Gauge Theories in the Derivative Expansion

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Abstract

Gauge theories with and without matter are formulated in the derivative expansion. Amplitudes are derived as a power series in the energy scales; there are simplifications as compared with the usual loop expansion. The incorporation and summation of Wilson loops is natural in this approach and allows for a derivation of the confining potential as well as the masses of the gauge invariant observables.
1 Introduction

In this work we derive, in the derivative expansion, the correlation functions and amplitudes of conventional QCD. The masses of the mesonic, hadronic and further states together with their dynamics are obtainable through this approach. In subsequent work we analyze a holographic version, incorporating gravity, although conventional quantum chromodynamics meshes with conventional gravity.

The composite operators of gauge theory, including matter, are,

\[ \prod_{i=1}^{n} \prod_{j=1}^{n_F} \nabla_{\mu_\sigma(i)} F_{\mu_\sigma(i),\nu_\sigma(j)} \prod_{i=1}^{n_\phi} \nabla_{\mu_\sigma(i)} \prod_{j=1}^{n_\sigma} \Phi \] (1.1)

\[ \times \prod_{i=1}^{n_\sigma} \nabla_{\mu_\sigma(i)} \prod_{j=1}^{n_\Psi} q_{\alpha_\sigma(i,j)} \] (1.2)

We focus on the composite operators, \( O_m \), of dimension \( n \), that are pure gluonic (glueballs) or quark-like.

The amplitudes may be constructed via two methods. The conventional methodology is to sum over the Feynman diagrams. Recently, the derivative expansion has been utilized to obtain the same information in scalar and spontaneously broken \( N = 4 \) gauge theory; we take the second approach in this work, together with non-perturbative corrections, in analyzing quantum chromodynamics. The derivative expansion has the advantage that the restricted set of diagrams that enter may all be integrated over, and that the non-perturbative corrections are easily implemented.

The masses and confining potentials of the nucleons may be found. Furthermore, the correlations of the quantum dressed composite operators are found. The methodology is the same as the one used in analyzing (and almost solving) the \( \phi^n \) theory, but with the complication that the integrals have massless particles.

The Langrangian we consider is

\[ \mathcal{L} = \int d^4x \left( -\frac{1}{4} F^a F_a + \bar{\psi} \gamma^u \gamma^u \nabla_u \psi_a + \sum m_j \bar{\psi}_j \psi_j \right) . \] (1.3)

In the case of QCD we have the usual doublet-triplets (t, b, c) and (s, d, u). The masses are, in terms of \( \Lambda (\Lambda/m_{pl})^{n_j/16} \), with \( \Lambda = 2 \) TeV,
\begin{align}
\begin{pmatrix}
1 & m_t \\
2 & m_b \\
3 & m_c \\
4 & m_s \\
5 & m_d \\
6 & m_u \\
\end{pmatrix}
\end{align}

and likewise for the leptons,

\begin{align}
\begin{pmatrix}
3 & m_\tau \\
4 & m_\mu \\
5 & m_e \\
\end{pmatrix}
\end{align}

The quantization of the masses follows from gravitational considerations \[2\].

The derivative expansion follows the simpler analysis of scalar field theory \[1\]. First we categorize the quantum generating functional of the S-matrix. In the case of gauge theory, the quantum vertices are non-polynomial due to zero-mass thresholds, and inverse powers of derivatives are present due to massless singularities. These two features, together with index structure, complicate the sewing that generates the S-matrix. However, the coefficients between different orders are not all independent due to known factorization identities.

The quantum generating functional is found from the general vertex of hard dimension,

\begin{align}
\prod_{i=1}^{p^A} \prod_{j=1}^{n^A_{i(i,j)}} \partial_{\mu_{\sigma(i,j)}} A_{\mu_{\beta(i,j)}} \prod_{i=1}^{p^\psi} \prod_{j=1}^{n^\psi_{i(i,j)}} \partial_{\mu_{\sigma(i,j)}} \psi_{\alpha_{\delta(i,j)}} m_i^A m_i^\psi
\end{align}

with \( \sum m_i^A = n^A \) and \( \sum m_i^\psi = n^\psi \)-point. The effective action is gauge invariant but gauge dependent (e.g. \( \lambda \text{Tr} F^4 \)) off shell, and the soft dimensional modifications are generic. The latter model the corrections to the classical scaling dimensions and have the form,

\begin{align}
\ln^m(\Box), \ln \ln \ldots \ln(\Box),
\end{align}

together with combinations. They are placed generically within the hard dimension vertex. Last, due to infra-red singularities (the form is known at multi-loop), we may
place a $1/\Box$ in front of $A^p$ with $p$ arbitrary. Massive particles have neither in the derivative expansion at low energy.

The polylogarithms encountered in multi-loop gauge theory calculations may be expanded upon this set of functions. Calculation of the integrals that contain these terms is facilitated by a useful analytic continuation \[3, 4,\]

\[\Box^\beta = \int_0^\infty dt t^{-1+\beta} e^{-t\Box},\] (1.8)

in which the exponential acts on the internal lines. The logarithms are found then by expanding,

\[\Box^\beta = 1 + \beta \ln(\Box^\beta) + \frac{1}{2} \beta^2 \ln^2(\Box) + \ldots .\] (1.9)

The nested log terms are handled via,

\[(\Box^{\beta_1})^{\beta_2} = 1 + \beta_2 \ln^{\beta_1}(\Box) + \ldots = 1 + \beta_2 (1 + \beta_1 \ln \ln(\Box) + \ldots) + \ldots ,\] (1.10)

with calculations done at integral values and then continued to obtain the logarithms. This analyticity has been useful in studying scalar field theory and quantum electrodynamics. The general effective theory is built from the complete set of polynomials and these two modifications. Furthermore, the color structure is generically multi-trace in these vertices, although a truncation to leading color (planar) is consistent.

With these sets of vertices, we may derive the recursion formulae to obtain the coefficients. In exchange for complicated multi-loop Feynman diagrams, which have never been computed at three-loops, there is complicated tensorial algebra; the integrals are all computable, however, and the algebra is suited for computer calculations.

The gauge boson sewing and recursion formulae have the form, at $m_l + m_r = n$-point,

\[
\sum_L \left[ \prod_{i=1}^{m_l} \left( \prod_{j=1}^{\theta(i, j)} \partial_{\mu_s(i, j)} A_{\mu_s(i)}(k_i) \right) \prod_{i=m_l+1}^{m_r} \left( \prod_{j=1}^{\theta(i, j)} \partial_{\mu_s(i, j)} A_{\mu_s(i)}(k_i) \right) \right] \times \left[ \prod_{i=1}^{L} \left( \prod_{j=1}^{m_l^{\delta}} \partial_{\mu_s(i, j)} \prod_{j=1}^{m_r^{\delta}} A_{\mu_s(i)} \right) \right] \] (1.11) 

\[
\times \left[ \prod_{i=1}^{L} \left( \prod_{j=1}^{m_l^{\delta}} \partial_{\mu_s(i, j)} \prod_{j=1}^{m_r^{\delta}} A_{\mu_s(i)} \right) \right] \] (1.12)
\[
\times \left[ \prod_{i=1}^{L} \left( \prod_{j=1}^{\hat{m}_i} \partial_{\mu_{(i,j)}} \prod_{j=1}^{\hat{m}_i^{A}} A_{\mu} \right) \right] \times t^{\mu_{a}, \mu_{i}, \hat{m}_i^{d}} t^{\mu_{a}, \mu_{i}, \hat{m}_i^{d}} + \text{perms} , \tag{1.13}
\]

with \( \sum m_i^A = \sum \hat{m}_i^A = L \). The coupling dependence in eqn. (1.13) is within the tensor \( t \), and is a power series,

\[
t^{\mu_{a}, \mu_{i}, \hat{m}_i^{d}} = \sum_{m=n-3}^{\infty} t^{\mu_{a}, \mu_{i}, \hat{m}_i^{d}} g_{YM}^{2m} , \tag{1.14}
\]

at \( n \)-point. The logarithmic modifications enter via the functions \( f_i(\Box) \) placed within the expression. The contraction of the internal lines is straightforward. The recursion formulae is derived by equating eqns. (1.13) with (1.6).

## 2 Non-perturbative

In the previous section the coupling of microscopic gauge fields and fermions was examined in perturbation theory, and the beta function controls the scaling of the coupling constant.

We examine more general configurations in this section and in particular, the contribution of the analog of the Wilson loop to the amplitudes. Summing these configurations generates a confining potential; these coherent loops generate the reggeization of the composite states through the Schrödinger equation, modeled by the composite operators. The resonances of the mesons, baryons, glueballs, and the like, are also potentially obtained.

The general correlation of the microscopic theory is

\[
\langle \prod_{i=1}^{m} A_{\mu_{i}}^{\alpha_{i}} \prod_{j=1}^{m} e^{-\frac{g}{2}} \oint A \rangle , \tag{2.1}
\]

where the multiple traces of the exponentials are implied. The coupling \( g \) appears in the coherent state, and \( \alpha \) is a dimensionful parameter chosen so that the calculated pion \((u\bar{d} - \bar{u}d)\) mass has the expected value. In asymptotically free theories the coupling \( g \) runs to zero; for simplicity we shall absorb it into \( \alpha \). Composite operator correlators are expectations,

\[
\langle \prod_{i=1}^{m} O^{(i)}_{(d_{i})} \prod_{j=1}^{m} e^{-\frac{g}{2}} f A \rangle \tag{2.2}
\]
The diagrams and integrals representing the perturbative calculations of the microscopic and gauge invariant correlatorion are different, although both resemble free-field theory. The correlation functions are found from the effective theory by sewing the composite operator’s external legs to other composite operators, without iteration. The full two-point function is to be utilized for this purpose. In general the soft dimensions of the operators that model the non-classical scaling dimensions are included.

First let’s examine the closed analog of the Wilson lines, as they have a dramatic effect on the amplitudes. The expression

$$e^{-\frac{\alpha^{1/2}}{g} \oint A e^{-\frac{\alpha^{1/2}}{g} \oint A}} \quad (2.3)$$

integrates, or contracts, to

$$e^{m\beta L(L+1)g^{2} \frac{\alpha}{\alpha x}} = e^{m\beta L(L+1)\frac{\alpha}{\alpha x}}. \quad (2.4)$$

The parameter $\beta$ is a number representing a constant factor obtained from the integration eqn. (2.4). The form in eqn. (2.4) is similar to a mass term, with mass $\alpha/beta$. The diagram is illustrated in fig. 1; in general, these closed loops intersect an arbitrary number $m$ times from 2 to $\infty$. These multiple points are to be integrated over, resulting in a factor of $|x|$, as the perturbative graphs are of ”thickness” $x_1 - x_2 = x$. These Wilson loops receive corrections via perturbation theory between the lines. Multiple non-interacting $e^{-\frac{\alpha^{1/2}}{g} \oint A}$ loops, as represented in figure 2, generate

$$e^{n(L+1)\frac{\alpha}{\alpha x}}. \quad (2.5)$$

A mass term is $e^{-mx}$, which does compare with the $e^{1/x^2}$. The summation over intersections between the Wilson line and the loop graph is straightforward: The loop has three choices upon encountering the graph, under/over or intersection. This results in a combinatoric factor of $3^p$ for the result in (2.5), when there are $p$ intersections. Count the number of loops via the intersection number: one example is a product of minimal loops (a of them) and another is one loop with $2a$ intersections. The expected length of the loop is the number of intersections divided by two, and times the distance $x_1 \cdot x_2$ in the diagram; this is a rough approximation, and we have neglected the interactions between the closed loops.
This formula in (2.5) suggests a weakening of the masses; however, we have an infinite summation of these closed loops, together with interactions between. The individual term in eqn. (2.5) when expanded at small $x$,

gives a $1/x$ term at small separation. The exponentials will be resummed, however, and the short distance singular Coulomb interaction comes from the perturbative results $\langle \psi \psi \ldots \psi \rangle$.

At loop $L$, there is a symmetry factor of $3^{2L}/L!$ that would effect the $L$ structure. The sum over the closed non-interacting loops is,

$$\sum_{n=0}^{\infty} e^{n(L+1)\frac{g^2}{\alpha x}} = \frac{1}{1 - e^{g^2/\alpha x}}.$$ 

This potential at small and large $x^2$ is,

$$V(x) = e^{-\beta(L+1)\frac{g^2}{\alpha x}} + \mathcal{O}(e^{-2/x}),$$  

$$V(x) = \beta(L + 1)\frac{g^2}{\alpha x} + \mathcal{O}(x^4).$$

These results modify the perturbative calculations. Different loop orders have different kinematic due to the integrations; however, as the $L$ dependence goes like $L/2^L$ due to combinatorics the large $x$ limit should not change much except for a factor.

The expected confining potential of the quarks and gluons is generated. At weak coupling, the exponential may be expanded in a power series. The first term is from tree-level perturbation theory ($e^\alpha = 1$) modeling the Coulomb interaction,

$$V(x) = \frac{g^2}{\sqrt{\alpha x}} + \frac{\sqrt{\alpha}}{g} + \ldots.$$ 

The constant factor is dropped, and in the general coupling case, we keep the $e^{-1/x}$. As a result of the potential, the composite operators have a well defined meaning in terms of partonic constituents.

There are modifications of the above result due to interactions between the closed lines, illustrated in figure (3). Via the confining potential, we find the excited (not lowest spin) states associated with binding $m$ quarks. Of course, there are modifications of the potential due to the perturbative loops, both in the derivative expansion diagrams and in the exponential.
In the scattering functions we integrate over $x$. The closed exponentiated loops are integrable at $x^2 \to 0$ due to the $e^{-1/|x|}$. This factor also modifies the high energy behavior by removing the ultra-violet divergences within the integrals.

The running of the coupling constant follows as in perturbation theory, including theories with different matter content. In the ultra-violet, if the coupling runs to zero then the Wilson loops become exponentially suppressed and break apart. However, the resummed series smooths out the ultra-violet divergences at intermediate couplings as a string would do.

The amplitudes and quark/gluon potentials depend on the details of the microscopic theory and the couplings within it.

3 Discussion

The derivative expansion in the context of gauge theories with general matter content is studied. The expansion is developed and recursion formulae are developed in order to determine the coefficients of the expansion. The integrals are much simpler than in the usual perturbative approach, in line with recent results pertaining to scalar and spontaneously broken supersymmetric gauge theories.

The interactions between the partonic constituents are examined upon including the analog of Wilson loops in this approach. The potential derived after summing the latter loops generates a confining potential and a reggeization of the composite states, including the glueballs and quark matter.

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