Bijections between generalized Catalan families of types A and C

Myrto Kallipoliti∗ and Eleni Tzanaki†

March 9, 2022

Abstract

Motivated by the relation \(N^m(C_n) = (mn+1)N^m(A_{n-1})\), holding for the \(m\)-generalized Catalan numbers of type A and C, the connection between dominant regions of the \(m\)-Shi arrangement of type \(A_{n-1}\) and \(C_n\) is investigated. More precisely, it is explicitly shown how \(mn+1\) copies of each element of the set \(R^m(A_{n-1})\) of dominant regions of the \(m\)-Shi arrangement of type \(A_{n-1}\), biject onto the set \(R^m(C_n)\) of type \(C_n\) such regions. This is achieved by exploiting two different viewpoints to express the representative alcove of each region: the Shi tableau and the abacus diagram. In the same line of thought, a bijection between \(mn+1\) copies of each \(m\)-Dyck path of height \(n\) and the set of \(N-E\) lattice paths inside an \(n \times mn\) rectangle is provided.

1 Introduction

The classical Catalan numbers, \(\text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}\), constitute one of the most ubiquitous number sequences in enumerative combinatorics, also appearing in several other contexts varying from algebra and representation theory \([12, 13, 18, 19]\) to discrete geometry \([3, 8, 17]\). In combinatorics only, we refer to \([21]\) Exercise 6.19] for a list of 66 families of objects enumerated by \(\text{Cat}(n)\). Such objects (also called Catalan objects) relevant to the present paper are, the set of dominant regions of the Shi arrangement \(\text{Shi}(A_{n-1})\), triangulations of a convex \((n+3)\)-gon, and Dyck paths of length \(n\). In \([3]\), Athanasiadis generalized Catalan numbers for every crystallographic root system \(\Phi\) and positive integer \(m\). More precisely, he defined the \(m\)-generalized Catalan number of type \(\Phi\) as

\[
\text{N}^m(\Phi) = \frac{1}{n+1} \prod_{i=1}^{n} e_i + mh + 1, \quad (1.1)
\]

where \(n\) is the rank, \(h\) is the Coxeter number and \(e_i\) are the exponents of \(\Phi\). In particular, he showed that \(\text{N}^m(\Phi)\) counts the number of dominant regions of the \(m\)-Shi arrangement associated to \(\Phi\) (see Section \([1]\) for the undefined notions). We note that the classical Catalan numbers \(\text{Cat}(n)\) are indeed a special case of \(\text{N}(1.1)\), since they occur when \(\Phi = A_{n-1}\) and \(m = 1\).

∗Fak. für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria, myrto.kallipoliti@univie.ac.at
†Department of Mathematics & Applied Mathematics, University of Crete, GR-700 13 Voutes, Heraklion, Greece, etzanaki@uoc.gr
More generally, if $\Phi = A_{n-1}$ or $C_n$, and $m$ is an arbitrary positive integer, the expression in (1.1) reduces respectively to

$$N^m(A_{n-1}) = \frac{1}{mn+1} \binom{(m+1)n}{n} \quad \text{and} \quad N^m(C_n) = \binom{(m+1)n}{n}.$$  

The numbers $N^m(A_{n-1})$, also known as Fuss-Catalan numbers, count a wealth of combinatorial objects, most of which can be seen as $m$-generalizations of type $A$ Catalan objects. For the families discussed above i.e., dominant regions of Shi arrangements, polygon triangulations and Dyck paths, such $m$-generalizations have been an object of research for more than a decade. In the same spirit, generalized $(m, \Phi)$-Catalan objects have been discovered, for every finite root system $\Phi$ and integer $m$ (see for instance [1, 4, 12]). In most cases, each root system $\Phi$ was studied separately, before a unified structure was discovered. Almost always, the starting point was the relation

$$N^m(A_{n-1})(mn + 1) = N^m(C_n), \quad (1.2)$$

holding between $m$-Catalan numbers of type $A$ and $C$. Occasionally, an $(m, C_n)$-Catalan object is a type $A$ one, of certain size and symmetry, where $mn + 1$ copies of the $(m, A_{n-1})$-Catalan object reside. Although most of the times the symmetry is rather natural to guess or understand, locating the $mn + 1$ copies of the $(m, A_{n-1})$-Catalan object in the corresponding $(m, C_n)$-type, can vary from easy to very complicated.

To give a motivating example, we describe a class of $(m, \Phi)$-Catalan objects where Relation (1.2) arises trivially. Consider the set $\mathcal{D}^m_n$ of $(m+2)$-angulations of a convex $(mn+2)$-gon $P$ i.e., dissections of $P$ by noncrossing diagonals into polygons each having $m+2$ vertices. Let also $\mathcal{E}^m_{2n}$ be the subset of $\mathcal{D}^m_{2n}$ consisting of centrally symmetric $(m+2)$-angulations of a $(2mn+2)$-gon. The sets $\mathcal{D}^m_n$ and $\mathcal{E}^m_{2n}$ are combinatorial realizations of the facets of the generalized cluster complex $\Delta^m(\Phi)$ for $\Phi = A_{n-1}$ and $C_n$ respectively [12]. From their description, Relation (1.2) is evident: we identify each centrally symmetric dissection $D$ in $\mathcal{E}^m_{2n}$ with the pair consisting of its diameter and one copy of the two $(m+2)$-angulations of the $(mn+2)$-gon into which the diameter divides the initial $(2mn+2)$-gon. Since the diameters are $mn+1$, the cardinality of $\mathcal{E}^m_{2n}$ is indeed $(mn+1)N^m(A_{n-1}) = N^m(C_n)$.

The aim of this work is to reveal two new instances of the identity $N^m(A_{n-1})(mn + 1) = N^m(C_n)$, which might lead to a better geometric understanding of the type $A$ and $C$ Shi arrangements. More precisely, we provide an explicit bijection between each of the following pair of sets:

- (Bj$_1$) the set containing $mn + 1$ copies of each dominant region of the $m$-Shi arrangement $\text{Shi}^m(A_{n-1})$ and that of dominant regions in $\text{Shi}^m(C_n)$
- (Bj$_2$) the set containing $mn + 1$ copies of each $m$-Dyck path of height $n$ and that of lattice paths from $(0, 0)$ to $(n,mn)$ in the grid $n \times mn$. 

2
We note that the second bijection is based on an idea in [16] while the first, which is the main contribution of this work, relies on the two different ways to view the representative alcove of a region in $\text{Shi}^m(\Phi)$: its Shi tableau [9] and its abacus diagram [2, 11]. Each of the bijections $Bj_1, Bj_2$ can stand on its own and the sections presenting them (Section 2 and 3 respectively) can be read independently. However, in the setting of dominant regions in $\text{Shi}^m(A_n)$, there exists previous work [9] which reveals a connection between the two bijections. Unifying previous and current results, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{dominant regions in } \text{Shi}^m(A_{n-1}) \times [mn + 1] & \xrightarrow{Bj_1} & \text{dominant regions in } \text{Shi}^m(C_n) \\
\xrightarrow{FKT_1} & & \xrightarrow{\exists} \\
\text{m-Dyck paths of height } n \times [mn + 1] & \xrightarrow{Bj_2} & \text{lattice paths from } (0, 0) \text{ to } (n, mn) \text{ in the grid } n \times mn \\
\xrightarrow{FKT_2} & & \xrightarrow{\exists} \\
(m + 2)-\text{angulations of an } (mn + 2)-\text{gon} \times [mn + 1] & \xrightarrow{Bj} & \text{centrally symmetric } (m + 2)-\text{angulations of an } (2mn + 2)-\text{gon}
\end{array}
\]

where FKT$_1$ and FKT$_2$ are bijections given in [9].

This paper is structured as follows. We end up this section by recalling basic facts on root systems and presenting the two families of Catalan objects we are dealing with. In Section 2 we introduce all necessary material and build our first bijection $Bj_1$. More precisely, in Section 2.1 we include background on Shi arrangements, discuss the notion of $m$-minimal alcoves and explain their connection to dominant regions in $\text{Shi}^m(\Phi)$. Subsequently, in Sections 2.2 and 2.3 we describe the two different viewpoints in which we can encode dominant alcoves: Shi tableaux and abacus diagrams. Section 2.4 serves to clarify the relation between them in the type A case and to show how this adjusts to the type $C$ case. Section 2.5 provides criteria for $m$-minimality in terms of the abacus representation. Finally, in Section 2.6 we construct the bijection $Bj_1$ and present an explicit example. We conclude with Section 3 where we prove $Bj_2$.

1.1 Preliminaries

**Root systems.** Let $V$ be an $n$-dimensional Euclidean space with inner product $(\cdot, \cdot)$. For each non-zero $\alpha \in V$, the reflection $r_\alpha$, is the linear map which sends $\alpha$ to its negative and fixes pointwise the hyperplane $H_\alpha$ orthogonal to $\alpha$. A finite root system $\Phi$ is a finite collection of non-zero vectors in $V$, called roots, which span $V$ and satisfy $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ and $r_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$. If, in addition, $2\frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$, then the root system $\Phi$ is called crystallographic. Any hyperplane in $V$ not orthogonal to any root in $\Phi$, partitions $\Phi$ into two sets; the set $\Phi^+$ of positive and $\Phi^-$ of negative roots, respectively. The set $\Pi$ of simple roots is a subset of $\Phi^+$ that spans $V$ and has the additional property that each positive root can be expressed as a linear combination of simple roots with non-negative coefficients. The rank of $\Phi$ is the dimension of the space generated by $\Phi^+$. 


Since in this paper we deal with root systems of type $A$ and $C$, we briefly describe their standard choice of positive $\Phi^+$ and simple $\Pi$ roots. In what follows we denote by $\varepsilon_1, \ldots, \varepsilon_{n+1}$ the standard basis of $\mathbb{R}^{n+1}$. For more information on root systems we refer the reader to [15].

For $\Phi = A_n$, we have $\Phi^+ = \{\alpha_{ij} := \varepsilon_i - \varepsilon_{j+1} \mid 1 \leq i \leq j \leq n\}$ and $\Pi = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n\}$ respectively. Then, each positive root can be written in terms of simple roots as:

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \text{ for } 1 \leq i \leq j \leq n. \quad (1.3)$$

For $\Phi = C_n$, we have $\Phi^+ = \{2\varepsilon_i, 2\varepsilon_i, \varepsilon_i \pm \varepsilon_{j+1} \mid 1 \leq i \leq j \leq n-1\}$ and $\Pi = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1}, \alpha_n := 2\varepsilon_n \mid 1 \leq i \leq n-1\}$ respectively. If, for $1 \leq i \leq j \leq n-1$, we set $\alpha_{ij} := \varepsilon_i - \varepsilon_{j+1}$, $\alpha_{in} := \varepsilon_i + \varepsilon_n$ and $\overline{\alpha}_{ij} := \varepsilon_i + \varepsilon_{j+1}$, each positive root can be written in terms of simple roots, as follows:

$$\alpha_{ij} = \alpha_i + \cdots + \alpha_j, \text{ for } 1 \leq i \leq j \leq n, \text{ and}$$

$$\overline{\alpha}_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{n-1}) + \alpha_n, \quad (1.4)$$

where $\alpha_{ii} := \alpha_i$ and the sum $\alpha_i + \cdots + \alpha_{j-1}$ is empty for $i = j$.

**Generalized Catalan objects.** We call generalized Catalan objects, families of combinatorial objects which are counted by generalized Catalan numbers. In this paragraph we present those which are relevant to this paper: the family of dominant regions of the $m$-Shi arrangement of type $A$ and $C$, that of $m$-Dyck paths of height $n$, and that of $N - E$ lattice paths inside an $n \times mn$ rectangle.

The $m$-Shi arrangement $\text{Shi}^m(\Phi)$ associated to a crystallographic root system $\Phi$ and positive integer $m$, is the collection of hyperplanes $\{H_{\alpha,k} \mid \alpha \in \Phi^+, \ -m < k < m\}$, where $H_{\alpha,k} = \{v \in V \mid \langle v, \alpha \rangle = k\}$, $\alpha \in \Phi$ and $k \in \mathbb{Z}$. The dominant chamber of $V$ is the intersection $\cap_{\alpha \in \Phi^+} \{v \in V \mid \langle v, \alpha \rangle \geq 0\}$. Every region contained in the dominant chamber is called dominant. We denote by $\mathcal{R}_m^m(\Phi)$ the set of dominant regions in $\text{Shi}^m(\Phi)$ and we note that $\mathcal{R}_m^m(\Phi)$ coincides with the set of dominant regions in the $m$-Catalan arrangement $\text{Cat}^m(\Phi)$. The number of regions in $\mathcal{R}_m^m(\Phi)$ is $N^m(\Phi)$. We refer the reader to [3] for more details.

A $N - E$ lattice path, is a lattice path in the grid $\mathbb{Z} \times \mathbb{Z}$ which takes only North $(0,1)$ and East $(1,0)$ steps. An $m$-Dyck path of height $n$, or an $(m,n)$-Dyck path for short, is a $N - E$ lattice path from $(0,0)$ to $(mn,n)$ that does not go below the line $y = \frac{1}{m}x$. Finally, an $N - E$ lattice path inside an $n \times mn$ rectangle, is an $N - E$ lattice path from $(0,0)$ to $(mn,n)$. The number of $(m,n)$-Dyck paths of height $n$ is $N^m(A_{n-1})$, while that of $N - E$ lattice paths inside an $n \times mn$ rectangle is $N^m(C_n)$. Although there is not a theory establishing type $(m,\Phi)$-lattice paths, for our purposes we consider the above sets as instances of type $A$ and $C$ Catalan objects, avoiding further generalizations.

2 Dominant regions and $\text{Bj}_1$

2.1 Shi arrangement and dominant alcoves

For $m \to \infty$, the arrangement $\text{Shi}^\infty(\Phi)$ is the infinite collection of hyperplanes $H_{\alpha,k}$, $\alpha \in \Phi^+, k \in \mathbb{Z}$. The arrangement $\text{Shi}^\infty(\Phi)$, which we simplify to $\text{Shi}(\Phi)$, is called the Shi arrangement of $\Phi$. All regions in $\text{Shi}(\Phi)$ are simplices, called alcoves. The set of dominant alcoves i.e., those lying in the dominant chamber, is denoted by $A_\Phi$. The fundamental alcove $A_0$ is the unique dominant alcove whose closure $\overline{A_0}$ contains the origin. For a fixed finite crystallographic root
Based on an idea of Shi, who arranges the positive roots of $\Phi$. The affine Weyl group $W_\alpha$ of $\Phi$ is the infinite Coxeter group generated by the reflections in $S_\alpha$. The group $W_\alpha$ acts simply transitively on the set of alcoves in $Shi(\Phi)$, thus one can identify each alcove $A$ with the unique $w \in W_\alpha$ for which $A = wA_0$. The above bijection restricts to one between the set $M_* (W_\alpha)$ of minimal length coset representatives in $W_\alpha/W$ and that of dominant alcoves in $Shi(\Phi)$. In other words, $wA_0 \in A_*(\Phi)$ if and only if $w \in M_* (W_\alpha)$.

Clearly, each dominant region $R \in R^m(\Phi)$ consists of one or more alcoves. However, among them there exists a unique, denoted by $A_R$, which is closest to the origin. We call this the $m$-minimal alcove of $R$. Every hyperplane $H_{\alpha,k}$ that supports a facet of $R$ is called a wall. We say that $H_{\alpha,k}$ is a separating wall if $R$ and $A_0$ lie in different halfspaces delimited by $H_{\alpha,k}$. It is proved in [4, Section 3] that each region $R$ in $Shi^m(\Phi)$ and its minimal alcove $A_R$ have the same set of separating walls. We refer the reader to [3], [4] for more details.

Our goal now is to exploit two different viewpoints of the dominant alcoves in $Shi(A_n-1)$ and $Shi(C_n)$: (i) the Shi tableau and (ii) the abacus diagram. These combinatorial structures, combined with the fact that each region in $Shi^m(\Phi)$ is represented by its $m$-minimal alcove, will be our tools for constructing $B_j$.

2.2 Combinatorial viewpoint I (Shi tableaux for dominant alcoves)

Based on an idea of Shi, who arranges the positive roots of $\Phi^+$ in diagrams [19], we assign to each dominant alcove $A$, a set $\{k_\alpha, \alpha \in \Phi^+\} \subset \mathbb{Z}$ of coordinates which describe its location in the affine Shi arrangement $Shi(A_{n-1})$. In particular, each $k_\alpha$ counts the number of positive integer translations of the hyperplane $H_{\alpha,0}$, which separate $A$ from the origin. The set of these coordinates, arranged according to the corresponding root diagram, is called the Shi tableau of the alcove $A$. We note that the entries of the Shi tableau yield the face defining inequalities of the alcove. More precisely, for each $\alpha \in \Phi^+$ we have

$$k < \langle x, \alpha \rangle < k + 1 \text{ for all } x \in A \text{ if and only if } k_\alpha = k. \quad (2.1)$$

The above inequalities imply that the coordinates of the Shi tableau have to satisfy the so-called Shi conditions i.e., for every $\alpha, \beta, \gamma \in \Phi^+$ with $\alpha + \beta = \gamma$

$$k_\gamma = k_\alpha + k_\beta + \delta, \text{ where } \delta = \delta(\alpha, \beta, \gamma) \in \{0, 1\}. \quad (2.2)$$

The Shi tableau of a dominant alcove $A$ in $Shi(A_n)$ can be seen as a staircase Young diagram of shape $(n, n-1, \ldots, 1)$ whose entries are non-negative integers $k_{ij}, 1 \leq i \leq j \leq n$ (see Figure 1 left). In particular, the coordinate $k_{ij}$ occupying the cell $(i, n-j+1)$ \(^{(1)}\) of the Shi tableau, indicates the number of integer translates of the hyperplane $H_{\alpha_{ij,0}}$ that separate the alcove $A$ from the origin. The Shi conditions in this case become:

$$k_{ij} = k_{i\ell} + k_{\ell+1,j} + \delta_{i\ell}, \text{ with } \delta_{i\ell} \in \{0, 1\}, \quad (2.3)$$

for every $1 \leq i < j \leq n$ and all $i \leq \ell < j$.

Traditionally, the Shi tableau of an alcove in $Shi(C_n)$ is represented by a shifted Young diagram. Below, we describe a different but equivalent way to write the Shi coordinates. We do this to emphasize and exploit its relation to the type $A$ case. We arrange the coordinates

\(^{(1)}\) The cell $(i,j)$ lies in the $i$-th row and $j$-th column of the diagram, where rows are counted from top to bottom and columns from left to right.
in a staircase diagram of shape \((2n - 1, 2n - 2, \ldots, 1)\) as follows: for \(1 \leq i \leq j \leq n - 1\) the coordinate \(k_{ij}\) occupies the cell \((i, j)\), whereas for \(1 \leq i \leq j \leq n\) the coordinate \(k_{ij}\) occupies the cell \((i, 2n - j)\). We finally fill the remaining cells so that the diagram is self-conjugate. Notice that the tableau obtained this way is a self-conjugate Shi tableau of type \(A_{2n-1}\).

![Tableau](image)

**Figure 1:** The Shi tableau for type \(A_4\) (left) and \(C_4\) (right). Replacing each coordinate \(k_{ij}\) by the corresponding root \(\alpha_{ij}\), we obtain the root diagram. Notice that, in both diagrams, the simple roots are those located on the diagonal, while each positive root is the sum of the simple roots lying below and to its right (this agrees with expressions (1.3) and (1.4)).

### 2.3 Combinatorial viewpoint II (The abacus diagram)

A standard way to realize the affine group \(\widetilde{A}_{n-1}\) is by identifying it with the set of all \(\mathbb{Z}\)-permutations i.e., all bijections \(w\) of \(\mathbb{Z}\) in itself such that:

1. \(w(x + n) = w(x) + n\) for all \(x \in \mathbb{Z}\), and
2. \(\sum_{i=1}^{n} w(i) = \binom{n+1}{2}\),

with composition as group operation. Clearly, each such permutation is uniquely determined by its values on \(n\) consecutive integers. Thus, for each \(k \in \mathbb{Z}\) we write \(w = [w(k+1), \ldots, w(k+n)]\) and call this a **window** of \(w\). In the special case where \(k = 0\), we say that \([w(1), \ldots, w(n)]\) is the **base window** of \(w\). In this paper we will often encode the base window of a permutation \(w \in \widetilde{A}_{n-1}\) in a more concise form, which is actually equivalent to the so-called **abacus representation** of \(w\) \([6, 11]\). To do this, for each \(a \in \mathbb{Z}\) we set \(r = a \mod n\), \(\ell = \left\lfloor \frac{a}{n} \right\rfloor\) and we write \(a := r^\ell\) to imply that \(a = r + n\ell\). We say that \(r\) is the **base** and \(\ell\) the **level** of \(a\). Note that, although \(n\) is suppressed in the base-level notation, it is implicit from the rank of the affine group in which the permutation \(w\) belongs. In other words, when we write \(w = [r_1^\ell, \ldots, r_n^\ell] \in \widetilde{A}_{n-1}\), we tacitly understand that \(r_i^\ell = r_i + n\ell_i\). The **abacus diagram** or simply **abacus** of \(w\) is the base window written in the concise form \([r_1^\ell, \ldots, r_n^\ell]\) \([2]\). Then, conditions (i) and (ii) imply that:

\[
\{r_1, \ldots, r_n\} = \{1, \ldots, n\} \quad \text{and} \quad \ell_1 + \cdots + \ell_n = 0.
\]

\(\text{(2.4)}\)

---

This diagram corresponds to the flush abacus with \(n\) runners, whose \(i\)-th runner has defining bead on level \(\ell_i\). Abacus models encode nicely the action of the affine group \(W_n\) in the set of alcoves \(\mathcal{A}(\Phi)\), when \(\Phi\) is a classical root system \([11]\). This justifies the term **abacus**, which we use with no further reference to its structure and properties.

---

6
Figure 2: The arrangement $\text{Shi}^3(A_2)$. The dashed alcove is a 3-minimal alcove since, among the two alcoves contained in $\mathcal{R}$, it is the one closest to the origin. The Shi tableaux of $\mathcal{R}$ and $\mathcal{A}_{\mathcal{R}}$, are shown on the right.

The level vector $\vec{n}(w) = (\beta_1, \ldots, \beta_n)$ of $w$ is the vector whose $i$-th coordinate $\beta_i$ is equal to the level of $i$ in the base window of $w$. Clearly $\{\beta_1, \ldots, \beta_n\} = \{\ell_1, \ldots, \ell_n\}$ and $\{1^{\beta_1}, \ldots, n^{\beta_n}\} = \{r_1^{\ell_1}, \ldots, r_n^{\ell_n}\}$. If, in addition, the entries of the base window are sorted i.e., $w(1) < \cdots < w(n)$, then $w$ belongs to the set $\mathcal{M}_*(\tilde{A}_{n-1})$ of minimal length coset representatives of $\tilde{A}_{n-1}/A_{n-1}$. In fact, it turns out that all permutations $w \in \tilde{A}_{n-1}$ whose base window is sorted, biject to the set $\mathcal{M}_*(\tilde{A}_{n-1})$ (see [17 Chapter 8.3]). In this special case, the level vector suffices to determine $w$, since its window consists of the numbers in $\{1^{\beta_1}, \ldots, n^{\beta_n}\}$ ordered increasingly. For instance, if $w \in \mathcal{M}_*(\tilde{A}_3)$ has level vector $\vec{n}(w) = (0, -2, 4, -2)$, then its abacus consists of the numbers in $\{1^0, 2^{-2}, 3^4, 4^{-2}\}$ ordered increasingly. Using base-level notation, we have $(1^0, 2^{-2}, 3^4, 4^{-2}) = (1, -6, 19, -4)$ and thus, the base window and abacus of $w$ are $[-6, -4, 1, 19]$ and $[2^{-2}, 4^{-2}, 1^0, 3^4]$ respectively.

A standard way to describe the affine group $\tilde{C}_n$ is by identifying it with the set of all mirrored $\mathbb{Z}$-permutations i.e., bijections $w$ of $\mathbb{Z}$ in itself such that:

(i) $w(x + N) = w(x) + N$ and

(ii) $w(-x) = -w(x)$ for all $x \in \mathbb{Z}$,

where $N = 2n + 1$ and with composition as group operation. Combining (i) and (ii), one can easily verify that $\sum_{i=1}^{2n+1} w(i) = \binom{2n+2}{2}$, hence $\tilde{C}_n$ is a subgroup of $\tilde{A}_{2n}$. As we did in the type $A$ case, we can write the abacus diagram of $w \in \tilde{C}_n$ in the concise form $[r_1^{\ell_1}, \ldots, r_{2n}^{\ell_{2n}}, r_{2n+1}^{\ell_{2n+1}}]$, where now $r^\ell := r + \ell N$. In this setting, (i) and (ii) imply that:

i. $w(0) = 0$ and $w(2n+1) = 2n+1$,

ii. $\{r_1, \ldots, r_{2n}\} = \{1, \ldots, 2n\},$

iii. $\ell_1 + \cdots + \ell_{2n} = 0$ and

iv. if $r_i + r_j = 2n + 1$ then $\ell_i = -\ell_j$.  

7
Notice that the first condition above implies that \( r_{2n+1} = 2n+1 \) and \( \ell_{2n+1} = 0 \) for every \( w \in \wtilde{C}_n \). Thus, the last coordinate of the abacus diagram is somewhat redundant. In Proposition 2.6, we rediscover and redefine type \( C \) abacus diagrams by exploiting the connection between type A and type C Shi tableaux. These diagrams are slightly different, in the sense that the last coordinate mentioned above does not appear. Strictly speaking, the above definition of the affine group \( \wtilde{C}_n \) is not necessary to us here. We refer the reader to [7, Chapter 8.4.8.5] and [14] for more details.

### 2.4 Switching from the Shi tableau to the abacus and vice versa

In this paragraph, we explain how, for each dominant alcove in \( \mathcal{A}_+(A_{n-1}) \), we go back and forth from its Shi tableau to its abacus diagram. We then exploit this, in order to apply the same for alcoves in \( \mathcal{A}_+(C_n) \) (see Proposition 2.6).

In [10] Section 2.7 Fishel et al., by rephrasing results of Shi [20], associate Shi tableaux with abacuses. In particular, they prove the following two propositions.

**Proposition 2.1.** [10] Consider a dominant alcove \( wA_0 \in \mathcal{A}(A_n) \) with abacus \( w = [\ell_1, \ldots, \ell_{n+1}] \).

Let \( T(w) \) be the Young tableau of shape \((n, n-1, \ldots, 1)\) whose entries \( k_{ij} \), \( 1 \leq i \leq j \leq n \), are:

\[
k_{ij} = \begin{cases} 
\ell_{j+1} - \ell_i, & \text{if } r_i < r_{j+1} \\
\ell_{j+1} - \ell_i - 1, & \text{if } r_i > r_{j+1}.
\end{cases}
\]  

(5)

The map \( T \) is a bijection between the set \( \mathcal{A}_+(A_n) \) of dominant alcoves in \( \text{Shi}(A_{n-1}) \) and the set of Shi tableaux of type \( A_n \).

Before continuing with the next proposition, a few remarks are in order. First, notice that the map above yields the same tableau if, instead of considering the abacus of the base window, we chose any other window of \( w \) (in base-level notation). This is natural, since the tableau \( T(w) \) should be independent of the way we represent the corresponding permutation \( w \). It is also clear from the above bijection that permutations whose abacus entries are sorted correspond to dominant regions (since, in this case, all \( k_{ij} \) are positive). Since sorted abacus entries correspond to minimal length coset representatives in \( \wtilde{A}_n/A_n \), we verify the fact that \( wA_0 \) is a dominant alcove in \( \text{Shi}(A_n) \) if and only if \( w \in \mathcal{M}_+(\wtilde{A}_n) \). In the remainder of the paper, we freely interchange the statements “\( wA_0 \in \mathcal{A}_+(A_n) \)” and “\( w \in \mathcal{M}_+(\wtilde{A}_n) \)”, according to which fits better in the situation. Thus, when we write \( w \in \mathcal{M}_+(\wtilde{A}_n) \) we sometimes use it to imply that \( wA_0 \) is a dominant alcove in \( \text{Shi}(A_n) \) or, other times, to imply that the entries of the abacus diagram of \( w \) are in increasing order.

The reverse map of \( T \) is rather involved; it sends each Shi tableau \( T \) to a unique dominant alcove \( wA_0 \). In Proposition 2.2, we determine \( wA_0 \) by providing a way to find the normalized window of \( w \), i.e., the window whose smallest entry is 0. Finally, in Lemma 2.3 we show how to shift from the normalized to the base window or equivalently the abacus diagram of \( w \).

**Proposition 2.2.** [10] Let \( T = \{k_{ij} : 1 \leq i < j \leq n\} \) be the Shi tableau of a dominant alcove in \( \mathcal{A}_+(A_n) \). For each \( 1 \leq j \leq n \) let

\[
(i) \ b_j := |\{(1, \ell, j), 1 \leq \ell < j : k_{ij} = k_{1 \ell} + k_{\ell+1 j} + 1\}|,
\]

and

\[
(ii) \ \sigma = [\sigma(1), \ldots, \sigma(n)] \text{ be the permutation of } \{1, \ldots, n\} \text{ with inversion table } (b_1, \ldots, b_n)
\]

(3)

\( (i) \) Given a permutation \( \sigma \) of \( \{1, \ldots, n\} \), its inversion table is a length \( n \) sequence \( b_1, \ldots, b_n \), where \( b_i \) is the number of elements that are smaller than \( i \) and appear to the right of \( i \) in \( \sigma \).
We define $B(T)$ to be the permutation $w \in \mathcal{M}_+(\tilde{A}_{n-1})$ whose normalized window has level vector 
\[
(0, k_{1,(1)}, k_{1,(2)}, \ldots, k_{1,(n)}).
\]
(2.6)

The map which sends the tableau $T$ to $wA_0$ is the reverse map of $T$.

**Lemma 2.3.** The base window of $w$ is obtained from its normalized window after subtracting $k_{11} + \cdots + k_{1n} - 1$ from each of its entries.

**Proof.** The entries of the normalized window of $w$ are the elements in $A = \{0^0, 1^{k_{1(1)}}, \ldots, n^{k_{1(n)}}\}$ arranged increasingly. To shift from the normalized to the base window, it suffices to add an integer $S$ to every element of $A$ so that the resulting set of numbers sum up to $1 + 2 + \cdots + n + (n+1)$. In other words, we seek an $S$ which satisfies:
\[
(0^0 + S) + (1^{k_{1(1)} + S}) + \cdots + (n^{k_{1(n)} + S}) = 1 + \cdots + (n+1).
\]
(2.7)

Making use of the base-level notation, the left-hand side of (2.7) is equal to:
\[
0^0 + 1^{k_{1(1)}} + \cdots + n^{k_{1(n)}} + (n+1)S = (n+1)^{-1} + \cdots + n^{k_{1(n)}} + 0^S
= (1 + \cdots + (n + 1)) + (n+1)(-1 + k_{1(1)} + \cdots + k_{1(n)} + S)
= (1 + \cdots + (n + 1)) + (n+1)(-1 + k_{11} + \cdots + k_{1n} + S).
\]

Equating the above with the right-hand side of (2.7), we deduce that indeed $S = -k_{11} - \cdots - k_{1n} + 1$.

In what follows, we present an explicit example of the maps described above.

**Example 2.4.** The dominant alcove $wA_0$ with $w = [5^{-2}, 2^{-1}, 4^0, 3^1, 1^2]$ has the Shi tableau $T$ shown below

\[
T = \begin{bmatrix}
3 & 2 & 1 & 0 \\
2 & 2 & 1 \\
1 & 0 \\
0
\end{bmatrix}
\]

We next illustrate the reverse map. According to Proposition 2.2, in order to find the normalized level-vector, for each $1 \leq j \leq 4$, we count the number of triples for which $k_{1j} = k_{1\ell-1} + k_{ij} + 1$. As indicated by the figure below, we have $b_4 = 3$, $b_3 = 1$ and $b_2 = 0$, ($b_1 = 0$ always).
Hence, the normalized level vector is \((0, k_{1\sigma(4)}, k_{1\sigma(1)}, k_{1\sigma(3)}, k_{1\sigma(2)}) = (0, 3, 0, 2, 1)\). This means that the elements of the set \(A = \{0^0, 1^3, 2^0, 3^2, 4^1\}\) arranged in increasing order give the normalized window of \(w\) i.e. \(w = [0^0, 2^0, 4^1, 3^2, 1^3]\). In view of Corollary 2.5, to shift from the normalized to the base window of \(w\), we need to subtract \(k_{11} + k_{12} + k_{13} + k_{14} - 1 = 5\) from each of the elements of \(A\). Using base-level notation, subtracting 5 is equivalent to adding \(0^{-1}\) or \(5^{-2}\) to each element of \(A\), from which we deduce that the abacus of \(w\) is indeed \([5^{-2}, 2^{-1}, 4^0, 3^1, 1^2]\).

As we have seen in Section 2.2, Shi tableaux of type \(A_n\) coincide with self-conjugate Shi tableaux of type \(A_{2n-1}\). We want to translate this in terms of level vectors and abacuses.

Let \(w\) be a minimal length coset representative in \(\mathcal{M}_s(\bar{A}_{2n-1})\) with antisymmetric level vector i.e. \(\bar{w}(w) = (\beta_1, \ldots, \beta_n, -\beta_n, \ldots, -\beta_1)\). Equivalently, the abacus diagram of \(w\) is the increasing rearrangement of the numbers in \(\{1^{β_1}, \ldots, n^{β_n}, (n + 1)^{-β_n}, \ldots, (2n)^{-β_1}\}\). We claim that the antisymmetry of the level-vector is equivalent to the fact that the abacus of \(w\) is balanced i.e., it is of the form \([r_1^{\ell_1}, \ldots, r_n^{\ell_n}, r_{n+1}^{\ell_{n+1}}, \ldots, r_{2n}^{\ell_{2n}}]\) where \(r_i + r_{2n+1-i} = 2n + 1\). This is the content of the next lemma.

**Lemma 2.5.** Let \(\mathcal{A}_0\) be a dominant alcove in \(\text{Shi}(A_{2n-1})\). Then \(\bar{w}(w) = (\beta_1, \ldots, \beta_n, -\beta_n, \ldots, -\beta_1)\) if and only if the abacus diagram of \(w\) is balanced.

**Proof.** To prove the forward direction, we assume that the level vector is antisymmetric. Then, recalling that \(α^\ell := α + \ell \cdot 2n\), we have:

\[
k_{β_k} < λ^{β_k} \iff (2n + 1 - k)^{β_k} < (2n + 1 - λ)^{β_k} \quad \text{and} \quad (2n + 1 - k)^{−β_k} < λ^{−β_k} \iff (2n + 1 - λ)^{−β_k} < k^{−β_k}.
\]

The above equivalences imply that the elements which precede \(k^{β_k}\) in the abacus diagram of \(w\) are as many as the elements which succeed \((2n + 1 - k)^{−β_k}\). This further implies that the diagram is balanced. The reverse direction is immediate from the definition of balanced. □

**Proposition 2.6.** The bijection \(T\) of Proposition 2.1 restricts to one between self-conjugate tableau of alcoves in \(\mathcal{A}_s(A_{2n-1})\) and minimal length coset representatives \(w\in \mathcal{M}_s(\bar{A}_{2n-1})\) whose level vector is antisymmetric i.e., \(β_i = −β_{2n+1-i}\) for all \(1 ≤ i ≤ n\).

**Proof.** Let \(w \in \mathcal{M}_s(\bar{A}_{2n-1})\) with antisymmetric level vector. In view of Lemma 2.5, \(w\) has balanced abacus diagram i.e., it can be written as \([r_1^{\ell_1}, \ldots, r_n^{\ell_n}, r_{n+1}^{−\ell_n}, \ldots, r_{2n}^{−\ell_1}]\). Thus, when applying (2.5) we have

\[
k_{2n-j}^{2n-i} = \begin{cases} \ell_{2n-i+1} - \ell_{2n-j}, & \text{if } r_{2n-j} < r_{2n-i+1} \\ \ell_{2n-i+1} - \ell_{2n-j} - 1, & \text{if } r_{2n-j} > r_{2n-i+1} \end{cases}
\]

\[
= \begin{cases} -\ell_i - (−\ell_{j+1}), & \text{if } 2n + 1 - r_{j+1} < 2n + 1 - r_i \\ -\ell_i - (−\ell_{j+1}) - 1, & \text{if } 2n + 1 - r_{j+1} > 2n + 1 - r_i \end{cases}
\]

\[
= \begin{cases} \ell_{j+1} - \ell_i, & \text{if } r_i < r_{j+1} \\ \ell_{j+1} - \ell_i - 1, & \text{if } r_i > r_{j+1} \end{cases}
\]

which means that \(T(w)\) is self-conjugate.
To prove the reverse, we use induction on \( n \), the claim for \( A_1 \) being trivial. Next, we assume that our claim holds for the group \( A_{2n-3} \) and we prove it for \( A_{2n-1} \). To this end, consider a self-conjugate tableau \( T \) of an alcove \( w \mathcal{A}_0 \in \mathcal{A}_s(\widetilde{A}_{2n-1}) \). Let \( T' \) be the tableau we obtain from \( T \) after deleting its first row and column. Clearly, \( T' \) is a self-conjugate Shi tableau of an alcove \( w' \mathcal{A}_0 \in \mathcal{A}_s(\widetilde{A}_{2n-3}) \) and induction implies that we can write its abacus diagram as \( w' = [r_1^{t_1}, \ldots, r_n^{t_n}, r_n, r_{n-1}, \ldots, r_2^{t_2}] \), where \( \{r_2, \ldots, r_{2n-2}\} = \{1, \ldots, 2n-1\} \) and \( r_i + r_{2n+1-i} = 2n - 1 \). By an appropriate shifting of the \( r_j \)'s we can assume that \( w = [r_1^{t_1}, r_2^{t_2}, \ldots, r_n^{t_n}, r_n, r_{n-1}, \ldots, r_2^{t_2}, r_2^{t_2}] \), where \( r_i + r_{2n+1-i} = 2n + 1 \) for all \( 2 \leq i \leq n - 1 \), i.e., that the abacus of \( w \) is balanced except possibly from the pair consisting of its first and last entry. Since the entries of the base window sum up to \( 1 + \cdots + 2n = n(2n + 1) \), we deduce that

\[
r_1^{t_1} + r_2^{t_2} = 2n + 1.
\] (2.8)

Our goal is to show that \( r_1 + r_2 = 2n + 1 \) and \( \ell_1 + \ell_2 = 0 \). To this end, for \( 2 \leq i \leq 2n \) we set \( \epsilon_i = 1 \) if \( r_i > r_1 \) and \( \epsilon_i = 0 \) otherwise. In view of (2.5), the sum of the entries of the first row of \( T \) is

\[
R_1 = \sum_{i=1}^{2n} k_{1i} = (\ell_2n - \ell_1 - \epsilon_2) + \sum_{i=2}^{n} (-\ell_i - \ell_1 - \epsilon_i) + \sum_{i=1}^{n-1} (\ell_i - \ell_1 - \epsilon_{i+n})
\]

\[
= (\ell_2n - \ell_1) - 2(n-1)\ell_1 - (r_1 - 1) \tag{2.9}
\]

where, in the last equation, we used the fact that \( \sum_i \epsilon_i = |\{i : r_i < r_1\}| = r_1 - 1 \). Analogously, for \( 1 \leq i \leq 2n - 1 \) we set \( \ell'_i = 1 \) if \( r_i > r_2 \) and \( \ell'_i = 0 \) otherwise. In view of (2.5), the sum of the entries of the first column of \( T \) is

\[
C_1 = \sum_{j=1}^{2n} k_{2j} = (\ell_2n - \ell_1 - \ell'_1) + \sum_{j=2}^{n} (\ell_2n - \ell_j - \ell'_j) + \sum_{j=2}^{n-1} (\ell_2n - \ell_j - \ell'_{j+n})
\]

\[
= (\ell_2n - \ell_1) - 2(n-1)\ell_2n - (2n - r_2) \tag{2.10}
\]

where again, in the last equation, we used the fact that \( \sum_j \ell'_j = |\{j : r_j > r_2\}| = 2n - r_2n \). Since the tableau \( T \) is self-conjugate we have that \( R_1 = C_1 \) and thus, using base-level notation for the expressions (2.9) and (2.10), we conclude that \( r_1^{t_1} + r_2^{t_2} = 2\ell_1 + 2\ell_2n + 1 + 2n \). In view of (2.8), this implies that \( \ell_1 + \ell_2 = 0 \) and thus \( r_1 + r_2 = 2n + 1 \).

As we have already pointed out, in Proposition 2.6 we recover a slightly modified description of the affine group \( \widetilde{C}_n \), compared to that given in Section 2.3. The difference is that, in the above proposition, we no longer require \( w(0) = 0 \) and thus all symmetry is encoded in the abacus diagram of size \( 2n \) i.e., in a subgroup of \( \widetilde{A}_{2n-1} \). This modification arises naturally, since the images through the map \( T \) are tableaux of the required size and symmetry. Thus, from now on we identify dominant alcoves in \( \text{Shi}(\widetilde{C}_n) \) with minimal length coset representatives \( w \in \mathcal{M}_s(\widetilde{A}_{2n-1}) \) whose abacus diagram is balanced i.e., \( w = [r_1^{t_1}, \ldots, r_n^{t_n}, r_n^{t_n}, r_{n+1}, \ldots, r_2^{t_2}] \) and \( r_i + r_{2n+1-i} = 2n + 1 \) for all \( i \).

---

\( \text{This claim is somewhat subtle. Induction implies that } \{r_2, \ldots, r_{2n-2}\} = \{1, \ldots, 2n - 1\} \text{ with } r_i + r_{2n+1-i} = 2n - 1 \text{ for all } 2 \leq i \leq n - 1. \text{ Then, depending on the value of } r_1, \text{ there is a unique way to shift the above pairs of } r_i \text{'s, either by } r_i \leftarrow r_i + 1 \text{ and } r_{2n+1-i} \leftarrow r_{2n+1-i} + 1 \text{ or by } r_i \leftarrow r_i \text{ and } r_{2n+1-i} \leftarrow r_{2n+1-i} + 2, \text{ so that } \{r_1, \ldots, r_{2n}\} = \{1, \ldots, 2n + 1\}, \text{ or } r_i + r_{2n+1-i} = 2n + 1 \text{ for all } 1 \leq i \leq n \text{ and again } T(w') = T'. \)
2.5 Criteria for $m$-minimalilty

As we mentioned in Section 2.1, each dominant region in $\text{Shi}^m(\Phi)$ is uniquely represented by its $m$-minimal alcove. This allows us to identify the set of regions in $R^+_m(\Phi)$ with that of $m$-minimal alcoves in $\mathcal{A}_m(\Phi)$, for which we can apply the bijections of Section 2.4 (involving Shi tableaux and abacus diagrams). However, to do this, we need to distinguish $m$-minimal among all alcoves in $\mathcal{A}_m(\Phi)$. Proposition 2.4 implies that it suffices to formulate criteria for $m$-minimality in the type $A$ case, since $m$-minimal alcoves in $\text{Shi}(C_n)$ correspond to $m$-minimal alcoves in $\text{Shi}(A_{2n-1})$ having self-conjugate Shi tableau.

The following theorem is equivalent to [2, Theorem 4.5], where criteria for $m$-minimality are expressed in terms of flush abacuses. To make the paper self-contained, we prove it using our own setup.

**Theorem 2.7.** Let $\mathcal{A} = wA_0 \in \mathcal{A}_m(\mathcal{A}_{n-1})$ be a dominant alcove with level vector $\tilde{n}(w) = (\beta_1, \ldots, \beta_n)$. Then, $\mathcal{A}$ is the $m$-minimal alcove of a dominant region $R$ in $\text{Shi}^m(A_{n-1})$ if and only if:

(i) $\beta_{i+1} - \beta_i \leq m$ for all $1 \leq i \leq n-1$, and

(ii) $\beta_1 - \beta_n - 1 \leq m$.

In the current setup, it is easier to prove Theorem 2.8, which is a reformulation of Theorem 2.7. In the proof, we repeatedly use Corollary A.2 of the Appendix, which allows us to compare the face defining inequalities of a region in $\text{Shi}^m(\Phi)$ with those of its $m$-minimal alcove.

**Theorem 2.8.** Let $\mathcal{A} = wA_0 \in \mathcal{A}_m(\mathcal{A}_{n-1})$ be a dominant alcove with abacus $w = [r^1_1, \ldots, r^n_n]$. Then, $\mathcal{A}$ is the $m$-minimal alcove of a dominant region $R \in \text{Shi}^m(A_{n-1})$ if and only if:

(i) $\ell_j - \ell_i \leq m$ for all $1 \leq i \leq j \leq n$ with $r_j = r_i + 1 \leq n$, and

(ii) $\ell_j - \ell_i - 1 \leq m$ if $r_j = 1$ and $r_i = n$.

**Proof.** Let $\mathcal{A} = wA_0$ be the $m$-minimal alcove of a dominant region $R$ in $\text{Shi}^m(A_{n-1})$ with abacus $w = [r^1_1, \ldots, r^n_n]$ and assume that one of the conditions in the statement of the lemma is violated. We separate cases:

(i) If there exist $r_{ij}, r_{ij}$ with $r_{ij} = r_{ij} + 1 \leq n$ for which $\ell_{ij} - \ell_{ij} > m$ then, since the entries in the abacus diagram are sorted, we can write $w = [r^1_1, \ldots, r^n_n, r^1_2, \ldots, r^1_n]$. Consider the affine permutation $w' = [r^1_1, \ldots, r^n_n, r^1_2, \ldots, r^1_n]$ where we have exchanged $r_{ij}$ and $r_{ij}$ without their levels, and notice that $w'$ is still sorted. Thus, $w'A_0$ is a dominant alcove. Moreover, from (2.5) one can check that its Shi tableau $\{k_{ij} : 1 \leq i \leq j \leq n-1\}$ has all $k_{ij} = k_{ij}'$ except from the entry $k_{ij}'$ for which $k_{ij}' = \ell_{ij} - \ell_{ij} - 1 = k_{ij} - 1$. Since we have assumed that $\ell_{ij} - \ell_{ij} - 1 \geq m$, Corollary A.2 implies that $w'A_0$ and $wA_0$ lie in the same region $R$.

(ii) If $r_{ij} = n$, $r_{ij} = 1$ and $\ell_{ij} - \ell_{ij} - 1 > m$ then, since the entries in the bracket are sorted, we can write $w = [r^1_1, \ldots, n^1, \ldots, 1^1, \ldots, r^n_n]$. Next, notice that for all $r^i_j$'s between $n^1$ and $1^1$ it is $\ell_{ij} < \ell_{ij} < \ell_{ij}$ (otherwise, the entries of the bracket would not be in increasing order). Thus, the bracket of $w' = [r^1_1, \ldots, 1^1, \ldots, n^1, \ldots, r^n_n]$ is also sorted and hence $w'A_0$ is a dominant alcove. Moreover, its Shi tableau $\{k_{ij}'' : 1 \leq i \leq j \leq n-1\}$ has all $k_{ij}'' = k_{ij}'$ except for $k_{ij}'$ for which $k_{ij}' = k_{ij} - 1 = \ell_{ij} - \ell_{ij} + 2 \geq m$. As before, Corollary A.2 implies that both $w'A_0$ and $wA_0$ lie in the same region $R$.
In both cases we have found an alcove \( w' A_0 \) in \( \mathcal{R} \) having all Shi coordinates equal to those of \( w A_0 \) except for one which is smaller. This contradicts the fact that \( w A_0 \) is the minimal alcove of \( \mathcal{R} \).

To prove the reverse, we assume that the conditions in the statement of the theorem hold for some \( w \) and we prove that all alcoves adjacent to \( w A_0 \) i.e., all those which share a facet with \( w A_0 \), either lie in another region or have larger coordinates. This immediately implies that \( w A_0 \) is \( m \)-minimal.

Two alcoves \( w A_0, w' A_0 \) are adjacent if they have all their Shi coordinates the same, except for one in which they differ by \( \pm 1 \). Arguing as above, this can only happen when, in the abacus diagram of \( w \), we either exchange pairs \( r_1, r_\nu \) with \( r_\nu = r_1 + 1 \leq n \) or the pair \( n, 1 \) with appropriate alteration of their levels. This indicates the following four cases, in which \( w A_0 \) and \( w' A_0 \) differ by exactly one Shi coordinate. 

**Case 1:** If \( r_\nu = r_1 + 1 \leq n \) then, from our assumption, \( \ell_\nu - \ell_1 \leq m \). We further distinguish cases. If \( \ell_\nu - \ell_1 \leq \mu_1 - \ell_1 \) then we switch from \( w = [r_1^{\ell_1}, \ldots, r_\mu^{\ell_\mu}, \ldots, r_\nu^{\ell_\nu}, \ldots, r_n^{\ell_n}] \) to \( w' = [r_1^{\ell_1}, \ldots, r_\nu^{\ell_\nu}, \ldots, r_\mu^{\ell_\mu}, \ldots, r_n^{\ell_n}] \).

The only different Shi coordinate of the alcoves \( w A_0 \) and \( w' A_0 \) is \( k_{\mu_\nu - 1} = \ell_\mu - \ell_\nu \leq m \) which becomes \( k_{\mu_\nu - 1} = \ell_\mu - \ell_1 + 1 \leq m \). In view of Corollary A.2 we deduce that \( w A_0 \) and \( w' A_0 \) lie in different regions.

If \( \ell_\nu - \ell_1 > m \) then we switch from \( [r_1^{\ell_1}, \ldots, r_\nu^{\ell_\nu}, \ldots, r_n^{\ell_n}] \) to \( [r_1^{\ell_1}, \ldots, r_\mu^{\ell_\mu}, \ldots, r_n^{\ell_n}] \).

The only different Shi coordinate of the alcoves \( w A_0 \) and \( w' A_0 \) is \( k_{\mu_\nu - 1} = \ell_\mu - \ell_\nu - 1 \) which becomes \( k_{\mu_\nu - 1} = \ell_\mu - \ell_\nu \). Since \( k_{\mu_\nu - 1} > k_{\mu_\nu - 1} \), the alcove \( w' A_0 \) does not lie closer to the origin than \( w A_0 \).

**Case 2:** If \( r_\nu = 1 \) and \( r_\mu = n \) then, from our assumption, \( \ell_\nu - \ell_1 - 1 \leq m \). We further distinguish cases.

If \( \ell_\nu - \ell_1 + 1 \) then we switch from the abacus diagram \( w = [r_1^{\ell_1}, \ldots, r_\nu^{\ell_\nu}, \ldots, r_n^{\ell_n}] \) to \( w' = [r_1^{\ell_1}, \ldots, r_\nu^{\ell_\nu}, \ldots, r_\mu^{\ell_\mu}, \ldots, r_n^{\ell_n}] \).

The only different Shi coordinate in the alcoves \( w A_0 \) and \( w' A_0 \) is \( k_{\mu_\nu - 1} = \ell_\mu - \ell_\nu \leq m \) which becomes \( k_{\mu_\nu - 1} = (\ell_\mu - 1) + (\ell_\mu + 1) = \ell_\nu - \ell_1 - 1 \leq m \). In view of Corollary A.2 we deduce that \( w A_0 \) and \( w' A_0 \) lie in different regions.

If \( \ell_\mu + 1 \leq \ell_\mu \) then, we switch from the abacus diagram \( w = [r_1^{\ell_1}, \ldots, r_\nu^{\ell_\nu}, \ldots, r_n^{\ell_n}] \) to \( w' = [r_1^{\ell_1}, \ldots, r_\mu^{\ell_\mu}, \ldots, r_\nu^{\ell_\nu}, \ldots, r_n^{\ell_n}] \).

The only different Shi coordinate in the alcoves \( w A_0 \) and \( w' A_0 \) is \( k_{\mu_\nu - 1} = \ell_\mu - \ell_\nu \) which becomes \( k_{\mu_\nu - 1} = (\ell_\mu + 1) - (\ell_\nu - 1) + 1 = \ell_\mu - \ell_\nu + 1 \). Since \( k_{\mu_\nu - 1} > k_{\mu_\nu - 1} \), the alcove \( w' A_0 \) does not lie closer to the origin than \( w A_0 \). \( \square \)

### 2.6 The bijection \( B_1 \)

We now have all the ingredients to describe the map \( B_1 \) and prove that it is a bijection. Let \( [w(1), \ldots, w(n)] \) be the base window of some \( w \in \mathcal{M}_r(\tilde{A}_{n-1}) \). For each \( k \in \mathbb{Z} \) we define the \( k \)-shift of \( w \), denoted by \( w[k] \), as the ordered collection of integers:

\[
  w[k] := (w(1) + k, \ldots, w(n) + k).
\]  

(2.11)

As will be shown below, there is a natural way to “expand” \( w[k] \) in order to obtain minimal length coset representatives in \( \tilde{A}_{2n-1}/A_{2n-1} \). To describe our expansion, let \( r_1 := (w(i) + k) \mod n \) be the base, \( \ell_i := \left\lfloor \frac{w(i) + k}{n} \right\rfloor \) be the level of each \( w(i) + k \) and rewrite (2.11) as:

\[
  w[k] = [r_1^{\ell_1}, \ldots, r_n^{\ell_n}].
\]  

(2.12)

Next, notice that \( \{r_1, \ldots, r_n\} = \{1, \ldots, n\} \). Indeed, condition (2.4) implies that, for each base window it is \( \{w(1) \mod n, \ldots, w(n) \mod n\} = \{1, \ldots, n\} \), hence also \( \{(w(1)+k) \mod n, \ldots, (w(n)+k) \mod n\} = \{1, \ldots, n\} \).


Let \( k \) mod \( n \) = \( \{1, \ldots, n\} \). This allows us to define the level vector of \( w [k] \) as we did in the case of base windows, regardless of the fact that the levels do not sum up to zero. Now, we are in a position to formulate the definition of our expansion.

**Definition 2.9.** Let \( w \) be an ordered collection of integers \([r_1^{a_1}, \ldots, r_n^{a_n}]\) such that \( \{r_1, \ldots, r_n\} = \{1, \ldots, n\} \). The antisymmetric expansion \( \alpha(w) \) of \( w \), is the sorted array \([\tilde{w}(1), \ldots, \tilde{w}(2n)]\), with elements:

\[
\{\tilde{w}(1) < \tilde{w}(2) < \ldots < \tilde{w}(2n)\} = \{r_i^{a_i}, (2n + 1 - r_i)^{-a_i} \text{ for } 1 \leq i \leq n\}.
\]

The next lemma shows that for each \( w \in M_+ (\overline{A}_{n-1}) \) and \( k \in \mathbb{Z} \), the antisymmetric expansion of its \( k \)-shift produces elements of \( M_+ (\overline{A}_{2n-1}) \) with balanced abacus.

**Lemma 2.10.** Let \( w \in M_+ (\overline{A}_n) \) with level vector \((\beta_1, \ldots, \beta_n)\). Then, for each \( k \in \mathbb{Z} \), the antisymmetric expansion \( \alpha(w[k]) \) of its \( k \)-shift, is a minimal length coset representative in \( M_+ (\overline{A}_{2n-1}) \) whose abacus diagram is balanced.

**Proof.** First, notice that if \( \{r_1, \ldots, r_n\} = \{1, \ldots, n\} \) then \( \{2n + 1 - r_1, \ldots, 2n + 1 - r_n, r_1, \ldots, r_n\} = \{1, \ldots, 2n\} \). Also, by the definition of the antisymmetric expansion, the levels in \( \alpha(w[k]) \) sum up to 0. Hence, both conditions in (2.4) are satisfied and thus \( \alpha(w[k]) \) is an element of \( \overline{A}_{2n-1} \). Finally, by construction, the level vector of \( w[k] \) is antisymmetric and thus, in view of Lemma 2.5, its abacus is balanced.

The first step towards Bij is the following proposition, which shows how, from any alcove in \( A_+ (A_{n-1}) \) and integer \( k \in \mathbb{Z} \) we can uniquely define an alcove in \( A_+ (C_n) \). The forward direction of the bijection follows naturally from the \( k \)-shift. The reverse, which extracts from a self-conjugate tableau of type \( A_{2n-1} \), a type \( A_{n-1} \) tableau appropriately shifted, is more complicated.

**Proposition 2.11.** The map \( \psi : A_+ (A_{n-1}) \times \mathbb{Z} \rightarrow A_+ (C_n) \) sending each pair \((w, k)\) to \( \alpha(w[k]) \) is a bijection. Its inverse map sends each \( \tilde{w} \) to \((w, k)\) where:

(i) \( k \) is the sum of the lower-half of the level vector \( \tilde{n}(\tilde{w}) = (\beta_1, \ldots, -\beta_n, \ldots, -\beta_1) \) of \( \tilde{w} \), and

(ii) \( w \) is the minimal length coset representative in \( \overline{A}_{n-1} \setminus A_{n-1} \) with level vector

\[
\tilde{n}(w) = (\beta_{n-r+1} + \ell + 1, \ldots, \beta_n + \ell + 1, \beta_1 + \ell, \ldots, \beta_{n-r} + \ell),
\]

where \( r = -k \) mod \( n \) and \( \ell = -k + r \).

**Proof.** Lemma 2.10 ensures that the map in the statement of the proposition is well defined. For the reverse, let \( w \in A_0 \) be an alcove in \( A_+ (C_n) \) with level vector \( \tilde{n}(\tilde{w}) = (\beta_1, \ldots, -\beta_n, \ldots, -\beta_1) \). Reversing the procedure of the antisymmetric expansion, we see that \( w[k] \) consists of the numbers \( \{1^{\beta_1}, \ldots, n^{\beta_n}\} \). In order to shift from \( w[k] \) to its base window \( w \), we have to subtract an integer \( k \) form each \( j^{\beta_j} \), so that the resulting set of numbers sum up to \( 1 + \cdots + n \). To determine
k, we set \( r := -k \mod n \) and \( \ell := \frac{r-k}{n} \), so that in the base-level notation \(-k = r\ell\) and we ask that:

\[
1 + \cdots + n = 1^{\beta_1} - k + \cdots + n^{\beta_n} - k
\]

\[
= (1^{\beta_1} + r\ell) + \cdots + (n^{\beta_n} + r\ell)
\]

\[
= (1 + r)^{\beta_1 + \ell} + (2 + r)^{\beta_2 + \ell} + \cdots + n^{\beta_n + \ell} + (n + r)^{\beta_n + \ell}
\]

\[
= (1 + r)^{\beta_1 + \ell} + (2 + r)^{\beta_2 + \ell} + \cdots + n^{\beta_n + \ell} + 1^{\beta_n + \ell + 1} + \cdots + r^{\beta_n + \ell + 1}.
\] (2.14)

The above equalities hold if the sum of the levels of the numbers in (2.14) is 0 i.e., if \((\beta_1 + \cdots + \beta_n) + n\ell + r = 0\). The last is equivalent to \( k = \beta_1 + \cdots + \beta_n \), which proves (i). It is also evident from (2.14) that the level vector of \( w \) is indeed the one described in item (ii) of the proposition.

In Figure 3 we illustrate two instances of the map \( \psi \) of Proposition 2.11.

[4^{-3}, 1^{-1}, 2^{2}, 3^{2}] \xrightarrow{\psi} [4^{-3}, 1^{-1}, 2^{2}, 3^{2}] \xrightarrow{\psi} [3^{-3}, 4^{-2}, 1^{2}, 2^{2}, 3^{2}]

[2^{-3}, 3^{-2}, 8^{-2}, 5^{-1}, 4^{1}, 1^{2}, 6^{2}, 7^{3}] \xrightarrow{\psi} [3^{-3}, 4^{-2}, 7^{-2}, 8^{-2}, 1^{2}, 2^{2}, 5^{2}, 6^{3}]

Figure 3: Two examples of the map \( \psi \) of Proposition 2.11.

In Figure 3 we illustrate two instances of the map \( \psi \) of Proposition 2.11. The tableau on the left corresponds to the 3-minimal alcove \( wA_0 \in A_+ (A_3) \) with \( w = [4^{-3}, 1^{-1}, 2^{2}, 3^{2}] \). The tableaux in the middle and right correspond to \( A_0 \) and \( A_0 \) respectively. Using the criterion for \( n \)-minimal alcoves, one can check that \( \psi(w, -2) \) is a 3-minimal alcove in \( A_+ (C_3) \) while \( \psi(w, -1) \) is not. It is therefore natural to inquire which integers \( k \) preserve \( m \)-minimality between the alcoves corresponding to \( w \) and \( \alpha(w_{[1]}) \). Since this is one of the main goals of this paper, we formulate it as a question.

**Question 1.** If \( wA_0 \) in an \( m \)-minimal alcove in \( A_+ (A_{n-1}) \), for which choices of \( k \in \mathbb{Z} \) does \( \alpha(w_{[1]}) \) correspond to an \( m \)-minimal alcove in \( A_+ (C_n) \)?

The answer is given in the next proposition.

**Proposition 2.12.** Let \( wA_0 \in A_+ (A_{n-1}) \) be an \( m \)-minimal alcove with \( \bar{n}(w) = (\beta_1, \ldots, \beta_n) \). If we write each integer \( k \) as \( k = r + n\ell \), where \( r \in \{0, 1, \ldots, n-1\} \) and \( \ell \in \mathbb{Z} \), then \( \alpha(w_{[1]}) \) corresponds to an \( m \)-minimal alcove in \( A_+ (C_n) \) if and only if one of the following conditions holds:

(i) \( r = 0 \) and \(-\left\lfloor \frac{m}{2} \right\rfloor - \beta_n \leq \ell \leq \left\lfloor \frac{m+1}{2} \right\rfloor - \beta_1 \), or
Proof. (i) Suppose first that \( r = 0 \). Then the \( k \)-shift of \( w \) has level vector
\[
\tilde{n}(w_{[k]}) = (\beta_1 + \ell, \ldots, \beta_n + \ell, -\beta_n - \ell, \ldots, -\beta_1 - \ell).
\]
Recalling that the level vector \( \tilde{n} \) of \( \alpha(w_{[k]}) \) is the antisymmetric expansion of \( \tilde{n}(w_{[k]}) \), we claim that it suffices to apply the criterion of Theorem \[2.7\] for the following two pairs:

First and last entry of \( \tilde{n} \): \((2n) - \beta_1 - \ell \) and \( 1\beta_1 + \ell \).

Two middle entries of \( \tilde{n} \): \( n\beta_n + \ell \) and \((n + 1) - \beta_n - \ell \).

Indeed, since all other pairs of consecutive integers in \( \tilde{n}(w_{[k]}) \) are shifted by \( \ell \), the differences to be checked are the same as those for \( \tilde{n}(w) \). These, however, satisfy Theorem \[2.7\] by the fact that \( wA_0 \) is \( m \)-minimal. Same applies for the upper half of \( \tilde{n} \) as well.

For the first pair we require that \( \beta_1 + \ell - (\beta_1 - \ell) - 1 \leq m \) or equivalently \( \ell \leq \frac{m+1}{2} - \beta_1 \), while for the second we require \(-\beta_n - \ell - (\beta_n + \ell) \leq m \) or equivalently \( \ell \geq -\frac{m}{2} - \beta_n \). Combining the above two inequalities and bearing in mind that \( \beta \) and \( \beta_n \) are shifted by \( \ell \), we arrive at the following range for \( \ell \)
\[
-\frac{m}{2} - \beta_n \leq \ell \leq \frac{m+1}{2} - \beta_1.
\]

(ii) Suppose now that \( r < n \). Then the lower half of the level vector \( \tilde{n} \) of \( \alpha(w_{[k]}) \) becomes
\[
(\beta_n + r + 1, \ldots, \beta_n + 1, \beta_1 + \ell, \beta_2 + \ell, \ldots, \beta_n - r + \ell).
\]
Arguing as before, it suffices to apply the criterion of Theorem \[2.7\] for the following two pairs:

First and last entry of \( \tilde{n} \): \((2n - m) - \beta_1 - \ell \) and \( 1\beta_1 + \ell \).

Two middle entries of \( \tilde{n} \): \( n\beta_n + \ell \) and \((n + 1) - \beta_n - \ell \).

For the first pair we require that \( \beta_n - r + 1 + 1 - (\beta_n - r - 1) - 1 \leq m \) or equivalently \( \ell \leq \beta_n - r + 1 + \frac{m}{2} \), while for the second we require \(-\beta_n + \ell - (\beta_n + r + \ell) \leq m \) or equivalently \( \ell \geq -\beta_n - \frac{m}{2} \). Combining the two inequalities, we deduce that the range of \( \ell \) in this case is
\[
-\beta_n - \frac{m}{2} \leq \ell \leq -\beta_n - m + 1 + \frac{m+1}{2}.
\]

For each \( m \)-minimal alcove \( wA_0 \) in \( A_r(A_{n-1}) \), we call the set \( \mathcal{X}_n(w) \) of integers \( k \) defined in Proposition \[2.12\] the \( m \)-admissible set of \( w \). Although the numbers in \( \mathcal{X}_w(m) \) seem to be rather random, they hold the answer to our bijection. They are as many as the should; each \( m \)-admissible set has \( mn + 1 \) elements. We prove this in the next proposition.

Proposition 2.13. For each \( m \)-minimal alcove \( wA_0 \) in \( A_r(A_{n-1}) \) there exist exactly \( mn + 1 \) distinct integer values \( k \in \mathbb{Z} \) such that \( \alpha(w_{[k]}) \) corresponds to an \( m \)-minimal alcove in \( A_r(C_n) \).

Proof. For \( 0 \leq r \leq n - 1 \), let \( c_r \) be the number of all possible integers \( k \) of the form \( r + n\ell \), \( \ell \in \mathbb{Z} \), for which \( \alpha(w_{[k]}) \) corresponds to an \( m \)-minimal alcove in \( A_r(C_n) \). Proposition \[2.12\] implies that
\[
c_r = \begin{cases} 
m + \beta_n - \beta_1 + 1, & \text{if } r = 0, \\
m + \beta_n - r - \beta_{n-r+1}, & \text{if } r \leq n - 1.
\end{cases}
\]
(2.15)

Summing over all \( r \) we have
\[
\sum_{r=0}^{n-1} c_r = m + \beta_n - \beta_1 + 1 + \sum_{r=1}^{n-1} (m + \beta_n - r - \beta_{n-r+1})
\]
\[
= m + \beta_n - \beta_1 + 1 + \sum_{r=1}^{n-1} (\beta_n - r - \beta_{n-r+1}) = mn + 1,
\]
which completes our proof. \( \square \)
Let $\mathcal{I}_m : \mathcal{K}_m(w) = \{k_1 < \cdots < k_{mn+1}\} \mapsto \{1, \ldots, mn+1\}$ be the bijection which sends each $k_i \in \mathcal{K}_m(w)$ to its index, when the elements of $\mathcal{K}_m(w)$ are in increasing order. Propositions 2.11 and 2.12 imply that, if we restrict the second argument of $\psi$ to $\mathcal{K}_m(m) \subseteq \mathbb{Z}$, $\psi$ maps each $m$-minimal alcove in $\mathcal{A}_+(A_n-1)$ to an $m$-minimal alcove in $\mathcal{A}_+(C_n)$. Identifying the set $\mathcal{K}_m(w)$ with that of its indices and bearing in mind that $m$-minimal alcoves in $\mathcal{A}_+(A_n-1)$ and $\mathcal{A}_+(C_n)$ are in bijection with dominant regions in $\text{Shi}^m(A_n-1)$ and $\text{Shi}^m(C_n)$ respectively, and we arrive to our main theorem.

**Theorem 2.14.** Let $(\mathcal{R}, i) \in \mathcal{R}_+^m(A_n-1) \times \{1, \ldots, mn+1\}$. Let $wA_0$ be the $m$-minimal alcove of the region $\mathcal{R}$ and $k_i$ be the $i$-th element of the $m$-admissible set $\mathcal{K}_m(w)$ of $w$. The map $B_{j_1}$ which sends each pair $(\mathcal{R}, i) \in \mathcal{R}_+^m(A_n-1) \times \{1, \ldots, mn+1\}$ to the region $\mathcal{R}' \in \mathcal{R}_+^m(C_n)$ whose $m$-minimal alcove corresponds to $w_{[i,]}$, is a bijection.

The map $B_{j_1}$ of Theorem 2.14 is a combination of previously defined bijections. Actually, if we ignore the first and last step that biject each region to its $m$-minimal alcove, the forward direction of $B_{j_1}$ can be stated more clearly: consider an $m$-minimal alcove $wA_0$ in $\mathcal{A}_+(A_n-1)$, compute the $m$-admissible set $\mathcal{K}_m(w) = \{k_1, \ldots, k_{mn+1}\}$ and send each $(w, i)$ to the antisymmetric expansion of its $k_i$-shift $w$. Identifying the set $\mathcal{K}_m(w)$ with that of its indices and bearing in mind that $m$-minimal alcoves in $\mathcal{A}_+(A_n-1)$ and $\mathcal{A}_+(C_n)$ are in bijection with dominant regions in $\text{Shi}^m(A_n-1)$ and $\text{Shi}^m(C_n)$ respectively, and we arrive to our main theorem.

In order to help the reader clarify the steps, we present an explicit example in Figure 4. The first and last step of the figure correspond to the way we associate each dominant region in $\text{Shi}^m(\Phi)$ to its $m$-minimal alcove. More precisely, each region is encoded by its region Shi tableau, which is a tableau defined in a way completely analogous to that of an alcove. Since this correspondence is not indispensable in our bijections, we describe it in Appendix A.

![Diagram](image)

**Figure 4:** The arrows indicate the steps for the inverse of the map $B_{j_1}$, each with reference to the required proposition/lemma.
Theorem 2.14 implies that ignoring the second argument of the map $\text{Bj}^{-1}$ we get a surjection.

**Corollary 2.15.** The map $\phi : R^m_+ (C_n) \to R^m_+ (A_{n-1})$ defined by $\phi(R) = \text{pr}_1(\text{Bj}^{-1}(R))$ is a surjection.

The map $\phi$ partitions $R^m_+ (C_n)$ into sets of regions each having the same image under $\phi$. Is there a way to geometrically understand this surjection? Since regions of $R^m_+ (A_{2n-1})$ are viewed as regions in $R^m_+ (A_{2n-1})$, is it reasonable to ask if $\phi$ corresponds to some affine projection from $R^{2n}$ to $R^n$. In Figure 5 we illustrate a simple example of this surjection.

![Figure 5: The arrangement $\text{Shi}^2(C_2)$, depicted above, has 15 dominant regions. The arrangement $\text{Shi}^2(A_1)$ has 3 dominant regions, with Shi tableau $[0, 1]$ and $[2]$. In view of Corollary 2.15 the regions in $\text{Shi}^2(C_2)$ are partitioned into three equinumerous sets, having the same image under $\phi$.](image)

We conclude this section by discussing the problems we encounter when we try to formulate an analogue of Theorem 2.14 for the type $B$ case. Without delving into details, we mention that Shi tableaux of type $B_n$ coincide with self-conjugate Shi tableaux of type $A_{2n}$ whose main diagonal is empty [19]. Since the map $T$ in Proposition 2.2 cannot be applied unless all Shi conditions are known, we seek a unique way to determine the empty entries of a type $B_n$ Shi tableau, so that Shi conditions are preserved. With this in mind, there is a natural way to go from the Shi tableau $T'$ of an alcove $w' A_0 \in A_+ (C_n)$ to the tableau $T$ of an alcove $w A_0 \in A_+ (B_n)$: if $\delta_1, \ldots, \delta_n$ are the entries of the main diagonal of $T'$, insert an $n$-th column and an $n$-th row to $T'$ with entries $\lfloor \delta_i - 1 \rfloor$ and then delete all $\delta_i$’s from the main diagonal (see Figure 6). It is not hard to see that the inserted entries preserve the Shi conditions on the new tableau $T$. In terms of abacus diagrams, the above procedure corresponds to inserting $n^0$ in the (central entry of the) diagram of $w'$ and replacing each $k^k$ with $k > n$ by $(k + 1)^k$. Two examples of this map are shown in Figure 6.

Unfortunately, the map described above in not a bijection. As one can see in Figure 6, two different tableaux $T'$ of alcoves in $A_+ (C_3)$ map to the same tableau in $A_+ (B_3)$. Even our hope of it being a bijection when restricted to $m$-minimal alcoves is dissolved by the same example; both type C tableau, which are 3-minimal, map on the same type B tableau. Thus, our wish to associate each $m$-minimal alcove in $A_+^m (C_n)$ with one such alcove in $A_+^m (B_n)$, cannot be worked out.
Maybe, the only approach to find the desired bijection, is to use the formal definition of \( \tilde{B}_n \) as the subgroup of even permutations of \( \tilde{C}_n \) \cite[Section 2]{2}, \cite[Section 8.5]{7}. In this case though, it is not evident how one could exploit evenness to produce self-conjugate Shi tableau with empty main diagonal.

\[
\begin{array}{c|c|c}
3 & 2 & 0 \\
2 & 1 & \quad \\
0 & \quad & \\
\end{array} \quad \xrightarrow{X} \quad \begin{array}{c|c|c}
3 & 2 & 1 & 0 \\
2 & 1 & 0 & \\
1 & 0 & \\
0 & \\
\end{array} \quad \begin{array}{c|c|c}
2 & 2 & 0 \\
2 & 1 & \\
0 & \\
0 & \quad \\
\end{array} \quad \xrightarrow{X} \quad \begin{array}{c|c|c}
3 & 2 & 1 & 0 \\
2 & 1 & 0 & \\
1 & 0 & \\
0 & \\
\end{array}
\]

\[w' = [4^{-2}, 3^{-1}, 2^1, 1^2]\]
\[w = [5^{-2}, 4^{-1}, 3^0, 2^1, 1^2]\]
\[w' = [1^{-1}, 3^{-1}, 2^1, 4^1]\]
\[w = [1^{-1}, 4^{-1}, 3^0, 2^1, 5^1]\]

Figure 6:

3 Lattice paths and \( B_{j_2} \)

Given \( n, m \geq 1 \), we denote by \( \mathcal{L}_n^m \) the set of all \( N - E \) lattice paths from \((0,0)\) to \((n, mn)\) and by \( \mathcal{D}_n^m \) the set of all \( m \)-Dyck paths of height \( n \). As we already mentioned in the introduction, the above sets are enumerated by Catalan numbers i.e., \(|\mathcal{D}_n^m| = N^m(A_{n-1})\) and \(|\mathcal{L}_n^m| = (\binom{mn+n}{n}) = N^m(C_n)\). Thus, the relation \(|\mathcal{L}_n^m| = |\mathcal{D}_n^m| = (mn+1)\) can be viewed as another instance of \([1,2]\) to be explained bijectively. In Theorem \([8,3]\) we provide a natural, but rather hidden, bijection between the sets \( \mathcal{L}_n^m \) and \( \mathcal{D}_n^m \times \{0, \ldots, mn\} \).

Before continuing, we introduce definitions and notation. Every path \( \mathcal{P} \in \mathcal{L}_n^m \) can uniquely be determined by its step sequence \((s_1, \ldots, s_n)\), where \( s_i \) denotes the number of east steps occurring before the \( i \)-th north step (see Figure \([7]\)). Thus, we can write \( \mathcal{P} = (s_1, \ldots, s_n) \) as well. Notice that \( 0 \leq s_1 \leq \cdots \leq s_n \leq mn \) for all paths in \( \mathcal{L}_n^m \), while \( \mathcal{P} \in \mathcal{D}_n^m \) if and only if \( s_i \leq m(i-1) \) for every \( 1 \leq i \leq n \).

Definition 3.1. For each \( \mathcal{P} = (s_1, \ldots, s_n) \in \mathcal{L}_n^m \) and \( 0 \leq i \leq n-1 \), we define its \( i \)-th permutation \( d_i(\mathcal{P}) \) to be the path in \( \mathcal{L}_n^m \) with step sequence \((0, s_{i+2} - s_{i+1}, \ldots, s_n - s_{i+1}, \bar{s}_1 - s_i, \bar{s}_2 - s_{i+1}, \ldots, \bar{s}_i - s_{i+1})\), where \( \bar{s}_j = s_j + mn + 1 \).

Pictorially, one may think of \( d_i(\mathcal{P}) \) as follows. For each \( \mathcal{P} \in \mathcal{L}_n^m \), let \( \mathcal{P}' \) be the path with step sequence \((s_1, s_2, \ldots, s_n, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n)\) i.e., \( \mathcal{P}' \) is the concatenation of \( \mathcal{P} \), an east step \( E \) and one more copy of \( \mathcal{P} \). Then, \( d_i(\mathcal{P}) \) corresponds to the subpath of \( \mathcal{P}' \) starting with its \( i \)-th and finishing before its \((n+i)\)-th north step. Although in general \( d_i(\mathcal{P}) \) is not an \( m \)-Dyck path, there exists a unique \( i_0 \) for which this is true. This is the content of the next Proposition, which is motivated from \([10\, \text{Chapter } 1.4]\).

Proposition 3.2. For each \( \mathcal{P} \in \mathcal{L}_n^m \) there exists a unique index \( 0 \leq i_0 \leq n - 1 \), for which \( d_{i_0}(\mathcal{P}) \in \mathcal{D}_n^m \).

Proof. For each \( \mathcal{P} \in \mathcal{L}_n^m \) let \( \mathcal{P}' \) be the path with step sequence \((s_1, s_2, \ldots, s_n, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n)\). Set \( O = (0, 0) \), \( Q_1 = (mn, n) \), \( Q_2 = (mn + 1, n) \) and \( R = (2mn + 1, 2n) \), so that \( \mathcal{P} \) is the subpath of \( \mathcal{P}' \) from \( O \) to \( Q_1 \) and \( \mathcal{P}' \), its shifted by \((mn + 1, n)\) copy, is the subpath from \( Q_2 \) to \( R \). Next, among all lines of slope \( \frac{1}{m} \), consider the unique \( \varepsilon_{k_0} : y = \frac{1}{m}x + k_0 \) tangent-to and containing \( \mathcal{P} \) in its closed upper half-plane \( \mathcal{H}_{k_0}^+ \). For every \( 0 \leq i \leq n - 1 \) set \( A_i = (s_i, i-1) \) and
Theorem 3.3. The forward direction of the map \( \phi \), we consider an

The reader is invited to follow the steps of the proof of Proposition 3.2 in Figure 7.

Figure 7: The path \( \mathcal{P} = \text{ENEEENEEEEEENENEENEN} \) from \( \mathcal{P} \) to \( Q_1 \) has step sequence (1, 4, 11, 12, 14) and belongs to \( \mathcal{L}^0_3 \). The path \( \mathcal{P}' \) from \( \mathcal{P} \) to \( R \), is the concatenation of \( \mathcal{P} \), \( E \) and \( \mathcal{P} \). Since \( m = 3 \), we consider the line \( \varepsilon \) with slope 1/3, tangent-to and containing \( \mathcal{P} \) in its upper half-space. Its first intersection point with \( \mathcal{P} \) is \( A_{i_0} \). In the present case it is \( A_3 \). Thus, we deduce that \( i_0 = 3 \) and \( d_{i_0}(\mathcal{P}) = (0, 1, 3, 6, 9) \).

The next theorem, which follows from Proposition 3.2 describes (the inverse of) \( \mathcal{B}_j_2 \).

Theorem 3.3. The map \( h : \mathcal{L}^m_3 \to \mathcal{D}^m_3 \times \{0, \ldots, mn\} \) with \( h(\mathcal{P}) = (d_{i_0}(\mathcal{P}), s_{i_0}) \) is a bijection.

Proof. The forward direction of the map \( h \) is immediate from Proposition 3.2. To prove the reverse, we consider an \( m \)-Dyck path \( \mathcal{P} = (s_1, \ldots, s_n) \) and an integer \( 0 \leq k \leq mn \). Let \( 1 \leq j_0 \leq n \) be the unique index for which \( s_j + k \leq mn \) for all \( j \leq j_0 \) and \( s_j > mn \) otherwise. We set

\[
\overline{\mathcal{P}} = (\overline{s}_1, \ldots, \overline{s}_n) := (s_{j_0+1} + k - mn - 1, \ldots, s_n + k - mn - 1, s_1 + k, \ldots, s_{j_0} + k)
\]

and show that \( \overline{\mathcal{P}} \in \mathcal{L}^m_3 \) (see Figure 8 for a pictorial description of \( \overline{\mathcal{P}} \)). By the choice of \( j_0 \) and \( k \) and recalling that \( s_i \leq m(n - i) \), one can easily deduce that \( 0 \leq \overline{s}_i \leq mn \) for all \( i \). Thus, for showing that \( \mathcal{P} \in \mathcal{L}^m_3 \), it remains to prove that \( \overline{s}_1 \leq \cdots \leq \overline{s}_n \). If \( j_0 = n \) the claim is immediate (since then \( \overline{\mathcal{P}} = \mathcal{P} \)), while if \( j_0 < n \) we only need to verify that \( s_{n} + k - mn - 1 \leq s_{1} + k \). Bearing
in mind that $s_1 = 0$ for all Dyck paths, the above simplifies to showing that $s_n \leq mn + 1$, which is true for all $P \in D_n^m$. We leave the reader to verify that the map sending each pair $(P, k)$ to $\overline{P}$ is indeed the reverse of $h$.

Figure 8: Let $P = (0, 1, 3, 6, 9) \in D_3^5$ and $k = 11$. Consider the path $PQ$ we obtain when shifting $P$ by 11 east steps, placed at the origin of a $15 \times 5$ grid (left). Let $AB$ be the first (horizontal) step of the path that exceeds the grid. The path $\overline{P}$ is obtained by switching the subpaths $PA$ and $BQ$, so that the latter begins at the origin and the former ends at the last point, of a $15 \times 5$ grid (right).

The following theorem completes the FKT$_1$ direction of the diagram of Section I.

**Theorem 3.4.** [9, Theorem 3.1] There exists an explicit bijection between dominant regions in $R_m^+ (A_{n-1})$ and $m$-Dyck paths in $D_m^m$.

Without delving into details, the idea of the bijection is as follows. We identify each dominant region in $R_m^+ (A_{n-1})$ with its Shi tableau. For each $m$-Dyck path, we ignore its first step(5) and we rewrite its step sequence as $\lambda_{n-1} \leq \cdots \leq \lambda_1$ with $0 \leq \lambda_i \leq (n - 1 - i)m$. Theorem 3.4 shows that there exists a unique way to construct a Shi tableau of a dominant region in $R_m^+ (A_{n-1})$, whose $i$-th row has coordinates which sum up to $\lambda_i$, for all $1 \leq i \leq n - 1$ (see Figure 9).

Figure 9: For $m = 3$, the bijection of Theorem 3.4 maps the Dyck path $(1, 3, 6, 9)$ to the region Shi tableau depicted on the right.

**Acknowledgments.**

The authors are grateful to Eli Bagno for helpful discussions and to Philippe Nadeau for bringing [16] to their attention. The first author was supported by a “Back-to-Research
Throughout the paper, we use the fact that each dominant region in \( \text{Shi}^m(\Phi) \) is represented by its \( m \)-minimal alcove, and build all our arguments upon this. In this section we show how

A Shi tableaux for dominant regions

Throughout the paper, we use the fact that each dominant region in \( \text{Shi}^m(\Phi) \) is represented by its \( m \)-minimal alcove, and build all our arguments upon this. In this section we show how
we encode each dominant region $\mathcal{R}$ in a tableau $T_\mathcal{R}$, and how we retrieve the Shi tableau of
its $m$-minimal alcove from $T_\mathcal{R}$.

The Shi tableau of a dominant region $\mathcal{R}$ in $\text{Shi}^m(\Phi)$ is the set $T_\mathcal{R} = \{r_\alpha, \alpha \in \Phi^+\} \subset \mathbb{N}$ of
coordinates, where $r_\alpha$ counts the number of integer translates $H_{\alpha,k}$ of $H_{\alpha,0}$, that separate $\mathcal{R}$
from the origin. Since, by definition of $\text{Shi}^m(\Phi)$, the integer $k$ is bounded by $m$, we deduce
that $0 \leq r_\alpha \leq m$. As before, the coordinates $r_\alpha$ are arranged in a tableau according to the root
system. The coordinates of $T_\mathcal{R}$ yield the face defining inequalities for the region $\mathcal{R}$. More
precisely, for each $\alpha \in \Phi^+$ it is $r_\alpha = r$ if and only if, for all $x \in \mathcal{R}$:

$$
\begin{align*}
  r < \langle x, \alpha \rangle < r + 1 & \quad \text{if } r < m, \\
  m < \langle x, \alpha \rangle & \quad \text{if } r = m.
\end{align*}
$$

(A.1)

The above inequalities imply the the following Shi conditions on a region: for $\alpha, \beta, \gamma \in \Phi^+$
with $\alpha + \beta = \gamma$, it holds:

$$
\begin{align*}
  r_\alpha &= \begin{cases} 
    r_\beta + r_\gamma + \delta_{\beta,\gamma} & \text{if } r_\beta + r_\gamma < m, \\
    m & \text{otherwise},
  \end{cases}
\end{align*}
$$

(A.2)

where $\delta_{\beta,\gamma} \in \{0, 1\}$.

As we mentioned in Section 2.1 each dominant region in $\text{Shi}^m(\Phi)$ is uniquely represented
by its $m$-minimal alcove. One can switch from the tableau of the region $\mathcal{R}$ to that of its
$m$-minimal alcove $A_\mathcal{R}$ and vice versa, as indicated by the following lemma (see also Figure 2).

**Lemma A.1.** [1 Section 3] Let $\mathcal{R}$ be a dominant region in $\text{Shi}^m(\Phi)$ with $m$-minimal alcove $A_\mathcal{R}$. Let also
$T_\mathcal{R} = \{r_\alpha : \alpha \in \Phi^+\}$ and $T = \{k_\alpha : \alpha \in \Phi^+\}$ be the Shi tableau of $\mathcal{R}$ and $A_\mathcal{R}$
respectively. Then, for each $\alpha \in \Phi^+$ the following relations hold:

(i) $r_\alpha = \min\{m, k_\alpha\}$ and

(ii) $k_\alpha = \max\{k_\beta + k_\gamma : \alpha = \beta + \gamma \text{ with } \beta, \gamma \in \Phi^+\}$.

The following corollary, which is a direct consequence of Lemma A.1, is used in the proof
of Theorem 2.8. We could alternatively have used arguments based on the fact that a region
as well as its $m$-minimal alcove have the same set of separating walls. We, however, add
this appendix in order to be able to present explicit examples associating regions to their
$m$-minimal alcove (see Figure 3 and Appendix B).

**Corollary A.2.** The dominant alcove $A$ with Shi tableau $T = \{k_{ij} : 1 \leq i \leq j \leq n - 1\}$ lies in
the dominant region $\mathcal{R}$ with Shi tableau $T_\mathcal{R} = \{r_{ij} = \min\{k_{ij}, m\} : 1 \leq i \leq j \leq n - 1\}$.

**B  Another example of $B_{j_1}$**

In this appendix we present one more example of the map $B_{j_1}$. More precisely, ignoring
the first and last step of $B_{j_1}$ (that biject each region to its $m$-minimal alcove), we consider
an $m$-minimal alcove $wA_0$ in $A_+(A_{n-1})$, we compute its $m$-admissible set $\mathcal{X}_w(m)$ and we
evaluate the Shi tableaux $T_i$ of the $m$-minimal alcoves in $A_+(C_n)$ corresponding to $w_{[k_i]}$, for
all $k_i \in \mathcal{X}_w(m)$. Our goal is to illustrate how a copy of the Shi tableau $T$ resides in each of the
$mn + 1$ tableaux $T_i$.

Let us consider the 2-minimal alcove $wA_0 \in A_+(A_3)$ with $w = [3^{-1}, 1^0, 2^0, 4^1]$, whose Shi
tableau $T$ is
Using Proposition 2.12, we compute the 2-admissible set \( \mathcal{K}_2(w) = \{-8, -4, -2, -1, 0, 2, 3, 4, 6\} \) of \( w \). For each \( k_i \in \mathcal{K}_2(w) \) we apply the antisymmetric expansion of the \( k_i \)-shift of \( w \) (i.e., the map of Proposition 2.11), and we obtain the tableaux listed in Figure 10. As illustrated in blue, a copy of the tableau \( T \) as well as one of its conjugate \( T' \), occupy certain rows and columns of each \( T_i \). Moreover, the values of \( k \in \mathcal{K}_2(w) \) have an interesting feature: as the \( k_i \)'s grow, the tableau \( T \) shifts gradually from the top rightmost entries of \( T_i \) to the bottom ones. So, for example, for the smallest value \( k_1 \), the tableau \( T \) occupies the top rightmost entries of \( T_1 \) (and \( T' \) occupies the bottom ones), while for the greater value \( k_9 \), the tableau \( T \) occupies the bottom entries of \( T_9 \) (and \( T' \) the top rightmost ones). The tableaux computed above correspond to 2-minimal alcoves of different regions in \( \mathcal{R}_2^+(C_4) \). Indeed, in view of Corollary A.2 if, in each tableau, we replace every entry \( k_{ij} \) by \( \min\{k_{ij}, 2\} \), we have no coincidences among the resulting tableaux, which verifies that the regions they represent are different.
\[
T_1 = \begin{bmatrix}
6 & 5 & 3 & 4 & 2 & 0 & 0 \\
5 & 4 & 4 & 3 & 1 & 0 \\
5 & 4 & 4 & 3 & 1 \\
4 & 3 & 3 & 2 \\
2 & 1 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[k_1 = -8 = 0^{-2}
\]
\[
\begin{bmatrix} 3^{-1} \end{bmatrix} \begin{bmatrix} 1^0 \end{bmatrix} \begin{bmatrix} 2^0 \end{bmatrix} \begin{bmatrix} 4^1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3^{-1} \end{bmatrix} \begin{bmatrix} 1^{-2} \end{bmatrix} \begin{bmatrix} 2^{-2} \end{bmatrix} \begin{bmatrix} 4^{-1} \end{bmatrix}
\]

\[
T_2 = \begin{bmatrix}
4 & 3 & 3 & 2 & 2 & 0 & 0 \\
3 & 2 & 2 & 1 & 1 & 0 & 0 \\
3 & 2 & 2 & 1 & 1 \\
2 & 1 & 1 & 0 \\
2 & 1 & 1 \\
0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[k_2 = -4 = 0^{-1}
\]
\[
\begin{bmatrix} 3^{-1} \end{bmatrix} \begin{bmatrix} 1^0 \end{bmatrix} \begin{bmatrix} 2^0 \end{bmatrix} \begin{bmatrix} 4^1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3^{-1} \end{bmatrix} \begin{bmatrix} 1^{-1} \end{bmatrix} \begin{bmatrix} 2^{-1} \end{bmatrix} \begin{bmatrix} 4^0 \end{bmatrix}
\]

\[
T_3 = \begin{bmatrix}
2 & 2 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & 1 & 0 & 0 & 0 \\
2 & 2 & 2 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 \\
2 & 1 & 1 \\
0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[k_3 = -2 = 2^{-1}
\]
\[
\begin{bmatrix} 3^{-1} \end{bmatrix} \begin{bmatrix} 1^0 \end{bmatrix} \begin{bmatrix} 2^0 \end{bmatrix} \begin{bmatrix} 4^1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1^{-1} \end{bmatrix} \begin{bmatrix} 3^{-1} \end{bmatrix} \begin{bmatrix} 4^{-1} \end{bmatrix} \begin{bmatrix} 2^1 \end{bmatrix}
\]

\[
T_4 = \begin{bmatrix}
2 & 2 & 2 & 1 & 0 & 0 & 0 \\
2 & 2 & 1 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
T_5 = \begin{bmatrix}
3 & 2 & 1 & 1 & 1 & 1 \\
2 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
T_6 = \begin{bmatrix}
3 & 2 & 1 & 1 & 1 & 1 \\
2 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[k_4 = -1 = 3^{-1}
\]
\[
\begin{bmatrix} 3^{-1} \end{bmatrix} \begin{bmatrix} 1^0 \end{bmatrix} \begin{bmatrix} 2^0 \end{bmatrix} \begin{bmatrix} 4^1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2^{-1} \end{bmatrix} \begin{bmatrix} 4^{-1} \end{bmatrix} \begin{bmatrix} 1^0 \end{bmatrix} \begin{bmatrix} 3^1 \end{bmatrix}
\]

\[
T_7 = \begin{bmatrix}
3 & 2 & 2 & 1 & 1 & 1 & 1 \\
2 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]
\[k_7 = 3 = 3^0
\]
\[
\begin{bmatrix} 3^{-1} \end{bmatrix} \begin{bmatrix} 1^0 \end{bmatrix} \begin{bmatrix} 2^0 \end{bmatrix} \begin{bmatrix} 4^1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2^0 \end{bmatrix} \begin{bmatrix} 4^0 \end{bmatrix} \begin{bmatrix} 1^1 \end{bmatrix} \begin{bmatrix} 3^2 \end{bmatrix}
\]

\[
T_8 = \begin{bmatrix}
3 & 2 & 2 & 1 & 1 & 1 \\
2 & 1 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
2 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]
\[k_8 = 4 = 0^1
\]
\[
\begin{bmatrix} 3^{-1} \end{bmatrix} \begin{bmatrix} 1^1 \end{bmatrix} \begin{bmatrix} 2^1 \end{bmatrix} \begin{bmatrix} 4^1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3^0 \end{bmatrix} \begin{bmatrix} 1^1 \end{bmatrix} \begin{bmatrix} 2^1 \end{bmatrix} \begin{bmatrix} 4^2 \end{bmatrix}
\]

\[
T_9 = \begin{bmatrix}
3 & 2 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 0 \\
2 & 0 \\
2 & 0 \\
1 & 0
\end{bmatrix}
\]
\[k_9 = 6 = 2^1
\]
\[
\begin{bmatrix} 3^{-1} \end{bmatrix} \begin{bmatrix} 1^0 \end{bmatrix} \begin{bmatrix} 2^0 \end{bmatrix} \begin{bmatrix} 4^1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1^1 \end{bmatrix} \begin{bmatrix} 3^1 \end{bmatrix} \begin{bmatrix} 4^1 \end{bmatrix} \begin{bmatrix} 2^2 \end{bmatrix}
\]

Figure 10:

\[
T_7 = \begin{bmatrix}
7 & 2 & 5 & 1 & 2^0 & 4^0 & 4^0 & 6^0 & 8^0 & 1^1 & 3^2 \\
6 & 5 & 3 & 1 & 2^0 & 4^0 & 4^0 & 6^0 & 8^0 & 1^1 & 3^2 \\
5 & 4 & 3 & 1 \\
4 & 3 & 2 \\
2 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

\[
T_8 = \begin{bmatrix}
8 & 2 & 5 & 1 & 6^1 & 3^0 & 7^0 & 1^1 & 2^1 & 4^2 \\
7 & 2 & 5 & 1 & 6^1 & 3^0 & 7^0 & 1^1 & 2^1 & 4^2 \\
6 & 5 & 3 & 1 & 2^0 & 4^0 & 4^0 & 6^0 & 8^0 & 1^1 & 3^2 \\
5 & 4 & 3 & 1 \\
4 & 3 & 2 \\
2 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

\[
T_9 = \begin{bmatrix}
6 & 5 & 7 & 1 & 8^1 & 1^1 & 3^1 & 4^1 & 2^1 \\
5 & 3 & 2 & 1 & 3^1 & 4^1 & 2^1 \\
4 & 3 & 1 \\
4 & 3 & 2 \\
3 & 2 & 1 \\
2 & 1 \\
1 & 0
\end{bmatrix}
\]