Probability Density Functions from the Fisher Information Metric

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ABSTRACT

We show a general relation between the spatially disjoint product of probability density functions and the sum of their Fisher information metric tensors. We then utilise this result to give a method for constructing the probability density functions for an arbitrary Riemannian Fisher information metric tensor. We note further that this construction is extremely unconstrained, depending only on certain continuity properties of the probability density functions and a select symmetry of their domains.

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1. Introduction

Information geometry is the study of the natural differential structures which arise on the space of families of probability density functions. The Fisher information metric defines a notion of the distance between two particular members of a family of probability density functions and is the natural measure arising out of the small change expansion of the Kullback-Liebler divergence [1]. The existence of such a distance measure is of obvious utility for answering questions related to, for example, the mutual information of two systems described by different probability density functions, the likely error made in approximating one distribution by another, and even a definition of a gradient descent algorithm consistent with the differential geometric structure of a probability space [2].

The study of information geometry was first expounded upon in detail by Shun’ichi Amari and the foundations were laid out in [3]. A great deal is now known about the geometric properties of information manifolds. In particular, given a family of probability density functions, the associated Fisher information metric may be stated as a concrete integral (or sum in the case of discrete variables). However, comparatively little is known about the ‘reverse’ operation. That is, given a Riemannian metric tensor, what can be said about the family of probability density functions which are naturally endowed with such a metric tensor? In this short note we show how one can, in theory, perform this inverse process and observe that it is far from one-to-one.
Our interest in the subject is not from the point of view of machine learning or information theory as such. In recent years, a new link has surfaced between information geometry and the study of space-time as an emergent phenomenon. Within string theory there has been much work over the last 15 years in the study of how the dynamics of interacting gauge theories in the limit of a large number of gauge degrees of freedom can give rise to emergent spacetimes of a variety of geometries. The most natural such structure arises out of a scale-free gauge theory providing, holographically, an anti-de Sitter space [4] – the so-called AdS/CFT correspondence. Coincidentally, the Euclidean version of anti-de Sitter space (a hyperbolic geometry) is a geometry which emerges frequently from a large class of different probability density functions. Indeed in the construction used by Hitchin [5], such a space arises naturally out of symmetry arguments when the Fisher information metric tensor is computed from the instanton moduli space in such gauge theories. In [6] these two ideas were tied together, showing how Information Geometry seemed to give a natural means for calculating emergent geometries in an AdS/CFT context. Interesting relationships between information geometry, quantum information and string theory/holography have been studied also in [7], [8], [9] and [10].

In what follows, we explore in more detail the link between information and geometry.

2. The Fisher information metric

2.1. Families of probability density functions and their associated geometries

For the purposes of this work, we will assume a narrow definition of a family of probability density functions. That is, when we write ‘family of probability density functions’ we will mean a family of continuous functions \( P_\theta : X \rightarrow \mathbb{R} \) for some domain \( X \subset \mathbb{R}^n \), parameterised over \( \theta \in M \subset \mathbb{R}^m \) (ie. an \( m \)-parameter family of distributions). Coordinatizing \( X \) by \( x = (x^1, \ldots, x^n) \) and the parameter space \( M \) by \( \theta = (\theta^1, \ldots, \theta^m) \), we will also further require that \( \partial_a P_\theta := \frac{\partial P_\theta}{\partial \theta^a} \) is continuous on \( X \) for all \( \theta \in M \). Furthermore, we will also require that every member of the family be normalised, that is,

\[
(\forall \theta \in M) \int_X P(x; \theta) \, dx = 1.
\]

All of this may be succinctly restated as \( \{ P_\theta \} \) being a parametrised family of normalised, continuous functions which changes ‘smoothly’ over parameter space. Finally, we will refer to \( X \) as the spatial domain and \( M \) as the parametric domain, and conventionally associate the spatial domain \( X \) to probability density function \( P_i \).
We now define the Fisher Information metric tensor on a finite dimensional statistical manifold. Given such a manifold, \( M \), whose points form a family of probability density functions with the properties listed above, there exists a Riemannian metric tensor on \( M \), viz.,
\[
g_{ab}(\theta) = \int_X P(x; \theta) \partial_a \ln P(x; \theta) \partial_b \ln P(x; \theta) \, dx. \tag{2.1.1}
\]
The central question addressed in this paper may thus be stated as: given a Riemannian metric tensor \( g \), under what circumstances can a family of probability density functions \( P \) be found such that the Fisher information metric tensor of \( P \) is \( g \).

2.2. Some examples

In order to build some intuition for the relationship between a family of probability density functions and their associated metrics, we give here two examples of the computation of the Fisher metric.

2.2.1. Univariate Normal Distribution

Here the family of probability density functions is given by
\[
P(x; \theta) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}.
\]
The distribution is parameterised by \( \mu \) and \( \sigma \), which we will collectively denote \( \theta \). Put another way, the manifold coordinates are given by \( \theta = (\mu, \sigma) \), and the random variable is \( x \in \mathbb{R} \). Note that the parametric domain is \( \mathbb{R} \times \mathbb{R}^0 \times \mathbb{R}^0 \). In order to compute \( g_{ab} \) we must compute \( \partial_a \ln P \)
\[
\ln P = -\left[ \frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2 + \ln \sigma + \ln \sqrt{2\pi} \right],
\]
\[
\frac{\partial}{\partial \mu} \ln P = \frac{1}{\sigma} \left( \frac{x-\mu}{\sigma} \right), \quad \frac{\partial}{\partial \sigma} \ln P = \frac{1}{\sigma} \left[ \left( \frac{x-\mu}{\sigma} \right)^2 - 1 \right].
\]
Then, using Equation 2.1.1, the Fisher metric for the univariate normal distribution has
\[
[g] = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^2} \end{bmatrix} \implies ds^2 = \frac{d\mu^2 + 2d\sigma^2}{\sigma^2}.
\]
Thus we see that the Fisher metric, in this case, describes the metric tensor of a two-dimensional hyperbolic geometry. The structure on this geometry can be intuitively understood by the properties of normal distributions. In particular, for distributions with \( \sigma \gg 1 \), the associated ‘difference’ between two distributions with means \( \mu_1 \) and \( \mu_2 \) is less pronounced — they are harder to distinguish. For two sharply peaked distributions (\( \sigma \ll 1 \)) with even similar \( \mu \), the difference will be very pronounced and so they are easy to distinguish. Hence the hyperbolic nature of the space.
2.2.2. Cauchy Distribution

The family of probability density functions for this distribution is given by

$$P(x; x_0, \gamma) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (x - x_0)^2}.$$ 

Thus, the parameter space for this family is spanned by the parameters $\theta = (x_0, \gamma) \in \mathbb{R} \times \mathbb{R}^{>0}$ and the calculation of the logarithmic derivatives gives

$$\ln P = \ln \gamma - \ln \left[\frac{\gamma}{\gamma^2 + (x - x_0)^2}\right] - \ln \pi,$$

$$\frac{\partial}{\partial x_0} \ln P = \frac{2(x - x_0)}{\gamma^2 + (x - x_0)^2}, \quad \frac{\partial}{\partial \gamma} \ln P = \frac{1}{\gamma} - \frac{2\gamma}{\gamma^2 + (x - x_0)^2}.$$ 

As such, it is a simple matter to verify that the Fisher metric for the Cauchy distribution is given by

$$g_{ab} = \frac{\delta_{ab}}{2\gamma^2} \implies ds^2 = \frac{1}{2} \left(\frac{dx_0^2 + d\gamma^2}{\gamma^2}\right).$$ 

The reader may wish to note that while we started with a very different distribution, the geometric structure described by its Fisher metric is very close to that of the normal distribution. In this sense, hyperbolic spaces (or Euclidean anti-de-Sitter spaces) appear ubiquitous in an information geometric context.

3. Reversing the Fisher information metric

It is not clear at first glance that it is at all possible to reverse the process of computing the Fisher metric in any meaningful way, as the exercise involves a definite integral of multiple powers of the underlying family of probability density functions. We present below a motivating example to suggest that under certain, constrained situations such a process is indeed possible. As a prototype for a more general construction, we demonstrate how to encode the metric tensor of $S^n$, for any $n \in \mathbb{N}$, in a family of one dimensional probability density functions.

3.1. The $n$–dimensional sphere, $S^n$

We begin our exploration of reversing the Fisher information computation with a one-dimensional family of probability density functions. In particular, we leverage the properties of orthonormal functions to produce a family of probability density functions which, with an appropriate set of functions $h^i$, give rise to the metric tensor of $S^n$.

Note that, for our purposes, a family of univariate, real-valued functions $\{f_i(x)\}_{i \in I}$ is said to be orthonormal with weight $w(x)$ over a domain $X$ if $\int_X f_i(x)f_j(x)w(x)dx = \delta_{ij}$. 

Proposition 3.1. Let $M \subset \mathbb{R}^n$ and $h^i \in C^1(M)$ such that $(\forall \theta \in M) h^i h^j \delta_{ij} = 4$ and $\{f_i(x)\}_{i}^n$ be a set of orthonormal, real-valued functions with positive semidefinite weight $w(x)$ over $X \subset \mathbb{R}$. Then the family of probability density functions

$$P(x; \theta) = \frac{1}{4} \left( \sum_{i=1}^{n} h^i(\theta) f_i(x) \right)^2 w(x), \quad (3.1.1)$$

gives the Fisher information metric tensor $g_{ab} = (\partial_a h^i)(\partial_b h^j) \delta_{ij}$.

Proof. That $P$ is normalised follows trivially from the orthonormality of $f_i$.

$$\frac{1}{4} \int_X \left( \sum_{i=1}^{n} h^i(\theta) f_i(x) \right)^2 w(x) \, dx = \frac{1}{4} \int_X \left( \sum_{i=1}^{n} \sum_{j=1}^{n} h^i h^j f_i f_j \right) w \, dx = \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_X h^i h^j f_i f_j w \, dx = \frac{1}{4} h^i h^j \delta_{ij} = 1.$$ 

A straightforward computation gives the desired result.

$$g_{ab} = \int_X P(\partial_a \ln P)(\partial_b \ln P) \, dx$$

$$= \int_X w \left( \sum_{i=1}^{n} h^i f_i \right)^2 \left( \sum_{i=1}^{n} (\partial_a h^i) f_i \right) \left( \sum_{i=1}^{n} (\partial_b h^i) f_i \right) \, dx$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \int_X (\partial_a h^i)(\partial_b h^j) f_i f_j w \, dx = (\partial_a h^i)(\partial_b h^j) \delta_{ij}.$$ 

Now we pause to note that we may view the above statement, $g_{ab} = (\partial_a h^i)(\partial_b h^j) \delta_{ij}$, as the result of applying the transition functions $h$ to the flat Euclidean metric $\delta$. As such, and noting that we required $h^i h^j \delta_{ij} = 4$, we immediately infer that

Corollary 3.2. The metric tensor of $S^n$ can be reached as the Fisher Information metric of the distribution Equation 3.1.1 where $h$ is the transition function from $\mathbb{E}^n$ to $4S^n$, the $n$-dimensional sphere of radius four.

In the above we have shown a general way to find a given metric tensor in terms of the transition functions from flat Euclidean space to a desired geometry. However, there is a specific condition on the $h^i$ given by $h^i h_i = 4$ which constrains these strongly. In what follows, we will generalise this result in a way which will remove this constraint.

\footnote{Here we use Einstein summation and the lowered and raised indices have no differential geometric interpretation other than to aid in the appropriate summations}
3.2. The Gaussian construction

Now that we have reason to believe that it is possible, at least in special cases, to pick a metric tensor and construct a family of probability density functions whose Fisher information metric is the selected metric, we attempt to extend our results to arbitrary Riemannian metrics.

Consider a family of probability density functions given by a product of \( n \), uncorrelated, disjoint, one-dimensional Gaussian probability density functions with unit variance. Explicitly,

\[
P(x; \theta) = \frac{1}{\sqrt{(2\pi)^n}} \exp \left( -\frac{1}{2} \sum_{i=1}^{n} (x^i - h^i(\theta))^2 \right),
\]

(3.2.1)

where \( M \), the parametric domain, is not yet fixed, \( X = \mathbb{R}^n \), and \( h^i \in C^1(M) \). From this, we may compute the Fisher information metric as follows

\[
g_{ab} = \frac{1}{\sqrt{(2\pi)^n}} \int_X dx e^{-\frac{1}{2} \sum_{i=1}^{n} (x^i - h^i)^2} \left( \sum_{j=1}^{n} (\partial_a h^j)(x^j - h^j) \right) \left( \sum_{k=1}^{n} (\partial_b h^k)(x^k - h^k) \right)
\]

\[
= \frac{1}{\sqrt{(2\pi)^n}} \int_X dx e^{-\frac{1}{2} \sum_{i=1}^{n} (x^i - h^i)^2} \left( \sum_{j=1}^{n} (\partial_a h^j)(\partial_b h^j)(x^j - h^j)^2 + \text{vanishing cross-terms} \right)
\]

\[
= \sum_i \left\{ (\partial_a h^i)(\partial_b h^i) \prod_k \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx^k e^{-\frac{1}{2} (x^k - h^k)^2} (x^k - h^k)^2 \right) \right\}.
\]

It is a simple matter to complete the computation to obtain

\[
g_{ab} = (\partial_a h^i)(\partial_b h^k) \delta_{jk}.
\]

(3.2.2)

This result allows us enough flexibility to be able to always give an \( h \) and \( M \) such that \( g_{ab} \) may be constructed as desired. In particular, we may begin at Equation 3.2.2 and read backwards to find Equation 3.2.1. In doing so, we fix a desired \( g_{ab} \) and accompanying manifold \( M \), and attempt to realise an \( h \) and \( M \) for which Equation 3.2.2 would hold. Unlike the case of Proposition 3.1, which came with the constraint \( h^i h_i = 4 \), this process is here always possible.

The Nash Embedding Theorem [11] tells us that there is an \( n \in \mathbb{N} \) such that \((M, g)\) may be \( C^1 \) isometrically embedded in \((\mathbb{E}^n, \delta)\). Specifically then, it tells us that there exists an \( h \) such that \( g = h^* \delta \). As such, interpreting Equation 3.2.2 as the statement that \( g \) is the pullback of \( \delta \) via \( h \) we see that we need only select an \( n \) large enough to accommodate the Nash embedding of the desired manifold \( M \) in \( \mathbb{E}^n \) (which is always possible) and we have \( h \) and \( M \) to satisfy the arrangement. Consequently, we have a family of probability density functions, given by Equation 3.2.1, whose Fisher information metric is the desired, arbitrary Riemannian metric.
Said another way, Equation 3.2.2 states simply that \( g_{ab} \) is the pullback from a higher dimensional flat space to a manifold embedded in that space, via \( h \). In the case of coincidence of dimensions between \( g \) and \( h \), the result bears the simple interpretation of \( h \) acting as a set of transition functions from \( \delta \) to \( g \).

### 3.2.1. The metric of \( S^2 \)

To cement the understanding of the importance and generality of Equation 3.2.2 we construct the metric tensor of \( S^2 \). Suppose we desire a family of probability density functions whose Fisher information metric is the metric tensor of \( S^2 \). Specifically, if the unit sphere has line element
\[
d s^2 = d\theta^2 + \sin^2 \theta d\phi^2,
\]
then we can proceed as outlined above, and write down a set of transition functions
\[
h = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi),
\]
from \( \mathbb{E}^3 \) to the embedded \( S^2 \). Applying the construction of Equation 3.2.1 we find
\[
P(x, y, z; \theta, \phi) = (2\pi)^{-3/2} e^{-\frac{1}{2}\left((x-\cos \theta \sin \phi)^2 + (y-\sin \theta \sin \phi)^2 + (z-\cos \phi)^2\right)}.
\]

This is easily recognisable as a product of three Gaussian probability density functions, each with a mean which is periodic in the parameters. This means that we have the geometry and topology of a sphere, where each point on the sphere corresponds to a three dimensional Gaussian distribution with unit variance and mean denoted by the point on the sphere. This exercise can be performed for any \( S^n \) by simply forming the appropriate \( h \).

The ease with which we are able to perform this construction is indicative of the power underlying Equation 3.2.2 and the accompanying statement that any Riemannian metric tensor may be reached via this construction.

### 3.3. The hyperbolic secant construction

In the previous subsection we gave a construction based upon a product of Gaussian probability density functions and demonstrated its flexibility. Now we demonstrate that the above-mentioned results are just as achievable with an entirely different family of probability density functions. Consider the family
\[
P = \frac{1}{\pi^n} \prod_{i=1}^{n} \text{sech} \left( x^i - h^i \sqrt{2} \right).
\]
Other than the functional dependence on $\sim x^i - h^i$, this is entirely different from the Gaussians discussed earlier. However, computing the Fisher information metric we find the result to be of that most general form
\[
g_{ab} = (\partial_a h^i)(\partial_b h^j)\delta_{ij}.
\]
Naturally, this bears the same interpretation as the previous result and serves to suggest that relatively little of the information about the original family of probability density functions is carried through to the metric tensor itself.

The careful reader will note that we now have two means to the same end, and may wonder just how many more ways we may achieve the above result. Indeed the following section serves to introduce a general framework which will show that the answer is that there is an infinite-fold degeneracy in the construction, and thus there is always an infinite to one mapping between families of PDFs and Riemannian metrics via the Fisher information metric.

## 4. General results

In this section we will elaborate on a more general set of statements which allow for definitions independent of dimensionality and functional dependence of the parameters of the PDF in question. We begin by showing how to construct a family of spatially disjoint probability density functions out of individual families of probability density functions.

**Definition 4.1.** The spatially disjoint product of two families of probability density functions on the same parametric domain, $P_1 = P_1(x^1, \ldots, x^k; \theta) : X_1 \times M \to \mathbb{R}$ and $P_2 = P_2(x^1, \ldots, x^n; \theta) : X_2 \times M \to \mathbb{R}$, is defined as
\[
(P_1 \circ P_2)(x^1, \ldots, x^{n+k}; \theta) = P_1(x^1, \ldots, x^n; \theta) \cdot P_2(x^{n+1}, \ldots, x^{n+k}; \theta).
\]
Note that $P_1 \circ P_2 : (X_1 \times X_2) \times M \to \mathbb{R}$ and we write $P^{\circ n}$ where we mean $\bigcirc_{i=1}^n P$.

Given this, we will here show how a special property of spatially disjoint products underpins all the general results achieved in this work. That is, the Fisher information metric transforms the spatially disjoint product of probability density functions into a sum of their corresponding, individually considered metric tensors.

**Theorem 4.2.** If $P = P(x; \theta)$ is a probability density function with a decomposition $P = \bigcirc P_i^{\circ e_i}$ for some $P_i$ and $e_i \in \mathbb{N}^+$ then $g_{ab}(\bigcirc P_i^{\circ e_i}) = \sum e_i g_{ab}(P_i)$.

**Proof.** Let us rewrite $P = \bigcirc \hat{P}_i^{\circ e_i} = \bigcirc P_j$ where each $\hat{P}_i$ has been accumulated into the spatially disjoint product $e_i$ times, that is, $\hat{P}_j = \hat{P}_i$ for $e_i$ many $j$. Then, in order to compute $g(P)$ we expand logarithmic derivatives to arrive at
\[
g_{ab}(P) = \sum_i \sum_j \int_X dx \frac{P}{P_i P_j} (\partial_a P_i)(\partial_b P_j).
\]
To proceed we must evaluate the double sum, and to do so we examine the cases $j = i$ and $j \neq i$ separately. In the event of the latter, $j \neq i$, we have

$$\int_X dx \frac{P_i P_j}{P_i P_j} (\partial_a P_i)(\partial_b P_j) = \left( \int_{X_i} dx^a \cdots dx^k \partial_a P_i \right) \left( \int_{X_j} dx^m \cdots dx^r \partial_b P_j \right),$$

where we have expanded the integral as a product over its disjoint spatial domains and have suppressed all other terms as they were of the form $\int_{X_i} dx^a \cdots dx^k P_i = 1$. Moreover, we note that $P_i$ satisfies the conditions (by the definition of the probability density function) for the exchange of integral and derivative and so

$$\int_{X_i} dx^a \cdots dx^k \partial_a P_i = \partial_a \int_{X_i} dx^a \cdots dx^k P_i = \partial_a (1) = 0.$$

Thus contributions from terms where $j \neq i$ is zero. On the other hand, the cases for which $i = j$ admit simple resolution as

$$\int_X dx \frac{P_i}{P_i^2} (\partial_a P_i)(\partial_b P_i) = \int_{X_i} dx^a \cdots dx^k (\partial_a P_i)(\partial_b P_i) \frac{1}{P_i} = g_{ab}(P_i),$$

where again we have expanded the integral as a product and suppressed all terms whose integral was one. Finally, we recall that we had exactly $e_i$ many $P_j$ such that $P_j = \hat{P}_i$ and so we collect $e_i$ many such contributions of $g_{ab}(P_i)$. \hfill \Box

**Remark 4.3.** That we essentially require $M_1 = M_2 = M$ in the definition of the spatially disjoint product is a matter of some subtlety. Consider that if $M_1 \neq M_2$ we would be within reason to set $M = M_1 \times M_2$ and reinterpret the definition as

$$(P_1 \odot P_2)(x^1, \ldots, x^{n+k}; \theta, \phi) = P_1(x^1, \ldots, x^n; \theta) \cdot P_2(x^{n+1}, \ldots, x^{n+k}; \phi).$$

In this case, however, $g(P)$ is not strictly the sum of $g(P_i)$ as the latter may all be of different dimension. Simply re-interpreting $P_i$ to have enlarged parametric domain $M$ will not solve this problem as then it may happen that $g(P_i)$ will no longer be non-degenerate and so not a metric tensor. Thus, the direct ability of the above result to “glue” together disjoint metric tensors is apparent, but nuanced and not an immediate consequence of the exposition given.

In effect then, care should be taken when examining the statement $g(\bigodot P_i) = \sum g(P_i)$ so as to ensure that it is done with the understanding that $g(P_i)$ is to have zero entries where appropriate for the purpose of the sum, but not when considered as its own metric tensor. More formally, we could write $g(\bigodot P_i) = \sum \tilde{g}(P_i)$ where $\tilde{g}$ is expressed precisely as $g$, but is extended to all of $M$ as suggested above, and is free from interpretation as a metric tensor. Hereafter, it is taken for granted that such nuances are appreciated by the reader. \hfill \Diamond

The importance of Theorem 4.2 cannot be overstated. From here on, it is simply a matter of finding convenient forms of $g_{ab}(P_i)$ for some parameterisation of $P_i$ so that we may take $\bigodot P_i$ and arrive at a desired metric tensor. That is, if we can find a $P_i$ such that
$g_{ab}(P_i) \propto (\partial_a h^i)(\partial_b h^i)$ then we can take $P = \bigcap P_i$ to find $g_{ab} \propto (\partial_a h^i)(\partial_b h^i)\delta_{ij}$ by the above. Here, the whole is more than the sum of its parts – given $g_{ab} \propto (\partial_a h^i)(\partial_b h^i)\delta_{ij}$ we are able to find an $h$ for our desired manifold and then create a desired $P$ out of constituent $P_i$, each containing some part of $\{h^i\}$. Beginning with disjoint $P_i$, however, the qualities which the individual distributions should exhibit, to attain a given $g$, are not clear. Furthermore, we note here that while $\bigcap P_i$ will yield the desired result, if we find multiple families of probability density functions, we may equally well combine them to achieve the same result.

Thus, what we really seek are simple forms of functional dependence of families of probability density functions upon our set of differentiable functions $h$ so that explicit computations may be made. Recall that we saw, in the calculations in subsections $3.2$ and $3.3$ that we may leverage reparameterisation invariance of spatial domains to our advantage. Such symmetries of the spatial domain allow us to essentially eliminate any functional dependence of the integrals upon the $h^i$ and produce multiplicative factors of $\partial_a h$ in the process. To that end, we explore a generalisation of the symmetry used in the above-mentioned subsections.

**Proposition 4.4.** Fix a one-dimensional probability density function $\hat{P}(x)$ on $X$ for which $X$ remains invariant under the change of variables $y = f(x; \theta)$, for some differentiable family of diffeomorphisms $f : X \times M \to X$ (the parameter space is $M$) and let $P(x; \theta) = f_x(x; \theta)\hat{P}(f(x; \theta))$ such that $\partial_a P \neq 0$ where we write $f_x$ for $\frac{\partial f}{\partial x}$ and $f_a$ for $\partial_a f$. Then

$$g_{ab}(P) = \int_X \frac{f_x f_{bx}}{(f_x)^2} \hat{P}(y) + \left(\frac{\partial (f_a f_b)}{\partial y} + f_a f_b \frac{\ln \hat{P}(y)}{dy} \right) \frac{d\hat{P}(y)}{dy} dy,$$

where we assume that we have written all functions in terms of $y = f(x; \theta)$ using the expression $x = f^{-1}(y; \theta)$ where necessary.

**Proof.** We first check that $P(x; \theta) = f_x(x; \theta)\hat{P}(f(x; \theta))$ is normalised. To that end, let $y = f(x; \theta)$

$$\int_X P dx = \int_X f_x \hat{P} dx = \int_X f_x \hat{P} \frac{dy}{f_x} = 1.$$

Then we compute the logarithmic derivatives necessary for the Fisher information metric

$$\partial_a \ln P = \frac{1}{f_x P(f)} \left(\frac{d\hat{P}(f)}{df}(f_a f_x) + \hat{P}(f)(f_{ax})\right).$$

We proceed with the computation by making the change of variables $y = f(x; \theta)$

$$g_{ab} = \int_X \frac{1}{(f_x)^2 P(f)} \left(\frac{d\hat{P}(y)}{dy}(f_a f_x) + \hat{P}(y)(f_{ax})\right) \left(\frac{d\hat{P}(y)}{dy}(f_b f_x) + \hat{P}(y)(f_{bx})\right) dy$$

$$= \int_X f_a f_b \frac{d\hat{P}(y)}{dy} \frac{d ln P(y)}{dy} + \frac{f_x f_{bx}}{(f_x)^2} P(y) + \left(\frac{f_a f_{bx} + f_b f_{ax}}{f_x} f_x \frac{d P(y)}{dy}\right) dy.$$

Finally, we recognise that $\frac{\partial}{\partial x} = f_x \frac{\partial}{\partial y}$ and that $f_a f_{bx} + f_b f_{ax} = \frac{\partial (f_a f_b)}{\partial x}$, and collect terms to arrive at the result. \qed
Of course, examining symmetry at such an abstract level cannot be expected to yield concrete answers immediately and so that the statement of Proposition 4.4 is opaque and not obviously useful is not surprising. Indeed, in what follows we make various simplifying assumptions about the functional form of the symmetry function \( f \) to arrive at generalisations of familiar results.

We begin by noticing that there is a term in Equation 4.4 which is proportional to \( f_a f_b \). If it could be arranged that \( f_a f_b \) be independent of \( y \), then we could simply extract a term proportional to \( f_a f_b \) from the result – a term whose importance we already know. Moreover, if we could ensure that the other terms vanish, we would have \( g_{ab} \propto f_a f_b \) and achieve our general result once more.

To that end, we choose to require that \( f_x \) be constant and \( f_{ax} = 0 \). Although this is likely not the only way to achieve our desired effect, it will certainly suffice. In this case, we see immediately that \( f(x; \theta) = cx + h(\theta) \) is the general solution – but this is nothing other than the statement of translation invariance. Thus, we may achieve the following results by means of Proposition 4.4.

**Proposition 4.5.** Fix a one-dimensional probability density function \( \hat{P} \) such that the change of variables \( y = x - h \) for \( h(\theta) \) a differentiable function on \( M \subset \mathbb{R}^m \) leaves the spatial domain \( X \) unchanged. Let \( P(x; \theta) = \hat{P}(x - h) \) then \( g_{ab} = (\partial_a h)(\partial_b h)D \) where

\[
D = \int_X dx \left( \frac{\partial P(x)}{\partial x} \right) \left( \frac{\partial \ln P(x)}{\partial x} \right).
\]

**Proof.** Apply Proposition 4.4 to \( f(x; \theta) = x - h(\theta) \). \( \square \)

**Corollary 4.6.** Fix one-dimensional probability density functions \( P_i \) and let \( h^i(\theta) \) be differentiable on \( M \subset \mathbb{R}^m \) and write \( y^i = x^i - h^i \) such that \( X_i \) is unchanged under this change of variables for all \( i \). \( P(x; \theta) = \bigotimes P_i(x^i - h^i) \) gives \( g_{ab}(P) = (\partial_a h^i)(\partial_b h^j)D_{ij} \) where

\[
D_{ij} = \begin{cases} 
eq \delta_{ij} & \text{if } i \neq j \\ \delta_{ij} & \text{if } i = j \end{cases}
\]

\[
D_{ij} = \left\{ \begin{array}{ll} c_i \int_{X_i} dx^i \left( \frac{\partial P_i}{\partial x^i} \right) \left( \frac{\partial \ln P_i}{\partial x^i} \right), & i = j \\ 0, & i \neq j \end{array} \right.
\]

**Proof.** Combine Proposition 4.5 and Theorem 4.2. \( \square \)

**Remark 4.7.** When \( P_i \) are all Gaussian, \( D_{ij} = \delta_{ij} \) and so the result of Equation 3.2.2 follows as a special case. \( \diamond \)

To demonstrate how one might achieve the encoding of an arbitrary Riemannian metric tensor into a spatially disjoint product of one-dimensional families of probability density functions, consider the following example.
Example 4.8. Suppose we desire a hyperbolic metric tensor \( g \) whose associated line element is given by \( \frac{1}{2}(d\alpha^2 + d\beta^2) \), on the open subset \( M = \{ (\alpha, \beta) \in \mathbb{R}^2 \mid \beta > 1 \} \subset \mathbb{H}^2 \). With some work, it can be shown that an isometric embedding of \( M \) into \( \mathbb{R}^3 \) can be achieved through the function

\[
h = \left( \frac{\cos \alpha}{\beta}, \frac{\sin \alpha}{\beta}, \ln \left( \beta + \sqrt{\beta^2 - 1} \right) - \frac{\sqrt{\beta^2 - 1}}{\beta} \right).
\]

That is, \( g = h^* \delta \). Moreover, it is evident that \( h \) is at least \( C^1 \) so we may apply our construction to it and write, for example,

\[
P = P_1(x - h^1) \odot P_2(y - h^2) \odot P_3(z - h^3),
\]

for any one-dimensional probability density functions \( P_i \) which satisfy translation invariance as outlined in Proposition 4.5. By Corollary 4.6 we then know that \( g(P) = h^* D \) and so the result follows in the case that \( D = \delta \).

In particular then, we may choose to let \( X_i = \mathbb{R} \) for \( i \in \{ 1, 2, 3 \} \) and put

\[
\hat{P}_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \hat{P}_2(x) = \frac{1}{\pi} \sech x, \quad \hat{P}_3(x) = \frac{1}{\pi (1 + x^2)},
\]

for which \( D_1 = 1 \) and \( D_2 = D_3 = \frac{1}{2} \). Thus, taking the values of \( D_i \) into account, we may write \( P(x, y, z; \alpha, \beta) = \hat{P}_1(x - h^1) \odot \hat{P}_2(y - \sqrt{2} h^2) \odot \hat{P}_3(z - \sqrt{2} h^3) \) to recover

\[
P(x, y, z; \alpha, \beta) = \left( \frac{2\pi}{\beta} \right)^{-1} \frac{\sech \left( x - \frac{\sqrt{2}\sin \alpha}{\beta} \right) e^{-\frac{1}{2} \left( y - \frac{\sqrt{2}\sin \alpha}{\beta} \right)^2}}{1 + \left[ z + \frac{\sqrt{2}\sin \alpha}{\beta} - \sqrt{2} \ln \left( \beta + \sqrt{\beta^2 - 1} \right) \right]^2},
\]

defined on \( \mathbb{R}^3 \times M \), and for which we know, due to Corollary 4.6, the metric tensor is \( g = \beta^{-2} \delta \). It may also be verified directly that, given,

\[
P_1(x; \alpha, \beta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( x - \frac{\sqrt{2}\sin \alpha}{\beta} \right)^2}, \quad P_2(x; \alpha, \beta) = \frac{1}{\pi} \sech \left( x - \frac{\sqrt{2}\sin \alpha}{\beta} \right),
\]

\[
P_3(x; \alpha, \beta) = \frac{\pi^{-1}}{1 + \left[ x + \frac{\sqrt{2}\sin \alpha}{\beta} - \sqrt{2} \ln \left( \beta + \sqrt{\beta^2 - 1} \right) \right]^2},
\]

we have

\[
g(P_1) = \frac{1}{\beta^4} \begin{bmatrix} \beta^2 \sin^2 \alpha & \beta \sin \alpha \cos \alpha & \sin \alpha \cos \alpha \cos^2 \alpha \\ \beta \sin \alpha \cos \alpha & \beta^2 \cos^2 \alpha & -\beta \sin \alpha \cos \alpha \sin^2 \alpha \end{bmatrix}, \quad g(P_2) = \frac{1}{\beta^4} \begin{bmatrix} \beta^2 \cos^2 \alpha & -\beta \sin \alpha \cos \alpha & \sin \alpha \cos \alpha \sin^2 \alpha \\ -\beta \sin \alpha \cos \alpha & \beta^2 \cos^2 \alpha & \beta \sin \alpha \cos \alpha \cos^2 \alpha \end{bmatrix},
\]

\[
g(P_3) = \frac{1}{\beta^4} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta^2 - 1 \end{bmatrix},
\]

whose sum is as desired – that is, \( g \left( \bigodot P_i \right) = \sum g(P_i) \) as Theorem 4.2 assured us. Thus, we have managed to encode a desired metric tensor as the Fisher information metric of a spatially disjoint product of three, one-dimensional families of probability density functions. \( \triangle \)
We can explore another possible simplifying form of transformation \( f \). Consider that were \( f(x; \theta) \propto x \), then every term in Equation 4.4 would contribute a factor proportional to \( f_a \). Again, this is a desirable result and so we explore the symmetry of scale invariance.

**Proposition 4.9.** Fix a one-dimensional probability density function \( \hat{P} \) such that the change of variables \( y = xe^h \) for \( h(\theta) \) a differentiable function on \( M \subset \mathbb{R}^m \) leaves the spatial domain \( X \) unchanged. Let \( P(x; \theta) = e^h \hat{P}(xe^h) \) then \( g_{ab} = (\partial_a h)(\partial_b h)E \) where

\[
E = \int_X P(x) \left( 1 + x \frac{\partial \ln P(x)}{\partial x} \right)^2 dx.
\]

**Proof.** We set \( f(x; \theta) = e^{h(\theta)x} \) and compute the required derivatives for Proposition 4.4 as follows

\[
f_a = \partial_a h xe^h, \quad f_x = e^h, \quad f_{ax} = \partial_a h e^h, \quad \frac{\partial (f_a f_b)}{\partial y} = 2(\partial_a h)(\partial_b h)y.
\]

The result follows straightforwardly. \( \square \)

**Corollary 4.10.** Fix one-dimensional probability density functions \( P_i \) and let \( h^i(\theta) \) be differentiable on \( M \subset \mathbb{R}^m \) and write \( y^i = x^i e^{h^i} \) such that \( X_i \) is unchanged under this change of variables for all \( i \). \( P(x; \theta) = \bigodot e^{h^i} P_i \left(x^i e^{h^i}\right)^{\otimes e_i} \) gives \( g_{ab}(P) = (\partial_a h^i)(\partial_b h^j)E_{ij} \) where

\[
E_{ij} = \begin{cases} e_i \int_{X_i} dx^i P_i \left( 1 + x \frac{\partial \ln P_i}{\partial x^i} \right)^2, & i = j \\ 0, & i \neq j \end{cases}
\]

**Proof.** Combine Proposition 4.9 and Theorem 4.2. \( \square \)

**Corollary 4.11.** Every Riemannian metric tensor may be reached as the result of the Fisher information metric acting upon a spatially disjoint product of families of one-dimensional probability density functions.

**Proof.** Apply either Corollary 4.10 or Corollary 4.6 to the desired \( C^1 \) pullback \( h \), which exists due to the isometric embedding of the desired manifold in \( \mathbb{E}^n \) via the Nash Embedding theorem. \( \square \)

It can now be seen that relatively simple computations give rise to highly useful results by way of Theorem 4.2. Indeed, to extend this work one need only find other families of probability density functions whose Fisher information metric can be made to be proportional to \( (\partial_a h)(\partial_b h) \) in order to combine them in the requisite multiplicity to allow \( h \) to be the pullback for a desired Riemannian metric tensor. That we made explicit use of spatial domain symmetries using Proposition 4.4 should be seen as merely a convenient and intuitive way of making use of Theorem 4.2 to construct desirable results.
5. Discussion

That we can associate a Reimannian information manifold with a well-defined metric to a given family of probability distribution functions is a remarkable thing. Indeed, the power of this statement immediately begs the question of how much statistical, or information theoretic properties can be captured in the language of differential geometry. It is clear that the Fisher metric captures only a small amount of information about the family of PDFs, however the metric is but one differential geometric structure, and one could imagine that more information may be translated into the language of form fields of different order.

What we have shown here is in line with the string theory ideas of holographic duality, which indicate that any scale-free gauge theory should give rise to a hyperbolic geometry. Different scale-free gauge theories should however give rise to different field contents, above and beyond the metric, depending on the operators which can be formed in the gauge theory. As discussed in the introduction, information geometry has already been used to go from: gauge theory → PDF → metric. Thus it would be interesting, both from the information theoretic point of view, as well as from the holographic point of view to see what more differential structure can be encoded in such mappings.

This article is our attempt to formulate a crisp statement about the uniqueness of the association of a metric to a probability distribution. We saw how the Fisher information metric took a spatially disjoint product of probability distributions to a sum of the individual metric tensors. We leveraged this result to entirely reverse the computation, in generality. In fact, we found that it is possible to explicitly construct any Riemannian metric via the spatially disjoint product of one-dimensional probability density functions exhibiting a select spatial domain symmetry. This symmetry in fact features in a crucial way in our construction to inject dependence upon the components of the pullback used to isometrically embed the desired metric in $E^n$. Moreover, up to the spatial domain symmetries mentioned and some mild conditions on the continuity of the probability density functions, we have shown that such a construction may be given in terms of arbitrary probability density functions.

While our results appear to be quite negative in terms of the amount of information encoded in the Fisher metric from a PDF, we propose to interpret it as a signal that, in order to fully capture a duality that seems to point to a one-to-one map between string theory on $AdS_5 \times S^5$ and maximally supersymmetric Yang-Mills theory on the $AdS$ boundary, a deeper understanding of information geometry is required. We leave this for future work.

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