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ON TWISTOR ALMOST COMPLEX STRUCTURES

MICHEL CAHEN, SIMONE GUTT, AND JOHN RAWNSLEY

Dedicated to our friend Kirill Mackenzie

Abstract. In this paper we look at the question of integrability, or not, of the two natural almost complex structures $J_\pm \nabla$ defined on the twistor space $J(M,g)$ of an even-dimensional manifold $M$ with additional structures $g$ and $\nabla$ a $g$-connection. We also look at the question of the compatibility of $J_\pm \nabla$ with a natural closed 2-form $\omega^{J(M,g,\nabla)}$ defined on $J(M,g)$. For $(M,g)$ we consider either a pseudo-Riemannian manifold, orientable or not, with the Levi Civita connection or a symplectic manifold with a given symplectic connection $\nabla$. In all cases $J(M,g)$ is a bundle of complex structures on the tangent spaces of $M$ compatible with $g$ and we denote by $\pi: J(M,g) \to M$ the bundle projection. In the case $M$ is oriented we require the orientation of the complex structures to be the given one. In the symplectic case the complex structures are positive.

The linear connection $\nabla$ on $M$ defines a horizontal space $H_\nabla j \cong T_{\pi(j)}M$ at any point $j$ in the twistor space so that $T_j J(M,g)$ is isomorphic to $H_\nabla j \oplus V_j$ where $V_j = \text{Ker} \pi_\ast j$ is the vertical space at $j$. Since both $V_j$ and $TM_{\pi(j)}$ carry complex structures defined by $j$, they add together to give the complex structure denoted by $(J_\pm \nabla)_j$ on $T_j J(M,g)$.

The almost complex structure denoted $(J_- \nabla)_j$ is defined by reversing the sign on the horizontal space.

We examine the integrability, or not, of the $J_\pm$ by looking at their Nijenhuis tensors $N^{J_\pm}$ and measure their non-integrability by the dimension of the span of the values of $N^{J_\pm}$.

The natural closed 2-form $\omega^{J(M,g,\nabla)}$ is defined on the twistor space as the trace of the curvature of a connection $D^E$ defined on the pull-back bundle bundle $E = \pi^{-1}TM$. This bundle $E$ is endowed with the complex vector bundle structure defined by the natural section $\Phi$ of $\text{End}(E)$ whose value at $j$ is $j$, and the connection $D^E$, built from the pullback connection $\pi^{-1}\nabla_E$, satisfies $\pi^{\text{End}}_E \Phi = 0$. We recall, as in Reznikov [10], when this 2-form is symplectic in the pseudo-Riemannian setting and we determine, in the pseudo-Riemannian and in the symplectic setting, when $\omega^{J(M,g,\nabla)}$ is of type $(1,1)$ with respect to $J_\pm$.

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Introduction

A twistor space over a manifold $M$ is a fibre bundle $\pi: Z \to M$ where each fibre is a complex manifold and each point $z$ in $Z$ defines a complex structure $J(z)$ on the tangent space $TM_{\pi(z)}$ ($M$ must be even-dimensional for this to be possible). An example is the bundle $J(M)$ of all complex structures $j$ on the tangent spaces of $M$. The case of interest here is the bundle $\pi: J(M,g) \to M$ of complex structures on the tangent spaces compatible with some geometric structure $g$ such as a pseudo-Riemannian metric (with an orientation or not) or a symplectic structure. Where we can we will treat those results common to the three cases together. The presentation we give of twistor spaces follows the Riemannian case in O’Brian–Rawnsley [8].

A linear connection $\nabla$ on $M$ preserving $g$ defines a horizontal space $H^\nabla_j$ at $j$ so that $T_j J(M,g)$ is isomorphic to $T_{\pi(j)} M \oplus V_j$ where $V_j = \ker \pi_*: T_j J(M,g) \to T_{\pi(j)} M$ is the vertical space at $j$. Since both $V_j$ and $TM_{\pi(j)}$ carry complex structures defined by $j$ (which we recall in Section 2), they add together to give a complex structure $(J^\nabla)_j$ on $T_j J(M,g)$. This almost complex structure $J^\nabla$ on $J(M,g)$ can sometimes be integrable producing a complex manifold which has been used in the pseudo-Riemannian setting to convert the Yang–Mills equations on $M$ into the Cauchy–Riemann equations on $J(M,g)$ in the 4-dimensional case, see [1]. Some twistor spaces over Riemannian manifolds have been a source of examples of non-Kählerian symplectic manifolds [6, 7, 10].

A second almost complex structure $J^{-\nabla}$ can be defined by reversing the sign on the horizontal bundle. This has had many uses in the study of harmonic maps of Riemann surfaces into $M$ when $M$ has a Riemannian structure $g$ and $\nabla$ is the Levi Civita connection of $g$ [5].

In this paper we look at the question of integrability, or not, of $J^\nabla$ and, when not integrable, examine their Nijenhuis tensors $N^{J^\nabla}$ to see how non-integrable they are, using as a measure of their non-integrability the dimension of the span of the values of $N^{J^\nabla}$, as in [4].
The bundle $\text{End}(E)$, where $E$ is the pull-back bundle $E = \pi^{-1}TM$ (which is isomorphic to the horizontal bundle $H^\nabla$ via $\pi_*$), has a section $\Phi$ whose value at $j$ is $j$. This makes $(E, \Phi)$ into a complex vector bundle with the multiplication by $\sqrt{-1}$ given by $\Phi$. This complex vector bundle has Chern classes $c_i(E, \Phi)$ in the de Rham cohomology of $J(M, g)$ represented by polynomials in the curvature of a connection on $E$ preserving $\Phi$. From the pullback connection $\pi^{-1}\nabla$, we get such a connection on $E$, called $D^E$, and construct a closed 2-form $\omega^{J(M, g, \nabla)}$ as the trace of the curvature of $D^E$. We write the conditions for this 2-form to be symplectic and we determine when $\omega^{J(M, g, \nabla)}$ is of type $(1, 1)$ with respect to $J^\pm$.

The results in the pseudo-Riemannian context include the following:

The almost complex structure $J^+\nabla$ is integrable in the pseudo-Riemannian context with no given orientation if and only if the Weyl component $C^\nabla$ of the Riemann curvature $R^\nabla$ vanishes (this is well known and proven in Proposition 5.2).

In the pseudo-Riemannian context with a given orientation, the results holds true (as is well known) in dimension $> 4$: $J^\nabla$ is integrable if and only if the Weyl component of the Riemann curvature vanishes, whether in dimension 4 it is integrable if and only if the Weyl component of the Riemann curvature tensor is self-dual when the signature is $(4, 0)$ or $(0, 4)$ and anti-self-dual when the signature is $(2, 2)$ (Proposition 5.3).

The almost complex structure $J^-\nabla$ is never integrable and the image of its Nijenhuis tensor always include the horizontal space: Image $N^J^-\nabla_j \supset H^-\nabla_j$.

If the space has non-vanishing constant sectional curvature, then the image of the Nijenhuis tensor associated to $J^-\nabla$ is the whole tangent space $T_jJ(M, g)$ at any point $j \in J(M, g)$.

More generally in the Riemannian case (Proposition 5.5), given any positive integer $n$, there exists an $\epsilon(n)$ such that, if the sectional curvature of a Riemannian manifold $(M, g)$ of dimension $2n$ is $\epsilon(n)$-pinched, the almost complex structure $J^-\nabla$ on the twistor space, defined using the Levi Civita connection $\nabla$, is maximally non-integrable (i.e. the image of the corresponding Nijenhuis tensor is the whole tangent space at every point).

Each of the complex structures $J^\pm\nabla$ is compatible with the closed 2-form $\omega^{J(M, g, \nabla)}$ if and only if the same condition as the integrability of $J^\nabla$ is satisfied, i.e. $\omega^{J(M, g, \nabla)}$ is of type $(1, 1)$ with respect to $J^\nabla$ (and automatically also to $J^-\nabla$) if and only if $C^\nabla = 0$ in the pseudo-Riemannian context with no orientation, or with an orientation if dim $M > 4$ and if and only if the Weyl component of the Riemann curvature tensor is self-dual when the signature is $(4, 0)$ or $(0, 4)$ and anti-self-dual when the signature is $(2, 2)$ (Proposition 5.8).

The results in the symplectic context include the following:

The almost complex structure $J^\nabla$ on the twistor space $J(M, \omega)$ of a symplectic manifold $(M, \omega)$ of dimension $2n \geq 4$, defined using a symplectic connection $\nabla$, is integrable if and only if the curvature of $\nabla$ is of Ricci-type (this was known and is proven in Proposition 6.5).

The almost complex structure $J^-\nabla$ is never integrable and the image of its Nijenhuis tensor at the point $j$ always include the horizontal space $H^-\nabla_j$. 
The closed 2-form $\omega^J(M,\omega,\nabla)$ is of type $(1, 1)$ for each of the $J^\pm_\nabla$ if and only if again the same condition as the integrability of $J^+_\nabla$ is satisfied, i.e. the curvature $R^\nabla$ is of Ricci type (Proposition 6.12).

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1. DESCRIPTION OF THE TWISTOR BUNDLE

Let $(M, g)$ be a $2n$-dimensional manifold endowed with a structure $g$ which can be either a (pseudo)-Riemannian structure of signature $(2p, 2q)$ where $n = p + q$, with an orientation or not, or a symplectic structure, or having no extra structure.

Let $F(M, g) \to M$ denote the corresponding frame bundle where a frame at a point $p \in M$ is a map $\xi : V \to T_pM$ which is a linear isomorphism from $V = \mathbb{R}^{2n}$, endowed with a standard structure $\tilde{g}_0$, to $(T_pM, g_p)$, where $\tilde{g}_0 = \begin{pmatrix} I_{p,q} & 0 \\ 0 & I_{p,q} \end{pmatrix}$ with $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ in the pseudo-Riemannian case, with an orientation or not, and $\tilde{g}_0 = \Omega_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ in the symplectic case.

The frame bundle is a principal bundle with structure group

$$G = \text{Gl}(V, \tilde{g}_0) = \left\{ \begin{array}{ll}
O(V, \tilde{g}_0) & \text{O}(2p, 2q; \mathbb{R}) \text{ in the pseudo-Riemannian setting} \\
SO(V, \tilde{g}_0) & \text{when there is furthermore an orientation} \\
Sp(V, \Omega_0) & \text{in the symplectic case} \\
\text{Gl}(V) = \text{Gl}(2n, \mathbb{R}) & \text{if there is no extra structure on } M.
\end{array} \right.$$

The twistor bundle, $J(M, g) \to M$, is the bundle whose fibre over a point $p$ of $M$ consists of all complex structures $j$ on $T_pM$ which are compatible with $g_p$ in the sense that there is a frame at the point $p$, $\xi$ in the fibre $F(M, g)_p$, in which the complex structure can be written $j = \xi \circ \tilde{j}_0 \circ \xi^{-1}$ where $\tilde{j}_0 := \begin{pmatrix} 0 & -Id_n \\ Id_n & 0 \end{pmatrix}$ (so we mean in particular positive compatible almost complex structures in the symplectic case, and we mean that $\tilde{j}_0$ is compatible with the orientation when an orientation is given in the pseudo-Riemannian case).

Observe that a complex structure $\tilde{j}$ on $V$ is compatible with $\tilde{g}_0$ if there exists a basis of $V$, compatible with $\tilde{g}_0$, in which the matrix associated to $\tilde{j}$ is $\tilde{j}_0$, hence $\tilde{j} = A\tilde{j}_0A^{-1}$ with $A \in G = \text{Gl}(V, \tilde{g}_0)$ and the space of such complex structures identifies with $\text{Gl}(V, \tilde{g}_0)/\text{Gl}(V, \tilde{g}_0, \tilde{j}_0)$ with

$$\text{Gl}(V, \tilde{g}_0, \tilde{j}_0) = \left\{ A \in G \mid A\tilde{j}_0 = \tilde{j}_0A \right\} \simeq \left\{ \begin{array}{ll}
U(p, q) & \text{in the pseudo-Riemannian setting} \\
U(n) & \text{in the symplectic case} \\
\text{Gl}(n, \mathbb{C}) & \text{if there is no extra structure}.
\end{array} \right.$$
The twistor bundle $J(M, g)$ can thus be seen as a quotient of the frame bundle:
\[ J(M, g) = F(M, g) \times_G \left( G/\text{Gl}(V, g_0, j_0) \right) = F(M, g)/\text{Gl}(V, g_0, j_0) \]  
(1.1)
and we shall denote by $\pi_1$ the natural projection (giving a $\text{Gl}(V, g_0, j_0)$-principal bundle structure):
\[ \pi_1 : F(M, g) \to J(M, g) = F(M, g)/\text{Gl}(V, g_0, j_0) : j \mapsto j \circ j_0 \circ \xi^{-1}. \]  
(1.2)

2. Almost complex structures on the twistor space

We shall denote by $\mathcal{V}$ the vertical tangent bundle to the twistor space
\[ \mathcal{V}_j := \text{Ker} \pi_1. \]

Note that a vector in $T_j J(M, g)$ is vertical if and only if it is tangent to the fibre, i.e. tangent to a curve $j_t$ of compatible complex structures on $T_p M$, with $p = \pi(j)$ and $j_0 = j$; hence:
\[ \mathcal{V}_j = \{ S \in \text{End}(T_p M) \mid S j + j S = 0, g_p(SX, Y) + g_p(X, SY) = 0, \forall X, Y \in T_p M \} \]
\[ = \{ [j, S'] \mid S' \in \text{End}(T_p M) \text{ and } g_p(S'^*X, Y) + g_p(X, S'^*Y) = 0 \}. \]  
(2.1)

(Indeed, given $S$ in the first set, one can define $S' = \frac{1}{2} S j$ in the second set). Let us denote by $\text{End}(TM, g)$ the bundle of infinitesimal isometries of the tangent bundle:
\[ \text{End}(TM, g)_p := \{ S \in \text{End}(T_p M) \mid g_p(SX, Y) + g_p(X, SY) = 0, \forall X, Y \in T_p M \} \]  
(2.2)
and consider the pullback bundles over $J(M, g)$:
\[ E := \pi^{-1}TM = \{ (j, X) \in J(M, g) \times TM \mid X \in T_p M \text{ with } p = \pi(j) \} \]  
(2.3)
\[ \text{End}(E, g) := \pi^{-1} \text{End}(TM, g) = \{ (j, S), j \in J(M, g)_p, S \in \text{End}(TM, g)_p, p \in M \}. \]

Clearly $\mathcal{V}$ is a subbundle of $\text{End}(E, g)$. The canonical section
\[ \Phi : J(M, g) \to \text{End}(E, g) : j \mapsto \Phi(j) := (j, j) \]  
(2.4)
defines the canonical (tautological) complex structure in the bundle $E$. Using equation (2.1), we can write
\[ \mathcal{V} = [\Phi, \text{End}(E, g)]. \]

We have a short exact sequence of bundles over $J(M, g)$:
\[ 0 \to \mathcal{V} \to T(J(M, g)) \xrightarrow{\pi} E \to 0. \]

The datum of a linear connection $\nabla$ on $M$ which preserves the structure $g$ (i.e. $\nabla g = 0$) gives a splitting
\[ T(J(M, g))_j = \mathcal{H}^\nabla_j \oplus \mathcal{V}_j \]
where the horizontal space $\mathcal{H}^\nabla_j$ is the projection by $\pi_1 \xi$ of the horizontal subspaces in the frame bundle: $H^\nabla_j = \text{Ker} \alpha^\nabla$ where $\alpha^\nabla$ is the Lie algebra $g$-valued connection 1-form on $F(M, g)$ associated to $\nabla$, with $g = \mathfrak{so}(2p, 2q, \mathbb{R})$, $\mathfrak{sp}(V, \Omega_0)$ or $\mathfrak{gl}(2n, \mathbb{R})$.

Since $\pi_1|_{\mathcal{H}^\nabla} : \mathcal{H}^\nabla_j \to T_{p=\pi(j)} M$ is an isomorphism, this splitting gives an isomorphism of bundles over $J(M, g)$:
\[ T(J(M, g)) = \mathcal{H}^\nabla \oplus \mathcal{V} \simeq E \oplus \mathcal{V} = E \oplus [\Phi, \text{End}(E, g)] \subset E \oplus \text{End}(E, g), \]  
(2.5)
the projection of $TJ(M,g)$ on $E$ being given by $\pi_\ast$.

Two natural almost complex structures $J^\pm_E$ are defined on $J(M,g)$ by:

$$
(J^\pm_E)_{J,\pi} (S) = j \circ S, \quad (J^\pm_E)_{J,\pi} = \pm (\pi_{\ast j} |_{H_\pi^E})^{-1} \circ j \circ (\pi_{\ast j} |_{H_\pi^E}).
$$

(2.6)

In other words,

$$
J^\pm_E |_E = \Phi.
$$

is left multiplication by $\Phi$ on $\mathcal{V}$ viewed as a subbundle of $\text{End}(E,g)$ and

$$
J^\pm_E |_E = \pm \Phi
$$

with $\Phi$ as a section of $\text{End}(E,g)$ acting on sections of $E$.

The almost complex structure $J^\pm_E$ was introduced by Eells and Salamon [5] as a first example of geometrically natural non-integrable almost complex structure.

2.1. Pullback connection and projection on the vertical bundle $\mathcal{V}$. The pullback connection $\pi^{-1}\nabla^E$ on $E$ is induced by the connection 1-form $p_2^* \alpha^\nabla$ on the pullback bundle $\pi^{-1}F(M,g)$, with

$$
p_2 : \pi^{-1}F(M,g) \subset J(M,g) \times F(M,g) \to F(M,g)
$$

the projection on the second factor. We denote by $p_1$ the bundle projection, i.e. the projection on the first factor $p_1 : \pi^{-1}F(M,g) \subset J(M,g) \times F(M,g) \to J(M,g)$.

Now $F(M,g)$ injects in $\pi^{-1}F(M,g)$ via

$$
i : F(M,g) \to \pi^{-1}F(M,g) : \xi \mapsto (\pi_1(\xi), \xi)
$$

and $i^* (p_2^* \alpha^\nabla) = \alpha^\nabla$.

The pullback $E'$ of a vector bundle associated with $F(M,g)$ for the representation $\rho$ of $G$ on $W$ (for instance $E' = E$ or $\text{End}(E,g)$) can be written as,

$$
E' := \pi^{-1} (F(M,g) \times_{G,\rho} W) = \pi^{-1}F(M,g) \times_{G,\rho} W \overset{p_1}{\to} J(M,g).
$$

A section $s$ of $E'$ can be viewed as the $G$-equivariant function $\hat{s}$ on the $G$-principal bundle $\pi^{-1}F(M,g)$ with values in $W$ so that $s(j) = [(j,\xi), \hat{s}(j,\xi)]$. It is completely determined by its restriction $\hat{s} := i^* \hat{s}$ defined on $F(M,g)$. Then

$$
(\pi^{-1}\nabla)^\prime_{\xi_j} s(j,\xi) = \Xi_{(j,\xi)} \hat{s} \text{ where } (p_2^* \alpha^\nabla)(\Xi) = 0 \text{ and } p_{1*}(\Xi) = \Xi \quad (2.7)
$$

and

$$
\frac{d}{dt} \hat{s}(j(t),\xi'(t))|_{t=0} \quad (2.8)
$$

with $j(t)$ a curve in $J(M,g)$ representing $\Xi_j \in T_jJ(M,g)$ and $\xi'(t)$ a curve in $F(M,g)$ representing $(\pi_{\ast j} \Xi)^\prime_\xi$, the horizontal lift in $H_\pi^E \subset T_\xi F(M,g)$ of $\pi_{\ast j} \Xi \in T_{pJ}M$, both curves projecting on the same curve $p(t)$ in $M$. This implies, since $X_\xi - \left(\alpha^\nabla_\xi (X_\xi)\right)^\ast$ is horizontal, for any $X_\xi \in T_\xi F(M,g)$, with $A^\ast$ the fundamental vector field associated to
the right action of $G$ on $F(M,g)$ (i.e. $A^*_\xi = d\xi \circ \exp tA|_{t=0}$) for any $A \in \mathfrak{g}$, and since
\[ \tilde{s}(\xi \exp tA) = \rho(\exp -tA)\tilde{s}(\xi) \]
\[
((\pi^{-1}\nabla)_{\pi_1 \xi}^E \xi s)(\xi) = ((\pi^{-1}\nabla)_{\pi_1 \xi}^E \xi s)(\pi_1 \xi, \xi) \\
= \frac{d}{dt}\tilde{s}(\pi_1 \xi(t), \xi(t))|_{t=0} - \frac{d}{dt}\tilde{s}(\pi_1 \xi(t), \xi(t))|_{t=0} - \tilde{s}(\pi_1 \xi(t), \xi(t))|_{t=0}
\]
where $\xi(t)$ is a curve in $F(M,g)$ representing $X_\xi$
\[
= X_\xi \tilde{s} + \rho_*(\alpha^E_\xi(X_\xi))(\tilde{s}(\xi))
\] (2.9)
Observe that the function $\tilde{\Phi}$ on $\pi^{-1}F(M,g)$ corresponding to the canonical section $\Phi$
of $\text{End}(E)$ is given by $\tilde{\Phi}(j, \xi) = \xi \circ j \circ \xi^{-1}$ so that its restriction to $F(M,g)$ is the constant function $\tilde{\Phi}(\xi) = \tilde{j}_0$. Hence
\[
((\pi^{-1}\nabla)_{\pi_1 \xi}^{\text{End}(E)} \xi \phi)(\xi) = \alpha^E_\xi(X_\xi))(\tilde{j}_0) = [\alpha^E_\xi(X_\xi), \tilde{j}_0].
\] (2.10)
If $\Xi_j$ is horizontal, we write $\Xi_j = \pi_1 \xi X_\xi$ with $\alpha^E_\xi(X_\xi) = 0$, so $(\pi^{-1}\nabla)_{\pi_1 \xi} X_\xi \Phi = 0$. If $\Xi_j$ is vertical, we write $\Xi_j = \pi_1 \xi \alpha^E_\xi$ with $A \in \mathfrak{g}$ such that $A \tilde{j}_0 + j_0 A = 0$; then
\[
\Xi_j = \frac{d}{dt}(\xi \circ \exp tA |_{t=0} \circ \xi^{-1})|_{t=0} = \xi \circ [A, \tilde{j}_0] \circ \xi^{-1},
\]
hence $\Xi_j(\xi) = [A, \tilde{j}_0]$ when we view the vertical tangent vector $\Xi_j$ as an element of $\text{End}(T_j M) = \text{End}(E)$. We also have $\alpha^E_\xi(A^\xi) = A$; hence

**Proposition 2.1.** The projection on the vertical tangent space $\mathcal{V}_j$
\[
P^j \mathcal{V}_j \mathcal{T}_j J(M, g) = \mathcal{H}^\mathcal{V}_j \oplus \mathcal{V}_j = [j, \text{End}(E, g)] = [\Phi, \text{End}(E, g)] \subset \text{End}(E, g)
\] (2.11)
is given in terms of the covariant derivative under the pullback connection of the canonical section $\Phi$ of $\text{End} E$ (defined by (2.4)) via
\[
(\pi^{-1}\nabla)_{\Xi_j}^{\text{End}(E), g} \Phi = P^j \mathcal{V}_j(\Xi_j).
\] (2.12)
Note that we differ here slightly from Proposition 3 in [8]; we follow their development, adapting to this difference.

Recall that the projection on $\mathcal{H}^\mathcal{V}$ identified with $E$ is given by $\pi_x$.

2.2. **A connection on** $TJ(M, g)$ preserving $J^\mathcal{V}_x$. We define a covariant derivative of sections of $E$ preserving $g$ so that the associated covariant derivative of sections of $\text{End}(E, g)$ preserves sections of $\mathcal{V}$; let
\[
D^E_\Xi Y := (\pi^{-1}\nabla)^E_\Xi Y + \frac{1}{2}(P^E(\Xi) \circ \Phi)(Y), \quad \Xi \in \Gamma(TJ(M, g)), \quad Y \in \Gamma(E)
\] (2.13)
where $\Phi$ and $P^E(\Xi)$ are viewed as sections of $\text{End}(E, g)$. This covariant derivative preserves the tautological complex structure $\Phi$ on the bundle $E$ since it is equal to
\[
D^E = (\pi^{-1}\nabla)^E - \frac{1}{2} \Phi \circ (\pi^{-1}\nabla)^{\text{End} E} \Phi.
\]
The associated covariant derivative of sections of $E$ is given by
\[
D^E_\text{End} E S := (\pi^{-1}\nabla)^E_\text{End} E S + \frac{1}{2}[P^E(\Xi) \circ \Phi, S], \quad S \in \Gamma(\text{End} E).
\] (2.14)
Since $\Phi$ anticommutes with any element of $\mathcal{V}$, and $\Phi^2 = -\text{Id}$ we have indeed

$$D^E\circ\Phi = (\pi^{-1}\nabla)^E\circ\Phi + \frac{1}{2}[P^\nabla(\Xi) \circ \Phi, \Phi] = P^\nabla(\Xi) - P^\nabla(\Xi) = 0.$$  \hfill (2.15)

Hence $D^E$ preserves sections of $\mathcal{V} = [\Phi, \text{End}(E, g)]$ and $D^E \oplus D^E$ induces a covariant derivative $D$ of sections of the tangent bundle $TJ(M, g)$. If $Y$ is a section of $\mathcal{H}^\nabla \simeq E$, then

$$D_\Xi(Y) = D_\Xi(\pm \Phi(Y)) = \pm \Phi(D_\Xi(Y)) = J_\Xi^\perp D_\Xi(Y);$$

if $S$ is a section of $\mathcal{V} \subset \text{End}(E, g)$ then

$$D_\Xi(J_\Xi^\perp S) = D_\Xi(\Phi \circ S) = \Phi \circ D_\Xi(S) = J_\Xi^\perp D_\Xi(S).$$

Hence

$$DJ_\Xi^\perp = 0.$$ \hfill (2.16)

Since $D$ preserves $\mathcal{H}^\nabla \simeq E$ and $\mathcal{V}$, the covariant derivative of the projections vanish:

$$DP^\nabla = 0 \quad D\pi_n = 0.$$ \hfill (2.17)

3. A closed 2-form on $J(M, g)$ associated to $\nabla$

Observe that $D^E$ preserves the tautological complex structure defined by $\Phi$ on the bundle $E$, hence can be used, following Reznikov [10] and Rawnsley [9], in the Chern–Weil construction of characteristic classes of $E$; the complex trace of the curvature of $D^E$,

$$\chi(J(M, g)) \ni \Xi, \Xi' \mapsto \text{Tr}_\mathbb{C} \left( D^E_\Xi \circ D^E_\Xi' - D^E_{\Xi'} \circ D^E_\Xi - D^E_{\Xi, \Xi'} \right)$$

is $-2\sqrt{-1}$ times a real closed 2-form on $J(M, g)$ representing $c_1(E, \Phi) \in H^2(J(M, g), \mathbb{R})$ which is the real first Chern class of the complex vector bundle $(E, \Phi)$.

**Proposition 3.1.** [10] Having chosen a torsion-free connection $\nabla$ preserving the pseudo-Riemannian or symplectic structure $g$, the 2-form $\omega^{J(M, g), \nabla}$ on $J(M, g)$ defined by

$$\omega^{J(M, g), \nabla}(\Xi, \Xi') := -2 \text{Tr}_\mathbb{R} \left( R^{\nabla}_{\pi(\jmath)}(\pi_j \Xi, \pi_j \Xi') \circ j \right) + i \text{Tr}_\mathbb{C} \left( [P^\nabla(\Xi), P^\nabla(\Xi')] \right), \hfill (3.1)$$

which represents $-8\pi c_1(E, \Phi)$, is symplectic if and only if, for any $p \in M$ and any $j \in J(M, g)_p$, the skew-symmetric bilinear form $\Omega^{\nabla, j}$ on $T_p M$

$$X, Y \mapsto \text{Tr}_\mathbb{R}(R^{\nabla}_p(X, Y) \circ j)$$ \hfill (3.2)

is non-degenerate.

**Proof.** Indeed, since $D^E = (\pi^{-1}\nabla)^E - \frac{1}{2} \Phi \circ ((\pi^{-1}\nabla)^E \Phi)$, we have

\[
\left( D^E_\Xi \circ D^E_{\Xi'} - D^E_{\Xi'} \circ D^E_\Xi - D^E_{\Xi, \Xi'} \right) = \pi^*(R^{\nabla}(\pi_j \Xi, \pi_j \Xi')) - \frac{1}{2} \Phi \left( \pi^*(R^{\nabla}(\pi_j \Xi, \pi_j \Xi')) \right)
\]

\[
\neq -\frac{1}{4} \left( (\pi^{-1}\nabla)^E_\Xi \circ (\pi^{-1}\nabla)^E_{\Xi'} \right)
\]

\[
= \frac{1}{2} \pi^*(R^{\nabla}(\pi_j \Xi, \pi_j \Xi')) - \frac{1}{2} \Phi \pi^*(R^{\nabla}(\pi_j \Xi, \pi_j \Xi')) \Phi
\]

\[
- \frac{1}{4} \left( P^\nabla(\Xi), P^\nabla(\Xi') \right)
\]

where $(\pi^*(R^{\nabla}(\pi_j \Xi, \pi_j \Xi'))(j) := R^{\nabla}_{\pi(\jmath)}(\pi_j \Xi_j, \pi_j \Xi'_j)$ is viewed as an endomorphism of $T_p M$, hence as an element of End$(E, g)_j$. 

Observe that \( V_1, V_2 \in T_j(Y_j) \rightarrow i \text{Tr}_C([V_1, V_2]) \) defines the usual symplectic structure on the fibre of \( J(M, g) \), i.e. the one induced by the isomorphism between a fibre and \( GL(V, \tilde{g}_0)/GL(V, \tilde{g}_0, j_0) \).

Hence the closed 2-form \( \omega^{(M, g, 4)}(\Xi, \Xi') = -4i \text{Tr}_C \left( D_{E}^{\prime} \circ D_{E}^{\prime} - D_{E}^{\prime} \circ D_{E}^{\prime} - D_{E}^{\prime} - D_{E}^{\prime} \right) \) is symplectic if and only if, for any \( p \in M \) and any \( j \in J(M, g)_p \), the bilinear form on \( T_p M \), \( X, Y \mapsto \text{Tr}_R(R^V_p(X, Y) \circ j) \) is non-degenerate. \( \square \)

**Lemma/Definition 3.2.** Each of the almost complex structures \( J^\pm \) is said to be compatible with the closed 2-form \( \omega^{(M, g, 4)} \) when

\[
\omega^{(M, g, 4)}(J^\pm \Xi, J^\mp \Xi^\prime) = \omega^{(M, g, 4)}(\Xi, \Xi^\prime) \tag{3.3}
\]

i.e. when \( \omega^{(M, g, 4)} \) is of type \((1,1)\) with respect to \( J^\pm \). This will be true if and only if

\[
\text{Tr}_R \left( R^V_p(jX, jY) \circ j \right) = \text{Tr}_R \left( R^V_p(X, Y) \circ j \right),
\]

\( \forall p \in M, X, Y \in T_p M, j \in J(M, g)_p. \) \tag{3.4}

### 4. The Nijenhuis Tensor for \( J^\pm \)

The torsion \( T^D \) of \( D \) can be used to compute the Nijenhuis tensor of \( J^\pm \). Now the vertical part of the torsion \( T^D \) is given by

\[
P^V T^D(\Xi, \Xi^\prime) = P^V \left( D_{\Xi} \Xi^\prime - D_{\Xi^\prime} \Xi - [\Xi, \Xi^\prime] \right) = D_{\Xi} \left( P^V(\Xi^\prime) \right) - D_{\Xi^\prime} \left( P^V(\Xi) \right) - P^V([\Xi, \Xi^\prime])
\]

\[
= D_{\Xi} \left( (\pi - 1) E \right) E F - D_{\Xi^\prime} \left( (\pi - 1) E \right) E F - (\pi - 1) E \right) E F
\]

\[
- (\pi - 1) E \right) E F + \frac{1}{2} \left[ P^V(\Xi) \circ F, P^V(\Xi^\prime) \right] - \frac{1}{2} \left[ P^V(\Xi^\prime) \circ F, P^V(\Xi) \right]
\]

\[
= [\pi^* (R^V(\pi, \Xi, \pi, \Xi^\prime))(\Xi), \Phi] + \frac{1}{4} \left[ [P^V(\Xi), P^V(\Xi^\prime)], \Phi \right]
\]

where \( (\pi^* (R^V(\pi, \Xi, \pi, \Xi^\prime))(\Xi), \Phi) \) is viewed as an endomorphism of \( T_p M \) hence as an element of \( \text{End}(E, g)_j \). The horizontal part of the torsion is given by

\[
\pi_* T^D(\Xi, \Xi^\prime) = \pi_* \left( D_{\Xi} \Xi^\prime - D_{\Xi^\prime} \Xi - [\Xi, \Xi^\prime] \right) = D_{\Xi} \left( \pi_* \Xi^\prime \right) - D_{\Xi^\prime} \left( \pi_* \Xi \right) - \pi_* ([\Xi, \Xi^\prime])
\]

\[
= (\pi - 1) E \right) E (\pi_* \Xi^\prime) - (\pi - 1) E \right) E (\pi_* \Xi) - \pi_* ([\Xi, \Xi^\prime])
\]

\[
+ \frac{1}{2} \left( P^V(\Xi) \circ \Phi \right) (\pi_* \Xi^\prime) - \frac{1}{2} \left( P^V(\Xi^\prime) \circ \Phi \right) (\pi_* \Xi)
\]

\[
= \pi^* (T^V(\pi, \Xi, \pi, \Xi^\prime)) - \frac{1}{2} \Phi \left( P^V(\Xi) \circ P^V(\Xi^\prime) \right)(\pi_* \Xi)
\]

where \( (\pi^* (T^V(\pi, \Xi, \pi, \Xi^\prime)) \) is an element of \( T_p M \) viewed as an element of \( E_j \).

Since \( DJ^\pm \) we know that

\[
T^V(JX, JY) - JT^V(JX, Y) - JT^V(X, JY) - T^V(X, Y) = -N^J(X, Y) \tag{4.1}
\]
hence

$$N^{J^\xi}_j(\Xi,\Xi') = -T^D(J^\xi_\xi \Xi, J^\xi_\xi \Xi') + J^\xi_\xi T^D(J^\xi_\xi \Xi, \Xi') + J^\xi_\xi T^D(\Xi, J^\xi_\xi \Xi') + T^D(\Xi, \Xi').$$

From the formulas above, since $\pi_*(J^\xi_\xi \Xi) = \pm \Phi(\pi_\xi \Xi)$ and $P^\nu(J^\xi_\xi \Xi) = \Phi \circ P^\nu(\Xi)$, we get

$$P^\nu(N^{J^\xi}_j(\Xi,\Xi')) = -[\pi^*(T^V(\Phi(\pi_\xi \Xi), \Phi(\pi_\xi \Xi'))), \Phi] \pm \Phi \circ [\pi^*(T^V(\Phi(\pi_\xi \Xi), \pi_\xi \Xi')), \Phi] + [\pi^*(T^V(\pi_\xi \Xi, \pi_\xi \Xi')), \Phi] \quad (4.2)$$

$$\pi_*(N^{J^\xi}_j(\Xi,\Xi')) = -\pi^*(T^V(\Phi(\pi_\xi \Xi), \Phi(\pi_\xi \Xi'))) + \Phi(\pi^*(T^V(\Phi(\pi_\xi \Xi), \pi_\xi \Xi')))
+ \Phi(\pi^*(T^V(\pi_\xi \Xi, \Phi(\pi_\xi \Xi')))) + \pi^*(T^V(\pi_\xi \Xi, \pi_\xi \Xi'))
\pm \text{Im}(\Phi(\pi^*P^\nu(\Xi)))(\Phi(\pi_\xi \Xi) - (\Phi \circ P^\nu(\Xi))(\Phi(\pi_\xi \Xi)))
\pm \text{Im}(\Phi(\pi^*P^\nu(\Xi)))\quad (4.3)$$

**Proposition 4.1.** The Nijenhuis tensor associated to the canonical almost complex structures $J^\xi_\xi$ on the twistor space $J(M, g)$ always vanishes on two vertical vector fields; $N^{J^\xi}_j$ vanishes on $\mathcal{V} \times H^\xi$ whereas $J^\xi_\xi$ is never integrable because

$$N^{J^\xi}_j(S, Y) = 2S_j(jY_j) = -2jS_j(jY_j), \quad \text{for } S \in \Gamma(\mathcal{V}) \subset \Gamma(\text{End}(E, g))$$

$$\text{and } Y \in \Gamma(H^\xi) = \Gamma(E) \quad (4.4)$$

so that $\text{Image } N^{J^\xi}_j \subset H^\xi$.

Choosing the connection $\nabla$ without torsion (which will be the Levi Civita connection in the pseudo-Riemannian setting) one sees that the horizontal part of $N^{J^\xi}_j$ vanishes on $H^\xi \times H^\xi$, hence Image $N^{J^\xi}_j \subset \mathcal{V}$.

The vertical part of the image of $N^{J^\xi}_j$ consists of all the endomorphisms of $T_pM$ with $p = \pi(j)$ given by

$$-[R^\nu_p(jX, jX'), j] \pm j \circ [R^\nu_p(jX, X'), j] \pm j \circ [R^\nu_p(X, jX'), j] + [R^\nu_p(X, X'), j]$$

$$= j \circ R^\nu_p(jX, jX') - R^\nu_p(jX, jX') \circ j \pm j \circ R^\nu_p(jX, X') \circ j \pm R^\nu_p(jX, X')$$

$$\pm j \circ R^\nu_p(X, jX') \circ j \pm R^\nu_p(X, jX') + R^\nu_p(X, X') \circ j - j \circ R^\nu_p(X, X')$$

$$= \text{Im}(\text{Id} - ij) \circ R^\nu_p((\text{Id} \pm ij)X, (\text{Id} \pm ij)X') \circ (\text{Id} + ij))$$

which is equal to Real part of $-j(\text{Id} - ij) \circ R^\nu_p((\text{Id} \pm ij)X, (\text{Id} \pm ij)X') \circ (\text{Id} + ij))$. 

We now proceed as in [8]: the vertical part of the image of \( N^J_p \) vanishes identically on all \( j's \in \pi^{-1}p \) if and only if the curvature \( \tilde{R} \), which is the expression (using a frame) of \( R_p^V \) as a 1,3 tensor on \( V \), satisfies
\[
(\text{Id} - i\tilde{j}) \circ \tilde{R} \left( (\text{Id} \pm i\tilde{j}) \cdot, (\text{Id} \pm i\tilde{j}) \cdot \right) \circ (\text{Id} + i\tilde{j}) = 0, \quad \forall \tilde{j} = A_{0\tilde{j}}A^{-1}
\]
where \( A \in G \). Hence for all \( A \in G \) and putting \( \tilde{j} = A_{0\tilde{j}}A^{-1} \)
\[
A(\text{Id} - i\tilde{j}_{0})A^{-1} \circ \tilde{R} \left( A(\text{Id} \pm i\tilde{j}_{0})A^{-1} \cdot, A(\text{Id} \pm i\tilde{j}_{0})A^{-1} \cdot \right) \circ A(\text{Id} + i\tilde{j}_{0})A^{-1} = 0, \quad \text{so}
\]
\[
(\text{Id} - i\tilde{j}_{0}) \circ A^{-1} \tilde{R} \left( (\text{Id} \pm i\tilde{j}_{0}) \cdot, A(\text{Id} \pm i\tilde{j}_{0}) \cdot \right) A \circ (\text{Id} + i\tilde{j}_{0}) = 0, \quad \text{so}
\]
\[
(\text{Id} - i\tilde{j}_{0}) \circ A^{-1} \cdot \tilde{R} \left( (\text{Id} \pm i\tilde{j}_{0}) \cdot, (\text{Id} \pm i\tilde{j}_{0}) \cdot \right) \circ (\text{Id} + i\tilde{j}_{0}) = 0,
\]
where \( A^{-1} \cdot \tilde{R} := A^{-1}\tilde{R}(A, A)A \) denotes the natural action of \( G \) on tensors, hence if and only if the curvature \( \tilde{R} \) takes values in the largest \( G \)-invariant subspace of tensors on \( V \) of (pseudo-Riemannian, symplectic or plain) curvature type for which
\[
(\text{Id} - i\tilde{j}_{0}) \circ \tilde{R} \left( (\text{Id} \pm i\tilde{j}_{0}) \cdot, (\text{Id} \pm i\tilde{j}_{0}) \cdot \right) \circ (\text{Id} + i\tilde{j}_{0}) = 0. \tag{4.5}
\]

There is a natural action of \( \tilde{j}_{0} \) on curvature type tensors given by
\[
(\tilde{j}_{0} \cdot \tilde{R})(U, V) = \tilde{j}_{0} \circ \tilde{R}(U, V) - \tilde{R}(\tilde{j}_{0}U, V) - \tilde{R}(U, \tilde{j}_{0}V) - \tilde{R}(U, V) \circ \tilde{j}_{0}.
\]
The action of \( \tilde{j}_{0} \) on \( V^C \) has \( \pm i \) as eigenvalues, the projection on the \( +i \)-eigenspace being given by \( \text{Id} - i\tilde{j}_{0} \). Hence the action on the space of tensors of curvature type has eigenvalues in \( \{0, \pm 2i, \pm 4i\} \); the projection on the \( 4i \)-eigenspace is given by
\[
(\text{Id} - i\tilde{j}_{0}) \circ \tilde{R} \left( (\text{Id} + i\tilde{j}_{0}) \cdot, (\text{Id} + i\tilde{j}_{0}) \cdot \right) \circ (\text{Id} + i\tilde{j}_{0}),
\]
thus (4.5) says that the vertical part of the image of \( N^J_p \) vanishes if and only if \( \tilde{R} \) takes values in the largest \( G \)-invariant subspace of curvature-type tensors on \( V \) for which \( 4i \) is not an eigenvalue of the action of \( \tilde{j}_{0} \).

Next we examine the decomposition of the space of curvature type tensors under the action of \( G \).

5. PSEUDO-RIEMANNIAN STRUCTURE OF SIGNATURE \( (2p, 2q) \) WITH (OR WITHOUT) A GIVEN ORIENTATION

In the case of a pseudo-Riemannian structure \( g \) of signature \( (2p, 2q) \) on a manifold \( M \), one uses the Levi Civita connection for \( \nabla \).

**Definition 5.1.** The space of curvature type tensors at the point \( p \in M \),
\[
\begin{align*}
\left\{ R \in \Lambda^2(V^*) \otimes \text{End}(V) \mid & \bigoplus_{X,Y,Z} R(X, Y)Z = 0, \quad g_p(R(X, Y)Z, T) = -g_p(R(X, Y)T, Z) \right\},
\end{align*}
\]
with \( V := T_pM \), will be denoted by \( \mathcal{R}(V, g_p) \) where \( \bigoplus_{X,Y,Z} \tilde{R}(X, Y)Z \) here and elsewhere denotes the sum over cyclic permutations of \( X, Y, Z \).
When \( G = O(2p, 2q) \) with \( 2p + 2q = 2n \), this space of curvature type tensors splits into 3-irreducible parts [3] so that:

\[
R^\nabla = S^\nabla + E^\nabla + C^\nabla,
\]

where \( S^\nabla \) is constructed algebraically using the metric tensor \( g \) and the scalar curvature \( \text{scal}(g) = \text{Tr} \rho^\nabla \) with \( g(X, \rho^\nabla Z) := \text{Ric}^\nabla(X, Z) := \text{Tr}[Y \to R^\nabla(X, Y)Z] \)

\[
g(S^\nabla(X, Y)Z, T) = \frac{\text{scal}(g)}{2n(2n - 1)} (g(X, Z)g(Y, T) - g(X, T)g(Y, Z)),
\]

where \( E^\nabla \) is the half traceless part constructed algebraically using the metric tensor and the traceless part of the Ricci tensor \( (\text{Ric}(X, Z) = \text{Ric}^\nabla(X, Z) - \frac{\text{scal}(g)}{2n} g(X, Z)) \):

\[
g(E^\nabla(X, Y)Z, T) = \frac{1}{2n - 2} \left( g(X, Z)\text{Ric}(Y, T) - g(X, T)\text{Ric}(Y, Z) + g(Y, T)\text{Ric}(X, Z) - g(Y, Z)\text{Ric}(X, T) \right)
\]

and where \( C^\nabla \) is the totally traceless part, the so-called Weyl tensor.

Since \( \tilde{g}_0(\tilde{j}_0X, Y) + \tilde{g}_0(X, \tilde{j}_0Y) = 0 \), the 4\( i \) eigenvalue can only arise in the Weyl tensor part and does so, hence the well known

**Proposition 5.2.** \( \tilde{J}_\nabla^\perp \) is integrable in the pseudo-Riemannian context with no given orientation if and only if \( C^\nabla = 0 \).

In the oriented case the decomposition of the curvature under the action of \( SO(2p, 2q) \) is the same as above in dimension greater than 4 but in dimension 4, there is a further splitting of the Weyl tensor into a self-dual and an anti-self-dual part. A Weyl tensor is said to be self-dual (respectively anti-self-dual), if, viewed as an endomorphism of \( \Lambda^2 T^* M \), it vanishes on the eigenspace of eigenvalue \(-1\) (respectively \(+1\)) of the Hodge * operator acting on 2-forms.

**Proposition 5.3.** \( J_\nabla^\perp \) is integrable in the pseudo-Riemannian context with a given orientation if and only if \( C^\nabla = 0 \) when \( 2n \geq 4 \); in dimension 4, it is integrable if and only if the the Weyl component of the Riemann curvature tensor is self-dual when the signature is \((4, 0)\) or \((0, 4)\) and anti-self-dual when the signature is \((2, 2)\).

**Proof (in dimension 4).** In an oriented pseudo-orthonormal basis \( \{e_1, \ldots, e_4\} \) with

\[
\tilde{g}_0(e_1, e_1) = \tilde{g}_0(e_3, e_3) = e_1 \text{ and } \tilde{g}_0(e_2, e_2) = \tilde{g}_0(e_4, e_4) = e_2 \text{ and with } \tilde{j}_0 = \begin{pmatrix}
0 & -\text{Id}_2 \\
\text{Id}_2 & 0
\end{pmatrix}
\]

as before, the eigenspace of eigenvalue \( \epsilon \) of the Hodge * operator is spanned by \( e_1 \wedge e_2 + \epsilon e_1 e_2 \wedge e_3 \wedge e_4 = e_1 \wedge e_2 + \epsilon e_1 e_2 \tilde{j}_0 e_1 \wedge \tilde{j}_0 e_2, e_1 \wedge e_3 - \epsilon e_2 \wedge e_4 \) and \( e_1 \wedge e_4 + \epsilon e_1 e_2 e_2 \wedge e_3 = e_1 \wedge e_4 + \epsilon e_1 e_2 \tilde{j}_0 e_1 \wedge \tilde{j}_0 e_4 \).

Hence, any tensor \( \tilde{R} \) vanishing on the eigenspace of eigenvalue \( \epsilon = -\epsilon_1 \epsilon_2 \) satisfies \( \tilde{R}(\tilde{j}_0 \cdots \tilde{j}_0 \cdot) = \tilde{R}(\cdot, \cdot) \), hence \( \tilde{R}((\text{Id} + i\tilde{j}_0) \cdots ((\text{Id} + i\tilde{j}_0) \cdot) = 0 \). The largest \( SO(V, g) \)-invariant subspace of Weyl tensors on \( V \) for which \( 4i \) is not an eigenvalue of the action of \( \tilde{j}_0 \) is thus the space of Weyl tensors vanishing on the eigenspace of eigenvalue \( \epsilon = -\epsilon_1 \epsilon_2 \) of the Hodge * operator.
Observe that
\[
g((\text{Id} - ij)S^\nabla((\text{Id} - ij)X,(\text{Id} - ij)Y)(\text{Id} + ij)Z,T) \\
= \frac{2\text{scal}(g)}{n(2n-1)} \left(g((\text{Id} - ij)X,Z)g((\text{Id} - ij)Y,T) - g((\text{Id} - ij)Y,Z)g((\text{Id} - ij)X,T)\right)
\]
hence

Imaginary part of \((\text{Id} - ij)\circ S^\nabla((\text{Id} - ij)X,(\text{Id} - ij)X') \circ (\text{Id} + ij))
\[
= \frac{2\text{scal}(g)}{n(2n-1)} \left(g(X',\cdot)jX + g(jX',\cdot)X - g(X,\cdot)jX' - g(jX,\cdot)X'\right)
\]
\[
= \frac{2\text{scal}(g)}{n(2n-1)} \left[j \cdot g(X',\cdot)X - g(X,\cdot)X'\right];
\]
and this shows that the vertical part of the image of \(N^\nabla_{\nabla} \) at \(j\) is the whole vertical tangent space \(\mathcal{V}_j = [j, \text{End}(E,g)] = [j, \text{End}(T_pM,g_p)]\) whenever the space has constant non-zero sectional curvature, i.e. when \(R^\nabla = S^\nabla\) and \text{scal}(g) \neq 0.

To summarise, we have

**Proposition 5.4.** For a pseudo-Riemannian manifold \((M,g)\) with no given orientation, the almost complex structure \(J^\nabla_{\nabla}\) on the twistor space \(J(M,g)\), defined using the Levi Civita connection \(\nabla\), is integrable if and only if the Weyl component of the Riemann curvature tensor vanishes, \(C^\nabla = 0\).

With a given orientation, the almost complex structure \(J^\nabla_{\nabla}\) on the twistor space \(J(M,g)\), defined using the Levi Civita connection \(\nabla\), is integrable if and only if the Weyl tensor \(C^\nabla\) vanishes when \(\dim M > 4\). In dimension 4, it is integrable if and only if the the Weyl component of the Riemann curvature tensor is self-dual when the signature is \((4,0)\) or \((0,4)\) and anti-self-dual when the signature is \((2,2)\).

The almost complex structure \(J^\nabla_{\nabla}\) is never integrable.

If the space has non-vanishing constant sectional curvature, then the image of the Nijenhuis tensor associated to \(J^\nabla_{\nabla}\) is the whole tangent space \(T_jJ(M,g)\) at any point \(j \in J(M,g)\).

Observe that in this case \((C^\nabla = 0, E^\nabla = 0\) and \text{scal}(g) \neq 0\), the closed 2-form on \(J(M,g)\) associated by \((3.1)\) to \(\nabla\), \(\omega^{J(M,g,\nabla)}\), is symplectic since \(\text{Tr}(R^\nabla(X,Y) \circ j) = \frac{\text{scal}(g)}{n(2n-1)} g(X,jY)\). Also in that case, the almost complex structures \(J^\nabla_{\nabla}\) are compatible with the symplectic 2-form, in the sense of equation \((3.3)\), i.e. \(\omega^{J(M,g,\nabla)}\) is of type \((1,1)\) with respect to \(J^\nabla_{\nabla}\); \(J^\nabla_{\nabla}\) is positive when \text{scal}(g) is positive and \(J^\nabla_{\nabla}\) is positive when \text{scal}(g) is negative.

Hence the twistor space \(J(M,g)\) on a pseudo-Riemannian manifold with non-vanishing constant sectional curvature has a natural symplectic structure \(\omega^{J(M,g,\nabla)}\) and two natural compatible almost complex structures, \(J^\nabla_{\nabla}\) yielding a pseudo-Kähler structure on this twistor space and \(J^\nabla_{\nabla}\) being maximally non-integrable in the sense that the image of the corresponding Nijenhuis tensor is the whole tangent space at every point.
More generally, for the twistor space on a Riemannian space, Reznikov [10] has proven that the closed 2-form $\omega^{\mathcal{J}(M,g,V)}$ (defined by (3.1)) is symplectic if the sectional curvature is sufficiently pinched. The proof relies on Berger’s inequalities [2], all components $R_{ijkl} := g_p(R_p(e_i,e_j)e_k,e_l)$ of the curvature tensor in an orthonormal basis $\{e_i; i \leq 2n\}$ of $T_pM$ are very small unless $\{i,j\} = \{k,l\}$. Hence the 2-form $X,Y \mapsto \frac{scal(g)}{n(2n-1)}g(X,jY)$ is very close to the 2-form $X,Y \mapsto R^\nabla(X,Y) \circ j$ is very close to the 2-form $X,Y \mapsto \frac{scal(g)}{n(2n-1)}g(X,jY)$ and is thus non-degenerate.

In a similar way, the endomorphism of $T_pM$ defined by

$$
\text{Imaginary part of } (\text{Id} - ij) \circ R^\nabla_p ((\text{Id} \pm ij)X, (\text{Id} \pm ij)X') \circ (\text{Id} + ij)
$$

is very close to $\frac{scal(g)}{n(2n-1)} [j,g(X',\cdot)X - g(X,\cdot)X']$ hence the vertical part of the image of $N_j^{\mathcal{J}_V}$ consists of all the endomorphisms $[j,A]$ of $T_pM$ where $p = \pi(j)$ and $A \in \text{End}(T_pM,g_p)$.

**Proposition 5.5.** Given any positive integer $n$, there exists an $\epsilon(n)$ such that, if the sectional curvature of a Riemannian manifold $(M,g)$ of dimension $2n$ is $\epsilon(n)$-pinched, the almost complex structure $\mathcal{J}_V$ on this twistor space, defined using the Levi Civita connection $\nabla$, is maximally non-integrable (i.e. the image of the corresponding Nijenhuis tensor is the whole tangent space at every point).

We shall now study when each of the almost complex structures $\mathcal{J}_V^\pm$ is compatible (in the classical sense of equation (3.3)) with the 2-form $\omega^{\mathcal{J}(M,g,V)}$ (defined by equation (3.1)); we have seen in Section 3 that it is the case if and only if equation (3.4) is satisfied: $\text{Tr}_R(R^\nabla_p(jX,jY) \circ j) = \text{Tr}_R(R^\nabla_p(X,Y) \circ j)$ for all $p \in M$, $X,Y \in T_pM$, $j \in J(M,g)_p$.

**Definition 5.6.** For $R \in \mathcal{R}(V,g_p)$ and $j \in J(M,g)_p$ let $\Omega^{R,j}_1(X,Y) = \text{Tr}_R(R(X,Y) \circ j)$ for $X,Y \in V$.

The condition of compatibility (3.4) is that $\Omega^{R,j}_1(jX,jY) = \Omega^{R,j}_1(X,Y)$ for all $X,Y \in V$ so if we define

**Definition 5.7.** $\Omega^{R,j}_2(X,Y) = \Omega^{R,j}_1(jX,jY) - \Omega^{R,j}_1(X,Y)$,

then the condition for compatibility becomes $\Omega^{R,j}_2 = 0$ for all $j \in J(M,g)_p$.

**Proposition 5.8.** Let $(M,g)$ be a pseudo-Riemannian manifold of dimension $2n \geq 4$ with Levi Civita connection $\nabla$. Condition (3.4) holds (i.e. $\mathcal{J}_V^\pm$ are compatible with the closed 2-form $\omega^{\mathcal{J}(M,g,V)}$) for $M$ non-oriented and $2n \geq 4$ or $M$ oriented and $2n \geq 6$ if and only if the Weyl component $C^\nabla$ of the curvature $R^\nabla$ vanishes. If $M$ is oriented and $2n = 4$, Condition (3.4) holds if and only if the Weyl component of the Riemann curvature tensor is self-dual when the signature is $(4,0)$ or $(0,4)$ and anti-self-dual when the signature is $(2,2)$.

**Proof.** Whenever the Weyl tensor vanishes, the remaining two terms $S^\nabla, E^\nabla$ satisfy

$$
\text{Tr}(S^\nabla_p(X,Y) \circ j) = \frac{scal(g)}{n(2n-1)}g_p(X,jY),
$$

$$
\text{Tr}_R(E^\nabla_p(X,Y) \circ j) = \frac{1}{n+1} \left( \widetilde{Ric}_p(X,jY) - \widetilde{Ric}_p(Y,jX) \right)
$$
for all \( p \in M, X, Y \in T_pM, j \in J(M, g)_p \), and both the right-hand sides satisfy condition (3.4) as was already mentioned in Proposition 5.4.

The remainder of this section is devoted to the proof of the converse; we use a construction from the analysis of the curvature in the (positive definite) almost Hermitian case due to Tricerri and Vanhecke [12, page 372] but which makes sense in the bundle \( J(M, g) \) of compatible almost complex structures where \( g \) is pseudo-Riemannian.

Fix \( p \in M \), let \( V = T_pM \), and \( j \in J(M, g)_p \). We set

\[
\mathcal{V}^j_3 = \{ S \in \wedge^2 V^* \mid S(jX, jY) = -S(X, Y) \quad \forall X, Y \in V \},
\]

then for \( S \in \mathcal{V}^j_3 \) and \( \psi_j(S) \in \wedge^2 V^* \otimes \text{End}(V) \) defined by

\[
g_p(\psi_j(S)(X, Y)Z, W) = 2g_p(X, jY)S(Z, jW) + 2g_p(Z, jW)S(X, jY) + g_p(X, jZ)S(Y, jW) + g_p(Y, jW)S(X, jZ) - g_p(X, jW)S(Y, jZ) - g_p(Y, jZ)S(X, jW),
\]

\( \psi_j(S) \) is in \( \mathcal{R}(V, g_p) \). With \( s \in \text{End} V \) defined by \( g(sX, Y) = S(X, Y) \), we have

\[
\psi_j(S)(X, Y)Z = -2g_p(X, jY)jsZ - 2S(X, jY)jZ - g_p(X, jZ)jsY - S(X, jZ)jY + S(Y, jZ)jX + g_p(Y, jZ)jsX.
\]

A simple computation shows that the Ricci trace of \( \psi_j(S) \) is zero for all \( S \in \mathcal{V}^j_3 \):

\[
\text{Tr}[Y \mapsto \psi_j(S)(X, Y)Z] = 2g_p(X, sZ) + 2S(X, Z) - g_p(X, jZ) \text{Tr}(js) - S(X, jZ) \text{Tr} j + S(jX, jZ) + g_p(jsX, jZ) = 2S(Z, X) + 2S(X, Z) - S(X, Z) + S(X, Z) = 0
\]

since \( j \) and \( js \) are traceless because \( g(s, \cdot) = S(\cdot, \cdot) \) and \( g(j \cdot, \cdot) \) are skew-symmetric. Hence \( \psi_j(S) \) lies in the space of Weyl tensors.

**Remark 5.9.** In [12], where only the positive definite metric case is discussed, the space \( \psi_j(\mathcal{V}_3^j) \) is one of the 10 irreducible components of the orthogonal Riemann curvature type tensors under the action of the unitary group and is there called \( \mathcal{W}_g \). It can be shown to be the only component with non-vanishing \( \Omega_{R,j}^2 \). For this reason we make the definition below in the pseudo-Riemannian case.

**Definition 5.10.** Put \( \mathcal{W}_g^j = \psi_j(\mathcal{V}_3^j) \) then:

**Lemma 5.11.** If \( R \in \mathcal{W}_g^j \) then \( \Omega_{R,j}^2(X, Y) = -8(n + 1)S(X, jY) \) where \( R = \psi_j(S) \) with \( S \in \mathcal{V}_3^j \).
Proof. If \( R \in \mathcal{V}_0^j \) then \( R = \psi_j(S) \) with \( S \) an antisymmetric bilinear form in \( \mathcal{V}_3^j \), with \( S(X,Y) = -S(Y,X) \) and we have

\[
\Omega_{2}^{R,j}(X,Y) = \text{Tr}(\psi_j(S)(X,Y))j
= -2g_p(X,jY)\text{Tr}(jsj) + 2S(X,jY)\text{Tr}(Id)
+ g_p(X,jsY) + g_p(jsX,J) - g_p(sY,jX) - g_p(Y,jsX)
= 4nS(X,jY) - S(Y,jX) + S(X,jY) - S(Y,jX) + S(X,Y)
= 4(n+1)S(X,jY)
\]

since \( S(Y,jX) = -S(jX,Y) = S(j^2X,jY) = -S(X,jY) \) and also \( jsj = s \) so \( \text{Tr}(jsj) = 0 \). Then

\[
\Omega_{2}^{R,j}(X,Y) = 4(n+1)S(X,j^2Y) - 4(n+1)S(X,jY) = -8(n+1)S(X,jY).
\]

\[\square\]

Let \( R \in \mathcal{R}(V,g_p) \) be any curvature and set \( S_{R,j}^j(X,Y) = \frac{1}{8(n+1)}\Omega_{2}^{R,j}(X,Y) \) then Lemma 5.11 implies \( R = \psi_j(S_R) \) when \( R \in \mathcal{V}_0^j \). We can then define \( P_j(R) = \psi_j(S_{R,j}^j) \) for any \( R \in \mathcal{R}(V,g_p) \). The following Lemma is obvious.

Lemma 5.12. Let \( j \in J(M,g) \) and \( h \in O(V,g_p) \) Then

- \( P_j \) is a linear endomorphism of the space \( \mathcal{R}(V,g_p) \) of curvature tensors with \( P_j^2 = P_j \) and with image in \( \mathcal{V}_0^j \) a subspace of Weyl tensors.
- \( P_j^{-1} = hP_jh^{-1} \) for the natural action of \( O(V,g_p) \) on curvature tensors.

We are now ready to complete the Proof of Proposition 5.8. It is a consequence of Lemma 5.12 that any curvature \( R \in \mathcal{R}(V,g_p) \) with \( \Omega_{2}^{R,j} = 0 \) is in the kernel of the projection \( P_j \) for each \( j \in J(M,g) \), and hence in the intersection of these kernels. This intersection will then be disjoint from the span \( \mathcal{W} \) of the images \( \mathcal{V}_0^j \) of \( P_j \) as \( j \) varies.

From the equivariance property of Lemma 5.12 it follows that \( \mathcal{W} \) is a non-zero \( O(V,g_p) \)-invariant subspace of the Weyl tensors. But the Weyl tensors are irreducible under the full orthogonal group when \( 2n \geq 4 \) [3, page 47] so \( R \) is of Ricci type. When there is an orientation, \( \mathcal{W} \) is a non-zero \( SO(V,g_p) \)-invariant subspace of the Weyl tensors. In dimension \( 2n > 4 \), the Weyl tensors are irreducible under \( SO(V,g_p) \). In dimension 4, we compute in a pseudo-orthonormal oriented basis \( \{e_1, \ldots, e_4\} \) in which \( g = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix} \) and

\[
j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \text{ then any } S \in \mathcal{V}_3^j \text{ has the form } S = \begin{pmatrix} 0 & A & 0 & B \\ -A & 0 & B & 0 \\ 0 & B & 0 & -A \\ -B & 0 & A & 0 \end{pmatrix}. \text{ The corresponding Weyl tensor } \psi_j(S)(X,Y) \text{satisfies}
\]

\[
\begin{align*}
\psi_j(S)(e_1,e_2) &= -\psi_j(S)(e_3,e_4) \\
\psi_j(S)(e_1,e_3) &= \epsilon_1\epsilon_2\psi_j(S)(e_2,e_4) \\
\psi_j(S)(e_1,e_4) &= -\psi_j(S)(e_2,e_3).
\end{align*}
\]
Since the Hodge star dual is given by
\[
\ast (e_1 \wedge e_2) = \epsilon_1 \epsilon_2 \ e_3 \wedge e_4 \\
\ast (e_1 \wedge e_3) = -e_2 \wedge e_4 \\
\ast (e_1 \wedge e_4) = \epsilon_1 \epsilon_2 \ e_2 \wedge e_3,
\]
we see that \(\psi_j(S)\) viewed as a map from \(\Lambda^2 T^*_p M\) into itself, vanishes on the \(\epsilon_1 \epsilon_2\)-eigenspace of the Hodge dual. This shows that \(W\) is the space of anti-self-dual Weyl tensors when \(\epsilon_1 \epsilon_2 = 1\) and the space of self-dual Weyl tensors when \(\epsilon_1 \epsilon_2 = -1\).

This completes the proof. \(\square\)

6. Symplectic structure

We consider a symplectic manifold \((M, \omega)\) of dimension \(2n \geq 4\); we shall use in this section the more classical notation of \(\omega\) (instead of \(g\)) for the symplectic structure. Let \(\Omega\) be a non-degenerate skew-symmetric bilinear form on a real vector space \(V\) of dimension \(2n\). A symplectic frame at a point \(p\) is a map \(\xi: V \to T_p M\) which is a linear isomorphism between \((V, \Omega)\) and \((T_p M, \omega_p)\); as mentioned in section 1 the bundle of symplectic frames \(F(M, \omega) \to M\) is a principal bundle with structure group \(G = Sp(V, \Omega)\), which is isomorphic to the simple split real Lie group \(Sp(2n, \mathbb{R})\) when one has chosen a basis of \(V\) in which the matrix associated to \(\Omega\) is \(\Omega_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\).

The twistor bundle \(J(M, \omega) \to M\) has fibre over the point \(p\) given by all complex structures \(j\) on \(T_p M\) which are compatible with \(\omega_p\) (i.e. \(\omega_p(jX, jY) = \omega_p(X, Y)\) for all \(X, Y \in T_p M\)) and positive (i.e. \(\omega_p(X, JX) > 0\) for all \(0 \neq X \in T_p M\)).

For the construction of the almost complex structures \(J^\pm \nabla\) on the twistor bundle \(J(M, \omega)\), one chooses a symplectic connection \(\nabla\); this is a linear torsion-free connection such that \(\nabla \omega = 0\); it is well known that those exist but are not unique on any symplectic manifold.

**Definition 6.1.** The space \(\mathcal{R}(T_p M, \omega_p)\) of symplectic curvature type tensors at a point \(p\) is isomorphic the subspace \(\mathcal{R}(V, \Omega)\) of elements \(\tilde{R} \in \Lambda^2 V^\ast \otimes \mathfrak{sp}(V, \Omega)\) satisfying the Bianchi identity
\[
\mathcal{R}(V, \Omega) = \left\{ \tilde{R} \in \Lambda^2 V^\ast \otimes \mathfrak{sp}(V, \Omega) \mid \bigoplus_{X,Y,Z} \tilde{R}(X, Y) Z = 0 \right\}.
\]
where \(\mathfrak{sp}(V, \Omega)\) is the Lie algebra of \(Sp(V, \Omega)\) and consists of endomorphisms \(\xi\) of \(V\) with \(\Omega(\xi X, Y) + \Omega(X, \xi Y) = 0\) for all \(X, Y\) in \(V\) or, equivalently, \(\Omega(\xi X, Y)\) is a symmetric bilinear form.

The adjoint representation of \(Sp(V, \Omega)\) on \(\mathfrak{sp}(V, \Omega)\) is isomorphic to the irreducible representation \(S^2 V^\ast\). The following elementary Lemma will be useful in constructing elements of \(\mathcal{R}(V, \Omega)\).

**Lemma 6.2.** Given an element \(A\) of \((\otimes^4 V)^\ast = \otimes^4 V^\ast\) satisfying

1. \(A(X, Y, Z, T)\) is anti-symmetric in \(X\) and \(Y\);
2. \(A(X, Y, Z, T)\) is symmetric in \(Z\) and \(T\);
3. \(\bigoplus_{X,Y,Z} A(X, Y, Z, T) = 0\)

The space \(\mathcal{R}(T_p M, \omega_p)\) of symplectic curvature type tensors at a point \(p\) is isomorphic to the subspace \(\mathcal{R}(V, \Omega)\) of elements \(\tilde{R} \in \Lambda^2 V^\ast \otimes \mathfrak{sp}(V, \Omega)\) satisfying the Bianchi identity
\[
\mathcal{R}(V, \Omega) = \left\{ \tilde{R} \in \Lambda^2 V^\ast \otimes \mathfrak{sp}(V, \Omega) \mid \bigoplus_{X,Y,Z} \tilde{R}(X, Y) Z = 0 \right\}.
\]
then there is a unique element $A \in \mathcal{R}(V, \Omega)$ such that $A(X, Y, Z, T) = \Omega(A(X, Y)Z, T)$.

Given an element $\tilde{R} \in \mathcal{R}(V, \Omega)$ we can form its Ricci trace $\text{Ric}(\tilde{R})$ given by

$$\text{Ric}(\tilde{R})(X, Y) = \text{Tr}(Z \mapsto \tilde{R}(X, Z)Y)$$

which is a symmetric bilinear form on $V$. This gives a linear map $\text{Ric}: \mathcal{R}(V, \Omega) \rightarrow S^2V^*$ which is equivariant for the natural actions of $Sp(V, \Omega)$. Given a symmetric bilinear form $r \in S^2V^*$ let $\rho^r \in \mathfrak{sp}(V, \Omega)$ be defined by

$$\Omega(\rho^r X, Y) = r(X, Y)$$

and $E(r)$ by

$$\Omega(E(r)(X, Y)Z, T) = \frac{-1}{2(n+1)}\left[2\Omega(X, Y)r(Z, T) + \Omega(X, Z)r(Y, T) - \Omega(Y, Z)r(X, T) + r(Y, Z)\Omega(X, T) - r(X, Z)\Omega(Y, T)\right]. \quad (6.1)$$

**Lemma/Definition 6.3.** $E(r)$ is in $\mathcal{R}(V, \Omega)$ and $E: S^2V^* \rightarrow \mathcal{R}(V, \Omega)$ is an equivariant linear map with $\text{Ric}(E(r)) = r$. $E(\text{Ric}(\tilde{R}))$ is called the Ricci component of $\tilde{R}$ and $W(\tilde{R}) = \tilde{R} - E(\text{Ric}(\tilde{R}))$ the Weyl component. If we define

$$\mathcal{E}(V, \Omega) = \{ \tilde{R} \in \mathcal{R}(V, \Omega) \mid E(\tilde{R}) = 0 \}$$

and $W(V, \Omega) = \{ \tilde{R} \in \mathcal{R}(V, \Omega) \mid E(\tilde{R}) = 0 \}$

then both subspaces are irreducible under the action of $Sp(V, \Omega)$ and

$$\mathcal{R}(V, \Omega) = \mathcal{E}(V, \Omega) \oplus W(V, \Omega).$$

**Proof.** To see that (6.1) defines a curvature term we check that the three properties in Lemma 6.2 hold which is straightforward. For the irreducibility see [13]. □

**Definition 6.4.** This gives a decomposition of the curvature $R^\nabla$ of a symplectic connection:

$$R^\nabla = E^\nabla + W^\nabla$$

where $E^\nabla$ is defined in terms of the Ricci tensor $Ric^\nabla(X, Y) = \text{Tr}[Z \rightarrow R^\nabla(X, Z)Y]$; it can be written as

$$E^\nabla(X, Y)Z = \frac{-1}{2(n+1)}\left[2\omega(X, Y)\rho^\nabla Z + \omega(X, Z)\rho^\nabla Y - \omega(Y, Z)\rho^\nabla X + \text{Ric}^\nabla(Y, Z)X - \text{Ric}^\nabla(X, Z)Y\right] \quad (6.2)$$

with $\omega(\rho^\nabla X, Y) = Ric^\nabla(X, Y)$ and of course the Weyl component is $W^\nabla = R^\nabla - E^\nabla$.

A symplectic connection $\nabla$ is said to be of Ricci-type if $W^\nabla = 0$, i.e. if $R^\nabla = E^\nabla$.

Since $\Omega_0(\tilde{0}_0 X, Y) + \Omega_0(X, \tilde{0}_0 Y) = 0$, the $4i$ eigenvalue can only arise in the $W(V, \Omega)$ tensor part and does so, hence $J^\nabla_+ \tilde{\nu}$ is integrable in the symplectic context if and only if $W^\nabla = 0$, as was observed by Vaisman [14].

If the symplectic connection is of Ricci-type, then

$$\text{Im} \quad \left( (\text{Id} - ij) \circ R^\nabla_p ((\text{Id} - ij)X, (\text{Id} - ij)Y) \circ (\text{Id} + ij) \right) \quad (6.3)$$

$$= \frac{-2}{n+1} \left[ ij, -x \otimes B^\nabla_j Y - B^\nabla_j Y \otimes X + Y \otimes B^\nabla_j X + B^\nabla_j X \otimes Y \right]$$
for any \( j \in J(M, \omega)_p \), where \( B = \rho_p^\nabla - j \rho_p^\nabla j \) and \( U = \omega_p(U, \cdot) \), and
\[
\text{Tr}_\nabla (R^\nabla_p (X, Y) \circ j) = -\frac{1}{n+1} \left( \omega_p(X, Y) \text{Tr}(\rho_p^\nabla \circ j) + \omega_p((\rho_p^\nabla \circ j + j \circ \rho_p^\nabla) X, Y) \right). \tag{6.4}
\]

**Proposition 6.5.** The almost complex structure \( J^\nabla \) on the twistor space \( J(M, \omega) \) of a symplectic manifold \( (M, \omega) \), of dimension \( 2n \geq 4 \), defined using a symplectic connection \( \nabla \), is integrable if and only if the curvature of \( \nabla \) is of Ricci-type, i.e. \( W^\nabla \) vanishes.

The almost complex structure \( J^\nabla \) is never integrable.

If the symplectic connection is of Ricci-type, then:

- the image of the Nijenhuis tensor associated to \( J^\nabla \) at any point \( j \in J(M, \omega) \), is the whole horizontal tangent space plus the part of the vertical tangent space given by the endomorphisms defined by formula (6.3);
- the closed 2-form on \( J(M, \omega) \) associated by (3.1) to \( \nabla \), \( \omega^{J(M, \omega, \nabla)} \), is symplectic if and only if
  \[
  \text{Tr}_\nabla (\rho_p^\nabla \circ j) \text{Id} + (\rho_p^\nabla \circ j + j \circ \rho_p^\nabla)
  \]
  has a vanishing kernel for all \( p \in M \) and all \( j \in J(M, \omega)_p \);
- the almost complex structures \( J^\nabla \) are compatible with the symplectic 2-form in the sense of equation (3.3).

The remainder of this section is devoted to the study of this compatibility (equation (3.4)) for a general symplectic connection. We define (as was done in Definitions 5.6 and 5.7) for an element \( R \in \mathcal{R}(V, \Omega) \) and a \( j \in J(V, \Omega) \) let \( \Omega^R_{1J}(X, Y) = \text{Tr}_\nabla (R(X, Y) \circ j) \) for \( X, Y \in V \) and let \( \Omega^R_{2J}(X, Y) = \Omega^R_{1J}(jX, jY) - \Omega^R_{1J}(X, Y) \). The compatibility condition becomes again \( \Omega^R_{2J} = 0 \) for all \( j \).

**Definition 6.6.** For \( j \in J(V, \Omega) \) we set
\[
\mathcal{V}(V, \Omega, j) = \{ S \in \Lambda^2(V^*) \mid S(jX, jY) = -S(X, Y) \}.
\]

**Remark 6.7.** As a representation of \( U(V, \Omega, j) \), \( \mathcal{V}(V, \Omega, j) \) is a real irreducible subspace of \( \Lambda^2(V^*) \) and its complexification is \( \Lambda^{(2,0)} \oplus \Lambda^{(0,2)} \).

**Definition 6.8.** For \( S \in \mathcal{V}(V, \Omega, j) \) define \( R(S, j)(X, Y) Z \in V \) by
\[
\Omega(R(S, j)(X, Y) Z, T) = -2\Omega(Z, jT) S(X, jY) + \Omega(X, jZ) S(Y, jT) + \Omega(X, jT) S(Y, jZ) - \Omega(Y, jT) S(X, jZ) - \Omega(Y, jZ) S(X, jT) \tag{6.5}
\]
for all \( T \in V \).

The left hand side \( \Omega(R(S, j)(X, Y) Z, T) \) is clearly antisymmetric in \( X \) and \( Y \), symmetric in \( Z \) and \( T \) and satisfies the Bianchi identity \( \sum_{X,Y,Z} R(S, j)(X, Y) Z = 0 \). A straightforward calculation shows it is Ricci flat and so in \( \mathcal{W}(V, \Omega) \), moreover we have \( \Omega^R_{2K}(X, Y) = -8(n - 1) S(X, jY) \). In summary:

**Lemma 6.9.** Formula (6.5) defines an element \( R(S, j) \in \mathcal{R}(V, \Omega) \) which is of Weyl type and \( S \mapsto R(S, j) \) is a \( U(V, \Omega, j) \) equivariant map \( \mathcal{V}(V, \Omega, j) \to \mathcal{R}(V, \Omega) \) with image in
the Weyl tensors. Moreover

\[ S(X, Y) = \frac{1}{8(n-1)} \Omega^R_{(S,j),j} (X, jY). \]

Under the action of \( h \in \text{Sp}(V, \Omega) \) we have

\[ h \cdot (\mathcal{V}(V, \Omega, j)) = \mathcal{V}(V, \Omega, hjh^{-1}) \quad \text{and} \quad h \cdot (R(S, j)) = (h \cdot R)(h \cdot S, hjh^{-1}). \]

**Definition 6.10.** For arbitrary \( R \in \mathcal{R}(V, \Omega) \) we define

\[ S^{R, j}(X, Y) = \frac{1}{8(n-1)} \Omega^R_{(S,j),j} (X, jY) \]

and

\[ P^j(R) = R(S^{R, j}, j). \]

**Lemma 6.11.** \( P^j \) is a linear map from \( \mathcal{R}(V, \Omega) \) to itself satisfying \( P^j \circ P^j = P^j \) and with image in the curvatures of Weyl type. \( j \mapsto P^j \) is \( \text{Sp}(V, \Omega) \)-equivariant.

**Proposition 6.12.** Let \((M, \omega)\) be a symplectic manifold of dimension \( 2n \geq 4 \) with a symplectic connection \( \nabla \). Then the closed 2-form \( \omega^{(M, \omega, \nabla)} \) is of type \((1, 1)\) for each of the \( J^\pm \) (i.e. equation (3.4) is satisfied) if and only if the curvature \( R^{\nabla} \) is of Ricci type.

**Proof.** If \( R^{\nabla} \) is of Ricci type then \( R^{\nabla} = E^{\nabla} \) and, as mentioned in Proposition 6.5, a direct calculation involving equation (6.4) shows that \( \Omega^R_{2, j} = 0 \) for all \( j \).

Conversely, assume \( \Omega^R_{2, j} = 0 \) for all \( j \) then as in the pseudo-Riemannian case this means \( R^{\nabla} \) is in the kernel of \( P_j \) for all \( j \) and by equivariance, replacing \( j \) by \( hjh^{-1} \) it follows that \( R^{\nabla} \) is in the intersection \( \cap_{h} \text{Ker} P_{hjh^{-1}} \) which is a subspace of \( \mathcal{R}(V, \Omega) \) disjoint from the span of the images of the \( P_{hjh^{-1}} \). This is a non-zero \( \text{Sp}(V, \Omega) \)-invariant subspace of the Weyl curvature tensors and by irreducibility must be the whole of the Weyl curvatures. Hence \( R^{\nabla} \) has no Weyl curvature so is of Ricci type. \( \square \)
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