A REMARK ON SEMI-LINEAR DAMPED $\sigma$-EVOLUTION EQUATIONS WITH A MODULUS OF CONTINUITY TERM IN NONLINEARITY

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Abstract. In this article, we indicate that under suitable assumptions of a modulus of continuity we obtain either the global (in time) existence of small data Sobolev solutions or the blow-up result of local (in time) Sobolev solutions to semi-linear damped $\sigma$-evolution equations with a modulus of continuity term in nonlinearity.

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1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the following Cauchy problem for the semi-linear damped $\sigma$-evolution equations with modulus of continuity term in nonlinearity:

\[
\begin{cases}
  u_{tt} + (-\Delta)^{\sigma} u + (-\Delta)^{\delta} u_t = |u|^{p^*(m,n)} \mu(|u|), \\
  u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),
\end{cases}
\]

where $\sigma \geq 1$, $\delta \in [0, \frac{\sigma}{2}]$, a given real number $p^*(m,n) := 1 + \frac{2m\gamma}{n-2\gamma}$, with $m \in [1,2)$ and $n \geq 1$, and the function $\mu = \mu(|u|)$ stands for some moduli of continuity. We are interested in studying two main equations in the present paper including $\sigma$-evolution equation with frictional damping $\delta = 0$ and that with structurally damping $\delta \in (0, \frac{\sigma}{2})$.

There are several recent papers (see, for instance, [3, 4, 6, 7]) concerning one of the most common cases of power nonlinearity $|u|^p$ to the semi-linear damped $\sigma$-evolution equations, that is, to the following Cauchy problems:

\[
\begin{cases}
  u_{tt} + (-\Delta)^{\sigma} u + (-\Delta)^{\delta} u_t = |u|^p \\
  u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),
\end{cases}
\]

with $\sigma \geq 1$ and $\delta \in [0, \frac{\sigma}{2}]$. In particular, the authors in [4, 6] used $(L^1 \cap L^2) - L^2$ estimates and $L^2 - L^2$ estimates for the solutions to the corresponding linear equations with vanishing right-hand side, i.e. the mixing of additional $L^1$ regularity for the data on the basis of $L^2 - L^2$ estimates to

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prove the global (in time) existence of small data solutions to (2) in the cases $\delta = \frac{\sigma}{2}$ and $\delta = 0$. Meanwhile, a different strategy appearing in [3] is to take account of additional $L^m \cap L^q$ regularity, with small $\eta$ and large $\tilde{q}$, in place of additional $L^1$ regularity. On the one hand, this strategy gives the global (in time) existence of small data solutions to the semi-linear models to (2) in the case $\delta \in (0, \frac{\sigma}{2})$. Besides, some blow-up results were obtained in the latter paper to really find critical exponent for $p$. Here, critical exponent $p_{crit} = p_{crit}(n)$ means that for some range of admissible $p > p_{crit}$ there exists a global (in time) Sobolev solution for small initial data from a suitable function space. Moreover, one may find suitable small data such that there exists no global (in time) Sobolev solution if $1 < p \leq p_{crit}$. In other words, we have, in general, only local (in time) Sobolev solutions under this assumption for the exponent $p$. The sharpness of the critical exponent in [3] to (2) is given by $p_{crit} = \sigma^*(1, n) = 1 + \frac{2\sigma}{n-2\sigma}$. For this reason, we can expect to look for a global (in time) result to (1) by assuming additional $L^m$ regularity for the data, with $m \in [1, 2]$, in the whole supercritical range $p > p^*(m, n)$.

Modulus of continuity is a well-known notation to describe the regularity of a function with respect to desired variables (see more [1, 11, 12]). Some linear Cauchy problems with low regular coefficients combined with modulus of continuity were considered in these references to study the uniqueness and the conditional stability. In the present paper, we investigate power nonlinearity linked to some moduli of continuity as another more complicated type of nonlinearity terms. On the one hand, the main difficulty appearing is to deal with estimating modulus of continuity terms in our proofs. Nevertheless, considering the nonlinearities combined with modulus of continuity to (1) brings some benefits to find $p_{crit}$. This means the connection is understood as an important approach to describe the behavior of critical exponent (see latter, Remarks 1.3 and 2.2). Hence, our goal is twofold. The first motivation of this paper is to derive a global (in time) existence of small data Sobolev solutions to (1) for all $\delta \in [0, \frac{\sigma}{2}]$ under a suitable assumption of moduli of continuity $\mu$ and by using additional $L^m$ regularity for the data, with $m \in [1, 2]$, in the whole supercritical range $p > p^*(m, n)$. Moreover, the second motivation of this paper is to verify the critical exponent $p_{crit} = p^*(1, 1)$ under a inverse assumption of moduli of continuity $\mu$ in the case of frictional damping with $\delta = 0$ and for all integers $\sigma \geq 1$.

Main results

First we state the global (in time) existence of small data energy solutions to (1) in the following theorem.

Theorem 1.1. Let $\sigma \geq 1$ and $\delta \in [0, \frac{\sigma}{2}]$. Let $m \in [1, 2)$ and $2m_0 \delta < n < 2\sigma$ with $\frac{1}{m_0} = \frac{1}{m} - \frac{1}{2}$. We assume the following conditions of modulus of continuity

$$s \mu'(s) \leq \mu(s),$$

and

$$\int_{C_0}^{\infty} \frac{\mu(\frac{s}{2})}{s} ds < \infty,$$

with a sufficiently large constant $C_0 > 0$. Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0, u_1) \in (L^m \cap H^\sigma) \times (L^m \cap L^2)$$

satisfying the assumption

$$\|u_0\|_{L^m \cap H^\sigma} + \|u_1\|_{L^m \cap L^2} \leq \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C([0, \infty), H^\sigma)$$
to (1). Moreover, the following estimates hold:

\[
\|u(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n}{2(\sigma - \delta)} + \frac{1}{2} - \frac{1}{n} + \frac{\delta}{\sigma - \delta}} (\|u_0\|_{L^m \cap H^\sigma} + \|u_1\|_{L^m \cap L^2}),
\]

\[
\|D^\sigma u(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n}{2(\sigma - \delta)} + \frac{1}{2} - \frac{1}{n} + \frac{\delta}{\sigma - \delta}} (\|u_0\|_{L^m \cap H^\sigma} + \|u_1\|_{L^m \cap L^2}),
\]

\[
\|u_t(t, \cdot)\|_{L^2} \leq (1 + t)^{-\frac{n}{2(\sigma - \delta)} + \frac{1}{2} - \frac{1}{n} + \frac{\delta}{\sigma - \delta}} (\|u_0\|_{L^m \cap H^\sigma} + \|u_1\|_{L^m \cap L^2}).
\]

Next we state the blow-up result to (1) in the following theorem.

**Theorem 1.2.** Let \(\sigma \geq 1\) be an integer number and \(\delta = 0\). Let \(n = 1\). We assume that the first datum \(u_0 = 0\), whereas the second datum \(u_1 \in L^1 \cap L^2\) satisfies the following relation:

\[
\int_{\mathbb{R}} u_1(x)\,dx > 0.
\]

Moreover, we suppose the following condition of modulus of continuity

\[
\int_{C_0}^{\infty} \frac{\mu(\frac{s}{\sigma})}{s}\,ds = \infty,
\]

with a sufficiently large constant \(C_0 > 0\). Then, there is no global (in time) energy solution to (1). In other words, we have only local (in time) Sobolev solutions to (1), that is, there exists \(T_0 < \infty\) such that

\[
\lim_{t \to T_0^-} \|(u, u_t)\|_{H^\sigma \times L^2} = +\infty.
\]

**Remark 1.1.** If we plug \(m = 1, n = 1\) and \(\delta = 0\) in Theorem 1.1, then from Theorem 1.2 it is clear that the critical exponent \(p_{\text{crit}}\) is given by \(p_{\text{crit}} = p^*(1, 1) = 1 + 2\sigma\), i.e. our result is optimal.

**Remark 1.2.** In comparison with the known result from the paper [16] concerning the power nonlinearity \(|u|^p\) to (1), we want to underline that the obtained critical exponent \(p_{\text{crit}}\) from Theorem 1.2 coincides with that in the cited paper.

**Remark 1.3.** From the assumption (4) in Theorem 1.1 and the assumption (6) in Theorem 1.2, here we want to emphasize that modulus of continuity comes into play to guarantee either the global (in time) existence or the blow-up result under its suitable conditions.

**The organization of this paper** is presented as follows: In Section 2, we collect some basic properties about modulus of continuity and the \((L^m \cap L^2) - L^2\) estimates and the \(L^2 - L^2\), with \(m \in [1, 2]\), for the solutions to the linear Cauchy problems as well. We shall apply these estimates to prove our main results in Section 3. In particular, we give the detail proof of the global (in time) existence of small data Sobolev solutions to (1) and indicate the optimality of the power exponent in the special case of frictional damping, respectively, in Section 3.1 and Section 3.2.

2. Preliminaries

In this section, we collect some preliminary knowledge needed in our proofs.

2.1. **Modulus of continuity.** First of all, we recall the significant properties and some typical examples about modulus of continuity as well (see, for instance, [1, 12] and the references therein).

**Definition 2.1.** Let \(\mu : [0, c] \to [0, c]\) be a continuous, concave an increasing function with a sufficiently small positive constant \(c\). Then \(\mu\) is called a modulus of continuity if it satisfies

\[
\mu(0) = 0.
\]
Remark 2.1. Here we want to underline that in some sense the above definition of a modulus of continuity could be extended into \([0, 1]\) or \([0, \infty)\) instead of \([0, c]\). However, the essential point of any modulus of continuity is its behavior near 0. This property also comes into play in comparison between two different moduli of continuity. For this reason, it is sufficient to only consider moduli of continuity on intervals \([0, c]\) with a sufficiently small positive constant \(c\).

Example 2.1. In the following list, we give some typical examples about moduli of continuity which are arranged according to their regularity from the highest one to the lowest one:

| modulus of continuity       | frequently called name          |
|-----------------------------|--------------------------------|
| \(\mu(s) = s\)             | Lipschitz-continuity           |
| \(\mu(s) = s\left(\log \left(\frac{1}{s}\right) + 1\right)\) | Log-Lip-continuity            |
| \(\mu(s) = s\left(\log \left(\frac{1}{s}\right) + 1\right) \log^{[m]} \left(\frac{1}{s}\right), \quad m \geq 1\) | Log-Log-\([m]\)-Lip-continuity |
| \(\mu(s) = s^{\alpha}, \quad \alpha \in (0, 1)\) | Hölder-continuity             |
| \(\mu(s) = \left(\log \left(\frac{1}{s}\right) + 1\right)^{-\alpha}, \quad \alpha \in (0, \infty)\) | Log-\(-\alpha\)-continuity    |

where \(\log^{[m]}(x) = \log \left(\log^{[m-1]}(x)\right) + 1\) for \(m \geq 2\), and \(\log^{[1]}(x) = \log(x) + 1\).

Remark 2.2. We can easily check that the moduli of continuity such as Lipschitz-continuity, Log-Lip-continuity, Log-Log-\([m]\)-Lip-continuity, Hölder-continuity and Log-\(-\alpha\)-continuity with \(\alpha \in (1, \infty)\) fulfill the assumption (4) in Theorem 1.1, whereas Log-\(-\alpha\)-continuity with \(\alpha \in (0, 1)\) satisfies the assumption (6) in Theorem 1.2.

2.2. Linear estimates. Let us consider the corresponding linear models with vanishing right-hand side in the following form:

\[ u_{ttt} + (-\Delta)^\sigma u + (-\Delta)^\delta u_t = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad (7) \]

with \(\sigma \geq 1\) and \(\delta \in [0, \frac{\sigma}{2}]\). Main goal of this section is to collect \((L^m \cap L^2) - L^2\) and \(L^2 - L^2\) estimates for the solutions and some of their derivatives to (2). These estimates play a crucial role to prove the global (in time) existence result to (1) in the next section.

2.2.1. The structural damping case \(\delta \in (0, \frac{\sigma}{2}]\). We obtained the following estimates from the previous paper [2] of the first author.

Proposition 2.1 (Proposition 2.1 and 2.2 in [2]). Let \(\delta \in (0, \frac{\sigma}{2}]\) in (7) and \(m \in [1, 2]\). The Sobolev solutions to (7) satisfy the \((L^m \cap L^2) - L^2\) estimates

\[
\|\partial_t^a |D|^a u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{a}{2(\sigma - 2\delta)}(1 - \frac{1}{m}) - \frac{a + 2\delta}{2(\sigma - 2\delta)(1 - \frac{1}{m})}} \|u_0\|_{L^m \cap H^{a+j} \sigma} + (1 + t)^{-\frac{a + 2\delta}{2(\sigma - 2\delta)(1 - \frac{1}{m})}} \|u_1\|_{L^m \cap H^{[a, (j-1)\sigma]} \cap H^{[\sigma, (j-1)\sigma]}},
\]

and the \(L^2 - L^2\) estimates

\[
\|\partial_t^a |D|^a u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{a + 2\delta}{2(\sigma - 2\delta)}} \|u_0\|_{H^{a+j} \sigma} + (1 + t)^{-\frac{a + 2\delta}{2(\sigma - 2\delta)}} \|u_1\|_{H^{[a, (j-1)\sigma]} \cap H^{[\sigma, (j-1)\sigma]}},
\]

for any non-negative number \(a, j = 0, 1\) and for all space dimensions \(n \geq 1\).

Remark 2.3. In Proposition 2.1 we state estimates for the solution and some its derivatives to (7) which hold for any space dimensions \(n \geq 1\). Moreover, we may prove a better result under a restriction to space dimensions \(n > 2m_0\delta\). We get the following sharper estimates.
Proposition 2.2 (Proposition 2.1 and 2.3 in [2]). Let $\delta \in (0, \frac{3}{2}]$ in (7) and $m \in [1, 2)$. The Sobolev solutions to (7) satisfy the $(L^m \cap L^2)$ – $L^2$ estimates
\[
\|e^t D^a u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)} - \frac{3}{2} - \frac{a}{2(\sigma - \delta)} - j}\|u_0\|_{L^m \cap H^{a+j}\sigma} + (1 + t)^{-\frac{a}{2(\sigma - \delta)} - j}\|u_1\|_{H^{a+(j-1)\sigma}^{a+j\sigma}},
\]
and the $L^2$ – $L^2$ estimates
\[
\|e^t D^a u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)} - \frac{a}{2(\sigma - \delta)} - j}\|u_0\|_{H^{a+j\sigma}} + (1 + t)^{-\frac{a}{2(\sigma - \delta)} - j}\|u_1\|_{H^{a+(j-1)\sigma}^{a+j\sigma}},
\]
for any non-negative number $a$, $j = 0, 1$ and for all space dimensions $n > 2m_0\delta$.

2.2.2. The frictional damping case $\delta = 0$. Our approach is based on the paper [6]. According to the treatment of Proposition 2.1 in [6], with minor modifications in the steps of the proofs we derive the following estimates.

Proposition 2.3. Let $\delta = 0$ in (7) and $m \in [1, 2)$. The Sobolev solutions to (7) satisfy the $(L^m \cap L^2)$ – $L^2$ estimates
\[
\|D^a u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)} - \frac{a}{2(\sigma - \delta)} - 1}\|u_0\|_{L^m \cap H^a} + (1 + t)^{-\frac{a}{2(\sigma - \delta)} - 1}\|u_1\|_{H^{a}^{a+\sigma}^{a+\sigma}},
\]
and the $L^2$ – $L^2$ estimates
\[
\|D^a u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)} - \frac{a}{2(\sigma - \delta)} - 1}\|u_0\|_{H^a} + (1 + t)^{-\frac{a}{2(\sigma - \delta)} - 1}\|u_1\|_{H^a},
\]
for any non-negative number $a$ and for all space dimensions $n \geq 1$.

Finally, combining Propositions 2.2 and 2.3 we may conclude the following estimates.

Proposition 2.4. Let $\delta \in [0, \frac{3}{2}]$ in (7) and $m \in [1, 2)$. The Sobolev solutions to (7) satisfy the $(L^m \cap L^2)$ – $L^2$ estimates
\[
\|e^t D^a u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)} - \frac{3}{2} - \frac{a}{2(\sigma - \delta)} - j}\|u_0\|_{L^m \cap H^{a+j}\sigma} + (1 + t)^{-\frac{a}{2(\sigma - \delta)} - j}\|u_1\|_{H^{a+(j-1)\sigma}^{a+j\sigma}},
\]
and the $L^2$ – $L^2$ estimates
\[
\|e^t D^a u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)} - \frac{a}{2(\sigma - \delta)} - j}\|u_0\|_{H^{a+j\sigma}} + (1 + t)^{-\frac{a}{2(\sigma - \delta)} - j}\|u_1\|_{H^{a+(j-1)\sigma}^{a+j\sigma}},
\]
for any non-negative number $a$, $j = 0, 1$ and for all space dimensions $n > 2m_0\delta$.

Remark 2.4. Here we want to underline that the decay estimates for the solutions to (7) from Proposition 2.4 are better than those from Proposition 2.2. Hence, it is reasonable to apply all statements from Proposition 2.4 in the steps of the proof to our global (in time) existence result.

3. Proofs for main results

3.1. Global existence for any $\sigma \geq 1$ and $\delta \in [0, \frac{3}{2}]$. For simplicity, in the following proof we use the abbreviation $p^* := p^*(m, n) = 1 + \frac{2m_0}{n - 2\sigma}$.

Proof of Theorem 1.1. We choose the data spaces
\[
(u_0, u_1) \in A_\sigma^m := (L^m \cap H^\sigma) \times (L^m \cap L^2)
\]
and introduce the solution space
\[
X(t) := C([0, t], H^\sigma),
\]
with the norm
\[ \|u\|_{X(t)} := \sup_{0 \leq \tau \leq t} \left( f(\tau)^{-1}\|u(\tau, \cdot)\|_{L^2} + g(\tau)^{-1}\|D^a u(\tau, \cdot)\|_{L^2} \right), \]
where
\[ f(\tau) = (1 + \tau)^{-\frac{n}{2(\sigma - \delta)} \left( \frac{1}{m} - \frac{1}{2} \right) + \frac{\delta}{\sigma - \delta}}, \quad g(\tau) = (1 + \tau)^{-\frac{n}{2(\sigma - \delta)} \left( \frac{1}{m} - \frac{1}{2} \right) + \frac{\delta}{m - \delta}}. \]
Denoting \( K_0(t, x) \) and \( K_1(t, x) \) as the fundamental solutions of the corresponding linear Cauchy problems (2) we may write the solutions to (2) in the following form:
\[ u^n(t, x) = K_0(t, x) * x u_0(x) + K_1(t, x) * x u_1(x). \]
Applying Duhamel’s principle gives the formal implicit representation of the solutions to (1) as follows:
\[ u(t, x) = u^n(t, x) + \int_0^t K_1(t - \tau, x) * x |u|^p \mu(|u|)d\tau =: u^n(t, x) + u^{nl}(t, x). \]
We define for all \( t > 0 \) the operator \( N : u \in X(t) \rightarrow Nu \in X(t) \) by the formula
\[ Nu(t, x) = u^n(t, x) + u^{nl}(t, x). \]
We will prove that the operator \( N \) satisfies the following two inequalities:
\[ \|Nu\|_{X(t)} \leq \|(u_0, u_1)\|_{A_0^n} + \|u\|_{X(t)}^p, \quad \text{(8)} \]
\[ \|Nu - Nv\|_{X(t)} \leq \|u - v\|_{X(t)} \|u\|_{X(t)}^{p - 1} + \|v\|_{X(t)}^{p - 1}. \quad \text{(9)} \]
Here we note that from the definition of the norm in \( X(t) \), by replacing \( a = 0 \) and \( a = \sigma \) in the statements from Proposition 2.3 we may conclude
\[ \|u^n\|_{X(t)} \leq \|(u_0, u_1)\|_{A_0^n}. \quad \text{(10)} \]
For this reason, in order to complete the proof of (8) we prove only the following inequality:
\[ \|u^{nl}\|_{X(t)} \leq \|u\|_{X(t)}^p. \quad \text{(11)} \]
Then, applying Banach’s fixed point theorem we obtain local (in time) existence results of large data solutions and global (in time) existence results of small data solutions as well.

First, let us prove the inequality (11). To control some estimates for \( u^{nl} \), our strategy is to use the \( (L^m \cap L^2) - L^2 \) estimates from Proposition 2.4 to get the following estimates for \( k = 0, 1 \):
\[ \left\| D^{k\sigma} u^{nl}(t, \cdot) \right\|_{L^2} \leq \int_0^t \left( 1 + \tau \right)^{-\frac{n}{2(\sigma - \delta)} \left( \frac{1}{m} - \frac{1}{2} \right) + \frac{\delta}{\sigma - \delta}} \|u(\tau, \cdot)\|_{L^m} \mu(|u(\tau, \cdot)|) \|u(\tau, \cdot)|\|_{L^{mp}} d\tau. \quad \text{(12)} \]
In order to estimate for \( |u(\tau, \cdot)| \mu(|u(\tau, \cdot)|) \) in \( L^m \cap L^2 \), we proceed as follows:
\[ \|u(\tau, \cdot)|\|_{L^m \cap L^2} \|u(\tau, \cdot)|\|_{L^{mp}} \mu(|u(\tau, \cdot)|) \|u(\tau, \cdot)|\|_{L^\infty}. \quad \text{(13)} \]
Since \( \mu \) is an increasing function, we obtain
\[ \|\mu(|u(\tau, \cdot)|)\|_{L^\infty} \leq \mu(\|u(\tau, \cdot)\|_{L^\infty}) \leq \mu(C_1 \|u(\tau, \cdot)\|_{H^\sigma}) \leq \mu(C_2 \|u(\tau, \cdot)\|_{L^2} + C_2 \|u(\tau, \cdot)\|_{H^\sigma}) \leq \mu(C_2 \|u(\tau, \cdot)\|_{L^2} + C_2 \|u(\tau, \cdot)\|_{H^\sigma}) \leq \mu(C_2\|u(\tau, \cdot)\|_{L^2} + C_2\|u(\tau, \cdot)\|_{H^\sigma}), \quad \text{(14)} \]
with some suitable positive constants \( C_1 \) and \( C_2 \). Here we used the embedding into \( L^\infty \) from Proposition 3.2 with the condition \( \sigma > \frac{n}{2} \). For the other interesting term of (13), we re-write
\[ \|u(\tau, \cdot)|\|_{L^m \cap L^2} = \|u(\tau, \cdot)|\|_{L^m} + \|u(\tau, \cdot)|\|_{L^2} = \|u(\tau, \cdot)|\|_{L^{mp}} + \|u(\tau, \cdot)|\|_{L^{2p}}. \]
Employing the fractional Gagliardo-Nirenberg inequality from Proposition 3.1 gives
\[
\|u(\tau, \cdot)\|_{L_{mp}^p}^p \lesssim (1 + \tau) \frac{n}{2m(\sigma - \delta)} (p^*-1) + \frac{p^* \delta}{\sigma - \delta} \|u\|_{X(\tau)}^p,
\]
\[
\|u(\tau, \cdot)\|_{L_{2p}^p}^p \lesssim (1 + \tau) \frac{n}{2m(\sigma - \delta)} (p^*-1) + \frac{p^* \delta}{\sigma - \delta} \|u\|_{X(\tau)}^p.
\]
As a result, we derive
\[
\|u(\tau, \cdot)\|_{L_{m\cap L^2}^p}^p \lesssim (1 + \tau) \frac{n}{2m(\sigma - \delta)} (p^*-1) + \frac{p^* \delta}{\sigma - \delta} \|u\|_{X(\tau)}^p
\]
(15)
Combining all the estimates from (12) to (15) we arrive at
\[
\|D|^{k\sigma} u^{nl}(t, \cdot)\|_{L^2} \lesssim \|u\|_{X(t)}^p \int_0^t (1 + t - \tau)(1 + \tau) \frac{n}{2m(\sigma - \delta)} (p^*-1) + \frac{p^* \delta}{\sigma - \delta} \times \mu \left(C_2 \varepsilon_0 (1 + \tau) \frac{n}{2m(\sigma - \delta)} \right) d\tau.
\]
By splitting the above integral into two parts, on the one hand we derive the following estimate:
\[
I_1 := \int_0^{t/2} (1 + t - \tau)(1 + \tau) \frac{n}{2m(\sigma - \delta)} \left(1 + \tau\right) \frac{n}{2m(\sigma - \delta)} (p^*-1) + \frac{p^* \delta}{\sigma - \delta} \times \mu \left(C_2 \varepsilon_0 (1 + \tau) \frac{n}{2m(\sigma - \delta)} \right) d\tau
\]
\[
\lesssim (1 + t) \frac{n}{2m(\sigma - \delta)} (p^*-1) + \frac{p^* \delta}{\sigma - \delta} \int_0^{t/2} (1 + \tau) \frac{n}{2m(\sigma - \delta)} (p^*-1) + \frac{p^* \delta}{\sigma - \delta} \times \mu \left(C_2 \varepsilon_0 (1 + \tau) \frac{n}{2m(\sigma - \delta)} \right) d\tau,
\]
(17)
where we used \((1 + t - \tau) \approx (1 + t)\) for any \(\tau \in [0, t/2]\). On the other hand, we can estimate the remaining integral as follows:
\[
I_2 := \int_{t/2}^t (1 + t - \tau)(1 + \tau) \frac{n}{2m(\sigma - \delta)} \left(1 + \tau\right) \frac{n}{2m(\sigma - \delta)} (p^*-1) + \frac{p^* \delta}{\sigma - \delta} \times \mu \left(C_2 \varepsilon_0 (1 + \tau) \frac{n}{2m(\sigma - \delta)} \right) d\tau
\]
\[
\lesssim (1 + t) \frac{n}{2m(\sigma - \delta)} (p^*-1) + \frac{p^* \delta}{\sigma - \delta} \int_{t/2}^t (1 + \tau) \frac{n}{2m(\sigma - \delta)} (p^*-1) + \frac{p^* \delta}{\sigma - \delta} \times \mu \left(C_2 \varepsilon_0 (1 + \tau) \frac{n}{2m(\sigma - \delta)} \right) d\tau,
\]
where we used \((1 + \tau) \approx (1 + t)\) for any \(\tau \in [t/2, t]\). Moreover, we pay attention that the following relations hold:
\[
- \frac{n}{2(\sigma - \delta)} \left(\frac{p^*-2}{m} + \frac{1}{2}\right) + \frac{(p^*-1)\delta}{\sigma - \delta} + \frac{k\sigma}{2(\sigma - \delta)} < 0,
\]
and
\[
- \frac{n}{2(\sigma - \delta)} \left(\frac{1}{m} - \frac{1}{2}\right) + \frac{\delta}{\sigma - \delta} < 0,
\]
due to the condition \(2m_0\delta < n < 2\sigma\). It is clear to see that \(1 + \tau \geq 1 + t - \tau\) for any \(\tau \in [t/2, t]\).

Consequently, we have
\[
(1 + \tau)^{-\frac{n}{2(\sigma - \delta)}(\frac{m}{m - \frac{1}{2}}) - \frac{k\sigma - 2\delta}{2(\sigma - \delta)}} < (1 + t - \tau)^{-\frac{n}{2(\sigma - \delta)}(\frac{m}{m - \frac{1}{2}}) - \frac{k\sigma - 2\delta}{2(\sigma - \delta)}},
\]

\[
(1 + \tau)^{-\frac{n}{2(\sigma - \delta)}(\frac{m}{m - \frac{1}{2}}) + \frac{\delta}{\sigma - \delta}} < (1 + t - \tau)^{-\frac{n}{2(\sigma - \delta)}(\frac{m}{m - \frac{1}{2}}) + \frac{\delta}{\sigma - \delta}}.
\]

Hence, we arrive at
\[
I_2 \lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)}(\frac{1}{m} - \frac{1}{2}) - \frac{k\sigma - 2\delta}{2(\sigma - \delta)}} \int_{t/2}^{t} (1 + t - \tau)^{-\frac{n}{2m(\sigma - \delta)}(p^* - 1) + \frac{\delta}{\sigma - \delta}} \times \mu(C_2\varepsilon_0(1 + t - \tau)^{-\frac{n}{2(\sigma - \delta)}(\frac{1}{m} - \frac{1}{2}) + \frac{\delta}{\sigma - \delta}}) d\tau.
\]

A standard change of variables leads to
\[
I_2 \lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)}(\frac{1}{m} - \frac{1}{2}) - \frac{k\sigma - 2\delta}{2(\sigma - \delta)}} \int_{0}^{t/2} (1 + \tau)^{-\frac{n}{2m(\sigma - \delta)}(p^* - 1) + \frac{\delta}{\sigma - \delta}} \times \mu(C_2\varepsilon_0(1 + \tau)^{-\frac{n}{2(\sigma - \delta)}(\frac{1}{m} - \frac{1}{2}) + \frac{\delta}{\sigma - \delta}}) d\tau.
\]

From (16) to (18) we may conclude
\[
\|D|^{k\sigma}u^{nl}(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)}(\frac{1}{m} - \frac{1}{2}) - \frac{k\sigma - 2\delta}{2(\sigma - \delta)}} \|u\|_{X(t)}^{p^*} \times \int_{0}^{t/2} (1 + \tau)^{-\frac{n}{2m(\sigma - \delta)}(p^* - 1) + \frac{\delta}{\sigma - \delta}} \times \mu(C_2\varepsilon_0(1 + \tau)^{-\frac{n}{2(\sigma - \delta)}(\frac{1}{m} - \frac{1}{2}) + \frac{\delta}{\sigma - \delta}}) d\tau
\]
\[
\lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)}(\frac{1}{m} - \frac{1}{2}) - \frac{k\sigma - 2\delta}{2(\sigma - \delta)}} \|u\|_{X(t)}^{p^*} \int_{0}^{\infty} (1 + \tau)^{-1} \mu(C_2\varepsilon_0(1 + \tau)^{-\alpha}) d\tau \lesssim \int_{0}^{\infty} \frac{\mu(1/s)}{s} ds \leq 1,
\]

Here we denoted a constant \(\alpha := \frac{n}{2(\sigma - \delta)}(\frac{1}{m} - \frac{1}{2}) - \frac{\delta}{\sigma - \delta} > 0\). Using again change of variable by denoting \(s^{-1} := C_2\varepsilon_0(1 + \tau)^{-\alpha}\) and a straightforward calculation give
\[
\int_{0}^{\infty} (1 + \tau)^{-1} \mu(C_2\varepsilon_0(1 + \tau)^{-\alpha}) d\tau \leq \int_{C_0\varepsilon_0}^{\infty} \frac{\mu(1/s)}{s} ds \leq 1,
\]

where \(C_0 = \frac{1}{C_2\varepsilon_0}\) is a sufficiently large constant. We notice that the assumptions (4) comes into play to guarantee the boundedness of the above integral. Therefore, from (19) and (20) we have shown that
\[
\|D|^{k\sigma}u^{nl}(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2(\sigma - \delta)}(\frac{1}{m} - \frac{1}{2}) - \frac{k\sigma - 2\delta}{2(\sigma - \delta)}} \|u\|_{X(t)}^{p^*}.
\]

From the definition of the norm in \(X(t)\), we may conclude immediately the inequality (11).

Next, let us prove the inequality (9). For two elements \(u\) and \(v\) from \(X(t)\), using again the \((L^m \cap L^2) - L^2\) estimates from Proposition 2.4 we get the following estimates with \(k = 0, 1\):
\[
\|D|^{k\sigma}(Nu - Nv)(t, \cdot)\|_{L^2} = \|D|^{k\sigma}(u^{nl} - v^{nl})(t, \cdot)\|_{L^2} \lesssim \int_{0}^{t} (1 + t - \tau)^{-\frac{n}{2(\sigma - \delta)}(\frac{1}{m} - \frac{1}{2}) - \frac{k\sigma - 2\delta}{2(\sigma - \delta)}} \|u(\tau, \cdot)|^{p^*}\mu(|u(\tau, \cdot)|) - |v(\tau, \cdot)|^{p^*}\mu(|v(\tau, \cdot)|)\|_{L^m \cap L^2} d\tau.
\]

By using the mean value theorem we get the integral representation
\[
|u(\tau, x)|^{p^*}\mu(|u(\tau, x)|) - |v(\tau, x)|^{p^*}\mu(|v(\tau, x)|) = (u(\tau, x) - v(\tau, x)) \int_{0}^{1} d|u|H(\omega u(\tau, x) + (1 - \omega)v(\tau, x)) d\omega.
\]
where $H(u) = |u|^{p^*} \mu(|u|)$. Thanks to the condition (3) of modulus of continuity, we can estimate

$$d_{|u|}H(u) = p^*|u|^{p^*-1} \mu(|u|) + |u|^{p^*} d_{|u|} \mu(|u|) \leq |u|^{p^*-1} \mu(|u|).$$

Hence, we obtain

$$|u(\tau, x)|^{p^*} \mu\left(|u(\tau, x)|\right) - |v(\tau, x)|^{p^*} \mu\left(|v(\tau, x)|\right)$$

$$\leq (u(\tau, x) - v(\tau, x)) \int_0^1 \omega u(\tau, x) + (1 - \omega)v(\tau, x) \right)^{p^*-1} \mu\left(\omega u(\tau, x) + (1 - \omega)v(\tau, x)\right) d\omega.$$

For this reason, we arrive at the following estimate:

$$\|u(\cdot, \cdot)\|^{p^*} \mu\left(|u(\tau, \cdot)|\right) - |v(\tau, \cdot)|^{p^*} \mu\left(|v(\tau, \cdot)|\right)\|_{L^m \cap L^2}^L$$

$$\leq \int_0^1 \|u(\tau, x) - v(\tau, x)\| \omega u(\tau, x) + |v(\tau, x)| \right)^{p^*-1} \mu\left(\omega u(\tau, x) + (1 - \omega)v(\tau, x)\right)\|_{L^m \cap L^2}^L d\omega.$$

In the similar approach to the proof of the inequality (11), we may conclude the inequality (9). Summarizing, the proof is completed. \(\square\)

3.2. Blow-up result for any integer $\sigma \geq 1$ and $\delta = 0$. The proof of blow-up result in this section is based on a contradiction argument by using the test function method (see, for example, [3, 5, 10, 17]). In general, this method cannot be directly applied to fractional Laplacian operators $(-\Delta)^\sigma$ and $(-\Delta)^\delta$ as well-known non-local operators. For this reason, the assumption for integers $\sigma$ comes into play to apply this method. Moreover, due to some techniques in our proof, we further assume $\delta = 0$ and $n = 1$, that is, we only consider the following semi-linear damped $\sigma$-evolution equations with frictional damping:

$$\begin{cases}
\ddot{u} + (-\partial_{xx})^\sigma u + u_t = |u|^{p_0} \mu(|u|), \\
u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),
\end{cases}$$

(21)

where $\sigma \geq 1$ and we denote $p_0 := p^*(1, 1) = 1 + 2\sigma$.

Proof of Theorem 1.2. First, we introduce test function $\varphi = \varphi(r)$ having the following properties:

$$\varphi \in C_0^\infty((0,)) \text{ and } \varphi(r) = \begin{cases} 1 \text{ for } r \in [0, 1/2], \\
0 \text{ for } r \in [1, \infty).
\end{cases}$$

Moreover, we assume that $\varphi = \varphi(r)$ is a decreasing function. We also introduce the function $\varphi^* = \varphi^*(r)$ satisfying

$$\varphi^*(r) = \begin{cases} 0 \text{ for } r \in [0, 1/2), \\
\varphi(r) \text{ for } r \in [1/2, \infty).
\end{cases}$$

Let $R$ be a large parameter in $[0, \infty)$. We define the following two functions:

$$\phi_R(t, x) = \left(\varphi\left(\frac{x^2 + t}{R}\right)\right)^{p_0}, \text{ and } \phi_R^*(t, x) = \left(\varphi^*\left(\frac{x^2 + t}{R}\right)\right)^{p_0}.$$

Then it is clear to see that

$$\text{supp}\phi_R \subset Q_R := \{(t, x) \in [0, R] \times [0, R^{1/2\sigma}]\},$$

$$\text{supp}\phi_R^* \subset Q^*_R := \{(t, x) \in [R/2, R] \times [(R/2)^{1/2\sigma}, R^{1/2\sigma}]\}.$$

Now we define the functional

$$I_R := \int_0^\infty \int_\mathbb{R} |u(t, x)|^{p_0} \mu(|u(t, x)|) \phi_R(t, x) \, dx \, dt = \int_{Q_R} |u(t, x)|^{p_0} \mu(|u(t, x)|) \phi_R(t, x) \, dx \, dt.$$
Let us assume that $u = u(t, x)$ is a global (in time) Sobolev solution to (1). After multiplying the equation (1) by $\phi_R = \phi_R(t, x)$, we carry out partial integration to derive

$$0 \leq I_R = -\int_{\mathbb{R}} u_1(x)\phi_R(0, x) \, dx + \int_{Q_R} u(t, x)\left(\partial_t^2 \phi_R(t, x) + (-\partial_{xx})\phi_R(t, x) - \partial_t \phi_R(t, x)\right) \, dx \, dt,$$

$$=: -\int_{\mathbb{R}} u_1(x)\phi_R(0, x) \, dx + J_R.$$

Because of assumption (5), there exists a sufficiently large constant $R_0 > 0$ such that for all $R > R_0$ it holds

$$\int_{\mathbb{R}} u_1(x)\phi_R(0, x) \, dx > 0.$$

Consequently, we obtain

$$0 \leq I_R < J_R \text{ for all } R > R_0. \quad (22)$$

In order to estimate $J_R$, firstly we have

$$|\partial_t \phi_R(t, x)| = \left|\frac{p_0}{R} \phi^2 \left(\frac{x^2\sigma + t}{R}\right) \varphi' \left(\frac{x^2\sigma + t}{R}\right) \right| \leq \frac{1}{R} \phi^2 \left(\frac{x^2\sigma + t}{R}\right)^{p_0 - 1}. \quad (23)$$

Besides, a further calculation leads to

$$|\partial_t^2 \phi_R(t, x)| = \left|\frac{p_0(p_0 - 1)}{R^2} \phi^2 \left(\frac{x^2\sigma + t}{R}\right)^{p_0 - 2} \varphi' \left(\frac{x^2\sigma + t}{R}\right) + \frac{p_0}{R^2} \phi^2 \left(\frac{x^2\sigma + t}{R}\right)^{p_0 - 1} \varphi'' \left(\frac{x^2\sigma + t}{R}\right) \right| \leq \frac{1}{R^2} \phi^2 \left(\frac{x^2\sigma + t}{R}\right)^{p_0 - 2}. \quad (24)$$

To control $(-\partial_{xx})^\sigma \phi_R(t, x)$, we shall apply Lemma 3.1 as a main tool. Indeed, we divide our consideration into two sub-steps as follows:

**Step 1:** Applying Lemma 3.1 with $h(z) = \varphi(z)$ and $f(x) = \frac{x^{2\sigma + t}}{R}$, we get

$$\left|\partial_{xx}^\alpha \varphi \left(\frac{x^{2\sigma + t}}{R}\right)\right| = \sum_{k=1}^{2\sigma} \varphi^{(k)} \left(\frac{x^{2\sigma + t}}{R}\right) \left(\sum_{\gamma_1 + \cdots + \gamma_k = \alpha, \gamma_i \geq 1} \frac{x^{2\sigma - \gamma_1} \cdots x^{2\sigma - \gamma_k}}{R}\right) \leq \sum_{k=1}^{2\sigma} \left(\frac{x^{2\sigma}}{R}\right)^k x^{-\alpha} \leq \frac{x^{2\sigma - \alpha}}{R} \quad \text{since } x^{2\sigma} \leq R \text{ in } Q_R. \quad (25)$$

**Step 2:** Applying Lemma 3.1 with $h(z) = z^{p_0}$ and $f(x) = \varphi \left(\frac{x^{2\sigma + t}}{R}\right)$, we obtain

$$\left|(-\partial_{xx})^\sigma \phi_R(t, x)\right| \leq \sum_{k=1}^{2\sigma} \left(\varphi \left(\frac{x^{2\sigma + t}}{R}\right)\right)^{p_0 - k} \left(\sum_{\gamma_1 + \cdots + \gamma_k = 2\sigma, \gamma_i \geq 1} \partial_{x}^\gamma \varphi \left(\frac{x^{2\sigma + t}}{R}\right) \cdots \partial_{x}^\gamma \varphi \left(\frac{x^{2\sigma + t}}{R}\right)\right) \leq \sum_{k=1}^{2\sigma} \left(\varphi \left(\frac{x^{2\sigma + t}}{R}\right)\right)^{p_0 - k} x^{2(2\sigma - 1) - 1} \left(\frac{x^{2\sigma - 2\sigma - \cdots - 2\sigma}}{R}\right) \leq \frac{1}{R} \varphi^2 \left(\frac{x^{2\sigma + t}}{R}\right)^{p_0 - 2\sigma} \quad \text{since } x^{2\sigma} \leq R \text{ in } Q_R. \quad (26)$$

From (23) to (26), we arrive at the following estimate:

$$|\partial_t^2 \phi_R(t, x) + (-\partial_{xx})^\sigma \phi_R(t, x) - \partial_t \phi_R(t, x)| \leq \frac{1}{R} \varphi^2 \left(\frac{x^{2\sigma + t}}{R}\right)^{p_0 - 2\sigma} = \frac{1}{R} \varphi^2 \left(\frac{x^{2\sigma + t}}{R}\right)^{p_0 - 2\sigma}. \quad (27)$$
Hence, we may conclude
\[ J_R = |J_R| \leq \frac{1}{R} \int_{Q_R} |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} d(x,t). \]  
(27)

Now we focus on our attention to estimate the above integral. To do this, we introduce the function \( \Psi(s) = s^{\rho_0} \mu(s) \). Then we derive
\[ \Psi \left( |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} \right) = |u(t,x)|^{\rho_0} \phi_R^+(t,x) \mu \left( |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} \right) \leq |u(t,x)|^{\rho_0} \phi_R^+(t,x) \mu \left( |u(t,x)| \right) \phi_R^+(t,x). \]  
(28)

Here we used the property of the increasing function \( \mu(s) \) and the relation \( 0 \leq \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} \leq 1 \). Moreover, it is clear to see that \( \Psi \) is a convex function. Applying Proposition 3.3 with \( h(s) = \Psi(s) \), \( f(t,x) = |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} \) and \( \gamma \equiv 1 \) gives the following estimate:
\[ \Psi \left( \frac{\int_{Q_R^+} |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} d(x,t)}{\int_{Q_R^+} 1 d(x,t)} \right) \leq \frac{\int_{Q_R^+} \Psi \left( |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} \right) d(x,t)}{\int_{Q_R^+} 1 d(x,t)}. \]

We can easily compute
\[ \int_{Q_R^+} 1 d(x,t) = R^{1+\frac{1}{\rho_0}}. \]

Hence, we get
\[ \Psi \left( \frac{\int_{Q_R^+} |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} d(x,t)}{R^{1+\frac{1}{\rho_0}}} \right) \leq \frac{\int_{Q_R^+} \Psi \left( |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} \right) d(x,t)}{R^{1+\frac{1}{\rho_0}}} \leq \frac{\int_{Q_R^+} \Psi \left( |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} \right) d(x,t)}{R^{1+\frac{1}{\rho_0}}}. \]  
(29)

Combining the estimates (28) and (29) we may arrive at
\[ \Psi \left( \frac{\int_{Q_R^+} |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} d(x,t)}{R^{1+\frac{1}{\rho_0}}} \right) \leq \frac{\int_{Q_R^+} \Psi \left( |u(t,x)| \right) \phi_R^+(t,x) d(x,t)}{R^{1+\frac{1}{\rho_0}}}. \]  
(30)

Since \( \mu(s) \) is an increasing function, it immediately follows that \( \Psi(s) \) is also increasing function. For this reason, from (30) it deduces
\[ \int_{Q_R} |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} d(x,t) \leq \int_{Q_R^+} |u(t,x)| \left( \phi_R^+(t,x) \right)^{\frac{1}{\rho_0}} d(x,t) \leq R^{1+\frac{1}{\rho_0}} \Psi^{-1} \left( \frac{\int_{Q_R^+} \Psi \left( |u(t,x)| \right) \phi_R^+(t,x) d(x,t)}{R^{1+\frac{1}{\rho_0}}} \right). \]  
(31)

From (22), (27) and (31) we may conclude
\[ I_R \lesssim R^{\frac{1}{\rho_0}} \Psi^{-1} \left( \frac{\int_{Q_R} \Psi \left( |u(t,x)| \right) \phi_R^+(t,x) d(x,t)}{R^{1+\frac{1}{\rho_0}}} \right), \]  
(32)
for all \( R > R_0 \). Next we introduce the following two functions:
\[ g(r) = \int_{Q_R} \Psi \left( |u(t,x)| \right) \phi_R^+(t,x) d(x,t), \quad \text{with } r \in (0, \infty), \]
and
\[ G(R) = \int_0^R g(r) r^{-1} dr. \]
Then we re-write
\[ G(R) = \int_0^R \left( \int_{Q_R} \Psi(\|u(t,x)\|) \phi^*_R(t,x) \, d(x,t) \right) r^{-1} \, dr \]
\[ = \int_{Q_R} \Psi(\|u(t,x)\|) \left( \int_0^R \left( \varphi^*(\frac{x^{2\sigma} + t}{r}) \right)^{p_0} r^{-1} \, dr \right) d(x,t). \]

Carrying out change of variables \( \tilde{r} = \frac{x^{2\sigma} + t}{r} \) we derive
\[ G(R) = \int_{Q_R} \Psi(\|u(t,x)\|) \left( \int_{\frac{1}{2}}^{2\sigma} (\varphi^*(\tilde{r}))^{p_0} \tilde{r}^{-1} \, d\tilde{r} \right) d(x,t) \]
\[ \leq \int_{Q_R} \Psi(\|u(t,x)\|) \left( \int_{\frac{1}{2}}^{1} (\varphi^*(\tilde{r}))^{p_0} \tilde{r}^{-1} \, d\tilde{r} \right) d(x,t) \quad (\text{since supp} \varphi^* \subset [1/2, 1]) \]
\[ \leq \int_{Q_R} \Psi(\|u(t,x)\|) \left( \int_{1/2}^{1} (\varphi(\tilde{r}))^{p_0} \tilde{r}^{-1} \, d\tilde{r} \right) d(x,t) \quad (\text{since } \varphi^* = \varphi \text{ in } [1/2, 1]) \]
\[ \leq \int_{Q_R} \Psi(\|u(t,x)\|) \left( \varphi \left( \frac{x^{2\sigma} + t}{R} \right) \right)^{p_0} \left( \int_{1/2}^{1} \tilde{r}^{-1} \, d\tilde{r} \right) d(x,t) \quad (\text{since } \varphi \text{ is decreasing}) \]
\[ \leq \log(1 + e) \int_{Q_R} \Psi(\|u(t,x)\|) \left( \varphi \left( \frac{x^{2\sigma} + t}{R} \right) \right)^{p_0} d(x,t) = \log(1 + e) I_R. \]

Moreover, it holds the following relation:
\[ G'(R)R = g(R) = \int_{Q_R} \Psi(\|u(t,x)\|) \phi^*_R(t,x) \, d(x,t) \quad (33) \]

From (32) to (33) we get
\[ \frac{G(R)}{\log(1 + e)} \leq I_R \leq C_1 R^{\frac{1}{2\sigma}} \left( G'(R) \right)^{-1} \left( \frac{G(R)}{R^{\frac{1}{2\sigma}}} \right), \]
for all \( R > R_0 \) and with a suitable positive constant \( C_1 \). This implies
\[ \Psi \left( \frac{G(R)}{C_2 R^{\frac{1}{2\sigma}}} \right) \leq \frac{G'(R)}{R^{\frac{1}{2\sigma}}}, \]
for all \( R > R_0 \) and \( C_2 := C_1 \log(1 + e) > 0 \). By the definition of the function \( \Psi \), the above inequality is equivalent to
\[ \left( \frac{G(R)}{C_2 R^{\frac{1}{2\sigma}}} \right)^{p_0} \mu \left( \frac{G(R)}{C_2 R^{\frac{1}{2\sigma}}} \right) \leq \frac{G'(R)}{R^{\frac{1}{2\sigma}}}, \]
for all \( R > R_0 \). Therefore, we have
\[ \frac{1}{C_3 R} \mu \left( \frac{G(R)}{C_2 R^{\frac{1}{2\sigma}}} \right) \leq \frac{G'(R)}{(G(R))^{p_0}}, \]
for all \( R > R_0 \) and \( C_3 := C_2^{p_0} > 0 \). Because \( G(R) \) is an increasing function, for all \( R > R_0 \) it immediately follows the following inequality:
\[ \frac{1}{C_3 R} \mu \left( \frac{G(R)}{C_2 R^{\frac{1}{2\sigma}}} \right) \leq \frac{G'(R)}{(G(R))^{p_0}}. \]

By denoting again \( \tilde{s} := R \) and integrating two sides on \( [R_0, R] \) we arrive at
\[ \frac{1}{C_3} \int_{R_0}^{R} \frac{1}{C_4 \tilde{s}^{\frac{1}{2\sigma}}} d\tilde{s} \leq \int_{R_0}^{R} \left( \frac{G'(\tilde{s})}{(G(\tilde{s}))^{p_0}} \right) d\tilde{s} = \frac{1}{2\sigma} \left( \frac{1}{(G(R_0))^{2\sigma}} - \frac{1}{(G(R))^{2\sigma}} \right) \leq \frac{1}{2\sigma (G(R_0))^{2\sigma}}, \]
where $C_4 := \frac{C_2}{G(R_0)} > 0$. Letting $R \to \infty$ leads to

$$\frac{1}{C_3} \int_{R_0}^\infty \frac{1}{s} \mu \left( \frac{1}{C_4 s^{2\sigma}} \right) ds \leq \frac{1}{2\sigma (G(R_0))^{2\sigma}}.$$  

Finally, using change of variables $s = C_4 \frac{2}{\sigma}$ we may conclude

$$C \int_{C_0}^{\infty} \frac{\mu \left( \frac{1}{s} \right)}{s} ds \leq \frac{1}{2\sigma (G(R_0))^{2\sigma}},$$

where $C := \frac{2\sigma}{C_4} > 0$ and $C_0 := C_4 \frac{2}{\sigma} > 0$ is a sufficiently large constant. This is a contradiction to the assumption (6). Summarizing, the proof is completed. \hfill\qed

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**Appendix A**

**A.1. Fractional Gagliardo-Nirenberg inequality**

**Proposition 3.1.** Let $1 < p, p_0, p_1 < \infty$, $\sigma > 0$ and $s \in [0, \sigma)$. Then, it holds the following fractional Gagliardo-Nirenberg inequality for all $u \in L^{p_0} \cap H^s_{p_1}$:

$$\|u\|_{H^s_{p}} \leq \|u\|_{p_0} \|u\|_{H^s_{p_1}}^{\theta},$$

where $\theta = \theta_{s,\sigma}(p, p_0, p_1) = \frac{1}{p_0} \frac{p_0 - \frac{s}{n}}{p_1 - \frac{s}{n}}$ and $\frac{s}{n} \leq \theta \leq 1$.

For the proof one can see [8, 9].

**A.2. Embedding into $L^\infty$**

**Proposition 3.2.** Let $q \in (1, \infty)$ and $s > \frac{n}{q}$. Then the following embedding holds:

$$H^s_q(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n).$$

The proof can be found in [14].

**A.3. A generalized Jensen’s inequality**

**Proposition 3.3.** Let $\Omega$ be a measurable set respecting a positive measure $\lambda$ so that $\lambda(\Omega)$ is the positive number. Let $f = f(z) : \Omega \to \mathbb{R}$ be a $\lambda$-integrable function with the image in $[a, b]$, and let $\gamma = \gamma(z) : \Omega \to \mathbb{R}$ be a positive $\lambda$-integrable function. Then each convex function $h = h(s) : [a, b] \to \mathbb{R}$ satisfies the following inequality:

$$h \left( \int_\Omega f(z) \gamma(z) d\lambda \right) \leq \int_\Omega (h(f(z)) \gamma(z) d\lambda).$$

The proof of this result can be found in [13].

**A.4. Usefull lemma**

**Lemma 3.1.** The following formula of derivative of composed function holds for any multi-index $\alpha$:

$$\partial^\alpha h(f(\xi)) = \sum_{k=1}^{|\alpha|} h^{(k)}(f(\xi)) \left( \sum_{|\gamma_1 + \cdots + \gamma_k| = |\alpha|, |\gamma_i| > 1} (\partial_{\xi}^{\gamma_1} f(\xi)) \cdots (\partial_{\xi}^{\gamma_k} f(\xi)) \right),$$

where $h = h(z)$ and $h^{(k)}(z) = \frac{d^k h(z)}{dz^k}$. 

The result can be found in [15] at page 202.

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