Geometry and classification of carved Monge surfaces

M Gil-oulbé1*, A J I Ndomilep
Department of Civil Engineering, RUDN University, Moscow, Russia.

*Email: gil-oulbem@hotmail.com

Abstract. There are more and more needs of new forms in the world architecture. The thin shells theory and the surfaces theory give enough materials to scientists and designers. This article is devoted to the geometric investigation of carved Monge surfaces. The geometry of these surfaces is inner and outer. For this, the coefficients of their fundamental forms are found and allow the classification of these surfaces. The kinetic method is used for their geometric investigation and classification. The parametric definitions of these multitude forms of carved Monge surfaces allow the 3D plotting of these surfaces by mean of the software MathCAD. The results obtained by this investigation are their vector, implicite, explicite and parametric definitions, the mathematic modelling of their inner and outer geometry, their 3D plotting and their classification. This variety of carved Monge surfaces can be used as the median surface of thin elastic shells design from them. The geometry these elastic shells, because their thin is the one of the carved Monge surfaces. The multitude of the designed thin elastic shells are expressive, durable and cover large spans. The multitude of their forms can find applications in the architecture of civil and industrial buildings and also in mechanical engineering.

Key words: Carved Monge surface, evolute surface, developing surface, directrix torso, fixed axoid, kinematic surfaces

1. Introduction

For the development and creation of the new urbanistic cities, designers need to pay more and more attention to such tendencies in the construction which allow us to use more expressive architectural images, to cut the consumption of materials, labor input of production and installation of designs, as well as to solve important functional problems. In this regard the efficiency of space structures’ usage is indisputable. But one shouldn’t pay attention only to architectural expressiveness of shell of difficult geometry [1].

The geometric investigations of surfaces [2] give many possibilities to design more and more new spaces structures. Each class of surfaces contains many different forms and their geometric study is concerned by its inner and outer geometry. For this, the definition of the surface is given and the coefficients of its fundamental forms are found. The inner geometry of a surface is the length of its segments, the angle between tow not parallel lines and is area. The outer geometry of a surface is 3D curvature. The inner geometry is determined using the coefficients of the first fundamental form and the outer ones by those of the second quadratic form. The definition of a surface can be vectorial, parametric, implicite and explicite. The parametric definition is most of the time the one used in softwares for 3D plot.

Carved surfaces are those for which the planes of one set of flat lines of curvature are orthogonal to the surface. The set of flat lines of curvature of the carved surface will be geodesic; therefore, the normals of these lines coincide with the normals of the surface. Thus, a carved surface can be characterized as a surface with a geodesic set of lines of curvature [3].

Studies showed that the carved surface, one of the evolutes of which is the torso, is called the carved surface of Monge, named after the geometer who first paid attention and examined surfaces of this type.
The evolute surface (evolute) of the carved surface is called its directrix surface [4]. Ivanov V. N. and Rizvan M [5, 6] investigated the geometry of carved Monge surfaces and construction of shells in 2002 and in 2003 studied Carved Monge surfaces and shell design. Stachel H. [7], published a course of lectures in descriptive geometry for students. Abdel-All, N. H. and Maher, A. [8, 9] generalized cylindrical coordinates and their application to the analysis of shells in the form of carved Monge surfaces. Krivoshapko S N and Ivanov V N [10, 11] in two encyclopedias presented the carved Monge surface as analytical surfaces. Ivanov V N [12] investigate the problems of the geometry and the architectural design of shells based on cyclic surfaces. More can be found in [13, 14, 15 and 16] about carved Monge surfaces.

The objective of the current study is a detailed investigation of the geometry of carved Monge surfaces, their classification and their 3D plotting using the software MathCAD.

2. Materials and methods
A carved Monge surface can be generated by the kinematic method of rolling without sliding a plane with a flat line (meridian) along a developing surface. If we take an arbitrary plane curve on the tangent plane $P$ of the developing surface $S$, and then start rolling this plane $P$ without friction on the surface of the fixed directrix torso $S$ (fixed axoid), then the generatrix curve will outline the carved Monge surface. Thus, the Monge surface is generated by the orthogonal trajectories of a one-parameter set of planes. Orthogonal trajectories of the points of the meridian are called parallels. All meridians of the carved surface are congruent. Meridians and parallels make up two sets of lines of curvature of the carved surface (figure 1).

The simplest example of a Monge surface is the surface of revolution, which can be considered as the degeneracy of the Monge surface. Here, a one-parameter set of planes carrying a meridian degenerates into a bunch of planes passing through the axis of rotation of the surface.

The kinematic method for constructing carved surfaces makes it possible to divide them into three groups depending on the type of directrix surface (fixed axoid): carved surfaces with cylindrical, conical and torso directrix surfaces.

G. Monge introduced into circulation carved surfaces that are generated by a flat generatrix of a curve lying on a plane, which, in turn, rolls without sliding along a cylinder or cone.

Moreover, at any moment of time, each point of the generatrix curve makes a motion orthogonal to this plane. Thus, a generally carved surface can be considered a surface generated by a flat generatrix when one point moves along an arbitrary directing curve. Moreover, the generatrix must always be in the normal plane of the directrix curve. In this case, all coordinate lines of one set will be congruent curves, and the other family of coordinate lines will be orthogonal to them. The resulting system of curvilinear coordinates will be conjugate.

Computer geometry as a new area of applied mathematics enables effective modeling of surfaces and create their images directly on the screen. The way to define the surface by displacements of curves in space is called kinematic. The method here is illustrated on the examples of several classes of carved Monge surfaces of kinematic modeling surfaces: nonruled and ruled.

3. Geometry
Let be $\alpha(u)x + \beta(u)y + \gamma(u)z = p(u)$ is the equation of a one-parameter set of planes; $x = x(u), y = y(u), z = z(u)$ - parametric equations of the orthogonal trajectory related to the same parameter $u$. The condition of orthogonality is the following conditions along the curve:
\[
\frac{dx}{du} = \lambda \alpha(u), \quad \frac{dy}{du} = \lambda \beta(u), \quad \frac{dz}{du} = \lambda \gamma(u), \quad \text{or} \quad \frac{dy}{dx} = \frac{\beta(u)}{\alpha(u)}, \quad \frac{dz}{dx} = \frac{\gamma(u)}{\alpha(u)}. \tag{3.1}
\]

Let \( \rho(u) = x(u)i + y(u)j + z(u)k \) be the radius vector of the directrix curve, and \( s = (x'^2 + y'^2 + z'^2)^{\frac{1}{2}} \) Then

\[
k = \left( \frac{x' \ y' \ z'}{ \sqrt{(x' ^2 + y' ^2 + z' ^2)} } \right) \left( \frac{x'^2 + y'^2 + z'^2}{z'^2} \right)^{\frac{1}{2}} \tag{3.2}
\]

Is the curvature of the directrix curve,

\[
\kappa = \left( \frac{x' \ y' \ z'}{ x'' \ y'' \ z'' } \right) \left( \frac{(y' z'' - z' y'')^2 + (z' x'' - x' z'')^2 + (x' y'' - y' x'')^2}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} \right) \tag{3.3}
\]

her torsion,

\[
\tau = (x'i + y'j + z'k)/s \tag{3.4}
\]

is the unit tangent vector of the directrix curve;

\[
\beta = [(y'z'' - y''z')i + (z'x'' - z''x')j + (x'y'' - x''y')k] / (ks^3) \tag{3.5}
\]

its unit binormal vector;

\[
v = [(y'(x'' - x'y') - z'(x'' - x'y'))i + [z'(y'' - y'z') - x'(y'' - y'x')]j + \ \tag{3.6}
\]

the unit normal vector of the directrix curve. The dashes show the differentiation with respect to the parameter \( u \), i.e.

\[
\ldots' = d\ldots/du; \quad \ldots'' = d^2\ldots/du^2. \tag{3.7}
\]

Knowing the unit vectors \( \tau, \beta, v \) of the directrix curve, we can write the vector equation of the curved surface of the general form [1]:

\[
r = r(u,v) = \rho(u) + X(v)e_0(u) + Y(v)g_0(u), \tag{3.8}
\]

where \( e_0(u) = \cos \theta + 0 + \beta \sin \theta; \quad g_0(u) = -\sin \theta + \beta \cos \theta, \theta = \theta(u) = \int ksdud + \theta_0 \)

is the angle between the normal vector \( n(u) \) of the directrix curve and the vector \( e_0 (u); X = X(v), Y = Y(v) \) are the parametric equations of the generatrix plane curve defined in the moving local coordinate system \( XoY \). The angle \( \theta(u) \) depends on the torsion of the directrix curve. For a plane directrix curve, \( \theta = \theta_0 = \text{const} \), i.e., the generatrix plane curve moves in the normal plane of the directrix curve without rotation.

For the surface (3.8), the coefficients of quadratic forms and its main curvatures will have the form:

\[
A = s\left[ 1 + k(X \cos \theta - Y \sin \theta) \right]; \quad F = 0; \quad B = \sqrt{X^2 + Y^2}; \tag{3.9}
\]

\[
L = (X \sin \theta + Y \cos \theta)sk \frac{A}{B}; \quad M = 0; \quad N = \frac{\dot{X}Y - \dot{Y}X}{B};
\]

where

\[
\ldots = \frac{d\ldots}{dv}, \quad \ldots'' = \frac{d^2\ldots}{dv^2}.
\]

\[
k_1 = k_u = \left( \frac{X \sin \theta + Y \cos \theta}{AB} \right) sk; \quad k_2 = k_v = \frac{\ddot{X}Y - \ddot{Y}X}{B^3}. \tag{3.10}
\]

4. Results and discussion

Consider an example. The carved sinusoidal surface is generated by a flat generatrix sinusoid

\[
X = X(v) = v, \quad Y = Y(v) = c \sin(dv), \tag{4.1}
\]
when one of its points moves along a flat directrix sinusoid

\[ x = x(u) = u, \quad y = y(u) = a\sin(bu). \]  

When moving, the generatrix sinusoid is always in the normal plane of the directrix sinusoid. To obtain the vector equation of the surface under consideration (figure 2), it is necessary to take in formulas (3.8)

\[ e_0(u) = \beta = k; \quad g_0(u) = -\nu = - (y' i - x' j)/s; \quad s = \sqrt{x'^2 + y'^2} = \sqrt{1 + a^2 b^2 \cos^2 bu}; \]

\[ \rho(u) = ui + a \sin(bu)j; \quad \theta = \pi/2; \quad \ldots' = d\ldots/du. \]

The vector equation is easily reduced to the parametric equations of the middle surface (figure 2):

\[ x = x(u,v) = x(u) - \frac{y'(u)}{s} Y(v) = u - \frac{y'(u)}{s} Y(v), \]

\[ y = y(u,v) = y(u) + \frac{x'(u)}{s} Y(v), \]

\[ z = z(v) = X(v) = v. \]

In figure 3 are shown some types of carved surfaces of a general form, more detailed information about which can be found in the book [11].
If the directrix surface of the carved surface of Monge is cylindrical, then all its parallels will be flat lines located in parallel planes orthogonal to the generatrix of the directrix cylinder. At points of the same parallel, all meridians (generatrix curves) have equal curvatures. Every two parallels cut off equal arcs on the geodesic lines of curvature (meridians). If the points of one of the parallels are taken as the origin of the arcs on all meridians, then the curvatures of the flat meridians will be the same functions of their arcs. Since the directrix surface is circular cylindrical, to obtain the equation of the carved Monge surface, one can use generalized cylindrical coordinates. Rectangular coordinates are expressed through the generalized cylindrical coordinates by the formulas:

\[ x = x(\alpha, \beta) = r \cos \alpha - u \sin \alpha, \quad y = y(\alpha, \beta) = r \sin \alpha + u \cos \alpha, \quad z = \beta, \quad (4.5) \]

where \( r \) is the radius of the generatrix of the cylinder (Fig. 4); \( u = u(\alpha, \beta) = f(\beta) - ra \).

The function \( f = u \) \((0, \beta)\) is given by the shape of the meridian. Based on the above parametric equations of the surface (4.5), we can obtain its coefficients of the first and second quadratic forms:

\[ A = (f - r \alpha), \quad F = 0, \quad B^2 = 1 + f'^2, \quad L = A/B, \quad M = 0, \quad N = -f''/B, \quad (4.6) \]

and to determine the main radii of curvature use the expression:

\[ R_2 = -B^2 f''/L, \quad R_1 = AB. \quad (4.7) \]

Having taken \( r = 0 \), we will have a surface of revolution. If you take the meridian in a straight line, i.e. \( f = c\beta + b \), then \( u = c\beta + b - ra \). In this case, the equation of the carved ruled surface of Monge (figure 5) can be represented in the form \([8]\):

\[ x = x(\alpha, \beta) = r \cos \alpha - (c\beta + b - r\alpha) \sin \alpha, \quad y = y(\alpha, \beta) = r \sin \alpha + (c\beta + b - r\alpha) \cos \alpha, \quad z = \beta. \quad (4.8) \]

Having these surface equations, it is easy to calculate its coefficients of the main quadratic forms:

\[ A = c\beta + b - ra, \quad F = 0, \quad B^2 = 1 + c^2, \quad N = M = 0, \quad L = A/B, \quad (4.9) \]

\[ R_1 = AB, \quad R_2 = \infty. \quad (4.10) \]

where \( e(\alpha) = i\cos \alpha + j\sin \alpha; \quad g(\alpha) = -i\sin \alpha + f\cos \alpha \) is a circular vector functions. Then the vector equation of the carved Monge surface with a circular cylindrical directrix surface will have the form (figure 4):

\[ r(\alpha, t) = r(e + (a_o - \alpha)g) + x(t)p(\alpha) + y(t)q(\alpha), \quad (4.11) \]

where

\[ p(\alpha) = g(\alpha)\cos \theta_0 + k \sin \theta_0; \quad q(\alpha) = -g(\alpha)\sin \theta_0 + k \cos \theta_0. \quad (4.12) \]

For this case, specifying the carved Monge surface, the formulas for calculating the coefficients of the basic quadratic forms of the surface take the form:

\[ A = r(a_o - \alpha) + x(t) \cos \theta_0 - y(t) \sin \theta_0, \quad F = 0, \quad B = \sqrt{x'^2(t) + y'^2(t)}, \quad (4.13) \]

\[ L = \frac{[x'(t) \sin \theta_0 + y'(t) \cos \theta_0]^A}{B}, \quad M = 0, \quad N = -\frac{x'(t)y'(t) - x'(t)y'(t)}{B^3}, \]

\[ k_1 = \frac{x'(t) \sin \theta_0 + y'(t) \cos \theta_0}{AB}, \quad k_2 = \frac{x'(t)y'(t) - x'(t)y'(t)}{B^3}. \]

Consider the carved Monge surface with a circular cylindrical directrix surface, where a square parabola is taken for the meridian. The parametric form of specifying this surface will be (figure 4):

\[ x = x(\alpha, t) = r \cos \alpha - [r(\alpha_0 - \alpha) + t \cos \theta_0 - at^2 \sin \theta_0] \sin \alpha, \]

\[ y = y(\alpha, t) = r \sin \alpha + [r(\alpha_0 - \alpha) + t \cos \theta_0 - at^2 \sin \theta_0] \cos \alpha, \quad (4.14) \]

\[ z = z(t) = t \sin \theta_0 + at^2 \cos \theta_0, \]
where \( r \) is the radius of the directrix cylinder; \( \theta_0 \) is the angle of inclination of the axis of the parabola to the axis of the guide cylinder; \( 0 \leq \alpha \leq \infty; -\infty \leq t \leq \infty \) (see figure 4).

With this method of specifying the vector equation of the directrix involute of a circle with radius \( r \), it is written in the form (4.10)

\[ \rho(\alpha) = r(e + (a_0 - \alpha)g), \quad (4.15) \]

or using parametric equations:

\[ x = x(\alpha) = r[\cos \alpha - (a_0 - \alpha)\sin \alpha], \]
\[ y = y(\alpha) = r[\sin \alpha + (a_0 - \alpha)\cos \alpha]. \quad (4.16) \]

The parametric equations of the parabola generatrix in the local coordinate system are presented in the form:

\[ x = t, \quad y = at^2. \quad (4.17) \]

The coefficients of the basic quadratic forms of the surface (4.14) and its main curvatures are written as

\[ A = r(\alpha_0 - \alpha) + t \cos \theta_0 - a_0^2 \sin \theta_0, \quad F = 0, \quad B^2 = 1 + 4a_0^2 t^2, \]
\[ L = -\frac{\sin \theta_0 + 2at \cos \theta_0}{B}, \quad M = 0, \quad N = -\frac{2a}{B} k_1 = -\frac{\sin \theta_0 + 2at \cos \theta_0}{AB}, \quad k_2 = -\frac{2a}{B^3}. \quad (4.18) \]

The surface is given in the lines of curvature \( t, \alpha \). One family of plane lines of curvature \( t \) coincides with the generatrix parabolas, and the second family of plane lines of curvature \( \alpha \) are the involutes of a circle of radius \( r \). In figure 6 shows the carved Monge surface with a cylindrical directrix surface and a parabolic meridian, for which

\[ r = 1 \text{ m}; a_0 = 0; a = 0.5 \text{ m}^{-1}; -2 \leq t \leq 2; \pi/2 \leq \alpha \leq 9\pi/2, \theta_0 = \pi/2. \]

The parametric form of specifying the same surface (Fig. 6) in cylindrical coordinates (4.5) will have the form:

\[ x = \rho(\alpha) = r \cos \alpha - c \sin \alpha, \quad y = \rho(\alpha, \beta) = r \sin \alpha + c \cos \alpha, \]
\[ z = \beta. \quad (4.19) \]

The coefficients of the basic quadratic surface forms (4.19) are written as

\[ A = \beta^2 [2p + c - r \alpha], \quad F = 0, \quad B^2 = (p^2 + \beta^2)/p, \quad (4.20) \]

The book [11] provides information on additional carved Monge surfaces of with a circular cylindrical directrix surface. These surfaces are shown in figure 5. A carved Monge surface with a conical directrix surface is generated by a plane curve lying in a plane that rolls without sliding along a circular cone. Thus, the determinant of the surface under consideration includes a straight circular cone as a non-moving axoid and a plane as a movable one. The method of surface formation allows us to conclude that these surfaces can also be considered as conical spiral snails (a subclass of rotational surfaces) that belong to the class of kinematic surfaces of a general form, or to relate them to the class of surfaces of congruent rotational sections.

Abdel-All, N. H. and Maher, A. [8, 9] suggest using the hyperbolic coordinates \( u, t, v \) (figure. 6), where \( t \) is the parameter of the moving plane \( \Omega_t \), \( t = \text{const} \) - the plane \( \Omega_t \); \( u, v \) - rectangular coordinates on the plane. The \( v \) axis is aligned with the direct contact of the plane \( \Omega_t \) and the cone \( \Omega^o \); the plane perpendicular to the axis of the cone \( \Omega_t \); the \( u \) axis is directed upward \( t \); \( u = \text{const} \) is a one-sheeted hyperboloid of revolution, for which the cone \( \Omega^o \) is asymptotic.
Functions of the form \( x = \phi(t, u, v), y = \psi(t, u, v), z = \xi(t, u, v) \) express the dependence of Cartesian rectangular coordinates on hyperbolic. They can be obtained by relating the same point \( A \) of the carved surface to a rectangular and hyperbolic coordinate system:

\[
x = x(u,t,v) = v \sin \alpha \cos t - u \sin t;
\]

\[
y = y(u,t,v) = v \sin \alpha \sin t + u \cos t;
\]

\[
z = z = v \cos \alpha.
\]

(4.21)

The hyperbolic coordinate system makes it possible to specify a carved surface, the fixed axoid of which is a circular cone, and the movable axoid is a plane, by the function \( v = v(u, t) \).

The inverse dependence of the hyperbolic coordinates on the Cartesian will look like this:

\[
t = 2 \arctg \left( \frac{y - \sqrt{y^2 + z^2 \tan^2 \alpha}}{(x + z \tan \alpha)} \right),
\]

\[
 u = \sqrt{x^2 + y^2 - z^2 \tan^2 \alpha},
\]

\[
v = z \sec \alpha.
\]

(4.22)

Figure 5. Carved Monge surfaces with a cylindrical directrix surface and different meridians:

- Carved Monge surface with a cylindrical directrix surface and a chain meridian.
- Carved Monge surface with a cylindrical directrix surface and a hyperbola-shaped meridian.
- Carved Monge surface with a cylindrical directrix surface and a meridian cycloid shaped.
- Carved Monge surface with a cylindrical directrix surface and a circle meridian.
- Carved Monge surfaces with a cylindrical directrix surface and sinusoids as a meridian.

Coefficients of the main quadratic surface forms (4.10):

\[
A^2 = \left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial y}{\partial u} \right)^2 + \left( \frac{\partial z}{\partial u} \right)^2 = 1 + \left( \frac{\partial v}{\partial u} \right)^2, \quad F = (v-u)\sin \alpha + \frac{\partial v}{\partial u} \frac{\partial v}{\partial t},
\]

(4.21)

\[
B^2 = \left( \frac{\partial x}{\partial t} \right)^2 + \left( \frac{\partial y}{\partial t} \right)^2 + \left( \frac{\partial z}{\partial t} \right)^2 = \left( \frac{\partial v}{\partial t} \right)^2 - 2u \sin \alpha \frac{\partial v}{\partial t} + u^2 + v^2 \sin^2 \alpha,
\]

\[
L = \frac{u \cos \alpha \frac{\partial^2 v}{\partial u^2}}{\sqrt{A^2 B^2 - F^2}}, \quad M = \cos \alpha - \frac{u \frac{\partial^2 v}{\partial t^2} - u \sin \alpha \left( \frac{\partial v}{\partial u} \right)^2 + v \sin \alpha \frac{\partial v}{\partial u} - \frac{\partial v}{\partial t}}{\sqrt{A^2 B^2 - F^2}},
\]

(4.22)

\[
N = \cos \alpha \frac{u \frac{\partial^2 v}{\partial t^2} - 2u \sin \alpha \frac{\partial v}{\partial t} \frac{\partial v}{\partial u} + \frac{\partial v}{\partial u} \left( v^2 \sin^2 \alpha + u^2 \right) - v \sin \alpha \frac{\partial v}{\partial t}}{\sqrt{A^2 B^2 - F^2}}.
\]

The normal planes of the parallels of the carved Monge surface with the conical directrix surface, like the tangent planes of the cone, pass through its apex. The parallels are thus spherical curves located on concentric spheres with centers at the top of the directrix cone. The converse is also true: a surface, one family of lines of curvature of which is located on concentric spheres, is a carved surface with a conical directrix.
A carved Monge surface with a conical directrix surface can be defined by parametric equations of the form

\[
x(u, v) = [l + \varphi(v)](\sin \theta \cos u \cos w - \sin u \sin w) - \psi(v)(\sin \theta \cos u \cos w + \sin u \cos w),
\]
\[
y(u, v) = [l + \varphi(v)](\sin \theta \sin u \sin w - \sin u \sin w) - \psi(v)(\sin \theta \sin u \sin w + \cos u \cos w),
\]
\[
z(u, v) = [l + \varphi(v)]\cos \theta \cos w - \psi(v)\cos \theta \sin w,
\]

where \(l\) is the distance along the line of contact of the cone and the plane from the top of the cone \(O\) to the beginning \(O_1\) of the moving coordinate system \(OXY\), with respect to which the plane generating curve is given

\[
X = X(v), \quad Y = Y(v).
\]

The moving coordinate system is located in a rolling plane, and its coordinate axes \(O; X, O; Y\) are rotated by an angle \(\omega\) with respect to the mutually orthogonal unit vectors \(e(u), g(u)\) (figure 9);

\[
\varphi(v) = X(v) \cos \omega - Y(v) \sin \omega, \quad \psi(v) = X(v) \sin \omega + Y(v) \cos \omega;
\]

\[
w(u) = R\omega/l = p\omega = \sin \theta - u; \quad p = R/l = \sin \theta;
\]

where \(\theta\) is the angle between the generatrix of the circular cone and its axis, \(u\) is the angle characterizing the rolling of the plane. Figure 9 shows the initial position of the cone and the plane \((u = 0)\). The coordinate lines \(u\) describe the spherical lines, and the coordinate lines \(v\) coincide with the generatrices of the plane curves \(m\) (figure 9). The coefficients of the main quadratic surface forms (4.23) have the form:

\[
A^2 = [l \sin w + \varphi(v) \sin w + \psi(v) \cos w] \left(1 - p^2\right), \quad F = 0, \quad B^2 = \varphi^2 + \psi^2 = X^2 + Y^2,
\]

\[
L = -\frac{\hat{\varphi} \psi A}{B}, \quad M = 0, \quad N = \frac{1}{B} \left(X\hat{Y} - \hat{X}Y\right), \quad k_1 = \frac{L}{A^2} = -\frac{\hat{\varphi} \psi}{AB}, \quad k_2 = \frac{L}{B^3} = k_0,
\]

where \(k_0\) is the curvature of the plane generatrix curve \(m\). Dots over functions denote differentiation with respect to the parameter \(v\). If we take the straight line as the production line (4.24) \(X = v, Y = bv\), then it is possible to obtain the parametric equations of a ruled conical spiral cochlea of revolution (figure 10):

**Figure 6.** The hyperbolic coordinates \(u, t, v\), where \(t\) is the parameter of the moving plane \(O'\), \(t = \text{const}\) - the plane \(O'\); \(u, v\) - rectangular coordinates on the plane.

**Figure 7.** The coordinate lines \(u\) describe the spherical lines, and the coordinate lines \(v\) coincide with the generatrices of the plane curves \(m\).
\[ y(u, v) = \left[ l + \varphi(v) \right] \sin \theta \sin u \cos w + \cos u \sin w \] 
\[ - \psi(v) \sin \theta \sin u \sin w - \cos u \cos w, \] 
\[ z(u, v) = \left[ l + \varphi(v) \right] \cos \theta \cos w - \psi(v) \cos \theta \sin w, \]

where \( \varphi(v) = v \cos \omega - bv \sin \omega; \ \psi(v) = v \sin \omega + b \cos \omega. \)

The parametric equations of a ruled conical rotation snail (4.28) are obtained from the general parametric equations for an arbitrary conical rotation snail after substituting the values in them \( X = v, \ Y = bv. \)

The coefficients of the main quadratic surface forms (4.28) have the form:

\[ A^2 = \left[ \sin w + \varphi(v) \sin w + \psi(v) \cos w \right]^2 \left( 1 - p^2 \right)^2, \ F = 0, \]
\[ B^2 = 1 + b^2. \]
\[ L = -\dot{\varphi} \psi A / B, \ M = 0, \ N = 0, \ k_1 = L / A^2 = -\dot{\psi} \psi / (AB), \]
\[ k_2 = 0, \ K = 0. \]  

Ruled conical snails of revolution will be surfaces of zero Gaussian curvature.

In some works, it is proposed to classify carved Monge surfaces into carved Monge surfaces of double curvature (Fig. 4;Fig.5) and carved ruled Monge surfaces (figure 5; figure 8).

The geometric investigation of carved Monge surfaces is the one of the surfaces of complex geometry. These surfaces can be the median ones of thin elastic shells. These surfaces can be used to design these thin elastic shell. Because these elastic shells are thin, their geometry is the one of their median carved Monge surfaces. The thin elastic shells design in the shapes of these surfaces can be expressive, durable, buckling resistant and cover large spans. They can find their application in the architecture of civil and industrial building and in mechanical engineering. The multitude of the expressive forms of carved Monge surfaces should call for their application in construction industry by architects.

5. Conclusions

1. As noted earlier, the kinematic method of generating carved surfaces automatically includes them in the class of surfaces of congruent sections. In addition, given that carved surfaces are generated by rolling without sliding the movable axoid (plane with a flat curve) along the fixed (cylinder, cone, torso), they can be introduced into the subclass “rotational surfaces”, which are themselves included in the class “kinematic surfaces of general kind”. However, taking into account the history of the issue, apparently, carved surfaces should be considered a separate class in the theory of surfaces.

2. Carved Monge surfaces, as it was shown earlier, have the following advantages as architectural expressiveness, the reliable design with high stiffness and durability, rather small consumption of materials (massive bearing walls aren’t required), technological effectiveness of construction that allows to carry out construction process directly on a building site, possibility of rational use of space, presence of smooth lines pleasant for eye perception, an opportunity to use shell for buildings of different functions.

3. The complexity of this geometric investigation generates as results many surfaces of multitudes complex forms. The coefficients of their fundamental forms are functions of their two curvilinear coordinates. This allows to classify the carved as surfaces of complex geometry.

4. The observation of the complexity of these multitudes forms shows that they can give rise to new complex forms when some parameters are changed inside the equations (geometric, parametric and algebraic) of these surfaces. This calls on additional geometric investigations.
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