Investigation of the slope stability problem using the Material Point Method

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Abstract. The Finite Element Method (FEM) has become a standard tool in engineering, although its shortcoming in large deformation analysis is apparent. Mesh distortions are common in this area. In order to be able to investigate applications including large deformations or displacements of the material, advanced numerical methods are needed. In this study we introduce the Material Point Method (MPM) as a powerful tool to simulate applications including large deformations. This capability of MPM plays an important role in the study of land and rockslides, avalanches, mudflow, etc. Here we will present a brief introduction on the governing equations and the solution procedure of MPM. Based on the presented formulation and using a two dimensional program developed, a slope stability problem is solved to show the capability of this method to be used in studying geohazard related phenomena. At the end some concluding remarks and recommendations are presented.

1. Introduction

The Finite Element Method (FEM) is a standard numerical tool in different fields of engineering especially in the field of geotechnical engineering. Some applications in this field include large deformations or displacements of the soil. FEM is not able to simulate these large deformations or displacements and suffers from what is called the “mesh distortion”. Examples of such applications are land and rockslides, avalanches, mudflow, etc. During the simulation of such applications the finite element mesh becomes so distorted that the determinant of the Jacobian matrix becomes negative. Remedies like using updated Lagrangian formulation introduce errors during the mapping process [1] and remeshing is computationally expensive.

In order to study these applications other suitable numerical methods should be used. Available numerical methods for the simulation of large deformations can be categorized into three major groups. In the first group, the methods use the advantages of the both Lagrangian and Eulerian descriptions of motion while avoiding their drawbacks. The Arbitrary Lagrangian Eulerian (ALE) method [2] and the Coupled Eulerian Lagrangian (CEL) method [3] are categorized in this group. Second group includes the mesh free methods like the Element Free Galerkin (EFG) method [4] and the Smooth Particle Hydrodynamic (SPH) method [5]. The last group contains the mesh based particle methods. The Particle in Cell (PIC) method [6], Fluid Implicit Particle (FLIP) method [7] and the Material Point Method (MPM) [8,9] are amongst others in this group.
MPM discretizes the continuum using the material points or particles and discretizes the space using an Eulerian background fixed mesh on which the equations of motion are solved (figure 1). This mesh should cover the whole space where the material may go during the simulation process. Material points are the integration points which can move during the simulation and using this property MPM is able to analyze large deformations or displacements of the material. Particles carry all the physical information (e.g. stresses, strain, etc.) during the simulation process and no permanent data are stored on the mesh.

Working procedure of MPM for one time step consists of three phases. First is the initialization phase where all the data are mapped from particles to the nodes of the background mesh using the nodal shape functions. Then is the Lagrangian phase in which the equations of motion are solved on the mesh. At the end is the convective phase. In this phase information are mapped back from the mesh to the material points and updates the data of the particles. Background mesh goes back to its original position (figure 2).

For the first time Sulsky et al. [8] applied the PIC method from fluid to solid mechanics and called it the material point method [9]. Bardenhagen et al. [10] introduced a frictional contact algorithm to the method based on the Coulomb friction law. Bardenhagen and Kober [11] presented the Generalized Interpolation Material Point Method (GIMP) to avoid the grid crossing error presented in the framework of original MPM. Quasi static formulation of MPM was first developed by Beuth et al. [12]. The abovementioned formulations use the explicit time integration scheme whereas Cummins and Brackbill [13], Guilkey and Weiss [14], Sulsky and Kaul [15] and Stolle et al. [16] presented unconditionally stable implicit time integration schemes for MPM. As MPM showed its capability to simulate large deformations especially in the field of geotechnical engineering, this method is used to study several applications in this area [17,18].

After the short introduction in chapter 1, an example of a one dimensional spring and mass system is solved in chapter 2 as a validation for the MPM formulation which is thoroughly discussed in appendix A. Next an example of a box which slides on a steep surface is simulated in chapter 2 as a validation for the contact algorithm which is discussed in appendix B. The problem of slope stability is analyzed using the developed code by the author based on the explained algorithms in chapter 3. At the end some concluding remarks are presented in chapter 4.

Figure 2. MPM working procedure for one time step.

2. One dimensional spring and mass system
To validate the complete formulation of MPM presented in appendix A, a one dimensional spring and mass example is simulated in this chapter using the code developed by the author. The spring is 1 m long and has a cross section of 0.1 m² and elastic modulus of $5 \times 10^3$ kN/m². The mass has a
length of 5 cm and the same cross section as the spring with a density of 1000 kg/m^3. Linear elastic material behavior is assumed for the spring and mass. Ten particles per element were used and the gravitational acceleration is assumed to be \( g = 10 \, m/s^2 \). Density of the spring is small in comparison with the mass to represent the massless spring.

![Analytical and numerical solution of the one dimensional spring and mass system.](image)

For this problem the analytical solution exists which can be written as

\[
y(t) = F_0 \frac{k}{k} [1 - \cos(\omega t)].
\]

(1)

By introducing the corresponding values it can be written that

\[
y(t) = 10^{-3} [1 - \cos(100t)].
\]

(2)

The simulation results are shown in figure 3 together with the analytical results. A good agreement can be seen between the curves and MPM can catch the analytical results very well.

3. Sliding of a box on the steep surface

In order to show the effectivity of the described contact algorithm from appendix B, a simple example of a box sliding on a steep surface on which the coefficient of friction is defined, is solved using the code developed by the author and the results are compared with the analytical ones.

Linear elastic material behavior is assumed for both the box and the surface. Ten particles per element were used for the MPM simulation. The surface is assumed to be 10 cm thick and 1 m long with an inclination angle of 30°. The box has a length of 15 cm and a height of 10 cm with a density of 200 kg/m^3. For this problem the analytical solution exists by assuming the coefficient of friction between the box and the surface to be 0.4 and the initial velocity and displacement to be zero which states

\[
x(t) = 0.768 t^2
\]

(3)

and for the coefficient of friction of 0.3 and the same initial conditions

\[
x(t) = 1.2 t^2.
\]

(4)
The results from the MPM simulation together with the analytical results are shown in figure 4, which shows a good agreement between them. With the coefficient of friction of 0.3 larger displacements are obtained than the case with the coefficient of friction of 0.4 which was expected from the analytical solution as well. Figure 5 shows the position of the box after one second of displacement before the box hits the wall for the case with the coefficient of friction equal to 0.4.

4. Stability of the slope
To show the applicability of the MPM to be used in the field of geotechnical engineering for the simulation of large deformations or displacements of the soil, the problem of slope stability is analyzed using the code developed by the author based on the described methods in the previous

![Figure 4. Analytical and numerical solution of the box sliding on the steep surface example.](image4.png)

![Figure 5. Position of the box after one second of movement.](image5.png)
chapters. In order to overcome the volumetric locking that happens in low order elements (here linear triangular elements) the enhanced volumetric strain method suggested by Detournay & Dzik [19] is used. This method is based on averaging the volumetric part of the strain on the nodes then averaging these values to be used for the particles inside the elements. During the averaging process volumetric part of the strain is smeared using the values from neighboring elements and this method helps to relax the elements from volumetric locking.

![Figure 6](image)

**Figure 6.** Geometry and boundary conditions of the soil.

Geometry and boundary conditions of the slope are shown in figure 6. The elastic-perfectly plastic Mohr-Coulomb material model is used to describe the behavior of the soil. In order to focus on the capabilities of MPM and exclude the effects of different material models on the final results, the simple failure criteria of Mohr-Coulomb is adopted. The soil of the slope is assumed to be sand with the friction angle of $30^\circ$, cohesion of 1 kpa, module of elasticity of $4 \times 10^4 \text{kN/m}^2$, Poisson ratio of 0.3 and density of $1800 \text{kg/m}^3$. Four particles per element were used for the MPM simulation. Total displacement of the particles is shown in figure 7. As can be seen in figure 7 the slope failure happens due to the self-weight of the slope under gravitational acceleration. The failure zone starts from the toe of the slope and goes up to the surface in a circular shape, an attitude which can also be seen in simulations and lab tests from other publications and case studies [20]. The failed soil slides down the slope and stops in the downhill.

In the nature during this sliding, the failed soil mass causes damage and destroys the buildings and structures in its way. How far the sliding soil will travel, can be predicted to prevent the damages caused during this phenomenon.

![Figure 7](image)

**Figure 7.** Total displacement of the particles.
5. Summary and conclusion

In the present work MPM is introduced as a powerful numerical tool which is able to simulate large deformations or displacements of the material without the mesh tangling problem which is present in Lagrangian FEM formulations. The governing equations of the method are presented briefly in appendix A and validated using a simple one dimensional spring and mass example. The contact algorithm based on the Coulomb friction law is described in appendix B and validated using a simple example of a box which slides on a steep surface for which a coefficient of friction is defined. Good agreement between the simulation results and the analytical solutions can be seen in the examples. To show the applicability of the method in the field of geotechnical engineering the problem of slope stability is analyzed. The simulation results show the potential of this numerical method to be used in order to study the large deformations and displacements of continua in the field of geotechnical engineering and scrutiny the geohazards related phenomena like land and rockslides, avalanches, mudflow, etc.

Further research on the development of the MPM is currently done at the Institute of Geotechnical Engineering (IGS) of the University of Stuttgart.

Appendix A: Governing equations of MPM

Starting point is the momentum balance equation

\[ \rho \ddot{u} = \nabla \cdot \sigma + \rho g \] \hspace{1cm} (A.1)

where \( \rho \) is the material density, \( u \) is the displacement vector, a superposed dot declares differentiation with respect to time, \( \sigma \) is the Cauchy stress tensor and \( g \) is the gravitational acceleration vector. The surface traction acting on the external boundary is denoted by \( t \) and \( n \) is the outward unit normal of the boundary.

Applying the virtual work principle to a domain of volume \( V \) which is surrounded by boundary \( S \) and using the integration by parts and the divergence theorem, the weak form of the momentum equation can be written as

\[
\int_V \delta u^T \rho \ddot{u} dV = -\int_V \delta \varepsilon^T \sigma dV + \int_V \delta u^T \rho g dV + \int_S \delta u^T t dS \] \hspace{1cm} (A.2)

where \( \delta \) denotes a virtual quantity, \( \varepsilon \) is the strain tensor and the superscript \( T \) specifies the transpose.

The displacement field is discretized using the nodal shape functions \( N \) and the nodal displacements \( a \)

\[
u = Na \] \hspace{1cm} (A.3)

which leads to the definition of the strain tensor

\[ \varepsilon = B a \hspace{1cm} B = LN \] \hspace{1cm} (A.4)

where \( B \) is the strain displacement matrix and \( L \) is the linear differential operator. Substituting equation (3) and equation (4) into equation (2) and doing some mathematical operations, the well-known equilibrium equation will be derived

\[ M \ddot{a} = (F^{ext} - F^{int}) \] \hspace{1cm} (A.5)
where $M$ is the mass matrix

$$M = \int_V \rho N^T N dV$$  \hspace{1cm} (A.6)$$

$F^{ext}$ is the external force vector

$$F^{ext} = \int_V \rho N^T g dV + \int_S N^T t dS$$  \hspace{1cm} (A.7)$$

and $F^{int}$ is the internal force vector

$$F^{int} = \int_V B^T \sigma dV.$$  \hspace{1cm} (A.8)$$

To increase the computational efficiency lumped mass matrix is used which is the row some of the entries of the consistent mass matrix. In the original formulation of MPM the density field is discretized using the Dirac delta function i.e.

$$\rho_{(x_i)} = \sum_{p=1}^{n_p} m_p \delta(x_i - x_i^p)$$  \hspace{1cm} (A.9)$$

where $x_i$ is an arbitrary position vector and $x_i^p$ is the position vector at particle $p$. Using Dirac delta function to discretize the density field indicates that the mass of the continuum is approximated to be in the particles and as long as the mass of the particles do not change during the simulation, conservation of mass is considered automatically.

Here the explicit time integration scheme of the discretized momentum equation is adopted via applying an Euler forward integration scheme

$$\ddot{\mathbf{a}}^t = [M]^t \dot{\mathbf{a}}^t, \quad \dot{\mathbf{a}}_p^{t+\Delta t} = \dot{\mathbf{a}}_p^t + \Delta t N_p \ddot{\mathbf{a}}^t$$  \hspace{1cm} (A.10)$$

where $\Delta t$ is the time increment, $\dot{\mathbf{a}}_p^{t+\Delta t}$ and $\dot{\mathbf{a}}_p^t$ are the particle velocities at time $t$ and $t + \Delta t$ respectively and $\ddot{\mathbf{a}}^t$ is the nodal accelerations at time $t$.

The nodal velocities $\ddot{\mathbf{a}}^{t+\Delta t}$ at time $t + \Delta t$ are then calculated from the updated particles velocities, solving the following equation

$$M_L^t \ddot{\mathbf{a}}^{t+\Delta t} \approx \sum_{p=1}^{n_p} m_p N_p^T \ddot{\mathbf{a}}_p^{t+\Delta t}.$$  \hspace{1cm} (A.11)$$

Vector of nodal displacements is calculated by using an Euler backward integration method. Then the position of particles are updated

$$\Delta \mathbf{a}^{t+\Delta t} = \Delta t \ddot{\mathbf{a}}^{t+\Delta t}, \quad \mathbf{X}_p^{t+\Delta t} = \mathbf{X}_p^t + N_p \Delta \mathbf{a}^{t+\Delta t}.$$  \hspace{1cm} (A.12)$$
The time step size of the explicit time integration scheme is restricted by the CFL condition in which
\[
\Delta t = \alpha_c \Delta t_{cr}, \quad \Delta t_{cr} = \frac{h_{\text{min}}}{c_d}
\]  
(A.13)

where \( h_{\text{min}} \) is the minimum representative distance in an element, \( c_d \) is the compression wave speed and \( \alpha_c \) is called the Courant’s number and is a value between one and zero.

**Appendix B: Contact algorithm for the material point method**

Due to the single valued velocity field in the framework of the original MPM formulation, interpenetration of the particles are excluded automatically. York et al. [21] proposed that when two bodies are colliding each other conventional formulation of MPM is used to avoid interpenetration of the particles and when the two bodies are separating from each other no restrictions should be applied in order to allow the free separation. Bardenhagen et al. [10] proposed a frictional contact algorithm based on the Coulomb friction law for MPM. His formulation is widely used and will be employed in this work as well. Since the formulation of frictional contact algorithm is proposed, many modifications to this algorithm are introduced by other researchers [20,22].

Contact algorithm of Bardenhagen et al. [10] for MPM is formulated on the nodes of the background mesh. First equations of motion for each body and for the whole system should be solved separately. Contact node is detected by comparing the nodal velocities of each body and the nodal velocities of the whole system. In figure 1 node p is a contact node while for this node states
\[
\dot{a}_{A,p}^{t+\Delta t} \neq \dot{a}_p^{t+\Delta t}
\]  
(B.1)

where \( \dot{a}_{A,p}^{t+\Delta t} \) is the velocity of node p from body A at time \( t + \Delta t \) and \( \dot{a}_p^{t+\Delta t} \) is the velocity of node p from the whole system (bodies A and B in figure 1) at time \( t + \Delta t \) (whereas point q is not a contact node). Contacting nodes on which the bodies are separating from each other do not need any corrections and the separation without any constraint should be allowed. For the contacting nodes on which the bodies are approaching each other it should be checked if the bodies are sliding on each other or not. This is done by comparing the tangential force in the contact node between the bodies and the maximum allowed tangential force by the Coulomb friction law. If the tangential force between bodies is less than the maximum force, the bodies are sticking to each other and no restriction is needed to be applied. But when the tangential force is more than the maximum value, the bodies slide on top of each other and two restrictions are needed to be applied. First restriction is applied on the normal velocities of each body to prevent the interpenetrations of the particles and the second one is applied on the tangential velocities of each body to satisfy the Coulomb friction law.

The same restrictions can also be applied by introducing forces into the equation of motion instead of correcting the velocities. In this work the correction of predicted velocities is adopted.

After correcting the velocities on the nodes of the background mesh, acceleration vector for each body is calculated and the conventional solution process of MPM is followed.

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