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Abstract. In this paper, by using the Alexandrov-Serrin method of moving plane combined with integral inequalities, we obtained the complete classification of positive solution for a class of degenerate elliptic system.

1. Introduction. We consider the following degenerate elliptic system:

\[
\begin{align*}
-yu_{yy} - au_y - \Delta_x u &= f(u, v), & \text{in } \mathbb{R}^{N+1}_+,
-yv_{yy} - av_y - \Delta_x v &= g(u, v), & \text{in } \mathbb{R}^{N+1}_+,
\end{align*}
\]

(1)

where \( a > 1 \) is a constant, \( u = u(x, y), v = v(x, y) \in C^2(\mathbb{R}^{N+1}_+) \) with \((x, y) \in \mathbb{R}^N \times \mathbb{R}^+ := \mathbb{R}^{N+1}_+\).

The main purpose of the present paper is the following: On the one hand, we show how the method of moving plane combined with integral inequalities, as developed in [4] (see also [9], [10], [6]) can be applied to produce simple proofs of Liouville type theorems for a class of degenerate elliptic systems with general nonlinearities, and no any boundary condition is imposed on the boundary \( y = 0 \). On the other hand, we give the complete classifications of positive solution for system (1). More precisely, we prove that system (1) has no positive solution when the nonlinearities are subcritical unless \[ f(u, v) = mu^{p_1}v^{q_1}, g(u, v) = lu^{p_2}v^{q_2} \]
with \( m, l \) being constants and \[ p_1 + q_1 = \frac{N+2a+2}{N+2a}, p_2 + q_2 = \frac{N+2a+2}{N+2a-2}. \] Moreover, in the later case, all the positive solutions of (1) take the form \[ u = a_1 U, v = b_1 U, \]
where \( a_1 = ma_1^{p_1}b_1^{q_1}, b_1 = la_1^{p_2}b_1^{q_2} \), and \( U > 0 \) is the unique positive solution (up to translation and scaling) of the following equation (see [7]):

\[
\begin{align*}
yU_{yy} + aU_y + \Delta_x U + U^{\frac{N+2a+2}{N+2a-2}} &= 0,
U(x, y) \geq 0, U(x, y) \in C^2(\mathbb{R}^{N+1}_+).
\end{align*}
\]

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As far as the system (1.1) is concerned, beside the difficulty caused by the degeneracy of the equations, the nonlinearities are assumed to be continuous, the method for classical solution can not be applied directly. Moreover, since the nonlinearities are coupled each other, we are not clear where we can start the method of moving plane. Furthermore, in the degenerate case, the system is not rotation invariant and new maximal principles are needed. Fortunately, motivated by the work of Huang [7] and Huang and Li [8], based on the Kelvin’s transform, we may use integral inequality in a narrow domain—as was used in Terracini in [9], [10] and Damascelli in [1], [4]—which helps us to start the method of moving plane.

Let \( f, g \) be continuous functions on \( \mathbb{R}^2 \),

\[ E = W^{1,2}_0(\mathbb{R}^{N+1}_+) \cap C^0(\mathbb{R}^{N+1}_+) \times W^{1,2}_0(\mathbb{R}^{N+1}_+) \cap C^0(\mathbb{R}^{N+1}_+) \]

and \( W = C^0_0(\mathbb{R}^{N+1}_+) \). We say \((u, v)\) is a solution of (1) in the sense of weak, if \((u, v) \in E\)

\[
\begin{align*}
\int_{\mathbb{R}^{N+1}_+} (-yu_{yy} - au_y - \Delta_x u) \varphi &= \int_{\mathbb{R}^{N+1}_+} f(u, v) \varphi, \quad \forall \varphi \in W, \\
\int_{\mathbb{R}^{N+1}_+} (-vy_{yy} - av_y - \Delta_x v) \varphi &= \int_{\mathbb{R}^{N+1}_+} g(u, v) \varphi, \quad \forall \varphi \in W.
\end{align*}
\]

One of our main results is:

**Theorem 1.1.** Let \((u, v) \in E\) be a weak solution of the following problem:

\[
\begin{align*}
(-\Delta_x - y \partial_y^2 + au_y - \Delta_x u)u(\mathbf{x}, \mathbf{y}) &= f(\mathbf{u}(\mathbf{x}, \mathbf{y})), \\
(-\Delta_x - y \partial_y^2 + av_y - \Delta_x v)v(\mathbf{x}, \mathbf{y}) &= g(\mathbf{w}(\mathbf{x}, \mathbf{y})), \\
u(\mathbf{x}, \mathbf{y}) &\geq 0,
\end{align*}
\]

\((x, y) \in \mathbb{R}^N \times \mathbb{R}^+ =: \mathbb{R}^{N+1}_+\). (4)

Suppose that \(a > 1, f, g : [0, \infty) \rightarrow \mathbb{R}^+\) are continuous functions satisfying:

(i) \( h(t), g(t) \) are nondecreasing in \((0, \infty)\);

(ii) \( h(t) = \frac{f(t)}{t^{N+2a+2}}, \quad k(t) = \frac{g(t)}{t^{N+2a+2}}\) are nonincreasing in \((0, \infty)\).

Then either \((u, v) = (C_1, C_2)\) for some constants \(C_1, C_2\) with \(f(C_2) = g(C_1) = 0\) or there exist positive constants \(m, l\) such that \(h(t) = m, k(t) = l\) and \((u, v)\) has the form

\[
u = a_1 \frac{[(N+2a)(N+2a-2)]^{N+2a-2}}{|s^2+4y+|x-x_0|^2|^{N+2a-2}}\]

\[
\begin{align*}
\rho &= b_1 \frac{[N+2a(N+2a-2)]^{N+2a-2}}{|x^2+4y+|x-x_0|^2|^{N+2a-2}}, \quad s \text{ is a positive parameter and } x_0 \text{ is a point in } \mathbb{R}^N, \quad \text{and } a_1, b_1 \text{ are constants satisfying } a_1 = mb_1, b_1 = la_1.
\end{align*}
\]

We remark that Theorem 1.1 can be extended to more general cases like (1) (see Section 3). For applications convenience, we have the following corollary.

**Corollary 1.2.** Let \((u, v) \in E\) be a weak solution of the problem:

\[
\begin{align*}
-(\Delta_x + y \partial_y^2 + au_y)u &= \sum_{i} u^{p_{ij_1}}v^{q_{ij_1}}, \\
-(\Delta_x + y \partial_y^2 + av_y)v &= \sum_{i} u^{p_{ij_2}}v^{q_{ij_2}},
\end{align*}
\]

in \( \mathbb{R}^{N+1}_+ \). (5)

Suppose that \( p_{ji}, q_{ji} \geq 0, \quad p_{ji} + q_{ji} \leq \frac{N+2a+2}{N+2a-2}, \quad j = 1, 2, \quad i = 1, 2, ..., n. \) Then (1.5) has no solutions unless \( p_{ji} + q_{ji} = \frac{N+2a+2}{N+2a-2}, j = 1, 2, i = 1, 2, ..., n. \)

Before the end of this introduction, we would like to mention that the non-existence results obtained in our paper will lead to an *a priori* estimates for positive
solutions of some semi-linear degenerate elliptic system in bounded domains, which arising from the study of geometry problem. For the proof of \textit{priori} bounds, one can use the blow up method, we refer to [5] for the corresponding analysis for Laplacian equation with Dirichlet boundary conditions and to [3] for a blow-up analysis for some mixed boundary problems in bounded domains. However, when the blow up point approach to the boundary, the problem becomes more complicated when the equation is degenerate and without any boundary conditions on the boundary, we refer to Huang [7] for the discuss for a single degenerate equation. For a general degenerate elliptic system, the problem will become more difficult, we will discuss this problem in forthcoming paper.

The paper is organized as follows. In Section 2, we discuss a simpler system \((4)\) and give a detailed proof of Theorem 1.1. We consider the extension system \((1)\) (Theorem 3.1) and \((27)\) (Theorem 3.2) in Section 3. We will show that how the techniques used in section 2 allow us to deal with other more general problems. We pay our attention on the proof of the analogous of those key lemmas in Section 2 and the uniqueness form of positive the solutions. We also state and prove a non-existence result (Theorem 3.9) under stronger smoothness assumptions on \(f, g\) but with simpler other conditions.

2. Preliminaries and the proof of Theorem 1.1. In this section, we first give some general facts for degenerate system \((4)\). We begin with some notations and comments. Let \(u, v \in C^2(\mathbb{R}_+^{N+1})\) be the solution of \((4)\), set \(x_{N+1} = 2\sqrt{y}\), we define

\[
\bar{u}(x_1, \ldots, x_N, x_{N+1}) = u(x_1, \ldots, x_N, y),
\]

\[
\bar{v}(x_1, \ldots, x_N, x_{N+1}) = v(x_1, \ldots, x_N, y).
\]

Then a direct computation shows that

\[
\begin{cases}
-\Delta \bar{u} - \frac{2a-1}{x_{N+1}} \frac{\partial \bar{u}}{\partial x_{N+1}} = f(\bar{v}) & \text{in } \mathbb{R}_+^{N+1}, \\
-\Delta \bar{v} - \frac{2a-1}{x_{N+1}} \frac{\partial \bar{v}}{\partial x_{N+1}} = g(\bar{u}) & \text{in } \mathbb{R}_+^{N+1},
\end{cases}
\]

and since \(u, v \in C^2(\mathbb{R}_+^{N+1})\), it holds that

\[
\frac{\partial \bar{u}}{\partial x_{N+1}}(x_1, \ldots, x_N, 0) = 0, \quad \frac{\partial \bar{v}}{\partial x_{N+1}}(x_1, \ldots, x_N, 0) = 0.
\]

Thus we can extend \(\bar{u}, \bar{v}\) to the whole space \(\mathbb{R}^{N+1}\) by

\[
\tilde{w}(x_1, \ldots, x_N, x_{N+1}) = \begin{cases}
\bar{u}(x_1, \ldots, x_N, x_{N+1}), & \text{if } x_{N+1} \geq 0, \\
\bar{u}(x_1, \ldots, x_N, -x_{N+1}), & \text{if } x_{N+1} < 0,
\end{cases}
\]

\[
\tilde{z}(x_1, \ldots, x_N, x_{N+1}) = \begin{cases}
\bar{v}(x_1, \ldots, x_N, x_{N+1}), & \text{if } x_{N+1} \geq 0, \\
\bar{v}(x_1, \ldots, x_N, -x_{N+1}), & \text{if } x_{N+1} < 0.
\end{cases}
\]

Then \(\tilde{w}, \tilde{z} \in C^2(\mathbb{R}^{N+1})\) with \(\frac{\partial \tilde{w}}{\partial x_{N+1}}(x_1, \ldots, x_N, 0) = 0, \frac{\partial \tilde{z}}{\partial x_{N+1}}(x_1, \ldots, x_N, 0) = 0\). We introduce their Kelvin’s transforms centered at the origin:

\[
w(x) = \frac{1}{|x|^{N+2a-2}} \tilde{w} \left( \frac{x}{|x|^2} \right), \quad z(x) = \frac{1}{|x|^{N+2a-2}} \tilde{z} \left( \frac{x}{|x|^2} \right), \quad x \in \mathbb{R}^{N+1} \setminus \{0\}.
\]
Obviously, \( w, z \) are continuous and strictly positive in \( \mathbb{R}^{N+1}_+ \setminus \{0\} \). And a direct computation yields:

**Lemma 2.1.** Let \( u, v \in C^2(\mathbb{R}^{N+1}_+) \) be a positive solution of system (4). Then \((w, z)\) satisfies the following system:

\[
\begin{aligned}
-\Delta_w(x) - \frac{2a-1}{x_{N+1}} \frac{\partial w}{\partial x_{N+1}} &= \frac{1}{|x|^{N+2a}} f(|x|^{N+2a-2}z(x)) \\
-\Delta_z(x) - \frac{2a-1}{x_{N+1}} \frac{\partial z}{\partial x_{N+1}} &= \frac{1}{|x|^{N+2a}} g(|x|^{N+2a-2}w(x)), \quad x \in \mathbb{R}^{N+1}_+ \setminus \{0\}. \\
\end{aligned}
\]

Moreover \((w, z)\) has the asymptotic behavior at \( \infty \):

\[
\begin{aligned}
 w(x) &= \frac{a_0}{|x|^{N+2a}} + \sum_{i=1}^{N+1} \frac{a_i x_i}{|x|^{N+2a}} + O(\frac{1}{|x|^{N+2a}}) \\
 \partial_i w(x) &= -\frac{(N+2a-2)a_i x_i}{|x|^{N+2a}} + O(\frac{1}{|x|^{N+2a}}) \\
 \partial_{ij} w(x) &= O(\frac{1}{|x|^{N+2a}}), \quad (i, j = 1, \ldots, N+1) \\
\end{aligned}
\]

and

\[
\begin{aligned}
 z(x) &= \frac{b_0}{|x|^{N+2a}} + \sum_{i=1}^{N+1} \frac{b_i x_i}{|x|^{N+2a}} + O(\frac{1}{|x|^{N+2a}}) \\
 \partial_i w(x) &= -\frac{(N+2a-2)b_i x_i}{|x|^{N+2a}} + O(\frac{1}{|x|^{N+2a}}) \\
 \partial_{ij} w(x) &= O(\frac{1}{|x|^{N+2a}}), \\
\end{aligned}
\]

where \( a_i, b_i (i = 1, 2, \ldots, N+1) \) are constants satisfying \( a_0, b_0 > 0 \). As a result \( w, z \in L^{\frac{2(N+2a)}{N+2a-2}}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^{N+1}_+ \setminus B_r(0)) \) for any \( r > 0 \).

In what follows, we shall use the method of moving plane. Without loss of generalization, we start by considering planes parallel to \( x_1 = 0 \), coming from \(+\infty\). For each \( \lambda > 0 \), we write \( x = (x_1, x') \) with \( x' = (x_2, \ldots, x_{N+1}) \in \mathbb{R}^N \) and define \( \Sigma_\lambda := \{ x = (x_1, x') | x_1 > \lambda \} \), \( T_\lambda := \partial \Sigma_\lambda = \{ x = (x_1, x') | x_1 = \lambda \} \). For \( x = (x_1, x') \in \Sigma_\lambda \), let \( x_\lambda = (2\lambda - x_1, x') \) be the reflected point with respect to the hyperplane \( T_\lambda \), and \( \Sigma_\lambda = \Sigma_\lambda \setminus \{ e_\lambda \} \) with \( e_\lambda = (2\lambda, 0, \ldots, 0) \). We define the reflected functions by:

\[
w_\lambda(x) = w(x), \quad z_\lambda(x) = z(x).
\]

Set

\[
W_\lambda(x) = w(x) - w_\lambda(x), \quad Z_\lambda(x) = z(x) - z_\lambda(x).
\]

**Lemma 2.2.** Set \( \bar{s}(a) = \frac{2(N+2a)}{N+2a-2} \), for any fixed \( \lambda > 0 \), \( w, z \in L^{\bar{s}(a)}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda) \), \( W^+_\lambda(x), Z^+_\lambda(x) \in L^{\bar{s}(a)}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda) \cap W^{1,2}(\Sigma_\lambda) \). Moreover, there exists \( C_\lambda > 0 \), nonincreasing in \( \lambda \), such that

\[
\begin{aligned}
||W^+_\lambda||_{L^{\bar{s}(a)}(\Sigma_\lambda)} &\leq C_\lambda ||x|^{-2(N+2a)}||_{L^1(A^{(1)})} ||Z^+_\lambda||_{L^{\bar{s}(a)}(\Sigma_\lambda)} + C_\lambda ||x|^{-\frac{2(N+2a)}{N+2a-2}}||_{L^1(A^{(1)})} ||W^+_\lambda||_{L^{\bar{s}(a)}(\Sigma_\lambda)}, \\
||Z^+_\lambda||_{L^{\bar{s}(a)}(\Sigma_\lambda)} &\leq C_\lambda ||x|^{-2(N+2a)}||_{L^1(A^{(1)})} ||W^+_\lambda||_{L^{\bar{s}(a)}(\Sigma_\lambda)} + C_\lambda ||x|^{-\frac{2(N+2a)}{N+2a-2}}||_{L^1(A^{(1)})} ||Z^+_\lambda||_{L^{\bar{s}(a)}(\Sigma_\lambda)},
\end{aligned}
\]
where $A^1_\lambda = \{ x \in \tilde{\Sigma}_\lambda : W_\lambda(x) \geq 0 \}, \ A^2_\lambda = \{ x \in \tilde{\Sigma}_\lambda : Z_\lambda(x) \geq 0 \}, \ W_\lambda^+ = \max \{ W_\lambda, 0 \}, Z_\lambda^+ = \max \{ Z_\lambda, 0 \}.$

Proof. We just prove (12), the proof of (13) is similar. For any fixed $\lambda > 0$, there exists $r > 0$ such that $\Sigma_\lambda \subset \mathbb{R}^{N+1} \setminus B_r(0)$, thus $w$ and $W_\lambda^+ \leq w \in L^{\frac{N}{N+2a+2}}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda)$ and \( \frac{1}{|x|^{N+2a+2}}, \frac{1}{|x|^{2(N+2a+2)}} \) are integrable in $\Sigma_\lambda$.

For $\epsilon > 0$ small, let $\eta_\epsilon \in C^1_0(\mathbb{R}^{N+1})$ be a cut-off function such that $0 \leq \eta_\epsilon \leq 1, \eta_\epsilon(x) = 1$ for $2\epsilon \leq |x - e\lambda| \leq \frac{|l|}{2}; \eta_\epsilon(x) = 0$ for $|x - e\lambda| \leq \epsilon$ or $|x - e\lambda| \geq \frac{|l|}{2}$, and $|\nabla \eta_\epsilon| \leq \frac{2}{\epsilon}$ for $\epsilon \leq |x - e\lambda| \leq 2\epsilon, |\nabla \eta_\epsilon| \leq 2\epsilon$ for $\frac{1}{\epsilon} \leq |x - e\lambda| \leq \frac{2}{\epsilon}$. Since $h(t) = \frac{f(t)}{t^{N+2a+2}}$, then for $x \in \tilde{\Sigma}_\lambda$, we have

\[
- \Delta w(x) - \frac{2a - 1}{x_{N+1}} \frac{\partial w}{\partial x_{N+1}} = h(|x|^{N+2a-2}z(x))z^{\frac{N+2a+2}{N+2a}}(x), \\
- \Delta w_\lambda(x) - \frac{2a - 1}{x_{N+1}} \frac{\partial w_\lambda}{\partial x_{N+1}} = h(|x\lambda|^{N+2a-2}z_\lambda(x))z^{\frac{N+2a+2}{N+2a}}(x),
\]

where \( \frac{\partial w}{\partial x_{N+1}}(x_1, ..., x_N, 0) = 0, \frac{\partial w_\lambda}{\partial x_{N+1}}(x_1, ..., x_N, 0) = 0, \ x \in \mathbb{R}^{N+1} \setminus \{ 0 \} \).

Now we test the above two equations in $\tilde{\Sigma}_\lambda$ with the function $\varphi_\epsilon = \eta_\epsilon^2 W_\lambda^+$, note that

\[
|\nabla W_\lambda^+ \eta_\epsilon|^2 = \nabla W_\lambda^+ \nabla \varphi_\epsilon + |W_\lambda^+|^2 |\nabla \eta_\epsilon|^2.
\]

By subtracting the two tested equations, we have

\[
\int_{\tilde{\Sigma}_\lambda} |\nabla W_\lambda^+ \eta_\epsilon|^2 = \int_{\tilde{\Sigma}_\lambda} [h(|x|^{N+2a-2}z(x))z^{\frac{N+2a+2}{N+2a}}(x) - h(|x\lambda|^{N+2a-2}z_\lambda(x))z^{\frac{N+2a+2}{N+2a}}(x)] \varphi_\epsilon + I^1_\epsilon + I^2_\epsilon,
\]

where $I^1_\epsilon = \int_{\Sigma_\lambda} (W_\lambda^+)^2 |\nabla \eta_\epsilon|^2, I^2_\epsilon = \int_{\Sigma_\lambda} (\frac{2a - 1}{x_{N+1}} \frac{\partial W_\lambda^+}{\partial x_{N+1}}) \varphi_\epsilon$.

Since $|x| > |x\lambda|$, $h$ is nonincreasing, if $z(x) \geq z(x\lambda) \geq 0$, then

\[
h(|x\lambda|^{N+2a-2}z_\lambda(x)) \geq h(|x|^{N+2a-2}z(x)).
\]

If $z(x) \leq z(x\lambda)$, noting that $f$ is nondecreasing and $h$ is nonincreasing, we have

\[
h(|x|^{N+2a-2}z(x))z^{\frac{N+2a+2}{N+2a}}(x) = \frac{f(|x|^{N+2a-2}z(x))}{|x|^{N+2a+2}} \leq \frac{f(|x\lambda|^{N+2a-2}z_\lambda(x))}{|x\lambda|^{N+2a+2}} \leq \frac{f(|x\lambda|^{N+2a-2}z_\lambda(x))}{|x\lambda|^{N+2a+2}} = h(|x\lambda|^{N+2a-2}z_\lambda(x))z^{\frac{N+2a+2}{N+2a}}(x),
\]

thus

\[
\int_{\tilde{\Sigma}_\lambda} |\nabla W_\lambda^+ \eta_\epsilon|^2 \leq \int_{\tilde{\Sigma}_\lambda} h(|x|^{N+2a-2}z(x))(z^{\frac{N+2a+2}{N+2a}} - z^{\frac{N+2a+2}{N+2a}})^+ \varphi_\epsilon + I^1_\epsilon + I^2_\epsilon \leq \int_{A^1_\lambda} h(|x|^{N+2a-2}z(x))(z^{\frac{N+2a+2}{N+2a}} - z^{\frac{N+2a+2}{N+2a}})^+ \varphi_\epsilon + I^1_\epsilon + I^2_\epsilon.
\]
Since \( v \) is positive and locally bounded, there are constants \( 0 < C_λ^1 < C_λ^2 \) such that

\[
0 < C_λ^1 < |x|^{N+2a-2} z(x) = v\left(\frac{x}{|x|^2}\right) < C_λ^2, \quad \forall x \in \Sigma_λ \subset \mathbb{R}^{N+1} \setminus B_ε(0),
\]

(18) hence

\[
0 < h(|x|^{N+2a-2} z(x)) \leq h(C_λ) := C_λ, \quad \forall x \in \Sigma_λ \subset \mathbb{R}^{N+1} \setminus B_ε(0).
\]

(19) Moreover, for \( 0 \leq z_λ \leq z \), since \( z \in L^∞(\Sigma_λ) \), and \( z \) decays at infinity as \( \frac{1}{|x|^{N+2a-2}} \), one has

\[
(z^{\frac{N+2a+2}{N+2a-2}} - z_λ^{\frac{N+2a+2}{N+2a-2}})^+ \leq \frac{N + 2a + 2}{N + 2a - 2} z^{\frac{N+2a-2}{N+2a-2}} (z - z_λ)^++ \leq C_λ \frac{1}{|x|^2}(z - z_λ)^+.
\]

(20) Here and in the following of the paper, we always use the same \( C_λ \) to stand for different constants.

Combining Hölder inequality with the previous estimates, we have

\[
\int_{\Sigma_λ} |\nabla W_λ^+ \eta_ε|^2 \leq \int_{A_λ^ε} h(|x|^{N+2a-2} z(x)) (z^{\frac{N+2a+2}{N+2a-2}} - z_λ^{\frac{N+2a+2}{N+2a-2}}) \varphi_ε + I^1_ε + I^2_ε
\]

\[
\leq C_λ \int_{A_λ^ε} \frac{1}{|x|^2} |\nabla z_λ^+| (z - z_λ)^+(w - w_λ)^++ + I^1_ε + I^2_ε
\]

\[
\leq C_λ \left( \int_{A_λ^ε} \frac{1}{|x|^{2(N+2a)}} \right)^{\frac{1}{2}} \left( \int_{\Sigma_λ} |(z - z_λ)^+|^{\frac{2}{a}} \right)^{\frac{1}{2}} \left( \int_{\Sigma_λ} |\nabla \eta_ε|^{N+2a} \right)^{\frac{1}{N+2a}}
\]

\[
\times \left( \int_{\Sigma_λ} |(w - w_λ)^+|^{\frac{2}{a}} \right)^{\frac{1}{2}} + I^1_ε + I^2_ε.
\]

(21) Now we estimate the term \( I^1_ε \). Let \( B_ε = \{ x \in \Sigma_λ : \epsilon < |x - e_λ| < 2\epsilon \ or \ \frac{1}{ε} < |x - e_λ| < \frac{1}{2}\epsilon \} \). Since \( \int_{B_ε} |\nabla \eta_ε|^{N+2a} \leq C \), we have

\[
I^1_ε \leq \left( \int_{B_ε} |(w - w_λ)^+|^{\frac{2}{a}} \right)^{\frac{1}{2}} \left( \int_{\Sigma_λ} |\nabla \eta_ε|^{N+2a} \right)^{\frac{1}{N+2a}} \rightarrow 0,
\]

as \( \epsilon \to 0 \) provided that \( (w - w_λ)^+ \in L^{2^*(a)}(\Sigma_λ) \).

Next we estimate the term \( I^2_ε \). In view of the asymptotic behavior of \( w(x) \) at \( \infty \) (see (10)), we have

\[
|I^2_ε| = \left| \int_{\Sigma_λ} \frac{2a - 1}{|x|^{N+1}} \frac{∂W_λ^+}{∂x_{N+1}} W_λ^+ dx \right|
\]

\[
\leq C_λ \left( \int_{A_λ^ε} \frac{1}{|x|^{2(N+2a)}} \right)^{\frac{1}{2}} \left( \int_{\Sigma_λ} |(Z_λ^+)^{\frac{1}{2^*(a)}}|^{\frac{2}{N+2a}} \right)^{\frac{1}{2}} \left( \int_{\Sigma_λ} |(W_λ^+)^{\frac{2}{N+2a-2}}|^{\frac{N+2a-2}{N+2a-2}} \right)^{\frac{1}{N+2a-2}}.
\]

(22) By using dominated convergence and Sobolev’s inequality, letting \( \epsilon \to 0 \) in (21), we get

\[
\left( \int_{\Sigma_λ} |W_λ^+|^{\frac{2(N+2a)}{N+2a-2}} \right)^{\frac{N+2a-2}{N+2a}} \leq C_λ \left( \int_{A_λ^ε} \frac{1}{|x|^{2(N+2a)}} \right)^{\frac{1}{2}} \left( \int_{\Sigma_λ} |Z_λ^+|^{\frac{2}{N+2a-2}} \right)^{\frac{N+2a-2}{2}} \times \left( \int_{\Sigma_λ} |W_λ^+|^{\frac{2}{N+2a-2}} \right)^{\frac{N+2a-2}{2N+2a}} + C_λ \left( \int_{A_λ^ε} \frac{1}{|x|^{2(N+2a)}} \right)^{\frac{1}{2}} \left( \int_{\Sigma_λ} |W_λ^+|^{\frac{2}{N+2a-2}} \right)^{\frac{N+2a-2}{2N+2a}}.
\]

(23)
Lemma 2.3. There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, $W_\lambda(x) \leq 0$ and $Z_\lambda(x) \leq 0$ for all $x \in \Sigma_\lambda$.

Proof. Since $\int_{\mathbb{R}^{N+1}} \frac{1}{|x|^{2(N+2a)}} \frac{1}{|x|^{2(N+2a+2)}} \in L^1(\mathbb{R}^{N+1} \setminus B_r)$ for any $r > 0$, hence
\[
\int_{A_1^\lambda} \frac{1}{|x|^{2(N+2a)}} \leq \int_{\Sigma_\lambda} \frac{1}{|x|^{2(N+2a)}} \to 0 \text{ as } \lambda \to +\infty,
\]
and
\[
\int_{A_1^\lambda} \frac{1}{|x|^{2(N+2a+2)}} \leq \int_{\Sigma_\lambda} \frac{1}{|x|^{2(N+2a+2)}} \to 0 \text{ as } \lambda \to +\infty.
\]
It follows that there exist $\lambda_0 > 0$, such that
\[
\frac{1}{|x|^{2(N+2a)}} < 1, \quad \frac{1}{|x|^{2(N+2a+2)}} < 1, \text{ for all } \lambda \geq \lambda_0,
\]
\[
\frac{1}{|x|^{2(N+2a)}} < 1, \quad \frac{1}{|x|^{2(N+2a+2)}} < 1, \text{ for all } \lambda \geq \lambda_0.
\]
By Lemma 2.2, we deduce that $||W_\lambda^+||_{L^{2^*}(-)} = 0$ and $||Z_\lambda^+||_{L^{2^*}(-)} = 0$ for all $\lambda \geq \lambda_0$, this implies that $W_\lambda \leq 0$ and $Z_\lambda \leq 0$ for all $x \in \Sigma_\lambda$ and $\lambda \geq \lambda_0$.

To proceed, we will need the following invariant of Maximum principle (see [7]). Consider the following elliptic operator:
\[
L(u) = \sum_{i=1}^{N+1} a_{ij}(x) D_{ij}u + \sum_{i=1}^{N} b_i(x) D_i u + \frac{a(x)}{x^{N+1}} D_{N+1}u.
\]

Lemma 2.4. Suppose that $a_{ij}(x), b_i(x), a(x) \in C(\mathbb{R}^{N+1}), a(x) \geq 0$ and the matrix $(a_{ij})$ is positive definite. Assume that $u \in C^2(B_1) \cap C(\bar{B}_1)$ with $\frac{\partial u}{\partial x_{N+1}} |_{(x_1, \ldots, x_N, 0)} = 0$ satisfying
\[
L(u) \geq 0 \text{ in } B_1.
\]
Then either $u$ is a constant or $u$ cannot attain its maximum in $B_1$, where $B_1$ is the unit ball in $\mathbb{R}^{N+1}$.

Lemma 2.5. Suppose that all the coefficients $a_{ij}, b_i, a$ are the same as in Lemma 2.4. Assume $u \in C^2(B_1) \cap C(\bar{B}_1)$ with $\frac{\partial u}{\partial x_{N+1}} |_{(x_1, \ldots, x_N, 0)} = 0$ satisfying
\[
L(u) \geq 0 \text{ in } B_1.
\]
If $u$ attains its maximum at $x_0 \in \partial B_1$, then either $u$ is a constant or $-\frac{\partial u}{\partial n} |_{x=x_0} < 0$, where $n$ is the outward normal to $\partial B_1$ at $x_0$.

Now we define
\[
\Lambda = \inf\{\lambda > 0 | W_\mu(x) \leq 0, Z_\mu(x) \leq 0, \forall x \in \Sigma_\mu, \mu > \lambda\}. \quad (24)
\]

Lemma 2.6. If $\Lambda > 0$, then $W_\lambda \equiv 0$ and $Z_\lambda \equiv 0$ for all $x \in \Sigma_\Lambda$.

Proof. By continuity, we see that $W_\lambda(x) \leq 0, Z_\lambda(x) \leq 0$ for $x \in \Sigma_\Lambda$.

By (16), for $x \in \Sigma_\Lambda$, we have
\[
h(|x|^{N+2a-2} z(x)) z(\frac{N+2a}{N+2a-2} x) \leq h(|x_\Lambda|^{N+2a-2} z_\Lambda(x)) z_\Lambda(x) \frac{N+2a+2}{N+2a-2},
\]
providing that \( z - z_A \leq 0 \). Hence
\[
-\Delta w - \frac{2a-1}{x_{N+1}} \frac{\partial w}{\partial x_{N+1}} = h(|x|^{N+2a-2} z(x)) \frac{N+2a+2}{N+2} \leq h(|x|^{N+2a-2} z_A(x)) \frac{N+2a+2}{N+2} = -\Delta w_A - \frac{2a-1}{x_{N+1}} \frac{\partial w_A}{\partial x_{N+1}}.
\]
That is \(-\Delta W_A - \frac{2a-1}{x_{N+1}} \frac{\partial W_A}{\partial x_{N+1}} \leq 0\). Note that \( W_A \leq 0 \) and by Lemma 2.4, either \( W_A \equiv 0 \) or \( W_A < 0 \) in \( \bar{\Sigma}_A \).

Now suppose \( W_A < 0 \) in \( \bar{\Sigma}_A \), let \( \chi_S \) be the characteristic function of the set \( S \).
Then \( \frac{1}{|x|^{N+2a}} \chi_{A_1^1} \) converges pointwisely to zero as \( \lambda \to \Lambda \) in \( \mathbb{R}^{N+1} \setminus (T_A \cup \{e_1\}) \).
Thus if \( 0 < \Lambda - \delta < \Lambda \), then
\[
\int_{A_1^1} \frac{1}{|x|^{2(N+2a)}} \chi_{A_1^1} \leq \frac{1}{|x|^{2(N+2a)}} \chi_{\Sigma_A - \delta} \in L^1(\Sigma),
\]
by dominate convergence, we get
\[
\int_{A_1^1} \frac{1}{|x|^{2(N+2a)}} \to 0 \text{ as } \lambda \to \Lambda.
\]
Similarly,
\[
\int_{A_2^2} \frac{1}{|x|^{2(N+2a)}} \to 0 \text{ as } \lambda \to \Lambda,
\]
and
\[
\int_{A_1^1} \frac{1}{|x|^{2(N+2a)}} \to 0, \quad \int_{A_1^1} \frac{1}{|x|^{2(N+2a)}} \to 0 \text{ as } \lambda \to \Lambda.
\]

Combine with the previous arguments, we can deduce that \( W_A(x) \leq 0 \) and \( Z_A(x) \leq 0 \) in \( \Sigma_A \) for \( \lambda < \Lambda \), which contradicts with the definition of \( \Lambda \).

Before the proof of Theorem 1.1, we first have:

**Proposition 2.7.** Let \( u, v \) and \( f, g \) be as in Theorem 1.1 and suppose that \((u, v)\) is positive in \( \mathbb{R}^{N+1}_+ \). Let \((w, z)\) be the Kelvin's transform of \((u, v)\) centered at a point \( p \). Then \((w, z)\) is radially symmetric with respect to some point \( q \) in \( \mathbb{R}^{N+1} \). Moreover if \( h, k \) are not constants in \((0, \sup_{x \in \mathbb{R}^{N+1}} v(x))\) and in \((0, \sup_{x \in \mathbb{R}^{N+1}} u(x))\), respectively, then \((w, z)\) is radially symmetric around the pole of Kelvin's transform, i.e. \( q = p \).

**Proof.** To prove that \( w \) and \( z \) are radially symmetric, we use the method of moving plane and prove the symmetry in every direction. Without loss of generality, for simplicity, we choose \( x_1 \) direction and prove that \( w, z \) are symmetric with respect to the \( x_1 \) direction.

If \( \Lambda = 0 \), we conclude by continuity that \( w(x) \leq w_0(x) \) and \( z(x) \leq z_0(x) \) for all \( x \in \Sigma_0 \). We perform the moving plane procedure from the left and find a corresponding \( \Lambda^l = 0 \) (again if \( \Lambda^l < 0 \), an analogue to Lemma 2.6 shows that \( w \) and \( z \) are symmetric with respect to \( \partial T^l_A \)), from which we get \( w_0(x) \leq w(x) \) and \( z_0(x) \leq z(x) \) for all \( x \in \Sigma_0 \). So \( w \) and \( z \) are symmetric with respect to the plane \( x_1 = 0 \). Therefore, if \( \Lambda = 0 \) for all directions, then \( w, z \) and hence \( u, v \) are radially symmetric around the point of the Kelvin’s transform.

Now we suppose that \( \Lambda > 0 \), then we have \( w \equiv w_A \) and \( z \equiv z_A \). This implies that \( w, z \) are regular at the origin, and hence \( u, v \) are regular at infinity. Since \( w \equiv w_A, z \equiv z_A \), we have
\[
-\Delta w - \frac{2a-1}{x_{N+1}} \frac{\partial w}{\partial x_{N+1}} = -\Delta w_A - \frac{2a-1}{x_{N+1}} \frac{\partial w_A}{\partial x_{N+1}}.
\]
which implies that
\[ h(|x|^N + 2a - 2z(x)) = h(|x_0|^N + 2a - 2z_0(x)). \]  
(25)

Note that for any \( x \in \Sigma_\Lambda, |x| > |x_\Lambda| \), and \( h \) is nonincreasing, it follows from (25) that \( h(t) \) is constant in a left neighborhood of \( t \) with the form \( t = |x|^N + 2a - 2z(x) = v(\frac{x}{|x|^N}), x_1 > \Lambda \). Similarly \( h \) is constant in any right neighborhood of \( t = v(\frac{x}{|x|^N}), x_1 < \Lambda \), in particular it is true for \( t \) is close to 0 since \( t = v(\frac{x}{|x|^N}) \) convergence to 0 at infinity. Therefore we conclude that if \( \Lambda > 0 \), \( h \) is constant in the range of \( v \). By the same arguments, we have that if \( \Lambda > 0 \) then \( k \) is constant in the range of \( u \).

**Proof of Theorem 1.1.** We first note that under the assumptions of the theorem, we have either \( u \equiv C_1, v \equiv C_2 \) or \( u, v \) are positive in \( \mathbb{R}^{N+1}_+ \). Indeed, since
\[
\frac{f(t)}{t^{N+2a-2}} \left( \frac{g(t)}{t^{N+2a-2}} \right) \text{ is nonincreasing and } f(t)g(t) \text{ is nondecreasing, we have } f(t) \geq f(0) \geq 0.
\]
Hence
\[-yu_{yy} - au_y - \Delta_x u = f(v) \geq 0,
\]
and therefore either \( u \equiv 0 \) or \( u > 0 \). Similarly, either \( v \equiv 0 \) or \( v > 0 \). Moreover, if \( u \equiv 0 \), then \(-yu_{yy} - au_y - \Delta_x v = f(u) = f(0) \), and \( v \) must be a constant. So either \((u, v) = (C_1, C_2)\) or \( u > 0, v > 0 \).

If \( u \equiv C_1, v \equiv C_2 \), the theorem is proved. Otherwise \( u, v \) are positive in \( \mathbb{R}^{N+1}_+ \).

By Proposition 2.7, we know that the Kelvin’s transform \( w, z \) of \( u, v \) centered at any point \( p \) are radially symmetric around some point \( q \). Moreover, if \( h, k \) are not constants on the value of \( u, v \) respectively, then \( p = q \), this implies that \( u, v \) are also radially symmetric around \( p \). It follows from the arbitrary choice of \( p \) that \( u, v \) are constants.

If \( h \) is constant, for example \( h = m \), then \( f(t) = mt^{N+2a-2} \) for \( t \in v(\mathbb{R}^{N+1}_+) \). By using the similar arguments as in the proof of Lemma 2.6, one has either for any pole \( p \), \( \Lambda = 0 \), which implies that \( v \) is constant, or there exists some pole \( p \) with \( \Lambda > 0 \). In this case, we have \( w \equiv w_\Lambda \) in \( \Sigma_\Lambda \), and hence 0 is not a singular point of \( w \) and \( u \) is regular at infinity and decay at infinity as \( \frac{1}{|x|^{N+2a-2}} \). Repeating the arguments as we did in the proof of Proposition 2.7, one has \( h, k \) are constants and \( u, v \) are radially symmetric, moreover one has
\[
u = a_1 \frac{[(N + 2a)(N + 2a - 2)s^2]^{N+2a-2}}{[s^2 + 4y + |x - x_0|^2]^{N+2a-2}}.
\]

In fact, this can be easily proved. For simplicity we assume \( l = m = 1 \), then
\[-yu_{yy} - au_y - \Delta_x u = v^{N+2a-2},
\]
\[-yv_{yy} - av_y - \Delta_x v = u^{N+2a-2}.
\]
Since \((u, v)\) is regular at infinity, we have
\[
\int_{\mathbb{R}^{N+1}} |\nabla(u - v)|^2 + \int_{\mathbb{R}^N} \frac{1}{2}(a + 1)(u - v)^2 + \int_{\mathbb{R}^N} (u - v)^2 y + \int_{\mathbb{R}^{N+1}} (u^{N+2a-2} - u^{N+2a-2})(u - v) = 0.
\]
Noting that \( y \in \mathbb{R}^+ \), we obtain \( u = v \). By the known result for a single equation (see [7]), we have \( u = v = \frac{[(N+2a)(N+2a-2)]^{\frac{N-2}{2}}}{|x^2+4y+|x-x_0|^2|^{\frac{N-2}{2}}} \). \( \square \)

3. Extensions. In this section, by using the similar techniques in Section 2, we consider a more general system:

\[
\begin{cases}
-yu_{yy} - au_y - \Delta_x u = f(u,v) \\
-yv_{yy} - av_y - \Delta_x v = g(u,v)
\end{cases}, \quad \text{for } (x, y) \in \mathbb{R}^N \times \mathbb{R}^+ =: \mathbb{R}^{N+1}_+ ,
\tag{26}
\]

and its extension version of

\[
\begin{cases}
-yu_{yy} - au_y - \Delta_x u = \sum_{i=1}^{m} f_i(u,v) \\
-yv_{yy} - av_y - \Delta_x v = \sum_{i=1}^{n} g_i(u,v)
\end{cases}, \quad \text{for } (x, y) \in \mathbb{R}^N \times \mathbb{R}^+ =: \mathbb{R}^{N+1}_+. \tag{27}
\]

We will use the same notations as the previous section.

Our main results are:

**Theorem 3.1.** Let \((u,v) \in E\) be a weak solution of the problem (26). Suppose that \(f, g : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}\) are continuous functions satisfying:

(i) there exist \(p_1 \geq 0, q_1 > 0, p_1 + q_1 = \frac{N+2a+2}{N+2a-2}\) such that \( \frac{f(s,t)}{sp_1t^{p_1}} \) is nonincreasing in \((s,t)\);

(ii) \(f(s,t)\) is nondecreasing in \(t\) and \(f > 0\) for \( s, t > 0 \);

(iii) there exist \(p_2 > 0, q_2 > 0, p_2 + q_2 = \frac{N+2a+2}{N+2a-2}\) such that \( \frac{g(s,t)}{sp_2t^{p_2}} \) is nonincreasing in \((s,t)\);

(iv) \(g(s,t)\) is nondecreasing in \(s\) and \(g > 0\) for \( s, t > 0 \).

Then either \((u,v) \equiv (C_1, C_2)\) for some constants \(C_1, C_2\) such that \(f(C_1, C_2) = g(C_1, C_2) = 0\) or there exist \(m, l > 0\) such that \(f(s,t) = ms^{p_1}t^{q_1}, g(s,t) = ls^{p_2}t^{q_2}\), and \(u, v\) are radially symmetric and regular at infinity. Moreover \(u = a_1 U, v = b_1 U\), where \(a_1 = ma_1^{p_1}b_1^{q_1}, b_1 = la_1^{p_2}b_2^{q_2}\) and \(U > 0\) satisfies \(yU_{yy} + aU_y + \Delta_x U + U^{\frac{N+2a+2}{N+2a-2}} = 0\) in \( \mathbb{R}^{N+1}_+ \).

**Theorem 3.2.** Let \((u,v) \in E\) be a weak solution of problem (27). Suppose that \(f = \sum_{i=1}^{m} f_i, g = \sum_{i=1}^{n} g_i, f_i, g_i : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}, i = 1, 2, ..., n\) are continuous functions satisfying:

(i) \(f(s,t)\) is nondecreasing in \(t\);

(ii) \(f(s,t)\) is nonincreasing in \((s,t)\);

(iii) for \(i \geq 1\), there exist \(p_{ii} \geq 0, q_{ii} > 0, p_{ii} + q_{ii} = \frac{N+2a+2}{N+2a-2}\) such that \( \frac{f_i(s,t)}{sp_{ii}t^{p_{ii}}} \) is nonincreasing in \((s,t)\), and \(f_i(s,t) \geq 0\) but not equal to zero at the same time;

(iv) \(g(s,t)\) is nondecreasing in \(s\);

(v) \(g(s,t)\) is nondecreasing in \((s,t)\);

(vi) for \(i \geq 1\), there exist \(p_{ii} > 0, q_{ii} > 0, p_{ii} + q_{ii} = \frac{N+2a+2}{N+2a-2}\) such that \( \frac{g_i(s,t)}{sp_{ii}t^{p_{ii}}} \) is nonincreasing in \((s,t)\) and \(g_i(s,t) \geq 0\) but not equal to zero at the same time.

Then either \((u,v) \equiv (C_1, C_2)\) for some constants \(C_1, C_2\) such that \(f(C_1, C_2) = g(C_1, C_2) = 0\) or there exist \(m_i, i > 0\) such that \(f_i(s,t) = m_i s^{p_{ii}}t^{q_{ii}}, g_i(s,t) = l_i s^{p_{ii}}t^{q_{ii}}, i = 0, 1, 2, ..., n, \) and \(u, v\) are radially symmetric and regular at infinity.
Moreover $u = a_1 u, v = b_1 U$, where $a_1 = f(a_1, b_1), b_1 = g(a_1, b_1)$ and $U > 0$ satisfies $y U_{yy} + a U_y + \Delta_x U + U = 0$ in $R^{n+1}_+$.  

**Remark 3.3.** Unlike Theorem 1.1, in Theorems 3.1 (also for Theorem 3.2), we consider only positive solutions, that is $u > 0, v > 0$. In fact, since $\frac{h(s,t)}{\eta^{1+}}$ is non-increasing in $t$, we have 

$$f(s, t) \geq \eta^{1+} f(s, 1), \text{ as } t \leq 1.$$  

Let $t \to 0$, one has $f(s, 0) \geq 0$. Note that $f(s, t)$ is nondecreasing in $t$, we get $f(s, t) \geq 0$ for any $s$, $t$. Thus $-y u_{yy} - a u_y - \Delta x u \geq 0$, it follows that $u \equiv 0$ or $u > 0$.

The same results are true for $v$. But we can not exclude the case of $u \equiv 0, v > 0$ or $v \equiv 0, u > 0$. And we consider only the positive solutions of (26) and (27).

But in Theorem 1.1, there are only two cases occur, namely, either $u \equiv C_1, v \equiv C_2$ or $u > 0, v > 0$. As far as system (26) is concerned, the situation is different. Suppose that $(u, v)$ is a nonnegative solution of (26). Then either $u \equiv 0$ or $u > 0$.

The proofs of Theorem 3.1 and Theorem 3.2 will consists in the usual steps in the method of moving plane and will be completed similarly to that of Theorem 1.1, once we prove the analogue of Lemma 2.2 and Lemma 2.6. For simple reason, we focus on the system (26). And the same trick works for the system (27). We use the same notations as in Section 2. Let $(u, v) \in E$ be a positive (weak) solution of system (26). Then $(u, v)$ satisfies (weakly) the following system:

$$\begin{aligned}
-\Delta w - \frac{2a_1 - 1}{x^{N+1}} \frac{\partial w}{\partial x^{N+1}} &= \frac{1}{|x|^{1+2N+2z}} f(|x|) |x|^{N+2a-2w, |x|^{N+2a-2z}} \\
-\Delta z - \frac{2a_1 - 1}{x^{N+1}} \frac{\partial z}{\partial x^{N+1}} &= \frac{1}{|x|^{1+2N+2z}} g(|x|) |x|^{N+2a-2w, |x|^{N+2a-2z}} \\
\end{aligned} \quad (28)$$

and

$$\begin{aligned}
-\Delta w_\lambda - \frac{2a_1 - 1}{x^{N+1}} \frac{\partial w_\lambda}{\partial x^{N+1}} &= \frac{1}{|x|^{1+2N+2z}} f(|x|) |x|^{N+2a-2w, |x|^{N+2a-2z}} \\
-\Delta z_\lambda - \frac{2a_1 - 1}{x^{N+1}} \frac{\partial z_\lambda}{\partial x^{N+1}} &= \frac{1}{|x|^{1+2N+2z}} g(|x|) |x|^{N+2a-2w, |x|^{N+2a-2z}} \\
\end{aligned} \quad (29)$$

**Lemma 3.4.** Under the conditions of Theorem 3.1, for any fixed $\lambda > 0$, $w, z \in L^2(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda), W^+_\lambda(x), Z^+_\lambda(x) \in L^2(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda) \cap W^{1,2}(\Sigma_\lambda)$. Moreover, there exist $C_\lambda > 0$ nonincreasing in $\lambda$ such that 

$$\begin{aligned}
||W^+_\lambda||^2_{L^{2*}(\Sigma_\lambda)} &\leq C_\lambda |x|^{-2(N+2a)} \frac{1}{[\lambda_{A_\lambda}]} \left(||W^+_{\lambda}||^2_{L^{2*}(\Sigma_\lambda)} + ||W^+_{\lambda}||^2_{L^{2*}(\Sigma_\lambda)} ||Z^+_{\lambda}||_{L^{2*}(\Sigma_\lambda)} \right) \\
+ C_\lambda |x|^{-2(N+2a)} \frac{1}{[\lambda_{A_\lambda}]} ||W^+_{\lambda}||^2_{L^{2*}(\Sigma_\lambda)}, \\
||Z^+_{\lambda}||^2_{L^{2*}(\Sigma_\lambda)} &\leq C_\lambda |x|^{-2(N+2a)} \frac{1}{[\lambda_{A_\lambda}]} \left(||Z^+_{\lambda}||^2_{L^{2*}(\Sigma_\lambda)} + ||Z^+_{\lambda}||^2_{L^{2*}(\Sigma_\lambda)} \right) \\
+ C_\lambda |x|^{-2(N+2a)} \frac{1}{[\lambda_{A_\lambda}]} ||Z^+_{\lambda}||^2_{L^{2*}(\Sigma_\lambda)}, \\
\end{aligned} \quad (30)$$

Where 

$$A_\lambda^1 = \{x \in \Sigma_\lambda : W_\lambda(x) \geq 0\}, \quad A_\lambda^2 = \{x \in \Sigma_\lambda : Z_\lambda(x) \geq 0\}.$$  

**Proof.** We just prove (30), the proof of (31) is similar. For any fixed $\lambda > 0$, there exists $r > 0$ such that $\Sigma_\lambda \subset R^{n+1} \setminus B_r(0)$, thus $w$ and $W^+_{\lambda} \leq w \in L^2(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda)$ and $\frac{1}{|x|^{2(N+2a)2}}, \frac{1}{|x|^{2(N+2a)2}}$ are integrable in $\Sigma_\lambda$.  

$$\begin{aligned}
\end{aligned} \quad (31)$$
For $\epsilon > 0$ small, let $\eta_\epsilon \in C^1(\mathbb{R}^{N+1})$ be a cut-off function such that $0 \leq \eta_\epsilon \leq 1$, $\eta_\epsilon(x) = 1$ for $2\epsilon \leq |x - e_\lambda| \leq \frac{1}{2}; \eta_\epsilon(x) = 0$ for $|x - e_\lambda| \leq \epsilon$ or $|x - e_\lambda| \geq \frac{2}{3}$, and $|\nabla \eta_\epsilon| \leq \frac{2}{N}$ for $\epsilon \leq |x - e_\lambda| \leq 2\epsilon$, $|\nabla \eta_\epsilon| \leq 2\epsilon$ for $\frac{1}{3} \leq |x - e_\lambda| \leq \frac{2}{3}$.

We will test the equations in (28), (29) with the function $\varphi_\epsilon = \eta_\epsilon^2 W_\epsilon^+$. Hence we may assume $w \geq w_\lambda$, so that $|x|^N + 2a - 2w \geq |x|^N + 2a - 2w_\lambda$ for any $\lambda > 0$.

If $z \leq z_\lambda$, we have

$$f(|x|^N + 2a - 2w_\lambda(x), |x|^N + 2a - 2z_\lambda(x))$$

$$\geq f(|x|^N + 2a - 2w_\lambda(x), |x|^N + 2a - 2z(x) w_\lambda(x))$$

$$\geq f(|x|^N + 2a - 2w(x), |x|^N + 2a - 2z(x)) \frac{|x|^N + 2a - 2w_\lambda(x)}{|x|^N + 2a - 2w(x)} p_1,$$

and

$$-\Delta (w - w_\lambda) - \frac{2a-1}{x_{N+1}} \frac{\partial (w-w_\lambda)}{\partial x_{N+1}}$$

$$\leq \frac{1}{|x|^N + 2a + 2} f(|x|^N + 2a - 2w(x), |x|^N + 2a - 2z(x))(1 - \frac{w_\lambda}{w}) \frac{N + 2a + 2}{N + 2a - 2}$$

$$\leq \frac{1}{|x|^N + 2a + 2} f(|x|^N + 2a - 2w(x), |x|^N + 2a - 2z(x))(1 - \frac{w_\lambda}{w}) \frac{N + 2a + 2}{N + 2a - 2}$$

$$\leq \frac{1}{|x|^N + 2a + 2} f(|x|^N + 2a - 2w(x), |x|^N + 2a - 2z(x)) \frac{N + 2a + 2}{N + 2a - 2} [1 - \frac{w_\lambda}{w} + (1 - \frac{z_\lambda}{z})]$$

$$\leq \frac{C_\lambda}{|x|^4} [(w - w_\lambda) + (z - z_\lambda)].$$

Hence

$$-\Delta (w - w_\lambda) - \frac{2a - 1}{x_{N+1}} \frac{\partial (w-w_\lambda)}{\partial x_{N+1}} \leq \frac{C_\lambda}{|x|^4} [(w - w_\lambda)^+ + (z - z_\lambda)^+], \text{ for } z \geq z_\lambda.$$
Lemma 3.5. Under the conditions of Theorem 3.1, there exists $\Lambda > 0$ such that for all $\lambda \geq \Lambda, W_{\lambda}(x) \leq 0$ and $Z_{\lambda}(x) \leq 0$ for all $x \in \tilde{\Sigma}_{\lambda}$.

Proof. By Lemma 3.4, we proceed as the same as the proof of Lemma 2.3.

Now we define $\Lambda = \inf\{\lambda > 0|\lambda W_{\mu}(x) \leq 0, Z_{\mu}(x) \leq 0, \forall x \in \tilde{\Sigma}_{\mu}, \mu > \lambda\}$.

Lemma 3.6. If $\Lambda > 0$ then $W_{\lambda} \equiv 0$ and $Z_{\lambda} \equiv 0$ for all $x \in \tilde{\Sigma}_{\lambda}$.

Proof. We prove that if $w = w_{\lambda}$ at some point $x_0 \in \tilde{\Sigma}_{\lambda}$, then in a neighborhood of $x_0$, $w = w_{\lambda}$, and hence $w \equiv w_{\lambda}$ in $\tilde{\Sigma}_{\lambda}$, by continuity.

Indeed, by continuity, we see that $W_{\lambda}(x) \leq 0, Z_{\lambda}(x) \leq 0$ for all $x \in \tilde{\Sigma}_{\lambda}$. Since $w(x_0) = w_{\lambda}(x_0)$, we have $|x|^{N+2a-2}w(x_0) > |x|^{N+2a-2}w_{\lambda}(x_0)$ for $x \in B_r := B_r(x_0)$ a neighborhood of $x_0$. By using the same arguments as in the proof of Lemma 3.4 and the fact that if $s > s', t > t'$, then

$$f(s', t') \geq f(s, t)(\frac{s'}{s})^{p_1}(\frac{t'}{t})^{q_1}, g(s', t') \geq g(s, t)(\frac{s'}{s})^{p_2}(\frac{t'}{t})^{q_2},$$
we have
\[
\begin{align*}
& f(|x|^N w_A, |x|^N z) + f(|x|^N w_A, |x|^N z) \frac{|x|^N w_A}{|x|^N z} \left( \frac{|x|^N w_A}{|x|^N z} \right)^{p_1} \left( \frac{|x|^N w_A}{|x|^N z} \right)^{q_1} \\
& = |x|^{N+2a+2} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z) \left( \frac{w_A}{z} \right)^{p_1},
\end{align*}
\]
and
\[
\begin{align*}
& -\Delta (w - w_A) - \frac{2a-1}{x^{N+1}} \frac{\partial (w - w_A)}{\partial x^{N+1}} \\
& = \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z) \\
& - \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z) \\
& \leq \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z) \\
& - \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z) \\
& \leq \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z) \\
& - \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z) \\
& \leq -C(w - w_A) \in B_r.
\end{align*}
\]

Where \(C > 0\) is a constant depending on \(x_0, r\). Hence
\[
\begin{align*}
& \left\{ \begin{array}{ll}
-\Delta W_A - \frac{2a-1}{x^{N+1}} \frac{\partial W_A}{\partial x^{N+1}} + CW_A \leq 0 \\
W_A \leq 0, W_A(x_0) = 0 \text{ in } B_r(x_0).
\end{array} \right.
\end{align*}
\]
By the Maximal principle, \(W_A \equiv 0\) in \(B_r(x_0)\).

Next, we claim that \(W_A \equiv 0\) in \(B_r(x_0)\). In fact, by the equations (28) and (29),
\[
\begin{align*}
& \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& = \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& \geq \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& \geq \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)).
\end{align*}
\]
Since \(f(s, t)\) is nondecreasing in \(t\), we deduce from the above inequality that
\[
|x|^{N+2a-2} z > |x|^{N+2a-2} z_A.
\]
On the other hand,
\[
\begin{align*}
& \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& = \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& \geq \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& \geq \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)).
\end{align*}
\]
Thus
\[
\begin{align*}
& \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& = \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& \geq \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)).
\end{align*}
\]

hence
\[
\begin{align*}
& f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) = f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& = f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& = f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& = f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)) \\
& = f(|x|^{N+2a-2} w_A, |x|^{N+2a-2} z(x)).
\end{align*}
\]
It follows from (36) that
\[ |x|^{N+2a-2}w \geq |x|^{N+2a-2}w, \]
\[ |x|^{N+2a-2}z \geq |x|^{N+2a-2}z, \]
By (38) and assumption (i) of Theorem 3.1,
\[ f(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z(x)) = f(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z(x)), \]...
As a consequence of (37) and (39), we have \( z^{q_1} = z^{q_1}, \) and hence \( z = z_\Lambda \) since \( q_1 > 0. \)

Suppose that \( W_\Lambda \neq 0 \) and \( Z_\Lambda \neq 0 \) in \( \Sigma_A, \) then \( w < w_\Lambda, z < z_\Lambda \) in \( \Sigma_A. \) Now let \( \chi_S \) be the characteristic function of set \( S. \) Then the above discuss shows that
\[ \frac{1}{|x|^{2(N+2a)}} \chi_A \] and \[ \frac{1}{|x|^{2(N+2a)}} \chi_A \] converge pointwisely to zero, respectively, as \( \lambda \to \Lambda \) in \( \mathbb{R}^{N+1} \setminus \{ T_\Lambda \cup \{ e_\Lambda \} \}. \) Thus if \( 0 < \Lambda - \delta < \Lambda, \) then
\[ \frac{1}{|x|^{2(N+2a)}} \chi_{\Lambda - \delta} \in L^1(\Sigma_\Lambda), \]
\[ \frac{1}{|x|^{2(N+2a)}} \chi_{\Lambda - \delta} \in L^1(\Sigma_\Lambda). \]
By dominate convergence, we get
\[ \int_{A^2_\Lambda} \frac{1}{|x|^{2(N+2a)}} d\lambda \to 0, \]
and hence
\[ C_1^1(\int_{A^2_\Lambda} \frac{1}{|x|^{2(N+2a)}})^{\frac{2}{N+2a}} < 1, \]
\[ C_2^1(\int_{A^2_\Lambda} \frac{1}{|x|^{2(N+2a)}})^{\frac{2}{N+2a}} < 1, \]
for \( \lambda \in (\Lambda - \delta, \Lambda). \)

Similarly, we have
\[ C_1^2(\int_{A^2_\Lambda} \frac{1}{|x|^{2(N+2a)}})^{\frac{2}{N+2a}} < 1, \]
\[ C_2^2(\int_{A^2_\Lambda} \frac{1}{|x|^{2(N+2a)}})^{\frac{2}{N+2a}} < 1, \]
for \( \lambda \in (\Lambda - \delta, \Lambda). \)

Combining with the previous arguments, we deduce that \( W_\Lambda(x) \leq 0 \) and \( Z_\Lambda(x) \leq 0 \) in \( \Sigma_\Lambda \) for \( \lambda \in (\Lambda - \delta, \Lambda), \) which contradicts with the definition of \( \Lambda. \)


Proof of Theorem 3.1. Suppose that \((u, v)\) is a positive solution of (3.1). Make the Kelvin’s transform around a point \( p \in \mathbb{R}^{N+1} \) and define \( \Lambda = \Lambda(p, \nu) \) as (24) for a direction \( \nu \in \mathbb{R}^{N+1}. \) If \( \Lambda(p, \nu) = 0 \) for all \( p \) and \( \nu, \) then \((u, v)\) is radially symmetric with respect to all \( p \in \mathbb{R}^{N+1}, \) and therefore must be constant. If \( \Lambda(p, \nu) > 0 \) for some \( p \) and \( \nu, \) then the corresponding Kelvin’s transform \((w, z)\) is radially symmetric with respect to a point \( q \) other than \( p \) and regular at the pole \( p, \) hence \((u, v)\) is regular at infinity, that is \( u(x) \sim \frac{w(0)}{|x|^{N+2a-2}}, v(x) \sim \frac{z(0)}{|x|^{N+2a-2}} \) as \( |x| \to \infty. \) Without loss of generality, we assume that \( p \) is the origin. If a point \( x \) is not on the line \( pq, \) then we can find two points \( x-, x+ \) such that
\[ |x- - q| = |x - q| = |x+ - q|, \]
\[ |x-| < |x| < |x+|. \]

We have
\[ w(x-) = w(x) = w(x+), \]
\[ \Delta w(x-) + \frac{2a - 1}{(x-)_{N+1}} \partial w(x-) = \Delta w(x) + \frac{2a - 1}{(x)_{N+1}} \partial w(x), \]
\[ \Delta w(x) = \Delta w(x+), \]
\[ z(x-) = z(x) = z(x+), \]
\[ w(x-) = w(x) = w(x+), \]
\[ \Delta w(x-) + \frac{2a - 1}{(x-)_{N+1}} \partial w(x-) = \Delta w(x) + \frac{2a - 1}{(x)_{N+1}} \partial w(x), \]
\[ \Delta w(x) = \Delta w(x+), \]
\[ z(x-) = z(x) = z(x+), \]
\[
\Delta z(x_+) + \frac{2a - 1}{(x_+)_{N+1}} \frac{\partial z(x_+)}{\partial x_{N+1}} = \Delta z(x) + \frac{2a - 1}{(x)_{N+1}} \frac{\partial z(x)}{\partial x_{N+1}},
\]
and
\[
f\left(\frac{|x_+|^{N+2a-2}w(x)}{|x-|^{N+2a-2}z(x)}\right) = f\left(\frac{|x_-|^{N+2a-2}w(x)}{|x_+|^{N+2a-2}z(x)}\right).
\]

Since \(f(s,t)\) is nonincreasing in \((s,t)\), it follows from (40) that \(f(s,t)\) is constant in the domain \([|x_-|^{N+2a-2}w(x), |x_+|^{N+2a-2}w(x)] \times [|x_-|^{N+2a-2}z(x), |x_+|^{N+2a-2}z(x)]\), a neighborhood of \((|x|^{N+2a-2}w(x), |x|^{N+2a-2}z(x)) = (u(x)\), \(v(x)\)) which implies that \(f(s,t)\) is constant in the whole range of \(\{(u(x), v(x))| x \in \mathbb{R}^{N+1}\}\). The same is true for the function \(g\). So the problem (3.1) reduces to
\[
\begin{align*}
- \Delta u - \frac{2a - 1}{2a - 1} \frac{\partial u}{\partial x_{N+1}} &= mu^p v^q, & u > 0 \quad \text{in } \mathbb{R}^N \\
- \Delta v - \frac{2a - 1}{2a - 1} \frac{\partial v}{\partial x_{N+1}} &= lv^p v^q, & v > 0 \quad \text{in } \mathbb{R}^N
\end{align*}
\]
where \(m, l\) are positive constants. Moreover \((u, v)\) is regular at infinity. We can apply the method of moving plane to \((u, v)\) and conclude that \((u, v)\) is radially symmetric. It remains to determine the form of \((u, v)\). Suppose that \((u, v)\) is radially symmetric with respect to the origin:
\[
u(x) = \tilde{u}(|x|), v(x) = \tilde{v}(|x|).
\]
Let \((w, z)\) be the Kelvin’s transform of \((u, v)\) with the pole \(p \neq 0\),
\[
w(x) = \frac{1}{|x-p|^{N+2a-2}} u\left(\frac{x-p}{|x-p|^2 + p}\right), z(x) = \frac{1}{|x-p|^{N+2a-2}} v\left(\frac{x-p}{|x-p|^2 + p}\right).
\]
\((w, z)\) is radially symmetric with respect to some point \(q\). Notice that for \(x = \lambda p + e, \lambda \in \mathbb{R}, (e, p) = 0\), we have
\[
|x-p|^2 = (\lambda - 1)^2|p|^2 + |e|^2, \quad \frac{x-p}{|x-p|^2} + p = \frac{1 + 2(\lambda - 1)|p|^2}{(\lambda - 1)^2|p|^2 + |e|^2} + |p|^2,
\]
and
\[
w(x) = \frac{1}{((\lambda - 1)^2|p|^2 + |e|^2)^{N+2a-2}} \tilde{u}\left((\lambda - 1)^2|p|^2 + |e|^2, |p|^2\right)\frac{1 + 2(\lambda - 1)|p|^2}{(\lambda - 1)^2|p|^2 + |e|^2} + |p|^2
\]
is a function of \(\lambda\) and \(|e|\), which implies that \(w\) is axisymmetric around the line \(op\), and \(q\) must be on this line. If \(q = p\), then \((u, v)\) is radially symmetric with respect to \(p\) too, hence constant. If \(q \neq p\), it follows from [3, lemma 7] that for some \(A, B, s = s(p, q) > 0\),
\[
\begin{align*}
&u(x) = \frac{A}{(s^2 + |x|^2)^{N+2a-2}} \\
v(x) = \frac{B}{(s^2 + |x|^2)^{N+2a-2}}
\end{align*}
\]
Let \(A = a_1(N+2a(N+2a-2)s^2)^{\frac{N+2a-2}{2}}, B = b_1(N+2a(N+2a-2)s^2)^{\frac{N+2a-2}{2}}\), then \(u = a_1 U, v = b_1 U\). Where \(U = \frac{(N+2a(N+2a-2)s^2)^{\frac{N+2a-2}{2}}}{(s^2 + |x|^2)^{\frac{N+2a-2}{2}}}\) satisfies \(yU_{yy} + aU_y + \Delta_x U + U = 0 \in \mathbb{R}^{N+1}_{+}\), and \(a_1, b_1\) satisfies
\[
\begin{align*}
a_1 &= ma^{p_1}b_1^{p_2}, \\
b_1 &= la^{p_1}b_1^{p_2}.
\end{align*}
\]
\]
Proof of Theorem 3.2. We proceed the similar technique arguments as in the proof of Theorem 3.1. For reader’s convenience, in the following, we only give the proof of how \( w \equiv w_A \) implies that \( z \equiv z_A \).

In fact, by the equations (28) and (29), we have

\[
\frac{1}{|x|^{N+2a+2}} \sum f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z(x)) \\
= \frac{1}{|x|^{N+2a+2}} \sum f(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x)) \\
\geq \sum \frac{1}{|x|^{N+2a+2}} f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x)) \\
> \sum \frac{1}{|x|^{N+2a+2}} f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x)) \\
= \frac{1}{|x|^{N+2a+2}} \sum f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x)).
\]

(42)

Since \( f(s, t) \) is increasing in \( t \), we deduce that

\[
|x|^{N+2a-2}z > |x|^{N+2a-2}z_A.
\]

(43)

Moreover,

\[
\frac{1}{|x|^{N+2a+2}} f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z(x)) \\
\geq \frac{1}{|x|^{N+2a+2}} f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x)),
\]

and

\[
\frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z(x)) \\
= \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x)) \\
\geq \frac{1}{|x|^{N+2a+2}} f(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x)) \\
\geq \sum \frac{1}{|x|^{N+2a+2}} f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x)) \\
= \frac{1}{|x|^{N+2a+2}} \sum f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x)),
\]

(44)

and

\[
\frac{1}{|x|^{N+2a+2}} f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z(x)) \\
= \frac{1}{|x|^{N+2a+2}} f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x)),
\]

(45)

By (44) and (45), for each \( i \), it holds that

\[
\frac{f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x))}{(|x|^{N+2a-2}w)^{p_i}(|x|^{N+2a-2}z_A)^{q_i}} = \frac{f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x))}{(|x|^{N+2a-2}w)^{p_i}(|x|^{N+2a-2}z_A)^{q_i}}.
\]

(46)

By (43), we have

\[
|x|^{N+2a-2}w \geq |x|^{N+2a-2}w_A = |x|^{N+2a-2}w, \\
|x|^{N+2a-2}z \geq |x|^{N+2a-2}z_A \geq |x|^{N+2a-2}z.
\]

Hence

\[
\frac{f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x))}{(|x|^{N+2a-2}w)^{p_i}(|x|^{N+2a-2}z_A)^{q_i}} = \frac{f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x))}{(|x|^{N+2a-2}w)^{p_i}(|x|^{N+2a-2}z_A)^{q_i}}, \ \forall i.
\]

(47)

That is

\[
\frac{f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x))}{|x|^{N+2a+2}} = \frac{f_i(|x|^{N+2a-2}w(x), |x|^{N+2a-2}z_A(x))}{|x|^{N+2a+2}}
\]

combining with (42), we see that \( z = z_A \) provided \( f_i > 0 \) for some \( i \). 

□
We observe that in Theorem 3.1 and 3.2, we only assume that \( f(s, t), g(s, t) \) are continuous functions. If \( f(s, t), g(s, t) \) are Lipschitz continuous functions with respect to \( t \) and \( s \) respectively, then the same results of theorem 3.1 hold but with simpler assumptions.

**Theorem 3.9.** Let \((u, v) \in E\) be a positive weak solution of problem (26). Suppose that \( f, g : [0, \infty) \times [0, \infty) \to \mathbb{R} \) are continuous functions satisfying

(i) \( f(\gamma x, \gamma t), g(\gamma x, \gamma t) \) are nonincreasing in \( \gamma \), and either

(ii) \( f(s, t) \) is increasing and locally Lipschitz continuous in \( t \), is

provided that \( m \geq t \geq t' > 0, m \geq s \geq 0 \). And \( g(s, t) \) is increasing and locally Lipschitz continuous in \( s \) in the sense that

\[ 0 \leq g(s, t) - g(s', t) \leq L_1(m)(s - s'), \]

or

(ii)' \( f(s, t) \) is increasing in \( t \) and nondecreasing in \( s \); \( g(s, t) \) is increasing in \( s \) and nondecreasing in \( t \).

Then either \((u, v) \equiv (C_1, C_2)\) for some constants \( C_1, C_2 \) such that \( f(C_1, C_2) = g(C_1, C_2) = 0 \) or \( f(s, t), g(s, t) \) are homogeneous of order in the range \( \{ (u(x), v(x)) \mid x \in \mathbb{R}^{N+1} \} \). Moreover \( u, v \) are radially symmetric and regular at infinity and \( u = a_1 U, v = b_1 U \), where \( a_1, b_1 \) satisfy

\[ f(\lambda_0 a_1, \lambda_0 b_1) = \lambda_0^{\frac{N+2}{N+2-2}} a_1, g(\lambda_0 a, \lambda_0 b) = \lambda_0^{\frac{N+2}{N+2-2}} b_1, \]

for \( \lambda_0 = \max U. U > 0 \) satisfies \( yU_{yy} + aU_y + \Delta U + U^{\frac{N+2}{N+2-2}} = 0, U(0) = \lambda_0 \).

**Proof.** It is sufficient to prove that the main estimates in the proof of Lemma 3.3 hold. Indeed,

(1) The condition (ii)' holds. Since for any fixed \( \lambda > 0, |x| \geq |x_\lambda| \), it follows that if \( z \leq z_\lambda \),

\[
\begin{align*}
-\Delta (w - w_\lambda) - \frac{2a-1}{|x|^N+2a-2} \frac{\partial (w - w_\lambda)}{|x|^N+2a-2} f(|x|^{N+2a-2} w_\lambda, |x|^{N+2a-2} z) &\leq \frac{1}{|x|^N+2a-2} f(|x|^{N+2a-2} w_\lambda, |x|^{N+2a-2} z) \\
&\leq \frac{1}{|x|^N+2a-2} f(|x|^{N+2a-2} w_\lambda, |x|^{N+2a-2} z) \leq \frac{1}{|x|^N+2a-2} (f(|x|^{N+2a-2} w_\lambda, |x|^{N+2a-2} z)(1 - \frac{w_\lambda}{w})^{\frac{N+2}{N+2-2}})
\end{align*}
\]

Where we apply the nonincreasing condition with \( \gamma = \frac{w_\lambda}{w} (\frac{|x|}{|x_\lambda|})^{N+2a-2} \leq 1 \).

If \( z \geq z_\lambda \),

\[
\begin{align*}
-\Delta (w - w_\lambda) - \frac{2a-1}{|x|^N+2a-2} \frac{\partial (w - w_\lambda)}{|x|^N+2a-2} f(|x|^{N+2a-2} w_\lambda, |x|^{N+2a-2} z) &\leq \frac{1}{|x|^N+2a-2} f(|x|^{N+2a-2} w_\lambda, |x|^{N+2a-2} z) \\
&\leq \frac{1}{|x|^N+2a-2} f(|x|^{N+2a-2} w_\lambda, |x|^{N+2a-2} z) \leq \frac{1}{|x|^N+2a-2} (f(|x|^{N+2a-2} w_\lambda, |x|^{N+2a-2} z)(1 - \frac{w_\lambda}{w})^{\frac{N+2}{N+2-2}})
\end{align*}
\]

\[
\frac{1}{|x|^N+2a-2} \left((w - w_\lambda) + (z - z_\lambda)\right).
\]
Where we apply the nonincreasing condition with \( \gamma = \frac{u_\lambda}{w_\lambda} \geq (\frac{|x|}{|z|})^{N+2a-2} \leq 1 \).

(2) The condition (ii) holds. Again, for any fixed \( \lambda > 0, |x| \geq |x_\lambda| \), if \( z \leq z_\lambda \), the proof is the same as (ii)\(^t\).

If \( z \geq z_\lambda \),

\[
-\Delta (w - w_\lambda) - \frac{2a-1}{|x|^{N+2a-2}} \frac{\partial (w-w_\lambda)}{\partial x} \leq \frac{1}{|x|^{N+2a-2}} f(|x|^{N+2a-2} w, |x|^{N+2a-2} \lambda)
\]

\[
= \frac{1}{|x|^{N+2a-2}} \left( f(|x|^{N+2a-2} w, |x|^{N+2a-2} \lambda) - \frac{1}{|x|^{N+2a-2}} [f(|x|^{N+2a-2} w, |x|^{N+2a-2} \lambda)] \right)
\]

\[
\leq \frac{1}{|x|^{N+2a-2}} \left( f(|x|^{N+2a-2} w, |x|^{N+2a-2} \lambda) - \frac{1}{|x|^{N+2a-2}} [f(|x|^{N+2a-2} w, |x|^{N+2a-2} \lambda)] \right)
\]

\[
\leq \frac{1}{|x|^{N+2a-2}} \left( L(m) |x|^{N+2a-2} (z - z_\lambda) + f(|x|^{N+2a-2} w, |x|^{N+2a-2} \lambda) \right)
\]

\[
\leq \frac{1}{|x|^{N+2a-2}} \left( L(m) |x|^{N+2a-2} (z - z_\lambda) + f(|x|^{N+2a-2} w, |x|^{N+2a-2} \lambda) \right)
\]

Proceeding the same arguments as in Lemma 3.4, we show that the integral inequality we need is true. The rest of the proof are the same as Theorem 3.1 with necessary modifications. \[\square\]

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