Multi-dimensional Jordan chain and Navier-Stokes Equation

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Abstract

Multi-dimensional Jordan chain is presented. It is shown that it admits the finite-component reductions to the Navier-Stokes equation and other important equations of continuous media theory.

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1. Introduction

The Navier-Stokes equation

\[ \rho \left( \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right) = -\nabla p + \eta \Delta \vec{u} + \left( \xi + \frac{\eta}{3} \right) \nabla (\text{div} \vec{u}) \]  

(1)

where \( \vec{u}, \rho, p \) are velocity, density, pressure and \( \eta, \xi \) are the viscosity coefficients is one of the basic equations in the theory of continuous media (see e.g. [1, 2]). It arises in the study of a number of phenomena in various branches of physics and has been addressed in large number of researches. It is well-known also that in spite of the apparent simplicity of the Navier-Stokes equation, an analysis of its properties is a tough problem.

In the present paper the relation between the Navier-Stokes equation and the multi-dimensional Jordan chain is established. The n-dimensional Jordan chain is an infinite set of quasi-linear equations of the form

\[ \frac{\partial u_{n+1}}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_{n+1}}{\partial x_k} + \frac{\partial u_{(i+1)n+1}}{\partial x_i} = 0 , \quad i = 1, ..., n; l = 0, 1, 2, ... \]  

(2)
It is shown that under the constraint
\[-\rho \frac{\partial u_n}{\partial x_i} = \frac{\partial \rho}{\partial x_k} + \eta \sum_{k=1}^{n} \frac{\partial^2 u_i}{\partial x_k^2} + (\xi + \eta) \frac{\partial}{\partial x_i} \left( \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} \right), \quad i = 1, ..., n \] (3)
the Jordan chain (2) is reduced to the n-dimensional Navier-Stokes equation (equation (1) at \(n = 3\)). Other important equations like multi-dimensional Euler equation and multi-dimensional Burgers equation also are the appropriate reductions of the Jordan chain (2).

It is demonstrated that a class of solutions of the Jordan chain (2) and its reductions is provided implicitly by the hodograph type equations
\[x_i = u_i t + f_i(u_1, u_2, ...) \quad , \quad i = 1, ..., n \]
(4)
\[0 = t + f_k(u_1, u_2, ...) \quad , \quad k = n + 1, n + 2, ...\]
with the certain constraints on the functions \(f_i(u_1, u_2, ...), i = 1, 2,...\).

The paper is organized as follows. In section 2 the N-component n-dimensional Jordan system is constructed. The Jordan chain (2) is its formal limit at \(N \to \infty\). Reductions of the Jordan chain to the n-dimensional Navier-Stokes equation and other equations are considered in section 3.

2. Multi-dimensional Jordan system

First we will construct the "N-component" n-dimensional Jordan system with arbitrary \(N\) and \(n\). The hodograph system \((N \geq 2)\)
\[x_i = u_i t + f_i(u_1, ..., u_{Nn}) \quad , \quad i = 1, ..., n, \]
(5)
\[0 = t + f_i(u_1, ..., u_{Nn}) \quad , \quad i = n + 1, ..., Nn \]
(6)
where \(f_i(u_1, ..., u_{Nn}), i = 1, ..., Nn\) are functions of \(Nn\) variables is our starting point.

Differentiating these equations with respect to \(x_k, k = 1, ..., n,\) one gets
\[\delta_{ik} = \sum_{l=1}^{Nn} \left( t k_l + \frac{\partial f_i}{\partial u_l} \right) \frac{\partial u_l}{\partial x_k}, \quad i, k = 1, ..., n \] (7)
and
\[
0 = \sum_{l=1}^{N_n} \frac{\partial f_i}{\partial u_l} \frac{\partial u_l}{\partial x_k} , \quad i = n + 1, ..., Nn; k = 1, ..., n
\]  
(8)

where \( \delta_{ik} \) is the Kroneker symbol. Introducing \( Nn \times Nn \) matrices \( \omega \) and \( A \) defines by
\[
\omega_{ik} = \delta_{ik}, \quad i, k = 1, ..., n, \quad \omega_{ik} = 0, \quad i = n + 1, ..., Nn \quad \text{and/or} \quad k = n + 1, ..., Nn
\]
and
\[
A_{ik} = t\omega_{ik} + \frac{\partial f_i}{\partial u_k} , \quad i, k = 1, ..., Nn
\]  
(9)

one rewrites the relations (7),(8) as
\[
\omega_{ik} = \sum_{l=1}^{N_n} A_{il} \frac{\partial u_l}{\partial x_k} , \quad i = 1, ..., Nn; k = 1, ..., n
\]  
(10)

Then, differentiating (5),(6) with respect to \( t \), one obtains
\[
-V_i = \sum_{l=1}^{N_n} A_{il} \frac{\partial u_l}{\partial t} , \quad i = 1, ..., Nn
\]  
(11)

where \( V_i = u_i, \quad i = 1, ..., n \) and \( V_i = 1, \quad i = n + 1, ..., Nn \). Hence, if \( \det A \neq 0 \), one has
\[
\frac{\partial u_i}{\partial x_k} = \sum_{l=1}^{N_n} (A^{-1})_{il} \omega_{lk} , \quad i = 1, ..., Nn; k = 1, ..., n
\]  
(12)

and
\[
\frac{\partial u_i}{\partial t} = -\sum_{l=1}^{N_n} (A^{-1})_{il} V_l , \quad i = 1, ..., Nn
\]  
(13)

Due to specific form of \( \omega \) and \( V \) these relations are equivalent to the following
\[
\frac{\partial u_i}{\partial x_k} = (A^{-1})_{ik} , \quad i = 1, ..., Nn; k = 1, ..., n
\]  
(14)

and
\[
\frac{\partial u_i}{\partial t} = -\sum_{k=1}^{n} (A^{-1})_{ik} u_k - \sum_{m=n+1}^{N_n} (A^{-1})_{im} , \quad i = 1, ..., Nn
\]  
(15)

The relations (14),(15) imply that a solution \( u_i(i = 1, ..., Nn) \) of the hodograph equations (5),(6) for any given functions \( f_1, ..., f_{Nn} \) obey the system of equations
\[
\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} + \sum_{m=n+1}^{N_n} (A^{-1})_{im} = 0 , \quad i = 1, ..., Nn
\]  
(16)
Now one imposes the constraints
\[
\sum_{m=n+1}^{Nn} (A^{-1})_{l\,n+i,m} = \frac{\partial u_{(l+1)n+i}}{\partial x_i} , \quad i = 1, \ldots, n; l = 0, 1, \ldots, N - 2, \tag{17}
\]
\[
\sum_{m=n+1}^{Nn} (A^{-1})_{(N-1)n+i,m} = 0 , \quad i = 1, \ldots, n. \tag{18}
\]
Under such constraints the system (16) assumes the form
\[
\frac{\partial u_{n+i}}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_{n+i}}{\partial x_k} + \frac{\partial u_{(l+1)n+i}}{\partial x_i} = 0 , \quad i = 1, \ldots, n; l = 0, 1, \ldots, N - 2, \tag{19}
\]
\[
\frac{\partial u_{(N-1)n+i}}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_{(N-1)n+i}}{\partial x_k} = 0 , \quad i = 1, \ldots, n. \tag{20}
\]
It is the "N-component" n-dimensional Jordan system.

The system (19), (20) is the multi-dimensional generalization of the one dimensional (\(n = 1\)) Jordan system introduced in [3]. The system similar to (19), (20) at \(N = 2\) has been constructed in [4] in a slightly different way. The case \(N = 1\) corresponds to the homogeneous Euler equation
\[
\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} = 0 , \quad i = 1, \ldots, n. \tag{21}
\]
The applicability of the hodograph equations (5) and their Lagrangian version to the system (21) has been demonstrated in [5, 6] and [7, 8].

Due to the relations (14) the constraints (17), (18) are equivalent to the following constraints on the elements of the matrix \(A^{-1}\)
\[
\sum_{m=n+1}^{Nn} (A^{-1})_{l\,n+i,m} = (A^{-1})_{(l+1)n+i,i} , \quad i = 1, \ldots, n; l = 0, 1, \ldots, N - 2, \tag{22}
\]
\[
\sum_{m=n+1}^{Nn} (A^{-1})_{(N-1)n+i,m} = 0 , \quad i = 1, \ldots, n.
\]
These constraints represent themselves the system of \(Nn\) nonlinear partial differential equations for \(Nn\) functions \(f_i(u_1, \ldots, u_{Nn})\). Any solution of this system
provides us implicitly, via the hodograph equations (5), (6), with the solutions of the Jordan system (19), (20).

Solutions of the Jordan system can be also constructed using the hodograph type equations

\[ x_i = u_i t + f_i (u_1, ..., u_N) \quad , \quad i = 1, ..., n, \]

\[ 0 = t + f_{n+1} (u_1, ..., u_N) \]

\[ 0 = g_k (u_1, ..., u_N) \quad , \quad k = n + 2, ..., Nn \]  

which obviously are equivalent to the system of equations (5), (6). At \( n = 1 \) the hodograph equations (23) has been used in [4].

Similar to the one-dimensional case [9] an imbedding of equations (21) into the Jordan system (19), (20) is a way to regularize the blow-ups of derivatives and gradient catastrophes for the homogeneous Euler equation. Indeed, due to the relations (14), (15) the first level blow-ups for the equations (21) occurs on the hypersurface defines by the equation

\[ \det A(N = 1) = 0 \]  

where

\[ A_{ik}(N = 1) = t \delta_{ik} + \frac{\partial x_i}{\partial u_k} \quad , \quad i, k = 1, ..., n. \]

After the imbedding into the "2-component" Jordan system (19), (20) (\( N = 2 \)) the condition (24) is no more critical and the corresponding first level blow-ups disappear. First level blow-ups for the "2-component" Jordan system which occur on the hypersurface \( \det A(N = 2) = 0 \) are regularized in the imbedding into the "3-component" Jordan system and so on. In detail such type of regularization of the blow-ups for the n-dimensional Euler equation (21) will be considered in a separate paper.
3. Multi-dimensional Jordan chain and reduction to Navier-Stokes equation

Now let us consider the formal limit \( N \to \infty \) of the system \((19),(20)\). In this limit it becomes an infinite set of equations of the form

\[
\frac{\partial u_{n+i}}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_{n+i}}{\partial x_k} + \frac{\partial u_{(i+1)n+i}}{\partial x_i} = 0, \quad i = 1, \ldots, n; l = 0, 1, 2, \ldots.
\]

(25)

We will refer to this set of equations as the \( n \)-dimensional Jordan chain. In the one-dimensional case \((n = 1)\) it has been introduced in different ways in the papers \([3, 10, 11]\).

Similar to the one-dimensional case the chain \((25)\) can be extended and used in various directions. Here we will consider only some of its finite-component reductions.

The first reduction is generated by the constraint

\[
\frac{\partial u_{n+i}}{\partial x_i} = -\frac{1}{\rho} \left( -\frac{\partial p}{\partial x_i} + \eta \Delta u_i + \left( \xi + \frac{\eta}{3} \right) \frac{\partial}{\partial x_i} \left( \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} \right) \right), \quad i = 1, \ldots, n
\]

(26)

where the density \( \rho \) obeys the continuity equation

\[
\frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\rho u_i) = 0.
\]

(27)

Here \( p \) is the pressure, \( \Delta \equiv \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} \) and \( \eta, \xi \) are arbitrary constants. Under this constraint the first equation \((l = 0)\) of the Jordan chain \((25)\) assumes the form

\[
\rho \left( \frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} \right) = -\frac{\partial p}{\partial x_i} + \eta \Delta u_i + \left( \xi + \frac{\eta}{3} \right) \frac{\partial}{\partial x_i} \left( \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} \right), \quad i = 1, \ldots, n.
\]

(28)

The subsequent equations with \( l = 1, 2, 3, \ldots \), i.e.

\[
\frac{\partial u_{(i+1)n+i}}{\partial x_i} = -\frac{\partial u_{n+i}}{\partial t} - \sum_{k=1}^{n} u_k \frac{\partial u_{n+i}}{\partial x_k}, \quad i = 1, 2, \ldots, n; l = 1, 2, 3, \ldots
\]

(29)
represent themselves the recurrent relation for the calculation of all derivatives \( \frac{\partial u_{i+1}}{\partial x_i} \), \( i = 1, ..., n \), \( l = 1, 2, 3, ... \) in terms of \( \frac{\partial u_{n+i}}{\partial x_i} \), \( i = 1, ..., n \) given by (26). The inclusion of equation (27) apparently does not produce additional constraints.

The system (28), (27) is the n-dimensional version of the classical basic equation (1), (27) \( (n = 3) \) of the theory of continuous media \([1, 2]\).

Constraint (26) is equivalent to the following constraint for the functions \( f_i \):

\[
\rho (A^{-1})_{n+i,i} + \eta \sum_{k=1}^{n} \sum_{m=1}^{\infty} \frac{\partial}{\partial u_m} (A^{-1})_{ik} (A^{-1})_{mk} + \\
\left( \xi + \frac{\eta}{3} \right) \sum_{k=1}^{n} \sum_{m=1}^{\infty} \frac{\partial}{\partial u_m} (A^{-1})_{kk} (A^{-1})_{mi} = \frac{\partial p}{\partial x_i}, \quad i = 1, ..., n.
\]

Here \( \rho \) obeys the equation (27) and \( p(x) \) is an arbitrary function.

This constraint together with the constraints (22) at \( N \to \infty \) characterizes those functions \( f_i, i = 1, 2, ..., \) for which the hodograph equations (5), (6) at \( N \to \infty \) provide us with the solutions of the n-dimensional Navier-Stokes equation (28).

Constraint (26) with \( \eta = \xi = 0 \) gives rise to the classical Euler equation which together with the continuity equation (27) describes the motion of inviscid compressible fluid \([1, 2]\).

In the approximation of incompressible inviscid fluid \( (\rho = \text{const}) \) for which

\[
\Delta p = -\rho \sum_{i,k=1}^{n} \frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_k}
\]

the constraint (26) assumes the form

\[
\sum_{k=1}^{n} \frac{\partial^2 u_{n+k}}{\partial x^2_k} = \sum_{i,k=1}^{n} \frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_k}.
\]

The corresponding constraint on \( A_{ik} \) is given by

\[
\sum_{k=1}^{n} \sum_{m=1}^{\infty} \frac{\partial}{\partial u_m} (A^{-1})_{m+k,k} (A^{-1})_{mk} = -\sum_{i,k=1}^{n} (A^{-1})_{ki} (A^{-1})_{ik}.
\]

Further, imposing the constraint

\[
\frac{\partial u_{n+i}}{\partial x_i} = \nu \Delta u_i, \quad i = 1, ..., n
\]
or equivalently

\[(A^{-1})_{n+i,i} = \nu \sum_{k=1}^{n} \sum_{m=1}^{\infty} \frac{\partial (A^{-1})_{ik}}{\partial u_m} (A^{-1})_{mk} , \quad i = 1, \ldots, n,\]

one reduces the Jordan chain to the n-dimensional Burgers equation

\[\frac{\partial u_i}{\partial t} + \sum_{k=1}^{n} u_k \frac{\partial u_i}{\partial x_k} + \nu \Delta u_i = 0 , \quad i = 1, \ldots, n. \quad (35)\]

This equation also has appeared in the study of various phenomena (see e.g. [12]).

At \(n = 1\) such a reduction has been considered in [10, 14].

Another type of finite-component reductions of the Jordan chain (5) is associated with the parametrization

\[u_{ln+i} = \frac{1}{ln+1} \sum_{\alpha=1}^{M} \epsilon_{\alpha} (V_{\alpha})^{ln+1} , \quad i = 1, \ldots, n; l = 0, 1, 2, \ldots, \quad (36)\]

where \(\epsilon_{\alpha}\) are arbitrary constants and \(M\) is arbitrary integer (\(\geq 1\)). Under such ansatz equations (25) become

\[\sum_{\alpha=1}^{M} \epsilon_{\alpha} (V_{\alpha})^{ln} \left\{ \frac{\partial V_{\alpha}}{\partial t} + \sum_{k=1}^{n} \sum_{\beta=1}^{M} \epsilon_{k\beta} V_{k\beta} \frac{\partial V_{\alpha}}{\partial x_k} + (V_{\alpha})^{n} \frac{\partial V_{\alpha}}{\partial x_i} \right\} = 0 , \quad (37)\]

\[i = 1, \ldots, n; l = 0, 1, 2, \ldots .\]

Hence, the variables \(V_{\alpha}\) obey the system of equations

\[\frac{\partial V_{\alpha}}{\partial t} + \sum_{k=1}^{n} \lambda_{\alpha,k} \frac{\partial V_{\alpha}}{\partial x_k} = 0 , \quad i = 1, \ldots, n; \alpha = 1, \ldots, M \quad (38)\]

where

\[\lambda_{\alpha,k} = \sum_{\beta=1}^{M} \epsilon_{k\beta} V_{k\beta} + \delta_{ik} (V_{\alpha})^{n} , \quad k, i = 1, \ldots, n; \alpha = 1, \ldots, M.\]

This n-dimensional hydrodynamic type system is the multi-dimensional generalization of the one-dimensional \(\epsilon\)-sistem discussed in [11].

It is noted that the system (38) apparently is not a reduction of the matrix n-dimensional homogeneous Euler equation considered in [13].

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The system (38) can be viewed as the source of solutions for the Jordan chain (25) and reduced equations presented above. Indeed, let us take a solution \( \{ V_{i\alpha}, i = 1, ..., n; \alpha = 1, ..., M \} \) of the system (38) and introduce the power sum variables \( u_{l_{n+i}}, i = 1, ..., n; l = 0, 1, ... \) defined by (39). Then the relation (37) implies that these \( u_{l_{n+i}} \) are the solutions of the Jordan chain (25). Varying \( M \) and \( \epsilon_{i\alpha} \), one obtains different classes of solutions for the Jordan chain.

Now let us consider the system (38) accompanied by the differential constraints

\[
\rho \sum_{\alpha=1}^{M} \epsilon_{i\alpha} (V_{i\alpha})^n \frac{\partial V_{i\alpha}}{\partial x_i} + \eta \sum_{\alpha=1}^{M} \epsilon_{i\alpha} \Delta V_{i\alpha} + \left( \xi + \frac{\eta}{3} \right) \sum_{k=1}^{n} \sum_{\alpha=1}^{M} \epsilon_{k\alpha} \frac{\partial^2 V_{k\alpha}}{\partial x_i \partial x_k} = \frac{\partial p}{\partial x_i}, \quad i = 1, ..., n, \]

\[
\frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \sum_{\alpha=1}^{M} \frac{\partial}{\partial x_i} (\rho \epsilon_{i\alpha} V_{i\alpha}) = 0. \tag{40}
\]

Then, due to the relation (26) solutions of the system (38)-(40) are the solutions of the Navier-Stokes equation (28) with \( u_i = \sum_{\alpha=1}^{M} \epsilon_{i\alpha} V_{i\alpha}, \quad i = 1, ..., n. \)

Classes of solutions of the multi-dimensional Burgers equation (35) are obtainable by the formula \( u_i = \sum_{\alpha=1}^{M} \epsilon_{i\alpha} V_{i\alpha}, \quad i = 1, ..., n \) where \( V_{i\alpha} \) are the solutions of the system composed by equation (38) and constraint

\[
\sum_{\alpha=1}^{M} \epsilon_{i\alpha} (V_{i\alpha})^n \frac{\partial V_{i\alpha}}{\partial x_i} + \nu \sum_{\alpha=1}^{M} \epsilon_{i\alpha} \Delta V_{i\alpha}, \quad i = 1, ..., n. \tag{41}
\]

The constraints (39), (41) are rather cumbersome. So the efficiency of such a procedure evidently depends on the degree of solvability of the systems of equations given above. This problem and that of applicability of the hodograph equations method to the system (38) definitely worth investigation.

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