THE PERRON-FROBENIUS THEOREM FOR MARKOV SEMIGROUPS

OMAR HIJAB

Abstract. Let \( P^V_t, t \geq 0 \), be the Schrödinger semigroup associated to a potential \( V \) and Markov semigroup \( P_t, t \geq 0, \) on \( C(X) \). Existence is established of a left eigenvector and right eigenvector corresponding to the spectral radius \( e^{\lambda_0 t} \) of \( P^V_t \), simultaneously for all \( t \geq 0 \). This is derived with no compactness assumption on the semigroup operators.

1. Introduction

Let \( X \) be a compact metric space and let \( P_t, t \geq 0, \) be a Markov semigroup on \( C(X) \) with generator \( L \). Given \( V \) in \( C(X) \) let \( P^V_t, t \geq 0, \) denote the Schrödinger semigroup on \( C(X) \) generated by \( L + V \). Then the principal eigenvalue \( \lambda_0(V) \equiv \lim_{t \to \infty} \frac{1}{t} \log \| P^V_t \| \)
is given by the Donsker-Varadhan formula [4]

\[
\lambda_0(V) = \sup_\mu \left( \int_X V \, d\mu - I(\mu) \right),
\]
where the supremum is over probability measures \( \mu \) on \( X \), and

\[
I(\mu) \equiv - \inf_{u \in D^+} \int_X \frac{L u}{u} \, d\mu.
\]

Here the infimum is over positive \( u \) in the domain \( D \) of \( L \).

An equilibrium measure is a measure \( \mu \) achieving the supremum in the Donsker-Varadhan formula. Let \( \lambda_0 = \lambda_0(V) \).

A ground state relative to \( \mu \) is a Borel function \( \psi \) satisfying \( \psi > 0 \) a.s. \( \mu \) and

\[
e^{-\lambda_0 t} P^V_t \psi = \psi, \quad \text{a.s.} \mu, t \geq 0.
\]

A ground measure is a measure \( \pi \) satisfying

\[
\int_X e^{-\lambda_0 t} P^V_t f \, d\pi = \int_X f \, d\pi, \quad t \geq 0
\]
for \( f \) in \( C(X) \).

Theorem 1. Suppose \( \pi \) and \( \mu \) are measures with \( \mu << \pi \). Suppose also \( \psi = d\mu/d\pi \) satisfies \( \psi \log \psi \in L^1(\pi) \). Then the following hold.

- If \( \pi \) is a ground measure and \( \psi \) is a ground state relative to \( \mu \), then \( \mu \) is an equilibrium measure.
• If $\pi$ is a ground measure and $\mu$ is an equilibrium measure, then $\psi$ is a ground state relative to $\mu$.

• If $\mu$ is an equilibrium measure and $\psi$ is a ground state relative to $\mu$, then $\pi$ is a ground measure.

Here is the Perron-Frobenius Theorem in this setting.

**Theorem 2.** Fix $V$ in $C(X)$, suppose
\[ e^{-\lambda_0 t} \| P^V_t \| \leq C, \quad t \geq 0, \]
for some $C > 0$, and let $\mu$ be an equilibrium measure. Then there is a ground measure $\pi$ satisfying $\mu << \pi$, $\psi = d\mu/d\pi$ is a ground state relative to $\mu$, and $\psi \log \psi \in L^1(\pi)$.

This says there is a nonnegative right eigenvector $\psi$ and a nonnegative left eigenvector $\pi$, corresponding to the spectral radius $e^{\lambda_0 t}$ of $P^V_t$, simultaneously for all $t \geq 0$.

Neither the hypothesis nor the conclusion hold when $L + V$ is a Jordan block: Take $X = \{0, 1\}$ and
\[ L = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

In this case every measure $\mu = (p, 1 - p)$ is an equilibrium measure and there is a unique ground measure $\pi = (0, 1)$.

In this generality, there is no guarantee of uniqueness of $\mu$, $\pi$ or $\psi$.

If $P_t$, $t \geq 0$, is self-adjoint in $L^2(\rho)$ for some measure $\rho$ on $X$, then, under suitable conditions, the Donsker-Varadhan formula reduces \cite{5} to the classical Rayleigh-Ritz formula for the principal eigenvalue, and every ground state $\psi$ relative to $\rho$ yields a ground measure $d\pi = \psi d\rho$ and an equilibrium measure $d\mu = \psi d\pi = \psi^2 d\rho$. In the self-adjoint case, existence of ground states is classical (Gross \cite{11}).

The Perron-Frobenius Theorem (with uniqueness) is known to hold in $L^p$ for positive operators in these cases: for finite irreducible matrices (Friedland \cite{7}, Friedland-Karlin \cite{8}, Sternberg \cite{19}), under a positivity improving property (Aida \cite{1}), under a spectral gap condition (Gong-Wu \cite{10}), and under uniform integrability or irreducibility (Wu \cite{20}).

The existence of ground measures for positive operators and the existence of ground states for compact positive operators are classical results due to Krein-Rutman \cite{13}, see also Schaefer \cite{16}, \cite{17}. In general, ground states do not exist pointwise everywhere on $X$, for example when $L \equiv 0$. The novelty of the above result is the handling of the general non-compact case by interpreting ground states as densities against ground measures, and the link with equilibrium measures.

The techniques used here are self-contained and follow the original papers \cite{4}, \cite{5}. Preliminaries are discussed in section \ref{2} equilibrium measures in section \ref{3} ground measures and ground states in section \ref{4} and entropy in section \ref{5}. The theorems are proved in section \ref{6}.

2. Preliminaries

Let $X$ be a compact metric space, let $C(X)$ denote the space of real continuous functions with the sup norm $\| \cdot \|$, and let $M(X)$ denote the space of Borel probability measures with the topology of weak convergence. Then $M(X)$ is a compact metric
space. Throughout $\mu(f)$ denotes the integral of $f$ against $\mu$ and all measures are probability measures.

A Markov semigroup on $C(X)$ is a strongly continuous semigroup $P_t : C(X) \to C(X)$, $t \geq 0$, preserving positivity, $P_tf \geq 0$, for $f \geq 0$, and satisfying $P_t1 = 1$.

The subspace $\mathcal{D} \subset C(X)$ of functions $f \in C(X)$ for which the limit

$$
\frac{d}{dt} \bigg|_{t=0} P_tf
$$

exists in $C(X)$ is dense. If $Lf$ is defined to be this limit, then the operator $L$ is the generator of $P_t$, $t \geq 0$, on $C(X)$.

Given $V$ in $C(X)$, the Schrodinger semigroup on $C(X)$ associated to $V$ is the unique strongly continuous semigroup $P_t^V : C(X) \to C(X)$, $t \geq 0$, preserving positivity, $P^V_tf \geq 0$, for $f \geq 0$, with generator $L+V$, in the sense the limit

$$
\frac{d}{dt} \bigg|_{t=0} P^V_tf
$$

exists in $C(X)$ iff $f \in \mathcal{D}$, and equals $Lf + Vf$. The Schrodinger semigroup may be constructed as the unique solution of

$$
P^V_tf = P_tf + \int_0^t P_{t-s}VP^V_s f \, ds, \quad t \geq 0.
$$

for $f \in C(X)$. For $f \geq 0$, \[2.2\] implies

$$
e^{\min_{Vt}}P_tf \leq P^V_tf \leq e^{\max_{Vt}}P_tf, \quad t \geq 0.
$$

Since $\|P^V_t\| = \max P^V_t1$, this implies

$$\min V \leq \lambda_0(V) \leq \max V.
$$

For $\mu$ in $M(X)$ and $\lambda_0 = \lambda_0(V)$, let

$$
I^V(\mu) \equiv I(\mu) - \int_X V \, d\mu + \lambda_0 = -\inf_{u \in \mathcal{D}^+} \int_X (L+V-\lambda_0)u \, d\mu
$$

Then $I^0(\mu) = I(\mu)$ and $I^V(\mu) = 0$ iff $\mu$ is an equilibrium measure for $V$.

**Lemma 2.1.** For $V$ in $C(X)$, $I^V$ is lower semicontinuous, convex, and nonnegative. In particular, $I$ is lower semicontinuous, convex, and nonnegative.

**Proof.** Lower semicontinuity and convexity follow from the fact that $I^V$ is the supremum of continuous affine functions. The Donsker-Varadhan formula implies $I^V$ is nonnegative.

Let $L^1(\mu)$ denote the $\mu$-integrable Borel functions on $X$ with norm

$$\|f\|_{L^1(\mu)} = \int_X |f| \, d\mu = \mu(|f|).
$$

For $V$ in $C(X)$, and let $P^V_t$, $t \geq 0$, be the Schrodinger semigroup on $C(X)$. Let $\mathcal{D}$ be the domain of the generator $L$ of $P_t$, $t \geq 0$, $\mathcal{D}^+$ the positive functions in $\mathcal{D}$, and $C^+(X)$ the positive functions in $C(X)$.

By positivity, there is a family $(t,x) \mapsto p^V_t(x,\cdot)$ of Borel measures on $X$ such that the Schrodinger semigroup may be written

$$
P^V_tf(x) = \int_X f(y) p^V_t(x,dy)
$$

\[2.5\]
for \( t \geq 0, x \in X, \) and \( f \in C(X). \) Hence \( P^V_t f(x) \leq +\infty \) is defined for nonnegative Borel \( f \) for all \( t \geq 0 \) and \( x \in X. \)

Let \( B(X) \) denote the bounded Borel functions on \( X \) and let \( B^+(X) \) denote the positive Borel functions \( u \) on \( X \) with \( f = \log u \in B(X). \) We say \( f_n \in B(X) \) converges \textit{boundedly} to \( f \in B(X) \) if \( f_n \rightarrow f \) pointwise everywhere and there is a \( C > 0 \) with \( |f_n| \leq C, \ n \geq 1. \)

**Lemma 2.2.** If \( f_n \rightarrow f \) boundedly then \( u_n = e^{f_n} \rightarrow u = e^f \) boundedly, \( P^V_t u_n \rightarrow P^V_t u \) boundedly, and
\[
\log\left(\frac{e^{-\lambda t}P^V_t u_n}{u_n}\right) \rightarrow \log\left(\frac{e^{-\lambda t}P^V_t u}{u}\right)
\]
boundedly.

**Proof.** If \( |f_n| \leq C, \ n \geq 1, \) then \( u_n \rightarrow u \) pointwise. Since \( e^{-C} \leq u_n \leq e^C, \ u_n \rightarrow u \) boundedly. Hence \( P^V_t u_n \rightarrow P^V_t u \) pointwise by the dominated convergence theorem. By \((2.4), \) \( P^V_t u_n \rightarrow P^V_t u \) boundedly. Since
\[
\left|\log\left(\frac{e^{-\lambda t}P^V_t u}{u}\right)\right| \leq t(\max V - \min V) + \log\left(\sup_{\inf u}\right),
\]
the last statement follows. \( \square \)

3. Equilibrium Measures

**Lemma 3.1.** For \( u \) in \( B^+(X), \)
\[
(3.1) \quad \int_X \log\left(\frac{e^{-\lambda t}P^V_t u}{u}\right) \, d\mu \geq -tI^V(\mu), \quad t \geq 0.
\]

**Proof.** By definition of \( I^V(\mu), \)
\[
(3.2) \quad \int_X \frac{(L + V - \lambda_0)u}{u} \, d\mu \geq -I^V(\mu), \quad u \in D^+.
\]

For \( t = 0, \) \((3.1)\) is an equality. Moreover for \( t > 0 \) and \( u \in D^+ , \) we have \( e^{-\lambda t}P^V_t u \in D^+ \) and
\[
\frac{d}{dt} \int_X \log\left(\frac{e^{-\lambda t}P^V_t u}{u}\right) \, d\mu = \int_X \frac{(L + V - \lambda_0)(e^{-\lambda t}P^V_t u)}{e^{-\lambda t}P^V_t u} \, d\mu \geq -I^V(\mu).
\]

This establishes \((3.1)\) for \( u \in D^+. \) Since \( D^+ \) is dense in \( C^+(X), \) \((3.1)\) is valid for \( u \) in \( C^+(X). \)

Now if \( \log u_n \rightarrow \log u \) boundedly and \((3.1)\) holds for \( u_n, \ n \geq 1, \) then by Lemma 2.2. \((3.1)\) holds for \( u. \) Thus the class of Borel functions \( f = \log u \) for which \((3.1)\) holds is closed under bounded convergence. Thus \((3.1)\) holds for all \( u \in B^+(X). \) \( \square \)

The following strengthening of Lemma 3.1 is necessary below.

**Lemma 3.2.** Let \( u > 0 \) Borel satisfy \( \log u \in L^1(\mu). \) Then for \( t \geq 0, \)
\[
(3.3) \quad tI^V(\mu) + \int_X \log^+\left(\frac{e^{-\lambda t}P^V_t u}{u}\right) \, d\mu \geq \int_X \log^+\left(\frac{e^{-\lambda t}P^V_t u}{u}\right) \, d\mu.
\]

Here the integrals may be infinite.
Proof. Without loss of generality, assume \( I^V(\mu) < \infty \).

Assume in addition \( u > \delta > 0 \) and let \( u_n = u \wedge n, n \geq 1 \). Then \( u_n \) is in \( B^+(X) \), (3.1) holds with \( u \), derived, Lemma 3.2. \( \square \)

Letting \( n \to \infty \) yields (3.1) hence (3.3) for \( \log u \in L^1(\mu) \), provided \( u \geq \delta > 0 \). Here the left side of (3.3) may be infinite, but the right side is finite.

Now for \( u > 0 \) Borel with \( \log u \in L^1(\mu) \), let \( u_\delta = u \vee \delta \). Then by what we just derived,

\[
I^V(\mu) + \int_X \log^+ \left( \frac{e^{-\lambda_0 t P_t^V u_\delta}}{u_\delta} \right) \, d\mu \geq \int_X \log^+ \left( \frac{e^{-\lambda_0 t P_t^V u_\delta}}{u_\delta} \right) \, d\mu
\]

so

\[
I^V(\mu) + \int_X \log^+ \left( \frac{e^{-\lambda_0 t P_t^V u_\delta}}{u_\delta} \right) \, d\mu \geq \int_X \log^+ \left( \frac{e^{-\lambda_0 t P_t^V u_\delta}}{u_\delta} \right) \, d\mu + \int_{u<\delta} \log u \, d\mu.
\]

We may assume

\[
\int_X \log^+ \left( \frac{e^{-\lambda_0 t P_t^V u_\delta}}{u_\delta} \right) \, d\mu < \infty,
\]

otherwise (3.3) is vacuously true. To establish (3.3), we pass to the limit \( \delta \downarrow 0 \) in (3.4). Since \( \log u \in L^1(\mu) \) and

\[
\log^+ \left( \frac{e^{-\lambda_0 t P_t^V u_\delta}}{u_\delta} \right), \quad \delta > 0,
\]

is an increasing sequence in \( \delta > 0 \), the right side of (3.4) converges to the right side of (3.3) as \( \delta \downarrow 0 \). Since \( u_\delta \leq u + \delta \),

\[
\log^+ \left( \frac{e^{-\lambda_0 t P_t^V u_\delta}}{u_\delta} \right) \leq \log^+ \left( \frac{e^{-\lambda_0 t P_t^V u}}{u} \right) + \log^+ u + C, \quad \delta \leq 1,
\]

hence the dominated convergence theorem applies. Thus the left side of (3.4) converges to the left side of (3.3) as \( \delta \downarrow 0 \).

If \( \psi > 0 \) is Borel, for \( u \geq 0 \) Borel, define

\[
P_t^{V,\psi} u \equiv e^{-\lambda_0 t P_t^V (\psi u)} \psi, \quad t \geq 0.
\]

Lemma 3.3. Suppose \( \log \psi \in L^1(\mu) \) and let \( u > 0 \) Borel satisfy \( \log u \in L^1(\mu) \). Then for \( t \geq 0 \),

\[
I^V(\mu) + \int_X \log^+ \left( \frac{P_t^{V,\psi} u}{u} \right) \, d\mu \geq \int_X \log^+ \left( \frac{P_t^{V,\psi} u}{u} \right) \, d\mu.
\]

Here the integrals may be infinite.

Proof. Since \( \log \psi \) is in \( L^1(\mu) \), \( \log(u \psi) \) is in \( L^1(\mu) \) iff \( \log u \) is in \( L^1(\mu) \). Now apply Lemma 3.2 \( \square \)

Lemma 3.4. Suppose \( \log \psi \in L^1(\mu) \). Then \( \mu \) is an equilibrium measure iff for all \( u > 0 \) Borel satisfying \( \log u \in L^1(\mu) \),

\[
\int_X \log^+ \left( \frac{P_t^{V,\psi} u}{u} \right) \, d\mu \geq \int_X \log^+ \left( \frac{P_t^{V,\psi} u}{u} \right) \, d\mu, \quad t \geq 0.
\]
By Fatou’s lemma, 

\[ \int_X \log^+ \left( \frac{e^{-\lambda_0 t} P_t V u}{u} \right) \, d\mu \geq \int_X \log^- \left( \frac{e^{-\lambda_0 t} P_t V u}{u} \right) \, d\mu. \]

For \( u \) in \( C^+(X) \), these integrals are finite hence 

\[ \int_X \log \left( \frac{e^{-\lambda_0 t} P_t V u}{u} \right) \, d\mu \geq 0. \]

Now for \( u \in D^+ \),

\[ e^{-\lambda_0 t} P_t V u = u + t(L + V - \lambda_0)u + o(t), \quad t \to 0, \]

uniformly on \( X \). Since \( u > 0 \),

\[ \frac{e^{-\lambda_0 t} P_t V u}{u} = 1 + t \frac{(L + V - \lambda_0)u}{u} + o(t), \quad t \to 0, \]

uniformly on \( X \). Thus

\[ \log \left( \frac{e^{-\lambda_0 t} P_t V u}{u} \right) = t \frac{(L + V - \lambda_0)u}{u} + o(t), \quad t \to 0, \]

uniformly on \( X \), hence

\[ \lim_{t \to 0} \frac{1}{t} \int_X \log \left( \frac{e^{-\lambda_0 t} P_t V u}{u} \right) \, d\mu = \int_X \frac{(L + V - \lambda_0)u}{u} \, d\mu. \]

This implies

\[ \int_X \frac{(L + V - \lambda_0)u}{u} \, d\mu \geq 0. \]

This implies \( I^V(\mu) \leq 0 \), hence \( I^V(\mu) = 0 \). \( \square \)

4. Ground Measures and Ground States

Lemma 4.1. If \( \pi \) is a ground measure, then \( e^{-\lambda_0 t} P_t V, \, t \geq 0, \) is a strongly continuous contraction semigroup on \( L^1(\pi) \) and \( \| f \| \) holds for \( f \) in \( L^1(\pi) \).

Proof. \( \| \) implies

\[ \| e^{-\lambda_0 t} P_t V f \|_{L^1(\pi)} = \int_X |e^{-\lambda_0 t} P_t V f| \, d\pi \leq \int_X e^{-\lambda_0 t} P_t V |f| \, d\pi = \int_X |f| \, d\pi = \| f \|_{L^1(\pi)} \]

for \( f \) in \( C(X) \). Let \( f \) be in \( L^1(\pi) \) and choose \( f_n \in C(X) \) converging to \( f \) in \( L^1(\pi) \). By Fatou’s lemma,

\[ \pi(e^{-\lambda_0 t} P_t V |f_n - f|) \leq \inf \pi(e^{-\lambda_0 t} P_t V |f_m - f_n|) = \inf m \pi(|f_n - f_m|) = \pi(|f_n - f|). \]

Thus \( P_t V f^+ \leq P_t V f \) and \( P_t V f = P_t V f^+ - P_t V f^- \) in \( L^1(\pi) \), and \( P_t V f_n \to P_t V f \) in \( L^1(\pi) \). Hence \( e^{-\lambda_0 t} P_t V, \, t \geq 0, \) is a strongly continuous contraction semigroup on \( L^1(\pi) \) satisfying \( \| \) for \( f \) nonnegative Borel. The invariance \( \| \) for \( f \) in \( L^1(\pi) \) follows. \( \square \)
Lemma 4.2. Suppose \( \pi \) and \( \mu \) are measures with \( \mu \ll \pi \), and let \( \psi = d\mu/d\pi \). Then \( e^{-\lambda t}P^V_t \), \( t \geq 0 \), is a strongly continuous contraction semigroup on \( L^1(\pi) \) iff \( P^V_t \), \( t \geq 0 \), is a strongly continuous contraction semigroup on \( L^1(\mu) \). Moreover, in this case, (5.1) holds for all \( \pi \) a supremum of continuous convex functions, in each variable. Proof. The lower-semicontinuity and convexity follow from the definition of \( \psi \), and let \( \psi = d\mu/d\pi \). Thus (5.1) holds for all \( \pi \) in \( L^1(\mu) \).

Lemma 4.3. Suppose \( \pi \) and \( \mu \) are measures with \( \mu \ll \pi \), and let \( \psi = d\mu/d\pi \). Suppose \( \psi \) is a ground state relative to \( \mu \). Then \( P^V_t \), \( t \geq 0 \), is a Markov semigroup on \( L^1(\mu) \).

Proof. \( P^V_t \) is a Markov semigroup on \( L^1(\mu) \) is a strongly continuous contraction semigroup \( Q_t, t \geq 0 \), on \( L^1(\mu) \) satisfying \( Q_t f \geq 0 \) a.s. \( \mu \) for \( f \geq 0 \) a.s. \( \mu \) and \( Q_t 1 = 1 \) a.s. \( \mu \).

5. Entropy

For \( \mu, \pi \) in \( M(X) \), the entropy of \( \mu \) relative to \( \pi \) is

\[
H(\mu, \pi) = \sup_f \left( \int_X f d\mu - \log \int_X e^f d\pi \right)
\]

where the supremum is over \( f \) in \( C(X) \).

Lemma 5.1. \( H(\mu, \pi) \geq 0 \) is finite iff \( \mu \ll \pi \) and \( \psi \log \psi \) is in \( L^1(\pi) \), where \( \psi = d\mu/d\pi \), in which case

\[
H(\mu, \pi) = \int_X \psi \log \psi d\pi.
\]

Moreover \( H \) is lower-semicontinuous and convex separately in each of \( \mu \) and \( \pi \).

Proof. The lower-semicontinuity and convexity follow from the definition of \( H \) as a supremum of continuous convex functions, in each variable \( \pi \), \( \mu \) separately.

Suppose \( H = H(\mu, \pi) < \infty \); then

\[
\int_X f d\mu - \log \int_X e^f d\pi \leq H
\]

for \( f \) in \( C(X) \). The class of Borel functions \( f \) for which (5.1) holds is closed under bounded convergence. Thus (5.1) holds for all \( f \in B(X) \). Inserting \( f = r1_A \), where \( \pi(A) = 0 \), we obtain

\[
r\mu(A) \leq r\mu(A) - \log(\pi(A^c)) \leq H.
\]
Let $r \to \infty$ to conclude $\mu \ll \pi$. Since $\psi = d\mu/d\pi \in L^1(\pi)$, let $0 \leq f_n \in C(X)$ with $f_n \to \psi$ in $L^1(\pi)$. By passing to a subsequence, assume $f_n \to \psi$ a.s. $\pi$. Insert $f = \log(f_n + \epsilon)$ into (5.1) to yield
\[
\int_X \log(f_n + \epsilon) \, d\mu - \log \int_X (f_n + \epsilon) \, d\pi \leq H.
\]
Let $n \to \infty$ followed by $\epsilon \to 0$. Since $f_n \to \psi$ in $L^1(\pi)$, let $0 \leq f_n \in C(X)$ with $f_n \to \psi$ in $L^1(\pi)$. By passing to a subsequence, assume $f_n \to \psi$ a.s. $\pi$. Insert $f = \log\left(\frac{f_n + \epsilon}{\epsilon}\right)$ into (5.1) to yield
\[
\int_X \log\left(\frac{f_n + \epsilon}{\epsilon}\right) \, d\mu - \log \int_X \left(\frac{f_n + \epsilon}{\epsilon}\right) \, d\pi \leq H.
\]
Thus, by Fatou’s lemma (twice),
\[
\int_X \log \psi \, d\pi - \log \int_X \psi \, d\pi \leq H.
\]
Since $\log \psi$ is bounded, this establishes $\psi \in L^1(\pi)$ and $\int_X \psi \, d\pi \leq H$.

Conversely, suppose $\psi = \frac{d\mu}{d\pi}$ exists and $\psi \log \psi \in L^1(\pi)$. By Jensen’s inequality,
\[
\int_X f \, d\mu \leq \log \int_X e^f \, d\mu
\]
for $f$ bounded Borel. Replace $f$ by $f - \log(\psi \wedge n + \epsilon)$ to get
\[
\int_X f \, d\mu - \log \int_X \left(\frac{e^f}{\psi \wedge n + \epsilon}\right) \, d\pi \leq \int_X \psi \log(\psi \wedge n + \epsilon) \, d\pi.
\]
Let $\epsilon \to 0$ followed by $n \to \infty$ obtaining
\[
\int_X f \, d\mu - \log \int_X e^f \, d\pi \leq \int_X \psi \log \psi \, d\pi.
\]
Now maximize over $f$ in $C(X)$ to conclude $H(\mu, \pi) \leq \int_X \psi \log \psi \, d\pi$. \hfill \Box

6. PROOFS OF THE THEOREMS

**Proof of Theorem 1** Assume $\pi$ is a ground measure and $\psi$ is a ground state relative to $\mu$. By Lemma 4.3, $P_t^{V,\psi}$, $t \geq 0$, is a Markov semigroup on $L^1(\mu)$. Suppose $\log u \in L^1(\mu)$. Then $P_t^{V,\psi}(\log u)$ is in $L^1(\mu)$, hence there is a set $N$ with $\mu(N) = 0$ and $P_t^{V,\psi}(\log u)(x) < \infty$ and $P_t^{V,\psi}(1) = 1$ for $x \notin N$. Jensen’s inequality applied to the integral $f \mapsto (P_t^{V,\psi} f)(x)$ implies
\[
\log \left(\frac{P_t^{V,\psi} u}{u}\right)(x) \geq P_t^{V,\psi}(\log u)(x) - (\log u)(x), \quad x \notin N.
\]
Thus the negative part of
\[
(6.1) \quad \log \left(\frac{P_t^{V,\psi} u}{u}\right)
\]
is in $L^1(\mu)$ and
\[
\int_X \log \left(\frac{P_t^{V,\psi} u}{u}\right) \, d\mu \geq \int_X \left(P_t^{V,\psi}(\log u) - \log u\right) \, d\mu = 0.
\]
By Lemma 3.3, this implies $\mu$ is an equilibrium measure, establishing the first claim.

Assume $\pi$ is a ground measure and $\mu$ is an equilibrium measure. Then $e^{-\lambda t} P_t^{V}$, $t \geq 0$, is a strongly continuous contraction semigroup on $L^1(\pi)$, hence $P_t^{V,\psi}$, $t \geq 0$,
is a strongly continuous contraction semigroup on $L^1(\mu)$. Since $1 \in L^1(\mu)$, $P^V_t \psi f \in L^1(\mu)$ hence $\log^+(P^V_t \psi f) \in L^1(\mu)$. By Lemma 2.3 $\log^-(P^V_t \psi f) \in L^1(\mu)$ hence $\log(P^V_t \psi f) \in L^1(\mu)$. By Jensen’s inequality, (4.1), and Lemma 3.4,

$$0 = \log(\mu(1)) = \log \left( \int_X P^V_t \psi f \, d\mu \right) \geq \int_X \log(P^V_t \psi f) \, d\mu \geq 0.$$ 

Since $\log$ is strictly concave, this can only happen if $P^V_t \psi f$ is $\mu$ a.s. constant. By (4.1), the constant is 1. Since $\psi > 0$ a.s. $\mu$ is immediate, this establishes the second claim.

Assume $\mu$ is an equilibrium measure and $\psi$ is a ground state relative to $\mu$. Then $P^V_t \psi f = 1$ a.s. $\mu$, hence for $u \in B^+(X)$,

$$\inf u \leq \frac{P^V_t \psi f u}{\sup u} \leq \sup u \inf u \quad \text{a.s.}$$

hence (6.1) is in $L^\infty(\mu)$. Thus by (3.6), for $f \in B(X)$,

$$\int_X \log \left( \frac{P^V_t \psi f e^f}{e^f} \right) \, d\mu \geq 0, \quad \epsilon > 0.$$ 

Since $P^V_t \psi f = 1$ a.s. $\mu$ and $e^f = 1 + \epsilon f + o(\epsilon)$,

$$\frac{1}{\epsilon} \log \left( \frac{P^V_t \psi f e^f}{e^f} \right) \to P^V_t \psi f - f, \quad \epsilon \to 0,$$

uniformly a.s. $\mu$ on $X$. Thus

$$\int_X \left( P^V_t \psi f - f \right) \, d\mu \geq 0$$

for $f \in B(X)$. Applying this to $\pm f$ yields

(6.2) $$\int_X e^{-\lambda_0 t} P^V_t (\psi f) \, d\pi = \int_X \psi f \, d\pi$$ 

for $f \in B(X)$. Now let $\psi_\delta = (\psi \vee \delta) \wedge (1/\delta)$. Then $\psi/\psi_\delta \to 1$ pointwise. For $f$ in $C(X)$, $f/\psi_\delta \in B(X)$; inserting this into (6.2) yields

(6.3) $$\int_X e^{-\lambda_0 t} P^V_t \left( \frac{\psi f}{\psi_\delta} \right) \, d\pi = \int_X \frac{\psi f}{\psi_\delta} \, d\pi.$$ 

Since

$$\frac{|f|}{\psi_\delta} \leq |f| + |f|\psi, \quad \delta \leq 1,$$

sending $\delta \to 0$ in (6.3) yields (4.1). Hence $\pi$ is a ground measure, establishing the third claim. \hfill \Box

**Proof of Theorem 2.** Let

$$M \equiv \sup_{t \geq 0} e^{-\lambda_0 t} \| P^V_t \|.$$ 

By (3.1),

$$\int_X \log \left( \frac{e^{-\lambda_0 t} P^V_t u}{u} \right) \, d\mu \geq -t I^V(\mu), \quad u \in C^+(X).$$
Thus for $f \in C(X)$,
\[
\int_X f \, d\mu - \int_X \log (e^{-\lambda_0 t P_t^V}) \, d\mu \leq t I^V(\mu), \quad f \in C(X).
\]
By Jensen’s inequality,
\[
\int_X f \, d\mu - \int_X \log (e^{-\lambda_0 t P_t^V}) \, d\mu \leq t I^V(\mu), \quad f \in C(X).
\]
Defining
\[
Z_t \equiv e^{-\lambda_0 t (P_t^V 1)}
\]
and
\[
\pi_t(f) \equiv \frac{e^{-\lambda_0 t (P_t^V f)}}{Z_t}
\]
yields
\[
\int_X f \, d\mu - \log \int_X e^f \, d\pi_t \leq t I^V(\mu) + \log Z_t, \quad f \in C(X).
\]
Taking the supremum over all $f$ yields
\[
H(\mu, \pi_t) \leq t I^V(\mu) + \log Z_t.
\]
Note $Z_t \leq M, t \geq 0$, hence
\[
H(\mu, \pi_t) \leq t I^V(\mu) + \log M, \quad t \geq 0.
\]
Now set
\[
\bar{\pi}_T(f) \equiv \frac{\int_0^T Z_t \pi_t(f) \, dt}{\int_0^T Z_t \, dt} = \frac{\int_0^T e^{-\lambda_0 t (P_t^V f)} \, dt}{\int_0^T Z_t \, dt}, \quad T > 0.
\]
Then $\pi_t$ is in $M(X)$ for $t > 0$ and $\bar{\pi}_T$ is in $M(X)$ for $T > 0$.

By convexity of $H$,
\[
H(\mu, \bar{\pi}_T) \leq \log M, \quad T > 0.
\]
By compactness of $M(X)$, select a sequence $T_n \to \infty$ with $\pi_n = \bar{\pi}_{T_n}$ converging to some $\pi$. By lower-semicontinuity of $H$, we have $H(\mu, \pi) \leq \log M$. Thus $\mu << \pi$ with $\psi = d\mu/d\pi$ satisfying $\psi \log \psi \varepsilon L^1(\pi)$. By Lemma 3.1,
\[
\log Z_t = \log \mu(e^{-\lambda_0 t P_t^V 1}) \geq \mu(\log(e^{-\lambda_0 t P_t^V 1})) \geq 0,
\]
hence $Z_t \geq 1, t \geq 0$. Since
\[
e^{-\lambda_0 t \mu(P_t^V (L + V - \lambda_0) f)} = \frac{d}{dt} e^{-\lambda_0 t \mu(P_t^V f)}, \quad f \in D,
\]
we have
\[
|\bar{\pi}_T((L + V - \lambda_0) f)| = \left|\frac{e^{-\lambda_0 T \mu(P_T^V f)} - \mu(f)}{\int_0^T Z_t \, dt}\right| \leq \frac{2M\|f\|}{T} \to 0, \quad T \to \infty.
\]
Thus
\[
\pi((L + V - \lambda_0) f) = 0, \quad f \in D,
\]
which implies
\[
e^{-\lambda_0 t \pi(P_t^V f)} = \pi(f), \quad t \geq 0,
\]
for $f \in D$, hence, by density, for $f$ in $C(X)$. Thus $\pi$ is a ground measure.

Clearly $\log \psi \in L^1(\mu)$ and $\psi > 0$ a.s. $\mu$. Since $\mu$ is an equilibrium measure and $\pi$ is a ground measure, Theorem 3.1 implies $\psi$ is a ground state relative to $\mu$. $\square$
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Department of Mathematics, Temple University, Philadelphia, PA 19122
E-mail address: hijab@temple.edu