Existence of discretely self-similar solutions to the Navier-Stokes equations for initial value in $L^2_{loc}(\mathbb{R}^3)$

Dongho Chae* and Jörg Wolf †

*Department of Mathematics
Chung-Ang University
Seoul 156-756, Republic of Korea
e-mail: dchae@cau.ac.kr

and

†Department of Mathematics
Humboldt University Berlin
Unter den Linden 6, 10099 Berlin, Germany
e-mail: jwolf@math.hu-berlin.de

Abstract

We prove the existence of a forward discretely self-similar solutions to the Navier-Stokes equations in $\mathbb{R}^3 \times (0, +\infty)$ for a discretely self-similar initial velocity belonging to $L^2_{loc}(\mathbb{R}^3)$.

AMS Subject Classification Number: 76B03, 35Q31

keywords: Navier-Stokes equations, existence, discretely self-similar solutions

1 Introduction

In this paper we study the existence of forward discretely self-similar (DSS) solutions to the Navier-Stokes equations in $Q = \mathbb{R}^3 \times (0, +\infty)$

\begin{align}
\nabla \cdot u &= 0, \\
\partial_t u + (u \cdot \nabla) u - \Delta u &= -\nabla \pi,
\end{align}

with the initial condition

\begin{equation}
\begin{aligned}
u &= u_0 \quad \text{on} \quad \mathbb{R}^3 \times \{0\}.
\end{aligned}
\end{equation}

Here $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ denotes the velocity of the fluid, and $u_0(x) = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$, while $\pi$ stands for the pressure. In case $u_0 \in L^2(\mathbb{R}^3)$ with
\[ \nabla \cdot u_0 = 0 \] in the sense of distributions the global in time existence of weak solutions to (1.1)-(1.3), which satisfy the global energy inequality for almost all \( t \in (0, +\infty) \)

\[(1.4) \quad \frac{1}{2} \|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds \leq \frac{1}{2} \|u_0\|_2^2 \]

has been proved by Leray \cite{8}. On the other hand, the important questions of regularity and uniqueness of solutions to (1.1)-(1.3) are still open. The first significant results in this direction have been established by Scheffer \cite{9} and later by Caffarelli, Kohn, Nirenberg \cite{2} for solutions \((u, \pi)\) that also satisfy the following local energy inequality for almost all \( t \in (0, +\infty) \) and for all nonnegative \( \phi \in C_c^\infty(Q) \)

\[(1.5) \quad \frac{1}{2} \int_{\mathbb{R}^3} |u(t)|^2 \phi(x, t) dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi dx ds \leq \frac{1}{2} \int_{\mathbb{R}^3} \int_0^t (|u|^2 + 2\pi) u \cdot \nabla \phi dx ds.

On the other hand, the space \( L^2(\mathbb{R}^3) \) excludes homogeneous spaces of degree \(-1\) belonging to the scaling invariant class. In fact we observe that \( u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t) \) solves the Navier-Stokes equations with initial velocity \( u_{0, \lambda}(x) = \lambda u_0(\lambda x) \), for any \( \lambda > 0 \). This suggests to study of the Navier-Stokes system for initial velocities in a homogenous space \( X \) of degree \(-1\), which means that \( \|v\|_X = \|v_\lambda\|_X \) for all \( v \in X \). Koch and Tataru proved in \cite{6} that \( X = BMO^{-1} \) is the largest possible space with scaling invariant norm which guarantees well-posedness under smallness condition. On the contrary, for self-similar (SS) initial data fulfilling \( u_{0, \lambda} = u \) for all \( \lambda > 0 \) a natural space seems to be \( X = L^{3,\infty}(\mathbb{R}^3) \). This space is embedded into the space \( L^2_{uloc}(\mathbb{R}^3) \), which contains uniformly local square integrable functions. Obviously, possible solutions to the Navier-Stokes equations with \( u_0 \in L^2_{uloc}(\mathbb{R}^3) \) do not satisfy the global energy equality, rather the local energy inequality in the sense of Caffarelli-Kohn-Nirenberg. Such solutions are called local Leray solutions. The existence of global in time local Leray solutions has been proved by Lemarié-Rieusset in \cite{7} (see also in \cite{5} for more details). This concept has been used by Bradshaw and Tsai \cite{11} for the construction of a discretely self-similar (\( \lambda \)-DSS, \( \lambda > 1 \)) local Leray solution for a \( \lambda \)-DSS initial velocity \( u_0 \in L^{3,\infty}(\mathbb{R}^3) \). This result generalizes the previous results of Jia and Šverák \cite{4} concerning the existence of SS local Leray solution, and the result by Tsai in \cite{10}, which proves the existence of a \( \lambda \)-DSS Leray solution for \( \lambda \) near 1. However, for the \( \lambda \)-DSS initial data it would be more natural to assume \( u_0 \in L^2_{uloc}(\mathbb{R}^3) \) instead \( L^{3,\infty}(\mathbb{R}^3) \). In general, such initial value does not belong to \( L^2_{uloc}(\mathbb{R}^3) \) and therefore it does not belong to the Morrey class \( M^{2,1} \), rather to the weighted space \( L^2_k(\mathbb{R}^3) \) of all \( v \in L^2_{loc}(\mathbb{R}^3) \) such that \( \frac{v}{(1+|x|^k)} \in L^2(\mathbb{R}^3) \) for all \( \frac{1}{2} < k < +\infty \).

Since the authors in \cite{11} work on the existence of periodic solutions to the time dependent Leray equation a certain spatial decay is necessary which can be ensured for initial data in \( L^{3,\infty}(\mathbb{R}^3) \). On the other hand, applying the local \( L^2 \) theory it would be
more natural to assume \( u_0 \in L^2(B_\lambda \setminus B_1) \) only. As explained in [1] their method even breaks down for initial data in the Morrey class \( M^{2,1}(\mathbb{R}^3) \), which is a much smaller subspace of \( L^{2,\text{loc}}(\mathbb{R}^3) \). By using an entirely different method we are able to construct a global weak solutions for such DSS initial data.

In the present paper we introduce a new notion of a local Leray solution satisfying a local energy inequality with projected pressure. To the end, we provide the notations of function spaces which will be use in the sequel. By \( L^s(G) \), \( 1 \leq s \leq \infty \) we denote the usual Lebesgue spaces. The usual Sobolev spaces are denoted by \( W^{k,s}(G) \) and \( W^{0,k,s}(G) \), \( 1 \leq s \leq +\infty, k \in \mathbb{N} \). The dual of \( W^{0,k,s}(G) \) will be denoted by \( W^{-k,s'}(G) \), where \( s' = \frac{n}{s-1}, 1 < s < +\infty \). For a general space of vector fields \( X \) the subspace of solenoidal fields will be denoted by \( X_\sigma \). In particular, the space of solenoidal smooth fields with compact support is denoted by \( C_{c,\sigma}^\infty (\mathbb{R}^3) \). In addition we define energy space

\[
V^2(G \times (0,T)) = L^\infty (0,T; L^2(G)) \cap L^2 (0,T; W^{1,2}(G)), \quad 0 < T \leq +\infty.
\]

We now recall the definition of the local pressure projection \( E_G^s : W^{-1,s}(G) \rightarrow W^{-1,s}(G) \) for a given bounded \( C^2 \)-domain \( G \subset \mathbb{R}^3 \), introduced in [12] based on the unique solvability of the steady Stokes system (cf. [3]). More precisely, for any \( F \in W^{-1,s}(G) \) there exists a unique pair \((v, p) \in W_{0,\sigma}^1(G) \times L^s_0(G)\) which solves weakly the steady Stokes system

\[
\begin{align*}
\nabla \cdot v &= 0 \quad \text{in} \ G, \\
-\Delta v + \nabla p &= F \quad \text{in} \ G, \\
v &= 0 \quad \text{on} \ \partial G.
\end{align*}
\]

Here \( W_{0,\sigma}^1(G) \) stands for closure of \( C_{c,\sigma}^\infty (\mathbb{R}^3) \) with respect to the norm in \( W^{1,s}(G) \), while \( L^s_0(G) \) denotes the subspace of \( L^s(G) \) with vanishing average. Then we set \( E_G^s(F) := \nabla p \), where \( \nabla p \) denotes the gradient function in \( W^{-1,s}(G) \) defined as

\[
\langle \nabla p, \varphi \rangle = -\int_G p \nabla \cdot \varphi \, dx, \quad \varphi \in W_{0}^{1,s'}(G).
\]

**Remark 1.1.** From the existence and uniqueness of weak solutions \((v, p)\) to (1.6) for given for any \( F \in W^{-1,s}(G) \) it follows that

\[
\|\nabla v\|_{s,G} + \|p\|_{s,G} \leq c\|F\|_{-1,s,G}, \tag{1.7}
\]

where \( c = \text{const} \) depending on \( s \) and the geometric properties of \( G \), and depends only on \( s \) if \( G \) equals a ball or an annulus which due to the scaling properties of the Stokes equation. In case \( F \) is given by \( \nabla \cdot f \) for \( f \in L^s(\mathbb{R}^3)^9 \) then (1.7) gives

\[
\|p\|_{s,G} \leq c\|f\|_{s,G}. \tag{1.8}
\]

According to the estimate \( \|\nabla p\|_{-1,s,G} \leq \|p\|_{s,G} \) and using (1.8), we see that the operator \( E_G^s \) is bounded in \( W^{-1,s}(G) \). Furthermore, as \( E_G^s(\nabla p) = \nabla p \) for all \( p \in L^s_0(G) \) we see that \( E_G^s \) defines a projection.
2. In case $F \in L^s(G)$, using the canonical embedding $L^s(G) \hookrightarrow W^{-1,s}(G)$, by the aid of elliptic regularity we get $E^*_G(F) = \nabla p \in L^s(G)$ together with the estimate
\begin{equation}
\|\nabla p\|_{s,G} \leq c\|F\|_{s,G},
\end{equation}
where the constant in (1.9) depends only on $s$ and $G$. In case $G$ equals a ball or an annulus this constant depends only on $s$ (cf. [3] for more details). Accordingly the restriction of $E^*_G$ to the Lebesgue space $L^s(G)$ appears to be a projection in $L^s(G)$. This projection will be denoted still by $E^*_G$.

**Definition 1.2** (Local Leray solution with projected pressure). Let $u_0 \in L^2_{loc}(\mathbb{R}^3)$. A vector function $u \in L^2_{loc,\sigma}(\mathbb{R}^3 \times [0, +\infty))$ is called a local Leray solution to (1.1)–(1.3) with projected pressure, if for any bounded $C^2$ domain $G \subset \mathbb{R}^3$ and $0 < T < +\infty$

1. $u \in V^2_\sigma(G \times (0, T)) \cap C_w([0, T]; L^2(G))$.

2. $u$ is a distributional solution to (1.2), i.e. for every $\varphi \in C^\infty_c(Q)$ with $\nabla \cdot \varphi = 0$
\begin{equation}
\int_Q -u \cdot \frac{\partial \varphi}{\partial t} - u \otimes u : \nabla \varphi + \nabla \varphi : \nabla \varphi dxdt = 0.
\end{equation}

3. $u(t) \to u_0$ in $L^2(G)$ as $t \to 0^+$.

4. The following local energy inequality with projected pressure holds for every nonnegative $\phi \in C^\infty_c(G \times (0, +\infty))$, and for almost every $t \in (0, +\infty)$
\begin{align}
\frac{1}{2} \int_G |v_G(t)|^2 \phi dx &+ \int_0^t \int_G |\nabla v_G|^2 \phi ds dx \\
\leq \frac{1}{2} \int_0^t \int_G |v_G|^2 \left(\Delta + \frac{\partial}{\partial t}\right) \phi + |v_G|^2 u \cdot \nabla \phi dx ds \\
&+ \int_0^t \int_G (u \otimes v_G) : \nabla^2 p_{h,G} \phi dx dt + \int_0^t \int_G p_{1,G} v_G \cdot \nabla \phi dx ds \\
&+ \int_0^t \int_G p_{2,G} v_G \cdot \nabla \phi dx ds.
\end{align}
(1.11)

where $v_G = u + \nabla p_{h,G}$, and
\begin{align*}
\nabla p_{h,G} &= -E^*_G(u), \\
\nabla p_{1,G} &= -E^*_G((u \cdot \nabla)u), \\
\nabla p_{2,G} &= E^*_G(\Delta u).
\end{align*}

**Remark 1.3.** 1. Note that due to $\nabla \cdot u = 0$ the pressure $p_{h,G}$ is harmonic, and thus smooth in $x$. Furthermore, as it has been proved in [12] the pressure gradient $\nabla p_{h,G}$ is continuous in $G \times [0, +\infty)$. 

4
2. The notion of local suitable weak solution to the Navier-Stokes equations satisfying the local energy inequality (1.11) has been introduced in [11]. As it has been shown there such solutions enjoy the same partial regularity properties as the usual suitable weak solutions in the Caffarelli-Kohn-Nirenberg theorem.

Our main result is the following

**Theorem 1.4.** For any λ-DSS initial data \( u_0 \in L^2_{loc,\sigma}(\mathbb{R}^3) \) there exists at least one local Leray solution with projected pressure \( u \in L^2_{loc,\sigma}(\mathbb{R}^3 \times [0, +\infty)) \) to the Navier-Stokes equations (1.1) - (1.3) in the sense of Definition 1.2, which is discretely self-similar.

## 2 Solutions of the linearized problem with initial velocity in \( L^2_{\lambda-DSS} \)

Let \( 1 < \lambda < +\infty \) be fixed. For \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) we denote \( f_\lambda(x) := \lambda f(\lambda x), x \in \mathbb{R}^3 \).

For a time dependent function \( f : Q \to \mathbb{R}^3 \) we denote \( f_\lambda(x, t) := \lambda f(\lambda x, \lambda^2 t), (x, t) \in \mathbb{R}^3 \times (0, +\infty) \). We now define for \( 1 \leq s \leq +\infty \)

\[
L^s_{\lambda-DSS}(\mathbb{R}^3) := \left\{ u \in L^1_{loc}(\mathbb{R}^3) \left| u \in L^s(B_\lambda \setminus B_1), u_\lambda = u \text{ a.e. in } \mathbb{R}^3 \right. \right\},
\]

\[
L^s_{\lambda-DSS}(Q) := \left\{ u \in L^1_{loc}(Q) \left| u \in L^s(Q_\lambda \setminus Q_1), u_\lambda = u \text{ a.e. in } Q \right. \right\}.
\]

Here \( B_r \) stands usual ball in \( \mathbb{R}^3 \) with center 0 and radius \( r > 0 \), while \( Q_r = B_r \times (0, r^2) \).

In the present section we consider the following linearized problem in \( Q \)

\[
(2.1) \quad \nabla \cdot u = 0,
\]

\[
(2.2) \quad \partial_t u + (b \cdot \nabla) u - \Delta u = -\nabla \pi
\]

with the initial condition

\[
(2.3) \quad u = u_0 \text{ on } \mathbb{R}^3 \times \{0\},
\]

where \( u_0 \) belongs to \( L^2_{\lambda-DSS}(\mathbb{R}^3) \) with \( \nabla \cdot u_0 = 0 \), and \( b \in L^3_{\lambda-DSS}(Q) \), \( 3 \leq s \leq 5 \) with \( \nabla \cdot b = 0 \) both in the sense of distributions. We give the following notion of a local solution with projected pressure for the linear system (2.1), (2.2).

**Definition 2.1** (Local solution with projected pressure to the linearized problem). Let \( u_0 \in L^2_{loc,\sigma}(\mathbb{R}^3) \) and let \( b \in L^3_{loc,\sigma}(\mathbb{R}^3 \times [0, +\infty)) \). A vector function \( u \in L^2_{loc,\sigma}(\mathbb{R}^3 \times [0, +\infty)) \) is called a local solution to (2.1) - (2.3) with projected pressure, if for any bounded \( C^2 \) domain \( G \subset \mathbb{R}^3 \) and \( 0 < T < +\infty \) the following conditions are satisfied

1. \( u \in V^2(G \times (0, T)) \cap C_w([0, T]; L^2(G)) \).

2. \( u \) is a distributional solution to (2.2), i.e. for every \( \varphi \in C^\infty_c(Q) \) with \( \nabla \cdot \varphi = 0 \)

\[
(2.4) \quad \int_Q -u \cdot \frac{\partial \varphi}{\partial t} - b \otimes u : \nabla \varphi + \nabla u : \nabla \varphi dx dt = 0.
\]
3. \( u(t) \to u_0 \) in \( L^2(G) \) as \( t \to 0^+ \).

4. the following local energy inequality with projected pressure holds for every non-negative \( \phi \in C_c^\infty(G \times (0, +\infty)) \), and for almost every \( t \in (0, +\infty) \)

\[
\frac{1}{2} \int_0^t \int_G |v_G(s, t)|^2 \phi dx + \int_0^t \int_G |\nabla v_G|^2 \phi dx ds \\
\leq \frac{1}{2} \int_0^t \int_G |v_G|^2 \left( \Delta + \frac{\partial}{\partial t} \right) \phi + |v_G|^2 b \cdot \nabla \phi dx ds \\
+ \int_0^t \int_G (b \otimes v_G) : \nabla^2 p_{h,G} \phi dt \\
+ \int_0^t \int_G p_{1,G} v_G \cdot \nabla \phi dx ds
\]

(2.5)

where \( v_G = u + \nabla p_{h,G} \), and

\[
\nabla p_{h,G} = -E_G^*(u), \\
\nabla p_{1,G} = -E_G^*((b \cdot \nabla)u), \\
\nabla p_{2,G} = E_G(\Delta u).
\]

**Theorem 2.2.** Let \( b \in L^3_{\lambda-DSS}(Q) \cap L^{3\frac{12}{13}}(0, T; L^3(B_1)) \), \( 0 < T < +\infty \), with \( \nabla \cdot b = 0 \) in the sense of distributions. Suppose that \( b \in L^3_{\text{loc}}(0, \infty; L^\infty(\mathbb{R}^3)) \). For every \( u_0 \in L^2_{\lambda-DSS}(\mathbb{R}^3) \) with \( \nabla \cdot u_0 = 0 \) in the sense of distributions, there exists a unique local solution with projected pressure \( u \in L^2_{\text{loc},\sigma}(\mathbb{R}^3 \times [0, +\infty)) \) to (2.1) – (2.3) according to Definition 2.1 such that for any \( 0 < \rho < +\infty \) and \( 0 < T < +\infty \) it holds

\[
(2.6) \quad u \in L^3_{\lambda-DSS}(Q), \\
(2.7) \quad u \in C([0, T]; L^2(B_\rho)), \\
(2.8) \quad \|u\|_{L^\infty(0,T;L^2(B_{\rho^\frac{2}{3}}))} + \|\nabla u\|_{L^2(B_{\rho^\frac{2}{3}} \times (0,T))} \leq C_0 K_0 \left( \rho^\frac{2}{3} + \|b\|^3 \max\{T^{\frac{14}{13}}, T^\frac{1}{2}\} \right), \\
(2.9) \quad \|u\|_{L^4(0,T;L^3(B_1))} \leq C_0 K_0 \left( 1 + \|b\|^3 \max\{T^{\frac{14}{13}}, T^\frac{1}{2}\} \right),
\]

where \( K_0 := \|u_0\|_{L^2(B_1)} \) and \( \|b\| = \|b\|_{L^{1\frac{12}{13}}(0,T;L^3(B_1))} \), while \( C_0 > 0 \) denotes a constant depending on \( \lambda \) only.

Before turning to the proof of Theorem 2.1, we show the existence and uniqueness of weak solutions to the linear system (2.1) – (2.3) for \( L^2_\rho \) initial data.

**Lemma 2.3.** Let \( b \in L^3_{\lambda-DSS}(Q) \cap L^{3\frac{12}{13}}(0, T; L^3(B_1)) \), \( 0 < T < +\infty \) with \( \nabla \cdot b = 0 \) in the sense of distributions. Suppose that \( b \in L^3_{\text{loc}}(0, \infty; L^\infty(B_1)) \). For every \( u_0 \in L^2_\rho(\mathbb{R}^3) \)
there exists a unique weak solution \( u \in V_0^2(Q) \cap C([0, +\infty); L^2(\mathbb{R}^3)) \) to (2.1)–(2.3), which satisfies the global energy equality for all \( t \in [0, +\infty) \)

\[
(2.10) \quad \frac{1}{2} \|u(t)\|_2^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx ds = \frac{1}{2} \|u_0\|_2^2.
\]

**Proof:** 1. **Existence:** By using standard linear theory of parabolic systems we easily get the existence of a weak solution \( u \in V_0^2(Q) \cap C([0, +\infty); L^2(\mathbb{R}^3)) \) to (2.1)–(2.3) which satisfies the global energy inequality for almost all \( t \in (0, +\infty) \)

\[
(2.11) \quad \frac{1}{2} \|u(t)\|_2^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx ds \leq \frac{1}{2} \|u_0\|_2^2.
\]

It is well known that such solutions have the property

\[
(2.12) \quad u(t) \to u_0 \quad \text{in} \quad L^2(\mathbb{R}^3) \quad \text{as} \quad t \to 0^+.
\]

On the other hand, from the assumption of the Lemma it follows that for all \( t_0 \in (0, T) \)

\[
\|b u\|_{L^2(\mathbb{R}^3 \times (t_0, T))} \leq \|b\|_{L^\infty(\mathbb{R}^3 \times (t_0, T))} \|u_0\|_2.
\]

Accordingly, \( u \in C((0, T]; L^2(\mathbb{R}^3)) \) and for all \( t_0 \in (0, T] \) and \( t \in [t_0, T] \) the following energy equality holds true

\[
(2.13) \quad \frac{1}{2} \|u(t)\|_2^2 + \int_{t_0}^t \int_{\mathbb{R}^3} |\nabla u|^2 dx ds = \frac{1}{2} \|u(t_0)\|_2^2.
\]

Now letting \( t_0 \to 0 \) in (2.13), and observing (2.12), we are led to (2.10).

By a similar argument, making use of (2.12) we easily prove the local energy inequality (2.3).

2. **Uniqueness:** Let \( v \in V_0^2(Q) \) be a second solution to (2.1)–(2.3) satisfying the global energy equality. As we have seen above this solution belongs to \( C([0, +\infty); L^2(\mathbb{R}^3)) \). Setting \( w = u - v \), by our assumption on \( b \) it follows that \( b \otimes w \in L^2(\mathbb{R}^3 \times (t_0, T)) \) for any \( t_0 \in (0, T] \). Accordingly, as above we get the following energy equality

\[
(2.14) \quad \frac{1}{2} \|w(t)\|_2^2 + \int_{t_0}^t \int_{\mathbb{R}^3} |\nabla w|^2 dx ds = \frac{1}{2} \|w(t_0)\|_2^2.
\]

Verifying that \( w(t_0) \to 0 \) in \( L^2(\mathbb{R}^3) \) as \( t_0 \to 0^+ \) from (2.14) letting \( t_0 \to 0^+ \) it follows that \( \|w(t)\|_2 = 0 \) for all \( t \in [0, T] \). This completes the proof of the uniqueness. \( \blacksquare \)

**Proof of Theorem 2.2** Since \( u_0 \) is \( \lambda \)-DSS we have \( \lambda u_0(\lambda x) = u_0(x) \) for all \( x \in \mathbb{R}^3 \).

We define the extended annulus \( \tilde{A}_k = B_{\lambda k} \setminus B_{\lambda k-3}, k \in \mathbb{N} \). Clearly, \( B_1 \cup (\cup_{k=1}^\infty \tilde{A}_k) = \mathbb{R}^3 \).

There exists a partition of unity \( \{\psi_k\} \) such that \( \text{supp} \psi_k \subset \tilde{A}_k \) for \( k \in \mathbb{N} \) and \( \text{supp} \psi_0 \subset \mathbb{R}^3 \)
$B_1$, and $0 \leq \psi_k \leq 1$, $|\nabla^2 \psi_k| + |\nabla \psi_k|^2 \leq \lambda^{-2k}$, $k \in \mathbb{N} \cup \{0\}$. We set $u_{0,k} = P(u_0 \psi_k)$, $k \in \mathbb{N} \cup \{0\}$. Clearly,

$$u_0 = \sum_{k=0}^{\infty} u_{0,k},$$

where the limit in (2.15) is taken in the sense of $L^2_{loc}(\mathbb{R}^3)$.

Let $k \in \mathbb{N} \cup \{0\}$ be fixed. Thanks to Lemma 2.3 we get a unique weak solution $u_k \in V^2_\sigma(Q)$ to the problem

\begin{align*}
(2.16) & \quad \nabla \cdot u_k = 0 \text{ in } Q, \\
(2.17) & \quad \partial_t u_k + (b \cdot \nabla) u_k - \Delta u_k = -\nabla \pi_k \text{ in } Q, \\
(2.18) & \quad u_k = u_{0,k} \text{ on } \mathbb{R}^3 \times \{0\},
\end{align*}

satisfying the following global energy equality for all $t \in [0, +\infty)$

$$\frac{1}{2} \|u_k(t)\|_2^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u_k|^2 dx ds = \frac{1}{2} \|u_{0,k}\|_2^2. \tag{2.19}$$

By using the transformation formula, we get

$$\|u_{0,k}\|_2^2 \leq \int_{\mathbb{R}^3} |u_0 \psi_k|^2 dx \leq \int_{\mathbb{A}_k} \|u_{0}\|_2^2 dx = \lambda^{3k} \int_{\mathbb{A}_1} \|u_0(\lambda^k x)\|_2^2 dx = \lambda^k \int_{\mathbb{A}_1} |\lambda^k u_0(\lambda^k x)|^2 dx = \lambda^k \int_{\mathbb{A}_1} |u_0(x)|^2 dx \leq cK^2_0 \lambda^k. \tag{2.20}$$

Combining (2.19) and (2.20), we are led to

$$\|u_k\|_{L^\infty(0,T;L^2)}^2 + \|\nabla u_k\|_{L^2(0,T;L^2)}^2 \leq cK^2_0 \lambda^k. \tag{2.21}$$

Next, let $\lambda^\frac{4}{3} k \leq r < \rho \leq \lambda^\frac{4}{3}(k+1)$ be arbitrarily chosen, but fixed. By introducing the local pressure we have

$$\frac{\partial v_{k,\rho}}{\partial t} + (b \cdot \nabla) u_k - \Delta v_{k,\rho} = -\nabla p_{1,k,\rho} - \nabla p_{2,k,\rho},$$

where $v_{k,\rho} = u_k + \nabla p_{h,k,\rho}$, and

$$\nabla p_{h,k,\rho} = -E^*_{B_\rho}(u_k), \quad \nabla p_{1,k,\rho} = -E^*_{B_\rho}((b \cdot \nabla) u_k), \quad \nabla p_{2,k,\rho} = E^*_{B_\rho}(\Delta u_k).$$
The following local energy equality holds true for all $\phi \in C^\infty_c(B_\rho)$ and for all $t \in [0,T]$,

\[
\frac{1}{2} \int \frac{v_{k,\rho}(t)^2}{B_\rho} \phi \, dx + \frac{1}{2} \int_0^t \int \frac{\nabla v_{k,\rho}}{B_\rho}^2 \phi \, dx \, ds = \frac{1}{2} \int \frac{v_{k,\rho}}{B_\rho}^{\Delta \phi^6} \phi \, dx \, ds \bigg|_{b} \int \frac{v_{k,\rho}}{B_\rho} \phi \, dx \bigg|_{b} \int \frac{v_{k,\rho}}{B_\rho} \phi \, dx \bigg|_{b} \int \frac{v_{k,\rho}}{B_\rho} \phi \, dx
\]

(2.22) \[= I + II + III + IV + V + VI.\]

Let $\phi \in C^\infty_c(\mathbb{R}^3)$ denote a cut off function such that $0 \leq \phi \leq 1$ in $\mathbb{R}^3$, $\phi \equiv 1$ on $B_r$, $\phi \equiv 0$ in $\mathbb{R}^3 \setminus B_\rho$, and $|\nabla^2 \phi| + |
abla \phi|^2 \leq c (\rho - r)^{-2}$ in $\mathbb{R}^3$.

Let $m \in \mathbb{N}$ be chosen so that $\lambda^{m-1} \leq \rho < \lambda^m$. Then we estimate

\[
\left\| \frac{b}{L^3(B_\rho \times (0,T))} \right\| = \lambda^{m} \int_{B_{\rho^{\lambda^{-m}}}^{T \lambda^{-2m}}} \left| b(\lambda^{-m} x, \lambda^{-2m} t) \right|^3 \, dx \, dt
\]

(2.23) \[\leq \lambda^{m} \int_{B_{\rho^{\lambda^{-m}}}^{T \lambda^{-2m}}} \left| b(\lambda^{-m} x, \lambda^{-2m} t) \right|^3 \, dx \, dt \]

where and hereafter the constants appearing in the estimates may depend on $\lambda$. The above estimate together with $\rho^{\frac{3}{2}} \leq \lambda^{k+1}$ yields

(2.24) \[\left\| \frac{b}{L^3(B_\rho \times (0,T))} \right\| \leq c \left\| b \right\| \lambda^{\frac{k}{12}} T^{\frac{1}{12}}.\]

In what follows we extensively make use of the estimate for almost all $t \in (0,T)$

(2.25) \[\left\| \nabla p_{h,k,\rho}(t) \right\|_{L^2(B_\rho)} \lesssim \left\| u_k(t) \right\|_{L^2(B_\rho)},\]

which is an immediate consequence of (1.9). In addition, we easily verify the inequality

Indeed, observing that

\[
\nabla^2 p_{h,k,\rho}(t) = \nabla (\nabla p_{h,k,\rho}(t) - u(t) B_\rho) = -\nabla \nabla^* E_{B_\rho}(u_k(t) - u_k(t) B_\rho)
\]

9
by means of elliptic regularity along with the Poincaré inequality we get
\[
\| \nabla^2 p_{h,k,\rho}(t) \|_{L^2(B_\rho)}^2 \leq c \rho^{-2} \| u_k(t) - u_k(t)_{B_\rho} \|_{L^2(B_\rho)}^2 + c \| \nabla u_k(t) \|_{L^2(B_\rho)}^2 \\
\leq c \| \nabla u_k(t) \|_{L^2(B_\rho)}^2.
\]
Whence, (2.25).

(i) With the help of (2.21) we easily deduce that
\[
I \leq c (\rho - r)^{-2} \int_0^t \int_{B_\rho} |u_k|^2 dx dt \leq c K_0^2 (\rho - r)^{-2} \lambda^k T.
\]

(ii) Next, using Hölder’s inequality and Young’s inequality together with (2.21), (2.23), (2.21) and (2.25), we estimate
\[
\| b \|_{L^1(B_\rho)} \| v_{k,\rho} \|_{L^5} \| T \|_{L^3} \leq c \| b \|_{L^1(B_\rho)} \| v_{k,\rho} \|_{L^5} \| T \|_{L^3} \\
+ c \| b \|_{L^1(B_\rho)} \| v_{k,\rho} \|_{L^5} \| T \|_{L^3} \leq c \| b \|_{L^1(B_\rho)} \| v_{k,\rho} \|_{L^5} \| T \|_{L^3}.
\]

(iii) In what follows we make use the following estimates using the fact that \( p_{h,k,\rho} \) is harmonic. By using the identity
\[
\int_{\mathbb{R}^3} |\nabla h|^2 \phi^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} h^2 \Delta \phi^2 dx
\]
for any harmonic function \( h \) on \( B_\rho \), and cut off function \( \phi \in C_0^\infty(B_\rho) \), we get
\[
\| \nabla^3 p_{h,k,\rho}(t) \|_2 \leq c (\rho - r)^{-1} \| \nabla^2 p_{h,k,\rho}(t) \|_2 \leq (\rho - r)^{-2} \| \nabla p_{h,k,\rho}(t) \|_{2,B_\rho}.
\]
By the aid of Sobolev’s inequality, together with (2.26), we get for almost every \( t \in (0, T) \)

\[
\|\nabla^2 p_{h,k,\rho}(t)\phi^3\|_a \leq c(\rho - r)^{-1}\|\nabla^2 p_{h,k,\rho}(t)\phi^2\|_{2,B_\rho} + c\|\nabla^3 p_{h,k,\rho}(t)\phi^3\|_2
\]

\[
\leq c(\rho - r)^{-1}\|\nabla^2 p_{h,k,\rho}(t)\phi^2\|_{2,B_\rho}
\]

\[
\leq c(\rho - r)^{-2}\|\nabla p_{h,k,\rho}(t)\|_{2,B_\rho}
\]

\[
\leq c(\rho - r)^{-2}\|u_k(t)\|_{2,B_\rho}.
\]

Integrating both sides of the above estimate, and estimating the right-hand side of of the resultant inequality by (2.21), we arrive at

\[
(2.27) \quad \|\nabla^2 p_{h,k,\rho}\phi^3\|_{L^2(0,T;L^6)} \leq c(\rho - r)^{-2}T^{\frac{1}{2}}K_0\lambda^{\frac{1}{2}}
\]

Arguing as above, and using (2.27), we find

\[
III \leq cT^{\frac{1}{2}}\|b\|_{L^3(0,T;L^3(B_\rho))}\|v_k\phi^3\|_{L^\infty(0,T;L^2)}\|\nabla^2 p_{h,k,\rho}\phi^3\|_{L^2(0,T;L^6)}
\]

\[
\leq cK_0(\rho - r)^{-2}T^{\frac{1}{2}}\lambda^{\frac{1}{2}}\|b\|_{L^3(0,T;L^3(B_\rho))}\|v_k\phi^3\|_{L^\infty(0,T;L^2)}
\]

\[
\leq c\|b\|\|K_0(\rho - r)^{-2}\lambda^{\frac{1}{2}}T^{\frac{13}{18}}\|v_k\phi^3\|_{L^\infty(0,T;L^2)}
\]

\[
\leq c\|b\|\|K_0(\rho - r)^{-1}\lambda^{\frac{1}{2}}T^{\frac{13}{18}}\|u_k\|_{L^\infty(0,T;L^2)}\|v_k\phi^3\|_{L^\infty(0,T;L^2)}
\]

\[
+ c\|b\|(\rho - r)^{-1}\lambda^{\frac{3}{2}}T^{\frac{13}{18}}\|v_k\phi^3\|_{L^\infty(0,T;L^2)}\|\nabla u_k\|_{L^2(0,T;L^2)}\|\nabla u_k\|_{L^2(0,T;L^2)}
\]

\[
\leq c\|b\|\|K_0(\rho - r)^{-1}\lambda^{\frac{7}{2}}T^{\frac{13}{18}}\|v_k\phi^3\|_{L^\infty(0,T;L^2)}
\]

\[
+ c\|b\|\|K_0(\rho - r)^{-1}\lambda^{\frac{7}{2}}T^{\frac{13}{18}}\|v_k\phi^3\|_{L^\infty(0,T;L^2)}\|\nabla u_k\|_{L^2(0,T;L^2)}^2
\]

\[
\leq c\|b\|^2K_0^2(\rho - r)^{-2}\lambda^{\frac{3}{2}}T^{\frac{13}{18}} + c\|b\|^6K_0^2(\rho - r)^{-6}\lambda^{\frac{3}{2}}T^{\frac{13}{18}}
\]

\[
+ \frac{1}{8}\|v_k\phi^3\|_{L^\infty(0,T;L^2)}^2 + \frac{1}{4}\|\nabla u_k\|_{L^2(0,T;L^2(B_\rho))}^2
\]

\[
\leq (1 + \|b\|^6)K_0^2(\rho - r)^{-6}\lambda^{\frac{3}{2}}\max\{T^{\frac{13}{18}}, T\}
\]

\[
+ \frac{1}{8}\|v_k\phi^3\|_{L^\infty(0,T;L^2)}^2 + \frac{1}{4}\|\nabla u_k\|_{L^2(0,T;L^2(B_\rho))}^2.
\]
(v) Recalling the definition of $p_{2,k,\rho}$, using (1.8), (2.21) and Young’s inequality, we get

$$V \leq c(\rho - r)^{-1}\|p_{2,k,\rho}\|_{L^2(0,T;L^2(B_\rho))}\|v_{k,\rho}\phi^3\phi_{3}\|_{L^2(0,T;L^2)}$$

$$\leq c(\rho - r)^{-1}T^{\frac{1}{2}}\left(\int_0^T \int_{B_\rho} |\nabla u_k|^2 dx dt\right)^{\frac{1}{2}}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)}$$

$$\leq cK_0^2(\rho - r)^{-2}\lambda^k T + \frac{1}{8}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)}^2$$

$$\leq cK_0^2(\rho - r)^{-2}\lambda^k T + \frac{1}{8}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)}^2.$$

(vi) It only remains to evaluate $VI$. Let $k \geq 9$. Then $\frac{3}{6}(k + 1) \leq k - 3$. Thus, $\text{supp}(\psi_k) \cap B_\rho = \emptyset$. In particular, $\psi_k u_0 = 0$ in $B_\rho$. This shows that, almost everywhere in $B_\rho$ it holds

$$u_{0,k} = \mathbb{P}(\psi_k u_0) - \psi_k u_0$$

which is a gradient field. Accordingly, almost everywhere in $B_\rho$

$$v_{0,k} = u_{0,k} - E_{B_\rho}(u_{0,k}) = u_{0,k} - u_{0,k} = 0.$$

Hence

$$VI = 0.$$

For $k \leq 8$ we find

$$VI \leq \|u_{0,k}\|_{L^2(B_\rho)}^2 \leq c \sum_{k=0}^8 \|u_0\psi_k\|_{L^2}^2 \leq c\|u_0\|_{L^2(B_{\lambda^8})}^2 \leq cK_0^2.$$

We now insert the above estimates of $I, \ldots, VI$ into the right-hand side of (2.23). This gives

$$\text{ess sup}_{t \in (0,T)} \int_{B_\rho} \left|v_{k,\rho}(t)\right|^2 \phi^6 dx + \int_0^T \int_{B_\rho} |\nabla v_{k,\rho}|^2 \phi^6 dx dt$$

$$\leq cK_0^2 \max\{8 - k, 0\} + c(1 + \|b\|_6)K_0^2 \max\{T^{\frac{1}{12}}, T\}(\rho - r)^{-6}\lambda^k T$$

$$+ \frac{1}{4} \int_0^T \int_{B_\rho} |\nabla u_k|^2 dx dt. \quad (2.28)$$

On the other hand, employing (2.26) and (2.21)

$$\int_{B_\rho} |\nabla^2 p_{h,k,\rho}|^2 \phi^6 dx dt \leq cK_0^2(\rho - r)^{-2}\lambda^k T,$$
we estimate
\[ \int_0^T \int_{B_r} |\nabla u_k|^2 \, dx \, dt \leq 2 \int_0^T \int_{B_\rho} |\nabla v_{k,\rho}|^2 \phi^6 \, dx \, dt + 2 \int_0^T \int_{B_\rho} |\nabla^2 p_{h,k,\rho}|^2 \phi^6 \, dx \, dt \]
\[ \leq 2 \int_0^T \int_{B_\rho} |\nabla v_{k,\rho}|^2 \phi^6 \, dx \, dt + cK_0^2 (\rho - r)^{-2} \lambda^k T \]
(2.29)

Combining (2.28) and (2.29), we are led to
\[ \int_0^T \int_{B_r} |\nabla u_k|^2 \, dx \, dt \leq cK_0^2 \max\{8 - k, 0\} + c(1 + \|b\|_6)K_0^2 \max\{T^{13/4}, T\}(\rho - r)^{-6} \lambda^{1/2} \lambda^k \]
\[ + \frac{1}{2} \int_0^T \int_{B_\rho} |\nabla u_k|^2 \, dx \, dt. \]
(2.30)

By virtue of a routine iteration argument from (2.30) we get for all \( \rho \in [\lambda^{1/2}, 2\lambda^{1/2}] \)
\[ \text{ess sup}_{t \in (0,T)} \int_{B_{\rho/2}} |v_{k,\rho}(t)|^2 \, dx + \int_0^T \int_{B_\rho} |\nabla u_k|^2 \, dx \, dt \leq cK_0^2 \max\{8 - k, 0\} + c(1 + \|b\|_6)K_0^2 \max\{T^{13/4}, T\}(\rho - r)^{-6} \lambda^{1/2} \lambda^k \]
\[ \leq cK_0^2 \max\{8 - k, 0\} + c(1 + \|b\|_6)K_0^2 \max\{T^{13/4}, T\}\lambda^{-1/2} \lambda^k. \]
(2.31)

In addition, by using the mean value property of harmonic functions along with (2.21), we estimate for almost all \( t \in (0,T) \)
\[ \|\nabla p_{h,k,\rho}(t)\|_{L^2(B_{\lambda^{1/2}})}^2 \leq c\lambda^{1/2} \|\nabla p_{h,k,\rho}(t)\|_{L^\infty(B_{\rho/2})}^2 \]
\[ \leq c\lambda^{-\frac{21}{20}k} \|\nabla p_{h,k,\rho}(t)\|_{L^2(B_\rho)}^2 \]
\[ \leq c\lambda^{-\frac{21}{20}k} \|u_k\|_{L^\infty(0,T;L^2(B_\rho))}^2 \leq cK_0^2 \lambda^{-\frac{1}{20}k}. \]

Combining this estimate with (2.31), we obtain
\[ \text{ess sup}_{t \in (0,T)} \int_{B_{\lambda^{1/2}}} |u_k(t)|^2 \, dx + \int_0^T \int_{B_{\lambda^{1/2}}} |\nabla u_k|^2 \, dx \, dt \leq cK_0^2 \left(1 + \|b\|_6 \max\{T^{13/4}, T\}\right)\lambda^{-\frac{1}{20}k}. \]
(2.32)
Next, let \( l \in \mathbb{N} \) be fixed. Then (2.32) implies for all \( k \geq l \)
\[
\| u_k \|_{L^\infty(0,T;L^2(B_{\lambda^{\frac{k}{l}}}))} + \| \nabla u_k \|_{L^2(B_{\lambda^{\frac{k}{l}}} \times (0,T))} \\
\leq \| u_k \|_{L^\infty(0,T;L^2(B_{\lambda^{\frac{k}{l}}}))} + \| \nabla u_k \|_{L^2(B_{\lambda^{\frac{k}{l}}} \times (0,T))} \\
\leq cK_0 \left( 1 + \| b \|_3^3 \max\{ T^{\frac{13}{18}}, T^{\frac{1}{2}} \} \right) \lambda^{-\frac{14}{18}k}.
\]
Thus, by means of triangular inequality we find for each \( N \in \mathbb{N} \), \( N > l \)
\[
\left\| \sum_{k=0}^N u_k \right\|_{L^\infty(0,T;L^2(B_{\lambda^{\frac{k}{l}}}))} + \left\| \sum_{k=0}^N \nabla u_k \right\|_{L^2(B_{\lambda^{\frac{k}{l}}} \times (0,T))} \\
\leq \sum_{k=0}^{l-1} \| u_k \|_{L^\infty(0,T;L^2(B_{\lambda^{\frac{k}{l}}}))} + \sum_{k=0}^{l-1} \| \nabla u_k \|_{L^2(B_{\lambda^{\frac{k}{l}}} \times (0,T))} \\
+ \sum_{k=l}^N \| u_k \|_{L^\infty(0,T;L^2(B_{\lambda^{\frac{k}{l}}}))} + \sum_{k=0}^N \| \nabla u_k \|_{L^2(B_{\lambda^{\frac{k}{l}}} \times (0,T))} \\
\leq cK_0 \lambda^{\frac{14}{18}} \lambda^{-\frac{14}{18}l} + cK_0 \left( 1 + \| b \|_3^3 \max\{ T^{\frac{13}{18}}, T^{\frac{1}{2}} \} \right) \\
\leq cK_0 \left( \lambda^{\frac{14}{18}} + \| b \|_3^3 \max\{ T^{\frac{13}{18}}, T^{\frac{1}{2}} \} \right).
\]
Therefore, \( u^N = \sum_{k=0}^N u_k \to u \) in \( V^2_{\text{loc}}(\mathbb{R}^3 \times [0, T]) \) as \( N \to \infty \). It is readily seen that \( u \) is a weak solution to (1.1)–(1.3), and by virtue of the above estimate we see that for every \( 1 \leq \rho < \infty \)
\[
\| u \|_{L^\infty(0,T;L^2(B_{\lambda^{\frac{k}{l}}}))} + \| \nabla u \|_{L^2(\mathbb{R}^3 \times (0,T))} \leq cK_0 \left( \rho^{\frac{3}{2}} + \| b \|_3^3 \max\{ T^{\frac{13}{18}}, T^{\frac{1}{2}} \} \right).
\]
In particular, in (2.34) taking \( \rho = 1 \), and using Sobolev’s embedding theorem, we get
\[
\| u \|_{L^4(0,T;L^3(B_{\lambda^{\frac{k}{l}}}))} + \| u \|_{V^2(B_{\lambda^{\frac{k}{l}}} \times (0,T))} \leq C_0 K_0 \left( 1 + \| b \|_3^3 \max\{ T^{\frac{13}{18}}, T^{\frac{1}{2}} \} \right)
\]
with a constant \( C_0 > 0 \) depending only on \( \lambda \).

It remains to show that \( u_\lambda = u \). Let \( N \in \mathbb{N} \), \( N \geq 4 \). We set \( w^N = u^N - u^N_\lambda \). Recalling that \( b = b_\lambda \), it follows that \( w^N \) solves the system
\[
\nabla \cdot w^N = 0 \quad \text{in} \quad Q_{\lambda^{-2}T},
\]
\[
\nabla \cdot (b \cdot \nabla) w^N - \Delta w^N = -\nabla \pi^N \quad \text{in} \quad Q_{\lambda^{-2}T},
\]
\[
w^N = w^N_0 \quad \text{on} \quad \mathbb{R}^3 \times \{0\},
\]
where
\[
w^N_0 = \sum_{k=0}^N \left( u^N_{0,k} - (u^N_{0,k})_\lambda \right) = \sum_{k=0}^N \left[ \mathbb{P}(u^N_0 \psi_k) - (\mathbb{P}(u^N_0 \psi_k))_\lambda \right] \\
= u^N_0 \sum_{k=0}^N \psi_k - \left( u^N_0 \sum_{k=0}^N \psi_k \right)_\lambda + \nabla \mathcal{N} \ast (u^N_0 \cdot \nabla \sum_{k=0}^N \psi_k) - \left( \nabla \mathcal{N} \ast (u^N_0 \cdot \nabla \sum_{k=0}^N \psi_k) \right)_\lambda \\
= u^N_0 \left( \sum_{k=0}^N \psi_k - \left( \sum_{k=0}^N \psi_k \right)_\lambda \right) + \nabla \mathcal{N} \ast (u^N_0 \cdot \nabla \sum_{k=0}^N \psi_k) - \left( \nabla \mathcal{N} \ast (u^N_0 \cdot \nabla \sum_{k=0}^N \psi_k) \right)_\lambda,
\]
where $N = \frac{1}{4\pi|x|}$ stands for the Newton potential. For obtaining the third line in the above equalities we used the fact that $(u_0)_\lambda = u_0$. Owing to $\sum_{k=0}^{N} \psi_k = 1$ in $B_{\lambda^{-4}}$ we have

$$
(\sum_{k=0}^{N} \psi_k - (\sum_{k=0}^{N} \psi_k)(\lambda \cdot)) = 0 \text{ in } B_{\lambda^{-4}}.
$$

Let $\lambda^{\frac{1}{3}} N \leq r < \rho \leq \lambda^{\frac{1}{3}} (N + 1)$ be arbitrarily chosen, but fixed. Let $\phi \in C^\infty_c(\mathbb{R}^3)$ denote a cut off function such that $0 \leq \phi \leq 1$ in $\mathbb{R}^3$, $\phi \equiv 1$ on $B_r$, $\phi \equiv 0$ in $\mathbb{R}^3 \setminus B_\rho$, and $|\nabla^2 \phi| + |\nabla \phi|^2 \leq c(\rho - r)^{-2}$ in $\mathbb{R}^3$. Without loss of generality we may assume that $\lambda^{\frac{1}{3}} (N + 1) \leq \lambda^{-1}$. Thus, in view of (2.39) we infer that $w_0^N$ is a gradient field in $B_\rho$, and therefore

$$
\|w_0^N\|_2^{L^2(0, \lambda^{-2}T; L^6(B_\rho))} + \int_0^{\lambda^{-2}T} \int_{B_\rho} |\nabla w_0^N|^2 dx dt
$$

$$
\leq cK_0^2(1 + \|b\|^6) \max\{T^{\frac{12}{5}}, T\}(\rho - r)^{-6} \lambda^{\frac{17}{5}N} + \frac{1}{2} \int_0^{\lambda^{-2}T} \int_{B_\rho} |\nabla w_0^N|^2 dx dt.
$$

Once more applying an iteration argument, together with the latter estimate, we deduce from (2.41)

$$
\|w_0^N\|_2^{L^2(0, \lambda^{-2}T; L^6(B_{\lambda^{\frac{1}{3}}N}))} \leq cK_0^2(1 + \|b\|^6) \max\{T^{\frac{12}{5}}, T\} \lambda^{-2}N.
$$

Accordingly, for all $0 < \rho < \infty$,

$$
w_0^N \rightarrow 0 \text{ in } L^2(0, \lambda^{-2}T; L^6(B_\rho)) \text{ as } N \rightarrow +\infty.
$$

On the other hand, observing that $w_0^N = u_0^N - (u_0^N)_\lambda \rightarrow u - u_\lambda$ in $L^2(0, \lambda^{-2}T; L^6(B_\rho))$ as $N \rightarrow \infty$, we conclude that $u = u_\lambda$. This completes the proof of the theorem.

### 3 Proof of Theorem 1.4

We divide the proof in three steps. Firstly, given a $\lambda$-DSS function $b \in L_{\text{loc}}^{\frac{18}{5}}([0, \infty); L_\text{loc}^3(\mathbb{R}^3))$ we get the existence of a unique $\lambda$-DSS local solution with projected pressure $u$ to the linearized system (2.21–2.23), replacing $b$ by $R_\varepsilon b$ therein (cf. appendix for the notion of the mollification $R_\varepsilon$). Secondly, based on the first step we may construct a mapping $\mathcal{T} : M \rightarrow M$, which is continuous and compact. Application of Schauder’s fixed point theorem gives a local suitable solution with projected pressure to the approximated Navier-Stokes equation. Thirdly, letting $\varepsilon \rightarrow 0^+$ in the weak formulation and in the
local energy inequality (2.5), we obtain the existence of the desired local Leray solution with projected pressure to (1.1)–(1.3).

We set
\begin{equation}
(3.1) \quad T := \min \left\{ \frac{1}{64C_0^2K_0^2}, \left( \frac{1}{64C_0^2K_0^2} \right)^{\frac{9}{13}} \right\}.
\end{equation}
Furthermore, set $X = L^3_{A-DSS}(Q) \cap L^3_{T_2}(0, T; L^3_{loc, \sigma}(\mathbb{R}^3))$ equipped with the norm
\[ \|v\| := \|v\|_{L^3_T L^3(B_1)}; \quad v \in X. \]
Then we define,
\[ M = \left\{ b \in X \mid \|b\| \leq 2C_0K_0 \right\}. \]
We now fix $0 < \varepsilon < \lambda - 1$. For $b \in M$ we set
\[ b_\varepsilon := R_\varepsilon b, \]
where $R_\varepsilon$ stands for the mollification operator defined in the appendix below. According to Theorem 2.2 there exists a unique $\lambda$-DSS solution $u \in X$ to (2.1)–(2.3) with $b_\varepsilon$ in place of $b$. Observing (2.3), it follows that
\begin{equation}
(3.2) \quad \|u\|_{L^1_T L^3(B_1)} + \|u\|_{V^2(B_1 \times (0, T))} \leq C_0K_0 \left( 1 + \|b\|^3 \max\{T^{\frac{3}{13}}, T^{\frac{2}{7}}\} \right).
\end{equation}
In view of (A.2) having $\|b_\varepsilon\|^3 \leq \lambda^2 \|b\|^3$, (3.2) together with (3.1) implies that
\[ \|u\| \leq 2C_0K_0, \]
and thus $u \in M$. By setting $\mathcal{T}_\varepsilon(b) := u$ defines a mapping $\mathcal{T}_\varepsilon : M \to M$.

$\mathcal{T}_\varepsilon$ is closed. In fact, let $\{b_k\}$ be a sequence in $M$ such that $b_k \to b$ in $X$ as $k \to \infty$, and let $u_k := \mathcal{T}_\varepsilon(b_k)$, $k \in \mathbb{N}$, such that $u_k \to u$ in $X$ as $k \to \infty$. From (3.2) it follows that $\{u_k\}$ is bounded in $V^2_\sigma(B_1 \times (0, T))$, and thus, eventually passing to a subsequence, we find $u_k \to u$ weakly in $V^2_\sigma(B_1 \times (0, T))$ as $k \to \infty$. Since $u_k$ solves (2.1)–(2.3) with $b_{k, \varepsilon} = R_\varepsilon b_k$ in place of $b$, from the above convergence properties we deduce that $u \in M \cap V^2_\sigma(B_1 \times (0, T))$ solves (2.1)–(2.3). Accordingly, $u = \mathcal{T}_\varepsilon(b)$.

$\mathcal{T}_\varepsilon(M)$ is relative compact in $X$. To see this, let $\{u_k = \mathcal{T}_\varepsilon(b_k)\} \subset \mathcal{T}_\varepsilon(M)$ be any sequence. Then $u_k \in L^2_{loc, \sigma}(\mathbb{R}^3 \times [0, \infty))$ is a $\lambda$-DSS local suitable weak solution with projected pressure to
\begin{align}
(3.3) & \quad \nabla \cdot u_k = 0 \quad \text{in} \quad Q, \\
(3.4) & \quad \partial_t u_k + (b_{k, \varepsilon} \cdot \nabla) u_k - \Delta u_k = -\nabla \pi_k \quad \text{in} \quad Q, \\
(3.5) & \quad u_k = u_0 \quad \text{on} \quad \mathbb{R}^3 \times \{0\}.
\end{align}
Introducing the local pressure, we have
\begin{equation}
(3.6) \quad \partial_t u_k + (b_{k, \varepsilon} \cdot \nabla) u_k - \Delta u_k = -\nabla \pi_{1, k} - \nabla \pi_{2, k} \quad \text{in} \quad B_2 \times (0, T),
\end{equation}
where $v_k = u_k + \nabla p_{h,k}$, and
\[
\nabla p_{h,k} = -E_{B_2}^*(u_k),
\]
\[
\nabla p_{1,k} = -E_{B_2}^*((b_{k}\varepsilon \cdot \nabla)u_k), \quad \nabla p_{2,k} = E_{B_2}^*(\Delta u_k).
\]

Thus, (3.11) implies that $v_k' = \nabla \cdot (-b_{k}\varepsilon \otimes u_k + \nabla u_k - p_{1,k}I - p_{2,k}I)$ in $B_2 \times (0, T)$. Since $b_k, u_k \in M$ we get the estimate
\[
|| -b_{k}\varepsilon \otimes u_k + \nabla u_k - p_{1,k}I - p_{2,k}I||_{L^2(0, T; L^2(B_2))} \leq c(1 + C_0^2 K_0^2).
\]

Furthermore, by means of the reflexivity of $L^2(0, T; W^{1, 2}(B_2))$, and using Banach-Alaoglu’s theorem we get a subsequence $\{u_{k_j}\}$ and a function $u \in M \cap V^2_{loc,\sigma}(\mathbb{R}^3 \times [0, T])$ such that
\[
u_{k_j} \to u \quad weakly \ in \ L^2(0, T; W^{1, 2}(B_2)),
\]
\[
u_{k_j} \to u \quad weakly^* \ in \ L^\infty(0, T; L^2(B_2)) \quad as \quad j \to \infty.
\]

In particular, we have for almost every $t \in (0, T)$
\begin{equation}
(3.7) \quad u_{k_j}(t) \to u(t) \quad weakly \ in \ L^2(B_2) \quad as \quad j \to \infty.
\end{equation}

In addition, verifying that $\{v_{k_j}\}$ is bounded in $V^2(B_2 \times (0, T))$, by Lions-Aubin’s compactness lemma we see that
\begin{equation}
(3.8) \quad v_{k_j} \to v \quad in \ L^2(B_2 \times (0, T)) \quad as \quad j \to +\infty,
\end{equation}
where $v = u + \nabla p_h$, and $\nabla p_h = -E^*(u)$. Now, let $t \in (0, T)$ be fixed such that (3.7) is satisfied. Then
\begin{equation}
(3.9) \quad \nabla p_{h,k_j}(t) \to \nabla p_h(t) \quad weakly \ in \ L^2(B_2) \quad as \quad j \to \infty.
\end{equation}

Since $p_{h,k}$ is harmonic in $B_2$, from (3.9) we deduce that
\begin{equation}
(3.10) \quad \nabla p_{h,k_j}(t) \to \nabla p_h(t) \quad a. e. \ in \ B_2 \quad as \quad j \to \infty.
\end{equation}

On the other hand, using the mean value property of harmonic functions, we see that $\{\nabla p_{h,k}\}$ is bounded in $L^\infty(B_1 \times (0, T))$. Appealing to Lebesgue’s theorem of dominated convergence, we infer from (3.10) that
\begin{equation}
(3.11) \quad \nabla p_{h,k_j} \to \nabla p_h \quad in \ L^2(B_1 \times (0, T)) \quad as \quad j \to \infty.
\end{equation}

Now combining (3.8) and (3.11), we obtain $u_{k_j} \to u$ in $L^2(B_1 \times (0, T))$. Recalling that $\{u_{k_j}\}$ is bounded in $V^2(B_1 \times (0, T))$, we get the desired convergence property $u_{k_j} \to u$ in $X$ as $j \to \infty$. To see this we argue as follows. Eventually passing to a subsequence, we may assume that $u_{k_j} \to u$ almost everywhere in $B_1 \times (0, T)$. Let $\varepsilon > 0$ be arbitrarily chosen. We denote $A_m = \{(x, t) \in B_1 \times (0, T) | \exists j \geq m : |u_{k_j}(x, t) - u(x, t)| > \varepsilon\}$. 

17
Clearly, \( \cap_{m=1}^{\infty} A_m \) is a set of Lebesgue measure zero. Thus \( \text{meas} A_m \to 0 \) as \( m \to \infty \).

We now get the following estimate
\[
\|u_{k_j} - u\|_{L^\infty_T (0,T;L^3(B_1))} = \\
\leq \|(u_{k_j} - u) \chi_{A_m}\|_{L^\infty_T (0,T;L^3(B_1))} + \|(u_{k_j} - u) \chi_{A_m^c}\|_{L^\infty_T (0,T;L^3(B_1))} \\
\leq \|u_{k_j} - u\|_{L^{32}_t (0,T;L^\infty(B_1))} \|\chi_{A_m}\|_{L^{504}_t (0,T;L^3(B_1))} + \|(u_{k_j} - u) \chi_{A_m^c}\|_{L^\infty_T (0,T;L^3(B_1))} \\
\leq c(\text{meas} A_m) + \varepsilon.
\]

This shows that \( \|u_{k_j} - u\| \to 0 \) as \( j \to \infty \). Applying Schauder’s fixed point theorem, we get a function \( u_\varepsilon \in M \) such that \( u_\varepsilon = T_\varepsilon(u_\varepsilon) \). Thus, \( u_\varepsilon \) is a local suitable weak solution with projected pressure to
\[
\begin{align*}
(3.12) & \quad \nabla \cdot u_\varepsilon = 0 \quad \text{in} \quad Q, \\
(3.13) & \quad \partial_t u_\varepsilon + (R_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon - \Delta u_\varepsilon = -\nabla \pi_\varepsilon \quad \text{in} \quad Q, \\
(3.14) & \quad u_\varepsilon = u_0 \quad \text{on} \quad \mathbb{R}^3 \times \{0\}.
\end{align*}
\]

In particular, we have the a-priori estimate
\[
\|u_\varepsilon\|_{L^4(0,T;L^3(B_1))} + \|u_\varepsilon\|_{V^2(B_1\times (0,T))} \leq 2C_0 K_0.
\]

Let \( \{\varepsilon_j\} \) be a sequence of positive numbers in \((0, \lambda - 1)\). Since \( u_\varepsilon \) is \( \lambda \)-DSS we have \( u_\varepsilon(x,t) = u_{\varepsilon,\lambda}(x,t) \) for almost every \((x,t) \in Q\). Thus, there exists a set of measure zero \( S \subset (0, +\infty) \) such that for all \( t \in [0, +\infty) \setminus S \)
\[
u_{\varepsilon_j}(x,t) = u_{\varepsilon_j,\lambda}(x,t) = \lambda^k u_{\varepsilon_j}(\lambda^k x, \lambda^2 t) \quad \text{for a.e.} \quad x \in \mathbb{R}^3, \quad \forall k \in \mathbb{Z}, \forall j \in \mathbb{N}.
\]

Clearly, \( t \in (0, +\infty) \setminus S \) iff \( \lambda^2 t \in (0, +\infty) \setminus S \). Indeed, let \( t \in N^c \). Then \( \lambda u_{\varepsilon_j}(\lambda x, t) = u_{\varepsilon_j}(\lambda^2 x, \lambda^2 t) \) for almost every \( x \in \mathbb{R}^3 \). By means of the reflexivity we get a sequence \( \varepsilon_j \to 0^+ \) as \( j \to \infty \) and \( u \in V^2_{loc,\sigma}(\mathbb{R}^3 \times [0, T]) \) such that
\[
u_{\varepsilon_j} \to u \quad \text{weakly in} \quad L^2(0,T;W^{1,2}(B_1)) \quad \text{as} \quad j \to +\infty,
\]
\[
u_{\varepsilon_j} \to u \quad \text{weakly}^* \quad \text{in} \quad L^\infty(0,T;L^2(B_1)) \quad \text{as} \quad j \to +\infty.
\]

Arguing as in the proof the compactness of \( T_\varepsilon \), we infer
\[
u_{\varepsilon_j} \to u \quad \text{in} \quad L^{18}_T (0,T;L^3(B_1)) \quad \text{as} \quad j \to 0^+.
\]

Note that \( u \) is DSS, since \( u \) is obtained as a limit of sequence DSS functions.

Together with Lemma \[A.3] we see that
\[
R_\varepsilon u_{\varepsilon_j} \to u \quad \text{in} \quad L^{18}_T (0,T;L^3(B_1)) \quad \text{as} \quad j \to 0^+.
\]

This shows that \( u \in L^2_{loc,\sigma}(\mathbb{R}^3 \times [0, +\infty)) \) is a local Leray solution with projected pressure to \((1.1)-(1.3)\).

\section*{Acknowledgements}

Chae was partially supported by NRF grants 2016R1A2B3011647, while Wolf has been supported supported by the German Research Foundation (DFG) through the project WO1988/1-1; 612414.
A Mollification for DSS functions

Let $1 < \lambda < +\infty$. Let $u \in L^s_{\lambda-DSS}(\mathbb{R}^3)$. Let $\rho \in C^\infty(\mathcal{B}_1)$ denote the standard mollifying kernel such that $\int_{\mathbb{R}^3} \rho dx = 1$. For $0 < \varepsilon < \lambda - 1$ we define

$$(R_\varepsilon u)(x, t) = \frac{1}{(\sqrt{t\varepsilon})^3} \int_{B_{\sqrt{t\varepsilon}}} u(x - y, t)\rho\left(\frac{y}{\sqrt{t\varepsilon}}\right)dy, \quad (x, t) \in Q.$$  

We have the following

**Lemma A.1.** $R_\varepsilon$ defines a bounded operator from $L^s_{\lambda-DSS}(Q)$ into itself. Furthermore, for all $u \in L^s_{\lambda-DSS}(Q)$ it holds for all $(x, t) \in Q$

$$(A.1) \quad |(R_\varepsilon u)(x, t)| \leq c\{\sqrt{t\varepsilon}\}^{-\frac{3}{2}}\|u(\cdot, t)\|_{L^s(B, \sqrt{t\varepsilon}(x))}$$

with an constant $c > 0$ depending on $s$ only.

**Proof:** Let $u \in L^s_{\lambda-DSS}(Q)$. First we will verify that $R_\varepsilon u$ is $\lambda$-DSS. Indeed, using the transformation formula of the Lebesgue integral, we calculate for any $(x, t) \in Q$,

$$\lambda(R_\varepsilon u)(\lambda x, \lambda^2 t) = \frac{1}{\lambda^2(\sqrt{t\varepsilon})^3} \int_{B_{\sqrt{t\varepsilon}}} u(\lambda x - y, \lambda^2 t)\rho\left(\frac{y}{\lambda \sqrt{t\varepsilon}}\right)dy,$$

$$= \frac{1}{(\sqrt{t\varepsilon})^3} \int_{\mathbb{R}^3} \lambda u(\lambda x - y, \lambda^2 t)\rho\left(\frac{y}{\sqrt{t\varepsilon}}\right)dy$$

$$= \frac{1}{(\sqrt{t\varepsilon})^3} \int_{\mathbb{R}^3} u(x - y, t)\rho\left(\frac{y}{\sqrt{t\varepsilon}}\right)dy = (R_\varepsilon u)(x, t).$$

Firstly, let $\lambda^{-2} < t \leq 1$. Noting that $(R_\varepsilon u)(\cdot, t) = u(\cdot, t) * \rho \sqrt{t\varepsilon}$, where $\rho \sqrt{t\varepsilon}(y) = \frac{1}{(\sqrt{t\varepsilon})^3}\rho\left(\frac{y}{\sqrt{t\varepsilon}}\right)$, recalling that $\varepsilon < \lambda - 1$, by means of Young’s inequality we find

$$\|(R_\varepsilon u)(\cdot, t)\|_{L^s(B_1)} \leq \|u(\cdot, t)\|_{L^s(B_1+\varepsilon)}\|\rho \sqrt{t\varepsilon}\|_{L^1} = \|u(\cdot, t)\|_{L^s(B_\lambda)}.$$  

Integrating the above inequality over $(\lambda^{-2}, 1)$, and using a suitable change of coordinates, we obtain

$$\|R_\varepsilon u\|_{L^s(B_1 \times (\lambda^{-2}, 1))} \leq \|u\|_{L^s(B_1 \times (\lambda^{-2}, 1))}$$

$$= \|u\|_{L^s(B_1 \times (\lambda^{-2}, 1))} + \|u\|_{L^s(B_1 \setminus \mathcal{B}_1 \times (\lambda^{-2}, 1))}$$

$$= \|u\|_{L^s(B_1 \times (\lambda^{-2}, 1))} + \lambda^{\frac{3}{2}} \|u\|_{L^s(B_1 \setminus \mathcal{B}_{\lambda^{-2}} \times (\lambda^{-4}, \lambda^{-2}))}.$$  

Secondly, for $0 < t \leq \lambda^{-2}$ we estimate

$$\|(R_\varepsilon u)(\cdot, t)\|_{L^s(B_1 \setminus \mathcal{B}_{\lambda^{-1}})} \leq \|u(\cdot, t)\|_{L^s(B_1 \setminus \mathcal{B}_{\lambda^{-1}})}\|\rho \sqrt{t\varepsilon}\|_{L^1} \|u(\cdot, t)\|_{L^s(B_1 \setminus \mathcal{B}_{\lambda^{-1}})}.$$  

19
Integration over $(0, \lambda^{-2})$ in time yields
\[
\|R_\varepsilon u\|_{L^s(B_1 \setminus B_{\lambda^{-1}} \times (0, \lambda^{-2}))} \leq \|u\|_{L^s(B_\lambda \setminus B_{\lambda^{-1}} \times (0, \lambda^{-2}))}
\]
\[
= \|u\|_{L^s(B_1 \setminus B_{\lambda^{-1}} \times (0, \lambda^{-2}))} + \|u\|_{L^s(B_\lambda \setminus B_1 \times (0, \lambda^{-2}))}
\]
\[
= \|u\|_{L^s(B_1 \setminus B_{\lambda^{-1}} \times (0, \lambda^{-2}))} + \lambda^{-\frac{s-3}{s}} \|u\|_{L^s(B_1 \setminus B_{\lambda^{-1}} \times (0, \lambda^{-4})}.}
\]
Combining the last two estimates, we get
\[
\|R_\varepsilon u\|_{L^s(Q_\lambda \setminus Q_{\lambda^{-1}})} \leq (1 + \lambda^{\frac{3}{s-3}}) \|u\|_{L^s(Q_\lambda \setminus Q_{\lambda^{-1}})}.
\]
This shows that $R_\varepsilon : L^s_{\lambda^{-DSS}}(Q) \to L^s_{\lambda^{-DSS}}(Q)$ is bounded.

The inequality $[(A.1)]$ follows immediately from the definition of $R_\varepsilon u$ with the help of Hölder’s inequality.

**Remark A.2.** Arguing as in the proof of Lemma $[A.1]$ we get for any $u \in L^3_{\lambda^{-DSS}}(Q) \cap L^{18}_{DSS}(0,T; L^3(B_1)), 0 < T < 1$

\[
(A.2) \quad \|R_\varepsilon u\|_{L^{18}_{DSS}(0,T; L^3(B_1))} \leq \lambda^{\frac{3}{s}} \|u\|_{L^{18}(0,T; L^3(B_1))}.
\]

**Lemma A.3.** Let $u \in L^3_{\lambda^{-DSS}}(Q) \cap L^{18}(0,T; L^3(B_1)), 0 < T \leq 1$. Then

\[
(A.3) \quad R_\varepsilon u \to u \quad \text{in} \quad L^{18}(0,T; L^3(B_1)) \quad \text{as} \quad \varepsilon \to 0^+.
\]

**Proof:** First by the absolutely continuity of the Lebesgue integral we see that for almost all $t \in (0,T)$

\[
(R_\varepsilon u)(\cdot,t) \to u(\cdot,t) \quad \text{in} \quad L^3(B_1) \quad \text{as} \quad \varepsilon \to 0^+.
\]
Let $A \subset (0,T)$ be any Lebesgue measurable set. By Young’s inequality of convolutions we get for almost all $t \in (0,T)$

\[
\int_A \|(R_\varepsilon u)(\cdot,t)\|_{L^3(B_1)}^{\frac{18}{15}} dt \leq \int_A \|u(\cdot,t)\|_{L^3(B_3)}^{\frac{18}{15}} dt
\]

Since $u \in L^{18}(0,T; L^3(B_3))$, the assertion $[(A.3)]$ follows by the aid of Vitali’s convergence lemma.

**B Weak trace for time dependent $\lambda$-DSS functions**

Let $1 < \lambda < +\infty$. A measurable function $u : Q \to \mathbb{R}^3$ is said to be $\lambda$-DSS, if for almost every $(x,t) \in Q$

\[
(B.1) \quad u(x,t) = \lambda u(\lambda x, \lambda^2 t).
\]
We denote by $M(u)$ the set of all $t \in [0, +\infty)$ such that for all $k \in \mathbb{Z}$

\[
(B.2) \quad u(x,t) = \lambda^k u(\lambda^k x, \lambda^{2k} t) \quad \text{for a.e.} \quad x \in \mathbb{R}^3.
\]
Lemma B.1. The set $[0, +\infty) \setminus M(u)$ is a set of Lebesgue measure zero.

Proof: For $m \in \mathbb{N}$ and $k \in \mathbb{N}$ by $A_{m,k}$ we denote the set of all $t \in [0, +\infty)$ such that

\[
\text{meas} \left\{ x \in \mathbb{R}^3 \mid u(x,t) = \lambda^k u(\lambda^k x, \lambda^{2k} t) \right\} \geq \frac{1}{m}.
\]

Since $u$ is discretely self-similar, we must have $\text{meas}(A_{m,k}) = 0$. Since $M(u) \setminus [0, +\infty) = \bigcup_{k \in \mathbb{Z}} \bigcup_{m=1} A_{m,k}$ the assertion follows.

Lemma B.2. For every $t \in [0, +\infty)$ it holds $t \in M(u)$ iff $\lambda^{2t} \in M(u)$.

Proof: Let $t \in M(u)$. There exists a set $P \subset \mathbb{R}^3$ with $\text{meas}(\mathbb{R}^3 \setminus P) = 0$ such that (B.2) holds for all $x \in P$. Define $P_k = \{ y = \lambda^k x \mid x \in P \}, k \in \mathbb{Z}$. Clearly, $\text{meas}(\mathbb{R}^3 \setminus \bigcap_{k \in \mathbb{Z}} P_k) = 0$. Let $x, \lambda^{-1} x \in P$, and therefore for all $k \in \mathbb{Z}$ we get $u(\lambda^{-1} x, t) = \lambda u(x, \lambda^2 t) = \lambda^{k+1} u(\lambda^k x, \lambda^{2k+2} t)$, which is equivalent to

\[
u(x,t) = \lambda^k u(\lambda^k x, \lambda^{2k} t).
\]

This shows that $\lambda^{2t} \in M(u)$. Similarly, we get the opposite direction.

As an immediate consequence of Lemma B.1 we see that

(B.3) \quad $t \in M(u) \iff \lambda^{2k} t \in M(u) \quad \forall k \in \mathbb{Z}.$

Let $\{v_j\}$ be a sequence in $L^2_{\text{loc}}(\mathbb{R}^3)$. We say

\[
v_j \to v \quad \text{weakly in} \quad L^2_{\text{loc}}(\mathbb{R}^3) \quad \text{as} \quad j \to +\infty
\]

if for every $0 < R < +\infty$

\[
v_j \to v \quad \text{weakly in} \quad L^2(B_R) \quad \text{as} \quad j \to +\infty.
\]

Lemma B.3. Let $\{v_j\}$ be a sequence in $L^2_{\text{loc}}(\mathbb{R}^3)$ such that for all $0 < R < +\infty$

(B.4) \quad $\sup_{j \in \mathbb{N}} \|v_j\|_{L^2(B_R)} < +\infty.$

Then there exists a subsequence $\{v_{j_m}\}$ and $v \in L^2_{\text{loc}}(\mathbb{R}^3)$ such that

\[
v_{j_m} \to v \quad \text{weakly in} \quad L^2_{\text{loc}}(\mathbb{R}^3) \quad \text{as} \quad m \to +\infty.
\]

Proof: By induction and the reflexivity of $L^2(B_m)$ we construct a sequence of subsequences $\{v_{j_k}^{(m)}\} \subset \{v_{j_{k-1}}^{(m-1)}\}$ and $\{v_{j_k}^{(m)}\} = \{v_j\}$ such that for some $v_m \in L^2(B_k)$ it holds

\[
v_{j_k}^{(m)} \to v_m \quad \text{in} \quad L^2(B_m) \quad \text{as} \quad k \to +\infty
\]

$(m \in \mathbb{N})$. Clearly, $v_m|_{B_{m-1}} = v_{m-1}$. This allows us to define $v : \mathbb{R}^3 \to \mathbb{R}$ be setting $v = v_m$ on $B_m$. Then by Cantor’s diagonalization principle the subsequence $v_{j_m} = v_{j_k}^{(m)}$ meets the requirements.

We denote $\mathcal{V} = L^\infty_{\text{loc}}([0, +\infty); L^2_{\text{loc}}(\mathbb{R}^3))$ the space of all measurable functions $u : Q \to \mathbb{R}$ such that $u \in L^\infty_{\text{loc}}(0, R^2, L^2(B_R))$ for all $0 < R < +\infty$. By $\mathcal{V}_{\lambda-DSS}$ we denote the space of all $\lambda$-DSS functions $u \in \mathcal{V}$. 21
Lemma B.4. Let \( u \in \mathcal{V}_{\lambda - \text{DSS}} \). We assume that \( \|u(t)\|_{L^2(B_R)} \leq \|u\|_{L^\infty(0,R^2;L^2(B_R))} \) for all \( t \in (0,R^2) \), \( 0 < R < +\infty \). There exists a constant \( C > 0 \) such that for every \( t \in M \)

\[
(B.5) \quad \|u(t)\|^2_{L^2(B_R)} \leq C \max \left\{ R\|u\|_{L^\infty(0,1;L^2(B_1))}, \|u(t)\|_{L^2(B_{\sqrt{\tau}})} \right\}.
\]

Proof: Let \( t \in M(u) \). Let \( k \in \mathbb{Z} \). Then by means of the transformation formula we get

\[
\int_{A_k} |u(x,t)|^2 \, dx = \lambda^k \int_{A_1} |u(\lambda^k x, t)|^2 \, dx = \lambda^k \int_{A_1} |\lambda^k u(\lambda^k x, \lambda^{2k} \lambda^{-2k} t)|^2 \, dx = \lambda^k \int_{A_1} |u(x, \lambda^{-2k} t)|^2 \, dx.
\]

In case \( \lambda^{2k} \geq t \) we get

\[
\|u(t)\|^2_{L^2(A_k)} \leq \lambda^k \|u\|_{L^\infty(0,1;L^2(B_1))}.
\]

On the contrary, if \( \lambda^{2k} < t \) we find

\[
\|u(t)\|^2_{L^2(B_k)} \leq \|u(t)\|_{L^2(B_{\sqrt{\tau}})}.
\]

Accordingly,

\[
\|u(t)\|^2_{L^2(B_k)} \leq c \max \left\{ \lambda^k \|u\|_{L^\infty(0,1;L^2(B_1))}, \|u(t)\|_{L^2(B_{\sqrt{\tau}})} \right\}.
\]

This yields \((B.5)\).

\[ \blacklozenge \]

Lemma B.5. Let \( u \in \mathcal{V}_{\lambda - \text{DSS}} \). Furthermore, let \( F_{ij}, g_i : Q \to \mathbb{R} \) such that \( F_{ij}, g_i \in L^1(Q_R) \) and for all \( 0 < R < +\infty \), \( i, j=1,2,3 \). We suppose for all \( t \in [0, +\infty) \) the function \( u(\cdot, t) \in L^\infty_{\text{loc}}(\mathbb{R}^3) \) with \( \nabla \cdot u(\cdot, t) = 0 \) in the sense of distributions, and that for all \( \varphi \in \mathcal{C}_c^\infty(Q) \) with \( \nabla \cdot \varphi = 0 \) the following identity holds true

\[
(B.6) \quad \int_Q \frac{\partial \varphi}{\partial t} \, dx \, dt = \int_Q F : \nabla \varphi + g \cdot \varphi \, dx \, dt.
\]

Then, eventually redefining \( u(t) \) for \( t \) in a set of measure zero, we have

\[
(B.7) \quad u \in C_w([0, +\infty); L^2(B_R)) \quad \forall 0 < R < +\infty,
\]

\[
(B.8) \quad M(u) = [0, +\infty).
\]

Proof: By \( L(u) \subset [0, +\infty) \) we denote the set of all Lebesgue points of \( u \), more precisely, we say \( t \in L(u) \), if for every \( 0 < R < +\infty \)

\[
\frac{1}{\varepsilon} \int_t^{t+\varepsilon} u(\cdot, \tau) \, d\tau \to u(\cdot, t) \quad \text{in} \quad L^2(B_R) \quad \text{as} \quad \varepsilon \to +\infty.
\]

22
By Lebesgue’s differentiation theorem we have $\text{meas}(0, +\infty \setminus L(u)) = 0$. Let $t \in L(u)$. By a standard approximation argument we deduce from (B.6) that for every $\varphi \in C^\infty_c(Q)$ with $\nabla \cdot u = 0$

\begin{equation}
(B.9) \quad -\int_{\mathbb{R}^3} u(t)\varphi(t)dx + \int_0^t \int_{\mathbb{R}^3} u \frac{\partial \varphi}{\partial t}dxds = \int_0^t \int_{\mathbb{R}^3} F : \nabla \varphi + g \cdot \varphi dx ds.
\end{equation}

Next, let $\{t_j\}$ be a sequence in $M(u) \cap L(u)$ such that $t_j \rightarrow t \in L(u)$ as $j \rightarrow +\infty$. Thank’s to Lemma B.3 there exists a subsequence $\{t_{j_n}\}$ and $v \in L^2_{loc}(\mathbb{R}^3)$ such that

$$u(t_{j_n}) \rightarrow v \quad \text{weakly in } L^2_{loc}(\mathbb{R}^3) \quad \text{as } m \rightarrow +\infty.$$  

Thus, (B.9) implies for all $\varphi \in C^\infty_c(Q)$ with $\nabla \cdot \varphi = 0$

\begin{equation}
(B.10) \quad -\int_{\mathbb{R}^3} v\varphi(t)dx + \int_0^t \int_{\mathbb{R}^3} u \frac{\partial \varphi}{\partial t}dxds = \int_0^t \int_{\mathbb{R}^3} F : \nabla \varphi + g \cdot \varphi dx ds.
\end{equation}

On the other hand, recalling that $t \in L(u)$, we have the same identity as (B.10) replacing $v$ by $u(t)$ therein. This shows that for all $\psi \in C^\infty_{c,\sigma}(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} (v - u(t)) \cdot \psi dx = 0.$$ 

Consequently, $v - u(t)$ is a harmonic function. On the other hand, by the lower semi continuity of the $L^2$ norm we obtain from (B.5) that

\begin{equation}
(B.11) \quad \|u(t) - v\|^2_{L^2(B_R)} \leq C \max \left\{ R\|u\|_{L^\infty(0,1;L^2(B_1))}, \|u(t)\|_{L^2(B_{\sqrt{t}})} \right\}.
\end{equation}

Whence, $v = u(t)$. In particular, $u(s) \rightarrow u(t)$ weakly in $L^2_{loc}(\mathbb{R}^3)$ as $s \in M(u) \cap L(u) \rightarrow t$.

Let $t \in [0, +\infty)$. There exists a sequence $\{t_j\}$ in $M(u) \cap L(u)$ such that $t_j \rightarrow t$ as $j \rightarrow +\infty$. Thank’s to Lemma B.3 there exists a subsequence $\{t_{j_n}\}$ and $v \in L^2_{loc}(\mathbb{R}^3)$ such that

$$u(t_{j_n}) \rightarrow v \quad \text{weakly in } L^2_{loc}(\mathbb{R}^3) \quad \text{as } m \rightarrow +\infty.$$ 

Observing (B.9) with $t_{j_n}$ in place of $t$ and letting $m \rightarrow +\infty$, we obtain for all $\varphi \in C^\infty_c(Q)$ with $\nabla \cdot \varphi = 0$

\begin{equation}
(B.12) \quad -\int_{\mathbb{R}^3} v\varphi(t)dx + \int_0^t \int_{\mathbb{R}^3} u \frac{\partial \varphi}{\partial t}dxds = \int_0^t \int_{\mathbb{R}^3} F : \nabla \varphi + g \cdot \varphi dx ds.
\end{equation}

On the other hand, by the lower semi continuity of the $L^2$ norm from (B.5) it follows that

\begin{equation}
(B.13) \quad \|v\|^2_{L^2(B_R)} \leq C \max \left\{ R\|u\|_{L^\infty(0,1;L^2(B_1))}, \|u(t)\|_{L^2(B_{\sqrt{t}})} \right\}.
\end{equation}
For a second subsequence \( \{t'_m\} \) with limit \( w \in L^2_{\text{loc}}(\mathbb{R}^3) \) we derive the same property as \( v \) which leads to the fact that for all \( \psi \in C^\infty_{c,0}(\mathbb{R}^3) \)

\[
\int_{\mathbb{R}^3} (v - w) \cdot \psi \, dx = 0.
\]

Consequently, \( v - w \) is a harmonic function. Now taking into account the estimate \((B.11)\), which is satisfied for \( w \) too, we infer \( v = w \). Thus, the limit is uniquely determined. In case \( t \notin M(u) \cap L(u) \) we set \( u(t) = v \). In particular, by the lower semi continuity of the norm we have for all \( v \)

\[
\text{Combining } (B.16) \text{ and } (B.15) \text{ and verifying } (B.13) \text{ for } w = v.
\]

Next, let \( t \in [0, +\infty) \), and let \( \{t_j\} \) be any sequence in \( [0, +\infty) \) with \( t_j \to t \) as \( j \to +\infty \). Once more applying Lemma \((B.3)\) we get a subsequence \( \{t_{j_m}\} \) and \( w \in L^2_{\text{loc}}(\mathbb{R}^3) \) such that

\[
u(t_{j_m}) \to w \quad \text{weakly in} \quad L^2_{\text{loc}}(\mathbb{R}^3) \quad \text{as} \quad m \to +\infty.
\]

Observing \((B.15)\) with \( t_{j_m} \) in place of \( t \) and letting \( m \to +\infty \), it follows that

\[
\int_{\mathbb{R}^3} (w \varphi(t) dx + \int_0^t \int_{\mathbb{R}^3} u \frac{\partial \varphi}{\partial t} dx ds = \int_0^t \int_{\mathbb{R}^3} F \cdot \nabla \varphi + g \cdot \varphi \, dx ds.
\]

Combining \((B.16)\) and \((B.15)\) and verifying \((B.13)\) for \( w \) by a similar reasoning as above, we conclude \( w = u(t) \). This shows that \( u \in C_w([0, +\infty); L^2_{\text{loc}}(\mathbb{R}^3)) \).

It only remains to prove that \( M(u) = [0, +\infty) \). To see this let \( \{t_j\} \) be a sequence in \( M(u) \) such that \( t_j \to t \). By using the transformation formula of the Lebesgue integral together with Lemma \((B.2)\) (cf. also \((B.3)\)), we calculate for all \( \psi \in C^\infty_c(\mathbb{R}^3) \)

\[
\int_{\mathbb{R}^3} u(x,t) \psi(x) dx = \lim_{j \to \infty} \int_{\mathbb{R}^3} u(x,t_j) \psi(x) dx
\]

\[
= \lambda^{-3k} \lim_{j \to \infty} \int_{\mathbb{R}^3} u(\lambda^{-k} x, t_j) \psi(\lambda x) dx
\]

\[
= \lambda^{-2k} \lim_{j \to \infty} \int_{\mathbb{R}^3} u(x, \lambda^{2k} t_j) \psi(\lambda x) dx
\]

\[
= \lambda^{-2k} \int_{\mathbb{R}^3} u(x, \lambda^{2k} t) \psi(\lambda x) dx = \int_{\mathbb{R}^3} \lambda^k u(\lambda x, \lambda^{2k} t) \psi(x) dx.
\]

This yields \( u(x,t) = \lambda^k u(\lambda x, \lambda^{2k} t) \) for almost every \((x,t) \in Q\), and thus \( t \in M(u) \).
References

[1] Z. Bradshaw and T.-P. Tsai, *Forward discretely self-similar solutions of the Navier-Stokes equations II*, arXiv:1510.07504v1 (2015), to appear in Ann. I. H. Poincaré-AN.

[2] L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math., 35 (1982), pp. 771–831.

[3] G. Galdi, C. Simader, and H. Sohr, *On the stokes problem in lipschitz domains*, Annali di Mat. pura ed appl. (IV), 167 (1994), pp. 147–163.

[4] H. Jia and V. Šverák, *Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions*, Invent. Math., 196 (2014), pp. 233–265.

[5] N. Kikuchi and G. A. Seregin, *Weak solutions to the cauchy problem for the Navier-Stokes equations satisfying the local energy inequality*, in Nonlinear equations and spectral theory, A. M. S. Transl., ed., no. 220 in Ser 2., Providence, RI, 2007, Amer. Math. Soc., pp. 141–164.

[6] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math., 157 (2001), pp. 22–35.

[7] P. G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*, vol. 431, Chapman Hall/CRC, Boca Raton, FL, 2002.

[8] J. Leray, *Sur le mouvement d’un liquide visqueux emplissant l’espace*, Acta Math., 63 (1934), pp. 193–284.

[9] V. Scheffer, *Partial regularity of solutions to the Navier-Stokes equations*, Pacific J. Math., 66 (1976), pp. 535–552.

[10] T.-P. Tsai, *Forward discretely self-similar solutions of the Navier-Stokes equations*, Comm. Math. Phys., 328 (2014), pp. 29–44.

[11] J. Wolf, *On the local regularity of suitable weak solutions to the generalized navier-stokes equations*, Ann Univ Ferrara, 61 (2015), pp. 149–171.

[12] ———, *On the local pressure of the navier-stokes equations and related systems*, to appear in Adv. Diff. Equs., (2016).