A DATABASE OF LOCAL FIELDS

JOHN W. JONES AND DAVID P. ROBERTS

Abstract. We describe our online database of finite extensions of $\mathbb{Q}_p$, and how it can be used to facilitate local analysis of number fields.

1. Introduction

1.1. Overview. Given a number field $K$, one has for each prime $p$ its associated $p$-adic algebra,

$$K \otimes \mathbb{Q}_p \cong \prod_{i=1}^{g} K_{p,i}. $$

Here the $K_{p,i}$ are fields, each a finite extension of $\mathbb{Q}_p$. For investigating some problems about number fields, it suffices to know just basic invariants of the $K_{p,i}$, such as ramification index and residual degree. For other investigations, it is essential to have much more refined information, such as local Galois groups and slopes measuring wildness of ramification.

To facilitate refined analysis of number fields, we have constructed a database of $p$-adic fields, available at http://math.asu.edu/~jj/localfields. Let $\mathcal{K}(p, n)$ be the set of isomorphism classes of degree $n$ extensions of $\mathbb{Q}_p$. The sets $\mathcal{K}(p, n)$ are finite, with general mass formulas counting these fields with certain weights being known [Se, Kr, PR]. Our database presents some of the sets $\mathcal{K}(p, n)$ in a complete and easy-to-use way. The philosophy behind the database is that the intricate local considerations needed to construct it should be done once and then recorded. Thereafter, a local result can be obtained by mechanical appeal to the database whenever it is needed in a global situation.

1.2. Fields in the database. When $n$ is not divisible by $p$, all fields in $\mathcal{K}(p, n)$ are tame, and so $\mathcal{K}(p, n)$ is relatively easy to describe. Our database treats these fields dynamically, without restriction on $p$ or $n$. The first case involving wild fields is $n = p$. This case is also relatively easy to describe in a way uniform in $p$; for example, $|\mathcal{K}(p, p)| = p^2 + 1$ for $p$ odd. Again, our database treats these fields without restriction on $p$.

The most visible parts of our database are tables explicitly describing $\mathcal{K}(p, n)$ for small $p$ and $n$. The numbers $|\mathcal{K}(p, n)|$ for $p < 30$ and $n < 10$ are listed in Table 1.1. The table for $\mathcal{K}(p, n)$ in the database has one line for each isomorphism class of $p$-adic field of degree $n$ and gives a defining polynomial for the field and many invariants of the field. Our tables provide many illustrations of the relatively easy cases discussed in the previous paragraph. However their main function is to cover the five harder cases with $n < 10$, namely $(p, n) = (2, 4), (2, 6), (2, 8), (3, 6)$, and $(3, 9).$
Table 1.1. The number $|\mathcal{K}(p, n)|$ of isomorphism classes of $p$-adic fields of degree $n$, for $p < 30$ and $n < 10$. The entries corresponding to the five cases which we treat individually are underlined.

| $n$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 |
|-----|---|---|---|---|----|----|----|----|----|----|
| 1   | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  |
| 2   | 7 | 3 | 3 | 3 | 3  | 3  | 3  | 3  | 3  | 3  |
| 3   | 2 | 10| 2 | 4 | 2  | 4  | 2  | 4  | 2  | 2  |
| 4   | 50| 5 | 7 | 5 | 7  | 7  | 5  | 7  | 5  | 7  |
| 5   | 2 | 2 | 26| 2 | 6  | 2  | 2  | 2  | 2  | 2  |
| 6   | 47| 75| 7 | 12| 7  | 12 | 7  | 12 | 7  | 7  |
| 7   | 2 | 2 | 2 | 2 | 50 | 2  | 2  | 2  | 2  | 2  |
| 8   | 1823| 8 | 11| 8 | 8  | 11 | 15 | 8  | 8  | 11 |
| 9   | 3 | 795| 3 | 7 | 3  | 7  | 3  | 13 | 3  | 3  |

1.3. Sections of this paper. Section 2 discusses how we found our lists of defining polynomials. It treats first the tame and $n = p$ cases systematically, and then describes our ad hoc approach to the five harder cases. All parts of the 2-adic quartic cases have received detailed attention previously, for example [We] for the one $A_4$ and the three $S_4$ extensions and [Na] for the thirty-six $D_4$ extensions. However even the mere listing of defining polynomials constitutes a new result in the remaining four cases. For example, while some classes of octic 2-adic fields have been studied completely by others [We, BR], these fields represent only a small subset of the full set of 1823 octic 2-adic fields.

Section 3 discusses how we computed the invariants for each field. We restrict discussion of details to the cases $n \leq 7$. The analogous details for 2-adic octics and 3-adic nonics are very much more intricate. We are treating these cases in separate papers, each of which will include sample applications to number fields of the same degree, where the same Galois theory applies.

Section 4 describes the two interactive features of our database, what we call the $p$-adic identifier and the Galois root discriminant calculator. These are designed to maximize the utility of our database for applications. One application we have in mind is to assist in matching number fields to automorphic forms of various sorts, as here complete understanding of ramification is very useful. Section 4 presents another application, one that stays within the confines of traditional algebraic number theory.

2. A complete irredundant list of defining polynomials

In this section, we describe how we chose the polynomials defining the fields in the database. Sections 2.1-2.3 deal with unramified, tamely ramified, and degree $p$ extensions of $\mathbb{Q}_p$, respectively. Section 2.4 deals with the remaining cases — wildly ramified extensions where the degree is composite.

2.1. Unramified extensions. Unramified extensions of $\mathbb{Q}_p$ are very simple, there being a unique one for each degree $n$, up to isomorphism. The only task is to choose a defining polynomial for each. In the sequel, we will sometimes drop qualifiers like “up to isomorphism,” as they are always present and our meaning is clear.
Since the extension of degree $n$ of $\mathbb{Q}_p$ corresponds to the unique degree $n$ extension of the residue field, one option is to use Conway polynomials for these extensions since they are considered the standard choices for defining $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$. However, Conway polynomials can be expensive to compute, primarily because they are required to satisfy a compatibility condition which is not used here.

Instead, we pick defining polynomials which are in the same spirit, but with fewer restrictions. We compute the “first” polynomial over $\mathbb{F}_p$ which has roots which are primitive, i.e., of multiplicative order $p^n - 1$. Here we use the same lexicographic ordering as for Conway polynomials. We write polynomials in the form $f(x) = x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} - \cdots$ and $g(x) = x^n - b_{n-1}x^{n-1} + b_{n-2}x^{n-2} - \cdots$ with $a_i$ and $b_i$ between 0 and $p - 1$ inclusive. Then we define $f < g$ iff there exists $k$ with $a_i = b_i$ for all $i > k$ and $a_k < b_k$. This normalization also defines how we will represent the polynomials in $\mathbb{Z}[x] \subset \mathbb{Q}_p[x]$. Note that for defining $\mathbb{Q}_p$ itself, our choice leads to the “degree 1 Conway polynomial” $x - r$, where $r$ is the first primitive root modulo $p$.

2.2. Tame extensions. Our starting point is the following standard result on totally ramified tame extensions. The statement here is a special case of Theorem 7.2 in [PR].

**Proposition 2.1.** Let $K^u$ be an unramified extension of $\mathbb{Q}_p$ with degree $f$. Let $\zeta \in K^u$ be a primitive $(p^f - 1)^{st}$ root of unity. Let $e$ be a positive integer with $p \nmid e$.

1. The totally ramified degree $e$ extensions of $K^u$, are given by roots of polynomials $h_{e,r}(x) = x^e - \zeta^r p$.
2. Two such polynomials $h_{e,r}$ and $h_{e,r'}$ yield $K^u$-isomorphic extensions iff $r \equiv r' \pmod{\gcd(e, p^f - 1)}$.
3. If a monic polynomial $g$ satisfies $g \equiv h_{e,r} \pmod{p^2}$, then $g$ defines the same extension as $h_{e,r}$.

To apply the proposition, we take $K^u = \mathbb{Q}_p[\alpha]/h(\alpha)$ where $h$ is the degree $f$ polynomial chosen in the previous subsection. We consider $x^e - \alpha^r p$, as the third part of Proposition 2.1. lets us replace $\zeta$ by $\alpha$.

Since $\text{Gal}(K^u/\mathbb{Q}_p)$ is generated by Frobenius $\sigma$, with

$$\sigma(\alpha) \equiv \alpha^p \pmod{p},$$

the polynomials $x^e - \alpha^r p$ give conjugate extensions for $r$ which differ multiplicatively by a power of $p$. Taking the norm of $x^e - \alpha^r p$ to $\mathbb{Q}_p$, we get an irreducible polynomial iff the orbit of $r$ in $\mathbb{Z}/(p^f - 1)\mathbb{Z}$ under multiplication by $p$ has length $f$.

Our recipe for picking defining polynomials of tamely ramified extensions with given $e$ and $f$ is as follows. Let $g = \gcd(e, p^f - 1)$ and partition $\mathbb{Z}/g\mathbb{Z}$ into orbits under multiplication by $p$. These will correspond to the desired extensions of $\mathbb{Q}_p$. For each orbit $O \subseteq \mathbb{Z}/g\mathbb{Z}$, we lift its elements to $\mathbb{Z}/(p^f - 1)\mathbb{Z}$ and consider them under multiplication by $p$. If there is an orbit of length $f$, take the smallest $r \geq 0$ contained in such an orbit. Then the norm of $x^e - \alpha^r p$ to $\mathbb{Q}_p$ will be irreducible. If there are no lifts to an orbit of length $f$ for our orbit $O$, we apply the following “root shift” procedure. We take the smallest $r \geq 0$ representing an element of the orbit and consider polynomials $(x + k\alpha)^e - \alpha^r p$ with $k = 1, 2, 3, \ldots$ and take their norms to $\mathbb{Q}_p[x]$. The first norm which is irreducible is our preferred defining polynomial.
For example, to generate the sextic tame extensions of $\mathbb{Q}_5$ with residue degree 2, we first construct the unramified quadratic extension of $\mathbb{Q}_5$. By the procedure described in [2, 1] we have $K^u = \mathbb{Q}_5[\alpha]/(\alpha^2 - \alpha + 2)$. Here $g = \gcd(e, p^f - 1) = \gcd(3, 5^2 - 1) = 3$. Multiplication by 5 on $\mathbb{Z}/3\mathbb{Z}$ has two orbits, $\{1, 2\}$ and $\{0\}$, so there will be two extensions. In the first case, $\{1, 5\} \subset \mathbb{Z}/24\mathbb{Z}$ is the prescribed lift, so we take the norm of $x^5 - 5\alpha$ to get $x^5 - 5x^3 + 50$. For the other orbit, the first orbit modulo 24 of length $f = 2$ reducing to $\{0\}$ is $\{3, 15\}$. Thus, we take the norm of $x^3 - 5\alpha^3$ to get $x^9 + 25x^3 + 200$.

As an example where the root shift procedure is necessary, consider degree 12 extensions of $\mathbb{Q}_5$ with $e = 6$ and $f = 2$, so that $g = \gcd(6, 24) = 6$. The orbit $\{0\} \subset \mathbb{Z}/6\mathbb{Z}$ has only lifts of size 1 in $\mathbb{Z}/24\mathbb{Z}$. However root shifting with $k = 1$ gives us the norm of $(x + \alpha)^6 - 5$, which is the irreducible polynomial $x^{12} + 6x^{11} + 27x^{10} + 80x^9 + 195x^8 + 366x^7 + 571x^6 + 702x^5 + 1005x^4 + 1140x^3 + 357x^2 - 138x + 44$.

2.3. Degree $p$ ramified extensions of $\mathbb{Q}_p$. The six ramified quadratic extensions of $\mathbb{Q}_2$ are given by $x^2 - D$ for $D = -4$, 12, $\pm 8$, and $\pm 24$, with $\ord_2(D)$ being the discriminantal exponent $c$. Each of these six extensions has two automorphisms. The rest of this subsection treats the case of $p$ odd, which is different as the generic degree $p$ extension of $\mathbb{Q}_p$ has just the identity automorphism.

Most of the information we need can then be extracted from [Am]. Table 2.2 summarizes these results, giving exactly one polynomial for each isomorphism class of degree $p$ extension of $\mathbb{Q}_p$. These come in three families, the main one with two parameters and the other two families each with one parameter. Table 2.2 gives our preferred defining polynomials, restrictions on the parameter(s), the exponent $c$ of the discriminant, and the Galois and inertia groups. To define $d_1$ and $d_2$, let $g = \gcd(p - 1, c)$. Then $d_1 = (p - 1)/g$ and $d_2 = (p - 1)/(\gcd((p - 1)/m, g))$ where $m$ is the order of $a\lambda$ in $F_p^\times$.

| Family | Parameters | $c$ | $G$ | $I$ |
|--------|------------|-----|-----|-----|
| $x^p + apx^\lambda + p$ | $0 < a \leq p - 1$ | $p + \lambda - 1$ | $C_p, C_d_2$ | $C_p, C_d_4$ |
| | $1 \leq \lambda \leq p - 1$ | $p + \lambda - 1$ | $C_p, C_d_2$ | $C_p, C_d_4$ |
| | $(\lambda, a) \neq (p - 1, p - 1)$ | $p + \lambda - 1$ | $C_p, C_d_2$ | $C_p, C_d_4$ |
| $x^p - px^{p-1} + p(1 + ap)$ | $0 \leq a \leq p - 1$ | $2p - 2$ | $C_p$ | $C_p$ |
| $x^p + p(1 + ap)$ | $0 \leq a \leq p - 1$ | $2p - 1$ | $C_p, C_{p-1}$ | $C_p, C_{p-1}$ |

2.4. Wild extensions of composite degree. The complexity of the unramified, tamely ramified, and degree $p$ cases just treated suggests that analogous recipes for the remaining cases would have to be quite complex indeed. So instead, we treat the five cases $(p, n) = (2, 4)$, $(2, 6)$, $(2, 8)$, $(3, 6)$, and $(3, 9)$ individually. The problem then becomes simply to find a defining polynomial for each degree $n$ extension of $\mathbb{Q}_p$ for the given $(p, n)$.

Pauli and Roblot give a general algorithm for solving this problem. One key ingredient is Panayi’s $p$-adic root finding algorithm [PR, Section 8] which lets one determine whether two degree $n$ fields $\mathbb{Q}_p[x]/f_1(x)$ and $\mathbb{Q}_p[x]/f_2(x)$ are isomorphic and similarly lets one compute the set of automorphisms of a given field $\mathbb{Q}_p[x]/f(x)$. Another key ingredient is the mass formula [PR, Theorem 6.1] which lets one determine when all fields have been found.
We used Pauli and Roblot’s approach for generating polynomials as needed. However, in many cases, we were able to generate the polynomials more efficiently by specialized methods. For example, to generate 2-adic fields of a given degree which have an index 2 subfield, we were able to look up all possible candidates for subfields from our database and construct the desired fields by taking square roots of suitably chosen elements. Similarly, when dealing with sextic fields we worked with the twin algebra \( [R_0] \). All except twelve 3-adic fields have reducible twin algebras. Thus, almost all sextic fields could be generated by sextic twinning using lower degree fields from the database.

3. Invariants associated to a given \( p \)-adic field

Let \( f(x) \in \mathbb{Z}[x] \) be a polynomial on one of our \( p \)-adic tables. In this section, we discuss the invariants the tables present for the corresponding field \( K = \mathbb{Q}_p[x]/f(x) \). Table 3.1 serves as an example for much of our discussion in this section.

### Table 3.1. The first six lines of the 2-adic quartic table, corresponding to the fields with \( c \leq 4 \).

| \( c \) | \( e \) | \( f \) | \( \epsilon \) | \( d \) | Polynomial | \( G \) | \( I \) | Wild Slopes | \( GMS \) | Deg 2 Subs |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 1 | * | \( x^4 + x + 1 \) | \( C_4 \) | \( < e > \) | 0 | * |
| 4 | 2 | 2 | -1 | 1 | \( x^4 + 8x^2 + 4 \) | \( V_4 \) | \( C_2 \) | 2 | 1 | * | -1 | -* |
| 4 | 2 | 2 | -1 | * | \( x^4 - x^2 + 5 \) | \( C_4 \) | \( C_2 \) | 2 | 1 | * |
| 4 | 2 | 2 | -i | -1 | \( x^4 + 2x^2 + 4x + 4 \) | \( D_4 \) | \( V_4 \) | 2, 2 | 3/2 | * |
| 4 | 2 | 2 | -i | -* | \( x^4 - 5 \) | \( D_4 \) | \( V_4 \) | 2, 2 | 3/2 | * |

3.1. Basic Data. The field discriminant of \( K \) as an ideal is \( (p^e) \subseteq \mathbb{Z}_p \). The largest unramified subfield of \( K^u \) of \( K \) has degree the residual degree \( f = [K^u : \mathbb{Q}_p] \). The ramification index is \( e = n/f = [K : K^u] \). The entry \( d \) in the fifth column is the field discriminant considered as an element of \( \mathbb{Q}_p^\times/\mathbb{Q}_p^\times 2 \). Here and elsewhere, \( * \in \mathbb{Q}_p^\times/\mathbb{Q}_p^\times 2 \) stands for a non-square unit. With this notational convention, \( Q_2^\times/\mathbb{Q}_2^\times 2 = \{1, *, -1, -*, 2, 2*, -2, -2*, 2*\} \) and otherwise \( Q_2^\times/\mathbb{Q}_2^\times 2 = \{1, *, p, ps\} \). The computer program gp [PARP] has commands to compute \( c, e, f, \) and \( d \).

3.2. Subfields and automorphisms. For each field, we give its subfields hyperlinked to their respective entries in the database. Quadratic subfields are listed by the codes described in the previous section. An unramified subfield of degree \( d \) is listed as simply \( U_d \). All other subfields are listed by their chosen defining polynomial. To determine if one field is a subfield of another, we make use of Panayi’s \( p \)-adic root finding algorithm mentioned in \([E2]\). Similarly, we use this algorithm to find the automorphisms of \( K \).

3.3. Root numbers. The root number \( \epsilon \) is a complex fourth root of unity. It is of use in distinguishing fields, especially when \( p = 2 \). However its principal use is in applications, for example to quadratic lifting, as explained in \([JR1] \) Section 2.1.
3.4. Galois groups. Let $K^g$ be a Galois closure of $K$. Our tables present the isomorphism type of the local Galois group $G = \text{Gal}(K^g/Q_p)$. Let $K^{9,u}$ be the maximum unramified subfield of $K^g$. Our tables also present the inertia group $I = \text{Gal}(K^g/K^{9,u})$, which is a normal subgroup of $G$ such that $G/I$ is cyclic. Let $K^{9,1}$ be the maximal tame subfield of $K^g$. Then the wild inertia group $I_w = \text{Gal}(K^g/K^{9,1})$ is the normal subgroup of $I$ of $p$-power order and prime-to-$p$ cyclic quotient. The quantity $t = |I/I_w|$, which appears in equation (2) below, can then easily be read off.

The computation of Galois groups over $Q_p$ is similar to the more familiar computation of Galois groups over $Q$. For example, knowledge of the automorphism group $\text{Aut}(K/Q_p)$ and complete knowledge of subfields of $K$ restricts the possibilities. Also factoring resolvents, now over $Q_p$, of course, is the principal technique for distinguishing between possible Galois groups. However some techniques for computing Galois groups over $Q$ are not available when working over $Q_p$. For example, one cannot use cycle types of Frobenius elements for varying primes $p$. In compensation, there are techniques which are particular to local fields. Certainly extensions of $Q_p$ are always solvable, as indeed one has the wild-tame-unramified filtration $I_w \leq I \leq G$. Also wild slopes, as discussed in the next subsection, can be used to get lower bounds on the size of $I_w$; in this sense they serve as substitute for Frobenius elements, which provide lower bounds in the global case.

The discriminant class $d$ determines the parity of the Galois group. This much suffices for $n = 3$, i.e. $G \cong A_3$ if $d = 1$ and $G \cong S_3$ if $d \neq 1$. Table 3.2 summarizes the computation in degrees $n = 4$ and 5. Here we use the number of automorphisms of the degree $n$ field $K$ to distinguish within $(C_4, D_4, S_4), (V_4, A_4),$ and $(C_5, D_5)$. Note that no resolvents are necessary here beyond using $d$ to determine the parity of $G$, which is equivalent to considering the factorization of $x^2 - d$.

| $G$ | $C_4$ | $V_4$ | $D_4$ | $A_4$ | $S_4$ | $C_5$ | $D_5$ | $F_5$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| Parity | $-$ | $+$ | $-$ | $+$ | $+$ | $+$ | $+$ |
| $|\text{Aut}(K)|$ | 4 | 4 | 2 | 1 | 1 | 5 | 1 | 1 |

There are 16 transitive subgroups of $S_6$ up to conjugation, 12 of which are solvable. The Algorithm 6.3.10 of [Co] starts by computing a resolvant polynomial for the twin algebra of $f$. Most cases are determined by factoring this twin and considering Galois groups of the irreducible factors, as shown by Table 3.3. Here, $C_1$ corresponds to linear factors in the twin polynomial. The two ambiguities remaining are distinguished by the parities of the groups: $T_7$ is even while $T_{11}$ is odd, and $T_{10} \cong C_2^3.C_4$ is even while $T_{13} \cong C_2^3.D_4$ is odd. Algorithm 6.3.10 mentioned above sometimes requires the computation of a second resolvant. Here that is unnecessary because of the absence of non-solvable groups. Also here the information on $|\text{Aut}(K)|$ is not needed to distinguish Galois groups; we present it because of the important role it plays in the mass formula.

In $S_7$, there are 7 transitive subgroups of which 4 are solvable, namely $C_7,C_d$ for $d = 1, 2, 3,$ and 6. One could apply Algorithm 6.3.11 of [Co] directly. It uses the discriminant and the factorization of a degree 35 resolvant. However, this high degree resolvant can be avoided since all cases are covered by Section 2. If $K$ is unramified, then $G = C_7$. If $G$ is tamely ramified, then $G = C_7.C_d$ with $d$ the order.
Table 3.3. Galois groups for $n = 6$.

| $G$ | T1 | T2 | T3 | T4 | T5 | T6 |
|-----|----|----|----|----|----|----|
| Twin | $C_2 C_3 C_1$ | $S_3 C_2$ | $S_3 C_2$ | $A_4 C_2^2$ | $S_3 C_3$ | $A_4 C_2$ |
| Parity | - | - | - | + | - | - |
| $|\text{Aut}(K)|$ | 6 | 6 | 2 | 2 | 3 | 2 |

| $G$ | T7 | T8 | T9 | T10 | T11 | T13 |
|-----|----|----|----|------|------|------|
| Twin | $S_1 C_2$ | $S_3 C_2$ | $S_3 C_3$ | Irred | $S_3 C_2$ | Irred |
| Parity | + | - | - | + | - | - |
| $|\text{Aut}(K)|$ | 2 | 2 | 1 | 1 | 2 | 1 |

of the subgroup of $\mathbb{F}_7^2$ generated by $p$. If $G$ is wildly ramified, then Table 2.2 covers the situation. In the much more complicated cases $n = 8$ and $n = 9$, we similarly use specifically $p$-adic phenomena to avoid large degree resolvents.

3.5. Slope filtration and root discriminants. For each field we compute its slope content as in [JR1], whose definition we briefly recall here. Let $G^\nu$ be the ramification groups with Artin upper numbering so that $G^0 = G$ and $G^1 = I$. Non-trivial quotients $Q^\nu = G^\nu/G^{\nu+1}$ contribute slope $\nu$. In terms of the wild-tame-unramified filtration mentioned above, $Q^0 = I/I_w$ corresponds to the tame part, and slopes greater than 1 contribute to the wild part. We have discussed finding $|Q^0|$ and $|Q^1|$ in the previous subsection, so our focus now is finding the wild slopes $\nu$. Our tables list each wild slope $\nu$, repeated according to its multiplicity $m = \text{ord}_p(|Q^\nu|)$. So the total number of wild slopes listed is $\text{ord}_p(|I|) = \log_p(|I_w|)$.

To compute slopes for a field, we use the following proposition.

**Proposition 3.4.** Let $K$ be a $p$-adic field. Say a subfield is distinguished iff all other subfields of the same degree have strictly larger discriminant exponent. Then the distinguished subfields form a chain

$$Q_p = K_0 \subset K_1 \subset \cdots K_{n-1} \subset K_n = K.$$

Say $K_i$ has degree $n_i$ and discriminant $p^{c_i}$. Then the

$$s_i = \frac{c_i - c_{i-1}}{n_i - n_{i-1}}$$

are weakly increasing. If $s_{j+1} = s_{j+2} = \cdots = s_i$, then $|Q^{s_i}| \geq n_i/n_j$.

This proposition follows from the basic facts of ramification theory as follows. An irreducible representation

$$\rho: G \to \text{Aut}(V)$$

of $G$ has a slope $s(\rho)$, namely the smallest $c \in [0, \infty)$ with $G^{>c}$ in the kernel of $\rho$. The Artin exponent $c(\rho)$ of $\rho$ is the slope $s(\rho)$ times the degree $\text{dim}(V)$. Artin exponents of arbitrary representations behave additively. If $\rho$ is induced from a permutation representation of $G$ on a finite set $X$, then the discriminant of the $p$-adic algebra corresponding to $X$ is $p^{c(\rho)}$. Take $X_i$ corresponding to $K_i$ so that the inclusions $K_{i-1} \subset K_i$ give surjections $X_i \to X_{i-1}$. The action of $G$ on $V = \mathbb{Q}[X_i]/\mathbb{Q}[X_{i-1}]$ must have a single slope as otherwise there would be a
distinguished subfield between $K_{i-1}$ and $K_i$. As the Artin exponent of $V$ is $c_i - c_{i-1}$ and the dimension of $V$ is $n_i - n_{i-1}$, one gets (1).

To analyze a field $K$, we begin by applying Proposition 3.4 directly to $K$. We refer to the wild slopes we see here as visible slopes. In many low degree cases, these suffice. For example, for a ramified degree $p$ extension of $\mathbb{Q}_p$, $p$ exactly divides the order of the inertia group. Thus, there is exactly one wild slope and it is visible from the degree $p$ extension, being just $c/(p-1)$. When we have not found enough slopes to account for all of $G$, we apply Proposition 3.4 to various resolvent fields, typically the same resolvents used to compute $G$.

For example, the first case of wild ramification with composite degree is 2-adic quartics. Here up to two wild slopes will be visible from the quartic field. When the Galois group is $C_4$, $V_4$, $A_4$, or $S_4$, this suffices, the visible slopes being the only slopes. The remaining case for 2-adic quartics is when the Galois group is $D_4$, where there are several viable approaches. One would be to compute the octic Galois field by taking the compositum of the defining quartic and $x^2 - d$. At this octic level, all slopes are visible.

When computing slopes of sextic fields, we make use of the sextic twin algebra, which we compute as part of the Galois group computation. In most cases, this is the product of smaller degree fields. On the one hand, we will have already have computed the slopes for these smaller fields. On the other hand, when considering the composita of the lower degree fields, slopes may combine in non-trivial ways, as discussed below in §4.3.

The twin algebra of a solvable sextic field is also a field when the Galois group of the normal closure is $C_2 \wr C_4$ or $C_3 \wr D_4$. These only appear for $p = 3$, where the first appears four times and the second eight times. In both cases, the chain of distinguished fields takes the form $\mathbb{Q}_3 \subset K_2 \subset K_6$, with the latter two fields now indexed by degree. Proposition 3.4 then says that $(c_6 - c_2)/4$ is a wild slope. Since neither group contains a normal subgroup of order 3, this slope must be repeated with multiplicity 2. This is an instance where group theory makes resolvent constructions unnecessary. The cases of degrees eight and nine involved group-theoretical arguments of this type, as well as actual resolvent constructions.

3.6. Galois mean slope. The root discriminant of $K$ is $p^{c/n}$. Also interesting is the corresponding quantity for the Galois closure $K^g$, which we write as $p^\beta$ with $\beta$ what we call the Galois mean slope. This Galois mean slope can be computed as a weighted sum of the slopes, with larger slopes counted more heavily. More precisely, let $s_1, \ldots, s_m$ be the slopes in decreasing order, so that $|I_w| = p^m$. Then

$$\beta = \left( \sum_{i=1}^{m} \frac{p-1}{p^i} s_i \right) + \frac{1}{p^m} \frac{t-1}{t},$$

with $t = |G/I|$. One has $\beta \geq c/n$ with equality iff $K^g \cong K \otimes U$ with $U$ an unramified extension of $\mathbb{Q}_p$, i.e. iff there are no hidden slopes. The Galois mean slopes $\beta$ play an important role in §4.3.

4. A sample use of the database

Our database has two interactive features, the $p$-adic identifier and the GRD calculator. Here we illustrate how they can be used in the study of number fields by means of a family of examples.
4.1. A family of number fields. Let
\[ f_t(x) = (x^6 + 12x^5 + 54x^4 + 176x^3 + 444x^2 + 624x + 552) \cdot (x^4 + 16x^3 - 36x^2 + 128x - 28)(x^3 - 12x^2 - 6x - 64) + t(3x^4 - 4x^3 + 12x^2 - 24x - 68)^2(x^4 + 16x^3 + 72x^2 + 128x + 188). \]

This one parameter family of polynomials was first introduced in [Ma]. Its polynomial discriminant is $2^{160}3^{114}(t^2 + 8)^{10}$. We are interested in the fields $K_t = \mathbb{Q}[x]/f_t(x)$ for $t \in \mathbb{Q}$. Let $K_t^7$ be the splitting field of $f_t(x)$ in $\mathbb{C}$. For generic $t$, the Galois group $\text{Gal}(K_t^7/\mathbb{Q})$ is isomorphic to the projective linear group $P\text{SL}_2(\mathbb{F}_3)$, of order $2^43^313 = 5616$. For these $t$, $K_{-t} = \mathbb{Q}[x]/f_{-t}(x)$ is the projective twin of $K_t$, meaning that if $K_t$ corresponds to an action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the projective space $\mathbb{P}^2(\mathbb{F}_3)$, then $K_{-t}$ corresponds to the induced action on the dual projective space $\mathbb{P}^2(\mathbb{F}_3)$. So $K_t$ and $K_{-t}$ are non-isomorphic fields with the same Dedekind zeta function, the same discriminant, and the same Galois closure $K_t^9 = K_{-t}^9$. If $p$ is such that the local decomposition group contains an element of order 13 then both $K_t \otimes \mathbb{Q}_p$ and $K_{-t} \otimes \mathbb{Q}_p$ are degree 13 fields, thus treated by the dynamic part of our database. Otherwise, by consideration of maximal subgroups of $P\text{SL}_3(\mathbb{F}_3)$, at least one of $K_t$ and $K_{-t}$ splits $p$-adically into a nonic times a quartic, and perhaps further.

4.2. The $p$-adic identifier. The $p$-adic identifier lets one input a polynomial $f(x) \in \mathbb{Z}[x]$ and a prime $p$. It uses Panayi’s root finding algorithm and returns the entries from the database corresponding to the factor fields of $\mathbb{Q}_p[x]/f(x)$. In the context of the family $f_t(x)$, it lets one study the $p$-adic completion of these fields, as a function of the parameter $t$, now considered $p$-adically. The interesting cases are $p = 2$ and $p = 3$, as otherwise ramification is tame. For $p = 2$, the behavior is locally constant for $t \in \mathbb{Q}_2$. Experiment quickly suggests that the factorization pattern is $4 + 4 + 2 + 2 + 1$ if $\text{ord}_2(t) \geq 4$ and otherwise $8 + 4 + 1$, with many possible fields arising for the octic. For $p = 3$, the behavior is locally constant for $t \in \mathbb{Q}_3 - \{ \sqrt[3]{-8}, -\sqrt[3]{8} \}$ with $\sqrt[3]{-8} = 1 + 3^2 + 3^3 + 2 \cdot 3^4 + 3^5 + 6^2 + 2 \cdot 3^6 + \cdots \approx 14293$. Here again the fields involved vary substantially with $t$. However experiment suggests that factor fields are always of degree $\leq 9$, unless $\text{ord}_3(t - \sqrt[3]{8}) \geq 3$, in which case they may be of degree 12.

As a specific example, consider $t = \pm 7/2$, which we choose because the corresponding fields have the smallest discriminant we are aware of for $P\text{SL}_2(\mathbb{F}_3)$ fields, namely $2^{28}3^{22}$. For comparison, the database [KM] has $2^{28}3^{24}$ as its smallest discriminant, and the two fields given there are $K_{\pm 1}$. Applying gp’s polredabs to $f_{\pm 7/2}(x)$ to get smaller coefficients, we have
\[ g_{-7/2}(x) = x^{13} - 2x^{12} - 8x^{10} + 55x^9 - 90x^8 - 108x^7 + 684x^6 - 1341x^5 + 1526x^4 - 1090x^3 + 468x^2 - 100x + 8 \]
\[ g_{7/2}(x) = x^{13} - x^{12} - 3x^{11} - 7x^{10} + 37x^9 - 9x^8 - 168x^7 + 24x^6 + 396x^5 + 20x^4 - 128x^3 + 192x^2 - 176x - 16. \]

Entering $f_{-7/2}(x)$ or equally well $g_{-7/2}(x)$, one gets the 2-adic information listed in Table [M]. One also gets that the polynomial factors 3-adically as $12+1$, not a
Table 4.1. Information output by the $p$-adic identifier for the polynomial $g_{-7/2}(x)$. The field factors 2-adically as $\mathbb{Q}_2 \times K_4 \times K_8$ with information on $K_4$ and $K_8$ being given in the first block. The field factors 3-adically as $K_4 \times K_9$ with information on $K_4$ and $K_9$ given in the second block.

| $c$ | $e$ | $f$ | $\epsilon$ | $d$ | Polynomial | $G$ | $I$ | Wild Slopes | GMS | Subfields |
|-----|-----|-----|-------------|-----|------------|-----|-----|-------------|-----|----------|
| 6   | 2   | 2   | $i$         | $-1$| $x^4 + 2x^2 - 4$ | $D_4$ | $V_4$ | 2, 3         | 2   | 3*       |
| 22  | 4   | 2   | $-i$        | $-1$| $x^8 + 10x^4 + 28$| $D_{8}$| $D_{19}$| 2, 3, 4      | 3   | $*, x^4 - 2x^2 + 4$ |
| 3   | 4   | 1   | $i$         | 3   | $x^3 + 3$          | $D_4$ | $C_4$ | $T_{19}$ | $T_{15}$ | $19/8, 19/8$, 53/24 | 3* |
| 19  | 9   | 1   | $i$         | 3   | $x^9 + 9x^3 + 3$   | $T_{19}$ | $T_{15}$ | 19/8, 19/8 | 53/24 | 3* |

surprise, as

$$\text{ord}_3(-7/2 - \sqrt{-8}) = \text{ord}_3(-7/2 - 14293) = \text{ord}_3(-3^4 \cdot 353/2) = 4.$$ 

Entering instead $f_{7/2}(x)$ or $g_{7/2}(x)$, one gets the same identification at 2 and now a 3-adic factorization of 9+4, with information as on Table 4.1.

4.3. The GRD calculator. A single numerical measure of ramification in a polynomial $f(x) \in \mathbb{Z}[x]$ is the root discriminant of its splitting field in $\mathbb{C}$, what we call its Galois root discriminant. The GRD calculator accepts a polynomial $f(x) \in \mathbb{Z}[x]$ as input. When all factors of all completions of $\mathbb{Q}[x]/f(x)$ are in the database, it returns lower and upper bounds on the Galois mean slope $\beta_p$ of each ramifying prime $p$, and hence bounds on the Galois root discriminant $\prod p^\beta_p$. In favorable cases, certainly when the $p$-adic algebra has only one wild factor, the lower and upper bounds on $\beta_p$ agree. In the remaining cases, it is typically easy to start with the bounds and continue by hand to exactly determine $\beta_p$, as our example will illustrate.

The reason the GRD calculator returns only bounds is as follows. We are assuming that all the $p$-adic factors $K_{p,i}$ of $K \otimes \mathbb{Q}_p$ are in our database. Thus we know the corresponding tame parts of inertia $t_i$ and the multiplicities $m_i(s)$ of any given wild slope $s > 1$. We need the corresponding information $t$ and $m(s)$ for the algebra $\prod_i K_{p,i}$. The tame index for the algebra is simply $t = \text{lcm}(t_i)$, and this formula is incorporated into the GRD calculator. The biggest possible Galois mean slope would arise if $m(s) = \sum_i m_i(s)$ for all $s$. The smallest possible Galois mean slope would arise if $m(s) = \max_i m_i(s)$ for all $s$. When these bounds disagree, i.e. when there is some overlap between the factor wild slope lists, there may be quite a number of possibilities in between, including slopes in the algebra which are not in any of the factors. For example, suppose $K_{p,1}$ and $K_{p,2}$ are distinct degree $p$ fields each with $c = 2p - 1$, thus each with unique wild slope $(2p - 1)/(p - 1)$. Then the slopes of $K_{p,1} \times K_{p,2}$ are $(2p - 1)/(p - 1)$ and $p/(p - 1)$.

The point of symmetry $t = 0$ of our family of polynomials is forced to be special with respect to Galois theory as $K_0$ is its own projective twin. It factors over $\mathbb{Q}$ as $6+4+3$ and its Galois group is $S_4$. The small size of the global Galois group limits
ramification at 2 and 3, and the GRD is the relatively small number $2^{11/4}3^{7/8} \approx 24.2367$. All this is recorded as the top entry of Table 4.2.

Table 4.2. The seven smallest GRDs found in the family $f_t(x)$. The first five correspond to globally reducible degree 13 polynomials, as indicated by the second block of columns. The last two correspond to irreducible degree 13 polynomials.

| $t$ | $K_{-t}$ | $K_t$ | GRD       | $|G|$ | $\text{ord}_3(t^2 + 8)$ |
|-----|----------|-------|------------|------|------------------------|
| 0   | 6, 4, 3  |       | $2^{11/4}3^{7/6} \approx 24.2367$ | 24   | 0                      |
| 23/5| 12, 1    | 9, 4  | $2^{11/4}3^{7/8}5^{1/2} \approx 39.3368$ | 144  | 6                      |
| 10  | 9, 4     | 12, 1 | $2^{29/8}3^{7/6} \approx 44.4504$ | 432  | 3                      |
| $6808/2209$ | 12, 1 | 9, 4  | $2^{37/8}11^{1/2} \approx 69.3853$ | 144  | 6                      |
| $10516/725$ | 12, 1 | 9, 4  | $2^{29/8}29^{1/2} \approx 66.4405$ | 432  | 15                     |
| $7/2$ | 13      | 13    | $2^{33/3}24 \approx 90.5175$ | 5616 | 4                      |
| 1   | 13      | 13    | $2^{11/4}3^{33/54} \approx 100.6849$ | 5616 | 2                      |

We have made a substantial search over carefully chosen rational $t$ and Table 4.2 presents the $t$ we have found with GRD less than 110. In each case, the lower bound returned by the GRD calculator is exact. For example, consider the 2-adic Galois mean slope in the case $t = \pm 7/2$, so that Table 1.1 applies. A local reason that one need not consider the quartic and its contribution of 2, 3 is that the quartic is a subfield of the octic. We plan to incorporate this sort of argument into the GRD calculator in the future, although many local arguments of this sort are more subtle. A global reason that one need not consider the quartic’s contribution is that the octic’s Galois group already has 16 elements and 16 exactly divides $5616 = 2^43^313$. Variants on this sort of global argument work in many situations.

Table 4.2 shows that for the four pairs $\pm t$ yielding the smallest GRD, the Galois group drops down from its generic size of 5616. The column $\text{ord}_3(t^2 + 8)$ gives the maximum of the numbers $\text{ord}_3(t + \sqrt{-8})$ and $\text{ord}_3(t - \sqrt{-8})$, the other being zero. Experiment suggests that ramification can be tame at 3 only if this number is $\geq 6$ and 3 can be unramified only if this number is $\geq 15$, while 2 is always wildly ramified with Galois mean slope at least 11/4; these observations guided our search for $t$. Many similar computations are presented in [JR2], where the “low GRD Galois drop” phenomenon visible in Table 4.2 is evident in similar strength. Detailed global analyses such as these would not be possible without quite complete control over local phenomena.

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