Vortices and domain walls: ‘wormholes’ in unconventional superconductors

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Abstract. In the framework of the 2D and 3D time-dependent Ginzburg-Landau model we study superconductors with multicomponent order parameter (d-pairing). We argue that topological defects inside the sample do affect its thermodynamic properties such as hysteresis loop, susceptibility, etc. Along with earlier known topological defects such as Abrikosov vortices, domain walls (DWs) which separate different magnetic phases and even vortices inside the DW, we found an interesting combination of DWs and vortices. Namely we show that equivalent magnetic phases may be linked together with a vortex going through the other magnetic phase. This configuration may correspond to a stable state even in a zero external magnetic field. We also mention that this configuration is topologically similar to the ‘wormholes’ in the quantum gravity.

1. Introduction
Recently, there has been an increased interest in studies of superconductors with multicomponent order parameter. Such superconductors include all ones with non-trivial pairing (p-, d-pairing, etc.), heavy-fermion systems (for example UPt3, that is considered to be a d-wave superconductor) and HTSC [1-4].

The fact that the order parameter has at least two complex components leads to a formation of not only ordinary Abrikosov vortices but also planar topological defects – domain walls (DWs) – and their complicated combinations as well [5, 6].

2. The Ginzburg-Landau model for d-wave superconductors
Let us introduce the Ginzburg-Landau (GL) functional for the case of d-pairing [3]:

\[ H = \frac{1}{2} \int dV \left( \sum_{\alpha} (\eta_\alpha^* \eta_\alpha)^2 + \beta_1 |\eta_1|^2 + \beta_2 |\eta_1|^2 + \beta_3 (|\eta_2|^4 + |\eta_3|^4) + K_1 D^*_i \eta_i^* D_i \eta_i ight) \]

where \( D_i = \nabla_i - igA_i, i = x, y; \) \( A \) is a vector potential; \( \eta = [\eta_1, \eta_2, \eta_3, \eta_4] \); \( \beta_1, \beta_2, \beta_3 \) and \( K_1 \) are phenomenological constants, \( \gamma = 1/8 \pi, \) and \( g = 2e/hc. \) \( \eta_1 \) and \( \eta_2 \) components are coefficients in spherical harmonic expansion of the Cooper pair wave function.
Namely, \( \Psi^{\text{pair}} = \eta_x Y_x + \eta_y Y_y \), where basis set \( Y_x \) and \( Y_y \) correspond to atomic orbitals \( d_{x^2-y^2} \) and \( d_{xy} \).

In the presence of an external magnetic field varying slowly in time, the dynamics of considered superconductor follow the sort of the GL equations [6-7] shown below:

\[
\frac{\partial \eta_j}{\partial t} = \{\Delta \eta_j - (-1)^j (2 A \nabla \eta_k + \eta_k \nabla A) - \eta_j [A^2 - 1 + S] + b_2 M \frac{\partial M}{\partial \eta_j} + b_1 L \frac{\partial L}{\partial \eta_j} \} + \gamma_\eta (-1)^j \{ j \Rightarrow k \}; \quad (j = 1 \ldots 4, \ k = j(-1)^j),
\]

\[
\frac{\partial A}{\partial t} = -\text{rot(rotA)} + \frac{1}{k^2} \left\{ [\eta_1 \nabla \eta_2 - \eta_2 \nabla \eta_1] + [\eta_3 \nabla \eta_4 - \eta_4 \nabla \eta_3] - AS \right\}.
\]

Where \( \xi = (K_i / a)^{1/2} \), \( b_2 = (2 \beta_2 + \beta_3) / (\beta_1 + \beta_2 + \beta_3) \), \( b_3 = \beta_3 / (\beta_1 + \beta_2 + \beta_3) \), \( S = |\eta_j|^2 \), \( M = (\eta_j \eta_j^* - \eta_j \eta_j^*) / 2i \), \( L = (\eta_j \eta_j^* + \eta_j \eta_j^*) / 2 \), \( P = |\eta_j|^2 - |\eta_j|^2 \), and \( k, \gamma_\eta, \gamma_A \) are parameters of the theory.

Let us consider the phase \( \eta = (1, i) \). This phase has a twofold degeneracy in the direction of the vector \( |\eta \eta^*| \) which is proportional to the orbital moment of the Cooper pair [2]. Thus in this phase DW separates superconductor domains where the spontaneous magnetizations have the opposite projections on the quantization axis. However, direct interaction between the spontaneous magnetization and an external magnetic field is negligibly weak [3, 4]. Nevertheless we show that topological features such as DW qualitatively change the behaviour of the superconductor in the magnetic field.

3. Results

We use finite-difference explicit algorithm (similar to that used in [6-7]) to integrate the set of equations (2) numerically. During the numerical experiment one can vary the value of an external homogeneous magnetic field \( H \). Increasing rate of the magnetic field was chosen to be slow enough to ensure good convergence of the method used. This is also in a good agreement with the quasi-stationary external field variation.

3.1. Two dimensional case

![Figure 1. Distribution of the order parameter in d-wave superconductor with magnetic field applied perpendicular to the plane of figure. Dark colour corresponds to the regions where the order parameter is low. In a zero field domain wall (DW) is strait (a). When the field reaches a value of critical field of DW, one-quantum vortices (V1) start to enter the DW (b). The external field with a value higher than the lower critical field leads to the penetration of two-quantum vortices (V2) inside the sample (c).](image)

In the 2D+t simulations the superconducting sample was chosen to have a circular shape. We set the initial conditions for the order parameter \( \eta(x, y)_{|t=0} \) so as to provide a formation of a DW along the circle’s diameter (figure 1a). As the external homogeneous magnetic field is applied along \( z \) perpendicular to the plane of the figure, the magnetic flux \( \Phi \) starts to penetrate into the surface layer of the sample. As a result a screening supercurrent appears and drags ends of the DW. When an external field reaches the certain value of \( H < H_c \) the magnetic flux starts to penetrate into the DW in the form...
of one-quantum vortices $\Phi_0$ (figure 1b). We decided to introduce a special notation ‘the critical magnetic field of DW’, or $H_{cW}$, for the magnetic field strength at which vortices start to penetrate into DWs.

However, when the external magnetic field reaches the value of $H_{c1}$, $2\Phi_0$-vortices start to cross the boundary of the sample at random places. Thus before the superconductivity collapses, one can observe a complicated configuration of different topological defects presented in the sample: $2\Phi_0$-vortices, DW and $\Phi_0$-vortices in it (figure 1c). They become closer to each other with further increasing of the magnetic field. When it reaches the value of $H_{c2}$, superconductivity is absolutely destroyed.

Figure 2. Hysteresis loop of $d$-wave superconductor. Arrows show the direction of magnetization-demagnetization.

Figure 2 shows the magnetic induction vs the low external magnetic field ($H<H_{c2}$). The arrows’ direction shows the way of the magnetization. This process (0-1-2-3) was described above. When reaches the point 3 the magnetic field starts decreasing. At the beginning of the demagnetization (section 3-4) one-quantum vortices are ejected from the DW and then two-quantum vortices start leaving the sample (section 4-5). The residual magnetization vanishes step-wise (point 5 of the curve).

Thus the 2D+t numerical experiment allows us to draw a conclusion that the correct lower critical field of a $d$-wave superconductor must be dependent on whether DWs are present in it or not. We expect that real experiments will show the splitting of the lower critical field. The magnitude of this splitting would give a chance to calculate some characteristics of DW, in particular, the value of $H_{cW}$.

3.2. Three dimensional case

In the 3D+t simulations the superconducting sample has cylindrical shape. At first we set random initial conditions for the order parameter $\eta(x, y, z)|_{t=0}$ so as to look for the possible topological defects present in the $d$-wave superconductor.

Figure 3. Some cross-sections of the $d$-wave superconductive cylindrical sample showing the distribution of the order parameter. Dark colour corresponds to the regions where the order parameter is low. Vortices (V) may bind different domain walls (DWs) or different points of a single DW.

The intermediate distribution of the order parameter relaxing to its equilibrium value in zero magnetic field is shown on figure 3. Though the configuration is rather complex, it is quite similar to that present in the two dimensional case. One can see the same elements – vortices and DWs and their
combinations – as in the two dimensional case. However, we found quite interesting picture in which vortices and DWs are involved. Namely, a mouth of a vortex may originate not only from a boundary of a sample but also from a DW. Thus two DWs may be linked together with a vortex while a vortex core passes through the homogeneous phase. In this case different points of DWs are bound together as if they were connected with a spring, because the vortex free energy is proportional to its length. This configuration is topologically equivalent to ‘wormholes’ known in the quantum gravity theory [9, 10].

We found initial conditions for the order parameter which are close to the equilibrium distribution and ensure a formation of such configurations. Namely, conditions shown below provide a vortex lying on cylinder axis and two DWs being perpendicular to it. We choose the coordinate system so that the origin point lies in the center of the cylinder and $z$ coincides with the cylinder axis.

$$\eta_j = \frac{1}{2} \left[ (-1)^j \delta_4 \left( F_j \tanh x + (1 - F_j) \tanh y \right) (1 - \tanh \rho) + F_j (1 + \tanh \rho) \right], j = 1…4 \quad (3)$$

where $\delta_4$ is a Kronecker delta, $F_j = \delta_j + \delta_4$, $\rho = |z| - h/2$ and $h$ is a vortex length. Similar initial conditions provide a vortex linking a DW with a sample base boundary.

Configurations discussed above are shown on figure 4. Surfaces shown were found using the fact that the order parameter is low inside DWs and vortices.

![Figure 4](image)

**Figure 4.** Domain walls (DWs) bound by vortex (a). Vortex binds DW to a boundary of a sample (b).

We note that the solution found is not stationary when the external magnetic field is zero. Vortex tends to contract in order to reduce the free energy. As a result DWs interacting via the vortex come closer to each other. However, applying an external magnetic field and pinning effect are not only ways for the system to become stabilized. Indeed, the free energy of a DW is proportional to its area. That is why sometimes DWs linked with vortices constitute a frustrated system. And the geometry of a sample is a very important aspect here.

4. Conclusion
In conclusion we mention that DWs and vortices do affect the properties of $d$-wave superconductor. And one must bear it in mind, because it seems that once spontaneously arose during a phase transition these defects stay in a sample quite often due to frustrations.

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