Center of $U(n)$, Cascade of Orthogonal Roots
and a Construction of Lipsman–Wolf

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Dedicated to Joe, a special friend and valued colleague

Abstract. Let $G$ be a complex simply-connected semisimple Lie group and let $\mathfrak{g} = \text{Lie } G$. Let $\mathfrak{g} = \mathfrak{n} - \mathfrak{h} + \mathfrak{n}$ be a triangular decomposition of $\mathfrak{g}$. One readily has that $\text{Cent } U(\mathfrak{n})$ is isomorphic to the ring $S(\mathfrak{n})^n$ of symmetric invariants. Using the cascade $\mathcal{B}$ of strongly orthogonal roots, some time ago we proved (see [K]) that $S(\mathfrak{n})^n$ is a polynomial ring $\mathbb{C}[\xi_1, \ldots, \xi_m]$ where $m$ is the cardinality of $\mathcal{B}$. The authors in [LW] introduce a very nice representation-theoretic method for the construction of certain elements in $S(\mathfrak{n})^n$. A key lemma in [LW] is incorrect but the idea is in fact valid. In our paper here we modify the construction so as to yield these elements in $S(\mathfrak{n})^n$ and use the [LW] result to prove a theorem of Tony Joseph.

Key words: cascade of orthogonal roots, Borel subgroups, nilpotent coadjoint action.
MSC (2010) codes: representation theory, invariant theory.

1. Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let

$$\mathfrak{g} = \mathfrak{n} - \mathfrak{h} + \mathfrak{n}$$

be a fixed triangular decomposition of $\mathfrak{g}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of $\mathfrak{h}$ roots in $\mathfrak{g}$. The Killing form $(x, y)$ on $\mathfrak{g}$, denoted by $\mathcal{K}$, induces a nonsingular bilinear form $(\mu, \nu)$ on $\mathfrak{h}^*$. For each $\varphi \in \Delta$ let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector. The root vectors can and will be chosen so that $(e_\varphi, e_{-\varphi}) = 1$ for all roots $\varphi$.

If $\mathfrak{s} \subset \mathfrak{g}$ is any subspace stable under $\text{ad } \mathfrak{h}$ let

$$\Delta(\mathfrak{s}) = \{\varphi \in \Delta \mid e_\varphi \in \mathfrak{s}\}.$$
Theorem A. There exists \( \xi_i \in S(n)^N, i = 1, \ldots, m \), so that
\[
S(n)^N = \mathbb{C}[\xi_1, \ldots, \xi_m]
\]
is a polynomial ring in \( m \)-generators. Furthermore,
\[
S(n)^N \cong \text{Cent } U(n)
\]
so that one has a similar statement for \( \text{Cent } U(n) \).

We will present an algebraic-geometric proof of a much stronger statement than Theorem A and relate it to a representation-theoretic construction, due to Lipsman–Wolf, of certain elements in \( S(n)^N \). See [K], [LW]. A key tool is the cascade \( \mathcal{B} = \{\beta_1, \ldots, \beta_m\} \) of orthogonal roots which will now be defined.

1.2. Let \( \Pi \subset \Delta_+ \) be the set of simple positive roots. For any \( \varphi \in \Delta_+ \) and \( \alpha \in \Pi \) there exists a nonnegative integer \( n_{\alpha}(\varphi) \) such that
\[
\varphi = \sum_{\alpha \in \Pi} n_{\alpha}(\varphi) \alpha.
\]
Let
\[
\Pi(\varphi) = \{\alpha \in \Pi \mid n_{\alpha}(\varphi) > 0\}.
\]
Then \( \Pi(\varphi) \) is a connected subset of \( \Pi \) and hence defines a simple Lie subalgebra \( g(\varphi) \) of \( g \). We will say that \( \varphi \) is locally high if \( \varphi \) is the highest root of \( g(\varphi) \). Obviously the highest roots of all the simple components of \( g \) are locally high.

Remark 1. If \( g \) is of type \( A_\ell \), but only in this case, are all \( \varphi \in \Delta_+ \) locally high.

Let \( \varphi \in \Delta_+ \) be locally high and let
\[
\Pi(\varphi)^o = \{\alpha \in \Pi(\varphi) \mid (\alpha, \varphi) = 0\};
\]
let \( g(\varphi)^o \) be the semisimple Lie algebra having \( \Pi(\varphi)^o \) as its set of simple roots. We will then say that a root \( \varphi' \in \Delta_+ \) is an offspring of \( \varphi \) if \( \varphi' \) is the highest root of a simple component of \( g(\varphi)^o \).

Remark 2. One notes that an offspring of a locally high root \( \varphi \) is again locally high and that it is strongly orthogonal to \( \varphi \).

A sequence of positive roots
\[
C = \{\beta'_1, \ldots, \beta'_k\}
\]
will be called a cascade chain if $\beta'_1$ is a highest root of a simple component of $\mathfrak{g}$, and if $1 < j \leq k$, then $\beta'_j$ is an offspring of $\beta'_{j-1}$. Now let $\mathcal{B}$ be the set of all positive roots $\beta$ which are members of some cascade chain. Let $W$ be the Weyl of $(\mathfrak{h}, \mathfrak{g})$.

**Theorem 1.** The cardinality of $\mathcal{B}$ is $m$ and

$$\mathcal{B} = \{\beta_1, \ldots, \beta_m\}$$

is a maximal set of strongly orthogonal roots. Furthermore, if $s_{\beta_i}$ is the $W$-reflection of $\mathfrak{h}$ corresponding to $\beta_i$, then the long element $w_o$ of $W$ may be given by

$$w_o = s_{\beta_1} \cdots s_{\beta_m}. \quad (1.1)$$

$\mathcal{B}$ is the cascade of orthogonal roots.

**1.3.** One has the vector space direct sum

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}. \quad (1.2)$$

Let $P : \mathfrak{g} \to \mathfrak{n}$ be the projection defined by (1.2). Since $\mathfrak{b}$ is the $K$-orthogonal subspace to $\mathfrak{n}$ in $\mathfrak{g}$ we may identify $\mathfrak{n}_-$ with the dual space $\mathfrak{n}^*$ to $\mathfrak{n}$, so that for $v \in \mathfrak{n}_-$ and $x \in \mathfrak{n}$, one has $\langle v, x \rangle = (v, x)$. The coadjoint action of $N$ on $\mathfrak{n}_-$ may then be given so that if $u \in N$, then on $\mathfrak{n}_-$

$$\text{Coad} u = P \text{Ad} u. \quad (1.3)$$

In fact, using (1.2) the coadjoint action of $N$ on $\mathfrak{n}_-$ extends to an action of $B$ on $\mathfrak{n}_-$, so that if $b \in B$ and $v \in \mathfrak{n}_-$, one has $b \cdot v = P \text{Ad} b(v)$. In addition we can regard $S(\mathfrak{n})$ as the ring of polynomial functions on $\mathfrak{n}_-$. Since $B$ normalizes $N$ the natural action of $N$ on $S(\mathfrak{n})$ extends to an action of $B$ on $S(\mathfrak{n})$ where if $f \in S(\mathfrak{n})$, $b \in B$, and $v \in \mathfrak{n}_-$, one has

$$(b \cdot f)(v) = f(b^{-1} \cdot v). \quad (1.4)$$

Recalling $m = \text{card} \mathcal{B}$, let $\mathfrak{r}$ be the commutative $m$-dimensional subalgebra of $\mathfrak{n}$ spanned by $e_{\beta}$ for $\beta \in \mathcal{B}$ and let $R \subset N$ be the commutative unipotent subgroup corresponding to $\mathfrak{r}$. In the dual space let $\mathfrak{r}_- \subset \mathfrak{n}_-$ be the span of $e_{-\beta}$ for $\beta \in \mathcal{B}$. For any $z \in \mathfrak{r}_-$, $\beta \in \mathcal{B}$, let $a_{\beta}(z) \in \mathbb{C}$ be defined so that

$$z = \sum_{\beta \in \mathcal{B}} a_{\beta}(z) e_{-\beta}, \quad (1.5)$$

and let

$$\mathfrak{r}^\times = \{\tau \in \mathfrak{r}_- \mid a_{\beta}(\tau) \neq 0, \forall \beta \in \mathcal{B}\}.$$
As an algebraic subvariety of $n_-$ clearly

$$\mathfrak{r}_-^\times \cong (\mathbb{C}^\times)^m. \quad (1.6)$$

Also for any $z \in n_-$ let $O_z$ be the $N$-coadjoint orbit containing $z$. Let $N_z \subset N$ be the coadjoint isotropy subgroup at $z$ and let $n_z = \text{Lie } N_z$. Since the action is algebraic, $N_z$ is connected and hence as $N$-spaces

$$O_z \cong N/N_z. \quad (1.7)$$

**Theorem 2.** Let $\tau \in \mathfrak{r}_-^\times$. Then (independent of $\tau$) $N_\tau = R$ so that (1.7) becomes

$$O_\tau \cong N/R. \quad (1.8)$$

In particular

$$\dim O_\tau = \dim n - m \quad (1.9)$$

and $O_\tau$ is a maximal dimensional coadjoint orbit of $N$.

Now consider the action of $B$ on $n_-$. In particular consider the action of $H$ on $n_-$. Obviously

$$\mathfrak{r}_-^\times \cong (\mathbb{C}^\times)^m, \quad (1.10)$$

and furthermore $\mathfrak{r}_-^\times$ is an orbit of $H$. In addition $H$ permutes the maximal $N$-coadjoint orbits $O_\tau$, $\tau \in \mathfrak{r}_-^\times$. More precisely,

**Theorem 3.** For any $a \in H$ and $\tau \in \mathfrak{r}_-^\times$, one has

$$a \cdot O_\tau = O_{a \cdot \tau}. \quad (1.11)$$

**1.4.** If $V$ is an affine variety, $A(V)$ will denote its corresponding affine ring of functions. Note that $S(n) = A(n_-)$. Let $Q(n_-)$ be the quotient field of $S(n)$.

**Theorem 4.** There exists a unique Zariski open nonempty orbit $X$ of $B$ on $n_-$. In particular

$$X = n_- \quad (1.12)$$

Furthermore $X$ is an affine variety so that

$$S(n_-) \subset A(X) \subset Q(n_-). \quad (1.13)$$

Moreover $n_-^\times \subset X$, and in fact one has a disjoint union

$$X = \bigsqcup_{\tau \in \mathfrak{r}_-^\times} O_\tau \quad (1.14)$$
so that all $N$-coadjoint orbits in $X$ are maximal and isomorphic to $N/R$.

Let $\Lambda \subset h^*$ be the $H$-weight lattice and let $\Lambda_{\text{ad}} \subset \Lambda$ be the root lattice. Let $\Lambda_B \subset \Lambda_{\text{ad}}$ be the sublattice generated by the cascade $B$. Since the elements of $B$ are mutually orthogonal note that

$$\Lambda_B = \bigoplus_{\beta \in B} \mathbb{Z} \beta$$

is a free $\mathbb{Z}$-module of rank $m$.

If $M$ is an $H$-module, let $\Lambda(M) \subset \Lambda$ be the set of $H$-weights occurring in $M$. Note that if $M$ is a $B$-module, then $M^N$ is still an $H$-module. Recalling the definition of $r_\infty^N$ and (1.6), note that

$$\Lambda(A(r_\infty^N)) = \Lambda_B$$

and each weight occurs with multiplicity 1.

We can now give more information about $X$ and its affine ring $A(X)$. Define a $B$ action on $r_\infty^N$ by extending the $H$-action so that $N$ operates trivially. Next define a $B$-action on $N/R$, extending the $N$-action by letting $H$ operate by conjugation, noting that $H$ normalizes both $N$ and $R$. With these structures and the original action on $X$, we have the following.

**Theorem 5.** One has a $B$-isomorphism

$$X \to N/R \times r_\infty^N$$

of affine varieties so that as $B$-modules

$$A(X) \cong A(N/R) \otimes A(r_\infty^N).$$

Furthermore, taking $N$-invariants, one has an $H$-module isomorphism

$$A(X)^N \cong A(r_\infty^N)$$

so that, by (1.16),

$$\Lambda(A(X)^N) = \Lambda_B$$

and each $H$-weight occurs with multiplicity 1.

Recalling (1.13) one has the $N$-invariant inclusions

$$S(n)^N \subset A(X)^N \subset Q(n_-)^N$$

of $H$-modules so that

$$\Lambda(S(n)^N) \subset \Lambda(A(X)^N) \subset \Lambda(Q(n_-)^N).$$
But since $S(n)$ is a unique factorization domain, any $u \in Q(n_-)$ may be uniquely written, up to scalar multiplication as

$$u = f/g$$

(1.22)

where $f$ and $g$ are prime to one another. Furthermore, it is then immediate (since $N$ is unipotent) that if $u$ is $N$-invariant, one has $f, g \in S(n)^N$. If, in addition, $u$ is an $H$-weight vector, the same is true of $f$ and $g$ so that, using Theorem 5, one readily concludes the following.

**Theorem 6.** Every $H$-weight in $\Lambda(S(n)^N)$ occurs with multiplicity 1 in $S(n)^N$. In fact $\Lambda(Q(n_-)) = \Lambda_B$ and every weight $\gamma$ in $\Lambda(Q(n_-))$ occurs with multiplicity 1 in $Q(n_-)^N$ and is of the form

$$\gamma = \nu - \mu$$

(1.23)

where $\mu, \nu \in \Lambda(S(n)^N)$.

For any $\gamma \in \Lambda_B$ let $\xi_{\gamma} \in Q_{n_-}^N$ be the unique (up to scalar multiplication) $H$-weight vector with weight $\gamma$. Thus if $\gamma \in \Lambda_B$, we may uniquely write (up to scalar multiplication)

$$\xi_{\gamma} = \xi_{\nu} / \xi_{\mu}$$

(1.24)

where $\mu, \nu \in \Lambda(S(n)^N)$ and $\xi_{\nu}$ and $\xi_{\mu}$ are prime to one another. Let

$$\Lambda_{\text{dom}} = \{ \lambda \in \Lambda \mid \lambda \text{ be a dominant weight} \}.$$

**Remark 3.** By the multiplicity 1-condition note that if $\nu \in \Lambda(S(n)^N)$, then $\xi_{\nu}$ is necessarily a homogeneous polynomial. Define $\deg \nu$ so that $\xi_{\nu} \in S^{\deg \nu}(n)$. Furthermore, clearly $\xi_{\nu}$ is then a highest weight vector of an irreducible $\mathfrak{g}$-module in $S^{\deg \nu}(\mathfrak{g})$ and in particular $\nu \in \Lambda_{\text{dom}}$. That is,

$$\Lambda(S(n)^N) \subset \Lambda_{\text{dom}} \cap \Lambda_B.$$

(1.25)

**1.5.** If $\nu \in \Lambda(S(n)^N)$, it follows easily from the multiplicity-1 condition and the uniqueness of prime factorization that all the prime factors of $\xi_{\nu}$ are again weight vectors in $S(n)^N$. Let

$$\mathcal{P} = \{ \nu \in \Lambda(S(n)^N) \mid \xi_{\nu} \text{ be a prime polynomial in } S(n)^N \}.$$  

(1.26)

We can then readily prove

**Theorem 7.** One has $\text{card } \mathcal{P} = m$ where, we recall $m = \text{card } B$, so that we can write

$$\mathcal{P} = \{ \mu_1, \ldots, \mu_m \}.$$  

(1.27)
Furthermore the weights $\mu_i$ in $\mathcal{P}$ are linearly independent and the set $P$ of prime polynomials, $\xi_{\mu_i}$, $i = 1, \ldots, m$, are algebraically independent. In addition, one has a bijection

$$\Lambda(S(n)^N) \to (\mathbb{N})^m, \; \nu \mapsto (d_1(\nu), \ldots, d_m(\nu))$$

(1.28)

such that, writing $d_i = d_i(\nu)$, up to scalar multiplication,

$$\xi_\nu = \xi_{\mu_1}^{d_1} \cdots \xi_{\mu_m}^{d_m}$$

(1.29)

and (1.29) is the prime factorization of $\xi_\nu$ for any $\nu \in \Lambda(S(n)^N)$. Finally,

$$S(n)^N = \mathbb{C}[\xi_{\mu_1}, \ldots, \xi_{\mu_m}]$$

(1.30)

so that $S(n)^N$ is a polynomial ring in $m$-generators.

**Remark 4.** One may readily extend part of Theorem 7 to weight vectors in $Q(n)^N$. In fact one easily establishes that there is a bijection

$$\Lambda(Q(n_-)^N) \to (\mathbb{Z})^m, \; \gamma \mapsto (e_1(\gamma), \ldots, e_m(\gamma))$$

so that writing $e_i(\gamma) = e_i$ one has

$$\xi_\gamma = \xi_{e_1}^{e_1_{\mu_1}} \cdots \xi_{e_m}^{e_m_{\mu_m}}.$$ 

(1.31)

Separating the $e_i$ into positive and negative sets yields $\xi_\nu$ and $\xi_\mu$ of (1.24).

1.6. Let $\nu \in \Lambda(S(n)^N)$. Then by Theorem 6 and (1.25) one has

$$\nu \in \Lambda_{\mathcal{B}} \cap \Lambda_{\text{dom}}$$

so that there exists nonnegative integers $b_{\beta}$, $\beta \in \mathcal{B}$ such that

$$\nu = \sum_{\beta \in \mathcal{B}} b_{\beta} \beta.$$ 

(1.31a)

**Remark 5.** The nonnegativity follows from dominance since one must have $(\nu, \beta) \geq 0$ for $\beta \in \mathcal{B}$.

We wish to prove

**Theorem 8.** One has

$$\sum_{\beta \in \mathcal{B}} b_{\beta} = \deg \nu,$$ 

(1.32)
and as a function $\xi_\nu \mid r_\nu^\times$ does not vanish identically and up to a scalar

$$\xi_\nu \mid r_\nu^\times = \prod_{\beta \in \mathcal{B}} e^{b_\beta}. \quad (1.33)$$

**Proof.** Let $S^\deg(\nu)(\mathfrak{n})$ be the $\nu$ weight space in $S^\deg(\nu)(\mathfrak{n})$. It does not reduce to zero since $\xi_\nu \in S^\deg(\nu)(\mathfrak{n})$. Let $\Gamma$ be the set of all maps $\gamma : \Delta_+ \rightarrow \mathbb{N}$ such that

$$\sum_{\varphi \in \Delta_+} \gamma(\varphi) = \deg \nu$$

$$\sum_{\varphi \in \Delta_+} \gamma(\varphi) \varphi = \nu. \quad (1.34)$$

Then if

$$e^\gamma = \prod_{\varphi \in \Delta_+} e^{\gamma(\varphi)},$$

the set $\{e^\gamma \mid \gamma \in \Gamma\}$ is clearly a basis of $S^\deg(\nu)(\mathfrak{n})$ and consequently unique scalars $s_\gamma$ exist so that

$$\xi_\nu = \sum_{\gamma \in \Gamma} s_\gamma e^\gamma. \quad (1.35)$$

But by Theorem 5 there exists $x \in X$ such that $\xi_\nu(x) \neq 0$. However since $X$ is $B$-homogeneous, the $H$-orbit $r_\nu^\times$ is contained in $X$ and there exists $t \in r_\nu^\times$ such that $x = u \cdot t$ for some $u \in N$. But since $\xi_\nu$ is $N$-invariant one has $\xi_\nu(t) \neq 0$. But from (1.34) this implies that

$$\sum_{\gamma \in \Gamma} s_\gamma e^\gamma(t) \neq 0. \quad (1.36)$$

But $e^\gamma(t) = 0$ for any $\gamma \in \Gamma$ such that $\gamma(\varphi) = 0$ for $\varphi \notin \mathcal{B}$. Thus there exists $\gamma' \in \Gamma$ such that

$$\gamma'(\varphi) = 0$$

for all $\varphi \notin \mathcal{B}$ and

$$e^{\gamma'}(t) \neq 0. \quad (1.37)$$

But by the independence of $\mathcal{B}$ one has that $\gamma'$ is unique and hence one must have $\gamma'(\beta) = b_\beta$. A similar argument yields (1.33). QED

2. A representation-theoretic construction, due to Lipsman–Wolf, of certain elements in $S(\mathfrak{n})^N$

2.1. Let $\lambda \in \Lambda_{\text{dom}}$ and let $V_{\lambda}$ be a finite-dimensional irreducible $\mathfrak{g}$-module with highest weight $\lambda$. Then, correspondingly, $V_{\lambda}$ is a $U(\mathfrak{g})$-module with respect to a surjection $\pi_\lambda : U(\mathfrak{g}) \rightarrow \text{End} V_{\lambda}$. Let $0 \neq v_\lambda \in V_{\lambda}$ be a highest weight vector. Also let $V_{\lambda}^*$ be
the contragredient dual $g$-module. The pairing of $V_\lambda$ and $V_\lambda^*$ is denoted by $\langle v, z \rangle$ with $v \in V_\lambda$ and $z \in V^*_\lambda$. (We will use this pairing notation throughout in other contexts.) But as one knows $V^*_\lambda$ is $g$-irreducible with highest weight $\lambda^* \in \Lambda_{\text{dom}}$ given by

$$\lambda^* = -w_0 \lambda. \quad (2.1)$$

But then by (1.1) and the mutual orthogonality of roots in the cascade

$$-\lambda^* = \lambda - \sum_{\beta \in B} \lambda(\beta^\vee) \beta.$$ 

That is

$$\lambda + \lambda^* = \sum_{\beta \in B} \lambda(\beta^\vee) \beta \quad (2.2)$$

and hence

$$\lambda + \lambda^* \in \Lambda_B \cap \Lambda_{\text{dom}}. \quad (2.3)$$

On the other hand, regarding $U(g)^*$ as a $g$-module (dualizing the adjoint action on $U(g)$) it is clear that if $f \in U(g)^*$ defined by putting, for $u \in U(g)$,

$$f(u) = \langle u v_\lambda, z_{\lambda^*} \rangle, \quad (2.4)$$

then

$$f \text{ is } n\text{-invariant and}$$

$$f \text{ is an } h \text{ weight vector of weight } \lambda + \lambda^*. \quad (2.5)$$

Now it is true (as will be seen below) that $\lambda + \lambda^* \in \Lambda(S(n)^N)$. It is the idea of Lipsman–Wolf to construct $\xi_{\lambda+\lambda^*}$ using $f$. The method in [L-W] is to symmetrize $f$ and restrict to $S(n)$. However Lemma 3.7 in [L-W] is incorrect (one readily finds counterexamples). But the idea is correct. One must modify $f$ suitably and this we will do in the next section.

**2.2.** Assume $\mathfrak{s}$ is a finite-dimensional Lie algebra. Let $U_j(\mathfrak{s})$, $j = 1, \ldots$, be the standard filtration of the enveloping algebra $U(\mathfrak{s})$. Let $0 \neq f \in U(\mathfrak{s})^*$. We will say that $k \geq -1$ is the codegree of $f$ if $k$ is maximal such that $f$ vanishes on $U_{k-1}(\mathfrak{s})$. But then if $k$ is the codegree of $f$ and if $x_i \in \mathfrak{s}$, $i = 1, \ldots, k$, and $\sigma$ is any permutation of $\{1, \ldots, k\}$, then $(x_1 \cdots x_k - x_{\sigma(1)} \cdots x_{\sigma(k)}) \in U_{k-1}(\mathfrak{s})$ so that

$$f(x_1 \cdots x_k) = f(x_{\sigma(1)} \cdots x_{\sigma(k)}). \quad (2.6)$$

But this readily implies that there exists a unique element $f_{(k)} \in S^k(\mathfrak{s})$ such that for any $u \in U_k(\mathfrak{s})$ one has

$$f_{(k)}(\bar{u}) = f(u) \quad (2.7)$$
where \( \tilde{u} \in S_k(s) \) is the image of \( u \) under the Birkhoff–Witt surjection \( U_k(s) \to S_k(s) \).

Now let \( s = g \) and let \( f \) be given by (2.4). Let \( k \) be the codegree of \( f \). Identify \( g \) with \( g^* \) using the Killing form. Then \( f_{(k)} \in (S^k(g))^N \) and is an \( H \)-weight vector of weight \( \lambda + \lambda^* \). On the other hand, by (1.2),

\[
U_k(g) = U_k(n_+) \oplus U_{k-1}(g)b. \tag{2.8}
\]

However \( b \cdot v_\lambda \subset C v_\lambda \) so that \( f \) vanishes on \( U_{k-1}(g)b \). But this readily implies \( f_{(k)} \in S(n)^N \). We have proved

**Theorem 9.** Let \( f \) be given by (2.4) and let \( k \) be the codegree of \( f \). Then \( \lambda + \lambda^* \in \Lambda(S(n)^N) \). Furthermore \( k = \deg(\lambda + \lambda^*) \) and up to scalar multiplication

\[
f_{(k)} = \xi_{\lambda + \lambda^*}. \tag{2.9}
\]

The inclusion (1.25) is actually an equality

\[
\Lambda(S(n)^N) = \Lambda_{dom} \cap \Lambda_B. \tag{2.10}
\]

This equality is due to Tony Joseph and I was not aware of it until read it in [J]. However, the equality (2.10) follows immediately from the modified Lipsman–Wolf construction Theorem 9. Indeed let \( \nu \in \Lambda_{dom} \cap \Lambda_B \). To show \( \nu \in \Lambda(S(n)^N) \), it suffices to show that

\[
e_i(\nu) \geq 0 \tag{2.11}
\]

in (1.31) for any \( i = 1, \ldots, m \). But putting \( \lambda = \nu \), one has \( \lambda + \lambda^* = 2\nu \) and by Theorem 9 one has all \( e_i(2\nu) \geq 0 \). But clearly \( e_i(2\nu) = 2e_i(\nu) \). This proves (2.11).

The results in this paper will appear in [K1] in Progress in Mathematics, in honor of Joe.

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