Effective action of gauged WZW model and exact string solutions

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We suggest how to derive the exact (all order in $\alpha'$) expressions for the background fields for string solutions corresponding to gauged WZW models directly at the 2$d$ field theory level. One is first to replace the classical gauged WZW action by the quantum effective one and then to integrate out the gauge field. We find the explicit expression for the gauge invariant non-local effective action of the gauged WZW model. The two terms (corresponding to the group and subgroup) which appear with the same coefficients in the classical action get different $k$-dependent coefficients in the effective one. The procedure of integrating out the gauge field is considered in detail for the $SL(2,R)/U(1)$ model and the exact expressions for the $D = 2$ metric and the dilaton (originally found in the conformal field theory approach) are reproduced.

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1. Introduction

Solutions of (tree level) string field equations are usually represented in terms of conformal invariant 2d theories. In the context of particle theory applications of string theory one is interested in string vacuum backgrounds which are direct products of non-trivial euclidean “internal” part and flat 4d Minkowski space-time. At the same time, interpreting string theory as a theory of (quantum) gravity one needs to study solutions which have curved space-time part, i.e. which should be described by conformal theories with Minkowski signature of a target space. In the context of gravitational applications of string theory it is important to be able to go beyond the leading orders of expansion in $\alpha'$ (the leading order solutions may not correctly describe the behaviour in the strong curvature regions near singularities, etc). Since the string effective equations contain terms of arbitrarily high order in $\alpha'$ their solutions will in general be non-trivial functions of $\alpha'$.

Given that the exact form of the string effective action is not known it is difficult to expect that such solutions can be found directly in the perturbative sigma model framework (there are of course special solutions for which higher order corrections cancel automatically as in the cases of group spaces \cite{1} or ‘plane wave’ backgrounds \cite{2}).

A number of new solutions of string equations can in principle be constructed by starting with gauged WZW models \cite{3} and also by using duality transformations of the known solutions (see e.g. \cite{5} for a review). However, both the standard procedure of extracting space-time backgrounds by integrating out gauge fields \cite{4} and (the leading

\footnote{The gauged WZW model is conformal only under a special prescription in which the group field $g$ and the gauge field $A$ are treated on an equal footing. Integrating out $A$ while keeping $g$ classical gives a model which is conformal only in the one-loop approximation. Though the formal transformations of the path integral may look preserving equivalence, to justify them one needs to introduce sources/backgrounds (as one is to do in order to compute e.g. $\beta$-functions) and also to account for the effects of Jacobians resulting from integrations.}
order form of) duality transformations preserve conformal invariance only in the leading 
α’ approximation. 2

It should be possible to determine the exact form of the background fields in the case 
when one knows explicitly the conformal theory corresponding to a particular solution. To 
give a space-time interpretation to a conformal theory (i.e. to find the couplings of the 
corresponding sigma model) is, in general, a non-trivial problem. One possible strategy is 
to consider perturbations of a given conformal theory by marginal operators considered as 
“probes” of geometry, interpreting the linear $L_0$-constraints as “Klein-Gordon” equations 
and trying to extract the values of the background fields from the explicit form of the 
differential operator $L_0$ (see e.g. [8]). This approach first applied to derivation of exact 
background for a particular $(SL(2,R)/U(1))$ model in [9] was developed in full generality 
(and with important understanding of a possibility to determine global geometry) in [10]. 
Though being consistent with perturbative solution of sigma model conformal invariance 
conditions (as was checked in the $SL(2,R)/U(1)$ case [9] up to three [11] and four [12] 
loops) the “$L_0$ - approach” is, however, rather heuristic being based on a number of 
implicit assumptions about the structure of the string effective action (or about how $L_0$ 
should depend on the dilaton, how the antisymmetric tensor should enter equations for 
higher level modes, etc).

Below we shall suggest the explicit method of determination of the background ge-
ometry directly at the $2d$ field theory level. The main idea is first to replace the classical 
gauged WZW action by a quantum effective one (with all α’ or $1/k$ corrections already 
accounted for) and then to eliminate the gauge field as in [4]. The resulting sigma model 
should have the exact space-time fields as its couplings.

2 The question of higher order corrections to sigma model duality transformations was 
addressed in [8]. Given that the exact form of these transformations is not known it is important 
to develop other methods of establishing the exact form of solutions. Since the leading order 
backgrounds corresponding to gauged WZW models (with abelian subgroup being gauged) can 
be represented as duality rotations of ungauged WZW models [7] understanding of how to find 
the exact backgrounds may shed light on exact form of duality transformations.
The reason why one should be able to find the exact background in the gauged WZW model directly in the $2d$ field theory framework is that being soluble in the conformal operator approach the model should be exactly solvable also in the Lagrangian approach, i.e. its effective action should be explicitly computable as a non-trivial function of the parameter $k$. As we shall see in Sec.2 the effective action in the ungauged WZW theory is just equal to the classical one with $k$ replaced by $k + \frac{1}{2}c_G$ ($c_G$ is the value of the second Casimir operator in the adjoint representation of $G$) so that the corresponding exact metric is just equal to the classical group space metric multiplied by $k + \frac{1}{2}c_G)^3$. At the same time, the classical action of the gauged WZW model (Sec.3) contains two different structures the coefficients of which get “renormalised” in different ways \[13\]. As a result, the effective action and hence the corresponding background fields are non-trivial functions of $k$ (or $\alpha'$).\footnote{The presence of this extra factor in the metric is consistent with the “hamiltonian” argument based on interpreting the $L_0$ - operator as a Klein-Gordon operator in a background.}

In Sec.4 we shall consider the $SL(2, R)/U(1)$ gauged WZW model and show that eliminating the gauge field from the corresponding effective action one reproduces the exact expressions for the background metric and dilaton found indirectly within the conformal field theory ($L_0$-operator) approach in \[9\]. It is interesting to note that if one modifies the coefficient of the $A^2$ term in the classical action of the $SL(2, R)/U(1)$ model $A^2 \to (1 + \frac{4}{k-2})A^2$ and then eliminates $A$ from the action one finds the exact metric of \[9\]. However, this naive ansatz for the “quantum” action is not gauge invariant and does not reproduce the exact form of the dilaton background \[9\]. The correct effective action found according to the general procedure described in Sec.3 corresponds to modifying (by $\frac{4}{k-2}$) of only the coefficient of the square of the transverse part of $A$.

Sec.5 contains some concluding remarks. In Appendix we discuss some actions which are related to effective action of the $SL(2, R)/U(1)$ model by duality transformations.\footnote{The backgrounds found in \[9\] provide first known examples of string solutions that are non-trivial functions of $\alpha'$. The $\alpha'$ - corrections, though absent in the supersymmetric case \[12\] \[10\], are still present in the heterotic one \[10\] \[14\].}
2. Effective action in WZW model

Our aim will be to show that the effective action of WZW theory \([1][15]\) is equal to the classical action with the level \(k\) replaced by \(k + \frac{1}{2}c_G\). Though such shift of \(k\) is a well-known quantum effect in the current algebra relations \([16]\) its origin within the Lagrangian field theory approach does not seem to be widely understood. Analogous shift of \(k\) in the effective action of related 3d Chern-Simons theory \([17]\) was discussed, e.g., in \([18][19]\). According to \([19]\) the shift of \(k\) in CS theory is essentially a one-loop phenomenon which takes place when one passes from the operator Wilson effective action to the standard \(c\)-number one.

An interesting analysis of WZW theory from the perturbative field theory point of view was recently presented in \([20]\). It was shown that the generating functional \(W(B)\) for correlators of chiral currents is given by its classical expression, i.e. does not receive quantum corrections. The correlator of two currents for zero external field is equal to the classical expression proportional to \(k\). For non-zero external field it contains both connected \(\left(\frac{\delta^2 W}{\delta B(z) \delta B(z')}\right)\) and disconnected \(-\frac{\delta W}{\delta B(z)} \frac{\delta W}{\delta B(z')}\) parts. Though \(W\) is classical, in the limit \(z \to z'\) the connected part (proportional to the Planck constant) takes effectively the form of a one-loop contribution. One term in the latter renormalizes the disconnected contribution (which is proportional to the energy-momentum tensor). This is the origin of the shift \(k \to k + \frac{1}{2}c_G\) in the corresponding OPE \(J(z)J(z') \sim (k + \frac{1}{2}c_G)T(z)\).

The results of \([20]\) suggest that the shift of \(k\) which from one point of view is due to a subtle relation between some connected and disconnected graphs may be reinterpreted as originating from a one-loop determinant. In fact, as we shall argue below the shift of \(k\) in WZW theory can be understood as appearing from a determinant \([15]\) identified as being a Jacobian of a change of variables which is necessary to perform in order to define the quantum effective action by a Legendre transformation.
Consider the following path integral\footnote{Though we shall use complex euclidean coordinate notation (\( \partial_a \partial^a = 4 \partial \bar{\partial} \), etc) we understand that in order to satisfy the natural reality conditions (especially in the context of gauged WZW theory) one should actually consider the case of Minkowski signature \cite{21}.}

\[
e^{-W(B)} = \int [dg] \ e^{-S(g) + BJ(g)} , \quad S = kI(g) , \quad (2.1)
\]

\[
I \equiv \frac{1}{2\pi} \int d^2 z \ \text{Tr} \ (\partial g^{-1} \bar{\partial} g) + \frac{i}{12\pi} \int d^3 z \ \text{Tr} \ (g^{-1} dg)^3 \ , \quad (2.2)
\]

\[
BJ(g) = \frac{k}{\pi} \int d^2 z \ \text{Tr} \ (B \bar{\partial} g g^{-1}) \ . \quad (2.3)
\]

Using the identity \cite{15}

\[
I(ab) = I(a) + I(b) - \frac{1}{\pi} \int d^2 z \ \text{Tr} \ (a^{-1} \partial a \ \bar{\partial} bb^{-1}) \ , \quad (2.4)
\]

it is easy to see that the generating functional for the correlators of the current \( \bar{J} \) is given by \cite{15, 22}

\[
W(B) = -kI(u) , \quad B = u^{-1} \partial u \ . \quad (2.5)
\]

Equivalently, that means that \( W \) does not receive quantum corrections, i.e. it is equal to the classical action evaluated on the classical solution depending on \( B \) \cite{20}.

Given that \( B \) plays the role of an external current associated with the composite field \( \bar{J} \) (i.e. \( u \) is a “source” corresponding to \( g \)) to get the effective action for \( g \) itself we still need to make the Legendre transformation of \( W(B) \). Instead of just solving the classical equations for \( B \) let us consider the path integral over it and define the functional \( \Gamma(g) \)

\[
e^{-\Gamma(g)} = \int [dB] \ e^{-W(B) + BJ(g)} . \quad (2.6)
\]

Since the quantum corrections in the corresponding path integral effectively cancel out \cite{20} the functional \( \Gamma \) will represent the required Legendre transform of \( W \). As follows from (2.1)

\[
e^{-\Gamma(g)} = \int [dg'] \ e^{-S(g')} \ \delta[\bar{J}(g') - \bar{J}(g)] , \quad (2.7)
\]
so that \( \Gamma \) has a natural “effective action” interpretation. To compute (2.6) we change the variable \( B = u^{-1} \partial u \) to \( u \). The resulting Jacobian is a determinant of a chiral operator. Since the original model is non-chiral the determinant should be defined in a left-right symmetric way (i.e. in a way preserving vector gauge symmetry) as the square root of the product \[15\][22]\[6\]

\[
\det (\partial + [B, \cdot]) \det (\bar{\partial} + [\bar{B}, \cdot]) = \exp [c_G I(v^{-1}u)] \det \partial \det \bar{\partial} ,
\]

(2.8)

\[
B = u^{-1} \partial u , \quad \bar{B} = v^{-1} \bar{\partial} v .
\]

In the present case of \( v = 1 \) we get

\[
[dB] = [du] \det (\partial + [B, \cdot]) = [du] \exp \left[ \frac{1}{2} c_G I(u) \right] (\det \partial \bar{\partial})^{1/2} .
\]

(2.9)

Then

\[
e^{-\Gamma(g)} = N \int [du] \exp \left[ (k + \frac{1}{2} c_G) I(u) + B(u) \bar{J}(g) \right] .
\]

(2.10)

As a result, we finish with (cf.(2.1),(2.5); \( k \rightarrow -(k + \frac{1}{2} c_G) \))

\[
\Gamma(g) = (k + \frac{1}{2} c_G) I(g) .
\]

(2.11)

The shift of \( k \) in the effective action is perfectly consistent with the presence of the shifted \( k \) in the quantum currents or in the quantum equations of motion in [16] \( (k + \frac{1}{2} c_G) \bar{\partial} g = \pi: \bar{J} g : , \) etc). Given that the energy-momentum tensor \( T \) should be represented by the

\[\text{The chiral determinant is not unambiguously defined. It was also suggested [23] that in the chiral case an additional wave function renormalisation should accompany the change } B \rightarrow u \]

\[
\int [dB] \exp [-W(B) + B \bar{J}(g)] = N \int [du] \exp [(k + c_G) I(u) + pB(u) \bar{J}(g)] ,
\]

where \( p = \sqrt{(k + c_G)/k} \).

\[\text{We suppress the wave-function renormalisation corresponding to } k \bar{\partial} g^{-1} \rightarrow (k + \frac{1}{2} c_G) \bar{\partial} g^{-1} . \]

Similar effective actions were discussed in the context of \( sl(2, R) \) approach to 2d quantum gravity in [24].
variation of the effective action over the background 2d metric eq. (2.11) is also consistent with the standard operator expression $T(z) = \frac{1}{k + \frac{1}{2}c_G} : J^2 :$.

The expression (2.11) for the effective action implies that the exact target space metric of the corresponding conformal sigma model is given by the group space metric multiplied by $k + \frac{1}{2}c_G$. This result is the same as one finds by interpreting the $L_0$ operator of the WZW conformal theory as a Klein-Gordon operator of a space-time theory \[^8\][^9][^10]. Though a particular value of the overall constant factor in the action (or in the target space fields) does not seem important, the observation that the effective action of the WZW model is multiplied by the shifted $k$ will have a non-trivial implication for the structure of the effective action in the gauged WZW model.

3. Effective action in gauged WZW model

Let us start with a review of the path integral quantisation of gauged WZW model following [21] (see also [25]). The action

$$I = \frac{1}{2\pi} \int d^2z \, \text{Tr} (\partial g^{-1} \bar{\partial} g) + \frac{i}{12\pi} \int d^3z \, \text{Tr} (g^{-1} dg)^3 , \quad (3.1)$$

is invariant under the global $G_L \times G_R$ transformations. The action invariant under the holomorphic vector gauge transformations $g \rightarrow u^{-1}(z)gu(\bar{z})$ (where $u$ belongs to a vector subgroup $H$) is

$$I_0(g, A) = I(g) - \frac{1}{\pi} \int d^2z \, \text{Tr} (A \bar{\partial} g g^{-1} - \bar{A} g^{-1} \partial g - g^{-1} Ag\bar{A}) . \quad (3.2)$$

To cancel the anomaly in the effective action one is to add the counterterm $\text{Tr} (A\bar{A})$ (see e.g. [26]). The final gauged action\[^8\]

$$I(g, A) = I(g) - \frac{1}{\pi} \int d^2z \, \text{Tr} (A \bar{\partial} g g^{-1} - \bar{A} g^{-1} \partial g - g^{-1} Ag\bar{A} + A\bar{A}) \quad (3.3)$$

\[^8\] In the case when one gauges the axial $U(1)$ subgroup the $A\bar{A}$ term should be added with the opposite sign, or, equivalently (after changing $A \rightarrow -\bar{A}$) the minus signs in the brackets in the second term in (3.3) should be replaced by the plus signs. Let us note that there exists a more general left–right asymmetric way of gauging found in the second reference in [27].
is invariant under the standard vector gauge transformations:

\[ g \rightarrow u^{-1}gu, \quad A \rightarrow u^{-1}(A + \partial)u, \quad \bar{A} \rightarrow u^{-1}(\bar{A} + \bar{\partial})u, \quad u = u(z, \bar{z}). \]

The invariance of (3.2) under the holomorphic gauge transformations indicates that the \( A\bar{A} \) term plays a special role in (3.3), i.e. that (3.3) should be considered as a sum of two different structures: \( I_0 (3.2) \) and \( \frac{1}{\pi} \int d^2z \text{ Tr } (A\bar{A}) \). In fact, as we shall see below, the coefficients of \( I_0 \) and \( A\bar{A} \) will become different in the quantum effective action corresponding to (3.3).

Parametrising \( A \) and \( \bar{A} \) in terms of \( h \) and \( \bar{h} \) which take values in \( H \) and transform as

\[ h \rightarrow u^{-1}h, \quad \bar{h} \rightarrow u^{-1}\bar{h}, \]

and using the identity (2.4) we can represent (3.3) as the difference of the two explicitly gauge invariant terms (given by WZW actions corresponding to \( G \) and \( H \)):

\[ I(g,A) = I(h^{-1}g\bar{h}) - I(h^{-1}\bar{h}). \quad (3.5) \]

Starting with the path integral (with the Haar measure for \( g \) and the canonical measure for \( A, \bar{A} \) including gauge fixing)

\[ Z = \int [dg][dA][d\bar{A}] \exp [-kI(g,A)] \quad (3.6) \]

and changing the variables using (2.8), i.e.

\[ \det (\partial + [A, ]\) det (\bar{\partial} + [\bar{A}, ]) = \exp [c_H I(h^{-1}\bar{h})] \det \partial \det \bar{\partial}, \quad (3.7) \]

we get

\[ Z = \int [dg][dh][d\bar{h}] \exp [-kI(h^{-1}g\bar{h}) + (k + c_H)I(h^{-1}\bar{h})]. \quad (3.8) \]

The last expression implies that \( I(g,A) \) can be quantised as the sum of the two WZW theories for the groups \( G \) and \( H \) with levels \( k_G = k \) and \( k_H = -(k + c_H) \). According to
our discussion in Sec.2 to get the effective action of WZW theory one is to replace $k$ by $k + \frac{1}{2} c_G$ (see (2.11)). This suggests that the effective action in the gauged WZW theory is explicitly computable and is given by

$$\Gamma(g, A) = (k + \frac{1}{2} c_G) I(h^{-1}g\bar{h}) - (k + \frac{1}{2} c_H) I(h^{-1}\bar{h}) \ .$$

(3.9)

The contribution of the Jacobian (3.7) is essential in order to make (3.9) a natural generalisation of both the classical gauged WZW action (3.5) and the effective action of the WZW theory (2.11) (note that $-(k + c_H) + \frac{1}{2} c_H = -(k + \frac{1}{2} c_H)$). The structure of (3.9) is in correspondence with that of the holomorphic energy-momentum tensor operator in the conformal field theory approach \[28\]

$$T(z) = \frac{1}{k + \frac{1}{2} c_G} : J_G^2 : - \frac{1}{k + \frac{1}{2} c_H} : J_H^2 : \ .$$

It should be noted that in (3.9) we have ignored field renormalisations.\[9\] The latter, however, are not relevant for our aim of deriving the exact backgrounds corresponding to the 2d action in which $h, \bar{h}$ are already eliminated.\[10\]

One can now re-write (3.9) in terms of the original variables $A, \bar{A}$ (note that since $\Gamma$ is the effective action, its arguments should be treated as classical fields). Using (3.5) we can represent the effective action (3.9) in the form

$$\Gamma(g, A) = (k + \frac{1}{2} c_G)[ I(g, A) + \frac{(c_G - c_H)}{2(k + \frac{1}{2} c_G)} \Omega(A) ] \ ,$$

(3.10)

where

$$\Omega(A) \equiv I(h^{-1}\bar{h}) = \omega(A) + \bar{\omega}(\bar{A}) + \frac{1}{\pi} \int d^2 z \ Tr (A\bar{A}) \ ,$$

(3.11)

$$\omega(A) \equiv I(h^{-1}) \ , \ \bar{\omega}(\bar{A}) \equiv I(\bar{h}) \ .$$

\[9\] The conjecture about the structure (3.9) of the effective action emerged as a result of collaboration with I. Bars and K. Sfetsos [13].

\[10\] The true reason why the effects of field renormalisations can be ignored here is that they introduce extra non-local terms in the action which can be ignored in the process of identifying the couplings in the classical sigma model action, see Note Added.
Ω is a non-local *gauge invariant* functional of $A$, $\bar{A}$ which represents the gauge-invariant part of $\int d^2z \ Tr (A\bar{A})$ (note that it is the coefficient of the gauge invariant part of the $A\bar{A}$ term in the classical action (3.3) that got modification different from that of the rest of the coefficients in the action).

Let us look at the structure of the two terms in (3.5) in more detail. Using the identity (2.4) and the definition (3.2) of $I_0$ we get

$$I(h^{-1}g\bar{h}) = I_0(g, A) + I(h^{-1}) + I(\bar{h}) \ , \quad (3.12)$$

$$I(h^{-1}\bar{h}) = \frac{1}{\pi} \int d^2z \ Tr (A\bar{A}) + I(h^{-1}) + I(\bar{h}) \ . \quad (3.13)$$

While the non-local terms $I(h^{-1}) + I(\bar{h})$ cancel out in the classical action (3.5) they survive in the effective one (3.9),(3.10).

In the case when $H$ is abelian the functional $\Omega$ (3.11) has the following explicit form

$$\Omega = \frac{1}{2\pi} \int \ Tr \ F \frac{\partial}{\partial \bar{\partial}} F \ , \quad F = \bar{\partial} A - \partial \bar{A} \ , \quad (3.14)$$

or, equivalently,

$$\Omega = \frac{1}{\pi} \int d^2z \ Tr (A\bar{A} - \frac{1}{2} A\frac{\partial}{\partial \bar{\partial}} A - \frac{1}{2} \bar{A} \frac{\partial}{\partial \partial} \bar{A}) \ . \quad (3.15)$$

Since the integrand of $\Omega$ (3.14),(3.15) is just the square of the transverse part of $A_a$ (i.e. $\Omega$ is proportional to the effective action of the Schwinger model) it can be rewritten also as the local $A^2_a$ term minus the square of the gauge-dependent longitudinal part of $A_a$

$$\Omega = \frac{1}{\pi} \int d^2z \ Tr (2A\bar{A} + \frac{1}{2} F' \frac{\partial}{\partial \bar{\partial}} F') \ , \quad F' \equiv \bar{\partial} A + \partial \bar{A} \ . \quad (3.16)$$

Note that the coefficient of the $A\bar{A}$ term in (3.16) has doubled. The *local* part of the effective action (3.10) in the abelian case ($c_H = 0$) then takes the form

$$\Gamma_{loc}(g, A) = (k + \frac{1}{2} c_G) \left[ I(g, A) + \frac{1}{\pi} \frac{c_G}{(k + \frac{1}{2} c_G)} \int d^2z \ Tr (A\bar{A}) \right]$$

---

11 As it is clear from (3.7) $\omega$ and $\bar{\omega}$ are effective actions corresponding to chiral (fermion) determinants.
\[
(I_0(g, A) - \frac{1}{\pi} (1 - \frac{c_G}{k + \frac{1}{2}c_G}) \int d^2z \, \text{Tr} \, (A \bar{A}) ) \ .
\] (3.17)

In general, it is possible to choose a gauge in which \( F' = 0 \) so that \( \Gamma \) in (3.10) is equal to (3.17). That means that (normalised) correlators of gauge-invariant operators computed using the full \( \Gamma \) (3.10) and its local part \( \Gamma_{loc} \) will be the same.

Having found the effective action \( \Gamma(g, A) \) one may follow \cite{4}, solving for the gauge field and eliminating it from the action (i.e. integrating it out in semiclassical approximation). This can be done in the two equivalent ways depending on whether one fixes a gauge on \( g \) or on \( A_a \). If we start directly with (3.10) and eliminate \( A_a \) we will get an effective action \( \Gamma(x) \) which will still be gauge invariant, i.e. will depend on gauge-invariant combinations \( x(g) \) of the fields \( g \). Elimination of the gauge field is more straightforward if one first fixes the gauge on \( A_a \) in which \( \Gamma \) (3.10) reduces to its local part (3.17). Since \( \Gamma_{loc} \) is local and quadratic in \( A_a \) the result will be a local action \( \Gamma'(g) \), which, however, will no longer be gauge invariant, i.e. it will depend on all components of \( g \). Integrating out the gauge degrees of freedom will lead us back to \( \Gamma(x) \).

The resulting model should be conformally invariant to all loop orders. A non-trivial question is whether its action \( \Gamma(x) \) will be local. Since the effective action (3.10) is non-local in the \( A_a \)-sector, it remains non-local in the gauge \( g = x \) so that it is not \textit{a priori} clear why integrating out \( A_a \) will give us a local action. If we start with (3.17) we get a local action \( \Gamma'(g) \) which may, however, reduce to a non-local once one decides to eliminate the gauge degrees of freedom.\cite{12}

In the next section we shall illustrate the procedure of integrating out the gauge field on the example of the \( SL(2, R)/U(1) \) gauged WZW model \cite{8,9}. We shall find that the target space metric corresponding to the local part of \( \Gamma(x) \) (or, what turns out to be equivalent, to \( \Gamma'(g) \) restricted to the gauge-invariant sector) reproduce the exact expression for the metric found in \cite{9}. To reproduce the exact expression for the dilaton one needs to start with the original non-local effective action (3.10),(3.14).

\footnote{One should, in fact, identify the effective action of the gauged WZW theory (after the gauge field is eliminated) with the \textit{effective} action of the corresponding sigma model. Then in order to determine the couplings of the sigma model which appear in its classical action one can ignore all the non-local terms that are present in the effective action, see Note Added.}
4. SL(2, R)/U(1) gauged WZW model: field-theoretic derivation of exact expressions for target space fields

4.1. Effective action and exact metric

Below we shall demonstrate how the above expressions (3.10),(3.17) for the effective action allow one to obtain the exact form of the corresponding target space backgrounds in the SL(2, R)/U(1) gauged WZW model. Let us follow [4][9] and consider the case when the axial U(1) subgroup of SL(2, R) is gauged. Using the parametrisation of SL(2, R) [9]

\[ g = e^{\frac{i}{2} \theta_L \sigma_2} e^{\frac{i}{2} r \sigma_1} e^{\frac{i}{2} \theta_R \sigma_2}, \quad \theta \equiv \frac{1}{2}(\theta_L - \theta_R), \quad \tilde{\theta} \equiv \frac{1}{2}(\theta_L + \theta_R), \] (4.1)

where \( r \) and \( \theta \) are invariant under the axial U(1) transformations (i.e. \( x = (r, \theta) \)) one can represent the classical action in the form (cf.(3.3))

\[ S(g, A) = k \left[ I(g) - \frac{1}{\pi} \int d^2z \text{Tr} \left( A \bar{g} g^{-1} + \bar{A} g^{-1} \partial g + g^{-1} A g \bar{A} + A \bar{A} \right) \right] \] (4.2)

\[ = \frac{k}{2\pi} \int d^2z \left[ \frac{1}{2} (\partial r \bar{r} - \partial \theta_L \bar{\partial} \theta_L - \partial \theta_R \bar{\partial} \theta_R) - 2 \cosh r \ \bar{\partial} \theta_L \partial \theta_R \right) \]

\[ + A(\bar{\partial} \theta_R + \cosh r \ \bar{\partial} \theta_L) + \bar{A}(\partial \theta_L + \cosh r \ \partial \theta_R) - A \bar{A}(\cosh r + 1) \] , (4.3)

or, equivalently, in terms of \( r, \theta, \tilde{\theta} \)

\[ S(r, \theta, \tilde{\theta}) = \frac{k}{2\pi} \int d^2z \left\{ \frac{1}{2} \partial r \bar{r} + a(r) \left[ \partial \theta \bar{\partial} \theta + (A - \bar{A}) \partial \theta - (\bar{A} - \bar{\partial} \theta) \bar{\partial} \theta \right] \]

\[ - b(r) (A - \bar{\partial} \theta)(\bar{A} - \bar{\partial} \theta) \right\} , \] (4.4)

\[ a \equiv \cosh r - 1 \quad \text{and} \quad b \equiv \cosh r + 1 . \] (4.5)

The gauge field was assumed to be proportional to \( \frac{1}{2} \sigma_2 \) (\( A \) and \( \bar{A} \) in (4.2) are the functions which remain after taking the trace). Solving for \( A_a \) and substituting it back in the classical action one finds that the gauge degree of freedom \( \tilde{\theta} \) drops out and so that one is left with the \( D = 2 \) model [4][29]

\[ S(r, \theta) = \frac{k}{4\pi} \int d^2z \left[ \partial r \bar{r} + 4 \tanh \frac{r}{2} \partial \theta \bar{\partial} \theta \right] . \] (4.6)
The effective action (3.10),(3.17) for the case when one gauges an abelian vector subgroup has an obvious analog in the case of the gauging of an abelian axial subgroup. The local part of the effective action (3.17) (corresponding to the gauge \( \partial a A_a = 0 \)) takes the form (in the present model \( c_G = -4, \ c_H = 0 \))

\[
\Gamma_{loc}(g, A) = (k - 2) \left[ I(g, A) - \frac{1}{\pi (k - 2)} \int d^2 z \ Tr (A\bar{A}) \right]
\]

\[
= (k - 2) \left[ I_0(g, A) - \frac{1}{\pi} (1 + \frac{4}{k - 2}) \int d^2 z \ Tr (A\bar{A}) \right]. \tag{4.7}
\]

Explicitly (cf.(4.3),(4.4))

\[
\Gamma_{loc}(g, A) = \frac{(k - 2)}{2\pi} \int d^2 z \left[ \frac{1}{2} (\partial r \bar{\partial} r - \partial \theta_L \bar{\partial} \theta_L - \partial \theta_R \bar{\partial} \theta_R - 2 \cosh r \bar{\partial} \theta_L \partial \theta_R) \right.
\]

\[
+ A(\bar{\partial} \theta_R + \cosh r \bar{\partial} \theta_L) + \bar{A}(\partial \theta_L + \cosh r \partial \theta_R) - A\bar{A}(\cosh r + 1 + \frac{4}{k - 2}) \right]
\]

\[
= \frac{(k - 2)}{2\pi} \int d^2 z \left\{ \frac{1}{2} \partial r \bar{\partial} r + a(r) \left[ \partial \theta \bar{\partial} \bar{\theta} + (A - \bar{A}) \bar{\partial} \theta - (\bar{A} - \bar{\partial} \bar{\theta}) \partial \theta \right] \right.
\]

\[
- b(r)(A - \bar{\partial} \bar{\theta})(\bar{A} - \bar{\partial} \bar{\theta}) - \frac{4}{k - 2} \bar{A} \bar{A} \right\}. \tag{4.8}
\]

Since the local part of the effective action is no longer invariant under the (non-holomorphic) axial transformations (because of the “quantum correction” \( \frac{4}{k - 2} \) in the coefficient of the \( A\bar{A} \) term) the field \( \bar{\theta} \) will no longer automatically decouple upon elimination of \( A \) from the action.

If we restrict consideration to “gauge invariant sector” \( \bar{\theta} = 0 \) then integrating out \( A \) we will get the following exact generalisation of the \( D = 2 \) semiclassical euclidean black hole action (4.6)

\[
\Gamma(r, \theta) = \frac{(k - 2)}{4\pi} \int d^2 z \left[ \partial r \bar{\partial} r + f(r) \partial \theta \bar{\partial} \bar{\theta} \right], \quad f(r) = \frac{4 \tanh^2 \frac{r}{\pi}}{1 - \frac{2}{k} \tanh^2 \frac{r}{\pi}}. \tag{4.10}
\]

In this way we reproduce the exact metric of ref.\[9\].\[14\]

---

13 If one integrates out \( A \) starting directly from the non-local effective action (3.10) one obtains an action which contains non-local terms in \( r \) but coincides with (4.10) for \( r = \text{const} \) (see below).

14 Note that \[9\] also used the restriction equivalent to \( \bar{\theta} = 0 \) in computing the \( L_0 \)-operator.
In general the action (4.9) can be rewritten in the form

$$\Gamma_{loc} = \frac{(k-2)}{2\pi} \int d^2z \left\{ \frac{1}{2} \partial r \partial r + a \partial r \partial \theta + a(B \partial r - \bar{B} \partial \theta) - b \bar{B} - \gamma(B + \partial \bar{\theta})(\bar{B} + \partial \bar{\theta}) \right\}, \quad (4.11)$$

$$B \equiv A - \partial \bar{\theta}, \quad \bar{B} \equiv \bar{A} - \bar{\partial} \bar{\theta}, \quad \gamma = \frac{4}{k-2}.$$

Eliminating $B$, $\bar{B}$ from the action we get

$$\Gamma(r, \theta, \bar{\theta}) = \frac{(k-2)}{4\pi} \int d^2z \left[ \partial r \partial r + f(r) \partial \theta \partial \bar{\theta} + E(r)(\partial \bar{\theta} \partial \bar{\theta} - \partial \theta \partial \bar{\theta}) - H(r) \partial \bar{\theta} \partial \bar{\theta} \right], \quad (4.12)$$

$$f = 2(a - \frac{a^2}{b + \gamma}) = \frac{4 \tanh^2 \frac{r}{2}}{1 - \frac{2}{k} \tanh^2 \frac{r}{2}}, \quad (4.13)$$

$$E = \frac{2 \gamma a}{b + \gamma} = \frac{2}{k} f, \quad H = \frac{2 \gamma b}{b + \gamma} = \frac{4}{k^2} f + \frac{8}{k} = \frac{8}{k(1 - \frac{2}{k} \tanh^2 \frac{r}{2})}. \quad (4.14)$$

The action (4.12) generalizes (4.10) to the case of $\tilde{\theta} \neq 0$ ($\bar{\theta}$ disappears from (4.12) in the limit $k \to \infty$ in which the action reduces to (4.6)). It is interesting to note that (4.12) coincides with the “charged black hole” action or semiclassical action of the $(SL(2, R) \times U(1))/U(1)$ model [30][31] (which is related by a duality transformation to the semiclassical action of the “black string” model $(SL(2, R)/U(1)) \times U(1)$ with a particular value of the ratio of the mass to the charge corresponding to the value $\lambda = \frac{2}{k-2}$ of the parameter in [31], see (A.6)). The action (4.12) can be represented also in the following form

$$\Gamma(r, \theta, \tilde{\theta}) = \frac{(k-2)}{4\pi} \int d^2z \left[ \partial r \partial r + G \partial(\theta - \tilde{\theta}')(\bar{\partial}(\theta + \tilde{\theta}) - 2k \partial \tilde{\theta} \partial \tilde{\theta}') \right], \quad (4.15)$$

$$\tilde{\theta}' \equiv \frac{2}{k} \bar{\theta},$$

or, equivalently,

$$\Gamma(r, \theta, \bar{\theta}) = \frac{(k-2)}{4\pi} \int d^2z \left[ \partial r \partial r - 2k \partial \bar{\theta} \partial \bar{\theta} + \frac{k^2}{4} H \partial(\theta - \bar{\theta}') \bar{\partial}(\theta + \bar{\theta}') \right]. \quad (4.16)$$

We see that for $r \neq \text{const}$ the field $\tilde{\theta}$ does not decouple from $\theta$ and hence cannot be easily integrated out: integration over $\tilde{\theta}$ gives extra non-local $O(\partial r)$ - dependent terms.
Let us now consider the integration over $A_a$ in the case when one starts directly with the original non-local effective action (3.10),(3.14), i.e. does not first impose the gauge condition on $A_a$ which reduces $\Gamma(g,A)$ to $\Gamma_{loc}(g,A)$. Introducing the fields $\rho_L$ and $\rho_R$ (corresponding to $h$ and $\bar{h}$ in (3.4))

$$A = \partial \rho_L , \quad \bar{A} = \bar{\partial} \rho_R , \quad \rho = \frac{1}{2}(\rho_L - \rho_R) , \quad \bar{\rho} = \frac{1}{2}(\rho_L + \rho_R) , \quad (4.17)$$

where $\rho$ and $\bar{\rho}$ represent the transverse and longitudinal parts of $A_a$, we get for the classical action (4.4)

$$S(r, \theta, \bar{\theta}, \rho, \bar{\rho}) = \frac{k}{2\pi} \int d^2 z \left\{ \frac{1}{2} \partial r \bar{\partial} r - a \partial \kappa \bar{\partial} \kappa - b \bar{\partial} \bar{\kappa} \partial \bar{\kappa} + a(\partial \bar{\kappa} \bar{\partial} \bar{\kappa} - \bar{\partial} \kappa \partial \bar{\kappa}) \right\} + 2 \partial \rho \bar{\partial} \rho , \quad (4.18)$$

$$\kappa \equiv \theta + \rho , \quad \bar{\kappa} \equiv \bar{\theta} - \bar{\rho} .$$

The gauge invariance manifests itself in that the action depends on only one combination $\bar{\kappa}$ of the longitudinal part of $A_a$ ($\bar{\rho}$) and $\bar{\theta}$. The two groups of terms in (4.18) correspond to the two terms in (3.5) (the terms in the square bracket represent the action of the $SL(2,R)$ WZW model). The effective action (3.10),(3.14) then takes the form

$$\Gamma(r, \theta, \bar{\theta}, \rho, \bar{\rho}) = \frac{(k-2)}{2\pi} \int d^2 z \left\{ \frac{1}{2} \partial r \bar{\partial} r - a \partial \kappa \bar{\partial} \kappa - b \bar{\partial} \bar{\kappa} \partial \bar{\kappa} \right\} + a(\partial \bar{\kappa} \bar{\partial} \bar{\kappa} - \bar{\partial} \kappa \partial \bar{\kappa}) + \frac{2k}{k-2} \partial \rho \bar{\partial} \rho \right\]$$

$$= \frac{(k-2)}{2\pi} \int d^2 z \left\{ \frac{1}{2} \partial r \bar{\partial} r - a \partial \kappa \bar{\partial} \kappa - b \bar{\partial} \bar{\kappa} \partial \bar{\kappa} \right\} + a(\partial \bar{\kappa} \bar{\partial} \bar{\kappa} - \bar{\partial} \kappa \partial \bar{\kappa}) + \frac{2k}{k-2}(\partial \kappa \bar{\partial} \bar{\kappa} - 2 \partial \bar{\kappa} \partial \theta + \partial \theta \partial \theta) \right\} ... \quad (4.19)$$

Note that it is the dependence on the transverse part $\rho$ of $A_a$ that is modified in the effective action. Eliminating $\kappa$ from the action we obtain (4.12) where $\bar{\theta}$ should be identified with $\bar{\kappa}$ (recall that in deriving (4.12) we have used the gauge $\bar{\rho}=0$). Integrating over $\kappa$ and $\bar{\kappa}$ we get (4.10) up to non-local terms depending on derivatives of $r$. 

16
It is possible to represent the result of elimination of $A_a$ in terms of a local action for the *three* fields: $r, \theta$ and an extra field $\chi$, i.e. in the form similar to (4.15),(4.16). Introducing an auxiliary scalar field $\chi$ one can replace the effective action (3.14)

$$\Gamma(r, \theta, A) = \frac{(k-2)}{2\pi} \int d^2 z \left[ \frac{1}{2} \partial r \partial r - b(r) A \bar{A} \right. $$

$$- \frac{1}{k-2} F \frac{1}{\partial \bar{\partial}} F + \frac{2k}{k-2} F \theta + \frac{2k}{k-2} \partial \bar{\partial} \theta \left. \right]$$

(4.20)

by the equivalent local functional

$$\Gamma(r, \theta, A, \chi) = \frac{(k-2)}{2\pi} \int d^2 z \left[ \frac{1}{2} \partial r \partial r - b(r) A \bar{A} \right. $$

$$- \frac{1}{k-2} \partial \chi \bar{\partial} \chi + \frac{2}{k-2} F(k \theta + \chi) + \frac{2k}{k-2} \partial \bar{\partial} \theta \left. \right]$$

(4.21)

We have fixed the gauge $\bar{\theta} = 0$ (in any case $\bar{\theta}$ drops out after the integration over $A_a$ because of the gauge invariance of $\Gamma$). Now the elimination of $A, \bar{A}$ from the action becomes straightforward and we are left with

$$\Gamma(r, \theta, \chi) = \frac{(k-2)}{2\pi} \int d^2 z \left[ \frac{1}{2} \partial r \partial r + \frac{2k}{k-2} \partial \bar{\partial} \theta - \frac{1}{k-2} \partial \chi \bar{\partial} \chi \right.$$

$$- \frac{4}{(k-2)^2 b(r)} \partial (\chi + k \theta) \bar{\partial} (\chi + k \theta) \left. \right]$$

(4.22)

Integration over $\chi$ leads us back to (4.10) up to non-local $O(\partial r)$ terms.

We conclude that the local part of the effective $D = 2$ action for $r$ and $\theta$ is given by (4.10).\textsuperscript{15} We thus reproduced the exact background of ref. [9] in the field-theoretic approach. The local $D = 2$ model (4.10) (with the dilaton coupling to be discussed below) is Weyl invariant (as was checked explicitly up to four leading orders in loop expansion\textsuperscript{[11],[12]})

\textsuperscript{15} The effective action is manifestly local only in terms of $D = 3$ representation (4.15),(4.16) or (4.21).
4.2. Dilaton field and measure factor

Though the “naive” form of the effective action ((4.9) with \( \bar{\theta} = 0 \)) is sufficient in order to obtain the expression for the metric (4.10), to derive the exact form of the dilaton coupling it is necessary to start with the full gauge invariant non-local effective action (3.10),(3.14) or (4.19),(4.20) defined on a curved 2d background. This provides an important check of the consistency of our approach.

As in the leading order approximation [4] the dilaton term originates from the determinant resulting from the integration over the gauge field. This determinant is not unambiguously defined (see [32]). The freedom of adding local counterterms should be fixed by using the condition that conformal invariance of the original theory should be preserved in the process of integrating out a subset of fields.\(^{16}\) The correct definition of the determinant (both at the semiclassical and exact levels) corresponds to treating \( A_a \) as being built out of the scalar fields \( \rho, \tilde{\rho} \) (4.17) with the latter considered as fundamental integration variables [33][6][32]\(^{17}\).

Let us first discuss the case which corresponds to integrating out the vector field in the classical action (4.3). Consider the following integral over the 2d vector field \( A_a \)

\[
Z = \int [dA_a] \exp \left[ -\frac{1}{2\pi} \int d^2z \sqrt{g} M(z) A_a A^a \right] ,
\]

where \( M \) is a given function. This integral is not well defined [32]. Using different definitions (regularisations) of (4.23) one will get expressions which will differ by local counterterms,

\[
Z = \exp[-(\text{Tr} \ln M)_{\text{reg}}]
\]

\(^{16}\) Put differently, conformal invariance of the original theory holds (without need to make additional field redefinitions) only within a specific regularisation scheme; it is that particular regularisation that should be used in computing the determinants resulting from integration over some fields.

\(^{17}\) Note that the effective action is local being expressed (4.19) in terms of \( \rho \) and \( \tilde{\rho} \).
\[ = \exp\left[-\frac{1}{8\pi} \int d^2z \sqrt{g} \left[ c_0 \Lambda^2 \ln M + c_1 (\partial_a \ln M)^2 + c_2 R \ln M \right] \right], \]  \tag{4.24}

where \( \Lambda \) is an UV cutoff, \( R \) is the curvature of 2d metric and \( c_i \) are finite coefficients. A choice of a particular definition of \( Z \) is dictated by some additional conditions which the total theory should satisfy. In the present case the preservation of the conformal invariance demands that \( c_1 = 0 \) and \( c_2 = -1 \) \footnote{There seems to exist some confusion in the literature concerning a relation between the dilaton term and the measure factor. The above remarks hopefully clarify the issue.}. The quadratically divergent term in (4.24) can be identified with a contribution to a local measure (regularized in a naive way) so that (4.24) can be represented in the form

\[ Z = \prod_z M^{-1}(z) \exp \left[ \frac{1}{8\pi} \int d^2z \sqrt{g} R \ln M(z) \right]. \tag{4.25}\]

It is important to stress that (4.25) is the product of two separate factors: a local measure factor and a (global) dilaton contribution,

\[ \phi = -\frac{1}{2} \ln M. \tag{4.26}\]

Note that since

\[ e^{2\phi(z)} = M^{-1}(z) \tag{4.27}\]

one may mis-interpret the measure factor as being related to the dilaton contribution (or vice versa). The two factors play, in fact, very different roles and should not be mixed up. For example, the dilaton factor is absent on a flat 2d background while the measure factor must be present irrespective of the value of the curvature of 2d metric (the measure factor is important for cancellation of quadratic divergences in the total theory while the dilaton contribution is essential for the absence of the Weyl anomalies)\footnote{There seems to exist some confusion in the literature concerning a relation between the dilaton term and the measure factor. The above remarks hopefully clarify the issue.}

In the case of the \( SL(2,R)/U(1) \) model treated in the semiclassical approximation it is the first factor in (4.25) that combines with the Haar measure on \( SL(2,R) \) to give the \( \sqrt{G} \) measure factor of the resulting \( D = 2 \) sigma model (4.6). In fact, after fixing the
gauge $\tilde{\theta} = 0$ and integrating out the gauge field the respective measure factors are given by

$$d\mu(r, \theta) = \prod_z dr(z) \ d\theta(z) \sinh r(z) \ \prod_z M^{-1}(z)$$

$$= N \ \prod_z dr(z) \ d\theta(z) \sqrt{G(z)} \ , \quad (4.28)$$

$$M = k(\cosh r + 1) \ , \quad G = 4k^2 \tanh^2 \frac{r}{2} \ , \quad N = \frac{1}{2} k^{-2} \ ,$$

so that the resulting sigma model partition function is $(x^\mu = (r, \theta))$

$$Z_{s.m.} = N \int \prod_z dx^\mu(z) \sqrt{G(z)} \exp\left[ -\frac{1}{4\pi} \int d^2z \sqrt{g} \ (G_{\mu\nu} \partial_\mu x^\nu \partial_\nu x^\rho + R \phi) \right] \ . \quad (4.29)$$

Let us now see how the same analysis goes through if we start with the effective action (4.19),(4.20). Since the squares of longitudinal and transverse parts of $A_a$ have different coefficients in the effective action, i.e.

$$MA_a^2 \rightarrow M_{||} A_0^2 + M_{\perp} A_\perp^2 \ ,$$

$$M_{||} = (k - 2)(\cosh r + 1) \ , \quad M_{\perp} = (k - 2)(\cosh r + 1 + \frac{4}{k - 2}) \ , \quad (4.30)$$

the result of integration over $A_a$ (i.e. over its transverse and longitudinal parts $\rho$ and $\tilde{\rho}$) is given by (4.25) with $M$ replaced by $\sqrt{M_{||}M_{\perp}}$, i.e.

$$Z = \prod_z M_{||}^{-\frac{1}{2}}(z) M_{\perp}^{-\frac{1}{2}}(z) \ \exp \left[ \frac{1}{16\pi} \int d^2z \sqrt{g} \ R \ \ln (M_{||}(z) M_{\perp}(z)) \right] \ . \quad (4.31)$$

Then the exact form of the measure factor (eq.(4.28)) is

$$d\mu(r, \theta) = \prod_z dr(z) \ d\theta(z) \sinh r(z) \ \prod_z M_{||}^{-\frac{1}{2}}(z) M_{\perp}^{-\frac{1}{2}}(z)$$

$$= N \ \prod_z dr(z) \ d\theta(z) \sqrt{G(z)} \ , \quad (4.32)$$

$$G = (k - 2)^2 \frac{4 \tanh^2 \frac{r}{2}}{1 - \frac{2}{k} \tanh^2 \frac{r}{2}} \ , \quad N = \frac{1}{2} k^{-1/2}(k - 2)^{-3/2} \ .$$
The dilaton contribution is given by
\[ \phi = \phi_0 - \frac{1}{4} \ln M_\parallel - \frac{1}{4} \ln M_\perp = \phi_0' - \frac{1}{2} \ln \sinh r + \frac{1}{4} \ln G , \]
(4.33)
i.e we reproduce the expression of [9].

Note that since
\[ e^{2\phi} = M_\parallel^{-\frac{1}{2}} M_\perp^{-\frac{1}{2}} \]
(4.34)
it follows from (4.32) that
\[ \sqrt{G} e^{-2\phi} = N^{-1} \sinh r . \]
(4.35)
This implies that in general \( \sqrt{G} e^{-2\phi} \) being proportional to the Haar measure (with proper gauge fixing) should be essentially \( k \)-independent (up to an overall constant factor) in agreement with previous suggestions [34][27] It is not clear if there is more than just a coincidence in the fact that \( \sqrt{G} e^{-2\phi} \) is, at the same time, the zero mode measure factor which appears in the sigma model partition function defined on a 2-sphere or in the corresponding tree level string effective action [35]
\[ Z \sim \int d^D x \sqrt{G} e^{-2\phi} . \]

5. Concluding remarks

As we have seen in Sec.4 the local part of the effective action of the \( SL(2, R)/U(1) \) model is given by the sigma model action (4.10) with the exact couplings of ref. [9]. The full effective action is local if expressed in terms of gauge invariant components (\( r, \theta \)) of \( g \) and an extra scalar field. The latter is also gauge invariant and can be identified either with \( \tilde{\kappa} \) in (4.18) (or, equivalently, \( \tilde{\theta} \) in (4.12)) or with an axiliary field \( \chi \) in (4.21). This field disappears from the action in the limit \( k \to \infty \), i.e. it is absent in the leading order approximation. Once we integrate over this extra field (i.e. integrate over all components

\^I am grateful to I. Bars and K. Sfetsos for an important discussion of the \( k \)-independence of the factor (4.35).
of the gauge field) we get the $D = 2$ action (4.10) for $x = (r, \theta)$ plus non-local $O(\partial r)$ terms (which vanish in the $k \to \infty$ limit).

The approach suggested in this paper should find applications in similar models. In particular, it is important to apply analogous method to derivation of exact backgrounds corresponding to $G/H$ gauged WZW models with non-abelian subgroups $H$.

It is straightforward to generalise the above analysis to the supersymmetric gauged WZW model. In the supersymmetric case there is no shift of $k$ in the effective action of WZW model (2.11) (because of an extra contribution of fermions). The effective action of the supersymmetric gauged WZW model is thus equal to the classical one (cf. (3.5), (3.9)) so that there are no $1/k$ corrections to the leading order form of the background fields in agreement with what one finds using the $L_0$ - argument.

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Note Added

The approach of this paper was further developed and clarified in our recent work. In particular, we pointed out that one should identify the effective action of the gauged WZW theory with the effective action of the corresponding sigma model. Then to find the sigma model couplings one needs to consider only the local part of the effective action. As a result, one can ignore non-local terms originating from the field renormalisations in (2.11) and (3.9). It is possible also to truncate the quantum term (3.11) in the effective action (3.10) to its quadratic part (3.15). As a result, one is able to derive the general expressions for the sigma model couplings (metric, dilaton and antisymmetric tensor) in the case of an arbitrary non-abelian subgroup $H$ (see also [36]).
6. Appendix

Below we shall consider several actions related to the actions from Sec.4 by duality transformations. In particular, we shall show that the effective action of the $SL(2, R)/U(1)$ model (4.12) is related through duality transformations to the action of the ungauged $SL(2, R)$ WZW model. Since the leading order form of the duality transformation which we shall use preserves conformal invariance only to one loop order the resulting actions are conformal invariant only in the one loop approximation and should not be expected to represent exact backgrounds. Even though making a duality transformation may simplify the structure of the actions (4.12),(4.22) the property of conformal invariance to all loop orders, in general, may be lost.

Let us consider the sigma model

$$I = \frac{1}{4\pi} \int d^2 z \sqrt{g} \left[ (G_{\mu\nu} + B_{\mu\nu})(g^{ab} + i\epsilon^{ab})\partial_a x^\mu \partial_b x^\nu + R\phi \right], \quad (A.1)$$

which is invariant under an abelian isometry $\delta x^\mu = \epsilon K^\mu$. Choosing coordinates $\{x^\mu\} = \{x^1 \equiv y, x^i\}$ in such a way that $G_{\mu\nu}, B_{\mu\nu}$ and $\phi$ are independent of the coordinate $y$ which is shifted by the isometry one finds the dual action $\tilde{I}$ for $\{\tilde{x}^\mu\} = \{\tilde{x}^1 \equiv \tilde{y}, x^i\}$ which has the form (A.1) with

$$\tilde{G}_{11} = M^{-1}, \quad \tilde{G}_{1i} = M^{-1}B_{1i}, \quad \tilde{G}_{ij} = G_{ij} - M^{-1}(G_{1i}G_{1j} - B_{1i}B_{1j}), \quad (A.2)$$

$$\tilde{B}_{1i} = M^{-1}G_{1i}, \quad \tilde{B}_{ij} = B_{ij} - M^{-1}(B_{1i}G_{1j} - G_{1i}B_{1j}), \quad M \equiv G_{11}, \quad (A.3)$$

Applying this transformation to the action of the ungauged $SL(2, R)$ WZW model (i.e. (4.4),(4.18) for $A = 0$) with $y = \tilde{\theta}$

$$S(r, \theta, \tilde{\theta}) = \frac{k}{2\pi} \int d^2 z \left[ \frac{1}{2} \partial r \partial r - a(r)\partial \theta \partial \tilde{\theta} - b(r) \partial \tilde{\theta} \partial \tilde{\theta} + a(r)(\partial \theta \partial \tilde{\theta} - \partial \tilde{\theta} \partial \theta) \right], \quad \phi = \phi_0, \quad (A.4)$$
we find

\[\hat{S}(r, \tau, \sigma) = \frac{k}{2\pi} \int d^2z \left[ \frac{1}{2} \partial r \partial r + 2\tanh^2 \frac{r}{2}\partial \tau \partial \tau - 2\partial \sigma \partial \sigma \right], \quad \phi = \phi_0' - \frac{1}{2} \ln (\cosh r + 1), \]

(A.5)

where \(r\) is the field dual to \(\tilde{\theta}\) and \(\sigma \equiv \theta - \tau\). This model corresponds to the “black string” background, i.e. is the direct product of the leading order \(D = 2\) black hole (4.6) and an extra free field. Though the \(SL(2,R)\) WZW model (A.4) we started with is conformal to all orders in \(\alpha'\) expansion the resulting model (A.5) is conformal only to the leading order in \(\alpha'\). As we have mentioned above, this is not surprising since (A.2),(A.3) represent only the leading order form of the duality transformation [6].

The action (A.5) has two commuting isometries. Acting on (A.5) by a particular \(O(2,2)\) transformation [31] we get the (euclidean) charged black hole metric

\[S = \frac{k}{2\pi} \int d^2z \left[ \frac{1}{2} \partial r \partial r + \frac{(1-\lambda)\sinh^2 \frac{r}{2}}{\cosh \frac{r}{2} - \lambda} \partial \tau_1 \partial \tau_1 \right.
\left. + \frac{\lambda \sinh \frac{r}{2}}{k(\cosh \frac{r}{2} - \lambda)}(\partial \tau_1 \partial \tau_2 - \partial \tau_2 \partial \tau_1) + \frac{\lambda \cosh \frac{r}{2}}{k^2(\cosh \frac{r}{2} - \lambda)} \partial \tau_2 \partial \tau_2 \right], \]

(A.6)

\[\phi = \phi_0'' - \frac{1}{2} \ln (\cosh r + 1 - 2\lambda), \]

(A.7)

Here \(\lambda\) is a free parameter related to the charge of black hole. If we now take \(\lambda = -\frac{2}{k-2}\) (A.6) becomes equivalent to the effective action (4.12). The dilaton factors are also equivalent as one can see by noting that the dilaton field corresponding to (4.14) originates from the integral over \(B, \bar{B}\) and thus is given by \(\phi = \phi_0 - \frac{1}{2} \ln [b(r) + \gamma]\) (see (4.26)). As a result, we discover that the \(\tau_2 = 0\) section of the metric corresponding to (A.6) coincides with the exact \(D = 2\) black hole metric (4.10) \((\frac{(1-\lambda)\sinh^2 \frac{r}{2}}{\cosh \frac{r}{2} - \lambda} = \frac{\tanh^2 \frac{r}{2}}{1-\frac{2}{k}\tanh^2 \frac{r}{2}}).\]

Note that the action (4.22) does not contain the antisymmetric tensor term. It can be generated by making a duality transformation of \(\chi\). One should be careful, however, not to mix \(\theta\) and \(\chi\) by field redefinitions since it is \(\chi\) that should be eventually integrated out to obtain a \(D = 2\) model.

20 The conformal field theories corresponding to the charged black hole and a compact black string are thus equivalent [31].

21 It is interesting to note that though we have used the leading order form of the duality transformation we got the exact \(D = 2\) black hole metric in the \(\tau_2 = 0\) section.
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