On equations defining the Ricci flows of manifolds

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Abstract

The examples of the Ricci flows of the four-dimensional manifolds which are determined by help of nonlinear differential equations of the type of Monge-Ampere are constructed. Their particular solutions are derived and their properties are discussed.

1 4D model of the Ricci-flow

We study the system of equations
\[ \frac{\partial}{\partial t} g_{ij}(\vec{x}, t) = -2R_{ij}(g) \] (1)
describing the Ricci flows of the four dimensional manifolds which are endowed by the metrics of the form
\[ ds^2 = A(x, y, t) du^2 + B(x, y, t) du dv + du dx + C(x, y, t) dv^2 + dv dy, \] (2)
where the components of metrics are dependent from two coordinates \((x, y)\) and from the parameter \(t\).

In this case the Ricci-tensor of the metric (1) has five components
\[ R_{uu}, R_{uv}, R_{ov}, R_{ux}, R_{vx}. \]

From the conditions of compatibility of the equations (1) we find that they are reduced to the equation
\[ 4 \left( \frac{\partial^2}{\partial x^2} h(x, y, t) \right) \frac{\partial^2}{\partial y^2} h(x, y, t) - 4 \left( \frac{\partial^2}{\partial x \partial y} h(x, y, t) \right)^2 - \frac{\partial}{\partial t} h(x, y, t) = 0 \] (3)
and components of the metric (2) take the form
\[ B(x, y, t) = \frac{\partial^2}{\partial x \partial y} h(x, y, t), \quad C(x, y, t) = -\frac{\partial^2}{\partial x^2} h(x, y, t), \quad A(x, y, t) = -\frac{\partial^2}{\partial y^2} h(x, y, t). \] (4)

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Thus the study of the properties of the Ricci flow on a 4-dimensional manifold with the metric (2) with the coefficients (4) lead to integration of the equation (3). Nyis equation is the first order relatively the parameter \( \tau \) and has form of the Monge-Ampere equation relatively of the variables \( x, y \).

## 2 Particular solutions

Let us consider some elementary solutions of equation (3).

After the substitution of the form

\[
h(x, y, t) = H \left( \frac{x}{y} - t \right) y^4
\]

the equation (3) takes the form

\[
48 \left( D^{(2)} \right) (H)(\eta)H(\eta) - 36 (D(H)(\eta))^2 + D(H)(\eta) = 0,
\]

where

\[
\eta = \frac{x}{y} - t.
\]

Solution of the equation (4) may be present in form

\[
\eta(H) = \mathcal{C}2 - 144 \frac{\sqrt{H}}{K^3} + \left( -48 \ln(1 - \sqrt{H}K) + 24 \ln(1 + \sqrt{H}K + \sqrt{H}K^2) + 48 \sqrt{3} \arctan \left( \frac{\sqrt{3} \sqrt{H}K}{2 + \sqrt{H}K} \right) \right) K^{-4},
\]

where \( K \) and \( \mathcal{C}2 \) are constants.

Remark that if the value \( K = -1 \) the function \( \eta(H) \) has a break.

The simplest singular solution of (4) is

\[
h(x, y) =
\]

\[
= \left( 6 \frac{B^2 x^4}{k} + \mathcal{C}1 \sqrt{x} \cos(1/2 \sqrt{23} \ln(x)) + \mathcal{C}2 \sqrt{x} \sin(1/2 \sqrt{23} \ln(x)) + yBx^3 + 1/24 y^2 k x^2 \right) (2kt + \mathcal{C}4)^{-1}
\]

More complicated solution has the form

\[
h(x, y, t) = -1/2 \mathcal{C}1 \mu \mathcal{C}1 \sqrt{c_1} \sqrt{2} \tan(1/2 \sqrt{c_1} (x - t) \sqrt{2}) \mathcal{C}2 \mu \mathcal{C}1 \sqrt{c_1} \sqrt{2} y^2 \tan(1/2 \sqrt{c_1} (x - t) \sqrt{2}) \mathcal{C}1 - 1/4 \mathcal{C}1 \tan(1/2 \sqrt{c_1} (x - t) \sqrt{2}) \mathcal{C}1.
\]

A more general solutions of the equation (4) are derived by the method of parametric representations of functions and their derivatives [1-2].

Let us apply its to the equation (4).

After the change of the variables, the function and its derivative

\[
h(x, y, t) \rightarrow u(x, \tau, t), \ y \rightarrow v(x, \tau, t), \ h_x \rightarrow u_x - \frac{u_x}{v_x} v_x = p, \ h_t \rightarrow u_t - \frac{u_x}{v_x} v_t = q,
\]
we derive from the equation
\[
\frac{\partial^2}{\partial x^2} h(x, y, t) \frac{\partial^2}{\partial y^2} h(x, y, t) - \left( \frac{\partial^2}{\partial x \partial y} h(x, y, t) \right)^2 - \frac{\partial}{\partial t} h(x, y, t) = 0
\]  
(7)

the relation between the functions \(u(x, \tau, t)\) and \(v(x, \tau, t)\), where variable \(\tau\) is considered as parameter
\[
\begin{align*}
&- \left( \frac{\partial^2}{\partial x^2} u(x, \tau, t) \right) \left( \frac{\partial}{\partial \tau} v(x, \tau, t) \right) \left( \frac{\partial}{\partial \tau} u(x, \tau, t) \right) \frac{\partial^2}{\partial \tau^2} v(x, \tau, t) - \\
&- \left( \frac{\partial}{\partial \tau} u(x, \tau, t) \right) \left( \frac{\partial^2}{\partial x^2} v(x, \tau, t) \right) \left( \frac{\partial}{\partial \tau} v(x, \tau, t) \right) \frac{\partial^2}{\partial \tau^2} u(x, \tau, t) + \\
&+ \left( \frac{\partial}{\partial \tau} u(x, \tau, t) \right)^2 \left( \frac{\partial^2}{\partial x^2} v(x, \tau, t) \right) \frac{\partial^2}{\partial \tau^2} v(x, \tau, t) - \\
&- \left( \frac{\partial^2}{\partial \tau \partial x} u(x, \tau, t) \right) \left( \frac{\partial}{\partial \tau} v(x, \tau, t) \right) \left( \frac{\partial}{\partial \tau} u(x, \tau, t) \right) \frac{\partial^2}{\partial \tau^2} v(x, \tau, t) - \\
&- \left( \frac{\partial}{\partial \tau} u(x, \tau, t) \right)^2 \left( \frac{\partial^2}{\partial \tau \partial x} v(x, \tau, t) \right) \left( \frac{\partial}{\partial \tau} v(x, \tau, t) \right)^4 + \\
&+ \left( \frac{\partial}{\partial \tau} u(x, \tau, t) \right) \left( \frac{\partial}{\partial t} v(x, \tau, t) \right) \left( \frac{\partial}{\partial \tau} v(x, \tau, t) \right)^3 = 0.
\end{align*}
\]  
(8)

Note that the relation (8) under the condition \(v(x, \tau, t) = \tau\) is equivalent to the equation
\[
\frac{\partial^2}{\partial x^2} u(x, t, \tau) \frac{\partial^2}{\partial \tau^2} u(x, t, \tau) - \left( \frac{\partial^2}{\partial \tau \partial x} u(x, t, \tau) \right)^2 - \frac{\partial}{\partial \tau} u(x, t, \tau) = 0
\]
similar to equation (7) and transformed into an equation of more general form if the functions \(u(x, \tau, t), (v(x, \tau, t)\) are dependent.

For example under the condition
\[
\begin{align*}
u(x, \tau, t) &= \tau \frac{\partial}{\partial \tau} \omega(x, \tau, t) - \omega(x, \tau, t), \quad v(x, \tau, t) = \frac{\partial}{\partial \tau} \omega(x, \tau, t)
\end{align*}
\]
from the (8) we obtain the p.d.e. for the function \(\omega(x, \tau, t)\)
\[
- \frac{\partial^2}{\partial x^2} \omega(x, \tau, t) + \left( \frac{\partial^2}{\partial \tau^2} \omega(x, \tau, t) \right) \frac{\partial}{\partial \tau} \omega(x, \tau, t) = 0.
\]  
(9)

The other condition
\[
\begin{align*}
v(x, \tau, t) &= \tau \frac{\partial}{\partial \tau} \omega(x, \tau, t) - \omega(x, \tau, t), \quad u(x, \tau, t) = \frac{\partial}{\partial \tau} \omega(x, \tau, t)
\end{align*}
\]
lead to the equation

\[
\frac{\partial^2}{\partial x^2} \omega(x, \tau, t) + \left( \frac{\partial^2}{\partial \tau^2} \omega(x, \tau, t) \right) \tau^3 \frac{\partial}{\partial t} \omega(x, \tau, t) = 0. \tag{10}
\]

Let us consider some solutions of the equation (9).

Under the substitution

\[
\omega(x, \tau, t) = A(x - kt, \tau)
\]

it is reduced at the equation

\[
\frac{\partial^2}{\partial \eta^2} A(\eta, \tau) + k \left( \frac{\partial^2}{\partial \tau^2} A(\eta, \tau) \right) \frac{\partial}{\partial \eta} A(\eta, \tau) = 0 \tag{11}
\]

where \(\eta = x - kt\).

This equation is reduced after the Legendre transformation to the Euler-Trikomi equation and therefore its solutions depend radically on the sign of the parameter \(k\).

Each solution (11) corresponds to the solution of the original equation (8) which is obtained by eliminating the parameter \(\tau\) from the correlations

\[
y - \tau \frac{\partial}{\partial \tau} \omega(x, \tau, t) + \omega(x, \tau, t) = 0, \quad h(x, y, t) - \frac{\partial}{\partial \tau} \omega(x, \tau, t) = 0. \tag{12}
\]

Here is an example of.

\section{Finite-dimensional reduction}

After the substitutions

\[
A(x, y, t) = A_0(t) + A_1(t)x + A_2(t)y + A_3(t)x^2 + A_4(t)yx + A_5(t)y^2, \\
B(x, y, t) = B_0(t) + B_1(t)x + B_2(t)y + B_3(t)x^2 + B_4(t)xy + B_5(t)y^2, \\
C(x, y, t) = C_0(t) + C_1(t)x + C_2(t)y + C_3(t)x^2 + C_4(t)xy + C_5(t)y^2, \\
B_4(t) = -2 A_3(t), \quad A_4(t) = -2 B_5(t), \quad C_4(t) = -2 B_3(t), \quad C_5(t) = A_3(t)
\]

the metric (2) takes the form

\[
ds^2 = \left( A_0(t) + A_1(t)x + A_2(t)y + A_3(t)x^2 - 2B_5(t)yx + A_5(t)y^2 \right) du^2 + \\
+ 2 \left( B_0(t) + B_1(t)x + B_2(t)y + B_3(t)x^2 - 2A_3(t)xy + B_5(t)y^2 \right) du dv + \\
+ \left( C_0(t) + C_1(t)x + C_2(t)y + C_3(t)x^2 - 2B_3(t)xy + A_3(t)y^2 \right) dv^2 + dx du + dy dv \tag{13}
\]

and from the system of equations (1) is obtained system of ODE’s

\[
\frac{d}{dt} B_3(t) = 24 B_3(t) A_3(t) - 24 C_3(t) B_5(t),
\]

\[
\frac{d}{dt} C_3(t) = -48 C_3(t) A_3(t) + 48 \left( B_3(t) \right)^2,
\]
Theorem 1

The flows of Ricci of the 4D-manifolds which are endowed by the metric with local coordinates \((x, y, z, t)\) the components of which are dependent from two coordinates \((x, t)\) and from the parameter \(\tau\)

\[
d s^2 = \left( \frac{\partial^2}{\partial x^2} f(x, t, \tau) \right) dx^2 + 2 \left( \frac{\partial^2}{\partial t \partial x} f(x, t, \tau) \right) dx \, dt + \left( \frac{\partial^2}{\partial t^2} f(x, t, \tau) \right) dy^2 + \\
+ 2 \left( \frac{\partial^2}{\partial t \partial y} f(x, t, \tau) \right) dy \, dz + \left( \frac{\partial^2}{\partial z^2} f(x, t, \tau) \right) dz^2 + \left( \frac{\partial^2}{\partial t^2} f(x, t, \tau) \right) dt^2
\]

is defined by the equation

\[
\frac{\partial}{\partial \tau} f(x, t, \tau) = \ln \left( \frac{\partial^2}{\partial t^2} f(x, t, \tau) \frac{\partial^2}{\partial x^2} f(x, t, \tau) - \left( \frac{\partial^2}{\partial t \partial x} f(x, t, \tau) \right)^2 \right) \quad (15)
\]

Let us consider some solutions of the equation \((15)\).

With aim of convenience we rewrite if in the form

\[
\left( \frac{\partial^2}{\partial x^2} f(x, y, \tau) \right) \frac{\partial^2}{\partial y^2} f(x, y, \tau) - \left( \frac{\partial^2}{\partial x \partial y} f(x, y, \tau) \right)^2 - e^{\frac{\partial}{\partial \tau} f(x, y, \tau)} = 0, \quad (16)
\]

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where the variable $t$ is changed on $y$.

After the $(u, v)$-transformation with condition $u(x, t, \tau) = t$ the equation (16) takes the form
\[
\left(\frac{\partial^2}{\partial x^2} v(x, t, \tau)\right) \frac{\partial^2}{\partial \tau^2} v(x, t, \tau) - \left(\frac{\partial^2}{\partial \tau \partial x} v(x, t, \tau)\right)^2 - e^{-\frac{\partial}{\partial \tau} v(x, t, \tau)} \left(\frac{\partial}{\partial t} v(x, t, \tau)\right)^4 = 0.
\]

Particular solutions of this equation are obtained with the help of additional conditions. For example in the case
\[
\frac{\partial}{\partial \tau} v(x, t, \tau) = \frac{\partial}{\partial t} v(x, t, \tau)
\]
we find that the function $v(x, t, \tau)$ has the form
\[
v(x, t, \tau) = F_1(x, \tau + t) = h(x, \eta),
\]
where the function $h(x, \eta)$ satisfies the equation
\[
\left(\frac{\partial^2}{\partial \eta^2} h(x, \eta)\right) \frac{\partial^2}{\partial x^2} h(x, \eta) - \left(\frac{\partial^2}{\partial \eta \partial x} h(x, \eta)\right)^2 - e^{-1} \left(\frac{\partial}{\partial \eta} h(x, \eta)\right)^4 = 0. \tag{17}
\]

After the $(u, v)$-transformation with the conditions
\[
v(x, t) = i \frac{\partial}{\partial t} \omega(x, t) - \omega(x, t), \quad u(x, t) = \frac{\partial}{\partial t} \omega(x, t)
\]
the equation (17) is reduced to the linear equation
\[
\frac{\partial^2}{\partial x^2} \omega(x, t) + \left(\frac{\partial^2}{\partial t^2} \omega(x, t)\right) e^{-1} = 0
\]
with general solution
\[
\omega(x, t) = F_1(t - i \sqrt{e^{-1}} x) + F_2(t + i \sqrt{e^{-1}} x)
\]
containing two arbitrary functions.

With the help of the function $\omega(x, t)$ we can obtain large class of solutions of the equation (16) in parametric form.

For example, in the case
\[
F_1(t - i \sqrt{e^{-1}} x) = \cosh(+ - i \sqrt{e^{-1}} x), \quad F_2(t + i \sqrt{e^{-1}} x) = \sinh(t + i \sqrt{e^{-1}} x)
\]
we get
\[
\omega(x, t) = \cosh(t) \cos(\sqrt{e^{-1}} x) + \sinh(t) \cos(\sqrt{e^{-1}} x)
\]
and elimination of the parameter $t$ from the relations
\[
\eta - t \cos(e^{-1/2} x) \cosh(t) - t \sinh(t) \cos(e^{-1/2} x) + \cosh(t) \cos(e^{-1/2} x) + \sinh(t) \cos(e^{-1/2} x) = 0,
\]
\[
h(x, \eta) - \cosh(t) \cos(e^{-1/2} x) - \sinh(t) \cos(e^{-1/2} x)
\]
lead to the function
\[
h(x, \eta) = e^{LambertW(\frac{\eta e^{-1}}{\cos(e^{-1/2} x)}) + 1} \cos(e^{-1/2} x)
\]
which is solution of the equation (17).

With the help of solutions of the equation (17) can be constructed solutions of the equation (16).
5 More general solution

The equation (16) after the \((u, v)\)-transformation with conditions

\[
-\left(\frac{\partial}{\partial \tau} u(x, t, \tau)\right) \frac{\partial}{\partial t} v(x, t, \tau) + \left(\frac{\partial}{\partial \tau} u(x, t, \tau)\right) \frac{\partial}{\partial \tau} v(x, t, \tau) = Ax,
\]

\[v(x, t, \tau) = F_1(x, u(x, t, \tau)) - Ax \tau\]
takes the form

\[
\left(\frac{\partial^2}{\partial \eta^2} h(x, \eta)\right) \frac{\partial^2}{\partial x^2} h(x, \eta) - \left(\frac{\partial^2}{\partial \eta \partial x} h(x, \eta)\right)^2 - e^{Ax} \left(\frac{\partial}{\partial \eta} h(x, \eta)\right)^4 = 0,
\]

where

\[\eta = u(x, t, \tau) - Ax \tau,\]
\[h(x, \eta) = F_1(x, u(x, t, \tau) - Ax \tau)\].

It is reduced to the equation

\[
e^{Ax} \frac{\partial^2}{\partial t^2} \omega(x, t) + \frac{\partial^2}{\partial x^2} \omega(x, t) = 0
\]
after the \((u, v)\)-transformation with conditions

\[v(x, t) = t \frac{\partial}{\partial t} \omega(x, t) - \omega(x, t), \quad u(x, t) = \frac{\partial}{\partial t} \omega(x, t)\].

Simplest solution of the equation (19) has the form

\[\omega(x, t) = (-C_1 \cos(\sqrt{-c_1} t) + C_2 \sin(\sqrt{-c_1} t)) \times
\]
\[\left(-C_3 BesselJ(0, 2 \sqrt{-c_1} \sqrt{e^{Ax}}) + C_4 BesselY(0, 2 \sqrt{\frac{-c_1}{A^2}} \sqrt{e^{Ax}})\right)
\]

and elimination of the parameter \(t\) from the relations

\[\eta + t \sin(\sqrt{-c_1} t) \sqrt{-c_1} BesselJ(0, 2 \sqrt{-c_1} \sqrt{e^x}) + \cos(\sqrt{-c_1} t) BesselJ(0, 2 \sqrt{-c_1} \sqrt{e^x}) = 0,
\]

\[h(x, \eta) + \sin(\sqrt{-c_1} t) \sqrt{-c_1} BesselJ(0, 2 \sqrt{-c_1} \sqrt{e^x}) = 0
\]
gives solution of the equation (18) defined from the relation

\[\eta \sqrt{-c_1} + \arcsin\left(\frac{h(x, \eta)}{\sqrt{-c_1 BesselJ(0, 2 \sqrt{-c_1} \sqrt{e^x})}}\right) h(x, \eta) +
\]
\[+ \sqrt{-c_1} \left(BesselJ(0, 2 \sqrt{-c_1} \sqrt{e^x})\right)^2 - (h(x, \eta))^2 = 0,
\]
(at the conditions \(A = 1\) and \(-C_2 = 0, -C_4 = 0, -C_1 = 1\)).

The solution of the equation (16) which corresponds the function \(h(x, \eta)\) is determined from the relation

\[y - h(x, f(x, y, \tau) - x \tau) = 0\]
and can be very complicated.

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