The spectrum of tightly knotted flux tubes in QCD

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Abstract. We fit the observed $J^{++}$ mesonic mass spectrum with one and two parameter models in terms of energies for tightly knotted and linked chromoelectric QCD flux tubes. We predict a possible new state at approximately 1190 MeV and many new states above 1690 MeV.

1. Introduction
Many experiments can be interpreted as signatures of unusual mesonic states, i.e., bosonic hadrons that are not pure $q\bar{q}$ [Nakamura et al., 2010]. Our interest is in one of the most studied and least understood classes of particles in QCD—glueballs—states with no valence quarks, e.g., string loops à la Nielsen-Olesen [Nielsen & Olesen, 1973] like closed flux tubes. Here we model all $J^{++}$ mesonic states, i.e., all $f_J$ and $f'_J$ states listed by the Particle Data Group (PDG) [Nakamura et al., 2010] ($f$ states, for brevity), as knotted or linked chromoelectric QCD flux tubes [Buniy & Kephart, 2003]. Hence we will use the term “glueball” loosely as a shorthand for any $J^{++}$ state in QCD.

2. Model
In [Buniy & Kephart, 2003] two of us argued how to generalize various classical ideas from plasma physics to a semiclassical model of knotted and linked configurations in QCD. Here we discuss a

1 We assume confinement is due to monopole condensation in terms of ’t Hooft’s dual confinement picture, with chromoelectric flux tubes of uniform cross section.
major reanalysis carried out recently by the present authors [Buniy et al., 2014]. Several recent developments are included in the new analysis. First and foremost are the new set of ideal knot and link lengths obtained in [Ashton et al., 2011]. In 2002 when [Buniy & Kephart, 2003] was written, only a handful of knot and link length were known, and some of those only to an accuracy in the 5–10% range. Ashton et al., [Ashton et al., 2011] provides us with hundreds of ideal knot and link lengths to better than 1% accuracy. Since we deal with knotted physical flux tubes, not ideal knots, there are curvature correction due to energy consideration from the bending of the tubes and these corrections have now been included [Buniy et al., 2014]. Angular momentum is treated differently because of the new assumption that the tube diameter is smaller and in the ∼ 0.1 fm range. More details can be found in [Buniy et al., 2014].

Let us first recall that Maxwell’s equations for an ideal plasma imply that flux lines are locked into the plasma flow. This means that if the flux lines are knotted or linked, then this flux line topology is conserved as the flow evolves. A classical consequence of conservation of topology involves the concept of helicity and its conservation in a plasma [Woltjer, 1958; Moffatt, 1969, 1985]. Helicity in our context corresponds to the degree of Gaussian linking of QCD flux tubes. Shortening while keeping flux tube topology fixed leads to tight knots [Katritch et al., 1996; Woltier, 1999; Pieranski & Przybyl, 2001; A. Stasiak & Pieranski, 1998] in QCD.

It is interesting to contrast knots in QCD with knots in plasma physics [Moffatt, 1990]. In the QCD case, which is a quantum (or at least semiclassical) system, the tube diameters are fixed. The length of a tube can shorten by loosing energy via particle emission. E.g., in a typical tightening process a loose knot $K^\ast$ would undergo a transition like $K^\ast \to K + N\pi$, i.e., to become a tight knot $K$ plus some number $N$ of pions. In the plasma case the knot can tighten and at the same time conserve energy by changing its diameter. Diameter change is allowed classically.

There is an infinite family of topologically different link types distinguished by topological invariants. Many of the simpler configurations have been tabulated; see [Doll & Hoste, 1991] and [Hoste et al., 1998]. However, regardless of which invariants are used to identify the configurations, the “frozen-in field” hypothesis implies that tube topology is conserved. Hence, we conjecture that, to zeroth order, the ground states of all systems of flux tubes, with any type of nontrivial linking, are the states with the shortest length tubes. We also consider self-linked (knotted) flux tubes, where the associated tight knot ground states are analogously defined. The simplest example of a nontrivially knotted flux tube has the form of the trefoil knot $3_1$. This configuration has dimensionless knot energy (i.e., length/diameter) which has been numerically calculated to be $\varepsilon_0(3_1) \approx 16.3715$ [Pieranski & Przybyl, 2012], but note that there are no known analytic forms for the lengths of any tight knots or links with nonplanar elements.

We consider a high energy hadron-hadron collision in the process of rehadronization, where there are baryons, mesons and quantized fluxes confined to tubes. If the tubes are open, with quarks and anti-quarks at their ends, then they are excited baryon or meson states. Our interest is in closed linked and knotted tubes. As a key part of our model, we identify all the $f$ states as knotted or linked QCD chromoelectric flux tubes. The topological quantum numbers are what stabilizes the configurations, so we assume that non-topological (i.e., unknotted/unlinked) closed flux tube are too unstable to have measurable widths. A tightly knot/link $K$ will have a corresponding state $f(K)$. Nontrivial knotting and linking leads to quasi-stable generalized minimum energy states. The generalized minimum energy state is the one that minimizes $\varepsilon_0(K)$. We will see below that the approximation of a fixed energy per unit length can be improved by an analysis of the effect of field rearrangement within a bent tube.

To be more specific we now summarize the model assumptions:

(i) There is a one-to-one correspondence between $f$ states and tightly knotted and linked chromoelectric flux tubes.

(ii) The flux is quantized with one flux quantum per tube.
(iii) Knotted and linked flux tubes are stabilized by topological quantum numbers.

(iv) The tube diameter is in the ~ 0.1 fm range. (This corresponds to a string tension of approximately 400 MeV, which agrees with lattice estimates.)

(v) The quantity \( J \) in an \( f_J \) or \( f_J' \) state is the intrinsic angular momentum of the associated knotted solitonic solution of the QCD field equations.

(vi) The relaxation to a tight state configuration (via processes where no topology change is involved) is faster than its decay rate (via processes with topology change) for an \( f \) state, i.e., \( \tau_{\text{relax}} \ll \tau_{\text{decay}} \).

One modification from [Buniy & Kephart, 2003] is that we now assume \( J \) is the intrinsic rather than the rotational angular momentum since the tube diameter is now assumed to be smaller and hence the rotational energy level spacing is larger, \( \sim 500 \text{ MeV} \), as opposed to a few MeV for the thicker tubes assumed in [Buniy & Kephart, 2003]. The other significant modification is that we correct the energy due to tube curvature and include estimated errors due to other physical corrections, see below.

To proceed with the model we identify ideal knot and link lengths with glueballs and/or predicted glueballs, where we include all \( f \) states. The lightest candidate is the \( f_0(500) \), which we identify with the shortest knot or link, i.e., the Hopf link \( 2^1_1 \); the \( f_0(980) \) is identified with the next shortest knot or link, in this case the trefoil knot \( 3_1 \), etc.

Our initial one-parameter fit of the data is shown in Figure 1. The slope is \( \Lambda_{\text{tube}} = 57 \text{ MeV} \) and \( \chi^2 = 84 \). (The \( \chi^2 \) test is a measure of goodness of fit, and small values on the order of one per data point indicate the fit assumptions are compatible with the data. We define \( \chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i} \) where the \( O_i \) are observed values and \( E_i \) are expected values from the assumptions of the fit. For more details see [Nakamura et al., 2010].) The fit is poor mainly because of the constraint imposed by the very small error bars on the masses of the \( f_2(1270) \) and the \( f_1(1285) \). We will now see how to improve the fit when ideal tubes are replaced with physical tubes. Note we are already assuming errors of 3% on the knot lengths in the figure which anticipates this replacement.

3. Curvature corrections
So far we have assumed uniform flux across the cross section of the tubes, but this is not necessarily the case for curved tubes. This leads us to define a new energy functional for tubes which we call the “flux tube energy” as discussed below.

The magnetic field of an ideal toroidal solenoid with fixed flux falls like \( 1/\rho \) from the symmetry axis. To see this, we hold the flux \( \Phi \) fixed and vary the field to find the functional form of the energy \( W \) for a toroidal solenoid. The general form of the energy in toroidal coordinates is \( W = \frac{1}{2} \int_B B^2 \rho \, dz \, d\rho \, da \). The \( dz \, d\rho \) integration runs over the cross section of the tube \( D \). Since \( B \) is independent of \( \alpha \), the \( \alpha \) integration gives \( W = \pi \int_B B^2 \rho \, dz \, d\rho \). The flux through \( D \) is \( \Phi = \int_B B \, dz \, d\rho \). Now we vary \( W \) with respect to \( B \) while holding \( \Phi \) fixed which is equivalent to considering \( \delta(W - \lambda \Phi) = 0 \), where \( \lambda \) is a Lagrange multiplier. For unit vector \( n \) normal to the cross section, the variation of \( B \) gives \( \int_B (2\pi B \rho - \lambda \alpha) \cdot \delta B \, dz \, d\rho = 0 \), which vanishes for arbitrary \( \delta B \) only if \( B(\rho) = \frac{\lambda \alpha}{2\pi \rho} \). We find \( \lambda \) from the requirement \( \Phi = \text{const} \), which gives \( B = \frac{\Phi}{2\pi} \).

\[ W = \frac{\pi \Phi^2}{r} \quad \text{with} \quad I = \int_B \frac{dz \, d\rho}{\rho}. \]

To calculate the integral \( I \) over the cross section of a torus of major radius \( R_2 \) and minor radius \( R_1 \), it is convenient to introduce polar coordinates \((r, \theta)\) with the origin at the center of disk \( D \), plus a toroidal angle \( \alpha \). The result of integration over \( D \) is \( I = 2\pi \left[ R_2 - \sqrt{R_2^2 - R_1^2} \right] \),
Figure 1. Fit using uncorrected knot/link lengths: Fit of the $f_J$ states data to tight knot and link lengths (ropelengths). Errors are shown for the states, but they are too small to be visible for the lengths of ideal knots and links, however we include a 3% error in the knot energies due to the fact that we are dealing with physical knots and links; see discussion in the text. Non-fitted knots and links are not shown. Here $\chi^2 = 84$ is rather high. This is due to the tension in the data caused by the incompatibility of the small experimental errors in the masses of the $f_0(980)$, the $f_2(1270)$ and the $f_2(1285)$ with the fit of the rest of the data.

Figure 2. The function $W(R_2)/W_0(R_2)$ (solid curve) and its approximation (dashed line) for $R_2/R_1 \gg 1$ and $R_1 = 1$.

which leads to

$$W(R_2) = \frac{\Phi^2}{2\left(R_2 - \sqrt{R_2^2 - R_1^2}\right)}.$$
The analogous result for the straight cylinder of length $2\pi R_2$ is $W_0(R_2) = \frac{3^2 R_2}{R_1^2}$, and so the ratio

$$ \frac{W(R_2)}{W_0(R_2)} = \frac{(R_1/R_2)^2}{2[1 - \sqrt{1 - (R_1/R_2)^2}]}, $$

the graph of which is plotted in Fig. 2, defines our new dimensionless energy $\varepsilon(K)$ for the toroidal solenoid.

More generally, for an embedded tube $K$ of fixed radius $R_1$ and parametric centerline curve $\gamma(s)$ with curvature $\kappa(s)$, we define the flux tube energy $\varepsilon(K)$ by the integral

$$ \varepsilon(K) = \frac{1}{2\pi R_1^2} \left( L + \int_0^L \sqrt{1 - R_1^2 \kappa^2(s)} \, ds \right), $$

where $L = \int_\gamma ds$ is the length of the center line. (In numerical studies of tight knots and links, it is observed that the integral in Eq. (1) is typically $\sim \sqrt{3L}$ which translates into an $\sim 7\%$ correction of the energy from the ropelength value.)

We minimize the original $\epsilon_0(K)$ energy numerically using ridgerunner [Ashton et al., 2011], and then compute the $\varepsilon(K)$ energy for the $\epsilon_0(K)$-minimizing configurations on the grounds that the difference between $\varepsilon(K)$-minimizing and $\epsilon_0(K)$-minimizing configurations are likely to be small. In Figure 3 we have histogrammed the shortest 72 knots and links after curvature corrections have been applied.

![Figure 3. Histogram of the magnitudes of the curvature corrections to the first 72 knot and link lengths.](image)

As an example of a case where corrections can be calculated exactly, consider the chain of three unknots $2^2_1 \# 2^2_1$ which has length $6\pi + 2 \approx 20.8496$ and has $R_2 = 2R_1$ in curved regions and $R_2 \to \infty$ in straight sections. We find an overall corrected value

$$ \varepsilon(2^2_1 \# 2^2_1) = \frac{1}{4(2 - \sqrt{3})}6\pi + 2 \approx 0.933013(6\pi + 2) \approx 19.4529 $$

In addition to the curvature corrections, tubes can be constricted due to being wrapped by another section of the tube or distorted by wrapping tightly around another section of tube (like a rope wrapped tightly around a post). We have approximated such corrections and have included them in our error estimates.
4. Results
In our model, the chromoelectric fields are confined to the knotted and linked tubes, each carrying one quantum of conserved flux. The energy is positive and, to first approximation, proportional to the length of the tube $l$ and thus the minimum of the energy is achieved by shortening $l$ (i.e., tightening the knot), subject to the curvature correction discussed above and other corrections discussed below.

We proceed to identify knotted and linked QCD flux tubes, i.e., curvature corrected physical flux tubes, with glueballs and/or predicted glueballs, where we include all $f$ states as described above, but by the fourth knot/link, the ordering begins to be reshuffled due to the curvature corrections, see [Buniy et al., 2014] for details.

All knot and link lengths have been calculated for states corresponding to energies well beyond 2 GeV. Above $\sim$ 2 GeV the number of knots and links grows rapidly, and so the corresponding hadronic states should become dense relative to their typical width. Hence we have confined our investigations to knot lengths corresponding to all known $f$ states below $\sim$ 2 GeV.

![Figure 4. Curvature-corrected lengths: The combined set of measured and fitted states (dots with error bars) and predicted states (circles) for the high $f_0(1370)$ one parameter fit. Note that the fit has been much improved over that in Figure [1]. $\chi^2$ has been reduced to 33. The curvature corrections have brought the $f_0(980)$, $f_2(1270)$ and $f_2(1285)$ in line with the rest of the data.](image-url)

We will give two interpretations of the data. The first possibility is with the $f_0(1370)$ identified with the $5_1$ knot which results in a prediction of a new state near 1190 MeV identified with the $4_1^2$ link. Alternatively we can identify the $f(1370)$ with the $4_1^2$ which gives our best fit.

We are now ready to discuss the details of the fit possibilities. Since the error on the $f_0(1370)$ mass is rather large ($\pm$ 150 MeV) it can be identified with several different knots and links, with minimal effect on $\chi^2$. However, the identifications of the other $f$-states are much more constrained due to the small errors on their masses and available nearby knot lengths. This in turn restricts the identification of the $f_0(1370)$ to two allowed choices with reasonable $\chi^2$s. We call the case discussed above the high-fit, where we identify the $f_0(1370)$ with the $5_1$ knot, and
Figure 5. Low $f_0(1370)$ one parameter fit for curvature-corrected lengths: This figure shows the combined set of measured and fitted states (dots with error bars) and predicted states (circles). The reordering of $f_0(1370)$ has provided one further improvement in the fit to $\chi^2 = 28$.

The low-fit leaves no gaps in the spectrum until we get near 1700 MeV. The $\chi^2$ is somewhat better than that for the high fit, 28 vs 33, and the value of $\Lambda_{\text{tube}}$ is similar to the high-fit case. We find

$$E(K) = \Lambda_{\text{tube}} \varepsilon(K), \quad \Lambda_{\text{tube}} = 65.16 \pm 0.61 \text{ MeV},$$

which is our best overall fit to the data. We have tabulated the fitted energies for the first 72 knots and links, identifying all $f$-states of energy less than 2 GeV, hence giving predictions of many new states at 1690 MeV and above. See [Buniy et al., 2014] for details.

Next let us discuss the statistical significance of our results. First we have done a number of tests to determine if our hypothesis that glueballs are knotted flux tubes is likely to be correct. For example, the well-known Kolmogorov-Smirnov test, which tests distribution functions $\hat{F}(y)$ and $F(y)$ by means of the quantity $\sup_y |\hat{F}(y) - F(y)|$, which in this case are the distribution functions of the glueball masses and calculated knot and link lengths. The resulting $p$-value for the Kolmogorov-Smirnov test is $p_{KS} = 0.95$, hence our model is in excellent agreement with the data. The Kolmogorov-Smirnov test is a measure of goodness-of-fit. A number of other goodness-of-fit tests as well as several variance tests all show excellent agreement between model and data [Buniy et al., 2014].

Finally, let us make one final comment about the model. A more conservative approach is to assume only the $0^{++}$ states, i.e., the $f_0$ states, correspond to knotted/linked flux tubes. In
that case there are only five states to fit and the result is displayed in Figure 6 for comparison, where the identified states are

\[ [f_0(500), f_0(980), f_0(1370), f_0(1500), f_0(1710)] \leftrightarrow [2_1^2, 3_1, 5_1, 5_2, 6_2]. \]

The slope is essentially unchanged, the \( R^2 = 0.998 \) value is roughly the same. The masses of the predicted states can be easily gotten by rescaling the knot lengths in the table with the new slope parameter \( \Lambda_{\text{tube}} = (65.50 \pm 1.81) \text{MeV} \). Note that the \( f_0(1370) \) has been moved in the ordering to improve the fit. More data is needed to distinguish between the fit of all \( f_J \) states and the restricted \( f_0 \) fit. The \( \chi^2 = 15.62 \) for the fit is not particularly good but it only takes

\[
E_1(f_j) \text{ [MeV]}
\]

\[
\varepsilon(K)
\]

**Figure 6.** The \( f_0 \) states data is fitted to the curvature corrected knot and link data. Errors are shown for the states and estimated to be 3% for the knot/link energy. Non-fitted knots and links are not shown.

replacing the new \( f_0(500) \) values with the old \( f_0(600) \) mass and error bars to get \( \chi^2 = 0.56 \). So if the shift to the new PDG \( f_0(500) \) mass value is due to mixing and one could extract the unmixed value for the pure gluonic state it is possible that the fit would improve again.

If a sufficient number of \( f_J \) states are found, so that they outnumber the total number of knots and links, then this would be evidence to support the restricted fit (since all short knots and links are presumed to be known). However, this is not the case at present.

5. **Discussions and conclusions**

We have considered hadronic collisions that produce some number of baryons and mesons plus gluonic states in the form of a closed QCD flux tubes. From an initial state, the fields in the flux tubes quickly relax to an equilibrium configuration, which is topologically equivalent to the initial state. (We assume topological quantum numbers are conserved during this rapid process.) The tube radius is set by the confinement scale, so to lowest order the energy of the final state depends only on the topology of the initial state and equals the length \( l_K \) of the tube times the average energy per unit length, or the dimensionless knot or link length \( \varepsilon_0(K) \) times the
energy scale parameter $\Lambda_{\text{tube}}$. While related to $\Lambda_{\text{QCD}}$ by constants of order unity, $\Lambda_{\text{tube}}$ can be more accurately determined (see above) and hence could be a useful dimensionful parameter in studying other properties of QCD, such as scattering and hadronization processes. The relaxation proceeds through minimization of the field energy. This process occurs via shrinking the tube length and the process halts to form a tight knot or link.

Details of knot excitations would be interesting to investigate, as would other quantum corrections, but at present we do not have a reliable way to estimate these effects, nor do we have a good way to calculate glueball decay rates. However, we do expect high mass glueball production to be suppressed because more complicated non-trivial topological field configurations are statistically disfavored and we also expect higher mass glueballs to be relatively less stable.

In addition to not fitting naturally into the quark model, glueballs have some other characteristic signatures, including enhanced production via gluon rich channels in the central rapidity region, branching fractions incompatible with $q\bar{q}$ decay, very weak coupling to $\gamma\gamma$, and OZI suppression. All the $f$-states have some or all of these properties. For instance, none have

Table 1. Low $f_0(1370)$ curvature corrected fit: Comparison of the glueball mass spectrum and fit energies for $\varepsilon(K)$ less than $\sim 29$. (See [Buniy et al., 2014] for an extended table.)

| State | Mass   | $K$         | $\varepsilon(K)$ | $E_1(K)$ |
|-------|--------|-------------|------------------|----------|
| $f_0(500)$ | 475 ± 75 | $2^1_1$      | 11.724           | 764      |
| $f_0(980)$ | 990 ± 20 | $3^1_1$      | 14.943           | 974      |
| $f_0(1370)$ | 1350 ± 150 | $4^1_1$   | 18.250           | 1189     |
| $f_2(1270)$ | 1275.1 ± 1.2 | $4^1_1$   | 19.411           | 1265     |
| $f_1(1285)$ | 1282.1 ± 0.6 | $2^1_1\#2^1_1$ | 19.556           | 1274     |
| $f_1(1420)$ | 1426.4 ± 0.9 | $5^1_1$    | 21.559           | 1405     |
| $f_2(1430)$ | $\approx$ 1430 | $2^1_1\#3^1_1$ | 22.697           | 1479     |
| $f_0(1500)$ | 1505 ± 6 | $5^2_2$      | 22.779           | 1484     |
| $f_1(1510)$ | 1518 ± 5 | $5^1_1$      | 22.866           | 1490     |
| $f_2(1525)$ | 1525 ± 5 | $6^3_3$      | 23.309           | 1519     |
| $f_2(1565)$ | 1562 ± 13 | $6^2_2$     | 24.854           | 1619     |
| $f_2(1640)$ | 1639 ± 6 | $7^2_7$      | 25.735           | 1677     |
|          |         | $6^2_2$      | 25.924           | 1689     |
|          |         | $6^1_1$      | 26.025           | 1696     |
|          |         | $2^2_2\#4^2_1$ | 26.046           | 1697     |
|          |         | $3^1_1\#3^1_1$ | 26.135           | 1703     |
|          |         | $3^1_1\#3^1_1$ | 26.151           | 1704     |
|          |         | $6^2_2$      | 26.158           | 1704     |
|          |         | $6^3_3$      | 26.327           | 1715     |
|          |         | $(2^2_2\#2^2_2\#2^1_1)_{kc}$ | 26.449           | 1723     |
|          |         | $2^2_2\#4^1_1$ | 26.466           | 1724     |
|          |         | $6^3_3$      | 26.567           | 1731     |
|          |         | $6^2_2$      | 26.590           | 1733     |
|          |         | $7^2_7$      | 26.720           | 1741     |
|          |         | $7^3_7$      | 26.963           | 1757     |
|          |         | $(2^2_2\#2^2_2\#2^1_1)_{kc}$ | 27.449           | 1788     |
| $f_0(1710)$ | 1720 ± 6 | $7^1_7$      | 28.018           | 1826     |
|          |         | $8^3_8$      | 28.152           | 1834     |
|          |         | $8^2_8$      | 28.458           | 1854     |
substantial branching fractions to $\gamma\gamma$. However, mixing with $q\bar{q}$ isoscalar states can obscure some of these properties. All these observations are in qualitative agreement with the model presented here.

Our high-fit model predicts one new state at 1190 MeV, twelve states concentrated near 1700 MeV and a tower of higher mass states with the next dense concentration starting near 1900 MeV. The low-fit model makes similar prediction except that there is no new state near 1200 MeV. We have argued that there is sufficient tension in the experimental data in these regions to allow the identification of many more states with knots and links. A careful statistical analysis of the data of all $f$-regions to resolve hidden states is needed.

It would be trivial to extend our predictions to states above 2 GeV. One just takes the knot of appropriate lengths from [Ashton et al., 2011] and scales them by the dimensionful parameter $\Lambda_{\text{tube}}$ from the fit. However, there is insufficient mesonic data above 2 GeV to improve or constrain our fit. More experimental data to test the model in this region would be very welcome.

To conclude, we have given two interpretations of the $f$-state data with a model of knotted chromoelectric flux tubes in QCD. The first possibility is where the $f(1370)$ is identified with the $5_1^1$ knot. This results in a prediction of a new state around 1190 MeV which is identified with the $4_1^2$ link. The other possibility—which also gives our best fit—is to identify the $f(1370)$ with the $4_1^2$ link to give a one-to-one matching of the first twelve $f_J$ states with the first twelve knots and links. Experiments could help to resolve which of these two possibilities is the correct choice.

Finally we should point out that there is considerable amount of tension in the $f_J$ data, as indicate by the $\chi^2$s of the individual states quoted by the PDG. We would not expect our fits to be better than the fits of the data on which they are based.

Table 2. High $f_0(1370)$ curvature corrected fit: Comparison of the glueball mass spectrum and fit energies for $\varepsilon(K)$ less than $\sim 30$. Except for the $f(4_1^2)$, this table contains only the fitted PDG states. Predictions for other states can be gotten by multiplying the knot energy by the appropriate value of $\Lambda_{\text{tube}}$ as given in [Buniy et al., 2014].

| State  | Mass   | $K$      | $\varepsilon(K)$ | $E_1(K)$ |
|--------|--------|----------|-----------------|----------|
| $f_0(500)$ | 475 ± 75 | $2_1^2$   | 11.724           | 763      |
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| $f(4_1^2)$ |          | $4_1^2$   | 18.250           | 1187     |
| $f_2(1270)$ | 1275.1 ± 1.2 | $4_1$     | 19.411           | 1263     |
| $f_1(1285)$ | 1282.1 ± 0.6 | $2_1^2\#2_1^1$ | 19.556           | 1272     |
| $f_1(1420)$ | 1426.4 ± 0.9 | $5_1$     | 21.559           | 1403     |
| $f_0(1500)$ | 1505 ± 6   | $2_1^2\#3_1$ | 22.697           | 1477     |
| $f_2(1430)$ | ≈ 1430    | $5_2$     | 22.779           | 1482     |
| $f_1(1510)$ | 1518 ± 5   | $5_1^1$   | 22.866           | 1488     |
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| $f_2(1565)$ | 1562 ± 13  | $6_1^1$   | 24.854           | 1617     |
| $f_2(1640)$ | 1639 ± 6   | $7_2^7$   | 25.735           | 1674     |
| $f_0(1370)$ | 1350 ± 150 | $6_2^2$   | 25.924           | 1687     |
| $f_0(1710)$ | 1720 ± 6   | $(2_1^2\#2_1^1\#2_1^1)_{k_c}$ | 26.449           | 1721     |
| $f_2(1810)$ | 1815 ± 12  | $7_1$     | 28.018           | 1823     |
| $f_2(1910)$ | 1903 ± 9   | $7_2$     | 29.330           | 1908     |
| $f_2(1950)$ | 1944 ± 12  | $2_1^1\#5_2$ | 29.840           | 1941     |
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References
A. Stasiak, J. Dubochet, V. K. & Pieranski, P. 1998 Ideal knots and their relation to the physics of real knots. World Sci. Pub., Singapore.
Ashtong, T., Cantarella, J., Piatek, M. & Rawdon, E. J. 2011 Knot Tightening by Constrained Gradient Descent. Experimental Mathematics 20, 57.
Breakstone, A. et al. 1990 The Reaction pomeron-pomeron → π⁺π⁻ and an unusual production mechanism for the f₂(1270). Z.Phys. C48, 569–576.
Buniy, R. V., Cantarella, J., Kephart, T. W. & Rawdon, E. 2014 The tight knot spectrum in QCD. Phys.Rev. D89, 054513.
Buniy, R. V. & Kephart, T. W. 2003 A Model of glueballs. Phys.Lett. B576, 127–134.
Doll, H. & Hoste, J. 1991 A tabulation of oriented links. Math. Computation 57, 747.
Hoste, J., Thistlethwaite, M. B. & Weeks, J. R. 1998 The first 1,701,936 knots. Math. Intelligencer 20, 33.
Katritch, V., Bednar, J., Michoud, D., Scharein, R. G., Dubochet, J. & Stasiak, A. 1996 Geometry and physics of knots. Nature 384, 142.
Moffatt, H. K. 1969 The degree of knottedness of tangled vortex lines. J. Fluid Mech. 35, 117.
Moffatt, H. K. 1985 A theorem on force-free magnetic fields. J. Fluid Mech. 159, 359.
Moffatt, H. K. 1990 The energy-spectrum of knots and links. Nature 347, 367.
Nakamura, K. et al. 2010 Review of particle physics. J.Phys. G37, 075021.
Nielsen, H. B. & Olesen, P. 1973 Vortex Line Models for Dual Strings. Nucl.Phys. B61, 45–61.
Pieranski, P. & Przybyl, S. 2001 Phys. Rev. E 64, 031801.
Pieranski, P. & Przybyl, S. 2012 High resolution portrait of the ideal trefoil knot. Unpublished.
Woltier, L. 1958 A theorem on force-free magnetic fields. Proc. Nat. Acad. Sci. 44, 489.
Woltier, L. 1999 Proc. Nat. Acad. Sci. 96, 4769.