ON GEOMETRIC AND ANALYTIC MIXING SCALES: COMPARABILITY AND CONVERGENCE RATES FOR TRANSPORT PROBLEMS

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Abstract. In this article we are interested in the geometric and analytic mixing scales of solutions to passive scalar problems. Here, we show that both notions are comparable after possibly removing large scale projections. In order to discuss our techniques in a transparent way, we further introduce a dyadic model problem.

In a second part of our article we consider the question of sharp decay rates for both scales for Sobolev regular initial data when evolving under the transport equation and related active and passive scalar equations. Here, we show that slightly faster rates than the expected algebraic decay rates are optimal.

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1. Introduction and Main Results

In this article we are interested in the mixing behavior of passive scalar problems

\[ \partial_t \rho + v \cdot \nabla \rho = 0, \]

\[ \rho \big|_{t=0} = \rho_0, \]

where \( v(t) \) is a given divergence-free vector field on \( \mathbb{R}^n \) or \( T^n \times \mathbb{R}^n \). Assuming sufficient regularity of \( v \), the flow preserves all \( L^p \) norms, i.e. \( \| \rho(t) \|_{L^p} = \| \rho_0 \|_{L^p} \) for all \( p \in [1, \infty] \) and all \( t > 0 \). However, if the flow is for instance ergodic and \( \rho_0 \) has
mean zero, then $\rho(t)$ weakly converges to zero as time tends to infinity and all $L^p$ norms strictly decrease in this weak limit. The solution is mixed as $t \to \infty$.

In order to quantify this limiting behavior, one commonly considers two different functionals:

**Definition 1.1** (Mixing scales; c.f. [Thi12]). Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a given measurable function. Then we call $\|\rho\|_{H^{-1}}$ the analytic mixing scale.

Furthermore, for given $r > 0$, we define the geometric mixing functionals

$$g_r[\rho] := \sup_{B_R(\xi); R \geq r} \frac{1}{|B_R|} \left| \int_{B_R(\xi)} \rho \right|.$$

(2)

If further $\rho \in L^\infty$, then for each $\kappa \in (0, 1)$ we define the geometric mixing scale as

$$\mathcal{G}_\kappa[\rho] := \inf \{ r : g_r[\rho] \leq \kappa \|\rho\|_{L^\infty} \}.$$

(3)

As one of the main results of this article, we show that while both notions are not equivalent, they are comparable in the sense that smallness of one implies smallness of the other.

**Theorem 1.1** (Comparison of mixing scales). Let $\rho \in L^2(\mathbb{R}^n)$ and $\|\rho\|_{L^2} \leq 1$. Then for all $0 < \epsilon \leq 1$ it holds that:

1. If $\|\rho\|_{H^{-1}} \leq \epsilon$ and $\rho$ is supported in $B_1$, then also $g_{\epsilon'}[\rho] \leq C\epsilon'$ for all $\epsilon' \geq \epsilon^\alpha$ and $g_{\epsilon'}[\rho] \leq C$ for all $\epsilon' \geq \epsilon^\beta$, where $\alpha = \frac{2}{n+2}$ and $\beta = \frac{2}{n+4}$ depend only on the dimension.

   In particular, supposing additionally that $\|\rho\|_{L^\infty} = 1$, it follows that

   $$\mathcal{G}_C[\rho] \leq \epsilon,$$

   $$\mathcal{G}_C[\rho] \leq \epsilon'.$$

2. If $g_r[\rho] \leq \epsilon$ and $\rho$ is supported in a compact set $K$, then also $\|\rho\|_{H^{-1}} \leq C_K \epsilon$.

These estimates are optimal in the powers of $\epsilon$.

In order to introduce our methods, we construct a dyadic Walsh-Fourier model on $L^2(\mathbb{T})$ in Section 3, where we introduce new dyadic analogues of both scales and show them to be equivalent when restricted to appropriate subspaces $E_j$. In particular, in that setting optimality of estimates is transparent. Subsequently, we discuss the continuous case as stated in Theorem 1.1 in Section 4.

A natural question here, of course, is whether

$$g_r[\rho] \leq C \epsilon$$

(4)

can be assumed in applications. Indeed, if $\rho(t)$ solves the passive scalar problem (1) and asymptotically converges weakly to a non-trivial state $\rho_\infty$, then we can generally not expect better control than

$$g_r[\rho(t)] \leq \|\rho_\infty\|_{L^\infty}.$$

However, as we discuss in Section 2, upon removing large scale projections (corresponding to asymptotic states) this assumption is natural and comparability holds in the above sense.

As a second part of our article, in Section 5, we consider the evolution of the mixing scales under transport-type equations and are interested in (sharp) upper
and lower bounds on decay rates of the scales. As a first model problem we consider the case of \( \rho(t) \) evolving under the free transport equation on \( \mathbb{T}^n \times \mathbb{R}^n \):

\[
\begin{align*}
\partial_t \rho + y \partial_x \rho &= 0 \text{ on } (0, \infty) \times \mathbb{T}^n \times \mathbb{R}^n, \\
\rho_{t=0} &= \rho_0 \text{ on } \mathbb{T}^n \times \mathbb{R}^n.
\end{align*}
\]

(5)

Here, we show that if the initial data is normalized in a Sobolev space \( H^s \), \( 0 \leq s \leq 1 \), with respect to \( y \), then the at first expected decay rates of \( t^{-s} \) turn out to be slightly suboptimal and instead decay rates of \( t^{-s} \log(1/t) \) are achieved.

**Theorem 1.2.** In the following, let \( 0 < s \leq 1 \), \( u_0 \in L^2(\mathbb{T}^n; H^s(\mathbb{R}^n)) \) with \( \int_{\mathbb{T}^n} u_0(x,y) dx = 0 \), and let

\[
u(t, x, y) = u_0(t, x - ty, y),
\]

be the solution of the free transport problem. For \( \sigma, s \in \mathbb{R} \) let \( H^\sigma \) denote the Hilbert space with norm

\[
\| u \|_{H^\sigma}^2 = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{2\sigma} \int_{\mathbb{R}^n} (\xi)^{2s} |\hat{u}(\xi, \eta)|^2 d\eta.
\]

1. There exists \( C_\sigma > 1 \) such that for all \( t \geq 1 \) and all initial data

\[
\| u(t) \|_{L^2 H^{-\sigma}} \leq Ct^{-\sigma} \| u_0 \|_{H^{-\sigma}}.
\]

2. Let \( \alpha_j > 0 \) with \( \| (\alpha_j)_j \|_{l^2} = 1 \). Then there exist \( c > 0 \), a sequence of times \( t_j \to \infty \) and initial data \( u_0 \) such that

\[
\| u(t_j) \|_{L^2 H^{-\sigma}} \geq c \alpha_j t_j^{-\sigma} \| u_0 \|_{H^{-\sigma}}.
\]

3. There exists no non-trivial initial data \( u_0 \in L^2(\mathbb{T}^n; H^s(\mathbb{R}^n)) \) such that

\[
\| u(t_j) \|_{L^2 H^{-\sigma}} \geq c t_j^{-\sigma} \| u_0 \|_{H^{-\sigma}}
\]

along some sequence \( t_j \to \infty \).

In the second statement, \( t_j \) can always be chosen larger and more rapidly increasing. For instance, we may chose \( t_j = \exp(\exp(\ldots \exp(j))) \) and \( \alpha_j = \frac{1}{j} = \ln(\ln(\ldots \ln(t_j))) \) as iterated exponentials and logarithms. Informally stated, the theorem hence shows that algebraic decay rates can be achieved along a subsequence up to an arbitrarily small loss. Conversely, the third statement shows that this loss is necessary and that the lower estimate is sharp in this sense.

We remark that in several works on (linear) inviscid damping, \([WZZ15]\), \([CZZ18]\), \([Zil16]\), \([BM15]\) or (linear) Landau damping \([BMM13]\), it is shown that perturbations to the Euler equations or Vlasov-Poisson equations scatter to solutions of the free transport problem as \( t \to \infty \). As a corollary, we hence obtain the optimality of the decay rates for these equations as well.

**Corollary 1.1.** Let \( U(y) \) be Lipschitz and \( U'' \in W^{2,\infty} \) with \( \| U'' \|_{W^{2,\infty}} \) sufficiently small. Then for any \( s \in (0, 1) \) and any \( \omega_0 \in H^s(\mathbb{T} \times \mathbb{R}) \) the solution \( \omega \) of the linearized Euler equations

\[
\partial_t \omega + U(y) \partial_x \omega - U''(y) \partial_x \Delta^{-1} \omega = 0,
\]

\[
\omega|_{t=0} = \omega_0
\]

satisfies the statements of Theorem 1.2.
Proof. In [Zil16], we have shown that solutions to the linearized Euler equations around such a shear flow $U(y)$ scatter in $H^s$. That is, for any $\omega_0 \in H^s$ there exists $W$ such that the associated solution $\omega$ with initial data $\omega_0$ satisfies

$$\omega(t, x - tU(y), y) \xrightarrow{H^s} W$$

as $t \to \infty$. Furthermore, the scattering map $\omega_0 \mapsto W$ is a small perturbation of the identity and hence an isomorphism. Thus, both the decay rates and the optimality follow from Theorem 1.2. $\square$

Concerning more general passive scalar problems, a recent active area of research, [ACM14], [Sei17], [CS17], is given by the study of upper and lower bounds on decay rates of mixing scales for solutions of (1)

$$\partial_t \rho + v \cdot \nabla \rho = 0,$$

where $v$ may be chosen arbitrarily under given constraints such as $\|v(t)\|_{W^{1,p}} \leq 1$. Here, our comparison result allows us to obtain an estimate of analytic mixing costs as a corollary of the results of [CDL08] on geometric mixing costs (c.f. Corollary 5.2).

**Corollary 1.2.** Let $p > 1$ and $\rho|_{t=0} = 1_{[0,1/2]}(x_2) \in L^1(T^2)$ and suppose that for $\epsilon > 0$ and some $0 < \kappa < \frac{1}{2}$ the solution $\rho$ of

$$\partial_t \rho + v \cdot \nabla \rho = 0,$$

$$\nabla \cdot \rho = 0.$$

satisfies

$$\|\rho|_{t=1} - \frac{1}{2}\|_{H^{-1}} \leq \epsilon.$$

Then for $p > 1$ the velocity field $v$ satisfies

$$\int_0^1 \|\nabla v\|_{L^p} dt \geq C|\log(\epsilon)|.$$

The remainder of the article is organized as follows:

- In Section 2, we discuss the comparability of both mixing scales using two prototypical examples and also discuss the role of large scale asymptotic profiles.
- In Section 3, we introduce a new dyadic Walsh-Fourier model of mixing scales. Due to improved orthogonality properties here we can establish our estimates in a transparent, accessible way.
- Subsequently, in Section 4 we show that most properties and estimates persist in the continuous setting despite the loss of beneficial additional structure.
- Finally, Section 5 considers (sharp) decay rates of mixing scales under passive scalar problems. Here, we first establish optimal rates for the free transport problem and then discuss more general dynamics.
2. Preliminaries and Prototypical Examples

In [LLN+12] two families of functions are constructed to highlight the differences of the analytic mixing scale (4) and the geometric mixing functionals (2). In order to introduce our ideas, we recall their construction and show that after removing the large scale weak limit of the second family both notions are comparable and thus motivate our choice of spaces and estimates.

2.1. The Analytic Mixing Scale Controls the Geometric Mixing Scale.

We briefly recall the construction from [LLN+12] Section IV.B. As a building block consider the “hat” function

\[ v(x) = \begin{cases} 1 - |x|, & \text{for } |x| \leq 1, \\ 0, & \text{else}. \end{cases} \]

Let \( \epsilon = 2^{-n} \) and let \( \alpha \in \mathbb{N} \) with \( \alpha < 2n - 1 \). We then built an odd function \( u \) on \([-1, 1]\) such that for \( x > 0 \)

\[ u(x) = \alpha \epsilon v\left(\frac{x}{\alpha \epsilon}\right) + \sum_{j=1}^{2n-2\alpha} \epsilon \phi\left(\frac{x - (2\alpha + 2j)\epsilon}{\epsilon}\right). \]

This function is a sawtooth function with one large tooth on the interval \((0, 2\alpha \epsilon)\) and smaller teeth of width \(2\epsilon\) on the remainder of \((0, 1)\). Furthermore, \( u \) is Lipschitz and \( \rho = u' \in \{-1, 1\} \) satisfies

\[ \|\rho\|_{H^{-1}}^2 = \|u\|_{L^2}^2 = \epsilon^2 \left(\frac{1}{3} - \frac{2}{3} \alpha \epsilon\right) + \frac{2}{3} \alpha^3 \epsilon^3 \approx \epsilon^2 \]

if \( \epsilon \) is small. Taking \( \alpha \approx \epsilon^{-1/3} \), we thus obtain that \( \|\rho\|_{H^{-1}} \approx \epsilon \) and that \( \rho = 1 \) on \( B_{2\epsilon^{1/3}}(0) \) and is hence geometrically mixed at most at scale \( \epsilon^{2/3} \).

Additionally, we compute that if we consider larger radii \( r \geq \epsilon^{2/5} \), then

\[ \frac{1}{|B_r|} \left| \int_{B_r(\xi)} \rho dxdy \right| \leq Cr \]

This example hence shows that an estimate of the type \( \epsilon \leadsto \epsilon \) is not possible, but \( \epsilon \leadsto \epsilon^n \) is for this case. In Sections 3 and 4 we show that such an estimate indeed holds for general functions and in higher dimensions and that our exponents of \( \epsilon \) are optimal. Roughly speaking, the loss of power in \( \epsilon \) here is due to the \( L^1 \) normalization of the characteristic functions and equivalence constants of (weighted) \( L^2 \) and \( L^\infty \) norms on finite-dimensional vector spaces (c.f. Section 3) and agrees with scaling (c.f. the proof of Theorem 4.1).

2.2. The Geometric Mixing Scale and Weak Limits.

Consider a periodic characteristic function \( u \in L^2(\mathbb{T}) \) with \( \int_{\mathbb{T}} u dx = \frac{1}{2} \) and for \( k \in \mathbb{N} \) define

\[ \rho_k(x) = \text{sgn}(x)u(k|x|) \in L^2((-1, 1)). \]

We note that \( \int \rho_k dx = 0, \|\rho_k\|_{L^\infty} = 1 \) and that

\[ \rho_k \overset{L^2}{\rightarrow} \frac{1}{2} \text{sgn}(x). \]
Hence, for any given ball $B_r(\xi) \subset (-1, 1)$,

$$\frac{1}{|B_r|} \left| \int_{B_r(\xi)} \rho \, dx \right| \to \frac{1}{|B_r|} \left| \int_{B_r(\xi)} \frac{1}{2} \text{sgn}(x) \, dx \right| \leq \frac{1}{2}$$

as $k \to \infty$. Using the periodicity to obtain more quantitative estimates, we obtain that for any $\kappa > \frac{1}{2}$ and any $r > 0$, we can achieve

$$\frac{1}{|B_r|} \left| \int_{B_r(\xi)} \rho_k \, dx \right| \leq \kappa \|\rho_k\|_{L^\infty}$$

provided $k$ is sufficiently large and that thus $G_\kappa[\rho_k] \to 0$ as $k \to \infty$.

However, this fails for $\kappa < \frac{1}{2}$ and by lower semicontinuity,

$$\liminf_{k \to \infty} \|u_k\|_{H^{-1}} \geq \frac{1}{2} \|\text{sgn}(x)\|_{H^{-1}} > 0.$$ 

Hence, this at first suggests that the geometric mixing scale is distinct from the analytic mixing scale. In view of our comparison result, Theorem 1.2, we instead suggest to interpret this example as

$$\|u_k\|_{H^{-1}} \leq C \max(\kappa, r)$$

and note that the lower bound on $\kappa \geq \frac{1}{2}$ here is due to large scale structures in the weak limit. Indeed, consider the functions $v_k$ obtained by projecting out large scales:

$$v_k(x) = u_k(x) - \frac{1}{2} \text{sgn}(x).$$

Then it holds that

$$\|v_k\|_{L^\infty} \geq \kappa,$$

$$\frac{1}{|B_r|} \left| \int_{B_r(\xi)} \rho_k \, dx \right| \leq \min(2\kappa, Cr) \text{ for } r \geq \frac{c}{K},$$

$$\|v_k\|_{H^{-1}} \leq \frac{C}{K},$$

$$v_k \overset{L^2}{\rightharpoonup} 0 \text{ as } k \to \infty.$$ 

**Remark 1.** When considering $u(t_k) = \rho(t_k)$ for $\rho(t)$ evolving under ergodic dynamics, the weak limit is given by a constant function. In that setting, we may assume that constant to be zero after normalization and hence, there without loss of generality both notions of geometric mixing coincide, i.e. $v_k = u_k$.

More generally, we don’t need to know a candidate for the weak limit (or even have a sequence), but rather consider the following setting:

If $\rho \in L^2 \cap L^\infty \cap H^{-1}$ is such that for all $r \geq r_0$ and all $x_0 \in \mathbb{R}^n$ it holds that

$$\frac{1}{|B_r|} \left| \int_{B_r(x_0)} \rho \, dx \right| \leq \kappa,$$

then we claim that (c.f. Lemma 4.2)

$$\rho_{r_0} = \rho - \left( \frac{1}{|B_{r_0}|} 1_{B_{r_0}(\xi)} * \rho \right)$$
satisfies

\[ \|\rho - \rho_{r_0}\|_{L^\infty} \leq \kappa, \]
\[ \frac{1}{|B_{r}|} \left| \int_{B_{r}(\xi)} \rho_{r_0} \, dx \right| \leq C r, \]
\[ \|\rho_{r_0}\|_{H^{-1}} \leq C r_0\|\rho\|_{L^2}, \]

for all \( r \geq r_0 \). Here, we stress that, while

\[ \frac{1}{|B_{r_0}|} 1_{B_{r_0}}(\xi) * \rho \to \rho \]

as \( r_0 \downarrow 0 \) for fixed \( \rho \), this is much more subtle for sequences \( \rho \) depending on \( r_0 \).

Indeed, letting \( u_k(x) \) be as above, we obtain that

\[ \frac{1}{|B_{1/8}|} 1_{B_{1/8}} * u_k(x) = \frac{1}{2} \operatorname{sgn}(x) \]

for all \( x \) with \( \text{dist}(x, \{0, \pi, -\pi\}) \geq \frac{2}{k} \).

3. A Walsh-Fourier Model

In order to introduce our ideas and establish sharpness of estimates, we first discuss and compare both mixing scales in a dyadic model setting. Here, we consider averages over dyadic intervals and replace the sin basis of \( L^2(T) \) by functions that are constant \(+1\) or \(-1\) on dyadic intervals. This setting is known in harmonic analysis as a Walsh-Fourier setting and associated with a “tile” characterization and Haar wavelet expansions, [MTT04], [Thi00a], [Thi00b].

In the following we briefly provide some definitions and statements. For a more in-depth introduction we refer the interested reader to [Thi06]. We remark that, for simplicity of notation and estimates, we here consider the setting of \( L^2([0,1)) \) instead of \( L^2(\mathbb{R}) \). The dyadic setting has the benefit of greatly simplifying estimates due to orthogonality and allows for explicit computations of newly introduced analogues of the mixing scales as Besov-type norms in terms of certain \( L^2 \) bases. Hence, here it is transparent what estimates are possible and whether they are optimal. In Section 4 we show that, with minor modifications, these results also extend to the continuous Sobolev setting.

3.1. Definitions, Tiles and Bases.

**Definition 3.1.** Let \([0,1)\) be the half-open unit interval. Then for each \( j \in \mathbb{N}_+ \), we define the set of dyadic intervals at scale \( 2^{-j} \) by

\[ D_j = \{ I_{k,j} := 2^{-j}[k,k+1) : k \in \{0, \ldots, 2^j - 1\} \}. \]

Associated with this partition of \([0,1)\), we introduce the \( L^2 \)-normalized characteristic functions

\[ \chi_I = \frac{1}{\sqrt{|I|}} 1_I \in L^2([0,1)). \]

We note that, if \( I, I' \in D_j \), then either \( I = I' \) or the intervals are disjoint. If the intervals are not of the same size, that is \( I \in D_j \) and \( I' \in D_{j'} \) with \( j \neq j' \), they are either disjoint or one is contained in the other.
In addition to the (normalized) characteristic functions, $\chi_I$, the following definition introduces a large family of oscillating $L^2$ normalized functions, which we use to define (fractional) Sobolev-type spaces.

**Figure 1.** Various tiles down to scale $2^{-4}$. The vertical gray tiles correspond to characteristic functions $\chi_I$. The horizontal gray lines correspond to our replacement of a Fourier basis, $\phi_{p_l}$. The tiles in green and blue in the upper right corner are at level $l = 1$. Plots of the corresponding wave packets of all colored tiles are given in Figure 2. By Lemma 3.1 wave packs $\phi_p, \phi_{p'}$ are $L^2$ orthogonal iff they are disjoint.

**Figure 2.** Walsh wave packets associated with the tiles of Figure 3.1

**Definition 3.2.** A tile $p$ is a dyadic rectangle of area one in $[0, 1) \times [0, \infty)$. That is,

$$p := I \times \omega = [2^{-j}k, 2^{-j}(k + 1)) \times [2^jl, 2^jl + 1),$$

where $k \in \{0, 2^j - 1\}, j \in \mathbb{N}_0, l \in \mathbb{N}_0$. If $l = 0$, we define the wave-packet $\phi_{p + 1}. For l > 0, we define $\phi_p$ recursively. That is, if $p$ is a tile at level $l$ and scale
$2^j$, we can express it as either upper or lower half the union of two tiles $p_l, p_r$ at level $\lfloor \frac{j}{2} \rfloor$ and scale $2^{-j-1}$. We thus define
\[
\phi_p = \frac{1}{\sqrt{2}} (\phi_{p_l} + \phi_{p_r}) \text{ if } l \text{ is even},
\]
\[
\phi_p = \frac{1}{\sqrt{2}} (\phi_{p_l} - \phi_{p_r}) \text{ if } l \text{ is odd}.
\]

This definition allows us to consider questions of orthogonality and basis expansions in a graphical way, a so-called cartoon (c.f. Figure 3.1). The following lemma summarizes some of the main properties we use in the following.

**Lemma 3.1** (c.f. Lemma 2.9 in [Thi00a]). For any two tiles $p, p'$ we have
\[
\left| \int_{[0,1)} \phi_p \phi_{p'} \right| = \sqrt{|p \cap p'|}
\]

In particular two wave packets are $L^2$-orthogonal if and only if the underlying tiles have empty intersection.

**Corollary 3.1** (c.f. Corollary 2.7 in [Thi00a]). Furthermore, two families of tiles $P, P'$ cover the same region in $[0,1) \times [0,\infty)$ if and only if the spans of $\{\phi_p : p \in P\}$ and $\{\phi_p : p \in P'\}$ are identical. In particular, denoting $p_l = [0,1) \times [l,l+1)$, we obtain that $\{\chi_I : I \in D_j\}$ and $\{\phi_{p_I} : I \in \{0,2^j-1\}\}$ are both orthonormal bases of the same space, which we denote by $E_j$. We further introduce the $L^2$ orthogonal projection operators $P_j$ onto $E_j$.

**Definition 3.3** (Mixing scales). Given $\rho \in L^2([0,1))$, we introduce the geometric mixing seminorms at scale $j$ as
\[
g_j[\rho] = \sup \left\{ \frac{1}{I} \left| \int_I \rho \right| : I \in D_j \right\} = \sqrt{2^j} \sup \{|\langle \chi_I, \rho \rangle| : I \in D_j\}.
\]

Furthermore, for $s \in [-1,1]$ we define analytic mixing seminorms up to scale $j$ by
\[
\|\rho\|_{h^s_j}^2 = \sum_{l=0}^{2^j-1} (1 + l^2)^s |\langle \rho, \phi_{p_I} \rangle|^2.
\]

We further define $\|\rho\|_{h^\infty_j} = \sup_j \|\rho\|_{h^s_j}$.

**3.2. Estimates.** We note that the geometric mixing seminorms can be expressed as
\[
g_j[\rho] = \sup_{I \in D_j} \sqrt{2^j} |\langle \chi_I, \rho \rangle|
\]
and are hence indeed norms on the space $E_j$. Likewise the analytic seminorms are weighted $l^2$ norms on the same space $E_j$ when expressed in the orthonormal basis $\{\phi_{p_I}\}$. Hence, both mixing seminorms are equivalent, with constants depending on $j$.

The following theorem establishes the corresponding estimates as well as related estimates between different scales with uniform constants.
Theorem 3.1. Let \( \rho \in L^2((0, 1)) \), then for all \( j \in \mathbb{N} \)

\[
\| \rho \|_{h_j^{-1}} \leq g_j[\rho], \\
\| \rho \|_{h_j^{-1}} \leq g_j[\rho] + 2^{-j}\|(1-P_j)\rho\|_{L^2}, \\
g_j[\rho] \leq 2^{\frac{j}{2}}\| \rho \|_{h_j^{-1}}.
\]

Furthermore, both seminorms depend on \( \rho \) only via its projection, that is \( \| \rho \|_{h_j^{-1}} = \| P_j \rho \|_{h_j^{-1}} \) and \( g_j[\rho] = g_j[P_j \rho] \).

We remark that the loss of the factor \( 2^{\frac{j}{2}} \) here can be interpreted as corresponding to the embedding \( H^{-1} \subset H^{1/2} \subset L^\infty \). Indeed, if \( \rho \in E_j \), then \( \rho \) is constant on each interval \( I \in D_j \) and thus \( g_j[\rho] = \| \rho \|_{L^\infty} \).

Corollary 3.2. Let \( \rho \in L^2((0, 1)) \), then

1. If \( \| \rho \|_{h_j^{-1}} \leq 2^{-j_0} \), then \( g_j[\rho] \leq C 2^j \| \rho \|_{L^2} \). In particular, if \( j \leq \frac{3}{4} j_0 \), then \( \rho \) is geometrically mixed at scale \( 2^{-j} \).
2. Furthermore, for \( j \leq \frac{3}{5} j_0 \), we obtain that \( g_j[\rho] \leq 2^{-j} \).
3. Conversely, if \( g_j[\rho] \leq 2^{-j} \), then

\[
\| \rho \|_{h_j^{-1}} \leq g_j[\rho] \leq 2^{-j}.
\]

If we additionally assume that \( \|(1-P_j)\rho\|_{L^2} \leq 1 \), then furthermore

\[
\| \rho \|_{h_j^{-1}} \leq g_j[\rho] \leq 2^{-j} + 2^{-j}\|(1-P_j)\rho\|_{L^2} \leq C 2^{-j}.
\]

Remark 2. Denoting \( 2^{-j_0} = \epsilon \), the above results show that \( \| \rho \|_{h_j^{-1}} \leq \epsilon \) implies \( g_{\log(2^{2/3})}[\rho] \leq C \) and \( g_{\log(\epsilon')}[\rho] \leq \epsilon' \) for \( \epsilon' \geq 2^{2/3} \).

Conversely, if \( g_{\log(\epsilon')}[\rho] \leq \epsilon' \) and we control \( \| \rho \|_{L^2} \), then also \( \| \rho \|_{h_j^{-1}} \leq \epsilon \).

The analytic and geometric mixing scales are hence comparable with a loss in the exponent in one direction (we hence do not use the word equivalent).

Furthermore, we show in Lemma 3.2 that this loss is optimal.

As mentioned in the introductory Section 2, in this article we hence stress the viewpoint that the examples constructed in [LLN+12] should instead be interpreted as showing the necessity of the control of \( g_{\log(\epsilon)}[\rho] \leq \epsilon \) instead of \( g_{\log(\epsilon)}[\rho] \leq \kappa \) and of the loss \( \epsilon \rightarrow \epsilon^{2/3} \).

We discuss this interpretation, scaling and the constructions further in Section 4.

Proof of Theorem 3.1. Let \( \rho \in L^2 \) be given and consider the basis expansions of \( P_j \rho \in E_j \):

\[
P_j \rho = \sum_I d_I \chi_I = \sum_I c_I \phi_{p_I}.
\]

Since both \( \chi_I \) and \( \phi_{p_I} \) are orthonormal bases of \( E_j \), it follows that

\[
\| d_I \|_{L^2} = \| c_I \|_{L^2}.
\]

Then we may estimate

\[
\| \rho \|_{h_j^{-1}} = \| \rho \|_{h_j^{-1}} \leq c_I \| c_I \|_{L^2} \leq \| d_I \|_{L^2} \leq \sqrt{2^j} \max |d_I|.
\]
On the other hand, the normalization of the geometric mixing functionals is such that

\[ g_j[\rho] = \max_\mathcal{T} \langle \chi_I, \rho \rangle = \sqrt{2^j} \max |d_I|. \]

For the converse estimate, we note that

\[ g_j[\rho] = \sqrt{2^j} \max |d_I| \leq \sqrt{2^j} \|d_I\|_2 \]
\[ = \sqrt{2^j} \|c_l\|_2 \leq \sqrt{2^j} 2^j \|c_l\|_2 \]
\[ \leq 2^{\frac{2}{3}j} \|\rho\|_{h^{-1}}. \]

If \( \rho \notin E_j \), we note that by definition

\[ \|\rho\|_{h^{-1}} = \sum_{l=0}^{2^j-1} \left| \frac{1}{1+t^2} \langle \rho, \phi_{p_l} \rangle \right|^2 \leq \sum_{l=0}^\infty \left| \frac{1}{1+t^2} \langle \rho, \phi_{p_l} \rangle \right|^2 = \|\rho\|_{h^{-1}}. \]

For the estimate of \( \|\rho\|_{h^{-1}} \), we thus split \( \rho \) using \( P_j, 1 - P_j \) and obtain

\[ \|\rho\|_{h^{-1}} = \|\rho\|_{h^{-1}} + \sum_{l=2^j}^{\infty} \left| \frac{1}{1+t^2} \langle \rho, \phi_{p_l} \rangle \right|^2 \]
\[ \leq g_j[\rho]^2 + 2^{-2j} \|\rho\|_{L^2}^2. \]

\[ \square \]

**Proof of Corollary 3.2.** We apply Theorem 3.1 to obtain that

\[ g_j[\rho] \leq C 2^{\frac{2}{3}j} - j_0. \]

The first statements hence follow by noting that

\[ \frac{3}{2}j - j_0 \leq 0 \iff j \leq \frac{2}{3} j_0, \]

and the second from

\[ \frac{3}{2}j - j_0 \leq -j \iff j \leq \frac{2}{5} j_0. \]

The last statement similarly follows as a direct corollary of Theorem 3.1. \( \square \)

The following lemma shows that these restrictions on \( j \) are optimal.

**Lemma 3.2.** There exists a family of functions \( \rho = \rho(j_0) \) such that \( \|\rho\|_{h^{-1}} \leq C 2^{-j_0} \) and which satisfies the following properties:

1. For any \( \alpha < \frac{2}{3} \) and \( j = \lfloor \alpha j_0 \rfloor \), it holds that \( g_j[\rho] = o(1) \) as \( j_0 \to \infty \). If instead \( \alpha > \frac{2}{3} \), then \( g_j[\rho] \to \infty \).
2. For any \( \alpha < \frac{2}{5} \) and \( j = \lfloor \alpha j_0 \rfloor \), \( 2^j g_j[\rho] = o(1) \) as \( j_0 \to \infty \). If instead \( \alpha > \frac{2}{5} \), then \( 2^j g_j[\rho] \to \infty \).

**Proof.** As shown in the preceding Theorem 3.1 and Corollary 3.2, we have

\[ g_j[\rho] = \sqrt{2^j} \|d_I\|_\infty \leq \sqrt{2^j} \|d_I\|_2 \leq 2^{\frac{2}{3}j} \|\rho\|_{h^{-1}} \leq 2^{\frac{2}{3}j} - j_0. \]

Hence, we note that \( \frac{3}{2}j - j_0 \leq 0 \iff j \leq \frac{3}{2} j_0 \) and that \( \frac{3}{2}j - j_0 \leq j \iff j \leq \frac{2}{5} j_0 \).
It hence only remains to show that these estimates are indeed sharp in the thresholds in $j$. For this purpose, consider

$$\rho = \sum_{2^{\alpha j_0} - 1 \leq l \leq 2^{\alpha j_0} - 1} \phi_{p_l},$$

with $\alpha \in (0, 1)$ to be chosen later (c.f. also Remark 3). Then we can compute

$$\|\rho\|_{H^{-1}} = \left( \sum_{2^{\alpha j_0} - 1 \leq l \leq 2^{\alpha j_0} - 1} l^{-2} \right)^{\frac{1}{2}} \approx 2^{-\alpha j_0 \frac{3}{2}}.$$

On the other hand, averaging over the interval $[0, 2^{-j})$, wave packets $\phi_{p_l}$ with $l > 2^{-j}$ are orthogonal, while wave packets with $l \leq 2^j$ are constantly equal to one when restricted to this interval. Hence, we obtain that for any $j \leq j_0$

$$g_j[\rho] = \sum_{2^{\alpha j_0} - 1 \leq l \leq 2^{\alpha j_0} - 1} 1 \approx 2^j$$

if $j > j_0 \alpha$. If we now multiply $\rho$ by $2^{-j_0(1 - \frac{\alpha}{2})}$, we are exactly in the setting described. That is,

$$\|2^{-j_0(1 - \frac{\alpha}{2})} \rho\|_{H^{-1}} \approx 2^{-j_0(1 - \frac{\alpha}{2})} 2^{-\alpha j_0 \frac{3}{2}} = 2^{-j_0}$$

and

$$g_j[2^{-j_0(1 - \frac{\alpha}{2})} \rho] \approx 2^{j - j_0(1 - \frac{\alpha}{2})}$$

for any $j$ with $\alpha j_0 \leq j \leq j_0$. Since the exponent is monotone in $j$, we only need to consider the case when $\alpha j_0 = j$ and thus the behavior of $(\alpha - (1 - \frac{\alpha}{2})j)$. This exponent is less or equal than zero if and only if $\alpha \leq \frac{2}{3}$ and less or equal than $-\alpha$ if and only if $\alpha \leq \frac{3}{5}$. The thresholds in $j$, respectively $\alpha$, are thus indeed optimal. □

**Remark 3.** We remark that $\langle \chi_{[0, 2^{-j})}, \phi_{p_l} \rangle = 2^{-j}$ for all $j = 0, \ldots, 2^j - 1$. Hence, $\sum_{0 \leq l \leq 2^{-j} - 1} \phi_{p_l} = 2^j 1_{[0, 2^{-j})}$. Thus, if $\alpha j_0$ is an integer, we obtain that

$$\sum_{2^{\alpha j_0} - 1 \leq l \leq 2^{\alpha j_0} - 1} \phi_{p_l} = 2^{j_0} 1_{[0, 2^{-j_0})} - 2^{\alpha j_0} 1_{[0, 2^{-\alpha j_0})},$$

which provides a more immediate view of the geometric mixing size. However, this explicit characterization is much less simple if $2^{\alpha j_0}$ is not a power of two and also intransparent in terms of the $H^{-1}$ norm.

### 4. The Continuous Setting

In the following we show that, with minor modifications, the estimates of the dyadic setting of Section 3 persist in the continuous setting. Here, additional key challenges are given by the lack of orthogonality and thus non-existence of spaces like $E_j$. 

4.1. Definition of Mixing Scales.

**Definition 4.1.** If \( \rho \in \dot{H}^{-1}(\mathbb{R}^n) \), we call \( \|\rho\|_{\dot{H}^{-1}} \) the analytic mixing scale.

Let \( \phi \in L^1(\mathbb{R}^n) \) with \( \phi \geq 0 \) and \( \|\phi\|_{L^1} = 1 \) and denote \( \phi_r(x) := \frac{\phi(rx)}{r^n} \). Then for any \( \rho \in L^1_{loc}(\mathbb{R}^n) \) and every \( \epsilon_0 > 0 \), we introduce the (nonlinear) functionals

\[
g_{\epsilon_0}[\rho] := \|\phi_r^* \rho\|_{L^\infty}.
\]

Here, the most common choice is given by \( \phi = \frac{1}{|B_1|} 1_{B_1} \), in which case

\[
g_{\epsilon_0}[\rho] = \sup_{r > \epsilon_0, x \in \mathbb{R}^n} |B_r(x)|^{-1} \int_{B_r(x)} \rho(y) dy.
\]

We say that a function \( \rho \in L^\infty(\Omega) \setminus L^1_{loc}(\Omega) \) is geometrically mixed by a factor \( \kappa \in (0, 1) \) up to scale \( \epsilon_0 > 0 \) if

\[
g_{\epsilon_0}[\rho] \leq \kappa \|\rho\|_{L^\infty}.
\]

For a given \( \kappa \), we denote

\[
G_\kappa[\rho] = \inf_{\epsilon_0 > 0} \left\{ \epsilon_0 > 0 : g_{\epsilon_0}[\rho] \leq \kappa \|\rho\|_{L^\infty} \right\},
\]

the infimum over all such \( \epsilon_0 \) and call it the geometric mixing scale.

We remark that, by Hölder’s inequality,

\[
g_{\epsilon_0}[\rho] \leq \|\rho\|_{L^\infty} \|\phi_r\|_{L^1} = \|\rho\|_{L^\infty},
\]

and that by Lebesgue integration theory

\[
\lim_{\epsilon_0 \downarrow 0} g_{\epsilon_0}[\rho] = \|\rho\|_{L^\infty}.
\]

The functionals \( g \) and the geometric mixing scale \( G \) thus describe the competition between cancellations in Hölder’s inequality and convergence of Dirac sequences.

**Remark 4.** The reason for our more general formulation in terms of \( \phi \in L^1 \) is that in later estimates optimality is easier to phrase and establish if we additionally require that \( \phi \in H^1 \). In particular, by duality

\[
\sup_{\rho \in \dot{H}^{-1} : \|\rho\|_{\dot{H}^{-1}} \leq 1} g_1[\rho] = \|\phi\|_{H^s},
\]

and hence an estimate like (6) is not possible unless \( \phi \) is sufficiently regular. However, for most estimates this only poses technical challenges (c.f. Lemma 4.4) in terms of the control of certain Fourier projections. In the dyadic setting of Section 3 these complications could be avoided by using orthogonality properties.

4.2. Comparison Estimates. Our main results are given by the following theorems and corollaries.

**Theorem 4.1 (Estimates).** Let \( \rho \in L^2 \cap \dot{H}^{-1}(\mathbb{R}^n) \) and suppose that \( \phi \in H^\lambda \cap L^1 \) for some \( \lambda \in [0, 1] \). Then for any \( \epsilon_0 > 0 \) it holds that

\[
g_{\epsilon_0}[\rho] \leq C \frac{\|\phi\|_{H^\lambda}}{\|\phi\|_{L^1}} \epsilon_0^{-n/(2-\lambda)} \|\rho\|_{H^{-\lambda}}.
\]

As a consequence, the geometric mixing scale of \( \rho \) can be estimated by

\[
G_\kappa(\rho) \leq \left( \frac{C_{\lambda,\phi} \|\rho\|_{H^{-\lambda}}}{\kappa \|\rho\|_{L^\infty}} \right)^{1/(n+2\lambda)}.
\]
In particular, if $\lambda = 1$, then
\begin{align}
    g_{\epsilon_0}[\rho] &\leq C \epsilon_0^{-n/2-1} \|\rho\|_{H^{-1}}, \\
    G_n(\rho) &\leq C \left( \frac{\|\rho\|_{H^{-1}}}{\kappa \|\rho\|_{L^\infty}} \right)^{\frac{1}{n+1}}.
\end{align}

Conversely, if $\rho$ is compactly supported and $C$ denotes the measure of a 1-neighborhood of the support, then for every $\epsilon_0 \leq 1$ it holds that
\begin{align}
    \|\rho\|_{H^{-1}} &\leq C g_{\epsilon_0}[\rho] + C \epsilon_0 \|\rho\|_{L^2}.
\end{align}

We note that in the estimate (7), assuming $G_\kappa[\rho] \leq \epsilon_0$ only yields a bound of $\|\rho\|_{H^{-1}} \leq \kappa$. Indeed, as explored in Section 2.2 for fixed $\kappa$ it is possible to find a sequence $\rho_n$ such that $G_\kappa[\rho_n] \to 0$, $\|\rho_n\|_{H^{-1}} \geq \kappa$, where the failure of decay of $\|\rho_n\|_{H^{-1}}$ was due to the persistence of structures at scale $\kappa$ (c.f. also Theorem 1.1 and the remarks thereafter).

As discussed in Remark 4, if $\phi = c_1 B_1$, we can not choose $\lambda = 1$ since $\phi \not\in H^1$. However, we may recover this estimate upon imposing further conditions on $\rho$ (c.f. Lemma 4.4).

As an application of the above estimates we derive comparability of both mixing scales. That is, while not equivalent (semi-)norms, smallness of one scale implies smallness of the other with a necessary loss in the exponents. For easier reference, we restate Theorem 1.1

**Theorem 4.2** (Comparison of mixing scales). Let $\rho \in L^2(\mathbb{R}^n)$ and $\|\rho\|_{L^2} \leq 1$ and let $\phi \in H^1$. Then for all $0 < \epsilon \leq 1$ it holds that:

1. If $g_\epsilon[\rho] \leq \epsilon$ and $\rho$ is supported in $B_1$, then also $\|\rho\|_{H^{-1}} \leq C \epsilon$.
2. If $\|\rho\|_{H^{-1}} \leq \epsilon$, then also $g_\epsilon[\rho] \leq C$ for all $\epsilon \geq \epsilon_0$ and $g_{\epsilon'}[\rho] \leq C \epsilon'$ for all $\epsilon' \geq \epsilon^\alpha$, with $\alpha = \frac{2}{n+2}$ and $\beta = \frac{2}{n+4}$ depends only on the dimension. In particular, supposing additionally that $\|\rho\|_{L^\infty} = 1$, it follows that
   \begin{align*}
   G_{\epsilon} &\leq \epsilon, \\
   G_{\epsilon'} &\leq \epsilon'.
   \end{align*}

These estimates are optimal in the powers of $\epsilon$.

In Section 3 we have seen that the loss of exponents is caused by the $(L^1, L^\infty)$ normalization in the geometric scale instead of $L^2$ normalization for the analytic scale and can also be seen as being due the Sobolev embedding into $L^\infty$. In this continuous setting, this is much less transparent due to the lack of spaces $E_j$. The necessity of the loss is established in Lemma 4.1 in analogy with Lemma 3.2 of the dyadic case.

**Corollary 4.1.** Let $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be such that
\begin{align*}
    \|\rho(t)\|_{H^{-1}} &\geq C e^{-C t}.
\end{align*}

Then, it follows that
\begin{align*}
    g_{e^{-C t}}[\rho(t)] &\geq C e^{-C t}.
\end{align*}
Remark 5. For example $u(t)$ may be given by the solution of a passive or active scalar problem, as in [CLS17]. As noted in Section 2, if one were instead to consider $\mathcal{G}_\epsilon[\rho(t)]$ for fixed $\kappa$, this function is not lower semi-continuous and there is no reason to expect any lower bound.

Proof of Theorem 4.1. As a first consistency check, we verify that the estimates of Theorem 4.1 scale correctly.

Let us now consider the proof of the theorem and let $\lambda \in [0,1]$ and $\phi \in H^\lambda$ be given. Then we may use duality to estimate

$$\mathcal{G}_\epsilon[\rho] \leq \frac{\|\phi\|_{H^\lambda}}{\|\phi\|_{L^1}} \|\rho\|_{H^{-\lambda}}$$

and use interpolation to control

$$\|\rho\|_{H^{-\lambda}} \leq C_\lambda \|\rho\|_{L^2}^{1-\lambda} \|\rho\|_{H^{-1}}^\lambda.$$  

We further note that by scaling

$$\frac{\|\phi_r\|_{H^\lambda}}{\|\phi_r\|_{L^1}} \leq C_r \|\phi\|_{H^\lambda} \|\phi\|_{L^1}$$

for $r \leq 1$ (for the homogeneous Sobolev norms we would have equality).

Combining both estimates, we obtain

$$\mathcal{G}_\epsilon[\rho] \leq C_\lambda \|\rho\|_{H^\lambda} \|\rho\|_{L^2}^{-\gamma} \|\rho\|_{H^{-1}}^\lambda.$$  

For the estimate on the geometric mixing scale, we have to show that for given $\kappa$ and all $\epsilon_0 \geq \mathcal{G}[\rho]

$$\mathcal{G}_\epsilon[\rho] \leq \kappa \|\rho\|_{L^\infty}.$$  

In view of the previous calculation this is implied by showing that

$$C_\lambda \phi \epsilon_0^{-n/2-\lambda/2} \|\rho\|_{L^2}^{1-\lambda} \|\rho\|_{H^{-1}}^\lambda \leq \kappa \|\rho\|_{L^\infty},$$  

with $C_\lambda = C r \|\phi\|_{H^\lambda}$. Dividing by $\kappa \|\rho\|_{L^\infty} > 0$ and taking a power $1/(n/2 + \lambda/2)$, we obtain that this holds if

$$\left(\frac{C_\lambda \|\rho\|_{L^2}^{1-\lambda} \|\rho\|_{H^{-1}}^\lambda}{\kappa \|\rho\|_{L^\infty}}\right)^{1/(n/2 + \lambda/2)} \leq \epsilon_0.$$
We thus obtain an upper bound on the geometric mixing scale by the left-hand-side, which concludes the proof. □

**Proof of Theorem 1.1.** We proceed as in the proof of Corollary 3.2 and consider $\epsilon_0 = \epsilon^t$ for $t \in (0, 1)$ to be determined. Then the estimate (6) of Theorem 4.1 implies that

$$g_{\epsilon^t} \leq C\epsilon^{t(-n/2-1)+1},$$

which yields the critical cases

$$t(-n/2 - 1) + 1 = 0 \Leftrightarrow t = \frac{2}{n+2}$$

$$t(-n/2 - 1) + t = \frac{2}{n+4}$$

□

The following lemmata consider questions of optimality and the removal of small scales discussed in Section 2.2.

**Lemma 4.1 (Counter example in the continuous setting).** There exists a sequence $\epsilon \downarrow 0$ and $\rho = \rho(\epsilon) \in L^2(\mathbb{R})$ with

$$\|\rho\|_{H^{-1}} \leq \epsilon,$$

but such that for every $\alpha < \frac{2}{3}$, it holds that

$$\mathfrak{g}_{\epsilon^\alpha}[\rho] \to \infty.$$

as $\epsilon \downarrow 0$ and such that for all $\beta < \frac{2}{5}$

$$\epsilon^{-\beta} \mathfrak{g}_{\epsilon^\beta}[\rho] \to \infty.$$

That is, the exponents in Theorem 1.1 are optimal.

**Proof of Lemma 4.1.** We follow a similar strategy as in the proof of Lemma 3.2 in the dyadic setting. Let $\epsilon = 2^{-j_0}$ and consider

$$\rho = 2^{j_0}1_{[0,2^{-j_0}]} - 2^{j_1}1_{[0,2^{-n_1}]},$$

with $j_1 = \alpha j_0$, $\alpha \in (0, 1)$. Then for any $j_1 \leq j \leq j_0$, we obtain

$$2^j \int_0^{2^{-j}} \rho dx = 2^j(2^{j_0} \min(2^{-j}, 2^{-j_0}) + 2^{j_1} \min(2^{-j}, 2^{-j_1})) = 2^j - 2^{j_1},$$

which is comparable to $2^j$ as long as $j > j_1$.

We hence conclude as in the proof of Lemma 3.2.
It hence remains to show the claim. We directly compute
\[
\hat{\rho}(\xi) = \int_{\mathbb{R}} e^{i\xi x} 2^{j_0} 1_{[0, 2^{-j_0}]} - 2^{j_1} 1_{[0, 2^{-j_1}]} \, dx = \frac{e^{i\xi 2^{-j_0}} - 1}{i\xi 2^{-j_0}} - \frac{e^{i\xi 2^{-j_1}} - 1}{i\xi 2^{-j_1}}.
\]
Both difference quotients are uniformly bounded by 1 and we distinguish the regions based on the size of $\xi 2^{-j_0}$ and $\xi 2^{-j_1}$.

If $\xi > c2^{j_0}$, we may roughly estimate
\[
\int_{\{\xi : \xi > c2^{j_0}\}} \frac{\hat{\rho}(\xi)^2}{|\xi|^2} \leq \int_{c2^{j_0}}^{\infty} \frac{4}{|\xi|^2} d\xi \leq C2^{-j_0},
\]
which is a very small contribution.

If $\xi < c2^{j_1}$ with $c$ small, we may use a Taylor expansion to estimate the error of the difference quotient:
\[
\frac{e^{i\xi 2^{-j_0}} - 1}{i\xi 2^{-j_0}} - \frac{e^{i\xi 2^{-j_1}} - 1}{i\xi 2^{-j_1}} = 1 + O(\xi 2^{-j_0}) - 1 + O(\xi 2^{-j_1}) = O(\xi 2^{-j_1}).
\]
The $H^{-1}$ energy for this segment can hence be estimated by
\[
\int_{\{\xi : \xi < c2^{j_1}\}} \frac{\hat{\rho}(\xi)^2}{|\xi|^2} \leq C \int_{\{\xi : \xi < c2^{j_1}\}} 2^{-2j_1} \leq C2^{-j_1}
\]
Finally, if $cj_1 \leq j \leq cj_0$ one difference quotient is about 1, while the other oscillates, but is bounded by 1. Thus the contribution can be estimated as
\[
\int_{\{\xi : c2^{j_1} \leq |\xi| \leq c2^{j_0}\}} \frac{\hat{\rho}(\xi)^2}{|\xi|^2} \approx \int_{c2^{j_1}}^{c2^{j_0}} 1 |\xi|^2 \, d\xi \approx 2^{-j_1}.
\]

\section*{Lemma 4.2}
Let $\rho \in L^2(\mathbb{R}^n)$ and $r > 0$ and define $\rho_r := \frac{1}{|B_r|} 1_{B_r} \ast \rho$. Then $\rho - \rho_r \in H^{-1}$ and there exists $C > 0$ depending only on the dimension $n$ such that
\[
\|\rho - \rho_r\|_{H^{-1}} \leq C r \|\rho\|_{L^2}.
\]

\section*{Proof of Lemma 4.2}
We consider the Fourier transform of $\phi_r$. Let thus $r > 0$ and $\xi \in \mathbb{R}^n$ be given and consider
\[
\frac{c_n}{r^n} \int_{B_r} e^{ix \cdot \xi} \, dx = \frac{1}{r^n} \int_{B_r} e^{ix \cdot |\xi|} = \frac{c_n}{|\xi|^n} \int_{B_{r|\xi|}} e^{ix \cdot |\xi|} = : \psi(|r\xi|).
\]
We remark that $\psi$ can explicitly computed in terms of Bessel functions (c.f. the proof of Theorem 5.2). Since $\psi(\cdot)$ is an average of $e^{ix_1}$ it follows that $|\psi| \leq 1$. Furthermore, by continuity of $e^{ix_1}$
\[
|\psi(|r\xi|) - 1| \leq \frac{c_n}{|r\xi|^n} \int_{B_{r|\xi|}} |e^{ix_1} - 1| \, dx \leq C r |\xi|.
\]
as $r|\xi| \downarrow 0$.

Hence, we can control
\[
|\mathcal{F}(\rho_r - \rho)|^2 = |(\psi(|r\xi|) - 1)\hat{\rho}(\xi)|^2 \leq \min(2, C r |\xi|) |\hat{\rho}|^2.
\]
and can estimate the $H^{-1}$ energy of $\rho - \phi_r \ast \rho$ by
$$\int \min(C^2|\xi|^2, 4) |\hat{\rho}(\xi)|^2 \leq C^2 \int |\hat{\rho}(\xi)|^2 = C^2 \|\rho\|_{L^2}^2.$$ \hfill \Box

**Lemma 4.3.** Let $\rho \in L^2(\mathbb{R}^n)$ with $\|\rho\|_{L^2} \leq 1$ be supported in $B_1(0)$ be such that $\|\frac{1}{r^{n-1}B_r} \ast \rho\|_{L^\infty} \leq \epsilon$, for some $0 < \epsilon < 1$. Then there exist $C$ depending only on the dimension $n$ such that the analytic mixing scale satisfies
$$\|\rho\|_{H^{-1}} \leq C \epsilon.$$

*Proof of Lemma 4.3.* By the triangle inequality
$$\|\rho\|_{H^{-1}} \leq \|\rho_c\|_{H^{-1}} + \|\rho - \rho_c\|_{H^{-1}}.$$ The second term can be estimated by $C\epsilon\|\rho\|_{L^2} \leq C\epsilon$ using Lemma 4.2, while for the first term we control
$$\|\rho_c\|_{H^{-1}} \leq \|\rho_c\|_{L^2} \leq |\text{supp}(\rho_c)|\|\rho_c\|_{L^\infty} \leq 2^n \epsilon,$$ where we estimated the support of $\rho_c$ by $B_{1+\epsilon} \subset B_2$. \hfill \Box

We remark that since the definition of $\rho$ is given in terms of local Lebesgue spaces some support or decay condition is necessary.

Indeed, consider $\rho \in L^2(\mathbb{R})$ which is compactly supported in $(0,1)$. Then for any $N \in \mathbb{N}$ we can define $\sigma(x) := \sum_{j=0}^N \rho(x + 2j)$. Due to the disjoint supports for all $\epsilon < \frac{1}{2}$ it holds that
$$\|\sigma_\epsilon\|_{L^\infty} = \|\rho_c\|_{L^\infty},$$
$$\|\sigma\|_{L^2} = \|\rho\|_{L^2}.$$ Furthermore, while $H^{-1}$ is non-local, we obtain that $\|\sigma\|_{H^{-1}} \approx \sqrt{N}\|\rho\|_{H^{-1}}$ for $N$ large.

The following lemma establishes the converse control of the geometric scale by the analytic scale. As noted in Remark 4, here regularity of $\phi$ allows for easier proofs. However, under additional assumptions, $\phi$ can also be chosen less regular such as $1_{B_1}$.

**Lemma 4.4.** Let $\rho \in H^{-1}(\mathbb{R}^n)$ with $\text{supp}(\hat{\rho}) \subset B_{r^{-1}}(0)$ and let $\phi \in L^1$. Then there exists a constant $C$ depending on $\rho$ such that
$$\mathfrak{g}_{\epsilon_0}[\rho] \leq C r^{n/2-1} \|\rho\|_{H^{-1}}.$$ If we require that $\phi \in H^1$, then the support assumption can be omitted.

*Proof of Lemma 4.4.* Using Plancherel’s theorem we compute
$$\phi_r \ast \rho(x) = \int \phi_r(x-y) \rho(y) dy = \int e^{ix\xi} \hat{\phi_r}(\xi) \hat{\rho}(\xi) d\xi.$$ Now recall that $\phi_r(x) = \frac{\phi(x)}{r^n}$ has constant $L^1$ norm and thus $\|\hat{\phi_r}\|_{L^\infty} \leq \|\phi_r\|_{L^1}$. We may hence control further by
$$\|\hat{\phi_r}\|_{L^\infty} \|\xi\|_{L^2(\text{supp}(\hat{\rho}))} \|\hat{\rho}\|_{L^2} \leq r^{-n/2-1} \|\rho\|_{H^{-1}},$$
where we used the support of $\hat{\rho}$ and that $\|\hat{\rho}\|_{L^\infty} \leq \|\phi_r\|_{L^1} = 1$.

If $\phi \in H^1$, we can instead directly estimate
\[
\|\phi_r \ast \rho\|_{L^\infty} \leq \|\phi_r\|_{H^1} \|\rho\|_{H^{-1}} = r^{-n/2-1} \|\phi\|_{H^1} \|\rho\|_{H^{-1}}.
\]
\[\square\]

Hence, the failure of estimates with $s \geq 1/2$ is due to the interaction of the “tail” of $\hat{\rho}$ for $|\xi| \geq r^{-1}$. We remark that in the dyadic setting of Section 3 this complication does not arise, since our seminorms project on spaces $E_j$ of lower frequency.

5. Damping Rates in Transport-type Equations

In this second part of our article, we are interested in the time dependence of mixing scales when $\rho(t)$ evolves under a passive scalar equation
\[
\partial_t \rho + v \cdot \nabla \rho = 0
\]
for a given divergence-free velocity field $v$. In particular, we are interested optimal decay rates of $\|\rho(t)\|_{H^{-1}}$ and $G_{\kappa(t)}[\rho(t)]$.

In Subsection 5.1, we consider the special case when $\rho(t)$ is advected by a given specific, regular, incompressible velocity field. This study is motivated by recent work of Crippa, Lucà and Schulze, [CLS17], who study the time behavior of both mixing scales under the evolution
\[
\rho(t, r, \theta) = \rho_0(r, \theta - tr),
\]
where $r, \theta$ are polar coordinates on $\mathbb{R}^2 \setminus \{0\}$, $\rho_0 \in L^1 \cap L^\infty$ and the angular averages $\langle \rho_0 \rangle_{\theta} = 0$ identically vanish. Adapting conformal polar coordinates $\theta, e^s = r$ this setting shares strong similarities with problems of inviscid damping in fluid dynamics.

Further examples of interest here are given by:

- Perturbations around shear flow solutions of Euler’s equations on $T \times \mathbb{R}$. In [Zil17a], [Zil16], we show that if $U(y)$ is, roughly speaking, close to affine, the linearized Euler equations in vorticity formulation asymptotically scatter in $H^s$ to the transport problem with $v = (U(y), 0)$. Using different, spectral methods [WZZ15] have further shown similar results under weaker conditions.

- When considering circular flows, [Zil17b], [CZZ18], and $v = u(r)e_\theta$ is an annular region or on $\mathbb{R}^2$ we similarly obtain stability, damping and scattering in weighted spaces and for more degenerate profiles.

- In the setting of Landau damping [BMM13] similarly one observes scattering to a transport problem.

The following results on the free transport equation
\[
\partial_t \rho(t, x, y) - y \partial_x \rho = 0,
\]
\[(t, x, y) \in (0, \infty) \times T^n \times \mathbb{R}^n,
\]

hence also extend by scattering to asymptotics for further equations exhibiting phase-mixing.

Finally, we discuss optimal mixing and stirring for more general passive scalar problems. Here, a recent active area of research, [ACM14], [Sci17], [CS17], [Bre03],
be the solution of the free transport problem.

where $v$ may be chosen arbitrarily under given constraints such as $\|v(t)\|_{W^{1,p}} \leq 1$. Using our comparison estimates of Theorem 1.1, we discuss implications of some known results.

5.1. On Sharp Decay Rates for the Free Transport Equation. Our main results for the analytic mixing scale are summarized in Theorem 1.2, which we restate in the following for easier reference. Using the estimates of Theorem 1.1, we also obtain control of the geometric scale, which we study in Theorem 5.2.

**Theorem 5.1.** Let $H^s H^s = H^s (\mathbb{T}^n ; H^s (\mathbb{R}^n))$ denote the Hilbert space with norm

$$
\|u\|_{H^s H^s}^2 = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{2s} \int_{\mathbb{R}^n} \langle \eta \rangle^{2s} |\widehat{u(k, \eta)}|^2 d\eta.
$$

In the following, let $0 < s \leq 1$, $u_0 \in L^2 (\mathbb{T}^n ; H^s (\mathbb{R}^n))$ with $\int_{\mathbb{T}^n} u_0(x, y) dx = 0$, and let

$$
u(t, x, y) = u_0(t, x - ty, y),$$

be the solution of the free transport problem.

1. There exists $C_\delta > 1$ such that for all $t \geq 1$ and all initial data $u_0$ with $\|u_0\|_{H^{-1}(\mathbb{T}^n ; H^s (\mathbb{R}^n))} = 1$.

2. Let $\alpha_j > 0$ with $\|(\alpha_j)_j\|_{l^2} = 1$. Then there exist $c > 0$, a sequence of times $t_j \to \infty$ and initial data $u_0$ such that

$$
\|u(t_j)\|_{H^{-1}} \geq c \alpha_j t_j^{-s} \|u_0\|_{H^{-1}(\mathbb{T}^n ; H^s (\mathbb{R}^n))}.
$$

3. There exists no non-trivial initial data $u_0 \in L^2 (\mathbb{T}^n ; H^s (\mathbb{R}^n))$ such that

$$
\|u(t_j)\|_{H^{-1}} \geq c \alpha_j t_j^{-s} \|u_0\|_{H^{-1}(\mathbb{T}^n ; H^s (\mathbb{R}^n))}
$$

along some sequence $t_j \to \infty$. In the second statement, $t_j$ can always be chosen larger and more rapidly increasing. For instance, we may chose $t_j = \exp(\exp(\ldots \exp(j)))$ and $\alpha_j = \frac{1}{j} = \ln(\ln(\ldots \ln(j)))$ as iterated exponential and logarithms. Informally stated, the theorem hence shows that algebraic decay rates can be achieved along a subsequence up to an arbitrarily small loss. Conversely, the third statement shows that this loss is necessary and that the lower estimate is sharp in this sense.

**Proof of Theorem 1.2.** We note that, for all $t$ the map $L^2 \ni u_0 \mapsto u(t) \in L^2$ is unitary and thus the statement holds for $s = 0$. Furthermore, we may use the explicit solution of the free transport problem and Plancherel’s identity with respect to $x$ to obtain that

$$
\|u(t)\|_{H^{-1}} = \sup_{\|\phi\|_{H^1} \leq 1} \sum_{k \neq 0} \int \overline{\phi(k, y)} e^{ikt} \overline{u_0(k, y)} dy
$$

$$
= \sup_{\|\phi\|_{H^1} \leq 1} \left| \int e^{ikt} \overline{\phi(k, y)} \overline{u_0(k, y)} ikt \right|
$$

$$
\leq \|u_0\|_{H^{-1}(\mathbb{T}^n ; H^s (\mathbb{R}^n))},
$$

where $\phi$ is given by the study of upper and lower bounds on decay rates of mixing scales for solutions of (1)

$$
\partial_t \rho + v \cdot \nabla \rho = 0,
$$

[CDL08] is given by the study of upper and lower bounds on decay rates of mixing scales for solutions of (1)
We remark that the Dirac in \(k\) will be frequency localized near \(0\). Similarly, for any \(j\), the following \(s\) follows by multiplying by \(e^{-it}\). Since the equation decouples with respect to \(k\), \(t = -\eta/k\) the frequency localization is near zero and hence any \(H^s\) norm is equivalent to the \(L^2\) norm for such a function.

Let thus \(\phi \in C^\infty\) be supported in a ball of radius 2 and \(L^2\) normalized and let \((\alpha_j)\) in \(L^2\) with \(\|\alpha_j\|_{L^2} = 1\). Suppose further that \(t_j\), to be determined precisely later, satisfies \(\min_j |t_j - t_j^0| > 4n\) and \(\min_j |t_j| > 4\). Then we define the function \(u_0 \in H^s(T^n \times \mathbb{R}^n)\) by its Fourier transform:

\[
\hat{u}_0(k, \eta) = \delta_{k=e_1} \sum_{j \in \mathbb{N}} \alpha_j (t_j)^{-s} \phi(\eta - t_je_1).
\]

We remark that the Dirac in \(k\) corresponds to assuming periodicity in \(x\). This construction also readily extends to the whole space case, \(\mathbb{R}^n \times \mathbb{R}^n\) if \(\delta_{k=e_1} \phi(\eta - t_je_1)\) is replaced by a bump function \(\phi(10k - e_1)\phi(\eta - kt_j)\).

By our assumption on \(t_j - t_{j+1}\), the functions \(\phi(\cdot - t_j)\) are disjointly supported and thus

\[
\|u_0\|_{H^s}^2 = \sum_j |\alpha_j|^2 \|\frac{\langle \cdot \rangle^{2s}}{(t_j)^{2s}} \phi(\cdot - t_je_1)\|_{L^2(B_2(0))}^2 = \sum_j |\alpha_j|^2 \|\frac{\langle -t_je_1 \rangle^{2s}}{(t_j)^{2s}} \phi(\cdot)\|_{L^2(B_2(0))}^2.
\]

Similarly, for any \(t \in \mathbb{R}\), it holds that

\[
\|u(t)\|_{H^s}^2 = \sum_j |\alpha_j|^2 \|\frac{\langle (t_j - t) e_1 + \cdot \rangle^{2s}}{(t_j)^{2s}} \phi(\cdot)\|_{L^2(B_2(0))}^2.
\]

By our assumptions on \(t_j\) and \(\phi\), in the first sum \(|\eta| \leq 2 \leq |t_j|\), and thus \(\|u_0\|_{H^s}^2\) is comparable (within a factor \(4^{2s}\)) to \(\sum_j |\alpha_j|^2 = 1\). One the other hand, for \(t = t_j\), the second sum is bounded from below by

\[
|\alpha_j|^2 \|\frac{\langle \cdot \rangle^{2s}}{(t_j)^{2s}} \phi(\cdot)\|_{L^2(B_2(0))}^2 \geq \frac{|\alpha_j|^2}{(t_j)^{2s}}.
\]

This concludes the proof of the second statement. We note that a similar lower bound can also be obtained for the homogeneous Sobolev spaces by choosing \(t = t_j + 4\) instead.

Finally, suppose that there exists \(u_0\) attaining the algebraic decay rates. Expressed in terms of \(u_0\) this implies that for a sequence \(t_j \to \infty\)

\[
\|t_j^s (\eta - t_j)\|_{L^2}^2 \|\langle \eta \rangle^s \langle k \rangle^{-s} \hat{u}_0\|_{L^2}^2 \geq C \|\langle \eta \rangle^s \langle k \rangle^{-s} \hat{u}_0\|_{L^2}^2.
\]

Since the equation decouples with respect to \(k\) and for easier notation, in the following we consider \(k\) arbitrary but fixed and omit the factors \(\langle k \rangle^{-s}\). The result of the Theorem then follows by multiplying by \(\langle k \rangle^{-s}\) and summing in \(k\).
Now let \( t = t_j \) and consider the sets \( \Omega_{C,t} = \left\{ (k, \eta) : \frac{|t^*|}{(\eta - (t_j) \langle \eta \rangle)^2} < \sqrt{C/2} \right\} \) and \( A_{C,t} \) their complements. Then the inequality (9) implies that

\[
\frac{t^*}{\langle \eta - k t_j \rangle \langle \eta \rangle^2} \| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2(A_{C,t})}^2 + \frac{C}{2} \| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2(\Omega_{C,t})}^2 \geq C \| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2(\Omega_{C,t})}^2
\]

(10)

\[
\Rightarrow \frac{t^*}{\langle \eta - k t_j \rangle \langle \eta \rangle^2} \| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2(A_{C,t})}^2 \geq C/2 \| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2}^2
\]

On the other hand

\[
\frac{t^*}{\langle \eta - k t_j \rangle \langle \eta \rangle^2} \leq \max \left( \frac{t^*}{1 \langle \eta \rangle^2}, \frac{t^*}{1 \langle \eta \rangle} \right) \leq 2^s,
\]

by considering \( |\eta| \leq t_j/2 \) and \( |\eta| > t_j/2 \) and using that \( 0 \leq s \leq 1 \).

Hence, it follows that, for a constant depending on \( s \), but independent of \( t \),

\[
\| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2(A_{C,t})}^2 \geq C_s \| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2}^2.
\]

By assumption, this holds for a sequence \( t_j \to \infty \). Upon passing to a subsequence, the sets \( A_{C,t_j} \) can be ensured to be mutually disjoint. Hence, by orthogonality

\[
\| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2}^2 \geq \sum_j \| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2(A_{C,t_j})}^2
\]

\[
\geq C_s \sum_j \| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2}^2 = \infty \| \langle \eta \rangle^s \tilde{u}_0 \|_{L^2}^2.
\]

This is a contradiction unless \( v = 0 \). \( \square \)

As a consequence of our comparison result, Theorem 1.1, we also obtain lower bounds on the decay of the geometric mixing scale. The following theorem instead provides a direct construction of a lower bound, where averages are taken over the scale \( \frac{1}{t_j} \) instead.

**Theorem 5.2.** Let \( 0 \leq s < \frac{1}{2} \), then there exists initial data \( u_0 \in L^2_\mu \mathcal{H}_0^s(\mathbb{T} \times [0, 2\pi]) \), so that along a sequence of times \( t_j = 2^{100+j} \), the solution \( u \) of the free transport problem satisfies

\[
\varrho_{\frac{1}{t_j}} [u] \geq \frac{\| u_0 \|_{L^2_\mu \mathcal{H}_0^s}}{\frac{1}{t_j} j^s}.
\]

That is, at scale \( r = \frac{1}{t_j} \), we have a lower bound by \( \frac{1}{t_j} j^s \).

We remark that as in the previous Theorem 1.2, \( t_j \) can be chosen to be increasing more rapidly and thus \( \frac{1}{t_j} = o(1) \) as \( t_j \to \infty \) can be chosen with very slow decay. Furthermore, the construction of our proof also extends to the \( n \) dimensional transport equations by extending constantly in the other directions.

**Proof of Theorem 5.2.** Consider the function

\[
u_0(x,y) = ce^{i \theta} \sum_{j=1}^{\infty} \frac{1}{j} e^{it_j y} (t_j)^s,
\]

as a function on \( \mathbb{T} \times [0, 2\pi] \), where \( c \in (\frac{1}{100}, 100) \) can be chosen such that this function is normalized since \( \frac{1}{j} \in L^2 \).
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Then, the solution \( u \) of the free transport problem is explicitly given by

\[
u(t, x, y) = c e^{ix} \sum_{j=1}^{\infty} \frac{1}{j} e^{i(t_j - t)y} / (t_j)^s,\]

For simplicity of notation and presentation we first consider averages over squares instead of balls, which allows for a simpler straightforward calculation. An extension to the latter setting is given at the end of this proof. Let thus \( t = t_{j_0} \) and consider a square \( S = I_x \times I_y \) of side length \( \frac{1}{100} > d > \frac{2}{t_{j_0}} \), which is centered close to a point where \( e^{ix} = 1 \) Then the integral of \( u \) over this square decouples by Fubini’s theorem and we may compute that

\[
\frac{1}{d} \int_{I_x} e^{ix} dx \approx 1
\]

and that

\[
\frac{1}{d} \int_{I_y} \sum_{j=1}^{\infty} \frac{1}{j} \cos((t_j - t)y) / (t_j)^s dy = \frac{1}{d} \int_{I_y} \frac{1}{j_0 (t_{j_0})^s} + \frac{1}{d} \sum_{j \neq j_0} \int_{I_y} \frac{1}{j} e^{i(t_j - t_{j_0})y} / (t_j)^s dy.
\]

We note that as an average over a constant function, the first term equals

\[
\frac{1}{j_0 (t_{j_0})^s} \approx \frac{1}{j_0 2^{j_0 s}}.
\]

On the other hand, by construction of \( t_j = 2^{100+j} \), for each \( j \neq j_0 \), \( |t_j - t_{j_0}| \geq \frac{1}{2} \max(t_j, t_{j_0}) \) and thus all further integrands are highly oscillatory. In particular,

\[
\left| \int_{S_y} \frac{1}{j} \cos((t_j - t_{j_0})y) / (t_j)^s \right| \leq \frac{1}{j (t_j)^s |t_j - t_{j_0}|} \leq \frac{1}{j 2^{j s} |2j - 2^{j_0}|},
\]

where we used integration by parts. Considering the sum in \( j \), we split into

\[
\sum_{j < j_0} \frac{1}{j 2^{j s} |2j - 2^{j_0}|} \leq \frac{2}{2^{j_0}} \sum_{j} \frac{1}{j 2^{j s}} \leq \frac{c_s}{2^{j_0}}.
\]

and

\[
\sum_{j > j_0} \frac{1}{j 2^{j s} |2j - 2^{j_0}|} \leq 2^{-j_0 s} \sum_{j > j_0} \frac{2}{2^{2j}} \leq 2^{-j_0 s} 2^{-j_0}.
\]

Dividing by \( d > \frac{1}{2^{j_0}} \), both terms will be smaller than the term in (11) by a large factor and hence

\[
\frac{1}{|S|} \int_S u(t_j) \geq c \frac{1}{j_0 (t_{j_0})^s},
\]

as claimed.

Let us next consider the original problem of averages over balls. In this case the integrals

\[
\frac{1}{j (t_j)^s |B_d|} \int_{B_d} e^{ix} e^{i(t_j - t)y}
\]
can be explicitly computed in terms of Bessel functions. That is, if the center of the ball is the point \((\xi_1, \xi_2)\), then after translating in \(x\) and \(y\), we obtain an exponential factor \(e^{i\xi_1 + i(t_j - t)\xi_2}\) and an integral over a ball centered in \((0, 0)\). We hence, need to compute

\[
\int_{B_d} e^{ix(1,t_j-t)} dx = d^n \int_{B_1} e^{ix(d,d(t_j-t))} dx.
\]

That is, the Fourier transform of the indicator function of a ball.

Using the rotation-invariance of \(B_1\), we compute

\[
\int_{B_1} e^{ix} dx = \int_{B_1} e^{ix_1|x|} dx_1 dx_2 = \int_{-1}^1 \sqrt{1 - x_1^2} e^{ix_1} dx_1 = c J_1(|\xi|)/|\xi|,
\]

where \(J_1\) denotes the Bessel function of the first kind and \(c \leq 10\). It hence follows that

\[
d^{-n} \int_{B_d} e^{ix(1,t_j-t)} dx \leq C_d \min \left(1, \frac{C}{d\sqrt{1+|t_j-t|}}\right).
\]

and that

\[
d^{-n} \int_{B_d} e^{ix(1,t_j-t)} dx \approx 1
\]

if \(t_j - t\) is small. Thus, the above estimates for squares extend to this case in a straightforward way. \(\square\)

### 5.2. On Lower Bounds for Mixing Costs.

Consider again the passive scalar problem

\[
\begin{align*}
\partial_t \rho + v \cdot \nabla \rho &= 0, \\
\nabla \cdot \rho &= 0.
\end{align*}
\]

(12)

In the previous section we considered \(v\) as given and asked about decay rates of mixing scales for \(\rho_0 \in H^s\) to be chosen freely.

As a related and in a sense inverse problem, one can ask about mixing costs. That is, you are given an explicit initial datum \(\rho_0 \in H^s\) and want it to be mixed to scale \(\epsilon\) by time 1. What kind of lower bound does this imply on Sobolev norms of \(v\) in space and time?

More precisely, the aim is to establish a lower bound of the type

\[
\int_0^1 \|v\|_{W^{1,p}} dt \geq C_p \log(\epsilon),
\]

when \(\rho(1)\) is geometrically mixed to scale \(\epsilon\). The case \(p > 1\) has been established in [CDL08] and the case \(p = 1\) is a conjecture of Bressan, [Bre03].

As an application of our comparison results, we consider the simplest case of \(p = \infty\), following the proof in [CDL08] via Gronwall’s estimate.

**Lemma 5.1.** Let \(1 > \epsilon > 0\) and \(\rho\) be a solution of (12) on \(\mathbb{R}^n\) be such that

\[
\|\rho\|_{t=0} H^{-1} = 1, \quad \|\rho\|_{t=1} H^{-1} = \epsilon > 0,
\]

then...
with $v \in W^{1,\infty}$. Then it holds that

$$\int_0^1 \|\partial_i v_j + \partial_j v_i\|_{L^\infty} dt \geq C|\log(\epsilon)|.$$

**Proof.** Since $v$ is divergence-free the solution operator $S(t_2, t_1)$ mapping $\rho|_{t_1}$ to $\rho|_{t_2}$ is unitary. Characterizing the $H^{-1}$ norm via duality we hence obtain that

$$\|\rho|_{t=0}\|_{H^{-1}} = \sup_{\|\psi\|_{H^1} \leq 1} \int_0^1 \psi \rho_t dt = \sup_{\|\psi\|_{H^1} \leq 1} \int \psi |\log(\epsilon)|.$$

and thus

(13) $$\|S(1,0)\|_{H^1 \rightarrow H^1} \geq \frac{1}{\epsilon}.$$

On the other hand, $S(1,0)$ conserves the $L^2$ norm and $\partial_j \rho$ satisfies

$$\partial_i \partial_j \rho + v \cdot \nabla \partial_j \rho + (\partial_j v) \cdot \nabla \rho = 0.$$

Testing against $\rho_j$ we thus obtain that

$$\frac{d}{dt} \|\nabla \rho\|_{L^2}^2 \leq 2 \sum_{i,j} \int (\partial_j \rho)(\partial_j v_i)(\partial_i \rho)$$

$$= \sum_{i,j} \int (\partial_j \rho)(\partial_j v_i + \partial_i v_j)(\partial_i \rho) \leq \|\partial_j v_i + \partial_i v_j\|_{L^\infty} \|\nabla \rho\|_{L^2}^2.$$

Gronwall's inequality thus implies that

$$\|S(1,0)\|_{H^1 \rightarrow H^1} \leq \exp \left( \int_0^1 \|\partial_j v_i + \partial_i v_j\|_{L^\infty} dt \right),$$

which in combination with the inequality (13) concludes the proof. □

As a corollary we obtain a lower bound on the geometric scale. While our comparison estimates of Section 4 can not be expected to be optimal due the different time dependence, we remark that lower bounds in terms of powers of $\epsilon$ yield the same logarithmic lower bounds. Hence, we may consider the assumptions of the following corollary to be equivalent to those of Lemma 5.1 for our purposes.

**Corollary 5.1.** Let $1 > \epsilon > 0$ and $\rho$ be a solution of (12) on $\mathbb{R}^n$ with $v \in W^{1,\infty}(\mathbb{R}^n)$. Suppose that $\|\rho|_{t=0}\|_{H^{-1}} = 1$ and that $\rho|_{t=1}$ is supported in $B_1$ and $G_\epsilon[\rho|_{t=1}] \leq \epsilon$.

Then it follows that

$$\int_0^1 \|\partial_i v_j + \partial_j v_i\|_{L^\infty} dt \geq C|\log(\epsilon)|.$$

**Proof.** By Theorem 1.1, $\rho$ also satisfies the assumptions of Lemma 5.1, which implies the result. □

For the case $p > 1$, in [CDL08] Crippa and De Lellis obtain the following mixing cost result. Unlike the setting $p = \infty$ this seminal result requires considerable effort to prove. In subsequent works we intend to study whether the comparability can be used to simplify steps of this proof. For this article, we only state a simple corollary of the established results.
Theorem 5.3 (Theorem 6.2 in [CDL08]). Let $p > 1$ and $\rho|_{t=0} = 1_{[0,1/2]}(x_2) \in L^1(\mathbb{T}^2)$ and suppose that for $\epsilon > 0$ and some $0 < \kappa < \frac{1}{2}$ the solution of (12) satisfies
\[
\kappa \leq \frac{1}{B_\epsilon} \int_{B_\epsilon} \rho|_{t=1} \leq 1 - \kappa.
\]
Then there exists a constant $C$ such that
\[
\int_0^1 \|\nabla v\|_{L^p} dt \geq C|\log(\epsilon)|
\]
for every $0 < \epsilon < \frac{1}{4}$.

Corollary 5.2. Let $p > 1$ and $\rho|_{t=0} = 1_{[0,1/2]}(x_2) \in L^1(\mathbb{T}^2)$ and suppose that for $\epsilon > 0$ and some $0 < \kappa < \frac{1}{2}$ the solution of (12) satisfies
\[
\left\|\rho|_{t=1} - \frac{1}{2}\right\|_{H^{-1}} \leq \epsilon.
\]
Then inequality (14) holds.

Proof. Theorem 4.1 implies that for $0 < \alpha < \frac{1}{2}$
\[
\left|\frac{1}{B_{\epsilon^{\alpha}}} \int_{B_{\epsilon^{\alpha}}} \rho|_{t=1} - \frac{1}{2}\right| \leq C\epsilon^{1/2-\alpha},
\]
where the upper bound on $\alpha$ is due to the regularity of $1_{B_1}$ as discussed in Remark 4. Denoting $\delta := C\epsilon^{1/2-\alpha}$ and adding $\frac{1}{2}$, we thus obtain
\[
\frac{1}{2} - \delta \leq \frac{1}{B_{\epsilon^{\alpha}}} \int_{B_{\epsilon^{\alpha}}} \rho|_{t=1} \leq \frac{1}{2} + \delta.
\]
Thus, we may apply the Theorem of Crippa and De Lellis with $\kappa \leq \frac{1}{2} - \delta$ and $\epsilon^{\alpha}$ to obtain that
\[
\int_0^1 \|\nabla v\|_{L^p} dt \geq C|\log(\epsilon^{\alpha})| = C\alpha|\log(\epsilon^{\alpha})|.
\]
\[\square\]

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