Discrete quantum gravity: continuum limit and the problem of state doubling

S.N. Vergeles

Landau Institute for Theoretical Physics, Russian Academy of Sciences, Chernogolovka, Moskov region, 142432 Russia

It is shown that in the theory of discrete quantum gravity defined on the irregular "breathing" lattice, if the macroscopic continuum phase is realized, the phenomenon of state doubling (even if it exists formally at kinematic level) actually is absent at experimentally accessible energies.

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1. In the recent my works one more version of the discrete quantum gravity has been introduced into consideration [1, 2]. This version of discrete quantum gravity manifests "naive" macroscopic continuum limit. But it does not mean that in the theory the macroscopic continuum phase indeed is realized dynamically. Moreover, some phenomena and problems are well known which prevent microscopic discrete quantum gravity from developing into macroscopic continuum phase with the properties close to the real one. For example, the phenomenon of state doubling (Wilson doubling) and the cosmological constant problem.

In the present and the subsequent letters I study the state doubling phenomenon and the cosmological constant problem under the assumption that the macroscopic continuum phase is realized dynamically in the theory of discrete gravity of the type [3]. I think that this is a very strong assumption: for realization of this assumption the microscopic theory must be determined very specially and, perhaps, uniquely (the correct set of fundamental fields and their interaction, and so on). Though all that is unknown, I apply a sort of phenomenological approach and study the mentioned problems in the macroscopic phase. It seems to me that the knowledge of some qualitative properties of the considered theories in the macroscopic phase can promote general progress of this branch of quantum field theory.

In the present letter I study the possible consequences due to the phenomenon of state doubling. The problem of graviton states doubling in discrete gravity was considered earlier in [4, 6]. The fermion state doubling (proper Wilson doubling, see, for example [7]) was investigated in [1]-[4]. For simplicity I study here the Dirac field on the lattice. But it is clear from the consideration given below that the conclusion remains valid with respect to all other fields.

At first sight it seems that the theories in which the state doubling phenomenon is present (I call them as anomalous theories) must be very different from the usual theories (without state doubling) in the continuum phase. Indeed, in anomalous theories the transitions from the states with normal modes into the states with anomalous modes (and vice versa) exist. But the such transitions are not registered at experimentally accessible energies.

The main result of the paper is the behaviour in the $x$-space of the propagator describing the propagation of anomalous modes

$$\langle \psi A(x) \overline{\psi} A(y) \rangle_{ee} \sim \exp \left[ -\frac{|x - y|}{a \Lambda} \right].$$

Here and below $a$ is a certain constant that has the length dimension and is of the order of the effective lattice step and $\Lambda$ is the number of the order of unit. But the propagators describing the propagation of normal modes have the conformal behaviour at the enough large energies. Thus, the propagators of anomalous modes decrease too fast as compared with the propagators of normal modes. Therefore the transitions between the states of normal and anomalous modes are strongly depressed at the scales much greater than $a$. The situation is similar to the following one: the electron loop corrections to the external smooth electromagnetic field die out at the scales much greater than $1/m$ ($m$ is the electron mass).

2. It is necessary to write out some formulæ from the work [3]. Here the notations are completely identical to that in [3].

Let $\mathcal{R}$ be a 4-dimensional simplicial complex such that the 3-dimensional complex $\mathcal{S} = \partial \mathcal{R}$ has the topology of 3-sphere $S^3$. To each vertex $a_i \in \mathcal{R}$, the Dirac spinors $\psi_i^a$ and $\bar{\psi}_i^a$ belonging to the complex Grassman algebra are assigned. To each oriented edge $a_i a_j \in \mathcal{R}$, an element of the group $Spin(4)$

$$\Omega_{ij} = \Omega^{-1}_{ji} = \exp \left( \frac{1}{2} \hat{\omega}_{ij}^a \sigma^{ab} \right), \quad \sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b],$$

and also an element $\hat{e}_{ij} \equiv e_{ij}^a \gamma^a$, such that

$$\hat{e}_{ij} = -\Omega_{ij} \hat{e}_{ji} \Omega^{-1}_{ij}, \quad -\infty < e_{ij}^a < \infty,$$

are assigned. The notations $\hat{\psi}_{Ai}$, $\psi_{Ai}$, $\hat{e}_{Ai}$, $\Omega_{Ai}$ and so on indicate that the edge $X_{Ai}^A = a_i a_j$ belongs to 4-simplex with index $A$. Let $a_{Ai}$, $a_{Aj}$, $a_{Ak}$, $a_{Al}$, and $a_{Am}$ be all five vertices of a 4-simplex with index $A$ and $\varepsilon_{Ai j k l m}$ be $\pm 1$ depending on whether the order of vertices...
\[ a_{Ai} a_{Aj} a_{Ak} a_{Al} a_{Am} \] defines the positive or negative orientation of this 4-simplex. The Euclidean action of the theory has the form

\[
I = \frac{1}{5 \times 24} \sum A \sum_{i,j,k,l,m} \varepsilon_{Aijklm} \operatorname{tr} \gamma^5 \times \left( -\frac{1}{2} \bar{\psi}_A \gamma^a \Omega_{Aij} \bar{\psi}_{Aj} \gamma^a \bar{\psi}_{Ai} \right) + \frac{\lambda}{4!} \varepsilon_{ijklm} \varepsilon_{Aijklm} \Omega_{Aij} \Omega_{Ajk} \Omega_{Akl} \Omega_{Alm} + \frac{i}{12} \varepsilon^a_a \left( \bar{\psi}_A \gamma^a \Omega_{Aij} \bar{\psi}_{Aj} \gamma^a \bar{\psi}_{Ai} \right) \varepsilon_{Amj} \Omega_{Akl} \Omega_{Alm}. \tag{4}
\]

The oriented volume of a 4-simplex with vertexes \( a_{Ai}, a_{Aj}, a_{Ak}, a_{Al}, a_{Am} \) is equal to

\[
V_A = \frac{1}{4!} \sum_{i,j,k,l,m} \varepsilon_{Aijklm} \varepsilon_{abcd} \varepsilon_{Aimb} \varepsilon_{Amj} \epsilon_{Aml} \epsilon_{Aml}. \tag{5}
\]

The quantity \( R_{ij}^2 = \sum_a (\varepsilon^a_{ij})^2 \) is the square of the length of the edge \( a_i a_j \).

The oriented volume of a 4-simplex with vertexes \( a_{Ai}, a_{Aj}, a_{Ak}, a_{Al}, a_{Am} \) is defined on the manifold \( \mathcal{V} \).

Then we can define the discrete action \( I \) transforms to the well known continuum action

\[
I = \int \varepsilon_{abcd} \left( -\frac{1}{2} R_{ab}^{\mu} \right) + \frac{i}{12} \left( \bar{\psi} \gamma^a D_{\mu} \psi - D_{\mu} \bar{\psi} \gamma^a \psi \right) d \varepsilon^a_{\mu} \times d \varepsilon^b_{\mu} \times d \varepsilon^c_{\mu} \times d \varepsilon^d_{\mu}, \tag{10}
\]

\[
d \omega^{ab} + \frac{\Omega_{Aim}}{2} \omega^{bc} = \frac{1}{4} R^{ab}. \tag{11}
\]

I emphasize that we obtain the action \( I \) only if the lowest derivatives of the fields are taken into account. It is important that in this case the information on the structure of the complex is lost. This is incorrectly if the highest derivatives of the fields are also taken into account.

Let’s denote by \( \mathcal{X} \) a 4-dimensional smooth manifold with topology of the complex \( \mathcal{G} \). Consider a set of maps \( \{ g \} \) from geometrical realization of the complex \( \mathcal{G} \) onto manifold \( \mathcal{X} \) which are not necessarily one-one maps. For a given local coordinates \( x^\mu, \mu = 1, 2, 3, 4 \) a map \( g \) defines the coordinates of images of vertexes \( a_{Ai} \). Define the four differentials

\[
d x^\mu_{g(A)} = x^\mu_{g(A)} - x^\mu_{g(A)}, \quad \psi(x) = \psi(x), \quad \psi(x) \]

Suppose the smooth fields \( \omega_{ab}^\mu(x), e_{ab}^\mu(x), \psi(x), \psi(x) \) are defined on the manifold \( \mathcal{X} \). Then we can define the discrete lattice variables according to the relations

\[
\omega_{ab}^\mu(x_{g(A)} m) d x^\mu_{g(A)} = \omega_{ab}^\mu, \quad e_{ab}^\mu(x_{g(A)} m) d x^\mu_{g(A)} = e_{ab}^\mu, \quad \psi(x_{g(A)} i) = \psi_{A i}. \tag{9}
\]

On the contrary, the discrete lattice variables in the right hand sides of Eqs. \( \psi \) define the values of the fields on the images of vertexes of the complex. It is clear \( \psi \) that the lattice variables which change enough smoothly along the complex we obtain the enough smooth fields. Moreover, in this case the discrete action \( I \) transforms to the well known continuum action

\[
I = \int \varepsilon_{abcd} \left( -\frac{1}{2} R_{ab}^{\mu} \right) + \frac{i}{12} \left( \bar{\psi} \gamma^a D_{\mu} \psi - D_{\mu} \bar{\psi} \gamma^a \psi \right) d \varepsilon^a_{\mu} \times d \varepsilon^b_{\mu} \times d \varepsilon^c_{\mu} \times d \varepsilon^d_{\mu}, \tag{10}
\]

\[
d \omega^{ab} + \omega^{ab} = \frac{1}{4} R^{ab}. \tag{11}
\]

Here, the sum in the parentheses is taken over any closed path formed by 1-simplices. Equations \( \psi \) indicates that the curvature and torsion are equal to zero. Thus, geometrical realization of the complex \( \mathcal{G} \) is in the four-dimensional Euclidean space, \( e_{ij}^a \) being the components of the vector in a certain orthogonal basis in this space, and if \( R^i_4 \) is the radius-vector of vertex \( a_i \), then \( e_{ij}^a = R^i_4 - R^j_4 \).

In this case one can take \( e_{ij}^a(x) = \delta_{ij}^a \). It is evident that Eqs. \( \psi \) are the only restrictions for the variables \( e_{ij}^a \).

Let us write down the action for the eigenmodes of the discrete Dirac operator in partial case \( \psi \). This operator is obtained by varying action \( I \) with respect to the variable \( \psi \). In the four-dimensional case, we have
anomalous modes, for certain neighboring pairs of vertices whose number is comparable with the total vertices number of the complex satisfy the conditions

$$|\psi_i^{s} - \psi_j^{s}| \sim |\psi_i^{s}| \sim N^{-1/2}. \quad (17)$$

Relations mean that anomalous modes generally change in spurts from a vertex to a neighboring vertex. Hence, the derivatives \( \partial_s \psi_i^A \) of anomalous modes also generally change in spurts from a vertex to a neighboring vertex. Otherwise, Eqs. (12) and (13) would provide the equation

$$i \gamma^a \partial_s \psi = \epsilon \psi,$$

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where \( \alpha^a(x) \) are random functions in the sense indicated below. Indeed, let \( \psi_j^{s(0)} \), \( s = 1, 2, \ldots \) be a complete set of zeroth anomalous modes [11]. Any linear combination of zeroth modes is evidently also a zeroth mode. For this reason, we seek a soft anomalous mode "growing" from zeroth anomalous modes in the form

$$\psi_j^A = \sum_s g_j^A \psi_j^{s(0)},$$

where the numerical field \( g_j^A \) is slowly varying. To derive an equation for the field \( g_j^A \), we substitute the expression

$$\psi_j = \psi_i + \epsilon_j \partial_s \psi_i,$$

which is almost true in the long-wavelength limit, i.e., for conventional modes. Here, \( x^a \) are the Cartesian coordinates in the same orthonormal basis in which the components of the vectors \( \epsilon_j^a \) are specified. It is again seen that information on the positions of vertices of the complex is completely lost in the long-wavelength limit; i.e., conventional long-wavelength modes lose information on the lattice structure. The eigenvalues of the modes of Dirac operator are \( \epsilon = \pm |k| \), where \( k^a \) is the wavevector of a mode. Thus, the eigenvalues of the usual modes can be arbitrarily small.

The main result of works [2, 3] is that the theory under consideration involves two types of lattices. On the lattices of one type, fermion doubling occurs, whereas it is absent on the lattices of the other type.

According to the definition, the lattice allows fermion state doubling if the discrete Dirac operator given by Eq. (12) allows two types of modes with eigenvalues approaching zero [10]. The qualitative difference between them is as follows. The normalized modes of the first type, or normal modes, satisfy the conditions

$$|\psi_i - \psi_j| \sim |\epsilon| \psi_j \sim |\epsilon| N^{-1/2},$$

where \( a_i \) and \( a_j \) are the neighboring vertices, \( \epsilon \) is the mode eigenvalue, and \( N \) is the number of the vertices of the complex. The normalized modes of the second type, or
Thus, the operator on the right-hand side of Eq. (18) is a first-order differential operator with variable matrix coefficients that acts on a multicomponent (or vector) function $g_s(x)$, $s = 1, 2, \ldots$.

Quantities (20) are irregular functions that depend strongly and locally on the positions of the vertices of the complex. This fact is demonstrated by the evident formulas in the two-dimensional theory (see [4]). I give here only qualitative description of the quantities (20) and their properties.

As it is known, the zeroth anomalous modes look like

$$\psi_{s j}^{A(0)} = \exp \left( \frac{2\pi i n_s(j)}{n_s} \right),$$  \hspace{1cm} (21)

where $n_s$ and $n_s(j)$ are whole numbers, $n_s > 1$, $n_s(j) = 0, \pm 1, \ldots, \pm (n_s - 1)$, and $n_s(j_1) \neq n_s(j_2)$ for the neighboring vertices $a_{j_1}$ and $a_{j_2}$. From here it follows that

$$\psi_{s j_1}^{A(0)} \neq \psi_{s j_2}^{A(0)}$$  \hspace{1cm} (22)

for the neighboring vertices $a_{j_1}$ and $a_{j_2}$. In consequence of Eqs. (21) and (22) the sums in the second square brackets in (20) become very irregular (12). Even the signs of these sums and the oriented volume $v_i$ in (11) are not correlated.

Thus, the components $\alpha^a(x)$ in Eq. (18) are discontinuous functions of $x$ and they depend strongly and locally on the integration variables $\{e_x\}$ in the integral (6). It is important that the variables $\{e_x\}$ in integral (6) in the quasi-classical phase can independently vary over a wide range without a change in the action (see the text between Eqs. (10) and (11)).

Further I assume that

$$\alpha^a(x) = \beta^a(x) + \lambda \rho^a,$$  \hspace{1cm} (23)

where $\rho^a_{s s'}$ is the constant matrix, $\lambda$ is the small numerical parameter, and all odd powers of $\beta(x)$ in the integrand in Eq. (6) are equal to zero:

$$\left\langle \beta(x_1) \cdots \beta(x_{2n+1}) \right\rangle_{e_x} = 0, \hspace{1cm} n = 0, 1, \ldots, (24)$$

Here the matrix product of matrices $\beta$ is averaged. The points $x_1, \ldots, x_{2n+1}$ can partially or completely coincide with each other. Averaging in Eq. (24) is performed by means of functional integral (9). The subscript $\{e_x\}$ indicates that integration is performed only with respect to the variables $\{e_x\}$. We also assume that the parameter $\lambda$ is small so that the expansion in this parameter is meaningful. The last assumption is in agreement with said above.

When the problem of the $S$ matrix (which is meaningful only in the continuum limit) is solved in the theory under consideration, all $S$ matrix elements must be averaged over the variables $\{e_x\}$ before the calculation of probabilities and cross sections. Therefore, if the vertices describing the interaction are universal (i.e., independent of the microstructure of the lattice), the propagators of the matter fields in such a theory must be averaged over the variables $\{e_x\}$. Indeed, in this case, the diagrammatic technique can be obtained in this case as a result of the expansion in a functional differential operator acting on the transition amplitude for the matter fields in the quadratic ("free") approximation against the background of the external field sources. In this case, the square transition amplitude must be averaged with a weight over the variables $\{e_x\}$ before the expansion in interaction.

So, it is necessary to obtain the propagator

$$\langle \psi_s^{A(0)} | \psi_s^{A(0)} \rangle_{e_x} = \psi_s^{A(0)} \langle ( - i \alpha^a \partial_a )^{-1} \rangle_{e_x} \psi_{s'}^{A(0)},$$  \hspace{1cm} (25)

which describes the propagation of anomalous modes. Thus, the problem is reduced to the study of the Green’s function that corresponds to the operator on the left-hand side of Eq. (18) and is averaged over the variables $\{e_x\}$ by taking into account conditions (24): (14):

$$\langle ( - i \alpha^a \partial_a )^{-1} \rangle_{e_x}.$$  \hspace{1cm} (26)

4. We first consider the case $\lambda = 0$. To perform the necessary averaging in Eq. (25), we use the well-known formula

$$\mathcal{P}_{\tau} \frac{1}{\tau} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ds \left[ e^{i s \tau} - e^{-i s \tau} \right].$$  \hspace{1cm} (27)

This formula implies that $\tau \neq 0$ or $\tau^{-1} \neq \pm \infty$. Using Eq. (27), we represent operator (26) (for the case $\lambda = 0$) before averaging over the variables $\{e_x\}$ in the form

$$\langle x | ( - i \beta^a \partial_a )^{-1} | y \rangle =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} ds \langle x | \left[ e^{i s \beta^a \partial_a} - e^{-i s \beta^a \partial_a} \right] | y \rangle.$$  \hspace{1cm} (28)

This relation is meaningful for $x \neq y$, because its left-hand side is finite in this case. However, representation (28) is meaningless for $x = y$. Then, let us average the terms in the square brackets in Eq. (28) over the variables $\{e_x\}$. In this case, it is only important that, owing to Eq. (24), the result of such averaging of each of these terms is a function of $(\pm s)^2 = s^2$ but not of $(\pm s)$. Therefore, the terms in the square brackets in Eq. (28) cancel each other after the averaging over the variables $\{e_x\}$ and thereby
the propagator describing the propagation of anomalous modes is proportional to the $\delta$ function of the variable $x$:

$$\langle \langle x | ( - i \beta^2 \partial_0 )^{-1} | y \rangle \rangle \sim a \delta^{(4)} ( x - y ).$$

(29)

I remind the reader that $a$ is a certain constant that has the length dimension and is of the order of the effective lattice scale.

Now let’s estimate quantity (26) for nonzero $\lambda$ by taking into account that it can be expanded in the parameter $\lambda$. This problem is easily solved by using field theory methods. Figure 1 shows the sum of the diagrams that corresponds to quantity (26). The solid line and cross correspond to unperturbed propagator (24) and the operator $i \rho \partial_0^2$, respectively. Then using the conventional diagrammatic technique rules, one can represent quantity (26) in the form of the series of diagrams shown in Fig. 1.

As a result, the $\delta$ function of the free propagator is “smeared” and the desired propagator describing the propagation of anomalous modes appears to be exponentially decreasing in the $x$ space according to the Eq. 16.

At the same time, according to Eq. 15, the free propagators describing the propagation of normal modes behave according to the power law in $x$ and momentum space.

5. We briefly summarize the conclusions as follows.

(i) On complexes that formally allow state doubling (of any fields), this phenomenon is really absent because normal and anomalous modes propagate differently and separately in spacetime. Normal modes (in the long-wavelength limit) have definite energy and momentum. On the contrary, anomalous modes cannot have definite energy and momentum and their propagators decrease exponentially in spacetime at scales comparable with the characteristic lattice scale.

(ii) Let’s give some general considerations concerning fermion (Wilson) doubling due to the above result.

Let us assume that one Weyl field is introduced on the lattice by introducing the projection operator $(1/2) (1 \pm \gamma^5)$ into the fermion action in Eq. 14. Since the lattice fermion action is invariant under global $\gamma^5$ transformations and the lattice fermion measure is invariant under local $\gamma^5$ transformations, the total fermion current is strictly conserved. However, this does not mean that each of the currents of normal and anomalous modes is conserved separately. On the contrary, it has been well known for a long time that, when $S$ matrix elements are calculated by using causal Feynman propagators, the Weyl-field current is not conserved due to the gauge anomaly. This phenomenon is interpreted in the theory under consideration as the mutual transfer of the axial charge between normal and anomalous modes. Here it is shown that this phenomenon can occur only on the lattice scales.

I also note that the theory under consideration does not exclude the case where the axial charges of normal and anomalous modes are conserved separately (see [3]). This means that the axial anomaly is absent. Such a regime can be realized only in problems where the use of Feynman propagators is incorrect, for example, at the universe inflation stage when the problem of the $S$ matrix is meaningless.

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[8] I mean here that the partition function is a functional of the values of the dynamic variables at the boundary $\partial \mathcal{M}$.
[9] The vertices $a_i$ and $a_j$ are called neighboring if the 1-simplex $a_i a_j$ exists.
[10] It is certainly assumed that the sizes of the lattice (i.e. the number of its simplexes) tend to infinity.
[11] Zeroth modes satisfy Eq. 12 with zero right-hand side.
[12] Note that for the normal zeroth modes due to the firs of Eqs. 12 the quantity 20 is equal to $\gamma^6$.
[13] Owing to the constraints 11, the variables $\{ \Omega \varepsilon \}$ are insignificant here.
[14] It is necessary to point to a fundamental difference of this computational procedure from that used in problem of the localization of particles in random potentials. The physical difference between these situations is that the parameters characterizing the randomness of a potential in the latter case are not dynamical degrees of freedom, whereas the positions of the vertices of the complex are dynamical variables.