Extra polarization states of cosmological gravitational waves in alternative theories of gravity

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Abstract
Cosmological gravitational waves (GWs) are usually associated with the transverse traceless part of the metric perturbations in the context of the theory of cosmological perturbations. These modes are just the usual polarizations ‘+’ and ‘×’ which appear in the general relativity theory. However, in the majority of the alternative theories of gravity, GWs can present more than these two polarization states. In this context, the Newman–Penrose formalism is particularly suitable for evaluating the number of non-null GW modes. In the present work we intend to take into account these extra polarization states for cosmological GWs in alternative theories of gravity. As an application, we derive the dynamical equations for cosmological GWs for two specific theories, namely a general scalar–tensor theory which presents four polarization states and a massive bimetric theory which is in the most general case with six polarization states for GWs. However, the mathematical tool presented here is quite general, so it can be used to study cosmological perturbations in all metric theories of gravity.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
The future detection of gravitational waves (GWs) of cosmological origin will strongly constrain the possible inflationary scenarios which have been proposed in the last decades. Also GWs will be useful to distinguish between the standard inflationary model and the alternative early Universe cosmologies, like the pre-big-bang scenario, since the predicted power spectrum of each model can present very different features.
In the general relativity theory (GRT), the usual procedure in order to evaluate the power spectrum of cosmological GWs is to expand small metric perturbations around the spatially homogeneous and isotropic Friedmann–Robertson–Walker (FRW) metric. The next step is to identify GWs with the transverse traceless (TT) part of the metric perturbations which do not couple with the perturbations of the perfect fluid. Thus, once it was generated, this radiative gravitational field can freely travel through the space and reach an observer today.

The observational effect of GWs is to generate relative tidal accelerations between test particles. The Riemann tensor determines these relative accelerations and it is the only locally observable imprint of gravity. To see that, consider a freely falling observer at any fiducial point \( P \) in the region. Let the observer set up an approximately Lorentz, normal coordinate system \( \{ x^\mu \} = \{ t, x^i \} \), with \( P \) as the origin. For a particle with spatial coordinates \( x^i \) at rest, the relative acceleration with respect to \( P \) is

\[
a_i = -R^i_{\alpha 0 j} x^j,
\]

where \( R^i_{\alpha 0 j} \) are the so-called electric components of the Riemann tensor due to waves or other external gravitational influences. When the linearized theory is considered, the Riemann tensor can be split into six algebraically independent components, but for the vacuum field equations of GRT they reduce to two, which represent the two polarization states of free GWs. These are the 'TT-modes', also called the + and \( \times \) polarizations.

Although these are the most studied modes in the GWs physics, when the framework of an alternative theory of gravity is considered, the number of non-null components of the Riemann tensor can be greater than two and the theory not only presents the TT-modes but other polarization states can also appear. This is a direct consequence of the new field equations which can generate other radiative modes. Thus, for a generic theory of gravity, GWs can present up to six polarization states corresponding to the six independent components of \( R^i_{\alpha 0 j} \).

Therefore, we should state that in order to work out the cosmological metric perturbations in an alternative theory of gravity, the first step is to find the number of independent polarizations of GWs in such a theory, i.e. the number of non-null components of the Riemann tensor. This can be done by a very elegant method which consists in the evaluation of the non-null Newman–Penrose (NP) quantities \([1, 2]\) of a given theory. These quantities are the irreducible parts of the Riemann tensor written in a complex tetrad basis which make them very useful to evaluate the polarizations of GWs in an unambiguous way.

Thus, the present paper intends to consider the general formalism of cosmological perturbations in the context of alternative theories of gravity, focusing on the dynamical equations of the GW modes. The theory of cosmological perturbations in GRT has been largely studied in the literature. Some classical examples are the works by Lifshitz \([3]\), Bardeen \([4]\), Peebles \([5]\), and Mukhanov, Feldman and Brandenberger \([6]\). For a recent review on cosmological dynamics see, e.g., \([7, 8]\). In the usual approach, the TT-modes of GWs are consistently described by the superadiabatic, or parametric, amplification mechanism in the GRT \([9]\). Furthermore, it was shown by Barrow and de Garcia Maia \([10, 11]\) that the same mechanism applies for modified theories of gravity as scalar–tensor theories and the so-called \( f(R) \) theories. Although they have analyzed only the evolution of the two TT-modes, it is known since the work by Eardley \textit{et al} \([1]\) that scalar–tensor theories present in addition at least one scalar GW polarization (more general scalar–tensor theories present two scalar GW polarizations) as a consequence of the additional degree of freedom included by the scalar field. The relic scalar GW production which arises from these kind of theories was recently discussed by Capozziello \textit{et al} \([12]\), and an upper limit was obtained from the amplitude of scalar perturbations in the Wilkinson microwave anisotropy probe (WMAP) data. But in their analysis, Capozziello \textit{et al} have considered only the vacuum field equations, this is a
limitation of their results since a coupling between scalar GWs and the scalar perturbations of the cosmological perfect fluid is expected as will be clear in our derivations.

In the case of \( f(R) \) theories, using the NP formalism, it was shown that a particular class of functions \( f(R) \) presents two scalar GW modes in addition to the + and \( \times \) modes, thus totaling four independent polarizations of GWs [13]. However, when the Palatini approach is used in the derivation of the field equations, the theory reveals only the usual two TT polarizations. It was also found that the scalar longitudinal mode which appears in \( f(R) \) theories is a massive mode which is potentially detectable by the future space GW interferometer LISA [14]. Again by considering only the vacuum equations, the production of the relic GWs of this particular mode was also considered and constraints using the WMAP data were established [15].

Moreover, the study of extra polarization states of cosmological GWs in the context of alternative theories of gravity can reveal new interesting features of these theories which do not appear in GRT. A remarkable example is the presence of vector longitudinal polarization modes of GWs in some theories. These modes give rise to a unusual Sachs–Wolf effect which leaves a vector signature on the CMB polarization [16]. Otherwise, vector perturbations in GRT decay too fast and it would not leave any signature on CMB polarization.

Therefore, it is clear that the future detection of GWs and the corresponding determination of the number of polarization modes are powerful tools to test the underlying gravity theory. Thus, the goal of the present paper is to furnish a general formalism to find the evolution equations of all the possible polarization modes which could appear in a generic theory of gravity. Once the number of independent polarization modes are found and the corresponding evolutionary equations could be obtained, one is in a position to obtain the power spectrum of each mode, finding the CMB signatures and constraining the additional modes. In order to show the application of the formalism, and to find some new features of the currently studied theories, we have chosen to obtain the dynamical equations for GWs in the context of two particular theories, namely a general scalar–tensor theory and a bimetric massive theory of gravity.

First proposed by Brans and Dicke [17] with the aim of making the theory of gravity compatible with Mach’s principle, the scalar–tensor theories are of a great interest since, as pointed out by several authors, a coupling between a scalar field and gravity seems to be a generic outcome of the low-energy limit of string theories (see, e.g., [18]). Another interest in the scalar–tensor models is that the \( f(R) \) theories can be written as the Einstein equations plus a scalar field, and thus we could in principle extend the same formalism applied for the scalar–tensor theories to the \( f(R) \) field equations. The bimetric massive theory we consider was proposed by Visser [19] with the aim of obtaining general covariant field equations with massive gravitons. His method was based on the introduction of a non-dynamical metric \((g_0)_{\mu \nu}\) besides the physical metric \(g_{\mu \nu}\). The resulting equations appear as a small modification of the Einstein field equations for which the massive gravitons and the metric \((g_0)_{\mu \nu}\) are present only in an additional energy–momentum tensor. Furthermore, our past studies have shown that Visser’s theory is a potential explanation for the current acceleration of the expansion of the Universe [20, 21].

In deriving the equations for GWs in the two theories, we will first review how to obtain the number of independent polarization modes for any theory following the Eardley et al approach [1]. In the case of the scalar–tensor models, the theory presents four polarization states in the more general case. Otherwise, Visser’s theory is a simple example of how a weak modification of gravity can produce six polarization modes. The subsequent analysis shows that all the polarization modes, apart from the usual + and \( \times \) polarizations, are dynamically ‘coupled’ to the perturbations of the cosmological perfect fluid. We argue that this kind
of coupling and the existence of additional polarization states could furnish distinguishable signatures of alternative theories in the power spectrum of the relic GWs.

The paper is organized as follows: in section 2 we present an overview of the NP formalism starting from the definition of the NP quantities which define the six possible polarization states for GWs. Then we find the non-vanishing parameters for the GRT, scalar–tensor theories and for Visser’s model. In section 3, considering a generic theory, we find general expressions for the perturbed Einstein tensor and for the energy–momentum tensor in the generalized harmonic coordinates. In section 4 we introduce a decomposition scheme which depends on the number of non-vanishing polarization modes of GWs which could appear in the various alternative theories. In sections 5 and 6 we apply the formalism of the preceding sections for two particular theories, the scalar–tensor theory and Visser’s bimetric model. Finally, we present our conclusions and discussions in section 7.

Throughout the paper we use units such that \( c = 1 \) unless otherwise mentioned.

2. An overview of the Newman–Penrose formalism

2.1. Tetrads’ components of tensors and null tetrads

At every point of the space one can introduce systems of four linearly independent vectors \( e^\mu_a \), which are known as tetrads. The index in parenthesis is the tetrad index which numbers the vectors from one to four. We can define the matrix:

\[
\mathbf{g}(a)(b) = e^\mu_a e^\nu_b g_{\mu\nu},
\]

which is an arbitrary symmetric matrix with negative-definite determinant. Its inverse \( g(c)(a) \), which is defined by

\[
g(c)(a)g(a)(b) = \delta(b)^{(c)}
\]

can be used to lower the tetrad indices:

\[
e^a_{(\mu)} = g(a)(b)e^b_{(\mu)},
\]

and to solve (2) for \( g_{\mu\nu} \):

\[
g_{\mu\nu} = g(a)(b)e^{a}_\mu e^b_{(\nu)},
\]

One can write any arbitrary vector or tensor as a linear combination of the four tetrad vectors:

\[
T^\mu\nu\cdots\gamma\sigma\cdots = T^{a(b)}_{(c)(d)}\epsilon^{a}_{(\mu)}\epsilon^{b}_{(\nu)}\cdots\epsilon^{c}_{(\gamma)}\epsilon^{d}_{(\sigma)}\cdots,
\]

where the quantities \( T^{a(b)}_{(c)(d)} \) are the tetrad components of the tensor. They are calculated according to

\[
T^{a(b)}_{(c)(d)} = T^{\mu\nu\cdots\gamma\sigma\cdots}_{(a)(b)}\epsilon_{(\mu)}^a\epsilon^b_{(\nu)}\cdots\epsilon^c_{(\gamma)}\epsilon^d_{(\sigma)}\cdots,
\]

which is consistent with (2) and (4). Tetrad indices are raised and lowered with \( g^{a(b)} \) and \( g(a)(b) \), respectively.

The advantages offered in many cases by the use of the tetrad components become clear when one examines their transformation properties and when one introduces tetrads which are appropriate to the particular problem being investigated. From equation (7) one can see that the tetrad components behave like scalars under coordinate transformations, i.e. the tetrad indices of tensors do not change under a coordinate transformation. Therefore, we have a good way of investigating the algebraic properties of tensors in a coordinate-independent fashion by the choice of the tetrads.
A possible choice is to identify the tetrad vectors with the base vectors of a Cartesian coordinate system in the local Minkowski system of the point concerned:

\[ g_{(a)(b)} = \delta^\mu_{(a)} \epsilon^\nu_{(b)} g_{\mu\nu} = \eta_{(a)(b)} = \text{diag}(-1, 1, 1, 1). \]  

(8)

The four tetrad vectors, which we shall call \((e_t, e_x, e_y, e_z)\), form an orthonormal system of one timelike and three spacelike vectors.

Another special case is the use of null vectors as tetrad vectors. A particular tetrad, known as the Newman–Penrose tetrad \([2]\), can be constructed from the orthonormal system introduced above; two of the four vectors \(k\) and \(l\) are real null vectors:

\[ k = \frac{1}{\sqrt{2}}(e_t + e_z), \quad l = \frac{1}{\sqrt{2}}(e_t - e_z), \]  

(9)

and the other two null vectors \(m\) and \(\overline{m}\) are complex conjugates of each other:

\[ m = \frac{1}{\sqrt{2}}(e_x + ie_y), \quad \overline{m} = \frac{1}{\sqrt{2}}(e_x - ie_y). \]  

(10)

It is easy to verify that the tetrad vectors obey the relations

\[ -k \cdot l = m \cdot \overline{m} = 1 \]  

(11)

\[ k \cdot m = k \cdot \overline{m} = l \cdot m = l \cdot \overline{m} = 0, \]  

(12)

and from (2) and (4) we obtain

\[ g_{(a)(b)} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}. \]  

(13)

A null tetrad basis is especially suitable for discussing null or nearly null waves.

2.2. Tetrad components of the Riemann tensor

The Riemann tensor \(R_{\lambda\mu\nu\kappa}\) can be split into the irreducible parts: the Weyl tensor, the traceless Ricci tensor and the curvature scalar (see, e.g., \([22]\)), whose tetrad components can be named, respectively, as \(\Psi\), \(\Phi\) and \(\Lambda\) following the notation of \([2]\). In general, in a four-dimensional space we have ten \(\Psi\)’s, nine \(\Phi\)’s and one \(\Lambda\) which are all algebraically independent. However, when we restrict ourselves to nearly plane waves, we find that the differential and algebraic properties of \(R_{\lambda\mu\nu\kappa}\) reduce the number of independent components to six \([1]\). Thus, following Eardley \textit{et al} \([1]\) we shall choose the set \(\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}\) to describe, in a given null frame, the six independent components of a wave in a generic metric theory. These NP quantities are related to the following components of the Riemann tensor in the null tetrad basis described earlier:

\[ \Psi_2 = -\frac{1}{6} R_{\hat{k}\hat{l}\hat{k}\hat{l}}, \]  

(14)

\[ \Psi_3 = -\frac{1}{2} R_{\hat{k}\hat{l}\hat{m}\hat{n}}, \]  

(15)

\[ \Psi_4 = -R_{\hat{m}\hat{n}\hat{m}\hat{n}}, \]  

(16)

\[ \Phi_{22} = -R_{\hat{m}\hat{n}\hat{m}\hat{n}}. \]  

(17)

Note that \(\Psi_3\) and \(\Psi_4\) are complex; thus, each represents two independent polarizations. One polarization for the real part and one for the imaginary part, thus totaling six components. Three are transverse to the direction of propagation, with two representing quadrupolar
deformations and one representing a monopolar ‘breathing’ deformation. Three modes are longitudinal, with one an axially symmetric stretching mode in the propagation direction, and one quadrupolar mode in each of the two orthogonal planes containing the propagation direction. Figure 1, which was taken from [23], shows the displacements induced on a ring of freely falling test particles by each of these modes. GRT predicts only the first two transverse quadrupolar modes (a) and (b).

Other useful expressions are the following relations for the Ricci tensor:

\begin{align}
R_{ik} &= R_{kiki}, \\
R_{ij} &= 2R_{iimi}, \\
R_{lm} &= R_{kilk}, \\
R_{im} &= R_{klim}, \\
R_{ii} &= R_{kikm},
\end{align}

and for the curvature scalar:

\begin{align}
R = -2R_{ik} = -2R_{kiki}.
\end{align}
The overall relative accelerations in a sphere of test particles is described by relation (1) and can be expressed in terms of the symmetric ‘driving-force matrix’ $S$ whose components are not but the electric components of the Riemann tensor $S_{ij} = R_{0i0j}$, where the latin indices represent spatial coordinates. These components can be written as a combination of the NP quantities in the following way:

$$
S = \begin{pmatrix}
-\frac{1}{2} (\Re \Psi_4 + \Phi_{22}) & \frac{1}{2} \Im \Psi_4 & -2 \Re \Psi_3 \\
\frac{1}{2} \Im \Psi_4 & \frac{1}{2} (\Re \Psi_4 - \Phi_{22}) & 2 \Im \Psi_3 \\
-2 \Re \Psi_3 & 2 \Im \Psi_3 & -6 \Psi_2
\end{pmatrix}.
$$

(23)

Since each NP amplitude is linearly independent, we can expand the components $S_{ij}$ as the sum

$$
S_{ij} = \sum_{r=1}^{6} p(r) E_{ij}^{(r)},
$$

(24)

where we have renamed the NP quantities as follows [1]:

$p(1) \equiv \Psi_2$,  
$p(2) \equiv \Re \Psi_3$,  
$p(3) \equiv \Im \Psi_3$,  
$p(4) \equiv \Re \Psi_4$,  
$p(5) \equiv \Im \Psi_4$,  
$p(6) \equiv \Phi_{22}$.

(25) - (30)

and $E_{ij}^{(r)}$ are the components of the ‘basis polarization matrices’ which are given by

$$
E_{(1)} = -6 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{(2)} = -2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
E_{(3)} = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{(4)} = -\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
E_{(5)} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{(6)} = -\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

(31)

Finally, analyzing the behavior of the set $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ under rotations, we see that they have the respective helicity values $s = \{0, \pm 1, \pm 2, 0\}$. These are all the possible helicity values for GWs in a general theory of gravitation.

### 2.3. Determining the polarizations of GWs for some specific theories

The procedure of evaluation of the number of independent polarizations involves examining the far-field, linearized, vacuum field equations of a theory, and then finding the non-null NP amplitudes. For GW sources sufficiently far from the observer, we hope that the external solutions of GWs approach that of vacuum and linearized regime; in this way, the method becomes an unambiguous tool for the determination of the propagating physical modes of GWs. In what follows we are going to evaluate the NP components of the Riemann tensor.
for three theories, namely the GRT, a general class of scalar–tensor theories and a bimetric massive theory.

In the case of the GRT, the field equations can be obtained from the Einstein–Hilbert action

\[ I = \frac{1}{16\pi G} \int \sqrt{-g} R \, d^4x + I_M, \]

(32)

where \( I_M \) is the action which describes the matter fields. From the Hamilton principle \( \delta I = 0 \) and the Einstein equations read

\[ G_{\mu\nu} = -8\pi GT_{\mu\nu}. \]

(33)

For the vacuum \( T_{\mu\nu} = 0 \) and (33) reduces to

\[ R_{\mu\nu} = 0. \]

(34)

Therefore, from the relations for the Ricci tensor and for the curvature scalar in the NP tetrad basis (18)–(22), we can show that

\[ R_{k\ell k\ell} = R_{k\ell k\ell m} = R_{m\ell m} = 0, \]

(35)

or equivalently

\[ \Psi_2 = \Psi_3 = \Phi_3 = 0, \]

(36)

and since we have no further constraints on the components of the Riemann tensor we conclude that

\[ \Psi_4 \neq 0. \]

(37)

Hence, as expected, a GW in the GRT presents two polarization states with helicity \( s = \pm 2 \).

The scalar–tensor theories were first proposed by Brans and Dicke \[17\] with the aim of making the theory of gravity compatible with Mach’s principle. These theories are of great interest since a coupling between a scalar field and gravity seems to be a generic outcome of the low-energy limit of string theories (see, e.g., \[18\]). Here, we will consider a class of theories characterized by the general action discussed by Wagoner \[24\] and Nordtvedt \[25\], which we write as found in \[26\]

\[ I = \frac{1}{16\pi} \int \sqrt{-g} d^4x \left[ -\phi R + \phi^{-1} \omega(\phi) \nabla_{\alpha}\phi \nabla^{\alpha}\phi - U(\phi) \right] + I_M, \]

(38)

where \( \phi \) is a scalar field, \( \omega(\phi) \) is the coupling parameter and \( U(\phi) \) can be interpreted as a potential associated with \( \phi \). The Brans–Dicke action can be obtained for the special case when \( \omega \) is constant and \( U(\phi) = 0 \). The field equations obtained by minimizing (38) with respect to the metric and with respect to the scalar field are

\[ G_{\mu\nu} = -8\pi T_{\mu\nu} - \frac{\omega}{\phi} \left( \phi,_{\mu} \phi,_{\nu} - \frac{1}{2} g_{\mu\nu} \phi,_{\alpha} \phi,^{\alpha} \right) - \frac{1}{\phi} \left( \phi,_{\mu\nu} - g_{\mu\nu} \Box \phi \right) - U g_{\mu\nu}, \]

(39)

\[ \left[ 3 + 2\omega(\phi) \right] \Box \phi - \phi \frac{dU}{d\phi} = 8\pi T - \frac{d\omega}{d\phi} \phi,_{\alpha} \phi^{\alpha}, \]

(40)

\[ T_{\mu\nu} = 0. \]

(41)

Note that the last condition, which expresses the conservation of the energy–momentum tensor, must be independently imposed since it is not a direct consequence of the field equations as in the case of the GRT. The vacuum field equations now read

\[ R_{\mu\nu} = - \frac{\omega(\phi)}{\phi^2} \phi,_{\mu} \phi,_{\nu} - \frac{1}{\phi} \left( \phi,_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \Box \phi \right) + g_{\mu\nu} U(\phi), \]

(42)
and

\[ R = -\frac{\omega}{\psi^2} \partial_\alpha \psi^{\alpha} - 3 \frac{\Box \psi}{\psi} + 4U. \]  

(43)

Let us first evaluate the polarizations for the case of the Brans–Dicke theory, for which equation (40) simply becomes

\[ \Box \psi = 0, \]  

(44)

with the solution

\[ \psi = \psi_0 + \psi_1 e^{i q_\alpha x^\alpha}. \]  

(45)

where \( \psi_0 \) is a constant obtained from the cosmological boundary conditions, \( \psi_1 \) is a small amplitude in such a way that we can work only to first order in \( \psi_1 \), and \( q_\alpha \) is the wave vector which is null for this particular case. It follows that the curvature scalar is null:

\[ R = 0 \Rightarrow R_{\mu\nu} = 0, \]  

(46)

and the Ricci tensor takes the form

\[ R_{\mu\nu} = -\frac{\psi_1}{\psi_0} e^{i q_\alpha x^\alpha} q_\mu q_\nu. \]  

(47)

Hence if the wave is propagating in the \( +z \) direction, the only non-null components of the Ricci tensor are \( R_{zz} \), \( R_{zt} \) and \( R_{tt} \), which means that \( R_{ll} \) is the only non-null component in the tetrad basis, leading us to conclude that

\[ R_{lkl} = 0 \quad \text{and} \quad R_{lml} \neq 0, \]  

(48)

and therefore

\[ \Psi_2 = \Psi_3 = 0, \quad \Psi_4 \quad \text{and} \quad \Phi_{22} \neq 0. \]  

(49)

That is, GWs in the Brans–Dicke theory present the two \( s = \pm 2 \) polarizations of GRT plus a perpendicular breathing polarization mode with helicity \( s = 0 \).

Otherwise, if the potential \( U(\psi) \) is not null, we have a more general scalar–tensor theory. Now, \( \Box \psi \) is not null in general, but obeys the following relation in the linearized regime [26]:

\[ \Box \psi - m_0^2 \psi = 0, \]  

(50)

where

\[ m_0^2 = \frac{\psi_0}{3 + 2\omega} \left( \frac{dU}{d\phi} \right)_{\phi = \phi_0}, \]  

(51)

and \( \psi_0 \) is the value of \( \psi \) for which the potential minimum is evaluated. The solution of (50) is given by (45), but now the wave vector satisfies

\[ q_\alpha q^\alpha = -m_0^2; \]  

(52)

thus, the curvature scalar does not vanish but gives

\[ R = -3m_0^2 \frac{\psi_1}{\psi_0} e^{i q_\alpha x^\alpha}. \]  

(53)

for first order in \( \psi_1 \), and the equation for the Ricci tensor is

\[ R_{\mu\nu} = -\frac{\psi_1}{\psi_0} e^{i q_\alpha x^\alpha} \left[ q_\mu q_\nu + \frac{1}{2} \eta_{\mu\nu} m_0^2 \right]. \]  

(54)

Therefore, from the evaluation of the tetrad components of the Ricci tensor and from the preceding relations for the Riemann tensor, we conclude that for a general scalar–tensor theory

\[ \Psi_3 = 0; \quad \Psi_2, \quad \Psi_4 \quad \text{and} \quad \Phi_{22} \neq 0. \]  

(55)
Now, we have a longitudinal scalar polarization mode besides the three modes which appear in the Brans–Dicke theory. Note also that the appearance of the longitudinal mode is related to the presence of a massive scalar field in the theory.

The next theory we are going to analyze is an alternative theory of gravity which takes into account massive gravitons by the action \[ I = \int d^4x \left[ \sqrt{-g} R - \frac{1}{16\pi G} m^2 \mathcal{L}_{\text{mass}}(g, g_0) \right] + I_{\text{M}}, \] (56)

where \( m \) is the graviton mass in natural units and the Lagrangian \( \mathcal{L}_{\text{mass}} \) is a function of the physical metric \( g \) and of a non-dynamical prior defined metric \( g_0 \). The massive Lagrangian proposed by Visser is

\[
\mathcal{L}_{\text{mass}}(g, g_0) = \frac{1}{2} \sqrt{-g_0} \left( (g_0^{-1})^{\mu\nu} (g - g_0)_{\mu\alpha} (g_0^{-1})^{\alpha\rho} \times (g - g_0)_{\rho\nu} - \frac{1}{2} \left[ (g_0^{-1})^{\mu\nu} (g - g_0)_{\mu\nu} \right]^2 \right). \]
(57)

By varying the action (56) with respect to the physical metric, we obtain the field equations

\[
G^{\mu\nu} - \frac{1}{2} m^2 M^{\mu\nu} = -8\pi GT^{\mu\nu}, \]
(58)

where the massive tensor \( M^{\mu\nu} \) reads

\[
M^{\mu\nu} = (g_0^{-1})^{\mu\sigma} \left( (g - g_0)_{\sigma\rho} - \frac{1}{2} (g_0)_{\sigma\rho} (g_0^{-1})^{\alpha\rho} (g - g_0)_{\alpha\sigma} \right) (g_0^{-1})^{\rho\nu}. \]
(59)

Among the existing bimetric theories of gravity (Rosen’s bimetric theory [27], for example), the most common criterion for the choice of the non-dynamical background metric is to impose the Riemann-flat condition \( R^{\mu\nu}(g_0) = 0 \) [23, 28]. Thus, the simpler choice is a flat metric which recovers the Minkowski metric by going to Cartesian coordinates [19, 20]. By introducing this background metric, Visser has constructed the Lagrangian (57) aiming a general covariant description of massive gravitons in such a way that the model circumvents the van Dam–Veltman–Zakharov discontinuity [29, 30], an inconsistency which plagues other massive terms.

Furthermore, Visser’s model could be an alternative explanation to the current acceleration of the expansion of the Universe as indicated by the recession of distant supernova [21]. In this sense, the massive tensor \( M^{\mu\nu} \) mimics the dark energy effects in large scales while the theory passes all the local tests of gravity. Hence, it is a matter of interest to distinguish between Visser’s theory and the Einstein gravity by dynamical tests such as GW observations.

From (58) we see that the vacuum field equations with massive gravitons read

\[
R^{\mu\nu} = m^2 \left( M^{\mu\nu} - \frac{1}{2} h^{\mu\nu} M \right), \]
(60)

and for the weak-field approximation the right-hand side of (60) is simply \( m^2 h_{\mu\nu} \), where \( h_{\mu\nu} \) is a metric perturbation around the flat background metric \( g_{\mu\nu} = (g_0)_{\mu\nu} + h_{\mu\nu} \) with \( |h_{\mu\nu}| \ll 1 \). Therefore, since there are no further restrictions on \( R_{\mu\nu} \), we conclude that all its components are non-null and we find that

\[
R_{ik}, R_{ilim}, R_{ilk} \quad \text{and} \quad R_{ilk} \neq 0, \]
(61)

and then

\[
\Psi_2, \Psi_3, \Psi_4 \quad \text{and} \quad \Phi_{22} \neq 0. \]
(62)

Thus, GWs in Visser’s theory present all the six possible polarization states showed in figure 1. This result was first obtained by de Paula et al [31] in a different but equivalent approach. Note that in the limit \( m \to 0 \), the polarization modes \( \Psi_2, \Psi_3 \) and \( \Phi_{22} \) vanish and we recover the GRT with the only non-null mode \( \Psi_4 \).
3. Cosmological perturbations without decomposition

Let us consider a general theory of gravity for which the field equations can be written in the form

\[ G_{\mu\nu} + F_{\mu\nu} = -8\pi G T_{\mu\nu}, \]  

(63)

where besides the Einstein tensor \( G_{\mu\nu} \) and the energy–momentum tensor for the matter fields \( T_{\mu\nu} \), we have a general tensor \( F_{\mu\nu} = F_{\mu\nu}(g^{\alpha\beta}, \gamma^{\alpha\beta}, \sigma^\alpha, \phi, \ldots) \) which could be a function of the physical metric \( g_{\mu\nu} \), of some prior defined metric \( \gamma_{\mu\nu} \), of vector fields \( \sigma_\mu \), of scalar fields \( \phi \) and of derivatives of these quantities. A prior defined metric is in general considered in bimetric theories of gravity, for which besides the dynamical metric \( g_{\mu\nu} \), a kind of ‘absolute’ geometry is specified through \( \gamma_{\mu\nu} \). One of the most known theories of this kind is Rosen’s theory [27] for which the second metric takes into account the effects of inertial forces. Another example is the bimetric massive theory considered by Visser which was introduced in the last section, and we will analyze it in more detail in section 6.

Supposing that there is a cosmological solution of such a theory, we can work out the perturbations on a cosmological background metric \( g_{\mu\nu} \). We adopt the metric \( g_{\mu\nu} \) as the Robertson–Walker metric written in Cartesian coordinates with zero spatial curvature \( k = 0 \).

With these considerations the line element reads

\[ ds^2 = a^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu, \]  

(64)

where the scale factor \( a(\eta) \) is a function of the conformal time \( \chi^0 = \eta \). The cosmic time \( t \) is related to the conformal time by the relation \( a(\eta) d\eta = dt \) and the Minkowski metric is \( \eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1) \).

Perturbing the metric (64) in the general form \( \delta g_{\mu\nu} = a^2(\eta) h_{\mu\nu}(x^\alpha) \) with \( |h_{\mu\nu}| \ll 1 \), we now have

\[ ds^2 = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu. \]  

(65)

In this form, the indices of \( h_{\mu\nu} \) are raised and lowered by the metric \( \eta_{\mu\nu} \). With the line element (65) in equations (63) we can obtain the perturbed field equations

\[ \delta G_{\mu\nu} + \delta F_{\mu\nu} = -8\pi [\delta T_{\mu\nu} + \delta GT_{\mu\nu}], \]  

(66)

where, in order to take into account the theories with varying Newtonian ‘constant’, we have included the perturbation \( \delta G \).

Hereafter we will use the generalized harmonic coordinates discussed by Bicak and Katz [32]. For this coordinate system, we have the condition

\[ g^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu} = 0, \]  

(67)

where \( \delta \Gamma_{\mu\nu}^\lambda \) is the metric connection perturbation. It is easy to show that condition (67) is equivalent to

\[ \nabla_\nu \delta g^{\mu\nu} = 0, \]  

(68)

where \( \delta g_{\mu\nu} = \delta g_{\mu\nu} - g_{\mu\nu} \delta g/2 \) with \( \delta g = g^{\mu\nu} \delta g_{\mu\nu} \). One can show that \( \delta g_{\mu\nu} = a^2 \tilde{h}_{\mu\nu} \) where the \( \tilde{h}_{\mu\nu} \) is the trace-reverse perturbation:

\[ \tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \]  

(69)

with \( h = \eta^{\mu\nu} h_{\mu\nu} \) and \( \tilde{h} = -h \).

With the metric perturbations given by (65) and with condition (68) we can evaluate the components of the perturbed Einstein tensor \( \delta G_{\mu\nu} \) to first order in \( h_{\mu\nu} \). A straightforward calculation leads to
\[\delta G^0_0 = \frac{1}{2a^2} \left[ -\dot{\bar{h}}^0 + \nabla^2 \bar{h}^0_0 - 2\mathcal{H}\ddot{\bar{h}}^0_0 + 3(3\mathcal{H}^2 - \mathcal{H}'\bar{h}^0_0 - (\mathcal{H}^2 - \mathcal{H}')\ddot{\bar{h}}^0_0 + 4a\mathcal{H}\partial_i\bar{h}^0_i \right], \quad (70)\]

\[\delta G^i_0 = \frac{1}{2a^2} \left[ -\dot{\bar{h}}^{''}_i + \nabla^2 \bar{h}^i_0 - 4\mathcal{H}\ddot{\bar{h}}^i_0 + (\mathcal{H}^2 - \mathcal{H}')\dot{\bar{h}}^i_0 + 2a^{-1}\mathcal{H}\eta^{-i}_j (\partial_j \bar{h}^i_0 - \partial_i \bar{h}^j_0) \right], \quad (71)\]

\[\delta G^j_i = \frac{1}{2a^2} \left[ -\dot{\bar{h}}^{''}_j + \nabla^2 \bar{h}^j_i - 2\mathcal{H}\dot{\bar{h}}^j_i + (\mathcal{H}^2 + \mathcal{H}')\bar{h}^i_j - (\mathcal{H}^2 - \mathcal{H}')\bar{h}^i_0 \delta^j_0 - 4a\mathcal{H}\eta^{ij} \partial_k (\bar{h}^i_0) \right]. \quad (72)\]

The prime in the above expressions denotes derivatives with respect to the conformal time \(\eta\), and we have defined the Hubble parameter for the conformal time as \(H \equiv a'/a\).

Now, it is necessary to evaluate the perturbations of the energy–momentum tensor for a perfect fluid:

\[T_{\mu\nu} = (\rho + P)U_{\mu} U_{\nu} + P g_{\mu\nu}, \quad (73)\]

where \(\rho\) and \(P\) are the energy density and the pressure, and \(U_{\nu} = -U^{\nu} = (1, 0, 0, 0)\) is the fluid four-velocity. Considering first-order perturbations in each one of these quantities, the perturbed energy–momentum tensor reads

\[T_{\mu\nu} \rightarrow \tilde{T}_{\mu\nu} = T_{\mu\nu} + \delta T_{\mu\nu}, \quad (74)\]

where

\[\delta T^\mu_\nu = (\rho + P)g^\mu\nu (U_{\mu} \delta U_{\nu} + \delta U_{\mu} U_{\nu}) + (\delta \rho + \delta P)U_{\mu} U_{\nu} + \delta P \delta^\mu_\nu - (\rho + P)\delta g^\mu_\nu U_{\mu} U_{\nu}. \quad (75)\]

And considering the perturbed metric as defined earlier we find the components of the perturbed quantity \(\delta T^\mu_\nu:\)

\[\delta \tilde{T}^0_0 = -\delta \rho, \quad \delta \tilde{T}^i_0 = (\rho + P) (V^i + a^{-2} \bar{h}^i_0), \quad \delta \tilde{T}^j_i = \delta P \delta^j_i, \quad (76)\]

where, for completeness, we have considered \(V^i = \delta U^i \ll 1\) as the spatial part of the perturbation of the fluid four-velocity as it appears in [6, 33]:

\[\tilde{U}^\nu = U^\nu + \delta U^\nu = \left(1 - \frac{1}{2} \delta g_{00}, V^i \right). \quad (77)\]

It is easy to see that the four-velocity is approximately the unit timelike vector since we assume \(V^i \ll 1\), and terms proportional to \(V^2\) and \(Vh\) can thus be neglected.

Then, from the calculation of the first-order perturbation of the tensor \(\delta F_{\mu\nu}\) and using the perturbed Einstein tensor and the perturbed energy–momentum tensor given above, one can write the perturbed field equations for a generic theory (see equation (66)).

As we will see in the following sections, the above equations are written in a suitable way to consider extra-polarization states of the GWs in any alternative theory of gravity with the field equations (63). They are also useful to find solutions without any decomposition as it was presented in the interesting work by Bicak et al [33]. Note that, despite the difference in notation, a similar set of equations can be found in that reference.

4. Metric perturbations with extra polarization states of GWs

Instead of considering the minimal decomposition which appears, for example, in [6], let us write the components of \(\delta g_{\mu\nu}\) in a more general form which was first presented by Bessada and Miranda [16]:

\[\delta g_{\mu\nu} = a^2(\eta) \begin{pmatrix} \phi & \delta S_i + \partial_i B \\ S_i + \partial_i B & \bar{h}_{ij} \end{pmatrix}, \quad (78)\]
where $\phi$ and $B$ are scalar perturbations and $S_i$ are the components of a divergence-less vector perturbation $\partial_i S^i = 0$. In our approach, the quantities $\phi$, $B$ and $S_i$ are dynamical quantities which appear in any theory. On the other hand, the way the perturbation $\bar{h}_{ij}$ must be decomposed depends on the number of independent GW polarization modes, and hence, it is theory dependent. Thus, the first step is to evaluate the number of non-null NP quantities and then we can decompose $\bar{h}_{ij}$ identifying the GW amplitudes in the expansion. In order to introduce the method, let us first consider the most general case for which all the NP quantities are non-null. For this particular case, $\bar{h}_{ij}$ can be expanded in terms of six components which represent the six metric amplitudes of GWs:

$$\bar{h}_{ij}(x, \eta) = \sum_{r=1}^{6} \epsilon^{(r)}_{ij} \bar{h}^{(r)}(x, \eta),$$

(79)

where $\epsilon^{(r)}_{\mu\nu}$ are the polarization tensors.

In what follows, without loss of generality, we can consider the wave vector of the GW oriented in the $+z$ direction. Thus, we can construct the six $\epsilon^{(r)}_{\mu\nu}$ by combinations of the three orthonormal vectors:

$$\ell^i = (1, 0, 0)$$
$$m^i = (0, 1, 0)$$
$$n^i = (0, 0, 1)$$

in the following way:

$$\epsilon^{(1)}_{ij} = n^i n^j,$$
$$\epsilon^{(2)}_{ij} = \ell^i n^j + \ell^j n^i,$$
$$\epsilon^{(3)}_{ij} = m^i n^j + m^j n^i,$$
$$\epsilon^{(4)}_{ij} = \ell^i \ell^j - m^i m^j,$$
$$\epsilon^{(5)}_{ij} = \ell^i m^j + \ell^j m^i,$$
$$\epsilon^{(6)}_{ij} = \ell^i \ell^j + m^i m^j.$$  

(80)

One can verify that the tensors (81)–(86) are linearly independent and form an orthogonal basis. Writing in a matricial form we have

$$[\epsilon^{(1)}_{ij}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [\epsilon^{(2)}_{ij}] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$[\epsilon^{(3)}_{ij}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad [\epsilon^{(4)}_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[\epsilon^{(5)}_{ij}] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\epsilon^{(6)}_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

(87)

We can see that, apart from a constant, these are just the basis polarization matrices (31) which we have described earlier. Now expansion (79) can be written in a more intuitive form:

$$\bar{h}_{ij} = \pi_{ij} + \tau_{ij} + \chi_{ij} + \psi_{ij},$$

(88)
where we have defined four new tensors following their helicity values $s$:

\begin{align*}
\pi_{ij} &\equiv \bar{h}_{ij}^{(1)} \quad \rightarrow \quad s = 0, \\
\tau_{ij} &\equiv \bar{h}_{ij}^{(2)} + \bar{h}_{ij}^{(3)} \quad \rightarrow \quad s = \pm 1, \\
\chi_{ij} &\equiv \bar{h}_{ij}^{(4)} + \bar{h}_{ij}^{(5)} \quad \rightarrow \quad s = \pm 2, \\
\psi_{ij} &\equiv \bar{h}_{ij}^{(6)} \quad \rightarrow \quad s = 0.
\end{align*}

(89)

And from the very definition of the quantities (89), we have the following properties:

\begin{align*}
\chi_i^i &= \tau_i^i = 0, \quad \partial_j \chi_j^i = \partial_j \psi_j^i = 0. \\
& \quad \text{(90)}
\end{align*}

Their associated helicity values can be evaluated from the behavior of the perturbations under rotations. Note also that we have chosen to define separately the two tensors for the $s = 0$ modes of GWs since one of them is a longitudinal mode ($\pi_{ij}$) and the other is transversal to the direction of propagation ($\psi_{ij}$). Furthermore, we have seen in section 2 that these two modes can appear for some theories (in a general scalar–tensor theory, for example), but other theories have a structure such that only one scalar mode appears (this is the case of the Brans–Dicke theory). This is a particular feature of the scalar modes, since it is not possible that a certain theory presents only one of the two tensor modes and not the other. In fact, there is no theory for which the $+ \text{ and } \times$ polarizations appear separated for vacuum GWs as it was evidenced by the construction of the NP amplitude $\Psi_4$ (see section 2). The same happens with the two vector modes which generate the same NP quantity $\Psi_3$.

It is instructive to write the NP quantities in terms of these metric perturbations considering an observer today located at the local Minkowskian spacetime. Thus, considering the Cartesian tetrad basis we have

\begin{align*}
\Psi_2 &= -\frac{1}{12} (\pi_{zz,00} - 2 \pi_{zz,zz} + \phi_{zz}) + \frac{1}{24} (\bar{h}_{,00} - \bar{h}_{zz}), \\
\Psi_3 &= \frac{1}{4\sqrt{2}} \left[ (\tau_{xz,zz} - \tau_{xz,00}) + i (\tau_{yz,00} - \tau_{yz,zz}) \right], \\
\Psi_4 &= \frac{1}{2} [\chi_{yy,00} - \chi_{xx,00} + 2i \chi_{xy,00}], \\
\Phi_{22} &= -\frac{1}{4} (\psi_{xx,00} + \psi_{yy,00} - \bar{h}_{,00}),
\end{align*}

(91-94)

where

\begin{equation}
\bar{h} = -\phi + \psi_{ij}^i + \pi_{ij}^i.
\end{equation}

(95)

Now, since the polarization modes of GWs are linearly independent, we can summarize the procedure to calculate the perturbed field equations for a given general theory of the form (63) in the following way.

**Scalar metric perturbations**

The general scalar metric perturbations are

\begin{equation}
\delta \bar{g}_{\mu\nu}^{(s)} = a^2(\eta) \left( \begin{array}{cc} \phi & \partial_i B \\ \partial_j B & h_{ij}^{(s)} \end{array} \right).
\end{equation}

(96)

If the theory has $\Psi_2 \neq 0$ and $\Phi_{22} \neq 0$, it means that an observer today can measure the two scalar GW modes ($s = 0$). Hence, in order to describe the evolution of these modes we write the term $h_{ij}^{(s)}$ in the form

\begin{equation}
h_{ij}^{(s)} = \pi_{ij} + \psi_{ij},
\end{equation}

(97)

with the constraint $\partial_j \psi_j^i = 0$ which guarantees the transversal propagation of the tensor $\psi_j^i$. 

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If the theory has $\Psi_2 = 0$ and $\Phi_{22} \neq 0$, no longitudinal scalar GW amplitude can be measured. Then $\pi_{ij}$ is suppressed, and in order to do not change the number of degrees of freedom of the metric perturbations, a scalar longitudinal component, say $D$, must be added. This new scalar component represents a dynamical perturbation of the cosmic gravitational potential rather than a radiative GW field. Consequently, an observer today measures only one scalar GW mode whose evolution can be described by $\psi_{ij}$ and $h_{ij}^{(s)}$ takes the form

$$h_{ij}^{(s)} = \partial_i \partial_j D + \psi_{ij},$$

(98)

where we have introduced $D$ using the spatial partial derivatives in order to represent the longitudinal behavior of this quantity. This is the same procedure of introducing longitudinal scalars in the minimal decomposition [6]. For those theories which can be found in this case (e.g. the Brans–Dicke theory), the only non-null scalar degree of freedom (in the local Minkowskian frame) is $\psi_{ij}$. In this sense, it is always possible to make $D = 0$ by gauge transformations in the reference frame of the local far-field observer, although in general $D$ does not cancel out when considering the whole cosmological evolution of the perturbations.

On the other hand, if the theory has $\Psi_3 = \Phi_{22} = 0$, the perturbations must be decomposed in the usual way, since there are no scalar GWs which could be observed today, that is,

$$h_{ij}^{(s)} = E \delta_{ij} + \partial_i \partial_j D,$$

(99)

where we have added the new scalar quantity $E$ to describe the evolution of the perturbations of the gravitational potential. The most important theory which can be found in this case is the Einstein theory for which we have only the two ‘pure tensor’ degrees of freedom for vacuum GWs. In Minkowski coordinates, it is always possible to cancel out all the scalar components by gauge transformations, but in the presence of the cosmic fluid they are important to describe the evolution of the perturbations.

**• Vector metric perturbations**

The general vector metric perturbations for any theory are

$$\delta \tilde{g}^{(v)}_{\mu\nu} = a^2(\eta) \begin{pmatrix} 0 & S_i \\ S_i & h_{ij}^{(v)} \end{pmatrix},$$

(100)

where the vector $S_i$ and $h_{ij}^{(v)}$ satisfy

$$\partial_i S^i = h_i^{(v)} = 0.$$

(101)

If the NP quantity $\Psi_3$ is non-null (Visser’s theory is an example), an observer today can measure the two vector modes ($s = \pm 1$) of GWs, and the metric perturbations $h_{ij}^{(v)}$ can be identified to the corresponding GW amplitude:

$$h_{ij}^{(v)} = \tau_{ij},$$

(102)

with $\tau^i_i = 0$.

On the other hand, if $\Psi_3 = 0$, there are no vector GWs today and we have the usual representation in terms of the vector quantity $Q_i$:

$$h_{ij}^{(v)} = \partial_i Q_j + \partial_j Q_i,$$

(103)

where from (101) $Q_i$ is divergenceless $\partial_i Q^i = 0$ and again, it is easy to verify that the number of the degrees of freedom of the metric perturbations does not change. The quantity $Q_i$ is a vector perturbation of the gravitational potential for which the dynamical equations gives, in general, a decaying mode in the context of the GRT (see, e.g., [4, 6]). As in the case of scalar perturbations we have discussed earlier, it is always possible to cancel $Q_i$ by gauge transformations in the local Minkowski frame when $\Psi_3 = 0$. 

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• Tensor metric perturbations

Finally, the tensor metric perturbations are constructed using a symmetric tensor $\chi_{ij}$ which satisfies the constraints

$$\chi_i^i = \partial_j \chi_j^i = 0.$$  \hfill (104)

Thus, the tensor component which corresponds to GWs with $s = \pm 2$ is written in the usual way in a theory-independent form:

$$\delta \tilde{g}_{\mu\nu}^{(ii)} = a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & \chi_{ij} \end{pmatrix}.$$  \hfill (105)

Counting the number of independent components we have used to construct $\delta g_{\mu\nu}$, and the number of constraints, we can see that we have four functions for scalar perturbations, four functions for vector perturbations and two functions for tensor perturbations. Thus, as expected, we have ten independent components of $\delta g_{\mu\nu}$.

In the context of the Einstein theory we have already found that $\Psi_4$ is the only non-null NP quantity; thus, it is direct to verify that the metric perturbations have the usual minimal decomposition

$$\delta \tilde{g}_{\mu\nu} = a^2(\eta) \left( \phi S_i + \partial_i B \quad E \delta_{ij} + \partial_i \partial_j D + \partial_i Q_j + \partial_j Q_i + \chi_{ij} \right),$$  \hfill (106)

and GWs are described only by the quantity $\chi_{ij}$ whose evolution equation can be derived from $\delta G_{\mu\nu}^{(ii)} = -8\pi G T_{\mu\nu}$ to obtain the very known result

$$\chi_j^{''} + 2H \chi_j^{'} - \nabla^2 \chi_j = 0.$$  \hfill (107)

In the following sections we will exemplify the above decomposition scheme for the two other theories which we have treated in section 2, namely a general scalar–tensor theory and Visser’s bimetric theory with massive gravitons. For each theory, after identifying the form of the metric perturbations, we will find the dynamical equations for GWs in the coordinate system defined by (67).

5. GWs in scalar–tensor theories

In this section we will consider perturbations of the general class of scalar–tensor theories introduced in section 2. With a glance at the field equations (39), we identify it with the general form (63) where $G(\phi) = \phi^{-1}$ and the generic function $F_{\mu\nu}$ takes the form

$$F_{\mu\nu} = \frac{\omega(\phi)}{\phi^2} \left( \phi \psi_{,\mu} \psi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi_{,\alpha} \right) + \frac{1}{\phi} (\psi_{,\mu\nu} - g_{\mu\nu} \Box \phi) + g_{\mu\nu} U(\phi).$$  \hfill (108)

Thus, disturbing the Newtonian ‘constant’

$$\delta G = \frac{dG}{d\phi} \delta \phi = -\frac{\delta \phi}{\phi^2},$$  \hfill (109)

and evaluating the components of the perturbation $\delta F_{\mu\nu}$, we can find the perturbed field equations (66) which for this case reads

$$\delta G_{\mu\nu} = -\frac{8\pi}{\psi} \left( \delta T_{\mu\nu} - \frac{\delta \psi}{\psi} T_{\mu\nu} \right) + \delta F_{\mu\nu}.$$  \hfill (110)

As we have already shown in section 2, the evaluation of the NP parameters for the most general scalar–tensor theory led us to conclude that an observer today would measure

$$\Psi_2 \neq 0, \quad \Psi_3 = 0, \quad \Psi_4 \neq 0 \quad \text{and} \quad \Phi_{22} \neq 0,$$  \hfill (111)
remembering that $\Psi_2 = 0$ if the potential associated with the scalar field is null $U(\varphi) = 0$. Thus, following the procedure of the last section we can write the perturbations for the general case which reads
\begin{equation}
\delta g_{\mu\nu} = a^2(\eta) \left( \frac{\phi}{S_i + \partial_i B} \pi_{ij} + \chi_{ij} + \psi_{ij} + \partial_i Q_j + \partial_j Q_i \right).
\end{equation}

Now, we can introduce the perturbations given above in the perturbed Einstein tensor and in the perturbed energy–momentum tensor calculated without decomposition in section 3, and we can also evaluate the perturbed quantity $\delta F_{\mu\nu}$. Thus, after a straightforward calculation, we find the dynamical equations which describe scalar and tensorial GWs in the context of the scalar–tensor theories:

**Scalar**

\begin{equation}
\phi'' + 2\mathcal{H}\phi' - (9\mathcal{H}^2 - 3\mathcal{H}')\phi - \nabla^2\phi - \left[ \frac{\omega}{2} \left( \frac{\varphi''}{\varphi} \right)^2 + \frac{\varphi'''}{\varphi} \right] (\phi + \xi_i) = -\frac{1}{2} [ (\mathcal{H}' - \mathcal{H}^2) ] \xi_i^j - 2a\mathcal{H}\nabla^2B = 16\pi a^2\varphi^{-1} \left( -\delta\rho + \frac{\delta\varphi}{\varphi} \right) - 2\Delta_1,
\end{equation}

\begin{equation}
(\vartheta^i B)'' + 4\mathcal{H}(\vartheta^i B)' - \nabla^2(\vartheta^i B) + (\mathcal{H}' - \mathcal{H}^2)\vartheta^i B + \left[ \frac{\varphi''}{\varphi} + \omega \left( \frac{\varphi'''}{\varphi} \right)^2 \right] \vartheta^i B = -2\mathcal{H}a^{-1} (2\vartheta^i \phi + \partial_i \pi^i k) = 16\pi Ga^2 (\rho + P) (V^i_0 + \vartheta^i B) + 2\Delta_3.',
\end{equation}

\begin{equation}
\xi_i'''' + 2\mathcal{H}\xi_i'' - \nabla^2\xi_i' + (\mathcal{H}' - \mathcal{H}^2)\xi_i^j \delta^j_i - 4\mathcal{H}a (\vartheta^i \vartheta^j B) + \left( \mathcal{H}' - \mathcal{H}^2 \right) \delta^j_i \frac{\varphi}{\varphi} \frac{\delta^j_i}{\varphi} = 16\pi a^2 \left[ \delta^j_i \frac{\rho^j_i}{\varphi} - \frac{\delta\varphi}{\varphi} \frac{\rho^j_i}{\varphi} \right] + 2\Delta_{3i}'
\end{equation}

\begin{equation}
\xi_i'''' + 2\mathcal{H} \xi_i'' + (\mathcal{H}' - \mathcal{H}^2) \xi_i^j \frac{\varphi}{\varphi} \frac{\delta^j_i}{\varphi} = 16\pi a^2 \left[ \delta^j_i \frac{\rho^j_i}{\varphi} - \frac{\delta\varphi}{\varphi} \frac{\rho^j_i}{\varphi} \right] + 2\Delta_{3i}'.
\end{equation}

**Tensor**

\begin{equation}
\chi_i'''' + (2\mathcal{H} + \frac{\varphi'}{\varphi}) \chi_i'' + 2\mathcal{H} \frac{\varphi'}{\varphi} \chi_i' - \nabla^2\chi_i' = 0.
\end{equation}

In the above equations we defined $\xi_i^j = \pi^j_i + \psi^j_i$, and $V^i_0$ is the component of $V^i$ which is parallel to the direction of propagation. The perturbed quantities $\Delta_1$, $\Delta_2$, and $\Delta_3$ take into account the perturbation of the scalar field $\varphi$ and its derivatives. They are defined as follows:

\begin{equation}
\Delta_1 = \omega \frac{\varphi'}{\varphi} \left( \frac{\delta\varphi}{\varphi} \right) \frac{\delta^j_i}{\varphi} + \frac{1}{2} \left( \frac{\varphi'}{\varphi} \right)^2 \delta\omega + 3\mathcal{H} \frac{\varphi'}{\varphi} \frac{\delta\varphi}{\varphi} + a^2 \left[ \frac{\delta(\varphi,0\varphi)}{\varphi} + \frac{\delta(\square \varphi)}{\varphi} \right],
\end{equation}

\begin{equation}
\Delta_2_i = \omega \frac{\varphi'}{\varphi} \frac{\vartheta_i}{\varphi} \left( \frac{\delta\varphi}{\varphi} \right) + \delta_i^j \frac{\delta(\varphi,0\varphi)}{\varphi},
\end{equation}

\begin{equation}
\Delta_3_i = \omega \frac{\varphi'}{\varphi} \frac{\delta\varphi}{\varphi} \frac{\delta^j_i}{\varphi} + \frac{1}{2} \left( \frac{\varphi'}{\varphi} \right)^2 \delta\omega \delta^j_i + \left( a^2 \frac{\varphi'}{\varphi} + \mathcal{H} \frac{\varphi'}{\varphi} \right) \frac{\delta\varphi}{\varphi} \delta^j_i + \frac{1}{\varphi} \left[ \delta(\varphi,0\varphi) \delta^j_i - a^2 \delta(\square \varphi) \delta^j_i \right].
\end{equation}

The equation for tensor perturbations in scalar–tensor theories (116) was studied in [10]. It represents the evolution of free GWs with helicity $s = \pm 2$ in such theories. These modes do
not couple with the perturbations of the perfect fluid and are generated quantum-mechanically due to vacuum perturbations in the very early Universe. They are amplified by the expansion of the Universe due to the process known as superadiabatic amplification [9]. In this case, the scalar field $\phi$ contributes for the cosmological potential which generates the amplification.

Regarding scalar perturbations, the driven equations for the NP modes $\Psi_2$ and $\Phi_{22}$ are given by equations (113), (114) and (115). These set of equations, considerably more complicated than the tensorial case, show some new physical features when compared with the usual metric decomposition. The most important difference is that these equations represent the evolution of radiative fields coupled to the evolution of the generalized Newtonian gravitational potential. The radiative fields are represented by the quantities $\pi_{ij}$ and $\psi_{ij}$ while the generalized Newtonian potential is described by the dynamical perturbations $\phi$ and $B$ or some particular combination of them. Thus, in some sense, we can say that if the scalar–tensor theory is the ‘correct’ theory we would have a new kind of cosmological GW background. In contrast to the tensorial case, this new background is coupled to the matter perturbations of the perfect fluid. This is expected since the usual scalar perturbations are also coupled to the fluid dynamics. But in the usual sense, GWs would be coupled to the matter only if a component of anisotropic stress would be present.

A direct consequence of such a coupling is that the whole evolution of the density perturbations of the cosmic fluid would depend not only on the evolution of $\phi$ and $B$ but also on the evolution of the amplitudes of the scalar GWs $\pi_{ij}$ and $\psi_{ij}$. Thus, from (113), for example, we are led to conclude that

$$\frac{\delta \rho}{\rho} = \frac{\delta \rho}{\rho}(\phi, B, \xi_i^i, \delta \phi),$$

where we have also included the dependence on the perturbations of the scalar field $\delta \phi$. Note that, in fact, the dependence of $\delta \rho/\rho$ with the GW amplitudes appears as the dependence on the trace of the overall contribution of the scalar GW amplitudes $\xi_i^i = \pi_i^i + \psi_i^i$. Therefore, a complete understanding of the evolution of the GWs with helicity $s = 0$ requires the knowledge of the evolution of the scalar perturbations and, similarly, these GW modes affect the evolution of the density perturbations.

Another issue of particular interest is how the presence of the scalar GWs would affect the angular pattern of the CMB. Since the GW amplitudes $\pi_{ij}$ and $\psi_{ij}$ now enter the geodesic equation for photons, it is expected that they leave a signature on the CMB due to the so-called Sachs–Wolfe effect, which can be understood as the shift of photon frequency along the line of sight. The small fluctuations in the CMB may be conveniently described by perturbations of the temperature parameter $T$ in the Planck distribution $f$. The computation of the contribution of the scalar GWs to the angular temperature inhomogeneities $\delta T/T$ of the CMB is out of the scope of this present paper and a rigorous treatment will appear elsewhere.

6. GW modes in a bimetric theory of gravity

Now let us turn our attention to the massive bimetric theory first considered by Visser [19] which we have introduced in section 2. Comparing the field equations (58) with the generic form (63) we identify $F^{\mu \nu} = -m^2 M^{\mu \nu}$.

Our explicit calculations in section 2, and the previous result by de Paula et al [31], have led us to the conclusion that all the NP quantities are non-null for Visser’s model:

$$\Psi_2 \neq 0, \quad \Psi_3 \neq 0, \quad \Psi_4 \neq 0 \quad \text{and} \quad \Phi_{22} \neq 0.$$
Thus, following the procedure of section 4, the perturbations now should be written in the form

$$\delta g_{\mu\nu} = a^2(\eta) \left( \frac{\phi}{S_i + \partial_i B} S_i + \partial_i B, \pi_{ij} + \tau_{ij} + \chi_{ij} + \psi_{ij} \right). \quad (122)$$

With (122) in the perturbed field equations (66), calculating the components of $\delta F_{\mu\nu}$ and with the help of the perturbed components of the Einstein tensor and of the energy–momentum tensor calculated in section 3, we obtain the perturbed field equations for Visser’s theory for each group of perturbations:

**Scalar**

$$\phi'' + 2\mathcal{H}\phi' - \nabla^2\phi - (9\mathcal{H}' - 3\mathcal{H}^2 - m^2a^2}\phi$$

$$\quad - [(a^2 - 1)\frac{\partial}{\partial \eta} a^2 B = -16\pi Ga^2\delta \rho, \quad (123)$$

$$\quad (\partial^i B)'' + 4\mathcal{H}(\partial^i B)' - \nabla^2(\partial^i B) + \frac{1}{2}[2(\mathcal{H}' - \mathcal{H}^2) + m^2a^4(3 - a^2)]\partial^i B$$

$$\quad - 2\mathcal{H}a^{-1}(2\partial^i \phi + \partial_i \pi^{ik}) = 16\pi Ga^2(\rho + P)(V'^{ij} + \partial^i B), \quad (124)$$

$$\xi'' + 2\mathcal{H}\xi' - \nabla^2\xi' - (\mathcal{H}' + \mathcal{H}^2)\xi \delta^i + \frac{1}{2}m^2a^4(a^2 + 1)\xi'' + 4\mathcal{H}a(\partial_i \partial^i B)$$

$$\quad - \frac{1}{2}[2(\mathcal{H}' - \mathcal{H}^2) - m^2a^2(a^2 - 1)]\phi \delta^i = 16\pi Ga^2\delta P \delta^i; \quad (125)$$

**Vector**

$$S'' + 4\mathcal{H}S' - \nabla^2S' + \frac{1}{2}[2(\mathcal{H}' - \mathcal{H}^2) - m^2a^4(a^2 - 3)]S'$$

$$\quad + 2\mathcal{H}a^{-1}\partial_i \tau^{ik} = 16\pi Ga^2(\rho + P)(V^{ij} + S'), \quad (126)$$

$$\tau'' + 2\mathcal{H}\tau' - \nabla^2\tau' + \frac{1}{2}m^2a^4(a^2 + 1)\tau' + 4\mathcal{H}a\eta^{ik} \partial_i \delta S_k = 0; \quad (127)$$

**Tensor**

$$\chi'' + 2\mathcal{H}\chi' - \nabla^2\chi' + \frac{1}{2}m^2a^4(a^2 + 1)\chi' = 0. \quad (128)$$

Similar to the tensorial equations for the GRT and for the scalar–tensor theory, this last equation describes the evolution of free GWs related to the NP mode $\Psi_3$. But now it is the new term which contains $m^2$ that contributes to the parametric amplification of GWs.

The equations for the scalar modes ($s = 0$) are now given by the set (123), (124) and (125). The argumentation for the scalar modes $\pi_{ij}$ and $\psi_{ij}$ is similar to the case of the scalar–tensor theories, except for the absence of the scalar field perturbation $\delta \phi$. Again the three equations must be solved simultaneously in order to find the evolution of the GWs with helicity $s = 0$ and to find the evolution of the density perturbation which is related to the metric perturbations through equation (123):

$$\frac{\delta \rho}{\rho} = \frac{\delta \rho}{\rho} (\phi, B, \xi'). \quad (129)$$

Conversely, we have now GWs with helicity $s = \pm 1$ which correspond to the mode $\Psi_3$. The equation which describes the evolution of the GW amplitudes for this mode is equation (127). Note the similarity of this equation to equation (128), except for the presence of the term containing $S'$ in equation (127). The presence of this term makes the vector GW modes coupled with the vector perturbations since $S'$ is coupled with the fluid vector perturbations through equation (126). Furthermore, the perpendicular part of the vector perturbation (which is a pure vector) is a function of the quantities $S'$ and $\tau^{ij}$:

$$V'^{ij} = V'^{ij}(S', \tau^{ij}). \quad (130)$$
Regarding the CMB anisotropy, the presence of the longitudinal vector modes of GWs does not yield the well-known version of the Sachs–Wolfe effect which appears in the GRT or in the scalar–tensor theory [34]. Theories which present vector GWs ($\Psi_1 \neq 0$) give rise to a nontrivial Sachs–Wolfe effect which leaves a vector signature of the quadrupolar form $Y_{2,\pm 1}$ on the CMB polarization (see the detailed discussion in [16]).

7. Final remarks

In the present work we have studied the evolution equations of cosmological GWs in alternative theories of gravity. Since the most part of the alternative theories present more than the two usual + and $\times$ polarizations of GWs, we have addressed the problem of how one could take into account the new polarization states in the cosmological metric perturbations.

First of all, we have presented an overview of the NP formalism since it is particularly suitable for evaluating the number of non-null GW modes of any theory. Then, we have proposed that the construction of the metric perturbations for a given theory should depend on the number of non-null NP parameters of the theory.

The formalism developed here is quite general and can be applied for a wide range of alternative theories of gravity. In order to show that, we have evaluated the evolution equations for GWs for two different theories: a class of scalar–tensor theories, for which Brans–Dicke theory is a particular case, and a massive bimetric theory. In the first case, the theory presents two scalar modes ($s = 0$) and two tensor modes ($s = \pm 2$) of GWs. In the case of the bimetric theory, GWs have in addition the two vector modes ($s = \pm 1$), totaling six polarization states of GWs, i.e. the most general case in the context of a four-dimensional theory of gravity.

A qualitative analysis of the governing equations of the scalar perturbations has shown that the evolution of the density perturbations of the cosmological fluid depends not only on the generalized Newtonian potential but also on the amplitudes of the scalar GWs. Another direct consequence of the presence of $\pi_{ij}$ and $\psi_{ij}$ is the possible signatures that this quantities would leave in the angular pattern of the CMB. Such a signature might impose strong limits in the amplitudes of the scalar GWs by the analysis of the CMB data. Moreover, a remarkable effect on the CMB which have already been studied in the literature [16] is a non-usual Sachs–Wolfe effect which appears due the presence of the longitudinal vector GW modes on the geodesic equation for photons. Such vector GW modes are present, for example, in the massive bimetric theory analyzed here.

The detection of GWs is a particularly challenging issue and it may be the final answer to the ‘correct’ theory of gravity. If GWs present non-tensorial polarization modes as discussed in this paper, we will have a stochastic cosmological background of GWs which is a mixture of all the polarization modes. If, in analyzing such a background, scalar and/or vector GWs could be found, the result would be disastrous for the Einstein theory.

The evaluation of the response function of the non-tensor polarization modes for interferometric GW detectors was carried out in [35] and [36]. Particularly, Nishizawa et al [35] have found that more than three detectors can separate the mixture of polarization modes in the detector outputs. But they have considered only separation between the three groups: scalar, vector and tensor modes of GWs. Furthermore, they have found that, statistically, the GW detectors have almost the same sensitivity to each polarization mode of the stochastic background of GWs. In the work by Corda [37], the detectability of a particular polarization was discussed, namely the longitudinal scalar component. It was also shown that the angular dependence of such a mode could, in principle, allow discrimination of this polarization with respect to that of GRT.
A positive detection for certain modes and a negative detection for others may exclude a particular theory or, at least, establish strong constraints to the alternative theories of gravity. But it is important to emphasize that the confirmation by the observation of the number of non-null GW modes is not enough to determine the ‘correct’ theory, since a number of theories can have the same number of non-null modes. It is also necessary to evaluate the spectrum of GWs for each mode and for each theory. The evaluation of the spectrum can only be done following the cosmological evolution of the GWs. In this sense, the equations given in this paper are the first step for such a computation. Finally, the analysis of the response of a given mode and the evaluation of the spectrum can bring crucial answers for the comprehension of the gravity in cosmological scales.

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