Non-equilibrium dynamics of disordered systems: renormalization flow towards an ‘infinite disorder’ fixed point at large times

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Abstract. To describe the non-equilibrium dynamics of random systems, we have recently introduced (Monthus and Garel, 2008 J. Phys. A: Math. Theor. 41 255002) a ‘strong disorder renormalization’ (RG) procedure in configuration space that can be defined for any master equation. In the present paper, we analyze in detail the properties of the large time dynamics whenever the RG flow is towards some ‘infinite disorder’ fixed point, where the width of the renormalized barrier distribution grows indefinitely upon iteration. In particular, we show how the strong disorder RG rules can be then simplified while keeping their asymptotic exactness, because the preferred exit channel out of a given renormalized valley typically dominates asymptotically over the other exit channels. We explain why the present approach is an explicit construction favoring the droplet scaling picture where the dynamics is governed by the logarithmic growth of the coherence length $l(t) \sim (\ln t)^{1/\psi}$, and where the statistics of barriers corresponds to a very strong hierarchy of valleys within valleys. As an example of application, we have followed numerically the RG flow for the case of a directed polymer in a two-dimensional random medium. The full RG rules are used to check that the RG flow is towards some infinite disorder fixed point, whereas the simplified RG rules allow us to study bigger sizes and to estimate the barrier exponent $\psi$ of the fixed point.
Non-equilibrium dynamics of disordered systems

**Keywords:** disordered systems (theory), energy landscapes (theory), slow dynamics and ageing (theory), slow relaxation and glassy dynamics

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1. Introduction

The non-equilibrium dynamics of disordered systems gives rise to a lot of striking properties that have been much studied both theoretically and experimentally (see [1]–[3] and references therein). The first property that has attracted a lot of attention is ‘ageing’ which can be seen as some ‘criticality in the time direction’, in the following sense: if the dynamics has taken place during the time interval $[0, t_w]$, the only relevant timescale for the dynamics at larger times $t > t_w$ is the time $t_w$ itself. A well known example is the case of phase ordering of finite dimensional systems when the dynamics tends towards equilibrium presenting some long-ranged order. It is useful to introduce some coherence length $l(t)$ that separates the smaller lengths $l < l(t)$ which are quasi-equilibrated from the bigger lengths $l > l(t)$ which are completely out of equilibrium. Then equilibrium is reached only when the coherence length reaches the macroscopic linear size $l(t_{eq}) = L$ of the system. In pure systems, these phenomena of phase ordering are well understood [4] and the coherence length grows algebraically

$$l_{\text{pure}}(t) \sim t^{1/z}$$

with some dynamical exponent $z$ [4]. It is important to note that for pure systems, domain growth is possible even at zero temperature because domain walls can still move and annihilate. In the presence of quenched disorder however, the dynamics requires thermal activation (in particular, at zero temperature, the dynamics stops at the first encountered local minimum). Within the droplet scaling theory proposed both for spin glasses [5,6] and for directed polymers in random media [7], the barriers grow as a power law of the length $l$

$$B(l) \sim l^\psi$$

with some barrier exponent $\psi > 0$. The typical time $t_{\text{typ}}(l)$ associated with scale $l$ grows as an exponential $\ln t_{\text{typ}}(l) \sim B(l) \sim l^\psi$. Equivalently, the characteristic length scale $l(t)$ associated with time $t$ grows only logarithmically in time

$$l(t) \sim (\ln t)^{1/\psi}.$$

In numerical studies, this logarithmic behavior has remained controversial, because the dynamics is very slow not only in real life but also in Monte Carlo simulations! As a consequence, the maximal equilibrated length $l_{\text{max}}$ measured at the end of the simulations is usually rather small, so that various fits of the data are possible. Some authors use the algebraic fit of equation (1) with a temperature and disorder dependent exponent $z(T, \epsilon)$ either for disordered ferromagnets [8] or for spin glasses [9,10], whereas logarithmic fits corresponding to equation (3) can be found in [11]–[13] for disordered ferromagnets and in [14,15] for spin glasses. For the case of an elastic line in a random medium, various authors have also used algebraic timescalings to fit ageing data [16], but more recently Kolton et al [17] have been able to exclude the power law $l(t) \sim t^{1/z}$ at large scales and to measure a barrier exponent $\psi > 0$ in equation (3) which is asymptotically size and time independent as it should be. However, since fits of numerical data in most disordered systems will probably remain controversial for a long time, we feel that more detailed theoretical arguments should be provided favoring either algebraic or logarithmic behavior. In this paper, we explain why the strong disorder renormalization (RG) approach in
configuration space introduced recently [18] is an explicit construction favoring the droplet logarithmic scaling of equation (3).

Besides ageing properties of disordered systems at a given temperature, physicists have been also interested in more complicated temperature cycling experiments that display rejuvenation and memory (see [1]–[3] and references therein for more details). The important point for the present discussion is that these phenomena require some hierarchical organization of valleys within valleys, where the rejuvenation due to short length scales does not destroy the memory of large length scales which are effectively frozen. Since this hierarchy is sometimes believed to be present only in mean-field models, we would like to stress here that the droplet logarithmic scaling of equation (3) effectively leads to a clear separation of timescales and to an effective hierarchy of valleys at large scales, as already argued in [1, 2]. The strong disorder RG procedure that we discuss in the present paper is in full agreement with these ideas, since we expect that for a very broad class of disordered systems in their glassy phase, the RG procedure flows towards some ‘infinite disorder fixed point’ that precisely describes a strong hierarchy of valleys within valleys.

The paper is organized as follows. In section 2, we recall the principles of strong disorder renormalization in configuration space introduced in [18] and discuss its properties. In section 3, we follow numerically the RG flow corresponding to the directed polymer in a two-dimensional random medium and find evidence of convergence towards an ‘infinite disorder’ fixed point. In section 4, we introduce simplified RG rules near ‘infinite disorder’ fixed point. In section 5, we present the numerical results based on simplified RG rules that allow us to study bigger sizes and to estimate the barrier exponent $\psi$ of the fixed point. In section 6, we discuss the physical meaning of the barrier exponent $\psi$ for the structure of renormalized valleys in the configuration space. Our conclusions are summarized in section 7.

2. Strong disorder RG rules for random master equations

Strong disorder renormalization (see [19] for a review) is a very specific type of RG that was first developed in the field of quantum spins: the RG rules of Ma and Dasgupta [20] have been put on firm grounds by Fisher who introduced the crucial idea of an ‘infinite disorder’ fixed point where the method becomes asymptotically exact, and who computed explicitly exact critical exponents and scaling functions for one-dimensional disordered quantum spin chains [21]. This method has thus generated a lot of activity for various disordered quantum models [19], and has been successfully applied to various classical disordered dynamical models, such as random walks in random media [22, 23], reaction–diffusion in a random medium [24], coarsening dynamics of classical spin chains [25], trap models [26], random vibrational networks [27], absorbing state phase transitions [28], zero-range processes [29] and exclusion processes [30]. In all these cases, the strong disorder RG rules have been formulated in real space, with specific rules depending on the problem. For more complex systems where the formulation of strong disorder RG rules has not been possible in real space, we have recently proposed in [18] a strong disorder RG procedure in configuration space that can be defined for any master equation. In the remainder of this section, we describe this procedure and discuss its properties in more detail.
2.1. Master equation defining the dynamics

In statistical physics, it is convenient to define the dynamics via a master equation describing the evolution of the probability $P_t(C)$ of being in a configuration $C$ at time $t$.

$$\frac{dP_t(C)}{dt} = \sum_{C'} P_t(C') W(C' \rightarrow C) - P_t(C) W_{\text{out}}(C). \quad (4)$$

The notation $W(C' \rightarrow C)$ represents the rate of transition per unit time from configuration $C'$ to $C$, and the notation

$$W_{\text{out}}(C) \equiv \sum_{C'} W(C \rightarrow C') \quad (5)$$

represents the total rate of exit out of configuration $C$. The two important properties of this master equation are the following.

(i) The time of exit $\tau$ from configuration $C$ is a random variable distributed with the law

$$P^\text{exit}_C(\tau) = W_{\text{out}}(C) e^{-\tau W_{\text{out}}(C)} \quad (6)$$

with the normalization $\int_0^{+\infty} d\tau P^\text{exit}_C(\tau) = 1$.

(ii) The new configuration $C'$ where the system jumps to at time $\tau$ when it leaves the configuration $C$ is chosen with the probability

$$\pi_C(C') = \frac{W(C \rightarrow C')}{W_{\text{out}}(C)} \quad (7)$$

normalized to $\sum_{C'} \pi_C(C') = 1$.

These two properties are the basis of faster-than-the-clock algorithms, called ‘Bortz–Kalos–Lebowitz algorithms’ [31] in physics (and ‘Gillespie algorithms’ [32] in chemistry), where each iteration leads to a movement. However, even if these algorithms avoid trapping in a given microscopic configuration, they do not avoid trapping in a valley of configurations. As a consequence, these algorithms which are usually very powerful for pure systems at low temperature become inefficient in the presence of frozen disorder because they face the ‘futility’ problem [33]: the number of distinct configurations visited during the simulation remains very small with respect to the accepted moves. The reason is that the system visits over and over again the same configurations within a given valley before it is able to escape towards another valley. This is why we propose in the following some renormalization procedure that allows us to work directly with the ‘valleys’ of configurations on larger and larger timescales.

2.2. Statement of the strong disorder renormalization rules

For dynamical models, the aim of any renormalization procedure is to integrate over ‘fast’ processes to obtain effective properties of ‘slow’ processes. The general idea of ‘strong renormalization’ for dynamical models consists in eliminating iteratively the ‘fastest’ process. The RG procedure introduced in [18] can be summarized as follows.
(1) Find the configuration $C^*$ with the biggest exit rate $W_{\text{out}}^*$ (i.e. the smallest exit time; see equation (6))

$$W_{\text{out}}^* = W_{\text{out}}(C^*) \equiv \max_{C} [W_{\text{out}}(C)].$$  (8)

(2) Find the neighbors $(C_1, C_2, \ldots, C_n)$ of configuration $C^*$, i.e. the configurations that were related via positive rates $W(C^* \rightarrow C_i) > 0$ and $W(C_i \rightarrow C^*) > 0$ to the decimated configuration $C^*$ (here we will assume for the simplicity of the discussion, and because it is usually the case in statistical physics models, that if a transition has a strictly positive rate, the reverse transition also has a strictly positive rate; but of course the renormalization rules can be simply extended to other cases). For each neighbor configuration $C_i$ with $i \in (1, \ldots, n)$, update the transition rate to go to the configuration $C_j$ with $j \in (1, \ldots, n)$ and $j \neq i$ according to

$$W_{\text{new}}(C_i \rightarrow C_j) = W(C_i \rightarrow C_j) + W(C_i \rightarrow C^*) \times \pi_{C^*}(C_j)$$  (9)

where the first term represents the ‘old’ transition rate (possibly zero), and the second term represents the transition via the decimated configuration $C^*$: the factor $W(C_i \rightarrow C^*)$ takes into account the rate of transition to $C^*$ and the term

$$\pi_{C^*}(C_j) = \frac{W(C^* \rightarrow C_j)}{W_{\text{out}}(C^*)}$$  (10)

represents the probability of making a transition towards $C_j$ when in $C^*$ (see equation (7)). The $2n$ rates $W(C^* \rightarrow C_i)$ and $W(C_i \rightarrow C^*)$ then disappear with the decimated configuration $C^*$. Note that the rule of equation (9) has been recently proposed in [34] to eliminate ‘fast states’ from various dynamical problems with two very separated timescales. The physical interpretation of this rule is as follows: the time spent in the decimated configuration $C^*$ is neglected with respect to the other timescales remaining in the system. The validity of this approximation within the present renormalization procedure will be discussed in detail below.

(3) Update the rates of exit out of the neighbors $C_i$, with $i = 1, \ldots, n$, either with the definition

$$W_{\text{out}}^{\text{new}}(C_i) = \sum_C W_{\text{out}}^{\text{new}}(C_i \rightarrow C)$$  (11)

or with the rule that can be deduced from equation (9)

$$W_{\text{out}}^{\text{new}}(C_i) = W_{\text{out}}(C_i) - W(C_i \rightarrow C^*) \frac{W(C^* \rightarrow C_i)}{W_{\text{out}}^*}$$  (12)

(since this rule contains a subtraction, it can be used numerically only with great care!). The physical meaning of this rule is the following. The rate of exit out of the configuration $C_i$ decays because the previous transition towards $C^*$ can lead to an immediate return towards $C_i$ with probability $\pi_{C^*}(C_i) = W(C^* \rightarrow C_i)/W_{\text{out}}^*$. After the decimation of the configuration $C^*$, this process is not considered as an ‘exit’ process any longer, but as a residence process in the configuration $C_i$. This point is very important for understanding the meaning of the renormalization procedure: the remaining configurations at a given renormalization scale are ‘formally’ microscopic configurations of the initial master equation (equation (4)), but each of these
remaining microscopic configurations actually represents some ‘valley’ in configuration space that takes into account all the previously decimated configurations.

(4) Return to point (1).

Note that in practice, the renormalized rates $W(C \to C')$ can rapidly become very small as a consequence of the multiplicative structure of the renormalization rule of equation (9). This means that the appropriate variables are the logarithms of the transition rates, that we will call ‘barriers’ in the remainder of this paper. The barrier $B(C \to C')$ from $C$ to $C'$ is defined by

$$B(C \to C') \equiv -\ln W(C \to C')$$

and similarly the barrier to exit out of configuration $C$ is defined by

$$B_{\text{out}}(C) \equiv -\ln W_{\text{out}}(C).$$

Note that a very important advantage of this formulation in terms of the renormalized transition rates of the master equation is that the renormalized barriers take into account the true ‘barriers’ of the dynamics, whatever their origin, which can be either energetic or entropic.

### 2.3. Notion of the ‘infinite disorder fixed point’ and asymptotic exactness of the RG rules

As mentioned above, the approximation made in the renormalization rule of equation (9) consists in neglecting the time spent in the decimated configuration $C^*$ with respect to the other timescales remaining in the system. In the present framework, this means that the maximal exit rate chosen in equation (8) should be well separated from the exit rates for the neighboring configurations $C_i$. The crucial idea of an ‘infinite disorder fixed point’ [19, 21] is that even if this approximation is not perfect during the first steps of the renormalization, this approximation will become better and better at large timescales if the probability distribution of the remaining exit rates becomes broader and broader upon iteration. More precisely, if the renormalization scale $\Gamma$ is defined as the exit barrier of the last eliminated configuration $C^*$

$$\Gamma = B_{\text{out}}(C^*) \equiv -\ln W_{\text{out}}^*$$

one expects that the probability distribution of the remaining exit barrier $B_{\text{out}} \geq \Gamma$ will converge towards some scaling form

$$P_\Gamma(B_{\text{out}} - \Gamma) \underset{\Gamma \to \infty}{\sim} \frac{1}{\sigma(\Gamma)} \hat{P} \left( \frac{B_{\text{out}} - \Gamma}{\sigma(\Gamma)} \right)$$

where $\hat{P}$ is the fixed point probability distribution, and where $\sigma(\Gamma)$ is the appropriate scaling factor for the width. The notion of the ‘infinite disorder fixed point’ means that the width $\sigma(\Gamma)$ grows indefinitely with the renormalization scale $\Gamma$

$$\sigma(\Gamma) \underset{\Gamma \to \infty}{\sim} +\infty.$$ 

Whenever this ‘infinite disorder fixed point’ condition is satisfied, the strong disorder renormalization procedure becomes asymptotically exact at large scales. In previously known cases of infinite disorder fixed points where calculations can be done explicitly [19],

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the scale $\sigma(\Gamma)$ has been found to grow linearly

$$\sigma(\Gamma) \sim \Gamma. \quad (18)$$

This behavior means that the cut-off $\Gamma$ is the only characteristic scale and thus describes some critical point [19]. For the present procedure concerning the dynamics in disordered models, this property means some ‘criticality in the time direction’, i.e. the absence of any characteristic timescale between the microscopic scale and the macroscopic equilibrium time of the full disordered sample. As explained at the beginning of the introduction, this ‘criticality in the time direction’ will naturally lead to ageing behaviors for two-time properties. An example where asymptotically exact two-time ageing properties have been explicitly computed via the strong disorder RG is the Sinai model [22].

When the width $\sigma(\Gamma)$ instead converges towards a finite value $\sigma(\infty) < +\infty$, one speaks of a ‘finite disorder fixed point’. However, if this constant $\sigma(\infty)$ is large, one speaks of a ‘strong disorder fixed point’, and the validity of the RG approach is of order $1/\sigma(\infty)$: we refer the reader to [23] where systematic expansions in $1/\sigma(\infty)$ with respect to the leading strong disorder RG have been explicitly computed. This notion of ‘strong disorder fixed point’ is very useful in studying the vicinity of the ‘infinite disorder fixed point’ in the space of parameters [19]. For instance in the Sinai model, the ‘infinite disorder fixed point’ is realized in the absence of drift where the diffusion is logarithmically slow, whereas the ‘strong disorder fixed point’ corresponds to the presence of a small drift where the diffusion is algebraic but with an anomalous exponent [22,23].

For the present strong disorder renormalization of a master equation, the convergence towards an ‘infinite disorder fixed point’ will depend on the initial condition of the transition rates, i.e. on the model (and on the temperature if there are phase transitions). However, the form of the RG rules of equation (9) is sufficiently similar to that of the usual Ma–Dasgupta rules [19] to think that the convergence towards some infinite disorder fixed point should be realized in a very broad class of disordered systems in their glassy phase. In practice, this should be checked numerically for each model of interest.

3. Numerical studies of the full RG procedure

3.1. Main numerical limitation: proliferation of neighbors

For dimension $d = 1$, strong disorder RG rules maintain the one-dimensional structure where each site has two neighbors, one on the left and one on the right, and this is why one can obtain explicit exact solutions [19]. For dimension $d > 1$, strong disorder RG rules change the local coordination numbers and usually lead to a proliferation of neighbors as already found in real-space strong disorder RG studies of quantum models [35,36]. With the present notation, the reason is clear from the RG rule of equation (9): if the decimated configuration $C^*$ has $n$ neighbors ($C_1, C_2, \ldots, C_n$), one eliminates $2n$ rates (the rates $W(C^* \rightarrow C_i)$ and $W(C_i \rightarrow C^*)$ for $i = 1, 2, \ldots, n$) but one can create up to $n(n-1)$ transition rates (the rates $W(C_i \rightarrow C_j)$ with $i = 1, 2, \ldots, n$ and $j \neq i$). The increase in the total number $N_{\text{rates}}$ of transitions rates when one decimates a configuration $C^*$ with $n$ neighbors is thus only bounded by

$$\Delta N_{\text{rates}} \leq n(n - 3). \quad (19)$$
In particular, each of the \( n \) neighbors \( C_i \) loses one neighbor (\( C^* \)), but can gain up to \((n-1)\) new neighbors, so that the increase of its coordination number \( z \) is only bounded by

\[
\Delta z \leq n - 2. \tag{20}
\]

For an initial master equation describing local single moves, the first applications of the strong disorder RG procedure will establish new links between configurations that were not initially related via single moves. As a consequence, the number \( N_{\text{rates}} \) of rates and the coordination \( z \) of the surviving configurations will increase during the first stages of the renormalization to describe moves made of two, three, \ldots elementary moves.

In real-space strong disorder RG studies of quantum models with couplings \( J_{ij} \), a numerical cut-off \( J_{\text{min}} \) is usually introduced to keep the new interactions only if they are above the cut-off \( J_{ij} > J_{\text{min}} \), whereas weaker bonds \( J_{ij} < J_{\text{min}} \) are disregarded \cite{35}. Within the present framework where transition rates are not symmetric \( (W(C_i \to C_j) \neq W(C_j \to C_i)) \) and where the renormalization concerns the rates of exit out of surviving configurations, the problem of simplifying the RG rules numerically is different. In the next section 4, we will propose simplified RG rules that are valid near ‘infinite disorder’ fixed points. But before studying these simplified RG flows, it is important to check that the full RG flow starting from an initial condition describing the dynamical models of interest does indeed flow towards some ‘infinite disorder’ fixed point. In the remainder of this section, we thus study numerically the full RG flow for the special case of a directed polymer in a two-dimensional random medium.

### 3.2. Example: directed polymer in a two-dimensional random medium

As an example of application, we consider the directed polymer in a two-dimensional random medium, a model first introduced to describe interfaces in random ferromagnets \cite{11} (see \cite{37} for a review). The statics is well described by the Fisher–Huse droplet theory \cite{7} as checked by detailed numerical studies \cite{7,38}. We consider a directed polymer of length \( L \) with a fixed origin: the \( \mathcal{N}_0 = 2^L \) possible configurations are given by the sequence of heights \((h_1, h_2, \ldots, h_L)\) that satisfy the chain constraints

\[
h_x - h_{x-1} = \pm 1 \tag{21}
\]

for \( x = 1, 2, \ldots, L \) with the boundary condition \( h_0 = 0 \). The energies of these configurations are given by

\[
E(C = (h_1, h_2, \ldots, h_L)) = \sum_{x=1}^{L} \epsilon(x, h_x) \tag{22}
\]

where the site random energies \( \epsilon(i, h) \) are frozen variables that represent the random medium. We consider the case where these energies are independent and drawn from the Gaussian distribution

\[
\rho(\epsilon) = \frac{1}{\sqrt{2\pi}} e^{-\epsilon^2/2}. \tag{23}
\]

For the directed polymer model, we are interested into the local Metropolis dynamics defined by the transition rates

\[
W(C \to C') = \delta_{|C,C'|} \frac{1}{L} \min \left( 1, e^{-(E(C')-E(C))/T} \right). \tag{24}
\]
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The first factor $\delta_{C,C'}$ means that the two configurations are related by a single move, the second factor $1/L$ ensures that the Monte Carlo time unit corresponds to $L$ attempts at local moves, and the last factor ensures the convergence towards thermal equilibrium at temperature $T$ via the detailed balance property

$$e^{-E(C)/T} W(C \rightarrow C') = e^{-E(C')/T} W(C' \rightarrow C).$$

(25)

In contrast with spin models, where a configuration of $N$ spins is related to exactly $N$ other configurations by single flips, a configuration of $L$ monomers of the directed polymer is usually not related to $L$ other configurations as a consequence of the chain constraints of equation (21). More precisely, if we call configuration $C[x]$ the configuration obtained from $C$ by the elementary move $h_x \rightarrow h_x \pm 2$, we note that, as a consequence of the chain constraint, this elementary move is possible only if the two neighbors are in the favorable positions $h_{x\pm 1} = h_x \pm 1$. The energy change associated with the elementary move $h_x \rightarrow h_x \pm 2$ reads in terms of the random energies introduced in equation (22)

$$E(C[x] \pm) - E(C) \equiv \epsilon(x, h_x \pm 2) - \epsilon(x, h_x).$$

(26)

With the full RG rules, where the problem of proliferation of neighbors discussed in section 3.1 is memory and time consuming, the linear sizes that we have been able to study are rather small, $L \leq 11$ (the number of configurations grows exponentially, $2^L \leq 2048$). The corresponding numbers $n_s(L)$ of disordered samples of length $L$ read

$$L = 5, 6, 7, 8, 9, 10, 11$$

$$n_s(L) = 7 \times 10^6, 10^6, 10^5, 14 \times 10^5, 15 \times 10^4, 14 \times 10^3, 500.$$

(27)

3.3. Analysis of the numerical data: two useful ensembles

To analyze the numerical results concerning the application of the strong disorder renormalization to many disordered samples, it is interesting to consider two different ‘ensembles’ corresponding to two types of averages as we now explain.

3.3.1. Averaging over disordered samples at fixed RG scale $\Gamma$. The first ensemble consists in collecting data at fixed RG scale $\Gamma$, where $\Gamma$ is the last decimated renormalized exit barrier remaining in the system. The advantage is that the comparison with theoretical statements is more straightforward, since many of the statements concern a fixed RG scale, in particular the probability distribution of equation (16) that defines the scaling properties of the barriers. However, since the RG scale $\Gamma$ is a continuous variable, one needs then to introduce some appropriate discretization numerically. For instance in the results presented below, we have chosen a window of width $\Delta \Gamma = 0.1$. In conclusion, ‘data at fixed RG scale $\Gamma$’ correspond to an average over the disordered samples where the last decimated renormalized exit barrier remaining in the system is within a window of width $\Delta \Gamma = 0.1$ around $\Gamma$.

3.3.2. Averaging over disordered samples at fixed number $N$ of surviving configurations (i.e. at fixed coherence length). However another way of analyzing numerical data, used for instance in strong disorder RG study of quantum models for dimension $d > 1$ [35] consists in collecting data corresponding to a fixed number of RG steps, or equivalently to a fixed

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number $\mathcal{N}$ of surviving configurations. This type of averaging can be considered as a fixed length ensemble as we now explain. In strong disorder RG study of quantum models for dimension $d > 1$ [35], the number $N$ of surviving spins when starting from $N_0 = L^d$ initial spins can be used to define a length scale $l$ via $N = (L/l)^d$. For the present renormalization where the number of initial configurations is $N_0 = 2^L$, we may similarly define a length $l$ via

$$
\mathcal{N} \equiv 2^{L/l}.
$$

(28)

This length $l$ represents some growing correlation length in the following sense: each segment of length $l$ of the polymer corresponds to one renormalized degree of freedom. Initially this length is 1 and corresponds to a single monomer, whereas at the end of the renormalization process where equilibrium is reached, the number of configurations reaches $\mathcal{N}_{eq} = 1$ and the coherence length reaches the total length $l_{eq} = L$ of the polymer. So besides the studies at fixed RG scale $\Gamma$ described above, it is also interesting to consider ‘data at fixed $\mathcal{N}$’ as in [35] where data are averaged over disordered samples having the same number $\mathcal{N}$ of surviving renormalized configurations.

### 3.4. Probability distribution of the renormalized exit barriers

As explained above in section 2.3, the first important observable to consider is the distribution of renormalized exit barriers of equation (16) to see whether the width $\sigma(\Gamma)$ grows indefinitely with $\Gamma$ (equation (17)). If this is the case, then the flow is towards some ‘infinite disorder fixed point’ and the strong disorder renormalization approach becomes asymptotically exact at large timescales.

We show in figure 1 our numerical results for a directed polymer of length $L = 9$ (corresponding to $2^9 = 512$ initial configurations). We find that the rescaled distribution of equation (16) is very close to the exponential form (see figure 1(a))

$$
\tilde{P}(x) \simeq e^{-x}
$$

(29)

and that the width $\sigma(\Gamma)$ grows linearly with the RG scale $\Gamma$

$$
\sigma(\Gamma) \sim \Gamma
$$

(30)

after an initial transient regime for smaller $\Gamma$ and a final finite size saturation regime at larger $\Gamma$. Note that the two properties of equations (29) and (30) seem extremely robust within strong disorder RG since they hold exactly in soluble models for $d = 1$ [19] and have been also found numerically in quantum models for dimension $d > 1$ [35].

The results shown in figure 1 have been obtained by averaging ‘at fixed RG scale $\Gamma$’, i.e. by collecting over many samples histograms of renormalized exit barriers when the last decimated exit barrier is within a window of width $\Delta \Gamma = 0.1$ around $\Gamma$ (see section 3.3.1 for more details). However as explained above in section 3.3.2, it is also interesting to make averages over disordered samples having the same number $\mathcal{N}$ of surviving renormalized configurations. For each $\mathcal{N}$, we have measured the average $\Gamma(\mathcal{N})$ of the minimal exit barrier remaining in the system and the width $\sigma(\mathcal{N})$ of all the remaining exit barriers. The parametric plot of $\sigma(\mathcal{N})$ as a function of $\Gamma(\mathcal{N})$ is shown in figure 2 for the four sizes $L = 8, 9, 10, 11$: the linear growth characterizing the infinite disorder fixed point only appears after an initial transient regime, as already noted in numerical studies of strong disorder RG of quantum models for dimension $d > 1$ [35].
Figure 1. Dynamics of the directed polymer in a two-dimensional random medium: numerical evidence for the convergence towards an infinite disorder fixed point (data obtained from the numerical application of the full RG rules to \(n_s = 15 \times 10^4\) disordered samples for a polymer of length \(L = 9\) with \(2^6 = 512\) initial configurations). (a) Flow of the probability distribution \(P(\Gamma)\) of the renormalized exit barriers (see equation (16)) as the RG scale grows, \(\Gamma = 3, 4, 5, 6, 7\); these distributions follow the exponential form (see equation (29)) with a scale dependent width \(\sigma(\Gamma)\). (b) The corresponding width \(\sigma(\Gamma)\) grows linearly with the RG scale \(\Gamma\).

3.5. Growth of the coherence length \(l_\Gamma\)

As explained above, it is convenient to define the coherence length \(l_\Gamma\) from the number \(N_\Gamma\) of surviving configurations via equation (28). The barrier exponent \(\psi\) of equations (2) and (3) of the introduction is the exponent governing the growth of the coherence length at large scale

\[
l_\Gamma \sim c \Gamma^{1/\psi}.
\]

We show in figure 3 our numerical results concerning the relation between the barrier scale and the length scale, within the two ensembles already introduced.

(a) The data corresponding to a fixed RG scale \(\Gamma\) (see section 3.3.1) are shown in figure 3(a). The growth of the coherence length \(l_\Gamma\) as a function of the RG scale \(\Gamma\) is shown for the three sizes \(L = 8, 9, 10\) (\(L = 11\) is not shown here because the data are too noisy): after a common growth, the curves separate because they saturate by construction at the value \(l_{eq} = L\).

(b) The data corresponding to a fixed number of \(N\) of surviving configurations, i.e. to a fixed coherence length \(l\) (via equation (28)), are shown in figure 3(b). The horizontal axis then corresponds to the average \(\Gamma(N)\) of the last decimated exit barrier.

The comparison of (a) and (b) shows that, for the coherence length, the numerical data obtained in the ‘fixed \(N\) ensemble’ display less finite size effect than data obtained in the ‘fixed \(\Gamma\) ensemble’ and are thus easier to analyze.
3.6. Statistics of the equilibrium time of finite systems

Within the strong disorder RG procedure, the equilibrium time $t_{eq}$ of a given disordered sample is determined by the renormalized exit barrier

$$\Gamma_{eq} = \ln t_{eq}$$

(32)

corresponding to the last decimation process where the two biggest metastable valleys merge into a surviving valley corresponding to thermal equilibrium of the whole sample. We have measured its probability distribution $Q_L(\Gamma_{eq} = \ln t_{eq})$ over the disordered samples of size $L$ as shown in figure 4(a). The convergence towards a fixed rescaled distribution

$$Q_L(\Gamma_{eq}) \sim \frac{1}{\Delta(L)} \hat{Q} \left( u \equiv \frac{\Gamma_{eq} - \overline{\Gamma_{eq}}(L)}{\Delta(L)} \right)$$

(33)

is rapid as shown in figure 4(b). However the sizes studied are not sufficient for obtaining a reliable measure of the barrier exponent $\psi$, the average $\overline{\Gamma_{eq}}(L) \sim L^\psi$. This is why we introduce in the next section simplified RG rules that are valid near infinite disorder fixed points and that allow us to study numerically bigger system sizes.

4. Simplified RG rules near infinite disorder fixed points

4.1. Dominance of the preferred exit channel

Whenever the flow is towards some ‘infinite disorder’ fixed point, where the distribution of renormalized exit barriers becomes broader and broader upon iteration (equations (16)
Figure 3. Growth of the coherence length $l$ with the RG scale $\Gamma$ of the renormalized barriers. (a) Data obtained at fixed RG scale $\Gamma$ for the sizes $L = 8, 9, 10$: the coherence length $l_\Gamma$ is obtained from the number $N_\Gamma$ of surviving configurations measured at RG scale $\Gamma$ via equation (28). (b) Data obtained at a fixed number $N$ of surviving configurations (i.e. at a fixed coherence length $l$), for a polymer of length $L = 8, 9, 10, 11$. The horizontal axis $\Gamma(N)$ represents the average of the minimal exit barriers remaining in the system.

Figure 4. Statistics of the equilibrium time $t_{eq}$ over the disordered samples for a directed polymer of length $L = 5, 6, 7, 8, 9$ in a two-dimensional random medium: (a) histograms $Q_L(\Gamma_{eq} = \ln t_{eq})$ of the last decimated renormalized exit barrier $\Gamma_{eq} = \ln t_{eq}$. (b) Same data in rescaled variables to obtain the rescaled distribution $\tilde{Q}(u)$ of equation (33).

and (17), one expects that the rate of exit out of the decimated configuration $C^*$

$$W_{out}(C^*) = \sum_{i=1}^{n} W(C^* \rightarrow C_i)$$

(34)
will actually be dominated by the preferred exit channel $i_{\text{pref}}$ having the biggest contribution in the sum of equation (34)

$$W_{\text{out}}(C^*) \simeq W(C^* \rightarrow C_{i_{\text{pref}}}),$$  \hspace{1cm} (35)

i.e. one expects that the probability distribution $\pi_{C^*}(C_j)$ of equation (10) will become a delta distribution on the preferred exit channel up to exponentially small terms

$$\pi_{C^*}(C_j) \simeq \delta_{j,i_{\text{pref}}} + \cdots.$$  \hspace{1cm} (36)

The dominance of the preferred exit channel (equation (35)) near an infinite disorder fixed point will be checked numerically below for the case of the directed polymer in a two-dimensional random medium (see section 5.6). However, we expect that it holds more generally for the following reasons. The RG rules with their characteristic multiplicative structure of equations (9) and (10) act directly on the transition rates $W(C_j \rightarrow C_i)$, between configurations, whereas the total exit rates $W_{\text{out}}$ are derived quantities obtained by summing over the possible exit channels. The notion of convergence towards an infinite disorder fixed point has been defined above by the property that the probability distribution of the remaining exit rates $W_{\text{out}}$ becomes broader and broader. However we expect that when it happens, it is because the probability distribution of the individual transition rates $W(C_j \rightarrow C_i)$ themselves becomes broader and broader, so that the sum in equation (34) is dominated by the biggest term. A simple one-dimensional example of this phenomenon is the Sinai model [22], where each renormalized configuration always has $n = 2$ neighbors: the rates of exit to the right and to the left surviving configurations follow the infinite disorder scaling form of equations (16) and (17): $W_{\text{right}} \sim e^{-\Gamma(1+\eta_{\text{right}})}$ and $W_{\text{left}} \sim e^{-\Gamma(1+\eta_{\text{left}})}$ where $\eta_{\text{right}}$ and $\eta_{\text{left}}$ are independent random variables of order 1 distributed with the exponential distribution $P(\eta) = e^{-\eta}$. As a consequence, the exit rate $W_{\text{out}} = W_{\text{right}} + W_{\text{left}}$ is dominated by the bigger of the two terms in the limit $\Gamma \rightarrow \infty$, since the probability of degeneracy is of order $1/\Gamma \rightarrow 0$. Such rare events where the dominance of the preferred exit channel is not realized will be discussed in more detail below (in section 4.3), but we first state precisely how the full RG rules can be simplified when the preferred exit channel dominates.

4.2. Simplified RG rules using the preferred exit channel

We thus introduce the following simplified RG procedure with respect to the full RG procedure described in the previous section.

(1) The first point is the same (equation (8)).

(2') Among the neighbors $(C_1, C_2, \ldots, C_n)$ of configuration $C^*$, find the preferred exit channel $i_{\text{pref}}$. Update the transitions rates from the $(n - 1)$ non-preferred neighbors $i \neq i_{\text{pref}}$ towards $i_{\text{pref}}$ using the approximated rule

$$W^{\text{new}}(C_i \rightarrow C_{i_{\text{pref}}}) \simeq W(C_i \rightarrow C_{i_{\text{pref}}}) + W(C_i \rightarrow C^*)$$  \hspace{1cm} (37)

where the probability distribution $\pi_{C^*}(C_j)$ of the full RG rule of equations (9) and (10) has been replaced by the leading delta function of equation (36). Update the rates of transition from $i_{\text{pref}}$, towards the $(n - 1)$ non-preferred neighbors $i \neq i_{\text{pref}}$, using the
full RG rule of equations (9) and (10)

\[ W_{\text{new}}(C_{\text{pref}} \to C_i) = W(C_{\text{pref}} \to C_i) + W(C_{\text{pref}} \to C^*) \times \frac{W(C^* \to C_i)}{W_{\text{out}}(C^*)}. \]  

(38)

Here the full rule is used because the ratio \( W(C^* \to C_i)/W_{\text{out}}(C^*) \) is small and should thus be evaluated correctly.

In contrast with rule (2), where the increase in the total number \( N_{\text{rates}} \) of transition rates was only bounded by equation (19), the rule (2') ensures that the total number of renormalized transitions rates always decreases

\[ \Delta N_{\text{rates}} \leq 2(n-1) - 2n = -2. \]  

(39)

Moreover, in contrast with rule (2), where the increase in the coordination number of the \( n \) neighbors \( C_i \) was only bounded by equation (20), the rule (2') ensures that the coordination numbers of the non-preferred neighbors do not grow

\[ \Delta z_{i \neq \text{pref}} \leq 0 \]  

(40)

and it is only the coordination number of the preferred neighbor that may grow up to

\[ \Delta z_{\text{pref}} \leq (n-1) - 1 = n - 2. \]  

(41)

(3') With the rule of equation (37), the rates of exit out of the \( n-1 \) non-preferred neighbors \( i \neq i_{\text{pref}} \) do not have to be updated since the rate of exit towards \( C^* \) has been completely transferred to \( i_{\text{pref}} \). So the only update of exit rate is for the preferred neighbor \( i_{\text{pref}} \) via the definition of equation (11) or with the equivalent rule of equation (12).

(4) Return to (1).

It is thus clear that these simplified RG rules correspond to a substantial gain from a computational point of view and will allow one to study bigger system sizes. We will describe in section 5 the numerical results that can be obtained for the directed polymer, and compare them with the numerical results concerning the full RG rules. However besides this numerical gain, these simplified rules have also important implications from a theoretical point of view as we now explain.

4.3. Interpretation in terms of quasi-equilibrium within metastable states

In the studies on slowly relaxing systems such as disordered systems, glasses or granular media, it is usual to separate the dynamics into two parts: there are ‘fast’ degrees of freedom which rapidly reach local quasi-equilibrium plus a slow non-equilibrium part. Within the present strong disorder renormalization in configuration space, these ideas can be applied directly as follows. With each time \( t \), one may associate a set of metastable states which are labeled by the surviving configurations at the RG scale \( \Gamma = \ln t \). Within each metastable state, configurations are quasi-equilibrated, whereas configurations belonging to different metastable states are still out of equilibrium. The slow non-equilibrium part of the dynamics corresponds to the evolution of the renormalized valleys with the RG scale: some valleys disappear and are absorbed by a neighboring valley.
Since at large scale, the RG flows towards an ‘infinite disorder’ fixed point, the different timescales are effectively very well separated. As a consequence, we may write, as in the Sinai model \cite{39}, that the probability $P(C_t|C_0)$ of being in configuration $C$ at time $t$ when starting in configuration $C_0$ at time $t=0$ is very well approximated by

$$P(C_t|C_0) \simeq \sum_{V_\Gamma} \frac{1}{Z_{V_\Gamma}} e^{-\beta E(C)} \theta_{V_\Gamma}(C) \theta_{V_\Gamma}(C_0)$$

(42)

where the sum is over all the renormalized valleys $V_\Gamma$ that are present in the system at the renormalization scale $\Gamma = \ln t$, and where $\theta_{V}(C)$ is the characteristic function of the valley $V$, i.e. $\theta_{V}(C) = 1$ if $C$ belongs to the valley and $\theta_{V}(C) = 0$ otherwise. The denominator represents the Boltzmann partition function over the valley $V_\Gamma$

$$Z_{V_\Gamma} = \sum_{C \in V_\Gamma} e^{-\beta E(C)}.$$

(43)

As discussed in detail in \cite{39}, the approximation of equation (42) breaks down only for rare events at large times near the infinite disorder fixed point. More precisely, the most important rare events that lead to temporary out-of-equilibrium situations for the set of thermal trajectories starting in the same configuration $C_0$ correspond to the following cases.

(i) When the valley $V_\Gamma$ containing $C_0$ is being decimated precisely at an RG scale of order $\Gamma = \ln t$: then the thermal packet is broken into two sub-packets; one has already jumped over the barrier, whereas the other has not jumped yet. Near the infinite disorder fixed point described by equation (16) and (30) for the distribution of renormalized exit barrier, these events occur with a vanishing probability of order 1/\Gamma = 1/(\ln t) at large times.

(ii) When the decimation of a valley corresponds to an accidental degeneracy between the second preferred exit channel and the first preferred exit channel: then the thermal packet is also broken into two sub-packets, one having jumped into the first preferred exit channel and the other having jumped into the second preferred exit channel. Again, near the infinite disorder fixed point these events occur with a vanishing probability of order 1/\Gamma = 1/(\ln t) at large times.

This discussion shows that the asymptotic dominance of the preferred exit channel near the infinite disorder fixed point is actually crucial for obtaining quasi-equilibrium within the visited region of phase space at a given large time $t$. In particular, if the degeneracy between the second preferred exit channel and the first preferred exit channel could occur with a finite probability, then finite contributions of out-of-equilibrium situations at all scales would ruin the quasi-equilibrium approximation of equation (42): the probability of being in a configuration $C$ at time $t$ would not depend only on its energy $E(C)$ and on the partition function $Z_{V_\Gamma}$ of the renormalized valley in which it belongs, but would be instead a very complicated function of all possible paths from $C_0$ to $C$ with their appropriate dynamical weights. To better understand the importance of this discussion, it is useful to recall here a well-identified exception of the quasi-equilibrium idea, namely the symmetric Bouchaud trap model in one dimension, where even in the limit of arbitrarily low temperature, the diffusion front in each sample consists in two delta peaks, which are completely out of equilibrium with each other \cite{26}: the weights
of these two delta peaks do not depend on their energies, but instead on the distances to the origin that determine the probability of reaching one before the other (see [26] for more details). In this trap model, the reason is clear: whenever the particle escapes from a trap, it jumps either to the right or to the left with equal probabilities (1/2), i.e. the two possible exit channels are degenerate by the very definition of the model that imposes this symmetry. In other disordered models where this degeneracy is not imposed by a symmetry of the model, this degeneracy can only occur accidentally with some probability. The question is then whether this probability of accidental degeneracy of the two preferred exit channels remains finite or becomes rare (i.e. decays to zero) at large times. Within the present strong disorder RG where the flow is towards some infinite disorder fixed point, the dominance of the preferred exit channel precisely means that the probability of these accidental degeneracies decays to zero, so the quasi-equilibrium approximation of equation (42) becomes asymptotically exact at large times.

5. Numerical studies using simplified RG rules

5.1. Numerical gain with respect to the full RG rules

As explained in section 3.1, the numerical applications of the full RG rules are limited to small sizes because the proliferation of neighbors is memory and time consuming. The simplified RG rules described in section 4, that preserve the asymptotic exactness near infinite disorder fixed points, allow one to study much bigger system sizes. For instance, for the directed polymer in a two-dimensional model introduced previously (section 3.2), the linear sizes $L$ and the corresponding numbers $n_s(L)$ of disordered samples that we have been able to study via simplified rules are

\[ L = 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18 \]
\[ n_s(L) = 7 \times 10^8, 3 \times 10^8, 10^6, 3 \times 10^7, 10^7, 3 \times 10^6, 7 \times 10^5, 18 \times 10^4, 4 \times 10^4, 8 \times 10^3, 1500, 600 \]  

which should be compared to the sizes given in equation (27) for the full RG rules. In particular, note the difference for the biggest sizes: the biggest size $L = 11$ for the full RG rules corresponds to an initial number of $2^{11} = 2048$ configurations, whereas the biggest size $L = 18$ for the simplified RG rules corresponds to an initial number of $2^{18} = 262144$ configurations. The numerical gain is thus substantial.

5.2. Choice of initial conditions for improving the convergence towards the fixed point

The simplified RG rules are based on the dominance of the preferred exit channel, which is realized only near ‘infinite disorder fixed points’, i.e. they will be good at large RG scales, but not during the first RG steps. As a consequence, it is important to stress here the two different aims of the numerical studies based on full and simplified RG rules respectively.

(i) The aim of the full RG rules is to study whether the true microscopic model of interest does indeed flow towards an ‘infinite disorder fixed point’ where the width of the renormalized barrier distribution grows without bounds. This is what we have checked in section 3 for the directed polymer, using as initial transition rates the ‘true’ Metropolis transition rates of equation (24).
The aim of the simplified RG rules is to study directly the properties of the ‘infinite disorder fixed point’. As a consequence here, we do not wish to use as initial transition rates the ‘true’ Metropolis transition rates of equation (24), but instead the initial conditions that reduce the transient as much as possible, i.e. the initial conditions that are the closest to the fixed point properties. As in the strong disorder RG studies of quantum models where the same strategy was used [35], one would like to choose an initial condition where the probability distribution of barriers is already exponential, i.e. of the same form as observed at large scale (see equation (29)) when applying the full RG rules to the Metropolis initial condition. For the spatial properties however, since the random correlated structure generated by the RG flow is difficult to characterize, one is restricted as in [35] to starting from the regular lattice structure of the microscopic model.

We now describe more precisely the initial conditions that we have used for our numerical studies for the directed polymer. Instead of the Gaussian energies of equation (23), we have drawn site energies from the exponential distribution

\[ \rho(\epsilon) = \theta(\epsilon \leq 0)e^{\epsilon}. \] (45)

Then to choose the transition rates, there is still some freedom within the detailed balance condition, since equation (25) simply means that

\[ e^{-E(\mathcal{C})/T} W(\mathcal{C} \rightarrow \mathcal{C}') = e^{-E(\mathcal{C}')/T} W(\mathcal{C}' \rightarrow \mathcal{C}) = \Delta(\mathcal{C}, \mathcal{C}'). \] (46)

where \( \Delta(\mathcal{C}, \mathcal{C}') = \Delta(\mathcal{C}', \mathcal{C}) \) represents some arbitrary symmetric barrier. Since the dominance of the preferred exit channel has a meaning only if all directions are ‘ascending’, we have chosen to avoid the presence of ‘descending’ directions in the initial condition. Since the site energies of equation (45) are all negative, the energies of the configurations inherit the same property, and we have thus chosen the following form for the symmetric barrier:

\[ \Delta(\mathcal{C}, \mathcal{C}') = e^{E(\mathcal{C})/T + E(\mathcal{C}')/T}. \] (47)

This corresponds to the following transition rates:

\[ W(\mathcal{C} \rightarrow \mathcal{C}') = e^{2E(\mathcal{C})/T + E(\mathcal{C}')/T}, \] (48)

i.e. the barriers of the initial condition are all positive and given by

\[ B(\mathcal{C} \rightarrow \mathcal{C}') = -2E(\mathcal{C})/T - E(\mathcal{C}')/T. \] (49)

We now describe in the remainder of this section the numerical results obtained by applying the simplified RG rules starting from this initial condition.

### 5.3. Probability distribution of the renormalized exit barriers

We show in figure 5 the histograms of renormalized exit barriers obtained via simplified RG rules for a polymer of length \( L = 15 \) (corresponding to \( 2^{15} = 32768 \) initial configurations), that should be compared with figure 1 showing the equivalent results obtained via the full RG rules for a polymer of length \( L = 9 \). The important point is the linear growth of the width \( \sigma(\Gamma) \sim \Gamma \) shown in figure 5(b). The deviations from the exponential distribution...
visible in figure 5(a), in particular the curvatures near the origin, indicate that our choice of initial conditions is not optimal: the problem is that the distribution of the initial barriers of equation (49) is not exponential near the origin. However, in the absence of a much better idea for the initial conditions that would reduce the transients, we have chosen to keep the initial conditions described in section 5.2.

5.4. Growth of the coherence length $l_\Gamma$

We show in figure 6 our numerical results concerning the relation between the barrier scale and the length scale, within the two ensembles already introduced (these data should be compared with the equivalent results of figure 3 for the full RG rules):

(a) The data corresponding to a fixed RG scale $\Gamma$ (see section 3.3.1) are shown in figure 6(a). The growth of the coherence length $l_\Gamma$ as a function of the RG scale $\Gamma$ is shown for the sizes $8 \leq L \leq 16$: the curvature is more and more pronounced before the finite size saturation at the value $l_{eq} = L$.

(b) The data corresponding to a fixed number of $N$ of surviving configurations, i.e. to a fixed coherence length $l$ (via equation (28)) are shown in figure 6(b). The horizontal axis then corresponds to the average $\overline{\Gamma(N)}$ of the last decimated exit barrier $\Gamma$.

5.5. Statistics of the equilibrium time of finite systems

We show in figure 7 the numerical results for the statistics of the equilibrium time $t_{eq}$ over disordered samples of a given length $L$ obtained via the simplified RG rules, that should
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Figure 6. Growth of the coherence length $l_\Gamma$ with the RG scale $\Gamma$ via simplified RG rules. (a) Data obtained at fixed RG scale $\Gamma$ for the sizes $8 \leq L \leq 16$: the coherence length $l_\Gamma$ is obtained from the number $\mathcal{N}_\Gamma$ of surviving configurations measured at RG scale $\Gamma$ via equation (28). (b) Data obtained at a fixed number $\mathcal{N}$ of surviving configurations (i.e. at a fixed coherence length $l$), for the sizes $4 \leq L \leq 18$. The horizontal axis $\Gamma(\mathcal{N})$ represents the average of the minimal exit barriers remaining in the system. (These data should be compared with the equivalent results shown in figure 3 obtained via full RG rules on smaller systems.)

be compared with the equivalent results of figure 4 obtained via the full RG rules. The rescaled distributions shown in figure 7(b) are very stable as the size $L$ changes, as in figure 4(b) for the full RG rules.

5.6. Comparison of the numerical obtained via full RG rules and via simplified RG rules

As explained above in section 5.2, we have used different initial conditions for our numerical studies of full RG rules and of simplified RG rules, so we cannot compare the ‘numbers’ obtained, but we should compare the stable properties of the fixed point that do not depend on the microscopic details, i.e. on the details of the initial condition. A good test is for instance the rescaled probability distribution of the equilibrium time $t_{eq}$ over disordered samples of a given length $L$, that was found to be stable with respect to the value of $L$ both for the full RG rules (see figure 4(b)) and for the simplified RG rules (see figure 7(b)). As shown in figure 8(a), these rescaled probability distributions obtained via the two RG rules do indeed coincide. This agreement is strong numerical evidence that the simplified RG rules capture correctly the properties of the fixed point. This is very important numerically, since the simplified RG rules allow us to study much bigger sizes than the full RG rules, and we compare for instance in figure 8(b) the data obtained via the two methods for the growth of the averaged equilibrium barrier $\Gamma_{eq}(L)$ with the length $L$. Whereas the sizes studied via full RG rules are not sufficient for obtaining a reliable measure of the asymptotic barrier exponent $\psi$, a direct two-parameter power law fit $\Gamma_{eq}(L) = a_0 L^\psi$ of the data obtained via simplified RG rules gives a value of order $\psi \sim 0.47$. (50)
Figure 7. Statistics of the equilibrium time $t_{\text{eq}}$ over the disordered samples of a given length $L$ using simplified RG rules. (a) Histograms $Q_L(\Gamma_{\text{eq}} = \ln t_{\text{eq}})$ of the last decimated renormalized exit barrier $\Gamma_{\text{eq}} = \ln t_{\text{eq}}$ for the four sizes $L = 9, 11, 13, 15$ (other sizes have not been shown for clarity). (b) Rescaled distribution $\tilde{Q}(u)$ (see equation (33)) for the sizes $8 \leq L \leq 15$. (These data should be compared with the equivalent results shown in figure 4 obtained via full RG rules on smaller systems.)

This estimate is of course not expected to be very precise, like any critical exponent measured for disordered samples of limited sizes, but it is nevertheless rather close to the best value $\psi \sim 0.49$ currently available that has obtained by Monte Carlo simulation of the Langevin dynamics [17]. Moreover, we have checked that the above value is quite stable when we analyze the various data on the coherence length presented above, either in the ensemble at fixed RG scale $\Gamma$ or in the ensemble at fixed $N$.

5.7. Debate on the value of the barrier exponent

An important physical issue is whether the barrier exponent $\psi$ is equal to the droplet exponent $\theta$ of the statics, which is exactly known to be $\theta = 1/3$ for the directed polymer in a two-dimensional medium. Although the assumption $\psi = \theta = 1/3$ has been made systematically in the literature since the very first paper [11] introducing the model, and is sometimes considered as established up to possible logarithmic corrections [40], we have explained elsewhere [41] why the equality $\psi = \theta$ is far from obvious within the droplet scaling theory where the only bounds are $\theta \leq \psi \leq d-1$ [6], and where already in the statics, free-energy fluctuations and energy fluctuations involve the different exponents $\theta = 1/3$ and $d_s/2 = 1/2$ [7]. Moreover in other disordered models like spin glasses, the barrier exponent $\psi$ is expected to be strictly bigger than the droplet exponent $\theta$, because they are distinct below the lower critical dimension: for dimension $d = 1$, the exact solution [5] yields $\psi_{1d} > 0 > \theta_{1d} = -1$, and for dimension $d = 2$, these two exponents do not have the same sign $\psi_{2d} > 0 > \theta_{2d}$. Despite their measure $\psi \sim 0.49$, the conclusion of the authors of [17], who believe in the identity $\psi = \theta = 1/3$, is that barriers contain
Figure 8. Comparison of data obtained via full RG rules and via simplified RG rules. (a) The rescaled histograms $\tilde{Q}(u)$ (see equation (33)) obtained via the full RG rules and via the simplified RG rules respectively, coincide: this shows that the simplified RG rules capture correctly the fixed point properties. (b) Growth of the averaged equilibrium barrier $\Gamma_{eq}(L)$ as a function of the system size $L$: the data obtained via full RG rules are limited to the sizes $4 \leq L \leq 10$, whereas the data obtained via simplified RG rules are for the sizes $4 \leq L \leq 18$. The numerical gain is thus substantial.

strong logarithmic corrections $B(L) \sim L^{1/3}(\ln L)^{\mu}$. Our interpretation is on the contrary that the measured value $\psi \sim 0.49$ in [17] and our present estimate of equation (50) could very well be of the correct order of magnitude, and thus strictly bigger than the droplet exponent $\theta = 1/3$.

6. Physical interpretation of the barrier exponent $\psi$

In this section, we explain how the barrier exponent $\psi$ that relates time and length scales (equation (2)) depends on the spatial connectivity of the renormalized degrees of freedom. In section 3.3.2, we have seen how to associate with each RG scale $\Gamma$ some coherence length $l_{\Gamma}$, such that the number $n_{\Gamma}$ of renormalized degrees of freedom is

$$n_{\Gamma} \equiv \frac{L}{l_{\Gamma}}$$

and the number of $N_{\Gamma}$ of surviving configurations at scale $\Gamma$ reads (equation (28))

$$N_{\Gamma} \equiv 2^{n_{\Gamma}}.$$ (52)

On one hand, the decrease of the number of surviving configurations reads

$$dN_{\Gamma} = -N_{\Gamma} \frac{v_{\Gamma}}{\Gamma} d\Gamma$$

where $v_{\Gamma} d\Gamma/\Gamma$ represents the probability decimated in a window of width $d\Gamma$ around $\Gamma$. The factor $1/\Gamma$ represents the probability of being decimated via a given exit channel near
the infinite disorder fixed point, and thus the additional factor $v_\Gamma$ can be interpreted as an effective number of independent exit channels that are in competition to be decimated. For instance in the one-dimensional Sinai model, this factor is simply $v_\Gamma(1D) = 2$ because there are exactly two independent neighbors at any stage of renormalization for $d = 1$, one on the left and one on the right. And this is why the number of renormalized valleys decays as $1/\Gamma^2$.

On the other hand, if we use the asymptotic expression of equation (31) for the coherence length $l_\Gamma \simeq c \Gamma^{1/\psi}$, we obtain the following decay for the number of surviving configurations of equation (52):

$$d \ln \mathcal{N}_\Gamma = d \left( \frac{L}{c \Gamma^{1/\psi}} \ln 2 \right) = - \frac{L}{c \Gamma^{1/\psi}} \frac{\ln 2 \, d\Gamma}{\psi} = - \frac{L}{l_\Gamma} \frac{\ln 2 \, d\Gamma}{\psi} = - L \ln 2 \, \frac{d\Gamma}{\psi \Gamma}. \quad (54)$$

The identification of equations (53) and (54) yields that the effective number $v_\Gamma$ of independent exit channels from a surviving configuration reads

$$v_\Gamma = \ln 2 \, \frac{L}{l_\Gamma} = \ln 2 \, \frac{n_\Gamma}{\psi}. \quad (55)$$

It is proportional to the number $n_\Gamma \equiv L/l_\Gamma$ of renormalized degrees of freedom. Since the numerical prefactor $\ln 2/\psi$ is finite for $\psi > 0$, this means that for a given renormalized degree, the number of independent directions that are in competition to be decimated remains effectively finite.

This discussion remains of course at a qualitative level, since a complete characterization of the random structure generated by the strong disorder RG flow remains a very challenging issue. However it is important to understand the meaning of the strong disorder RG procedure. The full RG rules give at first sight the impression that the proliferation of neighbors could ruin the method. We have seen that this is not the case, and that the RG flow can still be towards an infinite disorder fixed point. We have then explained how the preferred exit channel actually dominates over the others asymptotically. And the present discussion on the decay of the number $\mathcal{N}_\Gamma$ of renormalized configurations shows that for a given renormalized degree of freedom of size given by the coherence length $l_\Gamma$, the number of effective exit channels that are in competition to be decimated is effectively finite and proportional to $1/\psi$.

7. Conclusion

In this paper, we have analyzed in detail the strong disorder RG procedure in configuration space to study the non-equilibrium dynamics of random systems. In particular, we have shown that whenever the flow of the renormalized barriers is towards some ‘infinite disorder’ fixed point, the properties of the large time dynamics can be obtained via simplified RG rules that are asymptotically exact, because the preferred exit channel out of a given renormalized valley typically dominates over the other exit channels asymptotically. As an example of application, we have followed numerically the RG flow for the case of a directed polymer in a two-dimensional random medium. The full RG rules have been used to check that the RG flow is towards some infinite disorder fixed point, whereas the simplified RG rules that allow us to study bigger sizes have been used to estimate the barrier exponent $\psi \sim 0.47$ of the fixed point, in reasonable agreement.
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with the best numerical measure currently available, \( \psi \sim 0.49 \), obtained via Langevin dynamics [17].

From a theoretical point of view, we have explained why the RG flow towards an infinite disorder fixed point in configuration space gives strong support to the droplet scaling theory [6] where the dynamics is governed by the logarithmic growth of the coherence length \( l(t) \sim (\ln t)^{1/\psi} \), and where two-time \( (t_w, t_w + t) \) ageing properties involve the ratio \( l(t_w + t)/l(t_w) \) of the coherence lengths, i.e. the ratio \( \ln(t_w + t)/\ln(t_w) \). Moreover, the statistics of barriers corresponds to a very strong hierarchy of valleys within valleys, which is necessary to allow the coexistence of rejuvenation and memory effects in temperature cycling experiments [1]: the rejuvenation due to short length scales does not destroy the memory of large length scales which are effectively frozen.

Besides these ageing properties, another important issue is the response of disordered systems to an external force \( F \). This question is analyzed in detail in our recent work [42] where we explain how the ‘infinite disorder fixed point’ for \( F = 0 \) becomes a ‘strong disorder fixed point’ at small \( F \) with an exponential distribution of renormalized barriers, that leads to the existence of an anomalous zero-velocity phase for the motion of driven interfaces in random media.

From a numerical point of view, it is clear that the formulation of RG rules in configuration space has an exponential numerical price, since the number of initial configurations \( \mathcal{N}_0 \) grows exponentially with the number of degrees of freedom, i.e. grows exponentially with the volume \( L^d \) for a system of linear size \( L \) in dimension \( d \). This computational complexity is not surprising, since the determination of barriers for the dynamics is expected to be an NP-complete problem [43]. For the case of a directed polymer in a two-dimensional random medium considered in the present paper where \( \mathcal{N}_0 = 2^L \), we have been able to follow the full RG rules up to \( L \leq 11 \), and the simplified RG rules up to \( L \leq 18 \). So it is clear that the numerical study of higher dimensional disordered systems via strong disorder RG rules requires other decisive improvements. The most promising idea is to use the spatial locality of the dynamics and the fact that regions separated by a distance bigger than the coherence length \( l(t) \) are not yet dynamically correlated at time \( t \). This strategy of ‘quasi-factorization’ into patches of increasing length scale has been successfully applied recently in the context of Monte Carlo exact sampling of the two-dimensional Ising spin glass [44]. We thus hope that the numerical application of strong disorder RG in configuration space will become possible in the future for disordered models for two or more dimensions.

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Erratum

Equation (24) and the following sentence should read

\[ W(C \to C') = \delta_{(C,C')} \min \left( 1, e^{-\left( E(C') - E(C) \right)/T} \right). \] (24)

The first factor \( \delta_{(C,C')} \) means that the two configurations are related by a single move and the last factor ensures the convergence towards thermal equilibrium at temperature \( T \) via the detailed balance property of equation (25).