Infinite Ramsey-minimal graphs for star forests

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Abstract

For graphs $F$, $G$, and $H$, we write $F \to (G, H)$ if every red-blue coloring of the edges of $F$ produces a red copy of $G$ or a blue copy of $H$. The graph $F$ is said to be $(G, H)$-minimal if it is subgraph-minimal with respect to this property. The characterization problem for Ramsey-minimal graphs is classically done for finite graphs. In 2021, Barrett and the second author generalized this problem to infinite graphs. They asked which pairs $(G, H)$ admit a Ramsey-minimal graph and which ones do not. We show that any pair of star forests such that at least one of them involves an infinite-star component admits no Ramsey-minimal graph. Also, we construct a Ramsey-minimal graph for a finite star forest versus a subdivision graph. This paper builds upon the results of Burr et al. in 1981 on Ramsey-minimal graphs for finite star forests.

Key words: Ramsey-minimal graph, infinite graph, graph embedding, star forest, subdivision graph

2020 MSC: 05C55, 05C63, 05C35, 05C60, 05D10

1. Introduction

All our graphs are simple and undirected, and we allow uncountable graphs. We start by stating basic definitions. For graphs $F$, $G$ and $H$, we write $F \to (G, H)$ if every red-blue coloring of the edges of $F$ produces a red copy of $G$ or a blue copy of $H$. A red-blue coloring of $F$ is $(G, H)$-good if it produces neither a red copy of $G$ nor a blue copy of $H$. If $F \to (G, H)$ and every subgraph $F'$ of $F$ is such that $F' \not\to (G, H)$, then $F$ is $(G, H)$-minimal. The collection of all $(G, H)$-minimal graphs is denoted by $\mathcal{R}(G, H)$. A pair $(G, H)$ admits a Ramsey-minimal graph if $\mathcal{R}(G, H)$ is nonempty.

If $G$ and $H$ are both finite, then a $(G, H)$-minimal graph exists. Indeed, we can delete finitely many vertices and/or edges of $K_{r(G, H)}$ until it is $(G, H)$-minimal. This observation does not necessarily hold when at least one of $G$ and $H$ is infinite, even though there exists a graph $F$ such that $F \to (G, H)$ in the countable case by the Infinite Ramsey Theorem [20] and in general by the...
Erdős–Rado Theorem \[14\]. In fact, a pair of countably infinite graphs almost never admits a Ramsey-minimal graph—see Proposition \[2,4\]. In 2021, Barrett and the second author \[1\] introduced a characterization problem for pairs of graphs according to whether or not they admit a minimal graph.

**Main Problem** (\[1\]). Determine which pairs \((G, H)\) admit a Ramsey-minimal graph and which ones do not.

The primary motivation for posing the main problem is the classic problem of determining whether there are finitely or infinitely many \((G, H)\)-minimal graphs. This problem was first introduced in 1976 \[11, 17\], and it was studied for finite graphs in general by Nešetřil and Rödl \[18, 19\] and for various classes of graphs by Burr et al. \[5, 6, 7, 9, 12\]. A result by Burr et al. on Ramsey-minimal graphs for finite star forests is relevant to our discussion.

**Theorem 1.1** (\[8\]). The pair of star forests \((\bigcup_{i=1}^{s} S_{n_i}, \bigcup_{j=1}^{t} S_{m_j})\) admits infinitely many Ramsey-minimal graphs for \(n_1 \geq \cdots \geq n_s \geq 2\) and \(m_1 \geq \cdots \geq m_t \geq 2\) when \(s \geq 2\) or \(t \geq 2\).

The formulation of the main problem is also motivated by the more recent work of Stein \[22, 24, 25\] on extremal infinite graph theory. It is a subfield of extremal graph theory that developed after the notion of end degrees was introduced a few years prior \[4, 22\].

Barrett and the second author mainly studied the main problem for pairs \((G, H)\) in general. The following is one of the main results presented in their paper.

**Theorem 1.2** (\[1\]). Let \(G\) and \(H\) be graphs, and suppose that \(\mathcal{F}\) is a (possibly infinite) collection of graphs such that:

1. For all \(F \in \mathcal{F}\), we have \(F \rightarrow (G, H)\).
2. For every graph \(\Gamma\) with \(\Gamma \rightarrow (G, H)\), there exists an \(F \in \mathcal{F}\) that is contained in \(\Gamma\).

The following statements hold:

(i) If \(F\) is a \((G, H)\)-minimal graph, then \(F \in \mathcal{F}\) and \(F\) is non–self-embeddable.

(ii) Suppose that any two different graphs \(F_1, F_2 \in \mathcal{F}\) do not contain each other. A graph \(F\) is \((G, H)\)-minimal if and only if \(F \in \mathcal{F}\) and \(F\) is non–self-embeddable.

This paper instead focuses on pairs \((G, H)\) involving a **star forest**—a union of stars. Our first main result shows that any pair of star forests such that at least one of them involves an infinite-star component admits no Ramsey-minimal graph.

**Theorem 1.3.** Let \(G\) and \(H\) be star forests. If at least one of \(G\) and \(H\) contains an infinite-star component, then no \((G, H)\)-minimal graph exists.
Figure 1: The graph $F_d$. 

This theorem is in contrast to Theorem 1.1, which states that there are infinitely many $(G, H)$-minimal graphs when $G$ and $H$ are disconnected finite star forests with no single-edge components. Loosely speaking, the existence of infinitely many finite minimal graphs hence does not give an indication that a corresponding infinite minimal graph exists.

Similarly, the existence of only finitely many finite minimal graphs does not imply that there are only finitely many corresponding infinite minimal graphs. For $n \in \mathbb{N}$—where $\mathbb{N}$ is the set of positive integers—we denote the $n$-edge star by $S_n$. It is known from [10] that there are only finitely many $(nS_1, H)$-minimal graphs for $n \in \mathbb{N}$ and $H$ a finite graph. On the other hand, if $Z$ is the double ray—the two-way infinite path—then $(2S_1, Z \cup S_3)$ admits infinitely many minimal graphs. Indeed, we have $2F_d \in \mathcal{R}(2S_1, Z \cup S_3)$ for every $d \geq 3$, where $F_d$ is the graph illustrated in Figure 1.

A graph is leafless if it contains no vertex of degree one, and it is non–self-embeddable if it is not isomorphic to any proper subgraph of itself. Following [10, p. 79], we denote the subdivision graph of $G$ by $S(G)$, which is a graph obtained from $G$ by performing a subdivision on each one of its edges. For example, if $P_n$ denotes the $n$-vertex path, then $S(P_n) = P_{2n-1}$ for $n \in \mathbb{N}$.

For our second main result, we construct a Ramsey-minimal graph for a finite star forest versus the subdivision graph of a connected, leafless, non–self-embeddable graph. In 2020, subdivision graphs were used by Wijaya et al. [26] to construct new $(nS_1, P_4)$-minimal graphs.

**Theorem 1.4.** Let $G$ be a connected, leafless, non–self-embeddable graph. For any finite star forest $H$, there exists a $(S(G), H)$-minimal graph.

For future investigation, it would be interesting to consider whether every pair of non–self-embeddable graphs admits a minimal graph. If true, this would generalize the observation that a pair of finite graphs always admits a minimal graph, since finite graphs are non–self-embeddable.

**Question 1.5.** Is it true that every pair $(G, H)$ of non–self-embeddable graphs admits a Ramsey-minimal graph?

We give an outline of this paper. Section 2 discusses self-embeddable graphs and their relevance to the study of Ramsey-minimal graphs. In Section 3 we briefly discuss the Ramsey-minimal properties of $(G, H)$ when $H$ is a union of graphs. Finally, our two main theorems are proved in Sections 4 and 5.
2. Self-embeddable graphs

We first provide several preliminary definitions. A graph homomorphism \( \varphi: G \to H \) is a map from \( V(G) \) to \( V(H) \) such that \( \varphi(u)\varphi(v) \in E(H) \) whenever \( uv \in E(G) \). A graph homomorphism is an embedding if it is an injective map of vertices. Following [23], we write \( G \leq H \) if \( G \) embeds into \( H \); that is, there exists an embedding \( \varphi: G \to H \). Unlike in [23], however, we do not require that the graph image of \( \varphi \) is an induced subgraph of \( H \).

A graph \( G \) is self-embeddable if \( G \sim G' \) for some proper subgraph \( G' \) of \( G \), and the corresponding isomorphism \( \varphi: G \to G' \) is its self-embedding. Examples of self-embeddable graphs include the ray \( \mathbb{N} \)—the one-way infinite path—and a complete graph on infinitely many vertices. On the other hand, finite graphs and the double ray \( \mathbb{Z} \) are non–self-embeddable.

Proposition 2.1 provides a necessary and sufficient condition for a graph to be self-embeddable in terms of its components. This proposition is quite similar to [21, Theorem 2.5] for self-contained graphs, the “induced” version of self-embeddable graphs.

Proposition 2.1. A graph \( G \) is self-embeddable if and only if at least one of the following statements holds:

(i) There exists a self-embeddable component of \( G \).

(ii) There exists a sequence of distinct components \( (C_i)_{i \in \mathbb{N}} \) of \( G \) such that \( C_1 \leq C_2 \leq \cdots \).

Proof. The backward direction can be easily proved by defining a suitable self-embedding of \( G \) for each of the two cases; it remains to show the forward direction.

Suppose that \( G \) has a self-embedding \( \varphi \) that embeds \( G \) into \( G - p \), where \( p \) is either a vertex or an edge of \( G \), and \( G \) contains no self-embeddable component. Let \( C_0 \) be the component of \( G \) containing \( p \). We write \( v \simeq w \) if the vertices \( v \) and \( w \) belong to the same component, and we denote \( \varphi^k \) as the \( k \)-fold composition of \( \varphi \).

We claim that if \( u \in V(C_0) \), then for \( 0 \leq i < j \), we have \( \varphi^i(u) \neq \varphi^j(u) \). We use induction on \( i \). Let \( i = 0 \), and suppose to the contrary that \( u \) and \( \varphi^i(u) \), where \( j > 0 \), both belong to the same component, and we denote \( \varphi^k \) as the \( k \)-fold composition of \( \varphi \).

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so $\varphi^{j-i}(v) \in V(C)$ for every $v \in V(C)$. We now prove that $\varphi^{j-i}$ carries $C$ into $C - \varphi(u)$; this would contradict the non–self-embeddability of $C$. Suppose that $\varphi^{j-i}(v) = \varphi'(u)$ for some vertex $v$. We have $\varphi^{j-i-1}(v) = \varphi^{i-1}(u)$ by injectivity. By the induction hypothesis, we also have $\varphi^{i-1}(u) \not\equiv \varphi^{j-1}(u)$, so $\varphi^{j-i-1}(v) \not\equiv \varphi^{j-1}(u)$. It follows that $v \not\equiv \varphi'(u)$, and since $\varphi'(u) \in V(C)$, we infer that $v \not\in V(C)$. Therefore, $\varphi'(u)$ cannot be the image of a vertex of $C$ under $\varphi^{j-i}$, as desired.

Let $u \in V(C_0)$. Define a sequence $(C_i)_{i \in \mathbb{N}}$ such that $C_i$ is the component containing $\varphi'(u)$. This sequence consists of pairwise distinct components by the previous claim. It is clear that $\varphi$ carries $C_i$ to $C_{i+1}$, so $C_i \leq C_{i+1}$ for $i \in \mathbb{N}$, and we are done.

Proposition 2.1 implies, as an example, that the union of finite paths is self-embeddable, but the union of finite cycles with different lengths is not. Also, we obtain the following corollary.

**Corollary 2.2.** A star forest is self-embeddable if and only if it is infinite.

For nonempty graphs $G$, a stronger property than self-embeddability is the property that $G \leq G - e$ for all $e \in E(G)$. The ray and an infinite complete graph, for example, enjoy this stronger property. On the other hand, the disjoint union $\mathbb{N} \cup \mathbb{Z}$ is self-embeddable, but does not embed into $\mathbb{N} \cup (\mathbb{Z} - e)$, where $e$ is any edge of $\mathbb{Z}$. Thus $\mathbb{N} \cup \mathbb{Z}$ does not possess this stronger property.

**Proposition 2.3.** If $G$ is a nonempty graph such that $G \leq G - e$ for all $e \in E(G)$, then no $(G, H)$-minimal graph exists for any graph $H$.

**Proof.** We will prove that for every graph $F$ such that $F \rightarrow (G, H)$, we have $F - e \rightarrow (G, H)$ for some $e \in E(F)$. This would show that $(G, H)$ admits no minimal graph.

Let $F$ be a graph, and let $e$ be any one of its edges. Set $F' = F - e$. Suppose that $F' \not\rightarrow (G, H)$—there exists a $(G, H)$-good coloring $c'$ of $F'$. We show that $F \not\rightarrow (G, H)$. Define a coloring $c$ on $F$ such that $c|_{E(F')} = c'$ and $e$ is colored red. By this definition, no blue copy of $H$ is produced in $F$. We claim that $c$ does not produce a red copy of $G$ either. Suppose to the contrary that a red copy of $G$, say $\hat{G}$, is produced in $F$. Since $\hat{G} \leq \hat{G} - e$, we can choose a red copy of $G$ in $F$ that does not contain $e$; that is, there exists a red copy of $G$ in $F'$. This contradicts the $(G, H)$-goodness of $c'$. As a consequence, $c$ is a $(G, H)$-good coloring of $F$, and thus $F \not\rightarrow (G, H)$.

We note that Proposition 2.3 does not hold for self-embeddable graphs $G$ in general—see Example 3.3.

If $R$ is the Rado graph, then $R - e$ is also the Rado graph for every $e \in E(R)$ via [13, Proposition 2(b)]. As a result, the Rado graph satisfies the hypothesis of Proposition 2.3. Consequently, by [13], the following holds.

**Proposition 2.4.** For $H$ a fixed graph, almost all countably infinite graphs $G$ produce a pair $(G, H)$ which admits no Ramsey-minimal graph.
3. Graph unions

Before we focus on star forests proper, we provide a quick background on graph unions in general. Consider graphs \( G, H_1, \) and \( H_2 \); let \( F_i \in \mathcal{R}(G, H_i) \) for \( i \in \{1, 2\} \). Possible candidates for a \((G, H_1 \cup H_2)\)-minimal graph include \( F_1 \), \( F_2 \), and \( F_1 \cup F_2 \).

Although \( F_1 \cup F_2 \rightarrow (G, H_1 \cup H_2) \), it not necessarily true that \( F_1 \cup F_2 \in \mathcal{R}(G, H_1 \cup H_2) \). Indeed, let us take \( H_1 = H_2 = S_1 \). For \( G \) connected, we have \( 2G \in \mathcal{R}(G, 2S_1) \) provided that \( G \in \mathcal{R}(G, S_1) \). This was discussed in \[1\] but also follows from Proposition \[3.1\]. On the other hand, if \( G \) is disconnected, we have \( 3Z \in \mathcal{R}(2Z, 2S_1) \)—not \( 4Z \)—even though \( 2Z \in \mathcal{R}(2Z, S_1) \).

**Proposition 3.1.** Let \( G \) and \( H \) be nontrivial, connected graphs, and let \( n \in \mathbb{N} \).

If \( F_i \in \mathcal{R}(G, H) \) for \( 1 \leq i \leq n \), then

\[
\bigcup_{i=1}^{n} F_i \in \mathcal{R}(G, nH).
\]

Consequently, the existence of a \((G, nH)\)-minimal graph is assured provided that a \((G, H)\)-minimal graph exists.

**Proof.** The arrowing part is obvious, so we only show the minimality of \( \bigcup_{i=1}^{n} F_i \).

It is clear that \( F_i \not\rightarrow (G, 2H) \), since otherwise we would have \( F_i \notin \mathcal{R}(G, H) \). Let \( e \) be an edge of \( F_k \) for some \( 1 \leq k \leq n \). Color \( F_k - e \) by a \((G, H)\)-good coloring and \( F_i \), for \( i \neq k \), by a \((G, 2H)\)-good coloring. This coloring on \( \bigcup_{i=1}^{n} F_i - e \) is easily shown to be \((G, nH)\)-good from the connectivity of \( G \) and \( H \). Since \( e \) is arbitrary, the proposition is proved.

In contrast to Proposition \[3.1\] the following proposition considers \( F_i \) as a candidate for being in \( \mathcal{R}(G, H_1 \cup H_2) \). A sufficient condition is provided for a \((G, H_1)\)-minimal graph to be \((G, H_1 \cup H_2)\)-minimal.

**Proposition 3.2.** Let \( G, H_1, \) and \( H_2 \) be graphs, and let \( F \in \mathcal{R}(G, H_1) \). If \( F - V(\tilde{H}_1) \rightarrow (G, H_2) \) for every \( \tilde{H}_1 \) a copy of \( H_1 \) in \( F \), then \( F \in \mathcal{R}(G, H_1 \cup H_2) \).

**Proof.** We first prove that \( F \rightarrow (G, H_1 \cup H_2) \). Suppose \( c \) is a coloring on \( F \) that produces no red copy of \( G \). It follows from \( F \rightarrow (G, H_1) \) that \( c \) produces a blue copy of \( H_1 \), say \( \tilde{H}_1 \), in \( F \). Let \( F' = F - V(\tilde{H}_1) \). Since \( F' \rightarrow (G, H_2) \) and \( F' \) contains no red copy of \( G \), there exists a blue copy of \( H_2 \), say \( \tilde{H}_2 \), in \( F' \). We observe that \( \tilde{H}_1 \) and \( \tilde{H}_2 \) are disjoint, so \( c \) produces a blue copy of \( H_1 \cup H_2 \). Hence \( F \rightarrow (G, H_1 \cup H_2) \). Its minimality follows immediately from the \((G, H_1)\)-minimality of \( F \).

**Example 3.3.** Let

\[
\begin{align*}
G &= 2S_1, \\
H_1 &= \mathbb{Z}, \\
H_2 &= \mathbb{N}, \text{ and} \\
F &= 2\mathbb{Z}.
\end{align*}
\]
The graph $2Z$ is $(2S_1, Z)$-minimal, and $Z \rightarrow (2S_1, N)$, so we can conclude by Proposition 3.2 that $2Z \in R(2S_1, Z \cup N)$. This serves as an example of a pair $(G, H)$ involving a self-embeddable graph that admits a minimal graph. We note, however, that no $(S_1, Z \cup N)$-minimal graph exists since $Z \cup N$ is self-embeddable. Thus it is possible that a $(2G, H)$-minimal graph exists even though no $(G, H)$-minimal graph exists.

4. Proof of Theorem 1.3

We fix star forests $G$ and $H$ such that at least one of them contains a star component on infinitely many vertices. We prove in this section that $(G, H)$ admits no Ramsey-minimal graph.

Suppose that $F \rightarrow (G, H)$. Since one of $G$ and $H$ contains a vertex of infinite degree, there exists a vertex $v$ of infinite degree in $F$. We choose an arbitrary edge $e$ at $v$. We prove that $F' \rightarrow (G, H)$, where $F' = F - e$. Toward a contradiction, suppose that $F'$ admits a $(G, H)$-good coloring $c'$. Since $\deg(v)$ is infinite, there are two possible cases: $v$ is incident to infinitely many red edges or infinitely many blue edges under the coloring $c'$.

Suppose that $v$ is incident to infinitely many red edges. Define a coloring $c$ on $F$ such that $c |_{E(F)} = c'$ and $e$ is colored red. This coloring produces no blue copy of $H$, so by $F \rightarrow (G, H)$ it produces a red copy of $G$, say $\hat{G}$, in $F$. There exists a star component $S$ of $\hat{G}$ that contains $e$ since otherwise, $\hat{G} \subseteq F'$, which contradicts the $(G, H)$-goodness of $c'$.

If $S$ is infinite, then $F'$ clearly contains a red copy of $G$ by removing $e$ from $\hat{G}$. On the other hand, let us suppose that $S$ has $n$ vertices. We can pick a red star $S'$ on $n$ vertices that is centered on $v$ but does not contain $e$, since $v$ is incident to infinitely many red edges. The graph $F'$ can then be shown to contain a red copy of $G$ by exchanging $S$ from $\hat{G}$ for $S'$. In both cases, we obtain a contradiction.

The case when $v$ is incident to infinitely many blue edges can be handled similarly, so our proof of Theorem 1.3 is complete.

5. Subdivision graphs vs. star forests

5.1. Bipartite graphs

Recall that a graph is bipartite if its vertex set can be partitioned into two parts such that each part is an independent set. Let $K$ be a bipartite graph with bipartition $\{A, B\}$ such that $\deg(u) \neq \infty$ for all $u \in A$. Before we work on subdivision graphs $S(G)$, we construct for $n \in \mathbb{N}$, a graph $\Gamma(K, A, n)$ such that $\Gamma(K, A, n) \rightarrow (K, S_n)$.

We define $\Gamma(K, A, n)$ by adding additional vertices and edges to $K$. For every $u \in A$, we add vertices $u_1, ..., u_{\deg(u)n}$—each not already in $V(K)$—to $K$, where $m = \deg(u)$. We then insert an edge between each $u_i$ and a vertex $v$ of $K$ if $uv$ exists in $K$. We denote the resulting graph by $\Gamma(K, A, n)$. Also, for each $u \in A$, we define $A_u$ as the set $\{u, u_1, ..., u_{\deg(u)n-1}\}$. As a result, $\Gamma(K, A, n)$ admits a
bipartition \( \bigcup_{u \in A} A_u, B \). Figure 2 shows the result of this construction when \( K = \mathbb{Z} \) and \( n = 3 \).

There is a natural projection \( \pi: \Gamma(K, A, n) \to K \) that is also a homomorphism. It is defined as

\[
\pi(v) = \begin{cases} 
  u, & v \in A_u \text{ for some } u \in A, \\
  v, & v \in B.
\end{cases}
\]

**(Proposition 5.1)** Let \( K \) be a bipartite graph with bipartition \( \{A, B\} \) such that \( \deg(u) \neq \infty \) for all \( u \in A \). For \( n \in \mathbb{N} \), we have \( \Gamma(K, A, n) \to (K, S_n) \). Consequently for \( k \in \mathbb{N} \),

\[
\bigcup_{i=1}^k \Gamma(K, A, n_i) \to \left( K, \bigcup_{i=1}^k S_{n_i} \right),
\]

where \( n_1, \ldots, n_k \in \mathbb{N} \).

**Proof.** Suppose that \( c \) is a coloring on \( \Gamma(K, A, n) \) that produces no blue copy of \( S_n \). We prove that \( c \) produces a red copy of \( K \).

We claim that for all \( u \in A \), there exists \( v_u \in A_u \) such that \( v_u \) is incident to red edges only. By construction, the vertices in \( A_u \) share the same neighborhood \( N \) of \( m \) vertices, and \( |A_u| = m(n-1) + 1 \). If every vertex in \( A_u \) is incident to at least one blue edge, then the vertices in \( N \) in total are incident to at least \( m(n-1) + 1 \) blue edges. Since \( |N| = m \), there exists a vertex in \( N \) that is incident to at least \( n \) blue edges by the Pigeonhole Principle. This is impossible since \( \Gamma(K, A, n) \) does not contain a blue copy of \( S_n \). Therefore, \( A_u \) must contain a vertex that is incident to only red edges.

Hence we can define an embedding \( \varphi: K \to \Gamma(K, A, n) \) as

\[
\varphi(u) = \begin{cases} 
  v_u, & u \in A, \\
  u, & u \in B.
\end{cases}
\]
The graph image of \( \varphi \) is a red copy of \( K \) in \( \Gamma(K, A, n) \), as desired.

The graph \( \Gamma(K, A, n) \) is not necessarily \( (K, S_n) \)-minimal in general. For example, let us take \( K = S_k \) and \( A \) as the set of leaf vertices of \( S_k \). We have \( \Gamma(S_k, A, n) = S_{kn} \), which is not \( (S_k, S_n) \)-minimal for \( k, n \geq 2 \) since \( S_{k+n-1} \in \mathcal{R}(S_k, S_n) \). However, we potentially have \( \Gamma(K, A, n) \in \mathcal{R}(K, S_n) \) when \( K = S(G) \) for some graph \( G \) as stated in Theorem 1.4.

5.2. Proof of Theorem 1.4

Fix a connected, leafless, non–self-embeddable graph \( G \). Building upon Subsection 5.1, we prove that for \( n_1, \ldots, n_k \in \mathbb{N} \), we have

\[
\bigcup_{i=1}^{k} \Gamma(S(G), A, n_i) \in \mathcal{R} \left( S(G), \bigcup_{i=1}^{k} S_{n_i} \right),
\]

where \( A \) is taken as the set of vertices of \( S(G) \) that subdivide the edges of \( G \). We note that \( \deg(u) = 2 \) for all \( u \in A \). First, we show that the three properties of \( G \) transfer to \( S(G) \), and that \( S(G) \) is \( C_4 \)-free— it contains no 4-cycles. The following lemma can be verified using elementary means.

Lemma 5.2. Let \( G \) and \( H \) be connected, bipartite graph with bipartition \( \{A, B\} \) and \( \{C, D\} \), respectively. For any isomorphism \( \varphi: G \to H \), either \( \varphi(A) = C \) and \( \varphi(B) = D \), or \( \varphi(A) = D \) and \( \varphi(B) = C \).

Proposition 5.3. If \( G \) is a connected, leafless, non–self-embeddable graph, then \( S(G) \) is also a connected, leafless, non–self-embeddable graph. In addition, \( S(G) \) is \( C_4 \)-free.

Proof. The first two properties obviously transfer, and \( S(G) \) is \( C_4 \)-free since \( G \) contains no multiple edges. We now prove that \( G \) is self-embeddable given that \( S(G) \) is self-embeddable.

Suppose that \( \varphi \) is a self-embedding of \( S(G) \). Let \( A \) be the set of vertices of \( S(G) \) that subdivide the edges of \( G \), and let \( B = V(G) \). Since \( S(G) \) is connected and bipartite with bipartition \( \{A, B\} \), there are by Lemma 5.2 two cases to consider.

Case 1: \( \varphi(A) \subseteq A \) and \( \varphi(B) \subseteq B \). We claim that \( \varphi \), restricted to \( V(G) \), gives rise to a self-embedding \( \tilde{\varphi} \) of \( G \). It is straightforward to show that \( \tilde{\varphi} \) is an embedding, so we only prove that there is an edge of \( G \) not in the image of \( \tilde{\varphi} \). Suppose that \( u \), where \( u \in A \) and \( v \in B \), is an edge of \( S(G) \) not in the image of \( \varphi \), and suppose that \( u \) subdivides an edge \( vw \) of \( G \).

We prove that \( uv \) is not in the image of \( \tilde{\varphi} \). Suppose toward a contradiction that \( \tilde{\varphi}(a) = v \) and \( \tilde{\varphi}(b) = w \) for two adjacent vertices \( a, b \in V(G) \). Let \( c \) be the vertex that subdivides \( ab \). It is apparent that \( \{\varphi(c), v\} \) and \( \{\varphi(c), w\} \) are edges of \( S(G) \). Also, we cannot have \( \varphi(c) = u \) since \( uv \) is not in the image of \( \varphi \). But then the vertices in the set \( \{v, u, w, \varphi(c)\} \) induce a 4-cycle on \( S(G) \), which contradicts the fact that \( S(G) \) is \( C_4 \)-free.
Case 2: \( \varphi(A) \subseteq B \) and \( \varphi(B) \subseteq A \). The map \( \varphi^2 \) is a self-embedding of \( S(G) \) that carries \( A \) into \( A \), and \( B \) into \( B \). So by appealing to Case 1, we can obtain a self-embedding of \( G \).

Armed with Proposition 5.3, we are ready to prove Theorem 1.4. But first, let us provide a straightforward application of the membership statement of (2) that we will prove later.

Example 5.4. Choose \( G = \mathbb{Z} \) and \( H = S_3 \). Since \( \mathbb{Z} \) is connected, leafless, and non-self-embeddable, and \( S(\mathbb{Z}) = \mathbb{Z} \), the graph of Figure 2(b) is \((\mathbb{Z}, S_3)\)-minimal by (2).

Proof of Theorem 1.4. First, suppose

\[
H = \bigcup_{i=1}^{k} S_{n_i}, \text{ where } 1 \leq n_1 \leq \cdots \leq n_k.
\]

Let \( A \) be the set of vertices of \( S(G) \) that subdivide the edges of \( G \) so that \( \deg(u) = 2 \) for all \( u \in A \). Define \( \Gamma_i = \Gamma(S(G), A, n_i) \), and let \( \Gamma = \bigcup_{i=1}^{k} \Gamma_i \).

Denote the corresponding set to \( A \) that belongs to \( \Gamma_i \) by \( A_{u,i} \). We have \( |A_{u,i}| = 2n_i - 1 \). If \( B_i = V(\Gamma_i) \setminus \bigcup_{u \in A} A_{u,i} \), then \( \Gamma_i \) admits a bipartition \( \bigcup_{u \in A} A_{u,i}, B_i \).

We prove for each \( e \in E(\Gamma) \) that there is a \((S(G), H)\)-good coloring of \( \Gamma - e \). This, along with Proposition 5.3, would show that \( \Gamma \in \mathcal{R}(S(G), H) \).

Lemma 5.5. For every \( e \in E(\Gamma) \), there exists a coloring \( c \) on \( \Gamma - e \) such that both of the following statements hold:

(i) The coloring \( c \) produces no blue copy of \( H \).

(ii) There exists \( u \in A \) such that for \( 1 \leq i \leq k \), every vertex in \( A_{u,i} \) is incident to exactly one red edge.

Proof. Suppose that \( e \) is an edge of some \( \Gamma_j \), where \( 1 \leq j \leq k \), and that \( e \) is at a vertex \( v \in A_{u,j} \) for some \( u \in A \). We color each edge in every \( \Gamma_i \), minus the edge \( e \) when \( i = j \), by the following rules:

Case 1: \( i < j \). Recall that \( |A_{u,i}| = 2n_i - 1 \) and that all vertices in \( A_{u,i} \) share the same neighborhood \( \{a, b\} \). Arbitrarily partition \( A_{u,i} \) into sets \( S \) and \( T \) such that \( |S| = n_i \) and \( |T| = n_i - 1 \). Color all the edges in \( E(S, a) \cup E(T, b) \) blue, where \( E(S, a) \) denotes the set of all edges between the vertex set \( S \) and the vertex \( a \); this produces two blue stars of sizes \( n_i \) and \( n_i - 1 \), respectively. Color the rest of \( \Gamma_i \) red.

Case 2: \( i = j \). As before, let \( a \) and \( b \) be the vertices adjacent to each vertex in \( A_{u,j} \). Partition \( A_{u,j} \setminus v \) into sets \( S \) and \( T \) both of size \( n_j - 1 \). Similarly to Case 1, we color all the edges in \( E(S, a) \cup E(T, b) \) blue. This produces two blue stars of size \( n_j - 1 \). Color the rest of \( \Gamma_j \) red.

Case 3: \( i > j \). Let \( a \) be a vertex adjacent to each vertex in \( A_{u,i} \). Color \( E(A_{u,i}, a) \) blue; this produces a blue star of size \( 2n_i - 1 \). As previously, we color the rest of \( \Gamma_i \) red.
Denote the preceding coloring scheme by $c$. It is obvious from the preceding construction of $c$ that (ii) holds for our $u \in A$, so it remains to prove that (i) holds.

Let $j'$ be the least positive integer such that $n_{j'} = n_j$. Observe that we only produce blue stars of size at least $n_j$ in Case 3 and, if $j' < j$, in Case 1 also. Every $\Gamma_i$ such that $j' \leq i \leq k$ and $i \neq j$ contributes exactly one blue star of size at least $n_j$, so exactly $k - j'$ such blue stars are produced in $\Gamma - e$ in total. But $H$ contains $k - j' + 1$ stars of size at least $n_j$, so no blue copy of $H$ can be produced in $\Gamma - e$ by the coloring $c$.

We take the coloring $c$ of Lemma 5.5. To prove that $c$ is $(S(G), H)$-good, we need to show that $c$ does not produce a red copy of $S(G)$ in $\Gamma - e$.

Suppose to the contrary that there exists an embedding $\xi: S(G) \to \Gamma_i$ such that its graph image is a red copy of $S(G)$. Set $\varphi = \pi \circ \xi$, where $\pi: \Gamma_i \to S(G)$ is a projection that sends each vertex in $A_{u,i}$ to $u$ and is defined similarly to Eq. (1). We prove that $\varphi$ is a self-embedding of $S(G)$, which would contradict the non-self-embeddability of $S(G)$. For illustration, we provide the following commutative diagram of graph homomorphisms:

\[
\begin{array}{ccc}
S(G) & \xrightarrow{\xi} & \Gamma_i \\
\downarrow{\varphi} & & \downarrow{\pi} \\
S(G) & & 
\end{array}
\]

Suppose that $\varphi(a) = b$ for some vertices $a$ and $b$ of $S(G)$. If $b \in A$, then the vertex $\xi(a)$ belongs in $A_{b,i}$. Recall that $\deg(a) \geq 2$ since $S(G)$ is leafless. Since the graph image of $\xi$ is red, $\xi(a)$ needs to be incident to at least two red edges as a result. We infer that $b \neq u$, where $u \in A$ is taken from Lemma 5.5(ii). This shows that the vertex $u$ of Lemma 5.5(ii) is not in the image of $\varphi$.

Since $S(G)$ is $C_4$-free and $\xi$ is an embedding, there cannot be a $C_4$ in the graph image of $\xi$. We now prove that $\varphi$ is injective. Let $a$ and $b$ be distinct vertices of $S(G)$. Since $a$ and $b$ have degree at least two, the vertices $\xi(a)$ and $\xi(b)$ also have degree at least two. As a result, $\xi(a)$ and $\xi(b)$ cannot both belong in $A_{u,i}$ for some $u \in A$, since that would create a $C_4$ in the graph image of $\xi$. Therefore, $\varphi$ is injective. This completes the proof that $\varphi$ is a self-embedding and finishes our proof of Theorem 1.4.

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