Invisibility in $\mathcal{PT}$-symmetric complex crystals

Stefano Longhi

Dipartimento di Fisica, Politecnico di Milano, Piazza L. da Vinci 32, I-20133 Milano, Italy

E-mail: longhi@fisi.polimi.it

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Abstract

Bragg scattering in sinusoidal $\mathcal{PT}$-symmetric complex crystals of finite thickness is theoretically investigated by the derivation of exact analytical expressions for reflection and transmission coefficients in terms of modified Bessel functions of first kind. The analytical results indicate that unidirectional invisibility, recently predicted for such crystals by coupled-mode theory (Z Lin et al 2011 Phys. Rev. Lett. 106 213901), breaks down for crystals containing a large number of unit cells. In particular, for a given modulation depth in a shallow sinusoidal potential, three regimes are encountered as the crystal thickness is increased. At short lengths the crystal is reflectionless and invisible when probed from one side (unidirectional invisibility), whereas at intermediate lengths the crystal remains reflectionless but not invisible; for longer crystals both unidirectional reflectionless and invisibility properties are broken.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Physical phenomena described by reduced or effective non-Hermitian Hamiltonians are often encountered in a wide class of quantum or classical systems, for example in nuclear or condensed-matter physics of open systems [1–3] or in optical systems in the presence of optical gain or losses [4]. Among non-Hermitian Hamiltonians, great interest has been devoted in the past two decades to studying the properties of parity-time ($\mathcal{PT}$) invariant Hamiltonians, which possess a real-valued energy spectrum below a symmetry-breaking point in spite of non-Hermiticity. Such a class of non-Hermitian Hamiltonians was originally introduced by Bender in the framework of non-Hermitian extensions of quantum mechanics and quantum field theories [5–7], and recently has found an increased interest since the proposal of physical systems described by $\mathcal{PT}$-symmetric Hamiltonians, including optical [8–19] and electronic [20] systems.

Complex periodic potentials [21–30] realize a kind of synthetic complex crystals, which show rather unusual scattering and transport properties as compared to ordinary crystals.
Complex crystals have been investigated in different areas of physics, ranging from matter waves [25–29] to optics [10]. In optics, a complex crystal with $PT$ invariance is realized by the introduction of index and balanced gain/loss modulations in a dielectric medium [10]. Complex crystals can also be realized in atom optics experiments exploiting the interaction of near resonant light with an open two-level system. Scattering of matter waves from purely absorbing optical lattices was reported in a few earlier experiments [25, 26, 29]. As noted by Berry [31], although in such experiments on matter waves $PT$ symmetry is not strictly realized and the complex potentials are purely absorptive, the analysis is essentially the same, since the mean loss simply represents an overall exponential decay of the wave. From the theoretical side, Bragg scattering, diffraction and transport properties have been extensively investigated for sinusoidal $PT$-symmetric complex crystals [10, 21, 22, 32–36], revealing some interesting properties such as the violation of Friedel’s law of Bragg scattering [26, 27, 32], double refraction and non-reciprocal diffraction [10], and unidirectional Bloch oscillations [35]. In particular, in a recent work [36] it was predicted that a sinusoidal $PT$-symmetric sinusoidal crystal of finite length near the spontaneous $PT$-symmetry breaking point can act as a unidirectional invisible medium, i.e. the crystal is almost reflectionless when probed from one side, and transmission occurs as if the crystal were absent. Such an unidirectional invisibility of $PT$-symmetric Bragg scatters near the symmetry-breaking point was previously predicted to occur in [37] for waveguide Bragg gratings which combine matched periodic modulations of refractive index and loss/gain yielding asymmetrical mode coupling. In these previous studies [36, 37], invisibility was explained on the basis of a coupled-mode theory describing Bragg scattering and coupling of counter-propagating waves in the crystal, which is rather common in the optical context [38, 39]. Such an analysis predicts that, for a shallow grating near the $PT$-symmetry breaking point, the sinusoidal crystal appears to be invisible when probed from one side independently of the crystal length.

In this work, we re-consider the scattering properties of the sinusoidal $PT$-symmetric potential and derive exact analytical expressions for reflection and transmission coefficients. The analysis shows that the application of the coupled-mode theory in the standard form fails to predict the correct scattering properties in the case of long crystals. In particular, as at short lengths, the crystal is reflectionless and invisible when probed from one side (according to previous studies [36, 37]), at intermediate lengths the crystal remains reflectionless but not invisible. At even longer crystal lengths, both unidirectional reflectionless and invisibility properties are broken.

2. Bragg scattering in $PT$-symmetric sinusoidal potentials: general aspects and extended coupled-mode theory

2.1. The model

Let us consider the stationary Schrödinger equation for a quantum particle in a locally periodic and complex potential $V(x)$, which in the dimensionless form reads

$$\hat{H} \psi = -\frac{d^2 \psi}{dx^2} - V(x) \psi = E \psi,$$

where $E$ is the energy of the incident particle and $V(x)$ is the complex scattering potential with period $\Lambda$, which is non-vanishing in the interval $0 < x < L$. The crystal length $L$ is assumed to be an integer multiple of the lattice period $\Lambda$, i.e. $L = N\Lambda$, where $N$ is the number of unit cells in the crystal. As mentioned in the introduction, equation (1) describes Bragg scattering of matter waves from a complex potential in the non-interacting regime, which applies e.g. to
a dilute cold atomic beam (see, for instance, [29]). In this work, we will mainly focus our attention to the interaction of near resonant light with an open two-level system, and it is generally absorptive. In this work, we will mainly focus our attention to the PT-symmetric sinusoidal potential, assuming

\[ V(x) = V_0 [\cos(2\pi x / \Lambda) + i \sigma \sin(2\pi x / \Lambda)] \]  

for \( 0 < x < L \), and \( V(x) = 0 \) for \( x < 0 \) and \( x > L \), where \( V_0 \) is the lattice amplitude and \( \sigma \geq 0 \) measures the strength of the non-Hermitian part of the potential. The spectral properties of the PT-symmetric sinusoidal potential (3) have been investigated in [10, 22, 30, 32, 35]. For the infinitely extended crystal, the energy spectrum remains real valued for \( \sigma \leq 1 \), and breaking of the PT phase is attained at \( \sigma = \sigma_c = 1 \) [10, 22, 30, 32, 35]. Here, we consider Bragg scattering of incoming waves with the momentum \( p \) close to the Bragg value \( \pi / \Lambda \), i.e. with energy \( E = p^2 \) close to \( (\pi / \Lambda)^2 \), and typically will assume a modulation \( V_0 \) of the potential much smaller than the energy \( E \).

It should be noted that Bragg scattering of optical waves in one-dimensional Bragg grating structures, considered in [36, 37], is basically analogous to Bragg scattering of matter waves in the framework of equation (1). In fact, the electric field amplitude \( E(x) \) of an optical wave at frequency \( \omega \) that propagates along a dielectric medium with a spatially dependent relative dielectric constant \( \epsilon(x) = n_0^2 [1 + \Delta \epsilon(x)] \), where \( n_0 \) is the refractive index of the lossless medium and \( \Delta \epsilon(x + \Lambda) = \Delta \epsilon(x) \) accounts for the index and gain/loss modulation, satisfies the scalar Helmholtz equation, which can be written in the form

\[ -\frac{d^2 E}{dx^2} - E \Delta \epsilon(x) E = E \Delta \epsilon, \]  

where we have set \( E = k^2 \) and \( k = n_0 \omega / \epsilon_0 \). Note that equation (3) formally reduces to the Schrödinger equation provided that the following formal substitutions

\[ E \rightarrow \psi, \quad k \rightarrow p \]  

are made, with a complex scattering potential \( V(x) \) related to the modulation of the dielectric constant \( \Delta \epsilon(x) \) by the simple relation

\[ V(x) = E \Delta \epsilon(x). \]  

Hence, the only difference between scattering of matter waves in complex optical potentials and light waves in complex Bragg gratings is that, in the latter case, the complex scattering potential \( V(x) = E \Delta \epsilon(x) \) in the equivalent Schrödinger equation depends on the energy \( E \) of the incidence particle. However, for shallow gratings Bragg scattering occurs solely for optical fields with frequencies \( \omega \) very close to the Bragg frequency \( \omega_B = \omega_0 \pi / (n_0 \Lambda) \) (see, e.g., [37]), and thus one can safely assume \( V(x) \approx (\pi / \Lambda)^2 \Delta \epsilon(x) \), leaving out the dependence of the scattering potential from the energy. In the following, we will mainly focus our analysis on the determination of the reflection and transmission coefficients for scattering of matter waves in the framework of equation (1); however, similar results hold mutatis mutandis for reflection and transmission of optical waves in Bragg grating structures.

2.2. Scattering states, spectrum and reflection/transmission coefficients

Since \( V(x) = 0 \) for \( x < 0 \) and \( x > L \), the continuous spectrum of the Hamiltonian \( \hat{H} \) is the semi-infinite real axis of energies \( E = p^2 \geq 0 \), and the corresponding eigenfunctions are the scattered states, defined by the relations

\[ \psi(x) = \begin{cases} \alpha_1 \exp(ipx) + \beta_1 \exp(-ipx) & x \leq 0 \\ \alpha_2 \exp[ip(x - L)] + \beta_2 \exp[-ip(x - L)] & x \geq L, \end{cases} \]  

where we have set \( \Lambda = \pi / (n_0 \Delta \epsilon) \), and \( \beta_1 / \beta_2 \) is the reflection coefficient, with \( |\beta_1|^2 + |\beta_2|^2 = 1 \). For large \( \beta_1 \), the wave propagates along a dielectric medium with a spatially dependent relative dielectric constant \( \epsilon(x) = n_0^2 [1 + \Delta \epsilon(x)] \), where \( n_0 \) is the refractive index of the lossless medium and \( \Delta \epsilon(x + \Lambda) = \Delta \epsilon(x) \) accounts for the index and gain/loss modulation, satisfies the scalar Helmholtz equation, which can be written in the form

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where \( p \geq 0 \) is the momentum and \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) are the amplitudes of forward and backward propagating waves on the left-hand \((x < 0)\) and on the right-hand \((x > L)\) sides of the crystal, respectively. Such amplitudes are related by the algebraic equation (see, for instance, [13])

\[
\begin{pmatrix}
\alpha_2 \\
\beta_2
\end{pmatrix} = \mathcal{M}(p)
\begin{pmatrix}
\alpha_1 \\
\beta_1
\end{pmatrix},
\]

where the \(2 \times 2\) transfer matrix \(\mathcal{M}(p)\) is unimodular, i.e. \(\det \mathcal{M} = \mathcal{M}_{22} \mathcal{M}_{11} - \mathcal{M}_{12} \mathcal{M}_{21} = 1\). For a \(PT\)-symmetric potential, the further relation \(\mathcal{M}_{22}(p) = \mathcal{M}_{11}^*(p^*)\) holds. The transmission \((t)\) and reflection \((r)\) coefficients for left- \((l)\) and right-side \((r)\) incidence are related to the coefficients of the transfer matrix by the usual relations (see, for instance, [13])

\[
t^{(l)} = \frac{1}{\mathcal{M}_{22}}, \quad t^{(r)} = t^{(l)} \equiv t, \quad r^{(l)} = -\frac{\mathcal{M}_{21}}{\mathcal{M}_{22}}, \quad r^{(r)} = \frac{\mathcal{M}_{12}}{\mathcal{M}_{22}}.
\]

Note that the transmission coefficient does not depend on the incidence side like in an ordinary crystal, whereas generally one has \(|t^{(l)}| \neq |t^{(r)}|\), i.e. in reflection a complex crystal behaves differently for left- and right-side incidence. This is a rather general result of wave scattering from a complex potential barrier which was previously discussed e.g. in [40, 41]. If we indicate by \(Z(p)\) the fundamental matrix of equation (1) from \(x = 0\) to \(z = L\), which relates the values of \(\psi(x)\) and \((d\psi/dx)\) at the planes \(x = 0\) and \(x = L\), i.e.

\[
\begin{pmatrix}
\psi(L) \\
(d\psi/dx)(L)
\end{pmatrix} = Z(p)
\begin{pmatrix}
\psi(0) \\
(d\psi/dx)(0)
\end{pmatrix},
\]

it can be readily shown that the transfer matrix \(\mathcal{M}\) can be calculated as

\[
\mathcal{M}(p) = T^{-1}(p)Z(p)T(p),
\]

where we have set

\[
T(p) = \begin{pmatrix}
1 & 1 \\
ip & -ip
\end{pmatrix}.
\]

From a numerical viewpoint, the fundamental matrix \(Z(p)\) can be computed as follows. Let us cut the crystal into a sequence of \(N_0\) thin slices, of thickness \(\Delta x = L/N_0\), and let us indicate by \(Z_k(p)\) the fundamental matrix associated with the propagation at the \(k\)th slice \((k = 1, 2, \ldots, N_0)\), i.e. from \(x_k = (k - 1)\Delta x\) to \(x_{k+1} = k\Delta x\). Then, the fundamental matrix \(Z(p)\) can be calculated as the ordered product

\[
Z(p) = Z_N(p) \times Z_{N-1}(p) \times \cdots \times Z_2(p) \times Z_1(p).
\]

If \(\Delta x\) is much smaller than \(\Lambda\), the potential \(V(x)\) is almost constant in the interval \((x_k, x_{k+1})\), and thus \(Z_k(p)\) can be approximated as

\[
Z_k(p) \simeq \begin{pmatrix}
\cos(\lambda_k \Delta x) & (1/\lambda_k) \sin(\lambda_k \Delta x) \\
-\lambda_k \sin(\lambda_k \Delta x) & \cos(\lambda_k \Delta x)
\end{pmatrix},
\]

where we have set

\[
\lambda_k = \sqrt{p^2 + V(x_k)}.
\]

Note that, because of the periodicity of the crystal, one can limit to compute the fundamental matrix for the unit cell, \(Z^{(\text{cell})}(p)\), i.e. from \(x = 0\) to \(x = \Lambda\). The fundamental matrix of the crystal is then given by \(Z(p) = Z^{(\text{cell})N}(p)\), which can be computed using the relation

\[
Z(p) = Z^{(\text{cell})}(p)U_N - U_{N-2},
\]

where \(Z\) is the \(2 \times 2\) identity matrix, \(U_N = \sin[(N+1)\theta]/\sin \theta\), and the complex angle \(\theta\) is defined by \(\cos \theta = (1/2)\text{Tr}(Z^{(\text{cell})})\).
Besides scattering states, the Hamiltonian $\hat{H}$ can possess bound states belonging to the point spectrum, which are determined by the zeros of $M_{22}(p)$ (i.e. the poles of the transmission coefficient $\sigma$) in the half-complex plane $\text{Im}(p) > 0$. The zeros of $M_{22}(p)$ in the half-complex plane $\text{Im}(p) < 0$ are resonance states. The onset of $\mathcal{PT}$-symmetry breaking is detected by the appearance of a divergence (for some real value $p = p_0 \neq 0$ of the momentum) in the transmission coefficient as the non-Hermiticity parameter $\sigma$ is increased from zero. In fact, for $\sigma = 0$ the transmission and reflection coefficients are bounded from above and the zeros of $M_{22}(p)$ lie in the lower half-complex plane $\text{Im}(p) < 0$, i.e. they are resonances. As $\sigma$ increases, the resonances move toward the real axis $\text{Im}(p) = 0$, until at a critical value $\sigma = \sigma_c$ a resonance crosses the real axis, say at $p = p_0 \neq 0$ (real). Correspondingly, the transmission coefficient $t(p)$ diverges at $p = p_0$. Just above $\sigma_c$, a bound state with the complex energy $E = (p_0 + i\sigma_c)^2$ thus appears, which is the signature of $\mathcal{PT}$-symmetry breaking. For a sinusoidal crystal of finite length, the transmission and reflection coefficients at $\sigma = 1$ are bounded, and numerical calculations of the transfer matrix $M$ (using the procedure outlined above) indicate that symmetry breaking is attained at a value $\sigma_c$, larger than 1, with $\sigma_c \to 1^+$ as $L \to \infty$.

2.3. Coupled-mode theory

For a rather general class of $\mathcal{PT}$-symmetric complex potentials $V(x)$ describing Bragg scattering in shallow lattices, approximate expressions for the reflection and transmission coefficients can be derived by an asymptotic analysis of equation (1). In the optical context, such an analysis is generally referred to as the coupled-mode theory of Bragg scattering, which is known to provide accurate description of transmission and reflection coefficients for index-modulated shallow gratings (see, for instance, [38, 39]). Such an analysis applies to Bragg scattering of particles with the momentum $p$ close to $\pi/\Lambda$ (first-order Bragg scattering) provided that the particle energy $E \simeq (\pi/\Lambda)^2$ is much larger than the characteristic modulation depth $V_0$ of the complex crystal. For the sinusoidal crystal defined by equation (2), this means that the parameter $\alpha = \Lambda^2 V_0/\pi^2$ should be much smaller than 1. In [36, 37], it was shown that the application of coupled-mode theory to the sinusoidal $\mathcal{PT}$-symmetric crystal at $\sigma = 1$ gives the following expressions for the transmission and reflection coefficients:

$$t(p) = \exp(ipL), \quad r^{(i)}(p) = 0, \quad r^{(r)}(p) = \frac{iV_0\Delta \sin(\delta L)}{2\pi} \exp[i(p + \pi/\Lambda)L],$$

(15)

where we have set $\delta = p - \pi/\Lambda$. Such equations clearly indicate that, for left-side incidence, the crystal appears to be fully invisible, i.e. there are no reflected waves and the transmitted ones propagate as if the crystal were absent [36]. Conversely, for right-side incidence as the transmitted wave propagates again as if the crystal were absent, a reflected wave is generated, with a reflectance $R^{(r)} = |r^{(r)}|^2$ that grows quadratically with the crystal thickness $L$ at the Bragg resonance $\delta = 0$, i.e. for $p = \pi/\Lambda$. Such a physically relevant behavior was referred to as unidirectional invisibility in [36]. As is shown in appendix A and briefly mentioned in [36], in the framework of the coupled-mode theory, unidirectional invisibility is predicted to occur for a rather general class of complex potentials with a zero mean, $V(x) = \sum_{\alpha \neq 0} \Phi_{\alpha} \exp(2\pi i n x/\Lambda)$, provided that the condition $\Phi_{-1} = 0$ (or similarly $\Phi_1 = 0$) is satisfied. Note that the $\mathcal{PT}$-symmetric sinusoidal potential (2) at $\sigma = 1$ belongs to such a general class of complex potentials.

In the following section, we will derive exact expressions for transmission and reflection coefficients for the $\mathcal{PT}$-symmetric sinusoidal crystal, and we will show that the invisibility property of the crystal, as predicted by equation (15), occurs solely for short crystals and breaks down for long crystals. Here, we discuss the reasons for the failure of equation (15) to
predict the correct expressions of transmission and reflection coefficients in long crystals, and propose an extended version of coupled-mode theory to properly describe Bragg scattering in complex crystals. The derivation of coupled-mode equations is routinely done by means of averaging or multiple-scale asymptotic techniques, which is detailed in appendix A for the general case of a complex potential $V(x) = \sum_{n\neq 0} V_n \exp(2i\pi n x/\Lambda)$ with a zero mean. Here, we give explicit analytical results for the sinusoidal $PT$-symmetric crystal at $\sigma = 1$; however, as shown in appendix A, similar results are obtained for a more general complex crystal provided that the condition $\Phi_{-1} = 0$ is satisfied. For a small value of $\alpha$, a solution to equation (1) can be searched as a power series expansion

$$\psi(x) = \psi^{(0)}(x) + \alpha \psi^{(1)}(x) + \alpha \psi^{(2)}(x) + \cdots,$$

and multiple spatial scales $X_0 = x, X_1 = \alpha x, \ldots$ are introduced to satisfy solvability conditions at the various orders in the asymptotic analysis. If the analysis is pushed up to the order $\sim \alpha$, the solution to equation (1) inside the crystal for a small value of $\delta = p - \pi/\Lambda$ (of order $\sim \alpha$) can be written as

$$\psi(x) = \psi^{(0)}(x) + \alpha \psi^{(1)}(x) + o(\alpha^2),$$

where we have set

$$\psi^{(0)}(x) = u(x) \exp(i\pi x/\Lambda) + v(x) \exp(-i\pi x/\Lambda)$$

and where the amplitudes $u$ and $v$ satisfy the coupled-mode equations

$$i \frac{du}{dx} = -\delta u - \frac{V_0 \Lambda}{2\pi} v$$

and

$$i \frac{dv}{dx} = \delta v.$$

From equations (20) and (21), it follows that the amplitudes $u$ and $v$ at the planes $x = 0$ and $z = L$ are related by the relation $(u(L), v(L))^T = K(p)(u(0), v(0))^T$, where the matrix $K(p)$ reads explicitly

$$K(p) = \begin{pmatrix} \exp(i\delta L) & V_0 \frac{\sin(\delta L)}{2\pi} \\ 0 & \exp(-i\delta L) \end{pmatrix}.$$

In standard coupled-mode theory [38, 39], only the leading-order term $\psi^{(0)}(x)$ in the expansion (17) is considered for the computation of the transfer matrix $M$, and one simply has

$$M(p) = S(p) K(p),$$

where we have set

$$S(p) = \begin{pmatrix} \exp(i\pi L/\Lambda) & 0 \\ 0 & \exp(-i\pi L/\Lambda) \end{pmatrix}.$$
conditions would be \( u(0) = 1 \) and \( v(L) = 0 \). The key point is that, if expression (17) of \( \psi(x) \) up to the order \( \sim \alpha \) is now considered, the boundary condition \( v(L) = 0 \) does not exactly correspond to the absence of incident waves from the right side, just because of the (small) additional contribution to \( \psi(x) \) given by \( \alpha \psi^{(1)}(x) \) (equation (19)). Hence, a more accurate procedure should use the boundary conditions \( u(0) = 1 \) and \( v(L) = \epsilon \), where \( \epsilon \) is a small parameter (of order \( \alpha \)) to be determined such that \( (d\psi/dx) = ip\psi \) at \( x = L \). It is obvious that, if the solution to the coupled-mode equations (20)–(21) with the boundary conditions \( u(0) = 0 \) and \( v(0) = \epsilon \) (corresponding to probing the crystal from the right side with a small amplitude \( \epsilon \)) would remain small uniformly in the interval \((0, L)\), the additional contribution to the solution arising from taking \( \epsilon \neq 0 \) would just introduce a small correction to the solution corresponding to the boundary conditions \( u(0) = 1 \) and \( v(0) = 0 \). Hence, a small correction to the transmission and reflection coefficients (equation (15)) would be obtained. This case always occurs for short enough crystals. However, if the crystal length \( L \) is long enough such that the reflection \( \epsilon \) of right-side incidence becomes large (of the order or larger than \( \sim 1/\alpha \)), then the correction arising from the solution to the coupled-mode equations (20)–(21) with the boundary conditions \( u(0) = 0 \) and \( v(0) = \epsilon \) can no longer be neglected, and thus should be accounted for when calculating the reflection and transmission coefficients with the appropriate boundary conditions. The crystal length \( L \) at which the failure of equation (15) is expected to occur can be estimated by imposing \(|\epsilon(r)\| \lesssim 1/\alpha\). Taking for \(|\epsilon(r)\| \) its peak value at \( \delta = 0 \), i.e. \(|\epsilon(r)\| \sim V_0 AL/(2\pi)\), one then expects the failure of equation (15) for \( L \gtrsim L_c \), where

\[
L_c = \frac{2\pi^3}{V_0^2 A^3}.
\] (25)

Hence, for \( L \) of the order of larger than \( L_c \), equation (15) cannot be used to calculate the transmission and reflection coefficients of the crystal. A more appropriate procedure to compute the transfer matrix, and thus the transmission and reflection coefficients, is to use equation (10), in which the fundamental matrix \( Z \) is calculated using for \( \psi(x) \) the expression given by equations (17)–(19), i.e. \textit{including} the first-order correction term \( \alpha \psi^{(1)}(x) \). Such an extension of the ordinary coupled-mode theory will be referred to as the \textit{extended} coupled-mode theory.

3. Bragg scattering in sinusoidal \( \mathcal{PT} \)-symmetric crystals: exact analysis

Exact expressions for the reflection and transmission coefficients can be derived for the sinusoidal potential \( (2) \) at \( \alpha = 1 \) in terms of the modified Bessel functions of first kind. In fact, after the change of the variable \( y = (\Lambda\sqrt{V_0/\pi}) \exp(i\pi x/\Lambda) \), equation (1) reduces to the Bessel equation [35]:

\[
y^2 \frac{d^2\psi}{dy^2} + y \frac{d\psi}{dy} - (y^2 + q^2)\psi = 0,
\] (26)

where we have set

\[
q = \frac{p\Lambda}{\pi}.
\] (27)

Note that Bragg scattering for particles with momentum \( p \) close to \( \pi/\Lambda \) corresponds to \( q \sim 1 \).

For \( q \neq 1 \), two linearly independent solutions to equation (26) are \( I_q(y) \) and \( I_{-q}(y) \), where \( I_q(y) \) is the modified Bessel function \( I \) of first kind [43]. Hence, in the interval \( 0 < x < L \) two linearly independent solutions to equation (1) are given by

\[
\Phi_1(x) = I_q(\Delta \exp(i\pi x/\Lambda)), \quad \Phi_2(x) = I_{-q}(\Delta \exp(i\pi x/\Lambda)),
\] (28)
where we have set
\[ \Delta = \frac{\Lambda \sqrt{\nu_0}}{\pi} = \sqrt{\alpha}. \]  
(29)

Using equation (28), one can construct the fundamental matrix \( Z(p) \) of equation (1) from \( x = 0 \) to \( x = L \) as
\[ Z(p) = \begin{pmatrix} \Phi_1(L) & \Phi_2(L) \\ \Phi_1(L) & \Phi_2(L) \end{pmatrix} \times \begin{pmatrix} \Phi_1(0) & \Phi_2(0) \\ \Phi_1(0) & \Phi_2(0) \end{pmatrix}^{-1}, \]  
(30)
where the apex denotes the derivative with respect to \( x \). The transfer matrix \( M(p) \) is finally obtained after the substitution of equation (30) into equation (10). After some lengthy calculations, which are briefly detailed in appendix B, the following expressions for the transfer matrix coefficients are obtained:
\[ M_{11}(p) = \cos(pL) + i \frac{\Delta \sin(pL)}{2p \sin(q)} (p^2 Q_1 Q_2 - V_0 D_1 D_2) \]  
(31)
\[ M_{12}(p) = -i \frac{\Delta \sin(pL)}{2p \sin(q)} [V_0 D_1 D_2 + p^2 Q_1 Q_2 + p \sqrt{V_0} (D_1 Q_2 + D_2 Q_1)] \]  
(32)
\[ M_{21}(p) = i \frac{\Delta \sin(pL)}{2p \sin(q)} [V_0 D_1 D_2 + p^2 Q_1 Q_2 - p \sqrt{V_0} (D_1 Q_2 + D_2 Q_1)] \]  
(33)
\[ M_{22}(p) = \cos(pL) - i \frac{\Delta \sin(pL)}{2p \sin(q)} (p^2 Q_1 Q_2 - V_0 D_1 D_2), \]  
(34)
where we have set
\[ Q_1 = I_q(\Delta), \quad Q_2 = I_{-q}(\Delta), \quad D_1 = I_q(\Delta), \quad D_2 = I_{-q}(\Delta) \]  
(35)
and where \( q \) and \( \Delta \) are defined by equations (27) and (29), respectively. The reflection and transmission coefficients \( r(p) \) and \( t(p) \) are then obtained after the substitution of equations (31)–(34) into equation (8). In particular, for the transmission coefficient \( t(p) \) one explicitly obtains
\[ t(p) = \frac{1}{\cos(pL) - i F(p) \sin(pL)}, \]  
(36)
where we have set
\[ F(p) = \frac{\Delta}{2p \sin(q)} (p^2 Q_1 Q_2 - V_0 D_1 D_2). \]  
(37)
Note that equation (36) would reduce to the first part of equation (15), corresponding to unidirectional crystal invisibility, if the function \( F(p) \) were replaced by 1.

Equations (31)–(37) have been derived for Bragg scattering of matter waves in the framework of the Schrödinger equation (1); however, similar relations hold for Bragg scattering of light waves in a complex Bragg grating structure, governed by the similar equation (3). Specifically, in view of equations (4) and (5), for a grating structure with a sinusoidal \( PT \)-symmetric modulation of the complex relative dielectric constant \( \Delta \epsilon(x) = \Phi \exp(2i \pi x/\Lambda) \) and for incident light waves with frequency \( \omega \), the analytical expressions given above still hold, provided that the following formal substitutions \( p \rightarrow \omega n_0/c_0 \) and \( V_0 \rightarrow (\omega n_0/c_0)^2 \Phi \simeq (\pi/\Lambda)^2 \Phi \) are made, where \( n_0 \) is the refractive index of the lossless dielectric medium and \( c_0 \) is the speed of light in vacuum.
4. Unidirectional crystal invisibility

Let us now discuss the unidirectional invisibility of the sinusoidal $PPT$-symmetric crystal on the basis of the exact scattering results presented in the previous section. To study the exact behavior of $t(p)$ as given by equation (36) and breakdown of crystal transparency as the number of cells $N$ is increased, let us measure the length $x$ in units of $\Lambda/\pi$, i.e. let us set without loss of generality $\Lambda = \pi$. With such a scaling, one has $\alpha = V_0$, $q = p$, $L = N\pi$ and $\Delta = \sqrt{V_0}$. Using the identity $I_p(\Delta) = I_{p-1}(\Delta) - (p/\Delta)I_p(\Delta) = I_{p+1}(\Delta) + (p/\Delta)I_p(\Delta)$ for the derivative of modified Bessel functions [43], one can write

$$t(p) = \frac{1}{\cos(N\pi p) - iF(p) \sin(N\pi p)}$$

(38)

with

$$F(p) = \frac{\pi}{2p \sin(p\pi)} \left[ \sqrt{V_0} p (I_{p-1} + I_{p+1}) - V_0 (I_{p-1} I_{p+1}) \right]$$

(39)

and where the Bessel functions are calculated at $\sqrt{V_0}$. Note that $t(p)$ depends on two parameters solely: the number of crystal cells $N$ and the potential amplitude $V_0$. Note also...
that equation (38) is valid regardless of the smallness of $V_0$ and far from the Bragg resonance condition $p = 1$ as well. Similar expressions can be derived for the reflection coefficients $r^{(l,r)}(p)$ in terms of the modified Bessel functions. In the computation of the reflection and transmission coefficients, the modified Bessel functions have been calculated using a fast and highly accurate routine, discussed in [44]. The accuracy of our procedure has been tested by checking the agreement of the results obtained from the exact analytical prediction (equations (38) and (39)) and from the full numerical procedure outlined in section 2.2 (equations (10)–(14)).

Figures 1 and 2 show the behaviors of spectral transmittance $T(p) = |t(p)|^2$ and reflectance $R^{(l,r)}(p) = |r^{(l,r)}(p)|^2$ for left- and right-side incidence, as calculated by the exact analysis (solid curves), for increasing values of the number of cells $N$ and for $V_0 = 0.02$. In the figures, the behavior of the normalized phase time of transmitted waves, defined by $\tau_t(p) = (1/L)(d\phi_t/dp)$, is also depicted, where $\phi_t(p)$ is the phase of $t(p)$. Physically, $\tau_t(p)$ represents the traversal time of a narrow wave packet, with central momentum $p$, across the crystal, normalized to the transit time in vacuum (i.e., in the absence of the crystal). For a relatively small number of cells, as in figure 1, unidirectional invisibility is observed, and reflection for left-side crystal incidence is extremely small, according to the coupled-mode
Figure 3. Transmittance (left panels) and reflectance for left-side incidence (right panels) in a sinusoidal $PT$-symmetric crystal for $V_0 = 0.02$, $\Lambda = \pi$ and for increasing number of crystal cells: (a) $N = 10\,000$, (b) $50\,000$ and (c) $N = 1600\,000$.

The theory of section 2.3 and the results of [36, 37]. However, as the number of cells is increased to become comparable to or larger than $N_c = L_c / \Lambda$, given by (see equation (25))

$$N_c \sim \frac{L_c}{\Lambda} = \frac{2}{\pi \alpha^2}.$$ (40)
the invisibility regime breaks down near $p = 1$, with the appearance of oscillations of the transmittance and phase time, as one can clearly see in figures 2(a) and (c). In the figures, the predictions of spectral transmittance and reflectance computed by the extended coupled-mode theory, discussed at the end of section 2.3, are also depicted by the dotted curves. Note that, as the crystal is no longer transparent, the reflectance for left-side incidence remains extremely small (see figure 2(b)). In such a regime, the crystal is not invisible; however, it is still unidirectional reflectionless. As the number of cells is further increased, the transmittance grows in a narrow interval near the Bragg condition $p \simeq 1$, as well as the reflectance for right-side incidence. As the transmittance becomes large enough, such that $M_{22}$ becomes smaller and of the same order of magnitude as $M_{21}$, a very narrow resonance peak appears in the reflectance spectrum for left-side incidence. This is shown in figure 3, in which solid and dotted curves refer to the exact results and to the approximate ones based on the extended coupled-mode theory, respectively. In such a regime, both unidirectional invisibility and reflectionless properties of the complex crystal are thus broken. To estimate the number of cells $N'_c > N_c$ at which such a second transition occurs, we can apply the extended coupled-mode theory, discussed in section 2.3, to calculate an approximate expression of the reflection coefficient $r^{(l)}$ for left-side incidence. Using equations (17)–(21) for an approximate expression of $\psi(x)$ and applying the appropriate boundary conditions, corresponding to left-side incidence, at exact Bragg resonance ($p = \pi / \Lambda$) an approximate expression for $|r^{(l)}|$ can be derived, which reads explicitly $r^{(l)} \sim (\pi / 64) \alpha^3 (L / \Lambda)$. The critical number of cells $N'_c$ at which the crystal is no longer reflectionless for left-side incidence can be estimated by letting $|r^{(l)}| \sim 1$, which yields

$$N'_c \sim \frac{64}{\pi \alpha^3},$$

(41)

5. Conclusion

In this work, Bragg scattering in sinusoidal $\mathcal{P}\mathcal{T}$-symmetric complex crystals of finite thickness has been theoretically investigated, and exact analytical expressions for reflection and transmission coefficients have been derived in terms of modified Bessel functions. The analytical results indicate that unidirectional invisibility, recently predicted for such crystals by coupled-mode theory [36, 37], breaks down for crystals containing a large number of unit cells. In particular, for a given modulation depth in a shallow sinusoidal potential, three regimes have been found as the crystal length is increased. At short lengths the crystal is reflectionless and invisible when probed from one side (unidirectional invisibility), according to standard coupled-mode theory. As the number of cells is increased, the absence of reflection for one-side incidence is still observed; however, the crystal is no longer invisible because large oscillations in the transmittance and in the transmission phase time appear near the Bragg resonance. For still thicker crystals, both unidirectional reflectionless and invisibility properties are broken. An extension of coupled-mode theory has been proposed to properly model the scattering properties of complex crystals.

Appendix A. Derivation of coupled-mode equations

In this appendix, we briefly derive coupled-mode equations describing first-order Bragg scattering in a complex crystal with a shallow lattice in the framework of the Schrödinger equation (1). Let us consider Bragg scattering of a particle with the momentum $p$ close to the Bragg value $\pi / \Lambda$, i.e. with energy $E = p^2$ close to $(\pi / \Lambda)^2$, and let us assume for the
complex scattering potential $V(x)$ a rather general profile with zero mean, given by the Fourier expansion

$$V(x) = \sum_{n \neq 0} \Phi_n \exp(2i\pi nx/\Lambda) \quad \text{(A.1)}$$

for $0 < x < L$. The shallow lattice approximation implies that the Fourier amplitudes $\Phi_n$ are much smaller than $E$. Note that the $\mathcal{PT}$-symmetric sinusoidal potential (2) is obtained as a special case of equation (A.1) after setting $\Phi_1 = (V_0/2)(1 + \sigma)$, $\Phi_{-1} = (V_0/2)(1 - \sigma)$ and $\Phi_n = 0$ for $n \neq \pm 1$. To develop a perturbative analysis of equation (1) in the shallow grating approximation $V(x) \to 0$ and for $p \to \pi/\Lambda$, it is worth introducing a parameter $\alpha$ that measures the smallness of the various terms entering in the equations and rewriting equation (1) in the following form suited for an asymptotic analysis:

$$\frac{d^2\psi}{dx^2} + \left(\frac{\pi}{\Lambda}\right)^2 \psi = -\alpha [V(x)\psi + W\psi], \quad \text{(A.2)}$$

where we have set

$$W \equiv E - \left(\frac{\pi}{\Lambda}\right)^2 \simeq \frac{2\pi}{\Lambda} \left(p - \frac{\pi}{\Lambda}\right). \quad \text{(A.3)}$$

The problem is to construct an asymptotic approximation of the perturbed solution $\psi = \psi(x; \alpha)$ to equation (A.2) as $\alpha \to 0$. Therefore, we seek a perturbation expansion of $\psi$ in the form of a power series in $\alpha$:

$$\psi(x; \alpha) = \psi^{(0)}(x) + \alpha \psi^{(1)}(x) + \alpha^2 \psi^{(2)}(x) + \ldots \quad \text{(A.4)}$$

and introduce multiple scales for space $x$:

$$X_0 = x, \quad X_1 = \alpha x, \quad X_2 = \alpha^2 x, \ldots, \quad \text{(A.5)}$$

which are needed to satisfy the solvability conditions in the asymptotic expansion at various orders. Substitution of the ansatz (A.4) into equation (A.2) and using the derivative rule

$$\frac{d^2}{dx^2} = \frac{\partial^2}{\partial X_0^2} + 2\alpha \frac{\partial}{\partial X_0} \frac{\partial}{\partial X_1} + \alpha^2 \left(\frac{\partial^2}{\partial X_1^2} + 2 \frac{\partial}{\partial X_0} \frac{\partial}{\partial X_2}\right) + \ldots \quad \text{(A.6)}$$

yields a hierarchy of equations for successive corrections to $\psi$, which are obtained after collecting the terms of the same order in $\alpha$ in the equation so obtained. At leading order $\sim \alpha^0$, one has

$$\frac{\partial^2 \psi^{(0)}}{\partial X_0^2} + \left(\frac{\pi}{\Lambda}\right)^2 \psi^{(0)} = 0, \quad \text{(A.7)}$$

whose general solution is given by

$$\psi^{(0)}(X_0, X_1, X_2, \ldots) = u(X_1, X_2, \ldots) \exp(i\pi x_0/\Lambda) + v(X_1, X_2, \ldots) \exp(-i\pi x_0/\Lambda), \quad \text{(A.8)}$$

where the amplitudes $u$ and $v$ may vary over the slow spatial scales $X_1, X_2, \ldots$. At order $\sim \alpha$, one obtains

$$\frac{\partial^2 \psi^{(1)}}{\partial X_0^2} + \left(\frac{\pi}{\Lambda}\right)^2 \psi^{(1)} = g^{(1)}(X_0), \quad \text{(A.9)}$$

where the forcing term $g^{(1)}$ is given by

$$g^{(1)}(X_0) = -[V(X_0) + W] \psi^{(0)} - 2 \frac{\partial^2 \psi^{(0)}}{\partial X_0 \partial X_1}. \quad \text{(A.10)}$$

The solvability condition for equation (A.9) requires that the forcing term $g^{(1)}(X_0)$ does not contain terms oscillating like $\sim \exp(\pm i\pi x_0/\Lambda)$. After the substitution of equations (A.1) and (A.8) into equation (A.10) and letting equal to zero the coefficients of the terms oscillating
like ∼ \exp(\pm i\pi X_0/\Lambda) in the expression so obtained, the following equations for the evolution of the amplitudes \(u\) and \(v\) on the slow spatial scale \(X_1\) are then obtained:

\[
\frac{2\pi}{\Lambda} \frac{\partial u}{\partial X_1} = - W u - \Phi_1 v, \tag{A.11}
\]

\[
\frac{2\pi}{\Lambda} \frac{\partial v}{\partial X_1} = W v + \Phi_{-1} u, \tag{A.12}
\]

and the solution at order ∼ \(\alpha\) is given by

\[
\psi^{(1)}(X_0) = \frac{\Lambda^2}{\pi^2} \sum_{n \neq 0, -1} \frac{\Phi_n \exp[i\pi (2n + 1)X_0/\Lambda]}{(2n + 1)^2 - 1} + \frac{\Lambda^2}{\pi^2} \sum_{n \neq 0, 1} \frac{\Phi_n \exp[i\pi (2n - 1)X_0/\Lambda]}{(2n - 1)^2 - 1}.
\]

The evolution equations of the envelopes \(u\) and \(v\) at longer spatial scales \(X_2, X_3, \ldots\) are obtained similarly as solvability conditions at orders \(\alpha^2, \alpha^3, \ldots\) in the asymptotic expansion. The evolution of the amplitudes \(u\) and \(v\) in the physical spatial variable \(x\) is then given by \(du/dx = \alpha \partial_x u + \alpha^2 \partial_x^2 u + \cdots\) and \(dv/dx = \alpha \partial_x v + \alpha^2 \partial_x^2 v + \cdots\). If we limit our analysis to the order ∼ \(\alpha\), after setting \(\alpha = 1\) from equations (A.11) and (A.12) and using equation (A.3), one finally obtains the following coupled-mode equations for the envelopes \(u\) and \(v\):

\[
i \frac{du}{dx} = - \delta u - \rho_1 v \tag{A.14}
\]

\[
i \frac{dv}{dx} = \delta u + \rho_2 u, \tag{A.15}
\]

where we have set

\[
\delta = p - \frac{\pi}{\Lambda}, \quad \rho_1 = \frac{\Lambda \Phi_1}{2\pi}, \quad \rho_2 = \frac{\Lambda \Phi_{-1}}{2\pi}.
\]

In the framework of the standard coupled-mode theory [36, 37], unidirectional crystal invisibility is attained whenever the evolution of either one of the two amplitudes \(u\) or \(v\) is decoupled from the other one, i.e. for either \(\Phi_{-1} = 0\) or \(\Phi_1 = 0\). In particular, for the PT-symmetric crystal (equation (2)) at \(\sigma = 1\), one has \(\Phi_1 = V_0\) and \(\Phi_{-1} = 0\), which ensures unidirectional invisibility. In this case, the coupled-mode equations (A.14) and (A.15), as well as the first-order correction \(\psi^{(1)}\) as given by equation (A.13), reduce to equations (19), (20) and (21) given in the paper.

Appendix B. Derivation of the transfer matrix

The exact expression of the transfer matrix \(\mathcal{M}(p)\) is obtained using equation (10), where the fundamental matrix \(\mathcal{Z}(p)\) is calculated in terms of the modified Bessel functions according to equation (30). A simplified form of \(\mathcal{Z}(p)\) can be obtained after observing that, owing to the analytic continuation of the Bessel \(I\) function in the complex plane [43], one has

\[
\Phi_1(x+L) = \Phi_1(x) \exp(i\pi qL/\Lambda), \quad \Phi_2(x+L) = \Phi_2(x) \exp(-i\pi qL/\Lambda) \tag{B.1}
\]

and thus \(\Phi_{1,2}(L) = \Phi_{1,2}(0) \exp(\pm i\pi qL/\Lambda)\) and \(\Phi_{1,2}'(L) = \Phi_{1,2}'(0) \exp(\pm i\pi qL/\Lambda)\). Hence, one can write

\[
\begin{pmatrix} \Phi_1(L) & \Phi_2(L) \\ \Phi_1'(L) & \Phi_2'(L) \end{pmatrix} = \begin{pmatrix} \Phi_1(0) & \Phi_2(0) \\ \Phi_1'(0) & \Phi_2'(0) \end{pmatrix} \begin{pmatrix} \exp(i\pi qL/\Lambda) & 0 \\ 0 & \exp(-i\pi qL/\Lambda) \end{pmatrix}.
\]

(B.2)
Moreover, taking into account the property of the Wronskian of the modified Bessel functions [43], one has
\[
\begin{bmatrix}
\Phi_1(0) & \Phi_2(0) \\
\Phi_1'(0) & \Phi_2'(0)
\end{bmatrix} = \frac{i \pi \Delta}{\Lambda} \begin{bmatrix}
I_q(\Delta) & I_{-q}(\Delta) \\
I_q'(\Delta) & I_{-q}'(\Delta)
\end{bmatrix} = -\frac{i}{\Lambda} \sin(q\pi)
\] (B.3)
and hence
\[
\left(\begin{array}{cc}
\Phi_1(0) & \Phi_2(0) \\
\Phi_1'(0) & \Phi_2'(0)
\end{array}\right)^{-1} = \frac{\Lambda}{2 \sin(q\pi)} \left(\begin{array}{cc}
\Phi_2(0) & -\Phi_2(0) \\
-\Phi_1(0) & \Phi_1(0)
\end{array}\right) .
\] (B.4)
The substitution of equations (B.2) and (B.4) into equation (30) yields for the fundamental matrix \( \mathcal{Z}(p) \) the following simplified expression:
\[
\mathcal{Z}(p) = \frac{\Lambda}{2 \sin(q\pi)} \left(\begin{array}{cc}
\Phi_1(0) & \Phi_2(0) \\
\Phi_1'(0) & \Phi_2'(0)
\end{array}\right) \left(\begin{array}{cc}
\exp(i\pi qL/\Lambda) & 0 \\
0 & \exp(-i\pi qL/\Lambda)
\end{array}\right)
\times \left(\begin{array}{cc}
\Phi_2(0) & -\Phi_2(0) \\
-\Phi_1(0) & \Phi_1(0)
\end{array}\right),
\] (B.5)
where \( \Phi_1(0) = Q_1, \Phi_2(0) = Q_2, \Phi_1'(0) = i\sqrt{\nu_0}D_1, \Phi_2'(0) = i\sqrt{\nu_0}D_2 \) and \( Q_{1,2} \) and \( D_{1,2} \) are defined by equation (35) given in the paper. The substitution of equation (B.5) into equation (10) and using the expression of the matrix \( T(p) \) given by equation (11), after some lengthy but straightforward calculations one finally obtains for the coefficients of the transfer matrix \( M(p) \) the expressions given by equations (31)–(34).

References
[1] Moiseyev N 1998 Phys. Rep. 302 212
[2] Muga J G, Palao J P, Navarro B and Egusquiza I L 2004 Phys. Rep. 395 357
[3] Rotter I 2009 J. Phys. A: Math. Theor. 42 153001
[4] Kostenbauder A, Sun Y and Siegman A E 1997 J. Opt. Soc. Am. A 14 1780
[5] Bender C M and Boettcher S 1998 J. Phys. A: Math. Theor. 31 5243
[6] Bender C M 2007 Rep. Prog. Phys. 70 947
[7] Mostafazadeh A 2010 Phys. Scr. 82 038110
[8] Ruschhaupt A, Delgado F and Muga J G 2005 J. Phys. A: Math. Gen. 38 L171
[9] El-Ganainy R, Makris K G, Christodoulides D N and Musslimani Z H 2007 Opt. Lett. 32 2632
[10] Makris K G, El-Ganainy R, Christodoulides D N and Musslimani Z H 2008 Phys. Rev. Lett. 100 103904
[11] Klaasen S, G"unther U and Moiseyev N 2008 Phys. Rev. Lett. 101 080402
[12] Guo A, Salamo G I, Duchesne D, Morandotti R, Volatzer R, Ravat M, Aimez V, Siviloglou G A and Christodoulides D N 2009 Phys. Rev. Lett. 103 093902
[13] Mostafazadeh A 2009 Phys. Rev. Lett. 102 220402
[14] Longhi S 2010 Phys. Rev. Lett. 105 013903
[15] Ritter C E, Makris K G, El-Ganainy R, Christodoulides D N, Segev M and Kip D 2010 Nature Phys. 6 192
[16] Feng L, Ayache M, Huang J, Xu Y L, Lu M H, Chen Y F, Fainman Y and Scherer A 2011 Science 333 729
[17] Ramezani H, Kottos T, El-Ganainy R and Christodoulides D N 2010 Phys. Rev. A 82 043803
[18] Longhi S 2010 Phys. Rev. A 82 031801
[19] Cheng Y D, Ge L and Stone A D 2011 Phys. Rev. Lett. 106 093902
[20] Schindler J, Li A, Zheng M C, Ellis F M and Kottos T 2011 Phys. Rev. A 84 040101
[21] Cannata F, Junker G and Trost J 1998 Phys. Lett. A 246 219
[22] Bender C M, Dunne G V and Meisinger P N 1999 Phys. Lett. A 252 272
[23] Cervero J M 2003 Phys. Lett. A 317 26
[24] Shin K C 2004 J. Phys. A: Math. Gen. 37 8287
[25] Oberthaler M K, Abfalterer R, Bernet S, Schmiedmayer J and Zeilinger A 1996 Phys. Rev. Lett. 77 4980
[26] Keller C, Oberthaler M K, Abfalterer R, Bernet S, Schmiedmayer J and Zeilinger A 1997 Phys. Rev. Lett. 79 3327
[27] Berry M V 1998 J. Phys. A: Math. Gen. 31 3493
[28] Berry M V and O’Dell D H J 1998 J. Phys. A: Math. Gen. 31 2093
[29] Stützle R et al 2005 Phys. Rev. Lett. 95 110405
[30] Midya B, Roy B and Roychoudhury R 2010 Phys. Lett. A 374 2605
[31] Berry M V 2008 J. Phys. A: Math. Theor. 41 244007
[32] Longhi S 2010 Phys. Rev. A 81 022102
[33] Makris K G, El-Ganainy R, Christodoulides D N and Musslimani Z H Phys. Rev. A 81 063807
[34] Longhi S 2009 Phys. Rev. Lett. 103 123601
[35] Graefe E M and Jones H F 2011 Phys. Rev. A 84 013818
[36] Lin Z, Ramezani H, Eichelkraut T, Kottos T, Cao H and Christodoulides D N 2011 Phys. Rev. Lett. 106 213901
[37] Kulishov M, Laniel J M, Belanger N, Azana J and Plant D V 2005 Opt. Express 13 3068
[38] Sipe J E, Poladian L and de Sterke C M 1994 J. Opt. Soc. Am. B 11 1307
[39] Poladian L 1993 Phys. Rev. E 48 4758
[40] Ahmed Z 2001 Phys. Rev. A 64 042716
[41] Cannata F, Dedonder J P and Ventura A 2007 Ann. Phys., NY 322 397
[42] Griffiths D J and and Steinke C A 2001 Am. J. Phys. 69 137
[43] Abramowitz M and Stegun I A 1970 Handbook of Mathematical Functions (New York: Dover) p 374
[44] Amos D E 1986 ACM Trans. Math. Softw. 12 265