Floating bundles and their applications

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1 The homotopic functor, connected with floating algebra bundles

The aim of this section is to define the homotopic functor to category of Abelian groups, connected with the special classes of bundles with fiber matrix algebra or projective space. The results, which will be represented (without proofs\footnote{see some proofs in the articles (in Russian) [1] Ershov A. V. O gomotopicheskikh svoistvakh rassloenii so strukturoi gruppoi avtomorfizmov matrichnih algeb // Vestnik Moskovskogo universiteta, ser. Matematika. Mekhanika. 6, 1999, s. 56-58, [2] Ershov A. V. O K-teorii rassloenii na matrichnie algebri // UMN, T. 55, Vip. 2, 2000, c. 137-138.}) below, were obtained in the author’s dissertation.

Let $X$ be a finite CW-complex, $\tilde{M}_n := X \times M_n(\mathbb{C})$ be the product bundle over $X$ with fiber $M_n(\mathbb{C})$ (where $M_n(\mathbb{C})$ is the algebra of all $n \times n$ matrices over $\mathbb{C}$).

The next definition is motivated by well-known result that every vector bundle over compact base is the subbundle in a product bundle.

**Definition 1.** Let $A_k$ be a locally trivial bundle over $X$ with fiber $M_k(\mathbb{C})$. Assume that there exists a bundle map

\[
A_k \xrightarrow{\mu} \tilde{M}_{kl}
\]

such that for any point $x \in X$ the fiber $(A_k)_x \cong M_k(\mathbb{C})$ is embedded (by the restriction $\mu_x$ of $\mu$) into the fiber $(\tilde{M}_{kl})_x \cong M_{kl}(\mathbb{C})$ as a central simple subalgebra. Then the triple $(A_k, \mu, \tilde{M}_{kl})$ is called the *algebra bundle* (AB)
over $X$. Moreover if $k$ and $l$ are relatively prime (i.e. if their greatest common divisor $(k, l) = 1$), then the triple $(A_k, \mu, \widetilde{M}_{kl})$ is called the floating algebra bundle (FAB).

There exists the similar class of bundles with fiber projective space.

**Definition 2.** Let $X$ be a finite CW-complex; $P^{k-1}$ and $Q^{l-1}$ be two locally trivial bundles (over $X$) with fibers $\mathbb{C}P^{k-1}$ and $\mathbb{C}P^{l-1}$ respectively. By $\mathbb{C}P^{kl-1}$ denote the product bundle over $X$ with fiber $\mathbb{C}P^{kl-1}$. Assume that there exists a bundle map

\[
P^{k-1} \times Q^{l-1} \xrightarrow{\lambda} \mathbb{C}P^{kl-1}
\]

(where $P^{k-1} \times Q^{l-1}$ is the product of the bundles over $X$) such that for any point $x \in X$ the fiber $(P^{k-1} \times Q^{l-1})_x \cong \mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1}$ is embedded into the fiber $(\mathbb{C}P^{kl-1})_x \cong \mathbb{C}P^{kl-1}$ by the map of Segre. Then the bundle $P^{k-1} \times Q^{l-1}$ is called the bundle of Segre’s product (BSP), and under the condition $(k, l) = 1$ the floating bundle of Segre’s product (FBSP).

Note that the projective space $\mathbb{C}P^{n-1}$ and the matrix algebra $M_n(\mathbb{C})$ have the same group of automorphisms $\text{PGL}(n)$. There exists the special class of principal bundles such that any bundle satisfies the conditions of Definition 1 or 2 is associated with a bundle from this class. Therefore we get one-to-one correspondence between AB and BSP (respectively FAB and FBSP) over $X$.

A morphism from AB $(A_k, \mu, \widetilde{M}_{kl})$ to AB $(B_m, \nu, \widetilde{M}_{mn})$ (over $X$) is a pair $(f, g)$, where $f: A_k \hookrightarrow B_m$ and $g: \widetilde{M}_{kl} \hookrightarrow \widetilde{M}_{mn}$ are bundle maps such that their restricts to any fiber are monomorphisms of central simple algebras and the corresponding quadratic diagram

\[
\begin{array}{ccc}
\widetilde{M}_{kl} & \xrightarrow{g} & \widetilde{M}_{mn} \\
\mu \uparrow & \swarrow \nu \\
A_k & \xleftarrow{f} & B_m
\end{array}
\]
is commutative. Note that the morphism \((f, g)\) exists only if \(k \mid m, l \mid n\).

Morphisms of BSP may be defined analogously.

The set of all subalgebras in \(M_{kl}(\mathbb{C})\), that are isomorphic to \(M_k(\mathbb{C})\), is parametrized by the homogeneous manifold \(Gr'_{k,kl}\). This manifold may be considered as an analog of Grassmannian manifold. Moreover, the set of Segre’s maps \(\mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1} \hookrightarrow \mathbb{C}P^{kl-1}\) (where \(\mathbb{C}P^{kl-1}\) is fixed) is parametrized by the same manifold \(Gr'_{k,kl}\).

Note that the stable (i. e. \(\pi_r(Gr'_{k,kl})\) under the condition \(r < 2 \min\{k, l\}\)) homotopic groups of \(Gr'_{k,kl}\) are the same as for BSU.

Let us consider the canonical AB \((A'_k, \mu', \tilde{M}'_{kl})\) over \(Gr'_{k,kl}\), defined in the following way. The fiber of the bundle \(A'_k\) over a point \(x \in Gr'_{k,kl}\) is the subalgebra in \(M_{kl}(\mathbb{C})\), which corresponds to \(x\). Thus the bundle \(A'_k\) and its embedding into the product bundle \(\tilde{M}'_{kl}\) are uniquely defined. If \((k, l) = 1\), then \((A'_k, \mu', \tilde{M}'_{kl})\) is FAB.

The canonical BSP (if \((k, l) = 1\), then FBSP) \(P^{k-1} \times Q^{l-1}\) over \(Gr'_{k,kl}\) is defined in a similar way.

By the standard method of reduction of a structural group we can replace the noncompact manifold \(Gr'_{k,kl}\) by the homotopy-equivalent compact manifold \(\tilde{Gr}_{k,kl}\). Fibers of the canonical AB \((A_k, \mu, M_{kl})\) (respectively canonical BSP \(P^{k-1} \times Q^{l-1}\)) over \(Gr_{k,kl}\) have a concordant Hermitian (respectively Kählerian) structure.

Since the categories of AB and BSP (respectively FAB and FBSP) are equivalent, we shall formulate the next results only in terms of AB and FAB.

The following Proposition shows that if \((k, l) = 1\), then \(Gr_{k,kl}\) and the canonical FAB \((A_k, \mu, \tilde{M}_{kl})\) are the classifying space and the universal FAB (for FAB of the form \((A_k, \mu, \tilde{M}_{kl})\) over a finite CW-complexes) respectively.

**Proposition 3.** Let \(X\) be a finite CW-complex, \(\dim X < 2 \min\{k, l\}\), \((A_k, \mu, \tilde{M}_{kl})\) be a FAB over \(X\), i. e. \((k, l) = 1\). Then there exists a map (unique up to homotopy) \(\varphi: X \to Gr_{k,kl}\) such that \(\varphi^*(A_k, \mu, \tilde{M}_{kl}) \cong (A_k, \mu, \tilde{M}_{kl})\) (i. e. \(\varphi\) is a classifying map for \((A_k, \mu, \tilde{M}_{kl})\)).

Now let us consider an embedding \(i_{kl,mn}: Gr_{k,kl} \hookrightarrow Gr_{m,mn}\) induced by an embedding of algebras \(M_{kl}(\mathbb{C}) \hookrightarrow M_{mn}(\mathbb{C})\) (\(\Rightarrow k \mid m, l \mid n\)). Let the map \(i_{kl,mn}: \)
\[ \pi_r(Gr_{k,kl}) \to \pi_r(Gr_{m,mm}) \] be the homomorphism induced by \( i_{kl,mm} \). Let \( \gamma_{k,l} \) be the generator of \( \pi_{2s}(Gr_{k,kl}) \), then \( i_{kl,mm}(\gamma_{k,l}) = (m,n) \gamma_{m,n} \) (here we assume that \( s < \min\{k,l\} \); recall that in this case \( \pi_2(Gr_{k,kl}) = 0, \pi_{2s}(Gr_{k,kl}) = \pi_{2s}(BSU) = \mathbb{Z}, s > 1, \pi_{2s+1}(Gr_{k,kl}) = \pi_{2s+1}(BSU) = 0 \).

It follows from the last result that if \( (m, n) = 1 \) \( \Rightarrow (k, l) = 1 \) then the map \( i_{kl,mm} : Gr_{k,kl} \to Gr_{m,mm} \) induces an isomorphism of the stable homotopic groups. Therefore the homotopic groups are stabilized under the passage to the direct limit \( \lim_{(k,l)\to1} Gr_{k,kl} \). This explains the reason of separate study of FAB. In general case the localization of the homotopic groups occurs. Note that the theory of FAB (which will be stated below) cannot be reduced to the theory of usual principal bundles: embedding \( \mu \) plays an important role. Moreover a bundle \( A_k \), which forms a FAB \( (A_k, \mu, \tilde{M}_{kl}) \), has a special form.

We shall study only FAB below.

It may be proved that for any sequence \( \{k_j, l_j\}_{j \in \mathbb{N}} \) of pairs, satisfies the following conditions

\( (i) \ k_j, l_j \to \infty; \ (ii) \ k_j|k_{j+1}, l_j|l_{j+1}; \ (iii) \ (k_j, l_j) = 1, \)

the spaces \( \lim_{j \to \infty} Gr_{k_j,k_j,l_j} \) are homotopy-equivalent. By \( \lim_{(k,l)\to1} Gr_{k,kl} \) or \( Gr \) denote this unique homotopic type.

Let us consider a FAB of the form \( (\tilde{M}_k, \tau, \tilde{M}_{kl}) \). Let \( \tau \) be the map \( \tau : X \times M_k(\mathbb{C}) \to X \times M_{kl}(\mathbb{C}) \) such that \( \tau(x, T) = (x, T \otimes E_l) \) (where \( x \in X, T \in M_k(\mathbb{C}), E_l \) is the matrix unit of order \( l \times l \), and \( T \otimes E_l \) denotes the Kronecker product of matrix). Then the FAB \( (\tilde{M}_k, \tau, \tilde{M}_{kl}) \) is called trivial.

**Definition 4.** Isomorphism from FAB \( (A_k, \mu, \tilde{M}_{kl}) \) to FAB \( (C_k, \nu, \tilde{M}_{kl}) \) over \( X \) is a pair of bundle maps \( f : A_k \to C_k, g : \tilde{M}_{kl} \to \tilde{M}_{kl} \) such that the following conditions hold:

1) for any point \( x \in X \) the fiber \( (A_k)_x \) (respectively the fiber \( (\tilde{M}_{kl})_x \)) is embeded into the fiber \( (C_k)_x \) (respectively into the fiber \( (\tilde{M}_{kl})_x \)) by the restriction of \( f \) (respectively of \( g \)) as a central simple subalgebra (in particular \( f_x, g_x \) are homomorphisms).
2) the following diagram

\[
\begin{array}{c}
\tilde{M}_{kl} \\
\mu \quad \nu
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
\tilde{M}_{kl} \\
A_k \\
C_k
\end{array}
\]

is commutative.

**Definition 5.** FAB \((A_k, \mu, \tilde{M}_{kl})\) and \((B_m, \nu, \tilde{M}_{mn})\) are called *stable equivalent* if there exist a sequence of pairs of natural numbers \(\{t_i, u_i\}\), \(1 \leq i \leq s\) and a corresponding sequence of FAB \((A_{t_i}, \mu_{t_i}, \tilde{M}_{t_iu_i})\) such that the following conditions hold:

(i) \(\{t_1, u_1\} = \{k, l\}, \{t_s, u_s\} = \{m, n\}\);

(ii) \((t_it_{i+1}, u_iu_{i+1}) = 1\) for \(s > 1, 1 \leq i \leq s - 1\);

(1) \((A_{t_1}, \mu_1, \tilde{M}_{t_1u_1}) = (A_k, \mu, \tilde{M}_{kl}), (A_{t_s}, \mu_s, \tilde{M}_{t_su_s}) = (B_m, \nu, \tilde{M}_{mn})\);

(2) \((A_{t_i}, \mu_i, \tilde{M}_{t_iu_i}) \otimes (\tilde{M}_{t_{i+1}}, \tau, \tilde{M}_{t_{i+1}u_{i+1}}) \cong (A_{t_{i+1}}, \mu_{i+1}, \tilde{M}_{t_{i+1}u_{i+1}}) \otimes (\tilde{M}_{t_i}, \tau, \tilde{M}_{t_iu_i})\)

for \(s > 1, 1 \leq i \leq s - 1\), where \((\tilde{M}_{t_i}, \tau, \tilde{M}_{t_iu_i})\) is the trivial FAB.

The space \(\lim_{(k,l) \to 1} Gr_{k,kl}\) is the classifying space for FAB with the just defined relation of the stable equivalence. This motivates Definition 3.

By \([A_k, \mu, \tilde{M}_{kl}]\) denote the stable equivalence class of the FAB \((A_k, \mu, \tilde{M}_{kl})\).

Now let us show that the set of all stable equivalence classes of FAB over \(X\) is an Abelian group with respect to the operation, induced by the tensor product of FAB. For this we need the following Proposition.

**Proposition 6.** For any pair \(\{k,l\}\) such that (i) \((k,l) = 1\), (ii) \(2\min\{k,l\} > \dim X\) every equivalence class of FAB over \(X\) has a representative of the form \((A_k, \mu, \tilde{M}_{kl})\).
Suppose \((A_k, \mu, \tilde{M}_{kl})\) and \((B_m, \nu, \tilde{M}_{mn})\) are FAB over \(X\). If \((km, ln) = 1\), then it is clear that \((A_k \otimes B_m, \mu \otimes \nu, \tilde{M}_{klmn})\) is the FAB. Then by definition, put \([\langle A_k, \mu, \tilde{M}_{kl} \rangle] \cdot [\langle B_m, \nu, \tilde{M}_{mn} \rangle] = [\langle A_k \otimes B_m, \mu \otimes \nu, \tilde{M}_{klmn} \rangle]\). Otherwise, applying the previous Proposition, we can replace \((B_m, \nu, \tilde{M}_{mn})\) by another representative \((B'_m, \nu', \tilde{M}'_{mn'})\) of the same equivalence class such that \((km', ln') = 1\).

It is clear that the product of stable equivalence classes is well defined. The unit of this operation is the class of trivial FAB \([\langle \tilde{M}_k, \tau, \tilde{M}_{kl} \rangle]\). For given FAB \((A_k, \mu, \tilde{M}_{kl})\) consider the subbundle \(B_l\) in \(\tilde{M}_{kl}\) such that \((B_l)_x = Z(\tilde{M}_{kl})_x ((A_k)_x)\) (where \(Z_B(A)\) is the centralizer of a subalgebra \(A\) in an algebra \(B\)) for any point \(x \in X\). This also defines the embedding \(\nu: B_l \hookrightarrow \tilde{M}_{kl}\). Thus the FAB \((B_l, \nu, \tilde{M}_{kl})\) is defined. The inverse element for \([\langle A_k, \mu, \tilde{M}_{kl} \rangle]\) is the class of FAB \((B_l, \nu, \tilde{M}_{kl})\).

The next theorem sums the basic results obtained up to now.

**Theorem 7.** The set of all stable equivalence classes of FAB with respect to the operation, induced by the tensor product of FAB, defines the contravariant homotopic functor to the category of Abelian groups. This functor is denoted by \(\tilde{\text{AB}}^1\). Its represented space is the space \(\lim_{\substack{(k,l)\to\infty \infty}} \text{Gr}_{k,kl}\) with the structure of \(H\)-group, defined by the maps \(\text{Gr}_{k,kl} \times \text{Gr}_{m,mn} \to \text{Gr}_{km,kmln}\) for \((km, ln) = 1\) (the last maps are induced by the tensor product of algebras).

**Definition 8.** Let \((A_k, \mu, \tilde{M}_{kl})\) be an AB. The bundle \(A_k\) (that is considered as a locally trivial bundle with the structural group \(\text{Aut} M_k(\mathbb{C}) \cong \text{PGL}(k)\)) is called the base of the AB \((A_k, \mu, \tilde{M}_{kl})\).

In the next Lemma the important property of bundles (with fiber matrix algebra), which are bases of FAB, is established.

**Lemma 9.** Let \(X\) be a finite CW-complex, \(\dim X < 2 \min\{k, m\}\). Then the following conditions are equivalent:

(i) \(A_k\) is a base of a FAB over \(X\);

(ii) for any \(m, 2m > \dim X\), there exists a bundle \(B_m\) with fiber \(M_m(\mathbb{C})\) such that \(A_k \otimes \tilde{M}_m \cong B_m \otimes \tilde{M}_k\).
(iii) for some \( m, (k, m) = 1 \), an isomorphism \( A_k \otimes \tilde{M}_m \cong B_m \otimes \tilde{M}_k \) holds. Moreover for any pair of bundles \( (A_k, B_m) \) over \( X \) such that \( (k, m) = 1 \) and \( A_k \otimes \tilde{M}_m \cong B_m \otimes \tilde{M}_k \), there exists a unique stable equivalence class of FAB over \( X \) which has representatives of forms \( (A_k, \mu, \tilde{M}_k) \) and \( (B_m, \nu, \tilde{M}_m) \) (for any sufficiently large \( n \) such that \( (km, n) = 1 \)). This class is denoted by \([[(A_k, B_m)]\].

The most interesting implication in this Lemma is (iii) \( \Rightarrow \) (i). In its proof, which is based on the obstruction theory, the relatively primality of \( k \) and \( m \) plays the fundamental role.

Further in this section we describe a connection between the functor \( \tilde{\text{AB}} \) and the reduced K-functor \( \tilde{\text{KSU}} \).

Let \( X \) be a finite \( CW \)-complex, \( \xi_k \) be a complex vector \( \text{SU}(k) \)-bundle over \( X \) of a rank \( k \), \( 2k > \text{dim} \ X \). By \([n]\) denote the trivial bundle of rank \( n \) over \( X \). Take a positive integer \( m \) such that \( (k, m) = 1 \) and \( m > k \). Let us consider the pair

\[
\xi_k \otimes [k] - [k(k - 1)], (\xi_k \oplus [m - k]) \otimes [m] - [m(m - 1)]
\]

of virtual bundles of virtual dimensions \( k \) and \( m \) respectively. The condition \( \text{dim} \ X < 2k \) implies that there exists a unique (up to isomorphism) \( k \)-dimensional geometric representative of the stable equivalence class of \( \xi_k \otimes [k] - [k(k - 1)] \). By \( \xi_k^{\dagger} \) denote this geometric representative. Let \( (\xi_k \oplus [m - k])^{\dagger} \) be the analogous \( m \)-dimensional representative of the class of \( (\xi_k \oplus [m - k]) \otimes [m] - [m(m - 1)] \).

The next proposition is almost obviously.

**Proposition 10.** \( (\text{End} \xi_k^{\dagger}) \otimes \tilde{M}_m \cong (\text{End}(\xi_k \oplus [m - k])^{\dagger}) \otimes \tilde{M}_k \).

Let \([\xi_k]\) be the \( \tilde{\text{KSU}} \)-equivalence class of the bundle \( \xi_k \).

It follows from Lemma \( \text{[I]} \) that if \( (k, m) = 1 \), then there exists a unique stable equivalence class of FAB over \( X \), corresponding to the pair \( \xi_k^{\dagger}, (\xi_k \oplus [m - k])^{\dagger} \). Clearly, that this stable equivalence class of FAB is independent of a choice of representative of the equivalence class \([\xi_k]\). Moreover the following Proposition holds.
**Proposition 11.** Let \( k, m \) be the positive integers such that \( (k, m) = 1 \), \( 2 \min\{k, m\} > \dim X \). Then the map

\[
\widetilde{KSU}(X) \longrightarrow \widetilde{AB}^1(X), \quad [\xi_k] \longmapsto [(A_k, B_m)]
\]

(where \( A_k = \text{End} \xi_k^\dagger \), \( B_m = \text{End}(\xi_k \oplus [m - k])^\dagger \), and \([(A_k, B_m)]\) is the stable equivalence class of FAB, corresponding to the pair \((A_k, B_l)\) as in Lemma 9) is well defined bijection of the sets.

In particular, any base of FAB has the form \( \text{End} \xi_k^\dagger \) for some \( \text{SU}(k) \)-bundle \( \xi_k \).

It follows from the previous Proposition that the spaces \( \lim_{(k,l)\to 1} Gr_{k,kl} \) and \( \text{BSU} \) are homotopy-equivalent.

Denote by \( \text{BSU}_\otimes \) the space \( \text{BSU} \) with the structure of \( H \)-group, induced by the tensor product of virtual \( \text{SU} \)-bundles of virtual dimension 1.

**Theorem 12.** The \( H \)-group \( \lim_{(k,l)\to 1} Gr_{k,kl} \) with respect to the operation, induced by the tensor product of FAB, is isomorphic to the \( \text{BSU}_\otimes \) as an \( H \)-group.

Hence the group \( \widetilde{AB}^1(X) \) is isomorphic to the multiplicative group (in sense of the theory of formal groups) of the ring \( \widetilde{KSU}(X) \), i.e. to the group (since \( X \) is the finite \( CW \)-complex) \( \widetilde{KSU}(X) \) with respect to the operation \( \xi \circ \eta = \xi + \eta + \xi\eta \) (\( \xi, \eta \in \widetilde{KSU}(X) \)).

By means of FAB, the structure of \( H \)-group on \( \text{BSU}_\otimes \) may be described in geometric terms. Let us consider, for example, the finding of an inverse element. Recall that for any FAB \((A_k, \mu, \tilde{M}_{kl})\) there exists the FAB \((B_l, \nu, \tilde{M}_{kl})\) such that for any point \( x \in X \) its base \( B_l \) has the centralizer \( Z_{(\tilde{M}_{kl})_x}(A_k)_x \) as the fiber over \( x \) and \( \nu \) is the corresponding embedding. The equivalence class of the FAB \((B_l, \nu, \tilde{M}_{kl})\) is the inverse element for the class \( [(A_k, \mu, \tilde{M}_{kl})] \).

This shows that the matrix algebra's structure on fibers of considered bundles plays the same role that a metric in usual theory of vector bundles (recall that in usual theory of vector bundles for finding of a stable inverse class for a given class with a representative \( \xi \) we must embed the vector bundle \( \xi \) into a product bundle and take its orthogonal complement there).
Let us consider the bundle space $\mathcal{P}^{k-1} \times \mathcal{Q}^{l-1}, (k,l) = 1$ (which was defined on page 3). We also denote this space by $\widetilde{Gr}_{k,kl}$. It follows from Definition 2 that there exists the bundle map (over $Gr_{k,kl}$)

$$\lambda_{k,l}: \widetilde{Gr}_{k,kl} \hookrightarrow Gr_{k,kl} \times \mathbb{C}P^{kl-1}$$

such that its restriction to any fiber (isomorphic to $\mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1}$) is the Segre’s map $\mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1} \hookrightarrow \mathbb{C}P^{kl-1}$. If we take the composition of $\lambda_{k,l}$ with the projection onto $\mathbb{C}P^{kl-1}$, then we obtain the map

$$\kappa_{k,l}: \widetilde{Gr}_{k,kl} \hookrightarrow \mathbb{C}P^{kl-1}$$

such that its restriction to any fiber (isomorphic to $\mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1}$) is the classifying map for the exterior tensor product of the canonical line bundles over $\mathbb{C}P^{k-1}$ and $\mathbb{C}P^{l-1}$.

By $Gr$ denote the $H$-group $\lim_{(k,l)=1} Gr_{k,kl}$ (recall that $\lim_{(k,l)=1} Gr_{k,kl} \simeq BSU \otimes$ as $H$-groups). By $\widetilde{Gr}$ denote the direct limit $\lim_{(k,l)=1} \widetilde{Gr}_{k,kl}$ of the FBSP’s bundle spaces over $Gr_{k,kl}$ and by $\kappa$ denote the direct limit

$$\lim_{(k,l)=1} \kappa_{k,l}: \widetilde{Gr} \rightarrow \mathbb{C}P^{\infty}$$

of the maps $\kappa_{k,l}$, $(k,l) = 1$.

It can be proved that the maps

$$\phi_{kl, mn}: \widetilde{Gr}_{k,kl} \times \widetilde{Gr}_{m,mn} \rightarrow \widetilde{Gr}_{km,klmn}, \quad (km, ln) = 1$$

(which correspond to the multiplication of FBSP, consequently their restriction to any pair of fibers is the map

$$(\mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1}) \times (\mathbb{C}P^{m-1} \times \mathbb{C}P^{n-1}) \hookrightarrow \mathbb{C}P^{km-1} \times \mathbb{C}P^{ln-1}$$

induce the structure of $H$-group on the $\widetilde{Gr}$.

Consider the set of pairs $(\xi_k, \eta_l)$ of $k$ and $l$-dimensional $(k,l) = 1$ complex vector bundles (not necessarily with the structural group SU) over a finite $CW$-complex such that

$$\xi_k \otimes \eta_l \simeq \zeta \otimes [kl],$$

9
where ζ is some geometric line bundle. Define on this set the equivalence relation which is analogous to the FAB (or FBSP) stable equivalence relation (see Definition 3). The tensor product of pairs induces the group operation on the set of equivalence classes. It is easy to prove that the $H$-group \( \tilde{Gr} \) is the represented space for this homotopic functor to category of Abelian groups.

**Proposition 13.** The map \( \kappa: \tilde{Gr} \to \mathbb{C}P^\infty \) is the homomorphism of $H$-groups. Moreover \( \tilde{Gr} \) is isomorphic to \( BSU_\otimes \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty \) as an $H$-group (recall that \( \mathbb{C}P^\infty \) is the $H$-group with respect to the operation which corresponds to the tensor product of complex line bundles; this $H$-structure is induced by the Segre’s maps \( \mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1} \hookrightarrow \mathbb{C}P^{kl-1} \)).

Consider also the "one-half" \( \mathcal{P}^{k-1}_{Gr_{k,kl}} \times \mathcal{Q}^{l-1}_{Gr_{k,kl}} \) over \( Gr_{k,kl} \) (recall that \( \mathcal{P}^{k-1}_{Gr_{k,kl}} \times \mathcal{Q}^{l-1}_{Gr_{k,kl}} \) is the product over \( Gr_{k,kl} \) of the bundles \( \mathcal{P}^{k-1} \) and \( \mathcal{Q}^{l-1} \) with fibers \( \mathbb{C}P^{k-1} \) and \( \mathbb{C}P^{l-1} \) respectively). By \( \tilde{Gr}_{k,kl} \) we denote the bundle space \( \mathcal{P}^{k-1}_{Gr_{k,kl}} \) over \( Gr_{k,kl} \). Let \( \tilde{Gr} \) be the direct limit \( \lim_{(k,l)=1} \tilde{Gr}_{k,kl} \). Note that the $H$-structure on \( \tilde{Gr} \) may be restricted to the subspace \( \tilde{Gr} \). It is not hard to prove that the space \( \tilde{Gr} \) is the $H$-group with respect to the induced $H$-structure.

**Proposition 14.** $H$-groups \( \tilde{Gr} \) and \( BU_\otimes \) are isomorphic. Moreover they are isomorphic to the direct product \( BSU_\otimes \times \mathbb{C}P^\infty \).
2 Formal groups over Hopf algebras

The aim of this section is to define some generalization of the notion of formal group. More precisely, we consider the analog of formal groups with coefficients belonging to a Hopf algebra. We also study some example of a formal group over a Hopf algebra, which generalizes the formal group of geometric cobordisms.

Recently some important connections between the Landweber-Novikov algebra and the formal group of geometric cobordisms were established \[^2\].

Let \((H, \mu, \eta, \Delta, \varepsilon, S)\) be a (topological) Hopf algebra over ring \(R\) (where \(\mu: H \hat{\otimes} H \rightarrow H\) is multiplication, \(\eta: R \rightarrow H\) is unit, \(\Delta: H \rightarrow H \hat{\otimes} H\) is diagonal (comultiplication), \(\varepsilon: H \rightarrow R\) is counit, and \(S: H \rightarrow H\) is antipode).

**Definition 15.** A formal series \(\tilde{\mathcal{F}}(x \otimes 1, 1 \otimes x) \in H \hat{\otimes} H[[x \otimes 1, 1 \otimes x]]\) is called a formal group over the Hopf algebra \((H, \mu, \eta, \Delta, \varepsilon, S)\) if the following conditions hold:

1) (associativity)

\[
((\text{id}_H \otimes \Delta)\tilde{\mathcal{F}})(x \otimes 1 \otimes 1, 1 \otimes \tilde{\mathcal{F}}(x \otimes 1, 1 \otimes x)) = \\
((\Delta \otimes \text{id}_H)\tilde{\mathcal{F}})(\tilde{\mathcal{F}}(x \otimes 1, 1 \otimes x) \otimes 1, 1 \otimes 1 \otimes x);
\]

2) (unit)

\[
((\text{id}_H \otimes \varepsilon)\tilde{\mathcal{F}})(x \otimes 1, 0) = x \otimes 1, \\
((\varepsilon \otimes \text{id}_H)\tilde{\mathcal{F}})(0, 1 \otimes x) = 1 \otimes x;
\]

3) (inverse element) there exists the series \(\Theta(x) \in H[[x]]\) such that

\[
((\mu \circ (\text{id}_H \otimes S))\tilde{\mathcal{F}})(x, \Theta(x)) = 0 = ((\mu \circ (S \otimes \text{id}_H))\tilde{\mathcal{F}})(\Theta(x), x).
\]

If for a formal group \(\tilde{\mathcal{F}}(x \otimes 1, 1 \otimes x)\) over a commutative and cocommutative Hopf algebra \(H\) the equality \(\tilde{\mathcal{F}}(x \otimes 1, 1 \otimes x) = \tilde{\mathcal{F}}(1 \otimes x, x \otimes 1)\) holds, then it is called commutative. Below we shall deal only with the commutative case.

\[^2\]B. I. Botvinnik, V. M. Bukhshtaber, S. P. Novikov, S. A. Yuzvinskii Algebraicheskie aspekti teorii umnojenii v kompleksnikh kobordizmakh // UMN. – 2000. – T. 55, 4. S. 5–24.
Remark 16. Note that a formal group $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ over Hopf algebra $H$ over ring $R$ defines the formal group (in the usual sense) $F(x \otimes 1, 1 \otimes x) \in R[[x \otimes 1, 1 \otimes x]]$ over the ring $R$ in the following way. By $F(x \otimes 1, 1 \otimes x)$ denote the series $((\varepsilon \otimes \varepsilon)\mathfrak{F})(x \otimes 1, 1 \otimes x)$. If we identify $R \otimes R$ and $R$, we may assume that $F(x \otimes 1, 1 \otimes x) \in R[[x \otimes 1, 1 \otimes x]]$. Note that for any coalgebra $H$ the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\Delta} & H \otimes H \\
\varepsilon \downarrow & & \varepsilon \otimes \varepsilon \\
R & \xrightarrow{\cong} & R \otimes R
\end{array}
\]

is commutative. Using (2) and condition 1) of Definition 13, we get $F(x \otimes 1, 1 \otimes x) = F(F(x \otimes 1, 1 \otimes x) \otimes 1, 1 \otimes 1 \otimes x)$. Similarly the conditions $F(x \otimes 1, 0) = x \otimes 1$ and $F(0, 1 \otimes x) = 1 \otimes x$ may be verified. It is well known, that the existence of the inverse element (in the case of usual formal groups) follows from the proved conditions. However this may be deduced from the condition 3) of Definition 13 in the standard way. Moreover, the inverse element $\theta(x)$ in the formal group $F(x \otimes 1, 1 \otimes x)$ is equal to $(\varepsilon(\Theta))(x)$.

Therefore we may consider the formal group $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ over Hopf algebra $H$ as an extension of the usual formal group $F(x \otimes 1, 1 \otimes x)$ by the Hopf algebra $H$.

Remark 17. By definition, put $\tilde{\Delta}(x) = \mathfrak{F}(x \otimes 1, 1 \otimes x)$, $\tilde{\varepsilon}(x) = 0$, $\tilde{S}(x) = \Theta(x)$ and $\tilde{\Delta} |_\mu = \Delta$, $\tilde{\varepsilon} |_\mu = \varepsilon$, $\tilde{S} |_\mu = S$. We claim that $(H[[x]], \tilde{\mu}, \tilde{\eta}, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{S})$ is the Hopf algebra (here $\tilde{\mu}$, $\tilde{\eta}$ are evidently extensions of $\mu$, $\eta$). Indeed, the commutativity of the diagram

\[
\begin{array}{ccc}
H[[x]] & \xrightarrow{\Delta} & H[[x]] \otimes H[[x]] \\
\Delta \downarrow & & \downarrow \text{id}_{H[[x]]} \otimes \Delta \\
H[[x]] \otimes H[[x]] & \xrightarrow{\Delta \otimes \text{id}_{H[[x]]}} & H[[x]] \otimes H[[x]] \otimes H[[x]]
\end{array}
\]

follows from the equations

$$(\text{id}_{H[[x]]} \otimes \Delta)(\mathfrak{F}(x \otimes 1, 1 \otimes x)) = ((\Delta \otimes \text{id}_H)\mathfrak{F})(x \otimes 1 \otimes 1, 1 \otimes \mathfrak{F}(x \otimes 1, 1 \otimes x)) = ((\Delta \otimes \text{id}_H)\mathfrak{F})(\mathfrak{F}(x \otimes 1, 1 \otimes x) \otimes 1, 1 \otimes 1 \otimes x) = (\tilde{\Delta} \otimes \text{id}_{H[[x]]})(\mathfrak{F}(x \otimes 1, 1 \otimes x)).$$
The commutativity of the diagram

\[
\begin{array}{ccc}
R \otimes H[[x]] & \xrightarrow{\sim} H[[x]] \otimes H[[x]] & \xrightarrow{\bar\Delta} H[[x]] \otimes R \\
\xrightarrow{\Δ} & H[[x]] & \xrightarrow{\sim} \\
\end{array}
\]

follows from the equations

\[(\id_H[[x]] \otimes \bar\varepsilon) \circ \bar\Delta)(x) = ((\id_H \otimes \varepsilon) \bar\delta)((x \otimes 1) \otimes \bar\varepsilon(x)) = x \otimes 1,
\]

\[(\bar\varepsilon \otimes \id_H[[x]]) \circ \bar\Delta)(x) = ((\varepsilon \otimes \id_H) \bar\delta)(\bar\varepsilon(x) \otimes 1, 1 \otimes x) = 1 \otimes x.
\]

The axiom of antipode

\[(\bar\mu \circ (\id_H[[x]] \otimes \bar\delta) \circ \bar\Delta)(x) = (\bar\mu \circ (\bar\delta \otimes \id_H[[x]]) \circ \bar\Delta)(x) = (\bar\eta \circ \bar\varepsilon)(x) = 0
\]

follows from the condition 3) of Definition 15.

Remark 18. We may rewrite the conditions 1), 2), 3) of Definition 15 in terms of series \(\bar\delta(x \otimes 1, 1 \otimes x)\) in the next way. Let

\[
\sum_{i,j \geq 0} A_{i,j} x^i \otimes x^j = 
\]

\[
\sum_{i,j,k} (\sum_{k} a_{i,j}^k \otimes b_{i,j}^k) x^i \otimes x^j \in H \hat{\otimes} H[[x \otimes 1, 1 \otimes x]]
\]

be the series \(\bar\delta(x \otimes 1, 1 \otimes x)\). Then the condition 1) is equivalent to the following equality:

\[
\sum_{i,j \geq 0} (\sum_{k} a_{i,j}^k \otimes \Delta(b_{i,j}^k)) x^i \otimes \bar\delta(x \otimes 1, 1 \otimes x)^j =
\]

\[
\sum_{i,j \geq 0} (\sum_{k} \Delta(a_{i,j}^k) \otimes b_{i,j}^k) \bar\delta(x \otimes 1, 1 \otimes x)^i \otimes x^j
\]

The condition 2) is equivalent to

\[
\sum_{k} a_{i,0}^k \varepsilon(b_{i,0}^k) = 0, \quad \text{if} \quad i \neq 1, \quad \sum_{k} a_{1,0}^k \varepsilon(b_{1,0}^k) = 1,
\]

\[
\sum_{k} \varepsilon(a_{0,j}^k) b_{0,j}^k = 0, \quad \text{if} \quad j \neq 1, \quad \sum_{k} \varepsilon(a_{0,1}^k) b_{0,1}^k = 1.
\]

The condition 3) also may be rewritten in terms of series.
Let us consider some examples of defined objects.

**Example 19.** (Trivial extension) Let $F(x \otimes 1, 1 \otimes x)$ be a formal group (in the usual sense) over a ring $R$, and $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra over the same ring $R$. Then $\mathfrak{F}(x \otimes 1, 1 \otimes x) = ((\eta \otimes \eta)F)(x \otimes 1, 1 \otimes x) \in H \hat{\otimes} H[[x \otimes 1, 1 \otimes x]]$ is the formal group over the Hopf algebra $H$ (recall that we identify $R \otimes R$ and $R$).

**Example 20.** Now we construct a nontrivial extension $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ of the formal group of geometric cobordisms $F(x \otimes 1, 1 \otimes x) \in \Omega^*_U(pt)[[x \otimes 1, 1 \otimes x]]$ by the Hopf algebra $\Omega^*_U(Gr)$. For this let us consider the map (see page 10) $\hat{\phi}_{kl, mn} : \hat{Gr}_{k,kl} \times \hat{Gr}_{m,mn} \rightarrow \hat{Gr}_{km,klmn}$, $(km, ln) = 1$. By $x_{|km,ln}$ denote the cobordism’s class in $\Omega^*_U(\hat{Gr}_{km,klmn})$ such that its restriction to every fiber of the bundle

$$
\mathbb{C}P^{km-1} \hookrightarrow \hat{Gr}_{km,klmn} \quad \downarrow \quad \hat{Gr}_{km,klmn}
$$

is the standard generator in $\Omega^2_U(\mathbb{C}P^{km-1})$. Let $x_{|k,l}$ and $x_{|m,n}$ be analogously elements in $\Omega^2_U(\hat{Gr}_{k,kl})$ and $\Omega^2_U(\hat{Gr}_{m,mn})$ respectively. Then we obtain that

$$
\hat{\phi}_{kl, mn}^* (x_{|km,ln}) = \sum_{\emptyset \cup [l-k] \subseteq [m-1]} A_{i,j} (x_{|k,l})^i \otimes (x_{|m,n})^j,
$$

where $A_{i,j} \in \Omega^2_U(Gr_{k,kl} \times Gr_{m,mn})$. Applying the functor of unitary cobordisms to the following injective system of the spaces and their maps

$$
\begin{align*}
\hat{Gr}_{p,pq} \times \hat{Gr}_{t,tu} & \xrightarrow{\hat{\phi}_{pq,tu}} \hat{Gr}_{pt,pqtu} \\
\hat{Gr}_{k,kl} \times \hat{Gr}_{m,mn} & \xrightarrow{\hat{\phi}_{kl, mn}} \hat{Gr}_{km,klmn}.
\end{align*}
$$

(under the conditions $k \mid p, \quad l \mid q, \quad m \mid t, \quad n \mid u, \quad$ and $(pt, qu) = 1$), we obtain the formal series

$$
\mathfrak{F}(x \otimes 1, 1 \otimes x) = \sum_{i,j \geq 0} A_{i,j} x^i \otimes x^j \in \Omega^*_U(Gr) \hat{\otimes} \Omega^*_U(Gr)[[x \otimes 1, 1 \otimes x]]
$$

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such that $i_{kl}^* A_{i,j} = A_{i,j} |_{k,l}$ for injection $i_{kl}: Gr_{k,kl} \hookrightarrow Gr$ (for every pair $\{k, l\}$ such that $(k, l) = 1$).

By $R$ and $H$ denote the ring $\Omega_U^*(\text{pt})$ and the Hopf algebra $\Omega_U^*(Gr)$ (over the ring $\Omega_U^*(\text{pt})$) respectively (recall that we consider the space $Gr$ with the $H$-group structure, induced by the multiplication of FBSP).

**Proposition 21.** The series $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ is the formal group over the Hopf algebra $H$.

**Proof.** To prove $((\text{id}_H \otimes \Delta) \mathfrak{F})(x \otimes 1 \otimes 1 \otimes \mathfrak{F}(x \otimes 1, 1 \otimes x)) = (\Delta \otimes \text{id}_H) \mathfrak{F}(x \otimes 1, 1 \otimes x)$, we need the following commutative diagram $((kmt, lnu) = 1)$:

$$
\begin{array}{ccc}
\hat{Gr}_{k,kl} \times \hat{Gr}_{m,mn} \times \hat{Gr}_{t,tu} & \rightarrow & \hat{Gr}_{k,kl} \times \hat{Gr}_{nt,mntu} \\
\downarrow & & \downarrow \\
\hat{Gr}_{km,klmn} \times \hat{Gr}_{t,tu} & \rightarrow & \hat{Gr}_{kmt,klmntu}.
\end{array}
$$

(6)

To prove $((\text{id}_H \otimes \varepsilon) \mathfrak{F})(x \otimes 1, 0) = x \otimes 1$, we need the following commutative diagram $((km, ln) = 1)$:

$$
\begin{array}{ccc}
\hat{Gr}_{k,kl} \times \hat{Gr}_{m,mn} & \rightarrow & \hat{Gr}_{km,klmn} \\
\uparrow & & \uparrow \\
\hat{Gr}_{k,kl} \times \mathbb{C}^{P^{m-1}} & \leftarrow & \hat{Gr}_{k,kl} \times \{\text{pt}\},
\end{array}
$$

(7)

where right-hand vertical arrow is the standard inclusion.

To prove $((\mu \circ (\text{id}_H \otimes S)) \mathfrak{F})(x, \Theta(x)) = 0$, let us construct the fiber map $\hat{\nu}: \hat{Gr} \rightarrow \hat{Gr}$ such that the following two conditions are satisfied:

1) the restriction of $\hat{\nu}$ to any fiber ($\simeq \mathbb{C} P^\infty$) is the inversion in the $H$-group $\mathbb{C} P^\infty$;

2) $\hat{\nu}$ covers the $\nu: Gr \rightarrow Gr$ (where $\nu$ is the inversion in the $H$-group $Gr$).

Let us remember that $P_{Gr_{k,kl}}^{k-1} \times Q_{Gr_{k,kl}}^{l-1}$ is the canonical FBSP over $Gr_{k,kl}$ and we have denoted by $\hat{Gr}_{k,kl}$ the bundle space $P_{Gr_{k,kl}}^{k-1}$. Let $\hat{Gr}_{k,kl}'$ be the bundle space of the "second half" $Q_{Gr_{k,kl}}^{l-1}$ of the canonical FBSP over $Gr_{k,kl}$ ( $\lim_{(k, l) = 1} \hat{Gr}_{k,kl}'$ respectively).
First note that there exists the fiber isomorphism \( \hat{\nu}_{k,l} : \hat{Gr}_{k,kl} \to \hat{Gr}_{l,lk}' \) that covers the inverse map \( \nu_{k,l} : Gr_{k,kl} \to Gr_{l,lk} \) (in other words, the map \( \nu_{k,l} \) takes each subalgebra \( A_k \cong M_k(\mathbb{C}) \) in the \( M_{kl}(\mathbb{C}) \) to its centralizer \( Z_{M_{kl}(\mathbb{C})}(A_k) \cong M_l(\mathbb{C}) \) in the \( M_{kl}(\mathbb{C}) \)). Let \( c_{l,k} : \hat{Gr}_{l,lk}' \to \hat{Gr}_{l,lk}' \) be the fiber map such that the following two conditions are satisfied:

1) \( c_{l,k} \) covers the identity mapping of the base \( Gr_{l,lk} \);

2) the restriction of \( c_{l,k} \) to any fiber \( \cong \mathbb{C}P^{k-1} \) is the complex conjugation.

Let \( \hat{\nu}_{k,l} : \hat{Gr}_{k,kl} \to \hat{Gr}_{l,lk}' \) be the composition \( c_{l,k} \circ \hat{\nu}_{k,l}' \). It is easy to prove that the map \( \nu = \lim_{(k,l) \to 1} \hat{\nu}_{k,l} : \lim_{(k,l) \to 1} \hat{Gr}_{k,kl} \to \lim_{(l,k) \to 1} \hat{Gr}_{l,lk}' \) is required. In particular, there exists the fiber isomorphism between \( \hat{Gr} \) and \( \hat{Gr}' \).

The map \( \hat{\nu} \) defines (by the same way, as \( \hat{\phi} \) in the beginning of the example) the formal series \( \Theta(x) \in H[[x]] \) (note that \( \varepsilon(\Theta)(x) = \theta(x) \), where \( \theta(x) \in R[[x]] \) is the inverse element in the group of geometric cobordisms).

Now we claim that \( (\mu \circ (id_H \otimes S)) \mathfrak{F})(x, \Theta(x)) = 0 \). Indeed, this follows from the next commutative diagram:

\[
\begin{array}{cccccc}
\hat{Gr} & \xrightarrow{\text{diag}} & \hat{Gr} \times \hat{Gr} & \xrightarrow{id \times \hat{\nu}} & \hat{Gr} \times \hat{Gr} & \xrightarrow{\hat{\phi}} & \hat{Gr} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Gr & \xrightarrow{\text{diag}} & Gr \times Gr & \xrightarrow{id \times \nu} & Gr \times Gr & \xrightarrow{\phi} & Gr \\
\end{array}
\] (8)

(we see that the composition \( \hat{\phi} \circ (id \times \hat{\nu}) \circ \text{diag} \) is homotopic (in class of fiber homotopies) to the map \( \hat{Gr} \to \text{pt} \in \hat{Gr} \)).

In section 1 we considered the map \( \kappa : \hat{Gr} \to \mathbb{C}P^\infty \). It is the direct limit of the fiber maps (see p. 3)

\[
\begin{array}{ccc}
\hat{Gr}_{k,kl} & \xrightarrow{\kappa_{k,l}} & \mathbb{C}P^{kl-1} \\
\downarrow & & \downarrow \\
Gr_{k,kl} & \to & \text{pt}.
\end{array}
\] (9)

It defines (in the same way, as \( \hat{\phi} \) and \( \hat{\nu} \) above) the formal series \( \mathfrak{G}(x, y) = \sum_{i,j \geq 0} B_{i,j} x^i y^j \in H[[x, y]] \).
Proposition 22. \(\mathcal{G}(x, y) = ((\mu \circ (\text{id}_H \otimes S))\mathfrak{F})(x, y)\), i.e.

\[B_{i,j} = \sum_k a^k_{i,j}S(b^k_{i,j}).\]

Proof. Recall that in the proof of Proposition 21 the fiber maps \(\mathcal{V}'_{k,l}: \widetilde{Gr}_{k,kl} \rightarrow \widetilde{Gr}_{l,lk}\) were defined. By \(\mathcal{V}'\) denote the direct limit \(\lim_{(k,l)=1}\mathcal{V}'_{k,l}: \widetilde{Gr} \rightarrow \widetilde{Gr}\). Note that \(\mathcal{V}'\) covers the inversion \(\nu: Gr \rightarrow Gr\) in the \(H\)-group \(Gr\).

Now the proof follows from the next composition of the bundle maps:

\[
\begin{array}{cccc}
\widetilde{Gr} & \rightarrow & \widetilde{Gr} \times \widetilde{Gr} & \xrightarrow{id \times \mathcal{V}'} \rightarrow & \widetilde{Gr} \times \widetilde{Gr} & \xrightarrow{\phi} & \widetilde{Gr} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Gr & \xrightarrow{\text{diag}} & Gr \times Gr & \xrightarrow{id \times \nu} & Gr \times Gr & \xrightarrow{\phi} & Gr.
\end{array}
\] (10)

We see that the upper composition in fact is the map \(\widetilde{\mathcal{G}} \rightarrow \mathbb{C}P^\infty\) and it coincides with the map \(\kappa\). Let \(y\) be \(\mathcal{V}'^*(x)\). The upper composition gives \(x \mapsto \mathfrak{F}(x \otimes 1, 1 \otimes x) \mapsto ((\text{id}_H \otimes S)\mathfrak{F})(x \otimes 1, 1 \otimes y) \mapsto ((\mu \circ (\text{id}_H \otimes S))\mathfrak{F})(x \otimes 1, 1 \otimes y)\).

Without loss of sense we may write \(x\) and \(y\) instead of \(x \otimes 1\) and \(1 \otimes y\) respectively. □

The series \(\mathcal{G}(x, y)\) has the following interesting property.

Proposition 23.

\[(\Delta \mathcal{G})(\mathfrak{F}(x \otimes 1, 1 \otimes x), ((S \otimes S)\mathfrak{F})(y \otimes 1, 1 \otimes y)) = F(\mathcal{G}(x, y) \otimes 1, 1 \otimes \mathcal{G}(x, y)),\]

where \(F(x, y) \in R[[x, y]]\) is the formal group of geometric cobordisms.

Proof. We give two variants of the proof.

1).”Topological proof” follows from the commutative diagram

\[
\begin{array}{c}
\mathbb{C}P^{kl-1} \times \mathbb{C}P^{mn-1} \rightarrow \mathbb{C}P^{klnn-1} \\
\uparrow \hspace{1cm} \uparrow \\
\widetilde{Gr}_{k,kl} \times \widetilde{Gr}_{m,mn} \rightarrow \widetilde{Gr}_{km,klmn}
\end{array}
\] (11)

\(((km, ln) = 1)\) combining with the decomposition of the map \(\kappa\), which was obtained in previous proof.
2). By $\tilde{S}'$ denote the homomorphism $\tilde{\nu}^*: H[[x]] \to H[[y]]$ (recall that $\tilde{\nu}^*: H \to H$, where $S$ is the antipode). Let us consider the following composition of homomorphisms of Hopf algebras:

$$H[[x]] \xrightarrow{\tilde{\Delta}} H[[x]] \hat{\otimes}_R H[[x]] \xrightarrow{id \otimes \tilde{S}'} H[[x]] \hat{\otimes}_R H[[y]] \xrightarrow{(\mu)} H[[x, y]],$$

where $(\mu)$ is the homomorphism, induced by multiplication

$$\mu: H \hat{\otimes}_R H \to H.$$ It follows from the axiom of antipode

$$\mu \circ (id_H \otimes S) \circ \Delta = \eta \circ \varepsilon$$

that

$$(\mu) \circ (id_{H[[x]]} \otimes \tilde{S}') \circ \tilde{\Delta} |_H = \eta \circ \varepsilon.$$ Hence there exists the homomorphism of Hopf algebras

$$(\eta): R[[x]] \to H[[x, y]]$$

such that the following diagram

$$\begin{array}{ccc}
H[[x]] & \xrightarrow{\tilde{\Delta}} & H[[x]] \hat{\otimes}_R H[[x]] \\
\downarrow{(\varepsilon)} & & \downarrow{(\mu)} \\
R[[x]] & \xrightarrow{(\eta)} & H[[x, y]]
\end{array}$$

is commutative (here $(\varepsilon)$ is the homomorphism, induced by $\varepsilon$). Note that

$$(\eta)(x) = \mathfrak{G}(x, y).$$

This completes the proof that

$$\Delta_{R[[x]]}(x) = F(x \otimes 1, 1 \otimes x),$$

where $F(x \otimes 1, 1 \otimes x) \in R[[x \otimes 1, 1 \otimes x]]$ is the formal group of geometric cobordisms. □

It is very important that we consider the maps $\hat{\phi}$, $\hat{\nu}$, and $\kappa$ as fiber maps in this example. Otherwise instead of $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ we obtain the usual formal group of geometric cobordisms because the $H$-space $\hat{Gr}$ is isomorphic to the $H$-space $BSU_{\otimes} \times CP^\infty$ (see Proposition [14]).
It is well known\(^\text{3}\) that the formal group of geometric cobordisms is the universal formal group.

**Conjecture 24.** The formal group \(\mathcal{F}(x \otimes 1, 1 \otimes x)\) is the universal object in the category of formal groups over a (topological) Hopf algebras.

Let \(R'\) be a ring and \(F'(x \otimes 1, 1 \otimes x)\) be a formal group over \(R'\). Note that we may consider the \(R'\) as the Hopf algebra over \(R'\) with respect to the \(\Delta_{R'}: R' \cong R' \otimes_{R'} R', \eta_{R'} = \varepsilon_{R'} = S_{R'} = \text{id}_{R'}: R' \to R'\). If \(\chi: H \to R'\) is a homomorphism of the Hopf algebras from \((H, \mu, \eta, \Delta, \varepsilon, S)\) to \((R', \mu_{R'}, \eta_{R'}, \Delta_{R'}, \varepsilon_{R'}, S_{R'})\), then \(\chi = (\chi \circ \eta) \circ \varepsilon = \chi \mid_{R} \circ \varepsilon\). Hence there exists the natural bijection \(\text{Hom}_{\text{Hopf alg.}}(H, R') \leftrightarrow \text{Hom}_{\text{Ring}}(R, R')\). Therefore the Conjecture implies the universal property of the formal group of geometric cobordisms.

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\(^3\)Quillen D. On the formal group law of unoriented and complex cobordism theory, Bull. Amer. Math. Soc., 75:6 (1969), 1293–1298.