Jacobi equations using a variational principle

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Abstract
A variational principle is proposed for obtaining the Jacobi equations in systems admitting a Lagrangian description. The variational principle gives simultaneously the Lagrange equations of motion and the Jacobi variational equations for the system. The approach can be of help in finding constants of motion in the Jacobi equations as well as in analysing the stability of the systems and can be related to the vertical extension of the Lagrangian formalism. To exemplify two of such aspects, we uncover a constant of motion in the Jacobi equations of autonomous systems and we recover the well-known sufficient conditions of stability of two dimensional orbits in classical mechanics.

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Many classical dynamical systems have a variational formulation, for example, conservative mechanical systems, geodesics, classical field theory and even geometrical optics. All of them can be described using a Lagrangian function and a variant of Hamilton's principle [1]. Given such common factors we call them Lagrangian dynamical systems. The variational formulation of such Lagrangian systems is not only elegant and compact it also allows uncovering deep connections between the dynamical properties and other fields of physics and mathematics [1-6]. It is the goal of this short communication to formulate a generalized variational principle for Lagrangian dynamical systems, similar to Hamilton's, which is capable of producing not only the usual equations of motion but also the dynamical equations for deviations between two nearby trajectories: the so-called Jacobi variational equations [1,2].

We discuss a variational principle for what is known as the equation of geodesic deviations in gravitational physics and in Riemannian geometry [2,4,7,8] and give a brief outline of the possible applications of its consequences. Let us pinpoint that the Jacobi equations can be also regarded as the basic equations for deciding questions of dynamical stability, for evaluating the Lyapunov spectrum in dynamical systems [10,11], that they can be shown to occur naturally in the framework of the vertical extension of the Lagrangian formalism [3], and, possibly, may be used to recast the Jacobi field generated geodular structure of a newly connected manifolds [4].

Let us begin formulating the variational principle. Let us assume a N-degree of freedom system that can be described by a Lagrangian \( L(q;\dot{q};t) \), where \( q \) and \( \dot{q} \) stand, respectively, for the N generalized coordinates \( (q_1; q_2; \ldots; q_N) \) and the N generalized velocities \( (\dot{q}_1; \dot{q}_2; \ldots; \dot{q}_N) \). Using the Lagrangian of the system, define the function \( (q;\dot{q}; \ldots; t) \) as

\[
(q;\dot{q}; \ldots; t) \frac{\partial L}{\partial \dot{q}_a} + \frac{\partial L}{\partial q_a} \tag{1}
\]

Here, as in the rest of the article, the summation convention (summing from 1 to N) is implied for repeated indices. The N-vectors \( \alpha = (1; 2; \cdots; N) \) and \( \dot{\alpha} = (\dot{1}; \dot{2}; \cdots; \dot{N}) \) are to be regarded as describing deviations, and their corresponding velocities, from the motion described by \( q \) and \( \dot{q} \). That is, \( \dot{\alpha} \) plays the role of the Jacobi field associated with the trajectories of the original system [1,2,4]. Let us
notice the important property that \((q;\dot{q};;_;t)\) is an explicit function of time only when the Lagrangian is time-dependent (i.e., when it is non-autonomous).

Given the function defined in (1), let us introduce the functional

\[
Z_{t_2}^{t_1} \quad [q(t);(t)] = \int_{t_1}^{t_2} (q;\dot{q};;_;_t) dt \quad (2)
\]
of the paths joining two given configurations \((q_1;\dot{q}_1;;_;_;_1)\) and \((q_2;\dot{q}_2;;_;_;_2)\) of the varied system between two instants of time \(t_1\) and \(t_2\). The statement of the variational principle is now just that

\[
[q(t);(t)] = 0; \quad (3)
\]

when the path is varied with the endpoints and the time fixed. The \(2N\) conditions guaranteeing that the functional \([q(t);(t)]\) takes an extremum value are the associated Euler-Lagrange equations [1]

\[
\frac{d}{dt} \frac{\partial}{\partial \dot{q}_a} \frac{\partial}{\partial q_a} L = 0; \quad (4)
\]
or, using the definition (1) in the preceding equations, we obtain the \(N\) Lagrange equations of the original system

\[
\frac{d}{dt} \frac{\partial}{\partial q_a} \frac{\partial}{\partial q_a} L = 0; \quad a = 1;\ldots;N; \quad (5)
\]

plus the \(N\) equations:

\[
M_{ab} \dot{\dot{q}}_b + C_{ab} \dot{q}_b + K_{ab} \dot{q}_b = 0; \quad a = 1;\ldots;N; \quad (6)
\]

for the deviation, \(\ddot{q}\), between two nearby trajectories. The \(N\) matrices \(M\), \(C\) and \(K\), are defined by

\[
M_{ab} = \frac{\partial^2 L}{\partial q_a \partial q_b}; \quad C_{ab} = \frac{d}{dt} \frac{\partial^2 L}{\partial q_a \partial \dot{q}_b} + \frac{\partial^2 L}{\partial q_b \partial \dot{q}_a} \frac{\partial^2 L}{\partial q_a \partial q_b}; \quad K_{ab} = \frac{d}{dt} \frac{\partial^2 L}{\partial q_a \partial \dot{q}_b} \quad (7)
\]

\[a; b = 1;\ldots;N;\]
Equations (6) are the Jacobi variational equations for the original system [1, 2, 4].

If the Lagrangian of the system is time-independent, the system has a well-known constant of motion

\[ H = \frac{\partial L}{\partial \dot{q}_b} \dot{q}_b - L; \]  

(8)

in such time-independent case the Jacobi equations also admit hence the importance of the property mentioned above an analogous constant, namely

\[ h = \frac{\partial}{\partial \dot{q}_a} \dot{q}_a + \frac{\partial}{\partial \dot{q}_b} \dot{q}_b; \]  

(9)

using definition (1), the constant \( h \) can be cast in the form

\[ h = \frac{\partial H}{\partial \dot{q}_a} \dot{q}_a + \frac{\partial H}{\partial \dot{q}_b} \dot{q}_b; \]  

(10)

Equation (9) (or (10)) is an important conclusion from the variational formulation.

Notice that the variational equations can be written in first-order form provided that

\[ \det \frac{\partial^2 L}{\partial q_a \partial q_b} \neq 0; \]  

(11)

i.e. that the matrix \( M \) is invertible; this is also the condition for the existence of a Hamiltonian description of the system [1]. Granted such condition, the 2N variational equations can be written as

\[ \frac{dx}{dt} = J \quad x; \]

(12)

where the 2N -vector \( x \) and the 2N -matrix \( J \), are defined respectively by

\[ x = \quad ; \quad J = \begin{bmatrix} 0 & 1 \\ M^{-1}K & M^{-1}C \end{bmatrix} \]

(13)

where the 0 and the 1 stand, respectively, for \( N \times N \) zero and unit matrices.

We can now, for example, use the solutions to equations (12) (or to (5) and (6)) to evaluate the \( N \) Liapunov exponents, \( a \), in the standard way [8, 11]

\[ a = \lim_{t \to \infty} \frac{1}{t} \log \frac{\|x_a(t)\|}{\|x_a(0)\|}; \quad a = 1; \ldots; N; \]  

(14)
where $|v|$ is any norm of the N-vector $v$ and $x(0)$ is just the initial condition for a perturbation, in one of the N- appropriate directions, of the orbit $q(t)$ under analysis.

We emphasize that the variational formulation of the Jacobi equations can be of help for discovering constants of motion in the variational equations of particular system $s$ (using their symmetries, for example), like the one we derived here [equation (9)] for the autonomous case. The close relationship of the variational principle to the Lagrangian through definition (1) reflects the central relationship between $L(q; q; t)$ and the properties of the motion. The Lagrangian is thus, paraphrasing an apt description, the true gene of the motion [12].

The results derived from our variational principle have some bearings on matters of stability. For example, using our results, we can analyze the stability of particle orbits in two dimensions recovering well-established results in the process.

To illustrate the previous assertion, let us analyze the motion of a particle with unit mass under a particular time independent potential, $U(q)$, in two dimensions [9,13]. Select a particular orbit $O$ as a reference, the generalized coordinates can now be chosen as the distance, $z$, from the reference orbit to the particle and the arc length, $s$, from an arbitrary origin on $O$ to the point, also on the reference orbit $O$, from which $z$ is measured. The Lagrangian of the system and the function in these coordinates are (in obvious notation)

\[
L(s; z; z_s) = \frac{1}{2} z^2 + s^2 \left[ 1 + \frac{z^2}{s^2} \right] U(z; s);
\]

\[
(s; z; z_s) = z + s - \frac{z^2}{s} \left[ 1 + \frac{z^2}{s^2} \right] \frac{\partial U}{\partial z} z + \frac{\partial U}{\partial s} s;
\]

where $s$ is the radius of curvature of the reference orbit $O$ at $s$. The system is autonomous, hence the quantity

\[
h = sz + \frac{1}{2} z^2 + \frac{\partial U}{\partial z} z + \frac{\partial U}{\partial s} s;
\]

\[
(16)
\]

\[
(17)
\]
is a constant of the motion. Combining (16) and (17), using equations (5) and (6), and choosing 0 as the orbit under analysis, we can show that the equation governing the stability of that orbit is

\[ z + 3 \frac{\bar{z}^2}{\bar{z}} + \frac{\bar{z} U}{3 \bar{z}^2} z = \frac{2h}{3} ; \]

(18)

since the equation concerns itself only with deviations tangent to the original orbit. The stability of the two-dimensional orbits is easily established in the \( h = 0 \) case, which simply corresponds to analyzing varied trajectories in which the energy does not change respect to its value in the nonvaried orbit 0. In this homogeneous case, a well-known result [14] asserts that the above equation has oscillatory solutions (hence, the analyzed orbit is stable) if the quantity between parenthesis in (18) | called the coefficient of stability | is positive definite at every point on the orbit [13].

For the relationship to the stability of periodic orbits see, for example, [13,15]. Furthermore, we think the variational principle (3) can be useful for explaining in non-standard way some results concerning the relationship between singularities in the exponential map and the corresponding Jacobi fields in geodesic ow s on manifolds [2] with its natural geodular structures [4], and can have relevance in the study of both Lagrangian and time-dependent Hamiltonian mechanics and eld theories in the context of their vertical extensions [3].

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