Isotropic and Dominating Mixed Besov Spaces – a Comparison

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This paper is dedicated to the memory of Björn Jawerth.

Abstract. We compare Besov spaces with isotropic smoothness with Besov spaces of dominating mixed smoothness. Necessary and sufficient conditions for continuous embeddings will be given.

1. Introduction

For $t \in \mathbb{N}_0$ the isotropic Sobolev space $W^t_2(\mathbb{R}^d)$, built on $L^2(\mathbb{R}^d)$, is the collection of all functions $f \in L^2(\mathbb{R}^d)$ such that

$$\|f\|_{W^t_2(\mathbb{R}^d)} := \sum_{|\bar{\alpha}| \leq t} \|D^{\bar{\alpha}} f\|_{L^2(\mathbb{R}^d)} < \infty.$$  

The Sobolev space of dominating mixed smoothness $S^t_2W(\mathbb{R}^d)$ is the tensor product of the univariate Sobolev spaces $W^t_2(\mathbb{R})$, with other words

$$S^t_2W(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{S^t_2W(\mathbb{R}^d)} := \sum_{\|\bar{\alpha}\|_{\infty} \leq t} \|D^{\bar{\alpha}} f\|_{L^2(\mathbb{R}^d)} < \infty \right\}.$$  

Here $\bar{\alpha} = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$, $|\bar{\alpha}| = \alpha_1 + ... + \alpha_d$ and $\|\bar{\alpha}\|_{\infty} = \max_{i=1,...,d} |\alpha_i|$. Observe that the mixed derivative $D^{(t,...,t)} f$ has the highest order in this norm which is the reason for the name of these spaces. Spaces of dominating mixed smoothness have found applications in approximation theory since the early sixties, more recent in high dimensional approximation and information based complexity, see, e.g., [32] and [19,20,21]. Obviously we have the chain of continuous embeddings

$$W^{td}_2(\mathbb{R}^d) \hookrightarrow S^t_2W(\mathbb{R}^d) \hookrightarrow W^t_2(\mathbb{R}^d).$$  

Also easy to see is the optimality of these embeddings in various directions. We will discuss this below. These two types of Sobolev spaces $W^t_2(\mathbb{R}^d)$ and $S^t_2W(\mathbb{R}^d)$ represent particular cases of corresponding scales of Besov spaces, denoted by $B^{t}_{p,q}(\mathbb{R}^d)$.
(isotropic smoothness) and $S^t_{p,q} B (\mathbb{R}^d)$ (dominating mixed smoothness). Indeed, we have

$$W^t_2 (\mathbb{R}^d) = B^t_{2,2} (\mathbb{R}^d) \quad \text{and} \quad S^t_2 W(\mathbb{R}^d) = S^t_{2,2} B (\mathbb{R}^d)$$

in the sense of equivalent norms. In this paper we address the question under which conditions on $t,p,q$ the embedding

$$B^t_{p,q} (\mathbb{R}^d) \hookrightarrow S^t_{p,q} B (\mathbb{R}^d) \hookrightarrow B^t_{p,q} (\mathbb{R}^d) \quad (1.1)$$

holds true. In addition we shall discuss the optimality of these embeddings in various directions. Let us mention here that Schmeisser \[23] and Hansen \[7] have considered those embeddings as well. Below we will make a more detailed comparison.

Nowadays isotropic Besov spaces represent a well accepted regularity notion in various fields of mathematics. Besov spaces of dominating mixed smoothness are of increasing importance in approximation theory and information based complexity, we refer to \[32]. As a special case, the scale $S^t_{p,p} B (\mathbb{R}^d)$ contain the tensor products of the univariate Besov spaces $B^t_{p,p} (\mathbb{R})$, see \[27, 28]. It is the main aim of this paper to give a detailed comparison of these different extensions of univariate Besov spaces into the multi-dimensional situation.

Since a few years there is some strong motivation to study those spaces also for parameters $p,q < 1$. Let $\Phi := (\psi_j)$ denote a wavelet basis satisfying some additional smoothness, integrability, and moment conditions. We consider best $m$-term approximation with respect to $\Phi$, i.e., we investigate the quantity

$$\sigma_m (f, \Phi)_X := \inf \left\{ \| f - \sum_{j \in \Lambda} c_j \psi_j \| : |\Lambda| \leq m, \ c_j \in \mathbb{C}, \ j \in \Lambda \right\}, \ m \in \mathbb{N}_0.$$

Associated widths are defined as follows. Let $X$ and $Y$ be quasi-Banach spaces such that $Y \hookrightarrow X$. Then we define

$$\sigma_m (Y, X, \Phi) := \sup \left\{ \sigma_m (f, \Phi)_X : \| f \|_Y \leq 1 \right\}, \ m \in \mathbb{N}_0.$$

Usually one concentrates on $X = L_p (\mathbb{R}^d)$. Here we would like to recall the breakthrough result of DeVore, Jawerth and Popov \[5]. Let $0 < \tau < p$. A function $f$ belongs to the Besov space $B^d_{\tau, \tau} (\mathbb{R}^d)$ if, and only if it satisfies

$$\left( \sum_{m=1}^{\infty} m^{-1} \left[ m^{\tau - \frac{1}{p}} \sigma_m (f, \Phi)_{L_p (\mathbb{R}^d)} \right]^\tau \right)^{1/\tau} < \infty.$$

Hence, Besov spaces (in particular with $\tau < 1$) describe approximation spaces with respect to nonlinear approximation where the original question (behaviour of $\sigma_m (f, \Phi)_{L_p (\mathbb{R}^d)}$) has been asked for $L_p$-spaces with $p \geq 1$. For further results in this directions supporting the importance of spaces with $p,q < 1$, we refer to Jawerth and Milman \[11, 12]. Similar descriptions of $S^d_{\tau, \tau} B (\mathbb{R}^d)$ exist as well, see \[8, 9]. For us this motivates the investigation of our problem also for $p,q < 1$.

The paper is organized as follows. In Section 2 we recall the definition of the spaces $B^t_{p,q} (\mathbb{R}^d)$ and $S^t_{p,q} B (\mathbb{R}^d)$. Our main results are stated in Section 3. Proofs are concentrated in Section 4.
Notation. As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers, $\mathbb{C}$ refers to the complex numbers. For a real number $a$ we put $a_+ := \max(a, 0)$. If $k \in \mathbb{N}_0$, i.e., if $k = (k_1, \ldots, k_d)$, $k_\ell \in \mathbb{N}_0$, $\ell = 1, \ldots, d$, then we put $|k| := k_1 + \ldots + k_d$.

For $x \in \mathbb{R}^d$ we use $\|x\|_\infty := \max_{1 \leq \ell \leq d} |x_\ell|$. The symbols $c, c_1, c_2, \ldots, C, C_1, C_2, \ldots$ denote positive constants which are independent of the main parameters involved but whose values may differ from line to line. The symbol $A \asymp B$ means that there exist positive constants $C_1$ and $C_2$ such that $C_1 A \leq B \leq C_2 A$.

Let $X$ and $Y$ be two quasi-Banach spaces. Then the symbol $X \hookrightarrow Y$ indicates that the embedding is continuous. By $C^\infty_c(\mathbb{R}^d)$ the set of compactly supported infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{C}$ is denoted. Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. The topological dual, the class of tempered distributions, is denoted by $S'(\mathbb{R}^d)$ (equipped with the weak topology). The Fourier transform on $S(\mathbb{R}^d)$ is given by

$$\mathcal{F} \varphi(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\xi} \varphi(x) \, dx, \quad \xi \in \mathbb{R}^d.$$ 

The inverse transformation is denoted by $\mathcal{F}^{-1}$. We use both notations also for the transformations defined on $S'(\mathbb{R}^d)$.

Let $0 < p, q \leq \infty$. For an arbitrary countable index set $I$ we put

$$\| (f_k)_{k \in I} \|_{L^q_{\ell^p}(\mathbb{R}^d)} := \left( \sum_{k \in I} \left( \int_{\mathbb{R}^d} |f_k(x)|^p \, dx \right)^{q/p} \right)^{1/q}$$

(usual modification if $\max(p, q) = \infty$).

2. Besov spaces of isotropic and dominating mixed smoothness

2.1. Isotropic Besov spaces. Usually these spaces are defined by using the modulus of smoothness. Here we prefer to use the Fourier analytic descriptions since we are dealing also with negative smoothness.

Let $\phi_0 \in C^\infty_c(\mathbb{R}^d)$ be a non-negative function such that $\phi(x) = 1$ if $|x| \leq 1$ and $\phi(x) = 0$ if $|x| \geq \frac{3}{2}$. For $j \in \mathbb{N}$ we define

$$\phi_j(x) := \phi_0(2^{-j}x) - \phi_0(2^{-j+1}x), \quad x \in \mathbb{R}^d.$$ 

This yields

$$\sum_{j=0}^{\infty} \phi_j(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^d.$$ 

We shall call $(\phi_j)_{j=0}^{\infty}$ a smooth dyadic decomposition of unity.

**Definition 2.1.** Let $t \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then $B^t_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$\| f \|_{B^t_{p,q}(\mathbb{R}^d)}^\varphi = \left( \sum_{j=0}^{\infty} 2^{jtq} \| \mathcal{F}^{-1} [\phi_j \mathcal{F}(\cdot)]|L_p(\mathbb{R}^d)\|_q \right)^{1/q}$$

is finite (modification if $q = \infty$).
Remark 2.2. Besov spaces are discussed in various monographs, let us refer to [3, 18, 22] and [35]. They are quasi-Banach spaces (Banach spaces if $\min(p, q) \geq 1$) and they do not depend on the chosen generator $\varphi_0$ of the smooth dyadic decomposition (in the sense of equivalent quasi-norms). We call them isotropic because they are invariant under rotations. Characterizations in terms of differences can be found at various places, see, e.g., [35 2.5] or [36 3.5].

For us it will be convenient to switch to an equivalent quasi-norm. Let $\psi_0 \in C_0^\infty(\mathbb{R}^d)$ such that

$$\psi_0(x) = 1 \text{ if } \sup_{i=1, \ldots, d} |x_i| \leq 1 \text{ and } \psi_0(x) = 0 \text{ if } \sup_{i=1, \ldots, d} |x_i| \geq \frac{3}{2}.$$  

For $j \in \mathbb{N}$, we define

$$\psi_j(x) := \psi_0(2^{-j} x) - \psi_0(2^{-j+1} x), \quad x \in \mathbb{R}^d.$$  

Then we have

$$\supp \psi_j \subset \{ x : \sup_{i=1, \ldots, d} |x_i| \leq 3.2^{j-1} \} \setminus \{ x : \sup_{i=1, \ldots, d} |x_i| \leq 2^{j-1} \}, \quad j \in \mathbb{N}.$$  

As an easy consequence of [35 Proposition 2.3.2] one obtains the following.

Proposition 2.3. Let $t \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then $B_{p,q}^t(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$\|f|B_{p,q}^t(\mathbb{R}^d)||^\psi = \left( \sum_{j=0}^\infty 2^{jtq} \|\mathcal{F}^{-1}[\psi_j \mathcal{F} f](\cdot)|L_p(\mathbb{R}^d)\|^q \right)^{1/q}$$

is finite (modification if $q = \infty$). The quasi-norms $\|f|B_{p,q}^t(\mathbb{R}^d)||^\psi$ and $\|f|B_{p,q}^t(\mathbb{R}^d)||^\phi$ are equivalent.

In what follows we will work with the $\psi$-norm. Therefore we shall write $\|f|B_{p,q}^t(\mathbb{R}^d)||$ instead of $\|f|B_{p,q}^t(\mathbb{R}^d)||^\psi$.

2.2. Besov spaces of dominating mixed smoothness. Let $\varphi_0 \in C_0^\infty(\mathbb{R})$ satisfy $\varphi_0(\xi) = 1$ on $[-1, 1]$ and $\text{supp} \varphi \subset [-\frac{3}{2}, \frac{3}{2}]$. For $j \in \mathbb{N}$ we define

$$\varphi_j(x) = \varphi_0(2^{-j} x) - \varphi_0(2^{-j+1} x), \quad x \in \mathbb{R}.$$  

Hence, $(\varphi_j)_{j=0}^\infty$ forms a smooth dyadic decomposition of unity on $\mathbb{R}$. Now we switch to tensor products. For $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ the function $\varphi_k(x) \in C_0^\infty(\mathbb{R}^d)$ is defined by

$$\varphi_k(x) := \varphi_{k_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}(x_d), \quad x \in \mathbb{R}^d.$$  

Definition 2.4. Let $t \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then $S_{p,q}^t B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$\|f|S_{p,q}^t B(\mathbb{R}^d)||^\varphi = \left( \sum_{k \in \mathbb{N}_0^d} 2^{k_1tq} \|\mathcal{F}^{-1}[\varphi_k \mathcal{F} f](\cdot)|L_p(\mathbb{R}^d)\|^q \right)^{1/q}$$

is finite (modification if $q = \infty$).
Remark 2.5. (i) Besov spaces of dominating mixed smoothness are discussed in the monographs Amanov [1] and Schmeisser, Triebel [25], see also the booklet Vybiral [38]. They are quasi-Banach spaces (Banach spaces if min\(p, q\) ≥ 1) and they do not depend on the chosen generator \(\phi_0\) of the smooth dyadic decomposition (in the sense of equivalent quasi-norms). For characterizations in terms of differences we refer to [25, 2.3.4] and [37].
(ii) For us of certain importance will be the following observation. Besov spaces of dominating mixed smoothness have a cross-quasi-norm, i.e., if \(f_j \in B_{t,p,q}^d(\mathbb{R})\), \(j = 1, \ldots, d\) then
\[
f(x) = \prod_{j=1}^d f_j(x_j) \in S_{t,p,q}^d(\mathbb{R}^d) \quad \text{and} \quad \|f\|_{S_{t,p,q}^d(\mathbb{R}^d)} = \prod_{j=1}^d \|f_j\|_{B_{t,p,q}^d(\mathbb{R})}.
\]
(iii) For \(d = 1\) we have \(S_{t,p,q}^d(\mathbb{R}) = B_{t,p,q}^d(\mathbb{R})\).

3. The main results

We discuss these embeddings in (1.1) separately.

3.1. The embedding of dominating mixed spaces into isotropic spaces.

Theorem 3.1. Let \(0 < p, q \leq \infty\) and \(t \in \mathbb{R}\). Then we have
\[
S_{t,p,q}^d(\mathbb{R}^d) \hookrightarrow B_{t,p,q}^d(\mathbb{R}^d)
\]
if and only if one of the following conditions is satisfied
- \(t > 0\);
- \(t = 0\), \(0 < p < \infty\) and \(0 < q \leq \min(p, 2)\);
- \(t = 0\), \(p = \infty\) and \(q \leq 1\).

Remark 3.2. Sufficient conditions for embeddings as in (3.1) have been considered by Schmeisser [23] and Hansen [7]. Both used different methods than we do. Schmeisser used characterizations by differences and concentrated on the Banach space case. Hansen showed \(S_{t+\varepsilon,p,q}^d(\mathbb{R}^d) \hookrightarrow B_{t,p,q}^d(\mathbb{R}^d)\) with \(\varepsilon > 0\) and \(q_0, q\) arbitrary by applying wavelet characterizations.

We summarize what is known about the relation of \(B_{t,p,q}^d(\mathbb{R}^d)\) and \(S_{t,p,q}^d(\mathbb{R}^d)\) in the remaining cases. For subsets \(X, Y\) of \(S'(\mathbb{R}^d)\) we shall call not comparable if
\[
X \setminus Y \neq \emptyset \quad \text{and} \quad Y \setminus X \neq \emptyset.
\]

Proposition 3.3. (i) Let \(1 \leq p, q \leq \infty\), \(0 < q \leq \infty\) and \(t < 0\). Then we have
\[
B_{t,p,q}^d(\mathbb{R}^d) \hookrightarrow S_{t,p,q}^d(\mathbb{R}^d).
\]
(ii) Let \(t = 0\), \(p \neq 2\), \(1 < p < \infty\) and \(\min(2, p) < q < \max(2, p)\). Then \(B_{0,p,q}^d(\mathbb{R}^d)\) and \(S_{0,p,q}^d(\mathbb{R}^d)\) are not comparable.
(iii) Let \(t = 0\), \(p = 1\) and \(1 < q < \infty\). Then \(B_{t,1,q}^1(\mathbb{R}^d)\) and \(S_{t,1,q}^1(\mathbb{R}^d)\) are not comparable.
(iv) Let \(t = 0\), \(p = \infty\) and \(1 < q < \infty\). Then \(B_{\infty,q}^0(\mathbb{R}^d)\) and \(S_{\infty,q}^0(\mathbb{R}^d)\) are not comparable.
(v) Let \(t < 0\), \(0 < p < 1\) and \(0 < q \leq \infty\). Then \(B_{t,p,q}^d(\mathbb{R}^d)\) and \(S_{t,p,q}^d(\mathbb{R}^d)\) are not comparable.
Theorem 3.4. Let $0 < p_0, p, q_0, q \leq \infty$ and $t_0, t \in \mathbb{R}$. Let $p, q$ and $t$ be fixed. Within all spaces $S_{t_0}^{t_0} B(\mathbb{R}^d)$ satisfying

$$S_{t_0}^{t_0} B(\mathbb{R}^d) \hookrightarrow B_t^{t_0} B(\mathbb{R}^d)$$

the class $S_t^{t_0} B(\mathbb{R}^d)$ is the largest one.

Remark 3.5. Comparing Theorems 3.1 and 3.4 it is natural to ask also for the optimality of $S_t^{t_0} B(\mathbb{R}^d)$ in the other direction, i.e., we fix $S_t^{t_0} B(\mathbb{R}^d)$ and look for spaces $B_{p_0,q_0}^t (\mathbb{R}^d)$ such that (3.1) is true. For this we consider a special situation. Theorem 3.1 yields $S_2^{2} B(\mathbb{R}^d) \hookrightarrow B_2^{2} B(\mathbb{R}^d)$. On the other hand, a Sobolev-type embedding and Theorem 3.1 imply $S_2^{2} B(\mathbb{R}^d) \hookrightarrow S_{3/2}^{3/2} B(\mathbb{R}^d) \hookrightarrow B_{2,2}^{3/2} (\mathbb{R}^d)$, see the comments at the beginning of Subsection 4.10. But for $d \geq 2$ these isotropic Besov spaces $B_{1,2}^{2} (\mathbb{R}^d)$ and $B_{2,2}^{3/2} (\mathbb{R}^d)$ are not comparable. Hence, an optimality in such a wide sense is not true.

3.2. The embedding of isotropic spaces into dominating mixed spaces.

Theorem 3.6. Let $0 < p, q \leq \infty$ and $t \in \mathbb{R}$. Then we have

$$B_{p,q}^{td} (\mathbb{R}^d) \hookrightarrow S_t^{t_0} B(\mathbb{R}^d)$$

if and only if one of the following conditions is satisfied

- $t > \max(0, \frac{1}{p} - 1)$;
- $t = 0$, $1 < p \leq \infty$ and $\max(2, p) \leq q \leq \infty$;
- $0 < p \leq 1$, $t = \frac{1}{p} - 1$ and $q = \infty$.

Remark 3.7. Again we have to refer to Hansen [7] for an earlier result in this direction. He proved $S_{p,q}^{t_0} B(\mathbb{R}^d) \hookrightarrow S_t^{t_0} B(\mathbb{R}^d)$ with $\varepsilon > 0$ and $q_0, q$ arbitrary.

Also in this situation we summarize what is known about the relation of $B_{p,q}^{td} (\mathbb{R}^d)$ and $S_t^{t_0} B(\mathbb{R}^d)$ in the remaining cases.

Proposition 3.8. (i) Let $0 < p, q \leq \infty$ and $t < 0$. Then we have

$$S_t^{t} B(\mathbb{R}^d) \hookrightarrow B_{p,q}^{td} (\mathbb{R}^d).$$
(ii) Let $0 < p < 1$, $0 < t < \frac{1}{p} - 1$ and $0 < q \leq \infty$. Then $B_{p,q}^t(R^d)$ and $S_{p,q}^t B(R^d)$ are not comparable.

(iii) Let $0 < p < 1$, $t = 0$ and $0 < q \leq p$. Then $S_{p,q}^0 B(R^d) \hookrightarrow B_{p,q}^0(R^d)$ follows.

(iv) Let $0 < p < 1$, $t = 0$ and $p < q \leq \infty$. Then $B_{p,q}^0(R^d)$ and $S_{p,q}^0 B(R^d)$ are not comparable.

**Remark 3.9.** Obviously the case $t = 0$ is covered by Proposition 3.3. The set \{(p, 0) : 1 \leq p \leq \infty\} is part of the critical line of (3.2). Also for us it was surprising that the critical line for $0 < p < 1$ is given by $\frac{1}{p} - 1$.

These embeddings are optimal in the following sense.

**Theorem 3.10.** Let $0 < p_0, p, q_0, q \leq \infty$ and $t_0, t \in \mathbb{R}$. Let $p, q$ and $t$ be fixed. Within all spaces $B_{p_0,q_0}^{t_0}(R^d)$ satisfying

\[ B_{p_0,q_0}^{t_0}(R^d) \hookrightarrow S_{p,q}^t B(R^d) \]

the class $B_{p,q}^{t d}(R^d)$ is the largest one.

**Theorem 3.11.** Let $0 < p_0, p, q_0, q \leq \infty$ and $t_0, t \in \mathbb{R}$. Let $p, q$ and $t$ be fixed. Within all spaces $S_{p_0,q_0}^{t_0} B(R^d)$ satisfying

\[ B_{p,q}^{t d}(R^d) \hookrightarrow S_{p_0,q_0}^{t_0} B(R^d) \]

the class $S_{p,q}^t B(R^d)$ is the smallest one.

**Remark 3.12.** Let us come back to the chain of embeddings

\[ W_2^{t d}(R^d) = B_{2,2}^{t d}(R^d) \hookrightarrow S_2^{t d} B(R^d) = S_2^{t d} B(R^d) \hookrightarrow W_2^{t d}(R^d) = B_{2,2}^{t d}(R^d) \]

discussed in the Introduction. Employing Theorems 3.4, 3.10 and 3.11 we obtain the following optimality assertions.

- Within all spaces $S_{p_0,q_0}^{t_0} B(R^d)$ satisfying $S_{p_0,q_0}^{t_0} B(R^d) \hookrightarrow W_2^{t d}(R^d)$ the class $S_{p,q}^t B(R^d)$ is the largest one.
- Within all spaces $B_{p_0,q_0}^{t_0} (R^d)$ satisfying $B_{p_0,q_0}^{t_0} (R^d) \hookrightarrow S_2^{t d} B(R^d)$ the class $B_{2,2}^{t d}(R^d) = W_2^{t d}(R^d)$ is the largest one.
- Within all spaces $S_{p_0,q_0}^{t_0} B(R^d)$ satisfying $W_2^{t d}(R^d) \hookrightarrow S_{p_0,q_0}^{t_0} B(R^d)$ the class $S_2^{t d} B(R^d)$ is the smallest one.

**Figure 2.**

These embeddings are optimal in the following sense.
4. Proofs

To prove our main results we will apply essentially four different tools: Fourier multipliers; complex interpolation; assertions on dual spaces; some test functions. In what follows we collect what is needed.

4.1. Fourier multipliers. Let us recall some Fourier multiplier assertions. For a compact subset \( \Omega \subset \mathbb{R}^d \) we introduce the notation
\[
L^\Omega_p(\mathbb{R}^d) = \{ f \in S'(\mathbb{R}^d) : \text{ supp } f \subset \Omega, \ f \in L_p(\mathbb{R}^d) \}.
\]
For those spaces improved convolution inequalities hold, see [35 Proposition 1.5.1].

**Lemma 4.1.** Let \( \Omega \) and \( \Gamma \) be compact subsets of \( \mathbb{R}^d \). Let \( 0 < p \leq \infty \) and put \( u := \min(p, 1) \). Then there exists a positive constant \( c \) such that
\[
\| \mathcal{F}^{-1} M \mathcal{F} f | L_p(\mathbb{R}^d) \| \leq c \| \mathcal{F}^{-1} M | L_u(\mathbb{R}^d) \| \cdot \| f | L_p(\mathbb{R}^d) \|
\]
holds for all \( f \in L^\Omega_p(\mathbb{R}^d) \) and all \( \mathcal{F}^{-1} M \in L^u_\Gamma(\mathbb{R}^d) \).

Lemma 4.1 and the homogeneity properties of the Fourier transform yield the following.

**Lemma 4.2.** Let \( (\psi_j)_{j=0}^\infty \) and \( (\varphi_k)_{k \in \mathbb{N}_0^d} \) be the two decompositions of unity defined in (2.1) and (2.2), respectively. Let \( u := \min(1, p) \). Then there exists a positive constant \( C \) such that
\[
\| \mathcal{F}^{-1}(\psi_j \varphi_k \mathcal{F} f) | L_p(\mathbb{R}^d) \| \leq C \| \mathcal{F}^{-1} \varphi_k \mathcal{F} f | L_p(\mathbb{R}^d) \|
\]
and
\[
\| \mathcal{F}^{-1}(\varphi_k \psi_j \mathcal{F} f) | L_p(\mathbb{R}^d) \| \leq C \| \mathcal{F}^{-1} \psi_j \mathcal{F} f | L_p(\mathbb{R}^d) \|
\]
hold for all \( j \in \mathbb{N}_0 \), all \( k \in \mathbb{N}_0^d \) and all \( f \in S'(\mathbb{R}^d) \) with finite right-hand sides.

**Proof.** For \( \bar{k} \in \mathbb{N}_0^d \) and \( j \in \mathbb{N}_0 \) we put
\[
\Omega_{\bar{k}} = \{ x \in \mathbb{R}^d : |x_i| \leq 2^{k_i+1}, \ i = 1, \ldots, d \},
\]
\[
\Gamma_j = \{ x \in \mathbb{R}^d : \sup_{i=1, \ldots, d} |x_i| \leq 2^{j+1} \}.
\]

**Step 1.** Proof of (4.1). For \( f \in S'(\mathbb{R}^d) \) we put \( g := \mathcal{F}^{-1} \varphi_k \mathcal{F} f \). Then we have \( g \in L^{\Omega_{\bar{k}}}_p(\mathbb{R}^d) \) and \( g(2^{-j} \cdot) \in L^\Gamma_0(\mathbb{R}^d) \). Observe that
\[
\| \mathcal{F}^{-1} \psi_j \mathcal{F} g | L_p(\mathbb{R}^d) \| = 2^{-\frac{jd}{p}} \| (\mathcal{F}^{-1} \psi_j \mathcal{F} g)(2^{-j} \cdot) | L_p(\mathbb{R}^d) \|
\]
\[
= 2^{-\frac{jd}{p}} \| \mathcal{F}^{-1} (\psi_j (2^j \cdot) \mathcal{F} g(2^{-j} \cdot))) | L_p(\mathbb{R}^d) \|
\]
Let \( j \in \mathbb{N} \). Lemma 4.1 together with supp \( \psi_j (2^j \cdot) \subset \Gamma_0 \) yield
\[
\| \mathcal{F}^{-1} \psi_j \mathcal{F} g | L_p(\mathbb{R}^d) \| \leq c 2^{-\frac{jd}{p}} \| \mathcal{F}^{-1} (\psi_1 (2^{-j} \cdot)) | L_u(\mathbb{R}^d) \| \cdot \| g(2^{-j} \cdot) | L_p(\mathbb{R}^d) \|
\]
\[
\leq C \| \mathcal{F}^{-1} \psi_0 | L_u(\mathbb{R}^d) \| \cdot \| g | L_p(\mathbb{R}^d) \|
\]
A similar argument yields the estimate of \( \mathcal{F}^{-1} \psi_0 \mathcal{F} g \). This proves (4.1).

**Step 2.** To prove (4.2), we put \( h := \mathcal{F}^{-1} \psi_j \mathcal{F} f \). Then we have \( h \in L^{\Omega_{\bar{k}}}_p(\mathbb{R}^d) \), hence
\( h(2^{-j} \cdot) \in L^1_0(\mathbb{R}^d) \). In addition we know \( \text{supp} \varphi_{k}(2^j \cdot) \subset \Gamma_0 \). Let \( k = (k_1, \ldots, k_d) \) such that \( k_1, \ldots, k_d \neq 0 \). Using Lemma 4.1 we obtain

\[
\|\mathcal{F}^{-1} \varphi_k \mathcal{F} h\|_{L^p(\mathbb{R}^d)} = 2^{-\frac{j}{d}} \|\mathcal{F}^{-1} \varphi_k \mathcal{F} h(2^{-j} \cdot)\|_{L^p(\mathbb{R}^d)}
\]

(4.3)

\[
= 2^{-\frac{j}{d}} \|\mathcal{F}^{-1} [\varphi_k(2^j \cdot) \mathcal{F} h(2^{-j} \cdot)]\|_{L^p(\mathbb{R}^d)}
\]

\[
\leq c \|\mathcal{F}^{-1} [\varphi_k(2^j \cdot)]\|_{L^p(\mathbb{R}^d)} \| h(2^{-j} \cdot)\|_{L^p(\mathbb{R}^d)}
\]

\[
\leq c \|\mathcal{F}^{-1} [\varphi_k(2^j \cdot)]\|_{L^p(\mathbb{R}^d)} \| |h\|_{L^p(\mathbb{R}^d)}
\].

We put \( j := (j_1, \ldots, j_d) \). The homogeneity properties of the Fourier transform lead to

\[
\|\mathcal{F}^{-1} [\varphi_k(2^j \cdot)]\|_{L^p(\mathbb{R}^d)} = \|\mathcal{F}^{-1} [\varphi_{\frac{1}{2}^{-k+j+1}}(2^{-\frac{k}{2}+j+1} \cdot)]\|_{L^p(\mathbb{R}^d)}
\]

\[
\leq c j \|2^{j(\frac{d}{2}+(k-1))}\|_{L^p(\mathbb{R}^d)} \|\mathcal{F}^{-1} [\varphi_{\frac{1}{2}^{-1}}]\|_{L^p(\mathbb{R}^d)}
\].

Inserting this into (4.3) we get (4.2) for those \( k \). An obvious modification yields the estimate for the remaining \( k \). The proof is complete. \( \square \)

### 4.2. Complex interpolation.

For the basics of the classical complex interpolation method of Calderón we refer to the original paper [4] and the monographs [2, 3, 16, 34]. In the meanwhile it is well-known that the complex interpolation method extends to specific quasi-Banach spaces, namely those, which are analytically convex, see [13]. Note that any Banach space is analytically convex. The following Proposition, well-known in case of Banach spaces, see [3, Theorem 4.1.2], [16, Theorem 2.1.6] or [34, Theorem 1.10.3.1], can also be extended to the quasi-Banach case, see [13].

**Proposition 4.3.** Let \( 0 < \Theta < 1 \). Let \((X_1, Y_1)\) and \((X_2, Y_2)\) be two compatible couples of quasi-Banach spaces. In addition, let \( X_1 + X_2, Y_1 + Y_2 \) be analytically convex. If \( T \) is in \( \mathcal{L}(X_1, X_2) \) and in \( \mathcal{L}(Y_1, Y_2) \), then the restriction of \( T \) to \([X_1, Y_1]_\Theta\) is in \( \mathcal{L}([X_1, Y_1]_\Theta, [X_2, Y_2]_\Theta) \) for every \( \Theta \). Moreover,

\[
\|T : [X_1, Y_1]_\Theta \to [X_2, Y_2]_\Theta\| \leq \|T : X_1 \to X_2\|^{1-\Theta} \|T : Y_1 \to Y_2\|^\Theta.
\]

It is not difficult to see that all spaces \( B^t_{p,q}(\mathbb{R}^d) \) are analytically convex, see [17] or [13]. By means of Theorem 7.8 in [13] and the wavelet characterization of \( S^t_{p,q}B(\mathbb{R}^d) \), see [38], one can derive that also the spaces \( S^t_{p,q}B(\mathbb{R}^d) \) are analytically convex.

**Proposition 4.4.** Let \( t_i \in \mathbb{R}, 0 < p_i, q_i \leq \infty, i = 1, 2, \) and

\[
\min \left( \max(p_1, q_1), \max(p_2, q_2) \right) < \infty.
\]

If \( t_0, p_0 \) and \( q_0 \) are given by

\[
1 \frac{1}{p_0} = \frac{1}{p_1} + \frac{\Theta}{p_2}, \quad 1 \frac{1}{q_0} = \frac{1}{q_1} + \frac{\Theta}{q_2}, \quad t_0 = (1-\Theta) t_1 + \Theta t_2.
\]

Then

\[
B^{t_0}_{p_0,q_0}(\mathbb{R}^d) = [B^{t_1}_{p_1,q_1}(\mathbb{R}^d), B^{t_2}_{p_2,q_2}(\mathbb{R}^d)]_\Theta
\]

and

\[
S^{t_0}_{p_0,q_0}B(\mathbb{R}^d) = [S^{t_1}_{p_1,q_1}B(\mathbb{R}^d), S^{t_2}_{p_2,q_2}B(\mathbb{R}^d)]_\Theta.
\]
Remark 4.5. Complex interpolation of isotropic Besov spaces has been studied at various places, we refer to [3, Theorem 6.4.5] and [34, 2.4.1] as well as to the references given there. The extension to the quasi-Banach case has been done by Mendez and Mitrea [17], see also [13]. Vybiral [38, Theorem 4.6] has proved a corresponding result for sequence spaces associated to Besov spaces of dominating mixed smoothness. However, these results can be shifted to the level of function spaces by suitable wavelet isomorphisms, see [38, Theorem 2.12]. Here we would like to mention that all these extensions of the complex method to the quasi-Banach case in the particular situation of Besov spaces are based on investigations about corresponding Calderón products, an idea, which goes back to the fundamental paper of Frazier and Jawerth [6] (where Triebel-Lizorkin spaces are treated).

Later on we shall need also complex interpolation for some of the remaining cases not covered by Proposition 4.4. Let $X$ be quasi-Banach space of distributions. By $\hat{X}$ we denote the closure in $X$ of the set of all infinitely differentiable functions $f$ such that $D^\alpha f \in X$ for all $\alpha \in \mathbb{N}_0^d$.

Proposition 4.6. Let $t_i \in \mathbb{R}$, $\Theta \in (0,1)$ and $0 < q_i \leq \infty$, $i = 1, 2$. If $t_0$ and $q_0$ are defined as in (4.4), then

$$\hat{B}^{t_0}_{-\infty, q_0} (\mathbb{R}^d) = \left[ B^{t_1}_{-\infty, q_1} (\mathbb{R}^d), B^{t_2}_{-\infty, q_2} (\mathbb{R}^d) \right]_\Theta$$

and

$$\hat{S}^{t_0}_{-\infty, q_0} B (\mathbb{R}^d) = \left[ S^{t_1}_{-\infty, q_1} B(\mathbb{R}^d), S^{t_2}_{-\infty, q_2} B(\mathbb{R}^d) \right]_\Theta,$$

Proof. The formula (4.5) is proved in [39]. Concerning (4.6) one can argue in a similar way. □

4.3. Dual spaces. Next we will recall some results about the dual spaces of $B^t_{p,q} (\mathbb{R}^d)$ and $S^t_{p,q} B(\mathbb{R}^d)$. For $1 < p < \infty$ the conjugate exponent $p'$ is determined by $\frac{1}{p} + \frac{1}{p'} = 1$. If $0 < p \leq 1$ we put $p' = \infty$ and if $p = \infty$ we put $p' = 1$. For us it will be convenient to switch to the closure of $\mathcal{S}(\mathbb{R}^d)$ in these spaces.

Definition 4.7. (i) By $\hat{B}^t_{p,q} (\mathbb{R}^d)$ we denote the closure of $\mathcal{S}(\mathbb{R}^d)$ in $B^t_{p,q} (\mathbb{R}^d)$.  
(ii) By $\hat{S}^t_{p,q} B(\mathbb{R}^d)$ we denote the closure of $\mathcal{S}(\mathbb{R}^d)$ in $S^t_{p,q} B(\mathbb{R}^d)$.

Recall that

$$\hat{B}^t_{p,q} (\mathbb{R}^d) = B^t_{p,q} (\mathbb{R}^d) \iff \max(p, q) < \infty$$

and

$$\hat{S}^t_{p,q} B(\mathbb{R}^d) = S^t_{p,q} B(\mathbb{R}^d) \iff \max(p, q) < \infty.$$  

Because of the density of $\mathcal{S}(\mathbb{R}^d)$ in these spaces any element of the dual space can be interpreted as an element of $\mathcal{S}'(\mathbb{R}^d)$. Hence, a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the dual space $(\hat{B}^t_{p,q} (\mathbb{R}^d))'$ if and only if there exists a positive constant $c$ such that

$$|f(\varphi)| \leq c \|\varphi\| B^t_{p,q} (\mathbb{R}^d)$$

holds for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Similarly for $\hat{S}^t_{p,q} B(\mathbb{R}^d)$.

Proposition 4.8. Let $t \in \mathbb{R}$.

(i) If $1 \leq p < \infty$ and $0 < q \leq \infty$, then it holds

$$[\hat{B}^t_{p,q} (\mathbb{R}^d)]' = B^{-t}_{p',q'} (\mathbb{R}^d) \quad \text{and} \quad [\hat{S}^t_{p,q} B(\mathbb{R}^d)]' = S^{-t}_{p',q'} B(\mathbb{R}^d).$$
(ii) If $0 < p < 1$ and $0 < q \leq \infty$, then
\[
[B_{p,q}^t(\mathbb{R}^d)]' = B_{\infty,q}^{-\frac{t}{p}+\frac{1}{p}-1}(\mathbb{R}^d) \quad \text{and} \quad [S_{p,q}^t(\mathbb{R}^d)]' = S_{\infty,q}^{-\frac{t}{p}+\frac{1}{p}-1}(\mathbb{R}^d)
\]

**Proof.** The proof in the isotropic case can be found in [35 Section 2.11], see in particular Remark 2.11.2/2. For the dominating mixed smoothness we refer to [7 Subsection 2.3.8], at least if $0 < p, q < \infty$. Here we only give a proof in case $q = \infty$ for the Besov spaces of dominating mixed smoothness following essentially the arguments given in [33 2.5.1] for the isotropic case.

**Step 1.** We shall prove $[S_{p,\infty}^t(\mathbb{R}^d)]' = S_{p,1}^{-t}(\mathbb{R}^d)$.

**Substep 1.1.** Let $(\varphi_j)_{j \in \mathbb{N}_0}$ be the univariate smooth dyadic decomposition of unity used in the definition of the spaces. We put
\[
\tilde{\varphi}_j := \varphi_{j-1} + \varphi_j + \varphi_{j+1}, \quad j = 0, 1, \ldots,
\]
with $\varphi_{-1} \equiv 0$. For $k \in \mathbb{N}_0^d$ we define $\tilde{\varphi}_k := \tilde{\varphi}_{k_1} \otimes \cdots \otimes \tilde{\varphi}_{k_d}$. With $f \in S_{p,1}^{-t}(\mathbb{R}^d)$ and $\psi \in S(\mathbb{R}^d)$ we have
\[
|f(\psi)| = \left| \sum_{k \in \mathbb{N}_0^d} (F^{-1} \varphi_k F \psi)(\psi) \right| = \left| \sum_{k \in \mathbb{N}_0^d} (F^{-1} \tilde{\varphi}_k F^{-1} \varphi_k F \psi)(\psi) \right|
\]
\[
= \left| \sum_{k \in \mathbb{N}_0^d} (F^{-1} \varphi_k F \psi)(F^{-1} \tilde{\varphi}_k F \psi) \right|
\]
\[
\leq \|2^{-|k|t}F^{-1} \varphi_k F \psi\|_{\ell_\infty(L_p)} \cdot \|2^{|k|t}F^{-1} \tilde{\varphi}_k F \psi\|_{\ell_\infty(L_p)}.
\]
Observe that
\[
\|2^{|k|t}F^{-1} \tilde{\varphi}_k F \psi\|_{\ell_\infty(L_p)} \leq \sum_{||\tilde{\varphi}_k\|_{L_p} \leq 1} \|2^{|k|t}F^{-1} \varphi_k F \psi\|_{\ell_\infty(L_p)} \leq c \|\psi\|_{S_{p,\infty}^t(\mathbb{R}^d)}
\]
for some $c > 0$ independent of $\psi$ and $f$. Consequently
\[
|f(\psi)| \leq c \|f\|_{S_{p,1}^{-t}(\mathbb{R}^d)} \cdot \|\psi\|_{S_{p,\infty}^t(\mathbb{R}^d)}
\]
which means $f \in [S_{p,\infty}^t(\mathbb{R}^d)]'$.

**Substep 1.2.** Next we prove the reverse direction. We assume that the generator of our smooth dyadic decomposition of unity is an even function. Then $\varphi_k(-x) = \varphi_k(x)$ follows for all $x \in \mathbb{R}^d$ an all $k \in \mathbb{N}_0^d$.

Let $c_0(L_p)$ denote the space of all sequences $(\psi_k)_k$ of measurable functions such that
\[
\lim_{|k| \to \infty} \|\psi_k\|_{L_p(\mathbb{R}^d)} = 0
\]
equipped with the norm
\[
\| (\psi_k)_k \|_{c_0(L_p)} := \sup_{k \in \mathbb{N}_0^d} \|\psi_k\|_{L_p(\mathbb{R}^d)}.
\]
Observe that
\[
J : \ g \mapsto (2^{|k|t}F^{-1} \varphi_k F g)_k \in \mathbb{N}_0^d
\]
is isometric and bijective if $J$ is considered as a mapping from $S_{p,\infty}^t(\mathbb{R}^d)$ onto a closed subspace $Y$ of $c_0(L_p)$. Here we use the fact that
\[
\lim_{|k| \to \infty} \|2^{|k|t}F^{-1} \varphi_k F g\|_{L_p(\mathbb{R}^d)} = 0
\]
holds for all \( g \in \hat{S}_{p,\infty}^t B(\mathbb{R}^d) \).

Let \( f \in \hat{S}_{p,\infty}^t B(\mathbb{R}^d)' \). Hence, by defining

\[
\hat{f}((\psi_k)_k) := f\left( \sum_{k \in \mathbb{N}_0^d} 2^{-|k|t} \psi_k \right), \quad (\psi_k)_k \in Y,
\]

\( \hat{f} \) becomes a linear and continuous functional on \( Y \) satisfying \( \| \hat{f} | Y \to \mathbb{C} \| = \| f | [\hat{S}_{p,\infty}^t B(\mathbb{R}^d)]' \| \). Now, by the Hahn-Banach theorem, there exists a linear and continuous extension of \( \hat{f} \) to a continuous linear functional on the space \( c_0(L_p) \). It is known that \( [c_0(L_p)]' = \ell_1(L_{p'}) \) and any \( g \in [c_0(L_p)]' \) can be represented in the form

\[
g((\psi_k)_k) = \sum_{k \in \mathbb{N}_0^d} \int_{\mathbb{R}^d} g_k(x) \psi_k(x) \, dx, \quad (\psi_k)_k \in c_0(L_p),
\]

where the functions \( g_k \) satisfy

\[
\| g \|_{\ell_1(L_{p'})} = \sum_{k \in \mathbb{N}_0^d} \| g_k \|_{L_{p'}(\mathbb{R}^d)} < \infty,
\]

see \[34\] Lemma 1.11.1. Applying this with \( g = \hat{f} \) we find

\[
\| (f_k)_k \|_{\ell_1(L_{p'})} = \| \hat{f} | c_0(L_p) \| = \| f | [\hat{S}_{p,\infty}^t B(\mathbb{R}^d)]' \|
\]

for an appropriate sequence \( (f_k)_k \). In view of \[17\], the definition of \( \hat{f} \), the Plancherel identity and the symmetry condition with respect to \( (\varphi_k) \) we obtain

\[
f(\psi) = f\left( \sum_{k \in \mathbb{N}_0^d} \mathcal{F}^{-1} \varphi_k \mathcal{F} \psi \right) = \hat{f}\left( (2^{k|t|} \mathcal{F}^{-1} \varphi_k \mathcal{F} \psi) \right)_k
\]

\[
= \sum_{k \in \mathbb{N}_0^d} 2^{k|t|} \int f_k(x) (\mathcal{F}^{-1} \varphi_k \mathcal{F} \psi)(x) \, dx
\]

\[
= \sum_{k \in \mathbb{N}_0^d} 2^{k|t|} \int \psi(x) (\mathcal{F}^{-1} \varphi_k \mathcal{F} f_k)(x) \, dx
\]

\[
= \sum_{k \in \mathbb{N}_0^d} 2^{k|t|} \left( \mathcal{F}^{-1} \varphi_k \mathcal{F} f_k \right)_k(\psi)
\]

for any \( \psi \in \mathcal{S}(\mathbb{R}^d) \). This leads to the identity

\[
\varphi_k \mathcal{F} f = \sum_{\ell_j + k_j \geq 0, j \geq 1, \ldots, d} 2^{k + \ell|t|} \varphi_{k + \ell} \mathcal{F} f_{k + \ell},
\]
valid in $S'(\mathbb{R}^d)$. Consequently, by using a standard convolution inequality and a homogeneity argument, we have

$$\|F^{-1}\varphi_k Ff|L_{p'}(\mathbb{R}^d)\| \leq \sum_{\ell_j + k_j \geq 0, j=1, \ldots, d} 2^{k_j + \ell_j t} \|F^{-1}\varphi_k \varphi_{\ell + k} Ff_{\ell + k}|L_{p'}(\mathbb{R}^d)\|$$

$$\leq \sum_{\ell_j + k_j \geq 0, j=1, \ldots, d} 2^{k_j + \ell_j t} \|F^{-1}|S_t(p,\infty)B(\mathbb{R}^d)\| ≤ c_2 \|f|L_{p'}(\mathbb{R}^d)\|$$

follows. This together with (4.8) proves that

$$\|f|S_t(p,\infty)B(\mathbb{R}^d)\| ≤ c_2 \|[S_t(p,\infty)B(\mathbb{R}^d)]'\|$$

holds with a constant $c_2$ independent of $f$.

**Step 2.** We shall prove $[S_t(p,\infty)B(\mathbb{R}^d)]' = S_{\infty,1}^{-t+\frac{1}{p}-1}B(\mathbb{R}^d)$.

**Substep 2.1.** The embedding $S_t(p,\infty)B(\mathbb{R}^d) \hookrightarrow S_{1,\infty}^{-t+\frac{1}{p}+1}B(\mathbb{R}^d)$ implies

$$S_t(p,\infty)B(\mathbb{R}^d) \hookrightarrow [S_{1,\infty}^{-t+\frac{1}{p}+1}B(\mathbb{R}^d)]'.$$

Duality and Step 1 yields

$$S_{\infty,1}^{-t+\frac{1}{p}-1}B(\mathbb{R}^d) \hookrightarrow [S_{1,\infty}^{-t+\frac{1}{p}+1}B(\mathbb{R}^d)]'.$$

**Substep 2.2.** Let $f \in [S_t(p,\infty)B(\mathbb{R}^d)]'$. Following Hansen [7] page 75] we choose a point $x_k \in \mathbb{R}^d$ for any $k \in \mathbb{N}_0$ such that

$$1 \|F^{-1}\varphi_k Ff|L_\infty(\mathbb{R}^d)\| \leq \|(F^{-1}\varphi_k Ff)(x_k)\| \leq \|F^{-1}\varphi_k Ff|L_\infty(\mathbb{R}^d)\|$$

Then we define the function

$$\psi(x) := \sum_{|\ell| \leq n} a_\ell (F^{-1}\varphi_\ell)(x_\ell - x) 2^{\ell t}(-t+\frac{1}{p}-1), \quad x \in \mathbb{R}^d.$$
Obviously $\psi \in \mathcal{S}(\mathbb{R}^d)$. An easy calculation yields

$$\| \psi \|_{S^t_{p,\infty} B(\mathbb{R}^d)} = \sup_{k \in \mathbb{N}_0^d} 2^{|k| t} \left\| \mathcal{F}^{-1} \left( \sum_{|\ell| \leq 1, |k+\ell| \leq n} 2^{|\tilde{k}+\tilde{\ell}|(-t+\frac{1}{p}-1)} a_{k+\ell} \phi_{k+\ell}(-\xi) e^{-ix(k+\ell)\xi} \right) \right\|_{L_p(\mathbb{R}^d)}$$

$$\leq \sup_{k \in \mathbb{N}_0^d} 2^{|k| t} \left( \sum_{|\ell| \leq 1, |k+\ell| \leq n} 2^{|\tilde{k}+\tilde{\ell}|(-t+\frac{1}{p}-1)} a_{k+\ell} \mathcal{F}^{-1}[\phi_{k+\ell}(-\cdot)] \right) \| L_p(\mathbb{R}^d) \leq c_1 \sup_{k \in \mathbb{N}_0^d} 2^{|k| t} \left( \sum_{|\ell| \leq 1, |k+\ell| \leq n} 2^{|\tilde{k}+\tilde{\ell}|(-t+\frac{1}{p}-1)} a_{k+\ell} \mathcal{F}^{-1}[\phi_{k+\ell}(-\cdot)] \right) \| L_p(\mathbb{R}^d) \leq c_3 \sup_{|k| \leq 2n} |a_k| .$$

where the last inequality is a consequence of Lemma 1.1 and a homogeneity argument. Observe that

$$\| \mathcal{F}^{-1}[\phi_{k+\ell}(-\cdot)] \|_{L_p(\mathbb{R}^d)} = \| \mathcal{F} \phi_{k+\ell} \|_{L_p(\mathbb{R}^d)} = 2^{|k+\ell|(1-\frac{1}{p})} \| \mathcal{F} \phi_{\tilde{1}} \|_{L_p(\mathbb{R}^d)}$$

if $\tilde{k}_i + \tilde{\ell}_i \leq 1$ for all $i = 1, \ldots, d$. If $\min(\tilde{k}_i + \tilde{\ell}_i) = 0$ one has to modify this in an obvious way. Altogether we have found

$$\| \psi \|_{S^t_{p,\infty} B(\mathbb{R}^d)} \leq c_2 \sup_{k \in \mathbb{N}_0^d} \sum_{|\ell| \leq 1, |k+\ell| \leq n} |a_{k+\ell}| \leq c_3 \sup_{|k| \leq 2n} |a_k| .$$

This estimate can be used to derive

$$\sum_{|k| \leq n} a_k 2^{|k|(-t+\frac{1}{p}-1)} \| \mathcal{F}^{-1}\phi_k \mathcal{F} f \| \leq \| f \| \cdot \| \psi \|_{S^t_{p,\infty} B(\mathbb{R}^d)} \leq c_3 \| f \| \sup_{|k| \leq 2n} |a_k| .$$

Employing (4.9) and the fact that the $a_k$ can be chosen as we want, for instance such that

$$a_k (\mathcal{F}^{-1}\phi_k \mathcal{F} f)(x_k) = |\mathcal{F}^{-1}\phi_k \mathcal{F} f|(x_k),$$

we find

$$\sum_{|k| \leq n} 2^{|k|(-t+\frac{1}{p}-1)} \| \mathcal{F}^{-1}\phi_k \mathcal{F} f \|_{L_\infty(\mathbb{R}^d)} \leq c_3 \| f \| .$$

Here $c_3$ is independent of $f$ and $n$. For $n \to \infty$ we obtain

$$\| f \|_{S^{t-1+\frac{1}{p}-1}_{\infty, 1} \mathbb{R}^d} \leq c_3 \| f \| .$$

The proof is complete. \qed
4.4. Test functions. Let \( d \geq 2 \). Before we are going to define some test functions we mention a few more properties of our smooth decompositions of unity.

As a consequence of the definitions we obtain

\[
\varphi_k(x) = 1 \quad \text{if} \quad \frac{3}{4} 2^{k_j} \leq x_j \leq 2^{k_j}, \quad j = 1, \ldots, d,
\]

if \( \min_{j=1, \ldots, d} k_j > 0 \). In case \( \min_{j=1, \ldots, d} k_j = 0 \) the following statement is true:

\[
\varphi_k(x) = 1 \quad \text{if} \quad \frac{1}{4} 2^{k_j} \leq x_j \leq 2^{k_j} \quad \text{holds for all} \quad j \quad \text{such that} \quad k_j = 0 \quad \text{and} \quad 0 \leq x_j \leq 1 \quad \text{for the remaining components}.
\]

Now we switch to \( (\psi_t)_\ell \). For \( \ell \in \mathbb{N} \) it follows

\[
\psi_t(x) = 1 \quad \text{on the set} \quad \{ x : \sup_{j=1, \ldots, d} |x_j| \leq 2^\ell \} \setminus \{ x : \sup_{j=1, \ldots, d} |x_j| \leq \frac{3}{4} 2^\ell \}.
\]

**Example 1.** Let \( g \in C_0^\infty(\mathbb{R}^d) \) be a function such that \( \text{supp} \ g \in B(0, \epsilon) \) for \( \epsilon > 0 \) small enough \((0 < \epsilon < \frac{1}{8})\) and \( |\mathcal{F}^{-1}g(\xi)| > 0 \) on \([\pi, \pi]^d\). For \( \ell \in \mathbb{N} \) we define the family of functions \( f_\ell \) by

\[
(4.10) \quad \mathcal{F} f_\ell(\xi) := \sum_{j=1}^\ell a_j g(\xi_1 - \frac{7}{8} 2^j, \xi_2 - \frac{7}{8} 2^j, \xi_3, \ldots, \xi_d), \quad \xi \in \mathbb{R}^d,
\]

where the sequence \( (a_j)_{j=1}^\ell \) of complex numbers will be chosen later on. Note that

\[
\text{supp} \ g(\cdot - \frac{7}{8} 2^j, \cdot - \frac{7}{8} 2^j, \cdot, \cdot, \cdot) \subset \{ x : \varphi_k(x) = 1, \quad k = (\ell, j, 0, \ldots, 0) \}
\]

\[
\subset \{ x : \psi_t(x) = 1 \}, \quad j = 1, \ldots, \ell.
\]

It follows

\[
\mathcal{F}^{-1}[\psi_m \mathcal{F} f_\ell] = \delta_{m, \ell} f_\ell
\]

and

\[
\mathcal{F}^{-1}[\varphi_k \mathcal{F} f_\ell] = \delta_{k, (\ell, j, 0, \ldots, 0)} a_j \mathcal{F}^{-1}[g(\xi_1 - \frac{7}{8} 2^j, \xi_2 - \frac{7}{8} 2^j, \xi_3, \ldots, \xi_d)].
\]

Hence

\[
\| f_\ell \|_{L^p_{\text{loc}}(\mathbb{R}^d)} \leq 2^{\ell} \| \mathcal{F}^{-1}[\psi_t \mathcal{F} f_\ell(\cdot)|L_p(\mathbb{R}^d)] \|
\]

\[
= 2^{\ell} \left\| \sum_{j=1}^\ell a_j \mathcal{F}^{-1}[g(\xi_1 - \frac{7}{8} 2^j, \xi_2 - \frac{7}{8} 2^j, \xi_3, \ldots, \xi_d)] \right\|_{L_p(\mathbb{R}^d)}
\]

\[
(4.11) \quad \approx 2^{\ell} \left\| \mathcal{F}^{-1} g(x) \left( \sum_{j=1}^\ell a_j e^{\frac{j}{\pi}(2^{x_1} + 2^{x_2})} \right) \right\|_{L_p(\mathbb{R}^d)}
\]

For the last step we used that \( \mathcal{F}^{-1} g \) is rapidly decreasing, \( |\mathcal{F}^{-1}g(\xi)| > 0 \) on \([\pi, \pi]^d\) and that \( \mathbb{R}^d \) can be written as

\[
\mathbb{R}^d = \bigcup_{m \in \mathbb{Z}^d} [2m \pi, 2(m+1) \pi).
\]
In case $1 < p < \infty$ a Littlewood-Paley characterization of $L_p([-\pi, \pi]^2)$ yields

$$
\| f_\ell | B^t_{p,q}(\mathbb{R}^d) \| \approx 2^{\ell t} \left( \sum_{j=1}^{\ell} |a_j|^2 \right)^{1/2}.
$$

Similarly

$$
\| f_\ell | S^t_{p,q}(\mathbb{R}^d) \| = \left( \sum_{j=1}^{\ell} 2^{j^2 q} \left\| \mathcal{F}^{-1} [\varphi(\ell,j,0,...,0) \mathcal{F} f_\ell] (\cdot) | L_p(\mathbb{R}^d) \right\|^q \right)^{1/q}

= \left( \sum_{j=1}^{\ell} 2^{j^2 q} |a_j|^q \left\| \mathcal{F}^{-1} [g(\xi_1 - \frac{7}{8} 2^j, \xi_2 - \frac{7}{8} 2^j, ..., \xi_d)] | L_p(\mathbb{R}^d) \right\|^q \right)^{1/q}

= \left\| \mathcal{F}^{-1} g | L_p(\mathbb{R}^d) \right\| \left( \sum_{j=1}^{\ell} 2^{j^2 q} |a_j|^q \right)^{1/q}.
$$

**Example 2.** In case $p = \infty$ nontrivial periodic functions are contained in $B^\infty_{\infty,q}(\mathbb{R}^d)$ and $S^\infty_{\infty,q}(\mathbb{R}^d)$. So we can work directly with lacunary series. Let

$$
f_\ell(x) := \sum_{j=1}^{\ell} a_j e^{i(2^\ell x_1 + 2^j x_2)}, \quad x = (x_1, ..., x_d) \in \mathbb{R}^d.
$$

Then

$$
\mathcal{F}^{-1} [\varphi_m \mathcal{F} f_\ell] = \delta_{m,\ell} f_\ell
$$

and

$$
\mathcal{F}^{-1} [\varphi_k \mathcal{F} f_\ell] = \delta_{k,\ell} e^{i(2^\ell x_1 + 2^j x_2)}
$$

follow. For $a_j \geq 0$ for all $j$ this will allow us to calculate the quasi-norms in $B^\infty_{\infty,q}(\mathbb{R}^d)$ and $S^\infty_{\infty,q}(\mathbb{R}^d)$. We obtain in the first case

$$
\| f_\ell | B^\infty_{\infty,q}(\mathbb{R}^d) \| = 2^{\ell t} \left\| \mathcal{F}^{-1} [\varphi \mathcal{F} f] (\cdot) | L_\infty(\mathbb{R}^d) \right\|

= 2^{\ell t} \sup_{x \in \mathbb{R}^d} \left| \sum_{j=1}^{\ell} a_j e^{i(2^\ell x_1 + 2^j x_2)} \right|

= 2^{\ell t} \sum_{j=1}^{\ell} a_j.
$$

Concerning the dominating mixed smoothness we conclude

$$
\| f_\ell | S^\infty_{\infty,q}(\mathbb{R}^d) \| = \left( \sum_{j=1}^{\ell} 2^{j^2 q} \left\| \mathcal{F}^{-1} [\varphi(\ell,j,0,...,0) \mathcal{F} f_\ell] (\cdot) | L_\infty(\mathbb{R}^d) \right\|^q \right)^{1/q}

= \left( \sum_{j=1}^{\ell} 2^{j^2 q} |a_j|^q \right)^{1/q}.
$$
Example 3. Let us consider a function $g \in C_0^\infty(\mathbb{R})$ such that $\text{supp}\, g \subset \{x \in \mathbb{R} : 3/2 \leq |x| \leq 2\}$. For $j \in \mathbb{N}, \vec{k} \in \mathbb{N}^d$ we define
\[ g_j(t) = g(2^{-j+1}t) \quad \text{and} \quad g_{\vec{k}}(x) = g_{k_1}(x_1) \cdots g_{k_d}(x_d), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d. \]
Let
\[ \nabla_\ell := \{ \vec{k} \in \mathbb{N}^d, |\vec{k}|_\infty = \ell \}, \quad \ell \in \mathbb{N}. \]
Then, if $\vec{k} \in \nabla_\ell$, we have
\[ \text{supp}\, g_{\vec{k}} \subset \{ x : \phi_{\vec{k}}(x) = 1 \} \subset \{ x : \psi_{\ell}(x) = 1 \}. \]
We define the family of test functions
\[ f_\ell = \sum_{k \in \nabla_\ell} a_k \mathcal{F}^{-1}g_k, \quad \ell \in \mathbb{N}^d. \]
The coefficients $(a_k)_{\vec{k}}$ will be chosen later on. By construction we have
\[ \|f_\ell|S_{p,q}^0 B(\mathbb{R}^d)\| = \left( \sum_{k \in \nabla_\ell} \|F^{-1}[\phi_{\vec{k}}\mathcal{F}f_\ell(\cdot)]L_p(\mathbb{R}^d)\|^q \right)^{1/q} \]
\[ = \left( \sum_{k \in \nabla_\ell} |a_k|^q \|F^{-1}g_k|L_p(\mathbb{R}^d)\|^q \right)^{1/q}. \]
Observe that
\[ \|F^{-1}g_k|L_p(\mathbb{R}^d)\| = 2^{(|\vec{k}|_d-1)-(1-p)} \|F^{-1}g_1|L_p(\mathbb{R}^d)\| = C \cdot 2^{(|\vec{k}|_d-1)} \]
for an appropriate $C > 0$ (independent of $\ell$). Consequently we obtain
\[ \|f_\ell|S_{p,q}^0 B(\mathbb{R}^d)\| = C \left( \sum_{k \in \nabla_\ell} |a_k|^q \cdot 2^{(|\vec{k}|_d-1)q} \right)^{1/q}, \quad \ell \in \mathbb{N}. \]
Next we compute
\[ \|f_\ell|B_{p,q}^0(\mathbb{R}^d)\| = \left( \sum_{j=0}^\infty \|F^{-1}[\psi_j\mathcal{F}f_\ell(\cdot)]L_p(\mathbb{R}^d)\|^q \right)^{1/q} \]
\[ = \left\| \sum_{k \in \nabla_\ell} a_k F^{-1}g_k|L_p(\mathbb{R}^d)\right\|. \]
Recall, for $0 < p_0 < p < p_1 < \infty$ we have
\[ S_{p_0,p}^{1,\frac{1}{p}} B(\mathbb{R}^d) \hookrightarrow S_{p_0,2}^0 F(\mathbb{R}^d) \hookrightarrow S_{p_1,2}^{1/p} B(\mathbb{R}^d), \]
see [10], and $S_{p_0,2}^0 F(\mathbb{R}^d) = L_p(\mathbb{R}^d), \ 1 < p < \infty,$ see [15]. These arguments lead to
\[ \left\| \sum_{k \in \nabla_\ell} a_k F^{-1}g_k|L_p(\mathbb{R}^d)\right\| \leq C_1 \left( \sum_{k \in \nabla_\ell} 2^{j(\frac{d}{p^*} - \frac{1}{p})}|a_k|^p \|F^{-1}g_k|L_{p_0}(\mathbb{R}^d)\|^p \right)^{\frac{1}{p}} \]
\[ = C_2 \left( \sum_{k \in \nabla_\ell} 2^{j(\frac{d}{p^*} - \frac{1}{p})}|a_k|^p \cdot 2^{j(\frac{d}{p^*} - \frac{1}{p})} \right)^{\frac{1}{p}} \]
\[ = C_2 \left( \sum_{k \in \nabla_\ell} |a_k|^p 2^{j(\frac{d}{p^*} - \frac{1}{p})} \right)^{\frac{1}{p}}. \]
Similarly we have
\[
\left( \sum_{k \in \nabla} 2^{|k|/(n-\frac{1}{p})} |a_k|^p \|F^{-1}g_k\|_{L_p(\mathbb{R}^d)}^p \right)^{\frac{1}{p}} = C_3 \left( \sum_{k \in \nabla} |a_k|^{p_2^2 |k|(1-\frac{3}{p})} p \right)^{\frac{1}{p}} \leq C_4 \left( \sum_{k \in \nabla} |a_k|^{F^{-1}g_k|\|_{L_p(\mathbb{R}^d)}} \right).
\]
Altogether we have proved in case 1 \( p < \infty \)
\[
\|f_\ell|_{B^0_{p,q}(\mathbb{R}^d)} \| \asymp \left( \sum_{k \in \nabla} |a_k|^{2^{|k|/(1-\frac{3}{p})} p} \right)^{\frac{1}{p}},
\]
where the positive constants behind \( \asymp \) do not depend on \( \ell \in \mathbb{N} \).

**Example 4.** We consider the same basic functions \( g_k \) as in Example 3. This time we define
\[
f_\ell := \sum_{j=1}^{\ell} a_j F^{-1}g_j, \quad \bar{j} := (j, 1, \ldots, 1).
\]
As above we conclude
\[
\|f_\ell|S^t_{p,q}B(\mathbb{R}^d)\| = \left( \sum_{j=1}^{\ell} 2^{j}\Delta q |a_j|^q \|F^{-1}g_{\bar{j}}\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q} = C \left( \sum_{j=1}^{\ell} 2^{j}\Delta q |a_j|^q \right)^{1/q}
\]
and
\[
\|f_\ell|B^t_{p,q}(\mathbb{R}^d)\| = C \left( \sum_{j=1}^{\ell} 2^{j}\Delta q |a_j|^q \right)^{1/q}
\]
for an appropriate positive constant \( C \) (independent of \( \ell \)). Notice that we do not need the restriction \( 1 < p < \infty \) here. It is true for all \( p \).

**Example 5.** This will be one more modification of Example 3. Let \( g_k \) be defined as there. We put
\[
f_\ell := \sum_{j=1}^{\ell} a_j F^{-1}g_{\bar{j}} \quad \bar{j} := (j, \ldots, j).
\]
Then we have
\[
\|f_\ell|S^t_{p,q}B(\mathbb{R}^d)\| = \left( \sum_{j=1}^{\ell} 2^{j}\Delta q |a_j|^q \|F^{-1}g_{\bar{j}}\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q} = C \left( \sum_{j=1}^{\ell} 2^{j}\Delta q |a_j|^q \right)^{1/q}
\]
and
\[
\|f_\ell|B^t_{p,q}(\mathbb{R}^d)\| = C \left( \sum_{j=1}^{\ell} 2^{j}\Delta q |a_j|^q \right)^{1/q},
\]
where $C$ is as in Example 4. Also here the restriction $1 < p < \infty$ is not needed.

**Example 6.** This example is taken from [35 2.3.9]. Let $\varrho \in \mathcal{S}(\mathbb{R}^d)$ be a function such that $\text{supp} \mathcal{F} \varrho \subset \{ \xi : |\xi| \leq 1 \}$. We define
\[
 h_j(x) := \varrho(2^{-j}x), \quad x \in \mathbb{R}^d, \quad j \in \mathbb{N}.
\]
For all $p, q, t$ we conclude
\[
 \| h_j | B^t_{p,q} (\mathbb{R}^d) \| = \| h_j | L_p(\mathbb{R}^d) \| = 2^{jd/p} \| \varrho | L_p(\mathbb{R}^d) \|, \quad j \in \mathbb{N}.
\]
Similarly, also for all $p, q, t$, we obtain
\[
 \| h_j | S^t_{p,q} (\mathbb{R}^d) \| = \| h_j | L_p(\mathbb{R}^d) \| = 2^{jd/p} \| \varrho | L_p(\mathbb{R}^d) \|, \quad j \in \mathbb{N}.
\]
As an immediate consequence of these two identities we get the following.

**Lemma 4.9.** Let $0 < p_0, p_1, q_0, q_1 \leq \infty$ and $t_0, t_1 \in \mathbb{R}$.
(i) An embedding $S_{p_0,q_0}^{t_0} B(\mathbb{R}^d) \hookrightarrow B_{p_1,q_1}^{t_1}(\mathbb{R}^d)$ implies $p_0 \leq p_1$.
(ii) An embedding $B_{p_0,q_0}^{t_0}(\mathbb{R}^d) \hookrightarrow S_{p_1,q_1}^{t_1}(\mathbb{R}^d)$ implies $p_0 \leq p_1$.

4.5. **Proof of Theorem 4.1—sufficiency.** Step 1. Preparations. For $\bar{k} \in \mathbb{N}_0^d$ we define
\[
 \square \bar{k} := \{ j \in \mathbb{N}_0 : \text{supp} \psi_j \cap \text{supp} \varphi_{\bar{k}} \neq \emptyset \}
\]
and $j \in \mathbb{N}_0$
\[
 \Delta_j := \{ \bar{k} \in \mathbb{N}_0^d : \text{supp} \psi_j \cap \text{supp} \varphi_{\bar{k}} \neq \emptyset \}.
\]
The condition $\text{supp} \psi_j \cap \text{supp} \varphi_{\bar{k}} \neq \emptyset$ implies
\[
 (4.17) \quad \max_{i=1,\ldots,d} k_i - 1 \leq j \leq \max_{i=1,\ldots,d} k_i + 1.
\]
Consequently we obtain
\[
 (4.18) \quad |\square \bar{k}| \asymp 1, \quad \bar{k} \in \mathbb{N}_0^d \quad \text{and} \quad |\Delta_j| \asymp (1 + j)^{d-1}, \quad j \in \mathbb{N}_0.
\]
By definition we have
\[
 (4.19) \quad \psi_j(x) = \sum_{\bar{k} \in \Delta_j} \varphi_{\bar{k}}(x) \psi_j(x), \quad x \in \mathbb{R}^d.
\]
**Step 2.** Let $t > 0$ and let $u = \min(1, p)$. Employing (4.19) we find
\[
 \| f | B^t_{p,q} (\mathbb{R}^d) \|^q \leq \infty \sum_{j=0}^{\infty} 2^{jtq} \left( \sum_{\bar{k} \in \Delta_j} \| \mathcal{F}^{-1} \varphi_{\bar{k}} \psi_j \mathcal{F} f | L_p(\mathbb{R}^d) \| \right)^q 
\]
\[
 \leq \sum_{j=0}^{\infty} 2^{jtq} \left( \left| \sum_{\bar{k} \in \Delta_j} \| \mathcal{F}^{-1} \varphi_{\bar{k}} \psi_j \mathcal{F} f | L_p(\mathbb{R}^d) \| u \right| \right)^{q/u}.
\]
Using (4.11) it follows
\[
 (4.20) \quad \| f | B^t_{p,q} (\mathbb{R}^d) \|^q \leq C \sum_{j=0}^{\infty} \left( \sum_{\bar{k} \in \Delta_j} \left| 2^{j(|\bar{k}|)^t} \| \mathcal{F}^{-1} \varphi_{\bar{k}} \mathcal{F} f | L_p(\mathbb{R}^d) \| u \right| \right)^{q/u}.
\]
If $\frac{u}{u} \leq 1$ then we have
\[ \|f\|_{B_{p,q}^r(\mathbb{R}^d)}^q \leq C \sum_{j=0}^{\infty} \sum_{k \in \Delta_j} 2^{j|\bar{k}|} \|2^{j|\bar{k}|} |F^{-1}\varphi_k f| L_p(\mathbb{R}^d)\|_q \]

(4.21)

\[ \leq c_1 \sum_{k \in \mathbb{N}^d \setminus \{0\}} \sum_{j \in \mathbb{Z}} 2^{j|\bar{k}|} \|F^{-1}\varphi_k f| L_p(\mathbb{R}^d)\|_q. \]

The last inequality is due to \(2^{(j-1)|\bar{k}|}t_q \leq c_2 \) since \(t > 0\) and \(j - 1 \leq \max_i \bar{a}_i \leq j + 1\), see (4.17). In the case \(\frac{a}{u} > 1\) we use Hölder’s inequality with \(1 = \frac{a}{u} + (1 - \frac{a}{u})\). (4.20) implies

\[ \|f\|_{B_{p,q}^r(\mathbb{R}^d)}^q \leq C \sum_{j=0}^{\infty} \sum_{k \in \Delta_j} 2^{j|\bar{k}|} \|F^{-1}\varphi_k f| L_p(\mathbb{R}^d)\|_q \left( \sum_{k \in \Delta_j} 2^{(j-1)|\bar{k}|t_q} \right)^{\frac{q}{u}}. \]

Observe, for \(t > 0\) we have

\[ \sup_{j \in \mathbb{N}_0} \left( \sum_{k \in \Delta_j} 2^{(j-1)|\bar{k}|t_q} \right)^{\frac{q}{u}} < \infty, \]

see (4.17). Hence

(4.22) \[ \|f\|_{B_{p,q}^r(\mathbb{R}^d)}^q \leq c_3 \sum_{k \in \mathbb{N}^d \setminus \{0\}} \sum_{j \in \mathbb{Z}} 2^{j|\bar{k}|} \|F^{-1}\varphi_k f| L_p(\mathbb{R}^d)\|_q. \]

Finally, from (4.21), (4.22) together with \(\square_k \approx 1\) we conclude

\[ \|f\|_{B_{p,q}^r(\mathbb{R}^d)}^q \leq c_4 \sum_{k \in \mathbb{N}^d} 2^{j|\bar{k}|} \|F^{-1}\varphi_k f| L_p(\mathbb{R}^d)\|_q. \]

This proves (3.1).

**Step 3.** Let \(t = 0\).

**Substep 3.1.** First we assume that \(q \leq \min(p, 1)\). From (4.20) with \(t = 0\) we have

\[ \|f\|_{B_{p,q}^0(\mathbb{R}^d)}^q \leq c_1 \sum_{j=0}^{\infty} \sum_{k \in \Delta_j} \|F^{-1}\varphi_k f| L_p(\mathbb{R}^d)\|_q \]

\[ = c_1 \sum_{k \in \mathbb{N}^d} \sum_{j \in \mathbb{Z}} \|F^{-1}\varphi_k f| L_p(\mathbb{R}^d)\|_q. \]

Since \(\square_k \approx 1\) we obtain

\[ \|f\|_{B_{p,q}^0(\mathbb{R}^d)}^q \leq c_2 \|f\|_{S_{p,q}^0 B(\mathbb{R}^d)}^q. \]

**Substep 3.2.** Let \(1 < p < \infty\) and \(0 < q \leq \min(2, p)\). Our main tool will be the following Littlewood-Paley assertion. With \(1 < p < \infty\) there exist positive constants \(A, B\) such that

(4.23) \[ A \| f \|_{L_p(\mathbb{R}^d)} \leq \left\| \left( \sum_{k \in \mathbb{N}^d} |F^{-1}\varphi_k f|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \leq B \| f \|_{L_p(\mathbb{R}^d)} \]

holds for all \(f \in L_p(\mathbb{R}^d)\), see Lizorkin [13, 15] or Nikol’skij [18, 1.5.6]. This will be applied to \(f\) replaced by \(F^{-1}\psi_j f\). We proceed as in Step 1. Employing (4.19)
and (4.23) we find
\[ \|f|B_{p,q}^0(\mathbb{R}^d)\|_q = \sum_{j=0}^{\infty} \left\| \mathcal{F}^{-1} \psi_j \mathcal{F} f \right\|_{L_p(\mathbb{R}^d)}^q \leq \frac{1}{A^q} \sum_{j=0}^{\infty} \left( \sum_{k \in \Delta_j} |\mathcal{F}^{-1} \varphi_k \psi_j \mathcal{F} f|^2 \right)^{1/2} \|L_p(\mathbb{R}^d)\|_q^q. \]

Because of \( \| \cdot |L_p(\ell_2)\| \leq \| \cdot |\ell_{\min(2,p)}(L_p)\| \leq \| \cdot |\ell_q(L_p)\| \) we deduce
\[ \|f|B_{p,q}^0(\mathbb{R}^d)\|_q \leq \frac{1}{A^q} \sum_{j=0}^{\infty} \sum_{k \in \Delta_j} \|\mathcal{F}^{-1} \varphi_k \psi_j \mathcal{F} f|L_p(\mathbb{R}^d)\|_q \]
\[ \leq c \sum_{j=0}^{\infty} \sum_{k \in \Delta_j} \|\mathcal{F}^{-1} \varphi_k \mathcal{F} f|L_p(\mathbb{R}^d)\|_q \]
where we used in the last step (4.1). As in Step 1 we can continue the estimate by changing the order of summation and using \( |\Box_k| \asymp 1. \)

4.6. Proof of Theorem 3.6 – sufficiency. Step 1. Let us prove (3.2) in case \( t > \max(0, \frac{1}{p} - 1) \). We put \( u := \min(1, p) \). From we have
\[ \varphi_k(x) = \sum_{j \in \Box_k} \psi_j(x) \varphi_k(x), \quad x \in \mathbb{R}^d. \]
This identity yields
\[ \|f|\mathcal{S}_{p,q}^t(\mathbb{R}^d)\|_q^q = \sum_{k \in N_0^d} 2^{|k|q} \left\| \sum_{j \in \Box_k} \mathcal{F}^{-1} \psi_j \varphi_k \mathcal{F} f \right\|_{L_p(\mathbb{R}^d)}^q. \]
Applying \( |a + b|^u \leq a^u + b^u \) and (4.2) we find
\[ \|f|\mathcal{S}_{p,q}^t(\mathbb{R}^d)\|_q^q \leq \sum_{k \in N_0^d} 2^{|k|q} \left( \sum_{j \in \Box_k} \|\mathcal{F}^{-1} \psi_j \varphi_k \mathcal{F} f|L_p(\mathbb{R}^d)\|_u \right)^{q/u} \]
\[ \leq C \sum_{k \in N_0^d} 2^{|k|q} \left( \sum_{j \in \Box_k} (2^{(j-d-|k|)(\frac{1}{u} - 1)} \|\mathcal{F}^{-1} \psi_j \mathcal{F} f|L_p(\mathbb{R}^d)\|_u \right)^{q/u} \]
Because of (4.13) this implies
\[ \|f|\mathcal{S}_{p,q}^t(\mathbb{R}^d)\|_q^q \leq c \sum_{k \in N_0^d} \sum_{j \in \Box_k} 2^{|k|q} 2^{(j-d-|k|)(\frac{1}{u} - 1)q} \|\mathcal{F}^{-1} \psi_j \mathcal{F} f|L_p(\mathbb{R}^d)\|_q^q \]
Consequently
\[ \|f|\mathcal{S}_{p,q}^t(\mathbb{R}^d)\|_q^q \leq c \sum_{j=0}^{\infty} 2^{djq} \|\mathcal{F}^{-1} \psi_j \mathcal{F} f|L_p(\mathbb{R}^d)\|_q^q \sum_{k \in \Delta_j} 2^{(j-d-|k|)(\frac{1}{u} - 1)q} \]
It is easily derived from (4.17) and the restriction \( t > \frac{1}{u} - 1 \) that
\[ \sup_{j \in N_0} \sum_{k \in \Delta_j} 2^{(j-d-|k|)(\frac{1}{u} - 1)q} < \infty. \]
Hence
\[\|f\|_{S^{\frac{1}{(t-1)p}}_{p,q}B(\mathbb{R}^d)} \leq c_1 \sum_{j=0}^{\infty} 2^{jdq} \|F^{-1}\psi_j Ff|L_p(\mathbb{R}^d)\|\]
follows.

**Step 3.** Let \(t = 0, 1 < p \leq \infty\) and \(\max(2,p) \leq q \leq \infty\). We shall argue by duality. We have
\[S^{0}_{p',q'}B(\mathbb{R}^d) \hookrightarrow B^{0}_{p',q'}(\mathbb{R}^d),\]
see Theorem 4.1.4(i) can be used to prove the claim.

**Step 4.** Let \(0 < p \leq 1, t = \frac{1}{p} - 1\) and \(q = \infty\). Applying (4.21) we find
\[\|f\|_{S^{\frac{1}{(t-1)p}}_{p,q}B(\mathbb{R}^d)} = \sup_{k \in \mathbb{N}_0} 2^{\frac{k}{t-1}j} \|F^{-1}\varphi_k Ff|L_p(\mathbb{R}^d)\| \leq \sup_{k \in \mathbb{N}_0} 2^{\frac{k}{t-1}j} \sum_{j \in \square} F^{-1}\psi_j \varphi_k Ff|L_p(\mathbb{R}^d)\|.\]
Making use of (4.2), this implies
\[\|f\|_{S^{\frac{1}{(t-1)p}}_{p,q}B(\mathbb{R}^d)} \leq \sup_{k \in \mathbb{N}_0} 2^{\frac{k}{t-1}j} \left( \sum_{j \in \square} \|F^{-1}\psi_j \varphi_k Ff|L_p(\mathbb{R}^d)\|^p \right)^{\frac{1}{p}} \leq c_1 \sup_{k \in \mathbb{N}_0} 2^{\frac{k}{t-1}j} \left( \sum_{j \in \square} 2^{(j-|k|)(\frac{1}{t-1})p} \|F^{-1}\psi_j Ff|L_p(\mathbb{R}^d)\|^p \right) \leq c_1 \sup_{k \in \mathbb{N}_0} \left( \sum_{j \in \square} 2^{j(\frac{1}{t-1})p} \|F^{-1}\psi_j Ff|L_p(\mathbb{R}^d)\|^p \right)^{\frac{1}{p}}.\]
Taking into account (4.18), we obtain
\[\|f\|_{S^{\frac{1}{(t-1)p}}_{p,q}B(\mathbb{R}^d)} \leq c_2 \sup_{j \in \mathbb{N}_0} 2^{j(\frac{1}{t-1})p} \|F^{-1}\psi_j Ff|L_p(\mathbb{R}^d)\|.\]
The proof is complete.

**4.7. Proof of Theorem 3.1 — necessity.**

**Lemma 4.10.** (i) Let \(0 < p < \infty\) and \(0 < q \leq \infty\). Then the embedding
\[S^{0}_{p,q}B(\mathbb{R}^d) \hookrightarrow B^{0}_{p,q}(\mathbb{R}^d)\]
implies \(q \leq \min(2,p)\).

(ii) Let \(0 < q \leq \infty\). Then the embedding
\[S^{0}_{\infty,q}B(\mathbb{R}^d) \hookrightarrow B^{0}_{\infty,q}(\mathbb{R}^d)\]
implies \(q \leq 1\).

**Proof.** Step 1. We prove (i).

**Substep 1.1.** We show necessity of \(q \leq 2\). Temporarily we assume \(1 < p < \infty\). We use our test functions from Example 1, see (4.13). The embedding \(S^{0}_{p,q}B(\mathbb{R}^d) \hookrightarrow B^{0}_{p,q}(\mathbb{R}^d)\) implies the existence of a constant \(c\) such that
\[\left( \sum_{j=1}^{l} |a_j|^2 \right)^{1/2} \leq \|f_t |B^{0}_{p,q}(\mathbb{R}^d)\| \leq c \|f_t |S^{t}_{p,q}(\mathbb{R}^d)\| \leq c \left( \sum_{j=1}^{l} |a_j|^q \right)^{1/q}\]
where \( c \) does not depend on \( \ell \) and \((a_j)_j\), see \((4.12)\) and \((4.13)\). This requires \( q \leq 2 \).

Now we turn to \( 0 < p \leq 1 \). Again we shall work with Example 1. For any such \( p \) there exists some real number \( \Theta \in (0, 1) \) such that
\[
\frac{2}{3} = \frac{1 - \Theta}{p} + \frac{\Theta}{2}.
\]

Lyapunov’s inequality
\[
\| h \|_{L_3/2([-\pi, \pi]^2)} \leq \| h \|_{L_p([-\pi, \pi]^2)}^{1-\Theta} \| h \|_{L_2([-\pi, \pi]^2)}^\Theta
\]
valid for all \( h \in L_p([-\pi, \pi]^2) \cap L_2([-\pi, \pi]^2) \), in combination with the Littlewood-Paley characterization of \( L_{3/2} \) and \( L_2 \), leads us to
\[
\left( \sum_{j=1}^{\ell} |a_j|^2 \right)^{1/2} \leq c \left\| \sum_{j=1}^{\ell} a_j e^{\pm i(2^j x_1 + 2^j x_2)} \right\|_{L_p([-\pi, \pi]^2)}^{1-\Theta} \left( \sum_{j=1}^{\ell} |a_j|^2 \right)^{\Theta/2}
\]
with \( c \) independent of \( \ell \) and \((a_j)_j\). Hence
\[
\left( \sum_{j=1}^{\ell} |a_j|^2 \right)^{1/2} \leq c \left\| \sum_{j=1}^{\ell} a_j e^{\pm i(2^j x_1 + 2^j x_2)} \right\|_{L_p([-\pi, \pi]^2)}.
\]

Taking into account \((4.11)\) we can argue as in case \( 1 < p < \infty \).

Substep 1.2. We show necessity of \( q \leq p \). Therefore we use Example 3. In case \( 1 < p < \infty \) we choose \( a_k = 2^k |k|^{-\Theta} \). Then almost immediately we can conclude \( q \leq p \).

Now we turn to the remaining cases. Assume that there exist \( 0 < p \leq 1 \) and \( p < q \leq 2 \) such that
\[
S_{p,q}^0 B(\mathbb{R}^d) \hookrightarrow B_{p,q}^0(\mathbb{R}^d).
\]

In this situation we may choose a triple \((p_1, q_1, \Theta)\) such that
\[
1 < p_1 < q_1 \leq 2, \quad \Theta \in (0, 1), \quad \frac{1}{p_1} = \frac{\Theta}{p} + \frac{1 - \Theta}{2} \quad \text{and} \quad \frac{1}{q_1} = \frac{\Theta}{q} + \frac{1 - \Theta}{2}.
\]

Then it follows from Proposition \((4.4)\) that
\[
S_{p_1,q_1}^0 B(\mathbb{R}^d) = [S_{p,q}^0 B(\mathbb{R}^d), S_{2,2}^0 B(\mathbb{R}^d)]_\Theta
\]
and
\[
B_{p_1,q_1}^0(\mathbb{R}^d) = [B_{p,q}^0(\mathbb{R}^d), B_{2,2}^0(\mathbb{R}^d)]_\Theta.
\]

Proposition \((4.3)\) yields
\[
S_{p_1,q_1}^0 B(\mathbb{R}^d) \hookrightarrow B_{p_1,q_1}^0(\mathbb{R}^d).
\]

But this is a contradiction to Example 3.

Step 2. To prove (iii) we use Example 2, see \((4.14)\). The embedding \( S_{\infty,q}^0 B(\mathbb{R}^d) \hookrightarrow B_{\infty,q}^0(\mathbb{R}^d) \) implies the existence of a constant \( c \) such that
\[
\sum_{j=1}^{\ell} |a_j| = \| f \|_{B_{\infty,q}^0(\mathbb{R}^d)} \leq c \| f \|_{S_{\infty,q}^0(\mathbb{R}^d)} \geq \left( \sum_{j=1}^{\ell} |a_j|^q \right)^{1/q}
\]
where \( c \) does not depend on \( \ell \) and \((a_j)_j\), see \((4.15)\) and \((4.16)\). Choosing \( a_j = 1 \) it is obvious that this can happen only if \( q \leq 1 \). \( \square \)
Proposition 4.4 we obtain
and
Substep 2.1. We assume that 0
It remains to deal with Step 2.
Substep 2.2. Let 0

\[ \min(p, q) \]

Here \( \Theta = 1 \)

\[ \Theta \]

Therefore we shall use

\[ \Theta \]

Substep 2.3. We assume that 0

\[ \Theta \]

\[ \Theta \]

This contradicts Lemma 4.10.

Substep 2.2. Let 0 < p < \( \infty \) and \( q = \infty \). This time we use

\[ \Theta \]

Here \( \Theta = 1/2 \). Now Proposition 4.3 implies \( S_{p,q}^0 B(\mathbb{R}^d) \rightarrow B_{p,q}^0 (\mathbb{R}^d) \). This contradicts Theorem 3.1.

Substep 2.3. Let 0 < p < \( \infty \) and 0 < q \( \leq \) 1. We argue as in the previous step. Therefore we shall use

\[ \Theta \]

where we need \( (1 - \Theta)s + \Theta t = 0, \)

\[ \Theta \]

By choosing \( \Theta \) small we arrive at \( q_1 > 2 \). This contradicts Lemma 4.10.

Substep 2.4. We assume that \( p = \infty \) and 0 < q \( \leq \) \( \infty \). Proposition 4.6 yields

\[ \Theta \]

and

\[ \Theta \]

where \( (1 - \Theta)s + \Theta t = 0 \) and \( \frac{1}{q_1} = \frac{0}{q} \). We choose \( \Theta \) small enough such that \( q_1 > 1 \). Then, as a conclusion of Theorem 3.1 and Propositions 4.3 4.4 we get \( S_{p,q}^0 B(\mathbb{R}^d) \rightarrow \tilde{B}_{p,q}^0 (\mathbb{R}^d) \). Now we argue as in Step 3 of the proof of Lemma 4.10 by taking into account that our test functions from Example 2 are elements of \( \tilde{B}_{p,q}^0 (\mathbb{R}^d) \cap S_{p,q}^0 B(\mathbb{R}^d) \). \( \square \)
4.8. Proof of Theorem 3.6 – necessity. By means of the same arguments as used in proof of Lemma 4.10 the following dual assertion can be proved.

Lemma 4.11. Let $1 < p \leq \infty$ and $0 < q \leq \infty$. Then the embedding

$$B^0_{p,q}(\mathbb{R}^d) \hookrightarrow S^0_{p,q} B(\mathbb{R}^d)$$

implies $q \geq \max(p, 2)$.

Proof of Theorem 3.6 Step 1. Let $0 < p \leq 1$ and $t = \frac{1}{p} - 1$. Assume that there is some $q < \infty$ such that $B^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q} B(\mathbb{R}^d)$ holds. Then Proposition 4.8 yields $S^0_{\infty,q'} B(\mathbb{R}^d) \hookrightarrow B^0_{\infty,q'}(\mathbb{R}^d)$. In view of Lemma 4.11 this implies $q' \leq 1$, hence $q = \infty$.

Step 2. The necessity of the restrictions in case $t = 0$ follows by Lemma 4.11.

Step 3. It remains to deal with $t < \max(0, \frac{1}{p} - 1)$.

Step 3.1. Let $0 < p \leq \infty$ and $t < 0$. We employ the test functions from Example 1. Choosing $a_j = \delta_{j,\ell}$ in (4.11), we find

$$\| f_\ell | B^{td}_{p,q}(\mathbb{R}^d) \| = 2^{t \ell d} \| \mathcal{F}^{-1} g \ell \| L_p(\mathbb{R}^d) \|
$$

and

$$\| f_\ell | S^t_{p,q} B(\mathbb{R}^d) \| = 2^{t \ell} \| \mathcal{F}^{-1} g \ell \| L_p(\mathbb{R}^d) \|
$$

see (4.11) and (4.13). With $\ell \to \infty$ it becomes clear that

$$B^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q} B(\mathbb{R}^d)
$$

can not hold.

Step 3.2. Let $0 < p < 1$ and $0 \leq t < \frac{1}{p} - 1$. We assume $B^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q} B(\mathbb{R}^d)$. Proposition 4.8 yields

$$S^{t+\frac{1}{p} - 1}_{\infty,q'} B(\mathbb{R}^d) \hookrightarrow B^{(t+\frac{1}{p} - 1)}_{\infty,q'}(\mathbb{R}^d).
$$

Since $d(-t + \frac{1}{p} - 1) > -t + \frac{1}{p} - 1$ it is enough to use $g_k(x) := e^{ikx}, k \in \mathbb{Z}^d$, as test functions to disprove this embedding.

4.9. Proof of Propositions 3.3, 3.8. Proof of Proposition 3.3 Part (i) in case $1 \leq q \leq \infty$ follows by duality from Theorem 3.1, see Proposition 4.8. To cover also the cases $0 < q < 1$ we argue as in proof of Theorem 3.6 (sufficiency) replacing $B^{td}_{p,q}(\mathbb{R}^d)$ by $B^t_{p,q}$. Observe in this connection that

$$\sup_{\ell \in \mathbb{N}_0} \sum_{k \in \Delta_j} 2^{(t+\frac{1}{p})|k|} < \infty.
$$

Parts (ii)-(iv) are immediate consequences of Theorems 3.1, 3.6.

Now we turn to the proof of (v). Theorem 3.1 yields $S_{p,q}^t B(\mathbb{R}^d) \not\hookrightarrow B^t_{p,q}(\mathbb{R}^d)$. It remains to prove $B^t_{p,q}(\mathbb{R}^d) \not\hookrightarrow S_{p,q}^t B(\mathbb{R}^d)$. Therefore we employ Example 1 with $a_j := 2^{-j \ell}, j \in \mathbb{N}$. From (3.11) it follows

$$\| f_\ell | B^t_{p,q}(\mathbb{R}^d) \| \leq c 2^{\ell \ell} \left( \sum_{j=1}^\ell a_j e^{\pi(2^{j}x_1+2^{j}x_2)} \| L_p([-\pi, \pi]^2) \| \right)
$$

$$\leq c 2^{\ell} \left( \sum_{j=1}^\ell |a_j|^p \right)^{1/p} \approx 1.$$
On the other hand, (4.13) yields

\[ \| f_t | S_{p,q}^t (\mathbb{R}^d) \| = \| F^{-1} g | L_p (\mathbb{R}^d) \| \left( \sum_{j=1}^{\ell} 2^{jq} |a_j|^{q} \right)^{1/q} \approx \ell^{1/q}. \]

If we assume \( B_{p,q}^t (\mathbb{R}^d) \rightarrow S_{p,q}^t B (\mathbb{R}^d), \) \( q < \infty, \) then this leads to a contradiction. In case \( q = \infty \) we employ the same type of argument but choose \( a_j := 2^{-t} j^{-1}, \) \( j \in \mathbb{N}. \)

**Proof of Theorem 3.3.** To prove (i) we follow the arguments used in proof of Theorem 3.1 (sufficiency) by replacing \( B_{p,q}^t (\mathbb{R}^d) \) by \( B_{p,q}^{t \delta} (\mathbb{R}^d). \)

Concerning (ii)-(iv), observe that \( B_{p,q}^{t} (\mathbb{R}^d) \not\rightarrow S_{p,q} B (\mathbb{R}^d) \) follows from Theorem 3.6.

Now we split our investigations into two cases: \( 0 < t < \frac{1}{p} - 1, \) \( t = 0. \)

**Step 1.** Let \( 0 < p < 1 \) and \( 0 < t < \frac{1}{p} - 1. \) \( S_{p,q}^t B (\mathbb{R}^d) \not\rightarrow B_{p,q}^t (\mathbb{R}^d) \) follows from Example 1 with \( a_j := \delta_{j,t}, \) see (4.25) and (4.26).

**Step 2.** Let \( 0 < p < 1 \) and \( t = 0. \) Theorem 3.1 yields

\[ S_{p,q}^0 B (\mathbb{R}^d) \rightarrow B_{p,q}^0 (\mathbb{R}^d) \quad \iff \quad 0 < q \leq p. \]

Hence, in case \( q > p \) the spaces \( S_{p,q}^0 B (\mathbb{R}^d) \) and \( B_{p,q}^0 (\mathbb{R}^d) \) are not comparable. \( \square \)

4.10. **Proofs of the optimality assertions.** First we recall some well-known results about embeddings of Besov spaces. Let \( 0 < p \leq p_0 \leq \infty. \) Then \( B_{p,q}^t (\mathbb{R}^d) \rightarrow B_{p_0,q_0}^{t_0} (\mathbb{R}^d) \) holds if and only if either

\[ t_0 - \frac{d}{p_0} < t - \frac{d}{p} \quad \text{and} \quad q, q_0 \text{ are arbitrary} \]

or

\[ t_0 - \frac{d}{p_0} = t - \frac{d}{p} \quad \text{and} \quad q \leq q_0. \]

This result has a certain history. For the first time it has been proved by Taibleson in his series of papers [29, 31], see also [35, 2.7.1] and [26]. In case of the Besov spaces of dominating mixed smoothness the following is known. Again we suppose \( 0 < p \leq p_0 \leq \infty. \) Then \( S_{p,q}^t B (\mathbb{R}^d) \rightarrow S_{p_0,q_0}^{t_0} B (\mathbb{R}^d) \) holds if and only if either

\[ t_0 - \frac{1}{p_0} < t - \frac{1}{p} \quad \text{and} \quad q, q_0 \text{ are arbitrary} \]

or

\[ t_0 - \frac{1}{p_0} = t - \frac{1}{p} \quad \text{and} \quad q \leq q_0. \]

We refer to [24] and [10].

**Proof of Theorem 3.4.** Assuming \( S_{p_0,q_0}^t B (\mathbb{R}^d) \rightarrow B_{p,q}^t (\mathbb{R}^d) \) Lemma 4.9 implies \( p_0 \leq p. \) Next we apply Example 4 to derive some relations between \( p_0 \) and \( p. \)

We choose \( a_j := \delta_{j,t}, \) \( j = 1, \ldots, \ell. \) Then it follows

\[ \| f_t | B_{p,q}^t (\mathbb{R}^d) \| = C 2^{(t+1-1/p)} \quad \text{and} \quad \| f_t | S_{p_0,q_0}^t (\mathbb{R}^d) \| = C 2^{(t+1-1/p_0)}. \]

The assumed embedding implies

\[ t + 1 - \frac{1}{p} \leq t_0 + 1 - \frac{1}{p_0}. \]
In case $t - \frac{1}{p} = t_0 - \frac{1}{p_0}$ we employ Example 4 again with $a_j := 2^{-j}(t+1-1/p)$, $j = 1, \ldots, \ell$. As a consequence of the embedding we derive $q_0 \leq q$. This implies

$$S^{t_0}_{p_0,q_0} B(\mathbb{R}^d) \hookrightarrow S^t_{p,q} B(\mathbb{R}^d),$$

see the above comments to embeddings of Besov spaces. □

**Proof of Theorem 3.11** Assuming $B^{t_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow S^t_{p,q} B(\mathbb{R}^d)$ Lemma 4.9 implies $p_0 \leq p$. Next we employ Example 5 where the $a_j := \delta_j, j = 1, \ldots, \ell$. We obtain

$$\| f_\ell \|_{S^t_{p,q} B(\mathbb{R}^d)} = C 2^{d(t+1-\frac{1}{p})} \quad \text{and} \quad \| f_\ell \|_{B^{t_0}_{p_0,q_0}(\mathbb{R}^d)} = C 2^{d(t_0+1-\frac{1}{p_0})}$$

with $C > 0$ independent of $\ell$. The embedding $B^{t_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow S^t_{p,q} B(\mathbb{R}^d)$ yields

$$d \left( t_0 + 1 - \frac{1}{p_0} \right) \geq d \left( t + 1 - \frac{1}{p} \right) \iff t_0 - \frac{d}{p_0} \geq t - \frac{d}{p}.$$ 

Now, if $t_0 - \frac{d}{p_0} = dt - \frac{d}{p}$, we apply Example 5, again with $a_j := 2^{-j(t_0+1-1/p_0)}$, to obtain $q_0 \leq q$. All together we conclude

$$B^{t_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow B^{t_{ld}}_{p,q} B(\mathbb{R}^d),$$

see the above comments on embeddings. □

**Proof of Theorem 3.11** Assuming $B^{t_{ld}}_{p,q}(\mathbb{R}^d) \hookrightarrow S^{t_0}_{p_0,q_0} B(\mathbb{R}^d)$ Lemma 4.9 implies $p \leq p_0$. Example 5 with $a_j := \delta_j, j = 1, \ldots, \ell$, yields

$$\| f_\ell \|_{S^{t_0}_{p_0,q_0} B(\mathbb{R}^d)} = C 2^{d(t_0+1-\frac{1}{p_0})} \quad \text{and} \quad \| f_\ell \|_{B^{t_{ld}}_{p,q}(\mathbb{R}^d)} = C 2^{d(t+1-\frac{1}{p})}$$

with $C > 0$ independent of $\ell$. The embedding $B^{t_{ld}}_{p,q}(\mathbb{R}^d) \hookrightarrow S^{t_0}_{p_0,q_0} B(\mathbb{R}^d)$ implies

$$d \left( t_0 + 1 - \frac{1}{p_0} \right) \leq d \left( t + 1 - \frac{1}{p} \right) \iff t_0 - \frac{1}{p_0} \leq t - \frac{1}{p}.$$ 

Working with Example 5 in the case $t_0 - \frac{1}{p_0} = t - \frac{1}{p}$, choose $a_j := 2^{-j(t+1-1/p)}$, we obtain $q \leq q_0$. Taking into account the above comments on embeddings we arrive at $S^{t_0}_{p,q} B(\mathbb{R}^d) \hookrightarrow S^{t_0}_{p_0,q_0} B(\mathbb{R}^d)$. □

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