On Yang-Mills Stability Bounds and Plaquette Field Generating Function

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We consider the Yang-Mills (YM) QFT with group $U(N)$. We take a finite lattice regularization $\Lambda \subset a\mathbb{Z}^d$, $d = 2, 3, 4$, with $a \in (0, 1]$ and $L$ (even) sites on a side. Each bond has a gauge variable $U \in U(N)$. The Wilson partition function is used and the action is a sum of gauge-invariant plaquette (minimal square) actions times $a^{d-4}/g^2$, $g^2 \in (0,g_0^2]$, $0 < g_0^2 < \infty$. A plaquette action has the product of its four variables and the partition function is the integral of the Boltzmann factor with a product of $U(N)$ Haar measures. Formally, when $a \searrow 0$ our action gives the usual YM continuum action. For free and periodic b.c., we show thermodynamic and stability bounds for a normalized partition function of any YM model defined as before, with bound constants independent of $L$, $a$, $g$. The subsequential thermodynamic and ultraviolet limit of the free energy exist. To get our bounds, the Weyl integration formula is used and, to obtain the lower bound, a new quadratic global upper bound on the action is derived. We define gauge-invariant physical and scaled plaquette fields. Using periodic b.c. and the multi-reflection method, we bound the generating function of $r$-scaled plaquette correlations. A normalized generating function for the correlations of $r$ scaled fields is absolutely bounded, for any $L$, $a$, $g$, and location of the external fields. From the joint analyticity on the field sources, correlations are bounded. The bounds are new and we get $a^{-d}$ for the physical two-plaquette correlation at coincident points. Comparing with the $a \searrow 0$ singularity of the physical derivative massless scalar free field two-point correlation, this is a measure of ultraviolet asymptotic freedom in the context of a lattice QFT. Our methods are an alternative and complete the more traditional ones.

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I. INTRODUCTION

To show the existence and properties of an interacting relativistic quantum field theory (QFT) in spacetime dimension four is a fundamental problem in physics [1–4]. Many partial results have been obtained [4–7]. The quantum chromodynamics model (QCD) of interacting (anti)quarks and gauge, gluon fields is considered to be the best candidate for a four dimensional QFT model which rigorously exists. The action of this model is a sum of an interacting Fermi-gauge field part and a pure-gauge, self-interacting Yang-Mills (YM) field part.

In this paper, we will focus only on the pure-gauge YM model. In an imaginary-time functional integral formulation, a hypercubic lattice ultraviolet regularization $\Lambda$ is used. $\Lambda$ has $L \in \mathbb{N}$, $L$ even, sites on a side. The starting point is the Wilson plaquette action partition function. Stability bounds (see [8]) for the corresponding partition function have been proved in the seminal work of Balaban (see [4, 10] and Refs. therein), using renormalization group (RG) methods and the heavy machinery of multiscale analysis. Applying RG methods in the continuum spacetime and using momentum slices, the ultraviolet limit of the YM model in $d = 4$ with an infrared cutoff was treated in Ref. [11]. Using softer methods, in Ref. [12], the $d = 2$ YM model was solved exactly. It is expected that partition function stability bounds of Refs. [4, 10] lead to bounds on field correlations. Indeed, in the context of the RG, considering models which are small perturbations of the free field, the generating function of field correlations and the correlations can be obtained through a formula which involves the effective actions generated applying the RG transformations to the partition function (see e.g. [13]). However, unfortunately, in the case of gauge fields, this question, as well as the incorporation of fermion fields and the verification of the Osterwalder-Schrader-Seiler axioms [4], have never been completely analyzed up to now, for $d = 4$.

Recently, in the unpublished papers [14, 15], a simple proof of thermodynamic and ultraviolet stable (TUV) stability bounds is given by a direct analysis of the Wilson partition function with free boundary conditions (b.c.) in configuration space, starting with the model in a finite hypercubic lattice. The gauge group is taken as $G = U(N)$ or

\[ a_{\text{even}}(\mathbb{Z}, \ldots) \subset a_{\text{odd}}(\mathbb{Z}, \ldots). \]

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SU(\(N\)). For each lattice bond there is a bond variable \(U\) which is an element of the gauge group \(G\). The fields are elements of the Lie algebra. Also, by the spectral theorem, as \(U\) is unitary, there exists a unitary \(V\) which diagonalizes \(U\), i.e. \(V^{-1}UV = \text{diag}(e^{i\lambda_1}, \ldots, e^{i\lambda_N})\), \(\lambda_j \in (-\pi, \pi]\). The \(\lambda_j\) are called the angular eigenvalues of \(U\). A plaquette is a minimal square of the lattice and the action is a sum of plaquette actions. Each plaquette action involves the ordered product of the four bond variables comprising the sides of the plaquette.

The partition function is an integral over the Boltzmann factor (exponential of minus the action), with a product measure of \(U(\mathbb{N})\) Haar measures, one measure for each bond. Each plaquette action has a prefactor \(a^{d-4}/g^2\), where we take \(g^2 \in (0, g_0^2]\), \(0 < g_0 < \infty\). Our results hold for \(g^2\) in this range! Therefore, they are not restricted to small \(g^2\).

The hypercubic lattice \(\Lambda \subset a\mathbb{Z}^d\), \(d = 2, 3, 4\), with spacing \(a \in (0, 1]\) has \(L \in \mathbb{N}\) sites on each side, \(L\) even. In [14], the finite lattice partition function with free b.c. is denoted by \(Z_{\Lambda,a}\). These results were extended also to periodic b.c. by considering the partition function \(Z_{\Lambda,0}\), where \(\Lambda\) is left blank, for free b.c., and \(\Lambda = P\), for periodic b.c. A complete description of the Wilson action model is given in section [11]. We emphasize that the formal continuum limit \((a \searrow 0)\) of our Wilson action gives the well known YM classical continuum action. It is also worth noticing that the fixing of the enhanced temporal gauge is instrumental in this work. In this gauge, the temporal bond variables in \(\Lambda\) are set to the identity, as well as certain specified bond variables on the boundary \(\partial\Lambda\) of \(\Lambda\).

The upper and lower stability bounds we obtain have an interesting structure. They are both products of single-plaquette, single-bond variable partition functions. A new, global quadratic upper bound in the gluon fields, for the Wilson plaquette action, is proved. This bound gives rise to the factorized lower bound on the partition function. We denote by \(z_a(\zeta)\) the single-bond Haar integral partition functions describing the single-plaquette partition function for the upper (lower) stability bound on the partition function with periodic b.c. The integrands of \(z_a\) and \(z_\zeta\) are both class functions of the single variable \(U\), where we recall that a class function \(f(U)\) on the gauge group \(G\) satisfies the property \(f(U) = f(VUV^{-1})\), for all \(V \in G\). Thus, by Weyl’s integration formula [16–18], the \(N^2\)-dimensional (for \(G = U(\mathbb{N})\)) Haar integration over the group is reduced to a \(N\)-dimensional integration over the angular eigenvalues of \(U\). The probability density of the circular unitary ensemble (CUE) occurs in and the bounds on \(z_a\) and \(z_\zeta\) the probability density for the Gaussian unitary ensemble (GUE) of random matrix theory appears in a natural way (see Refs. [19, 20]).

Associated with these classical statistical mechanical model partition functions and its gauge-invariant correlations, there is a lattice quantum field theory. The Osterwalder-Seiler construction provides, via a Feynman-Kac formula, a quantum mechanical Hilbert space, self-adjoint mutually commuting spatial momentum operators and a positive energy operator. A key property in the construction is Osterwalder-Seiler reflection positivity, which is ensured here by choosing \(L\) to be even! (see Ref. [4]).

It is to be emphasized that the work of Ref. [14] concentrated only on the existence of a finite normalized free energy for the model, in the (subsequential) thermodynamic and continuum limits, respectively, \(\Lambda \not\rightarrow a\mathbb{Z}^d\) and \(a \searrow 0\). No other property of the model was considered. It is also worth noticing that the techniques and methods used in Ref. [14], combined with the results of Refs. [21, 22] could be used to prove the existence of a normalized free energy for a bosonic lattice QCD model, with the (anti)quark fields replaced with spin zero, multicomponent complex or real scalar fields. This is the content of Ref. [24].

In this paper, we also consider correlations of gauge invariant physical plaquette fields. As mentioned above, the continuum limit \(a \searrow 0\) of these fields are the usual continuum fields associated with the formal continuum limit of the Wilson YM action. To analyze these plaquette field correlations, we pass to globally scaled plaquette correlations where the scaled fields are related to the physical fields by a multiplicative factor which depends on the lattice spacing \(a\) but not on the position of the plaquette. The scaled field plaquette correlations are proved to be bounded, uniformly in \(a \in (0, 1]\). These bounds imply bounds on the singular behavior, in \(a\), of the physical plaquette correlations. For example, the bound implies that the physical plaquette-plaquette correlation has a singularity of at most \(a^{-d}\), when \(a \searrow 0\).

It is important to remark that the exponential decay of physical field correlations is the same as that of scaled field correlations. Hence, the associated energy-momentum spectrum is also the same.

Rather than bound the scaled plaquette correlations directly, we bound the generating function of \(r\)-scaled \((r \in \mathbb{N})\) field plaquette correlations using the multi-reflection method (see Ref. [4]). Using periodic b.c., based on the work of Ref. [22], we define a normalized generating function for the correlation of \(r \in \mathbb{N}\) gauge invariant scaled plaquette fields. The numerator is the periodic b.c. partition function with \(r\) additional source factors of strengths \(J_j, j = 1, \ldots, r\); the denominator is the periodic b.c. partition function \(Z_{\Lambda,0}^{r}\). Starting with the model with periodic b.c. which allows us to apply the multi-reflection method, in Theorem 4 below, we prove that this normalized generating function is absolutely bounded, with a bound that is independent of \(L, a, g,\) and the location and orientation of the \(r\) external plaquette fields. The generating function bound also has an interesting structure. The bound has only a product of single-plaquette, single bond-variable partition function \(z_a(J)\) with a source strength field \(J\) in the numerator; in the denominator only a product of \(z_\zeta\) (the same as in the preceding case!) occurs. In the bound for \(z_a(J)\) the probability density for the Gaussian symplectic ensemble appears (see [20]). The generating function is jointly...
analytic, entire function in the source strengths $J_1, \ldots, J_r$ of the $r$ plaquette fields. The $r$-plaquette field correlations admit a Cauchy integral representation and are bounded by Cauchy bounds. In particular, the coincident point plaquette-plaquette physical field correlation is bounded by $\text{const} \cdot a^{-d}$. The $a^{-d}$ factor at small $a$ behavior is the same as that of the physical or unscaled real derivative scalar free field two-point correlation (the physical free field correlation has a singular behavior $a^{-(d-2)}$). For the free field, these singular behaviors are a measure of ultraviolet asymptotic freedom, in the context of the lattice approximation to a continuum QFT.

In this way, we conclude that the singular behavior of the plaquette correlations is bounded by the singular behavior of the free derivative scalar field correlations in $d = 2, 3, 4$. For the physically relevant $d = 4$ case, we can say more. The coincident plaquette, physical plaquette-plaquette correlation is exactly $a^{-d} h(g)$, for some function $h(g)$ which is bounded.

For the free physical scalar field, locally scaled field correlations are bounded uniformly in $a \in (0, 1]$, such as no smearing of the fields is needed to achieve boundedness. The two-point correlation of physical fields for coincident points has an $a^{2-d}$ singular behavior, for $d = 3, 4$. If we consider correlations of physical derivative scalar fields, then the singular behavior is different. The two-point correlation of physical derivative scalar fields, at coincident points, has an $a^{-d}, a \searrow 0$ singularity, for $d = 2, 3, 4$; for the massless case the exact value is $a^{-d}/d$.

The relation between physical field or derivative scaled field quantities is developed in the Appendix, for the free scalar fields.

In this paper, we show detailed and much simplified proofs of the Theorems of Refs. [14, 15]. Besides these simplifications, and in order to make clear how our results are obtained, we incorporate an analysis of the special case of the abelian gauge group $U(1)$. For this group, the Haar measure is simpler as compared with $U(N > 1)$ and computations can be carried out more explicitly and transparently. We emphasize that the independence of our results on $a \in (0, 1]$ is already manifest in this case and the reader can better appreciate why this holds true.

For both, free and periodic b.c., our TUV stability bounds on the normalized partition functions (defined by extracting the $a \searrow 0$ singularity) lead to at least the existence of the subsequential thermodynamic and ultraviolet limits of the corresponding scaled free energies per effective degree of freedom. The existence of these subsequential continuum limits apply to any gauge model with the same Wilson action and free/periodic b.c.

This family of models, of course, encompasses both the trivial ultraviolet limit of a YM model as well as the nonabelian gauge models which are ultraviolet asymptotically free in $d = 4$, like YM and QCD. We show that the $a \searrow 0$ singularity we obtain, for $d = 4$, is compatible with these two types of models. However, we shall make clear that more work has to be done to prove their existence and better characterize their limiting models. Our method is not to be taken as a candidate to replace the well known multiscale analysis based on the RG. Indeed, both methods can be used together to accomplish more substantial progress in the field.

As our method is different from the multiscale analysis of the renormalization group, we give a brief description of it. We can describe our method as a change of field variables or a transformation of fields. The action and the configuration measure are transformed to new ones while the value of the partition function is unchanged. This is in contrast with the RG method, where the partition function is constant but there is a flow of the action. The flow is generated by successively integrating out fields with support on slices of high momentum scales. At each step, an effective action is generated which represents the contribution of the remaining lower momentum scales.

The transformation of field variables we consider is a site independent multiplication by a scaling factor, where the factor depends on the lattice spacing $a$, as well as on other model parameters. Concerning the functional integral appearing in the model definition, we choose our scaling factor so that the model action and field measure is more regular and amenable to analysis. We then analyze the transformed field partition function and generating functions directly.

The effect of field transformations on the generating functions or correlations is to make the transformed correlations more regular when $a \searrow 0$. For instance, correlations in the new fields are finite, independent of the lattice spacing $a$. In particular, they may become finite at coincident points.

Using the relation between the original and the transformed field correlations, we can obtain information on the singular behavior of the original field correlations. As we emphasized before, this is not a substitute to the RG but does give, at least, a simple way to obtain TUV and generating function bounds.

The paper is organized as follows. In Section II we define the model with the Wilson action for periodic and free b.c. In Section III we define and treat an approximate model. In the approximate model, we set to zero, in the Wilson action, plaquette actions corresponding to interior horizontal plaquettes (i.e., plaquettes orthogonal to the time direction), plus some specified plaquettes on the boundary $\partial \Lambda$ of $\Lambda$. Next, by a judicious integration procedure, we carry out all the remaining gauge bond variable integrations. In each integration, a factor is extracted which is a plaquette partition function depending only on a single bond variable. In this way, we obtain explicit and exact results for the approximate model partition function, free energy and plaquette correlations, as well as their continuum limits, in subsections III A, III B and III C. For the complete model, TUV stability bounds and bounds for the generating functions for the gauge invariant plaquette correlations are given in sections IV and V as our four main theorems.
These theorems are proved in section VII. Section VII is devoted to some concluding remarks. Finally, in the Appendix, considering the case of the real scalar free field $\phi$, we develop the relation between quantities expressed in terms of the unscaled or physical field $\phi^u(x)$ and locally scaled fields $\phi(x) = s(a) \phi^u(x)$. Comparing the $a \sim 0$ behavior of the free scalar case with the physical field coincident-point plaquette-plaquette correlation give us a measure of ultraviolet asymptotic freedom.

II. THE WILSON ACTION MODEL

We describe the partition function of the free and periodic b.c. models and their gauge invariance properties. The superscript $P$ will denote periodic b.c. quantities. For the lattice $\Lambda$, we denote by $\Lambda_\mu = L^d$ the total number of lattice sites. We let $x = (x^0, \ldots, x^{d-1})$ denote a site, and $x^0$ is the time direction.

**Free b.c. Bonds:** Let $e_i^\mu, \mu = 0, 1, \ldots, (d-1)$ denote the unit vector in the $\mu$-th Euclidean spacetime direction. $b_\mu(x)$ is the lattice bond with initial point $x$ and terminal point $x_\mu^a = x + ae^\mu \in \Lambda$. The number of free b.c. bonds in the lattice $\Lambda$ is $\Lambda_p = d(L-1)L^{d-1}$. Sometimes, we refer to the bonds in the time direction $x^0$ as vertical bonds. The other bonds are called horizontal.

**Periodic b.c. Bonds:** In addition to the above free b.c. lattice bonds, here, we have additional or extra bonds. An extra bond has initial point at the extreme right lattice site and terminal point at the extreme left lattice site, in each coordinate direction. If $\Lambda_e$ denotes the number of extra bonds, we have $\Lambda_e = dL^{d-1}$. The total number bonds in $\Lambda$ with periodic b.c. (henceforth called periodic bonds) is $\Lambda_p = \Lambda_\mu + \Lambda_e$.

**Free b.c. Plaquettes (Minimal Lattice Squares):** For $\mu, \nu = 0, \ldots, (d-1)$, let $p_{\mu\nu}(x)$ denote a plaquette in the $\mu\nu$-plane, with $\mu < \nu$ and with vertices at sites $x, x + ae^\mu, x + ae^\mu + ae^\nu, x + ae^\nu$ of $\Lambda$. These are the free plaquettes.

**Periodic b.c. Plaquettes:** In addition to the free b.c. lattice plaquettes, there are also extra plaquettes formed at least with one extra bond. The periodic b.c. plaquettes are comprised of all plaquettes that can be formed from the totality of periodic b.c. bonds. We denote the total number of free (periodic) plaquettes by $\Lambda_p (\Lambda_p^p)$. We have, $\Lambda_p = \Lambda_r$, for $d = 2; \Lambda_p \simeq 3L^3, 6L^2$, respectively, for $d = 3, 4$. $\Lambda_p^p$ is given by $\Lambda_p$ plus the number of boundary plaquettes.

Recalling that $a \in (0,1]$ and $g^2 \in (0, g_0^2)$, $0 < g_0^2 < \infty$, and letting $B = \text{blank or } P$, to denote free and periodic b.c., respectively, we represent the model partition function, with $B$-type b.c., by

$$Z_{\Lambda, a}^B = \int \exp \left[ -\frac{\alpha^{d-4}}{g^2} A^B \right] dg^2. \tag{1}$$

Here, for each lattice bond $b$, we assigned a unitary matrix $U \in U(N)$. These are the gauge bond variables. The measure $dg^2$ is the product over bonds $b$ of the single-bond gauge group Haar measures $d\sigma(U)$. For $p$ denoting any fixed plaquette, the model action is given by

$$A^B = \sum_p A_p, \tag{2}$$

where the four bond variable plaquette actions $A_p$ and where the sum $\sum_p$ is over plaquettes with the b.c. of type $B$.

To define $A_p$, we first recall some important facts about unitary matrices and their representation in terms of elements of the Lie algebra of self-adjoint matrices associated with the gauge group $G$. For an $N \times N$ matrix $M$ the Hilbert-Schmidt norm is $\|M\|_{\text{H-S}} = \|Tr(MM^\dagger)\|^{1/2}$, where $M^\dagger$ is the adjoint of $M$. Let $M_1$ and $M_2$ be $N \times N$ matrices. Then $(M_1, M_2) = Tr(M_1^\dagger M_2)$ is a sesquilinear inner product. We also have the following properties:

1. Let $X$ be a self-adjoint matrix. Define $\exp(iX)$ by the Taylor series expansion of the exponential. Then $\exp(iX)$ is unitary.

2. Given a unitary $N \times N$ matrix $U$, by the spectral theorem, there exists a unitary $V$ such that $V^{-1}UV = \text{diag}(e^{i\lambda_1}, \ldots, e^{i\lambda_N}), \lambda_j \in (-\pi, \pi]$. The $\lambda_j$ are the angular eigenvalues of $U$. Define $X = V^{-1}\text{diag}(\lambda_1, \ldots, \lambda_N)V$. Then, $X$ is self-adjoint, $U = \exp(iX)$, and the exponential map is onto (see [17]).

3. For $\alpha = 1, 2, \ldots, N$, let the self-adjoint $\theta_\alpha$ form a basis for the self-adjoint matrices (the $U(N)$ Lie algebra generators), with the normalization condition $Tr\theta_\alpha \theta_\beta = \delta_{\alpha\beta}$, with a Kronecker delta. Then, with $X$ being an $N \times N$ self-adjoint matrix, $X$ has the representation $X = \sum_{1 \leq \alpha \leq N^2} x_\alpha \theta_\alpha$, with $x_\alpha = \text{Tr} \theta_\alpha$, for $x_\alpha$ real.

4. For $U$ and $X$ related as in item 2, we have the important inequality:

$$\|X\|^2_{\text{H-S}} = \text{Tr} (X^\dagger X) = \sum_{1 \leq \alpha \leq N^2} |x_\alpha|^2 = |x|^2 = \sum_{1 \leq \alpha \leq N^2} \lambda_j^2 \leq N\pi^2, \quad \lambda_j \in (-\pi, \pi].$$

Thus, the exponential map is onto, for $|x| \leq N^{1/2}\pi$. 


For each bond $b$, we assign the gauge bond variable $U \in U(N)$. If we parametrize $U$ as $e^{i A_b}$, with $A_b$ self-adjoint, we call $A_b$ the physical gluon field associated with bond $b$. The physical gluon field $A_b$ has the representation $A_b = \sum_{\alpha=1, \ldots, N^2} A^b_{\alpha} \theta_{\alpha}$, and we refer to $A^b_{\alpha}$, $\alpha = 1, \ldots, N^2$, as the color or gauge components of $A_b$. If the plaquette $p$ is $p_{\mu \nu}(x)$, located in the $\mu \nu$ coordinate plane, define

$$U_p = e^{i A_p(x)} e^{i A_p(x+ae_{\mu})} e^{-i A_p(x+ae_{\mu})} e^{-i A_p(x)}.$$  

The plaquette action $A_p$ for the plaquette $p$ is defined by

$$A_p = ||U_p - 1||^2_{H-S} = 2 Re \text{Tr} (1 - U_p) = 2 Tr (1 - \cos X_p) = Tr (2 - U_p - U_p^\dagger),$$  

where $U_p = e^{i X_p}$. Obviously, $A_p$ is pointwise positive (nonnegative) and so is the total action for the model $A^B = \sum_p A_p$. For concreteness, we give the case of the gauge group $U(2)$ as an example. Here, $X = \sum_{\alpha=1, \ldots, 4} x_{\alpha} \theta_{\alpha}$, with $\text{Tr} \theta_{\alpha} \theta_{\beta} = \delta_{\alpha\beta}$, with a Kronecker delta, and, for $\sigma_j$, for $j = 1, 2, 3, 4$ being, respectively, the three $2 \times 2$ traceless and hermitian Pauli spin matrices and the $2 \times 2$ identity matrix $I$, we take $\theta_j = \sigma_j/\sqrt{2}$.

This completes the description of the model. Using the Baker-Campbell-Hausdorff formula, formally, it is shown in Ref. 2, for small lattice spacing $a > 0$, that

$$U_p = \exp \left[ i a^2 g F_{\mu \nu}^a(x) + R \right], \quad R = O(a^3),$$  

where $F_{\mu \nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + i g [A_\mu^a(x), A_\nu^a(x)]$ is the self-adjoint usual color 'electromagnetic' second order antisymmetric tensor with finite difference derivatives

$$\partial_\mu A_\nu^a(x) = a^{-1} \{ A_\nu(x + ae_\mu) - A_\nu(x) \},$$  

e etc, and $[\cdot, \cdot]$ denotes the Lie algebra commutator (Lie product) associated with the gauge group $G = U(N)$. Also it is shown that $A_p \simeq a^2 g^2 \text{Tr} \left[ F_{\mu \nu}^a(x) \right]^2$.

Each term in $[F_{\mu \nu}^a(x)]$ is self-adjoint. Hence, the square is a self-adjoint and positive matrix, and its trace is positive. The quantity $[a^{d-4}/g^2] \sum_p A_p$ is the Riemann sum approximation to the classical smooth field continuum YM action $\int \text{Tr} (F_{\mu \nu}^a(x))^2 d^d x$, when $\Lambda \gg aZ^d$ and after $a \downarrow 0$, formally, and the finite difference derivatives become ordinary partial derivatives.

We now discuss gauge invariance and gauge fixing. We take the global group as $\prod_{x \in \Lambda} G_x$, where $G_x = U_x \in G = U(N)$. It transforms the bond variables $U_p \rightarrow U_p e^{i X_p} x$ and its adjoint $U_p^\dagger \rightarrow U_p^\dagger e^{-i X_p} x$. The plaquette action and the total action are invariant under this gauge transformation. Due to the local gauge invariance of the action $A_p$, and so also $A = \sum_p A_p$, there is an excess of gauge variables in the definition of the partition function given by Eq. 1. By a gauge fixing procedure we eliminate gauge variables by setting them equal to the identity in the action and dropping the gauge bond variable integration. In this process of gauging away some of the gauge group, bond variables, the value of the partition function is unchanged, as long as we apply this procedure to bonds which do not form a closed loop in $\Lambda$ (see [4]). We will choose to work with what we call the enhanced temporal gauge. This gauge will be fixed to prove our main theorems.

In the enhanced temporal gauge, the temporal bond variables in $\Lambda$ are set to the identity, as well as certain specified bond variables on the boundary $\partial \Lambda$ of $\Lambda$. Letting $\Lambda_r$ denote the number of retained bonds, for free b.c., we have $\Lambda_r = \lfloor (L-1)/2 \rfloor$, $\lfloor (2L+1)(L-1)/2 \rfloor$, $\lfloor (3L^3 - L^2 - L - 1)(L-1) \rfloor$, respectively, for $d = 2, 3, 4$. Clearly $\Lambda_r \simeq (d-1) L^d$, for sufficiently large $L$, and $\Lambda_r \uparrow \infty$ as $\Lambda \uparrow aZ^d$. For periodic b.c., the same bond variables are gauged away; the number of non-gauged away bond variables is then $\Lambda_r + \Lambda_e$, where we recall that $\Lambda_e$ is the number of extra bonds we add to $\Lambda$ to implement periodic b.c.

The precise definition of gauged away bonds, for free b.c., is as follows (see page 4 of [24]). We label the sites of the $\mu$-th lattice coordinate by $1, 2, \ldots, L$. The enhanced temporal gauge is defined by setting in $\Lambda$ the following bond variables to 1. First, for any $d = 2, 3, 4$, we gauge away all temporal bond variables by setting $g_{b_1}(x) = 1$. For $d = 2$, take also $g_{b_2}(x^0 = 1, x^1) = 1$. For $d = 3$, set also $g_{b_1}(x^0 = 1, x^1, x^2) = 1$ and $g_{b_2}(x^0 = 1, x^1 = 1, x^2) = 1$. Similarly, for $d = 4$, set also to 1 all $g_{b_1}(x^0 = 1, x^1, x^2, x^3); g_{b_2}(x^0 = 1, x^1 = 1, x^2, x^3)$ and $g_{b_3}(x^0 = 1, x^1 = 1, x^2 = 1, x^3)$. For $d = 2$ the gauged away bond variables form a comb with the teeth along the temporal direction, and the open end at the maximum value of $x^0$. For $d = 3$, the gauged away bonds can be visualized as forming a scrub brush with bristles along the $x^0$ direction and the grip forming a comb. For any $d$, all gauged away bond variables are associated with bonds in the hypercubic lattice $\Lambda$ which form a maximal tree. Hence, by adding any other bond to this set, we form a closed loop.
III. TUV STABILITY AND PLAQUETTE FIELD CORRELATIONS FOR THE APPROXIMATE MODEL

In this section, we restrict our attention to a simplified lattice YM model. We simplify the YM model by setting to zero, in the Wilson action, the plaquette actions corresponding to internal horizontal plaquettes (i.e., those plaquettes orthogonal to the time direction), plus certain specified plaquettes on the boundary \( \partial \Lambda \) of the lattice \( \Lambda \). We refer to this model as the approximate model.

For the approximate model, the free-energy, plaquette field correlations and their thermodynamic limits, as well as their continuum limits, are obtained explicitly and exactly. The bounds obeyed in the approximate model are a good guide for the model without approximation.

In subsection III.1 we define the approximate model and treat stability. In subsection III.2 we obtain plaquette field correlations considering the gauge group \( U(1) \). The plaquette field correlation results are extended to \( U(N \geq 2) \) in subsection III.3.

The complete, non-approximate model is treated in the ensuing sections. For \( d = 2 \), the results obtained for the complete model and the approximate model coincide.

The physical gauge-invariant plaquette field plaquette-plaquette correlation is most singular for coincident points. The ultraviolet limit \( a \downarrow 0 \) singular behavior is \( (\text{const}/a^d) \). The same behavior occurs for the coincident-point derivative field correlations in the case of the real, massless scalar free field, as shown in the Appendix.

Of course, the abelian \( U(1) \) case and, for the model without approximation, the formal approximate model

\[
\begin{align*}
Z_{\Lambda,a} &= \int_{|A_b| \leq \pi \theta} \exp \left\{ -\frac{a^d}{g^2} \sum_{x,\mu<\nu} 2 \left[ 1 - \cos (\theta_{\mu\nu}(x)) \right] \right\} \prod_b d\theta_b, \\
Z_{\Lambda,a} &= \left( \frac{ag}{2\pi} \right)^{\Lambda_r} \int_{|A_b| \leq \pi/(ag)} \exp \left\{ -\frac{a^d}{g^2} \sum_{x,\mu<\nu} 2 \left[ 1 - \cos (a^2 g F_{\mu\nu}^a(x)) \right] \right\} \prod_b dA_b, \\
Z_{\Lambda,a} &= \left( \frac{g}{2\pi} a^{(4-d)/2} \right)^{\Lambda_r} \int_{|\chi_b| \leq \pi/g a^{(4-d)/2}} \exp \left\{ -\frac{a^d}{g^2} \sum_{x,\mu<\nu} 2 \left[ 1 - \cos \left( g a (4-d)/2 \chi_{\mu\nu}(x) \right) \right] \right\} \prod_b d\chi_b, \\
Z_{\Lambda,a} &= \left( \frac{g}{2\pi} a^{(4-d)/2} \right)^{\Lambda_r} \int_{|\chi_b| \leq \pi/g a^{(4-d)/2}} \exp \left\{ -\frac{a^d}{g^2} \sum_{x,\mu<\nu} 2 \left[ 1 - \cos \left( g a (4-d)/2 \chi_{\mu\nu}(x) \right) \right] \right\} \prod_b d\chi_b,
\end{align*}
\]

are obtained explicitly and exactly. The bounds obeyed in the approximate model are a good guide for the model without approximation.

Using our transformed field method, we obtain TUV stability bounds and bounds on the normalized free energy and also the boundedness of two-point plaquette scaled field correlation. For the group \( \mathcal{G} = U(1) \), the Haar measure is simpler, formulas are more familiar and the analysis becomes more transparent.

For \( d = 2 \), the results for the two-point plaquette field correlation are exact. For \( d = 3, 4 \), the results are also exact for the approximate model. This seemingly gross approximation gives the correct picture for bounds for the complete YM model with the nonabelian gauge group \( \mathcal{G} = U(N > 1) \).

1. Approximate Model: TUV Stability

Starting from the free b.c. partition function of Eq. (11), we set

\[
U_b = e^{i\theta_b} = e^{ia g A_b},
\]

where \( A_b \) is the physical gauge potential. For the plaquette \( p = p_{\mu\nu}(x) \), set

\[
\theta_{\mu\nu}(x) = \theta_{\mu}(x) + \theta_{\nu}(x_a^\dagger) - \theta_{\mu}(x^\dagger_a) - \theta_{\nu}(x^\dagger_b).
\]

Then, the finite lattice free b.c. partition function reads

\[
Z_{\Lambda,a} = \int_{|\theta_b| \leq \pi} \exp \left\{ -\frac{a^d}{g^2} \sum_{x,\mu<\nu} 2 \left[ 1 - \cos (\theta_{\mu\nu}(x)) \right] \right\} \prod_b d\theta_b,
\]

In terms of the physical fields \( A_b \), setting \( A_{\mu\nu}(x) \equiv a F_{\mu\nu}^a(x) \), where \( F_{\mu\nu}^a(x) \) is the usual field strength antisymmetric second order tensor, we have

\[
Z_{\Lambda,a} = \left( \frac{ag}{2\pi} \right)^{\Lambda_r} \int_{|A_b| \leq \pi/(ag)} \exp \left\{ -\frac{a^d}{g^2} \sum_{x,\mu<\nu} 2 \left[ 1 - \cos (a^2 g F_{\mu\nu}^a(x)) \right] \right\} \prod_b dA_b,
\]

Now, we transform to the locally scaled fields \( \chi_b \) defined by

\[
\chi_b = a^{(d-2)/2} A_b,
\]

and, in terms of these fields, the free b.c. partition function is

\[
Z_{\Lambda,a} = \left( \frac{g}{2\pi} a^{(4-d)/2} \right)^{\Lambda_r} \int_{|\chi_b| \leq \pi/g a^{(4-d)/2}} \exp \left\{ -\frac{a^d}{g^2} \sum_{x,\mu<\nu} 2 \left[ 1 - \cos \left( g a (4-d)/2 \chi_{\mu\nu}(x) \right) \right] \right\} \prod_b d\chi_b.
\]
Remark 1 We note that, instead of the above simple expression, in the nonabelian case $U(N > 1)$, the Haar measure presents also a weight function factor besides the product of Lebesgue measures.

Remark 2 In the $A_b$ variables, the Boltzmann factor, for $a \searrow 0$, is approximately
\[
\exp \left\{ -a^d \sum_{x, \mu < \nu} \left[ F_{\mu \nu}^a(x) \right]^2 \right\},
\]
for $d = 2, 3$ and, for $d = 4$ and $g \searrow 0$. In both cases, the action approximates the continuum model action.

In the $\chi_b$ variables, the Boltzmann factor, for $a \searrow 0$, is approximately
\[
\exp \left\{ -\sum_{x, \mu < \nu} \left[ \chi_{\mu \nu}(x) \right]^2 \right\},
\]
for $d = 2, 3$ and the same holds for $d = 4$ and $g \searrow 0$. Here, in both cases, the action is independent of the lattice spacing $a$. In the above quadratic approximation of the action, the model can be solved explicitly by diagonalizing the quadratic form of the action.

We now define more precisely and analyze our approximate model. We define our approximate model and give the bond gauge integration procedure. This is done case by case in the spacetime dimension $d$. For simplicity, we identify coordinates of a lattice site in each lattice direction with the labels $1, 2, \ldots, L$. We have:

- $d = 4$: For $x^0 = L, L - 1, \ldots, 2$, set the plaquette actions to zero in the planes parallel to the $\mu \nu = 12, 13, 23$ coordinate planes. For $x^0 = 1, x^3 = L, \ldots, 2$, set the plaquette actions to zero in the coordinate planes parallel to the 12-plane;
- $d = 3$: For $x^0 = L, L - 1, \ldots, 2$, set to zero the plaquette actions in the planes parallel to the 12-plane;
- $d = 2$: maintain all the plaquette actions.

Remark 3 We remark that a simpler approximate model can be defined by setting to zero all horizontal plaquette actions. Such a model can also be solved exactly with the same results as given here for our approximate model. Boundary effects disappear in the thermodynamic limit. In our approximate model, fewer plaquette actions are discarded.

Simplified, Approximate Model with the Abelian Gauge Group $G = U(1)$:

With these definitions, we now perform the bond integration. For ease of visualization, we carry it out explicitly for $d = 3$.

For $d = 3$, integrate over successive planes of horizontal bonds starting at $x^0 = L$ and ending at $x^0 = 2$. For the $x^0 = 1$ horizontal plane, integrate over successive lines of horizontal bonds in the coordinate direction two, starting at $x^1 = L$ and ending at $x^1 = 2$. For each horizontal bond variable integration, the bond variable appears in only one plaquette.

The simplification that occurs in our original model is that, in the approximate model, we can carry out all bond integrations. Besides, for each integration, we can extract a single plaquette partition function of a single bond variable.

Of course, in $d = 2$, the model can be solved without any approximation (see Ref. [12]).

After integration, each integral depends, in principle, on the other bond variables of the plaquette. However, as in Ref. [12], for $d = 2$, by a change of variables, the integral is independent of the other variables and their integrals are trivially done. Here, we are using the simplest case of the left and right invariance of the gauge group Haar measure (see e.g Refs. [17, 18]). In this way, a factor is extracted from the partition function and the factor is the partition function of a single plaquette of a single bond variable.

After the bond integration, we obtain
\[
Z_{\Lambda, a} = \left[ \frac{g}{2\pi a^{(d-4)/2}} \right]^{\Lambda_r} z^{\Lambda_r},
\]
where $z$ is the single bond partition function. Namely, we have
\[
z = \int_{|X| \leq \pi/g} \exp \left\{ -\frac{a^{d-4}}{g^2} \sum_{x, \mu < \nu} 2 \left[ 1 - \cos \left( g a^{(d-4)/2} X \right) \right] \right\} dX.
\]
Using the elementary inequalities (see e.g. Ref. [27] for a proof of the second one)

\[ 1 - \cos u \leq \frac{u^2}{2}, \quad u \in \mathbb{R} \]
\[ 1 - \cos u \geq \frac{2u^2}{\pi^2}, \quad u \in (-\pi, \pi], \]

we obtain the upper and lower bounds

\[ z \leq \int_{|X| \leq (\pi/g) a^{(d-4)/2}} \exp \left[ -\frac{4}{\pi^2} X^2 \right] dX, \quad (7) \]

and

\[ z \geq \int_{|X| \leq (\pi/g) a^{(d-4)/2}} e^{-X^2} dX \geq \int_{|X| \leq (\pi/g_0)} e^{-X^2} dX \equiv \hat{z}_f > 0, \quad (8) \]

for all \( a \in (0, 1] \) and \( 0 < g^2 \leq g_0^2 < \infty \).

We now define the normalized free b.c. partition function \( Z^n_{\Lambda, a} \), by extracting the \( a \searrow 0 \) singularity in Eq. (6). It reads

\[ Z^n_{\Lambda, a} = \left[ \frac{g}{2\pi a^{(d-4)/2}} \right]^{-\Lambda_r} Z^{\Lambda_r}_{\Lambda, a} = z^{\Lambda_r}. \quad (9) \]

In this way, in terms of \( Z^n_{\Lambda, a} \), we obtain the TUV stability bound

\[ 0 < \frac{z^{\Lambda_r}}{z_f} \leq Z^n_{\Lambda, a} \leq \frac{z^{\Lambda_r}}{z_f}, \]

so that, defining the normalized free energy per effective degree of freedom in the finite \( d \)-dimensional hypercubic lattice \( \Lambda \) by

\[ f^n_{\Lambda, a} = \frac{1}{\Lambda_r} \ln Z^n_{\Lambda, a} = \ln z, \quad (10) \]

the TUV bounds ensure, the thermodynamic limit \( \Lambda \searrow a \mathbb{Z}^d \) and the continuum limit \( a \searrow 0 \) exist (here, not only the subsequential limits as below!) and we obtain

\[ f^n \equiv \lim_{a \searrow 0} \lim_{\Lambda \searrow a \mathbb{Z}^d} f^n_{\Lambda, a} = \lim_{a \searrow 0} \ln z = \begin{cases} \int_{\mathbb{R}} e^{-X^2} dX = \sqrt{\pi/2}, \quad d = 2, 3, \\ \int_{|X| \leq \pi/g} e^{-2g^2[1 - \cos(gX)]} dX, \quad d = 4. \end{cases} \]

Besides, for \( d = 4 \), we have \( \lim_{g \searrow 0} f^n = \sqrt{\pi/2} \).

**Simplified Approximate Model with Gauge Group \( \mathcal{G} = U(N) \):**

Still considering the approximate model, here we extend our TUV bounds to the more general nonabelian \( U(N) \) case. Using the same bond integration procedure as in the above \( U(1) \) case, the simplified model, free b.c. partition function with the gauge group \( U(N) \) also factorizes as

\[ Z_{\Lambda, a} = z^{\Lambda_r}, \]

where

\[ z = \int_{U(N)} \exp \left[ -\frac{d-4}{g^2} \text{Tr} (2 - U - U^\dagger) \right] d\sigma(U). \]

Here, \( z \) is the partition function of a single plaquette with the single bond variable \( U \).
We explain how the factorization occurs, and we use the left and right invariance of the single bond Haar measure $d\sigma(U)$. We recall the invariance property (see e.g. [17, 18]): let $f(U)$ be a function of the bond variable $U \in U(N)$ and let $W \in U(N)$. Then,

$$\int_{U(N)} f(U) d\sigma(U) = \int_{U(N)} f(WU) d\sigma(U) = \int_{U(N)} f(UW) d\sigma(U).$$

Returning to the bond integration procedure, let $U_1, U_2, U_3, U_4$ be the plaquette $p$ bond variables and $U_p = U_1 U_2 U_3 U_4$. Consider the integration over $U_1$, where, in the partition function $Z_{\Lambda,a}$, $U_1$ only appears in the plaquette $p$. The integral over the bond variable $U_1$ is

$$\int_{U(N)} \exp \left\{ -\frac{a(d-4)}{g^2} \text{Tr} \left( 2 - U_p - U_p^\dagger \right) \right\} d\sigma(U_1).$$

By the Haar measure left and right invariance (take $W = U_2 U_3 U_4$ and $U = U_1$ above!), the integral is just the single bond partition function $z$, and is independent of the other bond variables. In this way, we extract the factors $z$ from the partition function $Z_{\Lambda,a}$.

To continue our analysis, we note that the integrand is a class function on $G$. For the $U(N)$ group integral of a class function, the $N^2$ dimensional integral over the group reduces to an $N$ dimensional integral over the angular eigenvalues of $U$, according to the Weyl integration formula [16–18].

The angular eigenvalues are defined as follows. If the eigenvalues of the unitary matrix $U$ are denoted by $\{e^{i\lambda_1}, \ldots, e^{i\lambda_N}\}$, with $\lambda_j \in (-\pi, \pi)$, $j = 1, \ldots, N$, then $\lambda = (\lambda_1, \ldots, \lambda_N)$ are called the angular eigenvalues of $U$. The Weyl integration formula then reads

$$\int_{U(N)} f(U) d\sigma(U) = \frac{1}{N!} \int_{(-\pi, \pi)^N} f(\lambda) \rho(\lambda) \frac{d\lambda}{(2\pi)^N},$$

where $d\lambda = d\lambda_1 \ldots d\lambda_N$ is a product measure of Lebesgue measures and the weight function or density $\rho(\lambda)$ arises from a squared Vandermonde determinant and is given by

$$\rho(\lambda) = \prod_{1 \le j < k \le N} |e^{i\lambda_j} - e^{i\lambda_k}|^2 = \prod_{1 \le j < k \le N} \{2 [1 - \cos(\lambda_j - \lambda_k)]\}.$$

In this way, applying the Weyl integration formula, we obtain

$$z = \frac{1}{N!(2\pi)^N} \int_{(-\pi, \pi)^N} \exp \left[ -\frac{2a d-4}{g^2} \sum_{j=1, \ldots, N} (1 - \cos \lambda_j) \right] \rho(\lambda) d\lambda.$$

Next, we use the previous simple bounds on $(1 - \cos u)$ and the bound, with $\hat{\rho}(\lambda) = \prod_{1 \le j < k \le N} |\lambda_j - \lambda_k|^2$,

$$\left( \frac{4}{\pi^2} \right)^{N(N-1)/2} \hat{\rho}(\lambda) \le \rho(\lambda) \le \tilde{\rho}(\lambda),$$

where the lower bound holds for all $|\lambda_j| \le \pi/2$ and there is no restriction for the upper bound. Besides, we make use of the changes of variables

$$y = \left( \frac{a d-4}{g^2} \right)^{1/2} \lambda ; \quad y = \left( \frac{a d-4}{g^2} \right)^{1/2} \frac{2}{\pi} \lambda,$$

respectively, in the lower and upper bounds. We then obtain the following bound on $z$

$$\frac{1}{N!(2\pi)^N} \left( \frac{a d-4}{g^2} \right)^{N^2/2} \int_\mathcal{L} \exp \left[ -\frac{2a d-4}{g^2} \sum_{1 \le j < N} y_j^2 \right] \hat{\rho}(\lambda) d\lambda \le z \le \frac{1}{N!(2\pi)^N} \int_\mathcal{U} \exp \left[ -\frac{2a d-4}{g^2} \sum_{1 \le j < N} y_j^2 \right] \tilde{\rho}(\lambda) d\lambda,$$

where we have the integration domains $\mathcal{L} = \{ y : |y_k| \le (\pi/2) (a d-4/g^2)^{1/2} \}$ and $\mathcal{U} = \{ y : |y_k| \le (a d-4/g^2)^{1/2} \}$.

We recognize the above integrands as being proportional to the well known (see e.g. Refs. [19, 20]) Gaussian Unitary Ensemble (GUE) probability density in $\mathbb{R}^N$ of random matrix theory.
Extracting the $a \searrow 0$ singularity and defining the normalized $U(N)$ approximate model finite lattice normalized partition with free b.c. by

$$Z_{\Lambda,a}^n = \left( \frac{a^{d-4}}{g^2} \right)^{N^2 \Lambda_r/2} Z_{\Lambda,a},$$

then $Z_{\Lambda,a}^n$ obeys the TUV bound

$$z_{\ell}^{\Lambda_r} \leq Z_{\Lambda,a}^n \leq z_u^{\Lambda_r},$$

with

$$z_n = \frac{1}{N!(2\pi)^N} \left( \frac{a^{d-4}}{g^2} \right)^{N^2/2} \int_{[-\pi,\pi]^N} \exp \left[ -\frac{2a^{d-4}}{g^2} \sum_{j=1,\ldots,N} (1 - \cos \lambda_j) \right] \rho(\lambda) d\lambda.$$

Also, we have $z_{\ell} = G((a^{d-4}/g^2)^{1/2} \pi/2)$ and $z_u = G((a^{d-4}/g^2)^{1/2} 2)$ where $G$ is the probability given in the GUE given by

$$G(u) = \frac{1}{N!(2\pi)^N} \int_{|y_k| \leq u} \exp \left[ -\sum_{1 \leq j \leq N} y_j^2 \right] \hat{\rho}(\lambda) d\lambda \leq G(\infty).$$

We now define a normalized finite lattice free energy by

$$f_{\Lambda,a}^n = \frac{1}{\Lambda_r} \ln Z_{\Lambda,a}^n,$$

such that the above TUV bounds ensure the existence of the thermodynamic and continuum limits given by

$$f_n = \lim_{a \searrow 0} \lim_{\Lambda \rightarrow \infty, d \rightarrow 4} f_{\Lambda,a}^n = \begin{cases} \ln G(\infty), & d = 2, 3 \\ \ln \left[ \frac{1}{N!(2\pi)^N} \int_{|y_k| \leq \pi/g} \exp \left[ -\frac{2g^{-2}}{a^2} \sum_{1 \leq j \leq N} (1 - \cos(gy_j)) \right] \hat{\rho}(y) dy \right], & d = 4. \end{cases}$$

Furthermore, for $d = 4$, we get $\lim_{g \searrow 0} f_n = \ln G(\infty) = 0$.

2. Approximate Model: Plaquette Field Correlations for $U(1)$

Here, first we take the gauge group to be $G = U(1)$. As shown below, in this simple abelian group case, we are able to compute the plaquette-plaquette correlation exactly for the approximate model and for vertical plaquettes (plaquettes with two vertical bonds). This computation allows us to show the boundedness of the scaled field plaquette-plaquette correlation. In the next subsection, we consider the nonabelian gauge groups $U(N \geq 2)$.

For the plaquette $p = p_{\mu\nu}(x)$ in the $\mu\nu$ coordinate plane, we define the physical gauge invariant plaquette field by, with $U_b = e^{iagA_b}$,

$$\mathcal{F}_p(U_p) = \frac{i}{2a^2g} (U_p^\dagger - U_p) = \frac{i}{a^2g} \sin \left[ a^2 g \rho_{\mu\nu}(x) \right],$$

where $U_p = e^{iagA_p}$ and $A_p = a F_{\mu\nu}^a$.

Next, considering a sufficiently small lattice spacing $a$, we show this plaquette field leads to the expected physical correlation. Using this field, for small $a$, we have

$$\mathcal{F}_p(U_p) \simeq F_{\mu\nu}^a = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Then, the gauge-invariant physical plaquette-plaquette correlation is defined by

$$\langle \mathcal{F}_{\mu\nu}(x) \mathcal{F}_{\rho\sigma}(y) \rangle = \frac{1}{N} \int_{|A_b| \leq \pi/(ag)} \left\{ \left[ \frac{1}{a^2g} \sin \left( a^2 g F_{\mu\nu}^a(x) \right) \right] \left[ \frac{1}{a^2g} \sin \left( a^2 g F_{\rho\sigma}^a(y) \right) \right] \right\} \times \exp \left\{ -\frac{a^{d-4}}{g^2} \sum_{z,\mu<\nu} 2 \left[ 1 - \cos \left( a^2 g F_{\mu\nu}^a(z) \right) \right] \right\} \prod_b dA_b. \quad (13)$$
For small $a$, the right-hand-side of Eq. (13) becomes

$$\frac{1}{\mathcal{N}} \int_{|A| \leq (\pi/a)} F_{\mu\nu}(x) F_{\rho\sigma}(y) \exp \left\{ -a^d \sum_{z, \mu \leq \nu} \left[ F_{\mu\nu}^{a}(z) \right]^2 \right\} \prod_b dA_b.$$ 

Note that the above action is the Riemann sum approximation to the smooth field classical continuum action $\int_{\mathbb{R}^4} d^4x \left( F_{\mu\nu}(x) \right)^2$, where the field strength antisymmetric tensor in the abelian case is $F_{\mu\nu}(x) = \partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x)$. Hence, we obtain the usual lattice approximation to the physical two-plaquette field correlation.

Now, for $a \in (0, 1]$, we define a scaled U(1) gauge invariant plaquette-plaquette correlation by

$$\langle M_{\mu\nu}(x) M_{\rho\sigma}(y) \rangle = \frac{1}{\mathcal{N}} \int_{|A| < \pi/(ag)} \left[ \left( \frac{a^{d-4}}{g^2} \right)^{1/2} \sin[a^2 g F^a_{\mu\nu}(x)] \right] \left[ \left( \frac{a^{d-4}}{g^2} \right)^{1/2} \sin[a^2 g F^a_{\rho\sigma}(y)] \right] \times \exp \left\{ -\frac{a^{d-4}}{g^2} \sum_{z, \mu \leq \nu} 2 (1 - \cos(a^2 g F^a_{\mu\nu}(z))) \right\} \prod_b dA_b$$

\begin{equation}
(14)
\end{equation}

where, here, $\mathcal{N}$ is the normalization

$$\mathcal{N} = \int_{|A| < \pi/(ag)} \exp \left\{ -\frac{a^{d-4}}{g^2} \sum_{x, \mu < \nu} \left[ (a^2 g F^a_{\mu\nu}(x)) \right] \right\} \prod_b dA_b.$$

Letting $\chi_b = a^{(d-2)/2} A_b$, we can rewrite the plaquette-plaquette correlation as

$$\langle M_{\mu\nu}(x) M_{\rho\sigma}(y) \rangle = \frac{1}{\mathcal{N}'} \int_{|\chi| < \pi(a^{d-4}/g^2)^{1/2}} \frac{a^{d-4}}{g^2} \sin \left[ \left( \frac{a^{d-4}}{g^2} \right)^{-1/2} \chi_{\mu\nu}(x) \right] \sin \left[ \left( \frac{a^{d-4}}{g^2} \right)^{-1/2} \chi_{\rho\sigma}(y) \right] \times \exp \left\{ -\frac{a^{d-4}}{g^2} \sum_{z, \mu \leq \nu} \left[ 2 (1 - \cos \left( \frac{a^{d-4}}{g^2} \right)^{-1/2} \chi_{\mu\nu}(z) \right) \right\} \prod_b d\chi_b,$$

where $\mathcal{N}'$ is the measure normalization constant.

Now, for the approximate model, we compute the plaquette-plaquette correlation exactly. We also show that its thermodynamic limit exists and that the correlation of Eq. (14) is bounded uniformly in $a \in (0, 1]$. The continuum limit of $\langle M_{\mu\nu}(x) M_{\rho\sigma}(y) \rangle$ also exists! [In the next subsection, we extend these results to the case of the nonabelian gauge group $\text{U}(N)$, $N \geq 2$.]

More precisely, for the approximate model, we will show that $\langle M_{\mu\nu}(x) M_{\rho\sigma}(x) \rangle$ is bounded uniformly in $a \in (0, 1]$ and $0 < g^2 < g_0^2 < \infty$. The importance of this bound is that it shows us that the coincident point $(x = y)$ physical plaquette-plaquette correlation behaves as const/a$d$. This behavior is analogous to what occurs if we transform the physical massless scalar field $\phi^a(x)$, by a local scaling factor, to a scaled field $\phi(x) = a^{(d-2)/2} (2d)^{1/2} \phi^a(x)$. The scaled field action is independent of the lattice spacing $a$. See Ref. [21] and the Appendix for more details. Moreover, the scaled field correlations are bounded at coincident points, uniformly in $a \in (0, 1]$, for $d = 3, 4$, and the unscaled derivative field two-point correlation has the exact value $2/(da^d)$, for dimensions $d = 2, 3, 4$.

In order to simplify the notation, like in Eq. (14), below $\mathcal{N}$ will mean the average of the identically 1 constant function with the relevant measure appearing in the associated integral, including the exponential density factor.

For the complete model with gauge group $\text{U}(N)$, the integrals do not factorize, but for the approximate model they do factorize, which makes much easier the plaquette-plaquette correlation analysis. For this reason, from now on, we deal with only the approximate model. Note that we also take the two plaquettes with external points to be vertical (at least one bond in the time direction).

To analyze the plaquette-plaquette correlation for the approximate model, we follow the same integration procedure employed before in our treatment of the partition function (see subsection III). The result is that all gauge integrals with densities given by the exponential of the actions, which do not contain the external points $x$ and $y$, factorize and correspond to single plaquette partition functions depending only on a single bond variable. They are present both in the numerator and the normalization integrals in the denominator in $\langle M_{\mu\nu}(x) M_{\rho\sigma}(y) \rangle$, and cancel out. After this partial cancellation, we are left in the numerator with integrals whose coordinate supports contain the $x$ and $y$ external points. However, since the single plaquette field correlation is zero by the $A \to -A$ symmetry, the only nonzero contribution occurs when the points $x$ and $y$ coincide.
For coincident points \( x = y \), the contributions depend on a single bond variable \( \chi_b(x) \). Taking into account the partial cancellation between the numerator and denominator of the normalized plaquette-plaquette correlation, the infinite volume limit can be then taken. By translation invariance, the remaining integral does not depend on the lattice site point \( x = y \) we fixed. Thus, we can suppress \( x \) and the bond lower index \( b \) in \( \chi_b(x) \) and simply write \( \chi \). Doing this, we obtain

\[
\langle [M_{\mu\nu}(x)]^2 \rangle = \frac{1}{N} \int_{|x|<\pi/g} a^{(d-4)/2} \left\{ \left( \frac{a^{d-4}}{g^2} \right) \sin \left( \frac{a^{d-4}}{g^2} \right) \right\} \chi \langle g \rangle \to 0
\]

\[
\times \exp \left\{ -\frac{a^{d-4}}{g^2} 2 \left[ 1 - \cos \left( \frac{a^{d-4}}{g^2} \right) \right] \right\} \ d\chi,
\]

where \( N \) denotes here the normalization with the integral over a single variable \( \chi \) given by

\[
N = \int_{|x|<\pi/g} \exp \left\{ -\frac{a^{d-4}}{g^2} 2 \left[ 1 - \cos \left( \frac{a^{d-4}}{g^2} \right) \right] \right\} d\chi.
\]

Using the elementary trigonometric inequalities employed in the previous sections, we have the bound, for \( a \in (0, 1] \) and \( 0 < g^2 \leq g_0^2 \leq \infty \),

\[
\langle [M_{\mu\nu}(x)]^2 \rangle \leq \frac{1}{N_1} \int_{|x|<\pi a^{(d-4)/2}/g} \chi^2 \exp \left[ -\frac{4}{\pi^2} \chi^2 \right] d\chi,
\]

where, for \( N_1 = \int_{|x|<\pi a^{(d-4)/2}/g} \exp \left( -\chi^2 \right) d\chi \), \( N_1 \) is defined as \( N \) but with \( g \) replaced by \( g_0 \) in the integral domain restriction.

Similarly, we obtain the lower bound

\[
\langle [M_{\mu\nu}(0)]^2 \rangle \geq \frac{4}{\pi^2} \int_{|x|<\pi/(2g)} \chi^2 e^{-\chi^2} d\chi \int_R e^{-(4/\pi^2)\chi^2} d\chi,
\]

where the numerator is bounded below taking the integration domain to be \( |\chi| \leq \lfloor (\pi a^{(d-4)/2})(2g_0) \rfloor \). Thus, we see that the scaled plaquette-plaquette correlation at coincident points is uniformly bounded for \( a \in (0, 1] \) and \( 0 < g^2 \leq g_0^2 \). Using these bounds and the relation given in Eq. (14), we see that, for coincident plaquettes, the singular behavior is exactly \( a^{-d} \) (rather than just an upper bound for the singular behavior!)

From these bounds, the continuum limit

\[
M^2(x) \equiv \lim_{g \to 0} \langle [M_{\mu\nu}(x)]^2 \rangle,
\]

exists and is given by

\[
M^2(x) = \begin{cases} 
\int_R \chi^2 e^{-\chi^2} d\chi \int_R e^{-\chi^2} d\chi = \frac{1}{2}, & d = 2, 3; \\
\int_{|\chi|<\pi/g} \left[ \sin(g\chi) \right] g e^{-2(1-\cos(g\chi))/g^2} d\chi \int_{|\chi|<\pi/g} e^{-2(1-\cos(g\chi))/g^2} d\chi, & d = 4.
\end{cases}
\]

Furthermore, from Eq. (14), for \( d = 4 \), the \( g \to 0 \) limit also exists and is \( 1/2 \).

In the next subsection, considering the approximate model, we extend these exact and explicit results to the nonabelian gauge group \( U(N) \), \( N \geq 2 \). In the following sections, we obtain boundedness results for the YM model without approximation. The nonabelian case \( N \geq 2 \) is more difficult than the abelian \( N = 1 \) case. One of the
difficulties is that the gauge group Haar measure is well more complicated than the Lebesgue measure of the abelian model. In our extension to the nonabelian case, rather than treat directly the correlations, we bound the two-point plaquette field normalized generating function (with the partition function in the denominator). Bounds on correlations follow from this using analyticity and Cauchy bounds for the source derivatives of the generating function at zero source field strengths.

To obtain bounds on the normalized generating function which are independent of the number of lattice sites, we use the well known multiple reflection method (see Ref. [16]). This method makes use of the Cauchy-Schwarz inequality in the quantum mechanical physical Hilbert space of the associated quantum field theory.

### 3. Approximate Model: Plaquette Field Correlations for \( U(N \geq 2) \)

In this subsection, we analyze the plaquette field correlations for the more general non-commutative case of the gauge unitary Lie group \( \mathcal{G} = U(N) \). We define a gauge-invariant plaquette field for the plaquette \( p = p_{\mu\nu}(x) \), in the \( \mu\nu \)-coordinate plane, by

\[
\text{Tr} \ F_{\mu\nu} = \frac{1}{a^2 g} \text{Im} \text{Tr} (U_p - 1) = -\frac{i}{2 a^2 g} \text{Tr} (U_p - U_p^\dagger) = \frac{1}{a^2 g} \sin X_p,
\]

where \( U_p = \exp\{i X_p\} \).

For the physical plaquette field, we parametrize \( U_b \) by \( U_b = \exp\{i a g A_b\} \). For small lattice spacing \( a \), we have

\[
\text{Tr} F_{\mu\nu}(x) \approx \text{Tr} F_{\mu\nu}^a(x),
\]

which for the gauge group \( U(1) \) becomes the familiar \( \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \) [see Eq. (1)].

We also define the gauge-invariant scaled plaquette field by

\[
\text{Tr} M_{\mu\nu}(x) = a^{d/2} \text{Tr} F_{\mu\nu} = \left( \frac{a^{d-4}}{g^2} \right)^{1/2} \text{Im} \text{Tr} (U_p - 1) = \left( \frac{a^{d-4}}{g^2} \right)^{1/2} \sin X_p.
\]

For small \( a \), we have that

\[
\text{Tr} M_{\mu\nu}(x) \approx a^{d/2} \text{Tr} F_{\mu\nu}^a(x),
\]

where, for \( \partial_\mu^a \) meaning a finite difference derivative with lattice spacing \( a \), in the \( \mu \) coordinate direction, we have

\[
F_{\mu\nu}^a(x) = \partial_\mu^a A_\nu(x) - \partial_\nu^a A_\mu(x) + i g [A_\nu(x), A_\mu(x)]
\]

and the bracket denotes the commutator in the gauge Lie algebra of \( U(N) \).

As explained in some detail in our plaquette-plaquette correlation analysis, with external points \( x \) and \( y \), in the case of the abelian gauge group \( U(1) \), for the approximate model, we have a factorization of single plaquette partition functions in the numerator and denominator of the plaquette-plaquette normalized correlations. Also a partial cancellation of these contributions occur between numerator and the normalization of the correlation, which allows us to take the infinite volume limit. Using the left-right invariance of the Haar measure, the integrals are again over a single bond Haar measure. By gauge integration properties, the only nonzero contributions are those with coincident points \( x = y \). Counting then all the plaquette-plaquette correlation becomes [check Eq. (12)]

\[
\langle (\text{Tr} M_{\mu\nu})^2 \rangle = \frac{1}{N^2} \int_{U(N)} \left[ \left( \frac{a^{d-4}}{g^2} \right)^{1/2} \text{Im} \text{Tr} (U - 1) \right]^2 \exp \left\{-2 \left( \frac{a^{d-4}}{g^2} \right) \text{Tr} (1 - U - U^\dagger) \right\} d\sigma(U)
\]

\[
= \frac{1}{N^2} \int_{[-\pi, \pi]^N} \left[ \left( \frac{a^{d-4}}{g^2} \right)^{1/2} \sum_{j=1,\ldots,N} \sin \lambda_j \right]^2 \exp \left\{-2 \left( \frac{a^{d-4}}{g^2} \right) \sum_{j=1,\ldots,N} (1 - \cos \lambda_j) \right\} \rho(\lambda) d\lambda.
\]

Note that the single plaquette correlation is obtained by replacing the squared bracket factor by the single bracket (power one!) in the above integrand. By the transformation of variables \( \lambda_j \to -\lambda_j \), the single plaquette correlation \( \langle \text{Tr} M_{\mu\nu} \rangle = 0 \), as asserted above.

In view of the recent result of Ref. [20], on the triviality of the continuum limit of the \( \phi^4 \) model, we investigate whether or not the continuum limit of the approximate model is Gaussian. For this, we also consider the normalized
four-plaquette correlation at coincident points. Following the same procedure as before, the thermodynamic limit of the $r$-th power of the plaquette field at coincident points, after passing to angular eigenvalues via the Weyl integration formula, reduces to

$$
\langle \langle \text{Tr} M_{ij} \rangle^r \rangle = \frac{1}{N^r} \int_{[-\pi,\pi]^N} \left[ \frac{g^d}{g^2} \frac{1}{2} \sum_{j=1}^{N} \sin \lambda_j \right]^r \exp \left\{ -2 \frac{g^{d-4}}{g^2} \sum_{j=1}^{N} (1 - \cos \lambda_j) \right\} \rho(\lambda) d\lambda,
$$

where the ratio is taken over single plaquette single variable bond variable integrals. Here, $N_r$ is a corresponding normalization constant and $\rho(\lambda)$ is given in Eq. (12). It is worth noticing that, for the abelian gauge group $U(1)$, we have $\rho(\lambda) = 1$.

We easily see the the $r$-correlation is zero if $r$ is odd.

Making a change of variables, using elementary inequalities and the well-known Lebesgue integral convergence theorems, we obtain that the continuum limit of the above coincident point truncated correlations exists. With

$$
\hat{\rho}(\lambda) = \prod_{1 \leq j < k \leq N} |\lambda_j - \lambda_k|^2
$$

and $T_\alpha(g) \equiv \lim_{\mu \to 0} (\langle \langle \text{Tr} M_{ij} \rangle^\alpha \rangle), for d = 2, 3, we obtain, letting $\lambda = (2a^{d-4}/g^2)^{-1/2} y,$

$$
T_\alpha(g) = \frac{2}{N^2} \int_{R^N} \left( \sum_{j=1,\ldots,N} y_j \right)^{\alpha} \hat{\rho}(y) \exp \left[ - \sum_{j=1}^{N} y_j^2 \right] d^N y,
$$

with an associated measure normalization $N_2$. For $d = 4$, letting $\lambda = gy$, we obtain

$$
T_\alpha(g) = \frac{1}{N_4} g^{N(N-1)/2} \int_{[-\pi/g,\pi/g]^N} \left[ \left( \sum_{j=1,\ldots,N} \frac{\sin(gy_j)}{g} \right)^\alpha \right] \rho(gy) \exp \left[ - \sum_{j=1}^{N} \frac{2[1 - \cos(gy_j)]}{g^2} \right] d^N y.
$$

For $d = 4$, the $g \to 0$ limit $T_\alpha$, of $T_\alpha(g)$, is

$$
T_\alpha = \frac{1}{N_4} \int_{R^N} \left( \sum_{j=1,\ldots,N} y_j \right)^{\alpha} \hat{\rho}(y) \exp \left[ - \sum_{j=1}^{N} y_j^2 \right] d^N y,
$$

with a normalization $N_4$.

Note that, for $\alpha = 2, 4$ the right-hand side of Eq. (17) is, respectively, $\sum_{i,j,k,\ell=1,\ldots,N} \langle y_i y_j \rangle_G$ and $\sum_{i,j,k,\ell=1,\ldots,N} \langle y_i y_j, y_k y_\ell \rangle_G$, where $\langle \cdot \rangle_G$ is the expectation in the GUE (Gaussian Unitary Ensemble) (see e.g. Refs. [17, 19, 20]).

Finally, for the case of an abelian gauge group $U(1)$, we then see that the continuum limit is Gaussian for $d = 2, 3$. For $d = 4$ the continuum limit followed by the $g \to 0$ limit is also Gaussian. From Eq. (17), for $d = 4$ and taking the gauge group $G = U(2)$, we have

$$
T_\alpha = \frac{1}{N_4} \int_{R^2} (y_1 + y_2)^\alpha (y_1 - y_2)^2 e^{-(y_1^2 + y_2^2)} dy_1 dy_2
$$

$$
= \frac{1}{\xi} \int_{R^2} (\sqrt{2}\eta)^\alpha (\sqrt{2}\epsilon)^2 e^{-(\eta^2 + \epsilon^2)} d\eta d\epsilon,
$$

with $\xi = \int_{R^2} e^{-(\eta^2 + \epsilon^2)} d\eta d\epsilon$, where we made the $(\pi/4)$ rotation change of variables $\sqrt{2}\epsilon = (y_1 - y_2)$ and $\sqrt{2}\eta = (y_1 + y_2)$. By performing the integrals in the denominator and the $\epsilon$ integral in the numerator, we obtain

$$
T_\alpha = \frac{2^{\alpha/2}}{\sqrt{\pi}} \int_R \eta^\alpha e^{-\eta^2} d\eta,
$$

which shows a Gaussian, non-interacting behavior. Whether or not this is the behavior we have for any gauge group $U(N > 2)$ is still to be analyzed.
IV. THERMODYNAMIC AND ULTRAVIOLET STABILITY BOUNDS: THE GENERAL $G = U(N \geq 1)$ CASE

We now obtain factorized stability bounds for the partition function $Z_{A,a}^B$ of the complete model defined in Eq. (11). In doing this, we are improving the proofs of [14, 24] and are extending the results to the periodic b.c. case. The bounds are factorized as a product. In the product, each factor is a single bond variable, single plaquette partition function. First, we give a Lemma which shows that the plaquette action $A_p$ has a global upper bound which is quadratic in each gluon bond variables. The lemma is used to obtain the factorized lower bound on $Z_{A,a}^B$.

Again, as an example, it is worth recalling that, for the abelian gauge group $U(1)$, the bound is obtained by elementary inequalities. Indeed, writing $U_p = \exp \{ i(\theta_1 + \theta_2 - \theta_3 - \theta_4) \}, |\theta_j| < \pi, j = 1, 2, 3, 4, $ and using the inequality $[2(1 - \cos u)] \leq u^2, u \in \mathbb{R}$, we obtain

$$A_p = 2[1 - \cos(\theta_1 + \theta_2 - \theta_3 - \theta_4)] \leq (\theta_1 + \theta_2 - \theta_3 - \theta_4)^2 \leq 4(\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2),$$

where, we have expanded the square in the first inequality and used the bound $2uv \leq u^2 + v^2, u, v \in \mathbb{R}$, to obtain the second inequality.

The following Lemma is the content of Lemma 2 of Ref. [24].

**Lemma 1** For the four retained bond plaquette, we have the global quadratic upper bound

$$A_p = \|U_p - 1\|^2_{H-S} \leq C^2 \sum_{1 \leq j \leq 4} |x_j|^2 = C^2 \sum_{1 \leq j \leq 4} |\lambda_j|^2, \quad C = 2\sqrt{N},$$

(18)

where $U_p = e^{iX_1}e^{iX_2}e^{iX_3}e^{iX_4}$. For $\alpha = 1, \ldots, N^2$, $X_j = \sum_\alpha x_\alpha x_\alpha^\dagger = \text{Tr} X_j \theta_\alpha$, and $\lambda_j = (\lambda_j, \ldots, \lambda_j, N)$, where, for $k = 1, \ldots, N$, $\lambda_j, k$ are the angular eigenvalues of $\exp\{iX_j\}$.

When there are only one, two or three retained bond variables in a plaquette, the sum over $j$ has, respectively, only one, two and three terms and the numerical factor 4 in $C^2 = 4N$ is replaced by 1, 2 and 3, respectively. For the total action $A^B = \sum_p A_p$, we have the global quadratic upper bound

$$A^B \leq 2(d - 1)C^2 \sum_b |x^b|^2 = 2(d - 1)C^2 \sum_b |\lambda_b|^2,$$

(19)

where the sum runs over all $A_r$ retained lattice bonds.

For completeness of the present paper, following Ref. [24], we give the proof of Lemma 1 in section VI. All the four theorems stated below are also proved there.

Our stability bounds on the partition function $Z_{A,a}^B$, leading to TUV stability bounds for the normalized partition function $Z_{A,a}^{B,n}$ are given by

**Theorem 1** The partition function $Z_{A,a}^B$ verifies the following stability bounds:

**Free b.c.**

$$z^{A_r}_r \leq Z_{A,a} \leq z^{A_r}_u,$$

(20)

**Periodic b.c.**

$$z^{A_r + A_\mu}_r \leq Z_{A,a}^P \leq Z_{A,a} \leq z^{A_r}_u,$$

(21)

where

$$z_u = \int \exp \left[ -2(a^{-d-4}/g^2) \Re \text{Tr} (1 - U) \right] d\sigma(U).$$

(22)

Also, we have $U = e^{iX}, C^2 = 4N, X = \sum_{\alpha=1}^{N^2} x_\alpha \theta_\alpha$ and then

$$\text{Tr} X^2 = \sum_{\alpha=1}^{N^2} x_\alpha^2 = \sum_{k=1}^{N} \lambda_k^2,$$

where $\lambda_1, \ldots, \lambda_N$ are the angular eigenvalues of $U$. Finally,

$$z_t = \int \exp \left[ -2C^2(a^{-d-4}/g^2) (d - 1) \text{Tr} X^2 \right] d\sigma(U).$$

(23)
where, recalling Eq. (25),
\[ \xi = \operatorname{exp}\left\{ -\frac{a^{d-4}}{g^2} \int \| U - 1 \|_H^2 \, d\sigma(U) \right\} \geq \operatorname{exp}\left\{ -2N \frac{a^{d-4}}{g^2} \right\}, \]
where we recall \( \Lambda_p \) is the number of plaquettes in \( \Lambda \). \( \Lambda_p = \Lambda_r \), for \( d = 2 \); \( \Lambda_p \approx 3L^3, 6L^4 \), respectively, for \( d = 3, 4 \). In Theorem 2 below, we obtain factorized lower and upper bounds with \( \Lambda_r = (d - 1)L^d \) factors. In both the upper and lower bound a factor of \( (a^{d-4}/g^2)^{-N^2/2} \) is extracted. This factor dominates the \( a, g^2 \) dependence.

We continue by giving more detailed bounds for \( z_u \) and \( z_\ell \). In these bounds, we extract a factor of \( (a^{d-4}/g^2)^{-N^2/2} \) from both \( z_u \) and \( z_\ell \). Note that the integrands of both \( z_u \) and \( z_\ell \) only depend on the angular eigenvalues of the gauge variable \( U \); they are class functions on \( G \). The \( N^2 \)-dimensional integration over the group can be reduced to an \( N \)-dimensional integration over the angular eigenvalues of \( U \) by the Weyl integration formula of Eq. (11) (see Refs. 19, 20). For the group \( U(N) \), it reads
\[ I_\beta(u) = \int_{\{-u, u\}}^N \exp\left\{ -(1/2) \beta \sum_{1 \leq j \leq N} y_j^2 \right\} \hat{\rho}^{\beta/2}(y) \, d^N y, \]
where \( \hat{\rho}(y) = \prod_{1 \leq j < k \leq N} (y_j - y_k)^2 \), \( I_\beta(u) < I_\beta(\infty) = N \hat{\rho} \), is the normalization constant for the GUE and the Gaussian Symplectic Ensemble (GSE) probability distributions for \( \beta = 2, 4 \), respectively. Explicitly, we have \( N_G = (2\pi)^{N/2} 2^{-N^2/2} \prod_{1 \leq j \leq N} j! \) and \( N_S = (2\pi)^{N/2} 4^{-N^2} \prod_{1 \leq j \leq N} (2j)! \).

For the upper bound on \( z_u \) and lower bound on \( z_\ell \), we have the following result.

**Theorem 2** Let \( C^2 = 4N \). Then, we have the bounds
\[ z_u = \mathcal{N}_C^{-1} \int_{\{-u, u\}}^N \exp\left\{ -2(a^{d-4}/g^2) \sum_{1 \leq j \leq N} (1 - \cos \lambda_j) \right\} \rho(\lambda) \, d^N \lambda \leq \left( a^{d-4}/g^2 \right)^{-N^2/2} (\pi/2)^N N_G(N)N_C^{-1}(N) \]
\[ = \left( a^{d-4}/g^2 \right)^{-N^2/2} e^{c_u}, \]
and
\[ z_\ell = \mathcal{N}_C^{-1} \int_{\{-u, u\}}^N \exp\left\{ -2C^2(d - 1)\left( a^{d-4}/g^2 \right) \sum_{1 \leq j \leq N} \lambda_j^2 \right\} \rho(\lambda) \, d^N \lambda \geq \left( a^{d-4}/g^2 \right)^{-N^2/2} N_C^{-1}(N) (4/\pi^2)^{N(N-1)/2} [2(d - 1)C^2]^{-N^2/2} I_\ell, \]
where, recalling Eq. (27), \( I_\ell \equiv I_2(\pi[2(d - 1)C^2]^{1/2}/(2g_0)). \) The constants \( c_u \) and \( c_\ell \) are real and finite, independent of \( a, \) \( a \in [0, 1] \) and \( g^2 \in [0, g_0^2], \) \( 0 < g_0 < \infty \).

Concerning the existence of the thermodynamic and continuum limits of the normalized free energy we define a normalized partition function by
\[ Z_{\Lambda,a}^{B,n} = \left( a^{d-4}/g^2 \right)^{(N^2/2)\Lambda_r} Z_{\Lambda,a}^B, \]
and a finite lattice normalized free energy by
\[ f_{\Lambda,a}^{B,n} = \Lambda_r^{-1} \ln Z_{\Lambda,a}^{B,n}. \]

Using Theorem 1 and Theorem 2 together with the Bolzano-Weierstrass theorem, we can directly prove the following Theorem.
Theorem 3 The normalized free energy $f_{a,n}^{B}$ converges subsequentially, at least, to a thermodynamic limit

$$f_{a}^{B} = \lim_{\lambda \to 0} f_{\lambda,a}^{B,n},$$

and, subsequently, again, at least subsequentially, to a continuum limit $f^{B,n} = \lim_{a \to 0} f_{a}^{B,n}$. Besides, $f_{a}^{B,n}$ satisfies the bounds

$$-\infty < c_{T} \leq f_{a}^{B,n} \leq c_{u} < \infty. \quad (30)$$

and so does $f^{B,n}$. The constants $c_{T}$ and $c_{u}$ are finite real constants independent of $a \in (0, 1]$ and $g^{2} \in (0, g_{0}^{2}]$, $0 < g_{0} < \infty$.

V. GENERATING FUNCTION FOR PLAQUETTE FIELD CORRELATIONS

Here, we obtain bounds for the generating function of gauge invariant plaquette field correlations. Bounds for the field correlations follow from analyticity in the source field strengths and using Cauchy estimates on the generating function. The same hypercubic lattice $\Lambda$ is maintained. We use periodic b.c. and the multiple reflection method. Our choice of correlations is guided by the energy-momentum spectral results from strong coupling (see [25]). We fix $a = 1$ and denote the plaquette coupling constant by $\gamma = a^{d-4}/2g^{2}$. For $0 < \gamma \ll 1$, a lattice quantum field theory is constructed via a Feynman-Kac formula. By polymer expansion methods, infinite lattice correlations exist and are analytic in $\gamma \in C, |\gamma| \ll 1$. In Ref. [23], it is shown that, for $0 < \gamma \ll 1$, associated with the truncated plaquette-plaquette correlation, there is an isolated particle (glueball) in the low-lying E-M spectrum, with mass of order $(-\ln \gamma)$. Furthermore, for an arbitrary gauge-invariant function with finite support, it is shown that the isolated dispersion curve of the glueball is the only low-lying spectrum that is present.

Returning to our model, we consider the generating function for the correlation of $r$ gauge-invariant real plaquette fields. Taking $p$ to be the plaquette $p_{\mu \nu}(x)$ in the $\mu \nu$ coordinate plane, with $U_{p} = e^{iX_{p}} \in U(N)$, we define the physical plaquette field by

$$\text{Tr} F_{\mu \nu}(x) = \frac{1}{a^{2}q} \text{Im} \text{Tr} (U_{p} - 1) = -\frac{i}{a^{2}g} \text{Tr} (U_{p} - U_{p}^{\dagger}) = \frac{2}{a^{2}g} \text{Tr} \sin X_{p} \approx \frac{1}{a^{2}g} \text{Tr} F_{\mu \nu}^{a}(x) = \frac{1}{a^{2}g} \text{Tr} \left[ \partial_{\mu}^{a} A_{\nu}(x) - \partial_{\nu}^{a} A_{\mu}(x) \right],$$

where $U_{b} = \exp \{ iagA_{b} \}$ and

$$F_{\mu \nu}^{a}(x) = \partial_{\mu}^{a} A_{\nu}(x) - \partial_{\nu}^{a} A_{\mu}(x) + ig \left[ A_{\mu}(x), A_{\nu}(x) \right],$$

with the brackets denoting the commutator in the Lie algebra of $U(N)$.

Next, define the gauge-invariant scaled plaquette field by

$$\text{Tr} M_{\mu \nu}(x) = \left( \frac{a^{d-4}}{g^{2}} \right)^{1/2} \text{Im} \text{Tr} (U_{p} - 1) = a^{d/2} \text{Tr} F_{\mu \nu} \approx a^{d/2} \text{Tr} \left[ \partial_{\mu}^{a} A_{\nu}(x) - \partial_{\nu}^{a} A_{\mu}(x) \right]. \quad (31)$$

With our choice of the scaling factor $(a^{d-4}/g^{2})^{1/2}$, the generating function for scaled plaquette field correlations is finite, uniformly in $a \in (0, 1]$. It may seem surprising that the generating function is pointwise bounded. However, it is known that a similar phenomenon occurs in the case of a free massless or massive scalar field in $d = 3, 4$. Namely, as analyzed in [22], if instead of the given physical field $\phi^{a}(x)$, we use a locally scaled field $\phi(x) \approx a^{(d-2)/2}\phi^{a}(x)$, then the $r$-point correlation function for the scaled $\phi$ fields is bounded pointwise, uniformly in $a \in (0, 1]$. No smearing by a smooth test function is needed to achieve boundedness! We give more details regarding the properties of scalar fields in the Appendix.

Remark 5 We can also define other plaquette fields and their associated scaled fields. For instance, we can also work with the field

$$\text{Tr} H_{\mu \nu}(x) = \frac{1}{a^{2}g} A_{p} \approx \text{Tr} \left[ F_{\mu \nu}^{a}(x) \right]^{2},$$

and the associated fields given by $\text{Tr} S_{\mu \nu}(x) \equiv a^{d} \text{Tr} H_{\mu \nu}(x)$. The results and proofs obtained below for the generating function of correlations of the scaled field $\text{Tr} M_{\mu \nu}(x)$ carry over to $\text{Tr} S_{\mu \nu}(x)$. 
The $r$-plaquette scaled field generating function, associated with the field of Eq. (31), is defined by

$$ G_{r,\Lambda,a}(J^{(r)}) = \frac{1}{Z_{\Lambda,a}^{P}} Z_{r,\Lambda,a}^{P}(J^{(r)}) , $$

where $Z_{r,\Lambda,a}^{P}(J^{(r)})$ is defined similarly to $Z_{\Lambda,a}^{P}$ (see Eq. (1)), but with the inclusion of $r$ local source factors in the integrand given by

$$ \exp \left[ \sum_{x \in \Lambda} \sum_{1 \leq j \leq r} J_{j}(x_{j}) \text{Tr} M_{p_{j}}(U_{p_{j}}) \right] , $$

where $J_{j}$, $j = 1, \ldots, r$, are source strengths. Here, we adopt the convention that the plaquette $p_{j}$ originates at the lattice point $x_{j}$. The $r$-plaquette correlation, with a set $y_{E} = (y_{1}, \ldots, y_{r})$ of $r$ lattice external points in $\Lambda$ is given by

$$ \left. \frac{\partial^{r}}{\partial J_{1}(y_{1}) \cdots \partial J_{r}(y_{r})} G_{r,\Lambda,a}(J^{(r)}) \right|_{J_{j}=0} . $$

Our factorized bound is given in the next Theorem. For simplicity of notation, from now on, we set $J_{i} \equiv J_{i}(y_{i})$.

**Theorem 4** Considering the model with periodic b.c., we have:

1. The $r$–plaquette scaled field generating function is bounded by

$$ |G_{r,\Lambda,a}(J^{(r)})| \leq \prod_{1 \leq j \leq r} |z_{u}(rJ_{j})|2^{N_{\Lambda}/(r\Lambda)} z_{\ell}^{2((d-1)/r)} . \tag{32} $$

2. From this, if $G_{r,a}(J^{(r)})$ denotes a sequential or subsequential thermodynamic limit $\Lambda \to a^{2d}$, then

$$ |G_{r,a}(J^{(r)})| \leq \prod_{1 \leq j \leq r} |z_{u}(rJ_{j})/z_{\ell}|2^{(d-1)/r} , $$

with

$$ |z_{u}(J)| = \int \exp \left[ |J| (a^{d-4}/g^{2})^{1/2} |\text{Im Tr}(U-1)| - (a^{d-4}/g^{2}) A_{p}(U) \right] d\sigma(U) $$

$$ = (N_{c})^{-1} \int \exp \left[ |J| (a^{d-4}/g^{2})^{1/2} \sum_{1 \leq j \leq N} |\sin \lambda_{j}| - 2a^{d-4}/g^{2} \sum_{1 \leq j \leq N} (1-\cos \lambda_{j}) \right] \rho(\lambda) d^{N}\lambda $$

$$ \leq \frac{(a^{d-4}/g^{2})^{-N/2} \pi^{N+1/2} N_{S}^{1/2}}{N_{c}} \exp((\pi^{2}/8)N|J|^{2}) $$

$$ \equiv (a^{d-4}/g^{2})^{-N/2} \exp(c_{u}^{\prime} + \pi^{2}/8N|J|^{2}) . $$

Recalling $C^{2} = 4N$ and using Eq. (27) of Theorem 2 we obtain

$$ z_{\ell} = N_{c}^{-1} \int_{(-\pi,\pi)^{N}} \exp \left\{ - \left[ 2C^{2}(d-1)a^{d-4}/g^{2} \right] \sum_{j=1,\ldots,N} \lambda_{j}^{2} \right\} \rho(\lambda) d^{N}\lambda $$

$$ \geq N_{c}^{-1} \left( \frac{2(d-1)C^{2}a^{d-4}}{g^{2}} \right)^{-N/2} \left( \frac{4}{\pi^{2}} \right)^{N(N-1)/2} I_{\ell} $$

$$ \equiv (a^{d-4}/g^{2})^{-N/2} e^{c_{\ell}} , $$

where $c_{\ell}$ is defined in Theorem 2 and $I_{\ell} \equiv I_{2}(\pi C \sqrt{2(d-1)/(2g_{0})})$ and $I_{2}$ is the function defined in Eq. (25).

Hence, from the bounds of Eqs. (31) and (33), it follows that $G_{r,\Lambda,a}(J^{(r)})$ is a jointly analytic, entire complex function of the source field strengths $J_{j} \in \mathbb{C}$.
3. Letting $G_r(J^{(r)})$ denote a sequential or subsequential continuum limit a $\gamma < 0$ of $G_{r,a}(J^{(r)})$, then

$$\left| G_r(J^{(r)}) \right| \leq \exp \left[ \frac{\varrho_d}{r} (d-1) (c'_a - c_d) + \left( \pi^2/8 \right) N_r \sum_{1 \leq j \leq r} |J_j|^2 \right].$$

This bound is independent of the location and orientation of the $r$ plaquettes, and independent of the value of $a \in (0,1]$ and $g^2$.

**Remark 6** The generating function extends to an entire jointly analytic function of the source strengths $J_i$, $i = 1, \ldots, r$ and, by Cauchy estimates can be used to bound the $r$-plaquette scaled field correlations. We use the $C^\infty$ version of the Cauchy bounds. Recall that, for $C$, if $f(z)$ is analytic in the disk $|z| < R$, $R > 0$, then $|(d^n f/dz^n)(z = 0)| \leq n! \sup_{z:|z|=R_0} |f(z)|/R_0^n$, for any $0 < R_0 < R$ (see e.g. Ref. [28]). In particular, the coincident point plaquette-plaquette physical field correlation is bounded by $\text{const} \ a^{-d}$. The $a^{-d}$ factor is the same small a behavior of the coincident point, two-point correlation of the derivative of the real scalar physical free field (see the Appendix). This singular behavior is a measure of the ultraviolet asymptotic freedom.

**Remark 7** In obtaining the bounds on the scaled plaquette field generating function and correlations, we have used the group bond variable parametrization $U_b = \exp \{ i g a^{-d/4} \} \chi_b$. In the physically relevant $d = 4$ case, $U_b = e^{i g \chi_b}$ and $\langle \{|\text{Tr} M|\} \rangle_{\Lambda,a,g}$ is independent of the lattice spacing $a$, so that

$$\langle \{|\text{Tr} M|\} \rangle_{\Lambda,a,g} \equiv \langle \{|\text{Tr} M|\} \rangle_{\Lambda,a,g} = a^{d a / 2} \langle \{|\text{Tr} F|\} \rangle_{\Lambda,a,g}.$$

For the thermodynamic limit or subsequential limit, we drop the subscript $\Lambda$, so that we have

$$\langle \{|\text{Tr} M|\} \rangle_g = a^{d a / 2} \langle \{|\text{Tr} F|\} \rangle_{a,g}.$$

Of course, the continuum limit of the left-hand-side is $\langle \{|\text{Tr} M|\} \rangle_g$ and

$$\langle \{|\text{Tr} F|\} \rangle_g = a^{-d a / 2} \langle \{|\text{Tr} M|\} \rangle_g,$$

which displays the exact dependence on the lattice spacing $a$ as a multiplicative factor.

Lemma 1 and Theorems 1 – 4 are proved in the next section.

**VI. PROOFS OF THE LEMMA AND THEOREMS**

Here, following Ref. [24], we give a proof of Lemma 1. We also prove Theorems 1 – 4. The enhanced temporal gauge is chosen for proving these theorems. The proof of the upper stability bound on the partition function actually does not depend on this choice.

1. **Proof of Lemma 1**

For simplicity, we consider the case where we have four retained bonds in a plaquette. The other cases, when only one, two, or three bonds are retained, are similar. We define, for $1 \leq j \leq 4$, $L_j = i \sum_{1 \leq \alpha \leq N^2} x^j_\alpha \theta_\alpha$, so that $U_j = e^{\delta L_j}$ and $U_p = U_1 U_2 U_3 U_4^\dagger$.

Since $||L_j|| \leq ||L_j||_{H-S} = |x^j|$ and letting $U_p(\delta) = U_1(\delta) U_2(\delta) U_3(\delta) U_4^\dagger(\delta)$, $U_j(\delta) = e^{\delta L_j}$, for $\delta \in [0,1]$, by the fundamental theorem of calculus, suppressing $\delta$,

$$U_p - 1 = \int_0^1 d\delta \left[ L_1 U_1 U_2 U_3 U_4^\dagger + U_1 L_2 U_2 U_3 U_4^\dagger - U_1 U_2 L_3 U_3 U_4^\dagger - U_1 U_2 U_3 L_4 U_4^\dagger \right].$$

Using the triangle and Cauchy-Schwarz inequalities, we obtain

$$||U_p - 1|| \leq \sum_{j=1}^4 ||L_j|| \leq \sum_{j=1}^4 ||L_j||_{H-S} = \sum_{j=1}^4 |x^j| \leq 2 \left[ \sum_{j=1}^4 |x^j|^2 \right]^{1/2}.$$
But, \(\|U_p - 1\| \geq N^{-1/2} \|U_p - 1\|_{H-S}\). Hence,

\[
A_p = \|U_p - 1\|^2_{H-S} \leq 4N \sum_{j=1}^{4} |x^j|^2.
\]

By considering the number of terms in the sum over \(j\), the factor 4 in \(C^2\) is replaced by 1, 2 and 3, respectively, when only one, two or three retained bond variables appear in a retained plaquette.

Using this upper bound on the single plaquette action, and summing over the retained plaquettes, the second inequality of Lemma 1 is easily proven.

2. Proof of Theorem 1

The Case of Free b.c.:

Upper Bound: For ease of visualization we carry it out explicitly for \(d = 3\). An upper bound is obtained by discarding all horizontal plaquettes from the action, except those with temporal coordinates \(x^0 = 1\). We now perform the horizontal bond integration. Integrate over successive planes of horizontal bonds starting at \(x^0 = L\) and ending at \(x^0 = 2\). For the \(x^0 = 1\) horizontal plane, integrate over successive lines in the \(\mu = 2\) direction, starting at \(x^1 = L\) and ending at \(x^1 = 2\). For each horizontal bond variable, integration appears in only one plaquette. After the integration, in principle, the integral still depends on the other bond variables of the plaquette. However, using the left or right invariance of the Haar measure, the integral is independent of the other variables. In this way, we extract a factor \(z_u\).

In the total procedure, we integrate over the \(\Lambda_r\) horizontal bonds, so that we extract a factor \(z_u^{\Lambda_r}\).

Lower Bound: Using Lemma 1 gives the factorization and \(z_u\).

The Case of Periodic b.c.:

Upper Bound: Considering the positivity of each term in the model action of Eq. (2), since \(A^P \geq A\), we have

\[
Z_{\Lambda,\alpha}^P \leq \int e^{-A} \, dg \leq Z_{\Lambda,\alpha} \leq z_u^{\Lambda_r}.
\]

Lower Bound: Use the global quadratic upper bound of Lemma 1 on all \(\Lambda_r \cup \Lambda_e\) bond variables. Thus, we have

\[
Z_{\Lambda,\alpha}^P \geq z_u^{\Lambda_r + \Lambda_e},
\]

where \(U = \exp(iX), X = \sum_\alpha x_\alpha \theta_\alpha\).

3. Proof of Theorem 2

In Theorem 2, the first line for \(z_u\) [see Eq. (24)] is the application of the Weyl integration formula of Eq. (23) (see Refs. 16, 18). Use the inequality (see 27) \((1 - \cos x) \geq 2x^2/\pi^2, x \in [-\pi, \pi]\), in the action, and the inequality \((1 - \cos x) \leq x^2/2\) in each factor of \(\rho(\lambda)\). After making the change of variables \(y = 2[a^{(d-4)/2}/(\pi g)] \lambda\) and using the monotonicity of the integral, the result follows.

To obtain Eq. (27) for \(z_\ell\), apply the Weyl integration formula and use the inequality \(2[1 - \cos(\lambda_j - \lambda_k)] \geq (4/\pi^2) (\lambda_j - \lambda_k)^2, |\lambda| < \pi/2\) in each factor of the density \(\rho(\lambda)\). Then, use the positivity of the integrand and restrict the domain of integration to \((-\pi/2, \pi/2)^N\). In making the change of variables \(y = [a^{(d-4)/2}/g] C \sqrt{2(d-1)} \lambda\), the integral \(I_2(a)(d-4)/2\) appears (see Eq. (25)). Since \(I_2(u)\) is monotone increasing, the integral assumes its smallest value for \(a = 1\) and \(g^2 = g_0^2\).
4. Proof of Theorem 3

For periodic b.c. and the lower bound, using Theorem 1, we have the finite volume lattice normalized free energy

\[
\begin{align*}
  f^{P,n}_{\Lambda,a} &= \frac{1}{\Lambda_r} \ln Z^{P,n}_{\Lambda,a} = \frac{1}{\Lambda_r} \ln \left[ \frac{a^{d-4}}{g^2} \right]^{N^2 \Lambda_r/2} + \frac{1}{\Lambda_r} \ln Z^{P}_{\Lambda,a} \\
  &\geq \frac{1}{\Lambda_r} \ln \left[ \frac{a^{d-4}}{g^2} \right]^{N^2 \Lambda_r/2} + \frac{1}{\Lambda_r} \ln z^{\Lambda_r+\Lambda_r}.
\end{align*}
\]

Continuing the inequality and using Theorem 2, we have

\[
\begin{align*}
  f^{P,n}_{\Lambda,a} &\geq \frac{1}{\Lambda_r} \ln \left[ \frac{a^{d-4}}{g^2} \right]^{N^2 \Lambda_r/2} + \frac{\Lambda_e + \Lambda_r}{\Lambda_r} \ln \left[ \frac{a^{d-4}}{g^2} \right]^{-N^2/2} e^{c\ell} \\
  &\geq \ln \left[ \frac{a^{d-4}}{g^2} \right]^{N^2/2} + \frac{\Lambda_e + \Lambda_r}{\Lambda_r} \ln \left[ \frac{a^{d-4}}{g^2} \right]^{-N^2/2} + c\ell \\
  \end{align*}
\]

which gives, when \( \Lambda \to a\mathbb{Z}^d \),

\[
  f^{P,n}_{a} \geq c\ell.
\]

A similar calculation for the upper bound, setting to zero the number of extra bonds in the lattice with periodic b.c., \( \Lambda_e = 0 \), proves the theorem for the upper bound. Of course, for free b.c., set \( \Lambda_e = 0 \) in the above calculations.

5. Proof of Theorem 4

To prove Theorem 4, first use the generalized Holder’s inequality to bound \( G_{r,\Lambda,a}(J^{(r)}) \) by a product of single plaquette generating functions, i.e.

\[
|G_{r,\Lambda,a}(J^{(r)})| \leq \prod_{1 \leq j \leq r} |G_{1,\Lambda,a}(rJ_j)|^{1/r}.
\]

Now, since we are adopting periodic b.c., we can apply the multi-reflection method (see 4) to bound each factor in the product. To this end, we make a shift in the lattice by \((1/2a)\) in each coordinate direction. Also, we use the \( \pi/2 \) lattice rotational symmetry and translational symmetry to put the single plaquette in the \( \mu\nu = 01 \) coordinate plane in the first quadrant, with lower left vertex at \((a/2,a/2,\ldots,a/2)\). Then, we apply the multi-reflection method to obtain the bound

\[
|G_{1,\Lambda,a}(rJ_j)| \leq |G_{\Lambda,a}(rJ_j)|^{2d/\Lambda_r},
\]

where \( G_{\Lambda,a}(J) = \left[ Z^{P}_{\Lambda,a} \right]^{-1} Z^{P}_{\Lambda,a}(J) \), with \( Z^{P}_{\Lambda,a}(J) \) denoting \( Z^{P}_{\Lambda,a} \) with a source of uniform source strength \( J \). The source factor is given by \( \exp[J \sum_{p} \text{Tr} M_{p}(U_{p})] \), where the sum is over an array of plaquettes. The array consists of planes of plaquettes that are parallel to the 01 coordinate plane. In each plane, they are only alternating, i.e. like considering only squares of a same color on a chessboard. We obtain a greater upper bound by noting that

\[
J \text{Tr} M_{p}(U_{p}) \leq |J| a^{(d-4)/2} \left[ \text{Im} \text{Tr} (U_{p} - 1) \right] \leq |J| a^{(d-4)/2} \left[ \text{Tr} (U_{p} - 1) \right] \leq |J| a^{(d-4)/2} N^{1/2} \|U_{p} - 1\|_{H-S},
\]

where we have used the Cauchy-Schwarz inequality in the Hilbert-Schmidt inner product.

We also increase the bound by summing over all plaquettes in the lattice \( \Lambda \) that are parallel to the 01 coordinate plane. We denote this sum by \( \sum' \). In this way, we obtain the upper bound

\[
|Z^{P}_{\Lambda,a}(J)| \leq \int \exp \left[ J a^{(d-4)/2} g^{-1} N^{1/2} \sum' \|U_{p} - 1\|_{H-S} - a^{d-4} A^{P}/g^{2} \right] d\phi^{P}.
\]
As in the proof of the upper stability bound, for the periodic model, given above, we discard plaquette actions in $A^P$, for plaquettes that are not in $\Lambda$ so that

$$|Z^P_{\Lambda,a}(J)| \leq \int \exp \left[ |J|a^{(d-4)/2}g^{-1}N^{1/2} \sum_p \|U_p - 1\|_{H-S} - a^{d-4}A/g^2 \right] dg.$$ 

We bound the integral as we did for the upper stability bound for the free b.c. case. In this manner, we obtain the factorized bound

$$|Z^P_{\Lambda,a}(J)| \leq |z_a(J)|^{N^*},$$

and the factorized bound of Theorem 3 for $G_{r\Lambda a}(J^{(r)})$ is proved. Here, we have used the factorized lower bound of Theorems 1 and 2 for $Z^P_{\Lambda,a}$. Now, recalling that $\Lambda_a = L^d$, $\Lambda_r \simeq (d-1)L^d$ and $\Lambda_c = dL^{d-1}$, the factorized bound for $G_{r\Lambda a}(J^{(r)})$ follows.

Application of the Weyl integration formula of Eq. (24) [16–18] gives the integral for free and periodic b.c. is given by

$$\text{for the Gaussian symplectic ensemble (see [19, 20]). Keeping track of the numerical factors gives the final inequality for } z_a(J) \text{ and the proof of Theorem 4 is complete.}$$

VII. CONCLUDING REMARKS

We consider the Yang-Mills relativistic quantum field theory in an imaginary-time functional integral formulation. In the spirit of the lattice approximation to the continuum, the Wilson partition function is used as an ultraviolet regularization, where the hypercubic lattice $\Lambda \subset aZ^d$, $d = 2, 3, 4$, $a \in (0, 1]$, has $L$ (even) sites on a side. We use both free and periodic b.c. and our lattice has $\Lambda_a = L^d$ sites.

If $x = (x^0, \ldots, x^{d-1})$ denotes a site of $\Lambda$ and $e^\mu$, $\mu = 0, \ldots, (d-1)$ is a unit vector in the positive $\mu$ direction (0 labels the time direction), the partition function for free and periodic b.c. is given by

$$Z^R_{\Lambda,a} = \int \exp[(-a^{d-4}/g^2) A^B] d\gamma^B,$$

where $B = P$, for periodic b.c., and, for free b.c., we omit the superscript. For each lattice bond $b$, we assign a gauge bond variable $U_b \in G$, where $G$ is the gauge group $U(N)$. We denote by $b_1(x)$ the bond with the lattice initial point $x$ and terminal point $x + ae^\mu$. The gauge (gluon) fields are the parameters of the $N^2$-dimensional Lie algebra of $U(N)$. 


Parametrizing the bond variable $U_b$, $b = b_{\mu}(x)$, by $\exp[iag A_{\mu}(x)]$, we call the self-adjoint gauge potential $A_{\mu}(x)$ the physical gluon field. A lattice plaquette (minimal square in $\Lambda$), on the $\mu \nu$ coordinate plane and with vertices $x$, $x + a\epsilon^\mu$, $x + a\epsilon^\mu + a\epsilon^\nu$, $x + a\epsilon^\nu$, $\mu < \nu$, is denoted by $p_{\mu \nu}(x)$ and the model action $A^B$ is a sum over all plaquettes of four variable bond plaquette actions $A_p$ of each plaquette $p = p_{\mu \nu}(x)$. Defining

$$U_p = e^{iag A_{\mu}(x)}e^{iag A_{\nu}(x+a\epsilon^\mu)}e^{-iag A_{\mu}(x+a\epsilon^\nu)}e^{-iag A_{\nu}(x)}$$

the plaquette action $A_p$ for the plaquette $p$ is the Wilson plaquette action given by

$$A_p = \|[U_p - 1]|_{H - g} = 2Re \text{Tr}(1 - U_p) = 2\text{Tr}(1 - \cos X_p),$$

where we used the ordinary Hilbert-Schmidt norm and $U_p = e^{iX_p}$. $A_p$ is pointwise nonnegative and so is the total Wilson action $A^B = \sum_p A_p$.

With this, the gauge group measure $d\tilde{\gamma}^B$ above is a product over single bond $\mathcal{G}$ Haar measures $d\sigma(U_b)$. Whenever periodic b.c. is employed, as usual, we add extra bonds to $\Lambda$ connecting the endpoints of the boundary $\partial \Lambda$ of $\Lambda$ to the initial points of the boundary $\partial \Lambda$ in each spacetime direction $\mu = 0, 1, \ldots, (d - 1)$. The periodic plaquettes are those that can be formed from the totality of periodic bonds.

Formally, for small lattice spacing $a \in (0, 1]$, $A_p \simeq a^d g^2 \text{Tr} [F^\mu_{\nu}(x)]^2$, where, with finite difference derivatives given in Eq. (1), we have $F^\mu_{\nu}(x) = \partial^\mu A_{\nu}(x) - \partial^\nu A_{\mu}(x) + ig [A_{\mu}(x), A_{\nu}(x)]$, where the commutator is taken over the Lie algebra of $\mathcal{G} = U(N)$. Thus,

$$\langle a^d g^2 \rangle \sum_p A_p \simeq a^d \sum_{x \in \Lambda} \sum_{\mu, \nu = 0, 1, \ldots, (d - 1); \mu < \nu} \text{Tr} [F^\mu_{\nu}(x)]^2,$$

is the Riemann sum approximation to the classical smooth field continuum Yang-Mills action $\sum_{\mu < \nu} \int_{[0, 1]^d} \text{Tr} [F_{\mu \nu}(x)]^2 d^dx$, where $F_{\mu \nu}(x)$ is defined as above, but with usual partial derivatives.

Associated with this classical statistical mechanical model partition function and its correlations, there is a lattice quantum field theory. The quantum field theory is constructed in [4], via a Feynman-Kac formula. An important ingredient in the construction is Osterwalder-Seiler reflection positivity which requires the number of lattice points $L$, in each spacetime direction, to be even, as given above. The construction provides a quantum mechanical Hilbert space, mutually commuting self-adjoint spatial momentum operators and a positive energy operator.

Here, we define a normalized partition function $Z^B_{\Lambda,a}$ related to $Z^B_{\Lambda,a}$ by a $g$ and $a$-dependent multiplicative factor, and show that $Z^B_{\Lambda,a}$ obeys thermodynamic and ultraviolet stability bounds. These bounds guarantee the existence of a normalized free energy, at least for a sequential or subsequent thermalistic limit $(\Lambda \searrow 0)$ and, subsequently, a subsequential continuum limit $(a \searrow 0)$. The proof given here has some improvements on the results of [14, 24] and also extends the results to the case when periodic boundary conditions are employed. The use of periodic conditions allows us to employ the multireflection method [3] to prove bounds for the plaquette fields generating function which we also analyze here. Fixing the gauge, that we call the enhanced temporal gauge, is instrumental in the proof of our stability results and all other results in the paper. In this gauge, the temporal bond variables in $\Lambda$ are set to the identity, as well as certain specified bond variables on the boundary $\partial \Lambda$ of $\Lambda$.

As a key ingredient for the lower bound on $Z^B_{\Lambda,a}$, we have found a new global upper bound for the four-bond variable Wilson plaquette action. The bound is local and quadratic in the gluon bond variables of the plaquettes. It is surprising since the naive small $a$ approximation to the action has positive quartic terms in the above range. Therefore, our results are not restricted to small $g^2$.

We also consider correlations. We define a gauge invariant plaquette field for the plaquette $p_{\mu \nu}(x)$ by

$$\text{Tr} F_{\mu \nu} = \frac{1}{a^2 g} \text{Im} \text{Tr}(U_p - 1) = -\frac{i}{2a^2 g} \text{Tr}(U_p - U_p^\dagger) = \frac{1}{a^2 g} \sin X_p,$$

where $U_p = \exp(iX_p)$.

For the physical plaquette field, we parametrize $U_b$ by $U_b = \exp(iag A_b)$. For small lattice spacing $a$, we have $\text{Tr} F_{\mu \nu} \approx \text{Tr} F^B_{\mu \nu}(x)$, which for the gauge group $U(1)$ becomes the familiar $\partial_{\mu} A_{\nu}(x) - \partial_{\nu} A_{\mu}(x)$ [check Eq. (2)].
Inspired by Ref. \cite{25}, we also define the gauge-invariant scaled plaquette field by

\[ \text{Tr} \, M_{\mu\nu}(x) = a^{d/2} \text{Tr} \, F_{\mu\nu} = \left( \frac{a^{d-4}}{g^2} \right)^{1/2} \text{Im} \, \text{Tr} \left( U_p - 1 \right) = \left( \frac{a^{d-4}}{g^2} \right)^{1/2} \sin X_p. \]

Using periodic b.c., we also obtain bounds for the normalized generating function for \( r \in \mathbb{N} \) scaled plaquette fields defined, with a collection of \( r \) source plaquette fields with source strengths \( J^{(r)} = \{ J_1, \ldots, J_r \} \), in the finite lattice \( \Lambda \), by

\[ G_{r,\Lambda,a}(J^{(r)}) = \frac{Z_{r,\Lambda,a}^P(J^{(r)})}{Z_{\Lambda,a}^F}, \]

where \( Z_{r,\Lambda,a}^P \) is the partition function \( Z_{\Lambda,a}^F \) with the inclusion of the usual exponential of the source factors, namely, \( \exp[\sum_{1 \leq j \leq r} J_j \text{Tr} \, M_p(U_p)] \). Here, \( p_j, j = 1, \ldots, r \) are plaquettes \( p_{\mu_1\nu_1}(x_j) \). For fixed \( \mu \) and \( \nu \), and the plaquette \( p_{\mu\nu}(x) \), the plaquette field we consider is approximately, for small \( a \),

\[ \text{Tr} \, M_p(U_p) \approx a^{d/2} \text{Tr} \, F_{\mu\nu}^a(x). \]

(\text{Note that the trace does not give zero since } \mu \text{ and } \nu \text{ are fixed}) The \( r \)-plaquette scaled field correlation, with plaquettes originating at the external points \( x_E = \{ x_1, \ldots, x_r \} \), is given by

\[ \mathcal{G}_{r,\Lambda,a}(x_E) = [\partial/\partial(J_1(x_1))] \ldots [\partial/\partial(J_r(x_r))] \left[ G_{r,\Lambda,a}(x_E) \right]_{J_1,\ldots,J_r=0}. \]

We also prove a factorized bound for \( \mathcal{G}_{r,\Lambda,a}(x_E) \) so that, denoting by \( G_{ra}(J^{(r)}) \) any sequential or subsequential thermodynamic limit, we have

\[ G_{ra}(J^{(r)}) \leq \prod_{j=1}^r \left[ \frac{z_u(J_j)}{z_l} \right]^{2d(d-1)/r}, \]

where \( z_u(J_j) \) is a single bond variable single plaquette partition function with a source of strength \( J_j \). It is shown that

\[ z_u(J) \leq \left( \frac{a^{d-4}}{g^2} \right)^{-N^2/2} e^{c'} e^{cJ^2}, \]

where \( c' \) and \( c \) are finite real constants, independent of \( a \) and \( g^2 \). Thus,

\[ G_{ra}(J^{(r)}) \leq \exp \left[ \frac{2d}{r} (d-1)(c' - c_l) + cr \sum_{1 \leq j \leq r} J_j^2 \right]. \]

The bounds extend to complex source strengths. The generating function is a jointly analytic, entire function of the source strengths \( J_1, \ldots, J_r \) of the \( r \) plaquette fields. The \( r \)-plaquette field correlations admit a Cauchy integral representation and are bounded by applying Cauchy bounds.

In particular, for two coincident plaquettes, the physical plaquette correlation is bounded by \( a^{-d} \). The continuum limit \( a \searrow 0 \) singularity coincides with the bound \( a^{-d/d} \) on the two-point derivative field correlation for the massive case of a scalar free field; the bound being the exact value in the massless limit. It must be emphasized that the bounds on the plaquette field generating functions and correlations are new.

For the sake of clarity and transparency, we analyze an approximate model where the plaquette actions are set to zero for interior plaquettes which are perpendicular to the temporal direction plus some specified plaquettes on the boundary \( \partial \Lambda \) of \( \Lambda \). The plaquette correlations, as well as their thermodynamic \((L \searrow \infty)\) and continuum limits are obtained exactly. The same holds for a normalized free energy.

In obtaining these bounds, we parametrize \( U_b \) by \( \bar{U}_b = \exp\{iga^{-(d-2)/2}\} \chi_b \), where we call \( \chi_b \) the scaled field. \( \chi_b \) is related to the physical field \( A_b \) according to Eq. \cite{4}.

Defining the difference

\[ \delta_{\mu} \phi(x) \equiv \phi(x^+_\mu) - \phi(x) = \phi(x + ae^\mu) - \phi(x), \] 

we have

\[ \delta_{\mu} \chi_b = a^{d/2} \partial_{\mu} a_b, \]
where $\delta \chi_b = \chi_b(x_\mu^+) - \chi_b(x^+)$, and $b$ is the lattice bond connecting $x$ to $x_\mu^+$.

We show in the Appendix that, on the $a\mathbb{Z}^d$ lattice, with $\phi^u(x)$ being the physical free scalar field and the scaled field
\begin{align}
\phi(x) &= a^{(d-2)/2} \phi^u(x), \\
\delta \chi_b(x) &= a^{d/2} \delta \phi^u(x),
\end{align}
that the physical derivative field two-point correlation $\langle \partial_{\mu} \phi^u(x) \partial_{\nu} \phi^u(x) \rangle^u$ is bounded for $d = 2, 3, 4$ and all lattice points $x$ and $y$. For the massless case and coincident points $x \equiv y$, we have the exact result $\langle \langle \partial_{\mu} \phi^u(x) \rangle^2 \rangle^u = (da^d)^{-1}$.

This singular behavior is a measure of ultraviolet asymptotic freedom in the context of a lattice quantum field theory. Thus, for the gauge case, the singular behavior of the coincident point plaquette-plaquette physical field correlation is bounded by the singular behavior of the coincident point correlation of the free zero-mass derivative field.

Our stability and generating function bound results hold for any lattice Yang-Mills model defined with the Wilson action. Of course, this class of lattice models encompasses both the trivial ultraviolet limit of a Yang-Mills model, as well as the nonabelian gauge models which are ultraviolet asymptotically free in $d = 4$, like QCD. Our results extend to the gauge group $\mathcal{G} = SU(N)$ and other connected, compact Lie groups $\mathcal{G}$. By the Bolzano-Weierstrass theorem, the stability bounds ensure the existence, at least in the subsequential sense, of a normalized free energy for the model, but do not give us information on any other model property and its the energy-momentum spectrum. The existence of the normalized free energy and boundedness of the generating function are the only questions we analyze here. More analysis is indeed needed e.g. to obtain interesting correlation properties, such as their decay rates. Moreover, we point out that our bounds hold whether or not a mass gap persists in the $a \searrow 0$ continuum limit.

Finally, we hope that our methods and techniques can be combined with the more traditional methods to provide a complete construction of the $d = 4$ Yang-Mills and QCD models, including the verification of the axioms.

**APPENDIX: Unscaled or Physical and Scaled Real Scalar Free Fields**

In this Appendix, considering the case of the real scalar free field, we develop the relation between quantities expressed in terms of the local unscaled or physical field $\phi^u(x)$ and locally scaled fields $\phi(x) = s \phi^u(x)$, with

$$s \equiv s(a) = [a^{d-2}(m^2 + 2d\kappa^2)]^{1/2},$$

where $m^u$ and $\kappa^u$ are the unscaled field mass and lattice hopping parameter defined below. We refer the reader to Ref. [24] for more details.

In the continuum limit, the unscaled two-point correlation is infinite at coincident points. By our choice of $s$, for $d = 3, 4$, the scaled field correlations are more regular in the continuum limit. More precisely, they are finite at coincident points! For the massless free scalar field, the $a$-dependence of the scaling factor is $a^{(d-2)/2}$. The $a \searrow 0$ singular behavior of the two-point correlation at coincident points is $a^{-(d-2)/2}$. For the derivative of scaled fields, the two-point correlation is finite for $d = 2, 3, 4$ and the coincident point singular behavior is $a^{-d}$.

These singular behaviors can be taken as a measure of ultraviolet asymptotic freedom for the lattice approximation to a continuum QFT.

In the case of YM, as discussed above, this same scaling factor relation between the physical gluon fields $A_{\mu}(x)$ and the scaled gluon fields $a^{(d-2)/2} A_{\mu}(x)$ makes the scaled plaquette correlations bounded, in the continuum limit.

Of course, these scaling transformations are not to be confused with the usual canonical scaling.

In the hypercubic lattice with free b.c., the unscaled or physical action for the real scalar free field is, up to boundary conditions and for $x_\mu^+ \equiv x + a e^\mu$,

$$A_{B,a}^u = \frac{\kappa^2}{2} a^{d-2} \sum_{x,\mu} [\phi^u(x_\mu^+) - \phi^u(x)]^2 + \frac{1}{2} m^2 a^d \sum_x [\phi^u(x)]^2$$

$$= -\kappa^2 a^{d-2} \sum_{x,\mu} \phi^u(x_\mu^+) \phi^u(x) + \frac{1}{2} (m^2 a^d + 2d\kappa^2 a^{d-2}) \sum_x [\phi^u(x)]^2.$$  \hspace{1cm} (A1)

$A_{B,a}^u$ is a sum of an unscaled hopping term, with an unscaled hopping parameter $\kappa^2 > 0$, and a mass term.

The thermodynamic limit of the unscaled two-point free field correlation exists and has the representation

$$C_{B,a}^u(x, y) = \frac{1}{2(2\pi)^d} \int_{(-\pi/a, \pi/a)^d} e^{i\phi(x-y)} D_a^{-1} d^d p.$$
where
\[ D_a = \frac{\kappa^2}{a^2} \sum_{\mu} (1 - \cos p_{\mu}a) + \left( m_a^2 / 2 \right). \]

The continuum limit \( C^u(x, y) \) of \( C^u_a(x, y) \) also exists, in the sense of distributions and is
\[ C^u(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{i(p-x)\cdot y}}{\kappa_a^2 \sum_{\mu} (p^\mu)^2 + m_a^2} q^d p, \]
with \( x, y \in \mathbb{R}^d \). Of course, \( C^u(x, y) \) is infinite at coincident points \( x = y \).

The formula for \( C^u_a(x, y) \) is obtained as the thermodynamic limit of the finite lattice two-point correlation which in turn is obtained from the spectral representation of the symmetric matrix \( M_{\lambda,a}^{u,B} \) associated with the quadratic form, i.e. \( S_{\lambda,a}^{u,B} = (\phi^u, M_{\lambda,a}^{u,B} \phi^u) \), with b.c. \( B \). \( B \) can be taken as free or periodic b.c.

The formula which relates the two-point correlation to the spectral representation is
\[ C^u_{\lambda,a}(x, y) = \left[ \int \phi^u(x) \phi^u(y) e^{-S_{\lambda,a}^{u,B} d\theta^u} \right] \left[ \int e^{-S_{\lambda,a}^{u,B} d\theta^u} \right]^{-1}, \]
\[ = \frac{1}{2} \left[ M_{\lambda,a}^{u,B} \right]^{-1}(x, y) = \frac{1}{2} \sum_{\upsilon} (\lambda_{\upsilon})^{-1} v_{\upsilon}^B(x) [v_{\upsilon}^B(y)]^t, \]
where \( t \) denotes the transpose and we write the spectral representation of \( M_{\lambda,a}^{u,B} \) as
\[ M_{\lambda,a}^{u,B}(x, y) = \sum_{\upsilon} \lambda_{\upsilon} v_{\upsilon}^B(x) [v_{\upsilon}^B(y)]^t, \]
with \( \lambda_{\upsilon} \) denoting an eigenvalue of \( M_{\lambda,a}^{u,B}(\cdot, \cdot) \) and \( v_{\upsilon}^B(\cdot) \) the corresponding eigenvector. The \( \upsilon \)'s, \( v = (v^0, \ldots, v^{d-1}) \), \( v^\mu \in (-\pi/a, \pi/a) \), that parametrize the sum depend on the b.c. but the thermodynamic limit \( C^u_{\lambda,a} \) is the same for free and periodic b.c. For \( m_a = 0 \), zero is (respectively, not) an eigenvalue of \( M_{\lambda,a}^{u,B}(M_{\lambda,a}^{u,B}) \).

To obtain a more regular, less singular behavior for the correlations, as well as for the model free energy, we introduce the above defined locally scaled fields \( \phi(x) \). With this scaling, unscaled field action \( A^u_{B,a} \) is transformed to the scaled action
\[ A_{B,a}(\tilde{\phi}) = -\kappa^2 \sum_{x,a} \phi(x) \phi(x) + \frac{1}{2} \sum_x |\phi(x)|^2 \]
\[ = \frac{\kappa^2}{2} \sum_{x,a} [\phi(x) - \phi(x)]^2 + \frac{1}{2} \left( \frac{m_a}{\kappa_a} \right)^2 \kappa^2 \sum_x |\phi(x)|^2, \tag{A2} \]
where the scaled hopping parameter \( \kappa^2 \) is given by
\[ \kappa^2 = \left[ 2d + \left( \frac{m_a}{\kappa_a} \right)^2 \right]^{-1}. \tag{A3} \]

The thermodynamic limit of the scaled two-point correlation is
\[ C_a(x, y) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{i(\xi x - y) / a} D^{-1} d^d \theta, \]
with \( D = 1 - 2\kappa^2 \sum_{\mu} \cos q_{\mu} \). \( C_a(x, y) \) is bounded uniformly in \( a \in (0, 1] \), for \( d = 3, 4 \), by the coincident-point value with \( a = 0 \), namely, \( C_0 \equiv C_0(0) = (2\pi)^{-d} \int_{[-\pi, \pi]^d} [1 - d^{-1} \sum_{\mu} \cos q_{\mu}]^{-1} d^d \theta \), which is finite.

As it is usual, to analyze a QFT four-point correlation using the so called Bethe-Salpeter kernel (see e.g. Ref. 29, 30 and references therein), consider the continuum limit of \( C_a(x, y) \). Decompose \( D^{-1} \) as
\[ D^{-1} = D_0^{-1} = [D_0^{-1} - D_0^{-1}], \]
where \( D_0 \) is the \( D \) value for \( a = 0 \). Using the Riemann-Lebesgue lemma on the integral of the first term and the Lebesgue dominated convergence theorem on the second term, for the sequence \( a_r \rightarrow 0 \), where \( x = na, y = ma, x_r = \ldots \)
\( m_a, y_r = m a_r, \) and \( m, n \to \infty, \) such that \( x_r \to x_c \) and \( y_r \to y_c, \) with \( x_c, y_c \in \mathbb{R}^d, \) shows that \( \lim_{a \to 0} C_a(x, y) = 0, \) for \( x \neq y. \)

Thus, the scaled free field correlations are not singular, even at coincident points. Furthermore, the scaled and unscaled two-point correlations are related by \( C_a(x, y) = s^2 C_a^u(x, y). \) Moreover, upon letting \( C_a(x - y) \equiv C_a(x, y), \) the two-point correlation decay rate is defined by

\[
\lim_{v \to \infty} \left(-1/v \right) \ln C_a(v),
\]

with \( v \equiv x^0. \) Thus, the decay rates are the same for \( C_a^u(x) \) and \( C_a(x). \) Considering the Osterwalder-Seiler construction of a lattice quantum field theory (see \([4, 7]\)), this decay rate is the same as the scalar particle mass. The mass is a

\[
\text{with } \nu
\]







\[
\text{with solution}
\]

\[
m = \frac{2}{a} \sinh^{-1} \left( \frac{m a}{2 \kappa_u} \right) = \frac{2}{a} \ln \left[ \frac{\sqrt{r}}{2} + \frac{\sqrt{4 + r}}{2} \right] = \frac{m_a}{\kappa_u} + O \left( a^2 \left( \frac{m_a}{\kappa_u} \right)^3 \right),
\]

where \( r = (m_a a/\kappa_u)^2. \) It is important to observe that \( m \) is jointly analytic in \( a \) and \( m_a. \)

The above results continue to hold for the thermodynamic limit in the massless case \( m_u = 0, \) for the case of free b.c., as above. For periodic b.c., take the thermodynamic limit first with \( m_u \neq 0 \) and then take the limit \( m_u = 0 \) to get the same result as for free b.c. The massless case is obtained by setting \( \kappa^2 = (1/2d) \) in the formula for \( C_a(x, y). \) In this case, the scaled field is related to the unscaled field by

\[
\phi(x) = a^{(d-2)/2} \sqrt{2d} \kappa_u \phi^u(x),
\]

and we note that the \( a \)-dependence of the scaling factor is \( a^{(d-2)/2}. \)

For clarity and transparency, we deduce the relation \( \langle \phi(x)\phi(y) \rangle = s^2 \langle \phi^u(x)\phi^u(y) \rangle^u. \) The same procedure is followed in the gauge model case in the above text.

For the physical action

\[
A^u(\phi^u) = \kappa_u^2 a^{d-2} \sum_{x, \mu} \left[ \phi^u(x_\mu) - \phi^u(x) \right]^2 + \frac{m_u^2 a^d}{2} \sum_x \left[ \phi^u(x) \right]^2,
\]

we have, with \( Z^u = \int \exp \{-A^u(\phi^u)\} \ d\phi^u, \)

\[
\langle \phi^u(x)\phi^u(y) \rangle^u = \frac{1}{Z^u} \int \phi^u(x) \phi^u(y) \exp \{-A^u(\phi^u)\} \ d\phi^u
\]

\[
= \frac{1}{s^2 Z} \int s\phi^u(x) s\phi^u(y) \exp \{-A^u(\phi^u)\} \ d\phi^u
\]

\[
= \frac{1}{s^2 Z} \int \phi(x) \phi(y) \exp \{-A(\phi)\} \ d\phi
\]

\[
= \frac{1}{s^2} \langle \phi(x)\phi(y) \rangle,
\]

with \( Z = \int \exp \{-A(\phi)\} \ d\phi, \) and where we made the change of variables \( \phi(x) = s \phi^u(x) \) in the second line and also set \( A(\phi) = A^u(\phi^u = s^{-1} \phi). \) The scaled action is

\[
A(\phi) = -\kappa^2 \sum_{x, \mu} \phi(x_\mu) \phi(x) + \frac{1}{2} \sum_x \left[ \phi(x) \right]^2.
\]

The difference in the gauge case is that the analogue of the correlation fields is not linear and the action is not quadratic in the field.

Next, we define the generating function for the \( r \)-point scaled free field correlations by

\[
\exp \left[ \frac{1}{2} \sum_{1 \leq j, k \leq r} J_j C_a(x_j, x_k) J_k \right] \leq \exp \left[ C_0 r \sum_{1 \leq j \leq r} J_j^2 \right].
\]
where $a$ is uniformly bounded in field, we define the derivative field two-point function by

$$
\left[ \int e^{(K,w)-(w,C^{-1}w)/2} d^r w \right] \left[ \int e^{-(w,C^{-1}w)/2} d^r w \right]^{-1} = e^{(K,C)K/2},
$$

for the generating function of $r$ source variables $w_1, \ldots, w_r$ with source strengths $K_1, \ldots, K_r$.

For the case of $r$ real variables $w_1, \ldots, w_r$, we use the conventional formula

$$
\exp \left[ (1/2) \sum_{j,k} a^{2d} f(x_j) (\Delta^u)^{-1}(x_j, y_k) f(y_k) \right],
$$

where

$$(\Delta^u)^{-1}(x,y) = \frac{1}{2(2\pi)^d} \int_{(-\pi/a,\pi/a)^d} e^{ip(x-y)} [D^u_a(p)]^{-1} d^d p,$$

and

$$D^u_a(p) = \left( \frac{K_u}{a} \right)^2 \sum_{\mu} (1 - \cos p^\mu a) + m^2_\mu / 2.$$

The pairing $(\phi^u, f)$ is the Riemann sum approximation to $(\phi^u, f)_2 = \int_{\mathbb{R}^d} \phi^u(x) f(x) d^d x$ and the $a \searrow 0$ limit of the generating function is

$$
\exp \left[ (1/2) \int_{\mathbb{R}^d} f(x) \Delta^{-1}(x,y) f(y) \right] d^d x d^d y = \exp \left[ (1/2) (f, \Delta^{-1} f)_2 \right],
$$

where

$$\Delta^{-1}(x,y) = \int_{\mathbb{R}^d} e^{i p(x-y)} \frac{1}{K_u \sum_{\mu} (p^\mu)^2 + m^2_\mu} d^d p.$$
and, with \( \delta_\mu \phi(x) \delta_\nu \phi(y) = [\phi(x_\mu^+) - \phi(x)] [\phi(y_\mu^+) - \phi(y)] \), we have

\[
\langle \delta_\mu \phi(x) \delta_\mu \phi(y) \rangle = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{iq(x-y)/a} \frac{[e^{iq_\mu} - 1]}{1 - 2\kappa^2 \sum_\mu \cos q_\mu} d^d q
\]

\[
= s^2 a^2 \langle \partial_\mu^\nu \phi^u(x) \partial_\mu^\nu \phi^u(y) \rangle^u .
\]

By the spectral representation, we see that this correlation is most singular at coincident points. Indeed, for \( d = 3, 4 \), we have

\[
\langle [\phi^u(0)]^2 \rangle^u = \frac{1}{s^2(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1}{1 - 2\kappa^2 \sum_\mu \cos q_\mu} d^d q
\]

\[
\leq \frac{1}{s^2(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1}{1 - d^{-1} \sum_\mu \cos q_\mu} d^d q
\]

\[
= \frac{2d}{s^2} (\Delta^1)^{-1}(0, 0) ,
\]

where \( \Delta^1 \) is the Laplacian operator in the unit spaced lattice with action, for \( x \in \mathbb{Z}^d \),

\[
[\Delta^1 f](x) = 2df(x) - \sum_{\mu=1,\ldots,d} [f(x_\mu^+ = x + e^\mu) + f(x_\mu^- = x - e^\mu)] .
\]

Also, for \( d = 2, 3, 4 \),

\[
\langle [\partial_\mu^\nu \phi^u(0)]^2 \rangle^u = \frac{1}{s^2 a^2(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{2 (1 - \cos q_\mu)}{1 - 2\kappa^2 \sum_\mu \cos q_\mu} d^d q
\]

\[
\leq \frac{2}{s^2 a^2} ,
\]

where, to obtain the above inequality, we used the bound \( \kappa^2 \leq (1/2d) \), and lattice \((\pi/2)\)-rotation transformation symmetry about each coordinate axis. For \( m^2_u = 0 \), the bound becomes an equality.

The above relations display explicitly the \( a \searrow 0 \) coincident point singular behavior of the unscaled field and derivative field two-point correlation.

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