The influence of noise on two- and three-frequency quasi-periodicity in a simple model system

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Abstract

We discuss the effect of noise on a system with a quasi-periodicity of different dimensions. As the basic model of our research we use the simplest three-dimensional map with two-frequency and three-frequency quasi-periodicity. Modification of the dynamical regimes at the influence of noise is considered with the help of Lyapunov chart method. The transformation of Lyapunov exponents plots characteristic for the quasi-periodic Hopf bifurcation of 3-torus birth at the presence of noise is examined.

1 Introduction

The effect of noise on various dynamical regimes is an important fundamental problem, since in real physical systems the presence of noise is unavoidable [1, 2, 3]. For systems with the possibility of chaos, noise destroys fine details of fractal structure in the phase space and parameter space, sometimes significantly modifying the observed picture. One of the interesting and less studied
questions is the case of systems with quasi-periodic dynamics. At discussion of this problem an important aspect is the possibility of quasi-periodicity with a different number of incommensurable frequencies, when in the phase space invariant tori of different dimensions are observed. Recently, new results have been obtained for autonomous systems with quasi-periodicity, including problems of quasi-periodic bifurcations [4, 5, 6, 7, 8, 9, 10]. In this report, we will consider the effect of noise on systems with two- and three-frequency quasi-periodicity and accompanying dynamical regimes.

2 Influence of additive noise on two-frequency quasi-periodicity. Circle map

For comparison with further results we give briefly the simplest illustrations of the noise influence on a system with a two-frequency quasi-periodicity. As it is known, the base model in this case is the circle map [1, 2]:

\[ x_{n+1} = x_n + r - \frac{K}{2\pi} \sin 2\pi x_n \pmod{1}. \]  

(1)

Here \( x \) is the dynamical variable (phase of oscillations), \( K \) and \( r \) are the amplitude and frequency parameters. The map (1) describes such phenomena as synchronization, quasi-periodic dynamics and its destruction at transition to chaos. In Fig.1a we can see numerically calculated Lyapunov chart in the \((r,K)\) parameter plane for the map (1) on which the regions of periodic regimes P, quasi-periodic two-frequency regimes \( T_2 \) and chaos C are indicated in different colors. (For the periodic regime Lyapunov exponent is negative, for the quasi-periodic is zero and for chaotic is positive one.) We can see the classical picture of Arnold’s tongues immersed in the region of quasi-periodicity. Above the critical line \( K = 1 \) tongues overlapping occurs in the system, and chaotic behavior becomes possible [1, 2].

Let us now add a random influence to the circle map [11]:

\[ x_{n+1} = x_n + r - \frac{K}{2\pi} \sin 2\pi x_n + \gamma \xi_n \pmod{1}. \]  

(2)

Here \( \gamma \) is the noise intensity, and \( \xi_n \) is a random sequence of values with zero mean \( \langle \xi_n \rangle = 0 \) and constant standard deviation \( \sigma = \sqrt{\langle \xi_n^2 \rangle} \). For numerical calculations we use computer generated quantities \( \xi_n \) uniformly distributed over the interval \([-0.5; 0.5]\). Note that if the noise amplitude is small and we
Figure 1: Lyapunov charts of the map (2), a) $\gamma = 0$, b) $\gamma = 0.01$, c) $\gamma = 0.05$, d) $\gamma = 0.08$.

examine dynamics of the model on large time scales, then the concrete form of the probability distribution for $\xi_n$ apparently will be not essential.

In Fig.1b,c,d Lyapunov charts corresponding to noise of different intensities are displayed. In the presence of noise the periodic or quasi-periodic dynamics is not realized in the exact sense, but the structure of the typical regions on Lyapunov charts remains visible. Therefore, we can talk about a "noisy periodic regime" when the Lyapunov exponent is negative; about the "noisy quasi-periodic regime" when it is close to zero; or about the "noisy chaotic regime" if Lyapunov exponent is positive. Lyapunov charts allow us to distinguish these regimes visually. The next conclusions follow from analysis of Lyapunov charts (Fig.1b,c,d):
At low noise intensities the structure of Arnold’s tongues is qualitatively preserved, and at large intensities it is destroyed. In the region of small values of parameter $K$ “relatively quasi-periodic regimes” dominate.

In the vicinity of the critical line there is a band of “relatively periodic regimes” expanding with increasing of noise intensity.

Above the critical line periodic regimes are destroyed and replaced by “relatively noisy chaotic regimes”.

Note that in the region of chaos fine details disappear, but in general this area does not enlarge with increasing of noise amplitude, but rather reduces in size.

3 Influence of additive noise on three-frequency quasi-periodicity. Torus map

Now let us consider the case of three-frequency quasi-periodicity. For its analysis, we use the discrete model presented recently in [12]. It is generated by finite difference discretization of flow equations describing one of the simplest generators of quasi-periodic oscillations [8, 9] and this is the simplest model with the required properties. This discrete model - the torus map - has the next form [12]:

\[
\begin{align*}
x_{n+1} &= x_n + h \cdot y_{n+1}, \\
y_{n+1} &= y_n + h \cdot ((\lambda + z_n + x_n^2 - \beta x_n^4)y_n - \omega_0^2 x_n), \\
z_{n+1} &= z_n + h \cdot (b(\varepsilon - z_n) - ky_n^2),
\end{align*}
\]

where $x, y, z$ are dynamical variables, $\lambda, \beta, \omega_0, b, \varepsilon, k$ are the set of control parameters (for details, see [8, 9, 12]), $h$ is the step of discretization.

In the center of Fig.2 Lyapunov chart in the $(\beta, \lambda)$ parameter plane for the map (3) is shown. Periodic regimes P are marked by red, two-frequency quasi-periodic regimes $T_2$ - by yellow, three-frequency quasi-periodicity $T_3$ - by blue, chaos $C$ - by black and hyper-chaos $HC$ - by lilac color. All listed regimes were determined by the values of Lyapunov exponents $\Lambda_i$ in accordance with their signature:

1) $P$: $\Lambda_1 < 0$, $\Lambda_2 < 0$, $\Lambda_3 < 0$,
2) $T_2$: $\Lambda_1 = 0$, $\Lambda_2 < 0$, $\Lambda_3 < 0$,
3) $T_3$: $\Lambda_1 = 0$, $\Lambda_2 = 0$, $\Lambda_3 < 0$, 
4) $C$: $\Lambda_1 > 0$, $\Lambda_2 < 0$, $\Lambda_3 < 0$, 
5) $HC$: $\Lambda_1 > \Lambda_2 > 0$, $\Lambda_3 < 0$.

Lyapunov chart in Fig. 2 has the following features. The line of quasi-periodic Hopf bifurcation $QH$ separates the regions of two- and three-frequency quasi-periodicity. From this line the bands of two-frequency regimes originate into the domain of three-dimensional tori, these bands are limited by lines of saddle-node bifurcations of two-frequency tori. Inside these bands the transverse streaks of periodic regimes - exact resonances - are built in. In the main region of two-frequency regimes periodic resonances are possible, the widest one corresponds to 10-period cycle. Also in Fig. 2 at certain points of parameter plane we give examples of phase portraits and Fourier spectra illustrating the characteristic types of regimes.

To account for noise in torus map (3) let us introduce random sequence $\xi_n$. We assume that $\xi_n$ represent discrete-time white noise, i.e., elements of sequence at different steps of time are independent:

$$
\begin{align*}
x_{n+1} &= x_n + h \cdot y_{n+1}, \\
y_{n+1} &= y_n + h \cdot ((\lambda + z_n + x_n^2 - \beta x_n^4)y_n - \omega_0^2 x_n), \\
z_{n+1} &= z_n + h \cdot (b(\varepsilon - z_n) - k y_n^2) + \gamma \xi_n,
\end{align*}
$$

(4)

here $\gamma$ is the noise intensity.

Figure 3 demonstrates Lyapunov charts in the $(\beta, \lambda)$ parameter plane for the torus map at several increasing values of noise intensity. As the amplitude of the noise increases, the following characteristics are observed:

- Small areas of periodic regimes inside the main two-frequency resonance band are destroyed, but in their place relatively large island of “noisy periodic regime” appears.

- The region of “noisy periodic regime” based on 10-period cycle is preserved even in case of large noise values.

- Three-frequency regimes are preserved at small noise, but at a sufficiently large value of noise they become two-frequency ones. Below we will illustrate this transition by means of Fourier spectra.

- Chaotic and hyper-chaotic regimes survives.

In Fig.4 we show the hierarchy of the Fourier spectra for noisy torus map (4) with increasing noise intensity at a parameter point corresponding to
Figure 2: Lyapunov chart in the $(\beta, \lambda)$ parameter plane for the torus map (central diagram). Fourier spectra and phase portraits are presented at several representative points. Discretization parameter $h = 0.1$, others parameters $b = 1$, $\varepsilon = 4$, $k = 0.02$, $\omega_0 = 2\pi$.

The three-frequency torus in the autonomous case. The Fourier spectrum in the absence of noise is a discrete set of components corresponding to incommensurable frequencies whose amplitude decreases on both sides of the basic frequency (Fig. 4a). The peak of the maximum height is at a frequency of 0.1, which is due to the value of the discretization parameter. With the growth of noise, numerous satellites at the combinational frequencies disappear (Fig. 4b-d). At noise intensity $\approx 10^{-1}$ Fourier spectrum becomes similar to the spectrum of a two-frequency quasi-periodic oscillations (Fig. 4e). These numerical illustrations are useful because they can be directly compared with spectra obtained as a result of physical experiment.

Now we discuss the influence of noise on the quasi-periodic Hopf bifurcation $QH$ transforming 2-torus to 3-torus. For this purpose in Fig. 5 we demon-
Figure 3: Lyapunov charts of model (4). Noise intensity: a) $\gamma = 10^{-2}$, b) $\gamma = 3 \cdot 10^{-2}$, c) $\gamma = 5 \cdot 10^{-2}$, d) $\gamma = 10^{-1}$.

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Figure 4: Fourier spectra of model (4) at point with coordinates $\beta = 0.081$, $\lambda = -3.08$, $b = 1$, $\varepsilon = 4$, $k = 0.02$, $\omega_0 = 2\pi$. Noise intensity $\gamma$ is: a) 0, b) $10^{-5}$, c) $10^{-4}$, d) $10^{-3}$, e) $10^{-2}$, f) $10^{-1}$.
	hen with the growth of noise they become unequal to each other. Thus the exponent $\Lambda_2$ grows, approaching zero, while the exponent $\Lambda_3$ reaches a maximum, which, however, is no longer equal to zero. Then exponent $\Lambda_3$ begins to decrease.

Fig. 5c illustrates also the behavior of a system with a saddle-node bifurcation of invariant tori under the influence of noise. Now Lyapunov exponent $\Lambda_2$ exhibits a “dip” into the negative values domain. But $\Lambda_2$ and $\Lambda_3$ are substantially not equal to each other, which is an attribute of saddle-node bifurcation of tori in the absence of noise [6]. Note that as the noise intensity increases, small features and the irregularity of the plots disappear (Fig. 5b, c), that corresponds to the destruction of small resonances.

4 Conclusion

Thus, we consider the effect of noise on the simplest system with two- and three-frequency quasi-periodicity. The three-frequency quasi-periodicity is
Figure 5: Lyapunov exponents of noisy torus map (4) versus parameter $\lambda$. Noise intensity: a) $\gamma = 0$, b) $\gamma = 10^{-2}$, c) $\gamma = 5 \cdot 10^{-2}$. Parameters $b = 1$, $\varepsilon = 4$, $k = 0.02$, $\omega_0 = 2\pi$, discretization step $h = 0.1$. Parameter $\beta$ is fixed, $\beta = 0.081$.

preserved for some noise amplitudes, but then turns into a two-frequency one. For Fourier spectra this process evolves according to the scenario of "smearing" of corresponding spectral components by noise components. Quasi-periodic bifurcations under influence of noise occupy certain intervals in the
parameter, but their main classification characteristics (equality or not) of corresponding Lyapunov exponents at a qualitative level are preserved.

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