Rigidity of Symplectic Translating Solitons

Hongbing Qiu

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Abstract
We obtain a rigidity result of symplectic translating solitons via the complex phase map. It indicates that we can remove the bounded second fundamental form assumption for symplectic translating solitons in Han, Sun (Translating solitons to symplectic mean curvature flows. Ann Glob Anal Geom 38(2):161–169, 2010).

Keywords Rigidity · Translating soliton · Complex phase map · Hyper-Lagrangian

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1 Introduction
Let \( X : M^n \to \mathbb{R}^{m+n} \) be an isometric immersion from an \( n \)-dimensional oriented Riemannian manifold \( M \) to the Euclidean space \( \mathbb{R}^{m+n} \). The mean curvature flow (MCF) in Euclidean space is a one-parameter family of immersions \( X_t = X(\cdot, t) : M^n \to \mathbb{R}^{m+n} \) with the corresponding image \( M_t = X_t(M) \) such that

\[
\begin{align*}
\frac{\partial}{\partial t} X(x, t) &= H(x, t), \quad x \in M, \\
X(x, 0) &= X(x),
\end{align*}
\]

is satisfied, where \( H(x, t) \) is the mean curvature vector of \( M_t \) at \( X(x, t) \) in \( \mathbb{R}^{m+n} \). \( M^n \) is said to be a translating soliton in \( \mathbb{R}^{m+n} \) if it satisfies

\[ H = -V^N_0, \]

\( \text{# Hongbing Qiu} \\ hbqiu@whu.edu.cn \\
1 School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China \\
2 Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, China \\
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where $V_0$ is a fixed vector in $\mathbb{R}^{m+n}$ with unit length and $V_0^N$ denotes the orthogonal projection of $V_0$ onto the normal bundle of $M^n$. The translating soliton plays an important role in analysis of singularities in MCF. It is not only a special solution to the MCF, but also it is one of the most important example of Type II singularities (see [1, 2, 11, 14, 15, 28, 29]). And the geometry of the translating soliton has been paid considerable attention during the past two decades, see the references (not exhaustive): [7, 9, 10, 16, 17, 20–23, 27, 30], etc.

Since the translating soliton can be regarded as a generalization of minimal submanifolds, it is natural to study the Bernstein type problem of translating solitons. For the codimension one case, Bao–Shi [3] showed a translating soliton version of the Moser’s theorem for minimal hypersurfaces [19], namely, if the image of the translating soliton $M^n$ under the Gauss map is contained in a regular ball, then such a complete translating soliton in $\mathbb{R}^{n+1}$ has to be a hyperplane. For higher codimensions, Kunikawa [17] generalized the result of [3] to the flat normal bundle case. In general, Xin [30] proved that if the $v$-function satisfies $v \leq v_1 < v_0 := \frac{2\sqrt{3}}{1+3\sqrt{3}}$, then any complete translating soliton $M^n$ in $\mathbb{R}^{m+n}$ $(m \geq 2)$ must be affine linear. Recently, this result was improved by using a new test function in [24].

One special class of translating solitons with higher codimension is the symplectic translating solitons, which are solutions to symplectic MCFs (see [6, 26]). By using the Omori–Yau maximum principle, Han–Li [12] gave an estimate of the kähler angle for symplectic standard translating solitons, where the second fundamental form $B$ satisfies $\max |B| = 1$. Afterward, Han–Sun [13] showed that if the cosine of the Kähler angle has a positive lower bound, then any complete symplectic translating soliton with nonpositive normal curvature and bounded second fundamental form has to be an affine plane. Inspired by the work of Colding–Minicozzi II [8] for self-shrinkers, it is natural to ask whether we can remove the bounded second fundamental form condition for symplectic translating solitons in the previous result of [13].

Recall that Leung–Wan [18] introduced the concept of hyper-Lagrangian manifolds which is a generalization of complex Lagrangian submanifolds: a submanifold $L^{2n}$ of a hyperkähler manifold $\tilde{M}^{4n}$ is called hyper-Lagrangian if each tangent space $T_x L \subset T_x \tilde{M}$ is a complex Lagrangian subspace with respect to $\Omega J(x)$ with varying $J(x) \in \mathbb{S}^2$. Such a map $x \rightarrow J(x)$ is called the complex phase map $J : L \rightarrow \mathbb{S}^2$. We observe that any oriented surface immersed in a hyperkähler 4-manifold is always hyper-Lagrangian, and in a hyperkähler 4-manifold, a surface being symplectic is equivalent to the condition that the image under the complex phase map is contained in an open hemisphere.

In this note, first, by applying the fact that there is always a hyper-Lagrangian structure on a 2-dimensional translating soliton in $\mathbb{R}^4$, we show that the complex phase map of a 2-dimensional translating soliton is a generalized harmonic map, which can be regarded as a counterpart of the Ruh–Vilms type result for 2-dimensional translating solitons (Theorem 1). Second, using Theorem 1 and a test function in terms of the mean curvature and the complex phase map, we prove that if the image of the complex phase map is contained in a regular ball in $\mathbb{S}^2$, then any complete symplectic translating soliton with nonpositive normal curvature has to be an affine plane (Theorem 2). Note that the image of the complex phase map being contained in a regular ball is
equivalent to the condition that the cosine of the Kähler angle has a positive lower bound. Therefore Theorem 2 implies that the rigidity result for symplectic translating solitons in [13] holds even without the bounded second fundamental form assumption.

2 Preliminaries

In this section, we give some notations that will be used throughout the paper and recall some relevant definitions and results.

Let $\tilde{M}^{4n}$ be a $4n$-dimensional hyperkähler manifold, i.e., there exists two covariant constant anti-commutative complex structures $J_1, J_2$, i.e., $J_1, J_2$ are parallel with respect to the Levi–Civita connection and $J_1 J_2 = -J_2 J_1$. Denote $J_3 := J_1 J_2$, then the following quaternionic identities hold

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -1.$$  

Every SO(3) matrix preserves the quaternionic identities, i.e., $\{ \tilde{J}_\alpha := \sum_{\beta=1}^{3} a_{\alpha\beta} J_\beta \}$ satisfies the quaternionic identities

$$\tilde{J}_1^2 = \tilde{J}_2^2 = \tilde{J}_3^2 = \tilde{J}_1 \tilde{J}_2 \tilde{J}_3 = -1.$$  

In particular, for every unit vector $(a_1, a_2, a_3) \in \mathbb{R}^3$, we get a covariant constant complex structure $\sum_{\alpha=1}^{3} a_{\alpha} J_\alpha$, and this implies that $\left( \tilde{M}, \sum_{\alpha=1}^{3} a_{\alpha} J_\alpha \right)$ is a Kähler manifold.

Let $\tilde{\mathcal{J}} = \sum_{\alpha=1}^{3} \lambda_{\alpha} J_\alpha$ be a complex structure on $\tilde{M}$. Let $\omega_{\tilde{\mathcal{J}}}$ be the Kähler form with respect to $\tilde{\mathcal{J}}$, then the associated symplectic 2-form $\Omega_{\tilde{\mathcal{J}}} \in \Omega^{2,0}(\tilde{M}, \tilde{\mathcal{J}})$ is given by

$$\Omega_{\tilde{\mathcal{J}}} = \omega_K + \sqrt{-1} \omega_K \tilde{J},$$

where $K = \sum_{\alpha=1}^{3} \mu_{\alpha} J_\alpha$ is a complex structure which is orthogonal to $\tilde{\mathcal{J}}$ in the sense that $\sum_{\alpha=1}^{3} \lambda_{\alpha} \mu_{\alpha} = 0$. If $\tilde{\mathcal{J}}$ is parallel, then $\Omega_{\tilde{\mathcal{J}}}$ is holomorphic with respect to the covariant constant complex structure $\tilde{\mathcal{J}}$.

Let $\omega_{\alpha}$ be the Kähler form associated with the complex structure $J_\alpha$, then $\left( \tilde{M}, J_1 \right)$ is a Kähler manifold and

$$\Omega_{J_1} = \omega_2 + \sqrt{-1} \omega_3 \in H^{2,0}(M, J_1)$$

is the associated holomorphic symplectic 2-form. We say that a $2n$-dimensional submanifold $L^{2n}$ of $\tilde{M}^{4n}$ is complex Lagrangian if for some covariant constant complex structure $\tilde{\mathcal{J}}$ of $\tilde{M}$ such that the associated holomorphic symplectic 2-form $\Omega_{\tilde{\mathcal{J}}}$ vanished everywhere on $L$. Without loss of generality, assume $\tilde{\mathcal{J}} = J_1$, then $L$ is a Kähler submanifold of the Kähler manifold $(\tilde{M}, J_1)$. In particular, $L$ is a minimal submanifold of $\tilde{M}$. Moreover, both $L \subset (\tilde{M}, J_2)$ and $L \subset (\tilde{M}, J_3)$ are Lagrangian immersions.

We say that $L^{2n}$ is a hyper-Lagrangian submanifold of $\tilde{M}^{4n}$ if there is a complex structure $\tilde{\mathcal{J}} = \sum_{\alpha=1}^{3} \lambda_{\alpha} J_\alpha$ such that the associated symplectic 2-form $\Omega_{\tilde{\mathcal{J}}}$ vanished everywhere on $L$. The map
\[ J : L \rightarrow \mathbb{S}^2, \quad x \mapsto J(x) := (\lambda_1, \lambda_2, \lambda_3) \]

is called the complex phase map. In other words, \( L \) is hyper-Lagrangian iff each \( T_x L \) is a complex Lagrangian subspace of \( T_x \tilde{M} \). Here, we say that \( T_x L \) is a complex Lagrangian subspace of \( T_x \tilde{M} \) if for some complex structure \( \tilde{J} = \sum_{\alpha=1}^{3} \lambda_\alpha J_\alpha \) we have

\[ \bar{g}(K \cdot, \cdot)|_{T_x L} = 0 \]

for all complex structures \( K = \sum_{\alpha=1}^{3} \mu_\alpha J_\alpha \) which are orthogonal to \( \tilde{J} \). Therefore, \( L \) is complex Lagrangian iff \( L \) is hyper-Lagrangian with the constant complex phase map.

The complex phase map \( J \) defines an almost complex structure \( \tilde{J} = \sum_{\alpha=1}^{3} \lambda_\alpha J_\alpha |_{TL \rightarrow TL} \) on \( L \) and an almost complex structure \( \tilde{J}^\perp = \sum_{\alpha=1}^{3} \lambda_\alpha J_\alpha |_{T^\perp L \rightarrow T^\perp L} \) on \( T^\perp L \), where \( T^\perp L \) is the normal bundle of \( L \) in \( \tilde{M} \). We denote the Levi–Civita connection on \( T \tilde{M}, TL \) by \( \nabla \), \( \nabla \) respectively. If there is no confusion, we also denote the normal connection on \( T^\perp L \) by \( \nabla \). For \( V \in \Gamma(TL) \), let \( \Delta_V := \Delta + \langle V, \nabla \cdot \rangle \), where \( \Delta \) is the usual Laplacian operator with respect to \( \nabla \) on \( L \).

The second fundamental form \( B \) of \( L^{2n} \) in \( \tilde{M}^{4n} \) is defined by

\[ B(U, W) := (\nabla_U W)^N \]

for \( U, W \in \Gamma(TL) \). We use the notation \((\cdot)^T \) and \((\cdot)^N \) for the orthogonal projections into the tangent bundle \( TL \) and the normal bundle \( T^\perp L \), respectively. For \( v \in \Gamma(T^\perp L) \) we define the shape operator \( A^v : TL \rightarrow TL \) by

\[ A^v(U) := -(\nabla_U v)^T. \]

Taking the trace of \( B \) gives the mean curvature vector \( H \) of \( L \) in \( \tilde{M}^{4n} \) and

\[ H := \text{trace}(B) = \sum_{i=1}^{2n} B(e_i, e_i), \]

where \( \{e_i\} \) is a local orthonormal frame field of \( L \).

The curvature tensors \( R(X, Y)Z \) and \( R(X, Y)\mu \) can be defined, corresponding to the connections in the tangent bundle and the normal bundle respectively, as follows:

\[ R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, \]
\[ R(X, Y)\mu = -\nabla_X \nabla_Y \mu + \nabla_Y \nabla_X \mu + \nabla_{[X,Y]} \mu, \]

where \( X, Y, Z \) are tangent vector fields, \( \mu \) is a normal vector field.

### 3 The Complex Phase Map

First, we give the following characterization of the hyper-Lagrangian condition:
Lemma 1 [25] Let $L^{2n}$ be a submanifold of a hyperkähler manifold $\tilde{M}^{4n}$. Then $L$ is hyper-Lagrangian iff

$$J_{\alpha}|_{TL} \rightarrow TL = \lambda_{\alpha} \tilde{J}, \quad \alpha = 1, 2, 3,$$

iff

$$J_{\alpha}|_{T^\perp L} \rightarrow T^\perp L = \lambda_{\alpha} \tilde{J}^\perp, \quad \alpha = 1, 2, 3.$$

Proof For the completeness, we write the details of the proof as follows:

Under the orthogonal decomposition $T_x \tilde{M} = T_x L \oplus T_x^\perp L$, we write

$$J_{\alpha} = \begin{pmatrix} A_{\alpha} & B_{\alpha} \\ -B^T_{\alpha} & C_{\alpha} \end{pmatrix}, \quad \alpha = 1, 2, 3.$$

Let $A = (a_{\alpha\beta})_{1 \leq \alpha, \beta \leq 3} \in \text{SO}(3)$ where $\lambda_{\alpha} = a_{1\alpha}$. Set $\tilde{J}_{\beta} = \sum_{\alpha=1}^{3} a_{\beta\alpha} J_{\alpha}$. Since $L$ is hyper-Lagrangian, we get

$$\sum_{\gamma=1}^{3} a_{\beta\gamma} A_{\gamma} = 0, \quad \beta = 2, 3.$$

Namely

$$\tilde{J}_{\beta} = \begin{pmatrix} 0 & \sum_{\alpha=1}^{3} a_{\beta\alpha} B_{\alpha} \\ -\sum_{\alpha=1}^{3} a_{\beta\alpha} B^T_{\alpha} & \sum_{\alpha=1}^{3} a_{\beta\alpha} C_{\alpha} \end{pmatrix}, \quad \beta = 2, 3.$$

Since $\tilde{J}_{\beta}^2 = -1$, we can conclude that

$$\sum_{\gamma=1}^{3} a_{\beta\gamma} C_{\gamma} = 0, \quad \beta = 2, 3.$$

Hence we have

$$\tilde{J}_2 = \begin{pmatrix} 0 & \sum_{\alpha=1}^{3} a_{2\alpha} B_{\alpha} \\ -\sum_{\alpha=1}^{3} a_{2\alpha} B^T_{\alpha} & 0 \end{pmatrix}, \quad \tilde{J}_3 = \begin{pmatrix} 0 & \sum_{\alpha=1}^{3} a_{3\alpha} B_{\alpha} \\ -\sum_{\alpha=1}^{3} a_{3\alpha} B^T_{\alpha} & 0 \end{pmatrix},$$

and

$$A_{\alpha} = \sum_{\beta, \gamma=1}^{3} a_{\alpha\beta} a_{\beta\gamma} A_{\gamma} = \sum_{\gamma=1}^{3} a_{1\alpha} a_{1\gamma} A_{\gamma} = a_{1\alpha} \tilde{J} = \lambda_{\alpha} \tilde{J}, \quad \alpha = 1, 2, 3,$$

$$C_{\alpha} = \sum_{\beta, \gamma=1}^{3} a_{\alpha\beta} a_{\beta\gamma} C_{\gamma} = \sum_{\gamma=1}^{3} a_{1\alpha} a_{1\gamma} C_{\gamma} = a_{1\alpha} \tilde{J}^\perp = \lambda_{\alpha} \tilde{J}^\perp, \quad \alpha = 1, 2, 3.$$

Conversely, if $A_{\alpha} = \lambda_{\alpha} \tilde{J} (\alpha = 1, 2, 3)$, it follows that
\[
\sum_{\gamma=1}^{3} a_{\beta\gamma} A_{\gamma} = \sum_{\gamma=1}^{3} a_{\beta\gamma} \lambda_{\gamma} \tilde{J} = \sum_{\gamma=1}^{3} a_{\beta\gamma} a_{1\gamma} \tilde{J} = \delta_{\beta1} \tilde{J} = 0, \quad \beta = 2, 3.
\]

Similarly, if \( C_{\alpha} = \lambda_{\alpha} \tilde{J} \perp (\alpha = 1, 2, 3) \),
\[
\sum_{\gamma=1}^{3} a_{\beta\gamma} C_{\gamma} = \sum_{\gamma=1}^{3} a_{\beta\gamma} \lambda_{\gamma} \tilde{J} \perp = \sum_{\gamma=1}^{3} a_{\beta\gamma} a_{1\gamma} \tilde{J} \perp = \delta_{\beta1} \tilde{J} \perp = 0, \quad \beta = 2, 3.
\]

Then, we get
\[
\tilde{J}_{\beta} = \left( \begin{array}{c} \sum_{\alpha=1}^{3} a_{\beta\alpha} A_{\alpha} \\ \sum_{\alpha=1}^{3} a_{\beta\alpha} B_{\alpha}^T \\ 0 \end{array} \right), \quad \beta = 2, 3.
\]

Since \( \tilde{J}_{\beta}^2 = -1 \), we can conclude that
\[
\sum_{\gamma=1}^{3} a_{\beta\gamma} A_{\gamma} = 0, \quad \beta = 2, 3.
\]

Hence, we complete the proof. \( \square \)

**Remark 1**

This Lemma claims that any oriented surface immersed in a hyper-kähler 4-manifold is always hyper-Lagrangian.

Recall that a map \( u \) from a Riemannian manifold \((M, g)\) to another Riemannian manifold \((N, h)\) is called a \( V \)-harmonic map if it solves
\[
\tau(u) + du(V) = 0,
\]
where \( \tau(u) \) is the tension field of the map \( u \), and \( V \) is a vector field on \( M \) (cf. [4, 5]). Clearly, it is a generalization of the usual harmonic map.

According to Remark 1, a 2-dimensional translating soliton \( \Sigma \) in \( \mathbb{R}^4 \) is hyper-Lagrangian. By applying this new structure on it, we show that the complex phase map of \( \Sigma \) is a \( -V_0^T \)-harmonic map as follows:

**Theorem 1**

Let \( X : \Sigma^2 \to \mathbb{R}^4 \) be a 2-dimensional translating soliton. Then, the complex phase map \( J : \Sigma \to S^2 \) satisfies
\[
\tau(J) = dJ \left( V_0^T \right).
\]

**Proof**

Since \( \Sigma \) is hyper-Lagrangian, by Lemma 1, for any \( Z \in \Gamma(T \Sigma) \), we get
\[
\lambda_{\alpha} \tilde{J} Z = (J_\alpha Z)^T. \quad (3.1)
\]
Then, we take the covariant differential on both sides of the above equation with any \( Y \in \Gamma (T \Sigma) \),
\[
\nabla_Y (\lambda_\alpha \tilde{J} Z) = \nabla_Y (J_\alpha Z)^T. \tag{3.2}
\]

Let \( \{ e_1, e_2 \} \) be a local orthonormal frame field of \( T \Sigma \), and \( \nabla e_i = 0 \) at the considered point. The Gauss formula and (3.1) imply that
\[
\nabla_Y (J_\alpha Z)^T = \nabla_Y (J_\alpha Z)^T - B(Y, (J_\alpha Z)^T)
\]
\[
= \nabla_Y \left( \sum_{j=1}^{2} (J_\alpha Z, e_j) e_j \right) - B(Y, \lambda_\alpha \tilde{J} Z)
\]
\[
= \sum_{j=1}^{2} \left[ (Y(J_\alpha Z, e_j)) e_j + (J_\alpha Z, e_j) \nabla_Y e_j \right] - B(Y, \lambda_\alpha \tilde{J} Z) \tag{3.3}
\]
\[
= (\nabla_Y (J_\alpha Z))^T + \sum_{j=1}^{2} (J_\alpha Z, B(Y, e_j)) e_j
\]
\[
+ \sum_{j=1}^{2} (J_\alpha Z, e_j) B(Y, e_j) - \lambda_\alpha B(Y, \tilde{J} Z).
\]

Since
\[
(\nabla_Y (J_\alpha Z))^T = ((\nabla_Y J_\alpha) Z + J_\alpha \overline{\nabla Y} Z)^T
\]
\[
= (J_\alpha \overline{\nabla Y} Z)^T = (J_\alpha \nabla_Y Z + J_\alpha B(Y, Z))^T, \tag{3.4}
\]

and
\[
\sum_{j=1}^{2} (J_\alpha Z, e_j) B(Y, e_j) = \sum_{j=1}^{2} ((J_\alpha Z)^T, e_j) B(Y, e_j)
\]
\[
= \sum_{j=1}^{2} (\lambda_\alpha \tilde{J} Z, e_j) B(Y, e_j) = \lambda_\alpha B(Y, \tilde{J} Z). \tag{3.5}
\]

Substituting (3.4) and (3.5) into (3.3), we obtain
\[
\nabla_Y (J_\alpha Z)^T = (J_\alpha \nabla_Y Z + J_\alpha B(Y, Z))^T + \sum_{j=1}^{2} \left( (J_\alpha Z)^N, B(Y, e_j) \right) e_j
\]
\[
= (J_\alpha \nabla_Y Z)^T + (J_\alpha B(Y, Z))^T + \sum_{j=1}^{2} \left( A(J_\alpha Z)^N(Y), e_j \right) e_j \tag{3.6}
\]
\[
= \lambda_\alpha \tilde{J} \nabla_Y Z + (J_\alpha B(Y, Z))^T + A(J_\alpha Z)^N(Y).
\]
By (3.2) and (3.6), we have
\[ Y(\lambda \alpha) \tilde{J}Z + \lambda \alpha (\nabla_Y \tilde{J})Z + \lambda \alpha \tilde{J} \nabla_Y Z = \nabla_Y (\lambda \alpha \tilde{J}Z) = \nabla_Y (J_\alpha Z)^T = \lambda \alpha \tilde{J} \nabla_Y Z + (J_\alpha B(Y, Z))^T + A^{(J_\alpha Z)^N}(Y). \]

Namely
\[ Y(\lambda \alpha) \tilde{J}Z + \lambda \alpha (\nabla_Y \tilde{J})Z = (J_\alpha B(Y, Z))^T + A^{(J_\alpha Z)^N}(Y). \]  \hspace{1cm} (3.7)

Multiplying \( \lambda \alpha \) on both sides of the above equality and take summation from \( \alpha = 1 \) to 3, we get
\[ Y \left( \sum_{\alpha=1}^{3} \lambda^2 \alpha \right) \tilde{J}Z + \sum_{\alpha=1}^{3} \lambda^2 \alpha (\nabla_Y \tilde{J})Z = \left( \sum_{\alpha=1}^{3} \lambda \alpha J_\alpha B(Y, Z) \right)^T + A \left( \sum_{\alpha=1}^{3} \lambda \alpha J_\alpha Z \right)^N(Y). \]

Note that \( \sum_{\alpha=1}^{3} \lambda^2 \alpha = 1 \) and \( \sum_{\alpha=1}^{3} \lambda \alpha J_\alpha = \tilde{J}_1 \), where \( \tilde{J}_1 \) is the same as the one in the proof of Lemma 1, we then derive
\[ (\nabla_Y \tilde{J})Z = (\tilde{J}_1 B(Y, Z))^T + A(\tilde{J}_1 Z)^N(Y). \]

Since \( \tilde{J}_1 = J = \tilde{J} \oplus \tilde{J}^\perp \), we have
\[ (\tilde{J}_1 Z)^N = 0 \quad (\tilde{J}_1 B(Y, Z))^T = 0. \]

Therefore
\[ \nabla \tilde{J} = 0. \]  \hspace{1cm} (3.8)

Combining (3.7) with (3.8), it follows that
\[ Y(\lambda \alpha) \tilde{J}Z = (J_\alpha B(Y, Z))^T + A^{(J_\alpha Z)^N}(Y). \]  \hspace{1cm} (3.9)

Let \( Z = e_j \) in (3.9), direct computation gives us
\[ 2Y(\lambda \alpha) = \sum_{j=1}^{2} Y(\lambda \alpha) \langle \tilde{J}e_j, \tilde{J}e_j \rangle = \sum_{j=1}^{2} \langle J_\alpha B(Y, e_j), \tilde{J}e_j \rangle + \sum_{j=1}^{2} \langle A^{(J_\alpha e_j)^N}(Y), \tilde{J}e_j \rangle \]
\[ = - \sum_{j=1}^{2} \langle B(Y, e_j), J_\alpha \tilde{J}e_j \rangle + \sum_{j=1}^{2} \langle A^{(J_\alpha e_j)^N}(Y), \tilde{J}e_j \rangle. \]  \hspace{1cm} (3.10)
Since \( \sum_{j=1}^{2} (B(Y, e_j), J_\alpha \tilde{e}_j) \) is independent of the choice of local orthonormal frame fields of \( \Sigma \) and \( \langle \tilde{e}_i, \tilde{e}_j \rangle = \delta_{ij} \), we obtain

\[
- \sum_{j=1}^{2} (B(Y, e_j), J_\alpha \tilde{e}_j) = \sum_{j=1}^{2} (B(Y, \tilde{e}_j), J_\alpha e_j) = \sum_{j=1}^{2} \left\langle B(Y, \tilde{e}_j), (J_\alpha e_j)^N \right\rangle = \sum_{j=1}^{2} \left\langle A^{(J_\alpha e_j)^N}(Y), \tilde{e}_j \right\rangle. \tag{3.11}
\]

From (3.10) and (3.11), we get

\[
Y(\lambda_\alpha) = \sum_{j=1}^{2} \left\langle A^{(J_\alpha e_j)^N}(Y), \tilde{e}_j \right\rangle = \sum_{j=1}^{2} \left\langle A^{(J_\alpha e_j)^N}(\tilde{e}_j), Y \right\rangle.
\]

It follows that

\[
\nabla \lambda_\alpha = \sum_{j=1}^{2} e_j(\lambda_\alpha)e_j = \sum_{i=1}^{2} A^{(J_\alpha e_i)^N}(\tilde{e}_i) \tag{3.12}
\]

and

\[
\Delta \lambda_\alpha = \sum_{j=1}^{2} e_j \left( \sum_{i=1}^{2} A^{(J_\alpha e_i)^N}(\tilde{e}_i), e_j \right) = \sum_{i,j=1}^{2} e_j \left( B(\tilde{e}_i, e_j), (J_\alpha e_i)^N \right) = \sum_{i,j=1}^{2} e_j \left( B(\tilde{e}_i, e_j), J_\alpha e_i \right) \tag{3.13}
\]

\[
= \sum_{i,j=1}^{2} \left\langle \nabla e_j B(\tilde{e}_i, e_j), J_\alpha e_i \right\rangle + \sum_{i,j=1}^{2} \left\langle B(\tilde{e}_i, e_j), \nabla e_j (J_\alpha e_i) \right\rangle.
\]

Using Weingarten formula in (3.13),

\[
\Delta \lambda_\alpha = \sum_{i,j=1}^{2} \left\langle -A B(\tilde{e}_i, e_j)(e_j) + \nabla e_j B(\tilde{e}_i, e_j), J_\alpha e_i \right\rangle + \sum_{i,j=1}^{2} \left\langle B(\tilde{e}_i, e_j), J_\alpha \nabla e_j e_i \right\rangle
\]

\[
= -\sum_{i,j=1}^{2} \left\langle B(e_j, (J_\alpha e_i)^T), B(\tilde{e}_i, e_j) \right\rangle + \sum_{i,j=1}^{2} \left\langle \nabla e_j B(\tilde{e}_i, e_j), J_\alpha e_i \right\rangle + \sum_{i,j=1}^{2} \left\langle B(\tilde{e}_i, e_j), J_\alpha B(e_i, e_j) \right\rangle. \tag{3.14}
\]
Since
\[(J_\alpha e_i)^T = \lambda_\alpha \tilde{J} e_i,\]
we get
\[B \left( e_j, (J_\alpha e_i)^T \right) = \lambda_\alpha B(e_j, \tilde{J} e_i). \tag{3.15} \]

By Lemma 1, for any $\xi \in \Gamma(T^\perp \Sigma)$,
\[(J_\alpha \xi)^N = \lambda_\alpha \tilde{J}^\perp \xi. \]

Thus
\[(J_\alpha B(e_i, e_j))^N = \lambda_\alpha \tilde{J}^\perp B(e_i, e_j). \tag{3.16} \]

Then we have
\[
\begin{aligned}
\langle B(\tilde{J} e_i, e_j), J_\alpha B(e_i, e_j) \rangle &= \langle B(\tilde{J} e_i, e_j), (J_\alpha B(e_i, e_j))^N \rangle \\
&= \lambda_\alpha \langle B(\tilde{J} e_i, e_j), \tilde{J}^\perp B(e_i, e_j) \rangle. \tag{3.17}
\end{aligned}
\]

Substituting (3.15) and (3.17) into (3.14), we obtain
\[
\Delta \lambda_\alpha = -\lambda_\alpha \sum_{i,j=1}^2 \left\{ B(\tilde{J} e_i, e_j), B(\tilde{J} e_i, e_j) - \tilde{J}^\perp B(e_i, e_j) \right\} \\
+ \sum_{i,j=1}^2 \langle \nabla e_j B(\tilde{J} e_i, e_j), J_\alpha e_i \rangle. \tag{3.18}
\]

By using the Gauss formula again,
\[
B(e_j, \tilde{J} e_i) = (\nabla e_j (\tilde{J} e_i))^N = \left( \sum_{\beta=1}^3 \nabla e_j (\lambda_\beta J_\beta e_i) \right)^N \\
= \left( \sum_{\beta=1}^3 e_j (\lambda_\beta) J_\beta e_i + \sum_{\beta=1}^3 \lambda_\beta J_\beta B(e_j, e_i) \right)^N \tag{3.19} \\
= \sum_{\beta=1}^3 e_j (\lambda_\beta) (J_\beta e_i)^N + \sum_{\beta=1}^3 \lambda_\beta (J_\beta B(e_j, e_i))^N.
\]

Since
\[(J_\beta e_i)^N = J_\beta e_i - (J_\beta e_i)^T = J_\beta e_i - \lambda_\beta \tilde{J} e_i\]
Substituting (3.16) and the above equality into (3.19), we get

\[ B(e_j, \tilde{J}e_i) = \sum_{\beta=1}^{3} e_j(\lambda_\beta)J_\beta e_i - \frac{1}{2} e_j \left( \sum_{\beta=1}^{3} \lambda_\beta^2 \right) \tilde{J}e_i + \sum_{\beta=1}^{3} \lambda_\beta^2 \tilde{J}^\perp B(e_j, e_i) \]

\[ = \sum_{\beta=1}^{3} e_j(\lambda_\beta)J_\beta e_i + \tilde{J}^\perp B(e_j, e_i). \]  

(3.20)

By (3.8), we derive

\[ \nabla e_j B(\tilde{J}e_i, e_j) = (\nabla e_j B)(\tilde{J}e_i, e_j) + B(\nabla e_j (\tilde{J}e_i), e_j) + B(\tilde{J}e_i, \nabla e_j e_j) \]

\[ = (\nabla e_j B)(\tilde{J}e_i, e_j) + B(\nabla e_j \tilde{J}e_i + \tilde{J} \nabla e_j e_i, e_j) + B(\tilde{J}e_i, \nabla e_j e_j) \]

\[ = (\nabla e_j B)(\tilde{J}e_i, e_j). \]

The Codazzi equation then implies

\[ \sum_{j=1}^{2} \nabla e_j B(\tilde{J}e_i, e_j) = \sum_{j=1}^{2} (\nabla e_j B)(\tilde{J}e_i, e_j) = \sum_{j=1}^{2} \left( \nabla \tilde{J} e_i B \right)(e_j, e_j) \]

\[ = \sum_{j=1}^{2} \nabla \tilde{J} e_i B(e_j, e_j) = \nabla \tilde{J} e_i H. \]  

(3.21)

Substituting (3.20) and (3.21) into (3.18), we obtain

\[ \Delta \lambda_\alpha = -\lambda_\alpha \sum_{i,j=1}^{2} \left\{ \sum_{\beta=1}^{3} e_j(\lambda_\beta)J_\beta e_i \right\} \left( \sum_{\beta=1}^{3} \lambda_\beta^2 \right) \tilde{J}e_i + \sum_{i=1}^{2} \left( \nabla \tilde{J} e_i H, J_\alpha e_i \right) \]

\[ = -\lambda_\alpha \sum_{i=1}^{2} \sum_{\beta=1}^{3} \left\{ B(\tilde{J}e_i, \nabla \lambda_\beta), J_\beta e_i \right\} + \sum_{i=1}^{2} \left( \nabla \tilde{J} e_i H, J_\alpha e_i \right). \]  

(3.22)

From (3.12) and (3.22), we have

\[ \Delta \lambda_\alpha = -\lambda_\alpha \sum_{i=1}^{2} \sum_{\beta=1}^{3} \left( A(J_\beta e_i)^N(\tilde{J}e_i), \nabla \lambda_\beta \right) + \sum_{i=1}^{2} \left( \nabla \tilde{J} e_i H, J_\alpha e_i \right) \]

\[ = -\lambda_\alpha \sum_{\beta=1}^{3} |\nabla \lambda_\beta|^2 + \sum_{i=1}^{2} \left( \nabla \tilde{J} e_i H, J_\alpha e_i \right). \]

Namely,

\[ (\tau(J))^\alpha = \sum_{i=1}^{2} \left( \nabla \tilde{J} e_i H, J_\alpha e_i \right). \]  

(3.23)
By (3.23), the translator equation (1.2) and the Weingarten formula, we get

\[
\left(\tau(J)\right)^{\alpha} = \sum_{j=1}^{2} \left\langle \nabla_{\tilde{e}_j} H, J_{\alpha} e_j \right\rangle = -\sum_{j=1}^{2} \left\langle \nabla_{\tilde{e}_j} V^N_0, J_{\alpha} e_j \right\rangle
\]

\[
= -\sum_{j=1}^{2} \left\langle \nabla_{\tilde{e}_j} V^N_0, J_{\alpha} e_j \right\rangle - \sum_{j=1}^{2} \left\langle A^N_0 (\tilde{e}_j), J_{\alpha} e_j \right\rangle
\]

\[
= -\sum_{j=1}^{2} \left\langle \nabla_{\tilde{e}_j} V_0, J_{\alpha} e_j \right\rangle + \sum_{j=1}^{2} \left\langle \nabla_{\tilde{e}_j} V^T_0, J_{\alpha} e_j \right\rangle - \sum_{j=1}^{2} \left\langle B(\tilde{e}_j, (J_{\alpha} e_j)^T), V^N_0 \right\rangle
\]

\[
(3.24)
\]

Since the translating soliton surface \( \Sigma \) is hyper-Lagrangian, by Lemma 1, we have

\[
(J_{\alpha} e_j)^T = \lambda_\alpha \tilde{e}_j. \tag{3.25}
\]

Thus we obtain

\[
\sum_{j=1}^{2} \left\langle B(\tilde{e}_j, (J_{\alpha} e_j)^T), V^N_0 \right\rangle = \sum_{j=1}^{2} \left\langle B(\tilde{e}_j, \lambda_\alpha \tilde{e}_j), V^N_0 \right\rangle
\]

\[
= \lambda_\alpha \left\langle H, V^N_0 \right\rangle = \lambda_\alpha \left\langle H, V_0 \right\rangle. \tag{3.26}
\]

On the other hand, by the Gauss formula, we derive

\[
\sum_{j=1}^{2} \left\langle \nabla_{\tilde{e}_j} V^T_0, J_{\alpha} e_j \right\rangle = \sum_{j=1}^{2} \left\langle \nabla_{\tilde{e}_j} \left( \sum_{k=1}^{2} (V_0, e_k) e_k \right), J_{\alpha} e_j \right\rangle
\]

\[
= \sum_{j,k=1}^{2} \left\langle V_0, \nabla_{\tilde{e}_j} e_k \right\rangle e_k, J_{\alpha} e_j \right\rangle + \sum_{j,k=1}^{2} \left\langle V_0, e_k \right\rangle \left\langle \nabla_{\tilde{e}_j} e_k, J_{\alpha} e_j \right\rangle \tag{3.27}
\]

\[
= \sum_{j,k=1}^{2} \left\langle V_0, B(\tilde{e}_j, e_k) \right\rangle e_k, J_{\alpha} e_j \right\rangle + \sum_{j,k=1}^{2} \left\langle V_0, e_k \right\rangle B(\tilde{e}_j, e_k), J_{\alpha} e_j \right\rangle
\]

\[
= \sum_{j=1}^{2} \left\langle V_0, B(\tilde{e}_j, (J_{\alpha} e_j)^T) \right\rangle + \sum_{j=1}^{2} \left\langle B(\tilde{e}_j, V^T_0), J_{\alpha} e_j \right\rangle.
\]
Using the formula (3.25) again, we then get
\[
\sum_{j=1}^{2} \left( \nabla_{\tilde{J}e_j} V_0^T, J_{\alpha} e_j \right) = \sum_{j=1}^{2} \left( V_0, B(\tilde{J}e_j, \lambda_{\alpha} \tilde{J}e_j) \right) + \sum_{j=1}^{2} \left( B(\tilde{J}e_j, V_0^T), (J_{\alpha} e_j)^N \right)
\]
\[
= \lambda_{\alpha} (V_0, H) + \sum_{j=1}^{2} \left( A(J_{\alpha} e_j)^N (\tilde{J}e_j), V_0^T \right).
\]
(3.28)

Substituting (3.26) and (3.28) into (3.24), and using (3.12), we have
\[
(\tau(J))^\alpha = \sum_{j=1}^{2} \left( \nabla_{\tilde{J}e_j} H, J_{\alpha} e_j \right) = \sum_{j=1}^{2} \left( A(J_{\alpha} e_j)^N (\tilde{J}e_j), V_0^T \right)
\]
\[
= \left( \nabla \lambda_{\alpha}, V_0^T \right) = (dJ(V_0^T))^\alpha, \quad \alpha = 1, 2, 3.
\]
Namely
\[
\tau(J) = dJ \left( V_0^T \right).
\]
\[
\square
\]

4 A Rigidity Result

Let \( \Sigma^2 \) be a 2-dimensional translating soliton in \( \mathbb{R}^4 \). Let \( \kappa^\perp := \langle R(e_1, e_2) v_1, v_2 \rangle \) be the normal curvature, where \( \{e_1, e_2\} \) is a local orthonormal frame field on \( \Sigma \), \( \{v_1, v_2\} \) is a local orthonormal frame normal field along \( \Sigma \), and the orientation of \( \{e_1, e_2, v_1, v_2\} \) is coincided with the one of \( \{e_1, \tilde{J}e_1, \tilde{J}e_2, \tilde{J}e_3\} \). Here \( \tilde{J}_2, \tilde{J}_3 \) is the same as the ones in the proof of Lemma 1.

Let \( V := -V_0^T \) and \( \Delta V := \Delta + \langle V, \nabla \cdot \rangle \).

Using Theorem 1 and gradient estimates, we obtain a rigidity result of translating solitons.

**Theorem 2** Let \( X : \Sigma^2 \to \mathbb{R}^4 \) be a 2-dimensional complete translating soliton with nonpositive normal curvature. Assume that the image of the complex phase map is contained in a regular ball in \( S^2 \), i.e., a geodesic ball \( B_R(q) \) disjoint from the cut locus of \( q \) and \( R < \frac{\pi}{2} \), then \( \Sigma \) has to be an affine plane.

**Proof** Since we can view \( \Sigma \) as a hyper-Lagrangian submanifold in \( \mathbb{R}^4 \) with respect to some complex structure \( \tilde{J} \). Let \( \{e_1, e_2 = \tilde{J}e_1\} \) be a local orthonormal frame field on \( \Sigma \) such that \( \nabla e_i = 0 \) at the considered point. Denote \( v_1 = \tilde{J}_2 e_1, v_2 = \tilde{J}^\perp v_1 = \tilde{J}_1 \tilde{J}_2 e_1 = \tilde{J}_3 e_1 \), where \( \tilde{J}_\beta (\beta = 1, 2, 3) \) is the same as the ones in the proof of Lemma 1. Then \( \{v_1, v_2\} \) is a local orthonormal frame normal field along \( \Sigma \).

From the translating soliton equation (1.2), we derive
\[
\nabla e_j H = -\left( \nabla e_j (V_0 - \langle V_0, e_k \rangle e_k) \right)^N = (V_0, e_k) B_{jk}
\]
and

$$\nabla_{e_i} \nabla_{e_j} H = \langle V_0, e_k \rangle \nabla_{e_i} B_{jk} - \langle H, B_{ik} \rangle B_{jk},$$

where $B_{jk} = B(e_j, e_k)$. Hence using the Codazzi equation, we obtain

$$\Delta V |H|^2 = \Delta |H|^2 + \langle V, \nabla |H|^2 \rangle$$

$$= 2 \langle \nabla_{e_i} \nabla_{e_i} H, H \rangle + 2 |\nabla H|^2 + \langle V, \nabla |H|^2 \rangle$$

$$= 2 \langle H, B_{e_i e_k} \rangle^2 + 2 \langle \nabla V_0^T H, H \rangle + 2 |\nabla H|^2 + \langle V, \nabla |H|^2 \rangle$$

$$= 2 \langle H, B_{e_i e_k} \rangle^2 - \nabla V_0 |H|^2 + 2 |\nabla H|^2 + \langle V, \nabla |H|^2 \rangle$$

$$= 2 \langle H, B_{e_i e_k} \rangle^2 + 2 |\nabla H|^2.$$

It follows that

$$\Delta V |H|^2 \geq 2 |\nabla H|^2 - 2 |B|^2 |H|^2. \quad (4.1)$$

By (3.20), we get

$$B(e_j, \tilde{J} e_k) - \tilde{J} B(e_j, e_k) = \sum_{\beta=1}^{3} e_j (\lambda_{\beta}) J_{\beta} e_k.$$  

The above equality implies that

$$|dJ|^2 = \sum_{j=1}^{2} |dJ(e_j)|^2 = \frac{1}{2} \sum_{j,k=1}^{2} |dJ(e_j) e_k|^2 = \frac{1}{2} \sum_{j,k=1}^{2} |B(e_j, \tilde{J} e_k)|^2$$

$$- \tilde{J} B(e_j, e_k) \rangle^2 = \frac{1}{2} \sum_{j,k=1}^{2} |B(e_j, \tilde{J} e_k)|^2 + \frac{1}{2} \sum_{j,k=1}^{2} |\tilde{J} B(e_j, e_k) |^2$$

$$= \sum_{j,k=1}^{2} \langle B(e_j, \tilde{J} e_k), \tilde{J} B(e_j, e_k) \rangle$$

$$= |B|^2 - \sum_{j,k=1}^{2} \langle B(e_j, \tilde{J} e_k), \tilde{J} B(e_j, e_k) \rangle.$$  

(4.2)

Applying the Ricci equation, we obtain

$$\sum_{j,k=1}^{2} \langle B(e_j, \tilde{J} e_k), \tilde{J} B(e_j, e_k) \rangle = \sum_{j,k,s=1}^{2} \langle B(e_j, \tilde{J} e_k), v_s \rangle \langle \tilde{J} B(e_j, e_k), v_s \rangle$$

$$= - \sum_{j,k=1}^{2} \langle B(e_j, \tilde{J} e_k), v_1 \rangle \langle B(e_j, e_k), v_2 \rangle$$

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\[ + \sum_{j, k=1}^{2} \langle B(e_j, \tilde{\nu} e_k), v_2 \rangle \langle B(e_j, e_k), v_1 \rangle \]
\[ = \sum_{k=1}^{2} \langle R(e_k, \tilde{\nu} e_k)v_1, v_2 \rangle = 2 \langle R(e_1, e_2)v_1, v_2 \rangle = 2 \kappa^\perp. \tag{4.3} \]

Substituting (4.3) into (4.2), we have
\[ |dJ|^2 = |B|^2 - 2 \kappa^\perp. \tag{4.4} \]

Let \( \rho \) be the distance function on \( S^2 \), and \( h \) the Riemannian metric of \( S^2 \). Define \( \psi = 1 - \cos \rho \), then \( \text{Hess}(\psi) = (\cos \rho)h \).

For any \( X = (x_1, \ldots, x_4) \in \mathbb{R}^4 \), let \( r = |X| \), then we have
\[ \nabla r^2 = 2X^T, \quad |\nabla r| \leq 1, \]
\[ \Delta r^2 = 4 + 2\langle H, X \rangle \leq 4 + 2r. \tag{4.5} \]

Since \( J(\Sigma) \subset B_R(q) \subset S^2 \), note that \( R < \frac{\pi}{2} \), so we can choose a constant \( b \), such that \( \psi(R) < b < 1 \). Let \( D_a(o) \) be the closed ball centered at the origin \( o \) with radius \( a \) in \( \mathbb{R}^4 \) and \( D_a(o) := \Sigma \cap B_a(o) \). Define \( f : D_a(o) \to \mathbb{R} \) by
\[ f = \frac{(a^2 - r^2)^2|H|^2}{(b - \psi \circ J)^2}. \]

Since \( f|_{\partial D_a(o)} = 0 \), \( f \) achieves an absolute maximum in the interior of \( D_a(o) \), say \( f \leq f(q) \), for some \( q \) inside \( D_a(o) \). Using the technique of support function we may assume that \( f \) is smooth near \( q \). We may also assume \( |H|(q) \neq 0 \). Then, from
\[ \nabla f(q) = 0, \]
\[ \Delta f(q) \leq 0, \]
we obtain the following at the point \( q \):
\[ -\frac{2\nabla r^2}{a^2 - r^2} + \frac{\nabla|H|^2}{|H|^2} + \frac{2\nabla(\psi \circ J)}{b - \psi \circ J} = 0, \tag{4.6} \]
\[ -\frac{2\Delta r^2}{a^2 - r^2} - \frac{2|\nabla r|^2}{(a^2 - r^2)^2} + \frac{\Delta r^2}{|H|^2} - \frac{|\nabla|H|^2|^2}{|H|^4} \\
+ \frac{2\Delta (\psi \circ J)}{b - \psi \circ J} + \frac{2|\nabla(\psi \circ J)|^2}{(b - \psi \circ J)^2} \leq 0. \tag{4.7} \]

Direct computation gives us
\[ |\nabla|H|^2|^2 = |2\langle \nabla H, H \rangle|^2 \leq 4|\nabla H|^2|H|^2, \tag{4.8} \]
\[ |\nabla(\psi \circ J)|^2 \leq |d\psi|^2 |dJ|^2 \leq |dJ|^2. \]  
\tag{4.9}

It follows from (4.1) and (4.8) that
\[ \frac{\Delta V|H|^2}{|H|^2} \geq \frac{|\nabla|H|^2|^2}{2|H|^4} - 2|B|^2. \]  
\tag{4.10}

From (4.6), we obtain
\[ \frac{|\nabla|H|^2|^2}{|H|^4} \leq \frac{4|\nabla r|^2}{(a^2 - r^2)^2} + \frac{8|\nabla r|^2|\nabla(\psi \circ J)|}{(a^2 - r^2)(b - \psi \circ J)} + \frac{4|\nabla(\psi \circ J)|^2}{(b - \psi \circ J)^2}. \]  
\tag{4.11}

By Theorem 1, \( \tau_V(J) = 0 \). Thus we get
\[ \Delta V(\psi \circ J) = \sum_{j=1}^{2} \text{Hess}(\psi)(dJ(e_j), dJ(e_j)) + d\psi(\tau_V(J)) = \cos \rho |dJ|^2. \]  
\tag{4.12}

Substituting (4.4), (4.5), (4.9), (4.10), (4.11), (4.12) into (4.7), we have
\[ \left( \frac{\cos \rho}{b - \psi \circ J} - 1 \right) |dJ|^2 - \frac{4r}{(a^2 - r^2)(b - \psi \circ J)} |dJ| - \frac{4(1 + r)}{a^2 - r^2} \]
\[ - \frac{8r^2}{(a^2 - r^2)^2} - 2\kappa \leq 0. \]

By the assumption that \( \kappa \leq 0 \), we derive
\[ \left( \frac{\cos \rho}{b - \psi \circ J} - 1 \right) |dJ|^2 - \frac{4r}{(a^2 - r^2)(b - \psi \circ J)} |dJ|
\]
\[ - \frac{4(1 + r)}{a^2 - r^2} - \frac{8r^2}{(a^2 - r^2)^2} \leq 0. \]

Note an elementary fact that if \( ax^2 - bx - c \leq 0 \) with \( a, b, c \) all positive, then
\[ x \leq \max\{2b/a, 2\sqrt{c/a}\}. \]

It is easy to see that there is a constant \( C > 0 \) such that \( \frac{\cos \rho}{b - \psi \circ J} - 1 > C \). Therefore, at the point \( q \),
\[ |dJ|^2 \leq \max \left\{ \frac{64r^2}{C^2(a^2 - r^2)^2(b - \psi \circ J)^2}, \frac{16(1 + r)}{C(a^2 - r^2)} + \frac{32r^2}{C(a^2 - r^2)^2} \right\}. \]  
\tag{4.13}

By (4.4) and \( \kappa \leq 0 \), we get
\[ |dJ|^2 \geq |B|^2 \geq \frac{|H|^2}{2}. \]
Thus we obtain, at the point \( q \),

\[
|H|^2 \leq 2 \max \left\{ \frac{64r^2}{C^2(a^2 - r^2)^2(b - \psi \circ J)^2}, \frac{16(1+r)}{C(a^2 - r^2)}, \frac{32r^2}{C(a^2 - r^2)^2} \right\}
\]

(4.14)

and

\[
f(q) \leq 2 \max \left\{ \frac{64a^2}{C^2(b - \psi(\tilde{R}))^4}, \frac{16(1+a)a^2}{C(b - \psi(\tilde{R}))^2}, \frac{32a^2}{C(b - \psi(\tilde{R}))^2} \right\}.
\]

Then for any point \( x \in D_{a/2}(o) \), we have

\[
|H|^2(x) \leq \frac{(b - \psi \circ J)^2}{(a^2 - r^2)^2} f(q)
\]

\[
\leq \frac{32b^2}{9a^4} \max \left\{ \frac{64a^2}{C^2(b - \psi(\tilde{R}))^4}, \frac{16(1+a)a^2}{C(b - \psi(\tilde{R}))^2}, \frac{32a^2}{C(b - \psi(\tilde{R}))^2} \right\}.
\]

(4.15)

Hence we may fix \( x \) and let \( a \to \infty \) in (4.15), we then derive that \( H \equiv 0 \). Then by Proposition 3.2 in [12], \( B \equiv 0 \). Namely, \( \Sigma \) is an affine plane. \( \square \)

**Remark 2**

1. Let \( \alpha \) be the Kähler angle of the translator, Theorem 2 implies that if \( \cos \alpha \) has a positive lower bound, then any complete symplectic translating soliton with nonpositive normal curvature has to be an affine plane. Han–Sun [13] showed that such a rigidity result holds under an additional bounded second fundamental form assumption.

2. The restriction of the image under the complex phase map in Theorem 2 is necessary. For example, the “grim reaper” \((x, y, -\ln \cos x, 0), |x| < \pi/2, y \in \mathbb{R}\) is a translating soliton to the symplectic MCF which translates in the direction of the constant vector \((0, 0, 1, 0)\), and \(J = (\cos x, 0, -\sin x)\) can not contained in any regular ball of \(S^2\). One can check that \(|B|^2 = |H|^2 = |dJ|^2 = \cos^2 x\) and the normal curvature is zero.

Due to the fact that, in a hyperkähler 4-manifold, a surface being symplectic is equivalent to the condition that the image under the complex phase map is contained in an open hemisphere while a surface being Lagrangian is equivalent to the condition that the image under the complex phase map is contained in a great circle. Theorem 2 implies the following:

**Corollary 1** Let \( X : \Sigma^2 \to \mathbb{R}^4 \) be a complete Lagrangian translating soliton with nonpositive normal curvature. If the cosine of the Lagrangian angle has a positive lower bound, then \( \Sigma \) has to be an affine plane.
Remark 3 Using the Gauss equation, we get
\[ |B|^2 = |H|^2 - 2\kappa, \]
where \( \kappa \) is the sectional curvature of \( \Sigma \). Then by the above equality and (4.4), we can conclude that
\[ |dJ|^2 = |H|^2 - 2(\kappa + \kappa^\perp). \]
For Lagrangian surfaces, the complex phase map \( J : \Sigma \to S^2 \) can be represented by \((\cos \theta, \sin \theta, 0)\), thus direct computation gives
\[
|dJ|^2 = \sum_j |dJ(e_j)|^2 = \sum_j |e_j(\cos \theta)J_1 + e_j(\sin \theta)J_2|^2
= \sum_j (-\sin \theta e_j(\theta))^2 + (\cos \theta e_j(\theta))^2 = |\nabla \theta|^2
= |\tilde{J}\nabla \theta|^2 = |H|^2.
\]
The above two equalities imply that for Lagrangian surfaces, we have \( \kappa + \kappa^\perp = 0 \). This shows that for Lagrangian surfaces, \( \kappa^\perp \leq 0 \) if and only if \( \kappa \geq 0 \). Therefore, Corollary 1 is equivalent to the Main Theorem 2 in [13].

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