TETRAHEDRON EQUATION AND THE ALGEBRAIC GEOMETRY

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Abstract. The tetrahedron equation arises as a generalization of the famous Yang—Baxter equation to the 2+1-dimensional quantum field theory and the 3-dimensional statistical mechanics. Very little is still known about its solutions. Here a systematic method is described that does produce non-trivial solutions to the tetrahedron equation with spin-like variables on the links. The essence of the method is the use of the so-called tetrahedral Zamolodchikov algebras.

0 Introduction

Considerable successes made in recent years in the 2-dimensional statistical mechanics and the 1+1-dimensional quantum field theory are closely connected with the studying of the Yang—Baxter equation, also called triangle equation, and finding new solutions to this equation. The tetrahedron equation is an attempt to approach the 3-dimensional problems in the same way. It is known now that there do exist nontrivial solutions to the tetrahedron equation [1,2,5,8,9]. Despite some setbacks in the existing solutions, such as negativity of some of their matrix elements, which does not allow to use them directly in statistical mechanics models, the tetrahedron equation is no doubt worth further studying. The guarantee is the mathematical beauty already revealed in this area.

Properly speaking, there are different modifications of the tetrahedron equation. Here I study (unlike the papers [1,2,10,11,12]) the equation “with variables on the links”. This means the equalness of two operator products in the tensor product of 6 complex linear spaces:

\[ S_{123} S_{145} S_{246} S_{356} = S_{356} S_{246} S_{145} S_{123}. \]

Each operator acts nontrivially only in the tensor product of 3 spaces numbered by its subscripts. This is illustrated by Fig. 1.

In describing the solutions to the tetrahedron equation, I follow the papers [8,9]. The way I came to these solutions was through the “tetrahedral Zamolodchikov algebras” (TZA’s) [3]. These are some structures lying between the triangle equation and the tetrahedron equation. As has been shown in [3], the TZA’s may be used, at least, for constructing a statistical mechanical model on
I begin with a study of the well-known Felderhof $\mathcal{L}$-operators \cite{6,4} and their products by means of algebraic geometry (section 1). This provides a large amount of TZA’s (section 2). The solutions to the tetrahedron equation are presented in section 3. I conclude with a discussion in section 4.

1 Vacuum Curves and Vacuum Vectors of the Felderhof $\mathcal{L}$-Operators’ Products

1.1 Vacuum curve and vacuum vectors of a Felderhof $\mathcal{L}$-operator

In this subsection and partly in the in the next I recall some facts from \cite{4}. Let $\mathcal{V}_0, \mathcal{V}_1, \ldots$ be 2-dimensional complex linear spaces with fixed bases. Felderhof $\mathcal{L}$-operator is a linear operator acting in the tensor product of two such spaces, e.g. $\mathcal{V}_0 \otimes \mathcal{V}_1$, of the following form:

$$L = \begin{pmatrix} a_+ & b_- & c & d \\ b_+ & c & d & c \\ d & c & b_+ & a_- \end{pmatrix},$$

where

Figure 1: The diagrammatic representation of the tetrahedron equation with variables on the links
\[ a_+a_- + b_+b_- = c^2 + d^2; \]
each \(2 \times 2\) block is an operator in \(\mathcal{V}_1\). Let us denote the Felderhof \(L\)-operators by the letters \(L, M, \ldots\). To emphasize the spaces in which the operator acts, I will write also \(L = L_{01}\) etc. It will be useful to consider the vectors from the spaces \(\mathcal{V}_k\) with the second coordinate equal to unity. Let us denote them as follows:

\[ U = \begin{pmatrix} u \\ 1 \end{pmatrix}, \quad V = \begin{pmatrix} v \\ 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ 1 \end{pmatrix}. \]

Let \(U, V \in \mathcal{V}_0, X, Y \in \mathcal{V}_1\). Under what conditions does the equality

\[ L(U \otimes X) = g(V \otimes Y) \]  \hspace{1cm} (1)

hold (\(g\) being a numerical factor)? The condition turns out to be the following connection between \(u\) and \(v\):

\[ u^2v^2 + \alpha u^2 + \delta v^2 + 1 = 0, \]  \hspace{1cm} (2)

where

\[ \alpha = \frac{a_+b_-}{cd}, \quad \delta = \frac{a_-b_+}{cd}. \]

Eq. (2) determines an elliptic curve \(\Gamma_L\), and to each point \((u, v) \in \Gamma_L\) corresponds a 1-dimensional space of the vectors proportional to \(X\). So the vectors proportional to \(X\)—let us call them the vacuum vectors of the operator \(L\)—make up a 1-dimensional vector bundle over \(\Gamma_L\). This bundle, as usually, may be determined by a divisor—e.g., by the pole divisor of the vector \(X\) first coordinate \(x\). Denote this divisor \(D_L\). One can find from Eq. (1) that it consists of 2 points \((u, v) = (u_L, v_L)\) and \((-u_L, -v_L)\), where

\[ u_L^2 = \frac{cb_+}{da_+}, \quad v_L = \frac{a_+}{b_+}u_L. \]

When necessary, I will add an index: \(X = X_L, x = x_L\) etc.

1.2 The vacuum curve and the vacuum vector bundle of the product of two Felderhof \(L\)-operators

Consider now two Felderhof \(L\)-operators \(L = L_{01}\) and \(M = M_{02}\). Consider their product \(LM\) acting in \(\mathcal{V}_0 \otimes \mathcal{V}_1 \otimes \mathcal{V}_2\) (Fig. 2). Let us investigate the vacuum vectors \(X_{LM} \in \mathcal{V}_1 \otimes \mathcal{V}_2\) satisfying the relation \(LM(U \otimes X_{LM}) = g(V \otimes Y_{LM})\). Dependence between \(u\) and \(v\) is found according to [4]. Let us write down the equations of the vacuum curves \(\Gamma_L\) and \(\Gamma_M\), and denote their points in the following way: \(\Gamma_L \ni (w, v), \Gamma_M \ni (u, w)\):

\[ \Gamma_L: \quad w^2v^2 + \alpha_L w^2 + \delta_L v^2 + 1 = 0, \]  \hspace{1cm} (3)
**Figure 2:** The diagrammatic representation of the product of two $\mathcal{L}$-operators

\[
\begin{align*}
\nu_0 & \quad L \quad M \\
\nu_1 & \quad \nu_2
\end{align*}
\]

\[\Gamma_M \quad \Gamma_L\]

\[X_M(u, w) \quad X_L(w, v)\]

\[X_M(u, -w) \quad X_L(-w, v)\]

\[u \quad v \quad -w\]

**Figure 3:** Vacuum curve and vacuum vectors of the product of two $\mathcal{L}$-operators

\[
\Gamma_M : \quad u^2 w^2 + \alpha_M u^2 + \delta_M w^2 + 1 = 0. \quad (4)
\]

The vacuum curve equation of the operator $LM$ is found by eliminating $w^2$ from Eqs. (3,4):

\[
\Gamma_{LM} : \quad (\delta_L - \alpha_M)u^2 v^2 + (1 - \alpha_L \alpha_M)u^2 + (\delta_L \delta_M - 1) v^2 + (\delta_M - \alpha_L) = 0. \quad (5)
\]

This time a 2-dimensional space of vectors $X_{LM}$ corresponds to each point $(u, v) \in \Gamma_{LM}$, as one can see in Fig. 3.

The 2-dimensional space contains the vectors

\[X_L(w, v) \otimes X_M(u, w) \quad (6)\]

and

\[X_L(-w, v) \otimes X_M(u, -w). \quad (7)\]
Theorem 1. The vacuum vector bundle of the operator $LM$ decomposes into a direct sum of two sub-bundles $E_1$ and $E_2$ of degree 2. The bundles $E_1$ and $E_2$ are, generally, non-isomorphic, yet each of them becomes isomorphic to the other after the transform $(u, v) \rightarrow (u, -v)$ of the curve $\Gamma_{LM}$.

Proof. Rewrite the vectors (6,7) together as a matrix

$$H = \begin{pmatrix} x_L^+ x_M^+ & x_L^- x_M^- \\ x_L^+ & x_L^- \\ x_M^+ & x_M^- \\ 1 & 1 \end{pmatrix}$$

where I use brief notations

$$X_L(\pm w, v) = \begin{pmatrix} x_L^+ \\ 1 \end{pmatrix}, \quad X_M(u, \pm w) = \begin{pmatrix} x_M^+ \\ 1 \end{pmatrix}.$$ 

One can get two other vectors forming a basis in the fiber by multiplying $H$ from the right by any non-degenerate $2 \times 2$ matrix $G$. Let us choose

$$G = \begin{pmatrix} x_L^+ x_M^+ & x_L^- x_M^- \\ 1 & 1 \end{pmatrix}^{-1}. $$

We find then

$$HG = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix},$$

where

$$K = (x_L^+ x_M^+ - x_L^- x_M^-)^{-1} \begin{pmatrix} x_L^+ - x_L^- & x_L^+ x_L^- (x_M^+ - x_M^-) \\ x_M^+ - x_M^- & x_M^+ x_M^- (x_L^+ - x_L^-) \end{pmatrix}.$$ 

Thus, the matrix $HG$ remains invariant under the change $w \leftrightarrow -w$ and depends only on $u$ and $v$, $(u, v) \in \Gamma_{LM}$. Now investigate its matrix elements’ poles. Thorough examination of all “suspicious” points shows that elements of the matrix $K$ have poles in the points where $x_L^+ x_M^+ - x_L^- x_M^- = 0$ except the points with $w = 0$ or $\infty$.

A routine consideration shows the degree of the matrix $K$ pole divisor to equal 4, the poles being located in some points

$$(u_1, v_1), (-u_1, -v_1), (u_2, v_2), (-u_2, -v_2).$$  

Let the ratios of the residues of the matrix $K$ first column elements to those of the second column be $\beta : 1$ in the first two points (8) and $\gamma : 1$ in the others.
(that is what is called a matrix divisor, see [4, 7]). Multiply $HG$ from the right by a matrix

$$G_1 = \begin{pmatrix} 1 & 1 \\ -\gamma & -\beta \end{pmatrix}.$$ 

Then the first column of $HGG_1$ has only two poles $(u_1, v_1)$ and $(-u_1, -v_1)$, i.e. its pole divisor $D_1 = (u_1, v_1) + (-u_1, -v_1)$. Similarly, the second column has a pole divisor $D_2 = (u_2, v_2) + (-u_2, -v_2)$. Evidently, $E = E_1 \oplus E_2$, the bundles $E_1$ and $E_2$ corresponding to the divisors $D_1$ and $D_2$.

To prove that $E_1$ (or $E_2$) transforms into a bundle isomorphic to $E_2$ (or $E_1$) by the isomorphism $(u, v) \rightarrow (u, -v)$ of the curve $\Gamma_{LM}$, let us multiply the matrix $H$ from the right by a matrix

$$G' = \begin{pmatrix} x_M^+ & x_M^- \\ 1 & 1 \end{pmatrix}$$

Again the elements of the matrix $HG'$ have 4 poles, but now those poles coincide with the poles of $x_L$. The 4 corresponding points are of the form $(\pm u_0, \pm v_0)$ with arbitrary signs chosen independently. The divisor $D$ of those poles can now be seen to be equivalent to $2D_u$, $D_u$ being a pole divisor of the function $u$ on $\Gamma_{LM}$.

The divisor $D$ corresponds to the determinant (see e.g. [7]) of the bundle $E$, thus $D \sim D_1 + D_2$, i.e.

$$D_1 + D_2 \sim 2D_u. \quad (9)$$

One can easily see that for each divisor $D_1$ of degree 2

$$D_1 + \tilde{D}_1 \sim 2D_u, \quad (10)$$

where $\tilde{D}_1$ is obtained from $D_1$ by changing $v \rightarrow -v$. From Eqs (9, 10) we find $D_2 \sim \tilde{D}_1$, and with this Theorem 1 is completely proved.

### 1.3 Vacuum vector bundle of the product of 3 Felderhof $L$-operators

Consider now a product $LMN = L_{01}M_{02}N_{03}$ of 3 Felderhof $L$-operators.

**Theorem 2.** The vacuum vector bundle $F$ of the operator $LMN$ is a direct sum $F = F_1 \oplus F_2 \oplus F_3 \oplus F_4$, where $F_1 \sim F_2, F_3 \sim F_4$, but $F_1$ is, generally, not isomorphic to $F_3$.

**Proof.** According to Theorem 1, the vacuum vectors of the operator $LM$ form the 2-dimensional bundle $E = E_1 \oplus E_2$, and each of $E_1$ and $E_2$ is of the same kind as if it were a vacuum vector bundle of some Felderhof $L$-operator. Having multiplied $LM$ by $N$, one obtains a 2-dimensional bundle, say $F_1 \oplus F_3$, out of the vectors of $E_1$, and another 2-dimensional bundle, say $F_2 \oplus F_4$, out of the
vectors of $\mathcal{E}_2$. Under the transform $(u, v) \rightarrow (u, -v)$ of the curve $\Gamma_{LMN}$, the bundles transform, up to the isomorphisms, as follows:

$$F_1 \leftrightarrow F_3, \quad F_2 \leftrightarrow F_4 \quad (11)$$

On the other hand, under the same transform of the curve $\Gamma_{LM}$, $E_1 \leftrightarrow E_2$, and consequently $(F_1 \oplus F_3) \leftrightarrow (F_2 \oplus F_4)$. One may assume that

$$F_1 \leftrightarrow F_4, \quad F_2 \leftrightarrow F_3. \quad (12)$$

Combine the formulae (11, 12), and the proof is completed.

**Corollary.** The linear space of endomorphisms of the bundle $F$ is 8-dimensional.

**Proof.** One can see from the Theorem 2 that each fiber of the bundle $F$ may be generated by linearly independent meromorphic vectors $X_1, X_2, X_3, X_4$, each of the pairs $X_1, X_2$ and $X_3, X_4$ having a common pole divisor of degree 2. In the basis $(X_1, X_2, X_3, X_4)$, the endomorphisms of the bundle $F$ are given by arbitrary constant matrices of the form

$$
\begin{pmatrix}
    a_{11} & a_{12} & 0 & 0 \\
    a_{21} & a_{22} & 0 & 0 \\
    0 & 0 & a_{33} & a_{34} \\
    0 & 0 & a_{34} & a_{44}
\end{pmatrix}
$$

acting from the right.

## 2 Tetrahedral Zamolodchikov Algebras

Now let the Felderhof $\mathcal{L}$-operators $L_{01}$ and $M_{02}$ satisfy the Yang—Baxter equation:

$$R_{12}L_{01}M_{02} = M_{02}L_{01}R_{12}. \quad (13)$$

The matrix $R_{12}$ is then known to be symmetrical: $R_{12}^T = R_{12}$. However (see, e.g., [4]), along with $R_{12} = R_{12}^0$ there exists a non-symmetrical matrix $R_{12}^\dagger$ such that

$$(R_{12}^\dagger)^T L_{01}M_{02} = M_{02}L_{01}R_{12}^\dagger.$$  

Examples are given in Section 3.

Let the pairs of operators $L_{01}, N_{03}$ and $M_{02}, N_{03}$ also satisfy equations of the kind of Eq. (13):

$$\tilde{R}_{13}L_{01}N_{03} = N_{03}L_{01}\tilde{R}_{13},$$

$$\tilde{R}_{23}M_{02}N_{03} = N_{03}M_{02}\tilde{R}_{23}.$$  

Here the tildes indicate that the $R$-operators not only act in different spaces but differ in their matrix elements. Somewhat freely, I will allow myself to omit
those tildes. Then I will re-denote $R_{13} = R_{13}^0$, $R_{23} = R_{23}^0$, and introduce $R_{13}^1$ and $R_{23}^1$ as before. Let us consider the operators $\mathcal{R} = \mathcal{R}_{123}$, $\mathcal{R} = \mathcal{R}_{123}$ which permute the $L$-operators as follows:

$$\mathcal{R}_{123} L_{01} M_{02} N_{03} = N_{03} M_{02} L_{01} \mathcal{R}_{123}. \quad (14)$$

One gets 8 such $\mathcal{R}$-operators (generally, linearly independent—see Section 4) in the form

$$\mathcal{R} = R_{123}^a R_{13}^b R_{23}^c$$

with $a, b, c = 0$ or 1. The corresponding $\hat{\mathcal{R}}$-operators are

$$\hat{\mathcal{R}} = (R_{23}^c R_{13}^b R_{12}^a)^T$$

(note that in [8,9] it was stated that $\hat{\mathcal{R}} = \mathcal{R}^T$, which is a mistake). On the other hand, there are also 8 $\mathcal{R}$-operators of the form

$$\mathcal{R} = R_{123}^I R_{13}^f R_{23}^d,$$

with corresponding

$$\hat{\mathcal{R}} = (R_{12}^d R_{13}^f R_{23}^I)^T.$$

It follows from Eq. (14) that $\mathcal{R}_{123}$ converts the vacuum vectors of $L_{01} M_{02} N_{03}$ into those of $N_{03} M_{02} L_{01}$. The corresponding vacuum vector bundles are isomorphic. Thus, it follows from the Corollary of Theorem 2 that the linear space of the operators $\mathcal{R}_{123}$ is 8-dimensional. This leads to the linear dependences of a tetrahedral Zamolodchikov algebra (Ref. 3):

$$P_{12}^a R_{13}^b R_{23}^c = \sum_{d,e,f=0}^1 S_{def}^{abc} R_{23}^e R_{13}^f R_{12}^d. \quad (15)$$

3 New Solutions to the Tetrahedron Equation

Consider a Felderhof $L$-operator of the form

$$L(\lambda) = \begin{pmatrix} 
\text{cn} \lambda & \text{sn} \lambda \text{dn} \lambda & k \text{sn} \lambda \\
\text{sn} \lambda \text{dn} \lambda & \text{dn} \lambda & \text{sn} \lambda \text{dn} \lambda \\
k \text{sn} \lambda \text{cn} \lambda & \text{sn} \lambda \text{dn} \lambda & \text{cn} \lambda 
\end{pmatrix}$$

$k$ being a modulus of elliptic functions. Let us fix 4 complex numbers $\lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$, and consider 4 operators

$$L_{01}(\lambda_1), L_{02}(\lambda_2), L_{03}(\lambda_3), L_{04}(\lambda_4).$$
Sometimes I will omit the arguments $\lambda_j$. Let us introduce the operators $R^e_{ij}(\lambda_i, \lambda_j), 1 \leq i < j \leq 4, e = 0, 1,$ by the following formulae:

$$R^0_{ij}(\lambda_i, \lambda_j) = f_0(\lambda_i, \lambda_j) \begin{pmatrix} a & d \\ b & c \\ d & a \end{pmatrix},$$

$$R^1_{ij}(\lambda_i, \lambda_j) = f_1(\lambda_i, \lambda_j) \begin{pmatrix} -a' & -b' & -d' \\ -b' & c' & b' \\ -d' & -c' & a' \end{pmatrix}. $$

Herein

$$a = \text{cn}(\lambda_i - \lambda_j), \quad b = \text{sn}(\lambda_i - \lambda_j)\text{dn}(\lambda_i - \lambda_j),$$

$$c = \text{dn}(\lambda_i - \lambda_j), \quad d = \text{ksn}(\lambda_i - \lambda_j)\text{cn}(\lambda_i - \lambda_j),$$

$$a' = \text{cn}(\lambda_i + \lambda_j), \quad b' = \text{sn}(\lambda_i + \lambda_j)\text{dn}(\lambda_i + \lambda_j),$$

$$c' = \text{dn}(\lambda_i + \lambda_j), \quad d' = \text{ksn}(\lambda_i + \lambda_j)\text{cn}(\lambda_i + \lambda_j),$$

$f_0(\lambda_i, \lambda_j)$ and $f_1(\lambda_i, \lambda_j)$ are arbitrary numerical factors. One can verify that the equations of the Section 3 hold:

$$R^0_{12}L_{01}L_{02} = L_{02}L_{01}R^0_{12}, \quad (R^1_{12})^T L_{01}L_{02} = L_{02}L_{01}R^1_{12}$$

etc.

Now let us introduce the 2-dimensional linear spaces $\mathcal{V}_{12}, \mathcal{V}_{13}, \mathcal{V}_{14}, \mathcal{V}_{23}, \mathcal{V}_{24}, \mathcal{V}_{34}$ and consider the matrix $S$ from Eq. (15) as a linear operator in $\mathcal{V}_{12} \otimes \mathcal{V}_{13} \otimes \mathcal{V}_{23}$, so that the indices $a$ and $d$ correspond to the space $\mathcal{V}_{12}$ and so on. It can be verified that for generic $k$ and $\lambda_j$ the matrix $S$ is determined from (15) uniquely, unlike the special case of the paper [3]. I will write

$$S = S_{12,13,23} = S_{12,13,23}(\lambda_1, \lambda_2, \lambda_3; k).$$

Consider then 3 more $S$-matrices $S_{12,14,24}, S_{13,14,34}, S_{23,24,34},$ whose definition is obvious (so that, e.g., $S_{12,14,24}$ acts in $\mathcal{V}_{12} \otimes \mathcal{V}_{14} \otimes \mathcal{V}_{24}$ and depends on $\lambda_1, \lambda_2, \lambda_4; k$). Does the tetrahedron equation

$$S_{12,13,23}S_{12,14,24}S_{13,14,34}S_{23,24,34} = S_{23,24,34}S_{13,14,34}S_{12,14,24}S_{12,13,23} \quad (16)$$

hold?

The answer is positive, but the calculations for generic $k$ are rather difficult and will be presented elsewhere. In this paper, I will restrict myself to the case $k \to 0$, in which I have actually obtained the matrix $S$ directly from Eq. (15) (one cannot just take $k = 0$ because of the linear dependence of the corresponding
For the elements of the $S$-matrices to have the simplest form, let us choose
\[ f_0(\lambda_i, \lambda_j) = \sin^{-1}(\lambda_i - \lambda_j), \]
\[ f_1(\lambda_i, \lambda_j) = \cos^{-1}(\lambda_i + \lambda_j) \]
and introduce new variables $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ by equalities
\[ \tanh \varphi_j = \tan^2 \lambda_j. \]

The matrix elements of $S = S_{12,13,23}$ are (for other matrices, make the obvious change of indices):
\[
\begin{align*}
S_{000}^{000} &= S_{011}^{011} = S_{101}^{101} = S_{110}^{110} = 1, \\
S_{010}^{001} &= \cotanh(\varphi_1 - \varphi_3) \tanh(\varphi_2 - \varphi_3), \\
S_{100}^{001} &= \tanh(\varphi_2 - \varphi_3), \\
S_{111}^{001} &= \cotanh(\varphi_1 - \varphi_3), \\
S_{001}^{010} &= S_{100}^{010} = S_{111}^{010} = 1, \\
S_{100}^{100} &= -\tanh(\varphi_1 - \varphi_2), \\
S_{000}^{100} &= \tanh(\varphi_1 - \varphi_2) \cotanh(\varphi_1 - \varphi_3), \\
S_{111}^{100} &= -\cotanh(\varphi_1 - \varphi_3), \\
S_{001}^{111} &= \tanh(\varphi_1 - \varphi_2), \\
S_{010}^{111} &= -\tanh(\varphi_1 - \varphi_2) \tanh(\varphi_2 - \varphi_3), \\
S_{100}^{111} &= -\tanh(\varphi_2 - \varphi_3),
\end{align*}
\]
all the other matrix elements are zeros.

The way I have proved that these $S$-matrices really satisfy the tetrahedron equations (16) was as follows. It can be seen that in the variables
\[ x_j = \tanh \varphi_j, \quad j = 1, \ldots, 4, \]
all the matrix elements of the $S$-matrices become rational functions with integer coefficients. After having been multiplied by its common denominator, Eq.(16) becomes an equality between two matrix polynomials in $x_1, \ldots, x_4$ with integer coefficients. Thus, it can be exactly verified by a computer, and that is what actually has been done. More “scientific” way of proving Eq. (16) will be presented elsewhere.

To conclude this section, note that $S$ depends only on the differences of the arguments $\varphi_j$. Less obvious observation is that $S$ is a reflection: $S^2 = 1$, with 2 eigenvalues equaling $-1$. 

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4 Discussion

It is shown in this paper that the tetrahedral Zamolodchikov algebras do lead to new solutions of the tetrahedron equation with variables on the links, at least in the particular case of Section 3. It would be very interesting to calculate the $S$-matrices for general Felderhof $L$-operators and to verify whether they satisfy the tetrahedron equation.

A certain disadvantage in the solutions of the tetrahedron equation known so far is that if their matrix elements are chosen to be real, some of them become incurably negative. This means that they cannot be directly used as Boltzmann weights of a statistical mechanical model on the cubic lattice. However, the experience obtained from studying the triangle equation suggests that the tetrahedron equation may as well have the most unexpected applications in various fields of mathematics.

REFERENCES

1. Zamolodchikov A.B. Tetrahedron equations and the relativistic $S$-matrix of straight-strings in 2+1-dimensions // Commun. Math. Physics. - 1981.V.79.- P. 489-505.
2. Baxter R.J. On Zamolodchikov’s solution of the tetrahedron equations // Commun. Math. Physics. - 1983. V.88, No.2. - P. 185-205.
3. Korepanov I.G. Tetrahedral Zamolodchikov algebra and the two-layer flat model in statistical mechanics // Modern Phys. Lett. B. - 1989. V.3, No 3. - P. 201-206.
4. Krichever I.M. Baxter equations and the algebraic geometry // Funk. analiz i pril. - 1981. V. 15, No. 2. - P. 22-35 (Russian).
5. Bazhanov V.V., Stroganov Yu.G. Free fermions on a three-dimensional lattice and tetrahedron equations // Nucl. Phys. B. - 1984. V. B230[FS10], No.4. - P.435-454.
6. Felderhof B.U. Diagonalization of the transfer matrix of the fermion model //Physica. - 1973. V. 66, No 2. - P. 279-298.
7. Tyurin A.N. Classification of the vector bundles over algebraic curves // Izvestia AN SSSR, ser. matem. - 1965. V. 29. - P. 658-680 (Russian).
8. Korepanov I.G. New solutions to the tetrahedron equation / Chelyabinsk, 1989. - 8p. - Dep. in the VINITI No. 1751-V89 (Russian).
9. Korepanov I.G. Applications of the algebro-geometrical constructions to the triangle and tetrahedron equations. Ph.D.Thesis /Leningrad: LOMI, 1989. - 87 p. (Russian).
10. Bazhanov V.V., Baxter R.J. New solvable lattice models in three dimensions. Preprint, 1992. Submitted to J. Stat. Phys.
11. Bazhanov V.V., Baxter R.J. Star-triangle relation for a three dimensional model. Preprint, 1992. Submitted to J. Stat. Phys.
12. Kashaev R.M., Mangazeev V.V., Stroganov Yu.G. Spatial symmetry, local integrability and tetrahedron equations in the Baxter-Bazhanov Model. IHEP
