The BV-algebra of a Jacobi manifold

Izu Vaisman

ABSTRACT. We show that the Gerstenhaber algebra of the 1-jet Lie algebroid of a Jacobi manifold has a canonical exact generator, and discuss duality between its homology and the Lie algebroid cohomology. We also give a new example of a Lie bialgebroid on Poisson manifolds.

1 Introduction

A Gerstenhaber algebra is a triple $(A = \bigoplus_{k \in \mathbb{Z}} A^k, \wedge, [\ , \ ])$ where $\wedge$ is an associative, graded commutative algebra structure (e.g., over $\mathbb{R}$), $[\ , \ ]$ is a graded Lie algebra structure for the shifted grades $[k] := k + 1$ (the sign := denotes a definition), and

\begin{equation}
[a, b \wedge c] = [a, b] \wedge c + (-1)^k b \wedge [a, c],
\end{equation}

$\forall a \in A^{k+1}, b \in A^j, c \in A$. If this structure is supplemented by an endomorphism $\delta : A \to A$, of grade $-1$, such that $\delta^2 = 0$ and

\begin{equation}
[a, b] = (-1)^k (\delta(a \wedge b) - \delta a \wedge b - (-1)^k a \wedge \delta b) \ (a \in A^k, b \in A),
\end{equation}

one gets an exact Gerstenhaber algebra or Batalin-Vilkovisky algebra (BV-algebra) with the exact generator $\delta$. If we also have a differential $d : A^k \to A^{k+1} (d^2 = 0)$ such that

\begin{equation}
d(a \wedge b) = (da) \wedge b + (-1)^k a \wedge (db) \ (a \in A^k, b \in A),
\end{equation}

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we will say that we have a differential BV-algebra. (Some authors include $d$ in the BV-structure [8, 21].) Finally, if

\[(1.4) \quad d[a, b] = [da, b] + (-1)^k[a, db] \quad (a \in \mathcal{A}^k, b \in \mathcal{A})\]

the differential BV-algebra is said to be strong [21].

On the other hand, a Jacobi manifold (e.g., [5]) is a smooth manifold $M^m$ (everything is of class $C^\infty$ in this paper) with a Lie algebra structure of local type on the space of functions $C^\infty(M)$ or, equivalently [5], with a bivector field $\Lambda$ and a vector field $E$ such that

\[(1.5) \quad [\Lambda, \Lambda] = 2E \wedge \Lambda, \quad [\Lambda, E] = 0.\]

In (1.5) one has usual Schouten-Nijenhuis brackets. If $E = 0$, $(M, \Lambda)$ is a Poisson manifold.

One of the most interesting examples of a BV-algebra is that of the Gerstenhaber algebra of the cotangent Lie algebroid of a Poisson manifold, described by Koszul [3] and Y. Kosman-Schwarzbach [8]. More generally, Xu [21] extends a result of Koszul [3] and proves that the exact generators of the Gerstenhaber algebra of a Lie algebroid $A \to M$ are provided by flat connections on $\wedge^r A$ ($r = \text{rank } A$), and Huebschmann [6] proves a corresponding result for Lie-Rinehart algebras.

The main aim of this note is to show that a Jacobi manifold also has a canonically associated, differential, BV-algebra (which, however, is not strong) namely, the Gerstenhaber algebra of the 1-jet Lie algebroid defined by Kerbrat-Benhammadi [7]. Then, we apply results of Xu [21], and Evens-Lu-Weinstein [3] to discuss duality between the homology of this BV-algebra and the cohomology of the Lie algebroid. (The homology was also independently introduced and studied by de León, Marrero and Padron [11].)

In the final section, we come back to a Poisson manifold $M$ with the Poisson bivector $Q$, and show that the infinitesimal automorphisms $E$ of $Q$ yield natural Poisson bivectors of the Lie algebroid $TM \oplus \mathbb{R}$. These bivectors lead to triangular Lie bialgebroids and BV-algebras in the usual way [8, 21].

Notice that BV-algebras play an important role in some recent researches of theoretical physics (e.g., [4]).

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2 The Jacobi BV-algebra

For any Lie algebroid $A \rightarrow M$ with anchor $\alpha : A \rightarrow TM$ one has a Gerstenhaber algebra $\mathcal{A}(A)$ defined by

\[(2.1) \quad \mathcal{A}(A) := (\oplus_{k \in \mathbb{N}} \Gamma \wedge^k A, \wedge, [ , ]_{SN}),\]

where $\Gamma$ denotes spaces of global cross-sections, and $SN$ denotes the Schouten-Nijenhuis bracket (e.g., [8], [21]; on the other hand, we refer the reader to [14, 8, 3], for instance, for the basics of Lie algebroids and Lie algebroid calculus). The BV-algebra which we want to discuss is associated with the 1-jet Lie algebroid of a Jacobi manifold $(M, \Lambda, E)$ defined in [7], which we present as follows.

We identify $M$ with $M \times \{0\} \subseteq M \times \mathbb{R}$, where $M \times \mathbb{R}$ is endowed with the Poisson bivector $[5]$

\[(2.2) \quad P := e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E) \quad (t \in \mathbb{R}).\]

Let $J^1M = T^*M \oplus \mathbb{R}$ be the vector bundle of 1-jets of real functions on $M$, and notice that $\Gamma J^1M$ is isomorphic as a $C^\infty(M)$-module with

\[(2.3) \quad \Gamma_0(M) := \{e^t(\alpha + fdt) / \alpha \in \wedge^1 M, f \in C^\infty(M)\} \subseteq \wedge^1(M \times \mathbb{R}).\]

A straightforward computation shows that $\Gamma_0(M)$ is closed under the bracket of the cotangent Lie algebroid of $(M \times \mathbb{R}, P)$ (e.g., [18]) namely,

\[(2.4) \quad \{e^t(\alpha + fdt), e^t(\beta + gdt)\}_P = e^t[L_{\sharp_{\Lambda} \alpha} \beta - L_{\sharp_{\Lambda} \beta} \alpha - d(\Lambda(\alpha, \beta))
+f L_E \beta - g L_E \alpha - \alpha(E)\beta + \beta(E)\alpha + \{f, g\} - \Lambda(df - \alpha, dg - \beta)dt],\]

where $< \sharp_{\Lambda} \alpha, \beta > := \Lambda(\alpha, \beta)\ (\alpha, \beta \in \wedge^1 M)$, and

\[\{f, g\} = \Lambda(df, dg) + f(Eg) - g(Ef)\quad (f, g \in C^\infty(M))\]
is the bracket which defines the Jacobi structure [3]. Therefore, (2.4) produces a Lie bracket on $\Gamma J^1 M$. Moreover, if $\sharp_P$ is defined similar to $\sharp_\Lambda$, we get

\begin{equation}
\sharp_P(e^t(\alpha + f dt)) = \sharp_\Lambda \alpha + f E - \alpha(E) \frac{\partial}{\partial t},
\end{equation}

and

\begin{equation}
\rho := (pr_{TM} \circ \sharp_P)_{t=0} : J^1 M \to TM
\end{equation}

has the properties of an anchor, since so does $\sharp_P$.

Formulas (2.4), (2.6) precisely yield the Lie algebroid structure defined in [3]. In what follows we refer to it as the 1-jet Lie algebroid. The mapping $f \mapsto e^t(df + f dt)$ is a Lie algebra homomorphism from the Jacobi algebra of $M$ to $\Gamma_0 M$.

2.1 Proposition. The Gerstenhaber algebra $A(J^1 M)$ is isomorphic to the subalgebra $A_0(M) := \oplus_{k \in \mathbb{N}} \wedge^k \Gamma_0(M)$ of the Gerstenhaber algebra $A(T^*(M \times \mathbb{R}))$.

Proof. The elements of $A_0^k(M)$ are of the form

\begin{equation}
\lambda = e^{kt}(\lambda_1 + \lambda_2 \wedge dt) \quad (\lambda_1 \in \wedge^k M, \lambda_2 \in \wedge^{k-1} M),
\end{equation}

and we see that $A_0(M)$ is closed by the wedge product and by the bracket $\{ , \}_P$ of general differential forms on the Poisson manifold $(M \times \mathbb{R}, P)$ (e.g., [13]). Accordingly, $(A(J^1 M), \wedge, \{ , \})$ and $(A_0(M), \wedge, \{ , \}_P)$ are isomorphic Gerstenhaber algebras since they are isomorphic at the grade 1 level, and the brackets of terms of higher degree are spanned by those of degree 1. Q.e.d.

2.2 Remark. Since $A_0(M)$ is a Gerstenhaber algebra, the pair $(A_0^0 = C^\infty(M), A_0^1 = \Gamma_0(M))$ is a Lie-Rinehart algebra [3].

Now, we can prove

2.3 Proposition. The Gerstenhaber algebra $A_0(M)$ has a canonical exact generator.

Proof. It is known that $A(T^*(M \times \mathbb{R}))$ has the exact generator of Koszul and Brylinski (e.g., [14])

\begin{equation}
\delta_P = i(P)d - di(P),
\end{equation}
where $P$ is the bivector (2.2). Hence, all we have to do is check that $\delta_P \lambda \in \mathcal{A}_{k-1}(M)$ if $\lambda$ is given by (2.7).

First, we notice that

$$i(P)(dt \wedge \mu) = e^{-t}(i(E)\mu + dt \wedge (i(\Lambda)\mu)) \quad (\mu \in \Lambda^* M).$$

Then, if we also introduce the operator $\delta_\Lambda := i(\Lambda)d - di(\Lambda)$ [1], and compute for $\lambda$ of (2.7), we get

$$\delta_P \lambda = e^{(k-1)t}[\delta_\Lambda \lambda_1 + (-1)^k L_E \lambda_2 + ki(E)\lambda_1$$

$$+ (\delta_\Lambda \lambda_2 + (-1)^k i(\Lambda)\lambda_1 + (k - 1)i(E)\lambda_2) \wedge dt].$$

Q.e.d.

Of course, $\delta_P$ of (2.10) translates to an exact generator $\delta$ of the Gerstenhaber algebra $\mathcal{A}(J^1 M)$, and the latter becomes a BV-algebra. This is the BV-algebra announced in Section 1, and we call it the Jacobi BV-algebra of the Jacobi manifold $(M, \Lambda, E)$. We can look at it under the two isomorphic forms indicated by Proposition 2.1.

It is easy to see that the Jacobi BV-algebra above has the differential

$$\bar{d}\lambda := e^{(k+1)t}d(e^{-kt}\lambda),$$

where $\lambda$ is given by (2.7). But, $\bar{d}$ is not a derivation of the Lie bracket $\{ , \}$ of $\mathcal{A}(J^1 M)$, and computations lead to

$$\delta_P \bar{d} + \bar{d}\delta_P)\lambda = e^{kt}[(k + 1)i(E)d\lambda_1$$

$$+ (L_E \lambda_2 + (k + 1)i(E)d\lambda_2 - (-1)^k \delta_\Lambda \lambda_1) \wedge dt],$$

where $\lambda$ is given by (2.7) again.

**2.4 Remark.** If we refer to the Poisson case $E = 0$, we see that both $T^* M$ and $J^1 M$ have natural structures of Lie algebroids. The Lie bracket and anchor map of $J^1 M$ are given by

$$(2.13) \quad \{e^\iota(\alpha + f dt), e^\iota(\beta + g dt)\} = e^\iota[\{\alpha, \beta\}\Lambda + ((\sharp_\Lambda \alpha)g - (\sharp_\Lambda \beta)f - \Lambda(\alpha, \beta)) dt],$$

$$(2.14) \quad \rho(e^\iota(\alpha + f dt)) = \sharp_\Lambda \alpha,$$

and the mapping $\alpha \mapsto e^\iota(\alpha + 0dt)$ preserves the Lie bracket, hence, $T^* M$ is a Lie subalgebra of $J^1 M$, and the latter is an extension of the former by the trivial line bundle $M \times \mathbb{R}$. $J^1 M$ was not yet used in Poisson geometry.
3 The homology of the Jacobi BV-algebra

We call the homology of the Jacobi BV-algebra of a Jacobi manifold \((M, \Lambda, E)\), with boundary operator \(\delta\), \textit{Jacobi homology} \(H^J_k(M, \Lambda, E)\). (Another Jacobi homology was studied in \([1]\).) Here, we look at this homology from the point of view of \([21]\) and \([3]\), and discuss duality between the Jacobi homology and the Lie algebroid cohomology of \(J^1M\), called \textit{Jacobi cohomology}.

Jacobi cohomology coincides with the one studied by de León, Marrero and Padrón in \([10]\). If \(C \in \Gamma \wedge^k (J^1M)^*\) is seen as a \(k\)-multilinear skew symmetric form on arguments \((2.7)\) of degree 1, at \(t = 0\), it may be written as

\[(3.1) \quad C = \tilde{C}_{/t=0} := e^{-kt}[(C_1 + \frac{\partial}{\partial t} \wedge C_2)_{/t=0} \quad (C_1 \in \mathcal{V}^kM, \ C_2 \in \mathcal{V}^{k-1}M)],\]

where \(\mathcal{V}^kM\) denotes the space of \(k\)-vector fields on \(M\). Furthermore, the coboundary, say \(\sigma\), is given by the usual formula

\[(3.2) \quad (\sigma C)(s_0, \ldots, s_k) = \sum_{i=0}^{k} (-1)^i (\rho s_i) C(s_0, \ldots, \hat{s_i}, \ldots, s_k) + \sum_{i<j}^{k} (-1)^{i+j} C(\{s_i, s_j\}, s_0, \ldots, \hat{s_i}, \ldots, \hat{s_j}, \ldots, s_k),\]

where \(\rho\) is given by \((2.6)\), and \(s_i \in \Gamma J^1M\). Again, if we see the arguments as forms \((2.7)\) with \(k = 1\), \(3.2\) becomes

\[(3.3) \quad (\sigma C) = [\sigma_P \tilde{C}]_{/t=0} = [P, \tilde{C}]_{/t=0},\]

where \(\sigma_P\) is the Lichnerowicz coboundary (e.g., \([19]\)). Up to the sign, \(3.3\) is the coboundary defined in \([10]\) namely,

\[(3.4) \quad \sigma C = [\Lambda, C_1] - kE \wedge C_1 - \Lambda \wedge C_2 - \frac{\partial}{\partial t} \wedge ([\Lambda, C_2] - (k-1)E \wedge C_2 + [E, C_1]).\]

We denote the Jacobi cohomology spaces by \(H^J_k(M, \Lambda, E)\).
3.1 Remark. [10] The anchor \( \rho \) induces homomorphisms \( \rho^\sharp : H^k_{de R}(M) \to H^k_J(M, \Lambda, E) \) given by

\[
(\rho^\sharp \lambda)(s_1, \ldots, s_k) = (-1)^k \lambda(\rho s_1, \ldots, \rho s_k) \quad (\lambda \in \wedge^k M, s_i \in \Gamma J^1 M).
\]

Now, we need a recapitulation of several results of [21] and [3].

For a Lie algebroid \( A \to M \) with anchor \( a \), an \( A \)-connection \( \nabla \) on a vector bundle \( E \to M \) consists of derivatives \( \nabla_s e \in \Gamma E \) \( (s \in \Gamma A, e \in \Gamma E) \) which are \( \mathbb{R} \)-bilinear and satisfy the conditions

\[
\nabla_{fs} e = f \nabla_s e, \quad \nabla_s(fe) = (a(s)f)e + f \nabla_s e \quad (f \in C^\infty(M)).
\]

For an \( A \)-connection, curvature may be defined as for usual connections.

Any flat \( A \)-connection \( \nabla \) on \( \wedge^r A \) \( (r = \text{rank} A) \) produces a Koszul operator \( D : \Gamma \wedge^k A \to \Gamma \wedge^{k-1} A \), locally given by

\[
DU = (-1)^{r-k+1} [i(\omega)\Omega + \sum_{h=1}^r \alpha^h \wedge (i(\omega)\nabla_{s_h} \Omega)],
\]

where \( \Omega \in \Gamma \wedge^r A, \omega \in \Gamma \wedge^{r-k} A^* \) is such that \( i(\omega)\Omega = U \), \( s_h \) is a local basis of \( A \), and \( \alpha^h \) is the dual cobasis of \( A^* \). Moreover, \( D \) is an exact generator of the Gerstenhaber algebra of \( A \), and every exact generator is defined by a flat \( A \)-connection as above. The operator \( D \) is a boundary, and yields a corresponding homology, called the homology of the Lie algebroid \( A \) with respect to the flat \( A \)-connection \( \nabla \), \( H_k(A, \nabla) \). For two flat connections \( \nabla, \bar{\nabla} \) such that \( D - \bar{D} = i(\alpha) \), where \( \alpha = d_A f \ (f \in C^\infty(M)) \), one has \( H_k(M, \nabla) = H_k(M, \bar{\nabla}) \). If \( \exists \Omega \in \Gamma \wedge^r A \) which is nowhere zero, and \( \nabla \Omega = 0 \), one has the duality \( H_k(A, \nabla) = H^{r-k}(A) \), defined by sending \( Q \in \Gamma \wedge^k A \) to \( *Q := i(Q)\Omega \).

These results may be applied to the cotangent Lie algebroid of an orientable Poisson manifold \( (N^n, Q) \). In this case, the flat connection \( \nabla_Q \Psi = \theta \wedge (df(\Psi)) (\theta \in T^* N, \Psi \in \wedge^n N) \) precisely has the Koszul operator \( \delta_Q \) and defines the known Poisson homology \( H_k(N, Q) \) (e.g., [19]). Finally ([21], Proposition 4.6 and Theorem 4.7), if \( N \) has the volume form \( \Omega \), which defines a connection \( \nabla_0 \) by \( \nabla_0 \Omega = 0 \), and if \( W^Q \) is the modular vector field which acts on \( f \in C^\infty(M) \) according to the equation \( L_{X_f^Q} \Omega = (W^Q f)\Omega \) \( (X_f^Q \) is the Hamiltonian field of \( f \)) [20], one has \( \delta_Q - \bar{D}_0 = i(W^Q) \). Accordingly, if
the modular field $W^Q$ is Hamiltonian (i.e., $(N, Q)$ is a unimodular Poisson manifold), $H_k(N, Q) = H^{n-k}(T^*N)$.

The case of a general, possibly non orientable, Poisson manifold is studied in [3]. The expression of $\nabla_\theta \Psi$ above can be seen as the local equation of a connection on $\wedge^n T^*N$, and it still defines the Koszul operator $\delta_Q$. The general duality Theorem 4.5 of [3] is

\[(*) \quad H_k(N, Q) = H^{n-k}(T^*N, \wedge^n T^*N),\]

where the right hand side is the cohomology of the Lie algebroid $T^*N$ with values in the line bundle $\wedge^n T^*N$. This means that the $k$-cocycles are spanned by cross sections $V \otimes \Psi, V \in \mathcal{V}^k N, \Psi \in \Gamma \wedge^n T^*N$, and the coboundary is given by

$$\partial(V \otimes \Psi) = [Q, V] \otimes \Psi + (-1)^k V \otimes \nabla \Psi.$$ 

Duality $(*)$ is again defined by the isomorphism which sends $V \otimes \Psi$ to $i(V)\Psi$.

With this recapitulation finished, we apply the results to Jacobi manifolds $(M^m, \Lambda, E)$. Consider the Poisson manifold $(M \times \mathbb{R}, P)$ which we already used before. Then $\delta_P$ is the Koszul operator of the $(T^*M \times \mathbb{R})$-connection

\[(3.6) \quad \nabla_\theta \Psi = \theta \wedge (di(P)\Psi) \quad (\theta \in T^*(M \times \mathbb{R}), \Psi \in \wedge^{m+1}(M \times \mathbb{R})).\]

In particular, if we take

\[(3.7) \quad \theta = e^t(\alpha + t dt), \quad \Psi = e^{(m+1)t}\Phi \wedge dt \quad (\alpha \in T^* M, \Phi \in \wedge^m M)\]

(and use (2.9)) we get

\[(3.8) \quad \nabla_\theta \Psi = e^{(m+1)t}[f di(E) \Phi - \alpha \wedge (di(\Lambda) \Phi + mi(E) \Phi)] \wedge dt,\]

and this formula may be seen as defining a $J^1 M$-connection on $\wedge^{m+1} J^1 M$. Clearly, the Koszul operator of this connection must be $\delta$ of (2.10). Therefore, we have

3.2 Proposition. The Jacobi homology of $(M, \Lambda, E)$ is equal to the homology of the Lie algebroid $J^1 M$ with respect to the flat connection (3.8) i.e.,

\[(3.9) \quad H^j_k(M, \Lambda, E) = H_k(J^1 M, \nabla).\]
Now, assume that $M$ has a volume form $\Phi \in \bigwedge^m M$. Then $\Omega := e^{(m+1)t} \Phi \wedge dt$ is a volume form on $M \times \mathbb{R}$, and one has a connection $\nabla_0$ defined by $\nabla_0 \Omega = 0$ with a Koszul operator $D_0$ such that
\begin{equation}
\delta_P - D_0 = i(W^P),
\end{equation}
where $W^P$ is the corresponding modular vector field i.e.,
\begin{equation}
L_{X^P} \Omega = (W^P \varphi) \Omega \quad (\varphi \in C^\infty(M \times \mathbb{R})).
\end{equation}

We need the interpretation of (3.10) at $t = 0$. To get it, we take local coordinates $(x^i)$ on $M$, and compute the local components of $W^P$ by using (3.11) for $\varphi = x^i$ and $\varphi = t$. Generally, we have
\begin{equation}
X^P = i(d \varphi)P = e^{-t}(\sharp_\Lambda d \varphi + \frac{\partial \varphi}{\partial t} E - (E \varphi) \frac{\partial}{\partial t}).
\end{equation}

On the other hand, on $M$, let us define a vector field $V$ and a function $\text{div}_\Phi E$ by
\begin{equation}
L^{\sharp_\Lambda df}_\Phi = (V f) \Phi, \quad L_E \Phi = (\text{div}_\Phi E) \Phi \quad (f \in C^\infty(M)).
\end{equation}
(The fact that $V$ is a derivation of $C^\infty(M)$ follows easily from the skew symmetry of $\Lambda$.) Then, the calculation of the local components of $W^P$ yield
\begin{equation}
W^P = e^{-t}[V - mE + (\text{div}_\Phi E) \frac{\partial}{\partial t}].
\end{equation}

At $t = 0$, (3.14) defines a section of $TM \oplus \mathbb{R}$ which we denote by $V^{(\Lambda,E)}$, and call the modular field (not a vector field, of course) of the Jacobi manifold.

As in the Poisson case, if $\Phi \mapsto a \Phi$ $(a > 0)$, $V^{(\Lambda,E)} \mapsto V^{(\Lambda,E)} + \sigma(\ln a)$ hence, what is well defined is the Jacobi cohomology class $[V^{(\Lambda,E)}]$, to be called the modular class. If the modular class is zero $(M, \Lambda, E)$ is a unimodular Jacobi manifold.

It is also possible to get the modular class $[V^{(\Lambda,E)}]$ from the general definition of the modular class of a Lie algebroid $\mathfrak{g}$. In the case of the algebroid $J^1 M$, the definition of $\mathfrak{g}$ means computing the expression
\begin{equation}
E := (L^{J^1 M}_{\sharp_\Phi(df + fdt)}(e^{mt} \Phi) \wedge (e^t dt)) \otimes \Phi
\end{equation}
\[ + (e^{(m+1)t} \Phi \wedge dt) \otimes (L_{\rho(e^{t(df + fdt))}} \Phi), \]

where \( \rho \) is given by (2.6), and

\[ L_{e^{t(df + fdt)}]}[e^{mt}\Phi] \wedge (e^t dt)] = \{e^t(df + f dt), (e^{mt}\Phi) \wedge (e^t dt) \} P. \]

If we decompose \( (e^{mt}\Phi) = \wedge_{i=1}^n (e^t \varphi_i), \varphi_i \in \Lambda^1 M, \) the result of the required computation turns out to be

\[ E = (2(V f) + 2f(div \Phi) - Ef)(e^{(m+1)t}\Phi \wedge dt) \otimes \Phi. \]

By comparing with (3.14), we see that the modular class in the sense of [3] is the Jacobi cohomology class of the cross section of \( TM \oplus \mathbb{R} \) defined by

\[ A^{(\Lambda,E)} = 2V^{(\Lambda,E)} - (2m + 1)E. \]

With all this notation in place, the recalled results of [21], Proposition 4.6, and [3], Theorem 4.5, yield

3.3 Proposition. If \( (M, \Lambda, E) \) is a unimodular Jacobi manifold one has duality between Jacobi homology and cohomology:

(3.15) \[ H_j^j(M, \Lambda, E) = H_j^{m-k+1}(M, \Lambda, E). \]

If \( (M, \Lambda, E) \) is an arbitrary Jacobi manifold, one has the duality

(3.15') \[ H_k^j(M, \Lambda, E) = H_j^{m-k+1}(J^1 M, \wedge^{m+1} J^1 M). \]

Proof. The right hand side of (3.15') is \textit{Jacobi cohomology with values in} \( \wedge^{m+1} J^1 M \), similar to that in \((*)\). The homologies and cohomologies of (3.15) and (3.15') are to be seen as given by subcomplexes of \( \oplus_k \Lambda^k (M \times \mathbb{R}) \), \( \oplus_k V^k(M \times \mathbb{R}) \) defined by (2.7) and (3.1). Then the result follows by the proofs of the theorems of [21], [3] quoted earlier, if we notice that

\[
\left[ i e^{-kt}(C_1 + \frac{\partial}{\partial t} \wedge C_2) \right] (e^{(m+1)t}\Phi \wedge dt)
\]

\[ = e^{(m-k+1)t}\left[ (-1)^m i(C_2) \Phi + i(C_1) \Phi \wedge dt \right]. \]

The notation is that of (3.1) and (3.7). Q.e.d.

To get examples, let us look at the \textit{transitive Jacobi manifolds} [4].
a). Let $M^{2n}$ be a locally or globally conformal symplectic manifold with the 2-form $\Omega = e^{\sigma} \Omega_\alpha$, where $\Omega_\alpha$ are symplectic forms on the sets $U_\alpha$ of an open covering of $M$, and $\sigma_\alpha \in C^\infty(U_\alpha)$. Then (e.g., [18]) $\{d\sigma_\alpha\}$ glue up to a global closed 1-form $\omega$, which is exact iff $\exists \alpha$, $U_\alpha = M$, and $\sharp \Lambda := \frac{\partial}{\partial \Omega}$. $E := \sharp \Lambda \omega$ define a Jacobi structure on $M$ [5]. It follows easily that $L_E \Omega = 0$ hence, $\text{div}_\Omega E = 0$. Furthermore,

$$L_{\sharp \Lambda(d\sigma)} \Omega^n = -n(n-1)df \wedge \omega \wedge \Omega^{n-1}.$$ 

Using the Lepage decomposition theorem ([12], pg.46) we see that $df \wedge \omega = \xi + \varphi \Omega$, where

$$\xi \wedge \Omega^{n-1} = 0, \quad \varphi = -\frac{1}{n} i(\Lambda)(df \wedge \omega) = Ef.$$

Hence, $V = -n(n-1)E$, and $V(\Lambda, E) = -n(2n-1)E$. Then, for $f \in C^\infty(M)$, (3.4) yields $\sigma f = \sharp \Lambda df - (Ef)(\partial/\partial t)$, and $\sigma f = E$ holds iff $\omega = df$. Thus, (3.15) holds on globally conformal symplectic manifolds. But, (3.15) may not hold in the true locally conformal symplectic case. For instance, it follows from Corollary 3.15 of [11] that the result does not hold on a Hopf manifold with its natural locally conformal Kähler structure. (Private correspondence from J. C. Marrero.)

b). Let $M^{2n+1}$ be a contact manifold with the contact 1-form $\theta$ such that $\Phi := \theta \wedge (d\theta)^n$ is nowhere zero. Then $M$ has the Reeb vector field $E$ where

$$i(E)\theta = 1, \quad i(E)d\theta = 0,$$

and $\forall f \in C^\infty(M)$ there is a Hamiltonian vector field $X_\theta^f$ such that

$$i(X_\theta^f)\theta = f, \quad i(X_\theta^f)d\theta = -df + (Ef)\theta.$$

Furthermore, if

$$\Lambda(df, dg) := d\theta(X_\theta^f, X_\theta^g) \quad (f, g \in C^\infty(M)),$$

$(\Lambda, E)$ is a Jacobi structure [5].

Now, let $(q^i, p_i, z) \ (i = 1, ..., n)$ be local canonical coordinates, such that $\theta = dz - \sum_i p_i dq^i$. Then

$$E = \frac{\partial}{\partial z}, \quad \Lambda = \sum_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{\partial}{\partial z} \wedge (\sum_i p_i \frac{\partial}{\partial p_i}).$$
This leads to \( \text{div}_x E = 0 \), \( V^{(\Lambda, E)} = nE \), and it follows that there is no \( f \in C^\infty(M) \) satisfying \( \sigma f = (nE \oplus 0) \).

We close this section by the remark that the identification of a manifold \( M \) with \( M \times \{0\} \subseteq M \times \mathbb{R} \) leads to other interesting structures too. For instance, if we define the spaces

\[
\wedge_0^k M := \{ e^i (\xi_1 + \xi_2 \wedge dt) / \xi_1 \in \wedge^k M, \xi_2 \in \wedge^{k-1} M \},
\]

the triple \((\oplus_k \wedge_0^k M, d, i(X + f(\partial/\partial t)))\) is a Gelfand-Dorfman complex \([2]\), and a Jacobi structure on \( M \) is equivalent with a Hamiltonian structure \([2]\) on this complex.

On the other hand, if we have a Jacobi manifold \((M, \Lambda, E)\), and put

\[
\mathcal{V}_0^k M := \{ e^{-(k-1)t} (Q_1 + \frac{\partial}{\partial t} \wedge Q_2) / Q_1 \in \mathcal{V}^k M, Q_2 \in \mathcal{V}^{k-1} M \},
\]

then \((\oplus_k \mathcal{V}_0^k M, [\cdot, \cdot], \sigma_P)\) \((P \text{ is defined by (2.2)})\) is a differential graded Lie algebra, the cohomology of which is exactly the 1-differentiable Chevalley-Eilenberg cohomology \( H^1_{1-\text{dif}}(M, \Lambda, E) \) of Lichnerowicz \([13]\). In particular, \( H^1_{1-\text{dif}}(M, \Lambda, E) \) is the quotient of the space of conformal Jacobi infinitesimal automorphisms by the space of the Jacobi Hamiltonian vector fields \([13]\).

4 Lie bialgebroid structures on \( TM \oplus \mathbb{R} \)

In the Poisson case, \( T^*M \) is a Lie bialgebroid \([8], [16]\) with dual \( TM \). This is not true for \( J^1M \) on Jacobi manifolds in spite of the fact that \((J^1M)^* = TM \oplus \mathbb{R} \) has a natural Lie algebroid structure, which extends the one of \( TM \). Namely, if we see \( X \in \Gamma(TM \oplus \mathbb{R}) \) as a vector field of \( M \times \mathbb{R} \) given by

\[
(4.1) \quad X = (X + f \frac{\partial}{\partial t})_{t=0} \quad (X \in \Gamma TM, f \in C^\infty(M)),
\]

we have the Lie bracket

\[
(4.2) \quad [X, Y]_0 := [X + f \frac{\partial}{\partial t}, Y + g \frac{\partial}{\partial t}] = [X, Y] + (Xg - Yf) \frac{\partial}{\partial t},
\]

and the anchor map \( a(X) := X \). If we would be in the case of a Lie bialgebroid, the bracket \( \{f, g\}_s := \langle df, dg \rangle \) \((f, g \in C^\infty(M))\), where \( d, d_s \) are the differentials of the Lie algebroids \( TM \oplus \mathbb{R} \) and \( J^1M \), respectively, would be Poisson \([8], [16]\). This is not true since one gets \( \{f, g\}_s = \Lambda(df, dg) \).

12
4.1 Remark. The differential $\bar{d}$ defined by (2.11) is the same as the differential $d$ of the Lie algebroid $TM \oplus \mathbb{R}$ with the bracket (4.2).

In Poisson geometry, the cotangent Lie bialgebroid structure is produced by a Poisson bivector $\Pi$ of $TM$ i.e., $[\Pi, \Pi] = 0$. It is natural to ask what is the structure produced by a Poisson bivector $\Pi$ of $TM \oplus \mathbb{R}$. We generalize this question a bit namely, we fix a closed 2-form $\Omega$ on $M$, and take the Lie bracket

$$(4.2') \quad [\mathcal{X}, \mathcal{Y}]_\Omega := [\mathcal{X}, \mathcal{Y}]_0 + \Omega(X, Y) \frac{\partial}{\partial t}.$$ 

The notation, and the anchor map $a$ are the same as for (4.2). It is known that $(4.2')$ defines all the transitive Lie algebroid structures over $M$ such that the kernel of the anchor is a trivial line bundle, up to an isomorphism $[15]$. A Poisson bivector $\Pi$ on $TM \oplus \mathbb{R}$ with the bracket $(4.2')$ will be called an $\Omega$-Poisson structure on $M$.

4.2 Proposition. An $\Omega$-Poisson structure $\Pi$ on $M$ is equivalent with a pair $(Q, E)$, where $Q$ is a Poisson bivector on $M$ (i.e., $[Q, Q] = 0$), and $E$ is a vector field such that

$$(4.3) \quad L_E Q = \sharp_Q \Omega.$$ 

Proof. Using the identification (4.1) of the cross sections of $TM \oplus \mathbb{R}$ with vector fields on $M \times \mathbb{R}$ for $t = 0$, and local coordinates $(x^i)$ on $M$, we may write

$$(4.4) \quad \Pi = Q + \frac{\partial}{\partial t} \wedge E = \frac{1}{2} Q^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \frac{\partial}{\partial t} \wedge (E^k(x) \frac{\partial}{\partial x^k}),$$

where $Q$ is a bivector field on $M$, $E$ is a vector field, and the Einstein summation convention is used.

Now, $[\Pi, \Pi]_\Omega = 0$ can be expressed by the known formula of the Schouten-Nijenhuis bracket of decomposable multivectors (e.g., [19], formula (1.12)), and $(4.2')$. The result is equivalent to $[Q, Q] = 0$ and (4.3). Q.e.d.

4.3 Corollary. If $(M, Q)$ is a Poisson manifold, $Q$ extends to a $\Omega$-Poisson structure for every closed 2-form $\Omega$, where the de Rham class $[\Omega]$ has zero $\sharp_Q$-image in the Poisson cohomology of $(M, Q)$, by taking $E$ such that (4.3) holds.

This is just a reformulation of Proposition 4.2.
It is well known that a Poisson bivector on a Lie algebroid \( A \) induces a bracket on \( \Gamma A^* \) such that \((A, A^*)\) is a triangular Lie bialgebroid \([8], [16]\). Namely, the Poisson bivector \( \Pi \) of (4.4) yields the following bracket

\[
\{ \alpha \oplus f, \beta \oplus g \}_\Omega := L_{\sharp \Omega (\alpha \oplus f)}^\Omega (\beta \oplus g) - L_{\sharp \Omega (\beta \oplus g)}^\Omega (\alpha \oplus f) - d_\Omega (\Pi (\alpha \oplus f, \beta \oplus g)),
\]

where \( \alpha \oplus f, \beta \oplus g \in \Gamma J^1 M \), and the index \( \Omega \) denotes the fact that the operators involved are those of the Lie algebroid calculus of (4.2\').

To make this formula explicit, notice that

\[
\sharp \Pi (\alpha + f dt) = \sharp Q \alpha + f E - \alpha (E) \frac{\partial}{\partial t},
\]

whence

\[
\Pi (\alpha + f dt, \beta + g dt) = Q (\alpha, \beta) + f \beta (E) - g \alpha (E).
\]

Then, by evaluation on a field of the form (4.1), and with (4.2'), we obtain

\[
L_{\sharp \Omega (\alpha + f dt)}^\Omega (\beta + g dt) = L_{\sharp \Omega (\alpha + f dt)}^\Omega (\beta + g dt) - g (\sharp \Omega \sharp Q \alpha) - f g (E) \Omega,
\]

where \( b_\Omega X := i (X) \Omega \). As a consequence, (4.5) becomes

\[
\{ \alpha \oplus f, \beta \oplus g \}_\Omega := \{ \alpha, \beta \}_Q + f (L_E \beta + b_\Omega \sharp Q \beta)
\]

\[
- g (L_E \alpha + b_\Omega \sharp Q \alpha) \} \oplus \{ (\sharp Q \alpha) g - (\sharp Q \beta) f + f (E g) - g (E f) \},
\]

where \((Q, E)\) are associated with \( \Pi \) as in Proposition 4.2.

The anchor map of the Lie algebroid \( J^1 M \) with (4.9) is \( \rho := pr_{TM} \circ \sharp \Pi \), and it is provided by (4.6).

In particular, Proposition 4.2 tells us that a pair \( (Q, E) \) which consists of a Poisson bivector \( Q \) and an infinitesimal automorphism \( E \) of \( Q \), to which we will refer as an enriched Poisson structure, provides a Poisson bivector \( \Pi \) on \( TM \oplus R \) with the bracket (4.2), and a Lie bialgebroid \((TM \oplus R, J^1 M = T^* M \oplus R)\).

An example (suggested by [14]) can be obtained as follows. Let \((M, \Lambda, E)\) be a Jacobi manifold. A time function is a function \( \tau \in C^\infty (M) \) which has no critical points and satisfies \( E \tau = 1 \). If such a function exists, \((\Lambda_0 := \Lambda - (\sharp \Lambda dt) \wedge E, E)\) is an enriched Poisson structure. Jacobi manifolds with time may be seen as generalized phase spaces of time-dependent
Hamiltonian systems. Namely, if $H \in C^\infty(M)$ is the Hamiltonian function, the trajectories of the system are the integral lines of the vector field $X^0_H := \sharp_{\lambda_0} df + E$.

Let us briefly indicate the important objects associated with the Lie algebroids $TM \oplus \mathbb{R}$, defined by the bracket (4.2'), and $J^1M$ with the bracket (4.9).

The cohomology of $TM \oplus \mathbb{R}$ is that of the cochain spaces

(4.10) \[ \wedge^k_\Omega M := \{ \lambda = \lambda_1 + \lambda_2 \wedge dt / \lambda_1 \in \wedge^k M, \lambda_2 \in \wedge^{k-1} M \} \]

with the corresponding coboundary, say $d_\Omega$. A straightforward evaluation of $d_\Omega \lambda$ on arguments $X_i + f_i(\partial/\partial t)$, in accordance with the Lie algebroid calculus [15], yields the formula

(4.11) \[ d_\Omega \lambda = d\lambda - (-1)^k \Omega \wedge \lambda_2. \]

The Poisson cohomology of $TM \oplus \mathbb{R}$ above i.e., the cohomology of the Lie algebroid $J^1M$ with (4.9), can be seen (with (4.1)) as having the cocycle spaces

(4.12) \[ C^k(M) := \{ C = C_1 + \frac{\partial}{\partial t} \wedge C_2 / C_1 \in \mathcal{V}^k M, C_2 \in \mathcal{V}^{k-1} M \}, \]

and the coboundary $\partial C = [\Pi, C]_\Omega$, with $\Pi$ of (4.4) and the $\Omega$-Schouten-Nijenhuis bracket. In order to write down a concrete expression of this coboundary, we define an operation $U \wedge_\Omega V(\alpha_1, \ldots, \alpha_{k+h-2}) = \frac{1}{(k-1)!(h-1)!} \sum_{\sigma \in S_{k+h-2}} [(\text{sign } \sigma) \sum_{i=1}^m U(e^i, \alpha_{\sigma_1}, \ldots, \alpha_{\sigma_k}) V(\lambda_\Omega e^i, \alpha_{\sigma_{k+1}}, \ldots, \alpha_{\sigma_{k+h-2}})]$, where $S$ is the symmetric group, $e_i$ is a local tangent basis of $M$, and $e_i^\ast$ is the corresponding dual cobasis. If $U, V$ are vector fields, $U \wedge_\Omega V = \Omega(U, V)$. By computing for decomposable multivectors $C_1, C_2$, we get

(4.13) \[ \partial C = [Q, C_1] + \frac{\partial}{\partial t} \wedge ([Q, C_2] + Q \wedge_\Omega C_1 - L_E C_1), \]
where the brackets are the usual Schouten-Nijenhuis brackets on $M$.

Furthermore, the exact generator of the BV-algebra of the Lie algebroid $J^1M$ is $\delta_{\Omega} := [i(\Pi), d_{\Omega}]$, and using (4.11) we get

\[(4.15) \quad \delta_{\Omega}(\lambda_1 + \lambda_2 \wedge dt) = \delta_{Q}\lambda_1 + (-1)^{k-1}([i(Q), e(\Omega)]\lambda_2
- di(E)\lambda_2) + (\delta_{Q}\lambda_2) \wedge dt,
\]

where $e(\Omega)$ is exterior product and $[\ , \ ]$ is the commutant of the operators.

Finally, let us discuss the modular class of the Lie algebroid $(J^1M, (4.9))$. For simplicity, we assume the manifold $M$ orientable, with the volume form $\Phi \in \Gamma \wedge^m M$. In the non orientable case, the same computations hold if $\Phi$ is replaced by a density $h$ $\Phi \in \Gamma| \wedge^m M$. Again, we denote by $W^Q$ the modular vector field defined by $L_{\rho_{Q}\Phi} = (W^Qf)\Phi$ (see Section 3).

There are two natural possibilities to define a modular class for the algebroid $J^1M$. One is by computing the Lie derivative:

\[(4.16) \quad L_{\zeta(\alpha + f dt)}(\Phi \wedge dt) = [L_{\alpha} \Phi + fL_{E} \Phi + df \wedge i(E)\Phi] \wedge dt.
\]

This result is obtained if the computation is done after $\Phi$ is decomposed into a product of $m$ 1-forms, and by using (4.6). Since $i(E)(df \wedge \Phi) = 0$, the last term in (4.16) is $(E f)\Phi$, and if we also use (3.13), (4.16) yields

\[(4.17) \quad L_{\zeta(df + f dt)}(\Phi \wedge dt) = (W^Q f + fdiv\Phi + Ef)(\Phi \wedge dt).
\]

Therefore, we get the modular field

\[(4.18) \quad W^\Pi := W^Q + E + (div\Phi)\frac{\partial}{\partial t}.
\]

If $\Phi$ is replaced by $h\Phi$ ($h \in C^\infty(M)$), it follows easily that the II-Poisson cohomology class $[W^\Pi]$ is preserved. This will be the modular class.

The second possibility is to apply the general definition of $\mathfrak{B}$. Similar to what we had for Jacobi manifolds in Section 3, this asks us to compute the flat connection $D$ on $(\wedge^{m+1}J^1M) \otimes (\wedge^m T^*M)$ given by

\[(4.19) \quad D_{(df + f dt)}[(\Phi \wedge dt) \otimes \Phi] = L^{J^1M}_{(df + f dt)}(\Phi \wedge dt) \otimes \Phi + (\Phi \wedge dt) \otimes (L_{\rho(df + f dt)}\Phi).
\]

From Lie algebroid calculus, we know that

\[(4.20) \quad L^{J^1M}_{(df + f dt)}(\Phi \wedge dt) = \{df + f dt, \Phi \wedge dt\}_\Omega,
\]
where the bracket is the Schouten-Nijenhuis extension of (4.9). If we look at a decomposition \( \Phi = \varphi_1 \wedge ... \wedge \varphi_n \) \((\varphi_i \in \wedge^1 M)\), (4.9) yields

\[
\{df + f dt, dt\}_\Omega = 0,
\]

\[
\{df + f dt, \varphi_i\}_\Omega = L_{\rho(df + f dt)} \varphi_i - \varphi_i(E)df + f\gamma_\Omega \sharp Q \beta - \left(\sharp Q \beta(f)\right)dt,
\]

and we get

(4.21) \( L_{(df + f dt)}^J M (\Phi \wedge dt) = [L_{(df + f dt)} \Phi - (Ef)\Phi + f tr(\gamma_\Omega \circ \sharp Q)\Phi] \wedge dt. \)

But, we also have

(4.22) \( L_{\rho(df + f dt)} \Phi = L_{\sharp Q df + fE} \Phi = [W^Q f + f div_\Phi E + Ef] \Phi. \)

By inserting (4.21), (4.22) into (4.19), we get another modular field namely,

(4.23) \( A_\Omega := (2W^Q + E) \oplus (div_\Phi E + tr(\gamma_\Omega \circ \sharp Q)) = (2W^\Pi - E) \oplus tr(\gamma_\Omega \circ \sharp Q). \)

From the general results of [3], it is known that the \( \Pi \)-Poisson cohomology class of this field is independent of the choice of \( \Phi \), and it is a modular class of \( J^1 M \).

As for the modular class of \( TM \oplus \mathbb{R} \) with the bracket (4.2'), it vanishes for reasons similar to those for the class of the tangent groupoid \( TM \) [3].

We finish by another interpretation of the enriched Poisson structures. If \( \mathcal{F} \), be an arbitrary associative, commutative, real algebra, we may say that \( f : M \to \mathcal{F} \) is differentiable if for any \( \mathbb{R} \)-linear mapping \( \phi : \mathcal{F} \to \mathbb{R} \), \( \phi \circ f \in C^\infty(M) \). Furthermore, an \( \mathbb{R} \)-linear operator \( \nu_x \) which acts on germs of \( \mathcal{F} \)-valued differentiable functions at \( x \in M \), and satisfies the Leibniz rule will be an \( \mathcal{F} \)-tangent vector of \( M \) at \( x \). Then, we have natural definitions of tangent spaces \( T_x(M, \mathcal{F}) \), differentiable \( \mathcal{F} \)-vector fields, etc. [17]. A bracket \( \{ \cdot , \cdot \} \) which makes \( C^\infty(M, \mathcal{F}) \) into a Poisson algebra will be called an \( \mathcal{F} \)-Poisson structure on \( M \).

Now, take \( \mathcal{F} \) to be the Studi algebra of parabolic dual numbers \( \mathcal{S} := \mathbb{R}[s / s^2 = 0] \), where \( s \) is the generator. An \( \mathcal{S} \)-Poisson structure \( \Pi \) in the above mentioned sense will be called a Studi-Poisson structure. The restriction of \( \Pi \) to real valued functions is a Poisson bivector \( Q \) on \( M \), and the Jacobi identity shows that the Hamiltonian vector field \( X_s^\Pi \) of the constant function \( s \) is an
infinitesimal automorphism $E$ of $Q$. Conversely, the pair $(Q, E)$ defines the Studi-Poisson bracket

$$\{f_0 + f_1 s, g_0 + g_1 s\} := \{f_0, g_0\}_Q + f_1(Eg_0) - g_1(Ef_0)$$

$$+ s(\{f_0, g_1\}_Q + \{f_1, g_0\}_Q + f_1(Eg_1) - g_1(Ef_1)).$$

Notice that we cannot say that $v_x s = 0$ for all $v_x \in T_x(M, \mathcal{F})$ since $v_x$ was linear over $\mathbb{R}$ only.

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Department of Mathematics,
University of Haifa, Israel.
E-mail: vaisman@math.haifa.ac.il