THE PATH-MISSING AND PATH-FREE COMPLEXES OF A DIRECTED GRAPH

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Abstract. We study two simplicial complexes arising from a directed graph $G = (V, E)$ with two chosen vertices $s$ and $t$: the path-free complex, consisting of all subsets $F \subseteq E$ that contain no path from $s$ to $t$, and the path-missing complex, its Alexander dual. Using discrete Morse theory, we prove that both complexes have well-behaved homotopy types – either contractible or homotopy-equivalent to spheres.

1. Introduction

1.1. General definitions. In the following, all graphs are directed multigraphs, and self-loops are allowed. Let $G = (V, E, s, t)$ be a directed graph with vertex set $V$, edge set $E$, and two distinguished vertices $s, t \in V$. A walk (of $G$) means a sequence $(v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n)$, where $n$ is a nonnegative integer and where $v_0, v_1, \ldots, v_n \in V$ and $e_1, e_2, \ldots, e_n \in E$ have the property that each $e_i$ is an edge from $v_{i-1}$ to $v_i$. This walk is said to have the vertices $v_0, v_1, \ldots, v_n$ and the edges $e_1, e_2, \ldots, e_n$, and it is furthermore said to be a walk from $v_0$ to $v_n$. A path means a walk that does not visit any vertex more than once. If $u, v \in V$, then a $u - v$-walk means a walk from $u$ to $v$. A $u - v$-path means a $u - v$-walk that is a path.

We shall use the language of (abstract) simplicial complexes. For an introduction to this language, see [Bjö95, Section 9] (but unlike [Bjö95], we shall not exclude the empty set in our simplicial complexes). We recall the most fundamental notions of this language: A simplicial complex means a pair $(W, \Delta)$ consisting of a finite set $W$ and a subset $\Delta$ of the powerset $2^W$ of $W$ such that the following axiom holds:

If $A \in \Delta$ and $B \subseteq A$, then $B \in \Delta$.

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1In terms of posets, this axiom says that $\Delta$ is an order ideal (i.e., a down-set) of the Boolean lattice $2^W$ (ordered by inclusion).
Thus, $\emptyset \in \Delta$ if $\Delta$ itself is not empty; but $\Delta$ is allowed to be empty. Unlike various authors, we do not require $\{w\} \in \Delta$ for all $w \in W$. We shall often denote the simplicial complex $(W, \Delta)$ by $\Delta$; that is, we omit $W$ from the notation. We will refer to the set $W$ as the ground set of the complex $(W, \Delta)$, and say that $(W, \Delta)$ is a complex on the ground set $W$. The elements of $\Delta$ (which are, themselves, subsets of $W$) are called the faces of the complex $\Delta$.

Given a graph $G$, we say that a subset $F \subseteq E$ contains a path $p$ if $F$ contains each edge of $p$. Let us now define the two simplicial complexes we are going to study.

**Definition 1.1.** The path-free complex $\mathcal{PF}(G)$ and the path-missing complex $\mathcal{PM}(G)$ are the following simplicial complexes on the ground set $E$:

$$
\mathcal{PF}(G) := \{ F \subseteq E : F \text{ contains no } s-t\text{-path} \};
$$

$$
\mathcal{PM}(G) := \{ F \subseteq E : E \setminus F \text{ contains an } s-t\text{-path} \}.
$$

**Example 1.2.** Let $G$ be the following directed graph:

![Graph Diagram]

Then the faces of the simplicial complex $\mathcal{PF}(G)$ are the sets

$$
\{b, c, e, f, g\}, \{a, c, e, f, g\}, \{b, c, d, g\}, \{a, c, d, f, g\}, \{a, b, e, f\}, \{a, b, d, f, g\}
$$

as well as all their subsets. Meanwhile, the faces of the simplicial complex $\mathcal{PM}(G)$ are the sets

$$
\{d, e, f, g\}, \{c, d, f\}, \{a, b, c, f, g\}, \{a, e, g\}
$$

as well as all their subsets.

### 1.2. Statements of the main results.

#### 1.2.1. The Euler characteristic.

The Euler characteristic $\chi(\Delta)$ of a simplicial complex $\Delta$ is defined to be the alternating sum $f_0 - f_1 + f_2 - f_3 + \cdots = \sum_{i \geq 0} (-1)^i f_i$, where $f_i$ denotes the number of faces $I \in \Delta$ having size $\#I = i + 1$. (The faces of $\Delta$ are themselves finite sets, so they have well-defined sizes. In combinatorial topology, the dimension of a face $I$ is defined to be its size minus 1, so that $f_i$ counts the faces $I \in \Delta$ of dimension $i$.)
The reduced Euler characteristic \( \tilde{\chi} (\Delta) \) of a simplicial complex \( \Delta \) is defined to be the alternating sum
\[
-f_{-1} + f_0 - f_1 + f_2 - f_3 + \cdots = \sum_{i \geq -1} (-1)^i f_i = \sum_{I \in \Delta} (-1)^{#I-1},
\]
where \( f_i \) denotes the number of faces \( I \in \Delta \) having size \( #I = i + 1 \). If \( \Delta \neq \emptyset \), then the two characteristics are related by the equality \( \tilde{\chi} (\Delta) = \chi (\Delta) - 1 \), since \( f_{-1} = 1 \).

A walk of \( G \) is said to be
- nontrivial if it contains at least one edge;
- closed if it starts and ends at one and the same vertex\(^2\);
- a cycle if it is nontrivial and closed and has the property that all but the last of its vertices are distinct (i.e., if \( v_0, v_1, \ldots, v_m \) are its vertices, then \( v_0, v_1, \ldots, v_{m-1} \) are distinct).

For example, the graph \( G \) in Example 1.2 has a cycle, which contains the vertices \( p, r, q, p \) and the edges \( b, g, f \).

A vertex \( v \) of \( G \) is said to be a nonsink if there exists at least one edge of \( G \) whose source is \( v \). We denote by \( V' \) the set of all nonsinks of \( G \).

A useless edge of \( G \) will mean an edge that belongs to no \( s-t \)-path. In particular, any self-loop is a useless edge. For example, the graph \( G \) in Example 1.2 has no useless edges, but if we remove its edge \( d \), then the resulting graph has \( f \) as a useless edge.

We can now describe the reduced Euler characteristics of \( P_M (G) \) and \( P_F (G) \).

**Theorem 1.3.**

(a) If \( G \) contains a useless edge or a cycle or satisfies \( (E = \emptyset \text{ and } s \neq t) \), then \( \tilde{\chi} (P_M (G)) = 0 \).

(b) Otherwise, \( \tilde{\chi} (P_M (G)) = (-1)^{#E-#V'+1} \).

**Theorem 1.4.** If \( E \neq \emptyset \), then:

(a) If \( G \) contains a useless edge or a cycle, then \( \tilde{\chi} (P_F (G)) = 0 \).

(b) Otherwise, \( \tilde{\chi} (P_F (G)) = (-1)^{#V'} \).

If \( E = \emptyset \), then \( \tilde{\chi} (P_F (G)) \) equals 0 if \( s = t \) and \( -1 \) otherwise.

We shall prove Theorem 1.3 and Theorem 1.4 in Section 6, after some preparations.

As a particularly simple-sounding corollary of the above two theorems, we can find the parities of the numbers of faces of \( P_F (G) \) and of \( P_M (G) \) (see the end of Section 6 for detailed proofs).

**Corollary 1.5.**

(a) If \( G \) contains a useless edge or a cycle or satisfies \( (E = \emptyset \text{ and } s \neq t) \), then \( # (P_M (G)) \) is even.

\(^2\)Here, of course, a walk is said to start at \( u \) and end at \( v \) if it is an \( u-v \)-walk.
Corollary 1.6.

(a) If $G$ contains a useless edge or a cycle or satisfies $(E = \emptyset$ and $s = t)$, then \(\#(\mathcal{PM}(G))\) is even.

(b) Otherwise, \(\#(\mathcal{PF}(G))\) is odd.

Remark 1.7. Theorem 1.3 and Theorem 1.4 show that the reduced Euler characteristics of $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$ always belong to \([-1, 0, 1]\). Not all combinatorially defined simplicial complexes are this well-behaved. For example, if we replaced directed graphs by undirected graphs throughout our definitions, then the same graph $G$ considered in Example 1.2 (but without the arrows on the edges) would satisfy $\bar{\chi}(\mathcal{PF}(G)) = 3$ and $\bar{\chi}(\mathcal{PM}(G)) = -3$.

1.2.2. The $f$-polynomials of $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$. If $\Delta$ is any simplicial complex on a ground set $W$, then we define the $f$-polynomial of $\Delta$ to be the polynomial

$$f_\Delta(x) := \sum_{I \in \Delta} x^{\#I} \in \mathbb{Z}[x].$$

An easy comparison of the definitions of $f_\Delta$ and $\bar{\chi}(\Delta)$ shows that the value of this polynomial at $-1$ is

$$f_\Delta(-1) = -\bar{\chi}(\Delta).$$

A quasi-cycle of $G$ will mean a subset $F$ of $E$ such that $F$ is either

- the set of edges of a cycle of $G$, or
- a 1-element set consisting of a single useless edge of $G$.

Theorem 1.3 (a) can be restated as claiming that if $G$ has a quasi-cycle, then the polynomial $f_{\mathcal{PM}(G)}$ is divisible by $1 + x$ (since this is tantamount to saying that $f_{\mathcal{PM}(G)}(-1) = -\bar{\chi}(\mathcal{PM}(G)) = 0$). Theorem 1.4 (a) says the same about $\mathcal{PF}(G)$ instead of $\mathcal{PM}(G)$. The next result is a natural strengthening of these corollaries.

**Theorem 1.8.** Let $k$ be a nonnegative integer. Assume that $G$ has at least $k$ disjoint quasi-cycles. Then the polynomials $f_{\mathcal{PM}(G)}(x)$ and $f_{\mathcal{PF}(G)}(x)$ are divisible by $(1 + x)^k$.

We shall prove this theorem in Section 4.

1.2.3. Homotopy types of $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$. The homotopy type of a simplicial complex is defined to be its equivalence class with respect to homotopy-equivalence. Our next goal is to describe the homotopy types of the complexes $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$ in terms of the structure of $G$.

First we consider $\mathcal{PM}(G)$. If $E = \emptyset$ and $s = t$, then $\mathcal{PM}(G) = \emptyset$, so we omit this case from the following theorem.
Theorem 1.9. Assume that $E \neq \emptyset$ or $s \neq t$. Then, the path-missing complex $\mathcal{P}M(G)$ has one of the following two homotopy types:

(a) If $G$ contains a useless edge or a cycle, then $\mathcal{P}M(G)$ is contractible.
(b) Otherwise, $\mathcal{P}M(G)$ is homotopy-equivalent to a sphere of dimension $\#E - \#V' - 1$.

Here and in the following, we say that a simplicial complex $(W, \Delta)$ is “homotopy-equivalent to a sphere of dimension $-1$” if $\Delta = \{\emptyset\}$. (Such a complex is often called an irrelevant complex, and its geometric realization is the empty space.) Moreover, we agree to consider the empty complex $(W, \emptyset)$ to be contractible.

Now we move on to $\mathcal{P}F(G)$. If $s = t$, then $\mathcal{P}F(G) = \emptyset$, since any subset of $E$ contains the trivial $s - t$-path in this case. If $E = \emptyset$ and $s \neq t$, then $\mathcal{P}F(G) = \{\emptyset\}$. In all other cases, the homotopy type of $\mathcal{P}F(G)$ is determined by the following theorem.

Theorem 1.10. Assume that $E \neq \emptyset$. Then, the path-free complex $\mathcal{P}F(G)$ has one of the following two homotopy types:

(a) If $G$ contains a useless edge or a cycle, then $\mathcal{P}F(G)$ is contractible.
(b) Otherwise, $\mathcal{P}F(G)$ is homotopy-equivalent to a sphere of dimension $\#V' - 2$.

Both Theorems 1.9 and 1.10 will be proved in Section 8. The proofs will rely on discrete Morse theory, a purely combinatorial approach to the homotopy of complexes, which yields more than just the homotopy types. See Section 7 for a brief introduction to discrete Morse theory, and [Koz20] and [For02] for deeper background.

2. Simple properties

Before we get to the proofs of the above theorems, let us explore some of the most elementary properties of the complexes $\mathcal{P}F(G)$ and $\mathcal{P}M(G)$.

If $\Delta$ is a simplicial complex on the ground set $W$, then the Alexander dual $\Delta^\vee$ of $\Delta$ is defined by

$$\Delta^\vee = \{F \subseteq W : W \setminus F \notin \Delta\}.$$ 

This is again a simplicial complex on the ground set $W$.

Lemma 2.1. The simplicial complexes $\mathcal{P}F(G)$ and $\mathcal{P}M(G)$ are Alexander duals of each other.

Proof. This is immediate from the definitions. \qed

Given a simplicial complex $\Delta$ with ground set $W$ and an element $w \in W$, we say that $\Delta$ is a cone with apex $w$ if it has the following property:

For any face $I$ of $\Delta$, we have $I \cup \{w\} \in \Delta$.

Note that the “dual” property (i.e., that $I \setminus \{w\} \in \Delta$ for any $I \in \Delta$) automatically holds for any simplicial complex $\Delta$. Thus, $\Delta$ is a cone with apex $w$ if and only if adding $w$ to any subset of $W$ does not change whether this subset is a face of $\Delta$. 


Lemma 2.2. Assume that $G$ has a useless edge $e$. Then, both $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$ are cones with apex $e$.

Proof. The edge $e$ is useless, and thus is not contained in any $s-t$-path. Hence, if $I$ is a subset of $E$, then any $s-t$-path contained in $I \cup \{e\}$ must also be contained in $I$. Therefore, if a subset $I$ of $E$ contains no $s-t$-path, then the subset $I \cup \{e\}$ contains no $s-t$-path either. In other words, for any $I \in \mathcal{PF}(G)$ it also holds that $I \cup \{e\} \in \mathcal{PF}(G)$. Thus, $\mathcal{PF}(G)$ is a cone with apex $e$. A similar argument applies to $\mathcal{PM}(G)$. (Alternatively, using Lemma 2.1, we can derive this from the following general fact: If a simplicial complex $\Delta$ is a cone with apex $w$, then $\Delta^\vee$ is a cone with apex $w$ as well.)

Let us observe a few more properties of $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$, which will not be used in the sequel. If $\Delta$ is a simplicial complex on ground set $W$, then the subsets of $W$ that do not belong to $\Delta$ are called the nonfaces of $\Delta$. A minimal nonface of $\Delta$ means a nonface of $\Delta$ that contains no smaller nonfaces of $\Delta$ as subsets. The codimension of a simplicial complex $\Delta$ on ground set $W$ is defined to be $\#W - \max_{F \in \Delta} \#F$.

Lemma 2.3.

(a) The minimal nonfaces of $\mathcal{PF}(G)$ are the $s-t$-paths of $G$.
(b) The minimal nonfaces of $\mathcal{PM}(G)$ are the minimal cut-sets of $G$, i.e., the minimal subsets of $E$ that intersect every $s-t$-path nontrivially.
(c) The codimension of $\mathcal{PF}(G)$ is the size of a smallest cut-set of $G$.
(d) The codimension of $\mathcal{PM}(G)$ is the length of a shortest $s-t$-path.

Proof. Parts (a) and (b) are immediate from the definitions. For parts (c) and (d), recall that the facets (i.e., maximal faces) of a complex are the complements of the minimal nonfaces of its Alexander dual. Hence the codimension of a complex is the minimum size of a minimal nonface of its Alexander dual.

3. Deletion-contraction for $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$

In preparation for the proofs, we will investigate ways to recursively disassemble simplicial complexes and graphs. To wit, we will introduce the deletion and link operations on simplicial complexes (Subsection 3.1), as well as the deletion and contraction operations on graphs (Subsection 3.2). We will then connect the former to the latter (Subsection 3.3), and finally study the effects of the latter on cycles and useless edges (Subsection 3.4).

3.1. Deletions and links. Given any simplicial complex $\Delta$ on the ground set $W$, we make the following definitions:
• If $w \in W$, then the deletion of $w$ from $\Delta$ is the simplicial complex $dl_\Delta(w)$ on the ground set $W \setminus \{w\}$ defined by
\[ dl_\Delta(w) := \{ F \subseteq W \setminus \{w\} : F \in \Delta \}. \]

• If $w \in W$, then the star of $w$ in $\Delta$ is the simplicial complex $st_\Delta(w)$ on the ground set $W$ defined by
\[ st_\Delta(w) := \{ F \in \Delta : F \cup \{w\} \in \Delta \}. \]

• If $w \in W$, then the link of $w$ in $\Delta$ is the simplicial complex $lk_\Delta(w)$ on the ground set $W \setminus \{w\}$ defined by
\[ lk_\Delta(w) := \{ F \subseteq W \setminus \{w\} : F \cup \{w\} \in \Delta \}. \]

As an exercise in working with the definitions, the simplicially innocent reader may prove the following fundamental fact.

**Proposition 3.1.** Let $\Delta$ be a simplicial complex on ground set $W$. Let $w \in W$. Then:

(a) The star $st_\Delta(w)$ is a simplicial complex on $W$, and is a cone with apex $w$.

(b) We have $dl_\Delta(w) \cup st_\Delta(w) = \Delta$ and $dl_\Delta(w) \cap st_\Delta(w) = lk_\Delta(w)$.

(c) We have $dl_\Delta(w) = \{ F \in \Delta : w \notin F \}$.

(d) We have
\[ lk_\Delta(w) = \{ F \in dl_\Delta(w) : F \cup \{w\} \in \Delta \} = \{ I \setminus \{w\} : I \in \Delta \text{ and } w \in I \}. \]

(e) The map $\{ I \in \Delta : w \in I \} \to lk_\Delta(w), \ I \mapsto I \setminus \{w\}$ is a bijection.

3.2. Deletion and contraction of edges. If $e$ is an edge in $E$, then we shall use the following notions:

• We let $G \setminus e$ denote the graph obtained from $G$ by deleting the edge $e \in E$ (that is, removing $e$ from the edge set of the graph). The vertices $s$ and $t$ remain the distinguished vertices of this new graph.

• We let $G/e$ denote the graph obtained from $G$ by contracting the edge $e$ (that is, identifying the source of $e$ with the target of $e$, and removing the edge $e$). If the two endpoints of $e$ are $u$ and $v$, then the two vertices $u$ and $v$ become one single vertex of $G/e$, which we denote by $u \sim v$ but will still refer to as $u$ or as $v$ (by abuse of notation); all other vertices of $G$ remain intact in $G/e$; and each edge of $G$ distinct from $e$ remains an edge of $G/e$ (except that if $u$ or $v$ was its source or target, then that source or target now changes to $u \sim v$). The distinguished vertices of $G/e$ are $s$ and $t$ again, or rather the vertices resulting from $s$ and $t$ upon the identification.

Thus, both graphs $G \setminus e$ and $G/e$ have edge set $E \setminus \{e\}$.

We shall next prove several graph-theoretical properties of $G \setminus e$ and $G/e$ that will allow us to argue recursively in the “deletion-contraction” paradigm.
Lemma 3.2. Let $e \in E$ be an edge whose source is $s$. Let $I$ be a subset of $E$ that contains $e$. Then, $I$ contains an $s-t$-path of $G$ if and only if $I \setminus \{e\}$ contains an $s-t$-path of $G/e$.

We notice that Lemma 3.2 would be false if we did not require that the source of $e$ is $s$: a simple counterexample is the graph $s \leftarrow t$ with $I = E$.

Proof of Lemma 3.2. $\Longrightarrow$: Assume that $I$ contains an $s-t$-path $p$ of $G$. We must prove that $I \setminus \{e\}$ contains an $s-t$-path of $G/e$. It suffices to show that $I \setminus \{e\}$ contains an $s-t$-walk of $G/e$ (since we can then obtain an $s-t$-path from such a walk by removing cycles). In other words, we must find an $s-t$-walk of $G/e$ contained in $I \setminus \{e\}$. If $p$ does not contain $e$, then $p$ is already such an $s-t$-walk (since all edges of $G$ except for $e$ are edges of $G/e$), and so we are done. If, however, $p$ does contain $e$, then we can obtain an $s-t$-walk of $G/e$ by removing $e$ from $p$, and this $s-t$-walk is clearly contained in $I \setminus \{e\}$.

This proves the “$\Longrightarrow$” direction of Lemma 3.2.

$\Longleftarrow$: Assume that $I \setminus \{e\}$ contains an $s-t$-path $q$ of $G/e$. We must prove that $I$ contains an $s-t$-path of $G$.

Let $s'$ be the target of $e$. Note that the contraction of $e$ identifies the two vertices $s$ and $s'$. Thus, the edges of $G/e$ starting at $s$ are (1) the edges of $G$ starting at $s$ except for the edge $e$, and (2) the edges of $G$ starting at $s'$ (if $s' \neq s$).

Now, let us regard the edges on the $s-t$-path $q$ as edges of the original graph $G$. In this way, it is no longer guaranteed that they still form an $s-t$-path, since $s$ and $s'$ are not necessarily identical in $G$; however, they still form either an $s-t$-path or an $s-t$-path (because $s$ and $s'$ are the only two vertices that get identified in $G/e$, and the $s-t$-path $q$ never revisits its starting vertex $s$ after leaving it). If they form an $s-t$-path, then we conclude that $I$ contains an $s-t$-path of $G$ (namely, the path that they form), and thus we are done. If they don’t, then they form an $s'-t$-path $q'$ that does not contain the vertex $s$, and we can again conclude that $I$ contains an $s-t$-path of $G$ (namely, the path obtained by prepending $e$ to $q'$). In either case, we have shown that $I$ contains an $s-t$-path of $G$. This proves the “$\Longleftarrow$” direction of Lemma 3.2. □

Lemma 3.3. Let $e \in E$ be an edge whose source is $s$. Let $F$ be a subset of $E \setminus \{e\}$. Then, $E \setminus F$ contains an $s-t$-path of $G$ if and only if $(E \setminus \{e\}) \setminus F$ contains an $s-t$-path of $G/e$.

Proof. This follows from Lemma 3.2 (applied to $I = E \setminus F$), because $e \in E \setminus F$ and $(E \setminus F) \setminus \{e\} = (E \setminus \{e\}) \setminus F$. □

3.3. Links and deletions of $\mathcal{P}\mathcal{M}(G)$ and $\mathcal{P}\mathcal{F}(G)$. The link and the deletion of an edge $e$ in $\mathcal{P}\mathcal{M}(G)$ and $\mathcal{P}\mathcal{F}(G)$ can be described in terms of smaller graphs at least when the source of $e$ is $s$.

Lemma 3.4. Let $e \in E$. Then:
(a) We always have $\text{lk}_{\mathcal{PM}(G)}(e) = \mathcal{PM}(G\backslash e)$.
(b) We have $\text{dl}_{\mathcal{PM}(G)}(e) = \mathcal{PM}(G/e)$ if the source of $e$ is $s$.

Proof. (a) An $s-t$-path of $G\backslash e$ is the same thing as an $s-t$-path of $G$ that does not contain $e$. Hence, for any subset $I$ of $E \backslash \{e\}$, we have the equivalence

$$ (I \text{ contains an } s-t\text{-path of } G) \iff (I \text{ contains an } s-t\text{-path of } G\backslash e) $$

(since an $s-t$-path of $G$ contained in $I$ cannot use the edge $e$ anyway, and thus must be an $s-t$-path of $G\backslash e$).

From the definition of $\text{lk}_{\mathcal{PM}(G)}(e)$, we have

$$ \text{lk}_{\mathcal{PM}(G)}(e) = \{F \subseteq E \backslash \{e\} : F \cup \{e\} \in \mathcal{PM}(G)\} $$

(by the definition of $\mathcal{PM}(G)$)

$$ = \{F \subseteq E \backslash \{e\} : E \backslash (F \cup \{e\}) \text{ contains an } s-t\text{-path of } G\} $$

(since $E \backslash (F \cup \{e\}) = (E \backslash \{e\}) \backslash F$ for any set $F$)

$$ = \{F \subseteq E \backslash \{e\} : (E \backslash \{e\}) \backslash F \text{ contains an } s-t\text{-path of } G\backslash e\} $$

(by the equivalence (3), applied to $I = (E \backslash \{e\}) \backslash F$)

$$ = \mathcal{PM}(G\backslash e) \quad (\text{by the definition of } \mathcal{PM}(G\backslash e)). $$

This proves Lemma 3.4 (a).

(b) Assume that the source of $e$ is $s$. From the definition of $\text{dl}_{\mathcal{PM}(G)}(e)$, we have

$$ \text{dl}_{\mathcal{PM}(G)}(e) = \{F \subseteq E \backslash \{e\} : F \in \mathcal{PM}(G)\} $$

(by the definition of $\mathcal{PM}(G)$)

$$ = \{F \subseteq E \backslash \{e\} : E \backslash F \text{ contains an } s-t\text{-path of } G\} $$

(by Lemma 3.3)

$$ = \mathcal{PM}(G/e) \quad (\text{by the definition of } \mathcal{PM}(G/e)). $$

This proves Lemma 3.4 (b).

The following is the $\mathcal{PF}(G)$-analogue of Lemma 3.4.

Lemma 3.5. Let $e \in E$. Then:

(a) We always have $\text{dl}_{\mathcal{PF}(G)}(e) = \mathcal{PF}(G\backslash e)$.
(b) We have $\text{lk}_{\mathcal{PF}(G)}(e) = \mathcal{PF}(G/e)$ if the source of $e$ is $s$.

Proof. (a) For any subset $I$ of $E \backslash \{e\}$, we have the equivalence

$$ (I \text{ contains no } s-t\text{-path of } G) \iff (I \text{ contains no } s-t\text{-path of } G\backslash e) $$

(Indeed, this follows from the equivalence (3) by negating both sides.)
From the definition of $\text{dl}_{\mathcal{P}F(G)}(e)$, we have

$$\text{dl}_{\mathcal{P}F(G)}(e) = \{ F \subseteq E \setminus \{e\} : F \in \mathcal{P}F(G) \}$$

$$= \{ F \subseteq E \setminus \{e\} : F \text{ contains no } s - t\text{-path of } G \}$$

(by the definition of $\mathcal{P}F(G)$)

$$= \{ F \subseteq E \setminus \{e\} : F \text{ contains no } s - t\text{-path of } G \setminus \{e\} \}$$

(by the equivalence (4), applied to $I = F$)

$$= \mathcal{P}F(G \setminus \{e\}) \quad \text{(by the definition of } \mathcal{P}F(G \setminus \{e\})\text{).}$$

This proves Lemma 3.5 (a).

(b) Assume that the source of $e$ is $s$. Let $F$ be a subset of $E \setminus \{e\}$. Hence, $F \cup \{e\}$ is a subset of $E$ that contains $e$. Thus, Lemma 3.2 (applied to $I = F \cup \{e\}$) shows that $F \cup \{e\}$ contains an $s - t$-path of $G$ if and only if $(F \cup \{e\}) \setminus \{e\}$ contains an $s - t$-path of $G/e$. Thus, we have the equivalence

$$(F \cup \{e\} \text{ contains an } s - t\text{-path of } G)$$

$$\iff ((F \cup \{e\}) \setminus \{e\} \text{ contains an } s - t\text{-path of } G/e)$$

$$\iff (F \text{ contains an } s - t\text{-path of } G/e)$$

(since $(F \cup \{e\}) \setminus \{e\} = F$). Negating both sides of this, we obtain the equivalence

$$(F \cup \{e\} \text{ contains no } s - t\text{-path of } G)$$

$$\iff (F \text{ contains no } s - t\text{-path of } G/e) .$$

(5)

Forget that we fixed $F$. We thus have proved the equivalence (5) for any subset $F$ of $E \setminus \{e\}$.

From the definition of $\text{lk}_{\mathcal{P}F(G)}(e)$, we have

$$\text{lk}_{\mathcal{P}F(G)}(e) = \{ F \subseteq E \setminus \{e\} : F \cup \{e\} \in \mathcal{P}F(G) \}$$

$$= \{ F \subseteq E \setminus \{e\} : F \cup \{e\} \text{ contains no } s - t\text{-path of } G \}$$

(by the definition of $\mathcal{P}F(G)$)

$$= \{ F \subseteq E \setminus \{e\} : F \text{ contains no } s - t\text{-path of } G/e \}$$

(by the equivalence (5))

$$= \mathcal{P}F(G/e) \quad \text{(by the definition of } \mathcal{P}F(G/e)\text{).}$$

This proves Lemma 3.5 (b). 

\[\square\]

### 3.4. Graph-theoretical properties of deletion and contraction

The next lemmas discuss how deletion and contraction of a single edge affect various properties of our graph, such as the existence of useless edges and cycles and the number of nonsinks.

**Lemma 3.6.** Any edge of $G$ that has target $s$ is useless.
Proof. An edge with target $s$ cannot appear in an $s-t$-path. Thus, it must be useless.

Lemma 3.7. Let $e \in E$ be an edge whose source is $s$. Let $s'$ be the target of $e$. Assume that $e$ is not the only edge with target $s'$. Then, $G/e$ has a useless edge.

Proof. We assumed that $e$ is not the only edge with target $s'$. Thus, there is another edge $f \neq e$ with target $s'$. In the graph $G/e$, this edge $f$ has target $s$ (since its target $s'$ is identified with $s$ in $G/e$) and thus is useless (by Lemma 3.6, applied to $G/e$ instead of $G$). This proves Lemma 3.7.

Lemma 3.8. Let $e \in E$ be an edge whose source is $s$. Assume that $e$ is not useless. Let $s'$ be the target of $e$. Assume further that $s' \neq t$, and that $e$ is the only edge with target $s'$. Then, $G/e$ has a useless edge.

Proof. The edge $e$ is not useless; thus, there exists an $s-t$-path that contains $e$. This $s-t$-path cannot end at $s'$ (since $s' \neq t$), and thus its last edge cannot be $e$. Hence, there exists an edge that follows $e$ in this $s-t$-path. Let $e'$ be this edge. Clearly, $e'$ has source $s'$.

We have assumed that $e$ is the only edge with target $s'$. Hence, there is no edge of $G/e$ that has target $s'$.

We shall show that $e'$ is useless in $G/e$. Indeed, assume the contrary. Thus, there exists some $s-t$-path of $G/e$ that uses this edge $e'$. This path must pass through the vertex $s'$ (since the edge $e'$ has source $s'$), and thus must start at $s'$ (since there is no edge of $G/e$ that has target $s'$). Since it is an $s-t$-path, this means that $s' = s$. Hence, the edge $e$ has target $s$ (since it has target $s'$), and thus is useless in $G$ (by Lemma 3.6). But this contradicts the fact that $e$ is not useless in $G$. This contradiction shows that our assumption was false. Hence, $G/e$ has a useless edge (namely, $e'$). This proves Lemma 3.8.

Lemma 3.9. Let $e \in E$ be an edge whose source is $s$. Assume that $e$ is not useless. Let $s'$ be the target of $e$. Assume further that $|E| > 1$, and that $e$ is the only edge with target $s'$. Then, $G/e$ has a useless edge.

Proof. If $s' \neq t$, then this follows from Lemma 3.8. Thus, for the rest of this proof, we WLOG assume that $s' = t$.

The edge $e$ is not useless; thus, it belongs to an $s-t$-path. This $s-t$-path thus contains at least one edge. Therefore, $s \neq t$.

But recall that $e$ is the only edge with target $s'$. In other words, $e$ is the only edge with target $t$ (since $s' = t$). Hence, the graph $G/e$ has no edge with target $t$. Therefore, the graph $G/e$ has no $s-t$-path (because $s \neq t$, so any $s-t$-path would have to end with an edge with target $t$). Therefore, any edge of $G/e$ is useless.

We have $|E| > 1$; hence, the set $E \setminus \{e\}$ is nonempty. In other words, $G/e$ has an edge (since $E \setminus \{e\}$ is the edge set of $G/e$). Hence, $G/e$ has a useless edge (since any edge of $G/e$ is useless). Lemma 3.9 is thus proved.
Lemma 3.10. Let $e \in E$ be an edge whose source is $s$. Let $s'$ be the target of $e$. Assume that $G$ has no useless edges. Then:

(a) The graph $G$ has no cycle containing $s$.
(b) We have $s' \neq s$.
(c) If $s' \neq t$, then there exists an edge of $G \setminus e$ whose source is $s'$.
(d) If $s' \neq t$, then the graph $G / e$ has exactly one fewer nonsink than $G$.
(e) If $e$ is the only edge of $G$ with target $s'$, and if $#E > 1$, then $s' \neq t$.

Proof. (a) Any cycle containing $s$ would contain an edge with target $s$. By Lemma 3.6, such an edge would be useless. This contradicts the assumption that $G$ has no useless edges. Thus, $G$ has no cycle containing $s$. This proves Lemma 3.10 (a).

(b) If we had $s' = s$, then the edge $e$ would be a self-loop and therefore useless (since every self-loop is useless). But this is impossible (since $G$ has no useless edges). Hence, Lemma 3.10 (b) is proven.

(c) Assume that $s' \neq t$. The edge $e$ of $G$ is not useless (since $G$ has no useless edges), and thus is contained in an $s - t$-path of $G$. This path cannot end with $e$ (since $s' \neq t$); thus, there exists an edge that follows $e$ in this path. Let $f$ be this edge. Then, the source of $f$ is $s'$, which is distinct from $s$ (since Lemma 3.10 (b) yields $s' \neq s$). Hence, $f \neq e$ (since the source of $e$ is $s$). Thus, $f$ is also an edge of $G \setminus e$. Hence, there exists an edge of $G \setminus e$ whose source is $s'$ (namely, $f$). This proves Lemma 3.10 (c).

(d) Assume that $s' \neq t$. Thus, Lemma 3.10 (c) shows that there exists an edge of $G \setminus e$ whose source is $s'$. Let $f$ be this edge. Then, $f$ is an edge of $G \setminus e$, thus also an edge of $G / e$ and of $G$. Hence, $G$ has an edge with source $s'$ (namely, $f$). In other words, $s'$ is a nonsink of $G$. Also, $s$ is a nonsink of $G$ (since the edge $e$ has source $s$).

The two nonsinks $s$ and $s'$ are distinct (by Lemma 3.10 (b)). In the graph $G / e$, these two nonsinks become identical, but still remain a nonsink of $G / e$ (since the edge $f$ has source $s'$ and is an edge of $G / e$). For any vertex $v \in V \setminus \{s, s'\}$, it is clear that $v$ is a nonsink of $G / e$ if and only if $v$ is a nonsink of $G$ (since the edges of $G / e$ with source $v$ are exactly the edges of $G$ with source $v$). Thus, when passing from $G$ to $G / e$, we only lose one nonsink (since the two nonsinks $s$ and $s'$ become identical). This proves Lemma 3.10 (d).

(e) Assume that $e$ is the only edge of $G$ with target $s'$, and that $#E > 1$. We must prove that $s' \neq t$.

Assume the contrary. Thus, $s' = t$. Hence, $e$ is the only edge of $G$ with target $t$ (since $e$ is the only edge of $G$ with target $s'$).

Since $#E > 1$, there exists at least one edge $f \in E$ distinct from $e$. This edge $f$ cannot be useless (since $G$ has no useless edges). In other words, $f$ is contained in an $s - t$-path $p$ of $G$. Consider this path $p$.

This path $p$ has at least one edge (since it contains $f$). Thus, it has a last edge. This last edge of $p$ must have target $t$ (since $p$ is an $s - t$-path), and thus must be
e (since e is the only edge of G with target t). Therefore, the second-to-last vertex of p is the source of e. In other words, the second-to-last vertex of p is s (since the source of e is s). Of course, the first vertex of p is s, too (since p is an s–t-path).

We know that both e and f are edges of p. Since f is distinct from e, this entails that p has at least two edges, thus at least three vertices. Hence, the first vertex and the second-to-last vertex of p are distinct (since the vertices of a path are distinct). But this contradicts the fact that both of these vertices are s. This contradiction shows that our assumption was false. Lemma 3.10 (e) is proved.

Lemma 3.11. Let e ∈ E be an edge whose source is s. Assume that G has a cycle but no useless edges. Then, both graphs G\e and G/e have cycles.

Proof. We have assumed that G has a cycle. This cycle cannot contain s (by Lemma 3.10 (a)), and thus cannot contain the edge e. Hence, this cycle is a cycle of G\e and also a nontrivial closed walk of G/e. Thus, both graphs G\e and G/e have nontrivial closed walks, and therefore have cycles. This proves Lemma 3.11.

Lemma 3.12. Let e ∈ E be an edge whose source is s. Assume that G has no cycles and no useless edges. Assume further that G\e has a useless edge. Then, the graph G/e has no cycles and no useless edges.

Proof. Let s′ be the target of e. Then, s′ ≠ s (by Lemma 3.10 (b)).

The graph G\e is a subgraph of G, and thus has no cycles (since G has no cycles).

We have assumed that G\e has a useless edge. Fix such an edge, and call it f. Thus, the edge f is distinct from e, and belongs to no s–t-path of G\e. But f is not a useless edge of G (since G has no useless edges); thus, there exists an s–t-path p of G that contains f. Consider this p. This path p contains e (since otherwise, p would be an s–t-path of G\e, which would contradict the fact that f belongs to no s–t-path of G\e).

First, we shall show that G/e has no cycles.

Indeed, assume the contrary. Thus, G/e has a cycle c. Consider each edge of c as an edge of G. Then, c is either a cycle of G, or an s′–s-path of G, or an s–s′-path of G (because the only way in which the cycle could break when lifted from G/e to G is if the cycle passes through the identified vertex s ∼ s'). The first of these three cases is impossible (since G has no cycles), and so is the second (by Lemma 3.6, since G has no useless edges). Thus, the third case must hold. In other words, c is an s–s′-path of G. Since c does not contain e (because c was originally a cycle of G/e), this shows that c is an s–s′-path of G\e. Now, replace the edge e by this s–s′-path c in the path p. The result is an s–t-path of G\e. This walk must be a path (since otherwise, it would contain a cycle, but G\e has no cycles), and thus is an s–t-path; furthermore, it contains f (since p contains f, and f survives the replacement of e by c because f is distinct from e). Hence, there exists an s–t-path of G\e that contains f (namely, this walk). This contradicts the fact that f is a
useless edge of $G/e$. This contradiction shows that our assumption was false. Hence, we have proven that $G/e$ has no cycles.

Next, we shall show that $G/e$ has no useless edges.

Let $h$ be any edge of $G/e$; we will show that $h$ is not useless. Consider $h$ as an edge of $G$. Then $h$ is distinct from $e$, and cannot be a useless edge of $G$ (since $G$ has no useless edges). Hence, there exists an $s-t$-path of $G$ containing $h$. Let $r$ be such an $s-t$-path. By removing the edge $e$ from $r$ (if it is contained in $r$), we obtain an $s-t$-walk of $G/e$. This latter $s-t$-walk must actually be an $s-t$-path (since $G/e$ has no cycles), and furthermore contains $h$ (because $r$ contains $h$, and because $h$ is distinct from $e$ and thus could not have been removed). Hence, there exists an $s-t$-path of $G/e$ that contains $h$. Thus $h$ is not useless. Since $h$ was an arbitrary edge of $G/e$, we have proven that $G/e$ has no useless edges.

 Altogether, we now know that $G/e$ has no cycles and no useless edges. This proves Lemma 3.12.

**Lemma 3.13.** Let $e \in E$ be an edge whose source is $s$. Assume that $\#E > 1$. Assume further that $G\setminus e$ has no useless edges.

Then, the graph $G\setminus e$ has the same nonsinks as the graph $G$.

**Proof.** The set of edges of the graph $G\setminus e$ is $E \setminus \{e\}$, and thus is nonempty (since $\#E > 1$). Thus, the graph $G\setminus e$ has at least one edge $f$. This edge $f$ is not useless (since $G\setminus e$ has no useless edges), and thus there exists an $s-t$-path of $G\setminus e$ that contains $f$. This $s-t$-path has at least one edge (since it contains $f$), and thus must have a first edge. This first edge has source $s$.

Hence, the graph $G\setminus e$ has an edge with source $s$ (namely, this first edge). In other words, $s$ is a nonsink of $G\setminus e$. Consequently, $s$ is a nonsink of $G$ as well.

For any vertex $v \in V \setminus \{s\}$, it is clear that $v$ is a nonsink of $G\setminus e$ if and only if $v$ is a nonsink of $G$ (since $G\setminus e$ differs from $G$ only in the edge $e$, whose source is $s$). This also holds for $v = s$ (since $s$ is a nonsink of $G$ and of $G\setminus e$ both). Thus, it holds for all $v \in V$. In other words, the graph $G\setminus e$ has the same nonsinks as the graph $G$. This proves Lemma 3.13. □

### 4. Proof of the f-polynomial

Let us first take aim at Theorem 1.8.

We begin with a recurrence for f-polynomials.

**Lemma 4.1.** Let $\Delta$ be a simplicial complex on the ground set $W$. Let $w \in W$. Then,

$$f_{\Delta}(x) = f_{\Delta \setminus \{w\}}(x) + xf_{\Delta \setminus \{w\}}(x).$$

**Proof.** The definition of $f_{\Delta}(x)$ yields

$$f_{\Delta}(x) = \sum_{I \subseteq \Delta} x^{\#I} = \sum_{I \subseteq \Delta, w \notin I} x^{\#I} + \sum_{I \subseteq \Delta, w \in I} x^{\#I}.$$ (6)
But the definition of $f_{dl\Delta}(w)(x)$ shows that
\begin{equation}
\sum_{I \in dl\Delta(w)} x^{|I|} = \sum_{I \in \Delta; \ w / \in I} x^{|I|}
\end{equation}
(since the $I \in dl\Delta(w)$ are precisely the $I \in \Delta$ satisfying $w \notin I$). The definition of $f_{lk\Delta}(w)(x)$ shows that
\begin{equation}
\sum_{I \in lk\Delta(w)} x^{|I\setminus\{w\}|} = \sum_{I \in \Delta; \ w \in I} x^{|I|}
\end{equation}
(since the map $\{I \in \Delta : w \in I\} \to lk\Delta(w), \ I \mapsto I \setminus \{w\}$ is a bijection). Multiplying the latter equality by $x$, we obtain
\begin{equation}
x f_{lk\Delta}(w)(x) = \sum_{I \in \Delta; \ w \in I} x^{1+|I\setminus\{w\}|} = \sum_{I \in \Delta; \ w \in I} x^{|I|}
\end{equation}
(since $1 + |I\setminus\{w\}| = |I|$ for any $I \in \Delta$ satisfying $w \in I$). Adding (7) and (8) together, and comparing the result with (6), we obtain Lemma 4.1.

As a consequence of Lemma 4.1, we get the following lemma.

**Lemma 4.2.** Let $\Delta$ be a simplicial complex on the ground set $W$. Let $w \in W$. Assume that $\Delta$ is a cone with apex $w$. Then:

(a) We have $dl\Delta(w) = lk\Delta(w)$.

(b) We have $f_\Delta(x) = (1 + x) f_{lk\Delta}(w)(x)$.

**Proof.** (a) Let $I \in dl\Delta(w)$. Then, $I \in \Delta$ and $w \notin I$ (by the definition of $dl\Delta(w)$). But $\Delta$ is a cone with apex $w$; hence, from $I \in \Delta$, we obtain $I \cup \{w\} \in \Delta$ (by the definition of a cone). Thus, $I$ is an $F \in dl\Delta(w)$ satisfying $F \cup \{w\} \in \Delta$. In other words, $I \in lk\Delta(w)$ (by the definition of $lk\Delta(w)$).

Forget that we fixed $I$. We thus have shown that $I \in lk\Delta(w)$ for each $I \in dl\Delta(w)$. In other words, $dl\Delta(w) \subseteq lk\Delta(w)$. Combining this with the obvious inclusion $lk\Delta(w) \subseteq dl\Delta(w)$, we obtain $dl\Delta(w) = lk\Delta(w)$. This proves Lemma 4.2 (a).

(b) Lemma 4.1 yields
\begin{align*}
f_\Delta(x) &= f_{dl\Delta}(w)(x) + x f_{lk\Delta}(w)(x) \\
&= f_{lk\Delta}(w)(x) + x f_{lk\Delta}(w)(x) \quad (\text{since part (a) yields } dl\Delta(w) = lk\Delta(w)) \\
&= (1 + x) f_{lk\Delta}(w)(x).
\end{align*}

Next, we start analyzing quasi-cycles of deletions and contractions.

**Lemma 4.3.** Let $e \in E$ be a useless edge. Let $k$ be a positive integer such that $G$ has at least $k$ disjoint quasi-cycles. Then, $G \setminus e$ has at least $k - 1$ disjoint quasi-cycles.
Proof. Any path of $G \setminus e$ is a path of $G$. Thus, any useless edge of $G$ other than $e$ is also a useless edge of $G \setminus e$. Furthermore, any cycle of $G$ that does not contain $e$ is also a cycle of $G \setminus e$. Combining these two statements, we conclude that any quasi-cycle of $G$ that does not contain $e$ is also a quasi-cycle of $G \setminus e$.

But $G$ has at least $k$ disjoint quasi-cycles. Clearly, at least $k - 1$ of them do not contain $e$, and thus are quasi-cycles of $G \setminus e$ as well. Hence, $G \setminus e$ has at least $k - 1$ disjoint quasi-cycles. □

**Lemma 4.4.** Let $e \in E$ be an edge whose source is $s$. Let $k$ be a nonnegative integer. Assume that $G$ has at least $k$ disjoint quasi-cycles but has no useless edges. Then, each of the two graphs $G \setminus e$ and $G/e$ has at least $k$ disjoint quasi-cycles.

Proof. Let $c$ be a quasi-cycle of $G$. Then, $c$ must be the set of the edges of a cycle of $G$ (since $G$ has no useless edges). This cycle cannot contain $s$ (by Lemma 3.10 (a)), and thus cannot contain the edge $e$. Hence, this cycle is still a cycle of $G \setminus e$ and also a nontrivial closed walk of $G/e$, which shows that it contains at least one cycle of $G/e$. Thus, the quasi-cycle $c$ is a quasi-cycle of $G \setminus e$ and contains at least one quasi-cycle of $G/e$.

Now, forget that we have fixed $c$. We thus have shown that each quasi-cycle $c$ of $G$ is a quasi-cycle of $G \setminus e$ and contains at least one quasi-cycle of $G/e$. Therefore, since $G$ has at least $k$ disjoint quasi-cycles, we conclude that each of the two graphs $G \setminus e$ and $G/e$ has at least $k$ disjoint quasi-cycles. □

**Proof of Theorem 1.8.** We proceed in multiple steps.

*Step 1:* We claim that Theorem 1.8 is true when $k = 0$.

Indeed, if $k = 0$, then any polynomial is divisible by $(1 + x)^k$ (since $(1 + x)^k = (1 + x)^0 = 1$); thus, Theorem 1.8 is proven in this case.

*Step 2:* We claim that Theorem 1.8 is true when $\#E = 0$.

Indeed, if $\#E = 0$, then $G$ has no edges and therefore no quasi-cycles; but this entails that $k = 0$ (since $G$ has at least $k$ disjoint quasi-cycles), and therefore Theorem 1.8 is true (according to Step 1). Hence, Theorem 1.8 is proven in the case when $\#E = 0$.

*Step 3:* We shall now prove Theorem 1.8 by induction on $\#E$.

The base case ($\#E = 0$) follows from Step 2.

Thus, we proceed to the induction step. We fix a directed graph $G = (V, E, s, t)$ with $\#E > 0$, and we assume (as induction hypothesis) that Theorem 1.8 is true for all graphs with exactly $\#E - 1$ edges. Thus, in particular, Theorem 1.8 is true for $G \setminus e$ and for $G/e$ whenever $e$ is an edge of $G$. We must now prove Theorem 1.8 for our graph $G$.

If $k = 0$, then Theorem 1.8 is true by Step 1; thus, we WLOG assume that $k$ is positive. Hence, $k - 1$ is a nonnegative integer.

We are in one of the following two cases:

*Case 1:* The graph $G$ has no useless edges.

*Case 2:* The graph $G$ has a useless edge.
Case 2: The graph $G$ has a useless edge.

Let us first consider Case 1. In this case, the graph $G$ has no useless edges. But $G$ has at least one edge (because $|E| > 0$). Let $f$ be such an edge. The edge $f$ cannot be useless (since $G$ has no useless edges), and thus belongs to some $s-t$-path. This $s-t$-path must have at least one edge (namely, $f$); let $e$ be its first edge. Thus, $e$ is an edge with source $s$.

Lemma 4.1 (applied to $W = E$, $\Delta = \mathcal{PM}(G)$ and $w = e$) yields

$$f_{\mathcal{PM}(G)}(x) = f_{\Delta_{\mathcal{PM}(G)}(e)}(x) + x f_{\text{lk}_{\mathcal{PM}(G)}(e)}(x)$$

(9)

(since $dl_{\mathcal{PM}(G)}(e) = \mathcal{PM}(G/e)$ by Lemma 3.4 (b), and since $lk_{\mathcal{PM}(G)}(e) = \mathcal{PM}(G/e)$ by Lemma 3.4 (a)).

Lemma 4.1 (applied to $W = E$, $\Delta = \mathcal{PF}(G)$ and $w = e$) yields

$$f_{\mathcal{PF}(G)}(x) = f_{\Delta_{\mathcal{PF}(G)}(e)}(x) + x f_{\text{lk}_{\mathcal{PF}(G)}(e)}(x)$$

(10)

(since $dl_{\mathcal{PF}(G)}(e) = \mathcal{PF}(G/e)$ by Lemma 3.5 (a), and since $lk_{\mathcal{PF}(G)}(e) = \mathcal{PF}(G/e)$ by Lemma 3.5 (b)).

But Lemma 4.4 shows that each of the two graphs $G/e$ and $G/e$ has at least $k$ disjoint quasi-cycles. By the induction hypothesis, we can thus apply Theorem 1.8 to each of these two graphs, and conclude that the polynomials $f_{\mathcal{PM}(G/e)}(x)$ and $f_{\mathcal{PF}(G/e)}(x)$ as well as the polynomials $f_{\mathcal{PM}(G/e)}(x)$ and $f_{\mathcal{PF}(G/e)}(x)$ are divisible by $(1 + x)^k$. Hence, all four addends on the right hand sides of (9) and (10) are divisible by $(1 + x)^k$. Hence, so are the left hand sides. In other words, both polynomials $f_{\mathcal{PM}(G)}(x)$ and $f_{\mathcal{PF}(G)}(x)$ are divisible by $(1 + x)^k$. Hence, we have proven Theorem 1.8 for our graph $G$ in Case 1.

Let us next consider Case 2. In this case, the graph $G$ has a useless edge. Let $e$ be such an edge. Lemma 2.2 shows that $\mathcal{PM}(G)$ is a cone with apex $e$. Hence, Lemma 4.2 (b) (applied to $W = E$, $\Delta = \mathcal{PM}(G)$ and $w = e$) shows that $f_{\mathcal{PM}(G)}(x) = (1 + x) f_{\text{lk}_{\mathcal{PM}(G)}(e)}(x)$. In view of Lemma 3.4 (a), this rewrites as $f_{\mathcal{PM}(G)}(x) = (1 + x) f_{\mathcal{PM}(G/e)}(x)$.

But Lemma 4.3 shows that the graph $G/e$ has at least $k - 1$ disjoint quasi-cycles. By the induction hypothesis, we can thus apply Theorem 1.8 to this graph with $k$ replaced by $k - 1$, and conclude that the polynomials $f_{\mathcal{PM}(G/e)}(x)$ and $f_{\mathcal{PF}(G/e)}(x)$ are divisible by $(1 + x)^{k - 1}$.

Now, the equality $f_{\mathcal{PM}(G)}(x) = (1 + x) f_{\mathcal{PM}(G/e)}(x)$ shows that the polynomial $f_{\mathcal{PM}(G)}(x)$ is divisible by $(1 + x)^k$ (since $f_{\mathcal{PM}(G/e)}(x)$ is divisible by $(1 + x)^{k - 1}$).

Furthermore, Lemma 2.2 shows that $\mathcal{PF}(G)$ is a cone with apex $e$. Hence, Lemma 4.2 (a) (applied to $W = E$, $\Delta = \mathcal{PF}(G)$ and $w = e$) shows that $dl_{\mathcal{PF}(G)}(e) = \text{lk}_{\mathcal{PF}(G)}(e)$. Also, Lemma 4.2 (b) (applied to $W = E$, $\Delta = \mathcal{PF}(G)$ and $w = e$) shows
that \( f_{\mathcal{PF}(G)}(x) = (1 + x) f_{\mathcal{lPK}_{\mathcal{PF}(G)}(e)}(x) \). In view of \( df_{\mathcal{PF}(G)}(e) = \mathcal{lK}_{\mathcal{PF}(G)}(e) \), this rewrites as \( f_{\mathcal{PF}(G)}(x) = (1 + x) f_{\mathcal{lPK}_{\mathcal{PF}(G)(e)}}(x) \). In view of Lemma 3.5 (a), this rewrites as \( f_{\mathcal{PF}(G)}(x) = (1 + x) f_{\mathcal{PF}(G\setminus e)}(x) \). Thus, the polynomial \( f_{\mathcal{PF}(G)}(x) \) is divisible by \((1 + x)^k\) (since \( f_{\mathcal{PF}(G\setminus e)}(x) \) is divisible by \((1 + x)^{k-1}\)).

We have now shown that both polynomials \( f_{\mathcal{PM}(G)}(x) \) and \( f_{\mathcal{PF}(G)}(x) \) are divisible by \((1 + x)^k\). Hence, we have proven Theorem 1.8 for our graph \( G \) in Case 2.

We have now proven Theorem 1.8 in both Cases 1 and 2; hence, Theorem 1.8 always holds for our graph \( G \). This completes the induction step, and with it the proof of Theorem 1.8. \( \square \)

5. The c-polynomial

5.1. Definition. Our following analysis of \( \mathcal{PM}(G) \) and \( \mathcal{PF}(G) \) will rely on certain properties of \( G \) that are most conveniently recorded under the umbrella of a polynomial. To define it, we need the following simple lemma.\(^3\)

**Lemma 5.1.** Assume that \( E \neq \emptyset \). Then, \( \#V' - 1 \) and \( \#E - \#V' \) are nonnegative integers.

**Proof.** The source of any edge \( e \in E \) is a nonsink of \( G \) (since it is the source of an edge), and thus belongs to \( V' \). Hence, the map

\[
E \to V',
\]

\[
e \mapsto \text{(the source of } e)\]

is well-defined. This map is furthermore surjective (since each \( v \in V' \) is a nonsink of \( G \), and thus (by definition) the source of some edge \( e \in E \)). Hence, we have found a surjective map from \( E \) to \( V' \). Thus, \( \#E \geq \#V' \). Therefore, \( \#E - \#V' \) is a nonnegative integer. It remains to prove that so is \( \#V' - 1 \).

There exists at least one edge \( e \in E \) (since \( E \neq \emptyset \)). Pick such an edge \( e \). Its source must be a nonsink of \( G \) (since it is the source of an edge), i.e., an element of \( V' \). Hence, the set \( V' \) has at least one element. In other words, \( \#V' \geq 1 \). Hence, \( \#V' - 1 \) is a nonnegative integer. This completes the proof of Lemma 5.1. \( \square \)

**Definition 5.2.** Assume that \( E \neq \emptyset \). Then, we define the c-polynomial of \( G \) to be the polynomial

\[
c_G(x, y) := \begin{cases} 
0, & \text{if } G \text{ has a useless edge or a cycle;} \\
\frac{x^{\#V' - 1} y^{\#E - \#V'}}{x^{\#E} y^{\#V'}}, & \text{otherwise}
\end{cases}
\]

in \( \mathbb{Z}[x, y] \). (This polynomial is well-defined, since Lemma 5.1 shows that both exponents \( \#V' - 1 \) and \( \#E - \#V' \) are nonnegative integers.)

\(^3\)Recall that \( V' \) denotes the set of all nonsinks of \( G \).
Note that \( c_G(x, y) \) depends not only on the underlying digraph \((V, E)\) but also on the vertices \(s\) and \(t\).

5.2. The recursion for c-polynomials. The most useful feature of the c-polynomial (to us) will be the following recursive formula.

**Lemma 5.3.** Assume that \( G \) has no useless edges, and that \(#E > 1\). Let \( e \in E \) be an edge whose source is \( s \). Then,

\[
c_G(x, y) = xc_{G/e}(x, y) + yc_{G\setminus e}(x, y).
\]

**Proof.** From \(#E > 1\), we obtain \( E \setminus \{e\} \neq \emptyset \). Hence, \( c_{G/e}(x, y) \) and \( c_{G\setminus e}(x, y) \) are defined.

If \( G \) has a cycle, then so do \( G/e \) and \( G\setminus e \) (by Lemma 3.11), and thus all three polynomials \( c_G(x, y) \) and \( c_{G/e}(x, y) \) and \( c_{G\setminus e}(x, y) \) equal 0 (by Definition 5.2). Hence, in this case, the claim of Lemma 5.3 boils down to \( 0 = x0 + y0 \), which is obvious.

Thus, we WLOG assume that \( G \) has no cycles. Hence, \( G \) has no useless edges and no cycles. Definition 5.2 thus yields

\[
c_G(x, y) = x#V' - 1y#E - #V'.
\]

Let \( s' \) be the target of the edge \( e \). We are in one of the following two cases:

- **Case 1:** The edge \( e \) is not the only edge of \( G \) with target \( s' \).
- **Case 2:** The edge \( e \) is the only edge of \( G \) with target \( s' \).

Let us first consider Case 1. In this case, the edge \( e \) is not the only edge of \( G \) with target \( s' \). Hence, Lemma 3.7 shows that \( G/e \) has a useless edge. Hence, Definition 5.2 yields

\[
c_{G/e}(x, y) = 0.
\]

Furthermore, the graph \( G\setminus e \) is a subgraph of \( G \), and thus has no cycles (since \( G \) has no cycles). If \( G\setminus e \) had a useless edge, then Lemma 3.12 would yield that the graph \( G/e \) has no cycles and no useless edges; but this would contradict the fact that \( G/e \) has a useless edge. Thus, \( G\setminus e \) has no useless edges.

Therefore, Lemma 3.13 shows that the graph \( G\setminus e \) has the same nonsinks as \( G \). In other words, the set of nonsinks of \( G\setminus e \) is \( V' \) (since the set of nonsinks of \( G \) is \( V' \)).

We now know that the graph \( G\setminus e \) has no useless edges and no cycles, and its set of nonsinks is \( V' \), whereas its edge set is \( E \setminus \{e\} \) (by its definition). Thus, Definition 5.2 yields

\[
c_{G\setminus e}(x, y) = x#V' - 1y#(E\setminus\{e\}) - #V' = x#V' - 1y#E - 1 - #V' \quad \text{(since } #(E\setminus\{e\}) = #E - 1).\]

Using this equality and using (12), we have

\[
x c_{G/e}(x, y) + y c_{G\setminus e}(x, y) = x0 + y x#V' - 1y#E - 1 - #V' = y x#V' - 1y#E - 1 - #V' = x #V' - 1y#E - #V'.
\]
Comparing this with (11), we obtain \( c_G(x, y) = xc_{G/e}(x, y) + yc_{G\setminus e}(x, y) \). Hence, Lemma 5.3 is proved in Case 1.

Let us now consider Case 2. In this case, the edge \( e \) is the only edge of \( G \) with target \( s' \). Moreover, the edge \( e \) of \( G \) is not useless (since \( G \) has no useless edges). Hence, Lemma 3.9 yields that \( G \setminus e \) has a useless edge. Therefore, Definition 5.2 yields

\[ c_{G\setminus e}(x, y) = 0. \]

Furthermore, Lemma 3.12 shows that the graph \( G/e \) has no cycles and no useless edges.

Let \( V'' \) be the set of all nonsinks of \( G/e \). Lemma 3.10 (e) yields that \( s' \neq t \). Thus, Lemma 3.10 (d) shows that the graph \( G/e \) has exactly one fewer nonsink than \( G \). In other words, \( \#V'' = \#V' - 1 \). Next, recall that \( G/e \) has no cycles and no useless edges, and the set of all nonsinks of \( G/e \) is \( V'' \), whereas the edge set of \( G/e \) is \( E\setminus\{e\} \). Thus, Definition 5.2 yields

\[ c_{G/e}(x, y) = x^{\#V'' - 1}y^{(\#(E\setminus\{e\}) - \#V'')} \]

\[ = x^{\#V' - 1}y^{(\#E - 1) - (\#V' - 1)} \]

\[ = x^{\#V' - 2}y^{\#E - \#V'}. \]

Using this equality and using (13), we obtain

\[ xc_{G/e}(x, y) + yc_{G\setminus e}(x, y) = x^{\#V' - 2}y^{\#E - \#V'} + y0 \]

\[ = x^{\#V' - 2}y^{\#E - \#V'} = x^{\#V' - 1}y^{\#E - \#V'}. \]

Comparing this with (11), we obtain \( c_G(x, y) = xc_{G/e}(x, y) + yc_{G\setminus e}(x, y) \). Hence, Lemma 5.3 is proved in Case 2.

We have now proved Lemma 5.3 in all cases.

\[ \square \]

6. Proof of the Euler characteristic

We shall now work towards computing the Euler characteristics of our complexes.

6.1. General facts about Euler characteristics. We begin with some general properties of Euler characteristics.

**Lemma 6.1.** Let \( \Delta \) be a simplicial complex on the ground set \( W \). Let \( w \in W \). Then,

\[ \tilde{\chi}(\Delta) = \tilde{\chi}(d\Delta(w)) - \tilde{\chi}(l\Delta(w)). \]

**Proof.** Lemma 4.1 yields \( f_\Delta(x) = f_{d\Delta(w)}(x) + xf_{l\Delta(w)}(x) \). Substituting \(-1\) for \( x \) on both sides of this equality, we find \( f_\Delta(-1) = f_{d\Delta(w)}(-1) - f_{l\Delta(w)}(-1) \). In view of (2), this rewrites as \( -\tilde{\chi}(\Delta) = -\tilde{\chi}(d\Delta(w)) - (-\tilde{\chi}(l\Delta(w))). \) Multiplying both sides of this equality by \(-1\), we obtain precisely the claim of Lemma 6.1. \( \square \)
Lemma 6.2. Let $\Delta$ be a simplicial complex on a nonempty ground set $W$. Then,

$$\tilde{\chi}(\Delta^\vee) = (-1)^{#W-1} \tilde{\chi}(\Delta).$$

Proof. By definition, the faces of $\Delta^\vee$ are the complements (in $W$) of the subsets of $W$ that do not belong to $\Delta$. Hence, the map $2^W \setminus \Delta \rightarrow \Delta^\vee$, $I \mapsto W \setminus I$ is a bijection. Thus,

$$\tilde{\chi}(\Delta^\vee) = \sum_{I \in \Delta^\vee} (-1)^{#I-1} = \sum_{I \in 2^W \setminus \Delta} (-1)^{(W \setminus I) - 1}$$

$$= (-1)^{#W-1} \sum_{I \in 2^W \setminus \Delta} (-1)^{#I}.$$  \hspace{1cm} (14)

Since $W$ is nonempty, we have $\sum_{I \in 2^W} (-1)^{#I} = 0$, and therefore

$$\sum_{I \in 2^W \setminus \Delta} (-1)^{#I} = - \sum_{I \in \Delta} (-1)^{#I} = \sum_{I \in \Delta} (-1)^{#I-1} = \tilde{\chi}(\Delta).$$

Substituting this into (14), we obtain Lemma 6.2. \hfill $\Box$

Lemma 6.3. Let $\Delta$ be a simplicial complex on a ground set $W$. Let $w \in W$. Assume that $\Delta$ is a cone with apex $w$. Then, $\tilde{\chi}(\Delta) = 0$.

Proof. Lemma 4.2 (b) yields $f_\Delta(x) = (1 + x) f_{lk_\Delta(w)}(x)$. Substituting $-1$ for $x$ on both sides of this equality, we obtain $f_\Delta(-1) = (1 + (-1)) f_{lk_\Delta(w)}(-1) = 0$. In view of (2), this rewrites as $-\tilde{\chi}(\Delta) = 0$. Thus, $\tilde{\chi}(\Delta) = 0$. This proves Lemma 6.3. \hfill $\Box$

6.2. The Euler characteristics of $\mathcal{P}M(G)$ and $\mathcal{P}F(G)$. Recall the c-polynomial $c_G(x, y)$ introduced in Definition 5.2. We shall now restate Theorem 1.3 in the form that is most convenient for our proof.

Lemma 6.4. Assume that $E \neq \emptyset$. Then,

$$\tilde{\chi}(\mathcal{P}M(G)) = -c_G(1, -1).$$

This lemma will quickly yield the original form of Theorem 1.3 (once we compute $c_G(1, -1)$ and handle the $E = \emptyset$ case by hand).

Proof of Lemma 6.4. We proceed in two steps:

Step 1: We claim that Lemma 6.4 is true when $G$ has a useless edge.

Indeed, assume that $G$ has a useless edge $e$. Then, Lemma 2.2 shows that $\mathcal{P}M(G)$ is a cone with apex $e$, and therefore Lemma 6.3 (applied to $W = E, \Delta = \mathcal{P}M(G)$ and $w = e$) shows that $\tilde{\chi}(\mathcal{P}M(G)) = 0$. Meanwhile, Definition 5.2 yields $c_G(x, y) = 0$ (since $G$ has a useless edge) and therefore $c_G(1, -1) = 0$. Therefore, $-c_G(1, -1) = -0 = 0$. Comparing this with $\tilde{\chi}(\mathcal{P}M(G)) = 0$, we see that $\tilde{\chi}(\mathcal{P}M(G)) = -c_G(1, -1)$. Hence, we have proved Lemma 6.4 in the case when $G$ has a useless edge.

Step 2: Let us now prove Lemma 6.4 in general.
We proceed by induction on the positive integer $\#E$ (this is a positive integer, since $E \neq \emptyset$).

**Base case:** We must show that Lemma 6.4 holds when $\#E = 1$.

Indeed, assume that $\#E = 1$. Thus, $E = \{e\}$ for some edge $e$ of $G$. Consider this edge $e$. If $G$ has a useless edge, then we already know (from Step 1) that Lemma 6.4 is true. Thus, we WLOG assume that $G$ has no useless edges. Hence, the edge $e$ is not useless. In other words, $e$ is contained in an $s - t$-path of $G$. This $s - t$-path cannot have any other edges beyond $e$ (since $E = \{e\}$), and thus must consist of the edge $e$ alone. Thus, the source of $e$ is $s$, whereas the target of $e$ is $t$. Moreover, the edge $e$ cannot be a self-loop (since it is contained in a path), and thus we have $s \neq t$. Thus, the digraph $G$ consists of the single edge $s \rightarrow t$ and a (possibly empty) set of other vertices but no other edges (since $E = \{e\}$). Thus, we can describe $\mathcal{P}M(G)$ explicitly: The only subsets of $E$ are $\emptyset$ and $\{e\}$ (since $E = \{e\}$), and we have $\emptyset \in \mathcal{P}M(G)$ (since $E \setminus \emptyset = E$ contains an $s - t$-path) but $\{e\} \notin \mathcal{P}M(G)$ (since $E \setminus \{e\} = \emptyset$ contains no $s - t$-path (because $s \neq t$)). Hence, $\mathcal{P}M(G) = \{\emptyset\}$, so that $\tilde{\chi}(\mathcal{P}M(G)) = \tilde{\chi}(\{\emptyset\}) = -1$. On the other hand, our description of $G$ shows that $G$ has no useless edges and no cycles, and has exactly 1 edge (namely, $e$) and exactly 1 nonsink (namely, $s$). Definition 5.2 thus shows that $c_G(x, y) = x^{1-1}y^{1-1} = x^0y^0 = 1$. Hence, $c_G(1, -1) = 1$, so that $-c_G(1, -1) = -1$. Comparing this with $\tilde{\chi}(\mathcal{P}M(G)) = -1$, we obtain $\tilde{\chi}(\mathcal{P}M(G)) = -c_G(1, -1)$. This shows that Lemma 6.4 holds for our $G$. Thus, Lemma 6.4 is proved when $\#E = 1$. This completes the base case.

**Induction step:** We fix a directed graph $G = (V, E, s, t)$ with $\#E > 1$, and we assume (as induction hypothesis) that Lemma 6.4 is true for all graphs with exactly $\#E - 1$ edges. We must now prove Lemma 6.4 for our graph $G$.

If $G$ has a useless edge, then we already know (from Step 1) that Lemma 6.4 is true. Thus, we WLOG assume that $G$ has no useless edges. However, $E \neq \emptyset$. Thus, there exists some $f \in E$. Consider this $f$. The edge $f$ cannot be useless (since $G$ has no useless edges), and thus is contained in an $s - t$-path of $G$. This path has at least one edge (since it contains $f$), and thus has a first edge. Let $e$ be this first edge. Then, the edge $e \in E$ has source $s$ (since it is the first edge of an $s - t$-path). Lemma 5.3 thus yields

$$c_G(x, y) = xc_{G/e}(x, y) + yc_{G\setminus e}(x, y).$$

Substituting 1 and $-1$ for $x$ and $y$ in this equality, we find

$$c_G(1, -1) = 1c_{G/e}(1, -1) + (-1)c_{G\setminus e}(1, -1)$$

$$= c_{G/e}(1, -1) - c_{G\setminus e}(1, -1).$$

(15)

However, $\#E > 1$ entails $E \not\subseteq \{e\}$, thus $E \setminus \{e\} \neq \emptyset$. The graph $G/e$ has edge set $E \setminus \{e\}$, whose size is $\#(E \setminus \{e\}) = \#E - 1$. Thus, by our induction hypothesis,
Lemma 6.4 is true for $G/e$ instead of $G$. In other words, we have

$$\bar{\chi}(\mathcal{P}M(G/e)) = -c_{G/e}(1, -1).$$

The same argument (applied to $G\backslash e$ instead of $G/e$) shows that

$$\bar{\chi}(\mathcal{P}M(G\backslash e)) = -c_{G\backslash e}(1, -1).$$

But Lemma 6.1 (applied to $W = E$, $\Delta = \mathcal{P}M(G)$ and $w = e$) yields

$$\bar{\chi}(\mathcal{P}M(G)) = \bar{\chi}(\text{dl}_{\mathcal{P}M(G)}(e)) - \bar{\chi}(\text{lk}_{\mathcal{P}M(G)}(e))$$

$$= \bar{\chi}(\mathcal{P}M(G/e)) - \bar{\chi}(\mathcal{P}M(G\backslash e))$$

(since $\text{dl}_{\mathcal{P}M(G)}(e) = \mathcal{P}M(G/e)$ by Lemma 3.4 (b) and since $\text{lk}_{\mathcal{P}M(G)}(e) = \mathcal{P}M(G\backslash e)$ by Lemma 3.4 (a)). In view of (16) and (17), we can rewrite this as

$$\bar{\chi}(\mathcal{P}M(G)) = \left( -c_{G/e}(1, -1) \right) - \left( -c_{G\backslash e}(1, -1) \right)$$

$$= - \left( c_{G/e}(1, -1) - c_{G\backslash e}(1, -1) \right) = -c_G(1, -1).$$

This shows that Lemma 6.4 holds for our $G$. This completes the induction step. Thus, Lemma 6.4 is proved by induction.

Proof of Theorem 1.3. We are in one of the following four cases:

Case 1: We have $E = \emptyset$ and $s = t$.

Case 2: We have $E = \emptyset$ and $s \neq t$.

Case 3: We have $E \neq \emptyset$, and the graph $G$ has a useless edge or a cycle.

Case 4: We have $E \neq \emptyset$, and the graph $G$ has no useless edges and no cycles.

Let us first consider Case 1. In this case, we have $E = \emptyset$ and $s = t$. Thus, the graph $G$ has no edges (since $E = \emptyset$). Moreover, $s = t$ shows that the set $\emptyset \setminus \emptyset$ contains an $s-t$-path (namely, the trivial path, with no edges at all). Hence, $\emptyset \in \mathcal{P}M(G)$, so that $\mathcal{P}M(G) = \{\emptyset\}$. Therefore, $\bar{\chi}(\mathcal{P}M(G)) = \bar{\chi}(\{\emptyset\}) = -1$. On the other hand, $G$ has no nonsinks (since $G$ has no edges); thus, $V' = \emptyset$ and therefore $\#V' = 0$. Combined with $\#E = 0$ (since $E = \emptyset$), this yields $(-1)^{\#E - \#V' + 1} = (-1)^{0-0+1} = -1$. Comparing this with $\bar{\chi}(\mathcal{P}M(G)) = -1$, we obtain $\bar{\chi}(\mathcal{P}M(G)) = (-1)^{\#E - \#V' + 1}$, which is precisely the value that Theorem 1.3 (b) predicts for $\bar{\chi}(\mathcal{P}M(G))$. Thus, Theorem 1.3 is proved in Case 1.

Let us next consider Case 2. In this case, we have $E = \emptyset$ and $s \neq t$. Thus, the graph $G$ has no edges (since $E = \emptyset$). Moreover, $s \neq t$ shows that any $s-t$-path must contain at least one edge. Hence, $G$ has no $s-t$-path (since $G$ has no edges). Therefore, $\mathcal{P}M(G) = \emptyset$, so that $\bar{\chi}(\mathcal{P}M(G)) = \bar{\chi}(\emptyset) = 0$. But this is precisely the value that Theorem 1.3 (a) predicts for $\bar{\chi}(\mathcal{P}M(G))$ (since $E = \emptyset$ and $s = t$). Thus, Theorem 1.3 is proved in Case 2.
Next, let us consider Case 3. In this case, we have \( E \neq \emptyset \), and the graph \( G \) has a useless edge or a cycle. Hence, Definition 5.2 yields \( c_G(x, y) = 0 \). Substituting 1 and \(-1\) for \( x \) and \( y \) in this equality, we find \( c_G(1, -1) = 0 \). However, Lemma 6.4 yields

\[
\tilde{\chi}(\mathcal{P}\mathcal{M}(G)) = -c_G(1, -1) = 0 \quad \text{(since } c_G(1, -1) = 0). \]

But this is precisely the value that Theorem 1.3 (a) predicts for \( \tilde{\chi}(\mathcal{P}\mathcal{M}(G)) \). Thus, Theorem 1.3 is proved in Case 3.

Finally, let us consider Case 4. In this case, we have \( E \neq \emptyset \), and the graph \( G \) has no useless edges and no cycles. Hence, Definition 5.2 yields \( c_G(x, y) = x^{#V' - 1}y^{#E - #V'} \). Substituting 1 and \(-1\) for \( x \) and \( y \) in this equality, we find \( c_G(1, -1) = (-1)^{#E - #V'} \). However, Lemma 6.4 yields

\[
\tilde{\chi}(\mathcal{P}\mathcal{M}(G)) = -c_G(1, -1) = -(-1)^{#E - #V'} = (-1)^{#E - #V' - 1}. \]

But this is precisely the value that Theorem 1.3 (b) predicts for \( \tilde{\chi}(\mathcal{P}\mathcal{M}(G)) \). Thus, Theorem 1.3 is proved in Case 4.

We have now proved Theorem 1.3 in all four cases. \( \square \)

**Proof of Theorem 1.4.** The case \( E = \emptyset \) is straightforward and left to the reader; so we WLOG assume that \( E \neq \emptyset \). Lemma 2.1 yields \( \mathcal{P}\mathcal{F}(G) = (\mathcal{P}\mathcal{M}(G))^\vee \). Hence, \( \tilde{\chi}(\mathcal{P}\mathcal{F}(G)) = \tilde{\chi}((\mathcal{P}\mathcal{M}(G))^\vee) = (-1)^{#E - 1}\tilde{\chi}(\mathcal{P}\mathcal{M}(G)) \) (by Lemma 6.2, applied to \( W = E \) and \( \Delta = \mathcal{P}\mathcal{M}(G) \)). Substituting the expression for \( \tilde{\chi}(\mathcal{P}\mathcal{M}(G)) \) given in Theorem 1.3 into this equation, we find an expression for \( \tilde{\chi}(\mathcal{P}\mathcal{F}(G)) \). This proves Theorem 1.4. \( \square \)

**Proof of Corollary 1.5.** If \( \Delta \) is any simplicial complex, then

\[
(18) \quad \#\Delta \equiv \tilde{\chi}(\Delta) \mod 2,
\]

because the definition of the reduced Euler characteristic (specifically, the right hand side of (1)) shows that

\[
\tilde{\chi}(\Delta) = \sum_{I \in \Delta} (-1)^{#I - 1} \equiv \sum_{I \in \Delta} 1 = \#\Delta \mod 2.
\]

Applying this to \( \Delta = \mathcal{P}\mathcal{M}(G) \), we obtain \( \#(\mathcal{P}\mathcal{M}(G)) \equiv \tilde{\chi}(\mathcal{P}\mathcal{M}(G)) \mod 2 \). Hence, Corollary 1.5 follows from Theorem 1.3. \( \square \)

**Proof of Corollary 1.6.** This is analogous to the proof of Corollary 1.5 (but relies on Theorem 1.4 instead of Theorem 1.3). \( \square \)
7. Discrete Morse theory

7.1. Definitions and topological meaning. In this section, we shall recall the basics of Forman’s discrete Morse theory (foreshadowed by Brown’s [Bro92]). We refer to [For02] and [Koz20] for deeper-going expositions of this subject. Here we shall only recall the basics that we need. We will follow the modern terminology of “acyclic matchings”, as in Kozlov’s [Koz20].

First, we introduce a basic set-theoretic notation.

**Definition 7.1.** Let \(A\) and \(B\) be two sets. Then, we write \(A \prec B\) if there exists some \(b \in B \setminus A\) such that \(B = A \cup \{b\}\). Equivalently, we write \(B \succ A\) in this case. Clearly, if \(A\) and \(B\) are two finite sets, then we have the equivalences
\[(A \prec B) \iff (B \succ A) \iff (A \subseteq B \text{ and } #(B \setminus A) = 1) \iff (A \subseteq B \text{ and } #B = #A + 1).\]

For instance, \(\{2, 5\} \prec \{2, 3, 5\}\) but not \(\{2, 5\} \prec \{2, 3, 4, 5\}\). The binary relations \(\prec\) and \(\succ\) are called “is covered by” and “covers”.

Next, we define the notion of a matching (following [Koz20, Definition 10.6]).

**Definition 7.2.** Let \(\Delta\) be a simplicial complex with ground set \(W\). A partial matching (or matching for short) on \(\Delta\) shall mean a pair \((M, \mu)\), where \(M\) is a subset of \(\Delta\) (that is, a set of faces of \(\Delta\)), and where \(\mu : M \to M\) is an involution (that is, a map satisfying \(\mu \circ \mu = \text{id}\)) with the property that each \(A \in M\) satisfies
\[\text{either } \mu(A) \prec A \text{ or } \mu(A) \succ A.\]

Note that \(M\) is uniquely determined by \(\mu\) (namely, as the domain of \(\mu\)), so that we will refer to \(\mu\) alone as a matching.

Given a matching \((M, \mu)\), we shall refer to the faces \(A \in M\) as the matched faces of this matching, while the faces \(A \in \Delta \setminus M\) will be called the unmatched faces of this matching.

**Example 7.3.** Let \(W = \{1, 2, 3\}\). Let \(\Delta\) be the simplicial complex with ground set \(W\) that contains all 8 subsets of \(W\) as faces. Consider the matching \((M, \mu)\) given by \(M = \Delta \setminus \{\emptyset, W\}\) and
\[
\begin{align*}
\mu(\{1\}) &= \{1, 2\}, & \mu(\{2\}) &= \{2, 3\}, & \mu(\{3\}) &= \{3, 1\}, \\
\mu(\{1, 2\}) &= \{1\}, & \mu(\{2, 3\}) &= \{2\}, & \mu(\{3, 1\}) &= \{3\}.
\end{align*}
\]
The unmatched faces of this matching are \(\emptyset\) and \(W\).

\(^4\)In [Koz20, Definition 10.6], Kozlov works in a more general setting, replacing a simplicial complex \(\Delta\) by an arbitrary poset. We do not need this generality here; the only posets we will be using are simplicial complexes \(\Delta\), ordered by inclusion (so that the relation \(\prec\) introduced in Definition 7.1 is precisely the covering relation of this poset).
Example 7.4. Let $n$ and $k$ be two nonnegative integers. Let $W$ be an $n$-element set. Let $\Delta$ be the simplicial complex consisting of all subsets $A$ of $W$ having size $\#A \leq k$. (This is called the $(k - 1)$-skeleton of the simplex on $W$.') Pick any element $w \in W$. Let $M \subseteq \Delta$ be the set of all subsets $A$ of $W$ that satisfy $\#(A \setminus \{w\}) < k$ (or, equivalently, that satisfy $\#A < k$ or $(\#A = k$ and $w \in A$)). We can then define a map $\mu : M \to M$ by setting

$$
\mu(A) = \begin{cases} 
A \cup \{w\}, & \text{if } w \notin A; \\
A \setminus \{w\}, & \text{if } w \in A 
\end{cases}
$$

for each $A \in M$.

(That is, the map $\mu$ inserts the element $w$ into any face that does not contain $w$, and removes it from any face that does.) It is easy to see that this map $\mu$ is well-defined and is a matching on $\Delta$ (or, more precisely, the pair $(M, \mu)$ is).

The unmatched faces of this matching are precisely the $k$-element subsets of $W$ that do not contain $w$. Their number is $\binom{n-1}{k}$.

Discrete Morse theory is interested in matchings with a special property, defined in terms of cycles ([Koz20, Definition 10.7]), that we describe now.

Definition 7.5. Let $\Delta$ be a simplicial complex with ground set $W$. Let $(M, \mu)$ be a matching on $\Delta$.

(a) A cycle of $\mu$ means an $n$-tuple $(F_1, F_2, \ldots, F_n)$ of distinct faces in $M$ such that $n \geq 2$ and

$$F_1 \succ \mu(F_1) \prec F_2 \succ \mu(F_2) \prec F_3 \succ \cdots \prec F_n \succ \mu(F_n) \prec F_1$$

(that is, such that each $i \in \{1, 2, \ldots, n\}$ satisfies $F_i \succ \mu(F_i) \prec F_{i+1}$, where $F_{n+1} := F_1$).

(b) The matching $\mu$ is said to be acyclic if it has no cycle.

Example 7.6.

(a) The matching $\mu$ constructed in Example 7.3 is not acyclic. Indeed, the 3-tuple

$$\{1, 2\} \succ \mu(\{1\}) \prec \{3, 1\} \succ \mu(\{3\}) \prec \{2, 3\} \succ \mu(\{2\}) \prec \{1, 2\}.$$ 

(b) The matching $\mu$ in Example 7.4 is acyclic. Indeed, the faces $F \in M$ satisfying $F \succ \mu(F)$ are precisely the faces $F \in M$ that contain $w$; therefore, a cycle $(F_1, F_2, \ldots, F_n)$ of $\mu$ would have to satisfy $w \in F_1$ and $w \in F_2$, which would easily yield $F_1 = \mu(F_1) \cup \{w\} = F_2$, contradicting the distinctness of $F_1, F_2, \ldots, F_n$.

\footnote{Indeed, each $A \in M$ satisfies $(A \cup \{w\}) \setminus \{w\} = A \setminus \{w\}$, so that $\#((A \cup \{w\}) \setminus \{w\}) = \#(A \setminus \{w\}) < k$ and therefore $A \cup \{w\} \in M$, and similarly $A \setminus \{w\} \in M$.}
Acyclic matchings are the main objects of discrete Morse theory, although different texts give different definitions whose equivalence is not always immediate\(^6\). They give highly useful information on the homotopy type of a complex, as the following theorem ([For02, Theorem 2.5], [Koz20, Theorem 11.2], actually a particular case of [Bro92, Proposition 1]) shows.

**Theorem 7.7.** Let \( \mu \) be an acyclic matching on a simplicial complex \( \Delta \). Then, \( \Delta \) is homotopy-equivalent to a CW complex \( X \) with the property that for each \( d \geq -1 \), the number of \( d \)-cells in \( X \) equals the number of unmatched \( d \)-dimensional faces of \( \Delta \).

Thus, acyclic matchings can be used as a combinatorial proxy for (certain kinds of) homotopy equivalences, particularly when they have few unmatched faces. It is via these proxies that we will prove Theorems 1.9 and 1.10.

### 7.2. The unmatched \( f \)-polynomial.

First, we shall show some basic properties of acyclic matchings. We will use the following polynomial fingerprint of matchings.

**Definition 7.8.** Let \( (M, \mu) \) be a matching on a simplicial complex \( \Delta \). Then, we define the **unmatched \( f \)-polynomial** of \( \mu \) to be the polynomial

\[
u(\mu)(x) := \sum_{I \in \Delta \setminus M} x^{|I|} \in \mathbb{Z}[x].\]

Note that the sum here ranges over all unmatched faces of \( \mu \).

For instance, the matching \( \mu \) in Example 7.3 has unmatched \( f \)-polynomial \( \nu(\mu)(x) = x^0 + x^3 \), whereas the one in Example 7.4 has unmatched \( f \)-polynomial \( \nu(\mu)(x) = \binom{n-1}{k} x^k \). A trivial example is the empty matching \((\emptyset, \emptyset)\), which exists for every simplicial complex \( \Delta \), and whose unmatched \( f \)-polynomial \( \nu(\emptyset)(x) \) is just the usual \( f \)-polynomial \( f(\emptyset)(x) \) (since \( \Delta \setminus \emptyset = \Delta \)).

A matching \( \mu \) whose unmatched \( f \)-polynomial \( \nu(\mu)(x) \) is a single monomial \( x^m \) is one that has only one unmatched face (which has size \( m \), that is, dimension \( m - 1 \)). For such matchings, Theorem 7.7 has the following consequence.

**Corollary 7.9.** Let \( \mu \) be an acyclic matching on a simplicial complex \( \Delta \).

\(^6\)The closest notion in Forman’s original work is that of a gradient vector field in [For02, §3]. Indeed, our partial matchings \( \mu \) correspond to Forman’s “discrete vector fields” as defined in [For02, Definition 3.3] (specifically, if \( (M, \mu) \) is a partial matching, then the set \{\((A, \mu(A)) \mid A \in M \text{ and } A \prec \mu(A)\)\} is a discrete vector field; our cycles are more or less Forman’s “closed V-paths” (at least those that cannot be broken up into shorter ones); thus, our acyclic partial matchings correspond to Forman’s “gradient vector fields of discrete Morse functions” (according to [For02, Theorem 3.5], which is proved in [Koz20, Proposition 14.11]). The notion of a discrete Morse function was regarded as fundamental when Forman originally conceived discrete Morse theory in 1995, but is nowadays considered as somewhat of a red herring.

We note that unmatched faces of a partial matching are called “critical simplices” in [For02].
(a) If \( u_\mu(x) = 0 \), then \( \Delta \) is contractible.
(b) If \( u_\mu(x) = x^m \) for some \( m \in \mathbb{N} \), then \( \Delta \) is homotopy-equivalent to a sphere of dimension \( m - 1 \).

We note that the condition \( u_\mu(x) = 0 \) in Corollary 7.9 (a) means that all faces of \( \Delta \) are matched; a simplicial complex \( \Delta \) with such a matching \( \mu \) is said to be collapsible (see [For02, Theorem 6.4]).

7.3. Two reduction lemmas. To construct acyclic matchings with simple unmatched f-polynomials, we shall use two basic lemmas. The first one guarantees the collapsibility of any cone ([Koz20, Proposition 10.11]).

**Lemma 7.10.** Let \( \Delta \) be a simplicial complex that is a cone. Then, \( \Delta \) has an acyclic matching \((M, \mu)\) satisfying \( u_\mu(x) = 0 \).

To keep this paper self-contained, we shall give a proof of this lemma in Appendix A.

The second lemma allows the recursive construction of acyclic matchings, based on deletions and links. In its essence, it appears to go back to Forman, and similar facts are found across the literature (e.g., [Koz20, Proposition 10.13], [Eng09, Lemma 2.4], [Koz08, Theorem 11.10]), but we have not been able to locate the following version.\(^7\)

**Lemma 7.11.** Let \( \Delta \) be a simplicial complex with ground set \( W \). Let \( w \in W \).

Let \((M_-, \mu_-)\) be an acyclic matching on \( \text{dl}_\Delta(w) \). Let \((M_+, \mu_+)\) be an acyclic matching on \( \text{lk}_\Delta(w) \). Then, \( \Delta \) has an acyclic matching \((M, \mu)\) satisfying
\[
u_\mu(x) = \nu_{\mu_-}(x) + xu_{\mu_+}(x).
\]

This lemma, too, will be proved in Appendix A.

8. Proof of the homotopy types

Our final goal is to prove the homotopy types of \( \mathcal{P} \mathcal{F}(G) \) and \( \mathcal{P} \mathcal{M}(G) \) (Theorems 1.9 and 1.10). In view of Corollary 7.9, it will suffice to prove the following two Morse-theoretic results.

**Theorem 8.1.** Assume that \( E \neq \emptyset \) or \( s \neq t \). Then:

(a) If \( G \) contains a useless edge or a cycle, then the complex \( \mathcal{P} \mathcal{M}(G) \) has an acyclic matching \((M, \mu)\) satisfying \( u_\mu(x) = 0 \).
(b) Otherwise, \( \mathcal{P} \mathcal{M}(G) \) has an acyclic matching \((M, \mu)\) satisfying \( u_\mu(x) = x^{#E-#V'} \).

**Theorem 8.2.** Assume that \( E \neq \emptyset \). Then:

\(^7\)Actually, Lemma 7.11 can be derived from [Koz08, Theorem 11.10] by taking \( P = \Delta \) and \( Q = \{0 < 1\} \) and letting \( \varphi : P \to Q \) be the map that sends each face \( F \in \Delta \) to 1 if \( w \in F \) and to 0 if \( w \notin F \). The acyclic matching \((M_-, \mu_-)\) on \( \text{dl}_\Delta(w) \) then becomes an acyclic matching of \( \varphi^{-1}(0) \), whereas the acyclic matching \((M_+, \mu_+)\) on \( \text{lk}_\Delta(w) \) can be converted into an acyclic matching of \( \varphi^{-1}(1) \) (by inserting \( w \) into each face).
(a) If $G$ contains a useless edge or a cycle, then the complex $\mathcal{PF}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = 0$.

(b) Otherwise, $\mathcal{PF}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = x^{|V|-1}$.

We will derive both of these from the following lemmas (stated in terms of the c-polynomial $c_G(x, y)$ from Definition 5.2).

**Lemma 8.3.** Assume that $E \neq \emptyset$. Then, the complex $\mathcal{PM}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = c_G(1, x)$.

**Lemma 8.4.** Assume that $E \neq \emptyset$. Then, the complex $\mathcal{PF}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = c_G(x, 1)$.

**Proof of Lemma 8.3.** Our proof is structurally similar to the proof of Lemma 6.4 above. We proceed in two steps:

Step 1: We claim that Lemma 8.3 is true when $G$ has a useless edge.

Indeed, assume that $G$ has a useless edge. Then, Definition 5.2 yields $c_G(x, y) = 0$. By specializing $x$ and $y$ to 1 and $x$ here, we obtain $c_G(1, x) = 0$. Meanwhile, Lemma 2.2 shows that $\mathcal{PM}(G)$ is a cone (since $G$ has a useless edge). Hence, Lemma 7.10 (applied to $W = E$ and $\Delta = \mathcal{PM}(G)$) shows that $\mathcal{PM}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = 0$. In other words, $\mathcal{PM}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = c_G(1, x)$ (since $c_G(1, x) = 0$). Hence, we have proved Lemma 8.3 in the case when $G$ has a useless edge.

Step 2: Let us now prove Lemma 6.4 in general.

We proceed by induction on the positive integer $|E|$ (this is a positive integer, since $E \neq \emptyset$).

Base case: We must show that Lemma 8.3 holds when $|E| = 1$.

Indeed, assume that $|E| = 1$. Thus, $E = \{e\}$ for some edge $e$ of $G$. Consider this edge $e$. If $G$ has a useless edge, then we already know (from Step 1) that Lemma 8.3 is true. Thus, we WLOG assume that $G$ has no useless edges. As in the proof of Lemma 6.4 above, we can now see that $\mathcal{PM}(G) = \{\emptyset\}$ and $c_G(x, y) = 1$. Substituting 1 and $x$ for $x$ and $y$ in the latter equality, we obtain $c_G(1, x) = 1$. However, $\mathcal{PM}(G) = \{\emptyset\}$ shows that the complex $\mathcal{PM}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = 1$ (namely, the empty matching $(\emptyset, \emptyset)$). In other words, $\mathcal{PM}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = c_G(1, x)$ (since $c_G(1, x) = 1$). This shows that Lemma 8.3 holds for our $G$. Thus, Lemma 8.3 is proved when $|E| = 1$. This completes the base case.

Induction step: We fix a directed graph $G = (V, E, s, t)$ with $|E| > 1$, and we assume (as induction hypothesis) that Lemma 8.3 is true for all graphs with exactly $|E| - 1$ edges. We must now prove Lemma 8.3 for our graph $G$.

If $G$ has a useless edge, then we already know (from Step 1) that Lemma 8.3 is true. Thus, we WLOG assume that $G$ has no useless edges. As in our above proof of Lemma 6.4, we can thus find an edge $e \in E$ with source $s$. Consider this edge $e$. 
Lemma 5.3 yields
\[ c_G(x, y) = xc_{G/e}(x, y) + yc_{G/e}(x, y). \]
Substituting 1 and \( x \) for \( x \) and \( y \) in this equality, we find
\[ c_G(1, x) = c_{G/e}(1, x) + xc_{G/e}(1, x). \]
However, \( |E| > 1 \) entails \( E \not\subseteq \{e\} \), thus \( E \setminus \{e\} \neq \emptyset \). The graph \( G/e \) has edge set \( E \setminus \{e\} \), whose size is \( |E \setminus \{e\}| = |E| - 1 \). Thus, by our induction hypothesis, Lemma 8.3 is true for \( G/e \) instead of \( G \). In other words, the complex \( \mathcal{P} \mathcal{M}(G/e) \) has an acyclic matching \((M_{f}, \mu_{f})\) satisfying \( u_{\mu_{f}}(x) = c_{G/e}(1, x) \). The same argument (applied to \( G \setminus \{e\} \)) shows that the complex \( \mathcal{P} \mathcal{M}(G/e) \) has an acyclic matching \((M_{s}, \mu_{s})\) satisfying \( u_{\mu_{s}}(x) = c_{G\setminus e}(1, x) \). Consider these two matchings \((M_{f}, \mu_{f})\) and \((M_{s}, \mu_{s})\). Thus, \((M_{f}, \mu_{f})\) is an acyclic matching of the complex \( \mathcal{P} \mathcal{M}(G/e) = d\mathcal{P} \mathcal{M}(G)(e) \) (by Lemma 3.4 (b)), whereas \((M_{s}, \mu_{s})\) is an acyclic matching of the complex \( \mathcal{P} \mathcal{M}(G/e) = l\mathcal{P} \mathcal{M}(G)(e) \) (by Lemma 3.4 (a)). Hence, Lemma 7.11 (applied to \( W = E \) and \( \Delta = \mathcal{P} \mathcal{M}(G) \) and \( w = e \) and \( (M_{s}, \mu_{s}) = (M_{f}, \mu_{f}) \) and \( (M_{f}, \mu_{f}) = (M_{s}, \mu_{s}) \)) shows that \( \mathcal{P} \mathcal{M}(G) \) has an acyclic matching \((M, \mu)\) satisfying
\[ u_{\mu}(x) = u_{\mu_{f}}(x) + x \quad u_{\mu_{s}}(x) = c_{G/e}(1, x) + xc_{G\setminus e}(1, x) = c_{G}(1, x) \]
(by (19)). This shows that Lemma 8.3 holds for our \( G \). This completes the induction step. Thus, Lemma 8.3 is proved by induction.

Proof of Lemma 8.4. This is analogous to our above proof of Lemma 8.3. The main difference is in the induction step: The matching \((M_{f}, \mu_{f})\) is now an acyclic matching of the complex \( \mathcal{P} \mathcal{F}(G/e) = l\mathcal{P} \mathcal{F}(G)(e) \) (by Lemma 3.5 (b)), whereas the matching \((M_{s}, \mu_{s})\) is now an acyclic matching of the complex \( \mathcal{P} \mathcal{F}(G/e) = d\mathcal{P} \mathcal{F}(G)(e) \) (by Lemma 3.5 (a)). Hence, Lemma 7.11 (applied to \( W = E \) and \( \Delta = \mathcal{P} \mathcal{F}(G) \) and \( w = e \) and \( (M_{s}, \mu_{s}) = (M_{f}, \mu_{f}) \) and \( (M_{f}, \mu_{f}) = (M_{s}, \mu_{s}) \)) shows that \( \mathcal{P} \mathcal{F}(G) \) has an acyclic matching \((M, \mu)\) satisfying
\[ u_{\mu}(x) = u_{\mu_{f}}(x) + x \quad u_{\mu_{s}}(x) = c_{G\setminus e}(x, 1) + xc_{G/e}(x, 1) \]
\[ = xc_{G/e}(x, 1) + c_{G\setminus e}(x, 1) = c_{G}(x, 1) \]
(again by a specialization of Lemma 5.3).

Proof of Theorem 8.1. (a) Assume that \( G \) has a useless edge or a cycle. Thus, Definition 5.2 yields \( c_{G}(x, y) = 0 \). Hence, \( c_{G}(1, x) = 0 \).
Our assumption shows furthermore that $G$ has at least one edge, so that $E \neq \emptyset$. Hence, Lemma 8.3 shows that the complex $\mathcal{PM}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = c_G(1, x)$. In other words, $\mathcal{PM}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = 0$ (since $c_G(1, x) = 0$). This proves Theorem 8.1 (a).

(b) Assume that $G$ has no useless edges and no cycles.

If $E = \emptyset$, then the claim of Theorem 8.1 (b) is easily checked by hand: Indeed, in this case, our “$E \neq \emptyset$ or $s \neq t$” assumption yields that $s \neq t$, and therefore $G$ has no $s-t$-path at all; therefore the complex $\mathcal{PM}(G) = \{\emptyset\}$ has an acyclic matching $(\emptyset, \emptyset)$. But this matches the claim of Theorem 8.1 (b), since $\#E = 0$ and $\#V' = 0$ and therefore $x \#E - \#V' = x^{0-0} = x^0 = 1$.

Having dealt with the case $E = \emptyset$, we thus WLOG assume that $E \neq \emptyset$. Hence, Definition 5.2 yields $c_G(x, y) = x^{\#V'-1}y^{\#E - \#V'}$ (since $G$ has no useless edges and no cycles). Substituting 1 and $x$ for $x$ and $y$ in this equality, we find $c_G(1, x) = 1^{\#V'-1}1^{\#E - \#V'} = x^{\#E - \#V'}$.

But Lemma 8.3 shows that the complex $\mathcal{PM}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = c_G(1, x)$. In other words, the complex $\mathcal{PM}(G)$ has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = x^{\#E - \#V'}$ (since $c_G(1, x) = x^{\#E - \#V'}$). This proves Theorem 8.1 (b). \hfill \Box

Proof of Theorem 8.2. This follows from Lemma 8.4 in the same way as Theorem 8.1 was derived from Lemma 8.3. (The proof is even simpler this time, since the case $E = \emptyset$ no longer needs to be handled.) \hfill \Box

As explained above, Theorem 1.9 and Theorem 1.10 follow (respectively) from Theorem 8.1 and Theorem 8.2 using Corollary 7.9.

9. Further directions

9.1. Combinatorial grapes. Marietti and Testa, in [MT08, Definition 3.2], introduced a certain well-behaved class of simplicial complexes: the combinatorial grapes. We recall their definition (translated into our language) next.

Definition 9.1. The combinatorial grapes are a class of simplicial complexes defined recursively:

- Any simplicial complex $(W, \Delta)$ with $\#W \leq 1$ is a combinatorial grape.
- Let $(W, \Delta)$ be a simplicial complex. If there exists an $a \in W$ such that both $\text{lk}_\Delta(a)$ and $\text{dl}_\Delta(a)$ are combinatorial grapes, and such that there is a cone $(W \setminus \{a\}, \Gamma)$ satisfying $\text{lk}_\Delta(a) \subseteq \Gamma \subseteq \text{dl}_\Delta(a)$, then $(W, \Delta)$ is a combinatorial grape.\(^8\)

\(^8\)Here, the statement $\text{lk}_\Delta(a) \subseteq \Gamma \subseteq \text{dl}_\Delta(a)$ means that each face of $\text{lk}_\Delta(a)$ is a face of $\Gamma$, and that each face of $\Gamma$ is a face of $\text{dl}_\Delta(a)$.
It is shown in [MT08, Proposition 3.3] that combinatorial grapes have a rather simple homotopy type (viz., they are disjoint unions of points or wedges of spheres).

An even more restrictive notion is that of a strong grape, which we define as follows.

**Definition 9.2.** The strong grapes are a class of simplicial complexes defined recursively:

- Any simplicial complex \((W, \Delta)\) with \(#W \leq 1\) is a strong grape.
- Let \((W, \Delta)\) be a simplicial complex. If there exists an \(a \in W\) such that both \(\text{lk}_\Delta(a)\) and \(\text{dl}_\Delta(a)\) are strong grapes, and such that at least one of \(\text{lk}_\Delta(a)\) and \(\text{dl}_\Delta(a)\) is a cone, then \((W, \Delta)\) is a strong grape.

Clearly, every strong grape is a combinatorial grape. We shall now show that both complexes \(\mathcal{PF}(G)\) and \(\mathcal{PM}(G)\) for a graph \(G\) are strong grapes.

**Proposition 9.3.** Both \(\mathcal{PF}(G)\) and \(\mathcal{PM}(G)\) are strong grapes.

**Proof.** We proceed by strong induction on \(#E\). Thus, we assume (as induction hypothesis) that both \(\mathcal{PF}(H)\) and \(\mathcal{PM}(H)\) are strong grapes whenever \(H\) is a graph with fewer edges than \(G\). We must now show that both \(\mathcal{PF}(G)\) and \(\mathcal{PM}(G)\) are strong grapes. This is obvious if \(#E \leq 1\), so we WLOG assume that \(#E > 1\).

The case when \(s = t\) is easy. Thus, we WLOG assume that \(s \neq t\).

The case when \(G\) has no \(s - t\)-path is easy. Thus, we WLOG assume that \(G\) has at least one \(s - t\)-path. This \(s - t\)-path has at least one edge (since \(s \neq t\)). Thus, its first edge is well-defined. This first edge must have source \(s\) and is not useless (since it belongs to an \(s - t\)-path). Hence, there exists an edge \(e\) with source \(s\) that is not useless. Consider this edge \(e\). Let \(s'\) be its target.

The graph \(G/e\) has fewer edges than \(G\). Hence, by the induction hypothesis, both \(\mathcal{PF}(G/e)\) and \(\mathcal{PM}(G/e)\) are strong grapes. Likewise, both \(\mathcal{PF}(G\setminus e)\) and \(\mathcal{PM}(G\setminus e)\) are strong grapes.

Lemma 3.5 (a) yields that \(\text{dl}_{\mathcal{PF}(G)}(e) = \mathcal{PF}(G\setminus e)\), whereas Lemma 3.5 (b) yields that \(\text{lk}_{\mathcal{PF}(G)}(e) = \mathcal{PF}(G/e)\). Likewise, Lemma 3.4 (a) yields that \(\text{lk}_{\mathcal{PM}(G)}(e) = \mathcal{PM}(G\setminus e)\), whereas Lemma 3.4 (b) yields that \(\text{dl}_{\mathcal{PM}(G)}(e) = \mathcal{PM}(G/e)\).

Recall that \(\mathcal{PF}(G/e)\) and \(\mathcal{PF}(G\setminus e)\) are strong grapes. In other words, \(\text{lk}_{\mathcal{PF}(G)}(e)\) and \(\text{dl}_{\mathcal{PF}(G)}(e)\) are strong grapes (since \(\text{dl}_{\mathcal{PF}(G)}(e) = \mathcal{PF}(G\setminus e)\) and \(\text{lk}_{\mathcal{PF}(G)}(e) = \mathcal{PF}(G/e)\)). Thus, if we can show that at least one of \(\text{lk}_{\mathcal{PF}(G)}(e)\) and \(\text{dl}_{\mathcal{PF}(G)}(e)\) is a cone, then we can conclude that \(\mathcal{PF}(G)\) is a strong grape (by the definition of a strong grape).

---

9Indeed, in this case, it is clear that any subset of \(E\) contains an \(s - t\)-path, and thus we have \(\mathcal{PF}(G) = \emptyset\) and \(\mathcal{PM}(G) = 2^E\). Thus, all that needs to be proved is that both \((E, 2^E)\) and \((E, \emptyset)\) are strong grapes. But this is straightforward to do by induction on \(#E\) (taking any \(a \in E\) in the induction step).

10Indeed, in this case, it can easily be checked that \(\mathcal{PF}(G) = 2^E\) and \(\mathcal{PM}(G) = \emptyset\). Thus, all that needs to be proved is that both \((E, 2^E)\) and \((E, \emptyset)\) are strong grapes. But this was done in the previous footnote.
Likewise, if we can show that at least one of $\text{lk}_{\mathcal{PM}(G)}(e)$ and $\text{dl}_{\mathcal{PM}(G)}(e)$ is a cone, then we can conclude that $\mathcal{PM}(G)$ is a strong grape (since $\text{lk}_{\mathcal{PM}(G)}(e) = \mathcal{PM}(G \setminus e)$ and $\text{dl}_{\mathcal{PM}(G)}(e) = \mathcal{PM}(G/e)$ are strong grapes).

We are in one of the following two cases:

Case 1: The edge $e$ is not the only edge with target $s'$. This entails that $\mathcal{PM}(G)$ is a strong grape. Furthermore, at least one of $\text{lk}_{\mathcal{PM}(G)}(e)$ and $\text{dl}_{\mathcal{PM}(G)}(e)$ is a cone (namely, $\text{lk}_{\mathcal{PM}(G)}(e)$). As we have seen above, this entails that $\mathcal{PM}(G)$ is a strong grape.

We thus have shown that both $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$ are strong grapes. This completes the induction step in Case 1.

Let us now consider Case 2. In this case, the edge $e$ is the only edge with target $s'$. Hence, Lemma 3.7 shows that $G/e$ has a useless edge. Let $e'$ be this useless edge. Thus, Lemma 2.2 (applied to $G/e$ and $e'$ instead of $G$ and $e$) shows that both $\mathcal{PF}(G/e)$ and $\mathcal{PM}(G/e)$ are cones with apex $e'$. In other words, both $\text{lk}_{\mathcal{PF}(G)}(e)$ and $\text{dl}_{\mathcal{PM}(G)}(e)$ are cones (since $\text{lk}_{\mathcal{PF}(G)}(e) = \mathcal{PF}(G/e)$ and $\text{dl}_{\mathcal{PM}(G)}(e) = \mathcal{PM}(G/e)$). As we have seen above, this entails that $\mathcal{PF}(G)$ is a strong grape. Furthermore, at least one of $\text{lk}_{\mathcal{PM}(G)}(e)$ and $\text{dl}_{\mathcal{PM}(G)}(e)$ is a cone (namely, $\text{dl}_{\mathcal{PM}(G)}(e)$). As we have seen above, this entails that $\mathcal{PM}(G)$ is a strong grape.

We thus have shown that both $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$ are strong grapes. This completes the induction step in Case 2.

We have now completed the induction step in each of the two Cases 1 and 2. Hence, the induction step is complete, and Proposition 9.3 is proved.

Remark 9.4. It is easy to see (recursively, using Lemma 7.11) that any strong grape has an acyclic matching $(M, \mu)$ satisfying $u_\mu(x) = 0$ or $u_\mu(x) = x^m$ for some $m \in \mathbb{N}$. However, it does not seem easy to describe which of these two alternatives holds, nor what $m$ is. Thus, this does not yield alternative proofs of Theorems 8.1 and 8.2.

9.2. Open questions. While the homotopy types of $\mathcal{PM}(G)$ and $\mathcal{PF}(G)$ answer many natural questions about these complexes, some further questions remain to be studied.

Question 9.5. Let $k$ be a nonnegative integer, and assume that $G$ has at least $k$ disjoint quasi-cycles. What can we say about the remainders of the polynomials $f_{\mathcal{PM}(G)}(x)$ and $f_{\mathcal{PF}(G)}(x)$ modulo $(1 + x)^{k+1}$?
**Question 9.6.** We proved our claims about $\mathcal{P}M(G)$ and $\mathcal{P}F(G)$ by induction. Can they be proved more directly? In particular, can the acyclic matchings we recursively constructed on $\mathcal{P}M(G)$ and $\mathcal{P}F(G)$ be described directly?

**Question 9.7.** We can generalize $\mathcal{P}M(G)$ and $\mathcal{P}F(G)$ as follows: Fix a positive integer $r$. Define two complexes

$$
\mathcal{P}F(G, r) := \{F \subseteq E : F \text{ contains no } r \text{ edge-disjoint } s-t \text{-paths}\},
$$

$$
\mathcal{P}M(G, r) := \{F \subseteq E : E \setminus F \text{ contains } r \text{ edge-disjoint } s-t \text{-paths}\}
$$
on the ground set $E$. What can we say about these two complexes? The Euler characteristics no longer restrict themselves to the value $s_0, 1, -1$ (for example, if $G$ consists of two vertices $s$ and $t$ and $k$ edges from $s$ to $t$, then $\tilde{\chi}(\mathcal{P}F(G, r)) = (-1)^r \binom{k-1}{r-1}$ and $\tilde{\chi}(\mathcal{P}M(G, r)) = (-1)^{k+r-1} \binom{k-1}{r-1}$). Still, can anything be said about these complexes? E.g., are they bouquets of spheres? combinatorial grapes?

The latter question is of particular interest in that any interesting homotopical properties of the complexes $\mathcal{P}F(G, r)$ and $\mathcal{P}M(G, r)$ would suggest a topological undercurrent in the theory of network flows. (Indeed, the existence of $r$ edge-disjoint $s-t$-paths is equivalent to the existence of an $s-t$-flow of value $r$; see [FF62, Theorem 4.2 and paragraph thereafter].)

**Appendix A. Proofs of Morse theory basics**

In this appendix, we shall give proofs for two lemmas left unproved in Section 7.

**Proof of Lemma 7.10.** Let $W$ be the ground set of $\Delta$. Let $w \in W$ be such that $\Delta$ is a cone with apex $w$. (Such a $w$ exists by assumption.)

For each subset $A$ of $W$ that contains $w$, we set $A^- := A \setminus \{w\}$. If $A \in \Delta$ contains $w$, then $A^- \in \Delta$ (since $A^- = A \setminus \{w\} \subseteq A$) and $w \notin A^-$ (by definition).

For each subset $A$ of $W$ that does not contain $w$, we set $A^+ := A \cup \{w\}$. If $A \in \Delta$ does not contain $w$, then $A^+ \in \Delta$ (since $\Delta$ is a cone with apex $w$, so that $A \in \Delta$ entails $A \cup \{w\} \in \Delta$) and $w \in A^+$ (by definition).

Define a map $\mu : \Delta \to \Delta$ by setting

$$
\mu(A) = \begin{cases} 
A^+, & \text{if } w \notin A; \\
A^-, & \text{if } w \in A
\end{cases}
$$

for each $A \in \Delta$.

This map $\mu$ is well-defined (by the previous two paragraphs) and is an involution (since each subset $A$ of $W$ satisfies $(A^-)^+ = A$ if it contains $w$, and $(A^+)^- = A$ if it does not). Moreover, each $A \in \Delta$ satisfies either $\mu(A) \prec A$ or $\mu(A) \succ A$ (indeed, if $w \in A$, then $\mu(A) = A^- = A \setminus \{w\} \prec A$, whereas otherwise we have...
\[ \mu(A) = A^+ = A \cup \{w\} \succ A. \] Thus, \((\Delta, \mu)\) is a matching on \(\Delta\). It has no unmatched faces. Thus, by the definition of \(u_\mu(x)\), we have
\[
u_\mu(x) = \sum_{f \in \Delta \setminus \Delta} x^{\#f} = \text{(empty sum)} = 0.
\]

It remains to show that this matching \((\Delta, \mu)\) is acyclic.

To show this, we let \((F_1, F_2, \ldots, F_n)\) be a cycle of \(\mu\). Thus, \(F_1, F_2, \ldots, F_n\) are distinct faces of \(\Delta\), satisfying \(n \geq 2\) and
\[
F_1 \succ \mu(F_1) \prec F_2 \succ \mu(F_2) \prec \cdots \prec F_n \succ \mu(F_n) \prec F_1.
\]

Our matching \(\mu\) has the property that the only faces \(A \in \Delta\) that satisfy \(A \succ \mu(A)\) are the faces that contain \(w\). Thus, from \(F_1 \succ \mu(F_1)\), we conclude that \(F_1\) contains \(w\). Similarly, \(F_2\) contains \(w\). The definition of \(\mu\) yields \(\mu(F_1) = F_1^-\) (since \(F_1\) contains \(w\)), so that \(\mu(F_1)\) does not contain \(w\).

However, \(\mu(F_1) \prec F_2\), so that the set difference \(F_2 \setminus \mu(F_1)\) has exactly 1 element. This element must be \(w\) (since \(F_2\) contains \(w\), but \(\mu(F_1)\) does not). Therefore, \(F_2 \setminus \mu(F_1) = \{w\}\). Since \(\mu(F_1) \subseteq F_2\) (because \(\mu(F_1) \prec F_2\)), this entails \(F_2 = \mu(F_1) \cup \{w\}\). On the other hand, \(F_1 = \mu(F_1) \cup \{w\}\) (since \(\mu(F_1) = F_1^- = F_1 \setminus \{w\}\)). Comparing these two equalities, we obtain \(F_1 = F_2\). This contradicts the fact that the faces \(F_1, F_2, \ldots, F_n\) are distinct.

Forget that we fixed \((F_1, F_2, \ldots, F_n)\). We thus have found a contradiction for every cycle \((F_1, F_2, \ldots, F_n)\) of \(\mu\). Hence, the matching \(\mu\) has no cycle, i.e., is acyclic. This completes the proof of Lemma 7.10.

Proof of Lemma 7.11. For each subset \(A\) of \(W\) that contains \(w\), we set \(A^- := A \setminus \{w\}\). This set \(A^-\) always satisfies \(w \notin A^-\) and \(A^- \prec A\) and \(#(A^-) = \#A - 1\). Moreover, if \(A \in \Delta\) contains \(w\), then \(A^- \in \Delta\) (since \(A^- = A \setminus \{w\} \subseteq A\)).

For each subset \(A\) of \(W\) that does not contain \(w\), we set \(A^+ := A \cup \{w\}\). This set \(A^+\) always satisfies \(w \in A^+\) and \(A^+ \succ A\) and \(#(A^+) = \#A + 1\). However, if \(A \in \Delta\) does not contain \(w\), then we don’t always have \(A^+ \in \Delta\).

Note that each subset \(A\) of \(W\) satisfies
\[
(A^-)^+ = A \quad \text{if } w \in A,
\]
and satisfies
\[
(A^+)^- = A \quad \text{if } w \notin A.
\]

Moreover, the operations \(A \mapsto A^-\) and \(A \mapsto A^+\) preserve covering relations: i.e.,

- If \(A\) and \(B\) are two subsets of \(W\) that contain \(w\), then
\[
A \prec B \implies A^- \prec B^-.
\]

Proof: Let \(A\) and \(B\) be two subsets of \(W\) that contain \(w\). Assume that \(A \prec B\). Now, \(A \prec B\) means that \(B = A \cup \{b\}\) for some \(b \in B \setminus A\). Consider this \(b\). From \(b \in B \setminus A\), we obtain \(b \in B\) and thus \(b \neq w\) (because \(b \in B\) and

...
\(w \notin B\). Combining \(b \in B\) with \(b \neq w\), we obtain \(b \in B \setminus \{w\} = B^-.\) Moreover, \(b \notin A\) (since \(b \in B \setminus A\)) and thus \(b \notin A \setminus \{w\} = A^-\). Combining \(b \in B^-\) with \(b \notin A^-,\) we obtain \(b \in B^- \setminus A^-\). Now, \(B^- = B \setminus \{w\} = (A \cup \{b\}) \setminus \{w\}\) (since \(B = A \cup \{b\}\)), so that \(B^- = (A \cup \{b\}) \setminus \{w\} = (A \setminus \{w\}) \cup \{b\}\) (since \(b \neq w\)). In view of \(A \setminus \{w\} = A^-\), we can rewrite this as \(B^- = A^- \cup \{b\}\). Since \(b \in B^- \setminus A^-\), this equality shows that \(A^- \prec B^-\). This proves (22).]

- If \(A\) and \(B\) are two subsets of \(W\) that don't contain \(w\), then

\[
A \prec B \implies A^+ \prec B^+.
\]

(23)

[Proof: Let \(A\) and \(B\) be two subsets of \(W\) that don't contain \(w\). Assume that \(A \prec B\). Now, \(A \prec B\) means that \(B = A \cup \{b\}\) for some \(b \in B \setminus A\). Consider this \(b\). From \(b \in B \setminus A\), we obtain \(b \notin A\) and thus \(b \neq w\) (because \(b \notin A\) and \(w \in A\)). Combining \(b \notin A\) with \(b \neq w\), we obtain \(b \notin A \cup \{w\} = A^+\). Moreover, \(b \in B \setminus A \subseteq B \subseteq B \cup \{w\} = B^+.\) Combining \(b \in B^+ \) with \(b \notin A^+,\) we obtain \(b \in B^+ \setminus A^+.\) Now, \(B^+ = B \cup \{w\} = (A \cup \{b\}) \cup \{w\}\) (since \(B = A \cup \{b\}\)), so that \(B^+ = (A \cup \{b\}) \cup \{w\} = (A \cup \{w\}) \cup \{b\} = A^+ \cup \{b\}\) (since \(A \cup \{w\} = A^+\)). Since \(b \in B^+ \setminus A^+,\) this equality shows that \(A^+ \prec B^+\). This proves (23).]

The set \(\Delta \setminus \text{dl}_\Delta (w)\) consists of all faces \(A \in \Delta\) that contain \(w\). Removing \(w\) from such a face yields a face of \(\text{lk}_\Delta (w)\). In other words, \(A^- \in \text{lk}_\Delta (w)\) for each \(A \in \Delta \setminus \text{dl}_\Delta (w)\). Conversely, inserting \(w\) into a face of \(\text{lk}_\Delta (w)\) yields a face \(A \in \Delta\) that contains \(w\), that is, a face in \(\Delta \setminus \text{dl}_\Delta (w)\). In other words, \(A^+ \in \Delta \setminus \text{dl}_\Delta (w)\) for each \(A \in \text{lk}_\Delta (w)\). Thus, we have two mutually inverse bijections

\[
\Delta \setminus \text{dl}_\Delta (w) \rightarrow \text{lk}_\Delta (w),
\]

\[
A \mapsto A^- = A \setminus \{w\}
\]

and

\[
\text{lk}_\Delta (w) \rightarrow \Delta \setminus \text{dl}_\Delta (w),
\]

\[
A \mapsto A^+ = A \cup \{w\}.
\]

Recall that \(M_+ \subseteq \text{lk}_\Delta (w) \subseteq 2^{W\setminus \{w\}}\), so that each face \(B \in M_+\) is a subset of \(W \setminus \{w\}\). In other words, each face \(B \in M_+\) is a subset of \(W\) that does not contain \(w\). Hence, we can set

\[
M_+^+ := \left\{ B^+ \mid B \in M_+ \right\}.
\]

Then, \(M_+^+ \subseteq \Delta\) (since each \(B \in M_+\) satisfies \(B \in M_+ \subseteq \text{lk}_\Delta (w)\) and thus \(B \cup \{w\} \in \Delta\), so that \(B^+ = B \cup \{w\} \in \Delta\)). Moreover, the set \(M_+^+\) only contains faces that contain \(w\) (since \(B^+\) contains \(w\) for each \(B \in M_+\)). Thus, every face \(A \in M_+^+\) satisfies \(w \in A\) and therefore \(A \notin \text{dl}_\Delta (w)\) (since \(A \in \text{dl}_\Delta (w)\) would mean that \(w \notin A\)), so that \(A \in \Delta \setminus \text{dl}_\Delta (w)\) (since \(A \in M_+^+ \subseteq \Delta\)). In other words, \(M_+^+ \subseteq \Delta \setminus \text{dl}_\Delta (w)\).
Furthermore, \( M_\leq \subseteq \text{dl}_\Delta (w) \subseteq 2^W \setminus \{w\} \). Hence, each face \( B \in M_\leq \) is a subset of \( W \setminus \{w\} \), and thus satisfies \( w \notin B \). In other words, \( M_\leq \) only contains faces that don’t contain \( w \).

We now define a set
\[
M := M_\leq \cup M_\geq^+.
\]

Note that \( M \subseteq \Delta \) (since \( M_\leq \subseteq \text{dl}_\Delta (w) \subseteq \Delta \) and \( M_\geq^+ \subseteq \Delta \)). Moreover, the union on the right hand side of (26) is a disjoint union (since the set \( M_\leq \) only contains faces that don’t contain \( w \), whereas the set \( M_\geq^+ \) only contains faces that contain \( w \)).

We observe the following:

Claim 1: Let \( A \in M \) be such that \( w \notin A \). Then, \( \mu_-(A) \in M \).

Proof of Claim 1. The face \( A \) does not contain \( w \) (since \( w \notin A \)). Hence, \( A \notin M_\geq^+ \) (since the set \( M_\geq^+ \) only contains faces that contain \( w \)). Combining this with \( A \in M \), we obtain \( A \in M \setminus M_\geq^+ \subseteq M_\leq \) (since \( M = M_\leq \cup M_\geq^+ \)). Thus, \( \mu_-(A) \) is well-defined and belongs to \( M_\leq \) (since \( \mu_- \) is a map \( M_- \to M_- \)). Therefore, \( \mu_-(A) \in M_\leq \subseteq M \) (since \( M = M_\leq \cup M_\geq^+ \)). This proves Claim 1. \( \square \)

Claim 2: Let \( A \in M \) be such that \( w \in A \). Then, \( (\mu_+(A^-))^+ \in M \).

Proof of Claim 2. The face \( A \) contains \( w \) (since \( w \in A \)). Hence, \( A \notin M_- \) (since the set \( M_- \) only contains faces that don’t contain \( w \)). Combining this with \( A \in M \), we obtain \( A \in M \setminus M_- \subseteq M_\geq^+ \) (since \( M = M_- \cup M_\geq^+ \)). Thus, \( A \in M_\geq^+ = \{B^+ \mid B \in M_+\} \). In other words, \( A = B^+ \) for some \( B \in M_+ \). Consider this \( B \). From \( A = B^+ \), we obtain \( A^- = (B^+)^- = B \in M_+ \). Thus \( \mu_+(A^-) \) is well-defined and belongs to \( M_+ \) (since \( \mu_+ \) is a map \( M_+ \to M_+ \)). Hence, \( \mu_+(A^-) \in M_+ \), so that \( (\mu_+(A^-))^+ \in M_\geq^+ \) (by the definition of \( M_\geq^+ \)), and therefore \( (\mu_+(A^-))^+ \in M_\geq^+ \subseteq M \) (since \( M = M_- \cup M_\geq^+ \)). This proves Claim 2. \( \square \)

Combining Claim 1 with Claim 2, see that
\[
\begin{cases}
\mu_-(A), & \text{if } w \notin A; \\
(\mu_+(A^-))^+, & \text{if } w \in A
\end{cases}
\]
\( \in M \) for each \( A \in M \).

This lets us construct a map
\[
\mu : M \to M,
\]
\[A \mapsto \begin{cases}
\mu_-(A), & \text{if } w \notin A; \\
(\mu_+(A^-))^+, & \text{if } w \in A.
\end{cases}
\]

We shall show that this map \( \mu \) is an acyclic matching on \( \Delta \). First, we argue that it is an involution:

Claim 3: We have \( \mu \circ \mu = \text{id} \).
Proof of Claim 3. It is enough to show that \( \mu(A) = A \) for each \( A \in M \). So let us fix \( A \in M \). We are in one of the following two cases:

Case 1: We have \( w \in A \).

Case 2: We have \( w \not\in A \).

First consider Case 1. In this case, we have \( w \in A \). The definition of \( \mu \) thus yields \( \mu(A) = (\mu_+(A^-))^+ \). Set \( C := \mu(A) \). Thus, \( C = \mu(A) = (\mu_+(A^-))^+ \), so that \( C^- = ((\mu_+(A^-))^+)^- = \mu_+(A^-) \) (by (21)). Hence, \( \mu_+(C^-) = \mu_+(\mu_+(A^-)) = A^- \) (since \( \mu_+ \circ \mu_+ = \text{id} \) (because \( \mu_+ \) is a matching)) and therefore \( (\mu_+(C^-))^+ = (A^-)^+ = A \) (by (20)).

Obviously, \( w \in (\mu_+(A^-))^+ \) (since \( w \in B^+ \) for any \( B \)). In other words, \( w \in C \) (since \( C = (\mu_+(A^-))^+ \)). Hence, the definition of \( \mu \) yields \( \mu(C) = (\mu_+(C^-))^+ = A \). Since \( \mu(A) = C \), we now have \( \mu(\mu(A)) = \mu(C) = A \). Thus, \( \mu(\mu(A)) = A \) is proved in Case 1.

Now, consider Case 2. Here, we have \( w \not\in A \). The definition of \( \mu \) thus yields \( \mu(A) = \mu_-(A) \in M_-(\text{since} \mu_- \text{is a map from } M_- \text{to } M_-) \). Thus, the face \( \mu(A) \) does not contain \( w \) (since the set \( M_- \) only contains faces that don’t contain \( w \)). In other words, \( w \not\in \mu(A) \). Hence, the definition of \( \mu \) yields \( \mu(\mu(A)) = \mu_-(\mu(A)) = \mu_-(\mu_-(A)) \) (since \( \mu(A) = \mu_-(A) \)). But \( \mu_-(\mu_-(A)) = A \) (since \( \mu_- \circ \mu_- = \text{id} \) (because \( \mu_- \) is a matching)). Thus, \( \mu(\mu(A)) = \mu_-(\mu_-(A)) = A \). Hence, \( \mu(\mu(A)) = A \) is proved in Case 2.

We have now proved \( \mu(\mu(A)) = A \) in both Cases 1 and 2. Hence, \( \mu(\mu(A)) = A \) always holds, and Claim 3 is proved.

Next, we argue that \( \mu \) is a matching:

Claim 4: Each \( A \in M \) satisfies either \( \mu(A) \prec A \) or \( \mu(A) \succ A \).

Proof of Claim 4. Let \( A \in M \). We are in one of the following two cases:

Case 1: We have \( w \in A \).

Case 2: We have \( w \not\in A \).

First consider Case 1. In this case, we have \( w \in A \). The definition of \( \mu \) thus yields \( \mu(A) = (\mu_+(A^-))^+ \). However, since \( \mu_+ \) is a matching, we must have either \( \mu_+(A^-) \prec A^- \) or \( \mu_+(A^-) \succ A^- \). Since both \( \mu_+(A^-) \) and \( A^- \) are subsets of \( W \) that don’t contain \( w \), we thus conclude using (23) that we have either \( (\mu_+(A^-))^+ \prec (A^-)^+ \) or \( (\mu_+(A^-))^+ \succ (A^-)^+ \). In view of \( (\mu_+(A^-))^+ = \mu(A) \) and \( (A^-)^+ = A \), we can rewrite this as follows: We have either \( \mu(A) \prec A \) or \( \mu(A) \succ A \). Thus, Claim 4 is proved in Case 1.

Now, consider Case 2. In this case, we have \( w \not\in A \). The definition of \( \mu \) thus yields \( \mu(A) = \mu_-(A) \). However, since \( \mu_- \) is a matching, we must have either \( \mu_-(A) \prec A \) or \( \mu_-(A) \succ A \). In view of \( \mu_-(A) = \mu(A) \), we can rewrite this as follows: We have either \( \mu(A) \prec A \) or \( \mu(A) \succ A \). Thus, Claim 4 is proved in Case 2.

We have now proved Claim 4 in both Cases 1 and 2; thus, Claim 4 always holds. \( \square \)
Claim 3 and Claim 4 reveal that \((M, \mu)\) is a matching on \(\Delta\). Next, we claim:

**Claim 5:** This matching \((M, \mu)\) is acyclic.

**Proof of Claim 5.** Let us first observe that any cycle of \(\mu\) can be rotated: If \((F_1, F_2, \ldots, F_n)\) is a cycle of \(\mu\), then \((F_i, F_{i+1}, \ldots, F_n, F_1, F_2, \ldots, F_{i-1})\) is also a cycle of \(\mu\) for each \(i \in \{1, 2, \ldots, n\}\) (because the definition of a cycle has rotational symmetry).

Let \((F_1, F_2, \ldots, F_n)\) be a cycle of \(\mu\). Thus, \(F_1, F_2, \ldots, F_n\) are distinct faces in \(M\), satisfying \(n \geq 2\) and

\[
F_1 \succ \mu(F_1) \prec F_2 \succ \mu(F_2) \prec F_3 \succ \cdots \prec F_n \succ \mu(F_n) \prec F_1.
\]

We will derive a contradiction.

Let us set \(F_{n+1} := F_1\), so that each \(j \in \{1, 2, \ldots, n\}\) satisfies

\[
F_j \succ \mu(F_j) \prec F_{j+1}\tag{27}
\]

(by (27)). We distinguish between the following two cases:

**Case 1:** Some \(i \in \{1, 2, \ldots, n\}\) satisfies \(w \in F_i\).

**Case 2:** No \(i \in \{1, 2, \ldots, n\}\) satisfies \(w \in F_i\).

We consider Case 1 first. In this case, some \(i \in \{1, 2, \ldots, n\}\) satisfies \(w \in F_i\). By rotating the cycle \((F_1, F_2, \ldots, F_n)\) to \((F_i, F_{i+1}, \ldots, F_n, F_1, F_2, \ldots, F_{i-1})\), we can ensure that this \(i\) becomes 1. Thus, we WLOG assume that \(i = 1\).

Hence, \(w \in F_1\) (since \(w \in F_i\)). Therefore, the definition of \(\mu\) yields \(\mu(F_1) = (\mu_+ (F_1^-))^+\). But \(w \in (\mu_+ (F_1^-))^+\) (since \(w \in B^+\) for any set \(B\)). In other words, \(w \in \mu(F_1)\) (since \(\mu(F_1) = (\mu_+ (F_1^-))^+\)). However, (27) also yields \(\mu(F_1) \prec F_2\), so that \(\mu(F_1) \subseteq F_2\) and therefore \(w \in \mu(F_1) \subseteq F_2\).

We have thus proved \(w \in F_2\) using (27) and \(w \in F_1\). Likewise, we can prove \(w \in F_3\) (using (27) and \(w \in F_2\)) and \(w \in F_4\) (using (27) and \(w \in F_3\)) and so on. This line of reasoning shows eventually (i.e., by induction on \(j\)) that \(w \in F_j\) for each \(j \in \{1, 2, \ldots, n\}\).

Now, let \(j \in \{1, 2, \ldots, n\}\). As we just showed, we have \(w \in F_j\), so that \(\mu(F_j) = (\mu_+ (F_j^-))^+\) (by the definition of \(\mu\)). Thus, \((\mu(F_j))^- = (\mu_+ (F_j^-))^+ = \mu_+ (F_j^-)\) (by (21)).

But \(w \in (\mu_+ (F_j^-))^+\) (since \(w \in B^+\) for any set \(B\)). In other words, \(w \in \mu(F_j)\) (since \(\mu(F_j) = (\mu_+ (F_j^-))^+)\)). Hence, \(w \in \mu(F_j) \subseteq F_{j+1}\) (since (28) says that \(\mu(F_j) \prec F_{j+1}\)). This shows that \(F_{j+1}\) is well-defined.

From (28), we know that \(F_j \succ \mu(F_j) \prec F_{j+1}\). Since \(F_j, \mu(F_j)\), and \(F_{j+1}\) are subsets of \(W\) that contain \(w\) (because \(w \in F_j\) and \(w \in \mu(F_j)\) and \(w \in F_{j+1}\)), we thus conclude using (22) that \(F_j^- \succ (\mu(F_j))^- \prec F_{j+1}^-\). In other words, \(F_j^- \succ \mu_+ (F_j^-) \prec F_{j+1}^-\) (since \((\mu(F_j))^- = \mu_+ (F_j^-))\).
Forget that we fixed \( j \). We thus have shown that \( F_j^- \succ \mu_+\left(F_j^-ight) \prec F_{j+1}^- \) for each \( j \in \{1, 2, \ldots, n\} \). In other words,

\[
(29) \quad F_1^- \succ \mu_+\left(F_1^-ight) \prec F_2^- \succ \mu_+\left(F_2^-ight) \prec F_3^- \succ \cdots \prec F_n^- \succ \mu_+\left(F_n^-ight) \prec F_1^-
\]

(since \( F_{n+1}^- = F_1^- \)). Moreover, the \( n \) sets \( F_1^-, F_2^-, \ldots, F_n^- \) are distinct and belong to \( M_+ \). Thus, they are \( n \) distinct faces of \( M_+ \). Since they satisfy (29) (and since \( n \geq 2 \)), we thus conclude that \( \left(F_1^-, F_2^-, \ldots, F_n^-ight) \) is a cycle of the matching \( (M_+, \mu_+) \). Thus, \( (M_+, \mu_+) \) has a cycle, i.e., is not acyclic. But this contradicts the assumption that \( (M_+, \mu_+) \) is acyclic. Thus, we have found a contradiction in Case 1.

Let us now consider Case 2. In this case, no \( i \in \{1, 2, \ldots, n\} \) satisfies \( w \in F_i \).

In other words, none of the \( n \) faces \( F_1, F_2, \ldots, F_n \) contains \( w \). Thus, these \( n \) faces \( F_1, F_2, \ldots, F_n \) belong to \( M_- \).

For each \( j \in \{1, 2, \ldots, n\} \), we have \( w \notin F_j \) (since none of the \( n \) faces \( F_1, F_2, \ldots, F_n \) contains \( w \)) and therefore \( \mu(F_j) = \mu_-(F_j) \) (by the definition of \( \mu \)). Hence, we can rewrite the chain (27) as follows:

\[
F_1 \succ \mu_-(F_1) \prec F_2 \succ \mu_-(F_2) \prec F_3 \succ \cdots \prec F_n \succ \mu_-(F_n) \prec F_1.
\]

This chain (combined with the fact that \( F_1, F_2, \ldots, F_n \) are \( n \) distinct faces in \( M_- \), and the fact that \( n \geq 2 \)) shows that \( (F_1, F_2, \ldots, F_n) \) is a cycle of the matching \( (M_-, \mu_-) \). Thus, \( (M_-, \mu_-) \) has a cycle, i.e., is not acyclic. But this contradicts the assumption that \( (M_-, \mu_-) \) is acyclic. Thus, we have found a contradiction in Case 2.

We have now found a contradiction in each case.

Forget that we fixed \( (F_1, F_2, \ldots, F_n) \). We thus have found a contradiction for each cycle \( (F_1, F_2, \ldots, F_n) \) of \( \mu \). Hence, \( \mu \) has no cycle. In other words, \( \mu \) is acyclic. Claim 5 is thus proved. \( \square \)

In preparation for our next step, we prove two more simple claims:

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11Proof. Assume the contrary. Thus, some \( i < j \) in \( \{1, 2, \ldots, n\} \) satisfy \( F_i^- = F_j^- \). Consider these \( i < j \). Then, (20) yields \( (F_i^-)^+ = F_i \) and \( (F_j^-)^+ = F_j \). Thus, \( F_i = (F_i^-)^+ = (F_j^-)^+ = (F_j^-)^+ \) (since \( F_i^- = F_j^- \)), so that \( F_i = (F_j^-)^+ = F_j \). But this contradicts the distinctness of \( F_1, F_2, \ldots, F_n \). This contradiction shows that our assumption was false.

12Proof. Let \( j \in \{1, 2, \ldots, n\} \). We must show that \( F_j^- \in M_+ \).

We have \( F_j \in M = M_- \cup M_+ \). However, the set \( F_j \) contains \( w \) (since \( F_j^- \) is defined in the first place), whereas the set \( M_- \) only contains faces that don’t contain \( w \). Thus, \( F_j \notin M_- \). Combining this with \( F_j \in M_- \cup M_+ \), we obtain \( F_j \in (M_- \cup M_+) \setminus M_- \subseteq M_+ = \{B^+ \mid B \in M_+\} \). In other words, \( F_j = B^+ \) for some \( B \in M_+ \). Consider this \( B \).

From \( F_j = B^+ \), we obtain \( F_j^- = (B^+)^- = B \) (by (21)), so that \( F_j^- = B \in M_+ \), qed.

13Proof. Let \( j \in \{1, 2, \ldots, n\} \). We must show that \( F_j \in M_- \).

We have \( F_j \in M = M_- \cup M_+ \). However, the set \( F_j \) does not contain \( w \) (since none of the \( n \) faces \( F_1, F_2, \ldots, F_n \) contains \( w \)), whereas the set \( M_+ \) only contains faces that contain \( w \). Thus, \( F_j \notin M_+ \). Combining this with \( F_j \in M_- \cup M_+ \), we obtain \( F_j \in (M_- \cup M_+) \setminus M_+ \subseteq M_- \), qed.
Claim 6: The map
\[ \text{lk}_\Delta (w) \setminus M_+ \to \{ I \in \Delta \setminus M : w \in I \}, \]
\[ A \mapsto A^+ \]
is well-defined and is a bijection.

Proof of Claim 6. We shall prove that this map is well-defined, injective and surjective.

Well-definedness: Let us first prove that this map is well-defined. To this purpose, we must show that \( A^+ \in \{ I \in \Delta \setminus M : w \in I \} \) for each \( A \in \text{lk}_\Delta (w) \setminus M_+ \).

Let us fix \( A \in \text{lk}_\Delta (w) \setminus M_+ \). Then, \( A \in \text{lk}_\Delta (w) \) and \( A \notin M_+ \).

From \( A \in \text{lk}_\Delta (w) \), we obtain \( A \subseteq W \setminus \{ w \} \) and \( A \cup \{ w \} \in \Delta \) (by the definition of \( \text{lk}_\Delta (w) \)). The definition of \( A^+ \) yields \( A^+ = A \cup \{ w \} \in \Delta \) and \( w \in A^+ \). Moreover, it is easy to see that \( A^+ \notin M_− \) and \( A^+ \notin M_+ \). Combining these two facts, we obtain \( A^+ \notin M_− \cup M_+ = M \) (by (26)). Combining this with \( A^+ \in \Delta \), we find \( A^+ \in \Delta \setminus M \). Thus, \( A^+ \) is a face \( I \in \Delta \setminus M \) satisfying \( w \in I \) (since \( w \in A^+ \)). In other words, \( A^+ \in \{ I \in \Delta \setminus M : w \in I \} \). This completes the proof that our map is well-defined.

Injectivity: Let us next prove that our map is injective.

For this purpose, we must show that any two faces \( A, B \in \text{lk}_\Delta (w) \setminus M_+ \) satisfying \( A^+ = B^+ \) must satisfy \( A = B \).

But this is easy: If \( A, B \in \text{lk}_\Delta (w) \setminus M_+ \) are two faces satisfying \( A^+ = B^+ \), then (21) yields \( A = \left( \frac{A^+}{B^+} \right)^- = (B^+)^- = B \) (by (21)). Thus, our map is injective.

Surjectivity: Let us now prove that our map is surjective.

Let \( C \in \{ I \in \Delta \setminus M : w \in I \} \). We must prove that \( C = A^+ \) for some \( A \in \text{lk}_\Delta (w) \setminus M_+ \).

Indeed, \( C \in \{ I \in \Delta \setminus M : w \in I \} \) shows that \( C \in \Delta \setminus M \) and \( w \in C \). From \( C \in \Delta \setminus M \), we obtain \( C \in \Delta \) and \( C \notin M \). From \( w \in C \), we conclude that \( C^- \) is well-defined. Moreover, (20) yields \( (C^-)^+ = C \in \Delta \). Now, we have \( C^- \subseteq W \setminus \{ w \} \) and \( C^- \cup \{ w \} = (C^-)^+ \in \Delta \), so that \( C^- \in \text{lk}_\Delta (w) \). If we had \( C^- \in M_+ \), then we

\[ ^{14}\text{Proof:} \text{Assume the contrary. Thus, } A^+ \in M_− \subseteq d\text{lk}_\Delta (w) \subseteq 2^{W \setminus \{ w \}}. \text{Hence, } A^+ \subseteq W \setminus \{ w \}, \text{so that } w \notin A^+. \text{But this contradicts } w \in A^+. \text{This contradiction shows that our assumption was false.} \]

\[ ^{15}\text{Proof. Assume the contrary. Thus, } A^+ \in M_+ = \{ B^+ : B \in M_+ \}. \text{In other words } A^+ = B^+ \text{ for some } B \in M_+. \text{Consider this } B. \text{Now, (21) yields } A = \left( \frac{A^+}{B^+} \right)^- = (B^+)^- = B \text{ (by (21))}. \]

Thus, \( A = B \in M_+ \), but this contradicts \( A \notin M_+ \). This contradiction shows that our assumption was false.
would have \((C^-)^+ \in M_+^+\) (by the definition of \(M_+^+\)), which would entail \(C = (C^-)^+ \in M_+^+ \subseteq M_- \cup M_+^+ = M\) (by (26)); but this would contradict \(C \notin M\). Thus, we cannot have \(C^- \in M_+\). Hence, we obtain \(C^- \notin M_+\). Combining this with \(C^- \in \text{lk}_\Delta (w)\), we obtain \(C^- \in \text{lk}_\Delta (w) \setminus M_+\). Since \(C = (C^-)^+\), we thus have \(C = A^+\) for some \(A \in \text{lk}_\Delta (w) \setminus M_+\) (namely, for \(A = C^-\)).

Forget that we fixed \(C\). We thus have shown that each \(C \in \{I \in \Delta \setminus M : w \in I\}\) can be written as \(C = A^+\) for some \(A \in \text{lk}_\Delta (w) \setminus M_+\). In other words, our map is surjective.

We have now proved that the map
\[
\text{lk}_\Delta (w) \setminus M_+ \to \{I \in \Delta \setminus M : w \in I\},
A \mapsto A^+
\]
is well-defined, injective and surjective. Hence, it is a bijection. Claim 6 is thus proved. \(\square\)

**Claim 7:** We have
\[
\{I \in \Delta \setminus M : w \notin I\} = \text{dl}_\Delta (w) \setminus M_-
\]

**Proof of Claim 7.** The sets \(\text{dl}_\Delta (w)\) and \(M_+\) are disjoint (since every face \(A \in \text{dl}_\Delta (w)\) satisfies \(w \notin A\), whereas every face \(A \in M_+^+\) satisfies \(w \in A\)). Hence, \(\text{dl}_\Delta (w) \setminus M_+^+ = \text{dl}_\Delta (w)\).

However,
\[
\{I \in \Delta \setminus M : w \notin I\} = \{I \in \Delta : I \notin M \text{ and } w \notin I\} = \{I \in \Delta : w \notin I\} \setminus M
\]
\[
= \text{dl}_\Delta (w) \setminus M = \text{dl}_\Delta (w) \setminus (M_+^+ \cup M_-)
\]
\[
= (\text{dl}_\Delta (w) \setminus M_+^+) \setminus M_- = \text{dl}_\Delta (w) \setminus M_-
\]
This proves Claim 7. \(\square\)

Finally, we compute the unmatched f-polynomial of the acyclic matching \(\mu\). The definition of the unmatched f-polynomial yields
\[
u_{\mu_-}(x) = \sum_{I \in \text{dl}_\Delta (w) \setminus M_-} x^{|I|}
\]
and
\[
u_{\mu_+}(x) = \sum_{I \in \text{lk}_\Delta (w) \setminus M_+} x^{|I|}
\]
and

\[(32) \quad u_\mu(x) = \sum_{I \in \Delta \setminus M} \#I = \sum_{I \in \Delta \setminus M; w \in I} x^I + \sum_{I \in \Delta \setminus M; w \notin I} x^I.\]

But Claim 6 yields that the map

\[\text{lk}_\Delta(w) \setminus M_+ \to \{I \in \Delta \setminus M : w \in I\}, \quad A \mapsto A^+\]

is a bijection. Hence, we can substitute \(A^+\) for \(I\) in the sum \(\sum_{I \in \Delta \setminus M; w \in I} x^I\). Thus, we obtain

\[\sum_{I \in \Delta \setminus M; w \in I} x^I = \sum_{A \in \text{lk}_\Delta(w) \setminus M_+} x^A = \sum_{A \in \text{lk}_\Delta(w) \setminus M_+} x^{A+} = x \sum_{I \in \text{dl}_\Delta(w) \setminus M_+} x^I = xu_\mu_+(x).\]

Moreover, the summation sign \(\sum_{I \in \Delta \setminus M; w \notin I}\) is equivalent to \(\sum_{I \in \text{dl}_\Delta(w) \setminus M_-}\) (by Claim 7). Thus,

\[\sum_{I \in \Delta \setminus M; w \notin I} x^I = \sum_{I \in \text{dl}_\Delta(w) \setminus M_-} x^I = u_\mu_-(x) \quad \text{(by (30))}.

Thus, (32) becomes

\[u_\mu(x) = \sum_{I \in \Delta \setminus M; w \in I} x^I + \sum_{I \in \Delta \setminus M; w \notin I} x^I = xu_\mu_+(x) + u_\mu_-(x) = u_\mu_-(x) + xu_\mu_+(x).\]

The proof of Lemma 7.11 is thus complete. \(\square\)

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