Abstract

In this paper we study the geometry of manifolds with vector cross product and its complexification. First we develop the theory of instantons and branes and study their deformations. For example they are (i) holomorphic curves and Lagrangian submanifolds in symplectic manifolds and (ii) associative submanifolds and coassociative submanifolds in $G_2$-manifolds.

Second we classify Kähler manifolds with the complex analog of vector cross product, namely they are Calabi-Yau manifolds and hyperkähler manifolds. Furthermore we study instantons, Neumann branes and Dirichlet branes on these manifolds. For example they are special Lagrangian submanifolds with phase angle zero, complex hypersurfaces and special Lagrangian submanifolds with phase angle $\pi/2$ in Calabi-Yau manifolds.

Third we prove that, given any Calabi-Yau manifold, its isotropic knot space admits a natural holomorphic symplectic structure. We also relate the Calabi-Yau geometry of the manifold to the holomorphic symplectic geometry of its isotropic knot space.
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1 Introduction

The vector product, or the cross product, in $\mathbb{R}^3$ was generalized by Gray ([1],[8]) to the product of any number of tangent vectors, called the vector cross product (abbrev. VCP). The list of Riemannian manifolds with VCP structures on their tangent bundles include symplectic (or Kähler) manifolds, $G_2$-manifolds and $Spin(7)$-manifolds. We introduce the complex analog of VCP in definition 23, called the complex vector cross product (abbrev. C-VCP). We show that there are only two classes of manifolds with C-VCP.

**Theorem 1** If $M$ is a closed Kähler manifold with a C-VCP, then $M$ must be either (i) a Calabi-Yau manifold, or (ii) a hyperkähler manifold.

We study the geometry of instantons which are submanifolds in $M$ preserved by the VCP. Instantons are always absolute minimal submanifolds in $M$. When an instanton is not a closed submanifold in $M$, we require its boundary to lie inside a brane in order to have a Fredholm theory for the free boundary value problem. For example, when $M$ is a symplectic manifold, then instantons and branes are holomorphic curves and Lagrangian submanifolds in $M$ respectively. These geometric objects play important roles in understanding the symplectic geometry of $M$.

| Manifolds w/ VCP | Symplectic Manifolds | $G_2$-Manifolds | $Spin(7)$-Manifolds | Oriented Manifolds |
|-------------------|----------------------|-----------------|---------------------|--------------------|
| Instantons        | Holomorphic curves   | Associative submanifolds | Cayley submanifolds | Open submanifolds  |
| Branes            | Lagrangian submanifolds | Coassociative submanifolds | N/A                | Hypersurfaces      |

Instantons $A$ in $M$ can be characterized by the condition

$$\tau|_A = 0$$

with $\tau \in \Omega^{r+1}(M, g_M)$. This is useful in giving a uniform description of the deformation theory of instantons (see section 2.3 for details.). For branes $C$ in $M$, its normal bundle is naturally identified with the space $\Lambda^r VCP T^*_C$ of VCP forms of degree $r$ on $C$. Infinitesimal deformations of branes are parametrized by such differential forms on $C$ which are closed. Furthermore they are always unobstructed, namely the moduli spaces of branes are always smooth. (See section 2.5 for details).

For the complex analog, there are two type of branes corresponding to the Dirichlet type and Neumann type boundary value problems for instantons. We
call them $D$-branes and $N$-branes respectively.

| Manifolds w/ $\mathbb{C}$-VCP | Calabi-Yau manifolds | hyperkähler manifolds |
|-------------------------------|----------------------|-----------------------|
| Instantons                   | Special Lagrangian $\theta=0$ | $I$-holomorphic curves |
| N-Branes                     | Complex Hypersurfaces | $J$-complex Lagrangians |
| D-Branes                     | Special Lagrangian $\theta=\pi/2$ | $K$-complex Lagrangians |

Using the Riemannian metric on $M$, any closed VCP determines a differential form $\phi$ as follows:

$$\phi(v_1, \ldots, v_r, v_{r+1}) = \langle v_1 \times \ldots \times v_r, v_{r+1} \rangle.$$

By transgressing $\phi$, we obtain a two form on the (multi-dimensional) knot space of $M$:

$$\mathcal{K}_\Sigma M = Map(\Sigma, M)_{emb}/Diff(\Sigma),$$

where $\Sigma$ is any smooth manifold of dimension $r - 1$. This gives a symplectic structure on $\mathcal{K}_\Sigma M$ which is compatible with the $L^2$-metric, namely an almost Kähler structure. For instance, when $M$ is a three manifold, $\mathcal{K}_\Sigma M$ is the space of knots in $M$. In this case, Brylinski showed that $\mathcal{K}_\Sigma M$ is indeed a Kähler manifold. He also studied relationship between the geometry of $\mathcal{K}_\Sigma M$ and the geometric quantization of Chern-Simons theory.

In general, VCP geometry on $M$ can be interpreted as the symplectic geometry on $\mathcal{K}_\Sigma M$. For example, for a disk $D$ in $Map(\Sigma, M)_{emb}$ which is horizontal in the principal bundle

$$Diff(\Sigma) \to Map(\Sigma, M)_{emb} \to \mathcal{K}_\Sigma M,$$

determines a map from $D \times \Sigma$ to $M$ and a disk $\hat{D} = \pi(D)$ in $\mathcal{K}_\Sigma M$. If $\hat{D}$ is a holomorphic disk in $\mathcal{K}_\Sigma M$, $D \times \Sigma$ gives an instanton in $M$. It has an extra property that the induced metric on $D \times \Sigma$ gives a Riemannian submersion from $D \times \Sigma$ to $D$. We call such an instanton a **tight instanton**. In section 4.1 we prove the following theorem.

**Theorem 2** Suppose $M$ is a Riemannian manifold with a closed differential form $\phi$. Then we have

1. $\phi$ is a VCP form on $M$ if and only if its transgression defines an almost Kähler structure on $\mathcal{K}_\Sigma M$;
2. For a normal disk $D$ in $Map(\Sigma, M)_{emb}$, $\hat{D}$ is a holomorphic disk in $\mathcal{K}_\Sigma M$ if and only if $D \times \Sigma$ gives a tight instanton in $M$ as above;
3. $\mathcal{K}_\Sigma C$ is a Lagrangian submanifold in $\mathcal{K}_\Sigma M$ if and only if $C$ is a brane in $M$.
When we consider the complex analog of the above theorem on a Calabi-Yau $n$-fold $M$, we choose any smooth manifold $\Sigma$ of dimension $n-2$, we might hope that $\text{Map}(\Sigma, M)_{\text{emb}} / \text{Diff}(\Sigma) \otimes \mathbb{C}$, if exists, is hyperkähler. Since the complexification of $\text{Diff}(\Sigma)$ does not exist, we should interpret the above quotient as a symplectic quotient $\text{Map}(\Sigma, M)_{\text{emb}} / / \text{Diff}(\Sigma)$, if exists. The problem arises because one needs to fix a background volume form on $\Sigma$ to define a symplectic structure on $\text{Map}(\Sigma, M)_{\text{emb}}$. We will explain in the last section on how to resolve this issue and prove the following theorem.

**Theorem 3** Suppose $M$ is a Calabi-Yau $n$-fold and $\Sigma$ is a closed manifold of dimension $n-2$. Then the isotropic knot space $\hat{K}_\Sigma M$ has a natural holomorphic symplectic structure.

Furthermore the Calabi-Yau geometry on $M$ can be interpreted as the holomorphic symplectic geometry on $\hat{K}_\Sigma M$. Namely we prove the following theorem in section 5.2.

**Theorem 4** Suppose $M$ is a Calabi-Yau $n$-fold. We have

1. $\hat{K}_\Sigma C$ is a $J$-complex Lagrangian submanifold in $\hat{K}_\Sigma M$ if and only if $C$ is a complex hypersurface in $M$;
2. $\hat{K}_\Sigma C$ is a $K$-complex Lagrangian submanifold in $\hat{K}_\Sigma M$ if and only if $C$ is a special Lagrangian submanifold in $M$ with phase $-\pi/2$.

Even though complex hypersurfaces and special Lagrangian submanifolds look very different inside a Calabi-Yau manifold, their isotropic knot spaces are both complex Lagrangian submanifolds in $\hat{K}_\Sigma M$. One reason is that any knot inside a special Lagrangian submanifold is automatically isotropic. To prove this theorem, we need to construct carefully certain appropriate deformations of isotropic knots inside $C$ so that $\hat{K}_\Sigma C$ being a $J$-complex Lagrangian (resp. $I$-complex Lagrangian) implies that the dimension of $C$ is at least $2n-2$ (resp. at most $n$).

## 2 Instantons and Branes

In this section we introduce and study instantons and branes on manifolds with (complex) vector cross products. Traditionally instantons refer to gradient flow lines of a Morse function $f$ on a Riemannian manifold $(M, g)$, as studied by Witten in [22]. Morse theory can be generalized to any closed one form $\phi$, because $\phi = df$ locally. Suppose that $\phi$ is nonvanishing, we can choose a Riemannian metric on $M$ such that it has unit length at every point. Then the gradient flow lines for the vector field $X$, defined by $\phi = \iota_X g$, can be reinterpreted as one dimensional submanifolds in $M$ calibrated by $\phi$.

Suppose that $M$ is a symplectic manifold with symplectic form $\omega$. By transgression on $\omega$, we obtain a closed one form on the free loop space of $M$. The instantons in the free loop space correspond to holomorphic curves in $M$ and they are calibrated by $\omega$. We continue to call these instantons and they play
important roles in the closed String theory (see e.g. [23]). In open String theory, we consider holomorphic curves in $M$ with boundaries lying on a Lagrangian submanifold in $M$, which we call it a brane.

In general instantons are submanifolds (of the smallest dimension) which are preserved by the VCP, and branes are the natural boundaries for the free boundary value problem for instantons.

2.1 Vector Cross Product

In this subsection we review the VCP as introduced by Gray. A basic example of VCP is the vector product in $\mathbb{R}^3$. The vector product of any two linearly independent vectors in a plane is a vector which is orthogonal to the plane and has the length equal to the area of the parallelogram spanned by those vectors. Actually, these two properties characterize VCP as in the following definition by Brown and Gray [1].

**Definition 5** On an $n$-dimensional Riemannian manifold $M$ with a metric $g$, an $r$-fold Vector Cross Product (VCP) is a smooth bundle map,

$$\chi : \wedge^r TM \to TM$$

satisfying

$$\left\{ \begin{array}{l} g (\chi (v_1, ..., v_r), v_i) = 0 , \ (1 \leq i \leq r) \\ g (\chi (v_1, ..., v_r), \chi (v_1, ..., v_r)) = \|v_1 \wedge ... \wedge v_r\|^2 \end{array} \right.$$ 

where $\|\cdot\|$ is the induced metric on $\wedge^r TM$.

We will also denote

$$v_1 \times ... \times v_r = \chi (v_1, ..., v_r).$$

The first condition in the above definition is equivalent to the following tensor $\phi$ being a skew symmetric tensor of degree $r + 1$, i.e. $\phi$ is a differential form,

$$\phi (v_1, ..., v_{r+1}) = g (v_1 \times ... \times v_r, v_{r+1})$$

Clearly an $r$-fold VCP $\chi$ on $M$ can be characterized by an appropriate differential form $\phi \in \Omega^{r+1} (M)$, which we call a VCP form.

**Definition 6** An VCP form of degree $r+1$ is a differential form $\phi \in \Omega^{r+1} (M)$ satisfying

$$|\iota_{e_1 \wedge e_2 \wedge ... \wedge e_r}(\phi)| = 1$$

for any orthonormal $e_1, ..., e_r \in TM_x$, all $x$ in $M$. 

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It is not difficult to see that a Hermitian almost complex structure is equivalent to a 1-fold \( VCP \) and the corresponding Kähler form is the corresponding \( VCP \) form. In fact the complete list of \( VCPs \) is surprisingly short. The classification of the linear \( VCPs \) on a vector space \( V \) with positive definite inner product \( g \), by Brown and Gray [11], can be summarized in the following.

(i) \( r = 1 \) : Let \( \chi : V \to V \) be a 1-fold \( VCP \), then \( |\chi(v)| = |v| \) implies that \( \chi \) is an orthogonal transformation. Polarizing \( \langle \chi(v), v \rangle = 0 \), we obtain \( \langle \chi(u), v \rangle + \langle u, \chi(v) \rangle = 0 \), that is \( \chi^* = -\chi \). Together we have \( \chi^2 = -id \). Namely a 1-fold \( VCP \) is equivalent to a Hermitian complex structure on \( V \). The symmetry group of \( \chi \) is isomorphic to \( U(m) = O(2m) \cap GL(m, \mathbb{C}) \). On the other hand, the isomorphism \( U(m) = O(2m) \cap Sp(2m, \mathbb{R}) \) reflects the fact that a 1-fold \( VCP \) is determined by its corresponding \( VCP \) form, or its Kähler form \( \phi \). The standard example is \( V = \mathbb{C}^m \) with

\[
\phi = dx^1 \wedge dy^1 + \cdots + dx^m \wedge dy^m.
\]

(ii) \( r = n - 1 \) : An \((n - 1)\)-fold \( VCP \) on an \( n \)-dimensional inner product space \( V \) is the Hodge star operator \( * \) given by \( g \) on \( \Lambda^{n-1}V \) and the VCP form of degree \( n \) is the induced volume form \( Vol_V \) on \( V \). The automorphism group \( Aut(V, Vol_V) \) is isomorphic to the group of linear transformations preserving \( g \) and \( Vol_V \), i.e. \( Aut(V, Vol_V) = O(n) \cap SL(n, \mathbb{R}) = SO(n) \). The standard example is \( V = \mathbb{R}^n \) and

\[
\phi = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.
\]

(iii) \( r = 2 \) : A 2-fold \( VCP \) on a 7-dimensional vector space \( \text{Im } \mathbb{O} \) is a cross product defined as \( a \times b = \text{Im } (ab) \) for any \( a, b \) in \( \text{Im } \mathbb{O} \). For coordinates \((x_1, \ldots, x_7)\) on \( \text{Im } \mathbb{O} \), the corresponding \( VCP \) form \( \Omega \) of degree 3 can be written as

\[
\Omega = dx^{123} - dx^{167} + dx^{145} + dx^{257} + dx^{246} - dx^{356} + dx^{347}
\]

where \( dx^{ijk} = dx^i \wedge dx^j \wedge dx^k \). Bryant [3] showed that the group of real-linear transformations of \( \text{Im } \mathbb{O} \) preserving the \( VCP \) form \( \Omega \) actually preserves \( g \) and \( VCP \) and more it is exactly \( G_2 \), the automorphism group of the octonion \( \mathbb{O} \), i.e. \( Aut(\text{Im } \mathbb{O}, \Omega) = G_2 \subset SO(\text{Im } \mathbb{O}) = SO(7) \).

(iv) \( r = 3 \) : A 3-fold \( VCP \) on an 8-dimensional vector space \( \mathbb{O} \) is a cross product defined as \( a \times b \times c = \frac{1}{3} (a(b(c) - c(ba)) \) for any \( a, b, \) and \( c \) in \( \mathbb{O} \). For coordinates \((x_1, \ldots, x_8)\) on \( \mathbb{O} \), the corresponding \( VCP \) form \( \Theta \) of degree 4 can be written as

\[
\Theta = -dx^{1234} - dx^{5678} - (dx^{21} + dx^{34}) (dx^{65} + dx^{78}) \\
- (dx^{31} + dx^{42}) (dx^{75} + dx^{86}) - (dx^{41} + dx^{23}) (dx^{85} + dx^{67})
\]

Bryant [3] also showed that the group of real-linear transformations of \( \mathbb{O} \) preserving the \( VCP \) form \( \Theta \) on \( \mathbb{O} \) preserves \( g \) and \( VCP \), and it is \( Spin(7) \). i.e. \( Aut(\mathbb{O}, \Theta) = Spin(7) \subset SO(8) \).
From the above classification of linear VCPs, the existence of a VCP on a Riemannian manifold $M$ is equivalent to the reduction of the structure group of the frame bundle from $O(n)$ to $U(m)$, $SO(n)$, $G_2$ and $Spin(7)$, for $r = 1$, $n-1$, 2 and 3 respectively. Using the obstruction theory in topology, the necessary and sufficient condition for the existence of such a reduction of structure group in the $r = n-1$, 2 and 3 cases are $w_1 = 0$, $w_1 = w_2 = 0$ and $w_1 = w_2 = p_1^2 - 4p_3 + 8 \chi = 0$ respectively. Here $w_i$, $p_i$ and $\chi$ are the Stiefel-Whitney class, the Pontrjagin class, and the Euler class of $M$, respectively (see e.g. [14]).

For a VCP on a Riemannian manifold $M$, that is a linear VCP on each tangent space of $M$, we would require certain integrability condition for its coherence. For example, a 1-fold VCP is a Hermitian almost complex structure on $M$. This defines a symplectic structure or Kähler structure on $M$ if the corresponding VCP form on $M$ is closed or parallel respectively. In general we have the following definition.

**Definition 7** Suppose that $M$ is a Riemannian manifold with a VCP $\chi$ and $\phi$ is its corresponding VCP form. We call $\chi$ a closed (resp. parallel) VCP if $d\phi = 0$ (resp. $\nabla \phi = 0$, where $\nabla$ is the Levi-Civita connection on $M$).

The classification of manifolds with closed/parallel VCP is presented in the following table.

| $r$  | Closed VCP                  | Parallel VCP               |
|------|-----------------------------|----------------------------|
| 1    | Almost Kähler manifolds     | Kähler manifolds           |
| $n-1$| Oriented manifolds          | Oriented manifolds         |
| 2    | Almost $G_2$-manifolds      | $G_2$-manifolds            |
| 3    | $Spin(7)$-manifolds         | $Spin(7)$-manifolds        |

Remark: A VCP form $\phi$ of degree $r+1$ on $M$ induces a VCP form of degree $r$ on any oriented hypersurface $H$ in $M$, namely the restriction of $\iota_\nu \phi$ to $H$ where $\nu$ is the unit normal vector field on $H$. However it usually does not preserve the closedness of the VCP form. For example, Calabi [9] and Gray [8] showed that
such a two form on any hypersurface in $\mathbb{R}^7 = \text{Im} \mathcal{O}$ is never closed unless it is an affine hyperplane. This result is generalized by Bryant [2] to any codimension two submanifold in $\mathcal{O}$, where he showed that closedness of such a two form is equivalent to the submanifold being a complex hypersurfaces for some suitable complex structure on $\mathcal{O}$.

Remark: We consider any nontrivial smooth map $f : M_1 \to M_2$ between two Riemannian manifolds with $r$-fold VCPs. If $f$ preserves their VCPs, i.e.

$$f_*(v_1 \times ... \times v_r) = (f_*v_1) \times ... \times (f_*v_r),$$

for any tangent vectors $v_i$'s to $M_1$, then Gray [9] showed that $f$ is actually an isometric immersion unless $r = 1$ and $f$ would be a holomorphic map. In particular any diffeomorphisms of $M$ preserving an $r$-fold VCP with $r \geq 2$ is automatically an isometry of $M$, i.e. $\text{Diff}(M, \chi) \subset \text{Isom}(M, g)$.

In the next section we study submanifolds in $M$ which are preserved by the VCP.

### 2.2 Instantons for Vector Cross Products

In a symplectic manifold with a compatible almost complex structure $J$, namely an almost Kähler manifold, an instanton is a two dimensional submanifold which is preserved by $J$, and a brane is a Lagrangian submanifold, i.e. a middle dimensional submanifold such that the restriction of the symplectic form vanishes. They play essential roles in symplectic geometry. These notions have natural analogs for manifolds with VCP.

**Definition 8** Let $M$ be a Riemannian manifold with a closed $r$-fold VCP $\chi$. An $r+1$ dimensional submanifold $A$ is called an **instanton** if it is preserved by $\chi$.

From the definition, two instantons in $M$ can never intersect along a codimension one subspace.

Suppose that $A$ is an instanton in $M$. At any given point $x$ in $A$ and any tangent vectors $u_1, ..., u_{r-1}$ in $T_xA$ and any normal vector $\nu$ in $N_{A/M,x}$, we have

$$u_1 \times ... \times u_{r-1} \times \nu \in N_{A/M,x}.$$  

This is because for any $u$ in $T_xA$,

$$g(u_1 \times ... \times u_{r-1} \times \nu, u) = \phi(u_1, ..., u_{r-1}, \nu, u)$$

$$= -\phi(u_1, ..., u_{r-1}, u, \nu)$$

$$= -g(u_1 \times ... \times u_{r-1} \times u, \nu)$$

$$= 0.$$  

The last equality follows from the fact that $A$ is preserved by $\chi$ and therefore $u_1 \times ... \times u_{r-1} \times u$ lies in $T_xA$. This implies that there is bundle map,

$$\Lambda^{r-1}T_A \otimes N_{A/M} \to N_{A/M}.$$ 

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For example, when \( r = 1 \), \( A \) is a holomorphic curve in an almost Kähler manifold \( M \). Then the above bundle map is simply the complex structure of the normal bundle of \( A \) in \( M \). In general the normal bundle to an instanton is always a twisted spinor bundle over \( A \) and the above bundle map is given by the Clifford multiplication. This description should be useful in understanding the deformation theory and moduli space of instantons.

An instanton in \( M \) is always a minimal submanifold with absolute minimal volume. This will follow from an equivalent characterization of instantons, namely they are those submanifolds in \( M \) calibrated by \( \phi \). The theory of calibration is developed by Harvey and Lawson in [10] to produce absolute minimal submanifolds. Recall that a closed differential form \( \psi \) of degree \( k \) on a Riemannian manifold \( M \) is called a \textit{calibrating form} if it satisfies

\[
\psi(x)|_V \leq Vol_V,
\]

for every oriented \( k \)-plane \( V \) in \( T_x M \), at each point \( x \) in \( M \). Here \( Vol_V \) is the volume form on \( V \) for the induced metric. A \textit{calibrated submanifold} \( A \) is a submanifold where \( \psi|_A \) is equal to the induced volume form on \( A \). An important observation is the following: any other submanifold \( B \), homologous to \( A \), satisfies

\[
Vol(B) \geq Vol(A) = \int_A [\psi],
\]

and the equality sign holds if and only if \( B \) is also a calibrated submanifold in \( M \). As we will discover that all the examples of calibrated submanifolds studied in [10] are either instantons or branes in manifolds with (complex) VCP.

The next lemma shows that instantons are calibrated.

**Lemma 9** Let \( M \) be a Riemannian manifold with a closed \( r \)-fold VCP \( \chi \) and we denote the corresponding VCP form as \( \phi \). Then we have (i) \( \phi \) is a calibrating form, (ii) an \((r+1)\)-dimensional submanifold \( A \) in \( M \) is calibrated by \( \phi \) if and only if it is an instanton.

**Proof.** The closed form \( \phi \) is a calibrating form because for any \( x \) in \( M \) and for any oriented orthonormal tangent vectors \( e_1, e_2, \ldots, e_{r+1} \) at \( x \), it satisfies,

\[
\phi(e_1, e_2, \ldots, e_{r+1}) = \langle \chi(e_1, e_2, \ldots, e_r), e_{r+1} \rangle \\
\leq |\chi(e_1, e_2, \ldots, e_r)| \cdot |e_{r+1}| \\
= \|e_1 \wedge e_2 \wedge \ldots \wedge e_r\| \cdot |e_{r+1}| \\
= 1.
\]

The equality signs hold if and only if \( \chi(e_1, e_2, \ldots, e_r) = e_{r+1} \). Namely the linear span of \( e_i \)'s is preserved by \( \chi \). ■

As a corollary of this lemma and basic properties of calibration, An \((r+1)\)-dimensional submanifold \( A \) in \( M \) is an instanton if and only if its total volume
with respect to the induced metric satisfies the following equality,

$$\text{Vol} (A) = \int_A \lfloor \phi \rfloor.$$  

Remark: As a matter of fact, each term in the expansion of $\exp (\phi)$ is a calibrating form and the corresponding calibrated submanifolds are preserved by the VCP $\chi$.

**Examples of instantons:** (i) Instantons is an oriented $n$-dimensional manifold (i.e. VCP form of degree $n$) are open subsets in $M$. (ii) Instantons in a Kähler manifold (i.e. parallel VCP form of degree 2) are holomorphic curves. (iii) Instantons in a $G_2$-manifold (i.e. parallel VCP form of degree 3) are called associative submanifolds. (iv) Instantons in a $\text{Spin}(7)$-manifold (i.e. parallel VCP form of degree 4, called the Cayley form) are called Cayley submanifolds (see [10]).

Suppose that $(M, g, \chi)$ has a torus symmetry group, say $M = X \times T^k$. Then an $r$-fold VCP on $M$ induces an $(r-k)$-fold VCP on $X$. Moreover a submanifold $B$ in $X$ is an instanton if and only if $B \times T^k$ is an instanton in $M$. For instanton if $\Sigma$ is a holomorphic curve is a Calabi-Yau threefold $X$, then $\Sigma \times S^1$ is an associative submanifold in the $G_2$-manifold $X \times S^1$ and vice versa.

**Proposition 10** Suppose that $\sigma$ is an isometric involution of $M$ preserving its $r$-fold VCP $\chi$. If the fix point set $M^\sigma$ has dimension $r+1$ then it is an instanton in $M$.

Proof: Given any tangent vectors $v_1, \ldots, v_r$ to $M^\sigma$ at any point $x$, we write

$$v_1 \times \cdots \times v_r = t + n,$$

where $t$ (resp. $n$) is tangent (resp. normal) to $M^\sigma$. Since $M^\sigma$ is the fix point set of an involution, we have $\sigma^* (t + n) = t - n$. On the other hand,

$$\sigma^* (v_1 \times \cdots \times v_r) = \sigma^* (v_1) \times \cdots \times \sigma^* (v_r)$$

$$= v_1 \times \cdots \times v_r$$

because $\sigma$ preserves $\chi$. This implies $n = 0$, thus $M^\sigma$ is preserved by $\chi$, namely an instanton in $M$. ■

Remark: Given any $r$-dimensional analytic submanifold $S$ in $M$, an $r$-fold VCP $\chi$ on $M$ determines a unique normal direction on $S$. Using Cartan-Kähler theory, we can always integrate out this direction and obtain an instanton in $M$ containing $S$ (see e.g. [10]).
2.3 Deformations of instantons

In order to describe instantons and their deformations effectively, we need to further develop the linear algebra of an inner product space $V$ with an $r$-fold VCP

$$\chi : \wedge^r V \to V,$$

or their associated VCP form $\phi \in \Lambda^{r+1}V^*$. We define

$$\tau : \Lambda^{r+1} V \to \Lambda^2 V$$

as the composition of the following homomorphisms:

$$\Lambda^{r+1} V \to V \otimes \Lambda^r V \to V \otimes V \to \Lambda^2 V,$$

where these maps are (i) the natural inclusion, (ii) $id \otimes \chi$, (iii) the natural projection. Explicitly we have

$$\tau(v_1, \ldots, v_{r+1}) = \frac{1}{\sqrt{r+1}} \sum_{k=1}^{r+1} (-1)^{k-1} v_k \wedge \chi (v_1, \ldots, \hat{v}_k, \ldots, v_{r+1}).$$

As a matter of fact, the image of $\tau$ lies inside a much small subspace in $\Lambda^2 V$.

We define $g \subset \mathfrak{so}(V) \cong \Lambda^2 V^*$ to be the space of infinitesimal isometries of $V$ preserving the VCP $\chi$. Namely $\zeta \in \mathfrak{so}(V) \subset \text{End}(V)$ lies inside $g$ if

$$\zeta(v_1 \times v_2 \times \ldots \times v_r) = \zeta(v_1) \times v_2 \times \ldots \times v_r + v_1 \times v_2 \times \ldots \times \zeta(v_r),$$

for any $v_i$'s in $V$. This is equivalent to

$$\sum_{i=1}^{r+1} \phi(v_1, v_2, \ldots, \zeta(v_i), \ldots, v_{r+1}) = 0,$$

for any $v_i$'s in $V$. The next lemma says that the image of $\tau$ is orthogonal to $g \subset \Lambda^2 V^*$ with respect to the natural pairing between $\Lambda^2 V$ and $\Lambda^2 V^*$.

**Lemma 11** Given any $r$-fold VCP $\chi$ on $V$, we have

$$\tau : \Lambda^{r+1} V \to g^\perp \subset \Lambda^2 V.$$

**Proof:** Given any $\zeta \in g \subset \mathfrak{so}(V) \subset \text{End}(V)$, we denote the corresponding two form in $\Lambda^2 V^*$ as $\tilde{\zeta}$. For any $v_i$'s in $V$, we compute

$$\langle \tau(v_1, \ldots, v_{r+1}), \tilde{\zeta} \rangle$$

$$= \tilde{\zeta} \left( \frac{1}{\sqrt{r+1}} \sum_{i=1}^{r+1} (-1)^{i-1} v_i \wedge (v_1 \times \ldots \times \hat{v}_i \times \ldots \times v_{r+1}) \right)$$

$$= \frac{1}{\sqrt{r+1}} \left( \sum_{i=1}^{r+1} (-1)^{i-1} (v_1 \times \ldots \times \hat{v}_i \times \ldots \times v_{r+1}), \zeta(v_i) \right)$$

$$= \frac{(-1)^r}{\sqrt{r+1}} \sum_{i=1}^{r+1} \phi(v_1, \ldots, \zeta(v_i), \ldots, v_{r+1})$$

$$= 0.$$
Hence the result. ■

**Proposition 12**

\[ |\phi(v_1, ..., v_{r+1})|^2 + |\tau(v_1, ..., v_{r+1})|^2 = |v_1 \wedge ... \wedge v_{r+1}|_{\Lambda^{r+1}}^2 \]

Proof: It suffices to assume that \( v_i \)'s are orthogonal to each others. Note that when \( l \neq m \), we have \( v_l \) perpendicular to both \( v_m \) and \( \chi(v_1, ..., \widehat{v_m}, ..., v_{r+1}) \) and therefore,

\[ \langle v_l \wedge \chi(v_1, ..., \widehat{v_l}, ..., v_{r+1}), v_m \wedge \chi(v_1, ..., \widehat{v_m}, ..., v_{r+1}) \rangle = 0. \]

We compute

\[ |\tau(v_1, ..., v_{r+1})|^2 \]

\[ = \frac{1}{r+1} \left| \sum_{k=1}^{r+1} (-1)^{k-1} v_k \wedge \chi(v_1, ..., \widehat{v_k}, ..., v_{r+1}) \right|^2 \]

\[ = \frac{1}{r+1} \sum_{k=1}^{r+1} |v_k \wedge \chi(v_1, ..., \widehat{v_k}, ..., v_{r+1})|^2 \]

\[ = \frac{1}{r+1} \sum_{k=1}^{r+1} \left( |v_1|^2 \cdots |v_{r+1}|^2 - \langle v_k, \chi(v_1, ..., \widehat{v_k}, ..., v_{r+1}) \rangle^2 \right) \]

\[ = \frac{1}{r+1} \sum_{k=1}^{r+1} \left( |v_1|^2 \cdots |v_{r+1}|^2 - |\phi(v_1, ..., v_{r+1})|^2 \right) \]

\[ = |v_1 \wedge ... \wedge v_{r+1}|^2 - |\phi(v_1, ..., v_{r+1})|^2. \]

Hence the result. ■

**Remark:** This proposition and the following corollary were first obtained by Harvey and Lawson in the \( G_2 \)- and \( \text{Spin}(7) \)-manifolds cases using special structures of the octonions [10]. As an immediately corollary of the above proposition, we have

**Corollary 13** Suppose that \( \chi \) is an \( r \)-fold VCP on \( V \). An \((r+1)\)-dimensional linear subspace \( P \subset V \) is preserved by \( \chi \), i.e an instanton, if and only if

\[ \tau(v_1, ..., v_{r+1}) = 0 \]

for any base \( v_i \)'s of \( P \).

We also denote the above condition as \( \tau(P) = 0 \). Assume this is the case, we denote the orthogonal decomposition of \( V \) as

\[ V = P \oplus N. \]

Similarly we have the following decomposition,

\[ \wedge^2 V \cong \wedge^2 P \oplus \wedge^2 N \oplus (P \otimes N). \]
Proposition 14 Suppose that $\chi$ is an $r$-fold VCP on $V$ with an orthogonal decomposition

$$V = P \oplus N$$

where $P$ is an $(r + 1)$-dimensional linear subspace of $V$ preserved by $\chi$. We have

$$\tau (p_1, ..., p_r, n) \in P \otimes N$$

for any $p_i$'s in $P$ and $n \in N$.

Proof: Note that $P$ being an instanton in $V$ implies that

$$\langle \chi (p_1, ..., p_{r-1}, n), pr \rangle = \pm \langle \chi (p_1, ..., p_{r-1}, pr), n \rangle = 0.$$ 

That is $\chi (p_1, ..., p_{r-1}, n) \in N$, or equivalently, $pr \wedge \chi (p_1, ..., p_{r-1}, n) \in P \otimes N$. However $\tau (p_1, ..., pr, n)$ is a linear combination of terms of this form and therefore $\tau (p_1, ..., pr, n) \in P \otimes N$. Hence the proposition. ■

We are going to use these linear algebra results to study the deformations of instantons. Given any VCP $\chi$ on a Riemannian manifold $M$, the spaces of infinitesimal automorphisms of $(T_M, \chi)$ on various fibers glue together to form a subbundle

$$g_M \subset \Lambda^2 T_M^*.$$ 

Similarly the vector cross product $\chi$ determines a tensor

$$\tau \in \Omega^{r+1} (M, g_M^\perp).$$

On the global level, the above corollary is equivalent to the following result.

Theorem 15 If $M$ is a Riemannian manifold with an $r$-fold VCP $\chi$, then a $(r + 1)$-dimensional submanifold $A$ is an instanton if and only if

$$\tau |_A = 0 \in \Omega^{r+1} (A, g_M^\perp).$$

This theorem is useful in studying the deformation of instantons. We first recall that any nearby submanifold to $A$ is the image of the exponential map

$$\exp_v : A \to M,$$

for some small normal vector field $v \in \Gamma (A, N_{A/M})$. Therefore, given any instanton $A$ in $M$, we can describe its nearby instantons as the zeros of the following map,

$$F : \Gamma (A, N_{A/M}) \to \Gamma (A, g_M^\perp)$$

$$F (v) = *_A \exp^*_v (\tau),$$

where $*_A$ is the Hodge star operator of $A$. Suppose that $A_t$ is a family of submanifolds in $M$ with $A = A_0$ an instanton and we denote its variation normal vector field as

$$v = \frac{dA_t}{dt} \bigg|_{t=0} \in \Gamma (A, N_{A/M}).$$

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By the proposition 14 and $A$ being an instanton, we have

$$\frac{d\tau|_{A_t}}{dt} \bigg|_{t=0} \in \Gamma \left( A, T^*_A \otimes N_{A/M} \cap g^+_M \right).$$

In particular the derivative of $F$ is given by

$$F'(0) : \Gamma \left( A, N_{A/M} \right) \rightarrow \Gamma \left( A, T^*_A \otimes N_{A/M} \cap g^+_M \right)$$

$$F'(0) \left( \frac{dA_t}{dt} \bigg|_{t=0} \right) = \frac{d\tau|_{A_t}}{dt} \bigg|_{t=0}.$$ 

By studying individual cases, we find out that $N_{A/M}$ is always a twisted spinor bundle over $A$ and the linearization of $F$ coincide with a twisted Dirac operator, i.e. $F'(0) = D$.

### 2.4 Branes for Vector Cross Products

In symplectic geometry, the natural free boundary condition for holomorphic curves require their boundaries lying inside a Lagrangian submanifold. The analog of a Lagrangian submanifold for VCP forms of higher degree is called a brane.

**Definition 16** Suppose $M$ is an $n$-dimensional manifold with a closed VCP form $\phi$ of degree $r+1$. A submanifold $C$ is called a **brane** if

$$\begin{cases} 
\phi |_{C} = 0 \\
\text{dim} C = (n + r - 1)/2.
\end{cases}$$

Remark on the dimension of a brane: Branes have the largest possible dimension among submanifolds $C$ satisfying $\phi|_{C} = 0$. To see this, it suffices to consider the linear case. Take any $r-1$ dimensional linear subspace $W$ in $C$, the interior product of $\phi$ by any orthonormal basis of $W$ determines a symplectic form on the orthogonal complement of $W$ in $C$, which we denote as $M/W$. Furthermore $C/W$ is an isotropic subspace in $M/W$ and therefore $\text{dim} C/W \leq (\text{dim} M/W)/2$. The equality sign holds exactly when $\text{dim} C = (n + r - 1)/2$.

As we recall in a symplectic manifold, a holomorphic disk intersects perpendicularly a Lagrangian submanifold along the boundary, we have the following lemma for intersection of an instanton and brane along the boundary of the instanton in a manifold with a closed VCP form.

**Lemma 17** Let $A$ be an instanton in an $n$-dimensional manifold $M$ with closed VCP form $\phi$ of degree $r+1$. Suppose the boundary of $A$ lies in a brane $C$, then $A$ intersect $C$ perpendicularly along $\partial A$.

**Proof.** For $x \in \partial A \subset C$, consider $u \in T_x A$ perpendicular to $\partial A$ and any $v \in T_x C$. Observe that there are $u_1, ..., u_r \in T_x (\partial A)$ such that $u = u_1 \times u_2 \times ... \times u_r$, where $\times$ denotes the cross product.
since $\phi|_A$ is the volume form on $A$. Then,

$$g(u, v) = g(u_1 \times u_2 \times \ldots \times u_r, v) = \phi(u_1, u_2, \ldots, u_r, v) = 0$$

because $u_i$'s and $v$ lie in $T_xC$ and $\phi|_C = 0$. That is $u$ is perpendicular to $C$. \qed

Note that we only need the assumption $\phi|_C = 0$ on $C$ in the above lemma.

The condition $\phi|_C = 0$ also implies that $[\phi] \in H^{r+1}(M, C)$. Any such instanton $A$ minimizes volume within the relative homology class $[A] \in H_{r+1}(M, C)$ and with volume equals to the pairing of $[\phi]$ and $[A]$. Furthermore any submanifold $(A', \partial A') \subset (M, C)$ with $[A'] = [A]$ and $vol(A') = vol(A)$ is also an instanton. However, if $\dim C < (n + r - 1)/2$, then finding instantons with boundaries lying on $C$ is an overdetermined system of equations.

Since the definition of a branes depends only on the closed VCP form $\phi$ instead of $\chi$, the image of any brane under an $\phi$-preserving diffeomorphism $f \in \text{Diff}(M, \phi)$ is again a brane. Infinitesimally, $v = df|_t|_{t=0} \in \text{Vect}(M, \phi)$ satisfies $L_v \phi = 0$. This implies that $\iota_v \phi$ is a closed form because $\phi$ is closed.

**Definition 18** Suppose that $\phi$ is a closed VCP form on $M$. An $\phi$-preserving vector field $v \in \text{Vect}(M, \phi)$ is called an $\phi$-Hamiltonian vector field if $\iota_v \phi$ is exact. That is

$$\iota_v \phi = d\eta$$

for some degree $r$ differential form $\eta$, which we call an $\phi$-Hamiltonian differential form.

We will discuss the $\phi$-Hamiltonian equivalence of branes in the next section.

**Examples of branes:** (i) Branes in an oriented $n$-dimensional manifold (i.e. VCP form of degree $n$) are hypersurfaces. (ii) Branes in a Kähler manifold (i.e. parallel VCP form of degree 2) are Lagrangian submanifolds. (iii) Branes in a $G_2$-manifold (i.e. parallel VCP form of degree 3) can be identified as those four dimensional submanifolds calibrated by $^*\phi$ (see [10]) and they are called coassociative submanifolds. (iv) The next proposition shows that there is no brane in any Spin($7$)-manifold.

**Proposition 19** Brane does not exist in any Spin($7$)-manifold. That is there is no 5-dimensional submanifold where the Cayley form vanishes.

**Proof.** Suppose that $C$ is any submanifold in a Spin($7$)-manifold $M$ where the Cayley form vanishes. This implies that $\chi(e_i, e_j, e_k)$ (denoted as $\chi_{ijk}$) is perpendicular to $C$ for any orthonormal tangent vectors $e_i$'s on $C$. Notice that these unit vectors satisfy

$$\chi_{ijp} \perp \chi_{ijq}.$$  

This is because

$$\|\chi(e_i, e_j, e_k + e_q)\| = \|e_p + e_q\|,$$
which implies that
\[ g(\chi(e_i, e_j, e_p), \chi(e_i, e_j, e_q)) = g(e_p, e_q) = 0. \]

If \( \dim C = 5 \), i.e. \( C \) is a brane in \( M \), then its normal bundle has rank three. However, by the above property, \( \chi_{123}, \chi_{124}, \chi_{134} \) and \( \chi_{234} \) are four orthonormal vectors normal to \( C \), which is a contradiction.

| Manifolds \( M \) \((\dim M)\) | VCP form \( \phi \) \((\text{degree of } \phi)\) | Instanton \( A \) \((\dim A)\) | Brane \( C \) \((\dim C)\) |
|---|---|---|---|
| Oriented mfd. \((n)\) | Volume form \((n)\) | Open Subset \((n)\) | Hypersurface \((n-1)\) |
| Kähler mfd. \((2m)\) | Kähler form \((2)\) | Holomorphic Curve \((2)\) | Lagrangian Submanifold \((m)\) |
| \( G_2 \)-manifold \((7)\) | \( G_2 \)-form \((3)\) | Associative Submanifold \((3)\) | Coassociative Submanifold \((4)\) |
| \( Spin(7) \)-mfd. \((8)\) | Cayley form \((4)\) | Cayley submfd. \((4)\) | N/A |

Table 2: Classification of instantons and branes

Remark on \( 0 \)-fold VCP: Even though we usually assume \( r \) is positive and exclude \( 0 \)-fold VCP in the classification, such a VCP or its corresponding VCP form is simply given by a closed one form \( \phi \) with unit pointwise length. When \( \phi \) has integral period, we can integrate it to obtain a function,

\[ f : M \to S^1 \text{ and } \phi = f^*d\theta. \]

Instantons are gradient flow lines for the Morse function \( f \) on \( M \). Branes are middle dimensional submanifolds in fibers of \( f \).

### 2.5 Deformation Theory of Branes

The intersections theory of branes plays an important role in describing the geometry of vector cross product, this is analogous to the role of the Floer’s Lagrangian intersection theory in symplectic geometry. Deformation theory of branes are essential in understanding both the intersections theory of branes and the moduli space of branes.

First we need to identify the normal bundle to any brane. Note that \( \phi|_C = 0 \) implies

\[ \chi : \Lambda^r T_C \to N_{C/M}. \]

When \( C \) has the maximum possible dimension, i.e. a brane, this is a surjective homomorphism onto \( N_{C/M} \). It is because, otherwise, there exists \( \nu \in N_{C/M} \)
perpendicular to the image of $\chi (\Lambda^r T_C)$, thus $\phi$ will vanish on the linear span of $T_C$ and $\nu$, i.e. a bigger space containing $C$, a contradiction.

By taking the dual on $\chi : \Lambda^r T_C \to N_{C/M}$, we obtain an injective map,

$$t : N_{C/M}^* \to \Lambda^r T_C^*$$

defined by

$$t (\alpha) (u_1, ..., u_r) := \alpha (\chi (u_1, ..., u_r))$$

for $\alpha \in N_{C/M}^*$ and $u_1, ..., u_r \in T_C$. Observe

$$\alpha (\chi (u_1, ..., u_r)) = g (\chi (u_1, ..., u_r), \tilde{\alpha}) = \phi (u_1, ..., u_r, \tilde{\alpha})$$

where $\tilde{\alpha} \in N_{C/M}$ such that its dual is $\alpha$. Then, for any $\alpha \in N_{C/M}^*$ with $|\alpha| = 1$, $t (\alpha)$ is a VCP form on $T_C$ of degree $r$. The reason is for any orthonormal vector $e_1, ..., e_{r-1} \in T_{C,x}$ and $x \in C$,

$$|t_{e_1 \wedge ... \wedge e_{r-1}} (t (\alpha))| = |t_{e_1 \wedge ... \wedge e_{r-1} \wedge \alpha} (\phi)| = 1$$

because $\phi$ is a VCP form and $\tilde{\alpha}$ is a unit normal vector. Therefore we proved the following proposition.

**Proposition 20** Suppose that $C$ is a brane in a manifold $M$ with a VCP form of degree $r + 1$, then the image of the map

$$t : N_{C/M}^* \to \Lambda^r T_C^*$$

is the subbundle spanned by VCP form of degree $r$ on $T_{C,x}$ for all $x \in C$.

We denote $N_{C/M}^*$ as $\Lambda_V^r T_C^*$. Using in the classification of branes below, $\Lambda_V^r T_C$ equals to (i) $T_C$ when $r = 1$, (ii) $\Lambda_2^r T_C$ when $r = 2$ (iii) $\Lambda_{n-1}^r T_C$ when $r = n - 1$. Note that brane does not exist when $r = 3$ (Proposition 19).

Using the exponential map, small deformations of $C$ correspond to sections of $N_{C/M}$ and the branes are the zeros of the following map,

$$F : \Gamma (N_{C/M}) \to \Omega^{r+1} (C)$$

$$F (v) = (\exp_v)^* \phi,$$

defined on a small neighborhood of the origin in $\Gamma (N_{C/M})$. We are going to study the deformation theory of branes following the approach by McLean in [17]. Under the identification $t : \Gamma (N_{C/M}) \cong \Omega_V^r (C)$, the differential of $F$ at 0 is given by the exterior derivative because

$$dF (0) (v) = L_v (\phi) |_C = d (i_v \phi) |_C = d (t (v)).$$

Recall $F (0) = 0$, we obtain $[F (v)] = [F (0)] = 0 \in H^{r+1} (C)$ because $C$ and $\exp_v (C)$ are homologous in $M$. Therefore we have

$$F : \Omega_V^r (C) \to d \Omega^r (C)$$

$$F (0) = 0,$$

$$dF (0) = d.$$
If we know that $d\Omega_{VCP}(C) = d\Omega(C)$, then using the implicit function theorem, we can show that $F^{-1}(0)$ is smooth near 0 and the tangent space is given by the kernel of $dF(0)$. The condition $d\Omega_{VCP}(C) = d\Omega(C)$ can be verified in each individual case, however the authors do not know of any general proof of this. In any case we have proved the following result.

**Proposition 21** Suppose that $\phi$ is a VCP form of degree $r+1$ on $M$. Then small deformations of any brane $C$ are parametrized by closed form in $\Omega_{VCP}^r(C)$. In particular the space of branes in $M$ is smooth.

The space of branes is usually of infinite dimensional. But quotienting out the equivalence relationship of $\phi$-Hamlitonian, the moduli space of branes of finite dimensional.

**Definition 22** Suppose that $C_1$ and $C_0$ are two branes in a manifold $M$ with a closed VCP form $\phi$ of degree $r+1$. They are called $\phi$-Hamiltonian equivalent to each other if they are joined by a family of branes $C_t$ such that their deformation vector fields $v_t = dC_t/dt \in \Gamma(N_{C_t}/M)$ satisfy

$$\iota_{v_t}\phi = d\eta_t,$$

for some $\eta_t \in \Omega^{r-1}(C)$.

Using the Hodge theory and the previous proposition, we have the tangent space to the moduli space of branes at any point $C$ equals $H^r_{VCP}(C)$, the space of harmonic forms in $\Omega_{VCP}^r(C)$. In particular, the moduli space is smooth and of finite dimensional. Its tangent space is given by (i) $H^1(C, \mathbb{R})$ when $r = 1$; (ii) $H^2_+(C, \mathbb{R})$ when $r = 2$ and (iii) $H^{n-1}(C, \mathbb{R}) \cong \mathbb{R}$ when $r = n - 1$. In the third case, i.e. $\phi$ is the volume form on $M$, two nearby hypersurfaces $C$ and $C'$ are $\phi$-Hamiltonian equivalent if there is a singular chain $B$ satisfying $\partial B = C - C'$ and $\text{Vol}(B) = 0$.

Lagrangian intersection theory in symplectic geometry plays the central role in the subject, and also plays very essential roles in mirror symmetry. Naively speaking we need to count the number of instantons bounding two Lagrangian submanifolds. It is natural to generalize this to other VCPs and count the number of instantons bounding two branes. This is a very difficult problem except when $r$ equals zero. In this case, suppose that $C_1$ and $C_2$ are two branes in $M$, i.e. $C_i \subset f^{-1}(\theta_i)$ for $i = 1, 2$ are middle dimensional submanifolds. Here we continue the notations in the previous remark. Since $f$ is a Riemannian submersion, $M$ is a Riemannian mapping cylinder, i.e.

$$M = X \times [0, 1] / \sim$$

for some isometry $h$ on $X$, identifying $X \times \{0\}$ and $X \times \{1\}$. Thus both $C_i$’s can be regarded as middle dimensional submanifolds in $X$. Then instantons in $M$ bounding $C_1$ and $C_2$ correspond to intersection points between $C_1$ and $h^k(C_2)$.
in $X$ for any integer $k$. Therefore the generating function for the number of instantons is given explicitly by the following topological sum,

$$\sum_{k=-\infty}^{\infty} \# (C_1 \cap h^k (C_2)) t^k.$$ 

### 3 Complex Vector Cross Product

#### 3.1 Classification of Complex Vector Cross Products

In this section we study vector cross products on complex vector spaces, or Hermitian complex manifolds. For the sake of convenience, the complex vector cross product ($\mathbb{C}$-VCP) will be defined in terms of complex vector cross forms on a Hermitian complex manifold. Recall that a Hermitian complex manifold is a Riemannian manifold $(M, g)$ with a Hermitian complex structure $J$, that is $g(Ju, Jv) = g(u, v)$ for any tangent vectors $u$ and $v$.

**Definition 23** On a Hermitian complex manifold $(M, g, J)$ of complex dimension $n$, an $r$-fold complex vector cross product (abbrev. $\mathbb{C}$-VCP) is a holomorphic form $\phi$ of degree $r + 1$ satisfying

$$|\epsilon_{e_1 \wedge e_2 \ldots \wedge e_r} (\phi)| = 2^{(r+1)/2},$$

for any orthonormal tangent vectors $e_1, \ldots, e_r \in T^1_x M$, for any $x$ in $M$.

A $\mathbb{C}$-VCP is called closed (resp. parallel) if $\phi$ is closed (resp. parallel with respect to the Levi-Civita connection) form.

Notice that if the manifold $M$ is a closed Kähler manifold, then every holomorphic form in $M$ is closed. For completeness we include the proof of this well-known fact.

**Lemma 24** Suppose $M$ is a closed Kähler manifold. Then every holomorphic form is a closed differential form.

**Proof.** Assume that $\psi$ is any holomorphic form of degree $k$ in $M$, that is $\psi \in \Omega^k(M)$ and $\partial \bar{\partial} \psi = 0$. We need to show that $\partial \psi = 0 \in \Omega^{k+1,0}(M)$. By the Riemann bilinear relation, the pairing

$$\int_M \eta_1 \wedge \bar{\eta}_2 \wedge \omega^{n-k-1}$$

is definite on $\eta_i \in \Omega^{k+1,0}(M)$. That is,

$$\int_M |\partial \psi|^2 \omega^n = C \int_M \partial \psi \wedge \bar{\partial} \psi \wedge \omega^{n-k-1}.$$  

Using integration by part on closed manifolds and holomorphicity of $\psi$, we have

$$\int_M |\partial \psi|^2 \omega^n = -C \int_M \partial \bar{\partial} \psi \wedge \bar{\psi} \wedge \omega^{n-k-1} = 0.$$
This implies that $\partial \psi = 0$, that is $\psi$ is a closed form on $M$.  ■

We are going to see that there are exactly two classes of Kähler manifolds with C-VCP, namely Calabi-Yau manifolds and hyperkähler manifolds. Furthermore every C-VCP is automatically parallel, in particular closed, provided that the manifold itself is closed.

Example: Calabi-Yau manifold (i.e. $(n-1)$-fold C-VCP). A linear complex volume form $\phi$ on $\mathbb{C}^n$ is an element in $\Lambda^{n,0}(\mathbb{C}^n)$ with $\phi \bar{\phi}$ equals the Riemannian volume form on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. This is because of the equality $|\det_{\mathbb{C}}(A)|^2 = \det(A_R)$ between a complex matrix $A$ and its realization $A_R$. It is given as follow, 

$$\phi = dz^1 \wedge dz^2 \wedge ... \wedge dz^n,$$

for a suitable choice of complex coordinate $z^j$'s on $V$. It is easy to see that $\phi$ defines a constant $(n-1)$-fold C-VCP. Similarly an $(n-1)$-fold C-VCP structure on a closed Kähler manifold $(M,g)$ is a holomorphic volume form $\Omega \in \Omega^{n,0}(M)$, 

$$\Omega \bar{\Omega} = C_n \omega^n,$$

where the constant $C_n$ equals $i^n (-1)^{(n-1)/2} 2^{-n}/n!$. This implies that the Ricci curvature of $M$ vanishes. Thus, using Bochner arguments, we can show that every holomorphic form on $M$ is parallel. In particular $\Omega$ is a parallel complex volume form on $M$ and therefore the holonomy group of $M$ lies inside $SU(n)$, i.e. a Calabi-Yau manifold. A celebrated theorem of Yau [24] says that any closed Kähler manifold with trivial first Chern class $c_1(M)$ admits Kähler metric with vanishing Ricci curvature. This implies that $M$ is a Calabi-Yau manifold if the canonical line bundle is trivial holomorphically.

Example: Hyperkähler manifold (i.e. 1-fold C-VCP). A hyperkähler manifold is a Riemannian manifold $(M,g)$ of dimension $n = 4m$ with its holonomy group lies inside $SU(n) = GL(m, \mathbb{H}) \cap SO(4m)$. Namely it has parallel Hermitian complex structures $I$, $J$ and $K$ satisfying the Hamilton relation,

$$I^2 = J^2 = K^2 = IJK = -Id.$$

These complex structures defines three different Kähler structures $\omega_I$, $\omega_J$ and $\omega_K$ on $(M,g)$ respectively. If we fix one of them, say $J$, then $\Omega = \omega_I - i \omega_K \in \Omega^{2,0}(M)$ is a parallel $J$-holomorphic symplectic form on $M$. These two descriptions of a hyperkähler manifold are equivalent and it is simply the global version of the isomorphism $Sp(m) = U(2m) \cap Sp(m, \mathbb{C})$. This form $\Omega$ is a parallel 1-fold C-VCP form on $M$. The reasoning is the same as the one in the real case. In the linear case, this is given as follow,

$$\Omega = dz^1 \wedge dz^2 + ... + dz^{2m-1} \wedge dz^{2m},$$

for some suitable choice of coordinates on $\mathbb{C}^{2m}$. Conversely, if $\Omega$ is a 1-fold C-VCP on a closed Kähler manifold $(M,g,J)$, then it is a holomorphic symplectic
form on $M$. Since $Sp(m) \subset SU(2m)$, any hyperkähler manifold is a Calabi-Yau manifold. This implies that $\Omega$ is indeed parallel as before. Therefore a hyperkähler structure is equivalent to a 1-fold $\mathbb{C}$-VCP on any closed Kähler manifold.

We remark that, as in the real setting, a constant $r$-fold $\mathbb{C}$-VCP on a complex vector space induces an $(r-1)$-fold $\mathbb{C}$-VCP on any of its complex hyperplane.

We are going to show that there is no other complex vector cross product besides the holomorphic volume form and the holomorphic symplectic form as discussed above. In particular, there is no complex analog of VCP for $G_2$-manifolds and $Spin(7)$-manifolds.

**Proposition 25** On a complex vector space $V$ of complex dimension $n$, there is an $r$-fold $\mathbb{C}$-VCP if and only if either (i) $r = 1$ and $n = 2m$ or (ii) $r = n - 1$ and $n$ arbitrary. The corresponding $\mathbb{C}$-VCP form is a holomorphic symplectic form and a holomorphic volume form respectively.

**Proof.** From the above two examples, there is an $(n-1)$-fold $\mathbb{C}$-VCP on $V$, and more if $n$ is an even number, a 1-fold $\mathbb{C}$-VCP exists on it. Now, we need to see there is no other type of $\mathbb{C}$-VCP on a complex vector space. For that matter, we claim that for $r \geq 2$ if there is an $r$-fold $\mathbb{C}$-VCP on a complex vector space $V$ of complex dimension $n$, $r$ must be $n-1$. At first, observe that for $r \geq 2$, an $r$-fold $\mathbb{C}$-VCP on a vector space induces an $(r - 1)$-fold $\mathbb{C}$-VCP on the complex hyperplane. Therefore, an $r$-fold $\mathbb{C}$-VCP on a complex vector space of complex dimension $n$ is reduced 2-fold $\mathbb{C}$-VCP on a complex $(n - r + 2)$-dimensional vector space. Now, to show claim, it is enough to verify if there is a 2-fold $\mathbb{C}$-VCP on a complex vector space $W$, then its complex dimension must be 3.

As in the one of the example of $\mathbb{C}$-VCP, when $\dim \mathbb{C} W = 3$, there is a 2-fold $\mathbb{C}$-VCP. Now, we need to show there is no higher complex vector space with 2-fold $\mathbb{C}$-VCP. Suppose $\dim \mathbb{C} W \geq 4$ with 2-fold $\mathbb{C}$-VCP $\phi$, and by choosing any unit holomorphic vector $z$ in $W$, consider a complex subspace $Z$ spanned by $z$ and $\bar{z}$. Then, $\iota_z(\phi)$ is a 1-fold $\mathbb{C}$-VCP on $Z^\perp$ and more $\dim \mathbb{C}(Z^\perp)$ is at least 4 because $\dim \mathbb{C}(Z^\perp) \geq 3$ and an even number so that $Z^\perp$ has a 1-fold $\mathbb{C}$-VCP.

Now, we may rewrite the 2-fold $\mathbb{C}$-VCP on $W$ as

$$\phi = z^* \land \iota_z(\phi) + \phi_1,$$

where $z^*$ is dual form of $z$ and $\phi_1$ is the sum of terms without $z^*$. Since $\dim \mathbb{C}(Z^\perp)$ is at least 4, 1-fold $\mathbb{C}$-VCP, $\iota_z(\phi)$ on $Z^\perp$ has of the form $a_1^* \land b_1^* + a_2^* \land b_2^*$. where $a_1, a_2, b_1$ and $b_2$ are orthonormal holomorphic vectors in $Z^\perp$.

We consider the following,

$$\iota_{(b_1 + b_2)}(a_1 + a_2)(\phi) = \iota_{(b_1)}a(\phi_1) + \iota_{(b_2)}a_1(\phi_2) + \iota_{(b_1)}a(\phi) + \iota_{(b_2)}a_1(\phi) = 2z^* + \iota_{(b_2)}a(\phi) + \iota_{(b_1)}a_1(\phi),$$

and

$$\iota_{-\sqrt{-1}(b_1 + b_2)}(-\sqrt{-1}a_1 + a_2)(\phi) = -\sqrt{-1}\iota_{(b_2)}a(\phi) - \sqrt{-1}\iota_{(b_1)}a_1(\phi).$$
Note that \( a_1 + a_2, b_1 + b_2, -\sqrt{-1}a_1 + a_2 \) and \(-\sqrt{-1}b_1 + b_2 \) are holomorphic vectors with the same length.

The interior product of any term from \( \iota_z(\phi) \) and \( \phi_1 \) is zero because it satisfies for example, \( |_{a_1,\partial b_1}(\phi)| = 1 \). This implies that \( \phi_1 \) does not have any term in \( \iota_z(\phi) \). Hence we have \( t_{b_2} \iota_{a_1}(\phi) = z^* \) and \( t_{b_2} \iota_{a_2}(\phi) = z^* \).

From the choice of all orthonormal \( z, a_1, b_1, a_2 \) and \( b_2 \), holomorphic vectors \( a_1 + a_2 \) and \( b_1 + b_2 \) are orthogonal to each other, and this is true between holomorphic vectors \( -\sqrt{-1}a_1 + a_2 \) and \(-\sqrt{-1}b_1 + b_2 \). So \( \iota_{(b_1+b_2)}(\iota_{(a_1+a_2)}(\phi)) \) and \( \iota_{(\sqrt{-1}b_1+b_2)}(\iota_{(\sqrt{-1}a_1+a_2)}(\phi)) \) are supposed to produce the same length by definition of \( \mathbb{C} \text{-VCP} \), but it can be checked that is impossible because \( z^* \) and \( \iota_{(b_2)}(\iota_{(a_2)}(\phi)) + \iota_{(b_1)}(\iota_{(a_1)}(\phi)) \) are perpendicular each other. From this contradiction, a complex vector space \( W \) is of complex dimension 3 so that it has a 2-fold \( \mathbb{C} \text{-VCP} \).

From this proposition and the examples of \( \mathbb{C} \text{-VCP} \), one can conclude the following theorem.

**Theorem 26** (Classification of \( \mathbb{C} \text{-VCP} \)) Suppose \( M \) is a closed Kähler manifold of complex dimension \( n \), with an \( r \)-fold \( \mathbb{C} \text{-VCP} \). Then either

(i) \( r = n - 1 \) and \( M \) is a Calabi-Yau manifold, or
(ii) \( r = 1 \) and \( M \) is a hyperkähler manifold.

### 3.2 Instantons for Complex Vector Cross Products

In this section, we introduce and study instantons and branes on a Kähler manifold \( M \) with a closed \( \mathbb{C} \text{-VCP} \( \phi \in \Omega^{r+1,0}(M) \). Recall that an instanton, in the real setting, is a \( (r+1) \)-dimensional submanifold \( A \) preserved by \( \chi \), or equivalently \( A \) is calibrated by the VCP form. In the complex setting, the real and imaginary parts of the complex VCP form are always calibrating forms and we called such calibrated submanifolds instantons.

**Lemma 27** Suppose \( \phi \) is a closed \( \mathbb{C} \text{-VCP} \) form of degree \( r + 1 \) on a Kähler manifold \( M \), then (i) \( \Re(e^{i\theta}\phi) \) is a calibrating form for any real number \( \theta \), and (ii) a \( (r+1) \)-dimensional submanifold \( A \) in \( M \) is calibrated by \( \Re(e^{i\theta}\phi) \) only if

\[
\Im(e^{i\theta}\phi)|_A = \omega|_A = 0.
\]

**Proof.** It suffices to check the linear case, namely \( M = \mathbb{C}^n \) with the standard complex structure \( J \). Consider any oriented orthonormal vectors \( a_1,...,a_{r+1} \) in \( \mathbb{R}^{2n} = \mathbb{C}^n \) and denote \( \xi = a_1 \wedge ... \wedge a_{r+1} \), then

\[
\{\Re(e^{i\theta}\phi) (\xi)\}^2 + \{\Im(e^{i\theta}\phi) (\xi)\}^2 = |(e^{i\theta}\phi)(\xi)|^2 = |\phi(\xi)|^2.
\]

Since \( \phi \) is of type \((r+1,0)\), we have

\[
|\phi(\xi)|^2 = 2^{-(r+1)}|\phi(\xi \otimes \mathbb{C})|^2.
\]

where \( \xi \otimes \mathbb{C} = \tilde{a}_1 \wedge ... \wedge \tilde{a}_{r+1} \), with \( \tilde{a}_i := (a_i - \sqrt{-1}Ja_i)/\sqrt{2} \). This is because \( dz_i(\tilde{a}_k) = \sqrt{2}dz_i(a_k) \).
If we denote the dual vector of any one form \( \eta \) as \( \eta^\# \), then

\[
2^{-(r+1)} |\phi (\xi \otimes \mathbb{C})|^2 = 2^{-(r+1)} |\phi (\tilde{a}_1, ..., \tilde{a}_{r+1})|^2
= 2^{-(r+1)} \left| \left( t_{\tilde{a}_1 \wedge ... \wedge \tilde{a}_r, \phi}^\#, \tilde{a}_{r+1} \right) \right|
\leq 2^{-(r+1)} \left| (t_{\tilde{a}_1 \wedge ... \wedge \tilde{a}_r, \phi})^\# \right| |\tilde{a}_{r+1}|
= 2^{-(r+1)} |t_{\tilde{a}_1 \wedge ... \wedge \tilde{a}_r, \phi}| \leq 1,
\]

because \( \phi \) is a \( \mathbb{C}\)-VCP and \( \tilde{a}_i \)'s are elements in \( T^{1,0}M \) of unit length. Therefore

\[
|\text{Re} \left( e^{i\theta} \phi \right) (\xi) | \leq 1,
\]

and when the equality sign holds, we have (i) \( |\text{Im} \left( e^{i\theta} \phi \right) (\xi) | = 0 \), (ii) \( (t_{\tilde{a}_1 \wedge ... \wedge \tilde{a}_r, \phi})^\# \) parallel to \( \tilde{a}_{r+1} \) and (iii) \( |t_{\tilde{a}_1 \wedge ... \wedge \tilde{a}_r, \phi}| = 2^{r+1} \). Since \( a_i \)'s are orthonormal, condition (iii) is equivalent to the \( \tilde{a}_i \)'s being orthonormal vectors in \( T^{1,0}M \). This happens exactly when the linear span of \( a_i \)'s is isotropic with respect to \( \omega \).

We remark that when \( \phi \) is the holomorphic volume form of a Calabi-Yau manifold, then a middle dimensional submanifold \( A \) in \( M \) is calibrated by \( \text{Re} \left( e^{i\theta} \phi \right) \) if and only if it satisfies \( |\text{Im} \left( e^{i\theta} \phi \right) |_A = \omega |_A = 0 \), and it is called a special Lagrangian submanifold with phase angle \( \theta \).

**Definition 28** On a closed Kähler manifold \( M \) with an \( r \)-fold \( \mathbb{C}\)-VCP \( \phi \), an \( r+1 \) dimensional submanifold \( A \) is called an **instanton** with phase \( \theta \in \mathbb{R} \) if it is calibrated by \( \text{Re} \left( e^{i\theta} \phi \right) \), i.e.

\[
\text{Re} \left( e^{i\theta} \phi \right) |_A = \text{vol} A.
\]

Equivalently, \( |\text{Im} \left( e^{i\theta} \phi \right) |_A = 0 \) and \( |t_{e_1 \wedge ... \wedge e_{r+1}} (\phi |_A) | = 1 \) for any orthonormal tangent vector \( e_i \)'s on \( A \).

Remark: Recall that the volume of a calibrated submanifold is topological. In this case, for the fundamental class \( [A] \in H_{r+1} (A, \mathbb{Z}) \) and \( [\text{Re} \left( e^{i\theta} \phi) \right] \in H^{r+1} (M, \mathbb{R}) \),

\[
\text{vol} (A) = \int_A \text{Re} \left( e^{i\theta} \phi \right) = [A] \cdot [\text{Re} \left( e^{i\theta} \phi \right)]
\]

Using the classification result of \( \mathbb{C}\)-VCPs in theorem 20, \( \phi \) must be either a holomorphic volume form in a Calabi-Yau manifold or a holomorphic symplectic form in a hyperkähler manifold. In the former case, instantons are called special Lagrangian submanifolds (see e.g. 20). In the latter case,

\[
\text{Re} \left( e^{i\theta} (\omega_I - i\omega_K) \right) = \cos \theta \omega_I + \sin \theta \omega_K = \omega_{(\cos \theta)I + (\sin \theta)K}
\]

so instantons are \( J_\theta \)-holomorphic curves where \( J_\theta = (\cos \theta) I + (\sin \theta) K \).
3.3 Dirichlet and Neumann Branes

Schoen’s school studies ([5], [19], [20]) free boundary value problem for special Lagrangian submanifolds in Calabi-Yau manifolds $M$. Suppose that $A$ is a special Lagrangian submanifold of zero phase, i.e. calibrated by $\text{Re} \, \Omega_M$, in $M$. If the boundary of $A$ is non-empty and lies on a submanifold $C$ in $M$, then (i) $C$ being a complex hypersurface in $M$ corresponds to Neumann boundary condition on $A$ and (ii) $C$ being a special Lagrangian submanifold of phase $\pi/2$ corresponds to Dirichlet boundary condition on $A$. Motivated from these, we have the following definitions of branes in Hermitian complex manifolds.

**Definition 29** On a Hermitian complex manifold $(M, \omega)$ of complex dimension $n$ with an $r$-fold $\mathbb{C}$-VCP $\phi \in \Omega^{r+1,0}(M)$,

(i) a submanifold $C$ is called a Neumann brane (abbrev. $N$-brane) if

$\dim (C) = n + r - 1$ and $\phi|_C = 0$,

(ii) an $n$-dimensional submanifold $C$ is called a Dirichlet brane (abbrev. $D$-brane) with phase $\theta \in \mathbb{R}$ if

$\omega|_C = 0$, $\text{Re} \, (e^{i\theta} \phi)|_C = 0$.

Even though branes are defined in Hermitian complex manifolds, for simplicity we will focus on branes in a closed K"ahler manifold.

As in the real setting, $N$-branes are submanifolds in $M$ with the biggest dimension on which $\phi$ vanishes. Furthermore $N$-branes are complex submanifolds in $M$.

**Proposition 30** Suppose that $M$ is an $n$-dimensional K"ahler manifold with an $r$-fold $\mathbb{C}$-VCP. Assume that $S$ is a submanifold in $M$ such that $\phi|_S = 0$, then

$\dim S \leq n + r - 1$.

When the equality sign holds, i.e. $S$ is a $N$-branes, then it is a complex submanifold in $M$.

**Proof.** It suffices to check the linear case, i.e. $T_xS$, $x \in S$. Consider a tangent vector $a \in T_x S$ and $\tilde{a} = (a - \sqrt{-1}Ja)/\sqrt{2}$. Since $\phi \in \Omega^{r+1,0}$, $\iota_{\tilde{a}} \phi = \sqrt{2} \mathbf{a} \phi$. Therefore, for $b_i$’s in $T_xS$.

$$\sqrt{2} \phi (a, b_1, ..., b_r) = \phi (\tilde{a}, b_1, ..., b_r).$$

So if $\phi(a, b_1, ..., b_r) = 0$, then $\phi(Ja, b_1, ..., b_r) = 0$, namely, $\phi|_S = 0$ implies $\phi|_{S+JS} = 0$. And more if $S$ has the maximal dimension, then $T_xS$ is complex linear subspace. In that case, as in the real setting, we have $\dim C S = (n + r - 1)/2$. \[\rule{0.5cm}{0.1cm}\]

The above proposition can also be verified case by case using the classification result of $\mathbb{C}$-VCP. The following classification of instantons and branes for $\mathbb{C}$-VCP is presented with the classification of $\mathbb{C}$-VCP on a closed Kähler manifold.

**Instantons and branes in CY manifold** : $(n-1)$-fold $\mathbb{C}$-VCP

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A closed Kähler manifold of complex dimension $n$ with $(n-1)$-fold $\mathbb{C}$-$\text{VCP}$ $\phi$ is a CY $n$-fold.

An instanton in the Calabi-Yau $n$-fold is a special Lagrangian submanifold with phase $\theta$ since it is calibrated by $\text{Re} (e^{i\theta} \phi)$. As in the previous proposition, an $N$-brane in the Calabi-Yau $n$-fold is a complex hypersurface, and a $D$-brane is a special Lagrangian submanifold with phase $\theta - \pi/2$, because it is calibrated by $\text{Re} (e^{i(\theta - \pi/2)} \phi) = \text{Im} (e^{i\theta} \phi)$.

**Instantons and branes in hyperkähler manifold : 1-fold $\mathbb{C}$-$\text{VCP}$**

A closed Kähler manifold of complex dimension $2n$ with a Kähler form $\omega_J$ and a 1-fold $\mathbb{C}$-$\text{VCP}$ $\phi$ is a hyperkähler manifold. Denote $\phi := \omega_I - \sqrt{-1} \omega_K$ and $J$, $I$, and $K$ as the complex structure corresponding to Kähler structures $\omega_J$, $\omega_I$ and $\omega_K$, respectively. And more, by putting $e^{i\theta} \phi$ in place of $\phi$, one can observe $\text{Re} e^{i\theta} \phi$ is another Kähler structure with a complex structure $J_{\theta} := \cos \theta I + \sin \theta K$.

Now, an instanton in a hyperkähler manifold is a $J_{\theta}$-holomorphic curve since it is calibrated by $\text{Re} e^{i\theta} \phi$, namely preserved by $J_{\theta}$. An $N$-brane in a hyperkähler manifold is a real $2n$-dimensional submanifold where $\phi$ vanishes, and as in the previous proposition, it is equivalently a $J$-complex Lagrangian which is a complex submanifold preserved by a complex structure $J$ with complex dimension $n$. A $D$-brane is a real $2n$-dimensional submanifold where $\omega$ and $\text{Re} (e^{i\theta} \phi)$ vanish. One can show that a $D$-brane is preserved by $J_{\theta + \pi/2}$ the almost complex structure corresponding to $-\text{Im} (e^{i\theta} \phi)$, i.e. $e^{i\theta} \phi = \omega_{J_{\theta}} - \sqrt{-1} \omega_{J_{\theta+\pi/2}}$. So a $D$-brane is equivalently, a $J_{\theta + \pi/2}$-complex Lagrangian.

The above classification of instantons, $N$-branes and $D$-branes in manifolds with $\mathbb{C}$-$\text{VCP}$ is summarized in the table in page 4.

## 4 Symplectic Geometry on Knot Spaces

Recall that a 1-fold VCP on $M$ is a symplectic structure. In general an $r$-fold VCP form on $M$ induces a symplectic structure on the space of embedded submanifolds $\Sigma$ in $M$ of dimension $r - 1$, which we simply call a (multi-dimensional) knot space $K_{\Sigma}M = Map(\Sigma, M) / Diff(\Sigma)$. For instance when $M$ is an oriented three manifold, $K_{\Sigma}M$ is the space of knots in $M$. In this case Brylinski [4] showed that $K_{\Sigma}M$ has a natural complex structure which makes it an infinite dimensional Kähler manifold and used it to study the problem of geometric quantization. For general $M$ with a VCP, we will relate the symplectic geometry of $K_{\Sigma}M$ to the geometry of branes and instantons in $M$.

When $M$ is a Kähler manifold with a $\mathbb{C}$-$\text{VCP}$, say Calabi-Yau manifold, one might try to complexify the above construction to define a holomorphic symplectic structure (i.e. 1-fold $\mathbb{C}$-$\text{VCP}$) on the symplectic quotient $Map(\Sigma, M) / Diff(\Sigma)$. However there are various difficulties due to the fact that $Map(\Sigma, M)$ does not have a natural symplectic structure unless we fix a background volume form on $\Sigma$. We will resolve this problem and define a natural holomorphic symplectic structure on the isotropic knot space $\hat{K}_{\Sigma}M$. We do
not know whether it is a hyperkähler structure or not because the symplectic form on \( \hat{K}_\Sigma M \) induced from the Kähler form on \( M \) may not be closed.

We will also show that both complex hypersurfaces (i.e. N-branes) and special Lagrangian submanifolds of phase \(-\pi/2\) (i.e. D-branes) in \( M \) correspond to complex Lagrangian submanifolds in \( \hat{K}_\Sigma M \), but with respect to different almost complex structures in the twistor \( S^2 \)-family for \( \hat{K}_\Sigma M \). For instance when \( M \) is a three dimensional Calabi-Yau manifold, \( \hat{K}_\Sigma M \) is roughly the space of loops, or strings, in \( M \) with an equivalence relation generated by deformations along complex directions (see remark 39 for details).

4.1 Symplectic Structure on Knot Spaces

Let \((M,g)\) be an \( n \)-dimensional Riemannian manifold \( M \) with a closed VCP form \( \phi \) of degree \( r + 1 \). Suppose \( \Sigma \) is any \( (r-1) \)-dimensional oriented closed manifold \( \Sigma \), we consider the mapping space of embeddings from \( \Sigma \) to \( M \),

\[
Map(\Sigma, M) = \{ f : \Sigma \to M \mid f \text{ is an embedding.} \}.
\]

Let

\[
ev : \Sigma \times Map(\Sigma, M) \to M
\]

be the evaluation map \( \ev(x,f) = f(x) \) and \( p_1, p_2 \) be the projection map from \( \Sigma \times Map(\Sigma, M) \) to its first and second factor respectively. We define a two form \( \omega_{Map} \) on \( Map(\Sigma, M) \) by taking the transgression of the VCP form \( \phi \),

\[
\omega_{Map} = (p_2)_*(\ev)^*\phi = \int_\Sigma \ev^*\phi.
\]

Explicitly, suppose \( u \) and \( v \) are tangent vectors to \( Map(\Sigma, M) \) at \( f \), that is \( u, v \in \Gamma(\Sigma, f^*(TM)) \), we have

\[
\omega_{Map}(u,v) = \int_\Sigma \iota_{u \wedge v}\phi.
\]

Since \( (\iota_{u \wedge v}\phi)|_\Sigma \) can never be a top degree form if \( u \) is tangent to \( \Sigma \), \( \omega_{Map} \) degenerates along tangent directions to the orbits of the natural action of \( Diff(\Sigma) \) on \( Map(\Sigma, M) \). Thus it descends to a two form \( \omega^K \) on the quotient space

\[
K_\Sigma M = Map(\Sigma, M) / Diff(\Sigma),
\]

the space of submanifolds in \( M \). For simplicity, we call it a (multi-dimensional) knot space. Note that tangent vectors to \( K_\Sigma M \) are sections of the normal bundle of \( \Sigma \) in \( M \).

On \( K_\Sigma M \) there is a natural \( L^2 \)-metric given as follow: Suppose \( [f] \in K_\Sigma M \) and \( u, v \in \Gamma(\Sigma, N_{\Sigma/M}) \) are tangent vectors to \( K_\Sigma M \) at \([f]\), then

\[
g^K_{[f]}(u,v) = \int_\Sigma g(u,v)\nu_{\Sigma}.
\]
where \( \nu_\Sigma \) is the volume form of \( \Sigma \) with respect to the induced metric on \( \Sigma \). We define an endomorphism \( J^K \) on the tangent space of \( K_\Sigma M \) as follow: Suppose \( [f] \in K_\Sigma M \) and \( u, v \in \Gamma(\Sigma, N_{\Sigma/M}) \) are tangent vectors to \( K_\Sigma M \) at \([f]\), then

\[
\omega^K_{[f]}(u, v) = g^K_{[f]}(J^K_{[f]}(u), v).
\]

**Proposition 31** Suppose \((M, g)\) is a Riemannian manifold with a VCP. Then \( J^K \) is a Hermitian almost complex structure on \( K_\Sigma M \), i.e. a 1-fold VCP.

**Proof.** Suppose \([f] \in K_\Sigma M \) and \( u, w \in \Gamma(\Sigma, N_{\Sigma/M}) \) are tangent vectors to \( K_\Sigma M \) at \([f]\). Let \( e_i \)'s be an oriented orthonormal base of \( \Sigma \) at a point \( x \). Then

\[
\iota_{u \wedge w}(ev^* \phi) = \phi(e_1, ... e_{r-1}, u, w) \nu_\Sigma = g(\chi(e_1, ... e_{r-1}, u), w) \nu_\Sigma.
\]

From the relationship \( \omega^K_{[f]}(u, v) = g^K_{[f]}(J^K_{[f]}(u), w) \) and the definitions of \( \omega^K_{[f]} \) and \( g^K_{[f]} \), we conclude that \( J^K \) on the tangent space of \( K_\Sigma M \) at \([f]\) is given by

\[
J^K : \Gamma(\Sigma, N_{\Sigma/M}) \rightarrow \Gamma(\Sigma, N_{\Sigma/M}) \chi(e_1, ... e_{r-1}, u).
\]

On the other hand, from the remark on the dimension of branes, we know that if \( \chi \) is an \( r \)-fold VCP on \( T_x M \), then \( \chi(e_1, ..., e_{r-1}, \cdot) \) defines a 1-fold VCP on the orthogonal complement to any oriented orthonormal vectors \( e_i \)'s at \( T_x M \). This implies that \( J^K \) is a 1-fold VCP on \( K_\Sigma M \). \( \blacksquare \)

Remark: The above proof actually shows that if \( J^K \) on \( K_\Sigma M \) is induced by an arbitrary differential form \( \phi \) on \( M \), then \( J^K \) is an Hermitian almost complex structure on \( K_\Sigma M \) if and only if \( \phi \) is a \( r \)-fold VCP form on \( M \).

When the VCP form \( \phi \) on \( M \) is closed and \( \Sigma \) is a closed manifold, then \( \omega_{\text{Map}} \) on \( \text{Map}(\Sigma, M) \) is also closed because

\[
d\omega_{\text{Map}} = d\int_{\Sigma} ev^* \phi = \int_{\Sigma} ev^* d\phi = 0.
\]

Therefore \( \omega_{\text{Map}} \) descends to a closed 1-fold VCP form on \( K_\Sigma M \).

Conversely the closedness of \( \omega_{\text{Map}} \) on \( \text{Map}(\Sigma, M) \), or on \( K_\Sigma M \), implies the closedness of \( \phi \) on \( M \). Since this type of localization arguments will be used several times in this paper, we include the proof of the following standard lemma.

**Lemma 32** (Localization) Let \( \Sigma \) be an \( s \)-dimensional manifold without boundary. A form \( \eta \) of degree \( k > s \) on a manifold \( M \) vanishes if the corresponding \((k - s)\)-form on \( \text{Map}(\Sigma, M) \) obtained by transgression vanishes.

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Proof. We need to show that

\[ \eta(p)(v_1, v_2, \ldots, v_k) = 0 \]

for any fixed \( p \in M \) and any fixed \( v_i \in T_p M \). For simplicity, we may choose \( v_i \)'s to be orthonormal vectors. We can find \( f \in \text{Map}(\Sigma, M) \) with \( p \in f(\Sigma) \) such that \( v_1, \ldots, v_s \) are along the tangential directions, and \( v_{s+1}, \ldots, v_k \) are along the normal directions at \( p \) in \( f(\Sigma) \). Moreover, we can choose sections \( \tilde{v}_{s+1}, \tilde{v}_{s+2}, \ldots, \tilde{v}_k \in \Gamma(\Sigma, f^*(T_M)) \) which equal \( v_{s+1}, \ldots, v_k \) at \( p \) respectively. By multiplying \( \tilde{v}_i \) with a sequence of functions on \( \Sigma \) approaching the delta function at \( p \), we obtain sections \( (\tilde{v}_{s+1})_\varepsilon \) which approach \( \delta(p) v_{s+1} \) as \( \varepsilon \to 0 \) where \( \delta(p) \) is Dirac delta function. Therefore,

\[ \eta(p)(v_1, v_2, \ldots, v_k) = \lim_{\varepsilon \to 0} (\int_{\Sigma} e^v \eta)((\tilde{v}_{s+1})_\varepsilon, \tilde{v}_{s+2}, \ldots, \tilde{v}_k). \]

From the given condition \( \int_{\Sigma} e^v \eta = 0 \), we conclude that

\[ \eta(p)(v_1, v_2, \ldots, v_k) = 0. \]

Hence the result. □

As a corollary of the above lemma and discussions, we have the following result.

**Proposition 33** Suppose \((M, g)\) is a Riemannian manifold with a differential form \( \phi \) of degree \( r + 1 \). Then \( \phi \) is a closed \( r \)-fold VCP form on \( M \) if and only if \( \omega^K \) is an almost Kähler structure on \( K_{\Sigma} M \), i.e. a closed 1-fold VCP form.

Remark: In general the transgression of any closed form \( \phi \) always gives a closed form on the knot space \( K_{\Sigma} M \) of degree \( r + 1 - s \) where \( \dim \Sigma = s \). However this can never be a VCP form unless \( s = r - 1 \). To see this, we can use the above localization method and the fact that there is no VCP form of degree bigger than two on any vector space with sufficiently large dimension.

When \( r = 1 \), that is \( M \) is a symplectic manifold, \( \text{Map}(\Sigma, M) \) is the same as \( M \) and therefore it is symplectic by trivial reasons.

Remark: Given any \( \phi \)-preserving vector field \( v \in \text{Vect}(M, \phi) \), it induces a vector field \( V \) of \( K_{\Sigma} M \) preserving \( \omega^K \). Furthermore, if \( v \) is \( \phi \)-Hamiltonian, that is

\[ \iota_v \phi = d\eta, \]

for some \( \eta \in \Omega^r(M) \), then we can define a function on the knot space,

\[ F_\eta : K_{\Sigma} M \to \mathbb{R} \]

\[ F_\eta(f) = \int_{\Sigma} f^* \eta. \]

It is easy to see that \( F_\eta \) is a Hamiltonian function on the symplectic manifold \( K_{\Sigma} M \) whose Hamiltonian vector field equals \( V \).
4.2 Holomorphic Curves and Lagrangians in Knot Spaces

In this subsection, we are going to show that holomorphic disks (resp. Lagrangian submanifolds) in the knot space $K_{\Sigma}M$ of $M$ correspond to instantons (resp. branes) in $M$. More generally the geometry of vector cross products on $M$ should be closely related to the symplectic geometry of its knot space $K_{\Sigma}M$. A natural problem is to understand the analog of the Floer’s Lagrangian intersection theory for manifolds with vector cross products. Note that if two branes $C_1$ and $C_2$ intersect transversely along a submanifold $\Sigma$, then the dimension of $\Sigma$ equals $r - 1$ and $[\Sigma]$ represents a transverse intersection point of (Lagrangians) $K_{\Sigma}C_1$ and $K_{\Sigma}C_2$ in $K_{\Sigma}M$. The converse is also true.

Because $K_{\Sigma}M$ is of infinite dimensional, a Lagrangian submanifold is defined as a subspace in $K_{\Sigma}M$ where the restriction of $\omega^K$ vanishes and with the property that any vector field $\omega^K$-orthogonal to $L$ is a tangent vector field along $L$, see [18] and [4]. In fact, it is easy to see that the latter condition for a submanifold $L$ to be Lagrangian is equivalent to the statement that $\omega^K$ will not vanish on any bigger space containing $L$. So, we refer the this condition as the maximally self $\omega^K$-perpendicular condition.

**Proposition 34** Suppose that $M$ is an $n$-dimensional manifold $M$ with a closed VCP form $\phi$ of degree $r + 1$ and $C$ is a submanifold in $M$. Then $K_{\Sigma}C$ is a Lagrangian submanifold in $K_{\Sigma}M$ if and only if $C$ is a brane in $M$.

**Proof.** For the only if part, we suppose that $K_{\Sigma}C$ is a Lagrangian in $K_{\Sigma}M$ and we want to show that $\phi$ vanishes along $C$ and $\dim C = (n + r - 1)/2$. For any fixed $[f] \in K_{\Sigma}C$, we have

$$0 = \omega^K_{[f]}(u, v) = \int_{\Sigma} t_{u \wedge v} (e v^* \phi)$$

where $u, v \in \Gamma(\Sigma, N_{\Sigma/C})$. By applying the localization method as in lemma 32 along $C$, we obtain the following: For any $x \in \Sigma$,

$$\phi(u(x), v(x), e_1, ..., e_{r-1}) = 0,$$

where $e_1, ..., e_{r-1}$ are any oriented orthonormal vectors of $T_x \Sigma$. By varying $u, v, f$ and $T_C$. One can show $\phi |_C = 0$.

Moreover, $K_{\Sigma}C$ being $\omega^K$-perpendicular in $K_{\Sigma}M$ implies that $C$ has the biggest possible dimension with $\phi|_C = 0$. As explained in the remark following the definition of branes that this gives $\dim C = (n + r - 1)/2$.

For the if part, we assume that $C$ is any brane in $M$. The condition $\phi|_C = 0$ implies that $\omega^K|_{K_{\Sigma}C} = 0$. To show that $K_{\Sigma}C$ is a Lagrangian in $K_{\Sigma}M$, we need to verify the maximally self $\omega^K$-perpendicular condition in $K_{\Sigma}M$. Recall that the tangent space at any point $[f] \in K_{\Sigma}C$ is $\Gamma(\Sigma, N_{\Sigma/C})$. Suppose there is a section $v$ in $\Gamma(\Sigma, N_{\Sigma/M})$ but not in $\Gamma(\Sigma, N_{\Sigma/C})$ such that it is $\omega^K$-perpendicular to $\Gamma(\Sigma, N_{\Sigma/C})$. By the localization arguments as in lemma 32 given any point $x \in f(\Sigma) \subset C$, $\phi$ vanishes on the linear space spanned by $v(x)$ and $T_x C$. This contradicts to the fact that $C$ is a brane. Hence the result. ■
Remark: $\phi$-Hamiltonian deformation of a brane $C$ in $M$ corresponds to Hamiltonian deformation of the corresponding Lagrangian submanifold $K_\Sigma C$ in the symplectic manifold $K_\Sigma M$. More precisely, if $v$ is a normal vector field to $C$ satisfying
\[ \iota_v \phi = d\eta, \]
for some $\eta \in \Omega^{r-1}(C)$, then the transgression of $\eta$ defines a function on $K_\Sigma C$ which generates a Hamiltonian deformation of $K_\Sigma C$.

Next we discuss holomorphic disks, i.e. instantons, in $K_\Sigma M$. We consider a two dimensional disk $D$ in $\text{Map}(\Sigma, M)$ such that for each tangent vector $v \in T_f [D]$, the corresponding vector field in $\Gamma(\Sigma, f^*T_M)$ is normal to $\Sigma$. We call such a disk $D$ as a normal disk. For simplicity we assume that the $r + 1$ dimensional submanifold
\[ A = \bigcup_{f \in D} f(\Sigma) \subset M, \]
is an embedding. This is always the case if $D$ is small enough. Notice that $A$ is diffeomorphic to $D \times \Sigma$. We will denote the corresponding disk in $K_\Sigma M$ as $\hat{D} =: \pi(D)$. We remark that the principal fibration
\[ \text{Diff}(\Sigma) \to \text{Map}(\Sigma, M) \xrightarrow{\pi} K_\Sigma M \]
has a canonical connection (see [4]) and $D$ being a normal disk is equivalent to it being an integral submanifold for the horizontal distribution of this connection.

In the following proposition, we describe the relation between a disk $\hat{D}$ in $K_\Sigma M$ given above and the corresponding $r + 1$ dimensional subspace $A$ in $M$.

**Proposition 35** Suppose that $M$ is a manifold with a closed $r$-fold VCP form $\phi$ and $K_\Sigma M$ is its knot space as before. For a normal disk $D$ in $\text{Map}(\Sigma, M)$, $\hat{D} =: \pi(D)$ is a $J^K$-holomorphic disk in $K_\Sigma M$, i.e. calibrated by $\omega^K$, if and only if $A$ is an instanton in $M$ and $A \to D$ is a Riemannian submersion.

We call such an $A$ a tight instanton.

**Proof.** For a fixed $[f] \in \hat{D}$, we consider $\nu, \mu \in T_{[f]}(\hat{D}) \subset \Gamma(\Sigma, N_{\Sigma/A})$. Since $\phi$ is a calibrating form, we have,
\[ \phi(\nu, \mu, e_1, ..., e_{r-1}) \leq \text{Vol}_A(\nu, \mu, e_1, ..., e_{r-1}) = |\nu \wedge \mu| \]
where $e_1, ..., e_{r-1}$ is any orthonormal frame on $f(\Sigma)$. In particular we have
\[ \int_{f(\Sigma)} \iota_{\nu \wedge \mu}(e^*\phi) \leq \int_{f(\Sigma)} |\nu \wedge \mu| \text{vol}_\Sigma, \]
and the equality sign holds for every $[f] \in \hat{D}$ if and only if $A$ is an instanton in $M$. We will simply denote $\int_{f(\Sigma)}$ by $\int_{\Sigma}$. Notice that the symplectic form on $K_\Sigma M$ is given by,
\[ \omega^K_{[f]}(\nu, \mu) = \int_{\Sigma} \iota_{\nu \wedge \mu}(e^*\phi). \]
Since $\omega^K$ is a 1-fold VCP form on $K_\Sigma M$, we have

$$\omega^K_{\gamma f}(\nu, \mu) \leq \left( |\nu|^2 |\mu|^2 - \langle \nu, \mu \rangle^2 \right)^{1/2},$$

where $\langle \nu, \mu \rangle_K =: g_K(\nu, \mu)$ and $|a|^2_K =: g_K(a, a)$. Furthermore the equality sign holds when $\hat{D}$ is a $J^K$-holomorphic disk in $K_\Sigma M$.

To prove the only if part, we suppose that $\hat{D}$ is a $J^K$-holomorphic disk in $K_\Sigma M$. From above discussions, we have

$$\int_{\Sigma} |\nu \wedge \mu| \geq \int_{\Sigma} \iota_{\nu \wedge \mu} (ev^* \phi) = \left( |\nu|^2 |\mu|^2 - \langle \nu, \mu \rangle^2 \right)^{1/2}$$

$$= \left( \int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2 - \left( \int_{\Sigma} \langle \nu, \mu \rangle \right)^2 \right)^{1/2}.$$ 

Combining with the Hölder inequality,

$$\left( \int_{\Sigma} |\nu \wedge \mu| \right)^2 + \left( \int_{\Sigma} \langle \nu, \mu \rangle \right)^2 \leq \int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2,$$ 

we obtain

(i) \quad \int_{\Sigma} \iota_{\nu \wedge \mu} (ev^* \phi) = \int_{\Sigma} |\nu \wedge \mu| \text{vol}_\Sigma \quad \text{and} \quad \int_{\Sigma} \langle \nu, \mu \rangle \quad \text{and} \\
(ii) \quad \left( \int_{\Sigma} |\nu \wedge \mu| \right)^2 + \left( \int_{\Sigma} \langle \nu, \mu \rangle \right)^2 = \int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2.$$

Condition (i) says that $A$ is an instanton in $M$. Condition (ii) implies that given any $[f]$, there exists constant $C_1$ and $C_2$ such that for any $x \in \Sigma$,

$$|\nu(x)| = C_1 |\mu(x)|, \quad \langle \nu(x), \mu(x) \rangle = C_2.$$ 

This implies that $A \rightarrow D$ is a Riemannian submersion.

For the if part, we notice that $A$ being an instanton in $M$ implies that

$$\int_{\Sigma} |\nu \wedge \mu| = \int_{\Sigma} \iota_{\nu \wedge \mu} (ev^* \phi) = \omega^K_{\gamma f}(\nu, \mu) \leq \left( |\nu|^2 |\mu|^2 - \langle \nu, \mu \rangle^2 \right)^{1/2}$$

$$= \left( \int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2 - \left( \int_{\Sigma} \langle \nu, \mu \rangle \right)^2 \right)^{1/2}.$$ 

Recall that the Riemannian submersion condition implies an equality,

$$\left( \int_{\Sigma} |\nu \wedge \mu| \right)^2 + \left( \int_{\Sigma} \langle \nu, \mu \rangle \right)^2 = \int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2,$$ 

so the above inequality turns into an equality so that it gives

$$\omega^K_{\gamma f}(\nu, \mu) = \int_{\Sigma} \iota_{\nu \wedge \mu} (ev^* \phi) = \left( |\nu|^2 |\mu|^2 - \langle \nu, \mu \rangle^2 \right)^{1/2}$$

i.e. $\hat{D}$ is $J^K$ holomorphic in $K_\Sigma M$.
5 Isotropic Knot Spaces of CY manifolds

Recall that any volume form on a manifold $M$ of dimension $n$ determines a closed 1-fold VCP on the knot space $\mathcal{K}_\Sigma M = \text{Map}(\Sigma, M) / \text{Diff}(\Sigma)$ where $\Sigma$ is any closed manifold of dimension $n-2$. It is natural to guess that the holomorphic volume form on any Calabi-Yau $n$-fold $M$ would determine a closed 1-fold $\mathbb{C}$-VCP on the symplectic quotient $\text{Map}(\Sigma, M) // \text{Diff}(\Sigma)$. However we need to choose a background volume form $\sigma$ on $\Sigma$ to construct a symplectic structure in $\text{Map}(\Sigma, M)$ which is only invariant under $\text{Diff}(\Sigma, \sigma)$, the group of volume preserving diffeomorphisms of $\Sigma$ ([7],[11]). Therefore we can only construct the symplectic quotient $\text{Map}(\Sigma, M) // \text{Diff}(\Sigma, \sigma) = \mu^{-1}(0) / \text{Diff}(\Sigma, \sigma)$. This larger space is not hyperkähler because the holomorphic two form $\int_\Sigma \Omega_M$ degenerates. We will show that a further symplectic reduction will produce a Hermitian integrable complex manifold $\hat{\mathcal{K}}_\Sigma M$ with a 1-fold $\mathbb{C}$-VCP, in particular a holomorphic symplectic structure. This may not be hyperkähler because even the Hermitian complex structure is integrable on $\hat{\mathcal{K}}_\Sigma M$, its Kähler form may not closed. $\hat{\mathcal{K}}_\Sigma M$ should be regarded as a modified construction for the non-existing space $\text{Map}(\Sigma, M) // \text{Diff}(\Sigma)$. We call this an isotropic knot space of $M$ for reasons which will become clear later.

Furthermore we will relate instantons, $N$-branes and $D$-branes in a Calabi-Yau manifold $M$ to holomorphic curves and complex Lagrangian submanifolds in the holomorphic symplectic manifold $\hat{\mathcal{K}}_\Sigma M$. These constructions is of particular interest when $M$ is a Calabi-Yau threefold (see remark 39 for details).

5.1 Holomorphic Symplectic Structures on Isotropic Knot Spaces

Let $M$ be a Calabi-Yau $n$-fold with a holomorphic volume form $\Omega_M$, i.e. a closed $(n-1)$-fold $\mathbb{C}$-VCP. To obtain a 1-fold $\mathbb{C}$-VCP on a certain knot space by transgression, we first fix an $n-2$ dimensional manifold $\Sigma$ without boundary and we let $\text{Map}(\Sigma, M)$ be the space of embeddings from $\Sigma$ to $M$ as before. For simplicity we assume that the first Betti number of $\Sigma$ is zero, $b_1(\Sigma) = 0$.

If we fix a background volume form $\sigma$ on $\Sigma$, then the Kähler form $\omega$ on $M$ induces a natural symplectic form on $\text{Map}(\Sigma, M)$ as follow: for any tangent vectors $X$ and $Y$ at a point $f$ in $\text{Map}(\Sigma, M)$, we define

$$\omega_{\text{Map}}(X, Y) = \int_{\Sigma} \omega(X, Y)\sigma$$

where $X, Y \in \Gamma(\Sigma, f^*(TM))$. Note that this symplectic structure on $\text{Map}(\Sigma, M)$ is not invariant under general diffeomorphisms of $\Sigma$. Instead it is preserved by the natural action of $\text{Diff}(\Sigma, \sigma)$, the group of volume preserving diffeomorphisms on $(\Sigma, \sigma)$.

As studied by Donaldson in [7] and Hitchin in [11], this action is Hamiltonian on the components of $\text{Map}(\Sigma, M)$ consisting of those $f$’s satisfying

$$f^*(|\omega|) = 0 \in H^2(\Sigma, \mathbb{R}) ,$$
and the moment map is given by

\[ \mu : \text{Map}_0(\Sigma, M) \to \Omega^1(\Sigma) \big/ d\Omega^0(\Sigma) \]

for any one form \( \alpha \in \Omega^1(\Sigma) \) satisfying \( d\alpha = f^*\omega \). Note that the dual of the Lie algebra of \( \text{Diff}(\Sigma, \sigma) \) can be naturally identified with \( \Omega^1(\Sigma) \big/ d\Omega^0(\Sigma) \). In particular \( \mu^{-1}(0) \) consists of isotropic embeddings of \( \Sigma \) in \( M \). Therefore the symplectic quotient

\[ \text{Map}(\Sigma, M) // \text{Diff}(\Sigma, \sigma) = \mu^{-1}(0) / \text{Diff}(\Sigma) \]

is almost the same as the moduli space of isotropic submanifolds in \( M \), which is \( \mu^{-1}(0) // \text{Diff}(\Sigma) \). Observe that the definition of the map \( \mu \) is independent of the choice of \( \sigma \) and therefore \( \mu^{-1}(0) \) is preserved by the action of \( \text{Diff}(\Sigma) \).

Remark: For the moment map \( \mu \) to be well-defined, the condition \( b_1(\Sigma) = 0 \) is necessary. However even in the case of \( b_1(\Sigma) \neq 0 \), which always happens when \( M \) is a Calabi-Yau threefold, there is modification of the symplectic quotient construction and we can obtain a symplectic manifold which is a torus bundle over the moduli space of isotropic submanifolds in \( M \) with fiber dimension \( b_1(\Sigma) \).

In order to construct a holomorphic symplectic manifold, we first consider a complex closed 2-form \( \Omega_{\text{Map}} \) on \( \text{Map}(\Sigma, M) \) induced from the holomorphic volume form \( \Omega_M \) on \( M \) by transgression.

\[ \Omega_{\text{Map}} = \int_{\Sigma} ev^* \Omega_M = \int_{\Sigma} ev^* \text{Re} \Omega_M + \sqrt{-1} \int_{\Sigma} ev^* \text{Im} \Omega_M = \omega_I - \sqrt{-1} \omega_K, \]

where \( \omega_I = \int_{\Sigma} ev^* (\text{Re} \Omega_M) \) and \( \omega_K = -\int_{\Sigma} ev^* (\text{Im} \Omega_M) \). We also define endomorphisms \( I \) and \( K \) on \( T_f(\mu^{-1}(0)) \) as follows:

\[ \omega_I(A, B) = g(IA, B), \quad \omega_K(A, B) = g(KA, B) \]

where \( A, B \in T_f(\mu^{-1}(0)) \) and \( g \) is the natural \( L^2 \)-metric on \( \mu^{-1}(0) \). Both \( \omega_I \) and \( \omega_K \) are degenerated two forms. We will define the isotropic knot space as the symplectic reduction of \( \mu^{-1}(0) \) with respect to either \( \omega_I \) or \( \omega_K \) and show that it has a natural holomorphic symplectic structure.

To do that, we define a distribution \( D \) on \( \mu^{-1}(0) \) by

\[ D_f = \{ X \in T_f(\mu^{-1}(0)) \subset \Gamma(\Sigma, f^*(TM)) \mid i_X \omega_I, f = 0 \} \subset T_f(\mu^{-1}(0)), \]

for any \( f \in \mu^{-1}(0) \).

**Lemma 36** For any \( f \in \mu^{-1}(0) \subset \text{Map}(\Sigma, M) \), we have,

\[ D_f = \Gamma(\Sigma, T\Sigma + JT\Sigma). \]
Proof. First, it is easy to see that $D_f \supset \Gamma (\Sigma, T_{\Sigma})$ since for any $X$ in $\Gamma (\Sigma, T_{\Sigma})$ and any $Y$ in $\Gamma (\Sigma, T_{\Sigma})$, $\iota_X \wedge Y \Re \Omega_M$ can not be a top degree form on $\Sigma$. By similar reasons, $D_f$ is preserved by the hermitian complex structure $J$ on $M$. Because $f$ is isotropic, we have $D_f \supset \Gamma (\Sigma, T_{\Sigma} + JT_{\Sigma})$. Now, as in lemma 32, we consider localization of $0 = \iota_X \omega_{I,f} = \int_{\Sigma} \iota_X e^\sigma \Re \Omega_M$, at $x$ in $\Sigma$ by varying $\Sigma$ and we obtain

$$0 = \iota_{x(x)} \wedge E_1 \wedge \ldots \wedge E_{n-2} \Re \Omega_M$$

where $E_1, \ldots, E_{n-2}$ is an orthonormal oriented basis of $(T_{\Sigma})_x$. This implies that $D_f = \Gamma (\Sigma, T_{\Sigma} + JT_{\Sigma})$, because for any $X$ in $T_M \setminus (T_{\Sigma} + JT_{\Sigma})$, there is $x$ in $\Sigma$ such that $\iota_X E_1 \wedge \ldots \wedge E_{n-2} \Re \Omega_M \neq 0$. Note that the same construction applied to $\omega_{K}$, instead of $\omega_{I}$, will give another distribution which is identical to $D_f$ because $f$ is isotropic. □

Observe that, for each $f \in \mu^{-1}(0)$, the rank of the subbundle $T_{\Sigma} + JT_{\Sigma}$ in $f^*T_M$ is $2(n-2)$, i.e. constant rank on $\mu^{-1}(0)$. That is $\omega_f$ is a closed two form of constant rank on $\mu^{-1}(0)$. From the standard theory of symplectic reduction, the distribution $D$ is integrable and the space of leaves has a natural symplectic form descended from $\omega_f$. We call this space as the isotropic knot space of $M$ and we denote it as

$$\hat{K}_{\Sigma}M = \mu^{-1}(0)/\langle D \rangle$$

where $\langle D \rangle$ are equivalence relations generated by the distribution $D$.

Remark: The isotropic knot space $\hat{K}_{\Sigma}M$ is a quotient space of $\mu^{-1}(0)/Diff (\Sigma)$, the space of isotropic submanifolds in $M$. If $f : \Sigma \rightarrow M$ parametrizes an isotropic submanifold in $M$, then deforming $\Sigma$ along $T_{\Sigma}$ directions simply changes the parametrization of the submanifold $f (\Sigma)$, namely the equivalence class in $\mu^{-1}(0)/Diff (\Sigma)$ remains unchanged. However if we deform $\Sigma$ along $T_{\Sigma} \otimes \mathbb{C}$ directions, then their equivalence classes in $\hat{K}_{\Sigma}M$ remains constant. Notice that the isotropic condition on $\Sigma$ implies that $T_{\Sigma} \otimes \mathbb{C} \cong T_{\Sigma} \cong J (T_{\Sigma}) \subset f^*T_M$. Therefore, roughly speaking, $\hat{K}_{\Sigma}M$ is the space of isotropic submanifolds in $M$ divided by $Diff (\Sigma) \otimes \mathbb{C}$. In particular, the tangent space of $\hat{K}_{\Sigma}M$ is given by

$$T_{[f]} (\hat{K}_{\Sigma}M) = \Gamma (\Sigma, f^*(TM) / (T_{\Sigma} + JT_{\Sigma}))$$

for any $[f]$ in $\hat{K}_{\Sigma}M$. Note a leaf is preserved by the induced almost Hermitian structure on $\mu^{-1}(0)$ from the Hermitian complex structure $J$ on $M$, because a tangent vector on it is a section to the subbundle $T_{\Sigma} + JT_{\Sigma}$. However $\hat{K}_{\Sigma}M$
may not be symplectic, because the symplectic form \(\omega_{Map}\) on \(\mu^{-1}(0)\) descends to a 2-form \(\omega^K\) on \(\hat{K}_\Sigma M\) which may not be closed.

We will show that \(\hat{K}_\Sigma M\) admits three almost complex structures \(I^K, K^K\) and \(J^K\) satisfying the Hamilton relation

\[(I^K)^2 = (J^K)^2 = (K^K)^2 = I^K J^K K^K = -id.\]

Furthermore the associated Kähler forms \(\omega^K_I\) and \(\omega^K_K\) are closed. In [12], Hitchin showed that existence of such structures on any finite dimensional Riemannian manifold implies that its almost complex structure \(J^K\) is integrable. Namely we obtain a 1-fold \(\mathbb{C}\)-VCP on a Hermitian complex manifold. If, in addition, \(\omega^K_J\) is closed then \(I^K, J^K\) and \(K^K\) are all integrable complex structures and we have a hyperkähler manifold.

We begin with the following lemma which holds true both on \(\mu^{-1}(0)\) and on \(\hat{K}_\Sigma M\).

**Lemma 37** On \(\mu^{-1}(0) \subset Map(\Sigma, M)\), the endomorphisms \(I, J\) and \(K\) satisfy the following relations,

\[I J = - J I \quad \text{and} \quad K J = - J K,\]

\[I = - K J \quad \text{and} \quad K = I J.\]

Proof: The formula \(I J = - J I\) can be restated as follows: for any \(f \in \mu^{-1}(0)\) and for any tangent vectors of \(Map(\Sigma, M)\) at \(f\), \(A, B \in \Gamma(\Sigma, f^*(T_M))\) we have

\[g_f (I_f J_f (A), B) = g_f (- J_f I_f (A), B).\]

For simplicity, we ignore the subscript \(f\). Since \(J^2 = -id\), we have

\[g (IJ (A), B) - g (- JI (A), B)\]

\[= g (IJ (A), B) - g (I (A), JB)\]

\[= \omega_f (JA, B) - \omega_f (A, JB)\]

\[= \int_\Sigma \iota_{J A A^\flat B} \Re \Omega_M - \int_\Sigma \iota_{A A^\flat J B} \Re \Omega_M\]

\[= \int_\Sigma \iota_{(J A A^\flat B - A A^\flat J B)} \frac{1}{2} (\Omega_M + \bar{\Omega}_M) = 0.\]

The last equality follows from the fact that \(\Omega_M\) and \(\bar{\Omega}_M\) vanish when each is contracted by an element in \(\Lambda^{1,1}T_M\). By replacing \(\Re \Omega_M\) with \(\Im \Omega_M\) in the above calculations, we also have \(K J = - J K\).

To prove the others formulas, we consider
therefore the above equality is equivalent to

\[ g(IA, B) - g(-KJA, B) = \omega_I(A, B) + \omega_K(JA, B) \]

\[ = \int_{\Sigma} \iota_{A\wedge B} \Re \Omega_M + \int_{\Sigma} \iota_{JA\wedge B} (-\Im \Omega_M) \]

\[ = \int_{\Sigma} \iota_{A\wedge B} \frac{1}{2} (\Omega_M + \bar{\Omega}_M) - \int_{\Sigma} \iota_{JA\wedge B} \frac{-\sqrt{-1}}{2} (\Omega_M - \bar{\Omega}_M) \]

\[ = \frac{1}{2} \int_{\Sigma} \iota_{(A + \sqrt{-1}JA)\wedge B} \Omega_M + \frac{1}{2} \int_{\Sigma} \iota_{(A - \sqrt{-1}JA)\wedge B} \Omega_M = 0 \]

since \( \iota_{(A + \sqrt{-1}JA)} \Omega_M = 0 \) and taking complex conjugation. This implies that \( g(IA, B) - g(-KJA, B) = 0 \) and therefore \( I = -KJ \). Finally \( J^2 = -Id \) and \( I = -KJ \) imply that \( K = JI \). Hence the results. \( \blacksquare \)

**Theorem 38** Suppose that \( M \) is a Calabi-Yau \( n \)-fold. For any \( n - 2 \) dimensional closed manifold \( \Sigma \) with \( b_1(\Sigma) = 0 \), the isotropic knot space \( \hat{K}_\Sigma M \) is an infinite dimensional integrable complex manifold with a natural 1-fold \( \mathbb{C} \)-VCP structure, in particular a natural holomorphic symplectic structure.

**Proof.** From the construction of \( \hat{K}_\Sigma M \), it has a Hermitian Kähler form \( \omega^K \) induced from that of \( M \) and a closed holomorphic sympletic form \( \Omega^K \) given by the transgression of the closed \((n - 1)\)-fold \( \mathbb{C} \)-VCP form \( \Omega_M \) on \( M \). As we have seen above, the induced holomorphic sympletic form is closed but the induced Hermitian Kähler form may not be closed. If their corresponding almost complex structures satisfy the Hamilton relation then this implies that \( J^K \) is integrable [12]. Namely \( \hat{K}_\Sigma M \) is a Hermitian integrable complex manifold with a 1-fold \( \mathbb{C} \)-VCP structure. In order to verify the Hamilton relation,

\[(I^K)^2 = (J^K)^2 = (J^K) = i^K J^K K^K = -Id,\]

we only need to show that \((I^K)^2 = -Id\) and \((K^K)^2 = -Id\) because of the previous lemma. Namely, \( I^K \) and \( K^K \) are almost complex structures on \( \hat{K}_\Sigma M \).

We consider a fixed \([f]\) in \( \hat{K}_\Sigma M \) and by localization method as in the proof of lemma 33 we can reduce the identities to the tangent space of a point \( x \) in \( \Sigma \). The transgression

\[ \int_{\Sigma} ev^* \Omega_M = \omega^K_{I,[f]} - \sqrt{-1} \omega^K_{K,[f]} \]

is descended to

\[ \iota_{E_1 \wedge \ldots \wedge E_{n-2}} \Omega_M = \omega^K_{I,[f],x} - \sqrt{-1} \omega^K_{K,[f],x} \]

where \( E_1, \ldots, E_{n-2} \) is an orthonormal oriented basis \((T_\Sigma)_x\). Since \( f \) is isotropic, the complexified vectors \( E_i - \sqrt{-1}JE_i \) of \( E_i \) can be defined over \((T_\Sigma + JT_\Sigma)_x\), therefore the above equality is equivalent to

\[ \iota_{(E_1 - \sqrt{-1}JE_1)/2 \wedge \ldots \wedge (E_{n-2} - \sqrt{-1}JE_{n-2})/2} \Omega_M = \omega^K_{I,[f],x} - \sqrt{-1} \omega^K_{K,[f],x} \]
i.e. a 1-fold C-VCP on $f^*(TM) / (T\Sigma + JT\Sigma) |_x$ which is $T_{[f],x} \hat{K}_\Sigma M$. Since this 1-fold C-VCP gives a hyperkähler structure on $T_{[f],x} \hat{K}_\Sigma M$, $T^R_{[f]}$ and $K^K_{[f]}$ satisfy the Hamilton relation at $x$ in $\Sigma$. That is

$$
\left( T^R_{[f],x} \right)^2 = -\text{Id} \quad \text{and} \quad \left( K^K_{[f],x} \right)^2 = -\text{Id}.
$$

Therefore we have $(T^R)^2 = -\text{Id}$ and $(K^K)^2 = -\text{Id}$. Hence the result. ■

**Remark 39** In String theory we need to compactify a ten dimensional spacetime on a Calabi-Yau threefold. When $M$ is a Calabi-Yau threefold, then $\Sigma$ is an one dimensional circle and therefore $b_1(\Sigma)$ is nonzero. In general, as discussed in [7] and [11], the symplectic quotient construction for $\text{Map}(\Sigma, M) // \text{Diff}(\Sigma, \sigma)$ can be modified to obtain a symplectic structure on a rank $b_1(\Sigma)$ torus bundle over the space of isotropic submanifolds in $M$. Roughly speaking this torus bundle is the space of isotropic submanifolds coupled with flat rank one line bundles (or gerbes) in $M$. In the Calabi-Yau threefold case, every circle $\Sigma$ in $M$ is automatically isotropic. Therefore $\hat{K}_\Sigma M$ is the space of loops (or string) coupled with flat line bundles in $M$, up to deformations of strings along their complexified tangent directions. We wonder whether this infinite dimensional holomorphic symplectic manifold $\hat{K}_\Sigma M$ has any natural physical interpretations.

### 5.2 Complex Lagrangians in Isotropic Knot Spaces

In this subsection we relate the geometry of C-VCP of a Calabi-Yau manifold $M$ to the holomorphic symplectic geometry of its isotropic knot space $\hat{K}_\Sigma M$. For example both N-branes (i.e. complex hypersurfaces) and D-branes (i.e. special Lagrangian submanifold with phase $-\pi/2$) in $M$ correspond to complex Lagrangian submanifold in $\hat{K}_\Sigma M$, but for different almost complex structures in the twistor family. First we discuss the correspondence for instantons.

In the following proposition, we use $e^{i\theta} \Omega_M$ instead of $\Omega_M$ to get a 1-fold C-VCP on $\hat{K}_\Sigma M$, and also $\text{Re}\left(e^{i\theta} \Omega_M\right)$ gives another symplectic form $\omega^R_{K,\theta}$ and corresponding Hermitian almost complex structure $J^K_{\theta}$ on $\hat{K}_\Sigma M$ defined as $J^K_{\theta} = \cos \theta T^R + \cos \theta K^K$.

**Proposition 40** Suppose that $M$ is a Calabi-Yau $n$-fold. Let $D$ be a normal disk in $\mu^{-1}(0)$ with an $n$-dimensional submanifold $A$ defined as $A = D \times \Sigma$, and assume $A \to D$ is a Riemannian submersion. We denote the reduction of $D$ in $\hat{K}_\Sigma M$ as $\hat{D}$.

Then $\hat{D}$ is an instanton i.e. a $J^K_{\theta}$-holomorphic curve in $\hat{K}_\Sigma M$ if and only if $A$ is an instanton i.e. a special Lagrangian with phase $\theta$ in $M$.

**Proof.** In the proposition VCP form $\phi$ plays a role of calibrating form rather than that of a VCP form. So by replacing $\phi$ by $\text{Re}\left(e^{i\theta} \Omega_M\right)$ in proposition readers can see this theorem can be proved in essentially the same manner. But since proposition is given for the parametrized knot space and this theorem is given by a reduction, we need to check $\hat{D}$ is a disk in $\hat{K}_\Sigma M$. Let $f$ be
the center of disk $D$. Since $D$ is a normal disk, a tangent vector $v$ at $f$ along $D \subset \mu^{-1}(0)$ is in $\Gamma(\Sigma, N_{\Sigma/M})$. And since $f$ is isotropic, $Jv$ is also in $\Gamma(\Sigma, N_{\Sigma/M})$ equivalently $v$ is in $\Gamma(\Sigma, N_{f^*TM})$. So $v \in \Gamma(\Sigma, f^*(TM)/(T\Sigma + JT\Sigma))$. This implies $T_f D$ can be identified with $T_{f|\hat{D}}$, i.e. $\hat{D}$ is a disk in $\hat{K}_\Sigma M$. ■

Next we are going to relate $N$- and $D$-branes in Calabi-Yau manifolds $M$ to complex Lagrangian submanifolds in $\hat{K}_\Sigma M$.

**Definition 41** For any submanifold $C$ in a Calabi-Yau manifold $M$, we define a subspace $\hat{K}_\Sigma C$ in $\hat{K}_\Sigma M = \mu^{-1}(0) / \langle D \rangle$ as follow: The equivalent relation $\langle D \rangle$ on $\mu^{-1}(0)$ restrict to one on $\text{Map}(\Sigma, C) \cap \mu^{-1}(0)$ and we define

$$
\hat{K}_\Sigma C = \{ \text{Map}(\Sigma, C) \cap \mu^{-1}(0) \} / \langle D \rangle.
$$

In the following theorem, we see that $C$ being a $N$-brane in $M$ corresponds to $\hat{K}_\Sigma C$ being a $J^K$-complex Lagrangian in $\hat{K}_\Sigma M$ which means that $\hat{K}_\Sigma C$ is both maximally self $\omega^K$-perpendicular and maximally self $\omega^K$-perpendicular in $\hat{K}_\Sigma M$.

**Theorem 42** Let $C$ be a connected analytic submanifold in a Calabi-Yau manifold $M$. Then the following two statements are equivalent:

(i) $C$ is an $N$-brane (i.e. a complex hypersurface) in $M$;

(ii) $\hat{K}_\Sigma C$ is a $J^K$-complex Lagrangian submanifold in $\hat{K}_\Sigma M$.

**Proof:** For the if part, we assume that $\hat{K}_\Sigma C$ is a $J^K$-complex Lagrangian in $\hat{K}_\Sigma M$. First we need to show that the dimension of $C$ is at least $2n - 2$ where $n$ is the complex dimension of $M$.

Before proving this claim for the general case, let us discuss a simplified linear setting where some of the key arguments become more transparent. Suppose that $M$ is a linear Calabi-Yau manifold, that is $M \cong \mathbb{C}^n$ with the standard Kähler structure and $\Omega_M = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n$. Let $\Sigma$ be a $(n-2)$-dimensional isotropic linear subspace in $M$ lying inside another linear subspace $C$ in $M$. For simplicity we assume that $\Sigma$ is the linear span of $x^1, x^2, \ldots, x^{n-2}$. Of course $M/\Sigma \oplus J\Sigma \cong \mathbb{C}^2$ is a linear holomorphic symplectic manifold with

$$
\int_{\Sigma} \Omega_M = d\bar{z}^{n-1} \wedge dz^n,
$$

which is the standard holomorphic symplectic form on $\mathbb{C}^2$. Suppose that

$$
C/ (\Sigma \oplus J\Sigma) \cap C \subset M/\Sigma \oplus J\Sigma
$$

is a complex Lagrangian subspace. Then there is a vector in $C$ perpendicular to both $\Sigma$ and $J\Sigma$, say $x^{n-1}$. If we denote the linear span of $x^2, \ldots, x^{n-2}, x^{n-1}$ as $\Sigma'$, then $\Sigma'$ is another $(n-2)$-dimensional isotropic linear subspace in $M$ lying inside $C$. Furthermore $x^1$ is a normal vector in $C$ perpendicular to $\Sigma' \oplus J\Sigma'$. This implies that $y^1 = Jx^1$ also lie in $C$. This is because $C/ (\Sigma' \oplus J\Sigma') \cap C \subset \mathbb{C}^2$.

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$M/\Sigma' \oplus J\Sigma'$ being a complex Lagrangian subspace implies that it is invariant under the complex structure on $M/\Sigma' \oplus J\Sigma'$ induced by $J$ on $M$. By the same reasoning, $y^j$ also lie in $C$ for $j = 1, 2, \ldots, n - 2$. That is $C$ contains the linear span of $\{x^j, y^j\}_{j=1}^{n-2}$. On the other hand it also contain $x^{n-1}$ and $y^{n-1}$ and therefore $\dim C \geq 2n - 2$.

We come back to the general situation where $\hat{K}_\Sigma C$ is a $J^K$-complex Lagrangian in $\hat{K}_\Sigma M$. One difficulty is to rotate the isotropic submanifold $\Sigma$ to $\Sigma'$ inside $M$.

We observe that the tangent space to $\hat{K}_\Sigma C$ at any point $[f]$ is given by

$$T_{[f]}(\hat{K}_\Sigma C) = \Gamma \left( \Sigma, \frac{f^*T_C}{T_\Sigma \oplus JT_\Sigma \cap f^*T_C} \right).$$

Since any $J^K$-complex Lagrangian submanifold is indeed an integrable complex submanifold, $T_{[f]}(\hat{K}_\Sigma C)$ is preserved by $J^K$. This implies that the complex structure on $T_M$ induces a complex structure on the quotient bundle $f^*T_C/[T_\Sigma \oplus JT_\Sigma \cap f^*T_C]$. For any $\nu \in T_{[f]}(\hat{K}_\Sigma C)$ we can regard it as a section of $f^*T_C$ over $\Sigma$, perpendicular to $T_\Sigma$ and $JT_\Sigma \cap f^*T_C$. In particular $J\nu$ is also such a section.

We need the following lemma which will be proven later.

**Lemma 43** In the above situation, we have $JT_\Sigma \subset f^*T_C$.

This lemma implies that the restriction of $T_C$ on $\Sigma$ contains the linear span of $T_\Sigma$, $JT_\Sigma$, $\nu$ and $J\nu$, which has rank $2n - 2$. (More precisely we consider the restriction to the complement of the zero set of $\nu$ in $\Sigma$.) Therefore $\dim C \geq 2n - 2$.

On the other hand, if $\dim C > 2n - 2$, then by using localizing method as in lemma 22, $\hat{K}_\Sigma C$ would be too large to be a complex Lagrangian submanifold in $\hat{K}_\Sigma M$. That is $\dim C = 2n - 2$.

In particular, this implies that $f^*T_C$ is isomorphic to the linear span of $T_\Sigma$, $JT_\Sigma$, $\nu$ and $J\nu$ outside the zero set of $\nu$. Therefore $T_C$ is preserved by the complex structure of $M$ along $\Sigma$. By varying the isotropic submanifold $\Sigma$ in $C$, we can cover an open neighborhood of $\Sigma$ in $C$ (for example using the gluing arguments as in the proof of the above lemma). This implies that an open neighborhood of $\Sigma$ in $C$ is a complex submanifold in $M$. By the analyticity of $C$, the submanifold $C$ is a complex hypersurface in $M$.

For the only if part, we suppose $C$ is complex hypersurface in $M$, then it is clear that $\omega_T^K$ and $\omega_K^K$ vanish along $\hat{K}_\Sigma C$ since $\Omega_M$ vanishes along $C$. Using similar arguments as in the proof of proposition 34, it is not difficult to verify that $\hat{K}_\Sigma C$ is maximally self $\omega_T^K$-perpendicular and maximally self $\omega_K^K$-perpendicular in $\hat{K}_\Sigma M$. That is $\hat{K}_\Sigma C$ is a $J^K$-complex Lagrangian submanifold in $\hat{K}_\Sigma M$. Hence the theorem. \(\blacksquare\)

We suspect that the analyticity assumption on $C$ is unnecessary. All we need is to deform the isotropic submanifold $\Sigma$ of $M$ inside $C$ to cover every point in $C$. 

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Proof of lemma: For any tangent vector $u$ of $\Sigma$ at a point $p$, i.e. $u \in T_p\Sigma \subset T_pC$, we need to show that $Ju \in T_pC$. We can assume that $\nu(p)$ has unit length. First we choose local holomorphic coordinate $z^1, z^2, \ldots, z^n$ near $p$ satisfying the following properties at $p$:

\[ z^i(p) = 0 \text{ for all } i = 1, \ldots, n, \]

\[ \Omega_M(p) = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n, \]

$T_p\Sigma$ is spanned by $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^{n-2}}$,

$\nu(p) = \frac{\partial}{\partial x^{n-1}}$ and $u = \frac{\partial}{\partial x^1}$.

where $z^j = x^j + iy^j$ for each $j$. We could use $x^1, \ldots, x^{n-2}$ to parametrize $\Sigma$ near $p$.

Recall that $\nu \in T_{[f]}(\hat{K}_\Sigma C) = \Gamma \left( \Sigma, \frac{f^*T_C}{T_\Sigma + JT_\Sigma \cap f^*T_C} \right)$.

For any smooth function $\alpha (x^1, \ldots, x^{n-2})$ on $\Sigma$, $\alpha \nu$ is again a tangent vector to $\hat{K}_\Sigma C$ at $[f]$. We are going to construct a particular $\alpha$ supported on a small neighborhood of $p$ in $\Sigma$ satisfying

\[ \frac{\partial \alpha}{\partial x^1}(0) = \infty, \]

\[ \frac{\partial \alpha}{\partial x^j}(0) = 0, \text{ for } j = 2, \ldots, n-2. \]

For simplicity we assume that this small neighborhood contain the unit ball in $\Sigma$. To construct $\alpha$, we fix a smooth even cutoff function $\eta : \mathbb{R} \rightarrow [0, 1]$ with $\text{supp}(\eta) \in (-1, 1), \eta(0) = 1$ and $\eta'(0) = 0$. We define $\alpha : \Sigma \rightarrow \mathbb{R}$ as follow:

\[ \alpha (x^1, x^2, \ldots, x^{n-2}) = (x^1)^{1/3} \cdot \eta(x^1) \cdot \eta \left( \sum_{j=2}^{n-2} (x^j)^2 \right). \]

For any small real number $\varepsilon$, we write

\[ \nu_\varepsilon = \varepsilon \alpha \nu \in T_{[f]}(\hat{K}_\Sigma C) \]

and we denote the corresponding family of isotropic submanifolds of $M$ in $C$ by $f_\varepsilon : \Sigma_\varepsilon \rightarrow M$. From $\alpha(0) = 0$, there is a family of points $p_\varepsilon \in \Sigma_\varepsilon$ with the property that

\[ |p_\varepsilon - p| = O \left( \varepsilon^2 \right). \]

Using the property $\partial \alpha / \partial x^1(0) = \infty$, we can find a family of normal vectors at $p_\varepsilon$,

\[ u_\varepsilon \in N_{\Sigma_{\varepsilon}/C}, \]
with the property
\[ |u_{\varepsilon} - \frac{\partial}{\partial x^1}| = O(\varepsilon^2). \]

Since \( T_{[f,]}(\hat{K}_{\Sigma}C) \) is preserved by \( f^\Sigma, f^\Sigma T_C / [T_{\Sigma} \oplus JT_{\Sigma} \cap f^\Sigma T_C] \) is preserved by \( J \), we have \( Ju_{\varepsilon} \in T_{p_{\varepsilon}}C \). By letting \( \varepsilon \) goes to zero, we have \( Ju \in T_{p}C \). Hence the lemma. \( \blacksquare \)

In the following theorem, we see that relationship between a \( D \)-brane \( L \) in \( M \) with respect to \( e^{i\theta} \Omega_M \) and a \( J_{\theta + \frac{\pi}{2}} \) complex Lagrangian \( \hat{K}_{\Sigma}L \) in \( \hat{K}_{\Sigma}M \), i.e. it is maximally self \( \omega^\Sigma \)-perpendicular and maximally self \( \omega^\Sigma \)-perpendicular.

**Theorem 44** Let \( L \) be a connected analytic submanifold of a Calabi-Yau manifold \( M \), then the following two statements are equivalent:

(i) \( L \) is a \( D \)-brane with phase \( \theta \), (i.e. a special Lagrangian with phase \( \theta - \frac{\pi}{2} \));

(ii) \( \hat{K}_{\Sigma}L \) is a \( J_{\theta + \frac{\pi}{2}} \)-complex Lagrangian submanifold in \( \hat{K}_{\Sigma}M \).

**Proof:** By replacing the holomorphic volume form on \( M \) from \( \Omega_M \) to \( e^{i\theta} \Omega_M \) if necessary, we can assume that \( \theta \) is zero.

For the only if part, we assume that \( L \) is a special Lagrangian submanifold in \( M \) with phase \( -\pi/2 \). Because the Kähler form \( \omega \) and \( \operatorname{Re} \Omega_M \) of \( M \) vanish along \( L \), it is clear that \( \omega^K \) and \( \omega_{T,0}^\Sigma \) vanishes along \( \hat{K}_{\Sigma}L \). Notice that \( L \) being a Lagrangian submanifold in \( M \) implies that any submanifold in \( L \) is automatically isotropic in \( M \), and moreover the equivalent relation \( \langle D \rangle \) on \( \mu^{-1}(0) \) restricted to \( \operatorname{Map}(\Sigma, L) \) is trivial locally. That is \( \hat{K}_{\Sigma}L \) is the same as \( K_{\Sigma}L = \operatorname{Map}(\Sigma, L) / \operatorname{Diff}(\Sigma) \) at least locally.

We claim that \( \hat{K}_{\Sigma}L \) is maximally self \( \omega^K \)-perpendicular in \( \hat{K}_{\Sigma}M \). Otherwise there is normal vector field \( \nu \in \Gamma(\Sigma, N_{\Sigma/M}) \) not lying in \( \Gamma(\Sigma, f^\Sigma T_S) \) such that \( \omega(\nu, u) = 0 \) for any \( u \in \Gamma(\Sigma, f^\Sigma T_S) \). Suppose that \( \nu(p) \notin N_{\Sigma/L,p} \), then this implies that \( \omega \) vanishes on the linear span of \( T_pL \) and \( \nu(p) \) inside \( T_pM \). This is impossible because the dimension of the linear span is bigger than \( n \).

Similarly \( \hat{K}_{\Sigma}L \) is maximally self \( \omega_{T,0}^\Sigma \)-perpendicular in \( \hat{K}_{\Sigma}M \). Otherwise \( \operatorname{Re} \Omega_M \) vanishes on the linear span of \( T_pL \) and \( \nu(p) \) inside \( T_pM \). This is again impossible because \( \Omega_{M,p} \) is a complex volume form on \( T_pM \cong \mathbb{C}^n \) and therefore can not vanish on any co-isotropic subspaces other than Lagrangians. Hence \( \hat{K}_{\Sigma}L \) is a \( J_{\pi/2} \)-complex Lagrangian in \( \hat{K}_{\Sigma}M \).

For the if part, we assume that \( \hat{K}_{\Sigma}L \) is a \( K^K \)-complex Lagrangian submanifold in \( \hat{K}_{\Sigma}M \). For any \( [f] \in \hat{K}_{\Sigma}L \), the tangent space of \( \hat{K}_{\Sigma}L \) (resp. \( \hat{K}_{\Sigma}M \)) at \( [f] \) is the section of the bundle \( f^\Sigma T_S / T_S \oplus JT_S \cap f^\Sigma T_S \) (resp. \( f^\Sigma (T_M) / (T_S + JT_S) \)) over \( \Sigma \). Let \( v \in T_{[f]}(\hat{K}_{\Sigma}L) \), we can regard \( v \) as a section of \( f^\Sigma T_L \) over \( \Sigma \), orthogonal to \( T_S \oplus JT_S \cap f^\Sigma T_L \).

Since \( f^\Sigma (T_M) / (T_S + JT_S) \) is a rank four bundle and \( \hat{K}_{\Sigma}L \) is a Lagrangian in \( \hat{K}_{\Sigma}M \), this implies that \( f^\Sigma T_L / T_S \oplus JT_L \cap f^\Sigma T_L \) must be a rank two bundle over \( \Sigma \). Therefore
\[
\dim L \geq n,
\]

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with the equality sign holds if and only if $JT_{\Sigma} \cap f^*T_L$ is trivial. Suppose that $\nu$ is a tangent vector of $K_{\Sigma}L$ at $f$

Assume that $JT_{\Sigma} \cap f^*T_L$ is not trivial, we can find a tangent vector $u$ to $f(\Sigma)$ at a point $p$ such that $Ju \in T_pL$. For simplicity we assume that $\nu$ has unit length at the point $p$.

As in the proof of the previous theorem, we can choose local holomorphic coordinates $z^i$'s of $M$ around $p$ such that

$z^i (p) = 0$ for all $i = 1, \ldots, n$,

$\Omega_M (p) = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n$,

$T_p\Sigma$ is spanned by $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^{n-2}},$

$\nu (p) = \frac{\partial}{\partial x^{n-1}}, u = \frac{\partial}{\partial x^1}$ and $Ju = \frac{\partial}{\partial y^1}$.

We choose a function $\alpha (x)$ as in the proof of the previous theorem, write

$\nu_\varepsilon = \varepsilon \alpha \nu \in T_{\hat{f}[\hat{K}_{\Sigma}L]}$ for any small real number $\varepsilon$, and we denote the corresponding family of isotropic submanifolds of $M$ in $L$ by $f_\varepsilon : \Sigma_\varepsilon \rightarrow M$ as before. From $\alpha (0) = 0$, there is a family of points $p_\varepsilon \in \Sigma_\varepsilon$ with the property that

$|p_\varepsilon - p| = O (\varepsilon^2)$.

Using the property $\partial \alpha / \partial x^1 (0) = \infty$, we can find two family of normal vectors at $p_\varepsilon$,

$u_\varepsilon, w_\varepsilon \in N_{\Sigma_\varepsilon / L},$

with the property

$\left| u_\varepsilon - \frac{\partial}{\partial x^1} \right| = O (\varepsilon^2)$ and $\left| w_\varepsilon - \frac{\partial}{\partial y^1} \right| = O (\varepsilon^2)$.

However, using localization arguments as before, $K_{\Sigma}L$ being an $\omega^K$-Lagrangian submanifold in $K_{\Sigma}M$ implies that $\omega (u_\varepsilon, w_\varepsilon) = 0$. By letting $\varepsilon$ goes to zero, we have $0 = \omega (\partial / \partial x^1, \partial / \partial y^1) = -1$, a contradiction. Hence $JT_{\Sigma} \cap f^*T_L$ is trivial and $\dim L = n$.

Since $\Gamma (\Sigma, f^*T_L / T_{\Sigma} \oplus JT_{\Sigma} \cap f^*T_L)$ is preserved by $K^K$, we have $K^K \nu (p) \in f^*T_L$. Note that $K^K \nu (p)$ is the tangent vector of $M$ which is the metric dual of the one form,

$\nu \omega^K_K (p) = \left( \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-2}} \wedge \nu \right) \wedge \text{Im} \Omega_M (p)$

$= \pm dy^n$,
since $\nu(p) = \partial/\partial x^{n-1}$. This implies that $T_pL$ is spanned by $\partial/\partial x^j$’s for $j = 1, 2, ..., n - 1$ and $\partial/\partial y^n$, that is a special Lagrangian subspace of phase $-\pi/2$.

As in the proof of the previous theorem, by deforming the isotropic submanifold $\Sigma$ in $L$ and using the fact that $L$ is an analytic submanifold of $M$, we conclude that $L$ is a special Lagrangian submanifold in $M$ with zero phase.

Hence the theorem. ■

6 Concluding Remarks

In this paper we study both real and complex vector cross products (VCP). Instantons in either settings are calibrated submanifolds. This gives a unified way to explain the calibrating property of many such examples, as studied by Harvey and Lawson in [10]. It is desirable to further study the calibration geometry from this point of view.

Manifolds with real VCP include symplectic/Kähler and $G_2$-manifolds. In section 4.1 we relate the geometry of VCP on the manifolds $M$ to the symplectic geometry of their knot spaces $K_\Sigma M$. Motivated from this relationship, it is natural to study the intersection theory of branes and count the number of instantons bounding them, similar to the Floer’s homology theory of Lagrangian intersections. For example, in the case of $G_2$-manifolds, counting associative submanifolds bounding nearby coassociative submanifolds is closely related to the Seiberg-Witten invariants of the four dimensional coassociative submanifolds [16]. Results along this line should be useful in understanding the M-theory in Physics.

Another interesting problem for VCP is the study of the submanifold geometry of branes. For example one would like to have a unified approach to the mean curvature flow for both hypersurfaces and Lagrangian submanifolds.

For manifolds with $C$-VCP $M$, we classify them in theorem 26, namely they must be either holomorphic volume forms in Calabi-Yau manifolds or holomorphic symplectic forms in hyperkähler manifolds. We study the geometry of instantons, Dirichlet branes and Neumann branes in $M$. In section 5.1 we construct an isotropic knot space $\hat{K}_\Sigma M$ for any Calabi-Yau manifold $M$ and show that it admits a natural holomorphic symplectic structure. We also relate the Calabi-Yau geometry of $M$ to the holomorphic symplectic geometry of $\hat{K}_\Sigma M$. This is particularly interesting when $M$ is a Calabi-Yau threefold.

There are many interesting questions arise from studying the geometry of C-VCPs. For example we would like to interpret the Strominger-Yau-Zaslow mirror transformation for Calabi-Yau manifolds $M$ as the twistor rotation for the holomorphic symplectic manifolds $\hat{K}_\Sigma M$.

Acknowledgments: This paper is partially supported by NSF/DMS-0103355. Authors express their gratitude to S.L. Kong, X.W. Wang for useful discussions.
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Address: School of Mathematics, University of Minnesota, Minneapolis, MN 55454, USA.
Email: JHLEE@MATH.UMN.EDU, LEUNG@MATH.UMN.EDU