Scattering from an AdS$_3$ bubble and an exact AdS$_3$

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Abstract

We investigate the close relationship between the potential and absorption cross section for test fields in an AdS$_3$ bubble (a 5D black hole) and an exact AdS$_3$. There are two solutions in type IIB string theory: an AdS$_3$ bubble corresponds to the dilatonic solution, while an exact AdS$_3$ is the non-dilatonic solution. In order to obtain the cross section for an AdS$_3$ bubble, we introduce the \{out\}-state scattering picture with the AdS$_3$-AFS matching procedure. For an exact AdS$_3$, one considers the \{in\}-state scattering picture with the AdS$_3$-AdS$_3$ matching. Here the non-normalizable modes are crucially taken into account for the matching procedure. It turns out that the cross sections for the test fields in an AdS$_3$ bubble take the same forms as those in an exact AdS$_3$. This suggests that in the dilute gas and the low energy limits, the S-matrix for an AdS$_3$ bubble can be derived from an exact AdS$_3$ space.
I. INTRODUCTION

There has been a great progress in string theory of the D1-D5 brane system with momentum modes along the string direction ($S^1$). This gives us a 5D black hole ($M_5$) with three charges ($Q_1, Q_5, Q_n$). The first progress was achieved in the Bekenstein-Hawking entropy [1]. Apart from the success of counting the microstates of a 5D black hole through D-brane physics, a dynamical consideration becomes an important issue [2–5]. This is so because the semiclassical absorption cross section (greybody factor) for a test field arises as a consequence of its potential barrier surrounding the horizon. That is, this is an effect of spacetime curvature. More precisely, it is worth noting that a semiclassical absorption cross section can be derived from a solution to the differential equation of a test field (most often, $\nabla^2 \phi = 0$) on the supergravity side.

The AdS/CFT correspondence states that string theory in the AdS is dual to a conformal field theory (CFT) defined on its remote boundary of AdS [6]. The semiclassical limit of spacetime physics is related to the large $N$ limit of the dual CFT. A 5D black hole ($M_5 \times S^1 \times T^4$) becomes $\text{AdS}_3 \times S^3 \times T^4$ near horizon but with an asymptotically flat space (AFS) [7]. Recently this is called an $\text{AdS}_3$ bubble in AFS and corresponds to the dilatonic solution [8]. In this case the matching procedure is crucial for obtaining an absorption coefficient and here we need to match an $\text{AdS}_3$ bubble to AFS.

On the other hand, one obtains $\text{AdS}_3 \times S^3 \times T^4$ as the other solution to the type IIB string theory. This is an exact $\text{AdS}_3$ with asymptotically $\text{AdS}_3$ and corresponds to the non-dilatonic solution [9]. Further it includes the BTZ black hole [10]. We point out that the near horizon limit of a 5D black hole yields the BTZ black hole, whereas this solution accommodates to the BTZ black hole spacetime as a whole.

It was understood that the greybody factor calculation makes sense when one finds an asymptotically flat region as in Sec.II [6]. Hence it may not be possible in an $\text{AdS}_3$ because there is no asymptotic state corresponding to particle at infinity of $\text{AdS}_3$. However, the authors in [9] calculated the greybody factor for a free scalar and the dilaton both
in $M_5 \times S^1 \times T^4$ (an AdS$_3$ bubble) and $\text{AdS}_3 \times S^3 \times T^4$ (an exact AdS$_3$) within the type IIB supergravity. Here in the exact AdS$_3$ calculation we choose the non-normalizable modes to obtain the greybody factor. This corresponds to the AdS$_3$-AdS$_3$ matching procedure. This expression denotes shorthand for a certain choice of boundary conditions where non-normalizable modes inject flux into AdS$_3$. It turns out that two results of a free scalar are exactly the same. And the results for the dilaton are the same up to a factor 3. These show a close relationship between two approaches. More recently, vacuum correlators of the dual CFT$_4$ were expressed as truncated n-point functions for the non-normalizable modes in AdS$_5$. One can interpret this result as an S matrix of an exact AdS$_5$ arising from a limit of scattering from an AdS$_5$ bubble [8]. This supports that our calculation based on the non-normalizable modes is correct.

In this paper we will show that the S-matrix of an AdS$_3$ bubble can be derived partly from an exact AdS$_3$ space. This is one of the current issues in the AdS/CFT correspondence. For this purpose, we investigate the close relationship between the potential and absorption cross section in an AdS$_3$ bubble and an exact AdS$_3$. Comparing an AdS$_3$ bubble with an exact AdS$_3$ leads to an assumption that the potential of an exact AdS$_3$ is the left-hand side of an AdS$_3$ bubble. For this study, we introduce the $\{\text{in}\}$ and $\{\text{out}\}$-state pictures for an AdS$_3$ bubble. For the exact AdS$_3$ study, one needs the $\{\text{in}\}$-state as well as non-normalizable modes. Further, we introduce the test fields for scattering analysis. These are in an AdS$_3$ bubble: a free scalar($\phi$) which, in the decoupling limit, relates to an (1,1) operator $\mathcal{O}$ in the holographically dual theory; two fixed scalars($\nu, \lambda$) to (2,2), (3,1), and (1,3) operators; two intermediate scalars($\eta, \xi$) to (1,2) and (2,1) operators. On the exact AdS$_3$ side, the test fields are a free scalar($\psi$), the dilaton($\Phi$), an intermediate scalar($\eta$), and the tachyon($T$).

The organization of this work is as follows. Sec.II is devoted to analyzing the scattering from an AdS$_3$ bubble within a 5D black hole. This corresponds to a conventional scattering study. We study the scattering of the test fields in an exact AdS$_3$ in Sec. III. In this case we are careful for obtaining the absorption coefficient because we cannot define the asymptotically flat space. Finally we discuss our results in Sec. IV.
II. SCATTERING FROM AN ADS$_3$ BUBBLE IN A 5D BLACK HOLE

Initially we introduce all perturbing modes in a 5D black hole background. It is pointed out that in s-wave calculation the fixed scalars are physically propagating modes and other fields belong to be redundant modes [5]. Hence we choose two fixed scalars ($\nu, \lambda$) and a free scalar ($\phi$) as the relevant modes. We begin with the 5D black hole with three charges,

$$ds_{5D}^2 = -hf^{-2/3}dt^2 + f^{1/3}(h^{-1}dr^2 + r^2d\Omega_3^2),$$  

(1)

where

$$f = f_1f_2f_n = \left(1 + \frac{r_1^2}{r_2^2}\right)\left(1 + \frac{r_5^2}{r_2^2}\right)\left(1 + \frac{r_n^2}{r_2^2}\right), \quad h = \left(1 - \frac{r_0^2}{r_2^2}\right).$$  

(2)

Here the radii are related to the boost parameters ($\alpha_i$) and the charges ($Q_i$) as

$$r_i^2 = r_0^2\sinh^2\alpha_i = \sqrt{Q_i^2 + \frac{r_0^4}{4} - \frac{r_0^2}{2}}, i = 1, 5, n.$$  

(3)

Hence the D-brane black hole depends on the four parameters ($r_1, r_5, r_n, r_0$). The event horizon (outer horizon) is clearly at $r = r_0$. When all three charges are nonzero, the surface $r = 0$ becomes a smooth inner horizon (Cauchy horizon). When at least one of the charges is zero, the surface $r = 0$ becomes singular. The extremal case corresponds to the limit of $r_0 \to 0$ with the boost parameters $\alpha_i \to \pm\infty$, keeping the charges ($Q_i$) fixed. We are interested in the limit of $r_0, r_n \ll r_1, r_5$, which is called the dilute gas limit. This is so because this limit corresponds to the near-outer horizon. Here we choose $Q_1 = r_1^2, Q_5 = r_5^2$, and $r_n = r_0\sinh\alpha_n$ with a finite $\alpha_n$. This corresponds to the near-extremal black hole and its thermodynamic quantities (energy, entropy, Hawking temperature) are given by

$$E_{next} = \frac{2\pi^2}{\kappa_5^2} \left[r_1^2 + r_5^2 + \frac{1}{2}r_0^2\cosh 2\alpha_n\right],$$  

(4)

$$S_{next} = \frac{4\pi^3r_0}{\kappa_5^2}r_1r_5\cosh \alpha_n,$$  

(5)

$$\frac{1}{T_{H, next}} = \frac{2\pi}{r_0}r_1r_5\cosh \alpha_n.$$  

(6)
where \( \kappa_5^2 \) is the 5D gravitational constant. The above energy and entropy are those of a gas of massless 1D particles. In this case the temperatures for left and right moving string modes are given by

\[
T_L = \frac{1}{2\pi} \left( \frac{r_0}{r_1 r_5} \right) e^{\alpha_n}, \quad T_R = \frac{1}{2\pi} \left( \frac{r_0}{r_1 r_5} \right) e^{-\alpha_n}.
\]  

(7)

This implies that the (left and right moving) momentum modes along the string direction are excited, while the excitations of D1-anti D1 and D5-anti D5-branes are suppressed. The Hawking temperature is given by their harmonic average

\[
\frac{2}{T_H} = \frac{1}{T_L} + \frac{1}{T_R}.
\]

(8)

### A. Potential analysis

For a free scalar \( \phi = \phi(r) e^{i\omega t} Y_l(\theta_1, \theta_2, \theta_3) \), the linearized equation \( \nabla^2 \phi = 0 \) in the background of (1) leads to [2,3,11]

\[
\left[ (hr^3 \partial_r)^2 + \omega^2 r^6 f - \frac{l(l+2)h^3}{r^2} \right] \phi = 0.
\]

(9)

The s-wave \( (l = 0) \) linearized equation for the fixed scalars takes the form [4,5]

\[
\left[ (hr^3 \partial_r)^2 + \omega^2 r^6 f - \frac{8hr^4r^4_{\pm}}{(r^2 + r^2_{\pm})^2} \left( 1 + \frac{r^2_0}{r^2_{\pm}} \right) \right] \phi_{\pm} = 0,
\]

(10)

where one gets \( \nu \) for \( \phi_+ \) and \( \lambda \) for \( \phi_- \). Here \( r^2_{\pm} = [r^2_1 + r^2_5 + r^2_n] \pm \sqrt{r^4_1 + r^4_5 + r^4_n - r^2_1 r^2_5 - r^2_1 r^2_n - r^2_5 r^2_n}/3 \). Considering \( N = r^{-3/2}\tilde{N} \), for \( N = \nu, \lambda, \phi \) and introducing a tortoise coordinate \( r^* = \int (dr/h) = r + (r_0/2) \ln \left| (r - r_0)/(r + r_0) \right| \) [2], then the equation takes the form

\[
\frac{d^2 \tilde{N}}{dr^{*2}} + (\omega^2 - \tilde{V}_N) \tilde{N} = 0.
\]

(11)

Here we take \( r_1 = r_5 = R \) and \( r_0 = r_n \) for simplicity. In the dilute gas limit \( (R \gg r_0), \tilde{V}_N(r) \) is given by
\[\tilde{V}_\nu(r) = -\omega^2(f - 1) + h \left[ \frac{3}{4r^2} \left( 1 + \frac{3r_0^2}{r^2} \right) + \frac{8R^4}{r^2(r^2 + R^2)^2} \right], \quad (12)\]

\[\tilde{V}_\lambda(r) = -\omega^2(f - 1) + h \left[ \frac{3}{4r^2} \left( 1 + \frac{3r_0^2}{r^2} \right) + \frac{8R^4}{r^2(3r^2 + R^2)^2} \right], \quad (13)\]

\[\tilde{V}_\phi(r) = -\omega^2(f - 1) + h \left[ \frac{3}{4r^2} \left( 1 + \frac{3r_0^2}{r^2} \right) + \frac{l(l + 2)}{r^2} \right], \quad (14)\]

where

\[f - 1 = \frac{r_0^2 + 2R^2}{r^2} + \frac{(2r_0^2 + R^2)R^2}{r^4} + \frac{r_0^2R^4}{r^6}. \quad (15)\]

FIG. 1. The graph of \((f - 1)\) in 5D black hole with \(r_0 = 0.01, R = 0.3\). A peak appears at outer horizon \((r = r_0)\).

We note that \(\tilde{V}_N\) depends on two parameters \((r_0, R)\) as well as the energy \((\omega)\). As \((14)\) stands, it is far from the Schrödinger-type equation. The \(\omega\)-dependence is a matter of peculiar interest to us compared with the Schwarzschild black hole potentials \((V_{RW}, V_Z, V_\psi)\) \([12]\). This makes the interpretation of \(\tilde{V}_N\) as a potential difficult. As is shown in Fig.\(1\), this
arises because \((f-1)\) is very large as \(10^6\) for \(r_0 = 0.01, R = 0.3\) in the near-horizon. In order for \(\tilde{V}_N\) to be a potential, it is necessary to take the low energy limit of \(\omega \to 0\). It is suitable to be \(10^{-3}\). And \(\omega^2(f-1)\) is of order \(\mathcal{O}(1)\) and thus it can be ignored in comparison to the remaining ones. Now we can define a potential \(V_N = \tilde{V}_N + \omega^2(f-1)\). Hence, in the low energy limit\((\omega \to 0)\), Eq.\((11)\) becomes as the Schrödinger-type equation. Further the last terms in \((12)-(13)\) are important to compare each other. After the partial fraction, these lead to

\[
\frac{8R^4}{r^2(r^2 + R^2)^2} = \frac{8}{r^2} - \frac{8}{r^2 + R^2} - \frac{8R^2}{(r^2 + R^2)^2},
\]

\[
\frac{8R^4}{r^2(3r^2 + R^2)^2} = \frac{8}{r^2} - \frac{24}{3r^2 + R^2} - \frac{24R^2}{(3r^2 + R^2)^2}.
\]

The last term of a free scalar in \((14)\) with \(l = 2\) keeps the first terms in \((16)-(17)\) only. One finds immediately the sequence

\[
V_{\phi_0} \ll V_{\lambda} \leq V_{\nu} \leq V_{\phi_2}.
\]

![FIG. 2. Four potential graphs \((V_{l=0}^\phi, V_{\nu}, V_{\lambda}, V_{l=2}^\phi)\) for an AdS\(_3\) bubble in a 5D black hole with \(r_0 = 0.01, R = 0.3\).](image)
Here $\phi_0$ denotes the s-wave ($l = 0$) free scalar and $\phi_2$ the free one with $l = 2$. This is also observed from the graphs of potential in Fig.2 with $r_0 = 0.01$, $R = 0.3$. Because the shape of their potentials takes nearly the same form ($V_\lambda \simeq V_\nu \simeq V_\phi^{l=2}$), these give us nearly the same reflection coefficient $R = |R|^2$ and absorption one $A = |A|^2$. For example, in the low energy limit of $\omega \to 0$, $\lambda, \nu, \phi_2$ take nearly the zero-absorption cross section. Furthermore all potentials go to zero, as $r$ approaches infinity. This implies the existence of the asymptotic states outside an AdS$_3$ bubble. The size of an AdS$_3$ bubble is from $r_0 = 0.01$ to $R = 0.3$.

**B. Scattering from an AdS$_3$ bubble**

We are interested in the scattering of the test fields off $V_N(r^*)$. It is well known that the scattering analysis is usually done by choosing a coordinate such as $-\infty \leq r^* \leq \infty$. It is always possible to visualize the black hole as presenting an effective potential barrier (or well) to an incoming test wave. One expects that some of the incident wave will be irreversibly absorbed by the black hole, while the remaining fraction will be scattered back to the infinity. In this scattering we can calculate the reflection and transmission (absorption) coefficients [13]. As is shown in Fig.3, we note that all potential barriers $V_N(r^*)$ take nearly the symmetric forms around $r^* = 0$ when they are rewritten by a tortoise coordinate $r^*$ [14]. Also they are localized at $r^* = 0$. Since $V_N(r^*) \to 0$ as $r^* \to \pm \infty$, one finds

$$d^2 \tilde{N}_{\pm \infty} / dr^{*2} + \omega^2 \tilde{N}_{\pm \infty} = 0.$$  \hspace{1cm} (19)

Asymptotically ($r^* \to \infty$) the solution is given by

$$N_{+\infty}^{\text{out}} = e^{i\omega r^*} + R_{N}^{\text{out}}(\omega)e^{-i\omega r^*}.$$  \hspace{1cm} (20)

Considering the time-part of $e^{i\omega t}$, the first is an incoming wave ($\leftarrow$) and the last is an outgoing wave ($\to$). Near the horizon it is purely incoming ($\leftarrow$) as

$$N_{-\infty}^{\text{out}} = A_{N}^{\text{out}}(\omega)e^{i\omega r^*}.$$  \hspace{1cm} (21)
We call this type of solution as \( \{ \text{out}\} \) and the corresponding vacuum state is defined as \( b_i |0\rangle_{\text{out}} = 0 \).

![Graph of four potential graphs \((V^{l=0}_\phi, V_\nu, V_\lambda, V^{l=2}_\phi)\) as functions of \(r^*\) for an AdS\(_3\) bubble in a 5D black hole with \(r_0 = 0.01, R = 0.3\).](image)

**FIG. 3.** Four potential graphs \((V^{l=0}_\phi, V_\nu, V_\lambda, V^{l=2}_\phi)\) as functions of \(r^*\) for an AdS\(_3\) bubble in a 5D black hole with \(r_0 = 0.01, R = 0.3\).

In order to study the Hawking radiation, we introduce another boundary condition. Asymptotically the wave is purely outgoing(\(\rightarrow\))

\[
N^{\text{in}}_{+\infty} = A^{\text{in}}_{N}(\omega) e^{-i\omega r^*} \tag{22}
\]

but near the horizon it has both outgoing(\(\rightarrow\)) and incoming(\(\leftarrow\)) parts as

\[
N^{\text{in}}_{-\infty} = e^{-i\omega r^*} + R^{\text{in}}_{N}(\omega) e^{i\omega r^*}. \tag{23}
\]

We call this type of solution as \( \{ \text{in}\} \) and its vacuum state is defined as \( a_i |0\rangle_{\text{in}} = 0 \). The vacuum states \( |0\rangle_{\text{out}} \) and \( |0\rangle_{\text{in}} \) form two different bases of which any state can be expanded in terms of the other. These are two different Fock space vacuum states and \( \{ \text{out}\} \) and \( \{ \text{in}\} \) are related to each other by the Bogoliubov transformation,
\[ b_i = \sum_j (\alpha_{ij} a_j - \beta_{ij} a_j^\dagger), \quad (24) \]
\[ b_i^\dagger = \sum_j (\alpha_{ij} a_j^\dagger - \beta_{ij} a_j). \quad (25) \]

The computation of Hawking shows in a semi-classical approximation that the thermal radiation from the black hole with temperature \( T_H \) is given by \cite{15}

\[ \langle N_\omega \rangle = \langle \text{in} | b_i^\dagger b_i | \text{in} \rangle = \sum_k |\beta_{ik}|^2 = \frac{\sigma_{5D}^N}{e^{\omega/T_H} - 1}, \quad (26) \]

with an absorption cross section \( \sigma_{5D}^N = |A_N^{\text{out}}|^2 \times 4\pi/\omega^3 = A_N^{\text{out}} 4\pi/\omega^3 \). Note that if \( \sigma_{5D}^N \) is a constant, \( \langle N_\omega \rangle \) is the same as that of a black body. Typically, \( \sigma_{5D}^N \) is not constant but varies.

The deviations from the black body spectrum have earned it the name “greybody factor”.

Here we define the S-matrix from \( \{\text{in}\}_N \) and \( \{\text{out}\}_N \) as \cite{14}

\[ S(\omega) = \begin{pmatrix} A_N^{\text{out}}(\omega) & A_N^{\text{in}}(\omega) \\ R_N^{\text{out}}(\omega) & R_N^{\text{in}}(\omega) \end{pmatrix}. \quad (27) \]

Here an incident wave \( e^{i\omega r^*} \) of unit amplitude from \( r^* = +\infty \) gives rise to \( A_N^{\text{out}}(\omega) \) and \( R_N^{\text{out}}(\omega) \). On the other hand, an incident wave \( e^{-i\omega r^*} \) of unit amplitude from \( r^* = -\infty \) gives rise to \( A_N^{\text{in}}(\omega) \) and \( R_N^{\text{in}}(\omega) \). The relation between these is given by

\[ A_N^{\text{out}}(\omega) = A_N^{\text{in}}(\omega) = A_N(\omega), \quad (28) \]
\[ \frac{R_N^{\text{out}}(\omega)}{A_N(\omega)} = -\frac{R_N^{\text{in}}(-\omega)}{A_N(-\omega)}, \quad \frac{R_N^{\text{out}}(-\omega)}{A_N(-\omega)} = -\frac{R_N^{\text{in}}(\omega)}{A_N(\omega)}, \quad (29) \]
\[ A_N^{*}(\omega) = A_N(-\omega), \quad R_N^{*}(\omega) = R_N^{*}(-\omega), \quad R_N^{*}(-\omega) = R_N^{*}(\omega). \quad (30) \]

The above relations establish the symmetry and unitarity of the S-matrix in the AdS\(_3\) bubble scattering. Actually \( A_N(\omega) \) can be calculated from the backscattering of an incident wave \( N \) off the potential \( V_N(r^*) \). It is not easy to find out the absorption amplitude \( A_N \) directly in the complicated potentials such as \( V_N(r^*) \). In this case one use the flux \( \mathcal{F} \) of the incoming wave to obtain the absorption coefficient

\[ A_N^{\text{out}} = \frac{\mathcal{F}(-\infty)}{\mathcal{F}(\infty)}, \quad (31) \]

where \( \mathcal{F}(-\infty)(\mathcal{F}(\infty)) \) are the fluxes at the horizon(infinity). In this way we can calculate the semiclassical absorption cross section.
C. Exact analysis of near-horizon (AdS$_3$ bubble)

First let us consider a free scalar. Using $z = h$, the wave equation (9) can be rewritten as

$$z(1 - z) \frac{d^2 \phi}{dz^2} + (1 - z) \frac{d\phi}{dz} + \left\{ -C + \frac{Q}{z} + \frac{E}{1 - z} \right\} \phi = 0,$$

(32)

where

$$C = \left( \frac{\omega r_1 r_5 r_n}{2 r_0^2} \right)^2 = \frac{\omega^2}{64 \pi^2} \left( \frac{1}{T_L} - \frac{1}{T_R} \right)^2,$$

(33)

$$E = -\frac{l(l+2)}{4} + \frac{\omega^2(r_1^2 + r_5^2 + r_n^2)}{4} \simeq -\frac{l(l+2)}{4},$$

(34)

$$Q = \left( \frac{\omega}{4 \pi T_H} \right)^2 \left[ \left( 1 + \frac{r_0^2}{r_1^2} + \frac{r_0^2}{r_5^2} \right) + 4 \pi^2 r_n^2 T_H^2 \right] \simeq \left( \frac{\omega}{4 \pi T_H} \right)^2.$$  

(35)

Here $\simeq$ means both the dilute gas limit ($r_0, r_n \ll r_1, r_5$) and low energy limit ($\omega \rightarrow 0$). Then one has $\omega r_0, \omega r_n \ll \omega r_1, \omega r_5 < 1$. In order to compare (32) with the hypergeometric equation, one has to transform it into the pole-free equation. With an unknown constant $A$, we find the ingoing mode at the horizon

$$\phi^a = A z^{-i \sqrt{Q}} (1 - z)^{(1 - \nu)/2} F(a, b, c; z),$$

(36)

where

$$\nu = \sqrt{(l+1)^2 - \omega^2(r_1^2 + r_5^2 + r_n^2)} \simeq l + 1,$$

(37)

$$a = \frac{1 - \nu}{2} - i \sqrt{Q} + i \sqrt{C} \simeq -\frac{l}{2} - i \frac{\omega}{4 \pi T_R},$$

(38)

$$b = \frac{1 - \nu}{2} - i \sqrt{Q} - i \sqrt{C} \simeq -\frac{l}{2} - i \frac{\omega}{4 \pi T_L},$$

(39)

$$c = 1 - 2i \sqrt{Q} \simeq 1 - i \frac{\omega}{2 \pi T_H}.$$  

(40)

The large $r$-behavior ($z \rightarrow 1$) of $\phi^a$ can be obtained from the ($z \rightarrow 1 - z$) transformation rule for the hypergeometric functions as

$$\phi^{a \rightarrow f} = \frac{A \Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} u^{\nu - 1} + \frac{A \Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} u^{-(\nu + 1)}.$$

(41)

For the fixed scalars($\nu, \lambda$), considering both (44), (47) and the near-horizon condition of $r \simeq r_0 \ll r_1, r_5$, one finds that (44) leads to (4) with $l = 2$. Thus one can obtain their near-horizon behaviors from $\phi_2$.  

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D. Asymptotic states

Let us first consider a free scalar. In the far region, we introduce \( \phi = \frac{\tilde{\phi}}{r} \) and \( u = \omega r \) and then (11) leads to

\[
\frac{d^2 \tilde{\phi}}{du^2} + \frac{1}{u} \frac{d\tilde{\phi}}{du} + \left[ 1 - \frac{\nu^2}{u^2} \right] \tilde{\phi} = 0.
\]  
(42)

The solution is given by the Bessel function when \( \nu \) is not an integer

\[
\phi^f = \left[ \alpha J_\nu \left( \frac{u}{2} \right) + \beta J_{-\nu} \left( \frac{u}{2} \right) \right],
\]  
(43)

where \( \alpha, \beta \) are unknown constants. From the large \( u \)-behavior \( (r \to \infty, \omega r \gg 1) \), one finds the asymptotic states

\[
\phi^f_\infty = \sqrt{\frac{1}{2\pi u^3/2}} e^{-iu} \left\{ \alpha e^{i(\nu+1/2)\pi/2} + \beta e^{-i(-\nu+1/2)\pi/2} \right\}
\]  
(44)

and its incoming flux

\[
\mathcal{F}(\infty) = -\frac{2}{\omega^2} \left| \alpha e^{i(\nu+1/2)\pi/2} + \beta e^{-i(-\nu+1/2)\pi/2} \right|^2.
\]  
(45)

The small \( u \)-behavior \( (\omega r < 1) \) of \( \phi^f \) is

\[
\phi^{f \to \text{inter}} = \frac{1}{u} \left[ \alpha \left( \frac{u}{2} \right)^\nu \frac{1}{\Gamma(\nu+1)} + \beta \left( \frac{u}{2} \right)^{-\nu} \frac{1}{\Gamma(-\nu+1)} \right].
\]  
(46)

On the other hand, the asymptotic behavior of the s-wave fixed scalars \( (\phi_\pm = \frac{\tilde{\phi}_\pm}{r}) \) is governed by

\[
\frac{d^2 \tilde{\phi}_\pm}{du^2} + \frac{1}{u} \frac{d\tilde{\phi}_\pm}{du} + \left[ 1 - \frac{\nu^2}{u^2} \right] \tilde{\phi}_\pm = 0
\]  
(47)

with \( \nu \simeq 1 \). Its solution is given by

\[
\phi^f_\pm \simeq \left[ \alpha_\pm \frac{J_\nu(u)}{u} + \beta_\pm \frac{J_{-\nu}(u)}{u} \right].
\]  
(48)

Here \( \alpha_\pm, \beta_\pm \) are unknown constants.
E. AdS₃-AFS matching procedure and absorption cross section

Here we use the matching procedure between an AdS₃ bubble (near-horizon of a 5D black hole) and asymptotically flat space to obtain an absorption coefficient. First consider the matching of a free scalar. Here the matching point resides on $0 < r^* < \infty$ in Fig.3. In the intermediate zone ($u < 1$), from (41) and (46) one finds

$$\alpha = A u_0 \frac{\Gamma(\nu + 1)\Gamma(\nu)\Gamma(c)u_0^{-\nu}}{\Gamma(c - a)\Gamma(c - b)}, \quad \beta = A u_0 \frac{\Gamma(-\nu + 1)\Gamma(-\nu)\Gamma(c)u_0^{-\nu}}{2\nu(a - 1)\Gamma(a)\Gamma(b)}.$$  (49)

Since $u_0 = \omega r_0 \ll 1$, one finds $\alpha \gg \beta$. In this case we take an incoming flux effectively. Furthermore the incoming flux at the horizon is found as

$$\mathcal{F}(0) = -8\pi r_0^2 \sqrt{Q}|A|^2.$$  (50)

The absorption coefficient is given by

$$A_{\phi}^{\text{out}} = \frac{\mathcal{F}(0)}{\mathcal{F}(\infty)} \approx 4\pi u_0^2 \sqrt{Q} \left|\frac{A}{\alpha}\right|^2.$$  (51)

The absorption cross section takes the form [11,16]

$$\sigma_{5D}^\phi = (l + 1)^2 \frac{4\pi}{\omega^3} A_{\phi}^{\text{out}} \approx \frac{A_{5D}^H}{[l!(l + 1)!]^2} (l + 1)^2 \left(\frac{\omega r_0}{2}\right)^{2l} \left|\frac{\Gamma(l + 2) - i\frac{\omega}{4\pi T_L}}{\Gamma(1 - i\frac{\omega}{2\pi T_H})}\right|^2.$$  (52)

with the area of horizon $A_{5D}^H = 2\pi r_1 r_3 r_n$. We have, for even $l$

$$\sigma_{5D}^\phi = (l + 1)^2 \frac{\pi^3}{2^{2l}} \frac{(r_1 r_3)^{2l + 2} \omega^{2l + 1}}{[l!(l + 1)!]^2} \left[\omega^2 + (2\pi T_L)^2 l^2\right] \cdots \left[\omega^2 + (2\pi T_L)^2 l^2\right] \frac{e^{\omega/T_H} - 1}{(e^{\omega/2T_L} - 1)(e^{\omega/2T_R} - 1)}.$$  (53)

Matching procedure for the s-wave $\phi_\pm$ is nearly the same as in a free scalar [7]. It leads to

$$\sigma_{5D}^{\phi_\pm} = \frac{\pi^3}{64 r_\pm^4} \omega \left(\omega^2 + 16\pi^2 T_L^2\right) \left(\omega^2 + 16\pi^2 T_R^2\right) \frac{e^{\omega/T_H} - 1}{(e^{\omega/2T_L} - 1)(e^{\omega/2T_R} - 1)}.$$  (54)

In the limit of $\omega \ll T_L, T_R, T_H$, the low-energy absorption cross sections are calculated as
\[ \sigma_{5D}^{\phi_0} = A_{5D}^H, \] (55)

\[ \sigma_{5D}^{\phi_2} = \frac{3}{16} A_{5D}^H (\omega r_0)^4 = \frac{3}{4} (\omega R)^4 \left\{ \frac{A_{5D}^H}{4} \left( \frac{r_0}{R} \right)^4 \right\}, \] (56)

\[ \sigma_{5D}^{\nu} = \frac{A_{5D}^H}{4} \left( \frac{r_0}{R} \right)^4, \] (57)

\[ \sigma_{5D}^\lambda = 9 \frac{A_{5D}^H}{4} \left( \frac{r_0}{R} \right)^4, \] (58)

where we impose the relation \( r_1 = r_5 = R, r_0 = r_n \). Here we find a sequence of cross section

\[ \sigma_{5D}^{\phi_0} \gg \sigma_{5D}^{\lambda} \geq \sigma_{5D}^{\nu} \geq \sigma_{5D}^{\phi_2}. \] (59)

This originates from the potential sequence in (18). It is consistent with our naive expectation that the absorption cross section increases, as the height of potential decreases. Here we wish to point out the difference between a free scalar and the fixed scalars. In the dilute gas limit \( (R \gg r_0) \) and the low energy limit \( (\omega \to 0) \), the s-wave cross section for a free scalar \( \sigma_{5D}^{\phi_0} \) goes to \( A_{5D}^H \) [3], while the s-wave cross sections for fixed scalars \( (\nu, \lambda) \) including \( \phi_2 \) approach zero [4]. Also this can be confirmed from Fig.2.

**III. SCATTERING FROM AN EXACT ADS_3 IN ADS_3(BTZ BLACK HOLE) \times S^3 \times T^4**

**A. Potential analysis**

Here we consider the geometry of an exact AdS_3(AdS_3 × S^3 × T^4) as the other solution to the type IIB string theory [3]. This corresponds to the non-dilatonic solution. This geometry can be led the BTZ black hole spacetime as a whole by the periodic identification.

A ten-dimensional minimally coupled scalar satisfies

\[ \Box_{10} \Psi = 0. \] (60)

\( \Psi \) can be decomposed into

\[ \Psi = e^{-i\omega t} e^{im\varphi} e^{iK_i x^i} Y_1(\theta_1, \theta_2, \theta_3) \psi(\rho). \] (61)
Then Eq. (60) leads to

$$\nabla^2_{\text{BTZ}} \psi(\rho) + \frac{\mu}{R^2} \psi(\rho) = 0 \quad (62)$$

with $\mu = -l(l+2) - K^2 r_5^2$. The $\mu = -8(l = 2)$ case contains both the dilaton($\Phi$) and a free scalar($\psi$) with $l = 2$. The $\mu = -3(l = 1)$ case corresponds to an intermediate scalar ($\eta$) and $\mu = 1$ leads to tachyon($T$). Here the BTZ black hole spacetime is given by [10]

$$ds^2_{\text{BTZ}} = -f^2 dt^2 + \rho^2 \left( d\varphi - \frac{J}{2\rho^2} dt \right)^2 + f^{-2} d\rho^2 \quad (63)$$

with $f^2 = \rho^2/R^2 - M + J^2/4\rho^2 = (\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)/\rho^2 R^2$. The mass, angular momentum, angular velocity at horizon and area of horizon are

$$M = (\rho_+^2 + \rho_-^2)/R^2, \quad J = 2\rho_+ \rho_- / R, \quad \Omega_H = J / 2\rho_+, \quad A^\text{BTZ}_H = 2\pi \rho_+. \quad (64)$$

Further one finds the relation between the BTZ and a 5D black holes as

$$T^\text{BTZ}_H = (\rho_+^2 - \rho_-^2)/2\pi R^2 \rho_- = T_H, \quad \frac{1}{T^\text{BTZ}_L/R} = \frac{1}{T^\text{BTZ}_H} \left( 1 \pm \frac{\rho_+}{\rho_-} \right) = \frac{1}{T_{L/R}}. \quad (65)$$

Its s-wave equation with $m = 0$ takes the form

$$\left[ f^2 \partial^2_\rho + \left\{ \frac{1}{\rho} \partial_\rho \left( \rho f^2 \right) \right\} \partial_\rho + \frac{\omega^2}{f^2} + \frac{\mu}{R^2} \right] \psi(\rho) = 0. \quad (66)$$

Defining $\psi(\rho) = \tilde{\psi}/\sqrt{\rho}$ and then (66) takes the form

$$f^2 \ddot{\tilde{\psi}} + \left( f^2 \right) \dot{\tilde{\psi}}' + \left[ \frac{f^2}{4\rho^2} - \frac{(f^2)'}{2\rho} + \frac{\omega^2}{f^2} + \frac{\mu}{R^2} \right] \tilde{\psi} = 0, \quad (67)$$

where the prime(‘) denotes the differentiation with respect to $\rho$. In order to obtain the Schrödinger-type equation, we introduce the tortoise coordinate $\rho^*$ as [17]

$$\rho^* = \int \frac{dp}{f^2} = \frac{R^2}{2(\rho_+^2 - \rho_-^2)} \left[ \rho_+ \ln \left( \frac{\rho - \rho_+}{\rho + \rho_+} \right) - \rho_- \ln \left( \frac{\rho - \rho_-}{\rho + \rho_-} \right) \right]. \quad (68)$$

We note that $\rho_+ \leq \rho \leq \infty$, while $-\infty \leq \rho^* \leq 0$ for outside horizon. On the other hand, in a 5D black hole one finds that $r_0 \leq r \leq \infty$ is mapped into $-\infty \leq r^* \leq \infty$. Then (67) leads to
\[
\frac{d^2 \tilde{\psi}}{d\rho^2} + \left( \omega^2 - V_\mu \right) \tilde{\psi} = 0, \quad (69)
\]

where the potential is given by
\[
V_\mu(\rho) = f^2 \left[ -\frac{f^2}{4\rho^2} + \frac{(f^2)'}{2\rho} - \frac{\mu}{R^2} \right]. \quad (70)
\]

Four potential graphs \((V_T, V_{\psi=0}, V_\eta, V_{\psi=2})\) in an exact AdS_3 background are shown in Fig.4. The parameters are chosen as \(\rho_+ = 0.01, \rho_- = 0.001, R = 0.3\). These are all monotonically increasing functions with \(\rho\), contrast to \(V_N\) for an AdS_3 bubble. This shows a peculiar property of AdS_3, which in this spacetime the asymptotic states cannot be defined. But the tachyon potential \((V_T)\) is a monotonically decreasing function. This field couples to the minimal weight primary operator with \((1/2,1/2)\) \([18]\). Also this satisfies both the stability condition for AdS_3 and the Dirichlet boundary condition \([19]\), which are clearly related to a shape of its potential.

![Four potential graphs (V_T, V_{\psi=0}, V_\eta, V_{\psi=2}) as functions of \(\rho\) for an exact AdS_3.](image-url)
B. Asymptotically AdS$_3$-behavior and non-normalizable modes

In the near-horizon ($\rho \to \rho_+ V_\mu \to 0$), (69) reduces to

$$\frac{d^2 \tilde{\psi}_{\text{NH}}}{d\rho^2} + \omega^2 \tilde{\psi}_{\text{NH}} = 0$$  \hspace{1cm} (71)

which leads to the plane-wave solution

$$\tilde{\psi}_{\text{EH}} = e^{-i\omega\rho^*} + R^n_\psi(\omega)e^{i\omega\rho^*}. \hspace{1cm} (72)$$

Here the first term is an outgoing mode ($\rightarrow$) and the second is an incoming mode ($\leftarrow$). Now let us discuss the asymptotically AdS$_3$-behavior. Near the timelike boundary ($\rho \to \infty, \rho^* \to 0$), one finds

$$\frac{d^2 \tilde{\psi}_{\infty}}{d\rho^2} - \left[\frac{3}{4} - \mu\right] \frac{\rho^2}{R^4} \tilde{\psi}_{\infty} = 0. \hspace{1cm} (73)$$

Here we introduce the relation between $\rho$ and $\rho^*$

$$\frac{\rho_+}{\rho(\rho^*)} = \frac{Y}{1 - \sigma^2} \sum_{n=0}^{\infty} a_n(\sigma)Y^{2n}, \hspace{0.5cm} Y = \tanh \lambda \rho^*, \hspace{0.5cm} \sigma = \frac{\rho_+}{\rho}, \hspace{0.5cm} \lambda = \frac{\rho_+}{R^2} (1 - \sigma^2), \hspace{0.5cm} a_0 = 1. \hspace{1cm} (74)$$

If $\sigma^2$ is very small and $\rho^* \to 0$, one finds

$$\rho = \frac{\rho_+}{Y} (1 - \sigma^2) = \rho_+ (1 - \sigma^2) \coth \lambda \rho^* \simeq \frac{R^2}{\rho^*}. \hspace{1cm} (75)$$

Using (75), (73) leads to

$$\frac{d^2 \tilde{\psi}_{\infty}}{d\rho^2} - \left[\frac{3}{4} - \mu\right] \frac{\tilde{\psi}_{\infty}}{\rho^*} = 0. \hspace{1cm} (76)$$

Its solution takes the form

$$\tilde{\psi}_{\infty} = \rho^* \frac{1 + \sqrt{1 - \mu}}{2}. \hspace{1cm} (77)$$

Finally we have

$$\psi_{\infty}(\rho^*) = \frac{\tilde{\psi}_{\infty}}{\sqrt{\rho}} \propto \rho^* (1 + \sqrt{1 - \mu}). \hspace{1cm} (78)$$
For the s-wave free scalar ($\mu = 0$), $\psi_\infty$ takes the form ($\rho^*^2$, constant) and for the dilaton field ($\mu = -8$), one finds ($\rho^*^4, 1/\rho^*^2$). We find ($\rho^*^{3/2}, \rho^*^{1/2}$) for a conformally coupled scalar ($\mu = 3/4$) \cite{17}. For the tachyon ($\mu = 1$), one has ($\rho^*, \rho^*$) and for an intermediate scalar ($\mu = -3$), one finds ($\rho^*^3, 1/\rho^*$). Instead of the plane-wave from, here one finds the power-law behaviors of $\rho^*$ \cite{20}. The non-normalizable modes are found as $1/\rho^*^2$ for the dilaton and $1/\rho^*$ for an intermediate scalar, which diverge as $\rho^*$ approaches the timelike boundary ($\rho^* \to 0$). The positive powers of $\rho^*$ all belong to be the normalizable modes, which converge at spatial infinity. We note that the tachyon takes only the normalizable modes \cite{19}. This can be easily conjectured from its shape of potential $V_T$. The s-wave free scalar takes a constant-behavior at infinity. This makes an difficulty in dividing $\psi_\infty$ into an ingoing and outgoing modes at spatial infinity \cite{21}. Actually it is impossible to define an ingoing wave and an outgoing wave at spatial infinity of an exact AdS$_3$. Instead in the asymptotically AdS$_3$ space it contains the normalizable as well as the non-normalizable modes. The latter will play an important role in calculation of the absorption coefficient $A_{\psi}$.

C. Scattering from an exact AdS$_3$

First we note that $\rho^*$ covers only the left-hand side ($-\infty \leq \rho^* \leq 0$) of the whole space. In the case of where the black hole geometry is asymptotically flat as a 5D black hole, the tortoise coordinate $r^*$ goes from $-\infty$ to $\infty$. Hence this is similar to the infinite string problem in which the initial data propagates towards left and right indefinitely \cite{17}. The initial data no longer enjoys this privilege when the background is asymptotically AdS$_3$ because the tortoise coordinate $\rho^*$ goes from $-\infty$ to 0 only. One may consider this as the semi-infinite string problem (or a finite cavity with reflecting walls in AdS$_5$ \cite{22}). Then the Dirichlet or Neumann boundary condition at spatial infinity ($\rho^* = 0$) is required to formulate the problem appropriately. However, we take a different point of view to attack an asymptotically AdS$_3$ problem. This is based on the observation of the shape of potential.
and the global structure of an exact $\text{AdS}_3$ \cite{10}. We first construct the potential $V_\mu(\rho^*)$ by replacing $\rho$ in (70) with $\rho \simeq \rho_+ \coth(\rho_+\rho^*/R^2)$. The potentials in Fig.3 look like the half of the $\text{AdS}_3$-bubble potentials. Especially, we observe that the free field potentials $V_{\psi}^{l=0}(\rho^*)$ and $V_{\psi}^{l=2}(\rho^*)$ in Fig.5 take nearly the same form as in $V_{\phi}^{l=0}(r^*)$ and $(V_{\phi}^{l=2}(r^*), V_{\nu}(r^*), V_{\lambda}(r^*))$ in the region of $-\infty \leq r^* \leq 0$ in Fig.3. Furthermore the Penrose diagram of an exact $\text{AdS}_3(-\infty \leq \rho^* \leq 0)$ is an half of the would-be whole diagram in $-\infty \leq \rho^* \leq \infty$ \cite{10}.

![Figure 5](image_url)

**FIG. 5.** Four potential graphs ($V_T, V_{\psi}^{l=0}, V_{\eta}, V_{\psi}^{l=2}$) as functions of $\rho^*$ for an exact $\text{AdS}_3$.

In this work we assume that the potential of an exact $\text{AdS}_3$ is the left-hand side one of an $\text{AdS}_3$ bubble. The important thing is to calculate the absorption cross section in the background of an exact $\text{AdS}_3$. Considering the $\{\text{in}\}_{\psi}$-state picture, it is not hard to calculate the absorption coefficient $A_{\psi}^{\text{in}}$. Although the $\{\text{out}\}_{\psi}$-state cannot be defined, we can derive $A_{\psi}^{\text{in}}$ from the backscattering of a test field $\psi$ off $V_{\psi}(\rho^*)$. If we choose the boundary condition appropriately, the of potential Fig.3 is enough to calculate the absorption coefficient, regardless of the would-be right-hand side ($0 \leq \rho^* \leq \infty$). In this backscattering process, the key point is to use an appropriate matching procedure between a near $\text{AdS}_3$ and an asymptotically
AdS$_3$. We remind the reader that $\rho^* = 0$ is a timelike boundary and thus information enter or exit from it. This is an exact middle point if one assumes a whole space of $-\infty \leq \rho^* \leq \infty$. Thus requiring the conventional boundary condition may lead to a wrong result in deriving the absorption coefficient. Instead of the Dirichlet condition of $\psi|_{\rho^*=0} = 0$, one may use the non-normalizable modes. The non-normalizable mode is a divergent quantity at $\rho^* = 0$ but its flux is finite at $\rho^* = 0$. Also it corresponds to specifying another boundary condition at spatial infinity. On the other hand, if one use the normalizable modes which satisfy the Dirichlet boundary condition, one may not succeed in obtaining the absorption coefficient in an exact AdS$_3$.

**D. AdS$_3$-AdS$_3$ matching procedure and absorption cross section**

In order to calculate the semiclassical absorption cross section, we have to solve the exact differential equation Eq.(62) with an appropriate boundary condition. Since it is difficult to solve Eq.(62) directly, one has to use the matching procedure between the near-horizon($\rho \sim \rho_+$) AdS$_3$ and the far-region($\rho \to \infty$) AdS$_3$. Here the matching point resides on $-\infty < \rho^* < 0$ in Fig.5.

In the far region Eq.(62) becomes

$$\psi''_{\infty} + \frac{3}{x} \psi'_{\infty} + \frac{\mu}{x^2} \psi_{\infty} = 0$$

(79)

with a dimensionless variable $x = \rho/R$. One easily finds the far-region solution

$$\psi_{\infty}(x) = \left[ \tilde{\alpha} x^{-1+\sqrt{1-\mu}} + \tilde{\beta} x^{-1-\sqrt{1-\mu}} \right]$$

(80)

with two unknown constants $\tilde{\alpha}, \tilde{\beta}$. The first term is a divergent quantity at $\rho = \infty$ but behaves well in the interior region. This corresponds to the non-normalizable modes and is coupled to the boundary operator $O$ at infinity. The second one is the normalizable mode and propagates in the bulk. This can be used to construct two, three and four-point functions. Although one cannot define an ingoing flux at infinity of AdS$_3$, one can calculate the total flux. The flux at spatial infinity is given by
\[ F(\infty) = -2\pi \sqrt{1 - \mu \left| \tilde{\alpha} - i\tilde{\beta} \right|^2}. \]  

(81)

We note here that for the tachyon with \( \mu = 1 \), \( F^T(\infty) = 0 \). This is because the tachyon takes only the normalizable modes which are to be zero at infinity. Thus we exclude it from our analysis.

In order to obtain the near-horizon behavior, we introduce the new variable \( z = (\rho^2 - \rho_0^2)/(\rho^2 - \rho_\pm^2) = (x^2 - x_\pm^2)/(x^2 - x_\pm^2) \). Then Eq. (82) leads to

\[ z(1-z)\frac{d^2\psi}{dz^2} + (1-z)\frac{d\psi}{dz} + \left( A_1 \frac{1}{z} + \frac{\mu/4}{1-z} - B_1 \right) \psi = 0, \]  

(82)

where

\[ A_1 = \left( \frac{\omega - m_\Omega_H}{4\pi T_H^2} \right)^2, \quad B_1 = -\frac{\rho_+^2}{\rho_\pm^2} \left( \frac{\omega - m_\Omega_H \rho_\pm^2/\rho_+^2}{4\pi T_H^2} \right)^2. \]

(83)

In the case of the s-wave propagation with \( m = 0 \), the near-horizon AdS$_3$ equation (82) leads exactly to the near-horizon equation (32) of a 5D black hole. Explicitly, the relationship between these is given by

\[ A_1 \rightarrow Q, \quad \frac{\mu}{4} \rightarrow E, \quad B_1 \rightarrow C. \]  

(84)

In this sense, (32) is called an AdS$_3$ bubble. The ingoing wave is given by the hypergeometric function

\[ \psi(z) = C_1 z^{-i\sqrt{A_1}} (1-z)^{(1-\sqrt{T-\mu})} F(a, b, c; z), \]  

(85)

where

\[ \sqrt{1-\mu} \simeq l + 1, \]  

(86)

\[ a = \frac{1-\sqrt{1-\mu}}{2} - i\sqrt{A_1} + i\sqrt{B_1} \simeq -\frac{l}{2} - i \frac{\omega}{4\pi T_H^2}, \]  

(87)

\[ b = \frac{1-\sqrt{1-\mu}}{2} - i\sqrt{A_1} - i\sqrt{B_1} \simeq -\frac{l}{2} + i \frac{\omega}{4\pi T_H^2}, \]  

(88)

\[ c = 1 - 2i\sqrt{A_1} = 1 - i \frac{\omega}{2\pi T_H^2}. \]  

(89)

Here \( \simeq \) means \( K^2 r_3^2 \simeq 0 \). The corresponding flux is
\[ F(0) = -8\pi \sqrt{A_1(x_+^2 - x_-^2)} |C_1|^2 \]  

(90)

with \( x_+^2 - x_-^2 = (r_0/R)^2 \ll 1 \). The absorption coefficient will be taken as

\[ A^\text{in}_\psi = \frac{F(0)}{F(\infty)} = \frac{4\sqrt{A_1(x_+^2 - x_-^2)}}{\sqrt{1 - \mu}} \frac{|C_1|^2}{|\tilde{\alpha} - i\tilde{\beta}|^2}. \]  

(91)

In order to obtain \( \tilde{\alpha} \) and \( \tilde{\beta} \), we use the matching procedure. It is important to remember that the present spacetime is an exact AdS\(_3\). Thus we have to match the near-AdS\(_3\) with asymptotically AdS\(_3\) to find an absorption coefficient. We know the far-region behavior of (85). This can be found from the hypegeometric function

\[ \psi_{n \rightarrow f}(x) = \left[ C_1 E_1(x_+^2 - x_-^2)^{(1 - \sqrt{1 - \mu})/2} x^{-1 + \sqrt{1 - \mu}} + C_1 E_2(x_+^2 - x_-^2)^{1 + \sqrt{1 - \mu}}/2 x^{-1 - \sqrt{1 - \mu}} \right], \]  

(92)

where

\[ E_1 = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, E_2 = \frac{\Gamma(c)\Gamma(-c + a + b)}{\Gamma(a)\Gamma(b)}. \]  

(93)

Matching (80) with (92) in the far-region \( x \gg 1 \) leads to

\[ \tilde{\alpha} = C_1 E_1 \left( \frac{r_0}{R} \right)^{1 - \sqrt{1 - \mu}}, \tilde{\beta} = C_1 E_2 \left( \frac{r_0}{R} \right)^{1 + \sqrt{1 - \mu}}. \]  

(94)

Considering \( R \gg r_0 \), one finds \( \tilde{\alpha} \gg \tilde{\beta} \) even for \( l = 0 \) case. Hence we can neglect \( \tilde{\beta} \) in favor of \( \tilde{\alpha} \). This amounts to taking the flux of the non-normalizable modes. The key point is an AdS\(_3\)-AdS\(_3\) matching in this backscattering process. Then the absorption coefficient is approximately given by

\[ A_\psi \approx \frac{4\sqrt{A_1(x_+^2 - x_-^2)}}{\sqrt{1 - \mu}} \left( \frac{r_0}{R} \right)^{2(\sqrt{1 - \mu} - 1)} \frac{1}{|E_1|^2}. \]  

(95)

The absorption cross section for AdS\(_3\times S^3\) with \( m = 0 \) leads to

\[ \sigma^\psi_{\text{AdS}} = \frac{A_\psi}{\omega} \approx \frac{\tilde{A}^{6\text{D}}_H}{l!(l + 1)!} \left( \frac{r_0}{R} \right)^{2l} \left| \Gamma\left(\frac{\ell + 2}{2} - i\frac{\omega}{4\pi T_{\text{BTZ}}^{2\text{D}}} \right) \Gamma\left(\frac{\ell + 2}{2} - i\frac{\omega}{4\pi T_{H}^{6\text{D}}} \right) \right|^2 \]  

(96)

with \( \tilde{A}^{6\text{D}}_H = A^{\text{BTZ}}_H \times 2\pi^2 R^3 \).

In the low energy limit \( \omega \to 0 \), it turns out that the 6D cross sections for an exact AdS-theory take the same form as (55) and (57).
\[
\sigma_{\text{AdS}}^\psi_0 = \tilde{A}^{6D}_H, \quad (97)
\]
\[
\sigma_{\text{AdS}}^\psi_2 = \sigma_{\text{AdS}}^\phi = \frac{1}{3} \frac{\tilde{A}^{6D}_H}{4} \left( \frac{r_0}{R} \right)^4. \quad (98)
\]

This 6D result is derived from AdS$_3 \times S^3$. In order for this to match with the cross section of a 5D black hole, it needs to introduce a compactified circle(S$^1$) in M$_5 \times S^1 \times T^4$. In this case one finds \( \tilde{A}^{6D}_H = A^{5D}_H \times 2\pi R \) with a radius of S$^1(R)$. We note that \( \phi_2, \nu(= \Phi) \) and \( \lambda \) give us slightly different cross sections in an AdS$_3$ bubble, whereas these(\( \psi_2, \Phi \)) do not make any distinction in an exact AdS-theory.

**IV. DISCUSSIONS**

It seems that the S-matrix cannot be extracted from the anti-de Sitter space even in a limit \[22,23\]. This is based on the fact that in an exact AdS$_3$ the asymptotic states cannot be defined, due to the timelike boundary and the periodicity of geodesics. However, the authors in \[8\] showed that the correlation functions of the dual CFT$_4$ to AdS$_5$ are considered as the bulk S-matrices. The vacuum correlators \( \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n) \rangle_{\text{CFT}_4} \) of the CFT$_4$ are expressed as truncated n-point functions convolved against the non-normalizable modes. These can be interpreted as an S-matrix for an exact AdS$_5$ arising from a limit of scattering from an AdS$_5$ bubble in asymptotically flat space.

In this work, we show that the S-matrix of an AdS$_3$ bubble can be derived from an exact AdS$_3$ in the dilute gas and low energy limits. We confirm this from the calculation of the absorption cross section. This originates from the fact that the near-horizon equations for an AdS$_3$ bubble \[32\] and an exact AdS$_3$ \[52\] are the same form, but they have different boundary conditions at infinity. In the AdS$_3$ bubble-calculation, one uses the AdS$_3$-AFS matching to obtain the absorption coefficient \( A_{\text{out}}^\psi_N \). On the other hand, in the exact AdS$_3$-calculation we use both the \{in\}-state picture and the non-normalizable mode to obtain the absorption coefficient \( A_{\text{in}}^\psi \). This amounts to taking the AdS$_3$-AdS$_3$ matching. The s-wave greybody factor of a free scalar of an AdS$_3$ bubble has exactly the same form as that of an
exact AdS$_3$. For the dilaton we find the same form of cross section $\sigma = c \tilde{A}^{6D} (r_0/R)^4$ but with $c = 1/4$ for an AdS$_3$ bubble and $c = 1/12$ for an exact AdS$_3$.

Let us compare our results with the others. The general formula for the greybody factor is derived from the vacuum two-point function $\langle \mathcal{O}(x)\mathcal{O}(0) \rangle_{\text{CFT}_2}$ of a boundary operator $\mathcal{O}$ in the effective string [24] and boundary CFT$_2$ approaches [25]. These give us the same result for a free scalar but for the dilaton, $c = 1/4$ as in an AdS$_3$ bubble. Consequently, the two-point correlator provides us the greybody factor in the dilute gas and low energy limits. This quantity takes exactly the same form in the CFT$_2$ and AdS$_3$ bubble approaches. Further, in the exact AdS$_3$ approach one finds the same form of the greybody factor. This means that the S-matrix can be derived from an exact AdS$_3$ space. It is obvious that the conformal limit of the gauge theory(CFT$_2$) corresponds with scattering from an exact AdS$_3$. Finally we present here a scattering picture in an exact AdS$_3$ and compare it with the scattering of an AdS$_3$ bubble in AFS.

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