Fractional cross intersecting families

Rogers Mathew¹, Ritabrata Ray², and Shashank Srivastava³

¹Department of Computer Science and Engineering, Indian Institute of Technology Kharagpur, Kharagpur 721302, India, rogersmathew@gmail.com
²Department of Electronics and Electrical Communication Engineering, Indian Institute of Technology Kharagpur, Kharagpur 721302, India, rayritabrata96@gmail.com
³Toyota Technological Institute at Chicago, Chicago 60615, USA, shashanksri47@gmail.com

Abstract

Let \( A = \{A_1, \ldots, A_p\} \) and \( B = \{B_1, \ldots, B_q\} \) be two families of subsets of \([n]\) such that for every \( i \in [p] \) and \( j \in [q] \), \(|A_i \cap B_j| = \frac{c}{d}|B_j|\), where \( \frac{c}{d} \in [0,1] \) is an irreducible fraction. We call such families \( \frac{c}{d} \)-cross intersecting families. In this paper, we find a tight upper bound for the product \(|A||B|\) and characterize the cases when this bound is achieved for \( \frac{c}{d} = \frac{1}{2} \). Also, we find a tight upper bound on \(|A||B|\) when \( B \) is \( k \)-uniform and characterize, for all \( \frac{c}{d} \), the cases when this bound is achieved.

1 Introduction

Let \([n]\) denote \(\{1,\ldots, n\}\) and let \(\mathcal{P}[n]\) denote the power set of \([n]\). We shall use \(\binom{[n]}{k}\) to denote the set of all \(k\)-sized subsets of \([n]\). Let \(\mathcal{F} \subseteq \mathcal{P}[n]\). The family \(\mathcal{F}\) is an intersecting family if every two sets in \(\mathcal{F}\) intersect with each other. The famous Erdős-Ko-Rado Theorem [1] states that \(|\mathcal{F}| \leq \binom{n-1}{k-1}\) if \(\mathcal{F}\) is a \(k\)-uniform intersecting family, where \(2k \leq n\). Several variants of the notion of intersecting families have been extensively studied in the literature. Given a set \(L = \{l_1, \ldots, l_s\}\) of non-negative integers, a family \(\mathcal{F} \subseteq \mathcal{P}[n]\) is L-intersecting if for all \(F_i, F_j \in \mathcal{F}, F_i \neq F_j, |F_i \cap F_j| \in L\). Ray-Chaudhuri and Wilson in [2] showed that if \(\mathcal{F}\) is \(k\)-uniform and L-intersecting, then \(|\mathcal{F}| \leq \binom{n}{s}\) and the bound is tight. Frankl and Wilson in [3] showed a tight upper bound of \(\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}\) if the restriction on the cardinalities of the sets of an L-intersecting family is relaxed. Further, if \(L\) is a singleton set, then Fisher inequality [4] gives an upper bound of \(|\mathcal{F}| \leq n\) for the cardinality of an L-intersecting family \(\mathcal{F}\). Recently, in [5], Balachandran et al. introduced a fractional variant of the classical L-intersecting families. For a survey on intersecting families, see [6].
Two families \( \mathcal{A}, \mathcal{B} \subseteq 2^{[n]} \) are cross-intersecting if \(|\mathcal{A} \cap \mathcal{B}| > 0\), \( \forall A \in \mathcal{A}, B \in \mathcal{B} \). Pyber in [7] showed that if \( n \geq 2k \), and \( \mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k} \) is a cross-intersecting pair of families, then \(|\mathcal{A}| |\mathcal{B}| \leq \left( \frac{n-k}{k-1} \right)^2 \). Frankl et al. in [8] showed that if \( \mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k} \) such that \(|A \cap B| \geq t\) for all \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), then for all \( n \geq (t+1)(k-t+1) \), \(|\mathcal{A}| |\mathcal{B}| \leq \left( \frac{n-k}{k-1} \right)^2 \), the cross-intersecting version of the Erdős-Ko-Rado Theorem. A cross-intersecting pair of families \( \mathcal{A}, \mathcal{B} \subseteq 2^{[n]} \) is said to be \( l \)-cross-intersecting if \( \forall A \in \mathcal{A}, B \in \mathcal{B}, |A \cap B| = l \), for some positive integer \( l \). Ahlswede, Cai and Zhang showed in [9], for all \( n \geq 2l \), a simple construction of an \( l \)-cross-intersecting pair \((\mathcal{A}, \mathcal{B})\) of families of subsets of \([n]\) with \(|\mathcal{A}| |\mathcal{B}| = \left( \frac{2}{l} \right)^{2^{n-2l}} = \Theta\left( \frac{2^n}{\sqrt{l}} \right) \). Later Alon and Lubetzky in [10] showed that the \( \Theta\left( \frac{2^n}{\sqrt{l}} \right) \) bound is tight and characterized the cases when the bound is achieved.

In this paper, we introduce a fractional variant of the \( l \)-cross-intersecting families. Let \( \mathcal{A} = \{A_1, \ldots, A_p\} \) and \( \mathcal{B} = \{B_1, \ldots, B_q\} \) be two families of subsets of \([n]\) such that for every \( i \in [p] \) and \( j \in [q] \), \(|A_i \cap B_j| = \frac{c}{d}\{|B_j|\} \), where \( \frac{c}{d} \in [0,1] \) is an irreducible fraction. We call such an \((\mathcal{A}, \mathcal{B})\) pair a \( \frac{c}{d} \)-cross-intersecting pair of families. Given \( c, d, n, \) and \( l \), let \( \mathcal{M}_{\frac{c}{d}}(n) \) denote the maximum value of \(|\mathcal{A}| |\mathcal{B}| \) where \((\mathcal{A}, \mathcal{B})\) is a \( \frac{c}{d} \)-cross-intersecting pair of families of subsets of \([n]\). We have the following results:

**Theorem 1.1.** \( \mathcal{M}_{\frac{c}{d}}(n) = 2^n \)

When \( \frac{c}{d} = 0 \), \( \mathcal{A} = 2^{[n]}, \mathcal{B} = \{\emptyset\} \) is a maximal pair. In fact, \( \mathcal{A} = 2^{[k]}, \mathcal{B} = \mathcal{P}(S) \), where \( \mathcal{P}(S) \) is the power set of \( S = \{k+1, \ldots, n\} \), are the only maximal pairs up to a relabelling of the elements, \( 0 \leq k \leq n \). When \( \frac{c}{d} = 1 \), \( \mathcal{A} = \{[n]\} \) and \( \mathcal{B} = 2^{[n]} \) is a maximal pair. In fact, \( \mathcal{B} = 2^{[k]}, \mathcal{A} = \{A : A = [k] \cup T, \text{ where } T \in \mathcal{P}(S)\} \), where \( \mathcal{P}(S) \) is the power set of \( S = \{k+1, \ldots, n\} \), are the only maximal pairs up to a relabelling of the elements, \( 0 \leq k \leq n \). In Theorem 1.2 we characterize all maximal pairs when \( \frac{c}{d} = \frac{1}{2} \).

**Theorem 1.2.** Let \((\mathcal{A}, \mathcal{B})\) be a \( \frac{1}{2} \)-cross-intersecting pair of families of subsets of \([n]\) with \(|\mathcal{A}| |\mathcal{B}| = 2^n \). Then \((\mathcal{A}, \mathcal{B})\) is one of the following \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \) pairs of families \((\mathcal{A}_k, \mathcal{B}_k), 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \), up to isomorphism.

\[
\mathcal{A}_0 = 2^{[n]} \quad \text{and} \quad \mathcal{B}_0 = \{\emptyset\}
\]

\[
\mathcal{A}_k = \{A \in 2^{[n]} : |A \cap \{2i - 1, 2i\}| = 1 \quad \forall i, 1 \leq i \leq k\}
\]

\[
\mathcal{B}_k = \{B \in 2^{[n]} : |B \cap \{2i - 1, 2i\}| \in \{0, 2\} \quad \forall i, 1 \leq i \leq k \quad \text{and} \quad \forall j > 2k, j \notin B\},
\]

where \( 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \).
It would be interesting to show a characterization theorem for any $\frac{c}{d} \in [0,1]$. We do have such a general characterization theorem (along with a new tight upper bound) in Theorem 1.3 for the case when $\mathcal{B}$ is $k$-uniform. The proof is a direct application of Theorem 1.1 in [10].

**Theorem 1.3.** Let $(\mathcal{A},\mathcal{B})$ be a $\frac{c}{d}$-cross intersecting pair of families of subsets of $[n]$. Let $\mathcal{B}$ be $k$-uniform. Then, there exists some $k_0 > 0$, such that for $k > k_0$ we have

$$|\mathcal{A}| |\mathcal{B}| \leq \left(\frac{2ck}{d}\right)^{2n-2ck}$$

and the bound is tight if and only if, either (a) or (b) hold:

(a) When $\frac{c}{d} = 1$, $\mathcal{A} = \{\{1,\ldots,\kappa\}\} \times 2^Y$, $\mathcal{B} = \{\binom{\kappa}{k}\}$ where $Y = \{\kappa + 1,\ldots,n\}$ and $\kappa \in \{2k-1,2k\}$ up to a relabelling of the elements of $[n]$.

(b) When $\frac{c}{d} \neq 1$:

(i) If $k$ is even, $c = 1, d = 2, \frac{ck}{d} = \lceil \frac{k}{2} \rceil$,

(ii) If $k$ is odd, $c = \frac{k+1}{2}, d = k, \frac{ck}{d} = \lceil \frac{k}{2} \rceil$,

and for both the cases(i) and (ii), there exists some $\tau$ such that, $k + \tau \leq n$ and up to a relabelling of the elements of $[n]$,

$$\mathcal{A} = \cup_{T \in J} T : J \subset \{\{1,k+1\},\ldots,\{\tau,k+\tau\},\{\tau+1\},\ldots,\{k\}\}, |J| = \lceil \frac{k}{2} \rceil \} \times 2^X$$

where $X = \{k+\tau+1,\ldots,n\}$ and

$$\mathcal{B} = \{L \cup \{\tau+1,\ldots,k\} : L \subset \{1,\ldots,\tau,k+1,\ldots,k+\tau\}, |L \cap \{i,k+i\}| = 1 \text{ for all } i \in [\tau]\}.$$

## 2 Notations and definitions

Given any $S \subseteq [n]$, we shall use $\chi(S)$ to denote the characteristic vector of $S$ which is a $0-1$ vector of size $n$ having its $i^{th}$ entry equal to 1 if and only if $i \in S$. The weight of a vector is the number of non-zero entries it has, and hence weight of $\chi(S)$ is the same as $|S|$.

For any family $\mathcal{A} \subseteq 2^{[n]}$, we shall (ab)use $\mathcal{A}$ to denote the collection of characteristic vectors of the members of $\mathcal{A}$ as well. The meaning will be clearly stated if not clear from the context.

Let $V$ be a collection of vectors in $\mathbb{F}_2^n$. Then, we define the following:
1. span(V): The collection of all the vectors that can be expressed as a linear combination in \( \mathbb{F}_2 \) of the vectors of V. We know that span(V) is a vector space over \( \mathbb{F}_2 \).

2. basis(V): We use basis(V) to denote the basis of span(V).

3. dim(V): \( \text{dim}(V) = |\text{basis}(V)| \)

Definition 1. V \( \subseteq \mathbb{F}_2^n \) is a linear code if \( V = \text{span}(V) \).

Definition 2. Given a linear code \( C \subseteq \mathbb{F}_2^n \), the dual code \( C^\perp \) is defined as,
\[
C^\perp = \{ x \in \mathbb{F}_2^n | \langle x, c \rangle = 0, \forall c \in C \}
\]
where \( \langle x, y \rangle \) is the standard inner product over \( \mathbb{F}_2 \).

The following is a well-known fact that is easy to verify.

Lemma 2.1. If \( C \subseteq \mathbb{F}_2^n \) is a linear code, then \( C^\perp \) is also a linear code.

Definition 3. Self orthogonal and self dual codes: A code \( C \) is self orthogonal if \( C \subseteq C^\perp \) and it is self dual if \( C = C^\perp \).

3 Bounding \( \mathcal{M}_{\frac{c}{d}}(n) \)

Let \( (\mathcal{A}, \mathcal{B}) \) be a \( \frac{c}{d} \)-cross-intersecting pair of families of subsets of \([n]\), where \( \frac{c}{d} \in [0, 1] \) is an irreducible fraction. We shall (ab)use \( \mathcal{A}, \mathcal{B} \) to denote the set of characteristic vectors of the sets in \( \mathcal{A}, \mathcal{B} \) respectively. For any \( a \in \mathcal{A}, b \in \mathcal{B} \), we observe that \( \langle a, b \rangle \equiv |A \cap B| \pmod{2} \), where \( a = \chi(A), b = \chi(B) \).

Partition the family \( \mathcal{B} \) into two parts as,
\[
\mathcal{B}_1 = \{ B \in \mathcal{B} : |B| \equiv 0 \pmod{2d} \} \quad (1)
\]
\[
\mathcal{B}_2 = \{ B \in \mathcal{B} : |B| \equiv d \pmod{2d} \} \quad (2)
\]

As all the sets \( B \in \mathcal{B} \) have their cardinality \( |B| \) divisible by \( d \), \( \{ \mathcal{B}_1, \mathcal{B}_2 \} \) is a valid partition of \( \mathcal{B} \). Therefore \( \forall a \in \mathcal{A}, b \in \mathcal{B} \), using the \( \frac{c}{d} \) intersection property, we have:
\[
\langle a, b \rangle = \begin{cases} 
1, & \text{if } b \in \mathcal{B}_2 \text{ and } c \text{ is odd} \\
0, & \text{otherwise}
\end{cases}
\]
Construction 1. Construct a set $B'_1$, by appending a 0 to the left of every vector in $B_1$, and a set $B'_2$ by appending a 1 to the left of every vector in $B_2$. Let $B' = B'_1 \cup B'_2$. Construct a set $A'$ by appending a 1 to the left of every vector in $A$.

We now have, the value of $\langle a, b \rangle = 0 \\forall a \in A', b \in B'$

So, $(\text{span}(A'), \text{span}(B'))$ is a pair of mutually orthogonal subspaces of $F_2^{n+1}$ over $F_2$. We thus have,

$$\dim(\text{span}(A')) + \dim(\text{span}(B')) \leq n + 1$$

So, it follows that

$$|\text{span}(A')| \cdot |\text{span}(B')| = 2^{\dim(\text{span}(A'))} \cdot 2^{\dim(\text{span}(B'))}$$

$$= 2^{\dim(\text{span}(A')) + \dim(\text{span}(B'))} \leq 2^{n+1} \quad (3)$$

Lemma 3.1. If the elements of a linear code $C \subseteq F_2^n$ are arranged as rows of a matrix $M_C$ with $n$ columns, then for each column, one of the following holds,

(i) All the entries in that column are 0

(ii) Exactly half the entries in that column are 0, and the rest are 1.

Proof. As $C$ is a linear code, if we pick any $a \in C$, and consider the set $S = \{a + x | x \in C\}$ where $a + x$ is the vector addition in $F_2^n$, then by the definition of a linear code $S = C$. Let $M_S$ be a matrix whose rows are the vectors of $S$, taken in any order. $M_S$ and $M_C$ have the same set of rows (only their order may differ).

Let $j \in [n]$. Column $j$ in $M_C$ and $M_S$ have the same number of 1's ( and 0's). Suppose (i) does not hold for column $j$ in $M_C$. Then, some row, say $a$, in $M_C$ has its $j^{th}$ entry as 1. Let $S$, and thereby $M_S$, be defined according to this vector $a$. From the definition of $S$, it is clear that the number of 1's in the $j^{th}$ column of $M_S$ is equal to the number of 1's in the $j^{th}$ column of $M_C$. Since adding $a$ to any $\{0, 1\}$ vector flips the $j^{th}$ coordinate of $v$, we conclude that (ii) holds for $M_c$.  

Corollary 3.2. $|\text{span}(A')| \geq 2|A'|$

Proof. The leftmost column of $M_{A'}$ does not contain any 0. As $\text{span}(A')$ is a linear code and $A' \subseteq \text{span}(A')$, by condition (ii) of Lemma 3.1 above, $\text{span}(A')$ must have at least $|A'|$ more elements having their leftmost entry as 0.  

Now we prove the main result of this section which is Theorem 1.

**Statement of Theorem 1.1.** \( \mathcal{M}_d(n) = 2^n \)

**Proof.** \( A = 2^{[n]}, B = \{\emptyset\} \) is a trivial example of a \( \frac{c}{d} \) cross-intersecting pair of families having \( |A||B| = 2^n \). Thus, \( \mathcal{M}_d(n) \geq 2^n \). The proof of the upper bound for \( \mathcal{M}_d(n) \) follows from Inequality (3) and Corollary 3.2. Let \( (A, B) \) be a \( \frac{c}{d} \) cross-intersecting pair of families of subsets of \([n]\). Let \( A', B' \) be constructed from \( A, B \), respectively, as explained in the beginning of this section. Note that \( |A'| = |A| \) and \( |B'| = |B| \) by construction.

\[
2^{n+1} \geq |\text{span}(A')| \cdot |\text{span}(B')| \quad \text{[from (3)]}
\]

\[
\geq 2 \cdot |A'| \cdot |\text{span}(B')| \quad \text{[from Corollary 3.2]}
\]

\[
\geq 2 \cdot |A'| \cdot |B'|
\]

\[
= 2 \cdot |A| \cdot |B| \quad \text{[by construction]}
\]

\( \square \)

4 Characterization of maximal pairs when \( \frac{c}{d} = \frac{1}{2} \)

**Definition 4.** Cross bisecting pair of families: A pair of families of subsets of \([n]\) is called a cross-bisecting pair if it is a \( \frac{1}{2} \) cross-intersecting pair. \( (A, B) \) is called a maximal cross bisecting or simply a maximal pair, if it is a cross bisecting pair and \( |A||B| = 2^n \).

For example, \( A = 2^{[n]} \) and \( B = \{\emptyset\} \) is a trivial maximal pair. In this section, we characterize all maximal pairs. Let \( (A, B) \) be a cross bisecting pair and let \( (A', B') \) be the associated pair constructed by appending bits as defined in the previous section.

**Definition 5.** Let \( f_A : A \rightarrow A' \) be a bijective mapping that maps every vector in \( A \) to its corresponding vector in \( A' \), and let \( g_A : A' \rightarrow A \) be its inverse. Likewise, define functions \( f_B \) and \( g_B \) between \( B \) and \( B' \). For any set \( V \subseteq A \), we shall use \( f_A(V) \) to denote \( \{f_A(A) \mid A \in V\} \) and for any \( V \subseteq A' \), we use \( g_A(A) \) to denote \( \{g_A(A) \mid A \in V\} \}. Similarly, for any \( V \subseteq B \), we use \( f_B(V) \) to denote \( \{f_B(B) \mid B \in V\} \) and for any \( V \subseteq B' \), \( g_B(V) \) to denote \( \{g_B(B) \mid B \in V\} \}

**Observation 1.** \( f_B(B_1) = B'_1 \) and \( f_B(B_2) = B'_2 \). Similarly, \( g_B(B'_1) = B_1 \) and \( g_B(B'_2) = B_2 \)
Suppose \((\mathcal{A}, \mathcal{B})\) is a maximal pair. Then from the proof of Theorem 1.1, we must have:

\[
|\text{span}(\mathcal{A}'))| = 2|\mathcal{A}'| \quad (4)
\]
\[
|\text{span}(\mathcal{B}'))| = |\mathcal{B}'| \quad (5)
\]
\[
\text{dim}(\text{span}(\mathcal{A}')) + \text{dim}(\text{span}(\mathcal{B}')) = n + 1 \quad (6)
\]

**Proposition 4.1.** \(\mathcal{B} = \text{span}(\mathcal{B})\). Further, \(f_\mathcal{B}\) is a linear map.

*Proof.* This follows from equation (5). Let \(x_1, x_2 \in \mathcal{B}\). We show that \(x_3 = x_1 + x_2 \in \mathcal{B}\). This would imply \(\mathcal{B}\) is closed under addition in \(\mathbb{F}_2^n\) over \(\mathbb{F}_2\), and hence \(\mathcal{B} = \text{span}(\mathcal{B})\).

Let \(x'_1 = f_\mathcal{B}(x_1)\) and \(x'_2 = f_\mathcal{B}(x_2)\). From Equation (5), we have, \(w = x'_1 + x'_2 \in \mathcal{B}'\). Since \(w\) and \(x_3\) agree on each of the rightmost \(n\) bits of \(x_3\), we have \(g_\mathcal{B}(w) = x_3\). Since \(w \in \mathcal{B}'\), from the definition of the function \(g_\mathcal{B}\) we have \(x_3 = g_\mathcal{B}(w) \in \mathcal{B}\).

Further, observe that \(f_\mathcal{B}(x_1) + f_\mathcal{B}(x_2) = w = f_\mathcal{B}(x_3) = f_\mathcal{B}(x_1 + x_2)\) and hence \(f_\mathcal{B}\) is a linear map. \(\square\)

That \(\mathcal{B}\) is a linear code from Proposition 4.1 implies closure of the family of subsets \(\mathcal{B}\) under symmetric difference. In fact, we have the following stronger result.

**Proposition 4.2.** Let vectors \(b_1, b_2 \in \mathcal{B}\). Then, \(b_1 + b_2 \in \mathcal{B}_1\) if and only if either \(b_1, b_2 \in \mathcal{B}_1\), or \(b_1, b_2 \in \mathcal{B}_2\). Otherwise, \(b_1 + b_2 \in \mathcal{B}_2\).

*Proof.* We prove the 2-way implication, and rest of the proposition follows from Proposition 4.1. Let \(b'_1 = f_\mathcal{B}(b_1), b'_2 = f_\mathcal{B}(b_2)\).

- \(b_1 + b_2 \in \mathcal{B}_1 \Rightarrow b'_1\) and \(b'_2\) are both from \(\mathcal{B}_1\), or both from \(\mathcal{B}_2\)
  
  Since \(f_\mathcal{B}\) is a linear map, we have \((b_1 + b_2 \in \mathcal{B}_1) \Rightarrow (f_\mathcal{B}(b_1 + b_2) = f_\mathcal{B}(b_1) + f_\mathcal{B}(b_2) = b'_1 + b'_2 \in \mathcal{B}_1)\). So, the leftmost bit of \(b'_1 + b'_2\) is 0. This means that the leftmost bit must be the same in \(b'_1\) and \(b'_2\), which directly implies that either \(b'_1, b'_2 \in \mathcal{B}_1\), or \(b'_1, b'_2 \in \mathcal{B}_2\).

- Either \(b_1, b_2 \in \mathcal{B}_1\), or \(b_1, b_2 \in \mathcal{B}_2 \Rightarrow b_1 + b_2 \in \mathcal{B}_1\)
  
  Since \(b'_1\) and \(b'_2\) agree upon the leftmost bit, \(b'_1 + b'_2\) has a 0 in its leftmost bit. So, \(b'_1 + b'_2 \in \mathcal{B}_1\). From the Observation 4.1 above, we have \(b_1 + b_2 \in \mathcal{B}_1\). \(\square\)

**Proposition 4.3.** \(\mathcal{B}\) is a self-orthogonal code.
Proof. We prove the proposition by showing that \( \forall b_1, b_2 \in B, \langle b_1, b_2 \rangle = 0 \). Let \( B_1, B_2 \) be the sets corresponding to the vectors \( b_1, b_2 \), respectively. Since we are operating in the field \( \mathbb{F}_2 \), it is enough to show that \( |B_1 \cap B_2| \) is even.

Let \( b_3 = b_1 + b_2 \). We observe that \( b_3 \) is the characteristic vector of \( B_3 = B_1 \Delta B_2 \), the symmetric difference of \( B_1 \) and \( B_2 \). We have,

\[
|B_3| = |B_1 \Delta B_2| = |B_1| + |B_2| - 2|B_1 \cap B_2| \tag{7}
\]

As \( \frac{2}{3} = \frac{1}{2}, \forall B \in B_1 \), we have \( |B| \equiv 0 \pmod{4} \). By Proposition 4.1, \( B_1 \Delta B_2 = B_3 \in B \) as \( B \) is a linear code. Taking equation (7) modulo 4, if \( B_3 \in B_1 \), then

\[
|B_1| + |B_2| - 2|B_1 \cap B_2| \equiv 0 \pmod{4}
\]

By Proposition 4.2, both \( B_1 \) and \( B_2 \) are either from \( B_1 \) or from \( B_2 \). In both cases, \( |B_1| + |B_2| \equiv 0 \pmod{4} \). Therefore, \( 2|B_1 \cap B_2| \equiv 0 \pmod{4} \) or \( |B_1 \cap B_2| \equiv 0 \pmod{2} \). If \( B_3 \in B_2 \), then

\[
|B_1| + |B_2| - 2|B_1 \cap B_2| \equiv |B_3| \equiv 2 \pmod{4}
\]

Again by Proposition 4.2, \( |B_1| + |B_2| \equiv 2 \pmod{4} \).

So, we have \( 2|B_1 \cap B_2| \equiv 0 \pmod{4} \) or \( |B_1 \cap B_2| \equiv 0 \pmod{2} \). Thus in both cases, \( |B_1 \cap B_2| \) is even, so \( B \) is a self-orthogonal code. \( \square \)

Lemma 4.4. Let \((A, B)\) be a maximal pair, then \( |B| \leq 2^\left\lfloor \frac{n}{2} \right\rfloor \)

Proof. It is a known result (see [11]) that for a linear code \( C \subseteq \mathbb{F}_2^n \) and its dual code \( C^\perp \),

\[
\dim(C) + \dim(C^\perp) = n \tag{8}
\]

For any self-orthogonal code \( C, C \subseteq C^\perp \). So,

\[
\dim(C) \leq \dim(C^\perp)
\]

Applying equation (8) in this inequality, we get

\[
n = \dim(C) + \dim(C^\perp) \geq 2\dim(C)
\]

Therefore, \( \dim(C) \leq \frac{n}{2} \)

Since \( B \) is a self-orthogonal code (Proposition 4.3), we get \( \dim(B) \leq \frac{n}{2} \). Hence,

\[
|B| \leq 2^\left\lfloor \frac{n}{2} \right\rfloor
\]

\( \square \)
**Proposition 4.5.** If a set $A$ bisects $B_1$, $B_2$ and $B_1 \Delta B_2$, then $A$ also bisects $B_1 \cap B_2$.

*Proof.*

\[
|A \cap (B_1 \Delta B_2)| = \frac{|B_1 \Delta B_2|}{2} \quad \Rightarrow |A \cap ((B_1 \setminus B_2) \cup (B_2 \setminus B_1))| = \frac{|B_1| + |B_2| - 2|B_1 \cap B_2|}{2} \\
\Rightarrow |A \cap (B_1 \setminus B_2)| + |A \cap (B_2 \setminus B_1)| = \frac{|B_1|}{2} + \frac{|B_2|}{2} - |B_1 \cap B_2| \\
\Rightarrow |A \cap B_1| - |A \cap (B_1 \cap B_2)| + |A \cap (B_2)| - |A \cap (B_1 \cap B_2)| = \frac{|B_1|}{2} + \frac{|B_2|}{2} - |B_1 \cap B_2| \\
\Rightarrow \frac{|B_1|}{2} + \frac{|B_2|}{2} - 2|A \cap (B_1 \cap B_2)| = \frac{|B_1|}{2} + \frac{|B_2|}{2} - |B_1 \cap B_2| \\
\Rightarrow 2|A \cap (B_1 \cap B_2)| = \frac{|B_1|}{2} + \frac{|B_2|}{2} - |B_1 \cap B_2| \\
\Rightarrow |A \cap (B_1 \cap B_2)| = \frac{|B_1 \cap B_2|}{2}
\]

\[\square\]

**Proposition 4.6.** $\mathcal{B}$ is closed under intersection.

*Proof.* Let $B_1, B_2 \in \mathcal{B}$. We show that $B_1 \cap B_2 \in \mathcal{B}$. By Proposition 4.1, $b_1 + b_2 \in \mathcal{B}$ i.e., $B_1 \Delta B_2 \in \mathcal{B}$. Let $A$ be any arbitrary member of $\mathcal{A}$. Now, $A$ bisects $B_1$, $B_2$ and $B_1 \Delta B_2$ as $(\mathcal{A}, \mathcal{B})$ is a cross bisecting pair. By Proposition 4.5, $A$ bisects $B_1 \cap B_2$. Since $(\mathcal{A}, \mathcal{B})$ is a maximal pair, we conclude that $B_1 \cap B_2 \in \mathcal{B}$. \[\square\]

Now, we prove the main result of this section, Theorem 1.2, the characterization of maximal pairs.

**Statement of Theorem 1.2.** Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{1}{2}$-cross intersecting pair of families of subsets of $[n]$ with $|\mathcal{A}| |\mathcal{B}| = 2^n$. Then $(\mathcal{A}, \mathcal{B})$ is one of the following $\left\lfloor \frac{n}{2} \right\rfloor + 1$ pairs of families $(\mathcal{A}_k, \mathcal{B}_k)$, $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, up to isomorphism.

$\mathcal{A}_0 = 2^{[n]}$ and $\mathcal{B}_0 = \{\emptyset\}$

$\mathcal{A}_k = \{A \in 2^{[n]} : |A \cap \{2i - 1, 2i\}| = 1 \quad \forall i, 1 \leq i \leq k\}$

$\mathcal{B}_k = \{B \in 2^{[n]} : |B \cap \{2i - 1, 2i\}| \in \{0, 2\} \quad \forall i, 1 \leq i \leq k$ and $\forall j > 2k, j \notin B\}$,
where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

By isomorphism, it is meant that for any maximal pair $(A, B)$, $\exists$ a bijective mapping $f : [n] \to [n]$ such that if every $A \in A$ is replaced by $A_f = \{ f(i) | i \in A \}$ and every $B \in B$ is replaced by $B_f = \{ f(i) | i \in B \}$ then the families $(A_f, B_f)$, where $A_f = \{ A_f | A \in A \}$ and $B_f = \{ B_f | B \in B \}$, is a maximal pair which is one of $(A_k, B_k), 0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

**Proof.** Consider a maximal pair $(A, B)$ where $B \neq \{\emptyset\}$. We write the elements of $B$ as rows of a $0-1$ matrix $M_0$. Suppose $n_0$ columns have only 0 entries in all the rows ($n_0$ may be 0). As the characterization is up to isomorphism, we may assume that these are the rightmost $n_0$ columns of the matrix $M_0$. In each of the remaining $n - n_0$ columns, from Lemma 3.1, there are exactly $\frac{|B|}{2}$ 1’s and $\frac{|B|}{2}$ 0’s as $B$ is a linear code. (by Proposition 4.1)

Define

$$B_1 = \bigcap_{B \in B} B$$

We write the $\frac{|B|}{2}$ rows containing 1 in the leftmost column of $M_0$ as the top $\frac{|B|}{2}$ rows to obtain a new matrix $M_1$. And $B_1$ is one of these rows according to Proposition 4.6. Moreover, as all intersections are of even cardinality (Proposition 4.3), $|B_1|$ is even.

Let $|B_1| = 2i_1, i_1 \geq 1$. So, there are $2i_1 - 1$ elements in $B_1$ other than the element 1. Due to isomorphism, we may assume them to be 2, 3, ..., $2i_1$.

If $2i_1 + 1 \leq n - n_0$, then define the set $B_2$ as:

$$B_2 = \bigcap_{B \in B} B$$

**Claim 4.7.** $1 \notin B_2$

**Proof.** Assume for the sake of contradiction, $1 \in B_2$. This implies that for all the $\frac{|B|}{2}$ sets which contain the element $2i_1 + 1$ also contain the element 1. From Lemma 3.1 (number of sets in $B$ that contain the element 1) = (number of sets in $B$ that contain the element $2i_1 + 1$) = $\frac{|B|}{2}$. Hence, for any $B \in B$, $1 \in B \iff 2i_1 + 1 \in B$. This implies that $2i_1 + 1 \in B_1$, which is a contradiction. Hence, $1 \notin B_2$ and therefore $B_2$ does not belong to the top $\frac{|B|}{2}$ rows of $M_1$.

**Claim 4.8.** $B_1 \cap B_2 = \emptyset$

**Proof.** Assume for the sake of contradiction, $x \in B_1 \cap B_2$. Then $x$ is present in the $\frac{|B|}{2}$ rows of the matrix $M_1$ whose intersection yields $B_1$. Since $x \in B_2$ and $B_2$ does not belong to these $\frac{|B|}{2}$ rows of $M_1$ (by Claim 4.7). Thus, we have the element $x$ present in at least $\frac{|B|}{2} + 1$ rows of $M_1$, contradicting Lemma 3.1.
We take the rows corresponding to the sets containing the \((2i_1 + 1)^{th}\) element that are not among the first \(\frac{|B|}{2}\) rows in \(M_1\) and arrange them below the top \(\frac{|B|}{2}\) rows to create a matrix called \(M_2\) from \(M_1\). Again from Proposition 4.3, \(|B_2|\) is even, say \(2i_2\). Due to isomorphism and Claim 4.8, we may assume that \(2i_1 + 1, \ldots, 2i_1 + 2i_2\) are these \(2i_2\) elements.

If \(2i_1 + 2i_2 + 1 \leq n - n_0\), then define,

\[
B_3 = \bigcap_{2i_1 + 2i_2 + 1 \in B} B
\]

**Claim 4.9.** \(1 \notin B_3 \) and \(2i_1 + 1 \notin B_3\).

The proof is similar to that of Claim 4.7.

**Claim 4.10.** \(B_1 \cap B_3 = \emptyset\) and \(B_2 \cap B_3 = \emptyset\).

The proof is again similar to that of Claim 4.8.

We take the rows corresponding to the sets containing the \((2i_1 + 2i_2 + 1)^{th}\) element that are not among the first \(r\) rows \((r > \frac{|B|}{2})\) in \(M_2\) which contain the elements 1 or \(2i_1 + 1\) and arrange them below the top \(r\) rows of \(M_2\) to create a matrix called \(M_3\) from \(M_2\). From Proposition 4.3 and the definition of \(B_3\), we have \(|B_3| = 2i_3\), \(i_3 \geq 1\). Due to isomorphism and Claim 4.10, we may assume that \(2i_1 + 2i_2 + 1, \ldots, 2i_1 + 2i_2 + 2i_3\) are these \(2i_3\) elements.

We continue in this manner for \(k\) steps by constructing sets \(B_1, \ldots, B_k\) and matrices \(M_1, \ldots, M_k\), where \(k \geq 1\), until we have \(2i_1 + \cdots + 2i_k = n - n_0\). Observe that \(B_1, \ldots, B_k\) and \(P = \{n - n_0 + 1, \ldots, n\}\) is a partition of \([n]\).

![Figure 1: Partitioning the universe and thereby the columns of \(M_k\)](image)

**Claim 4.11.** For any set \(B \in \mathcal{B}\), \(j \in [k]\), we have \(B \cap B_j \in \{\emptyset, B_j\}\). Further, \(B \cap P = \emptyset\).

**Proof.** From the definition of \(P\), we have \(B \cap P = \emptyset\). Let \(j \in [k]\). Since \(B_j\) is equal to the intersection of some \(\frac{|B|}{2}\) sets in \(\mathcal{B}\), we have \(B_j\) present as a subset of all these \(\frac{|B|}{2}\) sets. Applying Lemma 3.1, we can say that no element of \(B_j\) is present in any set in \(\mathcal{B}\) other than these \(\frac{|B|}{2}\) sets. Hence, the claim. \(\square\)
From Claim [4.11] observe that \( S = \{ B_1, \ldots, B_k \} \) forms a basis of the row space of the matrix \( M_k \). The advantage of such a “disjoint basis” is that the bisection in one part is independent of another.

![Figure 2: Basis for the code \( \mathcal{B} \)](image)

**Claim 4.12.** A set \( A \in \mathcal{A} \) bisects every set in \( \mathcal{B} \) if and only if it bisects every set in the basis \( S \) of \( \mathcal{B} \).

**Proof.** The forward direction is straightforward as \( S \subseteq \mathcal{B} \). For the opposite direction, let \( A \in \mathcal{A} \) be a set that bisects every member of \( S \). Since the sets corresponding to the members in \( S \) are disjoint, any \( B \in \mathcal{B} \) can be written as a union of some of these sets.

Let \( B = B_1 \cup \cdots \cup B_l \), where \( \{B_1, \ldots, B_l\} \subseteq S \). Then,

\[
|A \cap B| = |A \cap \left( \bigcup_{j=1}^{l} B_j \right)| = \sum_{j=1}^{l} |A \cap B_j| = \frac{\sum_{j=1}^{l} |B_j|}{2} = \frac{|\bigcup_{j=1}^{l} B_j|}{2} = \frac{|B|}{2}
\]

Since each set \( A \in \mathcal{A} \) bisects the sets \( B_1, \ldots, B_k \) and \( P \), from Claim 4.12 the set \( A \) may contain any of the \( 2^n \) subsets of \( P \), and \( |A \cap B_1| = i_1, \ldots, |A \cap B_k| = i_k \). Since \( \dim(\mathcal{B}) = k \), by Proposition 4.1 we have \( |\mathcal{B}| = 2^k \).

\[
|A||\mathcal{B}| = \left( 2^n \cdot \prod_{j=1}^{k} \binom{2i_j}{i_j} \right) \cdot 2^k
\]

Recall that \( \sum_{j=1}^{k} 2i_j = n - n_0 \). Right hand side of Equation (9) is equal to \( 2^n \) if and only if \( i_j = 1, \forall j \in [k] \).

Thus, if \( \mathcal{B} \neq \{\emptyset\} \), then \((\mathcal{A}_k, \mathcal{B}_k), k \geq 1, \) defined in the statement of the theorem are the only maximal pairs. This completes the proof of Theorem 1.2.
5 Tight upper bound on $M_{\frac{c}{d}}(n)$ when $B$ is $k$-uniform and characterization of the cases when the bound is achieved.

Let $(A, B)$ be a $\frac{c}{d}$ cross-intersecting pair of families of subsets of $[n]$, where $\frac{c}{d} \in [0, 1]$ is an irreducible fraction. In this section, we deal with the scenario when $B$ is $k$-uniform, where $0 < k \leq n$. Since $B$ is $k$-uniform, for any $A \in A$ and any $B \in B$, $|A \cap B| = \frac{ck}{d} = l$. Since $c$ is relatively prime with $d$, and $|A \cap B|$ is an integer, we have $k$ divisible by $d$. Therefore, we have a uniformly cross intersecting pair of families.

Alon and Lubetzky in [10] found a tight upper bound for the case of uniformly cross intersecting families and fully characterized the cases when the bound is achieved in the following theorem:

**Theorem 5.1.** [Theorem 1.1 in [10]] There exists some $l_0 > 0$ such that, for all $l \geq l_0$, every $l$-cross intersecting pair $A, B \subset 2^{[n]}$ satisfies:

$$|A||B| \leq \left(\frac{2l}{l}\right) 2^{n - 2l}$$

Furthermore, if $|A||B| = \left(\frac{2l}{l}\right) 2^{n - 2l}$, then there exists some choice of parameters $\kappa, \tau, n'$:

$$\kappa \in \{2l - 1, 2l\}, \tau \in \{0, \ldots, \kappa\}$$

such that up to a relabelling of the elements of $[n]$ and swapping $A, B$, the following holds:

- $A = \bigcup_{T \in J} T : J \subset \{1, \kappa + 1\}, \ldots, \{\tau, \kappa + \tau\}, \{\tau + 1\}, \ldots, \{\kappa\}, |J| = l \times 2^X$,
- $B = \{L \cup \{\tau + 1, \ldots, \kappa\} : L \subset \{1, \ldots, \tau, \kappa + 1, \ldots, \kappa + \tau\}, |L \cap \{i, \kappa + i\}| = 1 \text{ for all } i \in [\tau]\} \times 2^Y$,

where $X = \{\kappa + \tau + 1, \ldots, n'\}$ and $Y = \{n' + 1, \ldots, n\}$.

Let $(A, B)$ be a $\frac{c}{d}$ cross-intersecting family where $B$ is $k$-uniform. From Theorem 5.1 there exists a $k_0 > 0$ such that if $\frac{ck}{d} = l > k_0$, then $|A||B| \leq \left(\frac{2l}{l}\right) 2^{n - 2l}$. Consider the case when $B$ corresponds to $B$ of Theorem 5.1. If $|A||B| = \left(\frac{2l}{l}\right) 2^{n - 2l}$, then $n' = n$, $Y = \emptyset$, and $k = \kappa$ in the statement of Theorem 5.1. Since $l = \frac{ck}{d}$ and $k \in \{\frac{2ck}{d} - 1, \frac{2ck}{d}\}$, we have the following two cases:

**Case 1:** $k = \frac{2ck}{d} - 1$. Then, $(k + 1)d = 2ck$. Since $gcd(c, d) = 1$ and $gcd(k, k + 1) = 1$, we have $k|d|2k$. Thus, $d = k$ or $d = 2k$. We claim that $d = 2k$ is an invalid case.
This is because, when $d = 2k$, we have $c = k + 1$. Since $\gcd(c, d) = 1$, $k$ cannot be odd. And if $k$ is even, then $l = \frac{ck}{d} = \frac{k+1}{2}$ is not an integer. So, the only valid case is $d = k$, $c = \frac{k+1}{2} = l$ and $k$ is an odd integer.

**Case 2:** $k = \frac{2k}{d}$. Then, $\frac{c}{d} = \frac{1}{2}$, that is $(A, B)$ is a cross bisecting pair. Since $l = \frac{ck}{d} = \frac{k}{2}$ is an integer, $k$ must be even in this case.

If $B$ corresponds to $A$ of Theorem 1, we have $X = \emptyset$, $\tau = 0$, $B$ is $k$-uniform, $l = \frac{ck}{d}$ is an integer, and $A = \{1, \ldots, k\} \times 2^Y$ where $Y = \{\kappa + 1, \ldots, n\}$ and $B = \begin{pmatrix} \kappa \end{pmatrix}$, $\kappa \in \{2k - 1, 2k\}$ up to a relabelling of the elements.

This leads us to the main result of this section.

**Statement of Theorem 1.3** Let $(A, B)$ be a $\frac{c}{d}$-cross intersecting pair of families of subsets of $[n]$. Let $B$ be $k$-uniform. Then, there exists some $k_0 > 0$, such that for $k > k_0$ we have

$$|A||B| \leq \left(\frac{2}{\frac{c}{d}}\right) 2^{n - \frac{2k}{d}}$$

and the bound is tight if and only if, either (a) or (b) hold:

(a) When $\frac{c}{d} = 1$, $A = \{1, \ldots, \kappa\} \times 2^Y$, $B = \begin{pmatrix} \kappa \end{pmatrix}$ where $Y = \{\kappa + 1, \ldots, n\}$ and $\kappa \in \{2k - 1, 2k\}$ up to a relabelling of the elements of $[n]$.

(b) When $\frac{c}{d} \neq 1$:

(i) If $k$ is even, $c = 1$, $d = 2$, $\frac{ck}{d} = \left\lceil \frac{k}{2} \right\rceil$,

(ii) If $k$ is odd, $c = \frac{k+1}{2}$, $d = k$, $\frac{ck}{d} = \left\lceil \frac{k}{2} \right\rceil$,

and for both the cases((i) and (ii)), there exists some $\tau$ such that, $k + \tau \leq n$ and up to a relabelling of the elements of $[n]$,

$A = \{\cup_{T \in J} T : J \subset \{1, k + 1\}, \ldots, \{\tau, k + \tau\}, \{\tau + 1\}, \ldots, \{k\}, |J| = \left\lceil \frac{k}{2} \right\rceil\} \times 2^X$

where $X = \{k + \tau + 1, \ldots, n\}$ and $B = \{L \cup \{\tau + 1, \ldots, k\} : L \subset \{1, \ldots, \tau, k + 1, \ldots, k + \tau\}, |L \cap \{i, k + i\}| = 1$ for all $i \in [\tau]$.  

6 Discussion

What are those pairs of $\frac{c}{d}$-cross intersecting families $(A, B)$ which achieve $|A||B| = 2^n$ (equal to the upper bound for $M_{\frac{c}{d}}(n)$ proved in Theorem 1.1)? In the introduction we characterize such families when $\frac{c}{d} = 0$ and $\frac{c}{d} = 1$. In Theorem 1.2, we
characterize such families when $\frac{c}{d} = \frac{1}{2}$. From Theorem 1.3 we see that when $B$ is $k$-uniform, $|A||B|$ is maximized when $\frac{c}{d}$ is 1 or nearly $\frac{1}{2}$ (or $\frac{1}{2} + \frac{1}{2k}$). For $\frac{c}{d} \in (0,1)$, besides the case $A = 2^n$, $B = \varnothing$, is $|A||B| = 2^n$ achieved only when $\frac{c}{d}$ is close to $\frac{1}{2}$?

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