Towards a quantum-mechanical model for multispecies exclusion statistics

Stefan Mashkevich
Institute for Theoretical Physics, 252143 Kiev, Ukraine

and
Centre for Advanced Study of the Norwegian Academy of Science and Letters, Drammensveien 78, N-0271 Oslo, Norway

(May 21, 2018)

Abstract

It is shown how to construct many-particle quantum-mechanical spectra of particles obeying multispecies exclusion statistics, both in one and in two dimensions. These spectra are derived from the generalized exclusion principle and yield the same thermodynamic quantities as deduced from Haldane’s multiplicity formula.

PACS numbers: 03.65.-w, 05.30.-d, 05.70.Ce

Keywords: Exclusion statistics; Energy spectrum; Equation of state

*Present address. Email: mash@phys.ntnu.no
I. INTRODUCTION

Exclusion statistics was introduced by Haldane \[1\] by postulating a “generalized exclusion principle”, according to which adding a particle of species $b$ to a system reduces by $g_{ab}$ the number of single-particle states available for each subsequent particle of species $a$. Thus, the single-particle Hilbert space dimension is

$$D_a = D_a^{(0)} - \sum_b g_{ab} (N_b - \delta_{ab}),$$

where $D_a^{(0)}$ is that in the absence of particles and $N_a$ is the number of particles of species $a$; the many-particle dimension is taken to be

$$W = \prod_a \frac{(D_a + N_a - 1)!}{N_a!(D_a - 1)!}.$$ 

The quantities $g_{ab}$ are called mutual statistics parameters and constitute the statistics matrix $G = \{g_{ab} : a, b = 1, \ldots, s\}$, where $s$ is the total number of species.

Starting from these formulas, it is possible to calculate the thermodynamic quantities: the distribution function (at least implicitly) and the equation of state (at least in the form of a virial expansion); see Refs. [2,3] for the single-species case and Refs. [4] for the multispecies case. In particular, when the exponent $\sigma$ in the single-particle dispersion law $\varepsilon(p) = \varepsilon_0 + \Lambda p^\sigma$ is equal to the dimensionality of space, all the $G$ dependence of the equation of state is contained in the second order of the virial expansion [3–6].

The question to be addressed here is the following. What is the many-particle spectrum of noninteracting particles obeying exclusion statistics in a potential with a given single-particle spectrum? Understanding this is an important step towards solving a more general problem of finding multispecies integrable models that realize exclusion statistics. Thus far, there are two known kinds of such models: (i) the single-species one-dimensional Calogero model [7] (and the Sutherland model [8]); (ii) the single-species [9,10] and multispecies [11,4] lowest Landau level (LLL) anyon models. The spectrum of the Calogero model can be viewed either as that of interacting bosons/fermions or of free particles obeying exclusion
statistics \[12,13\], the $1/x^2$ interaction being traded for a “renormalization” (fractionization) of quantum numbers, which can be equivalently regarded as implied by the generalized exclusion principle. (In some way, this is reminiscent of the equivalence of Aharonov–Bohm interaction and anyon statistics [14].) The same structure of the spectrum occurs for the LLL anyon model, which can be also generalized to many species.

In the present Letter, we will point out a simple and natural way of constructing multispaces many-particle spectra (in a harmonic potential) for a symmetric statistics matrix \( \{g_{ab}\} \), that are to be viewed as a realization of exclusion statistics in the sense that they have the generalized exclusion principle built in and lead to the thermodynamics implied by Eqs. (1) and (2). After reviewing the one-dimensional case, we make the construction in two dimensions, where the underlying idea is to introduce the fractional exclusion principle “along one degree of freedom”.

II. ONE DIMENSION: CALOGERO MODEL

We start with recalling the known single-species one-dimensional case. The single-particle energy spectrum in a harmonic potential with frequency \( \omega \) is

\[
E_n = \left(n + \frac{1}{2}\right) \omega ,
\]

where \( n = 0, 1, 2, \ldots \). An \( N \)-boson state is characterized by a sequence of quantum numbers \( \{n_j : j = 1, \ldots, N\} \) such that \( n_{j+1} \geq n_j \), and its energy is \( E_{\{n_j\}}(0) = \sum_{j=1}^{N} E_{n_j} \). The \( N \)-boson partition function is

\[
Z_N(0) = \sum_{\{n_j\}} \exp[-\beta E_{\{n_j\}}(0)] = \frac{\exp[N^2x/2]}{\prod_{j=1}^{N}[\exp(jx) - 1]} ,
\]

where \( x = \beta \omega \). The spectrum of the Calogero model,

\[
E_{\{n_j\}}(g) = \left[ \sum_{j=1}^{N} \left(n_j + \frac{1}{2}\right) + \frac{N(N-1)}{2} g \right] \omega ,
\]

where \( g(g-1) \) is the coupling constant of the inverse square interaction, can be obtained in terms of free particles obeying exclusion statistics with parameter \( g \), in the following way:
Regard \( n \) in Eq. (3) as a continuous variable and postulate that every particle \( k \) "pushes" every particle \( j \) residing higher than \( k \), by \( g \) units up, in terms of \( n \). This is understood as \( g \) states being excluded for particle \( j \). Mathematically, this means that the quantum numbers are "renormalized": \( n_j \rightarrow \tilde{n}_j \), with

\[
\tilde{n}_j = n_j + \sum_{k=1}^{N} \theta(\tilde{n}_j - \tilde{n}_k)g ,
\]

where \( \theta \) is the step function. With \( n_{j+1} \geq n_j \), the solution to this equation is \( \tilde{n}_j = n_j + (j-1)g \) \[12\]; that is, the lowest particle remains in its place, the second lowest one gets shifted by \( g \) units up, the third one by \( 2g \) up, etc. On the other hand, there is no interaction now, so the many-body energy is a sum

\[
E_{\{n\}}(g) = \sum_{j=1}^{N} E_{\tilde{n}_j} .
\]

By summing (6) over \( j \), one sees that renormalizing the \( n_j \)'s increases the energy by \( g\omega \) per each pair of particles. Hence, Eq. (3) follows immediately. In particular, for \( g = 1 \) one obtains the fermionic spectrum. The partition function is \( Z_N(g) = \exp \left[-\frac{1}{2}N(N-1)gx\right] Z_N(0) \), and the thermodynamics is the same \[3\] as obtained starting from Eqs. (1)–(2).

### III. A MULTISPECIES GENERALIZATION

A state of \( s \) species of bosons is characterized by a set of quantum numbers \( \{n_{aj} : a = 1, \ldots, s, \ j = 1, \ldots, N_a\} \) with \( n_{a,j+1} \geq n_{aj} \), its energy is \( E_{\{n_{aj}\}}(0) = \sum_{a=1}^{s} \sum_{j=1}^{N_{a}} E_{n_{aj}} \), and the partition function is

\[
Z_{N_1 \ldots N_s}(0) = \prod_{a=1}^{s} Z_{N_a}(0) .
\]

We will assume the statistics matrix to be symmetric: \( g_{ab} = g_{ba} \)—an important restriction, which will be discussed later. The appropriate generalization of (6) is (cf. [13])

\[
\tilde{n}_{aj} = n_{aj} + \sum_{b=1}^{s} \sum_{k=1}^{N_{b}} \theta(\tilde{n}_{aj} - \tilde{n}_{bk})g_{ab} .
\]
This means that a particle of species $b$ “pushes” one of species $a$ by $g_{ab}$ units up whenever the latter resides higher than the former in the final state. Obviously, in contrast to the single-species case, the order of $\tilde{n}$’s may not coincide with that of $n$’s, so that Eq. (9) has to be solved in a self-consistent manner. However, assuming again the free-particle expression for the energy

$$E_{\{n_{aj}\}}(G) = \sum_{a=1}^{s} \sum_{j=1}^{N_a} E_{\tilde{n}_{aj}},$$

(10)

the form of Eq. (9) implies that, just like in the single-species case, each pair of particles, one of species $a$ and the other one of species $b$, contributes $g_{ab}\omega$ to the energy. Hence,

$$E_{\{n_{aj}\}}(G) = \left[ \sum_{a=1}^{s} \sum_{j=1}^{N_a} \left( n_{aj} + \frac{1}{2} \right) + \sum_{a=1}^{s} \frac{N_a(N_a-1)}{2} g_{aa} + \sum_{a<b}^{s} N_a N_b g_{ab} \right] \omega. \quad (11)$$

This is essentially the same as the LLL spectrum of multispecies anyons (with charges of the same sign) [11,4]. Moreover, a semiclassical treatment of that problem [4] leads to Eq. (9). The problem itself is two-dimensional, but the LLL restriction renders it effectively one-dimensional by reducing the phase space. In the genuine one-dimensional case, however, it is not known what Hamiltonian would be a multispecies generalization of the Calogero one and possess the spectrum (11). At least two possible such generalizations, one [16] involving and the other one [17] not involving a three-particle interaction, were considered in the literature, but neither of those serves our purpose.

**IV. THERMODYNAMICS**

Be that as it may, Eq. (11) by itself is enough to derive the thermodynamic functions. From the grand partition function $\Xi = \sum_{N_1 \ldots N_s} z_1^{N_1} \cdots z_s^{N_s} Z_{N_1 \ldots N_s}$ [where $z_a = \exp(\beta \mu_a)$], one can deduce the cluster expansion $\ln \Xi = \sum_{k_1 \ldots k_s} b_{k_1 \ldots k_s} z_1^{k_1} \cdots z_s^{k_s}$ and the dimensionless analog of the virial expansion $\ln \Xi = \sum_{k_1 \ldots k_s} A_{k_1 \ldots k_s} N_1^{k_1} \cdots N_s^{k_s} / Z_1^{k_1+\cdots+k_s-1}$, using the relation $N_a = z_a(\partial \ln \Xi / \partial z_a); \text{ the coefficients } A_{k_1 \ldots k_s} \text{ are just numbers. If there is a well-defined volume,} \text{ the usual virial expansion (the equation of state) } \beta P = \sum_{k_1 \ldots k_s} a_{k_1 \ldots k_s} \rho_1^{k_1} \cdots \rho_s^{k_s} \text{ follows, by } \ln \Xi = \beta PV, \ N_a = \rho_a V.$
The partition function that one obtains from Eq. (11) is

\[ Z_{N_1...N_s}(G) = \exp \left[ - \left( \sum_{a=1}^{s} \frac{N_a(N_a - 1)}{2} g_{aa} + \sum_{a,b=1}^{s} \frac{N_aN_b g_{ab}}{a < b} \right) x \right] Z_{N_1...N_s}(0) . \]  

(12)

This yields the cluster coefficients in the thermodynamic limit,

\[ b_{k_1...k_s}(G) = F_{k_1...k_s}(G)x^{-1} , \]  

(13)

\[ F_{k_1...k_s}(G) = (-1)^{s-1} \prod_{j=1}^{s} \prod_{l=1}^{k_j-1} \left[ 1 - \left( \sum_{n=1}^{s} k_n g_{jn} \right)/l \right] \prod_{n=1}^{s} k_n^2 \sum_{p_l q_1 ... p_{s-1} q_{s-1} = 1 \atop p_m < q_m} \prod_{j=1}^{s-1} k_{p_j} k_{q_j} g_{p_j q_j} ; \]  

(14)

in the last sum, all the pairs \((p_j, q_j)\) must be distinct, and each number from 1 to \(s\) must figure in the set \(\{p_1, \ldots, p_{s-1}, q_1, \ldots, q_{s-1}\}\) at least once. All the numbers \(k_1, \ldots, k_s\) are assumed to be different from zero (otherwise, an obvious invariance with respect to the renumbering of the species and the identity \(F_{k_1...k_r0...0} = F_{k_1...k_r}\) should be used). For example:

\[ F_{400}(G) = \frac{1}{16} (1 - 4g_{11})(1 - \frac{4}{3}g_{11}) ; \]

\[ F_{321}(G) = \frac{1}{6} \left[ 1 - (3g_{11} + 2g_{12} + g_{13}) \right] \left[ 1 - \frac{1}{2}(3g_{11} + 2g_{12} + g_{13}) \right] \left[ 1 - (3g_{12} + 2g_{22} + g_{23}) \right] \]

\[ \times (3g_{12}g_{13} + 2g_{12}g_{23} + g_{13}g_{23}) . \]

To compare, the general formula for particles obeying exclusion statistics in a \(D\)-dimensional harmonic potential, implied by Eqs. (1) and (2), is [4]

\[ b_{k_1...k_s}(G) = (k_1 + \cdots + k_s)^{-D} f_{k_1...k_s}(G) Z_1 , \]

(15)

and for a symmetric \(G\) it turns out that \(f_{k_1...k_s}(G) = (k_1 + \cdots + k_s) F_{k_1...k_s}(G)\). Since in the thermodynamic limit \(Z_1 = 1/x\), the system at hand does exhibit exclusion statistics. The dimensionless virial coefficients are

\[ A_2 = (2g_{11} - 1)/4 , \quad A_{11} = g_{12} ; \]

\[ A_k = B_{k-1}/k! \quad \text{for } k > 2 , \]

(16)

and all the others vanish. Thus, they only depend on the statistics at the second order [3].
The system considered is, in fact, thermodynamically equivalent to a system of particles with linear dispersion \( \varepsilon(p) = \Lambda p \) in a box (because the single-particle spectra are equidistant in both cases, and the renormalization will alter them in the same way) of a volume related to the harmonic frequency in such a way that the single-particle partition functions \( Z_1 \) in both cases match: \( 1/x = V/\pi \Lambda \beta \), or \( V = \pi \Lambda /\omega \). The virial coefficients of the latter system are \( a_{k_1...k_s} = A_{k_1...k_s} (\pi \Lambda \beta)^{k_1+...+k_s-1} \).

V. TWO DIMENSIONS

The single-particle spectrum in a two-dimensional harmonic potential is

\[
E_{mn} = (m + n + 1)\omega ,
\]

(17)

where \( m, n = 0, 1, 2, \ldots \). The quantum numbers \( m \) and \( n \) correspond to the two degrees of freedom (say, \( x \) and \( y \) motion). The idea is to implement the fractional exclusion principle “along one degree of freedom”, i.e., with respect to one quantum number. Thus, split up the spectrum into “sectors”, labeled by \( m \), each sector being the spectrum of a one-dimensional oscillator with a constant \((m + 1/2)\omega \) added to the levels, and postulate exclusion statistics in such a way: Particles within one sector behave like they do in the above multispecies one-dimensional model, while particles in different sectors do not influence one another.

Formally, it looks this way. A state of \( s \) species of bosons can be characterized by a set of pairs \( \{(m_{aj}, n_{aj}) : a = 1, \ldots, s, \ j = 1, \ldots, N_a\} \) such that either \( m_{a,j+1} > m_{aj} \) or \((m_{a,j+1} = m_{aj} \text{ and } n_{a,j+1} \geq n_{aj})\). That is, the next particle of the same species is put either into a higher sector than the previous one or into the same sector but not lower. Define a set of numbers \( \{N_{al} : a = 1, \ldots, s, \ l = 1, \ldots, l_{\text{max}}\} \), where \( N_{al} \) is the number of particles of species \( a \) in the \( l \)-th from below nonempty sector and \( l_{\text{max}} \) is the total number of nonempty sectors. One has \( N_a = \sum_{l=1}^{l_{\text{max}}} N_{al} \). Introduce now a quantity \( M_{al} = \sum_{l'=1}^{l} N_{a,l'} \); then for any \( a \), the \( l \)-th nonempty sector contains those particles \( (aj) \) for which \( M_{a,l-1} + 1 \leq j \leq M_{al} \).

Eq. (9), in accordance with the aforesaid, is replaced by
\[
\tilde{n}_{aj} = n_{aj} + \sum_{b=1}^{s} \sum_{k=1}^{N_b} \theta(\tilde{n}_{aj} - \tilde{n}_{bk}) \delta_{m_{aj}m_{bk}} \gamma_{ab},
\]
whereas \( \tilde{m}_{aj} = m_{aj} \). The free-particle expression for the energy
\[
E_{\{(m_{aj},n_{aj})\}}(G) = \sum_{a=1}^{s} \sum_{j=1}^{N_a} E_{\tilde{m}_{aj}\tilde{n}_{aj}}
\]
now yields
\[
E_{\{(m_{aj},n_{aj})\}}(G) = \sum_{l=1}^{l_{\text{max}}} \left[ \sum_{a=1}^{s} \mathcal{N}_{al} \left( m_{l} + \frac{1}{2} \right) + \sum_{a=1}^{s} \sum_{j=\mathcal{M}_{al},l-1+1}^{\mathcal{M}_{al}} \left( n_{aj} + \frac{1}{2} \right) \right]
+ \sum_{a=1}^{s} \mathcal{N}_{al}(\mathcal{N}_{al} - 1) \gamma_{aa} + \sum_{a,b=1}^{s} a<b \mathcal{N}_{al}\mathcal{N}_{bl} \gamma_{ab} \right] \nu, \tag{20}
\]
where \( m_{l} \) is the absolute number of the \( l \)-th nonempty sector, so that \( m_{aj} = m_{l} \) with \( l \) such that \( \mathcal{M}_{al},l-1+1 \leq j \leq \mathcal{M}_{al} \). The quantity under the outermost sum can be recognized as Eq. (11) applied within the \( l \)-th sector, plus a constant displacement determined by \( m_{l} \). For example, let in the case of Bose statistics there be particles of species 1 in the states \((0,1), (2,4), (3,1)\); particles of species 2 in the states \((0,0), (2,1), (2,2)\); particles of species 3 in the states \((1,2), (2,2), (3,2)\). The energy of the corresponding state with exclusion statistics will be \((3 + g_{12}) + (4) + (21 + 2g_{12} + g_{13}) + (2g_{22}) + (11 + g_{13}) = 39 + 3g_{12} + 2g_{13} + 2g_{22} + 2g_{23}\).

The partition function takes the form
\[
Z_{N_1...N_s}(G) = \sum_{\{N_{al}\}}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_{l+1}}^{\infty} \cdots \sum_{m_{l_{\text{max}}}}^{\infty} \prod_{l=1}^{l_{\text{max}}} \left\{ \prod_{a=1}^{s} Z_{N_{al}}(0) \right\}
\times \exp \left[ - \left( \sum_{a=1}^{s} \mathcal{N}_{al} \left( m_{l} + \frac{1}{2} \right) + \sum_{a=1}^{s} \mathcal{N}_{al}(\mathcal{N}_{al} - 1) \gamma_{aa} + \sum_{a,b=1}^{s} a<b \mathcal{N}_{al}\mathcal{N}_{bl} \gamma_{ab} \right) x \right] \right\}, \tag{21}
\]
The outermost sum is over all possible distributions of particles over sectors, with \( l_{\text{max}} \) determined by such a distribution.

The cluster coefficients that one obtains from this are
\[
b_{k_1...k_s}(G) = (k_1 + \cdots + k_s)^{-1} F_{k_1...k_s}(G) x^{-2} \tag{22}
\]
with \( F_{k_1...k_s}(G) \) as in Eq. (14). This is but again the general formula (15) for \( D = 2 \), so that exclusion statistics is present. The virial coefficients do not exhibit any specific pattern.
It is possible, then, to obtain the cluster coefficients of the same system in a box of volume $V$ rather than in a harmonic potential [11, 18, 19]:

$$b_{k_1\ldots k_s}^V(G) = \frac{Vx^2}{\lambda^2}(k_1 + \cdots + k_s)b_{k_1\ldots k_s}(G)$$ (23)

(the dispersion law being quadratic, $\varepsilon(p) = p^2/2m$, and $\lambda^2 = 2\pi\beta/m$). These coincide, up to a common factor, with the ones from Eq. (13), and the dimensionless virial coefficients are again those given by Eq. (17). (In fact, they will be the same for any system with exclusion statistics with a constant single-particle density of states.) The dimensional virial coefficients are $a_{k_1\ldots k_s}(G) = A_{k_1\ldots k_s}(G)\lambda^{2(k_1+\cdots+k_s-1)}$.

VI. DISCUSSION AND CONCLUSIONS

Essentially, what has been shown here is that (i) for a one-dimensional harmonic potential, “excluding $g_{ab}$ (single-particle) states” as required by the generalized exclusion principle has to be understood as renormalizing the quantum numbers so that the sum of those for a pair of particle $a$ and particle $b$ increases by $g_{ab}$; (ii) in two dimensions, the same procedure has to be followed within effectively one-dimensional subsets of the spectrum, corresponding to one degree of freedom. In fact, we have also verified that the same thermodynamics is obtained when in the two-dimensional problem, the sectors are made up of states with the same angular momentum, which appears to be a better choice since it preserves rotational invariance. It is a plausible assumption that any division by sectors is actually good, and that the same will be true even for the three-dimensional problem [where $E_{lmn} = (l+m+n+\frac{3}{2})\omega$ and the sectors would be labeled by $l$ and $m$].

It is, however, not clear at this point how to construct the spectrum if the matrix $g_{ab}$ is not symmetric. In deriving Eq. (1), or (20), it was implied that $\theta(x-y) + \theta(y-x) = 1$ always. However, putting $\theta(0) = 1/2$ and using it for a nonsymmetric matrix does not lead to the same thermodynamics as obtained from Eqs. (1) and (2). Perhaps some generalization of Eq. (9), or (18), is necessary in this case. Apparently, there is a relevant physical example:
Interpreting numerically found few-electron spectra in terms of fractional quantum Hall effect quasiparticles and comparing the multiplicities to Haldane’s formula \[ 20 \] seems to hint that the system of two species of anyons with opposite charges in the LLL should exhibit exclusion statistics with \( g_{12} = -g_{21} \). To prove (or disprove) this directly by finding the spectrum, i.e., by generalizing Eq. (11), remains an open problem.

For a symmetric statistics matrix, both the spectrum and the thermodynamics of the conjectured integrable models realizing multispecies exclusion statistics are now known, which provides one with an essential information helping to search for such models.

ACKNOWLEDGEMENTS

I would like to thank Serguei Isakov for many useful discussions and Diptiman Sen for a number of comments. I am grateful to the Centre for Advanced Study in Oslo, where this work was initiated, for kind hospitality and financial support. The analytic formulas for the cluster and virial coefficients were derived using Mathematica \[ 21 \].
REFERENCES

[1] F.D.M. Haldane, Phys. Rev. Lett. 67 (1991) 937.

[2] Y.-S. Wu, Phys. Rev. Lett. 73 (1994) 922; S.B. Isakov, Phys. Rev. Lett. 73 (1994) 2150; C. Nayak, F. Wilczek, Phys. Rev. Lett. 73 (1994) 2740.

[3] S.B. Isakov, D.P. Arovas, J. Myrheim, A.P. Polychronakos, Phys. Lett. A 212 (1996) 299.

[4] S.B. Isakov, S. Mashkevich, cond-mat/9701154; Nucl. Phys. B, to appear.

[5] M.V.N. Murthy, R. Shankar, Phys. Rev. Lett. 73 (1994) 3331.

[6] A. Dasnières de Veigy, S. Ouvry, Phys. Rev. Lett. 75 (1995) 352; M.V.N. Murthy, R. Shankar, Phys. Rev. Lett. 75 (1995) 353.

[7] F. Calogero, J. Math. Phys. 10 (1969) 2191; 10 (1969) 2197; 12 (1969) 419.

[8] B. Sutherland, J. Math. Phys. 12 (1971) 246; 12 (1971) 251; Phys. Rev. A 4 (1971) 2019; 5 (1972) 1372.

[9] A. Dasnières de Veigy, S. Ouvry, Phys. Rev. Lett. 72 (1994) 600.

[10] A. Dasnières de Veigy, S. Ouvry, Mod. Phys. Lett. A 10 (1995) 1.

[11] S.B. Isakov, S. Mashkevich, S. Ouvry, Nucl. Phys. B 448 (1995) 457.

[12] N. Kawakami, J. Phys. Soc. Japan 62 (1993) 4163.

[13] S.B. Isakov, Int. J. Mod. Phys. A 9 (1994) 2563.

[14] F. Wilczek, Phys. Rev. Lett. 48 (1982) 957.

[15] S.B. Isakov, S. Viefers, Int. J. Mod. Phys. A 12 (1997) 1895.

[16] C. Furtlehner, S. Ouvry, Mod. Phys. Lett. B 9 (1995) 503.

[17] D. Sen, Nucl. Phys. B 479 (1996) 554.
[18] A. Comtet, Y. Georgelin, S. Ouvry, J. Phys. A (Math.Gen.) 22 (1989) 3917.

[19] K. Olaussen, On the harmonic oscillator regularization of partition functions, Trondheim preprint No. 13 (1992), cond-mat/9207005.

[20] W.P. Su, Y.S. Wu, and J. Yang, Phys. Rev. Lett. 77 (1996) 3423; S.B. Isakov, G.S. Canright, and M.D. Johnson, Phys. Rev. B Phys. Rev. B 55 (1997) 6727.

[21] S. Wolfram, The *Mathematica* book, 3rd edition (Wolfram Media—Cambridge University Press, 1996).