Periods of the multiple Berglund–Hübsch–Krawitz mirrors

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Abstract
We consider the multiple Calabi–Yau mirror phenomenon which appears in Berglund–Hübsch–Krawitz (BHK) mirror symmetry. We show that for any pair of Calabi–Yau orbifolds that are BHK mirrors of a loop–chain-type pair of Calabi–Yau threefolds in the same weighted projective space the periods of the holomorphic nonvanishing form coincide.

Keywords Mirror symmetry · Calabi-Yau manifolds · Compactification

Mathematics Subject Classification 14J32 · 14J33 · 14J81 · 81T30 · 81T33

1 Introduction
Calabi–Yau manifolds (CY) arise in the context of spacetime supersymmetric compactifications of string theories. An important property of CY manifolds is mirror symmetry [1–3], which reflects geometric relation between pairs of CY manifolds. Namely, for a pair of n-dimensional Calabi–Yau manifolds X and Y, cohomologies are isomorphic, \( H^{p,q}(X, \mathbb{C}) = H^{n-p,q}(Y, \mathbb{C}) \).

Calabi–Yau manifolds possess the structure of complex and Kähler manifold which admit deformations (see [4]), giving rise to the moduli spaces \( M_C(X) \) and \( M_K(X) \). Mirror symmetry can be considered as matching of the special geometries [5] on the moduli spaces

In memory of Boris Dubrovin.

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Here, we focus on the class of CY manifolds, defined as hypersurfaces or orbifolds in weighted projective spaces $\mathbb{P}^4_{(k_1,k_2,k_3,k_4,k_5)}$ cut out by non-degenerate quasihomogeneous polynomials consisting of five monomials. The nondegeneracy requires that these polynomials must be a sum of the polynomials of three basic types, classified by Kreuzer and Skarke [6]. For this class of CY manifolds, it was suggested by Berglund and Hübsch in [7] that their mirrors are obtained as particular orbifolds of the projective spaces defined by the “transposed” polynomials. The Berglund–Hübsch mirror construction was generalized by Krawitz in [8,9]. Chiodo and Ruan [10] proved that the Berglund–Hübsch–Krawitz (BHK) mirrors form a mirror pair on the level of Chen-Ruan orbifold cohomology.

The multiple mirror phenomenon occurs when a given CY threefold possesses more than one mirror in different weighted projective spaces. More specifically, the phenomenon is the following. Some weighted projective spaces allow the existence of a few CY manifolds defined by the polynomials that belong to different types of Kreuzer-Skarke class. In such cases, the Berglund–Hübsch–Krawitz (BHK) mirrors of these two CY manifolds generally appear in two different weighted projective spaces. The examples of such phenomenon are given in Table 1, which shows all the cases when the weighted projective space allow the simultaneous occurrence of CY manifolds of loop and chain types of Kreuzer-Skarke class CY manifolds. The problem of birationality of multiple BHK mirrors has been investigated early in the case of one-parameter CY family [11–14], where it was shown that the different BHK mirrors are birational. The derived equivalence of such multiple BHK mirrors was obtained in [15]. In [16], the periods of the multiple mirrors were calculated in one case of multi-parameter CY family, and it was found that they match. In this paper, we prove that the periods of the nonvanishing holomorphic form $\Omega_1$ coincide for multiple BHK mirrors of the loop and chain types. We assume that using a similar technique, one can also prove coincidence of the periods for other types of multiple BHK mirrors.

In [17], it was shown using the technique of Griffiths-Dwork diagrams that if two invertible pencils $(\sum_j \prod_i x_{ij}^{M_{ij}} + t \prod_i x_i)$ with invertible $M$ have the same dual weights, then their Picard-Fuchs equations are the same. It is equivalent to the statement that one-parameter deformations of several BHK mirrors have the same Picard-Fuchs equations. This statement is consistent with ours as a special case. The paper [17] used another tools to show the coincidence of Picard-Fuchs equations for one-parameter deformation of invertible polynomials with multiple mirrors. Note that the coincidence of the periods does not follow automatically from this even in this particular case.

In Sect. 2, we briefly recall the BHK mirror construction. In Sect. 3, we consider the expressions for the periods of the holomorphic form $\Omega$ for two different BHK mirrors and show their coincidence.
\[ \mathbb{P}^4_{(k_1,k_2,k_3,k_4,k_5)} = \left\{ (x_1, \ldots, x_5) \in \mathbb{C}^5 \setminus \{0\} \mid x_i \sim \lambda^{k_i} x_i \right\} \quad (2.1) \]

by a quasi-homogeneous polynomial

\[ W^0_M(x) = \sum_{i=1}^{5} \prod_{j=1}^{5} x^{M_{ij}}. \quad (2.2) \]

Such polynomials obey the following properties:

1. Matrix \( M \) is integer, and invertible;
2. The polynomial \( W^0_M \) is quasi-homogeneous, i.e., there exist positive integers \( k_i \) (weights of the projective space), and \( d \) (degree), such that

\[ \sum_{j=1}^{5} M_{ij} k_j = d, \quad \forall i, \quad (2.3) \]

where \( d = \sum_{j=1}^{5} k_j \), which is CY condition for the orbifold in the weighted projective space.
3. The polynomial \( W^0_M \) is non-degenerate outside the origin.

From these conditions, it follows that the \( W^0_M \) is given by a sum of polynomials of one of three types [6]:

\[ x_1^{A_1} + x_2^{A_2} + \ldots + x_n^{A_n} - \text{Fermat} , \]
\[ x_1^{A_1} x_2 + x_2^{A_2} x_3 + \ldots + x_n^{A_n} - \text{Chain} , \]
\[ x_1^{A_1} x_2 + x_2^{A_2} x_3 + \ldots + x_n^{A_n} x_1 - \text{Loop} . \quad (2.4) \]

Then, it follows that the equation \( W^0_M = 0 \) defines a Calabi–Yau orbifold \( X^0_M \). We use below the following properties of the matrix \( M \). There exist positive integer numbers \( \bar{k}_j \) such that

\[ \sum_{i=1}^{5} \bar{k}_i M_{ij} = \bar{d}, \quad \forall j, \quad (2.5) \]

where \( \bar{d} = \sum_{i=1}^{5} \bar{k}_i \). The set \( \bar{k}_j \) is nothing but the set of weights of another weighted projective space \( \mathbb{P}^4_{(\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4,\bar{k}_5)} \) where \( X^\bar{M} \), i.e., the mirror of the CY orbifold \( X^0_M \), which will be defined below. Let \( B_{ij} \) denote the inverse matrix of \( M \), i.e., \( \sum_k B_{ik} M_{kj} = \delta_{ij} \). Then using (2.3) and (2.5), we get the relations (see e.g., [10]):

\[ \sum_{j=1}^{5} B_{ij} = \frac{k_i}{\bar{d}} \quad \text{and} \quad \sum_{i=1}^{5} B_{ij} = \frac{\bar{k}_j}{\bar{d}} . \quad (2.6) \]
The CY orbifold \( X_M^0 \) admits a deformation of the complex structure, which is realized by a polynomial \( W_M \), which is a deformation of the \( W_M^0 \). The full family \( X_M \) is given by zero locus of

\[
W_M = \sum_{i=1}^{5} \prod_{j=1}^{5} x_{ij}^{M} + \sum_{l=1}^{h} \varphi_l e_l(x),
\]

where \( e_l \) are some quasi-homogeneous monomials (see below). The \( \varphi_l \) are moduli of the complex structure of the family \( X_M \), and \( h := h_{21} \) is the Hodge number of the family \( X_M \), which is equal to the dimension of the complex structures moduli space of \( X_M \).

Berglund and Hübsch proposed [7] that the mirror partner \( X_M^0 \) of the orbifold \( X_M^0 \) is defined as a quotient of the zero locus of the polynomial

\[
W_M^0(z) = \sum_{i=1}^{5} \prod_{j=1}^{5} x_{ij}^{M^T} \quad (2.8)
\]

in the weighted projective space \( \mathbb{P}^4_{(k_1, k_2, k_3, k_4, k_5)} \). This quotient is realized by some subgroup of the phase symmetry group of \( W_M^0 \).

This approach has been generalized by Krawitz [8,9] as follows. Let \( \text{Aut}(M) \) be the group of the phase symmetries of \( W_M^0 = 0 \), and let \( \text{SL}(M) \) be its maximal subgroup, which preserves the nonvanishing holomorphic form \( \Omega \). We will call \( \text{SL}(M) \) the maximal admissible group. We denote by \( J_M \subseteq \text{SL}(M) \) the subgroup that consists of the phase symmetries induced by \( \mathbb{C}^* \) action on \( \mathbb{P}^4_{(k_1, k_2, k_3, k_4, k_5)} \). Choose a group \( G_0 \) to be some subgroup of \( \text{SL}(M) \) containing \( J_M \) and let \( G = G_0/J_M \). It can be shown that the quotient space \( Z(M, G) := X_M/G \) is a CY orbifold. The same can be applied to the transposed polynomial \( W_M^0 \) to get CY orbifold \( Z(M^T, G^T) := X_M^T/G^T \). Krawitz has shown how to choose the dual group \( G^T \) such that \( Z(M, G) \) and \( Z(M^T, G^T) \) form a mirror pair. Below we describe this mirror construction in more details.

Let CY hypersurface \( X_M^0 \) be defined, as above, by the zero locus of \( W_M^0(x) \) and let \( \omega = e^{2\pi i/d} \). The polynomial \( W_M^0(x) \), as well as the form \( \Omega \), is invariant under the action of the group \( J_M \) defined as

\[
x_i \mapsto \omega^{k_i} x_i, \quad \omega^d = 1. \quad (2.9)
\]

Let \( q_i(M) \) be generators of \( \text{Aut}(M) \), which act on the coordinates as

\[
q_i(M) : x_j \mapsto e^{2\pi i B_{ij}} x_j. \quad (2.10)
\]

In these terms, the generator of the group \( J_M \)

\[
\hat{J} = \prod_{i=1}^{5} q_i(M). \quad (2.11)
\]
Let \( G_0 \) be some admissible group such that \( J_M \subseteq G_0 \subseteq SL(M) \) and the quotient group

\[
G := G_0 / J_M. \tag{2.12}
\]

Then, the orbifold defined as \( Z(M, G) := X_M / G \) is a Calabi–Yau manifold. Note that \( X_M \) itself is a special case of \( Z(M, G) \) when \( G_0 = J_M \).

To define the monomials \( e_l \) (see (2.7)), we begin with the polynomial \( W^M_0 \) and some group \( G_0 \subseteq Aut(M) \). Let monomials

\[
e_l := \prod_{j=1}^{5} x_j^{S_{lj}}, \tag{2.13}
\]

where \( \sum_j S_{lj} k_j = d \), be all linearly independent elements of degree \( d \) of the \( G_0 \)-invariant subring of the Milnor ring \( \mathbb{C}[x_1, \ldots, x_5]/\langle \frac{\partial W^M_0}{\partial x_j} \rangle \) [18]. Then, the quotient \( Z(M; G) \) by the group \( G \) is defined by the zero locus of the polynomial

\[
W_M = \sum_{i=1}^{5} \prod_{j=1}^{5} x_j^{M_{ij}} + \sum_{l=1}^{h} \varphi_l \prod_{j=1}^{5} x_j^{S_{lj}}, \tag{2.14}
\]

where \( \varphi_l \) are moduli of complex structure deformations, and \( h := h_{21}(X) \) is Hodge number.\(^1\) We denote by \( e_h \) the monomial with \( S_{h,i} = 1 \), which plays a distinguished role,

\[
e_h = \prod_{i=1}^{5} x_i. \tag{2.15}
\]

Let \( \rho_s \) be the generators of the group \( G_0 \), then by definition

\[
\rho_s \cdot \prod_{j=1}^{5} x_j^{S_{lj}} = \prod_{j=1}^{5} x_j^{S_{lj}}, \quad l = 1, \ldots, h. \tag{2.16}
\]

Once we have defined \( G^T_0 \), we can also define similar data for the polynomial \( W^T_M \) and \( G^T_0 \). Taking the quotient \( G^T := G^T_0 / J_M^T \), we get the Calabi–Yau orbifold \( Z(M^T, G^T) \).

The way to choose a group \( G^T \) such that the manifold \( Z(M^T, G^T) \) will be the mirror for \( Z(M, G) \) is as follows [8,9]. The generators \( \rho^T_l \) of the group \( G^T_0 \) should be constructed using the exponents \( S_{li} \) of the invariant monomials (2.13) as follows:

\(^1\) With a slight abuse of notation, we use here \( h \) for the Hodge number of the orbifold, which we used before for the original manifold \( X_M \).
\[ \rho^T_l := \prod_{i=1}^{5} q_i(M^T)^{sl_i}, \quad (2.17) \]

where \( q_i(M^T) \) are the generators of \( \text{Aut}(M^T) \) acting on each coordinate \( z_j \) in \( \mathbb{P}^4(k_1,k_2,k_3,k_4,k_5) \) as

\[ q_i(M^T) : z_j \mapsto e^{2\pi i B^T_{ij}} z_j = e^{2\pi i b_{ij}} z_j. \quad (2.18) \]

It follows that the generators \( \rho^T_l \) of the group \( G^T_0 \) act on the coordinates \( z_j \) as

\[ \rho^T_l : z_j \mapsto e^{2\pi i \sum_i s_i b_{ij} z_j}. \quad (2.19) \]

The mirror Calabi–Yau family \( Z(M^T, G^T) \) is given by the zero locus of

\[ W_{M^T} := \sum_{i=1}^{5} \prod_{j=1}^{5} z^M_{ij}^{T^T} + \sum_{r=1}^{\tilde{h}} \psi_r \prod_{j=1}^{5} z^T_{rj} = \sum_{i=1}^{5} \prod_{j=1}^{5} z^M_{ij} + \sum_{r=1}^{\tilde{h}} \psi_r \prod_{j=1}^{5} z^T_{rj}, \quad (2.20) \]

where \( \psi_r \) are moduli of the complex structure of the family \( Z(M^T, G^T) \) and monomials \( \tilde{e}_r := \prod_{j=1}^{5} z^{T^T}_{rj} \) are invariant under the \( G^T_0 \) action (2.19):

\[ \rho^T_l \cdot \tilde{e}_r = e^{2\pi i \sum_i b_{ij} s_i t_{ij}} \prod_{j=1}^{5} z^T_{rj} = \prod_{j=1}^{5} z^T_{rj}. \quad (2.21) \]

It is also necessary to take into account the quasi-homogeneity condition

\[ \sum_{i=1}^{5} T_{mi} k_i = \tilde{d}. \quad (2.22) \]

It is convenient to consider the exponents \( s_{il} \), and \( t_{ij} \) as nonnegative integer five-component vectors \( (\tilde{S}_i)_{j} = s_{ij} \), and \( (\tilde{T}_r)_{i} = t_{ri} \). It was shown by Krawitz [8,9] that the condition (2.21) can be reformulated in terms of the pairing of these vectors, defined by the matrix \( B \), as follows\(^2\)

\[ (\tilde{S}_i, \tilde{T}_r) := \sum_{i,j=1}^{5} b_{ij} s_{ij} t_{ij} \in \mathbb{Z}. \quad (2.23) \]

The relations (2.22) and (2.23) are very restrictive because the entries of the matrices \( s_{il} \) and \( t_{ij} \) are integers, while the entries of the inverse matrix \( b_{ij} \) are rational. Therefore,

\(^2\) See also [19] for related discussion.
if the vectors $\vec{S}_r$ are defined as above, these relations have a finite number of solutions for the vectors $\vec{T}_r$. The CY orbifold cut out by $W_{M'^T}$ is the mirror for $Z(M, G)$, as it is shown in [8,9].

3 Periods of BHK mirrors and their coincidence

In this section, we use the relation (2.23) to prove our basic statement that the periods of multiple mirrors coincide. We will focus on the case when two hypersurfaces in the weighted projective space simultaneously come from two polynomials of the loop and chain types that “point” in the same direction. We will assume also that the admissible group $G_0 = J_M$ preserves both polynomials.

Let $X_M$ be the CY family $X_M$ as defined in Sect. 2. We regard $X_M$ as the family of CY manifolds, since the polynomial $W_M(x, \phi)$ is a function of two sets of variables $x$ and parameters $\{\phi_l\}$. Then the mirror family $Y_{M'^T}$ is defined in $\mathbb{P}_{\vec{k}}^4(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4, \vec{k}_5)$ by zeros of the polynomial

$$W_{M'^T}(x) = \sum_{i=1}^{5} \prod_{j=1}^{5} x^{M'T}_{ij} + \sum_{r=1}^{\vec{h}} \psi_r \prod_{j=1}^{5} x^{T}_{rj},$$

(3.1)

where it is assumed that the relationships $\sum_{j=1}^{5} T_{rj} \vec{k}_j = \vec{d}$ and $\sum_{ij} B_{ij} S_{ii} T_{ij} \in \mathbb{Z}$ are fulfilled.

Periods of the holomorphic form $\Omega$ for CY family $X_M$ can be rewritten as the contour integrals [22]

$$\sigma_{\vec{\mu}}(\phi) = \oint_{\Gamma_{\vec{\mu}}} e^{-W_M(x, \phi)} d^5 x .$$

(3.2)

The number of the contours $\Gamma_{\vec{\mu}}$ is equal to $\dim R(M) = 2h + 2$. Here, $R(M)$ is so-called chiral ring which is subring of polynomials in $x_1, \ldots, x_5$ modulo $\{\partial W_0^M / \partial x_i\}$ of degree 0, $d$, 2$d$ or 3$d$. The ring $R(M)$ consists of monomials

$$e_{\vec{\mu}}(x) = \prod_{i=1}^{5} x_i^{\mu_i},$$

(3.3)

where

$$\sum_{i=1}^{5} k_i \mu_i = 0, \ d, \ 2d, \ 3d$$

(3.4)

and $\mu_i$ are nonnegative integers satisfying some constraints.\(^3\)

\(^3\) Below $A_i$ are defined according to Eqs. (2.3) and (2.4).
In the Fermat case

\[ 0 \leq \mu_i \leq A_i - 2 . \] (3.5)

In the Loop case

\[ 0 \leq \mu_i \leq A_i - 1 . \] (3.6)

A convenient set of the chiral ring monomials in the Chain case was described in [20] (see also [21], [9]).

From (3.4), it follows that the ring \( R(M) \) is graded,

\[ R(M) = R^0(M) \oplus R^d(M) \oplus R^{2d}(M) \oplus R^{3d}(M) \] (3.7)

and \( \dim R^0(M) = \dim R^{3d}(M) = 1 \), while \( \dim R^d(M) = \dim R^{2d}(M) = h \). The monomials \( e_l(x) = \prod_{j=1}^{5} x_j^{S_{lj}} \), \( l = 1, \ldots, h \) are elements of the subspace \( R^d(M) \). Similar statements can be formulated for the mirror orbifold \( Y_{M^T} \).

Since

\[ W_M(x, \phi) = W^0_M(x) + \sum_{l=1}^{h} \phi_l e_l(x) , \] (3.8)

the periods have the form

\[
\sigma_{\vec{\mu}}(\phi_1, ..., \phi_h) = \int_{\Gamma_{\vec{\mu}}} d^5 x e^{-W^0_M(x)} e^{-\sum_{l=1}^{h} \phi_l e_l(x)} = \sum_{\{m_l\}} \prod_{l=1}^{h} \frac{\phi_l^{m_l}}{m_l!} \int_{\Gamma_{\vec{\mu}}} d^5 x e^{-W^0_M(x)} \prod_{j=1}^{5} x_j^{\sum_{l=1}^{h} m_l S_{lj}}. \] (3.9)

Using the standard cohomology technique [22], we obtain the following explicit expression:

\[
\sigma_{\vec{\mu}}(\phi_1, ..., \phi_h) \sim \sum_{\{m_l\} \in \Sigma_{\vec{\mu}}} C(\{m_l\}) \prod_{l=1}^{h} \frac{\phi_l^{m_l}}{m_l!} , \] (3.10)

where

\[
C(\{m_l\}) = \prod_{i=1}^{5} \Gamma \left( \sum_{j} (m_l S_{lj} + 1) B_{ji} \right). \] (3.11)
The canonical choice of $\Gamma_{\tilde{\mu}}$ defines it as a cycle dual to the element of the base of the chiral ring $e_{\tilde{\mu}}$

$$\int_{\Gamma_{\tilde{\mu}}} e_{\tilde{\nu}}(x)e^{-W_0(x)} d^5 x = \delta_{\tilde{\mu},\tilde{\nu}}. \quad (3.12)$$

It leads to the following definition of $\Sigma_{\tilde{\mu}}$ [22]. The set of nonnegative integers $\{m_l\}$, where $l = 1, \ldots, h$, belongs to $\Sigma_{\tilde{\mu}}$, if one can find such nonnegative integers $n_j$, $j = 1, \ldots, 5$ that

$$m_l S_{lj} = (\tilde{\mu})_l + \sum_j n_j M_{lj}. \quad (3.13)$$

From (3.11), we see that the periods are determined by the sets of $h$ five-component vectors $\sum_{j=1}^5 S_{lj} B_{lj}$, where $l = 1, \ldots, h$, as well as by the sets $\{m_l\}$, $l = 1, \ldots, h$.

Let’s now prove that the periods of CY orbifolds $Y_{M\mathbb{T}(1)}$ and $Y_{M\mathbb{T}(2)}$, defined in two different weighted projective spaces $\mathbb{P}_{k(1)}$ and $\mathbb{P}_{k(2)}$ are the same, because they are the BHK mirrors of the CY manifolds $X_M(1)$ and $X_M(2)$ located in the same projective space $\mathbb{P}_{\tilde{k}}$.

In order to simplify notations, we denote the initial loop matrix as $M(1)$, and the initial chain matrix as $M(2)$. Recall that they satisfy the relations

$$\sum_{j=1}^5 (M(1))_{ij} k_j = \sum_{j=1}^5 (M(2))_{ij} k_j = d = \sum_{j=1}^5 k_j. \quad (3.14)$$

We denote by $S_{lj}(1) \in \mathbb{Z}_{\geq 0}$ and $S_{lj}(2) \in \mathbb{Z}_{\geq 0}$ the vectors of the exponents of the monomials providing the deformation of the complex structure

$$e_l(1) = \prod_{j=1}^5 x_j^{S_{lj}(1)}, \quad e_l(2) = \prod_{j=1}^5 x_j^{S_{lj}(2)} \quad (3.15)$$

and

$$\sum_{j=1}^5 S_{lj}(1) k_j = \sum_{j=1}^5 S_{lj}(2) k_j = d. \quad (3.16)$$

The sets of vectors $\tilde{S}_l(1)$ and $\tilde{S}_l(2)$ are slightly different. It follows from the conditions for the sets of the chiral ring monomials in the Loop and Chain cases and also from the fact that $A_i(1) = A_i(2)$ for $i = 1, 2, 3, 4$ and $A_5(1) \neq A_5(2)$ in the considered case.
However, from Table 1 in the appendix, which lists all cases of weighted projective spaces in three-manifolds CY of types chain and loop appear simultaneously, we come to the following important observation:

When a loop polynomial $W_M(1)$ and a chain polynomial $W_M(2)$ appear in the same weighted projective space $\mathbb{P}(k_1, \ldots, k_5)$ and are connected by Kreuzer-Skarke cleaves [15], in all these 111 cases listed in Table 1 the weight $k_5 = 1$.

From this fact, we obtain that the sets of exponents of monomials of their chiral rings shifted by the vector $(1, \ldots, 1)$ not only belong to the same four-dimensional lattice, but also both contain the integral basis of the lattice, which consists of the four vectors $\vec{V}_a$, $a = 1, 2, 3, 4$

\[-1, 0, 0, k_1),
(0, -1, 0, 0, k_2),
(0, 0, -1, 0, k_3),
(0, 0, 0, -1, k_4).

Below we will use this observation to prove our main statement.

Recall that the mirrors $Y(1)$ and $Y(2)$ are CY orbifolds in the weighted projective spaces $\mathbb{P}_{\vec{k}(1)}$ and $\mathbb{P}_{\vec{k}(2)}$, respectively. The weights $\vec{k}(1)$ and $\vec{k}(2)$ satisfy the equations

$$\sum_{j=1}^{5} (M^T(\alpha))_{ij} \bar{k}_j(\alpha) = \bar{d}(\alpha) = \sum_{j=1}^{5} \bar{k}_j(\alpha), \quad \alpha = 1, 2. \quad (3.17)$$

The two sets of the vectors $T_{mj}(1)$ and $T_{mj}(2)$, defining the complex structure deformations, also satisfy the equations

$$\sum_{j=1}^{5} (T(\alpha))_{ij} \bar{k}_j(\alpha) = \bar{d}(\alpha), \quad \alpha = 1, 2. \quad (3.18)$$

From the BHK mirror conditions (2.23), it follows

$$(\vec{S}(\alpha)_l, \vec{T}(\alpha)_r) = \sum_{i,j=1}^{5} B(\alpha)_{ij} S(\alpha)_i T(\alpha)_j \in \mathbb{Z}, \quad \alpha = 1, 2. \quad (3.19)$$

where as before $B(\alpha)$ is the inverse matrix for $M(\alpha)$ and $B^T(\alpha)$ is its transpose. From these equations and from the fact that two sets of vectors $\vec{S}_l(1)$ and $\vec{S}_l(2)$ are linear combinations of the same basis vectors $\vec{V}_a$ with integer coefficients, it follows that

$$T_{ni}(1) B^T_{ij}(1) = T_{ni}(2) B^T_{ij}(2). \quad (3.20)$$

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4 This was calculated using Mathematica, the algorithm is based on the definition of the polynomials $W_M^0$ of chain and loop types given above. First, we find all the cases for these two types separately, then the intersection of the two sets is found.
This equality implies also the equivalence of the pairings
\[
(\vec{S}(1)_i, \vec{T}(1)_r) = (\vec{S}(2)_i, \vec{T}(2)_r),
\] (3.21)
which in what follows for briefness is denoted \((\vec{S}_i, \vec{T}_r)\).

For the periods of the two BHK mirrors, we have the following expressions similar to (3.11)
\[
\sigma(1)_{\vec{\mu}(1)}(\phi_1, ..., \phi_{\vec{h}}) \sim \sum_{\{m_r\} \in \Sigma(1)_{\vec{\mu}(1)}} C(1, \{m_r\}) \prod_{r=1}^{\vec{h}} \frac{\psi_r^{m_r}}{m_r!},
\] (3.22)
where
\[
C(1, \{m_r\}) = \prod_{i=1}^{5} \Gamma \left( \sum_{j} (m_r T(1)_{rj} + 1) B^T(1)_{ji} \right).
\] (3.23)

The condition \(\{m_r\} \subset \Sigma_{\vec{\mu}(1)}\) means that there exist nonnegative integers \(n_j, j = 1, ..., 5\) such that
\[
m_r T(1)_{rj} = (\vec{\mu}(1))_j + \sum_k n_k M^T(1)_{kj}.
\] (3.24)

From (3.20), it follows that \(\sum_j T(1)_{rj} B^T(1)_{ji}, i.e., the factor arising in the arguments of the gamma functions in the formula (3.23) for the period \(\sigma(1)_{\vec{\mu}(1)}\) coincides with \(\sum_j T(2)_{rj} B^T(2)_{ji}\) in the corresponding expression for the period \(\sigma(2)_{\vec{\mu}(2)}\) of the second BHK mirror.

The sets of \(m_r\) in the expressions for both mirrors also coincide, what is verified multiplying both equations by \(B(1)_{ji} S_{li}\).

Once we have done this, we obtain the linear system
\[
m_r \sum_{j=1} R(1)_{rj} B(1)_{ji} = \sum_{j=1} (\vec{\mu}(1))_j B(1)_{ji} + n_i(1).
\] (3.25)

Similarly, for the second BHK mirror, we get
\[
m_r \sum_{j=1} R(2)_{rj} B(2)_{ji} = \sum_{j=1} (\vec{\mu}(2))_j B(2)_{ji} + n_i(2).
\] (3.26)

Taking into account that \(\sum_{j=1} R_{ij}(1) B_{ji}(1) = \sum_{j=1} R_{ij}(2) B_{ji}(2)\) and \(\sum_{j=1} (\vec{\mu}(1))_j B(1)_{ji} = \sum_{j=1} (\vec{\mu}(2))_j B(2)_{ji}\), as shown above, we see that the coefficients in both systems of the linear equations are the same. So that the sets of integers \(m_r\) in the expressions for the periods of both BHK mirrors also coincide.

Together with the equality (3.20), it proves our main statement about the coincidence of the periods of two BHK mirrors.
4 Conclusion

In this paper, we showed that for any pair of three-dimensional Calabi–Yau orbifolds that are multiple BHK mirrors of a loop–chain-type pair of Calabi–Yau manifolds in the same weighted projective space, the periods of the holomorphic nonvanishing form coincide.

We assume that the key observation, leading to this statement, is true also for the \((n-2)\)-dimensional case. Namely, the following conjecture is true. Let \(W_1\) and \(W_2\) be a loop and a chain polynomials of the form as in (2.4) in the same weighted projective space \(\mathbb{P}(k_1, \ldots, k_n)\). Then, \(k_n = 1\). From this conjecture, we get that shifted exponents of the monomials of the chiral rings for the loop \(W_1\) and the chain \(W_2\) both contain the basis of the lattice

\[
\begin{align*}
&(-1, 0, 0, \cdots, 0, k_1), \\
&(0, -1, 0, \cdots, 0, k_2), \\
&\cdots \\
&(0, \cdots, 0, -1, k_{n-1}).
\end{align*}
\]

It is not clear for us whether the birationality of the multiple BHK mirrors is equivalent to the coincidence of their periods, we leave this question for further investigation. Even if it is so, we find that the statement about the coincidence of the multiple BHK mirrors is interesting, in particular taking into account rather straightforward way it is obtained.

It will be interesting to consider possible generalizations of this statement to a wider class of CY varieties.

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Appendix
Table 1 Here, $\{k_i\}$ are weights arising simultaneously in loop and chain types, with the corresponding exponents $\{A_j\}^C$ and $\{A_j\}^L$ in (2.4)

| N | $\{k_i(2)\}$ | $\{A_j\}^C \& \{A_j\}^{C(mir)}$ | $\{k_i\}$ | $\{A_j\}^L \& \{A_j\}^{L(mir)}$ | $\{k_i(1)\}$ |
|---|---|---|---|---|---|
| 1 | 126,42,70,13,1 | 2,3,3,14,239 | 92,55,74,17,1 | 2,3,3,14,147 | 77,26,43,8,1 |
| 2 | 270,90,150,13,17 | 2,3,3,30,31 | 12,7,10,1,1 | 2,3,3,30,19 | 157,58,91,8,17 |
| 3 | 6,2,2,1,1 | 2,3,5,10,11 | 4,3,2,1,1 | 2,3,5,10,7 | 83,36,31,16,25 |
| 4 | 216,36,66,61,53 | 2,6,6,6,7 | 3,1,1,1,1 | 2,6,6,6,4 | 97,25,37,35,53 |
| 5 | 64,48,52,51,41 | 4,4,4,4,5 | 1,1,1,1,1 | 4,4,4,4,4 | 1,1,1,1,1 |
| 6 | 96,32,20,43,1 | 2,3,8,4,149 | 52,45,14,37,1 | 2,3,8,4,97 | 62,21,13,28,1 |
| 7 | 132,44,20,61,7 | 2,3,11,4,29 | 10,9,2,7,1 | 2,3,11,4,19 | 83,30,13,40,7 |
| 8 | 160,40,28,73,19 | 2,4,10,4,13 | 5,3,1,3,1 | 2,4,10,4,8 | 89,27,17,45,19 |
| 9 | 108,27,63,17,1 | 2,4,3,9,199 | 82,35,59,22,1 | 2,4,3,9,117 | 63,16,37,10,1 |
| 10 | 168,42,98,17,11 | 2,4,3,14,29 | 12,5,9,2,1 | 2,4,3,14,17 | 93,26,57,10,11 |
| 11 | 324,36,204,37,47 | 2,9,3,12,13 | 6,1,4,1,1 | 2,9,3,12,7 | 151,22,109,20,47 |
| 12 | 120,24,54,31,11 | 2,5,4,6,19 | 8,3,4,3,1 | 2,5,4,6,11 | 64,15,31,18,11 |
| 13 | 224,32,104,43,45 | 2,7,4,8,9 | 4,1,2,1,1 | 2,7,4,8,5 | 34,7,19,8,15 |
| 14 | 180,15,115,49,1 | 2,12,3,5,311 | 146,19,83,62,1 | 2,12,3,5,165 | 95,8,61,26,1 |
| 15 | 450,15,295,121,19 | 2,30,3,5,41 | 20,1,11,8,1 | 2,30,3,5,21 | 221,8,151,62,19 |
| 16 | 126,18,78,29,1 | 2,7,3,6,223 | 100,23,62,37,1 | 2,7,3,6,123 | 69,10,43,16,1 |
| 17 | 324,18,210,73,23 | 2,18,3,6,25 | 12,1,7,4,1 | 2,18,3,6,13 | 157,10,109,38,23 |
| 18 | 288,24,184,49,31 | 2,12,3,8,17 | 8,1,5,2,1 | 2,12,3,8,9 | 137,14,97,26,31 |
| 19 | 112,16,52,43,1 | 2,7,4,4,181 | 80,21,34,45,1 | 2,7,4,4,101 | 62,9,29,24,1 |
| 20 | 128,16,60,49,3 | 2,8,4,4,69 | 31,7,13,17,1 | 2,8,4,4,38 | 23,3,11,9,1 |
| 21 | 160,16,76,61,7 | 2,10,4,4,37 | 17,3,7,9,1 | 2,10,4,4,20 | 83,9,41,33,7 |
| 22 | 272,16,132,103,21 | 2,17,4,4,21 | 10,1,4,5,1 | 2,17,4,4,11 | 44,3,23,18,7 |
| 23 | 180,45,21,113,1 | 2,4,15,3,247 | 94,59,11,82,1 | 2,4,15,3,153 | 111,28,13,70,1 |
| \(N\)  | \(\{k_i(2)\}\)             | \(\{A_j\}^C \& \{A_j\}^{C(mir)}\) | \(\{k_i\}\)    | \(\{A_j\}^L \& \{A_j\}^{L(mir)}\) | \(\{k_i(1)\}\) |
|-------|----------------------------|------------------------------------|----------------|-----------------------------------|----------------|
| 24    | \(\{300,75,21,193,111\}\) | \(\{2,4,25,3,37\}\)              | \(\{14,9,1,12,1\}\) | \(\{2,4,25,3,23\}\)         | \(\{181,48,13,120,11\}\)   |
| 25    | \(\{234,39,33,145,17\}\)  | \(\{2,6,13,3,19\}\)              | \(\{8,3,1,6,1\}\)  | \(\{2,6,13,3,11\}\)         | \(\{127,24,19,84,17\}\)    |
| 26    | \(\{100,20,36,41,3\}\)    | \(\{2,5,5,4,53\}\)               | \(\{22,9,8,13,1\}\) | \(\{2,5,5,4,31\}\)          | \(\{19,4,7,8,1\}\)         |
| 27    | \(\{210,15,81,113,1\}\)   | \(\{2,14,5,3,307\}\)             | \(\{144,19,41,102,1\}\)| \(\{2,14,5,3,163\}\)       | \(\{111,8,43,60,1\}\)      |
| 28    | \(\{480,15,189,257,19\}\) | \(\{2,32,5,3,37\}\)              | \(\{18,1,5,12,1\}\) | \(\{2,32,5,3,19\}\)         | \(\{237,8,97,132,19\}\)    |
| 29    | \(\{324,27,69,193,35\}\)  | \(\{2,12,9,3,13\}\)              | \(\{6,1,1,4,1\}\)   | \(\{2,12,9,3,7\}\)          | \(\{157,16,37,104,35\}\)   |
| 30    | \(\{220,20,84,89,27\}\)   | \(\{2,11,5,4,13\}\)              | \(\{6,1,2,3,1\}\)   | \(\{2,11,5,4,7\}\)          | \(\{35,4,15,16,9\}\)       |
| 31    | \(\{224,28,60,97,39\}\)   | \(\{2,8,7,4,9\}\)                | \(\{4,1,1,2,1\}\)   | \(\{2,8,7,4,5\}\)           | \(\{35,6,11,18,13\}\)      |
| 32    | \(\{24,24,16,7,1\}\)      | \(\{3,2,3,8,65\}\)               | \(\{14,23,19,8,1\}\)| \(\{3,2,3,8,51\}\)       | \(\{37,38,25,11,2\}\)      |
| 33    | \(\{27,27,18,7,2\}\)      | \(\{3,2,3,9,37\}\)               | \(\{8,13,11,4,1\}\) | \(\{3,2,3,9,29\}\)          | \(\{41,43,28,11,4\}\)      |
| 34    | \(\{33,33,22,7,4\}\)      | \(\{3,2,3,11,23\}\)              | \(\{5,8,7,2,1\}\)   | \(\{3,2,3,11,18\}\)         | \(\{49,53,34,11,8\}\)      |
| 35    | \(\{25,25,10,13,2\}\)     | \(\{3,2,5,5,31\}\)               | \(\{6,13,5,6,1\}\)  | \(\{3,2,5,5,25\}\)          | \(\{39,41,16,21,4\}\)      |
| 36    | \(\{32,48,20,27,1\}\)     | \(\{4,2,4,4,101\}\)              | \(\{15,41,19,25,1\}\)| \(\{4,2,4,4,86\}\)       | \(\{27,41,17,23,1\}\)      |
| 37    | \(\{72,108,30,43,35\}\)   | \(\{4,2,6,6,7\}\)                | \(\{1,3,1,1,1\}\)   | \(\{4,2,6,6,6\}\)           | \(\{53,97,25,37,35\}\)     |
| 38    | \(\{28,28,8,19,1\}\)      | \(\{3,2,7,4,65\}\)               | \(\{12,29,7,16,1\}\)| \(\{3,2,7,4,53\}\)       | \(\{45,46,13,51,2\}\)      |
| 39    | \(\{52,52,8,37,7\}\)      | \(\{3,2,13,4,17\}\)              | \(\{3,8,1,4,1\}\)   | \(\{3,2,13,4,14\}\)         | \(\{81,88,13,61,14\}\)     |
| 40    | \(\{9,9,2,5,2\}\)         | \(\{3,2,9,5,11\}\)               | \(\{2,5,1,2,1\}\)   | \(\{3,2,9,5,9\}\)           | \(\{67,77,16,41,20\}\)     |
| 41    | \(\{30,45,25,19,1\}\)     | \(\{4,2,3,5,101\}\)              | \(\{16,37,27,20,1\}\)| \(\{4,2,3,5,85\}\)       | \(\{25,38,21,16,1\}\)      |
| 42    | \(\{36,54,30,19,5\}\)     | \(\{4,2,3,6,25\}\)               | \(\{4,9,7,4,1\}\)   | \(\{4,2,3,6,21\}\)          | \(\{29,46,25,16,5\}\)      |
| 43    | \(\{18,99,39,59,1\}\)     | \(\{12,2,3,3,157\}\)             | \(\{8,61,35,52,1\}\)| \(\{12,2,3,3,149\}\)      | \(\{17,94,37,56,1\}\)      |
| 44    | \(\{9,54,21,32,1\}\)      | \(\{13,2,3,3,85\}\)              | \(\{4,33,19,28,1\}\)| \(\{13,2,3,3,81\}\)      | \(\{17,103,40,61,2\}\)     |
| 45    | \(\{9,63,24,37,2\}\)      | \(\{15,2,3,3,49\}\)              | \(\{2,19,11,16,1\}\)| \(\{15,2,3,3,47\}\)      | \(\{17,121,46,71,4\}\)     |
| 46    | \(\{9,81,30,47,4\}\)      | \(\{19,2,3,3,31\}\)              | \(\{1,12,7,10,1\}\) | \(\{19,2,3,3,30\}\)         | \(\{17,157,58,91,8\}\)     |
| 47    | \(\{15,45,20,17,8\}\)     | \(\{7,2,3,5,11\}\)               | \(\{1,4,3,2,1\}\)   | \(\{7,2,3,5,10\}\)          | \(\{25,83,36,31,16\}\)     |
Table 1 continued

| \(N\) | \(\{k_i(2)\}\) | \(\{A_i\}^C & \{A_i\}^{C(\text{mir})}\) | \(\{k_i\}\) | \(\{A_i\}^L & \{A_i\}^{L(\text{mir})}\) | \(\{k_i(1)\}\) |
|-------|----------------|-----------------|-----------|-----------------|----------------|
| 48    | \(\{66,99,15,83,1\}\) | \(\{4,2,11,3,181\}\) | \(\{24,85,11,60,1\}\) | \(\{4,2,11,3,157\}\) | \(\{57,86,13,72,1\}\) |
| 49    | \(\{126,189,15,163,11\}\) | \(\{4,2,21,3,31\}\) | \(\{4,15,1,10,1\}\) | \(\{4,2,21,3,27\}\) | \(\{107,166,13,142,11\}\) |
| 50    | \(\{21,42,9,32,1\}\) | \(\{5,2,7,3,73\}\) | \(\{8,33,7,24,1\}\) | \(\{5,2,7,3,65\}\) | \(\{37,75,16,57,2\}\) |
| 51    | \(\{39,78,9,62,7\}\) | \(\{5,2,13,3,19\}\) | \(\{2,9,1,6,1\}\) | \(\{5,2,13,3,17\}\) | \(\{67,141,16,111,14\}\) |
| 52    | \(\{27,81,12,59,10\}\) | \(\{7,29,3,13\}\) | \(\{1,6,1,4,1\}\) | \(\{7,2,9,3,12\}\) | \(\{47,151,22,109,20\}\) |
| 53    | \(\{7,14,3,8,3\}\) | \(\{5,2,7,4,9\}\) | \(\{1,4,1,2,1\}\) | \(\{5,2,7,4,8\}\) | \(\{15,34,7,19,8\}\) |
| 54    | \(\{30,75,21,53,1\}\) | \(\{6,2,5,3,127\}\) | \(\{12,55,17,42,1\}\) | \(\{6,2,5,3,115\}\) | \(\{27,68,19,48,1\}\) |
| 55    | \(\{15,45,12,31,2\}\) | \(\{7,2,5,3,37\}\) | \(\{3,16,5,12,1\}\) | \(\{7,2,5,3,34\}\) | \(\{27,83,22,57,4\}\) |
| 56    | \(\{72,48,84,11,1\}\) | \(\{3,3,2,12,205\}\) | \(\{56,37,94,17,1\}\) | \(\{3,3,2,12,149\}\) | \(\{52,35,61,8,1\}\) |
| 57    | \(\{168,112,196,11,17\}\) | \(\{3,3,2,28,29\}\) | \(\{8,5,14,1,1\}\) | \(\{3,3,2,28,21\}\) | \(\{116,83,141,18,17\}\) |
| 58    | \(\{60,24,78,17,1\}\) | \(\{3,5,2,6,163\}\) | \(\{48,19,68,27,1\}\) | \(\{3,5,2,6,115\}\) | \(\{42,17,55,12,1\}\) |
| 59    | \(\{168,48,228,23,37\}\) | \(\{3,7,2,12,13\}\) | \(\{4,1,6,1,1\}\) | \(\{3,7,2,12,9\}\) | \(\{104,35,157,16,37\}\) |
| 60    | \(\{32,24,52,19,1\}\) | \(\{4,4,2,4,109\}\) | \(\{23,17,41,27,1\}\) | \(\{4,4,2,4,86\}\) | \(\{25,19,41,15,1\}\) |
| 61    | \(\{80,48,136,23,33\}\) | \(\{4,5,2,8,9\}\) | \(\{2,1,4,1,1\}\) | \(\{4,5,2,8,7\}\) | \(\{18,13,35,6,11\}\) |
| 62    | \(\{88,16,124,35,1\}\) | \(\{3,11,2,4,229\}\) | \(\{72,13,86,57,1\}\) | \(\{3,11,2,4,157\}\) | \(\{60,11,85,24,1\}\) |
| 63    | \(\{184,16,268,71,13\}\) | \(\{3,23,2,4,37\}\) | \(\{12,1,14,9,1\}\) | \(\{3,23,2,4,25\}\) | \(\{120,11,181,48,13\}\) |
| 64    | \(\{132,24,186,35,19\}\) | \(\{3,11,2,6,19\}\) | \(\{6,1,8,3,1\}\) | \(\{3,11,2,6,13\}\) | \(\{84,17,127,24,19\}\) |
| 65    | \(\{64,24,116,35,17\}\) | \(\{4,8,2,4,13\}\) | \(\{3,1,5,3,1\}\) | \(\{4,8,2,4,10\}\) | \(\{45,19,89,27,17\}\) |
| 66    | \(\{48,60,114,29,37\}\) | \(\{6,4,2,6,7\}\) | \(\{1,1,3,1,1\}\) | \(\{6,4,2,6,6\}\) | \(\{35,53,97,25,37\}\) |
| 67    | \(\{24,32,44,19,1\}\) | \(\{5,3,2,4,101\}\) | \(\{16,21,38,25,1\}\) | \(\{5,3,2,4,85\}\) | \(\{20,27,37,16,1\}\) |
| 68    | \(\{140,56,26,197,1\}\) | \(\{3,5,14,2,223\}\) | \(\{60,43,8,111,1\}\) | \(\{3,5,14,2,163\}\) | \(\{102,41,19,144,1\}\) |
| 69    | \(\{75,30,13,106,1\}\) | \(\{3,5,15,2,119\}\) | \(\{32,23,4,59,1\}\) | \(\{3,5,15,2,87\}\) | \(\{109,44,19,155,2\}\) |
| 70    | \(\{85,34,13,121,2\}\) | \(\{3,5,17,2,67\}\) | \(\{18,13,2,33,1\}\) | \(\{3,5,17,2,49\}\) | \(\{123,50,19,177,4\}\) |
| 71    | \(\{105,42,13,151,4\}\) | \(\{3,5,21,2,41\}\) | \(\{11,8,1,20,1\}\) | \(\{3,5,21,2,30\}\) | \(\{151,62,19,221,8\}\) |
| N   | \(k_i(2)\)       | \(\{A_j\}^C\&\{A_j\}^C(\text{mir})\) | \(k_i\)  | \(\{A_j\}^L\&\{A_j\}^L(\text{mir})\) | \(k_i(1)\)  |
|-----|------------------|---------------------------------|---------|---------------------------------|-------------|
| 72  | (54,18,16,73,1)  | [3,6,9,2,89]                    | (25,14,5,44,1) | [3,6,9,2,64] | (77,26,23,105,2) |
| 73  | (78,26,16,109,5) | [3,6,13,2,25]                   | (7,4,1,12,1)  | [3,6,13,2,18] | (109,38,23,157,10) |
| 74  | (49,14,19,64,1)  | [3,7,7,2,83]                    | (24,11,6,41,1) | [3,7,7,2,59] | (69,20,27,91,2)  |
| 75  | (72,18,22,97,7)  | [3,8,9,2,17]                    | (5,2,1,8,1)   | [3,8,9,2,12] | (97,26,31,137,14) |
| 76  | (56,42,26,99,1)  | [4,4,7,2,125]                   | (24,29,6,62,1) | [4,4,7,2,101] | (45,34,21,80,1)  |
| 77  | (64,48,26,115,3) | [4,4,8,2,47]                    | (9,11,3,23,1) | [4,4,8,2,38] | (17,13,7,31,1)   |
| 78  | (88,66,26,163,9) | [4,4,11,2,21]                   | (4,5,1,10,1)  | [4,4,11,2,17] | (23,18,7,44,3)   |
| 79  | (50,30,34,83,3)  | [4,5,5,2,39]                    | (8,7,4,19,1)  | [4,5,5,2,31] | (13,8,9,22,1)    |
| 80  | (120,16,86,137,1)| [3,15,4,2,223]                  | (70,13,28,111,1) | [3,15,4,2,153] | (82,11,59,94,1)  |
| 81  | (216,16,158,245,13) | [3,27,4,2,31]                  | (10,1,4,15,1)  | [3,27,4,2,21] | (142,11,107,166,13) |
| 82  | (50,10,28,61,1)  | [3,10,5,2,89]                   | (27,8,9,44,1)  | [3,10,5,2,62] | (69,14,39,85,2)  |
| 83  | (55,10,31,67,2)  | [3,11,5,2,49]                   | (15,4,5,24,1)  | [3,11,5,2,34] | (75,14,43,93,4)  |
| 84  | (65,10,37,79,4)  | [3,13,5,2,29]                   | (9,2,3,14,1)   | [3,13,5,2,20] | (87,14,51,109,8) |
| 85  | (85,10,49,103,8) | [3,17,5,2,19]                   | (6,1,2,9,1)    | [3,17,5,2,13] | (111,14,67,141,16) |
| 86  | (84,14,34,109,11)| [3,12,7,2,13]                   | (4,1,1,6,1)    | [3,12,7,2,9] | (109,20,47,151,22) |
| 87  | (80,30,58,131,21)| [4,8,5,2,9]                     | (2,1,1,4,1)    | [4,8,5,2,7] | (19,8,15,34,7)   |
| 88  | (72,96,22,169,1) | [5,3,12,2,191]                  | (26,61,8,95,1) | [5,3,12,2,165] | (62,83,19,146,1) |
| 89  | (39,52,11,92,1)  | [5,3,13,2,103]                  | (14,3,3,4,51,1) | [5,3,13,2,89] | (67,90,19,159,2) |
| 90  | (45,60,11,107,2) | [5,3,15,2,59]                   | (8,19,2,29,1)  | [5,3,15,2,51] | (77,104,19,185,4) |
| 91  | (57,76,11,137,4) | [5,3,19,2,37]                   | (5,12,1,18,1)  | [5,3,19,2,32] | (97,132,19,237,8) |
| 92  | (42,70,26,113,1) | [6,3,7,2,139]                   | (16,43,10,69,1) | [6,3,7,2,123] | (37,62,23,100,1) |
| 93  | (54,90,26,149,5) | [6,3,9,2,35]                    | (4,11,2,17,1)  | [6,3,9,2,31] | (47,80,23,132,5) |
| 94  | (14,14,8,31,3)   | [5,4,7,2,13]                    | (2,3,1,6,1)    | [5,4,7,2,11] | (15,16,9,35,4)   |
| 95  | (18,78,58,97,1)  | [14,3,3,2,155]                  | (8,43,26,77,1) | [14,3,3,2,147] | (17,74,55,92,1)  |
| N  | \{k_i (2)\} | \{A_i\}^C & \{A_i\}^{C(mir)} | \{k_i\} | \{A_i\}^L & \{A_i\}^{L(mir)} | \{k_i (1)\} |
|----|-------------|----------------------|------|----------------------|----------------------|------|
| 96 | \{9,42,31,52,1\} | \{15,3,3,2,83\} | \{4,23,14,41,1\} | \{15,3,3,2,79\} | \{17,80,59,99,2\} |
| 97 | \{9,48,35,59,2\} | \{17,3,3,2,47\} | \{2,13,8,23,1\} | \{17,3,3,2,45\} | \{17,92,67,113,4\} |
| 98 | \{9,60,43,73,4\} | \{21,3,3,2,29\} | \{1,8,5,14,1\} | \{21,3,3,2,28\} | \{17,116,83,141,8\} |
| 99 | \{24,64,38,89,1\} | \{9,3,4,2,127\} | \{10,37,16,63,1\} | \{9,3,4,2,117\} | \{22,59,35,82,1\} |
| 100 | \{24,80,46,109,5\} | \{11,3,4,2,31\} | \{2,9,4,15,1\} | \{11,3,4,2,29\} | \{22,75,43,102,5\} |
| 101 | \{36,84,34,127,7\} | \{8,3,6,2,23\} | \{2,7,2,11,1\} | \{8,3,6,2,21\} | \{32,77,31,116,7\} |
| 102 | \{21,56,19,85,8\} | \{9,3,7,2,13\} | \{1,4,1,6,1\} | \{9,3,7,2,12\} | \{37,104,35,157,16\} |
| 103 | \{32,40,38,77,5\} | \{6,4,4,2,23\} | \{3,5,3,11,1\} | \{6,4,4,2,20\} | \{27,35,33,67,5\} |
| 104 | \{20,30,22,59,9\} | \{7,4,5,2,9\} | \{1,2,1,4,1\} | \{7,4,5,2,8\} | \{11,18,13,35,6\} |
| 105 | \{48,18,58,67,1\} | \{4,8,3,2,125\} | \{28,13,21,62,1\} | \{4,8,3,2,97\} | \{37,14,45,52,1\} |
| 106 | \{84,18,106,115,13\} | \{4,14,3,2,17\} | \{4,1,3,8,1\} | \{4,14,3,2,13\} | \{61,14,81,88,13\} |
| 107 | \{9,6,13,16,1\} | \{5,6,3,2,29\} | \{5,4,5,14,1\} | \{5,6,3,2,24\} | \{29,20,43,53,4\} |
| 108 | \{21,12,3,1,37,4\} | \{5,7,3,2,17\} | \{3,2,3,8,1\} | \{5,7,3,2,14\} | \{33,20,51,61,8\} |
| 109 | \{27,12,4,1,47,8\} | \{5,9,3,2,11\} | \{2,1,2,5,1\} | \{5,9,3,2,9\} | \{41,20,67,77,16\} |
| 110 | \{48,40,62,113,25\} | \{6,6,4,2,7\} | \{1,1,1,3,1\} | \{6,6,4,2,6\} | \{37,35,53,97,25\} |
| 111 | \{12,18,22,31,1\} | \{7,4,3,2,53\} | \{6,11,9,26,1\} | \{7,4,3,2,47\} | \{21,32,39,55,2\} |

The \{k_i (2)\} and \{A_i\}^{C(mir)} are the weights and the exponents of the mirror CY for the chain manifold \{A_i\}^C. The sets \{k_i (1)\} and \{A_i\}^{L(mir)} are the same for the loop manifold \{A_i\}^L. The notation \{A_i\}^C & \{A_i\}^{C(mir)} means that two sets coincide.
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