CLASSICAL AND QUANTUM SOLUTIONS OF CONFORMALLY RELATED MULTIDIMENSIONAL COSMOLOGICAL MODELS

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Abstract
We consider multidimensional universes $M = \mathbb{R} \times M_1 \times \ldots \times M_n$ with $D = 1 + \sum_{i=1}^{n} d_i$, where the $M_i$ of dimension $d_i$ have constant curvature, being compact for $i > 1$. For Lagrangian models $L(R, \phi)$ on $M$ which depend only on Ricci curvature $R$ and a scalar field $\phi$, there exists an explicit description of conformal equivalence, with the minimal coupling model and the conformal coupling model as distinguished representatives of a conformal class. For the conformally coupled model we study classical solutions and their relation to solutions in the equivalent minimally coupled model. The domains of equivalence are separated by certain critical values of the scalar field $\phi$. Furthermore the coupling constant $\xi$ of the coupling between $\phi$ and $R$ is critical at both, the minimal value $\xi = 0$ and the conformal value $\xi_c = \frac{D-2}{4(D-1)}$. In different noncritical regions of $\xi$ the solutions behave qualitatively different. For vanishing potential of the minimally coupled scalar field we find a multidimensional generalization of Kasner’s solution. Its scale factor singularity vanishes in the conformal coupling model. Static internal spaces in the minimal model become dynamical in the conformal one. The nonsingular conformal solution has a particular interesting region, where internal spaces shrink while the external space expands. While the Lorentzian solution relates to a creation of the universe at finite scale, its Euclidean counterpart is an (instanton) wormhole. Solving the Wheeler de Witt equation we obtain the quantum counterparts to the classical solutions. A real Euclidean quantum wormhole is obtained in a special case.

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1 Introduction

Recently gravitational models in multidimensional universes $M = \mathbb{R} \times M_1 \times \cdots \times M_n$, with $D = 1 + \sum_{i=1}^{n} d_i$, have received increasing interest. The geometry might be minimally coupled to a spatially homogeneous scalar field $\Phi$ with a potential $U(\Phi)$. This class of minisuperspace models is rich enough to study the relation and the imprint of internal compactified extra dimensions (like in Kaluza-Klein models\textsuperscript{1,2}) on the external space-time.

In order to obtain quantum cosmological solutions, within the framework of canonical quantum gravity, the Wheeler de Witt (WdW) equation has to be solved on the minisuperspace corresponding to multidimensional geometry minimally coupled to a spatially homogeneous scalar field $\Phi$ with a potential $U(\Phi)$.

In Ref. 3 a criterion of integrability for classes of multidimensional geometry has been found by analogy with Toda systems. E.g. when there is only one factor space, say $M_1$, which is non Ricci flat, the system is integrable. Furthermore in Refs. 4 and 5 interesting quantum solutions have been found, including quantum wormholes\textsuperscript{6}.

In Sec. 2 the theory for classical multidimensional universes $M = \mathbb{R} \times M_1 \times \cdots \times M_n$ with $D = 1 + \sum_{i=1}^{n} d_i$, is sketched, following the setup in Refs. 2, 7 and 8, where we are mainly interested in the case in which the $d_i$-dimensional spaces $M_i$ are of constant curvature and the internal spaces, with $i > 1$, are compact.

A special emphasis is put to compare existing natural time gauges\textsuperscript{9}, given by the choices of i) the synchronous time $t_s$ of the universe $M$, ii) the conformal time $\eta_i$ of a universe with the only spacial factor $M_i$, iii) the mean conformal time $\eta_i$, given differentially as some scale factor weighted average of $\eta_i$ for all $i$ and iv) the harmonic time $t_h$, which will be used as specially convenient in calculations on minisuperspace, since in this gauge the minisuperspace lapse function is $N \equiv 1$.

For a multidimensional universe $M$ the pure geometrical Einstein-Hilbert theory with Gibbons-Hawking boundary term, allowing a description on a minisuperspace, is minimally coupled to a spatially homogeneous scalar field $\Phi$ with a potential $U(\Phi)$, and this Lagrangian model is equivalent to a new one on an enlarged minisuperspace. The motivation for such an introduction is given by the request for an inflational cosmology.

In Sec. 3 we examine conformal transformations of Lagrangians on a $D$-dimensional space-time, first generally and then consider as example of special interest the conformal transformation between a model with minimally coupled scalar field and an equivalent conformal model with a conformally coupled scalar field, thus generalizing previous results from Refs. 10 and 11 obtained for $n = 1$ and $D = 4$. We compare the solutions of the minimal model to their conformal counterpart.
In the Ricci flat case with vanishing potential in the minimal coupling model a generalized Kasner solution is obtained. In the special case of only statical internal spaces in the minimal model, we get particularly interesting Lorentzian and Euclidean solutions in the conformal model. Internal spaces which are static in the minimal model show interesting dynamics against external space in the conformal model.

While in the minimal model time is harmonic, it is no longer harmonic in the conformal model. It is a characteristic feature that natural time gauges are not preserved under conformal transformation of Lagrangian models. The synchronous time pictures of the minimal and conformal coupling models are calculated.

In Sec. 4 we will investigate the quantum analogue of the classical solution for the particular model of Sec. 3 with all $M_i$ Ricci flat, and especially the degenerate case corresponding classically to static internal spaces in the minimal (coupling) model. We discuss also the quantum wave function corresponding to classical conformal model.

Examination of the transition to the Euclidean region provides in the case of real geometry a quantum wormhole solution according to the boundary conditions of Ref. 6.

In the Conclusion we resume the perspective of the present results.

2 Classical Multidimensional Universes

We consider a universe described by a (Pseudo-) Riemannian manifold

$$M = \mathbb{R} \times M_1 \times \ldots \times M_n,$$

with first fundamental form

$$g \equiv ds^2 = -e^{2\gamma} dt \otimes dt + \sum_{i=1}^{n} a_i^2 ds_i^2,$$  \hspace{1cm} (2.1)

where $a_i = e^{\beta_i}$ is the scale factor of the $d_r$-dimensional space $M_i$. In the following we assume $M_i$ to be an Einstein space, i.e. its first fundamental form

$$ds_i^2 = g_{kl}^{(i)} dx_{(i)}^k \otimes dx_{(i)}^l$$  \hspace{1cm} (2.2)

satisfies the equations

$$R_{kl}^{(i)} = \lambda_i g_{kl}^{(i)},$$  \hspace{1cm} (2.3)

and hence

$$R^{(i)} = \lambda_i d_i,$$  \hspace{1cm} (2.4)
Here the Ricci tensor and scalar are defined as usual by
\[ R_{\mu\nu} := R^\lambda_{\mu\lambda\nu} \quad \text{and} \quad R := R^\mu_\mu. \] (2.5)

Especially we will keep in mind the interesting subcase where \( M_i \) is of constant curvature. In this case
\[ ds^2_i = \frac{1}{(1 + \frac{1}{4}K_ir_i^2)^2} \sum_{k=1}^{d_i} dx^k_{(i)} \otimes dx^k_{(i)}, \] (2.6)
with radial variable \( r_i = \sqrt{\sum_{k=1}^{d_i} (x^k_{(i)})^2} \) and constant sectional curvature, normalized with \( K_i = \pm 1 \) for positive and negative \( K_i \) respectively. In the flat case \( K_i = 0 \). Then the Riemann tensor of \( M_i \) is
\[ R^{(i)}_{klmn} = K_i (g^{(i)}_{km}g^{(i)}_{ln} - g^{(i)}_{kn}g^{(i)}_{lm}). \] (2.7)

Ricci tensor and scalar are then given by Eq. (2.3) and (2.4) with
\[ \lambda_i \equiv K_i(d_i - 1). \] (2.8)

For the metric (2.1) the Ricci scalar curvature of \( M \) is
\[ R = e^{-2\gamma} \left\{ \sum_{i=1}^{n} (d_i\dot{\beta}^i)^2 + \sum_{i=1}^{n} d_i[(\dot{\beta}^i)^2 - 2\gamma\dot{\beta}^i + 2\ddot{\beta}^i] \right\} + \sum_{i=1}^{n} R^{(i)} e^{-2\beta^i}. \] (2.9)

Let us now consider a variation principle with the action
\[ S = S_{EH} + S_{GH} + S_M \] (2.10)
where \( S_M \) is a matter term,
\[ S_{EH} = \frac{1}{2\kappa^2} \int_M \sqrt{|g|} R \, dx \]
is the Einstein-Hilbert action and
\[ S_{GH} = \frac{1}{\kappa^2} \int_{\partial M} \sqrt{|h|} K \, dy \]
is the Gibbons-Hawking boundary term\(^{12}\), where \( K \) is the trace of the second fundamental form, which just cancels second time derivatives in the equation of motion.

Let us define a metric on the minisuperspace, which is spanned here in the coordinates \( \beta^i \). We set
\[ G_{ij} := d_i\delta_{ij} - d_id_j, \] (2.11)
thus defining the components \( G_{ij} \) of the minisuperspace metric
\[ G = G_{ij} d\beta^i \otimes d\beta^j. \] (2.12)
Furthermore we define the minisuperspace lapse function by

\[ N := e^\gamma - \sum_{i=1}^{n} d_i \beta^i \]  

(2.13)

and a minisuperspace potential \( V = V(\beta^i) \) via

\[ V := -\frac{\mu}{2} \sum_{i=1}^{n} R^{(i)} e^{-2\beta^i + \gamma + \sum_{j=1}^{n} d_j \beta^j}, \]

(2.14)

where

\[ \mu := \kappa^{-2} \prod_{i=1}^{n} \sqrt{\det g^{(i)}}. \]  

(2.15)

Then the variational principle of (2.10) is equivalent to a Lagrangian variational principle in minisuperspace,

\[ S = \int L dt, \quad \text{where} \quad L = N\{\frac{\mu}{2} N^{-2} G_{ij} \dot{\beta}^i \dot{\beta}^j - V\}. \]  

(2.16)

Here \( \mu \) is the mass of a classical particle in minisuperspace. Note that \( \mu^2 \) is proportional to the volumes of spaces \( M_i \).

Next let us compare different choices of time \( \tau \) in Eq. (2.1). The time gauge is determined by the function \( \gamma \). There exist few natural time gauges from the physical point of view.

i) The \textit{synchronous time gauge}

\[ \gamma \equiv 0, \]  

(2.17)

for which \( t \) in Eq. (2.1) is the proper time \( t_s \) of the universe. The clocks of geodesically comoved observers go synchronous to that time.

ii) The \textit{conformal time gauges} on \( \mathbb{R} \times M_i \subset M \)

\[ \gamma \equiv \beta^i, \]  

(2.18)

for which \( t \) in Eq. (2.1) is the conformal time \( \eta_i \) of \( M_i \) for some \( i \in \{1, \ldots, n\} \), given by

\[ d\eta_i = e^{-\beta^i} dt_s. \]  

(2.19)

iii) The \textit{mean conformal time gauge} on \( M \):

For \( n > 1 \) and \( \beta^2 \neq \beta^1 \) on \( M \) the usual concept of a conformal time does no longer apply. Looking for a generalized “conformal time” \( \eta \) on \( M \), we set

\[ d := D - 1 = \sum_{i=1}^{n} d_i \]  

(2.20)
and consider the gauge
\[ \gamma \equiv \frac{1}{d} \sum_{i=1}^{n} d_i \beta^i, \] (2.21)
which yields a time \( t \equiv \eta \) given by
\[ d\eta = \left( \prod_{i=1}^{n} a_i^{d_i} \right)^{-1/d} dt_s. \] (2.22)

Here \( \prod_{i=1}^{n} a_i^{d_i} \) is proportional to the volume of \( d \)-dimensional spacial sections in \( M \) and the relative time scale factor
\[ \left( \prod_{i=1}^{n} a_i^{d_i} \right)^{1/d} = e^{\frac{1}{d} \sum d_i \beta^i} \] (2.23)
is given by a scale exponent, which is the dimensionally weighted arithmetic mean of the spacial scaling exponents of spaces \( M_i \). It is
\[ (dt_s)^d = e^{\sum d_i \beta^i} d\eta^d. \] (2.24)

Since on the other hand by Eq. (2.19) we have
\[ (dt_s)^d = \otimes_{i=1}^{n} \left( e^{\beta^i} d\eta_i \right)^{d_i}, \] (2.25)
together with Eq. (2.24) we yield
\[ (d\eta)^d = e^{-\sum d_i \beta^i} \otimes_{i=1}^{n} \left( e^{\beta^i} d\eta_i \right)^{d_i}. \] (2.26)

So the time \( \eta \) is a mean conformal time, given differentially as a dimensionally scale factor weighted geometrical tensor average of the conformal times \( \eta_i \). An alternative to the mean conformal time \( \eta \) is given by a similar differential averaging like Eq. (2.26), but weighted by an additional factor of \( e^{(1-d) \sum d_i \beta^i} \). This is gauge is described in the following.

iv) The harmonic time gauge
\[ \gamma \equiv \gamma_h := \sum_{i=1}^{n} d_i \beta^i \] (2.27)
yields the time \( t \equiv t_h \), given by
\[ dt_h = \left( \prod_{i=1}^{n} a_i^{d_i} \right)^{-1} dt_s = \left( \prod_{i=1}^{n} a_i^{d_i} \right)^{-\frac{l_{-d}}{d}} d\eta. \] (2.28)

In this gauge any function \( \varphi \) with \( \varphi(t,y) = t \) is harmonic, i.e. \( \Delta[g] \varphi = 0 \), and the minisuperspace lapse function is \( N \equiv 1 \). The latter is especially convenient when we work in minisuperspace.
Here and in the following for any \( x \) we set
\[
\dot{x} := \frac{\partial x}{\partial t_h}
\] (2.29)
so the dot denotes the partial derivative w.r.t. harmonic time.

Then the equations of motion from Eq. (2.16) yield
\[
\mu G_{ij} \ddot{\beta}^j = -\frac{\partial V}{\partial \beta^i}
\] (2.30)
plus the energy constraint
\[
\frac{\mu}{2} G_{ij} \ddot{\beta}^i \dot{\beta}^j + V = 0.
\] (2.31)

Now we consider the Einstein equations for a universe (2.1), given by
\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}
\] (2.32)
with energy momentum tensor \( T_{\mu\nu} \) corresponding to \( S_M \). Let us assume we that the energy momentum tensor \( T_{\mu\nu} \) is of perfect fluid type, depending only on the matter density \( \rho \) in the universe and on the pressures \( p_i \) in the spaces \( M_i \). With equations of state
\[
p_i = p_i(\rho)
\] (2.33)
for the pressures \( p_i \) in \( M_i \) in terms of the matter density
\[
\rho = \rho(\beta^i),
\] (2.34)
the energy momentum tensor is a function of the dynamical variables \( \beta^i \).

The continuity equation \( \frac{\partial \rho}{\partial t} = d_j \ddot{\beta}^j (\rho + p_j) \) together with an equation of state
\[
p_j = \left( \frac{m_j}{d_j} - 1 \right) \rho
\] (2.35)
yields
\[
\rho = \rho(\beta^i) = M_{m_1 \cdots m_n} e^{-\sum_{j=1}^{n} m_j \beta^j}.
\] (2.36)

For tracefree \( T_{\mu\nu} \) we have
\[
\sum_{j=1}^{n} m_j = d = D - 1.
\] (2.37)
Let us consider the Ricci scalar curvatures \( R^{(i)} \) of \( M_i \) as the only sources of (stress-)energy. Then the density is
\[
\rho = -\sum_{i=1}^{n} R^{(i)} e^{-2\beta^i},
\] (2.38)
which is positive resp. negative, if all \( R^{(i)} \) are negative resp. positive semidefinite and at least \( R^{(1)} \neq 0 \). With such a density \( \rho(\beta^i) \) the minisuperspace potential of Eqs. (2.14) can be written as

\[
V = \frac{\mu}{2} \rho(\beta^i) e^{\gamma + \sum_{i=1}^n d_i \beta^i}. \tag{2.39}
\]

The equations of motion are known to be integrable if spaces \( M_i \) are flat for all \( i > 1 \). Therefore in the following we restrict to models with \( R^{(i)} = 0 \) for \( i > 1 \).

Up to now we have considered by (2.34) only a dependence of \( \rho \) on the geometrical data given by the \( \beta^i \) in Eq. (2.1). More generally we will admit in the following also a dependence on a scalar field \( \Phi \). This field shall be minimally coupled to the geometry of minisuperspace and have a potential \( U(\Phi) \).

The Lagrangian variational principle is given in this case by a Lagrangian

\[
L = \frac{1}{2} \mu e^{-\gamma + \sum_{i=1}^n d_i \beta^i} \left\{ \sum_{i=1}^n d_i (\tilde{\beta}^i)^2 - [\sum_{i=1}^n d_i \tilde{\beta}^i]^2 + \kappa^2 \dot{\Phi}^2 \right\} + \frac{1}{2} \mu e^{\gamma + \sum_{i=1}^n d_i \beta^i} R^{(1)} e^{-2\beta^1} - \mu \kappa^2 e^{\gamma + \sum_{i=1}^n d_i \beta^i} U(\Phi), \tag{2.40}
\]

where by Eq. (2.15) the mass \( \mu = \kappa^{-2} \prod_{i=1}^n \sqrt{|\det g^{(i)}|} \) in minisuperspace is actually determined by the volumes of the spaces \( M_i \).

3 Conformal Lagrangian Models and their Solutions

Generally we will have to distinguish between (1) conformal transformations of Lagrangian models and (2) transformations of the solutions of a fixed given Lagrangian model to a conformal coordinate frame.

(1) Conformal transformations of Lagrangian models:
We consider a differentiable manifold \( M \). Equipped with a Riemannian structure \( g_{ij} \) and scalar fields \( (\phi^1, \ldots, \phi^k) \) on \( M \) we obtain a Lagrangian model by imposing a Lagrangian variation principle

\[
\delta S = 0 \quad \text{with} \quad S = \int_M \sqrt{|g|} L d^D x \tag{3.1}
\]

given by a second order Lagrangian

\[
L = L(g_{ij}, \phi^1, \ldots, \phi^k; g_{ij}\!, \!\!\!, \phi^1\!, \!\!\!, \phi^1\!, \!\!\!, g_{ij}, \!\!\!, \ldots, g_{ij}, \!\!\!, \phi^k, \!\!\!, g_{ij}, \!\!\!, \ldots) \tag{3.2}
\]

Conformal transformation of the model keeps \( M \) fixed as a differentiable manifold, but varies its additional structures conformally

\[
(g_{ij}, \phi^1, \ldots, \phi^k) \to (\hat{g}_{ij}, \hat{\phi}^1, \ldots, \hat{\phi}^k), \tag{3.3}
\]
yielding a new variational principle by demanding
\[ \sqrt{|g|} L = \sqrt{|\hat{g}|} \hat{L} \]  
(3.4)
for the new Lagrangian
\[ \hat{L} = \hat{L}(\hat{g}_{ij}, \hat{\phi}^1, \ldots, \hat{\phi}^k; \hat{g}_{ij, l}, \hat{\phi}^1_{, l}, \ldots, \hat{\phi}^k_{, l}; \hat{g}_{ij, lm}) \]  
(3.5)
Therefore conformal transformation of models are performed in practice on a fixed coordinate patch \( x^i \) of \( M \).

(2) Conformal transformation of solutions to new coordinates:
We fix a Lagrangian model and transform the metric tensor components conformally,
\[ g_{ij}' = e^{2f(x)} g_{kl}, \]  
(3.6)
via a coordinate transform satisfying
\[ dx'^i = e^{-f(x)} dx^i \quad \text{or} \quad \frac{\partial x'^i}{\partial x^j} = e^{-f(x)} \delta^i_j. \]  
(3.7)
Here the first fundamental form
\[ ds^2 = g_{ij}' dx'^i dx'^j = g_{ij} dx^i dx^j, \]  
(3.8)
and therefore the inner geometry, remains invariant, in contrast to transformations under (1) above. Since all geometric invariants remain unchanged, the model is still the same, though looking different in different coordinate frames.

A special application of transformations (2) are time gauge transformations from arbitrarily given coordinates to one of the natural time gauges (i) to (iv). Via transformations (2) for any universe (2.1) e.g. there exists a frame which is w.r.t. time either synchronous (2.17) or harmonic (2.27), though in practice this frames may be difficult to compute explicitly. We will come back to this point later.

Now we want to study the effect of transformations (1) in more detail. One application of special interest is the transformation from a Lagrangian model with minimally coupled potential to a conformally equivalent one with nonminimal coupled potential and vice versa.

Let us follow Ref. 13 and consider an action of the kind
\[ S = \int d^D x \sqrt{|g|} (F(\phi, R) - \frac{\xi}{2}(\nabla \phi)^2). \]  
(3.9)
With
\[ \omega := \frac{1}{D-2} \ln(2\kappa^2 |\partial F/\partial R|) + C \]  
(3.10)
the conformal factor
\[ e^\omega = [2\kappa^2 \left| \frac{\partial F}{\partial R} \right|]^{\frac{1}{D-2}} e^C \] (3.11)
yields a conformal transformation from \( g_{\mu\nu} \) to the minimal metric
\[ \hat{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}. \] (3.12)

Especially let us consider in the following actions, which are linear in \( R \). With
\[ F(\phi, R) = f(\phi)R - V(\phi). \] (3.13)
the action is
\[ S = \int d^Dx \sqrt{|g|} (f(\phi)R - \frac{\epsilon}{2}(\nabla \phi)^2). \] (3.14)

The minimal metric is then related to the conformal one by (3.12) with
\[ \omega = \frac{1}{D-2} \ln(2\kappa^2|f(\phi)|) + C \] (3.15)
The scalar field in the minimal model is
\[ \Phi = \kappa^{-1} \int d\phi \left\{ \frac{\epsilon(D-2)f(\phi) + 2(D-1)(f'(\phi))^2}{2(D-2)f^2(\phi)} \right\}^{1/2} = \]
\[ = (2\kappa)^{-1} \int d\phi \left\{ \frac{2\epsilon f(\phi) + \xi_c^{-1}(f'(\phi))^2}{f^2(\phi)} \right\}^{1/2}, \] (3.16)
where
\[ \xi_c := \frac{D-2}{4(D-1)} \] (3.17)
is the conformal coupling constant.

For the following we define \( \text{sign}x \) to be \( \pm 1 \) for \( x \geq 0 \) resp. \( x < 0 \). Then with the new minimally coupled potential
\[ U(\Phi) = (\text{sign} f(\phi)) \left[ 2\kappa^2|f(\phi)| \right]^{-D/D-2} V(\phi) \] (3.18)
the corresponding minimal action is
\[ S = \text{sign} f \int d^Dx \sqrt{|\hat{g}|} \left( -\frac{1}{2} \left[ \hat{\nabla} \Phi \right]^2 - \frac{1}{\kappa^2} \hat{R} \right) - U(\Phi) \). \] (3.19)

Example 1:
\[ f(\phi) = \frac{1}{2} \xi \phi^2, \] (3.20)
\[ V(\phi) = -\lambda \phi^{2D-2}. \] (3.21)
Substituting this into Eq. (3.18) the corresponding minimal potential $U$ is constant,

$$U(\Phi) = (\text{sign}\xi) |\xi\kappa^2|^{-D/D-2} \lambda. \quad (3.22)$$

It becomes zero precisely for $\lambda = 0$, i.e. when $V$ is zero. With

$$f'(\phi) = \xi \phi \quad (3.23)$$

we obtain

$$\Phi = \kappa^{-1} \int d\phi \left\{ \left( \frac{(\frac{\xi}{\epsilon} + \frac{1}{\xi})}{\phi^2} \right)^{\frac{1}{2}} \right\} = \left( \kappa \sqrt{\xi} \right)^{-1} \sqrt{\frac{1}{\xi} + \frac{\epsilon}{\xi}} \int d\phi \frac{1}{\phi}$$

$$= \kappa^{-1} \sqrt{\frac{1}{\xi} + \frac{\epsilon}{\xi}} \ln |\phi| + C \quad (3.24)$$

for $-\frac{\xi}{\epsilon} \geq \xi_c$. Note that for

$$\frac{\xi}{\epsilon} = -\xi_c, \quad (3.25)$$

e.g. for $\epsilon = -1$ and conformal coupling, we have

$$\Phi = C. \quad (3.26)$$

Thus here the conformal coupling theory is equivalent to a theory without scalarfield.

For $-\frac{\xi}{\epsilon} < \xi_c$ the field $\Phi$ would become complex and, for imaginary $C$, purely imaginary.

In any case the integration constant $C$ may be a function of the coupling $\xi$ and the dimension $D$.

Example 2:

$$f(\phi) = \frac{1}{2}(1 - \xi \phi^2), \quad (3.27)$$

$$V(\phi) = \Lambda. \quad (3.28)$$

Then the constant potential $V$ has its minimal correspondence in a non constant $U$, given by

$$U(\Phi) = \pm \Lambda |\kappa^2(1 - \xi \phi^2)|^{-D/D-2} \quad (3.29)$$

respectively for $\phi^2 < \xi^{-1}$ or $\phi^2 > \xi^{-1}$.

Let us set in the following

$$\epsilon = 1. \quad (3.30)$$

Then with

$$f'(\phi) = -\xi \phi \quad (3.31)$$
we obtain
\[
\Phi = \kappa^{-1} \int d\phi \left\{ \frac{1 + c \xi \phi^2}{(1 - \xi \phi^2)^2} \right\}^{1/2},
\] (3.32)
where
\[
c := \frac{\xi}{\xi_c} - 1.
\] (3.33)

For \(\xi = 0\) it is \(\Phi = \kappa^{-1} \phi + A\), i.e. the coupling remains minimal.

To solved this integral for \(\xi \neq 0\), we substitute \(u := \xi \phi^2\).

To assure a solution of (3.32) to be real, let us assume \(\xi \geq \xi_c\) which yields \(c \geq 0\).

Then we obtain
\[
\Phi = \text{sign}(\phi) \frac{\text{sign}(\phi)}{2\kappa \sqrt{\xi}} \int \frac{\sqrt{u - 1} + c}{|1 - u|} du + C_<
\]
\[
= \frac{\text{sign}((1 - \xi \phi^2)\phi)}{2\kappa \sqrt{\xi}} \left\{ -\sqrt{\ln(2 \sqrt{1 + c \xi \phi^2} \sqrt{\xi} |\phi| + 2 c \xi \phi^2 + 1)} + \sqrt{1 + c \ln(2 \sqrt{1 + c \xi \phi^2} \sqrt{\xi} |\phi| + 2 c \xi \phi^2 + 1)} \right\} + C_<
\]
\[
= \frac{\text{sign}((1 - \xi \phi^2)\phi)}{2\kappa \sqrt{\xi}} \ln \left\{ \frac{2 \sqrt{1 + c \xi \phi^2} \sqrt{\xi} |\phi| + (2 c + 1) \xi \phi^2 + 1} {2 \sqrt{c \sqrt{1 + c \xi \phi^2} \sqrt{\xi} |\phi| + 2 c \xi \phi^2 + 1}} \right\}^{1/2} + C_<
\] (3.34)

The integration constants \(C_<\) for \(\phi^2 < \xi^{-1}\) and \(\phi^2 > \xi^{-1}\) respectively may be arbitrary functions of \(\xi\) and the dimension \(D\).

The singularities of the transform \(\phi \to \Phi\) are located at \(\phi^2 = \xi^{-1}\).

If the coupling is conformal \(\xi = \xi_c\), i.e. \(c = 0\), the expressions (3.34) simplify to
\[
\kappa \Phi = \frac{1}{\sqrt{\xi_c}} [\text{artanh} \sqrt{\xi_c} \phi + c_<]
\] (3.35)
for \(\phi^2 < \xi_c^{-1}\) and to
\[
\kappa \Phi = \frac{1}{\sqrt{\xi_c}} [\text{arcoth} \sqrt{\xi_c} \phi + c_>]
\] (3.36)
for \(\phi^2 > \xi_c^{-1}\).

In the following we restrict to this case of conformal coupling.
The inverse formulas expressing the conformal field $\phi$ in terms of the minimal field $\Phi$ are

$$\phi = \frac{1}{\sqrt{\xi}} \left[ \tanh(\sqrt{\xi} \kappa \Phi - c_<) \right]$$

(3.37)

with $\phi^2 < \xi^{-1}$ and

$$\phi = \frac{1}{\sqrt{\xi}} \left[ \coth(\sqrt{\xi} \kappa \Phi - c_>) \right]$$

(3.38)

with $\phi^2 > \xi^{-1}$ respectively.

The conformal factor is according to Eqs. (3.15) and (3.27) given by

$$\omega = \frac{1}{D - 2} \ln(\kappa^2 |1 - \xi_c \phi^2|) + C.$$  

(3.39)

In the following we want to compare the solutions of the minimal model to those of the corresponding conformal model. We specify the geometry for the minimal model to be of multidimensional type (2.1), with all $M_i$ Ricci flat, hence $R^{(i)} = 0$ for $i = 1, \ldots, n$. The minimally coupled scalar field is assumed to have zero potential $U \equiv 0$. In the harmonic time gauge (2.27) with harmonic time

$$\tau \equiv t_h^{(m)},$$  

(4.40)

we demand this model to be a solution for Eq. (2.40) with vanishing $R^{(1)}$ and $U(\Phi)$. We set $\beta^{n+1} := \kappa \Phi$ and and obtain as solution a multidimensional (Kasner like) universe, given by

$$\dot{\beta}^i = b^i \tau + c^i \text{ and } \dot{\gamma} = \sum_{i=1}^{n} d_i \dot{\beta}^i = \left( \sum_{i=1}^{n} d_i b^i \right) \tau + \left( \sum_{i=1}^{n} d_i c^i \right),$$  

(3.41)

with $i = 1, \ldots, n + 1$, where with $V \equiv 0$ the constraint Eq. (2.31) simply reads

$$G_{ij} b^i b^j + (b^{n+1})^2 = 0.$$  

(3.42)

With Eq. (3.39) the scaling powers of the universe given by Eqs. (3.41) with $i = 1, \ldots, n$ transform to corresponding scale factors of the conformal universe

$$\beta^i = \dot{\beta}^i - \omega$$

$$= b^i \tau + \frac{1}{2 - D} \ln |1 - \xi_c(\phi)^2| + c^i + \frac{2}{2 - D} \ln \kappa - C$$  

(3.43)

and

$$\gamma = \sum_{i=1}^{n} d_i \beta^i$$

$$= \left( \sum_{i} d_i b^i \right) \tau + \frac{1}{2 - D} \ln |1 - \xi_c(\phi)^2| + \left( \sum_{i} d_i c^i \right) + \frac{2}{2 - D} \ln \kappa - C.$$  

(3.44)
It should be clear from the remarks in Sec. 3 that the variable \( \tau \), when harmonic in the minimal model, in the conformal model cannot be expected to be harmonic either, i.e. in general
\[
\tau \neq t_h^{(c)}.
\] (3.45)

Let us take for simplicity
\[
C = \frac{2}{2 - D} \ln \kappa,
\] (3.46)
which yields the lapse function
\[
e^\gamma = e^{(\sum_i d_i b^i) \tau + (\sum_i d_i c^i) |1 - \xi_c(\phi)^2|^2} \frac{1}{\sqrt{\kappa}}
\] (3.47)
and for \( i = 1, \ldots, n \) the scale factors
\[
e^{\beta^i} = e^{b^i \tau + c^i |1 - \xi_c(\phi)^2|^2} \frac{1}{\sqrt{\kappa}}.
\] (3.48)

Let us further set for simplicity
\[
c_\leq = c_\geq = \sqrt{\xi_e} c^{n+1}.
\] (3.49)

The transformation of the scalar field from the solution (3.39) of the minimally coupled model
\[
\kappa \Phi(\tau) = b^{n+1} \tau + c^{n+1}
\] (3.50)
to the scalar field of the conformal model by Eqs. (3.37) or (3.38) and substitution of the latter in Eqs. (3.48) resp. (3.49) yields a lapse function
\[
e^\gamma = e^{(\sum_i d_i b^i) \tau + (\sum_i d_i c^i) \cosh \frac{\sqrt{\kappa}}{2} (\sqrt{\xi_e} b^{n+1} \tau)}
\] (3.51)
resp.
\[
e^\gamma = e^{(\sum_i d_i b^i) \tau + (\sum_i d_i c^i) \sinh \frac{\sqrt{\kappa}}{2} (\sqrt{\xi_e} b^{n+1} \tau)}
\] (3.52)
and, with \( i = 1, \ldots, n \), nonsingular scale factors
\[
e^{\beta^i} = e^{b^i \tau + c^i \cosh \frac{\sqrt{\kappa}}{2} (\sqrt{\xi_e} b^{n+1} \tau)}
\] (3.53)
resp. singular scale factors
\[
e^{\beta^i} = e^{b^i \tau + c^i \sinh \frac{\sqrt{\kappa}}{2} (\sqrt{\xi_e} b^{n+1} \tau)}
\] (3.54)
of the conformal model. The scale factor singularity of the minimal coupling model for \( \tau \to -\infty \) vanishes in the conformal model of Eqs. (3.51) and (3.53) for a scalar field \( \phi \) bounded according to (3.37). For \( D = 4 \) this result had already been indicated by Ref. 14.
On the other hand in the conformal model of Eqs. (3.52) and (3.54), with $\phi$ according to (3.38), though the scale factor singularity of the minimal model for $\tau \to -\infty$ has also disappeared, instead there is another new scale factor singularity at finite (harmonic) time $\tau = 0$.

Let us consider a special case of the nonsingular solution with $\phi^2 < \xi_c^{-1}$, where we assume the internal spaces to be static in the minimal model, i.e. $b^i = 0$ for $i = 2, \ldots, n$. Then in the conformal model, the internal spaces are no longer static. Their scale factors (3.54) with $i > 2$ have a minimum at $\tau = 0$. Remind that for solution (3.41) all spaces $M_i$, internal and external, $i = 1, \ldots, n$ have been assumed as flat. From Eq. (3.42) with $G_{11} = d_1(1 - d_1)$ we find that the scalar field is given by

$$(b^{n+1})^2 = d_1(d_1 - 1)(b^1)^2.$$  \hspace{1cm} (3.55)

With real $b_1$ then also

$$b^{n+1} = \pm \sqrt{d_1(d_1 - 1)b^1}$$  \hspace{1cm} (3.56)

is real and by Eq. (3.52) the scale $a_1$ of $M_1$ has a minimum at

$$\tau_0 = (\sqrt{\xi_c} b^{n+1})^{-1} \text{artanh} \left( \frac{2 - D}{2 \sqrt{\xi_c} b^n} \right),$$  \hspace{1cm} (3.57)

with $\tau_0 > 0$ for $b^1 < 0$ and $\tau_0 < 0$ for $b^1 > 0$.

The points $\tau = \tau_0$ and $\tau = 0$ are the turning points in the minimum for the factor spaces $M_1$ and $M_2, \ldots, M_n$ respectively. It is interesting to explain the creation of our Lorentzian universe by a "birth from nothing"\textsuperscript{15}, i.e. quantum tunneling from an Euclidean region. Let us first consider the geometry of this tunneling as usual for the external universe $\mathbb{R} \times M_1$. So if we cut $M$ along the minimal hypersurface at $\tau_0$ in 2 pieces, one of them, say $M'$, contains the hypersurface $\tau = 0$ where the internal spaces are minimal. We set $M'' := M \setminus M'$ to be the remaining piece. Then we can choose (eventually with time reversal $\tau \to -\tau$) either $M'$ or $M''$ as a universe $\tilde{M}$ that is generated at $\tau_0$ with initial minimal scale $a_1(\tau_0)$. In the usual quantum tunneling interpretation, at the scale $a_1(\tau_0)$ with $\dot{a}_1(\tau_0) = 0$ one glues smoothly a compact simply connected Euclidean space-time region to the Lorentzian $\tilde{M}$, yielding a joint differentiable manifold $\tilde{M}$. Then the sum of classical paths passing the boundary $\partial \tilde{M}$ from the Euclidean to the Lorentzian region can be interpreted as quantum tunneling from "nothing"\textsuperscript{15} to $\tilde{M}$.

According to Ref. 16 this interpretation has a direct topological correspondence in a projective blow up of a singularity of shape $M_2 \cdots \times M_n$ (the "nothing") to $S^{d_1}(a_1(\tau_0)) \times M_2 \cdots \times M_n$, where $S^{d_1}(a_1(\tau_0))$ denotes the $d_1$-dimensional sphere of radius $a_1(\tau_0)$.
For $\tilde{M} = M'$ the internal spaces shrink for (harmonic) time from $\tau_0$ towards $\tau = 0$ and expand from $\tau = 0$ onwards for ever, but for $\tilde{M} = M''$ the internal spaces expand for (harmonic) time from $\tau_0$ onwards for ever. So the decomposition of $M$ in $M'$ and $M''$ is highly asymmetric w.r.t. the internal spaces. For more realistic models it might be especially useful to consider the piece of $M'$ which lies between $\tau_0$ and $\tau = 0$, since it can describe a shrinking of internal spaces while the external space is expanding.

Remarkably the multidimensional geometries with $\tau < \tau_0$ and $\tau > \tau_0$ are $\tau$-asymmetric to each other. Taking one as contracting, the other as expanding w.r.t. $M_1$, the two are distinguished by a qualitatively different behavior of internal spaces $M_k$, $k \geq 2$.

The latter allows to choose the "arrow of time" in a natural manner determined by intrinsic features of the solutions. Note if there is at least one internal extra space, i.e. $n > 1$, then the minisuperspace w.r.t. scalefactors of geometry has Lorentzian signature $(-,+,\ldots,+)$. After diagonalization of (2.11) by a minisuperspace coordinate transformation $\beta^i \rightarrow \alpha^i$ ($i = 1,\ldots,n$), there is just one new scalefactor coordinate, say $\alpha^1$, which corresponds to the negative eigenvalue of $G$, and hence assumes the role played by time in usual quantum mechanics. (For $n = 1$ there are no internal spaces, but $G_11 < 0$ for $d_1 > 1$ still provides a negative eigenvalue that is distinguished at least against the additional positive eigenvalue from the scalar field.) This shows that, at least after diagonalization, that an "external" space is distinguished against the internal spaces, because its scale factor provides a natural "time" coordinate.

Upto now we have considered the smooth tunneling from an Euclidean region to the external universe $\mathbb{R} \times M_1$, where the external spaces have been considered as purely passive spectators of the tunneling process. As we have pointed out in contrast to models with only one (external) space factor $M_1$, the additional internal spaces $M_2,\ldots,M_n$ yield an asymmetry of $M$ w.r.t. (harmonic) time $\tau$ for $\tau_0 \neq 0$, which is according to Eq. (3.57) the case exactly when $D \neq 2$ and the external space is non static, i.e. $b_1 \neq 0$.

In the following we want to obtain a quantum tunneling interpretation for all of $M$, including the internal spaces. The picture becomes more complicated, since the extremal hypersurfaces of external space and internal spaces are located at different times $\tau = \tau_0$ resp. $\tau = 0$.

Let $M_1$ be the external space with $b_1 > 0$ and hence $\tau_0 < 0$. Let us start with an Euclidean region of complex geometry given by scale factors

$$a_k = e^{-ib^kt+\tilde{c}^k}\sin(\sqrt{\xi_{\nu}b^{\nu+1}}\tau)\sqrt{n-2}.$$}

Then we can perform an analytic continuation to the Lorentzian region with $\tau \rightarrow i\tau + \pi/(2\sqrt{\xi_{\nu}b^{\nu+1}})$, and we require $c^k = c^k - i\pi b^k/(2\sqrt{\xi_{\nu}b^{\nu+1}})$ to be the real constant of the real
geometry (3.48).

The quantum creation (via tunneling) of different factor spaces takes place at different values of $\tau$ (see Fig. 1).

First the factor space $M_1$ comes into real existence and after a time interval $\Delta \tau = |\tau_0|$ the internal factor spaces $M_2, \ldots, M_n$ appear in the Lorentzian region. Since $\Delta \tau$ is arbitrarily large, there is in principle an alternative explanation of the unobservable extra dimensions, independent from concepts of compactification and shrinking to a fundamental length in symmetry breaking. Similar to the spirit of the idea that internal dimensions might be hidden due to a potential barrier\textsuperscript{18}, they may have been up to now still in the Euclidean region and hence unobservable. This view is also compatible with their interpretation as complex resolutions of ADE symmetries\textsuperscript{16}.

Now let us perform a transition from Lorentzian time $\tau$ to Euclidean time $i\tau$. Then with a simultaneous transition from $b^k$ to $-ib^k$ for $k = 1, \ldots, n$ the geometry remains real, since $\hat{\beta}^k = b^k\tau + c^k$ is unchanged. But the analogue of Eq. (3.56) for the Euclidean region then becomes

$$b^{n+1} = \mp i\sqrt{d_1(d_1 - 1),}$$

Hence the scalar field is purely imaginary. This solution corresponds to a classical (instanton) wormhole. The sizes of the wormhole throats in the factor spaces $M_2, \ldots, M_n$ coincide with the sizes of static spaces in the minimal model, i.e. $\hat{a}_2(0), \ldots, \hat{a}_n(0)$ respectively.

With Eq. (3.56) replaced by (3.58), the Eq. (3.57) remains unchanged in the transition to the Euclidean region, and the minimum of the scale $a_1$ (unchanged geometry !) now corresponds to the throat of the wormhole.

If one wants to compare the synchronous time pictures of the minimal and the conformal m-solution, one has to calculate them for both metrics. In the minimal model we have

$$dt_s^{(m)} = e^{\hat{\gamma}}d\tau = e^{(\sum_i d_i b^i)\tau + (\sum_i d_i c^i)}d\tau,$$

which can be integrated to

$$t_s^{(m)} = (\sum_i d_i b^i)^{-1}e^{\hat{\gamma}} + t_0.$$  

The latter can be inverted to

$$\tau = (\sum_i d_i b^i)^{-1} \left\{ \ln(\sum_i d_i b^i) (t_s^{(m)} - t_0) - \sum_i d_i c^i \right\}. \quad \text{(3.61)}$$

Setting

$$B := \sum_{i=1}^n d_i b^i \text{ and } C := \sum_{i=1}^n d_i c^i, \quad \text{(3.62)}$$
this yields the scale factors

\[ \dot{a}_s^i = (t_s^{(m)} - t_0)^{b_i/B} e^{\frac{2\mu}{B}(\ln B - C) + c_i} \]  

(3.63)

and the scalar field

\[ \kappa \Phi = \frac{b^{n+1}}{B} \{ [\ln B(t_s^{(m)} - t_0)] - C \} + c^{n+1}. \]  

(3.64)

Let us define for \( i = 1, \ldots, n + 1 \) the numbers

\[ \alpha^i := \frac{b^i}{B}. \]  

(3.65)

With (3.62) they satisfy

\[ \sum_{i=1}^n d_i \alpha^i = 1, \]  

(3.66)

and by Eq. (3.42) also

\[ \alpha^{n+1} = \sqrt{1 - \sum_{i=1}^n d_i (\alpha^i)^2}. \]  

(3.67)

Eqs. (3.63) shows, that the solution (3.41) is really a generalized Kasner universe with exponents \( \alpha^i \) satisfying generalized Kasner conditions (3.66) and (3.67).

In the conformal model the synchronous time is given as

\[ t_s^{(c)} = \int e^{\gamma} d\tau = \int \cosh \frac{\gamma}{\sqrt{\xi}} (\sqrt{\xi} e^{B^{n+1} \tau}) e^{B \tau + C} d\tau \]  

(3.68)

resp.

\[ t_s^{(c)} = \int e^{\gamma} d\tau = \int \sinh \frac{\gamma}{\sqrt{\xi}} (\sqrt{\xi} e^{B^{n+1} \tau}) e^{B \tau + C} d\tau. \]  

(3.69)

Similarly one could also try to calculate other time gauges for both metrics.

4 Quantum Solutions from the WdW equation

In this section we investigate the quantum analogue of the classical solution for the particular model of Sec. 3 above with all \( M_i \) Ricci flat. The WdW equation for the minimal model reads

\[ \left[ \Gamma^{ij} \frac{\partial}{\partial \beta^j} \frac{\partial}{\partial \beta^i} + \frac{\partial^2}{\partial \Phi^2} \right] \Psi = 0, \]  

(4.1)

where the minimally coupled field \( \Phi \) is redefined by \( \Phi := \kappa \Phi \) as compared with the previous section, its potential is zero, \( U(\Phi) \equiv 0 \), and the WdW equation is written in harmonic time gauge, with components \( \Gamma^{ij} = \frac{\delta_{ij}}{d_i} - \frac{1}{2-D} \) of the inverse to the minisuperspace metric (2.11-12) in coordinates \( \beta \) corresponding to minimally coupled geometry.
The solutions of Eq. (4.1) are

\[ \Psi_\vec{b} = e^{iG_{kl}b^k\hat{\beta}^l} e^{ib_{n+1}\Phi}, \]  

(4.2)

where \( b_{n+1} = b^{n+1} \) and the quantum numbers \( b^k \) \((k = 1, \ldots, n+1)\) satisfy the constraint (3.42). From a formal point of view it is natural to set \( \hat{\beta}^{n+1} := \Phi \). Then \( \Psi_\vec{b} \) with \( \vec{b} \in \mathbb{R}^{n+1} \) are eigenfunctions of the momentum operators \( -i \frac{\partial}{\partial \beta^k} \) with eigenvalues \( \hat{p}_k = b_k = G_{kl}b^l \), i.e.

\[ -i \frac{\partial}{\partial \beta^k} \Psi_\vec{b} = b_k \Psi_\vec{b}, \quad b_k = 1, \ldots, n+1. \]  

(4.3)

For analogy with Sec. 3 let us investigate in more detail the degenerate case \( b^1 \neq 0 \), \( b^2 = \ldots = b^n = 0 \). With

\[ b_1 = -d_1(d_1 - 1)b^1, \quad \text{and} \quad b_k = -d_k(d_1b^1), \quad k = 2, \ldots, n \]  

(4.4)

we obtain

\[ b_{n+1} = b^{n+1} = \pm [d_1(d_1 - 1)]^{\frac{1}{2}} b^1 = \mp [d_1(d_1 - 1)]^{-\frac{1}{2}} b_1 \]

according to Eq. (3.56) from the constraint (3.42). We see that a nonvanishing scalar field in the minimal coupling model requires \( d_1 \neq 1 \).

The wave function (4.2) in the degenerate case is

\[ \Psi_{b_1,\pm} = \left[ V_{b_1,\pm} e^{\hat{\beta}^1 \mp \Phi} \right]^{ib_1}, \]  

(4.5)

where we have defined a new minimal field coordinate

\[ \hat{\Phi} := \Phi / \sqrt{d_1(d_1 - 1)}, \]

and

\[ V := \prod_{k=2}^n d_k^{d_k} \]  

(4.6)

is proportional to the volume of the internal spaces \( M_2 \times \ldots \times M_n \). For the classical solutions corresponding to the degenerate case this quantity is constant, i.e. the internal spaces are static.

Now we show that the wave function (4.5) describes an expanding universe if and only if \( b^1 > 0 \) (or likewise \( b_1 < 0 \)). The Hubble parameters of the spaces \( M_k \), w.r.t. harmonic time \( \tau \) and for a solution (3.41) of the minimal model, are given as

\[ \hat{h}_k = \frac{1}{a_i} \frac{\partial a_i}{\partial \tau} = b^k, \quad k = 1, \ldots, n. \]  

(4.7)
$\hat{h}_k > 0$ corresponds to an expanding factor space $M_k$. Since in the degenerate case only $M_1$ has nontrivial dynamics, an expanding universe corresponds to $b^1 > 0$ (or likewise $b_1 < 0$).

The classical momenta with $\mu \equiv 1$ in the harmonic gauge are $\hat{p}_k = \frac{\partial L}{\partial \dot{\beta}_k} = b_k$. Thus $\hat{p}_1 < 0$ corresponds to expanding $M_1$. Hence by Eq. (4.3) a wavefunction $\Psi_{b_1,\pm}$ describes an expanding universe if and only if $b_1 < 0$ (i.e. $b^1 > 0$), while the internal spaces $M_k$ with

$$-i\frac{\partial}{\partial \beta_k} \Psi_{b_1,\pm} = b_k \Psi_{b_1,\pm} = -d_k(d_1 b^1)\Psi_{b_1,\pm}$$

and $b^k = G^{kl}b_l = 0$ for $k = 2, \ldots, n$ are static.

The transformation to the conformal model has to be performed according to Eq. (3.43) and respectively either Eq. (3.35) or Eq. (3.36), substituting $\hat{\beta}_i = \beta_i + \omega(\phi)$ and $\hat{\Phi}(\phi)$ into Eq. (4.5), yielding

$$\Psi_{b_1,\pm} = \left[ e^{\frac{D-2}{4} + \omega(\phi)} V \frac{1}{\tau_1} e^{\beta_1 \mp \hat{\Phi}(\phi)} \right]^{ib^1}, \quad (4.8)$$

where $V := \prod_{k=2}^n d_k$.

The interpretation of the wave function (4.8) in the conformal model is complicated by the severe difficulty, that in contrast to (4.5), which is a solution of the WdW equation (4.1), for (4.8) it can not be expected, that it is the solution of a conformal WdW equation. A related difficulty is that $\Psi_{b_1,\pm}$ are only eigenfunctions of $\frac{\partial}{\partial \beta^k}$ but not of $\frac{\partial}{\partial \beta^e}$. As consequence the classical momenta $p_k = \frac{\partial L}{\partial \dot{\beta}^k}$ cannot be obtained as eigenvalues of a corresponding canonical operator. In contrast to the minimal model now for all $M_k$ with $k = 1, \ldots, n$ it is $p_k \neq 0$, and hence also the internal factor spaces are no longer static in the conformal model.

On classical level, the Hubble parameters of the nonsingular solution of the conformal model w.r.t. the time parameter $\tau$ are given as

$$h_k = \frac{1}{a_k} \frac{\partial a_i}{\partial \tau} = \hat{h}_k + \frac{2}{D-2} \sqrt{\xi c \xi^{n+1} \tanh(\sqrt{\xi c} b^{n+1} \tau)}, \quad (4.9)$$

with $\hat{h}_k$ from Eq. (4.7). For expanding $M_k$ it is $h_k > 0$. In the degenerate case with $\hat{h}_1 = b^1$ and $\hat{h}_k = 0$ for $k = 2, \ldots, n$ we yield $h_1 > 0$ for $\tau > \tau_0$ with $\tau_0$ given by Eq. (3.57). $M_1$ may expand for $b^1 > 0$ (in this case $\tau_0 < 0$) as well as for $b^1 < 0$ (in this case $\tau_0 > 0$). If $\tau > 0$ the factors $M_k$ ($k = 2, \ldots, n$) expand for all values of $b_1$. This classical investigation shows that the $\Psi_{b_1,\pm}$ may describe expanding spaces $M_k$ ($k = 1, \ldots, n$) for both $b^1 > 0$ and $b^1 < 0$.

Let us perform for the degenerate case the transition to the real Euclidean geometry, i.e. $b_1 \rightarrow -ib_1$ with $\tau \rightarrow i\tau$. Then the wave function goes from (4.5) to

$$\Psi_{b_1,\pm} = \left[ \hat{V} \frac{1}{\tau_1} e^{\beta^1 \mp \hat{\Phi}} \right]^{ib_1}. \quad (4.10)$$
The superposition

\[ \Psi_{\pm} := \sum_{b_1=0}^{\infty} \frac{(-1)^{b_1}}{b_1!} \Psi_{b_1, \pm} \] (4.11)

yields with (4.10) the wave function

\[ \Psi_{\pm} = e^{-\hat{V} \pi \frac{1}{\hat{a}_1}} \hat{a}_1 e^{\mp \hat{\phi}}, \] (4.12)

which satisfies the quantum wormhole boundary conditions:

1) it is exponentially damped for large spacial geometries (i.e. for \( \hat{a}_1 \to \infty \) or \( \hat{V} \to \infty \)).

2) it is regular, when the spacial geometry degenerates (i.e. when \( \hat{a}_1 \to 0 \) or \( \hat{V} \to 0 \)).

Thus (4.12) may be treated as quantum wormhole.
5 Conclusion

We have initially considered natural time gauges in multidimensional universes: (i) synchronous time, (ii) conformal times of different factor spaces, (iii) mean conformal time and (iv) harmonic time. Transitions between them are given by special conformal coordinate transformations. We have emphasized that conformal coordinate transformations have to be distinguished sharply from conformal transformations of geometrical Lagrangian models.

The conformal transformation of the minimally coupling model to the conformal coupling model has been performed in arbitrary dimensions $D$, with the conformal factor and scalar field in agreement with the result of Ref. 19. By Eq. (3.34) the proper generalization of the scalar field from the conformal coupling case to that of arbitrary coupling $\xi$ has been found (Note that Eq. (5) in Ref. 10 holds only for $\xi = \xi_c = \frac{1}{6}$ with $D = 4$).

We find a generalized Kasner solution for the minimally coupling model with $M_i$ flat and zero potential, having a scale factor singularity. Conformal transformation yields (w.r.t. harmonic time) a nonsingular solution (3.53) for $\phi^2 < \xi_c^{-1}$ and a singular solution (3.54) for $\phi^2 > \xi_c^{-1}$. This resolution of the scale factor singularity of the generalized Kasner solution of the minimal model in the corresponding conformal solution (3.53) confirms in arbitrary dimension $D$, what has been indicated in Ref. 14 for $D = 4$. At $\phi^2 = \xi_c^{-1}$ there is a singularity of the conformal transformation. The conformal equivalence between the models only holds separately in the range $\phi^2 < \xi_c^{-1}$ or $\phi^2 > \xi_c^{-1}$.

In the special case of static internal spaces in the minimal model, we find dynamical internal spaces with a nonzero minimum scale at $\tau = 0$ for the conformal model with external space having a minimal scale $a_1(\tau_0)$ at (harmonic) time $\tau_0$. In the internal spaces the conformal solution is highly asymmetric w.r.t. $\tau_0$. Cutting the solution at $\tau_0$, the resulting pieces allow to model the birth of universes at $\tau_0$ with different behaviour of internal spaces in harmonic time.

The region between $\tau_0$ and $\tau = 0$ is characterized by shrinking internal spaces, while external space expands. However further investigations will be required to yield a more detailed understanding of the dynamical behaviour of internal spaces. In Ref. 20 first investigations for the model from Eq. (2.40) in harmonic time gauge have shown how the dynamics of the factor spaces $M_i$ depends critically on the dimensions of $M_i$.

Besides the usual quantum creation of external space $M_1$ only, with internal spaces as spectators, we have pointed out the possibility to create both $M_1$ and the internal spaces by quantum tunneling from an Euclidean region. However the initial real time is different for the quantum creation of different factor spaces in general, and especially in the considered
model for $M_1$ and the internal factor spaces $M_i$, $i \geq 2$. If the time delay between creation of $M_1$ and internal spaces goes to infinity, $\Delta \tau = |\tau_0| \rightarrow \infty$, the internal spaces remain forever in the classically forbidden region, while external space is given by the real Lorentzian $M_1$. Hence extra dimensions are unobservable at any time.

Analytic continuation of this solution to the Euclidean time region (while pertaining real geometry) yields a purely imaginary scalar field. This solution corresponds to an (instanton) wormhole, where the minimal scale $a_1$ now indicates the throat of the wormhole w.r.t. external space, and the throats of the internal spaces are given by $\hat{a}_2(0), \ldots, \hat{a}_n(0)$.

For the minimal model we have obtained the corresponding quantum solution from the WdW equation. In the degenerate case, corresponding to static internal spaces, the solution describes a classically expanding factor $M_1$ if and only if the classical minisuperspace momentum satisfies $\hat{p}_1 = b_1 < 0$.

The wavefunction corresponding classically to the conformal model can not be interpreted as a solution of a conformal WdW equation. This is related to the fact that corresponding classical momenta $p_k$ are no longer eigenvalues. Corresponding classical internal spaces are no longer static, while the corresponding minisuperspace momenta $p_k$ are unrestricted. This wavefunction can describe a classically expanding $M_1$ for both $b_1 > 0$ and $b_1 < 0$.

Performing the transition to the real Euclidean region, a special solutions satisfying the quantum wormhole boundary conditions have been found.

**Acknowledgements**

This work was supported by WIP grant 016659 (U.B.), in part by DAAD and by DFG grant 436 UKR - 17/7/93 (A. Z.) and DFG grant Bl 365/1-1 (M.R.). A. Z. also thanks Prof. Kleinert and the Freie Universität Berlin as well as the members of the Gravitationsprojekt at Universität Potsdam for their hospitality. M. R. thanks the Projektgruppe Kosmologie at Universität Potsdam and H.-J. Schmidt for support and hospitality.
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Fig. 1: Quantum birth with compact Ricci flat spaces and birth time $\tau_0 \leq 0$ of external Lorentzian space $M_1$. The birth of internal factor spaces $M_2, \ldots, M_n$ is delayed by the interval $\Delta \tau = |\tau_0|$. For $\Delta \tau \to \infty$ the internal spaces remain for ever in the (unobservable) classically forbidden region.
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