(Un)boundedness of directional maximal operators through a notion of “Perron capacity” and an application

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Abstract

We introduce the notion of Perron capacity of a set of slopes $\Omega \subset \mathbb{R}$. Precisely, we prove that if the Perron capacity of $\Omega$ is finite then the directional maximal operator associated $M_\Omega$ is not bounded on $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$. This allows us to prove that the set

$$\Omega_e = \left\{ \frac{\cos n}{n} : n \in \mathbb{N}^* \right\}$$

is not finitely lacunary which answers a question raised by A. Stokolos.

1 Introduction

It is well known, since the original work of Jensen, Marcinkiewicz and Zygmund [16], that Lebesgue’s differentiation formula, i.e. the equality for a.e. $x \in \mathbb{R}^n$:

$$f(x) = \lim_{R \ni x \to 0} \frac{1}{|R|} \int_R f,$$  \hspace{1cm} (1)

holds along $n$-dimensional rectangles $R$ parallel to the coordinate hyperplanes (called standard rectangles), whenever one has $f \in L^{\log^{-1} n} L(\mathbb{R}^n)$ — the latter Orlicz space being, moreover, the largest for which the above formula holds if one does not assume any extra hypothesis on the rectangles $R$; the beautiful survey by A. Stokolos [21] provides a quite exhaustive state of the art on the topic — see also [5] for sufficient conditions (of geometric nature) on $\mathcal{R}$ ensuring that $L^{\log^{-1} n} L(\mathbb{R}^n)$ is the largest space for which (1) holds whenever $R$ is restricted to belong to the class $\mathcal{R}$.

In dimension $n = 2$, the nature of the problem changes when one allows rectangles $R$ to rotate around one of their vertices, requiring that the slope of their longest side...
belongs to a fixed set \( \Omega \) (call \( \mathcal{R}_\Omega \) the family of such rectangles in the plane). It has been shown e.g. by Cordoba and Fefferman [4] that when \( \Omega = \{ \omega_k : k \in \mathbb{N} \} \) is a lacunary sequence (see below for a definition), then formula (1) holds for any \( f \in L^p(\mathbb{R}^2) \) if one has \( p \in [2, +\infty[. \) This was later generalized for arbitrary \( 1 < p < \infty \) by Nagel, Stein and Wainger [19]. Given a set of slopes \( \Omega \), the question of finding Orlicz spaces (the largest possible) beyond \( L^p \) spaces, for all elements of which formula (1) holds is often delicate; issues in these directions have been studied in works involving some of the current authors, see [7] and [6] for example.

Concerning the possibility of (1) holding for all \( f \in L^p(\mathbb{R}^2) \), \( 1 < p \leq \infty \), it has been shown e.g. by Katz [17], Bateman and Katz [2], Hare [14] and more recently by Bateman [1] that Cantor sets (and even uncountable sets) never give rise to formula (1) for all \( f \in L^p(\mathbb{R}^2) \), \( 1 < p \leq \infty \) — Bateman’s work [1] even gives a complete characterization, even though not always simple to implement in practice (see below), of sets \( \Omega \) for which it does hold in any \( L^p(\mathbb{R}^2) \), \( 1 < p \leq \infty \): those are sets one will call finitely lacunary in the sequel.

Before to state more precisely Bateman’s result, let us mention an important tool in its proof: the possibility, when a set of slopes \( \Omega \) is too large, to find finite families of rectangles in \( \mathcal{R}_\Omega \) such that the union of their (centered) homothetic expansions occupies a much larger area in the plane than their union does; we call this a Kakeya blow.

When no restriction is made on directions of rectangles in play, the possibility of exhibiting a Kakeya blow is a standard result in measure theory (see Busemann and Feller’s work [3] for a first construction of this type):

**Theorem 1.1 (Kakeya Blow).** For any large \( A \gg 1 \), there exists a finite family of rectangles \( \{R_i\}_{i \leq N} \) such that we have

\[
A \left| \bigcup_{i \leq N} R_i \right| \leq \left| \bigcup_{i \leq N} 4R_i \right|.
\]

Here, \( 4R \) stands for the 4-fold dilation of \( R \) by its center.

Kakeya blows have deep implications in harmonic analysis: for example, a concrete construction proving the above theorem allowed Fefferman to disprove the Disc multiplier conjecture in [10].

A natural question, given a set of slopes \( \Omega \), then becomes the following: can we still construct Kakeya blows using only rectangles in \( \mathcal{R}_\Omega \), namely rectangles the longest side of which makes an angle \( \omega \in \Omega \) with the horizontal axis (see also Stokolos [20] and Hagelstein and Stokolos [13] concerning the links between directional maximal operators and multipliers)? In his study of the so-called directional maximal operator \( M_\Omega \) defined for a locally integrable \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( x \in \mathbb{R}^2 \) as

\[
M_\Omega f(x) := \sup_{x \in R \in \mathcal{R}_\Omega} \frac{1}{|R|} \int_R |f|,
\]

Bateman [1] introduced a notion we shall call finitely lacunary set and proved the following Theorem.
Theorem 1.2 (Bateman). The following conditions are equivalent

- the set $\Omega$ is not finitely lacunary;
- for any $A \gg 1$, there exists a finite family of rectangles $\{R_i\}$ such that the longest side of each makes an angle $\omega \in \Omega$ with the $x$-axis and also satisfying $A \left| \bigcup_{i \in I} R_i \right| \leq \left| \bigcup_{i \in I} 4R_i \right|$;
- the operator $M_\Omega$ is not bounded on $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$.

Moreover, the following conditions are also equivalent

- the set $\Omega$ is finitely lacunary;
- the operator $M_\Omega$ is bounded on $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$.

In the latter theorem, it needs to be explained what is meant by “finitely lacunary”. We recall it here, following the presentation by Kroc and Pramanik [18]. We start by defining the notion of lacunary sequence and then the notion of lacunary set of finite order.

Definition 1.3 (Lacunary sequence). We say that a sequence of real numbers $L = (L_k)$ is a lacunary sequence converging to $\ell \in \mathbb{R}$ when there holds

$$|\ell - \ell_{k+1}| \leq \frac{1}{2}|\ell - \ell_k|$$

for any $k$.

For example the sequences $L_1 := (\frac{1}{2^k})_{k \geq 2}$ and $L_2 := (\frac{1}{k})_{k \geq 4}$ are lacunary.

Definition 1.4 (Lacunary set of finite order). A lacunary set of order 0 in $\mathbb{R}$ is a set which is either empty or a singleton. Recursively, for $N \in \mathbb{N}^*$, we say that a set $\Omega$ included in $\mathbb{R}$ is a lacunary set of order at most $N+1$ — and write $\Omega \in \Lambda(N+1)$ — if there exists a lacunary sequence $L$ with the following properties: for any $a, b \in L$ such that $a < b$ and $(a, b) \cap L = \emptyset$, the set $\Omega \cap (a, b)$ is a lacunary set of order at most $N$ i.e. $\Omega \cap (a, b) \in \Lambda(N)$.

For example the set

$$\Omega := \left\{ \frac{1}{2^k} + \frac{1}{4^l} : k, l \in \mathbb{N}, l \leq k \right\}$$

is a lacunary set of order 2. In this case, observe that the set $\Omega$ cannot be re-written as $\Omega = \{\omega_k : k \in \mathbb{N}^*\}$ where $(\omega_k)$ is a monotone sequence, since it has several points of accumulation. We can finally give a definition of a finitely lacunary set.

Definition 1.5 (Finitely lacunary set). A set $\Omega$ in $\mathbb{R}$ is said to be finitely lacunary if there exists a finite number of set $\Omega_1, \ldots, \Omega_M$ which are lacunary of finite order such that

$$\Omega \subset \bigcup_{k \leq M} \Omega_k.$$
Figure 1: A representation of a lacunary sequence and a lacunary set of order 2.

Even if Theorem 1.2 gives a full characterization of the behavior of directional maximal operators, it is usually hard in practice to decide whether a set of slopes $\Omega$ is finitely lacunary. For example, it was known that the set of slopes $\Omega_0$ defined by

$$\Omega_0 := \left\{ \frac{1}{n} : n \in \mathbb{N}^* \right\}$$

generates a directional maximal operator $M_{\Omega_0}$ which is not bounded on $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$ (see e.g. [9, Theorem 2.1]); using Theorem 1.2 one can hence assert that the set of slopes $\Omega_0$ is not finitely lacunary. In the case where the set of directions $\Omega$ is a discrete set which can be ordered in $\mathbb{R}^*_+$, more precisely in the case where $\Omega = \left\{ \frac{1}{u} : u \in U \right\}$ and we can write $U = \{u_k : k \in \mathbb{N}^*\} \subset \mathbb{R}^*_+$ for an increasing sequence $(u_k)_{k \in \mathbb{N}^*}$, Hare and Rönning [15] provided a quantitative sufficient condition on $U$ which ensures that the maximal operator associated $M_U$ is not bounded on $L^p(\mathbb{R}^2)$. Precisely, they proved the following Theorem.

**Theorem 1.6** (Hare-Rönning [15]). Assume that $U = (u_k)_{k \in \mathbb{N}^*} \subset (0, \pi/2)$ is an increasing sequence of directions and that one has

$$G_U := \sup_{k \geq 1} \left( \frac{u_{k+2l} - u_{k+l}}{u_{k+l} - u_k} + \frac{u_{k+l} - u_k}{u_{k+2l} - u_{k+l}} \right) < \infty.$$

Then for any $p \in (1, \infty)$, one has $\|M_U\|_p = \infty$.

In particular, applying Bateman’s Theorem, it follows that if we have $G_U < \infty$ then $U$ is not finitely lacunary — other uses of Perron trees have also been made in works involving the current authors, see e.g. [8], [11] and [12]. Hence, usually one does not prove directly that a set of slopes $\Omega$ is not finitely lacunary (or that it is): in general, it is a consequence of the (un)-boundedness of the associated maximal operator $M_{\Omega}$ — obtained by other methods — and Bateman’s Theorem.

Our results are motivated by the following question raised by A. Stokolos: what can be said (for example) about the set of directions $\Omega_e$ defined as

$$\Omega_e := \left\{ \frac{\cos n}{n} : n \in \mathbb{N}^* \right\}.$$
2 Results

We denote by \( R \) the set containing all rectangles of the plane and for \( R \in R \), we denote by \( \omega_R \in [0, \pi) \) the angle that the longest side of the rectangle \( R \) makes with the \( x \)-axis. For any set \( \Omega \subset \mathbb{R} \) (which should be thought as a set of slopes) we define the family of rectangle \( \mathcal{R}_\Omega \) as

\[
\mathcal{R}_\Omega := \{ R \in \mathbb{R} : \tan(\omega_R) \in \Omega \}.
\]

We then define the directional maximal operator \( M_\Omega \) as

\[
M_\Omega f(x) := \sup_{x \in R \in \mathcal{R}_\Omega} \frac{1}{|R|} \int_R |f|.
\]

We finally define the Perron capacity of a set of slopes \( \Omega \) included in \( \mathbb{R} \) as follow.

**Definition 2.1** (Perron capacity of a set of slopes). For any set of slopes \( \Omega \subset \mathbb{R} \), we define its Perron capacity as

\[
P_\Omega := \lim_{N \to \infty} \inf_{U \subset \Omega} \#U = 2^N G_U \in [2, \infty]
\]

where \( G_U = \sup_{k \geq 1} \left( \frac{u_{k+2^l} - u_{k+l}}{u_{k+2^l} - u_k} + \frac{u_{k+l} - u_k}{u_{k+2^l} - u_{k+l}} \right) \) if \( U = \{ u_1, \ldots, u_{2^N} \} \) with \( u_1 < u_2 < \cdots < u_N \).

Our first result is the following: it gives a sufficient and quantitative condition on an arbitrary set \( \Omega \) which ensures that the associated maximal operator is unbounded on \( L^p(\mathbb{R}^2) \) for any \( 1 < p < \infty \). In contrast with Hare and Ronning Theorem, we do not assume that our set of slopes is ordered.

**Theorem 2.2.** Fix any set of slopes \( \Omega \subset \mathbb{R} \) and suppose that we have \( P_\Omega < \infty \). Then for any \( p \in (1, \infty) \), one has \( \|M_\Omega\|_p = \infty \).

Loosely speaking, the fact that \( P_\Omega < \infty \) indicates whether the set \( \Omega \) contains arbitrary large subsets which are (more or less) uniformly distributed in \( \mathbb{R} \). Our second result is an application of Theorem 2.2 and deals with the set of slopes \( \Omega_e \) defined as

\[
\Omega_e := \left\{ \frac{\cos n}{n} : n \in \mathbb{N}^* \right\}.
\]

Namely, we prove the following Theorem.

**Theorem 2.3.** The Perron capacity of the set \( \Omega_e \) is finite i.e. we have \( P(\Omega_e) < \infty \).

Hence, as a consequence of Theorems 2.3 and 2.2 we know that — for any \( 1 < p < \infty \) — we have \( \|M_{\Omega_e}\|_p = \infty \).

As an application of Bateman’s Theorem, we can hence say that the set \( \Omega_e \) is not finitely lacunary.
3 Proof of Theorem 2.2

Recall that for any set \( U := \{u_1, \ldots, u_{2^N}\} \subset \mathbb{R}_+ \) with \( u_1 < u_1 < \cdots < u_{2^N} \) (so that \( U \) has \( 2^N \) elements), one define its Perron factor as

\[
G_U := \sup_{k,l \geq 1 \atop k+2l \leq 2^N} \left( \frac{u_{k+2l} - u_{k+l}}{u_{k+l} - u_k} + \frac{u_{k+l} - u_k}{u_{k+2l} - u_{k+l}} \right) \in (0, \infty).
\]

The following proposition is a careful reading of Hare and Rönning’s work [15].

**Proposition 3.1.** There exists \( \varepsilon_0 \in (0, 1) \) such that for any \( \alpha \in [1 - \varepsilon_0, 1) \) there exists a set \( X \subseteq \mathbb{R}^2 \) for which one has:

\[
|X| \leq \left[ \alpha^{2N} + G_U (1 - \alpha)^2 \right] \left\{ M_U 1_X > \frac{1}{4} \right\}.
\]

Hence for any \( 0 < \alpha < 1 \) close enough to 1, this gives us the following lower bound for any \( p \in (1, \infty) \):

\[
\| M_U \|_p \gtrsim_p \frac{1}{\alpha^{2N} + G_U (1 - \alpha)^2},
\]

where \( \gtrsim_p \) means that the inequality holds up to a multiplicative constant that does only depend on \( p \).

Indeed, we have

\[
\| M_U 1_X \|_p \gtrsim_p \frac{1}{\alpha^{2N} + G_U (1 - \alpha)^2}.
\]

This allows us now to easily conclude the proof of Theorem 2.2. Indeed, fix an arbitrary set \( \Omega \subset \mathbb{R} \) and suppose that its Perron capacity is finite i.e. assume that one has:

\[
P_\Omega < \infty.
\]

By definition of \( P_\Omega \), there thus exists a strictly increasing sequence of integers \( \{N_k : k \in \mathbb{N}^*\} \) such that for any \( k \) there is a set \( U_k \subset \Omega \) such that \( G(U_k) < 2P_\Omega \) and \#\( U_k = 2^{N_k} \). Since there holds \( U_k \subset \Omega \), we obtain, for any \( 0 < \alpha < 1 \) sufficiently close to 1 and any \( k \geq 1 \):

\[
\| M_\Omega \|_p \gtrsim \| M_{U_k} \|_p \gtrsim \frac{1}{\alpha^{2N_k} + 2P_\Omega (1 - \alpha)^2}.
\]

Since this holds for any \( k \geq 1 \) and any \( \alpha \) close to 1, this implies that we have \( \| M_\Omega \|_p = \infty \).

4 Homogeneous sets

We fix an arbitrary set \( U \) in \( \mathbb{R}_+^* \) whose cardinal is \( 2^N \) for some \( N \in \mathbb{N}^* \). The following proposition shows that it is equivalent for \( U \) to be uniformly distributed and its Perron factor to equal 2.
Proposition 4.1. We have $G_U = 2$ if and only if the elements of $U$ are in arithmetic progression i.e. $U$ is of the form

$$U = \{a + k\delta : 1 \leq k \leq 2^N\}$$

for some $a \in \mathbb{R}$ and $\delta > 0$.

Proof. If $U$ is in arithmetic progression we have easily $G_U = 2$. On the other hand if we have $G_U = 2$ since $x + \frac{1}{x} = 2$ if and only $x = 1$ we have for any $k$ (taking $\ell = 1$)

$$u_{k+2} - u_{k+1} = u_{k+1} - u_k$$

which concludes the proof. \qed

We will be particularly interested by homogeneous sets that is to say sets $H$ of the form

$$H := H_{a,N} = \{ka : 1 \leq k \leq 2^N\}$$

for some integers $a \in \mathbb{N}^{\ast}$ and $N \in \mathbb{N}$. In particular, we wish to perturb a little an homogeneous set $H_{a,N}$ into a set $H'$ such that the Perron’s constant of $H'$ is still bounded bounded. More precisely, fix any $a, N \in \mathbb{N}^*$ and let $\varepsilon$ be an arbitrary function

$$\varepsilon : H_{a,N} \to (0,1).$$

Define then the set $H_{a,N}(\varepsilon)$ as

$$H_{a,N}(\varepsilon) := \{(1 + \varepsilon(x))x : x \in H_{a,N}\}.$$

![Figure 2: A representation of a uniformly distributed set and its perturbation.](image)

Proposition 4.2. Assume that $a \in \mathbb{R}^{\ast}, \ N \in \mathbb{N}^{\ast}$ and $\varepsilon : H_{a,N} \to (0,1)$ are given. If one has

$$2^N\|\varepsilon\|_\infty \leq \frac{1}{2},$$

then we have $G_{H_{a,N}} \leq 6$. 

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Proof. To simplify the notations, let for $1 \leq k \leq 2^N$, $h_k := [1 + \varepsilon(ka)]ka$. First, observe that for $1 \leq k \leq 2^N - 1$, one has:

$$h_{k+1} - h_k = \{1 + \varepsilon((k+1)a]\}(k+1)a - [1 + \varepsilon(ka)]ka \\
\geq a - ak\varepsilon(ka) = a[1 - k\varepsilon(ka)] \geq a[1 - (2^N - 1)\|\varepsilon\|_\infty] > 0,$$

so that there holds $h_1 < h_2 < \cdots < h_{2^N}$.

Now compute, for $1 \leq l \leq k$ such that $k + 2l \leq 2^N$ (implying in particular that one has $\frac{k}{l} \leq 2^N - 2$):

$$\frac{h_{k+2l} - h_{k+l}}{h_{k+l} - h_k} \leq \frac{la + 2^N a\|\varepsilon\|_\infty}{la - \|\varepsilon\|_\infty ka} \leq \frac{1 + 2^N\|\varepsilon\|_\infty}{1 - \|\varepsilon\|_\infty} \leq \frac{1 + 2^N\|\varepsilon\|_\infty}{1 - (2^N - 2)\|\varepsilon\|_\infty} \leq 3.$$

One obtains a similar inequality for the symmetric ratio. □

5 Proof of Theorem 2.3

Recall that we have defined

$$\Omega_e := \left\{ \frac{n}{\cos n} : n \in \mathbb{N}^* \right\}$$

and that we wish to prove that the Perron capacity of $\Omega_e$ is finite i.e. that we have $P(\Omega_e) < \infty$. To do so, we are simply going to prove that the set $\Omega_e$ contains “small perturbations” of arbitrarily long homogeneous sets. Specifically, for any $N \in \mathbb{N}$, we consider the set

$$E(N) := \{n \in \mathbb{N}^* : \exists m \in \mathbb{Z}, |n + 2\pi m| < 2^{-N}\}.$$

To begin with, we claim the following.

Claim 1. For any $N \in \mathbb{N}$ we have $\#E(N) = \infty$.

Proof of the claim. The claim follows from the fact that $G' = \{n + 2\pi m : n \in \mathbb{N}, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, which can be shown easily using standard techniques from basic real analysis. □

Claim 2. For any $n \in \mathbb{N}$ and any $n \in E(N)$, one has:

$$1 < \frac{1}{\cos n} \leq 1 + 2^{-2N}.$$
Proof of the claim. Choosing \( m \in \mathbb{Z} \) such that one has \(|n + 2\pi m| < 2^{-N}\) (which exists by definition since one has \( n \in E(N) \)), one obtains, using the inequality \( 1 - \cos x \leq \frac{1}{2}x^2 \) valid for any \( x \in \mathbb{R} \):

\[
1 - \cos n = 1 - \cos(n + 2\pi m) \leq \frac{1}{2}(n + 2\pi m)^2 \leq \frac{1}{2}2^{-2N}.
\]

Note that, in particular, this yields \( 0 < \cos n < 1 \).

Using the fact that one has \( \frac{1}{1-x} \leq 1 + 2x \) for any \( 0 \leq x \leq \frac{1}{2} \), one finally computes:

\[
\frac{1}{\cos n} = \frac{1}{1 - (1 - \cos n)} \leq 1 + 2(1 - \cos n) \leq 1 + 2^{-2N},
\]

as was announced. \( \Box \)

Fix an arbitrary large \( N \in \mathbb{N} \) and any integer \( a \in E(2N) \). We claim the following.

Claim 3. For any \( a \in E(2N) \), we have

\[ H_{a,N} \subset E(N). \]

Proof. Fix \( a \in E(2N) \); by definition we have \( m \in \mathbb{Z} \) such that

\[ |a + 2\pi m| < 2^{-2N}. \]

Hence for any \( 1 \leq k \leq 2^N \) we have

\[ |ka + 2\pi km| < k2^{-2N} \leq 2^{-N} \]

i.e. for any \( 1 \leq k \leq 2^N \) we have \( ka \in E(N) \). \( \Box \)

Fix now \( N \in \mathbb{N}^* \) and \( a \in E(2N) \) and define \( \varepsilon : H_{a,N} \to (0,1) \) by the formula

\[ 1 + \varepsilon(n) = \frac{1}{\cos n} \]

for \( n \in H_{a,N} \) — we hence see the set:

\[ \left\{ \frac{n}{\cos n} : n \in H_{a,N} \right\} = \left\{ [1 + \varepsilon(n)]n : n \in H_{a,N} \right\}, \]

as a perturbation of the above type of the set \( H_{a,N} \) — . We compute, using Claim 2

\[ 2^N \|\varepsilon\|_\infty \leq 2^{-N} \leq \frac{1}{2}, \]

so that Proposition 4.2 yields \( G_{H_{a,N}(\varepsilon)} \leq 6 \).

Finally, observe that we have by construction the inclusion

\[ H_{a,N}(\varepsilon) \subset \Omega. \]

Since this holds for any \( N \in \mathbb{N}^* \), it follows that we have \( P_\Omega \leq 6 \) as desired. \( \Box \)

Remark 5.1. The same conclusion holds if in Theorem 2.3 we replace \( \Omega_e := \left\{ \frac{n}{\cos n} : n \in \mathbb{N}^* \right\} \) with

\[ \Omega_s := \left\{ \frac{n}{\sin n} : n \in \mathbb{N}^* \right\}. \]

As an application of Bateman’s Theorem, we can hence say that also the set \( \Omega_s \) is not finitely lacunary. The argument is the same but, in this case, we approximate \( \frac{\pi}{2} \) by elements of \( G' \) and take \( 1 + \varepsilon(x) = \frac{1}{\sin x}. \)
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