LINEAR REPRESENTATIONS AND FROBENIUS MORPHISMS OF GROUPOIDS

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ABSTRACT. Given a morphism of (small) groupoids, we provide sufficient and necessary conditions under which the induction and co-induction functors between the categories of linear representations are naturally isomorphic. A morphism with this property is termed a Frobenius morphism of groupoids. As a consequence, an extension by a subgroupoid is Frobenius if and only if each fibre of the (left or right) pull-back biset has finitely many orbits. Our results extend and clarify the classical Frobenius reciprocity formulae in the theory of finite groups, and characterize Frobenius extension of algebras with enough orthogonal idempotents.

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1. Introduction

In this section we first explain the motivations behind this paper and give some general overviews on the theory hereby developed, thereafter we describe with some details the main results obtained here.

1.1. Motivation and overview. Either as abstract objects and/or geometrical ones, groupoids appear in different branches of mathematics and mathematical physics, see for instance the brief surveys [1, 25, 12]. The most common motivation in studying groupoids seems to have its roots in the concept of symmetry inquiring into the knowledge of its formalism. Apparently, groupoids do not only allow to consider symmetries of transformations of the object (i.e., automorphisms), but also the symmetry among the parts of objects. As was claimed in [8] not all the class of groupoids ought to be considered a proper generalization of the formal definition of symmetry, for which the equivalence relation and action groupoids serve as a middle ground to be reframed. From the abstract point of view, equivalence relation groupoids are too restrictive as they do not enjoy any non trivial isotropy group. In other words, there is no internal symmetry to be considered when these groupoids are employed. Concerning action groupoids and their linear representations, where the internal symmetry makes its presence, it is highlighted that they have been implicitly

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manifested in several physical situations for some time. The study of molecular vibration in terms of homogeneous vector bundles, is for instance a situation where linear representations of action groupoids are exemplified, see [24 §3.2, page 97] for more details in the specific case of the space of motions of carbon tetrachloride (see also [23] for others examples). In fact, any abstract homogeneous vector bundle \((E, \pi)\) over a given \(G\)-set \(M\) leads to a linear representation on the action groupoid \((G \times M, M)\) given by the \(\pi\)-equivariant \((\pi \times \pi)\)-vector bundle \((G \times E, E)\) \(\rightarrow (G \times M, M)\) of action groupoids. There is more to be said, namely, there is an equivalence of symmetric monoidal categories between that of homogeneous vector bundles over the \(G\)-set \(M \) and that of linear representations of the groupoid \((G \times M, M)\). Furthermore, the global sections functor can be identified with the induction functor attached to the canonical morphism of groupoids \((G \times M, M) \rightarrow (G, \{\ast\})\) (here the group \(G\) is considered as a groupoid with only one object \(\{\ast\}\)). As we will see here, in the context of groupoids the induction functor is related, via the (right) Frobenius reciprocity formula, with the restriction functor.

Frobenius reciprocity formula appears in the framework of groups under different forms, see for instance [24 Eq (3.4), p.109] or [24 Eq (3.7), p.111] and e.g., [15] Proposition 2.3.9[3] and was extended from finite groups to other classes of groups like locally compact groups [17, 21] or certain algebraic groups e.g., [10]. In the finiteness circumstances, this formula compare the dimensions of the vector spaces of homomorphisms spaces of linear representations over two different groups connected by a morphism of groups. In more conceptual terms, this remounts to say that for a given morphism of groups (not necessarily finite), the restriction functor has the induction functor as right adjoint while the co-induction functor as a left adjoint. From categorical point of view, these are well known constructions due to Kan, and termed, respectively, right and left Kan extensions [15]. In the same direction, if both groups are finite and the connecting morphism is injective, then the induction and co-induction functors are naturally isomorphic and the resulting morphism between the group algebras produces a Frobenius extension of unitary algebras [13] (this result becomes in fact a direct consequence of our main theorem, see the forthcoming subsection).

Apart from the interest they have been generating in algebra, geometry and topology, Frobenius unitary algebras are worthy objects to be studied by their own rights. For instance, commutative Frobenius algebras over fields, like group algebras of finite abelian groups, play a prominent role in 2-dimensional topological quantum field theory, as was corroborated in [14].

So far we have been dealing with situations where only finitely many objects are allowable. In other words, Frobenius unitary algebras and (or finite bundle of) groups are objects performed from categories with finitely many objects. Up to our knowledge, the general case of infinite objects still unexplored in the literature. As an illustration, the Frobenius formulæ for locally compact topological groupoids are far from being understood, since these formulæ are not even explicitly computed for the case of abstract groupoids.

Our motivation is to introduce the main ideas that underpin groupoids linear representation theory techniques in relation with their non-unitary algebras, which instead they posses enough orthogonal idempotents. Thus this paper pretends to set up in a very elementary way the basis tools to establishes Frobenius formulæ in the context of abstract groupoids and employ these formulæ in characterizing Frobenius extension of groupoids parallel to Frobenius extension of their paths algebras; hoping by this to fulfill the lack that is presented in the literature about this subject.

**General notations:** If \(C\) is a small category (that is the class of object is actually a set), and \(D\) any other category, then the symbol \([C, D]\) stands for the category whose objects are functors and morphisms are natural transformations between these ones. Since \(C\) is a small category, this resulting category is a Hom-set category, that is, the class of morphisms between any pair of objects form a set, and this set is denoted by \(\text{Nat}(F, G)\), for any pair of functors \(F, G\).

1.2. **Description of the main results.** Let \(k\) be a ground base field, the symbol \(\otimes\) denotes the tensor product between \(k\)-vector spaces and their \(k\)-linear maps. This category is denoted by \(\text{Vect}_k\), and the full
subcategory of finite dimensional vector space by vect, For any set S, we denote by kS the k-vector space whose basis is S, by convention this is a zero vector space whenever S is an empty set.

Given a (small abstract) groupoid G, we denote by Rep(G) its category of k-linear representations, that is, Rep(G) = [G, Vect] the category of functors from G to Vect. The path algebra, or Gabriel’s ring of G, is the k-ring with enough orthogonal idempotents B = ⊔_{x,x′ ∈ G} kG(x, x′). The multiplication of this ring is given by that of G and its set of orthogonal idempotents is provided by the images of the identities arrows of G, that is, by the set {1, ..., n}.

Let φ : H → G be a morphism of groupoids and denote by φi : Rep(G) → Rep(H) the associated restriction functor. We say that φ is a Frobenius morphism (see Definition 5.1 below) provided that the induction and the co-induction functors φ* and φ! are naturally isomorphic; these are right and left adjoint functor of φi, respectively (see Lemmas 3.7 and 3.11 for the precise definitions of these functors). In this situation we denote by A and B, the Gabriel rings of H and G, respectively, and consider φ : A → B, the associated canonical morphism of ring with enough orthogonal idempotents.

Our main result is the following theorem stated below as Theorem 5.2.

**Theorem A.** Given a morphism of groupoids φ : H → G, then the following are equivalent.

(i) φ is a Frobenius morphism;

(ii) There exists a natural transformation E = : G(φ(a), φ(v)) → kH(a, v) in Hop × H, and for every x ∈ G, there exists a finite set |{(u, b, c)}i=1,...,N ∈ kG(x, φ(u)) such that, for every pair of elements (b, b′) ∈ G(x, φ(u)) × G(φ(u), x), we have

\[ \sum_{i} E(b_i) c_i = b ∈ kG(x, φ(u)) \quad \text{and} \quad b’ = \sum_{i} b_i E(c_i b’ i) ∈ kG(φ(u), x). \]

(iii) For every x ∈ G, the left unital A-module AB occurs finitely generated and projective and there is a natural isomorphism B1n ≃ BHomn(A, AB, A1n), of left unital B-modules, for every u ∈ H.

Associated to the morphism φ : H → G, we have (see Example 2.10 below) the right pull-back (G, H)-biset \( \mathcal{U}^*(G) \) with structure maps \( \varsigma : \mathcal{U}^*(G) → G_n(a, a) \) and \( pr_r : \mathcal{U}^*(G) → H_n(a, a) \). Similarly, we have \( \mathcal{U}^*(G) \) with structure maps \( \varsigma : \mathcal{U}^*(G) → G_n(a, a) \) and \( pr_r : \mathcal{U}^*(G) → H_n(a, a) \).

The following result, which characterises the case of an extension by subgroupoids, is a corollary of Theorem 5.1 and stated below as Corollary 5.3. A groupoid therein is said to be finite if its has finitely many connected components and each of its isotropy group is finite.

**Corollary B.** Let φ : H → G be a morphism of groupoids with a faithful underlying functor (i.e., φi is an injective map). Then the following are equivalent:

(a) φ is a Frobenius extension;

(b) For any x ∈ G, the left H-set \( \varsigma^{-1}(\{x\}) \) has finitely many orbits;

(c) For any x ∈ G, the right H-set \( \varsigma^{-1}(\{x\}) \) has finitely many orbits.

In particular, any inclusion of finite groupoids is a Frobenius extension.

2. Abstract groupoids: General definition, basic properties and examples.

This section contains all the material: definitions, properties and examples of abstract groupoids that will be used in the course of the following sections. This material was recollected form [4][6][7] and from the references quoted therein. All groupoids discussed below are abstract and small ones, in the sense that the class of arrows is actually a set and do not enjoys any topological nor combinatorial structures.

2.1. Notations, basic notions and examples. A groupoid is a small category where each morphism is an isomorphism. That is, a pair of two sets \( G := (G_0, G_1) \) with diagram of sets,

\[ \xymatrix{ G_1 \ar@{.>}[rr] & & G_0 }, \]

where \( s \) and \( t \) are resp. the source and the target of a given arrow, and \( t \) assigns to each object its identity arrow; together with an associative and unital multiplication \( G_2 := G_1 × G_1 \to G_1 \), as well as a map \( G_1 \to G_0 \), which associated to each arrow its inverse. Notice, that \( t \) is an injective map, and so \( G_0 \) is identified with a subset of \( G_1 \). A groupoid is then a category with more structure, namely, the map which send any arrow to its inverse. We implicitly identify a groupoid with its underlying category.
Given a groupoid $\mathcal{G}$, consider two objects $x, y \in G_0$. We denote by $\mathcal{G}(x, y)$ the set of all arrows with source $x$ and target $y$. The isotropy group of $\mathcal{G}$ at $x$ is then the group:

$$\mathcal{G}^x := \mathcal{G}(x, x) = \{ g \in G_1 | s(g) = t(g) = x \}. \quad (1)$$

Clearly each of the sets $\mathcal{G}(x, y)$ is, by the groupoid multiplication, a left $\mathcal{G}^y$-set and right $\mathcal{G}^x$-set. In fact, each of the $\mathcal{G}(x, y)$'s is a $(\mathcal{G}^y, \mathcal{G}^x)$-biset, see [2] for pertinent definitions.

The (left) star of an object $x \in G_0$ is defined by $\text{Star}^l(x) := \{ y \in G_0 | s(g) = x \}$. The right star is defined using the source map, and the both left and right stars are in bijection. Now, given an arrow $g \in G$, we define the conjugation operation (or the adjoint operator) as the morphism of groups:

$$ad_g : \mathcal{G}^y \to \mathcal{G}^y, \quad (f \mapsto gf^{-1}). \quad (2)$$

A morphism of groupoids $\phi : \mathcal{H} \to \mathcal{G}$ is a functor between the underlying categories. Obviously any such a morphism induces homomorphisms of groups between the isotropy groups: $\phi_y : H_y \to G_{\phi y}$, for every $y \in H_0$. The family of homomorphisms $\{\phi_y\}_{y \in \mathcal{H}_0}$ is referred to as the isotropy maps of $\phi$. For a fixed object $x \in \mathcal{G}_0$, its fibre $\phi^{-1}_x(\{x\})$, if not empty, leads to the following "star" of homomorphisms of groups:

![Diagram](image)

where $y$ runs in the fibre $\phi^{-1}_x(\{x\})$.

**Example 2.1 (Trivial groupoid).** Let $X$ be a set. Then the pair $(X, X)$ is obviously a groupoid (in fact a small discrete category, i.e., with only identities arrows) with trivial structure. This know as the trivial groupoid.

**Example 2.2 (Action groupoid).** Any set $X$ can be seen as a set of objects of a groupoid where the only arrows are the identities ones, that is, a discrete category over $X$. This groupoid is called the trivial groupoid of $X$. On the other hand, any group $G$ can be considered as a groupoid by taking $G_1 = G$ and $G_0 = \{ * \}$ (a set with one element). Now if $X$ is a right $G$-set with action $\rho : X \times G \to X$, then one can define the so called the action groupoid: $G_x = X \times G$ and $G_0 = X$, the source and the target are $s = \rho$ and $t = pr_1$, the identity map sends $x \mapsto (e, x) = e$, where $e$ is the identity element of $G$. The multiplication is given by $(x, g)(x', g') = (x, gg')$, whenever $xg = x'$, and the inverse is defined by $(x, g)^{-1} = (xg, g^{-1})$. Clearly the pair of maps $(pr_1, +) : \mathcal{G} := (G_x, G_0) \to (G, +)$ defines a morphism of groupoids. For a given $x \in X$, the isotropy group $\mathcal{G}^x$ is clearly identified with the stabilizer $\text{Stab}_x := \{ g \in G | gx = x \}$ subgroup of $G$.

**Example 2.3 (Equivalence relation groupoid).** We expound here several examples which in fact belong to the same class, that of relation equivalence groupoids.

1. One can associated to a given set $X$ the so called the groupoid of pairs (called fine groupoid in [1] and simplicial groupoid in [2]), its set of arrows is defined by $G_1 = X \times X$ and the set of objects by $G_0 = X$; the source and the target are $s = pr_2$, and $t = pr_1$, the second and the first projections, and the map of identity arrows is $\iota$ the diagonal map. The multiplication and the inverse maps are given by

   $$(x, x')(x', x'') = (x, x''), \quad \text{and} \quad (x, x')^{-1} = (x', x).$$

2. Let $\nu : X \to Y$ be a map. Consider the fibre product $X \times_{\nu} X$, as a set of arrows of the groupoid $X \times X$, where as before $s = pr_2$, and $t = pr_1$, and the map of identity arrows is $\iota$ the diagonal map. The multiplication and the inverse maps are clear.

3. Assume that $\mathcal{R} \subseteq X \times X$ is an equivalence relation on the set $X$. One can construct a groupoid $\mathcal{R} \subseteq X \times X$, with structure maps as before. This is an important class of groupoids known as the groupoid of equivalence relation (or relation equivalence groupoid). Obviously $(\mathcal{R}, X) \to (X \times X, X)$ is a morphism of groupoid, see for instance [3] Exemple 1.4, page 301.

Notice, that in all these examples each of the isotropy groups is the trivial group.
Example 2.4 (Induced groupoid). Let \( \mathcal{G} = (G_i, G_0) \) be a groupoid and \( \varsigma : X \to G_0 \) a map. Consider the following pair of sets:
\[
G^* := X \times X, \quad G_i \times X = \{ (x, g, x') \in X \times G_i \times X \mid \varsigma(x) = t(g), \varsigma(x') = s(g) \}, \quad G^*_0 := X.
\]
Then \( \mathcal{G}^* = (G^*, G^*_0) \) is a groupoid, with structure maps: \( s = pr_1, t = pr_2, \iota = (\varsigma(x), t(x), s(x)) \), \( x \in X \).
The multiplication is defined by \( (x, g, y) (x', g', y') = (x, gg', y') \), whenever \( y = x' \), and the inverse is given by \( (x, g, y)^{-1} = (y, g^{-1}, x) \). The groupoid \( \mathcal{G}^* \) is known as the induced groupoid of \( \mathcal{G} \) by the map \( \varsigma \), or the pull-back groupoid of \( \mathcal{G} \) along \( \varsigma \), see [9] for dual notion). Clearly, there is a canonical morphism \( \phi^* := (pr_r, \varsigma) : \mathcal{G}^* \to \mathcal{G} \) of groupoids. A particular instance of an induced groupoid, is the one when \( \mathcal{G} = G \) is a groupoid with one object. Thus for any group \( G \) one can consider the Cartesian product \( X \times G \times X \) as a groupoid with set of objects \( X \).

Example 2.5 (Frame groupoid). Let \( \pi : \mathcal{Y} \to X \) be a surjective map, and write \( \mathcal{Y} = \bigcup_{i \in X} \mathcal{Y}_i \), where \( \mathcal{Y}_i := \pi^{-1}(i(x)) \). For any pair of elements \( x, u \in X \), we set
\[
\mathcal{G}(x, u) := \left\{ f : \mathcal{Y}_i \to \mathcal{Y}_j \mid f \text{ is bijective} \right\},
\]
then the pair \( (\mathcal{G}, G_0) := \left( \bigcup_{i \in X} \mathcal{G}(x, u), X \right) \) admits a structure of groupoid (possibly a trivial one), referred to as the frame groupoid of \( \pi \), see also [23]. In a more general setting, one can similarly define the frame groupoid of a given family \( \{ \mathcal{Y}_i \}_{i \in X} \) of objects in a certain category, indexed by the set \( X \).

Notice that, for a groupoid \( \mathcal{G} \), the disjoint union \( \bigcup_{i \in X} \mathcal{G}(s) \) of all isotropy groups form the set of arrows of a subgroupoid of \( \mathcal{G} \) whose source equal to its target, namely, the projection \( \varsigma : \bigcup_{i \in X} \mathcal{G}(s) \to G_0 \). We denote this groupoid by \( \mathcal{G}^0 \) and refer to as the isotropy groupoid of \( \mathcal{G} \). For instance, the isotropy groupoid of any equivalence relation groupoid is a trivial groupoid as in Example 2.4, see also Remark 2.5 below.

2.2. Groupoids actions, equivariant maps and the orbits sets. If we think of group as a groupoid with a single object, then an action of a group on a set \( \{ 1 \} \times X \), is nothing but a functor from the underlying category of such a groupoid to the core category of sets. Writing down this formulation for groupoids with several objects, will lead to the forthcoming definition, see the discussion in [23] Remark 2.6). This definition is in fact an abstract formulation of that given in [13] Definition 1.6.1 for Lie groupoids, and essentially the same definition based on the Sets-bundles notion given in [23] Definition 1.11].

Definition 2.6. Given a groupoid \( \mathcal{G} \) and a map \( \varsigma : X \to G_0 \). We say that \((X, \varsigma)\) is a right \( \mathcal{G} \)-set (with a structure map \( \varsigma \)), if there is a map (the action) \( \rho : X \times, \mathcal{G} \to X \) sending \((x, g)\) to \( xg \), satisfying the following conditions
\begin{enumerate}
\item \( s(g) = \varsigma(xg) \), for any \( x \in X \) and \( g \in \mathcal{G} \), with \( \varsigma(x) = t(g) \).
\item \( x \varsigma(x) = x \), for every \( x \in X \).
\item \( (xg)h = s(gh) \), for every \( x \in X, g, h \in \mathcal{G} \), with \( \varsigma(x) = t(g) \) and \( t(h) = s(g) \).
\end{enumerate}

A left action is analogously defined by interchanging the source with the target. In general a set with a (right or left) groupoid action is called a groupoid-set.

Obviously, any groupoid \( \mathcal{G} \) acts on itself over both sides by using the regular action, i.e. the multiplication \( \mathcal{G} \times \mathcal{G} \to \mathcal{G} \). That is, \((\mathcal{G}, s)\) is a right \( \mathcal{G} \)-set and \((\mathcal{G}, t)\) is a left \( \mathcal{G} \)-set with this action.

Let \((X, \varsigma)\) be a right \( \mathcal{G} \)-set, and consider the pair of sets \( X \bowtie \mathcal{G} := \{ X, \times, \mathcal{G}, X \} \) as a groupoid with structure maps \( s = \rho, t = pr_1, \iota = (x, t(x), s(x)) \). The multiplication and the inverse maps are defined by \( (x, g)(x', g') = (x, gg') \) and \( (x, g)^{-1} = (x, g^{-1}) \). The groupoid \( X \bowtie \mathcal{G} \) is known as the right translation groupoid of \( X \) by \( \mathcal{G} \). Furthermore, there is a canonical morphism of groupoids \( \alpha : X \bowtie \mathcal{G} \to \mathcal{G} \), given by the pair of maps \( \alpha = (\varsigma, pr_2) \).

A morphism of right \( \mathcal{G} \)-sets (or \( \mathcal{G} \)-equivariant map) \( F : (X, \varsigma) \to (X', \varsigma') \) is a map \( F : X \to X' \) such that the diagrams
\[
\begin{array}{c}
\begin{array}{c}
X \downarrow \quad \mathcal{G}_0 \\
\quad \mathcal{G}_1 \quad \downarrow \quad X
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
X \times, \mathcal{G}_1 \quad \downarrow \quad \mathcal{G}_1 \quad \downarrow \quad X
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
X' \downarrow \quad \mathcal{G}_0' \\
\quad \mathcal{G}_1' \quad \downarrow \quad X'
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
X' \times, \mathcal{G}_1' \quad \downarrow \quad \mathcal{G}_1' \quad \downarrow \quad X'
\end{array}
\end{array}
\]
commute. We denote by Hom\textsubscript{\mathcal{G}}(X, X') the set of all \mathcal{G}-equivariant maps from (X, \varsigma) to (X', \varsigma'). Clearly any such a \mathcal{G}-equivariant map induces a morphism of groupoids \text{\textit{F}} : X \times \mathcal{G} \rightarrow X' \times \mathcal{G}. A subset \mathcal{Y} \subseteq X of a right \mathcal{G}-set (X, \varsigma), is said to be \mathcal{G}\text{-invariant} whenever the inclusion \mathcal{Y} \hookrightarrow X is a \mathcal{G}\text{-equivariant map. For instance any left star \text{\textit{Star}}(x) of any object \(x \in \mathcal{G}, is a \mathcal{G}\text{-invariant subset of the right \mathcal{G}-set (G, s).}

**Example 2.7.** Let \(\phi : \mathcal{H} \rightarrow \mathcal{G}\) be a morphism of groupoids. Consider the triple (\mathcal{H}, \times, \mathcal{G}, 1, \theta), where \(\theta : \mathcal{H}_0 \times \mathcal{G} \rightarrow \mathcal{G}\) sends \((a, a) \rightarrow s(a), and pr_1 is the first projection. Then the following maps 

\[
\begin{array}{c}
\mathcal{H}_0 \times \mathcal{H}_0 \times \mathcal{G}, \mathcal{G} \\
\mathcal{H}_1 \times \mathcal{H}_0 \times \mathcal{G}, \mathcal{G} \\
\mathcal{H}_1 \times \mathcal{H}_1 \times \mathcal{G}, \mathcal{G}
\end{array}
\]

define, respectively, a structure of right \mathcal{G}\text{-sets and that of left \mathcal{H}\text{-set. Analogously, the maps}

\[
\begin{array}{c}
\mathcal{G}_1 \times \mathcal{H}_0 \times \mathcal{H}_0 \times \mathcal{G}, \mathcal{G} \\
\mathcal{G}_1 \times \mathcal{G}_0 \times \mathcal{H}_0 \times \mathcal{G} \\
\mathcal{G}_1 \times \mathcal{G}_1 \times \mathcal{G}_0 \times \mathcal{G}
\end{array}
\]

where \(\varsigma : \mathcal{G}_1 \times \mathcal{H}_0 \mathcal{G}_0 \rightarrow \mathcal{G}_0\) sends \((a, a) \rightarrow t(a), define, respectively, a left \mathcal{H}\text{-set and right \mathcal{G}\text{-set structures on \mathcal{G}_1, \mathcal{H}_0, and \mathcal{G}_0.} This in particular can be applied to any morphism of groupoids of the form \((X, X) \rightarrow (Y \times Y), (x, x') \rightarrow ((f(x), f(x)), f(x'))\) where \(f : X \rightarrow Y\) is any map. On the other hand, if \(f\) is a \mathcal{G}\text{-equivariant map, for some a group \mathcal{G} acting on both \(X\ and \(Y\), then the above construction applies to the morphism \((G \times X, X) \rightarrow (G \times Y, Y)\) sending \(((g, x), x') \rightarrow ((g, f(x)), f(x'))\) of action groupoids, as well.

Next we recall the notion of the orbit set attached to a right groupoid-set. This notion is a generalization of the orbit set in the context of group-sets. Here we use the (right) translation groupoid to introduce this set. First we recall the notion of the orbit set of a given groupoid. The orbit set of a groupoid \(\mathcal{G}\) is the quotient set of \(\mathcal{G}\) by the following equivalence relation: take an object \(x \in \mathcal{G}, define

\[
\mathcal{G}_\sim := \{ x \in \mathcal{G} \ | \ \exists y \in \mathcal{G} \ such \ that \ s(g) = x, t(g) = y \}
\]

which is equal to the set \(s^{-1}(\{x\})\). This is a non empty set, since \(x \in \mathcal{G}_\sim\). Two objects \(x, x' \in \mathcal{G}\) are said to be equivalent if and only if \(\mathcal{G}_\sim = \mathcal{G}_\sim\), or equivalently, two objects are equivalent if and only if there is an arrow connecting them. This in fact defines an equivalence relation whose quotient set is denoted by \(\mathcal{G}/\mathcal{G}\). In others words, this is the set of all connected components of \(\mathcal{G}\), which we denote by \(\pi_0(\mathcal{G}) := \mathcal{G}/\mathcal{G}\).

Given a right \mathcal{G}\text{-set (X, \varsigma), the orbit set X/\mathcal{G} of (X, \varsigma) is the orbit set of the (right) translation groupoid X \times \mathcal{G}. If \(\mathcal{G} = (X \times \mathcal{G}, X)\) is an action groupoid as in Example 2.2, then obviously the orbit set of this groupoid coincides with the classical set of objects \(X/G\). Of course, the orbit set of an equivalence relation groupoid \((\mathcal{R}, X)\), see Example 2.3 is precisely the quotient set \(X/\mathcal{R}\) by the equivalence relation \(\mathcal{R}\).

**Remark 2.8.** Let \(\mathcal{G}\) be a groupoid and denote by \(\mathcal{R}\) the equivalence relation defined by the action of \(\mathcal{G}\) on \(\mathcal{G}\), (thus the previous action defining the orbits set \(\pi_0(\mathcal{G})\)). Then we have a diagram of morphisms of groupoids:

\[
\begin{array}{c}
\mathcal{G}_0 \times \mathcal{G}_0 \\
\mathcal{G} \times \mathcal{G} \\
\mathcal{G}_0
\end{array}
\]

where the vertical morphism is the image of \((s, t)\). If the lower left hand map is an identity, i.e., if \(\mathcal{R} = \mathcal{G}_0 \times \mathcal{G}_0\), then \(\mathcal{G}\) posses only one connected component. Thus \(\pi_0(\mathcal{G})\) is a set with one element, and this happens if and only if \(\mathcal{G}\) is a transitive groupoid.

On the other hand, each isotropy group of a given groupoids as in Example 2.3 is a trivial group. That is, the isotropy groups of an equivalence relation \(\mathcal{R}\) are trivial, i.e., of the form \(\mathcal{R} = \{\cdot\}\), for any object \(x \in \mathcal{G}\). Conversely, any groupoid without parallel arrows is an equivalence relation groupoid. This is precisely the case when the vertical arrow in the above diagram is injective.
2.3. Bisets, two sided translation groupoid and the tensor product. Let $\mathcal{G}$ and $\mathcal{H}$ be two groupoids and $(X, \vartheta, \varsigma)$ a triple consisting of a set $X$ and two maps $\varsigma : X \to \mathcal{G}$, $\vartheta : X \to \mathcal{H}$. The following definitions are abstract formulations of those given in [11, 20] for topological and Lie groupoids, see also [3, 7].

**Definition 2.9.** The triple $(X, \vartheta, \varsigma)$ is said to be an $(\mathcal{H}, \mathcal{G})$-biset if there is a left $\mathcal{H}$-action $\alpha : \mathcal{H} \times \mathcal{G} \times X \to X$ and right $\mathcal{G}$-action $\rho : \mathcal{G} \times \mathcal{H} \times X \to X$ such that

1. For any $x \in X$, $h \in \mathcal{H}$, $g \in \mathcal{G}$, we have $\vartheta(xg) = \vartheta(x)g$ and $\varsigma(hx) = \varsigma(h)x$.

2. For any $x \in X$, $h \in \mathcal{H}$, and $g \in \mathcal{G}$, with $\varsigma(x) = t(g)$, $\vartheta(x) = s(h)$, we have $h(xg) = (hx)g$.

The two sided translation groupoid associated to a given $(\mathcal{H}, \mathcal{G})$-biset $(X, \varsigma, \vartheta)$ is defined to be the groupoid $\mathcal{H} \ltimes X \rtimes \mathcal{G}$ whose set of objects is $X$ and set of arrows is

$$\mathcal{H} \ltimes X \rtimes \mathcal{G} = \{ (h, x, g) \in \mathcal{H} \times X \times \mathcal{G} | s(h) = \vartheta(x), t(g) = \varsigma(x) \}.$$ 

The structure maps are:

$$s(h, x, g) = x, \quad t(h, x, g) = hxg^{-1} \quad \text{and} \quad \iota_x = (\iota_{\mathcal{H}}, x, \iota_{\mathcal{G}}).$$

The multiplication and the inverse are given by:

$$(h, x, g)(h', x', g') = (hh', x', gg'), \quad (h, x, g)^{-1} = (h^{-1}, h^{-1}xg^{-1}, g^{-1}).$$

The orbit space of the two translation groupoid is the quotient set $X/(\mathcal{H}, \mathcal{G})$ defined using the equivalence relation $x \sim x'$ if and only if there exist $h \in \mathcal{H}$ and $g \in \mathcal{G}$ with $s(h) = \vartheta(x)$ and $t(g) = \varsigma(x')$, such that $hx = x'g$.

**Example 2.10.** Let $\phi : \mathcal{H} \to \mathcal{G}$ be a morphism of groupoids. Consider, as in Example 2.7, the associated triples $(\mathcal{H}_{\text{iso}} \rtimes \mathcal{G}_{\text{iso}}$, $\vartheta_{\text{iso}}$, $\rho_{\text{iso}}$) and $(\mathcal{G}_{\text{iso}} \rtimes \mathcal{H}_{\text{iso}}$, $\rho_{\text{iso}}$, $\vartheta_{\text{iso}}$). Then the underlying sets are, respectively, an $(\mathcal{H}, \mathcal{G})$-biset and a $(\mathcal{G}, \mathcal{H})$-biset.

Next we recall the definition of the tensor product of two groupoid-bisets, see for instance [3, 7] or [5]. Fix three groupoids $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$. Given $(Y, \varsigma, \vartheta)$ and $(X, \vartheta', \varsigma')$, respectively, a $(\mathcal{G}, \mathcal{H})$-biset and a $(\mathcal{H}, \mathcal{K})$-biset. Consider the map $\omega : Y \times X \to \mathcal{H}$ sending $(y, x) \mapsto \vartheta(y) = \vartheta(x)$. Then the pair $(Y \times X, \omega)$ admits a structure of right $\mathcal{H}$-set with action

$$((y, x), h) \mapsto (yh, h^{-1}x).$$

Following the notation and the terminology of [3, Remark 2.12], we denote by $(Y \times X, \omega) / \mathcal{H} := Y \otimes_{\mathcal{H}} X$ the orbit set of the right $\mathcal{H}$-set $(Y \times X, \omega)$; we refer to $Y \otimes_{\mathcal{H}} X$ as the tensor product over $\mathcal{H}$ of $Y$ and $X$. It turns out that $Y \otimes_{\mathcal{H}} X$ admits a structure of $(\mathcal{G}, \mathcal{K})$-biset whose structure maps are given as follows. First, denote by $y \otimes_{\mathcal{H}} x$ the equivalence class of an element $(y, x) \in Y \times X$. That is, we have $yh \otimes_{\mathcal{H}} x = y \otimes_{\mathcal{H}} h^{-1}x$ for every $h \in \mathcal{H}$, with $\vartheta(y) = \vartheta(x)$. Second, one can easily check that, the maps

$$\overline{\varsigma} : Y \otimes_{\mathcal{H}} X \to \mathcal{K}, \quad (y \otimes_{\mathcal{H}} x) \mapsto \varsigma(y); \quad \overline{\vartheta} : Y \otimes_{\mathcal{H}} X \to \mathcal{K}, \quad (y \otimes_{\mathcal{H}} x) \mapsto \vartheta(x)$$

are well defined, in such a way that the following maps

$$(Y \otimes_{\mathcal{H}} X, x, \mathcal{K}) \to Y \otimes_{\mathcal{H}} X, \quad (y \otimes_{\mathcal{H}} x, k) \mapsto y \otimes_{\mathcal{H}} xk$$

$$G_1 \times X \to (Y \otimes_{\mathcal{H}} X, x, \mathcal{K}), \quad (g, y \otimes_{\mathcal{H}} x) \mapsto g \cdot y \otimes_{\mathcal{H}} x$$

define a structure of $(\mathcal{G}, \mathcal{K})$-biset on $Y \otimes_{\mathcal{H}} X$.

2.4. Normal subgroups and quotients. Given a morphism of groupoids $\phi : \mathcal{H} \to \mathcal{G}$, we define the kernel of $\phi$ and denote by $\text{Ker}(\phi)$ (or by $\phi^\circ : \text{Ker}(\phi) \to \mathcal{H}$), the groupoid whose underlying category is a subcategory of $\mathcal{H}$ given by following pair of sets:

$$\text{Ker}(\phi)_h = \mathcal{H}_h, \quad \text{Ker}(\phi)_x = \{ h \in \mathcal{H} | \phi(h) = \iota_{\mathcal{H}_h} = \iota_{\mathcal{H}_x} \}.$$ 

In other words, $\text{Ker}(\phi)$ is the subcategory of $\mathcal{H}$ whose arrows are all loops on which $\phi$ acts by identities. In particular, the isotropy groups of $\text{Ker}(\phi)$ coincide with the kernels of the isotropy maps. Thus

$$\text{Ker}(\phi)^\circ = \text{Ker}(\phi^\circ : \mathcal{H} \to \mathcal{G}^\circ), \quad \text{for any object } y \in \mathcal{H}_y.$$
Furthermore, for any arrow $g \in G$, we have that
\[ \text{ad}, (\ker(\phi)^{gg}) = \ker(\phi)^{g}, \]
where $\text{ad}$ is the adjoint operator of $g$ defined in (2). These properties motivate the following definition.

**Definition 2.11.** Let $H$ be a groupoid. A normal subgroupoid of $H$ is a subcategory $N \hookrightarrow H$ such that

(i) $N_e = H_e$;

(ii) For every $g \in H$, we have $\text{ad}, (N^{gg}) = N^{gg}$ as subgroups of $H^{gg}$.

Notice that given a normal subgroupoid $N$ of $H$, then each of the isotropy groups $N_x$ is a normal subgroup of $H$. In particular the isotropy groupoid $N^0$ of $N$, is a normal subgroupoid of the isotropy groupoid $H^0$ of $H$.

**Example 2.12.** As we have seen above the kernel of any morphism of groupoids is a normal subgroupoid. The converse also holds true (see Proposition 2.13 below). On the other hand, if $N \triangleleft G$ is a normal subgroup, then $(X \times N \times X, X)$ is clearly a normal subgroupoid of the induced groupoid $(X \times G \times X, X)$, see Example 2.4. Now taking $R$ any equivalence relation on a set $X$, and consider the associated groupoid as in Example 2.3. Then $(R, X)$ is a normal subgroupoid of the groupoid of pairs $(X \times X, X)$.

Next we recall the construction of the quotient groupoid from a given normal subgroupoid. Let $N$ be a normal subgroupoid of $H$. Clearly $(H_x, s)$ and $(H_x, t)$ are, respectively, right $N$-set and left $N$-set with actions given by the multiplication of $H$:

$H_x \times N \rightarrow H_x, \quad (h, e) \mapsto he$;

$N \times H_x \rightarrow H_x, \quad (e, g) \mapsto eg$.

Therefore $(H_x, s, t)$ is a $N$-biset in the sense of subsection 2.3. We denote its orbit set by $H_x / N$. That is, the quotient set of $H_x$ modulo the equivalence relation $h \sim g \Leftrightarrow \exists e \in N_x$ such that $eh = ge$. On the other hand, we can consider the quotient set of $H_x$ modulo the relation: $x \sim y \Leftrightarrow \exists e \in N_x$ such that $s(e) = x$ and $t(e) = y$. Denotes by $H_x / N$ the associated quotient set and by $H / N := (H_x / N, H_x / N)$ which going to be the quotient groupoid.

**Proposition 2.13.** Let $N$ be a normal subgroupoid of $H$. Then the pair of orbit sets $H / N$ admits a structure of groupoid such that there is a "sequence" of groupoids:

\[ N \rightarrow H \rightarrow H / N. \]

Furthermore, any morphism of groupoids $\phi : H \rightarrow G$ with $N \subseteq \ker(\phi)$, factors uniquely as

\[ \xymatrix{ H \ar[r]^-{\phi} & G \ar@{->>}[d]^-{\pi} \ar@{<-}[r]^-{\phi} & H / N } \]

**Proof.** The source, the target and the identity maps of $H / N$, are defined using those of $H$, that is, for a given arrow $\overrightarrow{h} \in H / N$, we set $s(\overrightarrow{h}) = s(h)$, $t(\overrightarrow{h}) = t(h)$ and $\omega = \pi$, for any $\overrightarrow{h} \in H / N$. These are well defined maps, since they are independent form the chosen representative of the equivalence class. The multiplication is defined by

\[ (H / N) \times (H / N) \rightarrow (H / N), \quad \overrightarrow{h} \cdot \overrightarrow{h'} \mapsto \overrightarrow{hh'} \]

This is a well defined associative multiplication thanks to condition (ii) of Definition 2.11. Lastly, the inverse of an arrow $\overrightarrow{h} \in H / N$ is given by the class of the inverse $h^{-1}$. The canonical map $(h, x) \mapsto (h, x)$ defines morphism of groupoids $H \rightarrow H / N$ whose kernel is $N \hookrightarrow H$. The proof of the rest of the statements are immediate. \qed

The fact that normal subgroups can be characterize as the invariant subgroups under the conjugation action, can be immediately extended to the groupoids context, as the following Lemma shows. But first let us observe that the conjugation operation of equation (2), induces a left $H$-action on the set of objects of
the isotropy groupoid $\mathcal{H}^0$, with the structure map $\varsigma : \mathcal{H}^0 = \sqcup_{x \in \mathcal{H}_0} \mathcal{H}^0 \to \mathcal{H}_0$ (source or the target of the isotropy groupoid $\mathcal{H}^0$). That is,
\[ \mathcal{H}_i \times \mathcal{H}^0 \xrightarrow{\quad} \mathcal{H}^0, \quad ((g, l) \mapsto glg^{-1}) \] (5)
defines a left $\mathcal{H}$-action on $\mathcal{H}^0$.

**Lemma 2.14.** Let $\mathcal{H}$ be a groupoid and $\mathcal{N} \hookrightarrow \mathcal{H}$ a subcategory with $\mathcal{N}_0 = \mathcal{H}_0$. Then $\mathcal{N}$ is a normal subgroupoid if and only if $\mathcal{N}^0$, is an $\mathcal{H}$-invariant subset of $\mathcal{H}^0$, with respect to the action of equation (5).

**Proof.** Straightforward.

3. **Linear representations of groupoid. Revisited.**

We provide in this section the construction and the basic properties of the induction, restriction and co-induction functors attached to a morphism of groupoids and connect the categories of linear representations. These properties are essential to follow the arguments presented in the forthcoming sections. The material presented here is probably well known to specialists, with the exception perhaps the result dealing with the characterization of linear representations of quotient groupoid that has its own interest. Nevertheless, we have preferred to give a self-contained and elementary exposition, which we think is accessible to wide range of the audience.

3.1. **Linear representations: Basic properties.** Given a groupoid $\mathcal{G}$, we denotes by $\text{Rep}_\mathbb{k}(\mathcal{G})$ the category of all $\mathbb{k}$-linear $\mathcal{G}$-representations. Thus an object in $\text{Rep}_\mathbb{k}(\mathcal{G})$ is a functor $\mathcal{V} : \mathcal{G} \to \text{Vect}_\mathbb{k}$, and morphism $\Phi : \mathcal{V} \to \mathcal{W}$ is a natural transformation (this is a set since all handled groupoids are small). From now on, a representation of $\mathcal{G}$ stands for a $\mathbb{k}$-linear $\mathcal{G}$-representation, i.e., an object in $\text{Rep}_\mathbb{k}(\mathcal{G})$. The $\mathbb{k}$-vector space of morphisms between two $\mathcal{G}$-representations will be denoted by $\text{Hom}_\mathbb{k}(\mathcal{V}, \mathcal{W})$.

To any representation $\mathcal{V}$ one associated the functor $\mathcal{V}^* : \mathcal{G} \to \text{Sets}$ by forgetting the $\mathbb{k}$-vector space structure of the representation. The same notation $\Phi^*$ will be used for any morphism $\Phi \in \text{Hom}_\mathbb{k}(\mathcal{V}, \mathcal{W})$. The image by $\Phi$ of an object $x \in \mathcal{G}_0$ is denoted by $\Phi(x)$, and the image of an arrow $g \in \mathcal{G}_1$ by $\Phi^g : \mathcal{V}_x \to \mathcal{V}_{gx}$.

As in the case of groups, a linear representation can be defined via a morphism of groupoids. Namely, given a $\mathcal{G}$-representation $\mathcal{V}$ and define its associated “vector bundle” as the pair $(\mathcal{V}, \pi_\mathcal{V})$ which consists of $\mathcal{V}_x = \bigcup_{g \in \mathcal{G}_0} \mathcal{V}_{gx}$, with projection $\pi_\mathcal{V} : \mathcal{V} \to \mathcal{G}_0$. Define the groupoid $\text{Iso}(\mathcal{V})$ whose set of objects is $\mathcal{G}_0$ and set of morphisms $\text{Iso}(\mathcal{V})(\mathcal{V}_x, \mathcal{V}_y) := \text{Hom}_\mathbb{k}(\mathcal{V}_x, \mathcal{V}_y)$ the set of all $\mathbb{k}$-linear isomorphisms from $\mathcal{V}_x$ to $\mathcal{V}_y$, for a given $x, y \in \mathcal{G}_0$. This groupoid is known as the frame groupoid of the vector bundle $(\mathcal{V}, \pi)$, see Example 2.5. In this way, it is clear that there is a morphism of groupoids $\mathcal{G}_\mathcal{V} : \mathcal{G} \to \text{Iso}(\mathcal{V})$. Conversely, given a vector bundle $\pi : E \to \mathcal{G}_0$ and morphism of groupoids $\mu : \mathcal{G} \to \text{Iso}(E)$, where as before $\text{Iso}(E)$ is the frame groupoid of $(E, \pi)$. Then one obtain a linear representation $\mathcal{E} : \mathcal{G} \to \text{Vect}$, acting on objects by $x \mapsto E_x$, (the fibre of $E$ at $x$) and on arrows by $g \mapsto [E^g = \mu(g) : E_{gx} \to E_{gx}]$.

On the other hand it is clear that for any $\mathcal{G}$-representation $\mathcal{V}$, the pair $(\mathcal{V}, \pi_\mathcal{V})$ with the map
\[ \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{V} \to \mathcal{V}, \quad ((g, v) \mapsto gv := \mathcal{V}(v)) \]
gives arise to a left $\mathcal{G}$-set structure on the bundle $\mathcal{V}$, in the sense of the left version of Definition 2.6. This in facts leads to a faithful functor from the category of $\mathcal{G}$-representations to the category of left $\mathcal{G}$-sets.

It is well known, see for instance [19], that the category $\text{Rep}_\mathbb{k}(\mathcal{G})$ is an abelian symmetric monoidal category with a set of small generators. The monoidal structure is extracted from that of $\text{Vect}_\mathbb{k}$, that is, for any two representations $\mathcal{U}$ and $\mathcal{V}$, their tensor product is $\mathcal{U} \otimes \mathcal{V} : \mathcal{G} \to \text{Vect}$, defined by $((\mathcal{U} \otimes \mathcal{V})_x = \mathcal{U}_x \otimes \mathcal{V}_x$ and $(\mathcal{U} \otimes \mathcal{V})^g = \mathcal{U}^g \otimes \mathcal{V}^g$, for every $x \in \mathcal{G}_0$ and $g \in \mathcal{G}_1$.

The category $\text{Rep}_\mathbb{k}(\mathcal{G})$, is in particular locally small, in the sense that the class of subobjects of any object is actually a set. The zero representation well be denoted by $\mathbf{0}$ and the identity representation (with respect to the tensor product), or the trivial representation, by $\mathbf{1}$. Moreover, to any representation one can associate its dual representation. Indeed, take a representation $\mathcal{V}$, for any object $x \in \mathcal{G}_0$, set $(\mathcal{V}^*), := \mathcal{V}_x^* = \text{Hom}_\mathbb{k}(\mathcal{V}_x, \mathbb{k})$ the linear dual of the vector space $\mathcal{V}_x$; and for an arrow set $g \in \mathcal{G}_1$, $(\mathcal{V}^*)^g := (\mathcal{V}^*)^g$. Then $\mathcal{V}^*$ is a representation with a canonical morphism of representations $\mathcal{V}^* \otimes \mathcal{V} \to \mathbf{1}$ fibrewise given by the evaluation maps $\mathcal{V}^*_x \otimes \mathcal{V}_x \to \mathbb{k}, \varphi \otimes v \mapsto \varphi(v)$, for every $x \in \mathcal{G}_0$. 
We say that a representation \( \mathcal{V} \in \text{Rep}(\mathcal{G}) \) is finite, when its image lands in the subcategory \( \text{vect}_k \) of finite dimensional \( k \)-vector spaces. The full subcategory of finite representations is then an abelian symmetric rigid monoidal category.

**Example 3.1.** For instance a finite representation where each one of its fibres is a one-dimensional \( k \)-vector space can be identified with a family of elements \( \{ \lambda_{s(0),0(0)} \}_{s \in \mathcal{G}} \) in \( k^\times \), the multiplicative group of \( k \), satisfying
\[
\lambda_{s(0),0(0)} \cdot \lambda_{t(0),0(0)} = \lambda_{s(0),s(0)} ,
\]
whenever \( t(g) = s(h) \), and \( \lambda_{s(e),e} = 1_e \), for every \( x \in \mathcal{G}_e \).

In the second condition, the term \( \lambda_{0,0} \) stands for the \( t_s \)'s projection of \( \lambda \). The family \( \{ \lambda_{s(0),0(0)} \}_{s \in \mathcal{G}} \) in \( k^\times \), where \( \lambda_{s(0),0(0)} = 1_s \), for every \( g \in \mathcal{G}_1 \), corresponds then to the trivial representation \( I \).

To any representation \( \mathcal{V} \in \text{Rep}(\mathcal{G}) \), one can consider the projective and the inductive limits of its underlying functor, since this one lands in the Grothendieck category \( \text{Vect}_k \). These \( k \)-vector spaces, are denoted by \( \lim_{\rightarrow \mathcal{G}} (\mathcal{V}) \) and \( \lim_{\leftarrow \mathcal{G}} (\mathcal{V}) \), respectively.

Given a representation \( \mathcal{V} \) we can define as follows its \( \mathcal{G} \)-invariant subrepresentation. For any \( x \in \mathcal{G}_x \), we set \( \mathcal{V}'_x \) the subspace of \( \mathcal{V} \), invariant under the action of the isotropy group \( \mathcal{G}_x \). That is,
\[
\mathcal{V}'_x = \{ v \in \mathcal{V} | \mathcal{V}(v) : = l v = v, \text{ for all } l \in \mathcal{G}_x \}.
\]

Now, take an arrow \( g \in \mathcal{G}_1 \), and a vector \( v \in \mathcal{V}'_{x_0} \). Then, for any \( q \in \mathcal{G}_0 \), we have that
\[
q(gv) = g((g^{-1} q)g)v = gv.
\]

Therefore, the image under the linear map \( \mathcal{V}' \) of any vector in \( \mathcal{V}'_{x_0} \) lands in \( \mathcal{V}'_{x_0} \). The same holds true interchanging \( g \) by \( g^{-1} \). This means that for any arrow \( g \in \mathcal{G}_1 \), we have a commutative diagram
\[
\mathcal{V}'_{x_0} \quad \xrightarrow{\mathcal{V}_g} \quad \mathcal{V}'_{x_0}.
\]

In this way we obtain a representation \( \mathcal{V}'' : \mathcal{G} \to \text{Vect}_k \) with a monomorphism \( \mathcal{V}'' \hookrightarrow \mathcal{V} \) in \( \text{Rep}(\mathcal{G}) \). This representation is referred to as the \( \mathcal{G} \)-invariant subrepresentation of \( \mathcal{V} \).

**Remark 3.2.** If \( \mathcal{G} \) is a groupoid with only one object, that is, a group, then \( \mathcal{V}'' = \text{Hom}_k ( I, \mathcal{V} ) \). In general, however, we can not directly relate this later vector space with the fibres of the \( \mathcal{G} \)-invariant representation. More precisely, we have that \( \lim_{\longrightarrow \mathcal{G}} (\mathcal{V}) = \text{Hom}_k ( I, \mathcal{V} ) \) as vector spaces, and the following commutative diagram of vector spaces:
\[
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\mathcal{V}_g} & \mathcal{V}_x \\
\downarrow \mathcal{V}_{\mathcal{G}} & & \downarrow \mathcal{V}_{\mathcal{G}} \\
\mathcal{V}_{x_0} & \xrightarrow{\mathcal{V}_g} & \mathcal{V}_{x_0}
\end{array}
\]

where for every \( g \in \mathcal{G}_x \), we have \( \pi_x = \mathcal{V}_x \circ p_{x_0} = p_{x_0} \) and the \( p \)'s are the canonical projections. Then the dashed monomorphism of vector spaces is not necessarily an isomorphism.

In a dual way, one defines the co-invariant representation. Specifically, let \( \mathcal{V} \) be a \( \mathcal{G} \)-representation, for any \( x \in \mathcal{G}_x \), we set the quotient \( k \)-vector space
\[
\mathcal{V}_{x'} := \mathcal{V}_x / \text{Span} \{ ev - v | e \in \mathcal{G} ', v \in \mathcal{V}_x \}.
\]
Given now an arrow \( g \in \mathcal{G}_1 \), we have that the linear map \( \mathcal{V}' \) extend to the quotients, that is, we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{V}_{\rho(g)} & \longrightarrow & \mathcal{V}_{\rho(g)}' \\
\downarrow & & \uparrow \\
\mathcal{V}_{\rho(g)} & \longrightarrow & \mathcal{V}_{\rho(g)}'.
\end{array}
\]

Therefore, the family \( \{(\mathcal{V}_g, \mathcal{V}_g')\}_{g \in \mathcal{G}_0} \) defines a representation which we denote by \( \mathcal{V}_g \), and we have a canonical epimorphism \( \mathcal{V} \rightarrow \mathcal{V}' \) in the category \( \text{Rep}_1(\mathcal{G}) \) given fibrewise by the linear map \( \pi_g : \mathcal{V}_g \rightarrow \mathcal{V}_g' \). In analogy with group theory context, the representation \( \mathcal{V}_g \) is refereed to as the \textit{coinvariant quotient representation} of \( \mathcal{V} \).

**Remark 3.3.** Similar to Remark 3.2 the coinvariant representation \( \mathcal{V}_g \) is related to the limit of the representation \( \lim_{\rightarrow \mathcal{G}} (\mathcal{V}) \) and also to the vector space \( \text{Hom}_g (\mathcal{V}, 1) \). More precisely, we have, form one hand, an isomorphism of \( k \)-vector spaces

\[
\left( \lim_{\rightarrow \mathcal{G}} (\mathcal{V}) \right)^* \longrightarrow \text{Hom}_g (\mathcal{V}, 1), \quad \phi \mapsto (\phi \circ \zeta)_\mu \colon \left( \lim_{\rightarrow \mathcal{G}} (\mathcal{V}) \right)^* \longrightarrow (\lim_{\rightarrow \mathcal{G}} (\mathcal{V}))^*, \quad \lim (f_\mu) \mapsto (f_\mu)_\mu
\]

where \( \zeta : \mathcal{V} \rightarrow \lim_{\rightarrow \mathcal{G}} (\mathcal{V})_1 \) are the structural maps of the stated limit. On the other hand, since each of the maps \( \zeta \), factors through the quotient \( \mathcal{V}_g' \), we have a family of maps \( \overline{\zeta} : \mathcal{V}_g' \rightarrow \lim_{\rightarrow \mathcal{G}} (\mathcal{V}) \) whose direct sum rends the following diagram

\[
\begin{array}{ccc}
\bigoplus_{g \in \mathcal{G}_1} \mathcal{V}_g(\mu) & \longrightarrow & \bigoplus_{\lambda \in \mathcal{G}_0} \mathcal{V}_{\lambda} \epsilon \longrightarrow \lim_{\rightarrow \mathcal{G}} (\mathcal{V}) \longrightarrow 0 \\
\downarrow & & \downarrow \theta_{g} \overline{\zeta} & & \downarrow \\
\bigoplus_{g \in \mathcal{G}_1} \mathcal{V}_g' & \longrightarrow & \bigoplus_{\lambda \in \mathcal{G}_0} \mathcal{V}_{\lambda}' \longrightarrow \lim_{\rightarrow \mathcal{G}} (\mathcal{V})' \longrightarrow 0
\end{array}
\]

commutative, where \( \tau \) is given by \( \tau_\mu = \tau_{\mu(g)} \circ \mathcal{V}' - \tau_{\mu(g)} \), for every \( g \in \mathcal{G}_1 \). The dashed epimorphism is then not necessarily an isomorphism.

We finish this subsection by the following observation.

**Lemma 3.4.** Let \( \mathcal{G} \) be a groupoid and \( \mathcal{V}, \mathcal{U} \) two representations in \( \text{Rep}_1(\mathcal{G}) \). Then the family of \( k \)-vector spaces \( \{\text{Hom}_g (\mathcal{U}, \mathcal{V})\}_{g \in \mathcal{G}_0} \) defines a representation in \( \text{Rep}_1(\mathcal{G}) \) denoted by \( \text{Hom}_g (\mathcal{U}, \mathcal{V}) \). In particular, if \( \mathcal{U} \) is a finite representation then

\[
\mathcal{V} \otimes \mathcal{U} \cong \text{Hom}_g (\mathcal{U}, \mathcal{V}) \,,
\]

a natural isomorphism in the category \( \text{Rep}_1(\mathcal{G}) \). Furthermore, for any \( \mathcal{U} \) and \( \mathcal{V} \), we have

\[
\lim_{\rightarrow \mathcal{G}} (\text{Hom}_g (\mathcal{U}, \mathcal{V})) = \text{Hom}_g (\mathcal{U}, \mathcal{V})
\]

**Proof.** The action of \( \text{Hom}_g (\mathcal{U}, \mathcal{V}) \) on a given arrow \( g \in \mathcal{G}_1 \), is defined by the following linear isomorphism

\[
\text{Hom}_g (\mathcal{U}_g, \mathcal{V}_g) \longrightarrow \text{Hom}_g (\mathcal{U}_g, \mathcal{V}_g), \quad (\sigma \mapsto \mathcal{V}' \circ \sigma \circ \mathcal{U}^{-1})
\]

This clearly defines an object in \( \text{Rep}_1(\mathcal{G}) \). If \( \mathcal{U} \) is a finite representation, then each fibre \( \text{Hom}_g (\mathcal{U}_g, \mathcal{V}) \) is linearly isomorphic to the \( k \)-vector space \( \mathcal{V}_g \otimes \mathcal{U}_g \). This family of linear isomorphisms leads in fact to the stated natural isomorphism.

For the proof of last statement, let us consider the following well defined map:

\[
\text{Hom}_g (\mathcal{U}, \mathcal{V}) \longrightarrow \text{Hom}_g (\mathcal{U}_g, \mathcal{V}_g)
\]
for every $x \in \mathcal{G}_s$. Each one of these maps is a $k$-linear map where $\text{Hom}_k(\mathcal{U}, \mathcal{V})$ is endowed with the structure of $k$-vector space fibrewise inherited from that of $\mathcal{V}$. This leads to a projective system which is in turn the universal one. Thus, $\text{Hom}_k(\mathcal{U}, \mathcal{V}) = \lim_{\rightarrow \mathcal{G}}(\text{Hom}_k(\mathcal{U}, \mathcal{V}))$ as claimed. □

3.2. The restriction functor. Let $\phi : \mathcal{H} \to \mathcal{G}$ be a morphism of groupoids. The restriction functor is the functor
\[
\phi_* : \text{Rep}(\mathcal{G}) \to \text{Rep}(\mathcal{H}), \quad (\mathcal{V} \to \mathcal{V} \circ \phi; \quad f \to f|_\mathcal{V})
\]
where the notation is the obvious one. In the subsequent we analyse the property of the restriction functor corresponding to a normal subgroupoid. Such a property is in fact a generalization of [15, Proposition 2.3.2] and of course has its own interest in groupoids context.

**Proposition 3.5** (Representations of quotients). Let $\mathcal{N}$ be a normal subgroupoid of $\mathcal{H}$ and denote by $\pi : \mathcal{H} \to \mathcal{G} = \mathcal{H}/\mathcal{N}$ the canonical projection. Then the restriction functor induces an isomorphism of categories between $\text{Rep}(\mathcal{G})$ and the full subcategory $\text{Rep}(\mathcal{H})^N$ of $\text{Rep}(\mathcal{H})$ whose representations are trivial on $\mathcal{N}$, that is, an object in $\text{Rep}(\mathcal{H})^N$ is a representation $\mathcal{V}$ of $\mathcal{H}$ such that $\mathcal{N} \subset \text{Ker}(\phi_\mathcal{N})$.

**Proof.** We know that $\pi_* : \text{Rep}(\mathcal{G}) \to \text{Rep}(\mathcal{H})$ has the image in the full subcategory $\text{Rep}(\mathcal{H})^N$. Let us denote also by $\pi_* : \text{Rep}(\mathcal{G}) \to \text{Rep}(\mathcal{H})^N$ the resulting functor. The inverse of this functor is construct with the help of Proposition 2.13. Explicitly, given a representation $(\mathcal{V}, \phi_\mathcal{V})$ in $\text{Rep}(\mathcal{H})^N$, thus a morphism of groupoids $\phi_\mathcal{V} : \mathcal{H} \to \text{Iso}(\mathcal{V})$ with $\mathcal{N} \subset \text{Ker}(\phi_\mathcal{N})$. Then by Proposition 2.13, we have a representation $\overline{\phi_\mathcal{V}} : \mathcal{G} = \mathcal{H}/\mathcal{N} \to \text{Iso}(\mathcal{V})$. This establishes a functor $\pi^* : \text{Rep}(\mathcal{H})^N \to \text{Rep}(\mathcal{G})$ which turns out to be the inverse of $\pi_*$. □

**Remark 3.6.** Let $\mathcal{N}$ be a normal subgroupoid of $\mathcal{H}$. Then for any representation $\mathcal{W} \in \text{Rep}(\mathcal{H})$, we can consider the assignment
\[
\mathcal{W}^\mathcal{N} : \mathcal{H}_0 \to \text{Vect}_k, \quad (u \mapsto \mathcal{W}^\mathcal{N} u),
\]
where the subspace $\mathcal{W}^\mathcal{N}$ of $\mathcal{W}_s$ consists of those vectors which are invariant under the action of loops in $\mathcal{N}$, that is, those $v \in \mathcal{W}_s$ such that $\mathcal{W}^\mathcal{N}(v) = e v = v$, for every $e \in \mathcal{N}$. It turns out that $\mathcal{W}^\mathcal{N}$ gives a well defined functor which acts by restriction on arrows, because $\mathcal{N}$ is normal. Indeed, take a vector $w \in \mathcal{W}^\mathcal{N}$ for some $h \in \mathcal{H}_0$ and $e \in \mathcal{N}^\mathcal{N}$. Then, we get that $e(hw) = h(h^{-1}eh)w = hw$. Therefore, $\mathcal{W}^\mathcal{N}$ is a sub-representation of the $\mathcal{H}$-representation $\mathcal{W}$.

3.3. The induction functor. Let $\phi : \mathcal{H} \to \mathcal{G}$ be a morphism of groupoids. Fix an object $x \in \mathcal{G}_s$ and consider the functor
\[
\phi^* : \mathcal{H} \to \text{Sets}, \quad (u \mapsto \mathcal{G}(x, \phi_\mathcal{H}(u)))
\]
which acts in the obvious way on arrows. That is, for any arrow $h \in \mathcal{H}_0$, we have $\phi^*(h) = \mathcal{G}(x, \phi_\mathcal{H}(h))$.

As a contravariant functor $\phi^* : \mathcal{G} \to [\mathcal{H}, \text{Sets}]$, it acts by $\phi_\mathcal{H}^* = \mathcal{G}(g, \phi_\mathcal{H}(u)) : \phi^*(u) \to \phi^*(v)$, for every object $u \in \mathcal{H}_0$ and arrow $g : x \to y$ in $\mathcal{G}_s$.

Given a representation $\mathcal{W}$ in $\text{Rep}(\mathcal{H})$, we can construct a family of $k$-vector spaces $[\text{Nat}(\phi^*, \mathcal{W})]_{x \in \mathcal{G}_s}$ by using the fibrewise $k$-vector space structure of $\mathcal{W}$, where $\mathcal{W} : \mathcal{H} \to \text{Sets}$ is the functor attached to $\mathcal{W}$. That is, for each $x \in \mathcal{G}_s$, the set of natural transformations $\text{Nat}(\phi^*, \mathcal{W})$ admits a canonical structure of $k$-vector space given componentwise by
\[
(\alpha + \beta)_u = \alpha_u + \beta_u, \quad (\lambda \alpha)_u = \lambda \alpha_u : \phi^*_x \to \phi^*_x,
\]
for every $u \in \mathcal{H}_0$ and $\alpha, \beta \in \text{Nat}(\phi^*, \mathcal{W})$. This leads to a functor
\[
\phi^*(\mathcal{W}) : \mathcal{G} \to \text{Vect}_k
\]
for $\mathcal{W} \in \text{Rep}(\mathcal{H})$, gives a well defined functor refereed to as the induction functor.
**Proof.** Given a morphism $\mathcal{E} : \mathcal{W} \to \mathcal{W}'$ in the category $\text{Rep}_1(\mathcal{H})$ and an arrow $g \in \mathcal{G}$, we need to check that the following diagram

$$
\begin{array}{ccc}
\text{Nat}(\phi^{\mathcal{E}_0}, \mathcal{W}) & \xrightarrow{\phi^*(\mathcal{E})} & \text{Nat}(\phi^{\mathcal{E}_0}, \mathcal{W}') \\
\text{Nat}(\phi^{\mathcal{E}_0}, \mathcal{W}) & \xrightarrow{\mathcal{H}(\mathcal{E})} & \text{Nat}(\phi^{\mathcal{E}_0}, \mathcal{W}')
\end{array}
$$

of vector spaces commutes. This follows from the equality

$$
\text{Nat}(\phi^{\mathcal{E}_0}, \mathcal{E}) \circ \text{Nat}(\phi^{\mathcal{E}_0}, \mathcal{W})(\alpha) = \mathcal{E} \circ \alpha \circ \phi^* \circ \text{Nat}(\phi^{\mathcal{E}_0}, \mathcal{E})(\alpha),
$$

for every $\alpha \in \text{Nat}(\phi^{\mathcal{E}_0}, \mathcal{E})$. \hfill \Box

**Example 3.8.** Let $\mathcal{H} := (G \times X, X)$ be an action groupoid as in Example 2.10 and consider the morphism $\phi := pr_1 : \mathcal{H} \to G$ of groupoids, where the group $G$ is considered as a groupoid with one object ($\ast$). Let $\mathcal{W} \in \text{Rep}_1(\mathcal{H})$ and set $\Gamma(\mathcal{W})$ the $k$-vector space of global sections of the vector bundle $\mathcal{W}$ attached to the representation $\mathcal{W}$. Then $\Gamma(\mathcal{W}) \in \text{Rep}_1(G)$ and the assignment $\mathcal{W} \mapsto \Gamma(\mathcal{W})$ establishes a functor from $\text{Rep}_1(\mathcal{H})$ to $\text{Rep}_1(G)$. Furthermore, we have a natural isomorphism

$$
\phi^*(\mathcal{W}) \cong \Gamma(\mathcal{W}).
$$

**Remark 3.9.** Let $\phi : \mathcal{H} \to \mathcal{G}$ be a morphism of groupoids. Denote by $\ast\mathcal{H}(\mathcal{G}) := \mathcal{H}_0 \times \mathcal{X}$, $\mathcal{G}$, the $(\mathcal{H}, \mathcal{G})$-biset of Example 2.10. Given an $\mathcal{H}$-representation $\mathcal{W}$ and consider it associated vector bundle $(\mathcal{W}, \pi_w)$ as a left $\mathcal{H}$-set. Then we have a natural isomorphism

$$
\begin{array}{ccc}
\text{Hom}_{\ast\mathcal{H}(\mathcal{G}), \ast\mathcal{W}}(\mathcal{W}) & \to & \prod_{x \in \mathcal{G}_0} \text{Nat}(\phi^*(\mathcal{W}), x) \\
\{f\} & \mapsto & \left\{ f_x : \phi^*(\mathcal{W}), (a \mapsto f(u, a)) \right\}_{x \in \mathcal{G}_0};
\end{array}
$$

This in fact comes from the natural isomorphism

$$
\begin{array}{ccc}
\text{Hom}_{\ast\mathcal{H}(\mathcal{G}), \ast\mathcal{W}}(\vartheta(x)^{-1}, \mathcal{W}) & \to & \text{Nat}(\phi^*(\mathcal{W}), x) \\
\{f\} & \mapsto & \left\{ f_x : \phi^*(\mathcal{W}), (a \mapsto f(u, a)) \right\}_{x \in \mathcal{G}_0};
\end{array}
$$

where $\vartheta : \ast\mathcal{H}(\mathcal{G}) \to \mathcal{G}_0$ is as before, $(u, a) \mapsto s(a)$.

**Remark 3.10 (Projection Formula).** Analogue to the group case, see for instance [15 Proposition 2.10.18], one can show that there is a natural isomorphism

$$
\phi^*(\mathcal{W} \otimes \phi^*(\mathcal{V})) \cong \phi^*(\mathcal{W} \otimes \mathcal{V}),
$$

for any pair of representation $\mathcal{V} \in \text{Rep}_1(\mathcal{G})$ and $\mathcal{W} \in \text{Rep}_1(\mathcal{H})$. At the level of objects this isomorphism is given by

$$
\begin{array}{ccc}
\text{Nat}(\phi^*(\mathcal{W}), \mathcal{V}) & \cong & \text{Nat}(\phi^*(\mathcal{W} \otimes \mathcal{V}), \mathcal{V}) \\
\eta \otimes \mathcal{V} & \mapsto & \left\{ \phi^*(u) \mapsto \mathcal{W}_u \otimes \mathcal{V}_{\mathcal{W}_{\mathcal{H}_0}}(b \mapsto \eta_{(b)} \otimes \mathcal{V}(v)) \right\}_{u \in \mathcal{H}_0}
\end{array}
$$

whose inverse is computed by fixing a dual basis $\{v_j\} \in \mathcal{V}$, and employing the dual basis $\{\mathcal{V}(v_j)\} \in \mathcal{V}_\mathcal{H}_0$, for any $b \in \phi^*(u)$ and $u \in \mathcal{H}_0$. The rest of verifications are lifted to the reader.

A more conceptual proof can be given using Mitchell’s Theorem [19 Theorem 4.5.2] employing the set of small projective generators that enjoy both categories $\text{Rep}_1(\mathcal{H})$ and $\text{Rep}_1(\mathcal{G})$. 

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3.4. The co-induction functor. Let \( \phi : \mathcal{H} \to \mathcal{G} \) be as before the morphism of groupoids. Denote by \( \mathcal{W}^*(\mathcal{G}) := \mathcal{G}_1 \times_{\mathcal{H}_0} \mathcal{H}_0 \) the underlying set of the \((\mathcal{G}, \mathcal{H})\)-biset described in Example 2.10 with the two structure maps \( \zeta : \mathcal{W}^*(\mathcal{G}) \to \mathcal{G}_0, (a, u) \mapsto (a, u) \) and \( \eta : \mathcal{W}^*(\mathcal{G}) \to \mathcal{G}_0, (a, u) \mapsto u \).

For a given \( x \in \mathcal{G}_0 \), we consider the fibre \( \zeta^{-1}(\{x\}) \) as an \( \mathcal{H} \)-invariant subset of \( \mathcal{W}^*(\mathcal{G}) \). We then perform the right translation groupoid \( \zeta^{-1}(\{x\}) \rtimes \mathcal{H} \) which we denote by \( \mathcal{H}^+ \). Associated to the canonical morphism of groupoids

\[
\gamma^{+,-} : \mathcal{H}^{+,-} = \zeta^{-1}(\{x\}) \rtimes \mathcal{H} \to \mathcal{H}, \quad \left((a, u), h \right) \mapsto (u, h),
\]

we consider, as in subsection 3.2, the restriction functor \( \gamma^{+,-}_s : \text{Rep}_s(\mathcal{H}) \to \text{Rep}_s(\mathcal{H}^{+,-}) \), for every \( x \in \mathcal{G}_0 \). In this way, for any \( x \in \mathcal{G}_0 \) and \( W \in \text{Rep}_s(\mathcal{H}) \), we set

\[
\gamma^{+,-}_s(W) := \lim_{\gamma^{+,-}_s} \left( \gamma^{+,-}_s(W) \right)
\]

and denote by \( \left[ \gamma^{+,-}_s \right] : \gamma^{+,-}_s(W)_{\mathcal{G}_0, x} \to \gamma^{+,-}_s(W)_{\mathcal{G}_0, a} \)

the structural \( \mathbb{k} \)-linear maps of this limit.

Now, given an arrow \( g \in \mathcal{G}_1 \), we have a diagram of morphism of groupoids

\[
\begin{array}{ccc}
\zeta^{-1}(\{g\}) \rtimes \mathcal{H} & \xrightarrow{\gamma^{+,-}_s(W_{\mathcal{G}_0, x})} & \mathcal{H} \\
\zeta^{-1}(\{g\}) \rtimes \mathcal{H} & \xrightarrow{\gamma^{+,-}_s(W_{\mathcal{G}_0, x})} & \mathcal{H}
\end{array}
\]

whose vertical arrow is the isomorphism sending \( ((a, u), h) \mapsto (ga, u) \). Therefore, by the definition of the limit of equation (8), there is a unique \( \mathbb{k} \)-linear map \( \gamma^{+,-}_s(W_{\mathcal{G}_0, x}) : \gamma^{+,-}_s(W)_{\mathcal{G}_0, x} \to \gamma^{+,-}_s(W)_{\mathcal{G}_0, a} \), rendering commutative the following diagrams

\[
\begin{array}{ccc}
\gamma^{+,-}_s(W_{\mathcal{G}_0, a}) & \xrightarrow{\gamma^{+,-}_s(W_{\mathcal{G}_0, x})} & \gamma^{+,-}_s(W_{\mathcal{G}_0, x}) \\
\gamma^{+,-}_s(W_{\mathcal{G}_0, a}) & \xrightarrow{\gamma^{+,-}_s(W_{\mathcal{G}_0, x})} & \gamma^{+,-}_s(W_{\mathcal{G}_0, x})
\end{array}
\]

for every \( (a, u) \in \zeta^{-1}(\{s\}(g)) \), where \( \gamma^{+,-}_s(W_{\mathcal{G}_0, a}) \) acts by identity, i.e.,

\[
\gamma^{+,-}_s(W_{\mathcal{G}_0, a}) : \gamma^{+,-}_s(W)_{\mathcal{G}_0, a} \to \gamma^{+,-}_s(W)_{\mathcal{G}_0, a} = \gamma^{+,-}_s(W)_{\mathcal{G}_0, a} \quad (w \mapsto w).
\]

Equations (8) and (9) lead then to a functor

\[
\begin{array}{ccc}
\gamma^{+,-}_s(W_{\mathcal{G}_0, a}) & \xrightarrow{\gamma^{+,-}_s(W_{\mathcal{G}_0, x})} & \gamma^{+,-}_s(W_{\mathcal{G}_0, x}) \\
\gamma^{+,-}_s(W_{\mathcal{G}_0, a}) & \xrightarrow{\gamma^{+,-}_s(W_{\mathcal{G}_0, x})} & \gamma^{+,-}_s(W_{\mathcal{G}_0, x})
\end{array}
\]

On the other hand, if \( \mathcal{F} : W \to W' \) is an \( \mathcal{H} \)-equivariant linear map. Then, for every \( x \in \mathcal{G}_0 \), we define the following \( \mathbb{k} \)-linear map

\[
\gamma^{+,-}_s(W_{\mathcal{G}_0, a}) : \gamma^{+,-}_s(W)_{\mathcal{G}_0, x} \to \gamma^{+,-}_s(W)_{\mathcal{G}_0, a}
\]

as the unique \( \mathbb{k} \)-linear map which renders commutative the following diagram of \( \mathbb{k} \)-vector spaces

\[
\begin{array}{ccc}
\gamma^{+,-}_s(W_{\mathcal{G}_0, a}) & \xrightarrow{\gamma^{+,-}_s(W_{\mathcal{G}_0, x})} & \gamma^{+,-}_s(W_{\mathcal{G}_0, x}) \\
\gamma^{+,-}_s(W_{\mathcal{G}_0, a}) & \xrightarrow{\gamma^{+,-}_s(W_{\mathcal{G}_0, x})} & \gamma^{+,-}_s(W_{\mathcal{G}_0, x})
\end{array}
\]

where \( \gamma^{+,-}_s(W_{\mathcal{G}_0, a}) : \gamma^{+,-}_s(W)_{\mathcal{G}_0, a} \to \gamma^{+,-}_s(W)_{\mathcal{G}_0, a} \) is the \( \mathbb{k} \)-linear map sending \( w \mapsto \mathcal{F}(w) \).

**Lemma 3.11.** Let \( \phi : \mathcal{H} \to \mathcal{G} \) be a morphism of groupoids. Then the assignment \( \gamma^{+,-}_s : \text{Rep}_s(\mathcal{H}) \to \text{Rep}_s(\mathcal{G}) \) described in equation (10), gives rise to a well defined functor, referred to as the co-induction functor.
**Proof.** Given a morphism \( f : W \to W' \) in the category \( \text{Rep}_\cdot(\mathcal{H}) \) and an arrow \( g \in \mathcal{G}_1 \), we need to check that the following diagram

\[
\begin{array}{ccc}
\phi(W) & \xrightarrow{\phi(f)} & \phi(W') \\
\downarrow \phi(u) & & \downarrow \phi(u') \\
\phi(W) & & \phi(W')
\end{array}
\]

of vector spaces, commutes. This remounts to show the commutativity of the following diagrams:

\[
\begin{array}{ccc}
\gamma^w_{s(x)}(W)_{pr(a)} & \xrightarrow{\xi^w_{s(x)}} & \gamma^w_{s(x)}(W')_{pr(a)} \\
\downarrow \xi^w_{pr(a)} & & \downarrow \xi^w_{pr(a)} \\
\gamma^w_{s(x)}(W)_{pr(a)} & & \gamma^w_{s(x)}(W')_{pr(a)}
\end{array}
\]

for any \((a, u) \in \gamma^{-1}(\{s(g)\})\). However, this is immediate from the definitions of the involved maps. \(\square\)

**Remark 3.12.** Given \( W \) an \( \mathcal{H} \)-representation, then one can consider, as in subsection 2.3, the tensor product \( \mathcal{V}^w(\mathcal{G}) \otimes_{\gamma^w} W' \), where as above \((\mathcal{W}, \pi_w)\) is the underlying vector bundle of \( W \) endowed within its canonical left \( \mathcal{H} \)-action. An equivalence class of an element \(((a, u), w) \in \mathcal{V}^w(\mathcal{G}) \otimes_{\gamma^w} W \) will be denoted by \((a, u) \otimes_{\gamma^w} w\). Now, if we consider the vector bundle \((\phi^w(W), \pi)\) endowed with its canonical left \( \mathcal{G} \)-action, then we obtain the following \( \mathcal{G} \)-equivariant map

\[
\begin{array}{ccc}
\mathcal{V}^w(\mathcal{G}) \otimes_{\gamma^w} W & \xrightarrow{\phi^w(W)} & \phi^w(W) \\
(a, u) \otimes_{\gamma^w} w & & \psi^w_{u,v}(w),
\end{array}
\]

which is not in general an isomorphism.

4. Frobenius reciprocity formulae.

In this section we prove the left and the right Frobenius reciprocity formulae. This mainly establishes, from one side an adjunction between the restriction and induction functors, and from another one an adjunction between the restriction and co-induction functors. Form a categorical point of view, this remount to the notions of left and right Kan extensions. Here we follow an elementary and direct exposition, making advantage of groupoids structure, without appealing to any heavy categorical notions.

4.1. Right Frobenius reciprocity formula. Let \( \phi : \mathcal{H} \to \mathcal{G} \) be a morphism of groupoids and consider two representations \( V \in \text{Rep}_\cdot(\mathcal{G}) \) and \( W \in \text{Rep}_\cdot(\mathcal{H}) \). Take a morphism \( \sigma \in \text{Hom}_\mathcal{G}(\phi^*V, W) \), and define, for every \( x \in \mathcal{G}_\mathcal{H} \), the linear map

\[
\Psi(\sigma)_x : V_x \longrightarrow \phi^*(W)_x = \text{Nat}(\phi^*V, W)_x, \quad \begin{pmatrix} v \\ p \end{pmatrix} \longmapsto \begin{pmatrix} \phi^*_x v \\ \sigma_x(pv) \end{pmatrix} \quad (12)
\]

**Lemma 4.1.** The family of linear map \( \{\Psi(\sigma)_x\}_{x \in \mathcal{G}_\mathcal{H}} \) stated in equation (12) defines a natural transformation. That is, \( \Psi(\sigma) \in \text{Hom}_\mathcal{G}(V, \phi^*(W)) \).

**Proof.** First let us check that each of the \( \Psi(\sigma)_x \)'s is well defined. So given an arbitrary arrow \( h \in \mathcal{H}_i \), we need to show that the diagram

\[
\begin{array}{ccc}
\phi^*_x v \longmapsto \Psi(\sigma)_x(\psi^x_{u,v})(v) & \xrightarrow{\Psi(\sigma)_x(\psi^x_{u,v})} & W_{xh} \\
\phi^*_x v \downarrow \phi^*_x \downarrow \Psi(\sigma)_x(\psi^x_{u,v}) & & \downarrow \psi^x_{u,v} \\
\phi^*_x v \downarrow \phi^*_x & & W_{xh}
\end{array}
\]
is commutative. So take an arrow \( p \in \phi_\varepsilon^{\alpha} = \mathcal{G}(x, \phi_\varepsilon(s(h))) \), then

\[
h \Psi(\sigma)_\varepsilon(v)_\alpha(p) = h \sigma_\alpha(p)v = \sigma_{\alpha\varepsilon}(\phi(h)p)v,
\]

because \( \sigma \) is \( \mathcal{H} \)-equivariant. On the other hand, we have that

\[
\Psi(\sigma)_\varepsilon(v)_\alpha \circ \phi_\varepsilon^{\alpha}(p) = \Psi(\sigma)_\varepsilon(v)_\alpha(\phi(h)p) = \sigma_{\alpha\varepsilon}(\phi(h)p).
\]

Therefore, \( \Psi(\sigma)_\varepsilon : \mathcal{V} \to \phi^*(\mathcal{W}) \), is a well defined linear map. Now take an arrow \( g \in \mathcal{G} \), we have to show that

\[
\Psi(\sigma)_\varepsilon \circ \gamma = \Gamma,
\]

which proves the commutativity of that diagram. □

Reciprocally, take a morphism \( \gamma \in \text{Hom}_\varepsilon(\mathcal{V}, \phi^*(\mathcal{W})) \), then for every object \( u \in \mathcal{H} \), we set

\[
\Phi(\gamma)_\alpha : \mathcal{V}_{\varepsilon\alpha} \longrightarrow \mathcal{W}_\varepsilon, \quad \left( v \mapsto \gamma_{\varepsilon\alpha}(v)_{\alpha}(\iota_{\varepsilon\alpha}) \right).
\]

**Lemma 4.2.** The family of linear maps \( \{ \Phi(\gamma)_\alpha \}_{\alpha \in \mathcal{H}} \) defines a natural transformation. That is, we have a morphism \( \Phi(\gamma) \in \text{Hom}_\varepsilon(\phi^*(\mathcal{V}), \mathcal{W}) \).

**Proof.** Let \( h \in \mathcal{H} \), and a vector \( v \in \mathcal{V}_{\varepsilon\alpha} \). Then, from one hand, we have

\[
\mathcal{W}^\varepsilon \circ \Phi(\gamma)_\alpha(v) = \mathcal{W}_\varepsilon \circ \gamma_{\varepsilon\alpha}(v)_{\alpha}(\iota_{\varepsilon\alpha}) = \gamma_{\varepsilon\alpha}(v)_\alpha(\phi(h)),
\]

where the second equality follows from the fact that \( \gamma_{\varepsilon\alpha}(v)_{\alpha}(\iota_{\varepsilon\alpha}) \in \text{Nat}(\phi_{\varepsilon\alpha}^{\varepsilon\alpha}, \mathcal{W}) \). On the other hand, since \( \gamma \in \text{Hom}_\varepsilon(\mathcal{V}, \phi^*(\mathcal{W})) \), we know that

\[
\gamma_{\varepsilon\alpha}(v)_\alpha(\phi(h)) = \gamma_{\varepsilon\alpha}(\phi(h)v)_\alpha, \quad \text{for every } u \in \mathcal{H} \text{ and } p \in \phi_{\varepsilon\alpha}^{\varepsilon\alpha} = \mathcal{G}(\phi_\varepsilon(t(h)), \phi_\varepsilon(u)).
\]

Substituting in equation (13), we get

\[
\Phi(\gamma)_\alpha \circ \gamma_{\varepsilon\alpha}(v)_{\alpha}(\iota_{\varepsilon\alpha}) = \gamma_{\varepsilon\alpha}(v)_\alpha(\phi(h)), \quad \text{for every } u \in \mathcal{H} \text{ and } v \in \mathcal{V}_{\varepsilon\alpha}.
\]

Therefore, for every \( h \in \mathcal{H} \) and \( v \in \mathcal{V}_{\varepsilon\alpha} \), this finishes the proof. □

**Proposition 4.3** (Right Frobenius reciprocity). Let \( \phi : \mathcal{H} \to \mathcal{G} \) be a morphism of groupoids and consider the restriction \( \phi_\varepsilon : \text{Rep}(\mathcal{G}) \to \text{Rep}(\mathcal{H}) \) and the induction \( \phi^* : \text{Rep}(\mathcal{H}) \to \text{Rep}(\mathcal{G}) \) functors. Then the maps \( \Psi \) and \( \Phi \) described, respectively, in Lemmata 4.1 and 4.2 define a natural isomorphism

\[
\text{Hom}_\varepsilon(\phi_\varepsilon(\mathcal{V}), \mathcal{W}) \cong \text{Hom}_\varepsilon(\mathcal{V}, \phi^*(\mathcal{W})),
\]

for every \( \mathcal{G} \)-representation \( \mathcal{V} \) and \( \mathcal{H} \)-representation \( \mathcal{W} \). In other words, the induction functor is a right adjoint functor of the restriction functor.
Proof. The naturality of both $\Psi$ and $\Phi$ are fulfilled by construction. Let us check that they are mutually inverse. So fixing $\sigma \in \Hom_{\mathcal{G}}(\phi_{\ast}(\mathcal{V}), \mathcal{W})$ and $\gamma \in \Hom_{\mathcal{H}}(\mathcal{V}, \phi_{\ast}(\mathcal{W}))$, for every $u \in \mathcal{H}$, and $v \in \mathcal{V}_{\phi(u)}$, we have that

$$
\Phi(\Psi(\sigma))(v) = \Psi(\sigma)(\gamma_{\phi(u)}(v)) = \sigma, (v),
$$

which implies that $\Phi \circ \Psi = \text{id}$. On the other way around, for every $x \in \mathcal{G}$, $w \in \mathcal{V}_{\ast}$, $u \in \mathcal{H}$ and $p \in \phi_{\ast}(u) = \mathcal{G}(x, \phi_{\ast}(u))$, we have that

$$
\Psi(\Phi(\gamma)) \circ \gamma_{\phi_{\ast}(u)}(p) = \Phi(\gamma)(p \circ \gamma_{\phi_{\ast}(u)}(p)) = \gamma_{\phi_{\ast}(u)}(p) = \gamma_{\phi_{\ast}(u)}(p)
$$

where in the third equality we have used the naturality of $\gamma$. Therefore, for every $x \in \mathcal{G}$ and $w \in \mathcal{V}_{\ast}$, we have checked that $\Psi(\Phi(\gamma))(w) = \gamma_{\phi_{\ast}(u)}(p)$. This means that $\Psi \circ \Phi = \gamma$, for an arbitrary $\gamma$, which implies that $\Phi \circ \Psi = \text{id}$ and this finishes the proof. \hfill \Box

4.2. Left Frobenius reciprocity formula. Keep the notations occurring in subsection [3.4]. Next we proceed to show that the co-induction functor is a left adjoint functor of the restriction functor. To this end, the subsequent lemma is needed. We consider then a morphism of groupoids $\phi : \mathcal{H} \to \mathcal{G}$.

**Lemma 4.4.** Let $\mathcal{V}$ and $\mathcal{W}$ be, respectively, a $\mathcal{G}$-representation and $\mathcal{H}$-representation. For any morphism $\theta \in \Hom_{\mathcal{G}}(\phi_{\ast}(\mathcal{W}), \mathcal{V})$, the family of $\mathcal{K}$-linear maps:

$$
\bigg\{ \Gamma(\theta)_{\ast} : \mathcal{W}_{\ast} = \mathcal{Y}_{\ast,0}(\mathcal{W})_{(\mathcal{V}_{\ast})_{0}} \xrightarrow{\mathcal{V}_{\ast}(\theta_{0})} \mathcal{V}_{\ast,0} \bigg\}
$$

defines a morphism $\Gamma(\theta) \in \Hom_{\mathcal{G}}(\mathcal{W}, \phi_{\ast}(\mathcal{V}))$.

**Proof.** Given an arrow $h \in \mathcal{H}$, we set $\overset{f}{\downarrow} = \eta_{(\mathcal{V}_{\ast})_{0}}^{(\mathcal{W}_{\ast})} \circ \mathcal{Y}_{\ast,0}(\theta_{0})$. Then, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{Y}_{\ast,0}(\mathcal{W})_{(\mathcal{V}_{\ast})_{0}} & \xrightarrow{\mathcal{V}_{\ast}(\theta_{0})} & \mathcal{V}_{\ast,0} \\
\eta^{h} & \downarrow{\overset{f}{\downarrow}} & \phi^{h} \\
\mathcal{Y}_{\ast,0}(\mathcal{W})_{(\mathcal{V}_{\ast})_{0}} & \xrightarrow{\mathcal{V}_{\ast}(\theta_{0})} & \mathcal{V}_{\ast,0}
\end{array}
$$

where the left hand square commutes, since the upper triangle is so by diagram [2], while the lower triangle commutes because of the limit defining $\mathcal{V}_{\ast,0}$ and because the $\zeta$ map acts by identities. This shows that $\Gamma(\theta) : \mathcal{W} \to \phi_{\ast}(\mathcal{V})$ is a natural transformation. \hfill \Box

In the other way around, consider $\delta \in \Hom_{\mathcal{G}}(\mathcal{W}, \phi_{\ast}(\mathcal{V}))$. For a fixed object $x \in \mathcal{G}_{s}$, we set the following family of $\mathcal{K}$-linear maps

$$
\bigg\{ \delta_{(\mathcal{V}_{\ast})_{0}}^{(\mathcal{W}_{\ast})} : \mathcal{Y}_{\ast,0}(\mathcal{W})_{(\mathcal{V}_{\ast})_{0}} = \mathcal{W}_{\ast} \xrightarrow{\delta_{(\mathcal{V}_{\ast})_{0}}} \mathcal{V}_{\ast,0} \bigg\}_{(\mathcal{V}_{\ast})_{0} \in \mathcal{K}_{\mathcal{H}}}
$$

where $\mathcal{K}_{\mathcal{H}}$ is, as in subsection [3.4] the right translation groupoid $\mathcal{H}^{-1}((x)) \rtimes \mathcal{H}$. It is from its own definition that the family $\bigg\{ \delta_{(\mathcal{V}_{\ast})_{0}}^{(\mathcal{W}_{\ast})} \bigg\}_{(\mathcal{V}_{\ast})_{0} \in \mathcal{K}_{\mathcal{H}}}$ is an inductive system of $\mathcal{K}$-vector spaces. Therefore, for every $x \in \mathcal{G}_{s}$, there is a unique $\mathcal{K}$-linear map

$$
\Sigma(\delta)_{x} = \lim_{(\mathcal{V}_{\ast})_{0} \in \mathcal{K}_{\mathcal{H}}} \bigg\{ \delta_{(\mathcal{V}_{\ast})_{0}}^{(\mathcal{W}_{\ast})} : \mathcal{Y}_{\ast,0}(\mathcal{W})_{(\mathcal{V}_{\ast})_{0}} \to \mathcal{V}_{\ast,0} \bigg\}
$$

(17)
such that $\Sigma(\delta) \circ v^\delta_{(a,u)} = \alpha^\delta_{(a,u)}$, for any object $(a, u)$ in $\mathcal{H}^{+\iota}$. Furthermore, given an arrow $g \in \mathcal{G}_l$ and an object $(a, u)$ in $\mathcal{H}^{+\iota}$, we have a commutative diagram

\[
\begin{array}{ccc}
V_{\delta g} & \xrightarrow{\gamma^\nu} & V_{\delta 0} \\
\sigma_{(\delta g)} & \downarrow & \sigma_{(\delta 0)} \\
\gamma_{(\delta g)0}(W)_{(a,u)} & \xrightarrow{g} & \gamma_{(\delta 0)0}(W)_{(a,u)}
\end{array}
\] (18)

where $\gamma^\nu_{(a,u)}$ is as in diagram (9).

**Lemma 4.5.** Let $\mathcal{W}$ and $\mathcal{V}$ be, respectively, an $\mathcal{H}$-representation and a $\mathcal{G}$-representation with a morphism $\delta \in \text{Hom}_\Sigma(\mathcal{W}, \phi_\Sigma(\mathcal{V}))$. Then the family of $k$-linear maps $\{\Sigma(\delta)_{\delta, \delta_0}\}$ defines a morphism $\Sigma(\delta) \in \text{Hom}_\varphi(\phi(\mathcal{W}), \mathcal{V})$.

**Proof.** For any arrow $g \in \mathcal{G}_l$ and an object $(a, u)$ in $\mathcal{H}^{+\iota}$, we have that the diagram

\[
\begin{array}{ccc}
V_{\delta g} & \xrightarrow{\gamma^\nu} & V_{\delta 0} \\
\Sigma(\delta g) & \downarrow & \Sigma(\delta 0) \\
\gamma_{(\delta g)0}(W)_{(a,u)} & \xrightarrow{g} & \gamma_{(\delta 0)0}(W)_{(a,u)}
\end{array}
\]

commutes by the definition of the involved maps and diagram (18). Therefore, the upper square should commutes as well, and this shows that $\Sigma(\delta)$ is compatible with the arrows of $\mathcal{G}$.

**Proposition 4.6 (Left Frobenius reciprocity).** Let $\phi : \mathcal{H} \rightarrow \mathcal{G}$ be a morphism of groupoids and consider the restriction $\phi_\Sigma : \text{Rep}_l(\mathcal{G}) \rightarrow \text{Rep}_l(\mathcal{H})$ and the co-induction $\phi' : \text{Rep}_l(\mathcal{H}) \rightarrow \text{Rep}_l(\mathcal{G})$ functors. Then the maps $\Gamma$ and $\Sigma$ described, respectively, in Lemmata 4.4 and 4.5, define a natural isomorphism

$$\text{Hom}_\varphi(\phi(\mathcal{W}), \mathcal{V}) \xrightarrow{\Sigma} \text{Hom}_\varphi(\phi(\mathcal{W}), \mathcal{V})$$

for every $\mathcal{G}$-representation $\mathcal{V}$ and $\mathcal{H}$-representation $\mathcal{W}$. In other words, the co-induction functor is a left adjoint functor of the restriction functor.

**Proof.** For a given $\delta \in \text{Hom}_\varphi(\phi(\mathcal{W}), \mathcal{V})$ and $u \in \mathcal{H}_\iota$, we know from Lemma 4.4 that

$$\Gamma(\Sigma(\delta))_u : V_{\delta 00}(W)_{(u, u)} \xrightarrow{v^\delta_{(u, u)}} \phi(\mathcal{W})_{(u, u)} \xrightarrow{\Sigma(\delta)_{00}} V_{\delta 00}$$

Therefore,

$$\Gamma(\Sigma(\delta))_u = \Sigma(\delta)_{00} \circ v^\delta_{(u, u)} = \gamma^\nu_{(u, u)} \circ \sigma_{(\delta 00)} = \delta_u$$

for every object $u \in \mathcal{H}_\iota$, and so $\Gamma \circ \Sigma = id$. Conversely, starting with a morphism $\theta \in \text{Hom}_\varphi(\phi(\mathcal{W}), \mathcal{V})$ and an object $x \in \mathcal{G}_l$, we have that

$$\Sigma(\Gamma(\theta))_x \xrightarrow{\lim_{(a,u) \in \mathcal{H}^{+\iota}}} (\mathcal{V}^\nu \circ \Gamma(\theta))_x \xrightarrow{\Sigma(\delta(\theta))_x} \mathcal{V}_{\delta 00}$$

Thus,

\[
\begin{align*}
\Sigma(\Gamma(\theta))_x & \xrightarrow{\lim_{(a,u) \in \mathcal{H}^{+\iota}}} (\mathcal{V}^\nu \circ \theta_{\delta(\theta)} \circ v^\delta_{(a,u)}) \\
& = \lim_{(a,u) \in \mathcal{H}^{+\iota}} (\theta_0 \circ \phi(\mathcal{W})' \circ v^\delta_{(a,u)}) \\
& = \lim_{(a,u) \in \mathcal{H}^{+\iota}} (\theta_0 \circ v^\delta_{(a,u)}) = \theta_0 \circ \lim_{(a,u) \in \mathcal{H}^{+\iota}} (v^\delta_{(a,u)}) = \theta_x
\end{align*}
\]
where in the third equality we have used the naturality of \( \theta \) and in the fourth one the diagram (2). This shows that \( \Sigma \circ \Gamma = id \) and finishes the proof.

**Remark 4.7.** As the expertise reader can observe, the construction of the induction and co-induction functors \( \phi^* \) and \( \phi^! \) corresponds, respectively, (up to natural isomorphisms) to the well known universal construction of the right and left Kan extensions of the functor \( \phi \), see [16] for more details. The proof presented here is somehow elementary and makes use of the groupoid structure, for instance the notion of translation groupoid among others. This also have the advantage of describing explicitly the natural isomorphisms establishing these adjunctions, which in fact is crucial to follow the arguments of the main result of the paper stated in the forthcoming section.

5. Frobenius extensions in groupoids context.

The main aim of this section is to characterize Frobenius morphism of groupoids, see Definition 5.1 below. This in fact is a kind of an universal definition which can be applied to any functor with left and right adjoints functors. Typical examples are Frobenius algebras over a given field (or commutative ring), where the forgetful functor form the category of modules to vector spaces has isomorphic left and right adjoint functors, namely, the tensor and the homs functors (see [13]).

**Definition 5.1.** Let \( \phi : \mathcal{H} \to \mathcal{G} \) be a morphism of groupoids. We say that \( \phi \) is a Frobenius morphism provided that the induction and the co-induction functors \( \phi^* \) and \( \phi^! \) are naturally isomorphic.

Before going on the characterization of a Frobenius morphism, it is convenient to recall the definition of paths algebras attached to groupoids, that is, the Gabriel rings with enough orthogonal idempotents associated to the underlying categories. So let \( \mathcal{G} \) be a groupoid and set

\[
B = \bigoplus_{x, x' \in \mathcal{G}_0} \mathbb{k}G(x, x')
\]

the direct sum of the \( \mathbb{k} \)-vector spaces \( \mathbb{k}G(x, x') \) with basis the set of arrows \( G(x, x') \). The image of any arrow \( b \in G(x, x') \) in \( B \) will be denoted by \( b \) itself, except for the identities arrow \( 1_x \) whose image will be denoted by \( 1_x \), for every \( x \in \mathcal{G}_0 \). The multiplication of \( B \) is given by the composition. Thus the multiplication \( b \cdot b' \) is the composition \( bb' \) if \( b \) and \( b' \) are composable, otherwise we set \( b \cdot b' = 0 \). In this way the set \( \{1_x\}_{x \in \mathcal{G}_0} \) forms a set of orthogonal idempotents which is a generating set for all others local units. That is, any local unit is a finite sum of the \( 1_x \)'s.

Let \( \phi : \mathcal{H} \to \mathcal{G} \) be a morphism of groupoids and denote by \( \phi : A \to B \) the associated morphism of rings, that is,

\[
\phi : A = \bigoplus_{u, u' \in \mathcal{H}_0} \mathbb{k}H(u, u') \longrightarrow B = \bigoplus_{x, x' \in \mathcal{G}_0} \mathbb{k}G(x, x'), \quad \left( \lambda h \mapsto \lambda \phi(h) \right)
\]

By scalars restriction, \( B \) is considered as an \( A \)-bimodule, although, this is not necessarily an unital one. The underlying vector spaces of the unital left, right \( A \)-module and \( A \)-bimodule parts of \( B \) are, respectively, given by the direct sums:

\[
AB = \bigoplus_{u \in \mathcal{H}_0, x \in \mathcal{G}_0} \mathbb{k}G(x, \phi(u)), \quad BA = \bigoplus_{u \in \mathcal{H}_0, x \in \mathcal{G}_0} \mathbb{k}G(\phi(u), x), \quad ABA = \bigoplus_{u, u' \in \mathcal{H}_0} \mathbb{k}G(\phi(u), \phi(u')). \quad (19)
\]

Since we are in groupoids context, it is clear that \( AB \cong BA \) as \( \mathbb{k} \)-vector spaces. We refer to [5], for details on unital modules and the notion of finitely generated and projective modules over rings with local units.

The following is the main result of this section and can be seen as a generalization to the groupoids context of the classical characterization of finite groups, see for instance [13].

**Theorem 5.2.** Given a morphism of groupoids \( \phi : \mathcal{H} \to \mathcal{G} \), then the following are equivalent.

(i) \( \phi \) is a Frobenius morphism;

(ii) There exists a natural transformation \( E : \mathcal{G}(\phi(u), \phi(v)) \longrightarrow \mathbb{k}H(u, v) \) in \( \mathcal{H}^{op} \times \mathcal{H} \), and for every \( x \in \mathcal{G}_0 \), there exists a finite set \( \{(u_i, b_i, c_i)\}_{i=1, \ldots, N} \in \mathcal{G}^{-1}((x)) \times \mathbb{k}G(x, \phi(u_i)) \) such that, for every pair of elements \( (b, b') \in \mathcal{G}(x, \phi(u)) \times \mathcal{G}(\phi(u), x) \), we have

\[
\sum_i E(bb_i)c_i = b \in \mathbb{k}G(x, \phi(u)) \quad \text{and} \quad b' = \sum_i b_iE(c_ib') \in \mathbb{k}G(\phi(u), x).
\]
(iii) For every \( x \in \mathcal{G}_u \), the left unital A-module \( AB_1 \) is finitely generated and projective and there is a natural isomorphism \( B_1 u \cong B \text{Hom}_{\mathcal{A}}(AB, A_1) \), of left unital B-modules, for every \( u \in \mathcal{H}_v \).

The proof of this theorem will done in several steps following the path: (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

5.1. The proof of (i) ⇒ (ii) in Theorem 5.2 Assume that there is a natural isomorphism \( \phi \cong \phi' \). This in particular implies that, for every \( x \in \mathcal{G}_u \), there is a natural isomorphism

\[
\text{Nat}(\phi, (-)) \cong \lim_{\eta \rightarrow \phi} (\mathcal{F}_\eta^+(-))
\]

(20)

Now, for a given object \( u \in \mathcal{H}_v \) let us denote by \( \mathcal{H}_u : \mathcal{H} \rightarrow \text{Vect} \), the \( \mathcal{H} \)-representation given over objects by \( v \rightarrow \mathcal{K} \mathcal{H}(u, v) \), and obviously defined over arrows. The natural isomorphism of (20) leads then to a family of bijections

\[
\text{Nat}(\phi', \mathcal{H}_u) \cong \lim_{\eta \rightarrow \phi} (\mathcal{F}_\eta^+(\mathcal{H}_u))
\]

natural in \( u \in \mathcal{H}_v \). In the previous situation we have

**Lemma 5.3.** For any \( x \in \mathcal{G}_u \) and \( u \in \mathcal{H}_v \), there is an isomorphism

\[
\lim_{\eta \rightarrow \phi} (\mathcal{F}_\eta^+(\mathcal{H}_u)) \cong \mathcal{K} \mathcal{G}(\phi(u), x),
\]

which is natural in both components \((u, x) \in \mathcal{H}_v \times \mathcal{G} \).

**Proof.** Fix for the moment \( x \) and \( u \) as in the statement. We need to check that the vector space \( \mathcal{K} \mathcal{G}(\phi(u), x) \) is the inductive limit of the inductive system

\[
\left\{ \mathcal{F}_\eta^+(\mathcal{H}_u)_{h_{\eta, \phi}} = \mathcal{K} \mathcal{H}(v, u) \right\}_{(h_{\eta, \phi}) \in \mathcal{C}^{1}_{1}(0, 0)}.
\]

Let us first define an inductive cone over \( \mathcal{K} \mathcal{H}(x, \phi(u)) \). So take \((b, v) \in \mathcal{C}^{1}_{1}(\{x\})\), that is, \(b \in \mathcal{G}(\phi(v), x)\), then we have a linear map

\[
\tau_{(b, v)} : \mathcal{K} \mathcal{H}(u, v) \rightarrow \mathcal{K} \mathcal{G}(\phi(u), x), \quad \left(\sum_i a_i \rightarrow \sum_i \lambda b \phi(a_i)\right)
\]

which is clearly compatible with the arrows of the groupoid \( \mathcal{H}^{o+} \), and this gives us the desired inductive cone. We need then to check that this is an initial object among others cones. So given an arbitrary cone \( \{\xi_{(b, v)} : \mathcal{K} \mathcal{H}(u, v) \rightarrow V\}_{(b, v) \in \mathcal{C}^{1}_{1}(0, 0)} \), for any \( b \in \mathcal{G}(\phi(u), x) \), we can consider the vector \( \xi_{(b, v)}(1_b) \in V \); whence we have a linear map

\[
\xi : \mathcal{K} \mathcal{G}(\phi(u), x) \rightarrow V, \quad \left(b \mapsto \xi_{(b, v)}(1_b)\right)
\]

It turns out that this is a morphism of inductive cones and this finishes the proof of the lemma. \( \square \)

By Lemma 5.3 the natural isomorphisms of equation (20), lead to natural isomorphism

\[
\text{Nat}(\phi', \mathcal{H}_u) \cong \mathcal{K} \mathcal{G}(\phi(u), x),
\]

for every \( x \in \mathcal{G}_u \) and \( u \in \mathcal{H}_v \). The following general lemma characterizes these kind of natural transformations.

**Lemma 5.4.** Let \( F : \mathcal{C} \rightarrow \mathcal{D} \) be a covariant functor between small categories. Then there is a natural isomorphism

\[
\text{Nat}\left(D(F(-), +), \text{Nat}\left(D(+, F(\bullet)), \mathcal{K} \mathcal{E}(-, \bullet)\right)\right) \cong \text{Nat}\left(D(F(\hat{\tau}), F(\hat{\xi})), \mathcal{K} \mathcal{E}(\hat{\tau}, \hat{\xi})\right)
\]

**Proof.** Is left to the reader. \( \square \)

In our case, we have that any natural isomorphism \( \Theta_{(b, v)} : \mathcal{G}(\phi(u), x) \rightarrow \text{Nat}(\phi', \mathcal{H}_u) \) gives a rise to a natural transformation

\[
E_{(b, v)} : \mathcal{G}(\phi(v), \phi(w)) \rightarrow \mathcal{K} \mathcal{H}(v, w), \quad \left(b \mapsto \Theta_{(b, v)}(b_\phi)(1_{\phi(w)})\right).
\]

(21)
On the other hand, the left hand functor in the isomorphism of equation (20), \( \text{Nat}(\varphi', (-)) \) should preserves colimits, since the involves categories are Grothendieck ones with a set of small projective generators. Namely, in the case of \( \text{Rep}(\mathcal{H}) \) this set is given by the family of representations \( \{\mathcal{H}_L\}_{L \in \mathcal{G}_0} \). In this way, saying that \( \text{Nat}(\varphi', (-)) \) preserves colimits, is equivalent to say that the object \( \mathcal{H} \varphi' : \mathcal{H} \rightarrow \text{Vect} \) is finitely generated and projective. Therefore, we are assuming that there exists a positive integer \( N \geq 1 \) and a split monomorphism \( \mathcal{H} \varphi' \hookrightarrow \oplus_{i=1,...,N}\mathcal{H}_{L_i} \). Hence, there is a set \( \{L_i\}_{i \in \mathcal{I}} \) such that each of the \( L_i \)'s belong to \( \mathcal{H}(x, \varphi(u)) \), and natural transformations \( \varphi'_c : \mathcal{H}(x, \varphi(-)) \rightarrow \mathcal{H}(u, \varphi(-)) \), such that, for every \( b \in \mathcal{H}(x, \varphi(u)) \), we have that

\[
\sum_i \varphi'_c(b_c) c_i = b,
\]
equality in the vector space \( \mathcal{H}(x, \varphi(u)) \). In this direction, one can consider the family of elements \( \{b_i\}_{i=1,...,N} \) where each \( b_i = \Theta^{-1}_{L_i}(\varphi'_c) \in \mathcal{H}(\varphi(u), x) \). Using the natural transformations \( E \) of (21) together with the properties which \( \Theta \) satisfies, we obtain

\[
\sum_i E_{\varphi_{ij}}(bb_i) c_i = b,
\]
for every \( b \) as above. Take an element \( b' \in \mathcal{G}(\varphi(u), x) \), then for every \( c \in \mathcal{G}(x, \varphi(v)) \) with \( v \in \mathcal{H} \), we have that

\[
\Theta_{\alpha,\nu} \left( \sum_i b_i E_{\alpha,\nu}(c_i b') \right)(c) = \sum_i \Theta_{\alpha,\nu}(b_i)(c) E_{\alpha,\nu}(c b')
\]

\[
= \sum_i \Theta_{\alpha,\nu}(\Theta^{-1}_{L_i}(\varphi'_c))(c) E_{\alpha,\nu}(c b')
\]

\[
= \sum_i \varphi'_c(c) E_{\alpha,\nu}(c b') = \sum_i E_{\alpha,\nu}(\varphi'_c c b')
\]

\[
= \Theta_{\alpha,\nu}(cb')(1_{\varphi_{ij}}) = \Theta_{\alpha,\nu}(b')(c).
\]

Thus, \( \Theta_{\alpha,\nu}(\sum_i b_i E_{\alpha,\nu}(c_i b')) = \Theta_{\alpha,\nu}(b') \) and so \( \sum_i b_i E_{\alpha,\nu}(c_i b') = b' \), which completes the proof of (ii).

5.2. The proof of (ii) \( \Rightarrow \) (iii) in Theorem 5.2. For every \( x \in \mathcal{G}_x \), it is clear that we have

\( AB1, = \Theta_{\alpha,\nu}\mathcal{H}(x, \varphi(u)) \).

Define left \( A \)-linear maps \( ^*e_i : AB1 \rightarrow A_{L_i} \) by sending \( b \in \mathcal{G}(x, \varphi(u)) \mapsto E_{\varphi_{ij}}(bb_i) \). Then, by hypothesis, the set \( \{^*e_i, c_i\}_{i=1,...,N} \) is a dual basis for the left \( A \)-module \( AB1 \). Each of the modules \( AB1 \) is finitely generated and projective, which is the first statement of part (iii).

Now fixing \( u \in \mathcal{H} \), we have a well defined left \( B \)-linear map

\[
\Psi_u : B_{A_{L_i}} \rightarrow B\text{Hom}_u(AB, A_{L_i}), \quad (b_{A_{L_i}} \mapsto [ab' \mapsto E(ab'b)]_{A_{L_i}})
\]

which by the naturality of \( E \) it is also natural. The maps \( \Psi \) are bijective as they have inverses given by: \( \text{U} : B\text{Hom}_u(AB, A_{L_i}) \rightarrow B_{A_{L_i}} \) sending \( a \mapsto \sum_i b_i \alpha(c_i) \). Indeed, for every \( a \in A \) and \( b' \in B \) with \( ab' \in AB1_{A_{L_i}} \), we have

\[
\Psi \text{U}(a)(ab') = \Psi(\sum_i b_i \alpha(c_i))(ab')
\]

\[
= \sum_i E(ab'b_i \alpha(c_i)) = \sum_i aE(b'_i \alpha(c_i)) = \sum_i aaE(b'_i c_i) = a(ab'),
\]

and so \( \Psi \text{U} = id \). In the other way around, we have that \( \text{U}\Psi(b_{A_{L_i}}) = \sum_i b_i \Psi(b_{A_{L_i}})(c_i) = \sum_i b_i E(c_i b) = b \).

This gives the desired isomorphism and finishes the proof of part (iii).
5.3. The proof of (iii) ⇒ (i) in Theorem 5.2. To show this implication, it is sufficient to check, from one hand, that each of the functors $\Nat(\phi', (-)) : \Rep_1(\mathcal{H}) \to \text{Vect}_\kappa$, with $x \in \mathcal{G}_x$, preserves colimits, and from another establish a natural isomorphism $\Gamma_{\mathcal{G}_x} : \mathcal{G}(\phi(x), x) \cong \Nat(\phi', \mathcal{H}_x)$, for every pair $(u, x) \in \mathcal{H}_x \times \mathcal{G}_x$.

The fact that $\Nat(\phi', (-))$ preserves colimits, is deduced by using the first statement of (iii) and the natural isomorphism

$$\text{Hom}_x(BA_1, M) \to \Nat(\phi', \mathcal{O}(M)), \quad \left(f \mapsto f'(M) : (b \mapsto f(b))\right)_{x \in \mathcal{G}_x}$$

(22) where $\mathcal{O'} : \mathcal{A}\text{-Mod} \to \Rep_1(\mathcal{H})$ is the inverse functor of the following (symmetric monoidal) isomorphism of categories

$$\mathcal{O}^{-1} : \Rep_1(\mathcal{H}) \to \mathcal{A}\text{-Mod}, \quad \left(W \to \oplus_{u \in \mathcal{H}_x} W_u\right),$$

where $\mathcal{A}\text{-Mod}$ denotes the category of unital left $\mathcal{A}$-modules. In particular, this induces a natural isomorphism

$$\text{Hom}_x(BA_1, A_{1\mathcal{A}}) \cong \Nat(\phi', \mathcal{H}_x).$$

As for the second condition, if we assume that there is a left $\mathcal{B}$-linear natural isomorphism $\Phi_\mathcal{B} : \mathcal{B}1_{\mathcal{A}} \cong B\text{Hom}_x(B, A_{1\mathcal{A}})$, for every $u \in \mathcal{H}_x$, then we can consider $\Gamma_{\mathcal{G}_x} : \mathcal{G}(\phi(u), x) \to \Nat(\phi', \mathcal{H}_x)$ to be defined by

$$\Gamma_{\mathcal{G}_x} : \mathcal{G}(\phi(u), x) \to \Nat(\phi', \mathcal{H}_x), \quad \left(b \mapsto \left[1_x, \Phi_\mathcal{B}(b) : \phi_x^* \to \mathcal{H}(u, v), \left(b' \mapsto \Phi_\mathcal{B}(b)(b'1_x)\right)\right]_{x \in \mathcal{G}_x}\right).$$

The fact that $\Gamma_{\mathcal{G}_x} \in \Nat(\phi', \mathcal{H}_x)$, follows directly from the fact that $\Phi_\mathcal{B}(b)$ is a left $\mathcal{A}$-linear for every $b \in \mathcal{G}(\phi(u), x)$. The naturality of $\Gamma$, that is, the commutativity of the diagrams of the form

$$\xymatrix{ \mathcal{G}(\phi(u), x) \ar[r] \ar[d] & \Nat(\phi', \mathcal{H}_x) \\
\mathcal{G}(\phi(u'), x') \ar[r] & \Nat(\phi', \mathcal{H}_x)}$$

for every pairs $a \in \mathcal{H}(u', u)$ and $g \in \mathcal{G}(x, x')$, is computed as follows. Take an element $b \in \mathcal{G}(\phi(u), x)$, then, for every object $v \in \mathcal{H}_x$ and $b' \in \phi_x^*$, $\mathcal{G}(x, v)$, we have

$$\Nat(\phi', \mathcal{H}_x) \circ \Gamma_{\mathcal{G}_x}(b) = \Gamma_{\mathcal{G}_x}(b) = \Phi_\mathcal{B}(b)(\phi_x^*) a = \Phi_\mathcal{B}(b)(b'1_a) = \Phi_\mathcal{B}(b(a1_{\mathcal{A}})(b'g)).$$

On the other hand, we have

$$\Gamma_{\mathcal{G}_x}(b' \circ \phi_x^* (gb\phi(a)), (b')) = \Phi_\mathcal{B}(gb\phi(a))(b') = \Phi_\mathcal{B}(b\phi(a))(b'g).$$

Comparing the two computations shows the commutativity of that diagram. Lastly, the inverse of $\Gamma_{\mathcal{G}_x}$ is provided by that of $1_{\Phi_\mathcal{B}}$, combined with the inverse of the natural isomorphism of equation (22).

The notion of a finite groupoid is vague, in the sense that there are several interrelated notions and all generalize that of a finite group. Next we adopt the following one: a groupoid $\mathcal{G}$ is said to be finite, provided that $\pi_0(\mathcal{G})$ is a finite set as well as each of its isotropy groups $\mathcal{G}^v$ when $x$ runs in $\mathcal{G}_x$. This is the case when for instance $\mathcal{G}$ is a finite set.

Given a morphism of groupoids $\phi : \mathcal{H} \to \mathcal{G}$, we recall from Example 2.10 that we have the $(\mathcal{G}, \mathcal{H})$-biset $\mathcal{B}^s(\mathcal{G}) : \mathcal{G}_x \times \mathcal{H}_x \to \mathcal{G}_x$, with structure maps $\zeta : \mathcal{B}^s(\mathcal{G}) \to \mathcal{G}, (a, u) \mapsto \phi(a)$ and $pr_1 : \mathcal{B}^s(\mathcal{G}) \to \mathcal{H}_x, (a, u) \mapsto u$. Similarly, we have $\mathcal{B}^s(\mathcal{G}) : \mathcal{H}_x \times \mathcal{G}_x \to \mathcal{G}_x$, an $(\mathcal{H}, \mathcal{G})$-biset with structure maps $\phi : \mathcal{B}^s(\mathcal{G}) \to \mathcal{G}, (u, a) \mapsto s(a)$ and $pr_1 : \mathcal{B}^s(\mathcal{G}) \to \mathcal{H}_x, (u, a) \mapsto u$. These are the right and the left pull-back biset.

The following corollary characterizes Frobenius extension by subgroupoids, in terms of finiteness of the orbits of the fibres of the pull-back biset, which in particular applies to the case of finite groupoids.

**Corollary 5.5.** Let $\phi : \mathcal{H} \to \mathcal{G}$ be a morphism of groupoids with a faithful underlying functor, that is, $\phi_x$ is an injective map. Then the following are equivalent:

(a) $\phi$ is a Frobenius extension;
(b) For any $x \in \mathcal{G}_x$, the left $\mathcal{H}$-set $\phi^{-1}((x))$ has finitely many orbits;
(c) For any $x \in \mathcal{G}_x$, the right $\mathcal{H}$-set $\phi^{-1}((x))$ has finitely many orbits.

In particular, any inclusion of finite groupoids is a Frobenius extension.
Proof. The equivalence between (c) and (b), follows directly from the isomorphism of left $\mathcal{H}$-sets:

$$\vartheta^{-1}((x)) \longrightarrow \varsigma^{-1}((x))$$

where $\varsigma^{-1}((x))$ is the opposite left $\mathcal{H}$-set of $\vartheta^{-1}((x))$.

The implication $(a) \Rightarrow (b)$ is derived as follows from the first condition of Theorem 5.2 (iii). For any $x \in \mathcal{G}_\alpha$, we know that

$$AB1_x = \bigoplus_{u \in \mathcal{G}_\alpha} \mathbb{k} \mathcal{G}(x, \varphi(u)) \cong \mathbb{k} \vartheta^{-1}((x)),$$

Therefore, as left $A$-module, $AB1_x$ can be decomposed as direct sum of cyclic $A$-submodules of the form

$$AB1_x = \bigoplus_{(u,q) \in \text{rep}_{\mathcal{H}}(\vartheta^{-1}((x)))} Aq$$

where $\text{rep}_{\mathcal{H}}(\vartheta^{-1}((x)))$ is a set of representative classes modulo the left $\mathcal{H}$-action on the fibre $\vartheta^{-1}((x))$, and $Aq$ is the $\mathbb{k}$-vector space spanned by the orbit set $\text{Orb}_p(u, q)$. Since $AB1_x$ is finitely generated and projective, this direct sum should be then finite, which means that $\text{rep}_{\mathcal{H}}(\vartheta^{-1}((x)))$ is a finite set and this is precisely condition (b).

Conversely, assume that for any $x \in \mathcal{G}_\alpha$, we know that $\vartheta^{-1}((x))$ has finitely many orbits. Choose a finite set of representatives classes $\{(u_i, q_i, q_{i+1})\}_{i=0}^n$ such that

$$\vartheta^{-1}((x)) = \sum_{i=1}^n \text{Orb}_p(u_i, q_i),$$

see [7]. Using the left $\mathcal{H}$-equivariant isomorphism of (23), we also have

$$\varsigma^{-1}((x)) = \sum_{i=1}^n \text{Orb}_p(q_{i-1}, u_i).$$

In this way, we obtain as above a decomposition of unital $A$-modules

$$AB1_x = \mathbb{k} \vartheta^{-1}((x)) = \bigoplus_{i=1}^n Aq_i \quad \text{and} \quad 1BA = \mathbb{k} \varsigma^{-1}((x)) = \bigoplus_{i=1}^n q_i^{-1}A.$$ (24)

Let us denote by $\varphi_i : AB1_x \rightarrow Aq_i$ and $\psi_i : 1BA \rightarrow q_i^{-1}A$ the canonical projections given by the decompositions of (24). Take $x$ to be of the form $\phi(u)$, for some $u \in \mathcal{H}_\alpha$. Then the element $(u, \iota_{u\alpha \beta}) \in \vartheta^{-1}((\phi(u)))$ and $(\iota_{u\alpha \beta}, u) \in \varsigma^{-1}((\phi(u)))$. Thus, if $\text{Orb}_{\phi}(u, \iota_{u\alpha \beta})$ is equal to some orbit of the form $\text{Orb}_p(u, q_{i, \alpha \beta})$, for some $i$, then necessarily this $q_{i, \alpha \beta}$ belongs to $\phi(\mathcal{H}_\alpha)$ (recall here that $\phi$ is assumed to be injective). Therefore, we can assume that at least one of the $q_{i, \alpha \beta}$'s is $\iota_{u\alpha \beta}$. In this way, we set $q_{1, \alpha \beta} := \iota_{u\alpha \beta}$. We are then lead to the following $A$-linear maps

$$\varphi_{1, \alpha \beta} : AB1_{\alpha \beta} \rightarrow A1_{\alpha \beta} \quad \text{and} \quad \psi_{1, \alpha \beta} : 1_{\alpha \beta}BA \rightarrow 1_{\alpha \beta}A.$$ (25)

If we take an element $p \in G(\phi(u), \phi(v))$, for $u, v \in \mathcal{H}_\alpha$, then we have $\varphi_{1, \alpha \beta}(p) \in \mathbb{k}\mathcal{H}(u, v)$. This in fact establishes a natural transformation

$$E_{\alpha \beta} : G(\phi(u), \phi(v)) \longrightarrow \mathbb{k}\mathcal{H}(u, v), \quad (p \mapsto \varphi_{1, \alpha \beta}(p)).$$ (26)

On the other hand, if we take an element $b' \in G(\phi(u), x)$, for some $x \in \mathcal{G}_\alpha$ and $u \in \mathcal{H}_\alpha$, then we can write $b' = \lambda q_{i, \alpha \beta}^{-1}a_i + \cdots + \lambda q_{n, \alpha \beta}^{-1}a_n$, where all the scalars $\lambda$’s vanish except the one which correspond exactly to the orbit that contains $b$. Now, each of the element $q_{i, \alpha \beta}$’s has the image $E(q_{i, \alpha \beta}) = \lambda a_i$. Therefore, we have that $b' = \sum_j q_{j, \alpha \beta}E(q_{j, \alpha \beta})$ as an element in the homogeneous component $\mathbb{k}G(x, \phi(u))$ of $B$. If we take now an element $b \in G(x, \phi(u))$, for some $x \in \mathcal{G}_\alpha$ and $u \in \mathcal{H}_\alpha$, then the same arguments will show that $b = \sum_j E(bq_{j, \alpha \beta})q_{j, \alpha \beta}$, as an element in the component $\mathbb{k}G(x, \phi(u))$. In summary, we have shown condition (ii) of Theorem 5.2 and thus $\phi$ is a Frobenius extension. The particular statement is now immediate, and this finishes the proof.

$\square$
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