Abstract

We prove a bijection between the triangulations of the 3-dimensional cyclic polytope $C(n,3)$ and persistent graphs with $n-2$ vertices. We show that under this bijection, the Stasheff-Tamari orders naturally translate to subgraph inclusion. Moreover, we describe a connection to the second higher Bruhat order $B(n-2,2)$. We also give an algorithm to efficiently enumerate all persistent graphs on $n-2$ vertices and thus all triangulations of $C(n,3)$.

Keywords: computational geometry, persistent graphs, time series visibility, Bruhat order, counting and enumeration

1 Introduction

In this work we reveal a new connection between cyclic polytopes and certain geometric graphs by proving a one-to-one correspondence between the triangulations of the 3-dimensional cyclic polytope and persistent graphs. Cyclic polytopes are natural generalizations of convex polygons to higher dimensions and are among the most studied classes of polytopes. They are neighborly and achieve the maximum number of faces according to the upper bound theorem of McMullen [17]. Triangulations of the $d$-dimensional cyclic polytope $C(n,d)$ are well-studied [20]. It is known that a triangulation is fully determined by the set of its $[d/2]$-dimensional faces [6]. For even dimension $d$, a combinatorial description of this set is known [18]. For odd dimension, however, no characterization is known so far.

We give a characterization for $d = 3$ by proving a bijection between the set of triangulations of $C(n,3)$ and the class of so-called persistent graphs. These (vertex-ordered) graphs are best known for their conjectured equality to the set of terrain visibility graphs. A graph $G$ is a terrain visibility graph (sometimes referred to as 1.5-dimensional terrain
visibility graph), if there exists a sequence of points \( p_i \in \mathbb{R}^2 \) with ascending \( x \)-coordinates (the vertices of \( G \)) such that there is an edge between \( p_i \) and \( p_j \) if and only if the line segment connecting \( p_i \) and \( p_j \) does not pass below any other point in between. Terrain visibility graphs are known to be persistent (i.e., they satisfy certain combinatorial properties). It is an open question, whether every persistent graph is also a terrain visibility graph. Our result opens new directions to answer this question and simultaneously provides new insights into the combinatorial structure cyclic polytope triangulations.

1.1 Related Work

For a general overview on triangulations, see the monograph by Loera et al. [16]. The triangulations of cyclic polytopes and their poset structures have received considerable interest [14, 7, 19, 20, 18]. Also, efforts have been made to determine the number of triangulations [3, 20, 13]. Thomas [21] gave a bijection between the triangulations of the cyclic polytope \( C(n, d) \) and so-called snug partitions of the set \([n−1]^d\).

Terrain visibility graphs are closely related to so-called orthogonal staircase polygons [5]. In this context, they were studied by Abello et al. [2], who proved that they are persistent (and claimed the converse implication, which is still open). A simplified proof of this result was more recently published by Evans and Saeedi [8], who also showed a connection to certain restricted 3-signotopes. Some graph-theoretic results regarding (forbidden) induced subgraphs of terrain visibility graphs and relation to other graph classes are known [10]. Interestingly, in the context of time series data, terrain visibility graphs (there called time series visibility graphs) have received a lot of attention as an analytical tool [15] (see also references in [10]). Also related classes such as terrain visibility graphs with uniform step length [1] and horizontal visibility graphs [12] have been individually studied (the latter are shown to be exactly the outerplanar graphs containing a Hamilton path).

1.2 Preliminaries

We introduce some notation, basic definitions and preliminary results.

**Notation.** We define \([n] := \{1, \ldots, n\}\) and denote the set of all size-2 subsets of \([n]\) by \(\binom{n}{2}\). The convex hull of a set \(S\) of points is denoted \(\text{conv}(S)\). We assume the reader to be familiar with the basics of the theory of polytopes (see e.g. Ziegler [22]). For a polytopal complex \(C\), we denote the set of \(i\)-dimensional faces of \(C\) as \(F_i(C)\) and we write \(f_i(C) := |F_i(C)|\). The \(i\)-skeleton of \(C\) is defined as \(\text{skel}_i(C) = \bigcup_{j=0}^{i} F_j(C)\). Note that the 1-skeleton defines a graph with vertices \(F_0(C)\) and edges \(F_1(C)\). Throughout this work, we always consider combinatorial faces and simplices, that is, we only consider the corresponding vertex sets.

**Cyclic Polytopes.** For an integer \(d \geq 1\), the \(d\)-dimensional cyclic polytope is defined via the \(d\)-th moment curve:

\[
\mu_d : \mathbb{R} \rightarrow \mathbb{R}^d, \ t \mapsto (t, t^2, \ldots, t^d).
\]
Let \( t_1 < t_2 < \ldots < t_n \) be \( n > d \) real numbers. Then,

\[
C(n, d) := \text{conv}\{\mu_d(t_1), \ldots, \mu_d(t_n)\}
\]

is the \( d \)-dimensional cyclic polytope with \( n \) vertices. It is well-known that the combinatorics of \( C(n, d) \) do not depend on the particular values of \( t_1, \ldots, t_n \) but just on the number \( n \). In this work, we consider \( C(n, 3) \) and denote its vertices by 0, 1, \ldots, \( n-1 \), ordered by their first coordinate. The faces of \( C(n, 3) \) are determined by Gale’s evenness criterion [11, Theorem 3] as follows (see Figure 1 for an example):

\[
F_1(C(n, 3)) = \{\{0, n-1\}, \{0, i\}, \{i, n-1\}, \{i, i+1\} | 0 < i < n-1\},
F_2(C(n, 3)) = \{\{0, i, i+1\}, \{i-1, i, n-1\} | 0 < i < n-1\}.
\]

A triangulation of \( C(n, 3) \) is a collection \( T = \{S_1, \ldots, S_m\} \) of 3-simplices (that is, tetrahedra) \( S_i = \{a, b, c, d\} \subseteq \{0, \ldots, n-1\} \), such that \( \bigcup_{i=1}^{m} \text{conv}(S_i) = C(n, 3) \) and each pair of 3-simplices intersects in a common (possibly empty) face. We denote the set of all triangulations of \( C(n, 3) \) by \( T_n \).

We proceed with some known results about characterizing triangulations of \( C(n, 3) \). We will use these in order to prove our main result. A circuit (also called a primitive Radon partition) is a pair \((X, Y)\) of disjoint minimal subsets of vertices of \( C(n, 3) \) such that \( \text{conv}(X) \cap \text{conv}(Y) \neq \emptyset \). The circuits of \( C(n, 3) \) are easily characterized as follows:

**Lemma 1.1** ([4]). The circuits of \( C(n, 3) \) are exactly the pairs \((\{u, v, w\}, \{x, y\})\) with \( u < x < v < y < w \).

The above result on circuits allows us to give the following characterization of a triangulation of \( C(n, 3) \) as a direct consequence of [19, Proposition 2.2].

**Proposition 1.2.** A set \( T \) of 3-simplices with vertices from \( C(n, 3) \) is a triangulation of \( C(n, 3) \) if and only if

1. for each \( S \in T \) and each facet \( F \) of \( S \) either \( F \in F_2(C) \) or there is another 3-simplex \( S' \in T \) of which \( F \) is a facet (Union-Property), and
2. there is no pair of 3-simplices \( S, S' \in T \) such that \( \{x_1, x_3, x_5\} \subseteq S \) and \( \{x_2, x_4\} \subseteq S' \) for any \( x_1 < x_2 < x_3 < x_4 < x_5 \) (Intersection-Property).

The next observation states that an internal edge of a triangulation is contained in at least three 3-simplices. It follows directly from the union-property of Proposition 1.2.
Observation 1.3. Let $T \in T_n$ and let $\{v, w\}$ be an internal edge, i.e., $\{v, w\} \in F_1(T) \setminus F_1(C(n, 3))$. Then, there are $k \geq 3$ vertices $x_1, \ldots, x_k, x_{k+1} = x_1$ such that $\{v, w, x_i, x_{i+1}\} \in T$ for all $i = 1, \ldots, k$.

This leads us to the following helpful lemma about internal edges.

Lemma 1.4. Let $T$ be a triangulation of $C := C(n, 3)$ and let $\{v, w\} \in F_1(T) \setminus F_1(C(n, 3))$ be an internal edge with $v < w$. Then, there are vertices $a, b, c$ with $a < v < b < w < c$ such that $\{v, w, a\}, \{v, w, b\}, \{v, w, c\} \subseteq F_2(T)$.

Proof. Note that $0 < v$ and $w < n - 1$ since $\{v, w\}$ is an internal edge. Let $x_1, \ldots, x_{k+1} = x_1$ be the $k \geq 3$ vertices given by Observation 1.3. Then $T' := \{v, w, x_i, x_{i+1}\} | i = 1, \ldots, k$ is a triangulation of $K := \text{conv}\{v, w, x_i, \ldots, x_k\}$. Since the facets of $T'$ that contain $\{v, w\}$ are exactly those of the form $\{v, w, x_i\}$ and since each of these appears in two 3-simplices in $T'$, it follows that $\{v, w\}$ is an internal edge of $T'$. Therefore, every plane $H \subset \mathbb{R}^3$ containing $v$ and $w$ (and thus $\text{conv}\{v, w\}$) divides $K$ into two nonempty 3-dimensional polytopes. Then, the two corresponding open half-spaces $H^+$ and $H^-$ must each contain a vertex from $\{x_1, \ldots, x_k\}$. Now, assume without loss of generality that the coordinates of $v$ are $(0, 0, 0)$.

If we take $H$ as the plane containing $\{v, w, (0, 0, 1)\}$ (that is, containing the $z$-axis and $w$) and $H^+$ as the open half-space not containing the vertex $0$, then $H^+ \cap F_0(C) = \{u \in F_0(C) | v < u < w\}$ (see Fig. 2 (left)). Since $H^+ \cap \{x_1, \ldots, x_k\} \subseteq H^+ \cap F_0(C)$ and $H^+ \cap \{x_1, \ldots, x_k\} \neq \emptyset$, there exists a vertex $b$ as claimed.

Figure 2: Schematic drawing of $C(7, 3)$ (dashed lines lying below/behind) with the vertex $v$ positioned at the origin. The thick lines indicate the respective planes $H$ containing the vertices $v$ and $w$. Left: Projection to the $x$-$y$-plane. The plane $H$ is chosen to contain the $z$-axis. Right: Projection to the $y$-$z$-plane. The plane $H$ is chosen to contain the $x$-axis.
Terrain Visibility and Persistent Graphs. Terrain visibility graphs are visibility graphs of 1.5-dimensional terrains, that is, $x$-monotone polygonal chains in the plane defined by a set $V \subseteq \mathbb{R}^2$ of terrain vertices with pairwise different $x$-coordinates. Two vertices $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ are adjacent if and only if they see each other, that is, there is no vertex between them that lies on or above the line segment connecting them. Formally, there exists an edge $\{v_1, v_2\}$, for $x_1 < x < x_2$, if and only if all terrain vertices $(x, y)$ with $x_1 < x < x_2$ satisfy

$$y < y_1 + \frac{(x - x_1)(y_2 - y_1)}{x_2 - x_1}.$$  

Figure 3 depicts an example. We denote the vertices by $1, \ldots, n$ in increasing order of their $x$-coordinates.

Terrain visibility graphs are known to be persistent where a graph $G = ([n], E)$ is called persistent if it satisfies the following three properties.

1. It contains a Hamilton path from $1, \ldots, n$, that is, $\{\{i, i + 1\} \mid 1 \leq i < n\} \subseteq E$

2. X-property: If $\{a, c\} \in E$ and $\{b, d\} \in E$ for some vertices $a < b < c < d$, then $\{a, d\} \in E$.

3. bar-property: For every edge $\{a, b\} \in E$ with $a < b - 1$, there exists a vertex $x$ with $a < x < b$ such that $\{a, x\} \in E$ and $\{x, b\} \in E$.

It is still open whether every persistent graph is also a terrain visibility graph. We denote the set of all persistent graphs with $n$ vertices by $\mathcal{P}_n$. 

Figure 3: A terrain visibility graph drawn in two different ways (with a corresponding terrain on the left).
Figure 4: Example of a triangulation $T$ of $C(7, 3)$ and the corresponding persistent graph $\Gamma(T)$ on five vertices. The 3-simplices are $T = \{\{0, 1, 2, 3\}, \{1, 2, 3, 6\}, \{0, 3, 4, 5\}, \{3, 4, 5, 6\}, \{0, 1, 5, 6\}, \{0, 1, 3, 5\}, \{1, 3, 5, 6\}\}$. The 3-simplex $\{1, 3, 5, 6\}$ (thick lines) yields the edges $\{1, 3\}$, $\{3, 5\}$, and $\{1, 5\}$ in $\Gamma(T)$. Conversely, this 3-simplex is obtained from the edge $\{3, 5\}$ according to the inverse map $\Xi$.

We prove the following elementary property about persistent graphs, which states that consecutive neighbors of a vertex are also neighbors of each other. Here, $N(v)$ denotes the neighborhood of vertex $v$ and $N[v] = N(v) \cup \{v\}$.

Lemma 1.5. Let $G = ([n], E)$ be a persistent graph and let $a, b, c$ be vertices such that $b < c$, $\{b, c\} \subseteq N(a)$, and there is no vertex $x \in N[a]$ with $b < x < c$. Then $\{b, c\} \in E$.

Proof. We assume that $a < b < c$ (the case $b < c < a$ is fully symmetric). By the bar-property, there exists a common neighbor $x$ of $a$ and $c$ with $a < x < c$. Note that $b < x$ is not possible by assumption. If $x = b$, then we are done. Otherwise, we have $a < x < b$ and the bar-property again implies the existence of a common neighbor $x'$ of $x$ and $c$ with $x < x' < c$. Now, if $b < x'$, then the X-property (applied to $\{a, b\}$ and $\{x, x'\}$) implies that $x'$ is a neighbor of $a$ which contradicts our assumption on $b$ and $c$. Thus, $a < x' \leq b$. Note that we can repeat the above argument again on $x'$ if $x' < b$. Since $G$ is finite, we can conclude that $b$ is a neighbor of $c$. \hfill \Box

2 A Bijection Between $\mathcal{T}_{n+2}$ and $\mathcal{P}_n$

In this section, we prove a bijection between triangulations of $C(n + 2, 3)$ and persistent graphs on $n$ vertices. The central observation is that the 1-skeleton of a triangulation restricted to the vertices $1, \ldots, n$ forms a persistent graph (see Figure 4 for an example). Formally, the bijection is defined as follows.

Definition 2.1. For $n \geq 2$, the map $\Gamma: \mathcal{T}_{n+2} \rightarrow \mathcal{P}_n$ is defined as

$$\Gamma(T) := ([n], F_1(T) \cap \binom{[n]}{2}),$$

that is, two vertices $i$ and $j$ are adjacent in $\Gamma(T)$ if and only if $\{i, j\} \subseteq S$ for some 3-simplex $S \in T$.

First, we show that $\Gamma$ is well-defined, that is, $\Gamma(T)$ is in fact a persistent graph.

Lemma 2.2. For every $T \in \mathcal{T}_{n+2}$, it holds $\Gamma(T) \in \mathcal{P}_n$. 

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Proof. Let $C := C(n+2,3)$. Clearly, $\Gamma(T)$ contains a Hamilton path from 1 to $n$ since $\{i,i+1\} \in F_1(C)$ and thus $\{i,i+1\} \in F_1(T)$ for each $i \in [n-1]$.

Next, we show that $\Gamma(T)$ satisfies the bar-property. Let $e = \{v,w\}$ be an edge of $\Gamma(T)$ with $v < w - 1$. Then, $e$ is an internal edge, that is, $e \in F_1(T) \setminus F_1(C)$. Hence, by Lemma 1.4, there exists a vertex $b$ with $v < b < w$ such that $\{v,w,b\} \in F_2(T)$. Therefore, $\{v,b\}$ and $\{b,w\}$ are edges of $\Gamma(T)$.

Now, for the X-property, assume towards a contradiction that $\Gamma(T)$ contains the edges $\{u,w\}$ and $\{v,x\}$ with $u < v < w < x$, but $\{u,x\} \notin E(\Gamma(T))$. Let $(u,v,w,x)$ be lexicographically minimal with this property. Note that $\{u,v\}$ and $\{v,x\}$ are both internal edges. Thus, Lemma 1.4 applied to $\{v,x\}$ implies that there exists a vertex $a < v$ such that $\{a,v,x\} \in F_2(T)$ (and thus $\{\{a,v\},\{a,x\}\} \subseteq E(\Gamma(T))$). By minimality of $v$, it follows $a \leq u$. If $a < u$, then $\{a,v,x\}$ and $\{u,w\}$ are subsets of two different 3-simplices of $T$, contradicting the intersection-property of Proposition 1.2 (since $(\{a,v,x\},\{u,w\})$ is a circuit). Thus, it follows $a = u$, that is, $\{u,x\} \in E(\Gamma(T))$, which is a contradiction. \qed

In order to show that $\Gamma$ is a bijection, we next define a map that maps a persistent graph to a triangulation. We then prove that this map is the inverse of $\Gamma$. To start with, we define the following auxiliary graph.

Definition 2.3. For a persistent graph $G = ([n],E)$, we define the supergraph $\hat{G} := ([0,\ldots,n+1],E \cup \{(0,n+1) \times \{0,\ldots,n+1\}\})$, that is, $G$ contains two additional vertices that are connected to all other vertices.

It is easy to see that $\hat{G}$ is a persistent graph since adding a vertex that is adjacent to all others cannot violate the X- or bar-property. Using Definition 2.3, we now introduce the inverse map $\Xi$.

Definition 2.4. Let $G = ([n],E)$ be a persistent graph. For $e = \{v,w\} \in E$, $v < w$, we define

$$\ell_G(e) := \max\{i \in V(\hat{G}) \mid i < v \land \{i,w\} \in E(\hat{G})\}, \quad \text{and}$$

$$r_G(e) := \min\{i \in V(\hat{G}) \mid w < i \land \{i,w\} \in E(\hat{G})\}.$$

Further, we define the 3-simplex $\xi_G(e) := \{\ell_G(e),v,w,r_G(e)\}$ and the map $\Xi: \mathcal{P}_n \rightarrow T_{n+2}$ as

$$\Xi(G) := \{\xi_G(e) \mid e \in E\}.$$

We omit the index $G$ whenever it is clear from the context.

Note that, by construction of $\hat{G}$, the vertices $\ell(e)$ and $r(e)$ always exist. Moreover, by Lemma 1.5, the vertices in $\xi(e)$ form a clique in $\hat{G}$. We now show that $\Xi$ is well-defined, that is, $\Xi(G)$ is indeed a triangulation. To this end, we show that $\Xi(G)$ satisfies the union-property and the intersection-property according to Proposition 1.2. We start with the intersection-property.

Lemma 2.5. Let $G$ be a persistent graph and let $a < b < c$ be vertices of a 3-simplex $S \in \Xi(G)$. Then, $G$ does not contain any edge $\{x,y\}$ with $a < x < b < y < c$. 

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Lemma 2.6. Let $G \in \mathcal{P}_n$, $C := C(n+2,3)$, and let $S \in \Xi(G)$ be a 3-simplex containing vertices $a < b < c$ such that $\{a,b,c\} \not\subseteq F_2(C)$. Then, there exists another 3-simplex $S' \in \Xi(G)$ with $\{a,b,c\} \subseteq S'$.

Proof. Fix an edge $e \in E(G)$ with $S = \xi(e)$ (recall that the vertices $\xi(e)$ form a clique in $\hat{G}$ by Lemma 1.3). By definition of $\xi$, it holds $b \in e$. Without loss of generality, we can assume that either $e = \{a,b\}$ or $e = \{b,b'\}$ with $b < b' < c$ (the cases $e = \{b,c\}$ or $e = \{b,b'\}$ with $a < b' < b$ are symmetric). The following case distinction yields the existence of a vertex $x$ with $a < x < b$ such that $x$ is a common neighbor of $b$ and $c$ in $\hat{G}$.

Case 1: $c = n+1$. Since $\{a,b,c\}$ is not a face of $C$, we have $a + 1 < b$. Clearly, the vertex $b - 1$ is a neighbor of $b$ and $c$ in $\hat{G}$ (by construction).

Case 2: $c < n+1$. If $\ell(\{b,c\}) = a$, then the 3-simplex $\xi(\{b,c\})$ also contains $\{a,b,c\}$ and we are done. Otherwise, $x = \ell(\{b,c\})$ is a common neighbor of $b$ and $c$ in $\hat{G}$ by Lemma 1.3.

In the following, we assume $x$ to be chosen minimally. By Lemma 2.5, $x$ has no neighbor between $b$ and $c$. Thus, $r(\{x,b\}) = c$. Furthermore, $b$ has no neighbor between $a$
and $x$ because, by the X-property, this would also be a neighbor of $c$, contradicting the minimality of $x$. Therefore, $\ell(\{x, b\}) = a$ and thus, $\xi(\{x, b\})$ contains $\{a, b, c\}$ (note that $\{x, b\} \not= e$).

Lemma 2.5 and 2.6 together with Proposition 1.2 now yield the following.

**Lemma 2.7.** For every $G \in \mathcal{P}_n$, it holds $\Xi(G) \in \mathcal{T}_{n+2}$.

Finally, we prove that $\Xi$ is the inverse of $\Gamma$.

**Theorem 2.8.** The map $\Gamma$ is a bijection with $\Gamma^{-1} = \Xi$ (and thus $\Xi$ is also a bijection).

**Proof.** First, we show that $\Gamma \circ \Xi = \text{id}$. Let $G = ([n], E)$ be a persistent graph. Note that, by definition, for each edge $e \in E$, the 3-simplex $\xi(e)$ contains $e$, that is, $e \in F_1(\Xi(G))$. Thus, $E \subseteq E(\Gamma(\Xi(G)))$. Moreover, since the vertices in $\xi(e)$ form a clique in $G$ (by Lemma 1.5), it follows that $(F_1(\xi(e)) \cap \binom{[n]}{2}) \subseteq E$. Thus, $E(\Gamma(\Xi(G))) \subseteq E$. Hence, $\Gamma(\Xi(G)) = G$.

To see that $\Xi \circ \Gamma = \text{id}$, let $T$ be any triangulation of $C = C(n + 2, 3)$ and let $S \in T$ be a 3-simplex. Let $a < b < c < d$ be the vertices of $S$. We claim that $a = \max\{i \mid 0 \leq i < b, \{i, c\} \in F_1(T)\}$. Assume towards a contradiction that there exists a vertex $x$ with $a < x < b$ and $\{x, c\} \in F_1(T)$. Then, $\{a, b, d\}$ and $\{a, c\}$ are subsets of two different simplices of $T$, contradicting the intersection-property of Proposition 1.2. By symmetry, we also obtain that $d = \min\{i \mid c < i \leq n + 1, \{i, b\} \in F_1(T)\}$. Now, since $\Gamma(T)$ contains the edge $\{b, c\}$, it follows that $\Xi(\Gamma(T))$ contains $S$. Thus, $T \subseteq \Xi(\Gamma(T))$. Since $T$ and $\Xi(\Gamma(T))$ are triangulations of $C$ (by Lemma 2.6), this implies $T = \Xi(\Gamma(T))$.

An interesting observation is that, for any $G \in \mathcal{P}_n$, the map $\xi_G: E(G) \to \Xi(G)$ is a bijection. Its inverse is given by the map $\{a, b, c, d\} \mapsto \{b, c\}$, where $a < b < c < d$. This implies that the number of edges in $G$ equals the number of 3-simplices in $\Xi(G)$.

To close this section, we compare our result for $d = 3$ with the characterization for even $d$ by Oppermann and Thomas [18]. They showed that for every triangulation $T$ of $C(n, 2k)$, the set of $k$-dimensional faces of $T$ that do not contain $\{i, i + 1\}$ for some $i$ contains exactly $\binom{n-k-1}{k}$ non-intertwining tuples from $\{0, \ldots, n-1\}^k$, where $(a_0, \ldots, a_k)$ intertwines $(b_0, \ldots, b_k)$ if $a_0 < b_0 < a_1 < b_1 < \cdots < a_k < b_k$. Conversely, they also proved that every non-intertwining set of size $\binom{n-k-1}{k}$ (which is maximal) defines a unique triangulation. For $k = 1$, this gives a one-to-one correspondence between triangulations of $C(n, 2)$ and maximal outerplanar graphs (which are chordal). Now, when moving to $d = 3$ dimensions, we lose planarity since edges can intertwine but have to satisfy the X-property. Also, chordality is lost and replaced by the bar-property.

## 3 Stasheff-Tamari Order on Persistent Graphs

A classic tool for the analysis of triangulations of cyclic polytopes are the first and second Stasheff-Tamari orders, which are certain partial orders on the set of triangulations. In this section we show how these partial orders translate to partial orders on persistent graphs. It is known that the first and second Stasheff-Tamari order are identical on $\mathcal{T}_n$ [2]. Hence, we will only define and use the first Stasheff-Tamari order here.
Let \( C = C(n, 3) \) and \( W := \{v_1 < v_2 < \cdots < v_5\} \) be a set of five vertices of \( C \). Note that \( \text{conv}(W) \) equals \( C(5, 3) \) and has exactly two triangulations:

\[
T^* := \{\{v_1, v_2, v_3, v_5\}, \{v_1, v_3, v_4, v_5\}\} \quad \text{and} \quad T_* := \{\{v_1, v_2, v_4, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_2, v_3, v_4, v_5\}\}.
\]

Now, let \( T \) be a triangulation of \( C \) with \( T_* \subseteq T \). Then, we obtain a new triangulation \( T' \) of \( C \) via \( T' := (T \setminus T_*) \cup T^* \). In this case, we say that \( T' \) is obtained from \( T \) by a \textit{bistellar up-flip}, and conversely, \( T \) is obtained from \( T' \) by a \textit{bistellar down-flip}. For any two triangulations \( T, T' \) of \( C \), we write \( T \preceq T' \) if \( T' \) is obtained from \( T \) by a sequence of bistellar up-flips. This defines a partial order called the first Stasheff-Tamari order [14]. Note that \( T \preceq T' \) implies that \( |T| \geq |T'| \).

The following theorem shows that a bistellar up-flip corresponds to removing a certain edge from the corresponding persistent graph.

**Theorem 3.1.** Let \( T, T' \in \mathcal{T}_n \). Then, \( T' \) is obtained from \( T \) by a bistellar up-flip if and only if \( E(\Gamma(T)) = E(\Gamma(T')) \cup \{e\} \) for some edge \( e \in E(\Gamma(T)) \). In particular, \( T \preceq T' \) if and only if \( \Gamma(T) \supseteq \Gamma(T') \).

**Proof.** Let \( T \preceq T' \) be related by a bistellar up-flip on the vertices \( v_1 < \cdots < v_5 \), that is,

\[
T \supseteq \{\{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}\} \quad \text{and} \quad T' \supseteq \{\{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_3, v_4\}, \{v_2, v_3, v_4, v_5\}\}.
\]

Then, \( E(\Gamma(T)) = E(\Gamma(T')) \cup \{\{v_2, v_4\}\} \).

Conversely, let \( G, G' \in \mathcal{P}_{n-2} \) with \( E(G) = E(G') \cup \{v, w\} \) with \( v < w \). Then, clearly \( v + 1 < w \). Thus, by the bar-property, there exists \( v < y < w \) with \( \{\{v, y\}, \{y, w\}\} \subseteq E(G') \). Moreover, \( y \) is unique because otherwise the X-property would imply that \( \{v, w\} \in E(G') \). In fact, the X-property even implies that \( \ell_G(\{y, w\}) = v \) and \( r_G(\{v, y\}) = w \). Let \( x := \ell_G(\{v, w\}) \) and \( z := r_G(\{v, w\}) \).

We claim that \( \xi_{G'}(\{v, y\}) = \{x, v, y, z\} \), that is, \( \ell_{G'}(\{v, w\}) = x \) and \( r_{G'}(\{v, w\}) = z \). First, note that, since \( r_{G'}(\{v, y\}) = w \), we must have \( r_{G'}(\{v, y\}) > w \) and thus \( r_{G'}(\{v, y\}) = r_{G'}(\{v, w\}) = z \). Now, if \( \ell_{G'}(\{v, y\}) > x \), then, by the X-property, \( \ell_{G'}(\{v, y\}) \) would also be a neighbor of \( v \) in \( G' \), contradicting \( x = \ell_{G'}(\{v, w\}) \). To see that \( G' \) contains the edge \( \{x, y\} \), note that otherwise \( x \) would have two consecutive neighbors \( a, a' \) with \( v < a < a' < w \). By Lemma [1.3] this implies that \( G' \) contains the edge \( \{a, a'\} \) (thus, \( \{a, a'\} \neq \{v, w\} \)). The X-property then implies that \( G' \) also contains the edges \( \{v, a'\} \) and \( \{a, w\} \). If \( a \neq v \), then this contradicts \( \ell_{G'}(\{v, y\}) = v < a \), and if \( a' \neq w \), then this contradicts \( r_{G'}(\{v, y\}) = w > a' \). Thus, we have \( \{x, y\} \in E(G') \) implying \( \ell_{G'}(\{v, y\}) = x \) (and thus also \( \ell_{G'}(\{v, w\}) = x \)). This proves the claim \( \xi_{G'}(\{v, y\}) = \{x, v, y, z\} \). Moreover, we clearly have \( \xi_{G'}(\{v, y\}) = \{x, v, y, w\} \). From symmetric arguments it follows that \( \xi_{G'}(\{y, w\}) = \{x, w, y, z\} \) and \( \xi_{G'}(\{y, w\}) = \{v, y, w, z\} \).

Finally, it is not difficult to check that \( \xi_G(e) = \xi_{G'}(e) \) for any edge \( e \in E(G') \) with \( e \not\subseteq \{v, y, w\} \). This is clear if \( e \cap \{v, w\} = \emptyset \). It is also clear if \( e \in \{v, w\} \times \{u \mid (u < v) \vee (w < u)\} \). For such an edge \( e = \{v, u\} \) with \( v < u < w \), note that \( u < y \) holds since otherwise \( G' \) would contain the edge \( \{v, u\} \) (by the X-property). Since \( r_{G'}(\{v, u\}) \leq y \), it follows \( r_{G'}(\{v, u\}) = r_{G'}(\{v, w\}) \), and thus \( \xi_G(e) = \xi_{G'}(e) \). Similarly, for an edge \( e = \{w, u\} \) with \( y < u < w \), we have \( \ell_G(\{w, u\}) = \ell_G(\{w, u\}) \geq y \), and thus \( \xi_G(e) = \xi_{G'}(e) \).
To sum up, we obtain

\[ \Xi(G') = (\Xi(G) \setminus \{\{x, v, y, w\}, \{x, v, w, z\}, \{v, y, w, z\}\}) \cup \{\{x, v, y, z\}, \{x, y, w, z\}\}, \]

that is, \( \Xi(G') \) is obtained from \( \Xi(G) \) by a bistellar up-flip on \( x < v < y < w < z \).

We close with observing a connection to higher Bruhat orders. Evans and Saeedi \[8, Theorem 3\] described a map \( \alpha: \mathcal{P}_n \to B(n, 2) \), where \( B(n, 2) \) is the second higher Bruhat order (which is isomorphic to the set of 3-signotopes \[9\]).

Moreover, Rambau \[19\] showed an order-preserving map \( f_d: B(n, d) \to \text{HST}_1(n + 2, d + 1) \) from the higher Bruhat order to the first higher Stasheff-Tamari order (see also \[20, Theorem 8.9\]). It is open whether this map is surjective. It can be observed that our bijection \( \Xi \) equals \( f_2 \circ \alpha \), which implies that \( f_2 \) is surjective.

4 Enumerating Triangulations

The bijection between \( T_{n+2} \) and \( \mathcal{P}_n \) has the practical implication that in order to enumerate all triangulations of \( C(n, 3) \), one can instead enumerate all persistent graphs on \( n - 2 \) vertices. Since these graphs are combinatorially simpler structures, we can thus improve upon previous enumeration efforts \[13\]. We present a simple and efficient algorithm for the enumeration of persistent graphs.

For given \( n \), let \( \mathcal{E} := \binom{[n]}{2} \setminus \{\{i, i + 1\} | i \in [n - 1]\} \) be the set of all potential edges that are not on the obligatory Hamilton path of a persistent graph. Further, we define a lexicographical order \( \preceq \) on \( \mathcal{E} \) by setting, for any \( x_1 < y_1 \) and \( x_2 < y_2 \),

\[ \{x_1, y_1\} \preceq \{x_2, y_2\} \iff (y_1 < y_2) \lor (y_1 = y_2 \land x_1 \leq x_2). \]

Starting from a path \( P_n \), Algorithm 1 processes the potential edges \( \mathcal{E} \) in ascending order and recurses on each edge, either adding or not adding it to the graph. Its efficiency arises mainly from the fact that we can quickly identify and skip edges whose addition would violate the X- or bar-property. We remark that, while the listing of Algorithm 1 assumes that all inputs are copied upon invocation, it is easy to modify the algorithm such that no copying of \( G \) is necessary.

The following proposition states the correctness.

**Proposition 4.1.** Let \( G = ([n], E) \) be a graph containing a path on \( 1, 2, \ldots, n \) and let \( e = \{x, k\}, 1 \leq x < k \leq n \), be such that the following properties hold:

- \( E \cap \{e' \in \mathcal{E} \mid e' > e\} = \emptyset. \)
- If \( e \notin E \), then \( G \) is persistent.
- If \( e \in E \), then either \( G \) is persistent or \( G \) satisfies the X-property and \( e \) is the only edge violating the bar-property.

Then \( \text{PersistentGraphs}(G, k, x) \) outputs exactly all graphs in the set

\[ \mathcal{P}_G^e := \{G' = ([n], E') \in \mathcal{P}_n \mid (E \subseteq E') \land ((E' \setminus E) \subseteq \{e' \in \mathcal{E} \mid e' \geq e\})\} \].
Algorithm 1 Enumerating persistent graphs on \( n \) vertices.

**Input:** A graph \( G = ([n], E) \), \( k \leq n \), and \( x < k \), such that \( E \cap \{ e' \in E \mid e' \succ \{ x, k \} \} = \emptyset \) and \( G \) is persistent except that the edge \( \{ x, k \} \) (if existing) may violate the bar-property.

**Output:** All persistent supergraphs of \( G \) obtainable by adding edges \( e' \succeq \{ x, k \} \).

1: function PersistentGraphs \(( G, k, x )\)
2: if \( x + 1 = k \) then
3: if \( k = n \) then
4: output \( G \)
5: else
6: PersistentGraphs \(( G, k + 1, 1 )\)
7: end if
8: return
9: end if
10: if \( \{ x, k \} \notin E \) then \( \triangleright \) edge \( \{ x, k \} \) does not exist
11: \( y \leftarrow \) rightmost neighbor of \( x \)
12: PersistentGraphs \(( G, k, y )\)
13: add \( \{ x, k \} \) to \( E \)
14: end if
15: for \( y = x + 1, \ldots, k - 1 \) do \( \triangleright \) edge \( \{ x, k \} \) does exist
16: if \( \{ x, y \} \in E \) then
17: add \( \{ y, k \} \) to \( E \)
18: PersistentGraphs \(( G, k, y )\)
19: remove \( \{ y, k \} \) from \( E \) (unless \( y = k - 1 \))
20: \( y \leftarrow \) rightmost neighbor of \( y \)
21: end if
22: end for
23: end function

**Proof.** If \( x + 1 = k \), then clearly \( e \in E \) and \( G \) is persistent. If now \( k = n \), then clearly \( \mathcal{P}_G^e = \{ G \} \), that is, Line 4 is correct. If \( k < n \), then \( \mathcal{P}_G^e = \mathcal{P}^{(1,k+1)}_G \). Thus, Line 6 is correct.

Now assume that \( x + 1 < k \). If \( e \notin E \) (Line 10), then \( G \) is persistent. The set \( \mathcal{P}_G^e \) can be partitioned into two sets:

\[
A := \{ G' = ([n], E') \in \mathcal{P}_G^e \mid e \notin E' \} \quad \text{and} \quad B := \{ G' = ([n], E') \in \mathcal{P}_G^e \mid e \in E' \}.
\]

Consider a graph \( G' = ([n], E') \in A \). Let \( y \) be the rightmost neighbor of \( x \) in \( G \) and note that \( y < k \) since \( E \) does not contain any edge \( e' \) with \( e' \succ e \). Due to the X-property, \( E' \) does not contain any edge \( \{ x', k \} \) with \( x < x' < y \). Thus, \( A = \mathcal{P}_G^{[y,k]} \) and all these graphs are produced by the recursive call in Line 12. As regards the set \( B \), note that \( B = \mathcal{P}_G^{e+e} \), where \( G + e := ([n], E \cup \{ e \}) \). Thus, we add \( e \) to \( E \) in Line 13 and then handle this case in Line 15.

If \( e \in E \), then, for every \( G' = ([n], E') \in \mathcal{P}_G^e \), there must be a minimal vertex \( y \) with \( x < y < k \) and \( \{ \{ x, y \}, \{ y, k \} \} \subseteq E' \) (by the bar-property). Since \( \{ x, y \} \prec e \), it
Table 1: Number of persistent graphs with $n \leq 16$ vertices. The values for $n \leq 13$ were already known \cite{13}.

| $n$ | $|\mathcal{P}_n| = |\mathcal{T}_{n+2}|$ | computation time |
|-----|---------------------------------|------------------|
| 1   | 1                               | $<0.1\text{ s}$ |
| 2   | 1                               | $<0.1\text{ s}$ |
| 3   | 2                               | $<0.1\text{ s}$ |
| 4   | 6                               | $<0.1\text{ s}$ |
| 5   | 25                              | $<0.1\text{ s}$ |
| 6   | 138                             | $<0.1\text{ s}$ |
| 7   | 972                             | $<0.1\text{ s}$ |
| 8   | 8 477                           | $<0.1\text{ s}$ |
| 9   | 89 405                          | $<0.1\text{ s}$ |
| 10  | 1 119 280                       | $<0.1\text{ s}$ |
| 11  | 16 384 508                      | $0.15\text{ s}$ |
| 12  | 276 961 252                     | $2\text{ s}$    |
| 13  | 5 349 351 298                   | $30\text{ s}$   |
| 14  | 116 985 744 912                 | $12\text{ m}$   |
| 15  | 2 873 993 336 097               | $4\text{ h 30 m}$ |
| 16  | 78 768 494 976 617              | $4\text{ d 23 h 2 m}$ |

follows that $\{x, y\} \in E$. Hence, $y$ has to be a neighbor of $x$ in $G$ with $x < y < k$. Furthermore, any neighbor $y'$ of $x$ with $x < y' < y$ cannot have a neighbor to the right of $y$, because the $X$-property would otherwise imply that also $\{y', k\} \in E'$, contradicting the minimality of $y$. That is, $y$ can only be neighbor of $x$ such that no other neighbor $y'$ of $x$ with $x < y' < y$ has a neighbor to the right of $y$. Let $Y$ denote the vertex set containing all these possible candidates. The for-loop in Line 15 iterates exactly over the candidates in $Y$. For a given $y \in Y$, let $A_y \subseteq \mathcal{P}_G$ be the subset of graphs $G' = ([n], E')$, where $y$ is the minimal vertex with $x < y < k$ and $\{(x, y), (y, k)\} \subseteq E'$, and note that $A_y = \mathcal{P}_{G+\{y,k\}}$. Moreover, $\{A_y \mid y \in Y\}$ is clearly a partition of $\mathcal{P}_G$. Hence, calling PersistentGraphs($G$, $k$, $y$) for each possible $y$, outputs exactly the graphs in $\mathcal{P}_G$.

Corollary 4.2. Let $G = ([n], E)$ be the path on $1, 2, \ldots, n$. Then PersistentGraphs($G$, 2, 1) outputs exactly $\mathcal{P}_{G}^{\{1,2\}} = \mathcal{P}_n$.

By Corollary 4.2 we can use Algorithm 1 to efficiently count the number of elements of $\mathcal{P}_n$ and thus of $\mathcal{T}_{n+2}$. The results for $n \leq 16$ are listed in Table 1. The computations were performed using an Intel Xeon W-2125 CPU.

5 Conclusion

Our results yield further insights into the structure of the triangulations of the $3$-dimensional cyclic polytope by relating their $1$-skeleton to persistent graphs. It remains open to characterize the structure of the $\lfloor d/2 \rfloor$-skeleton for arbitrary odd dimension $d$.

\footnote{Implementation available at \url{https://www.akt.tu-berlin.de/menue/software}.}
It is also open whether a closed formula for the number of triangulations of \( C(n, 3) \) can be given \[20\], Open Problem 9.2.

On the other side, the bijection might also lead to new insights about persistent graphs. Can the bijection be of help in resolving the conjecture that every persistent graph is a terrain visibility graph?

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