DERIVED AUTOEQUIVALENCES OF BIELLIPTIC SURFACES

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Abstract. We describe the group of exact autoequivalences of the bounded derived category of coherent sheaves on a bielliptic surface. We achieve this by studying its action on the numerical Grothendieck group of the surface.

1. Introduction

Let $X$ be a smooth projective variety over the complex numbers. We can construct the bounded derived category of coherent sheaves on $X$ denoted by $D(X) = D^b \text{Coh}(X)$. It is natural to study the symmetries of $D(X)$ which preserve the intrinsic structure: the group $\text{Aut} D(X)$ of exact $\mathbb{C}$-linear autoequivalences of $D(X)$ considered up to isomorphism as functors. We think of these autoequivalences as “higher” symmetries of the variety. Several autoequivalences of $D(X)$ arise naturally forming the subgroup

$$\text{Aut}_{st} D(X) = (\text{Aut} X \ltimes \text{Pic} X) \times \mathbb{Z}$$

of standard autoequivalences of $\text{Aut} D(X)$. This subgroup is generated by pulling back along automorphisms of $X$, tensoring by line bundles and by powers of the shift functor.

It is natural to ask if there are any others? When the (anti-)canonical bundle of $X$ is ample, Bondal and Orlov [4] showed that $\text{Aut} D(X) = \text{Aut}_{st} D(X)$, i.e. there are no extra autoequivalences of $D(X)$. The first example of a non-standard autoequivalence was observed by Mukai [13] for principally polarized abelian varieties. Many people have tried to understand non-standard autoequivalences of the derived category but the full group $\text{Aut} D(X)$ is only understood in a small number of cases. Orlov [14] computed the full group for Abelian varieties. Together with Bondal and Orlov’s result, this classifies the group of autoequivalences of the derived category of smooth projective curves. Broomhead and Ploog [8] computed the group for many rational surfaces (including most toric surfaces). Bayer and Bridgeland [3] described the group for K3 surfaces of Picard rank 1. Uehara [18] conjectured a description of the group for smooth projective elliptic surfaces of non-zero Kodaira

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dimension and proved the conjecture when each reducible fibre is a cycle of \((-2)\)-curves. Furthermore, he describes the group for elliptic ruled surfaces \([19]\). Ishii and Uehara \([11]\) computed the group for smooth projective surfaces (not necessarily minimal) of general type whose canonical model has at worst \(A_n\) singularities. No other examples are completely understood at this time. In this paper we describe the group \(\text{Aut} D(S)\) when \(S\) is a bielliptic surface.

Let \(S\) be a bielliptic surface, \(\tilde{S}\) the abelian surface which is the canonical cover of \(S\), and \(N(S)\) the numerical Grothendieck group of \(S\). Denote by \(O_\Delta(N(S))\) the subgroup of isometries of \(N(S)\) which preserve

\[
\Delta = \{ [E] \in N(S) \mid [E] = \pi([\tilde{E}]) \text{ for some } [\tilde{E}] \in N(\tilde{S}) \}
\]

where \(\pi : N(\tilde{S}) \to N(S)\) is the pushforward on \(K\)-theory. The main result is the following.

**Theorem 1.1.** There is an exact sequence

\[
1 \to (\text{Aut} S \ltimes \text{Pic}^0 S) \times \mathbb{Z} \to \text{Aut} D(S) \xrightarrow{\rho} O_\Delta(N(S))
\]

where \(\mathbb{Z}\) is generated by the second shift \([2]\). The map \(\rho\) is induced by the natural action of \(\text{Aut} D(S)\) on \(N(S)\) given by \(\rho(\Phi)[E] = [\Phi(E)]\). Furthermore, the image of \(\rho\) is a subgroup of \(O_\Delta(N(S))\) of index 4 if \(S\) is of type 2 or 4 and index 2 otherwise (see Table 1).

Moreover, we describe the generators of \(\text{Aut} D(S)\) in some cases.

**Theorem 1.2.** Suppose \(S\) is a split bielliptic surface (see Definition \[2,2\]). Then the group \(\text{Aut} D(S)\) is generated by standard autoequivalences and relative Fourier-Mukai transforms along the two elliptic fibrations.

The article is structured as follows. In section 2 we review preliminary material on bielliptic surfaces, the numerical Grothendieck group, canonical covers and relative Fourier-Mukai transforms. In section 3 we prove Theorem 1.1 by describing a collection of autoequivalences arising from moduli spaces of stable, special sheaves whose Chern character lies in \(\Delta\). In section 4 we prove Theorem 1.2.

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2. Preliminaries

All varieties will be over the complex numbers.

2.1. Bielliptic surfaces.

**Definition 2.1.** A bielliptic (or hyperelliptic) surface \( S \) is a minimal projective surface of Kodaira dimension zero with \( q = 1 \) and \( p_g = 0 \).

Bielliptic surfaces are constructed by taking the quotient of the product of two elliptic curves \( A \times B \) by a finite subgroup \( G \) of \( A \) acting on \( A \) by translations and on \( B \) by automorphisms, not all translations. These surfaces are classified by Bagnera and De Franchis into seven families [1, §V.5] determined by the subgroup \( G \) and the lattice \( \Gamma \) such that \( B = \mathbb{C}/\Gamma \) (see Table 1).

| Type | \( \Gamma \) | \( G \) | Action of \( G \) on \( B \) |
|------|-------------|--------|----------------------|
| 1    | Arbitrary   | \( \mathbb{Z}/2 \) | \( b \mapsto -b \) |
| 2    | Arbitrary   | \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) | \( b \mapsto -b \), \( b \mapsto b + \beta \), where \( 2\beta = 0 \) |
| 3    | \( \mathbb{Z} \oplus \mathbb{Z} \omega \) | \( \mathbb{Z}/3 \) | \( b \mapsto \omega b \) |
| 4    | \( \mathbb{Z} \oplus \mathbb{Z} \omega \) | \( \mathbb{Z}/3 \oplus \mathbb{Z}/3 \) | \( b \mapsto \omega b \), \( b \mapsto b + \beta \), where \( \omega \beta = \beta \) |
| 5    | \( \mathbb{Z} \oplus \mathbb{Z} i \) | \( \mathbb{Z}/4 \) | \( b \mapsto ib \) |
| 6    | \( \mathbb{Z} \oplus \mathbb{Z} i \) | \( \mathbb{Z}/4 \oplus \mathbb{Z}/2 \) | \( b \mapsto ib \), \( b \mapsto b + \beta \), where \( i\beta = \beta \) |
| 7    | \( \mathbb{Z} \oplus \mathbb{Z} \omega \) | \( \mathbb{Z}/6 \) | \( b \mapsto -\omega b \) |

Table 1. (\( \omega^3 = 1 \) and \( i^4 = 1 \) are complex roots of unity.)

**Definition 2.2.** We call a bielliptic surface split if it is of type 1, 3, 5, or 7 and non-split otherwise.

**Remark 2.3.** Associated to a bielliptic surface \( S \) are two elliptic fibrations:

\[
p_A: S \to A/G
\]
\[
p_B: S \to B/G
\]

with \( A/G \) an elliptic curve and \( B/G \cong \mathbb{P}^1 \).

Since the projection \( A \to A/G \) is étale, all the fibres of \( p_A \) are smooth. The fibre of \( p_B \) over a point \( P \in B/G \) is a multiple of a smooth elliptic curve. The multiplicity of the fibre of \( p_B \) at \( P \) is the same as the multiplicity of the projection \( B \to B/G \cong \mathbb{P}^1 \). As all smooth fibres of \( p_A \) (respectively \( p_B \)) are isomorphic to \( B \) (respectively \( A \)) we will denote the class of the smooth fibre of \( p_A \) and \( p_B \) in \( H^2(S, \mathbb{Q}) \) by \( B \) and \( A \) respectively.

The derived category of a bielliptic surface \( S \) is a strong invariant of the surface due to the following result of Bridgeland and Maciocia.
Proposition 2.4 ([7 Proposition 6.2]). Let $S$ be a bielliptic surface and $S'$ a smooth projective surface such that $D(S') \cong D(S)$.

Then $S$ is isomorphic to $S'$.

2.2. Numerical Grothendieck Group. The Grothendieck group $K(X)$ of a smooth projective variety $X$ is the free group generated by isomorphism classes of objects in $D(X)$ modulo an equivalence relation given by distinguished triangles [9 §5]. There is a natural bilinear form on this group, the Euler form, defined by

$$
\chi([E], [F]) = \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Hom}^i_{D(X)}(E, F)
$$

where $\text{Hom}^i_{D(X)}(E, F) = \text{Hom}_{D(X)}(E, F[i])$. This bilinear form is well defined as the Euler form is additive on distinguished triangles. We can consider the radical of the Euler form

$$\text{rad} \chi = \{ v \in K(X) | \chi(v, w) = 0 \text{ and } \chi(w, v) = 0 \text{ for all } w \in K(X) \}$$

and form the quotient $N(X) = K(X)/\text{rad} \chi$, which we call the numerical Grothendieck group of $X$. The Euler form descends to a non-degenerate bilinear form on $N(X)$. Recall that $\text{Num}(S)$ is the group of divisors on $S$ modulo numerical equivalence $\equiv$.

Proposition 2.5. Let $S$ be a bielliptic surface. The Chern character

$$ch: K(S) \to H^{2*}(S, \mathbb{Q})$$

identifies $N(S)$ with the group

$$H^0(S, \mathbb{Z}) \oplus \text{Num}(S) \oplus H^4(S, \mathbb{Z}) \cong \mathbb{Z} \oplus \text{Num}(S) \oplus \mathbb{Z}.$$ 

Under this identification, for $ch(E) = (r, D, s)$ and $ch(F) = (r', D', s')$ the Euler form becomes $\chi(E, F) = rs' + r's - D \cdot D'$.

Proof. For $v = (v_0, v_2, v_4) \in H^{2*}(S, \mathbb{Q})$ define $v' = (v_0, -v_2, v_4) \in H^{2*}(S, \mathbb{Q})$. Recall that the Mukai pairing on $H^{2*}(S, \mathbb{Q})$ is defined by

$$\langle v, v' \rangle = \int_X v' \cdot v'$$

where the product in the integral is the cup product of cohomology classes. As the Todd classes of abelian and bielliptic surfaces are $(1, 0, 0)$, by Hirzebruch-Riemann-Roch for $[E], [F] \in K(S)$

$$\chi([E], [F]) = \langle ch(E), ch(F) \rangle.$$

Thus the Euler form for $ch(E) = (r, D, s)$ and $ch(F) = (r', D', s')$ can be written as

$$\chi([E], [F]) = \langle (r, D, s), (r', D', s') \rangle = rs' + r's - D \cdot D'.$$

A class lies in the radical of the Euler form if and only if it lies in the radical of the Mukai pairing. As the Mukai pairing is non-degenerate
an element of $K(S)$ lies in the radical of the Euler form if and only if it has zero Chern Character. Hence $\ker(ch) = \text{rad } \chi$ and $\text{im}(ch) \cong N(S)$.

Using this alternative description of the Euler form, we see that the class of a numerically trivial divisor $D$, $[\mathcal{O}_S(D)]$ is equivalent to $[\mathcal{O}_S]$. Therefore, the image of the Chern character intersected with the group $H^2(S, \mathbb{Q})$ is the group $\text{Num}(S)$. Furthermore, by Hirzebruch-Riemann-Roch we have $ch_2(E) = \chi(E) \in \mathbb{Z}$ for all $E$. Thus we have an isomorphism

$$N(S) \cong H^0(X, \mathbb{Z}) \oplus \text{Num}(S) \oplus H^4(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \text{Num}(S) \oplus \mathbb{Z}.$$ 

\[ \square \]

**Remark 2.6.** Let $A$ be an Abelian surface. Then a similar argument to Proposition 2.5 shows that the Chern character induces an isomorphism $N(A) \cong \mathbb{Z} \oplus \text{Num}(A) \oplus \mathbb{Z}$.

**Remark 2.7.** We will study the group $\text{Aut } D(S)$ by studying its action on the numerical Grothendieck group given by the homomorphism

$$\rho: \text{Aut } D(S) \to \text{Aut}(N(S))$$

where $\rho(\Phi)([E]) = [\Phi(E)]$. Autoequivalences of $D(S)$ preserves the $\text{Hom}^i$ groups, thus the Euler form. Hence the image of $\rho$ is contained in the group of isometries $O(N(S))$ of $N(S)$.

2.3. Canonical covers of Bielliptic surfaces.

**Proposition 2.8** ([6, §2], [9, §7.3], [2, §7.2]). Let $X$ be a smooth projective variety whose canonical bundle $\omega_X$ has finite order, i.e. there exists $n$ such that $\omega_X^{\otimes n} \cong \mathcal{O}_X$. Then there exists a smooth projective variety $\tilde{X}$ with trivial canonical bundle, and an étale cover $\pi: \tilde{X} \to X$ of degree $n$ such that

$$\pi^*(\mathcal{O}_\tilde{X}) \cong \bigoplus_{i=0}^{n-1} \omega_X^{\otimes i}.$$ 

Furthermore, $\tilde{X}$ is uniquely defined up to isomorphism, and there is a free action of the cyclic group $\tilde{G} = \mathbb{Z}/n\mathbb{Z}$ on $\tilde{X}$ such that $\pi: \tilde{X} \to X = \tilde{X}/\tilde{G}$ is the quotient morphism.

The canonical cover of a bielliptic surface will play an important role in determining the group of autoequivalences. We list the following facts about the canonical cover of a bielliptic surface and leave the verification to the reader.

**Proposition 2.9.** Let $S$ be a bielliptic surface which is realized as the quotient of $A \times B$ be a finite group $G$ of order $nk$. Then there exists an abelian surface $\tilde{S}$ which is the canonical cover of $S$.

- If $S$ is split, then $k = 1$, $\tilde{S} \cong A \times B$ and $G \cong \tilde{G}$. 

• If $S$ is non-split, then $k > 1$ and $\tilde{S}$ can be realized as the quotient $\tilde{S} \cong (A \times B)/H$ where $H$ is the cyclic subgroup of $G$ of order $k$ acting on $A \times B$ purely by translations. We have $G \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$.

**Remark 2.10.** The canonical cover $\tilde{S}$ has two fibrations

\[ \tilde{p}_A : \tilde{S} \to A/H \]
\[ \tilde{p}_B : \tilde{S} \to B/H. \]

Both $\tilde{p}_A$ and $\tilde{p}_B$ are smooth fibrations with fibres isomorphic to $B$ and $A$ respectively. We will denote the class of these fibres by $\tilde{B}$ and $\tilde{A}$ in $\text{Num}(S)$ respectively. The degree of the intersection $\tilde{B} \cdot \tilde{A} = k = |H|$.

We summarize the description of $\text{Num}(S)$ given by Serrano [15, §1] in the following lemma.

**Lemma 2.11.** Let $|G| = nk$ and $n = \deg \pi$ where $\pi : \tilde{S} \to S$ is the canonical cover of $S$.

1. The second rational cohomology group $H^2(S, \mathbb{Q})$ is generated by $A$ and $B$.
2. Suppose $S$ is split. Then $k = 1$ and the group $\text{Num}(S)$ is generated by $\frac{1}{n}A$ and $B$.
3. Suppose $S$ is non-split. Then the group $\text{Num}(S)$ is generated by $\frac{1}{n}A$ and $\frac{1}{k}B$.

Consider the category $\text{Sp-Coh}(S)$ of coherent $\pi_*(\mathcal{O}_S)$-modules on $S$. A sheaf $E$ lies in $\text{Sp-Coh}(S)$ if and only if $E \otimes \omega_S \cong E$. We call such sheaves *special*. The following results from [6, §2], [2, §7.2] relate this category to the category of coherent sheaves on $\tilde{S}$.

**Lemma 2.12.** The functor

\[ \pi_* : \text{Coh}(\tilde{S}) \to \text{Sp-Coh}(S) \]

is an equivalence.

This descends to the level of derived categories in the following way:

**Proposition 2.13.** Let $E$ be an object of $D(S)$. Then there is an object $\tilde{E}$ of $D(\tilde{S})$ such that $R\pi_*(\tilde{E}) \cong E$ if and only if $E \otimes \omega_S \cong E$.

**Remark 2.14.** Recall $\pi_! : N(\tilde{S}) \to N(S)$ is defined by ([6, §5.2])

\[ \pi_![E] = \sum_{i \in \mathbb{Z}}(-1)^i[R^i\pi_*(E)]. \]

After taking Chern characters, $\pi_!$ coincides with the pushforward $\pi_*$ on cohomology by Grothendieck-Riemann-Roch. This is due to the Todd classes of $\tilde{S}$ and $S$ being $(1,0,0)$. 
On the level of the numerical Grothendieck group $N(S)$ consider the subgroup $\Delta$ of special classes

$$\Delta = \text{im}(\pi!) = \left\{ [E] \in N(S) \mid [E] = \pi_!([\tilde{E}]) \text{ for some } [\tilde{E}] \in N(\tilde{S}) \right\}.$$ 

Remark 2.15. The class $[E]$ of a special object $E \in D(S)$ lies in $\Delta$ by Proposition 2.13 as there exists $\tilde{E} \in D(\tilde{S})$ such that $[E] = [\pi_*(\tilde{E})] = \pi_![\tilde{E}]$.

The subgroup $\Delta$ is important because the image of autoequivalences of $D(S)$ under $\rho$ preserves $\Delta$.

Proposition 2.16. Let $\Phi \in \text{Aut} D(S)$. Then $\rho(\Phi)$ preserves $\Delta$.

Proof. Any autoequivalence $\Phi \in \text{Aut} D(S)$ lifts to an equivariant autoequivalence $\tilde{\Phi} \in \text{Aut} D(\tilde{S})$ by [5, Theorem 4.5] or [2, Theorem 7.13] such that $R\pi_* \circ \tilde{\Phi} \cong \Phi \circ R\pi_!$.

Consider $v \in \Delta$ and $w \in N(\tilde{S})$ such that $v = \pi_!(w)$. Then

$$\rho(\Phi)(v) = \rho(\Phi)(\pi_!(w)) = \pi_!(\rho(\tilde{\Phi})(w)) \in \Delta$$

Therefore $\rho(\Phi)(\Delta) \subset \Delta$. 

2.4. Relative Fourier-Mukai Transforms. Recall that a relatively minimal elliptic surface is a projective surface $X$ together with a fibration $\pi: X \to C$ with generic fibre isomorphic to an elliptic curve and with no $(-1)$-curves in the fibres. We will only consider relatively minimal elliptic surfaces.

For an elliptic surface $\pi: X \to C$ define $\lambda_\pi$ to be the smallest positive integer such that $\pi$ has a holomorphic $\lambda_\pi$-multisection. This is equivalent to

$$\lambda_\pi = \min\{f \cdot D > 0 \mid D \in \text{Num}(X)\},$$

where $f$ is the class of a smooth fibre of $\pi$.

Suppose $a > 0, b \in \mathbb{Z}$ with $\gcd(a\lambda_\pi, b) = 1$. Then we can construct the moduli space $J_X(a, b)$ of pure dimension 1 stable sheaves of class $(a, b)$ supported on a smooth fibre of $\pi$. Bridgeland constructed equivalences between the derived category of $X$ and the derived category of $J_X(a, b)$ [5]. We call these equivalences relative Fourier-Mukai transforms.

Theorem 2.17. [5, Theorem 5.3] Let $\pi: X \to C$ be an elliptic surface and take an element

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

such that $\lambda_X$ divides $d$ and $a > 0$. Let $Y$ be the elliptic surface $J_X(a, b)$ over $C$. Then there exists sheaves $P$ on $X \times Y$, flat and strongly simple over both factors such that for any point $(x, y) \in X \times Y$, $P_y$ has Chern
class \((0, af, b)\) on \(X\) and \(\mathcal{P}_x\) has Chern class \((0, af, c)\) on \(Y\). For any such sheaf \(\mathcal{P}\), the resulting functor \(\Phi = \Phi_{Y \to X}^\mathcal{P} : D(Y) \to D(X)\) is an equivalence and satisfies
\[
\begin{pmatrix}
  r(\Phi(E)) \\
  d(\Phi(E))
\end{pmatrix} = \begin{pmatrix}
  c & a \\
  d & b
\end{pmatrix} \begin{pmatrix}
  r(E) \\
  d(E)
\end{pmatrix}
\]
for all objects \(E\) of \(D(Y)\).

3. Proof of Theorem 1.1

To prove Theorem 1.1 we will compute the kernel and image of
\[
\rho : \text{Aut} \, D(S) \to O(N(S))
\]
given by \(\rho(\Phi)([E]) = [\Phi(E)]\).

We will need the following result concerning moduli spaces of sheaves on a bielliptic surface \(S\) which will give rise to autoequivalences of the derived category.

**Proposition 3.1.** Let \(S\) be a bielliptic surface and \(\pi : \tilde{S} \to S\) the canonical cover of \(S\). Take \(v \in \Delta\) such that \(\nu(v, v) = 0\) and there exists \(v' \in N(S)\) such that \(\nu(v, v') = 1\). Choose a generic ample line bundle \(H\) with respect to \(v\). Then there exists a two dimensional, projective, smooth, fine moduli space \(M\) of \(H\)-slope stable, special sheaves on \(S\) of class \(v\).

Moreover, the universal sheaf on \(M \times S\) induces an autoequivalence \(\Phi\) of \(D(S)\) such that \([\Phi(O_s)] = v\) for any closed point \(s \in S\).

**Proof.** Choose a generic ample divisor \(H\) which does not lie on a wall with respect to \(v\).

First we show that \(M\) is non-empty. As \(v \in \Delta\), there exists \(w \in N(\tilde{S})\) such that \(\pi_!(w) = v\). The moduli space of \(\pi^*H\)-Gieseker semistable sheaves of class \(w\) on the abelian surface \(\tilde{S}\) is non-empty by [10] §4.3].

Let \(F\) be a \(\pi^*H\)-Gieseker semistable sheaf of class \(w\). As \(\pi^*H\)-Gieseker semistable sheaves are \(\pi^*H\)-slope semistable, \(F\) is \(\pi^*H\)-slope semistable. By [17] Proposition 1.5] the pushforward \(\pi_*F\) is a \(H\)-slope semistable sheaf as \(\pi\) is finite étale. By construction, \([\pi_*F] = \pi_!(w) = v\). Therefore, the moduli space \(M_H^w\) of \(H\)-slope semistable sheaves of class \(v\) is non-empty.

As \(H\) was chosen not to lie on a wall and there exists \(v'\) such that \(\nu(v, v') = 1\), all \(H\)-slope semistable sheaves are \(H\)-slope stable. Therefore the moduli space \(M_H\) of \(H\)-slope stable sheaves is projective. By [10] Proposition 4.6] there exists a quasi-universal family on \(M_H \times S\). This family can be chosen to be universal due to the existence of \(v'\).

Let \(E\) be a \(H\)-stable sheaf of class \(v\) corresponding to a point of \(M_H\). As \(v = [E]\) is isotropic and \(E\) is stable, \(\dim \text{Hom}_S(E, E) = 1\) and
\[
\dim \text{Ext}_S^1(E, E) = 1 + \dim \text{Ext}_S^2(E, E).
\]
As $E$ is slope stable and $\omega_S$ is a numerically trivial, $E \otimes \omega_S$ is slope stable of the same slope. Thus $\dim \text{Hom}_S(E, E \otimes \omega_S) = 0$ or 1 Then by Serre Duality and the equality above, $\dim \text{Ext}_S^1(E, E) \leq 2$.

By construction, $M_H$ contains at least one closed point corresponding to a sheaf $F$ which is the pushforward of a semistable sheaf on the canonical cover. Thus $F$ is special by Proposition 2.13 so $F \otimes \omega_S \cong F$ and $\dim \text{Ext}_S^2(E, E) = 1$. Hence $\dim \text{Ext}_S^1(F, F) = 2$. By Serre Duality and [10] §4.5] $M_H$ is smooth at $F$ because the trace map on $\text{Ext}_S^2(F, F)$ has zero kernel due to $F$ being special.

As $M$ is smooth at $F$, $\dim M'_H = \dim \text{Ext}_S^1(F, F) = 2$ for some connected component $M'_H$ of $M_H$. Hence $\dim \text{Ext}_S^1(E, E) \geq 2$ for all sheaves $E$ corresponding to points of $M'_H$. So $\dim \text{Ext}_S^1(E, E) = 2$ for all such $E$. Thus $M'_H$ is smooth of dimension 2. Set $M = M'_H$.

As $E$ is stable and has the same slope as $E \otimes \omega_S$, any map between them is an isomorphism. So $E$ is special as $\dim \text{Hom}(E, E \otimes \omega_S) = \dim \text{Ext}_S^2(E, E) = 1$.

Thus $M$ is a two dimensional, projective, smooth, fine moduli space of $H$-slope stable, special sheaves on $S$ of class $v$.

By [7] Corollary 2.8] the universal sheaf $\mathcal{P}$ on $M \times S$ induces an equivalence

$$\Phi_\mathcal{P} : D(M) \to D(S).$$

By Proposition 2.4 $M$ is isomorphic to $S$. Thus the equivalence $\Phi_\mathcal{P}$ induces an autoequivalence $\Phi$ of $D(S)$ after choosing an isomorphism $M \cong S$. By construction $[\Phi(\mathcal{O}_s)] = [\mathcal{P}_s] = v$. \hfill \square

We now prove Theorem 1.1

Proof of Theorem 1.1 First we describe the kernel of $\rho$. Let $\Psi \in \ker \rho$. As $\Psi$ is an integral transform, by a theorem of Orlov [9] Theorem 5.14], $\Psi \cong \Psi_\mathcal{P}$ for some $\mathcal{P} \in D(S \times S)$. As $S$ is a bielliptic surface, $\mathcal{P}$ is isomorphic to a shift of a sheaf [10] Proposition 5.1]. Thus $\Psi(\mathcal{O}_s) = \mathcal{P}_s$ is a shift of a sheaf for any closed point $s \in S$. As $\Psi$ acts trivially on $N(S)$, $ch(\mathcal{P}_s) = ch(\mathcal{O}_s) = (0, 0, 1)$. Hence $\mathcal{P}_s$ is a shift of a skyscraper sheaf for any closed point $s \in S$. As $\Psi$ is a standard autoequivalence if and only if $\Psi(\mathcal{O}_s)$ is a shift of a skyscraper sheaf, $\Psi$ is a standard autoequivalence.

The only standard autoequivalences that act trivially on $N(S)$ are $(\text{Aut} S \times \text{Pic}^0 S) \times \mathbb{Z}[2]$. This is because the n-th power of the shift functor acts by $(-1)^n$ on $N(S)$. Tensoring by a line bundle $L$ act trivially on $N(S)$ if and only $c_1(L) = 0$, i.e. $L$ has degree zero. Automorphisms of $S$ act trivially on $N(S)$ because they preserve effective divisors and cannot exchange the fibres of the different elliptic fibrations as one has multiple fibres and the other does not.

We now characterize the image of $\rho$. Let $\varphi \in O_\Delta(N(S))$ and consider $v = \varphi(0, 0, 1) \in \Delta$. Then $v \in \Delta$, $v^2 = 0$ and there exists $v' = \varphi(1, 0, 0)$
such that \( \langle v, v' \rangle = 1 \). By Proposition 3.1 we can construct an autoequivalence \( \Phi \in \text{Aut } D(S) \) such that \( \rho(\Phi)(0, 0, 1) = v \). Consider the isometry
\[
\phi' = (\rho(\Phi))^{-1} \circ \phi.
\]
Then \( \phi'(0, 0, 1) = (0, 0, 1) \). As \( \phi'(1, 0, 0) = (1, D, s) \) is isotropic, \( D^2 = 2s \). Thus \( s = D^2/2 \) and \( \phi'(1, 0, 0) = (1, D, D^2/2) \) is the class of a line bundle \( L \) with \( c_1(L) = D \). Consider the isometry
\[
\phi'' = \rho(L^* \otimes (-)) \circ \phi'.
\]
Notice that \( \phi'' \) acts by
\[
id_{H^0} \otimes \psi \otimes id_{H^4}
\]
on \( N(S) \) where \( \psi \) is an isometry of \( \text{Num}(S) \). Note that \( \phi'' \) respects the grading and is an element of \( O_\Delta(N(S)) \) as it is a composite of elements of \( O_\Delta(N(S)) \).

The group \( \text{Num}(S) \) is isomorphic as a lattice to a single hyperbolic plane \( U \) with underlying group \( \mathbb{Z}^2 \) \[15] \S 1. \) The group of isometries \( O(U) \) is isomorphic to \( \mathbb{Z}/2 \times \mathbb{Z}/2 \). It is generated by the involutions \( \iota \), which acts by \( -id \) on \( U \), and \( \sigma \) which exchanges the two copies of \( \mathbb{Z} \).

Both of these give rise to isometries of \( N(S) \) by acting by the identity on \( H^0(S, \mathbb{Z}) \) and \( H^4(S, \mathbb{Z}) \) which we will denote by \( \iota \) and \( \sigma \) by an abuse of notation.

Suppose the isometry \( \iota \) is induced by an autoequivalence. As \( \iota \) fixes the class of a point and acts non-trivially on \( N(S) \), \( \iota \) is induced by a standard autoequivalence which acts non-trivially on \( N(S) \). But standard autoequivalences which act non-trivially on \( N(S) \) act by tensoring by \( \pm(1, D, D^2/2) \) for some line bundle \( L \) with \( c_1(L) = D \neq 0 \). However, \( \iota \) does not acts on \( N(S) \) in this way as \( \iota(1, 0, 0) = (1, 0, 0) \). Hence \( \iota \) is not induced by an autoequivalence. Similarly, \( \sigma \) and \( \iota \circ \sigma \) are not induced by autoequivalences. Thus the image of \( \rho \) intersected with \( O(\text{Num}(S)) \) is trivial.

Note that \( \iota \) preserves \( \Delta \). However, \( \sigma \) may not preserve \( \Delta \). The index of the image of \( \rho \) will 2 or 4 in \( O_\Delta(N(S)) \) depending on whether \( \sigma \) preserves \( \Delta \). As \( \sigma \) acts trivially on the two copies of \( \mathbb{Z} \) in \( N(S) \) it is sufficient to study the action on \( \text{Num}(S) \) by the following Lemma.

**Lemma 3.2.** A class \( (r, D, s) \in \Delta \) if and only if \( n \mid r \) and \( (0, D, 0) \in \Delta \). Thus \( \Delta = n \mathbb{Z} \oplus \pi_s(\text{Num}(\tilde{S})) \oplus \mathbb{Z} \subset \mathbb{Z} \oplus \text{Num}(S) \oplus \mathbb{Z} \cong N(S) \).

**Proof.** Suppose \( n \mid r \) and \( (0, D, 0) \in \Delta \). Then \( r = \tilde{r}n \) and there exists \( \tilde{D} \in \text{Num}(\tilde{S}) \) such that \( \pi_1(0, \tilde{D}, 0) = (0, \pi_s(\tilde{D}), 0) = (0, D, 0) \). Then
\[
\pi_1(\tilde{r}, \tilde{D}, s) = \pi_1(\tilde{r}, 0, 0) + \pi_1(0, \tilde{D}, 0) + \pi_1(0, 0, s) = (r, D, s)
\]
as \( \pi_1(0, 0, 1) = (0, 0, 1) \).
Suppose that \((r, D, s) \in \Delta\). Then there exists \([E] \in N(S)\) such that \(\pi([E]) = (r, D, s)\). By computing the Mukai pairing of \((r, D, s)\) with the classes \((1, 0, 0)\) and \((0, 0, 1)\) we see that \(ch_2([E]) = s\) and \(r = n \text{rk}(E)\). So \((r, 0, 0), (0, 0, s) \in \Delta\) as \(\pi(\text{rk}(E)[\mathcal{O}_S]) = (r, 0, 0)\) and \(\pi(s[\mathcal{O}_S]) = (0, 0, s)\). Then
\[
(r, D, s) - (0, 0, 0) - (r, 0, 0) = (0, D, 0) \in \Delta.
\]
\[\square\]

If \((r, D, s) \in \Delta\) then \(\sigma(r, D, s) = (r, \sigma(D), s) \in \Delta\) if and only if \((0, \sigma(D), 0) \in \Delta\). To determine whether \(\sigma\) preserves \(\Delta\) we reduce to studying classes of the form \((0, D, 0)\). By abuse of notation, we will denote the class \((0, D, 0) \in N(S)\) by \(D\) and we write \(D \in \Delta\) for \((0, D, 0) \in \Delta\).

**Lemma 3.3.** The classes \(A, B \in \Delta\) but \(\frac{1}{k}A \notin \Delta\). If \(S\) is non-split, then \(\frac{1}{k}B \notin \Delta\).

**Proof.** The classes \(A, B \in \Delta\) as \(\pi_*(\tilde{A}) = A\) and \(\pi_*(\tilde{B}) = B\).

Suppose that \(\frac{1}{n}A \in \Delta\). Then there exist \(0 \neq \tilde{D} \in \text{Num}(\tilde{S})\) such that \(\pi_*(\tilde{D}) = \frac{1}{n}A\). As \(\tilde{D} \cdot \tilde{D} = n(\pi_*D, \pi_*D) = n(\frac{1}{n}A, \frac{1}{n}A) = 0\), by [12, Proposition 2.3], \(D \equiv mE\) for some \(0 \neq m \in \mathbb{Z}\) and \(E\) an elliptic curve. Then by the push-pull formula we have

\[
0 = A \cdot \pi_*(mE) = \pi_*(\pi^*A \cdot mE).
\]

As the pushforward of points is injective on cohomology, we have

\[
0 = \pi^*A \cdot mE = n\tilde{A} \cdot mE = nm(\tilde{A} \cdot E).
\]

So \(\tilde{A} \cdot E = 0\). As \(E\) and \(\tilde{A}\) are irreducible curves, by [12, Proposition 2.1] \(E = T_{\tilde{A}}(\tilde{A})\), so \(E \equiv \tilde{A}\). But \(\pi_*(mE) = \pi_*(m\tilde{A}) = mA \neq \frac{1}{k}A\), which is a contradiction. Hence \(\frac{1}{k}A \notin \Delta\).

A similar argument holds for \(\frac{1}{k}B\) when \(S\) is a non-split bielliptic by replacing \(\tilde{A}\) by \(\tilde{B}\).
\[\square\]

Note that \(\sigma\) interchanges the generators of \(\text{Num}(S)\). We will consider separate cases to determine the index of the image of \(\rho\).

We will use the following repeatedly: A class \(D \in \Delta\) if and only if \(D' = D + (aA + bB) \in \Delta\) with \(a, b \in \mathbb{Z}\). Clearly if \(D \in \Delta\) then \(D' \in \Delta\). Conversely, if \(D' \in \Delta\), then \(D = D' - (aA + bB) \in \Delta\) as \(\Delta\) is a subgroup.

**Split Bielliptic:** Suppose that \(S\) is a split bielliptic surface. Then \(\sigma\) interchanges \(\frac{1}{n}A\) and \(B\). But by the above claim \(\frac{1}{n}A \notin \Delta\) but \(B \in \Delta\), so \(\sigma\) does not preserve \(\Delta\). Hence the index is 2.

**Bielliptic of type 2:** By Lemma 3.3 we have \(\frac{1}{2}A, \frac{1}{2}B \notin \Delta\) and \(A, B \in \Delta\). Consider \(D = \frac{1}{2}A + \frac{1}{2}B\) with \(a, b \in \mathbb{Z}\). Then \(\sigma(D) = \frac{1}{2}A + \frac{1}{2}B\). By adding or subtracting multiples of \(A\) and
we can reduce to the cases when \(a, b \in \{0, 1\}\). We have 3 cases:

1. If \(a = b = 0\) then \(D \in \Delta\) and \(\sigma(D) \in \Delta\).
2. Suppose \(a = 0\) and \(b = 1\). Then \(\sigma(D) = \frac{1}{2}A \notin \Delta\) and \(D = \frac{1}{2}B \notin \Delta\). A similar argument show that \(D, \sigma(D) \notin \Delta\) for \(a = 1\) and \(b = 0\).
3. Suppose that \(a = b = 1\). Then \(D = \frac{1}{2}A + \frac{1}{2}B = \sigma(D)\).

Hence \(D \in \Delta\) if and only if \(\sigma(D) \in \Delta\). Thus \(\sigma\) preserves \(\Delta\) and the index is 4.

**Bielliptic of type 4:** By Lemma 3.3 we have \(\frac{1}{3}A, \frac{1}{3}B \notin \Delta\) and \(A, B \in \Delta\). Consider \(D = \frac{2}{3}A + \frac{1}{3}B\) with \(a, b \in \mathbb{Z}\) and \(\sigma(D) = \frac{2}{3}A + \frac{2}{3}B\). By adding or subtracting multiples of \(A\) and \(B\) we can reduce to the cases when \(a, b \in \{0, 1, -1\}\). We have 4 cases:

1. If \(a = b = 0\). Then \(D \in \Delta\) and \(\sigma(D) \in \Delta\).
2. Suppose that \(a = b = 1\). Then \(\sigma(D) = \frac{1}{3}A + \frac{1}{3}B = D\).

Hence \(D \in \Delta\) if and only if \(\sigma(D) \in \Delta\). A similar argument works for \(a = b = -1\).
3. Suppose that \(m = a\) and \(b = 1\). Then \(D = \frac{1}{3}B \notin \Delta\) and \(\sigma(D) = \frac{1}{3}A \notin \Delta\). Similarly for \(a = 0\), \(b = -1\) and \(a = 1\), \(-1\), \(b = 0\) we have \(D \notin \Delta\) and \(\sigma(D) \notin \Delta\).
4. Suppose that \(a = 1\) and \(b = -1\). Then \(\sigma(D) = -\frac{1}{3}A + \frac{1}{3}B = -D\). As \(\Delta\) is a subgroup \(-D \in \Delta\) if and only if \(D \in \Delta\). Hence \(D \in \Delta\) if and only if \(\sigma(D) \in \Delta\). A similar argument works for \(a = -1\) and \(b = 1\).

Thus \(\sigma\) preserves \(\Delta\) and the index is 4.

**Bielliptic of type 6:** Note that \(\frac{1}{2}A \notin \Delta\) by a similar argument to Lemma 3.3. Then as \(\sigma\) interchanges \(\frac{1}{2}A\) and \(2(\frac{1}{2}B) = B\), \(\sigma\) does not preserve \(\Delta\). Hence the index is 2.

\[
\square
\]

### 4. Relative Fourier-Mukai Transforms and bielliptic surfaces

For a bielliptic surface \(S\), relative Fourier-Mukai transforms with respect to either elliptic fibration \(p_A\) or \(p_B\) give rise to autoequivalences of \(D(S)\) in the following way.

**Proposition 4.1.** Let \(S\) be a bielliptic surface and \(p_A: S \to A/G\) and \(p_B: S \to B/G\) its two relatively minimal elliptic fibrations. Then a relative Fourier-Mukai transform with respect to either fibration induces an autoequivalence on \(D(S)\) which is non-standard.

**Proof.** Let \(\Phi_{Rel}: D(Y) \to D(S)\) be a relative Fourier-Mukai transform induced by one of the two fibrations. By Proposition 2.4 Y is isomorphic to \(S\). After choosing an isomorphism \(g: Y \to S\), the composite
\( \Psi = \Phi_{\text{Rel}} \circ g^* \) is an autoequivalence of \( D(S) \). It is non-standard because \( ch(\Psi(O_s)) = (0, af, b) \) where \( f \) is the fibre of the elliptic fibration. \( \square \)

**Example 4.2.** Note that for either fibration \( p_A \) or \( p_B \) of \( S \) we have an autoequivalence corresponding to the matrix

\[
P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

given by Theorem 2.17. We have an autoequivalence \( \Psi_B \), constructed by composing the relative Fourier-Mukai transform along \( p_A \) associated to \( P \) and tensoring by a suitable line bundle, which acts on \( N(S) \) by

\[
(1, 0, 0) \mapsto (1, 0, 0) \\
(0, 0, 1) \mapsto (0, B, 1) \\
(0, B', 0) \mapsto (0, B', 0) \\
(0, A', 0) \mapsto (\lambda_{p_A}, A', 0).
\]

Note \( \Psi_B \) sends \((0, A, 0) \) to \((n, A, 0) \).

Suppose that \( S \) is split. Then the fibration \( p_A: S \to A/G \) admits a section, i.e. \( \lambda_{p_A} = 1 \). Then there is a relative Fourier-Mukai functor \( \hat{\Psi} \) that corresponds to the matrix

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

given by Theorem 2.17 which acts on \( N(S) \) by

\[
(1, 0, 0) \mapsto (0, (-1/n)A, 0) \\
(0, 0, 1) \mapsto (0, B, 0) \\
(0, B, 0) \mapsto (0, 0, 1) \\
(0, (1/n)A, 0) \mapsto (1, 0, 0).
\]

We now prove Theorem 1.2.

**Proof of Theorem 1.2.** As \( S \) is split, \( k = 1 \) and \( |G| = n = \deg \pi \) and \( \tilde{S} \cong A \times B \). Let \( \Phi \in \text{Aut} D(S) \). Consider \( v = \rho(\Phi)(0, 0, 1) \). Then \( v \in \Delta, v^2 = 0 \) and there exists \( v' = \rho(\Phi)(1, 0, 0) \) such that \( \langle v, v' \rangle = 1 \).

We will construct an autoequivalence \( \Psi \in \text{Aut} D(S) \) which is the composite of standard autoequivalences and relative Fourier-Mukai transforms along \( p_A \) and \( p_B \) such that \( \rho(\Psi)(0, 0, 1) = v \).

We separate the argument into three cases:

1. Suppose that \( v = \pm(0, 0, 1) \). Then \( \Psi = id \) or \([1]\).

2. Suppose that \( v = (0, D, s) \). As \( \langle v, v \rangle = 0 \), \( D = aA \) or \( bB \) for \( a, b \in \mathbb{Z}, a, b \neq 0 \). Suppose that \( D = aA \). As there exists
\(v' = \varphi(1, 0, 0) = (r', (a'/n)A + b'B, s')\) such that \(\langle v, v' \rangle = 1\), we have

\[a(B \cdot A)b' - sr' = 1.\]

As \(\lambda_{p_B} = B \cdot A\), \(\gcd(a\lambda_{p_B}, s) = 1\). Therefore there exists a relative Fourier-Mukai transform, \(\Phi\), along \(p_B\) such that \(\rho(\Phi)\) sends \((0, 0, 1)\) to \(v = (0, aA, s)\). Then set \(\Psi = \Phi\). A similar argument for \(D = bB\) will work to construct a relative Fourier-Mukai transform along \(p_A\) which sends \((0, 0, 1)\) to \((0, bB, s)\).

(3) Suppose that \(v = (r, aA + bB, s)\) with \(r \neq 0\). We can assume that \(r > 0\) after applying \(\rho([1])\). Then \(r = nc\) with \(c \in \mathbb{N}\), as \(v \in \Delta\). As \(v^2 = 0\) we have

\[v = (nc, aA + bB, ab/c).\]

Note one of \(a, b\) is non-zero as otherwise \(v\) would be divisible.

Suppose \(a = 0\), so \(v = (nc, bB, 0)\). Then we can apply the relative Fourier-Mukai transform \(\Psi\) which sends

\[(nc, bB, 0) \mapsto (0, -cA, b)\]

and reduce to case (2).

Suppose that \(a \neq 0\). After tensoring by \(A\) we can assume \(a > 0\). Let \(\gcd(c, a) = d\) for some \(d \in \mathbb{N}\). We can write \(c = dc'\) and \(a = da'\) with \(\gcd(a', c') = 1\). Thus \(v\) has the form

\[v = (ndc', da' A + bB, a'b/c').\]

We have two operations given by \(\rho(- \otimes (-1/n)A)\) and \(\rho(\Psi^{-1}_B)\) which act on \(ndc'\) and \(da'\) in the following way:

\[
\rho(- \otimes (-1/n)A) : (ndc', da') \mapsto (ndc', d(a' - c'))
\]

\[
\rho(\Psi^{-1}_B) : (ndc', da') \mapsto (nd(c' - a'), da').
\]

This is just the Euclidean algorithm on \(c'\) and \(a'\). Thus we can reduce \(a'\) to 1 and \(c'\) to 0 and proceed as in (2).

Consider the autoequivalence \(\Psi^{-1} \circ \Phi\) whose image under \(\rho\) sends \((0, 0, 1)\) to \((0, 0, 1)\). So \(\Psi^{-1} \circ \Phi\) is a standard autoequivalence. Thus we can express \(\Phi\) as a composite of standard autoequivalences and relative Fourier-Mukai transforms. \(\square\)

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