Geometrothermodynamics of black holes with a nonlinear source

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Abstract
We study thermodynamics and geometrothermodynamics of a particular black hole configuration with a nonlinear source. We use the mass as fundamental equation, from which it follows that the curvature radius must be considered as a thermodynamic variable, leading to an extended equilibrium space. Using the formalism of geometrothermodynamics, we show that the geometric properties of the thermodynamic equilibrium space can be used to obtain information about thermodynamic interaction, critical points and phase transitions. We show that these results are compatible with the results obtained from classical black hole thermodynamics.

Keywords Thermodynamics · Thermodynamic functions and equations of state · Phase transition · Riemannian geometries

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1 Introduction

Several regular black hole solutions have been found by coupling gravity to nonlinear electrodynamics theories. Among various regular models known to date, especially intriguing are the solutions to the coupled equations of nonlinear electrodynamics and general relativity [1,2]. The description of magnetically charged black hole provides
an interesting example of the system that could be both regular and extremal. In this same context are the theories of AdS-black holes that consider the so-called Power Maxwell Invariant (PMI) field \[3–6\] where an \(s\) parameter is introduced in order to represent a power term in the electromagnetic action, i.e. \((F_{\mu\nu}F^{\mu\nu})^s\), which reduces to Maxwellian field (linear electromagnetic source) when \(s = 1\). Nonlinear electrodynamic theories have been important at the low-energy limit of heterotic string theory or in the study of effects loop corrections in quantum electrodynamics [7].

On the other hand, black hole thermodynamics has been the subject of numerous researches in theoretical physics. The main attraction lies in the fact that a black hole is the best system to seek the aspects of quantum gravity and it is expected that the thermodynamics of black holes help us to know about its microscopic structure. The norm/gravity correspondence introduced by Maldacena [8], for example, considers the thermodynamic study of asymptotically AdS black holes relating black holes on the gravity side with the temperature on the field theory side. Hawking and Page’s research [9] showed that black holes could be assigned entropy through which it is possible to study thermodynamic properties such as phase transitions and interaction; for example, the Reissner–Nordström–AdS (RNAdS) black hole in \(n + 1\) dimensions [10], where it was found that it has phase transitions, or AdS black holes, where the cosmological constant is considered as a new thermodynamic variable [11,12]. However, the full implication of the gravitational—thermodynamic connection is not yet apparent [13].

On the other hand, the geometric description of the thermodynamic properties of black holes has been investigated by using two different approaches, namely, Weinhold and Ruppeiner’s approach of thermodynamic geometry [14–16] and the mathematical approach called geometrothermodynamics (GTD) [17]. Both formalisms use a Riemannian manifold to define a space of equilibrium states where the thermodynamic phenomena take place; however, the Weinhold and Ruppeiner approaches are not Legendre invariant[18], which implies that the properties of a given thermodynamic system can depend on the choice of thermodynamic potential. The formalism of GTD, on the contrary, is a geometric approach which considers the Legendre invariance and, therefore, describes the properties of the thermodynamic system independently of the thermodynamic potential, as in classical thermodynamics. In this work, we will use GTD in order to study the properties of black holes with a power Maxwell invariant field.

On the other hand, in black hole thermodynamics the degree of homogeneity of the fundamental equation is not always considered. The homogeneity of a thermodynamic system is very important because it allows us to know if the thermodynamic variables have a subextensive, extensive or supraextensive character. We show in this work how the GTD takes into account the homogeneity of the fundamental equation [19].

This work is organized as follows. In Sect. 2, we present the explicit form and review the main properties of a black hole with a power Maxwell invariant source, we analyze its fundamental equation and derive its main thermodynamic properties, considering the cosmological constant as an additional thermodynamic variable. In Sect. 3, we perform a geometrothermodynamic analysis of the corresponding 3-dimensional equilibrium manifold, and show that its thermodynamic curvature leads to results which are equivalent to the ones obtained from the analysis of the corresponding heat.
capacity. This proves the compatibility between classical black hole thermodynamics and GTD. Finally, in Sect. 4, we present the conclusions of our work.

2 Black holes with nolinear sources and theirs thermodynamics

The general action that describes Einstein-PMI gravity is given by the expression [3],

$$I = -\frac{1}{16\pi} \int_M d^{n+1}x \sqrt{-g} \left[ R + \frac{n(n-1)}{l^2} + \left( F_{\mu
u} F^{\mu
u} \right)^s \right],$$

(1)

where $F_{\mu\nu}$ represents the electromagnetic field tensor, $l$ is related to the cosmological constant by the expression $\Lambda = -\frac{3}{l^2}$ and $s$ is the parameter of nonlinearity. The action (1) is valid for any arbitrary dimension $n \geq 3$, however, it is possible to get different results depending on the value of the parameter $s$; for example, when $s = 1$, the PMI theory reduces to the Maxwell theory and for $s = (n + 1)/4$, with $(n + 1)$-dimensions, the PMI theory becomes conformally invariant [4,18]. Hereafter, we left the parameter $s$ arbitrary in order to analyze the thermodynamic quantities of this model and to determine the role of this parameter.

Considering a spherically symmetric spacetime with line element,

$$ds^2 = -f(r)dt^2 + \frac{dr}{f(r)} + r^2 d\Omega^2_{d-2},$$

(2)

where $d\Omega^2_{d-2}$ stands for the standard element on $S^d$. Using the equations obtained from the variation of the action (1), we get the next solution for a black hole with PMI source [4,20]

$$f(r) = 1 + \frac{r^2}{l^2} - \frac{m}{r^{n-2}} + \frac{(2s - 1)^2}{(n-1)(n-2)} \frac{(n-1)(2s-n)^2 q^2}{(2s-1)^2} \frac{r^{2s} - Q^{2s}}{r^{2(n-2)s} - Q^{2(n-2)s}}.$$ 

(3)

Here, $m$ and $q$ are related to the ADM mass $M$ and the electric charge $Q$ by means of the relation,

$$m = \frac{16\pi M}{(n-1)\omega_{n-1}},$$

(4)

$$q = \left[ \frac{8\pi}{\sqrt{2s\omega_{n-1}}} \right]^{\frac{1}{2s-1}} \left[ \frac{n-2}{n-1} \right]^{\frac{1}{2}} \left( 2s-1 \right)^{\frac{2s-2}{2s-1}} \left( 2s-n \right)^{\frac{2s-n}{2s-1}} \frac{Q^{2s}}{n-2s} - Q^{\frac{1}{2s-1}}.$$ 

(5)

The roots of the lapse function $f(r)$ ($g_{tt} = 0$) define the horizons $r = r_{\pm}$ of the spacetime. In particular, the null hypersurface $r = r_+$ can be shown to correspond to an event horizon, which in this case is also a Killing horizon, whereas the inner horizon at $r_-$ is a Cauchy horizon. Therefore, from $f(r_+) = 0$ we get the black hole mass, $M$. 

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\[ M(r_+, Q) = \frac{(n - 1)\omega_{n-1}}{16\pi} \left[ r_+^{n-2} + \frac{r_+^n}{l^2} - f_n r_+^{\frac{(2s-n)}{2s-1}} Q^{\frac{2s}{2s-1}} \right], \quad (6) \]

with

\[ f_n = \frac{(2s - 1)^{2-2s}(n - 1)^{s-1}(2s - n)^{2s-1}}{(n - 2)^s} \left[ \frac{8\pi}{\sqrt{2s\omega_{n-1}}} \right]^{\frac{2s}{2s-1}} \left[ \frac{n - 2}{n - 1} \right]^{\frac{s}{2s-1}} \frac{(2s - 1)^{\frac{2s(2s-2)}{2s-1}}}{(n - 2s)^{2s}}. \quad (7) \]

From the area-entropy relationship, \( S = \omega_{n-1} r_+^{n-1} \), Eq. (6) can be rewritten as

\[ M(S, Q) = \left[ 4 \frac{\omega_{n-1}^{\frac{n}{2}}}{\pi} \right]^{\frac{1}{n-1}} \omega_{n-1}^{\frac{1}{n-1}} \left[ S_{n-1}^{\frac{n-2}{n}} + S_{n-1}^{\frac{n}{n-1}} l^{-2} - \tilde{f}_n S \frac{2s-n}{(n-1)(2s-1)} Q^{\frac{2s}{2s-1}} \right], \quad (8) \]

where

\[ \tilde{f}_n = \left[ \frac{4}{\omega_{n-1}} \right]^{\frac{6s-2n-2}{(2s-1)(n-1)}} f_n, \quad (9) \]

with \( \omega_{n-1} = (2\pi^{n/2})/\Gamma(n/2) \).

The Eq. (8) is the fundamental equation for black holes with PMI source. This equation relates all the thermodynamic variables entering the black hole metric. In order to correctly describe the thermodynamic properties of black holes, it is necessary to impose the condition that the corresponding fundamental equation be a quasi-homogeneous function \([19,21]\). As we can observe, the fundamental equation (8) is an inhomogeneous function in the extensive variables \( S \) and \( Q \), e.i., the rescaling \( S \rightarrow S \lambda^{\beta_S} \) and \( Q \rightarrow Q \lambda^{\beta_Q} \), where \( \lambda \) and the \( \beta \)'s are real constants, does not fulfill the condition \( M(\lambda^{\beta_S} S, \lambda^{\beta_Q} Q) = \lambda^{\beta_M} M(S, Q) \). However, if we consider \( l \) also as a thermodynamic variable \([22–26]\), which rescales as \( l \rightarrow \lambda^{\beta_l} l \), the quasi-homogeneity condition \( M(\lambda^{\beta_S} S, \lambda^{\beta_Q} Q, \lambda^{\beta_l} l) = \lambda^{\beta_M} M(S, Q, l) \) holds if the relationships,

\[ \beta_S = \frac{n - 1}{n - 2}, \quad \beta_Q = \frac{(2s - 1)(n^2 - 2n - s + 1)}{s(n - 2)(2n - 1)}, \quad \beta_l = \frac{1}{n - 2}, \quad \beta_M = 1, \quad (10) \]

are fulfilled.

The physical parameters of the black hole with PMI source satisfy the first law of black hole thermodynamics \([13]\),

\[ dM = T dS + \Phi dQ + L dl, \quad (11) \]
where $T$ is the Hawking temperature, which is proportional to the surface gravity on the horizon, $\Phi$ is the electric potential and $L$ is the thermodynamic variable dual to $l$. The thermodynamic equilibrium conditions are given by the expressions,

$$T = \frac{\partial M}{\partial S}, \quad \Phi = \frac{\partial M}{\partial Q}, \quad L = \frac{\partial M}{\partial l}. \quad (12)$$

These expressions allow us to compute the explicit form of the corresponding intensive variables:

$$T = \sum_n \left[ \frac{n - 2}{n - 1} S^{\frac{1}{n - 1}} + \frac{n - 1}{n - 1} \frac{S^{\frac{1}{n - 1}}}{l^2} - \frac{(2s - n)}{(n - 1)(2s - 1)} S^{\frac{s}{n - 1}} \left( \frac{\sum_{n=1}^{s-1} \tilde{f}_n}{n^s} \right) \right], \quad (13)$$

$$\Phi = -\frac{2s \sum_n \tilde{f}_n}{(2s - 1)} S^{\frac{2s - n}{n - 1}} Q^{\frac{1}{n - 1}}, \quad (14)$$

$$L = -\frac{2\Omega_n}{l^3} S^{\frac{n}{n - 1}(2s - 1)}, \quad (15)$$

with $\Omega_n = \frac{[4]^{n - 1} (n - 1) \omega_{n - 1}}{\pi}$. The behavior of the temperature $T$, electrical potential $\Phi$ and the thermodynamic variable $L$ in terms of the entropy $S$ is shown in Figs. 1, 2, 3 and 4 for fixed values of the charge and the variable $l$.

It is easy to show that the temperature (13) coincides with the Hawking temperature. The temperature increases rapidly as a function of the entropy $S$ until it reaches its maximum value. Then, as the entropy increases, the temperature becomes a monotonically decreasing function until it reaches another point from which the temperature increases again. The heat capacity at constant values of $Q$ and $l$ is computed by the expression,
Fig. 2 Electrical Potential (left) and the thermodynamic variable $L$ (right) as a function of entropy $S$, where we have considered $n = 4, l = Q = 1$ and $s = 3$

Fig. 3 Temperature $T$ as a function of entropy $S$, in the ranges (left) $0 < S \leq 0.10$, (Center) $0 < S \leq 0.05$ and (right) $0.2 \leq S \leq 0.8$. The red line indicates the point where the temperature as a function of entropy reaches its maximum (minimum) value. Here $n = 5, l = Q = 1$ and $s = 4$

Fig. 4 Electrical Potential (left) and the thermodynamic variable $L$ (right) as a function of entropy $S$, where we have considered $n = 5, l = Q = 1$ and $s = 4$
Fig. 5 Heat capacity $C_{Q,l}$ as a function of entropy $S$ in the ranges (left) $0 < S \leq 0.4$, (center) $0 < S \leq 0.10$ and (right) $0.2 \leq S \leq 0.6$. The red line indicates the point where second order phase transitions occur. Here $n = 4, l = Q = 1$ and $s = 3$.

\[
C_{Q,l} = T \left( \frac{\partial S}{\partial T} \right)_{Q,l} = \frac{\left( \frac{\partial M}{\partial S} \right)_{Q,l}}{\left( \frac{\partial^2 M}{\partial S^2} \right)_{Q,l}},
\]

where the subscript indicates that derivatives are calculated keeping $Q$ and $l$ constants. The heat capacity that corresponds to the fundamental equation (8) is given by the expression

\[
C_{Q,l} = \frac{(n - 1) \left[ l^2 (2s - n) S_{\frac{4s - 2ns - 1}{n - 1}}^{\frac{2s}{2s - 1}} \tilde{f}_n - 2s - 1 \right] \left( n - 2 \right) l^2 + n S_{\frac{1}{n - 1}}^{\frac{1}{2s - 1}}}{(2s - 1) D_1},
\]

where

\[
D_1 = l^2 (n - 2) S_{\frac{n}{n - 1}}^{\frac{n}{n - 1}} - n S_{\frac{(n - 2)}{(n - 1)}}^{\frac{(n - 2)}{(n - 1)}}
+ \frac{(2s - n)(4s - 2sn - 1)}{(2s - 1)^2} l^2 \tilde{f}_n S_{\frac{6s - 4sn + n - 2}{(n - 1)(2s - 1)}}^{\frac{6s - 4sn + n - 2}{(n - 1)(2s - 1)}} Q_{\frac{2s}{2s - 1}}.
\]

According to Ehrenfest’s classification [27], second order phase transitions occur at those points where the heat capacity diverges, i.e., for $D_1 = 0$.

Unfortunately, it is not possible to solve equation (18) to know analytically the relationship that defines the points where the phase transitions take place. However, we can do a numerical analysis in order to describe its behavior. An example is shown in Fig. 5 for particular values: $n = 4, l = Q = 1$ and $s = 3$.

We can see from the graphics 1, 3, 5 and 7 that in the region with positive temperature the heat capacity is positive, indicating that the black hole is stable in this region. At the maximum (minimum) value of the temperature, the heat capacity diverges and changes spontaneously its sign from positive to negative. This indicates the presence of second order phase transitions, which are accompanied by a transition into a region of instability.
The Figs. 5 and 6 show some particular cases of the behavior of the heat capacity for values different from $s = 1$. We can see that for $s = 3$ and $s = 4$ there are phase transitions which occur, in general, in any dimension $n$.

If we consider the power $s = 1$, we obtain a theory with a linear source. Then, the heat capacity (17) takes the form,

$$C_{Q, I} = \frac{S \left( (n - 1) [(n - 2) l^2 + n] S^{\frac{2(n-2)}{(n-1)}} - l^2 Q^2 \left[ \frac{4}{\omega_{n-1}} \right]^{\frac{2(n-2)}{(n-1)}} \left[ \frac{8\pi}{\sqrt{2\omega_{n-1}}} \right]^2 \right)}{n(n - 1) S^2 - (n - 1)(n - 2) l^2 S^{\frac{2(n-2)}{(n-1)}} - (3 - 2n) l^2 Q^2 \left[ \frac{4}{\omega_{n-1}} \right]^{\frac{2(n-2)}{(n-1)}} \left[ \frac{8\pi}{\sqrt{2\omega_{n-1}}} \right]^2}.$$  

(19)

In this case, the heat capacity diverges at the points where the following equation is satisfied

$$n(n - 1) S^2 - (n - 1)(n - 2) l^2 S^{\frac{2(n-2)}{(n-1)}} - (3 - 2n) l^2 Q^2 \left[ \frac{4}{\omega_{n-1}} \right]^{\frac{2(n-2)}{(n-1)}} \left[ \frac{8\pi}{\sqrt{2\omega_{n-1}}} \right]^2 = 0,$$

(20)

which indicates the presence of phase transitions. Again, it is not possible to solve equation (20) analytically to find the points where the phase transitions take place. A numerical analysis for the particular values $l = Q = 1$ shows that there are no phase transitions.
transitions. This situation is illustrated in Fig. 7. The same happens when we choose the values $l = 2$ and $Q = 4$. The Fig. 8 depicts this particular behavior. This shows that the existence of phase transitions depends on the values of $l$ and $Q$. To exemplify this situation, let us consider the case $n = 3$ for which the Eq. (19) takes the form

$$C_{Q,l} = \frac{S \left[ l^2 + 3 \right] S - 2\pi l^2 Q^2}{3S^2 - l^2 S + 3\pi l^2 Q^2},$$

(21)

As we can see, there are second order phase transition at those points where $l$ and $Q$ satisfy the relationship $l^2 - 36\pi Q^2 \geq 0$. This corresponds to the case of AdS black holes, in which phase transitions have been found [28].

3 Geometrothermodynamics formalism

In brief, GTD is a formalism based on a space which is a $(2n + 1)$--dimensional Riemannian contact manifold $(T, \Theta, G)$, where $T$ represents a differential manifold, $\Theta$ is a contact form, i.e., $\Theta \wedge (d\Theta)^n \neq 0$, and $G$ a Riemannian metric. If we introduce in $T$ the coordinates $Z^A = \{ \Phi, E^a, I^a \}$ with $a = 1, \ldots, n$ and $A = 0, \ldots, 2n$, according to Darboux theorem, the contact form $\Theta$ can be expressed as $\Theta = d\Phi - \delta_{ab} E^a dE^b$. On the other hand, the main ingredient of GTD is the Legendre invariance, which is considered through the metric $G$ demanding that it must be invariant with respect to Legendre transformations [29]. In particular, three metrics have been found that are Legendre invariant [19]. One of them is used to describe the thermodynamics of black holes and can be written as

$$G = \Theta^2 + (\delta_{ab} E^a I^b)(\epsilon_{cda} E^c dI^d),$$

(22)
where $\delta_{ab} = \text{diag}(1, 1, \ldots, 1)$ and $\eta_{ab} = \text{diag}(-1, 1, \ldots, 1)$. Using the metric (22) GTD induces a Legendre invariant metric $g$ on an $n-$ dimensional submanifold $E \subset T$ by mean of the pullback $\varphi^*(G) = g$, which is associated with the smooth embedding map $\varphi : E \rightarrow T$ and fulfills the condition $\varphi^*(\Theta) = 0$. If we choose the set $E^a$ as coordinates of $E$, then the embedding reads $\varphi : \{E^a\} \rightarrow \{\Phi(E^a), E^a, I^a(E^a)\}$ so that $\Phi(E^a)$ is the fundamental equation and the induced metric becomes

$$g = \beta_\Phi \eta^b_\Phi \Phi_{,bc} dE^a dE^c,$$

(23)

with $\Phi_{,a} = \frac{\partial \Phi}{\partial \sigma^a}$, $\beta_\Phi$ is a constant and we have used the Euler identity in the form $\beta_a E^a \Phi_{,a} = \beta_\Phi \Phi$ for generalized homogeneous function [19].

We now apply the above formalism to the case of black holes with PMI source. Let us consider the fundamental equation (8) which, according to the analysis presented above, is a generalized homogeneous function of degree 1 that does not involve a redefinition of the thermodynamic variables, affecting the physical properties of the thermodynamic system [19]. The thermodynamic metric (23) is 3-dimensional and reduces to

$$g = \frac{M[4]^{n-1}}{\pi(n-1)} \left[ (n - 2)S^{-\frac{n}{n-1}} - \frac{n}{l^2} S^{-\frac{n-2}{n-1}} + \frac{(2s - n)(4s - 2sn - 1)}{(2s - 1)^2} S^{-\frac{6s - 4ns - n - 2}{(n-1)(2s-1)}} Q^{\frac{2s}{2s-1}} dS^2 - \frac{2s}{(2s - 1)^2} S^{-\frac{2s - n}{(n-1)(2s-1)}} Q^{-\frac{2(s-1)}{2s-1}} dQ^2 + \frac{6}{l^4} S^{-\frac{n}{(n-1)}} d\ell^2 \right].$$

(24)

The curvature scalar corresponding to the metric (24) takes the form,

$$R = \frac{N(S, Q, l)}{D_1^2 D_2},$$

(25)

where,

$$D_1 = l^2 (n - 2) S^{-\frac{n}{(n-1)}} - n S^{-\frac{(n-2)}{(n-1)}} + \frac{(2s - n)(4s - 2sn - 1)}{(2s - 1)^2} l^2 \tilde{f}_n S^{-\frac{6s - 4ns + n - 2}{(n-1)(2s-1)}} Q^{\frac{2s}{2s-1}},$$

(26)

$$D_2 = l^2 S^{-\frac{n-2}{n-1}} + S^{\frac{n}{n-1}} - l^2 \tilde{f}_n S^{-\frac{2s - n}{(n-1)(2s-1)}} Q^{\frac{2s}{2s-1}},$$

(27)

and $N(S, Q, l)$ is a function that is different from zero at the points where denominator vanishes and cannot be written in a compact form. There are two curvature singularities in this case. One of them occurs if $D_2 = 0$ and corresponds to $M = 0$, as follows from Eq. (8). This means that this singularity is nonphysical since no black hole is present in this case. Figure 9 shows numerically two of these roots. A second singularity is located at the roots of the equation $D_1 = 0$, according to the expression for the heat capacity (17). It coincides with the points where $C_{(l, Q)} \rightarrow \infty$, i.e., these are exactly
Fig. 9 The mass $M$ as a function of the entropy $S$. We have considered $n = 4$, $s = 3$ (left) and $n = 5$, $s = 4$ (right), in both cases $l = Q = 1$. The red line indicates the point where $M = 0$.

Fig. 10 Behavior of the curvature scalar $R$, as a function of entropy $S$, for different intervals. The red line indicates the points where curvature singularities occur. We have considered $n = 4$, $l = Q = 1$ and $s = 3$.

Fig. 11 Behavior of the curvature scalar $R$, as a function of entropy $S$, for different intervals. The red line indicates the points where curvature singularities occur. We have considered $n = 5$, $l = Q = 1$ and $s = 4$.

The roots that determine the critical points where phase transitions take place. This result is invariant in the sense that it does not depend on the choice of coordinates because it is based on the analysis of a scalar quantity. In Figs. 10 and 11, we show the numerical behavior of the curvature scalar for different sets of values of $n$, $l$ and $Q$. 
According to GTD, these results show that there exist curvature singularities at those points where second order phase transitions occur, because the denominators of the heat capacity and the curvature scalar coincide [30].

Figure 12 shows the behavior of the curvature scalar for a black hole with linear electromagnetic source ($s = 1$). As we can see in the linear electromagnetic case, with the particular values $l = Q = 1$ there are no singularities. Therefore, GTD also reproduces correctly this case.

On the other hand, a curvature different from zero is considered in GTD as a measure of thermodynamic interaction. In our case, the corresponding thermodynamic curvature (25) turned out to be nonzero in general, indicating the presence of thermodynamic interaction for this particular black hole configuration with a nonlinear source. According with [31], this result can be interpreted physically as due to the behavior of the black hole microstructure, where interaction between neighboring states exists. Moreover, the sign of the scalar curvature can be used to distinguish between repulsive and attractive interactions [32]. In our case, the resulting expression for the curvature cannot be represented in a compact and useful form as to determine its sign. Nevertheless, the particular numerical analysis carried out Figs. 10 and 11, shows that there are regions of positive and negative curvature depending on the values of the chosen parameters. This means that repulsive and attractive interactions can exist.

4 Conclusions

In this paper, we investigated the thermodynamics and geometrothermodynamics of a spherically symmetric AdS black hole with a PMI source. We analyzed the fundamental equation that relates the total mass, the entropy and charge. We showed that for the fundamental equation to be a generalized homogeneous function [19], it is necessary to consider the variable $l$, related to the cosmological constant, as an extensive thermodynamic variable, in order for the fundamental equation to depend on extensive variables only. As a result, we obtained a fundamental equation whose mathematical properties resemble those of classical thermodynamic systems. Considering the curvature radius $l$ as an extensive thermodynamic variable implies that the equilibrium space must be extended by one dimension. A similar result was obtained recently [33],
by assuming that the energy of a black hole is not represented by its total mass, but by the corresponding enthalpy, indicating that the cosmological constant is an intensive thermodynamic variable similar to the pressure. Our results corroborate from a more formal mathematical point of view the intuitive analysis performed in [33].

We investigated the properties of the extended 3-dimensional equilibrium space in the framework of GTD and we showed that in the space of equilibrium states of a black hole with PMI source, there exists a thermodynamic metric whose curvature turns out to be nonzero, indicating the presence of thermodynamic interaction. We also found that the curvature is singular at those points where phase transitions of the heat capacity occur. This has been shown by considering a particular metric in the thermodynamic phase space, and applying the formalism of geometrothermodynamics. An important property of our choice of thermodynamic metric is that it is invariant with respect to Legendre transformations so that the properties of our geometric description of thermodynamics are independent of the choice of thermodynamic potential and representation.

We conclude that the thermodynamic properties of this particular class of a black holes with no linear source are correctly described within the GTD formalism.

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