Duality Symmetry and Soldering in Different Dimensions

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Abstract

We develop a systematic method of obtaining duality symmetric actions in different dimensions. This technique is applied for the quantum mechanical harmonic oscillator, the scalar field theory in two dimensions and the Maxwell theory in four dimensions. In all cases there are two such distinct actions. Furthermore, by soldering these distinct actions in any dimension a master action is obtained which is duality invariant under a much bigger set of symmetries than is usually envisaged. The concept of swapping duality is introduced and its implications are discussed. The effects of coupling to gravity are also elaborated. Finally, the extension of the analysis for arbitrary dimensions is indicated.

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\begin{footnotesize}
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\end{footnotesize}
1 Introduction

The intriguing role of duality in different contexts is being progressively understood and clarified \[1, 2, 3, 4\]. Much effort has been given in sorting out several technical aspects of duality symmetric actions. In this context the old idea \[1, 5, 6\] of electromagnetic duality has been revived with considerable attention and emphasis \[7, 8, 9, 10\]. Recent directions \[7, 11, 12, 13\] also include an abstraction of manifestly covariant forms for such actions or an explicit proof of their equivalence with the nonduality symmetric actions, which they are supposed to represent. There are also different suggestions on the possible analogies between duality symmetric actions in different dimensions. In particular it has been claimed \[12\] that the two dimensional self dual action given in \[14\] is the analogue of the four dimensional electromagnetic duality symmetric action \[7\]. In spite of the recent spate of papers on this subject there does not seem to be a simple clear cut way of arriving at duality symmetric actions. Consequently the fundamental nature of duality remains clouded by technicalities. Additionally, the dimensionality of space time appears to be extremely crucial. For instance, while the duality symmetry in $D = 4k$ dimensions is characterised by the one-parameter continuous group $SO(2)$, that in $D = 4k + 2$ dimensions is described by a discrete group with just two elements \[10\]. Likewise, it has also been argued from general notions that a symmetry generator exists only in the former case. From an algebraic point of view the distinction between the dimensionalities is manifested by the following identities,

\[
**F = F \quad ; \quad D = 4k + 2 \\
= -F \quad ; \quad D = 4k
\]

where the $*$ denotes a usual Hodge dual operation and $F$ is the $\frac{D}{2}$ form. Thus there is a self dual operation in the former which is missing in the latter dimensions. This apparently leads to separate consequences for duality in these cases.

The object of this paper is to develop a method for systematically obtaining and investigating different aspects of duality symmetric actions that embrace all dimensions. A deep unifying structure is illuminated which also leads to new symmetries. Indeed we show that duality is not limited to field or string theories, but is present even in the simplest of quantum mechanical examples- the harmonic oscillator. It is precisely this duality which pervades all field theoretical examples as will be explicitly shown. The basic idea of our approach is deceptively simple. We start from the second order action for any theory and convert it to the first order form by introducing
an auxiliary variable. Next, a suitable relabelling of variables is done which induces an internal index in the theory. It is crucial to note that there are two distinct classes of relabelling characterised by the opposite signatures of the determinant of the $2 \times 2$ orthogonal matrix defined in the internal space. Correspondingly, in this space there are two actions that are manifestly duality symmetric. Interestingly, their equations of motion are just the self and anti-self dual solutions, where the dual field in the internal space is defined below in (2). It is also found that in all cases there is one (conventional duality) symmetry transformation which preserves the invariance of these actions but there is another transformation which swaps the actions. We refer to this property as swapping duality. This indicates the possibility, in any dimensions, of combining the two actions to a master action that would contain all the duality symmetries. Indeed this construction is explicitly done by exploiting the ideas of soldering introduced in [15] and developed by us [16, 17]. The soldered master action also has manifest Lorentz or general coordinate invariance. The generators of the symmetry transformations are also obtained.

It is easy to visualise how the internal space effectively unifies the results in the different $4k + 2$ and $4k$ dimensions. The dual field is now defined to include the internal index $(\alpha, \beta)$ in the fashion,

$$\tilde{F}^\alpha = \varepsilon^{\alpha\beta} F^\beta ; \quad D = 4k$$
$$\tilde{F}^\alpha = \sigma_1^{\alpha\beta} F^\beta ; \quad D = 4k + 2$$

(2)

where $\sigma_1$ is the usual Pauli matrix and $\varepsilon_{\alpha}\beta$ is the fully antisymmetric $2 \times 2$ matrix with $\varepsilon_{12} = 1$. Now, irrespective of the dimensionality, the repetition of the dual operation yields,

$$\tilde{\tilde{F}} = F$$

(3)

which generalises the relation (2). An immediate consequence of this is the possibility to obtain self and anti-self dual solutions in all even $D = 2k + 2$ dimensions. Their explicit realisation is one of the central results of the paper.

The paper is organised into five sections. In section 2 the above ideas are exposed by considering the example of the simple harmonic oscillator. A close parallel with the electromagnetic notation is also developed to illuminate the connection between this exercise and those given for the field theoretical models in the next two sections. The duality of scalar field theory in two dimensions is considered in section 3. The occurrence of a pair of actions is shown which exhibit duality and swapping symmetries. These are the analogues of the four dimensional electromagnetic duality symmetric actions. Indeed, from these expressions, it is a trivial matter to reproduce
both the self and anti-self dual actions given in [14]. Our analysis clarifies several issues regarding the intertwining roles of chirality and duality in two dimensions. The soldering of the pair of duality symmetric actions is also performed leading to fresh insights. The analysis is completed by including the effects of gravity. In section 4, the Maxwell theory is treated in great details. Following our prescription the duality symmetric action given in [7] is obtained. However, there is also a new action which is duality symmetric. Once again the soldering of these actions leads to a master action which contains a much richer structure of symmetries. Incidentally, it also manifests the original symmetry that interchanges the Maxwell equations with the Bianchi identity, but reverses the signature of the action. As usual, the effects of gravity are straightforwardly included. Section 5 contains the concluding comments.

2 Duality in 0 + 1 dimension

The basic features of duality symmetric actions are already present in the quantum mechanical examples as the present analysis on the harmonic oscillator will clearly demonstrate. Indeed, this simple example is worked out in some details to illustrate the key concepts of our approach and set the general tone of the paper. An extension to field theory is more a matter of technique rather than introducing truly new concepts. The Lagrangean for the one-dimensional oscillator is given by,

\[ L = \frac{1}{2} (\dot{q}^2 - q^2) \]  

leading to an equation of motion,

\[ \ddot{q} + q = 0 \]  

Introducing a change of variables,

\[ E = \dot{q} \quad ; \quad B = q \]  

so that,

\[ \dot{B} - E = 0 \]  
is identically satisfied, the above equations (4) and (5) are, respectively, expressed as follows;

\[ L = \frac{1}{2} (E^2 - B^2) \]
and,

\[ \dot{E} + B = 0 \]  

(9)

It is simple to observe that the transformations,

\[ E \to \pm B; \quad B \to \mp E \]  

(10)

swaps the equation of motion (9) with the identity (7) although the Lagrangean (8) is not invariant. The similarity with the corresponding analysis in the Maxwell theory is quite striking, with \( q \) and \( \dot{q} \) simulating the roles of the magnetic and electric fields, respectively. There is a duality among the equation of motion and the ‘Bianchi’ identity (7), which is not manifested in the Lagrangean.

In order to elevate the duality to the Lagrangean, the basic step is to rewrite (4) in the first order form by introducing an additional variable,

\[
L = p\dot{q} - \frac{1}{2}(p^2 + q^2)
= \frac{1}{2}(p\dot{q} - q\dot{p} - p^2 - q^2)
\]  

(11)

where a symmetrisation has been performed. There are now two possible classes for relabelling these variables corresponding to proper and improper rotations generated by the matrices \( R^+(\theta) \) and \( R^-(\phi) \) with determinant +1 and −1, respectively,

\[
\begin{pmatrix}
q \\
p
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]  

(12)

\[
\begin{pmatrix}
q \\
p
\end{pmatrix} =
\begin{pmatrix}
\sin \phi & \cos \phi \\
\cos \phi & -\sin \phi
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]  

(13)

leading to the distinct Lagrangeans,

\[
L_\pm = \frac{1}{2} \left( \pm x_\alpha \epsilon_{\alpha\beta} x_\beta - x_\alpha^2 \right)
= \frac{1}{2} \left( \pm B_\alpha \epsilon_{\alpha\beta} E_\beta - B_\alpha^2 \right)
\]  

(14)

\footnote{Note that these are just the discrete cases \((\alpha = \pm \frac{\pi}{2})\) for a general \(SO(2)\) rotation matrix parametrised by the angle \(\alpha\).}
where we have reverted back to the ‘electromagnetic’ notation introduced in (9). By these change of variables an index $\alpha = (1, 2)$ has been introduced that characterises a symmetry in this internal space, the complete details of which will progressively become clear. It is useful to remark that the above change of variables are succinctly expressed as,

$$
q = x_1 \ ; \ p = x_2 \\
q = x_2 \ ; \ p = x_1
$$

by setting the angle $\theta = 0$ or $\varphi = 0$ in the rotation matrices (12) and (13). Correspondingly, the Lagrangean (11) goes over to (14). Now observe that the above Lagrangeans (14) are manifestly invariant under the continuous duality transformations,

$$
x_\alpha \rightarrow R_{\alpha \beta}^+ x_\beta
$$

which may be equivalently expressed as,

$$
E_\alpha \rightarrow R_{\alpha \beta}^+ E_\beta \\
B_\alpha \rightarrow R_{\alpha \beta}^+ B_\beta
$$

where $R_{\alpha \beta}^+$ is the usual $SO(2)$ rotation matrix (12). The generator of the infinitesimal symmetry transformation is given by,

$$
Q^\pm = \pm \frac{1}{2} x_\alpha x_\alpha
$$

so that the complete transformations (16) are generated as,

$$
x_\alpha \rightarrow x_\alpha' = e^{-i\theta Q} x_\alpha e^{i\theta Q} \\
= R_{\alpha \beta}^+(\theta) x_\beta
$$

This is easy to verify by using the basic symplectic brackets obtained from (14),

$$
\{x_\alpha, x_\beta\} = \mp \epsilon_{\alpha \beta}
$$

Parametrising the angle by $\theta = \frac{\pi}{2}$ the discrete transformation is obtained,

$$
E_\alpha \rightarrow \epsilon_{\alpha \beta} E_\beta \\
B_\alpha \rightarrow \epsilon_{\alpha \beta} B_\beta
$$
This is the parallel of the usual constructions done in the Maxwell theory to induce a duality symmetry in the action.

Let us now comment on an interesting property, which is related to the existence of two distinct Lagrangeans (14), by replacing (17) with a new set of transformations,

\[
E_\alpha \rightarrow R^-_{\alpha \beta}(\varphi)E_\beta \\
B_\alpha \rightarrow R^-_{\alpha \beta}(\varphi)B_\beta
\]  

(22)

Notice that these transformations preserve the invariance of the Hamiltonian following from either \(L_+\) or \(L_-\). Interestingly, the kinetic terms in the Lagrangeans change signatures so that \(L_+\) and \(L_-\) swap into one another. This feature of duality swapping will subsequently recur in a different context and has important implications in higher dimensions.

The discretised version of (22) is obtained by setting \(\varphi = 0\),

\[
E_\alpha \rightarrow \sigma^\alpha_1 E_\beta \\
B_\alpha \rightarrow \sigma^\alpha_1 B_\beta
\]  

(23)

It is precisely the \(\sigma_1\) matrix that reflects the proper into improper rotations,

\[
R^+(\theta)\sigma_1 = R^-(\theta)
\]  

(24)

which illuminates the reason behind the swapping of the Lagrangeans in this example.

Since we have systematically developed a procedure for obtaining a duality symmetric Lagrangean, it is really not necessary to show its equivalence with the original Lagrangean, as was done in the Maxwell theory. Nevertheless, to complete the analogy, we show that (14) reduces to (8) or (4) by using the equation of motion,

\[
x_\alpha = \pm \epsilon_{\alpha \beta} \dot{x}_\beta
\]  

(25)

which can be reexpressed as,

\[
B_\alpha = \pm \epsilon_{\alpha \beta} E_\beta
\]  

(26)

to eliminate one component (say the variables with label 2) from (14). This immediately reproduces (8) while (21) reduces to (11).

An important point to stress is that there are actually two, and not one, duality symmetric actions \(L_{\pm}(14)\), corresponding to the signatures in the determinant of the transformation matrices. As shall be shown in subsequent sections this is also true for the scalar field theory in 1 + 1 dimensions and the electromagnetic theory in 3 + 1
dimensions. Usually, in the literature, only one of these is highlighted while the other is not mentioned. We now elaborate the implications of this property which will also be crucial in discussing field theoretical models. In the coordinate language these Lagrangeans correspond to two bi-dimensional chiral oscillators rotating in opposite directions. This is easily verified either by looking at the classical equations of motion or by examining the spectrum of the angular momentum operator,

\[ J_\pm = \pm \epsilon_{ij} x_i p_j = \pm H \]  

where \( H \) is the Hamiltonian of the usual harmonic oscillator. In other words the two Lagrangeans manifest the dual aspects of rotational symmetry in the two-dimensional internal space. Consequently it is possible to solder them by following the general techniques elaborated in \[16, 17\]. This soldering as well as its implications are the subject of the remainder of this section.

The soldering mechanism, it must be recalled, is intrinsically an operation that has no classical analogue. The crucial point is that the Lagrangeans (14) are now considered as functions of independent variables, namely \( L_+(x) \) and \( L_-(y) \), instead of the same \( x \). A naive addition of the classical Lagrangeans with the same variable is of course possible leading to a trivial result. If, on the other hand, the Lagrangeans are functions of distinct variables, a straightforward addition does not lead to any new information. The soldering process precisely achieves this purpose. Consider the gauging of the Lagrangeans under the following gauge transformations,

\[ \delta x_\alpha = \delta y_\alpha = \dot{\eta}_\alpha \]  

Then the gauge variations are given by,

\[ \delta L_\pm (z) = \epsilon_{\alpha\beta} \dot{\eta}_\alpha J^{(\pm)}_\beta (z) ; \ z = x, y \]  

where the currents are defined by,

\[ J^{(\pm)}_\alpha (z) = \pm \dot{z}_\alpha + \epsilon_{\alpha\beta} z_\beta \]  

Introducing a new field \( B_\alpha \) transforming as,

\[ \delta B_\alpha = \epsilon_{\beta\alpha} \dot{\eta}_\beta \]

that will effect the soldering, it is possible to construct a first iterated Lagrangian,

\[ L^{(1)}_\pm = L_\pm - B_\alpha J^{\pm}_\alpha \]
The gauge variation of (32) is easily obtained,

$$\delta L^{(1)}_\pm = -B_\alpha \delta J^\pm_\alpha$$

(33)

Using the above results we define a second iterated Lagrangean,

$$L^{(2)}_\pm = L^{(1)}_\pm - \frac{1}{2} B^2_\alpha$$

(34)

which finally leads to a Lagrangean,

$$L = L^{(2)}_+ (x) + L^{(2)}_- (y) = L_+ (x) + L_- (y) - B_\alpha (J^+_\alpha (x) + J^-_\alpha (y)) - B^2_\alpha$$

(35)

that is invariant under the complete set of transformations (28) and (31), i.e.;

$$\delta L = 0$$

(36)

It is now possible to eliminate the auxiliary $B_\alpha$ field by using the equation of motion, which yields,

$$B_\alpha = -\frac{1}{2} \left( J^+_\alpha (x) + J^-_\alpha (y) \right)$$

(37)

Inserting this solution back into (35), we obtain the final soldered Lagrangean,

$$L(w) = \frac{1}{4} \left( \dot{w}_\alpha^2 - w_\alpha^2 \right)$$

(38)

which is no longer a function of $x$ and $y$ independently, but only on their gauge invariant combination,

$$w_\alpha = (x_\alpha - y_\alpha)$$

(39)

The soldered Lagrangean just corresponds to a simple bi-dimensional oscillator. Thus, by starting from two Lagrangeans which contained the opposite aspects of a duality symmetry, it is feasible to combine them into a single Lagrangean which has a richer symmetry. A similar phenomenon also exists in the field theoretical examples, as shall be shown subsequently.

Let us now expose all the symmetries of the above Lagrangean. It is most economically done by recasting this Lagrangean in two equivalent forms,

$$L = \Omega^+_\alpha \Omega^-_\alpha = \bar{\Omega}^+_\alpha \bar{\Omega}^-_\alpha$$

(40)
where,
\[
\Omega^{\pm}_\alpha = \frac{1}{2}(\dot{w}_\alpha \pm \Lambda_{\alpha\beta} w_\beta)
\]
\[
\bar{\Omega}^{\pm}_\alpha = \frac{1}{2}(\Lambda_{\alpha\beta} \dot{w}_\beta \pm w_\alpha)
\]
\[
\Lambda_{\alpha\beta} = \begin{pmatrix} R^+_{\alpha\beta}, & R^-_{\alpha\beta} \end{pmatrix}
\] (41)

Now the Lagrangean (40) is manifestly symmetric under the following continuous dual transformations,
\[
w_\alpha \rightarrow R^\pm_{\alpha\beta} w_\beta (42)
\]
The transformation involving \( R^+ \) is just the original symmetry (17). Those involving the \( R^- \) matrices are the new symmetries. Recall that the latter transformations swapped the two independent Lagrangeans \( L^\pm \). The soldered Lagrangean contains both combinations and hence manifests both these symmetries. The corresponding symmetry group is therefore \( O(2) \). This is a completely new phenomenon. It also occurs in field theory with certain additional subtle features.

The generator of the infinitesimal transformations that leads to the \( SO(2) \) rotation in (42) is given by,
\[
Q = w_\alpha \epsilon_{\alpha\beta} \pi_\beta (43)
\]
so that,
\[
w_\alpha \rightarrow w'_\alpha = e^{-iQ} w_\alpha e^{iQ} = R^+_{\alpha\beta}(\theta) w_\beta (44)
\]
which is verified by using the canonical brackets,
\[
\{w_\alpha, \pi_\beta\} = \delta_{\alpha\beta} (45)
\]

It is worthwhile to point out the quantum nature of the above calculation by rewriting (38), after an appropriate scaling of variables, in the form of an identity,
\[
L(x - y) = L(x) + L(y) - 2x^+_\alpha y^-_\alpha
\]
\[
\dot{z}^\pm_\alpha = \frac{1}{\sqrt{2}}(\dot{z}_\alpha \pm \epsilon_{\alpha\beta} z_\beta) (46)
\]
This shows that the Lagrangean of the simple harmonic oscillator expressed in terms of the “gauge invariant” variables \( w = x - y \) is not obtained by just adding the independent contributions. Rather, there is a contact term which manifests the quantum
effect. Indeed, the above identity can be interpreted as the analogue of the well known Polyakov-Weigman \cite{18} identity in two dimensional field theory. As our analysis shows, such identities will always occur whenever dual aspects of some symmetry are being soldered or fused to yield a composite picture, irrespective of the dimensionality of space-time \cite{16}. In the Polyakov-Weigman case it was the chiral symmetry whereas here it was the rotational symmetry.

3 The Scalar Theory in 1+1 Dimensions

The ideas developed in the previous section are now implemented and elaborated in 1 + 1 dimensions. It is simple to realise that the scalar theory is a very natural example. For instance, in these dimensions, there is no photon and the Maxwell theory trivialises so that the electromagnetic field can be identified with a scalar field. Thus all the results presented here can be regarded as equally valid for the “photon” field. Indeed the computations will also be presented in a very suggestive notation which illuminates the Maxwellian nature of the problem. Consequently the present analysis is an excellent footboard for diving into the actual electromagnetic duality discussed in the next section. The effects of gravity are easily included in our approach as shown in a separate subsection.

The Lagrangean for the free massless scalar field is given by,

\[ \mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \right)^2 \] (47)

and the equation of motion reads,

\[ \ddot{\phi} - \phi'' = 0 \] (48)

where the dot and the prime denote derivatives with respect to time and space components, respectively. Introduce, as before, a change of variable using electromagnetic symbols,

\[ E = \dot{\phi} ; \quad B = \phi' \] (49)

Obviously, \( E \) and \( B \) are not independent but constrained by the identity,

\[ E' - \dot{B} = 0 \] (50)

In these variables the equation of motion and the Lagrangean are expressed as,

\[ \dot{E} - B' = 0 \]

\[ \mathcal{L} = \frac{1}{2} \left( E^2 - B^2 \right) \] (51)
It is now easy to observe that the transformations,

\[ E \rightarrow \pm B ; \quad B \rightarrow \pm E \]  

display a duality between the equation of motion and the ‘Bianchi’-like identity (52) but the Lagrangean changes its signature. Note that there is a relative change in the signatures of the duality transformations (10) and (52), arising basically from dimensional considerations. This symmetry corresponds to the improper group of rotations.

To illuminate the close connection with the Maxwell formulation, we introduce covariant and contravariant vectors with a Minkowskian metric \( g_{00} = -g_{11} = 1 \),

\[ F_\mu = \partial_\mu \phi ; \quad F^\mu = \partial^\mu \phi \]  

whose components are just the ‘electric’ and ‘magnetic’ fields defined earlier,

\[ F_\mu = (E, B) ; \quad F^\mu = (E, -B) \]  

Likewise, with the convention \( \epsilon_{01} = 1 \), the dual field is defined,

\[ *F_\mu = \epsilon_{\mu\nu} \partial^\nu \phi = \epsilon_{\mu\nu} F^\nu = (-B, -E) \]  

The equations of motion and the ‘Bianchi’ identity are now expressed by typical electrodynamical relations,

\[ \partial_\mu F^\mu = 0 \]  
\[ \partial_\mu *F^\mu = 0 \]  

To expose a Lagrangean duality symmetry, the basic principle of our approach to convert the original second order form (51) to its first order version and then invoke a relabelling of variables to provide an internal index, is adopted. This is easily achieved by first introducing an auxiliary field,

\[ \mathcal{L} = P E - \frac{1}{2} P^2 - \frac{1}{2} B^2 \]  

where \( E \) and \( B \) have already been defined. The following renaming of variables corresponding to the proper and improper transformations (see for instance (12) and (13) or (15)) is used,

\[ \phi \rightarrow \phi_1 \]  
\[ P \rightarrow \pm \phi_2' \]
where we are just considering the discrete sets (13) of the full symmetry (12) and (13). Then it is possible to recast (57) in the form,

\[ \mathcal{L} \rightarrow \mathcal{L}_\pm = \frac{1}{2} \left[ \pm \phi'_\alpha \sigma_1^{\alpha\beta} \dot{\phi}_\beta - \phi'^2_\alpha \right] = \frac{1}{2} \left[ \pm B_\alpha \sigma_1^{\alpha\beta} E_\beta - B^2_\alpha \right] \tag{59} \]

In the second line the Lagrangean is expressed in terms of the electromagnetic variables. This Lagrangean is duality symmetric under the transformations of the basic scalar fields,

\[ \phi_\alpha \rightarrow \sigma_1^{\alpha\beta} \phi_\beta \tag{60} \]

which, in the notation of \( E \) and \( B \), is given by,

\[ B_\alpha \rightarrow \sigma_1^{\alpha\beta} B_\beta \\
E_\alpha \rightarrow \sigma_1^{\alpha\beta} E_\beta \tag{61} \]

It is quite interesting to observe that, contrary to the harmonic oscillator example or the electromagnetic theory discussed in the next section, the transformation matrix in the \( O(2) \) space is not the epsilon, but rather a Pauli matrix. This result is in agreement with that found from general algebraic arguments \[7, 10\] which stated that for \( d = 4k + 2 \) dimensions there is a discrete \( \sigma_1 \) symmetry. Observe that (61) is a manifestation of the original duality (52) which was also effected by the same operation. It is important to stress that the above symmetry is only implementable at the discrete level. Moreover, since it is not connected to the identity, there is no generator for this transformation.

To complete the picture, we also mention that the following rotation,

\[ \phi_\alpha \rightarrow \epsilon_{\alpha\beta} \phi_\beta \tag{62} \]

interchanges the Lagrangeans (59),

\[ \mathcal{L}_+ \leftrightarrow \mathcal{L}_- \tag{63} \]

Thus, except for a rearrangement of the the matrices generating the various transformations, most features of the simple harmonic oscillator example are perfectly retained. The crucial point of departure is that now all these transformations are
only discrete. Interestingly, the master action constructed below lifts these symmetries from the discrete to the continuous.

Let us therefore solder the two distinct Lagrangeans to manifestly display the complete symmetries. Before doing this it is instructive to unravel the self and anti-self dual aspects of these Lagrangeans, which are essential to physically understand the soldering process. The equations of motion following from (59), in the language of the basic fields, is given by,

\[ \partial_\mu \phi_\alpha = \mp \sigma^1_{\alpha\beta} \epsilon_{\mu\nu} \partial_\nu \phi_\beta \]  

(64)

provided reasonable boundary conditions are assumed. Note that although the duality symmetric Lagrangean is not manifestly Lorentz covariant, the equations of motion possess this property. We will return to this aspect again in the Maxwell theory. In terms of a vector field \( F^\alpha_\mu \) and its dual \( *F^\alpha_\mu \) defined analogously to (54), (55), the equation of motion is rewritten as,

\[ F^\alpha_\mu = \pm \sigma^1_{\alpha\beta} *F^\beta_\mu = \pm \tilde{F}^\alpha_\mu \]  

(65)

where the generalised Hodge dual (\( \tilde{F} \)) has been defined in (2). This explicitly reveals the self and anti-self dual nature of the solutions in the combined internal and coordinate spaces. The result can be extended to any \( D = 4k + 2 \) dimensions with suitable insertion of indices.

We now solder the two Lagrangeans. This is best done by using the notation of the basic fields of the scalar theory. These Lagrangeans \( \mathcal{L}_+ \) and \( \mathcal{L}_- \) are regarded as functions of the independent scalar fields \( \phi_\alpha \) and \( \rho_\alpha \). Consider the gauging of the following symmetry,

\[ \delta \phi_\alpha = \delta \rho_\alpha = \eta_\alpha \]  

(66)

Following exactly the steps performed for the harmonic oscillator example the final Lagrangean analogous to (35) is obtained,

\[ \mathcal{L} = \mathcal{L}_+(\phi) + \mathcal{L}_-(\rho) - B_\alpha \left( J^+(\phi) + J^-(\rho) \right) - B^2_\alpha \]  

(67)

where the currents are given by,

\[ J^\pm_\alpha(\theta) = \pm \sigma^1_{\alpha\beta} \dot{\theta}_\beta - \theta'_\alpha \ ; \ \theta = \phi \ , \ \rho \]  

(68)

The above Lagrangean is gauge invariant under the extended transformations including (68) and,

\[ \delta B_\alpha = \eta'_\alpha \]  

(69)
Eliminating the auxiliary $B_\alpha$ field using the equations of motion, the final soldered Lagrangean is obtained from (67),

$$\mathcal{L}(\Phi) = \frac{1}{4} \partial_\mu \Phi_\alpha \partial^\mu \Phi_\alpha$$

(70)

where, expectedly, this is now only a function of the gauge invariant variable,

$$\Phi_\alpha = \phi_\alpha - \rho_\alpha$$

(71)

This master Lagrangean possesses all the symmetries that are expressed by the continuous transformations,

$$\Phi_\alpha \rightarrow R^{\pm}_{\alpha \beta}(\theta) \Phi_\beta$$

(72)

The generator corresponding to the $SO(2)$ transformations is easily obtained,

$$Q = \int dy \Phi_\alpha \epsilon_{\alpha \beta} \Pi_\beta$$

$$\Phi_\alpha \rightarrow \Phi'_\alpha = e^{-i\theta Q} \Phi_\alpha e^{i\theta Q}$$

(73)

where $\Pi_\alpha$ is the momentum conjugate to $\Phi_\alpha$. Observe that either the original symmetry in $\sigma_1$ or the swapping transformations were only at the discrete level. The process of soldering has lifted these transformations from the discrete to the continuous form. It is equally important to reemphasize that the master action now possesses the $SO(2)$ symmetry which is more commonly associated with four dimensional duality symmetric actions, and not for two dimensional theories. Note that by using the electromagnetic symbols, the Lagrangean can be displayed in a form which manifests the soldering effect of the self and anti self dual symmetries (65),

$$\mathcal{L} = \frac{1}{8} \left( F^\alpha_\mu + \tilde{F}^\alpha_\mu \right) \left( F^\mu_\alpha - \tilde{F}^\mu_\alpha \right)$$

(74)

where the generalised Hodge dual in $D = 4k + 2$ dimensions has been defined in (2).

An interesting observation is now made. Recall that the original duality transformation (52) switching equations of motion into Bianchi identities may be rephrased in the internal space by,

$$E_\alpha \rightarrow \mp R^{\pm}_{\alpha \beta} B_\beta$$

$$B_\alpha \rightarrow \mp R^{\pm}_{\alpha \beta} E_\beta$$

(75)
which is further written directly in terms of the scalar fields,

$$\partial_{\mu} \Phi_\alpha \rightarrow \pm R_{\alpha\beta}^\pm \epsilon_{\mu\nu} \partial^\nu \Phi_\beta$$  

(76)

It is simple to verify that under these transformations even the Hamiltonian for the theories described by the Lagrangeans $L_\pm$ (59) are not invariant. However the Hamiltonian following from the master Lagrangean (70) preserves this symmetry. The Lagrangean itself changes its signature. This is the exact analogue of the original situation. A similar phenomenon also occurs in the electromagnetic theory. This completes the discussion on the symmetries of the master Lagrangean.

It is now straightforward to give a Polyakov-Weigman type identity, that relates the “gauge invariant” Lagrangean with the non gauge invariant structures, by reformulating (70) after a scaling of the fields $(\phi, \rho) \rightarrow \sqrt{2}(\phi, \rho)$,

$$L(\Phi) = L(\phi) + L(\rho) - 2\partial_+ \phi_\alpha \partial_- \rho_\alpha$$  

(77)

where the light cone variables are given by,

$$\partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$$  

(78)

Observe that, as in the harmonic oscillator example, the gauge invariance is with regard to the transformations introduced for the soldering of the symmetries. Thus, even if the theory does not have a gauge symmetry in the usual sense, the dual symmetries of the theory can simulate the effects of the former. This leads to a Polyakov-Wiegman type identity which has an identical structure to the conventional identity.

Before closing this sub-section, it may be useful to highlight some other aspects of duality which are peculiar to two dimensions, as for instance, the chiral symmetry. The interpretation of this symmetry with regard to duality seems, at least to us, to be a source of some confusion and controversy. As is well known a scalar field in two dimensions can be decomposed into two chiral pieces, described by Floreanini Jackiw (FJ) actions [14]. These actions are sometimes regarded [12] as the two dimensional analogues of the duality symmetric four dimensional electromagnetic actions [7]. Such an interpretation is debatable since the latter have the $SO(2)$ symmetry (characterised by an internal index $\alpha$) which is obviously lacking in the FJ actions. Our analysis, on the other hand, has shown how to incorporate this symmetry in the two dimensional case. Hence we consider the actions defined by (59) to be the true analogue of the duality symmetric electromagnetic actions to be discussed later. Moreover, by
solving the equations of motion of the FJ action, it is not possible to recover the second order free scalar Lagrangean, quite in contrast to the electromagnetic theory [7]. Nevertheless, since the FJ actions are just the chiral components of the usual scalar action, these must be soldered to reproduce this result. But if soldering is possible, such actions must also display the self and anti-self dual aspects of chiral symmetry. This phenomenon is now explored along with the soldering process.

The two FJ actions defined in terms of the independent scalar fields \( \phi^+ \) and \( \phi^- \) are given by,

\[
L_{\pm} F J (\phi_{\pm}) = \pm \dot{\phi}_{\pm} \phi'_{\pm} - \phi''_{\pm} \tag{79}
\]

whose equations of motion show the self and anti self dual aspects,

\[
\partial_\mu \phi_{\pm} = \mp \epsilon_{\mu \nu} \partial_\nu \phi_{\pm} \tag{80}
\]

A trivial application of the soldering mechanism leads to,

\[
L(\Phi) = L_{+} F J (\phi^+) + L_{-} F J (\phi^-) + \frac{1}{8} \left( J^+ (\phi^+) + J^- (\phi^-) \right)^2
\]

\[
= \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi \tag{81}
\]

where the currents \( J_{\pm} \) and the composite field \( \Phi \) are given by,

\[
J_{\pm} = 2 \left( \pm \dot{\phi}_{\pm} - \phi'_{\pm} \right)
\]

\[
\Phi = \phi^+ - \phi^-
\]

Thus the usual scalar action is obtained in terms of the composite field. The previous analysis has, however, shown that each of the Lagrangeans (59) are equivalent to the usual scalar theory. Hence these Lagrangeans contain both chiralities described by the FJ actions (79). However, in the internal space, \( L_{\pm} \) carry the self and anti self dual solutions, respectively. This clearly illuminates the ubiquitous role of chirality versus duality in the two dimensional theories which has been missed in the literature simply because, following conventional analysis in four dimensions [6, 7], only one particular duality symmetric Lagrangean \( L^- \) was imagined to exist.

### 3.1 Coupling to gravity

It is easy to extend the analysis to include gravity. This is most economically done by using the language of electrodynamics already introduced. The Lagrangean for
the scalar field coupled to gravity is given by,

\[ L = \frac{1}{2} \sqrt{-g} g^{\mu\nu} F_\mu F_\nu \]  

(83)

where \( F_\mu \) is defined in (54) and \( g = \det g_{\mu\nu} \). Converting the Lagrangean to its first order form, we obtain,

\[ L = PE - \frac{1}{2\sqrt{-g} g^{00}} (P^2 + B^2) + \frac{g^{01}}{g^{00}} MB \]  

(84)

where the \( E \) and \( B \) fields are defined in (49) and \( P \) is an auxiliary field. Let us next invoke a change of variables mapping \((E, B) \rightarrow (E_1, B_1)\) by means of the \( O(2) \) transformation analogous to (58), and relabel the variable \( P \) by \( \pm B_2 \). Then the Lagrangean (84) assumes the distinct forms,

\[ L_{\pm} = \frac{1}{2} \left[ \pm B_\alpha \sigma^1_{\alpha\beta} E_\beta - \frac{1}{\sqrt{-g} g^{00}} B^2 \pm \frac{g^{01}}{g^{00}} \sigma^1_{\alpha\beta} B_\alpha B_\beta \right] \]  

(85)

which are duality symmetric under the transformations (51). As in the flat metric, there is a swapping between \( L_+ \) and \( L_- \) if the transformation matrix is \( \epsilon_{\alpha\beta} \). To obtain a duality symmetric action for all transformations it is necessary to construct the master action obtained by soldering the two independent pieces. The dual aspects of the symmetry that will be soldered are revealed by looking at the equations of motion following from (85),

\[ \sqrt{-g} F^\alpha_\mu = \mp g_{\mu\nu} \sigma^\alpha_{\beta\gamma} F^{\beta\gamma} \]  

(86)

The result of the soldering process, following from our standard techniques, leads to the master Lagrangean,

\[ L = \frac{1}{4} \sqrt{-g} g^{\mu\nu} F^\alpha_\mu F^\alpha_\nu \]  

(87)

where \( F^\alpha_\mu \) is defined in terms of the composite field given in (71). In the flat space this just reduces to the expression found previously in (70). It may be pointed out that, originating from this master action it is possible, by passing to a first order form, to recover the original pieces.

To conclude, we show how the FJ action now follows trivially by taking any one particular form of the two Lagrangeans, say \( L_+ \). To make contact with the conventional expressions quoted in the literature [19], it is useful to revert to the scalar field notation, so that,

\[ L_+ = \frac{1}{2} \left[ \phi_1' \phi_2 + \phi_2' \phi_1 + 2 \frac{g^{01}}{g^{00}} \phi_1' \phi_2' - \frac{1}{g^{00}} \phi_1' \phi_1' \right] \]  

(88)
This is diagonalised by the following choice of variables,

\[
\begin{align*}
\phi_1 &= \phi_+ + \phi_- \\
\phi_2 &= \phi_+ - \phi_-
\end{align*}
\]  

leading to,

\[
\mathcal{L}_+ = \mathcal{L}_+^{(+)}(\phi_+, \mathcal{G}_+) + \mathcal{L}_+^{(-)}(\phi_-, \mathcal{G}_-)
\]

with,

\[
\mathcal{L}_+^{(\pm)}(\phi_{\pm}, \mathcal{G}_\pm) = \pm \dot{\phi}_{\pm} \phi'_{\pm} + \mathcal{G}_{\pm} \phi'_{\pm} \phi'_{\pm}
\]

\[
\mathcal{G}_\pm = \frac{1}{g^{00}} \left( - \frac{1}{\sqrt{-g}} \pm g^{01} \right)
\]

These are the usual FJ actions in curved space as given in \[19\]. Such a structure was suggested by gauging the conformal symmetry of the free scalar field and then confirmed by checking the classical invariance under gauge and affine transformations \[19\]. Here we have derived this result directly from the action of the scalar field minimally coupled to gravity.

Observe that the explicit diagonalisation carried out in \[88\] for two dimensions is actually a specific feature of \(4k+2\) dimensions. This is related to the basic identity \(8\) governing the dual operation. If, however, one works with the master (soldered) Lagrangean, then diagonalisation is possible in either \(D = 4k + 2\) or \(D = 4k\) dimensions since the corresponding identity \(3\) always has the correct signature.

4 The Electromagnetic Duality

Exploiting the ideas elaborated in the previous sections, it is straightforward to implement duality in the electromagnetic theory. Let us start with the usual Maxwell Lagrangean,

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\]

which is expressed in terms of the electric and magnetic fields as\footnote{Bold face letters denote three vectors.}

\[
\mathcal{L} = \frac{1}{2} \left( \mathbf{E}^2 - \mathbf{B}^2 \right)
\]
where,
\[
E_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0 \\
B_i = \epsilon_{ijk} \partial_j A_k
\]  
(94)

The following duality transformation,
\[
E \rightarrow \mp B \; ; \; B \rightarrow \pm E
\]  
(95)
is known to preserve the invariance of the full set comprising Maxwell’s equations and the Bianchi identities although the Lagrangean changes its signature. To have a duality symmetric Lagrangean, we now know how to proceed in a systematic manner. The Maxwell Lagrangean is therefore recast in a symmetrised first order form,
\[
\mathcal{L} = \frac{1}{2} (\dot{P} \dot{A} - \dot{P} A) - \frac{1}{2} P^2 - \frac{1}{2} B^2 + A_0 \nabla \cdot P
\]  
(96)

Exactly as was done for the harmonic oscillator, a change of variables is invoked. Once again there are two possibilities which translate from the old set \((P, A)\) to the new ones \((A_1, A_2)\). It is, however, important to recall that the Maxwell theory has a constraint that is implemented by the Lagrange multiplier \(A_0\). The redefined variables are chosen which solve this constraint so that,
\[
P \rightarrow B_2 \; ; \; A \rightarrow A_1 \\
P \rightarrow B_1 \; ; \; A \rightarrow A_2
\]  
(97)

It is now simple to show that, in terms of the redefined variables, the original Maxwell Lagrangean takes the form,
\[
\mathcal{L}_\pm = \frac{1}{2} (\pm \dot{A}_\alpha \epsilon_{\alpha\beta} B_\beta - B_\alpha B_\alpha)
\]  
(98)

Adding a total derivative that would leave the equations of motion unchanged, this Lagrangean is expressed directly in terms of the electric and magnetic fields,
\[
\mathcal{L}_\pm = \frac{1}{2} (\pm B_\alpha \epsilon_{\alpha\beta} E_\beta - B_\alpha B_\alpha)
\]  
(99)

It is duality symmetric under the full \(SO(2)\) transformations mentioned in an earlier context. Note that one of the above structures (namely, \(\mathcal{L}_-\)) was given earlier in [7].

Once again, in analogy with the harmonic oscillator example, it is observed that the
transformation (22) involving the $R^-$ matrices switches the Lagrangeans $\mathcal{L}_+$ and $\mathcal{L}_-$ into one another. The generators of the $SO(2)$ rotations are given by,

$$Q^{(\pm)} = \mp \frac{1}{2} \int d^3x \ A^\alpha \cdot B^\alpha$$

so that,

$$A_\alpha \rightarrow A'_\alpha = e^{-iQ\theta} A_\alpha e^{iQ\theta}$$

This can be easily verified by using the basic brackets following from the symplectic structure of the theory,

$$[A^i_\alpha(x), e^{ijkl} \partial^k A^j_\beta(y)] = \pm i \delta^{ij} \epsilon_{\alpha\beta} \delta(x - y)$$

It is useful to digress on the significance of the above analysis. Since the duality symmetric Lagrangeans have been obtained directly from the Maxwell Lagrangean, it is redundant to show the equivalence of the former expressions with the latter, which is an essential perquisite in other approaches. Furthermore, since classical equations of motion have not been used at any stage, the purported equivalence holds at the quantum level. The need for any explicit demonstration of this fact, which has been the motivation of several recent papers, becomes, in this analysis, superfluous. A related observation is that the usual way of showing the classical equivalence is to use the equations of motion to eliminate one component from (98), thereby leading to the Maxwell Lagrangean in the temporal $A_0 = 0$ gauge. This is not surprising since the change of variables leading from the second to the first order form solved the Gauss law thereby eliminating the multiplier. Finally, note that there are two distinct structures for the duality symmetric Lagrangeans. These must correspond to the opposite aspects of some symmetry, which is next unravelled. By looking at the equations of motion obtained from (98),

$$\dot{A}_\alpha = \pm \epsilon_{\alpha\beta} \nabla \times A_\beta$$

it is possible to verify that these are just the self and anti-self dual solutions,

$$F^{\alpha}_{\mu\nu} = \pm \epsilon^{\alpha\beta*} F^{\beta}_{\mu\nu} \quad \Rightarrow F^{\beta}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} F^{\rho\lambda}_{\beta}$$

obtained by setting $A_0 = 0$. Recall that in the two dimensional theory the equation of motion naturally assumed a covariant structure. Here, on the other hand, the introduction of $A_0$ is necessary since this term gives a vanishing contribution to the
Lagrangean. This feature distinguishes a gauge theory from the non gauge theory discussed in the two dimensional example. It may be observed that the opposite aspects of the dual symmetry are contained in the internal space. Following our quantum mechanical analogy, the next task is to solder the two Lagrangeans (98). Consider then the gauging of the following symmetry,

\[ \delta H_\alpha = h_\alpha \ , \ \delta H = P, Q \quad (105) \]

where \( P \) and \( Q \) denote the basic fields in the Lagrangeans \( \mathcal{L}_+ \) and \( \mathcal{L}_- \), respectively. The Lagrangeans transform as,

\[ \delta \mathcal{L}_\pm = \epsilon_{\alpha\beta} \left( \nabla \times h_\alpha \right) J_{\beta}^\pm \quad (106) \]

with the currents defined by,

\[ J_\pm^\alpha (H) = \left( \mp H_\alpha + \epsilon_{\alpha\beta} \nabla \times H_\beta \right) \quad (107) \]

Next, the soldering field \( W_\alpha \) is introduced which transforms as,

\[ \delta W_\alpha = -\epsilon_{\alpha\beta} \nabla \times h_\beta \quad (108) \]

Following standard steps as outlined previously, the final Lagrangean which is invariant under the complete set of transformations (103) and (108) is obtained,

\[ \mathcal{L} = \mathcal{L}_+ (P) + \mathcal{L}_- (Q) - W^\alpha \left( J_+^\alpha (P) + J_-^\alpha (Q) \right) - W^2_\alpha \quad (109) \]

Eliminating the soldering field by using the equations of motion, the effective soldered Lagrangean following from (109) is derived,

\[ \mathcal{L} = \frac{1}{4} \left( \dot{G}_\alpha \cdot \dot{G}_\alpha - \nabla \times G_\alpha \cdot \nabla \times G_\alpha \right) \quad (110) \]

where the composite field is given by the combination,

\[ G_\alpha = P_\alpha - Q_\alpha \quad (111) \]

which is invariant under (103). It is interesting to note that, reinstating the \( G^\alpha_0 \) variable, this is nothing but the Maxwell Lagrangean with a doublet of fields,

\[ \mathcal{L} = -\frac{1}{4} G_{\mu\nu}^\alpha G^{\mu\nu}_\alpha \ ; \ G_{\mu\nu}^\alpha = \partial_\mu G^\alpha_\nu - \partial_\nu G^\alpha_\mu \quad (112) \]
In terms of the original $P$ and $Q$ fields it is once again possible, like the harmonic oscillator example, to write a Polyakov-Weigman like identity,

\[ \mathcal{L}(P - Q) = \mathcal{L}(P) + \mathcal{L}(Q) - 2W^+_{i,\alpha}(P)W^-_{i,\alpha}(Q) \]

\[ W^\pm_{i,\alpha}(H) = \frac{1}{\sqrt{2}} \left( F^\alpha_{0\alpha}(H) \pm \epsilon_{ijk}\epsilon_{\alpha\beta}\overline{F}^\beta_{jk}(H) \right) ; \quad H = P, Q \]

(113)

With respect to the gauge transformatins (105), the above identity shows that a contact term is necessary to restore the gauge invariant action from two gauge variant forms. This, it may be recalled, is just the basic content of the Polyakov-Weigman identity. It is interesting to note that the “mass” term appearing in the above identity is composed of parity preserving pieces $W^\pm_{i,\alpha}$, thanks to the presence of the compensating $\epsilon$-factor from the internal space.

Following the oscillator example, it is now possible to show that by reducing (112) to a first order form, we exactly obtain the two types of the duality symmetric Lagrangeans (99). This shows the equivalence of the soldering and reduction processes.

A particularly illuminating way of rewriting the Lagrangean (112) is,

\[ \mathcal{L} = -\frac{1}{8} \left( G^\alpha_{\mu\nu} + \epsilon^{\alpha\beta\gamma} G^\beta_{\mu\nu} \right) \left( G^\mu_\alpha - \epsilon_{\alpha\rho} G^\rho_{\mu\nu} \right) \]

\[ = -\frac{1}{8} \left( G^\alpha_{\mu\nu} + \tilde{G}^\alpha_{\mu\nu} \right) \left( G^\mu_\alpha - \tilde{G}^\mu_\alpha \right) \]

(114)

where, in the second line, the generalised Hodge dual in the space containing the internal index has been used to explicitly show the soldering of the self and anti self dual solutions. A similar situation prevailed in the two dimensional analysis. The above Lagrangean manifestly displays the following duality symmetries,

\[ A^\alpha_\mu \rightarrow R^\pm_{\alpha\beta} A^\beta_\mu \]

(115)

where, without any loss of generality, we may denote the composite field, of which $G_{\mu\nu}$ is a function, by $A$. The generator of the $SO(2)$ rotations is now given by,

\[ Q = \int dx \, \epsilon^{\alpha\beta} \Pi^\alpha \cdot A^\beta \]

(116)

Now observe that the master Lagrangian was obtained from the soldering of two distinct Lagrangeans (108). The latter were duality symmetric under both $A_\alpha \rightarrow \pm \epsilon_{\alpha\beta} A_\beta$, while the transformations involving the $\sigma_1$ matrix interchanged $\mathcal{L}_+$ with
\[ \mathcal{L}_-. \] The soldered Lagrangean is therefore duality symmetric under the transformations \((115)\). Furthermore, the discrete transformation related to the \(\sigma_1\) matrix has been lifted to its continuous form \(R^+\). The master Lagrangean, therefore, contains a bigger set of duality symmetries than \((98)\) and, significantly, is also manifestly Lorentz invariant. Furthermore, recall that under the transformations mapping the field to its dual, the original Maxwell equations are invariant but the Lagrangean changes its signature. The corresponding transformation in the \(SO(2)\) space is given by,

\[ G^\alpha_{\mu\nu} \rightarrow R^+_{\alpha\beta} G^\beta_{\mu\nu} \]

which, written in component notation, looks like,

\[ E^\alpha \rightarrow \mp \varepsilon^{\alpha\beta} B^\beta ; \quad B^\alpha \rightarrow \pm \varepsilon^{\alpha\beta} E^\beta \]

The standard duality symmetric Lagrangean fails to manifest this property. However, as may be easily checked, the equations of motion obtained from the master Lagrangean swap with the corresponding Bianchi identity while the Lagrangean flips sign. In this manner the original property of the second order Maxwell Lagrangean is retrieved. Note furthermore that the master Lagrangean possesses the \(\sigma_1\) symmetry (which is just the discretised version of \(R^-\)), a feature expected for two dimensional theories. A similar phenomenon occurred in the previous section where the master action in two dimensions revealed the \(SO(2)\) symmetry usually associated with four dimensional theories.

### 4.1 Coupling to gravity

To discuss how the effects of gravity are included, we will proceed as in the two dimensional example. The starting point is the Maxwell Lagrangean coupled to gravity,

\[ \mathcal{L} = -\frac{1}{4} \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \]

From our experience in the usual Maxwell theory we know that an eventual change of variables eliminates the Gauss law so that the term involving the multiplier \(A_0\) may be ignored from the outset. Expressing \((119)\) in terms of its components to separate explicitly the first and second order terms, we find,

\[ \mathcal{L} = \frac{1}{2} \dot{A}_i M^{ij} \dot{A}_j + M^i \dot{A}_i + M \]

24
where,

\[
M^{ij} = \sqrt{-g} \left( g^{0i} g^{0j} - g^{ij} g^{00} \right)
\]

\[
M^i = \sqrt{-gg^{0k} g^{ji} F_{jk}}
\]

\[
M = \frac{1}{4} \sqrt{-gg^{ij} g^{km} F_{im} F_{kj}}
\]  

(121)

Now reducing the Lagrangean to its first order form, we obtain,

\[
\mathcal{L} = P^i E_i - \frac{1}{2} P^i M_{ij} P^j - \frac{1}{2} M^i M_{ij} M^j + P^i M_{ij} M^j + M
\]  

(122)

where \( \dot{A}_i \) has been replaced by \( E_i \) and \( M_{ij} \) is the inverse of \( M^{ij} \),

\[
M_{ij} = \frac{-1}{\sqrt{-gg^{00} g_{ij}}}
\]  

(123)

with,

\[
g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda
\]  

(124)

Next, introducing the standard change of variables which solves the Gauss constraint,

\[
E_i \rightarrow E_i^{(1)}
\]

\[
P^i \rightarrow \pm B^{i(2)}
\]  

(125)

the Lagrangean (122) is expressed in the desired form,

\[
\mathcal{L}_\pm = \pm E_\alpha^\alpha \epsilon^{\alpha\beta} B_\beta^i + \frac{1}{\sqrt{-gg^{00} g_{ij}}} g_{ij} B^i_\alpha B^j_\beta
\]

\[
\pm \frac{g^{0k}}{g^{00}} \epsilon_{ijk} \epsilon^{\alpha\beta} B^i_\alpha B^j_\beta
\]  

(126)

Once again there are two duality symmetric actions corresponding to \( \mathcal{L}_\pm \). The enriched nature of the duality and swapping symmetries under a bigger set of transformations, the constructing of a master Lagrangean from soldering of \( \mathcal{L}_+ \) and \( \mathcal{L}_- \), the corresponding interpretations, all go through exactly as in the flat metric case. Incidentally, the structure for \( \mathcal{L}_- \) only was previously given in [1].
5 Conclusions

The present work revealed a unifying structure behind the construction of the various duality symmetric actions. The essential ingredient was the conversion of the second order action into a first order form followed by an appropriate redefinition of variables such that these may be denoted by an internal index. The duality naturally occurred in this internal space. Since the duality symmetric actions were directly derived from the original action the proof of their equivalence becomes superfluous. This is otherwise essential where such a derivation is lacking and recourse is taken to either equations of motion or some hamiltonian analysis. Obviously the most simple and fundamental manifestation of the duality property was in the context of the quantum mechanical harmonic oscillator. Since a field is interpreted as a collection of an infinite set of such oscillators, it is indeed expected and not at all surprising that all these concepts and constructions are almost carried over entirely for field theories. It may be remarked that the extension of the harmonic oscillator analysis to field theories has proved useful in other contexts and in this particular case has been really clinching. Furthermore, by invoking a highly suggestive electromagnetic notation for the harmonic oscillator analysis, its close correspondence with the field theory examples was highlighted.

A notable feature of the analysis was the revelation of a whole class of new symmetries and their interrelations. Different aspects of this feature were elaborated. To be precise, it was shown that there are actually two duality symmetric actions \( \mathcal{L}_\pm \) for the same theory. These actions carry the opposite (self and anti self dual) aspects of some symmetry and their occurrence was essentially tied to the fact that there were two distinct classes in which the renaming of variables was possible, depending on the signature of the determinant specifying the proper or improper rotations. To discuss further the implications of this pair of duality symmetric actions it is best to compare with the existing results. This also serves to put the present work in a proper perspective. It should be mentioned that the analysis for two and four dimensions are generic for \( 4k + 2 \) and \( 4k \) dimensions, respectively.

It is usually observed \[10\] that the invariance of the actions in different \( D \)-dimensions is preserved by the following groups,

\[
\mathcal{G}_d = Z_2 \ ; \ D = 4k + 2
\]  

and,

\[
\mathcal{G}_c = SO(2) \ ; \ D = 4k
\]  

\footnote{Note that usual discussions of duality symmetry consider only one of these actions, namely \( \mathcal{L}_- \).}

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which are called the “duality groups”. The $Z_2$ group is a discrete group with two elements, the trivial identity and the $\sigma_1$ matrix. Observe an important difference since in one case this group is continuous while in the other it is discrete. In our exercise this was easily verified by the pair of duality symmetric actions $L_\pm$. The new ingredient is that nontrivial elements of these groups are also responsible for the swapping $L_+ \leftrightarrow L_-$, but in the other dimensions. Thus the “duality swapping matrices” $\Sigma_s$ are given by,

$$\Sigma_s = \sigma_1 ; \quad D = 4k$$

$$= \epsilon ; \quad D = 4k + 2$$

(129)

It was next shown that $L_\pm$ contained the self and anti-self dual aspects of some symmetry. Consequently, following the ideas developed in [16, 17], the two Lagrangeans could be soldered to yield a master Lagrangean $L_m = L_+ \oplus L_-$. The master action, in any dimensions, was manifestly Lorentz or general coordinate invariant and was also duality symmetric under both the groups mentioned above. Moreover the process of soldering lifted the discrete group $Z_2$ to its continuous version. The duality group for the master action in either dimensionality therefore simplified to,

$$G = O(2) ; \quad D = 2k + 2$$

(130)

Thus, at the level of the master action, the fundamental distinction between the odd and even $N$-forms gets obliterated. It ought to be stated that the lack of usual Chern Simons terms in $D = 4k + 2$ dimensions to act as the generators of duality transformations is compensated by the presence of a similar term in the internal space. Thanks to this it was possible to explicitly construct the symmetry generators for the master action in either two or four dimensions.

We also showed that the master actions in any dimensions, apart from being duality symmetric under the $O(2)$ group, were factored, modulo a normalisation, as a product of the self and anti self dual solutions,

$$L = \left(F^\alpha + \tilde{F}^\alpha\right)\left(F^\alpha - \tilde{F}^\alpha\right) ; \quad D = 2k + 2$$

(131)

where the internal index has been explicitly written and the generalised Hodge operation was defined in [2]. The key ingredient in this construction was to provide a general definition of self duality that was applicable for either odd or even $N$ forms. Self duality was now defined to include the internal space and was implemented either by the $\sigma_1$ or the $\epsilon$, depending on the dimensionality. This naturally led to the universal structure (131).
Some other aspects of the analysis deserve attention. Specifically, the novel duality symmetric actions obtained in two dimensions revealed the interpolating role between duality and chirality. Furthermore, certain points concerning the interpretation of chirality symmetric action as the analogue of the duality symmetric electromagnetic action in four dimensions were clarified. We also recall that the soldering of actions to obtain a master action was an intrinsically quantum phenomenon that could be expressed in terms of an identity relating two “gauge variant” actions to a “gauge invariant” form. The gauge invariance is with regard to the set of transformations induced for effecting the soldering and has nothing to do with the conventional gauge transformations. In fact the important thing is that the distinct actions must possess the self and anti self dual aspects of some symmetry which are being soldered. The identities obtained in this way are effectively a generalisation of the usual Polyakov Weigman identity. We conclude by stressing the practical nature of our approach to duality which can be extended to other theories.

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