The Group-Theoretical Classification of Some Multiparticle States in the Presence of Magnetic Field and Periodic Potential

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The group-theoretical classification of multiparticle states (pairs of particles and charged excitons $X^\pm$) is based on considerations of products of irreducible projective representations of the two-dimensional translation group. The states of a pair particle-antiparticle are non-degenerate, whereas, for a given Born–von Kármán period $N$, degeneracy of pair states is $N$ and three-particle states are $N^2$-fold degenerated. The symmetrization of states with respect to particles transposition is considered. Three symmetry adapted bases for trions are considered: (i) the first is obtained from a direct conjugation of three representations; (ii) in the second approach the states of a electrically neutral pair particle-antiparticle are determined in the first step; (iii) the third possibility is to consider a pair of identical particles in the first step. In the discussion presented the Landau gauge $A = [0, Hx, 0]$ is used, but it is shown that the results obtained are gauge-independent. In addition the relation between changes of a chosen gauge and local basis transformations are discussed.

I. INTRODUCTION

The quantum Hall effect and high temperature superconductivity have given raise to interest in properties of the two-dimensional electron gas subjected to electric and magnetic fields. The observation of (negatively) charged excitons \cite{1} has recalled a forty-year old concept of excitons “trions” or “charged excitons” introduced by Lampert in 1958 \cite{2}. Recently, such excitons, consisting of two holes and an electron or two electrons and hole (denoted $X^\pm$, respectively), have been investigated both experimentally and theoretically \cite{3, 4, 5}.

In this paper classification based on translational symmetry in the presence of a periodic potential and an external magnetic field is presented. To perform this task the so-called magnetic translation operators, introduced by Brown \cite{6} and Zak \cite{7}, are used. These operators commute with the standard Hamiltonian of an electron in the magnetic field $H = \nabla \times A$ and a periodic potential $V(r)$

$$\mathcal{H} = \frac{1}{2m} \left( p + \frac{e}{c} A \right)^2 + V(r).$$

This paper exploits the fact that after imposing the Born–von Kármán (BvK) periodic conditions the magnetic translations form a finite-dimensional projective representation of the 2D translation group. Kronecker products of irreducible projective representations can be applied to description of multiparticle states \cite{8}.

The aim of this work is to present classification of three-particle states (strictly speaking states of particle-particle-antiparticle systems) when at the first step states of a pair of identical (charged) particles are constructed. Such an approach allow to discuss pair states, what may be important in considerations of high-\(T_c\) superconductors, where Cooper pairs are confined to Cu-O planes. A pair particle-antiparticle is also considered and a particle is added in the next step, what leads to the particle-particle-antiparticle system. States of such systems are also determined by means of direct conjugation of three representations. More detailed discussion of trions is also presented in another author’s article \cite{9}.

Investigating problems, which involve the magnetic field $H$ determined by the vector potential $A$, one has to keep in mind that some results may depend on a chosen gauge, though physical properties should be gauge-independent. Two gauges are most frequently used in description of the 2D electron systems: the Landau gauge with $A = [0, Hx, 0]$ and the antisymmetric one with $A = (H \times r)/2$. The relations between these gauges were discussed in the earlier article \cite{10}. For the sake of simplicity the considerations are limited the Landau gauge, but the previous results \cite{10} and the concept of rays \cite{11} together with application of a local basis transformation enable us to show that classification obtained here is gauge-independent. The presented results correspond to the limit of high magnetic fields, \textit{i.e.} there is no Landau level mixing.

II. MAGNETIC TRANSLATIONS AS PROJECTIVE REPRESENTATIONS

Magnetic translation operators $T(R)$ commuting with the Hamiltonian (1) form a projective representation of the translation group $\mathcal{T}$ \cite{6}, \textit{i.e.}

$$T(R_1)T(R_2) = T(R_1 + R_2)\mu(R_1, R_2);$$

where $\mu(R_1, R_2)$ denotes a phase factor for translation $R_1 + R_2$. The determination of the phase factor requires consideration of the Landau gauge, but it is shown that the results obtained are gauge-independent. In addition the relation between changes of a chosen gauge and local basis transformations are discussed.
The other special case, \( \mathbf{R}_1, \mathbf{R}_2 \in \mathcal{T}, \mu(\mathbf{R}_1, \mathbf{R}_2) \in U(1) \). It should be stressed that due to the presence of a magnetic field \( \mathbf{H} \) all interesting physical (as well as mathematical) features take place in the plane perpendicular to \( \mathbf{H} \). Therefore, we assume \( \mathbf{H} = H \hat{z} \) and a two-dimensional crystal lattice determined by vectors \( \mathbf{a}_1, \mathbf{a}_2 \) lying in the \( xy \)-plane is considered. The concept of magnetic translations can be generalized to a local gauge of the vector potential \( \mathbf{A} \), \( d \)-dimensional lattices, and the spatially inhomogeneous magnetic field \([10, 12, 13]\). The periodic boundary conditions give rise to the flux quantization:

\[
q\phi = \frac{l}{n}, \quad n|N, \, \gcd(l, n) = 1;
\]

\( q = Q/e \), where \( Q \) is a particle charge, \( \phi \) is a flux through the unit lattice cell, and \( N \) is the Born–von Kármán period. All possible pairs \( (l, n) \) label different irreducible representations of the translation group \( \mathcal{T} \). Two notes are in a place. At first, please note that the formula presented means, in fact, quantization of the product \( q\phi \) only. In the case \( n = N \) each \( l \), mutually prime with \( N \), determines a projective irreducible representation (irrep) in the unique way. In the other cases \( (1 \leq n < N) \) there is an additional pair of indices \( \mathbf{k} \equiv [k_1, k_2] \) \((0 \leq k_1 < (N/n), j = 1, 2)\), so there are \((N/n)^2\) \( n \)-dimensional nonequivalent projective irreps of \( \mathcal{T} \). Each of these irreps has the same factor system \( \mu(\mathbf{R}_1, \mathbf{R}_2) \) determined by the pair \( (l, n) \) \([14]\). This property is related to the concept of the so-called magnetic cells and magnetically periodic conditions \([6, 12]\).

### III. PRODUCT OF PROJECTIVE IRREPS

Assuming the Landau gauge, \( \mathbf{A} = [0, Hx, 0] \), the finite-dimensional projective irrep of the 2D translation group can be chosen as \( (n = N) \)

\[
D^l_{jk}[n_1, n_2] = \delta_{j, -n_2} \omega_N^{ln_1j},
\]

where \( \omega_N = \exp(2\pi i/N) \) and all expressions are calculated modulo \( N \); vectors of the translation group \( \mathcal{T} \supseteq \mathbb{R} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2 \) are replaced by their coordinates, \( n_1 \) and \( n_2 \), in the crystal basis. In the terms of the basis vectors of the irrep we obtain

\[
D^l[n_1, n_2][w] = \omega_N^{ln_1(w-n_2)}|w - n_2\rangle.
\]

The other special case, \( n = 1 \), leads to the \( N^2 \) standard (vector) irreps of the translation group labeled by wave vector \( \mathbf{k} \), \( i.e. \)

\[
D^k[n_1, n_2] = \omega_N^{kn_1+n_2}, \tag{2}
\]

where \( \mathbf{k} = [k_1, k_2], k_1, k_2 = 0, 1, 2, \ldots, N - 1. \)

The Kronecker product of representations \( D^l \) (for any \( n \)) corresponds to the addition of charges (for a fixed \( H \)) \([8, 15]\). In particular

\[
D^l \otimes D^{-l} = \bigoplus_k D^k; \tag{3}
\]

\[
D^l \otimes D^l \otimes D^{-l} = N^2 D^l \quad \text{for } n = N. \tag{4}
\]

In the further considerations we limit ourselves to the case, when one-particle states are described by representations with \( n = N \). It means that crystal and magnetic periods \([10]\) are identical and we consider products of representations \( D^l \) for \( l \) mutually prime with \( N \).

### IV. STATES OF PAIRS

#### A. States of particle-antiparticle pair

The particle-antiparticle pair is very simple to consider. Taking into account Eq. (3) one can construct symmetry adapted basis consisting of vectors

\[
|0\rangle_k = N^{-1/2} \sum_{s=0}^{N-1} \omega_N^{sk} |s\rangle_+ |s - xk_1\rangle_-, \tag{5}
\]
where \( x \equiv 1 \mod N \) (\( x = l^{-1} \mod N \)). The vector \(|0\rangle_k\) transform as the basis vector of the one-dimensional (vector) representation \( D^k \) determined in Eq. (2). The subscript ‘+’ (‘–’) denotes states of a particle (an antiparticle, respectively).

### B. Pair of identical particles

In the case of a pair of identical particles, i.e. considering the product \( D^l \otimes D^l \), one has to distinguish cases of odd and even \( N \). The first case (\( N \) odd) is easy to solve since

\[
D^l \otimes D^l = ND^{2l}.
\]

One of possible forms of the irreducible basis is

\[
|s\rangle_2 = |s + r\rangle_+ |s - r\rangle_+, \quad s, r = 0, 1, 2, \ldots, N - 1,
\]

where the subscript ‘2’ indices states of a pair and \( r \) is the repetition index.

The case \( N = 2M \) leads to the representations product in the following form

\[
D^l \otimes D^l = M(D^{[0,0]} \oplus D^{[1,0]} \oplus D^{[0,1]} \oplus D^{[1,1]}).
\]

The representations \( D^{[k_1,k_2]} \), \( k_1, k_2 = 0, 1 \) form a complete set of nonequivalent \( M \)-dimensional projective irreps of \( T \). They can be expressed by a general formula \[14\]

\[
D^{[k_1,k_2]} n_1, n_2 |s\rangle = \omega_N^{l_{k_1}(2s-2n+k_1)} (-1)^{k_2} |s-n\rangle_+,
\]

where \( k_1, k_2 = 0, 1, \eta = n_2 \mod M, \xi = 1 \) for \( M \leq s - n_2 \leq 2M \) and \( \xi = 0 \) otherwise. The irreducible bases can be chosen in the following way

\[
|s\rangle_2^{[k_1,k_2]} = 2^{-1/2} \left( |s + r\rangle_+ |s - r + k_1\rangle_+ + (-1)^{k_2} |M + s + r\rangle_+ |M + s - r + k_1\rangle_+ \right)
\]

with \( s, r = 0, 1, 2, \ldots, M - 1 \), where \( s \) labels vectors and \( r \) is the repetition index.

### C. Symmetrization of states

Having determined the basis \(|s\rangle_2\) for pairs of identical particle it is natural and obvious to investigate the symmetry properties related with the transposition of particles. As in all above problems the case of odd \( N = 2M + 1 \) is quite easy. It follows from Eq. (5) that \( r = 0 \) leads to symmetric states

\[
|0\rangle_+ |0\rangle_+, |1\rangle_+ |1\rangle_+, \ldots, |N-1\rangle_+ |N-1\rangle_+.
\]

The other \( N^2 - N = N(N - 1) \) vectors correspond to \( N - 1 = 2M \) representations labeled by \( r = 1, \ldots, N - 1 \). To construct symmetric (antisymmetric) states it is enough to take combinations

\[
|s\rangle_2^{r \pm} = 2^{-1/2} \left( |s + r\rangle_+ |s - r\rangle_\pm \pm |s - r\rangle_+ |s + r\rangle_+ \right),
\]

where \( r = 1, 2, \ldots, M \) now.

In the case of even \( N = 2M \) the product \( D^l \otimes D^l \) decomposes into \( M \) copies of four \( M \)-dimensional projective representations. Since the symmetrization of states has a slightly different form in each of these cases then they are considered separately.

The non-symmetrized basis of the representation \( D^{[0,0]} \) is given by Eq. (6) as

\[
|s\rangle_2^{r \pm} = 2^{-1/2} \left( |s + r\rangle_+ |s - r\rangle_\pm \pm |s - r\rangle_+ |s + r\rangle_+ \right). \]

The case \( r = 0 \) again gives \( M \) symmetric states. To consider the other \( M - 1 \) representations one has to check the parity of \( M \). If \( M = 2\mu \), then for \( r = \mu \) we have \( M - r = r \) and \( M + r = N - r \), so in this case all vectors are symmetric. The other vectors form symmetric and antisymmetric states in the standard way and it also concerns the case of odd \( M \).
The ‘additional’ symmetric state found above for \( N = 4 \mu \) is ‘lost’ considering \( D_{11}^{[1,0]} \). Its irreducible basis given by Eq. (6)

\[
|s\rangle_2^{[1,0]} = 2^{-1/2} (|s + r\rangle_+|s - r + 1\rangle_+ + |M + s + r\rangle_+|M + s - r + 1\rangle_+)
\]

have no symmetric states for \( M = 2 \mu \), but for odd \( M = 2 \mu - 1 \) and \( r = \mu \) the above formula reads

\[
|s\rangle_2^{[\mu,1]} = 2^{-1/2} (|s + \mu\rangle_+|s - \mu + 1\rangle_+ + |s - \mu + 1\rangle_+|s + \mu\rangle_+),
\]

so these states are symmetric.

The similar considerations have to be performed for representations \( D_{12}^{[k,1]} \) with special attention to the fact that

\[
|s - r\rangle_+|s + r\rangle_+ - |M + s - r\rangle_+|M + s + r\rangle_+ = -(|s + r\rangle_+|s - r\rangle_+ + |M + s + r\rangle_+|M + s - r\rangle_+),
\]

where \( r = 0, 1, \ldots, M - 1 \) and \( r' = M - r \).

V. STATES OF TRIONS

Trions \( X^\pm \) were introduced in 1958 by Lampert [2] as excitons consisting of two holes and an electron \((X^+)\) or two electrons and a hole \((X^-)\). This concept can be generalized to any system of two particles and one antiparticle.

A. Direct conjugation

In this case the Kronecker product of three representations presented in Eq. (4) should be considered. It can be verified that the symmetry adapted vectors can be constructed as

\[
|w\rangle_{pq}^\pm = |w + p\rangle_+|w + q\rangle_+|w + p + q\rangle_-,
\]

where a pair \((p, q)\), \( p, q = 0, 1, 2, \ldots, N - 1 \), plays a role of a repetition index and the subscript \( \pm \) denotes vectors of the irreps \( D_{12}^{\pm} \), respectively.

For \( p = q \) the states obtained are symmetric with respect to the transposition of identical particles. In the other cases \((p \neq q)\) symmetric and antisymmetric combinations can be formed:

\[
|w\rangle_{pq}^{\pm} = 2^{-1/2} (|w\rangle_{pq}^+ \pm |w\rangle_{pq}^-),
\]

where now \( q > p = 0, 1, \ldots, N - 1 \). One obtains \( N(N - 1)/2 \) antisymmetric states \( |w\rangle_{pq}^- \)
and \( N(N + 1)/2 \) symmetric ones \( |w\rangle_{pq}^+ \).

B. Conjugation via the neutral pair

In this case we start from the particle-antiparticle pair considered in Sec. IV A and the product of three representations is written as

\[
(D^j \otimes D^{-j}) \otimes D^j = \bigoplus_k D^k \otimes D^j.
\]

Since for each \( k \) one has \( D^k \otimes D^j = D^j \), then each such product yields states

\[
|w\rangle_{k}^j = \omega_N^{-wk_2}|0\rangle_k |w - xk_1\rangle_+.
\]

Equations obtained lead to the final expression \((xl = 1 \mod N, N^{-1/2} \text{ is a normalization factor})\)

\[
|w\rangle_{k}^j = N^{-1/2} \omega_N^{-wk_2} \sum_{s=0}^{N-1} \omega_N^{sk_2} |s\rangle_+ |s - xk_1\rangle_- |w - xk_1\rangle_+.
\]
In such a state there is a kind of symmetry between a particle and antiparticle, but there is no symmetry between two identical particles in the trion \( X^+ \). Since there are \( N^2 \) trion states labeled by \( w \) then it is possible to construct states symmetric and antisymmetric with respect to transposition of particles.

Scalar products with the vectors obtained previously are

\[
|k, l\rangle_\mathrm{trion}^\pm = N^{-1/2} \omega_N^{-p_{k,l}} \delta_{q+xk, l}\,.
\]

In the simplest case \( N = 2 \) this formula yields (the unique representation is obtained for \( l = x = 1 \));

\[
|w\rangle_0^0 = 2^{-1/2} \left( |w\rangle_0^0 + |w\rangle_0^1 \right), \quad |w\rangle_0^1 = 2^{-1/2} \left( |w\rangle_0^0 - |w\rangle_0^1 \right),
\]

\[
|w\rangle_1^0 = 2^{-1/2} \left( |w\rangle_1^0 - |w\rangle_1^1 \right), \quad |w\rangle_1^1 = 2^{-1/2} \left( |w\rangle_1^0 + |w\rangle_1^1 \right).
\]

Symmetrization of the formulas in the right column leads to the following expressions

\[
|w\rangle_0^0^+ = \left( |w\rangle_0^0^0 - |w\rangle_0^1 \right) / 2,
\]

\[
|w\rangle_0^0^- = \left( |w\rangle_0^0^0 - |w\rangle_0^1 \right) / 2.
\]

C. Conjugation via the pair of identical particles

In this case we use the results of Sec. IVB, so we start from products

\[
D^{2l} \otimes D^{-l} = ND^l, \quad D^{l,k_1,k_2} \otimes D^{-l} = MD^l.
\]

Note that in both cases we obtain copies of the representation \( D^l \).

The first case corresponds to \( N \) odd and again is very simple. One of possible choice of basis vectors is

\[
|w\rangle_0^0 = |w + v\rangle_2 |w + 2v\rangle_2, \quad w = 0, 1, 2, \ldots, N - 1,
\]

where \( v = 0, 1, 2, \ldots, N - 1 \) is the repetition index. Taking into account the previous results

\[
|w\rangle_0^0^+ = |w + v\rangle_2 |w + 2v\rangle_2 = |w + v + r\rangle_+ |w + v - r\rangle_+ |w + 2v\rangle_2.
\]

The states

\[
|w\rangle_0^pq = |w + p\rangle_+ |w + q\rangle_+ |w + p + q\rangle_-
\]

obtained previously correspond to \( p = v + r \) and \( q = v - r \) (calculated mod \( N \)). Note that for \( N = 2M \) these relations can not be inverted since \( v = (p + q)/2 \) and \( r = (p - q)/2 \) have no solutions mod \( N \) for odd \( p + q \) and \( p - q \), respectively.

In the case \( N = 2M \) considerations are a bit more difficult, but since one knows the final results, states \( |w\rangle_0^pq \), then they may serve as a useful hint. There are \( N^2 = 4M^2 \) different bases labeled by \( p, q = 0, 1, 2, \ldots, N - 1 \), however if \( p' = p + M \) and \( q' = q + M \) then states \( |w\rangle_0^pq \) and \( |w\rangle_0^{p'q'} \) have the same third element in the tensor products because

\[
|w\rangle_0^{p'q'} = |w + p + M\rangle_+ |w + q + M\rangle_+ |w + p + q\rangle_-
\]

Therefore, they can be gathered into \( 2M^2 = NM \) pairs

\[
|w\rangle_0^pq \quad \text{ and } \quad |w\rangle_0^{(p+M)(q+M)}.
\]

The ranges of indices have to be chosen in such a way that pairs \( pq \) and \( (p + M)(q + M) \) run over two separate sets. For each pair \( p, q \) one can form two new bases, labeled by ‘+’ and ‘−’, respectively,

\[
|w\rangle_0^{pq\pm} = |w\rangle_0^pq \pm |w\rangle_0^{(p+M)(q+M)}, \quad (8)
\]

so

\[
|w\rangle_0^{pq\pm} = |w + p + q\rangle_-(|w + p\rangle_+ |w + q\rangle_+ \pm |w + p + M\rangle_+ |w + q + M\rangle_+).
\]

These vectors have the second part (in parentheses) in the form resembling Eq. (6). Hence, one has to relate the repetition index \( r \) and a label \( u \) of a vector \( |u\rangle \) of the representation \( D^{-l} \) with the repetition indices \( p, q \) and \( \pm \) in Eq. (8). The solution of this problem leads to a proper choice of pairs \( pq \). This has been discussed in more details elsewhere [10].
VI. OTHER GAUGES

A trivial factor system \( \theta \) is determined by any mapping \( \phi: G \to U(1) \) as

\[
\theta(R, R') = \phi(R)\phi(R') / \phi(R + R').
\]

Let \( D \) be a projective representation with a factor system \( \mu \). A new representation \( D' \) determined as \( D'(R) = \phi(R)D(R) \) has an equivalent factor system \( \mu' = \theta \mu \). On the other hand, for any unitary operator \( S \) and \( D''(R) = SD(R)S^{-1} \) one obtains

\[
D''(R)D''(R') = \mu(R, R')D''(R + R').
\]

Therefore, equivalent representations have identical factor systems \([17]\). It means that equivalent factor systems, corresponding to different gauges \([10]\), lead to nonequivalent representations. However, it will be shown below, that these factors (so gauges, too) are related to a local transformation of state space.

The most popular gauges, i.e. the Landau and the symmetric ones, are the special cases of the so-called linear gauge \([13]\), which has the form

\[
A(r) = H[-\beta y, (1 - \beta)x].
\]

The relation between the vector potential \( A \) and the factor system of a projective representation \([10]\) yields that \( \beta = 0 \) corresponds to the representation considered above, whereas \( \beta = 1 \) and \( \beta = 1/2 \), determining the other form of the Landau gauge and the (anti)symmetric one \( A = (H \times r)/2 \) \([13, 18]\). Thus relation leads to matrices

\[
1D^l_{jk}[n_1, n_2] = \delta_{j-k-n_2}\omega_N^{ln_1k},
\]

\[
1/2D^l_{jk}[n_1, n_2] = \delta_{j-k-n_2}\omega^{jn_1(j+k)/2}.
\]

In a general case one obtains

\[
\beta D^l_{jk}[n_1, n_2] = \delta_{k-j,n_2}\omega_N^{ln_1(1-\beta)j+\beta k} = \delta_{k-j,n_2}\omega_N^{jn_1(j+\beta n_2)} = \omega_N^{jn_1n_2\beta}D^l_{jk}[n_1, n_2].
\]

The action of operators \( \beta D^l \) for vectors \([n_1, 0]\) and \([0, n_2]\) on the basis vectors yields

\[
\beta D^l[n_1, 0]|j\rangle = \omega_N^{ln_1j}|j\rangle, \quad \beta D^l[0, n_2]|j\rangle = |j - n_2\rangle.
\]

Therefore, they behave in the same way for all real numbers \( \beta \). The differences can be only noticed for general translations \([n_1, n_2]\) and, moreover, the non-zero matrix elements obtained for different \( \beta \) appear in the same places (it is controlled by the \( \delta_{k-j,n_2} \), but are multiplied by different powers of \( \omega_N^j \). Let us recall that projective representations are sometimes, especially in physics \([6]\), called ray representations. Since only a module of a bracket \(|j\rangle|k\rangle\) has a physical meaning, then vectors (complex functions) \(|j\rangle\) and \(|k\rangle\) are determined up to factors \( \lambda \in U(1) \) \([11]\). Therefore, a chosen state \(|j\rangle\) represents in fact a ray, i.e. the set \(|\langle j\rangle\rangle = \{\lambda|j\rangle | \lambda \in U(1)\}\). In a general case

\[
\beta D^l_{jk}[n_1, n_2] = \langle j|\beta D^l[n_1, n_2]|k\rangle = \delta_{k-j,n_2}\omega_N^{ln_1j\beta}\omega_N^{ln_1k\beta}.
\]

Therefore, replacing each vector \(|j\rangle\) by an element of the same ray \(|j'\rangle = \omega_N^{ln_1j'\beta} \) one obtains

\[
\langle j'|\beta D^l[n_1, n_2]|k'\rangle = \omega_N^{ln_1(j-k)\beta}\langle j'|\beta D^l[n_1, n_2]|k\rangle = \delta_{k-j,n_2}\omega_N^{ln_1j}\omega_N^{ln_1k} = \delta_{j-k,n_2}\omega_N^{ln_1j} = 0D^l_{jk}[n_1, n_2].
\]

However, this transformation is local, because it depends on \( n_1 \) and \( n_2 \), since \( k - j = n_2 \) for non-zero matrix elements. If can be verified that equations obtained above are invariant under this transformation, so all relations derived here for the Landau gauge are valid for other linear gauges.

VII. FINAL REMARKS

In this work the Kronecker products of irreducible projective representations of the two-dimensional translation group are applied to to the group-theoretical classification of multiparticle states in systems subjected to the external magnetic field. Such states may be useful in the considerations of quantum wells, high-\( T_c \) superconductors and the
fractional quantum Hall effect. In this last case, there is special interest in trions, which are a special case of considered here the system consisting of two particles and one antiparticle.

Due to internal structure of trion the degeneracy is higher and there are many possibilities to construct states $|w\rangle_t$; some of them have been discussed above. In these simplified considerations there are no interactions between trions or mixing of Landau levels and the spin or angular momentum numbers have not been considered. Taking into account spins will allow to construct states completely antisymmetric with respect to the permutational symmetry. Such problem has been discussed lately by Dzyubenko et al. [5] for the case of free trions (i.e. without a periodic potential, so there is no discrete translational symmetry). A sum of indices in the RHS of Eq. (7), taking into account signs of charges, is $(2w + p + q) - (w + p + q) = w$, what is equal to the index in the LHS of this equation. This is the same result as presented in [5], where the total angular momentum projection for holes ($j = 1, 2$) and an electron ($j = 3$); $n$ and $m$ are the Landau level and the oscillator quantum numbers, respectively. It is interesting that Dzyubenko et al. obtained their results in the antisymmetric gauge $\mathbf{A} = (\mathbf{H} \times \mathbf{r})/2$, whereas in the presented considerations the Landau gauge has been used. It confirms that the physical properties are gauge-independent. On the other hand, the actual form of wave functions is not discussed here, but the relations between representations and their product are taken into account only. These relations are independent of the matrix representations and, similarly, the form of resultant basis is independent of the function form: for a given Born–von Kármán period $N$ and any linear gauge irreducible projective representations are $N$-dimensional and their action on basis vectors are similar (up to a factor system) [6, 7, 10, 19]. This is proven by taking into account the notion of rays and ray representations.

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