On topologies on Lie braid groups

Yury Neretin

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To the memory of Vladimir Igorevich Arnold

We discuss groups corresponding to Kohno Lie algebra of infinitesimal braids and actions of such groups. We construct homomorphisms of Lie braid groups to the group of symplectomorphisms of the space of point configurations in \( \mathbb{R}^3 \) and to groups of symplectomorphisms of coadjoint orbits of \( \text{SU}(n) \).

1 Introduction

1.1. Group of pure braids. Denote by \( \text{Br}_n \) the group of pure braids of \( n \) strings (see e.g., [1], [12], [3]). It is a discrete group with generators \( A_{rs} \), where \( 1 \leq r < s \leq n \), and relations

\[
\begin{align*}
\{A_{rs}, A_{ik}\} &= 1, & \text{if } s < i \text{ or } k < r \\
\{A_{ks}, A_{ik}\} &= \{A_{is}^{-1}, A_{ik}\}, & \text{if } i < k < s \\
\{A_{rk}, A_{ik}\} &= \{A_{ik}^{-1}, A_{ir}^{-1}\}, & \text{if } i < r < k \\
\{A_{rs}, A_{ik}\} &= \{\{A_{is}^{-1}, A_{ir}^{-1}\}, A_{ik}\}, & \text{if } i < r < k < s
\end{align*}
\]

Here \( \{a, b\} := aba^{-1}b^{-1} \) denotes a commutator.

There is a natural homomorphism \( \text{Br}_n \to \text{Br}_{n-1} \), we forget the first string. It is easy to verify, that the kernel is the free group \( F_{n-1} \) of \( (n-1) \) generators. Therefore, \( \text{Br}_n \) is a semi-direct product

\[
\text{Br}_n = \text{Br}_{n-1} \rtimes F_{n-1}.
\]

Repeating the same argument, we get that \( \text{Br}_n \) is a product of its subgroups

\[
\text{Br}_n = F_1 \times F_2 \times \cdots \times F_{n-1}
\]

as a set,

i.e., each \( g \in \text{Br}_n \) admits a unique representation \( g = h_{n-1} \cdots h_2 h_1 \), where \( h_j \in F_j \) (we prefer the inverse order). More precisely,

\[
\text{Br}_n = F_1 \ltimes \left( F_2 \ltimes \left( F_3 \ltimes \cdots \ltimes \left( F_{n-2} \ltimes F_{n-1} \right) \right) \right)
\]

as a group and subgroups \( F_k \ltimes (\cdots \ltimes F_{n-1}) \) are normal.

1.2. Abstract Malcev construction (pro-unipotent completion). Let \( \Gamma \) be a group, \( A, B \) two subgroups. Denote by \( \{A, B\} \) their commutant, i.e., the subgroup generated by all commutators \( \{a, b\} \), where \( a \in A, b \in B \).

For a group \( \Gamma \) denote \( \Gamma_1 := \{\Gamma, \Gamma\}, \) and \( \Gamma_{j+1} := \{\Gamma_j, \Gamma\} \). A discrete group \( \Gamma \) is

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nilpotent if some $\Gamma_j = 1$ and virtually nilpotent if $\cap_j \Gamma_j = 1$. The group $\text{Br}_n$ is virtually nilpotent.

Malcev (see [13] and an exposition in [7]) proved that any nilpotent discrete group $\Gamma$ without torsion admits a canonical embedding $\Gamma \to G$ to a simply connected nilpotent Lie group $G$ as a uniform lattice\(^2\). He constructs 'coordinates' on $\Gamma$ in the following way. Choose a basis in each free Abelian group $\Gamma_j - 1 / \Gamma_j$, take representatives $a_1, a_2, \ldots, a_N$ of this basis elements in $\Gamma$, let an initial segment of this sequence corresponds to $\Gamma / \Gamma_1$, a next segment corresponds to $\Gamma_1 / \Gamma_2$ etc., Any element of $\Gamma$ admits a unique representation in the form $a_1^{q_1} a_2^{q_2} \ldots a_N^{q_N}$. Therefore, we can identify sets $\Gamma$ and $\mathbb{Z}^N$. Malcev proves that formulas for multiplication in 'coordinates' $(q_1, \ldots, q_N)$ are polynomial. We take real $q_j$ and get a nilpotent Lie group $G$, where a multiplication is given by the same formulas.

Similarly, any residually nilpotent group without torsion admits a canonical embedding to a certain projective limit of nilpotent Lie groups ([14]).

There is the following abstract interpretation (Quillen, [16]) of the Malcev construction. We consider the group algebra $\mathbb{C}[\Gamma]$ of a discrete residually nilpotent group $\Gamma$. Consider the ideal $I$ in $\mathbb{C}[\Gamma]$ generated by all elements $g^{-1}$, where $g$ ranges in $\Gamma$. Consider the following chain of quotients

$$\ldots \longleftrightarrow \mathbb{C}[\Gamma]/I^n \longleftrightarrow \mathbb{C}[\Gamma]/I^{n+1} \longleftrightarrow \ldots$$

Denote by $\mathbb{C}[\Gamma]$ the projective limit of this chain as $n \to \infty$. In the ring $\mathbb{C}[\Gamma]$, real powers

$$g^s := (1 - (1 - g))^s := \sum_{j=0}^{\infty} \frac{s(s-1)\ldots(s-j+1)}{j!} (1 - g)^j$$

are well defined. We consider the group $\mathfrak{G}$ generated by such powers. The corresponding Lie algebra $\mathfrak{g}$ is generated by elements $\ln g := \ln(1 - (1 - g))$.

This constructions implies the following functoriality: any representation of $\Gamma$ by unipotent matrices produces a representation of its completion.

1.3. Kohno algebra. Kohno [9] obtained an explicit description of the Lie algebra $\mathfrak{br}_n$ corresponding to the braid group $\text{Br}_n$. It is generated by elements $r_{ij}$, where $0 \leq i, j \leq n$ and $r_{ij} = r_{ji}$, the relations are

$$[r_{ij}, r_{kl}] = 0, \quad \text{if } i, j, k, l \text{ are pairwise distinct}, \quad (1.1)$$

$$[r_{ij}, r_{ik} + r_{jk}] = 0. \quad (1.2)$$

The Campbell–Hausdorff formula (see an introduction in Serre [18] and a treatise in Reutenauer [17]) produces a group structure on $\mathfrak{br}_n$,

$$x \cdot y := \ln(e^x e^y).$$

This is the corresponding Lie braid group $\overline{\text{Br}}_n$.

\(\text{\textsuperscript{2}}\text{i.e, } \Gamma \text{ is a discrete subgroup and the homogeneous space } G/\Gamma \text{ is compact}\)
Kohno obtained the Poincare series for dimensions of homogeneous subspaces of the universal enveloping algebra $\mathfrak{u}(\mathfrak{b} \mathfrak{r}_n)$ of $\mathfrak{b} \mathfrak{r}_n$,

$$\sum_{k \geq 0} t^k \cdot \dim \mathfrak{u}(\mathfrak{b} \mathfrak{r}_n)[k] = \prod_{j=1}^{n-1} (1 - jt)^{-1}. \quad (1.3)$$

This easily implies that

$$\dim \mathfrak{b} \mathfrak{r}_n[k] = \sum_{d|k} \mu(d) \left( \sum_{i=1}^{n-1} \frac{j^k}{d} \right),$$

where the summation is given over all divisors $d$ of $n$ and $\mu(d)$ is the M"obius function. In particular, $\dim \mathfrak{b} \mathfrak{r}_n[k]$ has exponential growth as $k \to \infty$.

Also, Kohno noted (see [10], [11], and below 3.4) that any finite-dimensional representation of the Lie algebra $\mathfrak{b} \mathfrak{r}_n$ produces a representation of the braid group in the same space.

1.4. Knizhnik–Zamolodchikov construction. Apparently the most important origin of representations of $\mathfrak{b} \mathfrak{r}_n$ is the following construction. Let $\mathfrak{g}$ be a semisimple Lie algebra. Consider the enveloping algebra $\mathfrak{u}(\mathfrak{g} \oplus \cdots \oplus \mathfrak{g})$ of direct sum of $n$ copies of $\mathfrak{g}$. Let $e_\alpha$ be an orthonormal basis in $\mathfrak{g}$. Let $e^i_\alpha$ be the same basis in the $j$-th copy of $\mathfrak{g}$. Then mixed Casimirs $\Delta_{ij} = \sum_\alpha e^i_\alpha e^j_\alpha$ satisfy the relations (1.1)–(1.2).

Therefore $\mathfrak{b} \mathfrak{r}_n$ acts in any tensor product $V_1 \otimes \cdots \otimes V_n$ of finite-dimensional representations of $\mathfrak{g}$.

Many finite-dimensional representations of $\mathfrak{b} \mathfrak{r}_n$ are known, we refer to [4], [10], [19], [15].

1.5. Klyachko’s spatial polygons. Consider the space $\mathbb{R}^3$ and the Poisson bracket on the space of functions determined by

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y$$

or, more precisely,

$$\{f, g\} = z \left( \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial x} \right) + x \left( \frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \cdot \frac{\partial g}{\partial y} \right) + y \left( \frac{\partial f}{\partial z} \cdot \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial z} \right).$$

Consider the space $(\mathbb{R}^3)^n$ whose points are collections of 3-vectors $(r_1, \ldots, r_n)$. Symplectic leaves of our Poisson structure are products of spheres $|r_j| = a_j$.

The following collection of Hamiltonians

$$\Delta_{ij} := \langle r_i, r_j \rangle = x_i x_j + y_i y_j + z_i z_j, \quad i \neq j,$$

3 if $d = p_1 \cdots p_l$ is a product of pairwise distinct primes, then $\mu(d) = (-1)^l$, otherwise $\mu(d) = 0.$
satisfies the Kohno relations (1.1)-(1.2).

In fact, Klyachko [8] considered the symplectic structure on the space of point configurations \((r_1, \ldots, r_n)\) such that

\[
\sum r_j = 0, \quad |r_j| = a_j
\]

up to rotations. This is the space of spatial polygons with given lengths of sides (this object has an unexpectedly rich geometry).

More generally, consider a semisimple Lie algebra \(\mathfrak{g}\). Consider the standard Poisson bracket on \(\mathfrak{g}^*\). If \(u_\alpha \in \mathfrak{g}\) is a basis, we write

\[
\{f, g\} := \sum_{i,j} [u_i, u_j] \frac{\partial f}{\partial u_i} \frac{\partial g}{\partial u_j}.
\]

Next, consider the product \(\mathfrak{g}^* \times \cdots \times \mathfrak{g}^*\), we denote its points as \((X_1, \ldots, X_n)\). Then the collection of Hamiltonians

\[
\Delta_{ij} := \text{tr} X_i X_j
\]

satisfies the Kohno relations (1.1)-(1.2).

In Klyachko’s example, \(\mathfrak{g} = \mathfrak{so}_3\).

Evidently, this construction is a classical limit of the Knizhnik–Zamolodchikov construction. The next construction is a classical limit of a construction discussed in [15].

1.6. Action of the Kohno algebra on coadjoint orbits. Consider the Poisson structure on the space of functions on \(\mathfrak{gl}(n)^*\) (we denote elements of this space as matrices \(\{x_{ij}\}\)). Symplectic leaves are coadjoint orbits. Then the collection of Hamiltonians

\[
\Delta_{ij} = 2x_{ij} x_{ji}
\]

form a representation of the Kohno algebra.

1.7. An example. Knizhnik–Zamolodchikov construction produces numerous examples of actions of \(\mathfrak{br}_n\) by second-order partial differential operators. For instance consider the product of \(n\) copies of \(\mathbb{R}^2\) with coordinates \((x_j, y_j)\). The operators

\[
\Delta_{kl} = (x_k y_l - x_l y_k) \left( \frac{\partial^2}{\partial x_k \partial y_l} - \frac{\partial^2}{\partial x_l \partial y_k} \right)
\]

(1.4)

satisfy the same commutation relations.

1.8. Purposes of the paper. As we have seen (1.2, 1.3, see also [11]) there are correspondences between finite-dimensional representations of the braid group \(\mathbb{B}r_n\) and representations of the Kohno algebra \(\mathfrak{br}_n\). However, representations of \(\mathfrak{br}_n\) do not produce representations of the Lie braid group \(\mathbb{B}r_n\) (since formal series diverge).

There is a well-known way to avoid convergence problems by passing to spaces over algebra of formal series. For instance, in the Klyachko case, we add
a formal variable $\hbar$ and take new generators $\tilde{\Delta}_{kl} := \hbar \Delta_{kl}$. Then any formal series in $\tilde{\Delta}_{kl}$ determines a well-defined operator in the space $C^\infty((\mathbb{R}^3)^n) \otimes \mathbb{C}[\hbar]$.

However Klyachko’s Hamiltonians $\Delta_{kl}$ and their linear combinations generate nice Hamiltonian flows, there arises a question about the group generated by all such flows.

In this paper, we define two versions of Lie braid groups. The first one $B_{br}^n$ acts in finite-dimensional representations of $br_n$ (this is a simple observation, see Section 4), the second group $B_{br}^!$ acts on compact analytic manifolds (Sections 5–6). I present numerous simple propositions for a clarification of the picture, because facts, which are obvious in finite-dimensional theory, require proofs and quite often are not correct. I also try to explain that Lie braid groups are hand-on (or at least semi-hand-on) objects (Sections 2–3).

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2 Groups $\overline{G}$

This section contains preliminary formalities.

2.1. Free associative algebra. Denote by $A_{fr} = A_{fr}[\omega]$ the free associative algebra with generators $\omega_1, \ldots, \omega_n$. The monomials $w = \omega_{i_1} \cdots \omega_{i_m}$ form a basis in $A_{fr}[\omega]$. By $A_{fr}^\circ$ we denote the corresponding algebra of formal series.

Denote by $fr = fr[\omega]$ the free Lie algebra with generators $\omega_j$. Then (see, e.g., [18], [17]) $A_{fr}$ is the universal enveloping algebra of $fr$, $A_{fr} \simeq U(fr[\omega])$.

2.2. Completed enveloping algebras. Let $g$ be a graded Lie algebra

$$
g = \bigoplus_{k>0} g^{[k]}, \quad [g^{[k]}, g^{[m]}] \subseteq g^{[k+m]},
$$

let $\dim g^{[k]} < \infty$, assume that $g^{[1]}$ generates $g$. We also fix a basis $\omega_1 \in g^{[1]}$ (our main objects, groups $\overline{G}$, $G^\circ$, and $G'$, do not depend on a choice of basis).

Our basic example is the Kohno Lie algebra $br_n$. Another example under discussion is the free Lie algebra $fr_m$ with $m$ generators.

Denote by $\overline{g}$ the Lie algebra whose elements are formal series

$$
x = x^{[1]} + x^{[2]} + \ldots, \quad \text{where } x^{[j]} \in g^{[j]}.
$$

4I cannot formulate any positive statement related to operators (1.4).

5Other examples of Lie algebras $g$ that can be interesting in this context are: braid algebras related to semisimple Lie algebras (see, e.g., [15]), triangular and quasitriangular algebras, see [2], and various algebras related to Teichmüller groups, see [6].
Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ equipped with the natural gradation arising from $\mathfrak{g}$. We expand an element $z \in U(\mathfrak{g})$ as a sum of homogeneous elements

$$z = z^{[0]} + z^{[1]} + z^{[2]} + z^{[3]} + \ldots,$$

where $z^{[j]} \in U(\mathfrak{g})^{[j]}$. (2.1)

Such sums are finite. Denote by $\overline{U}(\mathfrak{g})$ the space of infinite formal series (2.1). We equip the space $\overline{U}(\mathfrak{g})$ with the topology of direct product of finite-dimensional linear spaces, $\overline{U}(\mathfrak{g}) = \prod_j U(\mathfrak{g})^{[j]}$.

Denote by $U_+(\mathfrak{g})$ the space of all $z$ such that $z^{[0]} = 0$.

**Lemma 2.1**

a) $\overline{U}(\mathfrak{g})$ is an algebra.

b) For any formal series $h(\xi) = \sum_{j \geq 0} c_j \xi^j$, where $c_j \in \mathbb{C}$, and any $z \in U_+(\mathfrak{g})$, the formal series $h(z) \in \overline{U}(\mathfrak{g})$ is well defined.

c) For any $z \in U_+(\mathfrak{g})$ the exponent $\exp(z)$ and the logarithm $\ln(1 + z)$ are well defined.

This is obvious.

**2.3. Group $\mathfrak{G}$.**

**Proposition 2.2** The set $\mathfrak{G}$ of all elements $\exp(x) \in \overline{U}(\mathfrak{g})$, where $x$ ranges in $\mathfrak{g}$, is a group with respect to multiplication.

This follows from the Cambell–Hausdorff formula, see, e.g., [18] or [17].

**2.4. Characterization of $\mathfrak{G}$.** Consider the tensor product $\overline{U}(\mathfrak{g}) \otimes \overline{U}(\mathfrak{g})$. Consider the homomorphism (co-product) $\delta : \overline{U}(\mathfrak{g}) \to \overline{U}(\mathfrak{g}) \otimes \overline{U}(\mathfrak{g})$ by

$$\delta(x) = x \otimes 1 + 1 \otimes x,$$

for $x \in \mathfrak{g}$

The following statement is standard, see, e.g., [17].

**Proposition 2.3** Let $S \in \overline{U}(\mathfrak{g})$.

a) $S \in \mathfrak{G}$ if and only if $\delta(S) = S \otimes 1 + 1 \otimes S$.

b) $S \in \mathfrak{G}$ if and only if $\delta(S) = S \otimes S$.

Identify $\overline{U}(\mathfrak{g})$ with the symmetric algebra $S(\mathfrak{g})$ in the usual way, see, e.g., [18]. Let $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ be a monomial. Denote $p(x^\alpha) \in U(\mathfrak{g})$ the element obtained by symmetrization of $x^\alpha$. The following statement is obvious.

**Lemma 2.4**

$$\frac{1}{\alpha!} \delta(p(x^\alpha)) = \sum_{\beta, \gamma : \beta + \gamma = \alpha} \frac{1}{\beta!} p(x^\beta) \otimes \frac{1}{\gamma!} p(x^\gamma)$$

(2.2)
If $x_j$ is a basis in $\mathfrak{g}$, then elements $p_\alpha(x^\alpha)$ form a basis in $U(\mathfrak{g})$. Therefore elements $p_\alpha(x^\alpha) \otimes p_\alpha(x^\beta)$ form a basis in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Applying the identity (2.2) to a formal series $\sum c_\alpha p(x^\alpha)$, we immediately get an expansion of $\delta(\sum c_\alpha p(x^\alpha))$ in the basis. Emphasis that similar terms can not appear.

**Proof of Proposition 2.3.** a). Having a term $p(x^\alpha)$ in $S$ of degree $\geq 2$ we get a term in $\delta(S)$ of degree $(1,1)$.

b). Let $S = \sum b_\alpha p(x^\alpha)$ satisfy $\delta(S) = S \otimes S$. The expression $\delta(S)$ contains a subseries $\sum b_\alpha p(x^\alpha) \otimes 1$. The expression $S \otimes S$ contains a subseries $\sum b_\alpha p(x^\alpha) \otimes b_0$. Therefore, $b_0 = 1$. We write

\[ S = 1 + \sum s_k x_k + \left\{ \text{terms } p(x^\alpha) \text{ of higher degree} \right\}. \]

Then the term of $S \otimes S$ of degree $(1,1)$ is $\sum s_k x_k \otimes x_l$. This term must appear from $\delta(S)$. The only chance is

\[ S = 1 + \sum s_k x_k + \frac{1}{2!} \sum s_k s_l x_k x_l + \cdots \]

\[ = 1 + \sum s_k x_k + \frac{1}{2!} \left( \sum s_k x_k \right)^2 + \cdots = 1 + \sum s_k x_k + \frac{1}{2!} \sum s_k s_l p(x_k x_l) + \cdots. \]

Next, we examine the term of $S \otimes S$ of degree $(2,1)$, etc. Repeating this argument we come to

\[ S = \sum \frac{1}{k!} \left( \sum s_k x_k \right)^k. \]

2.5. **Inverse element.** Recall that any enveloping algebra $U(\mathfrak{g})$ has a canonical anti-automorphism $\sigma$ determined by

\[ \sigma(x) = -x \quad \text{for } x \in \mathfrak{g}, \quad \sigma(z)\sigma(u) = \sigma(uz). \]

**Lemma 2.5** For $S \in \mathcal{G}$, we have $S^{-1} = \sigma(S)$.

**Proof.** Indeed $S = \exp(x)$ for some $x \in \mathfrak{g}$. Then $\sigma(S) = \sigma(\exp(x)) = \exp(-x) = S^{-1}$. \qed

2.6. **Example.** **Free Lie groups** $\mathbb{F}_n$. Consider the group $\mathbb{F}_n$ corresponding to the free Lie algebra $\mathfrak{f}_n = \mathfrak{f}[\omega_1, \ldots, \omega_n]$. Let $S \in \mathbb{F}_n$, $S = \sum w c_w w$, where $w$ ranges in the set of words in the letters $\omega_1, \ldots, \omega_n$. Then the equation $\delta(S) = S \otimes S$ is equivalent to the system of quadratic equations

\[ c_v c_w = \sum_{u, \text{ where } u \text{ is a shuffle of } v \text{ and } w} c_u. \]

Recall that a shuffle of $v$ and $w$ is a word equipped with coloring of letters in black and white such that removing black letters we get the word $v$ and removing white letters we get the word $w$.

On other characterizations of $\mathbb{F}_n$, see Reutenauer’s book [17]. For instance, coefficients $c_v$ in the front of Lyndon words $v$ form a coordinate system on $\mathbb{F}_n$. 7
2.7. Ordered exponentials. Let \( t \) ranges in a segment \([0, a]\). Let

\[
\gamma(t) = \sum_{j=1}^{\infty} \gamma^j(t),
\]

be a map \([0, a] \to \mathfrak{g}\).

**Proposition 2.6** If each \( \gamma^j(t) \) is an integrable bounded function, then the differential equation

\[
\frac{d}{dt} E(t) = E(t) \gamma(t), \quad E(0) = 1
\]

has a unique solution (‘ordered exponential’) \( E(t) \in \mathfrak{U}(\mathfrak{g}) \). Moreover, \( E(t) \in \mathfrak{G} \).

**b)** Let \( \gamma_m(t) \) be a sequence. If for each \( j \) the sequence \( \gamma^j_m \) converges to \( \gamma^j \) in the \( L_1 \)-sense, then \( E_m(a) \) converges to \( E(a) \).

**Proof.** Expanding \( E(t) = \sum E(t)^j \) we come to the system of equations,

\[
\begin{cases}
\frac{d}{dt} E^j(t) = \gamma^j(t) + E^{[1]}(t) \gamma^{[j-1]}(t) + \cdots + E^{[j-1]}(t) \gamma^{[1]}(t) \\
E(0) = 1
\end{cases}
\]

We consequently find \( E^{[1]}, E^{[2]}, \ldots \). One can easily write closed expressions for \( E^{[j]}(t) \) in terms of repeated integrals, such expressions imply b).

If \( \mu(t) \) is piece-wise constant, then its ordered exponent has the form

\[
\prod_{j=1}^{\infty} \exp(x_j) \in \mathfrak{G}.
\]

Next, we approximate \( \gamma(t) \) by piece-wise function and pass to a limit. By Proposition 2.3, the group \( \mathfrak{G} \) is closed in \( \mathfrak{U}(\mathfrak{g}) \). \( \square \)

3 The Lie braid group \( \overline{\text{Br}}_n \)

3.1. A basis in \( U(\mathfrak{br}_n) \). Consider the Kohno Lie algebra \( \mathfrak{br}_n \) with generators \( r_{ij} \). We say that a word of \( r_{ij} \) is **good** if it has a form

\[
w_1 w_2 \ldots w_{n-1} \quad \text{(3.1)}
\]

where \( w_1 \) is a word composed of \( r_{12}, r_{13}, r_{14}, \ldots, r_{1n} \), a word \( w_2 \) is composed of \( r_{23}, r_{24}, \ldots, r_{2n} \), etc. The word \( w_{n-1} \) has the form \( r_{(n-1)n} \).

**Proposition 3.1** Good words form a basis in \( U(\mathfrak{br}_n) \).

In other words we get a canonical bijection

\[
U(\mathfrak{br}_n) \leftrightarrow \text{Asfr}_{n-1} \otimes \text{Asfr}_{n-2} \otimes \cdots \otimes \text{Asfr}_1
\]
PROOF. By definition, \( U(\mathfrak{br}_n) \) is the algebra with generators \( r_{ij} \) and quadratic relations
\[
\begin{align*}
  r_{ij}r_{kl} &= r_{kl}r_{ij} \quad \text{if } i, j, k, l \text{ are pairwise distinct}; \\  r_{ij}r_{jl} &= r_{jl}r_{ij} - r_{ij}r_{il} + r_{il}r_{ij}.
\end{align*}
\] (3.2) (3.3)
Let we have a word
\[
\ldots r_{ij}r_{kl} \cdots \in U(\mathfrak{br}_n)
\]
with \( \min(i, j) > \min(k, l) \). Then we transform \( r_{ij}r_{kl} \) according (3.2) or (3.3). Removing all such disorders we come to a linear combination of good words.
Denote by \( s_p \) the number of good words of degree \( p \). Obviously,
\[
\sum s_p t^p = \prod_{j=1}^{n-1} (1 - jt)^{-1}.
\]
Comparing with (1.3) we get that good words are linearly independent. \( \Box \)

The rule of multiplication is following. Let we wish to multiply
\[
r_{ij} \cdot w_1w_2 \ldots w_{n-1}
\]
(see (3.1)), let \( i < j \). We write
\[
\begin{align*}
  r_{ij} \cdot w_1w_2 \ldots w_{n-1} &= [r_{ij}, w_1]w_2 \ldots w_{n-1} + \ldots \\
  &\quad + w_1 \ldots [r_{ij}, w_{i-1}]w_i \ldots w_{n-1} + w_1 \ldots w_{i-1}(r_{ij}w_i) \ldots w_{n-1}
\end{align*}
\]
and evaluate commutators
\[
[r_{ij}, w_\alpha] = [r_{ij}, w_{\alpha,k_1}w_{\alpha,k_2} \ldots w_{\alpha,k_p}] =
\] \[
= [r_{ij}, w_{\alpha,k_1}]w_{\alpha,k_2} \ldots w_{\alpha,k_p} + \ldots + w_{\alpha,k_1}w_{\alpha,k_2} \ldots [r_{ij}, w_{\alpha,k_p}].
\]
A bracket can be non-zero only if \( k_\theta = j \) or \( \alpha = j \). In such case we get a sum of two good monomials.

Corollary 3.2 Let elements \( S \in U(\mathfrak{br}_n) \) be written as linear combinations of good monomials. Fix \( \alpha \). Then the substitution \( r_{ij} = 0 \) for all \( i \leq \alpha \) and \( j \leq \alpha \) determines a well-defined homomorphism \( U(\mathfrak{br}_n) \to U(\mathfrak{br}_{n-\alpha}) \).

3.2. Lie braid groups.

Proposition 3.3 An element \( S \in U(\mathfrak{br}_n) \) is contained in the group \( \mathfrak{Br}_n \) if and only if \( S \) can be represented as a product
\[
S = T_1T_2 \ldots T_{n-1},
\] (3.4)
where \( T_j \in \text{Asr}[r_{j(j+1)}, \ldots r_{jn}] \) is an element of the free Lie group \( \mathfrak{Fr}_{n-j} \).
Proof. We apply Proposition 2.3. Using the canonical basis we identify spaces

\[ \Xi : \mathcal{U}(\mathfrak{b} \mathfrak{r}_n) \rightarrow \mathcal{U}(\mathfrak{f} \mathfrak{r}_{n-1} \oplus \cdots \oplus \mathfrak{f} \mathfrak{r}_1). \]

Moreover, this identification is compatible with the co-product

\[ \delta \circ \Xi = (\Xi \otimes \Xi) \circ \delta. \]

For \( \mathcal{U}(\mathfrak{f} \mathfrak{r}_{n-1} \oplus \cdots \oplus \mathfrak{f} \mathfrak{r}_1) \) the required statement is obvious. \( \Box \)

Thus \( \mathfrak{b} \mathfrak{r}_n \) is a product of its subgroups (not a direct product)

\[ \mathfrak{b} \mathfrak{r}_n \simeq \mathfrak{f} \mathfrak{r}_{n-1} \times \mathfrak{f} \mathfrak{r}_{n-2} \times \cdots \times \mathfrak{f} \mathfrak{r}_1. \]

(3.5)

All subgroups \( \mathfrak{f} \mathfrak{r}_{n-1} \times \cdots \times \mathfrak{f} \mathfrak{r}_{n-j} \) are normal and we have homomorphisms \( \mathfrak{b} \mathfrak{r}_n \rightarrow \mathfrak{b} \mathfrak{r}_{n-j} \) for all \( j \).

The corresponding statement (see [22]) on the level of Lie algebras is

\[ \mathfrak{b} \mathfrak{r}_n \simeq \mathfrak{f} \mathfrak{r}_{n-1} \oplus \cdots \oplus \mathfrak{f} \mathfrak{r}_1 \]

(recall again, this is not an isomorphism of Lie algebras)

3.3. Decomposition of ordered exponentials. Our next remark: evaluation of ordered exponentials in \( \mathfrak{b} \mathfrak{r}_n \) is reduced to successive evaluation of the free Lie groups \( \mathfrak{f} \mathfrak{r}_1, \ldots, \mathfrak{f} \mathfrak{r}_{n-1} \). More precisely:

**Proposition 3.4** Let \( \gamma(t) \) be a way in \( \mathfrak{b} \mathfrak{r}_n \). Decompose it as

\[ \gamma(t) = \gamma_1(t) + \cdots + \gamma_{n-1}(t), \quad \text{where } \gamma_j(t) \in \mathfrak{f} \mathfrak{r}_j = \mathfrak{f} \mathfrak{r}_{(n-j)(n-j+1)}, \ldots, \mathfrak{r}_{(n-j)n} \]

Then the solution \( E(t) \) of the differential equation

\[ E'(t) = E(t) \gamma(t), \quad E(0) = 1 \]

admits a representation

\[ E(t) = U_{n-1}(t) \cdots U_1(t), \quad (3.6) \]

where \( U_j(t) \in \mathfrak{f} \mathfrak{r}_j = \mathfrak{f} \mathfrak{r}_{(n-j)(n-j+1)}, \ldots, \mathfrak{r}_{(n-j)n} \),

\[ U_1'(t) = U_1(t) \gamma_1(t), \quad U_1(0) = 1 \]

\[ U_2'(t) = U_2(t) \cdot [U_1(t) \gamma_2(t) U_1(t)^{-1}], \quad U_2(0) = 1 \]

\[ \ldots \]

\[ U_{n-1}'(t) = U_{n-1}(t) \cdot [U_{n-2}(t) \cdots U_1(t) \gamma_{n-1}(t) U_1(t)^{-1} \cdots U_{n-2}(t)^{-1}], \quad U_{n-1}(0) = 1 \]

and

\[ U_1(t) \gamma_2(t) U_1(t)^{-1} \in \mathfrak{f} \mathfrak{r}_2, \quad U_2(t) U_1(t) \gamma_3(t) U_1(t)^{-1} U_2(t)^{-1} \in \mathfrak{f} \mathfrak{r}_3, \ldots \]
Proof. For definiteness, set \( n = 4 \). Let us look for a solution in the form (3.6), by (3.5 it exists,
\[
U_1'U_2U_3 + U_2U_3U_1 + U_3U_1U_2 = U_3U_2U_1(\gamma_3 + \gamma_2 + \gamma_1).
\]
We apply the homomorphism \( \overline{Br}_4 \to \overline{Br}_2 \) to both sides of the equality and get
\[
U_1' = U_1 \gamma_1. \]
After a cancellation, we come to
\[
U_3U_2 + U_3U_2 + U_3U_1 = U_3U_2(U_1^{-1} \gamma_2 + U_1^{-1} \gamma_1).
\]
Next, we apply homomorphism \( \overline{Br}_4 \to \overline{Br}_3 \) to both sides of equality and get
\[
U_2' = U_2 \cdot U_1 \gamma_2. \]
After a cancellation, we find \( U_3 \).

3.4. Embedding of the braid group to \( \overline{Br}_n \). Denote by \( \Omega \) the space \( \mathbb{C}^n \) without diagonals \( \xi_i = \xi_j \). The fundamental group \( \pi_1(\Omega) \) is \( Br_n \). Consider the following differential 1-form on \( \Omega \) with values in \( U(br_n) \):
\[
d\omega = \sum_{k<l} r_{kl} d \ln(\xi_k - \xi_l)
\]
This form determines a flat connection (see Kohno [10], [11]). We take a reference point \( \xi \in \Omega \) and for any loop \( \mu(t) \) starting at \( \xi \) we write the ordered exponential of
\[
d\omega(\mu) = \sum_{k<l} r_{kl} d \ln(\mu_k(t) - \mu_l(t)) \in U(br_n).
\]
Thus we get a homomorphism \( \pi_1(\Omega) \simeq Br_n \to \overline{Br}_n \).

4 Groups \( G^o \)

4.1. Norm on free associative algebra. Let define norm on the \( Asfr[\omega] \). If \( z = \sum c_w w \) is the expansion of \( Asfr[\omega] \) in basis monomials, then
\[
\|z\| := \sum |c_w|.
\]
Obviously,
\[
\|z + u\| \leq \|z\| + \|u\|, \quad \|zu\| \leq \|z\| \cdot \|u\|, \quad \|[z, u]\| \leq 2\|z\| \cdot \|u\| \quad (4.1)
\]

4.2. Norm on \( U(g) \). Let us represent \( g \) as quotient of \( fr[\omega] \) with respect to an ideal \( I = \oplus I^[] \), by our assumptions \( \omega_j \) constitute a basis of \( g^{(1)} \). Denote by \( J \) the left ideal \( J \subset U(fr[\omega]) \) generated by \( I \).

Lemma 4.1 a) \( J \) is a two-side ideal.

b) \( fr[\omega] \cap J = I \).
This is obvious.

We define *norm* on $\mathbb{U}(\mathfrak{g})$ as the norm on the quotient space $\mathbb{U}(\mathfrak{g})/J$. In other words, for $z \in \mathbb{U}(\mathfrak{g})$ we consider all representations of $z$ as noncommutative polynomials in generators $\omega_j$,

$$z = \sum c_\omega \omega,$$

where $\omega$ are of the form $\omega_{j_1} \omega_{j_2} \ldots \omega_{j_p}$ and set

$$\| z \| = \inf \sum |c_\omega|.$$

This norm satisfies the same relations (4.1).

We use this norm only for homogeneous elements of $\mathbb{U}(\mathfrak{g})$.

4.3. Algebra $\mathbb{U}^\circ(\mathfrak{g})$. **Definition.** An element of $z = \sum z^p$ is contained in $\mathbb{U}^\circ(\mathfrak{g})$ if for any $C$

$$[z]_C := \sup_p (\| z^p \| e^{Cp}) < \infty. \tag{4.2}$$

Thus $\mathbb{U}^\circ(\mathfrak{g})$ is a Frechet space\(^\text{6}\) with respect to the family of semi-norms $[\cdot]_C$.

**Observation 4.2** For $z \in \mathbb{U}(\mathfrak{g})$ define the series

$$\Phi_z(\xi) := \sum \| z^p \| \xi^p. \tag{4.3}$$

Then $z \in \mathbb{U}^\circ(\mathfrak{g})$ if and only if $\Phi_z(\xi)$ is an entire function of the variable $\xi \in \mathbb{C}$.

**Lemma 4.3** a) If $z, u \in \mathbb{U}^\circ(\mathfrak{g})$, then $zu \in \mathbb{U}^\circ(\mathfrak{g})$.

b) The algebra $\mathbb{U}^\circ(\mathfrak{g})$ does not depend on a choice of a basis $\omega_j$ in $\mathfrak{g}^{[1]}$.

c) If $z \in \mathbb{U}^\circ_+(\mathfrak{g})$ and $f(\xi)$ is an entire function, then $f(z) \in \mathbb{U}^\circ(\mathfrak{g})$.

d) If $z \in \mathbb{U}^\circ_+(\mathfrak{g})$, then $\exp(z) \in \mathbb{U}^\circ(\mathfrak{g})$.

**Proof.** a) We have

$$\|(zu)^p\| \leq \sum_{q,r} \| z^q \| u^r \|.$$ 

Therefore Taylor coefficients of $\Phi_{zu}(\xi)$ are dominated by the Taylor coefficients of $\Phi_z(\xi)\Phi_u(\xi)$. Therefore $\Phi_{zu}$ is an entire function.

b) Let $\omega'_1, \ldots, \omega'_n$ be another basis in $\mathfrak{g}^{[1]}$. Denote by $A$ the transition matrix, by $\| \cdot \|$ another norm. Then

$$\| A^{-1} \|^p \| z^p \| \leq \| z^p \|' \leq \| A \|^p \| z^p \|.$$ 

---

\(^6\)A Frechet space is a complete locally convex space, whose topology is metrizable by a translation-invariant metric.
c) For \( f(\xi) = \sum a_m \xi^m \), we set \( F(\xi) = \sum |a_m| \xi^m \). We have
\[
\|f(z)^{[\alpha]}\| = \| \sum a_m ((z[1] + z[2] + z[3] + \ldots)^{[\alpha]} \| \leq \leq \sum_{m=1}^{\alpha} \left( m! |a_m| \right) \sum_{s_j = m} \frac{\|z[1]\|s_1! \ldots \|z[m]\|s_m!}{s_1! \ldots s_m!} .
\]
The last expression is the Taylor coefficient at \( \xi^\alpha \) of an entire function \( F(\Phi^z(\xi)) \).

\[\Box\]

4.4. The group \( G^\circ \). We define
\[ G^\circ := G \cap U^\circ(g), \quad g^\circ := g \cap U^\circ(g). \]
By Lemma 4.3 \( G^\circ \) is closed with respect to multiplications, by Lemma 2.5, \( G^\circ \) contains inverse elements.

By Lemma 4.3 we have a well-defined map \( \exp : g^\circ \to G^\circ \).

**Observation 4.4** Generally, this map is not surjective.

Indeed, let \( g = fr[\omega_1, \omega_2] \). We have \( \exp(\omega_1) \exp(\omega_2) \in G^\circ \). Opening brackets in
\[
\ln \left( \exp(\omega_1) \exp(\omega_2) \right) = \ln \left( 1 + \omega_1 + \sum_{k>1} \frac{\omega_k}{k!} \left( 1 + \omega_2 + \sum_{l>1} \frac{\omega_l}{l!} \right) \right),
\]
we find a summand \(- \frac{1}{2} \omega_1^2 \omega_2^k \) in the formal series, it is a unique summand with \( (\omega_1 \omega_2)^k \).

\[\Box\]

4.5. The group \( Br^\circ_\mathbb{n} \).

**Proposition 4.5** Represent \( S \in Br^\circ_\mathbb{n} \) as a product \( S = T_n \ldots T_1 \) of elements \( T_j \in Fr_j, \) see (3.4). Then all factors \( T_j \in Br^\circ_\mathbb{n} \).

**Proof** by induction. Denote by \( \pi \) the canonical homomorphism \( U(br_n) \to U(br_{n-1}) \), this is a quotient of \( U(br_n) \) by the ideal generated \( r_{12}, \ldots, r_{1n} \).
Evidently, for any homogeneous element \( z \), we have \( \|\pi(z)\| \leq \|z\|. \) Indeed, if \( z = \sum c_w w \) is a linear combination of monomials, then \( \pi(z) \) is obtained by removing of monomials containing letters \( r_{12}, \ldots, r_{1n} \). This diminish \( \sum |c_w| \).

Hence, for \( S \in Br^\circ_n \) we have \( \pi(S) \in Br^\circ_{n-1} \). Therefore, \( T_{n-1} = S\pi(S)^{-1} \in Br^\circ_{n-1} \).

Thus \( Br^\circ_n \) is a semidirect product of its free Lie subgroups. Obviously,
\[ Br^\circ_n \supset Fr^\circ_{n-1} \times \cdots \times Fr^\circ_1. \]

However, the author does not know is it \( \supset \) or \( = \).

4.6. Ordered exponentials. Now let \( \gamma : [0, a] \to g^\circ \) be a measurable function, let for each \( C \)
\[
\sup_{t \in [0, a], j \in \mathbb{N}} \| \gamma^{[j]}(t) \| e^{Cj} < \infty.
\]
Proposition 4.6  Under this condition the solution $E$ of the equation
\[ \frac{d}{dt} E(t) = E(t) \gamma(t), \quad E(0) = 1 \]  
(4.4)
is contained in $G^\circ$.

Proof. Denote by $V_C$ the completion of $U(g)$ with respect to the seminorm (4.2).

Lemma 4.7  For $\gamma \in g^\circ$, the linear operator
\[ Lz = z \gamma \]
is bounded in the Banach space $V_C$.

Proof of Lemma. Take $A > C$. We have $\sup \| \gamma[q] \| e^{Aq} < \infty$. Therefore
\[ \| (z\gamma)^[r] \| \leq \sum_{p,q : p+q=r} \| \gamma[q] \| \| z[p] \| \leq \text{const} \cdot \sum_{p,q : p+q=r} e^{-Aq} e^{-Cp} \leq \text{const} \cdot e^{-Cp}. \]

Proof of the proposition. The differential equation (4.4) is equivalent to the integral equation
\[ E(t) = 1 + \int_0^t E(\tau) \gamma(\tau) d\tau. \]  
(4.5)

The usual arguments (see any text-book on functional analysis, e.g., [20]) show that this equation has a unique solution in the Banach space of continuous functions $[0, a] \rightarrow V_C$.

Since this is valid for all $C$, we get $E(t) \in U^\circ(g)$. By Proposition 2.6, $E(t) \in G^\circ$. □

4.7. Representations of $G^\circ$. If the Lie algebra $g$ is real then we can consider real or complex enveloping algebra and therefore, we get two groups $G^\circ(R)$ and $G^\circ(C)$. For complex Lie algebras we have only the group $G^\circ(C)$.

Proposition 4.8 a) Let $\rho$ be a representation of $g$ in a finite-dimensional linear space $W$. Then $\rho$ can be integrated to a representation of $G^\circ(C)$ in $W$.

b) Let operators $\rho(\omega_j)$ be anti-selfadjoint. Then the group $G^\circ(R)$ acts in $W$ by unitary operators.

Proof. a) The enveloping algebra $U^\circ(g)$ acts in $W$.

b) We refer to Lemma 2.5, $\rho(S)^* = \rho(\sigma(S)) = \rho(S^{-1})$. □

Observation 4.9  Let $H$ be a non-compact real simple Lie group admitting highest weight representations\(^7\). Let $\rho_1, \ldots, \rho_n$ be unitary highest weight representations of $H$. Then the group $Br_n^\circ(R)$ acts in $\rho_1 \otimes \cdots \otimes \rho_n$ by unitary operators.

\(^7\)The list of such groups is: $SU(p,q)$, $Sp(2n, R)$, $SO^*(2n)$, $O(n,2)$ and two real forms of $E_6$ and $E_7$.  

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Proof. We apply the Knizhnik–Zamolodchikov construction (see above 1.4) and get a representation of \( \mathfrak{b}r_n \) in the tensor product. Decompose our tensor product with respect to the diagonal subgroup \( H \subset H \times \cdots \times H \),
\[
\rho_1 \otimes \cdots \otimes \rho_n = \bigoplus_{\mu} (V_\mu \otimes \mu),
\]
where \( \mu_j \) ranges in the set of highest weight representations of \( H \) and \( V_\mu \) are finite-dimensional spaces with trivial action of \( H \). The Lie algebra \( \mathfrak{b}r_n \) acts by \( H \)-intertwining operators, i.e., in the spaces \( V_\mu \). We apply Proposition 4.8. \( \square \)

5 Groups \( G' \)

5.1. Algebras \( U'(\mathfrak{g}) \). Fix \( C \). Denote by \( U'_a(\mathfrak{g}) \) the set of all elements \( z \in \overline{U}(\mathfrak{g}) \) satisfying
\[
[[z]]_a = \sup_p \|z^p\| \cdot p^a < \infty.
\]
Note that \( U'_a(\mathfrak{g}) \) is a Banach space with respect to the norm \( [[\cdot]]_a \).

We define \( U'(\mathfrak{g}) \) as
\[
U'(\mathfrak{g}) = \bigcup_a U'_a(\mathfrak{g}).
\]
Thus, \( U'(\mathfrak{g}) \) is an inductive limit of Banach spaces.

Lemma 5.1 a) \( U'(\mathfrak{g}) \) is an algebra.

b) Moreover if \( z \in U'_a(\mathfrak{g}) \), \( u \in U'_b(\mathfrak{g}) \), then \( zu \in U'_{a+b} \) and
\[
[[zu]]_{a+b} \leq [[z]]_a \cdot [[u]]_b.
\]

c) \( U'(\mathfrak{g}) \) does not depend on a choice of a basis in \( \mathfrak{g}^{[1]} \).

Proof. b) Let \( z \in U'_a(\mathfrak{g}) \), \( u \in U'_b(\mathfrak{g}) \), i.e.,
\[
\|z^p\| \leq \frac{a^p}{p!}, \quad \|u^q\| \leq \frac{a^q}{q!}.
\]
Therefore
\[
\|(zu)^r\| \leq \lambda \mu \sum_{p+q=r} \frac{a^p q^q}{p! q!} = \frac{\lambda \mu}{r!} (a+b)^r.
\]
Thus \( zu \in U'_{a+b}(\mathfrak{g}) \).

c) See proof of Lemma 4.3. \( \square \)

Example. Functions \( e^{\omega_1 \omega_2} \) and \( e^{[\omega_1, \omega_2]} \) are not contained in \( U'(\mathfrak{fr}_2) \). \( \square \)

5.2. Groups \( G' \). We define the group \( G' \) as
\[
G' := \overline{G} \cap U'(\mathfrak{g}).
\]
For any $a$ we define the group $G^a_a$ as the subgroup of $G^1$ generated by the subset $G \cap U^a(g)$. Evidently, if $a > b$, then $G^a_a \supset G^b_b$, and $G^1 = \cup G^a_a$.  

**Proposition 5.2** Represent $S \in Br^1_n$ as a product $S = T_{n-1} \cdots T_1$ of elements $T_j \in Fr_j$, see (3.4). Then all factors $T_j \in Br_n$.

See proof of Proposition 4.5.

**5.3. Ordered exponentials.** Let $\mu(t), t \in [0, T]$ be a continuously differentiable curve in $g^{[1]}$, 

$$\mu(t) = \sum \mu_j(t) \omega_j.$$  

**Proposition 5.3** The solution of the differential equation 

$$E'(t) = E(t) \mu(t), \quad E(0) = 1 \quad (5.1)$$  

is contained in $G^1$.

b) Moreover, the solution is contained in $\cap_a G^a_a$.

**Proof.** Let $\mu_\varepsilon(t)$ be a piecewise constant function such that $\| \mu(t) - \mu_\varepsilon(t) \| < \varepsilon$, let $\mu_\varepsilon = c_j$ on $(t_{j-1}, t_j)$. Denote by $E_\varepsilon(t)$ the solution of the corresponding differential equation. As we have seen above (Proposition 2.6) $E_\varepsilon(t)$ converges in $U(g)$ to $E(t)$. If $t_\alpha < t < t_{\alpha+1}$, then we have 

$$E_\varepsilon(t) = \exp \{(t_1 - t_0)c_1\} \exp \{(t_2 - t_1)c_2\} \cdots \exp \{(t - t_\alpha)c_{\alpha+1}\}$$

Keeping in mind Lemma 5.1 we get that $L_\varepsilon(t)$ is contained in the unit ball of the space $\sum_{j \in \mathbb{Z}} \| c_j \| U_{t_j}^{t_{j+1}}(g)$ and all $L_\varepsilon(t)$ are contained in the unit ball of a certain space $\cup_{j \in \mathbb{Z}} U_{t_j}^{t_{j+1}}(g)$. A limit as $\varepsilon \to 0$ is contained in the same ball.

b) We divide $[0, T]$ into small segments, solve equations 

$$E_j'(t) = E(t) \cdot \mu(t), \quad E_j(t_j) = 1.$$  

Then $E(T) = E_N(T) \cdots E_1(t_2)E_0(t_1)$.  

\[ \square \]

**6. Actions of groups $G^1_a(\mathbb{R})$ on manifolds**

Now we assume that the Lie algebra $g$ is real.

Let $M \subset \mathbb{C}^N$ be a closed complex submanifold satisfying the condition:

if $\xi \in M$, then the complex conjugate point $\bar{\xi}$ is contained in $M$. Let the intersection $M \cap \mathbb{R}^N$ be a smooth compact manifold.

Let the Lie algebra $g$ acts by holomorphic vector fields on $M$, denote by $\Omega_j$ vector fields corresponding to the generators $\omega_j$ of $g$. Assume that $\Omega_j$ are real on $M$.

Under these conditions we show that a certain group $G^1_a(\mathbb{R})$ acts on $M$ by analytic diffeomorphisms\footnote{The author does not knows, is it $G^1_a = G^1$?}.
6.1. Construction of the action. To any element of $z \in U(g)$ we assign the differential operator $\rho(z)$ on $M$,
\[\rho(\omega_{i_1}\omega_{i_2} \ldots \omega_{i_k}) = \Omega_{i_1}\Omega_{i_2} \ldots \Omega_{i_k}\]
For any element of $z = \sum z^{[p]} \in U(g)$ we assign the formal series $\rho(z) = \sum \rho(z^{[p]})$ of differential operators.

Denote by $\mathcal{A}(M)$ the algebra of entire functions on $M$ equipped with the topology of uniform convergence on compact subsets. More generally, for an open subset $V \subset M$ we denote by $\mathcal{A}(V)$ the algebra of analytic functions on $V$.

**Theorem 6.1**

a) For sufficiently small $a$ for any $z \in U_a^\prime(g)$ for any $F \in \mathcal{A}(M)$ the formal series $\rho(z)F$ uniformly converges in a neighborhood of $M$ in $M$.

b) Moreover, for $z \in G(\mathbb{R}) \cap U_a^\prime(g)$,
\[\rho(z)F(m) = F(Q_z(\xi))\]
for a certain analytic diffeomorphism $Q_z$ of $M$.

c) Represent $z \in G_a(\mathbb{R})$ as $z = z_1z_2 \ldots z_m$, where $z_j \in G(\mathbb{R}) \cap U_a^\prime(g)$. Then the formula
\[Q_z := Q_{z_m} \ldots Q_{z_2}\]
determines a well-defined homomorphism from $G_a(\mathbb{R})$ to the group of analytic diffeomorphisms of $M$.

d) Let $E(t) \in E$ be an ordered exponential as in Proposition 5.3. Let $E(a) = 1$. Consider a family of diffeomorphisms $R(t)$ such that
\[R(t)^{-1} \frac{d}{dt} R(t) = \rho(\mu(t)), \quad R(0) \text{ is identical}\]
Then the diffeomorphism $R(a)$ is identical.

6.2. Estimate of derivatives.

**Lemma 6.2**

There exist neighborhoods $U \supset V$ of $M$ in $\mathcal{M}$ and constants $c$, $A$ such that for any $F \in \mathcal{A}(M)$ for any collection $j_1, \ldots, j_p$ the following estimate holds
\[\sup_{m \in V} \Omega_{j_1} \ldots \Omega_{j_p} F(\xi) \leq A \cdot p!e^p \sup_{\xi \in U} |F(\xi)|. \quad (6.1)\]

**Proof.** Since the manifold $M$ is compact, the question is local. So let us write the vector fields $\Omega_j$ in coordinates
\[\Omega_j(\xi) = \sum_{k}^{\text{dim } M} \alpha_{j,k} \frac{\partial}{\partial \xi_k}\]
We open brackets in $\Omega_{j_1} \ldots \Omega_{j_p} F(\xi)$ and get $((\text{dim } M)^p$ summands. It is sufficient to obtain an estimate of the type (6.1) for each summand. Hence we assume that each $\Omega_j$ has the form
\[\Omega_j = \beta_j \frac{\partial}{\partial \xi_{k_j}}.\]
Next, we define operators 
\[ H_j(t) = F(\xi_1, \xi_2, \ldots) = F(\xi_1, \ldots, \xi_{k_j} + t, \xi_{k_j+1}, \ldots) \]
and write 
\[ \Omega_j \cdots \Omega_j F(\xi) = \left( \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_p} H_1(t_1) \cdots H_p(t_p) F(\xi) \right) \bigg|_{t_1=\cdots=t_p=0} \]
Let \( F, \beta_1, \beta_2, \ldots \) be holomorphic in the polydisk \( |\xi_\alpha| < \rho + \varepsilon \). Then 
\[ g(\xi, t) := H_1(t_1) \cdots H_p(t_p) F(\xi) \]
is holomorphic in the polydisk 
\[ |\xi_1| < \rho/2, \quad |\xi_1| < \rho/2, \ldots, |t_1| < \rho/2p, \ldots, |t_p| < \rho/2p. \]
We estimate derivatives by the multi-dimensional Cauchy formula (see, e.g., [5]).

\[ \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_p} g(\xi, t) \bigg|_{t_1=\cdots=t_p=0} = \left| \frac{(-1)^k}{(2\pi i)^k} \int_{|t_1|=\rho/2p} \cdots \int_{|t_p|=\rho/2p} \frac{g(\xi, s) \, ds_1 \cdots ds_p}{s_1^2 \cdots s_p^2} \right| \leq C \cdot \left( \frac{2p}{\rho} \right)^p \]
under the assumption \( |\xi_1| < \rho/2, \quad |\xi_1| < \rho/2, \ldots \) This is a desired estimate, recall that \( p! \sim \sqrt{2\pi p} (\frac{e p}{2})^p \). □

6.3. Proof of Theorem 6.1.b. The statement a) follows from Lemma 6.2. More precisely, there is a neighborhood \( V \subset \mathcal{M} \) and \( b \) such that for any \( z \in U_b^!(\mathfrak{g}) \) the operator \( \rho(z) \) is a well-defined map 
\[ \rho(z) : \mathcal{A}(\mathcal{M}) \to \mathcal{A}(V). \]
Moreover, for \( u, v \in U_b^!(\mathfrak{g}) \) we have 
\[ \rho(u) \rho(v) = \rho(uv). \]
The last equality is reduced to a permutation of absolutely convergent series.

Lemma 6.3 For \( S \in G^! \cap \overline{U}_{b/2}, F_1, F_2 \in \mathcal{A}(\mathcal{M}) \) we have 
\[ \rho(S)(F_1 \cdot F_2) = \rho(S)F_1 \cdot \rho(S)F_2. \]

Proof. Let \( \delta : \overline{U}(\mathfrak{g}) \to \overline{U}(\mathfrak{g}) \otimes \overline{U}(\mathfrak{g}) \) be the co-product in enveloping algebra as above. Let \( \delta(z) = \sum x_i \otimes y_i \). By the Leibnitz rule, 
\[ \rho(z)(F_1 F_2) = \sum \rho(x_i) F_1 \cdot \rho(y_i) F_2. \]
But \( S \) satisfies the condition \( \delta(S) = S \otimes S \). □
Let $S \in G^t \cap \overline{U}_{b/2}(g)$. Fix a point $v \in V$. We define a map $\theta_v : \mathcal{A}(\mathcal{M}) \to \mathbb{C}$, by

$$\theta_v(F) = \rho(S)F(v).$$

Then

$$\theta_v(F_1 + F_2) = \theta_v(F_1) + \theta_v(F_2), \quad \theta_v(F_1F_2) = \theta_v(F_1)\theta_v(F_2).$$

Thus we get a character $\mathcal{A}(\mathcal{M}) \to \mathbb{C}$. Since the manifold $\mathcal{M}$ is Stein\(^{10}\), we get

$$\theta_v(F) = F(w) \quad \text{for some } w = : \mathcal{Q}_S(v) \in \mathcal{M}. \quad \square$$

Therefore we get a map $\mathcal{Q}_S : V \to \mathcal{M}$. Note that $F \circ \mathcal{Q}_S$ is holomorphic for holomorphic $F$ and

$$F \circ \mathcal{Q}_S \circ \mathcal{Q}^{-1}_S = F = F \circ \mathcal{Q}^{-1}_S \circ \mathcal{Q}_S.$$ 

Therefore $\mathcal{Q}_S$ is a holomorphic embedding.

Moreover, $\mathcal{Q}_T \mathcal{Q}_S = \mathcal{Q}_ST$ for $S, T \in G^t(\mathbb{R}) \cap \overline{U}_{b/2}(g)$.

6.4. Proof of Theorem 6.1.c. The statement c) of Theorem 6.1 is a corollary of the following lemma and the statement d) is a corollary of c).

Lemma 6.4 Let $z_1, \ldots, z_m \in G^t_\tau(\mathbb{R})$. Let $z_1 \ldots z_m = 1$ in $G^t_\tau(\mathbb{R})$. Then $\mathcal{Q}_{z_1} \ldots \mathcal{Q}_{z_m} = 1$.

Proof of Lemma. For a non-zero $\tau \in \mathbb{C}^*$ we define the automorphism $A_\tau : \overline{U}(g) \to \overline{U}(g)$ by

$$A_\tau(z) = A_\tau(\sum z^{[p]}) = \sum \tau^p z^{[p]}.$$ 

Evidently, $A_\tau$ determines the automorphism of $G^t(\mathbb{C})$.

It can be easily checked that for $|\tau| < 1$ the map $\rho(A_\tau z) : \mathcal{A}(\mathcal{M}) \to \mathcal{A}(V)$ depends on $\tau$ holomorphically. Hence, for $S \in G^t_\tau(\mathbb{R})$ and $\tau \in (0, 1)$ the map $\mathcal{Q}_{A_\tau S}$ depends on $\tau$ real analytically.

Let $\tau \in (0, 1)$ and $z_j \in G^t_\tau(\mathbb{R})$ be as above. Then

$$(A_\tau z_1) \ldots (A_\tau z_1) = 1$$

On the hand $A_\tau z_j \in G^t_{\tau b}(\mathbb{R})$. Therefore, for $\tau < b/m$ we have

$$\mathcal{Q}_{A_\tau z_1} \ldots \mathcal{Q}_{A_\tau z_m} = 1.$$ 

But the left hand side depends analytically on $\tau$, therefore the identity holds for all $\tau \in (0, 1)$. \square

\(^{10}\)See the preamble to this section, on Stein manifolds see, e.g., [5].
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Math. Dept., University of Vienna, Nordbergstrasse, 15, Vienna, Austria

&

Institute for Theoretical and Experimental Physics, Bolshaya Cheremushkinskaya, 25, Moscow 117259, Russia

&

Mech. Math. Dept., Moscow State University, Vorob'yev Gory, Moscow

e-mail: neretin(at) mccme.ru

URL: www.mat.univie.ac.at/~neretin

www.th.itep.ru/~neretin