Phase Retrieval Using Unitary 2-Designs

Shelby Kimmel* and Yi-Kai Liu†

*Joint Center for Quantum Information and Computer Science (QuICS)
University of Maryland, College Park, MD, USA
†National Institute of Standards and Technology (NIST)
Gaithersburg, MD, USA

Abstract—We consider a variant of the phase retrieval problem, where vectors are replaced by unitary matrices, i.e., the unknown signal is a unitary matrix \( U \), and the measurements consist of squared inner products \( |\text{tr}(C^i U)|^2 \) with unitary matrices \( C \) that are chosen by the observer. This problem has been extensively studied in a number of contexts, including X-ray crystallography and optical imaging [1], [2].

In this paper we introduce a variant of the phase retrieval problem, where the vectors are replaced by unitary matrices: we want to reconstruct an unknown unitary matrix \( U \in \mathbb{C}^{d \times d} \) from measurements of the form \( |\text{tr}(C^i U)|^2 \) (for \( i = 1, \ldots, m \)), where the \( a_i \in \mathbb{C}^d \) are known vectors. This problem has been studied extensively in a number of contexts, including X-ray crystallography and optical imaging [1], [2].

From a mathematical point of view, this problem can be seen to be a special case of phase retrieval, where the measurements have some additional algebraic structure: in addition to being vectors in \( \mathbb{C}^d \), they are elements of the \( d \times d \) unitary group \( U(\mathbb{C}^d) \). This can be compared with previous work on phase retrieval, where the measurements were vectors in the unit sphere \( S^{d-1} \subset \mathbb{C}^d \). In particular, a recent line of work has led to tractable algorithms for phase retrieval, for measurements that are random vectors in \( S^{d-1} \), sampled from the Haar distribution, or from spherical 2-designs [6]–[10] (we will define these terms more precisely in the following sections.)

The main contribution of this paper is to prove analogous results when the measurements are random matrices in \( U(\mathbb{C}^{d \times d}) \), sampled from the Haar distribution, or from unitary 2-designs. In addition, this paper proves weaker recovery guarantees when the measurements are sampled from spherical and unitary 2-designs. (These measurements are of interest because they are easier to implement in experiments, e.g., for quantum process tomography.) In the following sections, we will describe these results in more detail.

As a side note, while we have argued that it is natural to consider measurement matrices \( C \) that are unitary, one may ask whether it is still possible to recover all matrices \( U \), and not just unitary ones? Unfortunately, the answer is no. For example, in the \( d = 2 \) case, consider the matrices

\[ U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

These matrices cannot be distinguished using any measurement of the form \( u \mapsto |\text{tr}(C^u U)|^2 \), where \( C \in \mathbb{C}^{2 \times 2} \) is unitary. (To see this, note that any such \( C \) can be written in the form

\[ C = \begin{pmatrix} \alpha & e^{i\varphi} \beta \\ \beta^* & e^{-i\varphi} \alpha^* \end{pmatrix}, \]

where \( \alpha, \beta \in \mathbb{C} \), \( |\alpha|^2 + |\beta|^2 = 1 \), and \( \varphi \in \mathbb{R} \). Hence we have \( |\text{tr}(C^u U)|^2 = |\alpha|^2 = |\text{tr}(C^v V)|^2 \), i.e., the measurement results are the same for \( U \) and \( V \). Thus, the best we can hope for is to reconstruct some large subset of matrices in \( \mathbb{C}^{d \times d} \), such as the set of all unitary matrices.

B. PhaseLift Algorithm

PhaseLift is a tractable algorithm for solving the phase retrieval problem, which works by “lifting” the quadratic problem in \( x \) to a linear problem in \( xx^\dagger \) [11], [12]. We propose the following variant of PhaseLift, for phase retrieval of unitary matrices.

Suppose we want to recover an unknown unitary matrix \( U \in \mathbb{C}^{d \times d} \). Our approach will be to solve for a Hermitian matrix \( \Gamma \in \mathbb{C}^{d^2 \times d^2} \), in such a way that the solution has the form

\[ \Gamma_{\text{ideal}} = \text{vec}(U)\text{vec}(U)^\dagger, \]

where \( \text{vec}(U) \) is the vectorization of the matrix \( U \). This is

\[ \Gamma = \begin{pmatrix} \alpha & e^{i\varphi} \beta \\ \beta^* & e^{-i\varphi} \alpha^* \end{pmatrix}, \]

where \( \alpha, \beta \in \mathbb{C} \), \( |\alpha|^2 + |\beta|^2 = 1 \), and \( \varphi \in \mathbb{R} \). Hence we have \( |\text{tr}(C^u U)|^2 = |\alpha|^2 = |\text{tr}(C^v V)|^2 \), i.e., the measurement results are the same for \( U \) and \( V \). Thus, the best we can hope for is to reconstruct some large subset of matrices in \( \mathbb{C}^{d \times d} \), such as the set of all unitary matrices.

B. PhaseLift Algorithm

PhaseLift is a tractable algorithm for solving the phase retrieval problem, which works by “lifting” the quadratic problem in \( x \) to a linear problem in \( xx^\dagger \) [11], [12]. We propose the following variant of PhaseLift, for phase retrieval of unitary matrices.

Suppose we want to recover an unknown unitary matrix \( U \in \mathbb{C}^{d \times d} \). Our approach will be to solve for a Hermitian matrix \( \Gamma \in \mathbb{C}^{d^2 \times d^2} \), in such a way that the solution has the form

\[ \Gamma_{\text{ideal}} = \text{vec}(U)\text{vec}(U)^\dagger, \]

where \( \text{vec}(U) \) is the vectorization of the matrix \( U \). This is

\[ \Gamma = \begin{pmatrix} \alpha & e^{i\varphi} \beta \\ \beta^* & e^{-i\varphi} \alpha^* \end{pmatrix}, \]

where \( \alpha, \beta \in \mathbb{C} \), \( |\alpha|^2 + |\beta|^2 = 1 \), and \( \varphi \in \mathbb{R} \). Hence we have \( |\text{tr}(C^u U)|^2 = |\alpha|^2 = |\text{tr}(C^v V)|^2 \), i.e., the measurement results are the same for \( U \) and \( V \). Thus, the best we can hope for is to reconstruct some large subset of matrices in \( \mathbb{C}^{d \times d} \), such as the set of all unitary matrices.
from which we can reconstruct $U$ (up to a global phase factor). Here, vec denotes the map $\text{vec} : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{d^2}$ that “flattens” a $d \times d$ matrix into a $d^2$-dimensional vector,

$$\text{vec}(U) = (U_{1,1}, U_{1,2}, U_{1,3}, \ldots, U_{d,d-1}, U_{d,d})^T.$$  \hspace{1cm} (I.4)

Note that this map preserves inner products: $\langle \text{vec}(U), \text{vec}(V) \rangle = \text{tr}(U^\dagger V)$.

We can describe our quadratic measurements of $U$ as follows. Let $C_1, \ldots, C_m \in \mathbb{C}^{d \times d}$ be the unitary measurement matrices introduced earlier. We use the same normalization convention as in previous work [9]; we work with re-scaled matrices $\sqrt{d}C_i$, which have roughly the same magnitude, in Frobenius or $\ell_2$ norm, as Gaussian random matrices. \(^1\)

We define the measurement operator $A : \mathbb{C}^{d^2 \times d^2} \rightarrow \mathbb{R}^m$ as follows:

$$A(\Gamma) = \left[ \text{vec}(\sqrt{d}C_i)^\dagger \text{vec}(\sqrt{d}C_i) \right]_{i=1}^m.$$  \hspace{1cm} (I.5)

Given the unknown matrix $U$, our measurement process returns a vector

$$y = A(\text{vec}(U)\text{vec}(U)^\dagger) + \varepsilon = \left[ d\text{tr}(C_i^\dagger U)^2 \right]_{i=1}^m + \varepsilon,$$

where $\varepsilon \in \mathbb{R}^m$ is an additive noise term.

Given the measurement results $y$, and an upper-bound $\eta \geq \|\varepsilon\|_2$ on the strength of the noise (measured using the $\ell_2$ norm), we will then find $\Gamma$ by solving a convex program:

$$\arg \min_{\Gamma \in \mathbb{C}^{d^2 \times d^2}} \text{tr}(\Gamma) \text{ such that }$$

$$\|A(\Gamma) - y\|_2 \leq \eta,$$

$$\text{tr}_1(\Gamma) = (I/d) \text{tr}(\Gamma),$$

$$\text{tr}_2(\Gamma) = (I/d) \text{tr}(\Gamma).$$  \hspace{1cm} (I.7)

Here, $\text{tr}_1(\Gamma) \in \mathbb{C}^{d \times d}$ and $\text{tr}_2(\Gamma) \in \mathbb{C}^{d \times d}$ denote the partial traces over the first and second indices of $\Gamma$, defined as follows. We view $\Gamma \in \mathbb{C}^{d^2 \times d^2}$ as a matrix whose entries $\Gamma(a,i),(b,j)$ are indexed by $(a,i),(b,j) \in \{1, \ldots, d\}^2$. Then $\text{tr}_1(\Gamma)$ and $\text{tr}_2(\Gamma)$ are the matrices whose entries are defined by

$$\text{tr}_1(\Gamma)_{i,j} = \sum_{a=1}^d \Gamma(a,i),(a,j), \quad \text{tr}_2(\Gamma)_{a,b} = \sum_{i=1}^d \Gamma(a,i),(b,i).$$  \hspace{1cm} (I.8)

The last three constraints in (I.7) have the effect of forcing $\Gamma$ to be a linear combination of terms $\text{vec}(V)\text{vec}(V)^\dagger$ where the $V$ are unitary. (This follows from some standard facts in quantum information theory. Let $|\Phi^+\rangle$ denote the maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \otimes |i\rangle \in \mathbb{C}^d$. Note that $\Gamma$ is proportional to the Jamiołkowski state $J(\mathcal{E}) = (\mathcal{E} \otimes I)(|\Phi^+\rangle \langle \Phi^+|) \in \mathbb{C}^{d^2 \times d^2}$ of some quantum process of $d \times d$ matrices $C_1, \ldots, C_m$, whose entries $G_{ij} \in \mathbb{C}$ are sampled independently from a complex Gaussian distribution with mean 0 and variance 1, has expected squared norm $\mathbb{E}[\|G\|^2_F] = d^2$.)

\(^1\)To justify this choice, we recall the definition of the Frobenius norm, $\|M\|_F = \text{tr}(M^\dagger M)^{1/2} = (\sum_{i,j} |M_{ij}|^2)^{1/2}$. For our re-scaled matrices $\sqrt{d}C_i$, we have $\|\sqrt{d}C_i\|_F^2 = \text{tr}(\sqrt{d}C_i^\dagger \sqrt{d}C_i) = d^2$. For comparison, we note that a Gaussian random matrix $G \in \mathbb{C}^{d \times d}$, whose entries $G_{ij} \in \mathbb{C}$ are sampled independently from a complex Gaussian distribution with mean 0 and variance 1, has expected squared norm $\mathbb{E}[\|G\|^2_F] = d^2$.}
in order to bound the 4th-moment tensor shown in equation (1.9).

Formally, we prove the following bound on the solution that is returned by the PhaseLift algorithm:

**Theorem 1.1.** There exists a numerical constant \( c_5 > 0 \) such that the following holds. Fix any \( c_0 > 2c_5 \). Consider the above scenario where \( m \) measurement matrices \( C_1, \ldots, C_m \) are chosen independently at random from a unitary 4-design \( \tilde{G} \) in \( \mathbb{C}^{d \times d} \).

Suppose that the number of measurements satisfies

\[
m \geq (64(4!)^2c_0)^2 \cdot d^2 \ln d. \tag{I.10}
\]

Then with probability at least \( 1 - \exp\left(-2m \left(4(4!)^{-4}\right)\right) \) (over the choice of the \( C_i \)), we have the following uniform recovery guarantee:

For any unitary matrix \( U \in \mathbb{C}^{d \times d} \), it is the case that any solution \( \Gamma_{opt} \) to the convex program (I.7) with noisy measurements (I.6) must satisfy:

\[
||\Gamma_{opt} - \text{vec}(U)\text{vec}(U)^\dagger||_F \leq \frac{128(4!)^2}{\sqrt{m}} \left(1 + \frac{2c_0}{c_0 - 2c_5}\right). \tag{I.11}
\]

We will present the proof in Section V.

**D. Phase Retrieval Using Unitary 2-Designs**

Next, we ask the question: how well does PhaseLift perform when the measurement matrices are sampled from a unitary 2-design, rather than a 4-design?

This question is motivated by a number of practical considerations. First, many experimental methods for quantum tomography make use of random Clifford operations [26], [27], which form a unitary 3-design, but not a 4-design [23]–[25]. Moreover, it is generally easier to construct unitary 2-designs, either using quantum circuits or group-theoretic methods [17], [19], [22]. The latter are particularly convenient for randomized benchmarking tomography, where the group structure is essential [48], [49].

We show that in this situation, PhaseLift still achieves approximate recovery of almost all unitary matrices. Here, the number of measurements \( m \) is still \( O(d^2 \text{poly}(\log d)) \), which is close to optimal. “Approximate recovery” means that the algorithm recovers the desired solution \( \text{vec}(U)\text{vec}(U)^\dagger \) up to an additive error of size \( \delta \|\text{vec}(U)\text{vec}(U)^\dagger\|_F = \delta d \), where \( \delta \ll 1 \) is some constant that is independent of the dimension \( d \). We show that this happens for all unitary matrices \( U \in \mathbb{C}^{d \times d} \), except for a subset that is small with respect to Haar measure.

In addition, we prove an analogous result for recovery of vectors using PhaseLift, when the measurements are sampled from a spherical 2-design. Again, we can show approximate recovery of almost all vectors in \( \mathbb{C}^d \), using \( m = O(d \text{poly}(\log d)) \) measurements.

These results give a clearer picture of the kinds of measurements that are sufficient for PhaseLift to succeed. In a sense, 2-designs are almost sufficient: while PhaseLift can sometimes fail using 2-design measurements, our results show that these failures occur very infrequently. (An explicit example of such a failure was previously shown in [7]; there exists a spherical 2-design \( D \) in \( \mathbb{C}^d \), and there exist vectors \( x \in \mathbb{C}^d \), such that phase retrieval of \( x \) requires \( m = \Omega(d^2) \) measurements.)

Our proofs require a new ingredient, which is a notion of non-spikiness of the unknown vector or matrix that one is trying to recover. Informally, we say that the unknown vector is “non-spiky” if it has small inner product with all of the possible measurement vectors. (This is reminiscent of previous work on low-rank matrix completion [34]–[36].)

Our proofs proceed in two steps: first, we show that almost all vectors are non-spiky, and then we prove that all non-spiky vectors can be recovered (uniformly) using PhaseLift. In the second step, the non-spikiness property plays a crucial role in bounding certain 4th-moment quantities, where we can no longer rely on the 4th moments of the measurement vectors, because the measurements are now being sampled from a 2-design.

We now state our results formally, focusing on the recovery of matrices. (We will describe our results on the recovery of vectors in Section III.)

Let \( \tilde{G} \) be a finite set of unitary matrices in \( \mathbb{C}^{d \times d} \). We say that a unitary matrix \( U \in \mathbb{C}^{d \times d} \) is non-spiky with respect to \( \tilde{G} \) (with parameter \( \beta \geq 0 \)) if the following holds:

\[
|\text{tr}(C^\dagger U)^2| \leq \beta, \quad \forall C \in \tilde{G}. \tag{I.12}
\]

Generally speaking, we will say that \( U \) is non-spiky when \( \beta \ll d \), e.g., \( \beta \leq \text{poly}(\log d) \). Our first result shows that, when the set \( \tilde{G} \) is not too large, almost all unitary matrices \( U \) are non-spiky with respect to \( \tilde{G} \).

**Proposition 1.2.** Choose \( U \in \mathbb{C}^{d \times d} \) to be a Haar-random unitary matrix. Then for any \( t \geq 0 \), with probability at least \( 1 - 4e^{-t} \) (over the choice of \( U \)), \( U \) is non-spiky with respect to \( \tilde{G} \), with parameter

\[
\beta = \frac{9n^2}{2} (t + \ln|\tilde{G}|). \tag{I.13}
\]

We will be interested in cases where the vectors in \( \tilde{G} \) form a unitary 2-design in \( \mathbb{C}^{d \times d} \). In these cases, the set \( \tilde{G} \) can be relatively small, i.e., sub-exponential in \( d \). As an example, let \( d = 2^n \), and let \( \tilde{G} \subset \mathbb{C}^{d \times d} \) be the set of Clifford operations on \( n \) qubits, so we have \( |\tilde{G}| \leq 2^{2n^2/3 + 3n} \) [26], [27]. Then with high probability, \( U \) is non-spiky with respect to \( \tilde{G} \), with parameter \( \beta = O(\log^2 d) \).

We then define a non-spiky variant of the PhaseLift algorithm. This consists of the convex program (I.7), together with the following additional constraint, which forces the solution to be non-spiky:

\[
0 \leq \text{vec}(C)^\dagger \text{vec}(C) \leq \beta, \quad \forall C \in \tilde{G}. \tag{I.14}
\]

We prove that the following recovery guarantee for this algorithm:

**Theorem 1.3.** There exists a numerical constant \( c_5 > 0 \) such that the following holds. Fix any \( \delta > 0 \) and \( c_0 > 2c_5 \). Consider the above scenario where \( m \) measurement matrices \( C_1, \ldots, C_m \) are chosen independently at random from a unitary 2-design \( \tilde{G} \subset \mathbb{C}^{d \times d} \). Let \( \beta \geq 0 \), and define \( \nu = \beta/\delta \).
Suppose that the number of measurements satisfies
\[ m \geq (8c_0 \nu^2)^2 \cdot d^2 \ln d. \] (I.15)

Then with probability at least \( 1 - \exp(-\frac{1}{128} m \nu^{-4}) \) (over the choice of the \( C_i \)), we have the following uniform recovery guarantee:

For any unitary matrix \( U \in \mathbb{C}^{d \times d} \) that is non-spiky with respect to \( G \) (with parameter \( \beta \), in the sense of (I.12)), it is the case that any solution \( \Gamma_{\text{opt}} \) to the convex program in (I.7) and (I.14), with noisy measurements (I.6), must satisfy:

\[ \| \Gamma_{\text{opt}} - \text{vec}(U)\text{vec}(U)^\dagger \|_F \leq \max \left\{ \delta \| \text{vec}(U)\text{vec}(U)^\dagger \|_F, \frac{16m^{\nu^2}}{\sqrt{m}} \left( 1 + \frac{2\delta}{c_0 - 2\delta} \right) \right\}. \] (I.16)

Note that \( m \), the number of measurements, again has almost-linear scaling with \( d^2 \), which is the number of degrees of freedom in the unknown matrix \( U \). In addition, \( m \) depends on the dimensionless quantity \( \nu = \beta/\delta \), where \( \beta \) measures the non-spikiness of the unknown matrix \( U \), and \( \delta \) controls the accuracy of the solution \( \Gamma_{\text{opt}} \). In typical cases, we will have \( \beta = O(\text{poly}(\log d)) \), and we will choose \( \delta \) to be constant, hence we have \( \nu = O(\text{poly}(\log d)) \).

We will present the proofs of these results for phase retrieval of matrices in Section IV, and for phase retrieval of vectors in Section III.

E. Application to Quantum Process Tomography

Finally, we describe an application of our results to quantum process tomography, in the case where the unknown process corresponds to a unitary operation \( U \in \mathbb{C}^{d \times d} \). This is sufficient to describe the dynamics of a closed quantum system, e.g., time evolution generated by some Hamiltonian, or the action of unitary gates in a quantum computer, including coherent (but not stochastic) errors. There exist fast methods for learning unitary processes, which achieve a “quadratic speedup” over conventional tomography, using ideas from compressed sensing and matrix completion [3]–[5], [40]–[44].

Here, we describe an alternative way of achieving this quadratic speedup, using phase retrieval. Our approach via phase retrieval has a significant advantage: the measurements can be performed in a way that is robust against state preparation and measurement errors (SPAM errors), by using randomized benchmarking techniques [45]–[49]. We discuss this in Section VI.

F. Outlook

In this paper we have studied a variant of the phase retrieval problem that seeks to reconstruct unitary matrices, and we have proposed a variant of the PhaseLift algorithm that solves this problem. We have proved strong reconstruction guarantees when the measurements are sampled from unitary 4-designs, as well as weaker guarantees when the measurements are unitary and spherical 2-designs. This leads to novel methods for quantum process tomography, when the unknown process is a unitary operation.

We mention a few interesting open problems. One is to prove error bounds that depend on the \( \ell_1 \) norm of the noise, rather than the \( \ell_2 \) norm, as in [8]. Another problem is to extend our recovery method to handle processes with Kraus rank \( r \) \( > 1 \) (e.g., stochastic errors), perhaps using the techniques in [9].

A third problem is to prove tighter bounds when the measurements are random Clifford operations. Clifford operations are particularly convenient for quantum tomography, and they play an essential role in randomized benchmarking methods. They are known to be a unitary 3-design, but not a 4-design [23]–[25]. However, our numerical simulations in Section VI seem to indicate that random Clifford measurements perform better than their classification as a unitary 3-design would suggest. This is supported by several recent results showing that random Clifford operations are “close to” a unitary 4-design [50], [51], and that random vectors chosen from a Clifford orbit perform well for phase retrieval of vectors [10]. Can one prove a similar result for phase retrieval of matrices, using random Clifford operations?

Finally, there has been progress in solving phase retrieval problems using gradient descent algorithms, such as Wirtinger Flow [52]. Can these methods be adapted to our matrix setting?

II. OUTLINE OF THE PAPER

In the rest of this paper, we will present the proofs of our theorems, as well as further details on quantum process tomography. In Section III, we begin with our simplest result, on phase retrieval of vectors, using measurements sampled from spherical 2-designs. In Section IV, we extend this to phase retrieval of unitary matrices, using measurements sampled from unitary 2-designs. In Section V, we extend this further, to handle measurements sampled from unitary 4-designs. Finally, in Section VI, we present further details and numerical simulations regarding quantum process tomography.

A. Notation

Let \( \mathbb{C}^{d \times d}_{\text{Her}} \) be the set of \( d \times d \) complex Hermitian matrices. Let \( \mathcal{L}(\mathbb{C}^{d \times d}, \mathbb{C}^{d \times d}) \) be the set of linear maps from \( \mathbb{C}^{d \times d} \) to \( \mathbb{C}^{d \times d} \). We will use calligraphic letters to denote these maps, e.g., \( \mathcal{U}, \mathcal{C} \). In general we will consider unitary maps, where \( \mathcal{U} : \rho \mapsto U \rho U^\dagger \) for a unitary \( U \), and \( \mathcal{C} : \rho \mapsto C \rho C^\dagger \) for a unitary \( C \). We will use \( \mathcal{U} \) to represent the unknown map we want to recover, and \( \mathcal{C} \) to represent the measurement maps we compare with \( \mathcal{U} \). Thus sometimes \( \mathcal{C} \) will represent an element of a unitary 2-design, and sometimes it will represent an element of a unitary 4-design.

For any matrix \( A \), let \( A^\dagger \) be its adjoint, let \( A^T \) be its transpose, and let \( A^* \) be its complex conjugate.

For any matrix \( A \), with singular values \( \sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_d(A) \geq 0 \), let \( \| A \| = \sigma_1(A) \) be the operator norm, let \( \| A \|_F = \sqrt{\sum_i \sigma_i^2(A)} \) be the Frobenius norm, and let \( \| A \|_* = \sum_i \sigma_i(A) \) be the nuclear or trace norm.

Because we use very similar approaches for the three cases of spherical 2-designs, unitary 2-designs, and unitary 4-designs, we will have similar notation in each section. In general, un-addorned notation (e.g. \( f, A \)) will be used for
spherical 2-designs, notation with tildes (e.g. \(\tilde{f}, \tilde{A}\)) will be used for unitary 2-designs, and notation with hats (e.g. \(\hat{f}, \hat{A}\)) will be used for unitary 4-designs.

III. Phase Retrieval and Low-Rank Matrix Recovery Using Spherical 2-Designs

We first consider phase retrieval of vectors. In fact, we follow the approach of [9], and consider the more general problem of low-rank matrix recovery. Whereas the authors of [9] showed an exact recovery result for measurements that are sampled from a spherical 4-design, we show an approximate, average-case recovery result for measurements that are chosen from a spherical 2-design.

We want to reconstruct an unknown matrix \(X \in \mathbb{C}^{d \times d}_{\text{Herm}}\), having rank at most \(r\), from quadratic measurements of the form

\[
y_i = w_i^\dagger X w_i + \varepsilon_i, \quad i = 1, 2, \ldots, m,
\]

where the measurement vectors \(w_i \in \mathbb{C}^d\) are known and are sampled independently at random from a spherical 2-design, and the \(\varepsilon_i \in \mathbb{R}\) are unknown contributions due to additive noise.

This problem has been studied previously, particularly in the rank-1 case (setting \(r = 1\)), where it corresponds to phase retrieval [7], [9]. Roughly speaking, it is known that spherical 4-designs are sufficient to recover \(X\) efficiently [9], whereas there exists a spherical 2-design that cannot recover \(X\) efficiently [7]. More precisely, \(X\) can be recovered (uniformly), via convex relaxation, from \(m = O(rd \log d)\) measurements chosen from any spherical 4-design; while on the other hand, \(X\) cannot be recovered, by any method, from fewer than \(m = \Omega(d^2)\) measurements chosen from a particular spherical 2-design.

Here, we show that 2-designs are sufficient to recover a large subset of all the rank-\(r\) matrices in \(\mathbb{C}^{d \times d}_{\text{Herm}}\). More precisely, we show that 2-designs achieve efficient approximate recovery of all low-rank matrices \(X\) that are non-spiky with respect to the measurement vectors (we will define this more precisely below). This implies that 2-designs are sufficient to recover generic (random) low-rank matrices \(X\), since these matrices satisfy the non-spikiness requirement with probability close to 1. Here, “efficient approximate recovery” means recovery up to an arbitrarily small constant error, using \(m = O(rd \log d)\) measurements, by solving a convex relaxation. This is reminiscent of results on non-spiky low-rank matrix completion [34]-[36].

A. Non-spikiness condition

Let \(G\) be a finite set of vectors in \(\mathbb{C}^d\), each of length 1. We say that \(X\) is non-spiky with respect to \(G\) (with parameter \(\beta \geq 0\)) if the following holds:

\[
|w^\dagger X w| \leq \frac{\beta}{d} \|X\|_s, \quad \forall w \in G.
\]  

(III.2)

Here, \(\|X\|_s = \text{tr}(|X|)\) denotes the nuclear norm. Generally speaking, we will say that \(X\) is non-spiky when the parameter \(\beta\) is much smaller than \(d\), e.g., of size \(\text{poly}(\log d)\).

We now show that, when the set \(G\) is not too large, almost all rank-\(r\) matrices \(X \in \mathbb{C}^{d \times d}_{\text{Herm}}\) are non-spiky with respect to \(G\).

Proposition III.1. Fix some rank-\(r\) matrix \(W \in \mathbb{C}^{d \times d}_{\text{Herm}}\). Construct a random matrix \(X\) by setting \(X = U W U^\dagger\), where \(U \in \mathbb{C}^{d \times d}\) is a Haar-random unitary operator.

Then for any \(t \geq 0\), with probability at least \(1 - 2e^{-t}\) (over the choice of \(U\)), \(X\) is non-spiky with respect to \(G\), with parameter

\[
\beta = 9\pi^{2}(t + \ln|G| + \ln r).
\]

(III.3)

Proof: Let \(S^{d-1}\) denote the unit sphere in \(\mathbb{C}^d\). Fix any vector \(w \in S^{d-1}\), and let \(x\) be a uniformly random vector in \(S^{d-1}\). Using Levy’s lemma [53], [54], we have that

\[
\Pr[|w^\dagger x| \geq \varepsilon] \leq 2 \exp\left(-\frac{\varepsilon^2}{2m}\right).
\]

(III.4)

Setting \(\varepsilon = \sqrt{\frac{9\pi^2}{d}(t + \ln|G| + \ln r)}\), we get that

\[
\Pr[|w^\dagger x| \geq \varepsilon] \leq 2e^{-t}|G|^{-1}r^{-1}.
\]

(III.5)

Now recall that \(X = U W U^\dagger\), and write the spectral decomposition of \(W\) as \(W = \sum_{i=1}^{r} \lambda_i v_i v_i^\dagger\). Then for any \(w \in G\), we can write \(w^\dagger X w\) as follows:

\[
w^\dagger X w = w^\dagger U W U^\dagger w = \sum_{i=1}^{r} \lambda_i |w^\dagger U v_i|^2.
\]

(III.6)

Each vector \(U v_i\) is uniformly random in \(S^{d-1}\), hence it obeys the bound in (III.5). Using the union bound over all \(w \in G\) and all \(v_1, \ldots, v_r\), we conclude that

\[
|w^\dagger U v_i| \leq \varepsilon, \quad \forall w \in G, \quad \forall i = 1, \ldots, r,
\]

(III.7)

with failure probability at most \(2e^{-t}\). Combining this with (III.6) proves the claim. \(\square\)

We will be interested in cases where the vectors in \(G\) form a spherical 2-design in \(\mathbb{C}^d\). In these cases, the set \(G\) can be relatively small, i.e., sub-exponential in \(d\). As an example, let \(d = 2^n\), and let \(G\) be the set of stabilizer states on \(n\) qubits, so we have \(|G| \leq 4^n\) [26]. Then with high probability, \(X\) is non-spiky with respect to \(G\), with parameter \(\beta = O(|\log^2 d|)\).

B. Convex relaxation (PhaseLift)

We now consider measurement vectors \(w_1, \ldots, w_m\) that are sampled independently from a spherical 2-design \(G \subset \mathbb{C}^d\). Using these measurements, we will seek to reconstruct those matrices \(X \in \mathbb{C}^{d \times d}_{\text{Herm}}\) that are low-rank, and non-spiky with respect to \(G\).

We assume we are given a bound on the nuclear norm of \(X\), say (without loss of generality):

\[
\|X\|_s \leq 1.
\]

(III.8)

As was done in [9], we will rescale the vectors \(w_i\) to be more like Gaussian random vectors, i.e., we will work with the vectors

\[
a_i = [(d + 1)d]^{1/4} w_i, \quad i = 1, \ldots, m.
\]

(III.9)
We also let $A$ be the renormalized version of the spherical 2-design $G$,
\[
A = \{(d+1)d^{1/4} w \mid w \in G\}. \quad (\text{III.10})
\]
Then $X$ will satisfy the following the non-spikiness conditions:
\[
|a^T X a| \leq \beta \sqrt{1 + \frac{1}{7^2}}, \quad \forall a \in A. \quad (\text{III.11})
\]
We define the sampling operator $\mathcal{A} : \mathbb{C}^{d \times d}_{\text{Herm}} \rightarrow \mathbb{R}^m$,
\[
\mathcal{A}(Z) = (a_i^T Z a_i)_{i=1}^m. \quad (\text{III.12})
\]
Using this notation, the measurement process returns a vector
\[
y = \mathcal{A}(X) + \varepsilon, \quad \|\varepsilon\|_2 \leq \eta, \quad (\text{III.13})
\]
where we assume we know an upper-bound $\eta$ on the size of the noise term.
We will solve the following convex relaxation, which is essentially the PhaseLift convex program, augmented with the noise term.
\[
\begin{align*}
\text{arg min} & \min_{Z \in \mathbb{C}^{d \times d}_{\text{Herm}}} \|Z\|_* \text{ such that } \\
& \|\mathcal{A}(Z) - y\|_2 \leq \eta, \\
& |a^T Z a| \leq \beta \sqrt{1 + \frac{1}{7^2}}, \quad \forall a \in A.
\end{align*}
\]
(over the choice of the $a_i$), we have the following uniform recovery guarantee:
For any rank-r matrix $X_{\text{true}} \in \mathbb{C}^{d \times d}_{\text{Herm}}$ that has nuclear norm $\|X_{\text{true}}\|_* \leq 1$, and is non-spiky with respect to $A$ (with parameter $\beta$, in the sense of (III.11)), it is the case that any solution $X_{\text{opt}}$ to the convex program (III.14) with noisy measurements (III.13) must satisfy:
\[
\|X_{\text{opt}} - X_{\text{true}}\|_F \leq \max\{\delta, \frac{16m\nu^2}{\gamma^2}(1 + \frac{1}{\eta})\}. \quad (\text{III.16})
\]
This error bound has similar scaling to the one in [9]: the number of measurements $m$ is only slightly larger than the number of degrees of freedom $O(rd)$, and the error $X_{\text{opt}} - X_{\text{true}}$ decreases like $\eta/\sqrt{m}$, until it reaches the limit $\delta$. However, now both of these bounds also involve the dimensionless quantity $\nu$, which in turn depends on the non-spikiness $\beta$ of the original matrix $X_{\text{true}}$, as well as the desired accuracy $\delta$ of the reconstructed matrix $X_{\text{opt}}$. In some sense, this is the price that one pays when one uses a weaker measurement ensemble, such as a spherical 2-design. It is an interesting question whether one can improve these bounds, to have a better dependence on $\nu$.

The proof follows the same strategy used in [9] to show low-rank matrix recovery using spherical 4-design measurements, by means of Mendelson’s small-ball method [30]. The key difference is that our measurements are spherical 2-designs only, so we do not have control over their fourth moments. Instead, we use the non-spikiness properties of $X_{\text{true}}$ and $X_{\text{opt}}$ to bound the fourth moments that appear in the proof. This allows us to show approximate recovery of $X_{\text{true}}$, up to an arbitrarily small constant error $\delta$.
We will present this proof in several steps.

D. Approximate recovery via modified descent cone
We begin by defining a modified version of the descent cone used in [9], [30]. Let $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper convex function, and let $x_{\text{true}} \in \mathbb{R}^d$ and $\delta \geq 0$. Then we define the modified descent cone $D'(f, x_{\text{true}}, \delta)$ as follows:
\[
D'(f, x_{\text{true}}, \delta) = \{y \in \mathbb{R}^d \mid \exists \tau > 0 \text{ such that } \|f(x_{\text{true}} + \tau y) - f(x_{\text{true}})\|_2 \geq \delta\}. \quad (\text{III.17})
\]
This is the set of all directions, originating at the point $x_{\text{true}}$, that cause the value of $f$ to decrease, when one takes a step of size at least $\delta$. Note that this is a cone, but not necessarily a convex cone.
We use this to state an approximate recovery bound, analogous to Prop. 7 in [9] and Prop. 2.6 in [30]. Let $y$ be a noisy linear measurement of $x_{\text{true}}$, given by $y = \Phi x_{\text{true}} + \varepsilon$, where $\Phi \in \mathbb{R}^{m \times d}$ and $\|\varepsilon\|_2 \leq \eta$. Then let $x_{\text{opt}}$ be a solution of the convex program
\[
\arg \min_{z \in \mathbb{R}^d} f(z) \text{ such that } \|f z - y\|_2 \leq \eta. \quad (\text{III.18})
\]
Then we have the following recovery bound:
\[
\|x_{\text{opt}} - x_{\text{true}}\|_2 \leq \max\left\{\delta, \frac{2\eta}{\lambda_{\min}(\Phi ; D'(f, x_{\text{true}}, \delta))}\right\}, \quad (\text{III.19})
\]
where $\lambda_{\min}(\Phi ; D'(f, x_{\text{true}}, \delta))$ is the conic minimum singular value,
\[
\lambda_{\min}(\Phi ; D'(f, x_{\text{true}}, \delta)) = \inf\{\|\Phi u\|_2 \mid u \in S^{d-1} \cap D'(f, x_{\text{true}}, \delta)\}. \quad (\text{III.20})
\]
This is proved easily, using the same argument as in [9], [30]. We now re-state this bound for the setup in Theorem III.2. Here the function $f : \mathbb{R}^{d \times d}_{\text{Herm}} \rightarrow \mathbb{R} \cup \{\infty\}$ is given by
\[
f(Z) = \begin{cases}
\|Z\|_* & \text{if } z \text{ is non-spiky in the sense of (III.11)}, \\
\infty & \text{otherwise},
\end{cases} \quad (\text{III.21})
\]
and we use
\[ D(f, X_{\text{true}}, \delta) = \{ Y \in \mathbb{C}^{d \times d}_{\text{Herm}} \mid \exists \tau > 0 \text{ such that} \]
\[ f(X_{\text{true}} + \tau Y) - f(X_{\text{true}}), \quad (\text{III.22}) \]
\[ \|\tau Y\|_F \geq \delta \}. \]

Our recovery bound is:
\[ \|X_{\text{opt}} - X_{\text{true}}\|_F \leq \max \left\{ \delta, \frac{2\eta}{\lambda_{\min}(A; D(f, X_{\text{true}}, \delta))} \right\}, \quad (\text{III.23}) \]
and we want to lower-bound the quantity \( \lambda_{\min} \):
\[ \lambda_{\min}(A; D(f, X_{\text{true}}, \delta)) \]
\[ = \inf_{Y \in E(X_{\text{true}})} \left[ \sum_{i=1}^{m} |\text{tr}(a_i a_i^\dagger Y)|^2 \right]^{1/2}, \]
where we define
\[ E(X_{\text{true}}) = \{ Y \in D(f, X_{\text{true}}, \delta) \mid \|Y\|_F = 1 \}. \quad (\text{III.24}) \]

E. Mendelson's small-ball method

In order to lower-bound \( \lambda_{\min} \), we will use Mendelson's small-ball method, following the steps described in [9] (see also [30]). We will prove a uniform lower-bound that holds for all \( X_{\text{true}} \) simultaneously, i.e., we will lower-bound \( \inf_{X_{\text{true}}} \lambda_{\min} \). We define
\[ E = \bigcup_{X_{\text{true}}} E(X_{\text{true}}), \quad (\text{III.25}) \]
where the union runs over all \( X_{\text{true}} \in \mathbb{C}^{d \times d} \) that have rank at most \( r \) and satisfy the non-spikiness conditions (III.11). We then take the infimum over all \( Y \in E \).

Using Mendelson’s small-ball method (Theorem 8 in [9]), we have that for any \( \xi > 0 \) and \( t \geq 0 \), with probability at least \( 1 - e^{-2u^2} \),
\[ \inf_{Y \in E} \left[ \sum_{i=1}^{m} |\text{tr}(a_i a_i^\dagger Y)|^2 \right]^{1/2} \geq \xi \sqrt{m Q_{2\xi}(E)} - 2W_m(E) - \xi t, \quad (\text{III.26}) \]
where we define
\[ Q_\xi(E) = \inf_{Y \in E} \left\{ \text{Pr}_{X_{\text{true}}} [\|\text{tr}(aa^\dagger Y)\| \geq \xi] \right\}, \quad (\text{III.27}) \]
\[ W_m(E) = \sup_{Y \in E} \left[ \text{tr}(HY) \right], \quad H = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \varepsilon_i a_i a_i^\dagger, \quad (\text{III.28}) \]
and \( \varepsilon_1, \ldots, \varepsilon_m \) are Rademacher random variables.

F. Lower-bounding \( Q_\xi(E) \)

We will lower-bound \( Q_\xi(E) \) as follows. Fix some \( Y \in E \). We know that \( Y \) is in \( E(X_{\text{true}}) \), for some \( X_{\text{true}} \in \mathbb{C}^{d \times d} \) that has rank at most \( r \) and is non-spiky in the sense of (III.11). Hence we know that \( \|Y\|_F = 1 \), and there exists some \( \tau > 0 \) such that \( X_{\text{true}} + \tau Y \) is non-spiky in the sense of (III.11), and we have
\[ \|X_{\text{true}} + \tau Y\|_F \leq \|X_{\text{true}}\|_F, \quad \|\tau Y\|_F \geq \delta. \quad (\text{III.29}) \]
Note that we have \( \|\tau Y\|_F = \tau \geq \delta \).

We will lower-bound \( \text{Pr}[\|\text{tr}(aa^\dagger Y)\| \geq \xi] \), for any \( \xi \in [0,1] \), using the Paley-Zygmund inequality, and appropriate bounds on the second and fourth moments of \( \text{tr}(aa^\dagger Y) \). Let us define a random variable
\[ S = |\text{tr}(aa^\dagger Y)|. \quad (\text{III.30}) \]

We can lower-bound \( \text{E}(S^2) \), using the same calculation as in Prop. 12 of [9]:
\[ \text{E}(S^2) = \text{tr}(\tau Y)^2 + \|\tau Y\|_F^2 \geq \tau^2. \quad (\text{III.31}) \]
To handle \( \text{E}(S^4) \), we need to use a different argument from the one in [9], since our \( a \) is sampled from a spherical 2-design, not a 4-design. Our solution is to upper-bound \( S \), using the non-spikiness properties of \( X_{\text{true}} \) and \( X_{\text{true}} + \tau Y \):
\[ S = |\text{tr}(aa^\dagger X_{\text{true}}) + \text{tr}(aa^\dagger (X_{\text{true}} + \tau Y))| \]
\[ \leq (2\beta) \sqrt{1 + \frac{1}{d}}. \quad (\text{III.32}) \]
This implies an upper-bound on \( \text{E}(S^4) \):
\[ \text{E}(S^4) \leq (2\beta)^2 (1 + \frac{1}{4}) \text{E}(S^2). \quad (\text{III.33}) \]
Putting it all together, we have:
\[ \text{Pr}_{a \sim A} [\|\text{tr}(aa^\dagger Y)\| \geq \xi] \]
\[ = \text{Pr}[S^2 \geq \tau^2 \xi^2] \geq \text{Pr}(S^2 \geq \xi^2 \text{E}(S^2)) \]
\[ \geq (1 - \xi^2)^2 \frac{\text{E}(S^2)^2}{\text{E}(S^4)} \]
\[ \geq (1 - \xi^2)^2 \frac{\tau^2}{(2\beta)^2 (1 + \frac{1}{d})}. \quad (\text{III.34}) \]
Finally, using the fact that \( \tau \geq \delta \), we have that
\[ Q_{1/2}(E) \geq (1 - \xi^2)^2 \frac{\delta^2}{(2\beta)^2 (1 + \frac{1}{d})}. \quad (\text{III.35}) \]

G. Upper-bounding \( W_m(E) \)

Next, we will upper-bound \( W_m(E) \), using essentially the same argument as in [9]. Recall that the argument in [9] showed an upper-bound on \( W_m(F) \), where \( F \) was a slightly different set than our \( E \), and the \( a_i \) were sampled from a spherical 4-design rather than a 2-design. The argument used the following steps:
\[ W_m(F) = \text{E} \sup_{Y \in F} \text{tr}(HY) \]
\[ \leq \text{E} 2\sqrt{r} \|H\| \]
\[ \leq 2\sqrt{r} \cdot (3.1049) \sqrt{d \log(2d)}, \quad (\text{III.36}) \]
using Lemma 10 and Prop. 13 in [9], and provided that \( m \geq 2d \log d \).

The first step still works to upper-bound \( W_m(E) \), because \( E \subset F \). To see this, recall that the set \( F \) was defined as follows:
\[ F = \bigcup_{X_{\text{true}}} F(X_{\text{true}}), \quad (\text{III.37}) \]
where the union was over all $X_{\text{true}} \in \mathbb{C}^{d \times d}_{\text{Herm}}$ with rank at most $r$, and

$$F(X_{\text{true}}) = \{ Y \in D(||X_{\text{true}}, 0) | |Y||_F = 1 \}. \quad \text{(III.38)}$$

This can be compared with our definition of the set $E$ in (III.25) and (III.24).

The second step still works for our choice of the $a_i$, because Prop. 13 in [9] does not use the full power of the spherical 4-design. In fact it only requires that the $a_i$ are sampled from a spherical 1-design (because the proof relies mainly on the Rademacher random variables $\varepsilon_i$). Thus we conclude that

$$W_m(E) \leq 2\sqrt{r} \cdot (3.1049) \sqrt{d} \log(2d). \quad \text{(III.39)}$$

### H. Final result

Combining equations (III.24), (III.26), (III.35) and (III.39), and setting $\xi = \frac{1}{4}$, $\nu = 2\beta/d$ and $t = \frac{1}{16} \sqrt{m} (\nu^2 (1 + \frac{1}{4}))^{-1}$, we get that

$$\inf \lambda_{\text{min}}(A; D(f, X_{\text{true}}, \delta)) \geq \frac{1}{2} \sqrt{m \frac{9}{16\nu^2 (1 + \frac{1}{4})}} - (12.44) \sqrt{rd \log(2d)} - \frac{1}{4} t \quad \text{(III.40)}$$

$$= \frac{1}{8} \sqrt{m \frac{1}{\nu^2 (1 + \frac{1}{4})}} - (12.44) \sqrt{rd \log(2d)}. \quad \text{(III.41)}$$

Now we set $m = (8C\nu^2 (1 + \frac{1}{4}))^2 \cdot rd \log(2d)$, which implies

$$\sqrt{m \frac{1}{8C\nu^2 (1 + \frac{1}{4})}} \geq \sqrt{rd \log(2d)}, \quad \text{(III.42)}$$

hence

$$\inf \lambda_{\text{min}}(A; D(f, X_{\text{true}}, \delta)) \geq \frac{1}{8} \sqrt{m \frac{1}{\nu^2 (1 + \frac{1}{4})}} \frac{C_\nu - 1}{C_{\nu}}. \quad \text{(III.43)}$$

This can be plugged into our approximate recovery bound (III.23). This finishes the proof of Theorem III.2. \qed

### IV. Phase Retrieval Using Unitary 2-Designs

Next, we extend our results from vectors to unitary matrices. First, we consider phase retrieval using measurements that are sampled from unitary 2-designs, as described in Section I-D.

#### A. Non-spikiness

First, let $\tilde{G} \subset \mathbb{C}^{d \times d}$ be a unitary 2-design. We prove Proposition I.2, showing that almost all unitary matrices are non-spiky with respect to $\tilde{G}$:

Proof: Fix any $C \in \tilde{G}$. Using Levy's lemma [56], and noting that the function $U \mapsto \text{tr}(U^4 C)$ has Lipschitz coefficient $\eta \leq ||C||_F = \sqrt{d}$, we have that

$$\text{Pr}[|\text{tr}(U^4 C)| \geq \varepsilon] \leq 4 \exp\left(-\frac{2}{9\pi^2} \varepsilon^2\right). \quad \text{(IV.1)}$$

Setting $\varepsilon = \sqrt{\frac{9\pi^2}{2^2} (t + \ln |\tilde{G}|)}$, we get that

$$\text{Pr}[|\text{tr}(U^4 C)| \geq \varepsilon] \leq 4e^{-t/|\tilde{G}|}. \quad \text{(IV.2)}$$

Using the union bound over all $C \in \tilde{G}$, we conclude that

$$|\text{tr}(U^4 C)| \leq \varepsilon, \quad \forall C \in \tilde{G}, \quad \text{(IV.3)}$$

with failure probability at most $4e^{-t}$. This proves the claim. \qed

#### B. Approximate recovery of non-spiky unitary matrices

Next, let $\tilde{A}$ denote the measurement operator defined in equation (I.5), where the measurement matrices $C_1, \ldots, C_m$ are chosen independently at random from the unitary 2-design $\tilde{G}.$

We now prove Theorem I.3, showing that all non-spiky unitary matrices can be recovered approximately via PhaseLift, given these measurements. We will present this proof in several steps, following the same strategy as in the previous section.

We will use the following notation. For any unitary matrix $U \in \mathbb{C}^{d \times d}$, let $U : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{d \times d}$ be the quantum process whose action is given by $U : \rho \mapsto U \rho U^\dagger.$ Let $J(U)$ be the corresponding Jamiolkowski state, as defined in equation (VI.3), which can be written as

$$J(U) = \frac{1}{d} \text{vec}(U) \text{vec}(U)^\dagger \in \mathbb{C}^{d^2 \times d^2}. \quad \text{(IV.4)}$$

Using this notation, we can write the desired solution of the PhaseLift convex program as $J(U)d$, and we can write the measurement operator $\tilde{A}$ as

$$\tilde{A}(\Gamma) = \left[\text{tr}(\Gamma J(C_i)d^2)\right]_i = 1. \quad \text{(IV.5)}$$

We want to recover the solution $J(U)d$, up to an additive error of size $\delta ||J(U)d||_F = \delta d.$

#### C. Modified descent cone

We define the function $\tilde{f} : \mathbb{C}^{d^2 \times d^2} \rightarrow \mathbb{R} \cup \{\infty\}$ as follows:

$$\tilde{f}(\Gamma) = \begin{cases} \text{tr}(\Gamma) & \text{if } \Gamma \text{ is non-spiky in the sense} \\
\infty & \text{of (I.14), and also satisfies the last three constraints in (I.7)}. \end{cases} \quad \text{(IV.6)}$$

Our recovery bound is:

$$||\Gamma_{\text{opt}} - J(U)d||_F \leq \max\left\{ \delta d, \frac{2\eta}{\lambda_{\text{min}}(\tilde{A}; D(\tilde{f}, J(U)d, \delta d))}\right\}, \quad \text{(IV.7)}$$

and we want to lower-bound the quantity $\lambda_{\text{min}}$:

$$\lambda_{\text{min}}(\tilde{A}; D(\tilde{f}, J(U)d, \delta d)) = \inf_{Y \in \tilde{E}(U)} \left[ \sum_{i=1}^m |\text{tr}(Y J(C_i)d^2)|^2 \right]^{1/2}, \quad \text{(IV.8)}$$

where we define

$$\tilde{E}(U) = \{ Y \in D(\tilde{f}, J(U)d, \delta d) | ||Y||_F = 1 \}. \quad \text{(IV.9)}$$
D. Mendelson’s small-ball method

We will use Mendelson’s small-ball method to lower-bound $\lambda_{\text{min}}$. We will prove a uniform lower-bound that holds for all $U$ simultaneously, i.e., we will lower-bound $\inf_U \lambda_{\text{min}}$. We define

$$\tilde{E} = \bigcup_U \tilde{E}(U),$$

where the union runs over all unitary processes $U : \rho \mapsto U\rho U^\dagger$ such that $U \in \mathbb{C}^{d \times d}$ satisfies the non-spikiness conditions (I.12). We then take the infimum over all $Y \in \tilde{E}$.

We will then lower-bound this quantity, using Mendelson’s small-ball method (Theorem 8 in [9]): for any $\xi > 0$ and $t \geq 0$, with probability at least $1 - e^{-2t^2}$, we have that

$$\inf_{Y \in \tilde{E}} \left[ \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |\text{tr}(Y J(C_i) d^2)| \right]^{1/2} \geq \xi \sqrt{\text{tr}(\tilde{E})} - 2 \tilde{W}_m(\tilde{E}) - \xi t,$$

(IV.11)

where we define

$$\tilde{Q}_\xi(\tilde{E}) = \inf_{Y \in \tilde{E}} \left\{ \text{Pr} \left[ \left| \text{tr}(Y J(C) d^2) \right| \geq \xi \right] \right\},$$

(IV.12)

$$\tilde{W}_m(\tilde{E}) = \mathbb{E}_{C \sim \tilde{G}} \left[ \sup_{Y \in \tilde{E}} \text{tr}(Y \tilde{H}) \right],$$

(IV.13)

and $\varepsilon_1, \ldots, \varepsilon_m$ are Rademacher random variables.

E. Lower-bounding $\tilde{Q}_\xi(\tilde{E})$

We will lower-bound $\tilde{Q}_\xi(\tilde{E})$ as follows. Fix some $Y \in \tilde{E}$. We know that $Y$ is in $\tilde{E}(U)$, for some unitary process $U : \rho \mapsto U\rho U^\dagger$ such that $U$ is non-spiky in the sense of (I.12). (As a result, $J(U) d$ also satisfies the last three constraints in (I.7).)

Hence, we know that $||Y||_F = 1$, and there exists some $\tau > 0$ such that $J(U) d + \tau Y$ is non-spiky in the sense of (I.14) and satisfies the last three constraints in (I.7), and we have

$$\text{tr}(J(U) d + \tau Y) \leq \text{tr}(J(U) d), \quad ||\tau Y||_F \geq \delta d.$$  

(IV.14)

Note that we have $||\tau Y||_F = \tau \geq \delta d$.

We will lower-bound $\text{Pr}[\left| \text{tr}(Y J(C) d^2) \right| \geq \xi]$, for any $\xi \in [0, 1]$, using the Paley-Zygmund inequality, and appropriate bounds on the second and fourth moments of $\text{tr}(Y J(C) d^2)$. To simplify the notation, let us define a random variable

$$\tilde{S} = \tau |\text{tr}(Y J(C) d^2)|.$$  

(IV.15)

1) Lower-bounding $\mathbb{E}(\tilde{S}^2)$: We will first put $\tilde{S}$ into a form that is easier to work with. We can write $Y$ in the form $Y = J(Y) d$, which is the rescaled Choi matrix of some linear map $Y \in \mathcal{L}(\mathbb{C}^{d \times d}, \mathbb{C}^{d \times d})$. Using the relationship between the trace of Choi and Liouville representations (see Appendix A), we have

$$\tilde{S} = \tau |\text{tr}(J(C) J(Y) d^2)| = \tau d |\text{tr}(C^T Y L^2)|.$$

(IV.16)

We can calculate $\mathbb{E}(\tilde{S}^2)$ by using the fact that $C \in \tilde{G}$ is chosen from a unitary 2-design:

$$\mathbb{E}(\tilde{S}^2) = \tau^2 d^2 \int_{\text{Haar}} \left| \text{tr} \left( (U \otimes U^\dagger \otimes U \otimes U^*) (Y_L \otimes Y_L) \right) \right| dU$$

$$\geq \tau^2 d^2 \left| \text{tr} \left( \int_{\text{Haar}} (U \otimes U^\dagger \otimes U \otimes U^*) dU (Y_L \otimes Y_L) \right) \right|.$$  

(IV.17)

Now using Weingarten functions [31]–[33], we have that

$$\int_{\text{Haar}} (U \otimes U \otimes U^\dagger \otimes U^*) dU = \frac{P_{3412} + P_{4321}}{d^2 - 1} - \frac{P_{3421} + P_{4312}}{d(d^2 - 1)}$$

(IV.18)

where

$$P_{\sigma(1), \sigma(2), \ldots, \sigma(t)} = \sum_{j_1, j_2, \ldots, j_t} |j_1\rangle\langle j_1| \otimes |j_2\rangle\langle j_2| \otimes \cdots \otimes |j_t\rangle\langle j_t|,$$

(IV.19)

is a permutation of the registers.

Let $T_2(\cdot)$ transpose the second half of the indices of a matrix in $\mathbb{C}^{d^2 \times d^2}$. That is, for any matrix

$$X = \sum_{ijkl} x_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|,$$

(IV.20)

we have

$$T_2(X) = \sum_{ijkl} x_{ijkl} |i\rangle\langle j| \otimes |l\rangle\langle k|.$$  

(IV.21)

Then note that

$$P_{1324} T_2 \left( \int_{\text{Haar}} (U \otimes U \otimes U^\dagger \otimes U^*) dU \right) P_{1324}$$

$$= \int_{\text{Haar}} (U \otimes U^* \otimes U \otimes U^*) dU.$$  

(IV.22)

Combining with Eq. (IV.17) and Eq. (IV.18), we have

$$\mathbb{E}(\tilde{S}^2) \geq \tau^2 d^2 \left| \text{tr} \left( \frac{P_{3412} + P_{4321}}{d^2 - 1} - \frac{P_{3421} + P_{4312}}{d(d^2 - 1)} \right) P_{1324} (Y_L \otimes Y_L) \right|.$$  

(IV.23)

Because addition commutes with permutation and transposition and trace, we need to calculate

$$\text{tr} \left( P_{1324} T_2(P_\sigma) P_{1324}(Y_L \otimes Y_L) \right)$$  

(IV.24)
for $\sigma \in \{3412, 4312, 3421, 4321\}$. Letting $\sigma = (abcd)$, we have

$$
\text{tr} \left( P_{1324} T_2 (P_{abcd} P_{1324}(Y^L \otimes Y^L)) \right) \\
= \text{tr} \left( P_{1324} T_2 \left( \sum_{j_1, j_2, j_3, j_4} |j_1\rangle |j_2\rangle |j_3\rangle |j_4\rangle |j_6\rangle \otimes |j_6\rangle |j_3\rangle |j_4\rangle |j_2\rangle |j_1\rangle \right) P_{1324}(Y^L \otimes Y^L) \right) \\
= \text{tr} \left( P_{1324} \sum_{j_1, j_2, j_3, j_4} |j_1\rangle |j_2\rangle |j_3\rangle |j_4\rangle |j_6\rangle |j_3\rangle |j_4\rangle |j_1\rangle |j_2\rangle |j_6\rangle P_{1324}(Y^L \otimes Y^L) \right) \\
= \text{tr} \left( \sum_{j_1, j_2, j_3, j_4} |j_1\rangle |j_2\rangle |j_3\rangle |j_4\rangle |j_6\rangle |j_3\rangle |j_4\rangle |j_1\rangle |j_2\rangle |j_6\rangle (Y^L \otimes Y^L) \right) \\
= \sum_{j_1, j_2, j_3, j_4} \langle j_6 | j_1 | j_2 | j_3 | j_4 | j_6 | Y^L | j_1 | j_2 | j_3 | j_4 | j_6 \rangle. 
$$

(IV.25)

We know that $Y = J(Y) d$ satisfies the last three constraints in (I.7). Because of these constraints, we have

$$
\sum_{i_1, i_2, i_3} \langle i_2 i_3 | Y^L | i_1 \rangle = \sum_{i_1, i_2, i_3} \langle i_2 i_1 | J(Y) | i_3 \rangle \\
= \sum_{i_1, i_2, i_3} d \langle i_2 | \text{tr}_2 (J(Y)) | i_3 \rangle \\
= d \text{tr}(J(Y)), 
$$

(IV.26)

and

$$
\sum_{i_1, i_2, i_3} \langle i_1 | Y^L | i_2 i_3 \rangle = \sum_{i_1, i_2, i_3} d \langle i_1 | \text{tr}_1 (J(Y)) | i_3 \rangle \\
= \sum_{i_1, i_2, i_3} d \langle i_2 | \text{tr}_1 (J(Y)) | i_3 \rangle \\
= d \text{tr}(J(Y)). 
$$

(IV.27)

We now go through the four permutations of Eq. (IV.23):

- **P$_{3412}$**: In this case, $a = 3$, $b = 4$, $c = 1$, and $d = 2$. Plugging into Eq. (IV.25) and using Eqs. (IV.27) and (IV.26), we have

$$
\text{tr} \left( P_{1324} T_2 (P_{3412} P_{1324}(Y^L \otimes Y^L)) \right) = d^2 \text{tr}(J(Y))^2. 
$$

(IV.28)

- **P$_{4312}$**: In this case, $a = 4$, $b = 3$, $c = 2$, and $d = 1$. Plugging into Eq. (IV.25), we have

$$
\text{tr} \left( P_{1324} T_2 (P_{4312} P_{1324}(Y^L \otimes Y^L)) \right) \\
= \sum_{j_1, j_2, j_3, j_4} \langle j_4 j_3 | Y^L | j_1 j_2 \rangle \langle j_4 j_3 | Y^L | j_2 j_1 \rangle \\
= \sum_{j_1, j_2, j_3, j_4} \langle j_4 j_3 | Y^L | j_1 j_2 \rangle \langle j_4 j_3 | Y^L | j_2 j_1 \rangle \\
= \sum_{j_1, j_2, j_3, j_4} \langle j_4 j_3 | Y^L | j_1 j_2 \rangle^* \langle j_4 j_3 | Y^L | j_2 j_1 \rangle \\
= \| Y^L \|_F^2 \\
= d^2 \| Y \|_F^2. 
$$

(IV.29)

where we used Eq. (A.4).

- **P$_{4321}$**: In this case, $a = 4$, $b = 3$, $c = 2$, and $d = 1$. Plugging into Eq. (IV.25) and using Eqs. (IV.27) and (IV.26), we have

$$
\text{tr} \left( P_{1324} T_2 (P_{4321} P_{1324}(Y^L \otimes Y^L)) \right) = d^2 \text{tr}(J(Y))^2. 
$$

(IV.30)

- **P$_{4312}$**: In this case, $a = 4$, $b = 3$, $c = 1$, and $d = 2$. Plugging into Eq. (IV.25) and using Eqs. (IV.27) and (IV.26), we have

$$
\text{tr} \left( P_{1324} T_2 (P_{4312} P_{1324}(Y^L \otimes Y^L)) \right) = d^2 \text{tr}(J(Y))^2. 
$$

(IV.31)

Putting it all together, we have

$$
E(\hat{S}^2) \geq \tau^2 d^4 \left( \frac{\text{tr}(J(Y))^2 + \| J(Y) \|_F^2}{d^2 - 1} - \frac{2 \text{tr}(J(Y))^2}{d(d^2 - 1)} \right) \\
\geq \tau^2 d^4 (1 - \frac{2}{d}) \text{tr}(J(Y))^2 + \tau^2 d^2 \| J(Y) \|_F^2 \\
\geq 0 + \tau^2 d^2 \| J(Y) \|_F^2 = \tau^2 \| J(Y) \|_F^2. 
$$

(IV.32)

2) Upper-bounding $E(\hat{S}^4)$: To handle $E(\hat{S}^4)$, we need to use a different argument from that in [9], since we are sampling from a unitary 2-design, not a 4-design. Our solution is to use an upper-bound $\tilde{S}$, using the non-spikiness properties of $J(U) d$ and $J(U) d + \tau Y$:

$$
\tilde{S} = | - \text{tr}(J(U) d J(C) d^2) + \text{tr}(J(U) d + \tau Y) J(C) d^2 | \\
\leq \beta d. 
$$

(IV.33)

This implies an upper-bound on $E(\hat{S}^4)$:

$$
E(\hat{S}^4) \leq \beta^2 d^2 E(\hat{S}^2). 
$$

(IV.34)
Putting it all together, and using the Paley-Zygmund inequality, for any $\xi \in [0,1]$, we have:

$$\Pr[|\text{tr}(YJ(C)d^2)| \geq \xi] = \Pr[S^2 \geq \xi^2 \tau^2] \geq \Pr[S^2 \geq \xi^2 \mathbb{E}(S^2)] \geq (1 - \xi^2)^2 \frac{\mathbb{E}(S^2)^2}{\mathbb{E}(S^4)} \geq (1 - \xi^2)^2 \frac{\tau^2}{\beta^2 d^2}.$$  

(IV.35)

Thus, using the fact that $\tau \geq \delta$, we have that

$$\hat{Q}_\xi(\tilde{E}) \geq (1 - \xi^2)^2 \frac{\delta^2}{\beta^2}.$$  

(IV.36)

F. Upper-bounding \(\hat{W}_m(E)\)

In this section, we will bound

$$\hat{W}_m(E) = \mathbb{E} \sup_{Y \in E} \text{tr}(Y \tilde{H}), \quad \hat{H} = \frac{1}{\sqrt{m}} \sum_{i=1}^m \xi_i J(C_i)d^2.$$  

(IV.37)

where the $\xi_i$ are Rademacher random variables, and the expectation is taken over the $\xi_i$ and the choice of the unitaries $C_i$. For this bound, we will only need the fact that the $C_i$ are chosen from a unitary 1-design, rather than a unitary 2-design.

We start by following the argument in [9], where it is shown that

$$\mathbb{E} \sup_{Y \in F_r} \text{tr}(Y \tilde{H}) \leq \mathbb{E} 2\sqrt{\tau\|\tilde{H}\|}$$  

(IV.38)

where

$$F_r = \bigcup_{X_{\text{true}}} F(X_{\text{true}}),$$  

(IV.39)

and the union is over all $X_{\text{true}} \in \mathbb{C}^{d\times d}_{\text{Herm}}$ with rank at most $r$, and

$$F(X_{\text{true}}) = \{Y \in D(\|\cdot\|, X_{\text{true}}, 0) | \|Y\|_F = 1\}.$$  

(IV.40)

This can be compared with our definition of the set $\tilde{E}$ in (IV.10) and (IV.9).

In our case, we have $\tilde{E} = \bigcup_j \tilde{E}(U)$, where the union is over all processes $U$ whose unnormalized Choi state $J(U)d \in \mathbb{C}^{d^2 \times d^2}_{\text{Herm}}$ has rank 1, and where $U$ satisfies the non-spikiness condition (1.14). $\tilde{E}(U)$ is defined similarly to $F(X_{\text{true}})$, but using the function $f$ from (IV.6) rather than the trace norm. While $f$ involves the trace $\text{tr}(\cdot)$ instead of the trace norm $\|\cdot\|$, because we are considering only positive semidefinite matrices, it can be replaced by the trace norm. Hence $\tilde{E} \subset F_1$, and so

$$\mathbb{E} \sup_{Y \in \tilde{E}} \text{tr}(Y \tilde{H}) \leq \mathbb{E} \sup_{Y \in F_1} \text{tr}(Y \tilde{H}) \leq 2\mathbb{E}\|\tilde{H}\|.$$  

(IV.41)

Now we analyze

$$\mathbb{E}_{\xi, C} \left| \frac{1}{\sqrt{m}} \sum_{i=1}^m \xi_i J(C_i)d^2 \right|^2,$$  

(IV.42)

using similar tools to what is used in [9]. Since the $\xi_i$’s form a Rademacher sequence and $J(C_i)$ are Hermitian, we can apply the non-commutative Khintchine inequality [55], [57]:

$$\mathbb{E}_{\xi, C} \left| \frac{1}{\sqrt{m}} \sum_{i=1}^m \xi_i J(C_i)d^2 \right|^2 \leq \mathbb{E}_C \sqrt{2 \ln(2d^2)/m} \left( \sum_{i=1}^m \|J(C_i)\|^2 \right)^{1/2} d^2$$

(IV.43)

where in the second line we used Jensen’s inequality, and we used the fact that $J(C_i)$ is a rank one projector with trace 1, so $\|J(C_i)\|^2 = J(C_i)$.

We now apply the matrix Chernoff inequality of Theorem 15 in [9] to $\mathbb{E}_C \|\sum_{i=1}^m J(C_i)\|$. To apply the theorem, we need to bound

$$\left\| \mathbb{E}_C \sum_{i=1}^m J(C_i) \right\|$$  

and  

$$\|J(C_i)\|.$$  

(IV.44)

Since $J(C_i)$ corresponds to a quantum state, its maximum eigenvalue is 1, so $\|J(C_i)\| = 1$. Next, notice

$$\mathbb{E}_C \sum_{i=1}^m J(C_i) = \sum_{i=1}^m \mathbb{E}_C J(C_i) = \frac{m}{d^2} I.$$  

(IV.45)

Because the $C_i$ are drawn from a unitary 2-design, $\mathbb{E}_C J(C_i)$ is the state that results when a depolarizing channel is applied to one half of a maximally entangled state. (Randomly applying operations from a unitary 1-design results in the depolarizing channel.) The resulting state is the maximally mixed state $I/d^2$ in $\mathbb{C}^{d^2 \times d^2}$. Thus

$$\left\| \mathbb{E}_C \sum_{i=1}^m J(C_i) \right\| = \frac{m}{d^2}.$$  

(IV.46)

Then the Matrix Chernoff Inequality (Theorem 15 in [9]) tells us that for any $\gamma > 0$,

$$\mathbb{E} \left| \sum_{j=1}^m J(C_j) \right| \leq (e^{\gamma} - 1)m + d^2 \log d^2 \gamma.$$  

(IV.47)

Minimizing $\gamma$ gives

$$\mathbb{E} \left| \sum_{j=1}^m J(C_j) \right| \leq c_4 m/d^2$$  

(IV.48)

for some numerical constant $c_4 > 0$. Plugging this expression into Eq. (IV.43), we obtain the desired bound:

$$\hat{W}_m(\tilde{E}) \leq c_5 \sqrt{\ln d \cdot d},$$  

(IV.49)

where $c_5 > 0$ is some numerical constant.
G. Final result

Combining equations (IV.8), (IV.11), (IV.36) and (IV.49), and setting \( \xi = 1/4, \nu = \beta/\delta \) and \( t = \frac{1}{16}\sqrt{m} \), we get that:

\[
\inf_{\tilde{A}} \lambda_{\min}(\tilde{A}; D(\bar{f}, J(U)d, d)) \geq \frac{1}{4} \sqrt{m} \frac{9}{16t^2} - 2c_5 \ln d \cdot d - \frac{1}{4} t \tag{IV.50}
\]

Now we set \( m \geq (8c_20)^2 \cdot d^2 \ln d \), which implies

\[
\frac{1}{8\sqrt{m} d_0} \sqrt{m} \geq \sqrt{\ln d \cdot d}, \tag{IV.51}
\]

hence

\[
\inf_{\tilde{A}} \lambda_{\min}(\tilde{A}; D(\bar{f}, J(U)d, d)) \geq \frac{\sqrt{m}}{8t^2} \left( 1 - \frac{2c_5}{c_0} \right). \tag{IV.52}
\]

This can be plugged into our approximate recovery bound (IV.7). This finishes the proof of Theorem 1.3. \( \square \)

V. Phase Retrieval Using Unitary 4-Designs

In this section, we again consider the PhaseLift algorithm for recovering an unknown unitary matrix \( U \in \mathbb{C}^{d \times d} \). However, we now consider a stronger measurement ensemble: we let the measurement matrices \( C_1, \ldots, C_m \) be chosen from a unitary 4-design \( G \) in \( \mathbb{C}^{d \times d} \). This leads to a stronger recovery guarantee: PhaseLift achieves exact recovery of all unitaries \( U \). This claim was presented in Theorem I.1. We show the proof here.

We use the same notation as in the previous section, except that here we call the measurement operator \( \tilde{A} \) rather than \( \tilde{A} \). We can write the desired solution of the PhaseLift convex program as \( J(U)d \), and we can write the measurement operator \( \tilde{A} \) as

\[
\tilde{A}(\Gamma) = \left[ \text{tr}(\Gamma J(C_i)d^2) \right]_{i=1}^m. \tag{V.1}
\]

We want to recover the solution \( J(U)d \), up to an additive error of size \( \delta \|J(U)d\|_F = \delta d \).

A. Descent cone

For the case of unitary 4-designs, we use a similar descent cone to the one used in [9], [30] — that is, for \( f : \mathbb{C}^{d \times d} \rightarrow \mathbb{R} \cup \{ \infty \} \) a proper convex function, and \( X_{\text{true}} \in \mathbb{C}^{d \times d} \), we use the descent cone \( D(f, X_{\text{true}}, 0) \). This is the set of all directions, originating at the point \( X_{\text{true}} \), that cause the value of \( f \) to decrease.

We define the function \( \bar{f} : \mathbb{C}^{d \times d} \rightarrow \mathbb{R} \cup \{ \infty \} \) as follows:

\[
\bar{f}(\Gamma) = \begin{cases} \text{tr}(\Gamma) & \text{if } \Gamma \text{ satisfies the last three constraints in (I.7),} \\ \infty & \text{otherwise.} \end{cases} \tag{V.2}
\]

By Prop. 7 in [9], our recovery bound is:

\[
\|\Gamma_{\text{opt}} - J(U)d\|_F \leq \frac{2\eta}{\lambda_{\min}(\tilde{A}; D(\bar{f}, J(U)d, d))}, \tag{V.3}
\]

and we want to lower-bound the quantity \( \lambda_{\min} \):

\[
\lambda_{\min}(\tilde{A}; D(\bar{f}, J(U)d, d)) = \inf_{Y \in E(U)} \left[ \sum_{i=1}^m \text{tr}(Y J(C_i)d^2) \right]^{1/2}, \tag{V.4}
\]

where we define

\[
\hat{E}(U) = \{ Y \in D(\bar{f}, J(U)d, 0) | \|Y\|_F = 1 \}. \tag{V.5}
\]

B. Mendelson’s small-ball method

We will use Mendelson’s small-ball method to lower-bound \( \lambda_{\min} \). We will prove a uniform lower-bound that holds for all \( U \) simultaneously, i.e., we will lower-bound \( \inf_U \lambda_{\min} \). We define

\[
\hat{E} = \bigcup U \hat{E}(U), \tag{V.6}
\]

where the union runs over all unitary processes \( U \in \mathcal{L}(\mathbb{C}^{d \times d}, \mathbb{C}^{d \times d}) \) of the form \( U : \rho \mapsto U \rho U^\dagger \). We then take the infimum over all \( Y \in \hat{E} \).

We will then lower-bound this quantity, using Mendelson’s small-ball method (Theorem 8 in [9]): for any \( \xi > 0 \) and \( t \geq 0 \), with probability at least \( 1 - e^{-2t^2} \), we have that

\[
\inf_{Y \in E} \left[ \sum_{i=1}^m \text{tr}(Y J(C_i)d^2) \right]^{1/2} \geq \xi \sqrt{m} \hat{Q}_\xi(\hat{E}) - 2\hat{W}_m(\hat{E}) - \xi t, \tag{V.7}
\]

where we define

\[
\hat{Q}_\xi(\hat{E}) = \inf_{Y \in E} \left\{ \Pr \left[ \text{tr}(Y J(C)d^2) \geq \xi \right] \right\}, \tag{V.8}
\]

\[
\hat{W}_m(\hat{E}) = \mathbb{E}_{\nu \sim \pm 1 \in \mathcal{G} \cup \{ \infty \}} \left[ \sup_{Y \in E} \text{tr}(Y \hat{H}) \right], \quad \hat{H} = \frac{1}{\sqrt{m}} \sum_{i=1}^m \nu_i J(C_i)d^2, \tag{V.9}
\]

and \( \nu_1, \ldots, \nu_m \) are Rademacher random variables.

C. Lower-bounding \( \hat{Q}_\xi(\hat{E}) \)

We will lower-bound \( \Pr_{\nu \sim \pm 1 \in \mathcal{G}}[\text{tr}(Y J(C)d^2) \geq \xi] \), for any \( \xi \in [0, 1] \), using the Paley-Zygmund inequality, and appropriate bounds on the second and fourth moments of \( \text{tr}(Y J(C)d^2) \). To simplify the notation, let us define a random variable

\[
\hat{S} = |\text{tr}(Y J(C)d^2)|. \tag{V.10}
\]

We will first put \( \hat{S} \) into a form that is easier to work with. We can write \( Y \) in the form \( Y = J(Y)d \), which is the rescaled Choi matrix of some linear map \( \gamma \in \mathcal{L}(\mathbb{C}^{d \times d}, \mathbb{C}^{d \times d}) \). Using the relationship between the trace of Choi and Liouville representations (see Appendix A), we have

\[
\hat{S} = |\text{tr}(J(Y)d J(C)d^2)| = |d| \text{tr}((C^\dagger)^y L^y) = |d| \text{tr}((C^\dagger \otimes C^y) L^y)). \tag{V.11}
\]
1) Lower-bounding $\mathbb{E}(\hat{S}^2)$: Because we are working with a unitary 4-design, and $\hat{S}^2$ only depends on the second moment of the distribution, the bound will be the same as for a unitary 2-design, and we can use our bound from Section IV-E1, Eq. (IV.32) (with $\tau = 1$):

$$
\mathbb{E}(\hat{S}^2) \geq d^2(1 - \frac{2}{d}) \text{tr}(J(Y))^2 + d^2\|J(Y)\|_F^2 \\
\geq \frac{1}{2}d^2 \max \{\text{tr}(J(Y))^2, \|J(Y)\|_F^2\} ,
$$

and also (using Eq. (IV.32)):

$$
\mathbb{E}(\hat{S}^2) \geq d^2\|J(Y)\|_F^2 = \|Y\|_F^2 = 1.
$$

2) Upper-bounding $\mathbb{E}(\hat{S}^4)$: Now we would like to bound $\mathbb{E}[\hat{S}^4]$. Because we are considering a unitary 4-design, which has the same fourth moments as the Haar measure on unitaries, instead of taking the average $\mathbb{E}[\hat{S}^4]$ over the 4-design, we will take the average over Haar random unitaries. We have

$$
\mathbb{E}[\hat{S}^4] = d^4 \int_{\text{Haar}} dU|\text{tr}((U \otimes U^*)Y^L)|^4
$$

$$
= d^4 \int_{\text{Haar}} dU
\text{tr}((U \otimes U^*)Y^L) \text{tr}((U^* \otimes U)(Y^L)^*)^2
$$

$$
= d^4 \int_{\text{Haar}} dU
\text{tr}((U \otimes (U^*)^2 \otimes U)^2(Y^L \otimes (Y^L)^*)^2).
$$

Using similar tricks as in the case of the second moment, we have that

$$
\int_{\text{Haar}} dU (U \otimes (U^*)^2 \otimes U)^2
$$

$$
= P_{15842673}T_2 \left( \int_{\text{Haar}} dUU^4 \otimes (U^\dagger)^4 \right) P_{15842673}.
$$

(V.15)

We will define $P^* = P_{15842673}$. Then we use Weingarten functions [31]-[33] to obtain an expression for the integral on the right hand side of Eq. (V.15) in terms of permutation operators:

$$
\int_{\text{Haar}} dUU^\otimes4 \otimes (U^\dagger)^\otimes4 = \sum_{\sigma,\tau \in S_4} W_g(d, 4, \sigma\tau^{-1})P_{\sigma,\tau},
$$

(V.16)

where $S_4$ is the symmetric group of 4 elements, and

$$
P_{\sigma,\tau} = P_{\tau(1)+4,\tau(2)+4,\tau(3)+4,\tau(4)+4,\sigma^{-1}(1),\sigma^{-1}(2),\sigma^{-1}(3),\sigma^{-1}(4)}.
$$

(V.17)

Combining Eqs. (V.14), (V.15), and (V.16), we have

$$
\mathbb{E}[\hat{S}^4] = d^4 \sum_{\sigma,\tau \in S_4} W_g(d, 4, \sigma\tau^{-1}) \times
$$

$$
\text{tr}\left(P^* T_2(P_{\sigma,\tau})P^* (Y^L \otimes (Y^L)^*)^2\right).
$$

(V.18)

The permutations $P_{\sigma,\tau}$ in the sum will be of the form $P_{wxyzabcd}$ (where $(wxyz)$ is a nonrepeating sequence of elements in the set $\{5, 6, 7, 8\}$ and $(abcd)$ is a non-repeating sequence of elements in the set $\{1, 2, 3, 4\}$). Each term in the above sum is therefore of the form:
\[
\text{tr} \left( P^* T_2^* (P_{wxzyabcd}) P^* (Y^L \otimes (Y^L)^*)^{\otimes 2} \right)
\]
\[
= \text{tr} \left( P^* T_2^* \left( \sum_{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8 \in J} |j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8 \rangle \langle j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8| P^* (Y^L \otimes (Y^L)^*)^{\otimes 2} \right) \right)
\]
\[
= \text{tr} \left( \sum_{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8} |j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8 \rangle \langle j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8| (Y^L \otimes (Y^L)^*)^{\otimes 2} \right)
\]
\[
= \sum_{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8} \langle j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8| (Y^L \otimes (Y^L)^*)^{\otimes 2} \rangle
\]
\[
= \sum_{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8} \langle j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8| Y^L j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8 \rangle
\]

(V.19)

where in the second to last line we have used Eq. (A.4), and in the last line, we have reordered the elements of $Y^L$ as

\[
\langle j_t, j_s| Y^L | j_t, j_u \rangle = \langle j_t, j_s| Y| j_u, j_t \rangle.
\]

(V.20)

Now we will use a graphical representation of the matrices $\hat{Y}$ in order to evaluate these terms:

\[
\begin{align*}
- \frac{t}{s} \begin{array}{c}
\hat{Y}
\end{array} & = \langle j_t, j_s| \hat{Y}| j_u, j_r \rangle \\
\frac{r}{s} \begin{array}{c}
\hat{Y}
\end{array} \begin{array}{c}
\hat{Y}
\end{array} & = \langle j_t, j_s| \hat{Y}\hat{Y}| j_m, j_n \rangle = \sum_{u, r} \langle j_t, j_s| \hat{Y}| j_u, j_r \rangle \langle j_u, j_r| \hat{Y}| j_m, j_n \rangle
\end{align*}
\]

(V.21)

\[
\sum_{j_r, j_s} \langle j_t, j_s| \hat{Y}| j_t, j_s \rangle = \text{tr}(\hat{Y})
\]

(V.22)

Furthermore, because of Eq. (IV.26) and Eq. (IV.27),

\[
\begin{align*}
\frac{r}{s} \begin{array}{c}
\hat{Y}
\end{array} & = \frac{r}{s} \begin{array}{c}
\hat{Y}
\end{array} \begin{array}{c}
\hat{Y}
\end{array} = \frac{r}{s} \begin{array}{c}
\hat{Y}
\end{array} = d \text{tr}(J(Y)).
\end{align*}
\]

(V.23)

We see that a single self-loop on either register forces the other register to also have a self loop. Because of this simplifying effect, we can enumerate all possible configurations of commutative diagrams that arise from choices of (wxyzabc) in the last line of Eq. (V.19). We have the following 7 diagrams that can represent the last line of Eq. (V.19):
These commutative diagrams were found in the following way. First, there is the case that all four $\mathcal{Y}$'s have self loops. This corresponds to Diagram I. Next, we consider the case that three $\mathcal{Y}$'s have self loops, but in this case, the only way to connect up the final $\mathcal{Y}$'s tensor legs is to create self loops, so we are back to Diagram I. In the case that two $\mathcal{Y}$'s have self loops, the only way to connect the remaining two $\mathcal{Y}$'s without giving them self loops is as shown in Diagram II. Continuing in this way, we can enumerate all possible diagrams.

We ignore diagrams that are identical to diagrams that are depicted, but which have one or more loops reversed in direction, or that have dashed and solid arrows switched. This is acceptable, because by reordering the elements of the matrix, as we did in Eq. (V.20), we can create a new figure which looks identical to ones shown above, but involving a new matrix $\mathcal{Y}'$ whose elements are the same as $\mathcal{Y}$. In our analysis below, we will ultimately see that our bounds on the contributions due to each figure will depend only on the Frobenius norm of $\mathcal{Y}'$, which only depends on the elements of the matrix, and not on their ordering. (We also have contributions that depend on the $\text{tr}(\mathcal{Y})$, but these terms only come from self-loops about a single element, for which reordering does not produce a new term.)

For a square matrix $A$ and integer $i > 1$, we will use the following bound:

$$
\text{tr}(A^i) = \text{tr}(A^{i-1}A) = \text{vec}((A^{i-1})^T)\text{vec}(A) \\
\leq \|A^{i-1}\|_F \|A\|_F \\
\leq \|A\|_F^i
$$

(V.28) where we have used Cauchy-Schwarz, the submultiplicative property of the Frobenius norm, and the fact that $\|A^T\|_F = \|A\|_F$.

We now bound the contribution due to each diagram.

**I:** We read off the contribution of $\text{tr}(\mathcal{Y})^4 = d^4 \text{tr}(J(\mathcal{Y}))^4$. 

**II:** 

**III:** 

**IV:** 

**V:** 

**VI:** 

**VII:**
II: We have a contribution of

\[ d^2 \text{tr}(J(Y))^2 \leq d^2 \text{tr}(J(Y))^2 \| \hat{J} \|^2_F = d^4 \text{tr}(J(Y))^2 \| J(Y) \|^4_F. \] (V.29)

III: We have a contribution of

\[ d \text{tr}(J(Y)) \text{tr}(\hat{J}^3) \leq d \text{tr}(J(Y)) \| \hat{J} \|^2 \| J(Y) \|^3_F = d^4 \text{tr}(J(Y))^2 \| J(Y) \|^4_F. \] (V.30)

IV: We have a contribution of

\[ \text{tr}(\hat{F}^4) \leq d^4 \| J(F) \|^4_F. \] (V.31)

V: We reprint the diagram here, but with labels:

\[ \langle \hat{J}_\phi | \hat{J}_\beta | \hat{J}_\gamma | \hat{J}_\delta | \hat{J}_\phi \rangle \]

VI: We reprint the diagram here, but with labels:

\[ \langle \hat{J}_\phi | K | \hat{J}_\chi | \hat{J}_\alpha | \hat{J}_\beta | \hat{J}_\gamma | \hat{J}_\delta | \hat{J}_\phi \rangle \]

Then the contribution of the diagram is bounded by

\[ \sum_{J_\phi, J_\beta, J_\gamma, J_\delta} \langle \hat{J}_\phi | J_\chi | J_\alpha | J_\beta | J_\gamma | J_\delta | \hat{J}_\phi \rangle \]

\[ \leq \sum_{J_\phi, J_\beta} \left[ \left( \sum_{J_\chi} \langle \hat{J}_\chi | \hat{J}_\phi | J_\phi \rangle \right)^2 \right] \]

\[ \times \left[ \left( \sum_{J_\alpha, J_\beta} \langle \hat{J}_\alpha | J_\phi | J_\gamma \rangle \right)^2 \right] \]

\[ \times \left[ \left( \sum_{J_\beta, J_\gamma} \langle \hat{J}_\beta | J_\phi | J_\gamma \rangle \right)^2 \right] \]

\[ \leq \sum_{J_\phi, J_\chi} \langle \hat{J}_\phi | K | J_\chi | J_\phi \rangle \langle \hat{J}_\chi | J_\phi | J_\chi \rangle \langle \hat{J}_\phi | K | J_\phi \rangle \]

\[ = \text{tr}(K^4) \]

\[ \leq \| K \|^4_F \]

\[ = \left( \sum_{J_\alpha} \sum_{J_\beta} \left( \langle \hat{J}_\alpha | J_\beta | J_\beta \rangle \right)^2 \right)^2 \]

\[ = \| \hat{J} \|^4_F \]

\[ = d^4 \| J(Y) \|^4_F. \] (V.34)

Let \( K \) be the \( d \times d \) matrix such that

\[ \langle j_a | K | j_b \rangle = \sqrt{\sum_{J_\alpha, J_\beta} \left( \langle j_a, J_\alpha | \hat{J}_b, J_\beta \rangle \right)^2}. \] (V.33)

Let \( K' \) be the \( d \times d \) matrix such that

\[ \langle j_a | K' | j_b \rangle = \sqrt{\sum_{J_\alpha, J_\beta} \left( \langle j_a, J_\alpha | \hat{J}_b, J_\beta \rangle \right)^2}. \] (V.36)
Then the contribution of the diagram is bounded by
\[
\sum_{j_{\alpha}, j_{\beta}, j_{\gamma}, j_{\delta}} \langle j_{\alpha}, j_{\beta} | \hat{Y} | j_{\delta}, j_{\alpha} \rangle \langle j_{\delta}, j_{\alpha} | \hat{Y} | j_{\gamma}, j_{\delta} \rangle \\
\times \langle j_{\gamma}, j_{\delta} | \hat{Y} | j_{\delta}, j_{\gamma} \rangle \\
\leq \sum_{j_{\alpha}, j_{\beta}} \left( \sum_{j_{\delta}} \left( \langle j_{\alpha}, j_{\beta} | \hat{Y} | j_{\delta}, j_{\alpha} \rangle \right)^2 \right) \\
\times \sum_{j_{\gamma}, j_{\delta}} \left( \langle j_{\gamma}, j_{\delta} | \hat{Y} | j_{\delta}, j_{\gamma} \rangle \right)^2 \\
\times \sum_{j_{\alpha}, j_{\beta}} \left( \langle j_{\alpha}, j_{\beta} | \hat{Y} | j_{\delta}, j_{\beta} \rangle \right)^2 \\
\leq \sum_{j_{\alpha}, j_{\beta}, j_{\gamma}, j_{\delta}} \langle j_{\alpha} | K' | j_{\delta} \rangle \langle j_{\gamma} | K' | j_{\beta} \rangle \langle j_{\alpha} | j_{\beta} \rangle \langle j_{\gamma} | K' | j_{\delta} \rangle \langle j_{\beta} | K' | j_{\alpha} \rangle \\
= \text{tr}(K'^2)^2 \\
\leq \| K' \|_{F}^4 \\
= \left( \sum_{j_{\beta}} \sum_{j_{\alpha}, j_{\delta}, j_{\alpha}} \left( \langle j_{\alpha}, j_{\delta} | \hat{Y} | j_{\delta}, j_{\alpha} \rangle \right)^2 \right)^2 \\
= \| \hat{Y} \|_{F}^4 \\
= d^4 \| J(Y) \|_{F}^4. \\
\text{(V.37)}
\]

**VII: We have a contribution of**
\[
\text{tr}(\hat{Y}^2) \text{tr}(\hat{Y}^2) \leq \| \hat{Y} \|_{F}^4, \\
\leq d^4 \| J(Y) \|_{F}^4. \\
\text{(V.38)}
\]

Looking at all possible diagrams, we see that the total contribution of any diagram is less than
\[
d^4 \max \{ \text{tr}(J(Y))^4, \| J(Y) \|_{F}^4 \}. \\
\text{(V.39)}
\]

Hence this bounds the size of any term in Eq. (V.18). Each term in Eq. (V.18) is multiplied by a Weingarten function $W_\phi(d, 4, \sigma t^{-1})$. The largest Weingarten term is
\[
W_\phi(d, 4, (1234)) = \\
d^4 - 8d^2 + 6 \\
\frac{(d+3)(d+2)(d+1)(d-1)(d-2)(d-3)d^2}{4} \\
< \frac{2}{d^4} \\
\text{(V.40)}
\]

for $d > 3$. There are $(4!)^2$ total terms in the sum. Putting it all together, we have
\[
\mathbb{E}[\hat{S}^4] \leq 2(4!)^2 d^4 \max \{ \text{tr}(J(Y))^4, \| J(Y) \|_{F}^4 \} \\
\text{(V.41)}
\]

for $d > 3$. Now we combine Eqs. (V.12), (V.13), and (V.41), and the Payley-Zygmund inequality to obtain
\[
\mathbb{P} \left[ \hat{S} \geq \xi \right] \geq \mathbb{P} \left[ \hat{S}^2 \geq \xi^2 \mathbb{E}[\hat{S}^2] \right] \\
\geq \left( 1 - \xi^2 \right)^2 \mathbb{E}[\hat{S}^2] \\
\geq \left( 1 - \xi^2 \right)^2 \frac{4^2 d^4 \max \{ \text{tr}(J(F))^4, \| J(F) \|_{F}^4 \} }{2(4!)^2 d^4 \max \{ \text{tr}(J(F))^4, \| J(F) \|_{F}^4 \}} \\
\geq \frac{1 - \xi^2}{8(4!)}^2. \\
\text{(V.42)}
\]

Thus
\[
\hat{Q}_\xi(\hat{E}) \geq \frac{1 - \xi^2}{8(4!)}^2. \\
\text{(V.43)}
\]

**D. Upper-bounding $\hat{W}_m(\hat{E})$**

In this section, we will bound
\[
\hat{W}_m(\hat{E}) = \mathbb{E} \sup_{Y \in \mathcal{E}} \text{tr}(Y \hat{H}), \\
\hat{H} = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \delta_i \mathcal{C}_i d^2, \\
\text{(V.44)}
\]

where the $\delta_i$ are Rademacher random variables, and the expectation is taken both over the $\delta_i$ and the choice of the unitaries $\mathcal{C}_i$. Because a unitary 4-design is also a unitary 2-design, a nearly identical argument to that used in Sec. IV-F holds, and we have
\[
\hat{W}_m(\hat{E}) \leq c_5 \sqrt{\ln d \cdot d}, \\
\text{(V.45)}
\]

where $c_5 > 0$ is some numerical constant.

**E. Final result**

Combining equations (V.4), (V.7), (V.43) and (V.45), and setting $\xi = 1/4$ and $t = \frac{1}{16} : \frac{1}{4} c_5 \sqrt{m}$, we get that:
\[
\inf_{\mathcal{U}} \lambda_{\min}(\hat{A}; D(\hat{f}, J(\mathcal{U})d, 0)) \\
\geq \frac{1}{4} \sqrt{m} \left( \frac{9}{16} \right) - 2c_5 \sqrt{\ln d \cdot d - \frac{1}{4}} t \\
= \frac{1}{2} \sqrt{m} \left( \frac{9}{16} \right) - 2c_5 \sqrt{\ln d \cdot d}. \\
\text{(V.46)}
\]

Now we set $m \geq (64(4!)^2 c_0)^2 \cdot d^2 \ln d$, which implies
\[
\frac{1}{64(4!)^2 c_0} \sqrt{m} \leq \sqrt{\ln d \cdot d}, \\
\text{(V.47)}
\]

hence
\[
\inf_{\mathcal{U}} \lambda_{\min}(\hat{A}; D(\hat{f}, J(\mathcal{U})d, 0)) \\
\geq \frac{\sqrt{m}}{64(4!)^2} \left( 1 - \frac{2c_5}{c_0} \right). \\
\text{(V.48)}
\]

This can be plugged into our approximate recovery bound (V.3). This finishes the proof of Theorem 1.1. □
VI. QUANTUM PROCESS TOMOGRAPHY

A. Motivation

Quantum process tomography is an important tool for the experimental development of large-scale quantum information processors. Process tomography is a means of obtaining complete knowledge of the dynamical evolution of a quantum system. This allows for accurate calibration of quantum gates, as well as characterization of qubit noise processes, such as dephasing, leakage, and cross-talk. These are the types of error processes that must be understood and ameliorated in order to build scalable quantum computers with error rates below the fault-tolerance threshold.

Formally, a quantum process $\mathcal{E}$ is described by a completely-positive trace-preserving linear map $\mathcal{E}: \mathbb{C}^{d \times d} \to \mathbb{C}^{d \times d}$, which maps density matrices to density matrices. We want to learn $\mathcal{E}$ by applying it on known input states $\rho$, and measuring the output states $\mathcal{E}(\rho)$.

Process tomography is challenging for (at least) two reasons. First, it requires estimating a large number of parameters: for a system of $n$ qubits, the Hilbert space has dimension $d = 2^n$, and a general quantum process has approximately $d^4 = 2^{4n}$ degrees of freedom. For example, to characterize the cross-talk between a pair of two-qubit gates acting on $n = 4$ qubits, using the most general approach, one must estimate $\sim 2^{16}$ parameters. To address this issue, several authors have proposed methods based on compressed sensing, which exploit sparse or low-rank structure in the unknown state or process, to reduce the number of measurements that must be performed [3], [5], [40]–[44].

The second challenge is that, in most real experimental setups, one must perform process tomography using state preparation and measurement devices that are imperfect. These devices introduce state preparation and measurement errors (“SPAM errors”), which limit the accuracy of process tomography. This limit is encountered in practice, and often produces non-physical estimates of processes [49]. Perhaps surprisingly, there are methods that are robust to SPAM errors, such as randomized benchmarking [45]–[49], and gate set tomography [37]–[39].

Here, we show how our phase retrieval techniques can address both of these challenges. We focus on the special case where we want to learn a unitary quantum process. This is a process $U: \mathbb{C}^{d \times d} \to \mathbb{C}^{d \times d}$ that has the form

$$U: \rho \mapsto U\rho U^\dagger,$$  

where $U \in \mathbb{C}^{d \times d}$ is a unitary matrix. This describes the dynamics of a closed quantum system, e.g., time evolution generated by some Hamiltonian, or the action of unitary gates in a quantum computer. Using our phase retrieval techniques, we devise methods for learning $U$ that are both fast and robust to SPAM errors (at least in principle).

Whereas a generic quantum process would have $\sim d^4$ degrees of freedom, $U$ has only $\sim d^2$ degrees of freedom. Our phase retrieval techniques work by estimating $m = O(d^2 \text{poly}(|\log d|))$ parameters of $U$, hence they achieve a quadratic speedup over conventional tomography. Furthermore, we show how these measurements can be implemented using commonly-available quantum operations. In particular, they can be performed using randomized benchmarking protocols, which are robust to SPAM errors.

B. Measurement Techniques

In order to learn the unknown process $U$, we would like to measure the quantities $|\text{tr}(C_i^\dagger U)|^2$, where $C_1, \ldots, C_m$ are chosen at random from a unitary 4- or 2-design in $\mathbb{C}^{d \times d}$. Then we can apply the PhaseLift algorithm to reconstruct $U$.

We remark that these measurements are quite different from the Pauli measurements that were used in earlier works on compressed sensing for quantum tomography [3]–[5]. In those earlier works, one would estimate the Pauli expectation values of the Jamiolkowski state $J(U)$. This led to measurements of the form $\text{tr}((P_1 \otimes P_2)J(U)) = \frac{1}{d} \text{tr}(P_1 U P_2 U^\dagger)$, where $P_1$ and $P_2$ were multi-qubit Pauli operators.

There are (at least) two ways of measuring the quantity $|\text{tr}(C^\dagger U)|^2$, for some prescribed unitary $C \in \mathbb{C}^{d \times d}$. The first way works by estimating the Choi-Jamiolkowski state of $U$. This method is straightforward, and fairly efficient, but it requires reliable and error-free state preparations and measurements.

To apply this method, recall that any quantum process $E$ can be equivalently described by its Choi-Jamiolkowski state,

$$J(E) = (E \otimes I)(\Phi^+)(\Phi^+) = \mathbb{C}^{d^2 \times d^2},$$  

which is obtained by applying $E$ to one half of the maximally entangled state $\Phi^+ = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle \in \mathbb{C}^d$. In the case of a unitary process $U$, this is equivalent to the “lifted” representation of $U$ used in PhaseLift:

$$J(U) = (U \otimes I)(\Phi^+)(\Phi^+) = \frac{1}{d} \text{vec}(U)\text{vec}(U)^\dagger.$$  

In particular, this implies that

$$|\text{tr}(C^\dagger U)|^2 = d^2 \text{tr}(J(C)J(U)) = d^2 |\text{tr}((C^\dagger \otimes I)(U \otimes I)\Phi^+)|^2.$$  

Thus, the quantity $|\text{tr}(C^\dagger U)|^2$ can be estimated via the following procedure: (1) prepare a maximally entangled state $\Phi^+$ on two registers; (2) apply the unknown unitary operation $U$ on the first register; (3) apply the prescribed unitary operation $C^\dagger$ on the first register; (4) measure both registers in the Bell bases; (5) repeat the above steps, and count the number of times that the outcome $\Phi^+$ is observed.

The second way of estimating $|\text{tr}(C^\dagger U)|^2$ makes use of a more sophisticated technique known as randomized benchmarking tomography [48], [49]. This method is robust to SPAM errors, but it is also quite resource-intensive. In addition, this method requires that the measurement matrix $C$ be chosen from some group that has a computationally efficient description, such as the Clifford group [26], [27].

In randomized benchmarking tomography, one implements a sequence that alternates between applying a random Clifford
and applying the unknown unitary map $U$. This alternation repeats $L$ times, and then is followed by a single recovery Clifford operation. For example, the recovery operation could be the Clifford that inverts the action of the $L$ randomly applied Cliffords in the sequence, so the total effect of the $L + 1$ applied Cliffords (ignoring the applications of $U$) would be the identity.

This protocol, when repeated many times, and for many different lengths $L$, produces a measurement signature of a decaying exponential in $L$, where the rate of decay depends only on $\text{tr}(U)$. By choosing different recovery operations, one can cause the rate of decay to depend on $\text{tr}(UC^\dagger)$ for any Clifford $C$. (This analysis assumes that Clifford operations can be implemented perfectly. If the Clifford operations contain errors, this procedure still works, and it produces a characterization of the process $U \circ \Lambda$, where $\Lambda$ is the average Clifford error.)

To summarize, randomized benchmarking tomography allows us to estimate quantities of the form $\text{tr}(UC^\dagger)$, where we can choose $C : \rho \mapsto C\rho C^\dagger$ to be any Clifford operation. We can rewrite this as

$$\text{tr}(UC^\dagger) = d^2 \text{tr}(J(U)J(C)) = |\text{tr}(C^\dagger U)|^2. \quad \text{(VL5)}$$

C. Numerical Simulation

We ran numerical simulations to assess the performance of the PhaseLift algorithm for reconstructing an unknown unitary operation $U : \rho \mapsto U\rho U^\dagger$ (where $U \in \mathbb{C}^{d \times d}$), using random Clifford measurements $C_1, \ldots, C_m \in \mathbb{C}^{d \times d}$. In our simulations, we chose $U$ at random, by sampling from the Haar distribution on the unitary group. We then sampled the $C_i$ from the uniform distribution on the Clifford group. We compared this with a second scenario, where the $C_i$ were sampled from the Haar distribution on the unitary group, since this is the “best” measurement ensemble for phase retrieval.

We simulated the measurement procedure (I.6) on $U$, in the noiseless case ($\epsilon = 0$). Then we solved the PhaseLift convex program (I.7), again in the noiseless case ($\eta = 0$), in order to obtain an estimate $U'$ of $U$. Here we omitted the non-spikiness constraints (I.14), in order to make the convex program easier to solve. Finally, we computed the reconstruction error in the Frobenius norm, $\|J(U') - J(U)\|_F$.

The results are shown in Figure 1. These results suggest that phase retrieval using random Clifford measurements performs nearly as well as phase retrieval using Haar-random unitary measurements. In both cases, the number of measurements scales as $m \approx Cd^2$, where $C \approx 4.8$. This suggests that, although random Clifford operations are not a unitary 4-design, they are “close enough” to ensure accurate reconstruction of unitary matrices via phase retrieval.

Acknowledgments

The authors thank Richard Kueng and David Gross for helpful discussions about this problem; Easwar Magesan for sharing his Clifford sampling code; Aram Harrow and Steve Flammia for clarifications regarding the results in [13]; and several anonymous reviewers for suggestions which improved this paper. Contributions to this work by NIST, an agency of the US government, are not subject to US copyright.

REFERENCES

[1] R. P. Millane, “Phase retrieval in crystallography and optics,” J. Optical Society of America A 7(3), pp.394-411 (1990).

[2] Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao and M. Segev, “Phase Retrieval with Application to Optical Imaging: A contemporary overview,” IEEE Signal Processing Magazine, vol. 32, no. 3, pp. 87-109, May 2015.

[3] D. Gross, Y.-K. Liu, S. T. Flammia, S. Becker, and J. Eisert. Quantum state tomography via compressed sensing. Phys. Rev. Lett., 105:150401, 2010.

[4] Y.-K. Liu. Universal low-rank matrix recovery from Pauli measurements. In Advances in Neural Information Processing Systems 24: 25th Annual Conference on Neural Information Processing Systems 2011, pages 1638–1646, 2011.

[5] S. T Flammia, D. Gross, Y.-K. Liu, and J. Eisert. Quantum tomography via compressed sensing: error bounds, sample complexity and efficient estimators. New Journal of Physics, 14(9):095022, 2012.

[6] E. J. Candés, T. Strohmer, and V. Voroninski. Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming. Commun. Pure and Applied Math., 66(8):1241–1274, 2013.

[7] D. Gross, F. Krahmer, and R. Kueng. A partial derandomization of phaselift using spherical designs. J. Fourier Analysis and Applications, 21(2):229–266, 2015.

[8] E. J. Candés and X. Li. Solving quadratic equations via phaselift when there are about as many equations as unknowns. Found. Comput. Math., 14(5):1017–1026, 2014.

[9] R. Kueng, H. Rauhut, and U. Terstiege. Low rank matrix recovery from rank one measurements. Applied and Comput. Harmonic Analysis 42(1), pp.88-116, January 2017.

[10] R. Kueng, H. Zhu, and D. Gross. Low rank matrix recovery from Clifford orbits. arXiv:1610.08070.
For example from the complete positivity constraint, we have that $J(X)$ is Hermitian, so
\[
\langle ki | J(X) | lj \rangle = \langle lj | J(X)^* | ki \rangle.
\] (A.3)
Converting this into Liouville representation, we have
\[
\langle kl | X_L | ij \rangle = \langle lk | (X_L)^* | ji \rangle.
\] (A.4)
Using this fact, we can show that for completely positive superoperators $F$ and $K$,
\[
\text{tr}(J(F)J(K)) = \frac{1}{d^2} \text{tr}(F^L(K^L)^\dagger).
\] (A.5)
To see this, we calculate
\[
\begin{align*}
\text{tr}(J(F)J(K)) &= \frac{1}{d^2} \text{tr} \left( \sum_{ij} F(\langle i | j \rangle) K(\langle j | i \rangle) \right) \\
&= \frac{1}{d^2} \sum_{ijkl} \langle k | F(\langle i | j \rangle) | l \rangle \langle l | K(\langle j | i \rangle) | k \rangle \\
&= \frac{1}{d^2} \sum_{ijkl} \langle kl | F^L(\langle i | j \rangle) | kj \langle k | (K^L)^* | l \rangle \\
&= \frac{1}{d^2} \text{tr}(F^L(K^L)^\dagger).
\end{align*}
\] (A.6)
Additionally, we note that for unitary maps $U : \rho \mapsto U \rho U^\dagger$, where $U$ is a unitary operation, the corresponding Liouville representation takes the form
\[
U_L = U \otimes U^*,
\] (A.7)
because
\[
U(|i\rangle\langle j|) = U|i\rangle\langle j|U^\dagger
\] (A.8)
Using Eq. (A.5) and Eq. (A.7), we have that for a map $U$ representing a unitary $U$ and a map $C$ representing a unitary $C$, that
\[
\text{tr}(J(U)J(C)) = \frac{1}{d^2} |\text{tr}(U^* C)|^2.
\] (A.9)