MAPPING CLASS GROUPS OF COVERS WITH
BOUNDARY AND BRAID GROUP EMBEDDINGS

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Abstract. We consider finite-sheeted, regular, possibly branched covering spaces of surfaces with boundary and the associated liftable and symmetric mapping class groups. In particular, we classify when either of these subgroups coincides with the entire mapping class group of the surface. As a consequence, we construct infinite families of non-geometric embeddings of the braid group into mapping class groups in the sense of Wajnryb. Indeed, our embeddings map standard braid generators to products of Dehn twists about curves forming chains of arbitrary length. As key tools, this paper uses the Birman-Hilden theorem and the action of the mapping class group on a particular fundamental groupoid of the surface.

1. Introduction

The mapping class group $\text{Mod}(\Sigma, B)$ of a compact orientable surface $\Sigma$ with finitely many marked points $B$ is the group of orientation-preserving homeomorphisms of $\Sigma$ preserving $B$ setwise, up to isotopies that preserve $B$. If $\Sigma$ has non-empty boundary, then homeomorphisms and isotopies must fix the boundary pointwise. We may sometimes denote a genus $g$ surface with $m$ boundary components by $\Sigma^m_g$.

Leveraging covering spaces to study mapping class groups has been a fruitful endeavour. See the work of Aramayona-Leininger-Souto [2], Bigelow-Budney [3], Brendle-Margalit [7], Brendle-Margalit-Putman [6], Endo [10], Morifuji [22], and Stukow [25, 26] to name only a few.

Let $p : \Sigma \to \Sigma$ be a finite-sheeted, regular covering space of surfaces with deck group $D$, possibly branched at $B \subset \Sigma$. Recall that a homeomorphism $f : \Sigma \to \Sigma$ is called fibre-preserving if $p(x) = p(y)$ implies $pf(x) = pf(y)$. Let $\text{SMod}(\Sigma) < \text{Mod}(\Sigma)$ be the subgroup consisting of isotopy classes of fibre-preserving homeomorphisms, called the symmetric mapping class group. Let $\text{LMod}(\Sigma, B) < \text{Mod}(\Sigma, B)$ be the subgroup consisting of isotopy classes of homeomorphisms that lift to boundary preserving homeomorphisms of $\Sigma$, called the liftable mapping class group.

Surveying the landscape of results, one notices that the fertile ground often occurs when at least one of the liftable or symmetric mapping class groups coincides with the mapping class groups $\text{Mod}(\Sigma, B)$ or $\text{Mod}(\Sigma)$ respectively. This leads to the following natural questions:
When does $\text{LMod}(\Sigma, \mathcal{B}) = \text{Mod}(\Sigma, \mathcal{B})$?

When does $\text{SMod}(\tilde{\Sigma}) = \text{Mod}(\tilde{\Sigma})$?

If equality does not hold, when are these subgroups finite-index?

These questions are answered in Theorems 1.1 and 1.2 of this paper in the case where the surfaces have non-empty boundary.

The first author and Winarski classified all cyclic branched covers of the sphere with the property that $\text{LMod}(\Sigma_0, \mathcal{B}) = \text{Mod}(\Sigma_0, \mathcal{B})$ [12, Theorem 1.1]. Birman and Hilden proved that if $g \geq 3$, there are no finite cyclic covers $p : \tilde{\Sigma} \to \Sigma_0$ of the sphere such that $\text{SMod}(\tilde{\Sigma}) = \text{Mod}(\tilde{\Sigma})$ [4, Theorem 6]. After completing the proof, they make the following remark:

*The possibility remains that if we relax the requirements on $(p, \Sigma_0, \tilde{\Sigma})$ to admit coverings of other Riemann surfaces, or to admit all regular coverings, or to admit non-regular coverings that we will have better luck. (However we conjecture that all such efforts will fail).*

As we will see, our results agree with their sentiment.

1.1. Main Results. From now on, let $p : \tilde{\Sigma} \to \Sigma$ be a finite-sheeted, regular cover of a surface $\Sigma$ with non-empty boundary, possibly branched at $\mathcal{B} \subset \Sigma$.

The Burau covers. Let $D_n$ be a disk with $n$ points removed, and enumerate the punctures. Pick a point $x \in \partial D_n$ and let $\gamma_i \in \pi_1(D_n, x)$ be the homotopy class of a loop surrounding only the $i$th puncture anti-clockwise. Then $\{\gamma_1, \ldots, \gamma_n\}$ generates $\pi_1(D_n, x)$. For each $k \geq 2$, define a homomorphism $q_k : \pi_1(D_n, x) \to \mathbb{Z}/k\mathbb{Z}$ by $q_k(\gamma_i) = 1$ for all $i$. The kernel of $q_k$ determines a $k$-sheeted cyclic branched cover $p_k : \Sigma^m_g \to \Sigma_0^m$ branched at $n$ points. Here $m = \gcd(n, k)$ and $g = 1 - \frac{1}{2}(k + n + m - nk)$. We will call such a cover a $k$-sheeted Burau cover. The 2-sheeted Burau covers are usually referred to as hyperelliptic covers of a disk as in each case the non-trivial element of the deck group is a hyperelliptic involution.

We note that the Burau covers have previously been considered by McMullen to study certain unitary representations of the braid group [21].

**Theorem 1.1.** Let $p : \tilde{\Sigma} \to \Sigma$ be a finite-sheeted, regular covering space of surfaces with boundary, possibly branched at $\mathcal{B} \subset \Sigma$. Then

(i) $\text{LMod}(\Sigma, \mathcal{B}) = \text{Mod}(\Sigma, \mathcal{B})$ if and only if $p$ is a Burau cover, and

(ii) $\text{LMod}(\Sigma, \mathcal{B})$ is always finite-index in $\text{Mod}(\Sigma, \mathcal{B})$.

For a general covering space $p : \tilde{X} \to X$, a homeomorphism $f : X \to X$ lifts to a homeomorphism $\tilde{f} : \tilde{X} \to \tilde{X}$ if and only if $f_*\pi_1(\tilde{X}, \tilde{x}) = p_*\pi_1(\tilde{X}, \tilde{x})$. It is therefore tempting to characterise liftable mapping classes by their action on the fundamental group of the base space. This approach ultimately falls short since it may be the case that a homeomorphism of the base space $\Sigma$ lifts to a homeomorphism of $\tilde{\Sigma}$, but none of its lifts fix the boundary of $\tilde{\Sigma}$ pointwise. Indeed, consider the 2-sheeted unbranched
Figure 1. One of the lifts of a Dehn twist $T \in \text{LHomeo}^+(A)$ that does not preserve the boundary of $\tilde{A}$ pointwise.

cover of an annulus $A$ by an annulus $\tilde{A}$. Any representative Dehn twist of $T \in \text{Mod}(A)$ acts trivially on $\pi_1(A,x)$, however neither of its lifts preserve the boundary of $\tilde{A}$ pointwise (see Figure 1). Therefore $T \notin \text{LMod}(A)$. On the other hand, $T^2 \in \text{LMod}(A)$. In order to get around this issue, we study the action of $\text{Mod}(\Sigma, \mathcal{B})$ on a particular fundamental groupoid of $\Sigma$. A key piece of the proof of Theorem 1.1 is a characterisation of elements of $\text{LMod}(\Sigma, \mathcal{B})$ in terms of their action on the fundamental groupoid in Theorem 4.1. The corresponding result for the symmetric mapping class group is as follows.

**Theorem 1.2.** Let $p : \tilde{\Sigma} \to \Sigma$ be a finite-sheeted, regular, possibly branched covering space of surfaces with boundary. Then

(i) $\text{SMod}(\tilde{\Sigma}) = \text{Mod}(\tilde{\Sigma})$ if and only if $\tilde{\Sigma}$ is a disk, an annulus, or $p : \Sigma_1 \to \Sigma_0^1$ is the hyperelliptic cover, otherwise

(ii) $\text{SMod}(\tilde{\Sigma})$ is infinite-index in $\text{Mod}(\tilde{\Sigma})$.

As shown, the property that $\text{SMod}(\tilde{\Sigma}) = \text{Mod}(\tilde{\Sigma})$ is rare, echoing Birman and Hilden’s sentiment in the case of surfaces with boundary.

The techniques used in this paper unfortunately do not naturally extend to surfaces without boundary. What the precise analogous statements of Theorems 1.1 and 1.2 should be remains an intriguing open question.

### 1.2. Braid group embeddings.

The braid group $B_n$ on $n$ strands is isomorphic to the mapping class group $\text{Mod}(D_n)$, where $D_n$ is a disk with $n$ punctures. There is a standard embedding of the braid group in the mapping class group, that sends each standard generator to a Dehn twist. Such an embedding is called geometric.

A question of Wajnryb asks whether there are non-geometric embeddings of the braid group in a mapping class group [28]. Non-geometric embeddings have since been constructed in the works of Bödigheimer-Tillman [5], Kim-Song [17], Song [23], Song-Tillman [24], and Szepietowski [27].
Using Theorem 1.1 and the Birman-Hilden theorem, we construct a family of non-geometric embeddings of the braid group. Let $\sigma_1, \ldots, \sigma_{n-1}$ be the standard braid generators for $B_n$.

A $k$-chain on a surface $\Sigma$ is a sequence of simple closed curves $\{a_1, \ldots, a_k\}$ on $\Sigma$ such that $i(a_i, a_j) = 1$ if $|i - j| = 1$ and $i(a_i, a_j) = 0$ otherwise. Here $i(a_i, a_j)$ is the geometric intersection number of the curves $a_i$ and $a_j$.

A $k$-chain twist is the mapping class $T_{a_1} \cdots T_{a_k}$ where $\{a_1, \ldots, a_k\}$ is a $k$-chain. It is true that if $k \geq 2$, then a $k$-chain twist is not a single Dehn twist.

**Theorem 1.3.** Let $n \geq 2$. For each $k \geq 2$, there exists a surface $\Sigma$ and an injective homomorphism $\beta_k : B_n \rightarrow \text{Mod}(\Sigma)$ such that $\beta_k(\sigma_i)$ is a $(k - 1)$-chain twist for all $i$.

When $k = 2$, this embedding coincides with the standard geometric embedding. The embedding when $k = 3$ was independently arrived at by Kim-Song [17].

Using these embeddings as inspiration, we define a combinatorial condition on two $k$-chains that imply their respective chain twists satisfy a braid relation. The condition is defined in Section 5.4 and proven to be sufficient in Proposition 5.6. Figure 2 shows two 3-chains satisfying the combinatorial condition, so their chain twists satisfy a braid relation.

### 1.3. Outline of the paper

In Section 2 we review some facts about lifting and projecting homeomorphisms and the Birman-Hilden theorem. In Section 3 we review some basic facts about groupoids and prove some useful lemmas regarding fundamental groupoids and covering spaces, including a version of the Birman-Hilden theorem for automorphisms of groupoids (Lemma 3.6). The main classification theorems are proved in Section 4. The non-geometric embeddings of the braid groups arising from the Burau covers are investigated in Section 5. The paper concludes with a list of open questions relating to the braid group embeddings.
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2. The Birman-Hilden theorem with boundary

Let \( \Sigma \) be an orientable surface, possibly with boundary \( \partial \Sigma \) and finitely many marked points \( B \subset \Sigma \setminus \partial \Sigma \). Denote by \( \text{Homeo}(\Sigma) \) and \( \text{Homeo}^+(\Sigma) \) the group of homeomorphisms \( \Sigma \) and the group of orientation-preserving homeomorphisms of \( \Sigma \) respectively. If \( B \) appears in the argument of \( \text{Homeo} \), then we require the homeomorphisms to fix \( B \) setwise. If \( \partial \Sigma \) appears, we require the homeomorphisms fix \( \partial \Sigma \) pointwise. Note that \( \text{Homeo}^+(\Sigma, \partial \Sigma) = \text{Homeo}(\Sigma, \partial \Sigma) \).

Let \( p: \tilde{\Sigma} \to \Sigma \) be a regular cover with finite deck group \( D \subset \text{Homeo}(\tilde{\Sigma}) \), possibly branched at \( B \subset \Sigma \setminus \partial \Sigma \). A homeomorphism \( f \in \text{Homeo}^+(\tilde{\Sigma}) \) is 
\textit{fibre-preserving} if whenever \( p(x) = p(y) \), \( pf(x) = pf(y) \) for all \( x, y \in \tilde{\Sigma} \). Let \( \text{SHomeo}^+(\tilde{\Sigma}) \) be the subgroup consisting of fibre-preserving homeomorphisms. There is a homomorphism \( \Pi : \text{SHomeo}^+(\tilde{\Sigma}) \to \text{Homeo}(\Sigma, B) \) given by \( \Pi(\tilde{f})(x) = p\tilde{f}(\tilde{x}) \) for any \( \tilde{x} \) such that \( p(\tilde{x}) = x \). It follows that \( \ker(\Pi) = D \). If \( \Pi(\tilde{f}) = f \), then the square
\[
\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\tilde{f}} & \tilde{\Sigma} \\
\downarrow{p} & & \downarrow{p} \\
\Sigma & \xrightarrow{f} & \Sigma
\end{array}
\]
commutes. That is, \( p\tilde{f} = fp \). Furthermore, it is true that \( \text{SHomeo}^+(\tilde{\Sigma}) \) is the normaliser of \( D \) in \( \text{Homeo}(\tilde{\Sigma}) \).

We say a homeomorphism \( f \in \text{Homeo}(\Sigma, B) \) \text{\textit{lifts}} if there exists a homeomorphism \( \tilde{f} \in \text{Homeo}(\Sigma) \) such that \( pf = p\tilde{f} \). Let \( \text{LHomeo}^+(\Sigma, B) \) be the subgroup of \( \text{Homeo}(\Sigma, B) \) consisting of homeomorphisms that lift. Then the image of \( \Pi \) is \( \text{LHomeo}^+(\Sigma, B) \) so \( \text{SHomeo}^+(\tilde{\Sigma})/D \cong \text{LHomeo}^+(\Sigma, B) \).

Lifting and projecting with boundary. Suppose \( \tilde{\Sigma} \) and \( \Sigma \) have non-empty boundary \( \partial \tilde{\Sigma} \) and \( \partial \Sigma \) respectively. We wish to lift and project homeomorphisms that preserve the boundary pointwise, to homeomorphisms that preserve the boundary pointwise. To that end, let \( \text{SHomeo}(\Sigma, \partial \tilde{\Sigma}) = \text{SHomeo}^+(\tilde{\Sigma}) \cap \text{Homeo}(\Sigma, \partial \Sigma) \).

**Proposition 2.1.** \( \text{SHomeo}(\Sigma, \partial \tilde{\Sigma}) = C \cap \text{Homeo}(\Sigma, \partial \Sigma) \) where \( C \) is the centraliser of the deck group \( D \) in \( \text{Homeo}(\Sigma) \).

**Proof.** It suffices to show that any homeomorphism \( \tilde{f} \) that fixes the boundary and is in the normaliser of \( D \) is in the centraliser of \( D \). Let \( \tilde{x} \in \partial \tilde{\Sigma} \) and
let $d \in D$. It is clear then that $d(\tilde{x}) \in \partial \tilde{\Sigma}$. Since $\tilde{f}$ fixes the boundary pointwise we have $f^{-1}d\tilde{f}(\tilde{x}) = \tilde{f}^{-1}d(\tilde{x}) = d(\tilde{x})$. Since $\tilde{f}$ is in the normaliser of $D$, $f^{-1}df \in D$. Since the deck group acts freely on $\partial \tilde{\Sigma}$ we have $\tilde{f}^{-1}d\tilde{f} = d$, completing the proof. ■

Any fibre-preserving homeomorphism of $\tilde{\Sigma}$ that fixes $\partial \tilde{\Sigma}$ pointwise must project to a homeomorphism of $\Sigma$ that fixes $\partial \Sigma$ pointwise. Since the only element of $D$ that fixes $\partial \tilde{\Sigma}$ pointwise is the identity, restricting the domain of $\Pi$ gives us an injective homomorphism $\Pi : \text{SHomeo}(\tilde{\Sigma}, \partial \tilde{\Sigma}) \to \text{LHomeo}(\Sigma, \partial \Sigma) \cap \text{Homeo}(\Sigma, \partial \Sigma)$.

Define $\text{LHomeo}(\Sigma, \partial \Sigma, B) = \Pi(\text{SHomeo}(\tilde{\Sigma}, \partial \tilde{\Sigma}))$. That is, the set of homeomorphisms of $\Sigma$ fixing $\partial \Sigma$ pointwise that lift to a homeomorphism of $\tilde{\Sigma}$ fixing $\partial \tilde{\Sigma}$ pointwise.

While it is tempting to define $\text{LHomeo}(\Sigma, \partial \Sigma, B) = \text{LHomeo}(\Sigma, \partial \Sigma)$ as $\text{LHomeo}(\Sigma, \partial \Sigma) \cap \text{Homeo}(\Sigma, \partial \Sigma)$, in general there exist boundary preserving homeomorphisms of $\Sigma$ that lift to homeomorphisms of $\tilde{\Sigma}$ that do not fix the boundary.

Let $\tilde{\Psi} : \text{Homeo}(\tilde{\Sigma}, \partial \tilde{\Sigma}) \to \text{Mod}(\tilde{\Sigma})$ and $\Psi : \text{Homeo}(\Sigma, \partial \Sigma, B) \to \text{Mod}(\Sigma, B)$ be the natural quotient maps. Define the symmetric mapping class group and the liftable mapping class group by $\text{SMod}(\tilde{\Sigma}) = \tilde{\Psi}(\text{SHomeo}(\tilde{\Sigma}, \partial \tilde{\Sigma}))$ and $\text{LMod}(\Sigma, B) = \Psi(\text{LHomeo}(\Sigma, \partial \Sigma, B))$ respectively.

2.1. The Birman-Hilden Theorem. Suppose $\tilde{\Sigma}$ is closed with genus at least 2. The Birman-Hilden theorem states that projecting homeomorphisms induces an isomorphism $\text{SMod}(\tilde{\Sigma})/D \cong \text{LMod}(\Sigma, B)$. Furthermore, $\text{SMod}(\tilde{\Sigma})$ is the normaliser of $D$ in $\text{Mod}(\tilde{\Sigma})$. The Birman-Hilden theorem was first proved for solvable covers [4], and later proved for all finite-sheeted, regular branched covers [19]. A version of the Birman-Hilden theorem was proved by Winarski for possibly irregular, fully-ramified covers [29], and by Aramayona, Leininger, and Souto in for irregular unbranched covers [2]. We refer the reader to a survey by Margalit-Winarski of the Birman-Hilden theorem [20].

When $\tilde{\Sigma}$ and $\Sigma$ have non-empty boundary, the deck group $D$ is no longer a subgroup of $\text{Mod}(\tilde{\Sigma})$ since no non-trivial element of $D$ fixes the boundary components pointwise. In this case, the Birman-Hilden theorem takes the following form.

**Theorem 2.2** (Birman-Hilden with boundary). Let $p : \tilde{\Sigma} \to \Sigma$ be a finite-sheeted regular covering space of surfaces with boundary, possibly branched at $B \subset \Sigma \setminus \partial \Sigma$. Then projecting homeomorphisms induces an isomorphism $\text{SMod}(\tilde{\Sigma}) \cong \text{LMod}(\Sigma, B)$.

This result is well known and a proof can be found in the thesis of the second author. Rephrased, **Theorem 2.2** states that the isomorphism $\Pi : \text{SHomeo}(\tilde{\Sigma}, \partial \tilde{\Sigma}) \to \text{LHomeo}(\Sigma, \partial \Sigma, B)$ induces an isomorphism $\Pi : \text{SMod}(\tilde{\Sigma}) \to \text{LMod}(\Sigma, B)$. 
3. The Fundamental Groupoid

In this section we review some facts about groupoids before studying the automorphism groups of groupoids related to covering spaces.

3.1. Groupoids. Here we survey the relevant results about groupoids. See the books [14] and [8] for more details.

A groupoid is a small category where every morphism is an isomorphism. Equivalently, a groupoid \( G \) is a disjoint collection of sets \( \{G_{ij}\}_{i,j \in I} \) together with an associative partial operation \( \cdot : G_{ij} \times G_{jk} \rightarrow G_{ik} \) such that

- For each \( i \in I \) there is an identity \( e_i \in G_{ii} \) such that \( e_i f = f \) and \( g e_i = g \) for all \( f, g \) such that the products \( e_i f \) and \( g e_i \) are defined, and
- For each \( g \in G_{ij} \) there is an inverse \( g^{-1} \in G_{ji} \) such that \( g g^{-1} = e_i \) and \( g^{-1} g = e_j \).

We will call \( I \) the object set of \( G \). If \( |I| = 1 \) (equivalently if the category has one object), then \( G \) is a group. Define the source and target maps, \( s, t : G \rightarrow I \) by \( s(g) = i \) and \( t(g) = j \) for all \( g \in G_{ij} \).

A groupoid is connected if \( G_{ij} \neq \emptyset \) for all \( i, j \in I \). Notice that \( G_{ii} \) is a group for all \( i \in I \), and if \( G \) is connected then \( G_{ii} \cong G_{jj} \) for all \( i, j \in I \). The groups \( G_{ii} \) will be called vertex groups. From now on we will assume \( G \) is a connected groupoid.

Fix an \( i_0 \in I \). For each \( i \in I \) choose an element \( \iota_i \in G_{i_0 i} \) with \( \iota_{i_0} = e_{i_0} \). Then \( G \) is generated by the vertex group \( G_{i_0 i} \) and \( \{\iota_i\}_{i \in I} \). In fact, every element in \( G_{ij} \) is uniquely written as \( \iota_i^{-1} g \iota_j \) for some \( g \in G_{i_0 i_0} \). We call \( \{\iota_i\}_{i \in I} \) a star based at \( i_0 \).

A subgroupoid \( H < G \) is a collection of subsets \( \{H_{ij} \subset G_{ij}\}_{i,j \in J} \) for some non-empty \( J \subset I \) such that \( H \) is a groupoid with the operation from \( G \). A subgroupoid is wide if \( J = I \). A subgroupoid \( H < G \) is normal if \( f^{-1} H_{ii} f \subset H_{jj} \) for all \( f \in G_{ij} \). It follows that normal subgroupoids of connected groupoids are wide, and \( h \mapsto f^{-1} h f \) is an isomorphism of groups \( H_{ii} \cong H_{jj} \).

Let \( H \) be a connected normal subgroupoid of \( G \). Construct the quotient groupoid \( G/H \) to be a groupoid with one object, or a group, as follows. Put an equivalence relation \( \sim \) on \( G \) by \( a \sim b \) if there exists \( x, y \in H \) such that \( a = xby \). The equivalence classes are called the cosets of \( H \) in \( G \), and these are the elements of \( G/H \). Define an operation on the cosets by \( \left[ a \right] \left[ b \right] = \left[ ab \right] \) for any \( a \in H \) with \( s(x) = t(a) \) and \( t(x) = s(b) \). This is a well defined group operation on \( G/H \).

Although we will not need it, the quotient groupoid can be defined for disconnected normal subgroupoids of connected groupoids. The only difference is that there is one object for each connected component of \( H \) (see [14 Chapter 12]).

3.2. Automorphisms of groupoids. Let \( G \) and \( H \) be groupoids with object sets \( I \) and \( J \) respectively. A morphism \( \phi : G \rightarrow H \) is a functor from
To set notation, the group operation on $G$ is given by $g_i \cdot g_j = g_{ij}$ for all $i, j \in I$ such that $\hat{\phi}(a) \cdot \hat{\phi}(b) = \hat{\phi}(ab)$ for all $a \in G_{ij}, b \in G_{jk}$. It follows that $\hat{\phi}(e_i) = e_{\hat{\phi}(i)}$ and $\hat{\phi}(g^{-1}) = \hat{\phi}(g)^{-1}$ for all $i, j \in I$ and $g \in G_{ij}$. Given a groupoid automorphism $\phi$, we will abuse notation and denote the maps $\hat{\phi}$ and $\hat{\phi}_j$ by $\phi$. Let $s(\phi(g)) = \phi(s(g))$ and $t(\phi(g)) = \phi(t(g))$ for all $g \in G$.

An automorphism of $G$ is a morphism $\phi : G \to G$ with a two-sided inverse. The set of automorphisms of $G$ forms a group under composition, denoted $\text{Aut}(G)$.

We now restrict our attention to connected groupoids with finite object set. Let $G$ be a group and consider the semi-direct product $G^n \rtimes \text{Aut}(G)$. To set notation, the group operation on $G^n \rtimes \text{Aut}(G)$ is given by

$$(g_1, \ldots, g_n, \psi)((h_1, \ldots, h_n), \varphi) = ((\psi(h_1)g_1, \ldots, \psi(h_n)g_n), \psi \varphi)$$

for all $g_i, h_i \in G$ and $\psi, \varphi \in \text{Aut}(G)$.

Define the pure automorphism group of $G$ by

$$\text{PAut}(G) := \{ \phi \in \text{Aut}(G) : \phi(i) = i \text{ for all } i \in I \}.$$ 

Let $G$ be a connected groupoid with object set $I = \{0, 1, \ldots, n\}$. Let $G = G_{00}$ be the vertex group at $0 \in I$. Choose a star $\{\iota_i\}_{i \in I} \subset G$ based at $0 \in I$ and let $g_0 = e_0 \in G$.

**Lemma 3.1.** The map $\theta : G^n \rtimes \text{Aut}(G) \to \text{PAut}(G)$ given by

$$\theta(((g_1, \ldots, g_n), \psi))((i^{-1}_i \iota_j), \psi) = i^{-1}_i g_i^{-1} \psi(a) g_j \iota_j$$

is an isomorphism.

**Lemma 3.1** is proved in [1, §3]. Note that the isomorphism $\theta$ depends on the choice of star.

Let $\mathcal{H} < G$ be a normal subgroupoid. If $\phi \in \text{Aut}(G)$ is such that $\phi(\mathcal{H}) \subset \mathcal{H}$, then $\phi$ induces an automorphism $\overline{\phi} \in \text{Aut}(G/\mathcal{H})$ by $\overline{\phi}([a]) = [\phi(a)]$. Define the subgroup $\text{LAut}_\mathcal{H}(G) < \text{PAut}(G)$ by

$$\text{LAut}_\mathcal{H}(G) = \{ \phi \in \text{PAut}(G) : \phi(\mathcal{H}) = \mathcal{H} \text{ and } \overline{\phi} = \text{id} \in \text{Aut}(G/\mathcal{H}) \}.$$ 

Our goal is to prove, with certain restrictions on $G$ and $\mathcal{H}$, that $\text{LAut}_\mathcal{H}(G)$ is finite-index in $\text{PAut}(G)$.

As above, suppose that $G$ is a connected groupoid with object set $I = \{0, 1, \ldots, n\}$ and let $G = G_{00}$ be the vertex group at $0 \in I$. Let $\mathcal{H}$ be a connected normal subgroupoid with vertex group $H = H_{00}$, which is a normal subgroup of $G$.

Define the subgroup $K < G^n \rtimes \text{Aut}(G)$ by

$$K = \{(g_1, \ldots, g_n, \psi) \in G^n \rtimes \text{Aut}(G) : \psi \in \text{LAut}_\mathcal{H}(G), g_i \in H \text{ for all } i \}.$$ 

Here, $\text{LAut}_\mathcal{H}(G)$ is defined by considering a group as a groupoid with one object.
Lemma 3.2. Choose a star $S = \{s_i\}_{i \in I} \subset H$ based at $0 \in I$. Consider the isomorphism $\theta : G^n \rtimes \text{Aut}(G) \to \text{PAut}(G)$ from Lemma 3.1 defined by $S$. Then $\theta(K) = \text{LAut}_H(G)$.

Proof. Let $k = ((h_1, \ldots, h_n), \psi) \in K$, let $h_0 = e_0 \in H$ and let $i^{-1}_g g_j$ be an arbitrary element in $\mathcal{G}$. Then $\theta(k)(i^{-1}_g g_j) = i^{-1}_h h^{-1}_i \psi(g) h_{ij}$. Since $\psi(g) \in H$ if and only if $g \in H$, $\theta(k)(i^{-1}_g g_j) \in H$ if and only if $g \in H$. Therefore $\theta(k)(\mathcal{H}) = \mathcal{H}$. In $\mathcal{G}/\mathcal{H}$, $[i^{-1}_g g_j] = [g]$ for any $g \in G$. Therefore $\theta(k)([i^{-1}_g g_j]) = \theta(k)([g]) = [\psi(g)]$. Since $\psi \in \text{LAut}_H(G)$, $[\psi(g)] = [g]$ implying $\theta(k) \in \text{LAut}_H(G)$.

Conversely, suppose $k = ((g_1, \ldots, g_n), \psi) \in G^n \rtimes \text{Aut}(G)$ is such that $\theta(k) \in \text{LAut}_H(G)$. We have $\theta(k)([g]) = [\psi(g)] = [g]$ for all $g \in G$, so $\psi \in \text{LAut}_H(G)$. Let $h_{ij}$ be an arbitrary element of $H_{0j}$. Then $\theta(k)(h_{ij}) = \psi(h)g_{ij}$. For $\theta(k)(h_{ij})$ to be in $\mathcal{H}$, we must have $g_j \in H$. Therefore $k \in K$, completing the proof. \hfill \blacksquare

Lemma 3.3. Let $\mathcal{G}$ be a connected groupoid with object set $I = \{0, 1, \ldots, n\}$ and $\mathcal{H}$ a connected normal subgroupoid. Let $G = G_0$ and $H = H_0$ as above. Suppose $G$ is finitely generated and $H$ is finite-index in $G$. Then $\text{LAut}_H(\mathcal{G})$ is finite-index in $\text{PAut}(\mathcal{G})$.

Proof. By Lemma 3.2, it suffices to show that $K$ is finite-index in $G^n \rtimes \text{Aut}(G)$. It is easily checked that $((g_1, \ldots, g_n), \psi)$ and $((h_1, \ldots, h_n), \varphi)$ are in the same right coset of $K$ if and only if $[h_i] = [g_i]$ in $G/H$ for all $i$ and $\psi$ and $\varphi$ are in the same right coset of $\text{LAut}_H(G)$ in $\text{Aut}(G)$. The result then follows from the fact that if $G$ is finitely generated and $H$ is finite-index in $G$, $\text{LAut}_H(G)$ is finite-index in $\text{Aut}(G)$. \hfill \blacksquare

3.3. The fundamental groupoid. The groupoids of interest in this paper will be fundamental groupoids of surfaces with boundary. We will briefly state the definition and some properties without proof that will be useful later on. For a full treatment, see [8, Chapter 6] or [14, Chapter 6].

Let $X$ be a topological space and $A \subset X$ a subset. The fundamental groupoid $\pi_1(X, A)$ is the set of homotopy classes of paths $\delta : ([0, 1], \{0, 1\}) \to (X, A)$ relative to the endpoints. Equipped with concatenation of paths as the partial operation, $\pi_1(X, A)$ is a groupoid with object set $A$. The source and target maps are given by $s([\delta]) = \delta(0)$ and $t([\delta]) = \delta(1)$. When $A = \{x\}$, we recover the fundamental group $\pi_1(X, x)$. It is helpful to think of the fundamental groupoid as a fundamental group with multiple basepoints.

Like the fundamental group, the fundamental groupoid provides a functor from the category of pairs of topological spaces to groupoids. In particular, if $f : X \to X$ is a homeomorphism such that $f(A) = A$, then $f_* : \pi_1(X, A) \to \pi_1(X, A)$ is an automorphism of the groupoid $\pi_1(X, A)$. Furthermore, if $f, g : (X, A) \to (Y, B)$ are homotopic relative to $A$, then $f_* = g_* : \pi_1(X, A) \to \pi_1(Y, B)$. 
We now shift our focus to covering spaces. For a groupoid \(G\) with object set \(I\), define the sets \(S(i) := \{ g \in G : s(g) = i \}\) and \(T(i) := \{ g \in G : t(g) = i \}\). The next lemma will be useful throughout the rest of the paper.

**Lemma 3.4.** Let \(p : \widetilde{X} \to X\) be a covering space, \(A \subset X\) a subset. Consider the induced groupoid morphism \(p_* : \pi_1(\widetilde{X}, p^{-1}(A)) \to \pi_1(X, A)\). The restriction maps \(p_* : S(x) \to S(p(x))\) and \(p_* : T(x) \to T(p(x))\) are bijections for all \(x \in p^{-1}(A)\).

The lemma follows from the path and homotopy lifting properties for covering spaces, and the details are left to the reader. It is worth noting that Lemma 3.4 says that \(p_*\) is a covering morphism (see [8, Section 10.2]).

Let \(p : \widetilde{X} \to X\) be a finite-sheeted, regular covering space with deck group \(D\). Let \(A = \{x_1, \ldots, x_k\} \subset X\), and \(B = p^{-1}(A) \subset \widetilde{X}\). For each \(i \in \{1, \ldots, k\}\), choose \(\tilde{x}_i \in p^{-1}(x_i)\). Let \(\tilde{A} = \{\tilde{x}_1, \ldots, \tilde{x}_k\}\).

Define the following groupoids.

\[
G := \pi_1(X, A), \quad H := p_*(\pi_1(\widetilde{X}, \tilde{A})), \quad K := \pi_1(\widetilde{X}, B).
\]

Since \(p\) is a regular cover, \(H\) is a normal subgroupoid of \(G\). Recall the definition of \(\text{LAut}_H(G)\) from Section 3.2. It will be useful for us to identify when two elements of \(G\) are in the same coset of \(H\).

**Lemma 3.5.** Let \(g_1, g_2 \in G\) and let \(\tilde{g}_1, \tilde{g}_2 \in K\) be the unique elements such that \(s(\tilde{g}_1), s(\tilde{g}_2) \in \tilde{A}\). Then \(t(\tilde{g}_1) = d_1(\tilde{x}_i)\) and \(t(\tilde{g}_2) = d_2(\tilde{x}_j)\) for some \(\tilde{x}_i, \tilde{x}_j \in \tilde{A}\) and some \(d_1, d_2 \in D\). The elements \(g_1\) and \(g_2\) are in the same coset of \(H\) if and only if \(d_1 = d_2\).

**Proof.** Suppose \(d_1 = d_2\). Choose any \(z \in \pi_1(\widetilde{X}, \tilde{A})\) such that \(s(z) = s(\tilde{g}_1)\) and \(t(z) = s(\tilde{g}_2)\). Let \(y = d^{-1}(\tilde{g}_2^{-1}z^{-1}\tilde{g}_1)\). Note \(y \in \pi_1(X, A)\). Then \(\tilde{g}_1 = \tilde{z}\tilde{g}_2t(y)\) and \(g_1 = p_*(z)g_2p_*(y)\) so \(g_1\) and \(g_2\) are in the same coset of \(H\).

Conversely, suppose \(g_1 = zg_2y\) for some \(z, y \in H\). Let \(\tilde{z}\) and \(\tilde{y}\) be such that \(p_*(\tilde{z}) = z\), \(p_*(\tilde{y}) = y\), and \(s(\tilde{z}), t(\tilde{z}), s(\tilde{y}), t(\tilde{y}) \in \tilde{A}\). Then by Lemma 3.4, \(\tilde{g}_1 = \tilde{z}\tilde{g}_2d_2(\tilde{y})\). Therefore \(t(\tilde{g}_1) = d_2(t(\tilde{y}))\), completing the proof.

Since \(D\) acts freely on \(B\), \(D\) injects into \(\text{Aut}(K)\). Abusing notation, we will identify \(D\) with its image in \(\text{Aut}(K)\). Define the group

\[
\text{SAut}(K) := \{ \phi \in \text{PAut}(K) : [\phi, d] = 1 \text{ for all } d \in D\},
\]

that is, \(\text{SAut}(K)\) is the intersection of \(\text{PAut}(K)\) with the centraliser of \(D\) in \(\text{Aut}(K)\). Define a map \(\Pi : \text{SAut}(K) \to \text{LAut}_H(G)\) by \(\Pi(\phi)(\tilde{g}) = p_*(\phi(\tilde{g}))\) where \(\tilde{g} \in p_*^{-1}(g)\) is any choice of lift of \(g\). The next lemma is a kind of Birman-Hilden theorem for groupoid automorphisms. It will be of particular importance in Sections 4 and 5.

**Lemma 3.6.** The group homomorphism

\[
\Pi : \text{SAut}(K) \to \text{LAut}_H(G)
\]

is an isomorphism.
Before embarking on the proof, we must set some notation. For \( g \in \mathcal{G} \), denote by \([g]_{(x)}\) the unique lift of \( g \) such that \( s([g]_{(x)}) = x \). Such a lift exists by Lemma 3.4.

Proof of Lemma 3.6. Let \( \phi \in \text{SAut}(\mathcal{K}) \) and \( g \in \mathcal{G} \). To see \( \Pi(\phi) \) is a well defined set map, note that if \( \tilde{g}_1, \tilde{g}_2 \in p_*^{-1}(g) \), then \( \tilde{g}_1 = d\tilde{g}_2 \) for some \( d \in D \). Then \( p_*\phi(\tilde{g}_1) = p_*\phi(d\tilde{g}_2) = p_*\phi(\tilde{g}_2) \).

For \( g, h \in \mathcal{G} \) with \( s(h) = t(g) \), choose lifts \( \tilde{g} \) and \( \tilde{h} \) such that \( s(\tilde{h}) = t(\tilde{g}) \). Then \( \tilde{g}h \in p_*^{-1}(gh) \) and \( p_* \phi(\tilde{g}h) = p_* \phi(\tilde{g})p_* \phi(\tilde{h}) \). Therefore \( \Pi(\phi) : \mathcal{G} \to \mathcal{G} \) is a well defined groupoid morphism. It is easily checked that \( \Pi(\phi^{-1}) \) is a two-sided inverse for \( \Pi(\phi) \). We may finally conclude that \( \Pi : \text{SAut}(\mathcal{K}) \to \text{Aut}(\mathcal{G}) \) is a well defined group homomorphism.

Define a map which we optimistically label \( \Pi^{-1} \) defined in \( \mathcal{K} \) since \( \psi \) preserves each coset of \( \mathcal{H} \) in \( \mathcal{G} \). Set \( \tilde{g} = [g]_{(x_i)} \) for some \( i \). Then \( t(\tilde{g}) = d(x_j) \) for some \( j \) and some \( d \in D \). Since \( \phi \in \text{PAut}(\mathcal{K}) \), \( s\phi(\tilde{g}) = x_i \) and \( t(\phi(\tilde{g})) = d(x_j) \). Therefore by Lemma 3.5, \( g \) and \( p_* \phi(\tilde{g}) \) are in the same coset of \( \mathcal{H} \). We may now conclude \( \Pi^{-1} \) is a well defined set map. For \( \phi, \varphi \in \text{SAut}(\mathcal{K}) \), \( \Pi(\phi)\Pi(\varphi)(g) = p_*\phi(p_*\varphi(\tilde{g})) = p_*\varphi p_*\phi (\tilde{g}) = \Pi(\phi\varphi)(g) \) so \( \Pi : \text{SAut}(\mathcal{K}) \to \text{LAut}(\mathcal{G}) \) is a well defined group homomorphism.

Define a map which we optimistically label \( \Pi^{-1} : \text{LAut}(\mathcal{H})(\mathcal{G}) \to \text{SAut}(\mathcal{K}) \) by \( \Pi^{-1}(\psi)(k) = [\psi p_* (k)]_{(s(k))} \) for all \( k \in \mathcal{K} \).

For \( k, l \in \mathcal{K} \) with \( s(l) = t(k) \), the product \( [\psi (p_* (k)]_{(s(k))} [\psi (p_* (l)]_{(s(l))} \) is, by Lemma 3.5, defined in \( \mathcal{K} \) since \( \psi \) preserves each coset of \( \mathcal{H} \) in \( \mathcal{G} \). Furthermore, \( p_*[\psi (p_* (k)]_{(s(k))} = p_* \left( [\psi (p_* (k)]_{(s(k))} [\psi (p_* (l)]_{(s(l))} \right) \) so \( \Pi^{-1}(\psi) : \mathcal{K} \to \mathcal{K} \) is a groupoid homomorphism by Lemma 3.4.

It is easily checked that \( \Pi^{-1}(\psi^{-1}) \) is a two-sided inverse for \( \Pi^{-1}(\psi) \). Furthermore, by the definition of \( \Pi^{-1}(\psi) \), \( s(\Pi^{-1}(\psi)(k)) = s(k) \) for all \( k \in \mathcal{K} \) so \( \Pi^{-1}(\psi) \in \text{PAut}(\mathcal{K}) \).

We have \( p_*[\psi (p_* (d(k)))]_{(s(dk))} = p_* d \left( [\psi (p_* (k)]_{(s(k))} \right) \) for all \( d \in D \) and \( s \left( [\psi p_* (d(k))]_{(s(dk))} \right) = s \left( d \left( [\psi (p_* (k)]_{(s(k))} \right) \right) = d s(k) \). So by Lemma 3.4, \( \Pi^{-1}(\psi)(d(k)) = d \Pi^{-1}(\psi)(k) \) implying \( \Pi^{-1}(\psi) \in \text{SAut}(\mathcal{K}) \).

It remains to show, as the notation suggests, that \( \Pi^{-1} \) is a two-sided inverse for \( \Pi \). We have \( \Pi \Pi^{-1}(\psi)(g) = p_* [\psi (p_* (\tilde{g}))]_{(s(\tilde{g}))} = \psi (g) \) for all \( \psi \in \text{LAut}(\mathcal{H})(\mathcal{G}) \) and \( g \in \mathcal{G} \).

On the other hand, first note that \( p_*[p_* \phi (k)]_{(s(k))} = p_* \phi (k) \) and \( s \left( [p_* \phi (k)]_{(s(k))} \right) = s(\phi (k)) = s(k) \). Therefore by Lemma 3.4, \( [p_* \phi (k)]_{(s(k))} = \phi (k) \). We now have \( \Pi \Pi^{-1}(\phi)(k) = [p_* \phi (k)]_{(s(k))} = \phi (k) \).

Since \( \Pi^{-1} \) is a two-sided inverse for \( \Pi \), we may finally conclude that \( \Pi : \text{SAut}(\mathcal{K}) \to \text{LAut}(\mathcal{H})(\mathcal{G}) \) is an isomorphism. ■
4. Proof of Classification Results

Given a covering space \( p : \widetilde{\Sigma} \to \Sigma \) of a surface branched at \( B \subset \Sigma \), let \( \widetilde{\Sigma}^o = \widetilde{\Sigma} \setminus p^{-1}(B) \) and \( \Sigma^o = \Sigma \setminus B \). Abusing notation, denote the resulting unbranched cover \( p : \widetilde{\Sigma}^o \to \Sigma^o \). It is easy to see that restricting homeomorphisms gives isomorphisms \( \text{LMod}(\Sigma, B) \cong \text{LMod}(\Sigma^o) \) and \( \text{SMod}(\Sigma) \cong \text{SMod}(\Sigma^o) \). With this in mind, we will move back and forth between the branched and unbranched covers without much indication.

As hinted at above, to characterise when a homeomorphism lifts to a homeomorphism that fixes boundary components, we must look at the action of a homeomorphism on the fundamental groupoid \( \pi_1(\Sigma^o, A) \) for a specific choice of basepoints \( A \subset \Sigma^o \).

4.1. Action on the fundamental groupoids. Suppose \( \partial \Sigma^o \) has \( m \) components. Let \( A = \{x_0, x_1, \ldots, x_{m-1}\} \subset \partial \Sigma^o \) be such that each component contains exactly one of the \( x_i \). For each \( x_i \), choose a point \( \tilde{x}_i \in p^{-1}(x_i) \) and let \( \tilde{A} = \{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{m-1}\} \subset \partial \widetilde{\Sigma}^o \). Let \( B = p^{-1}(A) \) and denote the fundamental groupoids \( \mathcal{G} = \pi_1(\Sigma^o, A) \), \( \mathcal{H} = p_*\pi_1(\Sigma^o, \tilde{A}) \), and \( \mathcal{K} = \pi_1(\widetilde{\Sigma}^o, B) \) as in Section 3.3.

We have a useful commutative diagram

\[
\begin{array}{ccc}
\text{SMod}(\Sigma) & \xrightarrow{\Psi} & \text{SAut}(\mathcal{K}) \\
\cong \downarrow \Pi & & \cong \downarrow \Pi \\
\text{LMod}(\Sigma, B) & \xleftarrow{\Psi} & \text{LAut}_{\mathcal{H}}(\mathcal{G})
\end{array}
\]

where the horizontal injections are given by the action of the mapping class group on the fundamental groupoid (see [15, Theorem 3.1.1]), and the vertical maps are the Birman-Hilden isomorphisms from Theorem 2.2 and Lemma 3.6. Note that the horizontal maps are actually the composition of the action on the fundamental groupoid with the isomorphisms \( \text{SMod}(\Sigma) \cong \text{SMod}(\Sigma^o) \) and \( \text{LMod}(\Sigma, B) \cong \text{LMod}(\Sigma^o) \).

4.2. The case where everything lifts. We now move on to proving Theorem 1.1. As such, we return to the setting of the original branched cover \( p : \Sigma \to \Sigma \) branched at \( B \subset \Sigma \setminus \partial \Sigma \). If \( [f] \in \text{Mod}(\Sigma, B) \) we will abuse notation and denote by \( f_* \in \text{Aut}(\pi_1(\Sigma^o, A)) \) the automorphism induced by any representative homeomorphism for \( [f] \). The abuse of notation is legal since any representative homeomorphism for \( [f] \) fixes \( A \) pointwise, and isotopic homeomorphisms induce the same groupoid automorphism.

**Theorem 4.1.** \( \text{LMod}(\Sigma, B) = \{[f] \in \text{Mod}(\Sigma, B) : f_* \in \text{LAut}_{\mathcal{H}}(\mathcal{G})\} \).

**Proof.** By the commutative diagram above, it suffices to show that if \( f_* \in \text{LAut}_{\mathcal{H}}(\mathcal{G}) \), then \( [f] \in \text{LMod}(\Sigma, B) \). The homeomorphism \( f \) lifts since \( f_*(p_*\pi_1(\Sigma, \tilde{x}_0)) = p_*\pi_1(\Sigma, \tilde{x}_0) \). Let \( \tilde{f} \) be the lift of \( f \) such that \( \tilde{f}(\tilde{x}_0) = \tilde{x}_0 \). It remains to show \( \tilde{f} \) fixes \( \partial \widetilde{\Sigma} \) pointwise.
Let \( \tilde{y} \in \partial \tilde{\Sigma} \). Note that \( y = p(\tilde{y}) \) is in the same component of \( \partial \Sigma \) as \( x_i \) for some \( i \). Choose \( \sigma \in \pi_1(\Sigma, \partial \Sigma) \) such that \( s(\sigma) = y, t(\sigma) = x_i \) and \( \sigma \) is represented by an arc completely contained in \( \partial \Sigma \) so \( f_\ast(\sigma) = \sigma \). Choose \( \tilde{\delta} \in \pi_1(\tilde{\Sigma}, \partial \tilde{\Sigma}) \) such that \( s(\tilde{\delta}) = \tilde{x}_0 \) and \( t(\tilde{\delta}) = \tilde{y} \). Let \( \delta = p_\ast(\tilde{\delta}) \) and note that \( \delta \sigma = \tilde{\sigma} \in \tilde{G} \).

We have \( p_\ast \tilde{f}_\ast(\tilde{\delta}\tilde{\sigma}) = f_\ast(\delta \sigma), \) \( p_\ast(\tilde{\delta}\tilde{\sigma}) = \delta \sigma, \) \( s(\tilde{f}_\ast(\tilde{\delta}\tilde{\sigma})) = s(\tilde{\delta}\tilde{\sigma}) = \tilde{x}_0, \) and \( t(f_\ast(\delta \sigma)) = t(\delta \sigma) \). Since \( f_\ast(\delta \sigma) \) and \( \delta \sigma \) are in the same coset of \( H \), Lemma 3.5 implies \( t(f_\ast(\tilde{\delta}\tilde{\sigma})) = t(\delta \sigma) \), implying \( t(\tilde{f}_\ast(\tilde{\sigma})) = t(\tilde{\sigma}) \). Since \( p_\ast \tilde{f}_\ast(\tilde{\sigma}) = p_\ast(\tilde{\sigma}) \), we have \( \tilde{f}_\ast(\tilde{\sigma}) = \tilde{\sigma} \) by Lemma 3.4. Therefore \( \tilde{f}(\tilde{y}) = s(\tilde{f}_\ast(\tilde{\sigma})) = s(\tilde{\sigma}) = \tilde{y} \), completing the proof.

If \( \Sigma \) has one boundary component we get the following well-known corollary.

**Corollary 4.2.** Suppose \( \Sigma \) has one boundary component. Choose a basepoint \( x \in \partial \Sigma^o \) and \( \tilde{x} \in p^{-1}(x) \). Then

\[
\text{LMod}(\Sigma, B) = \{ [f] \in \text{Mod}(\Sigma, B) : qf_\ast = q \}
\]

where \( q : \pi_1(\Sigma^o, x) \to \pi_1(\Sigma^o, \tilde{x})/p_\ast(\pi_1(\Sigma^o, \tilde{x})) \) is the quotient map and \( f_\ast \) is the induced map on \( \pi_1(\Sigma^o, x) \).

**Proof.** The condition \( qf_\ast = q \) is equivalent to \( f_\ast \) acting trivially on the cosets of \( p_\ast(\pi_1(\Sigma^o, \tilde{x})) \) in \( \pi_1(\Sigma^o, x) \). The result then follows from Theorem 4.1. ■

The next proposition gives a direct way to check whether or not an element of \( \text{Mod}(\Sigma, B) \) is in \( \text{LMod}(\Sigma, B) \). To that end, choose a point \( x_0 \in \partial \Sigma^o \) and a lift \( \tilde{x}_0 \in p^{-1}(x_0) \). Choose a generating set \( \{ \gamma_1, \ldots, \gamma_n \} \) of \( \pi_1(\Sigma^o, x_0) \). Since the cover is regular, \( \pi_1(\Sigma^o, x_0)/p_\ast(\pi_1(\Sigma^o, \tilde{x}_0)) \cong D \). Choose an isomorphism and let \( q : \pi_1(\Sigma^o, x_0) \to D \) be the quotient map.

Suppose there are \( m \) components of \( \partial \Sigma^o \). Enumerate the components not containing \( x_0 \) from 1 to \( m - 1 \). For each \( i \in \{1, \ldots, m - 1\} \), choose an arc \( \sigma_i : [0, 1] \to \Sigma^o \) such that \( \sigma_i(0) = x_0 \) and \( \sigma_i(1) \) is in the \( i \)th boundary component. Let \( x_i = \sigma_i(1) \in \partial \Sigma^o \).

Let \( A = \{ x_0, x_1, \ldots, x_{m-1} \} \subset \partial \Sigma^o \). Then the \( \gamma_i \) and \( \{ \sigma_j \} \) are all elements of \( \pi_1(\Sigma^o, A) \). Given an element \( [f] \in \text{Mod}(\Sigma^o) \), \( f_\ast(\sigma_j) = a_j[\sigma_j] \) for some \( a_j \in \pi_1(\Sigma^o, x_0) \).

**Proposition 4.3.** A mapping class \( [f] \) is in \( \text{LMod}(\Sigma, B) \) if and only if \( qf_\ast(\gamma_i) = q(\gamma_i) \) for all \( i \) and \( a_j \in \ker q \) for all \( j \).

**Proof.** Choose a lift \( \tilde{x}_0 \in p^{-1}(x_0) \). For all \( i \) choose lifts \( \tilde{\sigma}_i \) of \( \sigma_i \) such that \( \tilde{\sigma}_i(0) = \tilde{x}_0 \). Let \( \tilde{\sigma}_i = \tilde{\sigma}_i(1) \) and let \( \tilde{A} = \{ \tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{m-1} \} \). Let \( \tilde{G} = \pi_1(\Sigma^o, A) \) and \( \tilde{H} = p_\ast\pi_1(\tilde{\Sigma}^o, \tilde{A}) \). Then by Theorem 4.1, \( [f] \in \text{LMod}(\Sigma, B) \) if and only if \( f_\ast \in \text{LAut}_\tilde{H}(\tilde{G}) \) where \( \text{LAut}_\tilde{H}(\tilde{G}) \) is defined in Section 3.2.

The condition \( qf_\ast(\gamma_i) = q(\gamma_i) \) for all \( i \) is equivalent to \( f_\ast \) acting trivially on the cosets of \( p_\ast\pi_1(\tilde{\Sigma}^o, \tilde{x}_0) \) in \( \pi_1(\Sigma^o, x_0) \). The condition \( a_j \in \ker q \) implies \( a_j \in p_\ast\pi_1(\tilde{\Sigma}^o, \tilde{x}_0) \) for all \( i \). The result follows from observing that \( \{ [\sigma_1], \ldots, [\sigma_{m-1}] \} \) is a star in \( \tilde{H} \) and applying Lemma 3.2. ■
Figure 3. A basic generating set of \( \pi_1(\Sigma^1_{1,2}, x_0) \) and an essential generating set of \( \pi_1(\Sigma^2_{1,1}, x_0) \).

For any element \( c \) of \( \pi_1(\Sigma^\circ, x_0) \) we will abuse notation by writing \( c \) for a loop in \( \Sigma^\circ \) representing \( c \). Furthermore, if \( c \) is a simple loop, we will write \( c \) for the corresponding unique free isotopy class of simple closed curves in \( \Sigma^\circ \).

If \( m > 1 \) let \( \{\iota_1, \ldots, \iota_{m-1}\} \) be a star of the fundamental groupoid \( \pi_1(\Sigma^\circ, A) \) based at \( x_0 \) where each representative of \( \iota_i \) terminates at a point \( x_i \) on a unique boundary component \( \partial_i \) of \( \Sigma^\circ \). We now give two types of generating set for the fundamental group of a surface with boundary.

**Basic and essential generating sets.** If \( m = 1 \) we call a finite generating set \( S = \{a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n\} \) of \( \pi_1(\Sigma^\circ, x_0) \) a basic generating set if each element is represented by a simple loop, each \( \gamma_i \) bounds a subsurface homeomorphic to a punctured disk, and

\[
i(a_i, a_j) = i(b_i, b_j) = 0 \quad \text{for all } i, j,
\]

\[
i(a_i, b_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}
\]

If \( m > 1 \) we call a finite generating set \( S \) of \( \pi_1(\Sigma^\circ, x_0) \) essential with respect to the star \( \{\iota_i\}_{i=1}^{m-1} \) if for all \( c \in S \) there exists an \( i \in \{1, \ldots, m - 1\} \) such that \( i(\iota_i, c) = 1 \). Here, the name comes from the fact that each generator has essential intersection with at least one element of the star. The fact that basic generating sets exist is well known; see [13, Section 1.2]. The next lemma shows the existence of essential generating sets of fundamental groups.

**Lemma 4.4.** Let \( \Sigma^\circ \) be a surface with \( m > 1 \) boundary components and let \( x_0 \in \partial \Sigma^\circ \). There exists an essential generating set \( S \) of \( \pi_1(\Sigma^\circ, x_0) \) with respect to a star \( \{\iota_i\}_{i=1}^{m-1} \).

**Proof.** It follows from [13, Section 1.2] that we have a generating set

\[
\{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_{m-1}, \gamma_1 \ldots \gamma_n\},
\]

where each \( c_i \) is represented by a simple loop around a boundary component other than the unique component \( \partial_0 \) containing \( x_0 \) and each \( \gamma_i \) is represented by a simple loop around a puncture. Furthermore, we can choose elements \( c_i \) such that \( i(\iota_i, c_i) = 1 \) for all \( i \in \{1, \ldots, m - 1\} \). To see this we will abuse
arguments can be used to show that
\( \iota \) defines to be a representative of
\( \iota \) isotopy class of simple closed curve represented in a regular neighbourhood

Proof of Theorem 1.1. We first prove that \( LMod(\Sigma) \) is a Burau cover. Let \( \Sigma \) be the associated unbranched cover of a Burau cover. Let \( x_0 \in \partial D \) and let \( \{ \gamma_1, \ldots, \gamma_n \} \) be a basic generating set for \( \pi_1(D, x_0) \). Let \( H_{c_i} \in \text{Mod}(D) \) be a half twist whose support (a disk with two punctures) intersects all representative loops of \( \gamma_i \) and \( \gamma_{i+1} \) and is disjoint from representatives of \( \gamma_j \) for all \( j \neq i, i+1 \).

If we consider each \( H_{c_i} \) as an automorphism of \( \pi_1(D, x_0) \) we can assume that \( H_{c_i}(\gamma_i) = \gamma_i \gamma_{i+1} \gamma_i^{-1} \) and \( H_{c_i}(\gamma_{i+1}) = \gamma_i \). From the definition of Burau covers we have that

\[
q H_{c_i}(\gamma_i) = q(\gamma_i \gamma_{i+1} \gamma_i^{-1}) = q(\gamma_i) q(\gamma_{i+1}) q(\gamma_i^{-1}) = 1,
\]

\[
q H_{c_i}(\gamma_i) = q(\gamma_i) = 1, \quad \text{and}
\]

\[
q H_{c_i}(\gamma_{i+1}) = q(\gamma_{i+1}) = 1 \quad \text{for all } j \neq i, i+1.
\]

By Corollary 4.2 we have that \( H_{c_i} \in \text{LMod}(D) \). It follows from the fact that the set \( \{ H_{c_1}, \ldots, H_{c_{n-1}} \} \) generates \( \text{Mod}(D) \) that every mapping class lifts, that is, \( \text{LMod}(D) = \text{Mod}(D) \). As discussed at the beginning of Section 4 this is equivalent to showing that \( \text{LMod}(\Sigma, \mathcal{B}) = \text{Mod}(\Sigma, \mathcal{B}) \).

To prove the other direction we first assume that \( \Sigma \) has \( m > 1 \) boundary components. Let \( \{ \iota_i \}_{i=1}^{m-1} \) be a star in the fundamental groupoid and let \( S \) be an essential generating set of \( \pi_1(\Sigma, x_0) \) as in Lemma 4.4. Let \( c \in S \) and let \( T_c \) be the corresponding Dehn twist about a simple closed curve freely isotopic to \( c \). Without loss of generality we have that \( T_c(\iota_i) = c \iota_i \), as an element of the fundamental groupoid for some \( i \). If \( T_c \in \text{LMod}(\Sigma) \) then from Proposition 4.3 it follows that \( c \in p_* \pi_1(\Sigma, x_0) \) and since \( c \in S \) was arbitrary we have that every element of \( S \) belongs to \( p_* \pi_1(\Sigma, x_0) \), hence the image of the quotient map is trivial. This is a contradiction and so we have shown that \( \Sigma \) must have a single boundary component.

We now assume that \( \Sigma \) has positive genus \( g \). Let \( S \) be a basic generating set of \( \pi_1(\Sigma, x_0) \), we denote by \( \gamma_1, \ldots, \gamma_n \in S \) the simple loops around the removed branch points. For all \( i \in \{1, \ldots, n\} \) we can find an element
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\[ f \in \text{Mod}(\Sigma^o) \] such that \( f_*(\gamma_i) = \gamma_1. \) It follows from Corollary 4.2 that \( q(\gamma_i) = qf_*(\gamma_i) = q(\gamma_1) \) for all \( i. \)

We label the 2g elements of \( S \) that do not bound a disk by \( a_1, b_1, \ldots, a_g, b_g. \) Each representative loop belongs to a different free isotopy class of non-separating simple closed curves and so \( a_i, b_i \) belong to the Mod(\( \Sigma^o \))-orbit of \( a_1 \) for all \( i = 1, \ldots, g. \) Similarly, as representative loops of the elements \( \gamma_1 b_1 a_1 \) and \( b_1 a_1 \) belong to free isotopy classes of non-separating simple closed curves we have that there exist \( [f], [h] \in \text{Mod}(\Sigma^o) \) such that \( f_*(a_1) = \gamma_1 b_1 a_1 \) and \( h_*(b_1) = b_1 a_1. \) We now have that

\[
q(b_1) = qh_*(b_1) = q(b_1 a_1) = q(b_1)q(a_1).
\]

Hence \( q(a_1), \) and therefore each \( q(a_i) \) and \( q(b_i), \) is equal to the identity element of the deck group \( D \) for any value of \( i. \) Furthermore, it follows that

\[
q(\gamma_1) = q(\gamma_1)q(b_1)q(a_1) = q(\gamma_1 b_1 a_1) = qf_*(a_1) = q(a_1).
\]

Again, this implies that the image of the quotient map is trivial, which is a contradiction. The genus of \( \Sigma \) must therefore be zero and, as shown above, \( \Sigma \) has a single boundary component, that is, \( \Sigma \) is a disk. We have already shown that \( q(\gamma_i) = q(\gamma_1) \) for all \( i = 1, \ldots, n \) and so it follows that \( p: \tilde{\Sigma} \to \Sigma \) is a Burau cover.

We would also like to show that LMod(\( \Sigma, \mathcal{B} \)) is always finite-index in \( \text{Mod}(\Sigma, \mathcal{B}). \) Recall the definitions of the fundamental groupoids \( \mathcal{H} \) and \( \mathcal{G} \) from the beginning of Section 4. Let \( \Psi: \text{Mod}(\Sigma, \mathcal{B}) \to \text{PAut}(\mathcal{G}) \) be the injective homomorphism given by the action of \( \text{Mod}(\Sigma, \mathcal{B}) \) on the fundamental groupoid \( \mathcal{G}. \) By Theorem 4.1 it follows that \( \Psi(\text{LMod}(\Sigma, \mathcal{B})) \) is contained in \( \text{LAut}_H \mathcal{G}. \) We have

\[
[\text{Mod}(\Sigma, \mathcal{B}), \text{LMod}(\Sigma, \mathcal{B})] = [\Psi(\text{Mod}(\Sigma, \mathcal{B})), \Psi(\text{LMod}(\Sigma, \mathcal{B}))]
\leq [\text{PAut}(\mathcal{G}): \text{LAut}_H \mathcal{G}]
< \infty,
\]

where the last inequality is by Lemma 3.3.

While we have shown that there are infinitely many covering spaces with the property that \( \text{LMod}(\Sigma, \mathcal{B}) = \text{Mod}(\Sigma, \mathcal{B}), \) it is clear that this occurs only in a distinct minority of cases. We will see in the next section that the conditions for a covering space to satisfy \( \text{SMod}(\tilde{\Sigma}) = \text{SMod}(\Sigma) \) are even more severe.

4.3. The case where everything is symmetric. In this section we prove Theorem 1.2. In particular, we show that the symmetric mapping class group \( \text{SMod}(\Sigma) \) coincides with the mapping class group \( \text{Mod}(\Sigma) \) in a very small number of cases. To prove the result we make use of the mapping class group action on homology. Recall that the Lefschetz fixed point theorem for
smooth manifolds states

\[ \sum_{p \in \text{fix}(f)} i(f, p) = \sum_{i=0}^{\infty} (-1)^i \text{tr}(f_* : H_i(\Sigma; \mathbb{Q}) \to H_i(\Sigma; \mathbb{Q})). \]

where \( i(f, p) \) is the index of the fixed point \( p \) of the homeomorphism \( f \). We will apply this result to our context of surfaces with boundary.

**Lemma 4.5.** Let \( \Sigma \) be an oriented surface with boundary. Let \( f \) be a finite-order, orientation-preserving homeomorphism of \( \Sigma \). Then the fixed points of \( f \) are isolated and the number of fixed points is equal to

\[ 1 - \text{tr}(f_* : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z})). \]

**Proof.** We will first prove that the fixed points are isolated. Let \( f \) have order \( k \) and let \( \mu \) be a Riemannian metric on \( \Sigma \). Define the Riemannian metric \( \overline{\mu} := k \sum_{i=1}^{k} (f^k)^* \mu \).

Then \( f^* \overline{\mu} = \overline{\mu} \) and so \( f \) is an isometry. Since \( f \) is orientation-preserving its fixed points must be isolated. Let \( p \in \Sigma \) be such a fixed point and let \( T_p \Sigma \cong \mathbb{R}^2 \) be the tangent space. Now, all orientation-preserving isometries of \( \mathbb{R}^2 \) that fix the origin are rotations about the origin. We therefore have that \( f \) induces a rotation \( T_p \Sigma \to T_p \Sigma \) and so \( i(f, p) = 1 \).

For a surface with boundary, \( H_i(\Sigma; \mathbb{Q}) \cong \{0\} \) for all \( i \geq 2 \). Furthermore, \( H_0(\Sigma; \mathbb{Q}) \cong \mathbb{Q} \) and \( f_* : H_0(\Sigma; \mathbb{Q}) \to H_0(\Sigma; \mathbb{Q}) \) is the identity map. It follows that \( \text{tr}(f_* : H_0(\Sigma; \mathbb{Q}) \to H_0(\Sigma; \mathbb{Q})) = 1 \). Note that since the first homology group is free abelian we may replace the coefficients with \( \mathbb{Z} \). Finally, since the index of each fixed point is 1, we have that the number of fixed points is equal to

\[ 1 - \text{tr}(f_* : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z})). \]

completing the proof.

**Corollary 4.6.** Suppose \( \Sigma \) is an orientable surface of genus \( g \) with \( m \geq 1 \) boundary components other than a disk or an annulus. Let \( f \in \text{Homeo}^+(\Sigma) \) be a finite-order, orientation-preserving homeomorphism of \( \Sigma \). Then \( f \) acts non-trivially on \( H_1(\Sigma; \mathbb{Z}) \).

**Proof.** If \( f \) acts trivially on \( H_1(\Sigma; \mathbb{Z}) \) then \( \text{tr}(f_* : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z})) = 2g + m - 1 \) by choice of a natural basis of \( H_1(\Sigma; \mathbb{Z}) \). By Lemma 4.5, we must have \( 1 - (2g + m - 1) \geq 0 \) and so \( 2g + m \leq 2 \). This only occurs when \( g = 0 \) and \( m = 1, 2 \), or equivalently, when \( \Sigma \) is a disk an annulus.

The next result shows that a hyperelliptic involution has a unique action on homology up to conjugation. The proof follows from Lemma 4.5 and an argument similar to that of \[11\] Proposition 7.15.
Lemma 4.7. Let \( \Sigma \) be a surface of genus \( g \geq 1 \) with a single boundary component. Suppose \( f_1, f_2 \in \text{Homeo}^+(\Sigma) \) are order 2 homeomorphisms such that \((f_1)_* = (f_2)_* = -I : H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z})\). Then \( f_1 \) and \( f_2 \) are conjugate in \( \text{Homeo}^+(\Sigma) \).

Throughout the proof of Theorem 1.2 we will repeatedly use the fact that if \( c \) is an isotopy class of simple closed curves then every power of the Dehn twist \( T_c \) is an element of \( \text{SMod}(\tilde{\Sigma}) \) if and only if \( d(c) = c \) for all \( d \in D \), where \( D \) is the deck group.

Proof of Theorem 1.2. We start by proving that \( \text{SMod}(\tilde{\Sigma}) = \text{Mod}(\tilde{\Sigma}) \) in the three cases stated in the theorem. First, if \( \tilde{\Sigma} \) is a disk then \( \text{SMod}(\tilde{\Sigma}) = \text{Mod}(\tilde{\Sigma}) \) trivially. If \( \tilde{\Sigma} \) is an annulus, let \( c \) be the unique unoriented isotopy class of an essential simple closed curve. For every homeomorphism \( f \) of \( \tilde{\Sigma} \) we have that \( f(c) = c \). Since \( \text{Mod}(\tilde{\Sigma}) = \langle T_c \rangle \) it follows that \( \text{SMod}(\tilde{\Sigma}) = \text{Mod}(\tilde{\Sigma}) \). Finally, suppose \( \tilde{\Sigma} \) is a torus with a single boundary component, and let \( \iota \in \text{Homeo}^+(\tilde{\Sigma}) \) be a hyperelliptic involution. There exist two simple closed curves \( a \) and \( b \) whose isotopy classes are fixed by \( \iota \) such that the Dehn twists \( T_a, T_b \) generate \( \text{Mod}(\tilde{\Sigma}) \). Therefore we have that \( \text{SMod}(\tilde{\Sigma}) = \text{Mod}(\tilde{\Sigma}) \).

Conversely, suppose \( \text{SMod}(\tilde{\Sigma}) = \text{Mod}(\tilde{\Sigma}) \) and \( \tilde{\Sigma} \) is neither a disk nor an annulus. Suppose \( \tilde{\Sigma} \) is a surface of genus \( g \) with \( m \geq 1 \) boundary components. There is a generating set \( S = \{a_1, \ldots, a_{2g}, x_0, \ldots, x_{m-1}\} \) of \( H_1(\tilde{\Sigma}; \mathbb{Z}) \) where each generator \( a_i \) is represented by an essential simple closed curve and each \( x_i \) is the homology class of a curve isotopic to a boundary component.

Let \( d \) be a non-trivial element of the deck group \( D \). It must be that \( d \) preserves the unoriented isotopy class of every essential simple closed curve and so we see that \( d_* : H_1(\tilde{\Sigma}; \mathbb{Z}) \to H_1(\tilde{\Sigma}; \mathbb{Z}) \) is given by the diagonal matrix

\[
\begin{bmatrix}
\epsilon_1 \\
\vdots \\
\epsilon_{2g+m-1}
\end{bmatrix}
\]

with respect to the generating set \( S \), where \( \epsilon_i = \pm 1 \) for all \( i \). However, since \( d \) is orientation-preserving, it must preserve the orientation of every boundary component, therefore \( \epsilon_i = 1 \) for all \( i > 2g \).

We now argue that \( \tilde{\Sigma} \) must have exactly one boundary component. Since \( d \) is not the identity, Lemma 4.6 implies that there must be at least one \( i \in \{1, \ldots, 2g\} \) such that \( \epsilon_i = -1 \). If \( m \geq 2 \) then there is at least one element in \( S \) that is the homology class of a boundary component. Consider the homology classes \( x_0 + a_i \) and \( x_0 - a_i \). One of these is the homology class of an essential simple closed curve \( c \). Since \( d_*(x_0 \pm a_i) = x_0 \mp a_i \) we have that \( d_*(c) \neq \pm c \) and so the unoriented isotopy class of \( c \) is not preserved by \( d \). Therefore \( T_c \notin \text{SMod}(\tilde{\Sigma}) \) and so \( \text{SMod}(\tilde{\Sigma}) \neq \text{Mod}(\tilde{\Sigma}) \), a contradiction. It follows then that \( m = 1 \).
The next step is to show $d$ is a hyperelliptic involution. Suppose not, then by Lemma 4.6 there are $i, j \in \{1, \ldots, 2g\}$ such that $\epsilon_i = 1$ and $\epsilon_j = -1$. Similar to the above argument, we can find a curve $c$ such that $T_c \notin \text{SMO}d(\tilde{\Sigma})$. This implies that

$$d_* = -I : H_1(\tilde{\Sigma}; \mathbb{Z}) \to H_1(\tilde{\Sigma}; \mathbb{Z})$$

for all non-trivial $d \in D$. By Lemma 4.7 we may conclude that $D$ is generated by a hyperelliptic involution $\iota$. Finally, if $g \geq 2$ then we can find a curve that is not fixed by $\iota$ (see Figure 4), completing the proof of the first statement.

If $\text{SMO}d(\tilde{\Sigma}) \neq \text{Mod}(\tilde{\Sigma})$ then we may choose a Dehn twist $T_c \notin \text{SMO}d(\tilde{\Sigma})$ and note that each power belongs to a different coset of $\text{SMO}d(\tilde{\Sigma})$ in $\text{Mod}(\tilde{\Sigma})$. Since Dehn twists have infinite order, it follows that $\text{SMO}d(\tilde{\Sigma})$ is infinite-index in $\text{Mod}(\tilde{\Sigma})$. ■

Remark 4.8. Suppose $p : \tilde{\Sigma} \to \Sigma$ is a finite-sheeted, regular, possibly branched cover of surfaces without boundary with deck group $D$. Combining the proof of Theorem 4 in [4] with the Neilsen realisation theorem for finite groups [16] allows one to conclude that $\text{SMO}d(\tilde{\Sigma})$ is the normaliser of $D$ in $\text{Mod}(\tilde{\Sigma})$.

When the surfaces in question have boundary, then $D$ is not a subgroup of $\text{Mod}(\tilde{\Sigma})$. However, $D$ and $\text{Mod}(\Sigma)$ are both subgroups of $\text{Aut}(\mathcal{K})$, where $\mathcal{K}$ is the groupoid defined in Section 4.1.

In light of both the normaliser result just stated for closed surfaces, and Theorem 4.1 for $\text{LMO}d(\Sigma, \mathcal{B})$, we conjecture that $\text{SMO}d(\Sigma) = \{ [f] \in \text{Mod}(\Sigma) : f_* \in \text{SAut}(\mathcal{K}) \}$. Unfortunately, a proof seems out of reach at the moment.

5. Non-geometric embeddings of braid groups

In this section we will investigate a family of injective homomorphisms from the braid group to mapping class groups. We will refer to such a homomorphism as a braid group embedding. We first recall the definition of the Burau covers from Section 4.
Burau covers. Pick a point \( x \in \partial D_n \) and let \( \gamma_i \in \pi_1(D_n, x) \) be the homotopy class of a loop surrounding solely the \( i \)th puncture anti-clockwise. Then \( \{\gamma_1, \ldots, \gamma_n\} \) generates \( \pi_1(D_n, x) \). For each \( k \geq 2 \), define a homomorphism
\[
q_k : \pi_1(D_n, x) \to \mathbb{Z}/k\mathbb{Z} \\
\gamma_i \mapsto 1
\]
for all \( i \). The kernel of \( q_k \) determines a \( k \)-sheeted cyclic branched cover \( p_k : \Sigma \to \Sigma_1 \) branched at \( n \) points. Here \( m = \gcd(n, k) \) and \( g = 1 - \frac{1}{2}(k + n - nk + m) \).

In Theorem 1.1 it was shown that \( \text{LMod}(\Sigma, \mathcal{B}) = \text{Mod}(\Sigma, \mathcal{B}) \) if and only if \( \Sigma \) is a disk and \( p_k : \Sigma \to \Sigma_1 \) is a \( k \)-sheeted Burau cover. We can therefore define the following braid group embedding;
\[
\beta_k : B_n \cong \text{Mod}(\Sigma_1, \mathcal{B}) = \text{LMod}(\Sigma_1, \mathcal{B}) \cong \text{SMod}(\Sigma_1) \hookrightarrow \text{Mod}(\Sigma_1).
\]
The first isomorphism is well known, the equality comes from Theorem 1.1, and the second isomorphism is a consequence of the Birman-Hilden theorem.

Let \( \{\sigma_1, \ldots, \sigma_{n-1}\} \) be the standard generators of \( B_n \). It is known that \( \beta_2(\sigma_i) = T_{c_i} \) where \( c_i \) is some non-separating curve for all \( i \in \{1, \ldots, n-1\} \). Furthermore, the deck group \( D \cong \mathbb{Z}/2\mathbb{Z} \) is generated by a hyperelliptic involution [11, Section 9.4].

In this section we will describe the image of the standard braid generators under \( \beta_k \) where \( k \geq 3 \). In particular we show that \( \beta_k \) is a non-geometric embedding of the braid group, that is, \( \beta_k(\sigma_i) \) is not a Dehn twist. In order to describe the image of a single braid generator it suffices to consider the embeddings
\[
\beta_{2g+1} : B_2 \hookrightarrow \text{Mod}(\Sigma_g^1) \quad \text{and} \quad \beta_{2g+2} : B_2 \hookrightarrow \text{Mod}(\Sigma_g^2),
\]
for integers \( g > 0 \). In other words, we will study the Burau covers
\[
p_{2g+1} : \Sigma_g^1 \to \Sigma_0^1 \quad \text{and} \quad p_{2g+2} : \Sigma_g^2 \to \Sigma_0^1,
\]
in each case branched at two points. We will deal with the two cases separately although the techniques used in each case are similar.

5.1. Od Burau. First we consider the braid group embedding \( \beta_{2g+1} \) given above. We will define an element \( N \) of \( \text{Mod}(\Sigma_g^1) \) and then prove that the isomorphism
\[
\Pi : \text{SMod}(\Sigma_g^1) \to \text{LMod}(\Sigma_0^1, \mathcal{B})
\]
sends \( N \) to the standard generator of \( \text{Mod}(\Sigma_0^1, \mathcal{B}) \). Recall that we can represent a closed surface of genus \( g \) by a regular \((4g + 2)\)-gon, centred at the origin, with opposite sides identified. If we remove an open disk about the centre we arrive at a representation of \( \Sigma_g^1 \). Label this representation \( P \) and let \( \iota \) be the anti-clockwise rotation of \( P \) about its centre by \( 2\pi/(2g+1) \). The two unique vertices of \( P \) are fixed by \( \iota \). We see that the quotient space \( \Sigma_g^1/\langle \iota \rangle \) is homeomorphic to \( \Sigma_0^1 \), and the quotient map is a covering map branched at two points. Furthermore, around both fixed points \( \iota \) is locally a rotation
by $2\pi/(2g + 1)$ anti-clockwise, therefore the associated covering space is the $(2g + 1)$-sheeted Burau cover of $\Sigma^1_g$ with deck group $D \cong \mathbb{Z}/(2g + 1)\mathbb{Z}$.

We will write $p_{2g+1}: \Sigma^1_g \rightarrow D_2$ for the associated unbranched cover. Let $x \in \partial D_2$ and let $a$ and $b$ be elements of $\pi_1(D_2, x)$ such that $a$ is represented by a loop that surrounds a single marked point and $b$ is represented by a loop isotopic to $\partial D_2$ as in Figure 6(i).

The elements $a$ and $b$ generate $\pi_1(D_2, x)$. Denote the full preimage $p_{2g+1}^{-1}(x)$ by $\{\bar{x}_i\}$ indexed by elements of $\mathbb{Z}/(2g + 1)\mathbb{Z}$ such that $\iota(\bar{x}_i) = \bar{x}_{i+1}$. Similarly we define $\{p_{2g+1}^{-1}(a) = \{a_i\} \text{ and } (p_{2g+1})^{-1}(b) = \{b_i\}\}$ such that $\iota_*(a_i) = a_{i+1}$ and $\iota_*(b_i) = b_{i+1}$, see Figure 6(ii). The set $\{a_i, b_i\}$, indexed by elements of $\mathbb{Z}/(2g + 1)\mathbb{Z}$, generates the fundamental groupoid $\pi_1(\Sigma^1_g, \{\bar{x}_i\})$, a fact which follows from Lemma 3.4.

The odd notch. We let $N$ denote the mapping class in $\text{Mod}(\Sigma^1_g)$ represented by the homeomorphism that rotates the edges of $P$ by $2\pi/(4g + 2)$ and fixes the single boundary component at the centre. See Figure 7 for an image of $N$ when $g = 1$.

In the following lemma we will write $H$ for the half twist in $\text{Mod}(\Sigma^1_0, \mathcal{B})$ and for the induced automorphism of $\pi_1(D_2, x)$ such that $H(a) = ba^{-1}$.

**Lemma 5.1.** Given the Burau cover $p_{2g+1}: \Sigma^1_g \rightarrow \Sigma^1_0$ the half twist $H \in \text{Mod}(\Sigma^1_0, \mathcal{B})$ lifts to the mapping class $N \in \text{Mod}(\Sigma^1_g)$.

**Proof.** Let $\mathcal{G} = \pi_1(D_2, x)$ and let $\mathcal{K}$ be the fundamental groupoid $\pi_1(\Sigma^1_g, \{\bar{x}_i\})$. We will abuse notation by writing $N$ for its image in $\text{Aut}(\mathcal{K})$ and $H$ for its image in $\text{Aut}(\mathcal{G})$ under the injective natural homomorphisms. We need to show that $N \in \text{SAut}(\mathcal{K})$ as defined in Section 3.3. Since the deck group $D$ is generated by $\iota$, this is equivalent to showing that $N\iota = \iota N$ as automorphisms of $\mathcal{K}$. It can be seen from Figure 7 that $N(a_i) = b_i a_{i+1}^{-1}$ and that $N(b_i) = b_i$.

It follows then that

\[ N\iota(a_i) = N(a_{i+1}) = b_{i+1} a_{i+2+g}^{-1} = \iota(b_i a_{i+1+g}^{-1}) = \iota N(a_i), \quad \text{and} \]

\[ N\iota(b_i) = N(b_{i+1}) = b_{i+1} = \iota(b_i) = \iota N(b_i). \]

Since the set $\{a_i, b_i\}$ generates the fundamental groupoid, we are done.
We will now show that the image of \( N \) in \( \text{LAut}_H(G) \) under the isomorphism \( \Pi \) of Lemma 3.6 is equal to \( H \). This makes sense since \( \text{LMod}(\Sigma_1^0, B) = \text{Mod}(\Sigma_1^0, B) \) and so from Theorem 4.1 we conclude that \( H \in \text{LAut}_H(G) \). We now have
\[
\Pi(N)(a) = p_*N(a_i) = p_*(b_ia_{i+1+g}) = ba^{-1} = H(a), \quad \text{and}
\]
\[
\Pi(N)(b) = p_*N(b_i) = p_*(b_i) = b = H(b).
\]
So \( \Pi(N) = H \) and since the diagram
\[
\text{SMod}(\Sigma_g^1) \xrightarrow{\Psi} \text{SAut}(\mathcal{K}) \xrightarrow{\Pi} \text{Mod}(\Sigma_0^1, B) \xrightarrow{\Psi} \text{LAut}_H(G)
\]
commutes, the mapping class \( N \) is indeed the lift of the half twist \( H \).

5.2. Even Burau. We will now move on to the braid group embedding \( \beta_{2g+2} : B_2 \hookrightarrow \text{Mod}(\Sigma_g^2) \) given above. As in the odd case we will define an element of \( \text{Mod}(\Sigma_g^2) \) and then prove that it is the lift of a half twist \( H \). We take \( H \) to be the half twist such that \( H_*(a) = ba^{-1} \) for \( a, b \in \pi_1(D_2, x) \) as before. We want to find a polygonal representation of \( \Sigma_g^2 \). We take a regular \((4g + 2)\)-gon with opposite sides identified. This time, we remove two open disks as shown in Figure 5 and label the representation \( P \). We define an order \( 2g + 2 \) homeomorphism \( \iota \) as follows:
1. Cut \( P \) along a straight line connecting the top and bottom vertices and label the resulting \((2g + 2)\)-gons \( P_L \) and \( P_R \).
2. Rotate both \( P_L \) and \( P_R \) anti-clockwise by \( 2\pi/(2g + 2) \) and re-attach them along the straight line connecting top and bottom vertices.
3. Rotate \( P \) by \( \pi \).

While this homeomorphism of \( \Sigma_g^2 \) is substantially more complicated than the one described in the Section 5.1, it shares many properties. Both vertices of \( P \) are fixed by \( \iota \) however, this time, locally \( \iota \) is a clockwise rotation by
Figure 7. The mapping class $H \in \text{Mod}(D_2)$ and its lifts in the 3 and 4-sheeted Burau covers. Lemma 5.1 shows that $N$ is the lift of $H$.

2π/(2g + 2). It follows that the quotient space $\Sigma^2_g/\langle \iota \rangle$ is homeomorphic to $\Sigma^1_{0}$ and the associated covering space is the (2g + 2)-sheeted Burau cover of $\Sigma^1_{0}$ with deck group $D \cong \mathbb{Z}/(2g + 2)\mathbb{Z}$.

**The even notch.** We define $N \in \text{Mod}(\Sigma^2_g)$ to be the mapping class represented by the homeomorphism that rotates the edges of both $P_L$ and $P_R$ by $2\pi/(2g + 2)$ and fixes the boundary components. See Figure 7 for an image of $N$ when $g = 1$.

Using the same method as the proof of Lemma 5.1 we arrive at the following result.

**Lemma 5.2.** Given the Burau cover $p_{2g+2} : \Sigma^2_g \to \Sigma^1_{0}$ the half twist $H \in \text{Mod}(\Sigma^2_g, B)$ lifts to the mapping class $N \in \text{Mod}(\Sigma^1_{0})$.

The proof of Lemma 5.2 is identical to the odd case, except that while $N(b_i) = b_i$ as before, we now have $N(a_i) = b_i a_i^{-1}$.

**5.3. Chain twists.** We will now describe the two maps defined in the previous section as products of Dehn twists. We will often abuse notation by referring to an isotopy class of curves by the name of a single representative curve.

**Chains.** Recall that a sequence of curves $\{c_1, c_2, \ldots, c_k\}$ is called a $k$-chain if $i(c_i, c_j) = 1$ if $j = i \pm 1$ and $i(c_i, c_j) = 0$ otherwise. If $k = 2g$ for some $g$ then the closed neighbourhood of $\cup c_i$ is a subsurface homeomorphic to $\Sigma^1_g$ with boundary component isotopic to the curve $d$. Furthermore, if $k = 2g + 1$ then the closed neighbourhood of $\cup c_i$ is a subsurface homeomorphic to $\Sigma^2_g$ with boundary components $d_1$ and $d_2$ (see Figure 8). By considering the braid group embedding $\beta_2 : B_{k+1} \hookrightarrow \tilde{\Sigma}$ it can be shown that

$$(T_{c_1}T_{c_2} \cdots T_{c_{2g}})^{4g+2} = T_d$$

and

$$(T_{c_1}T_{c_2} \cdots T_{c_{2g+1}})^{2g+2} = T_{d_1}T_{d_2},$$

see Farb-Margalit [11] Section 4.4 for more details. Given a $k$-chain $C = \{c_1, c_2, \ldots, c_k\}$, we call the product $T_C := T_{c_1}T_{c_2} \cdots T_{c_k}$ a $k$-chain twist (or
a chain twist). Now, let $p_{2g+1} : \Sigma_g^1 \to \Sigma_0^1$ be a Burau cover and let $N$ be the lift of the half twist as discussed in Lemma 5.1. If the curve $d$ is isotopic to the boundary of $\Sigma_g^1$ then it is clear that

$$N^{4g+2} = T_d = (T_C)^{4g+2}$$

for any $2g$-chain $C$ in $\Sigma_g^1$. Similarly, suppose $p_{2g+2} : \Sigma_g^2 \to \Sigma_0^1$ is a Burau cover and $N$ is the lift of the half twist discussed in Lemma 5.2. If the curves $d_1, d_2$ are isotopic to the boundary components of $\Sigma_g^2$ then we have

$$N^{2g+2} = T_{d_1}T_{d_2} = (T_C)^{2g+2}$$

for any $(2g+1)$-chain $C$ in $\Sigma_g^2$. In Proposition 5.3 we will prove that as well as having the same power as a chain twist, the notch $N$ is in fact equal to a chain twist in both the odd and even cases, proving Theorem 1.3. This implies that there exist chains $A, B$ (of any length) whose corresponding chain twists $T_A, T_B$ satisfy the braid relation.

In Section 5.4 we will give the explicit combinatorial data required for two $k$-chains to admit chain twists satisfying the braid relation. Furthermore, we show that this data encodes the braid relation on the level of Dehn twists.

**Proposition 5.3.** Given a Burau cover $p_k : \tilde{\Sigma} \to \Sigma_0^1$ the half twist $H \in \text{Mod}(\Sigma_0^1, B)$ lifts to a $(k-1)$-chain twist.

**Proof.** Given Lemmas 5.1 and 5.2 we need only show that the mapping class $N \in \text{Mod}(\Sigma_g^m)$ is equal to a $(k-1)$-chain twist $T_C$, for some $(k-1)$-chain $C$. To do this we will show that the image of $T_C$ and $N$ are equal in the group $\text{Aut}(K)$ where $K$ is a fundamental groupoid of $\Sigma_g^m$ with basepoints on all boundary components.

To that end, let $k = 2g+1$ and let $K$ be the fundamental groupoid of $\Sigma_g^1$ generated by the set depicted in Figure 6(ii). Note that this set was also used to define a fundamental groupoid of $\Sigma_g^{10}$. In this setting however, the vertices of the polygon are not punctures. Furthermore, in order to facilitate the proof we change the indexing so that $a_{2i} := a_i$. We define the curve $c_0$ uniquely by the groupoid element $a_0a_1b_0^{-1}$. We then define $c_i$ to be $N^i(c_0)$ for all $i \in \mathbb{Z}/(2g+1)\mathbb{Z}$, see Figure 9(ii). Now, we define the $2g$-chain $C := \{c_1, \ldots, c_{2g}\}$. In fact, we may choose $C$ to be any $2g$-chain consisting of the curves $c_i$. Now, by construction it can be seen that
Figure 9. The 2-chain and 3-chain shown have corresponding chain twists equal to the notch $N$ coming from the 3-fold and 4-fold Burau covers respectively.

\[ T_{c_i}(\alpha_i) = N(\alpha_i), \]
\[ T_{c_j}(\alpha_i) = \alpha_i \text{ for } j > i, \]
\[ T_{c_j}N(\alpha_i) = N(\alpha_i) \text{ for } j < i. \]

It follows then that for any $i \in \{1, \ldots, 2g\}$ we have

\[ T_C(\alpha_i) = T_{c_1}T_{c_2} \cdots T_{c_{2g}}(\alpha_i) \]
\[ = T_{c_1}T_{c_2} \cdots T_{c_i}(\alpha_i) \]
\[ = T_{c_1}T_{c_2} \cdots T_{c_{i-1}}N(\alpha_i) \]
\[ = N(\alpha_i). \]

It remains to show that $T_C(\alpha_0) = N(\alpha_0)$. The curve $c_{2g}$ intersects the representative of $\alpha_0$ once and by definition $i(c_i, c_{i+1}) = 1$. It therefore follows that the product $T_C$ adds a copy of each of the $c_i$ to $\alpha_0$. It is shown in Figure 10 that this is in fact equal to $N(\alpha_0)$ in the case where $g = 1$, and indeed, this is true for any $g > 0$. It follows that $T_C = N$ as groupoid automorphisms, and hence, they are equal as elements of $\text{Mod}(\Sigma_g^0)$. The case where $k = 2g + 2$ is similar to that of $2g + 1$. We can use the groupoid generators shown in Figure 6(iii) and the details are left to the reader.

Note that Proposition 5.3 proves Theorem 1.3 in that it completely determines the image of each braid group generator $\sigma_i$ by the homomorphism $\beta_k$.

5.4. Intersection data. In Section 5.3 we saw that half twists lift to $(k-1)$-chain twists with respect to any $k$-sheeted Burau cover where $k \geq 3$. We will now explicitly describe a sufficient combinatorial condition for two chains that implies their chain twists satisfy the braid relation. We will assume that $k \geq 3$ for the remainder of this section.

Bracelets. Let $C = \{c_1, \ldots, c_{k-1}\}$ be a $(k-1)$-chain and let $c_0 = T_C(c_{k-1})$. We call the set $\{c_i : i \in \mathbb{Z}/k\mathbb{Z}\}$ a $k$-bracelet (or a bracelet). Such a bracelet is called the bracelet completion of the chain $C$. 
Figure 10. All arcs on the bottom row of hexagons are isotopic to the arc shown in the hexagon on the top right. The arcs $T_c(\alpha_0)$ and $N(\alpha_0)$ are isotopic and so are equal as elements of the fundamental groupoid.

**Lemma 5.4.** Let $C = \{c_1, \ldots, c_{k-1}\}$ be a chain and $\{c_i : i \in \mathbb{Z}/k\mathbb{Z}\}$ the bracelet completion of $C$. Then

1. $i(c_i, c_j) = \begin{cases} 1 & \text{if } i = j - 1, j + 1 \\ 0 & \text{otherwise} \end{cases}$
2. $T_{c_i}T_{c_{i+1}} \cdots T_{c_{i-2}} = T_{c_{i+1}}T_{c_{i+2}} \cdots T_{c_{i-1}}$ for all $i \in \mathbb{Z}/k\mathbb{Z}$.

**Proof.** Note that $T_{c_0} = T_{c_1} \cdots T_{c_{k-2}}T_{c_{k-2}}^{-1} \cdots T_{c_1}^{-1}$. Any solution to the word problem for the braid group (Dehornoy’s handle reduction [9] for example) can be used to show $[T_{c_0}, T_{c_i}] = 1$ if $i \neq 1, k - 1$ and $T_{c_0}T_{c_i}T_{c_0} = T_{c_i}T_{c_0}T_{c_i}$ if $i = 1, k - 1$. This proves property (i).

For (ii), note $T_{c_0} \cdots T_{c_{k-2}} = T_{c_1} \cdots T_{c_{k-1}}$ from the definition of $c_0$. Suppose now $T_{c_j} \cdots T_{c_{j-2}} = T_{c_{j+1}} \cdots T_{c_{j-1}}$ for some $j \in \mathbb{Z}/k\mathbb{Z}$. Then

\[
T_{c_{j+2}} \cdots T_{c_j} = T_{c_{j+2}}^{-1} T_{c_{j+1}} T_{c_{j+2}} \cdots T_{c_{j-1}} T_{c_j} \\
= T_{c_{j+2}} T_{c_{j+1}} T_{c_{j+2}} \cdots T_{c_{j-1}} T_{c_j} \\
= T_{c_j} T_{c_{j+1}} T_{c_j}^{-1} T_{c_{j+2}} \cdots T_{c_{j-1}} T_{c_j} \\
= T_{c_j} \cdots T_{c_{j-2}}
\]

completing the proof. ■

Note that property (ii) in Lemma 5.4 implies that a $k$-bracelet is the completion of any of the $(k-1)$-chains obtained by deleting a curve. Abusing notation, suppose $C$ is a $k$-bracelet. In light of this fact, define the **bracelet twist** $T_C$ as the chain twist about any of the $(k-1)$-chains obtained by deleting a curve from $C$.

**Mesh intersection.** Let $A = \{a_i : i \in \mathbb{Z}/k\mathbb{Z}\}$ and $B = \{b_j : j \in \mathbb{Z}/k\mathbb{Z}\}$ be two $k$-bracelets. We say $A$ and $B$ have mesh intersection if there exists
\[ t \in \mathbb{Z}/k\mathbb{Z} \] such that
\[ i(a_i, b_{j+t}) = \begin{cases} 1 & \text{if } i = j, j+1 \\ 0 & \text{otherwise.} \end{cases} \]

In practice, we may simply relabel the curves in \( \mathcal{B} \) and assume \( t = 0 \).

Fix \( k \geq 3 \) and let \( \beta_k : B_3 \to \text{Mod}(\Sigma_g^m) \) be the embedding of the braid group arising from the \( k \)-sheeted Burau cover. Proposition 5.3 shows that each standard generator is sent to a \((k-1)\)-chain twist. The proof proceeds by first constructing a set of \( k \) curves, and then arbitrarily discarding one. The set of \( k \) curves constructed is in fact the completion of the \((k-1)\)-chain. Furthermore, it can be checked that \( \beta_k \) sends the two standard generators to bracelet twists about bracelets with mesh intersection. In fact, in the discussion following the statement of Proposition 5.6 we will see that if \( A \) and \( B \) are two bracelets with mesh intersection such that all \( 2k \) curves are distinct, then there exists a Burau cover that lifts two half twists satisfying a braid relation to \( T_A \) and \( T_B \). It follows from Theorem 1.1 that \( T_A \) and \( T_B \) satisfy the braid relation. By leveraging the algebraic properties of Dehn twists, we may arrive at the same conclusion without mention of such a covering space.

**Theorem 5.5.** If two \( k \)-bracelets \( A \) and \( B \) have mesh intersection then \( T_A T_B T_A = T_B T_A T_B \).

**Proof.** Be relabelling the curves in \( \mathcal{B} \), we may assume \( t = 0 \) in the definition of mesh intersection. We first show that \( T_A T_B T_A = T_B T_A T_B \) as follows;
\[
T_A T_B T_A = T_{a_{i+1}} \cdots T_{a_{i-1}} T_{b_i} \cdots T_{b_{i-2}} T_{a_i}
= T_{a_{i+1}} T_{b_i} T_{a_{i+2}} \cdots T_{a_{i-1}} T_{b_{i+1}} \cdots T_{b_{i-2}} T_{a_i}
= T_{a_{i+1}} T_{b_i} T_{a_{i+2}} \cdots T_{a_{i-1}} T_{b_{i+1}} \cdots T_{b_{i-2}}
= T_{a_{i+1}} T_{b_i} T_{a_{i+1}} \cdots T_{a_{i-1}} T_{b_{i+1}} \cdots T_{b_{i-2}}
= T_{b_i} T_{a_{i+1}} T_{b_i} T_{a_{i+2}} \cdots T_{a_{i-1}} T_{b_{i+1}} \cdots T_{b_{i-2}}
= T_{b_i} T_{a_{i+1}} T_{a_{i+2}} \cdots T_{a_{i-1}} T_{b_i} \cdots T_{b_{i-2}}
= T_{b_i} T_{A} T_{B}.
\]

The second and third equalities come from the intersection data of curves \( b_i \) and \( a_{i-1} \) respectively. The fourth equality comes from property (ii) of Lemma 5.4. The fifth and sixth equalities come from property (i) of Lemma 5.4 applied to \( \mathcal{B} \).

This allows us to achieve the braid relation as follows:
\[
T_A T_B T_A = T_A T_B T_{a_1} \cdots T_{a_k-2}
= T_{b_0} T_A T_B T_{a_1} \cdots T_{a_k-2}
= \cdots
= T_{b_0} T_{b_1} \cdots T_{b_{k-2}} T_A T_B
= T_B T_A T_B. \]
When \( k \geq 4 \), it can be shown that if two \( k \)-bracelets have mesh intersection, then all \( 2k \) curves in question are distinct. However, this is not the case when \( k = 3 \). Figure 11 shows two 3-bracelets \( A = \{a_0, c, a_2\} \) and \( B = \{b_0, b_1, c\} \) with mesh intersection such that \( a_1 = b_2 \).

**Intersection data for chains.** We now shift our attention to finding a sufficient combinatorial condition for two \((k-1)\)-chains \( A \) and \( B \) to have the property that their bracelet completions have mesh intersection. We will then be able to conclude, by Theorem 5.5, that the two chain twists \( T_A \) and \( T_B \) satisfy a braid relation.

Suppose \( \alpha_1, \alpha_2, \alpha_3 \) are curves on a surface in minimal position such that \( i(\alpha_i, \alpha_j) = 1 \) if \( i \neq j \). The graph given by the three curves defines two triangles and a hexagon on the surface. Suppose one of the triangles bounds a disk \( D \). We say the triple \((\alpha_1, \alpha_2, \alpha_3)\) bounds a positively oriented triangle if you can traverse \( \partial D \) in an anti-clockwise direction from the intersection point \( x \in \alpha_1 \cap \alpha_3 \) and travel along a segment of \( \alpha_1 \), then a segment of \( \alpha_2 \), then a segment of \( \alpha_3 \) in that order and return to \( x \). See Figure 12 for a local picture of three curves bounding a positively oriented triangle.

We say a triple \((c_1, c_2, c_3)\) of isotopy classes of curves bounds a positively oriented triangle if there exist representatives \( \gamma_i \) of \( c_i \) such that \((\gamma_1, \gamma_2, \gamma_3)\) bounds a positively oriented triangle.

Note that if \((c_1, c_2, c_3)\) bounds a positively oriented triangle and \( \sigma \in S_3 \) is a permutation, then \((c_{\sigma(1)}, c_{\sigma(2)}, c_{\sigma(3)})\) bounds a positively oriented triangle if and only if \( \sigma \) is an even permutation.

The importance of the definition of a triple bounding a positively oriented triangle is that if \((c_1, c_2, c_3)\) bounds a positively oriented triangle, then \( i(T_{c_1}(c_3), c_2) = 0 \). This can be seen in Figure 12.

**Proposition 5.6.** Suppose \( A = \{a_1, \ldots, a_{k-1}\} \) and \( B = \{b_1, \ldots, b_{k-1}\} \) are two \((k-1)\)-chains with the property that

(i) \( i(a_i, b_j) = \begin{cases} 1 & \text{if } i = j, j+1 \\ 0 & \text{otherwise} \end{cases}, \) and

(ii) The triples \((a_i, b_i, a_{i+1})\) and \((b_i, a_{i+1}, b_{i+1})\) bound positively oriented triangles for all \( i \in \{1, \ldots, k-2\} \).

Then the chain twists \( T_A \) and \( T_B \) satisfy a braid relation.
If each of the $2k-2$ curves in $A$ and $B$ are distinct then we may view them as depicted Figure 13.

Defining $\Sigma_g^m$ to be the regular neighbourhood of the curves and triangles it can be seen that $m = \gcd(3, k)$. Furthermore, an Euler characteristic argument shows that $g = k - 2$ if $m = 3$ and $g = k - 1$ if $m = 1$. These are precisely the values of $m$ and $g$ that give rise to a $k$-sheeted Burau cover $p_k : \Sigma_g^m \to \Sigma_1^1$ with three branch points. By using a variation of the change of coordinates principle (see [11, Section 1.3.2]) we may conclude that $T_A$ and $T_B$ are lifts of half twists that satisfy a braid relation. Hence from Theorem 1.1 we conclude that $T_A$ and $T_B$ satisfy a braid relation.

Unlike the discussion above, the following proof of Proposition 5.6 does not make use of the Birman-Hilden Theorem. As such, it provides a more intrinsic perspective of chain twists satisfying a braid relation, and deals with the case when the curves in $A$ and $B$ are not distinct.

**Proof of Proposition 5.6.** Let $a_0 = T_A(a_{k-1})$ and $b_0 = T_B(b_{k-1})$. To ease notation let $\Delta = T_{b_k} \cdots T_{b_{k-2}}$ and $\nabla = T_{b_2} \cdots T_{b_{k-1}}$. Note that $T_{a_0} = \Delta T_{b_{k-1}} \Delta^{-1} = \nabla^{-1} T_{b_1} \nabla$.

By Theorem 5.5 it suffices to show

\[
\begin{align*}
  i(a_0, b_j) &= \begin{cases} 1 & \text{if } j = 0, k - 1, \\
 0 & \text{otherwise} \end{cases} \quad \text{and} \quad i(a_i, b_0) &= \begin{cases} 1 & \text{if } i = 0, 1, \\
 0 & \text{otherwise}. \end{cases}
\end{align*}
\]

Let $i \neq 0, 1$. Since $(b_j, a_{j+1}, b_{j+1})$ bounds a positively triangle for all $j \in \{1, \ldots, k-2\}$, we have $i(T_{b_j}(b_{j+1}), a_{j+1}) = 0$ so $[T_{b_j} T_{b_{j+1}} T_{b_j}^{-1}, T_{a_{j+1}}] = 1$. Rearranging and relabelling we get

\[
T_{b_i}^{-1} T_{b_{i-1}}^{-1} T_{a_i} T_{b_{i-1}} T_{b_i} = T_{b_{i-1}}^{-1} T_{a_i} T_{b_{i-1}}^{-1}
\]
for all $i \neq 0, 1$. We have

$$[T_{b_0}, T_{a_i}] = \Delta T_{b_{k-1}} \Delta^{-1} T_{a_i} \Delta T_{b_{k-1}} \Delta^{-1} T_{a_i}$$

$$= \Delta T_{b_{k-1}} T_{b_{k-2}} \cdots T_{b_{i+1}} T_{b_i} \Delta^{-1} T_{b_{i-1}} T_{a_i} \Delta^{-1} T_{a_i}$$

$$= \Delta T_{b_{k-1}} \Delta^{-1} T_{b_{k-2}} \cdots T_{b_{i+1}} T_{b_i} \Delta^{-1} T_{b_{i-1}} T_{a_i} \Delta^{-1} T_{a_i}$$

$$= T_{b_1} \cdots T_{b_{i-1}} T_{b_i} T_{b_{i-1}} \Delta^{-1} T_{a_i} \Delta^{-1} T_{a_i} \Delta^{-1} T_{b_{i-1}} T_{b_i} \cdots T_{b_{k-2}} T_{b_{k-1}} \Delta^{-1} T_{a_i} \Delta^{-1} T_{a_i}$$

$$= T_{b_1} \cdots T_{b_{i-1}} T_{a_i} T_{b_{i-1}} T_{b_i} \cdots T_{b_{k-2}} T_{b_{k-1}} \Delta^{-1} T_{a_i} \Delta^{-1} T_{a_i}$$

$$= 1.$$  

Therefore $i(a_i, b_0) = 0$. When $i = 1$ we have

$$T_{a_1} T_{b_0} T_{a_1} = T_{a_1} \nabla^{-1} T_{b_1} \nabla T_{a_1}$$

$$= \nabla^{-1} T_{a_1} T_{b_1} T_{a_1} \nabla$$

$$= \nabla^{-1} T_{b_1} T_{a_1} T_{b_1} \nabla$$

$$= \nabla^{-1} T_{b_1} \nabla T_{a_1} \nabla^{-1} T_{b_1} \nabla$$

$$= T_{b_0} T_{a_1} T_{b_0}$$

so $i(a_1, b_0) = 1$. Similar arguments show $i(a_0, b_j) = 0$ for $j \neq 0, k - 1$ and $i(a_0, b_{k-1}) = 1$.

It remains to show $i(a_0, b_0) = 1$. We have

$$T_{a_0} T_{b_0} T_{a_0} = T_{a_0} \Delta T_{b_{k-1}} \Delta^{-1} T_{a_0}$$

$$= \Delta T_{a_0} T_{b_{k-1}} T_{a_0} \Delta^{-1}$$

$$= \Delta T_{b_{k-1}} T_{a_0} T_{b_{k-1}} \Delta^{-1}$$

$$= \Delta T_{b_{k-1}} \Delta^{-1} T_{a_0} \Delta T_{b_{k-1}} \Delta^{-1}$$

$$= T_{b_0} T_{a_0} T_{b_0}$$

completing the proof. ☐
See Figure 2 for two 3-chains on $\Sigma_3^1$ satisfying the conditions of Lemma 5.6. The positively oriented triangles are shaded in grey.

5.5. **Open Questions.** Here are a few natural questions relating to the braid group embeddings constructed above. Recall that for each $k \geq 3$ and $n \geq 2$ we have constructed an embedding $\beta_k : B_n \hookrightarrow \text{Mod}(\Sigma^m_g)$ arising from the $k$-sheeted Burau cover. Here, $m = \gcd(n, k)$ and $g = 1 - \frac{1}{2}(k+n+m-nk)$.

**Necessity of mesh intersection.** When two simple closed curves $a$ and $b$ on a surface intersect once, then $T_a$ and $T_b$ satisfy a braid relation. In fact, this condition is necessary. That is, $T_a T_b T_a = T_b T_a T_b$ if and only if $i(a, b) = 1$ (see [11, §3.5]). The next question asks the analogous question for chain twists.

**Question 5.7.** Suppose $A$ and $B$ are $k$-chains for $k \geq 3$, and let $T_A$ and $T_B$ be the corresponding chain twists. Is it true that if $T_A T_B T_A = T_B T_A T_B$, then the bracelet completions of $A$ and $B$ have mesh intersection?

**Automorphisms of free groups.** For a surface $\Sigma$ with non-empty boundary, there is a homomorphism $\text{Mod}(\Sigma) \to \text{Aut}(\pi_1(\Sigma))$ given by the action of $\text{Mod}(\Sigma)$ on the fundamental group of $\Sigma$ with a basepoint on the boundary. For a surface of genus $g$ and $m$ boundary components, $\pi_1(\Sigma^m_g) \cong F_{2g+m-1}$. Precomposing with the braid group embeddings above, we get an induced homomorphism from the braid group into the automorphism group of a free group.

**Question 5.8.** Let $k \geq 3$. For each $n \geq 2$ there is a homomorphism $\phi_{n,k} : B_n \to F_{(n-1)(k-1)}$. What can be said about this family of homomorphisms? Do they give rise to new embeddings of the braid group in $\text{Aut}(F_n)$?

**Triviality of the induced map on stable homology.** There is a geometric embedding $B_{2g} \hookrightarrow \text{Mod}(\Sigma^1_g)$ for each $g$. This family of embeddings gives a map from $B_\infty = \lim_{g \to \infty} B_{2g}$ to $\Gamma_\infty = \lim_{g \to \infty} \text{Mod}(\Sigma_g)$. In the 1980s J. Harer conjectured that the induced map on stable homology $H_*(B_\infty; \mathbb{Z}/2\mathbb{Z}) \to H_*(\Gamma_\infty; \mathbb{Z}/2\mathbb{Z})$ is trivial. The conjecture was proved by Song and Tillman in [24, Theorem 1.1]. A stronger version of Harer’s conjecture was proved for a large family of non-geometric embeddings of the braid group in [5].

**Question 5.9.** Fix $k > 3$. Is the map on stable homology $H_*(B_\infty; \mathbb{Z}/2\mathbb{Z}) \to H_*(\Gamma_\infty; \mathbb{Z}/2\mathbb{Z})$ induced by the embeddings $\beta_k : B_n \hookrightarrow \text{Mod}(\Sigma^m_g)$ trivial?

Note that this question is answered affirmatively for stable homology with any coefficients when $k = 3$ by Kim-Song [17, Theorem 3.4].

**Classifying braid embeddings.** There are now infinite families of non-geometric embeddings of braid groups in mapping class groups.

**Question 5.10.** Is there a classification of all possible conjugacy classes of embeddings of the braid group in the mapping class group?
The proof of Theorem 5.5 suggests a way to construct more examples as follows.

Suppose we have two subsets of mapping classes \( \{\phi_i\} \) and \( \{\theta_i\} \) indexed by \( \mathbb{Z}/k\mathbb{Z} \) such that
\[
\phi_i \phi_j = \begin{cases} 
\phi_j \phi_i & \text{if } j = i - 1, i + 1 \\
\phi_j \phi_i & \text{otherwise},
\end{cases}
\]
\[
\theta_i \theta_j = \begin{cases} 
\theta_j \theta_i & \text{if } j = i - 1, i + 1 \\
\theta_j \theta_i & \text{otherwise}.
\end{cases}
\]
Suppose further that the following relations are satisfied;
\[
\phi_i \theta_j = \begin{cases} 
\theta_j \phi_i & \text{if } i = j, j + 1 \\
\theta_j \phi_i & \text{otherwise},
\end{cases}
\]
and for any \( i \in \mathbb{Z}/k\mathbb{Z} \) we have
\[
\Phi := \phi_i \ldots \phi_{i-2} = \phi_{i+1} \ldots \phi_{i-1} \quad \text{and} \quad \Theta := \theta_i \ldots \theta_{i-2} = \theta_{i+1} \ldots \theta_{i-1}.
\]
Then the products \( \Phi \) and \( \Theta \) satisfy the braid relation, that is, \( \Phi \Theta \Phi = \Theta \Phi \Theta \).

We conjecture however, that this is only possible when each \( \phi_i \) and \( \theta_i \) is a Dehn twist and the corresponding sets of curves are bracelets with mesh intersection. There is no particular reason to assume otherwise, except to satisfy our own insatiable desire for pattern.

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