Equivariant quantization of Poisson homogeneous spaces and Kostant’s problem

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Abstract

We find a partial solution to the longstanding problem of Kostant concerning description of the so-called locally-finite endomorphisms of highest weight irreducible modules. The solution is obtained by means of its reduction to a far-reaching extension of the quantization problem. While the classical quantization problem consists in finding \( \star \)-product deformations of the commutative algebras of functions, we consider the case when the initial object is already a noncommutative algebra, the algebra of functions within \( q \)-calculus.

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1 Introduction

Let \( \tilde{U}_q \mathfrak{g} \) be the quantized universal enveloping algebra “of simply connected type” that corresponds to a finite dimensional split semisimple Lie algebra \( \mathfrak{g} \). Let \( L(\lambda) \) be the irreducible highest weight \( \tilde{U}_q \mathfrak{g} \)-module of highest weight \( \lambda \). The aim of this paper is to show that for certain values of \( \lambda \), the action map \( \tilde{U}_q \mathfrak{g} \to (\text{End} \ L(\lambda))_{\text{fin}} \) is surjective. Here \( (\text{End} \ L(\lambda))_{\text{fin}} \) stands for the locally finite part of \( \text{End} \ L(\lambda) \) with respect to the adjoint action of \( \tilde{U}_q \mathfrak{g} \). For the Lie-algebraic case \( (q = 1) \), this problem is known as the classical Kostant’s problem, see [3, 4, 10, 11]. The complete answer to it is still unknown even in the \( q = 1 \) case. However, there are examples of \( \lambda \) for which the action map \( U(\mathfrak{g}) \to (\text{End} \ L(\lambda))_{\text{fin}} \) is not surjective. Such examples exist even in the case \( \mathfrak{g} \) is of type \( A \) [12].

The main idea of our approach to Kostant’s problem, both in the Lie-algebraic and quantum group cases, is that \( (\text{End} \ L(\lambda))_{\text{fin}} \) has two other presentations. First, it follows from the results of [8] that \( (\text{End} \ L(\lambda))_{\text{fin}} \) is canonically isomorphic to \( \text{Hom}_U(L(\lambda), L(\lambda) \otimes F) \), where \( U \) is \( U(\mathfrak{g}) \) (resp. \( \tilde{U}_q \mathfrak{g} \)), and \( F \) is the algebra of (quantized) regular functions on the connected simply connected algebraic
group $G$ corresponding to the Lie algebra $\mathfrak{g}$. In other words, $F$ is spanned by matrix elements of finite dimensional representations of $U$ with an appropriate multiplication.

One more presentation of the algebra $(\text{End} \ L(\lambda))_{\text{fin}}$ comes from the fact that $\text{Hom}_U\left(L(\lambda), L(\lambda) \otimes F\right)$ is isomorphic as a vector space to a certain subspace $F'$ of $F$. The subspace $F'$ can be equipped with a $\ast$-multiplication obtained from the multiplication on $F$ by applying the so-called reduced fusion element. Then $(\text{End} \ L(\lambda))_{\text{fin}}$ is isomorphic as an algebra to $F'$ with this new multiplication. For certain values of $\lambda$, the same $\ast$-multiplication on $F'$ can be defined by applying the universal fusion element, that yields the affirmative answer to Kostant’s problem in such cases.

More exactly, consider the triangular decomposition $U = U^- U^0 U^+$. We have $L(\lambda) = M(\lambda)/K_\lambda \mathbf{1}_\lambda$, where $M(\lambda)$ is the corresponding Verma module, $\mathbf{1}_\lambda$ is the generator of $M(\lambda)$, and $K_\lambda \subset U^-$. Consider also the opposite Verma module $\tilde{M}(\lambda)$ with the lowest weight $-\lambda$ and the lowest weight vector $\tilde{1}_{-\lambda}$. Then its maximal $U$-submodule is of the form $\tilde{K}_\lambda \cdot \tilde{1}_{-\lambda}$, where $\tilde{K}_\lambda \subset U^+$. We have $F' = F[0]^{K_\lambda + \tilde{K}_\lambda}$ — the subspace of $U^0$-invariant elements of $F$ annihilated by both $K_\lambda$ and $\tilde{K}_\lambda$. The $\ast$-product on $F[0]^{K_\lambda + \tilde{K}_\lambda}$ has the form

$$f_1 \ast_\lambda f_2 = \mu \left(J^{\text{red}}(\lambda)(f_1 \otimes f_2)\right),$$

where $\mu$ is the multiplication on $F$, and the reduced fusion element $J^{\text{red}}(\lambda) \in U^- \hat{\otimes} U^+$ is computed in terms of the Shapovalov form on $L(\lambda)$. Notice that for generic $\lambda$ the element $J^{\text{red}}(\lambda)$ is equal up to an $U^0$-part to the fusion element related to the Verma module $M(\lambda)$, see for example [1].

We also investigate limiting properties of $J(\lambda)$. In particular, for some values of $\lambda_0$ we can guarantee that $f_1 \ast_\lambda f_2 \to f_1 \ast_{\lambda_0} f_2$ as $\lambda \to \lambda_0$. Also, for any $\lambda_0$ having a “regularity property” of this kind, the action map $U \to (\text{End} \ L(\lambda_0))_{\text{fin}}$ is surjective. This gives the affirmative answer to the (quantum version of) Kostant’s problem.

For some values of $\lambda$, the subspace $F[0]^{K_\lambda + \tilde{K}_\lambda}$ is a subalgebra of $F[0]$, and can be considered as (a flat deformation of) the algebra of regular functions on some Poisson homogeneous space $G/G_1$. In those cases, the algebra $(F[0]^{K_\lambda + \tilde{K}_\lambda}, \ast)$ is an equivariant quantization of the Poisson algebra of regular functions on $G/G_1$.

This paper is organized as follows. In Section 2 we recall the definition of the version of quantized universal enveloping algebra used in this paper, and some related constructions that will be useful in the sequel. Section 3 is the core of the paper. We provide there a construction of a star-product on $F[0]^{K_\lambda + \tilde{K}_\lambda}$ in terms of the Shapovalov form on $L(\lambda)$. Finally, in Section 4 we study limiting properties of fusion elements and the corresponding star-products.
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2 Algebra $\tilde{U}_q\mathfrak{g}$

Let $\mathbb{k}$ be the field extension of $\mathbb{C}(q)$ by all fractional powers $q^{1/n}$, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. We use $\mathbb{k}$ as the ground field.

Let $(a_{ij})$ a finite type $r \times r$ Cartan matrix. Let $d_i$ be relatively prime positive integers such that $d_ia_{ij} = d_ja_{ji}$. For any positive integer $k$, define

$$[k]_i = \frac{q^{kd_i} - q^{-kd_i}}{q^{d_i} - q^{-d_i}} , \quad [k]_i! = [1]_i [2]_i \ldots [k]_i.$$ 

The algebra $U = \tilde{U}_q\mathfrak{g}$ is generated by the elements $t_i, t_i^{-1}, e_i, f_i$, $i = 1, \ldots, r$, subject to the relations

$$t_it_i^{-1} = t_i^{-1}t_i = 1 ,$$
$$t_ie_it_i^{-1} = q^{d_i\delta_{ij}}e_j ,$$
$$t_if_it_i^{-1} = q^{-d_i\delta_{ij}}f_j ,$$
$$e_if_j - fjej = \delta_{ij} \frac{k_i - k_i^{-1}}{q^{d_i} - q^{-d_i}} , \text{ where } k_i = \prod_{j=1}^r t_{ij}^{a_{ij}} ,$$

$$\sum_{m=0}^{1-a_{ij}} \frac{(-1)^m}{[m]_i! [1 - a_{ij} - m]_j} e_i^m e_j e_i^{1-a_{ij}-k} = 0 \text{ for } i \neq j ,$$
$$\sum_{m=0}^{1-a_{ij}} \frac{(-1)^m}{[m]_i! [1 - a_{ij} - m]_j} f_i^m f_j f_i^{1-a_{ij}-k} = 0 \text{ for } i \neq j .$$

Notice that $k_ie_jk_i^{-1} = q^{d_{aij}}e_j$, $k_if_jk_i^{-1} = q^{-d_{aij}}f_j$.

The algebra $U$ is a Hopf algebra with the comultiplication $\Delta$, the counit $\varepsilon$, and the antipode $\sigma$ given by

$$\Delta(t_i) = t_i \otimes t_i , \quad \varepsilon(t_i) = 1 , \quad \sigma(t_i) = t_i^{-1} ,$$
$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i , \quad \varepsilon(e_i) = 0 , \quad \sigma(e_i) = -k_i^{-1}e_i ,$$
$$\Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i , \quad \varepsilon(f_i) = 0 , \quad \sigma(f_i) = -f_i k_i .$$

In what follows we will sometimes use the Sweedler notation for comultiplication.

Let $U^0$ be the subalgebra of $U$ generated by the elements $t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}$. Let $U^+$ and $U^-$ be the subalgebras generated respectively by the elements $e_1, \ldots, e_r$ and $f_1, \ldots, f_r$. We have a triangular decomposition $U = U^- U^0 U^+$. 

3
Denote by \( \theta \) the involutive automorphism of \( U \) given by \( \theta(e_i) = -f_i, \theta(f_i) = -e_i \), \( \theta(t_i) = t_i^{-1} \). Notice that \( \theta \) gives an algebra isomorphism \( U^- \rightarrow U^+ \). Set \( \omega = \sigma \theta \), i.e., \( \omega \) is the involutive antiautomorphism of \( U \) given by \( \omega(e_i) = f_i k_i, \omega(f_i) = k_i^{-1} e_i, \omega(t_i) = t_i \).

Let \( (\mathfrak{h}, \Pi, \Pi^\vee) \) be a realization of \( (a_{ij}) \) over \( \mathbb{Q} \), that is, \( \mathfrak{h} \) is a (rational form of) a Cartan subalgebra of the corresponding semisimple Lie algebra, \( \Pi = \{ \alpha_1, \ldots, \alpha_r \} \subset \mathfrak{h}^* \) the set of simple roots, \( \Pi^\vee = \{ \alpha_1^\vee, \ldots, \alpha_r^\vee \} \subset \mathfrak{h} \) the set of simple coroots. Let \( R \) be the root system, \( R_+ \) the set of positive roots, and \( W \) the Weyl group. Denote by \( s_\alpha \in W \) the reflection corresponding to a root \( \alpha \). For \( w \in W \), \( \lambda \in \mathfrak{h}^* \) we set \( w \cdot \lambda = w(\lambda + \rho) - \rho \). Let \( u_1, \ldots, u_l \in \mathfrak{h} \) be the simple coweights, i.e., \( \langle \alpha_i, u_j \rangle = \delta_{ij} \). We denote by \( \rho \) the half sum of the positive roots. For \( w \in W \) and \( \lambda \in \mathfrak{h}^* \), we set \( w \cdot \lambda = w(\lambda + \rho) - \rho \).

Let
\[
Q_+ = \sum_{\alpha \in \Pi} \mathbb{Z}_+ \alpha.
\]
For \( \beta = \sum_j c_j \alpha_j \in Q_+ \), denote \( \text{ht} \beta = \sum_j c_j \in \mathbb{Z}_+ \). For \( \lambda, \mu \in \mathfrak{h}^* \) we set \( \lambda \geq \mu \) iff \( \lambda - \mu \in Q_+ \).

Take an invariant scalar product \( (\cdot | \cdot) \) on \( \mathfrak{h}^* \) such that \( (\alpha | \alpha) = 2 \) for any short root \( \alpha \). Then \( d_i = \frac{(\alpha_i | \alpha_i)}{2} \).

Denote by \( T \) the multiplicative subgroup generated by \( t_1, \ldots, t_r \). Any \( \lambda \in \mathfrak{h}^* \) defines a character \( \Lambda : T \rightarrow \mathbb{K} \) given by \( t_i \mapsto q^{d_i(\lambda, u_i)} \). We will write \( \Lambda = q^\lambda \).

Notice that \( q^{\lambda}(k_i) = q^{d_i(\lambda, \alpha_i^\vee)} \). We extend \( q^\lambda \) to the subalgebra \( U^0 \) by linearity.

We say that an element \( x \in U \) is of weight \( \lambda \) if \( ttx^{-1} = q^\lambda(t)x \) for all \( t \in T \).

For a \( U \)-module \( V \), we denote by
\[
V[\lambda] = \{ v \in V \mid tv = q^\lambda(t)v \text{ for all } t \in T \}
\]
the weight subspace of weight \( \lambda \). We call the module \( V \) admissible if \( V \) is a direct sum of finite-dimensional weight subspaces \( V[\lambda] \).

The Verma module \( M(\lambda) \) over \( U \) with highest weight \( \lambda \) and highest weight vector \( 1_\lambda \) is defined in the standard way:
\[
M(\lambda) = U^- 1_\lambda, \quad U^+ 1_\lambda = 0, \quad t 1_\lambda = q^\lambda(t) 1_\lambda, \quad t \in T.
\]
The map \( U^- \rightarrow M(\lambda), y \mapsto y 1_\lambda \) is an isomorphism of \( U^- \)-modules.

Set \( U_+^\pm = \text{Ker} \varepsilon|_{U^\pm} \) and denote by \( x \mapsto (x)_0 \) the projection \( U \rightarrow U^0 \) along \( U^- \cdot U + U \cdot U_+^\pm \). For any \( \lambda \in \mathfrak{h}^* \) consider \( \pi_\lambda : U^+ \otimes U^- \rightarrow \mathbb{K}, \pi_\lambda(x \otimes y) = q^\lambda((\sigma(x)y)_0), \) and \( S_\lambda : U^- \otimes U^- \rightarrow \mathbb{K}, S_\lambda(x \otimes y) = \pi_\lambda(\theta(x) \otimes y) = q^\lambda((\omega(x)y)_0) \). We call \( S_\lambda \) the Shapovalov form on \( U^- \) corresponding to \( \lambda \). We can regard \( S_\lambda \) as a bilinear form on \( M(\lambda) \).

Set
\[
K_\lambda = \{ y \in U^- \mid \pi_\lambda(x \otimes y) = 0 \text{ for all } x \in U^+ \},
\]
\[ \tilde{K}_\lambda = \{ x \in U^+ | \pi_\lambda(x \otimes y) = 0 \text{ for all } y \in U^- \}. \]

Clearly, \( K_\lambda \) is the kernel of \( S_\lambda \), \( \tilde{K}_\lambda = \theta(K_\lambda) \). Notice that \( K(\lambda) = K_\lambda \cdot 1_\lambda \) is the largest proper submodule of \( M(\lambda) \), and \( L(\lambda) = M(\lambda)/K(\lambda) \) is the irreducible \( U \)-module with highest weight \( \lambda \). Denote by \( \overline{T}_\lambda \) the image of \( 1_\lambda \) in \( L(\lambda) \).

The following propositions are well known for the Lie-algebraic case. They also hold for the case of \( U = \hat{U}_q \mathfrak{g} \). Proposition 1 follows from a simple \( U_q \mathfrak{sl}(2) \) computation. For Propositions 2, 3, see [5, 9].

**Proposition 1.** Assume that \( \lambda \in \mathfrak{h}^* \) satisfies \( \langle \lambda + \rho, \alpha_i^\vee \rangle = n \in \mathbb{N} \) for a simple root \( \alpha_i \). Then \( f_i^n \) is in \( K_\lambda \).

**Proposition 2.** Let \( \lambda \in \mathfrak{h}^* \) be dominant integral. Then the \( U \)-module \( L(\lambda) \) is finite dimensional, and \( \dim(L(\lambda)) = \dim(L(\lambda))[w\mu] \) for any \( \mu \in \mathfrak{h}^* \) and \( w \in W \).

**Proposition 3.** Assume that \( \lambda \in \mathfrak{h}^* \) satisfies \( \langle \lambda + \rho, \alpha_i^\vee \rangle = n \in \mathbb{N} \) for some \( \alpha \in \mathbb{R}_+ \) and \( \langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{N} \) for all \( \beta \in \mathbb{R}_+ \setminus \{ \alpha \} \). Then \( K_\lambda \) is generated by a single element of weight \( -n\alpha \).

**Proposition 4.** Let \( \lambda \in \mathfrak{h}^* \) be dominant integral, \( \langle \lambda + \rho, \alpha_i^\vee \rangle = n_i \), \( i = 1, \ldots, r \). Then \( K_\lambda \) is generated by the elements \( f_i^{n_i} \), \( i = 1, \ldots, r \).

In the sequel we need some properties of the universal \( R \)-matrix of \( U \). Namely, let \( V_1 \), \( V_2 \) be \( U \)-modules such that \( V_1 \) is a direct sum of highest weight modules or \( V_2 \) is a direct sum of lowest weight modules. Then the \( R \)-matrix induces an isomorphism \( \tilde{R} : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1 \) of \( U \)-modules. Moreover, if \( V_1 \) is a highest weight module with highest weight \( \lambda \) and highest weight vector \( 1_\lambda \), and \( V_2 \) is a lowest weight module with lowest weight \( \mu \) and lowest weight vector \( 1_\mu \), then
\[ \tilde{R}(1_\lambda \otimes \overline{1}_\mu) = q^{-\langle \lambda + \rho, \alpha_i^\vee \rangle} \overline{1}_\mu \otimes 1_\lambda. \]

Let \( F = \mathbb{k}[G]_q \) be the quantized algebra of regular functions on a connected simply connected algebraic group \( G \) that corresponds to the Cartan matrix \( (a_{ij}) \) (see [3, 6]). We can consider \( F \) as a Hopf subalgebra in the dual Hopf algebra \( U^* \). We will use the left and right regular actions of \( U \) on \( F \) defined respectively by the formulae \( (a f)(x) = f(xa) \) and \( (f a')(x) = f(ax) \). Notice that \( F \) is a sum of finite-dimensional admissible \( U \)-modules with respect to both regular actions of \( U \) (see [9]).

### 3 Star products and fusion elements

#### 3.1 Algebra of intertwining operators

Let us denote by \( U_{\text{fin}} \subset U \) the subalgebra of locally finite elements with respect to the right adjoint action of \( U \) on itself. We will use similar notation for any (right) \( U \)-module.
For any (left) $U$-module $M$ we equip $F$ with the left regular $U$-action and consider the space $\text{Hom}_{U}(M, M \otimes F)$. For any $\varphi, \psi \in \text{Hom}_{U}(M, M \otimes F)$ define

$$\varphi \ast \psi = (\text{id} \otimes \mu) \circ (\varphi \otimes \text{id}) \circ \psi,$$

where $\mu$ is the multiplication in $F$. We have $\varphi \ast \psi \in \text{Hom}_{U}(M, M \otimes F)$, and this definition equips $\text{Hom}_{U}(M, M \otimes F)$ with a unital associative algebra structure.

Consider the map $\Phi : \text{Hom}_{U}(M, M \otimes F) \to \text{End}_{U}M$, $\varphi \mapsto u_{\varphi}$, defined by $u_{\varphi}(m) = (\text{id} \otimes \varepsilon)(\varphi(m))$; here $\varepsilon(f) = f(1)$ is the counit in $F$. Consider $U_{\text{fin}}$, \text{Hom}_{U}(M, M \otimes F)$ and $\text{End}_{U}M$ as right $U$-module algebras: $U_{\text{fin}}$ via right adjoint action, $\text{Hom}_{U}(M, M \otimes F)$ via right regular action on $F$ (i.e., $(\varphi \cdot a)(m) = (\text{id} \otimes \varepsilon)(\varphi(m))$), and $\text{End}_{U}M$ in a standard way (i.e., $u \cdot a = \sum_{(a)} \sigma(a_{1})Mua(2)_{M}$).

Then $\text{Hom}_{U}(M, M \otimes F)_{\text{fin}} = \text{Hom}_{U}(M, M \otimes F)$, and $\Phi : \text{Hom}_{U}(M, M \otimes F) \to (\text{End}_{U}M)_{\text{fin}}$ is an isomorphism of right $U$-module algebras (see [8, Proposition 6]).

Now we apply this to $M = M(\lambda)$ and $M = L(\lambda)$. Since $U_{\text{fin}} \to (\text{End}_{U}M(\lambda))_{\text{fin}}$ is surjective (see [3, 6]), we have the following commutative diagram

$$\begin{array}{ccc}
\text{Hom}_{U}(M(\lambda), M(\lambda) \otimes F) & \longrightarrow & \text{Hom}_{U}(L(\lambda), L(\lambda) \otimes F) \\
\Phi_{M(\lambda)} & & \Phi_{L(\lambda)} \\
(\text{End } M(\lambda))_{\text{fin}} & \longrightarrow & (\text{End } L(\lambda))_{\text{fin}}
\end{array}$$

(see [8, Proposition 9]).

For any $\varphi \in \text{Hom}_{U}(L(\lambda), L(\lambda) \otimes F)$ the formula $\varphi(T_{\lambda}) = T_{\lambda} \otimes f_{\varphi} + \sum_{\mu < \lambda} v_{\mu} \otimes f_{\mu}$, where $v_{\mu}$ is of weight $\mu$, defines a map $\Theta : \text{Hom}_{U}(L(\lambda), L(\lambda) \otimes F) \to F[0], \varphi \mapsto f_{\varphi}$.

**Theorem 5.** $\Theta$ is an embedding, and its image equals $F[0]^{K_{\lambda} + \widetilde{K}_{\lambda}}$.

To prove Theorem 5 we need some preparations.

In the sequel $V$ stands for an $U$-module which is a direct sum of finite dimensional admissible $U$-modules.

For an admissible $U$-module $M$ we will denote by $M^{*}$ its restricted dual.

Let $\widetilde{M}(\lambda)$ be the “opposite Verma module” with the lowest weight $\lambda \in \mathfrak{h}^{*}$ and the lowest weight vector $\widetilde{1}_{\lambda}$. It is clear that $\widetilde{K}_{-\lambda} \cdot \widetilde{1}_{\lambda}$ is the largest proper submodule in $\widetilde{M}(\lambda)$.

**Lemma 6.** $\text{Hom}_{U^{-}}(M(\lambda), V) = (V \otimes M(\lambda))^{U^{-}}, \text{Hom}_{U^{+}}(\widetilde{M}(\lambda), V) = (V \otimes \widetilde{M}(\lambda)^{*})^{U^{+}}$.

**Proof.** For any $\varphi \in \text{Hom}_{U^{-}}(M(\lambda), V)$ the image of $\varphi$ is equal to the finite-dimensional $U^{-}$-submodule $U^{-}\varphi(1_{\lambda})$. Therefore for any $x \in U^{-}$ such that $x1_{\lambda}$ is a weight vector whose weight is large enough we have $\varphi(x1_{\lambda}) = x\varphi(1_{\lambda}) = 0$. Thus $\varphi$ corresponds to an element in $(V \otimes M(\lambda)^{*})^{U^{-}}$.

The second part of the lemma can be proved similarly. □
Choose vectors $1^*_\lambda \in M(\lambda)^*[-\lambda]$ and $\tilde{1}^*_\lambda \in \tilde{M}(-\lambda)^*\langle\lambda\rangle$ such that $\langle 1^*_\lambda, 1_\lambda \rangle = \langle \tilde{1}^*_\lambda, \tilde{1}_\lambda \rangle = 1$. Define maps $\zeta : \text{Hom}_U(M(\lambda), V \otimes \tilde{M}(-\lambda)^*) \to V[0]$ and $\tilde{\zeta} : \text{Hom}_U(M(-\lambda), V \otimes M(\lambda)^*) \to V[0]$ by the formulae $\varphi(1_\lambda) = \zeta \otimes \tilde{1}^*_\lambda + \text{lower order terms}$, $\varphi(\tilde{1}_\lambda) = \tilde{\zeta} \otimes 1^*_\lambda + \text{higher order terms}$.

Consider also the natural maps

\[ r : \text{Hom}_U(M(\lambda) \otimes \tilde{M}(-\lambda), V) \to \text{Hom}_U(M(\lambda), V \otimes \tilde{M}(-\lambda)^*), \]
\[ \tilde{r} : \text{Hom}_U(M(\lambda) \otimes \tilde{M}(-\lambda), V) \to \text{Hom}_U(\tilde{M}(-\lambda), V \otimes M(\lambda)^*). \]

**Proposition 7.** Maps $\zeta$, $\tilde{\zeta}$, $r$, and $\tilde{r}$ are vector space isomorphisms, and the diagram

\[
\begin{array}{ccc}
\text{Hom}_U(M(\lambda) \otimes \tilde{M}(-\lambda), V) & \xrightarrow{r} & \text{Hom}_U(M(\lambda), V \otimes \tilde{M}(-\lambda)^*) \\
\downarrow{\tilde{r}^{-1}} & & \downarrow{q^{-\langle\lambda\rangle}} \\
\text{Hom}_U(\tilde{M}(-\lambda) \otimes M(\lambda), V) & \xrightarrow{\tilde{r}} & \text{Hom}_U(\tilde{M}(-\lambda), V \otimes M(\lambda)^*)
\end{array}
\]

is commutative.

**Proof.** First of all notice that we have the natural identification

\[ \text{Hom}_U(M(\lambda), V \otimes \tilde{M}(-\lambda)^*) = (V \otimes \tilde{M}(-\lambda)^* U^+)^{\langle\lambda\rangle}, \]

Further on, we have

\[ \text{Hom}_U(M(\lambda) \otimes \tilde{M}(-\lambda), V) = \text{Hom}_U(M(\lambda), \text{Hom}(\tilde{M}(-\lambda), V)) = \text{Hom}_U(\tilde{M}(-\lambda), V)[\lambda] = V[0]. \]

On the other side, $\text{Hom}_U(\tilde{M}(-\lambda), V) = (V \otimes \tilde{M}(-\lambda)^* U^+)^{\langle\lambda\rangle}$ by Lemma 3. Now it is clear that the map $r$ (resp. $\zeta$) corresponds to the identification $\text{Hom}_U(M(\lambda) \otimes \tilde{M}(-\lambda), V) = (V \otimes \tilde{M}(-\lambda)^* U^+)^{\langle\lambda\rangle}$ (resp. $(V \otimes \tilde{M}(-\lambda)^*)^{U^+}[\lambda] = V[0]$).

The second part of the proposition concerning $\tilde{r}$ and $\tilde{\zeta}$ can be verified similarly.

Finally, since $\tilde{r}^{-1}(\tilde{1}^- \otimes 1_\lambda) = q^{-\langle\lambda\rangle} 1_\lambda \otimes \tilde{1}^- \lambda$, the whole diagram is commutative. \qed

Now note that the pairing $\pi_\lambda : U^+ \otimes U^- \to \mathbb{k}$ naturally defines a pairing $\tilde{M}(-\lambda) \otimes M(\lambda) \to \mathbb{k}$. Denote by $\chi_\lambda : M(\lambda) \to \tilde{M}(-\lambda)^*$ the corresponding morphism of $U$-modules. The kernel of $\chi_\lambda$ is equal to $K(\lambda) = K_\lambda \cdot 1_\lambda$, and the image of $\chi_\lambda$ is $(\tilde{K}_\lambda \cdot \tilde{1}_\lambda) L(\lambda)$. Therefore $\chi_\lambda$ can be naturally represented as $\chi_\lambda^\vee \otimes \chi_\lambda^\prime$, where

\[ M(\lambda) \xrightarrow{\chi_\lambda} L(\lambda) \xrightarrow{\chi_\lambda^\prime} \tilde{M}(-\lambda)^*. \]
The morphisms $\chi'_\lambda$ and $\chi''_\lambda$ induce the commutative diagram of embeddings

$$
\begin{array}{c}
\text{Hom}_U(L(\lambda), V \otimes L(\lambda)) \\
\downarrow \\
\text{Hom}_U(L(\lambda), V \otimes \widetilde{M}(-\lambda)^*) \\
\downarrow \\
\text{Hom}_U(L(\lambda), V \otimes \tilde{M}(-\lambda)^*) \\
\end{array}
\rightarrow
\begin{array}{c}
\text{Hom}_U(M(\lambda), V \otimes L(\lambda)) \\
\downarrow \\
\text{Hom}_U(M(\lambda), V \otimes \widetilde{M}(-\lambda)^*) \\
\downarrow \\
\text{Hom}_U(M(\lambda), V \otimes \tilde{M}(-\lambda)^*).
\end{array}
$$

It is clear that the following lemma holds:

**Lemma 8.** The image of $\text{Hom}_U(L(\lambda), V \otimes L(\lambda))$ in $\text{Hom}_U(M(\lambda), V \otimes \tilde{M}(-\lambda)^*)$ under the embedding above consists of the morphisms $\varphi : M(\lambda) \to V \otimes \tilde{M}(-\lambda)^*$ such that $\varphi(K\lambda 1_\lambda) = 0$ and $\varphi(M(\lambda)) \subset V \otimes (K\lambda \tilde{1}_{-\lambda})^\perp$. □

**Proposition 9.** Let $\varphi \in \text{Hom}_U(M(\lambda), V \otimes \tilde{M}(-\lambda)^*)$. Then $\varphi(M(\lambda)) \subset V \otimes (K\lambda \tilde{1}_{-\lambda})^\perp$ iff $K\lambda \zeta_\varphi = 0$.

**Proof.** First notice that $\varphi(M(\lambda)) \subset V \otimes (K\lambda \tilde{1}_{-\lambda})^\perp$ iff $\varphi(1_\lambda) \in V \otimes (K\lambda \tilde{1}_{-\lambda})^\perp$. Indeed, for any $x \in U$ we have $\varphi(x1_\lambda) = \sum (x_1 \otimes x_2) \varphi(1_\lambda)$ and $U.(K\lambda \tilde{1}_{-\lambda})^\perp = (K\lambda \tilde{1}_{-\lambda})^\perp$.

Denote by $\psi$ the element in $\text{Hom}_{U^+}(\tilde{M}(-\lambda), V)$ that corresponds to $\varphi(1_\lambda) \in (V \otimes \tilde{M}(-\lambda)^*)^U^\perp$ (see Lemma 6). Under this notation $\varphi(1_\lambda) \in V \otimes (K\lambda \tilde{1}_{-\lambda})^\perp$ iff $\psi(K\lambda \tilde{1}_{-\lambda}) = 0$. On the other hand, $\zeta_\varphi = \psi(\tilde{1}_{-\lambda})$ and $\psi(K\lambda \tilde{1}_{-\lambda}) = K\lambda \psi(\tilde{1}_{-\lambda}) = \tilde{K}\lambda \zeta_\varphi$. This completes the proof. □

**Proposition 10.** Let $\varphi \in \text{Hom}_U(M(\lambda), V \otimes \tilde{M}(-\lambda)^*)$. Then $\varphi(K\lambda 1_\lambda) = 0$ iff $K\lambda \zeta_\varphi = 0$.

**Proof.** Consider $\hat{\varphi} = r^{-1}(\varphi) \in \text{Hom}_U(M(\lambda) \otimes \tilde{M}(-\lambda), V)$, $\varphi = \hat{\varphi} \circ \hat{R}^{-1} \in \text{Hom}_U(M(-\lambda) \otimes M(\lambda), V)$, and $\bar{\varphi} = \bar{r}(\varphi) \in \text{Hom}_U(M(-\lambda), V \otimes M(\lambda)^{\perp^*})$ (see Proposition 7). Since $K\lambda 1_\lambda \otimes \tilde{M}(-\lambda)$ is an $U \otimes U$-submodule in $M(\lambda) \otimes M(-\lambda)$, one has $\varphi(K\lambda 1_\lambda) = 0$ iff $\bar{\varphi}(M(-\lambda) \otimes K\lambda 1_\lambda) = 0$ iff $\bar{\varphi}(\tilde{M}(-\lambda)) \subset V \otimes (K\lambda 1_\lambda)^\perp$.

Arguing as in the proof of Proposition 9 we see that $\bar{\varphi}(\tilde{M}(-\lambda)) \subset V \otimes (K\lambda 1_\lambda)^\perp$ iff $K\lambda \zeta_{\bar{\varphi}} = 0$. Now it is enough to notice that $\zeta_{\bar{\varphi}} = q^{-(\lambda|\lambda)} \zeta_\varphi$ by Proposition 7 and therefore $K\lambda \zeta_{\bar{\varphi}} = 0$ iff $K\lambda \zeta_\varphi = 0$. □

Define maps $u : \text{Hom}_U(L(\lambda), L(\lambda) \otimes V) \to V[0]$ and $v : \text{Hom}_U(L(\lambda), V \otimes L(\lambda)) \to V[0]$ via $\varphi \mapsto u_\varphi$, where $\varphi(\tilde{1}_\lambda) = \tilde{T}_\lambda \otimes u_\varphi + \text{lower order terms}$, and $\psi \mapsto v_\psi$, where $\psi(\tilde{1}_\lambda) = v_\psi \otimes \tilde{T}_\lambda + \text{lower order terms}$.

**Proposition 11.** The map $v$ defines the isomorphism $\text{Hom}_U(L(\lambda), V \otimes L(\lambda)) \simeq V[0]^{K\lambda + K\lambda}$. 8
Remark 1. For \( \beta = 0 \) we have \( y^0 = 1 \) and \( f^{\beta,i} = f \).

Proposition 13. \( f^{\beta,i} = \sum_j \left( \mathcal{S}_\lambda^{-1} \right)_{ij} \theta \left( y^i \right) f. \)

Proof. For any \( \beta = \sum_j c_j \alpha_j \in Q_+ \) set \( k_\beta = \prod_j k^{c_j}_j \in T \) and \( \Lambda_\beta = q^\lambda(k_\beta) = \prod_j q^{d_j c_j(\lambda, \alpha)} \).

Set \( \xi = \varphi(\overline{T}_\lambda) \). Clearly, \( \xi \) is a singular element in \( L(\lambda) \otimes F \). In particular, \( (k_i \otimes k_i) \xi = q^{d_j(\lambda, \alpha)} \xi \) and \( (e_i \otimes 1 + k_i \otimes e_j) \xi = 0 \). Thus \( (e_i \otimes 1) \xi = q^{\delta_i(\lambda, \alpha)^{-1}}(1 \otimes \sigma^{-1}(e_i)) \xi \). By induction we get \( (x \otimes 1) \xi = \Lambda_\beta(1 \otimes \sigma^{-1}(x)) \xi \) for any \( x \in U^+[\beta] \).
Let $\omega'$ be the involutive antiautomorphism of $U$ given by $\omega'(e_i) = f_i$, $\omega'(f_i) = e_i$, $\omega'(t_i) = t_i$. Set $x^i_\beta = \omega'(y^i_\beta)$. Then we have

$$\left( S^\lambda \otimes \text{id} \right) \left( T^\lambda \otimes (x^i_\beta \otimes 1) \xi \right) = \Lambda^\beta \left( S^\lambda \otimes \text{id} \right) \left( T^\lambda \otimes (1 \otimes \sigma^{-1}(x^i_\beta)) \xi \right). \quad (2)$$

It is easy to show by induction on $\text{ht} \beta$ that $\omega(x^i_\beta) = q^{c(\beta)}y^i_\beta k^i_\beta$ and $\sigma^{-1}(x^i_\beta) = q^{c(\beta)}\theta(y^i_\beta)k^i_\beta$ for a certain $c(\beta)$. (Actually $c(\beta) = d_1 + \ldots + d_i - \frac{1}{2}(\beta, d_1 \alpha_1' + \ldots + d_i \alpha_i')$.) Hence the l. h. s. of $(2)$ equals

$$\sum_i S^\lambda \left( \omega(x^i_\beta) T^\lambda, y^i_\beta T^\lambda \right) f^{\beta,i} = q^{c(\beta)}\Lambda^\beta \sum_i S^\lambda \left( y^i_\beta T^\lambda, y^i_\beta T^\lambda \right) f^{\beta,i}$$

and the r. h. s. of $(2)$ equals

$$\Lambda^\beta \sigma^{-1}(x^i_\beta) f = q^{c(\beta)}\Lambda^\beta \theta(y^i_\beta) f.$$ 

Combining these together we get

$$\sum_i S^\lambda \left( y^i_\beta T^\lambda, y^i_\beta T^\lambda \right) f^{\beta,i} = \theta(y^i_\beta) f,$$

and the proposition follows. \hfill $\square$

For any $\lambda \in \mathfrak{h}^*$ consider

$$J^{\text{red}}(\lambda) = \sum_{\beta \in \mathbb{Q}^+} \sum_{i,j} \left( S^\lambda \right)^{-1}_{ij} y^i_j \otimes \theta(y^j_\beta). \quad (3)$$

One can regard $J^{\text{red}}(\lambda)$ as an element in a certain completion of $U^- \otimes U^+$.

**Remark 2.** This element $J^{\text{red}}(\lambda)$ is not uniquely defined (e.g., because $U^- \to L(\lambda)$ has a kernel), but this does not affect our further considerations.

**Remark 3.** For $f \in F[0]^{K_\lambda + \tilde{K}_\lambda}$ and $\varphi = \Theta^{-1}(f)$ one has $\varphi(T^\lambda) = J^{\text{red}}(\lambda)(T^\lambda \otimes f)$.

Let us define an associative product $\ast_\lambda$ on $F[0]^{K_\lambda + \tilde{K}_\lambda}$ by means of $\Theta$, i.e., for any $f_1, f_2 \in F[0]^{K_\lambda + \tilde{K}_\lambda}$ we define $f_1 \ast_\lambda f_2 = \Theta(\varphi_1 \ast \varphi_2)$, where $\varphi_1 = \Theta^{-1}(f_1)$, $\varphi_2 = \Theta^{-1}(f_2)$, and $\ast$ is the product on $\text{Hom}_U(L(\lambda), L(\lambda) \otimes F')$ given by (1). By this definition, we get a right $U$-module algebra $(F[0]^{K_\lambda + \tilde{K}_\lambda}, \ast_\lambda)$.

**Theorem 14.** We have

$$f_1 \ast_\lambda f_2 = \mu \left( J^{\text{red}}(\lambda)(f_1 \otimes f_2) \right). \quad (4)$$
Proof. Observe that

\[(\varphi_1 \ast \varphi_2)(\mathbf{1}_\lambda) = (\text{id} \otimes \mu)(\varphi_1 \otimes \text{id})(\varphi_2(\mathbf{1}_\lambda)) = \]

\[(\text{id} \otimes \mu)(\varphi_1 \otimes \text{id}) \left( \mathbf{1}_\lambda \otimes f_2 + \sum_{\beta \in Q_+ \setminus \{0\}} \sum_i y_\beta^i \cdot \mathbf{1}_\lambda \otimes f_2^{\beta,i} \right) = \]

\[(\text{id} \otimes \mu) \left( \varphi_1(\mathbf{1}_\lambda) \otimes f_2 + \sum_{\beta \in Q_+ \setminus \{0\}} \sum_i (\Delta(y_\beta^i)\varphi_1(\mathbf{1}_\lambda)) \otimes f_2^{\beta,i} \right) = \]

\[\mathbf{1}_\lambda \otimes \left( f_1 f_2 + \sum_{\beta \in Q_+ \setminus \{0\}} \sum_i \left( \overleftarrow{y_\beta f_1} \right) f_2^{\beta,i} \right) + \text{lower order terms}, \]

where in the last equation we use the fact that for any \( y \in U_+ \) we have \( \Delta(y) = 1 \otimes y + \sum_k y_k \otimes z_k \) with \( y_k \in U_+ \). Therefore

\[ f_1 \ast_\lambda f_2 = f_1 f_2 + \sum_{\beta \in Q_+ \setminus \{0\}} \sum_i \left( \overleftarrow{y_\beta f_1} \right) f_2^{\beta,i} = \sum_{\beta \in Q_+} \sum_i \left( \overleftarrow{y_\beta f_1} \right) f_2^{\beta,i}. \]

To finish the proof it is enough to apply Proposition 13 to \( f_2 \). \qed

Remark 4. Theorem 14 together with results of [8] implies that the algebras \( \text{Hom}_U(L(\lambda), L(\lambda) \otimes F) \), \( \text{End}_{\mathcal{L}}(\lambda) \) \(), \text{fin} \), and \((F[0]^{K_\lambda + \tilde{K}_\lambda}, \ast_\lambda)\) are isomorphic as right Hopf module algebras over \( U \).

4 Limiting properties of the fusion element

We say that \( \lambda \in \mathfrak{h}^* \) is \textit{generic} if \( \langle \lambda + \rho, \beta^\vee \rangle \not\in \mathbb{N} \) for all \( \beta \in \mathbb{R}_+ \). In this case \( L(\lambda) = M(\lambda) \), and we set \( J(\lambda) = J^{\text{reg}}(\lambda) \). Notice that \( J(\lambda) \) up to a \( U_0 \)-part equals the fusion element related to the Verma module \( M(\lambda) \) (see, e.g., [1]).

4.1 Regularity

Let \( \lambda_0 \in \mathfrak{h}^* \). Since \( J(\lambda) \) is invariant w. r. to \( \tau(\theta \otimes \theta) \) (where \( \tau \) is the tensor permutation), one can easily see that the following conditions on \( \lambda_0 \) are equivalent: 1) for any \( U^- \)-module \( M \) the family of operators \( J(\lambda)^M : M \otimes F[0]^{K_{\lambda_0}} \to M \otimes F \) naturally defined by \( J(\lambda) \) is regular at \( \lambda = \lambda_0 \), 2) for any \( U^+ \)-module \( N \) the family of operators \( J(\lambda)_N : F[0]^{K_{\lambda_0}} \otimes N \to F \otimes N \) naturally defined by \( J(\lambda) \) is regular at \( \lambda = \lambda_0 \). We will say that \( \lambda_0 \) is \( J \)-\textit{regular} if these conditions are satisfied. Clearly, any generic \( \lambda_0 \) is \( J \)-regular.

Proposition 15. Assume that \( \lambda_0 \in \mathfrak{h}^* \) is \( J \)-regular. Then \( F[0]^{K_{\lambda_0}} = F[0]^{\tilde{K}_{\lambda_0}} = F[0]^{K_{\lambda_0} + \tilde{K}_{\lambda_0}} \).
Proof. Let $g \in F[0]^{\tilde{K}_{\lambda_0}}$. If $\lambda \in \mathfrak{h}^*$ is generic, then the element $J(\lambda)^{M(\lambda)}(1_{\lambda} \otimes g)$ is a singular vector of weight $\lambda$ in $M(\lambda) \otimes F$. Therefore $Z := \lim_{\lambda \to \lambda_0} J(\lambda)^{M(\lambda)}(1_{\lambda} \otimes g)$ is a singular vector of weight $\lambda_0$ in $M(\lambda_0) \otimes F$, and hence we have $\varphi_Z \in \text{Hom}_U(M(\lambda_0), M(\lambda_0) \otimes F)$, $\varphi_Z(1_{\lambda_0}) = Z$.

Under the natural map $\text{Hom}_U(M(\lambda_0), M(\lambda_0) \otimes F) \to \text{Hom}_U(L(\lambda_0), L(\lambda_0) \otimes F)$ we have $\varphi_Z \mapsto \varphi_Z$, where $\varphi_Z(T_{\lambda_0}) = Z$ is the projection of $Z$ onto $L(\lambda_0) \otimes F$. Now notice that $g = \Theta(\varphi_Z) \in F[0]^{K_{\lambda_0} + \tilde{K}_{\lambda_0}}$.

The proof of $F[0]^{K_{\lambda_0}} = F[0]^{K_{\lambda_0} + \tilde{K}_{\lambda_0}}$ is similar. □

Proposition 16. Assume that $\lambda_0 \in \mathfrak{h}^*$ is $J$-regular. Then the natural map $\text{Hom}_U(M(\lambda_0), M(\lambda_0) \otimes F) \to \text{Hom}_U(L(\lambda_0), L(\lambda_0) \otimes F)$ is surjective.

Proof. We have the isomorphism

$$\Theta : \text{Hom}_U(L(\lambda_0), L(\lambda_0) \otimes F) \to F[0]^{K_{\lambda_0} + \tilde{K}_{\lambda_0}} = F[0]^{\tilde{K}_{\lambda_0}}.$$ 

Now take $g \in F[0]^{\tilde{K}_{\lambda_0}}$. Consider $Z = \lim_{\lambda \to \lambda_0} J(\lambda)^{M(\lambda)}(1_{\lambda} \otimes g) \in M(\lambda_0) \otimes F$. Since $Z$ is a singular vector of weight $\lambda_0$, we have $\varphi_Z \in \text{Hom}_U(M(\lambda_0), M(\lambda_0) \otimes F)$, $\varphi_Z(1_{\lambda_0}) = Z$. Clearly, under the mapping $\text{Hom}_U(M(\lambda_0), M(\lambda_0) \otimes F) \to \text{Hom}_U(L(\lambda_0), L(\lambda_0) \otimes F)$ the image of $\varphi_Z$ equals to $\Theta^{-1}(g)$, which proves the proposition. □

Proposition 17. Assume that $\lambda_0 \in \mathfrak{h}^*$ is $J$-regular. Then the action map $U_{\text{fin}} \to (\text{End } L(\lambda_0))_{\text{fin}}$ is surjective.

Proof. Recall that we have the isomorphisms

$$\text{Hom}_U(M(\lambda_0), M(\lambda_0) \otimes F) \simeq (\text{End } M(\lambda_0))_{\text{fin}},$$

$$\text{Hom}_U(L(\lambda_0), L(\lambda_0) \otimes F) \simeq (\text{End } L(\lambda_0))_{\text{fin}}.$$ 

It is well known that the action map $U_{\text{fin}} \to (\text{End } M(\lambda_0))_{\text{fin}}$ is surjective for any $\lambda_0 \in \mathfrak{h}^*$ (see [5, 6]). Since by Proposition 16 the map $(\text{End } M(\lambda_0))_{\text{fin}} \to (\text{End } L(\lambda_0))_{\text{fin}}$ is surjective, the map $U_{\text{fin}} \to (\text{End } L(\lambda_0))_{\text{fin}}$ is also surjective. □

Proposition 18. Assume that $\lambda_0 \in \mathfrak{h}^*$ is $J$-regular. Then for any $f, g \in F[0]^{K_{\lambda_0}}$ we have $\overrightarrow{J(\lambda)}(f \otimes g) \to \overrightarrow{J^{\text{red}}(\lambda_0)}(f \otimes g)$ as $\lambda \to \lambda_0$.

Proof. For any $\lambda \in \mathfrak{h}^*$ we may naturally identify $M(\lambda)$ with $U^-$ as $U^-$-modules. Therefore we know by definition of $J$-regularity that $J(\lambda)^{M(\lambda)}(1_{\lambda} \otimes g)$ is regular at $\lambda = \lambda_0$. Thus $J(\lambda)^{M(\lambda)}(1_{\lambda} \otimes g) \to Z \in M(\lambda_0) \otimes F$ as $\lambda \to \lambda_0$. In an arbitrary basis $y_{\beta} \in U^{-}[\beta]$ we have

$$J(\lambda)^{M(\lambda)}(1_{\lambda} \otimes g) = \sum_{\beta \in Q_+} \sum_{i,j} (S_{\lambda}^{\beta})^{-1}_{i,j} y_{\beta}^i 1_{\lambda} \otimes \overrightarrow{\theta(y_{\beta}^j)g},$$

12
and

\[ Z = \sum_{\beta \in Q_+} \sum_{i,j} a_{ij}^\beta \cdot y_\beta^i 1_{\lambda_0} \otimes \theta (y_\beta^j) g \]

for some coefficients \( a_{ij}^\beta \).

Now choose a basis \( y_\beta^i \in U^-[-\beta] \) in the following way: first take a basis in \( K_{\lambda_0}[-\beta] = K_{\lambda_0} \cap U^-[-\beta] \) and then extend it arbitrarily to a basis in the whole \( U^-[-\beta] \). In this basis the projection \( Z \in L(\lambda_0) \otimes F \) of the element \( Z \) is given by

\[ Z = \sum_{\beta \in Q_+} \sum_{y_\beta^i \notin K_{\lambda_0}[-\beta]} a_{ij}^\beta \cdot y_\beta^i 1_{\lambda_0} \otimes \theta (y_\beta^j) g. \quad (5) \]

Now notice that \( Z \), being the limit of singular vectors of weight \( \lambda \) in \( M(\lambda) \otimes F \), defines the intertwining operator \( \varphi_Z \in \text{Hom}_U(M(\lambda_0), M(\lambda_0) \otimes F) \), \( \varphi_Z(1_{\lambda_0}) = Z \).

Under the natural map \( \text{Hom}_U(M(\lambda_0), M(\lambda_0) \otimes F) \to \text{Hom}_U(L(\lambda_0), L(\lambda_0) \otimes F) \) we have \( \varphi_Z \mapsto \varphi_Z^{-1} \), where \( \varphi_Z^{-1}(1_{\lambda_0}) = Z^{-1} \). Therefore \( Z = J^{\text{red}}(\lambda_0)^{-1} \). Comparing this with \( (5) \) we conclude that for all \( i, j \) such that \( y_\beta^i, y_\beta^j \notin K_{\lambda_0}[-\beta] \) we have \( a_{ij}^\beta = (S_{\lambda_0}^\beta)^{-1}_{ij} \).

Finally,

\[ \sum_{\beta \in Q_+} \sum_{y_\beta^i \notin K_{\lambda_0}[-\beta]} a_{ij}^\beta \rightarrow (y_\beta^i f) \otimes \theta (y_\beta^j) g = \]

\[ \sum_{\beta \in Q_+} \sum_{y_\beta^i \notin K_{\lambda_0}[-\beta]} a_{ij}^\beta f \otimes \theta (y_\beta^j) g = \]

\[ \sum_{\beta \in Q_+} \sum_{y_\beta^i \notin K_{\lambda_0}[-\beta]} (S_{\lambda_0}^\beta)^{-1}_{ij} y_\beta^i f \otimes \theta (y_\beta^j) g = J^{\text{red}}(\lambda_0)(f \otimes g) \]

as \( \lambda \to \lambda_0 \).

\[ \square \]

**Corollary 19.** Assume that \( \lambda_0 \in \mathfrak{h}^* \) is \( J \)-regular. Let \( f_1, f_2 \in F[0]^{K_{\lambda_0}} \). Then \( f_1 \ast_\lambda f_2 \to f_1 \ast_{\lambda_0} f_2 \) as \( \lambda \to \lambda_0 \).

\[ \square \]

### 4.2 One distinguished root case

**Theorem 20.** Let \( \alpha \in \mathbb{R}_+ \). Consider \( \lambda_0 \in \mathfrak{h}^* \) that satisfies \( \langle \lambda_0 + \rho, \alpha^\vee \rangle = n \in \mathbb{N} \), \( \langle \lambda_0 + \rho, \beta^\vee \rangle \notin \mathbb{N} \) for all \( \beta \in \mathbb{R}_+ \setminus \{ \alpha \} \). Then \( \lambda_0 \) is \( J \)-regular.

**Proof.** Fix an arbitrary line \( l \subset \mathfrak{h}^* \) through \( \lambda_0 \), \( l = \{ \lambda_0 + t\nu \mid t \in \mathbb{R} \} \), transversal to the hyperplane \( \langle \lambda + \rho, \alpha^\vee \rangle = n \).

Identify \( M(\lambda) \) with \( U^- \) in the standard way. Recall that we have a basis \( y_\beta^i \in U^-[-\beta] \) for \( \beta \in Q_+ \). Let \( L(S_{\lambda}^\beta) \in \text{End} U^-[-\beta] \) be given by the matrix \((S_{\lambda}^\beta)_{ij}\).
in the basis $y^\alpha$. Notice that $\text{Ker} \, L \left( S^\beta_{\lambda_0} \right) = \text{Ker} \, S^\beta_{\lambda_0} = K_{\lambda_0}[-\beta] = K_{\lambda_0} \cap U^{-}[-\beta]$. For any $\lambda \in l$ sufficiently close to $\lambda_0$, $\lambda \neq \lambda_0$ we have $M(\lambda)$ is irreducible, and $L \left( S^\beta_{\lambda} \right)$ is invertible for any $\beta \in Q_+$. In this notation we have

$$J(\lambda) = \sum_{\beta \in Q_+} \sum_j L \left( S^\beta_{\lambda} \right)^{-1} y^\beta_j \otimes \theta \left( y^\beta_j \right).$$

Take $\lambda = \lambda_0 + t\nu \in l$. Fix any $\beta \in Q_+$ and set $V = U^{-}[-\beta]$, $A_t = L \left( S^\beta_{\lambda} \right)$, $V_0 = \text{Ker} \, A_0 = K_{\lambda_0}[-\beta] \subset V$. Write $A_t = A_0 + tB_t$, where $B_t$ is regular at $t = 0$.

Since $J(\lambda)$ may have at most simple poles (see, e.g., [2]) we have $A_t^{-1} = \frac{1}{t} C + D_t$, where $D_t$ is regular at $t = 0$.

**Lemma 21.** $\text{Im} \, C \subset V_0$.

**Proof.** We have $A_t A_t^{-1} = \text{id}$ for any $t \neq 0$, i.e., $\frac{1}{t} A_0 C + A_0 D_t + B_t C + tB_t D_t = \text{id}$. Since the left hand side should be regular at $t = 0$, we have $A_0 C = 0$, which proves the lemma. □

For $t \neq 0$ set $J_t = \sum_j A_t^{-1} y_j \otimes \theta(y_j)$ (from now on we are omitting the index $\beta$ for the sake of brevity). By Lemma 21 we have $C y_j \in V_0 = K_{\lambda_0}[-\beta]$. Hence for $f \in F[0]^{K_{\lambda_0}}$ we have $\overrightarrow{C y_j f} = 0$. Therefore $A_t^{-1} y_j f = \frac{1}{t} \overrightarrow{C y_j f} + \overrightarrow{D_t y_j f}$. This proves the regularity of $(J_t)_N(f \otimes \cdot)$ at $t = 0$, i.e., the regularity of $J_0(\lambda)_N(f \otimes \cdot)$ at $\lambda = \lambda_0$. □

### 4.3 Subset of simple roots case

**Theorem 22.** Let $\Gamma \subset \Pi$. Consider $\lambda_0 \in \mathfrak{h}^*$ that satisfies $\langle \lambda_0 + \rho, \alpha^\vee \rangle \in \mathbb{N}$ for all $\alpha_i \in \Gamma$, $\langle \lambda_0 + \rho, \beta^\vee \rangle \notin \mathbb{N}$ for all $\beta \in \mathbb{R}_+ \setminus \text{Span} \, \Gamma$. Then $\lambda_0$ is $J$-regular.

**Proof.** Recall that the only singularities of $J(\lambda)$ near $\lambda_0$ are (simple) poles on the hyperplanes $\langle \lambda - \lambda_0, \alpha^\vee \rangle = 0$ for $\alpha \in \mathbb{R}_+ \cap \text{Span} \, \Gamma$ (see, e.g., [2]). Therefore it is enough to show that for any $f \in F[0]^{K_{\lambda_0}}$ the operator $J(\lambda)_N(f \otimes \cdot)$ has no singularity at any such hyperplane.

Take $\alpha \in \mathbb{R}_+ \cap \text{Span} \, \Gamma$ and consider a hyperplane $\langle \lambda - \lambda_0, \alpha^\vee \rangle = 0$. Take an arbitrary $\lambda' \in \mathfrak{h}^*$ such that $\langle \lambda' - \lambda_0, \alpha^\vee \rangle = 0$, and $\langle \lambda' + \rho, \beta^\vee \rangle \notin \mathbb{N}$ for all $\beta \in \mathbb{R}_+ \setminus \{\alpha\}$.

**Lemma 23.** $K_{\lambda'} \subset K_{\lambda_0}$.

**Proof.** First, assume that $\lambda_0$ is dominant integral, i.e., $\Gamma = \Pi$. Set $n = \langle \lambda_0 + \rho, \alpha^\vee \rangle$. Consider the irreducible $U$-module $L(\lambda_0)$ with a highest weight vector $\overrightarrow{T_{\lambda_0}}$. The inclusion $K_{\lambda'} \subset K_{\lambda_0}$ is equivalent to the equality $K_{\lambda'} \overrightarrow{T_{\lambda_0}} = 0$. By Proposition 3 $K_{\lambda'}$ is generated by an element $x_{\lambda'} \in U^{-}$ of weight $-n\alpha$. Hence the vector $x_{\lambda'} \overrightarrow{T_{\lambda_0}}$ has weight $\lambda_0 - n\alpha$. Thus it suffices to show that $\lambda_0 - n\alpha$
is not a weight of $L(\lambda_0)$. Indeed, easy computations show that $s_\alpha(\lambda_0 - n\alpha) = \lambda_0 + (\rho, \alpha^*)\alpha > \lambda_0$, and since the set of weights of $L(\lambda_0)$ is $W$-invariant, the lemma is proved in this case.

Now, let us consider the general case. Let $\lambda_0$ be an arbitrary weight satisfying the assumptions of the theorem. Set $n_i = \langle \lambda_0 + \rho, \alpha_i^* \rangle$. Denote by $K'_{\lambda_0}$, the right ideal in $U^-$ generated by the elements $f_i^{n_i}$, $\alpha_i \in \Gamma$. By Proposition [10] we have $K'_{\lambda_0} \subset K_{\lambda_0}$.

Denote by $X(\lambda_0)$ the set of dominant integral weights $\mu$ such that $\langle \mu + \rho, \alpha_i^* \rangle = n_i$ for all $\alpha_i \in \Gamma$. We have proved that $K'_{\lambda_0} \subset K_\mu$ for any $\mu \in X(\lambda_0)$. Thus $K_{\lambda'} \subset \bigcap_{\mu \in X(\lambda_0)} K_\mu$.

For $\mu \in X(\lambda_0)$, Proposition [4] yields that $K_\mu$ is generated by $K'_{\lambda_0}$ and the elements $f_j^{\langle \mu + \rho, \alpha_j^* \rangle}$, $j \in \Pi \setminus \Gamma$. Notice that choosing $\mu \in X(\lambda_0)$, all the numbers $\langle \mu + \rho, \alpha_j^* \rangle$ can be made arbitrary large. Hence, $K'_{\lambda_0} = \bigcap_{\mu \in X(\lambda_0)} K_\mu$. Therefore, $K_{\lambda'} \subset K'_{\lambda_0} \subset K_{\lambda_0}$. The lemma is proved.

The lemma implies that $F[0]K_{\lambda'} \supset F[0]K_{\lambda_0}$, and applying Theorem [20] to $\lambda'$ we complete the proof of the theorem.

Remark 5. In fact, it is possible to prove that if $\lambda_0$ satisfies the assumptions of Theorem [22] then $K_{\lambda_0}$ is generated by the elements $f_i^{\langle \lambda_0 + \rho, \alpha_i^* \rangle}$, $\alpha_i \in \Gamma$. Indeed, this fact is well-known in the classical case $q = 1$, see [3]. Let us consider the highest weight module $V_q = M_q(\lambda_0)/K_{\lambda_0}$ (we have used at this point the notation $M_q$ for the Verma modules over $\hat{U}_q\mathfrak{g}$). We have to prove that it is irreducible under assumptions of the theorem. If not, there exists some weight space $V_q[\mu]$ such that the determinant of the restriction of the Shapovalov form on $V_q$ to $V_q[\mu]$ is zero. Let us denote the Shapovalov form on $V_q$ by $S_q$ and its restriction to $V_q[\mu]$ by $S_q[\mu]$. Taking limit $q \to 1$ (it can be done in the same way as in [5] Sections 3.4.5–3.4.6] we see that $S_q[\mu] \to S[\mu]$ and det $S_q[\mu] \to$ det $S[\mu]$. However, the latter determinant is non-zero because $V_1$ is irreducible. Hence, $V_q$ is also irreducible.

### 4.4 Application to Poisson homogeneous spaces

Let $\Gamma \subset \Pi$. Assume that $\lambda \in \mathfrak{h}^*$ is such that $\langle \lambda, \alpha^* \rangle = 0$ for all $\alpha \in \Gamma$, and $\langle \lambda + \rho, \beta^* \rangle \notin \mathbb{N}$ for all $\beta \in \mathbb{R}_+ \setminus \text{Span}\Gamma$. By Theorem [22] $\lambda$ is $J$-regular. In particular, $F[0]K_{\lambda + \rho} = F[0]K_\lambda$.

In what follows it will be more convenient to write $F_q$, $J_q$, and $K_{q,\lambda}$ instead of $F$, $J$, and $K_\lambda$. We will also need the classical limits $F_1 = \lim_{q \to 1} F_q$ and $K_{1,\lambda} = \lim_{q \to 1} K_{q,\lambda}$. They can be defined in the same way as in [5] Sections 3.4.5–3.4.6].

Clearly, $F_1$ is the algebra of regular functions on the connected simply connected group $G$, whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{h}$ be a reductive subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$ and is defined by $\Gamma$, $K$ the corresponding subgroup of $G$, and $F(G/K)$
the algebra of regular functions on the homogeneous space \( G/K \). According to [8, Theorem 33], we have \( F(G/K) = F_1[0]^{K_{1,\lambda}} \). Therefore we get

**Proposition 24.** \( \lim_{q \to 1} F_q[0]^{K_q,\lambda} = F(G/K) \).

Furthermore, since \( F_q[0]^{K_q,\lambda} \) is a Hopf module algebra over \( U \), \( F(G/K) \) is a Poisson homogeneous space over \( G \) equipped with the Poisson-Lie structure defined by the Drinfeld-Jimbo classical \( r \)-matrix \( r_0 = \sum_{\alpha \in \mathbb{R}_+^+} e_\alpha \wedge e_{-\alpha} \).

All such structures on \( G/K \) were described in [7]. It follows from [7] that any such Poisson structure on \( G/K \) is uniquely determined by an intermediate Levi subalgebra \( n \) satisfying \( \mathfrak{k} \subset n \subset \mathfrak{g} \) and some \( \lambda \in \mathfrak{h}^* \) which satisfies certain conditions, in particular, \( \langle \lambda, \alpha^\vee \rangle = 0 \) for \( \alpha \in \Gamma \) and \( \langle \lambda, \beta^\vee \rangle \not\in \mathbb{Z} \) for \( \beta \in \text{Span} \Gamma \setminus \text{Span} \Gamma_n \). Here \( \Gamma_n \) is the set of simple roots defining \( n \).

Now we can describe the Poisson bracket on \( G/K \) defined by \( \star_{\lambda} \)-multiplication on \( F_q[0]^{K_q,\lambda} \).

**Theorem 25.** Assume that \( \langle \lambda_0, \alpha^\vee \rangle = 0 \) for \( \alpha \in \Gamma \) and \( \langle \lambda_0, \beta^\vee \rangle \not\in \mathbb{Z} \) for \( \beta \in \mathbb{R}_+ \setminus \text{Span} \Gamma \). Then the classical limit of \( (F_q[0]^{K_q,\lambda_0}, \star_{\lambda_0}) \) is the algebra \( F(G/K) \) of regular functions on \( G/K \) equipped with the Poisson homogeneous structure defined by \( n = \mathfrak{g} \) and \( \lambda_0 \).

**Proof.** By Theorem [22], \( \lambda_0 \) is \( J \)-regular, i.e., for \( f_1, f_2 \in F_q[0]^{K_q,\lambda_0} \) we have

\[
f_1 \star_{\lambda_0} f_2 = \lim_{\lambda \to \lambda_0} f_1 \star_{\lambda} f_2 = \mu \left( J_q(\lambda)(f_1 \otimes f_2) \right).
\]

Take \( q = e^{-\frac{\hbar}{2}} \). Then \( J_q(\lambda) = 1 \otimes 1 + hj(\lambda) + O(h^2) \), and \( r(\lambda) = j(\lambda) - j(\lambda)^2 \) is the standard trigonometric solution of the classical dynamical Yang-Baxter equation (see, e.g., [1]). Thus the Poisson bracket on \( F(G/K) \) that corresponds to \( (F_q[0]^{K_q,\lambda_0}, \star_{\lambda_0}) \) is given by

\[
\{f_1, f_2\} = \lim_{\lambda \to \lambda_0} \mu \left( r(\lambda)(f_1 \otimes f_2) \right)
\]

for \( f_1, f_2 \in F(G/K) \). By [7], this is exactly the Poisson structure defined by \( n = \mathfrak{g} \) and \( \lambda_0 \).

Notice that an analogous result for simple Lie algebras of classical type was obtained in [13] using reflection equation algebras.

**References**

[1] P. Etingof and O. Schiffmann. Lectures on the dynamical Yang-Baxter equations. In: Quantum groups and Lie theory (Durham, 1999): 89–129, London Math. Soc. Lecture Note Ser., 290, Cambridge Univ. Press, Cambridge, 2001.
[2] P. Etingof and K. Styrkas. Algebraic integrability of Macdonald operators and representations of quantum groups. *Compositio Math.*, **114** (1998), 125–152.

[3] J.C. Jantzen. *Einhüllende Algebren halbeinfacher Lie-Algebren*. Springer-Verlag, Berlin, 1983.

[4] A. Joseph. *Kostant’s problem, Goldie rank and the Gelfand-Kirillov conjecture*. *Invent. Math.*, **56** (1980), 191–213.

[5] A. Joseph. *Quantum groups and their primitive ideals*. Springer-Verlag, New York, 1995.

[6] A. Joseph and G. Letzter. Verma modules annihilators for quantized enveloping algebras. *Ann. Sci. Ecole Norm. Sup.*, **28** (1995), 493–526.

[7] E. Karolinsky, K. Muzykin, A. Stolin, and V. Tarasov. Dynamical Yang-Baxter equations, quasi-Poisson homogeneous spaces, and quantization. *Lett. Math. Phys.*, **71** (2005), 179–197.

[8] E. Karolinsky, A. Stolin, and V. Tarasov. Irreducible highest weight modules and equivariant quantization. *Advances in Math.*, **211** (2007), 266–283.

[9] L. Korogodski and Y. Soibelman. *Algebras of functions on quantum groups*. American Mathematical Society, 1998.

[10] V. Mazorchuk. A twisted approach to Kostant’s problem. *Glasgow Math. J.*, **47** (2005), 549–561.

[11] V. Mazorchuk, C. Stroppel. Categorification of (induced) cell modules and the rough structure of generalized Verma modules. *Advances in Math.*, **219** (2008), 1363–1426.

[12] V. Mazorchuk, C. Stroppel. Categorification of Wedderburns basis for $C[S_n]$. *Arch. Math.*, **91** (2008), 1–11.

[13] A. Mudrov. Quantum conjugacy classes of simple matrix groups. *Comm. Math. Phys.*, **272** (2007), 635–660.

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