An example for nonequivalence of symplectic capacities

Ursula Hamenstädт *
Mathematisches Institut der Universität Bonn
Beringstraße 1, D-53115 Bonn, Germany
e-mail: ursula@math.uni-bonn.de

Abstract

We construct an open bounded star-shaped set \( \Omega \subset \mathbb{R}^4 \) whose cylindrical capacity is strictly bigger than its proper displacement energy.  

1 Introduction

Consider the standard \( 2n \)-dimensional euclidean space \( \mathbb{R}^{2n} \) equipped with the euclidean symplectic form \( \omega_0 = \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i} \). In this paper we are interested in symplectic invariants of nonempty open subsets of \( (\mathbb{R}^{2n}, \omega_0) \). One example of such an invariant is a relative or nonintrinsic capacity [MS] which associates to every open subset \( \Omega \) of \( \mathbb{R}^{2n} \) a number \( c(\Omega) \in [0, \infty] \). This number \( c(\Omega) \) measures the symplectic size of \( \Omega \) in such a way that the following three properties hold.

A1 Monotonicity: \( c(\Omega) \leq c(D) \) if there is a global symplectomorphism of \( \mathbb{R}^{2n} \) which maps \( \Omega \) into \( D \).

A2 Conformality: \( c(a\Omega) = a^2 c(\Omega) \) for all \( a > 0 \).

A3 Nontriviality: \( c(B^{2n}(1)) = 1 = c(Z^{2n}(1)) \) for the open normalized ball \( B^{2n}(1) \) of radius \( 1/\pi \) and the open symplectic cylinder \( Z^{2n}(1) = B^{2}(1) \times \mathbb{R}^{2n-2} \) in the standard space \( (\mathbb{R}^{2n}, \omega_0) \).

Here we use coordinates \( (x_1, \ldots, x_{2n}) \) in \( \mathbb{R}^{2n} \) and we write \( B^{2n}(r) = \{ x \in \mathbb{R}^{2n} \mid |x|^2 < r/\pi \} \) and \( Z^{2n}(r) = B^{2}(r) \times \mathbb{R}^{2n-2} = \{ x \in \mathbb{R}^{2n} \mid x_1^2 + x_2^2 < r/\pi \} \) for the ball and cylinder of capacity \( r > 0 \) in \( \mathbb{R}^{2n} \).

The celebrated non-squeezing lemma of Gromov [G] shows that for \( r > 1 \) the ball \( B^{2n}(r) \) does not admit a symplectic embedding into the cylinder \( Z^{2n}(1) \). This implies that relative capacities do exist, and in fact there are many ways...

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to define them. The resulting invariants do not coincide in general. We will consider the following four examples of such relative capacities.

The Gromov width assigns to an open set \( \Omega \subset \mathbb{R}^{2n} \) the supremum \( c_0(\Omega) \) of all numbers \( r > 0 \) such that there is a symplectic embedding of the ball \( B^{2n}(r) \) into \( \Omega \). By monotonicity, the Gromov width is the smallest capacity which means that if \( c' \) is any relative capacity, then \( c_0(\Omega) \leq c'(\Omega) \) for every open set \( \Omega \subset \mathbb{R}^{2n} \).

Let \( \mathcal{O} \) be the family of nonempty open bounded subsets of \( \mathbb{R}^{2n} \). Our second example is the cylindrical capacity which associates to \( \Omega \in \mathcal{O} \) the infimum \( c_p(\Omega) \) of all numbers \( r > 0 \) for which there is a symplectomorphism of \( \mathbb{R}^{2n} \) which maps \( \Omega \) into the cylinder \( \mathbb{R}^{2n}(r) \) [P]. If \( \Omega \subset \mathbb{R}^{2n} \) is unbounded then we define \( c_p(\Omega) = \sup \{ c_p(\Omega') \mid \Omega' \subset \Omega \} \). By monotonicity, the cylindrical capacity is the biggest relative capacity which means that if \( c' \) is any relative capacity, then \( c'(\Omega) \leq c_p(\Omega) \) for every \( \Omega \in \mathcal{O} \).

Third the displacement energy is defined as follows. Recall that a compactly supported smooth time dependent function \( H(t, x) \) on \([0, 1] \times \mathbb{R}^{2n} \) induces a time-dependent Hamiltonian flow on \( \mathbb{R}^{2n} \). Its time-one map \( \varphi \) is then a symplectomorphism of \( \mathbb{R}^{2n} \). The group \( \mathcal{D} \) of compactly supported symplectomorphisms obtained in this way is called the group of compactly supported Hamiltonians [HZ].

The Hofer-norm on the group \( \mathcal{D} \) assigns to \( \varphi \in \mathcal{D} \) the value

\[
\| \varphi \| = \inf_H \left( \sup_{t \in [0,1]} \left( \sup_{x \in \mathbb{R}^{2n}} H(t, x) - \inf_{x \in \mathbb{R}^{2n}} H(t, x) \right) \right)
\]

where \( H \) ranges over the set of all compactly supported time dependent functions whose Hamiltonian flows induce \( \varphi \) as their time-one map. The Hofer-norm \( \| \cdot \| \) induces a bi-invariant distance function \( d \) on the group \( \mathcal{D} \) by defining \( d(\varphi, \psi) = \| \varphi \circ \psi^{-1} \| \), in particular we have \( \| \varphi \| > 0 \) for \( \varphi \neq 1d \) [HZ].

For a bounded set \( A \subset \mathbb{R}^{2n} \) we define the displacement energy \( d(A) \) to be the infimum of the Hofer norms \( \| \varphi \| \) of all those \( \varphi \in \mathcal{D} \) which displace \( A \), i.e. for which we have \( \varphi(A) \cap A = \emptyset \). If \( A \subset \mathbb{R}^{2n} \) is unbounded then we define \( d(A) = \sup \{ d(A') \mid A' \subset A, A' \text{ bounded} \} \).

Since the cube \( (0, 1) \times (0, a) \subset \mathbb{R}^2 \) of area \( a > 0 \) is displaced by the time-one map of the Hamiltonian flow induced by the time-independent function \( H(t, x, y) = ax \), the displacement energy of the cylinder \( (0, 1) \times (0, a) \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n} \) in \( \mathbb{R}^{2n} \) is not bigger than its capacity \( a > 0 \). This implies in particular that \( d(\Omega) \leq c_p(\Omega) \) for every open bounded subset of \( \mathbb{R}^{2n} \). On the other hand, the displacement energy of an euclidean ball of capacity \( a \) is not smaller than \( a \) (this was first shown by Hofer; we refer to [HZ] and [LM] for proofs and references). Since moreover clearly \( d(\Omega') \leq d(\Omega) \) if \( \Omega' \subset \Omega \), the displacement energy is a relative capacity.

Following [HZ] we call two subsets \( A, B \) of \( \mathbb{R}^{2n} \) properly separated if there is a symplectomorphism \( \Psi \) of \( \mathbb{R}^{2n} \) such that \( \Psi(A) \subset \{ x_1 < 0 \} \) and \( \Psi(B) \subset \{ x_1 > 0 \} \). Define the proper displacement energy \( e(\Omega) \) of a set \( \Omega \in \mathcal{O} \) to be the infimum of the Hofer-norms \( \| \varphi \| \) of all those \( \varphi \in \mathcal{D} \) for which \( \varphi(\Omega) \) and \( \Omega \) are properly separated. If \( \Omega \subset \mathbb{R}^{2n} \) is unbounded we define \( e(\Omega) = \sup \{ e(\Omega') \mid \Omega' \subset \Omega, \Omega' \text{ bounded} \} \).
As before, the proper displacement energy is a relative capacity. We have the inequalities $c_0(\Omega) \leq d(\Omega) \leq e(\Omega) \leq c_p(\Omega)$ for every set $\Omega \in \mathcal{O}$.

Even for star-shaped subsets of $\mathbb{R}^{2n}$ ($n \geq 2$) our above capacities define different symplectic invariants. The earliest result known to me in this direction is due to Hermann [He]. He constructed for every $n \geq 2$ star-shaped Reinhardt-domains in $\mathbb{R}^{2n}$ with arbitrarily small volume and hence arbitrarily small Gromov width whose displacement energy is bounded from below by 1. For the estimate of the displacement energy he uses a remarkable result of Chekanov [C] who showed that the displacement energy of a closed Lagrangian submanifold of $\mathbb{R}^{2n}$ is positive.

The displacement energy of closed Lagrangian submanifolds is not the only obstruction for embeddings of a star-shaped set $\Omega$ into a cylinder of small capacity. The purpose of this note is to show.

**Theorem:** There is an open bounded starshaped subset $\Omega$ of $\mathbb{R}^4$ with $e(\Omega) < c_p(\Omega)$.

A modification of our construction can be used to obtain for every $n \geq 2$ examples of open bounded subsets $\Omega$ of $\mathbb{R}^{2n}$ with $e(\Omega) < c_p(\Omega)$.

The organization of this note is as follows. In Section 2 we collect some results on symplectic embeddings and symplectic isotopies which are needed for the proof of our Theorem. The theorem is proved in Section 3, and Section 4 contains some additional remarks on our capacities.

## 2 Extensions of symplectic embeddings and isotopies

In this section we formulate some versions of well known existence results for extensions of symplectic embeddings which are needed for the construction of our example.

Denote again by $\mathcal{O}$ the collection of all open bounded subsets of $\mathbb{R}^{2n}$. Define a *proper symplectic embedding* of an open bounded set $\Omega \in \mathcal{O}$ into an open (not necessarily bounded) subset $C$ of $\mathbb{R}^{2n}$ to be a symplectic embedding of a neighborhood of $\Omega$ in $\mathbb{R}^{2n}$ into $C$.

The *neighborhood extension theorem* of Banyaga [B] (see also [MS] for a proof) gives a sufficient condition for the existence of a symplectomorphism of $\mathbb{R}^{2n}$ extending a given proper symplectic embedding of a suitable set $\Omega \in \mathcal{O}$.

**Theorem 2.1:** Let $\Omega \in \mathcal{O}$ be an open bounded set such that $H^1(\overline{\Omega}, \mathbb{R}) = 0$. Then for every symplectic embedding $\varphi$ of a neighborhood of $\overline{\Omega}$ into $\mathbb{R}^{2n}$ there is a symplectomorphism of $\mathbb{R}^{2n}$ which coincides with $\varphi$ near $\overline{\Omega}$.

Let again $\Omega \in \mathcal{O}$. A *strict symplectomorphism* of $\Omega$ is a symplectomorphism $\varphi$ of $\Omega$ which equals the identity near the boundary of $\Omega$. A *strict isotopy* of
\( \Omega \) is a 1-parameter family \( \varphi_t \) of symplectomorphisms of \( \Omega \) which coincides with the identity near the boundary of \( \Omega \) and such that \( \varphi_0 = Id \). We did not find the precise formulation of the following lemma in the literature, so we include the easy proof for convenience.

**Lemma 2.2:** Let \( \Omega \subset \mathbb{R}^{2n} \) be open, bounded and star-shaped. Then any two strict symplectomorphisms of \( \Omega \) are strictly isotopic.

**Proof:** Let \( \Omega \subset \mathbb{R}^{2n} \) be open, bounded and star-shaped with respect to the origin. We have to show that every strict symplectomorphism \( \Psi \) of \( \Omega \) is strictly isotopic to the identity.

For this we use the arguments of Banyaga. Namely, since the support of \( \Psi \) is compact there is an isotopy \( \Psi_t \) of the identity with compact support in \( \mathbb{R}^2 \) for some \( a \geq 1 \) and such that \( \Psi_1 = \Psi \) (see [MS]). Define \( \varphi_t(x) = \frac{1}{a} \Psi_t(ax) \). Then \( \varphi_t \) is a strict isotopy of \( \Omega \) such that \( \varphi_t(x) = \frac{1}{a} \Psi(ax) \). For \( t \in [0, 1] \) write moreover \( \varphi_t(x) = \frac{1}{a(1-t)+t} \Psi((a(1-t)+t)x) \); then \( \varphi_0 = \varphi_1 \) and \( \varphi_1 = \Psi \) and therefore the composition of the isotopies \( \varphi_t \) and \( \varphi_t \) define a strict isotopy of \( \Omega \) as required.

q.e.d.

Two proper embeddings \( \psi_1, \psi_2 \) of a set \( C \in \mathcal{O} \) into an open subset \( \Omega \) of \( \mathbb{R}^{2n} \) are strictly isotopic if there is a strict isotopy \( \varphi_t \) of \( \Omega \) such that \( \varphi_t|C = \psi_2|C \).

As a corollary of Lemma 2.2 we obtain.

**Corollary 2.3:** Let \( \Omega \subset \mathbb{R}^{2n} \) be open, bounded and star-shaped, and let \( B \subset \mathbb{R}^{2n} \) be open and bounded and such that \( H^1(B, \mathbb{R}) = 0 \). Then any two proper embeddings \( \psi_1, \psi_2 \) of \( B \) into \( \Omega \) are strictly isotopic.

**Proof:** The case \( n = 1 \) is well known, so assume that \( n \geq 2 \). Let \( \Omega \subset \mathbb{R}^{2n} \) be open and star-shaped with respect to the origin. Then \( \Omega \) is simply connected and the same is true for \( \mathbb{R}^{2n} - \Omega \). Let \( B \subset \mathbb{R}^{2n} \) be open and bounded and such that \( H^1(B, \mathbb{R}) = 0 \). Let \( \psi_1, \psi_2 : B \to \Omega \) be proper embeddings. Choose an open neighborhood \( U \supset \overline{B} \) of \( B \) such that \( \psi_i \) is defined on \( U \) (i = 1, 2). Since \( \mathbb{R}^{2n} - \Omega \) is simply connected and \( H^1(\psi_i(B), \mathbb{R}) = 0 \) there is by Theorem 2.1 a symplectomorphism \( \Psi \) of \( \mathbb{R}^{2n} \) whose restriction to \( \mathbb{R}^{2n} - \Omega \) equals the identity and whose restriction to a neighborhood of \( \psi_1(B) \) which is contained in \( \psi_1(U) \) coincides with \( \psi_2 \circ \psi_1^{-1} \). Lemma 2.2 then shows that \( \Psi \) is isotopic to the identity with an isotopy which equals the identity on \( \mathbb{R}^{2n} - \Omega \).

q.e.d.

**Corollary 2.4:** Let \( \Omega \subset \mathbb{R}^{2n} \) be open, bounded and star-shaped and let \( U \subset V \subset \mathcal{O} \) be such that \( \overline{U} \subset V \) and that \( \overline{U} \) and \( \overline{V} \) are simply connected. Let \( \varphi : U \to \Omega \) be a proper embedding. If there is a proper embedding \( \zeta : V \to \Omega \) then there is a proper embedding \( \tilde{\zeta} : V \to \Omega \) whose restriction to \( U \) coincides with \( \varphi \).

**Proof:** Let \( \varphi : U \to \Omega \) and \( \zeta : V \to \Omega \) be proper embeddings. By Corollary 2.3 there is a symplectomorphism \( \Psi \) of \( \Omega \) which equals the identity near the boundary and such that \( \Psi \circ \varphi = \zeta(U) \). Then \( \Psi^{-1} \circ \zeta \) is a proper embedding of \( V \) whose restriction to \( U \) coincides with \( \varphi \). q.e.d.
3 Cylindrical capacity and proper displacement energy

Using the assumptions and notations from the introduction and the beginning of Section 2, the goal of this section is to show.

**Theorem 3.1:** There is an open bounded starshaped set $\Omega \subset \mathbb{R}^4$ such that $c_p(\Omega) > e(\Omega)$.

For the proof of our theorem we will need the following simple lemma.

**Lemma 3.2:** Let $h : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R}^{2n-2} \to \mathbb{R}$ be smooth functions with Hamiltonian flows $\varphi_t, \eta_s$. View $h$ and $f$ as functions on $\mathbb{R}^{2n}$ which only depend on the first two and last $2n-2$ coordinates respectively. Let $\nu_t$ be the Hamiltonian flow on $\mathbb{R}^{2n}$ of the function $hf$; then $\nu_t(x,z) = (\varphi_{tf(z)}(x), \eta_{h(x)}(z))$ for every $x \in \mathbb{R}^2$ and every $z \in \mathbb{R}^{2n-2}$.

**Proof:** Let $Z_h, Z_f$ be the Hamiltonian vector fields of $h, f$ as functions on $\mathbb{R}^{2n}$ only depending on the first two and last $2n-2$ coordinates respectively. Then $Z_h$ is a section of the 2-dimensional subbundle of $T\mathbb{R}^{2n}$ spanned by the basic vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$, and $Z_f$ is a section of the $2n-2$-dimensional subbundle of $T\mathbb{R}^{2n}$ spanned by the basic vector fields $\frac{\partial}{\partial x_i}$ for $i \geq 3$. Moreover $f Z_h + h Z_f$ is the Hamiltonian vector field of the function $hf$. We denote by $\nu_t$ its Hamiltonian flow.

The functions $h, f$ induce Hamiltonian flows $\varphi_t, \eta_s$ on $\mathbb{R}^2, \mathbb{R}^{2n-2}$. Since $h$ is constant along the orbits of $\varphi_t$ and $f$ is constant along the orbits of $\eta_s$, for every $x \in \mathbb{R}^2$ and every $z \in \mathbb{R}^{2n-2}$ we have $\nu_t(x,z) = (\varphi_{tf(z)}(x), \eta_{h(x)}(z))$ which shows the lemma. \textbf{q.e.d.}

Using our lemma we can now determine the cylindrical capacity of a special open bounded star-shaped set as follows.

**Example 3.3:** Consider $\mathbb{R}^4$ with the standard symplectic form $\omega_0$. Define $Q_1 = \{(0,s,t,0) \mid -\frac{1}{2} \leq s \leq \frac{1}{2}, 0 \leq t \leq 2\}$. For a small number $\delta < 1/4$ let $L$ be the convex cone in the $(x_2,x_3)$-plane with vertex at the origin whose boundary consists of the ray $\ell_1$ through 0 and the point $(0, \delta, 2, 0)$ and the ray $\ell_2$ through 0 and $(0, \frac{1}{2}, 2, 0)$. Let $\tau > 2$ be the unique number with the property that the line $\{(0,s,\tau,0) \mid s \in \mathbb{R}\}$ intersects the cone $L$ in a segment of length 1. Define $Q_2 = \{(0,x_2,x_3,0) \in L \mid x_3 \leq \tau\}$; then the boundary of $Q_2$ is a triangle with one vertex at the origin, a second vertex $z_1 \neq 0$ on the line $\ell_1$ and the third vertex $z_2 \neq 0$ on the line $\ell_2$. Let $\ell_3$ be the line through $z_2$ which is parallel to $\ell_1$. The lines $\ell_1, \ell_3$ bound a strip $S$ which is foliated into line segments of length 1 which are parallel to the $x_2$-coordinate axis. Choose a large number $M > \tau + 2$ and define $Q_3 = \{(0,x_2,x_3,0) \in S \mid \tau \leq x_3 \leq M\}$.

By construction, the set $Q = Q_1 \cup Q_2 \cup Q_3$ is star-shaped with respect to 0, and for every $t > 0$ the line $\{x_3 = t, x_1 = x_4 = 0\}$ intersects $Q$ in a connected segment of length at most 1.
Let $P_0 \subset \mathbb{R}^3 = \{x_1 = 0\}$ be the set which we obtain by rotating $Q$ about the origin in the $(x_3, x_4)$-plane. The set $P = [-1/2, 1/2] \times P_0$ is star-shaped with respect to the origin and it contains the cube $[-1/2, 1/2]^2 \times D$ where $D$ is the disc of capacity $4\pi > 1$ in the $(x_3, x_4)$-plane. Thus the Gromov-width of $P$ is not smaller than 1.

We claim that the cylindrical capacity of $P$ equals 1. For this let $\epsilon > 0$ and choose a smooth function $\sigma : [0, \infty) \to [0, \infty)$ with the property that we have $Q \subset \{(0, s, t, 0) \mid t \geq 0, \sigma(t) - 1/2 \leq s \leq \sigma(t) + 1/2 + \epsilon\}$. Such a function $\sigma$ exists by the definition of $Q$, and we may assume that it vanishes identically on $[0, 2]$. The Hamiltonian vector field of the restriction of $f$ to each of the at most two components of $(0, 2)$ is of capacity $4\pi$ in the $(x_3, x_4)$-plane. The Hamiltonian vector field of the restriction of $f$ to each of the at most two components of $(0, 2)$ preserves the concentric circles about the origin. Define $h(x_1, x_2, x_3, x_4) = -x_1\sigma(\sqrt{x_3^2 + x_4^2})$. By Lemma 3.2 and the fact that $P$ is invariant under rotation about the origin in the $(x_3, x_4)$-plane we conclude that the image of $P$ under the time-one map of the Hamiltonian flow of the function $h$ equals the set $\tilde{P} = \{(x_1, x_2, x_3, x_4) \mid (x_1, x_2 + \sigma(\sqrt{x_3^2 + x_4^2}), x_3, x_4) \in P\}$ which is contained in the subset $[-1/2, 1/2 + \epsilon]^2 \times B^2(\pi M^2)$ of the cylinder $[-1/2, 1/2 + \epsilon]^2 \times \mathbb{R}^2 \subset \mathbb{R}^4$. Since $\epsilon > 0$ was arbitrary, the cylindrical capacity of $P$ is not bigger than 1 and hence it coincides with the Gromov width of $P$.

Now we can complete the proof of Theorem 3.1. Let $Q \subset \{x_1 = 0, x_4 = 0\}$ be as in Example 3.3. Reflect $Q$ along the line $\{x_3 = 0\}$ in the $(x_2, x_3)$-plane. We obtain a set $\tilde{Q}$ which is star-shaped with respect to the origin. Define $\tilde{P}_0 \subset \{x_1 = 0\}$ to be the set which be obtain by rotating $Q$ about the origin in the $(x_3, x_4)$-plane. Let $\tilde{P} = [-1/2, 1/2] \times \tilde{P}_0$. Then $\tilde{P}$ is star-shaped with respect to the origin and contains $P$ as a proper subset.

We claim that $\epsilon(\tilde{P}) = 1$. To see this notice that for every $t > 0$ the intersection of $\tilde{Q}$ with the line $L_t = \{(0, s, t, 0) \mid s \in \mathbb{R}\}$ consists of at most 2 segments of length at most 1 each. Thus for every $\epsilon > 0$ we can find a smooth function $f_\epsilon$ on the half-plane $\{x_3 > 0\}$ which satisfies $\sup_{z \in \tilde{Q}} f_\epsilon(z) - \inf_{z \in \tilde{Q}} f_\epsilon(z) \leq 1 + \epsilon$ and such that for every $t \geq 0$ its restriction to each of the at most two components of $L_t \cap \tilde{Q}$ equals a translation. We may choose $f_\epsilon$ in such a way that $f_\epsilon(x_2, x_3) = x_2$ for $0 < x_3 < 2$.

Extend the function $f_\epsilon$ to a function $\tilde{f}$ on $\mathbb{R}^4$ which does not depend on the first coordinate and is invariant under rotation about the origin in the $(x_3, x_4)$-plane. The Hamiltonian vector field of the restriction of $\tilde{f}$ to our set $\tilde{P}$ is of the form $\frac{\partial}{\partial x_4} + Z$ where the vector field $Z$ is tangent to the concentric circles about the origin in the $(x_3, x_4)$-plane. Since $\tilde{P}$ is invariant under rotation about the origin in the $(x_3, x_4)$-plane we conclude that for every $s > 0$ the image of $\tilde{P}$ under the time-$s$ map of the Hamiltonian flow of $\tilde{f}$ equals the set $[-1/2 + s, 1/2 + s] \times P_0$. This means that the time-$(1 + \epsilon)$ map of the Hamiltonian flow of $\tilde{f}$ properly displaces $\tilde{P}$. Via multiplying $\tilde{f}$ with a suitable cutoff-function we deduce that the proper displacement energy of $\tilde{P}$ is not bigger than $(1 + \epsilon)^2$. Since $\epsilon > 0$ was arbitrary and since $C_\epsilon(\tilde{P}) \geq 1$ we have $\epsilon(\tilde{P}) = 1$.

We are left with showing that $c_p(\tilde{P}) > 1$. For this define $A \subset \mathbb{R}^2$ to be the closed annulus $B^2(\pi M^2) - B^2(\pi \tau^2)$ of area $\pi(M^2 - \tau^2) \geq 4\pi$. By the discussion
in Example 3.3 there is a small number \( \rho > 0 \) depending on the choice of \( \delta \) in the construction of the set \( Q \) with the following properties.

1. For every \( \epsilon \in (0, \rho) \) there is a symplectic embedding \( \psi_\epsilon \) of a neighborhood of the star-shaped set \( P \subset \tilde{P} \) into the cylinder \( B^2(1 + \epsilon) \times \mathbb{R}^2 \) with the property that \( \psi_\epsilon(P) \supset B^4(1 - \epsilon) \cup B^2(1 - \epsilon) \times A \) and \( \psi_\epsilon(P \cap \{x_2 > 0\}) \supset B^2(\rho) \times B^2(\pi \rho^2) \).

2. There is a proper symplectic embedding of the standard ball \( B^4(1/2 - \rho) \) into \( \tilde{P} - P \) whose image \( \tilde{B} \) is strictly isotopic in \( P \cap \{x_2 < 0\} \subset \tilde{P} - P \) to a standard ball embedded in \( B^4(1 - \epsilon) \cap \{x_2 < 0\} \).

Assume to the contrary that \( c_p(\tilde{P}) = 1 \). Then there is for every \( \epsilon \in (0, \rho) \) a proper symplectic embedding of \( \tilde{P} \) into the cylinder \( B^2(1 + \epsilon) \times \mathbb{R}^2 \). Since the closure of the disconnected set \( P \cup B \subset \tilde{P} \) is simply connected and since the standard cylinder \( B^2(1 + \epsilon) \times \mathbb{R}^2 \) is star-shaped with respect to the origin, we can apply Corollary 2.4 to proper embeddings of \( P \cup B \) into \( B^2(1 + \epsilon) \times \mathbb{R}^2 \). This means that there is a proper symplectic embedding \( \Psi \) of \( \tilde{P} \) into \( B^4(1 + \epsilon) \times \mathbb{R}^2 \) whose restriction to \( P \subset \tilde{P} \) coincides with \( \psi_\epsilon \), and which maps \( \tilde{B} \) to a standard euclidean ball \( B \) which is contained in \( B^2(1 + \epsilon) \times (\mathbb{R}^2 - B^2(\pi \rho^2)) \) and can be obtained from \( B^4(1/2 - \rho) \) by a translation.

Let \( \omega_1 \) be a standard volume form on the sphere \( S^2 \) whose total area is bigger than but arbitrarily close to \( 1 + \epsilon \). Embed the disc \( B^2(1 + \epsilon) \) symplecticly into \( S^2 \). The image in \( S^2 \) of the annulus \( B^2(1 + \epsilon) - B^2(1 - \epsilon) \) is contained in a closed round disc \( D \subset S^2 \) of area bigger than but arbitrarily close to \( 2 \epsilon \). The complement of \( D \) in \( S^2 \) is the disc \( B^2(1 - \epsilon) \). The complement of the disc \( B^2(\rho) \) in \( S^2 \) is area-preserving equivalent to a closed disc in \( \mathbb{R}^2 \).

Our embedding of \( B^2(1 + \epsilon) \) into \( (S^2, \omega_1) \) extends to a symplectic embedding of \( B^2(1 + \epsilon) \times \mathbb{R}^2 \) into \( (S^2 \times \mathbb{R}^2, \omega = \omega_1 + \omega_2) \). Thus if \( \tilde{P} \) admits a proper symplectic embedding into \( B^2(1 + \epsilon) \times \mathbb{R}^2 \) then the standard linear embedding of \( B^4(1/2 - \rho) \) onto a ball in \( S^2 \times (\mathbb{R}^2 - B^2(\pi \rho^2)) \subset S^2 \times \mathbb{R}^2 \) is strictly isotopic in \( S^2 \times (\mathbb{R}^2 - B^2(\pi \rho^2)) \cup D \cup B^2(1 + \epsilon) \times B^2(\pi \rho^2) \subset S^2 \times \mathbb{R}^2 \) to the standard inclusion of \( B^4(1/2 - \rho) \) into \( B^4(1 + \epsilon) \times B^4(1/2 - \rho) \).

However a suitable version of the symplectic camel theorem in dimension 4 [MDT] shows that for sufficiently small \( \epsilon \) this is not possible. We formulate this version as a proposition which completes the proof of our Theorem 3.1.

**Proposition 3.4:** Let \( D \subset S^2 \) be an open round disc of area \( \frac{1}{4} \) in a standard sphere \( (S^2, \omega_1) \) of area 1. Let \( \omega = \omega_1 + \omega_2 \) be a standard symplectic form on \( S^2 \times \mathbb{R}^2 \). For every \( R \in (\frac{1}{4}, 1) \) a standard embedding of the ball \( B^4(R) \) into \( S^2 \times (\mathbb{R}^2 - B^2(3)) \) is not properly isotopic in \( S^2 \times (\mathbb{R}^2 - B^2(3)) \cup D \times \partial B^2(2) \cup B^2(1) \times B^2(2) \subset S^2 \times \mathbb{R}^2 \) to a standard embedding of \( B^4(R) \) into \( S^2 \times S^2 \).

**Proof:** Using the notation from the proposition, let \( R \in (\frac{1}{4}, 1) \) and let \( \varphi_0 \) be a standard embedding of the ball \( B^4(R) \) of capacity \( R \) into \( B^2(2) \times (\mathbb{R}^2 - B^2(3)) \). Let moreover \( \varphi_1 \) be a standard embedding of \( B^4(R) \) into \( B^2(1) \times B^2(2) \subset S^2 \times B^2(1) \). We argue by contradiction and we assume that \( \varphi_0 \) can be connected to \( \varphi_1 \) by a proper isotopy \( \varphi_t \ (t \in [0, 1]) \) whose image is
contained in the subset $S^2 \times (\mathbb{R}^2 - B^2(2)) \cup D \times \partial B^2(2) \cup B^2(1) \times B^2(2)$ of the manifold $S^2 \times \mathbb{R}^2$.

We follow [MDT] and arrive at a contradiction in three steps.

**Step 1**

Let $c$ be the boundary of the disc $B^2(2)$. Denote by $\nu$ the boundary circle of the disc $D \subset S^2$. Then $T = \nu \times c$ is a Lagrangian torus embedded in $S^2 \times \mathbb{R}^2$. For a fixed point $y$ on $c$, the circle $\nu \times \{y\}$ bounds the embedded disc $D \times \{y\} \subset S^2 \times \{y\}$. We call such a disc a **standard flat disc**. It defines a homotopy class of maps of pairs from a closed unit disc $(D_0, \partial D_0) \subset \mathbb{R}^2$ into $(S^2 \times \mathbb{R}^2, T)$.

Let $\mathcal{J}$ be the space of all smooth almost complex structures $J$ on $S^2 \times \mathbb{R}^2$ which **calibrate** the symplectic form $\omega$ (i.e. such that $g(v, w) = \omega(v, Jw)$ defines a Riemannian metric on $S^2 \times \mathbb{R}^2$). In the sequel we mean by a pseudoholomorphic disc a disc which is holomorphic with respect to some structure $J \in \mathcal{J}$. For $J \in \mathcal{J}$ define a $J$-**filling** of the torus $T$ to be a 1-parameter family of disjoint, $J$-holomorphic discs which are homotopic as maps of pairs to the standard flat disc, whose boundaries foliate $T$ and whose union $F(J)$ is diffeomorphic to $D \times S^1$ and does not intersect $(S^2 - D) \times c$. The set $F(J)$ then necessarily disconnects $\Omega = S^2 \times \mathbb{R}^2 - (S^2 - D) \times c$.

For $t \in [0, 1]$ let $J_t \in \mathcal{J}$ be an almost complex structure depending continuously on $t$. We require that $J_0 = J_1$ is the standard complex structure and that the restriction of $J_t$ to $\varphi_t^* B^4(R)$ coincides with $(\varphi_t)_* J_0$ (where by abuse of notation we denote by $J_0$ the natural complex structure on $\mathbb{R}^4$ and on $S^2 \times \mathbb{R}^2$). Such structures exist since $\mathcal{J}$ is the space of smooth sections of a fibre bundle over $S^2 \times \mathbb{R}^2$ with contractible fibre.

Assume that for every $t \in [0, 1]$ there is a unique $J_t$-filling $F(J_t)$ of $T$ depending continuously on $t$ in the Hausdorff topology for closed subsets of $S^2 \times \mathbb{R}^2$. Then

$$X = \{ (t, x) \mid x \in F(J_t), 0 \leq t \leq 1 \}$$

is a closed subset of $[0, 1] \times \Omega$. Since each filling $F(J_t)$ disconnects $\Omega$, the set $X$ disconnects $[0, 1] \times \Omega$. Now $J_0$ and $J_1$ are standard and the standard filling of $T$ by flat discs separates $\varphi_0^* B^4(R)$ from $\varphi_1^* B^4(R)$. Thus the points $(0, \varphi_0(0))$ and $(1, \varphi_1(0))$ are contained in different components of $[0, 1] \times \Omega - X$. Therefore the path $(t, \varphi_t(0))$ must intersect $X$. In other words, for some $t$, there is a $J_t$-holomorphic disc $C$ through $\varphi_t(0)$ with boundary on $T$ and which is contained in the homotopy class of the standard flat disc. The connected component of $\varphi_t^{-1} C \cap B^4(R)$ containing 0 is a planar holomorphic curve with respect to the standard integrable complex structure whose boundary is contained in the boundary of $B^4(R)$. Thus this surface is a minimal surface with boundary on the boundary of $B^4(R)$ and therefore its area is not smaller than $R |G|$. On the other hand, the torus $T$ is Lagrangian and hence the area of a $J_t$-holomorphic disc with boundary on $T$ only depends on the free homotopy class of the boundary curve. In particular, since the curve $C$ is homotopic to the standard flat disc its area equals the area $\frac{4}{\pi}$ of the standard flat disc. This contradicts our assumption that $\frac{4}{\pi} < R$ (compare [MDT] p.178).
By the above it is now enough to construct for every $t \in [0, 1]$ an almost complex structure $J_t \in \mathcal{J}$ whose restriction to $\varphi_t^* B^4(R)$ coincides with $(\varphi_t)_* J_0$ and such that for every $t \in [0, 1]$ the torus $T$ admits a unique $J_t$-filling depending continuously on $t \in [0, 1]$ in the Hausdorff topology. For this we follow again [MDT].

**Step 2**

Let $H$ be an oriented hypersurface in an almost complex 4-manifold $(N, J)$. There is a unique two-dimensional subbundle $\xi$ of the tangent bundle of $H$ which is invariant under $J$. We call $H$ J-convex if for one (and hence any) one-form $\alpha$ on $H$ whose kernel equals $\xi$ and which defines together with the restriction of $J$ to $\xi$ the orientation of $H$ and for every $0 \neq v \in \xi$ we have $d\alpha(v, Jv) > 0$.

Let $H \subset S^2 \times \mathbb{R}^2$ be a smooth hypersurface which contains the Lagrangian torus $T$ and bounds an open domain $U_H \subset S^2 \times \mathbb{R}^2$ which contains the open disc bundle $D \times c - T$. We equip $H$ with the orientation induced by the outer normal of $U_H$. Denote by $\mathcal{J}_H \subset \mathcal{J}$ the set of all almost complex structures $J \in \mathcal{J}$ for which $H$ is J-convex and which coincide with the standard complex structure $J_0$ near $T$. If the set $\mathcal{J}_H$ is not empty then it follows again from the fact that $\mathcal{J}$ is the space of smooth sections of a fibre bundle over $S^2 \times \mathbb{R}^2$ with contractible fibre that we can find some $\tilde{J} \in \mathcal{J}_H$ which coincides with $J_0$ on a neighborhood of the disc bundle $D \times c$. Then the standard flat discs define a $\tilde{J}$-filling of $T$.

Let $J_t \subset \mathcal{J}_H \ (t \in [1, 2])$ be a differentiable curve (there is some subtlety here about the differentiable structure of $\mathcal{J}_H$ which will be ignored in the sequel; a discussion of this problem is contained in [MDT]). Assume that for every $t \in [1, 2]$ there is a holomorphic disc $D_t$ of $J_t$ with boundary on $T$ depending continuously on $t$ in the Hausdorff topology and such that $D_1$ is a standard flat disc. Since $H$ is $J_1$-convex, Lemma 2.4 of [MD2] and Proposition 3.2 in [MDT] show that each of the discs $D_t$ meets the hypersurface $H$ transversely at its boundary, and its interior is contained in $U_H$.

By assumption, each of the structures $J_t$ coincides with the standard complex structure near the torus $T$. Thus there is an open neighborhood $V$ of $T$ in $S^2 \times \mathbb{R}^2$ and a $J_t$-antiholomorphic involution in $T$ on $V$ (see [MDT]). Namely, under the usual identification of $\mathbb{R}^4$ with $\mathbb{C}^2$, the torus $T$ is just the cartesian product of the boundary of a disc of radius $r_1 > 0$ with the boundary of a disc of radius $r_2 > 0$. The map $(z_1, z_2) \to (\frac{r_1^2}{z_1}, \frac{r_2^2}{z_2})$ is an antiholomorphic reflection in $\mathbb{C}^2$ which fixes $T$ pointwise. This reflection restricts to a reflection on an open neighborhood $V$ of $T$ which is antiholomorphic with respect to any of the almost complex structures $J_t$.

Using this involution we can double our domain $U_H$ near $T$ [MDT] and use the Schwarz reflection principle to extend our holomorphic discs $D_t$ to holomorphic spheres in the double of $U_H$ [MDT]. Intersection theory for pseudoholomorphic spheres in almost complex manifolds then shows that each of our discs $D_t$ is embedded. Moreover any two different such discs for the same structure $J \in \mathcal{J}_H$ do not intersect [MD1].

Now Gromov’s compactness theorem is valid for pseudoholomorphic discs
with Lagrangian boundary condition [O]. The area of each pseudoholomorphic disc with boundary on \( T \) and in the homotopy class of the standard flat disc coincides with the area \( \frac{1}{4} \) of the standard flat disc. Moreover, since our torus \( T \) is rational [P] and \( \frac{1}{4} \) is the generator of the subgroup of \( \mathbb{R} \) induced by evaluation of \( \omega_0 \) on \( \pi_2(\mathbb{R}^4, T) \), \( \frac{1}{4} \) is the minimal area of any pseudoholomorphic disc whose boundary is contained in \( T \). This implies that for discs in our given homotopy class, bubbling off of holomorphic spheres and holomorphic discs can not occur. Therefore we can use standard Fredholm theory for the Cauchy Riemann operator [MDT] to compute for a dense set of points in \( J_H \) the parameter space of pseudoholomorphic discs with boundary on \( T \) and which are homotopic to the standard flat disc. As a consequence [MDT], for every \( J \in J_H \) which can be connected to our fixed structure \( \tilde{J} \) by a differentiable curve in \( J_H \) there is a unique \( J \)-filling \( F(J) \) of \( T \) depending continuously on \( J \in J_H \) in the Hausdorff topology for closed subsets of \( S^2 \times \mathbb{R}^2 \) (here uniqueness means uniqueness of the image and hence we divide the family of all holomorphic discs by the group of biholomorphic automorphisms of the unit disc in \( \mathbb{C} \)).

Together with Step 1 above we conclude that our proposition follows if we can construct a hypersurface \( H \) in \( S^2 \times \mathbb{R}^2 \) with the following properties.

1. \( H \) contains \( T \) and bounds an open set \( U_H \) containing the open disc bundle \( D \times c - T \).

2. \( U_H \) contains a neighborhood of \( \cup_{t \in [0,1]} \varphi_t B^4(R) \).

3. \( J_H \neq \emptyset \).

Namely, for such a hypersurface \( H \) we can choose a fixed almost complex structure \( J \in J_H \) whose restriction to the disc bundle \( D \times c \) coincides with the standard structure. For each \( t \in [0,1] \) we modify \( J \) near \( \varphi_t B^4(R) \) in such a way that the modified structure \( J_t \) is unchanged near the hypersurface \( H \) and coincides with \( (\varphi_t)_* J_0 \) on \( \varphi_t B^4(R) \). We can do this in such a way that \( J_t \) depends differentiably on \( t \). For each of the structures \( J_t \) there is then a unique \( J_t \)-filling of \( T \) depending continuously on \( t \).

By our assumption, the closure of the set \( \cup_{t \in [0,1]} \varphi_t B^4(R) \) is contained in \( S^2 \times (\mathbb{R}^2 - B^2(2)) \cup D \times \partial B^2(2) \cup B^2(1) \times B^2(2) \) and hence it is enough to find a hypersurface \( H \) in the symplectic manifold

\[
N = S^2 \times (\mathbb{R}^2 - B^2(2)) \cup B^2(1) \times \partial B^2(2) \cup \mathbb{R}^2 \times B^2(2)
\]

with properties 1-3. In the third step of our proof we construct such a hypersurface.

**Step 3**

Let \( A \subset \mathbb{R}^2 \) be a closed circular annulus containing the circle \( c \) in its interior. We assume that \( A \) is small enough so that the closure of the set \( \cup_{t \in [0,1]} \varphi_t B^4(R) \) intersects \( S^2 \times A \) in \( B^2(\frac{1}{4}) \times A \). We also require that there is some \( a > 0 \) such that \( S^2 \times A \subset S^2 \times \mathbb{R}^2 \) is symplectomorphic to the quotient of the bundle \( S^2 \times [-a,a] \times \mathbb{R} \) under a translation \( \tau \) in the plane \( \mathbb{R}^2 \) in such a way that the
torus $T$ is the quotient of the standard circle bundle $\partial B^2(\frac{1}{2}) \times \{0\} \times \mathbb{R}$. The standard complex structure on $S^2 \times \mathbb{R}^2$ is invariant under the translation $\tau$ and projects to an integrable complex structure on a neighborhood of $S^2 \times A$ in $\mathbb{R}^4 \times \mathbb{R}^2$ which calibrates $\omega$.

For small $\sigma \in [0, a)$ write $\ell_\sigma = \{(0, 0, -\sigma, s) \mid s \in \mathbb{R}\} \subset \mathbb{R}^4$. The circle bundle $\partial B^2(\frac{1}{4}) \times \{0\} \times \mathbb{R}$ is contained in the boundary $\partial U_\sigma$ of a tubular neighborhood $U_\sigma$ of some radius $r(\sigma) > 1/2\sqrt{\pi}$ about the line $\ell_\sigma$. The hypersurface $\partial U_\sigma$ with its orientation as the boundary of $U_\sigma$ is convex with respect to the euclidean metric.

Let $X_\sigma$ be the gradient of the function $z \mapsto \frac{1}{2}\text{dist}(z, \ell_\sigma)^2$. Then $X_\sigma$ is perpendicular to the hypersurface $\partial U_\sigma$ and can be written down explicitly by $X_\sigma = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + (x_3 + \sigma) \frac{\partial}{\partial x_3}$. If we denote by $\iota_{X_\sigma} \omega_0$ the $1$-form $\omega_0(X_\sigma, \cdot)$ then $\iota_{X_\sigma} = x_1 dx_2 - x_2 dx_1 + (x_3 + \sigma) dx_4$ and $d(\iota_{X_\sigma} \omega_0) = 2dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. This implies that $\partial U_\sigma$ is convex with respect to $J_0$.

As a first step towards the construction of a hypersurface $H$ with properties $1$-$3$ above we construct a smooth embedded hypersurface $E \subset \mathbb{R}^4 - \ell_\sigma$ with the following properties.

a) $E$ divides $\mathbb{R}^4$ into two components and is invariant under the translations in direction of the $x_4$-axis.

b) There is some $\sigma \in (0, a/2)$ such that $E$ is everywhere transverse to the vector field $X_\sigma$ and contains a neighborhood of $T$ in $\partial U_\sigma$.

For this let $E_0$ be a hypersurface which is contained in $\{0 \leq x_3 \leq a/2\}$ and which bounds a noncompact convex set $V$ containing $\{x_3 > a/2\}$. We assume that $E_0$ is invariant under translations along the lines parallel to $\ell_\sigma$, and we orient $E_0$ as the boundary of $V$.

Assume that the intersection of $E_0$ with the hyperplane $\{x_3 = 0\}$ equals the line $\ell_0 = \{(0, 0, 0, s) \mid s \in \mathbb{R}\}$ and that the hypersurface $E_0$ is strictly convex in directions transverse to the lines parallel to the $x_4$-axis. Since the vector field $X_0$ is tangent to the lines in $\mathbb{R}^4$ which meet $\ell_0$ orthogonally, by convexity $X_0$ is everywhere transverse to $E_0 - \ell_0$. More precisely, for every $z \in E_0 - \ell_0$ the nonoriented angle between $X_0(z)$ and the outer normal of $E_0$ (as the boundary of $V$) at $z$ is smaller than $\pi/2$. An explicit example of such a hypersurface can be obtained as follows. Choose an even strictly convex function $f : \mathbb{R} \rightarrow [0, a/2]$ which has a (necessarily unique) minimum $0$ at $0$ and define $E_0$ to be the solution of the equation $f(x_1^2 + x_2^2 + x_4^2) - x_3 = 0$.

By construction, there is a tubular neighborhood of the line $\ell_0$ which is invariant under the translation $\tau$ and contained in each of the sets $U_\sigma$ for all small $\sigma \geq 0$. This implies that for a suitable choice of $E_0$ and for sufficiently small $\sigma > 0$ the vector field $X_\sigma$ is everywhere transverse to $E_0 - U_\sigma$. More precisely, for such a $\sigma$ and for every compact set $K$ there is a number $\delta(K) > 0$ such that for every $z \in K \cap (E_0 - U_\sigma)$ the nonoriented angle at $z$ between $X_\sigma$ and the outer normal of $E_0$ is contained in the interval $[0, \pi/2 - \delta(K)]$. This means that for such a $\sigma$ the vector field $X_\sigma$ is everywhere transverse to the boundary of the set $U_\sigma \cup V \cup \{x_3 < -\sigma/2\} = W$, and its nonoriented angle with the outer
normal of this boundary is strictly smaller than \( \pi/2 \). In particular, after a small perturbation of our sets near the intersections \( \partial U_\sigma \cap E_0 \) and \( \partial U_\sigma \cap \{ x_3 = -\sigma/2 \} \) we may assume that the boundary of our set \( W \) is a smooth hypersurface \( \partial W \) which is contained in \( \{ -\sigma/2 \leq x_3 \leq a/2 \} \), is invariant under the translation \( \tau \) and satisfies properties (a), (b) above.

The boundary \( \partial W \) of \( W \) is contained in \( \{ -\sigma/2 \leq x_3 \leq a/2 \} \) and hence it projects to a smooth hypersurface in \( \mathbb{R}^2 \times A \) which bounds an open set \( W \). The set \( W \) in turn projects to a set \( \tilde{W} \subset N \). By our explicit construction we may assume that \( \tilde{W} \) contains the closure of the set \( \bigcup_{t \in [0,1]} \mathcal{B}_1^{\sigma}(R) \).

The vector field \( X_\sigma \) is transverse to \( \partial W \) and therefore the kernel of the 1-form \( \iota_{X_\sigma} \omega_0 \) intersects the tangent bundle of \( E \) in a two-dimensional subbundle \( \xi \). The restriction of \( \omega_0 \) to \( \xi \) is non-degenerate and hence there is an almost complex structure \( J_\sigma \) on \( \xi \) which calibrates \( \omega_0 \). This structure \( J_\sigma \) extends to an almost complex structure \( J_\sigma \) near \( E \) which calibrates \( \omega_0 \). Since a neighborhood of \( T \) in \( E \) is contained in \( \partial U_\sigma \), the almost complex structure \( J_\sigma \) can be chosen to coincide with the standard complex structure near \( T \). Moreover, \( E \) is \( J_\sigma \)-convex. Now \( X_\sigma \) and \( \partial W \) are invariant under the translation \( \tau \) and hence we may assume that the same is true for the almost complex structure \( J_\sigma \). Therefore this almost complex structure projects to an almost complex structure on a neighborhood of \( \partial W \) which we denote again by \( J_\sigma \). The hypersurface \( \partial W \) is \( J_\sigma \)-convex.

By construction, there is a circle \( \gamma \) in \( \mathbb{R}^2 - B^2(2) \) such that the intersection of \( \partial W \) with the set \( (\mathbb{R}^2 - B^2(1/2)) \times (\mathbb{R}^2 - B^2(2)) \) is contained in the hypersurface \( \mathbb{R}^2 \times \gamma \). But this just means that \( \partial W \) projects to a smooth hypersurface \( H \) in the set \( N \).

Denote by \( \alpha \) the restriction of the 1-form \( \iota_{X_\sigma} \omega_0 \) to the hyperplane \( Q = \{ x_3 = -\sigma/2 \} \). Let \( (r, \theta) \) be polar coordinates about 0 in the \( (x_1, x_2) \)-plane. By our explicit formula for \( X_\sigma \) the form \( \alpha \) can be written in coordinates \( (r, \theta, x_1) \) for \( Q \) as \( \alpha = rd\theta + \frac{\sigma}{2} dx_4 \). Then \( d\alpha = dr \wedge d\theta \).

Now if \( \beta \) is any one-form on \( \{ x_3 = -\sigma/2 \} \) which is invariant under rotation about the origin in the \( (x_1, x_2) \)-plane and the translations along the lines parallel to \( \ell_\sigma \) then in our above coordinates we can write \( \beta = \varphi(r) dr + \rho(r) dx_4 \) for functions \( \varphi, \rho \) on \( (0, \infty) \). The one-form \( \beta \) vanishes nowhere if and only if the functions \( \varphi, \rho \) do not have a common zero. Now \( d\beta = \varphi'(r) dr \wedge d\theta + \rho'(r) dr \wedge dx_4 \) and hence the restriction of \( d\beta \) to the kernel of \( \beta \) vanishes nowhere if and only if we have \( \varphi'(r) \rho(r) - \rho'(r) \varphi(r) > 0 \).

Let \( \varphi, \rho \) be functions on \( (0, \infty) \) with the following properties.

1. \( \varphi(t) = t, \rho(t) = \sigma/2 \) for \( t \leq 1/2 \).

2. \( \varphi' \rho - \rho' \varphi > 0 \).

3. \( \varphi \) and \( \rho \) do not have a common zero.

4. \( \varphi(1) = 0 \).

Such functions \( \varphi, \rho \) can easily be constructed. We replace the one-form \( \alpha \) on \( Q \) by the one-form \( \beta = \varphi(r) dr + \rho(r) dx_4 \). Then \( \beta \) is a contact form on \( Q \) which coincides with the contact form \( \alpha \) on \( B^2(1/2) \times \mathbb{R}^2 \cap Q \). Moreover, this
contact form projects to a contact form on the hypersurface $\mathbb{R}^2 \times \gamma$ which we denote again by $\beta$. The kernel of $\beta$ on $\partial B^2(1) \times \gamma$ is spanned by the vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ and consequently this kernel projects to a plane bundle on the projection of $(B^2(1) - B^2(1/2)) \times \gamma$ to $N$ which we obtain by mapping each circle $\partial B^2(1) \times \{y\}$ to a point.

In other words, there is a modification $\tilde{J}_\sigma$ of the almost complex structure $J_\sigma$ which coincides with the integrable complex structure near the torus $T$ and which projects to an almost complex structure on a neighborhood of $H$ in $N$. This structure is the restriction of a smooth almost complex structure $\hat{J}_\sigma$ on $N$ which calibrates $\omega$. The oriented hypersurface $H$ is the boundary of an open set $U$ containing $\cup_{t \in [0,1]} \varphi_t B^4(R)$, and it is $\tilde{J}_\sigma$-convex.

Together this means that $H$ and $\hat{J}_\sigma$ satisfy the properties 1-3 above. This finishes the proof of our proposition. q.e.d.

4 Displacement and proper displacement

In this short section we collect some easy properties of the displacement energy and the proper displacement energy. We continue to use the assumptions and notations from Section 2-4.

Recall that the displacement energy $d(\Omega)$ of an open bounded set $\Omega \subset \mathbb{R}^{2n}$ equals the infimum of the Hofer norms of all symplectomorphisms $\Psi$ of $\mathbb{R}^{2n}$ such that $\Psi \Omega \cap \Omega = \emptyset$. For every $\Omega \in \mathcal{O}$ the displacement energy $d(\overline{\Omega})$ of the closure $\overline{\Omega}$ of $\Omega$ is not smaller than the proper displacement energy $e(\Omega)$ of $\Omega$. In the case $n = 1$ equality $d(\overline{\Omega}) = e(\Omega)$ always holds.

The following lemma is an easy corollary of the neighborhood extension theorem of Banyaga [B] (see Theorem 2.1).

**Lemma 4.1:** Let $\Omega \in \mathcal{O}$ be such that $H^1(\overline{\Omega}, \mathbb{R}) = 0$; then $d(\overline{\Omega}) = e(\Omega)$.

**Proof:** Since we always have $e(\Omega) \geq d(\overline{\Omega})$ we have to show the reverse inequality under the assumption that $H^1(\overline{\Omega}, \mathbb{R}) = 0$. For this we only have to consider the case $n \geq 2$.

Let $\epsilon > 0$ and let $\Psi \in \mathcal{D}$ be a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ of Hofer norm smaller than $d(\overline{\Omega}) + \epsilon$ and such that $\Psi(\Omega) \cap \Omega = \emptyset$. Then there is an open neighborhood $U$ of $\overline{\Omega}$ such that $\Psi(U) \cap U = \emptyset$.

Assume without loss of generality that $U \subset \{x_1 < 0\}$. Let $e_1$ be the first basis vector of the standard basis of $\mathbb{R}^{2n}$, choose some $\mu < \inf \{x_1(z) \mid z \in U\}$ and define $W = U \cup (U - \mu e_1)$. Then $W$ contains two copies of $\overline{\Omega}$ in its interior which are separated by the hyperplane $\{x_1 = 0\}$. Moreover the set $\Omega \cup \Psi \Omega$ admits a natural proper symplectic embedding into $W$ whose restriction to $\Omega$ is just the inclusion.

Since $H^1(\overline{\Omega} \cup \overline{\Psi \Omega}, \mathbb{R}) = 0$, by the Banyaga extension theorem this proper symplectic embedding of $\Omega \cup \Psi \Omega$ into $W$ can be extended to a symplectomorphism $\eta$ of $\mathbb{R}^{2n}$. Then $\eta \circ \Psi \circ \eta^{-1}$ is a symplectomorphism of Hofer-norm
smaller than $d(\Omega) + \epsilon$ which properly displaces $\Omega = \eta(\Omega)$. This shows that $e(\Omega) \leq d(\Omega) + \epsilon$, and since $\epsilon > 0$ was arbitrary the lemma follows. \textbf{q.e.d.}

Finally we look at the relation between the displacement energy of a set $\Omega \in \mathcal{O}$ and the displacement energy of its closure $\overline{\Omega}$. We first give an easy example which shows that the equality $d(\Omega) = d(\overline{\Omega})$ does not even hold for open bounded topological balls with smooth boundary.

\textbf{Example 4.2:} Let $\epsilon \in (0,1/2)$ and define

$$Q_\epsilon = B^2(4) \times [0,1]^2 \cup (B^2(8) - B^2(4)) \times [1 - \epsilon, 2 - \epsilon] \times [0,1].$$

Then $Q_\epsilon$ is a closed topological ball with piecewise smooth boundary whose interior we denote by $U_\epsilon$. By construction, the sets $U_\epsilon$ and $U_\epsilon + (0,0,1,0)$ are disjoint and hence $U_\epsilon$ can be displaced by the time-one map of the Hamiltonian flow of the function $f(x_1, x_2, x_3, x_4) = -x_4$. This implies that $d(U_\epsilon) \leq 1$. Since $U_\epsilon$ contains the open cylinder $B^2(4) \times (0,1)^2$ of displacement energy 1 we conclude that $d(U_\epsilon) = 1$.

On the other hand, the closure $Q_\epsilon$ of $U_\epsilon$ contains the split Lagrangian torus $T^2 = \partial B^2(4) \times \partial([0,2-\epsilon] \times [0,1])$ of displacement energy $2 - \epsilon$ and therefore we have $d(Q_\epsilon) > d(U_\epsilon)$. Via replacing the squares in our construction by discs with smooth boundary we can also find an example of a topological ball $\Omega$ with smooth boundary $\partial \Omega$ and such that $d(\Omega) > d(\Omega)$.

Recall that a (not necessarily smooth) hypersurface $H$ in $\mathbb{R}^{2n}$ is of \textit{contact type} if there is a conformal vector field $\xi$ defined near the hypersurface $H$ (i.e. such that the Lie-derivative of $\omega_0$ with respect to $\xi$ coincides with $\omega_0$) which is transverse to $H$ in the sense that the flow lines of $\xi$ intersect $H$ transversely. Let $\varphi_t$ be the local flow of $\xi$ and assume that there is some $\epsilon > 0$ such that $\varphi_t$ is defined near $H$ for all $t \in (-\epsilon, \epsilon)$. If $H$ is the boundary of a bounded open set $\Omega \in \mathcal{O}$ then for $t \in (0,\epsilon)$ the set $\varphi_t H$ is the boundary of a neighborhood of $\overline{\Omega}$, and for $t \in (-\epsilon,0)$ the set $\varphi_t H$ is contained in $\Omega$. The hypersurface $H$ is called of \textit{restricted contact type} if the conformal vector field $\xi$ can be defined on all of $\mathbb{R}^{2n}$. For example, if $\Omega \in \mathcal{O}$ is starshaped with respect to 0 and if the lines through 0 intersect the boundary $\partial \Omega$ transversely then $\partial \Omega$ is of restricted contact type.

Our last lemma shows that the difficulty encountered in our example 4.2 does not occur for open sets with boundary of restricted contact type.

\textbf{Lemma 4.3:} Let $\Omega$ be an open bounded set in $\mathbb{R}^{2n}$ ($n \geq 2$). If the boundary of $\Omega$ is of restricted contact type then $d(\Omega) = d(\overline{\Omega})$.

\textbf{Proof:} Let $\Omega \in \mathcal{O}$ be an open bounded subset of $\mathbb{R}^{2n}$ whose boundary is of restricted contact type.

Let $\xi$ be a conformal vector field on $\mathbb{R}^{2n}$ which intersects the boundary of $\Omega$ transversely. Assume that there is a neighborhood $U$ of $\Omega$ and a number $\epsilon > 0$ such that the local flow $\varphi_t$ of $\xi$ is defined on $(-2\epsilon, 2\epsilon) \times U$. Then the
image of $\Omega$ under the time-$\epsilon$ map of the flow $\phi_t$ of $\xi$ is a neighborhood of $\overline{\Omega}$. Since $\phi_t^*\omega_0 = e^t\omega_0$ for all $t$ and wherever this is defined, by conformality the displacement energy of $\phi^*\Omega$ is not bigger than $\epsilon^d(\Omega)$. But this means that $d(\Omega) \leq \epsilon^d(\Omega)$, and since $\epsilon > 0$ was arbitrary we conclude that $d(\Omega) = d(\Omega)$. q.e.d.

5 References

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