A new result on backward uniqueness for parabolic operators

DANIELE DEL SANTO and MARTINO PRIZZI
Dipartimento di Matematica e Informatica, Università di Trieste
Via A. Valerio 12/1, 34127 Trieste, Italy

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Abstract

Using Bony’s paramultiplication we improve a result obtained in [7] for operators having coefficients non-Lipschitz-continuous with respect to $t$ but $C^2$ with respect to $x$, showing that the same result is valid when $C^2$ regularity is replaced by Lipschitz regularity in $x$.

1 Introduction

In this note we consider the following backward parabolic operator

$$L = \partial_t + \sum_{i,j=1}^{n} \partial_{x_j} (a_{jk}(t,x) \partial_{x_k}) + \sum_{j=1}^{n} b_j(t,x) \partial_{x_j} + c(t,x).$$ (1.1)

We assume that all coefficients are defined in $[0,T] \times \mathbb{R}^n_x$, measurable and bounded; $(a_{jk}(t,x))_{jk}$ is a real symmetric matrix for all $(t,x) \in [0,T] \times \mathbb{R}^n_x$ and there exists $\lambda_0 \in (0,1]$ such that

$$\sum_{j,k=1}^{n} a_{jk}(t,x)\xi_j \xi_k \geq \lambda_0 |\xi|^2$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n_x$ and $\xi \in \mathbb{R}^n_x$.

Given a functional space $\mathcal{H}$ (in which it makes sense to look for the solutions of the equation $Lu = 0$) we say that the operator $L$ has the $\mathcal{H}$–uniqueness property if, whenever $u \in \mathcal{H}$, $Lu = 0$ in $[0,T] \times \mathbb{R}^n_x$ and $u(0,x) = 0$ in $\mathbb{R}^n_x$, then $u = 0$ in $[0,T] \times \mathbb{R}^n_x$.
We choose \( \mathcal{H} \) to be the space of functions
\[
\mathcal{H} = H^1((0, T), L^2(\mathbb{R}^n_x)) \cap L^2((0, T), H^2(\mathbb{R}^n_x)).
\]
(1.2)
This choice is natural, since it follows from elliptic regularity results (see e.g.

Theorem 8.8 in [10]) that the domain of the operator \(- \sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(t, x) \partial_{x_k})\)
in \(L^2(\mathbb{R}^n)\) is \(H^2(\mathbb{R}^n)\) for all \(t \in [0, T]\).

The problem we are interested in is the following: find the minimal regularity on the coefficients \(a_{jk}\) ensuring the \(\mathcal{H}\)-uniqueness property to \(L\).

A classical result of Lions and Malgrange [11] (see for related or more
general results [14], [1], [9]) shows that a sufficient condition for backward
uniqueness is given by the assumption that the map \(t \mapsto a_{jk}(t, \cdot)\) be Lipschitz
continuous from \([0, T]\) to \(L^\infty(\mathbb{R}^n)\).

On the other hand the well known example of Miller [14] (where an op-
erator, having coefficients which are Hölder–continuous of order 1/6 with
respect to \(t\) and \(C^\infty\) with respect to \(x\), does not have the uniqueness property) shows that a certain amount of regularity on the \(a_{jk}\)'s with respect to \(t\) is necessary for the \(\mathcal{H}\)-uniqueness.

In our previous paper [7], we proved the \(\mathcal{H}\)-uniqueness property for the
operator (1.1) when the coefficients \(a_{jk}\) are \(C^2\) in the \(x\) variables and non–
Lipschitz–continuous in \(t\). The regularity in \(t\) was given in terms of a modulus
of continuity \(\mu\) satisfying the so called Osgood condition
\[
\int_0^1 \frac{1}{\mu(s)} \, ds = +\infty.
\]

This uniqueness result was a consequence of a Carleman estimate in
which the weight function depended on the modulus of continuity; such kind
of weight functions in Carleman estimates were introduced by Tarama [15]
in the case of second order elliptic operators. In obtaining our Carleman esti-
mate, the integrations by parts, which couldn’t be used since the coefficients
were not Lipschitz–continuous, was replaced by a microlocal approximation
procedure.

In [7] a technical difficulty in the estimate of a commutator led to im-
posing on \(a\) the the \(C^\mu\) regularity with respect to \(t\), together with the \(C^2\)
regularity with respect to \(x\). In [6] this statement was improved, as it was
shown that under the Osgood condition for \(\mu\), the \(C^\mu\) regularity with respect
to \(t\), together with the Hölder \(C^{1,\varepsilon}\) regularity with respect to \(x\), is sufficient
for the same uniqueness result. The proof followed the same pattern as the
one in [7], the only difference being in the introduction of a paradifferential
operator (actually a simple paramultiplication) in place of the second order
part of the operator $L$. In the present paper, we further improve the result of [6], showing that $C^{1,\varepsilon}$ regularity can be replaced by Lipschitz regularity in $x$. In order to achieve our result, we introduce a modified paramultiplication. We obtain a Carleman estimate in a space $H^{-s}$, with $0 < s < 1$, instead of the classical estimate in $L^2$. However, with such estimate we can repeat the arguments of [7] and regain the desired uniqueness property. The estimate of the commutator, in [6] and in the present case, is made more effective by a theorem due to Coifman and Meyer [3, Th. 35] (see also, for a similar use of that theorem, [5, Prop. 3.7]).

2 Definitions and result

**Definition 2.1.** A function $\mu$ is said to be a *modulus of continuity* if $\mu$ is continuous, concave and strictly increasing on $[0,1]$, with $\mu(0) = 0$ and $\mu(1) = 1$. Let $I \subseteq \mathbb{R}$ and let $\varphi : I \to \mathcal{B}$, where $\mathcal{B}$ is a Banach space. We say that $\varphi \in C^\mu(I, \mathcal{B})$ if $\varphi \in L^\infty(I, \mathcal{B})$ and

$$
\sup_{0 < \gamma < 1; t, s \in I} \frac{\|\varphi(t) - \varphi(s)\|_\mathcal{B}}{\mu(|t - s|)} < +\infty.
$$

It is immediate to verify the following properties

- $\mu(s) \geq s$ for all $s \in [0,1]$;
- the function $s \mapsto \mu(s)/s$ is decreasing on $[0,1]$;
- there exists $\lim_{s \to 0^+} \mu(s)/s$;
- the function $\sigma \mapsto \sigma \mu(1/\sigma)$ is increasing on $[1, +\infty[$;
- the function $\sigma \mapsto 1/(\sigma^2 \mu(1/\sigma))$ is decreasing on $[1, +\infty[$.

**Definition 2.2.** A modulus of continuity is said to satisfy the *Osgood condition* if

$$
\int_0^1 \frac{1}{\mu(s)} ds = +\infty. \quad (2.3)
$$

**Theorem 2.3.** Let $L$ be the operator

$$
L = \partial_t + \sum_{j,k=1}^n \partial_{x_j} (a_{jk}(t,x) \partial_{x_k}) + \sum_{j=1}^n b_j(t,x) \partial_{x_j} + c(t,x), \quad (2.4)
$$

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where all the coefficients are supposed to be defined in $[0, T] \times \mathbb{R}^n_x$, measurable and bounded; let the coefficients $b_j$ and $c$ be complex valued; let $(a_{jk}(t, x))_{jk}$ be a real symmetric matrix for all $(t, x) \in [0, T] \times \mathbb{R}^n_x$ and suppose that there exists $\lambda_0 \in (0, 1)$ such that
\[
\sum_{j,k=1}^{n} a_{jk}(t, x)\xi_j\xi_k \geq \lambda_0 |\xi|^2,
\]
for all $(t, x) \in [0, T] \times \mathbb{R}^n_x$ and $\xi \in \mathbb{R}_x^n$. Let $H$ be the space of functions
\[
\mathcal{H} = H^1((0, T), L^2(\mathbb{R}_x^n)) \cap L^2((0, T), H^2(\mathbb{R}_x^n)).
\]
Let $\mu$ be a modulus of continuity satisfying the Osgood condition. Suppose that
\[
a_{jk} \in C^\mu([0, T], L^\infty(\mathbb{R}_x^n)) \cap C([0, T], Lip(\mathbb{R}_x^n)),
\]
for all $j, k = 1, \ldots, n$.

Then $L$ has the $\mathcal{H}$-uniqueness property, i.e. if $u \in \mathcal{H}$, $Lu = 0$ in $[0, T] \times \mathbb{R}^n_x$ and $u(0, x) = 0$ in $\mathbb{R}^n_x$, then $u = 0$ in $[0, T] \times \mathbb{R}^n_x$.

**Remark.** The choice of the space $\mathcal{H}$ is natural, since it follows from elliptic regularity results (see e.g. Theorem 8.8 in [10]) that the domain of the operator $-\sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(t, x)\partial_{x_k})$ in $L^2(\mathbb{R}^n)$ is $H^2(\mathbb{R}^n)$ for all $t \in [0, T]$.

## 3 Proof

### 3.1 Modulus of continuity and Carleman estimate

Theorem [2.3] will follow from a Carleman estimate in Sobolev spaces with negative index. The weight function in the Carleman estimate will be obtained from the modulus of continuity. The crucial idea of linking the weight function to the regularity of the coefficients goes back to the paper [15] in which a uniqueness result for elliptic operators with non-Lipschitz-continuous coefficients was proved.

We define
\[
\phi(t) = \int_0^t \frac{1}{\mu(s)} \ ds.
\]

The function $\phi$ is a strictly increasing $C^1$ function. From [2.3] we have $\phi([1, +\infty]) = [0, +\infty]$; moreover $\phi'(t) = 1/(t^2\mu(1/t)) > 0$ for all $t \in [1, +\infty]$. We set
\[
\Phi(\tau) = \int_0^\tau \phi^{-1}(s) \ ds.
\]
We obtain $\Phi'(\tau) = \phi^{-1}(\tau)$ and consequently $\lim_{\tau \to +\infty} \Phi'(\tau) = +\infty$. Moreover
\[
\Phi''(\tau) = \left(\Phi'(\tau)\right)^2 \mu\left(\frac{1}{\Phi'(\tau)}\right)
\]
for all $\tau \in [0, +\infty[$ and, as the function $\sigma \mapsto \sigma \mu(1/\sigma)$ is increasing on $[1, +\infty[$, we deduce that
\[
\lim_{\tau \to +\infty} \Phi''(\tau) = \lim_{\tau \to +\infty} \left(\Phi'(\tau)\right)^2 \mu\left(\frac{1}{\Phi'(\tau)}\right) = +\infty.
\]

Now we state the Carleman estimate.

**Proposition 3.1.** For all $s \in (0, 1)$, there exist $\gamma_0, C > 0$ such that
\[
\int_0^T e^{2\gamma \Phi(\gamma(T-t))} \left\| \partial_t u + \sum_{j,k=1}^n \partial_{x_j} (a_{jk}(t,x) \partial_{x_k} u) \right\|^2_{H^{-s}} dt \\
\geq C \gamma^2 \int_0^T e^{2\gamma \Phi(\gamma(T-t))} \left(\|\nabla_x u\|^2_{H^{-s}} + \gamma^2 \|u\|^2_{H^{-s}}\right) dt,
\]

for all $\gamma > \gamma_0$ and for all $u \in C_0^\infty(\mathbb{R}^{n+1})$ such that $\text{supp } u \subseteq [0, T/2] \times \mathbb{R}^n_x$ (the symbol $\nabla_x f$ denotes the gradient of $f$ with respect to the $x$ variables).

The way of obtaining the $H$-uniqueness from the inequality (3.10) is a standard procedure, the details of which, in the case of a Carleman estimate in $L^2$, can be found in [7, Par. 3.4].

### 3.2 Paraproducts
#### 3.2.1 Littlewood-Paley decomposition
We review some known results on Littlewood-Paley decomposition and related topics. More can be found in [2], [13, Ch. 4 and Ch. 5] and [5, Par. 3].

Let $\chi \in C_0^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$, even and such that $\chi(s) = 1$ for $|s| \leq 11/10$ and $\chi(s) = 0$ for $|s| \geq 19/10$. For $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$, let us consider $\chi_k(\xi) = \chi(2^{-k}|\xi|)$, let’s denote $\tilde{\chi}_k(x)$ its inverse Fourier transform and let’s define the operators
\[
S_{-1}u = 0, \quad S_k u = \tilde{\chi}_k * u = \chi_k(D_x)u,
\]
\[
\Delta_0 u = S_0 u, \quad \text{and, for } k \geq 1, \quad \Delta_k u = S_k u - S_{k-1} u.
\]

In the following propositions we recall the characterization of Sobolev spaces and Lipschitz-continuous functions via Littlewood-Paley decomposition (see [13 Prop. 4.1.11], [5 Prop. 3.1 and Prop. 3.2] and [8 Lemma 3.2].
Proposition 3.2. Let $s \in \mathbb{R}$. A temperate distribution $u$ is in $H^s$ if and only if the following two conditions hold

i) for all $k \geq 0$, $\Delta_k u \in L^2$;

ii) the sequence $(\delta_k)_k$, with $\delta_k = 2^k \|\Delta_k u\|_{L^2}$, is in $l^2$.

Moreover there exists $C_s \geq 1$ such that, for all $u \in H^s$,

$$\frac{1}{C_s}\|u\|_{H^s} \leq \left( \sum_{k=0}^{+\infty} \delta_k^2 \right)^{1/2} \leq C_s \|u\|_{H^s}.$$ 

Proposition 3.3. Let $s \in \mathbb{R}$ and $R > 0$. Let $(u_k)_k$ a sequence of functions in $L^2$ such that

i) the support of the Fourier transform of $u_0$ is contained in $\{ |\xi| \leq R \}$ and the support of Fourier transform of $u_k$ is contained in $\{ \frac{1}{R} 2^k \leq |\xi| \leq R 2^k \}$, for all $k \geq 1$;

ii) the sequence $(\delta_k)_k$, with $\delta_k = 2^k \|u_k\|_{L^2}$, is in $l^2$.

Then the series $\sum_k u_k$ is converging, with sum $u$, in $H^s$ and the norm of $u$ in $H^s$ is equivalent to the norm of $(\delta_k)_k$ in $l^2$.

When $s > 0$ it is sufficient to assume the Fourier transform of $u_k$ to be contained in $\{ |\xi| \leq R 2^k \}$, for all $k \geq 1$.

Proposition 3.4. A bounded function $a$ is in Lip, the space of bounded Lipschitz continuous functions defined on $\mathbb{R}_+^N$, if and only if

$$\sup_{k \in \mathbb{N}} \|\nabla_x (S_k a)\|_{L^\infty} < +\infty.$$ 

Moreover there exists $C > 0$ such that if $a \in \text{Lip}$, then

$$\|\Delta_k a\|_{L^\infty} \leq C \|a\|_{\text{Lip}} 2^{-k} \quad \text{and} \quad \|\nabla_x (S_k a)\|_{L^\infty} \leq C \|a\|_{\text{Lip}}$$

(where $\|f\|_{\text{Lip}} = \|f\|_{L^\infty} + \|\nabla_x f\|_{L^\infty}$).
3.2.2 Bony’s modified paraproduct

Let \( a \in L^\infty \). The Bony’s paraproduct of \( a \) and \( u \in H^s \) (see [2, Par. 2]) is defined as
\[
T_a u = \sum_{k=3}^{+\infty} S_{k-3} a \Delta_k u.
\]

We modify the definition of paraproduct introducing the following operator
\[
T^m_a u = S_{m-1} a S_{m+1} u + \sum_{k=m+2}^{+\infty} S_{k-3} a \Delta_k u.
\]

where \( m \in \mathbb{N} \) (remark that \( T_a = T^0_a \)). Useful properties of the (modified) paraproduct are contained in the following propositions (see also [13, Prop. 5.2.1], [5, Prop. 3.4]).

**Proposition 3.5.** Let \( m \in \mathbb{N}, s \in \mathbb{R} \) and \( a \in L^\infty \).
Then \( T^m_a \) maps \( H^s \) into \( H^s \) and
\[
\|T^m_a u\|_{H^s} \leq C_{m,s} \|a\|_{L^\infty} \|u\|_{H^s}.
\] (3.11)

Let \( m \in \mathbb{N}, s \in (0,1) \) and \( a \in \text{Lip} \).
Then \( u \mapsto au - T^m_a u \) maps \( H^{-s} \) into \( H^{1-s} \) and
\[
\|au - T^m_a u\|_{H^{1-s}} \leq C_{m,s} \|a\|_{\text{Lip}} \|u\|_{H^{-s}}.
\] (3.12)

**Proof.** We prove only the second part of the statement. We have
\[
au - T^m_a u = \sum_{k=\max\{3,m\}}^{+\infty} \Delta_k a S_{k-3} u + \sum_{k=m}^{+\infty} \sum_{j \geq 0} \Delta_k a \Delta_j u.
\]

We remark that the support of the Fourier transform of \( \Delta_k a S_{k-3} u \) is contained in \( \{2^{k-2} \leq |\xi| \leq 2^{k+2}\} \). Moreover, by Proposition 3.4 we have that
\[
\|\Delta_k a S_{k-3} u\|_{L^2} \leq \|\Delta_k a\|_{L^\infty} \|S_{k-3} u\|_{L^2} \leq C\|a\|_{\text{Lip}} 2^{-k} \sum_{j=0}^{k-3} 2^{js} \delta_j
\]

where \( \delta_j = 2^{-js} \|\Delta_j u\|_{L^2} \). From Proposition 3.2 we have that \( (\delta_j)_j \in l^2 \) and its \( l^2 \) norm is equivalent to the \( H^{-s} \) norm of \( u \). On the other hand, setting
\[
\tilde{\delta}_k = \sum_{j=0}^{k} 2^{(j-k)s} \delta_j,
\]
we have that \((\tilde{\delta}_k)_k \in l^2\) and \(\| (\tilde{\delta}_k)_k \|_{l^2} \leq C_s \| (\delta_k)_k \|_{l^2}\). Consequently
\[
\| \Delta_k a S_k - 3 u \|_{L^2} \leq C_s \| a \|_{Lip} 2^{-k(1-s)} \tilde{\delta}_k,
\]
and then, by Proposition 3.3 we have that \(\sum_{k=\max\{3,m\}}^{+\infty} \| \Delta_k a S_k - 3 u \|_{H^{1-s}} \leq C_{m,s} \| a \|_{Lip} \| u \|_{H^{1-s}}\). Consequently

Next, we see that, for \(k \geq 2\),
\[
\sum_{k=m}^{+\infty} ( \sum_{j \geq 0} \Delta_k a \Delta_j u ) = \sum_{k=m}^{+\infty} \Delta_k a \Delta_{k-2} u + \cdots + \sum_{k=m}^{+\infty} \Delta_k a \Delta_{k+2} u,
\]
with a slight modification in the case \(k = 0, 1\). We have that the support of the Fourier transform of \(\Delta_k a \Delta_{k-2} u\) is contained in \(\{ |\xi| \leq 2^{k+2} \}\) and similarly for the other four terms, e.g., the support of the Fourier transform of \(\Delta_k a \Delta_{k+2} u\) is contained in \(\{ |\xi| \leq 2^{k+4} \}\). Moreover
\[
\| \Delta_k a \Delta_{k-2} u \|_{L^2} \leq \| \Delta_k a \|_{L^\infty} \| \Delta_{k-2} u \|_{L^2} \leq C_s \| a \|_{Lip} 2^{-k(1-s)} \tilde{\delta}_{k-2}.
\]
Again from Proposition 3.3 we have that \(\sum_{k=m}^{+\infty} \| \Delta_k a \Delta_{k-2} u \|_{H^{1-s}} \leq C_{m,s} \| a \|_{Lip} \| u \|_{H^{-s}}\).

Arguing similarly for the other terms we have that \(\sum_{k=m}^{+\infty} ( \sum_{j \geq 0} \Delta_k a \Delta_j u ) \in H^{1-s}\) and

\[
\| \sum_{k=m}^{+\infty} \Delta_{k-2} a \Delta_k u \|_{H^{1-s}} \leq C_{m,s} \| a \|_{Lip} \| u \|_{H^{-s}}\.
\]

The conclusion of the proof of the proposition is reached putting together (3.13) and (3.14).

As pointed out in [5, Par. 3], the positivity of the function \(a\) does not imply, for all \(m \geq 0\), the positivity of \(T^m_a\). Nevertheless the following proposition holds (see [5, Cor. 3.12]).
Proposition 3.6. Let $a \in \text{Lip}$ and suppose that $a(x) \geq \lambda_0 > 0$ for all $x \in \mathbb{R}^n$. Then there exists $m$ depending on $\lambda_0$ and $\|a\|_{\text{Lip}}$ such that

$$\text{Re}(T_m^a u, u)_{L^2} \geq \frac{\lambda_0}{2} \|u\|_{L^2}, \quad (3.15)$$

for all $u \in L^2$ (here $(\cdot, \cdot)_{L^2}$ denotes the scalar product in $L^2$). A similar result is valid for vector valued functions when $a$ is replaced by a positive symmetric matrix $(a_{jk})_{j,k}$.

We state now a property of commutation which will be crucial in the proof of the Carleman estimate (see [5, Prop.3.7]).

Proposition 3.7. Let $m \in \mathbb{N}$, $a \in \text{Lip}$, $s \in (0, 1)$ and $u \in H^{1-s}$.

Then

$$\left( \sum_{\nu=0}^{+\infty} 2^{-2\nu s} \| \partial_{x_j}(\lfloor \Delta_\nu, T_m^a \rfloor \partial_{x_h} u) \|_{L^2}^2 \right)^{1/2} \leq C_{m,s} \|a\|_{\text{Lip}} \|u\|_{H^{1-s}} \quad (3.16)$$

(where $[A, B]$ denotes the commutator between the operators $A$ and $B$, i.e. $[A, B]w = A(Bw) - B(Aw)$).

Proof. We start remarking that

$$[\Delta_\nu, T_m^a]w = [\Delta_\nu, S_{m-1}a]S_{m+1}w + \sum_{k=m+2}^{+\infty} [\Delta_\nu, S_{k-3}a] \Delta_k w,$$

and consequently

$$\partial_{x_j}(\lfloor \Delta_\nu, T_m^a \rfloor \partial_{x_h} u) = \partial_{x_j}(\lfloor [\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h} u) \rfloor)$$

$$+ \partial_{x_j}(\sum_{k=m+2}^{+\infty} [\Delta_\nu, S_{k-3}a] \Delta_k (\partial_{x_h} u)).$$

In fact $\Delta_\nu$ and $\Delta_k$ commute so that

$$\Delta_\nu(S_{m-1}aS_{m+1}w) - S_{m-1}aS_{m+1}(\Delta_\nu w)$$

$$= \Delta_\nu(S_{m-1}aS_{m+1}w) - S_{m-1}a\Delta_\nu(S_{m+1}w)$$

and similarly for the other term.

Let’s consider

$$\partial_{x_j}(\lfloor [\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h} u) \rfloor) \quad \text{and} \quad \partial_{x_j}(\lfloor [\Delta_\nu, S_{m-1}a] \partial_{x_h} (S_{m+1}u) \rfloor).$$
Looking at the support of the Fourier transform, it is easy to see that this term is identically equal to 0 if \( \nu \geq m + 4 \). Moreover the support of the Fourier transform is contained in \( \{ |\xi| \leq 2^{m+3} \} \). From Bernstein’s inequality we have

\[
\| \partial x_j ([\Delta_\nu, S_{m-1}a] \partial x_h (S_{m+1}u)) \|_{L^2} \leq 2^{m+3}\| [\Delta_\nu, S_{m-1}a] \partial x_h (S_{m+1}u) \|_{L^2}.
\]

On the other hand, using the result of \([3, \text{Th. 35}]\) (see also \([16, \text{Par. 3.6}]\)) we deduce that

\[
\| [\Delta_\nu, S_{m-1}a] \partial x_h (S_{m+1}u) \|_{L^2} \leq C ||a||_{Lip} \| S_{m+1}u \|_{L^2} \leq C ||a||_{Lip} \| u \|_{L^2}.
\]

Consequently

\[
\| \partial x_j ([\Delta_\nu, S_{m-1}a] S_{m+1}(\partial x_h u)) \|_{L^2} \leq C 2^{m+3} ||a||_{Lip} \| u \|_{L^2},
\]

and, since \( s \in (0, 1) \),

\[
\sum_{\nu=0}^{+\infty} 2^{-2\nu s} \| \partial x_j ([\Delta_\nu, S_{m-1}a] S_{m+1}(\partial x_h u)) \|_{L^2}^2
\]

\[
= \sum_{\nu=0}^{+\infty} 2^{-2\nu s} \| \partial x_j ([\Delta_\nu, S_{m-1}a] S_{m+1}(\partial x_h u)) \|_{L^2}^2 \leq C_{m,s} \| a \|_{Lip}^2 \| u \|_{H^{1-s}}^2.
\]

(3.17)

Let’s consider

\[
\partial x_j \left( \sum_{k=m+2}^{+\infty} [\Delta_\nu, S_{k-3}a] \Delta_k (\partial x_h u) \right) = \partial x_j \left( \sum_{k=m+2}^{+\infty} [\Delta_\nu, S_{k-3}a] \partial x_h (\Delta_k u) \right).
\]

Again looking at the support of the Fourier transform, it is possible to see that \( [\Delta_\nu, S_{k-3}a] \partial x_h (\Delta_k u) \) is identically 0 if \( |k - \nu| \geq 4 \). Consequently the sum is on at most 7 terms: \( \partial x_j ([\Delta_\nu, S_{\nu-6}a] \partial x_h (\Delta_{\nu-3} u)) + \cdots + \partial x_j ([\Delta_\nu, S_{\nu}a] \partial x_h (\Delta_{\nu+3} u)) \), each of them having the support of the Fourier transform contained in \( \{ |\xi| \leq C 2^\nu \} \). Let’s consider one of these terms, e.g. \( \partial x_j ([\Delta_\nu, S_{\nu-3}a] \partial x_h (\Delta_{\nu} u)) \), the computation for the others being similar. We have, from Bernstein’s inequality

\[
\| \partial x_j ([\Delta_\nu, S_{\nu-3}a] \partial x_h (\Delta_{\nu} u)) \|_{L^2} \leq C 2^\nu \| [\Delta_\nu, S_{\nu-3}a] \partial x_h (\Delta_{\nu} u) \|_{L^2},
\]

and consequently, using again \([3, \text{Th. 35}]\),

\[
\| \partial x_j ([\Delta_\nu, S_{\nu-3}a] \partial x_h (\Delta_{\nu} u)) \|_{L^2} \leq C 2^\nu ||a||_{Lip} \| \Delta_{\nu} u \|_{L^2}.
\]
Since \( u \in H^{1-s} \) and consequently the sequence \( (2^{\nu(1-s)}\|\Delta_\nu u\|_{L^2})_\nu \) is in \( l^2 \) then the same is valid for \( (2^{-\nu s}\|\partial_{x_j}(\Delta_\nu, S_{\nu-3}a)\partial_{x_h}(\Delta_\nu u)\|_{L^2})_\nu \) and
\[
\sum_{\nu=0}^{+\infty} 2^{-2\nu s}\|\partial_{x_j}(\Delta_\nu, S_{\nu-3}a)\partial_{x_h}(\Delta_\nu u)\|_{L^2}^2 \leq C_{m,s}\|a\|_{Lip}^2\|u\|_{H^{1-s}}^2.
\]

The computation of the other terms being similar we obtain
\[
\sum_{\nu=0}^{+\infty} 2^{-2\nu s}\|\partial_{x_j}(\sum_{k=m+2}^{+\infty} [\Delta_\nu, S_k-a]\partial_{x_h}(\Delta_k u))\|_{L^2}^2 \leq C_{m,s}\|a\|_{Lip}^2\|u\|_{H^{1-s}}^2.
\]

The estimate (3.16) follows from (3.17) and (3.18), concluding the proof.

We end this subsection with a result on the adjoint of \( T^m_a \) (see [5, Prop. 3.8 and Prop. 3.11].

**Proposition 3.8.** Let \( m \in \mathbb{N}, \ a \in \text{Lip} \) and \( u \in H^s \). Then
\[
\|(T^m_a - (T^m_a)^*)\partial_{x_j}u\|_{L^2} \leq C_m\|a\|_{Lip}\|u\|_{L^2}.
\]  
(3.19)

**Proof.** We remark that
\[
(T^m_a - (T^m_a)^*)\partial_{x_j}u = [S_{m-1}a, S_{m+1}]\partial_{x_j}u + \sum_{k=m+2}^{+\infty} [S_{k-3}a, \Delta_k]\partial_{x_j}u.
\]

From [3] Th. 35] we deduce that
\[
\|[S_{m-1}a, S_{m+1}]\partial_{x_j}u\| \leq C\|\nabla_x(S_{m-1}a)\|_{L^\infty}\|u\|_{L^2},
\]
and hence, from Prop [3.4] we obtain
\[
\|[S_{m-1}a, S_{m+1}]\partial_{x_j}u\| \leq C\|a\|_{Lip}\|u\|_{L^2}.
\]  
(3.20)

On the other hand we have that the support of Fourier transform of \([S_{k-3}a, \Delta_k]\partial_{x_j}u\) is contained in \( \{2^{k-2} \leq |\xi| \leq 2^{k+2}\} \). Moreover it is easy to see that
\[
[S_{k-3}a, \Delta_k]\partial_{x_j}u = [S_{k-3}a, \Delta_k]\partial_{x_j}((\Delta_{k-1} + \Delta_k + \Delta_{k+1})u).
\]

Again from [3] Th. 35] and Proposition [3.4] we have
\[
\|[S_{k-3}a, \Delta_k]\partial_{x_j}u\|_{L^2} = \|[S_{k-3}a, \Delta_k]((\Delta_{k-1} + \Delta_k + \Delta_{k+1})u)\|_{L^2}
\leq C\|a\|_{Lip}(\|\Delta_{k-1}u\|_{L^2} + \|\Delta_k u\|_{L^2} + \|\Delta_{k+1} u\|_{L^2}).
\]
From Proposition 3.3 we finally obtain that \( \sum_{k=m+2}^{+\infty} [S_{k-3}a, \Delta_k] \partial_{x_j} u \in L^2 \) and
\[
\| \sum_{k=m+2}^{+\infty} [S_{k-3}a, \Delta_k] \partial_{x_j} u \| \leq C_m \| a \|_{Lip} \| u \|_{L^2}.
\]
(3.21)
The estimate (3.19) follows from (3.20) and (3.21).

### 3.3 Approximation and Carleman estimate

We set \( v(t, x) = e^{\frac{1}{\gamma} \Phi(\gamma(T-t))} u(t, x) \). The inequality (3.10) becomes: for all \( s \in (0, 1) \), there exist \( \gamma_0, C > 0 \) such that
\[
\int_0^T \left\| \partial_t v + \sum_{j,k=1}^n \partial_{x_j} (a_{jk}(t, x) \partial_{x_k} v) + \Phi'(\gamma(T-t))v \right\|_{H^{-s}}^2 \, dt \\
\geq C \gamma \frac{1}{2} \int_0^T \left( \| \nabla x v \|_{H^{-s}}^2 + \gamma \frac{1}{2} \| v \|_{H^{-s}}^2 \right) \, dt,
\]
(3.22)
for all \( \gamma > \gamma_0 \) and for all \( v \in C^\infty(\mathbb{R}^{n+1}) \) such that \( \text{supp} \, v \subseteq [0, T/2] \times \mathbb{R}^n \).

Using the Proposition 3.6 we fix the parameter \( m \) in such a way that the modified paraproduct associated to \((a_{jk})_{j,k}\) is a positive matrix operator. From the second part of Proposition 3.5 (estimate (3.12)), the inequality (3.22) will be deduced from the following
\[
\int_0^T \left\| \partial_t v + \sum_{j,k=1}^n \partial_{x_j} (T^m_{a_{jk}} \partial_{x_k} v) + \Phi'(\gamma(T-t))v \right\|_{H^{-s}}^2 \, dt \\
\geq C \gamma \frac{1}{2} \int_0^T \left( \| \nabla x v \|_{H^{-s}}^2 + \gamma \frac{1}{2} \| v \|_{H^{-s}}^2 \right) \, dt,
\]
(3.23)
as the quantity \( 2\| \sum_{j,k=1}^n \partial_{x_j} ((a_{jk} - T^m_{a_{jk}}) \partial_{x_k} v) \|_{H^{-s}}^2 \) can be absorbed by the right hand side part of (3.22), possibly taking different \( C \) and \( \gamma_0 \).

Let’s go back to the Littlewood-Paley decomposition; a consequence of Proposition 3.2 is that, denoting from now on \( \Delta_k u \) by \( u_k \), there exists \( K_s > 0 \) such that
\[
\frac{1}{K_s} \sum_{\nu} 2^{-2\nu s} \| u_\nu \|_{L^2}^2 \leq \| u \|_{H^{-s}}^2 \leq K_s \sum_{\nu} 2^{-2\nu s} \| u_\nu \|_{L^2}^2
\]
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for all $u \in H^{-s}$. We have
\[
\int_0^T \| \partial_t v + \sum_{j,k=1}^n \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v) + \Phi'(\gamma(T-t))v \|^2_{H^{-s}} dt \\
\geq \frac{1}{K_s} \int_0^T \sum_{\nu} 2^{-2\nu s} \| \Delta_{\nu} (\partial_t v + \sum_{j,k=1}^n \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v) + \Phi'(\gamma(T-t))v) \|^2_{L^2} dt \\
\geq \frac{1}{K_s} \int_0^T \sum_{\nu} 2^{-2\nu s} \| \partial_t v + \sum_{j,k=1}^n \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v) + \Phi'(\gamma(T-t))v \|^2_{L^2} dt \\
+ \frac{1}{K_s} \int_0^T \sum_{j,k=1}^n 2^{-2\nu s} \| \sum_{\nu} \partial_{x_j} ([\Delta_{\nu}, T_{a_{jk}}^m \partial_{x_k} v]) \|^2_{L^2} dt.
\]
From the result of Proposition 3.7 is then immediate that (3.23) will be deduced from the same estimate from below for
\[
\int_0^T \sum_{\nu} 2^{-2\nu s} \| \partial_t v + \sum_{j,k=1}^n \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v) + \Phi'(\gamma(T-t))v \|^2_{L^2} dt,
\]
again with possibly different $C$ and $\gamma_0$. We have
\[
\int_0^T \sum_{\nu} 2^{-2\nu s} \| \partial_t v + \sum_{j,k} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v) + \Phi'(\gamma(T-t))v \|^2_{L^2} dt \\
= \int_0^T \sum_{\nu} 2^{-2\nu s} (\| \partial_t v \|^2_{L^2} + \| \sum_{j,k} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v) + \Phi'(\gamma(T-t))v \|^2_{L^2} \\
+ \gamma \Phi''(\gamma(T-t)) \| v \|^2_{L^2} + 2 \Re \langle \partial_t v, \sum_{j,k} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v) \rangle_{L^2} \rangle dt.
\]
We approximate the last term in the above equality using a well known technique which goes back to [1]. Let $\rho \in C_0^{\infty}(\mathbb{R})$ with supp $\rho \subseteq [-1/2, 1/2]$, $\int_{\mathbb{R}} \rho(s) ds = 1$ and $\rho(s) \geq 0$ for all $s \in \mathbb{R}$; we set
\[
a_{j,k}(t, x) = \int_{\mathbb{R}} a_{j,k}(s, x) \frac{1}{\varepsilon} \rho\left(\frac{t-s}{\varepsilon}\right) ds
\]
for $\varepsilon \in ]0,1/2]$. We obtain from (2.7) that there exist \( C \) such that

\[
|a_{jk, \varepsilon}(t, x) - a_{jk}(t, x)| \leq C \mu(\varepsilon) \tag{3.24}
\]

and

\[
|\partial_t a_{jk, \varepsilon}(t, x)| \leq C \frac{\mu(\varepsilon)}{\varepsilon}, \tag{3.25}
\]

for all \( j,k = 1, \ldots, n \) and for all \((t, x) \in [0, T] \times \mathbb{R}^n_x \). We have

\[
\int_0^T 2 \text{Re} \left( \partial_t v_\nu, \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_\nu) \right)_{L^2} dt
\]

\[
=-2 \text{Re} \int_0^T \sum_{jk} \langle \partial_{x_j} \partial_t v_\nu, T_{a_{jk}}^m \partial_{x_k} v_\nu \rangle_{L^2} dt
\]

\[
=-2 \text{Re} \int_0^T \sum_{jk} \langle \partial_{x_j} \partial_t v_\nu, (T_{a_{jk}}^m - T_{\tilde{a}_{jk}, \varepsilon}^m) \partial_{x_k} v_\nu \rangle_{L^2} dt
\]

\[
-2 \text{Re} \int_0^T \sum_{jk} \langle \partial_{x_j} \partial_t v_\nu, T_{\tilde{a}_{jk}, \varepsilon}^m \partial_{x_k} v_\nu \rangle_{L^2} dt.
\]

We remark that \( T_{a_{jk}}^m - T_{\tilde{a}_{jk}, \varepsilon}^m = T_{a_{jk} - \tilde{a}_{jk}, \varepsilon}^m \) and consequently, from (3.11) and (3.24), we have that

\[
\| (T_{a_{jk}}^m - T_{\tilde{a}_{jk}, \varepsilon}^m) \partial_{x_k} v_\nu \|_{L^2} = \| T_{a_{jk} - \tilde{a}_{jk}, \varepsilon}^m \partial_{x_k} v_\nu \|_{L^2} \leq C \mu(\varepsilon) \| \partial_{x_k} v_\nu \|_{L^2}.
\]

Moreover \( \| \partial_{x_j} \partial_t v_\nu \|_{L^2} \leq 2^{\nu+1} \| v_\nu \|_{L^2} \) and \( \| \partial_{x_j} \partial_t v_\nu \|_{L^2} \leq 2^{\nu+1} \| \partial_t v_\nu \|_{L^2} \) for all \( \nu \in \mathbb{N} \). Hence

\[
|2 \text{Re} \int_0^T \sum_{jk} \langle \partial_{x_j} \partial_t v_\nu, (T_{a_{jk}}^m - T_{\tilde{a}_{jk}, \varepsilon}^m) \partial_{x_k} v_\nu \rangle_{L^2} dt|
\]

\[
\leq 2C \mu(\varepsilon) \int_0^T \sum_{jk} \| \partial_{x_j} \partial_t v_\nu \|_{L^2} \| \partial_{x_k} v_\nu \|_{L^2} dt
\]

\[
\leq \frac{C}{N} \int_0^T \| \partial_t v_\nu \|^2_{L^2} dt + CN 2^{4(\nu+1)} \mu(\varepsilon) \int_0^T \| v_\nu \|^2_{L^2} dt
\]

for all \( N > 0 \) (note that \( \mu(\varepsilon)^2 \leq \mu(\varepsilon) \)). On the other hand \( \partial_t (T_{a_{jk}, \varepsilon}^m w) = T_{\partial_t a_{jk}, \varepsilon} w + T_{a_{jk}, \varepsilon} \partial_t w \), then, using also the fact that the matrix \( (a_{jk})_{j,k} \) is
real and symmetric,

\[ -2 \text{Re} \int_0^T \sum_{jk} \langle \partial_{x_j} \partial_t v_\nu, T_{a_{jk}, \varepsilon}^m \partial_{x_k} v_\nu \rangle_{L^2} dt \]

\[ = \int_0^T \sum_{jk} \langle \partial_{x_j} v_\nu, T_{a_{jk}, \varepsilon}^m \partial_{x_k} v_\nu \rangle_{L^2} + \langle (T_{a_{jk}, \varepsilon}^m) - (T_{a_{jk}, \varepsilon}^m)^* \rangle \partial_{x_j} v_\nu, \partial_{x_k} \partial_t v_\nu \rangle_{L^2} dt. \]

From (3.11) and (3.25) we deduce

\[ | \int_0^T \sum_{jk} \langle \partial_{x_j} v_\nu, T_{a_{jk}, \varepsilon}^m \partial_{x_k} v_\nu \rangle_{L^2} dt | \leq C 2^{2(\nu+1)} \frac{\mu(\varepsilon)}{\varepsilon} \int_0^T \| v_\nu \|_{L^2}^2 dt, \]

and, from (3.19) and (3.24),

\[ | \int_0^T \sum_{jk} \langle (T_{a_{jk}, \varepsilon}^m) - (T_{a_{jk}, \varepsilon}^m)^* \rangle \partial_{x_j} v_\nu, \partial_{x_k} \partial_t v_\nu \rangle_{L^2} dt | \]

\[ \leq 2C \mu(\varepsilon) \int_0^T \sum_{jk} \| \partial_{x_j} v_\nu \|_{L^2} \| \partial_{x_k} \partial_t v_\nu \|_{L^2} dt \]

\[ \leq C N \int_0^T \| \partial_t v_\nu \|_{L^2}^2 dt + CN 2^{4(\nu+1)} \mu(\varepsilon) \int_0^T \| v_\nu \|_{L^2}^2 dt \]

for all \( N > 0 \). Choosing suitably \( N \), we finally obtain

\[ \int_0^T \| \partial_t v_\nu + \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_\nu) + \Phi'(\gamma(T - t)) v_\nu \|_{L^2}^2 dt \]

\[ \geq \int_0^T (\| \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_\nu) + \Phi'(\gamma(T - t)) v_\nu \|_{L^2}^2 + \gamma \Phi''(\gamma(T - t)) \| v_\nu \|_{L^2}^2 - C(2^{4(\nu+1)} \mu(\varepsilon) + 2^{2(\nu+1)} \frac{\mu(\varepsilon)}{\varepsilon}) \| v_\nu \|_{L^2}^2 \) dt. \]

(3.26)

### 3.4 End of the proof of the Carleman estimate

From now on the proof is exactly the same as in [7, Par. 3.2]. We detail it for the reader’s convenience. Let \( \nu = 0 \). From (3.9) we can choose \( \gamma_0 > 0 \) such that \( \Phi''(\gamma(T - t)) \geq 1 \) for all \( \gamma > \gamma_0 \) and for all \( t \in [0, T/2] \). Taking
now $\varepsilon = 1/2$ we obtain from (3.26) that

$$\int_0^T \| \partial_t v_0 + \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_0) + \Phi'(\gamma(T - t))v_0 \|_{L^2}^2 \, dt \leq \int_0^T (\gamma - 16C\mu(\gamma^2))\|v_0\|_{L^2}^2 \, dt$$

for all $\gamma > \gamma_0$. Possibly choosing a larger $\gamma_0$ we have, again for all $\gamma > \gamma_0$,

$$\int_0^T \| \partial_t v_0 + \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_0) + \Phi'(\gamma(T - t))v_0 \|_{L^2}^2 \geq \frac{\gamma}{2} \int_0^T \|v_0\|_{L^2}^2 \, dt.$$  \hspace{1cm} (3.27)

Let now $\nu \geq 1$. We take $\varepsilon = 2^{-2\nu}$. We obtain from (3.26) that

$$\int_0^T \| \partial_t v_\nu + \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_\nu) + \Phi'(\gamma(T - t))v_\nu \|_{L^2}^2 \, dt \leq \int_0^T \left( \| \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_\nu) + \Phi'(\gamma(T - t))v_\nu \|_{L^2}^2 \right.
\left. + \gamma \|v_\nu\|_{L^2}^2 - K 2^{4\nu} \mu(2^{-2\nu})\|v_\nu\|_{L^2}^2 \right) \, dt$$

$$\geq \int_0^T \left( \| \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_\nu) \|_{L^2}^2 - \| \Phi'(\gamma(T - t))v_\nu \|_{L^2}^2 \right)^2
\left. + 2^{4\nu} \mu(2^{-2\nu})\|v_\nu\|_{L^2}^2 \right) \, dt$$

where $K = 16C$. On the other hand, from (3.15), recalling that in this case $\|\nabla v_\nu\| \geq 2^{\nu-1}\|v_\nu\|$, we have

$$\| \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_\nu) \|_{L^2} \|v_\nu\|_{L^2} \geq \left| \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_\nu), v_\nu \right|_{L^2} \geq \frac{\lambda_0}{2} \|\nabla v_\nu\|_{L^2}^2 \geq \frac{\lambda_0}{8} \|v_\nu\|_{L^2}^2.$$  \hspace{1cm} (3.28)

Suppose first that $\Phi'(\gamma(T - t)) \leq \frac{\lambda_0}{8} 2^{2\nu}$. Then from (3.28) we deduce that

$$\| \sum_{jk} \partial_{x_j} (T_{a_{jk}}^m \partial_{x_k} v_\nu) \|_{L^2} - \| \Phi'(\gamma(T - t))v_\nu \|_{L^2} \geq \frac{\lambda_0}{16} 2^{2\nu} \|v_\nu\|_{L^2}^2$$
and then, using also the fact that $\Phi''(\gamma(T-t)) \geq 1$, we obtain that
\[
\int_0^T \left( \left\| \sum \partial_{x_j} (T_{a_{jk}} \partial_{x_k} v_\nu) \right\|_{L^2}^2 - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2}^2 \right)^2 
+ \gamma \Phi''(\gamma(T-t))\|v_\nu\|_{L^2}^2 - K 2^{4\nu}\mu(2^{-2\nu})\|v_\nu\|_{L^2}^2 \right) dt 
\geq \int_0^T \left( \frac{\lambda_0}{16} 2^{2\nu} \right)^2 + \gamma - K 2^{4\nu}\mu(2^{-2\nu})\|v_\nu\|_{L^2}^2 \right) dt 
\geq \int_0^T \left( \frac{1}{2} \left( \frac{\lambda_0}{16} \right)^2 - K \mu(2^{-2\nu}) \right) 2^{4\nu} + \frac{\gamma}{3} \right)\|v_\nu\|_{L^2}^2 dt 
+ \int_0^T \left( \frac{1}{2} \left( \frac{\lambda_0}{16} \right)^2 2^{4\nu} + \frac{2}{3} \right)\|v_\nu\|_{L^2}^2 dt.
\]
Since $\lim_{\nu \to +\infty} \mu(2^{-2\nu}) = 0$, there exists $\gamma_0 > 0$ such that
\[
\left( \frac{1}{2} \left( \frac{\lambda_0}{16} \right)^2 - K \mu(2^{-2\nu}) \right) 2^{4\nu} + \frac{\gamma}{3} \geq 0
\]
for all $\gamma \geq \gamma_0$ and for all $\nu \geq 1$. Consequently there exist $\gamma_0$ and $c > 0$ not depending on $\nu$ such that
\[
\int_0^T \left( \left\| \sum \partial_{x_j} (T_{a_{jk}} \partial_{x_k} v_\nu) \right\|_{L^2}^2 - \Phi'(\gamma(T-t))\|v_\nu\|_{L^2}^2 \right)^2 
+ \gamma \Phi''(\gamma(T-t))\|v_\nu\|_{L^2}^2 - K 2^{4\nu}\mu(2^{-2\nu})\|v_\nu\|_{L^2}^2 \right) dt 
\geq \int_0^T \left( \frac{1}{2} \left( \frac{\lambda_0}{16} \right)^2 2^{4\nu} + \frac{2}{3} \gamma\right)\|v_\nu\|_{L^2}^2 dt \geq \int_0^T \left( \frac{\gamma}{2} + c \gamma \right) 2^{4\nu}\|v_\nu\|_{L^2}^2 dt \tag{3.29}
\]
for all $\gamma \geq \gamma_0$.

If on the contrary $\Phi'(\gamma(T-t)) \geq \frac{\lambda_0}{16} 2^{2\nu}$ then, using \textit{(3.8)}, the fact that $\lambda_0 \leq 1$ and the properties of $\mu$, \centerline{$$
\Phi''(\gamma(T-t)) = \left( \frac{\Phi'(\gamma(T-t))^2}{\Phi'(\gamma(T-t))} \right)^{-1} \mu \left( \frac{1}{\Phi'(\gamma(T-t))} \right) \geq \left( \frac{\lambda_0}{16} \right)^2 2^{4\nu} \mu(2^{-2\nu}) \geq \left( \frac{\lambda_0}{16} \right)^2 2^{4\nu} \mu(2^{-2\nu}).$$}
Hence also in this case there exist $\gamma_0$ and $c > 0$ such that
\[
\int_0^T \left( \left\| \sum_{jk} \partial_{x_j} (T_{m_{jk}} \partial_{x_k} v_\nu) \right\|_{L^2}^2 - \Phi' (\gamma (T - t)) ||v_\nu||_{L^2}^2 \right) dt \\
+ \gamma \Phi'' (\gamma (T - t)) ||v_\nu||_{L^2}^2 - K 2^{4\nu} \mu (2^{-2\nu}) ||v_\nu||_{L^2}^2 \right) dt \\
\geq \int_0^T \left( \frac{\gamma}{2} + \frac{\gamma_0^2}{2} - K 2^{4\nu} \mu (2^{-2\nu}) \right) ||v_\nu||_{L^2}^2 dt \\
\geq \int_0^T \left( \frac{\gamma}{2} + c \gamma 2^{2\nu} \right) ||v_\nu||_{L^2}^2 dt
\]
for all $\gamma \geq \gamma_0$ and for all $\nu \geq 1$. Putting together (3.29) and (3.30) we have that there exist $\gamma_0$ and $c > 0$ such that
\[
\int_0^T \left( \left\| \sum_{jk} \partial_{x_j} (T_{m_{jk}} \partial_{x_k} v_\nu) \right\|_{L^2}^2 + \Phi' (\gamma (T - t)) ||v_\nu||_{L^2}^2 \right) dt \\
\geq \int_0^T \left( \frac{\gamma}{2} + c \gamma 2^{2\nu} \right) ||v_\nu||_{L^2}^2 dt
\]
for all $\nu \geq 1$ and for all $\gamma \geq \gamma_0$.

Form (3.27) and (3.31) we get that there exist $\gamma_0$ and $c > 0$ such that
\[
\int_0^T \left( \sum_{\nu} 2^{-2\nu s} ||\partial_t v_\nu + \sum_{jk} \partial_{x_j} (T_{m_{jk}} \partial_{x_k} v_\nu) + \Phi' (\gamma (T - t)) v_\nu ||_{L^2}^2 \right) dt \\
\geq c \gamma^2 \int_0^T \left( \sum_{\nu} 2^{-2\nu s} \left( ||\nabla v_\nu||_{L^2}^2 + \gamma^2 ||v_\nu||_{L^2}^2 \right) dt
\]
for all $\gamma \geq \gamma_0$.

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