Context-Aware Temporal Logic for Probabilistic Systems

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Abstract. In this paper, we introduce the context-aware probabilistic temporal logic (CAPTL) that provides an intuitive way to formalize system requirements by a set of PCTL objectives with a context-based priority structure. We formally present the syntax and semantics of CAPTL and propose a synthesis algorithm for CAPTL requirements. We also implement the algorithm based on the PRISM-games model checker. Finally, we demonstrate the usage of CAPTL on two case studies: a robotic task planning problem, and synthesizing error-resilient scheduler for micro-electrode-dot-array digital microfluidic biochips.

Keywords: Markov-decision process, temporal logic, model checking, probabilistic systems, synthesis

1 Introduction

The correct-by-design paradigm in Cyber-Physical Systems (CPS) has been a central concept during the design phase of various system components. This paradigm requires the abstraction of both the system behavior and the design requirements. Typically, the system behavior is modeled as a discrete Kripke structure, with nondeterministic transitions representing various actions or choices that need to be resolved. In systems where probabilistic behavior is prevalent, formalisms such as Markov decision processes (MDPs) are best suited. The applications of correct-by-design synthesis paradigm span CPS fields such as robot path and behavior planning, smart power grids, safety-critical medical devices, and autonomous vehicles.

Temporal logic (TL) can be utilized to formalize CPS design requirements. For example, Linear Temporal Logic (LTL) is used to capture safety and reachability requirements over Boolean predicates defined over the state space. Similarly, computation tree logic (CTL) allows for expressing requirements over all computations branching from a given state. Probabilistic computation tree logic (PCTL) can be viewed as a probabilistic variation of CTL to reason about the satisfaction probabilities of temporal requirements.

The choice of which TL to use is both a science and an art. Nevertheless, fundamental factors include expressiveness (i.e., whether the design requirements
of interest can be expressed by the logic), and the existence of model checkers that can verify the system model against the design requirement, synthesize winning strategies, or generate counterexamples. Although prevalent TLs can be inherently expressive, two notions are oftentimes overlooked, namely, how easy it is to correctly formalize the design requirements, and whether existing model checkers are optimized for such requirements. The more complex it becomes to formalize a given requirement, the more likely it is that human error is introduced in the process.

In particular, we focus in this paper on requirements that are naturally specified as a set of various objectives with an underlying priority structure. For instance, the objective of an embedded controller might be focused on achieving a primary task. However, whenever the chances of achieving such task fall below a certain threshold, the controller shall proceed with a fail-safe procedure. Such requirement, while being easy to state and understand, can prove challenging when formalized for two reasons. First, multiple objectives might be involved with a priority structure, i.e., one objective takes priority over another. Second, the context upon which the objectives are switched is of probabilistic nature, i.e., it requires the ability to prioritize objectives based on probabilistic invariants.

To this end, in this work we consider the problem of modeling and synthesis of CPS modeled as MDPs, with context-based probabilistic requirements, where a context is defined over probabilistic conditions. We tackle this problem by introducing the context-aware probabilistic temporal logic (CAPTL). CAPTL provides intuitive means to formalize design requirements as a set of objectives with a priority structure. For example, a requirement can be defined in terms of primary and secondary objectives, where switching from the former to the latter is based upon a probabilistic condition (i.e., a context). The ability to define context as probabilistic conditions sets CAPTL apart from similar TLs.

In addition to providing the syntax and semantics of CAPTL for MDPs, we investigate the problem of synthesizing winning strategies based on CAPTL requirements. Next, we demonstrate how the synthesis problem can be reduced to a set of PCTL-based synthesis sub-problems. Moreover, for deterministic CAPTL requirements with persistence objectives, we propose an optimized synthesis algorithm. Finally, we implement the algorithm on top of PRISM-games [19], and we show experimental results for two case studies where we synthesize a robotic task planner, and an error-resilient scheduler for microfluidic biochips.

**Organization.** The rest of this section discusses related work. Preliminaries and a motivating example are provided in Sec. 2. In Sec. 3 we introduce the syntax and semantics of CAPTL. The CAPTL-based synthesis problem is introduced in Sec. 4, where we first explore how a CAPTL requirement can be approached using PCTL, followed by our proposed synthesis algorithm. For evaluation, we consider two case studies in Sec. 5. Finally, we conclude the paper in Sec. 6.

**Related Work.** The problem of multi-objective model checking and synthesis has been studied in literature, spanning both MDPs and stochastic games, for various properties, including reachability, safety, probabilistic queries, and reward-based requirements [8,11,13,14]. Our work improves upon the multi-objective
One prevalent workaround is to define multiple reward structures, where states are assigned tuples of real numbers depicting how favorable they are with respect to multiple criteria. The synthesis problem is then reduced to an optimization problem over either a normalized version of the rewards (i.e., assigning weights), or one reward with logical constraints on the others [17]. Results are typically presented as Pareto curves, depicting feasible points in the reward space [14]. Our work differs in two aspects. First, we use probabilities as means to define priorities rather than reward structures. Second, the mechanics needed to define context-based priorities are an integral part of CAPTL.

Perhaps the closest notion to our context-based prioritization scheme are probabilistic invariant sets (PIS) [17]. Both CAPTL and PIS involve the identification of state-space subsets that maintain a probability measure within specific bounds. While prevalent in the field of probabilistic programs [3], PIS was not considered in the field of CPS synthesis, despite the fact that (non-probabilistic) invariant sets are used in controller design [4]. The problem of merging strategies for MDPs that correspond to different objectives has been investigated [5,27]. Our approach, however, is primarily focused on formalizing the notion of context-based priorities within the specification logic itself rather than altering the original model. While one can argue that PCTL alone can be used to define priorities by utilizing nested probabilistic operators, the nesting is typically limited to qualitative operators [20]. In contrast, CAPTL relaxes such limitation by allowing quantitative operators as well. Moreover, CAPTL-based synthesis provides an insight into which objective is being pursued at a given state.

2 Problem Setting

Preliminaries. For a measurable event $E$, we denote its probability by $\Pr(E)$. The powerset of $A$ is denoted by $\mathcal{P}(A)$. We use $\mathbb{R}$ and $\mathbb{B}$ for the set of reals and booleans, respectively. For a sequence or a vector $\pi$, we write $\pi[i]$, $i > 0$, to denote the $i$-th element of $\pi$.

We formally model the system as an MDP. MDPs feature both probabilistic and nondeterministic transitions, capturing both uncertain behaviors and nondeterministic choices in the modeled system, respectively. We adopt the following definition for a system model as an MDP [2].

Definition 1 (System Model). A system model is an MDP $\mathcal{M} = (S, \text{Act}, P, s_0, AP, L)$ where $S$ is a finite set of states; $\text{Act}$ is a finite set of actions; $P : S \times \text{Act} \times S \rightarrow [0,1]$ is a transition probability function s.t. $\sum_{s' \in S} P(s,a,s') \in \{0,1\}$ for $a \in \text{Act}$; $s_0$ is an initial state; $AP$ is a set of atomic propositions; and $L : S \rightarrow \mathcal{P}(AP)$ is a labeling function.

Given a system $\mathcal{M}$, a path is a sequence of states $\pi = s_0s_1 \ldots$, such that $P(s_i,a_i,s_{i+1}) > 0$ where $a_i \in \text{Act}(s_i)$ for all $i \geq 0$. The trace of $\pi$ is defined as $\text{trace}(\pi) = L(s_0)L(s_1) \cdots$. We use $\text{FPath}_{\mathcal{M},s}$ ($\text{IPath}_{\mathcal{M},s}$) to denote the set of all finite (infinite) paths of $\mathcal{M}$ starting from $s \in S$. We use $\text{Paths}_{\mathcal{M},s}$ to denote the set of all finite and infinite paths starting from $s \in S$. If $P(s,a,s') = p$ and...
We will denote such queries as \( \text{Path}(s) \) to denote that, with probability \( p \), taking action \( a \) in state \( s \) will yield to state \( s' \). We define the cardinality of \( M \) as \( |M| = |S| + |P| \), where \( |P| \) is the number of non-zero entries in \( P \).

A strategy (also known as a policy or a scheduler) defines the behavior upon which nondeterministic transitions in \( M \) are resolved. A memoryless strategy uses only the current state to determine what action to take, while a memory-based strategy uses previous states as well. We focus in this work on pure memoryless strategies, which are shown to suffice for PCTL reachability properties [2].

**Definition 2 (Strategy).** A (pure memoryless) strategy of \( M = (S, \text{Act}, P, s_0, AP, L) \) is a function \( \sigma : S \rightarrow \text{Act} \) that maps states to actions.

By composing \( M \) and \( \sigma \), nondeterministic choices in \( M \) are resolved, reducing the model to a discrete-time Markov chain (DTMC), denoted by \( M^\sigma \). We use \( \text{Pr}_M^\sigma \) to denote the probability measure defined over the set of infinite paths \( \text{Paths}_M^\sigma \). The function \( \text{Reach}(M, s, \sigma) \) denotes the set of reachable states in \( M \) starting from \( s \in S \) under strategy \( \sigma \), while \( \text{Reach}(M, s) \) denotes the set of all reachable states from \( s \) under any strategy.

We use probabilistic computation tree logic (PCTL) to formalize system objectives as temporal properties with probabilistic bounds, following the grammar:

\[
\Phi ::= \top \mid a \mid \neg \Phi \mid \Phi \land \Phi \mid \text{Pr}_J[\varphi], \quad \varphi ::= X\Phi \mid \Phi U \Phi \mid \Phi U^k \Phi,
\]

where \( J \subseteq [0,1] \), and \( X \) and \( U \) denote the next and until temporal modalities, respectively. Other derived modalities include \( \diamond \) (eventually), \( \square \) (always), and \( W \) (weak until). Given a system \( M \) and a strategy \( \sigma \), the PCTL satisfaction semantics over \( s \in S \) and \( \pi \in \text{Paths}_M^\sigma \) is defined as follows [2,12]:

\[
s, \sigma \models a \iff a \in \text{Act}(s) \\
s, \sigma \models \neg \Phi \iff s \not\models \Phi \\
s, \sigma \models \Phi_1 \land \Phi_2 \iff s \models \Phi_1 \land s \models \Phi_2 \\
s, \sigma \models \text{Pr}_J[\varphi] \iff \text{Pr}\{\pi \mid \pi \models \varphi\} \in J \\
\pi, \sigma \models X\Phi \iff \pi[1] \models \Phi \\
\pi, \sigma \models \Phi_1 U \Phi_2 \iff \exists j \in \mathbb{N}, j \geq 0, \pi[j] \models \Phi_2 \land (\pi[k] \models \Phi_1, \forall k \in \mathbb{N}, 0 \leq k < j) \\
\pi, \sigma \models \Phi_1 U_j^k \Phi_2 \iff \exists j \in \mathbb{N}, 0 \leq j \leq n, \pi[j] \models \Phi_2 \land (\pi[k] \models \Phi_1, \forall k \in \mathbb{N}, 0 \leq k < j)
\]

PCTL can be extended with quantitative queries of the form \( \text{P}_{\min}^{\varphi}(\text{P}_{\max}[\varphi]) \) to compute the minimum (maximum) probability of achieving \( \varphi \) [12,26], i.e.,

\[
\text{P}_{\min}[\varphi] = \inf_{\sigma \in \Sigma} \text{Pr}_M^\sigma \left( \{\pi \mid \pi \models \varphi\} \right), \quad \text{P}_{\max}[\varphi] = \sup_{\sigma \in \Sigma} \text{Pr}_M^\sigma \left( \{\pi \mid \pi \models \varphi\} \right).
\]

We will denote such queries as \( \text{P}_{\text{opt}} \) (read: optimal), where \( \text{opt} \in \{\text{min}, \text{max}\} \).

**Motivating Example.** Consider the simple grid-world shown in Fig. 1(left).

The robot can move between rooms through doorways where obstacles can be probabilistically encountered (e.g., closed doors), requiring the robot to consume more power. The robot state is captured as a tuple \( s = (g, b, x, y) \), where \( g \in \{\text{on}, \text{sleep}, \text{error}\} \) is the robot’s status, \( b \in \{0, 1, ..., 10\} \) is the robot’s battery level, and \( x \) and \( y \) are its current coordinates. As shown in Fig. 1(right), the system can be modeled as \( M = (S, \text{Act}, P, s_0, AP, L) \), where \( \text{Act} = \{\text{N}, \text{S}, \text{E}, \text{W}, \text{sleep}, \text{error}\} \), and \( s_0 = (0, 10, 1, 1) \). Suppose that the main objective for the robot is to reach the
goal with a charge \( h > 3 \) (objective A). However, if the probability of achieving objective A is less than 0.8, the robot should prioritize reaching the charging station and switch to sleep mode (objective B). Moreover, if the probability of achieving objective B falls below 0.7, the robot should stop and switch to err mode, preferably in one of the safe zones (objective C).

Now let us examine how such requirements can be formalized. Let \( \varphi_A = \Diamond (goal \land (h > 3) \land on) \), \( \varphi_B = \Diamond (chrg \land (h > 3) \land on) \), and \( \varphi_C = \Diamond (error) \). One can use PCTL to capture each objective separately as the reachability queries \( \Phi_A = \text{P}_{\text{max}}[\varphi_A] \), \( \Phi_B = \text{P}_{\text{max}}[\varphi_B] \), and \( \Phi_C = \text{P}_{\text{max}}[\varphi_C] \). A multi-objective query \( \Phi_1 = \Phi_A \lor \Phi_B \lor \Phi_C \) does not capture the underlying priority structure in the original requirements. In fact, an optimal strategy for \( \Phi_1 \) always chooses the actions that reflect the objective with the highest probability of success, resulting in a strategy where the robot simply signals an error from the very initial state. Similarly, the use of \( \Phi_2 = \text{P}_{\text{max}}[\varphi_A \land \varphi_B] \) does not provide means to specify the context upon which switching from \( \varphi_A \) to \( \varphi_B \) occurs. Attempts featuring multi-objective queries with nested operators, such as \( \Phi_3 = \text{P}_{\text{max}}[\varphi_A \land \text{P}_{\text{max}} \geq 0.8 [\varphi_A]] \lor \text{P}_{\text{max}}[\varphi_B \land \text{P}_{\text{max}} < 0.8 [\varphi_A]] \), have several drawbacks. First, correctly formalizing the requirement is typically cumbersome and hard to troubleshoot. Second, to the best of our knowledge, nested queries in the form of \( \text{P}_{\text{opt}} \in J \) are not supported by model checkers. Third, the semantics of the formalized requirement is potentially different from the original one. For instance, \( \Phi_3 \) allows the system to pursue \( \varphi_A \) even after switching to \( \varphi_B \) if the probability of achieving \( \varphi_A \) rises again above \( 0.8 \) — a behavior that was not called for in the original requirement.

Consequently, in this paper we focus on two problems: the formalization of PCTL objectives with an underlying context-based priority structure, and the synthesis of strategies for such objectives. The first problem is addressed by introducing CAPTL in Sec. 3 while the second is addressed in Sec. 4. We will use this motivating example as a running one throughout the rest of this paper.

3 Context-Aware Temporal Logic

CAPTL Syntax. CAPTL features two pertinent notions, namely, objectives and contexts. Let \( \mathcal{M} \) be our system model, and let \( \Xi \) be the set of all possible PCTL path formulas defined for \( \mathcal{M} \). In CAPTL, we define an objective \( q \) as a conjunctive optimization query \( q = \bigwedge_{i=1}^m \text{P}_{\text{opt}}[\varphi_i], \varphi_i \in \Xi, m > 0 \). When \( m = 1 \), \( q \) resembles a multi-objective optimization query in the conjunctive form. Otherwise, in the simplest form where \( m = 1 \), \( q \) is a single-objective query.
For the running example, Fig. 2 shows an example of a CAPTL requirement $A$.

Notice that objectives can have a single context, e.g., $\varphi_1$. Notice that contexts are used as labels for the conditional transition relation. In the rest of this paper, we will overload the notation and use $W(q) = W(q_1) \cup W(q_2) \cup \cdots$ to denote the set of contexts emerging from a given objective. We will also use $\varphi_1$ for $(\text{goal} \land (h > 3) \land \text{on})$, $\varphi_2$ for $(\text{chrg} \land (h > 3) \land \text{sleep})$, $\varphi_3$ for $(\text{safe} \land \text{sleep})$, and $\varphi_0$ for $(\text{error})$.

**Fig. 2.** The CAPTL requirement for the running example.

A context $w(q,q')$ marks a state where switching from objective $q$ to objective $q'$ is required. Formally, we define a context $w$ over $\Xi$ as a set of satisfaction queries in the disjunctive normal form $w = \bigvee_{j=1}^{J} \bigwedge_{i=1}^{n} P_{\text{opt}}(\varphi_{ij}), \varphi_{ij} \in \Xi, J \subseteq [0,1]$. Intuitively, in a state where $w(q,q')$ is satisfied, the system switches from $q$ to $q'$. Notice that the context definition utilizes the operator $P_{\text{opt}}$ without intervals, i.e., a quantitative optimization query that can return a numerical value in $[0,1]$.

A **CAPTL requirement** defines a set of objectives to be satisfied, in addition to a set of contexts, representing the probabilistic conditions upon which objectives are prioritized. Formally, we define the syntax of a CAPTL requirement as follows.

**Definition 3 (CAPTL Requirement).** Given a set of PCTL path formulas $\Xi$, a CAPTL requirement is a tuple $A = (Q,W,\Xi,\rightarrow,q_0)$ where

- $Q \subseteq \big\{ \bigwedge_{i=1}^{m} P_{\text{opt}}(\varphi_i) \mid \varphi_i \in \Xi \big\}$ is a finite nonempty set of objectives over $\Xi$,
- $W \subseteq \big\{ \bigvee_{j=1}^{J} \bigwedge_{i=1}^{n} P_{\text{opt}}(\varphi_{ij}) \mid \varphi_{ij} \in \Xi, J \subseteq [0,1] \big\}$ is a set of contexts,
- $\rightarrow \subseteq Q \times W \times Q$ is a conditional transition relation, and
- $q_0 \in Q$ is an initial objective.

In a CAPTL requirement $A$, each state $q \in Q$ represents an objective, i.e., an optimization query to be satisfied. The conditional transition relation $\rightarrow$ defines how objectives are allowed to change. For instance, if $q \xrightarrow{w} q'$, a shorthand for $(q,w,q') \in \rightarrow$, then the objectives are switched from $q$ to $q'$ if $w$ is satisfied. Notice that contexts are used as labels for the conditional transition relation. In the rest of this paper, we will overload the notation and use $W : Q \rightarrow P(W)$ to denote the set of contexts emerging from a given objective. We will also use $Q(q,w) = q'$ to denote that objective $q$ has a context $w$ that leads to $q'$.

**Example 1.** For the running example, Fig. 2 shows an example of a CAPTL requirement $A$ where $Q = \{q_0,q_1,q_2,q_3\}$, $W = \{w_{01},w_{02},w_{13},w_{23}\}$, and $\rightarrow = \{(q_0,w_{01},q_1), (q_0,w_{02},q_2), (q_1,w_{13},q_3), (q_2,w_{23},q_3)\}$. The requirement starts by prioritizing $q_0 = P_{\text{max}}[\varphi_0]$. If $P_{\text{max}}[\varphi_0] \in [0.75,0.85]$, the context $w_{01}$ becomes true, and by executing $q_0 \xrightarrow{w_{01}} q_1$, $q_1 = P_{\text{max}}[\varphi_1]$ is prioritized. Similarly, if $P_{\text{max}}[\varphi_0] \in [0,0.75]$, $w_{02}$ becomes true, executing $q_0 \xrightarrow{w_{02}} q_2$ where $q_2 = P_{\text{max}}[\varphi_2]$ is prioritized. Notice that objectives can have a single context, e.g., $W(q_1) = \{w_{13}\}$; multiple contexts, e.g., $W(q_0) = \{w_{01},w_{02}\}$; or none, e.g., $W(q_3) = \emptyset$. 
CAPTTL Semantics for MDPs. We progressively define CAPTTL semantics for MDPs by first defining the satisfaction semantics for objectives and contexts. Let $q = P_{\text{max}}[\varphi]$ be the objective at state $s$, and let $\Sigma$ be the set of all strategies for $M$. We say that $s, \sigma^* \models q$ if $\sigma^* \in \Sigma$ such that
\[
Pr_{M}^{\sigma^*,s} = \sup_{\sigma \in \Sigma} \Pr_{M,s}^{\sigma} (\{ \pi \in \text{Paths}_{M,s}^{\sigma} | \pi \models \varphi \}). \tag{1}
\]
In that case, we call $\sigma^*$ a local strategy, i.e., an optimal strategy w.r.t. $(q, s)$.

**Definition 4 (Local Strategy)**. Let $q_i = P_{\text{opt}}[\varphi_i]$ be an objective. A local (optimal) strategy for $(q_i, s_i)$ is a strategy $\sigma_{(q_i, s_i)} \in \Sigma$ such that
\[
Pr_{M,s_i}^{\sigma_{(q_i, s_i)}} = \text{opt} \Pr_{M,s_i}^{\sigma} (\{ \pi \in \text{Paths}_{M,s_i}^{\sigma} | \pi \models \varphi_i \})
\]

Next, let $(q, w, q') \in \rightarrow$, where $w = P_{\leq_i}[\varphi]$. Let $s_k \in \text{Reach}(M, s, \sigma^*)$, where $\sigma^*$ is the local strategy for $(q, s)$. We say that $s_k \models w$ if
\[
\sup_{\sigma \in \Sigma} \Pr_{M,s_k}^{\sigma} (\{ \pi \in \text{Paths}_{M,s_k}^{\sigma} | \pi \models \varphi \}) \leq c. \tag{2}
\]
Note that contrary to (1), the set of paths $\{ \pi \}$ in (2) is not limited to those induced by the local strategy $\sigma^*$. Moreover, if $\exists \pi = s...s_i...s_k \in \text{Paths}_{M,s}^{\sigma}$ s.t. $s_i \models w$, and $s_i \not\models w$ for all $i < k$, then $s_k$ is called a switching state, i.e., the first state on a path $\pi$ to satisfy $w$, triggering a switch from $q$ to $q'$.

**Definition 5 (Switching Set)**. Let $q = P_{\text{opt}}[\varphi]$ and $\sigma^* \in \Sigma$ such that $s_0, \sigma^* \models q$. The corresponding switching set $S_q \subseteq \text{Reach}(M, s_0, \sigma^*)$ is defined as
\[
S_q = \left\{ s_k \mid \exists \pi = s_0...s_i...s_k \in \text{Paths}_{M,s_0}^{\sigma^*} \text{ s.t. } s_i \not\models \bigvee_{w \in W(q)} w, \forall i < k; s_k \models \bigvee_{w \in W(q)} w \right\}.
\]
We use $S_q^f$ to denote the set of switching states from $q$ to $q'$.

An objective is active in a state $s$ if it is being pursued at that state.

**Definition 6 (Active Objective)**. Let $A = (Q, W, \Xi, \rightarrow, q_0)$ and $M = (S, Act, P, s_0, AP, L)$. An activation function $g: S \rightarrow \mathcal{P}(Q)$ is defined inductively as:
(i) $g(s_0) \ni q_0$; and (ii) $g(s) \ni q'$ if $g(s) \ni q$ and $s \in S_q^f$. We say objective $q \in Q$ is active at state $s \in S$ if $g(s) \ni q$.

As captured in Definition 4, local strategies are tied to their respective objectives. Consequently, a local strategy is switched whenever an objective is switched as well, and the new local strategy substitutes its predecessor. We call the set of local strategies a strategy profile, and the resulting behavior a protocol.

**Definition 7 (Protocol)**. Let $A = (Q, W, \Xi, \rightarrow, q_0)$ and $M = (S, Act, P, s_0, AP, L)$. Given a strategy profile $\sigma = \{\sigma_{(q,s)},\ldots\}$, the induced (optimal) protocol is a (partial) function $\Pi : Q \times S \rightarrow \text{Act} \cup \mathcal{P}(W)$ such that
- $\Pi(q,s) = \sigma_{(q,s)}(s) \in \text{Act}$ iff $q \in g(s)$ and $s \not\in S_q$; and
- $\Pi(q,s) \ni w_{(q,q')} \in W$ iff $q \in g(s)$ and $s \in S_q^f$.

Given $(q, s)$, a protocol assigns either an optimal action based on the local strategy associated with $q$, or a context to switch the active objective itself. We will use $\mathcal{P}$ to denote the set of all possible protocols.
Definition 8 (System-Protocol Composition). Let $M = (S, Act, P, s_0, AP, L)$ and $\Pi : Q \times S \rightarrow Act \cup P(W)$ be a compatible protocol. Their composition is defined as $M^{\Pi} = (\hat{Q}, Act \cup W, \hat{P}, \hat{s}_0, \hat{L})$ where $\hat{Q} \subseteq Q \times S$, $\hat{s}_0 = (q_0, s_0)$, and

$$P((q, s), a, (q', s')) = \begin{cases} P((s, a, s')) & \text{if } \Pi(q, s) = a, q' = q, \\ 1 & \text{if } \Pi(q, s) = w, s' = s, q' = Q(q, w), \\ 0 & \text{otherwise.} \end{cases}$$

We now define the CAPTL satisfaction semantics as follows.

Definition 9 (CAPTL Satisfaction Semantics). Let $A = (Q, W, \Xi, \rightarrow, q_0)$, $M = (S, Act, P, s_0, AP, L)$, and $\Pi : Q \times S \rightarrow Act \cup P(W)$. The CAPTL satisfaction semantics is defined inductively as follows:

$$M, \Pi \models q \iff P_{\Pi}^{\max} \left( \{ \pi \in Paths_{M^{\Pi}} | \text{last}(\pi) = (q, s'), s' \models q \} \right) \geq 1,$$

$$M, \Pi \models c A \iff P_{\Pi}^{\max} \left( \{ \pi \in Paths_{M^{\Pi}} | \text{last}(\pi) = (q, s'), s' \models q, q \in Q \} \right) = c,$$

$$M, \Pi \models A \iff M, \Pi \models \exists A \iff M, \Pi \models \geq 1 A.$$ 

CAPTL semantics dictate that $M$ and $\Pi$ satisfy $A$ if every path $\pi \in Paths_{M^{\Pi}}$ ends with a state $s \in S$ where $q \ni g(s)$ and $s \models q$, i.e., the system reaches some state $s$ where some objective $q$ is both active and satisfied.

CAPTL Fragments. A CAPTL requirement is nondeterministic if for some $q \in Q, \exists w_i, w_j \in W(q)$ such that $S_i^q \cap S_j^q \neq \emptyset$. That is, at least one objective has two or more contexts that can be active at the same state. If that is not the case, then the CAPTL requirement is deterministic. We now identify a fragment of deterministic CAPTL requirements where the following two conditions are met. First, every $q \in Q$ is a quantitative PCTL persistence objective. Second, every $w \in W(q)$ is a qualitative PCTL persistence objective over the same persistence set as in $q$. This is formally captured in the following definition.

Definition 10 (Persistence CAPTL). A CAPTL requirement $A = (Q, W, \Xi, \rightarrow, q_0)$ is persistent if every $q \in Q$ is of the form $q = P_{\max}[\mathbf{\exists} \Box B]$; and if $W(q) \neq \emptyset$ then $\forall w(q, q_i) \in W(q)$, $w(q, q_i) = P_{\max \in J_i}[\mathbf{\exists} \Box B]$ such that $(J_i)$ are disjoint intervals where $\cup_{j} J_j = [0, c)$, $0 < c \leq 1$.

A persistence CAPTL (P-CAPTL) requirement allows for defining persistence objectives, where each objective maximizes the probability of (i.e., prioritizes) reaching a corresponding persistence set. Contexts in this case can be understood as lower bounds of their respective objectives. That is, an objective is pursued as long as, at any transient state, the probability of achieving such objective does not drop below a certain threshold. The requirement also ensures that at most one context is satisfied at any state, eliminating any nondeterminism in $A$.

Example 2. Continuing Example 1, Fig. 3 shows the persistence CAPTL requirement for the robot. Notice that all objectives are in the form $P_{\max}[\mathbf{\exists} \Box B]$. Also, the intervals $[0.75, 0.85]$ and $[0.075, 0.75]$ of $w_{01}$ and $w_{02}$, respectively, are disjoint, hence at most one context in $W(q_0) = \{w_{01}, w_{02}\}$ can be satisfied at any state.
Context-Aware Temporal Logic for Probabilistic Systems

\[ q_0 : P_{\text{max}}[\varphi_0] \]
\[ w_{01} : P_{\text{max}} \in [0.75, 0.85] [\varphi_0] \]
\[ q_1 : P_{\text{max}}[\varphi_1] \]
\[ w_{13} : P_{\text{max}} < 0.7 [\varphi_1] \]
\[ q_2 : P_{\text{max}}[\varphi_2] \]
\[ w_{23} : P_{\text{max}} < 0.8 [\varphi_2] \]
\[ q_3 : P_{\text{max}}[\varphi_3] \]

\[ \varphi_0 : \Diamond \Box (\text{goal} \land (h > 3) \land \text{on}) \]
\[ \varphi_1 : \Diamond \Box (\text{chrg} \land (h > 3) \land \text{sleep}) \]
\[ \varphi_2 : \Diamond \Box (\text{safe} \land \text{sleep}) \]
\[ \varphi_3 : \Diamond \Box (\text{error}) \]

Fig. 3. The persistence CAPTL requirement for the running example.

4 CAPTL-Based Synthesis

In this section we first define the synthesis problem for CAPTL requirements. Next, we examine a general procedure for deterministic CAPTL where the synthesis problem is reduced to solving a set of PCTL-based strategy synthesis problems. Finally, we utilize the underlying structure of persistence properties to propose a synthesis procedure optimized for P-CAPTL requirements.

**CAPTL Synthesis Problem.** In the rest of this section, let \( \mathcal{M} = (S, \text{Act}, P, s_0, AP, L) \) and \( \mathcal{A} = (Q, W, \Xi, \rightarrow, q_0) \). We assume that a probabilistic model checker is given (e.g., PRISM-games [19] or Uppaal Stratego [9]) that can accept an MDP-based model \( \mathcal{M} \) and a PCTL formula \( \Phi \) as inputs, and provides the following functions:

- \( \text{REACH} : (\mathcal{M}, s) \mapsto R \subseteq S \) returns \( R = \text{Reach}(\mathcal{M}, s) \).
- \( \text{VERIFY} : (\mathcal{M}, s, \Phi) \mapsto b \in \mathbb{B} \) returns \( \top \) iff \( \mathcal{M}, s \models \Phi \), and \( \bot \) otherwise.
- \( \text{SYNTH} : (\mathcal{M}, s, \Phi) \mapsto (\sigma, c) \) finds \( \sigma \in \Sigma \) s.t. \( \Pr (\mathcal{M}^\sigma \models \Phi) = c \in [0, 1] \).

We also assume that the model checker functions terminate in finite time and return correct answers. We now define the CAPTL synthesis problem as follows.

**Definition 11 (CAPTL Synthesis Problem).** Given \( \mathcal{M} = (S, \text{Act}, P, s_0, AP, L) \) and \( \mathcal{A} = (Q, W, \Xi, \rightarrow, q_0) \), the CAPTL synthesis problem seeks to find a protocol \( \Pi : Q \times S \rightarrow \text{Act} \cup W \) such that \( \mathcal{M}, \Pi \models \mathcal{A} \).

**PCTL-Based Approach.** The synthesis problem can be reduced to solving a set of PCTL-based synthesis queries as demonstrated in Algorithm 1. Starting with \( (q_0, s_0) \), the algorithm verifies whether any context \( w \in W(q_0) \) is satisfied, and if true, adds \( w \) to the protocol and switches to the next objective. If no context is satisfied, the algorithm synthesizes a local strategy and adds the corresponding optimal action to the protocol.

**Proposition 1.** Algorithm 1 terminates; and returns \( \Pi, c \) iff \( \mathcal{M}, \Pi \models c \mathcal{A} \).

**Synthesis for P-CAPTL** We now propose a synthesis algorithm optimized for persistence CAPTL. To this end, we show that for a given persistence objective, synthesizing a local strategy in the initial state suffices. In a manner similar to switching states (see Definition 5), we devise a partition of reachable states for every objective. We will use those concepts to define a system-CAPTL composition and show that it is bisimilar to \( \mathcal{M}^\Pi \).
Algorithm 1: PCTL-Based Synthesis

Input: $M = (S, Act, P, s_0, AP, L)$, $\mathcal{A} = (Q, W, \Xi, \leadsto, q_0)$
Result: $\Pi, c$ such that $M, \Pi \models c$ $\mathcal{A}$

1. foreach $q \in Q$ do
   $\hat{S}_q \leftarrow \emptyset$, $\check{S}_q \leftarrow \emptyset$
2. $\Pi \leftarrow \emptyset$, $\hat{S}_{q_0} \leftarrow \{s_0\}$, $q \leftarrow q_0$, $C \leftarrow 0_{Q \times S} \in [0, 1]^{Q \times S}$, repeat $\leftarrow \top$
3. while $\hat{S}_q \neq \emptyset$ do
   4. Let $s \in \hat{S}_q$, $\hat{S}_q \leftarrow \hat{S}_q \setminus\{s\}$, $\check{S}_q \leftarrow \check{S}_q \cup\{s\}$
   5. while repeat do repeat $\leftarrow \perp$
   6. foreach $w \in W(q)$ do
   7. if $\text{Verify}(M, s, w) = \top$ then
   8. $\Pi \leftarrow \Pi \cup \{(s, q, w)\}$, $q \leftarrow Q(q, w)$, repeat $\leftarrow \top$, break
   9. $(\sigma, C(q, s)) \leftarrow \text{Synth}(M; s, q)$, $\Pi \leftarrow \Pi \cup \{(s, q, \sigma(s))\}$
   10. $\hat{S}_q \leftarrow \hat{S}_q \cup (\text{Post}(M, s, \sigma(s)) \setminus \check{S}_q)$
11. $c \leftarrow \text{Verify}(M^\Pi, (q_0, s_0), P \left[\diamond \bigvee_{q \in Q} (\langle q, s \rangle \land C(q, s) = 1)\right])$

Let $R = \text{Reach}(M, s_0)$. We first note that given $M$ and $q = P_{\text{opt}}[\diamond \Box B]$, existing model checking and synthesis algorithms typically compute a least fixed point (LFP) vector $x_q \in [0, 1]^{|R|}$, where $x_q[s]$ is the optimal probability of satisfying $\diamond \Box B$ at state $s \in R$ (e.g., see [2,16]). That is, when $\text{Synth}(M, s_0, q)$ is called, $x_q$ is computed, but only $c = x_q[s_0]$ is returned (i.e., the value at the initial state). We exploit this fact by implementing a function $\text{ReachP} :: (M, s, q) \rightarrow x_q$ that returns the LFP vector $x_q$ associated with $q$.

Lemma 1 (Local Strategy Dominance). Let $\mathcal{M} = (S, Act, P, s_0, AP, L)$ and $q = P_{\text{opt}}[\diamond \Box B]$. For all $s \in \text{Reach}(M, s_0)$, $\sigma(q, s) = \sigma(q, s_0)\big|_{\text{Reach}(M, s)}$.

Lemma 1 signifies that a local strategy for $q$ in the initial state (i.e., $\sigma(q, s_0)$) subsumes all local strategies for the same probabilistic reachability objective in every $s \in R$. Next, for every $q \in Q$, let us define the following partition of $R$:
- $R_q^q = \{s \in R \mid \forall w = P_{\text{max}} \in J[\diamond \Box B] \in W(q), x_q[s] \notin J\}$, i.e., the states in $R$ where, if $q$ is active, keep pursuing $q$.
- $R_q^{q'} = \{s \in R \mid \exists w = P_{\text{max}} \in J[\diamond \Box B] \in W(q), x_q[s] \in J, Q(q, w) = q'\}$, i.e., the states in $R$ where, if $q$ is active, switch to $q'$.

Lemma 2 (Partitioning). Let $\mathcal{M} = (S, Act, P, s_0, AP, L)$, $\mathcal{A} = (Q, W, \Xi, \leadsto, q_0)$, and $R = \text{Reach}(M, s_0)$. For every $q \in Q$, $\bigcup_{q' \neq q} R_q^{q'} = R$; and $R_q^q \cap R_q^{q'} = \emptyset$ for every $q' \neq q''$.

Proof Sketch. From Definition 10, the intervals $(J_w)_{w \in W(q)}$ are disjoint; hence $(R_q^q)_{q' \neq q}$ are disjoint as well, and that $R_q^q = R \setminus \bigcup_{q' \neq q} R_q^{q'}$. □

Example 3. Returning to the P-CAPTL requirement specified in the running example (see Fig. 3), Fig. 4 depicts the partitioning of the state-space based on $q_0$, $q_1$, $q_2$ and $q_3$. Notice that for any $q \in Q$, the sets $(R_q^{q'})_{q' \in Q}$ are pairwise disjoint, where $\bigcup_{q' \in Q} R_q^{q'} = \text{Reach}(M, s_0)$. For example, $R_{q_0}^{q_0}$, $R_{q_0}^{q_1}$ and $R_{q_0}^{q_2}$ do not intersect, and their union spans $R = \text{Reach}(M, s_0)$. In this case, $R_{q_0}^{q_3} = \emptyset$ since there is no direct context emerging from $q_0$ to $q_3$. 
Definition 12 (System-CAPTL Composition). Let $\mathcal{M} = (S, \text{Act}, P, s_0, AP, L)$, $A = (Q, W, \varepsilon, \rightarrow, q_0)$, and $\sigma = \{\sigma(q, a) \mid q \in Q\}$. Their composition is defined as the automaton $\mathcal{M}_A = (V, \text{Act}, P, s_0, \rightarrow', v_0, AP, \bar{L})$ where $V \subseteq S \times Q \times \Gamma$, and $\Gamma = \{\{1\}, \{2\}\}; \overline{\text{Act}} = \text{Act} \cup W \cup \{\tau\}$, where $\tau$ is a stutter action; $v_0 = (s_0, q_0, \{1\})$; $L : V \rightarrow P(AP)$ such that $\bar{L}(s, q, \gamma) = L(s)$; and the transition relation $\rightarrow'$ is defined using the following compositional rules:

\begin{align*}
[R1] \quad & s \xrightarrow{a, p} s' \wedge \sigma(q, a)(s) = a \\
[R2] \quad & (s, q, \{1\}) \xrightarrow{a, p, \tau}(s', q, \{2\}) \\
[R3] \quad & s \xrightarrow{\omega, s'} (s, q, \{2\}) \xrightarrow{\tau'(s, q, \{1\})}
\end{align*}

The rules in Definition 12 are interpreted as follows. The state space $V$ is partitioned into $V_{\{1\}}$ (where $\mathcal{M}$ actions are allowed) and $V_{\{2\}}$ (where $\mathcal{A}$ actions are allowed), resembling a turn-based 2-player game. $[R1]$ ensures that, if $q$ is active in $s$, then only the transitions with the optimal action $\sigma(q, a)(s)$ are allowed. $[R2]$ ensures that, if $s \in R^q_1$, the active objective remains unchanged. If $s \in R^q_2$, however, $[R3]$ enforces switching the active objective to $q'$. The action $\tau$ is a stutter since $\forall v \xrightarrow{\tau} v'$, $\bar{L}(v) = \bar{L}(v')$.

**Lemma 3 (Induced DTMC).** $\mathcal{M}_A$ constructed using Definition 12 is a DTMC.

Lemma 3 dictates that the probability measure $Pr_{\mathcal{M}_A}$ is well-defined. We will now use the notion of stutter equivalence [2] to prove that $\mathcal{M}_A$ is bisimilar to $\mathcal{M}^H$. Basically, two paths $\pi_1$ and $\pi_2$ are stutter-equivalent, denoted by $\pi_1 \cong \pi_2$, if there exists a finite sequence $A_0 ... A_n \in (P(\text{Act}))^+$ such that $\text{tr}ace(\pi_1), \text{tr}ace(\pi_2) \in A_0^+ A_1^+ ... A_n^+$, where $A^+ = \{A, AA, ...\}$ is the set of finite, non-empty repetitions.

**Theorem 1 (Stutter-Equivalence).** Let $\mathcal{M}$, $\mathcal{A}$, and $\Pi \in \mathfrak{P}$ be such that $\mathcal{M}, \Pi \models \mathcal{A}$. For every $\pi \in \text{Path}_{\mathcal{M}, \mathcal{A}}$ there exists $\hat{\pi} \in \text{Path}_{\mathcal{M}_A, \mathcal{A}}$ such that $\pi \cong \hat{\pi}$ and $\text{Pr}_{\mathcal{M}^{\mathcal{A}}}(\pi) = \text{Pr}_{\mathcal{M}_A}(\hat{\pi})$. For every $\hat{\pi} \in \text{Path}_{\mathcal{M}_A, \mathcal{A}}$, where last($\hat{\pi}$) $\in V_{\{2\}}$, there exists $\hat{\pi} \in \text{Path}_{\mathcal{M}_{\mathcal{A}}}$ such that $\hat{\pi} \cong \pi$ and $\text{Pr}_{\mathcal{M}_{\mathcal{A}}}(\hat{\pi}) = \text{Pr}_{\mathcal{M}^{\mathcal{A}}}(\pi)$.

**Proof Sketch.** We show that for every execution fragment $q_1 = (s, q) \xrightarrow{a, p}(s, q')$ there exists $\hat{q}_1 = (s, q, \{2\}) \xrightarrow{a, p, \tau}(s, q, \{1\}) \xrightarrow{\tau'}(s', q, \{2\})$. Moreover, for every $q_2 =

Fig. 4. Partitioning the state-space of the running example using $q_8$, $q_9$, $q_1$, and $q_3$. For example, $(q_1, q_9, h_5, 2, 1) = \triangleright$ indicates that $s : (g, h, x, y) = (0, 5, 2, 1) \in R^q_3$. 

\begin{align*}
S \backslash R & \quad \quad \quad \quad R^q_1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad R^q_2 \\
\text{[0x0]} & \quad \quad \quad \quad \text{[0x0]} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{[0x0]}
\end{align*}
there exists \( \hat{q}_2 = (s, q, \begin{array}{c} \end{array}) \rightarrow (s, q', \begin{array}{c} \end{array}) \). Using induction, we show that for every arbitrary execution \( \hat{q} \) there exists \( \hat{q}_2 \) such that \( \hat{q} \triangleq \hat{q}_2 \), where

\[
\text{trace}(\hat{q}) = (A_0 + A_0A_0) (A_1 + A_1A_1) \ldots (A_n + A_nA_n) \in (\mathcal{P}(AP))^+ \\
\text{trace}(\hat{q}) = (A_0A_0) (A_1A_1) \ldots (A_nA_n) \in (\mathcal{P}(AP))^+
\]

and \( \Pr(\hat{q}) = \Pr(\hat{q}_2) \). Similarly, the other direction can be shown for every last(\( \hat{q}_2 \)) that ends with last(\( \hat{q}_2 \)) \( \in V_{\hat{q}_2} \).

We use both Lemma \( 4 \) and Theorem \( 1 \) to devise the protocol synthesis procedure summarized in Algorithm \( 2 \). In the first part (lines 1–8), the procedure starts by synthesizing a local strategy \( \sigma_{(q_0, s_0)} \) and obtaining the associated LFP vector \( x_{q_0} \in [0, 1]^R \). Next, \( R \) is partitioned using \( x_{q_0} \) to obtain \( (R_{q_0}^q)_{q \in Q} \). If \( R_{q_0}^q = \emptyset \) for some \( q \neq q_0 \), the same procedure is repeated for \( q \) to obtain \( (q, s_0) \), \( x_q \) and \( (R_q^q)_{q \in Q} \). In the second part (lines 9–16), three modules are constructed based on Definition \( 12 \). The resulting parallel composition constitutes \( \mathcal{M}^c_{A_q} \), which mimics a stochastic 2-player game between \( \mathcal{M} \) (player 1) and \( \hat{A} \) (player 2), where the players’ choices are already resolved by \( \hat{\sigma} \). Finally, \( \Pi \) is populated by a query that checks for the CAPTL satisfaction condition (line 17), i.e., a state \( \langle s, q_1, \gamma \rangle \) is reached where \( q_1 = P_{\max} [\Delta B_3] \) is active, and \( \square B_3 \) holds. Notice that, based on the results from Lemma \( 4 \), Algorithm \( 2 \) synthesizes a local strategy at most once for every \( q \in Q \), compared to Algorithm \( 1 \) where synthesis is performed at every reachable state.

**Theorem 2.** Algorithm \( 2 \) terminates; and returns \( \Pi, c \) if \( \mathcal{M}, \Pi \models_c A \).

**Example 4 (Protocol Synthesis).** For the CAPTL requirement in Example \( 2 \) (see Fig. 3), Fig. 4 shows a visual representation of the protocol synthesized using Algorithm \( 2 \), where blue markers indicate actions in Act, and red markers indicate actions in \( W \). While pursuing \( q_0 \), the robot can achieve the task by moving N(\( \triangleleft \)), N(\( \bigtriangledown \)), E(\( \bigtriangledown \)) if no obstacles are encountered, or if obstacles are encountered only once while moving E(\( \bigtriangledown \)). Switching from \( q_0 \) to \( q_1 \) via \( w_{01}(\bigtriangledown) \) occurs in one state \( (0, 7, 1, 2) \); while switching from \( q_0 \) to \( q_2 \) via \( w_{02}(\bigtriangledown) \) occurs in four states \( (0, 8, 1, 1), (0, 4, 3, 1) \), \( (2, 7, 2, 3) \) and \( (0, 4, 1, 3) \).

**5 Experimental Evaluation**

We demonstrate the use of CAPTL for protocol synthesis and analysis on two case studies. The first extends the robot task planning problem introduced in Sec. 2. The second considers the problem of synthesizing an error-resilient scheduler.
Algorithm 2: Synthesis Procedure for P-CAPTL

Input: $\mathcal{M} = (S, \text{Act}, \mathbf{P}, s_0, A^P, L), \mathcal{A} = (Q, W, \Xi, \leftarrow, q_0)$
Result: $\Pi, c$ such that $\mathcal{M}, \Pi \models_c \mathcal{A}$

1. **foreach** $(q, q') \in Q \times Q$ do $R^q_i \leftarrow \emptyset$ // Initialize
2. $\Pi \leftarrow \emptyset, \bar{Q} \leftarrow \{q_0\}, Q \leftarrow \emptyset, R \leftarrow \text{Reach}(\mathcal{M}, s_0)$
3. **while** $\bar{Q} \neq \emptyset$ do // Partition $R$
4. Let $q \in \bar{Q}$. $Q \leftarrow Q \setminus \{q\}$, $Q \leftarrow Q \cup \{q\}$, $R^q_i \leftarrow R$
5. $\sigma_{(q, s_0)} \leftarrow \text{Synth}(\mathcal{M}; s_0, q), x_q \leftarrow \text{ReachP}(\mathcal{M}, s_0, \sigma_{(q, s_0)})$
6. **foreach** $w \in W(q)$ where $q' = Q(q, w)$ do
7. $\bar{R}^w_q \leftarrow \{ s \mid x_q[s] \in J_w \}$, $R^q_i \leftarrow R^q_i \setminus \bar{R}^w_q$
8. If $R^w_q \neq \emptyset \wedge q' \notin Q$ then $\bar{Q} \leftarrow \bar{Q} \cup \{q'\}$
9. **construct** $\mathcal{M}$ module such that // Construct $\mathcal{M}^\sigma_A$
10. **foreach** $[a] s \rightarrow p_i : (s')$ do add $[a] s \wedge \langle 1 \rangle \rightarrow p_i : (s') \wedge (\bigcirc)$
11. **construct** $\mathcal{A}$ module such that
12. **foreach** $q \in Q$ do add $[\tau] q_{\text{act}} = q \wedge \bigcirc \wedge L(R^q_i : s) \rightarrow (q_{\text{act}} = q) \wedge (\bigcirc)$
13. **foreach** $q \rightarrow w' q'$ do add $[w] q_{\text{act}} = q \wedge \bigcirc \wedge L(R^q_i : s) \rightarrow (q_{\text{act}} = q') \wedge (\bigcirc)$
14. **construct** $\hat{\sigma}$ module such that
15. **foreach** $\sigma_{(q, s_0)} \neq \emptyset$ and $s \in R$ do add $[\sigma_{(q, s_0)}(s)] q_{\text{act}} = q \wedge s \rightarrow \top$
16. $\mathcal{M}^\sigma_A \leftarrow \mathcal{M} \parallel \mathcal{A} \parallel \hat{\sigma}$
17. $(\Pi, c) \leftarrow \text{Synth}(\mathcal{M}^\sigma_A, (q_0, s_0, \bigcirc), \mathbb{P}[\forall_{q_i \in Q} (q_{\text{act}} = q_i) \wedge B_i])$

for digital microfluidic biochips. To this end, we implemented Algorithm 2 in MATLAB on top of a modified version of PRISM-games [19] (v.4.4), where ReachP functionality was added. The experiments presented in this section were run on an Intel Core i7 2.6GHz CPU with 16GB RAM.

**Robotic Task Planner.** Table 1 summarizes the performance results for running Algorithms 2 on various sizes of the running example. Notice that the number of choices in $\mathcal{M}^\sigma_A$ always matches the number of states, which agrees with the results from Lemma 3. In the three models, $q_0$ is always active in $s_0$, and thus is always verified. As the grid size grows larger, the probability of reaching the goal — and hence satisfying $q_0$ — becomes lower, dropping below 0.85 at the initial state in both (6x6) and (9x9). As a result, $q_1$ is never active (and hence is never verified) in the second and third models. We also notice that the total time required to run Algorithm 2 does not necessarily grow as the size of the problem grows. In fact, the total time required for (6x6) and (9x9) is lower than the one for (3x3). This is primarily due to the fact that $q_3$ is never reached or verified in the second and third models as we described. When comparing the model size for $\mathcal{M}$ and $\mathcal{M}^\sigma_A$, we notice that $|\mathcal{M}^\sigma_A| < |\mathcal{M}|$, with the difference being in orders of magnitude for larger models. However, the time required to construct $\mathcal{M}^\sigma_A$ is longer than the time required to construct $\mathcal{M}$.

**MEDA-Biochip Scheduler.** We now consider synthesizing error-resilient scheduler for micro-electrode-dot-array (MEDA) digital microfluidic biochips, where we borrow examples from [10, 21]. A biochip segment consists of a $W \times H$ matrix of on-chip actuators and sensors to manipulate microfluidic droplets, and
is further partitioned into $3 \times 3$ blocks. Two reservoirs are used to dispense droplets A and B. Various activation patterns can be applied to manipulate the droplets, including moving (moving droplets individually), flushing (moving both droplets at the same time in the same direction) and mixing (merging two droplets occupying the same block). As the biochip degrades, the actuators become less reliable, and an actuation command may not result in the droplet moving as expected. The probability of an error occurring is proportional to the total number of errors occurred in the same block.

Fig. 6 shows part of the segment scheduler (left) and the droplet (right) models. In the initial state, the scheduler can dispense both droplets through the \textit{dispense} action, where the droplet location $(x, y)$ can probabilistically deviate from the dispenser location $(x_0, y_0)$ with error $\epsilon$. Subsequently, droplets can be individually manipulated via \textit{mvA}$[d]$ and \textit{mvB}$[d]$ actions where $d$ is the direction, or together via \textit{flush}. The probability of successful manipulation (i.e., $1 - p(e_\ell)$) depends on both the number of errors occurred within the same block (i.e., $e_\ell$) and the activation pattern used. The scheduler executes \textit{update} to sense droplet locations and register errors.

The primary task of the scheduler is to perform a mixing operation within the given segment ($q_0$). However, if the droplets are dispensed and (due to faulty blocks) the probability of a successful mixing operation is below 0.85 ($w_{01}$), salvaging the dispensed droplets by moving them to an adjacent segment is prioritized ($q_1$). If the mixing probability drops below 0.7 ($w_{02}$), or if the salvaging probability drops below 0.7 ($w_{12}$), the scheduler is to abort the operation ($q_2$). The aforementioned requirements are formalized using CAPTL as shown in Fig. 7.

The set of objectives is $Q = \{q_0, q_1, q_2\}$, and the set of contexts is defined as $W = \{w_{01}, w_{02}, w_{12}\}$. The performance results for running Algorithm 2 on three different segment sizes is reported in Table 1.

6 Conclusion

In this paper we have introduced context-aware probabilistic temporal logic (CAPTL). The logic provides intuitive means to formalize requirements that comprises a number of objectives with an underlying priority structure. CAPTL allows for defining context (i.e., probabilistic conditions) as the basis for switching
between two different objectives. We have presented CAPTL syntax and semantics for Markov-decision processes (MDPs). We have also investigated the CAPTL synthesis problem, both from PCTL and CAPTL-based approaches, where we have shown that the latter provides significant performance improvements. To demonstrate our work, we have presented two case studies.

As this work has primarily considered CAPTL semantics for MDPs, further investigation is required to generalize the results for stochastic multi-player games. Another research direction involves expanding the results to include PCTL fragments beyond persistence objectives, such as safety, bounded reachability and reward-based objectives.

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A Proofs

Proposition 1 Algorithm 1 terminates; and returns $\Pi$, $c$ iff $M, \Pi \models c A$.

Proof. We break the proof into two parts: termination and correctness.

Termination. We first note that $|S|, |Q|, |W| < \infty$ by definition.

- The foreach loop (line 6) terminates either by exhausting all $w \in W(q)$ (which is finite), or by breaking whenever repeat $= \top$. The loop can only run indefinitely if $|W(q)| = \infty$.
- For the inner while loop (line 5), the only way to remain indefinitely in that loop is for repeat $= \top$ to always hold, which is only set whenever $M, s \models w$. However, whenever $M, s \models w$ holds, $q$ is updated (line 8). Since $Q$ is finite and $A$ is acyclic, recursion over $Q$ ends in a finite number of loops, ending with a $q \in Q$ where $W(q) = \emptyset$. Hence, $M, s \models w$ cannot hold indefinitely.
- For the outermost while loop (line 3), line 4 dictates that the set $\bar{S}_q$ shrinks by one state $s$ each and every loop, which is also added to $\bar{S}_q$. Hence, for $S_q \neq \emptyset$ to hold indefinitely for some $q \in Q$, $Post(M, s, \sigma(s)) \setminus \bar{S}_q \neq \emptyset$ must always hold (line 10). However, this mandates that $\bar{S}_q$ can grow indefinitely. Since $|S| < \infty$ by definition, $\bar{S}_q$ cannot grow indefinitely, and the loop eventually terminates in a finite number of iterations.

Correctness. Initially, $s = s_0$ and $q = q_0$. We identify the following cases:

- (a) Case $W(q) = \emptyset$. Then repeat $= \bot$, $\sigma$ is synthesized such that $M_0^\sigma \models q$, and $(s, q, \sigma(s))$ is added to $\Pi$.
- (b) Case $\forall w \in W(q), M, s \not\models w$. Then repeat $= \bot$, $\sigma$ is synthesized such that $M_0^\sigma \models q$, and $(s, q, \sigma(s))$ is added to $\Pi$.
- (c) Case $\exists w \in W(q), M, s \models w$. Then from Definition 10 we conclude that $\forall \tilde{w} \in W(q) \setminus \{w\}$ it holds that $M, s \not\models \tilde{w}$. Consequently, repeat $= \top$, $(s, q, w)$ is added to $\Pi$, and $q$ is updated. Since $A$ is finite and acyclic, the loop eventually halts with condition (a) or (b) becoming true.

\[\Box\]

Lemma 1 (Local Strategy Dominance). Let $M = (S, Act, P, s_0, AP, L)$ and $q = \max[\Diamond \Box B]$. For all $s \in Reach(M, s_0)$, $\sigma_{(q, s)} = \sigma_{(q, s_0)}|_{Reach(M, s)}$.

Proof. In the first part of the proof, we establish the used notation. In the second part, we show that if $s \in Reach(M, s_0)$, then the domain of $\sigma_{(q, s)}$ is subset of the domain of $\sigma_{(q, s_0)}$. In the last part, we show that $\sigma_{(q, s)} = \sigma_{(q, s_0)}|_{Reach(M, s)}$

Notation. For a function $f : A \rightarrow B$, we will use $dom(f)$ to denote the domain of $f$, and $f(a) \Downarrow a \in A$, $f(a) \Uparrow a \not\in A$. We will use $\Sigma_s : S \rightarrow Act$ to denote the set of all possible (pure memoryless) strategies from state $s \in S$. We assume that for every $\sigma \in \Sigma_s$, $\sigma(s')$ is defined for every $s' \in Reach(M, s)$.
Well-Definedness. Now, let us consider an arbitrary strategy $\sigma_s \in \Sigma_s$. Hence, $\text{dom}(\sigma_s) = \text{Reach}(M, s)$. Moreover, consider an arbitrary state $s \in \text{Reach}(M, s_0)$, and an associated strategy $\sigma_s \in \Sigma_s$. In such case, $\text{dom}(\sigma_s) = \text{Reach}(M, s)$. Assume we can find a state $s' \in S$ such that $\sigma_s(s') \downarrow$ and $\sigma_{s_0}(s') \uparrow$. The assumptions imply that

$$\frac{\sigma_s(s') \downarrow}{s' \in \text{Reach}(M, s)}, \quad \frac{\sigma_{s_0}(s') \uparrow}{s' \notin \text{Reach}(M, s_0)}$$

(3)

However,

$$s \in \text{Reach}(M, s_0)$$

(4)

Since the inferences from (3) and (4) contradict each other, we conclude that $\sigma_s(s') \downarrow \implies \sigma_{s_0}(s') \downarrow$. That is, for every $s \in \text{Reach}(M, s_0)$, $\text{dom}(\sigma_s) \subseteq \text{dom}(\sigma_{s_0})$.

Equivalence. We now prove that $\sigma_{(q, s)}(s') = \sigma_{(q, s_0)}(s')$ for every $s' \in \text{Reach}(M, s)$.

To this end, we first recall a well-established result on the existence of memoryless strategies for probabilistic reachability requirements. For $M$ and $q = \max_{s} [\Diamond B]$, let $R = \text{Reach}(M, s_0)$. The maximum probability of reaching the target set $B$ from a state $s \in R$ can be formulated as

$$p_{\max}(s, q) = \sup_{\sigma \in \Sigma} \Pr_{M, \sigma}^s \{ s \ldots t_0 t_1 \ldots \in \text{Paths}_{M, \sigma}^s | \forall i \geq 0, t_i \in B \}.$$ 

As the computation of $p_{\max}(s', q)$ at state $s' \in \text{Reach}(M, s_0)$ is independent of the path that lead to $s'$, the optimal action associated with $s'$ is also independent of such path. Therefore, $\sigma_{(q, s)}(s') = \sigma_{(q, s_0)}(s')$.

□

Lemma 2 (Partitioning). Let $M = (S, \text{Act}, P, s_0, AP, L)$, $A = (Q, W, \Xi, \rightarrow, q_0)$, and $R = \text{Reach}(M, s_0)$. For every $q \in Q$, $\bigcup_{q' \in Q} R^q_{q'} = R$; and $R^q \cap R^q' = \emptyset$ for every $q' \neq q''$.

Proof. For every $q \in Q$, we identify two cases:
- Case $W(q) = \emptyset$. In this case, $R^q_q = \emptyset$ for every $q' \neq q$, and $R^q_q = R \setminus \bigcap_{q' \neq q} R^q_{q'} = R$. Therefore, the lemma holds.
- Case $W(q) \neq \emptyset$. By definition, $s \in R^q_q$ implies that $x_q[s] \notin J'$ for every $w_{(q, q')} = \max_{s \in J'} [\Diamond B] \in W(q)$. Therefore, $R^q_q \cap R^q_{q'} = \emptyset$ holds for every $q' \neq q$ (a). Next, let us assume that we find $s \in R$ such that $s \in R^q_q$ and $s \in R^q''_{q''}$, where $q' \neq q''$ and $w_{(q, q'')} = \max_{s \in J''} [\Diamond B] \in W(q)$. This implies that $J' \cap J'' = \emptyset$, which contradicts Definition 10. Hence, such a does not exist, and $R^q_q \cup R^q_{q''} = \emptyset$ holds for every $q' \neq q''$ (b). From (a) and (b), we conclude that the lemma holds.

□

Lemma 3 (Induced DTMC). $M^P_\alpha$ constructed using Definition 12 is a DTMC.
Proof. Assume that there exists a state \( v \in V \) with at least two actions \( \{\alpha_1, \alpha_2\} \in \mathcal{A}(v) \). We identify two disjoint subsets of \( V \), namely, \( V_1 = \{v \in V \mid v = (s, q, \mathbb{1})\} \) and \( V_2 = \{v \in V \mid v = (s, q, \mathbb{2})\} \). In case \( v = (s, q, \mathbb{1}) \in V_1 \), only transitions defined by \( R1 \) are allowed. Since for any \( q \) at most one action \( \sigma(q, s_0)(s) \) is allowed, we conclude that \( v \notin V_1 \). In case \( v = (s, q, \mathbb{2}) \in V_2 \), only transitions defined by \( R2 \) and \( R3 \) are allowed. Our assumption requires that \( s \in S'_q \) and \( s \notin S'_q \). Since \( S'_q \cap S'_q = \emptyset \), therefore, \( v \notin V_2 \). Consequently, \( v \notin V \), which contradicts our assumption. We conclude that \( \mathcal{M}_A^\sigma \) has no nondeterministic choices.

\[ \eta \]

\textbf{Theorem I (Stutter-Equivalence).} Let \( \mathcal{M}, \mathcal{A}, \) and \( \Pi \in \mathcal{P} \) be such that \( \mathcal{M}, \Pi \models \mathcal{A} \). For every \( \pi \in \text{FPath}_{\mathcal{M},\Pi} \) there exists \( \tilde{\pi} \in \text{FPath}_{\mathcal{M}_A^\sigma} \) such that \( \pi \equiv \tilde{\pi} \) and \( \Pr_{\mathcal{M},\Pi}(\pi) = \Pr_{\mathcal{M}_A^\sigma}(\tilde{\pi}) \). For every \( \tilde{\pi} \in \text{FPath}_{\mathcal{M}_A^\sigma} \), where last(\( \tilde{\pi} \)) \( \in V_2 \), there exists \( \hat{\pi} \in \text{FPath}_{\mathcal{M},\Pi} \) such that \( \hat{\pi} \equiv \pi \) and \( \Pr_{\mathcal{M},\Pi}(\hat{\pi}) = \Pr_{\mathcal{M}_A^\sigma}(\tilde{\pi}) \).

Proof. The transitions of \( \mathcal{M}^\Pi \) can be partitioned into two subsets \( \rightarrow_{\mathcal{A}ct} \) and \( \rightarrow_W \), where the transitions take the forms \( (s, q) \xrightarrow{\alpha} (s', q) \) and \( (s, q) \xrightarrow{w} (s, q') \), respectively. Starting from \( (s_0, q_0) \), let us assume that \( (s_0, q_0, \mathbb{1}) \xrightarrow{a,p} (s_1, q_0) \), which is based on \( \sigma(s_0, q_0)(s) \). Similarly, \( \mathcal{M}_A^\sigma \) exhibits the execution fragment

\[ (s_0, q_0, \mathbb{1}) \rightarrow (s_0, q_0, \mathbb{1}) \xrightarrow{a,p} (s_1, q_0) \]

Since \( \hat{a} = \sigma(s_0, q_0)(s) \), we conclude that \( \hat{a} = a \), and hence \( \hat{p} = p \) and \( \hat{s}_1 = s_1 \). From Lemma 1, we know that \( \sigma(s_0, q_0)(s) = \sigma(s, q)(s) \). Hence, for every execution fragment \( \varrho_{\mathcal{A}ct} \) in \( \mathcal{M}^\Pi \), where no objective switching occurs, we can find an execution \( \hat{\varrho}_{\mathcal{A}ct} \) in \( \mathcal{M}_A^\sigma \) such that

\[ \varrho_{\mathcal{A}ct} = (s_0, q_0) \]
\[ \hat{\varrho}_{\mathcal{A}ct} = (s_0, q_0, \mathbb{1}) \xrightarrow{a_1,p_1} (s_1, q_0) \xrightarrow{a_2,p_2} \cdots (s_i, q_0) \]

\[ \text{trace}(\varrho_{\mathcal{A}ct}) = L(s_0) \xrightarrow{1 \text{time}} L(s_1) \xrightarrow{1 \text{time}} \cdots L(s_i) \xrightarrow{1 \text{time}} \]
\[ \text{trace}(\hat{\varrho}_{\mathcal{A}ct}) = L(s_0) \xrightarrow{2 \text{times}} L(s_1) \xrightarrow{2 \text{times}} \cdots L(s_i) \xrightarrow{2 \text{times}} \]

\[ \Pr(\varrho_{\mathcal{A}ct}) = (p_0) \cdot (p_1) \cdots (p_i) \]
\[ \Pr(\hat{\varrho}_{\mathcal{A}ct}) = (p_0) \cdot (p_1) \cdots (p_i) \]

Therefore, \( \varrho_{\mathcal{A}ct} \equiv \hat{\varrho}_{\mathcal{A}ct} \). Now, consider an execution fragment that ends with switching the active objective. In that case, for every execution fragment \( \varrho \) we can find \( \hat{\varrho} \) such that

\[ \varrho_w = (s, q) \xrightarrow{w(q,q')} (s, q') \]
\[ \hat{\varrho}_w = (s, q, \mathbb{1}) \xrightarrow{w(q,q')} (s, q', \mathbb{2}) \]
Context-Aware Temporal Logic for Probabilistic Systems

\[ \text{trace}(\varrho_W) = \frac{L(s)L(s)}{2-\text{times}} \]
\[ \text{trace}(\hat{\varrho}_W) = \frac{L(s)L(s)}{2-\text{times}} \]

and

\[ \Pr(\varrho_W) = 1 \]
\[ \Pr(\hat{\varrho}_W) = 1 \]

Therefore, \( \varrho_W \triangleq \hat{\varrho}_W \). Using induction, we can show that for every arbitrary execution \( \varrho \) there exists \( \hat{\varrho} \) such that \( \varrho \triangleq \hat{\varrho} \), where

\[ \text{trace}(\varrho) = (A_0 + A_0A_0) (A_1 + A_1A_1) \ldots (A_n + A_nA_n) \in (P(AP))^+ \]
\[ \text{trace}(\hat{\varrho}) = (A_0A_0) (A_1A_1) \ldots (A_nA_n) \in (P(AP))^+ \]

and \( \Pr(\varrho) = \Pr(\hat{\varrho}) \).

\[ \square \]

**Theorem 2.** Algorithm \( \mathcal{A} \) terminates; and returns \( \Pi, c \) iff \( \mathcal{M}, \Pi \models_c \mathcal{A} \).

**Proof.** We break the proof into two parts: termination and correctness.

**Termination.** We first note that \( |S|, |Q|, |W| < \infty \) by definition.
- The **foreach** loop (line 6) terminates by exhausting all \( w \in W(q) \) (which is finite). The loop can only run indefinitely if \( |W(q)| = \infty \).
- For the **while** loop (line 3), line 4 dictates that the set \( \hat{Q} \) shrinks by one element \( q \) each and every loop, which is also added to \( Q \). Moreover, line–8 dictates that an objective \( q' \) is added to \( Q \) only if it is not in \( \hat{Q} \). That is, every objective \( q \in Q \) can be added at most once to \( Q \). Since \( |Q| < \infty \) by definition, the condition \( \hat{Q} = \emptyset \) is met in a finite number of iterations.
- Since \( |M| < \infty \), every **construct** code blocks also terminates in a finite number of iterations.

**Correctness.** From Theorem 1 we know that the paths in \( \mathcal{M}^\Pi \) and \( \mathcal{M}^{\hat{\Pi}} \) are stutter equivalent and probabilistically bisimilar. Hence, for every \( q_i \in Q \), the two probability measures

\[ \Pr_{\mathcal{M}^\Pi} \left( \{ \pi \mid \pi \models \Diamond(q_i, s) \land C(q_i, s) = 1 \} \right) \]
\[ \Pr_{\mathcal{M}^{\hat{\Pi}}} \left( \{ \hat{\pi} \mid \hat{\pi} \models \Diamond(q_{\text{act}} = q_i) \land B_i \} \right) \]

are equivalent. Hence, the algorithm returns a correct answer.

\[ \square \]