The stochastic Schwarz lemma on Kähler manifolds by couplings and its applications

Myeongju Chae\(^1\)  |  Gunhee Cho\(^2\)  |  Maria Gordina\(^3\)  |  Guang Yang\(^4\)

\(^1\)School of Applied Mathematics and Computer Engineering, Hankyong National University, Anseong, Republic of Korea
\(^2\)Department of Mathematics, University of California, Santa Barbara, Isla Vista, California, USA
\(^3\)Department of Mathematics, University of Connecticut, Storrs, Connecticut, USA
\(^4\)Department of Mathematics, Purdue University, West Lafayette, Indiana, USA

Correspondence
Gunhee Cho, Department of Mathematics, University of California, Santa Barbara, 552 University Rd, Isla Vista, CA 93117, USA.
Email: gunheecho@ucsb.edu

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Abstract
We first provide a stochastic formula for the Carathéodory distance in terms of general Markovian couplings and prove a comparison result between the Carathéodory distance and the complete Kähler metric with a negative lower curvature bound using the Kendall–Cranston coupling. This probabilistic approach gives a version of the Schwarz lemma on complete non-compact Kähler manifolds with a further decomposition Ricci curvature into the orthogonal Ricci curvature and the holomorphic sectional curvature, which cannot be obtained by using Yau–Royden’s Schwarz lemma. We also prove coupling estimates on quaternionic Kähler manifolds. As a by-product, we obtain an improved gradient estimate of positive harmonic functions on Kähler manifolds and quaternionic Kähler manifolds under lower curvature bounds.

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1 | INTRODUCTION

The Carathéodory pseudodistance $c_M$ on a complex manifold $M$ is defined as

$$c_M(x, y) := \sup_{f \in \text{Hol}(M, D)} \rho_D(f(x), f(y)).$$

Here, $\text{Hol}(M, D)$ is the collection of all holomorphic functions from $M$ to $D$, and we denote the Poincaré distance on a unit disk $D$ in $\mathbb{C}^1$ by

$$\rho_D(z, z') := \frac{|z - z'|}{1 - \bar{z}'z}.$$  

Note that $c_M$ is called a pseudodistance because there could be $x \neq y, x, y \in M$ with $c_M(x, y) = 0$. The triangle inequality, however, is always satisfied. In the rest of this paper, we will simply refer to $c_M$ as Carathéodory distance when there is no risk of confusion.

Little is known about properties of holomorphic functions on noncompact complex manifolds except for special bounded domains in $\mathbb{C}^n$. For instance, even existence of bounded holomorphic functions on such manifolds is an open question. Existence of nonconstant holomorphic functions can be shown on noncompact Kähler manifolds by solving the $\bar{\partial}$-equation with Hörmander’s $L^2$-estimate, but it is difficult to deal with boundedness as in [14]. In this regard, the Carathéodory distance plays a prominent role, as it provides quantitative information about nonconstant bounded holomorphic functions. Moreover, among the invariant distances that satisfy the distance decreasing property, the Carathéodory distance is the smallest invariant distance (see, e.g., [16]).

A fundamental tool for studying the metric and distance of negatively curved complex manifolds is the Yau–Royden’s Schwarz lemma [24, 28]. However, although Yau–Royden’s Schwarz lemma implies that on a complex manifold, the Carathéodory–Reiffen metric $\gamma_M$ is bounded above by a complete Kähler metric $\omega$ with a negative Ricci curvature negative lower bound, and therefore, the Carathéodory distance is bounded by the geodesic distance $d_\omega$ of $\omega$, such a comparison is not sharp in general. For example, nonsharpness of such a comparison between $c_M$ and $d_\omega$ follows from the fact that the Carathéodory distance $c_M$ could be different from the inner-Carathéodory pseudodistance (see Section 2.2). On the other hand, it is known that sharp Laplacian and volume
comparisons with complex space forms usually require a decomposition of the Ricci curvature into the orthogonal Ricci curvature and the holomorphic sectional curvature as in [22], whereas Yau–Royden’s Schwarz lemma requires the Chern–Lu formula [28], which is not easily adjusted to the further decomposition of the Ricci curvature.

In this paper, we derive a new stochastic formula of the Carathéodory distance on complete Kähler manifolds using couplings of Brownian motions. A version of the Schwarz lemma on Kähler manifolds is provided, which can be adjusted under the change of the Ricci curvature in the way described earlier. To our best knowledge, this is the first case of probabilistic techniques being used in the study of complex geometry.

We briefly explain some basic definitions of couplings techniques in order to state our main results. More details can be found in Section 3. For a diffusion $W_t$ on $\mathbb{M}$, we say that $(X_t, Y_t)$ on $\mathbb{M} \times \mathbb{M}$ is a coupling of $W_t$ if the marginal processes $X_t$ and $Y_t$ have the same distribution as $W_t$. A coupling is called Markovian if $\{(X_t, Y_t)\}$ is a Markov process with respect to its natural filtration $\mathcal{F}_s = \sigma\{(X_s, Y_s), s \leq t\}$. We define the coupling time

$$\tau(X, Y) := \inf \{t > 0 : X_t = Y_t\}.$$ 

It is always assumed that $X_t = Y_t$ for all $t > \tau(X, Y)$.

We are now ready to state our main results.

**Theorem 1.1.** On a complete Kähler manifold $\mathbb{M}$, for any $x, y \in \mathbb{M}$ with $x \neq y$, we consider a Markovian coupling $(X_t, Y_t)$ of Brownian motions with the coupling time $\tau(X, Y)$. Then, for all $t \geq 0$, we have

$$c_{\mathbb{M}}(x, y) = \sup_{f \in \text{Hol}(\mathbb{M}, \mathbb{D}), f(y) = 0} |\mathbb{E}^{x, y}[f(X_t) - f(Y_t)] \cdot 1(\tau(X, Y) > t)|,$$  

where $1(\tau(X, Y) > t)$ is the indicator function of the event $\{\tau(X, Y) > t\}$. In particular,

$$c_{\mathbb{M}}(x, y) \leq 2p^{x, y}(\tau(X, Y) > t),$$

and by letting $t \to \infty$,

$$c_{\mathbb{M}}(x, y) \leq 2p^{x, y}(\tau(X, Y) = \infty).$$

**Remark 1.2.** Historically, the study of Carathéodory–Reiffen metric and the Carathéodory distance is split into upper and lower bound estimates on mostly special classes of pseudoconvex domains in $\mathbb{C}^n$. For instance, the upper bound was studied for various settings in [1, 12, 13, 19]. The lower bound of integrated Carathéodory–Reiffen metric on complete simply connected noncompact Kähler manifolds was recently established by one of the authors in [11]. On the other hand, our result (2) gives an equality for Carathéodory distance, which provides more precise information.

When $\mathbb{M}$ is a unit disk $\mathbb{D}$ in $\mathbb{C}^1$, Theorem 1.1 leads to the following interesting version of a stochastic Schwarz lemma.

**Corollary 1.3** (Stochastic Schwarz lemma on a disk). On a unit disk $\mathbb{D}$ in $\mathbb{C}^1$, for any $x \in \mathbb{D} \setminus \{0\}$ with $y = 0$ and any $t \geq 0$, consider a Markovian coupling $(X_t, Y_t)$ of Brownian motions with the coupling...
time $\tau(X, Y)$. Then, for any holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$

$$|\mathbb{E}^{X,Y}[f(X_t) - f(Y_t)] \cdot 1(\tau(X, Y) > t)| \leq |\mathbb{E}^{X,Y}[X_t - Y_t] \cdot 1(\tau(X, Y) > t)|. \quad (5)$$

If the equality holds for some $x, y$, then $f$ must be a rotation.

For the next theorem, we use the Lindvall–Rogers (Kendall–Cranston) coupling. In particular, Theorem 1.4 improves the estimate in [8] because of the stronger curvature assumptions as we point out in Remark 6.2. See Section 2.1 for definitions of $H$ and $\text{Ric}^\perp$, and Section 6.1 for $\tau_{(B(x_0,2\delta))(X)}$.

**Theorem 1.4.** Let $(\mathbb{M}, g)$ be a complete noncompact Kähler manifold of the complex dimension $n$. Assume that $H \geq 4k_1$ and $\text{Ric}^\perp \geq (2n-2)k_2$ for some $k_1, k_2 \in \mathbb{R}$ with $k_1, k_2 < 0$. Let $\{X_t\}$ be the Markov process with the infinitesimal generator $L = \frac{1}{2}\Delta_g + Z$, where $Z$ is a smooth vector field for which there is a constant $m$ such that

$$|Z(x)| \leq m, x \in \mathbb{M}. \quad (6)$$

Let $(X, Y)$ be the Lindvall–Rogers coupling. Then there is a constant $c = c(k_1, k_2, n)$ such that for all $x, y \in B(x_0, \delta)$

$$\mathbb{P}^{x,y}(\tau(X, Y) > \tau_{B(x_0,2\delta)}(X) \wedge \tau_{B(x_0,2\delta)}(Y)) \leq c \left(\frac{1}{\delta} + 1\right) \rho_{\mathbb{M}}(x, y), \quad (7)$$

where $\rho_{\mathbb{M}}$ is the geodesic distance of $g$ on $\mathbb{M}$. Furthermore,

$$\mathbb{P}^{x,y}(\tau(X, Y) = \infty) \leq \left(8((n-1)\sqrt{|k_2|} + \sqrt{|k_1|}) + 2m\right) \rho_{\mathbb{M}}(x, y). \quad (8)$$

By combining Theorems 1.1 and 1.4, we have the comparison between the Carathéodory distance and the geodesic distance on a complete Kähler manifold with a negative curvature lower bound.

**Corollary 1.5** (Stochastic Schwarz lemma on Kähler manifolds). Let $(\mathbb{M}, g)$ be a complete noncompact Kähler manifold of the complex dimension $n$. Under the settings of Theorem 1.4 with $m \equiv 0$, we have for any $x, y \in \mathbb{M}$

$$c_{\mathbb{M}}(x, y) \leq 4 \left(4((n-1)\sqrt{|k_2|} + \sqrt{|k_1|})\right) \rho_{\mathbb{M}}(x, y). \quad (9)$$

**Remark 1.6.** It is worth mentioning that, by Yau–Royden’s lemma [24, 28], one can give a comparison between the Carathéodory distance and the complete Kähler metric with negative Ricci curvature lower bound. Our previous formula with a coupling time estimate can be applied under the further decomposition of the Ricci tensor. Thus, (9) can be viewed as a Schwarz lemma on complete Kähler manifolds based on coupling methods. However, it is restricted to the comparison with the Carathéodory distance only.

In addition, as a by-product of the coupling technique, we prove a gradient estimate for real-valued harmonic functions.
We emphasize that due to the Ricci curvature decomposition, our result is stronger than the same type of estimate on Riemannian manifolds [18].

**Corollary 1.7** (Gradient estimates for harmonic functions on Kähler manifolds). *Suppose the manifold $(\mathcal{M}, g)$ is as in Theorem 1.4. If $Lu = 0$ on $\mathcal{M}$ and $u$ is bounded and positive, then*

$$ |\nabla u(x)| \leq \left( 8((n - 1)\sqrt{|k_2|} + \sqrt{|k_1|}) + 2m \right) ||u||_{\infty}. \tag{10} $$

We remark that for the local gradient estimate, $||u||_{\infty}$ in the right-hand side of (10) can be replaced by $|u(x)|$ by applying known results of local gradient estimate of positive harmonic function, for example, Cheng–Yau [6], and improved local version by O. Munteanu [21], and Cranston’s coupling approach [8] and the different stochastic approach of A. Thalmaie and F-Y-Wang [25].

Along similar lines, we can also provide coupling estimates and their consequences on quaternionic Kähler manifolds.

**Theorem 1.8.** Let $(\mathcal{M}, g)$ be a complete noncompact quaternionic Kähler manifold of the complex dimension $n$. Suppose $k_1, k_2 \in \mathbb{R}$ with $k_1, k_2 < 0$. Assume that $Q \geq 12k_1$ and that $\text{Ric}^\perp \geq (4n - 4)k_2$. Let $(X_t)$ be the Markov process corresponding with the infinitesimal generator $L = \frac{1}{2} \Delta_g + Z$, where $Z$ is a smooth vector field for which there is a constant $m$ such that

$$ |Z(x)| \leq m, \quad x \in \mathcal{M}, \tag{11} $$

and let $(X, Y)$ be the Lindvall–Rogers mirror coupling. Then

$$ P^{X,Y}(T(X, Y) = \infty) \leq \left( 8(n - 1)\sqrt{|k_2|} + 24\sqrt{|k_1|} + 2m \right) \rho_{\mathcal{M}}(x, y). \tag{12} $$

**Corollary 1.9** (Gradient estimates for harmonic functions on quaternionic Kähler manifolds). *Let $(\mathcal{M}, g)$ be as in Theorem 1.8. If $Lu = 0$ on $\mathcal{M}$ and $u$ is bounded and positive, then*

$$ |\nabla u(x)| \leq \left( 8((n - 1)\sqrt{|k_2|} + \sqrt{|k_1|}) + 2m \right) ||u||_{\infty}. $$

2 | **GEOMETRIC PRELIMINARIES: KÄHLER AND QUATERNIONIC KÄHLER MANIFOLDS**

In this section, for the sake of completeness, we give the definitions we will be using in this paper. We refer to [3] for more details. Throughout the paper, let $(\mathcal{M}, g)$ be a smooth complete Riemannian manifold. Denote by $\nabla$ the Levi–Civita connection on $\mathcal{M}$.

2.1 | **Kähler manifolds**

After [22] and [23], we will be considering the following type of curvatures on Kähler manifolds. Let

$$ R(X, Y, Z, W) = g((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}Z)Z, W) $$

be the Riemannian curvature tensor on $(\mathcal{M}, g)$. 
We have two natural connections on complex manifolds. The Chern connection $\nabla^c$ is compatible with the Hermitian metric and the complex structure $J$, and the Levi–Civita connection $\nabla$ is a torsion-free connection compatible with the induced Riemannian metric. The two connections coincide precisely when a hermitian metric on a complex manifold is Kähler.

**Definition 2.1** (Kähler manifold). The manifold $(\mathcal{M}, g)$ is called a Kähler manifold if there exists a smooth $(1,1)$ tensor $J$ on $\mathcal{M}$ that satisfies the following properties.

- For every $x \in \mathcal{M}$, and $X, Y \in T_x \mathcal{M}$, $g_x(J_x X, Y) = -g_x(X, J_x Y)$.
- For every $x \in \mathcal{M}$, $J^2_x = -\text{Id}_{T_x \mathcal{M}}$.
- $\nabla J = 0$.

The map $J$ is called a complex structure.

We decompose the complexified tangent bundle $T_{\mathbb{R}}\mathcal{M} \otimes_{\mathbb{R}} \mathbb{C} = T'\mathcal{M} \oplus T''\mathcal{M}$, where $T'\mathcal{M}$ is the eigenspace of $J$ with respect to the eigenvalue $\sqrt{-1}$ and $T''\mathcal{M}$ is the eigenspace of $J$ with respect to the eigenvalue $-\sqrt{-1}$. We can identify $v, w$ as real tangent vectors, and $\eta, \xi$ as corresponding holomorphic $(1,0)$ tangent vectors under the $\mathbb{R}$-linear isomorphism $T_{\mathbb{R}}\mathcal{M} \to T'\mathcal{M}$, that is, $\eta = \frac{1}{\sqrt{2}}(v - \sqrt{-1}Jv), \xi = \frac{1}{\sqrt{2}}(w - \sqrt{-1}Jw)$. The components of the curvature 4-tensor of the Chern connection associated with the Hermitian metric $g$ of a (complex) $n$-dimensional complex manifold are given by

$$R_{ijkl}^c := R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_l}\right)$$

$$= g\left(\nabla^c_{\frac{\partial}{\partial z_i}} \nabla^c_{\frac{\partial}{\partial z_j}} \frac{\partial}{\partial z_k} - \nabla^c_{\frac{\partial}{\partial z_j}} \nabla^c_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_k} - \nabla^c \left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right] \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_l}\right)$$

$$= -\frac{\partial^2 g_{ij}}{\partial z_k \partial z_l} + \sum_{p,q=1}^n g^{pq} \frac{\partial g_{ij}}{\partial z_k} \frac{\partial g_{pq}}{\partial z_l},$$

where $i, j, k, l \in \{1, \ldots, n\}$ and $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ is a standard (local) basis for $T'\mathcal{M}$.

The holomorphic sectional curvature of the Kähler manifold $(\mathcal{M}, g, J)$ is defined as

$$H(X) = \frac{R(X, JX, JX, X)}{g(X, X)^2}.$$

The orthogonal Ricci curvature of the Kähler manifold $(\mathcal{M}, g, J)$ is defined for a vector field $X$ such that $g(X, X) = 1$ by

$$\text{Ric}^\perp(X, X) = \text{Ric}(X, X) - H(X),$$

where $\text{Ric}$ is the usual Riemannian Ricci tensor of $(\mathcal{M}, g)$. The table below shows the curvature of the Kähler model spaces $\mathbb{C}^m$, $\mathbb{C}P^m$ and $\mathbb{C}H^m$, see [3] (Tables 1 and 2).

Unlike the Ricci tensor, $\text{Ric}^\perp$ does not admit polarization, so we never consider $\text{Ric}^\perp(u, v)$ for $u \neq v$. For a real vector field $v$, we can write

$$\text{Ric}^\perp(v, v) = \sum R(v, E_i, E_i, v),$$
Table 1 Curvatures of Kähler model spaces.

| $\mathbb{M}$ | $H$ | $\text{Ric}^{\perp}$ |
|--------------|-----|-------------------|
| $\mathbb{C}^m$ | 0   | 0                 |
| $\mathbb{C}P^m$ | 4   | $2m - 2$         |
| $\mathbb{C}H^m$ | $-4$ | $-(2m - 2)$     |

Table 2 Curvatures of the quaternionic Kähler model spaces.

| $\mathbb{M}$ | $Q$ | $\text{Ric}^{\perp}$ |
|--------------|-----|-------------------|
| $\mathbb{H}^m$ | 0   | 0                 |
| $\mathbb{H}P^m$ | 12  | $4m - 4$         |
| $\mathbb{H}H^m$ | $-12$ | $-(4m - 4)$     |

where $\{e_i\}$ is any orthonormal frame of $\{v, Jv\}^\perp$. We will assign index 1,2 to $v$ and $Jv$ in this summation expression for complex $n$-dimensional Kähler manifold $M^n$, and use indices from 3 to $2n$ for orthonormal frames $\{E_i\}$ of $\{v, Jv\}^\perp$. Denote by $F_i = \frac{1}{\sqrt{2}}(E_i - \sqrt{-1}J(E_i))$ a unitary frame such that $E_1 = v/|v| =: \tilde{v}$ by following the convention $E_{n+i} = J(E_i)$, then

$$\frac{1}{|v|^2} \text{Ric}^{\perp}(v, v) = \text{Ric}^{\perp}(\tilde{v}, \tilde{v}) = \text{Ric}(\tilde{v}, \tilde{v}) - R(\tilde{v}, J\tilde{v}, \tilde{v}, J\tilde{v})$$

$$= \text{Ric}(F_1, \overline{F}_1) - R(F_1, \overline{F}_1, F_1, \overline{F}_1) = \sum_{j=2}^{n} R(F_1, \overline{F}_1, F_j, \overline{F}_j).$$

In particular, we have $\text{Ric}(F_i, \overline{F}_i) = \text{Ric}(E_i, E_i)$, $\text{Ric}^{\perp}(\tilde{v}, \tilde{v}) = \text{Ric}(F_1, \overline{F}_1) - R_{1\overline{1}1\overline{1}}.$

2.2 Invariant metrics

Given any complex manifold $\mathbb{M}$, for each $p \in \mathbb{M}$ and a tangent vector $v$ at $p$, define the Carathéodory–Reiffen metric by

$$\gamma_{\mathbb{M}}(p; v) := \sup \{|df(p)(v)|; f : \mathbb{M} \to \mathbb{D}, f(p) = 0, f \text{ holomorphic}\}.$$

The inner-Carathéodory pseudodistance on $\mathbb{M}$ is defined by

$$c_{\mathbb{M}}^i(x, y) := \inf \{l^F(\sigma)(x, y)\},$$

where the infimums are taken over all piece-wise $C^1$ curves in $\mathbb{M}$ joining $x$ and $y$ and the inner-Carathéodory length of a piecewise $C^1$ curve $\sigma : [0, 1] \to \mathbb{M}$ are given by

$$l^F(\sigma) := \int_0^1 \gamma_{\mathbb{M}}(\sigma, \sigma'),$$
The following relation is true in general:

\[ 0 \leq c_M \leq c_M'. \]

Moreover, \( c_M \) and \( c_M' \) can be different in general (for one such a class of pseudoconvex domains [26]). In such a class of Kähler manifolds, \( c_M \) and the geodesic distance of the Kähler–Einstein metric of Ricci curvature \(-1\) must be different due to the Yau–Royden’s Schwarz lemma (e.g., see [7]).

A classical Schwarz lemma on unit disk \( \mathbb{D} \) in \( \mathbb{C}^1 \) has a very natural geometric interpretation: if \( f : \mathbb{D} \to \mathbb{D} \) is a holomorphic function, then

\[ \rho_{\mathbb{D}}(f(z), f(z')) \leq \rho_{\mathbb{D}}(z, z'), \] for any \( z, z' \in \mathbb{D} \),

which tells us that holomorphic map of the unit disk into itself decreases the Poincaré distance between points. As a consequence, any automorphism must be the isometry with respect to the Poincaré metric. This observation from the Schwarz lemma suggests a natural definition of an invariant metric on a complex manifold \( \mathbb{M} \); that is, for \( F \) be either a Finsler metric or a hermitian metric. We say that \( F \) is an invariant metric if for any (holomorphic) automorphism \( f : M \to M \),

\[ f^*F = F. \]

The Carathéodory–Reiffen metric, the Kobayashi–Royden metric, the complete Kähler–Einstein metric of the negative scalar curvature, and the Bergman metric are well-known examples of invariant metrics on arbitrary complex manifolds.

For example, the Poincaré metric

\[ \omega_p = \frac{\sqrt{-1}}{2} \partial \overline{\partial} \log \left( \frac{n!}{\pi^n} (1 - |z|^2)^{-n+1} \right), \tag{13} \]

which has the negative constant holomorphic sectional curvature \( -\frac{4}{n+1} \) on the \( n \)-dimensional complex hyperbolic space \( H^n(\mathbb{C}) = \{ z \in \mathbb{C}^n : ||z|| < 1 \} \) coincides with those four invariant metrics up to a dimensional constant.

Indeed, for \( \mathbb{M} = H^n(\mathbb{C}) \),

\[ \gamma_{H^n(\mathbb{C})}(p; v) = \frac{||v||^2}{1 - ||p||^2} \left[ \frac{||v||^2}{1 - ||p||^2} + \frac{|(p, v)|^2}{(1 - ||p||^2)^2} \right]^{\frac{1}{2}}, \]

here \( ||.|| \) means the complex Euclidean norm in \( \mathbb{C}^n \) (see [16, p43, Cor 2.3.5]). Thus, \( \gamma_{H^n(\mathbb{C})}(0; v) = ||v|| \). On the other hand, \((i, \bar{j})\)th component of \( \omega_p(0) \) is \((n + 1)\delta_{ij}\), and the automorphism group of \( H^n(\mathbb{C}) \) acts on \( H^n(\mathbb{C}) \) transitively as a group of isometries on both \( \omega_p \) and \( \gamma_{H^n(\mathbb{C})} \); consequently, two metrics are the same up to the constant \( \sqrt{n + 1} \). In particular, the Carathéodory distance coincides with the Poincaré distance on \( H^n(\mathbb{C}) \) up to a dimensional constant.
2.3 Quaternionic Kähler manifolds

Definition 2.2. The manifold $(\mathbb{M}, g)$ is called a \textit{quaternionic Kähler manifold} if there exists a covering of $\mathbb{M}$ by open sets $U_i$ and for each $i$, 3 smooth $(1,1)$ tensors $I, J, K$ on $U_i$ such that

- for every $x \in U_i$, and $X, Y \in T_x\mathbb{M}$, $g_x(I_xX, Y) = -g_x(X, I_xY)$, $g_x(J_xX, Y) = -g_x(X, J_xY)$, $g_x(K_xX, Y) = -g_x(X, K_xY)$;
- for every $x \in U_i$, $I^2_x = J^2_x = K^2_x = I_xJ_xK_x = -\text{Id}_{T_x\mathbb{M}}$;
- for every $x \in U_i$, and $X \in T_x\mathbb{M}$, $\nabla X I, \nabla X J, \nabla X K \in \text{span}\{I, J, K\}$;
- for every $x \in U_i \cap U_j$, the vector space of endomorphisms of $T_x\mathbb{M}$ generated by $I_x, J_x, K_x$ is the same for $i$ and $j$.

On quaternionic Kähler manifolds, we will be considering the following curvatures. As above, let

$$R(X, Y, Z, W) = g((\nabla X \nabla Y - \nabla Y \nabla X - \nabla_{[X,Y]}Z)Z, W)$$

be the Riemannian curvature tensor of $(\mathbb{M}, g)$. We define the quaternionic sectional curvature of the quaternionic Kähler manifold $(\mathbb{M}, g, I, J)$ as

$$Q(X) = \frac{R(X, IX, IX, X) + R(X, JX, JX, X) + R(X, KX, KX, X)}{g(X, X)^2}.$$

We define the orthogonal Ricci curvature of the quaternionic Kähler manifold $(\mathbb{M}, g, I, J)$ for a vector field $X$ such that $g(X, X) = 1$ by

$$\text{Ric}^\perp(X, X) = \text{Ric}(X, X) - Q(X),$$

where Ric is the usual Riemannian Ricci tensor of $(\mathbb{M}, g)$. The table below shows the curvature of the quaternion-Kähler model spaces $\mathbb{H}^m, \mathbb{H}P^m,$ and $\mathbb{H}H^m$, see [3].

3 PROBABILISTIC PRELIMINARIES: COUPLINGS

We gather some basic materials of coupling methods in this section. Let $\{W_t\}_{t \geq 0}$ be stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking value in a measure space $(E, \mathcal{E})$.

Definition 3.1 (Coupling of stochastic processes). A coupling of $\{W_t\}_{t \geq 0}$ is a $(E \times E, \mathcal{E} \times \mathcal{E})$-valued stochastic process $\{(X_t, Y_t)\}_{t \geq 0}$ on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, such that the marginal processes $\{X_t\}_{t \geq 0}$ and $\{Y_t\}_{t \geq 0}$ have the same distribution as $\{Z_t\}_{t \geq 0}$. A coupling is called Markovian if $\{(X_t, Y_t)\}_{t \geq 0}$ is a Markov process with respect to its natural filtration $\mathcal{F}_t = \sigma\{(X_s, Y_s), s \leq t\}$.

We will only consider Markovian couplings in this paper. It is often beneficial to have a coupling with a fixed starting point $(x, y) \in E \times E$. This can be achieved by choosing the underlying probability, which will be denoted as $\mathbb{P}^{x,y}$, so that

$$\mathbb{P}^{x,y}(X_0 = x, Y_0 = y) = 1.$$
Another crucial object in coupling method is the next.

**Definition 3.2.** Consider a Markovian coupling \( \{(X_t, Y_t)\}_{t \geq 0} \). The coupling time is defined as

\[
\tau(X, Y) := \inf \{t > 0 : X_t = Y_t\}.
\]

We assume that \( X_t \) and \( Y_t \) will stick together and move as a single process after the coupling time.

**Definition 3.3** (Successful coupling). A Markovian coupling of two diffusion processes \( \{X_t\}_{t \geq 0} \) is said to be successful if \( \mathbb{P}\{\tau < \infty\} = 1 \).

In terms of the coupling time having a successful coupling is equivalent to saying that

\[
\mathbb{P}\{\tau \geq t\} \xrightarrow{t \to \infty} 0.
\]

## 4 PROOF OF THEOREM 1.1

**Lemma 4.1.** Given a complex manifold \( \mathbb{M} \), for any distinct two points \( x, y \in \mathbb{M} \), there exists the extremal map \( f \in \text{Hol}(\mathbb{M}, \mathbb{D}) \) with respect to the Carathéodory pseudodistance, that is, there exists a holomorphic function \( f : \mathbb{M} \to \mathbb{D} \) satisfying

\[
c_M(x, y) = \rho_{\mathbb{D}}(f(x), f(y)).
\]

**Proof.** Suppose that \( \{f_i\}_{i \geq 1} \) is a sequence of holomorphic functions for which \( \rho_{\mathbb{D}}(f_i(x), f_i(y)) \) converges to \( c_M(x, y) \). Since a family of holomorphic functions from \( \mathbb{M} \) to \( \mathbb{D} \) are all uniformly bounded, by Montel’s theorem [20], \( \{f_i\}_{i \geq 1} \) must be normal. Hence, \( \{f_i\}_{i \geq 1} \) converges to a holomorphic function \( f \) and \( c_M(x, y) = \rho_{\mathbb{D}}(f(x), f(y)) \). \( \square \)

**Proof of Theorem 1.1.** By Lemma 4.1, for any two points in a complex manifold \( \mathbb{M} \), there exists a holomorphic function \( f : \mathbb{M} \to \mathbb{D} \) satisfying \( c_M(x, y) = \rho_{\mathbb{D}}(f(x), f(y)) \). By acting an automorphism of \( \mathbb{D} \) and the fact that Carathéodory distance is invariant under automorphisms, we may assume that \( f(y) = 0 \). From the formula of the Poincaré distance \( \rho_{\mathbb{D}}(a, b) = \frac{|a-b|}{1-\bar{a}b} \), we deduce \( \rho_{\mathbb{D}}(f(x), f(y)) = |f(x)| \).

Since \( f \) is harmonic, we have for any \( t \geq 0 \) that

\[
c_M(x, y) = |f(x)| = |f(x) - f(y)| = |E^{X,Y}[f(X_t) - f(Y_t)]|, \]

\[
= |E^{X,Y}[f(X_t) - f(Y_t)] \cdot 1(\tau(X, Y) > t)|, \]

\[
\leq 2\|f\|_{\infty} E^{X,Y}(\tau(X, Y) > t), \]

\[
\leq 2\|f\|_{\infty} E^{X,Y}(\tau(X, Y) > t). \tag{14}
\]

Equations (3) and (4) follow immediately.

On the other hand, for any holomorphic function \( h : M \to \mathbb{D} \) with \( h(y) = 0 \), we have

\[
c_M(x, y) \geq \rho_{\mathbb{D}}(h(x), h(y)) = |h(x)| = |E^{X,Y}[h(X_t) - h(Y_t)]|, \]

\[
= |E^{X,Y}[f(X_t) - f(Y_t)] \cdot 1(\tau(X, Y) > t)|. \tag{15}
\]

This proves (2). \( \square \)
Proof of Corollary 1.3. On the four-dimensional complex hyperbolic space $\mathbb{D}$, the Carathéodory distance coincides with the Poincaré distance with Gaussian curvature $-1$. Thus,

$$c_{\mathbb{D}}(f(x), f(y)) = \rho_{\mathbb{D}}(f(x), f(y)) = \rho_{\mathbb{D}}(f(x), 0) = |f(x)|,$$

then we can use (14) to get the left-hand side of (5). Similarly,

$$c_{\mathbb{D}}(x, y) = \rho_{\mathbb{D}}(x, 0) = |x|,$$

and apply (14) with the identity map on $\mathbb{D}$ to establish the right-hand side of (5). The rest of conclusion follows from a classical Schwarz lemma on the unit disk. □

Proof of Corollary 1.7. With a positive real valued $u$ with $Lu = 0$, (14) can be changed as follows:

$$|u(x) - u(y)| = |E^{X,Y}[u(X_t) - u(Y_t)]|$$

$$= |E^{X,Y}[u(X_t) - u(Y_t)] \cdot 1(\tau(X, Y) > t)|$$

$$\leq ||u||_\infty P^{X,Y}(\tau(X, Y) > t)$$

$$\leq P^{X,Y}(\tau(X, Y) > t).$$

Now Corollary 1.7 follows from Theorem 1.4, □

5 | INGREDIENTS FOR THE KENDALL–CRANSTON COUPLING

We gather some basic results that we will need in the next section. Let $(\mathbb{M}, g)$ be a complete Riemannian manifold with real dimension $n$ and denote by $d$ the Riemannian distance on $\mathbb{M}$. The index form of a vector field $X$ (with not necessarily vanishing endpoints) along a geodesic $\gamma$ is defined by

$$I(\gamma, X, X) := \int_0^T (\langle \nabla_{\dot{\gamma}} X, \nabla_{\dot{\gamma}} X \rangle - \langle R(\dot{\gamma}, X)X, \dot{\gamma} \rangle)dt,$$

where $\nabla$ is the Levi–Civita connection and $R$ is the Riemann curvature tensor of $\mathbb{M}$.

We will denote by

$$\text{Cut}(\mathbb{M}) := \{(x, y) \in \mathbb{M} \times \mathbb{M}, x \notin \text{Cut}(y)\}.$$

Here, Cut(y) means the cut locus at $y$, and for $(x, y) \notin \text{Cut}(\mathbb{M})$, we denote

$$I(x, y) = \sum_{i=1}^{n-1} I(y, Y_i, Y_i),$$

where $\gamma$ is the unique length parameterized geodesic from $x$ to $y$ and $\{Y_1, \ldots, Y_{n-1}\}$ are Jacobi fields such that at both $x$ and $y$, $\{\gamma', Y_1, \ldots, Y_{n-1}\}$ is an orthonormal frame.
Throughout the paper, we consider the comparison function:

\[
G(k, r) = \begin{cases} 
-2 \sqrt{k} \tan \frac{\sqrt{k}r}{2} & \text{if } k > 0, \\
0 & \text{if } k = 0, \\
2 \sqrt{|k|} \tanh \frac{\sqrt{|k|}r}{2} & \text{if } k < 0.
\end{cases}
\] (15)

5.1 Index comparison theorems

Let \((\mathbb{M}, g, J)\) be a complete Kähler with complex dimension \(m\) (i.e., the real dimension is \(2m\)). The holomorphic sectional curvature of \(\mathbb{M}\) will be denoted by \(H\) and the orthogonal Ricci curvature by \(\text{Ric}^\perp\) as introduced in Section 2.1.

**Theorem 5.1.** Let \(k_1, k_2 \in \mathbb{R}\). Assume that \(H \geq 4k_1\) and that \(\text{Ric}^\perp \geq (2m - 2)k_2\). For every \((x, y) \notin \text{Cut}(\mathbb{M})\), one has

\[
I(x, y) \leq (2m - 2)G(k_2, d(x, y)) + 2G(k_1, 2d(x, y)).
\]

**Proof.** [2, Theorem 2.1].

**Theorem 5.2.** Let \(k_1, k_2 \in \mathbb{R}\). Assume that \(Q \geq 12k_1\) and that \(\text{Ric}^\perp \geq (4m - 4)k_2\). For every \((x, y) \notin \text{Cut}(\mathbb{M})\),

\[
I(x, y) \leq (4m - 4)G(k_2, d(x, y)) + 6G(k_1, 2d(x, y)).
\]

**Proof.** [2, Theorem 2.3].

5.2 Laplacian comparison theorems

We introduce the following function.

\[
F(k, r) = \begin{cases} 
\sqrt{k} \cot \sqrt{k}r & \text{if } k > 0, \\
\frac{1}{r} & \text{if } k = 0, \\
\sqrt{|k|} \coth \sqrt{|k|}r & \text{if } k < 0.
\end{cases}
\] (16)

The following theorems follows from proofs in [3, 22] with a slight modification with \(k_1, k_2\) (also see [9]).

**Theorem 5.3.** Let \(k \in \mathbb{R}\). Assume that \(H \geq 4k_1\) and that \(\text{Ric}^\perp \geq (2m - 2)k_2\). Let \(x_0 \in \mathbb{M}\) and denote \(r(x) = d(x_0, x)\). Then, pointwise outside of the cut-locus of \(x_0\), and everywhere in the sense of distribution, one has

\[
\Delta r \leq (2m - 2)F(k_2, r) + 2F(k_1, 2r).
\] (17)
Theorem 5.4. Let $k \in \mathbb{R}$. Assume that $Q \geq 12k_1$ and that $\text{Ric}^\perp \geq (4m - 4)k_2$. Let $x_0 \in \mathbb{M}$ and denote $r(x) = d(x_0, x)$. Then, pointwise outside of the cut-locus of $x_0$, and everywhere in the sense of distribution, one has

$$\Delta r \leq (4m - 4)F(k_2, r) + 6F(k_1, 2r).$$

(18)

5.3 The Kendall–Cranston coupling on complete Riemannian manifolds

Let $(\mathbb{M}, g)$ be a complete Riemannian manifold. We denote by $O(\mathbb{M})$ the orthonormal frame bundle of $\mathbb{M}$ and $\pi: O(\mathbb{M}) \to \mathbb{M}$ the projection onto $\mathbb{M}$. The Levi–Civita connection gives rise to the horizontal lift

$$H: T\mathbb{M} \to TO(\mathbb{M}).$$

For any $u \in O(\mathbb{M})$ and $e \in \mathbb{R}^d$, $H_v(u)$ is the horizontal lift of $ue \in T_{\pi u}\mathbb{M}$.

For a bounded smooth vector filed $Z$ on $\mathbb{M}$, we consider the horizontal diffusion given by

$$du_t = \sqrt{2} \sum_{i=1}^{d} H_{e_i}(u_t) dB_t + H_Z(u_t) dt, \quad u_0 \in O(\mathbb{M}),$$

where $\{e_i\}_{1 \leq i \leq d}$ is the standard basis of $\mathbb{R}^d$ and $B_t$ is a $d$-dimensional Brownian motion. The projection process $X_t = \pi u_t$ is the diffusion on $\mathbb{M}$ generated by $\frac{1}{2} \Delta + Z$.

The Kendall–Cranston coupling is a way to construct Markovian couplings of $X_t$ on $\mathbb{M}$. Let $(x, y) \in \text{Cut}(\mathbb{M})$ that can be connected by a (unique) minimizing geodesic. We define the “mirror map” $m_{X_t, Y_t} : T_x\mathbb{M} \to T_y\mathbb{M}$ as follows: carry each tangent vector in $T_x\mathbb{M}$ to $T_y\mathbb{M}$ by parallel transport along the unique geodesic between $x$ and $y$, and then reflect it in the hyperplane normal to the geodesic in $T_y\mathbb{M}$.

The Kendall–Cranston coupling $(X_t, Y_t)$ of diffusion generated by $\frac{1}{2} \Delta + Z$ with starting points $(x, y)$ is explicitly given by the following system of Stochastic Differential Equations

$$\begin{align*}
    &du_t = \sqrt{2} \sum_{i=1}^{d} H_{e_i}(u_t) dB_t + H_Z(u_t) dt \\
    &dv_t = \sqrt{2} \sum_{i=1}^{d} H_{e_i}(v_t) dB_t + H_Z(v_t) dt \\
    &dW_t = (v_t^{-1}m_{X_t, Y_t} u_t) dB_t \\
    &X_t = \pi u_t, \quad X_0 = x \\
    &Y_t = \pi v_t, \quad Y_0 = y.
\end{align*}$$

(19)

Note that $W_t$ is another $d$-dimensional Brownian motion due to the fact that $m_{X_t, Y_t}$ is an isometry.

With everything prepared, we can now have the following.

Theorem 5.5 (Theorem 3 in [8]). With the same settings as above, let $\rho_{\mathbb{M}}$ be the geodesic distance on $(\mathbb{M}, g)$. Then, the following inequality holds:

$$d\rho_{\mathbb{M}}(X_t, Y_t) \leq 2\beta_t + I(X_t, Y_t) dt + [\langle Z(Y_t), T_t \rangle - \langle Z(X_t), T_t \rangle] dt,$$

(20)
where \( (\beta_t)_{t \geq 0} \) is a Brownian motion on \( \mathbb{R} \) with its quadratic variation \( \langle \beta \rangle_t = 2t \) and \( T_t \) is the tangent vector to \( \gamma \) and the integral is along the geodesic \( \gamma \).

6  |  KENDALL–CRANSTON COUPLING

With the index form estimates of the previous section in hands, we can use the reflection coupling method by M. Cranston, M. F. Chen, and F. Y. Wang [5, 8, 27] (see also [15, Section 6.7]) to get coupling time estimates.

6.1  |  Kähler case

Proof of Theorem 1.4. Define \( \rho_t \) by

\[
d\rho_t = d\beta_t + (4(n-1)\sqrt{|k_2|} + 4\sqrt{|k_1|} + 2m)dt
\]

for the same Brownian motion appearing in (20) with the initial condition

\[
\rho_0 = \rho_M(x, y).
\]

We have

\[
I(X_t, Y_t) \leq 4(n-1)\sqrt{|k_2|} + 4\sqrt{|k_1|}
\]

from (15) and Theorem 5.1 and

\[
\langle Z(Y_t), T_t \rangle - \langle Z(X_t), T_t \rangle \leq 2m.
\]

Thus, by a comparison theorem, it follows that

\[
\rho_M(X_t, Y_t) \leq \rho_t \quad \text{for all } t > 0 \ \text{a.s.}
\]

(Recall that the integrands in the \( dt \) terms are 0 if \( Y \) is inside of the cut-locus of \( X_t \).) Now if

\[
\sigma_a(\rho_M) \equiv \inf \{ t > 0 : \rho_M(X_t, Y_t) = a \},
\]

then

\[
\tau(X, Y) = \sigma_0(\rho_M)
\]

so that

\[
\tau(X, Y) \leq \sigma_0(\rho)
\]

and

\[
\sigma_2(\rho_M) \geq \sigma_2(\rho) \ \text{a.s. by the above comparison.}
\]
Thus,

\[ P^{x,y}(\tau(X, Y) = \infty) \leq P^{\rho_0}(\sigma(\rho) = \infty). \tag{21} \]

Also,

\[
\begin{align*}
P^{x,y}(\tau(X, Y) > \tau_{B(x, \delta)}(X) \land \tau_{B(x, \delta)}(Y)) \\
= P^{x,y}(\tau(X, Y) > \tau_{B(x, \delta)}(X) \land \tau_{B(x, \delta)}(Y) \land \sigma_{2\delta}(\rho_M)) \\
\leq P^{x,y}(\tau(X, Y) > \tau_{B(x, \delta)}(X) \land \sigma_{2\delta}(\rho_M)) + P^{x,y}(\tau(X, Y) > \tau_{B(x, \delta)}(Y) \land \sigma_{2\delta}(\rho_M)) \\
\leq P^{x,y}(\sigma_0(\rho) \geq \sigma_\delta(\rho_M(X, x) \land \sigma_{2\delta}(\rho)) + P^{x,y}(\sigma_0(\rho) \geq \sigma_\delta(\rho_M(Y, x) \land \sigma_{2\delta}(\rho)). \tag{22} \end{align*}
\]

Focusing first on the inequality (21), define

\[ u(\rho_0) = P^{\rho_0}(\sigma_0(\rho) = \infty). \]

We need the following lemma to estimate \( u(\rho_0) \).

**Lemma 6.1** Lemma 2.1 in [27]. Let \((r_t)_{t \geq 0}\) be the one-dimensional diffusion process generated by \(a \frac{d^2}{dr^2} + b(r) \frac{d}{dr}\), where \(a > 0\) is a constant and \(b \in C^1(\mathbb{R})\). Let \(r_0 > 0\) and \(\tau_0 := \{t \geq 0 : r_t = 0\}\). Let

\[
\begin{align*}
\xi(r) &:= \int_0^r \exp \left[ -\frac{1}{a} \int_0^s b(t) dt \right] ds, \quad r \in \mathbb{R}, \\
c(u) &:= \frac{1}{a} \sup_{t \in [0, u]} \int_0^t b(s) ds, \quad u > 0.
\end{align*}
\]

We have

\[ P(\tau_0 > t) \leq \xi(r_0) \inf_{s > r_0} \left\{ \frac{1}{\xi(s)} + \frac{e^{c(s)}}{\sqrt{a\pi t}} \right\}, \quad t > 0. \tag{23} \]

In the context of the paper, we apply the lemma to \(\rho_t\) generated by \(\frac{1}{2} \frac{d^2}{dr^2} + b \frac{d}{dr}\) and \(\tau_0 = \sigma_0(\rho)\), where

\[ b = 4(n - 1) \sqrt{|k_2|} + 4 \sqrt{|k_1|} + 2m. \]

Then \(\xi(\rho_0) = \frac{a}{b} (1 - e^{-\frac{a}{b} \rho_0})\) and the right hand side of (23) is

\[ \xi(\rho_0) \frac{b}{a} \left( 1 + \frac{1}{(a\pi t)^{\frac{1}{4}}} + \frac{(a\pi t)^{\frac{3}{2}} + 1}{(a\pi t)^{\frac{1}{2}}} \right) \]

for sufficiently large \(t\). By passing \(t \to \infty\), we have

\[ u(\rho_0) \leq \frac{b}{a} \rho_0 = [8((n - 1) \sqrt{|k_2|} + \sqrt{|k_1|}) + 2m] \rho_0, \]
which proves (8). For (7), it suffices to handle one term on the extreme right-hand side of (22), the other being similar. According to Kendall [17], there is a local time term $L^x$, that is, an increasing process supported on $C(X)$ such that

$$
d\rho_M(X_t, x) = d\omega_t + \left(\frac{1}{2} \triangle + Z\right)\rho_M(X_t, x)dt - dL_t^x,
$$

where $\{\omega_t : t \geq 0\}$ is $BM(\mathbb{R}^1)$. If $X_t$ is outside of the cut locus of $x$, by the Laplacian comparison theorem,

$$
\triangle \rho_M(X_t, x) \leq (2m - 2)F(k_1, \rho_M(X_t, x)) + 2F(k_2, 2\rho_M(X_t, x)).
$$

We take $(\frac{1}{2} \triangle + Z)\rho_M(X_t, x) = 0$ for $X_t \in C(x)$. Thus, using the same $BM(\mathbb{R}^1)$, $w$, appearing in (29) to define $\eta_t$ by

$$
d\eta_t = dw_t + ((m - 1)\sqrt{|k_1|} \coth \sqrt{|k_1|} |\eta_t| + \sqrt{|k_2|} \coth 2\sqrt{|k_2|} |\eta_t| + m)dt, \quad \eta_0 = 0.
$$

By comparison again and (16), one has

$$\rho_M(X_t, x) \leq \eta_t \text{ for all } t \geq 0 \text{ a.s.}
$$

(As before, the $dt$ coefficients for $\rho_M(X_t, x)$ are set equal to 0 if $X_t \in C(x)$.) Thus,

$$
\tau_{B(x, \delta)}(X) : = \sigma_{\delta}(\rho_M(X, x)) \geq \sigma_{\delta}(\eta) \text{ a.s.}
$$

Using (25) in the first term on the far right-hand side of (22), it appears that

$$
P^{x,y}(\sigma_0(\rho) \geq \sigma_{\delta}(\rho_M(X, x) \wedge \sigma_{2\delta}(\rho))) \leq P(\rho_0, 0)(\sigma_0(\rho) > \sigma_{\delta}(\eta) \wedge \sigma_{2\delta}(\rho)).
$$

The superscript on the probability on the right means $\rho_0$ is the starting point for $\rho, 0$ for $\eta$. Set $B_1 = 4(m - 1)\sqrt{|k_2|} + 4\sqrt{|k_1|} + m$, $B_2(\eta) = (m - 1)F(k_1, \eta) + F(k_2, 2\eta) + m$. From Itô’s formula, we have

$$
h(\rho_t, \eta_t) = h(\rho_0, 0) + \int_0^t \nabla h(\rho_s, \eta_s)d(2b_s, w_s)
$$

$$
+ \int_0^t (2h_{\rho\rho} + h_{\rho\eta} + B_1 h_{\rho} + B_2(\eta) h_{\eta})ds + \int_0^t 2h_{\rho\eta}(\rho_s, \eta_s)d\langle b, w \rangle_s
$$

for a nice function $h, P^{\rho_0, 0}$ a.s. The right-hand side of (26) may be viewed as $E^{(\rho_0, 0)} h(\rho_t, \eta_t)$, where

$$
\tau = inf \{t > 0 : \rho_t \notin (0, \delta) \text{ or } \eta_t \notin [0, \delta]\},
$$

and $A = \{(2\delta, \eta) : 0 \leq \eta \leq \delta\} \cup \{(-\rho, \delta) : 0 < \rho \leq 2\delta\}$. Thus, $E^{\rho_0, 0} h(\rho_{\tau}, \eta_{\tau})$ may be estimated by selecting a nice $h \geq 0$ for which $h|_A \leq 1_A$ and evaluating $E^{\rho_0, 0} h(\rho_{\tau}, \eta_{\tau})$. 
To this end, define
\[
h(\rho, \eta) = \begin{cases} 
1 & \text{on } A, \\
\rho/2\delta & \text{on } \{(\rho, \eta) : 0 < \rho < 2\delta, 0 < \eta < \delta/2\}, 
\end{cases}
\]
and then extend \(h\) so that
\[
|h_\eta| \vee |h_\rho| \vee \delta|h_{\rho\rho}| \vee \delta|h_{\eta\eta}| \vee \delta|h_{\rho\eta}| \leq \frac{c}{\delta}.
\]
(28)

Note that the choice of \(h\) also resolves the singularity of \(F(k, \eta)\) at \(\eta = 0\) by achieving \(h_\eta = 0\) near \(\eta = 0\). Thus, using (27) and (28), it follows that
\[
P_{\rho_0,0}(\sigma_0 > \sigma_{\delta}(\rho) \wedge \sigma_{2\delta}(\rho)) \leq h(\rho_0, 0) + \frac{c}{\delta^2}E_{\rho_0,0}\tau \\
\leq \left(\frac{c}{\delta} + c(K, d, m)\right)\rho_0.
\]

For \(E_{\rho_0,0}\tau \leq c\rho_0\delta\), we use the reflection property of \(\rho\), the Brownian motion with a drift [17]. The proof is complete. \(\square\)

Remark 6.2. \(H \geq 4k_1\) and \(\text{Ric}^L \geq (2n - 2)k_2\) imply the Ricci lower bound \(\text{Ric} \geq 4k_1 + (2n - 2)k_2\) and we always have
\[
(2n - 2)G(k_2, r) + 2G(k_1, 2r) \leq (2n - 1)G\left(\frac{4k_1 + (2n - 2)k_2}{2n - 1}, r\right)
\]
by concavity of \(k \rightarrow G(k, r)\). Combining with Theorem 5.5, (15) with the same argument in the proof, this reduces to the same coupling result of M. Cranston for Riemannian manifolds with Ricci curvature lower bound [8].

6.2 | Quaternion Kähler case

The proof works as same except modifying \(\rho_t\) by
\[
d\rho_t = d\beta_t + (4(n - 1)\sqrt{|k_2|} + 4\sqrt{|k_1|})dt,
\]
and we apply the lemma to \(\rho_t\) generated by \(\frac{1}{2} \frac{d^2}{dr^2} + b \frac{d}{dr}\) and \(\tau_0 = \sigma_0(\rho)\), where
\[
b = 4(m - 1)\sqrt{|k_2|} + 4\sqrt{|k_1|}.
\]
Then, \(\xi(\rho_0) = \frac{a}{b}(1 - e^{-\frac{a}{b}\rho_0})\) and the right-hand side of (23) is
\[
\xi(\rho_0)\frac{b}{a}\left(1 + \frac{1}{(a\pi t)^{\frac{1}{2}}} + \frac{(a\pi t)^{\frac{1}{2}} + 1}{(a\pi t)^{\frac{1}{2}}}\right)
\]
for sufficiently large $t$. By passing $t \to \infty$, we have

$$u(\rho_0) \leq \frac{b}{a} \rho_0 = 8((m - 1) \sqrt{|k_2|} + \sqrt{|k_1|}) \rho_0,$$

which proves (8). For (7), it suffices to handle one term on the extreme right-hand side of (22), the other being similar. According to Kendall, there is a local time term $L^x$, that is, an increasing process supported on $C(X)$ such that

$$d\rho_M(X_t, x) = dw_t + \left(\frac{1}{2} \triangle + Z\right) \rho_M(X_t, x)dt - dL^x_t,$$  \hspace{1cm} (29)

where $\{w_t : t \geq 0\}$ is $BM(\mathbb{R}^1)$. If $X_t$ is outside of the cut locus of $x$, by the Laplacian comparison theorem,

$$\triangle \rho_M(X_t, x) \leq (2m - 2)F(k_1, \rho_M(X_t, x)) + 2F(k_2, 2\rho_M(X_t, x)).$$

We take $(\frac{1}{2} \triangle + Z)\rho_M(X_t, x) = 0$ for $X_t \in C(x)$. Thus, using the same $BM(\mathbb{R}^1)$, $w$, appearing in (29) to define $\eta_t$ by

$$d\eta_t = dw_t + ((m - 1)\sqrt{|k_1|} \coth \sqrt{|k_1|} \eta_t + \sqrt{|k_2|} \coth 2\sqrt{|k_2|} \eta_t + m)dt, \hspace{1cm} \eta_0 = 0.$$

By comparison again and (16), one has

$$\rho_M(X_t, x) \leq \eta_t \text{ for all } t \geq 0 \text{ a.s.}$$

7 COUPLINGS ON $H^2(\mathbb{C})$

In this section, we study Markovian couplings of Brownian motions on $H^2(\mathbb{C})$ as an example. It is known that properties of these couplings are closely related to the harmonic functions. We state the next result and refer to [4, page 38] for a proof on Euclidean space; generalization to Riemannian manifolds is straightforward.

**Proposition 7.1.** Let $\mathbb{M}$ be a Riemannian manifold. If there exists successful coupling of Brownian motions for any pair of starting points $(x, y) \in \mathbb{M} \times \mathbb{M}$, then any bounded harmonic function on $\mathbb{M}$ must be constant.

We know that there are bounded nonconstant harmonic functions on $H^2(\mathbb{C})$, so by previous proposition, we cannot have successful coupling of Brownian motions for all different starting points. In fact, we can get a stronger result and show that all Markovian couplings with different starting points are unsuccessful.

**Proposition 7.2.** Let $(X_t, Y_t)$ be a Markovian coupling of Brownian motions on $H^2(\mathbb{C})$ with starting point $(x, y)$. If $x \neq y$, then the coupling is not successful. In particular,

$$\mathbb{P}(\tau = \infty) \geq \frac{\rho^2(x, y)}{4},$$

where $\rho(x, y)$ is the Euclidean distance between $x$ and $y$. 
We prepare some materials regarding Brownian motions on $H^2(\mathbb{C})$. It was proved in [10] that a Brownian motion on $H^2(\mathbb{C})$ satisfies the following SDE:

$$dX_1(t) = 2\sqrt{1 - |X(t)|^2} \left( 1 - \frac{|X_1(t)|^2}{1 - \sqrt{1 - |X(t)|^2}} \right) dB_1(t) - \frac{X_1(t)X_2(t)}{1 - \sqrt{1 - |X(t)|^2}} dB_2(t),$$

$$dX_1(t) = 2\sqrt{1 - |X(t)|^2} \left( \frac{-\overline{X_1(t)}X_2(t)}{1 - \sqrt{1 - |X(t)|^2}} dB_1(t) + \frac{1 - |X_1(t)|^2}{1 - \sqrt{1 - |X(t)|^2}} dB_2(t) \right),$$

where $|X(t)|^2 = |X_1(t)|^2 + |X_2(t)|^2$. By identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ through the canonical isomorphism, $X_t$ can be regarded as a diffusion in $\mathbb{R}^4$ given by

$$dX_t = A(X_t) dB_t, \quad (30)$$

where $B_t$ is a standard Brownian motion in $\mathbb{R}^4$ and for $x = (a_1, b_1, a_2, b_2) \in \mathbb{R}^4$, and $A(x)$ is given by

$$A(x) = 2\sqrt{1 - |x|^2} \left( \begin{array}{cccc} 1 - \frac{a_1^2 + b_1^2}{1 + \sqrt{1 - |x|^2}} & 0 & -(a_1a_2 + b_1b_2) & b_1a_2 - a_1b_2 \\ 0 & 1 - \frac{a_1^2 + b_1^2}{1 + \sqrt{1 - |x|^2}} & a_1a_2 - b_1b_2 & -a_1a_2 + b_1b_2 \\ -(a_1a_2 + b_1b_2) & a_1a_2 - b_1b_2 & 1 - \frac{a_2^2 + b_2^2}{1 + \sqrt{1 - |x|^2}} & 0 \\ b_1a_2 - a_1b_2 & -a_1a_2 + b_1b_2 & 0 & 1 - \frac{a_2^2 + b_2^2}{1 + \sqrt{1 - |x|^2}} \end{array} \right).$$

The symmetric matrix $A(x)$ is positive definite with eigenvalues $2\sqrt{1 - |x|^2}$, $2(1 - |x|^2)$, both of multiplicity 2. But it gets closer to being degenerate as $|x|$ approaches 1.

If $(X_t, Y_t)$ is a Markovian coupling of Brownian motions, it must admit a generator of the form

$$L = \frac{1}{2} \sum_{i,j=1}^{8} a(x,y)_{i,j} \frac{\partial^2}{\partial i \partial j},$$

where $a(x,y) \in \mathbb{R}^{8 \times 8}$ is a symmetric matrix given by

$$\begin{pmatrix} A(x)A^T(x) & C(x,y) \\ C^T(x,y) & A(y)A^T(y) \end{pmatrix},$$
for some $C(x, y) \in \mathbb{R}^{4 \times 4}$. More precisely, one has

$$C(x, y) = A(x) \cdot \frac{d(B_t, W_t)}{dt} \big|_{(x, y)} \cdot A^T(y),$$

where $\langle B_t, W_t \rangle$ denotes the covariance process of the underlying Brownian motions of $X_t$ and $Y_t$. By standard properties of diffusion generators, $a(x, y)$ must be nonnegative definite in $H^2(\mathbb{C}) \times H^2(\mathbb{C}) \subset \mathbb{R}^8$. As a result,

$$\hat{A}(x, y) = A(x)A^T(x) + A(y)A^T(y) - 2C(x, y)$$

is also nonnegative definite. In particular, $Tr(\hat{A}(x, y)) > 0$.

**Proof of Proposition 7.2.** It is readily checked that

$$L\rho^2(X_t, Y_t) = Tr(\hat{A}(x, y)).$$

We define a sequence of stopping times

$$\tau_n(X_t, Y_t) = \inf \{t \geq 0 : \rho(X_t, Y_t) \leq \frac{1}{n}\}, \quad n \in \mathbb{N}^+.$$ 

Obviously, $\tau(X_t, Y_t) = \lim_{n \to \infty} \tau_n(X_t, Y_t)$. By Itô’s formula,

$$\mathbb{E}\rho^2(X_{\tau_n \wedge t}, Y_{\tau_n \wedge t}) = \rho^2(X_0, Y_0) + \mathbb{E} \int_0^{\tau_n \wedge t} Tr(\hat{A}(X_s, Y_s)) ds.$$

The above equation can be recast as

$$\mathbb{E} (\mathbb{1}(\tau_n > t)\rho^2(X_t, Y_t)) + \frac{1}{n^2} \mathbb{P}(\tau_n < t) = \rho^2(X_0, Y_0) + \mathbb{E} \int_0^{\tau_n \wedge t} Tr(\hat{A}(X_s, Y_s)) ds.$$

Since $X_t, Y_t$ never leave $H^2(\mathbb{C}) \subset \mathbb{R}^4$ and $Tr(\hat{A}(X_s, Y_s))$ is nonnegative, we deduce

$$4\mathbb{P}(\tau_n > t) + \frac{1}{n^2} \mathbb{P}(\tau_n < t) \geq \rho^2(X_0, Y_0).$$

Sending $n \to \infty$ gives

$$4\mathbb{P}(\tau > t) \geq \rho^2(X_0, Y_0) \quad \text{for all } t \geq 0,$$

and we conclude

$$\mathbb{P}(\tau = \infty) \geq \frac{\rho^2(X_0, Y_0)}{4} = \frac{\rho^2(x, y)}{4} > 0.$$

Thus, the coupling $(X_t, Y_t)$ is not successful.
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ORCID
Gunhee Cho https://orcid.org/0000-0002-8474-4749

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