$ with $d = 1, 2, 3, 4, 7$, generalizing [1.6] and [1.4]. Denote by $W_{d,p}(h)$ the density of representable pairs $(a, a + h)$ in $\mathbb{Z}/p^k \mathbb{Z}$, $k \to \infty$, for $p \in R_d$, and $Y_{d,h}(n)$ the dependance term which must be taken into consideration. We extract the asymptotic term depending on $n$ exactly as was done in Section [5]
\[
\frac{1}{Y_{d,h}(n)} \prod_{p \in R_d} W_{d,p}(h) \prod_{p \in Q_d, p \leq n} \frac{1 - p^{-(m_p(h)+1)}}{1 + p^{-1}} \\
\sim \frac{1}{Y_{d,h}(n)} \prod_{p \in R_d} W_{d,p}(h) \prod_{p \in Q_d, p \leq n} \frac{1 - p^{-(m_p(h)+1)}}{1 + p^{-1}} \left( \frac{L_d(n)}{\mathcal{M}_d(n)/y_d(n)} \right)^2 \\
\sim \frac{1}{\log n} \left( \frac{y_d^2(n)}{Y_{d,h}(n)} \right) \prod_{p \in R_d} W_{d,p}(h) \prod_{p \in Q_d, p \leq n} \frac{1 - p^{-(m_p(h)+1)}}{1 + p^{-1}} \cdot \frac{(1 - p^{-2})^{-1}}{2 \neq p \in Q_d} \\
\sim \prod_{p \in Q_d, p \mid h} \frac{1 - p^{-(m_p(h)+1)}}{1 - p^{-1}} \cdot S_d
\]

where \( S_d = \begin{cases} 
1, & d = 1, 2, 4, 7 \\
\frac{1}{4}, & d = 3
\end{cases} \).

Again the conjecture is that for all \( h \)

\[
\frac{y_d^2(n)}{Y_{d,h}(n)} \to 1
\]
as \( n \to \infty \), and so assuming the validity of this conjecture the density of pairs of the form \( x^2 + dy^2 \) is given by

\[
\frac{1}{n} B_h(d, n) = \frac{1}{\log n} \cdot \mathcal{J}_{d,h}
\]

where

\[
\mathcal{J}_{d,h} = c_d \prod_{p \in R_d} W_{d,p}(h) \prod_{p \in Q_d, p \mid h} \frac{1 - p^{-(m_p(h)+1)}}{1 - p^{-1}}
\]

and

\[
c_d = \frac{\delta^2}{\pi \varphi(2 \mid u)} \prod_{p \in R_d} \frac{1}{w^2_d(p)} S_d
\]

It is left to compute \( c_d \) and \( W_{d,p}(h) \) case by case. Dirichlet’s class number formula (see [10]) gives

\[
L_1(1) = \frac{\pi}{4}, \quad L_2(1) = \frac{\pi}{2 \sqrt{2}}, \quad L_3(1) = \frac{\pi}{2 \sqrt{3}}, \quad L_4(1) = \frac{\pi}{4}, \quad L_7(1) = \frac{\pi}{2 \sqrt{7}}
\]
and so plugging all the different term we have

\[ (6.4) \quad c_1 = 2, \quad c_2 = 2\sqrt{2}, \quad c_3 = \frac{2}{\sqrt{3}}, \quad c_4 = 2, \quad c_7 = \frac{2\sqrt{7}}{3}. \]

In addition, calculations similar to those shown for sums of squares in Section 4 give

\[
W_{1,2}(h) = \begin{cases} 
\frac{1}{4} & , m_2(h) = 0 \\
\frac{1}{2^{m_2(h)+1}} - 3 & , m_2(h) \geq 1
\end{cases}, \quad W_{2,2}(h) = \begin{cases} 
\frac{1}{4} & , m_2(h) = 0, 1 \\
\frac{1}{2^{m_2(h)+1}} - 3 & , m_2(h) \geq 2
\end{cases}, \\
W_{3,3}(h) = \frac{1}{2} \cdot \frac{3^{m_3(h)+1} - 2}{3^{m_3(h)+1}}, \quad W_{4,2}(h) = \begin{cases} 
\frac{1}{8} & , m_2(h) = 0 \\
0 & , m_2(h) = 1 \\
\frac{4}{16} & , m_2(h) = 2 \\
\frac{3^{m_2(h)-1} - 3}{2^{m_2(h)+1}} & , m_2(h) \geq 3
\end{cases}, \\
W_{7,2}(h) = \begin{cases} 
\frac{1}{4} & , m_2(h) = 0, 1 \\
\frac{3}{4} & , m_2(h) \geq 2
\end{cases}, \quad W_{7,7}(h) = \frac{1}{2} \cdot \frac{7^{m_7(h)+1} - 4}{7^{m_7(h)+1}}.
\]

6.4. Distribution in short intervals - the second moment. We wish to generalize our result from Section 2 concerning the second moments of the distribution of representable integers in short intervals.

We are interested in the distribution of values of \( B(d, m + \alpha_d) - B(d, m) \), which stands for the number of integers of the form \( x^2 + dy^2 \) in the interval \((m, m + \alpha_d)\), for \( m \leq n \) and \( \alpha_d \sim \frac{\lambda}{\beta_d} \sqrt{\log n} \). Assuming \((6.2)\) we wish to show that the second moment of this distribution is consistent with a Poissonian distribution with parameter \( \lambda \), and so we prove Gallagher’s Lemma for integers of the form \( x^2 + dy^2 \), \( d = 1, 2, 3, 4, 7 \) and \( k = 2 \).

**Theorem 14.** The singular series \( \mathcal{T}_{d,h} \) has mean value \( \beta_d \) for \( d = 1, 2, 3, 4, 7 \) as defined in \((6.7)\). More explicitly

\[ \sum_{1 \leq d_1 \neq d_2 \leq H} \mathcal{T}_{d_1,h} = 2 \sum_{1 \leq h \leq H-1} (H-h) \mathcal{T}_{d,h} = \beta_d^2 H^2 + O_\varepsilon(H^{1+\varepsilon}) \]

as \( H \to \infty \), for all \( \varepsilon > 0 \).

**Proof.** We follow the proof described in Section 2. First we normalize \( \mathcal{T}_{d,h} \) by defining \( a_d(h) = \frac{\mathcal{T}_{d,h}}{\mathcal{T}_{d,1}} \), which is now multiplicative. Then we show that the corresponding Dirichlet series \( D_d(s) \) has a simple pole at \( s = 1 \) with residue \( \frac{\beta_d}{\mathcal{T}_{d,1}} \), as was done for sums of squares. Following the exact same steps detailed in Section 2 we have

\[ D_d(s) = R_d(s) P_d(s) Q_d(s) \]

where
\[ R_d(s) = \prod_{p \in R_d} \left( 1 + \sum_{k=1}^{\infty} \frac{a_d(p^k)}{p^{ks}} \right) \]

\[ P_d(s) = \prod_{p \in P_d} (1 - p^{-s})^{-1} \]

\[ Q_d(s) = \prod_{p \in Q_d} \left( 1 + \frac{1}{1 - p^{-1}} \frac{p^{-s}}{1 - p^{-1}} - \frac{p^{-1}}{1 - p^{-1}} \right) \]

It can be shown \( D_d(s) = A_d(s)\zeta(s) \) with \( A_d(s) \) analytic, and so to calculate the residue of \( D(s) \) at \( s = 1 \) it is left to calculate \( A_d(1) \) which gives

\[ A_d(1) = \lim_{s \to 1} \frac{D_d(s)}{\zeta(s)} = \prod_{p \in R_d} \frac{1 + \sum_{k=1}^{\infty} \frac{a_d(p^k)}{p^{ks}}}{(1 - p^{-1})^{-1}} \prod_{p \in Q_d} (1 - p^{-2})^{-1}. \]

Recall

\[ \beta_d^2 = \delta_d^2 \frac{L_d(1) \cdot 2|d|}{\pi \varphi(2|d|)} \prod_{p \in Q_d} (1 - p^{-2})^{-1} \]

and so it remains to show that indeed for \( d = 1, 2, 4, 7 \)

\[ \prod_{p \in R_d} \frac{1 + \sum_{k=1}^{\infty} \frac{a_d(p^k)}{p^{ks}}}{(1 - p^{-1})^{-1}} = \frac{\delta_d^2 L_d(1) \cdot 2|d|}{\pi \varphi(2|d|)} \]

We continue exactly as was detailed in Section 2 for sums of squares. Plugging in all the relevant constants, all computed above, we arrive at the desired result.

\[ \square \]

7. Numerical Computations

The approach taken in [1] as well as ours to the pair correlation conjecture for integers representable as the sum of two squares, stated in (1.6), is essentially heuristic, and so some numerical computations are in place in order to support our conjecture. The conjecture as stated here is that as \( n \to \infty \)

\[ \frac{y(n)^2}{Y_h(n)} \to 1 \]

which, as shown in (5.1), can be calculated by taking the ratio between the numeric density of pairs and the conjectured pair correlation function:

\[ \frac{y(n)^2}{Y_h(n)} = \frac{1}{\log n} \cdot \frac{1}{\# \{ m \leq n \mid m \text{ and } m + h \text{ are representable} \}} \cdot 2W_2(h) \prod_{\substack{p \equiv 3 \pmod{4} \\ p \not| h}} \frac{1 - p^{-(m_p(h) + 1)}}{1 - p^{-1}} \]
In Figure 7.1 we present some calculations of this ratio for various $h$:

\[ y(n) \]

Figure 7.1. $\frac{y(n)^2}{\gamma_k(n)}$ for $1 \leq h \leq 25$ at $n = 10^6, 10^8$

Examining different values of $h$ for which the primes $2$ and $p \equiv 3 \pmod{4}$ appear with equal multiplicity, such as $h = 1, 5, 17, 25$ or $h = 4, 20$, one can see they take very similar values. This was checked for many more values of $h$ which are not shown here and so strengthens our belief that the pair correlation depends only on the multiplicity of these primes in $h$.

One can also see that the fluctuations between different values of $h$ diminish for larger $n$, where the peaks in the above graph are obtained at values of $h$ for which $m_2(h) = 1, 2$ or $m_3(h) = 1$, since the small primes are the most dominant in our computations.

We must not be discouraged by the extremely slow decay to 1, for it is consistent with the large error term which appears in Landau’s theorem in (1.3). In fact the convergence implied in Landau’s theorem, or more precisely

\[ \beta(n)^2 = \left( \frac{\# \{m \leq n \mid m \text{ is representable} \}}{\beta \frac{n}{\sqrt{\log n}}} \right)^2 \to 1 \]

as $n \to \infty$, shows similar behavior as shown in Figure 7.2, in which the values for the ratio $\frac{y(n)^2}{\gamma_k(n)}$ are calculated for $h = 1$. The reason we compare the rate of convergence to that of $\beta(n)^2$ and not to $\beta(n)$ is that we look at pairs of representable integers. The values for the ratio $\frac{y(n)^2}{\gamma_k(n)}$ are calculated for $h = 1$. 

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The generalizations presented in Section 6 for integers of the form $x^2 + dy^2$ show similar numeric results. Figure 7.3 is the equivalent of Figure 7.1 for integers representable by $x^2 + 2y^2$. 

Figure 7.2. $\frac{y(n)^2}{Y_1(n)}$ and Landau’s $\beta^2$ convergence

Figure 7.3. $\frac{y_2(n)^2}{Y_2,h(n)}$ for $1 \leq h \leq 25$ at $n = 10^6, 10^8$
We have obtained results of this type for the other forms in question where the main difference between the forms is the location of the peaks, which occur at values of $h$ with small $m_p(h)$ for small primes $p \in Q_d \cup R_d$.

To conclude we have arrived with numerical results which are consistent with our expectation regarding the dependency on the prime decomposition of $h$, and regarding the rate of convergence. Note that the numerical data presented here improves previous computations by a factor of 10 for $1 \leq h \leq 25$ as appears in Figure 7.1 and by 1000 for $h = 1$ as appears in Figure 7.3.

8. further Directions

The work presented here may be expanded by producing conjectures for $k$–correlation functions for the set of representable pairs for $k \geq 3$, as described in (1.4). For example, following the methods presented for the calculation of the mean density and the pair correlation one can derive the following result for the density of representable triplets of the form $(m, m + 1, m + 2)$ for $m \leq n$, given by

$$\frac{1}{\log^2 n} \cdot \frac{1}{8\beta} \cdot \prod_{p \equiv 3 (4)} \left( \frac{1}{p} \right)^2 \approx \frac{0.11698}{\log^2 n}$$  \hspace{1cm} (8.1)

It seems possible to generalize this result for triplets $(m, m + h_1, m + h_2)$ and so on for higher degrees, though it would be difficult to obtain a general $k$–correlation function this way because of the inductive element of our approach. Also when comparing the expression for the density of representable triplets (8.1) to the expression for the density of representable pairs (1.6) one can easily notice that the product for the latter depends only on primes dividing $h$, where in the case of the triplets the product is over all primes $p \equiv 3 (4)$ and so the manipulation of such expressions is bound to be more complicated.

A second direction, assuming a $k$–correlation function is obtained, is to prove (1.5), which is a version of Gallagher’s Lemma (1.2) for sums of two squares. Gallagher’s approach in [5], and similarly the approach taken by Ford in [4] when proving the Lemma in the case of the primes, would apparently not do in the case of sums of two squares. In addition it is important to note that our proof of Gallagher’s Lemma for sums of two squares and $k = 2$ uses the methods of the analytic theory of Dirichlet series, and these methods become extremely difficult in higher dimensions. This means that even for $k = 3, 4$ a new approach for Gallagher’s Lemma for sums of squares must be found. For these reasons we did not pursue any additional $k$–correlation conjectures.

It is important to note that the main difference between the case of the set of primes and the case of the set of integers representable as a sum of two squares is the $k$–correlation conjectures. Hardy and Littlewood’s conjecture for primes (1.1) depends only on $\nu_d(p)$, which stands for the number of
residue classes modulo $p$ occupied by $d_1, \ldots, d_k$, which in the case of $k = 2$ is equivalent to whether or not $p$ divides $d_2 - d_1$ or in other words whether or not $m_p(d_2 - d_1)$ is 0. In the case of the sums of squares Connors and Keating’s conjecture (1.6) and the numerical work presented in Section (7) provide evidence of dependence also on the values of $m_p(d_2 - d_1)$.

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