Sen-Witten orthonormal three-frame and gravitational energy quasilocalization

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Abstract. Expression for the Witten-Nester 4-spinor 3-form of the Hamiltonian density of gravitational field in the asymptotically flat space-time in terms of the Sommers-Sen spinors, direct with a certain orthonormal three-frame connect, is obtained. A direct connection between the one and the ADM Hamiltonian density in the Sen-Witten frame is established on this basis.

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1. Introduction

The equivalence principle excludes a possibility for existence of the gravitational energy density, however, there is possible its quasilocalization in the Penrose conception [1]. This conception is realized in several proposals for the quasilocal energy-momentum [2-7]. According to the Nester and coauthors approach, for each gravitational energy momentum pseudotensor there is Hamiltonian boundary term, and the energy-momentum in a domain, bounded by close 2-surface, depends on the field values and the frame of reference on the 2-surface. Various criteria are insufficient and, most probably [6], will be insufficient for selecting a unique Hamiltonian boundary term. Variety of these terms is characterized by different choices of dynamic variables (metric, orthonormal frame, spinors), boundary conditions and reference configurations. According to this there appears a problem of the different Hamiltonians comparing [3, 8].

Among criteria that must be satisfied by the quasilocal energy-momentum density, at least in asymptotically Minkowskian space [9], must be positivity. It can be ensured by finding the locally non-negative Hamiltonian density dependent on the Sen-Witten spinor according to Witten [10], or by applying the ADM Hamiltonian and the Nester special orthonormal frame (SOF) [16].

In the asymptotically flat space the Hamiltonian is of the general form [12]

\[ H(N) = \int_\Sigma (N H + N^a H_a) dV + \oint_{\partial \Sigma} B \]  

and includes the Regge-Teitelboim boundary term [13] at spacial infinity.

Grounding and developing the Wittenian proof of the positive energy theorem (PET), Nester [14] proposed an expression for the Hamiltonian density as the 4-covariant quadratic spinor 3-form:

\[ \mathcal{H}(\psi) := 2 \left[ D(\bar{\psi} \wedge \gamma_5 \gamma) \wedge D\psi - D\bar{\psi} \wedge D(\gamma_5 \gamma \psi) \right] \]

where

\[ D\psi = d\psi + \frac{1}{2} \omega^{\mu\nu} \sigma_{\mu\nu} \psi, \quad \sigma_{\mu\nu} = \frac{1}{2} [\gamma_{\mu}, \gamma_{\nu}], \quad \gamma = \gamma_{\mu} \theta^{\mu} \]

\[ \gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = 2g_{\mu\nu}, \quad \gamma_5^2 = -E, \quad \gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3. \]

From expression (2) one can obtain the following expression for \( \mathcal{H}(\psi) \):

\[ \mathcal{H}(\psi) = 4D\bar{\psi} \wedge \gamma_5 \gamma \wedge D\psi = 4 \nabla_\pi \psi \left( \gamma^\mu \sigma^{\pi\rho} + \sigma^{\pi\rho} \gamma^\mu \right) \nabla_\rho \psi d\Sigma_\mu \]

where \( d\Sigma_\mu = \frac{1}{3!} \sqrt{|g|} \varepsilon_{\mu\nu\pi\sigma} dx^\nu \wedge dx^\pi \wedge dx^\sigma \).

In the Gaussian normal system of coordinates in the neighborhood of arbitrary spacelike hypersurface \( \Sigma \), the Hamiltonian density (3) can be written as a sum of positive and negative definite components [14]

\[ \mathcal{H}(\psi) = -4g^{ab} D_a \psi^+ D_b \psi + D_a \bar{\psi} \left( \gamma^d \gamma^a \gamma^b + \gamma^a \gamma^b \gamma^d \right) D_b \psi d\Sigma_0 \]
and be locally non-negative if $SL(4, \mathbb{C})$-spinor $\psi$ on the spacelike hypersurface $\Sigma$ satisfies the Sen-Witten equation (SWE)

$$\gamma^a D_a \psi = 0.$$  \hspace{1cm} (5)

Expression (4) cannot give the true positive energy density for the gravitational field because $\psi$, as solution of the SWE, is a nonlocal functional on the initial data $(h, K, \Sigma)$ set; therefore, a concept of the locally non-negative density of the gravitational energy is treated as the locally non-negative functional on the set of initial data $(h, K, \Sigma)$ and the boundary values of function $\psi$. The gravitational Hamiltonian density (4) has significant advantages in comparison with the other ones: except a fact that it is explicitly 4-covariant, the gravitational Hamiltonian, which includes it, allows to prove that the total 4-momentum and the Bondi 4-momentum are timelike. To its liabilities Nester and Tung had referred the physical mysteriousness of the Sen-Witten spinor field, and absence of the direct relation to the Hamiltonian density in the SOF method [15, 16, 11]. For establishing such relation, Nester and Tung [3] had developed a new method of proving the PET and the gravitational energy localization, which employees the 3-dimensional spinors and a new identity connecting the 3-dimensional scalar curvature to the spinor expression in the Hamiltonian. The Einstein 3-spinor Hamiltonian with a zero shift the authors obtained in the form \(\dagger\)

$$H = \int_{\Sigma} \left[ \varphi^{+} \varphi g^{-\frac{1}{2}} \left( \pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) - 4 \left( \sigma^{ab} \nabla_a \varphi^{+} \nabla_b \varphi^{+} + \nabla_a \varphi^{+} \sigma^b \nabla_b \varphi^{+} \right) \right] \, d^3x$$  \hspace{1cm} (6)

from which follows a conclusion that the density is non-negative definite, if on the maximal hypersurface the asymptotically constant spinor $\varphi$ satisfies the Dirac equation in the 3-dimensional space

$$\sigma^a \nabla_a \varphi = 0.$$  \hspace{1cm} (7)

The main result of the Nester, Tung [3] and the Nester, Tung, Zhytnikov [17] works is formulated in the form of a statement that between the localization method, based on the 4-covariant spinor Hamiltonian, and the SOF-based method there exists a close connection owing to the 3-spinor Hamiltonian [6].

Such statement is grounded on the two circumstances: 1) among terms of which the 4-covariant spinor density consists, the 3-spinorial density is present; 2) between the 3-spinor field variables there exists, as Nester and Tung declared, a close relation, since from the 3-dimensional Dirac equation (7) it follows that

$$\sigma^a \nabla_a \varphi = \sigma^a \varphi_a - \frac{1}{2} \tilde{q}_b \sigma^b \varphi + \frac{1}{4} i * q \varphi = 0$$  \hspace{1cm} (8)

where forms $q$ and $\tilde{q}$ are defined in the following way:

$$q = \theta_\dot{a} \wedge d\theta^{\dot{a}}, \quad \tilde{q} = i_\dot{a} d\theta^{\dot{a}}$$  \hspace{1cm} (9)

\(\dagger\) In this formula and in some next formulas we change the signs, comparing with the original papers, according to the chosen here convention that a signature is $(+ - - -)$. 
and fix SOF on the asymptotically flat surfaces by means of the Nester gauge

\[ *q = 0, \quad \tilde{q} = d \ln \Phi, \quad (10) \]

where \( \Phi \) is arbitrary everywhere positive function. Nester, Tung and Zhytnikov results do not connect the Dirac equation itself and the Nester gauge by the equivalence relationship of a certain type, and do not establish the explicit and unique connection in all points of \( \Sigma \) (see, for example, [13, 14, 11]) between the variables of the 3-spinor field and the SOF variables. That is why a search for the valuable grounding of a statement about existence of a close correlation between both approaches remains topical.

We propose further a new insight on the problems of this correlation that is fully correct for the case of maximal hypersurface and is grounded asymptotically in the case of quite arbitrary hypersurfaces. The reason is known: the linear equations for the spinor fields become nonlinear after transition to the respective tensor functions.

2. Direct link between the 4-covariant spinor 3-form and the Einstein Hamiltonian

In [20] we have proved the two theorems:

**Theorem 3.** Let:

- a) initial data set \((h, K, \Sigma)\) be asymptotically flat by Reula [21];
- b) everywhere on \(\Sigma\) the matrix of the spinorial tensor

\[ C_A^B := \frac{\sqrt{\beta}}{4} D_A^B K + \frac{1}{4} \varepsilon_A^B \left( 2K^2 + \frac{1}{2} \varepsilon_{\pi\mu} K^{\pi\nu} + \mu \right) \quad (11) \]

has, at least, one non-negative eigenvalue, for definiteness \(C_0\);
- c) \(\Re \lambda^0_\infty \) or \(\Im \lambda^0_\infty \) asymptotically nowhere are equal to zero.

Then the asymptotically constant nontrivial solution \(\lambda^C\) to the SWE does not have the knot points on \(\Sigma_t\).

Here \(\lambda^A_\infty\) — asymptotic value of the \(SU(2)\)-spinor field \(\lambda^A\), which is a result of the Sommers and Sen reducing \([22, 23]\) the \(SL(2, \mathbb{C})\) structure to the \(SU(2)\) structure on the spacelike hypersurface \(\Sigma\) with the unit normal one-form \(n\).

**Theorem 4.** Let the conditions of Theorem 3 be fulfilled. Then everywhere on \(\Sigma\) there exists a two-to-one correspondence between the Sen-Witten spinor field \(\lambda^A\) and the Sen-Witten orthonormal frame \(\theta^a\).

Here the Sen-Witten orthonormal frame (SWOF) we call a 3-frame \(\hat{\theta}^a\) distinguished by the gauge conditions

\[ \varepsilon^{\hat{a}\hat{b}\hat{c}} \omega_{\hat{a}\hat{b}\hat{c}} \equiv *q = 0, \quad \omega^{\hat{a}}_{\hat{1}\hat{a}} \equiv -\tilde{q}_1 = -F_1, \quad \omega^{\hat{a}}_{\hat{2}\hat{a}} = -\tilde{q}_2 = -F_2, \quad \omega^{\hat{a}}_{\hat{3}\hat{a}} = -\tilde{q}_3 = -K - F_3, \quad (12) \]

where \(\omega_{\hat{a}\hat{b}\hat{c}}\) are the connection one forms coefficients, \(F = d \ln \lambda\), and \(\lambda = \lambda_A \lambda^A; "hats"\) distinguish orthonormal frame indices.

The SWOF generalizes the Nester SOF because gauge (10) can be written as

\[ *q := \varepsilon^{\hat{a}\hat{b}\hat{c}} \omega_{\hat{a}\hat{b}\hat{c}} = 0, \quad \tilde{q}_b := -\omega^{\hat{a}}_{\hat{b}\hat{a}} = \partial_b \ln \Phi. \quad (13) \]
The above mentioned correspondence between the Sen-Witten spinor and SWOF is of the form:

\[ \theta^1 = \frac{\sqrt{2}}{2\lambda} (L + \tilde{L}), \quad \theta^2 = \frac{\sqrt{2}}{2\lambda} (L - \tilde{L}), \quad \theta^3 = \tilde{L} \]  

(14)

where \( L = -\lambda_A \lambda_B \) and \( \tilde{L} = |L|^{-1} \cdot (L \wedge L) \) is the nonzero spatial one-form.

In other words, we can say that if the spinor field \( \lambda_A \) satisfies the SWE and conditions of Theorem 3, then everywhere on \( \Sigma_t \) exists the orthonormal 3-frame which satisfies conditions (12), and conversely.

Frauendiener \[24\] represented "squared neutrino equation", obtained by Sommers \[22\] from Weyl equation by means of transformation (14), as a conditions for a set of three mutually orthogonal fields of equal length on \( \Sigma \). Our conditions (12) are the other form of "squared zero-modes neutrino equation" and the Frauendiener conditions. Such form of these conditions will permit us further to apply them efficiently for transformation of the ADM Hamiltonian along Nester's line. More important is the following consideration. As it was pointed by Dimakis and Müller-Hoissen \[18, 19\] (see also \[25\]), the spinorial field as a solution to elliptic equation may have zeros (knot points). Respectively, the Sommers transformation does not exist on the knot surfaces, lines and points on \( \Sigma \); on such submanifolds "squared zero-modes neutrino equation" and the Frauendiener conditions will be not satisfied, and any correspondence between the Sen-Witten spinor field and the SWOF will not exist. Our Theorem 4 solves this problem, establishing the conditions for existence of transformation (14) everywhere on \( \Sigma \).

Taking into account that the Hamiltonian density (2) and the SWE were obtained by the spinor parameterization for the Hamiltonian displacement, we write in terms of the Sommers-Sen spinors

\[ N^\mu = \lambda^A \lambda^A = \lambda^{A \lambda B} + N^\mu r_B n^{AB} = \lambda^{A \lambda B} + \frac{1}{\sqrt{2}} \lambda_D \lambda^{D+} \varepsilon^{AB}. \]  

(15)

That is why \( N \equiv N^0 = \lambda_A \lambda^A \) and \( F = d \ln N \). Note, that the Nester SOF approach does not limit a choice of the dependence \( N = N(\Phi) \) \[11\].

We will further give the 4-covariant Hamiltonian density in terms of the Sen-Witten spinor using the SWE in the form

\[ \mathcal{D}_{BC} \lambda^C = 0. \]  

(16)

An action of operator \( \mathcal{D}_{AB} \) on the spinor fields is

\[ \mathcal{D}_{AB} \lambda_C = D_{AB} \lambda_C + \frac{\sqrt{2}}{2} K_{ABC} D^D \lambda_D \]

where \( D_{AB} \) — the spinorial form of the derivative operator \( D_\alpha \) compatible with metric \( h_{\mu\nu} \) on the \( C^\infty \) hypersurface \( \Sigma_t \), \( K_{ABCD} \) — the spinorial tensor of the extrinsic curvature of hypersurface \( \Sigma \).
The standard substitution transforms (3) to the form
\[ \mathcal{H}(\varphi, \chi) = \left[-2\sqrt{2}\left(n_{\dot{A}A} D_{\mu} \varphi^{A} D^\mu \varphi^{\dot{A}} + n_{\dot{A}A} D_{\mu} \chi^{A} D^\mu \chi^{\dot{A}}\right) + 2 \left(n^{B\dot{C}} D_{BA} \varphi^{A} D^\mu \varphi^{\dot{A}} + n^{B\dot{C}} D_{BA} \chi^{A} D^\mu \chi^{\dot{A}}\right)\right] d^3 \Sigma. \] (17)

Let us take into account that

\[ h^\mu_{\dot{A}} n_{\dot{A}A} D_{\mu} \varphi^{A} D^\nu \varphi^{\dot{A}} = \varepsilon^{BD} \varepsilon^{BD} n_{\dot{A}A} \left(D_{BB} \varphi^{A}\right) \left(D_{DD} \varphi^{\dot{A}}\right) = 2 n^{RB} n_{\dot{A}A} \left(D_{BB} \varphi^{A}\right) \left(D_{DD} \varphi^{\dot{A}}\right) = \left(D_{B} \varphi^{A}\right) \left(D_{R} \varphi^{\dot{A}}\right) n_{\dot{A}A}, \] (18)

and

\[ \left(D_{R} \varphi^{\dot{A}}\right) n_{\dot{A}A} = \frac{1}{\sqrt{2}} \left[-D_{BR} \varphi_{A}^{+} - \sqrt{2} \left(D_{BR} n_{\dot{A}A}\right) \varphi^{\dot{A}}\right], \] (19)

\[ D_{BR} n_{\dot{A}A} = K_{BR A} + \frac{\sqrt{2}}{2} F_{AA} \varepsilon^{BR} = K_{BR A}. \] (20)

Then

\[ n_{\dot{A}A} h^\mu_{\dot{A}} \left(D_{\mu} \varphi^{A} D^\nu \varphi^{\dot{A}} + D_{\mu} \chi^{A} D^\nu \chi^{\dot{A}}\right) = \left(D_{BR} \varphi^{A}\right) \left(\frac{\sqrt{2}}{2} D_{BR} \varphi^{+A} + K^{BRA}_{S} \varphi^{+S}\right) \] + \left(D_{BR} \chi^{A}\right) \left(\frac{\sqrt{2}}{2} D_{BR} \chi_{A}^{+} + K^{BRA}_{S} \chi^{+S}\right). \] (21)

For transformation of the other terms we will use the identity

\[ D_{BA} \varphi^{\dot{A}} = D_{BA} \left(2 n^{AC} n_{CC} \varphi^{\dot{C}}\right) = -\frac{2}{\sqrt{2}} D_{BA} \left(n^{AC} \varphi_{C}^{+}\right) = -\frac{2}{\sqrt{2}} \left(K_{BA} \varphi_{C}^{+} + \frac{1}{\sqrt{2}} D_{B} \varphi_{C}^{+}\right) = -\frac{2}{\sqrt{2}} \left(K_{B} \varphi_{C}^{+} + \frac{1}{\sqrt{2}} D_{B} \varphi_{C}^{+}\right), \] (22)

and, therefore,

\[ n^{B\dot{C}} D_{BA} \varphi^{\dot{A}} D_{BA} \varphi^{A} = \frac{\sqrt{2}}{2} \left(D_{A}^{B} \varphi^{A}\right) \left(D_{BC} \varphi^{+C}\right) - \frac{1}{2} K_{BC} \varphi^{+C} D_{A}^{B} \varphi^{A}. \] (23)

The final expression for \( \mathcal{H}(\varphi, \chi) \) we will give in the form

\[ \mathcal{H}(\varphi, \chi) = \sqrt{2} \left\{ \left(D_{BR} \varphi^{A}\right) \left(\sqrt{2} D_{BR} \varphi^{+A} + K^{BRA}_{S} \varphi^{+S}\right) \right. \] + \left(D_{A}^{B} \varphi^{A}\right) \left(-D_{BC} \varphi^{+C} + K_{BC} \varphi^{+C}\right) \] + \left(D_{BR} \chi^{A}\right) \left(\sqrt{2} D_{BR} \chi_{A}^{+} + K_{S}^{BRA} \chi^{+S}\right) \] + \left(D_{A}^{B} \chi^{A}\right) \left(-D_{BC} \chi^{+C} + K_{BC} \chi^{+C}\right) \} dV. \] (24)

The Hamiltonian 3-form \( \mathcal{H}(\varphi, \chi) \) \[24\] in comparison with the Hamiltonian 3-form, obtained by Ashtekar and Horowitz \[25\], contains the terms with the external curvature of hypersurface \( \Sigma \).

The first and the second terms are positive definite, and the next ones turn to zero if on hypersurface \( \Sigma \) the spinor fields \( \varphi^{A} \) and \( \chi^{A} \) satisfy the SWE \( \{16\} \).
On the other hand, the ADM Hamiltonian density, parameterized with orthonormal 3-frames $\theta^a$, is of the form \[ \mathcal{H}(N) = -2|h|^{\frac{3}{2}} q^a \partial_a N + N|h|^{\frac{3}{2}} (K^{ab} \mathcal{K}_{ab} - K^2) - 2|h|^{\frac{3}{2}} (K^a_b - \delta^a_b \mathcal{K}) D_a N^b + N|h|^{\frac{3}{2}} \left[ q^{ab} q_{ab} + \frac{1}{2} |\tilde{q}|^2 q_a - \frac{1}{6} (\ast q)^2 \right] \] (25)

where the symmetric tensor $q_{ab}$, vector $\tilde{q}_a$, and scalar $\ast q$ are defined by irreducible decomposition

\[ C^a_{bc} = q^{ad} \varepsilon_{dcb} + \frac{1}{2} (\delta_c^a \tilde{q}_b - \delta_b^a \tilde{q}_c) + \frac{1}{3} \ast q \varepsilon^{ac}. \]

Varying the lapse in (25), we obtain the super-Hamiltonian constraint in the form

\[ 2 \partial_k \left( |h|^{\frac{3}{2}} k^k \right) + \frac{1}{2} |h|^{\frac{3}{2}} q^k = |h|^{\frac{3}{2}} \left[ K^{mn} K_{mn} - K^2 + q^{mn} q_{mn} + \frac{1}{2} |\tilde{q}|^2 q_a - \frac{1}{6} (\ast q)^2 \right] = 0. \]

(26)

If the spinor fields $\varphi^A$ and $\chi^A$ satisfy the SWE and conditions of Theorem 3, then condition (12) be fulfilled, and vice versa. Then, on the one hand, $\mathcal{H}(\varphi, \chi)$ will be positive, and, on the other hand, this will permit us to write $\mathcal{H}(N)$ in the SWOF, under the necessary in this context limitation for $F$, and at $N^a = 0$, in the form

\[ \mathcal{H}^{SWOF}(N) = N|h|^{\frac{3}{2}} \left( -\frac{3}{2} h^{mn} \partial_m \ln N \partial_n \ln N - K \partial_3 \ln N \right. \]

\[ \left. \quad - \frac{3}{2} K^2 + K^{mn} K_{mn} + q^{mn} q_{mn} \right) \].

(27)

Here the lapse is determined by the super-Hamiltonian constraint

\[ 2 \partial_m \left( |h|^{\frac{3}{2}} h^{mn} \partial_n \ln N \right) + 2 \partial_m \left( |h|^{\frac{3}{2}} \theta^3 m \mathcal{K} \right) + |h|^{\frac{3}{2}} \left( \frac{1}{2} h^{mn} \partial_m \ln N \partial_n \ln N + 2 \mathcal{K} \partial^3 m \partial_m \ln N \right. \]

\[ \left. - \frac{3}{2} K^2 + K^{mn} K_{mn} + q^{mn} q_{mn} \right) = 2 \partial_m \left( |h|^{\frac{3}{2}} h^{mn} \partial_m \ln N \right) + |h|^{\frac{3}{2}} \left( \frac{1}{2} h^{mn} \partial_m \ln N \partial_n \ln N \right. \]

\[ \left. + 2 \mathcal{K} \partial_3 \ln N - 2 \partial_3 \mathcal{K} + \frac{1}{2} K^2 + K^{mn} K_{mn} + q^{mn} q_{mn} \right) = 0. \]

(28)

Let us consider first of all the especially simple case of a maximal spacial Cauchy hypersurface. Then the Hamiltonian density (27) takes the form

\[ \mathcal{H}^{SWOF}(N) = N|h|^{\frac{3}{2}} \left( -\frac{3}{2} h^{mn} \partial_m \ln N \partial_n \ln N + K^{mn} K_{mn} + q^{mn} q_{mn} \right), \]

(29)

and will be everywhere positive definite if on $\Sigma$ exists an appropriate solution of the super-Hamiltonian constraint

\[ 2 \partial_m \left( |h|^{\frac{3}{2}} h^{mn} \partial_m \ln N \right) + |h|^{\frac{3}{2}} \left( \frac{1}{2} h^{mn} \partial_m \ln N \partial_n \ln N + K^{mn} K_{mn} + q^{mn} q_{mn} \right) = 0. \]

(30)

Unique positive solution $N$ of this equation exists because Nester’s gauge enjoys the property of conformal invariance and thus fits into the Lichnerowicz–Choquet-Bruhat–York initial-value problem analysis. Therefore, we conclude, that owing to the correspondence between the SWE and the Nester gauge on the maximal hypersurface...
there exists the direct relationship between the Hamiltonian based positivity localization in the 4-covariant spinor method and in the ADM method based on the SOF for the case \( N = \Phi^4 \).

Now, let us consider the hypersurface \( \Sigma \) which is not maximal, and let be asymptotically \( N = a + O(r^{-1}) \), \( \partial_m N = O(r^{-2}) \). Then the super-Hamiltonian constraint \( (28) \) for enough large \( r \) can be written as

\[
2\partial_m \left( |h|^{\frac{1}{2}} h^{mn} \partial_m N \right) + N |h|^{\frac{1}{2}} \left( -2 \partial_3 \mathcal{K} + \frac{1}{2} \mathcal{K}^2 + \mathcal{K}^{mn} \mathcal{K}_{mn} + q^{mn} q_{mn} \right) = 0. \tag{31}
\]

The Dirichlet problem for equation \( (31) \) has the unique solution, if

\[
C(x) = |h|^{\frac{1}{2}} \left( -2 \partial_3 \mathcal{K} + \frac{1}{2} \mathcal{K}^2 + \mathcal{K}^{mn} \mathcal{K}_{mn} + q^{mn} q_{mn} \right) \geq 0. \tag{32}
\]

The same condition, and the condition that \( N \) is positive on the boundary or asymptotically, ensure the non-occurrence of the knot points of equation \( (31) \), since the knot submanifolds of elliptic equation of second order are closed or their ends lie on the boundary. That is why everywhere \( N > 0 \), and we can choose \( a = 1 \).

Further, a general theorem for the elliptic second-order system claims \[26\] that its solutions continuously depend on coefficients, domain and values of the functions on the boundary, therefore the Hamiltonian density \( \mathcal{H}^{SWOF}(N(\mathcal{K}), N) \) \( (27) \) continuously depends on \( \mathcal{K} \) and thus is non-negative on the hypersurfaces which satisfy the condition

\[
-2 \partial_3 \mathcal{K} + \frac{1}{2} \mathcal{K}^2 \geq 0 \tag{33}
\]

and lie in some neighborhood of the maximal one. The presence of the terms \(-2 \partial_3 \mathcal{K}\) and \(\frac{1}{2} \mathcal{K}^2\) in the right side of relationship \( (32) \) is caused just by a fact that we used the SWOF; the application of Nester’s gauge does not give a possibility to prove the existence of this class of hypersurfaces, on which the Hamiltonian density in the SOF is non-negative.

In order to establish a correspondence between conditions \((11)\) and \((33)\) we write following a space spinors definition

\[
D_A^{\,\dot{B}} \mathcal{K} = -\sqrt{2} n_\dot{\alpha} \sigma_{\dot{A}}^{\dot{\alpha}} \sigma^B A \partial_3 \mathcal{K} = -\sqrt{2} \sigma^\dot{B} A \sigma^\dot{A} B \partial_3 \mathcal{K}, \tag{34}
\]

and obtain that the diagonal elements of matrix

\[
\frac{\sqrt{2}}{4} D_A^{\,\dot{B}} \mathcal{K} + \frac{1}{2} \epsilon_A^{\,\dot{B}} \mathcal{K}^2
\]

are

\[
\frac{1}{4} \partial_3 \mathcal{K} + \frac{1}{2} \mathcal{K}^2 \quad \text{and} \quad -\frac{1}{4} \partial_3 \mathcal{K} + \frac{1}{2} \mathcal{K}^2.
\]

Therefore, the second of them is non-negative on the hypersurfaces which satisfy condition \((33)\). This means that under fulfilling of condition \((33)\) there is also fulfilled condition b) of Theorem 3.

So, if the SWE and conditions a) and c) of Theorem 3 are fulfilled, then on hypersurfaces, which satisfy condition \((33)\) and lie in some neighborhood of the
maximal one, the Hamiltonian density $\mathcal{H}(\varphi, \chi)$ \cite{21} and the ADM Hamiltonian density $\mathcal{H}^{SWOF}(N)$ \cite{27} are locally nonnegative simultaneously.

Let us note that just an absence of the result about connection between the SWE equation and the Nester gauge (a theorem like Theorem 3 and Theorem 4) did not permit Nester and Tung to obtain a direct relationship between the 4-spinor 3-form of the Hamiltonian density under fulfilling of the SWE and the Hamiltonian density in the SOF formalism, both on the enough general hypersurfaces and even on the maximal ones. The 3-spinor formalism, developed by these authors and Zhytnikov, ensure the partial solving of this problem; in particular, the energy is guaranteed to be locally non-negative only on the maximal hypersurfaces.

3. Conclusions

Generalization of the SOF by the SWOF allows us to remove two liabilities of the SOF method: necessity of the restriction to the maximal hypersurfaces, and impossibility of extension to the future null infinity and, hence, description of the Bondi 4-momentum. Therefore, it follows that for the quasilocal Hamiltonian density \cite{22} investigation are suitable not the Dirac equation and the 3-spinors, but the SWE and the space spinors introduced by Sommers \cite{22}. Although the 3-dimensional Dirac equation and the SWE are very similar, we see that fixing of the spinor field by the Dirac equation or by the SWE leads to different physical consequences. The mathematical consequences for application of these gauge conditions for the spinor field are also different; in particular, the conditions for existence of solutions existence differ in domains of finite measure \cite{20}.

The equivalence of the Sen-Witten spinor field and the SWOF (Sec. 2), under the reasonable from the physical point of view fulfilling of conditions of Theorem 3, permits to establish that the method of the 4-covariant quadratic spinor Hamiltonian and the SOF method are very close. The spinor parameterization of the Hamiltonian displacement and correlations \cite{14} are a key for the orthonormal frame interpretation of the Hamiltonian 4-covariant spinor form \cite{2} and the spinor interpretation of the ADM Hamiltonian density even in the case when the spinor field or the orthonormal frame are not fixed.

Note at the end that conditions of \cite{11} and \cite{28} type are the only sufficient ones, and we expect to weaken then significantly or to exclude completely.

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