CRYSTAL BASES OF QUANTUM AFFINE ALGEBRAS AND AFFINE KAZHDAN-LUSZTIG POLYNOMIALS

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Abstract. We present a fast version of the algorithm of Lascoux, Leclerc, and Thibon for the lower global crystal base for the Fock representation of quantum affine \( \mathfrak{sl}_n \). We also show that the coefficients of the lower global crystal base coincide with certain affine Kazhdan-Lusztig polynomials. Our algorithm allows fast computation of decomposition numbers for tilting modules for quantum \( \mathfrak{sl}_k \) at roots of unity, and for the Hecke algebra of type \( A_n \) at roots of unity.

1. Introduction

Lascoux, Leclerc, and Thibon in [LLT] conjectured a connection between the decomposition numbers for the Hecke algebra \( H_n(q; \mathbb{C}) \) of type \( A_{n-1} \), for \( q \) a primitive \( l \)-th root of unity in \( \mathbb{C} \), and the lower global crystal base for the simple highest weight module \( M(\Lambda_0) \) of the affine quantum group \( U_v(\mathfrak{sl}_l) \). More explicitly, they conjectured that the coefficients \( d_{\lambda,\mu}(v) \) of the lower global crystal base, which a priori lie in \( \mathbb{Q}[v] \), satisfy

1. \( d_{\lambda,\mu}(1) = d_{\lambda,\mu} \), where \( d_{\lambda,\mu} = (S^\lambda : D^\mu) \) is the multiplicity of the simple \( H_n(q; \mathbb{C}) \) module \( D^\mu \) in a composition series for the Specht module \( S^\lambda \).

2. \( d_{\lambda,\mu}(v) \in \mathbb{N}[v] \) (positivity).

The Specht module is a canonical indecomposable module of \( H_n(q; \mathbb{C}) \), which has, when \( \lambda \) is \( l \)-regular, a unique simple quotient \( D^\lambda \); see [DJ1, DJ2]. The first of these conjectures was proved by Ariki [Ar]. Varagnolo and Vasserot [VV] have recently proved a more general conjecture of Leclerc and Thibon [LT] concerning \( q \)-Schur algebras, and have obtained a proof of the positivity conjecture (2) as well. A recent review of Geck [G] discusses the work of LLT and Ariki.

Lascoux, Leclerc, and Thibon also gave a recursive algorithm for computing the lower global crystal base in [LLT]; we will consider improvements of their algorithm in this paper.

A similar connection between certain affine Kazhdan-Lusztig polynomials and decomposition numbers for the quantum group \( U_q(\mathfrak{g}) \), where \( \mathfrak{g} \) is any simple Lie algebra over \( \mathbb{C} \) and \( q \) is a root of unity, was conjectured by Soergel in [S1] and proved in [S2]. Soergel’s theorem is that the affine Kazhdan-Lusztig polynomials \( n_{\lambda,\mu}(v) \) described in [S1] satisfy

\[
n_{\lambda+\rho,\mu+\rho}(1) = (T_\mu : \nabla_\lambda),
\]
where $T(\mu)$ is the indecomposable tilting module of $U_q(g)$ with highest weight $\mu$ and $\nabla(\lambda)$ is the “good” module with highest weight $\lambda$. We quickly review the relevant definitions from [An1]: A good module $\nabla(\mu)$ is one induced from the character $\mu$ of the Borel subalgebra $U_q^0U_q^+$. A Weyl module $\Delta(\lambda)$ is $\nabla(-w_0\lambda)^*$, where $w_0$ is the longest element of the Weyl group of $g$. A tilting module is one with a filtration by good modules and by Weyl modules; there exists a unique indecomposable tilting module $T(\mu)$ with highest weight $\mu$. Finally ($T_{\mu} : \nabla(\lambda)$) is the number of subquotients isomorphic to $\nabla(\lambda)$ in a good filtration of $T(\mu)$.

The affine Kazhdan-Lusztig polynomials are also computed by a recursive algorithm, which is described in [S1].

In this paper, we present a fast version of the algorithm of [LLT] for the coefficients of the lower global crystal base. We also show that the two polynomial analogues of decomposition numbers coincide (for Lie type A),

$$d_{\lambda,\mu}(v) = n_{\lambda+\rho,\mu+\rho}(v).$$

The first idea of our modified LLT algorithm is to fix an integer $k$ and compute the $d_{\lambda,\mu}(v)$ only for Young diagrams $\lambda, \mu$ with no more than $k$ rows; in this way, we can identify Young diagrams with dominant integral weights of $U_q(slk)$. We wish to compute a formal sum:

$$\tilde{G}(\mu) = \sum\{d_{\lambda,\mu}(v) : \ell(\lambda) \leq k\}.$$  

The method of the original LLT algorithm is the following: One considers a certain path in the Weyl chamber of $sl_k$ from $\rho$ to $\mu + \rho$, and all “conjugate” paths obtained by reflections in hyperplanes of the affine Weyl group of $sl_k$ acting at level $l$. (Here $\rho$ is one-half the sum of positive roots.) The conjugate paths, which end at diagrams $\lambda + \rho$ which are in the orbit of $\mu + \rho$ under the action of the affine Weyl group, are assigned Laurent polynomial weights according to a rule involving the local geometry at points where the path meets or leaves reflection hyperplanes. These weights provide a first approximation to $\tilde{G}(\mu)$:

$$\tilde{A}(\mu) = \sum\{\alpha_{\lambda,\mu}(v) : \ell(\lambda) \leq k\}.$$  

The next step is a “trimming” operation: The approximation is corrected by subtracting certain multiples of $\tilde{G}(\lambda)$ for $\lambda + \rho$ an endpoint of a conjugate path.

In our modified algorithm, we take instead a path to $\mu + \rho$ which begins at a nearby point $\nu + \rho$ which is as singular as possible, that is, has large isotropy for the action of the affine Weyl group. The recursive computation of $\tilde{G}(\mu)$ then involves $\tilde{G}(\nu)$ as well as the $\tilde{G}(\lambda)$ for $\lambda + \rho$ the endpoint of a conjugate path. However, as the path is short, it has few conjugates; furthermore, $\tilde{G}(\nu)$ is relatively easy to compute because of the large isotropy group of $\nu + \rho$.

When the point $\mu + \rho$ is sufficiently far from the walls of the Weyl chamber, our algorithm (which was suggested by our previous work [GW]) is roughly equivalent to Soergel’s algorithm [S1] for affine Kazhdan-Lusztig polynomials (which involves the periodic Kazhdan-Lusztig module). The main advantage of our approach comes for diagrams which are near the boundary of the Weyl chamber.

Regarding the equality of the polynomial analogues of the decomposition numbers, note that Schur-Weyl duality [Du] already suggests that the decomposition numbers coincide: $d_{\lambda,\mu} = \vdots$
CRYSTAL BASES AND KAZHDAN-LUSZTIG POLYNOMIALS

Our proof of the equality $d_{\lambda,\mu}(v) = n_{\lambda+\rho,\mu+\rho}(v)$ is combinatorial; we show that the $d_{\lambda,\mu}(v)$ satisfy the same recursive relations which define the Kazhdan-Lusztig polynomials. Varagnolo and Vasserot have recently given a representation theoretic proof that the $d_{\lambda,\mu}(v)$ coincide with Kazhdan-Lusztig polynomials [VV], Theorem 9.3. Steen Ryom-Hansen has informed us that he has also proved the coincidence of the two polynomial analogues of decomposition numbers [R-H].

We intend to address generalizations of this work to other Lie types, as well as relations to tensor ideals for tilting modules, in future publications.

2. Preliminaries

Fix a positive integer $l$. Write $[n] = \text{Mod}(n,l) = n - \lfloor n/l \rfloor l$; thus $n = \lfloor n/l \rfloor l + [n]$, and $0 \leq [n] \leq l - 1$.

We adopt the conventions of [M] regarding partitions, Young diagrams and tableaux. The Young diagram of a partition $\lambda$ is the set of points $(i,j) \in \mathbb{N} \times \mathbb{N}$ such that $1 \leq j \leq \lambda_i$; we identify the partition with its diagram.

For a point $(i,j) \in \mathbb{N} \times \mathbb{N}$, the content of $(i,j)$ is $i - j$, and the $l$-residue is $[i - j]$; in particular these quantities are defined for the nodes of a Young diagram. A Young diagram is called $l$-regular if it has no more than $l - 1$ rows of any length. The length $\ell(\lambda)$ of $\lambda$ is the number of (non-empty) rows. A tableau is regarded alternatively as a numbering of the nodes of a Young diagram, or as an increasing sequence of Young diagrams.

A node $(i,\lambda_i) \in \lambda$ is called removable if $i = \ell(\lambda)$ or if $i < \ell(\lambda)$ and $\lambda_{i+1} < \lambda_i$; that is, if one obtains a Young diagram by removing the node from $\lambda$. The node is called a removable $r$-node if it is removable and if its $l$-residue is equal to $r$.

A point $(i,\lambda_i + 1)$ is called an indent node of $\lambda$ if $i = 1$ or if $i > 1$ and $\lambda_{i-1} > \lambda_i$, that is, if one obtains a Young diagram by adding a new node to $\lambda$ at $(i,\lambda_i + 1)$. Note that $(\ell(\lambda) + 1,1)$ counts as an indent node. The node is called an indent $r$-node of $\lambda$ if it is an indent node and if its $l$-residue is equal to $r$.

We will frequently use the notion of a face of a closed convex set of a Euclidean space $\mathbb{R}^k$. A face $F$ of a closed convex set $C$ is an extreme convex subset of $C$; this means that if a point $f \in F$ is a convex combination of points $a,b \in C$, then $a,b \in F$. The boundary $\partial F$ of a face $F$ is the union of all proper subfaces; this does not generally coincide with the topological boundary of $F$ in $\mathbb{R}^k$. The interior of a face $F$ is $F \setminus \partial F$. An open face is the interior of a closed face.

We will denote the unit vector in $\mathbb{R}^k$ with a 1 in the $i$th coordinate and zeros elsewhere by $\epsilon_i$.

3. The LLT Algorithm

We first recall the setting for the algorithm of Lascoux, Leclerc, and Thibon. The affine quantum universal enveloping algebra $U_v(\hat{\mathfrak{sl}}_l)$, defined over $\mathbb{Q}(v)$, has generators $e_i$ and $f_i$, for $0 \leq i \leq l - 1$, and $v^h$, for $h$ in the dual weight lattice $P^*$ of $\hat{\mathfrak{sl}}_l$. We refer, for example to [LLT],
Section 4, for the list of relations satisfied by the generators. Write $U^-$ for the subalgebra generated by the $f_i$.

The algebra $U_v(\hat{\mathfrak{sl}}_l)$ has a representation on the “Fock space” $\mathcal{F}$, which is the $Q(v)$-vector space spanned by Young diagrams of all sizes. The generators $q^h$ are diagonal on the basis of Young diagrams. The generators $e_i$ act as “annihilation operators,” removing nodes from a Young diagram, and the generators $f_i$ act as “creation operators,” adding nodes to a Young diagram. This representation was described by Hayashi [H] and Misra and Miwa [MM]; see also [LLT], Section 4. We will need the explicit description of the action only for the $f_i$: 

\[ f_i \lambda = \sum_{\nu} v^{N(\lambda, \nu)} \nu, \]

where the sum is over all diagrams $\nu$ which can be obtained from $\lambda$ by adding one node of $l$-residue $i$, and $N(\lambda, \nu)$ is given as follows. If $\nu$ is obtained from $\lambda$ by adding a node of $l$-residue $i$ in a certain row $r$, then $N(\lambda, \nu)$ is the number of indent $i$-nodes of $\lambda$ in rows $r' < r$, less the number of removable $i$-nodes of $\lambda$ in rows $r' < r$. Of course, $f_i \lambda = 0$ if $\lambda$ has no indent $i$-nodes.

We write $M$ for the cyclic submodule of $\mathcal{F}$ generated by the empty diagram $\emptyset$, $M = U_v(\hat{\mathfrak{sl}}_l) \emptyset = U^- \emptyset$. Then $M$ is isomorphic to the simple integrable highest weight module $M(\Lambda_0)$. Let $A$ be the subring of $Q(v)$ of rational functions with no poles at $v = 0$, and let $L$ denote the $A$-span of all Young diagrams. Let $U^-_Q$ denote the $Q[v, v^{-1}]$-subalgebra of $U^-$ generated by all divided powers $f_i^{(m)} = f_i^m / [m]!$. Here $[m]$ is the $v$-integer, $[m] = \frac{v^m - v^{-m}}{v - v^{-1}}$, and $[m]! = [m][m-1]\cdots[1]$. Let $M_Q$ denote the cyclic $U^-_Q$ module, $M_Q = U^-_Q \emptyset$.

There is an involution $a \mapsto \bar{a}$ of $U_v(\hat{\mathfrak{sl}}_l)$ such that the $e_i$ and $f_i$ are self-dual, $\bar{v} = v^{-1}$, and $\bar{v}^h = v^{-h}$. This induces an involution on $M$ by $(a \emptyset)^- = \bar{a} \emptyset$.

The following theorem is due to Kashiwara (see [Ka1, Ka2]).

**Theorem 3.1.** There is a unique $Q[v, v^{-1}]$-basis $\{G(\mu)\}$ of $M_Q$ indexed by $l$-regular Young diagrams $\mu$, and satisfying:

\begin{itemize}
  \item[(G1)] $G(\mu) \equiv \mu \pmod{L}$; and
  \item[(G2)] $G(\mu)$ is self-dual.
\end{itemize}

This basis is called the *lower global crystal basis*. The algorithm given by Lasoux, Leclerc and Thibon computes the elements of this basis.

As observed in [LLT], in order to obtain the lower global crystal basis of $M(\Lambda_0)$, it is convenient to first compute for each $l$-regular Young diagram $\mu$ an element $A(\mu) \in M_Q$ satisfying:

\begin{itemize}
  \item[(A1)] $A(\mu) = \sum_{\lambda} \alpha_{\lambda \mu} \mu$, where $\alpha_{\mu \nu} = 1$, $\alpha_{\lambda \mu} \in \mathbb{Z}[v, v^{-1}]$, and $\alpha_{\lambda \mu} = 0$ unless $\lambda \subseteq \mu$, and
  \item[(A2)] $A(\mu)$ is self-dual.
\end{itemize}
Such elements \((A_\mu)\) are by no means unique.

The \(G(\mu)\) can be computed recursively from the \(A(\mu)\) by a triangular reduction algorithm (Gaussian elimination). First, if \(\mu\) is the smallest \(l\)-regular Young diagram of a given size \(n\) in lexicographic order, then \(G(\mu) = A(\mu) = \mu\), as follows from (A1) and (G1). Next, given an \(l\)-regular Young diagram of size \(n\), one can compute \(G(\mu)\) using \(A(\mu)\) and the \(G(\mu')\) for \(\mu' < \mu\) in lexicographic order. Namely, if there is some \(\mu' < \mu\) such that the coefficient \(\alpha_{\mu'\mu}(v) \notin v\mathbb{Z}[v]\), then take the lexicographically largest such \(\mu'\). Necessarily \(\mu'\) is \(l\)-regular.

Writing \(\alpha_{\mu'\mu}(v) = \sum_{n=-N}^{N} \alpha_{\mu'\mu}^n v^n\), put \(\gamma_{\mu'\mu} = -\sum_{n=-N}^{N} \alpha_{\mu'\mu}^n (v^n + v^{-n}) + \alpha_{\mu'\mu}^0\), and replace \(A(\mu)\) by \(A(\mu) - \gamma_{\mu'\mu} G(\mu')\). Continue in this way until all coefficients in \(A(\mu)\) for diagrams \(\mu' < \mu\) lie in \(v\mathbb{Z}[v]\). Then one has \(A(\mu) = G(\mu)\) by uniqueness of \(G(\mu)\).

Next we consider how to compute elements \(A(\mu)\) satisfying (A1) and (A2).

**Definition 3.2.** Let \(T\) be a standard (skew) tableau of size \(n\). The \(l\)-residue sequence associated to \(T\) is described as follows. For each \(i, 1 \leq i \leq n\), let \(a_i\) be the \(l\)-residue of the node containing \(i\) in the tableau \(T\). Now write
\[
(a_1, \ldots, a_n) = (r_{m_1}, r_{m_2}, \ldots, r_{m_s}).
\]

More precisely, consider the maximal constant subsequences of the sequence \((a_i)\) and let \(r_i\) be the constant value of the \(i^{th}\) constant subsequence and \(m_i\) be the length of the \(i^{th}\) constant subsequence.

**Definition 3.3.** We say that two standard tableaux are \(l\)-conjugate if they have the same \(l\)-residue sequence. Similarly we say that two standard skew tableaux are \(l\)-conjugate if they have the same starting shape and the same \(l\)-residue sequence.

Note that \(l\)-conjugate (skew) tableaux can be of different shapes.

**Definition 3.4.** Let \(T\) be a standard (skew) tableau. Define a corresponding element \(f(T) \in U_\mathbb{Q}\) by
\[
f(T) = f_{r_{m_1}} \cdots f_{r_{m_s}},
\]
where \((r_{m_1}, r_{m_2}, \ldots, r_{m_s})\) is the \(l\)-residue sequence of \(T\).

**Definition 3.5.** If \(T\) is a standard tableau (i.e. not a skew tableau), define \(A(T) = f(T)\emptyset\).

Note that the element \(A(T)\) is always self-dual.

The following combinatorial lemma from [LLT] is important for understanding the properties of the elements of the form \(A(T)\).
Lemma 3.6. For any Young diagram \( \lambda \),
\[
 f^{(m)}_{r} \lambda = \sum_{\mu} v^{N(\lambda, \mu)} \mu,
\]
where the sum is over all Young diagrams \( \mu \) such that \( \mu \setminus \lambda \) consists of \( m \) nodes all with \( l \)-residue \( r \), and \( N(\lambda, \mu) \) is computed as follows: Let
\[
 T_{0} = (\lambda = \mu^{(0)} \subseteq \mu^{(1)} \subseteq \cdots \subseteq \mu^{(m)} = \mu)
\]
be the standard skew tableau of shape \( \mu \setminus \lambda \) in which the \( m \) nodes of \( \mu \setminus \lambda \) are filled from top to bottom. Put
\[
 N(T_{0}) = \sum_{i} N(\mu^{(i-1)}, \mu^{(i)}),
\]
where now \( N(\mu^{(i-1)}, \mu^{(i)}) \) is as in the explanation following Equation 3.4. Then
\[
 N(\lambda, \mu) = N(T_{0}) + \binom{m}{2}.
\]

Proof. See the proof of Lemma 6.2 in [LLT]. \( \square \)

It is evident that \( N(\lambda, \mu) \) depends only on the position of the \( m \) nodes of \( \mu \setminus \lambda \) relative to other possible indent \( r \)-nodes of \( \lambda \) and possible removable \( r \)-nodes of \( \lambda \).

Lemma 3.7. Suppose that every removable \( r \)-node of \( \lambda \) lies below every indent \( r \)-node of \( \lambda \) and furthermore that \( \mu \setminus \lambda \) consists of the first (highest) \( m \) indent \( r \)-nodes of \( \lambda \). Then \( N(\lambda, \mu) = 0 \).

Proof. In this case \( N(T_{0}) = -\binom{m}{2} \); compare the proof of Lemma 6.4 in [LLT]. \( \square \)

We describe a map from standard (skew) tableaux to certain (row- and column-strict) semi-standard (skew) tableaux of the same shape, as follows. Let \( T \) be a standard (skew) tableau of shape \( \lambda \setminus \nu \), with \( l \)-residue sequence \((r_{1}^{m_{1}}, r_{2}^{m_{2}}, \ldots, r_{s}^{m_{s}})\). The corresponding semi-standard tableau is
\[
 S(T) = (\nu = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(s)} = \lambda),
\]
where \( \lambda^{(i)} \setminus \nu \) is the subdiagram of \( \lambda \setminus \nu \) containing the numbers 1 through \( m_{1} + \cdots + m_{i} \). Thus \( \lambda^{(i)} \setminus \lambda^{(i-1)} \) is a skew diagram consisting of \( m_{i} \) nodes all with \( l \)-residue equal to \( r_{i} \). As there are \( m_{i}! \) different standard fillings of \( \lambda^{(i)} \setminus \lambda^{(i-1)} \) for each \( i \), \( S(T) \) is the image of \( \prod_{i} m_{i}! \) different standard (skew) tableaux. Define
\[
 \text{wt}(T) = \text{wt}(S(T)) = v \sum_{i} N(\lambda^{(i-1)}, \lambda^{(i)}),
\]
where \( N(\lambda^{(i-1)}, \lambda^{(i)}) \) is as in Lemma 3.4

The operator \( f(T) \) applied to the empty diagram may be thought of as generating all semi-standard tableaux \( S(T') \), where \( T' \) is a standard tableau \( l \)-conjugate to \( T \). Then
\[
 A(T) = \sum S \text{wt}(S) \lambda(S),
\]
where the sum is over all $S = S(T')$ with $T'$ $l$-conjugate to $T$, and $\lambda(S)$ denotes the shape of the semi-standard tableau $S$.

Let $T$ be a standard (skew) tableau of shape $\lambda \setminus \nu$, let $(r_1^{m_1}, r_2^{m_2}, \ldots, r_s^{m_s})$ be the $l$-residue sequence of $T$ and let

$$S(T) = (\nu = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(s)} = \lambda)$$

the semi-standard tableau associated to $T$. We require a technical condition on the tableau $T$ which will insure that $A(T)$ will satisfy properties (A1) and (A2). The following is a sufficient condition:

(L) For every $i$, each removable $r_i$ node of $\lambda^{(i-1)}$ lies below each indent $r_i$ node of $\lambda^{(i-1)}$, and $\lambda^{(i)} \setminus \lambda^{(i-1)}$ consists of the first $m_i$ indent $r_i$-nodes of $\lambda^{(i-1)}$.

Lemma 3.8. Let $T$ be a standard tableau with property (L) and with shape $\mu$.

1. If $T'$ is a standard tableau $l$-conjugate to $T$, with shape $\mu'$, then $\mu' \subseteq \mu$.
2. $A(T)$ satisfies properties (A1) and (A2).

Proof. Point (a) is immediate from the definition of property (L). All elements of the form $A(T)$ satisfy (A2). Finally property (A1) follows from Lemmas 3.6 and 3.7. \hfill \Box

Lascoux, Leclerc, and Thibon describe for each $l$-regular Young diagram $\mu$ a certain standard tableau $T(\mu)$ of shape $\mu$ which has property (L), namely the standard tableau which fills $\mu$ from top to bottom along “$l$-ladders.” So putting

$$A(\mu) = A(T(\mu))$$

for each $l$-regular Young diagram $\mu$ gives a family $(A(\mu))_{\mu}$ with the desired properties, by Lemma 3.8.

4. A Modified Algorithm

In this section, we explain a fast variant of the LLT algorithm. The LLT algorithm is relatively slow because the $A(\mu)$ produced in the first step tend to have contributions from many diagrams $\mu'$ which do not contribute to $G(\mu)$; this means that the recursive computation during the triangular reduction step becomes very complicated. So the key to a more efficient computation is to find a way to compute $A(\mu)$ which have fewer extraneous contributions, and so to reduce the complexity of the recursion.

We fix an integer $k$, and consider only Young diagrams of length no more than $k$. So for an $l$-regular Young diagram $\mu$, we consider the problem of computing the truncation of $G(\mu)$

$$\tilde{G}(\mu) = \sum \{ (d_{\lambda, \mu}(v) 4 : \ell(\lambda) \leq k \}.$$

It is more or less clear that the LLT algorithm, modified only by throwing out diagrams with more than $k$ rows, computes the $\tilde{G}(\mu)$. Formally, we note that the span $F'_k$ of diagrams with more than $k$ rows is a $U-$submodule of $F$. So we work in the quotient module $F_k = F/F'_k$. 
and the cyclic submodule \( M_k = M / (M \cap F_k') \). The quotient module \( F_k \) has its natural basis labelled by Young diagrams with no more than \( k \) rows; and we regard the \( \tilde{G}(\mu) \) as elements of \( M_k \). Note that \( M \cap F_k' \) is also invariant under the involution of \( M \), because the \( G(\mu) \) with \( \ell(\mu) > k \) is a self-dual basis, so the involution also passes to the quotient \( M_k \). Let \( L_k \subseteq F_k \) denote the \( A \)-span of diagrams with no more than \( k \) rows, i.e. the image of the \( A \)-lattice \( L \) in \( F_k \), and let \( M_{\mathbb{Q},k} \) denote the image of \( M_{\mathbb{Q}} \) in \( F_k \).

**Lemma 4.1.** The set of \( \tilde{G}(\mu) \) where \( \mu \) is an \( l \)-regular Young diagram with no more than \( k \) rows is the unique \( \mathbb{Q}[v, v^{-1}] \) basis of \( M_{\mathbb{Q},k} \) satisfying:

- (\( \tilde{G}1 \)) \( \tilde{G}(\mu) \equiv \mu \) (mod \( L_k \))
- (\( \tilde{G}2 \)) \( \tilde{G}(\mu) \) is self-dual.

**Proof.** The set of \( \tilde{G}(\mu) \) does have the stated properties. For uniqueness, if \( \{ \tilde{H}(\mu) \} \) is another such basis, then each \( \tilde{H}(\mu) \) has a self-dual pre-image \( H(\mu) \) in \( M_{\mathbb{Q}} \), and the set of \( H(\mu) \) together with the set of \( G(\mu) \) with \( \ell(\mu) > k \) is then a basis of \( M_{\mathbb{Q}} \) satisfying (\( \tilde{G}1 \)) and (\( \tilde{G}2 \)). By uniqueness of the lower global crystal basis, one has \( H(\mu) = G(\mu) \), and hence \( \tilde{H}(\mu) = \tilde{G}(\mu) \). \( \square \)

In order to compute the \( \tilde{G}(\mu) \), it will suffice to find elements \( \tilde{A}(\mu) \in M_k \), for \( l \)-regular diagrams of length no more than \( k \), satisfying

- (\( \tilde{A}1 \)) \( \tilde{A}(\mu) = \sum \{ \alpha_{\lambda\mu}(v) \lambda : \ell(\lambda) \leq k \} \), where \( \alpha_{\mu\mu} = 1 \), \( \alpha_{\lambda\mu} \in \mathbb{Z}[v, v^{-1}] \), and \( \alpha_{\lambda\mu} = 0 \) unless \( \lambda \sqsubseteq \mu \); and
- (\( \tilde{A}2 \)) \( \tilde{A}(\mu) \) is self-dual.

The \( \tilde{G}(\mu) \) can then be obtained from the \( \tilde{A}(\mu) \) by triangular reduction, as before.

We now consider how to compute such elements \( \tilde{A}(\mu) \). Recall that the \( l \)-residue sequence, and the operator \( f(T) \) are defined for standard skew tableaux as for standard tableaux. Henceforth, all Young diagrams will have no more than \( k \) rows, so we will no longer mention this.

**Definition 4.2.** If \( T \) is a standard skew tableau of some shape \( \mu \setminus \nu \), define \( \tilde{A}(T) = f(T)\tilde{G}(\nu) \).

The proof of the following lemma is left to the reader.

**Lemma 4.3.** Let \( T \) be a standard skew tableau with property (L) and with shape \( \mu \setminus \nu \). Then \( \tilde{A}(T) \) satisfies properties (\( \tilde{A}1 \)) and (\( \tilde{A}2 \)).

We identify Young diagrams with dominant integral weights of \( \mathfrak{gl}_k \), by

\[
\lambda \mapsto \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \Lambda_i + \lambda_k \Lambda_k,
\]
where the $\Lambda_i$ are the fundamental weights. The fundamental weight $(1 \leq i \leq k)$, corresponds to the diagram with $i$ nodes arranged in one column. The half-sum $\rho$ of positive roots of $\mathfrak{sl}_k$ corresponds to the partition $(k-1, k-2, \ldots, 1, 0)$. The positive Weyl chamber $C$ of $\mathfrak{sl}_k$ is the cone generated by \{ $\Lambda_i : 1 \leq i \leq k-1$ \} in the dual $\mathfrak{h}^*$ of the diagonal subalgebra.

For an element $A = \sum \alpha \lambda \in F_k$, let us write $\Lambda_k + A = \sum \alpha \lambda (\lambda + \Lambda_k)$.

**Proposition 4.4.** If both $\mu$ and $\mu + \Lambda_k$ are $l$-regular, one has

$$\check{G}(\mu + \Lambda_k) = \Lambda_k + \check{G}(\mu).$$

**Proof.** We prove this lemma here only under the assumption that $k < l$. In this case, all Young diagrams are $l$-regular. The general case follows from Proposition 5.14.

It follows from the LLT algorithm for the lower global crystal base that the diagrams $\lambda$ appearing in the expansion of $\check{G}(\mu)$ have the same $l$-core as $\mu$ and satisfy $\lambda \leq \mu$. Therefore if $\mu$ is the lexicographically smallest diagram of a given size with a given $l$ core, then $\check{G}(\mu) = \mu$.

The same holds for $\mu + \Lambda_k$, so $\check{G}(\mu + \Lambda_k) = \mu + \Lambda_k = \Lambda_k + \check{G}(\mu)$. This argument applies in particular to the empty diagram.

Now let $\mu$ be a non-empty diagram and suppose inductively that the assertion holds for all diagrams with fewer nodes, and for all diagrams with the same number of nodes which are lexicographically smaller. Let $\mu'$ be the diagram obtained by removing the highest removable node of $\mu$ (i.e., that with the least row index), let $r$ be the residue and $i$ the row index of that node. By the induction, hypothesis, we have $\check{G}(\mu' + \Lambda_k) = \Lambda_k + \check{G}(\mu')$. The diagram $\mu'$ has no indent $r$-nodes and no removable $r$-nodes in rows $j < i$, so we can take $\check{A}(\mu) = f_r \check{G}(\mu')$. For the same reason,

$$\check{A}(\mu + \Lambda_k) = f_{r+1} \check{G}(\mu' + \Lambda_k) = f_{r+1}(\Lambda_k + \check{G}(\mu')) = \Lambda_k + f_r \check{G}(\mu) = \Lambda_k + \check{A}(\mu).$$

The conclusion now follows from the induction hypothesis. \qed

**Corollary 4.5.** If $\mu$ is an $l$-regular Young diagram with $\mu_k \neq 0$, then

$$\check{G}(\mu) = \mu_k \Lambda_k + \check{G}(\mu - \mu_k \Lambda_k).$$

It now suffices to give an algorithm for computing $\check{G}(\mu)$ when $\mu$ is an $l$-regular diagram with $\mu_k = 0$. Our strategy will be to choose for any $l$-regular Young diagram $\mu$ an $l$-regular Young diagram $\nu \subseteq \mu$ and a standard skew tableau $T$ of shape $\mu \setminus \nu$, and to define $\check{A}(\mu) = \check{A}(T) = f(T) \check{G}(\nu)$. In the original LLT algorithm, one always takes $\nu$ to be the empty diagram and $T$ to be the standard ladder tableau. Here we will choose $\nu$ and $T$ in a way which makes the recursive computation of the $\check{G}(\mu)$ more efficient.

**Case 1. Critical and Interior Diagrams.**

A Young diagram $\mu$ is called $k$-critical if $\mu_i - \mu_{i+1} + 1$ is divisible by $l$ for $i = 1, 2, \ldots, k-1$. Equivalently, $(\mu + \rho, \alpha)$ is divisible by $l$ for all roots $\alpha$ of $\mathfrak{sl}_k$. The Steinberg weight $(l-1)\rho$ is the smallest $k$-critical dominant integral weight. We call a Young diagram $\mu$ interior if it
lies above the Steinberg weight, i.e., $\mu - (l - 1)\rho$ is a dominant integral weight. All interior diagrams are $l$-regular, and all $k$-critical diagrams are interior.

The fundamental box $B$ is the set

$$\{\sum_{i=1}^{k-1} m_i \Lambda_i : 0 \leq m_i < l\}.$$ 

The set of interior Young diagrams is tiled by translates of the fundamental box $B$ by $k$-critical Young diagrams; i.e. for each interior Young diagram $\mu$, there is a unique $k$-critical Young diagram $\mu_c$ such that $\mu \in \mu_c + B$.

**Proposition 4.6.** If $\mu$ is a $k$-critical diagram, then $\tilde{G}(\mu) = \mu$.

**Proof.** It is possible to give a simple combinatorial proof, adapting arguments from [GW]. Here we will make use of Ariki’s theorem, and the positivity result of Varagnolo and Vasserot, in the interest of brevity.

It is well known that the decomposition numbers for the Hecke algebra $H_n(q; \mathbb{C})$ satisfy $d_{\lambda, \mu} = 0$ for all Young diagrams $\lambda \neq \mu$ with no more than $k$-rows; see, for example [GW]. It follows from $d_{\lambda, \mu}(v) \in \mathbb{N}[v]$ and $d_{\lambda, \mu}(1) = d_{\lambda, \mu}$, that $d_{\lambda, \mu}(v) = 0$ as well for all Young diagrams $\lambda \neq \mu$ with no more than $k$-rows.

Let $\mu$ be an interior Young diagram and let $\mu_c$ be the associated critical diagram, such that $\mu \in \mu_c + B$. Let $\mu - \mu_c = \sum_{i=1}^{k-1} d_i \Lambda_i$, where $0 \leq d_i < l$ for each $i$. Consider the skew tableau $T$ from $\mu_c$ to $\mu$:

$$\mu^{(k)} = \mu_c \rightarrow$$
$$\mu^{(k)} + \Lambda_{k-1} \rightarrow \cdots \rightarrow \mu^{(k)} + d_{k-1} \Lambda_{k-1} = \mu^{(k-1)}$$
$$\mu^{(k-1)} + \Lambda_{k-2} \rightarrow \cdots \rightarrow \mu^{(k-1)} + d_{k-2} \Lambda_{k-2} = \mu^{(k-2)}$$
$$\vdots$$
$$\mu^{(2)} + \Lambda_1 \rightarrow \cdots \rightarrow \mu^{(2)} + d_1 \Lambda_1 = \mu,$$

where at each stage the $i$ cells of $\Lambda_i$ are added from top to bottom.

It is easy to see that the skew tableau $T$ has property (L), so by Lemma 3.8 $\tilde{A}(\mu) = \tilde{A}(T) = f(T)\mu$ satisfies (A1) and (A2).

**Remark 4.7.** This construction was suggested by our approach to the decomposition numbers in [GW]. For $\mu$ an interior diagram, the definition of the element $\tilde{A}(\mu)$ is a ‘$q$-version’ of the construction in section 5 of [GW]. There we defined $N(\lambda, \mu)$ to be the number of paths from $\mu_c$ to $\lambda$ which are conjugate to the skew tableau $T$, and

$$n(\lambda, \mu) = \frac{N(\lambda, \mu)}{\prod_{i=1}^{k-1} (i!)^{d_i}},$$

(4.2)
One has $n(\lambda, \mu) = \alpha_{\lambda, \mu}(1)$, where $\tilde{A}(\mu) = \sum_{\lambda} \alpha_{\lambda, \mu}(v) \lambda$. We showed in [GW] that $n(\lambda, \mu)$ is an upper bound for the decomposition number $d_{\lambda, \mu}$.

Using section 5 of this paper, Soergel’s theorem [S2], and results on tensor ideals for tilting modules [O], one can show that the algorithm for $\tilde{G}(\mu)$, or a minor modification of it, for interior diagrams $\mu$ has the properties:

1. The only diagrams $\mu'$ such that $\tilde{G}(\mu')$ is used in the recursive calculation of $\tilde{G}(\mu)$ are interior diagrams.
2. The complexity of the algorithm for interior diagrams $\mu$ is uniformly bounded, for fixed $k$ and $l$, independent of the size of the diagram $\mu$. The dependence on $l$ can also be eliminated using Proposition 5.14.

We do not have a direct elementary proof of these statements.

**Case 2. Non-interior diagrams.**

We now consider how to define $\tilde{A}(\mu)$ when $\mu$ is not an interior diagram. For simplicity, we assume that $k \leq l$, so that all Young diagrams are $l$-regular (aside from the diagrams $n\Lambda_k$, if $k = l$.) This assumption is not essential, and we will indicate afterwards how to modify the procedure for $k > l$.

If $\mu$ is a non-interior Young diagram, then $\mu + \rho \in a + B$ for some diagram $a$ located on one or more boundary hyperplanes of the positive Weyl chamber $C$, and satisfying $(a, \alpha_i)$ is divisible by $l$ for all simple roots $\alpha_i$.

If $a = 0$, i.e., if $\mu + \rho$ is contained in the fundamental box $B$, then we compute $\tilde{A}(\mu)$ by the LLT algorithm; that is, take $T$ to be the standard ladder tableau of shape $\mu$ and put $\tilde{A}(\mu) = f(T)\emptyset$.

If $a \neq 0$, we proceed as follows: Let $A^+$ denote the lowest $l$-alcove in the positive Weyl chamber $C$, namely

$$A^+ = \{x: x_1 > x_2 \geq \cdots > x_k, \text{ and } x_1 - x_k < l\}.$$ 

There is a unique closed face $F$ of $(a + A^+)^-$ of smallest dimension such that $a \in F$, and the interior $F^0$ of the face lies in the interior of the positive Weyl chamber. In fact, let $I = \{i_1, \ldots, i_s\} \subseteq \{1, 2, \ldots, k-1\}$ be the complete list of indices $i$ such that the simple root $\alpha_i$ satisfies $(a, \alpha_i) = 0$. Then $F$ is the convex hull of $\{a\} \cup \{a + l\Lambda_i : i \in I\}$. Because we are assuming $k \leq l$, there exist integer points on $F^0$; to be definite, take the point $p = a + \Lambda_{i_1} + \Lambda_{i_2} + \cdots + \Lambda_{i_s}$. We have $\mu + \rho = p + \sum_i d_i \Lambda_i$, where $0 \leq d_i < l$, and necessarily $d_i > 0$ for $i \in I$. Hence $\mu + \rho = p + \sum_i d_i \Lambda_i$, where $d_i = d_i$ if $i \not\in I$ and $d_i = d_i - 1$ if $i \in I$. Since $p$ is in the interior of the Weyl chamber, it has the form $\nu + \rho$ for some Young diagram $\nu$, and we have $\mu = \nu + \sum_i d_i \Lambda_i$. Now we take the skew tableau of shape $\mu \setminus \nu$ of the same
sort as before, namely
\begin{equation}
\mu^{(k)} = \nu \rightarrow \\
\mu^{(k)} + \Lambda_{k-1} \rightarrow \cdots \rightarrow \mu^{(k)} + d'_{k-1} \Lambda_{k-1} = \mu^{(k-1)} \rightarrow \\
\mu^{(k-1)} + \Lambda_{k-2} \rightarrow \cdots \rightarrow \mu^{(k-1)} + d'_{k-2} \Lambda_{k-2} = \mu^{(k-2)} \rightarrow \\
\vdots \\
\mu^{(2)} + \Lambda_1 \rightarrow \cdots \rightarrow \mu^{(2)} + d'_1 \Lambda_1 = \mu.
\end{equation}

This skew tableau has property (L), so we can define \( \tilde{A}(\mu) = f(T) \tilde{G}(\nu) \).

**Case 3. Non-interior diagrams on a critical face.**

We continue to use the notation of the previous case. Next consider the case that \( \mu \) is a non-interior Young diagram, that \( \mu + \rho \) is not contained in the fundamental box \( B \), but \( \mu + \rho \) already lies in the interior \( F^0 \) of the face \( F \) of \( a + A^+ \). Put \( J = \{1, 2, \ldots, k-1\} \setminus I \). Note that \( a \) lies on the face \( H \) of the Weyl chamber \( C \) generated by \( \{\Lambda_j : j \in J\} \). If there is some \( j \in J \) such that \( \mu - l \Lambda_j \) is in the interior of the Weyl chamber, then take the greatest such \( j \) and put \( \nu = \mu - l \Lambda_j \). (Thus, in going from \( \mu + \rho \) to \( \nu + \rho \), one moves toward the origin, parallel to the face \( H \) of \( C \), and \( \nu + \rho \) lies on an open face of the same type as \( F^0 \).) Furthermore, take \( T \) to be the skew tableau of shape \( \mu \setminus \nu \)
\[ \nu \rightarrow \nu + \Lambda_j \rightarrow \cdots \rightarrow \nu + l \Lambda_j = \mu, \]
where at each stage, the \( j \) nodes of \( \Lambda_j \) are added from top to bottom. Then \( T \) has property (L), and we put \( \tilde{A}(\mu) = f(T) \tilde{G}(\nu) \).

Finally, if there is no \( j \in J \) such that \( \mu - l \Lambda_j \) is in the interior of the Weyl chamber, then compute \( \tilde{A}(\mu) \) by the original LLT algorithm; that is, take \( T \) to be the standard ladder tableau of shape \( \mu \) and put \( \tilde{A}(\mu) = f(T)\emptyset \).

**Case 4. \( k > l \)**

When \( l < k \), only faces of dimension \( \leq l - 1 \) contain dominant integral weights, and not all dominant integral weights are \( l \)-regular. No modification to the algorithm is necessary in the interior region, but the algorithm for the boundary regions must be modified as follows. In case 2, compute \( \nu \) as before, and take the first \( \nu' = \mu k - j + s \Lambda_{k-j+1} \) on the canonical tableau from \( \nu \) to \( \mu \) which is \( l \)-regular. Take \( T' \) to be the tail of this tableau from \( \nu' \) to \( \mu \) and put \( \tilde{A}(\mu) = f(T') \tilde{G}(\nu') \). In case \( \mu = \nu' \), proceed as in case 3. This completes the description of the algorithm.

Empirically, the modified algorithm produces enormous improvements in efficiency. To demonstrate this, we include some timing experiments comparing the original LLT algorithm with the modified algorithm. We also compare the algorithm given in [S1]; cf. Section 5.

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1 The algorithms were encoded in Mathematica, and run on a 266 mhz Apple Macintosh G3 computer.
2 The Soergel algorithm may have been somewhat disadvantaged by inefficient programming.
5. The lower global crystal base and affine Kazhdan-Lusztig polynomials

In this section we will demonstrate that the coefficients of the lower global crystal base for $U_v(\mathfrak{sl}_k)$ coincide with certain affine Kazhdan-Lusztig polynomials. We first recall the definition of these Kazhdan-Lusztig polynomials, following [5].

Let $R \subseteq \mathfrak{h}^* \cong \mathbb{R}^{k-1}$ denote the root system of $\mathfrak{sl}_k$ contained in the dual of the diagonal subalgebra. Let $W \cong S_k$ denote the Weyl group. The affine Weyl group is the semi-direct product $W = W \rtimes \mathbb{Z}R$, which acts on $\mathfrak{h}^*$. Fix a positive integer $l$. We consider the level $l$ action of the affine Weyl group on $\mathfrak{h}^*$, i.e., the action via the natural isomorphism of $W$ with its subgroup $W^{(l)} = W \rtimes l\mathbb{Z}R$. $W^{(l)}$ is generated by reflections in certain affine hyperplanes in $\mathfrak{h}^*$. The connected components of the complement of the union of all of these reflection hyperplanes are called the alcoves at level $l$. The set of all alcoves is denoted by $A$, and the set of all alcoves which are contained in the positive Weyl chamber $C$ is denoted by $A^+$. The alcove $A^+$ is the unique element of $A^+$ which contains the origin $0$ in its closure. Let $S$ denote the set of reflections in the walls of $A^+$. Then $(W, S)$ is a Coxeter group with generating set $S$.

The affine Hecke algebra $\mathcal{H} = \mathcal{H}(W, S)$ is the associative algebra with identity element $1$ over the ring of Laurent polynomials $\mathbb{Z}[v, v^{-1}]$ with generators $\{T_s : s \in S\}$ which satisfy the braid relations and the quadratic relation $T_s^2 = v^{-2}1 + (v^{-2} - 1)T_s$, for $s \in S$. Using instead the generators $H_s = vT_s$, one has instead the quadratic relations $H_s^2 = H_s + (v - v^{-1})1$. The Hecke algebra has a basis $\{H_x : x \in W\}$ satisfying $H_xH_y = H_{xy}$ in case $\ell(xy) = \ell(x) + \ell(y)$, and $H_xH_s = H_x + (v - v^{-1})H_{sx}$ in case $\ell(sx) = \ell(x) - 1$, for $x \in W$ and $s \in S$. The Hecke algebra has an involution $d : a \mapsto \bar{a}$ defined by $\bar{v} = v^{-1}$ and $(H_x)^- = (H_{x^{-1}})^{-1}$. An element fixed by this involution is called self-dual. Fundamental self-dual elements are the $C_s = H_s + v$.

Let $S_0 \subseteq S$ be the set of reflections fixing the origin. The finite Weyl group $W$ is the Coxeter subgroup of $W$ with generating set of reflections $S_0$. Let $\mathcal{H}_f = \mathcal{H}(W, S_0) \subseteq \mathcal{H}$ denote its Hecke algebra. Consider the sign representation of $\mathcal{H}_f \to \mathbb{Z}[v, v^{-1}]$ which takes each $H_s$ to $-v^{-1}$. Let $\mathcal{N}$ denote the induced right $\mathcal{H}$-module

$$\mathcal{N} = \mathbb{Z}[v, v^{-1}] \otimes_{\mathcal{H}_f} \mathcal{H}$$

Let $W^f \subseteq W$ be the coset representatives of minimal length of the right cosets of $W$ in $W$. Then $\mathcal{N}$ has a basis $N_x = 1 \otimes H_x$, for $x \in W^f$, and the operation of the $C_s$ for $s \in S$ on this
basis has the following form:

\[
\begin{cases}
N_{xs} + vN_x & \text{if } xs \in W^f \text{ and } xs > x; \\
N_{xs} + v^{-1}N_x & \text{if } xs \in W^f \text{ and } xs < x; \ \\
0 & \text{if } xs \not\in W^f,
\end{cases}
\]

where the inequality signs refer to the Bruhat order on \( W \). The involution on \( H \) induces an involution on \( \mathcal{N} \) defined by \( a \otimes b \mapsto \bar{a} \otimes \bar{b} \).

The following result is discussed in [S1], following [D1], [D2].

**Theorem 5.1.** \( \mathcal{N} \) has a unique basis \( \{ N_x : x \in W^f \} \) satisfying

1. \( N_x \in N_x + \sum_{y<x} vZ[v]N_y \).
2. \( N \) is self-dual.

**Definition 5.2.** The affine Kazhdan-Lusztig polynomials \( n_{y,x} \) are defined by

\[
N_x = N_x + \sum_{y<x} n_{y,x}N_y
\]

The affine Weyl group \( \mathcal{W} \) acts freely and transitively on alcoves, so there is a bijection \( \mathcal{W} \to \mathcal{A} \) given by \( w \mapsto wA^+ \), where \( A^+ \) is the unique alcove in \( \mathcal{A}^+ \) containing 0 in its closure. Under this bijection, the elements of \( W^f \) correspond to alcoves contained in the positive Weyl chamber. One also has an action of \( \mathcal{W} \) on the right on \( \mathcal{A} \) given by \( (wA^+)x = wxA^+ \).

Using the bijection between \( W^f \) and \( \mathcal{A}^+ \), one can rename the distinguished elements of the right \( \mathcal{H} \) module \( \mathcal{N} \) using alcoves \( A \in \mathcal{A}^+ \) rather than coset representatives \( x \in W^f \). Thus if \( x, y \in W^f \) correspond to \( A, B \in \mathcal{A}^+ \), then we write \( N_A \) for \( N_x \), \( \underline{N}_A \) for \( \underline{N}_x \), and \( n_{A,B} \) for \( n_{x,y} \).

The right action of \( \mathcal{H} \) is then given by

\[
\begin{cases}
\underline{N}_A = \underline{N}_A + vN_A & \text{if } As \subset A^+ \text{ and } As > A; \\
\underline{N}_A + v^{-1}N_A & \text{if } As \subset A^+ \text{ and } As < A; \ \\
0 & \text{if } As \not\subset A^+,
\end{cases}
\]

where now the inequalities have a geometric interpretation: \( As > A \) if \( As \) is on the positive side of the hyperplane separating the two alcoves. We remark that the \( \underline{N}_A \) are computed by a recursive scheme reminiscent of the computation of the lower global crystal base. One has \( \underline{N}_{A^+} = N_{A^+} \). Given \( A \neq A^+ \), one can choose \( s \in S \) such that \( As \in A^+ \) and \( As < A \). As a first approximation to \( \underline{N}_A \) one takes

\[
N_{As} = N_A + \sum_{B < A} f_{B,A}(v)N_B.
\]

This element is self-dual, but may have coefficients with non-zero constant term. So one corrects these coefficients by subtracting a self-dual linear combination of \( \underline{N}_B \) for \( B < A \).
Finally, we want to rename the elements of $N$ one last time, using dominant integral weights of $\mathfrak{sl}_k$. We fix an integer $l$ and take the level $l$ action of the affine Weyl group on $\mathbb{R}^k$, generated by reflections in the hyperplanes $x_i - x_j = ml$. Suppose first that a weight $\mu$ lies in an open alcove. Write $a(\mu)$ for the alcove of $\mu$ and define

$$n_{\lambda,\mu} = \begin{cases} n_{a(\lambda),a(\mu)} & \text{if } \lambda \text{ is in the } W \text{ orbit of } \mu; \\ 0 & \text{otherwise,} \end{cases}$$

and $N_\mu = \sum_\lambda n_{\lambda,\mu}$.

Now consider a weight $\mu$ which lies on one or more affine hyperplanes, and define $a^+(\mu)$ to be the unique open alcove which contains $\mu$ in its closure and which lies on the positive side of all hyperplanes containing $\mu$. In this case we put

$$n_{\lambda,\mu} = \begin{cases} n_{a^+(\lambda),a^+(\mu)} & \text{if } \lambda \text{ is in the } W \text{ orbit of } \mu; \\ 0 & \text{otherwise,} \end{cases}$$

and again $N_\mu = \sum_\lambda n_{\lambda,\mu}$.

(Note that for these definitions, it is quite unnecessary to assume $k \leq l$ so that the open alcoves in fact contain integral weights.)

**Theorem 5.3.** Fix integers $k$ and $l$. Let $\lambda$ and $\mu$ be Young diagrams of the same size, both with no more than $k$ rows, and with $\mu$ $l$-regular. Then $d_{\lambda,\mu}(v) = n_{\lambda+\rho,\mu+\rho}(v)$.

The proof of this result, while straightforward, will require a number of intermediate lemmas and observations. We begin by recalling a geometric interpretation of tableaux and of conjugacy of tableaux (with no more than $k$ rows.) To each Young diagram $\lambda$ there corresponds the dominant integral weight $\tilde{\lambda}$ of $\mathfrak{sl}_k$ given by $\tilde{\lambda}_i = \lambda_i - k$.

To a standard (skew) tableau

$$T = (\lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(s)}),$$

there corresponds the path

$$(\tilde{\lambda}^{(0)} \subseteq \tilde{\lambda}^{(1)} \subseteq \cdots \subseteq \tilde{\lambda}^{(s)}),$$

in the positive Weyl chamber; one might picture the successive $\tilde{\lambda}^{(i)}$ as begin connected by affine line segments, so that the path in fact becomes a piecewise linear curve. The tableau may be recovered from the path and the initial diagram $\lambda^{(0)}$, so we will not distinguish between tableaux and paths. The tableau $T + \rho$ is that obtained by adding the half sum of positive roots $\rho$ to each diagram $\lambda^{(i)}$. Note that the length $\lambda_i - i + k$ of the $i$th row of $\lambda + \rho$ is the content of the node $(i,\lambda_i)$ of $\lambda$, plus $k$. Therefore, the weight of a tableau $T$, computed in Section 3 in terms of contents of nodes of diagrams along the path $T$, can be computed instead in terms of row lengths of diagrams along the path $T + \rho$. As the hyperplanes of the affine reflection group $W$ at level $l$ are the loci of points having two particular coordinates conjugate modulo $l$, two standard tableaux $S,T$ are $l$-conjugate if and only if the corresponding paths $S + \rho, T + \rho$ are related by reflections in such hyperplanes. In particular, the end-points of such paths are in the same $W$ orbit. For our present purposes it will suffice to consider standard skew tableaux whose residue sequences (cf. Definition 3.2) have all multiplicities $m_i$ equal to 1.
In the following, a face will always mean a face of $A$, where $A \in A^+$ is an open alcove. An open face is the interior of such a face; cf. Section 2. Given faces $F$ and $F'$ in a $W$-orbit, write $F' \triangleleft F$ if for all $\mu \in F$ and $\mu' \in W\mu \cap F'$, one has $\mu' \triangleleft \mu$.

**Lemma 5.4.** Let $T$ be a standard skew tableau such that the corresponding path $T + \rho$ is contained in some open face. Then the weight of $T$ is
\[ \text{wt}(T) = 1. \]
Furthermore, $T$ has no $l$-conjugates (other than itself).

**Proof.** At the $i^{\text{th}}$ step in the tableau $T$ a node of some residue $r_i$ is added to a diagram $\lambda^{(i-1)}$. It follows from the assumption that $T$ stays in the open face $F$, that $\lambda_i$ has no other indent nodes or removable nodes of the same residue. Therefore the weight of $T$ is 1 and $T$ has no $l$-conjugates other than itself. \qed

We will work for a while under the assumption that $k \leq l$, so that open faces of all dimensions contain dominant integral weights and all Young diagrams of length $\leq k$ (except for $n\Lambda_k$) are $l$-regular.

**Corollary 5.5.** Let $\mu, \lambda, \mu', \lambda'$ be Young diagrams, such that $\lambda + \rho$ is in the $W$ orbit of $\mu + \rho$ and $\lambda' + \rho$ is in the $W$ orbit of $\mu' + \rho$. If $\mu + \rho$ and $\mu' + \rho$ lie on the same open face, and $\lambda + \rho$ and $\lambda' + \rho$ lie on the same open face, then
\[ d_{\lambda, \mu}(v) = d_{\lambda', \mu'}(v). \]

**Proof.** One can assume without loss of generality that $\mu' \subseteq \mu$, and that there is a standard skew tableau $T$ of shape $(\mu + \rho) \setminus (\mu' + \rho)$ which lies entirely contained in the open face containing the two endpoints. It follows from Lemma 5.4 that $\tilde{G}(\mu) = f(T)\tilde{G}(\mu') = \sum_{\lambda} d_{\lambda', \mu'}(v)f(T)\lambda' = \sum_{\lambda} d_{\lambda', \mu'}(v)\lambda$, where the first sum is over the $W$ orbit of $\mu'$ and the second over the $W$ orbit of $\mu$. \qed

**Definition 5.6.** Let $F$, $F'$ be open faces in the same $W$ orbit such that $F' \triangleleft F$. Suppose $\lambda$ and $\mu$ are Young diagrams of the same size, $\mu + \rho \in F$, and $\lambda + \rho \in F' \cap W\mu$. Define $d_{F', F}(v)$ to be $d_{\lambda, \mu}(v)$, and
\[ \tilde{G}(F) = \sum_{F'} d_{F', F}F'. \]

This makes sense according to Corollary 5.5.

**Lemma 5.7.** Let $F$ be an open face of dimension $d < k - 1$. Let $I \subseteq \{1, 2, \ldots, k\}$ be a subset of cardinality $\geq 2$ which is maximal with respect to the property that for all $\mu \in F$ $\mu_i \equiv \mu_j$ (mod 1) if $i, j \in I$. Then there is a $\mu \in F$ such that for all $i \in I$, $\mu + \epsilon_i$ lies in an open face of dimension $d + 1$. 
Proof. Left to the reader. □

Let $F$ be an open face of dimension $d < k - 1$, and let $I \subseteq \{1, 2, \ldots, k\}$ be a subset with the property described in the lemma. For each $i \in I$ let $F_i$ be the open face of dimension $d + 1$ which contains $\mu + t \epsilon_i$ for $\mu \in F$ and for $t > 0$ small. Set $F_+(I) = F_{i_0}$, where $i_0$ is the least element of $I$. $F_+(I)$ has the property: For each reflection hyperplane $H$ of $W$ which contains $F$ but which does not contain $F_+(I)$, the face $F_+(I)$ lies on the positive side of $H$.

Let $F'$ be an open face such that $F'$ lies in the $W$ orbit of $F$ and $F' \subseteq F$. Let $I'$ be the set of indices which plays the same role for $F'$ as does $I$ for $F$. Namely, if $w \in W$ is an element such that $wF = F'$, let $w(x_1, \ldots, x_k) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(k)}) + w(0)$ for all $(x_1, \ldots, x_k) \in \mathbb{R}^k$, and put $I' = \sigma(I)$. Note that $\sigma$ is not uniquely determined, but $I'$ is uniquely determined by $F$ and $F'$, and $I$. Define $F'_{+}(I')$ in the same way as $F_+(I)$, namely $F'_{+}(I') = F'_{j_0}$, where $j_0$ is the least element of $I'$.

**Lemma 5.8.**

$$d_{F',F}(v) = d_{F'_+(I'),F_+(I)}(v).$$

**Proof.** Adopt the notation of the two paragraphs preceding the statement of the lemma. Let $\mu$ be a Young diagram such that $\mu + \rho \in F$ and such that $\mu + \rho + \epsilon_i \in F_+(I)$, as is possible by Lemma 5.7. Let $r$ be the residue of the indent node of $\mu$ in the row $i_0$. By the assumption on $\mu$, the Young diagram $\mu$ has exactly $|I|$ indent $r$-nodes and no removable $r$-nodes, and the same holds for any $\lambda$ such that $\lambda + \rho$ is in the $W$ orbit of $\mu + \rho$. For any such $\lambda$, with $\lambda + \rho$ lying on an open face $F'$, one has

$$f_{r,\lambda} = \sum_{i \in I'} v^{N(\lambda, \lambda+\epsilon_i)}(\lambda + \epsilon_i),$$

and furthermore $N(\lambda, \lambda + \epsilon_i) \geq 0$, with equality if and only if $\lambda + \epsilon_i \in F'_+(I')$. It follows that

$$f_{r,\tilde{G}}(\mu) = \tilde{G}(\mu + \epsilon_{i_0}),$$

and furthermore

$$d_{F'_+(I'),F_+(I)}(v) = d_{\lambda + \epsilon_{i_0},\mu + \epsilon_{i_0}}(v) = d_{\lambda,\mu}(v) = d_{F',F}(v).$$

□

For a face $F$, let $a^+(F)$ denote the unique alcove $A$ such that $F$ is contained in the closure of $A$ and $A$ lies on the positive side of all hyperplanes containing $F$.

**Corollary 5.9.** For any face $F$ and any face $F'$ in the $W$ orbit of $F$ such that $F' \subseteq F$, one has

$$d_{F',F}(v) = d_{a^+(F'),a^+(F)}(v)$$

**Proof.** This follows by induction from the previous lemma. □
To complete the proof of Theorem 5.3 in the case $k \leq l$, it remains to show that $d_{B,A} = n_{B,A}$ for alcoves $B, A$. The $n_{B,A}$ are defined by a recursion involving crossing of walls, so we will have to see that the $d_{B,A}$ satisfy the same recursion.

By a wall of an alcove, we mean a face of the alcove of dimension $k-2$, that is the non-empty intersection of the closure of the alcove with a reflection hyperplane of $W$.

**Lemma 5.10.** Let $A$ and $B$ be adjacent alcoves, separated by an open wall $F$, and let $T$ be a skew tableau such that $T + \rho$ which starts at a diagram $\nu + \rho \in A$ and ends in a diagram $\mu + \rho \in B$, passing through $F$. Denote the reflection of $\mu + \rho$ in $F$ by $\mu' + \rho$.

1. If $A \prec B$ then $f(T)\nu = \mu + v\mu'$.
2. If $A \succ B$ then $f(T)\nu = \mu + v^{-1}\mu'$.

**Proof.** $T$ has one $l$-conjugate tableau $T'$ which ends in $\mu'$. It has to be shown that $T$ has weight $\text{wt}(T) = 1$, while $T'$ has weight $\text{wt}(T') = v$ in case (a), and $\text{wt}(T') = v^{-1}$ in case (b). To verify this, one has to check what happens when a path hits a wall from above or below, or when a path leaves a wall towards the positive or negative side. The various cases to be checked are listed in the following lemma.

**Lemma 5.11.** Let $T$ be a skew tableau.

1. If $T + \rho$ begins in an alcove $A$ and ends in an open wall $F$ of the alcove, and $A$ lies on the negative side of the hyperplane containing $F$, then $\text{wt}(T) = 1$.
2. If $T + \rho$ begins in an alcove $A$ and ends in an open wall $F$ of the alcove, and $A$ lies on the positive side of the hyperplane containing $F$, then $\text{wt}(T) = v^{-1}$.
3. If $T + \rho$ begins in an open wall $F$ of an alcove $A$ and ends in the alcove $A$, and $A$ lies on the negative side of the hyperplane containing $F$, then $\text{wt}(T) = v$.
4. If $T + \rho$ begins in an open wall $F$ of an alcove $A$ and ends in the alcove $A$, and $A$ lies on the positive side of the hyperplane containing $F$, then $\text{wt}(T) = 1$.

**Proof.** The proofs of parts (a)-(d) are all similar, so we prove part (b) and leave the rest to the reader. Let 

\[ T = (\lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(s)}), \]

where $\lambda^{(i)} + \rho \in A$ for $i < s$ and $\lambda^{(s)} + \rho \in F$. It follows from Lemma 5.4 that $\text{wt}(T) = v^{N(\lambda^{(s-1)}, \lambda^{(s)})}$. Say the wall $F$ is given by $x_a - x_b = ml$, where $a < b$. Then $\lambda^{(s)}$ is obtained from $\lambda^{(s-1)}$ by filling an indent node of some residue $r$ in row $b$ (since $A$ lies above $F$), and furthermore $\lambda^{(s-1)}$ has no other indent $r$-nodes, but has a removable $r$-node in row $a$. It follows that $N(\lambda^{(s-1)}, \lambda^{(s)}) = -1$ and $\text{wt}(T) = v^{-1}$.

**Lemma 5.12.** For all alcoves $A, B$, 

\[ d_{B,A} = n_{B,A} \]
Proof. Define a map $\Phi : F_k \to N$ by putting $\Phi(\mu) = N_A$ if $\mu + \rho$ is in the open alcove $A$, and $\Phi(\mu) = 0$ if $\mu$ is not contained in an open alcove. The assertion of the lemma is equivalent to $\Phi(G(\mu)) = N_A$, if $\mu + \rho \in A$. If $\mu + \rho \in A^+$, then $\Phi(G(\mu)) = N_{A^+} = N_{A^+}$. The proof proceeds by induction on $A^+$ with respect to $\prec$. For an alcove $A \neq A^+$, choose $s \in S$ such that $As \prec A$, and let $F$ be the open wall separating $As$ and $A$. One can choose a skew tableau $T$ of shape $\mu \setminus \nu$ such that $\nu + \rho \in As$, and $\mu + \rho \in A$, and $T + \rho$ crosses from $As$ to $A$ through $F$. Put $\tilde{A}(\mu) = f(T)G(\nu)$. One has

$$
\Phi(\tilde{A}(\mu)) = \Phi(G(\nu))C_s = N_{As}C_s,
$$

where the first equality comes from Lemma 5.10, and the definition of the right action of $C_s$ (Equation 5.2), and the second equality from the induction hypothesis. In particular $\tilde{A}(\mu) = \sum\lambda \alpha_{\lambda,\mu} \lambda$, where $\alpha_{\lambda,\mu} \in \mathbb{N}[v]$. Therefore the rectification of $\tilde{A}(\mu)$ to $G(\mu)$ takes the simple form

$$
G(\mu) = \tilde{A}(\mu) - \sum_{\lambda<\mu} \alpha_{\lambda,\mu}(0)G(\lambda).
$$

Thus

$$
\Phi(G(\mu)) = N_{As}C_s - \sum_{\lambda<\mu} \alpha_{\lambda,\mu}(0)N_{s(\lambda+\rho)} = N_A,
$$

using the induction hypothesis. $\square$

This lemma completes the proof of Theorem 5.3, in case $k \leq l$. To handle the case $l < k$, we show that the polynomials $d_{\lambda,\mu}(v)$ are independent of $l$ in a certain sense.

Let $F^l$ denote the set of all open faces for the action of the affine Weyl group $W^{(l)}$ at level $l$. If $l_1 < k < l_2$, then the map $x \mapsto l_2l_1^{-1}x$ from $C$ to $C$ induces a bijection $\psi : F^{l_1} \to F^{l_2}$. For $\mu \in F \in F^{l_1}$ and $\nu \in \psi(F)$, define a map $\psi^\mu_{\nu}$ from the set of $\mu'$ such that $\mu' + \rho$ is in the $W^{(l_1)}$ orbit of $\mu + \rho$ to the set of $\nu'$ such that $\nu' + \rho$ is in the $W^{(l_2)}$ orbit of $\nu + \rho$ by $\psi^\mu_{\nu}(\mu') = \nu'$ if $\mu' + \rho \in H \in F^{l_1}$ and $\nu' + \rho \in \psi(H)$. Extend $\psi^\mu_{\nu}$ linearly to the $\mathbb{Z}[v,v^{-1}]$-modules spanned by such diagrams.

Lemma 5.13. Let $\mu$ be an $l_1$-regular diagram, and $F$ the open face containing $\mu + \rho$. Let $T$ be the standard ladder tableau with shape $\mu$. Put $\tilde{A}(\mu) = \tilde{A}(T)$. There exists a $\bar{\mu}$ with $\bar{\mu} + \rho \in \psi(F)$ such that $\psi^\mu_{\bar{\mu}}(\tilde{A}(\mu))$ is self-dual.

Proof. The proof goes by induction on the number of nodes of $\mu$. If $\mu$ is the empty diagram, let $F$ the the open face containing $\mu + \rho$ and let $\bar{\mu}$ be any diagram such that $\bar{\mu} + \rho \in \psi(F)$. Then $\psi^\mu_{\bar{\mu}}(\tilde{A}(\mu)) = \bar{\mu}$. But, since $\bar{\mu}$ is the lowest diagram in its $W^{(l_2)}$ orbit, $\bar{\mu} = G(\bar{\mu})$. Thus $\psi^\mu_{\bar{\mu}}(\tilde{A}(\mu))$ is self-dual.

Now fix an $l_1$-regular $\mu \in F \in F^{l_1}$, and assume the assertion holds for all $l_1$-regular diagrams with fewer nodes. Let $T'$ be the tableau obtained by removing the last node of $T$, $\mu'$ the shape of $T'$, and $F'$ the open face containing $\mu' + \rho$. Note that $\mu'$ is $l_1$-regular. By the induction hypothesis, there is a diagram $\bar{\mu}'$ with $\bar{\mu}' + \rho \in \psi(F')$ such that $\psi^\mu_{\bar{\mu}'}(\tilde{A}(T'))$ is self-dual.
Let \( r \) be the residue of the node \( \mu \setminus \mu' \) and let \( c = v^{N(\mu', \mu)} \). Then

\[
\tilde{A}(\mu) = c^{-1} f_r \tilde{A}(T')
\]

We assert the existence of a skew tableau \( \tilde{S} \) such that \( \tilde{S} + \rho \) starts at \( \tilde{\mu} + \rho \in \psi(F') \) and ends in some diagram \( \tilde{\mu} + \rho \in \psi(F) \).

If \( F' = F \), then there is nothing to show. Otherwise, one might have \( F \) an open face of some dimension \( d \) and \( F' \) an open face of dimension \( d - 1 \) contained in the boundary of the closure of \( F \), or \( F' \) a face of dimension \( d + 1 \) such that \( F \) is contained in the boundary of its closure. In either case, \( \psi(F') \) and \( \psi(F) \) stand in the same relation, and a path of the desired type exists. Or one might have \( F \) and \( F' \) both \( d \) dimensional boundary faces of a \( d + 1 \) dimensional face \( H \). In this case, a path from \( \tilde{\mu} + \rho \in \psi(F') \) to \( \psi(F) \) might necessarily contain integral points of \( \psi(H) \), while the one step path from \( \mu' + \rho \) to \( \mu + \rho \) has no points in \( H \); this difference, however, has no effect.

In all cases, one has for all \( \lambda' \) such that \( \lambda' + \rho \) is in the \( W^{(l_1)} \) orbit of \( \mu' + \rho \),

\[
\psi_{\mu}^R(f_r \lambda') = f(\tilde{S}) \psi_{\mu}^R(\lambda'),
\]

because the effect of \( f(\tilde{S}) \) is determined entirely by the faces which \( \tilde{S} + \rho \) enters and leaves.

Consequently, one has

\[
\psi_{\mu}^R(\tilde{A}(\mu)) = c^{-1} f(\tilde{S}) \psi_{\mu}^R(\tilde{A}(T')).
\]

The desired conclusion follows from this. \( \Box \)

**Proposition 5.14.** Let \( \mu \) be an \( l_1 \) regular diagram and let \( F \) be the open face containing \( \mu + \rho \). For every diagram \( \nu \) such that \( \nu + \rho \in \psi(F) \), \( \psi_{\mu}^R(\tilde{G}(\mu)) = \tilde{G}(\nu) \).

**Proof.** Given \( \mu \), let \( \tilde{A}(\mu) \) and \( \tilde{\mu} \) be as in the lemma, and put \( \tilde{A}(\tilde{\mu}) = \psi_{\mu}^R(\tilde{A}(\mu)) \). It follows from the lemma that \( \tilde{A}(\tilde{\mu}) \) satisfies properties \( (\tilde{A}1) \) and \( (\tilde{A}2) \).

If \( \mu + \rho \) is the lexicographically smallest diagram in its \( W^{(l_1)} \) orbit such that \( \mu \) is \( l_1 \)-regular, then \( \tilde{A}(\mu) = \tilde{G}(\mu) \). It follows that \( \tilde{A}(\tilde{\mu}) \equiv \tilde{\mu} \pmod{L_k} \), so \( \tilde{A}(\tilde{\mu}) = \tilde{G}(\tilde{\mu}) \), by the uniqueness of \( \tilde{G}(\tilde{\mu}) \). Therefore, for such \( \mu \), we have \( \psi_{\mu}^R(\tilde{G}(\mu)) = \tilde{G}(\tilde{\mu}) \). But then, Corollary 5.3 implies \( \psi_{\mu}^R(\tilde{G}(\mu)) = \tilde{G}(\nu) \) for all \( \nu \) such that \( \nu + \rho \in \psi(F) \).

Now fix \( \mu \) and assume that the result holds for all \( \mu' < \mu \) such that \( \mu' + \rho \) is in the \( W^{(l_1)} \) orbit of \( \mu + \rho \). Write

\[
\tilde{G}(\mu) = \tilde{A}(\mu) - \sum_{i} \gamma_i(\nu) \tilde{G}(\mu^{(i)}),
\]

where \( \mu^{(i)} < \mu \) and \( \mu^{(i)} + \rho \) is in the \( W^{(l_1)} \) orbit of \( \mu + \rho \). Let \( F_i \) be the open face containing \( \mu^{(i)} + \rho \), and let \( \nu^{(i)} \) be the diagram with \( \nu^{(i)} + \rho \in \psi(F_i) \) and \( \nu^{(i)} + \rho \) in the \( W^{(l_2)} \) orbit of \( \tilde{\mu} + \rho \). Then one has

\[
\psi_{\mu}^R(\tilde{G}(\mu^{(i)})) = \psi_{\mu^{(i)}}^R(\tilde{G}(\mu^{(i)})) = \tilde{G}(\nu^{(i)}),
\]
with the last equality coming from the induction hypothesis. Applying this to Equation 5.3 gives
\[
\psi_\mu^\beta(\tilde{G}(\mu)) = \tilde{A}(\bar{\mu}) - \sum_i \gamma_i(\nu) \tilde{G}(\nu^{(i)}) = \tilde{G}(\bar{\mu}).
\]
Now applying Corollary 5.5 again gives \(\psi_\nu^\mu(\tilde{G}(\mu)) = \tilde{G}(\nu)\) for all \(\nu\) such that \(\nu + \rho \in \psi(F)\). \(\square\)

This proposition completes the proof of Theorem 5.3.

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