Stable Vortex Solitons in Nonlocal Self-Focusing Nonlinear Media

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We reveal that spatially localized vortex solitons become stable in self-focusing nonlinear media when the vortex symmetry-breaking azimuthal instability is eliminated by a nonlocal nonlinear response. We study the main properties of different types of vortex beams and discuss the physical mechanism of the vortex stabilization in spatially nonlocal nonlinear media.

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Vortices are fundamental objects which appear in many branches of physics. In optics, vortices are usually associated with phase singularities of diffracting optical beams, and they can be generated experimentally in different types of linear and nonlinear media. However, optical vortices become highly unstable in self-focusing nonlinear media due to the symmetry-breaking azimuthal instability, and they decay into several fundamental solitons. In spite of many theoretical ideas to stabilize optical vortices in specific nonlinear media, no stable optical vortices created by coherent light were readily observed in experiment. Thus, the important challenge remains to reveal physical mechanisms which would allow the first experimental observation of stable vortices in realistic nonlinear media.

In this Letter, we reveal that the symmetry-breaking azimuthal instability of the vortex beams can be eliminated in a medium characterized by a nonlocal nonlinear response. This observation allows us to suggest a simple and realistic way to generate experimentally the first stable spatially localized vortices in self-focusing nonlinear media. We study the main properties and stability of different types of vortex beams, and discuss the physical mechanism of their stabilization in spatially nonlocal nonlinear media.

We notice that there are many physical systems characterized by nonlocal nonlinear response. In particular, a nonlocal response is induced by heating and ionization, and it is known to be important in plasmas. Nonlocal response is a key feature of the orientational nonlinearities due to long-range molecular interactions in nematic liquid crystals. An interatomic interaction potential in Bose-Einstein condensates with dipole-dipole interactions is also substantially nonlocal. In all such systems, nonlocal nonlinearity can be responsible for many novel features such as the familiar effect of the collapse arrest.

We consider propagation of the electric field envelope \( E(X,Y,Z) \) described by the paraxial wave equation:

\[
2ik_0 \frac{\partial E}{\partial Z} + \frac{\partial^2 E}{\partial X^2} + \frac{\partial^2 E}{\partial Y^2} + k_0^2 n_T \Theta E = 0,
\]

(1)

where \( k_0 \) is the wave number, and the function \( \Theta \) characterizes a nonlinear, generally nonlocal, medium response. For example, in the case of the wave beam propagation in partially ionized plasmas, \( \Theta = T_e/T \) is the relative electron temperature perturbation, \( T \) being unperturbed temperature, the coupling coefficient \( n_T = +1 \). Stationary temperature perturbation obeys the equation:

\[
\alpha^2 \Theta - l_e^2 \left[ \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} + \frac{\partial^2 \Theta}{\partial Z^2} \right] = |E|^2/E_c^2,
\]

(2)

where \( E_c^2 = 3mT(\omega_0^2 + \nu_e^2)/\epsilon^2 \), \( \nu_e \) is the electron collision frequency, \( \omega_0 \) is the wave frequency, \( m \) is electron mass, \( \alpha^2 \) characterizes the relative portion of the energy that electron deliver to a heavy particle during single collision. The second term describes thermal diffusion with the characteristic spatial scale \( l_e^2 \). Note, that the model identical to Eqs. (1) and (2) has been employed in Ref. to study two-dimensional bright solitons observed experimentally in nematic liquid crystals. In this case, the field \( \Theta \) describes the spatial distribution of the molecular director.

Rescaling the variables, \( (X,Y,Z) = l_e(x,y,z) \), \( Z = 2l_e z/\epsilon \) and the fields, \( E = (E_c \epsilon/\sqrt{n_T}) \Psi(x,y,z) \) and \( \Theta = (\epsilon^2/n_T) \Theta(x,y,z) \), where \( \epsilon = (k_0 l_e)^{-1} \), we present Eq. (2) in the dimensionless form,

\[
\alpha^2 \theta - \Delta_{\perp} \theta - \frac{\epsilon^2}{4} \frac{\partial^2 \theta}{\partial z^2} = |\Psi|^2,
\]

(3)

where \( \Delta_{\perp} = \partial^2/\partial x^2 + \partial^2/\partial y^2 \) is the transverse Laplacian. For the analysis performed below, we omit in Eq. (3) the term proportional to \( \epsilon^2 \).
Thus, the basic dimensionless equations describing the propagation of the electric field envelope $\Psi(x, y, z)$ coupled to the temperature perturbation $\theta(x, y, z)$ become

$$
i \frac{\partial \Psi}{\partial z} + \Delta_\perp \Psi + \theta \Psi = 0,$$
$$\alpha^2 \theta - \Delta_\perp \theta = |\Psi|^2.$$  \hspace{1cm} (4)

In the limit $\alpha^2 \gg 1$, we can neglect the second term in the equation for the field $\theta$ in Eq. (4) and reduce this system to the standard local nonlinear Schrödinger (NLS) equation with cubic nonlinearity. The opposite case, i.e. $\alpha^2 \ll 1$, can be referred to as a strongly nonlocal regime of the beam propagation.

We look for stationary solutions of the system (4) in the form, $\Psi(x, y, z) = \psi(r) \exp(i m \varphi + i \Lambda z)$, where $\varphi$ and $\psi(r)$, are the azimuthal angle and radial coordinate, respectively, and $\Lambda$ is the beam propagation constant. Such solutions describe either the fundamental optical soliton, when $m = 0$, or the vortex soliton with the topological charge $m$, when $m \neq 0$.

The beam radial profile $\psi(r)$ and accompanied temperature field $\theta(r)$ can be found by solving the system of ordinary differential equations,

$$-\lambda \psi + \Delta_{(m)} \psi + \theta \psi = 0,$$
$$\alpha^2 \theta - \Delta_{(0)} \theta = |\psi|^2,$$

where $\Delta_{(m)} = d^2/dr^2 + (1/r)(d/dr) - (m^2/r^2)$, and we rescale $\psi$, $\theta$, $1/r^2$, and $\alpha^2$ by the factor $\Lambda$, and choose the unity propagation constant. Boundary conditions are: for the localized vortex field, $\psi(\infty) = \psi(0) = 0$, and for the temperature field, $d\theta/dr|_{r=0} = 0$ and $\theta(\infty) = 0$.

The second equation of the system (4) can be readily solved for axially symmetric intensity distribution $|\Psi|^2$,

$$\theta(r, z) = \int_0^{+\infty} |\Psi(\xi, z)|^2 G_0(r, \xi; \alpha) d\xi,$$

where $G_0$ is the Green’s function defined at $\nu = 0$ from the general expression

$$G_\nu(\xi_1, \xi_2; a) = \begin{cases} K_\nu(a\xi_2)I_\nu(a\xi_1), & 0 \leq \xi_1 < \xi_2, \\
I_\nu(a\xi_2)K_\nu(a\xi_1), & \xi_2 < \xi_1 < +\infty, \end{cases}$$

and $I_\nu$ and $K_\nu$ are the modified Bessel functions of the first and second kind, respectively. Thus, Eqs. (4) are equivalent to a single integro-differential equation obtained from (5) when $\theta(r)$ is eliminated, or to a single integral equation:

$$\psi(r) = \int_0^{+\infty} \theta(\eta)\psi(\eta) G_m(r, \eta; \sqrt{\lambda}) d\eta,$$

where $G_m$ is defined by Eq. (7) and $\theta$ is given by Eq. (5).

We solve the nonlinear integral equation (5) using the stabilized relaxation procedure similar to that employed in Ref. [12]. Figure 2 shows several examples of the solutions of the system (5) found numerically for different values of nonlocality parameter $\alpha$. To characterize these solutions, we define the effective radii $r_\psi$ and $r_\theta$ of the intensity distribution $|\psi|^2$ and the temperature perturbation distribution $\theta$, respectively, as follows,

$$r_\psi^2 = \frac{1}{N} \int r^2 |\psi(r)|^2 d^2r, \quad r_\theta^2 = \frac{\int r^2 \theta(r) d^2r}{\int \theta(r) d^2r}.$$  \hspace{1cm} (8)

Figure 2(a) shows the radii $r_\psi$ and $r_\theta$ as functions of the nonlocality parameter $\alpha$. Both $r_\psi$ and $r_\theta$ decrease monotonically when the nonlocality parameter grows. In the local limit ($\alpha \gg 1$), $r_\psi$ and $r_\theta$ saturate at the same finite value, which increases with the topological charge. Figure 2(b) shows the beam power $P = \int |E|^2 d^2r$ as a function of the nonlocality parameter $\alpha$.

The important information on stability of the vortex solitons can be obtained from the analysis of small perturbations of the stationary states. The basic idea of such a linear stability analysis is to represent a perturbation as the superposition of the modes with different azimuthal symmetry. Since the perturbation is assumed to be small, stability of each linear mode can be studied independently. Presenting the nonstationary solution in the vicinity of the stationary mode as follows,

$$\Psi(r, z) = \left\{ \psi(r) + \varepsilon_+(r)e^{i\omega z+IL\varphi} + \varepsilon_-^*(r)e^{-i\omega z-IL\varphi} \right\} e^{i\lambda z},$$
where $|\varepsilon_\pm| \ll \psi$, $|\vartheta_\pm| \ll \theta$, $\theta$, $\psi$ are assumed to be real without loss of generality, we linearize Eqs. \ref{eq:linearization} and obtain the system of linear equations of the form:

$$
\pm \left\{ -\lambda + \Delta_r^{(m\pm L)} + \theta(r) + \hat{g}_L \right\} \varepsilon_\pm \pm \hat{g}_L \varepsilon_\mp = \omega \varepsilon_\pm, \tag{9}
$$

where

$$
\hat{g}_L = \psi(r) \int_0^\infty \xi \psi(\xi) G_L(r, \xi; \alpha) e^{i \alpha} \xi \xi d\xi.
$$

The Hankel spectral transform was applied to reduce the integro-differential eigenvalue problem \ref{eq:linear-eigenvalue} to a linear algebraic one. The maximum growth rate $|\omega|$ of linear perturbation modes is shown in Fig. \ref{fig:eigenvalues} for a single-charge vortex ($m = 1$). The symmetry-breaking modes can become unstable only for $L = 1, 2, 3$. The growth rates saturate in the local regime $\alpha \gg 1$. The largest growth rate as well as the widest instability region has the azimuthal mode with the number $L = 2$. The real and imaginary parts of the eigenvalues $\omega$ for this most dangerous mode are shown in Fig. \ref{fig:eigenvalues}a. Importantly, there exhibits a bifurcation point $\alpha_{\text{cr}} \approx 0.12$ below which the growth rate $\Re(\omega)$ vanishes. Thus, the symmetry-breaking azimuthal instability is eliminated in a highly-nonlocal regime: all growth rates vanish provided $\alpha < \alpha_{\text{cr}}$.

We also perform the linear stability analysis for multi-charge vortices with the topological charges $m = 2$ and $m = 3$. Figure \ref{fig:eigenvalues}c shows the growth rate of the linear perturbation modes for the vortex with $m = 2$. Importantly, the growth rate of the $L = 2$ mode always remains nonzero, and the same result holds for the vortices with $m = 3$. Therefore, the linear stability analysis predicts the existence of stable single-charge vortex in a highly-nonlocal regime, while the multi-charge vortices are shown to be unstable with respect to decay into the fundamental solitons, even in the limit $\alpha \to 0$.

Our stability analysis has been verified with direct simulations of the propagation dynamics of perturbed vortex solitons by employing the split-step Fourier method to solve Eqs. \ref{eq:linearization} numerically. The results of our linear stability analysis agree well with the numerical simulations. In particular, the symmetry-breaking instabilities have been observed in the region predicted by the linear stability analysis, and some examples of the vortex decay instability are presented in Fig. \ref{fig:decay} for the vortices with $m = 1$. If a perturbation is applied to a single-charge vortex in the strongly nonlocal regime (such that all azimuthal instabilities are completely suppressed by the nonlocality), the vortex beam evolves in a quasi-periodic fashion: the effective radii and amplitudes oscillate with $z$. Thus, our numerical simulations indicate that single-charge vortex solitons become stable if the nonlocality parameter $\alpha$ is below some critical value which is very close to the value $\alpha_{\text{cr}} \approx 0.12$ predicted by the linear stability analysis. In experiments, the input beam may differ essentially from the exact stationary solution. Therefore, we perform additional numerical simulations for singular Gaussian input beams of the form $\Psi(r, 0) = h r \exp(-r^2/w^2) + i \varphi$ and, as follows from Fig. \ref{fig:decay}, observe that such beams are indeed stabilized below the critical value of nonlocality, and the beam effective intensity, defined as $|\Psi(r)|^2 = N^{-1} \int |\Psi|^4 d^2r$, undergoes large-amplitude oscillations.

The physical mechanism for suppressing the symmetry-breaking azimuthal instability of the vortex beam in a nonlocal nonlinear medium can be understood as being associated with effective diffusion processes introduced by a nonlocal response. Indeed, if a small azimuthal perturbation of the radially-symmetric vortex deforms its shape in some region, the corresponding temperature distribution along the vortex ring becomes nonuniform. As a result, the intrinsic thermodiffusion processes would smooth out this inhomogeneity and suppress its further growth, leading to the complete vortex stabilization in a highly-nonlocal regime.

In conclusion, we have studied the basic properties and stability of spatially localized vortex beams in a self-focusing nonlinear medium with a nonlocal response. We have found that single-charge optical vortices can be stabilized with respect to any symmetry-breaking azimuthal instability in the regime of strong nonlocal response, whereas multi-charged vortices remain unstable and decay into the fundamental solitons for any degree of nonlocality. Our numerical simulations confirm that the stable propagation of single-charge vortex is indeed possible in a nonlocal nonlinear medium. We expect that these results will stimulate the first experimental observation of stable vortices in self-focusing nonlinear media.
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FIG. 1: Examples of the stationary vortex solutions for $m = 1$ and $m = 2$ at different values of the nonlocality parameter $\alpha$. Shown are the fields $\psi(r)$ (solid) and $\theta(r)$ (dotted).

FIG. 2: (a) Effective radii of the field intensity distribution $r_\psi$ (solid) and the temperature field $r_\theta$ (dotted) vs. the nonlocality parameter $\alpha$, for $m = 1, 2$. (b) Power $P$ vs. $\alpha$.

FIG. 3: Maximum growth rate of linear perturbation modes vs. the nonlocality parameter $\alpha$ for the vortices with (a) $m = 1$ and (c) $m = 2$. Numbers near the curves stand for the azimuthal mode numbers $L$. (b) Real and imaginary parts of the eigenvalue $\omega$ of the most dangerous azimuthal mode with $L = 2$ and $m = 1$. 
FIG. 4: Evolution of the beam intensity $|\Psi|^2$ (upper row) and temperature field $\theta$ (lower row) of a perturbed single-charge vortex soliton for (a) $\alpha = 5$ and (b) $\alpha = 1$.

FIG. 5: (a) Stable propagation of a single-charged vortex generated by a singular Gaussian beam with $m = 1$ ($h = 0.14$, $w = 4.2$, $\alpha = 0.07$) shown for the beam intensity $|\Psi|^2$ and temperature field $\theta$. (b) Oscillatory dynamics of the amplitude of the intensity field $\langle |\Psi|^2 \rangle$ vs. $z$ during the beam propagation.