Estimation of anthracnose dynamics by nonlinear filtering

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Abstract

In this paper, we apply the nonlinear filtering theory to the estimation of the partially observed dynamics of anthracnose which is a phytopathology. The signal here is the inhibition rate and the observations are the fruit volume and the rotten volume. We propose stochastic models based on the deterministic models given in the references [21, 22], in order to represent the noise introduced by uncontrolled variation on parameters and errors on the measurements. Under the assumption of Brownian noises we prove the well-posedness the models either they take into account the space variable or not. The filtering problem is solved for the non-spatial model giving Zakai and Kushner-Stratonovich equations satisfied respectively by the unnormalized and the normalized conditional distribution of the signal with respect to the observations. A prevision problem and a discrete filtering problem are also studied for the realistic cases of discrete and possibly incomplete observations. We illustrate the filter behaviour through numerical simulations corresponding to different scenarios

KeyWords — Anthracnose modelling, State estimation, Nonlinear filtering.

AMS Classification — 60H15, 60H10, 93E11, 93E10.

1 Introduction

Anthracnose is a phytopathology which occurs on several commercial tropical crops. Among them the coffee is concerned by the coffee berry disease (CBD) caused by the Colletotrichum kahawae which is an ascomycete fungus [4, 5, 13, 27, 35, 39, 47]. In order to understand, predict and control the disease dynamics, several models have been proposed in the literature [16, 18, 19, 34, 35, 36, 37, 38, 47]. Recently, in [21] and [23], an evolution model with spatial diffusion has been studied for anthracnose control. Optimal strategies were computed with respect to given cost functionals. The general model surveyed in [21] was given by the following equations:

\[
\frac{\partial \theta}{\partial t} = \alpha (t, x) (1 - w(t, x) \theta) + \text{div} (A(t, x) \nabla \theta), \quad \text{on} \quad \mathbb{R}^*_+ \times U
\]

\[
\frac{\partial v}{\partial t} = \frac{\beta (t, x, \theta)}{\eta (t, x) v_{\max}} \left( \eta (t, x) v_{\max} - \frac{v}{1 - \theta} \right)
\]

\[
\frac{\partial v_r}{\partial t} = \frac{\gamma (t, x, \theta)}{v} (v - v_r)
\]

\[
\theta (0, x) \in [0, 1], \quad x \in \overline{U} \subseteq \mathbb{R}^3
\]

\[
(v (0, x), v_r (0, x)) \in [0, v_{\max}] \times [0, v_{\max}], \quad x \in \overline{U} \subseteq \mathbb{R}^3
\]

and

\[
\langle A(t, x) \nabla \theta (t, x), n(x) \rangle = 0, \quad \text{on} \quad \mathbb{R}^*_+ \times \partial U
\]

where \( n(x) \) denotes the normal vector on the boundary at \( x \) and

\[
w(t, x) = \frac{1}{1 - \sigma u(t, x)}.
\]

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In the model above $\theta$ denotes the inhibition rate. The state variables $v$ and $v_r$ are respectively the fruit volume density and the rot volume density. The density $v$ is upper bounded by a value $v_{\text{max}}$ which models the natural fact that the fruit growth is limited. Nonnegative functions $\alpha, \beta, \gamma$ characterize the effects of environmental and climatic conditions on the rate of change of inhibition rate, fruit volume, and infected fruit volume respectively \[16, 18, 19\]. There is a control parameter $u$ representing the chemical strategy consisting on the effects after application of fungicides. The parameter $1 - \sigma \in [0, 1]$ models the positive inhibition rate corresponding to epidermis penetration by hyphae. Once the epidermis has been penetrated, the inhibition rate cannot fall below this value, even under maximum control effort ($u = 0$). Without any control effort ($u = 0$), the inhibition rate should increase towards 1. The environmental and climatic conditions affect the maximum fruit volume through the $[0, v_{\text{max}}]$-valued function $\eta$. The term $\text{div}(A\nabla \theta)$ refers to the spatial spread of the disease in the open domain $U \subset \mathbb{R}^3$ which is assumed of class $C^1$. The boundary condition $\langle A\nabla \theta, n \rangle = 0$ where $A$ is a $3 \times 3$-matrix $(a_{ij})$ could be understood as the law steering migration of the disease between $U$ and its exterior. For instance, if $A$ is reduce to $I$ the identity matrix then $\langle \nabla \theta, n \rangle = 0$ means that the domain $U$ has no exchange with its exterior. The model in \[23\] has a similar form with the model given above. However the authors added a new control strategy by impulses representing the harvesting of pathogens with a given frequency.

In several cases, especially for the results in \[21, 23\] on anthracnose disease, optimal control strategies are given such as a feedbacks and need to know the current state of the system and parameters values. It is difficult in general to know exactly the trajectory of dynamic system. Unfortunately the dynamics of the inhibition rate of anthracnose is not exempt from that fact. However, it is more easier to observe volumes $v$ and $v_r$. On the other hand, the global evolution of the system is subject to pertubations coming from several sources. For instance, parameters of the models vary depending on random climatic conditions and are often estimated such as statistical averages. We can also mention errors occurring even during every measurement of any output of the pathosystem. Those perturbations could be taken into account through stochastic noises. The stochastic framework presents several advantages related to the use of large tools developed in probability theory. As said before a probabilistic model enables to introduce the randomness for some events. It also permits based on several observations to smooth the model with better parameters. Another interest of stochastic model is the possibility of estimation either of parameters or hidden states using the displayed other states of the system. That last issue has been widely studied in the framework of hidden Markov processes [20]. The corresponding attempts of solution in the large literature on the topic has been regrouped on the name "filtering" \[2, 20, 43\].

The aim of this paper is to propose a stochastically noised model of anthracnose and to apply the filtering theory for the estimation of the inhibition rate assuming that volumes $v$ and $v_r$ are observed. In the remainder there is the following organization. In the Section 2 we recall useful definitions adopt some notations that will be used later. The Section 3 focuses on modelling and studying the well-posedness of the noised dynamics of anthracnose either for the spatially distributed model or not. In the Section 4 we apply the filtering theory in order to determine the law of the inhibition rate conditionally to the fruit volume and the rotted volume. However, we first give in the Subsection 4.1 an equivalent model which is more suitable for the filtering procedure. Filtering equations are derived into the Subsection 4.2. The Section 5 is concerned by resolution of a prevision problem in Subsection 5.1 and a discrete filtering problem in Subsection 5.2. We realize and discuss several simulations in Section 6 in order to illustrate the behaviour of the filter for different scenarios. Finally, the paper ends with a global discussion in Section 7.

## 2 Preliminaries

In this Subsection and the remaining of the paper we consider a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t \geq 0}$ such that $\mathcal{F}_0$ contains all negligible sets. Let $E$ denote a Banach space, $\mathcal{B}_E$ (or simply $\mathcal{B}$ when there is not ambiguity) the Borel $\sigma$-algebra on $E$ and $\lambda_E$ the Lebesgue’s measure. In order to alleviate notations we will note $(\Omega, \mathcal{F}, P)$ and $(E, \mathcal{B}_E, \lambda_E)$ simply by $\Omega$ and $E$. When $E \subseteq \mathbb{R}$ we
simply note $\lambda_E$ by $\lambda$. If $F$ is another Banach space then we note $L(E;F)$ the space of linear continuous applications from $E$ to $F$, $L(E) = L(E;E)$, $E' = L(E;\mathbb{R})$ and $L^0(\Omega;E)$ the set of $E$-valued random variables.$^4$

**Definition 2.1** Let $X \in L^0(\Omega;E)$, $p \in ]0,\infty]$ and $t \geq 0$.

(i) $X \in L^p(\Omega;E)$ if $E \left[\|X\|^p\right] = \int_{\Omega} \|X(\omega)\|^p dP(\omega) < \infty$.

(ii) $L^p(\Omega;E)$ is the set of classes in $L^p(\Omega;E)$ such that $[X] = [Y]$ if $E \left[\|X - Y\|^p\right] = 0$.

(iii) $X \in L^p_t(\Omega;E)$ if $X \in L^p(\Omega;E)$ and $X$ is $\mathcal{F}_t$-measurable.

**Definition 2.2** Let $X \in L^\infty(\Omega;E)$ and $t \geq 0$.

(i) $X \in L^\infty(\Omega;E)$ if there is a negligible set $\mathcal{N} \in \mathcal{F}$ and a positive number $m$ such that $\forall \omega \in \Omega \setminus \mathcal{N}$, $\|X(\omega)\|_E \leq m$.

(ii) $L^\infty(\Omega;E)$ is the set of classes in $L^\infty(\Omega;E)$ such that $[X] = [Y]$ if there is a negligible set $\mathcal{N} \in \mathcal{F}$ such that $\forall \omega \in \Omega \setminus \mathcal{N}$, $\|X(\omega) - Y(\omega)\|_E = 0$.

(iii) $X \in L^\infty_t(\Omega;E)$ if $X \in L^\infty(\Omega;E)$ and $X$ is $\mathcal{F}_t$-measurable.

**Definition 2.3** Let $I \subseteq \mathbb{R}_+$ and $X = (X_t)_{t \in I}$ such that $\forall t \in I$, $X_t : (\mathcal{O}, \mathcal{F}, P) \rightarrow (E, \mathcal{B}_E)$ is a random variable. Then $X$ is called a stochastic process. $X$ is said $(\mathcal{F}_t)$-adapted if $\forall t \in I$, $X_t$ is $\mathcal{F}_t$-measurable.

**Definition 2.4** A stochastic process $X$ is said progressively measurable if $\forall t \geq 0$, $X_t$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable.

**Definition 2.5** A process $(B_t)_{t \geq 0}$ is called a Brownian motion$^2$ on $E'$ the dual space of $E$ if the following conditions are satisfied.

(i) $\forall t \geq 0$, $B_t$ is a linear form on $E'$.

(ii) $\forall x \in E'$, the process $(B_t(x))_{t \geq 0}$ is a real Brownian motion.$^2$

(iii) There is a self adjoint positive linear and continuous operator $K : E' \rightarrow E$ such that $\forall x, y \in E'$, $\forall s, t \geq 0$,

\[ E \left[(B_t(x) - B_s(x))(B_t(y) - B_s(y))\right] = \langle x, K y \rangle (t-s) \]

$K$ is called the associated covariance operator.

**Definition 2.6** Let $V$ and $H$ be two hilbert spaces such that $V \subseteq H$ and $H$ is identified with its dual space. Let consider $F : \Omega \times \mathbb{R}_+ \times V \rightarrow H$, $G : \Omega \times \mathbb{R}_+ \times V \rightarrow H \otimes E'$, $(B_t)_{t \geq 0}$ a Brownian motion$^2$ on $E'$ and stochastic differential equation:

\[
\begin{cases}
    dX_t = F(t, X_t) \, dt + \langle G(t, X_t), dB_t \rangle \\
    X_0 = \xi
\end{cases}
\]

A progressively measurable process $X$ is called a (strong) solution of (8) on $[0,T]$ if it satisfies

\[
\int_0^t \|F(s, X_s)\|_H^2 \, ds + \int_0^t \|G(s, X_s)^* G(s, X_s)\|_H^2 \, ds < \infty
\]

and

\[
X_t = \xi + \int_0^t F(s, X_s) \, ds + \int_0^t \langle G(s, X_s), dB_s \rangle.
\]

\[ ^1 \text{See [11] for the definition and properties of random variables valued in Banach spaces.} \]

\[ ^2 \text{See [11] for more details on the topic. Also see [15] page 134, for Hilbert valued Brownian motions.} \]

\[ ^3 \text{See Chapter 1, Section 1.3 in [11].} \]

\[ ^4 \text{See [11] for more details on the topic. Also see [15] page 143.} \]
If $E$ has a finite dimension $n$ and $\Phi : E \to \mathbb{R}$ is a functional of class $C^k$ then we note $D^k\Phi$ the differential of order $l = (l_1, \cdots, l_n) \in \mathbb{N}^n$ with $\sum_{i=1}^n l_i \leq k$. When $n = 1$ we simply note $D\Phi$ instead of $D^1\Phi$.

3 Modelling of the anthracnose noised dynamics

In this section we construct stochastic (partial) differential equation models which reflect the random behaviour of the anthracnose dynamics. As we said before, that dynamics is subject to many random perturbations and measurements on the system are also noised. We make the common choice to represent the randomness of the system by Brownian motions. Indeed, the Brownian motion has some good properties and they are several well-known results in the literature concerning stochastic differential equations with Brownian noise. For instance the Brownian motion has a continuous version and is a martingale. Those properties are useful for the regularity of the solution and the last one is particularly useful for filtering. We formally note $\theta = (\theta_t)_{t \geq 0}$ the stochastic process such that $\forall \omega \in \Omega$, $\theta_t(\omega)$ is a space dependent function defined on $U \subseteq \mathbb{R}^3$ and representing the spatial distribution of anthracnose inhibition rate. In the same manner we note $(\nu_t)_{t \geq 0}$ and $(\nu^r_t)_{t \geq 0}$ the spatial processes of fruits volumes and rotted volumes. We set $\rho_t \nu_t = \nu^r_t$.

We adopt the following model for every $(t, x) \in \mathbb{R}_+^* \times U$,

$$
\begin{align*}
\frac{d\theta_t(x)}{dt} &= (f_1(t, x, \theta_t(x)) + \mathcal{L}_t \theta_t(x)) dt + g_1(t, x, \theta_t(x)) dB^1_t(x) \\
\frac{d\nu_t(x)}{dt} &= f_2(t, x, \nu_t(x), \theta_t(x)) dt + g_2(t, x, \nu_t(x)) dB^2_t(x) \\
\frac{d\rho_t(x)}{dt} &= f_3(t, x, \nu_t(x), \rho_t(x), \theta_t(x)) dt + g_3(t, x, \rho_t(x)) dB^3_t(x)
\end{align*}
$$

where $\forall \omega \in \Omega$, $\forall x \in U$, $\forall y = (y_1, y_2, y_3) \in \mathbb{R}^3$, $\forall i \in \{1, 3\}$,

$$
\begin{align*}
g_1(t, x, y_i) &= \delta_i(t, x) \kappa_i(y_i) \\
g_2(t, x, y_2) &= \delta_2(t, x) \kappa_2 \left( \frac{y_2}{v_{\text{max}}} \right) \\
f_1(t, x, y_1) &= \alpha(t, x) (1 - y_1 w(t, x)) \\
f_3(t, x, y, z) &= \gamma(t, x, y) (1 - y_3) \\
\mathcal{L}_t \theta_t(\omega)(x) &= \text{div} \left( A(t, x) \nabla \theta_t(\omega)(x) \right)
\end{align*}
$$

and

$$
f_2(t, x, y) = \frac{\beta(t, x, y_1)}{\eta(t, x) v_{\text{max}}} \left( \eta(t, x) v_{\text{max}} - \frac{y_2}{1 + \varepsilon - y_1} \right).
$$

The positive term $\varepsilon$ (very smaller than 1) has been already introduced in the reference [22] and models the fact that even the inhibition rate is near to its maximal value the volume of the fruit is remains greater than a smallest value. We guess that lower bound value is in the neighborhood of $\varepsilon v_{\text{max}} \min \{ \eta(t) \}$. On the other hand the term $\varepsilon$ permits to avoid singularities in the model. In order to take into account the impacts of random climatic changes in the model, the parameters are assumed to depend on the time. For $i$ belonging to $\{1, 2, 3\}$, $(B^i_t)_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$-adapted cylindrical Brownian motion on the dual of an Hilbert space to make precise later and its covariance operator is the identity operator. The system of initial conditions both with the Brownian motions is assumed independent. Each $\delta_i$ is positive function giving the range of noises and $\kappa_i$ is a nonnegative locally lipschitz continuous functions modelling the dependence of the noises with respect to the state.
3.1 A lumped model

In this subsection we survey the model (11) – (15) assuming that the diffusion operator \( L_t \) is null. That can correspond to a situation where disease spreading is limited either by natural climatic and relief conditions or by a control strategy. Since there is not diffusion, the study is restricted at each point and the space variable can be forgotten. Each Brownian motion \( (B_t^i)_{t \geq 0} \) is assumed to be a standard real Wiener process starting from zero. The model is then simpler to study and however could display average behaviours and give an idea on the way to study the general model. We then keep the same notations, omit the space variable and remove the diffusion term in equation (11). We assume that all the parameters of the model are not random. The following assumptions are considered.

**Assumption 3.1** \( \forall i \in \{1, 2, 3\}, \delta_i, \alpha \in L^\infty_{loc} (\mathbb{R}_+; \mathbb{R}_+) \).

**Assumption 3.2** \( u, \eta \in L^\infty (\mathbb{R}_+; [0, 1]) \) and \( \forall t \geq 0, \inf \{ \eta (s); s \in [0; t] \} > 0 \).

**Assumption 3.3** \( \forall i \in \{1, 2, 3\}, \kappa_i \) is a nonnegative locally Lipschitz continuous function which is positive on the set \([0, 1]\) and null on \( \mathbb{R} \setminus [0, 1] \).

**Assumption 3.4** \( \beta \in L^\infty_{loc} (\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}_+) \) and \( \gamma \in L^\infty_{loc} (\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}_+) \) satisfies

(i) \( \gamma \) is a measurable with respect to the two first parameters and locally Lipschitz continuous with respect to the third parameter,

(ii) \( \gamma (t, \ldots, \cdot) \) is increasing with respect to the first parameter and \( \gamma (t, 0, \ldots) \) is decreasing with respect to the last parameter. Moreover, \( \gamma (t, \ldots, 0) \) is nonnegative and such that

\[
\gamma (t, 0, \ldots) = 0 = \gamma (t, \ldots, 0) .
\]

The assumption (3.4) – (i) guarantees that while the berry has a null volume (without berry) or the disease has not started, the rot volume remains null. The assumption (3.4) – (ii) means that the rot volume increases with the inhibition rate; when there is a not inhibition the volume of rot does not increase while the fruit grows better and therefore the proportion \( \rho \) decreases. \( \gamma \) could be chosen with the form \( \gamma (t, y_1, y_2, y_3) = (\gamma_1 (t) y_1 - \gamma_2 (t) y_3)^2 y_2 \), with \( \gamma_1, \gamma_2 : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}_+ \). Since all the coefficients of the simplified version of the model (11) – (15) are Lipschitz continuous with respect to state variables we can apply Theorem 5.2.1 in [10] (page 66) to conclude that there is unique solution (in the sense of indistinguishability) defined on a maximal time set \([0, T]\) with \( T \in \mathbb{R}_+^* \cup \{ \infty \} \). In the remainder of the subsection we will establish that the solution is bounded in \([0, 1]^3\) and therefore that \( T = \infty \).

**Lemma 3.1** Let \( (\theta_0, \frac{m_0}{r_{\max}}, \rho_0) \in [0, 1]^3, P\text{-almost surely. If } ((\theta_t, v_t, p_t))_{t \in [0, T]} \text{ is the solution of the lumped model (11) – (15)} \) then \( P\text{-almost surely, } \forall t \in [0, T], (\theta_t, \frac{m_t}{r_{\max}}, p_t) \in [0, 1]^3 \).

**Proof.** Let \( \varphi \in C^2 (\mathbb{R}) \) be a nonnegative function which is null on \([0, 1]\) and positive elsewhere, decreases on \( ]0, 0[ \text{ but increases on } ]1, \infty[ \). An example of such a function \( \varphi \) is the map

\[
x \mapsto \begin{cases} 
-x^3, & x \leq 0 \\
0, & 0 < x < 1 \\
(x - 1)^3, & x \geq 1 
\end{cases}
\]

Using the Itô formula we have

\[
d \varphi (\theta_t) = D \varphi (\theta_t) d\theta_t + \frac{1}{2} D^2 \varphi (\theta_t) d\theta_t \cdot d\theta_t \\
= f_1 (t, \theta_t) D \varphi (\theta_t) dt + \frac{1}{2} D^2 \varphi (\theta_t) (g_1 (t, \theta_t))^2 dt \\
+ D \varphi (\theta_t) g_1 (t, \theta_t) dB_t^1 \\
= f_1 (t, \theta_t) D \varphi (\theta_t) dt
\]

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We can easily check that \( f_1 (t, \theta_t) D\varphi (\theta_t) \) is not positive and
\[
\frac{d\varphi (\theta_t)}{dt} \leq 0
\]
The last inequality and the fact that \( \varphi (\theta_0) = 0 \) imply that \( \varphi (\theta_t) \) is not positive and therefore null since \( \varphi \) is a nonnegative function. Using the definition of \( \varphi \) we deduce that necessarily \( \theta_t \in [0, 1] \). Using similar arguments and the fact that almost surely \( \forall t \in [0, T], \theta_t \in [0, 1] \) we also obtain that \( \forall t \in [0, T], \varphi \left( \frac{\nu_t}{\nu_{\text{max}}} \right) = \varphi (\rho_t) = 0 \). Therefore \( (\rho_t, \frac{\nu_t}{\nu_{\text{max}}}) \in [0, 1]^2 \). ■

**Proposition 3.1** The lumped model has a unique (in the sense of indistinguishability) solution defined on \( \mathbb{R}_+ \).

**Proof.** Using Lemma 3.1 and the Theorem 5.2.1 in [40] the result follows. ■

**Lemma 3.2** Let \( \left( \theta_0, \frac{\nu_0}{\nu_{\text{max}}}, \rho_0 \right) \in [0, 1]^3 \), \( P \)-almost surely. If \( \left( (\theta_t, \nu_t, \rho_t) \right)_t \geq 0 \) is the solution of the lumped model then \( P \)-almost surely, \( \forall t \geq 0 \), \( \left( \theta_t, \frac{\nu_t}{\nu_{\text{max}}}, \rho_t \right) \in [0, 1]^3 \).

Before giving a proof for the Lemma 3.2 we first recall a particular version of the general comparison Proposition 3.12 in [41] (page 149) :

**Proposition 3.2** Let \( f, \tilde{f} : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) and \( g : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^k \) be three progressively measurable processes with respect to the first two variables and continuous with respect to the third one. Let \( W \) be a \( k \) dimensional Brownian motion. Assume that \( \forall t \geq 0 \), \( P \)-almost surely the following inequalities hold :
\[
\int_0^t |g (s, X_s)|^2 ds + \int_0^t \left| g \left( s, \tilde{X}_s \right) \right|^2 ds < \infty
\]
and
\[
\int_0^t |f (s, X_s)| ds + \int_0^t \left| \tilde{f} \left( s, \tilde{X}_s \right) \right| ds < \infty
\]
where \( X \) and \( \tilde{X} \) are solution of the following stochastic differential equations :
\[
X_t = X_0 + \int_0^t f (s, X_s) ds + \int_0^t \langle g (s, X_s), dW_s \rangle
\]
and
\[
\tilde{X}_t = \tilde{X}_0 + \int_0^t \tilde{f} (s, \tilde{X}_s) ds + \int_0^t \langle g \left( s, \tilde{X}_s \right), dW_s \rangle.
\]
Also assume that there are two progressively measurable processes \( L, \ell : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \) such that almost surely \( \forall t \geq 0 \),
\[
\max \left\{ \int_0^t \ell_s^2 ds, \int_0^t L_s ds \right\} < \infty
\]
and \( d\lambda \otimes dP \)-almost everywhere, \( \forall x, y \in \mathbb{R} \),
\[
|f (t, x) - f (t, x)| \leq L_t |x - y|
\]
and
\[
\left| g \left( t, X_t \right) - g \left( s, \tilde{X}_t \right) \right| \leq \ell_t \left| X_t - \tilde{X}_t \right|.
\]
Finally, assume that \( P \)-almost surely \( X_0 \geq \tilde{X}_0 \) and \( d\lambda \otimes dP \)-almost everywhere on \( \Omega \times \mathbb{R}_+ \), \( f (t, x) \geq \tilde{f} (t, x) \). Then
(i) $P$-almost surely, $\forall t \geq 0$, $X_t \geq \tilde{X}_t$ and $X$ is the unique solution of (23).

(ii) If moreover there exist $A \in \mathcal{F}$ and a stopping time $\tau > 0$ such that $\forall \omega \in A$, $X_0(\omega) > \tilde{X}_0(\omega)$ or

$$\int_0^{\tau(\omega)} \left( f(\omega, s, X_s) - \tilde{f}(\omega, s, \tilde{X}_s) \right) ds > 0$$

then $\forall \omega \in A$, $X_t(\omega) > \tilde{X}_t(\omega)$; in particular if $\forall \omega \in A$, $X_0(\omega) > \tilde{X}_0(\omega)$ then $\forall (\omega, t) \in A \times \mathbb{R}_+$, $X_t(\omega) > \tilde{X}_t(\omega)$.

**Proof.** (of the Lemma 3.2)

Let $(\theta^1_t)_{t \geq 0}$, $(\theta^2_t)_{t \geq 0}$, $(\theta^3_t)_{t \geq 0}$ be three solutions of the equation (11) with the respective initial conditions $0$, $\theta_0^1 \in [0, 1]$ and 1. Since $\alpha, \omega, \delta \in L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ and $\kappa_1$ is Lipschitz continuous we use Lemma 3.1 and apply Proposition 3.2 with $f = \tilde{f} = f_1$ and $g = g_1$. Then $P$-almost surely $\forall t \geq 0$,

$$0 \leq \theta^1_t < \theta^2_t < \theta^3_t \leq 1$$

That gives the result for $\theta$. With the same manner we also establish it for $\nu$ and $\rho$; we just have to take respectively $f = \tilde{f} = f_1(\cdot, \theta)$ and $g = g_i$ for $i \in \{1, 2\}$. ■

Now let us give a stronger result useful for the filtering. For that we need an additional assumption:

**Assumption 3.5** There is a time $T^* > 0$ such that there is a nonempty interval $I \subset ]0, T^*[ satisfying $\forall t \in I$, $\alpha(t), \beta(t), \gamma(t) > 0$.

The Assumption 3.5 seems restrictive but is still realistic because we are interested by the disease dynamics since favourable conditions are fulfilled even for a small time.

**Lemma 3.3** Let at least one of the initial conditions be $P$-almost surely null. Under the Assumption 3.5 if $((\theta_t, \nu_t, \rho_t))_{t \geq 0}$ is the solution of the lumped model then $P$-almost surely $\forall t > 0$, $\left( \theta_t, \frac{\nu_t}{\nu_{\max}}, \rho_t \right) \in [0, 1]^3$.

**Proof.** Following Lemmas 3.1 and 3.2 it is sufficient to establish that if $P$-almost surely $\theta_0$, $\frac{\nu_0}{\nu_{\max}}$, $\rho_0$ are all null then $P$-almost surely $\forall t \in ]0, T[$, $\left( \theta_t, \rho_t, \frac{\nu_t}{\nu_{\max}} \right) \in [0, 1]^3$. We first give the proof that if $P$-almost surely $\theta_0$ is null then $P$-almost surely, $\forall t \in ]0, T[$, $\theta_t \in [0, 1]$. Let consider the stopping time

$$\tau_1 = \inf \{ t \geq 0; \theta_t > 0 \}$$

Since $\theta$ is continuous, $\forall t \geq 0$, $\theta_{t\wedge \tau_1}$ and $E \left[ \int_0^{t \wedge \tau_1} g_1(s, \theta_s) dB^1_s \right]$ are all null and necessarily

$$E \left[ \int_0^{t \wedge \tau_1} f_1(s, \theta_s) ds \right] = 0$$

Let $T^*$ be the random time given the Assumption 3.6

$$E \left[ \int_0^{T^*} \alpha_1(s \leq \tau_1) ds \right] = E \left[ \int_0^{T^* \wedge \tau_1} \alpha(s) ds \right]$$

$$= E \left[ \int_0^{T^* \wedge \tau_1} \alpha(s) (1 - w(s) \theta_s) ds \right]$$

$$= E \left[ \int_0^{T^* \wedge \tau_1} f_1(s, \theta_s) ds \right]$$

$$= 0$$
By the Assumption 3.5, $\forall t \in I$, $\alpha(t)$ is positive and necessarily $1_{\{t \leq \tau_1\}}$ is null. Hence, $\tau_1 = 0$. In the same manner let consider the stopping times

$$\tau_2 = \inf \{ t \geq 0; v_t > 0 \}$$

(27)

and

$$\tau_3 = \inf \{ t \geq 0; \rho_t > 0 \}.$$  

(28)

Using the Assumption 3.5 and the fact that $\forall t \in I$, $\theta_t \in [0,1]$ we have

$$E \left[ \int_0^{T^* \wedge \tau_2} f_2 (s, 0, \theta_s) \, ds \right] = 0 = E \left[ \int_0^{T^* \wedge \tau_3} f_3 (s, v_s, 0, \theta_s) \, ds \right].$$

and $\beta(t, \theta_t)$ and $\gamma_t(t, \theta_t, v_t, \rho_t)$ are positive. Necessarily $\tau_i = 0$, $\forall i \in \{2,3\}$ and the result follows. ■

It seems important to mention that the Lemma 3.1, the Lemma 3.2 and the Lemma 3.3 remain true even if $g_i$ is indentically null.

3.2 The distributed parameters model

This subsection is devoted to the study of the full model (11) − (15) with the diffusion term. We expect to generalize results obtained for the lumped model. Each process $(B^i_t)_{t \geq 0}$ is now assumed to satisfy the general definition (2.3) on the dual space of the Sobolev space $H^2(U)$ and is identified to an $H^2(U)$-valued process. As said before, we set identity operator as the common covariance operator of each Brownian motion. We do not identify the Hilbert space $H^2(U)$ with its dual space while we identify the Lebesgue space $L^2(U)$ with its dual space. By the Maurin Theorem, the embedding of $H^2(U)$ in $L^2(U)$ is of Hilbert-Schmidt type and therefore the restriction on $L^2(U)$ of each Brownian motion has a nuclear covariance operator. From the results in [24] (Theorems 5-8 in chapter 1) and the Friedrichs Theorem (Theorem 9.2 and Corollary 9.8 in [12]) those covariance operators have kernels as bilinear forms on $L^2(U)$. The following assumptions are considered.

**Assumption 3.6** $\forall i \in \{1,2,3\}$, $\delta_i$, $\alpha \in L^\infty_{loc}(\mathbb{R}_+ \times \overline{U}; \mathbb{R}_+)$.  

**Assumption 3.7** $u, \eta \in L^\infty(\mathbb{R}_+ \times \overline{U}; [0,1])$ and $\forall t \geq 0$, $\inf \{ \eta(s,x) ; s \in [0,t], x \in U \} > 0$.  

**Assumption 3.8** $\forall i \in \{1,2,3\}$, $\kappa_i$ is a nonnegative locally Lipschitz continuous function which is positive on the set $[0,1]$ and null on $\mathbb{R} \setminus [0,1]$.  

**Assumption 3.9** $\beta \in L^\infty_{loc}(\mathbb{R}_+ \times \overline{U} \times \mathbb{R}; \mathbb{R}_+)$ and $\gamma \in L^\infty_{loc}(\mathbb{R}_+ \times \overline{U} \times \mathbb{R}^3; \mathbb{R}_+)$. satisfies

(i) $\gamma$ is a measurable with respect to the two first parameters and locally Lipschitz continuous with respect to the third parameter,

(ii) $\gamma(t,x,\ldots)$ is increasing with respect to the first parameter and $\gamma(t,x,0,\ldots)$ is decreasing with respect to the last parameter. Moreover, $\gamma(t,x,\ldots,0)$ is nonnegative and such that

$$\gamma(t,x,0,0,\ldots) = 0 = \gamma(t,x,\ldots,0).$$

(29)

similarly to the lumped model, $\gamma$ could be chosen such as $\gamma(t,x,y_1,y_2,y_3) = (\gamma_1(t,x)y_1 - \gamma_2(t,x)y_3)y_2$,

with $\gamma_1, \gamma_2 : \mathbb{R}_+ \times \overline{U} \times \mathbb{R}^3 \to \mathbb{R}_+$.  

**Assumption 3.10** $\forall i,j \in \{1,2,3\}$, $a_{ij} \in L^\infty_{loc}(\mathbb{R}_+ \times \overline{U}; \mathbb{R})$.  

---

See [14].

We refer to [24] for the properties of nuclear and Hilbert-Schmidt operators.
we will show in the sequel that the solution is bounded and therefore is defined for every time. Note that the solution is just defined on a maximal set of time. However, that condition is definitely satisfied using the inequality (30), the boundedness of $c, \lambda$.

Moreover, Proposition 3.3

There is a process $\phi$ and dense in $F$:

There is a process $\phi$ and dense in $F$:

Assumption 3.11

$\forall T \geq 0, \exists C(T) \in \mathbb{R}^*_+$ such that $\forall t \in [0, T], \forall h \in \mathbb{R}^3, \forall x \in U$,

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij}(t, x) h_i h_j \geq C \sum_{i=1}^n h_i^2. \quad (30)$$

Before giving the first result of this subsection we state a particular version of the Theorem 2.1 (page 93) proved in [11].

Theorem 3.12

Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ denote three separable Hilbert spaces such that $\mathcal{X}$ is continuously embedded and dense in $\mathcal{Y}$ which is identified with its dual space. Let $\mathcal{X}'$ and $\mathcal{Z}'$ denote respective dual spaces of $\mathcal{X}$ and $\mathcal{Z}$, $p$ be a real number in $[1, \infty]$ and $T \in \mathbb{R}_+$. Let also consider the stochastic differential equation given such as $\phi_0 \in L^2(\Omega \times [0, T]; \mathcal{Y})$ and $\forall t \in [0, T]$,

$$d\phi_t + F(t, \phi_t) dt + G(t, \phi_t) dB_t = f(t, \cdot) dt \quad (31)$$

where $(B_t)_{t \geq 0}$ is a Brownian motion on $\mathcal{Z}'$, $f \in L^p(\Omega \times [0, T]; \mathcal{X}')$ is non anticipative, the operators $F: \mathcal{X} \to \mathcal{X}'$ and $G: \mathcal{Y} \to L(\mathcal{Z}; \mathcal{Y})$ are not necessarily linear but satisfy for almost every $t \in [0, T]$ and independently on the choice of $t$ the following conditions:

(i) There are three real constants $c, \lambda$ and $\nu$ such that $c > 0$ and $\forall \psi \in \mathcal{X}$,

$$2 \langle F(t, \psi), \psi \rangle + \lambda \|\psi\|_{\mathcal{Y}}^2 + \nu \geq c \|\psi\|_{\mathcal{X}}^2 + \|G(t, \psi)\|_{L(\mathcal{Z}; \mathcal{Y})}^2,$$

(ii) $\forall \psi, \varphi \in \mathcal{X}$, $\langle F(t, \psi) - F(t, \varphi), \psi - \varphi \rangle + \lambda \|\psi - \varphi\|_{\mathcal{Y}}^2 \geq 0$,

(iii) There is a constant $\mu \geq 0$ such that $\forall \psi \in \mathcal{X}$, $\|F(t, \psi)\|_{\mathcal{X}'} \leq \mu \|\psi\|_{\mathcal{X}}^{-1}$,

(iv) $\forall \psi, \varphi, \phi \in \mathcal{X}$, the application $\xi \in \mathbb{R} \mapsto \langle F(t, \psi + \xi \varphi), \phi \rangle$ is continuous,

(v) $\forall \psi \in \mathcal{X}$, $\forall \varphi \in \mathcal{Y}$, the applications $t \in [0, T] \mapsto F(t, \psi)$ and $t \in [0, T] \mapsto G(t, \varphi)$ are Lebesgue-measurable,

(vi) $G(t, 0) = 0$ and for every bounded subset $S \subseteq \mathcal{Y}$, there is a constant $C(S)$ such that $\forall \psi, \varphi \in S$,

$$\|G(t, \psi) - G(t, \varphi)\|_{L(\mathcal{Z}; \mathcal{Y})} \leq C \|\psi - \varphi\|_{\mathcal{Y}}.$$

Then the equation (31) has a unique adapted solution $\phi \in L^p(\Omega \times [0, T]; \mathcal{X}) \cap L^2(\Omega; C([0, T]; \mathcal{Y}))$.

Proposition 3.3

There is a process $(\theta_t)_{t \geq 0}$ valued in $H^1(U)$ which is the unique solution of the equations [11], [11] and [15].

Proof. We set $\mathcal{X} = H^1(U; \mathbb{R}), \mathcal{Y} = L^2(U), \mathcal{Z} = H^2(U; \mathbb{R}), F(t, \psi) = (t,.) w(t,.) \psi - L_t(\psi), G(t, \psi): \varphi \in H^2(U; \mathbb{R}) \mapsto g_1(t, \varphi) \varphi \in L^2(U), f(t,.) = (t,.) \alpha$ and $p = 2$. Recall that $\alpha, f_1$ and $g_1$ are bounded. Moreover, $f_1(t,.)$ and $g_1(t,.)$ are Lipschitz continuous with respect to $\theta_t$ and $\kappa_t(0) = 0$. Hence conditions (ii) – (vi) of the Theorem 3.12 are fulfilled. To get the result it suffices to show that there are constants $c, \lambda$ and $\nu$ such that $\alpha > 0$ and $\forall \psi, \varphi \in H^1(U; \mathbb{R}), \forall t \geq 0$,

$$- 2 \langle L_t(\psi), \psi \rangle_{L^2(U)} + 2\alpha(t,.) w(t,.) \psi(t,.) + \lambda \|\psi\|^2_{L^2(U)} \geq c \|\psi\|^2_{H^1(U; \mathbb{R})} + \|G(t, \psi)\|^2_{L^2(H^2(U; \mathbb{R}); L^2(U))}. \quad (32)$$

That condition is definitively satisfied using the inequality (30), the boundedness of $\delta_1$ and the fact that $\kappa_1$ is locally Lipschitz continuous. Note that the solution is just defined on a maximal set of time. However, we will show in the sequel that the solution is bounded and therefore is defined for every time. ■

Corollary 3.1

There is a process $((\theta_t, v_t, \rho_t))_{t \geq 0}$ valued in $H^1(U) \times L^2(U) \times L^2(U)$ which is the unique solution of [11] – [15].
Theorem 2.16 in [12], page 285.
with $c_1$ the Lipschitz constant of $\varphi$ and

$$C = \sup \{ D^2 \varphi(x) ; x \in \mathbb{R} \}.$$ 

Indeed, $\forall h \in H^1(U)$ the operators $D\Phi (h)$ and $D^2 \Phi (h)$ have respectively the kernels $D\varphi \circ h$ and $D^2 \varphi \circ h$.

We can now apply the Itô formula\(^8\) as in [11] (Theorem 4.2, page 65) and obtain

\[
\Phi (\theta_t) = \Phi (\theta_0) + \int_0^t D\Phi (\theta_s) (f_1(s,.,\theta_s) + \mathcal{L}_s \theta_s) \, ds \\
+ \int_0^t D\Phi (\theta_s) G(s,\theta_s) dB_s^1 + \frac{1}{2} Tr \int_0^t D^2 \Phi (\theta_s) G(s,\theta_s) QG^* (s,\theta_s) \, ds \\
= \Phi (\theta_0) + \int_0^t \int_U D\varphi (\theta_s(x)) (f_1(s,x,\theta_s(x)) + \mathcal{L}_s \theta_s (x)) \, dx \, ds \\
+ \sum_{j=1}^\infty \int_0^t \lambda_j \int_U D\varphi (\theta_s(x)) g_1 (s,x,\theta_s(x)) e_j (x) \, dx \, dB_s^{1,j} \\
+ \frac{1}{2} \sum_{j=1}^\infty \int_0^t \lambda_j^2 \int_U D^2 \varphi (\theta_s(x)) g_1 (s,x,\theta_s(x)) e_j^2 (x) \, dx \\
\times \int_U g_1 (s,y,\theta_s(y)) e_j^2 (y) \, dy \, ds
\]

The integration with respect to the Brownian motion is guaranteed since $(g_1(s,.,\theta_s(.)))_{s \geq 0}$ is measurable and bounded\(^\frac{9}{4}\). We can easily check that $\Phi (\theta_0) = 0$, $f_1^1(\theta_t) D\varphi (\theta_t)$ is not positive and the last two terms in the right side of the equality are null since $\kappa_1$ has been assumed null on $\mathbb{R} \setminus [0,1[$. Hence,

\[
\Phi (\theta_t) = \int_0^t \int_U (f_1(s,x,\theta_s(x)) + \mathcal{L}_s \theta_s (x)) \, D\varphi (\theta_s(x)) \, dx \, ds \\
\leq \int_0^t \int_U D\varphi (\theta_s(x)) \, \mathcal{L}_s \theta_s (x) \, dx \, ds \\
= - \int_0^t \int_U \langle A_s (x) \nabla \theta_s, \nabla D\varphi (\theta_s) \rangle \, dx \, ds \\
= - \int_0^t \int_U \langle A_s (x) \nabla \theta_s, D^2 \varphi (\theta_s) \nabla \theta_s \rangle \, dx \, ds \\
\leq 0
\]

The last inequality implies that $\Phi (\theta_t)$ is not positive. Using the definition of $\Phi$ necessarily $\Phi (\theta_t)$ is null and therefore $\theta_t(x) \in [0,1]$ for almost every $x \in U$. \hfill \blacksquare

**Proposition 3.4** Let \( \left( \frac{v_{\max}}{v_{\min}}, \rho_t \right) \) be valued in $[0,1]^2$, $P$-almost surely. If \( ((\theta_t, v_t, \rho_t))_{t \in [0,T]} \) is the solution of the model (11) - (15) then $P$-almost surely $\forall t \geq 0$, \( \left( \frac{v_{\max}}{v_{\min}}, \rho_t \right) \) is valued in $[0,1]^2$.

**Proof.** Using the particular form of equations (12) - (13) we can fix the space variable and conclude as in Lemma 3.2. \hfill \blacksquare

It seems difficult to generalize the Lemma 3.2 and the Lemma 3.3 in spatial case. However, we are able to give some results similar to the Lemma 3.2.

Let

$$m = \inf \{ \min \{ \theta_0(x), 1 - \sigma u_t(x) \} ; t \geq 0, x \in U \}$$

and

$$2M = \inf \{ \min \{ 1 - \theta_0(x), \sigma u_t(x) \} ; t \geq 0, x \in U \} .$$

\(^8\)See also [17] (Theorem 7.21, page 133).

\(^9\)See [11] and references therein.
Remembering that \( \omega \) is to give a general representation of the noised dynamics of the process (signal with respect to an observation. In the present case the signal is generated by the two dimensional process (of the model (11).

### 4.1 Another modelling of the noised observations dynamics

Another modelling of the noised observations dynamics is to give a general representation of the noised dynamics of anthracnose. Although the previous modelling given in Section 3 seems natural and displays good properties, it is less practical for the estimation we aim to carry out in the sequel of this work. On the other hand, the fact that \( \omega \) makes difficult the filtering procedure. Hence, let \( \overline{\eta}, \overline{\eta} \) satisfies the 'deterministic' parts of the equations (12) and (13) with the initial conditions \( (\overline{\eta}_0, \overline{\eta}_0) = (v_0, \rho_0) \); that is

\[
d\overline{\eta}_t = f^2_t (x, \overline{\eta}_t (x), \theta_t (x)) 1_{\{v_t > 0\}} dt \tag{35}
\]
\[ d\tilde{\rho}_t = \begin{cases} f(t, \Xi_t, \theta_t) \, dt + \delta_2(t) \, dB_t^2 & \text{if } 0 < \tilde{\rho}_t < v_{\max} \\ 0, & \text{otherwise} \end{cases} \]  

(36)

We can prove using arguments similar with the Section \[3\] that if \((\tilde{v}_0, \tilde{\rho}_0)\) is valued in \([0, v_{\max}] \times [0, 1]\) then the all process \((\tilde{\rho}_t, \tilde{\rho}_t)\) is also valued in \([0, v_{\max}] \times [0, 1]\). Conditionally upon \(\theta_t\), the expectation of \((v_t, \rho_t)\) is given by \((\bar{\rho}_t, \bar{\rho}_t)\). The term \(1_{\{v_t > 0\}}\) in the equation (35) ensures the realistic property that \(\bar{\rho}\) remains null while \(v\) is null. The term \(1_{\{\rho_t > 0\}}\) plays a similar role in the equation (36).

If we set
\[ \bar{X}_t = \left\{ \begin{array}{ll} \ln \left( \frac{\tilde{\rho}_t}{v_{\max} - \tilde{\rho}_t} \right), & \text{if } 0 < \tilde{\rho}_t < v_{\max} \\ 0, & \text{otherwise} \end{array} \right. \]  

(37)

and
\[ \bar{Y}_t = \left\{ \begin{array}{ll} \ln \left( \frac{\tilde{\rho}_t}{1 - \tilde{\rho}_t} \right), & \text{if } 0 < \tilde{\rho}_t < 1 \\ 0, & \text{otherwise} \end{array} \right. \]  

(38)

then \(\bar{X}\) and \(\bar{Y}\) satisfies when \(0 < \tilde{\rho} < v_{\max}\) and \(0 < \tilde{\rho} < 1\) the following equations:

\[ d\bar{X}_t(x) = \left( \eta(t, x) \left( 1 + \exp(-\bar{X}_t) \right) - \frac{1}{1 + \varepsilon - \theta_t} \right) \beta(t, x, \theta_t) \left( 1 + \exp(\bar{X}_t) \right) \frac{\eta(t, x) v_{\max}^2}{\eta(t, x) v_{\max}} \, dt \]  

and

\[ d\bar{Y}_t(x) = \left( 1 + \exp(-\bar{Y}_t) \right) \gamma(t, x, \theta_t, v_{\max} \exp(\bar{X}_t), \exp(\bar{Y}_t)) \]  

(39)

(40)

A common additive introduction a Brownian noise in the dynamics of \((\bar{X}, \bar{Y})\) leads to a diffusion process \((X, Y)\) satisfying

\[ dX_t(x) = d\bar{X}_t(x) + \delta_2(t, x) \, dB_t^2(x) \]  

(41)

and

\[ dY_t(x) = d\bar{Y}_t(x) + \delta_3(t, x) \, dB_t^3(x) \]  

(42)

The terms \(\varepsilon, B_t^2\) and \(B_t^3\) have the same definitions given in the Section \[3\].

Similarly to the relation between \((\bar{\rho}, \bar{\rho})\) and \((\bar{X}, \bar{Y})\), we could assume that when \(v_t\) and \(\rho_t\) are not null they satisfy respectively

\[ v_t = \frac{v_{\max} \exp(X_t)}{1 + \exp(X_t)} \]  

(43)

and

\[ \rho_t = \frac{\exp(Y_t)}{1 + \exp(Y_t)}. \]  

(44)

Note that it is useless to start the filtering while \(v = 0\) since there is not fruit. In the same order of idea, when \(v = 0\) and \(\rho = 0\) we can restrict ourselves to the informations brought by the dynamics of \(X\). Indeed, when \(v\) and \(\rho\) are respectively null the variation of \(X\) and \(Y\) are reduced to a Brownian noise.

### 4.2 State estimation with continuous observations

In this subsection, we assume that at each time \(t \geq 0\) all the observations \((\bar{v}_s, \bar{\rho}_s)\) are really known. However, instead of \((\bar{v}_s, \bar{\rho}_s)\) \(s \geq t\geq 0\) we will use the equivalent process \((\bar{X}_s, \bar{Y}_s)\) \(s \geq t\geq 0\) which satisfies the equations

\[ dX_t = f(t, \bar{X}_t, \theta_t) \, dt + \delta_2(t) \, dB_t^2 \]  

(45)

and

\[ dY_t = g(t, \bar{X}_t, \bar{Y}_t, \theta_t) \, dt + \delta_3(t) \, dB_3^3. \]  

(46)

where

\[ f(t, \bar{X}_t, \theta_t) = \left( \eta(t) \left( 1 + \exp(-\bar{X}_t) \right) - \frac{1}{1 + \varepsilon - \theta_t} \right) \beta(t, \theta_t) \left( 1 + \exp(\bar{X}_t) \right) \frac{\eta(t) v_{\max}}{\eta(t) v_{\max}} \]  

(47)
and
\[ g(t, X_t, Y_t, \theta_t) = (1 + \exp(-Y_t)) \gamma \left( t, \theta_t, \frac{\nu_{\max}(X_t)}{1 + \exp(X_t)}, \frac{\exp(Y_t)}{1 + \exp(Y_t)} \right). \] (48)

We make the following necessary assumption until the end of the section:

**Assumption 4.1** \( \forall i \in \{1, 2, 3\}, \delta_i \in L_{loc}^\infty(\mathbb{R}; \mathbb{R}_+) \) and \( \forall t \geq 0, \inf \{\delta_i(s); s \in [0, t]\} > 0. \)

Let adopt \( \forall t \geq 0, \) the formal definition
\[
Z_t = \exp \left( -\frac{1}{2} \int_0^t \left( \frac{f^2(s, X_s, \theta_s)}{\delta_2(s)} + \frac{g^2(s, X_s, Y_s, \theta_s)}{\delta_3(s)} \right) ds \right) \tag{49}
\]
\[ \times \exp \left( -\int_0^t \left( \frac{f(s, X_s, \theta_s)}{\delta_2(s)} dB_s^2 + \frac{g(s, X_s, Y_s, \theta_s)}{\delta_3(s)} dB_s^3 \right) \right) \]

The following lemma holds.

**Lemma 4.1** If \( \left( \theta_0, \frac{\nu_{\max}}{\max}, \rho_0 \right) \in [0, 1]^3 \) then under the probability \( P, Z \) is an \( (\mathcal{F}_t) \)-martingale. Moreover, \( \forall t \geq 0, Z_t \) is the Radon-Nikodym derivative of the restriction of a probability \( \tilde{P} \) on \( \mathcal{F}_t^{23} \) with respect to the restriction of \( P \) on \( \mathcal{F}_t^{23} \):
\[
Z_t = \frac{d\tilde{P}_{|\mathcal{F}_t^{23}}}{dP_{|\mathcal{F}_t^{23}}}. \tag{50}\]

**Proof.** The process \( (Z_t)_{t \geq 0} \) is \( \mathcal{F}_t^{23} \)-adapted. Using the properties of the solution \( (\phi, \psi) \) of the equations \( \ref{52} \) and \( \ref{53} \), the relations \( \ref{47} \) and \( \ref{48} \), and the properties of functions \( f \) and \( g \) given by \( \ref{47} \) and \( \ref{48} \) the following Novikov condition is satisfied for every \( t \geq 0 \):
\[
E \left[ \exp \left( -\frac{1}{2} \int_0^t \left( \frac{f^2(s, X_s, \theta_s)}{\delta_2(s)} + \frac{g^2(s, X_s, Y_s, \theta_s)}{\delta_3(s)} \right) ds \right) \right] < \infty. \tag{51}\]

Therefore using Proposition 2.50 in [44] (page 124) and the Itô formula to compute \( dZ_t \), we deduce that \( (Z_t) \) is an \( (\mathcal{F}_t^{23}) \)-martingale which satisfies \( E[Z_t] = 1 \) and there are probabilities \( \tilde{P}_t \) such that \( \forall t \geq 0, \)
\[
Z_t^n = \frac{d\tilde{P}_t}{dP_{|\mathcal{F}_t^{23}}} \tag{52}\]

Using the Daniell-Kolmogorov-Tulcea Theorem A.12 stated in [2] (page 302) there is a probability \( \tilde{P} \) on \( \mathcal{F} \) such that its restriction on \( \mathcal{F}_t^{23} \) is \( \tilde{P}_t \). \( \blacksquare \)

The Lemma \ref{4.1} gives a change of probability which will be very useful in the remainder of the subsection. If we set
\[
\tilde{B}_t^2 = \int_0^t \frac{dX_s}{\delta_2(s)} \tag{53}
\]
and
\[
\tilde{B}_t^3 = \int_0^t \frac{dY_s}{\delta_3(s)} \tag{54}
\]
then \( (\tilde{B}_t^2, \tilde{B}_t^3)_{t \geq 0} \) is a Brownian motion under the probability \( \tilde{P} \). Let
\[
\tilde{Z}_t = \exp \left( -\frac{1}{2} \int_0^t \left( \frac{f^2(s, X_s, \theta_s)}{\delta_2(s)} + \frac{g^2(s, X_s, Y_s, \theta_s)}{\delta_3(s)} \right) ds \right) \tag{55}
\]
\[ \times \exp \left( \int_0^t \left( \frac{f(s, X_s, \theta_s)}{\delta_2(s)} dB_s^2 + \frac{g(s, X_s, Y_s, \theta_s)}{\delta_3(s)} dB_s^3 \right) \right). \]
Under $\tilde{P}$, $\tilde{Z}_t$ has the same properties of $Z_t$ under $P$ and
\[ \tilde{Z}_t = \frac{dP|_{\mathcal{F}_{23}^t}}{dP|_{\mathcal{F}_{23}^t}}. \] (56)

Moreover, $\left( E \left[ \tilde{Z}_t|_{\mathcal{F}_{\infty}^{23}} \right] \right)_{t \geq 0}$ is an $(\mathcal{F}_{23}^t)$-martingale under $\tilde{P}$ and has a continuous version (see Proposition 2.3.1 in [43]).

In the following, we set $\forall t \geq 0$, $\pi_t(\varphi) \equiv E \left[ \tilde{Z}_t \varphi(\theta_t)|_{\mathcal{F}_{23}^t} \right]$ where $\varphi$ is a measurable function such that
\[ E \left[ |\varphi(\theta_t)| \right] = \tilde{E} \left[ \tilde{Z}_t | \varphi(\theta_t) \right] < \infty. \] (57)

Now we recall the useful Proposition in [2, 43].

**Proposition 4.1 (Kallianpur-Striebel)**

If $\varphi$ satisfies the condition (57) then $P$ and $\tilde{P}$-almost surely the following equality holds:
\[ \pi_t(\varphi) = \frac{\tilde{E} \left[ \tilde{Z}_t \varphi(\theta_t)|_{\mathcal{F}_{23}^t} \right]}{\tilde{E} \left[ \tilde{Z}_t|_{\mathcal{F}_{23}^t} \right]} . \] (58)

There are instructive comments on a more general but similar process $(\pi_t)_{t \geq 0}$ in [2] especially in the Theorem 2.1 (page 14). For arbitrary $\varphi \in L^\infty(\mathbb{R}; \mathbb{R})$ and $\forall t \geq 0$, let
\[ \zeta_t = \tilde{E} \left[ \tilde{Z}_t|_{\mathcal{F}_{23}^t} \right] \] (59)
and
\[ \varsigma_t(\varphi) = \zeta_t \pi_t(\varphi). \] (60)

Then the equality (58) becomes
\[ \pi_t(\varphi) = \frac{\varsigma_t(\varphi)}{\varsigma_t(1)}. \] (61)

Let $\left( \mathcal{F}_{23}^t \right)$ denote the filtration generated by the two dimensional process $\left( \left( \tilde{B}_2^t, \tilde{B}_3^t \right) \right)_{t \geq 0}$. Naturally, we have $\tilde{\mathcal{F}}_{23}^t \subseteq \mathcal{F}_{23}^t$ and conversely $\mathcal{F}_{23}^t \subseteq \tilde{\mathcal{F}}_{23}^t$ holds since the following equations has unique solutions:
\[ X_t = \int_0^t \delta_2(s) d\tilde{B}_s^2 \] (62)
and
\[ Y_t = \int_0^t \delta_3(s) d\tilde{B}_s^3. \] (63)

Since $\tilde{\mathcal{F}}_{23}^t = \mathcal{F}_{23}^t$ and $\left( \left( \tilde{B}_2^t, \tilde{B}_3^t \right) \right)_{t \geq 0}$ is a Brownian motion under $\tilde{P}$ we have
\[ \varsigma_t(\varphi) = \tilde{E} \left[ \tilde{Z}_t \varphi(\theta_t)|_{\mathcal{F}_{\infty}^{23}} \right]. \] (64)

The unnormalized law $\varsigma$ of $\theta$ is given by the following

**Theorem 4.2 (The Zakai equation)**

If $O \subseteq \mathbb{R}$ is an open set containing $[0, 1]$ and $\varphi \in C^2(O)$ then $\forall t \geq 0$,
\[ \varsigma_t(\varphi) = \varsigma_0(\varphi) + \int_0^t \varsigma_s(A_s^1 \varphi) ds + \int_0^t \varsigma_s(A_s^2 \varphi) dX_s + \int_0^t \varsigma_s(A_s^3 \varphi) dY_s \] (65)
where

\[ A_1^1 \varphi(x) = f_1(t, x) \varphi'(x) + \frac{1}{2} g_1^2(t, x) \varphi''(x), \]  

(66)

\[ A_1^2 \varphi(x) = \frac{f(t, \overline{X}_t, \theta_t)}{\delta_x^2(t)} \varphi(x), \]  

(67)

and

\[ A_1^3 \varphi(x) = \frac{g(t, \overline{X}_t, \overline{Y}_t, \theta_t)}{\delta_x^2(t)} \varphi(x). \]  

(68)

Before giving the proof of the Theorem 4.2 we first state an adapted version of the Lemma 2.2.4 proved in [43] (page 83).

**Lemma 4.2** Let \((\xi_t)_{t \geq 0}\) be an \(\mathcal{F}_t\)-progressive process such that \(\forall t \geq 0\),

\[ E \left[ \int_0^t \xi_s^2 ds \right] < \infty \]

then

\[ \tilde{E} \left[ \int_0^t \xi_s dB_s^1 | \mathcal{F}_\infty^{23} \right] = 0, \]

\[ \tilde{E} \left[ \int_0^t \xi_s dX_s | \mathcal{F}_\infty^{23} \right] = \int_0^t \tilde{E} \left[ \xi_s | \mathcal{F}_\infty^{23} \right] dX_s \]

and

\[ \tilde{E} \left[ \int_0^t \xi_s dY_s | \mathcal{F}_\infty^{23} \right] = \int_0^t \tilde{E} \left[ \xi_s | \mathcal{F}_\infty^{23} \right] dY_s. \]

**Proof.** (of Theorem 4.2)

Let consider the probability \(\tilde{P}\).

\[ d\varphi(\theta_t) = A_1^1 \varphi(\theta_t) dt + g_1(t, \theta_t) \varphi'(\theta_t) dB_1^1, \]

(69)

\[ d\tilde{Z}_t = A_1^2 \tilde{Z}_t dX_t + A_1^3 \tilde{Z}_t dY_t, \]

(70)

and

\[ \tilde{Z}_t \varphi(\theta_t) = \tilde{Z}_0 \varphi(\theta_0) + \int_0^t \tilde{Z}_s A_1^1 \varphi ds + \int_0^t \tilde{Z}_s g_1(s, \theta_s) \varphi'(\theta_s) dB_1^1 \]

\[ + \int_0^t \tilde{Z}_s A_1^2 \varphi dX_s + \int_0^t \tilde{Z}_s A_1^3 \varphi dY_s. \]

(71)

We now use Lemma 4.2 since \((\theta, X, Y)\) is continuous and bounded, \(\varphi \in C^2(O)\) and the parameters of the model are locally bounded with respect to the time and Lipschitz continuous with respect to the other
variables.
\[ \zeta_t (\varphi) = \tilde{E} \left[ \tilde{Z}_t \varphi (\theta_t) \big| \mathcal{F}_\infty^t \right] \]
\[ = \tilde{E} \left[ \tilde{Z}_0 \varphi (\theta_0) \big| \mathcal{F}_\infty^t \right] + \tilde{E} \left[ \int_0^t \tilde{Z}_s A_s^1 \varphi (\theta_s) \, ds \big| \mathcal{F}_\infty^t \right] + \tilde{E} \left[ \int_0^t \tilde{Z}_s g^1 (s, \theta_s) \varphi' (\theta_s) \, dB_s^1 \big| \mathcal{F}_\infty^t \right] \]
\[ + \tilde{E} \left[ \int_0^t \tilde{Z}_s A_s^2 \varphi (\theta_s) \, dX_s \big| \mathcal{F}_\infty^t \right] \]
\[ = \zeta_0 (\varphi) + \int_0^t \tilde{E} \left[ \tilde{Z}_s A_s^1 \varphi (\theta_s) \big| \mathcal{F}_\infty^t \right] ds + \int_0^t \tilde{E} \left[ \tilde{Z}_s A_s^2 \varphi (\theta_s) \big| \mathcal{F}_\infty^t \right] dX_s \]
\[ + \int_0^t \tilde{E} \left[ \tilde{Z}_s A_s^3 \varphi (\theta_s) \big| \mathcal{F}_\infty^t \right] dY_s \]
\[ = \zeta_0 (\varphi) + \int_0^t \zeta_s (A_s^1 \varphi) ds + \int_0^t \zeta_s (A_s^2 \varphi) dX_s + \int_0^t \zeta_s (A_s^3 \varphi) dY_s. \]

The normalized law \( \pi \) of \( \theta \) is given by the following

**Theorem 4.3** (The Kushner-Stratonovich equation)

If \( O \subseteq \mathbb{R} \) is an open set containing \([0, 1]\) and \( \varphi \in C^2 (O) \) then \( \forall t \geq 0, \)
\[ \pi_t (\varphi) = \pi_0 (\varphi) + \int_0^t \pi_s (A_s^1 \varphi) ds + \int_0^t \pi_s (\varphi) \left( \pi_s^2 (A_s^2 \varphi) + \pi_s^2 (A_s^3 \varphi) \right) ds \\
- \int_0^t \left( \pi_s (A_s^2 \varphi) \pi_s (A_s^2 \varphi) + \pi_s (A_s^3 \varphi) \pi_s (A_s^3 \varphi) \right) ds \\
+ \int_0^t \left( \pi_s (A_s^2 \varphi) - \pi_s (\varphi) \pi_s (A_s^2 \varphi) \right) dX_s \\
+ \int_0^t \left( \pi_s (A_s^3 \varphi) - \pi_s (\varphi) \pi_s (A_s^3 \varphi) \right) dY_s. \]  

**Proof.** Using Theorem 4.2

\[ \zeta_t = \zeta_0 (1) \]
\[ = \zeta_0 (1) + \int_0^t \zeta_s (A_s^2 \varphi) dX_s + \int_0^t \zeta_s (A_s^3 \varphi) dY_s \]
\[ = 1 + \int_0^t \zeta_s (A_s^2 \varphi) dX_s + \int_0^t \zeta_s (A_s^3 \varphi) dY_s. \]

It follows that

\[ \zeta_t = \exp \left( -\frac{1}{2} \int_0^t \pi_s^2 (A_s^2 \varphi) ds + \int_0^t \pi_s (A_s^3 \varphi) ds \right) \times \exp \left( \int_0^t \pi_s (A_s^2 \varphi) dX_s + \int_0^t \pi_s (A_s^3 \varphi) dY_s \right) \]

and

\[ \zeta_t = \exp \left( -\frac{1}{2} \int_0^t \pi_s^2 (A_s^2 \varphi) ds + \int_0^t \pi_s (A_s^3 \varphi) ds + \int_0^t \pi_s (A_s^2 \varphi) dX_s + \int_0^t \pi_s (A_s^3 \varphi) dY_s \right). \]

We can also compute

\[ d \left( \frac{1}{\zeta_t} \right) = \frac{1}{\zeta_t} \left( \zeta_t^2 (A_t^2 \varphi) + \zeta_t^2 (A_t^3 \varphi) \right) dt - \frac{1}{\zeta_t^2} \left( \zeta_t (A_t^2 \varphi) dX_t + \zeta_t (A_t^3 \varphi) dY_t \right) \]
\[ = \frac{1}{\zeta_t} \left( \pi_t^2 (A_t^2 \varphi) dt + \pi_t^2 (A_t^3 \varphi) dt \right) - \frac{1}{\zeta_t} \left( \pi_t (A_t^2 \varphi) dX_t + \pi_t (A_t^3 \varphi) dY_t \right) \]

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and therefore,
\[
\begin{align*}
d\pi_t (\varphi) &= d \left( \frac{\bar{q}(\varphi)}{\zeta_t} \right) \\
&= \pi_t (A^1_t \varphi) dt + \pi_t (A^2_t \varphi) dX_t + \pi_t (A^3_t \varphi) dY_t + \pi_t (\varphi) (\pi^2_t (A^1_t 11) dt + \pi^2_t (A^1_t) dt) \\
&\quad - \pi_t (\varphi) (\pi_t (A^2_t 1) dX_t + \pi_t (A^3_t 1) dY_t) - \pi_t (A^2_t \varphi) \pi_t (A^1_t 1) dt - \pi_t (A^3_t \varphi) \pi_t (A^1_t 1) dt \\
&= (\pi_t (A^1_t \varphi) + \pi_t (\varphi) \pi^2_t (A^1_t) + \pi_t (\varphi) \pi^2_t (A^3_t 1)) dt - \pi_t (A^2_t \varphi) \pi_t (A^1_t 1) dt \\
&\quad + \pi_t (A^2_t \varphi) \pi_t (A^1_t 1) dt + \pi_t (A^3_t \varphi) dX_t - \pi_t (\varphi) \pi_t (A^3_t 1) dY_t \\
&\quad - \pi_t (\varphi) \pi_t (A^3_t 1) dY_t.
\end{align*}
\]

We end this subsection with the following useful

**Theorem 4.4** Assume that $P$-almost surely $(\theta_0, v_0, \rho_0) \in [0,1]^3$. Let $O$ in the Theorem 4.3 be bounded and $\varphi \in L^2 (O)$. If there is $T > 0$ such that $f_1 \in L^\infty ([0,T] \times \Omega; W^{1,\infty} (O; \mathbb{R}))$, $g_1 \in L^\infty ([0,T] \times \Omega; W^{2,\infty} (O; \mathbb{R}))$, $f \in L^\infty ([0,T] \times \Omega; L^\infty (\mathbb{R} \times O; \mathbb{R}))$ and $g \in L^\infty ([0,T] \times \Omega; L^\infty (\mathbb{R}^2 \times O; \mathbb{R}))$ then the solution of the equation (65) is unique and $\varsigma$ can be identified to an element of $L^\infty ([0,T] \times \Omega; H^1 (O))$.

Before giving the a proof for the Theorem 4.4 we first state a result based on the Theorem 3.2.4 and Remark 3.2.6 in [43] (pages 105 and 106).

**Theorem 4.5** Let $\mathcal{X}$ and $\mathcal{Y}$ denote two Hilbert spaces such that $\mathcal{X}$ is continuously embedded and dense in $\mathcal{Y}$ which is identified with its dual space. Let $\mathcal{X}'$ denote dual space of $\mathcal{X}$, $F \in L^\infty ([0,T] \times \Omega; \mathcal{L}(\mathcal{X}, \mathcal{X}'))$, $f \in L^2 ([0,T] \times \Omega; \mathcal{X}'')$, $g \in L^2 ([0,T] \times \Omega; \mathcal{L}(\mathcal{Y}^N, \mathcal{Y}))$, $G \in L^\infty ([0,T] \times \Omega; \mathcal{L}(\mathcal{X}; \mathcal{L}(\mathcal{Y}^N, \mathcal{Y})))$. We identify $\mathcal{L}(\mathcal{Y}^N, \mathcal{Y})$ with $\mathcal{Y}^N$ and assume that there are constants $c, C > 0$ such that for all $\forall \psi \in \mathcal{X}$, $\forall t \in [0,T]$,

\[
2 \langle F(t) \psi, \psi \rangle + C \| \psi \|^2_\mathcal{Y} \geq c \| \psi \|^2_\mathcal{X} + \sum_{i=1}^N \| G_i (t, \psi) \|^2_\mathcal{Y}.
\]

Let $B$ denote the standard $\mathcal{Y}^N$-valued Brownian motion. Then there is a unique $\Phi \in L^2 ([0,T] \times \Omega; \mathcal{X})$ satisfying $\forall t \in [0,T]$,

\[
\Phi(t) = \phi - \int_0^t (F(s) \Phi(s) + f(s)) ds + \int_0^t (G(s, \Phi(s)) + g(s)) dB_s.
\]

Looking at the proof of the Theorem 4.4 given in [43] it is clear that it still remains true if $f$ and $G$ have a Lipschitz continuous dependence on $\Phi$.

**Proof.** (of the Theorem 4.4)

We use the Theorem 4.3 setting $\mathcal{X} = H^1 (O)$ and $\mathcal{Y} = L^2 (O)$. Note that the inclusions $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{X}'$ hold with continuous dense injections. Let $T > 0$ be an arbitrary fixed time and $A^1_t, A^2_t, A^3_t$ be the respective adjoint operators of $A^1_t, A^2_t, A^3_t : \mathcal{X} \rightarrow \mathcal{Y}$. Omitting $\varphi$ in (65) we have the following SPDE

\[
\varsigma_t = \varsigma_0 + \int_0^t A^1_t \varsigma_s ds + \int_0^t A^2_s \varsigma_s dX_s + \int_0^t A^3_s \varsigma_s dY_s.
\]
Let $\psi \in \mathcal{X}$ and $t \in [0, T]$.

$$-2 \langle A^1_t \psi, \psi \rangle = -2 \langle \psi, A^1_t \psi \rangle$$

$$= -2 \int_U \left( f_1 (t, x) \psi' (x) + g_1^2 (t, x) \psi'' (x) \right) \psi (x) \, dx$$

$$= \int_U f_1' (s, x) \psi^2 (x) \, dx + \int_U \left( g_1^2 (t, x) \psi (x) \right)' \psi' (x) \, dx$$

$$= \int_U f_1' (s, x) \psi^2 (x) \, dx + \int_U g_1^2 (t, x) \left( \psi' (x) \right)^2 \, dx + \int_U \left( g_1^2 (t, x) \right)' \psi (x) \psi' (x) \, dx$$

$$\geq -C_1^{T,O} \| \psi \|^2_Y + C_2^{T,O} \| \psi' \|^2_Y$$

with

$$C_1^{T,O} = \text{ess sup}_{x \in U, s \in [0, T]} \left| f_1' (s, x) - \frac{1}{2} \left( g_1^2 (t, x) \right)' \right| \geq 0$$

and

$$C_2^{T,O} = \inf_{t \in [0, T]} \{ \delta_1 (t) \} C_3^{T,O} > 0$$

The existence of $C_3^{T,O}$ is guaranteed because $\psi \in \mathcal{X} \mapsto \| \kappa_1 (x) \psi' \|^2_Y$ is a norm equivalent to the usual norm since the Assumption $[X]$ is satisfied and we necessarily have $\| \kappa_1 (x) \psi' \|^2_Y > 0$ when $\| \psi' \|^2_Y > 0$. $A^1_t$ and $A^3_t$ are self-adjoint and

$$\| A^2_t \psi (x) \|^2_Y + \| A^3_t \psi (x) \|^2_Y \leq C_4^{T,O} \int_U \psi^2 (x) \, dx$$

with

$$C_4^{T,O} = \text{ess sup}_{x \in U, t \in [0, T]} \left( \frac{f_2^2 (t, \bar{X}_t, \theta_t)}{\delta_3^4 (t)} + \frac{g_2^2 (t, \bar{X}_t, \bar{Y}_t, \theta_t)}{\delta_3^4 (t)} \right) \geq 0$$

The existence of $C_4^{T,O}$ is due to the boundedness of $f_i$, $\forall i \in \{ 2, 3 \}$ and Lemma 3.2. Hence, the condition is satisfied with $C > C_1^{T,O} + C_4^{T,O}$ and $c \leq \min \{ C_1^{T,O} + C_4^{T,O}, C_2^{T,O} \}$. Theorefore $\varsigma_t \in \mathcal{X}$ is the unique solution of (76).

Theorem 4.4 gives conditions under which $(\varsigma_t)_{t \geq 0}$ and therefore $(\pi_t)_{t \geq 0}$ are uniquely defined by their respective equations. Note that the domain of $(\varsigma_t)_{t \geq 0}$ and $(\pi_t)_{t \geq 0}$ has been extended from $C^2 (O)$ to $L^2 (O)$.

## 5 State estimation with discrete time observations

In this section we, consider a realistic case where observations are discretely made with respect to an increasing sequence of nonnegative stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \tau_n = \infty$. That situation occurs frequently when following a phenomenon since it is difficult to collect data continuously. The problem here is to find $E \left[ \phi (\theta_t) \ | \ F^{23}_{\tau_n} \right]$, $\forall t \in [\tau_n, \tau_{n+1} [ \, , \ \forall \phi \in L^\infty_{loc} (\mathbb{R})$. We may mention here that $(F^{23}_{\tau_n})$ is the discrete filtration generated by the process $((v_{\tau_n}, \rho_{\tau_n}))_{n \in \mathbb{N}}$, parameters and all $P$-null sets. We can distinguish two cases. Indeed, if $t \in [\tau_n, \tau_{n+1} [ \, , \ assuming that the law of $\theta_{\tau_n}$ is known then we have to solve a prediction problem. The case $t = \tau_n$ corresponds to a discrete filtering problem. We study those two situations in the following.

---

10See Proposition 8.13 in [12] (page 218) on Poincaré’s inequality and the open mapping Theorem 2.6 (page 35 ).
5.1 Prediction problem

In this subsection, we assume that at each time $t \geq 0$ only the observations $((v_{s\wedge \tau}, \rho_{s\wedge \tau}))_{0 \leq s \leq t}$ are available with $\tau$ a stopping time. To deal with the prediction problem we can just assume as in [33] that after $\tau$ the observations are reduced to a new independent Brownian motion. That is $\forall t \geq 0$,

$$\hat{X}_t = \mathbf{X}_t + \int_0^t 1_{\{s \leq \tau\}} \delta_2(s) dB_s^2 + W_t^2 - W_{t\wedge \tau}^2$$

(77)

and

$$\hat{Y}_t = \mathbf{Y}_t + \int_0^t 1_{\{s \leq \tau\}} \delta_3(s) dB_s^3 + W_t^3 - W_{t\wedge \tau}^3.$$  

(78)

where $(W^2, W^3) = W$ is an independent two-dimensional Brownian motion. We can easily check that $(\hat{X}_{t\wedge \tau}, \hat{Y}_{t\wedge \tau}) = (X_{t\wedge \tau}, Y_{t\wedge \tau})$ and $(\hat{X}_{t\wedge \tau'}, \hat{Y}_{t\wedge \tau'}) = (X_{t\wedge \tau'}, W_{t\wedge \tau'}^2 - W_{t\wedge \tau'}^3, Y_{t\wedge \tau'} + W_{t\wedge \tau'}^3 - W_{t\wedge \tau'}^3)$. That permits us to use a similar approach with the Subsection 4.2. If $(\hat{F}_t^{23})$ is the filtration generated by $(\hat{X}, \hat{Y})$, parameters and all $P$-null sets then by the independence of $W$ we have $E \left[ \theta_t | \hat{F}_{t\wedge \tau}^{23} \right] = E \left[ \theta_t | \hat{F}_t^{23} \right]$. Let $\forall \varphi \in L^\infty(\mathbb{R}; \mathbb{R})$, $\forall t \geq 0$,

$$\hat{Z}_t = \hat{Z}_{t\wedge \tau}$$

(79)

$$\hat{\pi}_t(\varphi) = E \left[ \varphi(\theta_t) | \hat{F}_t^{23} \right]$$

(80)

$$\hat{\zeta}_t = \hat{E} \left[ \hat{Z}_t | \hat{F}_t^{23} \right] = \zeta_{t\wedge \tau}$$

(81)

$$\hat{\varsigma}_t(\varphi) = \hat{E} \left[ \hat{Z}_t \varphi(\theta_t) | \hat{F}_t^{23} \right] = \hat{\varsigma}_t \hat{\pi}_t(\varphi)$$

(82)

Note that if $t \leq \tau$ then $\hat{\varsigma}_t = \varsigma_t$ and $\varsigma_t = \hat{\varsigma}_t$.

The dynamics of the unnormalized law $\hat{\varsigma}$ is given by the

Theorem 5.1 (The Zakai prediction equation)

If $O \subseteq \mathbb{R}$ is an open set containing $[0, 1]$ and $\varphi \in C^2(O)$ then $\forall t > \tau$,

$$\hat{\varsigma}_t(\varphi) = \varsigma_0(\varphi) \varsigma_t(1) + \int_0^t \hat{\varsigma}_s (A_1^3 \varphi - \varsigma_0(\varphi) A_1^3 1) ds + \int_0^t \varsigma_0 (A_3^2 \varphi - \varsigma_0(\varphi) A_3^2 1) dX_s$$

(83)

$$+ \int_0^t \varsigma_0 (A_3^3 \varphi - \varsigma_0(\varphi) A_3^3 1) dY_s$$

where $\forall i \in \{1, 2, 3\}$, $A_i^t$ and $\varsigma$ are given in the Theorem 4.2.

Proof. Using the Lemma 4.2 and integration by part formula we have

$$\hat{\varsigma}_t(\varphi) = \hat{E} \left[ \hat{Z}_t \varphi(\theta_t) | \hat{F}_t^{23} \right]$$

$$= \hat{E} \left[ \hat{Z}_t \varphi(\theta_0) | \hat{F}_t^{23} \right] + \hat{E} \left[ \hat{Z}_t \int_0^t A_1^1 \varphi(\theta_s) ds | \hat{F}_t^{23} \right] + \hat{E} \left[ \hat{Z}_t \int_0^t g^1(\theta_s) D \varphi(\theta_s) dB_s^1 | \hat{F}_t^{23} \right]$$

$$= \varsigma_0(\varphi) \varsigma_t(1) + \hat{E} \left[ \hat{Z}_t \int_0^t A_1^1 \varphi(\theta_s) ds | \hat{F}_t^{23} \right] + \hat{E} \left[ \int_0^t \hat{Z}_s A_1^1 \varphi(\theta_s) ds | \hat{F}_t^{23} \right]$$

$$= \varsigma_0(\varphi) \varsigma_t(1) + \hat{E} \left[ \int_0^t \hat{Z}_s A_1^1 \varphi(\theta_s) ds | \hat{F}_t^{23} \right] + \int_0^t \hat{E} \left[ \hat{Z}_s A_1^1 \varphi(\theta_s) ds | \hat{F}_t^{23} \right]$$

$$+ \hat{E} \left[ \int_0^t A_1^3 \varsigma_t \int_0^s A_1^1 \varphi(\theta_r) dr dY_s | \hat{F}_t^{23} \right] + \int_0^t \hat{E} \left[ \hat{Z}_s A_1^1 \varphi(\theta_s) ds | \hat{F}_t^{23} \right] ds$$

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\[
\begin{align*}
= s_0(\varphi) &\mathcal{C}_\tau(1) + \int_0^t \mathbb{E} \left[ \tilde{Z}_s A_s^1 \varphi(\theta_s) | \mathcal{F}^2_{\infty} \right] ds + \int_0^\tau \mathbb{E} \left[ (\varphi(\theta_s) - \varphi(\theta_0)) A_s^2 \tilde{Z}_s | \mathcal{F}^2_{\infty} \right] dX_s \\
+ \int_0^\tau \mathbb{E} \left[ (\varphi(\theta_s) - \varphi(\theta_0)) A_s^3 \tilde{Z}_s | \mathcal{F}^2_{\infty} \right] dY_s \\
= s_0(\varphi) &\mathcal{C}_\tau(1) + \int_0^t \tilde{c}_s (A_s^1 \varphi) ds + \int_0^\tau s_s (A_s^2 \varphi - s_0(\varphi) A_s^2 1) dX_s \\
+ \int_0^\tau s_s (A_s^3 \varphi - s_0(\varphi) A_s^3 1) dY_s
\end{align*}
\]

The dynamics of the normalized law \( \hat{\pi} \) is given by the

**Theorem 5.2** *(The Kushner-Stratonovich prediction equation)*

If \( O \subseteq \mathbb{R} \) is an open set containing \([0,1] \) and \( \varphi \in C^2(O) \) then \( \forall t > \tau \),

\[
\hat{\pi}_t(\varphi) = \pi_0(\varphi) + \int_0^\tau \hat{\pi}_s (A_s^1 \varphi) ds - \int_0^\tau \pi_s (A_s^2 \varphi - \pi_0(\varphi) A_s^2 1) \pi_s (A_s^2 1) ds
\]

\[+ \int_0^\tau \pi_s (A_s^3 \varphi - \pi_0(\varphi) A_s^3 1) \pi_s (A_s^3 1) ds\]  
(84)

**Proof.** Let consider the probability \( \hat{P} \) and \( t > \tau \). Using integration by part formula we have

\[
\begin{align*}
\hat{\pi}_t(\varphi) = s_0(\varphi) &\mathcal{C}_\tau(1) + \frac{1}{\mathbb{C}_\tau} \int_0^\tau \tilde{c}_s (A_s^1 \varphi) ds + \frac{1}{\mathbb{C}_\tau} \int_0^\tau \tilde{c}_s (A_s^1 \varphi) ds \\
+ \frac{1}{\mathbb{C}_\tau} \int_0^\tau s_s (A_s^2 \varphi - s_0(\varphi) A_s^2 1) dX_s + \frac{1}{\mathbb{C}_\tau} \int_0^\tau s_s (A_s^3 \varphi - s_0(\varphi) A_s^3 1) dY_s \\
= s_0(\varphi) &- \int_0^\tau \pi_s (A_s^2 \varphi - s_0(\varphi) A_s^2 1) \pi_s (A_s^2 1) ds \\
- \int_0^\tau \pi_s (A_s^3 \varphi - s_0(\varphi) A_s^3 1) \pi_s (A_s^3 1) ds \\
+ \int_0^\tau \left( \int_0^s \mathcal{C}_\tau (A_s^2 \varphi - s_0(\varphi) A_s^2 1) ds \right) d\left( \frac{1}{\mathbb{C}_s} \right) \\
+ \int_0^\tau \left( \int_0^s \mathcal{C}_\tau (A_s^3 \varphi - s_0(\varphi) A_s^3 1) ds \right) d\left( \frac{1}{\mathbb{C}_s} \right) \\
+ \int_0^\tau \hat{\pi}_s (A_s^1 \varphi) ds + \int_0^\tau \left( \int_0^s \mathcal{C}_\tau (A_s^1 \varphi) ds \right) d\left( \frac{1}{\mathbb{C}_s} \right)
\end{align*}
\]
We already know properties of the operators \( A \).

Proof. The proof is similar to the one of Theorem 4.2. We also use Theorem 4.5 setting \( L \) are satisfied and the result follows.

Since \( \varrho \) is Markovian and therefore the discrete process \( \theta, v, \rho \) can be identified to an element of \( L^2(\Omega) \) is unique and \( \hat{\zeta} \) can be identified to an element of \( L^2([0, T] \times \Omega; H^1_0(\omega)) \).

5.2 Discrete filtering problem

In this subsection, we consider the discrete filtering problem mentioned above. The process \( (\theta, v, \rho) \) is Markovian and therefore the discrete process \( ((\theta_{n+1}, v_{n+1}, \rho_{n+1}))_{n \in \mathbb{N}} \) is a Markov chain. To achieve our objective we will make some approximations in order to have a discrete filtering problem. To make simple the notations, we introduce when there is not ambiguity the index \( n \) to play the role of \( \tau_n \). If \( \Delta_n \) denotes the difference \( \tau_{n+1} - \tau_n \) and it is sufficiently small then the following approximations hold for a given \( \vartheta \in [0, 1] \):

\[
\Delta \theta_n \equiv \theta_{n+1} - \theta_n \\
\simeq \Delta \tau_n f_1(n (1 - \vartheta) \theta_n + \vartheta \theta_{n+1}) + \sqrt{\Delta \tau_n g_1(n (\theta_n) \xi_{1,n, n}},
\]

\[
\Delta X_n \equiv X_{n+1} - X_n \\
\simeq \sqrt{\Delta \tau_n} \theta_{2,n} \xi_{2,n} \\
+ \Delta \tau_n f_2((1 - \vartheta) \theta_n + \vartheta \theta_{n+1}) (1 - \vartheta) \theta_n + \vartheta \theta_{n+1})
\]

and

\[
\Delta Y_n \equiv Y_{n+1} - Y_n \\
\simeq \sqrt{\Delta \tau_n} \xi_{3,n} \\
+ \Delta \tau_n g_2((1 - \vartheta) Y_n + \vartheta Y_{n+1} (1 - \vartheta) \theta_n + \vartheta \theta_{n+1})
\]

with \( (\xi_n)_{n \in \mathbb{N}} = ((\xi_{1,n, \xi_{2,n}, \xi_{3,n}}))_{n \in \mathbb{N}} \) a sequence of independent identically distributed centred and normalized gaussian vectors. The use of the term \( \vartheta \) corresponds to the well-known theta method in the large literature of numerical analysis. It is justified by the fact that the mathematical expectation of \( (\theta_t, v_t, \rho_t)_{t \geq 0} \) is differentiable and we can use the finite increments formula. That cannot be applied to the Brownian term if we want to keep safe the properties of the Itô integral. We refer to the works in [23, 26, 29, 45] and references therein to know further about stochastic numerical schemes.
Let $\forall n \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $\overline{Z}_0 = \overline{X}_0 = 1$,

$\overline{X}_n (x,y,z) = \exp \left( -\Delta \tau_n f_n^2 ((1-\vartheta) \overline{X}_n + \vartheta \overline{X}_{n+1}, (1-\vartheta) \theta_n + \vartheta x) / 2\delta^2_{2,n} \right) \times \exp \left( -\Delta \tau_n g_n^2 ((1-\vartheta) \overline{X}_n + \vartheta \overline{X}_{n+1}, (1-\vartheta) \overline{Y}_n + \vartheta \overline{Y}_{n+1}, (1-\vartheta) \theta_n + \vartheta x) / 2\delta^2_{2,n} \right) \times \exp \left( z_n ((1-\vartheta) \overline{X}_n + \vartheta \overline{X}_{n+1}, (1-\vartheta) \overline{Y}_n + \vartheta \overline{Y}_{n+1}, (1-\vartheta) \theta_n + \vartheta x) / \delta^2_{3,n} \right)$

(89)

and

$\mathcal{Z}_n = \prod_{i=0}^{n} \overline{X}_i (\theta_{i+1}, \Delta X_i, \Delta Y_i)$

(90)

By the Girsanov theorem, the discrete process $(\mathcal{Z}_n)_{n \in \mathbb{N}}$ is an $(\mathcal{F}_{n}^{23})$-martingale and there is a probability $\mathcal{P}$ such that

$\mathcal{Z}_n = \frac{d\mathcal{P} |_{\mathcal{F}_n^{23}}}{d\mathcal{P} |_{\mathcal{F}_n^{23}}}.$

(91)

Note that parameters of the model and $\theta$ keep the same law either under $P$ or $\mathcal{P}$. Moreover, under $\mathcal{P}$ parameters and $\theta$ are independent with the process $(\sqrt{\Delta \tau_n \frac{g_{n} \mathcal{X}_n (\vartheta \mathcal{X}_n, \vartheta \mathcal{Y}_n)}{n, \vartheta \mathcal{X}_n, \vartheta \mathcal{Y}_n}})_{n \in \mathbb{N}}$ which is a sequence of independent identically distributed centered and normalized gaussian vectors.

Let also define $\forall n \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $\forall \varphi \in L^\infty (\mathbb{R}; \mathbb{R})$

$\zeta_n = \mathbb{E} \left( [\mathcal{Z}_n | \mathcal{F}_n^{23}] \right),$

(92)

$\pi_n (\varphi) = \mathbb{E} \left( [\varphi (\theta_n) | \mathcal{F}_n^{23}] \right),$

(93)

$s_{n,n+1} (\varphi) = \mathbb{E} \left( [\mathcal{Z}_n \varphi (\theta_{n+1}) | \mathcal{F}_{n+1}^{23}] \right),$  

(94)

$\zeta_n (\varphi) = \mathbb{E} \left( [\mathcal{Z}_n \varphi (\theta_n) | \mathcal{F}_{n}^{23}] \right) = \zeta_n \pi_n (\varphi)$

(95)

and

$P_n (x, \varphi) = \mathbb{E} \left( [\varphi (\theta_{n+1}) | \theta_n = x] \right).$

(96)

The main result of this subsection is the following

**Theorem 5.4** $\forall n \in \mathbb{N}$, $\forall \varphi \in L^\infty (\mathbb{R}; \mathbb{R})$,

$s_{n+1} (\varphi) = s_n \left( P_n (\cdot, \overline{X}_{n+1} (\cdot, x, y) \varphi) \right) |_{x=\Delta X_n, y=\Delta Y_n}$

(97)

and

$\zeta_{n+1} = s_n \left( P_n (\cdot, \overline{X}_{n+1} (\cdot, x, y)) \right) |_{x=\Delta X_n, y=\Delta Y_n}$

(98)

where $\zeta_0 = 1$, $s_0$ is assumed known and

$P_n (x, \varphi) = E \left[ \varphi \left( \frac{x + f_{1,n} ((1-\vartheta) x)}{1 + \alpha_n w_n \vartheta} \Delta \tau_n + \frac{g_{1,n} (x) \sqrt{\Delta \tau_n} \xi_n}{1 + \alpha_n w_n \vartheta} s_{n+1} \right) \right].$

(99)

**Proof.**

$s_{n+1} (\varphi) = \mathbb{E} \left( [\mathcal{Z}_{n+1} \varphi (\theta_{n+1}) | \mathcal{F}_{n+1}^{23}] \right)$

$= \mathbb{E} \left( [\mathcal{Z}_n \overline{X}_{n+1} (\theta_{n+1}, \Delta X_n, \Delta Y_n) \varphi (\theta_{n+1}) | \mathcal{F}_{n+1}^{23}] \right)$

$= \mathbb{E} \left( [\mathcal{Z}_n \mathbb{E} \left( [\overline{X}_{n+1} (\theta_{n+1}, \Delta X_n, \Delta Y_n) \varphi (\theta_{n+1}) | \theta_{n+1}, \mathcal{F}_{n+1}^{23}] | \mathcal{F}_{n+1}^{23} \right) | \mathcal{F}_{n}^{23}] \right)$

$= \mathbb{E} \left( [\mathcal{Z}_n \mathbb{E} \left( [P_n (\theta_n, \theta_{n+1}, \Delta X_n, \Delta Y_n) \varphi (\theta_{n+1}) | \theta_n, \mathcal{F}_{n+1}^{23}] | \mathcal{F}_{n+1}^{23} \right) | \mathcal{F}_{n}^{23}] \right)$

$= \mathbb{E} \left( [\mathcal{Z}_n P_n (\theta_n, \overline{X}_{n+1} (\cdot, x, y) \varphi (\cdot)) |_{x=\Delta X_n, y=\Delta Y_n} | \mathcal{F}_{n}^{23}] \right) | \mathcal{F}_{n}^{23}$

$= s_n \left( P_n (\cdot, \overline{X}_{n+1} (\cdot, x, y) \varphi) \right) |_{x=\Delta X_n, y=\Delta Y_n}$
The equation (98) is obtained when we apply simply the formula \( \zeta_{n+1} = \zeta_{n} (1) \).

\[
P_n (x, \varphi) = E[\varphi (\theta_{n+1}) | \theta_n = x] \\
= E[\varphi (\theta_{n+1}) | \theta_n = x] \\
= E \left[ \varphi \left( \frac{x + \alpha_n (1 - w_n (1 - \vartheta)) x}{1 + \alpha_n w_n \vartheta} \Delta \tau_n + g_{1,n} (x) \sqrt{\Delta \tau_n} \epsilon_1 \right) \right].
\]

\[
\quad
\]

6 Numerical illustrations of the time continuous filtering

In this section, we carry out some simulations in order to have an idea on the behaviour of the optimal filter we have theoretically studied in previous sections. We use the Theorem 4.2 to solve the SPDE (76). For the reasons of stability, memory space and simulation time, we take relatively big space stepsize \( \Delta x = 0.1 \) and time stepsize \( \Delta t = 10^{-3} \) for the resolution of the equation (76). We simply use the well-known Euler’s numerical scheme\(^{11}\). The parameters are taken following [21, 22]. The control strategy \( u \) is given for every time \( t \geq 0 \) by

\[
u (t) = \sin^2 \left( \omega_1 (t - \varphi_1)^2 \right) \exp \left( -\omega_2 (t - \varphi_2)^2 \right).
\]

The functions \( \alpha, \beta \) and \( \gamma \) are taken with the following form

\[
\alpha (t) = p_1 (t) + b_1 (1 - \cos (c_1 t)) (t - d_1)^2, \forall t \in \mathbb{R}_+, \hspace{1cm} (101)
\]

\[
\beta (t, x) = b_2 (1 - \cos (c_2 t)) (t - d_2)^2 p_2 (x), \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \hspace{1cm} (102)
\]

and

\[
\gamma (t, x_1, x_2, x_3) = b_3 (1 - \cos (c_3 t)) (t - d_3)^2 (x_1 - \kappa x_3) x_2, \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3. \hspace{1cm} (103)
\]

\( p_1 \) is a nonnegative function of the time \( t \) and \( p_2 \) is a real positive function of \( x \). \( \forall i \in \{1, 2, 3\}, b_i, c_i, \) and \( d_i \) are positive coefficients corresponding respectively to the maximal amplitude, the pulsation and the global maximum of \( \alpha, \beta \) and \( \gamma \). \( \kappa \) is a constant positive function of the evolution of the rot volume with respect to the inhibition rate. The terms \( 1 - \cos (c_i t) \) represent the seasonality probably due to climatic and environmental variations. Concerning the random parts of the equations, the functions \( \delta_i \) are assumed constant (the upper bound for instance) and \( \kappa_i (x) = x (1 - x) \) if \( x \in [0, 1] \) and is null elsewhere. The initial conditions are taken such as \( \theta (0) \in \{0.05, 0.75\}, \nu (0) \in \{0.25, 0.50\} \) and \( \rho (0) \in \{0.25, 0.75\} \).

The following table gives the assumed parameters values.

| Parameters | Values | Source | Parameters | Values | Source |
|-----------|--------|--------|-----------|--------|--------|
| \( b_1 \) | \( 5 \ln (10) \) | [22] | \( v_{\text{max}} \) | 1 | [22] |
| \( b_2 \) | \( v_{\text{max}} \ln (10^5 v_{\text{max}} (1 - \varepsilon^*)) / 2 \) | [22] | \( \varepsilon \) | \( 10^{-4} \) | [22] |
| \( b_3 \) | \( v_{\text{max}} \ln (10^5 v_{\text{max}}) \) | [22] | \( \sigma \) | 0.9 | [21, 22] |
| \( c_i, i = 1, 2, 3 \) | 10\( \pi \) | [21, 22] | \( \kappa \) | 1 | [22] |
| \( d_i, i = 1, 2, 3 \) | \( 7.5 \times 10^{-1} \) | [21, 22] | \( \Delta t \) | \( 10^{-3} \) | Assumed |
| \( \omega_1 \) | 25\( \pi \) | [22] | \( \eta (t) \) | 1 / (1 + \varepsilon) | [22] |
| \( \omega_2 \) | 10 | [22] | \( p_1 (t) \) | 0 | [22] |
| \( \varphi_1 \) | 0.4 | [22] | \( p_2 (x) \) | 1 | Assumed |
| \( \varphi_2 \) | 0.6 | [22] | \( \delta_i = \delta_i, i = 2, 3 \) | 10^{-2} | Assumed |

Table 1: Simulation parameters for the filtering

---

\( ^{11} \)See the reference [29].

---
\( \theta (0) = v (0) = 0.05 \) \hspace{1cm} \theta (0) = 0.05 \text{ and } v (0) = 0.5

\( \theta (0) = 0.75 \text{ and } v (0) = 0.05 \) \hspace{1cm} \theta (0) = 0.75 \text{ and } v (0) = 0.5

Figure 1: Inhibition rate and optimal filter
\( \theta (0) = v (0) = 0.05 \) \quad \theta (0) = 0.05 \text{ and } v (0) = 0.5

\( \theta (0) = 0.75 \text{ and } v (0) = 0.05 \) \quad \theta (0) = 0.75 \text{ and } v (0) = 0.5

Figure 2: Relative absolute estimation error of the optimal filter
Looking at the simulations, the filter display fairly good behaviour. The estimation seems better when started soon, that is $\theta$, $v$, $\rho$ are relatively small. We also note that the variance of the absolute relative error is often big and we think it is due to the strong nonlinearity of the model, the small size of parameters $\delta_i$, $i = 1, 2, 3$ and even the stepsizes of the numerical scheme.

7 Discussion

This work is concerned by a filtering problem on anthracnose disease dynamics. The aim has been to provide an estimation of the inhibition rate based on the assumption that the fruit volume and the rotted volume are easier to know. Our used approach is similar with the one in the references \cite{22} except that we assumed a noised dynamics. The noise has been modelled by Brownian motions in order to keep a certain regularity on the solutions although taking into account uncontrolled parameters variations (changes on at least climate and environment) and errors on measurements. We have proposed and proved the well-posedness for two modelling of the noised dynamics of the observations trying to remain realistic. That work has been done both for a within host version and a space distributed version. The first modelling seems to be more natural but presents some singularities in the noise. Those singularities make difficult the application of classical filtering theory. We have then proposed through a logistic transformation the second modelling which keeps roughly speaking the same properties is easier to manage.

The filtering procedure has consisted into the determination of the law of the inhibition rate at each time conditionally upon the fruit volume and the rotted volume measurements. We have derived for that objective the Zakai and the Kushner-Stratonovich equations respectively for the unnormalized and the normalized conditional distributions. Unfortunately, we have restricted ourselves to the non-spatial model because the spatial distributed model requires more sophisticated technical tools. Indeed, the problem consists in that case to find a measure valued process operating on a functional space, since at each fixed time the inhibition rate is not anymore a real but a function of the space variable. However, we think that it might be possible to deal with that problem if we consider gaussian spaces \cite{12} and existing works such as \cite{11, 31, 8, 9, 10} on resolution of Fokker-Plank equations on infinite dimensional spaces. Additionally to the main filtering problem, we have also study related realistic problems such as prevision and discrete filtering. That has appeared important to the authors since the observations are often discrete and incomplete.

In order to illustrate numerically the filter behaviour, we have carried out several simulations solving a stochastic partial differential equation corresponding to the unnormalized conditional distribution. Following the literature \cite{6, 7, 28, 46, 49, 51}, the filter is more effective as the size of the noise is weaker. Unfortunately, that induces an increase of the variance of the filter since there is a division by the variance of the observation noise. Moreover, it makes more difficult the computations in terms of stability of the numerical scheme, time and memory required. We suggest based on the theory of Luenberger-like observers (see \cite{32}) to multiply the terms coming from the observations in filtering equations by an adequate constant. We could also replace those terms by the minimum between them and an adequate constant. That changes may permit to reduce the variance of the filter and unfortunately could neglect the informations brought by the observations. We expect in future studies to survey rigourously the properties of our filters since as far as we know that has been tried in very restrictive cases in the literature (See for instance references \cite{6, 7, 28, 46, 49, 51}).

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\footnote{See the Chapter 5 of the book \cite{33}.}
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