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Uniform asymptotics of area-weighted Dyck paths

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Using the generalized method of steepest descents for the case of two coalescing saddle points, we derive an asymptotic expression for the bivariate generating function of Dyck paths, weighted according to their length and their area in the limit of the area generating variable tending towards 1. The result is valid uniformly for a range of the length generating variable, including the tricritical point of the model. © 2015 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4917052]

I. INTRODUCTION

A Dyck path is a trajectory of a directed random walk on the two-dimensional square lattice above the diagonal, starting at the origin and ending on this diagonal. More precisely, the random walk starts at the point $(0,0)$ and, from any given point $(x,y)$, the random walker can only step towards $(x+1,y)$ and $(x,y+1)$. Steps from $(x,y)$ to $(x-1,y)$ or to $(x,y-1)$ are forbidden. Furthermore, each point on the trajectory $(x,y)$ must satisfy $x \leq y$, which means that the random walker always stays above the main diagonal $x = y$, and its final position $(x,y)$ must be on the main diagonal. Figure 1 shows an example of a Dyck path as it is usually drawn with the lattice oriented such that the main diagonal lies horizontally in the image. As a possible physical system which can be described by Dyck paths, one can think of two-dimensional vesicles attached to a surface or of discrete trajectories of charged particles moving in an external magnetic field. Since we will only deal with Dyck paths in the following, we will refer to them simply as “paths.”

The length of a path is the number of steps it consists of. Since the number of horizontal steps must equal the number of vertical steps, it follows that the length of a path is always even. Further, the area of a path is the number of entire unit squares enclosed between the trajectory and the main diagonal. For example, the path in Figure 1 has length $n = 18$ and area $m = 10$ (occasionally, an alternative definition of the area under a path as the number of triangular plaquettes enclosed by the trajectory and the main diagonal is used; see, e.g., Ref. 3).

The generating function of paths of length $2n$, weighted according to their area is defined as

$$Z_n(q) = \sum_{m=1}^{\infty} c_{m,n} q^m,$$

where $c_{m,n}$ is the number of paths of length $2n$ and area $m$ and $q$ is the weight associated to the area. In physical terms, one can interpret $Z_n(q)$ as the canonical partition function of two-dimensional surface-attached vesicles with perimeter $4n$ and area-fugacity $q$. Note that the maximum area of a path of given finite length is bounded; therefore, the sum on the RHS of (1) is finite for finite values of $n$. As a physical consequence, phase transitions can only occur in the thermodynamic limit $n \to \infty$.

One can also define the generating function of paths of area $m$, weighted with respect to their length as

$$Q_m(t) = \sum_{n=0}^{\infty} c_{m,n} t^n,$$

where $c_{m,n}$ is the number of paths of area $m$ and length $2n$ and $t$ is the weight associated to the length.

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where \( t \) is the weight conjugate to the length. The generating function of paths weighted according to both their area and their length is defined as

\[
G(t,q) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{m,n} q^m t^n. \tag{3}
\]

With (1), this can be rewritten as

\[
G(t,q) = \sum_{n=1}^{\infty} Z_n(q) t^n \tag{4}
\]

and with (2), we can write

\[
G(t,q) = \sum_{m=1}^{\infty} G_m(t) q^m. \tag{5}
\]

By a standard factorization argument, one obtains the functional equation

\[
G(t,q) = 1 + t G(t,q) G(qt,q), \tag{6}
\]

which can be solved by using the ansatz,

\[
G(t,q) = \frac{H(qt)}{H(t)}. \tag{7}
\]

Here,

\[
H(t) = \sum_{n=0}^{\infty} q^{n^2-n} (-t)^n \frac{1}{(q; q)_n}, \tag{8}
\]

and we have used the standard notation for the q-Pochhammer symbol,

\[
(z; q)_n = \prod_{k=0}^{n-1} (1 - zq^k), \tag{9}
\]

which is a q-generalization of the Pochhammer symbol. The function \( H(qt) = Ai_q(t) \) is a q-Airy function. For \( q = 1 \), (6) is solved by the generating function of the Catalan numbers (see, e.g., Ref. 14),

\[
G(t,1) = \frac{1}{2t} (1 - \sqrt{1 - 4t}). \tag{10}
\]

In Figure 2, we show how the radius of convergence \( t_\infty \) of the series on the RHS of Eq. (4) behaves qualitatively as a function of \( q \). This picture, which is also called the “phase diagram” of the system, is typical for lattice polygon models. The radius of convergence is defined by a decreasing line of pole singularities for \( q < 1 \) and zero for \( q > 1 \). From (10), the value of \( t_\infty \) for \( q = 1 \) can be deduced to be \( t_c = 1/4 \). The point \( (t,q) = (t_c,1) \) is called the tricritical point of the model (see, e.g., Ref. 7) and the area below \( t_\infty(q) \) is called the finite size region.
FIG. 2. The qualitative behaviour of the radius of convergence $t_\infty$ of $G(t,q)$ as a function of $q$. The small circle marks the tricritical point.

In Ref. 3, the general form of continued fraction expressions for generating functions of Dyck and Motzkin paths has been discussed. In particular,

$$G(t,q) = \frac{1}{1 - \frac{t}{1 - \frac{tq}{1 - \frac{tq^2}{1 - \frac{tq^3}{1 - \ldots}}}}}. \quad (11)$$

This expression enables us to continue $G(t,q)$ analytically beyond the finite size region.

The asymptotic behaviour of $G(t,q)$ for $q \to 1^-$ as one approaches the tricritical point has so far not been derived rigorously. The aim of this paper is to close this gap by rigorously deriving an asymptotic expression for the generating function $G(t,q)$ in the limit $q \to 1^-$ which is valid uniformly for a range of values of $t$ including the critical point $t_c$. A similar calculation has been carried out in Ref. 8 for staircase polygons.

Our main result is given in Proposition 2.1 and Corollary 2.2.

II. RESULTS

According to Eq. (7), the function $G(t,q)$ is given as a quotient of two alternating $q$-series. In order to obtain its asymptotic behaviour, we first derive the asymptotic behaviour of both the numerator and the denominator separately. Taking the fraction of the two obtained expressions will then lead us to the asymptotic behaviour of $G(t,q)$. We will start with the asymptotic expansion of $H(t)$.

A. Uniform asymptotic expansion of $H(t)$

The first step in our calculation is to express $H(t)$ as a contour integral. We prove

**Lemma 2.1.** For complex $t$ and $0 < q < 1$,

$$H(t) = \frac{(q; q)_\infty}{2\pi i} \int_C \frac{z^{(1+\log_q z)/2-\log_q t}}{(z; q)_\infty} dz, \quad (12)$$

where $C$ is a contour as shown in Figure 3(b).

**Proof.** For complex $q \neq 0$ and $n \in \mathbb{N}_0$, we have
The contours $0 < \varphi < \pi$ and $0 < \psi < \pi$, from which it follows that

$$(-1)^n q^{\frac{n^2}{2}} \frac{(q; q)_n(q; q)_\infty}{(q; q)_\infty} = -\text{Res} \left[ (z; q)_\infty^{-1} ; z = q^{-n} \right],$$

from which it follows that

$$(-1)^n q^{n^2-n} \frac{(q; q)_n(q; q)_\infty}{(q; q)_\infty} = -\text{Res} \left[ \frac{z^{(1+\log_q z)/2 - \log_q t}}{(z; q)_\infty} ; z = q^{-n} \right].$$

Suppose now that $0 < q < 1$. Then, the contour $C_N = C_N^1 \cup C_N^2 \cup C_N^3$, where

$$C_N^1 = \{ \rho + \lambda e^{-i\varphi} \mid 0 < \lambda < q^{-N-1/2} \} \quad (15)$$

$C_N^2 = \{ \rho + \lambda e^{i\varphi} \mid 0 < \lambda < q^{-N-1/2} \} \quad (15)$

$C_N^3 = \{ \rho + q^{-N-1/2} e^{i\psi} \mid -\psi < \theta < \varphi \} \quad (15)$

$0 < \rho < 1$ and $(\varphi, \psi) \in \{0, \pi\}^2$, surrounds exactly the $N$ leftmost singularities of the integrand on the RHS of Eq. (12)—see Figure 3(a). We can therefore write

$$\sum_{n=0}^{N} q^{n^2-n} (-1)^n \frac{(q; q)_n}{(q; q)_\infty} = \frac{(q; q)_\infty}{2\pi i} \oint_{C_N} \frac{z^{(1+\log_q z)/2 - \log_q t}}{(z; q)_\infty} dz,$$

where the integration is performed in clockwise sense, as indicated by the arrows in Figure 3(a). Combining (8) and (16), we obtain

$$H(t) = \lim_{N \to \infty} \frac{(q; q)_\infty}{2\pi i} \oint_{C_N} \frac{z^{(1+\log_q z)/2 - \log_q t}}{(z; q)_\infty} dz.$$  

It is left to show that in the limit $N \to \infty$, the contribution of the circle segment $C_N^3$ to contour integral (17) vanishes, such that the contour $C_N$ can be replaced by the contour shown in Figure 3(b). On $C_N^3$, we can estimate the denominator of the integrand on the RHS of (17) as

$$\left| (z, q)_\infty \right| = \prod_{n=0}^{\infty} \left( 1 - q^{-\frac{N}{2} + n} e^{i\varphi} \right) \geq \prod_{n=0}^{\infty} \left( 1 - q^{-N-1/2+n} \right)$$

$$= \prod_{n=0}^{\infty} (1 - q^{-1/2+n}) \cdot \prod_{n=1}^{N} (1 - q^{-1/2-n}) \geq c_1 \prod_{n=1}^{N} q^{-1/2-n} = c_1 |q|^{-N^2/2-N},$$

where $c_1$ is a constant independent of $N$. Furthermore, the absolute value of the enumerator has for $z \in C_N^3$ the upper bound

$$\left| z^{(1+\log_q z)/2 - \log_q t} \right| \leq c_2 |q|^{N^2/2} |t|^N,$$

where $c_2$ is another constant independent of $N$. Therefore, we can estimate
where \( c_3 \) is a third constant independent of \( N \). Since the expression on the right hand side tends to zero as \( N \to \infty \) for \( q < 1 \), it follows that

\[
\lim_{N \to \infty} \frac{1}{2\pi i} \oint_{C_N} \frac{z^{(1+\log_q z)/2-\log_q t}}{(z; q)_\infty} \, dz = \frac{1}{2\pi i} \int_C \frac{z^{(1+\log_q z)/2-\log_q t}}{(z; q)_\infty} \, dz,
\]

(21)

where

\[
C = \{ \rho + \lambda e^{-i\phi} \mid 0 < \lambda < \infty \} \cup \{ \rho + \lambda e^{i\phi} \mid 0 < \lambda < \infty \},
\]

(22)

and where the integration is carried out as indicated by the arrows in Figure 3(b). Combining (21) with (17), we obtain (12).

In Ref. 8, it was shown by applying the Euler-Maclaurin summation formula that

\[
\ln(z; q)_\infty = \frac{1}{\log(q)} \text{Li}_2(z) + \frac{1}{2} \ln(1 - z) + \ln(q) R(z, q),
\]

(23)

where the remainder satisfies the non-uniform bound

\[
|R(z, q)| \leq \frac{1}{6} \left( \ln |1 - z| + \frac{\text{Re}(z)}{\text{Im}(z)} \arctan \frac{\text{Im}(z)}{1 - \text{Re}(z)} \right).
\]

(24)

Here, \( \text{Li}_2 \) denotes the Euler dilogarithm, which can be defined as

\[
\text{Li}_2(z) = -\int_0^z \frac{\ln(1 - s)}{s} \, ds.
\]

(25)

Combining (12) with (23) and defining

\[
f(z, t) = \ln(t) \ln(z) + \text{Li}_2(z) - \frac{1}{2} \ln(z)^2,
\]

(26a)

\[
g(z) = \left( \frac{z}{1 - z} \right)^{1/2},
\]

(26b)

we can write

\[
H(t) = \frac{\langle q; q \rangle_\infty}{2\pi i} \int_C \exp \left( \frac{1}{\epsilon} f(z, t) + \epsilon R(z, q) \right) g(z) \, dz,
\]

(27)

where \( \epsilon = -\ln(q) \).

The function \( f(z, t) \) is analytic for \( \arg(z) < \pi \) and \( \arg(1 - z) < \pi \) and has the two saddle points

\[
z_1(t) = \frac{1}{2} \left( 1 + \sqrt{1 - 4t} \right), \quad z_2(t) = \frac{1}{2} \left( 1 - \sqrt{1 - 4t} \right),
\]

(28)

which coalesce for \( t = t_c = 1/4 \).

From the identity

\[
\text{Li}_2(\lambda e^{i\phi}) = -\frac{1}{2} \ln(-\lambda e^{i\phi})^2 - \frac{\pi^2}{6} - \text{Li}_2\left( \frac{1}{\lambda} e^{-i\phi} \right)
\]

(29)

(see, e.g., Ref. 10), we obtain

**Lemma 2.2.** For \( 0 < |\phi| \leq \pi \),

\[
\text{Li}_2(\lambda e^{i\phi}) \sim -\frac{1}{2} \ln(-\lambda e^{i\phi})^2,
\]

(30a)

\[
\text{Re}[\text{Li}_2(\lambda e^{i\phi})] \sim -\frac{1}{2} \ln(\lambda)^2,
\]

(30b)

\[
\text{Im}[\text{Li}_2(\lambda e^{i\phi})] \sim -\frac{1}{2} \text{Im}[\ln(-\lambda e^{i\phi})^2]
\]

(30c)

as \( \lambda \to \infty \).
Consequently, we have the following.

**Corollary 2.1.** For complex $t$ and $0 < \phi < \pi$,

$$f(\lambda e^{i\phi}, t) \sim -\ln(\lambda)^2 - i\psi \ln(\lambda) \quad \text{as} \quad \lambda \to \infty,$$  

(31)

where $\psi = 2\phi + \pi$ for $\phi < 0$ and $\psi = 2\phi - \pi$ for $\phi > 0$.

The remainder $R(z, q)$ is not uniformly bounded with respect to $z$; therefore, it is not immediately clear that it can be neglected in the limit $\epsilon \to 0^+$. However, from the asymptotic behaviour of $f(z, t)$, one can conclude that the tails of the integration contour do not contribute to the asymptotics of the integral. Therefore, we have

**Lemma 2.3.** For complex $t$ and $0 < q < 1$,

$$H(t) \sim \frac{(q; q)_\infty}{2\pi i} \int_C \exp \left( \frac{1}{\epsilon} f(z, t) \right) g(z) \, dz \quad \text{as} \quad \epsilon \to 0^+. \tag{32}$$

It was shown in Ref. 11 (see also Refs. 12 and 13) that for a function $f(z, t)$, which is analytic with respect to both $z$ and $t$ and which has two saddle points $z_1(t)$ and $z_2(t)$, there is a unique transformation $u : z \mapsto u(z)$, such that

$$f(z, t) = \frac{1}{3} u^3 - \alpha(t) u + \beta(t), \quad \text{as} \quad t \to \pm \infty,$$  

(33)

which is regular and one-to-one in a domain containing $z_1(t)$ and $z_2(t)$ if $t$ lies in some small domain containing $t_c$. Moreover,

$$u(z_1(t)) = |\alpha(t)|^{1/2}, \quad u(z_2(t)) = -|\alpha(t)|^{1/2}. \tag{34}$$

Combining (33) with (34), one gets the explicit form

$$u(z) = \left[ \left( \frac{3}{2} f(z, t) - \beta \right)^2 + \left( \frac{3}{2} f(z, t) - \beta \right)^2 - \frac{1}{2} \alpha^3 \right]^{1/3} +$$

$$+ \alpha \left[ \left( \frac{3}{2} f(z, t) - \beta \right)^2 + \left( \frac{3}{2} f(z, t) - \beta \right)^2 - \frac{1}{2} \alpha^3 \right]^{1/2} - \frac{1}{3}, \tag{35}$$

where the two parameters are obtained by inserting (34) into (33) as

$$\alpha(t) = \left( \frac{3}{4} \left[ f(z_2, t) - f(z_1, t) \right] \right)^{2/3},$$

$$\beta(t) = \frac{1}{2} \left( f(z_1, t) + f(z_2, t) \right) = \frac{1}{2} \ln(t^2 + \frac{\pi^2}{6}). \tag{36}$$

Note that for $t \to t_c$, one has $\alpha(t) \sim 1 - 4t$. From Corollary 2.1, it follows that

$$u(\rho + i\lambda) \sim \exp \left( \pm \frac{\pi}{3} \sqrt{\lambda} \right) \frac{3}{2} \ln |\lambda|^{1/3}, \tag{37}$$

for $\lambda \to \pm \infty$. This leads us to

**Lemma 2.4.** For complex $t$,

$$H(t) \sim \frac{(q; q)_\infty}{2\pi i} \int_C \frac{1}{\epsilon} \left( \frac{1}{3} u^3 - \alpha u + \beta \right) g(z(u)) \frac{dz}{du} \, du \tag{38}$$

as $\epsilon \to 0^+$, where $C$ is a contour as shown in Figure 3(b) with $\varphi = \psi = \pi/3$. 

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It is possible to write
\[
g(z(u)) \frac{dz}{du} = \sum_{m=0}^{\infty} (p_m + u q_m)(u^2 - \alpha)^m
\] (39)
and insert this expansion into (38). Interchanging the order of integration and summation, we obtain the asymptotic expansion,
\[
H(t) \sim \frac{(q; q)_{\infty}}{2\pi i} \sum_{m=0}^{\infty} \int_C (p_m + u q_m)(u^2 - \alpha)^m e^{\frac{1}{3}u^3 - \alpha u + \beta} \, du.
\] (40)
The two leading coefficients can be obtained from (33) and (34). We get
\[
2g(z_t) \sqrt{\frac{\alpha}{f''(z_t)}} = p_0 + \alpha^{1/2} q_0, \quad 2g(z_2) \sqrt{\frac{\alpha}{f''(z_2)}} = p_0 - \alpha^{1/2} q_0,
\] (41)
and these two equations can be solved with respect to \(p_0\) and \(q_0\), respectively, to obtain
\[
p_0 = \left(\frac{\alpha}{d}\right)^{1/2} \left(z_{1/2}^2 + z_{2/2}^2\right), \quad q_0 = \left(\frac{1}{\alpha d}\right)^{1/2} \left(z_{1/2}^3 - z_{2/2}^3\right).
\] (42)
Here, we have set \(d = 1 - 4t\).
Inserting (42) into (40), we arrive at an asymptotic expression for \(H(t)\) in terms of the Airy function,
\[
\operatorname{Ai}(z) = \int_C \exp\left(\frac{w^3}{3} - zw\right) \, dw.
\] (43)
This presents the main result of this section.

**Lemma 2.5.** For complex \(t\),
\[
H(t) \sim \frac{(q; q)_{\infty}}{2\pi i} \left(\frac{1}{\alpha d}\right)^{1/2} \exp\left(\frac{\beta}{\epsilon}\right) \left(\alpha^{1/2} e^{1/3}(z_{1/2}^3 + z_{2/2}^3)\operatorname{Ai}(\alpha \epsilon^{-2/3}) + \epsilon^{2/3}(z_{2/2}^3 - z_{1/2}^3)\operatorname{Ai}'(\alpha \epsilon^{-2/3})\right) \quad \text{as} \ \epsilon \to 0^+.
\] (44)

**B. Uniform asymptotic expansion of** \(H(qt)\)

The approach used in Sec. II A can be applied in a completely analogous way to \(H(qt)\). The function \(f(z, t)\) remains the same, whereas now
\[
g(z) = \left(\frac{1}{z(1 - z)}\right)^{1/2}.
\] (45)
This changes the leading coefficients of the expansion towards
\[
p_0 = \left(\frac{\alpha}{d}\right)^{1/2} \left(z_{1/2}^2 + z_{2/2}^2\right), \quad q_0 = \left(\frac{1}{\alpha d}\right)^{1/2} \left(z_{1/2}^2 - z_{2/2}^2\right),
\] (46)
and we obtain the following.

**Lemma 2.6.** For complex \(t\),
\[
H(qt) \sim \frac{(q; q)_{\infty}}{2\pi i} \left(\frac{1}{\alpha d}\right)^{1/2} \exp\left(\frac{\beta}{\epsilon}\right) \left(\alpha^{1/2} e^{1/3}(z_{1/2}^3 + z_{2/2}^3)\operatorname{Ai}(\alpha \epsilon^{-2/3}) + \epsilon^{2/3}(z_{2/2}^3 - z_{1/2}^3)\operatorname{Ai}'(\alpha \epsilon^{-2/3})\right) \quad \text{as} \ \epsilon \to 0^+.
\] (47)
In Figure 5, we have plotted

\[ G(t, q) \]

where the critical exponents are

\[ a = \frac{1}{2} \quad \text{and} \quad b = 1. \]

We can further show that the convergence is uniform in the Catalan numbers; hence, our result is consistent with (10). By applying Dini’s theorem, one can easily show that for \( t < t_c \), though not in a uniform sense due to the occurrence of poles in the denominator. This fact is illustrated in Figure 4, where we have plotted \( G(t, q) \) against the uniform asymptotic expression (48) for \( \epsilon = 10^{-2} \) and \( t \) ranging between 0 and 1/2 (horizontal axis).

C. Uniform asymptotics of \( G(t, q) \)

Combining Lemmas 2.5 and 2.6, we arrive at the following.

**Proposition 2.1.** For complex \( t \) and \( q \to 1^- \),

\[
G(t, q) \sim \frac{\alpha^{1/2}(\epsilon^{1/2} + \epsilon^{3/2})}{\alpha^{1/2}(\epsilon^{1/2} - \epsilon^{3/2})} \text{Ai}(\alpha \epsilon^{2/3}) + (\epsilon^{1/2} - \epsilon^{3/2}) \text{Ai}'(\alpha \epsilon^{-2/3})
\]

(48)

where \( \epsilon = -\ln(q) \),

\[
z_1 = \frac{1}{2}(1 + \sqrt{1 - 4t})
\]

\[
z_2 = \frac{1}{2}(1 - \sqrt{1 - 4t})
\]

and \( \alpha(t) \) is given by Eq. (36).

One easily shows that for \( t \leq t_c \) and \( q \to 1^- \), \( G(t, q) \) tends towards the generating function of the Catalan numbers; hence, our result is consistent with (10). By applying Dini’s theorem, one can further show that the convergence is uniform for \( t \in [0, t_c] \). However, (48) is also valid for \( t > t_c \), though not in a uniform sense due to the occurrence of poles in the denominator. This fact is illustrated in Figure 4, where we have plotted \( G(t, q) \) against the uniform asymptotic expression (48) for \( \epsilon = 10^{-2} \) and \( t \in [0, 1/2] \).

Defining the tricritical scaling function

\[
F(s) := \frac{\text{Ai}'(s)}{\text{Ai}(s)},
\]

(49)

we can conclude

**Corollary 2.2.** For fixed \( s = (1 - 4t) \cdot (1 - q)^{-\phi} \) and \( q \to 1^- \),

\[
G(t, q) \sim 2 \left[ 1 + (1 - q)^{-\gamma_0} F((1 - 4t)(1 - q)^{-\phi}) \right],
\]

(50)

where the critical exponents are \( \phi = 2/3 \) and \( \gamma_0 = -1/3 \).

In particular,

\[
G(t_c, q) \sim 2 \cdot \left( 1 + A_0 \cdot (1 - q)^{-\gamma_0} \right) \quad (q \to 1^-),
\]

(51)

where \( A_0 = \text{Ai}'(0)/\text{Ai}(0) = -0.72 \ldots \)

Both (48) and (50) can be rearranged in order to obtain an asymptotic expression for \( F(s) \). In Figure 5, we have plotted \(-2 F(s)\) against the expressions obtained from (48) and (50) for \( \epsilon = 10^{-3}, \ 10^{-4}, \) and \( 10^{-5} \). For fixed value of \( q \), (48) provides a more accurate approximation than (50).
D. Finite size scaling

Given observed scaling behaviour (50) of the generating function, we now aim to calculate an expression for the finite size scaling function of fixed-area partition function (2). For this purpose, we first derive a further asymptotic expression for $G(t,q)$.

We define the singular part of $G(t,q)$ as

$$G_{\text{sing}}(t,q) = G(t,q) - \frac{1}{2t}$$

and prove

**Proposition 2.2.** Let $I = [t_0, t_c]$, where $0 < t_0 \leq t_c$. Then,

$$G_{\text{sing}}(t,q) \sim \frac{1}{2t} \left( 1 - q \right)^{-\gamma_0} F((1 - 4t)(1 - q)^{-\phi}) \quad (q \to 1^-),$$

uniformly for $t \in I$.

**Proof.** By using that $F(s) \sim -\sqrt{s}$ for $s \to \infty$, one easily sees that for $t \in I$, the RHS of (53) converges in a pointwise sense towards $C_{\text{sing}} := -\sqrt{1 - 4t/2t}$, which is a continuous function on $I$.

One also proves easily that for all $t \in I$, the RHS of (53) decreases monotonically with $q$. It therefore follows from Dini’s theorem that the RHS converges uniformly to $C_{\text{sing}}(t)$. It is also clear that the same holds for the LHS, $G_{\text{sing}}(t,q)$. Hence, both sides of (53) have the same uniform asymptotic expression and are therefore uniformly asymptotic to each other. \qed

It is possible to derive the Hadamard product expression,

$$A_i(s) = A_i(0) \exp(-A_0 s) \prod_{k=1}^{\infty} \left( 1 - \frac{s}{s_k} \right),$$

where $s_k$ is the $k$th zero of the Airy function. Taking the derivative of the logarithm of this expression and exercising some careful analysis, we get

$$F(s) = -\frac{1}{s} \sum_{j=1}^{\infty} Z(j) s^j,$$

where we have used the Airy Zeta function,

$$Z(j) = \sum_{k=1}^{\infty} \left( \frac{1}{s_k} \right)^j$$

and inserted the conjectured value $Z(1) = -A_0(0)/A_0(0)$.

FIG. 5. Plot of the scaling function $F(s)$ (black) against asymptotic expressions (48) (a) and (50) (b) (grey) for $\epsilon = 10^{-3}, 10^{-4}$, and $10^{-5}$ (the smallest value of $\epsilon$ corresponds to the closest approximation).
Inserting (55) into (53), we obtain that for \( m \to \infty \),
\[
G_{\text{sing}}(t, q) \sim -\sum_{j=0}^{\infty} Z(j+1)(1-4t)^j(1-q)^{-2/3j+1/3}
\]
\[
= -\sum_{m=0}^{\infty} \sum_{j=0}^{\infty} Z(j+1)(1-4t)^j \left( m + \frac{2}{3} j - \frac{4}{3} \right) q^m.
\]  
(57)

For \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \), Theorem VI.1 from Ref. 4 states that
\[
[z^n](1-z)^{-\alpha} = \binom{n + \alpha - 1}{n} \sim n^{\alpha-1} \frac{1 + \alpha(\alpha-1)}{2n}.
\]  
(58)

Inserting the leading order of this expansion into (57) and extracting the \( m \)th coefficient, one formally arrives at
\[
Q_m(t) \sim m^{-4/3} \sum_{j=0}^{\infty} \frac{Z(j+1)}{\Gamma\left(\frac{2}{3} j - \frac{4}{3}\right)} m^{2/3j} (1-4t)^j \quad (m \to \infty).
\]  
(59)

Defining the finite size scaling function
\[
\phi(s) := \sum_{j=0}^{\infty} \frac{Z(j)}{\Gamma\left(\frac{2}{3} j - \frac{4}{3}\right)} s^j,
\]  
(60)

we can rewrite (59) as
\[
Q_m(t) \sim m^{-4/3} \phi((1-4t)m^{2/3}).
\]  
(61)

This expression is of the generic form expected for models which exhibit tricritical scaling.\(^7\)^\(^{17}\)

### III. CONCLUSION

We have calculated an asymptotic expression for the generating function of Dyck paths, weighted with respect to both their perimeter and their area in the limit of the area generating variable tending towards 1. The result is valid uniformly for a range of values of the perimeter generating variable, including the tricritical point.

In the limit of both the perimeter and the area generating variable tending towards their critical values, we have shown the existence of a scaling function, expressible via Airy functions and their derivatives. The same type of scaling expression has been proven before to hold in the case of staircase polygons.\(^8\)

Note in particular that the scaling function is obtained as a particular limit of the uniform asymptotic expansion. This is in contrast to the behaviour found in Ref. 8, where the scaling function and the uniform asymptotic expansion are related by a local variable transformation. In Ref. 17, uniform asymptotic expansions for tricritical phase transitions were also constructed from scaling functions using such transformations.

From the scaling function of the generating function, we derived an expression for the scaling function of the finite-area partition function.

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