Time-reparametrization-invariant dynamics of a relativistic string

B.M. Barbashov, V.N. Pervushin

Joint Institute for Nuclear Research, 141980, Dubna, Russia.

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Abstract

The time-reparametrization-invariant dynamics of a relativistic string is studied in the Dirac generalized Hamiltonian theory by resolving the first class constraints. The reparametrization-invariant evolution parameter is identified with the time-like coordinate of the "center of mass" of a string which is separated from local degrees of freedom by transformations conserving the group of diffeomorphisms of the generalized Hamiltonian formulation and the Poincare covariance of local constraints. To identify the "center of mass" time-like coordinate with the invariant proper time (measured by an observer in the comoving frame of reference), we apply the Levi-Civita - Shanmugadhasan canonical transformations which convert the global (mass-shell) constraint into a new momentum, so that the corresponding gauge is not needed for the Hamiltonian reduction.

The resolving of local constraints leads to an "equivalent unconstrained system" of the type of the Röhrlich string. Our classical Hamiltonian formalism naturally provides this approach to quantum theory of relativistic string.

1. Introduction

The group of diffeomorphisms of the Hamiltonian description of relativistic systems (particles, string, branes, general relativity) [1]-[11] contains the Abelian subgroup of the reparametrization of the coordinate time [9]. All known descriptions of a relativistic string [12, 13] are based on the reduction of the extended phase space by the fixation of gauge [4, 7] which breaks reparametrization - invariance from very beginning. The questions arise: Can one describe the time - reparametrization - invariant dynamics of a relativistic string dynamics directly in the terms of reparametrization - invariant variables, and what is a difference of this description from the gauge-fixing method?

To answer these questions, in the present paper, we apply a method of a reparametrization - invariant Hamiltonian description developed for gravitation [2, 3, 5, 14].

The method of a reparametrization - invariant description is based on the reduction of an action by the explicit resolving of the first class constraints. An important element of the invariant reduction is the Levi-Civita - Shanmugadhasan canonical transformation [15, 16] that linearizes the energy constraint as the generator of reparametrizations of the coordinate time.

The content of the paper is the following. In Section 2 we formulate the method of the invariant Hamiltonian reduction using the simplest examples of classical mechanics and relativistic particle. Section 3 is devoted to the generalized Hamiltonian approach to a relativistic string and the statement of the problem. In Section 4, local excitations are separated from the "center of mass" coordinates of the string. In Section 5, the Levi-Civita transformations and the invariant Hamiltonian reduction
are performed to resolve the global constraint and to convert the time-like variable of the global motion into the proper time. In Section 6, the classical and quantum dynamics of local excitations are described in terms of the proper time. Section 7 is devoted to the generating functional for the Green functions.

2. Invariant Hamiltonian Reduction

2.1. Mechanics

To illustrate the time-reparametrization-invariant Hamiltonian reduction \[3\] and its difference from the gauge-fixing method, let us consider an extended form of a classical-mechanical system

\[
W = \int_{\tau_1}^{\tau_2} d\tau \left( p\dot{q} - \Pi_0\dot{Q}_0 - \lambda[-\Pi_0 + H(p, q)] \right),
\]

that is invariant under reparametrizations of the coordinate evolution parameter \( \tau \) and "lapse" function \( \lambda \)

\[
\tau \rightarrow \tau' = \tau'(\tau), \quad \lambda \rightarrow \lambda' = \lambda \frac{d\tau'}{d\tau}.
\]

The problem of the classical description is to obtain the evolution of the physical variables of the world space \( q, Q_0 \) in terms of the geometric time \( T \) defined as

\[
dT := \lambda d\tau, \quad T = \int_0^\tau d\tau' \lambda(\tau').
\]

that is also invariant under reparametrizations (2).

The second problem (connected with quantization) is to present the effective action of the equivalent unconstrained theory directly in terms of \( T \), the equations of which reproduce this evolution. The solution of the second problem will be called the invariant Hamiltonian reduction.

The resolving of the first problem for the considered system is trivial, as the equations of motion of this system

\[
\dot{q} = \lambda \partial_p H, \quad \dot{p} = -\lambda \partial_q H, \quad \dot{Q}_0 = \lambda, \quad \dot{\Pi}_0 = 0
\]

are completely equivalent to the equations of the conventional unconstrained mechanics in the reduced phase space \( (p, q) \)

\[
W_{\text{reduced}} = \int_{T(\tau_1) = T_1}^{T(\tau_2) = T_2} dT \left( p \frac{dq}{dT} - H(p, q) \right).
\]

The problem is how to derive this system from the extended one (1) to apply the simplest Hamiltonian quantization with a clear physical interpretation of the invariant quantities.

The solution of the problem of the invariant Hamiltonian reduction considered in the present review is the explicit resolving of three equations of the extended system (1):

i) for the variable \( \lambda \) (treated as constraint)

\[
\frac{\delta W}{\delta \lambda} = -\Pi_0 + H(p, q) = 0,
\]
ii) for the momentum $\Pi_0$ with a negative contribution to the constraint (7)

$$\frac{\delta W}{\delta \Pi_0} = 0 \Rightarrow \frac{dQ_0}{d\tau} = \lambda ,$$

(8)

and iii) for its conjugate variable $Q_0$

$$\frac{\delta W}{\delta Q_0} = \frac{d\Pi_0}{d\tau} = 0 .$$

(9)

(We call these three equations (7) - (9) the geometric sector.)

The resolving of the constraint (7) expresses the "ignorable" momentum $\Pi_0$ through $H(p,q)$ with a positive value $\Pi_0 = H(p,q) > 0$. The second equation (8) identifies the dynamic evolution parameter $Q_0$ with the proper time $T$ $Q_0 = T$. It is not the gauge but the invariant solution of the equation of motion (8). The third equation (9) is the conservation law.

As a result of the invariant Hamiltonian reduction (i.e., a result of the substitution of $\Pi_0 = H$ and $Q_0 = T$ into the initial action (1)) this action is reduced to the one of the conventional mechanics (6) in terms of the proper time $T$ where the role of the nonzero Hamiltonian of evolution in the proper time $T$ is played by the constraint-shell value of the "ignorable" momentum $\Pi_0 = H(p,q)$. In other words, this constraint-shell action $W(\text{constraint}) = W^M$ determines the nonzero Hamiltonian $H(p,q)$ in the proper time $T$, instead of the zero generalized Hamiltonian in the coordinate time $\tau$ in $\lambda(-\Pi_0 + H)$.

Thus, the equivalent unconstrained system was constructed without any additional constraint of the type:

$$\lambda = 1, \quad \tau = T$$

(10)

which confuse quantities of the measurable sector with noninvariant ones. This confusion is contradiictable. The "gauge-fixing" identification of the coordinate evolution parameter $\tau$ and the geometric time $dT = \lambda d\tau$ in the form of the gauges (10) contradicts to the difference of their Hamiltonians $\lambda(-\Pi_0 + H) \neq H(p,q)$.

The second difference of the "gauge-fixing" from the invariant Hamiltonian reduction is more essential, namely, the formulation of the theory in terms of the invariant geometric time (3) is achieved by the explicit resolving of the constraint (7) and equation of motion (8), as a result of which "ignorable" variables $\Pi_0, Q_0$ are excluded from the phase space.

2.2. Special Relativity

Let us apply the invariant Hamiltonian reduction to relativistic particle.

To answer the question: Why is the reparametrization-invariant reduction needed?, let us consider relativistic mechanics in the Hamiltonian form (8)

$$W[P,X|N|\tau_1,\tau_2] = \int_{\tau_1}^{\tau_2} d\tau [-P_\mu \dot{X}^\mu - \frac{N}{2m}(-P_\mu^2 + m^2)] ,$$

(11)

which is classically equivalent to the conventional square root form

$$W[X|\tau_1,\tau_2] = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\dot{X}^{\mu} \dot{X}_\mu}$$

(12)

Both these action is invariant with respect to reparametrizations of the coordinate evolution parameter

$$\tau \rightarrow \tau' = \tau'(\tau), \quad N' d\tau' = N d\tau$$

(13)
given in the one-dimensional space with the invariant interval

\[ dT := Nd\tau, \quad T = \int_0^\tau d\tau N(\tau) \quad (14) \]

We called this invariant interval the geometric time \( N \) whereas the dynamic variable \( X_0 \) (with a negative contribution in the constraint) we called dynamic evolution parameter.

In terms of the geometric time \( (14) \) the classical equations of the generalized Hamiltonian system \( (11) \) takes the form

\[ \frac{dX_\mu}{dT} = \frac{P_\mu}{m}, \quad \frac{dP_\mu}{dT} = 0, \quad P_\mu^2 - m^2 = 0. \quad (15) \]

The classical problem is to find the evolution of the world space variables with respect to the geometric time \( T \).

The quantum problem is to obtain the equivalent unconstrained theories directly in terms of the invariant times \( X_0 \) or \( T \) with the invariant Hamiltonians of evolution. The solution of the second problem is called the dynamic (for \( X_0 \)), or geometric (for \( T \)) reparametrization-invariant Hamiltonian reductions.

The dynamic reduction of the action \( (11) \) means the substitution of the explicit resolving of the energy constraint \( (-P_\mu^2 + m^2) = 0 \) with respect to the momentum \( P_0 \) into this action

\[ \frac{\delta W}{\delta N} = 0 \quad \Rightarrow \quad P_0 = \pm \sqrt{m^2 + P_i^2}. \quad (16) \]

In accordance with two signs of the solution \( (16) \), after the substitution of \( (16) \) into \( (11) \), we have two branches of the dynamic unconstrained system

\[ W(\text{constraint})_\pm = \int_{X_0(\tau_1) = X_0(1)}^{X_0(\tau_2) = X_0(2)} dX_0 \left[ P_i \frac{dX_i}{dX_0} \mp \sqrt{m^2 + P_i^2} \right]. \quad (17) \]

The role of the time of evolution, in this action, is played by the variable \( X_0 \) that abandons the Dirac sector of ”observables” \( P_i, X_i \), but not the sector of ”measurable” quantities. At the same time, its conjugate momentum \( P_0 \) converts into the corresponding Hamiltonian of evolution, values of which are energies of a particle.

This invariant reduction of the action gives an ”equivalent” unconstrained system together with definition of the dynamic evolution parameter \( X_0 \) corresponding to a nonzero Hamiltonian \( P_0 \).

Thus, we need the reparametrization-invariant Hamiltonian reduction to determine the dynamic evolution parameter and its invariant Hamiltonian for a reparametrization-invariant system and to apply the symplest canonical quantization to it.

In quantum relativistic theory, we get two Schrödinger equations

\[ i \frac{d}{dX_0} \Psi(\pm)(X|P) = \pm \sqrt{m^2 + P_i^2} \Psi(\pm)(X|P), \quad (18) \]

with positive and negative values of \( P_0 \) and normalized wave functions

\[ \Psi(\pm)(X|P) = \frac{A_\pm^2 \theta(\pm P_0)}{(2\pi)^{3/2} \sqrt{2P_0}} \exp(-iP_\mu X^\mu), \quad ([A_P^-, A_P^+] = \delta^3(P_i - P_i')) \quad (19). \]
The continual limit of the multiple integral with the integral representation for parameter, which leads to the arrow of the geometric time (20) and to the causal Green function

In other words, instead of changing the sign of energy, we change that of the dynamic evolution with positive energy. The physical states are formed by action of these operators on the vacuum

The coefficient $A_i$, in the secondary quantization, is treated as the operator of creation of a particle with positive energy; and the coefficient $A_i$, as the operator of annihilation of a particle also with positive energy. The physical states are formed by action of these operators on the vacuum $0$ in the form of out-state ($|P> = A_P0$) with positive frequencies and in-state ($<0|P> = A_P0$) with negative frequencies. This treatment means that positive frequencies propagate forward ($X_{02} > X_{01}$); and negative frequencies, backward ($X_{01} > X_{02}$), so that the negative values of energy are excluded from the spectrum to provide the stability of the quantum system in QFT [17]. For this causal convention the geometric time (14) is always positive in accordance with the equations of motion (15)

$$\left(\frac{dT}{dX_0}\right)_\pm = \pm \frac{m}{\sqrt{P^2 + m^2}} \Rightarrow T(X_{02}, X_{01}) = \pm \frac{m}{\sqrt{P^2 + m^2}} (X_{02} - X_{01}) \geq 0 \quad (20)$$

The question appears: How to construct the path integral without gauges?

To obtain the reparametrization-invariant form of the functional integral adequate to the considered gauge-less reduction (17) and the causal Green function (21), we use the version of composition law for the commutative Green function with the integration over the whole measurable sector $X_{1\mu}$

$$G_+(X - X_0) = \int G_+(X - X_1)\tilde{G}_+(X_1 - X_0)dX_1 \quad \tilde{G}_+ = \frac{G_+}{2\pi\delta(0)} \
\quad (23)$$

where $\delta(0) = \int dN$ is the infinite volume of the group of reparametrizations of the coordinate $\tau$. Using the composition law $n$-times, we got the multiple integral

$$G_+(X - X_0) = \int G_+(X - X_1) \prod_{k=1}^n \tilde{G}_+(X_k - X_{k+1})dX_k \quad (X_{n+1} = X_0) \quad (24)$$

The continual limit of the multiple integral with the integral representation for $\delta$-function

$$\delta(P^2 - m^2) = \frac{1}{2\pi} \int dN \exp[iN(P^2 - m^2)]$$

can be defined as the path integral in the form of the average over the group of reparametrizations

$$G_+(X) = \int_{X(\tau_1) = 0}^{X(\tau_2) = X} \frac{dN(\tau_2)dP(\tau_2)}{(2\pi)^3} \prod_{\tau_1 \leq \tau < \tau_2} \left\{ d\tilde{N}(\tau) \prod_{\mu} \left( \frac{dP_\mu(\tau)dX_\mu(\tau)}{2\pi} \right) \right\} \exp(iW[P, X|N|\tau_1, \tau_2]), \quad \tilde{N} = N/2\pi\delta(0), \quad W \text{ is the initial extended action (1)}.$$
2.3. Geometric unconstrained system for a relativistic particle

The Hamiltonian of the unconstrained system in terms of the geometric time $T$ can be obtained by the canonical Levi-Civita - type transformation \[15, 16, 22\]

$$(P_{\mu}, X_{\mu}) \Rightarrow (\Pi_{\mu}, Q_{\mu})$$

(26)

to the variables $(\Pi_{\mu}, Q_{\mu})$ for which one of equations identifies $Q_0$ with the geometric time $T$. This transformation converts the constraint into a new momentum

$$\Pi_0 = \frac{1}{2m}[P_0^2 - P_i^2], \quad \Pi_i = P_i, \quad Q_0 = X_0 \frac{m}{P_0}, \quad Q_i = X_i - X_0 \frac{P_i}{P_0}$$

(27)

and has the inverted form

$$P_0 = \pm \sqrt{2m\Pi_0 + \Pi_i^2}, \quad P_i = \Pi_i, \quad X_0 = \pm Q_0 \frac{\sqrt{2m\Pi_0 + \Pi_i^2}}{m}, \quad X_i = Q_i + Q_0 \frac{\Pi_i}{m}.$$  

(28)

After transformation (27) the action (11) takes the form

$$W = \int_{\tau_1}^{\tau_2} d\tau \left[ -\Pi_{\mu} \dot{Q}^\mu - N(-\Pi_0 + \frac{m}{2}) - \frac{d}{d\tau}S^{lc} \right], \quad S^{lc} = (Q_0\Pi_0).$$

(29)

The invariant reduction is the resolving of the constraint $\Pi_0 = m/2$ which determines a new Hamiltonian of evolution with respect to the new dynamic evolution parameter $Q_0$, whereas the equation of motion for this momentum $\Pi_0$ identifies the dynamic evolution parameter $Q_0$ with the geometric time $T$ ($dQ_0 = dT$). The substitution of these solutions into the action (29) leads to the reduced action of a geometric unconstrained system

$$W(\text{constraint}) = \int_{\tau_1}^{\tau_2} dT \left( \Pi_i \frac{dQ_i}{dT} - \frac{m}{2} - \frac{d}{dT}(S^{lc}) \right), \quad S^{lc} = (Q_0\Pi_0).$$

(30)

where variables $\Pi_i, Q_i$ are cyclic ones and have the meaning of initial conditions in the comoving frame

$$\frac{\delta W}{\delta \Pi_i} = \frac{dQ_i}{d\tau} = 0 \Rightarrow Q_i = Q_i^{(0)}, \quad \frac{\delta W}{\delta Q_i} = \frac{d\Pi_i}{d\tau} = 0 \Rightarrow \Pi_i = \Pi_i^{(0)}.$$  

(31)

The substitution of all geometric solutions

$$Q_0 = T, \quad \Pi_0 = \frac{m}{2}, \quad \Pi_i = \Pi_i^{(0)} = P_i, \quad Q_i = Q_i^{(0)}$$

(32)

into the inverted Levi-Civita transformation (28) leads to the conventional relativistic solution for the dynamical system

$$P_0 = \pm \sqrt{m^2 + P_i^2}, \quad P_i = \Pi_i^{(0)}, \quad X_0(T) = T \frac{P_0}{m}, \quad X_i(T) = X_i^{(0)} + T \frac{P_i}{m}.$$  

(33)

The Schrödinger equation for the wave function

$$\frac{d}{dT}\Psi(T, Q_i|\Pi_i) = \frac{m}{2} \Psi(T, Q_i|\Pi_i), \quad \Psi(T, Q_i|\Pi_i) = \exp(-iT \frac{m}{2}) \exp(i\Pi_i^{(0)}Q_i)$$

(34)
contains only one eigenvalue $m/2$ degenerated with respect to the cyclic momentum $\Pi_i$. We see that there are differences between the dynamic and geometric descriptions. The dynamic evolution parameter is given in the whole region $-\infty < X_0 < +\infty$, whereas the geometric one is only positive $0 < T < +\infty$, as it follows from the properties of the causal Green function (21) after the Levi-Civita transformation (27)

$$G_c(Q_\mu) = \int_{-\infty}^{+\infty} d^4\Pi_\mu \frac{\exp(iQ_\mu^\alpha \Pi_\mu)}{2m(\Pi_0 - m/2 - i\epsilon/2m)} = \frac{\delta^3(Q)}{2m} \theta(T), \quad T = Q_0.$$ 

Two solutions of the constraint (a particle and antiparticle) in the dynamic system correspond to a single solution in the geometric system.

Thus, the reparametrization-invariant content of the equations of motion of a relativistic particle in terms of the geometric time is covered by two "equivalent" unconstrained systems: the dynamic and geometric. In both the systems, the invariant times are not the coordinate evolution parameter, but variables with the negative contribution into the energy constraint. The Hamiltonian description of a relativistic particle in terms of the geometric time can be achieved by the Levi-Civita-type canonical transformation, so that the energy constraint converts into a new momentum. Whereas, the dynamic unconstrained system is suit for the secondary quantization and the derivation of the causal Green function that determine the arrow of the geometric time.

3. Relativistic String

3.1. The generalized Hamiltonian formulation

We begin with the action for a relativistic string in the geometrical form [18]

$$W = -\gamma \int d^2u \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\mu, \quad u_\alpha = (u_0, u_1)$$

where the variables $x_\mu$ are string coordinates given in a space-time with a dimension $D$ and the metric $(x_\mu x^\mu := x_0^2 - x_i^2)$; $g_{\alpha\beta}$ is a second-rank metric tensor given in the two-dimensional Riemannian space $u_\alpha = (u_0, u_1)$.

To formulate the Hamiltonian approach, one needs to separate the two-dimensional Riemannian space $u_\alpha = (u_0, u_1)$ on the set of space-like lines $\tau = $ constant in the form of the Dirac-Arnovitt-Deser-Misner parametrization of the two-dimensional metric

$$g_{\alpha\beta} = \Omega^2 \left( \begin{array}{cc} \lambda_1^2 & \lambda_2^2 \\ \lambda_1 & -1 \end{array} \right), \quad \sqrt{-g} = \Omega^2 \lambda_1$$

with the invariant interval [1]

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta = \Omega^2[\lambda_1^2 d\tau^2 - (d\sigma + \lambda_2 d\tau)^2], \quad u_\alpha = (u_0 = \tau, u_1 = \sigma)$$

where $\lambda_1$ and $\lambda_2$ are known in general relativity (GR) as the lapse function and shift "vector", respectively [19, 20]. The action (35) after the substitution (37) does not depend on the conformal factor $\Omega$ and takes the form

$$W = -\gamma \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1}^{\sigma_2(\tau)} d\sigma \left[ (D_\tau x)^2 - \lambda_1 x^2 \right]$$

(38)
where

\[ D_\tau x_\mu = \dot{x}_\mu - \lambda_2 x'_\mu \quad (\dot{x} = \partial_\tau x, \ x' = \partial_\sigma x) \quad (39) \]

is the covariant derivative with respect to the two-dimensional metric (37). The metric (37), the action (38), and the covariant derivative (39) are invariant under the transformations (see Appendix A)

\[ \tau \Rightarrow \tilde{\tau} = f_1(\tau), \quad \sigma \Rightarrow \tilde{\sigma} = f_2(\tau, \sigma). \quad (40) \]

A similar group of transformations in GR is well-known as the "kinemetric" group of diffeomorphisms of the Hamiltonian description [9].

The variation of action (38) with respect to \( \lambda_1 \) and \( \lambda_2 \) leads to the equations

\[ \frac{\delta W}{\delta \lambda_2} = \frac{x'D_\tau x}{\lambda_1} = 0 \Rightarrow \lambda_2 = \frac{x'}{x^2}; \quad (41) \]

\[ \frac{\delta W}{\delta \lambda_1} = \frac{(D_\tau x)^2}{\lambda_1^2} + x'^2 = 0 \Rightarrow \lambda_1^2 = \frac{(\dot{x}x')^2 - \dot{x}^2x'^2}{(x')^2} \]

The solutions of these equations convert the action (38) into the standard Nambu-Gotto action of a relativistic string [13, 21]

\[ W = -\gamma \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2x'^2}. \]

The generalized Hamiltonian form [3] is obtained by the Legendre transformation [8] of the action (38)

\[ W = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma (-p_\mu D_\tau x^\mu + \lambda_1 \phi_1) = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma (\dot{x}^\mu x'_\mu + \lambda_1 \phi_1 + \lambda_2 \phi_2), \quad (42) \]

where

\[ \phi_1 = \frac{1}{2\gamma} [p_\mu^2 + (\gamma x'_\mu)^2], \quad \phi_2 = x'^\mu p_\mu, \quad (43) \]

and the generalized Hamiltonian

\[ \mathcal{H} = \lambda_1 \phi_1 + \lambda_2 \phi_2 \quad (44) \]

is treated as the generator of evolution with respect to the coordinate time \( \tau \), and \( \lambda_1, \lambda_2 \) play the role of variables with the zero momenta

\[ P_{\lambda_1} = 0, \quad P_{\lambda_2} = 0 \quad (45) \]

considered as the first class primary constraints [13, 18]. The equations for \( \lambda_1, \lambda_2 \)

\[ \frac{\delta W}{\delta \lambda_1} = \phi_1 = 0; \quad \frac{\delta W}{\delta \lambda_2} = \phi_2 = 0 \quad (46) \]

are known as the first class secondary constraints [3, 18, 18]. The Hamiltonian equations of motion take the form

\[ \frac{\delta W}{\delta x^\mu} = \dot{\phi}_\mu - \partial_\sigma [\gamma \lambda_1 x'_\mu + \lambda_2 p_\mu] = 0, \quad \frac{\delta W}{\delta p_\mu} = p_\mu + \gamma \frac{D_\tau x_\mu}{\lambda_1} = 0 \quad (47) \]

The problem is to find solutions of the Hamiltonian equations of motion (47) and constraints (46) which are invariant with respect to the kinematic transformations (40).
There is the problem of the solution of the linearized "gauge-fixing" equation in terms of the evolution parameter $\tau$ (as the object reparametrizations in the initial theory) being adequate to the initial kinemetic invariant and relativistic invariant system. In particular, the constraints mix the global motion of the "center of mass" coordinates with local excitations of a string $\xi_\mu$, which contradicts to the relativistic invariance of internal degrees of freedom of a string. In this context, it is worth to clear up a set of questions: Is it possible to introduce the reparametrization-invariant evolution parameter for the string dynamics, instead of the non-invariant coordinate time ($\tau$) used as the evolution parameter in the gauge-fixing method? Is it possible to construct the observable nonzero Hamiltonian of evolution of the "center of mass" coordinates? What is relation of the "center of mass" evolution to the unitary representations of the Poincare group?

4. The separation of the "center of mass" coordinates

To apply the reparametrization-invariant Hamiltonian reduction discussed before to a relativistic string, one should define the proper time in the form of the reparametrization-invariant functional of the lapse function (of type (14)), and to point out, among the variables, a dynamic evolution parameter, the equation of which identifies it with the proper time of type (8). As any extended object admits to define the coordinates of its center of mass, we identify this dynamic evolution parameter with the time-like coordinate of the center of mass of a string

$$X_\mu(\tau) = \frac{1}{l(\tau)} \int^{\sigma_2(\tau)}_{\sigma_1(\tau)} d\sigma x_\mu(\tau, \sigma), \quad l(\tau) = \sigma_2(\tau) - \sigma_1(\tau).$$ \hspace{1cm} (48)

We see that the invariant reduction requires to separate the "center of mass" variables before variation of the action. This separation is fulfilled by the substitution of

$$x_\mu(\tau, \sigma) = X_\mu(\tau) + \xi_\mu(\tau, \sigma)$$ \hspace{1cm} (49)

into the action (38), which takes the form

$$W = -\frac{\gamma}{2} \int^{\tau_2}_{\tau_1} d\tau \left\{ \dot{X}^2(\tau) + 2\dot{X}_\mu \int d\sigma \frac{D_\tau \xi_\mu}{\lambda_1}\frac{\sigma_2(\tau)}{\sigma_1(\tau)} + \int d\sigma \frac{(D_\tau \xi)^2}{\lambda_1} - \lambda_1 \xi'^2 \right\},$$ \hspace{1cm} (50)

where the global lapse function $N_0(\tau)$ is defined as the functional of $\lambda_1(\tau, \sigma)$

$$\frac{1}{N_0(\lambda_1)} = \frac{1}{l(\tau)} \int^{\sigma_2(\tau)}_{\sigma_1(\tau)} d\sigma \frac{1}{\lambda_1(\tau, \sigma)}.$$ \hspace{1cm} (51)

From definition (48) and equality (49) it follows that the local variables $\xi_\mu$ are given in the class of functions (with the nonzero Fourier harmonics) which satisfy the conditions

$$\int^{\sigma_2(\tau)}_{\sigma_1(\tau)} d\sigma \xi_\mu(\tau, \sigma) = 0.$$ \hspace{1cm} (52)

The formulation of the Hamiltonian approach (consistent with (48)) supposes the similar separation of the conjugate momenta $p_\mu$ defined by equation (47). If we substitute the definition (49) in these
equations, we get
\[ p_\mu(\tau, \sigma) = -\gamma \left( \frac{X_\mu(\tau)}{\lambda_1} + \frac{D_\tau \xi_\mu(\tau, \sigma)}{\lambda_1} \right). \] (53)

Defining the total momentum of a string \( P_\mu \)
\[ P_\mu = \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma p_\mu(\tau, \sigma) = -\gamma \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left( \frac{X_\mu(\tau)}{\lambda_1} + \frac{D_\tau \xi_\mu(\tau, \sigma)}{\lambda_1} \right), \] (54)
and taking into account (51) we obtain the following expression
\[ P_\mu = -\gamma \frac{X_\mu}{N_0(\tau)} - \gamma \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{D_\tau \xi_\mu(\tau, \sigma)}{\lambda_1}, \] (55)
therefore the equality
\[ \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{D_\tau \xi_\mu(\tau, \sigma)}{\lambda_1} = \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{\pi_\mu(\tau, \sigma)}{\lambda_1} = 0. \] (56)
should be valid. This separation conserves the group of diffeomorphisms of the Hamiltonian [3] and leads to the Bergmann-Dirac generalized action
\[ W = \int_{\tau_1}^{\tau_2} d\tau \left[ \left( \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left[ -\pi_\mu D_\tau \xi_\mu - \lambda_1 H \right] \right) - P_\mu X_\mu + N_0 P_\mu^2 \right], \] (57)
where \( H \) is the Hamiltonian of local excitations
\[ H = -\frac{1}{2\gamma} [\pi_\mu^2 + (\gamma \xi_\mu')^2]. \] (58)
The variation of the action (57) with respect to \( \lambda_1 \) results in the equation
\[ \frac{\delta W}{\delta \lambda_1} = H - \left( \frac{1}{l\lambda_1} \right)^2 \frac{P_\mu^2}{2\gamma} = 0, \] (59)
where
\[ \bar{\lambda}_1(\tau, \sigma) = \frac{\lambda_1(\tau, \sigma)}{N_0(\tau)} \] (60)
is the reparametrization-invariant component of the local lapse function. Here we have used the variation of the functional \( N_0[\lambda_1] \) (51)
\[ \frac{\delta N_0[\lambda_1]}{\delta \lambda_1} = \frac{1}{l(\tau)\lambda_1^2}. \]
In accordance with our separation of dynamic variables onto the global and local sectors, the first class constraint (54) has two projections onto the global sector (zero Fourier harmonic) and the local one. The global part of the constraint (59) can be obtained by variation of the action (57) with respect to \( N_0 \) (after the substitution of (60) into (57))
\[ \frac{\delta W}{\delta N_0} = \frac{P_\mu^2}{2\gamma} - H = 0, \] (61)
or, in another way, by the integration over $\sigma$ of (59) multiplied by $\lambda_1$. Then, the local part of the constraint (59) can be obtained by the substitution of (61) into (59)

$$\bar{\lambda}_1 \mathcal{H} - \frac{1}{l\lambda_1} \int d\sigma \bar{\lambda}_1 \mathcal{H} = 0.$$  \hspace{1cm} (62)

The integration of the local part over $\sigma$ is equal to zero if we take into account the normalization of the local lapse function

$$\frac{1}{l(\tau)} \int d\sigma \frac{1}{\lambda_1} = 1.$$ \hspace{1cm} (63)

This follows from the definition of the global lapse function (51).

Finally, we can represent the action (57) in the equivalent form

$$W = \int_{\tau_1}^{\tau_2} d\tau \left[ \left( \int d\sigma \left[ -\pi_\mu D_\tau \xi^\mu \right] \right) - P_\mu \dot{X}^\mu - N_0 (-\frac{P_\mu^2}{2\gamma} + H) \right],$$ \hspace{1cm} (64)

where the global lapse function $N_0$ and the local one $\bar{\lambda}_1$ are treated as independent variables, with taking the normalization (63) into account after the variation.

According to (40) and (51) the invariant proper time $T$ measured by the watch of an observer in the "center of mass" frame of a string is given by the expression

$$\sqrt{\gamma}dT := N_0 d\tau; \hspace{1cm} \sqrt{\gamma}T = \int_0^{\tau} d\tau' \left[ \frac{1}{l(\tau')} \int d\sigma \frac{1}{\lambda_1(\tau', \sigma)} \right]^{-1}.$$ \hspace{1cm} (65)

We include the constant $\sqrt{\gamma}$ to provide the dimension of the time measured by the watch of an observer.

Now we can see from (64) that the dynamics of the local degrees of freedom $\pi, \xi$, in the class of functions of nonzero harmonics (52), is described by the same kinematic invariant and relativistic covariant equations (47) where $x, p$ are changed by $\xi, \pi$, with the set of the first class (primary and secondary) constraints

$$P_{\lambda_1} = 0, \hspace{0.5cm} P_{\lambda_2} = 0, \hspace{0.5cm} \pi_\mu \xi'^\mu = 0, \hspace{0.5cm} \bar{\lambda}_1 \mathcal{H} - \frac{1}{l\lambda_1} \int d\sigma \bar{\lambda}_1 \mathcal{H} = 0.$$ \hspace{1cm} (66)

The separation of the "center of mass" (CM) variables on the level of the action removes the interference terms which mix the CM variables with the local degrees of freedom; as a result, the new local constraints (66) do not depend on the total momentum $P_\mu$, in contrast to the standard ones. In other words, there is the problem: when can one separate the CM coordinates of a relativistic string; before the variation of the action or after the variation of the action? The relativistic invariance dictates the first one, because an observer in the CM frame (which is the preferred frame for a string) cannot measure the total momentum of the string.

The first class local constraints (66) can be supplemented by the second class constraints

$$\bar{\lambda}_1 - 1 = 0, \hspace{0.5cm} \lambda_2 = 0, \hspace{0.5cm} n^\mu \xi_\mu = 0, \hspace{0.5cm} n^\mu \pi_\mu = 0 ,$$ \hspace{1cm} (67)
where \( n_\mu \) is an arbitrary time-like vector. In particular, for \((n_\mu = (1, 0, 0, 0))\) the equations of the local constraint-shell action

\[
W(\text{loc.constrs.}) = \int_{\tau_1}^{\tau_2} d\tau \left[ \left( \int_{\varphi_1(\tau)}^{\varphi_2(\tau)} d\varphi \pi_\mu \xi_\mu \right) - P_\mu \dot{X}^\mu - N_0 \left( -\frac{P_\mu^2}{2\gamma} + H \right) \right]
\]

(68)

coincide with the complete set of equations and the same constraints (66), (67) of the extended action, i.e., the operations of constraining and variation commute. The substitution of the global constraint (61) with \( \bar{\lambda}_1 = 1 \) into the action (68) leads to the constraint-shell action

\[
W_D^{\pm} = \int_{X_0(\tau_1)}^{X_0(\tau_2)} dX_0 \left[ \left( \int_{\varphi_1(X_0)}^{\varphi_2(X_0)} d\varphi \pi_\mu \frac{d\xi_\mu}{dX_0} \right) + P_i \frac{dX_i}{dX_0} \mp \sqrt{P_i^2 + 2\gamma H} \right].
\]

(69)

This action describes the dynamics of a relativistic string with respect to the time measured by an observer in the rest frame with the physical nonzero Hamiltonian of evolution. However, in this system, equations become nonlinear. To overcome this difficulty, we pass to the “center of mass” frame.

5. Levi-Civita geometrical reduction of a string

To express the dynamics of a relativistic string in terms of the proper time (65) measured by an observer in the comoving (i.e. "center of mass") frame, we use the Levi-Civita-type canonical transformations [15, 22] (as in Section 2.3)

\[
(P_\mu, X_\mu) \Rightarrow (\Pi_\mu, Q_\mu);
\]

they convert the global part of the constraint (61) into a new momentum \( \Pi_0 \)

\[
\Pi_0 = \frac{1}{2\gamma} \left[ P_0^2 - P_i^2 \right], \quad \Pi_i = P_i, \quad Q_0 = X_0 \frac{\tilde{\gamma}}{P_0}, \quad Q_i = X_i - X_0 \frac{P_i}{P_0}.
\]

(70)

The inverted form of these transformations is

\[
P_0 = \pm \sqrt{2\gamma \Pi_0 + \Pi_i^2}, \quad P_i = \Pi_i, \quad X_0 = \mp Q_0 \frac{\sqrt{2\gamma \Pi_0 + \Pi_i^2}}{\tilde{\gamma}}, \quad X_i = Q_i + Q_0 \frac{\Pi_i}{\tilde{\gamma}}.
\]

(71)

As a result of transformations (70), the extended action (64) in terms of the Levi-Civita geometrical variables takes the form (compare with (1))

\[
W = \int_{\tau_1}^{\tau_2} d\tau \left[ \left( \int_{\varphi_1(\tau)}^{\varphi_2(\tau)} d\varphi \pi_\mu \xi_\mu \right) - \Pi_\mu \dot{Q}_\mu - N_0 \left( -\Pi_0 + H \right) - \frac{d}{d\tau}(Q_0 \Pi_0) \right].
\]

(72)

The Hamiltonian reduction means to resolve constraint (61) with respect to the momentum \( \Pi_0 \)

\[
\frac{\delta W}{\delta N_0} = 0 \Rightarrow \Pi_0 = H.
\]

(73)

The equation of motion for the momentum \( \Pi_0 \)

\[
\frac{\delta W}{\delta \Pi_0} = 0 \Rightarrow \frac{dQ_0}{d\tau} = N_0 \quad (i.e., \frac{dQ_0}{d\tau} = N_0 d\tau := \sqrt{\gamma} dT)
\]

(74)
identifies (according to our definition (65)) the new variable \( Q_0 \) with the proper time \( T \), whereas the equation for \( Q_0 \)
\[
\frac{\delta W}{\delta Q_0} = 0 \Rightarrow \frac{d\Pi_0}{dT} = 0, \quad \text{i.e.,} \quad \frac{dH}{dT} = 0,
\]
(75)
in view of (73), gives us the conservation law.

Thus, resolving the global energy constraint \( \Pi_0 = H \), we obtain, from (72), the reduced action for a relativistic string in terms of the proper time \( T \)
\[
W^G = \int_{T_1}^{T_2} dT \left[ \left( \int_{\sigma_1}^{\sigma_2} d\sigma \left[ -\pi_\mu D_T \xi^\mu \right] \right) + \Pi_i \frac{dQ_i}{dT} - H - \frac{d}{dT}(TH) \right],
\]
(76)
where in analogy with (60) we introduced the factorized "shift-vector" \( \lambda_2 = N_0 \lambda_2 / \sqrt{\gamma} \); in this case the covariant derivative (39) takes the form
\[
D_T \xi_\mu = \partial_T \xi_\mu - \lambda_2 \xi'_\mu = \frac{D_T \xi_\mu N_0}{\sqrt{\gamma}}.
\]
(77)
The reduced system (76) has trivial solutions for the global variables \( \Pi_i, Q_i \)
\[
\frac{\delta W^R}{\delta \Pi_i} = 0 \Rightarrow \frac{dQ_i}{dT} = 0; \quad \Pi_i = \text{const};
\]
\[
\frac{\delta W^R}{\delta Q_i} = 0 \Rightarrow \frac{d\Pi_i}{dT} = 0, \quad \Pi_i = \text{const}
\]
(78)
which have the meaning of initial data.

If the solutions of equations (73), (74), and (78) for the system (76)
\[
\Pi_0 = H := \frac{M^2}{2\gamma}, \quad \Pi_i = P_i, \quad Q_0 = T \sqrt{\gamma}, \quad Q_i = X_i(0),
\]
(79)
are substituted into the inverted Levi-Civita canonical transformations (71)
\[
P_0 = \pm \sqrt{M^2 + P_i^2}, \quad X_0(T) = T \frac{P_0}{\sqrt{\gamma} \ell}, \quad X_i(T) = Q_i + T \frac{P_i}{\sqrt{\gamma} \ell},
\]
(80)
the initial extended action (64) can be described in the rest frame of an observer who measures the energy \( P_0 \) and the time \( X_0 \) and sees the rest frame evolution of the "center of mass" coordinates
\[
X_i(X_0) = Q_i + X_0 \frac{P_i}{P_0}.
\]
(81)
The Lorentz scheme of describing a relativistic system in terms of the time and energy \((X_0, P_0)\) in the phase space \( P_i, X_i, \pi_\mu, \xi_\mu \) is equivalent to the above-considered the Levi-Civita scheme in terms of the proper time and the evolution Hamiltonian \((T, H)\) in the phase space \( \Pi_i, Q_i, \pi_\mu, \xi_\mu \), where the variables \( \Pi_i, Q_i \) are cyclic.
6. Dynamics of the local variables

6.1. Reparametrization - invariant reduction for an open string

We restrict ourselves to an open string with the boundary conditions

$$\sigma_1(T) = 0, \quad \sigma_2(T) = \pi, \quad l(T) = \pi .$$  \hfill (82)

In the gauge-fixing method, by using the kinemetric transformation, we can put

$$\bar{\lambda}_1 = 1, \quad \bar{\lambda}_2 = 0 .$$  \hfill (83)

This requirement does not contradict the normalization of $\bar{\lambda}_1$ (63).

In view of (66), it means that the reduced Hamiltonian $H$ (61) coincides with its density (58)

$$\bar{\phi}_1 = H - \frac{1}{\pi} \int_0^\pi d\sigma H = 0, \quad \bar{\phi}_2 = \pi_{\mu} \xi'_{\mu} = 0$$  \hfill (84)

In this case, the reparametrization-invariant equations for the local variables obtained by varying the action (76)

$$\delta W_R^{\mu} = 0 \Rightarrow \partial_T \pi_{\mu} - \partial_\sigma (\bar{\lambda}_2 \pi_{\mu}) = \gamma \partial_\sigma (\bar{\lambda}_1 \xi'_{\mu}), \quad \delta W_R^{\mu} = 0 \Rightarrow \gamma D_T \xi_{\mu} = \bar{\lambda}_1 \pi_{\mu}$$  \hfill (85)

lead to the D’Alambert equations

$$\partial_T^2 \xi_{\mu} - \partial_\sigma^2 \xi_{\mu} = 0 .$$  \hfill (86)

The general solution of these equations of motion in the class of functions (52) with the boundary conditions (82) is given by the Fourier series

$$\xi_{\mu}(T, \sigma) = \frac{1}{2 \sqrt{\pi \gamma}} [\psi_{\mu}(z_+) + \psi_{\mu}(z_-)], \quad \psi_{\mu}(z) = i \sum_{n \neq 0} e^{(-inz)} \frac{\alpha_{n\mu}}{n}, \quad z_\pm = T \sqrt{\gamma} \pm \sigma .$$  \hfill (87)

$$\xi'_{\mu}(T, \sigma) = \frac{1}{2 \sqrt{\pi \gamma}} [\psi'_{\mu}(z_+) - \psi'_{\mu}(z_-)], \quad \pi_{\mu}(T, \sigma) = \frac{1}{2 \sqrt{\pi}} \gamma [\psi'_{\mu}(z_+) + \psi'_{\mu}(z_-)] .$$

The total coordinates $Q^{(0)}_{\mu}$ and momenta $P_{\mu}$ are determined by the reduced dynamics of the “center of mass” (78), (79), (80), and the string mass $M$ obtained from (61)

$$P_{\mu}^2 = M^2 = 2 \pi \gamma H = 2 \pi \gamma \int_0^\pi d\sigma H .$$  \hfill (88)

The substitution of $\xi_{\mu}$ and $\pi_{\mu}$ from (87) into (58) leads to the Hamiltonian density

$$H = -\frac{1}{4 \pi} \left[ \psi'^2_{\mu}(z_+) + \psi'^2_{\mu}(z_-) \right] ,$$

and from (88) we obtain, for the mass, the expression

$$M^2 = -2 \pi \gamma \bar{L}_0 = -\frac{\gamma}{2} \int_0^\pi d\sigma \left[ \left( \psi'_{\mu}(z_+) \right)^2 + \left( \psi'_{\mu}(z_-) \right)^2 \right] .$$  \hfill (89)
The second constraint (84) in terms of the vector $\psi'^{\mu}$ in (87) takes the form
\[ \xi'_{\mu} = \frac{1}{4\pi} \left[ \psi'^{2}_{\mu}(z_{+}) - \psi'^{2}_{\mu}(z_{-}) \right] = 0 \Rightarrow \psi'^{2}_{\mu}(z_{+}) = \psi'^{2}_{\mu}(z_{-}) = \text{const.} , \] (90)
and the first constraint (84) $\bar{\phi}_{1} = 0$ is satisfied identically. After the substitution of the constant value (90) into (89) we obtain that $\text{const.} = -M^{2}/\pi \gamma$; thus, finally the reparameterization-invariant constraint takes the form
\[ P^{2}_{\mu} + \pi \gamma \psi'^{2}_{\mu}(z_{\pm}) = 0 \quad (P^{2}_{\mu} = M^{2}) . \] (91)

Unlike this constraint, the gauge-fixing reparameterization-noninvariant constraint \[12, 13\]
\[ (P_{\mu} + \sqrt{\pi \gamma} \psi'^{2}_{\mu})^{2} = 0 \] (92)
contains the interference of the local and global degrees of freedom $\psi'^{2}_{\mu} P^{\mu}$. The latter violates the relativistic invariance of the local excitations which form the mass and spin of a string.

The constraint (91) in terms of the Fourier components (87) takes the form
\[ \psi'^{2}_{\mu}(z_{\pm}) = \sum_{k,m \neq 0} \alpha_{k\mu}^{m} e^{-i(k+m)z_{\pm}} = 2 \sum_{n = -\infty}^{\infty} \bar{L}_{n} e^{-inz_{\pm}} = -\frac{M^{2}}{\pi \gamma} , \] (93)
where $\bar{L}_{n}$ are the contributions of the nonzero harmonics
\[ \bar{L}_{0} = -\frac{1}{2} \sum_{k \neq 0} \alpha_{k\mu}^{k} , \quad \bar{L}_{n \neq 0} = -\frac{1}{2} \sum_{k \neq 0,n} \alpha_{k\mu}^{n-k} . \] (94)

From (93) one can see that the zero harmonic of this constraint determines the mass of a string
\[ M^{2} = -2\pi \gamma \bar{L}_{0} = -\pi \gamma \sum_{k \neq 0} \alpha_{k\mu} \alpha_{-k\mu} \] (95)
and coincides with the gauge-fixing value. However, the nonzero harmonics of constraint (93)
\[ \bar{L}_{n \neq 0} = -\frac{1}{2} \sum_{k \neq 0,n} \alpha_{k\mu}^{n-k} = 0 , \quad \bar{L}_{n} = \bar{L}_{n}^{*} \] (96)
(as we discussed above) strongly differ from the nonzero harmonics of the gauge-fixing constraints (92). The latter (in the contrast to (91)) contains the mixing the global motion of the center of mass $P_{\mu}$ with the local excitations $\psi_{\mu}$. It is clear that this mixing the global and local motions violates the the Poincare invariance of the local degrees of freedom.

The algebra of the local constraints (96) of the reparameterization-invariant dynamics of a relativistic string is not closed, as it does not contains the zero Fourier harmonic of the energy constraint (which has been resolved to express the dynamic equations in terms of the proper time).

The reparameterization-invariant dynamics of a relativistic string in the form of the first and second class constraints \[64, 65\] coincides with the Röhrlich approach to the string theory \[23\]. This approach is based on the choice of the gauge condition
\[ p_{\mu} \xi^{\mu} = 0, p_{\mu} \pi^{\mu} = 0 \quad \Rightarrow \quad G_{\mu} = P_{\mu} \alpha_{\mu}^{n} = 0 , \quad n \neq 0 \] ,
instead of (67). As consequence of this gauge the constraints (91), (92) became equivalent. In quantum theory, this condition is used for eliminating the states with negative norm in the "center of mass" (CM) frame (in our scheme, the CM frame appears as a result of the geometric Levi-Civita reduction). This reference frame is the only preferred frame for quantizing such a composite relativistic object as the string, as only in this frame one can quantize the initial data. This is a strong version of the principle of correspondence with classical theory: the classical initial data become the quantum numbers of quantum theory.
6.2. Quantum theory

Thus, our classical Hamiltonian reparametrization - invariant formalism provides the quantization of the string as in Röhrlich gauge.

The Röhrlich approach distinguishes two cases: \( M^2 = 0 \) and \( M^2 \neq 0 \).

The first case, in our scheme, the equality \( M^2 = 0 \) together with the local constraints \( (96) \) form the Virasoro algebra. The reparametrization-invariant version of the Virasoro algebra (with all its difficulties, including the \( D = 26 \) problem and the negative norm states) appears only in the case of the massless string \(-2\pi\gamma L_0 = M^2 = 0\).

In the second case \( M^2 \neq 0 \), the Röhrlich gauge \( \alpha_{n,0} = 0 \) allows us to exclude the time Fourier components \( \alpha_{n,0} \), and it is just these components that after quantization

\[ [\alpha_{n,\mu}, \alpha_{n,\nu}^+] = -m\eta_{\mu,\nu}\delta_{m,n}; \quad (n, m \neq 0, \quad \eta_{00} = -\eta_{ii} = 1) \]

lead to the states with negative norm because of the system being unstable. This means that the state vectors in the CM frame are constructed only by the action on vacuum of the spatial components of the operators \( a_{n_i}^+ = \alpha_{-n_i}/\sqrt{n}, n > 0 \) of

\[ [\Phi_{\nu}]_{CM} = \frac{\infty}{n=1} \frac{(a_{n_x}^+)_{\nu_{nx}}(a_{n_y}^+)_{\nu_{ny}}(a_{n_z}^+)_{\nu_{nz}}}{\sqrt{\nu_{nx}} \sqrt{\nu_{ny}} \sqrt{\nu_{nz}}} |0> , \quad (97) \]

where the three-dimensional vectors \( \nu_{n} = (\nu_{nx}, \nu_{ny}, \nu_{nz}) \) have only nonnegative integers as components. These state vectors automatically satisfy the constraint

\[ \alpha_{n0} [\Phi_{\nu}]_{CM} = 0, \quad n > 0 \] (98)

The physical states \( (97) \) are subjected to further constraints with \( n \geq 0 \)

\[ \vec{L}_n [\Phi_{\nu}]_{CM} = 0, \quad n > 0, \quad P^2 = M^2 = \pi\gamma <\Phi_{\nu} \sum_{m \neq 0} \alpha_{-m,i} a_{m,i} |\Phi_{\nu} > , \quad (99) \]

where \( \vec{L}_n \) can be represented in the normal ordering form

\[ \vec{L}_{n>0} = \sum_{k=1}^{\infty} a_{k,i}^+ a_{n+k,i} + \frac{1}{2} \sum_{k=1}^{n-1} \alpha_{k,i} a_{n-k,i}. \quad (100) \]

Constraints \( (98) \) and \( (99) \) are the first class constraints, in accordance with the Dirac classification because they form a closed algebra for \( n, m > 0 \)

\[ [G_n, G_m] = 0, \quad [\vec{L}_n, \vec{L}_m] = (n - m) \vec{L}_{n+m}, \quad [G_n, \vec{L}_m] = nG_{m+n} . \quad (101) \]

Therefore the conditions \( (98) \) eliminating the ghosts and the conditions \( (99) \) defining the physical vector states are consistent. Note that the commutator \( [\vec{L}_n, \vec{L}_m] \) does not contain a c-number since suffices \( n \geq 0 \) and \( m \geq 0 \) in the Virasoso operators \( \vec{L}_n \) do not lead to the central term.

On the operator level, equations determining the resolution of the constraints are fulfilled in a weak sense, as only the “annihilation” part of the constraints is imposed on the state vectors.

In quantum theory, one can introduce a complete set of eigen functions satisfying equations

\[ H[\pi_i, \xi_i] <\xi |\nu> = \frac{M^2}{2\pi\gamma} <\xi |\nu>, \quad (102) \]

where

\[ <\xi |\nu> = <\xi |\Phi_{\nu}>, \quad \sum_{\nu} <\xi_1 |\nu> <\nu |\xi_2> = \prod_\sigma \delta^3(\xi_1 - \xi_2). \]
7. The causal Green functions

Now we can construct the causal Green function for a relativistic string as the analogy of the causal Green function for a relativistic particle (23) - (25) discussed in Section 2.

The Veneziano-type causal Green function is the spectral series with the Hermite polynomials $\langle \xi | \nu \rangle$ over the physical state vectors $| \Phi \rangle = | \nu \rangle$

$$G_c(X|\xi_1,\xi_2) = G_+(X|\xi_1,\xi_2)\theta(X_0) + G_-(X|\xi_1,\xi_2)\theta(-X_0) =$$

$$i \int \frac{d^4P}{(2\pi)^4} \exp(-iPX) \sum_\nu \frac{\langle \xi_1 | \nu \rangle \langle \nu | \xi_2 \rangle}{P^2 - M^2_\nu - i\varepsilon}. \tag{103}$$

The commutative Green function for a relativistic string $G_+(X|\xi_2,\xi_1)$ can be represented in the form of the Faddeev-Popov functional integral [24] in the local gauge (67)

$$G_+(X|\xi_2,\xi_1) = \int_{X(\tau_1)=0}^{X(\tau_2)=X} dN_0(\tau_2) d^4P(\tau_2) \prod_{\tau_1 \leq \tau < \tau_2} \left\{ dN_0(\tau) \prod_\mu \left( \frac{dP_\mu(\tau)dX_\mu(\tau)}{2\pi} \right) \right\} F_+(\xi_2,\xi_1), \tag{104}$$

using the representation of the spectral series

$$F_+(\xi_2,\xi_1) = \sum_\nu \langle \xi_2 | \nu \rangle \exp \{ iW[P,X,N_0,M_\nu] \} \langle \nu | \xi_1 \rangle = \tag{105}$$

in the form of the functional integral

$$F_+(\xi_2,\xi_1) = \int_{\xi_1}^{\xi_2} D(\xi,\pi)\Delta_{fp} \exp \{ iW_{fp} \} , \tag{106}$$

$W[P,X,N_0,M_\nu]$ is the action (11) with the mass $M_\nu$

$$W_{fp} = \int_0^{\tau(X_0)} d\tau \left[ - \int_0^{\pi} d\sigma \pi_\mu \dot{\xi}_\mu - P_\mu \dot{X}^\mu - N_0 \left( - \frac{P^2}{2\pi\gamma} + H \right) \right], \tag{107}$$

is the constraint-shell action [28],

$$D(\xi,\pi) = \prod_{\tau,\sigma} \prod_\mu \frac{d\xi_\mu d\pi_\mu}{2\pi}, \tag{108}$$

and

$$\Delta_{fp} = \prod_{\tau,\sigma} \delta(\phi_1)\delta(\pi_0)\delta(\phi_2)\delta(\xi_0) \det B^{-1}, \quad \det B = \det \{ \phi_1, \phi_2, \pi_0, \xi_0 \} \tag{109}$$

is the FP determinant given in the monograph [7].

8. Conclusion

To describe the invariant dynamics of constrained relativistic string we used the universal method of the Hamiltonian reduction of their actions by resolving the energy constraint, so that one of variables of the extended phase space (with a negative contribution to the energy constraint) converts into the
invariant evolution parameter, and its conjugate momentum becomes the invariant Hamiltonian of evolution.

This method allows us to find integrals of motion by the Levi-Civita canonical transformations which converts the energy constraint into a new momentum, and the time-like variable of the world space into the proper time interval. For a particle and a string the Levi-Civita transformations are the Hamiltonian form of the Lorentz transformations which describe pure relativistic effects of the transition from the rest frame of reference to the comoving one.

We have shown that a relativistic string can be described directly in terms of the reparametrization-invariant parameter of evolution with the nonzero Hamiltonians of evolution in agreement with the equations of motion of the initial system.

A crucial point in our approach is the separation of the "center of mass" coordinates on the level of the action. The definition of the proper time with the nonzero Hamiltonian of evolution consistent with the group of diffeomorphisms of the Hamiltonian description requires to separate the "center of mass" coordinates before varying the action, whereas in the standard gauge-fixing method, the "center of mass" coordinates are separated after varying the action. The operations of separation of the "center of mass" coordinates and variation of the action do not commute. The relativistic invariance dictates the reparametrization-invariant way, as an observer in the comoving frame cannot measure the components of the total momentum of a string. Unique admissible gauge is the Röhrlich gauge that leads directly to the quantum theory of a string without a critical dimension.

Thus, we can formulate the novelty of this work: i) the separation of the "center of mass" coordinates on the level of the action, ii) finding of the integrals of motion by the Levi-Civita transformation, iii) deriving of the nonzero Hamiltonian of evolution of a string with respect to the proper time with the new algebra of the Poisson brackets, that provides the Röhrlich gauge, and iv) constructing of new reparametrization-invariant path integral representations of the causal Green functions for relativistic particle and string.

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**Appendix A: Kinemetric transformations**

The kinemetric transformations of the differentials

\[ \tilde{\tau} = \dot{f}_1(\tau)d\tau, \quad d\tilde{\sigma} = \dot{f}_2(\tau,\sigma)d\tau + f_2'(\tau,\sigma)d\sigma \]

correspond to transformations of the string coordinates

\[ x_\mu(\tau,\sigma) = \tilde{x}_\mu(\tilde{\tau},\tilde{\sigma}), \quad x'_\mu(\tau,\sigma) = \tilde{x}'_\mu(\tilde{\tau},\tilde{\sigma})f'_2(\tau,\sigma), \]

\[ \dot{x}_\mu(\tau,\sigma) = \dot{x}_\mu(\tilde{\tau},\tilde{\sigma})\dot{f}_1(\tau) + \tilde{x}'_\mu(\tilde{\tau},\tilde{\sigma})\dot{f}_2(\tau,\sigma), \]

From these equations, we can derive the transformation law for \(\lambda_1, \lambda_2\) taking into account (11)

\[ \lambda_1(\tau,\sigma) = \frac{\sqrt{(\tilde{x}'_\tau)^2 - \tilde{x}'_\tau^2}}{x'^2(\tau,\sigma)} = \frac{\sqrt{(\tilde{x}'_\tau)^2 - \tilde{x}'_\tau^2}}{\tilde{x}'_\tau^2(\tilde{\tau},\tilde{\sigma})} \frac{\dot{f}_1}{f'_2} = \tilde{\lambda}_1 \frac{\dot{f}_1(\tau)}{f'_2(\tau,\sigma)}. \]

\[ \lambda_2(\tau,\sigma) = \frac{\dot{x}'_\tau}{x'^2} = \frac{(\tilde{x}'\tilde{\tau})\dot{f}_1 f'_2 + \tilde{x}'_\tau \dot{f}_2 f'_2}{\tilde{x}'_\tau^2 f'^2_2} = \tilde{\lambda}_2 \frac{\dot{f}_1}{f'_2} + \frac{\dot{f}_2}{f'_2}. \]
The kinemetric-invariance of the interval (37) with respect to (40) follows from these transformation laws and the transformation of the conformal factor

$$ \Omega(\tau, \sigma) = f'_2(\tau, \sigma)\tilde{\Omega}(\tilde{\tau}, \tilde{\sigma}) $$

The covariant derivative (39) is transformed under (40) as

$$ D_\tau x_\mu = \dot{x}_\mu - \lambda_2 \dot{x}'_\mu = \dot{f}_1(\tau) \left[ \dot{x}_\mu - \tilde{\lambda}_2 \dot{x}'_\mu \right] = \dot{f}_1(\tau)D_{\tilde{\tau}}\tilde{x}_\mu. $$

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