MCLEAN’S SECOND VARIATION FORMULA REVISITED

HÔNG VÂN LÊ AND JIŘÍ VANŽURA

Abstract. We revisit McLean’s second variation formulas for calibrated submanifolds in exceptional geometries, and correct his formulas concerning associative submanifolds and Cayley submanifolds, using a unified treatment based on the (relative) calibration method and Harvey-Lawson’s identities.

Contents

1. Introduction 1
2. Proof of the Main Theorem 4
3. Second variation formula for associative and coassociative submanifolds 6
   3.1. Associative and coassociative submanifolds 6
   3.2. The normal bundle of an associative submanifold and its associated Dirac operator 7
   3.3. Second variation of the volume of an associative submanifold 8
   3.4. Second variation of the volume of a coassociative submanifold 9
4. Second variation formula for Cayley submanifolds 11
   4.1. Cayley submanifolds in Spin(7)-manifolds and cross products 11
   4.2. The normal bundle of a Cayley submanifold and its associated Dirac type operator 12
   4.3. Second variation of the volume of a Cayley submanifold 13
Acknowledgement 14
References 14

Key words: calibrated submanifold, second variation, Harvey-Lawson’s identity

1. INTRODUCTION

Calibrated geometry has been invented by Harvey-Lawson in 1982 [HL1982] motivated by rich theories of complex manifolds, exceptional geometries and

Date: August 2, 2016.
2010 Mathematics Subject Classification. Primary 53C40, 53C38.
HVL and JV are partially supported by RVO: 67985840.
minimal submanifolds. We refer the reader to [Morgan2009] for an extensive survey on calibration method. In 1998 McLean published a paper on deformation of calibrated submanifolds [McLean1998], inspired by similarities between calibrated submanifolds and complex submanifolds. One important part of his study is the second variation of volume of compact calibrated submanifolds, which is also the subject of our note. McLean distinguished two families of calibrated submanifolds in exceptional geometries. The first family consists of special Lagrangian and coassociative submanifolds. The second family consists of associative and Cayley submanifolds. In the first family the normal bundle of a calibrated submanifold is isomorphic to a vector bundle intrinsic to the submanifold, namely the normal bundle of a special Lagrangian submanifold $L$ is isomorphic to the tangent bundle $TL$ (or the cotangent bundle $T^*L$ via the metric) and the normal bundle of a coassociative submanifold $L$ is isomorphic to the bundle of self-dual two-forms. From a computational point of view, special Lagrangian and coassociative submanifolds $L$ can be defined in terms of vanishing of closed forms on $L$. Moreover deformation of calibrated submanifolds in this family is unobstructed. In the second family the normal bundle of a calibrated submanifold is not intrinsic, namely the normal bundle of an associative submanifold $L$ is trivial (Lemma 3.6) and the normal bundle of a Cayley submanifold is a twisted spinor bundle [McLean1998 Section 6]. From a computational point of view, associative and Cayley submanifolds cannot be defined in terms of the vanishing of closed forms, but they can be defined in terms of the vanishing of certain vector valued forms. In particular, deformation theory for calibrated submanifolds in the second family has a different character than the one for the first family.

In [McLean1998 Theorem 2.4, p. 711], using moving frame method, McLean derived a general formula for the second variation of the volume of a compact calibrated submanifold. Applying this formula to calibrated submanifolds of the first and second family he obtained formulas which are similar to Simons’ second variation formula for Kähler submanifolds [Simons1968 p. 78]. McLean’s second variation formula for special Lagrangian submanifolds has been revisited by Lê-Schwachhöfer in [LS2014], where they extended the relative calibration method developed by Lê in [Le1989, Le1990] to derive the second variation formula for compact Lagrangian submanifolds in strict nearly Kähler 6-manifolds. They also indicated how their method can be applied to calibrated submanifolds, whose corresponding calibration satisfies the long version of Harvey-Lawson’s identity, see Remark 1.2. As an example, they analyzed the second variation formula for special Lagrangian submanifolds.

In this note we revisit McLean’s second variation formula for associative and coassociative submanifolds in $G_2$-manifolds, and Cayley submanifolds in Spin(7)-manifolds. Our main observation is that all calibrations under consideration satisfy the following Harvey-Lawson’s identity.
**Definition 1.1.** A calibration $\varphi \in \Lambda^k(\mathbb{R}^n)^*$ is said to satisfy Harvey-Lawson’s identity, if there exists a vector valued $k$-form $\Psi \in \Lambda^k(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ such that
\begin{equation}
(1.1) \quad |\varphi(\xi)|^2 + |\Psi(\xi)|^2 = |\xi|^2 \text{ for all } \xi \in Gr_k(\mathbb{R}^n).
\end{equation}
Let $M$ be a Riemannian manifold. A calibration $\varphi \in \Omega^k(M^n)$ is said to satisfy Harvey-Lawson’s identity, if there exists a Riemannian vector bundle $E$ over $M^n$ and an $E$-valued $k$-form $\Psi \in \Omega^k(M, E)$ such that for all $x \in M^n$ we have
\begin{equation}
(1.2) \quad |\varphi(\xi)|^2 + |\Psi(\xi)|^2 = |\xi|^2 \text{ for all } \xi \in Gr_k(T_xM^n).
\end{equation}

**Remark 1.2.** Harvey-Lawson’s identity appears many times in [HL1982], but instead of $|\Psi(\xi)|^2$ Harvey-Lawson usually wrote its long version $\sum_i |\Psi_i(\xi)|^2$, see Question 6.5 and Formula (6.6) in [HL1982, p. 68], as well as Formulas (6.16) (p. 71), (6.17) (p.73), Theorem 1.7 (p. 88) of the cited paper and Lemmas 3.1, 3.2, 4.1 below. All calibrated submanifolds considered in McLean’s paper have corresponding calibrations that satisfy Harvey-Lawson’s identity, see also Remark 2.3 below.

In this note we prove the following.

**Theorem 1.3 (Main Theorem).** Let $\varphi$ be a calibration on a Riemannian manifold $M$ and $\Psi \in \Omega^*(M, E)$ such that $\varphi$ and $\Psi$ satisfy Harvey-Lawson’s identity. Assume that $L$ is a compact oriented $\varphi$-calibrated submanifold and $V$ is normal vector field on $L$. Then the second variation of the volume of $L$ with variation field $V$ is given by
\begin{equation*}
\frac{d^2}{dt^2} \big|_{t=0} \text{vol}(L_t) = \int_L |\nabla_{\partial t}|_{t=0} \Psi((\exp tV)_*(\xi(x)))|^2 dvol_x.
\end{equation*}
Here
- $\xi(x)$ is the unit decomposable $k$-vector that is associated to $T_xL$,
- $\exp tV$ denotes the flow on a neighborhood of $L$ that is generated by a vector field whose value at $x \in L$ is equal to $V$, and $L_t = \exp tV(L),$
- $\Psi((\exp tV)_*(\xi(x))) \in E_{\exp tV(x)}$ and $\nabla_{\partial t}\Psi((\exp tV)_*(\xi(x)))$ denotes the covariant derivative of the section $\Psi((\exp tV)_*(\xi(x))) \in E|_{\exp tV(x)}$ of the restriction of the vector bundle $E$ to the curve $[\exp tV(x)] \subset M$,
- we assume that the covariant derivative $\nabla_{\partial t}$ of $E$ along the curve $[\exp tV(x)]$ preserves the metric on $E$.

Note that the equation $\nabla_{\partial t}|_{t=0} \Psi((\exp tV)_*(\xi(x))) = 0$ describes the equation for $V$ to be an infinitesimal deformation of $\varphi$-calibrated submanifold $L$. Thus the second variation formula for calibrated submanifolds in Theorem 1.3 follows from the equation for Zariski tangent vectors of the moduli space of calibrated submanifolds under consideration.

From our Main Theorem we obtain immediately the following.

**Corollary 1.4.** Let $L$ be a $\varphi$-calibrated submanifold in Theorem 1.3. Then a normal vector field $V$ on $L$ is an infinitesimal deformation of $\varphi$-calibrated
submanifolds if and only if $V$ is a Jacobi vector field on $L$, regarding $L$ as a minimal submanifold.

In fact, Corollary 1.4 holds for any compact calibrated submanifold without validity of Harvey-Lawson’s identity, see Remark 2.3 below.

As applications of the Main Theorem, we shall derive simple second variation formulas for associative, coassociative and Cayley submanifolds respectively, which agree with McLean’s formulas up to a multiplicative constant.

Our note is organized as follows. In Section 2 we give a proof of Theorem 1.3 and discuss a slight generalization of it in Remark 2.3. In Section 3 we give a new proof of McLean’s second variation formula for associative, coassociative submanifolds in $G_2$-manifolds and derive from it a second variation formula for special Lagrangian submanifolds in Calabi-Yau 6-manifolds. In Section 4 we give a new proof of McLean’s second variation formula for Cayley submanifolds. (Our formulas for associative and Cayley submanifolds differ from McLean’s formulas by a scaling factor). At the end of our note we explain where McLean did mistakes in his computations (Remark 4.6).

2. Proof of the Main Theorem

Proof of Theorem 1.3. Let us keep notations in the previous section, in particular, in Theorem 1.3. Abusing the notation, denote by $V$ a vector field in a neighborhood of $L$ whose value at $L$ is the given normal vector field $V$, see explanation in Theorem 1.3. Set

$$\xi_t(x) := (\exp tV)_*(\xi(x)),$$

and

$$g_t|_L := (\exp tV)^*g|_{\exp tV(L)},$$

where $g|_{\exp tV(L)}$ denotes the metric on $\exp tV(L)$ induced from the ambient metric on $M$. Denote by $\text{vol}_t$ the induced volume form on $L$ associated to $g_t$. Since $\text{vol}_t(x) = \det(g_{ij})^{1/2}dx = |\xi_t(x)| \cdot \text{vol}_0(x)$, taking into account the minimality of $L$, we observe that for all $x \in L$

$$|\xi_0(x)| = 1 \text{ and } \frac{d}{dt}|_{t=0}|(\xi_t(x))| = 0.$$ 

Hence

$$\frac{d^2}{dt^2}|_{t=0}\text{vol}(\exp tV(L)) = \int_L \frac{d^2}{dt^2}|_{t=0}|(\xi_t(x))| \, d\text{vol}_x$$

$$= \frac{1}{2} \int_L \frac{d^2}{dt^2}|_{t=0}|(\xi_t(x))|^2 \, d\text{vol}_x.$$

To simplify notation, we write

$$D(V)(x) := \nabla_{\xi_0}|_{t=0}\Psi(\xi_t(x)).$$
Lemma 2.1. For all $x \in L$ we have
\[
\frac{d^2}{dt^2}|_{t=0}|\Psi(\xi_t(x))|^2 = 2|D(V)|^2(x).
\]

Proof. We compute
\[
\frac{d^2}{dt^2}|_{t=0}|\Psi(\xi_t(x))|^2 = 2\frac{d}{dt}|_{t=0}(\nabla_{\partial_t}(\Psi(\xi_t)), \Psi(\xi_t))(x)
= 2|D(V)|^2(x) + 2\langle\Psi(\xi_t(x)), \nabla_{\partial_t} \nabla_{\partial_t}(\Psi(\xi_t))(x)|_{t=0}\rangle.
\]
Since $\Psi(\xi_t(x)) = 0$ by Harvey-Lawson’s identity, this completes the proof of Lemma 2.1. □

Lemma 2.2. We have
\[
\frac{d^2}{dt^2}|_{t=0}\int_L \varphi(\xi_t)^2 \, d\text{vol}_x = 0.
\]

Proof. Since $L$ is a $\varphi$-calibrated submanifold, by [Le1989, Proposition 2.2. (ii)], see also [Le1990, Proposition 1.2 (ii)], we have
\begin{equation}
(V|\varphi)|_L = 0.
\end{equation}
Using $d\varphi = 0$, we obtain from (2.3)
\begin{equation}
(L_V\varphi)|_L(x) = 0 \text{ for all } x \in L.
\end{equation}

Now let us compute
\begin{equation}
\frac{d^2}{dt^2}|_{t=0}\int_L \varphi(\xi_t)^2 \, d\text{vol}_x = 2\frac{d}{dt}|_{t=0}\int_L \varphi(\xi_t) \cdot \frac{d}{dt}(\exp tV)^*\varphi(\xi_t) \, d\text{vol}_x.
\end{equation}
Using (2.4), we obtain from (2.5), noting that $\varphi(\xi(x)) = 1$
\[
\frac{d^2}{dt^2}|_{t=0}\int_L \varphi(\xi_t)^2 \, d\text{vol}_x = 2\int_L L_V d(V|\varphi) = 0.
\]
This completes the proof of Lemma 2.2. □

Now let us complete the proof of Theorem 1.3. Using (2.2), Harvey-Lawson’s identity (1.2) and Lemmas 2.1 2.2 we obtain
\begin{equation}
\frac{d^2}{dt^2}|_{t=0}\text{vol}(\exp tV(L)) = \frac{1}{2}\frac{d^2}{dt^2}|_{t=0}\int_L \langle\Psi(\xi_t(x)), \Psi(\xi_t(x))\rangle d\text{vol}_x
= \int_L |D(V)|^2 d\text{vol}_x.
\end{equation}
This completes the proof of Theorem 1.3. □

Remark 2.3. Any calibration $\varphi$ on a Riemannian manifold $M$ satisfies a weak version of Harvey-Lawson’s identity (1.2), where we replace $\Phi \in \Omega^k(M, E)$ by a real function, also denoted by $\Phi$, on the Grassmannian of oriented $k$-decomposable vectors in $TM$. In this case, using the argument of the proof of Theorem 1.3 the function under integral in the RHS of the formula in Theorem 1.3 is replaced by $(\partial_t|_{t=0}\Phi(\xi_t(x)))^2$. Thus Corollary 1.4 also holds for any calibrated submanifold.
3. SECOND VARIATION FORMULA FOR ASSOCIATIVE AND COASSOCIATIVE SUBMANIFOLDS

3.1. Associative and coassociative submanifolds. In this subsection we recall basic definitions of associative 3-submanifolds and coassociative 4-submanifolds in a $G_2$-manifold $(M^7, \varphi, g)$ and show that the associated calibrations satisfy Harvey-Lawson’s identity (Lemmas 3.1, 3.2).

Let $\mathbb{O}$ denote the octonion algebra. Denote by $\langle \cdot, \cdot \rangle$ the scalar product on $\mathbb{O}$ and by $\cdot$ the octonion multiplication. Recall that the associative 3-form $\varphi$ on $\text{Im} \mathbb{O}$ is defined as follows [HL1982] (1.1), IV.1.A, p. 113]

$$\varphi(x, y, z) := \langle x \cdot y, z \rangle = \langle x, y \cdot z \rangle.$$ 

Let $\text{Im} \mathbb{O} = \mathbb{R}^7$ have coordinates $(x^1, \ldots, x^7)$. We abbreviate $dx^i \wedge dx^j \wedge dx^k$ as $x^{ijk}$. The fundamental associative 3-form $\varphi$ can be written in coordinate expression as follows [HL1982] (1.2), p. 113]

$$\varphi = x^{123} + x^{145} - x^{167} + x^{246} + x^{257} + x^{347} - x^{356}.$$

Its dual

$$\ast \varphi = x^{4567} + x^{2367} - x^{2345} + x^{1357} + x^{1346} + x^{1256} - x^{1247}$$

is called the coassociative form.

It is well-known that $G_2$, the automorphism group of $\mathbb{O}$, is also the subgroup of $\text{GL}(\mathbb{R}^7)$ that preserves $\varphi$ (resp. $\ast \varphi$). Let $g_0$ denote the standard Euclidean metric on $\mathbb{R}^7$. We call $(\varphi_0, g_0)$ the standard $G_2$-structure.

Let $M^7$ be an oriented 7-manifold and $\varphi$ a 3-form on $M^7$. A 3-form $\varphi$ is called a $G_2$-structure on $M^7$ if for each $p \in M^7$, there exists an oriented linear isomorphism $I_p$ between $T_pM^7$ and $\mathbb{R}^7$ identifying $\varphi_p$ with $\varphi_0$. Then $\varphi$ induces the metric $g_\varphi$ by pulling back $g_0$ via $I_p$. Since $G_2$ is a subgroup of $\text{SO}(7)$ the metric $g_\varphi$ does not depend on the choice of $I_p$.

In our paper we are concerned only with $G_2$-manifolds $(M^7, \varphi, g)$, i.e. the $G_2$-structure on $(M^7, \varphi, g)$ is torsion-free, equivalently $d\varphi = 0$ and $d\ast \varphi = 0$.

A 3-submanifold $L \subset M^7$ is called associative, if $\varphi|_L = \text{vol} L$. A 4-submanifold $L \subset M^7$ is called coassociative, if $\ast \varphi|_L = \text{vol} L$.

We shall show that $\varphi$ and $\ast \varphi$ satisfy Harvey-Lawson’s identity. We set ([HL1982] p. 114) [HL1982] Definition IV.1.11, Proposition IV.1.14, p. 116]

$$\langle \chi(x, y, z), w \rangle := \ast \varphi(x, y, z, w).$$

We regard $\chi$ as an element in $\Omega^3(M^7, TM^7)$.

The following Lemma is a Harvey-Lawson’s identity.

**Lemma 3.1.** ([HL1982] Theorem IV.1.6, p. 114) For all $x, y, z \in TM^7$ we have

$$\varphi(x, y, z)^2 + |\chi(x, y, z)|^2 = |x \wedge y \wedge z|^2.$$ 

Now we set for $x, y, z, w \in TM^7$ ([HL1982] (1.17), Theorem 1.18, p. 117])

$$\tau(x, y, z, w) := -(\varphi(y, z, w)x + \varphi(z, x, w)y + \varphi(x, y, w)z + \varphi(y, x, z)w).$$

The following Lemma is also a Harvey-Lawson’s identity
Lemma 3.2. ([HL1982] Theorem IV.1.18, p. 117) For all \( x, y, z, w \in TM^7 \) we have

\[ \ast \varphi(x, y, z, w)^2 + |\tau(x, y, z, w)|^2 = |x \wedge y \wedge z \wedge w|^2. \]

We regard \( \tau \) as an element in \( \Omega^4(M^7, TM^7) \), see also Remark 4.2.

Example 3.3. ([Joyce2007] 12.2.1, p. 260], [CHNP2012] p. 43] Let \( (M^6, \omega, \Omega) \) be a Calabi-Yau manifold with a fundamental 2-form \( \omega \) and a complex volume form \( \Omega \), see e.g. [CS2002] for characterization of \( SU(3) \)-manifolds via \( (\omega, \Omega) \). Denote by \( g \) the associated Calabi-Yau metric on \( (M^6, \omega, \Omega) \) Then \( (S^1 \times M^6, d\theta \wedge \omega + Re \Omega, d\theta^2 + g) \) is a \( G_2 \)-manifold. If \( L \) is a special Lagrangian submanifold in \( (M^6, \omega, \Omega) \), then \( L_{\theta} := \{ \theta \} \times L \) is an associative submanifold in \( S^1 \times M^6 \) for any \( \theta \in S^1 \). If \( C \) is a complex curve in \( M^6 \), then \( S^1 \times C \) is an associative submanifold in \( S^1 \times M^6 \).

Example 3.4. ([Joyce2007] 12.2.1, p. 260]) Let \( (M^6, \omega, \Omega) \) be a Calabi-Yau manifold as above. If \( L \) is a special Lagrangian submanifold in \( (M^6, \omega, \Omega) \), then \( S^1 \times L \) is a coassociative submanifold in \( (S^1 \times M^6, d\theta \wedge \omega + Re \Omega, d\theta^2 + g) \). If \( C \) is a complex surface in \( M^6 \), then \( C_{\theta} := \{ \theta \} \times C \) is a coassociative submanifold in \( S^1 \times M^6 \) for any \( \theta \in S^1 \).

We refer the reader to [Lotay2012] [Kawai2014a] [Kawai2014b] for consideration of homogeneous associative submanifolds in nearly \( G_2 \)-manifolds.

Remark 3.5. An associative 3-form \( \varphi \) defining a \( G_2 \)-structure on a 7-manifold \( M^7 \) can be expressed in terms of the cross product \( T M^7 \times TM^7 \rightarrow TM^7 \) defined as follows [HL1982] Definition B. 1, Appendix IV.B, p. 145]

\[ \varphi(x, y, z) = \langle x \times y, z \rangle. \]

3.2. The normal bundle of an associative submanifold and its associated Dirac operator. We recall known facts necessary for understanding Formula (3.6) that enters in the proof of Theorem 3.9. Our exposition follows [CHNP2012] §5, p. 38-40], and [Gayet2010] (1)-(5)], see also Remarks ??, [4.6] for comparison with McLean’s formula.

Let \( L \) be an associative 3-fold in a \( G_2 \)-manifold \( (M^7, \varphi, g) \). Since \( L \) is orientable, it is parallelizable\(^1\) so we identify \( TL \) with \( L \times \text{Im} \mathbb{H} \). Since \( \text{rank} \ NL > \text{dim} \ L \), therefore there is a non-trivial section of \( NL \). Using the cross product \( TL \times NL \rightarrow NL \), we obtain the following

Lemma 3.6. ([CHNP2012] Lemma 5.1, §5, p. 38]). The normal bundle \( NL \) of an associative submanifold \( L \) is differentiably trivial.

Let \( \nabla \) denote the Levi-Civita connection defined by the metric \( g \) on \( M^7 \). Denote by \( \nabla^\perp \) the induced connection in the normal bundle \( NL \).

\(^1\)the assertion is well-known for compact orientable 3-manifolds. For the proof of the case of non-compact orientable 3-manifolds we refer the interested reader to [http://math.stackexchange.com/questions/1107682/elementary-proof-of-the-fact-that-any-orientable-3-manifold-is-parallel](http://math.stackexchange.com/questions/1107682/elementary-proof-of-the-fact-that-any-orientable-3-manifold-is-parallel)
Using this, we express the Dirac operator $\mathcal{D} : \Gamma(NL) \to \Gamma(NL)$ as follows. For any $x \in L$ let $e_1, e_2, e_3$ denote a positive orthonormal basis of $T_xL$ and for $V \in \Gamma(NL)$ we set
\[
\mathcal{D}(V)_x := \sum_{i=1}^{3} e_i \times (\nabla^\perp_{e_i} V).
\]

Example 3.7. ([Gayet2010, Proposition 4.7], cf. [CHNP2012, p. 43]) Let $L$ be a special Lagrangian submanifold in a Calabi-Yau manifold $(M^6, \omega, \Omega)$. Using notations in Example 3.3, for $\theta \in S^1$ we have
\[
NL_\theta = \mathbb{R} \oplus NL,
\]
where $NL$ is the normal bundle of $L$ in $M^6$. Then we identify $\Gamma(NL_\theta) \ni V = f_V \oplus \alpha_V \in \Omega^0(L) \oplus \Omega^1(L)$ where $f_V \in \Omega^0(L)$ and $\alpha_V \in \Omega^1(L)$ is dual to $JV \in \Gamma(NL)$ w.r.t. the associated Riemannian metric, equivalently $\alpha_V = V|_\omega$, see [McLean1998, Theorem 3.13, p. 723]. Using this identification we rewrite the Dirac operator $\mathcal{D} : \Gamma(NL_\theta) \to \Gamma(NL_\theta)$ as follows
\[
\mathcal{D}(f_V, \alpha_V) = (\ast d \ast \alpha_V, -df_V - d \ast \alpha_V).
\]
(The formula in (3.5) is identical with the formula in [Gayet2010] Proposition 4.7] and differs from the one in [CHNP2012, p.43] by the sign (-1), noting that $d^\ast \alpha_V = - \ast d^\ast \alpha_V$.)

3.3. Second variation of the volume of an associative submanifold.
In this subsection we give a new proof of McLean’s second variation formula for associative submanifolds (Theorem 3.9), correcting a coefficient in RHS of Formula (5.7) in [McLean1998, p. 737]), which is twice larger than our coefficient. Then we derive from Theorem 3.9 the McLean second variation formula for special Lagrangian submanifolds in Calabi-Yau 6-manifolds (Example 3.10).

We assume that $L$ is a closed associative submanifold in a $G_2$-manifold $M$. To compute the second variation of the volume of $L$, by Theorem 1.3 and Lemma 3.1 it suffices to have the following.

Lemma 3.8. ([Gayet2010 (1)-(5)]) Let $\xi(x)$ denote the unit decomposable 3-vector associated with the tangent space $T_xL$. Then for any $V \in NL$ we have
\[
\nabla_{\partial t}|_{t=0}(\chi(\exp(tV))\ast(\xi(x))) = \mathcal{D}(V)(x) \in N_xL.
\]

Theorem 3.9. (cf. [McLean1998 Theorem 5.3]) Let $L$ be an associative submanifold in a $G_2$-manifold $(M^7, \varphi, g)$. For any normal vector field $V$ on
with compact support, the second variation of the volume of $L$ with the variation field $V$ is given by

$$
\left. \frac{d^2}{dt^2} \right|_{t=0} \text{vol}(L_t) = \int_L \langle D(V), D(V) \rangle \text{dvol}_x.
$$

**Proof.** Clearly Theorem 3.9 follows from Theorem 1.3 and Lemmas 3.1, 3.8. □

**Example 3.10.** We shall derive a formula for the second variation of the volume of a special Lagrangian submanifold $L$ in a Calabi-Yau manifold $(M^6, \omega, \Omega)$ from Theorem 3.9, using notations and formulas in Example 3.7. Let $V$ be a normal vector field on $L$ in $(M^6, \omega, \Omega)$. Then $V$ is also a normal vector field of the associative submanifold $L_0 \subset S^1 \times M^6$. Let $\Phi_t$ denote the variation associated to $V$ in $M^6$. Then $\tilde{\Phi}_t := \text{Id} \times \Phi_t$ is the associated variation of $L_0 \subset S^1 \times M^6$. Since $\tilde{\Phi}_t(L_0)$ is isometric to $\Phi_t(L)$, we have

$$
\left. \frac{d^2}{dt^2} \right|_{t=0} \text{vol}(\Phi_t(L)) = \left. \frac{d^2}{dt^2} \right|_{t=0} \text{vol}(\tilde{\Phi}_t(L_0)).
$$

Applying Theorem 3.9, taking into account (3.5), we obtain

$$
\left. \frac{d^2}{dt^2} \right|_{t=0} \text{vol}(\Phi_t(L)) = \int_L (|d^* \alpha_V|^2 - |d\alpha_V|^2) \text{dvol}_x
$$

(3.8)

Our formula (3.10) agrees with the formula in [McLean1998, Theorem 3.13, p. 723].

### 3.4. Second variation of the volume of a coassociative submanifold.

In this subsection, using Theorem 1.3 we give a new proof of McLean’s second variation formula for coassociative submanifolds (Theorem 3.12).

Let $L$ be a coassociative submanifold in a $G_2$-manifold $(M^7, \varphi, g)$. We identify the normal bundle $NL$ with the bundle $\Lambda_+^2 T^* L$ as follows ([JS2005, Theorem 2.5], cf [McLean1998, Theorem 4.5]). Let us denote by $\Lambda_+^2 T^* L$ the bundle of self-dual 2-forms on $L$. We define the following map

$$
NL \ni V \mapsto \alpha_V := (V|\varphi)_L \in \Lambda_+^2 T^* L.
$$

The following Lemma is due to McLean.

**Lemma 3.11.** (cf. [McLean1998, Theorem 4.5], cf. [JS2005, Theorem 2.5]) Assume that $L$ is a coassociative submanifold in $(M^7, \varphi, g)$ and $V \in \Gamma(NL)$ is a normal vector field. Then

$$
\left. \frac{d}{dt} \right|_{t=0} ((\exp tV)^* \varphi)_L = d\alpha_V.
$$

Now we are ready to give a new proof of the following Theorem due to McLean.
Theorem 3.12. (McLean1998, Theorem 4.9, p. 731) Let $L$ be a coassociative submanifold in a $G_2$-manifold $(M^7, \varphi, g)$. For any normal vector field $V \in \Gamma(NL)$ with compact support we have

$$d^2_{t=0} \text{vol}(L_t) = \int_L \langle d\alpha_V, d\alpha_V \rangle d\text{vol}_x.$$

Proof. Recall that $\tau$ is defined in (3.4). As before we denote by $\xi_t(x) := (\exp tV)^*(\xi(x))$, where $\xi(x)$ is the decomposable 4-vector associated with $T_x L$.

Lemma 3.13. Let $L$ be a compact coassociative submanifold and $V \in \Gamma(NL)$. Then for all $x \in L$ we have

$$\langle \nabla_{\partial t}(\tau(\xi_t(x))), \nabla_{\partial t}(\tau(\xi_t(x))) \rangle_{t=0} = \langle d\alpha_V, d\alpha_V \rangle(x).$$

Proof. As before we set $L_t := \exp(tV)(L)$. Define the Poincare duality map for any $y \in L_t P_t : T_y L_t \to \Lambda^3(T_y^* L_t)$, $v \mapsto v \cdot \text{vol}_{L_t}(y)$.

Note that $\tau(\xi_t(y)) \in T_{\exp tV(x)} L_t$. Using $P_t$ and denoting $y := \exp tV(x)$, we rewrite the relation (3.4) as follows

$$P_t(\tau(\xi_t(y))) = \varphi|_{L_t} \cdot |\xi_t(y)|.$$

Using (3.11), noting that $\varphi|_L = 0$ and $|\xi_0(x)| = 1$, we obtain

$$\nabla_{\partial t}(\tau(\xi_t))|_{t=0}(x) = (P_0)^{-1}(\nabla_{\partial t}(\varphi|_{L_t})|_{t=0}(x)).$$

Denote by $\Pi_t$ the parallel transportation from $\exp tV(x)$ to $x$ along the curve $\exp tV(x)$ that is induced by the connection $\nabla$, and abbreviate

$$\varphi_t(x) := \varphi(\exp tV(x)|_{L_t}, D_t := (\exp tV)^*.$$

Then we have

$$\nabla_{\partial t}(\varphi|_{L_t})|_{t=0}(x) = \frac{d}{dt}|_{t=0}[\Pi_t \circ D_t^{-1} \circ D_t(\varphi_t(x))] = \frac{d}{dt}|_{t=0}[D_t(\varphi_t(x))]$$

(3.13)

since $\varphi_0(x) = 0$. From (3.13) we obtain

$$\nabla_{\partial t}(\varphi|_{L_t})|_{t=0}(x) = \frac{d}{dt}|_{t=0}[(\exp tV)^*(\varphi)]|_L.$$

Using Lemma 3.11 noting that $P_0$ is an isometry, we derive Lemma 3.13 from (3.12) and (3.14) immediately. 

Continuation of the proof of Theorem 3.12. Clearly Theorem 3.12 follows from Theorem 1.3 and Lemmas 3.13 and 3.12. □
Remark 3.14. In [McLean1998, p. 736] McLean gave a short proof of the following formula

\begin{equation}
\frac{d}{dt}|_{t=0} (\exp(tV)^*(\chi))|_L = \mathcal{L}(V) \cdot \text{vol}_L \in \Omega^3(L, NL).
\end{equation}

This formula, which looks like (3.6), was important for McLean’s computation of infinitesimal deformations of associative submanifolds. Unfortunately, in his proof McLean applied the Cartan formula \( \mathcal{L}(V)(\phi) = d(V\cdot\phi) + V\cdot d\phi \) for scalar valued differential forms \( \phi \) to the tangent bundle valued forms \( \varphi \). Using the argument in the proof of Lemma 3.13 we can easily prove (3.15). To prove (3.15) was one of our motivations to revisit McLean’s second variation formulas.

4. Second variation formula for Cayley submanifolds

In this section we give a new proof of McLean’s second variation formula for a compact Cayley submanifold in a Spin(7)-manifold (Theorem 4.5), correcting a coefficient in the RHS of Formula (6.16) in [McLean1998, p. 743], which is twice larger than our coefficient.

4.1. Cayley submanifolds in Spin(7)-manifolds and cross products.

In this subsection we recall basic facts concerning Cayley submanifolds in Spin(7)-manifolds that are important for understanding of our proof of McLean’s second variation formula for Cayley submanifolds. Our main sources are [HL1982], [Fernandez1986], [McLean1998], [Ohst2014].

Let \((x_1, \ldots, x_8)\) be coordinates of \(\mathbb{R}^8\). Define a 4-form \(\Phi_0\) on \(\mathbb{R}^8\) by

\(\Phi_0 = dx^{1234} + dx^{1256} - dx^{1278} + dx^{1357} + dx^{1458} - dx^{1467}
- dx^{2358} + dx^{2367} + dx^{2457} + dx^{2468} - dx^{3456} + dx^{3478} + dx^{5678},\)

where \(dx^{i_1\cdots i_4}\) is an abbreviation of \(dx^{i_1} \wedge \cdots \wedge dx^{i_4}\). The subgroup of GL(8, \(\mathbb{R}\)) preserving \(\Phi_0\) is Spin(7). Let \(g_0\) denote the standard metric on \(\mathbb{R}^8\). We call \((\Phi_0, g_0)\) the standard Spin(7)-structure.

Let \(M^8\) be an oriented 8-manifold and \(\Phi\) be a 4-form on \(M^8\). A 4-form \(\Phi\) is called a Spin(7)-structure on \(M^8\) if for each \(p \in M\), there exists an oriented isomorphism \(I_p\) between \(T_pM^8\) and \(\mathbb{R}^8\) identifying \(\Phi_p\) with \(\Phi_0\). Then \(\Phi\) induces the metric \(g_\Phi\) by pulling back the metric \(g_0\) using \(I_p\). Since Spin(7) is a subgroup of SO(8), \(g_\Phi\) does not depend on the choice of \(I_p\). In our paper we shall consider only Spin(7)-manifolds \((M^8, \Phi, g)\), i.e. manifolds with \(d\Phi = 0\).

The 4-form \(\Phi_0\) has been discovered by Harvey-Lawson in [HL1982], where they call it the Cayley calibration.

\(^2\)there are many choices of coordinates on \(O\), which result in seemingly different \(\Phi_0\) in different papers on Spin(7)-geometry. Here we consistently follow [Ohst2014], which agrees with [HL1982, Corollary 3.1, p. 120]
On a Spin(7)-manifold \((M^8, \Phi, g)\) we define a triple cross product \(P \in \Omega^3(M^8, TM^8)\) as follows
\[
\Phi(x, y, z, w) = \langle x, P(y, z, w) \rangle
\]
(4.1)

We shall show that the Cayley calibration \(\Phi\) on any Spin(7)-manifold \((M^8, \Phi, g)\) satisfies Harvey-Lawson’s identity. To define a bundle \(E\) on \(M^8\) and \(\Psi \in \Omega^4(M^8, E)\) such that \((\Phi, \Psi)\) satisfy (1.2), we need recall the notion of the cross product on \(M^8\).

First we need the following (point-wise) splitting on \((M^8, \Phi, g)\)
\[
\Lambda^2T^*M^8 = \Lambda^2_1T^*M^8 \oplus \Lambda^2_2T^*M^8,
\]
where \(\Lambda^2_1T^*M^8\) corresponds to an irreducible Spin(7)-module of dimension \(k\) in the Spin(7)-module \(\Lambda^2T^*M^8\).

For a tangent vector \(v \in TM^8\), define a cotangent vector \(v^\flat \in T^*M^8\) by \(v^\flat = g(v, \cdot)\). Define a 2-fold cross product \(TM^8 \times TM^8 \to \Lambda^2_1T^*M^8\) by
\[
v \times w = 2\pi_7(v^\flat \wedge w^\flat) = \frac{1}{2} \left( v^\flat \wedge w^\flat - *\left(v^\flat \wedge w^\flat \wedge \Phi\right) \right)
\]
for \(v, w \in TM^8\), where \(\pi_7\) denotes the projection to \(\Lambda^2_1T^*M^8\) according to the above splitting of \(\Lambda^2T^*M^8\).

Now we set \(\tau \in \Omega^4(M^8, \Lambda^2T^*M^8)\) as follows \([Ohst2014, (2.7)]\).
\[
(4.2) \quad \tau(a, b, c, d) := -a \times P(b, c, d) + \langle a, b\rangle(c \times d) + \langle a, c\rangle(d \times b) + \langle a, d\rangle(b \times c).
\]

The following Lemma asserts that \(\Phi\) satisfies Harvey-Lawson’s identity.

**Lemma 4.1.** \([HL1982, Theorem 1.28, p. 119]\) For all \(x, y, z, w \in TM^8\) we have
\[
\Phi(x \wedge y \wedge z \wedge w)^2 + |\tau(x, y, z, w)|^2 = |x \wedge y \wedge z \wedge w|^2.
\]

**Remark 4.2.** \([HL1982, Proposition IV.B.14, p. 149]\) Let \((M^7, \varphi, g)\) be a \(G_2\)-manifold. Then \((S^1 \times M^7, d\theta \wedge \varphi + *\varphi, d\theta^2 + g)\) is a Spin(7)-manifold. Furthermore, for any \(\theta \in S^1\), the restriction of \(\tau\) on \(M^8\) to \(\{\theta\} \times M^7\) is equal to the 4-form \(\tau\) defined in (3.4). Thus we use the notation \(\tau\) for the form on \(M^7\) as well as for the form on \(M^8\).

Recall that a 4-submanifold \(L\) in a Spin(7)-manifold \((M^8, \Phi, g)\) is called Cayley, if \(L\) is calibrated by \(\Phi\), i.e. \(\Phi|_{L} = \text{vol}_L\).

**Example 4.3.** Assume that \((M^7, \varphi, g)\) be a \(G_2\)-manifold and \(L\) is a coassociative submanifold in \((M^7, \varphi, g)\). Then \((S^1 \times M^7, dt \wedge \varphi + *\varphi, dt^2 + g)\) is a Spin(7)-manifold and \(\{\theta\} \times L\) is its Cayley submanifold for any \(\theta \in S^1\).

4.2. The normal bundle of a Cayley submanifold and its associated Dirac type operator. We collect known results from \([McLean1998, Section 6]\) and \([Ohst2014, §2, 3]\).

Let \(L\) be a Cayley submanifold in a Spin(7)-manifold \((M^8, \Phi, g)\). Then the bundle \(\Lambda^2T^*L\) of anti-self dual 2-forms on \(L\) is isomorphic to a subbundle of the bundle \(\Lambda^2T^*M^8|_L \subset \Lambda^2T^*M^8|_L\) via the following embedding...
(McLean1998, Section 6], see also [Ohst2014, Section 2])

\[ \Lambda_2^2 T^* L \to \Lambda_2^2 T^* M^8|_L, \alpha \mapsto 2\pi_7(\alpha) = \frac{1}{2}(\alpha - * (\alpha \wedge \Phi)), \]

where we extend \( \alpha \in \Lambda_2^2 T^* L \) to \( \Lambda_2^2 T^* M^8|_L \) by \( v \lvert \alpha = 0 \) for all \( v \in NL \), and \( \pi_7 \) is defined above. Let \( E_L \) denote the orthogonal complement of \( \Lambda_2^2 T^* L \) in \( \Lambda_2^2 T^* M^8|_L \), i.e.

\[ \Lambda_2^2 T^* M^8|_L \cong \Lambda_2^2 T^* L \oplus E_L. \]

Note that \( E_L \) has rank 4. Furthermore the cross product restricts to \( TL \times NL \to E_L \). Now we define a Dirac type operator \( D : \Gamma(NL) \to \Gamma(E_L) \) as follows (cf. Subsection 3.2)

\[ D(s) := \sum_{i=1}^{4} e_i \times \nabla_{e_i} s, \]

where \( e_i \) denote a positive orthonormal basis of \( T_x L \) and \( \nabla_{e_i} \) is the induced connection on the normal bundle.

4.3. Second variation of the volume of a Cayley submanifold. To derive the second variation formula we use the following Lemma, which is an analogue of Lemma 3.8.

Lemma 4.4. ([Ohst2014, Theorem 3.1]) Let \( \xi(x) \) denote the unit decomposable 4-vector associated with the tangent space \( T_x L \). Then

\[ \nabla_{\partial t} (\tau((\exp TV)_x, \xi(x)))|_{t=0} = D(V)(x) \in E_L(x). \]

Theorem 4.5. ([McLean1998, Theorem 6.4, p. 743]) Let \( L \) be a compact Cayley submanifold in a Spin(7)-manifold \( (M^8, \Phi, g) \). For any normal vector field \( V \) on \( L \) with compact support, the second variation of the volume of \( L \) with the variation field \( V \) is given by

\[ \frac{d^2}{dt^2} |_{t=0} \text{vol}(L_t) = \int_L \langle D(V), D(V) \rangle dvol_x. \]

Proof. Theorem 4.5 is a consequence of Theorem 1.3 and Lemmas 4.1, 4.4. \( \square \)

Remark 4.6. We would like to explain where McLean made mistakes leading to his Theorems [McLean1998, Theorem 5.3] and [McLean1998, Theorem 6.4] concerning the second variation formulas for associative and Cayley submanifolds. His general formula [McLean1998, Theorem 2.4, p. 711] for second variation of calibrated submanifolds seems to be correct, at least we do not find any mistake in McLean’s application of that formula to the special Lagrangian and coassociative submanifolds. His computation before the end of the proof of Theorem 5.3 in [McLean1998, p. 737] also agrees with our formula, but McLean suddenly added a coefficient 2 to his formula in [McLean1998, Theorem 5.3], referring to [McLean1998, Theorem 2.4], which has been misprinted there as Theorem 1.4. The same mistake has been repeated in McLean’s proof of [McLean1998, Theorem 6.4] in
The last formula in McLean’s proof of McLean’s Theorem 6.4 agrees with ours, but McLean added a coefficient 2 to his formula, referring to an irrelevant formula (2.13) in his paper. We guess that McLean worked on several versions of his paper and did not check all formulas carefully, see also McLean’s citation of Simon’s formulas in p. 707, 717, which are not consistent.

Acknowledgement. We thank Kotaro Kawai for a helpful comment on an early version of this paper and an anonymous referee for his suggestions which improve the exposition of our note.

References

[CS2002] S. Chiossi and S. Salamon, The intrinsic torsion of SU(3) and G2-structures, Differential geometry, Valencia, 2001, 115-133, World Sci. Publ., River Edge, NJ, 2002.
[CHNP2012] A. Corti, M. Haskins, J. Nordström and T. Pacini, G2-manifolds and associative submanifolds via semi-fano 3-folds, Duke Math. J. 164, no. 10 (2015), 1971-2092. arXiv:1207.4703v3.
[Fernandez1986] M. Fernandez, A Classification of Riemannian Manifolds with Structure Group Spin(7), Annali di Matematica Pura ed Applicata, 143(1986), 101-122.
[Gayet2010] D. Gayet, Smooth moduli spaces of associative submanifolds, Q. J. Math. 65(2014), 1213-1240.
[HL1982] R. Harvey and H. B. Lawson, Calibrated geometry, Acta Math. 148(1982), 47-157.
[JS2005] D. Joyce and S. Salur, Deformations of asymptotically cylindrical coassociative submanifolds with fixed boundary, Geometry & Topology Volume 9 (2005) 1115-1146.
[Joyce2007] D. Joyce, Riemannian holonomy groups and calibrated geometry, Oxford, 2007.
[Kawai2014a] K. Kawai, Deformations of homogeneous associative submanifolds in nearly parallel G2-manifolds, to appear in Asian J. of Math., arXiv:1407.8046.
[Kawai2014b] K. Kawai, Some associative submanifolds of the squashed 7-sphere, Q. J. Math. 66, (2015), 861-893.
[Le1989] H. V. Lé, Minimal Φ-Lagrangian surfaces in almost Hermitian manifolds, Math USSR Sbornik, 67 (1990), 379-391.
[Le1990] H. V. Lé, Relative calibration and the problem of stability of minimal surfaces, Lect. Notes in Math., Springer-Verlag, 1990, v1453, 245-262.
[LS2014] H. V. Lé and L. Schwachöfer, Lagrangian submanifolds in strict nearly Kähler 6-manifolds, arXiv:1408.6433.
[Lotay2012] J. D. Lotay, Associative Submanifolds of the 7-Sphere, Proc. Lond. Math. Soc. (3), 105, (2012), 1183-1214.
[McLean1998] R. McLean, Deformations of Calibrated Submanifolds, Comm. in Analysis and Geom. 6 (1998), 705-747.
[Morgan2009] F. Morgan, Geometric Measure Theory: a Beginner’s Guide (4th ed.), London: Academic Press, 2009.
[Ohst2014] M. Ohst, Deformations of Compact Cayley Submanifolds with Boundary, arXiv:1405.7886.
[Simons1968] J. Simons, Minimal varieties in Riemannian manifolds, Annals of Math., 88(1968), 62-105.