LOCALITY PROPERTIES OF STANDARD HOMOGENIZATION COMMUTATOR

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ABSTRACT. In the present work we study how the standard homogenization commutator, a random field that plays a central role in the theory of fluctuations, quantitatively decorrelates on large scales.

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1. Introduction

This work amounts to homogenization theory for uniformly elliptic linear equations in divergence-form. That is, we consider

$$\nabla \cdot \left( a \left( \frac{x}{\varepsilon} \right) \nabla u_{\varepsilon} + f \right) = 0, \quad \text{in} \ \mathbb{R}^d \quad (1.1)$$

with $f \in C^\infty_c(\mathbb{R}^d)$ and $a$ being random (not necessarily symmetric) coefficients that satisfy

$$\xi \cdot a(x) \xi \geq \lambda |\xi|^2, \quad \xi \cdot a^{-1}(x) \xi \geq |\xi|^2 \quad \text{for any} \ x, \xi \in \mathbb{R}^d$$

for some positive constant $\lambda$. In the following we denote by $\langle \cdot \rangle$ the expectation with respect to the underlying measure on $a$’s.

Since the works of Papanicolaou and Varadhan [15] and Kozlov [12] we know that in stationary and ergodic random environments, the equation (1.1) homogenizes as $\varepsilon \to 0$ to an equation

$$\nabla \cdot (\bar{a} \nabla \bar{u} + f) = 0, \quad \text{in} \ \mathbb{R}^d \quad (1.2)$$

where the coefficients $\bar{a}$ are constant and deterministic. More precisely, the effective coefficients $\bar{a}$ are given by $\bar{a} e_i := \langle a(\nabla \phi_i + e_i) \rangle$ where the corrector $\phi_i$ is the (up to a random additive constant) unique a.s. solution of the equation $\nabla \cdot a(\nabla \phi_i + e_i) = 0$ in $\mathbb{R}^d$, with $\nabla \phi_i$ being stationary, centered and having finite second moments. Furthermore, the aforementioned qualitative theory states that $\nabla u_{\varepsilon}$ weakly converges to $\nabla \bar{u}$

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with the oscillations of \( u_\varepsilon \) being captured by those of the so-called two-scale expansion 
\( (1 + \varepsilon \phi_i \left( \frac{x}{\varepsilon} \right) \partial_i) \bar{u} \) in a strong norm.

The quantitative theory of stochastic homogenization for (1.1) (namely, the study of the error for the approximation \( \nabla u_\varepsilon \approx \nabla((1 + \varepsilon \phi_i \left( \frac{x}{\varepsilon} \right) \partial_i) \bar{u}) \)), has been also well-developed during the last decade. For that a suitable quantification of the ergodicity assumption is needed. Most of the developments are based on either a spectral gap inequality or on a finite range of dependence assumption. Here we adopt the spectral gap inequality approach which means that we have a version of Poincaré’s inequality in infinite dimensions. Roughly speaking, we assume that the variance of an observable defined on the space of coefficient fields described above, can be estimated by a suitable norm of its functional derivative with respect to \( a \), which describes the sensitivity of an observable (for instance \( \nabla u_\varepsilon \) or \( \nabla \phi_i \)) under changes on \( a \)’s. This direction of research was initiated by Gloria and Otto in [8] and [9], inspired by the strategy introduced by Nadaff and Spencer in [14] . On the other hand, finite range of dependence and mixing conditions have been introduced by Yurinskii in [16] and further studied by Armstrong and Smart in [3] (we refer the reader to [2] for a detailed description of the progress in this direction).

Next to the spatial oscillations of \( \nabla u_\varepsilon \) described above, stochastic homogenization also studies the random fluctuations of observables of the form \( \int g \cdot \nabla u_\varepsilon \). One of the first results in this direction is given in [10] where the authors show that \( \varepsilon^{-d/2} \int g \cdot (\nabla u_\varepsilon - \langle \nabla u_\varepsilon \rangle) \) converges in law to a Gaussian random variable. In the same work the authors showed that the four-tensor \( Q \) introduced in [13] describes explicitly the leading-order of the variance of \( \varepsilon^{-d/2} \int g \cdot \nabla u_\varepsilon \). Moreover, they observed that the limiting variance of \( \varepsilon^{-d/2} \int g \cdot \nabla u_\varepsilon \) is not captured by that of \( \varepsilon^{-d/2} \int g \cdot \nabla \left((1 + \varepsilon \phi_i \left( \frac{x}{\varepsilon} \right) \partial_i) \bar{u} \right) \) as one would naturally expect. As discovered in [4] (for the random conductance model) and [5] (in the continuum Gaussian setting) a reasonable quantity to look at, when it comes to fluctuations, is the homogenization commutator

\[
\Xi^1_\varepsilon[\nabla u_\varepsilon] := \left( a \left( \frac{\cdot}{\varepsilon} \right) - \bar{a} \right) \nabla u_\varepsilon.
\]

This notion first introduced in [1] and it is highly related to H-convergence which is in fact equivalent to \( \Xi^1_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). The motivation to consider \( \Xi^1_\varepsilon \) while studying fluctuations of \( \int g \cdot \nabla u_\varepsilon \) comes from the following observation: for the Lax-Milgram solution \( \bar{v} \) to the dual equation \( \nabla \cdot (\bar{a}^* \nabla \bar{v} + g) = 0 \) we have

\[
\int g \cdot \nabla u_\varepsilon = - \int \nabla \bar{v} \cdot \bar{a} \nabla u_\varepsilon = \int \nabla \bar{v} \cdot \left( a \left( \frac{\cdot}{\varepsilon} \right) - \bar{a} \right) \nabla u_\varepsilon + \int \nabla \bar{v} \cdot f.
\]

That is, the quantity of interest can be written in terms of homogenization commutator plus a deterministic term (which does not contribute to fluctuations). Subsequently, in [4] and [5] the authors turned their attention to the study of \( \Xi^1_\varepsilon \) to realize that its
fluctuations are captured by those of its two-scale expansion. More precisely, for ː = ∫ g ⋅ [(a (z/ε) − ̄a) ∇u - (a (z/ε) − ̄a) (ei + ∇φ (z/ε)) ∂i ⃗u], it holds (for d ≥ 3)

\[ ε^{-d/2} (⟨\bar{F}_ε - (\tilde{F}_ε)⟩)^{1/2} \lesssim ε. \]

This reveals the special role that the so-called standard homogenization commutator,

\[ Ξ^0,1_ε := (a (z/ε) − ̄a) (Id + ∇φ (z/ε)), \]

plays in the theory of fluctuations. In [4] and [5] the limiting covariance structure of ∫ g(X) − ̄g(ε) is quantitatively characterized and a (quantitative) CLT-type result is obtained for that quantity.

In the present work our aim is to explain how Ξ^0,1 decorrelates when averaged over balls which are far enough. From now on we set, for convenience, ε = 1 and work with macroscopic observables. More precisely, we consider the following macroscopic test functions with supports that are quantitatively “far”,
in the estimation of the r.h.s. of the covariance estimate will be the stochastic moment bounds for the correctors and the large-scale regularity theory.

In addition to the above estimate, we also study how the order of the decay is improved when we employ higher-order homogenization theory. More precisely, we consider the higher-order standard homogenization commutator

\[ \Xi_{ij} := e_j \cdot (a - \bar{a})^1(\nabla \phi_1^1 + e_i) - \sum_{k=2}^{n} (-1)^{k-1} a_{ij_1...i_k-2}^* \partial^{k-1}_{i_1...i_k-1} \nabla \phi_1^1 \] (1.6)

where, \( 1 \leq i, j \leq d \) and \( 1 < n \leq \tilde{d} \) for \( \tilde{d} \) being the smallest integer larger than \( \frac{d}{2} \) and \( a_{ij_1...i_k-2}^* \) the higher-order effective coefficients (see subsection 2.2 for precise definitions).

Moreover, we assume that we are in a Gaussian framework characterized by a covariance function with integrable decay of order \( o(|x|^{-d-\alpha_0}) \), for \( 0 < \alpha_0 \leq \frac{d}{2} \) (see subsection 2.1 for a precise description of the class of ensembles we consider). We derive that the correlation has order \( o\left(R^{-\frac{d}{2}}L^{-\frac{d}{2}-\alpha_0}\right) \) (up to a logarithmic correction) as well. That is, we see that the standard homogenization commutator inherits the property of \( a \)'s being weakly correlated, keeping the order of decay as well. The following is our main result.

**Theorem 1** (Main Theorem). If \( d > 2 \) is odd then

\[ P_{ijml}^{o,\frac{d}{2}} := \left\langle \int_{\mathbb{R}^d} g(x)\Xi_{ij}^{o,\frac{d}{2}}(x) dx \int_{\mathbb{R}^d} g'(y)\Xi_{ml}^{o,\frac{d}{2}}(y) dy \right\rangle \lesssim R^{-\frac{d}{4}}L^{-\frac{d}{4}-\alpha_0} \ln \left(\frac{L}{R}\right). \]

If \( d \geq 2 \) is even then

\[ P_{ijml}^{o,d} := \left\langle \int_{\mathbb{R}^d} g(x)\Xi_{ij}^{o,d}(x) dx \int_{\mathbb{R}^d} g'(y)\Xi_{ml}^{o,d}(y) dy \right\rangle \lesssim R^{-\frac{d}{4}}(\ln R)^{\frac{1}{2}}L^{-\frac{d}{4}-\alpha_0} \ln \left(\frac{L}{R}\right). \]

Here \( g \) and \( g' \) are as described in (1.3), \( L >> R >> 1 \) and \( \alpha_0 \) is so that (2.1) holds.

2. Preliminaries

2.1. Assumptions on the ensemble. We first describe the framework we adopt and the main ingredients we need for our analysis which hold true in this framework.

Let \( \langle \cdot \rangle \) be a stationary and centered Gaussian ensemble of scalar fields \( G \) on \( \mathbb{R}^d \) characterized by its covariance function \( c(x) = \langle G(x)G(0) \rangle \) which is assumed to satisfy the following

\[ |c(x)| \leq \frac{C_0}{(1 + |x|)^{d+\alpha_0}} \] (2.1)

for some constants \( 0 < \alpha_0 \leq \frac{d}{2} \) and \( C_0 > 0 \). Moreover, we assume that the (always non-negative) Fourier transform of \( c \) satisfies

\[ \mathcal{F}c(k) \leq \frac{C_1}{(1 + |k|)^{d+2\alpha_1}} \] (2.2)
for some $\alpha_1 \in (0, 1)$ and some constant $C_1 > 0$. Next we identify $\langle \cdot \rangle$ with its push-forward under the map: $G \mapsto a$, where $a(x) := A(G(x))$, $A : \mathbb{R} \to \mathbb{R}^{d \times d}$ a Lipschitz function and $\mathbb{R}^{d \times d}_\lambda$ the space of $\lambda$-elliptic matrices.

Note that in the framework adopted here, one can ensure that a spectral gap inequality (see for instance Lemma 3.1 in [11]). Furthermore, Helffer-Sjöstrand representation formula

$$\text{cov}[F, H] = \left\langle \int \int \frac{\partial F}{\partial G(x)} c(x - y)(1 + \mathcal{L})^{-1} \frac{\partial H}{\partial G(y)} \, dy \, dx \right\rangle$$

holds for every suitable random variables $F$ and $H$ (we refer the reader to section 4 in [5] for precise statements and the definition of the differential operator $\mathcal{L}$ - here we only use the fact that this operator is bounded to get (2.5)). We denote by $\frac{\partial F_{ij}^a}{\partial a(x)} = \frac{\partial F}{\partial a(x)} A'(G(x))$, where the random tensor $\frac{\partial F_{ij}^a}{\partial a(x)}$ stands for the functional derivative of $F$ defined through

$$\lim_{t \to 0} \frac{F(a + t\delta a) - F(a)}{t} = \int \frac{\partial F}{\partial a(x)} : \delta a(x) \, dx. \quad (2.4)$$

One of the main ingredients we use in this paper is the following covariance estimate (which is an immediate consequence of (2.3) and of the fact $\left| \frac{\partial F}{\partial G(x)} \right| \lesssim \left| \frac{\partial F}{\partial a(x)} \right|$)

$$\text{cov}[F, H] \lesssim \int \left( \left| \frac{\partial F}{\partial a(x)} \right|^2 \right)^{1/2} \int |c(x - y)| \left( \left| \frac{\partial H}{\partial a(y)} \right|^2 \right)^{1/2} \, dy \, dx. \quad (2.5)$$

Finally, let us also mention that for the class of ensembles we consider here (in particular, because of (2.2)) we can show that realizations $G$ (thus $a$’s) are Hölder continuous with Hölder norms having bounded moments, that is,

$$\langle ||a||_{C^{\alpha'}(B_r)}^q \rangle_{\alpha}^{1/q} \lesssim_{q, \alpha'} 1 \quad (2.6)$$

for any $0 < \alpha' < \alpha_1$ and $q \geq 1$ (see Appendix A in [11] for a proof).

Note that by $\lesssim$ we mean $\leq$ times a constant which depends only on $d, \lambda, \alpha_0, \alpha_1, ||A||_{C^1}$ and on quantities related to $c$. Moreover, note that we use the Einstein’s summation convention.

2.2. Higher-order theory. For reader’s convenience let us first introduce the notions of higher-order correctors and effective coefficients and their main properties that we use in the following (see Definition 2.1 and Proposition 2.2 in [5]).

**Definition 2.** Let $\tilde{d}$ be the smallest integer larger than $\frac{d}{2}$. The correctors $(\phi^0)_{0 \leq n \leq \tilde{d}}$, the flux correctors $(\sigma^0)_{0 \leq n \leq \tilde{d}}$ and the effective coefficients $(\tilde{a}^0)_{1 \leq n \leq \tilde{d}}$ are inductively defined as follows

- $\phi^0 := 1$ and $\phi^n := (\phi^a_{i_1 \ldots i_n})_{1 \leq i_1 \ldots i_n \leq d}$ for any $1 \leq n \leq \tilde{d}$, with $\phi^a_{i_1 \ldots i_n}$ a scalar field satisfying

$$-\nabla \cdot a \nabla \phi^a_{i_1 \ldots i_n} = \nabla \cdot \left( (a \phi^{n-1}_{i_1 \ldots i_{n-1}} - \sigma^{n-1}_{i_1 \ldots i_{n-1}}) e_{i_n} \right) \quad (2.7)$$
with \( \nabla \phi_{i_1 \ldots i_n} \) being stationary, centered and having finite second moments.

- \( \tilde{a}^n := (\tilde{a}_{i_1 \ldots i_{n-1}}^n)^{1 \leq i_1, \ldots, i_{n-1} \leq \tilde{d}} \) for any \( 1 \leq n \leq \tilde{d} \), with \( \tilde{a}_{i_1 \ldots i_{n-1}}^n \) the matrix given by
  \[
  \tilde{a}_{i_1 \ldots i_{n-1}}^n e_{i_n} := \left\langle a \left( \nabla \phi_{i_1 \ldots i_n}^n + \phi_{i_1 \ldots i_{n-1}}^{n-1} e_{i_n} \right) \right\rangle.
  \] (2.8)

- \( \sigma^0 := 0 \) and \( \sigma^n := (\sigma_{i_1 \ldots i_n}^n)^{1 \leq i_1, \ldots, i_n \leq \tilde{d}} \) for any \( 1 \leq n \leq \tilde{d} \), with \( \sigma_{i_1 \ldots i_n}^n \) a skew-symmetric matrix satisfying
  \[
  \nabla \cdot \sigma_{i_1 \ldots i_n}^n := a \nabla \phi_{i_1 \ldots i_n}^n + \left( \alpha_{i_1 \ldots i_{n-1}}^{n-1} - \sigma_{i_1 \ldots i_{n-1}}^{n-1} \right) e_{i_n} - \tilde{a}_{i_1 \ldots i_{n-1}}^n e_{i_n}
  \] (2.9)
with \( \nabla \sigma_{i_1 \ldots i_n}^n \) being stationary, centered and having finite second moments. Here we denote by \( (\nabla \cdot \sigma_{i_1 \ldots i_n}^n)_j = \sum_{k=1}^{d} \partial_k (\sigma_{i_1 \ldots i_n}^n)_{jk}, 1 \leq j \leq d \).

**Proposition 3.** All the quantities in definition 2 exist and they satisfy for all \( 1 \leq n \leq \tilde{d} \) and \( p \geq 1 \)
\[
|\alpha^n| \leq 1, \quad \left\langle |\nabla \phi^n|^p \right\rangle^{1/p} \leq_{p,n} 1, \quad \left\langle |\phi^n(x)|^p \right\rangle^{1/p} + \left\langle |\sigma^n(x)|^p \right\rangle^{1/p} \leq_{p,n} \mu_{d,n}(x).
\] (2.10)
where
\[
\mu_{d,n}(x) := \begin{cases} 
1, & \text{if } n < \tilde{d} \\
\ln^{1/2}(2 + |x|), & \text{if } n = \tilde{d} \text{ and } d \text{ even} \\
1 + |x|^{1/2}, & \text{if } n = \tilde{d} \text{ and } d \text{ odd}.
\end{cases}
\]

Next we explain why definition (1.6) is reasonably derived from the definitions of higher-order commutators given in [5]. The homogenization commutator \( \Xi^n[\nabla w] = (a - \tilde{a}) \nabla w \) naturally extends to the higher-order as
\[
\Xi^n[\nabla w] := (a - \sum_{k=1}^{n} \tilde{a}_{i_1 \ldots i_{k-1}}^k \partial_{i_1 \ldots i_{k-1}} \nabla w) = (a - \tilde{a}) \nabla w - \sum_{k=2}^{n} \tilde{a}_{i_1 \ldots i_{k-1}}^k \partial_{i_1 \ldots i_{k-1}} \nabla w.
\] (2.11)

Then the standard homogenization commutator \( \Xi_{i,j}^{o,n}[\nabla \tilde{w}] \) is given by \( \Xi^n \) applied to the nth-order Taylor polynomial of the nth-order two-scale expansion of \( \tilde{w} \). However a more explicit formula for \( \Xi_{i,j}^{o,n}[\nabla \tilde{w}] \) is available (see Lemma 3.5 in [5]) and if this formula is applied to the linear functions \( \tilde{w}(x) = x_i \) we get
\[
\Xi_{i,j}^{o,n} = e_j \cdot \Xi^n[\nabla \phi_i^1 + e_i].
\]

Now for our analysis, especially when deriving representation formulas for the Malliavin derivatives, it is more convenient to work with (1.6) which is defined through the transposes of the higher-order effective coefficients. For (1.6), we use the following alternative representation of \( \Xi^n \)
\[
e_j \cdot \Xi^n[\nabla w] = a^* e_j \cdot \nabla w - \sum_{k=1}^{n} (-1)^{k-1} \partial_{j_{i_1 \ldots i_{k-2}}}^{k-1} e_{i_{k-1}} \cdot \partial_{i_1 \ldots i_{k-1}}^{k-1} \nabla w.
\] (2.12)
The above is a consequence of Lemma 2.4 in [5] which extends the fact \( \tilde{a}^{*,1} = (\tilde{a}^{1})^{*} \) to the higher-order,

\[
\operatorname{Sym}_{i_{1}...i_{n}}(e_{j} \cdot \tilde{a}_{i_{1}...i_{n-1}}^{*} e_{i_{n}}) = (-1)^{n+1} \operatorname{Sym}_{i_{1}...i_{n}}(e_{i_{n}} \cdot \tilde{a}_{j_{1}...j_{n-2}}^{*,n} e_{i_{n-1}}) \quad (2.13)
\]

where \( \operatorname{Sym}_{i_{1}...i_{k}} T_{i_{1}...i_{k}} := \frac{1}{k!} \sum_{\sigma \in S_{k}} T_{i_{\sigma(1)}...i_{\sigma(k)}} \), \( T \) a kth-order tensor and \( S_{k} \) the set of permutations of \( \{1, \ldots k\} \).

3. Proof of Main Theorem

Since we intend to bound quantity (1.4) via the covariance estimate (2.5), we first derive a suitable representation formula for the derivative of \( F_{ij}^{o,n} := \int_{\mathbb{R}^{d}} g(x) \Xi_{ij}^{o,n}(x) \, dx \). Our intention is to get as many derivatives as possible for the r.h.s of the equation that the term \( \nabla h_{j} \) satisfies. We show the following

**Proposition 4** (Representation formula).

\[
\frac{\partial F_{ij}^{o,n}}{\partial a} = (\nabla \phi_{i}^{1} + e_{i}) \otimes \left( \sum_{k=0}^{n} \partial_{i_{1}...i_{k}}^{k} g(\nabla \phi_{j_{1}...i_{k}}^{*,k} + e_{i_{k}} \phi_{j_{1}...i_{k-1}}^{*,k}) + \nabla h_{j} \right) \quad (3.1)
\]

with \( h_{j} \) solving

\[
-\nabla \cdot a^{*} \nabla h_{j} = \nabla \cdot \left( (a^{*} \phi_{j_{1}...i_{n-1}}^{*,n} - \sigma_{j_{1}...i_{n-1}}^{*,n}) \nabla \phi_{i_{1}...i_{n-1}}^{n} g \right) . \quad (3.2)
\]

Note that in the sum appears in (3.1), we use the convention that the \( k = 0 \)-term is just \( \nabla \phi_{j}^{*,1} + e_{j} \), while the \( k = n \)-term is just \( \partial_{i_{1}...i_{n}}^{n} g e_{i_{n}} \phi_{j_{1}...i_{n-1}}^{*,n} \) (see definition 2).

**Proof.** We have (integrating by parts)

\[
\frac{F_{ij}^{o,n}(a + t\delta a) - F_{ij}^{o,n}(a)}{t} = \frac{1}{t} \int g e_{j} \cdot (a + t\delta a - \tilde{a}) (\nabla \phi_{i}^{1}(a + t\delta a) + e_{i})
\]

\[
- \frac{1}{t} \int \sum_{k=2}^{n} (-1)^{2(k-1)} \partial_{i_{1}...i_{k-1}}^{k-1} g \tilde{a}_{j_{1}...i_{k-2}}^{*,k} e_{i_{k-1}} \cdot \nabla \phi_{i}^{1}(a + t\delta a)
\]

\[
- \frac{1}{t} \int g e_{j} \cdot (a - \tilde{a}) (\nabla \phi_{i}^{1}(a) + e_{i})
\]

\[
+ \frac{1}{t} \int \sum_{k=2}^{n} (-1)^{2(k-1)} \partial_{i_{1}...i_{k-1}}^{k-1} g \tilde{a}_{j_{1}...i_{k-2}}^{*,k} e_{i_{k-1}} \cdot \nabla \phi_{i}^{1}(a)
\]

\[
= \int g e_{j} \cdot \delta a \left( e_{i} + \nabla \phi_{i}^{1}(a + t\delta a) \right) + \int g e_{j} \cdot (a - \tilde{a}) \frac{\nabla \phi_{i}^{1}(a + t\delta a) - \nabla \phi_{i}^{1}(a)}{t}
\]

\[
- \int \sum_{k=2}^{n} \partial_{i_{1}...i_{k-1}}^{k-1} g \tilde{a}_{j_{1}...i_{k-2}}^{*,k} e_{i_{k-1}} \cdot \nabla \phi_{i}^{1}(a + t\delta a) - \nabla \phi_{i}^{1}(a)
\]

\[
- \int \sum_{k=2}^{n} \partial_{i_{1}...i_{k-1}}^{k-1} g \tilde{a}_{j_{1}...i_{k-2}}^{*,k} e_{i_{k-1}} \cdot \nabla \phi_{i}^{1}(a + t\delta a) - \nabla \phi_{i}^{1}(a)
\]
where $\nabla \phi_i^1(a + t\delta a) - \nabla \phi_i^1(a) := \nabla \psi_i(a, t\delta a)$, with $-\nabla \cdot (a + t\delta a) \sum_{i=1}^{n} = \nabla \cdot \delta a(\nabla \phi_i^1(a) + e_1)$ (see section 3.4 in [6]). Thus letting $t \to 0$ we get

$$
\lim_{t \to 0} \frac{F_{ij}^{\alpha,n}(a + t\delta a) - F_{ij}^{\alpha,n}(a)}{t} = \int g e_j \cdot \delta a (e_i + \nabla \phi_i^1) + \int g e_j \cdot (a - \bar{a}^1) \nabla \delta \phi_i
$$

$$
- \int \sum_{k=2}^{n} \partial_{1}^{k-1} \cdots \partial_{1}^{k-1} g i_{k} a_{j_{1} \cdots i_{k-1}} e_{i_{k-1}} \cdot \nabla \phi_i
$$

(3.3)

where

$$
- \nabla \cdot a \nabla \delta \phi_i = \nabla \cdot \delta a(\nabla \phi_i^1 + e_i).
$$

(3.4)

Next we further analyze the r.h.s of (3.3) to get the desired representation formula. The main ingredient we use is the following relation between the correctors (see (2.9) in definition 2)

$$
(a^k_{i_1 \cdots i_{k-1}} - a^k_{i_1 \cdots i_{k-1}}) e_{i_k} = -a \nabla \phi_{i_1 \cdots i_k}^1 + \nabla \cdot \sigma_{i_1 \cdots i_k}^1 - a^k_{i_1 \cdots i_k} e_{i_k}
$$

(3.5)

which for $k = 1$ reduces to the well known $(a - \bar{a}^1)e_i = -a \nabla \phi_i^1 + \nabla \cdot \sigma_i^1$.

We show by induction on $n$ the following

$$
\lim_{t \to 0} \frac{F_{ij}^{\alpha,n}(a + t\delta a) - F_{ij}^{\alpha,n}(a)}{t} = \int \sum_{k=0}^{n} \partial_{1}^{k} \cdots \partial_{1}^{k} g (\nabla \phi_{j_{1} \cdots i_{k-1}}^{*,k+1} + e_{i_k} \phi^{*,k}_{j_{1} \cdots i_{k-1}}) \cdot \delta a(\nabla \phi_i^1 + e_i)
$$

$$
+ \int (a^*_{j_{1} \cdots i_{n-1}} - \sigma_{j_{1} \cdots i_{n-1}}^*) \nabla \phi_i^{n-1} g \cdot \nabla \delta \phi_i.
$$

(3.6)

Note that the above gives the result (via (2.4)). Indeed, testing equation (3.2) with $\delta \phi_i$ the second term of the r.h.s of (3.6) turns into $-\int a^* \nabla h_j \cdot \nabla \delta \phi_i = -\int \nabla h_j \cdot a \nabla \delta \phi_i = \int \nabla h_j \cdot \delta a(\nabla \phi_i^1 + e_i)$. Where the last equality is obtained by testing equation (3.4) with $h_j$.

Now for the induction we start with $n = 1$. In that case (3.3) reduces to

$$
\lim_{t \to 0} \frac{F_{ij}^{\alpha,1}(a + t\delta a) - F_{ij}^{\alpha,1}(a)}{t} = \int g e_j \cdot \delta a (e_i + \nabla \phi_i^1) + \int g e_j \cdot (a - \bar{a}^1) \nabla \delta \phi_i
$$

We work with the second term of the r.h.s.

$$
\int g e_j \cdot (a - \bar{a}^1) \nabla \delta \phi_i = \int g (a^* - \bar{a}^{*,1}) e_j \cdot \nabla \delta \phi_i = \int (a^* \nabla \phi_j^{*,1} + g \nabla \cdot \sigma^*_{j}^1) \cdot \nabla \delta \phi_i
$$

$$
= \int \left(-a^* \nabla (g \phi_j^{*,1}) + a^* \phi_j^{*,1} \nabla g - \sigma^*_{j}^1 \nabla g \right) \cdot \nabla \delta \phi_i
$$

where the first two terms result from Leibniz rule and the last from the following property of $\sigma_i$

$$
\nabla \cdot (g \nabla \sigma_i) = -\nabla \cdot (\sigma_i \nabla g), \quad \text{for any smooth enough } g,
$$
which is an easy consequence of the skew-symmetry of $\sigma_i$. Thus we have

$$
\lim_{t \to 0} \frac{F_{ij}^{0,1}(a + t\delta a) - F_{ij}^{0,1}(a)}{t} = \int ge_j \cdot \delta a (e_i + \nabla \phi_1^i) + \int -a^* \nabla (g\phi_j^{*,1}) \cdot \nabla \delta i + \int \left( a^* \phi_j^{*,1} - \sigma_j^{*,1} \right) \nabla g \cdot \nabla \delta i.
$$

Note that testing equation (3.4) with $g\phi_j^{*,1}$ we get $-a^* \nabla (g\phi_j^{*,1}) \cdot \nabla \delta i = \int \nabla (g\phi_j^{*,1}) \cdot \delta a (\nabla \phi_i^1 + e_i)$. Then

$$
\lim_{t \to 0} \frac{F_{ij}^{0,1}(a + t\delta a) - F_{ij}^{0,1}(a)}{t} = \int \left( e_j + \nabla \phi_j^{*,1} \right) g + \phi_j^{*,1} \nabla g \cdot \delta a (e_i + \nabla \phi_i^1)
$$

$$
+ \int \left( a^* \phi_j^{*,1} - \sigma_j^{*,1} \right) \nabla g \cdot \nabla \delta i.
$$

which is exactly (3.6) for $n = 1$. Next assume that (3.6) holds for $n - 1$. We show that it is true for $n$. Indeed, by (3.3) we see that

$$
\lim_{t \to 0} \frac{F_{ij}^{0,n}(a + t\delta a) - F_{ij}^{0,n}(a)}{t} = \lim_{t \to 0} \frac{F_{ij}^{0,n-1}(a + t\delta a) - F_{ij}^{0,n-1}(a)}{t}
$$

$$
- \int \partial_{11...i_{n-1}}^{n-1} g \partial_{j1...i_{n-2}}^{*,n} e_{i_{n-1}} \cdot \nabla \delta i
$$

$$
= \int \sum_{k=0}^{n-1} \partial_{11...i_{n-2}}^{n-1} g (\nabla \phi_j^{*,k+1} + e_k \phi_j^{*,k}) \cdot \delta a (\nabla \phi_i^1 + e_i)
$$

$$
+ \int \left( a^* \phi_j^{*,n-1} - \sigma_j^{*,n-1} \right) \partial_{11...i_{n-2}}^{n-1} g e_{i_{n-1}} \cdot \nabla \delta i
$$

$$
- \int \partial_{11...i_{n-1}}^{n-1} g \partial_{j1...i_{n-2}}^{*,n} e_{i_{n-1}} \cdot \nabla \delta i.
$$

We use again (3.5) for the middle term

$$
\int \partial_{11...i_{n-1}}^{n-1} g \left( a^* \phi_j^{*,n-1} - \sigma_j^{*,n-1} \right) e_{i_{n-1}} \cdot \nabla \delta i
$$

$$
= \int \partial_{11...i_{n-1}}^{n-1} g \left( -a^* \nabla \phi_j^{*,n} + \nabla \cdot \sigma_j^{*,n} - \partial_{j1...i_{n-2}}^{*,n} e_{i_{n-1}} \right) \cdot \nabla \delta i.
$$

Hence

$$
\lim_{t \to 0} \frac{F_{ij}^{0,n}(a + t\delta a) - F_{ij}^{0,n}(a)}{t} = \int \sum_{k=0}^{n-1} \partial_{11...i_{n-2}}^{n-1} g (\nabla \phi_j^{*,k+1} + e_k \phi_j^{*,k}) \cdot \delta a (\nabla \phi_i^1 + e_i)
$$

$$
- \int a^* \nabla (\partial_{11...i_{n-1}}^{n-1} g \phi_j^{*,n}) \cdot \nabla \delta i
$$

$$
+ \int \left( a^* \phi_j^{*,n} - \sigma_j^{*,n} \right) \partial_{11...i_{n-1}}^{n-1} \nabla \phi_j^{*,n} g - \sigma_j^{*,n} \nabla \partial_{11...i_{n-1}}^{n-1} g \cdot \nabla \delta i
$$

where we used again Leibniz rule and the skew-symmetry of $\sigma_j^{*,n}$. For the middle term we use equation (3.4) tested with $\partial_{11...i_{n-1}}^{n-1} g \phi_j^{*,n}$ to get

$$
- \int a^* \nabla (\partial_{11...i_{n-1}}^{n-1} g \phi_j^{*,n}) \cdot \nabla \delta i.
\[= \int \nabla (\partial_{i_1 \ldots i_{n-1}} g \phi_{j_1 \ldots i_{n-1}}^{n}) \cdot \delta a(\nabla \phi_i^j + e_i)\]

Substituting in (3.7) and recalling that for the sum \(\sum_{k=0}^{n-1} \sigma_j^k \) we have the convergence that the \((n-1)\)-term is just \(\partial_{i_1 \ldots i_{n-1}} g \phi_{j_1 i_2 \ldots i_n} \). We then have using the Lipschitz estimate (or mean-value property) of Theorem 1 in [1] which contains \(\sigma_i^j \) and recalling that for the sum \(\sum_{k=0}^{n-1} \sigma_j^k \), we estimate \(\tilde{h}_{1} = 0 \). Next we study the solution \(h_j \) of (3.2) deriving some bounds that will be useful for the proof of the main theorem. In the sequel we assume that \(d \) is odd and we denote by \(\tilde{d} := \frac{d+1}{2} \) (note that the proof when \(d \) is even is similar - the slightly different bound comes from the stochastic moment bounds of the correctors).

**Lemma 5.** Let \(h_j \) be a solution of (3.2) with \(n = \tilde{d} \), then for any \(|x| > \frac{L}{2} \) it holds

\[\left\langle \left| \nabla h_j(x) \right|^{4} \right\rangle^{1/4} \lesssim R^{-d/2} L^{-d}. \tag{3.8}\]

**Proof.** Recall that \(h_j \) satisfies the equation \(-\nabla \cdot a^* \nabla h_j = \nabla \cdot f_j \), where \(f_j := (a^* \phi_{ji_1 \ldots i_{n-1}}^\tilde{d} - \sigma_{ji_1 \ldots i_{n-1}}^\tilde{d}) \nabla \phi_i^j \). Note that \(\supp f_j \subset B_{R} \), in particular \(h_j \) is \(a^*\)-harmonic in \(B_R^c \) which contains \(B_L^{c/2} \) (recall that \(L >> R \)). For any \(|x| > \frac{L}{2} \), we bound first \(f_{B_{L/2}^{c}(x)} |\nabla h_j|^{2} \) using the Lipschitz estimate (or mean-value property) of Theorem 1 in [7]. To achieve this we consider the auxiliary function \(v \) which solves

\[-\nabla \cdot a \nabla v_j = \nabla \cdot \tilde{f}_{j}, \quad \text{where} \quad \tilde{f}_{j} := \chi_{B_{L/4}}^{c} \nabla h_{j}.\]

We then have

\[\int_{B_L^{c/4}} |\nabla h_j|^{2} = \int \tilde{f}_{j} \cdot \nabla h_j = \int a \nabla v_j \cdot \nabla h_j = \int \nabla v_j \cdot f_j \]

\[\lesssim \left( \int_{B_{R}} |\nabla v_{j}|^{2} \right)^{1/2} \left( \int_{B_{R}} |f_{j}|^{2} \right)^{1/2}. \]

Denoting by \(R_{1}^{*} = \max \{r^{*}(x), R \} \), where \(r^{*} \) the minimal random radius of Theorem 1 in [7], we estimate

\[\left( \int_{B_{R}} |\nabla v_{j}|^{2} \right)^{1/2} \lesssim (R_{0}^{*})^{d/2} \left( \int_{B_{R_{0}^{*}}} |\nabla v_{j}|^{2} \right)^{1/2}\]

\[\lesssim (R_{0}^{*})^{d/2} \left( \int_{B_{L/2}} |\nabla v_{j}|^{2} \right)^{1/2} \quad (v \text{ is } a\text{-harmonic in } B_{L/2})\]

\[\lesssim \frac{(R_{0}^{*})^{d/2}}{L^{d/2}} \left( \int |\tilde{f}_{j}|^{2} \right)^{1/2} \quad \text{ (by the energy estimate)}.\]
Thus we get

\[
\int_{B_{\frac{L}{4}}^c} |\nabla h_j|^2 \lesssim \frac{(R^*_h)^{d/2}}{L^{d/2}} \left( \int_{B_{\frac{L}{4}}^c} |\nabla h_j|^2 \right)^{1/2} \left( \int_{B_R} |f_j|^2 \right)^{1/2}
\]

that is,

\[
\left( \int_{B_{L/4}(x)} |\nabla h_j|^2 \right)^{1/2} \lesssim \frac{1}{L^{d/2}} \left( \int_{B_{L/4}^c} |\nabla h_j|^2 \right)^{1/2} \lesssim \frac{(R^*_h)^{d/2}}{L^{d/2}} \left( \int |f_j|^2 \right)^{1/2}.
\]

Next we estimate \( \left\langle |\nabla h_j(x)|^4 \right\rangle^{1/4} \) for every \( |x| > \frac{L}{2} \). By small-scale regularity we have, for \( |x| > \frac{L}{2} \),

\[
|\nabla h_j(x)| \lesssim C(a) \left( \int_{B_{R/4}(x)} |\nabla h_j|^2 \right)^{1/2},
\]

where \( C(a) \) denotes the Hölder constant of the coefficient field \( a \) (note that for \( L > 4R \), \( B_{R/4}(x) \subset B_{L/4}^c \) and \( h_j \) is \( a^* \)-harmonic in \( B_{L/4}^c \)). Then we apply Lipschitz estimate once more to get (assuming that \( L > 4R \))

\[
|\nabla h_j(x)| \lesssim C(a) \frac{(R^*_h)^{d/2}}{R^{d/2}} \left( \int_{B_{R/4}(x)} |\nabla h_j|^2 \right)^{1/2} \lesssim C(a) \frac{(R^*_h)^{d/2}}{R^{d/2}} \frac{(R^*_j)^{d/2}}{L^d} \left( \int |f_j|^2 \right)^{1/2}.
\]

Then we take expectation, we use the fact that both \( C(a) \) (see (2.6)) and \( r^*(x) \) have uniformly bounded stochastic moments and choosing the worst scenario for the factor we get

\[
\left\langle |\nabla h_j(x)|^4 \right\rangle^{1/4} \lesssim \frac{R^{d/2}}{L^d} \left\langle \left( \int |f_j|^2 \right)^8 \right\rangle^{1/16}.
\]

It remains to estimate the r.h.s using Minkowski’s integral inequality and Proposition 3 (recall that \( f_j := (a^* \phi^s_{j_1 \ldots j_{d-1}} - \sigma^s_{j_1 \ldots j_{d-1}}) \nabla \partial_{j_1 \ldots j_{d-1}}^{n-1} g \)). We have

\[
\left\langle |\nabla h_j(x)|^4 \right\rangle^{1/4} \lesssim \frac{R^{d/2}}{L^d} \left( \int \langle |f_j|^16 \rangle^{1/8} \right)^{1/2} \lesssim \frac{R^{d/2}}{L^d} \left( \int_{B_R} |D^d g|^2 \langle |a^* \phi^s_{j_1 \ldots j_{d-1}} - \sigma^s_{j_1 \ldots j_{d-1}}|^16 \rangle^{1/8} \right)^{1/2} \lesssim \frac{R^{d/2}}{L^d} \left( \int_{B_R} |D^d g|^2 |z| \right)^{1/2} \lesssim \frac{R^{d/2}}{L^d} R^{d/2} = R^{-d/2} L^{-d}. \]

\( \square \)
Remark 6. Note that by translation we easily see that the solution $h'_i$ of

$$-\nabla \cdot a^s \nabla h'_i = \nabla \cdot \left( (a^s_1 \phi_{i_1 \ldots i_{d-1}}^s - a^s_{i_1 \ldots i_{d-1}}) \nabla \delta_{i_1 \ldots i_{d-1}}^d \right)$$

satisfies, for any $|x - Le| > \frac{L}{2}$,

$$\left| \left\langle \nabla h'_i(x) \right\rangle \right|^{1/4} \lesssim R^{-d/2} L^{-\frac{d}{4}}.$$  \hspace{1cm} (3.9)

Note also that the above bounds can be rephrased (assuming that $L > 4R$) as

$$\left| \left\langle \nabla h_j(x) \right\rangle \right|^{1/4} \lesssim R^{-d/2} |x|^{-d}, \text{ for any } |x| > 2R \hspace{1cm} (3.10)$$

$$\left| \left\langle \nabla h'_j(x) \right\rangle \right|^{1/4} \lesssim R^{-d/2} |x - Le|^{-d}, \text{ for any } |x - Le| > 2R. \hspace{1cm} (3.11)$$

A last observation that will be useful in the sequel is the following

$$\left( \int \left| \nabla h_j \right|^{4} \right)^{1/2} \lesssim \left( \int \left| f_j \right|^{4} \right)^{1/2} \lesssim R^{1/2} \left( \int |D^g g|^2 \right)^{1/2} \lesssim R^{1/2} R^{-\frac{d}{2}} = R^{-d}. \hspace{1cm} (3.12)$$

which is a consequence of annealed Calderon-Zygmund estimates (see Proposition 7.1 in [11]) and Proposition 3

$$\left( \int \left| \nabla h_j \right|^{4} \right)^{1/2} \lesssim \left( \int \left| f_j \right|^{4} \right)^{1/2} \lesssim R^{1/2} \left( \int |D^g g|^2 \right)^{1/2} \lesssim R^{1/2} R^{-\frac{d}{2}} = R^{-d}. \hspace{1cm} (3.12)$$

A similar bound holds for $h'_i$ as well.

We are now ready to prove our main theorem.

Proof of Theorem 1. Combining estimate (2.5) with Proposition 4 we get

$$P_{ijml}^{s,0,\tilde{d}} \lesssim \int \left[ \left| (\nabla \phi^1_i(x) + e_i) \otimes g(x) \left( \nabla \phi^1_j(x) + e_j \right) \right|^{2} \right]^{1/2}$$

$$+ \left[ \left| (\nabla \phi^1_i(x) + e_i) \otimes \sum_{k=1}^{d-1} \partial \phi^1_i \otimes g(x) \left( \nabla \phi_{j_1 \ldots j_k}^{s+1} + e_i \phi_{j_1 \ldots j_k}^{s+1} \right) \right|^{2} \right]^{1/2}$$

$$+ \left[ \left| (\nabla \phi^1_i(x) + e_i) \otimes \nabla h_j(x) \right|^{2} \right]^{1/2}$$

$$\times \int |c(x - y)|^{\text{same terms with } i \leftrightarrow m, j \leftrightarrow l, x \leftrightarrow y \text{ and } g \leftrightarrow g'} \, dy \, dx.$$

Using Proposition 3 we may estimate

- $$\left[ \left| (\nabla \phi^1_i + e_i) \otimes g(\nabla \phi^1_j + e_j) \right|^{2} \right]^{1/2} \lesssim |g| \left[ \left| (\nabla \phi^1_i + e_i) \right|^{4} \right]^{1/4} \left[ \left| (\nabla \phi^1_j + e_j) \right|^{4} \right]^{1/4} \lesssim |g|.$$

- $$\left[ \left| (\nabla \phi^1_i + e_i) \otimes \sum_{k=1}^{d-1} \partial \phi^1_i \otimes g(\nabla \phi_{j_1 \ldots j_k}^{s+1} + e_i \phi_{j_1 \ldots j_k}^{s+1}) \right|^{2} \right]^{1/2}$$
\[
\left\langle \left| \nabla \phi_i^1 + e_i \right|^4 \right\rangle^{1/4} \sum_{k=1}^{\tilde{d}-1} |\partial_k^{\tilde{d}} g| \left( \left\langle \left| \nabla \phi_{j1}^{\ast k+1} \right|^4 \right\rangle^{1/4} + \left\langle \left| \phi_{j1}^{\ast k} \right|^4 \right\rangle^{1/4} \right) \\
\lesssim \sum_{k=1}^{\tilde{d}-1} |\partial_k^{\tilde{d}} g|. \quad \text{(note that } k < \tilde{d})
\]

-  \( \left\langle \left| (\nabla \phi_i^1 + e_i) \otimes \partial_{\tilde{d}} \phi_{j1}^{\ast} g \right|^2 \right\rangle^{1/2} \lesssim |\partial_k^{\tilde{d}} g| \left\langle \left| \nabla \phi_i^1 + e_i \right|^4 \right\rangle^{1/4} \left\langle \left| \phi_{j1}^{\ast} g \right|^4 \right\rangle^{1/4} \lesssim |z|^{1/2} |\partial_k^{\tilde{d}} g|.

-  \( \left\langle \left| (\nabla \phi_i^1 + e_i) \otimes \nabla h_j \right|^2 \right\rangle^{1/2} \lesssim \left\langle \left| \nabla \phi_i^1 + e_i \right|^4 \right\rangle^{1/4} \left\langle \left| \nabla h_j \right|^4 \right\rangle^{1/4} \approx \left\langle \left| \nabla h_j \right|^4 \right\rangle^{1/4}.

Now we return to (3.13), we apply the above estimates and multiply to get

\[
P_{ijml}^{\alpha, \tilde{d}} \lesssim \int \left( \sum_{k=0}^{\tilde{d}-1} |\partial_k^{\tilde{d}} g(x)| + |x|^{1/2} |\partial_k^{\tilde{d}} g(x)| \right) \\
\times \int |c(x - y)| \left( \sum_{k=0}^{\tilde{d}-1} |\partial_k^{\tilde{d}} g(y)| + |y - Le|^{1/2} |\partial_k^{\tilde{d}} g(y)| \right) \, dy \, dx \\
+ \int \left\langle \left| \nabla h_j(x) \right|^4 \right\rangle^{1/4} \int |c(x - y)| \left( \sum_{k=0}^{\tilde{d}-1} |\partial_k^{\tilde{d}} g(y)| + |y - Le|^{1/2} |\partial_k^{\tilde{d}} g(y)| \right) \, dy \, dx \\
+ \int \left( \sum_{k=0}^{\tilde{d}-1} |\partial_k^{\tilde{d}} g(x)| + |x|^{1/2} |\partial_k^{\tilde{d}} g(x)| \right) \int |c(x - y)| \left\langle \left| \nabla h_i^j(y) \right|^4 \right\rangle^{1/4} \, dy \, dx \\
+ \int \left\langle \left| \nabla h_j(x) \right|^4 \right\rangle^{1/4} \int |c(x - y)| \left\langle \left| \nabla h_i^j(y) \right|^4 \right\rangle^{1/4} \, dy \, dx.
\]

Next we focus on bounding each of these four terms. Starting with the first one we observe that it is enough to estimate the term

\[
\int |g(x)| \int |c(x - y)| |g(y)| \, dy \, dx \lesssim L^{-d-\alpha_0} \lesssim R^{-d/2} L^{-d/2-\alpha_0}
\]

where we used that \( x \in B_R \) and \( y \in B_R(Le) \) because of the supports of \( g \) and \( g' \) respectively. Then \( |x - y| \geq L/2 \) which gives the estimate if we use the integrable decay (2.1) we have assumed for \( c \).

We proceed with the second term where we see once again that it is enough to estimate the "subterm" with the "worst" behaviour, that is

\[
\int \left\langle \left| \nabla h_j(x) \right|^4 \right\rangle^{1/4} \int |c(x - y)||g'(y)| \, dy \, dx.
\]

For, we split the domain of integration into two in order to be able to apply the estimate of Lemma 5 to \( h_j \). So for the first term we apply bound (3.8) and Young's convolution inequality, while for the second term we use estimate (3.12) and the integrable decay
of $c$ together with Minkowski’s integral inequality,

\[
\int \left( \int |\nabla h_j(x)|^4 \right)^{\frac{1}{4}} \int |c(x-y)||g'(y)| \ dy \ dx \\
\lesssim R^{-d/2} L^{-d} \int_{B_{L/2}^c} \int |c(x-y)||g'(y)| \ dy \ dx \\
+ \int_{B_{L/2}^c} \int \frac{|g'(y)|}{(1 + |x-y|)^{d+\alpha_0}} \left( \int \langle |\nabla h_j(x)|^4 \rangle^{\frac{1}{4}} \ dy \ dx \right) \\
\lesssim R^{-d/2} L^{-d} \langle |g'| \rangle_{L^1} \langle |c| \rangle_{L^1} \\
+ \left( \int_{B_{L/2}^c} \left( \int_{B_{R(L)}^c} \frac{|g'(y)|}{(1 + |x-y|)^{d+\alpha_0}} \right)^2 \ dy \ dx \right)^{\frac{1}{2}} \left( \int \left( \int \langle |\nabla h_j(x)|^4 \rangle^{\frac{1}{4}} \right)^2 \ dx \right)^{\frac{1}{2}} \\
\lesssim R^{-d/2} L^{-d} + R^{-d} \int_{B_{R(L)}^c} |g'(y)| \left( \int_{B_{L/2}^c} \frac{1}{(1 + |x-y|)^{2d+2\alpha_0}} \ dx \right)^{\frac{1}{2}} \ dy \\
\lesssim R^{-d/2} L^{-d} + R^{-d} L^{-d-\alpha_0} \lesssim R^{-d/2} L^{-d-\alpha_0}.
\]

It remains to bound the fourth term which is the most challenging. In order to be able to use the bounds of Lemma 5 and Remark 6 we divide the domains of integration as follows

\[
\int \left( \int |\nabla h_j(x)|^4 \right)^{\frac{1}{4}} \int |c(x-y)||\nabla h'_i(y)| \ dy \ dx \\
\lesssim R^{-d/2} L^{-d} \int_{B_{3R}(Le)} \left( \int |c(x-y)|^2 \ dy \right)^{\frac{1}{2}} \left( \int \left( \int |\nabla h'_i(y)|^4 \right)^{\frac{1}{4}} \ dy \right)^{\frac{1}{2}} \ dx \\
+ R^{-d/2} L^{-d} \left( \int_{B_{3R}^c(L)} \left( \int |\nabla h_j(x)|^4 \right)^{\frac{1}{4}} \ dx \right)^{\frac{1}{2}} \left( \int_{B_{3R}^c(L)} \left( \int |c(x-y)|^2 \ dy \right)^{\frac{1}{2}} \ dx \right)^{\frac{1}{2}} \\
+ \int_{B_{3R}^c(L)} \left( \int |\nabla h_j(x)|^4 \right)^{\frac{1}{4}} \int_{B_{3R}^c} |c(x-y)||\nabla h'_i(y)| \ dy \ dx \\
\lesssim R^{-d/2} L^{-d} + \int_{B_{3R}^c(L)} \left( \int |\nabla h_j(x)|^4 \right)^{\frac{1}{4}} \int_{B_{3R}^c} |c(x-y)||\nabla h'_i(y)| \ dy \ dx
\]

using estimates (3.8), (3.9) and (3.12). Now for the last term we need to further divide the domains. Precisely, we split the $x$-integral into $B_{3R}$ and $B_{3R}^c$ and the $y$-integral into $B_{3R}(Le)$ and $B_{3R}^c(Le)$. This produces four new terms that we estimate in the following. For the first one we may apply estimates (3.10) and (3.11). Then we divide the domain of $y$-integral once again and use the integrable decay (2.1) of $c$

\[
\int_{B_{3R}^c(L) \cap B_{3R}} \left( \int |\nabla h_j(x)|^4 \right)^{\frac{1}{4}} \int_{B_{3R}^c(L) \cap B_{3R}} |c(x-y)||\nabla h'_i(y)| \ dy \ dx \\
\lesssim R^{-d} \int_{B_{3R}^c(L) \cap B_{3R}} \int_{B_{3R}(Le) \cap B_{3R}} |x|^{-d} |y - Le|^{-d} |c(x-y)| \ dy \ dx
\]
We then calculate

\[
R^{-d} \int_{B_{3R}(Le) \cap B_{3R}^c} |x|^{-d} |x - Le|^{-d} \, dx
\]

\[
= 2R^{-d} \int_{\{ |x| < |x - Le| \} \cap B_{3R}^c} |x|^{-d} |x - Le|^{-d} \, dx
\]

\[
\lesssim R^{-d} \int_{\{ |x| < |x - Le| \} \cap B_{3L}^c} |x|^{-2d} \, dx + R^{-d} \int_{\{ |x| < |x - Le| \} \cap (B_{3L} \setminus B_{3R})} |x|^{-d} \, dx
\]

\[
\lesssim R^{-d} L^{-d} \ln \frac{L}{R} \lesssim R^{-d/2} L^{-d/2-\alpha_0} \ln \frac{L}{R}.
\]

Similarly \(\int_{B_{3R}(Le) \cap B_{3R}^c} |x|^{-d} |x - Le|^{-d-\alpha_0+1} \, dx \lesssim R^{-d} L^{-d-\alpha_0+1} \ln \frac{L}{R}\) (note that in that term we could get rid of the logarithmic correction when \(d \geq 4\)).

For the next term we split the \(y\)-integral into \(|x - y| > L/4\) and \(|x - y| \leq L/4\). So in the first case we use the integrable decay (2.1) of \(c\) to gain the power we need on \(L\) together with estimate (3.12). In the second case we see that \(|x| \geq L/2\) which allows to apply Lemma 5 which we then combine with (3.12). Indeed,

\[
\int_{B_{3R}(Le) \cap B_{3R}^c} \left\langle |\nabla h_j(x)|^4 \right\rangle^{\frac{1}{4}} \int_{B_{3R}(Le) \cap B_{3R}^c} |c(x - y)| \left\langle |\nabla h_t'(y)|^4 \right\rangle^{\frac{1}{4}} \, dy \, dx
\]

\[
\lesssim \left( \int \left\langle |\nabla h_j(x)|^4 \right\rangle^{\frac{1}{2}} \, dx \right)^{1/2}
\]

\[
\times \left( \int_{B_{3R}(Le) \cap B_{3R}^c} \left( \int_{B_{3R}(Le) \cap \{|x-y| > L/4\}} |c(x - y)| \left\langle |\nabla h_t'(y)|^4 \right\rangle^{\frac{1}{4}} \, dy \right)^2 \, dx \right)^{1/2}
\]

\[
+ \int_{B_{3R}(Le) \cap B_{3R}^c} \left\langle |\nabla h_j(x)|^4 \right\rangle^{\frac{1}{4}} \int_{B_{3R}(Le) \cap \{|x-y| \leq L/4\}} |c(x - y)| \left\langle |\nabla h_t'(y)|^4 \right\rangle^{\frac{1}{4}} \, dy \, dx
\]
can be treated analogously. Finally, it remains to bound

\[ R^{-d} \int_{B_{3R}^c(Le)} \frac{1}{2} \left( \int_{B_{3R}^c(Le) \cap B_{3R}^c([x-y]>L/4)} |c(x-y)|^2 \, dx \right)^{1/2} dy \\
+ R^{-d/2} L^{-d} \int_{B_{3R}^c(Le) \cap B_{3R}^c} |c(x-y)| \left( |\nabla h_j^i(y)|^4 \right)^{1/4} dy \, dx \\
\lesssim R^{-d} R^{d/2} R^{-d/2-\alpha_0} + R^{-d/2} L^{-d/2} R^{-d} ||c||_{L^1} \lesssim R^{-d/2} L^{-d/2-\alpha_0}.
\]

Note that the term

\[ \int_{B_{3R}^c(Le) \cap B_{3R}} \left( |\nabla h_j(x)|^4 \right)^{1/4} \int_{B_{3R}^c(Le) \cap B_{3R}^c} |c(x-y)| \left( |\nabla h_j^i(y)|^4 \right)^{1/4} dy \, dx 
\]

can be treated analogously. Finally, it remains to bound

\[ \int_{B_{3R}^c(Le) \cap B_{3R}} \left( |\nabla h_j(x)|^4 \right)^{1/4} \int_{B_{3R}^c(Le) \cap B_{3R}^c} |c(x-y)| \left( |\nabla h_j^i(y)|^4 \right)^{1/4} dy \, dx \\
\lesssim \left( \int \left( |\nabla h_j(x)|^4 \right)^{1/4} \, dx \right)^{1/2} \left( \int_{B_{3R}} \left( \int_{B_{3R}(Le)} |c(x-y)| \left( |\nabla h_j^i(y)|^4 \right)^{1/4} \, dy \right)^2 \, dx \right)^{1/2} \\
\lesssim R^{-d} R^{d/2} L^{-d-\alpha_0} R^{d/2} \left( \int \left( |\nabla h_j^i(y)|^4 \right)^{1/4} dy \right)^{1/2} \lesssim R^{-d/2} L^{-d/2-\alpha_0}
\]

where we used estimate (3.12), the integrable decay (2.1) of $c$ together with the fact that $|x-y| \gtrsim L$ when $x \in B_{3R}$ and $y \in B_{3R}(Le)$.

\[ \square \]

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