MODULI SPACES OF GERMS OF SEMIQUASIHOMOGENEOUS LEGENDRIAN CURVES

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ABSTRACT. We construct a moduli space for Legendrian curves singularities which are contactomorphic-equivalent and equisingular through a contact analogue of the Kodaira-Spencer map for curve singularities. We focus on the specific case of Legendrian curves which are the conormal of a plane curve with one Puiseux pair.

1. Introduction

Greuel, Laudal, Pfister et all (see [5], [8]) constructed moduli spaces of germs of plane curves equisingular to a plane curve \{y^k + x^n = 0\}, (k, n) = 1. Their main tools are the Kodaira Spencer map of the equisingular semiuniversal deformation of the curve and the results of [6]. We extend their results to Legendrian curves.

Let Y be the germ of a plane curve that is a generic plane projection of a Legendrian curve L. The equisingularity type of Y does not depend on the projection (see [12]). Two Legendrian curves are equisingular if their generic plane projections are equisingular. We say that an irreducible Legendrian curve L is semiquasihomogeneous if its generic plane projection is equisingular to a quasihomogeneous plane curve \{y^k + n^n = 0\}, for some k, n such that (k, n) = 1. Hence the generic plane projection of L is a semiquasihomogeneous plane curve.

In section 2 we recall the main results of relative contact geometry. In section 3 we construct the microlocal Kodaira Spencer map and study its kernel \(L_B\), a Lie algebra of vector fields over the base space \(C^B\) of the semiuniversal equisingular deformation of the plane curve \{y^k + n^n = 0\}. We use \(L_B\) in order to construct a Lie algebra of vector fields \(L_C\) over the base space \(C^C\) of the microlocal semiuniversal equisingular deformation of \{y^k + n^n = 0\}. In section 4 we recall some results of [6]. In section 5 we study the stratification of \(C^\omega\) induced by \(L_C\) and show that the conormals of two fibers \(F_b, F_c\) of the microlocal semiuniversal equisingular deformation of \{y^k + n^n = 0\} are isomorphic if and only if b and c are in the same integral manifold of \(L_C\). Moreover, we construct the moduli spaces. The final section is dedicated to presenting an example.

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2. Relative contact geometry

Let \( q : X \to S \) be a morphism of complex spaces. We can associate to \( q \) a coherent \( \mathcal{O}_X \)-module \( \Omega^1_{X/S} \), the sheaf of relative differential forms of \( X \to S \), and a differential morphism \( d : \mathcal{O}_X \to \Omega^1_{X/S} \) (see [7] or [9]).

If \( \Omega^1_{X/S} \) is a locally free \( \mathcal{O}_X \)-module, we denote by \( \pi = \pi_{X/S} : T^*(X/S) \to X \) the vector bundle with sheaf of sections \( \Omega^1_{X/S} \). We say that \( T(X/S) \) is the relative tangent bundle [cotangent bundle] of \( X \to S \).

Let \( \varphi : X_1 \to X_2 \), \( q_i : X_i \to S \) be morphisms of complex spaces such that \( q_2 \varphi = q_1 \). There is a morphism of \( \mathcal{O}_{X_i} \)-modules

\[
\varphi^* \Rightarrow \mathcal{O}_{X_2/S} = \mathcal{O}_{X_1} \otimes_{\varphi^* \mathcal{O}_{X_2}} \varphi^* \Omega^1_{X_2/S} \to \Omega^1_{X_1/S}.
\]

If \( \Omega^1_{X_i/S}, i = 1, 2 \), and the kernel and cokernel of (2.1) are locally free, we have a morphism of vector bundles

\[
\rho \varphi : X_1 \times X_2 T^*(X_2/S) \to T^*(X_1/S).
\]

If \( \varphi \) is an inclusion map, we say that the kernel of (2.2), and its projectivization, are the conormal bundle of \( \varphi \) relative to \( X \) of \( \mathcal{O}_{X/S} \). We will replace "\( M \times S \)" by "\( M|S \)". Let \( r \) be the projection \( M \times S \to M \).

Notice that \( \Omega^1_{M|S} \to r_{*} \mathcal{O}_M \) is a locally free \( \mathcal{O}_{M \times S} \)-module. Moreover, \( T^*(M|S) = T^*M \times S \)

We say that \( \Omega^1_{M|S} \) is the sheaf of relative differential forms of \( M \) over \( S \). We say that \( T^*(M|S) \) is the relative cotangent bundle of \( M \) over \( S \).

Let \( N \) be a complex manifold of dimension \( 2n - 1 \). Let \( S \) be a complex space. We say that a section \( \omega \) of \( \Omega^1_{N|S} \) is a relative contact form of \( N \) over \( S \) if \( \omega \wedge dw^{n-1} \) is a local generator of \( \Omega^1_{N|S} \). Let \( \mathcal{C} \) be a locally free subsheaf of \( \Omega^1_{N|S} \). We say that \( \mathcal{C} \) is a structure of relative contact manifold on \( N \) over \( S \) if \( \mathcal{C} \) is locally generated by a relative contact form of \( N \) over \( S \). We say that \( (N \times S, \mathcal{C}) \) is a relative contact manifold over \( S \). When \( S \) is a point we obtain the usual notion of contact manifold.

Let \( (N_1 \times S, \mathcal{C}_1), (N_2 \times S, \mathcal{C}_2) \) be relative contact manifolds over \( S \). Let \( \chi \) be a morphism from \( N_1 \times S \) into \( N_2 \times S \) such that \( q_{N_2} \circ \chi = q_{N_1} \). We say that \( \chi \) is a relative contact transformation of \( (N_1 \times S, \mathcal{C}_1) \) into \( (N_2 \times S, \mathcal{C}_2) \) if the pull-back by \( \chi \) of each local generator of \( \mathcal{C}_2 \) is a local generator of \( \mathcal{C}_1 \).

We say that the projectivization \( \pi_{X/S} : \mathbb{P}^*(X/S) \to X \) of the vector bundle \( T^*(X/S) \) is the projective cotangent bundle of \( X \to S \).

Let \( (x_1, ..., x_n, \xi_1, ..., \xi_n) \) be a partial system of local coordinates on an open set \( U \) of \( X \). Let \( (x_1, ..., x_n, \xi_1, ..., \xi_n) \) be the associated partial system of symplectic coordinates of \( T^*(X/S) \) on \( V = \pi^{-1}(U) \). Set \( p_{i,j} = \xi_i \xi_j^{-1}, i \neq j, \)

\[
V_i = \{(x, \xi) \in V : \xi_i \neq 0\}, \quad \omega_i = \xi_i^{-1} \theta, \quad i = 1, ..., n.
\]
each $\omega_i$ defines a relative contact form $dx_j - \sum_{i\neq j} p_{i,j} dx_i$ on $\mathbb{P}^*(X/S)$, endowing $\mathbb{P}^*(X/S)$ with a structure of relative contact manifold over $S$.

Let $\omega$ be a germ at $(x, o)$ of a relative contact form of $\mathcal{C}$. A lifting $\tilde{\omega}$ of $\omega$ defines a germ $\tilde{\mathcal{C}}$ of a relative contact structure of $N \times T_o S \to T_o S$. Moreover, $\tilde{\mathcal{C}}$ is a lifting of the germ at $o$ of $\mathcal{C}$.

Let $(N \times S, \mathcal{C})$ be a relative contact manifold over a complex manifold $S$. Assume $N$ has dimension $2n - 1$ and $S$ has dimension $\ell$. Let $\mathcal{L}$ be a reduced analytic set of $N \times S$ of pure dimension $n+\ell-1$. We say that $\mathcal{L}$ is a relative Legendrian variety of $N \times S$ over $S$ if for each section $\omega$ of $\mathcal{C}$, $\omega$ vanishes on the regular part of $\mathcal{L}$. When $S$ is a point, we say that $\mathcal{L}$ is a Legendrian variety of $N$.

Let $\mathcal{L}$ be an analytic set of $N \times S$. Let $(x, o) \in \mathcal{L}$. Assume $S$ is an irreducible germ of a complex space at $o$. We say that $\mathcal{L}$ is a relative Legendrian variety of $N$ over $S$ at $(x, o)$ if there is a relative Legendrian variety $\mathcal{L}$ of $(N, x)$ over $(T_o S, 0)$ that is a lifting of the germ of $\mathcal{L}$ at $(x, o)$. Assume $S$ is a germ of a complex space at $o$ with irreducible components $S_i, i \in I$. We say that $\mathcal{L}$ is a relative Legendrian variety of $N$ over $S$ at $(x, o)$ if $S_i \times S \mathcal{L}$ is a relative Legendrian variety of $S_i \times S N$ over $S_i$ at $(x, o)$, for each $i \in I$.

We say that $\mathcal{L}$ is a relative Legendrian variety of $N \times S$ if $\mathcal{L}$ is a relative Legendrian variety of $N \times S$ at $(x, o)$ for each $(x, o) \in \mathcal{L}$.

Let $Y$ be a reduced analytic set of $M$. Let $\mathcal{Y}$ be a flat deformation of $Y$ over $S$. Set $X = M \times S \setminus \mathcal{Y}_{\text{sing}}$. We say that the Zariski closure of $\mathbb{P}^* Y_{\text{reg}}(X/S)$ in $\mathbb{P}^*(M/S)$ is the conormal $\mathbb{P}^*_Y(M/S)$ of $\mathcal{Y}$ over $S$.

**Theorem 2.1.** The conormal of $\mathcal{Y}$ over $S$ is a relative Legendrian variety of $\mathbb{P}^*(M/S)$. If $\mathcal{Y}$ has irreducible components $\mathcal{Y}_1, ..., \mathcal{Y}_r$,

\[
\mathbb{P}^*_Y(M/S) = \bigcup_{i=1}^r \mathbb{P}^*_Y(M/S).
\]

**Theorem 2.2.** Let $\mathcal{L}$ be an irreducible germ of a relative Legendrian analytic set of $\mathbb{P}^*(M/S)$. If the analytic set $\pi(\mathcal{L})$ is a flat deformation over $S$ of an analytic set of $M$, $\mathcal{L} = \mathbb{P}^*_\pi(\mathcal{L})(M/S)$.

Let $\theta = \xi dx + \eta dy$ be the canonical 1-form of $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$. Hence $\pi = \pi_{\mathbb{C}^2} : \mathbb{P}^* \mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ is given by $\pi(x, y; \xi : \eta) = (x, y)$. Let $U [V]$ be the open subset of $\mathbb{P}^* \mathbb{C}^2$ defined by $\eta \neq 0 [\xi \neq 0]$. Then $\theta/\eta \theta/\xi$ defines a contact form $dy - pdx [dx - qdy]$ on $U [V]$, where $p = -\xi/\eta, q = -\eta/\xi$. Moreover, $dy - pdx$ and $dx - qdy$ define the structure of contact manifold on $\mathbb{P}^* \mathbb{C}^2$.

If $L$ is a germ of a Legendrian curve of $\mathbb{P}^* M$ and $L$ is not a fiber of $\pi_M$, $\pi_M(L)$ is a germ of plane curve with irreducible tangent cone and $L = \mathbb{P}^*_\pi_M(L)$.

Let $Y$ be the germ of a plane curve with irreducible tangent cone at a point $o$ of a surface $M$. Let $L$ be the conormal of $Y$. Let $\sigma$ be the only point of $L$ such that $\pi_M(\sigma) = o$. Let $k$ be the multiplicity of $Y$. Let $f$ be a defining function of $Y$. In this situation we will always choose a system of
Lemma 2.3. The following statements are equivalent:

1. $\text{mult}_0(L) = \text{mult}_o(Y);$ 
2. $C_\sigma(L) \nsubseteq (D\pi(\sigma))^{-1}(0,0);$ 
3. $f \in (x^2, y)^k;$ 
4. if $t \mapsto (x(t), y(t))$ parametrizes a branch of $Y$, $x^2$ divides $y$.

Definition 2.4. Let $S$ be a reduced complex space. Let $Y$ be a reduced plane curve. Let $Y'$ be a deformation of $Y$ over $S$. We say that $Y'$ is generic if its fibers are generic. If $S$ is a non reduced complex space we say that $Y'$ is generic if $Y'$ admits a generic lifting.

Given a flat deformation $Y$ of a plane curve $Y$ over a complex space $S$ we will denote $P^*_Y(C^2|S)$ by $\text{Con}(Y)$.

Theorem 2.5 (Theorem 1.3, [2]). Let $\chi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a germ of a contact transformation. Let $L$ be a germ of a Legendrian curve of $\mathbb{C}^3$ at the origin. If $L$ and $\chi(L)$ are in generic position, $\pi(L)$ and $\pi(\chi(L))$ are equisingular.

Definition 2.6. Two Legendrian curves are equisingular if their generic plane projections are equisingular.

Lemma 2.7. Assume $Y$ is a generic plane curve and $Y \hookrightarrow Y'$ defines an equisingular deformation of $Y$ with trivial normal cone along its trivial section. Then $Y'$ is generic.

Definition 2.8. Let $L$ be (a germ of) a Legendrian curve of $\mathbb{C}^3$ in generic position. Let $\mathcal{L}$ be a relative Legendrian curve over (a germ of) a complex space $S$ at $o$. We say that an immersion $i : L \hookrightarrow \mathcal{L}$ defines a deformation

$$L \hookrightarrow \mathbb{C}^3 \times S \to S$$

of the Legendrian curve $L$ over $S$ if $i$ induces an isomorphism of $L$ onto $\mathcal{L}_o$ and there is a generic deformation $Y$ of a plane curve $Y$ over $S$ such that $\chi(\mathcal{L})$ is isomorphic to $\text{Con}Y$ by a relative contact transformation verifying (2.6).

We say that the deformation (2.3) is equisingular if $Y$ is equisingular. We denote by $\text{Def}_{es}L$ the category of equisingular deformations of $L$.

Remark 2.9. We do not demand the flatness of the morphism (2.3).

Lemma 2.10. Using the notations of definition 2.8 given a section $\sigma : S \to \mathcal{L}$ of $\mathbb{C}^3 \times S \to S$, there is a relative contact transformation $\chi$ such that $\chi \circ \sigma$ is trivial. Hence $\mathcal{L}$ is isomorphic to a deformation with trivial section.

Consider the maps $i : X \hookrightarrow X \times S$ and $q : X \times S \to S$. 

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Theorem 2.11. Assume $\mathcal{Y}$ defines an equisingular deformation of a generic plane curve $Y$ with trivial normal cone along its trivial section. Let $\chi : X \times S \to X \times S$ be a relative contact transformation verifying
\[ \chi \circ i = i, \quad q \circ \chi = q \quad \text{and} \quad \chi(0,s) = (0,s) \quad \text{for each} \ s. \]
Then $\mathcal{Y}^\chi = \pi(\chi(\text{Con}(\mathcal{Y})))$ is a generic equisingular deformation of $Y$.

Definition 2.12. Let $\text{Def}^{es,\mu}_f$ (or $\text{Def}^{es,\mu}_Y$) be the category given in the following way: the objects of $\text{Def}^{es,\mu}_f$ are the objects of $\text{Def}^{es}_f$; two objects $Y, Z$ of $\text{Def}^{es,\mu}_f(T)$ are isomorphic if there is a relative contact transformation $\chi$ over $T$ such that $Z = Y^\chi$.

Lemma 2.13. Assume $f \in \mathbb{C}\{x,y\}$ is the defining function of a generic plane curve $Y$. Let $L$ be the conormal of $Y$. For each $\ell \geq 1$ there is $h_\ell \in \mathbb{C}\{x,y\}$ such that
\[ ((\ell + 1)p^\ell f_x + \ell p^{\ell+1} f_y) \equiv h_\ell \mod I_L. \]
Moreover, $h_\ell$ is unique modulo $I_Y$.

Definition 2.14. Let $f$ be a generic plane curve with tangent cone $\{y = 0\}$. We will denote by $I_f$ the ideal of $\mathbb{C}\{x,y\}$ generated by the functions $g$ such that $f + \varepsilon g$ is equisingular over $T_\varepsilon$ and has trivial normal cone along its trivial section. We call $I_f$ the equisingularity ideal of $f$.

We will denote by $I^\mu_f$ the ideal of $\mathbb{C}\{x,y\}$ generated by $f, (x,y)f_x, (x^2,y)f_y$ and $h_\ell, \ell \geq 1$.

Theorem 2.15 (9). Assume $Y$ is a generic plane curve with conormal $L$, defined by a power series $f$. Assume $f$ is SQH or $f$ is NND. If $g_1,\ldots,g_n \in I_f$ represent a basis of $I_f/I^\mu_f$ with Newton order $\geq 1$, the deformation $G$ defined by
\[ G(x,y,s_1,\ldots,s_n) = f(x,y) + \sum_{i=1}^{n} s_i g_i \]
is a semiuniversal deformation of $f$ in $\text{Def}^{es,\mu}_f$.

Lemma 2.16. Let $S$ be the germ of a complex space. Assume $F$ defines an object $\mathcal{F}$ in $\text{Def}^{es}_f(S)$. Given $\gamma \geq 1$ there are $H^\gamma \in \mathcal{O}_S\{x,y\}$ such that
\[ H^\gamma \equiv p^\gamma \partial_x F \mod I_{\text{Con}(\mathcal{F})} + \Delta_F. \]
If $f$ has multiplicity $k$, $H^\gamma \equiv 0$ for $\gamma \geq k - 1$.

Proof. Let us first show that
\[ H^\gamma \equiv (\gamma + 1)p^\gamma \partial_x F + \gamma p^{\gamma+1} \partial_y F \mod I_{\text{Con}(\mathcal{F})}. \]
This is a relative version of Lemma 7.2 of [9]. Since $\mathcal{F}$ is equisingular, the multiplicity and the conductor are constant. Moreover, there are parametrizations of each component of $\mathcal{F}$. Therefore, we can generalize the argument in the proof of the quoted Lemma.
Now it is enough to show that

\[(2.5) \quad \partial_x F + p \partial_y F \equiv 0 \mod I_{\text{Con}(F)}.\]

Assume \( \mathcal{F} \) is irreducible. Let \((t, s) \mapsto (X, Y, P)\) be a parametrization of \(\text{Con}(\mathcal{F})\). Since \(F(X, Y) = 0\) we conclude that

\[\partial_x F \partial_t X + \partial_y F \partial_t Y = 0.\]

Since \(P = \partial_t Y / \partial_t X\), (2.5) holds. \(\square\)

Let \(T_\varepsilon\) be the complex space with local ring \(\mathbb{C}\{\varepsilon\}/(\varepsilon^2)\). Let \(I, J\) be ideals of the ring \(\mathbb{C}\{s_1, \ldots, s_m\}\). Assume \(J \subset I\). Let \(X, S, T\) be the germs of complex spaces with local rings \(\mathbb{C}\{x, y, p\}, \mathbb{C}\{s\}/I, \mathbb{C}\{s\}/J\). Consider the maps \(i: X \hookrightarrow X \times S, j: X \times S \hookrightarrow X \times T\) and \(q: X \times S \rightarrow S\).

Let \(m_X, m_S\) be the maximal ideals of \(\mathbb{C}\{x, y, p\}\), \(\mathbb{C}\{s\}/I\). Let \(n_S\) be the ideal of \(O_{X \times S}\) generated by \(m_X m_S\).

Let \(\chi: X \times S \rightarrow X \times S\) be a relative contact transformation. If \(\chi\) verifies

\[(2.6) \quad \chi \circ i = i, \quad q \circ \chi = q \quad \text{and} \quad \chi(0, s) = (0, s) \quad \text{for each} \ s.\]

there are \(\alpha, \beta, \gamma \in n_S\) such that

\[(2.7) \quad \chi(x, y, p, s) = (x + \alpha, y + \beta, p + \gamma, s).\]

**Theorem 2.17.** (1) Let \(\chi: X \times S \rightarrow X \times S\) be a relative contact transformation that verifies (2.6). Then \(\gamma\) is determined by \(\alpha\) and \(\beta\). Moreover, there is \(\beta_0 \in n_S + pO_{X \times S}\) such that \(\beta\) is the solution of the Cauchy problem

\[(2.8) \quad \left(1 + \frac{\partial \alpha}{\partial x} + p \frac{\partial \alpha}{\partial y}\right) \frac{\partial \beta}{\partial p} - p \frac{\partial \alpha}{\partial p} \frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial p} \frac{\partial \beta}{\partial x} = p \frac{\partial \alpha}{\partial p},\]

\(\beta + pO_{X \times S} = \beta_0\).

(2) Given \(\alpha \in n_S\), \(\beta_0 \in n_S + pO_{X \times S}\), there is a unique relative contact transformation \(\chi\) that verifies (2.6) and the conditions of statement (a). We denote \(\chi\) by \(\chi_{\alpha, \beta_0}\).

(3) If \(S = T_\varepsilon\) the Cauchy problem (2.8) simplifies into

\[(2.9) \quad \frac{\partial \beta}{\partial p} = p \frac{\partial \alpha}{\partial p}, \quad \beta + pO_{X \times T_{\varepsilon}} = \beta_0.\]

Consider the contact transformations from \(\mathbb{C}^3\) to \(\mathbb{C}^3\) given by

\[(2.10) \quad \Phi(x, y, p) = (\lambda x, \lambda \mu y, \mu p), \quad \lambda, \mu \in \mathbb{C}^\ast,\]

\[(2.11) \quad \Phi(x, y, p) = (ax + bp, y + \frac{ac}{2} x^2 + \frac{bd}{2} p^2 + bcxp, cx + dp), \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1,\]

**Theorem 2.18.** (See [2] or [10].) Let \(\Phi: (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0)\) the the germ of a contact transformation. Then \(\Phi = \Phi_1 \Phi_2 \Phi_3\), where \(\Phi_1\) is of type (2.10), \(\Phi_2\) is of type (2.11) and \(\Phi_3\) is of type (2.7), with \(\alpha, \beta, \gamma \in \mathbb{C}\{x, y, p\}\). Moreover,
there is $\beta_0 \in \mathbb{C}\{x,y\}$ such that $\beta$ verifies the Cauchy problem \((2.8)\), $\beta - \beta_0 \in (p)$ and

\[(2.12) \quad \alpha, \beta, \gamma, \beta_0, \frac{\partial \alpha}{\partial x}, \frac{\partial \beta}{\partial x}, \frac{\partial \beta}{\partial p}, \frac{\partial^2 \beta}{\partial x \partial p} \in (x, y, p).\]

If $D\Phi(0)(\{y = p = 0\}) = \{y = p = 0\}$, $\Phi_2 = id_{\mathbb{C}^3}$.

**Proposition 2.19.** Let $f$ and $g$ be two microlocally equivalent SQH or NND generic plane curves. Then, $f$ and $g$ have equisingular semiuniversal microlocal deformations with isomorphic base spaces.

**Proof.** Let $X, Y$ denote the germs of analytic subsets at the origin of $\mathbb{C}^3$ defined by $Conf$ and $Cong$ respectively. Let $\chi : \mathbb{C}^3 \to \mathbb{C}^3$ be a contact transformation such that $\chi(Y) = X$ and $\mathcal{X} := (i, \Phi) : X \hookrightarrow \mathbb{C}^3 \times \mathbb{C}^\ell \to \mathbb{C}^\ell$ be a semiuniversal equisingular deformation of $X$ (to see that such an object exists see Theorem 2.15). Let us show that $(i \circ \chi, \Phi)$ is a semiuniversal equisingular deformation of $Y$:

Let $\mathcal{Y} := (j, \Psi) : Y \hookrightarrow \mathbb{C}^3 \times \mathbb{C}^k \to \mathbb{C}^k$ be an equisingular deformation of $Y$. Because $\mathcal{X}$ is versal there is $\varphi : \mathbb{C}^k \to \mathbb{C}^\ell$ such that $\varphi^* \mathcal{X} \cong (j \circ \chi^{-1}, \Psi)$.

\[(2.13) \quad \begin{array}{ccc}
Y & \xleftarrow{\chi^{-1}} & X \\
\downarrow j & & \uparrow \varphi^* i \\
\mathbb{C}^3 \times \mathbb{C}^k & \cong & \mathbb{C}^3 \times \mathbb{C}^k \\
\downarrow \Psi & & \uparrow \varphi^* \Phi \\
\mathbb{C}^k & \cong & \mathbb{C}^k
\end{array}\]

Then, $(\varphi^* i \circ \chi, \varphi^* \Phi) \cong (j, \Psi)$ which means that $\varphi^*(i \circ \chi, \Phi) \cong \mathcal{Y}$. The result follows from the fact that a semiuniversal deformation is unique up to isomorphism (see Lemma 2.1.12 of [4]).

Recall that, for a SQH or NND generic plane curve $f$, there is a semiuniversal microlocal equisingular deformation with base space $\mathbb{C}^k$, where $k$ is the the dimension as vector space over $\mathbb{C}$ of $I_f/I^\mu_f$. So, because of Proposition 2.19 and Proposition 2.2.17 of [4], the following defines an invariant between microlocally equivalent fibers of $F$.

**Definition 2.20.** Let $f$ be a SQH or NND generic plane curve. Then

$$\widehat{\tau}(f) := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{I^\mu_f}$$

is the microlocal Tjurina number of $f$. 

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3. The microlocal Kodaira-Spencer map

Assume $k, n$ are coprime integers, $0 < 2k < n$. Set $f = y^k - x^n$, $\mu = (n - 2)/(k - 2)$. Consider in $\mathbb{C}[x, y]$ the grading given by $o(x^iy^j) = ki + nj$, $(i,j) \in \mathbb{N}^2$. Set $\omega = o(x^{n-2}y^{k-2}) - kn$, $\varpi = o(x^{n-k}y^{k-2}) - kn$, $e(x^iy^j) = (i,j) \in \mathbb{N}^2$.

$$B = \{(i,j) \in \mathbb{N}^2 : i \leq n - 2, j \leq k - 2\},$$
$$C = \{(i,j) \in B : i + j \leq n - 2\},$$
$$D = \{(i,j) \in B : o(x^iy^j) - kn \leq \varpi\},$$
$$A_0 = \{(i,j) \in A : ki + nj > kn\}, \text{ for each } A \subseteq B.$$

Let $m_1, \ldots, m_\mu$ be the family $x^iy^j$, $(i,j) \in B$, ordered by degree. Set $b = \#B_0$. If $\mu - b + 1 \leq \ell \leq \mu$, set $o(\ell) = o(m_\ell) - kn$ and $o(s_{o(\ell)}) = -o(\ell)$.

Let $A \subseteq B$. Set $I_A = \{\ell : e(m_\ell) \in A_0\}$, $s_A = (s_{o(\ell)})_{\ell \in I_A}$. Set $A^A = \mathbb{C}^{#A^0}$ with coordinates $s_A$. Notice that $I_B = \{\mu - b + 1, \ldots, \mu\}$. Moreover,

$$F_A = f + \sum_{\ell \in I_A} s_{o(\ell)}m_\ell$$

is homogeneous of degree $kn$.

Let $Y$ be the plane curve defined by $f$. Let $\Gamma$ be the conormal of $Y$. Let $F_A$ be the deformation defined by $F_A$. Notice that

- $F_B$ is a semiuniversal equisingular deformation of $Y$,
- $F_C$ is a semiuniversal equisingular microlocal deformation of $Y$,
- if $C \subseteq A \subseteq B$, $F_A$ is a complete equisingular microlocal deformation of $Y$.

Let $\Delta_{F_A}$ be the ideal of $\mathbb{C}[s_A]$ generated by $\partial_x F_A$ and $\partial_y F_A$. Assume $o(p) = n - k$ in order to guarantee that the contact form $dy - pdx$ is homogeneous.

**Lemma 3.1.** Assume $C \subseteq A \subseteq B$ and $\gamma \geq 1$. There is $H_A^\gamma \in \mathbb{C}[s_A]\{x,y\}$ such that $H_A^\gamma \equiv p^\gamma \partial_x F_A \mod I_{\text{Con}(F_A)} + \Delta_{F_A}$ where $H_A^\gamma$ is homogeneous of degree $\gamma(n - k) + kn - k$. If $\gamma \geq k - 1$, $H_A^\gamma \in \Delta_{F_A}$. If $C \subseteq A' \subseteq A \subseteq B$, $H_A' = H_A^\gamma |_{C \cdot A'}$.

**Proof.** Set $\psi_0 = \theta$, where $\theta^k = -1$. There are $\psi_i \in (s_A)\mathbb{C}[s_A]$, $i \geq 1$, such that

$$X(t, s_A) = t^k, \quad Y(t, s_A) = \sum_{i \geq 0} \psi_i t^{n+i}$$

defines a parametrization $\Phi$ of $F_A$. Setting $P(t, s_A) = \sum_{i \geq 0} \frac{n+i}{k+i} \psi_i t^{n-k+i}$, $X, Y, P$ defines a parametrization $\Psi$ of $\text{Con}(F_A)$. Since $x$ is homogeneous of degree $k$ and $x = t^k$, we assume $t$ homogeneous of degree 1. Let us show

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that $Y$ is homogeneous of degree $n$. The $\mathbb{C}^*$-action acts on $\Phi$ by

$$a \cdot \Phi(t, s_A) = \left( a^k t^k, a^n (\theta t^n + \sum_{i \geq 1} (a \cdot \psi_i) a^i t^{n+i} ) \right).$$

Since $F_A$ is homogeneous, for each $s_A$,

$$t \mapsto \Phi_a(t, s_A) = \left( t^k, \theta t^n + \sum_{i \geq 1} (a \cdot \psi_i) a^i t^{n+i} \right)$$

is another parametrization of the curve defined by $(x, y) \mapsto F_A(x, y, s_A)$. Since the first term of both parametrizations coincide, $\Phi_a = \Phi$, $a \cdot \psi_i = a^{-1} \psi_i$ and $\Phi$ is homogeneous. Therefore, $\Psi$ is homogeneous.

There is an integer $c$ such that $\Phi^*(\Delta_{F_A}) \supset t^c \mathcal{C}[s_A]\{t\}$. Remark that $p^\gamma \partial_x F_A$ is homogeneous of degree $\gamma(n - k) + kn - k$. We construct $H_A^\gamma$ in the following manner. There is a monomial $ax^iy^j$, $a \in \mathcal{C}[s_A]$ such that the monomials of lowest $t$-order $\Phi^*(ax^iy^j)$ and $\Psi^*(p^\gamma \partial_x F_A)$ coincide. Replace $p^\gamma \partial_x F_A$ by $p^\gamma \partial_x F_A - ax^iy^j$ and iterate the procedure. After a finite number of steps we construct $H_A^\gamma$ such that

$$\Psi^*(p^\gamma \partial_x F_A - H_A^\gamma) \in t^c \mathcal{C}[s_A]\{t\}.$$ 

Therefore,

$$p^\gamma \partial_x F_A - H_A^\gamma \in I_{\text{Con}(F_A)} + \Delta_{F_A}.$$ 

Remark that the monomial $ax^iy^j$ is homogeneous of degree $\gamma(n - k) + kn - k$. 

Set $\Theta_B = \text{Der}_{\mathcal{C}} \mathcal{C}[s_B]$, $\partial_{o(\ell)} = \partial_{s_{o(\ell)}}$ and $o(\partial_{s(\ell)}) = o(\ell)$ for each $\ell \in I_B$. Assume $C \subseteq A' \subseteq A \subseteq B$. Let $\Theta_{A, A'}$ be the $\mathcal{C}[s_A]$-submodule of $\Theta_B$ generated by $\partial_{s(\ell)}$, $\ell \in I_{A'}$. Set $\Theta_A = \Theta_{A, A'}$. There are maps

$$\Theta_A \leftarrow \Theta_{A, A'} \stackrel{r_{A, A'}}{\rightarrow} \Theta_{A'},$$

where $r_{A, A'}$ is the restriction to $\mathcal{C}^{A'}$.

**Definition 3.2.** Let $I_F^\mu$ be the ideal of $\mathcal{C}[s_B][[x, y]]$ generated by $F_B$, $\Delta F_B$ and $H_B^\gamma$, $\gamma = 1, \ldots, k - 2$. We say that the map

$$\rho : \Theta_B \rightarrow \mathcal{C}[s_B][[x, y]]/I_F^\mu,$$

given by $\rho(\delta) = \delta F_B + I_F^\mu$ is the microlocal Kodaira-Spencer map of $f$. We will denote the kernel of $\rho$ by $\mathcal{L}_B$.

Assume we have defined $\mathcal{L}_A$. We set

$$\mathcal{L}_{A, A'} = \mathcal{L}_A \cap \Theta_{A, A'} \text{ and } \mathcal{L}_{A'} = r_{A, A'}(\mathcal{L}_{A, A'}).$$

Let $L$ be a Lie subalgebra of $\Theta_A$. Consider in $\mathcal{C}^A$ the binary relation $\sim$ given by $p \sim q$ if there is a vector field $\delta$ of $L$ and an integral curve $\gamma$ of $\delta$ such that $p$ and $q$ are in the trajectory of $\gamma$. We denote by $\mathcal{L}$ the equivalence relation
generated by $\sim$. We say that a subset $M$ of $\mathbb{C}^4$ is an integral manifold of $L$ if $M$ is an equivalence class of $L$.

Assume $C \subseteq A \subseteq B$. The family $m_\ell, 1 \leq \ell \leq \mu$, defines a basis of the $\mathbb{C}[s_A]$-module

$$R_A = \mathbb{C}[s_A][[x,y]]/\Delta F_A.$$ 

Set $H^0_A = F_A$. The relations

$$m_\ell H_A^\gamma = \sum_{v=1}^\mu c^\gamma_{\ell,v} m_v \mod \Delta F_A$$

define $c^\gamma_{\ell,v} \in \mathbb{C}[s_A]$ for each $0 \leq \gamma \leq k-2, 1 \leq \ell, v \leq \mu$. Assume $A = B$ and set

$$\delta^\gamma_\ell = \sum_{v=\mu-b+1}^\mu c^\gamma_{\ell,v} \partial_{s_{o(v)}}, \quad \ell = 1,\ldots,\mu, \quad \gamma = 0,\ldots, k-2.$$ 

If $m_\ell = x^i y^j$ we will also denote $\delta^\gamma_\ell$ by $\delta^\gamma_{i,j}$. For $1 \leq \gamma \leq k-2$, set

$$\alpha^0_\ell = o(m_\ell), \quad \alpha^\gamma_\ell = \alpha^0_\ell + \gamma(n-k) - k, \quad \ell = 1,\ldots,\mu,$$

$$\alpha^0_{i,j} = o(x^i y^j), \quad \alpha^\gamma_{i,j} = \alpha^0_{i,j} + \gamma(n-k) - k, \quad (i,j) \in B.$$ 

**Lemma 3.3.** With the previous notations, we have that:

1. The vector fields $\delta^\gamma_\ell$ ($\delta^\gamma_{i,j}$) are homogeneous of degree $\alpha^\gamma_\ell$ ($\alpha^\gamma_{i,j}$), $0 \leq \gamma \leq k-2, 1 \leq \ell \leq \mu$ ($i,j \in B$).
2. $\delta^\gamma_{i,j}(0) \neq 0$ if and only if $\gamma \geq 1$, $i \leq \gamma - 1, \gamma + j \leq k-2$.
3. $\delta^\gamma_{i,j} = 0$ if $\alpha^\gamma_{i,j} > \omega$.
4. The Lie algebra $L_B$ is generated as $\mathbb{C}[s_B]$-module by $\{\delta^\gamma_\ell : 0 \leq \gamma \leq k-2, \alpha^\gamma_\ell \leq \omega\}$.
5. If $\sigma > \omega$, $\partial_{s_\sigma} \in L_B$.
6. If $(u,v) \in B \setminus C$ there is $\delta \in L_B$ such that $\delta = \partial_{s_\sigma} + \varepsilon$ is homogeneous of degree $\sigma = ku + nv - kn$, where $\varepsilon$ is a linear combination of $\partial_{s_{o(v)}}, i \in I_B, i > \sigma$, with coefficients in $\mathbb{C}[s_B]$.

**Proof.** (3): Just notice that if $\alpha^0_{i,j} > \omega = o(m_\mu) - kn$ then $o(m_{i,j} F_B) = o(m_\mu)$. Now, because $n > 2k$, $o(H^0_B) > kn = o(F_B)$ for any $\gamma = 1,\ldots, k-2$, the result holds for $\gamma > 0$.

(4): For $\gamma = 0$ ($1 \leq \gamma \leq k-2$) and each $\ell = 1,\ldots,\mu$ such that $o(m_\ell) \leq \omega$, we have that $\rho(\delta^\gamma_\ell) = \delta^\gamma_\ell F_B + I^\mu_F = m_\ell F_B + I^\mu_F (m_\ell H^\gamma_B + I^\mu_F) = 0 + I^\mu_F$. So, $\{\delta^\gamma_\ell : 0 \leq \gamma \leq k-2, \alpha^\gamma_\ell \leq \omega\} \subseteq L_B$.

Now, let

$$\delta = \sum_{v=\mu-b+1}^\mu w_v \partial_{s_{o(v)}} \in \Theta_B$$

such that $\rho(\delta) = 0$. Then

$$\delta F_B = \sum_{v=\mu-b+1}^\mu w_v m_v = M_0 F_B + M_1 H^1_B + \ldots + M_{k-2} H^{k-2}_B \mod \Delta F_B,$$
with \( M_0, \ldots, M_{k-2} \in \mathbb{C}[s_B][[x,y]] \). Suppose

\[
M_0 = \sum_{\ell=1}^{\mu} M_{0,\ell m_{\ell}} \Delta F_B, \\
\ldots \\
M_{k-2} = \sum_{\ell=1}^{\mu} M_{k-2,\ell m_{\ell}} \Delta F_B,
\]

where the \( M_{\gamma,\ell} \in \mathbb{C}[s_B] \) for each \( \ell = 1, \ldots, \mu, \gamma = 0, \ldots, k-2 \). Then

\[
M_0 F_B = M_{0,1} m_1 F_B + \ldots + M_{0,\mu} m_{\mu} F_B \mod \Delta F_B \\
= M_{0,1} m_1 F_B + \ldots + M_{0,\mu} m_{\mu} F_B \mod \Delta F_B \\
= M_{0,1} \delta^0 F_B + \ldots + M_{0,\mu} \delta^0 F_B \mod \Delta F_B.
\]

Similarly, for any \( \gamma = 1, \ldots, k-2 \)

\[
M_\gamma H_B = M_{\gamma,1} m_1 H_B + \ldots + M_{\gamma,\mu} m_{\mu} H_B \mod \Delta F_B \\
= M_{\gamma,1} \delta^\gamma F_B + \ldots + M_{\gamma,\mu} \delta^\gamma F_B \mod \Delta F_B.
\]

So,

\[
\delta F_B = \sum_{\gamma=0}^{k-2} \sum_{\ell=1}^{b} M_{\gamma,\ell} \delta^\gamma F_B \mod \Delta F_B,
\]

which means that

\[
\delta = \sum_{\gamma=0}^{k-2} \sum_{\ell=1}^{b} M_{\gamma,\ell} \delta^\gamma.
\]

\( \square \)

Let \( L_B \) be the Lie algebra generated by \( \delta^\gamma, \gamma = 0, \ldots, k-2, \ell = 1, \ldots, b \). Remark that \( \mathbb{C}^B / L_B \cong \mathbb{C}^B / L_B \). Consider a matrix with lines given by the coefficients of the vector fields \( \delta^\gamma, \gamma = 0, \ldots, k-2, \ell = 1, \ldots, b \). After performing Gaussian diagonalization we can assume that:

- For each \( \sigma \in I_B \setminus I_C \) there is a line corresponding to a vector field \( \partial_{\psi(\sigma)} + \varepsilon \), where \( \varepsilon \in \Theta_{B,C} \).
- The remaining lines correspond to vector fields \( \delta^\ell, \ell \in J \), of \( \Theta_{B,C} \).

The vector fields \( \delta^\ell, \ell \in J \), generate \( L_{B,C} \) as a \( \mathbb{C}[s_B] \)-module. Let \( \delta^\ell \) be the restriction of \( \delta^\ell \) to \( \mathbb{C}^C \) for each \( \ell \in J \). The vector fields \( \delta^\ell, \ell \in J \), generate \( L_C \) as \( \mathbb{C}[s_C] \)-module. Note that \( \{\delta^\ell, \ell \in J\} \) is in general not uniquely determined but the \( \mathbb{C}[s_C] \)-module generated by them is. Let \( L_C \) be the Lie algebra generated by \( \{\delta^\ell, \ell \in J\} \). Since \( L_C \subseteq L_B \) the inclusion map \( \mathbb{C}^C \hookrightarrow \mathbb{C}^B \) defines a map \( \mathbb{C}^C / L_C \rightarrow \mathbb{C}^B / L_B \). By statement (6) of Lemma [3.3] this map is surjective.

Assume there is a vector field \( \delta\ell, \ell \in J, \) of order \( \alpha \). Let \( \{\delta^{\alpha, i} : i \in I_\alpha\} \) be the set of vector fields \( \delta\ell, \ell \in J, \) of order \( \alpha \), with \( I_\alpha = \{1, \ldots, \#I_\alpha\} \). If
there is \( \ell_0 \) such that \( \delta^{\alpha,j}(s_i) = 0 \) for \( \ell \leq \ell_0 \) and \( \delta^{\alpha,i}(s_{\ell_0}) \neq 0 \), we assume that \( i < j \). If \( I_\alpha = \{1\} \), set \( \delta^\alpha = \delta^{\alpha,1} \).

**Remark 3.4.** If \( k = 7, n = 15 \), we have that a semiuniversal equisingular microlocal deformation of \( f \) by

\[
F_C = y^7 + x^{15} + s_2x^{11}y^2 + s_3x^9y^3 + s_4x^7y^4 + s_5x^5y^5 + s_6x^3y^6 + s_7x^2y^7 + s_8x^6y^8 + s_9x^4y^9 + s_{10}x^3y^{10} + s_{11}x^8y^4 + s_{12}x^6y^5 + s_{13}x^4y^7 + s_{14}x^2y^8 + s_{15}x^4y^8 + s_{16}x^2y^8 + s_{17}x^4y^8 + s_{18}x^2y^8.
\]

Notice that the vector fields \( \delta^0 \) and \( \delta^1 \) give origin to the linearly independent vector fields

\[
\delta^{15,1} = 3s_3\partial_{s_{18}} + 4s_4\partial_{s_{19}} + \cdots.
\]

and

\[
\delta^{15,2} = \left( \frac{7^2}{15} - 3 \left( \frac{15}{7} \right)^2 s_4 \right) \partial_{s_{19}} + \cdots.
\]

**Theorem 3.5.** The map \( \mathbb{C}^C/L_C \to \mathbb{C}^B/L_B \) is bijective.

**Proof.** Let \( I_p \) be the subset of \( I_B \) that contains \( I_C \) and the \( p \) smallest elements of \( I_B \setminus I_C \). Set \( C_p = \{(i,j) \in B : ki + nj - kn \in I_p\} \). The Lie algebra \( L_{C_p} = L_{C_p} \cup L_B \) generates \( L_{C_p} \) as \( \mathbb{C}[s_{C_p}] \)-module. There is \( p \) such that \( C_p = D \). By statement (5) of Lemma 3.3 the integral manifolds of \( L_B \) are of the type \( M \times \mathbb{C}^{B \setminus D} \), where \( M \) is an integral manifold of \( L_{C_p} \). Therefore, \( \mathbb{C}^D/L_D \cong \mathbb{C}^B/L_B \). Assume \( \mathbb{C}^{C_p+1}/L_{C_p+1} \cong \mathbb{C}^B/L_B \) and \( I_{C_{p+1}} \setminus C_p = \{\sigma\} \). The Lie algebra \( L_{C_{p+1}} \) is generated by \( L_{C_p} \) and a vector field \( \partial_{s_{\sigma}} + \varepsilon \), where \( \varepsilon \in L_{C_{p+1}} \). Consider the flow of \( \partial_{s_{\sigma}} + \varepsilon \) with initial condition at a point of \( \mathbb{C}^{C_p} \). We can use this flow to construct an homogeneous affine isomorphism of \( \mathbb{C}^{C_p+1} \) into itself that equals the identity on \( \mathbb{C}^{C_p} \) and rectifies \( \partial_{s_{\sigma}} + \varepsilon \), leaving invariant \( L_{C_p} \). Hence, \( \mathbb{C}^{C_p}/L_{C_p} \cong \mathbb{C}^{C_{p+1}}/L_{C_{p+1}} \).

\( \square \)

**Remark 3.6.** Let us denote by \( P(s_{C}) \) the restriction of \( P \in \mathbb{C}[s_{C}][[x,y]] \) to \( \mathbb{C}^C \). Then, \( F_B(s_{C}) = F_C, \Delta F_B(s_{C}) = \Delta F_C \) and \( H_B^{\gamma}(s_{C}) = H_C^{\gamma} \) for each \( \gamma = 1, \ldots, k - 2 \). Let \( \{\delta_{\ell,\mu}, \ell \in J\} \subset \text{Der} \mathbb{C}[s_{C}] \) be the set of vector fields obtained if we proceed as in the definition of \( \{\delta^\ell, \ell \in J\} \), now with \( C \) in the place of \( B \). Then \( \langle \delta_{\ell,\mu} \rangle \rangle = \langle \delta_\ell \rangle \rangle \) as \( \mathbb{C}[s_{C}] \)-modules. To see this just notice that, if

\[
m_iF_B = \sum_{j=1}^{\mu} c^0_{ij}m_j \text{ mod } \Delta F_B
\]

\[
m_iH_B^\gamma = \sum_{j=1}^{\mu} c^\gamma_{ij}m_j \text{ mod } \Delta F_B
\]
then
\[ m_i F_B(s_C) = \sum_{j=1}^{\mu} c_{i,j}^0(s_C) m_j \mod \Delta F(s_C) \]
\[ m_i H_B^\gamma(s_C) = \sum_{j=1}^{\mu} c_{i,j}^\gamma(s_C) m_j \mod \Delta F(s_C). \]

4. Geometric Quotients of Unipotent Group Actions

An affine algebraic group is said to be unipotent if it is isomorphic to a group of upper triangular matrices of the form \( \text{Id} + \varepsilon \), where \( \varepsilon \) is nilpotent. If \( G \) is unipotent its Lie algebra \( L \) is nilpotent and the map \( \exp : L \to G \) is algebraic. Given a nilpotent Lie algebra \( L \), there is a unipotent group \( G = \exp L \) such that \( L \) is the Lie algebra of \( G \).

Let \( A \) be a Noetherian \( \mathbb{C} \)-algebra. A linear map \( D : A \to A \) is a derivation of \( A \) if
\[ D(fg) = fD(g) + gD(f). \]
A derivation \( D \) of \( A \) is nilpotent if for each \( f \in A \) there is \( n \) such that \( D^n(f) = 0 \). Let \( \text{Der}^{\text{nil}}(A) \) denote the Lie algebra of nilpotent derivations of \( A \). Here, we set \( A = \mathbb{C}[s_C] \).

Let \( G \) be an algebraic group acting algebraically on an algebraic variety \( X \). If \( Y \) is an algebraic variety and \( \pi : X \to Y \) a morphism then \( \pi \) is called a geometric quotient, if
1. \( \pi \) is surjective and open,
2. \( (\pi_* \mathcal{O}_X)^\mathcal{G} = \mathcal{O}_Y \),
3. \( \pi \) is an orbit map, i.e. the fibres of \( \pi \) are orbits of \( \mathcal{G} \).

If a geometric quotient exists it is uniquely determined and we just say that \( X/\mathcal{G} \) exists. Here, \( \mathcal{G} \) will act on each strata of \( \mathbb{C}^c = \text{Spec} A \) through the action of \( \mathcal{G} \) on each fiber of \( G \). On Theorem 5.3 we prove that \( \mathbb{C}^c/L_c \) is a classifying space for germs of Legendrian curves with generic plane projection \( \{y^k + x^n = 0\} \). The integral manifolds of \( L_c \) are the orbits of the action of \( \mathcal{G}_0 := \exp L_c \). Set \( L := [L_c, L_c] \) and \( \mathcal{G} = \exp L \). Note that \( L \) is nilpotent (\( \mathcal{G} \) unipotent) and \( L_c/L \cong \mathbb{C}^\delta_0 \), where \( \delta_0 \) is the Euler field.

Definition 4.1. Let \( \mathcal{G} \) be a unipotent algebraic group, \( Z = \text{Spec} A \) an affine \( \mathcal{G} \)-variety and \( X \subseteq Z \) open and \( \mathcal{G} \)-stable. Let \( \pi : X \to Y := \text{Spec} A^\mathcal{G} \) be the canonical map. A point \( x \in X \) is called stable under the action of \( \mathcal{G} \) with respect to \( A \) (or with respect to \( Z \)) if the following holds:
There exists an \( f \in A^\mathcal{G} \) such that \( x \in X_f = \{y \in X, f(y) \neq 0\} \) and \( \pi : X_f \to Y_f := \text{Spec} A^\mathcal{G}_f \) is open and an orbit map. If \( X = Z = \text{Spec} A \) we call a point stable with respect to \( A \) just stable.

Let \( X^s(A) \) denote the set of stable points of \( X \) (under \( \mathcal{G} \) with respect to \( A \)).

Proposition 4.2 ([6]). With the previous notations, we have that:
1. \( X^s(A) \) is open and \( \mathcal{G} \)-stable.
2. \( X^s(A)/\mathcal{G} \) exists and is a quasiaffine algebraic variety.
(3) If $V \subset \text{Spec } A^G$ is open, $U = \pi^{-1}(V)$ and $\pi : U \to V$ is a geometric quotient then $U \subset X^s(A)$.

(4) If $X$ is reduced then $X^s(A)$ is dense in $X$.

**Definition 4.3.** A geometric quotient $\pi : X \to Y$ is **locally trivial** if an open covering $\{V_i\}_{i \in I}$ of $Y$ and $n_i \geq 0$ exist, such that $\pi^{-1}(V_i) \cong V_i \times A^\text{nil}_{n_i}$ over $V_i$.

We use the following notations:

Let $L \subseteq \text{Der}^{\text{nil}}(A)$ be a nilpotent Lie-algebra and $d : A \to \text{Hom}_C(L, A)$ the differential defined by $\delta(a) = \delta(a)$. If $B \subset A$ is a subalgebra then $\int B := \{a \in A : \delta(a) \in B\}$ for all $\delta \in L$. If $\mathfrak{a} \subset A$ is an ideal, $V(\mathfrak{a})$ denotes the closed subscheme $\text{Spec } A/\mathfrak{a}$ of $\text{Spec } A$ and $D(\mathfrak{a})$ the open subscheme $\text{Spec } A - V(\mathfrak{a})$.

Let $A$ be a noetherian $C$-algebra and $L \subseteq \text{Der}^{\text{nil}}(A)$ a finite dimensional nilpotent Lie-algebra. Suppose that $A = \bigcup_{i \in \mathbb{Z}} F^i(A)$ has a filtration

$$F^i : 0 = F^{-1}(A) \subset F^0(A) \subset F^1(A) \subset \ldots$$

by sub-vector spaces $F^i(A)$ such that

$$\delta F^i(A) \subset F^{i-1}(A)$$

for all $i \in \mathbb{Z}$ and all $\delta \in L$.

Assume, furthermore, that

$$Z_* : L = Z_0(L) \supseteq Z_1(L) \supseteq \ldots \supseteq Z_\ell(L) \supseteq Z_{\ell+1}(L) = 0$$

is filtered by sub-Lie-algebras $Z_j(L)$ such that

$$[L, Z_j(L)] \supseteq Z_{j+1}(L)$$

for all $j \in \mathbb{Z}$.

The filtration $Z_*$ of $L$ induces projections

$$\pi_j : \text{Hom}_C(L, A) \to \text{Hom}_C(Z_j(L), A).$$

For a point $t \in \text{Spec } A$ with residue field $\kappa(t)$ let

$$r_i(t) := \dim_{\kappa(t)} \text{Ad } F^i(A) \otimes_A \kappa(t) \quad i = 1, \ldots, \rho,$$

with $\rho$ minimal such that $\text{Ad } F^\rho(A) = \kappa(t)$,

$$s_i(t) := \dim_{\kappa(t)} \text{Ad } F^i(A) \otimes_A \kappa(t) \quad j = 1, \ldots, \ell,$$

such that $s_j(t)$ is the orbit dimension of $Z_j(L)$ at $t$.

Let $\text{Spec } A = \bigcup U_\alpha$ be the flattening stratification of the modules

$$\text{Hom}_C(L, A)/\text{Ad } F^i(A), \quad i = 1, \ldots, \rho$$

and

$$\text{Hom}_C(Z_j(L), A)/\pi_j(\text{Ad } A), \quad j = 1, \ldots, \ell.$$

**Theorem 4.4.** Each stratum $U_\alpha$ is invariant by $L$ and admits a locally trivial geometric quotient with respect to the action of $L$. The functions $r_i(t)$ and $s_j(t)$ are constant along $U_\alpha$. Let $x_1, \ldots, x_\rho \in A_1, \delta_1, \ldots, \delta_q \in L$ satisfying the following properties:

1. The functions $r_i(t)$ and $s_j(t)$ are constant along $U_\alpha$.
2. The functions $x_1, \ldots, x_\rho$ are constant along $U_\alpha$.
3. The functions $\delta_1, \ldots, \delta_q$ are constant along $U_\alpha$.
4. The functions $x_1, \ldots, x_\rho$ and $\delta_1, \ldots, \delta_q$ are constant along $U_\alpha$.

Then $U_\alpha$ is locally trivial with respect to the action of $L$.

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• there are $\nu_1, \ldots, \nu_\rho, 0 \leq \nu_1 < \ldots < \nu_\rho = p$, such that $dx_1, \ldots, dx_{\nu_1}$ generate the $A$-module $AdF^i(A)$;
• there are $\mu_0, \ldots, \mu_\ell, 1 = \mu_0 < \mu_1 < \ldots < \mu_\ell$ such that $\delta_{\mu_1}, \ldots, \delta_m \in Z_{j}(L)$ and $Z_{j}(L) \subseteq \sum_{i \geq \mu_j} A\delta_i$.

Then
\begin{align}
(4.1) \quad \text{rank} (\delta_{\alpha}(x_{\beta})(t))_{\beta \leq \nu_i} = r_i(t) \quad i = 1, \ldots, \rho, \\
(4.2) \quad \text{rank} (\delta_{\alpha}(x_{\beta})(t))_{\alpha \geq \mu_j} = s_j(t) \quad j = 1, \ldots, \ell.
\end{align}

The strata $U_{\alpha}$ are defined set theoretically by fixing (4.1) and (4.2).

5. Filtrations and Strata

Set $L = [L_C, L_C]$. Fix a integer $a$ such that $k \geq a \geq 0$. For each $i \in \mathbb{Z}$ let $F^i_a$ be the $\mathbb{C}$-vector space generated by monomials in $\mathbb{C}[s_C]$ of degree $\geq -(a + ik)$. Since $o(\delta) \geq k$ for each homogeneous vector field of $L$, $LF^i_a \subseteq F^{i-1}_a$ for each $j$. For each $m \in \mathbb{Z}$ let $I^m_a$ be the ideal of $\mathbb{C}[[x, y]]$ generated by the monomials of degree $\geq a + mk$. Let $\rho$ be the smallest $i$ such that $dF^i_a$ generates $\mathbb{C}[s_C]d\mathbb{C}[s_C]$ as a $\mathbb{C}[s_C]$-module.

Given $\alpha \in \mathbb{Z}$, set $\alpha : = nk - k^2 - 2n - \alpha$. For each integer $j$ set $S_j = \{\alpha : s_\alpha \in F^\rho_{a-j}, \alpha \neq 0\}$ and let $Z^j_a$ be the sub-Lie algebra of $L$ generated by the homogeneous vector fields $\delta \in L$ such that $o(\delta) \in S_j$. Remark that

$$Z^1_a = L, Z^0_{a+1} = 0 \quad \text{and} \quad [L, Z^a_j] \subseteq Z^a_{j+1}.$$

For each $t \in \mathbb{C}^C$ let $I^m_t$ be the ideal of $\mathbb{C}[[x, y]]$ generated by $F_t, \Delta F_t$ and $H^1_t, \ldots, H^{k-2}_t$. Set

$$\tau^m_{a,1}(t) = \text{dim}_C \mathbb{C}[[x, y]]/(I^m_t, I^m_a), \quad \tau^m_{a,2}(t) = \text{dim}_C \mathbb{C}[[x, y]]/(\Delta F_t, (F_t, H^1_t, \ldots, H^{k-2}_t) \cap I^a_{\rho-1+2n-m}),$$

for $m = n, \ldots, n + \rho$ and

$$\tau^m_{a}(t) = (\tau^m_{a,1}(t), \ldots, \tau^{n+\rho}_{a,1}(t); \tau^m_{a,2}(t), \ldots, \tau^{n+\rho}_{a,2}(t)).$$

We say that $\tau^m_{a}(t)$ is the microlocal Hilbert function of $X_t$. Set

$$\hat{\mu} = \#C = \mu - (k - 2)(k - 1)/2, \quad \hat{\mu}^1_t = \hat{\mu} - \#\{m_\ell \in I^a_\mu : \ell \in I_C\},$$

$$\hat{\mu}^2_t = \mu - \#\{m_\ell \in I^a_{\rho-1+2n-k} : \ell \in I_B \setminus C\}.$$

We only define $\tau^m_{a}(t)$ for $m = n, \ldots, n + \rho$ because

$$\tau^m_{a,1}(t) = \tau^m_{a,2}(t) = \hat{\tau}(X_t)$$

(the microlocal Tjurina number of $X_t$) if $m$ is big and

$$\tau^m_{a,1}(t) = \text{dim}_C \mathbb{C}[[x, y]]/I^m_a, \quad \tau^m_{a,2}(t) = \hat{\mu}^m$$

(hence independent of $t$) if $m$ is small.

Let $\{U^a_{\alpha}\}$ be the flattening stratification of $\mathbb{C}^C$ corresponding to $F^\bullet_a$ and $Z^a$. It follows from Theorem 4.4 that $U^a_{\alpha} \to U^a_{\alpha}/L$ is a geometric quotient.
Moreover, $L_C/L \cong \mathbb{C}^*$ acts on $U^a_\alpha/L$ and $U^a_\alpha/L_C = U^a_\alpha/L_C$ is a geometric quotient of $U^a_\alpha$ by $L_C$. For $t \in \mathbb{C}^C$ let us define

$$e^a(t) = (u^a_0(t), \ldots, u^a_\rho(t); v^a_0(t), \ldots, v^a_\rho(t)) \in \mathbb{N}^{2\rho+2},$$

where

$$u^a_j(t) = \text{rank}(\delta(s'_\beta)(t))_{a(\tau(\beta))t \leq a+j}, \quad j = 0, \ldots, \rho,$$

and

$$v^a_j(t) = \text{rank}(\delta(s'_\beta)(t))_{a(\delta)t \leq a+j}, \quad j = 0, \ldots, \rho.$$  

**Lemma 5.1.** The function $t \mapsto e^a(t)$ is constant on $U^a_\alpha$ and takes different values for different $\alpha$. The analytic structure of $U^a_\alpha$ is defined by the corresponding subminors of $(\delta(s_C)(t))$. Moreover, $u^a_j(t) = \tilde{\rho}_1^{a+j} - \tilde{\tau}^n_{a,j}(t)$ and $v^a_j(t) = \tilde{\mu}_2^{a+j} - \tilde{\tau}^{n,j}_{a,2}(t)$. In particular, $u^a_0(t) = v^a_0(t) = \tilde{\mu} - \tilde{\tau}(X_t)$ where $\tilde{\tau}(X_t)$ is the microlocal Tjurina number of the curve singularity $X_t$.

**Proof.** That $e^a(t)$ is constant on $U^a_\alpha$ and takes different values for different $\alpha$ is a consequence of Theorem [4.4] as is the claim about the analytic structure of each strata.

Let $t \in U^a_\alpha$ and consider for each $m \in \{n, \ldots, n+\rho\}$ the induced $\mathbb{C}$-base $\{m_{t \in J_m(t)} = \{m_{t \in J_m}\}$ of $\mathbb{C} \{x, y\}/(\Delta F_t, I^m_\alpha)$. Then, for each $\ell \in J_m$

$$m_\ell F_t = \sum_{j=1}^b \delta^0_j(s_{o(j)}(t)m_{\mu^{-b+j}} \text{ mod } (\Delta F_t, I^m_\alpha)$$

and

$$m_\ell H^\gamma_t = \sum_{j=1}^b \delta^\gamma_j(s_{o(j)}(t)m_{\mu^{-b+j}} \text{ mod } (\Delta F_t, I^m_\alpha)$$

for $\gamma = 1, \ldots, k-2$. Then, by definition of $\tilde{\tau}^a(t)$ and from the definition of $\{\delta^a\}$, $u^a_j(t) = \tilde{\mu}_1^{a+j} - \tilde{\tau}^n_{a,1}(t)$.

The proof of the claim about the $v^a_j(t)$ is similar with the difference that we’re now interested in the relations mod $\Delta F_t$ between the $m_\ell F_t, m_\ell H^\gamma_t$ that belong to $I^{a-1+2n-m}_a$ for each $m \in \{n, \ldots, n+\rho\}$. Note that $m_\ell F_t, m_\ell H^\gamma_t \in I^{-1+2n-m}_a$ if and only if $a_1^0, a^\gamma \in S_{m-n}$.

**Lemma 5.2.** If $a, b \in \mathbb{C}^B$ are such that $\text{Con}(F_a) \cong \text{Con}(F_b)$, there is $\psi : \mathbb{C} \rightarrow \mathbb{C}^B$ microlocally trivial such that $\psi(0) = a$ and $\psi(1) = b$.

**Proof.** Let $\chi_0$ be a contact transformation given by $\alpha, \beta_0$ such that $F_b = uF_a^{x_0}$ for some unit $u \in \mathbb{C}\{x, y\}$. We can assume $\deg \chi_0 > 0$. There is a relative contact transformation $\chi(t)$ over $\mathbb{C}$ such that $\chi(0) = id_{\mathbb{C}^3}$ and $\chi(1) = \chi_0$. Then

$$G(t) = u(tx, ty)F_B^y(x, y, a)$$
is an unfolding of $F_a$ such that $G(1) = F_b$. By versality of $F_B$ and because $F_a$ is semiquasihomogeneous ($j(F_a) = (F_a, j(F_a))$) there is a relative coordinate transformation

$$\Phi : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C} \times \mathbb{C}^2$$

$$(t, x, y) \mapsto (t, \Phi_1, \Phi_2)$$

and $\psi : \mathbb{C} \to \mathbb{C}^B$ such that

$$\Phi(G(t)) = F_{\psi(t)}.$$  

(see Remark 1.1 and Corollary 3.3 of [?]). Now, because $F_B$ is semiversal (hence does not contain trivial subfamilies with respect to right equivalence) $\Phi(1) (G(1)) = \Phi(1) (F_b) = F_{\psi(1)}$ implies that $\psi(1) = b$.

**Theorem 5.3.** Given $a, b \in \mathbb{C}^C$, $\text{Con}(F_a) \cong \text{Con}(F_b)$ if and only if $a$ and $b$ are in the same integral manifold of $L_C$.

**Proof.** By Theorem 3.5 we can replace $C$ by $B$.

Let us first prove sufficiency. Let $C \subset A \subset B$ and $S$ be a complex space. We say that a holomorphic map $\psi : S \to \mathbb{C}^A$ is trivial if for each $o \in S$, $\psi^* F_A$ is a trivial deformation of $\text{Def}^{f^{es, \mu}}(S, o)$. Assume $\psi : (\mathbb{C}, 0) \to \mathbb{C}^B$ is the germ of an integral curve of a vector field $\delta$ in $\mathcal{L}_B$. Set $q = \psi(0)$. Let $\psi : T_\varepsilon \to \mathbb{C}^B$ be the morphism induced by $\psi$. There are $a_0, a_1, \ldots, a_l, \alpha_0, \beta_0 \in C\{s_B\}[[x, y]]$ such that

$$\delta F_B = a_0 F_B + \sum_{j=1}^\ell a_j H^j_B + \alpha_0 \partial_x F_B + \beta_0 \partial_y F_B.$$  

Set $u = 1 + \varepsilon a_0(q)$, $\alpha = \alpha(q) + \sum_{j=1}^\ell a_j(q)p^j$ and $\beta = \beta(q) + \sum_{j=1}^\ell \frac{1}{j+1} a_j(q)p^{j+1}$. By Theorem 2.17 there is $\gamma \in C\{x, y, p\}$ such that

$$(x, y, p, \varepsilon) \mapsto (x + \alpha \varepsilon, y + \beta \varepsilon, p + \gamma \varepsilon, \varepsilon)$$

defines a relative contact transformation $\chi$ over $T_\varepsilon$. Let $G \in \mathbb{C}\{x, y, p, \varepsilon\}$ be defined by $G(x, y, p, \varepsilon) = F_B(x + \alpha \varepsilon, y + \beta \varepsilon, q)$. Since $\psi^* F_B \equiv uG \text{ mod } (\varepsilon)$ and

$$\partial_\varepsilon \psi^* F_B \equiv \partial_\varepsilon uG \text{ mod } I_{\text{Con}(F_a)} + (\varepsilon),$$

we have that

$$\psi^* F_B \equiv uG \text{ mod } I_{\chi^e(\text{Con}(F_a))} + (\varepsilon^2).$$

Therefore, $\psi^* F_B$ is a trivial deformation of $\text{Def}^{f^{es, \mu}}(T_\varepsilon)$. Then $\psi^* F_B$ is a trivial deformation of $\text{Def}^{f^{es, \mu}}(\mathbb{C}, 0)$ (see the proof of Theorem 2.15).

Conversely, assume that there is a germ of contact transformation $\chi$ such that $(F_\chi^1) = (F_b)$. We can assume $\deg \chi_1 > 0$. If $\chi_1$ is of type $\frac{2.10}{17}$, by Lemma 5.2 there is a trivial curve $\psi : \mathbb{C} \to \mathbb{C}^B$ such that $\psi(0) = a$ and $\psi(1) = b$. Moreover, $\psi$ is an integral curve of the Euler vector field. Since the derivative of $\chi_1$ leaves $\{y = p = 0\}$ invariant, we can assume by Theorem 2.18 that $\chi_1$ is of type $\frac{2.7}{17}$. Set $\chi = \chi_{\varepsilon a_1 b_0}$. There is a curve
with polynomial coefficients \( \psi : \mathbb{C} \to \mathbb{C}^b \) such that \( F_0^x = \psi^* F_B, \psi(0) = a \) and \( \psi(1) = b \).

Let \( \Omega \) be an open set of \( \mathbb{C} \). Let \( \psi : \Omega \to \mathbb{C}^b \) be a trivial curve. Let us show that \( \psi \) is contained in an integral manifold of \( \mathcal{L}_B \). Let \( U \) be the union of the strata \( U_0 \) such that, for each \( c \in U \) the microlocal Tjurina number of \( F_0 \) equals the microlocal Tjurina number of \( F_a \). Remark that the trajectory of \( \psi \) is contained in \( U \). By Theorem 4.4 \( \mathcal{L}_B[U] \) verifies the Frobenius Theorem. Hence, it is enough to show that, for each \( t_0 \in \Omega \), there is \( \delta \in \mathcal{L}_B \) such that \( \psi'(t_0) = \delta(\psi(t_0)) \). We can assume \( t_0 = 0 \). Since \( \psi \) is trivial, there are a relative contact transformation \( \chi \) and \( u \in \mathbb{C}\{x,y,t\} \) such that \( u(x, y, 0) = 1 \) and

\[
F(x, y, \psi(t)) \equiv uF(x, y, q) \mod I_{\chi(<c_{(u,F)})}.
\]

If \( \chi \) is of type \( \text{[2.10]} \) we can assume \( \delta \) is the Euler field. Hence we can assume that \( \chi \) is of type \( \text{(2.7)} \). Therefore there are \( \ell \geq 1 \) and \( a, b, a_i \in \mathbb{C}\{x,y\}, 1 \leq i \leq \ell \), such that

\[
F(x, y, \psi(t)) = uF(x, y, q) + \sum_{\ell=1}^{k-2} a_\ell t H^\ell_{x} + abt \partial_x F_q + bt \partial_y F_q \mod (t^2).
\]

Deriving in order to \( t \) and evaluating at 0, there is \( a_0 \in \mathbb{C}\{x,y\} \) such that

\[
\sum_{(i,j) \in C_0} \psi'_{i,j}(0)x^i y^j = a_0 F_q + \sum_{\ell=1}^{k-2} a_\ell H^\ell_{x} + a \partial_x F_q + b \partial_y F_q.
\]

There are \( \delta \in L_B \) and \( \varepsilon \in \Delta_{F_B} \) such that

\[
\delta F_B = a_0 F_B + \sum_{\ell=1}^{k-2} a_\ell H^\ell_{B} + \varepsilon.
\]

Hence

\[
\sum_{(i,j) \in B_0} \psi'_{i,j}(0)x^i y^j - \delta(q) F_B = \varepsilon(q) + a \partial_x F_q + b \partial_y F_q.
\]

If \( \delta = \sum_{(i,j) \in B_0} a_{i,j} \partial s_{i,j}, a_{i,j} (\psi(0)) = \psi'_{i,j}(0) \) for each \( (i,j) \in B_0 \).

\[\square\`

**Theorem 5.4.** (1) Let \( \varepsilon = (e_1, \ldots, e_\rho) \in \mathbb{N}^{\rho + 1} \) and let \( U_{E}^\sigma \) denote the unique stratum (assumed to be not empty) such that \( \varepsilon^\sigma(t) = \varepsilon \) for each \( t \in U_{E}^\sigma \). The geometric quotient \( U_{E}^\sigma / \mathcal{L} \) is quasifinite and of finite type over \( \mathbb{C} \). It is a coarse moduli space for the functor which associates to any complex space \( S \) the set of isomorphism classes of flat families (with section) over \( S \) of plane curve singularities with fixed semigroup \( \langle k,n \rangle \) and fixed microlocal Hilbert function \( \tau_{\pi}^\bullet \).

(2) Let \( T_{\pi_{\min}} \) be the open dense set defined by singularities with minimal microlocal Tjurina number \( \tau_{\min} \). Then the geometric quotient \( T_{\pi_{\min}} / \mathcal{L}_C \) exists and is a coarse moduli space for curves with semigroup \( \langle k,n \rangle \) and microlocal Tjurina number \( \tau_{\min} \). Moreover, \( T_{\pi_{\min}} / \mathcal{L}_C \) is locally isomorphic to an open subset of a weighted projective space.

**Proof.** It follows from Lemma 5.1 and Theorems 4.4 and 5.3. \[\square\]
Choosing a \( f \) is a semiuniversal equisingular microlocal deformation of \( f = y^6 + x^{13} \). Let \( Y_s \) denote the germ at the origin of \( \{ F = 0 \} \) in \( \mathbb{C}^3 \). In the previous example, \( Y_{s, 0} = \langle s_2, s_3, s_4, s_9, s_{10} \rangle = \langle s_2, s_3, s_4, s_9, s_{10}, s_{16} \rangle \). We prove that \( Y_{s, 0} = \langle s_2, s_3, s_4, s_9, s_{10}, s_{16} \rangle \) by fixing the parameter \( \alpha \). For \( \alpha = 1 \), we get \( \langle s_2, s_3, s_4, s_9, s_{10}, s_{16} \rangle = \langle s_2, s_3, s_4, s_9, s_{10}, s_{16} \rangle \), and the stratification \( \{ U^a \} \) given by fixing \( \varepsilon^a(t) = (u_0^a(t), u_1^a(t), u_2^a(t); v_0^a(t), v_1^a(t), v_2^a(t)) \) is given by

\[
U_1 = \{ t = (t_2, t_3, t_4, t_9, t_{10}, t_{16}) \in \text{Spec } \mathbb{C}[s_C] : \varepsilon^a(t) = (1, 3, 4; 1, 3, 4) \}
\]
\[
= \{ t : 9t_2^2 - 8t_2t_4 + \frac{116}{39}t_2^3 \neq 0 \}.
\]

\[
U_2 = \{ t \in \text{Spec } \mathbb{C}[s_C] : \varepsilon^a(t) = (1, 2, 3; 1, 2, 3) \}
\]
\[
= \{ t : 9t_2^2 - 8t_2t_4 + \frac{116}{39}t_2^3 = 0 \text{ and } t_3 \neq 0 \text{ or } t_3 \neq 0 \text{ or } t_4 \neq 0 \}.
\]

\[
U_3 = \{ t \in \text{Spec } \mathbb{C}[s_C] : \varepsilon^a(t) = (0, 1, 2; 0, 1, 2) \}
\]
\[
= \{ t : t_2 = t_3 = t_4 = 0 \text{ and } t_{10} \neq 0 \}.
\]

\[
U_4 = \{ t \in \text{Spec } \mathbb{C}[s_C] : \varepsilon^a(t) = (0, 1, 1; 0, 0, 0) \}
\]
\[
= \{ t : t_2 = t_3 = t_4 = t_{10} = 0 \text{ and } t_9 \neq 0 \}.
\]

\[
U_5 = \{ t \in \text{Spec } \mathbb{C}[s_C] : \varepsilon^a(t) = (0, 0, 1; 0, 0, 1) \}
\]
\[
= \{ t : t_2 = \cdots = t_{10} = 0 \text{ and } t_{16} \neq 0 \}.
\]

\[
U_6 = \{ t \in \text{Spec } \mathbb{C}[s_C] : \varepsilon^a(t) = (0, 0, 0; 0, 0, 0) \}
\]
\[
= \{ t : t_2 = \cdots = t_{16} = 0 \}.
\]

\[U_1\] is the stratum with minimal microlocal Tjurina number.

Let us present detailed calculations concerning the generators of \( L_c \). Let \( Y \) denote the germ at the origin of \( \{ F_C = 0 \} \).
The relative conormal $L$ of $Y$ can be parametrized by

$$
\begin{align*}
x &= -t^6, \\
y &= t^{13} + \psi_2 t^{15} + \psi_3 t^{16} + \psi_4 t^{17} + \psi_5 t^{18} + \psi_6 t^{19} + \psi_7 t^{20} + \psi_8 t^{21} + \psi_9 t^{22} + \cdots, \\
p &= -\frac{13}{6} t^7 - \frac{5}{2} \psi_2 t^9 - \frac{8}{3} \psi_3 t^{10} - \frac{17}{6} \psi_4 t^{11} - 3 \psi_5 t^{12} - \frac{19}{6} \psi_6 t^{13} - \frac{10}{3} \psi_7 t^{14} - \frac{7}{2} \psi_8 t^{15} \\
&\quad - \frac{11}{3} \psi_9 t^{16} + \cdots,
\end{align*}
$$

where $\psi_i \in (s_C)C[s_C]$ are homogeneous of degree $-i$. These are the $a_i$ such that the polynomial in $C[t]$ given by the following SINGULAR session is zero:

```plaintext
> ring r=(0,a2,a3,a4,a5,a6,a7,a8,a9,s2,s3,s4,s9,s10,s16),(x,y,t),dp;
> poly F=y6+x13+s2*x9y2+s3*x7y3+s4*x5y4+s9*x8y3+s10*x6y4+s16*x7y4;
> subst(F,x,-t6);
-t^78+(-s2)*y^2*t^54+(s9)*y^3*t^48+(-s16)*y^4*t^42+(-s3)*y^3*t^42+(s10)*y^4*t^36+(-s4)*y^4*t^30+y^6
> subst(-t^78+(-s2)*y^2*t^54+(s9)*y^3*t^48+(-s16)*y^4*t^42+(-s3)*y^3*t^42+(s10)*y^4*t^36+(-s4)*y^4*t^30+y^6,y,t^13+a2*t^15+a3*t^16+a4*t^17+a5*t^18+a6*t^19+a7*t^20+a8*t^21+a9*t^22)
```

As we’ll see, the only $\psi_1$ we actually need to find the generators of $L_C$ is

$$
\psi_2 = s_2/6.
$$
Let us calculate the vector fields generating $L_C$. Here, all equalities are $\text{mod } \Delta F_C$ and in the vector fields we identify, by abuse of language, the monomials and the corresponding $\partial$’s:

- $\delta_{14}$:
  
  $$xH_2 = xp^2 \partial_x F_C = 13p^2 x^{13} + 9s_2p^2 x^9 y^2 + \cdots$$

  Notice that, as a consequence of Lemma 3.3, the monomials occurring with order bigger than $\deg(x^7 y^4)$ can be ignored in this calculation. From now on, whenever we use the symbol $\cdots$ we mean that bigger order monomials can be ignored. Now, continuing the previous SINGULAR session:

  ```
  > poly p=(-13t7-15*a2*t9-16*a3*t10-17*a4*t11-18*a5*t12-19*a6*t13-20*a7*t14
  -21*a8*t15-22*a9*t16)/6;
  > poly X=-t6;
  > poly Y=t13+a2*t15+a3*t16+a4*t17+a5*t18+a6*t19+a7*t20+a8*t21+a9*t22;
  > p^2*X^13-(13/6)^2*X^11*Y^2;
  > poly X=-t6;
  > poly Y=t13+a2*t15+a3*t16+a4*t17+a5*t18+a6*t19+a7*t20+a8*t21+a9*t22;
  > p^2*X^13-(13/6)^2*X^11*Y^2;
  (-35*a9^2)/4*t^-110+(-293*a8*a9)/18*t^-109+(-271*a7*a9-136*a8^2)/18*t^-108+(-249*a6*a9-251*a7*a8)/18*t^-107+(-454*a5*a9-460*a6*a8-231*a7^2)/36*t^-106+(-205*a4*a9-209*a5*a8-211*a6*a7)/18*t^-105+(-183*a3*a9-188*a4*a8-191*a5*a7-96*a6^2)/18*t^-104+(-161*a2*a9-167*a3*a8-171*a4*a7-173*a5*a6)/18*t^-103+(-292*a2*a8-302*a3*a7-308*a4*a6-155*a5^2)/36*t^-102+(-131*a2*a7-135*a3*a6-137*a4*a5-117*a9)/18*t^-101+(-116*a2*a6-119*a3*a5-60*a4^2-104*a8)/18*t^-100
  +(-101*a2*a5-103*a3*a4-91*a7)/36*t^-99+(-71*a2*a4-87*a3^2-156*a6)/36*t^-98+13/6)^2\quad x^{11} y^2 + \frac{13\psi_2}{9} x^7 y^4 + \cdots
  $$

  we see that

  $$p^2 x^{13} = \left(\frac{13}{6}\right)^2 x^{11} y^2 + \frac{13\psi_2}{9} x^7 y^4 + \cdots$$

  Now, $\delta_{13}$, given by $yH_1$, which has the same order as $xH_2$ can be used to, through elementary operations, eliminate from $\delta_{14}$ the monomial $x^{11} y^2$. Thus,

  $$\delta_{14} = s_2 x^7 y^4.$$  

- $\delta_{13}$:

  $$-6.13y F_C = 2s_2 x^9 y^3 + 3s_3 x^7 y^4 + \cdots$$

  But, as

  $$H_3 = \left(\frac{13}{6}\right)^3 \cdot 13 x^9 y^3 + \cdots$$

  we see that, through elementary operations involving $\delta_{13}^3$, we can eliminate from $\delta_{13}^3$ the monomial $x^9 y^3$. Thus,

  $$\delta_{13} = 3s_3 x^7 y^4.$$  

- $\delta_{12}$:

  $$-6.13x^2 F_C = 2s_2 x^{11} y^2 + 3s_3 x^9 y^3 + 4s_4 x^7 y^4 + \cdots$$
through elementary operations involving $\delta^1_3$ and $\delta^3_0$ we can eliminate the monomials $x^{11}y^2$ and $x^9y^3$ from $\delta^0_2$ and get:

$$\delta^0_2 = (4s_4 + *s^2_2)x^7y^4, \quad * \in \mathbb{C}.$$ 

Finally, using $\delta^{14}$ to eliminate $*s^2_2x^7y^4$, we have that

$$\delta^{12} = 4s_4x^7y^4.$$

For $\delta^7$:

$$xH_1 = x\partial_x F_C = 13px^{13} + 9s_2px^9y^2 + 7s_3px^7y^3 + 5s_4px^5y^4 + 8s_9px^4y^3 + \cdots$$

and

$$px^{13} = \frac{13}{6}x^{12}y + \frac{s_2}{18}x^8y^3 + \frac{s_3}{12}x^6y^4 + \cdots$$

Remark 6.1. The reason why we can ignore in $px^{13}$ the monomials that occur after $x^6y^4$ is that

1. All monomials after $x^6y^4$, except for $x^7y^4$, can be eliminated because of Lemma 3.3 and through elementary operations involving $\delta^3_1$ and $\delta^0_6$.

2. Even $x^7y^4$ can be ignored, observing that $px^{13}$ is homogeneous of degree 7 and as such, the only variables involved in the coefficient (in $\mathbb{C}[s_C]$) of $x^7y^4$ may be $s_2$, $s_3$ or $s_4$. Now, using $\delta^{14}$, $\delta^{13}$ and $\delta^{12}$ we can eliminate, through elementary operations, the monomial $x^7y^4$ from $\delta^7$.

From

$$y\partial_x F_C = 13x^{12}y + 9s_2x^8y^3 + 7s_3x^6y^4 + 5s_4x^4y^5 + 8s_9x^7y^4 + \cdots$$

we get that

$$\frac{13}{6}x^{12}y = -\frac{3}{2}s_2x^8y^3 - \frac{7}{6}s_3x^6y^4 - \frac{5}{6}s_4x^4y^5 - \frac{8}{6}s_9x^7y^4 + \cdots$$

Reasoning as in remark 6.1 we see that $s_4x^4y^5$ can be ignored. Thus,

$$13px^{13} = 13\left((-\frac{3}{2}s_2 + \frac{s_2}{18})x^8y^3 + \left(-\frac{7}{6}s_3 + \frac{s_3}{12}\right)x^6y^4 - \frac{8}{6}s_9x^7y^4 + \cdots\right)$$

Now,

$$px^9y^2 = \frac{13}{6}x^8y^3 + \cdots$$

$$px^7y^3 = \frac{13}{6}x^6y^4 + \cdots$$

$$px^8y^3 = \frac{13}{6}x^7y^4 + \cdots$$
Once again, the monomials ignored can be eliminated, reasoning as in Remark 6.1. So,

\[ xH_1 = \left( 13 \left( -\frac{3}{2}s_2 + \frac{s_9}{18} \right) + 9 \frac{13}{6}s_2 \right)x^8y^3 + \left( 13 \left( -\frac{7}{6}s_3 + \frac{s_9}{12} \right) + 7 \frac{13}{6}s_3 \right)x^6y^4 + \]
\[ + \left( -\frac{13}{6}s_9 + 8 \frac{13}{6}s_9 \right)x^7y^4 \]
\[ = \frac{13}{18}s_2x^8y^3 + \frac{13}{12}s_3x^6y^4. \]

We get that

\[ \delta^7 = \frac{s_2}{3}x^8y^3 + \frac{s_3}{2}x^6y^4. \]

• \( \delta^6 \):

\[-6.13xF_C = 2s_2x^{10}y^2 + 3s_3x^8y^3 + 4s_4x^6y^4 + 9s_6x^4y^5 + 10s_{10}x^7y^4 + \cdots \]

Because (monomials ignored as in Remark 6.1)

\[ H_2 = p^2\partial_x F_C = 13p^2x^{12} + 9s_2p^2x^8y^2 + \cdots, \]
\[ p^2x^{12} = \left( \frac{13}{6} \right)^2x^{10}y^2 + \frac{13}{6}9s_2x^6y^4 + \cdots, \]

and

\[ p^2x^8y^2 = \left( \frac{13}{6} \right)^2x^6y^4 + \cdots \]

we get

\[-6.13xF_C - 2s_2 \left( \frac{6}{13} \right)^2 \frac{H_2}{13} = \]
\[ = 3s_3x^8y^3 + \left( 4s_4 - 2s_2 \left( \frac{6}{13} \right)^2 \frac{13}{6}9s_2 - 2s_2 \left( \frac{6}{13} \right)^2 \left( \frac{9}{13} \right)^2s_2 \right)x^6y^4 + 10s_{10}x^7y^4 = \]
\[ = 3s_3x^8y^3 + \left( 4s_4 - \frac{58}{39}s_2 \right)x^6y^4 + 10s_{10}x^7y^4. \]

So,

\[ \delta^6 = 3s_3x^8y^3 + \left( 4s_4 - \frac{58}{39}s_2 \right)x^6y^4 + 10s_{10}x^7y^4. \]

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