CRITERIA ON THE EXISTENCE AND STABILITY OF PULLBACK EXPONENTIAL ATTRACTORS AND THEIR APPLICATION TO NON-AUTONOMOUS KIRCHHOFF WAVE MODELS

ZHJIANG YANG* AND YANAN LI
School of Mathematics and Statistics, Zhengzhou University
No.100, Science Road, Zhengzhou 450001, China

(Communicated by Irena Lasiecka)

Abstract. In this paper, we are concerned with the existence and stability of pullback exponential attractors for a non-autonomous dynamical system. (i) We propose two new criteria for the discrete dynamical system and continuous one, respectively. (ii) By applying the criteria to the non-autonomous Kirchhoff wave models with structural damping and supercritical nonlinearity we construct a family of pullback exponential attractors which are stable with respect to perturbations.

1. Introduction. It is well known that pullback attractor and pullback exponential attractor are two basic concepts to study the longtime dynamics of infinite dimensional non-autonomous dynamical system. To be more precise, a process acting on the Banach space $E$ is a two-parametrical family of operators $\{U(t,\tau) : E \to E| t, \tau \in \mathbb{R}, t \geq \tau\}$ (or $t, \tau \in \mathbb{Z}$ for discrete time) satisfying

$$U(t,s)U(s,\tau) = U(t,\tau), \quad U(\tau,\tau) = I \text{ (identity operator)}, \quad t, s, \tau \in \mathbb{R}, \quad t \geq s \geq \tau.$$ 

A family of nonempty compact subsets $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ in $E$ is said to be a pullback attractor of the process $\{U(t,\tau)\}$ if it is invariant, i.e., $U(t,s)\mathcal{A}(s) = \mathcal{A}(t), \quad t \geq s$, and it pullback attracts all bounded subsets of $E$, i.e., for every bounded subset $D \subset E$ and $t \in \mathbb{R}$,

$$\lim_{s \to +\infty} \text{dist}_E\{U(t,t-s)D, \mathcal{A}(t)\} = 0. \quad (1)$$

Here, $\text{dist}_E\{\cdot, \cdot\}$ is the Hausdorff semidistance in $E$, i.e.,

$$\text{dist}_E\{A, B\} = \sup_{x \in A} \inf_{y \in B} \text{dist}_E\{x, y\}, \quad A, B \subset E.$$

Pullback attractor is usually used to describe the longtime behavior of a non-autonomous dynamical system (cf. [2, 10]). However, the pullback attractor may have some drawbacks: (i) the rate of convergence in (1) may be slow, which leads to the fact that it is difficult to estimate the pullback attracting rate in term of the physical parameters of the system; (ii) in many situations, one cannot show

* Corresponding author: Zhijian Yang.

2010 Mathematics Subject Classification. Primary: 37L15, 37L30; Secondary: 35B40, 35B41.

Key words and phrases. Pullback exponential attractor, stability with respect to perturbations, non-autonomous Kirchhoff wave equations, structural damping, supercritical nonlinearity.

This work is supported by National Natural Science Foundation of China (No.11671367).
the finiteness of the fractal dimension for the sections of pullback attractor, which results in that the pullback attractor may be unobservable in experiments or in numerical simulations.

In order to overcome these drawbacks, Efendiev et al [13] proposed the concept of pullback exponential attractor, which contains pullback attractor, pullback attracts every bounded subset at an exponential rate and is of finite fractal dimension.

**Definition 1.1.** A family of nonempty compact subsets \( \{ \mathcal{M}(t) \}_{t \in \mathbb{R}} \) is said to be a pullback exponential attractor of the process \( \{ U(t, \tau) \} \) acting on the Banach space \( E \) if

(i) it is semi-invariant, i.e., \( U(t, s) \mathcal{M}(s) \subset \mathcal{M}(t), t \geq s \);
(ii) the fractal dimension in \( E \) of the sections \( \mathcal{M}(t), t \in \mathbb{R} \) is uniformly bounded, i.e.,

\[
\sup_{t \in \mathbb{R}} \dim_f(\mathcal{M}(t), E) < +\infty,
\]

where \( \dim_f(A, E) = \limsup_{\epsilon \to 0} \frac{\ln N(A, \epsilon)}{\ln(1/\epsilon)} \) and \( N(A, \epsilon) \) denotes the cardinality of the minimal covering of the set \( A \) by the closed subsets of diameter \( \leq 2\epsilon \);
(iii) it pullback attracts every bounded subset \( B \) in \( E \) at an exponential rate, i.e.,

\[
\text{dist}_E \{ U(t, t-s)B, \mathcal{M}(t) \} \leq C(\| B \|_E) e^{-\beta s}, \tag{2}
\]

for some \( \beta > 0 \), where \( t, s \in \mathbb{R}, s \geq 0, \| B \|_E = \sup_{x \in B} \| x \|_E \).

Lately, Efendiev et al [14] once again gave more general definitions on non-autonomous dynamical system and pullback exponential attractor which will be used in the present paper.

**Definition 1.2.** [14] Let \( E \) be a Banach space with the norm \( \| \cdot \|_E \), \( M \) be a subset of \( E \), which is a metric space equipped with the distance \( d(x, y) = \| x - y \|_E \), the family \( \{ U(t, \tau) \} \) be a process acting on \( M \). Then the triple \( (U(t, \tau), M, E) \) is said to be a non-autonomous dynamical system, \( M \) and \( E \) are said to be the phase space and the universal space, respectively.

**Definition 1.3.** [14] A family \( \{ \mathcal{M}(t) \}_{t \in \mathbb{R}} \) of subsets of \( M \) is said to be an exponential attractor of the non-autonomous dynamical system \( (U(t, \tau), M, E) \), if

(i) it is semi-invariant, i.e., \( U(t, s) \mathcal{M}(s) \subset \mathcal{M}(t), t \geq s \);
(ii) each section \( \mathcal{M}(t) \) is a compact set of \( E \) and its fractal dimension in \( E \) is uniformly bounded, i.e.,

\[
\sup_{t \in \mathbb{R}} \dim_f(\mathcal{M}(t), E) < +\infty;
\]

(iii) it pullback attracts every bounded subset \( B \) in \( M \) at an exponential rate, i.e.,

\[
\text{dist}_E \{ U(t, t-s)B, \mathcal{M}(t) \} \leq C(\| B \|_E) e^{-\beta s}, \quad t \in \mathbb{R}, \quad s \geq T(\| B \|_E),
\]

for some \( \beta > 0 \).

Obviously, \( M \) coincides with \( E \) in Definition 1.1. In addition, the existence of pullback exponential attractor implies the existence of finite dimensional pullback attractor (cf. [5]).

But there is also a question: pullback exponential attractors lose their uniqueness and may be sensitive to perturbations, that is, they may change dramatically under small perturbations of the system. However, a given system is usually an approximation of reality and it is thus essential that the pullback exponential attractors are robust under small perturbations. Since the pullback exponential attractors of
a process are not unique, the optimal choice of a pullback exponential attractor which is robust under small perturbations is very important. So does the pullback attractor.

Generally speaking, in contrast to pullback attractor, it is expected that pullback exponential attractor is more robust under perturbations and numerical approximations because of its exponentially pullback attractability for any bounded subset in phase space.

In order to discuss the stability of pullback exponential attractors on the perturbations, it is imperative to construct a family of pullback exponential attractors \( \{ M_{\sigma}(t) \} \) for the family of processes \( \{ U_{\sigma}(t, \tau) \} \), \( \sigma \in \Sigma \) (\( \Sigma \) denotes an index set or a symbol space) such that the map \( \sigma \mapsto M_{\sigma}(t) \) is, in some sense, stable. Efendiev et al [13] proposed first criterion to construct pullback exponential attractors which are robust for the discrete process (the criterion is a development of that for discrete semigroup (cf. [12])), and they gave an explicit algorithm for the discrete process and an application to non-autonomous reaction-diffusion systems. Langa et al [18], Czaja and Efendiev [10] extended the existence results in [13] to the continuous process, but the assumptions in [10, 18] require the strong regularity on time \( t \) of the process, which seems typical for parabolic problems but is hard to realize for hyperbolic problems. Carvalho and Sonner [3] and Efendiev et al [14] further gave some alternative criteria on the existence of pullback exponential attractors for the continuous process, which remove the requirement for strong regularity on time \( t \) as in [10, 18] and give some more relaxed assumptions of asymptotical compactness. In [4], Carvalho and Sonner applied the abstract criterion in [3] to the non-autonomous damped wave equations

\[
\frac{d^2 u}{dt^2} + \beta(t) \frac{du}{dt} - \Delta u - f(u) = 0,
\]

(3)

to obtain the existence of pullback exponential attractors of the process related to Eq. (3).

We mention that there have been extensive investigations on the existence of pullback exponential attractors (see for example [1, 11, 25] and references therein). But all those investigations do not give any results on the stability of pullback exponential attractors with respect to perturbations except [13].

Recently, based on the assumptions on the stability and quasi-stability estimate, Chueshov [9] has established an abstract criterion on the existence of exponential attractors for a discrete semigroup, which includes many of others before as its special case because of its weaker assumptions.

Motivated by the idea in [9, 13], we establish two new abstract criteria on the existence and stability of a family of pullback exponential attractors for the discrete dynamical system and continuous one, respectively (see Theorem 2.3 and Theorem 3.1), which are of more relaxed assumptions and applicability and are the developments of construction in [13]. By applying these criteria to non-autonomous Kirchhoff wave models with structural damping and supercritical nonlinearity

\[
\frac{d^2 u}{dt^2} - M(\| \nabla u \|^2) \Delta u + (-\Delta)^{\alpha} u_t + f(u) = g(x, t),
\]

(4)

with \( \alpha \in (1/2, 1) \), we construct a family of pullback exponential attractors \( \{ M_{\sigma}(t) \} \) and show their stability on the perturbations \( g \in \Sigma \) (see Theorem 4.4).

For the physical background of Eq. (4) and related researches, one can refer to [7, 8, 15, 22, 23, 24] in detail. Recently, for the autonomous Kirchhoff wave model (4), i.e., \( g(x, t) \equiv g(x) \), Yang et al [24] have found an optimal supercritical
exponent $\alpha$ (rather than $\bar{p} = \frac{N+2\alpha}{(N-2)^\alpha}$ as known before in [7], where $a^+ = \max\{a,0\}$). By the way, here the growth exponent $p^*$, with $p^* = \frac{N+2\alpha}{N-2}$ ($< p_\alpha$), $N \geq 3$, is said to be critical relative to the natural energy space $X = H_0^1(\Omega) \cap L^{p+1}(\Omega) \times L^2(\Omega)$ for $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ as $p \leq p^*$. It is also shown in [24] that when the growth exponent $p$ of the nonlinearity $f(u)$ is up to the supercritical range $1 \leq p < p_\alpha$,

(i) the well-posedness and longtime behavior of solutions of Eq. (4) are of characteristics of parabolic equations. In particular, the solutions are of higher global regularity (rather than higher partial one as usual) as $t > 0$;

(ii) the related solution semigroup $S(t)$ has in $X$ a global and an exponential attractor, respectively.

A challenging question is that whether the similar results on pullback exponential attractors hold for more complicated non-autonomous Kirchhoff model (4)? Unfortunately, the previous theory developed by Carvalho and Sonner [3] cannot be applied to the existence of pullback exponential attractor of the particular example considered in the current paper because its quasi-linear structure and supercritical nonlinearity cause that the traditional decomposition method of the evolution process as used in [4] ceases to be effective.

In this paper, the treatment for non-autonomous Eq. (4) is a continuation of researches in [24], under the same assumptions as in [24] except $g \in H_0^1(\mathbb{R};L^2)$, by virtue of the criterion established above, we prove the existence of pullback exponential attractor and show their stability with respect to perturbations. To the best of the authors’ knowledge, it is the first result on the existence and stability of pullback exponential attractors for the non-autonomous Kirchhoff wave models with structural damping and supercritical nonlinearity.

The main contributions of the paper are that

(i) We establish two new criteria on the existence and stability of pullback exponential attractors for the non-autonomous discrete dynamical system (see Theorem 2.3 and the simplified proof for Theorem 2.4) and continuous one (see Theorem 3.1), respectively. Compared with the already published literature on this topic, the importance of these criteria lies in that they are based on recently developed quasi-stability method rather than traditional decomposition of the evolution process, which makes that they are of greater applicability because of their more relaxed assumptions than before. A model example (see Section 4) shows that they can be used to deal with more complicated quasilinear hyperbolic problem with supercritical nonlinearity. The paper provides answers for examples not tractable with the existing theory.

(ii) By applying above criterion, we construct a family of pullback exponential attractors $\{\mathcal{M}_g(t)\}_{t \in \mathbb{R}}$ for a family of processes $\{U_g(t,\tau)\}$, $g \in \Sigma$ generated by the Kirchhoff wave model (4), which are stable with respect to perturbations $g \in \Sigma$ (symbol space) (see Theorem 4.4).

The paper is arranged as follows. In Section 2 and Section 3, we discuss two abstract criteria on the existence and stability of pullback exponential attractors for the non-autonomous discrete dynamical system and continuous one, respectively. In Section 4, we apply the criterion to non-autonomous Kirchhoff wave model (4) to construct a family of pullback exponential attractors which are stable with respect to perturbations.
2. Criterion 1 (Discrete case). In this section, we give a criterion on the existence and stability of pullback exponential attractors \( \{M^\sigma(n)\}_{n \in \mathbb{Z}} \) for a discrete non-autonomous dynamical system \((U_\sigma(m, n), M, E)\).

**Definition 2.1.** A seminorm \( n(\cdot) \) on the Banach space \( E \) is said to be compact if any bounded sequence \( \{x_m\} \subset E \) contains a subsequence \( \{x_{m_k}\} \) such that

\[ n(x_{m_k} - x_{m_l}) \to 0 \text{ as } k, l \to \infty. \]

**Definition 2.2.** Let \( B \) be a bounded subset in Banach space \( E \). The Kuratowski \( \alpha \)-measure of noncompactness \( \alpha(B) \) of \( B \) is defined by

\[ \alpha(B) = \inf \{ \delta > 0 | B \text{ has a finite cover by sets of diameter } \leq \delta \}. \]

**Theorem 2.3.** (Discrete case) Let \( \Sigma \) be an index set or a symbol space, \( M \) be a bounded closed subset of the Banach space \( E \), which is equipped with the distance \( d(x, y) = \|x - y\|_E \), and \((U_\sigma(m, n), M, E)\) be a discrete non-autonomous dynamical system for each \( \sigma \in \Sigma \). And assume that

1. the mapping \( U_\sigma(n) = U_\sigma(n + 1, n) : M \to M \) is Lipschitz continuous and there exists a positive constant \( L_1 \) such that
   \[ \sup_{\sigma \in \Sigma} \sup_{n \in \mathbb{Z}} \|U_\sigma(n)x_1 - U_\sigma(n)x_2\|_E \leq L_1 \|x_1 - x_2\|_E, \quad \forall x_1, x_2 \in M; \]  
   \[ \text{(5)} \]
2. there exist a Banach space \( Z \) and a compact seminorm \( n_Z(\cdot) \) on \( Z \), and there exists a mapping \( K^\sigma_n : M \to Z \) for each \( \sigma \in \Sigma, n \in \mathbb{Z} \) such that
   \[ \sup_{\sigma \in \Sigma} \sup_{n \in \mathbb{Z}} \|K^\sigma_n x_1 - K^\sigma_n x_2\|_Z \leq L \|x_1 - x_2\|_E, \]  
   \[ \|U_\sigma(n)x_1 - U_\sigma(n)x_2\|_E \leq \eta \|x_1 - x_2\|_E + n_Z(K^\sigma_n x_1 - K^\sigma_n x_2), \quad \forall x_1, x_2 \in M, \]  
   \[ \text{(6)} \]
   where \( \eta \in (0, 1), L > 0 \) are constants independent of \( \sigma \) and \( n \).

Then for each \( \theta \in (\eta, 1), \sigma \in \Sigma \) there exists a family \( \{M^\sigma_\theta(n)\}_{n \in \mathbb{Z}} \) of compact subsets of \( M \) possessing the following properties:

1. (semi-invariance)
   \[ U_\sigma(m, n)M^\sigma_\theta(n) \subset M^\sigma_\theta(m), \quad m \geq n; \]  
   \[ \text{(7)} \]
2. (boundedness of the fractional dimension)
   \[ \sup_{n \in \mathbb{Z}} \text{dim} \{M^\sigma_\theta(n), E\} \leq C_1 \equiv \left[ \ln \left( \frac{1}{\theta} \right) \right]^{-1} \ln m_Z(\frac{2L}{\theta - \eta}) < +\infty, \]  
   \[ \text{(8)} \]
   where \( m_Z(R) \) is the maximal number of elements \( z_i \) in the ball \( \{z \in Z | \|z\|_Z \leq R\} \) such that \( n_Z(z_i - z_j) > 1 \);
3. (pullback exponential attractability)
   \[ \sup_{n \in \mathbb{Z}} \text{dist}_E \{U_\sigma(n, n - k)M, M^\sigma_\theta(n)\} \leq C \theta^k, \quad \forall k \in \mathbb{N}, \]  
   \[ \text{(9)} \]
   where \( C \) is a positive constant.

That is, the family \( \{M^\sigma_\theta(n)\}_{n \in \mathbb{Z}} \) is a pullback exponential attractor of the non-autonomous dynamical system \((U_\sigma(m, n), M, E)\). Moreover,

4. (stability w.r.t. perturbations) for any \( \sigma_0 \in \Sigma \), if \( \sigma \in \Sigma \) satisfies
   \[ \Gamma(\sigma, \sigma_0) \equiv \sup_{n \in \mathbb{Z}} \sup_{x \in M} \|U_\sigma(n)x - U_{\sigma_0}(n)x\|_E < 1, \]
   then
   \[ \sup_{n \in \mathbb{Z}} \text{dist}^{\text{sym}} \{M^\sigma_\theta(n), M^{\sigma_0}_\theta(n)\} \leq C[\Gamma(\sigma, \sigma_0)]^3, \]  
   \[ \text{(10)} \]
Proof. Condition (6) implies that the operator $U_\sigma(n)$ is an $\alpha$-contraction (cf. [9]), i.e.,
\[
\alpha(U_\sigma(n)M) \leq \eta \alpha(M), \quad \forall n \in \mathbb{Z}, \sigma \in \Sigma.
\]
Without loss of generality we assume that $\alpha(M) > 0$ because one can easily deduce from above inequality that $\alpha(U_\sigma(n, n - k)M) \leq \eta^k \alpha(M) < 1$ for $k$ suitably large, which means $N(M, 1/2) < \infty$, where $N(B, \epsilon)$ denotes the cardinality of the minimal covering of the set $B(\subset E)$ by the closed subsets of diameter $\leq 2\epsilon$.

Let $c_0 = 1/2, \theta \in (\eta, 1), \{F_i\}_{i=1}^{N(M, c_0)}$ be the minimal covering of $M$ by its closed subsets of diameter $\leq 2c_0$. That is, $M = \bigcup_{i=1}^{N(M, c_0)} F_i$. For every $n \in \mathbb{Z}$ and $\sigma \in \Sigma$, we define the pseudometric on $M$:
\[
\rho_\sigma^n(x, y) = n\epsilon(K_\sigma^n x - K_\sigma^n y), \quad x, y \in M.
\]
Obviously, $\rho_\sigma^n(\cdot, \cdot)$ is compact on $M$, so there exists a maximal subset $\{x_j\}_{j=1}^n$ of $F_i(\subset M)$ such that
\[
\rho_\sigma^n(x_j, x_k) > (\theta - \eta)\epsilon_0, \quad j \neq k,
\]
and
\[
n_i = m_{\rho_\sigma^n}(F_i, (\theta - \eta)\epsilon_0) \leq \exp\{\Lambda_{\rho_\sigma^n}(M, \theta - \eta)\}, \tag{11}
\]
where $m_{\rho_\sigma^n}(B, \epsilon)$ is the maximal cardinality of a subset $\{z_k\}$ in $B$ such that $\rho_\sigma^n(z_k, z_l) > \epsilon$ and
\[
\Lambda_{\rho_\sigma^n}(M, \theta - \eta) = \sup_{0 < \epsilon < 1} \{\ln m_{\rho_\sigma^n}(F, (\theta - \eta)\epsilon)\mid F \subset M, \text{ diam} F \leq 2\epsilon\}.
\]
We claim that
\[
\sup_{n \in \mathbb{Z}, \sigma \in \Sigma} \Lambda_{\rho_\sigma^n}(M, \theta - \eta) \leq \ln m_Z\left(\frac{2L}{\theta - \eta}\right) < \infty. \tag{12}
\]
Indeed, let $\epsilon \in (0, 1), B \subset M, \text{ diam} B \leq 2\epsilon, B_\sigma^n = \{K_\sigma^n x \mid x \in B\}$. By (6),
\[
\text{ diam} B_\sigma^n = \sup_{x, y \in B} \|K_\sigma^n x - K_\sigma^n y\|_Z \leq L\text{ diam} B \leq 2\epsilon L,
\]
so there exists a point $y_\sigma^n \in B_\sigma^n$ such that
\[
B_\sigma^n \subset B_{2\epsilon L}(y_\sigma^n) = \{z \in Z \mid \|z - y_\sigma^n\|_Z \leq 2\epsilon L\}.
\]
Therefore, by virtue of the linearity of the seminorm,
\[
m_{\rho_\sigma^n}(B, (\theta - \eta)\epsilon) = \mathbb{N}\{z_i \in B_\sigma^n \mid \|z_i - z_j\|_Z > (\theta - \eta)\epsilon, i \neq j\}
\leq \mathbb{N}\{z_i \in B_{2\epsilon L}(y_\sigma^n) \mid \|z_i - z_j\|_Z > (\theta - \eta)\epsilon, i \neq j\}
= \mathbb{N}\{z_i \in B_{2\epsilon L}(0) \mid \|z_i - z_j\|_Z > 1, i \neq j\} = m_Z\left(\frac{2L}{\theta - \eta}\right),
\]
where $\mathbb{N}\{\cdots\}$ denotes the maximal number of elements with the given properties. By the arbitrariness of $\epsilon$ and $B(\subset M)$, we get (12), i.e., the claim is valid.

Obviously,
\[
F_i \subset \bigcup_{j=1}^{n_i} B_j^i, \quad B_j^i = \{x \in F_i \mid \rho_\sigma^n(x, x_j) \leq (\theta - \eta)\epsilon_0\}.
\]
Thus (see (6)),
\[ U_\sigma(n)M \subset \bigcup_{i=1}^{N(M,\epsilon_0)} \bigcup_{j=1}^{n_i} U_\sigma(n)B^j_i, \quad \text{diam}(U_\sigma(n)B^j_i) \leq \eta \text{diam}F_i + 2(\theta - \eta)\epsilon_0 \leq 2\theta\epsilon_0. \]

Therefore (see (11)-(12)),
\[ N(U_\sigma(n)M,\theta\epsilon_0) \leq N(M,\epsilon_0) \exp\{A_{\rho^*}(M,\theta - \eta)\} \leq N(M,\epsilon_0)m_Z\left(\frac{2L}{\theta - \eta}\right). \tag{13} \]
Replacing \( M \) in (13) by \( U_\sigma(n, n - k)M(\subset M) \), we obtain
\[ N(U_\sigma(n+1, n - k)M,\theta\epsilon_0) \leq N(U_\sigma(n, n - k)M,\epsilon_0)m_Z\left(\frac{2L}{\theta - \eta}\right) \leq N(M,\epsilon_0)m_Z\left(\frac{2L}{\theta - \eta}\right). \]
Let \( \epsilon_k = \theta^k\epsilon_0 \) (with \( \epsilon_0 = 1/2 \), \( k \in \mathbb{N} \)). We deduce that
\[ N(U_\sigma(n+1, n - k)M,\epsilon_{k+1}) \leq N(M,\epsilon_0)\left[m_Z\left(\frac{2L}{\theta - \eta}\right)\right]^{k+1}, \tag{14} \]
which implies that there exists a family of finite sets \( \{V_{\sigma}^k(n)\} \) such that
\[ V_{\sigma}^k(n) \subset U_\sigma(n, n - k)M \quad \text{and} \quad U_\sigma(n, n - k)M \subset \bigcup_{h \in V_{\sigma}^k(n)} B(h, \theta^k), \tag{15} \]
where \( B(h, \theta^k) = \{x \in E||x - h||_E \leq \theta^k\} \), and
\[ \text{Card}V_{\sigma}^k(n) \leq N(M,\epsilon_0)\left[m_Z\left(\frac{2L}{\theta - \eta}\right)\right]^k. \tag{16} \]
For any \( \sigma_0 \in \Sigma \), we split the set \( \Sigma \) into the union \( \Sigma = \Sigma_1 \cup \Sigma_2 \), where
\[ \Sigma_1 = \{\sigma \in \Sigma|\Gamma(\sigma, \sigma_0) \geq 1\} \cup \{\sigma_0\}, \quad \Sigma_2 = \{\sigma \in \Sigma|0 < \Gamma(\sigma, \sigma_0) < 1\}. \]

(1) When \( \sigma \in \Sigma_1 \), let
\[ E^n_k = V^n_k, \quad E^n_k = V^n_k \cup U_\sigma(n - 1)E^n_{k-1}(n - 1), \]
\[ M^n_k(n) = \left[\bigcup_{k \geq 1} E^n_k(n)\right]_E, \quad n \in \mathbb{Z}, \theta \in (\eta, 1). \tag{17} \]
Obviously (see (15)-(16)),
\[ E^n_k(n) = \bigcup_{l=0}^{k-1} U_\sigma(n, n - l)V^n_{k-l}(n - l) \subset U_\sigma(n, n - k)M, \]
\[ M^n_k(n) \subset M, \quad U_\sigma(n)E^n_k(n) \subset E^n_{k+1}(n + 1), \]
\[ \text{Card}E^n_k(n) \leq \sum_{l=0}^{k-1} \text{Card}V^n_{k-l}(n - l) \leq N(M,\epsilon_0)\sum_{l=0}^{k-1} \left[m_Z\left(\frac{2L}{\theta - \eta}\right)\right]^{k-l} \leq CN(M,\epsilon_0)\left[m_Z\left(\frac{2L}{\theta - \eta}\right)\right]^{k+1}, \quad \forall \sigma \in \Sigma. \tag{18} \]
We show that the family of sets \( \{M^n_k(n)\}_{n \in \mathbb{Z}} \) is of properties (i)-(iii) of Theorem 2.3.
Consequently, by virtue of (18), we get

\[
\hat{\text{dist}}_{\mathcal{M}_\sigma}(n) = 0, \quad \forall n \geq 1.
\]

(ii) (Pullback exponential attractability) By (17) and (15),

\[
\text{dist}_E\{U_\sigma(n, n - k)M, M^\sigma_k(n)\} \leq \text{dist}_E\{U_\sigma(n, n - k)M, V^\sigma_k(n)\} \leq \theta^k, \quad k \geq 1.
\]

(iii) (Boundedness of the fractal dimension) For any \(\epsilon \in (0, \epsilon_0)\), there exists a \(k(\epsilon) \in \mathbb{N}\) such that \(\epsilon/4 \in [\theta^{k(\epsilon)}, \theta^{k(\epsilon) - 1} \cdot \epsilon_0]\). By (18) and (15),

\[
E^\sigma_k(n) \subset U_\sigma(n, n - k)M \subset U_\sigma(n, n - k(\epsilon))M \subset \bigcup_{h \in V^\sigma_k(n)} B(h, \frac{\epsilon}{2}), \quad k \geq k(\epsilon),
\]

\[
M^\sigma_k(n) = \left[ \bigcup_{k < k(\epsilon)} E^\sigma_k(n) \right] \cup \left[ \bigcup_{k \geq k(\epsilon)} E^\sigma_k(n) \right] \subset \bigcup_{k < k(\epsilon)} E^\sigma_k(n)
\]

\[
\cup \left[ \bigcup_{h \in V^\sigma_k(n)} B(h, \frac{\epsilon}{2}) \right] \subset \bigcup_{k < k(\epsilon)} E^\sigma_k(n) \cup \bigcup_{h \in V^\sigma_k(n)} B(h, \epsilon).
\]

Therefore, by (16) and (18),

\[
N(M^\sigma_k(n), \epsilon) \leq \sum_{k < k(\epsilon)} \text{Card} E^\sigma_k(n) + \text{Card} V^\sigma_k(n)
\]

\[
\leq N(M, \epsilon_0) \left[ \sum_{k < k(\epsilon)} \sum_{l=0}^{k-1} \left[ m_Z \left( \frac{2L}{\theta - \eta} \right) \right]^{k-l} + \left[ m_Z \left( \frac{2L}{\theta - \eta} \right) \right]^{k(\epsilon)} \right]
\]

\[
= N(M, \epsilon_0) \left[ m_Z \left( \frac{2L}{\theta - \eta} \right) \right]^{k(\epsilon)} \left[ \sum_{k < k(\epsilon)} (k(\epsilon) - k) \left[ m_Z \left( \frac{2L}{\theta - \eta} \right) \right]^{-(k(\epsilon) - k)} + 1 \right]
\]

\[
\leq N(M, \epsilon_0) \left[ m_Z \left( \frac{2L}{\theta - \eta} \right) \right]^{k(\epsilon)} \left[ 1 + \frac{m_Z \left( \frac{2L}{\theta - \eta} \right)}{m_Z \left( \frac{2L}{\theta - \eta} \right) - 1} \right].
\] (20)

By the arbitrariness of \(\epsilon \in (0, \epsilon_0)\), we see from (20) that \(M^\sigma_k(n) \subset M\) is a compact set in \(E\) for each \(n \in \mathbb{Z}\). Due to \(\epsilon/4 < \theta^{k(\epsilon) - 1}/2\), which means \(k(\epsilon) \leq \ln(2/\epsilon) \ln^{-1}(1/\theta) + 1\), a simple calculation shows that

\[
\dim_f(M^\sigma_k(n), E) = \limsup_{\epsilon \to 0} \frac{\ln N(M^\sigma_k(n), \epsilon)}{\ln(1/\epsilon)} \leq \left[ \ln \frac{1}{\theta} \right]^{-1} \ln m_Z \left( \frac{2L}{\theta - \eta} \right).
\]

(2) When \(\sigma \in \Sigma_2\), due to \(E^\sigma_k(n) \subset U_\sigma(n, n - k)M\) for \(k \geq 1\), there exists a set \(\hat{E}_k(n - k) \subset M\) such that

\[
U_\sigma(n, n - k)\hat{E}_k(n - k) = E^\sigma_0(n), \quad \text{Card} \hat{E}_k(n - k) = \text{Card} E^\sigma_0(n).
\] (21)

Consequently, by virtue of (18), we get

\[
U_\sigma(n + 1, n - k)\hat{E}_k(n - k)
\]

\[
= U_\sigma(n)U_\sigma(n - k)\hat{E}_k(n - k) = U_\sigma(n)E^\sigma_0(n)
\]

\[
\subset E^\sigma_0(n + 1, n - k)\hat{E}_{k+1}(n - k) = U_\sigma(n + 1, n - k)\hat{E}_{k+1}(n - k).
\] (22)
Without loss of generality we assume that 
\[ \tilde{E}_k(n-k) \subset \tilde{E}_{k+1}(n-k), \quad k \geq 1. \]

Let 
\[ \tilde{E}_n^\sigma(n) = U_\sigma(n,n-k)\tilde{E}_k(n-k) (\subset U_\sigma(n,n-k)M), \quad k \geq 1. \]

By (21) and (18),
\[
\text{Card} \tilde{E}_n^\sigma(n) \leq \text{Card} \tilde{E}_k(n-k) = \text{Card} E_{k0}^\sigma(n) \leq CN(M, \epsilon_0) \left[ \sup_{n,z} \frac{2L}{\theta - \eta} \right]^{k+1}.
\]

Due to (5) and the fact
\[
\sup_{x \in M} \|U_\sigma(n)x - U_{\sigma_0}(n)x\|_E \leq \Gamma(\sigma, \sigma_0), \quad n \in \mathbb{Z},
\]
we have, for every \( n \in \mathbb{Z}, k \geq 1 \) and \( x \in M, \)
\[
\|U_\sigma(n,n-k)x - U_{\sigma_0}(n,n-k)x\|_E \leq \|U_\sigma(n-1)U_\sigma(n,n-k)x - U_{\sigma_0}(n-1, n-k)x\|_E \]
\[
+ \|U_\sigma(n,n-1, n-k)x - U_{\sigma_0}(n-1, n-k)x\|_E + \|U_\sigma(n,n-1, n-k)x - U_{\sigma_0}(n-1, n-k)x\|_E \]
\[
\leq \L_1 \|U_\sigma(n-1,n-k)x - U_{\sigma_0}(n-1,n-k)x\|_E \quad \Gamma(\sigma, \sigma_0)
\]
\[
\leq \cdots \leq \Gamma(\sigma, \sigma_0)(L_1^{k-1} + \cdots + 1) \leq CL(\sigma, \sigma_0)L_1^k,
\]
where \( C = 1/(L_1 - 1). \) Therefore, we have
\[
\sup_{x \in M} \sup_{n \in \mathbb{Z}} \|U_\sigma(n,n-k)x - U_{\sigma_0}(n,n-k)x\|_E \leq CTL_1^k,
\]
and
\[
\sup_{n \in \mathbb{Z}} \text{dist}_E \{ \tilde{E}_n^\sigma(n), E_{k0}^\sigma(n) \}
\]
\[
= \sup_{n \in \mathbb{Z}} \text{dist}_E \{ U_\sigma(n,n-k)\tilde{E}_k(n-k), U_{\sigma_0}(n,n-k)\tilde{E}_k(n-k) \} \leq \Gamma \Gamma^k, \quad 1 \leq k \leq k_\Gamma.
\]

where
\[
\Gamma = \Gamma(\sigma, \sigma_0), \quad k_{\Gamma} = \frac{\log L_1(1/\Gamma)}{1 + \log L_1(1/\theta)}, \quad \lambda = \frac{\log L_1(1/\theta)}{1 + \log L_1(1/\theta)},
\]
which means
\[
\Gamma L_1^{k_{\Gamma}} = \theta^{k_{\Gamma}} = \Gamma^\lambda.
\]

By the symmetry of \( \Gamma(\sigma, \sigma_0), \tilde{E}_n^\sigma(n) \) and \( E_{k0}^\sigma(n) \) in (24), we obtain
\[
\sup_{n \in \mathbb{Z}} \text{dist}_E \{ E_{k0}^\sigma(n), \tilde{E}_n^\sigma(n) \} \leq C\Gamma^{\lambda}, \quad 1 \leq k \leq k_{\Gamma}.
\]

For any \( b \in M, \) there exists \( b_0 \in E_{k0}^\sigma(n) \) such that (see (19))
\[
\text{dist}_E \{ U_{\sigma_0}(n,n-k)b, b_0 \} = \text{dist}_E \{ U_{\sigma_0}(n,n-k)b, E_{k0}^\sigma(n) \} \leq \text{dist}_E \{ U_{\sigma_0}(n,n-k)M, V_{k0}^\sigma(n) \} \leq \theta^k, \quad \forall n \in \mathbb{Z}, \quad k \geq 1.
\]
The combination of (24)-(26) gives
\[
\text{dist}_E\{U_\sigma(n, n - k)b, \hat{E}_k\}(n) \leq \text{dist}_E\{U_\sigma(n, n - k)b, U_{\sigma_0}(n, n - k)b\} + \text{dist}_E\{U_{\sigma_0}(n, n - k)b, b_0\} + \text{dist}_E\{b_0, \hat{E}_k\}(n) \leq C\eta + \eta^k + C\gamma^k, \quad 1 \leq k \leq k_G.
\]

By the arbitrariness of \(b \in M\), we have
\[
\text{dist}_E\{U_\sigma(n, n - k)M, \hat{E}_k\}(n) \leq C\eta^k, \quad 1 \leq k \leq k_G, \quad n \in \mathbb{Z}. \tag{27}
\]

Let
\[
E^\sigma_k(n) = \begin{cases} \hat{E}_k\sigma(n), & 1 \leq k \leq k_G, \\ V_k\sigma(n) \cup U_\sigma(n - 1)E^\sigma_{k-1}(n - 1), & k > k_G, \end{cases}
\]

\[
M^\sigma_\theta(n) = \left[ \bigcup_{k \geq 1} E^\sigma_k(n) \right].
\]

Taking account of the fact
\[
E^\sigma_k(n) = \left[ \bigcup_{l=0}^{k - \lfloor k_G \rfloor - 1} U_\sigma(n, n - l)V_{k-l}\sigma(n - l) \right] \cup U_\sigma(n, n - k + k_G)\hat{E}_{k_G}\sigma(n - k + [k_G]),
\]

we infer from (18) and (23) that
\[
E^\sigma_k(n) \subset U_\sigma(n, n - k)M, \quad M^\sigma_\theta(n) \subset M,
\]

\[
\text{Card}E^\sigma_k(n) \leq \sum_{l=0}^{k - \lfloor k_G \rfloor - 1} \text{Card}V_{k-l}\sigma(n - l) + \text{Card}\hat{E}_{k_G}\sigma(n - k + [k_G]) \leq C N(M, \epsilon_0) \left[ m_Z \left( \frac{2L}{B - \theta} \right) \right]^{k+1}, \quad k \geq 1, n \in \mathbb{Z}, \tag{28}
\]

where \([k_G]\) denotes the integer part of \(k_G\), \(C > 0\) is a constant independent of \(k, n\) and \(\sigma\).

Now, we show that the family of sets \(\{M^\sigma_\theta(n)\}_{n \in \mathbb{Z}}\) is of properties (i)-(iv) of Theorem 2.3 for every \(\sigma \in \Sigma_2\).

(i) (Semi-invariance) When \(k > k_G\), obviously, \(U_\sigma(n)E^\sigma_k(n) \subset E^\sigma_{k+1}(n+1)\). When \(1 \leq k \leq k_G\), due to \(\hat{E}_{k}(n - k) \subset \hat{E}_{k+1}(n - k)\), we obtain
\[
U_\sigma(n)E^\sigma_k(n) = U_\sigma(n)\hat{E}_k\sigma(n) = U_\sigma(n + 1, n - k)\hat{E}_k(n - k) \subset U_\sigma(n + 1, n - k)\hat{E}_{k+1}(n - k) = E^\sigma_{k+1}(n+1),
\]

then by the continuity of \(U_\sigma(n)\) on \(M\),
\[
U_\sigma(m, n)M^\sigma_\theta(n) \subset \left[ \bigcup_{k \geq 1} U_\sigma(m, n)E^\sigma_k(n) \right] \subset \left[ \bigcup_{k \geq 1} \left[ \bigcup_{l=0}^{k - [k_G]} U_\sigma(m, n)E^\sigma_{k-l}(n - l) \right] \right] \subset \left[ \bigcup_{k \geq 1} \left[ \bigcup_{l=0}^{k - [k_G]} E^\sigma_{k-l+m-n}(m) \right] \right] \subset \left[ \bigcup_{k \geq 1} E^\sigma_k(m) \right] = M^\sigma_\theta(m), \quad m \geq n.
\]
(ii) (Pullback exponential attractability) By (27) and (15), we have
\[ \text{dist}_E \{ U_\sigma(n, n - k)M, \mathcal{M}_\sigma^0(n) \} \leq \text{dist}_E \{ U_\sigma(n, n - k)M, \mathcal{E}_k^\sigma(n) \} \]
\[ \leq \begin{cases} \text{dist}_E \{ U_\sigma(n, n - k)M, \mathcal{E}_k^\sigma(n) \}, & 1 \leq k \leq k_1, \\ \text{dist}_E \{ U_\sigma(n, n - k)M, \mathcal{V}_k^\sigma(n) \}, & k > k_1, \end{cases} \leq C\theta^k, \quad k \geq 1, \quad n \in \mathbb{Z}. \]

(iii) (Boundedness of the fractal dimension) Based on (28), repeating the same proof as in case (1): (iii) we obtain estimate (8) and the compactness of \( \mathcal{M}_\sigma^0(n) \) in \( E \) for \( \sigma \in \Sigma_2 \).

(iv) (Stability w.r.t. perturbations) For any \( \sigma \in \Sigma_2, a \in \bigcup_{k \geq 1} E_k^\sigma(n) \), there must be \( a \in E_k^\sigma(n) \) for some \( k \). When \( 1 \leq k \leq k_1 \), by (24),
\[ \text{dist}_E \{ a, \mathcal{M}_\sigma^0(n) \} \leq \text{dist}_E \{ a, E_k^\sigma(n) \} \leq \text{dist}_E \{ \mathcal{E}_k^\sigma(n), E_k^\sigma(n) \} \leq CT^\lambda. \]
When \( k > k_1 \), due to
\[ a \in E_k^\sigma(n) \subset U_\sigma(n, n - k)M = U_\sigma(n, n - [k_1])U_\sigma(n - [k_1], n - k)M, \]
there exists an element \( b \in U_\sigma(n - [k_1], n - k)M \subset M \) such that \( a = U_\sigma(n, n - [k_1])b \), so by (24) and (19),
\[ \text{dist}_E \{ a, \mathcal{M}_\sigma^0(n) \} \leq \text{dist}_E \{ U_\sigma(n, n - [k_1])b, E_k^\sigma(n) \} \]
\[ \leq \text{dist}_E \{ U_\sigma(n, n - [k_1])b, U_{\sigma_0}(n, n - [k_1])b \} \]
\[ + \text{dist}_E \{ U_{\sigma_0}(n, n - [k_1])b, E_k^\sigma(n) \} \]
\[ \leq CTL_1^{[k_1]} + \theta^{k_1} \leq CT L_1^{k_1} + \frac{1}{\theta^{k_1}} \leq CT^\lambda. \]
By the arbitrariness of \( a \in \bigcup_{k \geq 1} E_k^\sigma(n) \), we obtain
\[ \text{dist}_E \{ \mathcal{M}_\sigma^0(n), \mathcal{M}_\sigma^0(n) \} = \text{dist}_E \bigcup_{k \geq 1} E_k^\sigma(n), \mathcal{M}_\sigma^0(n) \leq CT^\lambda. \quad (29) \]
Similarly, repeating the proof of (29) (changing the position of \( \sigma \) and \( \sigma_0 \)) and making use of (25), we obtain
\[ \text{dist}_E \{ \mathcal{M}_\sigma^0(n), \mathcal{M}_\sigma^0(n) \} \leq CT^\lambda. \quad (30) \]
The combination of (29) and (30) gives (10). \quad \square

By virtue of Theorem 2.3 here, one can easily deduce the criterion in [13].

**Theorem 2.4.** [13] Let \( H \) and \( H_1 \) be two Banach spaces, \( H_1 \) compactly embed into \( H \), and \( B \) be a bounded subset of \( H_1 \). For given positive constants \( \delta \) and \( K \), we define a class \( S_\delta(K) \) of nonlinear operators \( S : H_1 \rightarrow H_1 \) as follows

(i) \( S : \mathcal{O}_\delta(B) \rightarrow B \), where \( \mathcal{O}_\delta(B) \) is a \( \delta \)-neighborhood of the set \( B \) in \( H_1 \);

(ii) \( \| Sh_1 - Sh_2 \|_{H_1} \leq K \| h_1 - h_2 \|_H, \quad \forall h_1, h_2 \in \mathcal{O}_\delta(B). \)

Then the non-autonomous dynamical system \( (U(m, n), B, H_1) \) possesses a pullback exponential attractor \( \{ \mathcal{M}_U(n) \}_{n \in \mathbb{Z}} \), where \( B \) is equipped with the distance \( d(x, y) = \| x - y \|_{H_1} \). Moreover, the map \( U \mapsto \mathcal{M}_U(n) \) is uniformly Hölder continuous in the following sense: for each process \( U_1 \) and \( U_2 \) satisfying \( U_i(n) \in S_\delta(K)(B), n \in \mathbb{Z}, i = 1, 2 \), we have
\[ \text{dist}_{H_1} \{ \mathcal{M}_{U_1}(n), \mathcal{M}_{U_2}(n) \} \leq C_3 \sup_{l \in (-\infty, n)} \{ e^{-\beta(n-l)} \| U_1(l) - U_2(l) \|_H^2 \}, \quad (31) \]
where the positive constants $C_i, i = 1, 2, 3, \alpha, \beta$ and $\kappa$ depend only on $B, H, H_1, \delta$ and $K$, and are independent of $n$ and of the specific choice of the $U_i$,
\[
\|S_1 - S_2\|_\delta = \sup_{x \in O_i(B)} \|S_1x - S_2x\|_{H_1}, \quad S_1, S_2 \in \mathcal{S}_{\delta,K}(B).
\]

**Proof.** Under the assumptions of Theorem 2.4, taking
\[
E = Z = H_1, \quad n_Z(\cdot) = K||| \cdot ||_{H_1}, \quad M = [O_{\delta/2}(B)]_{H_1},
\]
then, for every process $U$ acting on $H_1$ and satisfying $U(n) \in \mathcal{S}_{\delta,K}(B)$ for every $n \in \mathbb{Z}$, we have that
(a) $U(n) : M \to M$ for $M \subset O_{\delta}(B)$ and $U(n)M \subset U(n)O_{\delta}(B) \subset B \subset M$;
(b) by (ii), for any $h_1, h_2 \in M$,
\[
\|U(n)h_1 - U(n)h_2\|_{H_1} \leq K\|h_1 - h_2\|_{H_1} \leq CK\|h_1 - h_2\|_{H_1},
\]
\[
\|U(n)h_1 - U(n)h_2\|_{H_1} \leq \eta\|h_1 - h_2\|_{H_1} + n_Z(K^U_{n}h_1 - K^U_{n}h_2),
\]
where $\eta \in (0, 1)$ and $K^U_{n} : M \to Z$ is Lipschitz continuous, i.e.,
\[
\|K^U_{n}h_1 - K^U_{n}h_2\|_{Z} = \|h_1 - h_2\|_{H_1}.
\]

Then by Theorem 2.3, the non-autonomous dynamical system $(U(m, n), M, H_1)$ possesses a pullback exponential attractor $\{M_{U_{n}}(n)\}_{n \in \mathbb{Z}}$, and stability estimate (10) holds, i.e.,
\[
\sup_{n \in \mathbb{Z}} \text{dist}^{symm}_{H_1} \{M_{U_{n}}(n), M_{U_{n+1}}(n)\} \leq C \sup_{n \in \mathbb{Z}} \|U_1(n)x - U_2(n)x\|_{H_1}.
\]

**Remark 1.** (i) Stability estimate (32) is a conclusion of (31), which means that it is compatible with estimate (31). Indeed, one can easily deduce from (31) that
\[
\sup_{n \in \mathbb{Z}} \text{dist}^{symm}_{H_1} \{M_{U_{n+1}}(n), M_{U_{n+2}}(n)\} \leq C_3 \sup_{n \in \mathbb{Z}} \sup_{t \in (-\infty, n]} \{e^{-\beta(n-t)}||U_1(t) - U_2(t)||_{H_1}\}
\]
\[
\leq C_3 \sup_{n \in \mathbb{Z}} ||U_1(n) - U_2(n)||_{H_1}\sup_{n \in \mathbb{Z}} \sup_{t \in (-\infty, n]} \{e^{-\beta(n-t)}\}
\]
\[
\leq C_3 \sup_{n \in \mathbb{Z}} \sup_{x \in O_{1}(B)} \|U_1(n)x - U_2(n)x\|_{H_1}.
\]

(ii) One sees from the above proof that the conditions of Theorem 2.3 are greatly weaker than those of Theorem 2.4, which leads to the fact that Theorem 2.3 is of greater applicability.

3. **Criterion 2 (Continuous case).** In this section, on the basis of Theorem 2.3, we further establish a criterion on the existence of a class of robust pullback exponential attractors for the time-continuous non-autonomous dynamical system $(U_{\sigma}(t, \tau), M, E)$. It is more applicable for a large class of evolution problem.

**Theorem 3.1.** (Continuous case) Let $\Sigma$ be an index set or a symbol space, $M$ be a bounded closed subset of the Banach space $E$, which is equipped with the distance $d(x, y) = \|x - y\|_{E}$, and $(U_{\sigma}(t, \tau), M, E)$ be a non-autonomous dynamical system for each $\sigma \in \Sigma$. And assume that
(i) there exist constants \( T > 0, L_T > 0 \) such that, for any \( \tau \in \mathbb{R}, x_1, x_2 \in M, \)

\[
\bigcup_{\sigma \in \Sigma} U_\sigma(t + \tau, \tau)M \subset M, \quad t \geq T,
\]

\[
\sup_{t \in [0, T]} \sup_{\sigma \in \Sigma} \|U_\sigma(t + \tau, \tau)x_1 - U_\sigma(t + \tau, \tau)x_2\|_E \leq L_T \|x_1 - x_2\|_E; \tag{33}
\]

(ii) there exist a Banach space \( Z \) and a compact seminorm \( n_Z(\cdot) \) on \( Z \), and there exists a mapping \( K_n^\sigma : M \to Z \) for each \( \sigma \in \Sigma, n \in \mathbb{Z} \) such that for any \( x_1, x_2 \in M, \)

\[
\sup_{\sigma \in \Sigma} \sup_{n \in \mathbb{Z}} \|K_n^\sigma x_1 - K_n^\sigma x_2\|_Z \leq L \|x_1 - x_2\|_E, \tag{34}
\]

\[
\|U_\sigma((n + 1)T, nT)x_1 - U_\sigma((n + 1)T, nT)x_2\|_E \leq \eta \|x_1 - x_2\|_E + nZ(K_n^\sigma x_1 - K_n^\sigma x_2), \tag{35}
\]

where \( \eta \in (0, 1), L > 0 \) are constants independent of \( \sigma \) and \( n \).

Then, for each \( \theta \in (\eta, 1), \sigma \in \Sigma \), the dynamical system \((U_\sigma(t, \tau), M, E)\) possesses a pullback exponential attractor \( \{M_\theta^\sigma(t)\}_{t \in \mathbb{R}} \). Moreover, the map \( \sigma \mapsto M_\theta^\sigma(t) \) is stable in the following sense: for any \( \sigma_0 \in \Sigma \), if \( \sigma \in \Sigma \) satisfies

\[
\Gamma(\sigma, \sigma_0) \equiv \sup_{s \in [0, T]} \sup_{\tau \in \mathbb{R}} \sup_{x \in M} \|U_\sigma(s + \tau, \tau)x - U_{\sigma_0}(s + \tau, \tau)x\|_E < 1, \tag{36}
\]

then

\[
\sup_{t \in \mathbb{R}} \text{dist}_{E}^{E \Sigma} \{M_\theta^\sigma(t), M_\theta^{\sigma_0}(t)\} \leq C[\Gamma(\sigma, \sigma_0)]^\lambda, \tag{36}
\]

where \( C > 0, \beta > 0 \) and \( \lambda : 0 < \lambda < 1 \) are constants independence of \( \sigma \).

Proof. For any \( \sigma \in \Sigma \), we define a discrete process \( \{U_\sigma(m, n)\} \) acting on the phase space \( M \):

\[
U_\sigma(m, n) = U_\sigma(mT, nT), \quad m, n \in \mathbb{Z}, \quad m \geq n, \tag{37}
\]

where \( T \) is the positive constant as shown in (33). Obviously, the family of processes \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \) satisfies the conditions of Theorem 2.3, thus for any \( \sigma \in \Sigma, \theta \in (\eta, 1), \) there exists a family \( \{M_\theta^\sigma(n)\}_{n \in \mathbb{N}} \) of compact subsets of \( M \) processing the properties (i)-(iv) of Theorem 2.3. Let

\[
M_\theta^\sigma(t) = U_\sigma(t, nT)M_\theta^\sigma(n), \quad \forall t \in [nT, (n + 1)T), \quad n \in \mathbb{Z}, \quad \sigma \in \Sigma, \tag{38}
\]

(i) (Semi-invariance) For any \( \sigma \in \Sigma, t \geq s \), let \( t = n_1T + t_1, s = n_2T + s_1 \) for some \( n_1, n_2 \in \mathbb{Z} \) \((n_1 \geq n_2)\) and \( t_1, s_1 \in [0, T) \). Then by the semi-invariance of \( \{M_\theta^\sigma(n)\}_{n \in \mathbb{N}}, \) we have

\[
U_\sigma(t, s)M_\theta^\sigma(s) = U_\sigma(t, s)U_\sigma(s, n_2T)M_\theta^\sigma(n_2) = U_\sigma(t, n_2T)M_\theta^\sigma(n_2)
= U_\sigma(t, n_1T)U_\sigma(n_1T, n_2T)M_\theta^\sigma(n_2) \subset U_\sigma(t, n_1T)M_\theta^\sigma(n_1) = M_\theta^\sigma(t).
\]

(ii) (Boundedness of the fractal dimension) For \( t \in [nT, (n + 1)T), n \in \mathbb{Z} \) and \( \sigma \in \Sigma, \) by the Lipschitz continuity of \( U_\sigma(t, \tau) \) (see (33)) we know that \( M_\theta^\sigma(t) \) is a compact subset of \( E, \)

\[
\dim_f(M_\theta^\sigma(t), E) \leq \dim_f(M_\theta^\sigma(n), E) \leq \left[ \frac{1}{\theta} \right]^{-1} \ln m_Z\left( \frac{2L}{\theta - \eta} \right),
\]
and
\[
\mathcal{M}_\sigma^\alpha(t) = U_\sigma(t,nT)M_\sigma^\alpha(n) \subset \left[ \bigcup_{k \geq 1} U_\sigma(t,nT)E_k^\alpha(n) \right]_E \\
\subset \left[ \bigcup_{k \geq 1} U_\sigma(t,nT)U_\sigma(nT,(n-k)T)M \right]_E \subset M.
\]

(iii) (Pullback exponential attractability) Let \( \sigma \in \Sigma, s \in \mathbb{R} \) with \( s \geq 2T \), and let \( t = nT + t_1 \) with \( n \in \mathbb{Z}, t_1 \in [0,T) \), \( s = kT + 2T + s_1 \) with \( k \in \mathbb{N}, s_1 \in [0,T) \). We infer from condition (33), estimate (9) and definition (38) that
\[
\text{dist}_E\{U_\sigma(t, t-s)M, \mathcal{M}_\sigma^\alpha(t)\} = \text{dist}_E\{U_\sigma(t,nT)U_\sigma(nT,t)M, U_\sigma(t,nT)M_\sigma^\alpha(n_1)\} \\
\leq L_T \text{dist}_E\{U_\sigma(nT,(n_1-k)T)U_\sigma((n_1-k)T,t-s)M, M_\sigma^\alpha(n_1)\} \\
\leq L_T \text{dist}_E\{U_\sigma(nT,(n_1-k)T)M, M_\sigma^\alpha(n_1)\} \\
\leq C_L T^\theta \leq C_2 e^{-\beta t},
\]
where we have used the fact: \( (n_1-k)T \geq t-s+T \), and where \( C_2 = C_L T^{-3}, \beta = -\ln(\theta^{1/T}) \).

(iv) (Stability w.r.t. perturbations) For any \( \sigma_0 \in \Sigma, \sigma \in \Sigma \) with \( \Gamma(\sigma, \sigma_0) < 1 \), and \( t \in \mathbb{R} \), let \( t = nT + t_1 \) for some \( n \in \mathbb{Z} \) and \( t_1 \in [0,T) \). Then, by conditions (33) and (10),
\[
\text{dist}_E^\text{symm}\{M^\alpha_\sigma(t), M^\alpha_\sigma(0)\} = \text{dist}_E^\text{symm}\{U_\sigma(t,nT)M^\alpha_\sigma(n), U_\sigma(t,nT)M^\alpha_\sigma(0)\} \\
\leq \text{dist}_E^\text{symm}\{U_\sigma(t,nT)M^\alpha_\sigma(n), U_\sigma(t,nT)M^\alpha_\sigma(0)\} \\
+ \text{dist}_E^\text{symm}\{U_\sigma_0(t,nT)M^\alpha_\sigma(n), U_\sigma_0(t,nT)M^\alpha_\sigma(0)\} \\
\leq \Gamma(\sigma, \sigma_0) + L_T \text{dist}_E^\text{symm}\{M^\alpha_\sigma(n), M^\alpha_\sigma_0(0)\} \\
\leq (1 + L_T C_3) \left[ \Gamma(\sigma, \sigma_0) \right],
\]
which means (36).

\[\square\]

Corollary 1. Under the assumptions of Theorem 3.1, if \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \) is also a family of processes acting on the Banach space \( E \), and the bounded closed set \( M \) in Theorem 3.1 is a uniformly (w.r.t. \( \tau \in \mathbb{R} \)) absorbing set of the process \( \{U_\sigma(t, \tau)\} \) for each \( \sigma \in \Sigma \), i.e., for any bounded subset \( B \) in \( E \), there exists a \( T = T(B) > 0 \) such that \( \bigcup_{\tau \in \mathbb{R}} U_\sigma(t+\tau, \tau)B \subset M \) for all \( t \geq T \), then the pullback exponential attractor \( \{\mathcal{M}^\alpha_\sigma(t)\}_{t \in \mathbb{R}} \) of the non-autonomous dynamical system \( (U_\sigma(t, \tau), M, E) \) as shown in Theorem 3.1 is also a pullback exponential attractor of the dynamical system \( (U_\sigma(t, \tau), E) \).

4. Application to non-autonomous Kirchhoff wave models. We consider the existence and stability of pullback exponential attractors of the following non-autonomous Kirchhoff model with structure damping
\[
u_{tt} - M(\|\nabla u\|^2)\Delta u + (-\Delta)^\alpha u_t + f(u) = g(x, t), \quad x \in \Omega, \ t \geq \tau, \ \tau \in \mathbb{R},
\]
\[
u_{\partial \Omega} = 0, \quad u(x, \tau) = u_0(x), \quad u_t(x, \tau) = u_1(x),
\]
where \( \alpha \in (1/2, 1), \Omega \) is a bounded domain in \( \mathbb{R}^N(N \geq 1) \) with the smooth boundary \( \partial \Omega \), and the nonlinearity \( f(u) \) and external force term \( g \) will be specified later.
For brevity, we use the following abbreviations:

\[ L^p = L^p(\Omega), \quad H^k = H^k(\Omega), \quad V_t = H^1_0, \quad V_t^{-1} = H^{-1}, \quad \| \cdot \| = \| \cdot \|_{L^2}, \quad \| \cdot \|_p = \| \cdot \|_{L^p}, \]

with \( p \geq 1, H^k \) are the \( L^2 \)-based Sobolev spaces. The notation \((\cdot, \cdot)\) for the \( L^2 \)-inner product will also be used for the notation of duality pairing between dual spaces, the sign \( H_1 \hookrightarrow H_2 \) denotes that the functional space \( H_1 \) continuously embeds into \( H_2 \) and \( H_1 \hookrightarrow \hookrightarrow H_2 \) denotes that \( H_1 \) compactly embeds into \( H_2 \), and \( C(\cdot, \cdot) \) stands for positive constants depending on the quantities appearing in the parenthesis.

Rewriting Eq. (41) at an abstract level, we obtain

\[
\begin{align*}
&u_t + M(\|u\|_{V_1}^2)A u + A^\alpha u_t + f(u) = g(t), \\
&(u(\tau), u_t(\tau)) = (u_0^\tau, u_1^\tau) = \xi_\alpha(\tau),
\end{align*}
\]

where \( A = -\Delta \), with Dirichlet boundary condition. Obviously, the operator

\[ A : V_1 \rightarrow V_{-1}, \quad (Au, v) = (\nabla u, \nabla v) \quad \text{for any} \quad u, v \in V_1. \]

\( A \) is self-adjoint in \( L^2 \) and strictly positive on \( V_1 \). Then we can define the power \( A^s \) of \( A (s \in \mathbb{R}) \), and the spaces \( V_s = D(A^{\frac{s}{2}}) \) are the Hilbert spaces with the scalar products and the norms

\[ (u, v)_s = (A^{\frac{s}{2}}u, A^{\frac{s}{2}}v), \quad \|u\|_{V_s} = \|A^{\frac{s}{2}}u\|, \]

respectively. We define the phase spaces

\[ X = V_1 \cap L^{p+1} \times L^2, \quad X_{1+\alpha} = V_{1+\alpha} \times V_\alpha, \quad X_\alpha = V_\alpha \times V_{-\alpha}, \]

with \( \alpha \in (1/2, 1) \), which are equipped with the usual graph norms, for example,

\[ \|(u, v)\|_X^2 = \|u\|_{V_1}^2 + \|u\|_{L^{p+1}}^2 + \|v\|^2. \]

Obviously, they are the Banach spaces, and

\[ X_{1+\alpha} \hookrightarrow \hookrightarrow X \hookrightarrow X_\alpha \quad \text{for} \quad 1 \leq p < p_\alpha = \frac{N + 4\alpha}{(N - 4\alpha)^+}. \]

In particular, \( X = V_1 \times L^2 \) when \( 1 \leq p < p^* \equiv \frac{N + 2}{(N - 2)^+} \) for \( V_1 \hookrightarrow L^{p+1} \).

**Assumption (H)**

(i) \( M \in C^1(\mathbb{R}^+), M'(s) \geq 0, M(0) \equiv M_0 > 0; \)

(ii) \( f \in C^1(\mathbb{R}), f(0) = 0. \) When \( N \geq 2 \), either

\[
\mu_f = \liminf_{|s| \to +\infty} \frac{f(s)}{s} > -\lambda_1 M_0, \quad |f'(s)| \leq C(1 + |s|^{p-1}), \quad (45)
\]

where \( \lambda_1(> 0) \) is the first eigenvalue of the operator \( A \), and \( 1 \leq p < +\infty \) if \( N = 2; 1 \leq p \leq p^* \equiv \frac{N + 2}{N - 2} \) if \( N \geq 3 \), in particular,

\[ f'(s) > -C_1, \quad s \in \mathbb{R} \quad \text{if} \quad p = p^*, \]

or else, there exist constants \( C_0 > 0, C_1 \geq 0 \) such that

\[
C_0 |s|^{p-1} - C_1 \leq f'(s) \leq C(1 + |s|^{p-1}) \quad \text{with some} \quad p^* < p < p_\alpha; \quad (46)
\]

(iii) \( g_0 \in H^1_b(\mathbb{R}; L^2) = \{ \phi \in L^2_b(\mathbb{R}; L^2) | \partial_t \phi \in L^2_b(\mathbb{R}; L^2) \}, \)

where

\[ L^2_b(\mathbb{R}; L^2) = \{ \phi \in L^2_{loc}(\mathbb{R}; L^2) | \sup_{t \in \mathbb{R}} \int_t^{t+1} \| \phi(s) \|^2 ds < \infty \}; \]

(iv) \( \xi_\alpha(\tau) = (u_0^\tau, u_1^\tau) \in X \) with \( \|\xi_\alpha(\tau)\|_X \leq R. \)

Let

\[ \Sigma = \{T(h)g_0 | h \in \mathbb{R} \} \]

(47)
be a symbol space, where \( \{T(h)\}_{h \in \mathbb{R}} \) is a translation group acting on \( L_{loc}^2(\mathbb{R}; L^2) \),
\[ T(h) : L_{loc}^2(\mathbb{R}; L^2) \to L_{loc}^2(\mathbb{R}; L^2), \quad (T(h)g_0)(t) = g_0(t+h), \quad h, t \in \mathbb{R}. \]
Obviously, \( \Sigma \) is invariant with respect to \( \{T(h)\}_{h \in \mathbb{R}} \), i.e., \( T(h)\Sigma = \Sigma \), \( h \in \mathbb{R} \).

**Lemma 4.1.** [21] Let \( X \) be a Banach space with dual \( X' \), \( u, g \in L^1(a, b; X) \). Then the following three conditions are equivalent:

(i) There exists a \( \xi \in X \) such that
\[ u(t) = \xi + \int_a^t g(s)ds \quad \text{for a.e.} \quad t \in [a, b]. \]

(ii)
\[ \int_a^b u(t)\phi'(t)dt = -\int_a^b g(t)\phi(t)dt, \quad \forall \phi \in C_0^\infty(a, b). \]

(iii) For each \( \eta \in X' \),
\[ \frac{d}{dt} \langle u, \eta \rangle = \langle g, \eta \rangle \]
in the scalar distribution sense on \( (a, b) \).

If one of the conditions (i)-(iii) holds, then \( g = \partial_t u \) is the \( (X\text{-valued}) \) distribution derivative of \( u, u \in C([a, b]; X) \) and
\[ \sup_{a \leq t \leq b} \| u(t) \|_X \leq C \| u \|_{W^{1,1}(a, b; X)}, \]
where the constant \( C = C(b-a) \).

**Lemma 4.2.** Let \( g_0 \in H^1_0(\mathbb{R}; L^2) \). Then for every \( g \in \Sigma \) (see (47)), the \( (L^2\text{-valued}) \) distribution derivative \( \partial_t g \in L^2_0(\mathbb{R}; L^2) \), and
\[ \sup_{g \in \Sigma} \| \partial_t g \|_{L^2_0(\mathbb{R}; L^2)} \leq \| \partial_t g_0 \|_{L^2(\mathbb{R}; L^2)} \]
\[ \sup_{g \in \Sigma} \| g \|_{C_b(\mathbb{R}; L^2)} \leq C \| g_0 \|_{H^1_0(\mathbb{R}; L^2)}, \]
that is, \( \Sigma \) is a bounded set in \( C_b(\mathbb{R}; L^2) \).

**Proof.** For every \( g \in \Sigma, g = T(h)g_0 \) for some \( h \in \mathbb{R} \). For any \( [a, b] \subset \mathbb{R} \), due to \( T(h)g_0, T(h)\partial_t g_0 \in L^2(a, b; L^2) \), we have, for each \( \phi \in C_0^\infty(a, b), \)
\[ \int_a^b g(t)\phi'(t)dt = \int_a^b (T(h)g_0)(t)\phi'(t)dt = \int_a^b g_0(t+h)\phi'(t)dt \]
\[ = -\int_a^b \partial_t g_0(t+h)\phi(t)dt = -\int_a^b T(h)\partial_t g_0(t)\phi(t)dt, \]
which means (by the arbitrariness of \( [a, b] \subset \mathbb{R} \)),
\[ \partial_t g = \partial_t(T(h)g_0) = T(h)(\partial_t g_0) \in L^2_{loc}(\mathbb{R}; L^2), \sup_{g \in \Sigma} \| \partial_t g \|_{L^2_{loc}(\mathbb{R}; L^2)} \leq \| \partial_t g_0 \|_{L^2_{loc}(\mathbb{R}; L^2)}. \]
For any \( a \in \mathbb{R}, \) it follows from Lemma 4.1 that \( g \in H^1(a, a+1; L^2) \hookrightarrow C([a, a+1]; L^2) \),
\[ \sup_{t \in [a, a+1]} \| g(t) \| \leq C \| g \|_{H^1(a, a+1; L^2)} \leq C \| g_0 \|_{H^1(\mathbb{R}; L^2)}. \]
By the arbitrariness of \( a \in \mathbb{R} \), we have \( g \in C_b(\mathbb{R}; L^2) \) and
\[ \sup_{g \in \Sigma} \| g \|_{C_b(\mathbb{R}; L^2)} \leq C \| g_0 \|_{H^1(\mathbb{R}; L^2)}. \]
Therefore, \( \Sigma \) is a bounded set in \( C_b(\mathbb{R}; L^2) \).
Theorem 4.3. Let Assumption (H) be valid and \( g \in \Sigma \). Then problem \((43)-(44)\) admits a unique weak solution \( u \), with \((u, u_t) \in L^\infty(\tau, T; X) \cap C(\tau, T; X)\), and

\[
\|(u, u_t)(t)\|_X^2 + \|u_{xt}(t)\|_{V_{-2a}}^2 + \int_\tau^t \|u_x(s)\|_{V_a}^2 \, ds \leq \tilde{C}, \quad t \in [\tau, T],
\]

where \( \tilde{C} \equiv C(R, t - \tau, \|g_0\|_{H^1_\alpha(\mathbb{R}, L^2)}) \). Moreover, the solution \( u \) processes the following properties:

(i) (Global regularity for \( t > \tau \)) For any \( a \in (\tau, T), T > \tau, (u_t, u_{tt}) \in L^\infty(a, T; X_a) \cap L^2(a, T; V_{1} \times H) \), and

\[
\|(u_t, u_{tt})(t)\|_{X_a}^2 + \int_a^t \left( \|u_t(s)\|_{V_1}^2 + \|u_{tt}(s)\|^2 + b_0 \int_\Omega |u|^{p-1} |u_t|^2 \, dx \right) \, ds \leq \left( 1 + \frac{1}{(t - \tau)^2} \right) \tilde{C}, \quad t > \tau,
\]

where \( b_0 = 0 \) if \( 1 \leq p \leq p^* \); \( b_0 = 1 \) if \( p^* < p < p_\alpha \). Furthermore, for any \( b \in (a, T) \), \( u \in L^\infty(a, T; V_{1+a}) \), and

\[
\|u(t)\|_{V_{1+a}}^2 \leq \left( 1 + \frac{1}{(t-a)^{1/(1-a)}} \right) \tilde{C}, \quad t > a.
\]

(ii) (Energy identity) The following energy identity

\[
E(\xi_a(t_2)) + \int_{t_1}^{t_2} \left( \|u_t(s)\|_{V_0}^2 - \int_\Omega g u \, dx \right) \, ds = E(\xi_a(t_1))
\]

holds for every \( t_2 > t_1 \geq \tau \), where \( \xi_a = (u, u_t) \),

\[
E(u, v) = \frac{1}{2} \left( \|v\|^2 + \int_0^v u \, ds \right) + \int_\Omega F(u) \, dx, \quad (u, v) \in X.
\]

(iii) (Weak stability and weak quasi-stability) The following stability estimate holds in weaker space \( X_a(= V_a \times V_{-a}) \):

\[
\|(z, z_t)(t)\|^2_{X_a} + \int_\tau^t \left( \|z(s)\|_{V_1}^2 + \|z_t(s)\|^2 + b_0 \int_\Omega (|u|^{p-1} + |u|^r-1) |z(s)|^2 \, dx \right) \, ds \
\leq \tilde{C} e^{k(t-\tau)} \left( \|(z, z_t)(\tau)\|^2_{X_a} + \|g_1 - g_2\|_{L^2(\tau, t; L^2)}^2 \right), \quad t \geq \tau.
\]

Furthermore,

\[
\|(z, z_t)(t)\|^2_{X_a} \leq \tilde{C} e^{-\kappa(t-\tau)} \|(z, z_t)(\tau)\|^2_{X_a}
\]

\[
+ \tilde{C} \int_\tau^t e^{-\kappa(t-s)} \left( \|z(s)\|^2 + \|z_t(s)\|_{V_{-2a}}^2 + \|(g_1 - g_2)(s)\|^2 \right) \, ds, \quad t \geq \tau,
\]

where \( z = u^1 - u^2, u^1 \) and \( u^2 \) are two weak solutions of Eq. \((43)\) corresponding to symbol \( g_1, g_2 \in \Sigma \) and initial data \( \xi^1_a(\tau), \xi^2_a(\tau) \in X \), with \( \|\xi^1_a(\tau)\|_X + \|\xi^2_a(\tau)\|_X \leq R \), respectively, \( k > 0 \), and \( \kappa \) is a small positive constant.

Proof. Similar to the arguments to the autonomous case (see [24]), one can easily obtain the conclusions of Theorem 4.3 except property (i). Thus we only prove property (i) here.

Formally differentiating Eq. \((43)\) with respect to \( t \) we receive that \( v = u_t \) solves

\[
v_{tt} + M(\|u\|_{V_1}^2) A v + A^\alpha v_t + 2M'(\|u\|_{V_1}^2)(A^{1/2} u, A^{1/2} v) A v + f'(u) v = g_t.
\]
Using the multiplier $A^{-\alpha}v_t + \epsilon v$ in (53), we have
\[
\begin{align*}
\frac{d}{dt} H_1(\xi_v) &+ (1 - \epsilon)\|v_t\|^2 + \epsilon f'(u)v + A^{-\alpha}v_t \\
&+ \epsilon \left(M(||u||_{V_1})||v_t||^2 + 2M'(||u||_{V_1})(A^\frac{1}{2}u, A^\frac{1}{2}v)^2\right) \\
&= -2M'(||u||_{V_1})(A^\frac{1}{2}u, A^\frac{1}{2}v)(Au, A^{-\alpha}v_t) \\
&- M(||u||_{V_1})(Av, A^{-\alpha}v_t) + (g_t, A^{-\alpha}v_t + \epsilon v),
\end{align*}
\]
where $\xi_v = (v, v_t)$, and
\[
H_1(\xi_v) = \frac{1}{2} \left[ \|v_t\|_{V_\alpha}^2 + \epsilon \left(\|v\|_{V_\alpha}^2 + 2(v, v_t)\right) \right] \sim \|v_t\|_{V_\alpha}^2 + \|v\|_{V_\alpha}^2
\]for $\epsilon > 0$ suitably small. Repeating the similar arguments as in [24], one easily obtains
\[
\begin{align*}
\frac{d}{dt} H_1(\xi_v) &+ \kappa H_1(\xi_v) + \frac{1}{2}\|v_t\|^2 + \kappa \left(\|v\|_{V_1}^2 + b_0 \int_\Omega |u|^{p-1}v^2dx\right) \\
&\leq C(\|v\|^2 + \|v_t\|^2_{V_{\alpha-\nu}}) + C\|g_t\|^2.
\end{align*}
\]
When $t : \tau < t \leq \tau + 1, \tau \in \mathbb{R}$, multiplying (55) by $(t - \tau)^2$, we have
\[
\begin{align*}
\frac{d}{dt} &\left((t - \tau)^2H_1(\xi_v)\right) + \kappa(t - \tau)^2H_1(\xi_v) \\
&+ \kappa(t - \tau)^2 \left(\|v\|_{V_1}^2 + b_0 \int_\Omega |u|^{p-1}v^2dx\right) + \frac{1}{2}(t - \tau)^2\|v_t\|^2 \\
&\leq C(t - \tau)^2 \left(\|v\|^2 + \|v_t\|^2_{V_{\alpha-\nu}} + \|g_t\|^2\right) + C(t - \tau) \left(\|v_t\|^2_{V_{\alpha-\nu}} + \|v\|_{V_\alpha}^2\right) \\
&\leq C \left(\|v\|^2_{V_\alpha} + \|v_t\|^2_{V_{\alpha-\nu}} + \|g_t\|^2\right) + \frac{1}{4}(t - \tau)^2\|v_t\|^2,
\end{align*}
\]
where we have used the fact
\[
C(t - \tau)\|v_t\|^2_{V_{\alpha-\nu}} \leq C(t - \tau)\|v_t\|_{V_{\alpha-\nu}} \leq \frac{1}{4}(t - \tau)^2\|v_t\|^2 + C\|v_t\|^2_{V_{\alpha-\nu}}.
\]
It follows from (56) and (48) that
\[
\begin{align*}
(t - \tau)^2 H_1(\xi_v) &\leq C \int_\tau^t e^{-\kappa(t-s)} \left(\|u_t\|^2_{V_\alpha} + \|u_{tt}\|^2_{V_{\alpha-\nu}}\right)ds \leq \tilde{C}, \\
\|u_t(t)\|_{V_\alpha}^2 + \|u_{tt}(t)\|_{V_{\alpha-\nu}}^2 &\leq \frac{1}{(t - \tau)^2}\tilde{C}, \quad \tau < t \leq \tau + 1, \tau \in \mathbb{R}.
\end{align*}
\]
When $t > \tau + 1$, applying the Gronwall lemma to (55) over $(\tau + 1, t)$ and exploiting (57) yield
\[
\|u_t(t)\|_{V_\alpha}^2 + \|u_{tt}(t)\|_{V_{\alpha-\nu}}^2 \leq \tilde{C}, \quad t > \tau + 1, \tau \in \mathbb{R},
\]
where we have used the fact
\[
\begin{align*}
\int_{\tau+1}^t e^{-\kappa(t-s)}|g_t(s)|^2ds \leq \sum_{k=0}^\infty \int_{t-k}^{t-k-1} e^{-\kappa(t-s)}|g_t(s)|^2ds \\
\leq \sum_{k=0}^\infty e^{-\kappa k} \int_{t-k}^{t-k-1} \|g_t(s)\|^2ds \leq \tilde{C}.
\end{align*}
\]
Integrating (56) over \((t, t + 1)\) (with \(t > \tau\)) and using (48), (57)-(58), we obtain
\[
\int_{t}^{t+1} \left( \|u_t(s)\|_{V_1} + \|u_{tt}(s)\|^2 \right) + b \int_{\Omega} |u|^p \|u_t|^2 \, dx \leq \left(1 + \frac{1}{(t-\tau)^2}\right) \tilde{C}. \tag{59}
\]

The combination of (57)-(59) gives (49).

Using the multiplier \(Au\) in Eq. (43), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{V^{1+\alpha}} + M_0 \|u\|^2_{V_2} + (f'(u), |\nabla u|^2) \\
\leq -(u_{tt} - g, Au) \leq \frac{M_0}{4} \|u\|^2_{V_2} + C(\|u_{tt}\|^2 + \|g\|^2).
\]

When \(1 \leq p < p^*,\) taking account of the Sobolev embedding: \(V_{1-\delta} \hookrightarrow L^{p+1}\) for \(\delta > 0 < \delta < 1\) and the interpolation theorem, we have
\[
\|(f'(u), |\nabla u|^2)\| \leq C(1 + \|u\|^{p-1}_{p+1}) \||\nabla u||^2_{p+1} \leq C\|u\|^2_{V^{1-\delta}} \leq C\|u\|^2_{V_1} + \frac{M_0}{4} \|u\|^2_{V_2},
\]

and when \(p^* < p < p_0, (f'(u), |\nabla u|^2) \geq -C\|u\|^2_{V_1}\) (see (46)), then we obtain
\[
\frac{d}{dt} \|u\|^2_{V^{1+\alpha}} + M_0 \|u\|^2_{V_2} \leq C(\|u\|^2_{V_1} + \|u_{tt}\|^2 + \|g\|^2). \tag{60}
\]

For any \(a > \tau,\) when \(a < t \leq a + 1,\) multiplying (60) by \((t-a)^{1-\alpha}\) and using the interpolation theorem, we obtain
\[
\frac{d}{dt} \left((t-a)^{1-\alpha} \|u\|^2_{V^{1+\alpha}}\right) + M_0(t-a)^{1-\alpha} \|u\|^2_{V_2} \\
\leq C(\|u\|^2_{V_1} + \|u_{tt}\|^2 + \|g\|^2) + \frac{1}{1-\alpha} (t-a)^{\alpha-\alpha} \|u\|^2_{V_1} \|u\|^2_{V_1} (1-\alpha) \\
\leq \frac{M_0}{2} (t-a)^{1-\alpha} \|u\|^2_{V_2} + C(\|u\|^2_{V_1} + \|u_{tt}\|^2 + \|g\|^2).
\]

Therefore,
\[
\|u(t)\|^2_{V^{1+\alpha}} \leq \frac{C}{(t-a)^{1-\alpha}} \int_{a}^{t} e^{-\kappa(t-s)} (\|u(s)\|^2_{V_1} + \|u_{tt}(s)\|^2 + \|g(s)\|^2) \, ds \tag{61}
\]
\[
\leq \frac{1}{(t-a)^{1-\alpha}} \tilde{C}.
\]

When \(t > a + 1,\) applying the Gronwall lemma to (60) over \((a+1, t)\) and exploiting (61), we have
\[
\|u(t)\|^2_{V^{1+\alpha}} \leq C \int_{a+1}^{t} e^{-\kappa(t-s)} (\|u(s)\|^2_{V_1} + \|u_{tt}(s)\|^2 + \|g(s)\|^2) \, ds \\
+ C\|u(a+1)\|^2_{V^{1+\alpha}} \leq \tilde{C}, \tag{62}
\]

where we have used the fact
\[
\int_{a+1}^{t} e^{-\kappa(t-s)} \|u_{tt}(s)\|^2 \, ds = \sum_{k=1}^{\lfloor \frac{t-a}{a+k+1} \rfloor} \int_{a+k}^{a+k+1} e^{-\kappa(t-s)} \|u_{tt}(s)\|^2 \, ds \leq \tilde{C}.
\]

The combination of (61) and (62) gives (50).
For every \( \tau \in \mathbb{R}, g \in \Sigma \), we define the operator
\[ U_g(t, \tau) : X \to X, \quad U_g(t, \tau) \xi = (u(t), u(t)), \quad t \geq \tau, \]
where \( u \) is the weak solution of problem (43)-(44). Theorem 4.3 shows that \( U_g(t, \tau) \)
is well-defined and \( \{U_g(t, \tau)\}, g \in \Sigma \) constitutes a family of processes on \( X \), and the
following translation identity holds:
\[ U_{T(h)g}(t, \tau) = U_g(t + h, \tau + h), \quad h \geq 0, t \geq \tau, \tau, \tau \in \mathbb{R}. \]

**Theorem 4.4.** Let Assumption (H) be valid. Then
(i) for each \( g \in \Sigma \), the non-autonomous dynamical system \((U_g(t, \tau), X)\) has a
pullback exponential attractor \( \{M^g(t)\}_{t \in \mathbb{R}} \), where the sections \( M^g(t) \) are
uniformly (w.r.t. \( g \in \Sigma \) and \( t \in \mathbb{R} \)) bounded in \( X_{1+\alpha} \);
(ii) there exists a \( \delta > 0 \) such that for any \( g \in \Sigma \), with \( \|g - g_0\|_{L^2_x} \leq \delta \),
\[ \sup_{t \in \mathbb{R}} \text{dist}_X \{M^g(t), M^{g_0}(t)\} \leq C \|g - g_0\|^\nu_{L^2_x}, \]
where \( C > 0 \) and \( \nu \in (0, 1) \) are some constants independent of \( g \).

In order to prove Theorem 4.4, we first give a lemma.

**Lemma 4.5.** Under the assumptions of Theorem 4.4, there exists a closed subset \( B \) in \( X_{1+\alpha} \), which is bounded in \( X_{1+\alpha} \) and satisfies
\[ \bigcup_{g \in \Sigma} \bigcup_{\tau \in \mathbb{R}} U_g(t + \tau, \tau)B \subset B \quad \text{for } t \geq T_1, \]
where \( T_1 > 0 \) is a constant. Moreover, for each \( g \in \Sigma \), \( B \) is a uniformly (w.r.t. \( \tau \in \mathbb{R} \)) absorbing set of the non-autonomous dynamical system \((U_g(t, \tau), X)\).

**Proof.** Using the multiplier \( u_t + \epsilon u \) in Eq. (43), one easily obtains
\[ \frac{d}{dt} H_2(\xi_u) + \kappa \left( \|u\|^2_{V_1} + b_0 \|u\|^{p+1}_{p+1} + \|u_t\|^2 \right) \leq C(\|g_0\|_{H^1_x} + 1), \]
where \( \xi_u = (u, u_t) \) and
\[ H_2(\xi_u) = \frac{1}{2} \|u\|^2 + \frac{1}{2} \int_0^t M(s)ds + (F(u), 1) + \epsilon((u, u_t) + \frac{1}{2} \|u\|^2_{V_1}). \]

A simple calculation shows that there exist two constants \( k_1, k_2 > 0 \) such that
\[ \phi_1(\xi_u) \leq H_2(\xi_u) \leq \phi_2(\xi_u), \]
for \( \epsilon > 0 \) suitably small, where
\[ \phi_1(\xi_u) = k_1 \Phi(\xi_u) - C = k_1 \left( \|u\|^2_{V_1} + b_0 \|u\|^{p+1}_{p+1} + \|u_t\|^2 \right) - C, \]
\[ \phi_2(\xi_u) = k_2 \Phi(\xi_u) + \frac{1}{2} M(\|u\|^2_{V_1}) \|u\|^2_{V_1}. \]

Therefore,
\[ \frac{d}{dt} H_2(\xi_u) + \kappa \Phi(\xi_u) \leq C(\|g_0\|_{H^1_x} + 1). \]

Since the functionals \( H_2(\xi_u), \Phi(\xi_u) \) and \( \phi_j(\xi_u) (j = 1, 2) \) are continuous on \( X \), and
\[ \lim_{\|\xi_u\|_{X} \to \infty} |\Phi(\xi_u)| = +\infty, \quad \lim_{\|\xi_u\|_{X} \to \infty} |\phi_j(\xi_u)| = +\infty, \quad j = 1, 2, \]
by virtue of the standard “barrier” method (cf. [6]), we infer from (66)-(68) that the family of processes \( \{U_g(t, \tau)\}, g \in \Sigma \) has a uniformly (w.r.t. \( g \in \Sigma \)) absorbing ball \( B_0 = \{x \in X : \|x\|_X \leq R_0\} \).
Obviously, there exists a $T_0 > 0$ such that $\bigcup_{g \in \Sigma} U_g(t, 0) \mathcal{B}_0 \subset \mathcal{B}_0$ for $t \geq T_0$. Let
\[
\mathcal{B} = \left[ \bigcup_{g \in \Sigma} \bigcup_{t \geq 1 + T_0} U_g(t, 0) \mathcal{B}_0 \right]_{X_a}. \tag{69}
\]
We claim that $\mathcal{B}$ is the desired absorbing set. Indeed, (i) for every bounded set $\mathcal{B} \subset X$, there exists a time $t_0 = t_0(\mathcal{B}) > 0$ such that $\bigcup_{g \in \Sigma} U_g(t, 0) \mathcal{B} \subset \mathcal{B}_0$ for $t \geq t_0$. We have known that for any $\tau \in \mathbb{R}$, $g \in \Sigma$ and $t \geq t_0'$, with $t_0' = t_0 + 1 + T_0$, there exist two elements $g^1, g^2 \in \Sigma$ (cf. Lemma 2.1 in [19]) such that
\[
U_g(t + \tau, \tau) = U_{g^1}(t, 0), \quad U_{g^1}(t, t_0) = U_{g^2}(t - t_0, 0),
\]
\[
U_g(t + \tau, \tau) \mathcal{B} = U_{g^1}(t, 0) \mathcal{B} = U_{g^1}(t, t_0) U_{g^2}(t_0, 0) \mathcal{B}
\subset U_{g^2}(t - t_0, 0) \mathcal{B}_0 \subset \mathcal{B},
\]
i.e., $\mathcal{B}$ is a uniformly (w.r.t. $\tau \in \mathbb{R}$) absorbing set of the non-autonomous dynamical system $(U_g(t, \tau), X)$.

(ii) Due to $\bigcup_{g \in \Sigma} U_g(t, 0) \mathcal{B}_0 \subset \mathcal{B}_0$ for $t \geq T_0$, we have $\mathcal{B} \subset [\mathcal{B}_0]_{X_a} = \mathcal{B}_0$. Then
\[
\bigcup_{g \in \Sigma} \bigcup_{\tau \in \mathbb{R}} U_g(t + \tau, \tau) \mathcal{B} \subset \bigcup_{g \in \Sigma} \bigcup_{\tau \in \mathbb{R}} U_g(t + \tau, \tau) \mathcal{B}_0 = \bigcup_{g \in \Sigma} U_g(t, 0) \mathcal{B}_0 \subset \mathcal{B} \quad \text{for} \quad t \geq T_1,
\]
where $T_1 = T_0 + 1$, i.e., (64) holds.

(iii) We show that $\mathcal{B}$ is bounded in $X_{1+\alpha}$. By the lower semi-continuity of the norm, it is enough to show that
\[
\bigcup_{g \in \Sigma} \bigcup_{t \geq 1 + t_0} U_g(t, 0) \mathcal{B}_0 \quad \text{is bounded in} \quad X_{1+\alpha}. \tag{70}
\]
For every $g \in \Sigma$ and $t \geq 1 + T_0$,
\[
U_g(t, 0) \mathcal{B}_0 = U_g(t, t - 1) U_g(t - 1, 0) \mathcal{B}_0 \subset U_g(t, t - 1) \mathcal{B}_0.
\]
Then
\[
\bigcup_{g \in \Sigma} \bigcup_{t \geq 1 + T_0} U_g(t, 0) \mathcal{B}_0 \subset \bigcup_{g \in \Sigma} \bigcup_{t \in \mathbb{R}} U_g(t, t - 1) \mathcal{B}_0.
\]
For any $g \in \Sigma$, $\xi = (u_0, u_1) \in \mathcal{B}_0$, let $U_g(t, t - 1)\xi = \xi u(t) = (u(t), u_1(t))$. We infer from (49)-(50) that (taking $\tau = t - 1$ and $\alpha = t - 1/2$ there)
\[
\|u(t)\|_{H_{1+\alpha}}^2 + \|u_1(t)\|_{H_{1+\alpha}}^2 \leq C(R_0, \|g_0\|_{H^1(\mathbb{R}; L^2)}) \tag{71}, \quad t \in \mathbb{R}.
\]
By the arbitrariness of $t \in \mathbb{R}$, $g \in \Sigma$ and $\xi = (u_0, u_1) \in \mathcal{B}_0$, (70) holds. Therefore, $\mathcal{B}$ is a bounded set in $X_{1+\alpha}$. \hfill \Box

**Proof of Theorem 4.4.** (i) We first show that the family of non-autonomous dynamical systems $(U_g(t, \tau), \mathcal{B}, X_a)$ ($\mathcal{B}$ is equipped with the distance $d(x, y) = \|x - y\|_{X_a}$ and $g \in \Sigma$) has a robust family of pullback exponential attractors.

By estimate (48) and Lemma 4.5,
\[
\sup_{g \in \Sigma} \sup_{\xi \in \mathcal{B}} \|U_g(t + \tau, \tau)\xi\|_{X_a}^2 \leq C(R_0, \|g_0\|_{H^1(\mathbb{R}; L^2)}) \equiv C^* \quad \text{for} \quad t \geq 0, \quad \tau \in \mathbb{R}. \tag{71}
\]
For every $\xi^1, \xi^2 \in \mathcal{B}, g_1, g_2 \in \Sigma$ and $\tau \in \mathbb{R}$, let
\[
(z(t + \tau), z(t + \tau)) = U_{g_1}(t + \tau, \tau)\xi^1 - U_{g_2}(t + \tau, \tau)\xi^2
\]
\[
= (u^1(t + \tau), u_1^1(t + \tau)) - (u^2(t + \tau), u_1^2(t + \tau)), \quad t \geq 0.
\]
Then \( z \) solves
\[
\begin{align*}
  z_{tt} + M_{12}(t + \tau)A\delta z + A^\alpha z_t + f(u^1) - f(u^2) \\
  = -\tilde{M}_{12}(t + \tau)(A^\frac{1}{2}(u^1 + u^2), A^\frac{1}{2} z)A(u^1 + u^2) + g_1 - g_2, \\
  (z(\tau), z_t(\tau)) = \xi^1 - \xi^2,
\end{align*}
\]
where
\[
M_{12}(t + \tau) = \frac{1}{2}[M(\|u^1(t + \tau)\|_{V_1}) + M(\|u^2(t + \tau)\|_{V_1})],
\]
\[
\tilde{M}_{12}(t + \tau) = \frac{1}{2} \int_0^1 M'(\lambda\|u^1(t + \tau)\|_{V_1} + (1 - \lambda)\|u^2(t + \tau)\|_{V_1}) d\lambda.
\]
It follows from (51)-(52) that
\[
\begin{align*}
  &\|\langle z(t + \tau) \rangle^2_X\|_{V_{2\alpha}} \\
  &+ \int_\tau^{t+\tau} \left(\|z(s)\|^2_{V_1} + \|z_t(s)\|^2 + b_0 \int_\Omega \left(\|u^1|^{p-1} + \|u^2|^{p-1}\right) dx ds \right) ds \\
  &\leq C^* e^{\tau t} \left(\|\xi^1 - \xi^2\|^2_{X_{\alpha}} + \|g_1 - g_2\|^2_{L^2(\tau, \tau + \tau; L^2)} \right), \\
  &\quad t \geq 0, \\
  \end{align*}
\]
(73)

\[
\begin{align*}
  &\|\langle z(t + \tau) \rangle^2_X\|_{V_{2\alpha}} \\
  &+ C^* \int_\tau^{t+\tau} \left(\|\langle z(s) \rangle^2_{V_{2\alpha}} + \|g_1 - g_2\|^2_{L^2(\tau, \tau + \tau; L^2)} \right) ds, \\
  &\quad t \geq 0.
\end{align*}
\]
(74)

Let \( r > \max\{N/2, 2\alpha\} \), which means \( L^2 \hookrightarrow V_{2\alpha-r} \) and \( L^1 \hookrightarrow V_{r} \). By Eq. (72) and estimate (73),
\[
\begin{align*}
  &\int_\tau^{t+\tau} \|z_{tt}(s)\|^2_{V_{r}} ds \\
  &\leq C^* \int_\tau^{t+\tau} \left(\|A\delta z\|^2_{V_{r}} + \|A^\alpha z_t\|^2_{V_{r}} + \|f(u^1) - f(u^2)\|^2_{V_{r}} \right) ds \\
  &\leq C^* \int_\tau^{t+\tau} \left(\|z\|^2_{V_1} + \|z_t\|^2 + b_0 \int_\Omega \left(\|u^1|^{p-1} + \|u^2|^{p-1}\right) dx ds \right) ds \\
  &\leq C^* e^{\tau t} \left(\|\xi^1 - \xi^2\|^2_{X_{\alpha}} + \|g_1 - g_2\|^2_{L^2(\tau, \tau + \tau; L^2)} \right), \\
  &\quad t \geq 0.
\end{align*}
\]
(75)

Taking \( T_2 > 0 \) large enough such that \( \eta^2 \equiv C^* e^{-\kappa T_2} < 1 \), and letting \( T = \max\{T_1, T_2\} \), where \( T_1 \) is as shown in Lemma 4.5, we infer from estimate (73) that
\[
\begin{align*}
  &\|U_{t}(t + \tau, \tau)\xi^1 - U_{t}(t + \tau, \tau)\xi^2\|_{X_{\alpha}} \leq L_T\|\xi^1 - \xi^2\|_{X_{\alpha}}, \\
  &\|U_{t}(t + \tau, \tau)\xi - U_{t}(t + \tau, \tau)\xi\|_{X_{\alpha}} \leq L_T\sqrt{T + 1}\|g_1 - g_2\|_{L^2(0, T; L^2)},
\end{align*}
\]
(76)

for all \( \xi, \xi^1 \in \mathcal{B}, g, g_1 \in \Sigma \) (\( i = 1, 2 \)), \( \tau \in \mathbb{R} \) and \( t \in [0, T] \), where \( L_T = C^* e^{\kappa T} \), i.e., estimate (33) holds.

Define the space
\[
Z = \{(u, u_t) \in L^2(0, T; X_{\alpha})| u_{tt} \in L^2(0, T; V_{r})\}
\]
equipped with the norm
\[
\| (u, u_t) \|_Z^2 = \int_0^T \left(\|u(t)\|^2_{V_{2\alpha}} + \|u_t(t)\|^2_{V_{2\alpha}} + \|u_{tt}(t)\|^2_{V_{r}} \right) dt, \quad (u, u_t) \in Z.
\]
Obviously, $Z$ is a Banach space. Define the mapping, for each $n \in \mathbb{Z}$ and $g \in \Sigma$,
\[ K_n^g : B \to Z, \quad K_n^g \xi_u = U_g(\cdot + nT, nT)\xi = (u(\cdot + nT), u_t(\cdot + nT)), \]
where $u(\cdot + nT)$ means $u(t + nT), t \in [0, T]$. By estimates (73) and (75),
\[ \|K_n^g\xi^1 - K_n^g\xi^2\|^2_2 = \int_0^{nT} \left( \|z(s)\|^2_{V_a} + \|z_t(s)\|^2_{V_{-\alpha}} + \|z_{tt}(s)\|^2_{V_{-\alpha}} \right) ds 
\leq \int_0^{nT} \left( L^2_\eta \|\xi^1 - \xi^2\|^2_{V_a} + \|z_t(s)\|^2_{V_{-\alpha}} \right) ds 
\leq L^2_\eta (T + 1) \|\xi^1 - \xi^2\|^2_{V_a} = L^2 \|\xi^1 - \xi^2\|^2_{V_a}, \]
for all $g \in \Sigma$, $\xi^1, \xi^2 \in B$ and $n \in Z$, where $L^2 = L^2_\eta (T + 1)$, i.e., condition (34) holds. Let
\[ n_Z(u, u_t) = C_\ast \|(u, u_t)\|_{L^2(0,T;L^2 \times V_{-\alpha})}, \quad (u, u_t) \in Z. \]
Obviously (cf. [20]), $n_Z(\cdot)$ is a compact seminorm on $Z$, and (74) means that, for any $\xi^1, \xi^2 \in B, n \in \mathbb{Z}$ and $g \in \Sigma$,
\[ \|U_g((n + 1)T, nT)\xi^1 - U_g((n + 1)T, nT)\xi^2\|_{V_a} \leq \eta \|\xi^1 - \xi^2\|_{V_a} + n_Z(K_n^g\xi^1 - K_n^g\xi^2). \]
Therefore, by Theorem 3.1, for each $g \in \Sigma$ and $\theta \in (\eta, 1)$, the dynamical system $(U_g(t, \tau), B, X_a)$ has a pullback exponential attractor $\{M^g_\theta(t)\}_{t \in \mathbb{R}}$. Moreover, it is stable in the following sense: for every $g \in \Sigma$ satisfying $\|g - g_0\|_{L^2(\mathbb{R}; L^2)} \leq 1/(L_T \sqrt{T + 1})$,
\[ \sup_{t \in \mathbb{R}} \text{dist} \text{span}_{X_a} \{M^g_\theta(t), M^{g_0}_\theta(t)\} \leq C(\|g - g_0\|_{L^2(\mathbb{R}; L^2)}^\lambda), \]
where we have used the fact
\[ \Gamma(g, g_0) = \sup_{\tau \in [0,T]} \sup_{\xi_u \in B} \sup_{s \in [0,T]} \|U_g(\tau + s, \tau)\xi_u - U_{g_0}(\tau + s, \tau)\xi_u\|_{V_a} \leq L_T \sqrt{T + 1} \|g - g_0\|_{L^2(\mathbb{R}; L^2)}. \]
(ii) We show that the family of sets $\{M^g_\theta(t)\}_{t \in \mathbb{R}}$ is just a pullback exponential attractor of the non-autonomous dynamical system $(U_g(t, \tau), X)$, which is stable in the sense of (63).

By the interpolation theorem, for every $(u_0, u_1) \in B$,
\[ \|u_0\|_{V_1} + \|u_0\|_{1+\beta} \leq C\|u_0\|_{V_2} \leq C\|u_0\|_{V_{-\alpha}} \leq C_\ast\|u_0\|_{V_a}^{1-\alpha}, \]
\[ \|u_1\| \leq C\|u_1\|_{V_{-\alpha}}^{1/2} \|u_1\|_{V_a}^{1/2} \leq C_\ast\|u_1\|_{V_a}^{1-\alpha}, \]
i.e.,
\[ \|(u_0, u_1)\|_{X_a} \leq C_\ast\|(u_0, u_1)\|_{V_a}^{1-\alpha}, \quad (u_0, u_1) \in B. \]

Define the mapping
\[ I : B(\subset X_a) \to X, \quad I\xi_u = \xi_u, \quad \forall \xi_u \in B. \]
Estimate (81) shows that the mapping $I$ is $(1 - \alpha)$-Hölder continuous. Therefore, (i) $\mathcal{M}^g_\theta(t) = IM^g_\theta(t)(\subset B)$ (the image of a compact subset in $X_a$) is compact in $X$ and bounded in $X_{1+\alpha}$, and
\[ \sup_{g \in \Sigma} \sup_{t \in \mathbb{R}} \dim_f(M^g_\theta(t), X) \leq \frac{1}{1-\alpha} \sup_{g \in \Sigma} \sup_{t \in \mathbb{R}} \dim_f(M^g_\theta(t), X_a) \]
\[ \leq \frac{1}{1-\alpha} \left[ \ln \frac{1}{\theta} \right]^{-1} \ln m_Z \left( \frac{2L}{\theta - \eta} \right); \]
(ii) \[
\text{dist}_X \{ U_g(t, t-s)B, \mathcal{M}^\theta(t) \} \leq C^* \left[ \text{dist}_X \{ U_g(t, t-s)B, \mathcal{M}^\theta(t) \} \right]^{1-\alpha} \\
\leq C e^{-\mu s}, \quad s > 2T,
\]
where \( \mu = (1-\alpha)\beta \). Therefore, \( \{ \mathcal{M}^\theta(t) \}_{t \in \mathbb{R}} \) is the desired pullback exponential attractor because \( B \) is a uniformly (w.r.t. \( \tau \in \mathbb{R} \)) absorbing set of the non-autonomous dynamical system \( (U_g(t, \tau), X) \) (see Corollary 1). And
\[
\sup_{t \in \mathbb{R}} \text{dist}^{symm}_X \{ \mathcal{M}^\theta(t), \mathcal{M}^\theta_0(t) \} \leq C^* \sup_{t \in \mathbb{R}} \left[ \text{dist}^{symm}_X \{ \mathcal{M}^\theta(t), \mathcal{M}^\theta_0(t) \} \right]^{1-\alpha} \\
\leq C \| g - g_0 \|_{L^\infty(\mathbb{R}; L^2)}^{\nu},
\]
where \( \nu = \lambda(1-\alpha) \), i.e., (63) holds. \( \square \)

**Remark 2.** Under the assumptions of Theorem 4.4, one can easily infer from the standard theory on pullback attractor (cf. [5]) that the non-autonomous dynamical system \( (U_g(t, \tau), X) \) has a finite dimensional pullback attractor \( \{ A_g(t) \}_{t \in \mathbb{R}} \) for each \( g \in \Sigma \).

**Acknowledgments.** The authors thank the referee for his/her valuable comments and suggestions which helped improving the original manuscript.

**REFERENCES**

[1] S. Bosia and S. Gatti, Pullback exponential attractor for a Cahn-Hilliard-Navier-Stokes system in 2D, *Dynamics of PDE*, 11 (2014), 1–38.
[2] A. N. Carvalho, J. A. Langa, J. C. Robinson and A. Suárez, Characterization of non-autonomous attractors of a perturbed infinite-dimensional gradient system, *J. Differential Equations*, 236 (2007), 570–603.
[3] A. N. Carvalho and S. Sonner, Pullback exponential attractors for evolution processes in Banach spaces: theoretical results, *Commun. Pure Appl. Anal.*, 12 (2013), 3047–3071.
[4] A. N. Carvalho and S. Sonner, Pullback exponential attractors for evolution processes in Banach spaces: properties and applications, *Commun. Pure Appl. Anal.*, 13 (2014), 1114–1165.
[5] A. N. Carvalho, J. A. Langa and J. C. Robinson, *Attractors for Infinite-Dimensional Non-Autonomous Dynamical Systems*, Applied Mathematical Sciences, 182, Springer, New York, 2013.
[6] I. Chueshov, *Introduction to the Theory of Infinite-Dimensional Dissipative Systems*, AKTA, Kharkiv, 1999.
[7] I. Chueshov, Global attractors for a class of Kirchhoff wave models with a structural nonlinear damping, *J. Abstr. Differ. Equ. Appl.*, 1 (2010), 86–106.
[8] I. Chueshov, Long-time dynamics of Kirchhoff wave models with strong nonlinear damping, *J. Differential Equations*, 252 (2012), 1229–1262.
[9] I. Chueshov, *Dynamics of Quasi-Stable Dissipative Systems*, Springer, 2015.
[10] R. Czaja and M. Efendiev, Pullback exponential attractors for nonautonomous equations Part I: Semilinear parabolic problems, *J. Math. Anal. Appl.*, 381 (2011), 748–765.
[11] R. Czaja, Pullback exponential attractors with admissible exponential growth in the past, *Nonlinear Analysis*, 104 (2014), 90–108.
[12] M. Efendiev, A. Miranville and S. Zelik, Exponential Attractors for a Nonlinear Reaction-Diffusion System in \( \mathbb{R}^3 \), *C. R. Acad. Sci. Paris Sr. I Math.*, 330 (2000), 713–718.
[13] M. Efendiev, S. Zelik and A. Miranville, Exponential attractors and finite-dimensional reduction for nonautonomous dynamical systems, *Proceedings of the Royal Society of Edinburgh*, 135 (2005), 703–730.
[14] M. Efendiev, Y. Yamamoto and A. Yagi, Exponential attractors for non-autonomous dissipative systems, *Journal of the Mathematical Society of Japan*, 63 (2011), 647–673.
[15] G. Kirchhoff, Vorlesungen über Mechanik, (German) [Lectures on Mechanics], Teubner, Stuttgart, 1883.
[16] P. Kloeden, Pullback attractors of nonautonomous semidynamical systems, Stoch. Dyn., 3 (2003), 101–112.
[17] K. Kuratowski, Sur les espaces complets, Fund. Math., 15 (1930), 301–309.
[18] J. A. Langa, A. Miranville and J. Real, Pullback exponential attractors, Discrete Contin. Dyn. Syst., 26 (2010), 1329–1357.
[19] S. S. Lu, H. Q. Wu and C. K. Zhong, Attractors for nonautonomous 2D Navier-Stokes equations with normal external forces, Discrete Contin. Dyn. Syst., 13 (2005), 701–719.
[20] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl., 146 (1986), 65–96.
[21] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1988.
[22] Y. H. Wang and C. K. Zhong, Upper semicontinuity of pullback attractors for nonautonomous Kirchhoff wave models, Discrete Cont. Dyn. Sys., 33 (2013), 3189–3209.
[23] Z. J. Yang and Y. Q. Wang, Global attractor for the Kirchhoff type equation with a strong dissipation, J. Differential Equations, 249 (2010), 3258–3278.
[24] Z. J. Yang, P. Y. Ding and L. Li, Longtime dynamics of the Kirchhoff equation with fractional damping and supercritical nonlinearity, J. Math. Anal. Appl., 442 (2016), 485–510.
[25] S. F. Zhou and X. Y. Han, Pullback exponential attractors for non-autonomous lattice systems, J Dyn. Diff. Equat., 24 (2012), 601–631.

Received May 2017; revised November 2017.

E-mail address: yzjzzut@tom.com
E-mail address: LYN20112110109@163.com