Quantum fields on timelike curves

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Abstract

A quantum field $\Phi(x)$ exists at an event $x \in M$ of space-time $(M, g)$ in general only as a quadratic form $\Phi(x)$. Only after smearing $\Phi(x)$ with a smooth test function $f$ we get an operator $\Phi(f)$. In this paper the question is considered whether it is possible as well to smear $\Phi(x)$ with a singular test function $T$ (i.e. test distributions) supported by a smooth timelike curve $\gamma$. It is shown that this is always possible if $\Phi(x)$ satisfies the microlocal spectrum condition and $T$ belongs to a special class of distributions which retain some regularity in timelike directions (i.e. along $\gamma$). In the free field case these results are used to define some kind of time-translation along $\gamma$ which generalizes global space-time translations of Minkowski space.

1 Introduction

The most fundamental object of algebraic quantum field theory is a net $O \mapsto \mathcal{A}(O)$ which associates to each open and bounded region $O$ of space-time $(M, g)$ a *-algebra $\mathcal{A}(O)$ (in most cases a C* or von Neumann algebra) in such a way that the self adjoint elements of $\mathcal{A}(O)$ describe local observables measurable in $O$ (see [16] and the references therein for details). Many measurements in general relativity are, however, observer dependent and it seems to be reasonable to associate observables not only to space-time regions but also to worldlines. To demonstrate this with a simple example, consider the free scalar field $\Phi(x)$ in Minkowski space $(\mathbb{R}^4, \eta)$ and an observer with worldline $\gamma(t) = (t, x(t))$. (Note that this is in general not the proper time parameterization.) Quantities of the form

$$C(\gamma, f) = \Phi(T)^* \Phi(T)$$

with $\Phi(T) = \int_{\mathbb{R}^4} T(t, x) \Phi(t, x) dt dx$ and $T(t, x) = \delta(x - x(t)) f(t)$
where \( f : \mathbb{R} \to \mathbb{C} \) denotes a Schwartz function, describe simple, point-like particle detectors moving along \( \gamma \) (see [13] for a more detailed discussion of such models). From the works of Fulling [15], Unruh [29], Bisognano and Wichmann [2] and others we know that a uniformly accelerated observer \( \gamma_a \) sees the free field vacuum as a thermal state at finite temperature while it is of course a ground state with respect to an inertial observer \( \gamma_i \). With appropriately chosen \( f \) the observables \( C(\gamma_a, f) \) and \( C(\gamma_i, f) \) can therefore be used to distinguish the worldlines \( \gamma_a \) and \( \gamma_i \). More generally we can say that at least some information about the geometry of worldlines is reflected in the structure of the observable families \( \{C(\gamma, f) \mid f \in \mathcal{S}(\mathbb{R})\} \).

Hence for the study of observer dependent effects in quantum field theory it is reasonable to consider quantum fields “smeared” by distributional test “functions” \( T(x) \) with \( \text{supp} \, T \subset \text{Ran} \, \gamma \) and to study their relation to the geometry of \( \gamma \). A quantum field, however, is a very singular object, i.e. \( \Phi(x) \) exists in general only as quadratic form. It is therefore not clear whether objects like \( \Phi(T) \) in Equation (1) exists as an operator and how they should be defined in a mathematically precise way. For a free field in a globally hyperbolic space-time an analysis of this kind is partially carried out by Wollenberg [32, 33], showing that the algebras

\[
\mathcal{A}(\gamma) = \bigcup_{\mathcal{O} \supset \text{Ran} \, \gamma} \mathcal{A}(\mathcal{O}).
\]

which can be associated to any (i.e. not necessarily timelike) curve, contain some information about the causal character of \( \gamma \).

In this paper we are using methods from micro local analysis, in particular wave front sets of distributions, to consider a much bigger class of fields. The concept of wave front sets, originally introduced by Duistermaat and Hörmander [18, 11] in the context of hyperbolic partial differential equations, was recently applied with great success in quantum field theory: In [25] Radzikowski has shown that Hadamard states can be characterized in a very elegant way in terms of the wave front set of its two-point function. Based on this result Brunetti, Fredenhagen and Köhler gave in [5, 4] a micro local generalization of the spectrum condition. One reason why micro local methods are so useful in quantum field theory depends on the fact that wave fronts provides conditions under which the product of two distributions is defined. The existence of products of distributions is, however, closely related to the problem raised in the last paragraph. To explain this remark in greater detail consider a coordinate system \( u : M \supset M_u \ni p \mapsto u(p) := (t, x^1, \ldots, x^{n-1}) \in \mathbb{R}^n \) of space-time, such that \( t \mapsto u^{-1}(t, 0, \ldots, 0) \in M_u \subset M \) coincides with the (parametrized) world-line \( \gamma \), and a distribution of the form

\[
T(f) = \int_{\mathbb{R}} \sum_{|\alpha| \leq l} a_\alpha(t) \frac{\partial^{|\alpha|} f(t, 0)}{\partial x^\alpha} dt, \quad a_\alpha \in \mathcal{D}(\mathbb{R}) \forall |\alpha| \leq l
\]  

supported by \( \gamma \). The corresponding generalization of the expression \( \Phi(T) \) from
Equation (3) is now given by

\[ \Phi(T) = \int_{\mathbb{R}} \sum_{|\alpha| \leq l} a_\alpha(t) \frac{\partial^{|\alpha|} \Phi(t,0)}{\partial x^\alpha} \, dt, \tag{3} \]

i.e. \( \Phi(T) \) is, as before, the quantum field “smeared” by the singular test function \( T \). If we disregard for a moment the fact that \( \Phi \) is operator valued and not just a (numerical) distribution we can give the formal expression in Equation (3) a mathematical precise meaning by defining \( \Phi(T) := (\Phi T)(1) \), where \( \Phi T \) denotes the product of the distributions \( \Phi \) and \( T \) (which we assume to exist, of course) and \( 1 \in \mathcal{E}(M) \) is the function with \( 1 \equiv 1 \) (note that the support of the distribution \( \Phi T \) is compact if the support of \( T \) is compact, hence \( (\Phi T)(1) \) exists). Applying this idea to Wightman distributions and combining it with the reconstruction theorem of quantum field we can show (Theorem 5.1) that for a quantum field satisfying the micro local spectrum condition all the operators \( \Phi(T) \) exist with a common dense domain if the distribution \( T \) is regular or of the form given in Equation (2).

The second main result presented in this paper concerns a new look at representations of space-time translations. It is well known and in fact one of the major problems of quantum field theory in curved space-times that there is in a generic space-time no replacement for global space-time translations of Minkowski space. The possibility however to restrict a quantum field \( \Phi \) to a timelike curve \( \gamma \) leads naturally to the more general question whether there is a way to translate \( \Phi \) along \( \gamma \). We will show that at least in the free field case, this is indeed possible. To outline the corresponding construction, consider the C*-algebra \( \mathcal{A}^1(\gamma) \) generated by unitary operators \( \exp(i\Phi(T)) \) with test distributions \( T \) which are supported by \( \gamma \) and appropriately regular (more precisely \( T \) is given as in Equation (3) with \( l = 1 \); see Sect. 3 and 5 for details). Using a result of Demoen et al [7] we can show that there exists a family \( \alpha_t \) of completely positive maps on \( \mathcal{A}(\gamma) \) such that the translated Weyl operators \( \alpha_t[\exp(i\Phi(T))] \) coincide up to a numerical factor with \( \exp(i\Phi(T_t)) \) where \( T_t \) is the distribution pushed forward in the following way\(^1\)

\[ T_t(f) = \int_{\mathbb{R}} \sum_{|\alpha| \leq l} a_\alpha(t' - t) \frac{\partial^{|\alpha|} f(t',0)}{\partial x^\alpha} \, dt. \]

It is obvious that this is the most direct generalization of the space-time translation automorphisms of the Minkowski space theory. It is therefore reasonable to claim that the \( \alpha_t \) reflect the history of the observer with worldline \( \gamma \).

The paper is organized as follows: In Section 2 we will give a short summary of some well known material about quantum fields which will be used throughout the paper. Section 3 contains some general considerations about smearing quantum fields with distributions which are applied in Section 5 and 6.

\(^1\)This construction seems to depend on the coordinate system \((M, u)\) given above. We will see however that only a reference frame is needed to define \( T_t \) and this is a physically satisfactory dependency.
special class of distributions which is introduced in Section 4 (cf. Equation (2)). In Section 7 and 8 the free scalar field is treated as an example and in Section 9 the structures developed so far are used to define the time-translutions along worldlines outlined in the last paragraph. In the last Section we will discuss some ideas how the results of this paper can be used to study observer dependent aspects of quantum field theory. In the Appendix we have postponed some more technical proofs (Appendices B and D) and given some background material on micro local analysis, differential operators and jet-bundles, and the global Hadamard condition (Appendices A, C and E).

2 Quantum fields

We will start with a brief review of some well known material on quantum fields. Hence consider a (strongly causal) space-time \((M,g)\) and an hermitian quantum field \(\Phi(f)\), i.e. a map \(D(M) \ni f \mapsto \Phi(f) \in L(D_0, H)\) from the set \(D(M)\) of smooth, compactly supported, complex valued functions on \(M\) into the space \(L(D_0, H)\) of (unbounded) operators on a Hilbert space \(H\) with dense domain \(D_0 \subset H\). As usual \(\Phi\) should satisfy the following additional conditions 1. \(f \mapsto \langle u, \Phi(f)v \rangle\) is a distribution on \(M\) for all \(u,v \in D_0\), 2. The domain \(D_0\) is invariant, i.e. \(\Phi(f)D_0 \subset D_0\) for all \(f \in D(M)\), 3. There is a vector \(\Omega \in D_0\) cyclic for the *-algebra generated by all \(\Phi(f)\), 4. \(\Phi(f) = \Phi(f^*)D_0\) for all \(f \in D(M)\) and 5. \([\Phi(f), \Phi(h)]_u = 0\) for all \(u \in D_0\) and all \(f,h \in D(M)\) with spacelike separated supports. Cyclicity of the vacuum \(\Omega\) implies immediately that the span of expressions of the form \(\Phi(f_1) \cdots \Phi(f_n)\Omega\) defines a domain \(\tilde{D}_0 \subset D_0\), which is, as well as \(D_0\), dense and invariant. We will assume therefore without loss of generality that \(\tilde{D}_0 = D_0\) holds throughout this paper.

Consider now the Borchers-Uhlmann algebra, i.e. the topological tensor algebra

\[ A = 1 \oplus D(M) \oplus D(M^2) \oplus \cdots \oplus D(M^n) \oplus \cdots, \]

which is together with the map \(f^*(x_1, \ldots, x_n) = f(x_n, \ldots, x_1)\) a topological *-algebra. Each quantum field defines a state or Whightman functional \(W : A \to \mathbb{C}\) on \(A\) by

\[ W = 1 \oplus W^{(1)} \oplus W^{(2)} \oplus \cdots \]

where the \(n\)-point distributions \(W^{(n)}\) are given by

\[ W^{(n)}(f_1 \otimes \cdots \otimes f_n) = \langle \Omega, \Phi(f_1) \cdots \Phi(f_n)\Omega \rangle. \]

Each state \(W\) defines on the other hand a unique quantum field such that (4) and (5) hold. The cyclic representation \((H, \Phi, \Omega)\) of \(A\) related to \(\Phi\) by \(\Phi(f_1 \otimes \cdots \otimes f_n) = \Phi(f_1) \cdots \Phi(f_n)\) is the (unbounded operator version of the) well known GNS representation corresponding to \(W\).

\(^2\)A state is in this context a positive, continuous, linear functional. Note that in contrast to C*-algebras continuity is on \(A\) not implied by positivity.
From a physical point of view of greater importance as the fields itself are local von Neumann algebras to which the $\Phi(f)$ are affiliated. The most simple way to define such algebras is given if the common domain $D_0$ of the $\Phi(f)$ is a domain of essential self adjointness for all $\Phi(f)$ with real valued test function $f$. In this case we can associate to each open, relatively compact region $O \subset M$ of space-time the von Neumann algebra

$$R(O) := \left\{ e^{i\Phi(f)} \mid \mathcal{F} = f, \text{ supp } f \subset O \right\}'.$$  \hspace{1cm} (6)

The $R(O)$ form obviously an isotone family, i.e. if $O_1 \subset O_2$ holds, $R(O_1) \subset R(O_2)$ holds as well. This means that the family $(R(O))_{O \subset B(M)}$ forms a net of von Neumann algebras. Here $B(M)$ denotes the set of all admissible regions, i.e. $B(M) := \{ O \subset M \mid O \text{ open and compact } \}$. We will assume in addition that $(R(O))_{O \subset B(M)}$ is a causal net, i.e. the two algebras $R(O_1)$ and $R(O_2)$ commute if the corresponding regions are spacelike separated. Note that this property is not implied by the corresponding assumption on the fields (see [26], Sec. VIII.5). However under some additional assumptions (e.g. quasianalyticity of the vacuum vector and some kind of Reeh Schlieder theorem [1, Prop. 13.2.3]) causality of the net $(R(O))_{O \subset B(M)}$ can be derived from causality of the fields (see [1, Sec. 13.2.2] for details). Physically the $R(O)$ are interpreted in terms of bounded, local observables of the field. More precisely each self adjoint element of $R(O)$ describes a bounded, local observable of the fields measurable in the space-time region $O \subset M$.

If self adjointness of the $\Phi(f)$, as described above, is not given, the definition in Eq. (6) is not applicable. In this case we should use the weak commutant, which is given for an arbitrary set $P \subset \text{L}(D_0, \mathcal{H})$ of (unbounded) operators by

$$P'_w := \{ A \in B(\mathcal{H}) \mid \langle B^\ast u, Av \rangle = \langle A^\ast u, Bv \rangle \forall B \in P \forall u, v \in D_0 \}.$$  \hspace{1cm} (7)

Now we can define von Neumann algebras $R(O)$ alternatively by

$$R(O) := \left\{ (\{ \Phi(f) \mid \text{ supp } f \subset O \})'_w \right\}'.$$  \hspace{1cm} (7)

If the $\Phi(f)$ are essentially self adjoint for real valued $f$ Eqs. (6) and (7) are equivalent [1, Sec. 13.2.1, 13.2.2], which justifies the usage of the same symbol. The advantage of (7) is that it works without additional assumptions. In passing we will note here that it is at least under special assumptions on the geometry of $(M, g)$ (e.g. existence of a transitively acting isometry group; see [1, Ch. 14]) also possible to start with a net $(R(O))_{O \subset B(M)}$ and to construct the fields. However we will not use these results here. Instead we will always assume that the fields exist in the described way.

We have not yet talked about symmetries and translational invariance of the vacuum, which plays a central role in Minkowski space quantum field theory. The reason is that these concepts are, due to the lack of a nontrivial isometry group, almost useless in a generic space-time. However there are some promising ideas to replace at least the spectrum condition by some assumptions on the wave front.
set of the n-point distributions of the field $\Phi(f)$. A good choice for our purposes is the micro local spectrum condition introduced by Brunetti, Fredenhagen and Köhler.\footnote{These two conditions are not present in the original definition. However they only single out degenerate cases: A graph $G$ with no edges makes obviously not much sense in the current context and edges with $s(e) = t(e)$ do not contribute to the sum in Equation (8), because $k(e, x_i)$ and $k(e^{-1}, x_i) = -k(e, x_i)$ occurs there due to items 4 and 6.}

**Definition 2.1.** Let us consider the set $\text{Gr}_n$ of finite, unordered graphs with vertices $\{1, \ldots , n\}$ whose edges always occur in both admissible directions. Assume further that no graph $G \in \text{Gr}_n$ has an empty set of edges and that no edge has the same vertex as source and target.\footnote{These two conditions are not present in the original definition. However they only single out degenerate cases: A graph $G$ with no edges makes obviously not much sense in the current context and edges with $s(e) = t(e)$ do not contribute to the sum in Equation (8), because $k(e, x_i)$ and $k(e^{-1}, x_i) = -k(e, x_i)$ occurs there due to items 4 and 6.} An immersion of $G \in \text{Gr}_n$ is a triple $(x, \gamma, k)$ of maps such that:

1. $x$ maps vertices of $G$ to points of $M$.
2. $\gamma$ maps edges of $G$ to piecewise smooth curves in $M$ with source $s(\gamma(e)) = x(s(e))$ and target $t(\gamma(e)) = x(t(e))$.
3. $k$ maps edges to covariantly constant, causal covector fields $k(e)$ along $\gamma(e)$.
4. The edge $e^{-1}$ with the opposite direction of $e$ is mapped by $\gamma$ to the curve $\gamma(e^{-1})$ inverse to $\gamma(e)$.
5. $k$ is future pointing iff the source $s(e)$ of $e$ is smaller than its target $t(e)$ and
6. $k(e^{-1}) = -k(e)$.

A quantum field $\Phi$ satisfies the microlocal spectrum condition (μSC) if the wave front sets of its $n$–point functions satisfy

$$\text{WF}(\mathcal{W}^{(n)}) \subset \{(x_1, k_1; \ldots ; x_n, k_n) \in T^* M^n \setminus \{0\} | \exists G \in \text{Gr}_n \exists \text{ immersion } (x, \gamma, k) \text{ of } G \text{ such that } x(i) = x_i \forall i = 1, \ldots , m$$

$$\text{and } k_i = \sum_{s(e) = i} k(e; x_i)\}.$$  \hspace{1cm} (8)

It is shown in\footnote{These two conditions are not present in the original definition. However they only single out degenerate cases: A graph $G$ with no edges makes obviously not much sense in the current context and edges with $s(e) = t(e)$ do not contribute to the sum in Equation (8), because $k(e, x_i)$ and $k(e^{-1}, x_i) = -k(e, x_i)$ occurs there due to items 4 and 6.} that at least Wick ordered products of free fields (including free fields itself) belong to this class. Applications concerning interacting fields, especially renormalizability, can be found in\footnote{These two conditions are not present in the original definition. However they only single out degenerate cases: A graph $G$ with no edges makes obviously not much sense in the current context and edges with $s(e) = t(e)$ do not contribute to the sum in Equation (8), because $k(e, x_i)$ and $k(e^{-1}, x_i) = -k(e, x_i)$ occurs there due to items 4 and 6.}.

### 3 Quantum fields with singular test functions

It is well known that quantum fields do not exist as operator valued fields but only as operator valued distributions. This means that it is in general impossible to define something like $\Phi(x) = \Phi(\delta_x)$ as an operator (here $\delta_x$ denotes the delta-distribution at $x \in M$). However this does not imply that it is impossible to...
evaluate a quantum field on any distribution. A natural way to define $\Phi(T)$ for a distribution $T \in \mathcal{E}'(M)$ is to consider limits of the form $\lim_{k \to \infty} \Phi(T_k)u$ for an element $u \in D$ and a sequence $\mathbb{N} \ni k \mapsto T_k \in \mathcal{D}(M)$ of smooth functions converging weakly to $T$. However this idea has the drawback that it is not clear whether limits of this kind (if they exist) depend on the chosen sequence. It is therefore more reasonable to consider convergence in the space $\mathcal{D}'(M)$ of distributions with wave front set contained in a closed cone $\Gamma \subset T^*M$ (See Appendix A for a review of some material about wave front sets). If we have an additional closed cone $\Sigma$ which contains the wave front sets of all distributions $f \mapsto \langle u, \Phi(f)v \rangle$ and satisfies $\Gamma \oplus \Sigma := \{ (x, \xi_1 + \xi_2) \in T^*M \mid (x, \xi_1) \in \Gamma, (x, \xi_2) \in \Sigma \} \subset T^*M \setminus \{0\}$, (9)

(i.e. there is no element of the form $(x, 0)$ in $\Gamma \oplus \Sigma$) it follows from Theorem A.9 that limits $\lim_{k \to \infty} \Phi(T_k)u$ (if they exist) depend on $T$ but not on the sequence $\mathbb{N} \ni k \mapsto T_k \in \mathcal{D}(M)$. Hence we can define:

**Definition 3.1.** Consider a compactly supported distribution $T \in \mathcal{E}'(M)$ such that $\text{WF}(T) \subset \Gamma$ holds with a closed cone $\Gamma$ satisfying (9) and a sequence $\mathbb{N} \ni k \mapsto T_k \in \mathcal{D}(M)$ converging in $\mathcal{D}'(M)$ to $T$. We define $\Phi(T) : D_0 \to \mathcal{H}$ as the unique (if it exists) operator satisfying $\| \cdot \| - \lim_{k \to \infty} \Phi(T_k)u = \Phi(T)u$ for all $u$.

Consider now a linear subspace $\mathfrak{D}$ of $\mathcal{E}'(M)$ such that $\mathcal{D}(M) \subset \mathfrak{D}$ and such that $\Phi(T)$ exists for each $T \in \mathfrak{D} \setminus \mathcal{D}(M)$. We can define in analogy to the Borchers-Uhlmann algebra $\mathfrak{A}$ a tensor algebra $\mathfrak{A}(\mathfrak{D})$ generated by “test-distributions”

$$\mathfrak{A}(\mathfrak{D}) := \mathbb{C} \oplus \mathfrak{D} \oplus (\mathfrak{D} \otimes \mathfrak{D}) \oplus \cdots \oplus \mathfrak{D}^n \cdots .$$

(10)

Note that all tensor products and direct sums in this expression are **purely algebraic** (to avoid topological difficulties), i.e.

$$\mathfrak{D}^n := \text{span}\{T_1 \otimes \cdots \otimes T_n \mid T_j \in \mathfrak{D} \} \subset \mathcal{D}'(M^n)$$

and the direct sum means **finite direct sums**.

The purpose of this *-algebra is to carry an extension $\mathfrak{W}$ of the Wightman functional $W$ which is simply given by

$$\mathfrak{W}^{(n)}(T_1 \otimes \cdots \otimes T_n) = \langle \Omega, \Phi(T_1) \cdots \Phi(T_n) \Omega \rangle .$$

(11)

However this functional is defined only if the operators $\Phi(T)$ can be extended to an invariant, dense domain $D \subset \mathcal{H}$. If the existence of such a domain is not a priori known it is more convenient to define $\mathfrak{W}^{(n)}$ directly as a continuous extension of $W^{(n)}$ to a distribution space $\mathcal{D}'_\Gamma(M^n)$, where $\Gamma \subset T^*M$ is a closed cone with $\Gamma \oplus \text{WF}(W^{(n)}) \subset T^*M^n \setminus \{0\}$ (cf. Theorem A.3 and Theorem A.5). This idea motivates the following definition.
Definition 3.2. To each monomial $T = T^{(1)} \otimes \cdots \otimes T^{(n)} \in \mathcal{D}^{\otimes n}$ we can associate a closed cone $\Gamma(T) \subset T^* M^n \setminus \{0\}$ which is recursively defined by the following properties

1. $n = 1$ implies $WF(T) = \Gamma(T)$,
2. if $T \in \mathcal{D}^{\otimes n}$ and $S \in \mathcal{D}^{\otimes m}$ the following recursion relation holds (cf. Equation (54))

$$
\Gamma(T \otimes S) = \Gamma(T) \odot \Gamma(S) = (\Gamma(T) \times \Gamma(S)) \cup ([M^n \times \{0\}] \times \Gamma(S) \cup \Gamma(T) \times [M^m \times \{0\}])
$$

A quantum field $f \mapsto \Phi(f)$ with $n$–point functions $W^{(n)}$ is called extendible to a distribution space $\mathcal{D} \subset \mathcal{E}'(M)$ with $\mathcal{D}(M) \subset \mathcal{D}$ if $\Gamma(T) \odot WF(W^{(n)}) \subset T^* M^n \setminus \{0\}$ is satisfied for all $n \in \mathbb{N}_0$ and all $T \in \mathcal{D}^{\otimes n}$.

The family of cones $\Gamma(T)$ is chosen in such a way that 1. $WF(T) \subset \Gamma(T)$ holds and 2. the sequence $N \ni j \mapsto T_j \otimes S_j \in \mathcal{D}(M^{n+m})$ converges in $\mathcal{D}'(M^{n+m})$ to $T \otimes S$ if $j \mapsto T_j$ and $j \mapsto S_j$ converge in $\mathcal{D}'(M^n)$ respectively $\mathcal{D}'(M^m)$ to $T$ respectively $S$ (cf. Proposition A.7). Hence we can define a functional $\mathfrak{W}$ on the algebra $\mathfrak{A}(\mathfrak{D})$ by

$$
\mathfrak{W}^{(n)}(T^{(1)} \otimes \cdots \otimes T^{(n)}) := \lim_{l \to \infty} \mathcal{W}^{(n)}(T^{(1)}_l \otimes \cdots \otimes T^{(n)}_l),
$$

where the sequences $N \ni l \mapsto T^{(j)}_l \in \mathcal{D}(M), \quad j = 1, \ldots, n$

converge in $\mathcal{D}'(T^{(1)} \otimes \cdots \otimes T^{(n)})$ to $T^{(j)}$. By Theorem A.8 this limit exists and depends only on $T = T^{(1)} \otimes \cdots \otimes T^{(n)}$. If we consider in particular a regular $T$, i.e. $T \in \mathcal{D}(M^n)$ we can choose the constant sequence $l \mapsto T$ in Equation (13) which converges in $\mathcal{D}'(M^n)$ to $T$ for any $\Gamma$. Hence we get $\mathcal{W}^{(n)}(T) = \mathfrak{W}^{(n)}(T)$ in this case, and this means that $\mathfrak{W}$ is really an extension of $\mathcal{W}$. Summarizing this discussion we get the following:

Proposition 3.3. If the quantum field $f \mapsto \Phi(f)$ is extendible to a distribution space $\mathcal{D} \subset \mathcal{E}'(M)$ its $n$–point functions $W^{(n)}$ can be extended to $\mathcal{D}^{\otimes n}$ in exactly one way such that

$$
\mathfrak{W}^{(n)}(T) = \lim_{l \to \infty} \mathcal{W}^{(n)}(T_l),
$$

holds for any sequence $N \ni l \mapsto T_l \in \mathcal{D}(M^n)$ converging in $\mathcal{D}'(T)(M^n)$ to $T$. 

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Let us reconsider now our original definition of $W$ in Equation (11). It is natural to ask whether it coincides with the expression given in Proposition 3.3. This includes in particular the question whether extendibility of $\Phi(f)$ in the sense of Definition 3.2 implies the existence of $\Phi(T)$ for all $T \in \mathcal{D}$ (cf. Definition 3.1) and of an invariant dense domain $D \subset H$. The following theorem states that this is indeed the case.

**Theorem 3.4.** A quantum field $f \mapsto \Phi(f)$ which is extendible to the distribution space $\mathcal{D} \subset \mathcal{E}'(M)$ has the following properties

1. $\Phi(T)$ exists for all $T \in \mathcal{D}$ in the sense of Def. 3.1.
2. There is a dense, invariant (i.e. $\Phi(T)D \subset D$ for all $T \in \mathcal{D}$) domain $D$ with $\Omega \in D$ and $D \subset D(\Phi(T))$ for all $T \in \mathcal{D}$.
3. For each sequence $\mathbb{N} \ni l \mapsto T_l \in D(O)$ converging in $D'_O(T)$ to a monomial $T = (T^{(1)} \otimes \cdots \otimes T^{(n)}) \in \mathcal{D} \otimes^n$ the limit
   
   $$\lim_{l \to \infty} \Phi(T_l)\Omega = \Phi(T_l^{(1)})\cdots \Phi(T_l^{(n)})\Omega$$

   exists and coincides with $\Phi(T)\Omega = \Phi(T^{(1)})\cdots \Phi(T^{(n)})\Omega$.

The proof of this theorem is somewhat lengthy and technical. Therefore we have postponed it to Appendix B. Let us consider now local von Neumann algebras. In analogy to Eq. (7) we can define

$$\tilde{\mathcal{R}}(O) := \left( \{ \Phi(T) \mid T \in \mathcal{D}(O) \} \right)_{\text{w}}' \text{ with } \mathcal{D}(O) = \{ T \in \mathcal{D} \mid \text{supp } T \subset O \} \quad (15)$$

Obviously we have $\mathcal{R}(O) \subset \tilde{\mathcal{R}}(O)$. The next proposition says that even equality holds.

**Proposition 3.5.** For a quantum field $\Phi(f)$ extendible to $\mathcal{D} \subset \mathcal{E}'(M)$ we have

$$\mathcal{R}(O) = \left( \{ \Phi(T) \mid T \in \mathcal{D}(O) \} \right)_{\text{w}}'.$$

where $\mathcal{R}(O)$ denotes the local von Neumann algebra defined according to Equation (13) and $\mathcal{D}(O)$ is given in Equation (15).

**Proof.** We have to show that

$$\langle A^*u, \Phi(f)v \rangle = \langle \Phi(f)^*u, Av \rangle \quad \forall u, v \in D_0 \quad \forall f \in \mathcal{D}(O) \quad (16)$$

is equivalent to

$$\langle A^*u, \Phi(T)v \rangle = \langle \Phi(T)^*u, Av \rangle \quad \forall u, v \in D \quad \forall T \in \mathcal{D}(O) \quad (17)$$

The implication (17) $\Rightarrow$ (16) is trivial because we have $\mathcal{D}(O) \subset \mathcal{D}(O)$ and $D_0 \subset D$. To prove the other direction note that there are, according to Theorem 3.4 item 3 (cf. also 3.4), sequences $l \mapsto T_l = T^{(1)}_l \otimes \cdots \otimes T^{(n)}_l \in \mathcal{D}(M^n)$ and
because \( \Phi(\mathbf{T}) \) follows.

Proof. See [19, Theorem 2.3.5].

In addition we have \( \lim_{t \to \infty} \Phi(T)x = \Phi(T)x \) and \( \lim_{t \to \infty} \Phi(T)x = \Phi(T)x = \Phi(T)^*x \) for each \( x \in D \). If \( A \) satisfies Equation (16) we get

\[
\langle A^* \Phi(T) \Omega, \Phi(T) \Phi(S_l) \Omega \rangle = \langle \Phi(T) \Phi(S_l) \Omega, A \Phi(T) \Omega \rangle
\]

because \( \Phi(T) \Omega \in D_0 \) and \( \Phi(S_l) \Omega \in D_0 \). Taking the limit \( l \to \infty \) Equation (17) follows.

4 Distribution supported by smooth curves

The purpose of this paper is the study of quantum fields which are concentrated on timelike curves, or, using the terminology of the last section, to extend quantum fields to distributions which are supported by such curves. To proceed in this direction, it is useful to discuss first some properties of this special kind of distribution fields to distributions which are supported by such curves. To proceed in this direction, it is useful to discuss first some properties of this special kind of distributions. Hence let us consider a (not necessarily timelike) smooth curve \( \gamma : (a, b) \to M \) and a compactly supported distribution \( T \) with \( \text{supp} T \subset \text{Ran}(\gamma) \).

In an appropriate coordinate system \( T \) can be represented obviously by a distribution \( \tilde{T} \in \mathcal{E}'(\mathbb{R}^n) \) with support on \( \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}^{n-1} \). Therefore the following theorem tells us something about the basic structure of \( T \):

**Theorem 4.1.** Consider a compactly supported distribution \( T \in \mathcal{E}'(\mathbb{R}^n) \) of order \( k \) with support contained in \( \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}^{n-1} \). Then we have for a smooth test function \( \mathbb{R} \times \mathbb{R}^{n-1} \ni (t, x) \mapsto f(t, x) \in \mathbb{C} \)

\[
T(f) = \sum_{|\alpha| \leq k} T_\alpha(f_\alpha) \quad \text{with } f_\alpha(t) = \frac{\partial^{|\alpha|} f(t, 0)}{\partial x^\alpha},
\]

where the \( T_\alpha \) are compactly supported distributions on \( \mathbb{R} \) of order \( k - |\alpha| \).

*Proof.* See [13, Theorem 2.3.5].

This result gives us a lower bound on the wave front set of \( T \).

**Proposition 4.2.** Consider a distribution \( T \in \mathcal{E}'(M) \) of finite order and with support contained in the image of the curve \( \gamma \). Then we have

\[
\text{WF}(T) \supset \{ (\gamma(t), \theta) \in T^*M \mid \gamma(t) \in \text{supp} T, \ \theta \cdot \gamma'(t) = 0 \}.
\]

*Proof.* Without loss of generality we can assume that \( T \in \mathcal{E}'(\mathbb{R}^n) \) holds with \( \text{supp} T \subset \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}^{n-1} \). Hence to calculate the wave front set of \( T \) we have to consider its Fourier transform:

\[
\mathbb{R} \times \mathbb{R}^{n-1} \ni (\rho, \xi) \mapsto \hat{T}(\rho, \xi) = T(e^{-i(\rho, \xi)}).
\]

According to the definition of the wave front set we have to calculate the Fourier transform of \( fT \) for appropriate functions in \( \mathcal{D}(\mathbb{R}^n) \). However this would not affect the proof substantially, in other words we can assume without loss of generality that \( f \equiv 1 \) holds.
Using Theorem 4.1 this leads to

$$\hat{T}(\rho, \xi) = \sum_{|\alpha| \leq k} (-i)^{|\alpha|} \xi^\alpha \hat{T}_\alpha(\rho)$$

(19)

where the $\hat{T}_\alpha$ are, as in Theorem 4.1, distributions on $\mathbb{R}$. This equation shows that the function $\mathbb{R}^+ \ni \lambda \mapsto \hat{T}(0, \lambda \xi)$ grows polynomially for each $\xi \in \mathbb{R}^{n-1}$. Hence $(t, 0; 0, \xi) \in (\mathbb{R} \times \mathbb{R}^{n-1}) \times (\mathbb{R} \times \mathbb{R}^{n-1})$ can not be a regular directed point whenever $(t, 0) \in \text{supp} \ T$.

The discussion of the last section shows that the class of quantum fields which are extendible to a special distribution space is bigger if the wave front set of these distributions is as small as possible. Hence we will concentrate in the following our discussion to those $T$, where $\text{WF}(T)$ exactly coincides with the lower bound derived in the last proposition. Hence let us define:

**Definition 4.3.** A distribution $T \in \mathcal{E}'(M)$ of finite order with $\text{supp} \ T \subset \text{Ran}(\gamma)$ for a smooth curve $\gamma : (a, b) \rightarrow M$ is called as regular as possible if

$$\text{WF}(T) = \{ (\gamma(t), \theta) \in T^* M | \gamma(t) \in \text{supp} \ T, \ \theta \cdot \gamma'(t) = 0 \}$$

holds. We will denote the space of all distributions of this kind with $\mathcal{D}^\infty(\gamma)$ or simply $\mathcal{D}(\gamma)$. The subspace of all order $l$ distributions in $\mathcal{D}(\gamma)$ is denoted by $\mathcal{D}^l(\gamma)$.

For the rest of this section we will develop a special “parametrization” of $\mathcal{D}^l(\gamma)$ in terms of jet-bundles (see Appendix C for a short review of this concept). The first step in this direction is the following proposition.

**Proposition 4.4.** The spaces $\mathcal{D}^l(\gamma)$ just defined can be characterized alternatively by: $T \in \mathcal{D}^l(\gamma) \iff T = P^\dagger T_\gamma$, i.e. $T(f) = T_\gamma(Pf)$, where $T_\gamma$ is the distribution given by

$$T_\gamma(f) = \int_a^b f(\gamma(t)) dt,$$

$P$ is a $l^{th}$ order differential operator defined around $\gamma$ and $P^\dagger$ denotes its formal adjoint.

*Proof.* As in the proof of Proposition 4.2 we can assume without loss of generality that $T \in \mathcal{E}'(\mathbb{R}^n)$ holds, with $\text{supp} \ T \subset \mathbb{R} \times \{0\}$. In this case we have $T = PT_\gamma$ with a differential operator $P$ iff all the distributions $T_\alpha$ in Equation (18) are regular, i.e. $T$ has the form (using the notations of Theorem 4.1)

$$Tf = \sum_{|\alpha| \leq l} \int_\mathbb{R} a_\alpha(t) \frac{\partial^{|\alpha|} f(t, 0)}{\partial x^\alpha} dt,$$

where the $a_\alpha$ are smooth, compactly supported functions on $\mathbb{R}$. It is easy to see (cf. Prop. A.12 and A.13) that each distribution of this kind has wave front set as in Equation (20).
To prove the other implication let us assume that \( T \) has the general form of \( \text{Theorem 4.4} \) and that its wave front set satisfies \( \text{Equation (19)} \). Hence the Fourier transform of \( T \) is given by \( \text{Equation (19)} \). Since \( \text{WF}(T) \) does not contain an element of the form \((t,0;\rho,0)\) we get for \( \hat{T} \) (cf. \text{Footnote 3})

\[
|\hat{T}(\rho,0)| = |\hat{T}_0(\rho)| \leq \frac{C_N}{(1 + |\rho|)^N} \quad \forall \rho \in \mathbb{R} \quad \forall N \in \mathbb{N},
\]

where \( \hat{T}_0 \) denotes \( \hat{T}_\alpha \) with \( \alpha = 0 \) and \( C_N \) is a constant. This implies obviously that \( T_0 \) is regular. For an higher order multiindex \( \alpha \) we can modify this argument by considering the Fourier transform of the product \( p_\alpha T \) where \( p_\alpha \) denotes the monomial \( \mathbb{R} \times \mathbb{R}^{n-1} \ni (t,x) \rightarrow p_\alpha(t,x) = i^{|\alpha|}t^\alpha \). Since \( \text{WF}(p_\alpha T) \subset \text{WF}(T) \) (cf. \text{Proposition A.13}) we get similar to the case \( \alpha = 0 \) the inequality

\[
|\hat{p_\alpha T}(\rho,0)| = \left| \frac{\partial^{|\alpha|} \hat{T}(\rho,0)}{\partial \xi^\alpha} \right| = |\hat{T}_\alpha(\rho)| \leq \frac{C_{N,\alpha}}{(1 + |\rho|)^N} \quad \forall \rho \in \mathbb{R} \quad \forall N \in \mathbb{N}.
\]

Hence all the distributions \( T_\alpha \) are regular, i.e. \( T_\alpha \in \mathcal{D}(\mathbb{R}) \) as stated.

The differential operator \( P \) of \text{Proposition 4.4} can be expressed according to \text{Proposition C.1} by a section \( \psi \) of the vector bundle \( \mathcal{J}^l(U, \mathbb{C}) \), where \( U \) is an open neighbourhood of \( \gamma \). To determine the distribution \( T \) it is even sufficient to know the value of \( \psi \) only on the curve \( \gamma \). Hence the distribution space \( \mathcal{D}^l(\gamma) \) can be parametrized by \( l \)-jets as described in the following theorem.

**Theorem 4.5.** Consider the space \( \mathcal{D}(\text{Ran}(\gamma), E^l(\gamma)) \) of smooth, compactly supported sections of the vectorbundle \( E^l(\gamma) := \mathcal{J}^l(M, \mathbb{C}) |\gamma \) which in turn denotes the restriction of the dual of the \( l \)-jet bundle \( \mathcal{J}^l(M, \mathbb{C}) \) to \( \text{Ran}(\gamma) \). Then there is a surjective linear map

\[
\mathcal{D}(M, E^l(\gamma)) \ni \psi \mapsto T_\psi \in \mathcal{D}^l(\gamma) \quad \text{with} \quad T_\psi(f) = \int_a^b \psi(t) j^l_{\gamma(t)} f(t) \, dt.
\]

Hence we have

\[
\mathcal{D}^l(\gamma) = \{ T_\psi \in \mathcal{D}(M) | \psi \in \mathcal{D}(M, E^l(\gamma)) \} \quad \text{and} \quad \mathcal{D}(\gamma) = \bigcup_{l \in \mathbb{N}} \mathcal{D}^l(\gamma). \quad (21)
\]

**Proof.** The statement is an immediate conqueence of \text{Proposition A.4} and C.1.

The map \( \psi \mapsto T_\psi \) just defined is, as stated, surjective but not injective, in other words we can not associate a unique section \( \psi \in \mathcal{D}(M, E^l(\gamma)) \) to a distribution \( T \in \mathcal{D}^l(\gamma) \). To understand the reason consider again \( T \in \mathcal{E}'(\mathbb{R}^n) \) and \( \gamma(t) = (t,0) \in \mathbb{R} \times \mathbb{R}^{n-1} \). A \( j \)-th order differential operator along \( \gamma \) is given by

\[
(Pf)(t) = \sum_{|\alpha| + j \leq l} a_{\alpha,j}(t) \frac{\partial^{|\alpha| + j} f(t,0)}{\partial x^\alpha \partial t^j}
\]
the corresponding distribution is therefore
\[ T(f) = \int_{\mathbb{R}} \sum_{|\alpha| + j \leq l} a_{\alpha,j}(t) \frac{\partial^{|\alpha|+j} f(t,0)}{\partial x^\alpha \partial t^j} dt. \]

By partial integration this is equivalent to
\[ T(f) = \int_{\mathbb{R}} \sum_{|\alpha| + j \leq l} (-1)^j a_{\alpha,j}(t) \frac{\partial^{|\alpha|} f(t,0)}{\partial t^j} \partial x^\alpha dt. \]

Hence the operator
\[ (\tilde{P} f)(t) = \sum_{|\alpha| + j \leq l} (-1)^j a_{\alpha,j}(t) \frac{\partial^{|\alpha|} f(t,0)}{\partial t^j} \partial x^\alpha \]
leads to the same distribution. This observation motivates the next proposition.

**Proposition 4.6.** Consider a coordinate chart \((M_a, u)\) defined around \(\gamma\) (i.e. \(\text{Ran}(\gamma) \subset M_a\)) and an open neighbourhood \(V\) of \(0 \in \mathbb{R}^{n-1}\) with the following properties: \(u(M_a) = (a, b) \times V\) and \(u(\gamma(t)) = (t, 0)\). For each distribution \(T \in \mathcal{D}'(\gamma)\) there is exactly one \(\psi \in \mathcal{D}(\text{Ran} \gamma, E^l(\gamma))\) with \(T = T_\psi\) and with local representative of the form \((f_u := f \circ u\) with \(f \in \mathcal{D}((a, b) \times V)):\)
\[
(P_\psi f_u)(t) = (\psi \circ j^i f_u)(t) = \sum_{|\alpha| \leq l} a_{\alpha}(t) \frac{\partial^{|\alpha|} f(t,0)}{\partial x^\alpha} \tag{22}
\]

with \((t,x) \in (a,b) \times V, f_u = f \circ u\) and \(f \in \mathcal{D}(\text{Ran} \gamma \times V)\).

**Proof.** We have just seen that an operator of this kind exist for all \(T \in \mathcal{D}'(\gamma)\). To prove the uniqueness identify \(T\) with its local representative, i.e. assume \(T \in \mathcal{E}'((a,b) \times V)\) and consider for each \(t \in (a,b)\) the restriction \(T|\{(t) \times V\}\) of \(T\) to the submanifolds \(\{t\} \times V\), which exists due to Theorem [A.2]. \(T|\{(t) \times V\}\) is for all \(t \in (a,b)\) a distribution in \(\mathcal{E}'(V)\) with support equal to \(\{0\}\). Hence \(T|\{(t) \times V\}\) is a finite linear combination of derivatives of the delta-distribution.

In other words
\[
(T|\{(t) \times V\})(h) = \sum_{|\alpha| \leq l} b_{\alpha} \frac{\partial^{|\alpha|} h(0)}{\partial x^\alpha}
\]

with test function \(h \in \mathcal{D}(V)\). However if \(T\) is given as in Equation (22) we get
\[
(T|\{(t) \times V\})(h) = \sum_{|\alpha| \leq l} a_{\alpha}(t) \frac{\partial^{|\alpha|} h(0)}{\partial x^\alpha}.
\]

This leads to the condition \(a_{\alpha} = b_{\alpha}\) which fixes the differential operator \(P_\psi\) and therefore the section \(\psi\). \(\square\)
It is useful to note at this point that the condition on \( \psi \in D(\text{Ran} \gamma, E^l(\gamma)) \) which makes the map \( \psi \mapsto T \phi \) unique, is coordinate dependent. More precisely, the class of sections defined implicitly in Proposition 4.6 depends on the \( l \)-jet \( j^l u|_\gamma \) of the coordinate map along \( \gamma \). For us, this is not a problem, because we will always have a canonical choice for \( j^l u|_\gamma \).

**Corollary 4.7.** Consider a timelike curve \( \gamma \) and an orthonormal frame \( e_\nu, \nu = 0, \ldots, 3 \) along \( \gamma \) with \( e_0(t) = \gamma'(t) \). Assume in addition without loss of generality that \( 0 \in \text{Dom} \gamma \). Then we can choose an open neighbourhood \( U \) of \( 0 \in \mathbb{R}^4 \) such that the map

\[
\hat{U} \ni (t, x) \mapsto \exp_{\gamma(t)} \left( \sum_{i=1}^3 x^i e_i(t) \right) \in U \subset M
\]

defines a coordinate system around \( \text{Ran} \gamma \cap U \). For each \( T \in D^l(\gamma) \) there is exactly one \( \psi \in D(\text{Ran} \gamma, E^l(\gamma)) \) such that Proposition 4.6 holds with the coordinate system just introduced.

**Proof.** The statement is a simple consequence of elementary properties of the exponential map and of Proposition 4.6.

\[\square\]

5 Quantum fields along worldlines

Now we are able to consider quantum fields which are concentrated on timelike curves. This implies especially that \( \gamma \) denotes a timelike curve for the rest of the paper. The space of “test distributions” \( D \) to which quantum fields should be extended (cf. Definition 3.2) is naturally defined as the union of \( D(M) \) and all possible \( D(\gamma) \), i.e.

\[
D(M) := D(M) \cup \bigcup_{\gamma \text{ smooth, timelike}} D(\gamma).
\]

Note that in the union on the left hand side of this equation all possible smooth, timelike curves occur, including those which are only reparametrizations of one another. However it is not necessary to consider only distinguished parametrizations (e.g. only proper time parametrizations) because the spaces \( D^l(\gamma) \) are independent of the parametrization of \( \gamma \).

It is reasonable to assume that physically realistic models can be extended to this special space of test distributions because we can show that this is true for all quantum fields satisfying \( \mu \text{SC} \) (see Sec. 2):

**Theorem 5.1.** Each quantum field satisfying \( \mu \text{SC} \) can be extended in the sense of Prop. 3.4 to the test distribution space \( D(M) \) defined in Eq. (24).

**Proof.** According to Definition 3.2 we have to check that there is no element of the form \( (x_1, \ldots, x_n; 0, \ldots, 0) \) in \( WF(W^{(n)}) \oplus \Gamma(T) \) where \( T := T^{(1)} \otimes \ldots \otimes T^{(n)} \) is an element of \( D(M)^{\otimes n} \).
To compute \( \Gamma(\mathcal{T}) \) let us consider first the case \( n = 2 \). By Equation (12) we have

\[
\Gamma(T^{(1)} \otimes T^{(2)}) =
\]

\[
(\text{WF}(T^{(1)}) \times \text{WF}(T^{(2)})) \cup ([\text{supp}(T^{(1)}) \times \{0\}] \times \text{WF}(T^{(2)})) \cup
\]

\[
(\text{WF}(T^{(1)}) \times [\text{supp}(T^{(2)}) \times \{0\}]).
\]

The elements of \( \mathcal{D}(M) \) are either regular (i.e. smooth function) or concentrated on smooth timelike curves \( \gamma_1, \gamma_2 \) (i.e. \( T^{(j)} \in \mathcal{D}^{(j)}(\gamma) \)). Hence we get, due to Proposition A.12 and A.13 in combination with Proposition 4.4:

\[
\Gamma(T^{(1)} \otimes T^{(2)}) =
\]

\[
(N(\gamma_1) \times N(\gamma_2)) \cup ([M \times \{0\}] \times N(\gamma_2)) \cup (N(\gamma_1) \times [M \times \{0\}]),
\]

where \( N(\gamma) \subset T^*M \) denotes the normal bundle of the curve \( \gamma \), i.e.

\[
N(\gamma) = \{ k \in T^*M \mid k(\gamma') = 0 \}.
\]

Assume now that \( \mathcal{T} \) is an \( n \)-fold tensor product. By applying the arguments just discussed recursively we see (although an explicit calculation is somewhat involved) that \( (x_1, \ldots, x_n; k_1, \ldots, k_n) \in \Gamma(\mathcal{T}) \) implies \( k_j = 0 \) or \( k_j \) spacelike for all \( j = 1, \ldots, n \).

The wave front set of \( \mathcal{W}^{(n)}(\gamma) \) is, according to \( \mu SC \), given by Definition 2.4. Hence consider a finite graph \( \mathcal{G} \in \text{Gr}_n \) and an immersion \((x, \gamma, k) \in \mathcal{G} \). Since the set of nodes of \( \mathcal{G} \) is finite there is at least one node \( j \in \{1, \ldots, n\} \) such that 1. the set of edges starting in \( j \) is not empty and 2. for each edge \( e \) with \( s(e) = j \) we have \( t(e) > s(e) \). (In Definition 2.4 we have excluded graphs without edges and the case \( s(e) = t(e) \); see footnote 3.) Together with item 5 of Definition 2.1 and Equation (8) this implies that there is for each \( (x_1, \ldots, x_n; k_1, \ldots, k_n) \in \text{WF}(\mathcal{W}^{(n)}) \) at least one \( j = 1, \ldots, n \) such that \( k_j \in T^*M \) is causal. This shows together with the property of \( \text{WF}(\mathcal{T}) \) derived in the last paragraph that there is no element of the form \( (x_1, \ldots, x_n, 0, \ldots, 0) \in \text{WF}(\mathcal{W}^{(n)}) \oplus \text{WF}(\mathcal{T}) \) and this completes the proof.

Let us consider now the extension \( \mathcal{W}^{(n)} \) of the \( n \)-point function \( \mathcal{W}^{(n)}(\gamma) \) defined in Proposition 3.3. The parametrization of \( \mathcal{D}^{(1)}(\gamma) \) in terms of jet-bundles allows us to interpret the \( \mathcal{W}^{(n)} \) as a family of vector bundle valued distributions. To do this let us introduce the \textit{external tensor product} of \( k \) respectively \( j \) copies of the bundles \( E^{\gamma}(\gamma) \) and \( M \times \mathbb{C} \):

\[
E^{(\gamma)} \otimes (M \times \mathbb{C}) = \bigcup_{(t_1, \ldots, t_k) \in (a, b)^k, (x_1, \ldots, x_j) \in M^j} E^{\gamma(t_1)}(\gamma) \otimes \cdots \otimes E^{\gamma(t_k)}(\gamma) \otimes \{x_1\} \times \mathbb{C} \otimes \cdots \otimes \{x_j\} \times \mathbb{C}.
\]

\[\text{Note that the usual tensor product symbol } \otimes \text{ is occupied in the context of vector bundles already for another construction (the ordinary tensor product).}\]
where $E_{l}(t_{i})(\gamma)$ denotes the fiber of $E^{l}(\gamma)$ over $\gamma(t_{i})$. Using in addition the map $D(\text{Ran } \gamma, E^{l}(\gamma)) \ni \psi \mapsto T_{\psi} \in D^{l}(\gamma)$ defined in Theorem 4.3, we get the following:

**Proposition 5.2.** The linear map

$$D((\text{Ran } \gamma)^{k} \times M^{j}, E^{l}(\gamma)^{\otimes k} \boxtimes (M \times \mathbb{C})^{\otimes j}) \ni \psi_{1} \boxtimes \cdots \boxtimes \psi_{k} \boxtimes f_{1} \boxtimes \cdots \boxtimes f_{j} \mapsto \mathcal{W}_{l}(T_{\psi_{1}} \otimes \cdots \otimes T_{\psi_{k}} \otimes f_{1} \otimes \cdots \otimes f_{j})$$

(25)

is a vector bundle valued distribution.

**Proof.** The statement follows from the fact that the functional defined in Equation (25) coincides in the case $l = 0$ with the restriction of $\mathcal{W}(l; n)(\gamma)$ to $(\text{Ran } \gamma)^{k} \times M^{j}$. If $l > 0$ we have to decompose the functional in terms of a natural frame of the vector bundle $E^{l}(\gamma)^{\otimes k} \boxtimes (M \times \mathbb{C})^{\otimes j}$ (cf. Equation (61)). Each component we get in this way is a differential of the restriction considered in the $l = 0$ case, hence a distribution.

Of particular importance for us is the case $j = 0$, because we get a family of distributions

$$\mathcal{W}_{l}(\gamma)(\psi_{1} \boxtimes \cdots \boxtimes \psi_{n}) = \mathcal{W}(n)(T_{\psi_{1}} \otimes \cdots \otimes T_{\psi_{n}})$$

(26)

which defines a state on the *-algebra

$$\mathfrak{A}^{l}(\gamma) := \mathbb{C} \oplus D(\text{Ran } \gamma, E^{l}(\gamma)) \oplus D((\text{Ran } \gamma)^{2}, E^{l}(\gamma) \boxtimes E^{l}(\gamma)) \oplus \cdots ,$$

(27)

where the *-operation is given by

$$(\theta_{1} \boxtimes \cdots \boxtimes \theta_{n})^{*} := \overline{\theta_{n}} \boxtimes \cdots \boxtimes \overline{\theta_{1}}$$

and

$$\overline{\theta(j(t), f)} := \overline{\theta(j(t)_{f})}$$

Constructing the generalized GNS-representation (cf. Section 3) we get a $E^{l}(\gamma)$-valued quantum field $\Phi_{\gamma,l}$. The following Theorem shows how it is related to the original field $\Phi$.

**Theorem 5.3.** Consider the Hilbert space

$$\mathcal{H}^{l}(\gamma) := (\text{span}(\Phi(T_{1}) \cdots \Phi(T_{n})\Omega \mid T_{1} \otimes \cdots \otimes T_{n} \in D^{l}(\gamma)^{\otimes n}, \ n \in \mathbb{N})\| - \text{cl})$$

and the operators $\Phi^{l}_{\gamma}(\psi) := \Phi(T_{\psi})|\mathcal{H}^{l}(\gamma)$. Then the map

$$D(\text{Ran } \gamma, E^{l}(\gamma)) \ni \psi \mapsto \Phi^{l}_{\gamma}(\psi) := \Phi(T_{\psi})$$

defines an $E^{l}(\gamma)$-valued quantum field with $n$-point function (26).

**Proof.** This statement is an easy consequence of the definitions and of Proposition 5.2.  \(\square\)
Note that the quantum field $\Phi_\gamma$ carries less information as the map $D_l(\gamma) \ni T \mapsto \Phi(T)$. However $\Phi_\gamma$ is much easier to study, because it is described completely in terms of the Wightman functional $W_l^\gamma$. The full field operators $\Phi(T)$ always need the knowledge of $W_l$, which is a functional on $D(M)$ rather than $D_l(\gamma)$. Hence a lot of geometry not directly related to $\gamma$ comes in. To avoid this complication is the major reason for us to introduce the fields $\Phi_l^\gamma$.

6 Local algebras

Let us consider now local von Neumann algebras. In analogy to Eq. (7) we can define for each smooth timelike curve $\gamma : (a,b) \to M$ and for each $k \in \mathbb{N} \cup \{\infty\}$ a von Neumann algebra $R_l^\gamma$ by

$$R_l^\gamma := \left(\{\Phi(T) \mid T \in D_l(\gamma)\}_w\right)'$$

Since $J_l(M,\mathbb{C})^*$ is for $l < k$ a subbundle of $J_k(M,\mathbb{C})$ we have $D((\text{Ran} \gamma), E_l^k(\gamma)) \subset D((\text{Ran} \gamma), E_k^l(\gamma))$ and therefore $D_l^\gamma \subset D_k^\gamma$. Hence the definition of the $R_l^\gamma$ implies immediately that $R_l^\gamma$ is a subalgebra of $R_k^\gamma$ if $l < k$ holds. In other words we get the infinite sequence

$$R^0(\gamma) \subset R_1^\gamma \subset \cdots \subset R_l^\gamma \subset \cdots \subset R^\infty(\gamma) \subset R(\gamma)$$

where we have introduced the additional algebra

$$R(\gamma) = \bigcap_{\mathcal{O} \in \mathcal{B}(\gamma)} R(\mathcal{O}), \quad \mathcal{B}(\gamma) := \{\mathcal{O} \in \mathcal{B}(M) \mid \gamma((a,b)) \subset \mathcal{O}\}.$$  

The relations $R_l^\gamma \subset R(\gamma)$ for all $l \in \mathbb{N} \cup \{\infty\}$ stated in Eq. (29) follow directly from Prop. 3.5.

To interpret the algebras $R_l^\gamma$ and $R(\gamma)$ let us review first some simple material about timelike curves and worldlines (for a detailed exposition of observers and reference frame in general relativity see the book of Sachs and Wu [27]). First it is useful to distinguish between parametrized curves, i.e. smooth maps $\gamma : (a,b) \to M$ and paths (curves without a distinguished parametrization) i.e. an equivalence class of parametrized curves (where two curves are defined to be equivalent if there is a smooth, strictly monotone reparametrization). Each parametrized curve $\gamma$ determines a unique path which we will denote by $\gamma$ as well, as long as confusion can be omitted. A timelike path describes physically the worldline of an observer, while the choice of a special parametrization fixes the clock used by the observer. Among all possible parametrizations of a given path the proper time parametrizations are distinguished by the condition $g(\gamma'(t), \gamma'(t)) = -1$ for all $t \in (a,b)$. Physically proper time is measured by so called standard clocks which are experimentally approximated up to a very high degree of accuracy by atomic clocks.

Let us come back now to the algebras $R_l^\gamma$ and $R(\gamma)$. Since $R(\mathcal{O})$ contains observables measurable in the region $\mathcal{O}$ the definition of $R(\gamma)$ in Eq. (30) implies
immediately that $A = A^* \in \mathcal{R}(\gamma)$ is an observable which is measurable in any region containing $\gamma$, in other words measurable along $\gamma$. This means we can interpret self adjoint elements of $\mathcal{R}(\gamma)$ as bounded observables the observer $\gamma$ can measure in the time-interval $(a, b)$. In the special case of $A = A^* \in \mathcal{R}^l(\gamma)$ with finite $l$ only observables depending at most on $l^{th}$ derivatives of the field are considered. The algebras $\mathcal{R}^\infty(\gamma)$ lie somewhat intermediate between $\mathcal{R}(\gamma)$ and $\mathcal{R}^l(\gamma)$ with $l \in \mathbb{N}$. It is reasonable to conjecture that $\mathcal{R}(\gamma) = \mathcal{R}^\infty(\gamma)$ holds for physically realistic models (cf. Section 8).

The set of $T \in D^l(\gamma)$ is not changed by reparametrizations and hence the algebras $\mathcal{R}^l(\gamma)$ are identical for worldlines belonging to the same path. This is very plausible from a physical point of view, because the set of observables measurable during certain period should not depend on the clock with which time is measured. From a more formal point of view this means that we should consider the set of smooth timelike path as the index set of the $\mathcal{R}^l(\gamma)$ (and not parametrized curves). If we introduce in addition the ordering relation $\gamma_1 \subset \gamma_2 \iff \text{Ran} \gamma_1 \subset \text{Ran} \gamma_2$ we get again an isotone family of von Neumann algebras which is causal in the same way as the family $\mathcal{R}(\mathcal{O})$. Hence the $\mathcal{R}^l(\gamma)$ form as well as the $\mathcal{R}^l(\mathcal{O})$ a causal net of von Neumann algebras.

Let us consider now the state $\mathcal{R}^l(\gamma) \ni A \mapsto \langle \Omega, A \Omega \rangle$. Its GNS representation is $(\mathcal{H}^l(\gamma), \eta^l, \Omega)$ with

$$\mathcal{R}^l(\gamma) \ni A \mapsto \eta^l(A) = P^l_\gamma AP^l_\gamma \in B(\mathcal{H}^l(\gamma)), \quad (31)$$

where $P^l_\gamma$ denotes the projection onto $\mathcal{H}^l(\gamma)$ (cf. Theorem 5.3). This representation is related to the fields $\Phi^l_\gamma$ by

$$\mathcal{M}^l_\gamma(\mu) := \eta^l_\gamma(\mathcal{R}^l(\mu))^\prime \prime = \{ \Phi^l_\gamma(\psi) \mid \psi \in D(\text{Ran} \gamma, E^l(\mu)) \}_w^\prime, \quad (32)$$

with $\mu \subset \gamma$. A state $\rho$ on $\mathcal{R}^l(\gamma)$ is normal with respect to $\eta^l_\gamma$ if the observer $\gamma$ can prepare $\rho$ out of the vacuum using only operations from $\mathcal{R}^l(\gamma)$. We will see that this is a serious restriction as long as $l$ is finite. Hence considering the algebras $\mathcal{M}^l_\gamma(\mu)$ instead of $\mathcal{R}^l(\mu)$ means to consider limited measuring possibilities of the observer. However the $\mathcal{M}^l_\gamma(\mu)$ are easier to study than the $\mathcal{R}^l(\mu)$, due to their direct relation to the fields $\Phi^l_\gamma$ and the corresponding Wightman functionals $W^l_\gamma$, cf. the discussion at the end of Section 5.

## 7 The free scalar field

Let us consider now the free scalar field on a globally hyperbolic $(M, g)$ spacetime as a particular example. We will start with a short review of this model (see [31] and the references therein for details).

As a consequence of global hyperbolicity there exist unique advanced and retarded fundamental solutions $G^\pm$ of the Klein-Gordon equation (see Appendix). This means there are continuous operators $G^\pm : D(M, \mathbb{R}) \to E(M, \mathbb{R})$

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with \( G^\pm (\Box - m^2)f = (\Box - m^2)G^\pm f = f \) and \( \text{supp} \, G^\pm (f) \subset J^\pm (\text{supp} \, f) \) for all \( f \in \mathcal{D}(M, \mathbb{R}) \). Here \( \Box \) denotes d’Alembertian on \((M, g)\) and \( J^\pm \) denotes the causal past/future. \( G^\pm \) gives rise to an antisymmetric bilinear form \( G : \mathcal{D}(M, \mathbb{R}) \times \mathcal{D}(M, \mathbb{R}) \) by

\[
G(f, h) = \int_M Gf(x)h(x)\lambda_g(x),
\]

where \( Gf = G^+ f - G^- f \). Since \( G \) is degenerate it is not a symplectic form on \( \mathcal{D}(M, \mathbb{R}) \). In other words, it is necessary to consider the quotient space \( \mathcal{P} := \mathcal{D}(M, \mathbb{R})/\ker G \). This leads to the real symplectic vector space \((\mathcal{P}, G)\) and to the corresponding CCR-algebra \( \text{CCR}(\mathcal{P}, G) \). Alternatively we can consider an arbitrary (smooth) Cauchy surface \( \Sigma \subset M \) and the space of compactly supported, smooth initial data, i.e. \( \mathcal{D}(\Sigma, \mathbb{R}^2) \) which is a symplectic space too, if we equip it with the symplectic form

\[
\tilde{G}(f_1, k_1; f_2, k_2) = \int_\Sigma f_1(x)k_2(x)\lambda_\Sigma(x) - \int_\Sigma f_2(x)k_1(x)\lambda_\Sigma(x),
\]

where \( \lambda_\Sigma \) denotes the natural (i.e. induced by the metric) volume form on \( \Sigma \). A symplectic isomorphism between \( \mathcal{P} \) and \( \mathcal{D}(\Sigma, \mathbb{R}^2) \) is given by the map

\[
\mathcal{P} \ni f \mapsto \iota_\Sigma(f) := \left( (Gf)|\Sigma, [\partial_t (Gf)]|\Sigma \right) \in \mathcal{D}(\Sigma, \mathbb{R}^2).
\]

A quasi-free, regular state \( \omega \) on this C*-algebra is given by a “one particle structure” i.e. a real linear function \( K : \mathcal{P} \to \mathcal{K} \) into a separable Hilbert space \( \mathcal{K} \), which is symplectic: \( \text{Im} (K(f), K(h)) = G(f, h) \) and has the property that \( \text{Ran} \, K + i \text{Ran} \, K \) is dense in \( \mathcal{K} \). The state \( \omega \) is then defined by its generating functional \( \omega (W(f)) = \exp (-\frac{1}{4}\| K(f) \|^2) \). Consider now the GNS representation \((\mathcal{H}, \pi, \Omega)\) of \( \omega \) and the unitary operators \( \pi (W(f)) \) (for \( f \in \mathcal{P} \)). It is well known that we can identify \( \mathcal{H} \) with the symmetric Fock space \( \mathcal{F}_+(\mathcal{K}) \), the cyclic vector \( \Omega \) with the corresponding Fock vacuum and the Weyl operators \( \pi (W(f)) \) with the exponentials \( \exp [i\Phi (K(f))] \) of Segal operators \( \Phi (K(f)) \). The free scalar quantum field \( \Phi (f) \) on \((M, g)\) in the (quasi free) state \( \omega \) is given by complex linear extension of the map \( \mathcal{D}(M, \mathbb{R}) \ni f \mapsto \Phi (f) \). It has the two-point function

\[
W^{(2)}(f \otimes h) = \langle K(f), K(h) \rangle = \text{Re} \langle K(f), K(h) \rangle + ig(f, h).
\]

For real valued \( f \) the operators are essentially self adjoint. Hence the von Neumann algebras \( \mathcal{R}(\mathcal{O}) \) can be defined according to Equation (1). However for this particular model there is a more direct way to define the \( \mathcal{R}(\mathcal{O}) \) without explicit use of the fields. If we introduce first a causal net of C*-subalgebras of \( \text{CCR}(\mathcal{P}, G) \) by

\[
\mathcal{B}(M) \ni \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) := \text{C}^\ast \{ W(f) \in \text{CCR}(\mathcal{P}, G) \mid \text{supp} \, f \subset \mathcal{O} \} \subset \text{CCR}(\mathcal{P}, G),
\]

### Footnote

1If confusion can be avoided we will identify in the following functions \( f \in \mathcal{D}(M, \mathbb{R}) \) and the corresponding equivalence classes \( f \in \mathcal{P} \).
we get
\[ \mathcal{R}(\mathcal{O}) = \{ \pi(W(f)) | \text{supp } f \subset \mathcal{O} \}^{''} = \pi(\mathcal{A}(\mathcal{O}))^{''}. \quad (37) \]

To complete the description of the model we have to restrict the class of admissible states \( \omega \), because there is, due the lack of a translation group as in Minkowski space, no distinguished vacuum state. As a replacement we have to assume that \( \omega \) is, as a physically reasonable state, a “Hadamard state”. This means that \( \omega \) has to satisfy the global Hadamard condition, first introduced in mathematically rigorous way by Kay and Wald in [21]. Physically this condition says that the two point function \( W^{(2)} \) has the same short distance behavior as the Minkowski vacuum. The exact definition of a Hadamard state is however somewhat involved. Therefore we have postponed it to the Appendix E.

An equivalent, but simpler characterisation of Hadamard states, introduced by Radzikowski [25] can be given in terms of the wave front set of \( W^{(2)} \): A quasi-free, regular state \( \omega \) is a Hadamard state iff \( \text{WF}(W^{(2)}) \) is given by
\[ \text{WF}(W^{(2)}) = \{ (x_1, k_1; x_2, k_2) \in T^*(M \times M) \setminus 0 | (x_1, k_1) \sim (x_2, -k_2), k_1 \text{ future pointing } \}, \quad (38) \]
where \( (x_1, k_1) \sim (x_1, -k_1) \) means that \( (x_1, k_1) \) and \( (x_2, -k_2) \) are on the same null geodesic strip (cf. Appendix D).

Eq. (38) and Definition 2.1 imply immediately that a Hadamard state satisfies the microlocal spectrum condition. Hence we can apply Theorem 5.1 which implies that the algebras \( \mathcal{R}^l(\gamma) \) exist and we can use all the machinery introduced in Section 5. However in this particular case there is a more direct way to define these von Neumann algebras because we can associate to each smooth worldline \( \gamma \) and each \( l \in \mathbb{N} \cup \{ \infty \} \) a C*-subalgebra \( \mathcal{A}^l(\gamma) \) of \( \text{CCR}(\mathcal{P}, G) \) such that \( \mathcal{R}^l(\gamma) = \pi(\mathcal{A}^l(\gamma))^{''} \) holds. The explicit construction is described by the following proposition.

**Proposition 7.1.** Consider the unique, weakly continuous extension of \( G^\pm \) to \( \mathcal{E}'(M, \mathbb{R}) \), i.e. \( G^\pm(T)(f) = T(G^\pm(f)) \) (cf. Appendix E) and the subspace \( \mathfrak{D}(M, \mathbb{R}) \) consisting of real valued elements of \( \mathfrak{D}(M) \).

1. \( GT = G^+T - G^-T \) is for each \( T \in \mathfrak{D}(M, \mathbb{R}) \) a smooth function with support contained in \( J^+(\text{supp } T) \cup J^-(\text{supp } T) \).

2. The distribution \( G \in \mathcal{D}'(M \times M, \mathbb{R}) \) is extendible to \( \mathfrak{D}(M, \mathbb{R}) \otimes \mathfrak{D}(M, \mathbb{R}) \) and has the form
   \[ G(T \otimes S) = S(G(T)) = -T(G(S)) = \frac{1}{2i}(\mathfrak{W}^{(2)}(T \otimes S) - \mathfrak{W}^{(2)}(S \otimes T)), \]
   where \( \mathfrak{W}^{(2)} \) is the (extension to \( \mathfrak{D}(M) \) of the) two-point function associated to a Hadamard state.
3. For each $T \in \mathcal{D}(M, \mathbb{R})$ there is an $f \in \mathcal{D}(M, \mathbb{R})$ such that $GT \equiv Gf$. Hence the symplectic space $(\mathcal{D}(M, \mathbb{R})/\ker G, G)$ can be identified with $(\mathcal{P}, G)$ in a natural way.

Proof. Item 1. From Corollary D.3 we know that $\WF(G \pm T) \subset \WF(T)$, hence $\WF(GT) \subset \WF(T)$ holds as well. But $GT$ is at the same time a solution of the homogeneous Klein-Gordon equation. Together with Theorem D.2 this implies that each $\theta \in \WF(GT)$ is a null covector. Consequently the characterization of $T \in \mathcal{D}((\Ran \gamma, \mathbb{R})$ in Definition 4.3 leads to $\WF(GT) = \emptyset$, which is equivalent to regularity of $GT$.

Item 2. From Equation (33) we know that $G$ is simply the imaginary part of the two-point function. Together with Proposition A.11 this implies $\WF(G) \subset \WF(W^{(2)})$. Hence we see by Lemma B.3 and the characterization of Hadamard states in Equation (33) that $G(T \otimes S)$ exists and can be defined by $G(T \otimes S) = \lim_{k \to \infty} G(T_k \otimes S_k)$, where $N \ni k \mapsto T_k \in \mathcal{D}(M, \mathbb{R})$, $N \ni l \mapsto S_l \in \mathcal{D}(M, \mathbb{R})$ are sequences converging in $\mathcal{D}_{\WF}(T)$ respectively $\mathcal{D}_{\WF}(S)$ to $T$ and $S$. On the other hand weak continuity of the map $\mathcal{E}'(M, \mathbb{R}) \ni T \mapsto G(T) \in \mathcal{D}'(M, \mathbb{R})$ implies that

$$
\lim_{k \to \infty} \int_M G(T_k)(x)S_l(x)\lambda(x) = \int_M G(T)(x)S_l(x)\lambda(x)
$$

holds where $G(T)$ denotes the smooth solution of the Klein-Gordon equation consider in item 1. Since $S$ is compactly supported, the latter integral equals $S(G(T))$, which was to show.

Item 3. By 2 we know that $GT$ is a smooth solution of the Klein-Gordon equation with support contained in $J^+(\text{supp } T) \cup J^-(\text{supp } T)$. Hence $GT$ has smooth, compactly supported initial data $((GT)|_{\Sigma}, [\partial_t(GT)]|_{\Sigma})$ on each Cauchy surface $\Sigma$. This implies together with bijectivity of the map $\iota_\Sigma$ from Equation (34) that $GT = Gf$ holds for each $f \in \mathcal{D}(M, \mathbb{R})$ with $\iota_\Sigma(f) = ((GT)|_{\Sigma}, [\partial_t(GT)]|_{\Sigma})$.  

The last proposition shows that we can define a Weyl element $W(T) \in \CCR\mathcal{P}, G)$ for each distribution $T \in \mathcal{D}'(M, \mathbb{R})$. It is therefore natural to associate the $C^*$-Algebra

$$
\mathcal{A}^l(\gamma) := C^*\{W(T) | T \in \mathcal{D}'(\gamma)\} \subset \CCR\mathcal{P}, G)
$$

(39)
to each $\gamma$ and $l$. Hence the von Neumann algebras $\mathcal{R}^l(\gamma)$ are given by $\mathcal{R}^l(\gamma) = \pi(\mathcal{A}'^l(\gamma))''$, in analogy to Equation (37). To get the algebras $\mathcal{M}_\gamma^l(\mu)$ defined in Equation (32) we only have to consider $\pi^l_\gamma(\mathcal{A}(\mu))' = \mathcal{M}^l_\gamma(\mu)$, where $\pi^l_\gamma$ denotes the GNS representation of the state $\omega^l_\gamma = \omega^l_\gamma \mathcal{A}(\gamma)$. The fields $\Phi(T)$ and $\Phi^l_\gamma(T)$ can be derived from the representations $\pi$ and $\pi^l_\gamma$ by the conditions $\exp(i\Phi(T)) = \pi(W(T))$ and $\exp(i\Phi^l_\gamma(T)) = \pi^l_\gamma(W(T))$.

If the subspace $\mathcal{P}^l(\gamma)$ generated by equivalence classes of distributions $T \in \mathcal{D}'(\gamma)$ is a symplectic subspace of $\mathcal{P}$ (i.e. if the restriction of $G$ to $\mathcal{P}^l(\gamma) \times$
functions and tempered distributions. The Hadamard form given in Appendix E. E.g. for the previous sections. In our context it mainly serves as an explicit example for the abstract concepts introduced in Chapter P.\cr
\mbox{}

\textbf{Algebra CCR}\(\mathcal{P}_l(\gamma)\) is non-degenerate) we can look at \(\omega_1\) as a quasi-free state on the CCR-algebra CCR(\(\mathcal{P}_l(\gamma),G\)) with a two point function which can be derived from the Hadamard form given in Appendix E\(^8\). E.g. for \(l = 0\) this leads to

\[
\mathcal{W}_\gamma^{(0:2)}(f \otimes h) = \lim_{\epsilon \to 0} \int_{\mathbb{R} \times \mathbb{R}} \left[ \frac{1}{(2\pi)^2} \frac{\sqrt{\Delta_\gamma(t_1,t_2)}}{\sigma_\gamma(t_1,t_2) + 2i\epsilon(t_1 - t_2) + \epsilon^2} + V_\gamma^{(n)}(t_1,t_2) \ln(\sigma_\gamma(t_1,t_2) + 2i\epsilon(t_1 - t_2) + \epsilon^2) \right] + H_{n,\gamma}(t_1,t_2) f(t_1) h(t_2) dt_1 dt_2,
\]

where \(\sigma_\gamma, \Delta_\gamma, V_\gamma^{(n)}\) and \(H_{n,\gamma}\) are given in terms of the corresponding functions from Appendix E\(^8\), i.e. \(\sigma_\gamma(t_1,t_2) = \sigma(\gamma(t_1),\gamma(t_2)), \Delta_\gamma(t_1,t_2) = \Delta(\gamma(t_1),\gamma(t_2)), V_\gamma^{(n)}(t_1,t_2) = V^{(n)}(\gamma(t_1),\gamma(t_2))\) and \(H_{n,\gamma}(t_1,t_2) = H_{(n,\gamma)}(\gamma(t_1),\gamma(t_2))\). If \(\gamma\) is a proper time parameterized geodesic the expression is simplified by \(\sigma_\gamma(t_1,t_2) = (t_1 - t_2)^2\). However even in this case \(\mathcal{W}_\gamma^{(0:2)}\) (and therefore the fields \(\Phi^{(0)}\)) contains geometric information of space time \((\mathbb{M},g)\), because the functions \(\Delta_\gamma\) and \(V_\gamma^{(n)}\) depend on derivatives of \(g\) of higher order along \(\gamma\) (cf. the discussion of the fundamental solutions in Appendix E\(^8\)). Another problem arises from the fact that it is not clear in general whether \(\mathcal{P}_l(\gamma)\) is really a symplectic subspace of \(\mathcal{P}\). To demonstrate this consider again \(l = 0\). In this case the restriction of \(G\) to \(\mathcal{P}_0(\gamma) \times \mathcal{P}_0(\gamma)\) is non-degenerate if the restriction of the function \(GT\) to \(\text{Ran}\,\gamma\) does not vanish for \(T \neq 0 \in \mathcal{P}\). It seems to be plausible that the latter condition is true at least if we are considering only a very small neighborhood of an event \(p \in \mathbb{M}\) (such that we are in some sense “close” to Minkowski space). Unfortunately it is difficult to derive a precise proof from this intuition. If \(G\) is really degenerate on \(\mathcal{P}_l(\gamma)\) the C*-closure of the *-algebra generated by \(W(T), T \in \mathcal{D}(\gamma,\mathbb{R})\), is not unique and we need additional information about the embedding of \(\mathcal{A}(\gamma)\) into CCR(\(\mathcal{P},G\)).

\section{A simple example in Minkowski space}

To make the discussion of the last section more transparent, we will discuss in the following a simple example in Minkowski\(^8\) space \((\mathbb{R}^4,\gamma)\). Hence let us introduce some notations first. The one particle Hilbert space \(\mathcal{K}\) is given by \(L^2(\mathbb{R}^3,d\xi)\) and the commutator function \(G\) is \(G(f \otimes h) = \text{Im}(Kf,Kh)\) where\(^4\)

\(^{4}\)Please note that most of the material presented in this section is quite well known \(^8\). In our context it mainly serves as an explicit example for the abstract concepts introduced in the previous sections.

\(^{8}\)Since we are on Minkowski space it is reasonable to consider in this section Schwartz functions and tempered distributions.
$K : \mathcal{S}(\mathbb{R}^4, \mathbb{R}) \to \mathcal{K}$ is the real linear function

\[
(Kf)(\xi) = \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^4} \frac{e^{i(\lambda(\xi)t - \xi \cdot x)}}{\sqrt{\lambda(\xi)}} f(t, x) dt dx.
\]  

(40)

with $\lambda(\xi) = \sqrt{|\xi|^2 + m^2}$. The generating functional of the Minkowski vacuum is given as well in terms of $K$ and has the form

\[
\mathcal{P} \ni [f] \mapsto \omega(W([f])) = \exp \left( -\frac{1}{4} \|K(f)\|_2^2 \right) = \langle \Omega, \exp(i\Phi_S(Kf))\Omega \rangle,
\]

where $\Phi_S(Kf)$ denotes the Segal operator on the Fock space $\mathcal{F}_+(\mathcal{K})$. Therefore we get $\Phi(f) = \Phi_S(Kf)$ for the free field. Alternatively we can write

\[
\Phi(f) = \int_{\mathbb{R} \times \mathbb{R}^3} \Phi(t, x)f(t, x) dt dx,
\]

with the quadratic form

\[
\Phi(t, x) = \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} \left( A(\xi)e^{i(\lambda(\xi)t + \xi \cdot x)} + A^*(\xi)e^{-i(\lambda(\xi)t - \xi \cdot x)} \right) \frac{d\xi}{\sqrt{2\lambda(\xi)}}.
\]

$A^*(\xi), A(\xi)$ denote here the well known creation and annihilation “operators”\textsuperscript{10} at $\xi \in \mathbb{R}^3$.

Let us consider now an inertial observer $\gamma_i : \mathbb{R} \ni t \mapsto (t, 0) \in \mathbb{R}^4$. A distribution $T$ of $\mathcal{D}^i(\gamma_i)$ can be expressed by

\[
T(f) = \sum_{|\alpha| \leq l} \int_{\mathbb{R}} a_{\alpha}(t) \frac{\partial^{|\alpha|} f(t, 0)}{\partial x^\alpha} dt
\]

(cf. Proposition 4.6). Therefore the map $K$ can be extended to $\mathcal{D}^i(\gamma_i)$ by

\[
KT(\xi) = \sum_{|\alpha| \leq l} (-i)^{|\alpha|} \hat{a}_{\alpha}(\lambda(\xi)) \sqrt{\lambda(\xi)} \xi^\alpha.
\]  

(41)

This follows easily by calculating the Fourier transform of $T$ and restricting $\hat{T}$ to the mass shell $\{(\lambda(\xi), \xi) \in \mathbb{R}^4 | \xi \in \mathbb{R}^3\}$. This implies for the field operators $\Phi(T) = \Phi_S(KT)$ and therefore:

\[
\Phi(T) = \sum_{|\alpha| \leq l} a_{\alpha}(t) \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} (-i)^{|\alpha|} \xi^\alpha \left( A(\xi)e^{i\lambda(\xi)t} + A^*(\xi)e^{-i\lambda(\xi)t} \right) \frac{d\xi}{\sqrt{2\lambda(\xi)}}
\]

\[
= \sum_{|\alpha| \leq l} a_{\alpha}(t) \frac{\partial^{|\alpha|} \phi(t, 0)}{\partial x^\alpha}.
\]

\textsuperscript{10} $A^*(\xi)$ is defined only as a quadratic form on $\mathcal{H} = \mathcal{F}_+(\mathcal{K})$. 

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The local von Neumann algebras $\mathcal{R}^l(\gamma_i)$ are given by

$$\mathcal{R}^l(\gamma_i) = \{ \exp(i\Phi_S(f)) \mid f \in K^l(\gamma_i) \}''$$

with

$$K^l(\gamma_i) = \{ KT \mid T \in D^l(\gamma_i, \mathbb{R}) \} \subset K.$$  \hspace{1cm} (42)

These subspaces are best studied in polar coordinates $\mathbb{R}^+ \times S^2 \ni (r, \kappa) \mapsto r\kappa \in \mathbb{R}^3 \setminus \{0\}$. If we identify $L(\mathbb{R}^3, d^3\xi)$ and $L(\mathbb{R}^+, r^2 dr) \otimes L(S^2, d\Omega(\kappa))$ (where $d\Omega$ denotes the standard surface element of $S^2$) with respect to the unitary transformation $L(\mathbb{R}^3, d^3\xi) \ni f \mapsto \hat{f} \in L(\mathbb{R}^+, r^2 dr) \otimes L(S^2, d\Omega)$ given by $\hat{f}(r, \kappa) = f(r\kappa)$ we get the following lemma

**Lemma 8.1.** The closure of the real linear subspace $K^l(\gamma_i)$ defined in Equation (12) is complex linear and has the form

$$\overline{K^l(\gamma_i)} = L(\mathbb{R}^+, r^2 dr) \otimes \mathcal{Y}_l,$$

where $\mathcal{Y}_l \subset L(S^2, d\Omega(\kappa))$ is the (finite dimensional) subspace generated by spherical harmonics up to the order $l$.

**Proof.** Let us consider first the space

$$\tilde{Z}_j = \{ Z_j(f) \mid f \in S(\mathbb{R}, \mathbb{R}) \} \subset L^2(\mathbb{R}^+, r^2 dr)$$  \hspace{1cm} (43)

with

$$\mathbb{R}^+ \ni r \mapsto (\tilde{Z}_j f)(r) = \frac{\hat{f}(\lambda(r))}{\sqrt{\lambda(r)}} r^j \in \mathbb{C},$$

where we have used $\lambda(r) = \sqrt{r^2 + m^2}$ in slight abuse of notation. $Z_j$ is a complex linear subspace of $L(\mathbb{R}^+, r^2 dr)$, although only real valued Schwartz functions are used in Equation (43). To see this, note that $\overline{f} = f \iff \hat{f}(\lambda) = \hat{f}(-\lambda) \forall \lambda \in \mathbb{R}$ holds for any Schwartz function $f \in S(\mathbb{R}, \mathbb{C})$ and that only the restriction of $\hat{f}$ to $\mathbb{R}^+_m := (m, \infty)$ enters in $Z_j f$. Therefore if $f \in S(\mathbb{R}, \mathbb{R})$ we can replace $i\hat{f}$ by an $h \in S(\mathbb{R}, \mathbb{C})$ such that $h(\lambda) = i\hat{f}(\lambda)$ and $h(-\lambda) = h(\lambda) = -i\hat{f}(-\lambda)$ holds for all $\lambda \in \mathbb{R}^+_m$. On the interval $[−m, m]$ we can choose $h$ freely as long as it remains smooth and the condition $h(-\lambda) = \hat{h}(\lambda)$ is satisfied. Hence we get $\tilde{Z}_j h = i\hat{f}$ which shows that $\tilde{Z}_j$ is complex linear.

Consider now the unitary operator $U : L^2(\mathbb{R}^+, r^2 dr) \ni h \mapsto U h \in L^2(\mathbb{R}^+_m, d\lambda)$ with

$$(Uh)(\lambda) = \sqrt{\lambda r(\lambda)} h(r(\lambda)) \text{ and } r(\lambda) = \sqrt{\lambda^2 - m^2}.$$  

Combining it with $\tilde{Z}_j$ we get the map $S(\mathbb{R}) \ni f \mapsto Z_j f \in L^2(\mathbb{R}^+_m, d\lambda)$ with

$$(Z_j f)(\lambda) = r(\lambda)^{j+1/2} \hat{f}(\lambda).$$

Hence we have to show that $\{ Z_j f \mid f \in S(\mathbb{R}) \}$ is dense in $L^2(\mathbb{R}^+_m, d\lambda)$. To this end note first that $\{ h|_{\mathbb{R}^+_m} \mid h \in S(\mathbb{R}) \}$ is dense
in \( L^2(\mathbb{R}^+, d\lambda) \), because \( \mathcal{S}(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}, d\lambda) \). Therefore it is sufficient to approximate an arbitrary \( h|_{\mathbb{R}^+_m}, h \in \mathcal{S}(\mathbb{R}) \) by a sequence \( \mathbb{N} \ni k \to Z_j f_k \) with \( f_k \in \mathcal{S}(\mathbb{R}) \). With a sequence \( k \to b_k \) of smooth, positive functions satisfying \( b_k(\lambda) = 0 \) for \( m \leq \lambda \leq m + 1/k \) and \( b_k(\lambda) = 1 \) for \( \lambda > m + 2/k \) we can choose \( f_k = (b_k r^{-j-1/2} h)^\vee \in \mathcal{S}(\mathbb{R}) \). Hence we get

\[
\|h - Z_j f_k\|^2 = \int_0^\infty |h(\lambda) - b_k(\lambda) h(\lambda)|^2 d\lambda \leq \frac{2}{k} \sup_{\lambda \in \mathbb{R}} |h(\lambda)|^2.
\]

Since \( h \in \mathcal{S}(\mathbb{R}) \) it is bounded, which implies \( Z_j f_k \to h \), or in other words: \( \tilde{Z}_j \) is dense in \( L^2(\mathbb{R}^+, r^2 dr) \).

The next step concerns the space

\[
Z_j = \{Z_j(f) \mid f \in D(\mathbb{R}, \mathbb{R}) \} \subset \tilde{Z}_j.
\]

To prove that it is dense in \( \tilde{Z}_j \), and therefore dense in \( L^2(\mathbb{R}^+, r^2 dr) \) as well, we choose a polynomial \( p(\lambda) \) such that \( r(\lambda)^{j+1/2} \leq p(\lambda) \) holds for all \( \lambda \in \mathbb{R}^+_m \). With \( f \in \mathcal{S}(\mathbb{R}, \mathbb{R}) \) and \( h \in D(\mathbb{R}, \mathbb{R}) \) we get

\[
\|Z_j f - Z_j h\| = \int_0^\infty |\hat{f}(\lambda) - \hat{h}(\lambda)|^2 r(\lambda)^{2j+1} d\lambda
\leq \int_\mathbb{R} |p(\lambda)(\hat{f}(\lambda) - \hat{h}(\lambda))|^2 d\lambda
\leq \int_\mathbb{R} |p(D)f(x) - p(D)h(x)|^2 dx,
\]

where \( p(D) \) is the partial differential operator with \( p \) as its symbol. Since \( D(\mathbb{R}, \mathbb{R}) \) is dense in \( \mathcal{S}(\mathbb{R}, \mathbb{R}) \) with respect to each Sobolev norm we can choose for all \( f \in \mathcal{S}(\mathbb{R}, \mathbb{R}) \) and all \( \epsilon > 0 \) a \( h \in D(\mathbb{R}, \mathbb{R}) \) such that the integral in Equation (43) is smaller than \( \epsilon \). Hence \( Z_j \) is dense in \( L^2(\mathbb{R}^+, r^2 dr) \).

Now consider \( \mathcal{K}^l(\gamma_i) = \bigoplus_{j=0}^l Z_j \otimes Y_j \), where \( Y_j \subset L^2(S^2, d\Omega(\kappa)) \) denotes the space generated by restrictions of \( j \)th order homogeneous polynomials to \( S^2 \subset \mathbb{R}^3 \). Up to now we have shown that

\[
\mathcal{K}^l(\gamma_i) = \bigoplus_{j=0}^l L^2(\mathbb{R}^+, r^2 dr) \otimes Y_j = L^2(\mathbb{R}^+, r^2 dr) \otimes \left( \bigoplus_{j=0}^l Y_j \right).
\]

However the space \( \bigoplus_{j=0}^l Y_j \) is generated by restrictions of (not necessarily homogeneous) \( j \)th order polynomials to \( S^2 \), which is known to coincide with \( \mathcal{Y}_l \) (see 23.38.3.4)).

This result immediately implies the following statements concerning the algebras \( \mathcal{R}^l(\gamma_i) \).

**Theorem 8.2.** Consider the free scalar field in Minkowski space \( (\mathbb{R}^4, g) \) and the inertial observer with worldline \( \gamma_i : \mathbb{R} \ni t \mapsto (t, 0) \in \mathbb{R}^4 \).
1. The family of von Neumann algebras $\mathcal{R}^l(\gamma_i)$, $l \in \mathbb{N}_0$ is strictly increasing.

2. The inductive limit $\mathcal{R}^\infty(\gamma_i) = \left( \bigcup_{l \in \mathbb{N}_0} \mathcal{R}^l(\gamma_i) \right)$ coincides with $\mathcal{B}(\mathcal{H})$.

Proof. This is an immediate consequence of Lemma 8.1 and the fact that the map $K \ni f \mapsto \exp(i\Phi_S(f))$ is strongly continuous.

Another consequence of Lemma 8.1 concerns the subspace $\mathcal{P}^l(\gamma_i)$ of $\mathcal{P}$ (cf. Section 7). It is easy to show that $\mathcal{P}^l(\gamma_i)$ is a symplectic subspace of $\mathcal{P}$.

**Proposition 8.3.** The set $\mathcal{P}^l(\gamma_i) = \{ [T] \in \mathcal{P}^l(\gamma_i) | T \in \mathcal{D}^l(\gamma_i) \} \subset \mathcal{P}$ is a symplectic subspace of $\mathcal{P}$.

Proof. According to Lemma 8.1 the range of the map

$$K : \mathcal{P}^l(\gamma_i) \to L^2(\mathbb{R}^+, r^2dr) \otimes \mathcal{Y}_l$$

is dense. This implies that each $T \in \mathcal{P}^l(\gamma_i)$ and each $\epsilon > 0$ admit a $S \in \mathcal{P}^l(\gamma_i)$ with $\|iK(T) - K(S)\| \leq \epsilon$. On the other hand we know that $K$ is symplectic; hence:

$$|G(T, S) - \text{Im}(K(T), iK(T))| = |\text{Im}(K(T), K(S) - iK(T))| \leq \epsilon\|K(T)\|.$$  

Since $T \neq 0$ implies $\text{Im}(K(T), iK(T)) = \|K(T)\|^2 > 0$ we can choose $\epsilon > 0$ and $S$ in such a way that $|G(T, S)| > 0$ holds, which was to show.

The last proposition shows that the C*-algebras $\mathcal{A}_l(\gamma_i)$ are defined as CCR-algebras of the symplectic spaces $\mathcal{P}^l(\gamma_i)$. Hence they do not depend on the embedding $\mathcal{A}_l(\gamma_i) \subset \text{CCR}(\mathcal{P}, G)$ as it was discussed in the last Section.

9 Time translations

At the end of this paper we want to apply the quasi free description described in Section 8 to generalize space-time translations of Minkowski space theories. To this end recall first that in the special relativistic case just discussed there is a representation $\mathbb{R}^4 \ni v \mapsto \alpha_v$ of $\mathbb{R}^4$ by $^*$-automorphisms of the CCR-algebra $\text{CCR}(\mathcal{P}, G)$, where the $\alpha_v$ are given in terms of Bogolubov transformations by $\alpha_v(W(f)) = W(f(\cdot - v))$. If $g(v, v) = -1$ and $\gamma_i$ is the worldline of the inertial observer with four velocity $v$, it is easy to see that $t \mapsto \alpha_{tv}$ provides a one-parameter group of automorphisms of $\mathcal{A}_l(\gamma_i)$, which can be interpreted physically as follows: Consider a subcurve $\mu = \gamma_i[(a, b)]$ and a self adjoint element $A \in \mathcal{A}_l(\mu)$ describing an observable measurable by the observer $\gamma_i$ in the time interval $(a, b)$. During the measurement $\gamma_i$ and his measuring equipment is at rest in the global inertial frame described by the flow $\mathbb{R} \times \mathbb{R}^4 \ni (t, x) \mapsto x + tv \in \mathbb{R}^4$, or equivalently by the vector field $Z \equiv v$. The observable $\alpha_{tv}(A) \in \mathcal{A}_l(\mu + t)$ with $\mu + t = \gamma_i[(a + t, b + t)]$ represents the same measurement, i.e. it is carried out by the same observer $\gamma_i$ with the same equipment, in the same inertial frame $Z -$ but $t$ time units later.
On a space-time without non-trivial symmetries this construction does not work, although the physical interpretation just given would still make sense. The following Theorem of Demoen et. al. can be used to get (under some conditions) at least a family of completely positive maps between algebras $A_{\gamma}(\gamma)$.

**Theorem 9.1.** Consider a real symplectic space $(\mathcal{S},\mathcal{S})$, the corresponding CCR-algebra CCR$(\mathcal{S},\mathcal{S})$ and $a$, not necessarily symplectic, linear map $L : \mathcal{S} \to \mathcal{S}$.

1. $\mathcal{S}_{L}(x,y) = \mathcal{S}(x,y) - \mathcal{S}(Lx, Ly)$ defines an (in general degenerate) antisymmetric bilinear form on $\mathcal{S}$. The corresponding CCR-algebra (equipped with the minimal C*-norm; cf. [2]) will be denoted by CCR$(\mathcal{S},\mathcal{S}_{L})$.

2. For each state $\rho$ on CCR$(\mathcal{S},\mathcal{S}_{L})$ there is a completely positive map $\alpha_{L} : \text{CCR}(\mathcal{S},\mathcal{S}) \to \text{CCR}(\mathcal{S},\mathcal{S})$ with $\alpha_{L}(W(x)) = \rho(W(x))W(Lx)$ for all $x \in \mathcal{S}$.

To get a useful generalization of the time translations $\alpha_{t\alpha}$ in terms of this theorem we have to specify how distributions $T \in \mathcal{D}(\gamma)$ should be pushed forward along $\gamma$. Hence consider again a generic (globally hyperbolic) space time $(M,g)$ and an arbitrary reference frame, i.e. a timelike vector field $Z$ normalized to $-1$ $(G(Z,Z) = -1)$, with flow $(t,p) \mapsto F_{t}(p)$ and assume that $\gamma$ is one of its integral curve. For each subcurve $\mu \subset \gamma$ and each admissible $t \in \mathbb{R}$ with $\mu + t := F_{t} \circ \mu \subset \gamma$ the $l$-jet extension $j_{l}^{\gamma}F_{t}$ of the flow leads to a bundle map $(j_{l}^{\gamma}F_{t}^{*})^{-1} : E_{l}(\mu) \to E_{l}(\mu + t)$ (see Appendix C for details). Hence if we define the domain (which depends in contrast to the map $\Xi_{l\gamma}^{l}$ below not on $Z$)

$$\text{Dom}(\Xi_{l\gamma}^{l}) = \{(t,T) \in \mathcal{D}(\gamma,\mathbb{R}) | F_{t}(\text{supp} T) \subset \text{Ran} \gamma\}$$

we get the following one-parameter family of linear maps

$$\text{Dom}(\Xi_{l\gamma}^{l}) \ni (t, T_{\psi}) \mapsto \Xi_{l\gamma}^{l} \cdot T_{\psi} := T_{(j_{l}^{\gamma}F_{t}^{-1})}\psi \in \mathcal{D}(\gamma,\mathbb{R}). \quad (46)$$

Here $\psi \in \mathcal{D}(\text{Dom} \gamma, E_{l}(\mu))$ is a section which satisfies the condition from Corollary 4.7 and $T_{\psi} \in \mathcal{D}(\gamma,\mathbb{R})$ denotes the distribution which is defined by $\psi$ according to Theorem 4.3.

Note however that $\Xi_{l\gamma}^{l}T_{\psi}$ does not coincide with the usual push forward $F_{t}T_{\psi}$ of $T_{\psi}$ with $F_{t}$. This can be seen most easily if we recall the fact that a regular distribution is basically not a smooth function but a smooth $n$-form $\lambda$ (with $n = \dim M$). If $\lambda_{g}$ denotes the volume element associated to the metric and $f \in \mathcal{E}(M)$, the usual push-forward of $f \lambda_{g}$ is given by $(F_{t}^{*}f)(F_{t}^{*}\lambda_{g})$, while the natural generalization of the construction given in Equation (46) to regular distributions would leave $\lambda_{g}$ fixed, i.e. $f \lambda_{g} \mapsto (F_{t}^{*}f)\lambda_{g}$. In other words the definition of $\Xi_{l}^{l}Z_{l}T$ depends on a distinguished volume form (namely $\lambda_{g}$) while $F_{t}^{*}$ is an invariant construction. However since $\lambda_{g}$ is a natural object on a Lorentzian manifold, this is not a significant drawback. On the other hand we have the advantage that $\Xi_{l\gamma}^{l}T$ for $T \in \mathcal{D}(\gamma)$ depends only on $j_{l}^{\gamma}F_{t}[\gamma]$, while

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11 This means that $(t,\mu(s))$ should be for all $s \in \text{Dom}(\mu)$ in the domain of the flow $F$. In the following we will assume this implicitly if nothing else is stated.

12 This is necessary to get a well defined map, cf. the discussion at the end of Section 4.
$F_1, T$ would depend on $j^{l+1} F_1 \big| \gamma$ (since the determinant of the differential of $F_1$ comes in).

This last remark leads us directly to another point which is worth to note: The map $T \mapsto \Xi^l_{\gamma t} T$ depends as just stated only on $l$-jets of $F_1$ along $\gamma$, or in other words only the $l$-jet of $Z$ along $\gamma$ is relevant for the construction. This implies in particular that in the case $l = 0$ there is a unique map $\text{Dom}(\Xi^0) \ni (t, T) \mapsto \Xi^0_{\gamma t} \in \mathcal{D}(\gamma, \mathbb{R})$, with $\Xi^0_{\gamma t} = \Xi^0_{\gamma t}$ for all $Z$ admitting $\gamma$ as an integral curve. Hence in the case $l = 0$ the construction do not depend on the reference frame. If $l = 1$ holds this not true, however there is a natural choice: $\Xi^1_{\gamma t}$ depends in this case on one-jets of $F_1$, i.e. the tangent maps $T_{\gamma(s)} F_1$. Hence it is natural to assume that $T_{\gamma(s)} F_1: T_{\gamma(s)} M \to T_{\gamma(s+1)} M$ coincides with parallel transport along $\gamma$.

Now we need the following lemma to proceed.

**Lemma 9.2.** Consider an analytic space-time $(M, g)$ and a smooth (i.e. not necessarily analytic) timelike, normalized vector field $Z$ with flow $(t, p) \mapsto F_1(p)$.

For each point $p$ of $(M, g)$ there is an $\epsilon > 0$ such that the following statement holds: For each proper subcurve $\mu$ of $(-\epsilon, \epsilon) \ni s \mapsto \gamma(s) := F_\mu(p) \in M$, the kernel of the map

$$\mathfrak{D}^l(\mu, \mathbb{R}) \ni T \mapsto G T \in \mathcal{E}(M).$$

is given by (setting $\mathfrak{D}^j(\mu, \mathbb{R}) = \{0\}$ for $j < 0$)

$$\kappa(\mu, l) = \{(\Box - m^2)T \mid T \in \mathfrak{D}^{l-2}(\mu, \mathbb{R})\}.$$ (48)

**Proof.** From $G T = 0$ we get $G^+ T = G^- T$ and due to the support properties of $G^\pm$ we see that sup $G^+ T \subset \text{Dom}(\mu)$ holds where $\text{Dom}(\mu)$ denotes the double cone generated by $\mu$, i.e. $\text{Dom}(\mu) = I^+(F_{-\epsilon}(p)) \cap I^-(F_{\epsilon}(p))$. According to Theorem 0.4 we can assume that the restriction of $G^+ T = G^- T$ to the submanifold

$$\Sigma(0, s, \epsilon) := [(H^+(\gamma(s)) \setminus \{\gamma(s)\}) \cap [I^+(\gamma(-\epsilon)) \cup I^-(\gamma(\epsilon))]].$$

defined in equation 138 is analytic. On the other hand we have $G^+ T \setminus \Sigma(0, s, \epsilon) \setminus \text{Dom}(\mu) \equiv 0$, hence $G^+ T | \Sigma(0, s, \epsilon) \equiv 0$ due to analyticity and the fact that $\Sigma(0, s, \epsilon) \setminus \text{Dom}(\mu)$ is not empty (since $-\epsilon < a < b < \epsilon$). This implies that the support of $G^+ T$ is contained in $\text{Dom}(\mu)$. Since $(\Box - m^2)G^+T = T$ and $T$ is of order $l$ we get $G^+ T \in \mathfrak{D}^{l-2}(\mu, \mathbb{R})$ (see Definition 0.3). In other words the kernel of the map from Equation (17) is given by Equation (18) as stated.

To define cp-maps on the algebras $\mathfrak{A}^l(\gamma)$ by the method given in Theorem 0.1 we need a vector field $Z$ such that the map $\Xi^l_{\gamma t} : \mathfrak{D}^l(\mu, \mathbb{R}) \to \mathfrak{D}^l(\mu + t, \mathbb{R})$ defined in Equation (10) maps the kernel $\kappa(\mu, l)$ to $\kappa(\mu + t, l)$. To this end let us give the following definition:

**Definition 9.3.** Consider a timelike vector field $F_1$ with flow $F_1$ and an integral curve $\gamma$ of it. $Z$ is called an $l^{th}$-order infinitesimal symmetry along $\gamma$ if $j^{2}_{\gamma(s)}(F_1 g) = j^{l}_{\gamma(s)} g$ holds for all $t, s - t \in \text{Dom}(\gamma)$. The curve $\gamma$ is called in this case $l^{th}$-order symmetric with respect to $Z$. 28
So, roughly speaking, a vector field is \( l \)-th order infinitesimal symmetric along some curve \( \gamma \) if the restriction of the \( l \)-jet extension of the metric to the range of \( \gamma \) is invariant under the flow of \( Z \). It is easy to see that each time-like curve is \( 0 \)-th order symmetric with respect to an appropriate vector field which can be constructed locally as follows: Consider the coordinate map

\[
\bar{U} \ni (t, x) \mapsto \exp_{\gamma(t)} \left( \sum_{i=1}^{3} x^i(t) e_i(t) \right) \in U \subset M
\]

deﬁned in Corollary 4.7 and Equation (23). The basis vector ﬁelds \( (\partial_\nu)_{\nu=0,\ldots,3} \) associated to this chart coincide obviously along \( \gamma \) with the \( e_\nu \), i.e. \( \partial_\nu(\gamma(t)) = e_\nu(t) \). Hence the ﬂow \( F_t \) of \( \partial_0 \) has the desired property \( (F_t^* g)_{\gamma(s)} = g_{\gamma(s)} \). Using the fact that ﬁrst order derivatives of the metric (in an appropriate coordinate system) vanishes along \( \gamma \) if it is a geodesic, we can show in the same way that each geodesic is ﬁrst order symmetric. Therefore we have just shown the following proposition

**Proposition 9.4.** Each smooth timelike curve is \( 0 \)-th order symmetric and each timelike geodesic is ﬁrst order symmetric.

With similar arguments we can check now that the kernel \( \kappa(\mu,l) \) deﬁned in Equation (48) is really invariant under the ﬂow of inﬁnitesimal symmetric vector ﬁelds.

**Lemma 9.5.** Consider again an analytic space-time, a smooth timelike curve \( \gamma \) and a subcurve \( \mu = \gamma \) such that Lemma 9.3 holds. For each vector ﬁeld \( Z \), inﬁnitesimally symmetric of order \( l-1 \) along \( \gamma \) and each admissible \( t \) we have

\[
\Xi_{\gamma Z t} \cdot \kappa(\mu,l) = \kappa(\mu + t,l).
\]

For \( l = 0,1 \) the statement is satisﬁed for any vector ﬁeld with \( \gamma \) as integral curve.

**Proof.** If \( l = 0,1 \) is satisfied, \( \kappa(\mu,l) \) is trivial which implies immediately the assertion. Hence assume \( l \geq 2 \). Each element \( T \) of \( \kappa(\mu,l) \) has according to Lemma 9.2 the form

\[
T(f) = T_\psi(\square - m^2)f = \int_\mathbb{R} \psi(\gamma(t)) \frac{d^{l-2}}{dt^{l-2}}\left( (\square - m^2)f \right) dt
= \int_\mathbb{R} (P_\psi \cdot (\square - m^2)f)(\gamma(t)) dt,
\]

where \( \psi \in \mathcal{D}(\text{Ran}\, \gamma, E_{\mu-2}^l(\gamma)) \), \( T_\psi \) denotes the corresponding distribution in \( \mathcal{D}^{l-2}(\gamma, \mathbb{R}) \) (cf. Theorem 4.3) and \( P_\psi \) is the differential given by a smooth extension of \( \psi \) to a small neighbourhood of \( \text{Ran}\, \gamma \) (cf. Proposition C.1 and Equation (62)). Proposition C.2 and Equation (46) imply that

\[
(\Xi_{\gamma Z t} T)(f) = \int_\mathbb{R} (F_t^* P_\psi (\square - m^2) F_t f)(\gamma(t)) dt
= \int_\mathbb{R} (F_t^* P_\psi F_t^* F_t (\square - m^2) F_t f)(\gamma(t)) dt
\tag{49}
\]
holds. The Klein-Gordon operator depends on the metric \( g \) and its first derivatives. Since \( P_\psi \) is a differential operator of order \( l - 2 \) this implies that the expression in Equation (49) depends on differentials of \( g \) up to order \( l - 1 \). However we only need to know its value along \( \gamma \) and \( F_t \) leaves by assumption the \( l - 1 \) jet of \( g \) along \( \gamma \) invariant. Therefore we get

\[
(\Xi^l_{\gamma, Z}T)(f) = \left\{ F_t^*P_\psi F_t((\Box - m^2)f)\right\}(\gamma(t))dt = T_{(J_0 - 2F_t)}(\alpha_\gamma F_t, ((\Box - m^2)f),
\]

and therefore \( \Xi^l_{\gamma, Z}T \in \kappa(\mu + t, l) \) which was to show.

Now we are ready to prove the following theorem, which is the main result of this section.

**Theorem 9.6.** Consider an analytic space-time \((M, g)\) and a smooth (i.e. not necessarily analytic) curve \( \gamma \). Each \( s \in \text{Dom} \gamma \) admits a subcurve \( \mu \subset \gamma \) with \( s \in \text{Dom} \mu \) such that the following statements hold:

1. If \( l \geq 2 \) there exists for each \((l - 1)\)th order infinitesimal symmetry along \( \gamma \) and each admissible \( t \in \mathbb{R} \) a functional \( \alpha^l_{\gamma, Z} : D_{l}(\gamma, \mathbb{R}) \to C \) and a cp-map \( \alpha^l_{\gamma, Z} : A_{l}(\mu) \to A_{l}(\mu + t) \) with \( \alpha^l_{\gamma, Z} : W(T) = \Xi^l_{\gamma, Z}(T)W(\Xi^l_{\gamma, Z}T) \).

2. If \( l = 0, 1 \) the same is true for any vector field with \( \gamma \) as an integral curve.

3. If \( l = 0 \) the construction is independent of \( Z \). Hence we will write \( \alpha^0_{\gamma, t} \) respectively \( \alpha^0_{\gamma, Z} \).

**Proof.** According to Lemma 9.5 the linear map \( P(l) \ni [T] \mapsto \Xi^l_{\gamma, Z}T \in P(l + t) \subset P \) is well defined. If \( P(l) \) is a symplectic subspace of \( P \) Theorem 9.1 implies immediately the assertion. However we can not assume in general that \( G \) is non-degenerate on \( P(l) \) (cf. the corresponding discussion in Section 7). Hence we need an additional argument: By Zorn’s lemma we can find an algebraic complement \( \tilde{P} \) of \( P(l) \) in \( \tilde{P} \) and we can extend the linear map just defined by \( P(l) \oplus \tilde{P} \ni (T, S) \mapsto F_tT \in \tilde{P} \) to \( \tilde{P} \) Now Theorem 9.1 applies, which proves item 1 and 3. Item 3 is then an immediate consequence of the fact that \( \Xi^l_{\gamma, Z}T \) is independent of \( Z \) as well (cf. the discussion after before Lemma 9.2).

**10 Conclusions**

Let us discuss now some applications of our results to the description of observer dependent effects in quantum field theory. Maybe the most important one is particle creation due to acceleration, or more generally: observer dependent particle concepts. To get an idea how to proceed in this direction, restrict, for simplicity, the analysis to the case \( l = 0 \) and consider again the free field on Minkowski space \((\mathbb{R}^4, g)\) together with the inertial observer \( \gamma_i \) from the last section. The subgroup \( \mathbb{R}^4 \ni (x^0, x) \mapsto F_t(x^0, x) := (x^0 + t, x) \in \mathbb{R}^4 \) of the translation group is represented in a canonical way by Bogolubov automorphisms \( \alpha_t \).
of CCR(\(\mathcal{P}, G\)). The C*-algebras \(\mathcal{A}^0(\gamma)\) are on the other hand invariant under the \(\alpha_t\) and \(\alpha_t|_{\mathcal{A}^0(\gamma)}\) coincides with \(\alpha^0_{\gamma,t}\) if we choose \(\alpha^0_{\gamma,t} \equiv 1\) (cf. Theorem \(\ref{thm:9.6}\)). The free field vacuum \(\omega_0\) is (obviously) a ground state with respect to the group \(\alpha_t\). Hence the restriction of \(\omega_0\) to \(\mathcal{A}^0(\gamma_i)\) is a ground state with respect to the one parameter group \(\alpha^0_{\gamma_i,t}\). In a similar way \(\omega_0\) becomes a KMS-state with respect to \(\alpha^0_{\gamma_i,t}\) if \(\gamma_i\) is a uniformly accelerating observer; this is a consequence of the well known result of Bisognano and Wichmann \(\cite{2}\).

This observation suggests that a reasonable generalization of notions like ground state, KMS-state and particle should be based on properties of the cp-maps \(\alpha^0_{\gamma,t}\). The most naive approach is to consider for an arbitrary observer \(\gamma\) and a state \(\omega\) in a general space-time \((M, g)\) the functions \(\mathbb{R} \ni t \mapsto \omega(A\alpha^0_{\gamma,t}(B)) \in \mathbb{C}, \ A, B \in \mathcal{A}^0(\gamma)\) and to generalize the original definitions of ground and KMS states in the most direct way, e.g. to look at analyticity properties of these functions. However this is most probable to naive. It is in particular not very likely that an observer with non-constant acceleration really sees a thermal particle spectrum (with constant temperature) during a finite measuring interval. Maybe more realistic is to assume that the properties of ground and KMS-states can be retained only in an approximation, e.g. some kind of Taylor developments around each instant of proper time of the observer.

In this context it is interesting that we can use the idea of the proof of Theorem \(\ref{thm:9.6}\) to identify \(\alpha^0_{\gamma,t}\) with \(\alpha^0_{\gamma_i,t} (= \alpha_t)\) at least in a certain local sense. More precisely consider the cp maps (we ignore temporarily the restriction to small pieces of the curves, wich is necessary in the proof of Theorem \(\ref{thm:9.6}\))

\[
\mathcal{A}_0(\gamma) \ni W(T) \mapsto \beta^0_{\gamma,\gamma_i}(W(T)) = \beta^0_{\gamma,\gamma_i}(T)W(T) \in \mathcal{A}^0(\gamma_i)
\]

and

\[
\mathcal{A}^0(\gamma_i) \ni W(T) \mapsto \beta^0_{\gamma_i,\gamma}(W(T)) = \beta^0_{\gamma_i,\gamma}(T)W(T) \in \mathcal{A}^0(\gamma),
\]

defined according to Theorem \(\ref{thm:9.1}\) by an obvious identification of distributions from \(\mathcal{D}^0(\gamma)\) and \(\mathcal{D}^0(\gamma_i)\), and appropriate functionals \(\beta^0_{\gamma,\gamma_i}, \beta^0_{\gamma_i,\gamma}\). Hence if we set

\[
\beta^0_{\gamma_i,\gamma}(T + t)\beta^0_{\gamma,\gamma_i}(T) = \alpha^0_{\gamma,t},
\]

\(\mathcal{A}^0(\gamma) \ni A \mapsto \beta^0_{\gamma_i,\gamma} \circ \alpha_t \circ \beta^0_{\gamma,\gamma_i} \in \mathcal{A}(\gamma)\) is a cp-map of the form discussed in Theorem \(\ref{thm:9.6}\). This implies that we can investigate properties of the state \(\omega\) on \(\mathcal{A}^0(\gamma)\) by considering \(\omega \circ \beta^0_{\gamma,\gamma_i}\) on \(\mathcal{A}^0(\gamma_i)\), which is a great advantage because the \(\alpha_t\) form in contrast to the \(\alpha^0_{\gamma,t}\) an automorphism group which is easier to study. In addition we have the possibility to compare \(\omega\) directly with the Minkowski vacuum \(\omega_0\). This requires of course a detailed study of \(\beta^0_{\gamma_i,\gamma}\) and \(\beta^0_{\gamma,\gamma_i}\). However this task is simplified by the fact that both functionals are, according to Theorem \(\ref{thm:9.4}\), generating functionals of states on appropriate CCR algebras.
A different complex of questions to which these ideas can be applied as well concerns averaged energy conditions, which are considered recently in a number of papers (see e.g. [12, 30] and the references therein). The problem discussed there arises from the fact that the expectation value of the energy momentum tensor of a quantum field does not satisfy the usual energy conditions, which breaks many theorems in general relativity (e.g. singularity theorems). In some situations however positivity of energy is not required at each event of space-time, but only averaged along timelike or null curves. At least in the timelike case the quantum fields $\Phi^l\gamma$ and the corresponding algebras $\mathcal{A}^l(\gamma)$ provide an optimal framework to study this kind of problem. This is in particular the case for the scheme outlined in the last paragraph: The identification of each $\mathcal{A}^l(\gamma)$ with the Minkowski space algebra $\mathcal{A}(\gamma_i)$ given by the cp-maps $\beta_{\gamma\gamma_i}$ and $\beta_{\gamma_i\gamma}$ might lead to an easy generalization of results which are currently only available in the flat case.

Another interesting application concerns the question whether an observer $\gamma$ can determine its acceleration by quantum measurements. More precisely: is it possible for $\gamma$ (and how) to decide (at least) whether he is in free fall or not by measuring observables from $\mathcal{A}^l(\gamma)$. In Minkowski space we can pose the related but much simpler question, how to distinguish between an inertial observer $\gamma_i$ and a uniformly accelerated one $\gamma_a$. According to our discussion at the beginning of this section, one possible answer is to count particles in the Minkowski vacuum. The temperature of the observed spectrum (which should be thermal of course) is a direct measure for the acceleration. In the general case however this scheme is not applicable, because there is no distinguished global state which we can use as a reference. Hence the observer $\gamma$ (possibly together with “helpers” traveling on neighboring wordlines) has to perform a more complex protocol consisting of several steps, e.g.: First prepare some states locally by distinguished procedures and then count particles. In any case however the knowledge of approximate ground and KMS states, as outlined above, seems to be necessary.

A closely related, but slightly different question is: Is it possible to determine the space-time uniquely (up to isomorphisms) by a quantum field theory? At least a partial answer is available can be found in [22, 24] and [33] (see also [1] for a completely different approach) In these papers it is shown (using commutation relations in [22, 23] and algebras of curves in [33]) that the conformal structure of space-time can be fixed uniquely by a net of local C*-algebras under quite general conditions. Hence the discussion of the last paragraph, the knowledge of all geodesics in space-time, provides exactly the missing information. However the result of Theorem 9.6 might lead to a simpler approach: According to Proposition 9.4 each timelike geodesic is first order symmetric. The converse however is not true, because each integral curve of a timelike Killing vector field is $l^{th}$ order symmetric for any $l \in \mathbb{N}$. Hence the question is, whether conformally equivalent metrics can be distinguished by their sets of first order symmetric curves. If the answer is true, we can ask whether first symmetric curves can be characterized by the existence (and possibly particular properties) of the first order time-translation maps $\alpha_{\gamma t}^1: \mathcal{A}^1(\mu) \rightarrow \mathcal{A}^1(\mu + t)$. 

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Acknowledgments
I like to thank A. S. Holevo for pointing out to me reference [7].

A Wave front sets of distributions
In this appendix we will summarize some material about wave front sets of distributions used throughout the paper. A detailed presentation can be found in Chapter VIII of Hörmander’s book [3]. Let us start with a distribution $T$ on $\mathbb{R}^n$. A pair $(x_0,\xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is called a regular directed point of $T$, if there is a neighbourhood $U$ of $x_0$ and a conic neighbourhood $V$ of $\xi_0$ such that for all $f \in \mathcal{D}(U,\mathbb{R})$ and each $N \in \mathbb{N}$ there is a constant $C_{f,N}$ with

$$|(T, e^{-i\langle \cdot, \xi \rangle} f)| \leq \frac{C_{f,N}}{(1 + |\xi|)^N} \quad \forall \xi \in V.$$

Definition A.1. The wave-front set of $WF(T)$ is the set of all $(x_0,\xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ which are not regular directed.

$WF(T)$ is closely related to the singular support $\text{singsupp} T$ of $T$. More precisely an element $x_0 \in \mathbb{R}^n$ is in $\text{singsupp}(T)$ iff there is a $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ with $(x_0,\xi_0) \in WF(T)$. In contrast to $\text{singsupp}(T)$ however the wave front set does not only tell us where the singularity of $T$ is located but also by which high frequency oscillations it is caused.

Consider now a smooth manifold $M$. Distributions on $M$ can be defined, as in the flat case, as continuous linear functionals on the space $\mathcal{D}(M)$ of smooth, complex valued and compactly supported functions on $M$, equipped with the usual topology. However in contrast to euclidean space there is no natural embedding of $\mathcal{D}(M)$ into its topological dual $\mathcal{D}'(M)$. There are basically two ways to solve this problem: First we can introduce in addition the space $\mathcal{D}(M,\Lambda^nT^*M)$ of smooth, compactly supported, complex densities on $M$ and its dual space $\mathcal{D}'(M,\Lambda^nT^*M)$. Now we can embed $\mathcal{D}(M,\Lambda^nT^*M)$ naturally into $\mathcal{D}'(M)$ by mapping a density $\theta$ to the distribution $\theta(f) := \int_M f(x)\theta(x).$ In a similar way we can embed $\mathcal{D}(M)$ into $\mathcal{D}'(M,\Lambda^nT^*M)$. Alternatively we can select a distinguished volume $\lambda$ form (this is possible only if $M$ is orientable) and to identify $\mathcal{D}(M)$ with $\mathcal{D}(M,\Lambda^nT^*M)$ by the isomorphism $\mathcal{D}(M) \ni f \mapsto f\lambda \in \mathcal{D}(M,\Lambda^nT^*M).$ Since we are considering exclusively orientable Lorentzian manifolds in this paper a distinguished volume form is naturally given. Therefore we will use in most cases the second possibility and interpret $\mathcal{D}(M)$ as a subspace of $\mathcal{D}'(M)$. However we should keep in mind that such an identification depends basically on the volume form $\lambda.$

Let us come back to wave front sets now. If $(M_u,u)$ is a coordinate chart of $M$ we can consider the push forward forward $(u_* T, f) = (T, (f \circ u))$ of $T \in \mathcal{D}'(M)$ and get a distribution $u_* T$ on $u(M_u) \subset \mathbb{R}^n,$ which has the wave front set

\footnote{Note that the Fourier transform of a compactly supported distribution is always smooth.}
WF\left( u^*T_u \right). Using different coordinate systems, it turns out that the elements of WF\left( u^*T_u \right) transforms like co-vectors (cf. [19, Theorem 8.2.4]). Hence we can define WF\left( T \right) as follows:

**Definition A.2.** The wave front set WF\left( T \right) of a distribution T on the manifold M is the unique subset of the cotangent bundle \( T^*M \setminus \{0\} \) satisfying

\[
WF(T)\bigr|_{M_u} = (T^*u)^{-1} WF(u^*T_u), \quad \forall \text{ coordinate charts } (M_u, u).
\]

Here \( (T^*M_u, T^*u) \) denotes the coordinate chart of \( T^*M \) induced by \( (M_u, u) \).

The notion of wave front set allows us to extend a number of operations to distributions. Of special importance for us are products of distributions and restrictions to submanifolds. The basic idea to do that is to extend those operations in a continuous way from spaces of smooth functions to distribution spaces. This concept requires however a special form of convergence because the weak topology is not appropriate. Let us discuss the euclidean case first, since the generalization to manifolds is straightforward.

**Definition A.3.** Consider an open subset \( U \subset \mathbb{R}^n \) and a closed cone \( \Gamma \subset U \times \mathbb{R}^n \setminus \{0\} \) and define

\[ D'_\Gamma(U) = \{ T \in D'(U) \mid WF(T) \subset \Gamma \}. \]  

We say that a sequence \( N \ni j \mapsto T_j \in D'_\Gamma(U) \) is converging in \( D'_\Gamma(U) \) to \( T \in D'_\Gamma(U) \) if

1. \( j \mapsto T_j \) converges weakly to \( T \),
2. for each \( (x_0, \xi_0) \in (U \times \mathbb{R}^n \setminus \{0\}) \setminus \Gamma \) there is a function \( f \in D(U) \) with \( f(x_0) \neq 0 \) and a conic neighbourhood \( V \) of \( \xi_0 \) such that

\[
\lim_{j \to \infty} \sup_{\xi \in V} |\xi|^N |\hat{f}T_j(\xi) - \hat{f}T(\xi)| \to 0 \quad \forall N \in \mathbb{N}_0 \]  

Note that \( T \in D'_\Gamma(U) \) is equivalent to the following: For each \( (x_0, \xi_0) \in (U \times \mathbb{R}^n \setminus \{0\}) \setminus \Gamma \) there are \( f, V \) as above with

\[
\sup_{\xi \in V} |\xi|^N |\hat{f}T(\xi)| < \infty \quad \forall N \in \mathbb{N}_0.
\]

This implies especially that each constant sequence is converging. Note in addition that due to weak convergence \( \hat{f}T_j \) converges uniformly on each compact set. This implies that (51) is equivalent to

\[
\sup_{j \in \mathbb{N}} \sup_{\xi \in V} |\xi|^N |\hat{f}T_j(\xi)| < \infty \quad \forall N \in \mathbb{N}_0.
\]

To generalize this definition to distributions on a manifold \( M \) we can use again coordinate representations. If in particular \( WF(T) \subset \Gamma \) holds for \( T \in D'(M) \) and a closed cone \( \Gamma \subset T^*M \setminus \{0\} \) we have

\[
WF(u^*T) \subset T^*u(T^*M_u \cap \Gamma) =: T^*u \cdot \Gamma,
\]
where \((M_u, u)\) is a coordinate chart of \(M\) and \((T^*M_u, T^*u)\) the corresponding coordinate system of \(T^*M\). Hence we can define:

**Definition A.4.** Consider in analogy to Equation (50) the set

\[
D'_\Gamma(M) = \{ T \in D'(M) | WF(T) \subset \Gamma \},
\]

where \(M\) is a smooth manifold and \(\Gamma \subset T^*M \setminus \{0\}\) a closed cone. A sequence \(N \ni j \mapsto T_j \in D'_\Gamma(M)\) converges in \(D'_\Gamma(M)\) to \(T \in D'_\Gamma(M)\) if \(j \mapsto u_* T_j\) converges for each coordinate chart \((M_u, u)\) in \(D_{T^*u\Gamma}(u(M_u))\) to \(u_* T\).

Now we are ready to discuss restrictions of distributions \(T \in D'(M)\) to submanifolds \(\Sigma \subset M\). Note in this context that we need volume forms on \(M\) and on \(\Sigma\) to embed \(D(M)\) and \(D(\Sigma)\) into \(D'(M)\) respectively \(D'(\Sigma)\). However if \((M, g)\) is a Lorentzian manifold and \(\Sigma\) is non-null (i.e. the pull back of \(g\) to \(\Sigma\) is nondegenerate) this is no problem, because there are canonical choices on \(M\) and on \(\Sigma\) (the volume forms induced by the corresponding metrics).

**Theorem A.5.** Let \(M\) be a manifold and \(\Sigma\) a submanifold with normal bundle

\[
N(\Sigma) := \{ \lambda \in T^*M | S | \lambda(v) = 0 \ \forall v \in T_{\pi(\lambda)}\Sigma \}.
\]

For every distribution \(T \in D'(M)\) with \(WF(T) \cap N(\Sigma) = \emptyset\) the restriction \(T|\Sigma\) can be defined for every distribution \(T \in D'(M)\) with \(WF(T) \cap N(\Sigma) = \emptyset\) in exactly one way such that

1. it coincides with usual restrictions for \(T \in D(M)\),
2. the wave front set of the restriction satisfies

\[
WF(T|\Sigma) \subset i^{*}_\Sigma WF(T) = \{ \theta \in T^*\Sigma | \exists \lambda \in WF(T) with \theta(v) = \lambda(Ti^*_\Sigma v) \ \forall v \in T_{\pi(\lambda)}\Sigma \}
\]

3. and for each closed cone \(\Gamma \subset T^*M \setminus \{0\}\) the map \(D_{T^*\Gamma}(\Sigma) \ni T \mapsto T|\Sigma \in D_{i^{*}\Sigma \Gamma}(\Sigma)\) is sequential continuous, i.e. this means that for each sequence \(N \ni j \mapsto T_j\) converging in \(D_{T^*\Gamma}(M)\) to \(T\) the sequence \(j \mapsto T_j|\Sigma\) converges in \(D_{i^{*}\Sigma \Gamma}(\Sigma)\) to \(T|\Sigma\).

The proof of this theorem can be found in [19] (see Theorem 8.2.4, cf. also Corollary 8.2.7). Using restrictions of distributions we are now able to define products as well, because a product is simply the restriction of the tensor product to the diagonal. Hence we need the following result about the wave front set of tensor products (cf. Theorem 8.2.9 of [19]).

**Proposition A.6.** For \(T, S \in D'(M)\) we have

\[
WF(T \otimes S) \subset (WF(T) \times WF(S)) \cup (WF(S) \times [\{0\} \times WF(T))] \cup (WF(T) \times [WF(T) \times \{0\}])
\]

for the wave front set of the tensor product.
Proposition A.7. Consider now two sequences \( \mathbb{N} \ni j \mapsto T_j \in \mathcal{D}'_1(M) \) and \( \mathbb{N} \ni k \mapsto S_k \in \mathcal{D}'_0(M) \) converging in \( \mathcal{D}'_1(M) \) respectively in \( \mathcal{D}'(M) \) to \( S \) and \( T \). The sequence \( k \mapsto T_k \otimes S_k \) of tensor products converges to \( T \otimes S \) in \( \mathcal{D}'(\mathcal{O}) \) where \( \mathcal{O} \) denotes the cone

\[
\mathcal{O} := (\mathcal{O} \times \mathcal{O}) \cup ((\mathcal{O} \times \{0\}) \times \mathcal{O}) \cup (\mathcal{O} \times (\mathcal{O} \times \{0\})).
\]

Proof. Since this statement is not explicitly proved in [19], we will give a sketch of a proof here. First of all let us consider only the corresponding Euclidean problem, i.e. \( M \) is an open subset of \( \mathbb{R}^n \) and \( \mathcal{O}, \mathcal{O} \) are closed cones in \( M \times \mathbb{R}^n \). According to Definition A.4 it is easy to derive the more general statement from this special case.

We have to check now whether the sequence \( j \mapsto T_j \otimes S_j \) satisfies the conditions from Definition A.3. Weak convergence to \( T \otimes S \) is obvious, which implies that only item 2 has to be shown. Hence we have to show that for \( (x_1, x_2, \xi_1, \xi_2) \in (M^2 \times (\mathbb{R}^{2n} \setminus \{0\})) \setminus (\mathcal{O} \times \mathcal{O}) \) a function \( f \in \mathcal{D}(M^2) \) with \( f(x_1, x_2) \neq 0 \) and a conic neighbourhood \( V \subset \mathbb{R}^{2n} \) of \((\xi_1, \xi_2)\) exist such that Equation (55) holds. The definition of \( \mathcal{O} \otimes \mathcal{O} \) in Equation (54) implies that the only nontrivial case is \((x_1, \xi_1) \in \mathcal{O} \) and \((x_2, \xi_2) \notin \mathcal{O} \) (or vice versa).

Consider first that \((x_1, \xi_1) \in \mathcal{O} \). Hence we can not assume that item 2 of Definition A.3 is satisfied for \( j \mapsto T_j \) and \((x_1, \xi_1)\). However \( f_1 T_j \) and \( f_1 T \) are compactly supported distributions for each \( f_1 \in \mathcal{D}(M) \), i.e. \( f_1 T_j, f_1 T \in \mathcal{E}'(M) \). Hence by the Paley-Wiener theorem we get constants \( C_j, N_j \) such that

\[
|\hat{f_1 T_j}(\eta_1)| \leq C_j (1 + |\eta_1|)^{N_j} \forall \eta_1 \in \mathbb{R}^n
\]

holds. In addition we have, due to weak convergence, \( f_1 T_j(h) \to f_1 T(h) \) for \( j \to \infty \) and for all \( h \in \mathcal{E}(M) \). Therefore the set \( \{ |f_1 T_j(h)| : j \in \mathbb{N} \} \) is bounded for all \( h \in \mathcal{E}(M) \). But \( \mathcal{E}(M) \) is a Fréchet space and the uniform boundedness principle applies. This means that

\[
|f_1 T_j(h)| \leq C \max_{|\alpha| \leq N} |D^\alpha h|
\]

holds for a compact subset \( K \subset \mathbb{R}^n \) and some constants \( C, N \) independent of \( j \).

Applying this inequality to \( h(\eta) = \exp(i x \cdot \eta) \) we see that Equation (55) holds as well for constants \( C, N \) independent of \( j \).

Since \((x_2, \xi_2) \notin \mathcal{O} \) we can assume that there is a function \( f_2 \in \mathcal{D}(M) \) with \( f_2(x_2) \) and a conic neighbourhood \( V \subset \mathbb{R}^n \) of \( \xi_2 \) such that

\[
\sup_{j \in \mathbb{N}} \sup_{\eta_2 \in V} \sup_{\lambda \in \mathbb{R}^+} \lambda^{\tilde{N}} |\hat{f_2 S_j(\lambda \eta_2)}| < \infty \forall \tilde{N} \in \mathbb{N}_0
\]

is satisfied. Together with Equation (55) and an appropriately chosen \( V_1 \) we get

\[
\sup_{j \in \mathbb{N}} \sup_{(\eta_1, \eta_2) \in V_1 \times V} \sup_{\lambda \in \mathbb{R}^+} \lambda^{\tilde{N}} |\hat{f_1 T_j(\lambda \eta_1)}||\hat{f_2 S_j(\lambda \eta_2)}| \leq
\]

\[
\sup_{j \in \mathbb{N}} \sup_{\eta_2 \in V} \sup_{\lambda \in \mathbb{R}^+} \lambda^{\tilde{N}} (1 + \lambda^{\tilde{N}}) |\hat{f_2 S_j(\lambda \eta_2)}| < \infty
\]

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and this completes the proof.

Combining the last two propositions with Theorem A.3 we get the following statement about products of distributions (cf. Theorem 8.2.10 of [19] and Theorem 2.5.10 of [18]):

**Theorem A.8.** Consider two distributions $T, S \in \mathcal{D}'(M)$ with the following property

$$(x, 0) \notin \text{WF}(T) \oplus \text{WF}(S) := \{(x, k_1 + k_2) \in T^* M \mid (x, k_1) \in \text{WF}(T) \text{ and } (x, k_2) \in \text{WF}(S)\}.$$

Then the product $TS \in \mathcal{D}'(M)$ can be defined in a unique way such that the following condition holds:

1. If $T, S$ are regular, $TS$ coincides with the usual product of functions.
2. For each pair of closed cones $\Gamma, \Theta \subset T^* M \setminus \{0\}$ with $\Gamma \oplus \Theta \subset T^* (M \times M) \setminus \{0\}$ the map $\mathcal{D}'_\Gamma(M) \times \mathcal{D}'_\Theta(M) \ni (T, S) \mapsto TS \in \mathcal{D}'(M)$ is sequentially continuous, i.e. if $j \mapsto T_j$ converges in $\mathcal{D}'_\Gamma(M)$ to $T$ and $j \mapsto S_j$ converges in $\mathcal{D}'_\Theta(M)$ the sequence of tensor products converges weakly in $\mathcal{D}'(M)$.

The wave front set of $TS$ is given by

$$\text{WF}(TS) \subset \text{WF}(T) \cup \text{WF}(S) \cup (\text{WF}(T) \oplus \text{WF}(S)).$$

In Section 5 we are mainly interested in limits of the form $\lim_{k \to \infty} T(S_k)$ where $S_k \to S$ in $\mathcal{D}'_\Theta(M)$ for compactly supported $S$ and an appropriate cone $\Theta$. Heuristically this means that we want to calculate

$$T(S) = \int_M T(x)S(x)\lambda_g = \int_M T(x)S(x)1(x)\lambda_g = (TS)(1), \quad (56)$$

where $\lambda_g$ denote the volume element on $M$ induced by $g$ and $1 : M \to \mathbb{R}$ is the function with $1(x) = 1, \forall x \in M$. Note that the last equality in Equation (56) follows from the fact that $TS$ is compactly supported if $S$ is, and that $TS$ can, in this case, be extended to a linear functional on $\mathcal{E}(M)$. If the sequence $\mathbb{N} \ni k \mapsto S_k \in \mathcal{D}(M)$ converges as described to $S$ we get (again heuristically)

$$(TS)(1) = \int_M T(x)S(x)1(x)\lambda_g = \lim_{k \to \infty} \int_M T(x)S_k(x)1(x)\lambda_g(dx) = \lim_{k \to \infty} \int_M T(x)S_k(x)\lambda_g(dx) = \lim_{k \to \infty} T(S_k).$$

This indicates that $\lim_{k \to \infty} T(S_k)$ exists if the product $TS$ exists and is in this case equal to $TS(1)$ (and therefore independent from the sequence $S_k$). The following theorem shows that this conjecture is true.
Theorem A.9. Consider two distributions $T \in \mathcal{D}'(M)$, $S \in \mathcal{E}'(M)$ (i.e. $S$ is compactly supported) with the same property as in Theorem A.8 ($(x,0) \notin WF(S) \oplus WF(T)$) and a sequence $\mathbb{N} \ni k \mapsto S_k \in \mathcal{D}(M)$ converging in $\mathcal{D}'_\Theta(M)$ to $S$, where $\Theta$ satisfies $WF(T) \oplus \Theta \subset T^*M \setminus \{0\}$.

1. The product $TS$ exists and is compactly supported. This implies $TS \in \mathcal{E}'(M)$.

2. The limit $T(S) := \lim_{k \to \infty} T(S_k)$ exists and is equal to $TS(1)$, which implies in particular that $T(S)$ does not depend on the sequence $S_k$.

Proof. The first statement is an immediate consequence of Theorem A.8 and the fact that $S$ is compactly supported. To prove the second one consider the sequence $j \mapsto (T,S_j) \in \mathcal{D}'_{WF(T)}(M) \times \mathcal{D}'_\Theta(M)$ which converges obviously to $(T,S) \in \mathcal{D}'_{WF(T)}(M) \times \mathcal{D}'_\Theta(M)$. Theorem A.8 implies that the sequence of products $j \mapsto TS_j$ converges weakly to $TS$. Since the $S_j$ are compactly supported and converge to a compactly supported distribution $S$ there is a compact subset $K$ of $M$ with $supp S_j \subset K$ for all $j \in \mathbb{N}$ and $supp S \subset K$. Weak convergence of the sequence $j \mapsto TS_j$ implies in addition $S_j T(f) \to ST(f)$ for a smooth compactly supported function $f$ with $f(x) = 1$ for all $x \in K$. Hence we get $T(S) = S_j T(1) \to ST(1)$ for $j \to \infty$ as stated in the theorem.

For the rest of this appendix we will provide some methods, which are needed in the paper to calculate some wave front sets. The first one concerns distributions of the type $Kf$ where $K : \mathcal{D}(M) \to \mathcal{D}'(N)$ is a continuous linear map (and $M,N$ are manifolds). According to [19, Theorem 8.2.12] we have

Theorem A.10. Let $M,N$ be manifolds and $K \in \mathcal{D}'(N \times N)$. If the corresponding linear transformation from $\mathcal{D}(M)$ to $\mathcal{D}'(N)$ is denoted by $K$ we have:

$$WF(Kf) \subset \{(x,\xi) | (x,y,\xi,0) \in WF(K) \text{ for } y \in supp f\}.$$ 

In addition we need the following result about sums of distributions and their wave front sets (see [19, Ch. VII]).

Proposition A.11. Consider the sum $T+S$ of two distributions $T,S \in \mathcal{D}'(M)$. Its wave front set can be estimated from above by $WF(T+S) \subset WF(T)+WF(S)$.

And finally two results which are needed to calculate the wave front set of the distributions $T_\psi$ defined in section 5 (cf. Section VIII.2 of [19] for proofs).

Proposition A.12. The wave front set of a distribution $S$ given by

$$S(f) = \int_{\Sigma} f(x)\lambda(x)$$

with a volume element $\lambda$ on a submanifold $\Sigma \subset M$ is contained in the normal bundle $N(\Sigma)$ of $\Sigma$ (see Equation (53)).

Proposition A.13. For each distribution $T$ on $M$ and each differential operator $P$ we have $WF(PT) \subset WF(T)$.
B Proof of the extension theorem

Let us discuss now the proof of Theorem 3.4 which we have postponed in section 3 to this appendix. Hence consider a quantum field $f \mapsto \Phi(f)$ which is extendible in the sense of Definition 3.2 to a distribution space $\mathcal{D}$. We have to show first that for each $T \in \mathcal{D}$ the operator $\Phi(T)$ exists as described in Definition 3.1. The following lemma gives a necessary condition.

**Lemma B.1.** For each $T \in \mathcal{D}$ there is a closed cone $\Gamma \subset T^*M \setminus \{0\}$ which contains the wave front sets of all distributions $f \mapsto \langle u, \Phi(f) v \rangle$, $u, v \in D_0$ and satisfies in addition the relation $WF(T) \oplus \Gamma \subset T^*M \setminus \{0\}$

**Proof.**

We can choose $u, v \in D_0$ such that $\Phi(F)\Omega = u$ and $\Phi(G)\Omega = v$ holds with $F = f_1 \otimes \cdots \otimes f_n \in \mathcal{D}(M^n)$ and $G = g_1 \otimes \cdots \otimes g_m \in \mathcal{D}(M^m)$. Hence we have $f \mapsto \langle u, \Phi(f) v \rangle = W^{(n+m+1)}(F^* \otimes f \otimes G)$. This implies together with Theorem A.10 that the wave front set of the distribution $\langle u, \Phi(\cdot) v \rangle$ is contained in the set

$$\Gamma := \{(x, \xi) | (x_1, \ldots, x_n, x, x_{n+1}, \ldots, x_{n+m}; 0, \ldots, 0, \xi, 0, \ldots, 0) \in WF(W^{(n+m+1)}(M^n \times \{0\}) \times WF(T) \times (M^m \times \{0\}) \subset T^*M^{n+m+1}.$$

Hence we get by Definition 3.2 $WF(T) \oplus \Gamma \subset T^*M \setminus \{0\}$ as stated.

This lemma implies by Theorem A.3 that $\Phi(T)$ exists for each $T \in \mathcal{D}$ as a quadratic form. To prove that $\Phi(T)$ exists even as an operator we will use the following result.

**Lemma B.2.** Consider two manifolds $M, N$, distributions $S \in \mathcal{D}'(M \times N)$, $T = T^{(1)} \otimes T^{(2)} \in \mathcal{E}'(M \times N)$ such that $WF(S) \oplus (WF(T^{(1)}) \otimes WF(T^{(2)})) \subset T^*(M \times N) \setminus \{0\}$ holds, where $WF(T^{(1)}) \otimes WF(T^{(2)})$ is defined as in Equation (54) and two sequences $N \ni k \mapsto T^{(1)}_k \in \mathcal{D}(M)$, $N \ni l \mapsto T^{(2)}_l \in \mathcal{D}(N)$ converging in $\mathcal{D}_{WF(T^{(1)})}(M)$ respectively $\mathcal{D}_{WF(T^{(2)})}(N)$ to $T^{(1)}$ respectively $T^{(2)}$. Then the double sequence $N^2 \ni (k, l) \mapsto S(T^{(1)}_k \otimes T^{(2)}_l) \in \mathbb{C}$ converges uniformly in $k, l$ to $TS(1)$, i.e. for each $\epsilon > 0$ there is a $N_\epsilon \in \mathbb{N}$ such that

$$|S(T^{(1)}_k \otimes T^{(2)}_l) - (TS)(1)| < \epsilon \forall k, l > N_\epsilon.$$

**Proof.**

Only the statement concerning the uniformity of the convergence of $(k, l) \mapsto S(T^{(1)}_k \otimes T^{(2)}_l)$ is not an immediate consequence of Theorem A.3. Hence
assume the statement is false, i.e. there is an $\epsilon > 0$ such that for each $N \in \mathbb{N}$ there are $(k_N, l_N) \in \mathbb{N}^2$ with $k_N > N$ and $l_N > N$ and
\[
|S(T_{k_N}^{(1)} \otimes T_{l_N}^{(2)}) - TS(1)| \geq \epsilon \forall N \in \mathbb{N}.
\] (57)

However the sequences $N \mapsto T_{k_N}^{(1)}$ and $N \mapsto T_{l_N}^{(2)}$ are subsequences of $k \mapsto T_k^{(1)}$ and $l \mapsto T_l^{(2)}$ which implies that they converge in $D'_{WF(T^{(1)})}(M)$ respectively $D'_{WF(T^{(2)})}(M)$ to $T^{(1)}$ and $T^{(2)}$. Hence Equation (57) contradicts Theorem A.3.

Now we are able to prove the existence of the operators $\Phi(T)$ as stated in item 3 of Theorem B.3.

**Proposition B.3.** If the quantum field $f \mapsto \Phi(f)$ is extendible to the distribution space $\mathcal{D} \subset \mathcal{E}'(M)$ the operators $\Phi(T)$ exist for each $T \in \mathcal{D}$ in the sense of Definition 3.3.

**Proof.** Consider $T \in \mathcal{D}$ and a sequence $N \ni k \mapsto T_k \in \mathcal{D}(M)$ which converges in $D'_{WF(T)}(M)$ to $T$. We have seen in lemma B.1 that $k \mapsto (u, \Phi(T_k)v)$ converges as well and depends only on $T$, not on the sequence $k \mapsto T_k$. Now we have to show convergence of the sequence $N \ni k \mapsto \Phi(T_k)u \in \mathcal{H}$ for all $u \in D_0$. To this end let us assume first that $T$ and the $T_k$ are real valued. Then we have
\[
\|\Phi(T_k)u - \Phi(T_l)u\|^2 = \\
\|\Phi(T_k)\|^2 + \|\Phi(T_l)\|^2 - (\Phi(T_k)u, \Phi(T_l)u)
\]
and therefore
\[
\|\Phi(T_k)u - \Phi(T_l)u\|^2 = \\
\mathcal{W}^{(2n+2)}(f^* \otimes T_{l} \otimes T_{l} \otimes f) + \mathcal{W}^{(2n+2)}(f^* \otimes T_{k} \otimes T_{k} \otimes f) \\
- \mathcal{W}^{(2n+2)}(f^* \otimes T_{k} \otimes T_{l} \otimes f) - \mathcal{W}^{(2n+2)}(f^* \otimes T_{l} \otimes T_{k} \otimes f),
\] (58)

where we have chosen, as in the proof of B.1, $u, v \in D_0$ such that $\Phi(f)\Omega = u$ and $\Phi(G)\Omega = v$ holds with $f = f_1 \otimes \cdots \otimes f_n \in \mathcal{D}(M^n)$ and $G = g_1 \otimes \cdots \otimes g_m \in \mathcal{D}(M^m)$. By assumption we have $WF(f^* \otimes T \otimes T \otimes f) \subset \Theta$ and $\Theta \oplus WF(V^{(2n+2)}) \subset T^*(M^{2n+2}) \setminus \{0\}$ with $\Theta = (M^n \times \{0\}) \times (WF(T) \otimes\otimes WF(T)) \times (M^n \times \{0\})$ (cf. Definition 3.2). Hence convergence of $k \mapsto T_k$ implies together with B.2 that there is a $W \in \mathbb{C}$ such that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ with
\[
\left|\mathcal{W}^{(2n+2)}(f^* \otimes T_{k} \otimes T_{l} \otimes f) - W\right| < \epsilon \quad \forall k, l > N.
\]

In other words each summand on the left hand side of equation (58) differs from $W$ only by $\epsilon/4$ provided $k, l$ are big enough; i.e we have
\[
\left|\mathcal{W}^{(2n+2)}(f^* \otimes T_{l} \otimes T_{l} \otimes f) - W\right| + \left|\mathcal{W}^{(2n+2)}(f^* \otimes T_{k} \otimes T_{k} \otimes f) - W\right| \leq \frac{\epsilon}{2},
\]
\[
\left|\mathcal{W}^{(2n+2)}(f^* \otimes T_{k} \otimes T_{l} \otimes f) - W\right| + \left|\mathcal{W}^{(2n+2)}(f^* \otimes T_{l} \otimes T_{k} \otimes f) - W\right| \leq \frac{\epsilon}{2}.
\]
This implies \( \| \Phi(T_k)u - \Phi(T_l)u \|^2 \leq \epsilon/2 \) for \( k, l > N \), i.e. \( k \mapsto \Phi(T_k)u \) is a Cauchy sequence, converging to an element \( \Phi(T)u \) of \( \mathcal{H} \).

This defines an operator \( \Phi(T) \) with domain \( D \subset \mathcal{H} \) because independence of the limit \( \lim_{k \to \infty} \Phi(T_k)u \) from the sequence \( k \mapsto T_k \) follows from the corresponding property of \( \lim_{k \to \infty} \langle v, \Phi(T_k)u \rangle \) for all \( v \in D_0 \) (cf. Definition 3.4).

If \( T \) is complex valued we can apply the arguments just discussed to the real and imaginary part separately, and we get again an operator \( \Phi(T) \). This completes the proof. \( \square \)

The next step in the proof concerns the existence of an invariant domain \( D \). If \( D \) exists, its elements are given by linear combinations of expressions of the form \( \Phi(T^{(1)}) \cdots \Phi(T^{(n)})\Omega \) with \( T = T^{(1)} \otimes \cdots \otimes T^{(n)} \in \mathcal{D}^\otimes n \). Hence it is natural to consider limits of the form \( \lim_{k \to \infty} \Phi(T_k)\Omega \) where \( \mathbb{N} \ni k \mapsto T_k = T^{(1)}_k \otimes \cdots \otimes T^{(n)}_k \in \mathcal{D}(M^n) \) converges in \( \mathcal{D}_\Gamma(T)(M^n) \) to \( T = T^{(1)} \otimes \cdots \otimes T^{(n)} \in \mathcal{D}^\otimes n \).

1. The limit \( \lim_{k \to \infty} \Phi(T_k)\Omega =: u(T) \in \mathcal{H} \) exists and depends only on \( T \).
2. If \( u(T) = 0 \) holds, we get \( u(S \otimes T) = 0 \) for all \( S \in \mathcal{D} \).

**Proof.** 1. We use the same idea as in the first part of Proposition 3.3. Hence consider \( l, k \in \mathbb{N}^n \) and
\[
\| \Phi(T_k)\Omega - \Phi(T_l)\Omega \|^2 = \| \Phi(T_k) \|^2 + \| \Phi(T_l) \|^2 - \langle \Phi(T_l)\Omega, \Phi(T_k)\Omega \rangle - \langle \Phi(T_k)\Omega, \Phi(T_l)\Omega \rangle
\]
which leads to
\[
\| \Phi(T_k)\Omega - \Phi(T_l)\Omega \| = \mathcal{W}^{(2n)}(T^*_k \otimes T_k) + \mathcal{W}^{(2n)}(T^*_l \otimes T_l) - \mathcal{W}^{(2n)}(T^*_l \otimes T_k) - \mathcal{W}^{(2n)}(T^*_k \otimes T_l). \quad (59)
\]
Extendibility of the quantum field implies together with lemma 3.2 that each of the terms \( \mathcal{W}^{(2n)}(\cdot) \) converges to the same value \( W \). Hence we have
\[
\left| \mathcal{W}^{(2n)}(T^*_l \otimes T_l) - W \right| + \left| \mathcal{W}^{(2n)}(T^*_l \otimes T_k) - W \right| \leq \frac{\epsilon}{2}
\]
\[
\left| \mathcal{W}^{(2n)}(T^*_k \otimes T_l) - W \right| + \left| \mathcal{W}^{(2n)}(T^*_k \otimes T_k) - W \right| \leq \frac{\epsilon}{2},
\]
provided \( l, k \) are big enough. This implies together with Equation (59) convergence of \( \Phi(T_l)\Omega \) to an element \( u(T) \in \mathcal{H} \). The fact that \( u(T) \) depends only on \( T \) and not on the sequence \( k \mapsto T_k \) follows with the same argument as in Proposition 3.3.
2. By assumption we have \( \Phi(T_l) \Omega \to 0 \) as \( l \to \infty \) for each sequence \( \mathbb{N} \ni l \mapsto T_l \in D(M^n) \) converging to \( T \) as described above. This implies

\[
W^{n+m+1}(f^* \otimes S_k \otimes T_l) = \langle v, \Phi(S_k)u(T_l) \rangle \to 0 \quad l \to \infty \tag{60}
\]

for each \( v = u(f) \) with \( f \in D(M^m) \), each sequence \( \mathbb{N} \ni k \mapsto S_k \in D(M) \) converging in \( D_{WF(S)}(M) \) to \( S \) and each fixed \( l \in \mathbb{N} \). On the other hand we can consider the multisequence \( (k,l) \mapsto S_k \otimes T_l \) which converges in \( D_{\Gamma(S \otimes T)}(M^{n+1}) \) to \( S \otimes T \). This implies that \( (k,l) \mapsto \Phi(S_k)\Phi(T_l)\Omega \) converges uniformly in \( (k,l) \) to \( u(S \otimes T) \). Hence to prove that \( u(S \otimes T) = 0 \) it is sufficient to show that for each \( v \in \mathcal{D} \) there is a subsequence \( \mathbb{N} \ni j \mapsto \Phi(S_{k_j})\Phi(T_{l_j}) \) such that

\[
\langle v, \Phi(S_{k_j})\Phi(T_{l_j}) \rangle \to 0 \quad k \to \infty
\]

holds. However this is a direct consequence of Equation (60). Therefore we get \( \Phi(S)u(S \otimes T) = 0 \) as stated.

Now we are ready to prove the existence of an invariant dense domain \( D \subset \mathcal{H} \) for all \( \Phi(T) \) (see item 2 of Theorem 3.4).

**Proposition B.5.** The operators \( \Phi(T) \) can be extended to the invariant domain

\[
D := \text{span}\{u(T) | T \in \mathcal{D}^n, \ n \in \mathbb{N}\},
\]

which is a subset of the domain \( D(\overline{\Phi(T)}) \) of the closure of \( \Phi(T) \).

**Proof.** Obviously \( D_0 \subset D \) hence \( D \) is dense in \( \mathcal{H} \). In addition we have \( \Phi(T)u(f) = u(T \otimes f) \) for all \( f \in D(M^n) \) and \( T \in \mathcal{D} \). Hence we can define an extension of \( \Phi(T) \) to the domain \( D \) by \( \Phi(T)u(T) := u(T \otimes T) \) provided the left hand site depends only on \( u(T) \in \mathcal{D}^n \) and not on \( T \), but this is a consequence of item 2 of Lemma B.4.

To complete the proof of item 2 we have to show that all \( \Phi(T) \) are closed and \( D \) is a subspace of the domain of the closure of \( \Phi(T) \). To prove closedness consider \( \Phi(T)u \) for \( u \in D_0 \). Obviously we have \( \Phi(T)u = \lim_{l \to \infty} \Phi(T_l) \) where \( T_l \) converges weakly to \( T \). But \( \Phi(T_l) \subset \Phi(T_l)^* \) and this implies \( \Phi(T) \subset \Phi(T)^* \). Hence the domain of \( \Phi(T)^* \) is dense, which implies closability of \( \Phi(T) \). Since the construction of the operator \( \Phi(T) : D \to \mathcal{H} \) given above implies immediately that its graph is contained in the closure of the graph of \( \Phi(T) : D_0 \to \mathcal{H} \), which in turn coincides with the graph of the closure of \( \Phi(T) : D_0 \to \mathcal{H} \), we get \( D \subset D(\overline{\Phi(T)}) \) and this completes the proof of the second statement.

The last statement of Theorem 3.4 (item 3) is now a simple consequence of Proposition B.5 and Lemma B.4. Hence Theorem 3.4 is proved.
C Jet bundles and differential operators

In this appendix we want to summarize some material about jet bundles used throughout the paper. For a more detailed presentation we want to refer to the book of Saunders [28].

Hence consider two manifolds \( M, N \), a point \( x \in M \), the set \( \mathcal{D}(x, N) \) of smooth maps from an open neighbourhood of \( x \) to \( N \) and a nonnegative integer \( l \). Two such maps \( f, h \) coincide in \( x \) up to the \( l \)th order, iff \( f(x) = h(x) \) and their Taylor expansion (in coordinates around \( x \) and \( f(x) \)) coincide up to the \( l \)th order. It is easy to check that this defines an equivalence relation. The corresponding equivalence class of \( f \) is called the \( k \)-jet of \( f \) at \( x \in M \) and denoted by \( j^k_x f \); the set of all equivalence classes is denoted by \( J^k_x(M, N) \). If we take the union

\[
J^k(M, N) := \bigcup_{x \in M} J^k_x(M, N)
\]

we get a manifold which is a fiber bundle with respect to the source-projection

\[
J^k(M, N) \ni j^k_x f \mapsto \pi^k(j^k_x f) = x \in M
\]

and as well with respect to the target projection

\[
J^k(M, N) \ni j^k_x f \mapsto \pi^k_0(j^k_x f) = f(x) \in N.
\]

An adapted coordinate system is given in terms of a chart \( (M_u, u) \) of \( M \) and \( (N_v, v) \) of \( N \) by

\[
J^l(M_u, N_v) \ni j^l_x f \mapsto (u(x), v[f(x)], \hat{f}(u(x)), \ldots, \hat{f}^{(l)}(u(x)))
\]

\[
\in u(M_u) \times v(N_v) \times L^1(\mathbb{R}^m, \mathbb{R}^n) \times L^2(\mathbb{R}^m, \mathbb{R}^n) \times \ldots \times L^l(\mathbb{R}^m, \mathbb{R}^n)
\]

(61)

where \( \hat{f} = v \circ f \circ u^{-1} \) denotes the local representative of \( f \), \( m = \dim(M) \), \( n = \dim(N) \) and \( L^k(\mathbb{R}^m, \mathbb{R}^n) \) is the space of \( k \)-linear, symmetric maps \( (\mathbb{R}^m)^k \to \mathbb{R}^n \). Each smooth function \( f \) on \( M \) defines a section, the \( l \)-jet extension of \( f \), by

\[
j^l f : M \to J^l(M, K), \quad x \mapsto j^l f(x) := j^l_x f.
\]

However not all sections of \( J^l(M, K) \) are of this form.

To discuss the relation of jet bundles to (linear) differential operators consider \( J^l(M, \mathbb{K}) \) with \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \). It is a vector bundle with respect to the source projection \( \pi^l \), i.e. each fiber \( J^l_x(M, \mathbb{K}) \) of \( \pi^l : J^l(M, \mathbb{K}) \to M \) is a \( \mathbb{K} \)-vector space. A linear differential operator \( P : \mathcal{E}(M, \mathbb{K}) \to \mathcal{E}(M, \mathbb{K}) \) of order \( l \) defines a morphism \( \psi_P \) of vector bundles (i.e. \( \psi_P \) is linear on each fiber) by

\[
\psi_P : J^l(M, \mathbb{K}) \to M \times \mathbb{K}, \quad j^l_x f \mapsto \psi_P(j^l_x f) := (p, (P f)(x)).
\]

If on the other hand a vector bundle morphism \( \psi : J^l(M, \mathbb{K}) \to M \times \mathbb{K} \) is given, we get immediately a differential operator

\[
P_\psi : \mathcal{E}(M, \mathbb{K}) \to \mathcal{E}(M, \mathbb{K}), \quad f \mapsto P_\psi f := \psi \circ j^l f
\]

(62)
In other words differential operators $P$ and morphisms $\psi$ are in a one to one correspondence. Since the morphism $\psi$ defines a linear map $\psi_\ast$ for each $x \in M$ by $\psi(j^1_x f) = (x, \psi_\ast(j^1_x f))$ we can identify $\psi$ with a section of the vector bundle $J^1(M, \mathbb{K})^\ast$ dual to $J^1(M, \mathbb{K})$, i.e. $J^1(M, \mathbb{K}) = \bigcup_{x \in M} J^1_x(M, \mathbb{K})^\ast$. Hence we have proved the following

**Proposition C.1.** The space of smooth sections $\Gamma(J^1(M, \mathbb{K})^\ast)$ coincides with the space of linear, 1th order, $\mathbb{K}$-valued differential operators on $M$.

Let us consider now a (smooth) diffeomorphism $F$ of $M$. It defines the pull back $F^\ast f = f \circ F$ and the push forward $F_\ast f = f \circ F^{-1}$ of the smooth function $f : M \to \mathbb{K}$. A simple calculation shows that $J^1 F(j^1_x f) := j^1_x(F \circ f)$ is well defined, i.e. does not depend on the representative $f \in j^1_x$. Hence we get an automorphism $J^1 F : J^1(M, \mathbb{K}) \to J^1(M, \mathbb{K})$ of the vector bundle $J^1(M, \mathbb{K})$ which covers $F : M \to M$, i.e. the following diagram commutes

$$
\begin{array}{ccc}
J^1(M, \mathbb{K}) & \xrightarrow{J^1 F} & J^1(M, \mathbb{K}) \\
\pi_0 \downarrow & & \pi_0 \downarrow \\
M & \xrightarrow{F} & M.
\end{array}
$$

If we consider now for each $x \in M$ the dual $J^1_x F^\ast : J^1_{F(x)}(M, \mathbb{K})^\ast \to J^1_x(M, \mathbb{K})^\ast$ of the linear isomorphism $J^1_x(M, \mathbb{K}) \ni j^1_x f \mapsto J^1_x F(j^1_x f) \in J^1_{F(x)}(M, \mathbb{K})$, we get the vector bundle isomorphism $J^1 F^\ast : J^1(M, \mathbb{K})^\ast \to J^1(M, \mathbb{K})^\ast$ dual to $J^1 F$, which given by $J^1 F^\ast|_{J^1_x(M, \mathbb{K})^\ast} = J^1_x F^\ast$. In contrast to $J^1 F$ it covers $F^{-1}$ rather than $F$:

$$
\begin{array}{ccc}
J^1(M, \mathbb{K})^\ast & \xrightarrow{J^1 F^\ast} & J^1(M, \mathbb{K})^\ast \\
\downarrow & & \downarrow \\
M & \xrightarrow{F^{-1}} & M.
\end{array}
$$

If $P$ is a differential operator of order $l$, we can apply $J^1 F^\ast$ to the section $\psi_P \in \Gamma(J^1(M, \mathbb{K})^\ast)$ associated to $P$ by Proposition C.1. We get obviously another section of $J^1(M, \mathbb{K})^\ast$ and therefore a new differential operator, which is in the following way associated to $P$:

**Proposition C.2.** If $Pf = \psi_P \circ j^1 f$ holds for a section $\psi_P$ of $J^1(M, \mathbb{K})$ and a linear differential operator of 1th order, the pull back $F^* P$ of $P$ by a diffeomorphism $F : M \to M$, given by $(F^* P)f := F^* (P(F, f))$ satisfies $(F^* P)f = (J^1 F^\ast \circ \psi_P \circ F) \cdot j^1 f$.

\footnote{Note that stars as sub- and superscripts appear in the context of vectorbundles customarily in two different meanings: They denote dual maps and dual spaces on the one hand and pull backs and push forwards on the other.}
D Fundamental solutions of the Klein-Gordon equation

In this appendix we want to discuss solutions of the inhomogeneous Klein-Gordon equation with distributions $\mathcal{T} \in \mathcal{D}'(\gamma)$ as source term. As a first step let us recall some well known properties of the retarded and the advanced fundamental solutions (see [10] and the references therein for details).

**Theorem D.1.** Consider the Klein-Gordon operator $\Box_g - m^2$ on a globally hyperbolic space-time $(M, g)$. There are distributions $G^\pm \in \mathcal{D}'(M \times M)$, which are uniquely determined by the following properties ($f \in \mathcal{D}(M)$):

1. The distributions $h \mapsto G^\pm(f, h)$ are regular. The corresponding continuous maps $\mathcal{D}(M) \to \mathcal{E}(M)$ will be denoted by $G^\pm$ as well.
2. The function $G^\pm f$ solves the inhomogeneous equation $(\Box_g - m^2)h = f$, i.e. we have $G^\pm(\Box_g - m^2)f = (\Box_g - m^2)G^\pm f = f$.
3. The support of $G^\pm f$ is contained in the causal future/past of $\text{supp} f$, i.e. $\text{supp} G^\pm f \subset J^\pm(\text{supp} f)$.
4. Hence the support of the distribution $f \mapsto G^\pm_{\mathcal{F}}(f) := (G^\pm f)(p)$ ($p \in M$) satisfies $\text{supp} G^\pm_{\mathcal{F}} \subset J^\pm(p)$.

The $G^\pm$ can be extended in exactly one way to continuous maps $G^\pm : \mathcal{E}'(M) \to \mathcal{D}'(M)$: For $T \in \mathcal{E}'(T)$ the distribution $G^\pm T$ is given by $(G^\pm T)(f) = T(G^\pm f)$, where $f \in \mathcal{D}(M)$. As in the regular case $G^\pm T$ solves the inhomogeneous Klein-Gordon equation with $T$ as source term. More precisely: $G^\pm T$ is the unique weak solution of $(\Box_g - m^2)G^\pm T = T$ with support in $J^\pm(\text{supp} T)$.

One mayor problem in the theory of (linear) partial differential equations is to determine the singularities of the solution if the singularities of the initial data and/or the inhomogeneity is given. In the hyperbolic case a complete answer is given by the propagation of singularities theorem of Duistermaat and Hörmander, which we will state here in the special Klein-Gordon case (see [11] for a proof and for the general statement). To this end it is necessary to introduce some terminology first: $L \subset T^*M$ denotes the set of nonvanishing null covectors, i.e.

$$L = \{\theta \in T^*M \setminus \{0\} \mid g(\theta, \theta) = 0\}$$

and $\sim \subset L \times L$ is the bicharacteristic relation associated to $\Box_g - m^2$, i.e.$(p_1, \theta_1) \sim (p_2, \theta_2)$ iff 1. $p_1$ and $p_2$ can be connected by a null geodesic $\nu$ and 2. the vectors $\theta^\#_j \in T^*_pM$ with $g(\theta^\#_j, \cdot) = \theta_j$ are tangent to $\nu$.

**Theorem D.2.** Consider a weak solution $S \in \mathcal{D}'(M)$ of the inhomogeneous Klein-Gordon equation $(\Box_g - m^2)S = T$. Then we have

1. $\text{WF}(S) \setminus \text{WF}(T) \subset L$ and
2. \((p_1, \theta_1) \in \text{WF}(S) \setminus \text{WF}(T)\) and \((p_1, \theta_1) \sim (p_2, \theta_2)\) imply \((p_2, \theta_2) \in \text{WF}(S) \setminus \text{WF}(T)\) i.e. \(\text{WF}(S) \setminus \text{WF}(T)\) is invariant under the bicharacteristic flow.

Let us consider now a smooth timelike curve \(\gamma\) and distributions \(T \in \mathcal{D}'(\gamma)\) \((l \in \mathbb{N}_0)\). The propagation of singularities theorem allows us immediately to locate the singularities of \(G^\pm T\).

**Corollary D.3.** If \(T \in \mathcal{D}'(\gamma)\) holds, we get \(\text{WF}(G^\pm T) \subset \text{WF}(T)\). Hence the singular support of \(G^\pm T\) is contained in the range of \(\gamma\).

**Proof.** We have to show that \(\text{WF}(G_0^\pm T) \setminus \text{WF}(T)\) is empty. Hence assume \((p_1, \theta_1) \in \text{WF}(G^\pm T) \setminus \text{WF}(T)\). It defines a unique, maximally extended null geodesics \(\nu\) by \(\nu(0) = p_1\) and \(g(\nu'(0), \cdot) = \theta_1\). Since \(\text{WF}(T) \cap \mathbb{L} = \emptyset\) holds, the bicharacteristic strip given by \(\nu\), i.e. \(t \mapsto (\nu(t), g(\nu'(t), \cdot))\), lies entirely in \(\text{WF}(G^\pm T) \setminus \text{WF}(T)\). Hence by Theorem D.2 we get \((\nu(t), g(\nu'(t), \cdot)) \in \text{WF}(G^\pm T) \setminus \text{WF}(T)\) for all \(t\) in the domain of \(\nu\).

Due to global hyperbolicity there is in addition a Cauchy surface \(C\) in the past (in the \(G^+\) case) or future (for \(G^-\)) of \(\text{supp} T\). Since \(C\) is a Cauchy surface, \(\nu\) hits it in a point \(\nu(t_2) = p_2\). But \(p_2 \not\in \text{supp} G^\pm T\) holds, due to the condition \(\text{supp}(G^\pm T) \subset \mathcal{J}^\pm(\text{supp} T)\). This implies obviously \((p_2, g(\nu'(t_2), \cdot)) \notin \text{WF}(G^\pm T) \setminus \text{WF}(T)\), in contradiction to the result of the last paragraph. Hence \(\text{WF}(G^\pm T) \setminus \text{WF}(T) = \emptyset\) as stated. \(\square\)

Hence we have seen that the weak solutions \(G^\pm T\) are regular on \(M \setminus \text{Ran} \gamma\). In Section 7 however this information is not quite sufficient, because analyticity properties of \(G^\pm T\) are used in the proof of Theorem D.2. To proceed in this direction it is necessary to determine the distributions \(G^\pm_p\) defined in item \((b)\) of Theorem D.2 more explicitly. Let us introduce some notations first: If \(O \subset M\) is an open convex set the square of the geodesic distance \(\Gamma(p,q)\) between two points \(p, q \in O\) is a well defined smooth function \(O \times O \ni (p,q) \mapsto \Gamma(p,q) \in \mathbb{R}\). For each fixed \(p \in O\) the function \(\Gamma_p(\cdot, \cdot) := \Gamma(p, \cdot, \cdot)\) gives rise to a pair of distributions \(\delta_\pm(\Gamma_p)\) defined formally as the composition of \(\Gamma_p\) with the delta distribution. More precisely for a test function \(f\) with support in \(O\) we define

\[
\langle \delta_\pm(\Gamma_p), f \rangle := \frac{1}{2} \int_{\mathbb{R}^3} \frac{(u_p f)(\pm |x|, x) \sqrt{|\det [(u_p g)^{\mu \nu}(\pm |x|, x)]|}}{|x|} dx,
\]

where \(O \ni q \mapsto u_p(q) = (x^0, x)\) is a normal coordinate system around \(p\) with the \(x^0\) axis future directed, and \(u_p f, (u_p g)^{\mu \nu}\) are the local representatives\(^{15}\) of \(f\) respectively \(g\) in this chart. Finally we need the so called van Vleck–Morette determinant \(\Delta : O \times O \rightarrow \mathbb{R}\) which is defined in an arbitrary coordinate system \(u : O \rightarrow u(O) \subset \mathbb{R}^4\) by

\[
\frac{1}{16} \frac{1}{\sqrt{|\det [(u_\ast g)^{\mu \nu}(x)] \det [(u_\ast g)^{\mu \nu}(y)]|}} \left| \frac{\partial^2 [(u_\ast g)^{\mu \nu}] (x, y)}{\partial x^\rho \partial y^\sigma} \right|. \tag{64}
\]

\(^{15}\) i.e. \(u_\ast f, u_\ast g\) denote the push forwards of \(f\) and \(g\) by the coordinate map \(u\).
Theorem D.4. Consider an analytic space-time and an open convex set $O \subset M$ which is at the same time globally hyperbolic.

1. The square of the geodesic distance $\Gamma$ and the van Vleck – Morette determinant $\Delta$ ([64]) are analytic functions on $O \times O$.

2. For each $p \in O$ we have

$$G^\pm_p \big| O = \frac{1}{2\pi} \sqrt{\Delta(p, \cdot)} \delta^\pm(\Gamma_p) + \chi_{J^\pm(p,O)} V(p, \cdot),$$

where $\delta^\pm(\Gamma_p)$ is the distribution from Equation ([63]), $V : O \times O \to \mathbb{R}$ is a smooth function and $\chi_{J^\pm(p,O)}$ denotes the characteristic function of $J^\pm(p,O) := J^\pm(p) \cap O$.

3. Each $p \in O$ has a neighbourhood $Q_p \subset O$ such $V(p, \cdot)$ is given on $Q_p$ by an absolutely convergent power series

$$V(p,q) = \sum_{j=0}^\infty V_j(p,q) \frac{\Gamma^j(p,q)}{j!}.$$  

Hence $V(p, \cdot)$ is analytic on $Q_p$.

4. The coefficient functions $V_j$ are defined by the Hadamard recursion relations:

$$V_j(p,q) = -\frac{1}{4} \Delta(p,q) \int_0^1 \frac{\mathbb{I} \otimes (\Box_g - m^2) V_{j-1}(p,\nu(s))}{\Delta(p,\nu(s))} s^j ds$$

for $j \in \mathbb{N}_0$ and $V_{-1} = \Delta$. The integrations are carried out along the unique geodesic $\nu : [0,1] \to O$ with $\nu(0) = p$ and $\nu(1) = q$.

5. The neighborhood $Q_p$ depends continuously on $p$, i.e. $Q_p$ can be chosen in such a way that $V(p', \cdot)$ is analytic on $Q_p$ for all $p' \in Q_p$.

This is a well known result which follows directly from Hadamards work on second order, hyperbolic, partial differential equations [17]; see also [14, Chapter 4.3] and [8]. In the non-analytic case the series ([66]) does not converge and leads therefore only to an asymptotic expansion of $V$; see [14] and the references therein for details. However we do not want to proceed in this direction, because our main interest concerns the following extension of Corollary D.3.

Theorem D.5. Consider an analytic manifold $(M,g)$ and a smooth (but not necessarily analytic) timelike curve $\gamma : (a,b) \to M$. Each $t \in (a,b)$ has a neighborhood $I_{t,\epsilon} := (t-\epsilon, t+\epsilon) \subset (a,b)$ such that for each $T \in \mathcal{D}(\gamma(t-\epsilon, t+\epsilon))$ the function $G^\pm T \setminus \text{Ran} \gamma$ is analytic on

$$\Sigma(t,\epsilon) := [H^\pm(\gamma(s)) \setminus \{\gamma(s)\}] \cap [I^+(\gamma(t-\epsilon)) \cup I^-(\gamma(t+\epsilon))].$$
Here \( H^\pm(p) = J^\pm(p) \setminus I^\pm(p) \) denotes the future/past horizon of \( p \in M \) (cf. [24, Chapter 14]), i.e. the manifold generated by future/past pointing null geodesics starting in \( p \).

**Proof.** We will look only at the \( G^+ \) case, since \( G^- \) can be treated similarly. Due to Corollary [D.3] the distribution \( G^+T \) is regular on \( M \setminus \text{Ran} \\gamma \). This means it can be identified with a smooth function which we can restrict to the submanifolds \( \Sigma(t, \epsilon) \). To investigate the analyticity behaviour of this restriction, let us consider now an open convex normal neighbourhood of \( \gamma(t) \) and choose \( \epsilon > 0 \) in such way that the double cone

\[
\mathcal{O} := I^+(\gamma(t - \epsilon)) \cup I^-(\gamma(t + \epsilon))
\]

is contained in it. Using Equation (63) we can decompose the restriction of \( G^+T \) to \( \mathcal{O} \) into the sum \((G^+T)(f) = (G^+_1T)(f) + (G^+_2T)(f)\), where \( f \) is a smooth test function with support in \( \mathcal{O} \) and \( G^+_j(f), j = 1, 2 \) are the functions

\[
G^+_1(f)(p) = \Delta(p, \cdot)\delta_+(\Gamma_p)(f) \quad \text{and} \quad G^+_2(f)(p) = \int_{I^+(p, \mathcal{O})} V(p, q)f(q)\lambda(q).
\]

Before we start to calculate \( G^+_1(f) \) note that we can assume without loss generality that \( T \in \mathcal{D}^0(\gamma) \) holds with

\[
T(f) := \int_{-\varepsilon}^{\varepsilon} h(s)f(\gamma(s + t))\,ds,
\]

where \( h : (-\varepsilon, \varepsilon) \to \mathbb{R} \) is a compactly supported smooth function, because the higher order case differs only by additional derivatives in the \( p \) variable, which do not affect analyticity. Consider now the coordinate system \( \mathcal{O} \setminus \text{Ran} \\gamma \ni p \to u(p) \in \mathcal{O} \) which is defined as follows: 1. Identify the tangent spaces \( T_{\gamma(t)}M \) for all \( s \in (t - \varepsilon, t + \varepsilon) \) with the \( \mathbb{R}^4 \) such that parallel transport becomes constant and the \( x^0 \)-axis is future pointing. 2. Set \( u^{-1}(x^0, x) = \exp_{\gamma(t, x^0)}(|x|, x) \) for all \( (x^0, x) \) with \( x^0 \in (-\varepsilon, \varepsilon) \) and with \( \exp_{\gamma(t, x^0)}(|x|, x) \in \Sigma(t + x^0, \epsilon) \). The restriction of this coordinate map to the future horizon is analytic (since it coincides with the exponential map there) although the curve is only assumed to be smooth. According to Equation (63) \( T(G^+_1(f)) \) can be expressed in this chart by:

\[
T(G^+_1(f)) = \int_{-\varepsilon}^{\varepsilon} h(s)\int_{\mathbb{R}^3} \frac{\langle u_\ast f(s, x)\rangle \langle [\text{Id} \times u_\ast], \Delta \rangle(\gamma(t + s); s, x)k(s, x)}{|x|} \,dx\,ds,
\]

where \( u, f \) and \([\langle \text{Id} \times u_\ast \rangle, \Delta \rangle(\gamma(t + s), \cdot) \) are the local representatives of \( f \in \mathcal{D}(\mathcal{O}) \) and \([\langle \text{Id} \times u_\ast \rangle, \Delta \rangle(\gamma(t + s), \cdot) \) in the coordinate system \( u \). Furthermore \( k(s, x) \) is defined by

\[16\langle \text{Id} \times u_\ast \rangle, \Delta \rangle \text{ denotes the push-forward of } \Delta \text{ by the diffeomorphism } \mathcal{O} \times \mathcal{O} \ni (p, q) \to (\text{Id} \times u)(p, q) = (p, u(q)) \in \mathcal{O} \times u(\mathcal{O}).\]
\[ k(s, x) := \sqrt{\left| \det \left[ (u_{\gamma(t+s)}g)_{\mu\nu}(x, x) \right] \right|}, \]

where \((u_{\gamma(t+s)}g)_{\mu\nu}\) denotes the representative of \(g\) in normal coordinates \(u_{\gamma(t+s)} : \mathcal{O} \to u_{\gamma(t+s)}(\mathcal{O}) \subset \mathbb{R}^4\) around \(\gamma(t+s)\); cf. Equation (63). Hence the Distribution \(G^+_1 T\) coincides on \(\mathcal{O} \setminus \text{Ran} \gamma\) with the smooth function

\[
(G^+_1 T)(u^{-1}(s, x)) = h(s) \frac{(u_s f)(s, x)[(\text{Id} \times u)_* \Delta]\!(\gamma(t + s); s, x)k(s, x)}{|x| \sqrt{\left| \det \left[ (u_s g)_{\mu\nu}(s, x) \right] \right|}} dx ds.
\]

Since the functions \(\mathcal{O} \times \mathcal{O} \ni (p, q) \mapsto u_p(q)\) and \(\Sigma(t + s, \varepsilon) \ni q \mapsto u(q)\) are analytic, \(G^+_1 T|\Sigma(t + s, \varepsilon)\) is analytic as well.

Consider now the second term. According to Proposition [4.4] \(T_{G^+_2}(f)\) is given by

\[
\int_{t-\varepsilon}^{t+\varepsilon} \int_{\gamma(s), \mathcal{O}} h(s)V(\gamma(s), q)f(q)\lambda(q)ds,
\]

where \(\lambda(q)\) denotes the volume form defined by the metric. Hence the distribution \(G^+_2 T\) can be identified with the \(L^1_{\text{loc}}\) function

\[
(G^+_2 T)(q) = \int_{s(q)}^{t+\varepsilon} h(s)V(\gamma(s), q)ds,
\]

where \(s(q)\) is defined by \(q \in \Sigma(s(q), \varepsilon)\). This function is smooth on \(\mathcal{O} \setminus \text{Ran} T\) since the distributions \(G^+_T\) and \(G^+_1 T\) are regular and \(G^+_2 T = G^+_T - G^+_1 T\). By Theorem [4.4] we can choose \(\varepsilon\) in such a way that \(V(\gamma(s), \cdot)\) is analytic on \(I^+(\gamma(t-\varepsilon)) \cup I^-(\gamma(t+\varepsilon))\) for each \(s \in (t-\varepsilon, t+\varepsilon)\). Hence \(G^+_2 T|\Sigma(t + s, \varepsilon)\) is analytic too, and this completes the proof.

\[\square\]

### E The global Hadamard condition

In this appendix we will give the original definition of global Hadamard states which we have postponed in Sec. 7 because it is somewhat involved.

Consider first a smooth Cauchy surface \(\Sigma\) of space-time \((M, g)\). A causal normal neighbourhood of \(\Sigma\) is an open neighbourhood \(N\) of \(\Sigma\) such that \(\Sigma\) is a Cauchy surface for \(\Sigma\) and such that for all \(x_1, x_2 \in N\) with \(x_1 \in J^+(x_2)\) there is a convex normal neighbourhood which contains \(J^-(x_1) \cap J^+(x_2)\). This implies that the square of the geodesic distance is a well defined smooth function on \(N \times N\). The existence of \(N\) is guaranteed by Lemma 2.2 of [21].

In addition we need two open regions \(\mathcal{O}, \mathcal{O}^\prime \subset M \times M\) with the following properties: \(\overline{\mathcal{O}} \subset \mathcal{O}\), \(\mathcal{O}^\prime \subset N \times N\) and \(\mathcal{O}\) is a neighbourhood of the set of causally related points \((x_1, x_2) \in M \times M\) such that \(J^+(x_1) \cap J^-(x_2)\) and \(J^+(x_2) \cap J^-(x_1)\) are contained in a convex normal neighbourhood. On \(\mathcal{O}\) the square of
the geodesic distance $\Gamma(x_1, x_2)$ is again well defined and smooth such that we get a function $V^{(n)} \in \mathcal{D}(\mathcal{O}, \mathbb{C})$

\[ V^{(n)}(x_1, x_2) := \sum_{j=0}^{n} V_j(x_1, x_2)\Gamma^j \]

for each $n \in \mathbb{N}$, where the $V_j$ are the functions from Equation (67).

Finally we need a smooth, global time function $T : \mathcal{M} \to \mathbb{R}$, a smooth function $\chi \in \mathcal{E}(N \times N, \mathbb{R})$ with

\[ \chi(x_1, x_2) = \begin{cases} 
0 & \text{for } (x_1, x_2) \notin \mathcal{O} \\
1 & \text{for } (x_1, x_2) \in \mathcal{O}'
\end{cases} \]

and the van Vleck-Morette determinant $\Delta : N \times N \to \mathbb{R}$ (64). This leads for each $n \in \mathbb{N}$ and $\epsilon > 0$ to a complex valued function $G^{T,n}_\epsilon$:

\[ G^{T,n}_\epsilon(x_1, x_2) := \frac{1}{(2\pi)^2} \left( \frac{\sqrt{\Delta(x_1, x_2)}}{\Gamma + 2i\epsilon(T(x_1) - T(x_2)) + \epsilon^2} \right. \\
\left. + \chi^{(n)}(x_1, x_2) \ln \left( \Gamma + 2i\epsilon(T(x_1) - T(x_2)) + \epsilon^2 \right) \right) \]

where the branch-cut of the logarithm is taken to be on the negative real axis.

Now we are ready to give the following definition

**Definition E.1.** A global Hadamard state is a quasi-free regular state $\omega$ on the CCR-algebra $\text{CCR}(\mathcal{S}, E)$ whose two-point function $W^{(2)}$ has the following structure: There exists a sequence of functions $H^{(n)} \in C^n(N \times N, \mathbb{R})$ such that for all $f_1, f_2 \in \mathcal{D}(N, \mathbb{C})$ and for all $n \in \mathbb{N}$ we have

\[ W^{(2)}(f_1 \otimes f_2) = \lim_{\epsilon \to 0} \int_{N \times N} W^{T,n}_\epsilon(x_1, x_2)f_1(x_1)f_2(x_2)\lambda_\epsilon(x_1) \wedge \lambda_\epsilon(x_2) \]

with

\[ W^{T,n}_\epsilon(x_1, x_2) := \chi(x_1, x_2)G^{T,n}_\epsilon(x_1, x_2) + H^{(n)}(x_1, x_2). \]

This definition does not depend on the choice of $\Sigma, N, \chi$ and $T$ (see [21]).
| Symbol | Description |
|--------|-------------|
| $\mathcal{E}'(M)$ | complex valued, compactly supported distributions on the manifold $M$ |
| $\mathcal{E}'(M,\mathbb{R})$ | real valued, compactly supported distributions on the manifold $M$ |
| $\mathcal{E}'(M,E)$ | compactly supported, distributional sections of the vectorbundle $E \to M$ |
| $\mathcal{D}(M)$ | complex valued, compactly supported, smooth functions on the manifold $M$ |
| $\mathcal{D}(M,\mathbb{R})$ | real valued, compactly supported, smooth functions on the manifold $M$ |
| $\mathcal{D}(M,E)$ | compactly supported, smooth sections of the vectorbundle $E \to M$ |
| $\mathcal{D}'(M)$ | complex valued distributions on the manifold $M$ |
| $\mathcal{D}'(M,\mathbb{R})$ | real valued distributions on the manifold $M$ |
| $\mathcal{D}'(M,E)$ | distributional sections of the vectorbundle $E \to M$ |
| $\mathcal{L}(D_0,\mathcal{H})$ | (unbounded) operators on the Hilbert space $\mathcal{H}$ with domain $D_0 \subset \mathcal{H}$. |
| $f \mapsto \Phi(f)$ | hermitian quantum field (Section 2) |
| $D_0$ | common, dense domain of the quantum field $\Phi$ (Section 2) |
| $\Omega$ | vacuum vector of the quantum field $\Phi$ |
| $\mathfrak{A}$ | Borchers-Uhlmann algebra (Section 2) |
| $\mathcal{W}, \mathcal{W}^{(n)}$ | Wightman functional and $n$-point function (Section 2) |
| $\mathcal{R}(\mathcal{O})$ | local von Neumann algebra associated to space-time region $\mathcal{O} \subset M$ (Section 2) |
| $\mathcal{B}(M)$ | open, relatively compact subsets of the manifold $M$ (Section 2) |
| $\text{WF}(T)$ | wave front set of distribution $T$ (Definition A.1) |
| $\mathcal{D}_\Gamma(M)$ | $(M$ manifold, $\Gamma$ closed cone) Definition A.4 |
| $\Gamma \oplus \Sigma$ | $(\Gamma, \Sigma \subset T^* M$ closed cones) Equation (9), Theorem A.8 |
| $\mathfrak{D} \subset \mathcal{D}(M)$ | space of test distributions (Section 3) |
| $\Phi(T), T \in \mathfrak{D}$ | extended quantum field (Definitions 3.1, 3.2 and Theorem 3.4) |
| $D \subset \mathcal{H}$ | common, dense domain of the extended quantum field (Theorem 3.4) |
\( \mathfrak{A}(\mathcal{D}) \)  
Equation (11)

\( \mathfrak{W}, \mathfrak{W}^{(n)} \)  
extended Wightman functional and \( n \)-point function  
(Proposition 3.3)

\( \Gamma \odot \Sigma \)  
(\( \Gamma, \Sigma \subset T^*M \) closed cones)  
Equation (54)

\( \Gamma(T) \)  
(\( T \in \mathcal{D}'(M^n) \))  
Definition 4.2

\( \mathfrak{D}(\gamma), \mathfrak{D}^{\infty}(\gamma), \mathfrak{D}^l(\gamma) \)  
(\( \gamma \) smooth curve)  
Definition 4.3

\( J^l(M,C) \to M \)  
l-jet bundle over manifold \( M \)  
(Appendix C)

\( E^l(\gamma) \)  
(\( \gamma \) smooth curve)  
Theorem 4.5

\( T_{\psi} \in \mathfrak{D}^l(\gamma) \)  
Theorem 4.5

\( \mathfrak{D}(M) \)  
Equation (24)

\( E \boxtimes F \)  
exterior tensor product of vector bundles \( E,F \)  
(Section 5)

\( \mathfrak{A}^l(\gamma) \)  
(\( l \in \mathbb{N}, \gamma \) smooth, timelike curve)  
Equation (27)

\( \mathcal{W}^{l,n}(\gamma) \)  
(\( l,n \in \mathbb{N}, \gamma \) smooth, timelike curve)  
Equation (29)

\( \mathcal{H}^l(\gamma) \subset \mathcal{H} \)  
Theorem 5.3

\( \Phi^l(\gamma) \)  
Theorem 5.3

\( \mathcal{R}^l(\gamma) \)  
(\( l \in \mathbb{N}, \gamma \) smooth, timelike curve)  
Equation (28)

\( \mathcal{R}(\gamma) \)  
(\( \gamma \) smooth, timelike curve)  
Equation (30)

\( \eta^l_\gamma \)  
(\( l \in \mathbb{N}, \gamma \) smooth, timelike curve)  
Equation (31)

\( \mathcal{M}^l_\gamma(\mu) \)  
(\( l \in \mathbb{N}, \gamma \) smooth, timelike curve, \( \mu \subset \gamma \))  
Equation (32)

\( G \)  
retarded minus advanced solution  
(Section 5)

\( (\mathcal{P}, G) \)  
symplectic space defined by \( G \)  
(Section 5)

\( \text{CCR}(\mathcal{P}, G) \)  
CCR-algebra associated to symplectic space \( (\mathcal{P}, G) \)  
(Section 5)

\( \iota_\Sigma \)  
(\( \Sigma \) Cauchy surface)  
Equation (34)

\( K \)  
one particle structure  
(Section 7)

\( \mathcal{A}(\mathcal{O}) \)  
local C*-algebra  
(Equation (36)

\( \mathcal{A}^l(\gamma) \)  
(\( l \in \mathbb{N}, \gamma \) smooth, timelike curve)  
Equation (33)

\( \mathcal{P}^l(\gamma) \subset \mathcal{P} \)  
(\( l \in \mathbb{N}, \gamma \) smooth, timelike curve)  
Section 5

\( \gamma_i \)  
inertial observer in Minkowski space  
(Section 8)

\( \Xi^l_{\gamma, Z t} \)  
Equation (46)

\( \alpha^l_{\gamma, Z t} \)  
Theorem 9.6

\( \overline{\alpha}_{\gamma, Z t} \)  
Theorem 9.6
Theorem 9.6

References

[1] H. Baumgärtel and M. Wollenberg. *Causal nets of operator algebras*. Akademie Verlag, Berlin (1992).

[2] J. J. Bisognano and E. H. Wichmann. *On the duality condition for a hermitian scalar field*. J. Math. Phys. **16**, 985–1007 (1975).

[3] H.-J. Borchers. *Field operators as $C^\infty$ functions in spacelike directions*. Nuovo Cimento **33**, 1600–1613 (1964).

[4] R. Brunetti and K. Fredenhagen. *Interacting quantum fields in curved space: Renormalizability of $\varphi^4$*. In *Operator algebras and quantum field theory. Proceedings of the conference dedicated to Daniel Kastler in celebration of his 70th birthday* (S. Doplicher et. al., editor), pages 546–563. Cambridge, MA: International Press (1997).

[5] R. Brunetti, K. Fredenhagen and M. Köhler. *The microlocal spectrum condition and Wick polynomials of free fields on curved spacetimes*. Commun. Math. Phys. **180**, 633–652 (1996).

[6] D. Buchholz, O. Dreyer, M. Florig and S. J. Summers. *Geometric modular action and spacetime symmetry groups*. [math-ph/9805026] (1998).

[7] B. Demoen, P. Vanheuverzwijn and A. Verbeure. *Completely positive maps on the CCR-algebra*. Lett. Math. Phys. **2**, 161–166 (1977).

[8] B. S. DeWitt and R. W. Brehme. *Radiation damping in a gravitational field*. Ann. Phys. (NY) **9**, 220–259 (1960).

[9] J. Dieudonné. *Treatise on analysis. VII*. Academic Press, Boston (1988).

[10] J. Dimock. *Algebras of local observables on a manifold*. Commun. Math. Phys. **77**, 219–228 (1980).

[11] J.J Duistermaat and L. Hörmander. *Fourier integral operators. II*. Acta Math. **128**, 183–269 (1972).

[12] C. J. Fewster. *A general worldline inequality*. Class. Quant. Grav. **17**, 1897–1911 (2000).

[13] K. Fredenhagen and R. Haag. *Generally covariant quantum field theory and scaling limits*. Commun. Math. Phys. **108**, 91–105 (1987).

[14] F. G. Friedlander. *The wave equation on a curved space-time*. Cambridge Univ. Press, Cambridge (1975).
[15] S. A. Fulling. *Nonuniqueness of canonical field quantization in riemannian space-time*. Phys. Rev. D **7**, 2851–2862 (1973).

[16] R. Haag. *Local quantum physics*. Springer, Berlin (1992).

[17] J. Hadamard. *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*. Hermann, Paris (1932).

[18] L. Hörmander. *Fourier integral operators. I*. Acta Math. **127**, 79–183 (1971).

[19] L. Hörmander. *Analysis of linear partial differential operators. I*. Springer, Berlin (1983).

[20] J. Manuceau J., M. Sirugue, D. Testard and A. Verbeure. *The smallest C*-algebra for canonical commutation relations*. Commun. Math. Phys. **32**, 231–243 (1973).

[21] B. S. Kay and R. M. Wald. *Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate killing horizon*. Phys. Rep. **207**, 49–136 (1991).

[22] M. Keyl. *Causal spaces, causal complements and their relations to quantum field theory*. Rev. Math. Phys. **8**, 229–270 (1996).

[23] M. Keyl. *On causal compatibility of quantum field theories and space-times*. Commun. Math. Phys. **195**, 15–28 (1998).

[24] B. O’Neill. *Semi-Riemannian geometry*. Academic Press, New York (1983).

[25] M. J. Radzikowski. *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*. Commun. Math. Phys. **179**, 529–553 (1996).

[26] M. Reed and B. Simon. *Methods of modern mathematical physics. I*. Academic Press, San Diego (1980).

[27] R. K. Sachs and H. Wu. *General relativity for mathematicians*. Springer, Berlin (1977).

[28] D. J. Saunders. *The geometry of jet bundles*. Cambridge University Press, Cambridge (1989).

[29] W. G. Unruh. *Notes on black hole evaporation*. Phys. Rev. **D14**, 870–892 (1976).

[30] R. Verch. *The averaged null energy condition for general quantum field theories in two dimensions*. J. Math. Phys. **41**, 206–217 (2000).

[31] R. M. Wald. *Quantum field theory in curved spacetime and black hole thermodynamics*. Univ. Chicago Press, London (1994).
[32] M. Wollenberg. *On the relation between conformal structures in space-time and nets of local algebras of observables.* Lett. Math. Phys. **31**, 195–203 (1994).

[33] M. Wollenberg. *On conformal structure in space-time and nets of local algebras of observables.* Math. Nachr. **193**, 235–242 (1998).