CONTINUUM LINE-OF-SIGHT PERCOLATION ON POISSON–VORONOI TESSELLATIONS

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Abstract

In this work, we study a new model for continuum line-of-sight percolation in a random environment driven by the Poisson–Voronoi tessellation in the $d$-dimensional Euclidean space. The edges (one-dimensional facets, or simply 1-facets) of this tessellation are the support of a Cox point process, while the vertices (zero-dimensional facets or simply 0-facets) are the support of a Bernoulli point process. Taking the superposition $Z$ of these two processes, two points of $Z$ are linked by an edge if and only if they are sufficiently close and located on the same edge (1-facet) of the supporting tessellation. We study the percolation of the random graph arising from this construction and prove that a 0–1 law, a subcritical phase, and a supercritical phase exist under general assumptions. Our proofs are based on a coarse-graining argument with some notion of stabilization and asymptotic essential connectedness to investigate continuum percolation for Cox point processes. We also give numerical estimates of the critical parameters of the model in the planar case, where our model is intended to represent telecommunications networks in a random environment with obstructive conditions for signal propagation.

Keywords: Percolation; Cox process; Poisson–Voronoi tessellation; coarse-graining arguments; simulation

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1. Introduction

1.1. Background and motivation

Bernoulli bond percolation, introduced in the late fifties by Broadbent and Hammersley [16], is one of the simplest mathematical models featuring phase transition. Since then, this model has been generalized in many different ways, making percolation theory a broader and still very active research topic today.

Since the seminal work [24] of Gilbert, random graphs have been a key mathematical tool for the modeling of telecommunications networks. Good connectivity of the network is then
interpreted as percolation of its associated connectivity graph. Over the years, Gilbert’s original model has been refined, and lots of mathematical models for telecommunications networks are now available in the literature.

In a recent work [30], a new mathematical model for the so-called device-to-device wireless networks in obstructive urban environments was proposed and studied numerically. This model is meant to represent direct wireless connections between users (and possibly some relays) taking place in an urban environment, with limited-range connections possible only within the line of sight along the streets. Such obstructive conditions for signal propagation are sometimes called urban canyon shadowing. In this paper, we study the percolation of this new model via a more theoretical approach.

More precisely, we model the system of streets as the edges (i.e. one-dimensional facets, or simply 1-facets) of the Poisson–Voronoi tessellation (PVT) in $d$-dimensional Euclidean space. These edges (which will also be called streets) form the random support of a Cox point process modeling users of the network, with a constant density $\lambda$ of users per unit length of street. Moreover, the vertices of the PVT (i.e. zero-dimensional facets, which will also be called crossroads) are the support of a Bernoulli point process modeling relays. These relays are assumed to be conditionally independent of the users given the realization of the PVT, and the (conditional) probability of having a relay on a given PVT vertex is denoted by $p$. The superposition of these two point processes (users and relays), denoted by $Z$, defines the nodes of the connectivity graph, denoted by $G$, with edges existing between any two nodes that are located on the same edge of the PVT and closer to each other than some threshold distance $r$, called the connectivity range.

This new percolation model can be seen as a superposition of some more basic models. Indeed, the Bernoulli process of the relays alone, with infinite connectivity range ($r = \infty$), corresponds to the PVT site percolation model that has already attracted some attention in the literature; see e.g. [6, 36]. On the other hand, taking all relays ($p = 1$) with a finite connectivity range ($r < \infty$) and no users on streets ($\lambda = 0$) corresponds to some PVT bond percolation model with an edge open if its length is smaller than $r$, or equivalently to the Gilbert graph [24], considered here on the point process of the vertices of the PVT. To the best of our knowledge, this model, which we call PVT hard-geometric bond percolation, has not been considered before. Finally, adding users (taking $\lambda > 0$) introduces random, conditionally independent, openings of some arbitrarily long streets, where the opening probability depends (in some particular way) on the street length. This is similar to the classical random-connection model [32, Chapter 6], considered here on the point process of the vertices of the PVT. Again, to the best of our knowledge, this model, which we call PVT soft-geometric bond percolation, has not been considered before. Both of these models (PVT hard-geometric and soft-geometric bond percolation) can be seen as random connection models in a line-of-sight environment, given the random support $S$.

Our main findings regarding the general connectivity graph $G$ can be summarized as follows:

- **0–1 law for the percolation probability of the connectivity graph $G$:** The probability that the connectivity graph $G$ percolates is either 0 or 1. This is a consequence of the fact that the superposition $Z$ of the point processes of users and relays is mixing, and hence ergodic.

- **Critical probability of the Bernoulli relay process:** There exists a minimal value of the parameter of the Bernoulli process $p^* \in (0, 1)$ under which percolation of the connectivity graph $G$ cannot happen with positive probability, regardless of all other
parameters. This is a consequence of the nontriviality of the PVT site percolation threshold. Although it has been estimated numerically many times in the literature (see e.g. [6, 36]), we were not able to find any proof of this fact in the literature. Note that the PVT site percolation model should not be confused with the Voronoi tiling percolation model, which consists in coloring each cell of a PVT in black independently from all other cells with some fixed probability \( p \) and investigating the random tiling of black cells. The percolation of this latter model is studied in [4], and the critical probability in the planar case has been proven to be 1/2 in [14].

- **Critical connectivity range:** For \( p > p^* \), there exists a critical connectivity range \( r^* = r^*(p) \) separating the following two connectivity regimes:
  - **Permanently supercritical** \( G \): For \( r > r^*(p) \) the graph \( G \) percolates with positive probability for all \( \lambda \geq 0 \).
  - **User critical** \( G \): For \( r < r^*(p) \) the graph \( G \) exhibits a nontrivial phase transition in \( \lambda \); i.e. it does not percolate for small \( \lambda > 0 \), and it percolates with positive probability for large enough \( \lambda < \infty \).

We prove that the critical range \( r^*(p) \), numerically estimated in [30], is nontrivial in the sense that \( 0 < r^*(p) < \infty \) for \( p > p^* \) large enough, including some \( p < 1 \).

- As a corollary we obtain the existence of a nontrivial phase transition in the PVT hard-geometric bond percolation model.

The rest of this paper is organized as follows. We begin by recalling some related works in Subsection 1.2. Then, in Section 2, we present the details of our network model and introduce convenient notation. In Subsection 2.3, we state our theoretical results in more detail. In Subsection 2.4, we present the results of numerical simulations of our model in the planar case, so as to illustrate our main mathematical results. Then we proceed with the proofs of our results in Section 3. Finally, we conclude and give perspectives for future work in Section 4.

1.2. Related works

In [24], Gilbert introduced percolation in a continuum setting by considering a planar homogeneous Poisson point process where two points are joined by an edge if and only if they are separated by a distance gap less than a given threshold. This model was at the time considered a good candidate for representing a telecommunications network, with the range of the stations being taken into account as a parameter. The Poisson case has now been studied extensively [32], and Gilbert’s model has recently been extended to other types of point processes, including sub-Poisson [9, 10, 11], Ginibre [23], and Gibbsian point processes [28, 39].

The study of percolation processes living in random environments has only been considered recently: it has been noted that many standard techniques from Bernoulli or continuum percolation cannot be applied in such cases. As a matter of fact, new tools and techniques have had to be introduced. In this regard, the papers of Balister and Bollobás [4] and Bollobás and Riordan [14] on the threshold of Voronoi tiling percolation in the plane are pioneering. Later on, [1, 40] contributed additional results concerning this model. Other percolation models [42], tessellations [15], and other random graphs [3, 5, 7] have also been considered. A more general study of Bernoulli and first-passage percolation on random tessellations was conducted in [43, 44].

A natural extension of Gilbert’s model in a random environment setting is obtained by considering a Cox point process, i.e. a Poisson point process with a random intensity
measure. Percolation of Gilbert’s model in such a setting was studied theoretically for the first
time in [26].

In Gilbert’s original model, connectivity between two network nodes depends only on their mutual Euclidean distance. This assumption has proven to be quite simplistic for the modeling of real telecommunications networks, where physical phenomena such as interference, fading, or shadowing are at stake, making the occurrence of connectivity between two nodes depend on other factors. As a matter of fact, other extensions of Gilbert’s work have been considered for more accurate modeling of telecommunications networks. In particular, percolation of the signal-to-interference-plus-noise ratio (SINR) model on the plane was studied theoretically in [20]. In the SINR model, connection between a pair of points depends not only on their relative distance but also on the positions of all the other nodes of the network. SINR percolation for Cox point processes has only been explored very recently [41]. From a more applied perspective, random tessellations have turned out to yield good fits of real street systems, as proven in [25]. Percolation thresholds of the Gilbert graph of Cox processes supported by random tessellations were numerically investigated in [17], yielding other interesting applications for telecommunications networks.

Recently, mathematical models of so-called line-of-sight (LOS) networks have been introduced, modeling telecommunications networks in environments with regular obstructions, such as large urban environments or indoor environments. Nodes of the network are then connected when they are sufficiently close and when they have line-of-sight access to one another, in other words if no physical obstacle stands between them. In [22], asymptotically tight results on $k$-connectivity of the connectivity graphs arising from such models were studied. Bollobás, Janson and Riordan [13] extended these results by introducing a line-of-sight site percolation model on the discrete square lattice $\mathbb{Z}^2$ and the two-dimensional $n$-torus $[n] \times [n] := [1, n] \times [1, n]$, and asymptotical results for the critical probability were derived as well. Interesting connections to Gilbert’s continuum percolation model were also investigated. However, the study of line-of-sight percolation in a continuum setting with a random environment has not, as far as we know, been studied yet.

It is in light of these recent developments that we introduced in our previous work [30] a new percolation model (originally in the planar case) for Cox processes supported by PVTs.

2. Model and main results

2.1. Connectivity graph

Let $d \geq 2$, $\lambda_S > 0$, and let $X_S$ be a homogeneous Poisson point process in the state space $\mathbb{R}^d$ with intensity $\lambda_S \in (0, \infty)$. Consider the Poisson–Voronoi tessellation (PVT) $S$ associated with $X_S$. In particular, $S$ is stationary and isotropic. By analogy with a telecommunications network, $S$ will be called the street system from now on.

Denote by $E := (e_i)_{i \geq 1}$ the edge-set of $S$ and by $V := (v_i)_{i \geq 1}$ the vertex-set of $S$. In other words, the elements of $E$ are the 1-facets of $S$ and the elements of $V$ are the 0-facets of $S$. Furthering the analogy with a telecommunications network, the elements of $E$ (respectively $V$) will be called streets (respectively crossroads).

Let $\Lambda(dx) := \nu_1(S \cap dx)$, where $\nu_1$ denotes the one-dimensional Hausdorff measure of $\mathbb{R}^d$. Observe that $\Lambda$ is a stationary random measure on $\mathbb{R}^d$ ($\Lambda(B)$ is the total edge length of $S$ contained in any Borel set $B \subset \mathbb{R}^d$) with finite non-null intensity $\gamma := \mathbb{E}[\Lambda[0, 1]^d] \in (0, \infty)$. Note that $\gamma$ may also be interpreted as the total edge length per unit volume: for this reason, we call $\gamma$ the street intensity. In the planar case ($d = 2$), it is known that $\gamma = 2\sqrt{\lambda_S}$ (see
For $\lambda > 0$ consider a Cox point process $X^\lambda$ driven by the random intensity measure $\lambda \Lambda$. In other words, conditioned on a given realization of the street system $S$, $X^\lambda$ is a Poisson point process with mean measure $\lambda \Lambda$. In accordance with the telecommunication interpretation of the PVT as a street system, the points of $X^\lambda$ model locations of users (equipped with mobile devices). In particular, the number of users on a given street $e \in E$ is a Poisson random variable with mean $\lambda \nu_1(e)$, and the numbers of users on two disjoint subsets of $E$ are independent random variables.

We consider yet another point process on the PVT, namely a doubly stochastic Bernoulli point process $Y$ on the set of crossroads $V$ with parameter $p$: conditioned on $\Lambda$ (or, equivalently, on the PVT $S$), points of $Y$ are placed on the crossroads $V$ of $S$ independently with probability $p$. The points of $Y$ will be called (fixed) relays. We also assume that the processes of users and of relays are conditionally independent given their random support, i.e. $X^\lambda \cup Y \mid \Lambda$. We denote by $Z := X^\lambda \cup Y$ the superposition of users and relays.

We consider an undirected connectivity graph $\mathcal{G}$ with the set of vertices given by the points of the point process $Z$, and an edge $Z_i \leftrightarrow Z_j$, $i \neq j$, if and only if $Z_i$ and $Z_j$ are located on the same street and of mutual Euclidean distance less than $r$:

$$\forall i \neq j, \quad Z_i \leftrightarrow Z_j \iff \begin{cases} \exists e \in E, \ Z_i \in e \text{ and } Z_j \in e, \\ \|Z_i - Z_j\| \leq r. \end{cases}$$

Figure 1 illustrates a realization of our network model and of its connectivity graph in the planar case ($d = 2$).

In this paper, we study the percolation properties of the graph $\mathcal{G}$. More precisely, we identify regimes (i.e. sets of model parameters $p, \lambda, r$) where $\mathcal{G}$ percolates (i.e. has an infinite component) with positive probability.

### 2.2. Dimensionless model parameters

While our original percolation model parameters are $\gamma, p, \lambda$, and $r$, it is customary to introduce the following dimensionless model parameters by proceeding as follows. Such parameters turn out to be much more convenient for numerical simulations, as they are also scale-invariant.

Denote by $\bar{l}$ the mean length of the typical street (that is to say, the mean length of the typical point of the process of 1-facets of $S$; see [37]). A general formula for $\bar{l}$ in any dimension $d \geq 2$ is available in [33, 35]. In particular, there exists a positive constant $\kappa := \kappa(d)$ such that $\bar{l} = \kappa(d)\lambda^{-1/d}$. Now, introduce the following dimensionless parameters:

$$U := \bar{l} \lambda \quad \text{and} \quad H := \frac{\bar{l}}{r}.$$  

$U$ corresponds to the mean number of users per typical street, while $H$ is the mean number of hops (of length $r$) required to traverse the typical street. It is easily shown that for all $d \geq 2$, $\bar{l} = \infty \iff \gamma = 0$ and $\bar{l} = 0 \iff \gamma = \infty$. In what follows, we will thus denote by $\mathcal{G}_{p, U, H}$ the connectivity graph $\mathcal{G}$ as a function of the model parameters $(p, U, H)$ in the domain $p \in [0; 1]$, $U \geq 0$, $H \geq 0$, with $H = 0$ interpreted as $r = \infty$ for some $0 < \gamma < \infty$ and $U = 0$ interpreted as $\lambda = 0$ for some $0 < \gamma < \infty$. Note that since $S$ is stationary and isotropic, percolation of the connectivity graph $\mathcal{G}$ does not actually depend on the parameter $\gamma$. If needed, we can thus fix a value $\gamma$ once and for all (e.g. $\gamma = 1$), so that there is a one-to-one map $(p, \lambda, r) \mapsto (p, U, H)$ between the original and the dimensionless parameters.
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Figure 1: Illustration of the connectivity graph $G$ in a bounded simulation window of the plane $\mathbb{R}^2$. The dashed blue lines illustrate the PVT supporting the Cox point process of users represented by red circular points. The Bernoulli point process of relays is illustrated by the green triangular points. Finally, the edges of the largest connected component of $G$ in the simulation window are the thicker solid segments highlighted in orange, while the edges of smaller connected components are the thinner solid segments highlighted in black. These connected components illustrate the connectivity mechanism given by (1).

2.3. Results

Define

$$P(p, U, H) := \mathbb{P}(G_{p, U, H} \text{ percolates}),$$

and observe that $P$ is increasing in $p$ and $U$ and decreasing in $H$.

Our first result is an ergodicity result. More precisely, we have the following.

**Proposition 1.** The superposition $Z$ of the point processes of users and relays is mixing, and hence ergodic.

Since the percolation of the connectivity graph $G$ is a translation-invariant event, a straightforward consequence of the previous result is the following 0–1 law.

**Corollary 1.** For every $\gamma > 0$, $p \in [0, 1]$, $U \geq 0$, $H \geq 0$,

$$P(p, U, H) \in \{0, 1\}.$$ 

In other words, percolation of the connectivity graph $G$ is either an almost sure or almost impossible event.

For given $p \geq 0$, $H \geq 0$, consider the following critical value of the mean number of users per street $U$:

$$U_c(p, H) := \inf\{U \geq 0 : P(p, U, H) > 0\},$$

with $U_c(p, H) := \infty$ if $P(p, U, H) = 0$ for all $U \geq 0$. 

We aim to show that there is a region (i.e. a connected subset) of parameters \((p, H)\) such that \(0 < U_c(p, H) < \infty\). This is the region where the percolation of \(G\) exhibits nontrivial phase transition in the density of users. Existence of this region follows from the next two results.

**Theorem 1.** (Existence of subcritical intensities of users.) For large enough \(H \in [0, \infty)\) and small enough \(U > 0\) (with the thresholds for \(H\) and \(U\) not depending on one another) we have \(P(1, U, H) = 0\) and, consequently, \(P(p, U, H) = 0\) for any \(p \in [0, 1]\).

**Theorem 2.** (Existence of supercritical intensities of users.) For any \(H \in [0, \infty)\), for large enough \(p \in (0, 1)\) and \(U < \infty\) (the thresholds for \(p\) and \(U\) depend on \(H\) but not on one another) we have \(P(p, U, H) > 0\).

The proofs are presented in Section 3.

**Remark 1.** There are three and may be up to five different ranges of parameters \((p, H)\) of interest in our model.

The range of parameters \((p, H)\) where \(0 < U_c(p, H) < \infty\) (the upper-right range schematically presented in orange in Figure 2) can be seen as the critical range of \((p, H)\), in the sense that it separates the following two ranges of \((p, H)\): the (permanently) subcritical range (the lower range schematically presented in blue in Figure 2), where \(G\) does not percolate, however large the density of users \((U_c(p, H) = \infty)\), and the (permanently) supercritical range (the upper-left range schematically presented in red in Figure 2), where \(G\) percolates with positive probability, however small the density of users \((U_c(p, H) = 0)\). We cannot exclude the possibility that this latter range contains a non-empty subset of \((p, H)\) such that \(G\) does not percolate without users \((U = 0)\) but percolates with positive probability for an arbitrarily small density of users, as depicted in Figure 3. Moreover, we do not know whether the permanently subcritical range contains some \(p > p^*\), as also depicted in Figure 3. Note that we do not know the exact shapes.
FIGURE 3: Phase transition diagram of $G$ with hypothetical ranges of $(p, H)$. Estimated values for $H \mapsto p_c(H)$, $H_c$, and $p^*$ are obtained in the planar case via Monte Carlo simulations; see Subsection 2.4 and the earlier publication [30].

of the curves separating these ranges except that they are monotonic. Even continuity is not known.

In what follows we discuss some special cases of our percolation model.

2.3.1 PVT site percolation. Note that for $H = 0$ ($r = \infty$ for some $\gamma > 0$), $P(p, U, 0) = P_{PVT}(p)$ does not depend on $U$ and corresponds to the probability of the (independent and identically distributed (i.i.d.)) site percolation model on the (dimensionless) planar PVT. Denote the critical parameter of this model by

$$p^* := \inf\{p \in [0, 1] : P_{PVT}(p) > 0\}.$$

Clearly, by the monotonicity of the model, $G_{p, U, H}$ does not percolate for $p < p^*$, whatever the values of $U \geq 0$, $H \geq 0$. Moreover, as a consequence of Theorem 2 and of standard percolation arguments, we obtain the nontriviality of the PVT site percolation threshold, as follows.

**Proposition 2.** We have $p^* \in (0, 1)$.

2.3.2. PVT hard-geometric bond percolation. For $U = 0$ (no mobile users), $G_{1, 0, H}$ corresponds to a (non-standard) inhomogeneous bond percolation model on the PVT, in which the edges of the PVT are open or closed depending whether their length is smaller or larger than some threshold. We call this **PVT hard-geometric bond percolation**. It seems that this model has not been studied in the literature.

Define the critical bond parameter of this model as

$$H_c := \sup\{H \geq 0 : P(1, 0, H) > 0\}.$$
Note that $H_c \leq H_0 := \sup\{H \geq 0 : U_c(1, H) = 0\}$ and, by Theorem 1, $H_0 < \infty$. This observation, combined with the following result, ensures that there is a nontrivial phase transition in the PVT hard-geometric bond percolation model, as stated in the introduction.

**Theorem 3.** (Existence of the permanently supercritical range.) *For large enough $p < 1$ and small enough $H > 0$ (with the thresholds for $p$ and $H$ not depending on one another), we have that $P(p, 0, H) > 0$.*

As a consequence of Theorem 3 and the monotonicity of $P(p, 0, H)$ with $p$, we immediately have the following result.

**Corollary 2.** (Existence of a supercritical phase in the PVT hard-geometric bond percolation.) *For small enough $H > 0$, we have that $P(1, 0, H) > 0$. Therefore, $H_c > 0$.***

### 2.3.3 PVT soft-geometric bond percolation

Considering $U > 0$ introduces to our model the possibility of opening some long edges, which are not open in the PVT hard-geometric bond percolation. Note that this is equivalent to yet another bond percolation, in which the edges of the PVT are open independently with probabilities depending on their lengths. We call this *soft-geometric bond percolation*. It seems that this model has not been studied in the literature either.

### 2.3.4 $\mathcal{G}$ as a superposition of three percolation models

Note that $\mathcal{G}_{p, U, H}$ is a superposition of the three independent (given the PVT) percolation models: the site model, the hard-geometric bond model, and the soft-geometric bond model. The reason we cannot exclude the hypothetical ranges of $(p, H)$ such that $U_c(p, H) = \infty$ for $p > p^*$ (see Figure 3) is that we do not know whether for all $p > p^*$ the percolation of $\mathcal{G}$ can be preserved when lowering the distance threshold of the hard-geometric bond percolation (increasing $H$) by increasing the probabilities of edge opening (increasing $U$) in the soft-geometric percolation model. This is possible only for large enough $p$ (see Theorem 2).

### 2.4. Numerical observations in the planar case

In what follows, for the sake of visualization, we briefly recall some numerical findings in the planar case (i.e. setting $d = 2$) that were obtained in an earlier publication [30], to which we refer the reader for further information regarding the simulation methodology. Note that in the planar case, it is known (see e.g. [37]) that we have

$$\gamma = 2\sqrt{\lambda_S} \quad \text{and} \quad \bar{l} = \frac{2}{3\sqrt{\lambda_S}};$$

as a consequence of the two previous formulae, we have $\kappa(2) = \frac{2}{3}$, and the length of the typical edge can be written as $\bar{l} = 4/(3\gamma)$.

The estimates for the critical values $p^*$ and $H_c$ were found to be close to $p^* \approx 0.713$ and $H_c \approx 0.743$. Recall that $H_c$ concerns a model which, as far as we know, has not yet been studied in the literature. As for the estimate for $p^*$, our value differs only slightly from the most recent estimate available in the literature [6], which is $p^* \approx 0.71410 \pm 0.00002$. While the authors providing this estimate proceeded with Monte Carlo simulations with periodic boundary conditions and investigated the growth of the largest cluster, we chose a crossing-window method to obtain our estimate of $p^*$.

For $H < H_c$ define

$$p_c(H) := \inf\{p > 0 : P(p, 0, H) > 0\}.$$
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This is hence the lower boundary of the strictly supercritical range of \((p, H)\). Some estimated values of the function \(p_c(H)\) are presented in Figure 2.

Finally, Figure 4 presents some estimated values of \(U_c(p, H)\) for some selected parameters \((p, H)\) (mainly in the critical range).

3. Proofs

3.1. General approach

Theorems 1–3 and Corollary 2 have been stated in terms of the dimensionless parameters \((p, U, H)\) introduced in Subsection 2.2. This allowed us to introduce the different connectivity regimes and their frontiers in a more applied fashion, as illustrated by Figures 2 and 3.

However, when working on the proofs of the aforementioned results, it will be much more convenient to come back to the original parameters \((\gamma, p, \lambda, r)\). This is mostly due to the fact that \(H\), being inversely proportional to \(r\), is less easy to work with when we are considering particular events related to connectivity in the network graph. Moreover, recalling that percolation of the connectivity graph \(G\) does not depend on the parameter \(\gamma\), we can set \(\gamma = 1\) without loss of generality in what follows. Switching back to the original network parameters \((\gamma = 1, p, \lambda, r)\), we will therefore now refer to the connectivity graph as \(G = G_{p, \lambda, r}\) and prove the following equivalent formulations of Theorems 1–3.

**Reformulation 1.** (Reformulation of Theorem 1.) For small enough \(r \in (0, \infty]\) and small enough \(\lambda > 0\) (with the thresholds for \(r\) and \(\lambda\) not depending on one another), \(G_{1, \lambda, r}\) does not percolate; consequently, neither does \(G_{p, \lambda, r}\) for any \(p \in [0, 1]\).

**Reformulation 2.** (Reformulation of Theorem 2.) For any \(r \in (0, \infty]\), for large enough \(p \in (0, 1)\) and \(\lambda < \infty\) (the thresholds for \(p\) and \(\lambda\) depend on \(r\) but not on one another), \(G_{p, \lambda, r}\) percolates.

**Reformulation 3.** (Reformulation of Theorem 3.) For large enough \(p < 1\) and large enough \(r < \infty\) (with the thresholds for \(p\) and \(r\) not depending on one another), \(G_{p, 0, r}\) percolates.
The general idea of the proofs of the above reformulations is to use a coarse-graining argument. This consists in the following steps:

1. Map the percolation of $G$ to a discretized percolation process on a rescaled integer lattice.

2. As needed, relate the percolation (or absence of percolation) of $G$ to the percolation (or absence of percolation) of the discretized process.

3. Prove that the discretized process features only short-range dependencies. More precisely, we will refer to the notion of $k$-dependence, which will be introduced in due course; see Definition 5.

4. Apply a result from Liggett, Schonmann and Stacey ([31, Theorem 0.0]) to conclude that the discretized process is dominated by a Bernoulli percolation process, and thereby draw conclusions on $G$.

3.2. Preparation

We begin by introducing some notation and definitions that will be useful for the purposes of our development.

We will use the following convenient notation for the length $|s|$ of a street segment $s$ (which is a connected, topologically closed subset of some $e \in E$). For $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^d$, we denote the Euclidean distance between $A$ and $B$ (as customary) by $\text{dist}(A, B) := \inf\{\|x - y\|_2, \ x \in A, \ y \in B\}$.

For $x \in \mathbb{R}^d$, $a > 0$ we denote by $Q_a(x) := x + [-a/2, a/2]^d$ the $d$-dimensional cube of side length $a$ centered at $x$. We note that this is exactly the definition of the closed ball $B(x, a/2)$ with center $x$ and radius $a/2$ for the infinite norm of $\mathbb{R}^d$:

$$Q_a(x) = \{y \in \mathbb{R}^d : \|y - x\|_\infty \leq a/2\} = B(x, a/2).$$

For simplicity, whenever $a = n \in \mathbb{N} \setminus \{0\}$, we will write $Q_n$ to mean $Q_n(0)$.

We denote by $\mathbf{M}$ the space of Borel measures on $\mathbb{R}^d$, equipped with the evaluation $\sigma$-algebra [29, Section 13.1], which is the smallest $\sigma$-algebra making the mappings $\mathbf{M} \ni \Xi \mapsto \Xi(B)$ measurable for all Borel sets $B \subset \mathbb{R}^d$. For a (possibly random) Borel measure $\mu$ on $\mathbb{R}^d$ and $A \subset \mathbb{R}^d$, we denote the restriction of $\mu$ to $A$ by $\mu_A(\cdot) := \mu(A \cap \cdot)$. We also adapt the definition of the support of a measure as follows: let $\mu$ be a (possibly random) Borel measure on $\mathbb{R}^2$. The support of $\mu$ is the following set:

$$\text{supp}(\mu) := \{x \in \mathbb{R}^d : \forall \epsilon > 0, \ \mu(Q_\epsilon(x)) > 0\}.$$

We will also need the concepts of stabilization and asymptotic essential connectedness, both introduced in [26] for investigating spatial dependencies of random measures.

**Definition 1.** [26, Definition 2.3.] A random measure $\Xi$ on $\mathbb{R}^d$ is called stabilizing if there exists a random field of stabilization radii $R = \{R_s\}_{s \in \mathbb{R}^d}$ defined on the same probability space as $\Xi$ and $\Xi$-measurable, such that

1. $(\Xi, R)$ are jointly stationary; 

2. $\lim_{n \uparrow \infty} \mathbb{P}\left(\sup_{y \in Q_n \cap Q^d} R_s < n\right) = 1$;
(3) for all \( n \geq 1 \), the random variables

\[
\left\{ f \left( \Xi_{Q_n(x)} \right) \mathbb{1} \left\{ \sup_{y \in Q_n(x) \cap \mathbb{R}^d} R_y < n \right\} \right\}_{x \in \varphi}
\]

are independent for all bounded measurable functions

\[ f : \mathbb{M} \to [0, +\infty) \]

and finite \( \varphi \subset \mathbb{R}^d \) such that \( \forall x \in \varphi, \ \text{dist}(x, \varphi \setminus \{x\}) > 3n \).

For the sake of simplicity, we slightly modify the definition of asymptotic essential connectedness given in [26], using the following definition instead.

**Definition 2.** Let \( \Xi \) be a random measure on \( \mathbb{R}^d \). Then \( \Xi \) is asymptotically essentially connected if there exists a random field \( R = \{ R_x \}_{x \in \mathbb{R}^d} \) such that \( \Xi \) is stabilizing with \( R \) as in Definition 1, and if for all \( n \geq 1 \), whenever

\[
\sup_{y \in Q_{2n} \cap \mathbb{Q}^d} R_y < n/2,
\]

the following assertions hold:

1. \( \text{supp}(\Xi_{Q_n}) \neq \emptyset \);
2. \( \text{supp}(\Xi_{Q_n}) \) is contained in a connected component of \( \text{supp}(\Xi_{2Q_n}) \).

The following result is stated in [26, Example 3.1] for a slightly modified version of Definition 2. It is easy to check that it adapts to our case as follows.

**Proposition 3.** Let \( \Lambda = \nu_1(S \cap dx) \), where \( S \) is the PVT generated by a homogeneous stationary Poisson point process. Then \( \Lambda \) is stabilizing and asymptotically essentially connected with the following stabilization field:

\[
\forall x \in \mathbb{R}^d, \quad R_x := \inf \{ \| x - X_{S,i} \|_2, \ X_{S,i} \in X_S \},
\]

where \( X_S \) is the Poisson point process generating \( S \).

For simplicity, for \( x \in \mathbb{R}^d \) and \( n \in \mathbb{N} \setminus \{0\} \) we define

\[
R(Q_n(x)) := \sup_{y \in Q_n(x) \cap \mathbb{Q}^d} R_y.
\]

Finally, we define the openness and closedness of crossroads and street segments (including whole streets) as follows.

**Definition 3.** (Open/closed crossroad.) We say a crossroad \( v \in V \) is open if it is an atom of the point process \( Y \), i.e. \( Y(\{v\}) = 1 \). We say \( v \) is closed if it is not open.

**Definition 4.** (Open/closed street segment.) Let \( e \in E \) be a street and let \( s \subseteq e \) be a non-empty street segment. We say \( s \) is open if either of the two following sets of conditions are satisfied:
1. $|s| \leq r$, 
   OR
2. \[
\begin{cases}
|s| > r, \\
\forall c \subset s, \ (|c| = r, \ c \text{ connected and } c \text{ topologically closed}) \Rightarrow X^c(c) \geq 1.
\end{cases}
\]

We say that $s$ is closed if $s$ is not open.

We are now ready to proceed with the proofs of Theorems 1–3.

3.3. Proof of Proposition 1

To prove that $Z$ is mixing, we will work on the canonical space; it will thus suffice to show that

$$
\lim_{\|x\|_2 \to \infty} \mathbb{P}(A \cap S_xB) = \mathbb{P}(A)\mathbb{P}(B)
$$

for all events $A, B \in \sigma(Z)$ that are measurable with respect to the sigma-algebra $\sigma(Z)$ generated by $Z$, where $\{S_x\}_{x \in \mathbb{R}^d}$ denotes the natural shift on $\mathbb{R}^d$.

Note first that by [19, Lemma 12.3.11] (and as done in the proof of [19, Proposition 12.3.6]), it suffices to check the mixing condition (2) for local events, i.e. events of the form $A \in \sigma(Z \cap W_A)$ and $B \in \sigma(Z \cap W_B)$, where $W_A$ and $W_B$ are compact observation windows in $\mathbb{R}^d$. Thus, let $A$ and $B$ be such events. We will show that for all $\epsilon > 0$, we can find $x$ with $\|x\|_2$ sufficiently large so that $|\mathbb{P}(A \cap S_xB) - \mathbb{P}(A)\mathbb{P}(B)| \leq \epsilon$.

Take any $\epsilon > 0$. By Condition 2 in the definition of stabilization (Definition 1), we can find sufficiently large $n \geq 1$ such that $\mathbb{P}(R(Q_n) \geq n) \leq \epsilon/3$. Moreover, such $n$ can be chosen so as to satisfy $W_A \subseteq Q_n$ and $W_B \subseteq Q_n$. Fix such $n$.

Since $A \in \sigma(Z \cap W_A)$, $A$ depends only on the configuration of $Z$ inside $W_A$. In the same way, $B$ depends only on the configuration of $Z$ inside $W_B$, and $S_xB$ depends only on the configuration of $Z$ inside $S_xW_B = W_B - x$. Since $B \subseteq Q_n$, we have that $S_xW_B \subseteq Q_n - x = Q_n(-x)$. Take $x$ with $\|x\|_2 \geq 6n\sqrt{2}$. Then we have $Q_n \cap Q_n(-x) = \emptyset$ and thus $W_A \cap S_xW_B = \emptyset$, so that the events $A$ and $S_xB$ depend on the configuration of $Z$ in disjoint sets. Since the conditional distribution of $Z$ given the random support $\Lambda$ is that of a superposition of a Bernoulli process and of a Poisson point process, the events $A$ and $S_xB$ are conditionally independent given $\Lambda$. Hence,

$$
\mathbb{P}(A \cap S_xB) = \mathbb{E}\left[\mathbb{E}((1\{A\}1\{S_xB\} | \Lambda))\right] = \mathbb{E}\left[\mathbb{E}(1\{A\} | \Lambda)\mathbb{E}(1\{S_xB\} | \Lambda)\right].
$$

Now, since $A \in \sigma(Z \cap W_A)$ with $W_A \subseteq Q_n$, we can write $\mathbb{E}(1\{A\} | \Lambda) = f(\Lambda_{Q_n})$ as a bounded deterministic function of $\Lambda_{Q_n}$. In the same way, we can write $\mathbb{E}(1\{S_xB\} | \Lambda) = g(\Lambda_{Q_n(-x)})$ as a bounded deterministic function of $\Lambda_{Q_n(-x)}$. By (3), we thus get

$$
\mathbb{P}(A \cap S_xB) = \mathbb{E}\left[f(\Lambda_{Q_n})g(\Lambda_{Q_n(-x)})\right]
= \mathbb{E}\left[f(\Lambda_{Q_n})g(\Lambda_{Q_n(-x)})1\{R(Q_n) < n\}1\{R(Q_n(-x)) < n\}\right]
+ \mathbb{E}\left[f(\Lambda_{Q_n})g(\Lambda_{Q_n(-x)})1\{(R(Q_n) \geq n) \cup (R(Q_n(-x)) \geq n)\}\right].
$$

Let us first deal with the second term appearing in the right-hand side of (4). Using the fact that both $f$ and $g$, being conditional expectations of indicator functions, are bounded above by 1, we get

$$
\mathbb{E}\left[f(\Lambda_{Q_n})g(\Lambda_{Q_n(-x)})1\{(R(Q_n) \geq n) \cup (R(Q_n(-x)) \geq n)\}\right]
\leq \mathbb{P}\left[(R(Q_n) \geq n) \cup (R(Q_n(-x)) \geq n)\right]
\leq \mathbb{P}\left[R(Q_n) \geq n\right] + \mathbb{P}\left[R(Q_n(-x)) \geq n\right],
$$
where we have used the union bound in (5). Now, by stationarity of the stabilization field \( \{R_i\}_{i \in \mathbb{R}^d} \), we get that the right-hand side in (5) is equal to \( 2 \mathbb{P}[R(Q_n) \geq n] \). In all, we thus get

\[
\mathbb{E} \left[ f(\Lambda_{Q_n}) g(\Lambda_{Q_n(-x)}) \mathbb{1} \{ (R(Q_n) \geq n) \cup (R(Q_n(-x)) \geq n) \} \right] \leq 2\varepsilon/3. \quad (6)
\]

We now deal with the first term appearing in the right-hand side of (4). Note that since \( \|x\|_2 > 6n \sqrt{2} \), the set \( \varphi := \{0, -x\} \subset \mathbb{R}^d \) satisfies \( \forall y \in \varphi, \text{dist}(y, \varphi \setminus \{y\}) > 3n \), and so, by Condition 3 in the definition of stabilization, the random variables \( f(\Lambda_{Q_n}) \mathbb{1} \{R(Q_n) < n\} \) and \( g(\Lambda_{Q_n(-x)}) \mathbb{1} \{R(Q_n(-x)) < n\} \) are independent. Thus, the first term appearing in the right-hand side of (4) becomes

\[
\mathbb{E} \left[ f(\Lambda_{Q_n}) g(\Lambda_{Q_n(-x)}) \mathbb{1} \{ R(Q_n) < n \} \mathbb{1} \{ R(Q_n(-x)) < n \} \right]
= \mathbb{E} \left[ f(\Lambda_{Q_n}) \mathbb{1} \{ R(Q_n) < n \} \right] \mathbb{E} \left[ g(\Lambda_{Q_n(-x)}) \mathbb{1} \{ R(Q_n(-x)) < n \} \right]. \quad (7)
\]

Now, using the fact that \( f(\Lambda_{Q_n}) : = \mathbb{E} (\mathbb{1}[A] | \Lambda) \) and noting that the event \( \{R(Q_n) < n\} \) is \( \Lambda \)-measurable, we can put everything back into a single expectation and get

\[
\mathbb{E} \left[ f(\Lambda_{Q_n}) \mathbb{1} \{ R(Q_n) < n \} \right] = \mathbb{E} \left[ \mathbb{1}[A] | \Lambda \right] \mathbb{1} \{ R(Q_n) < n \}
= \mathbb{E} \left[ \mathbb{1}[A] \mathbb{1} \{ R(Q_n) < n \} | \Lambda \right]
= \mathbb{E} \left[ \mathbb{1}[A] \mathbb{1} \{ R(Q_n) < n \} \right]
= \mathbb{P}(A \cap \{ R(Q_n) < n \}).
\]

In the same way, we get

\[
\mathbb{E} \left[ g(\Lambda_{Q_n(-x)}) \mathbb{1} \{ R(Q_n(-x)) < n \} \right] = \mathbb{P}(S_x B \cap \{ R(Q_n(-x)) < n \}).
\]

Thus, (7) yields

\[
\mathbb{E} \left[ f(\Lambda_{Q_n}) g(\Lambda_{Q_n(-x)}) \mathbb{1} \{ R(Q_n) < n \} \mathbb{1} \{ R(Q_n(-x)) < n \} \right]
= \mathbb{P}(A \cap \{ R(Q_n) < n \}) \mathbb{P}(S_x B \cap \{ R(Q_n(-x)) < n \})
= \mathbb{P}(A \cap \{ R(Q_n) < n \}) \mathbb{P}(B \cap \{ R(Q_n) < n \})
= \mathbb{P}(A \cap \{ R(Q_n) < n \}) \mathbb{P}(B \cap \{ R(Q_n) < n \}),
\]

where we have used the stationarity assumption to get the last line. Finally, using the fact that \( \mathbb{P}(R(Q_n) < n) \geq 1 - \varepsilon/3 \), we get

\[
|\mathbb{P}(A \cap \{ R(Q_n) < n \}) \mathbb{P}(B \cap \{ R(Q_n) < n \}) - \mathbb{P}(A) \mathbb{P}(B)| \leq \varepsilon/3
\]

and thus

\[
|\mathbb{E} \left[ f(\Lambda_{Q_n}) g(\Lambda_{Q_n(-x)}) \mathbb{1} \{ R(Q_n) < n \} \mathbb{1} \{ R(Q_n(-x)) < n \} \right] - \mathbb{P}(A) \mathbb{P}(B)| \leq \varepsilon/3. \quad (8)
\]

Using (6) and (8) to put everything back together in (4) and using the triangle inequality, we finally get

\[
|\mathbb{P}(A \cap S_x B) - \mathbb{P}(A) \mathbb{P}(B)| \leq 2\varepsilon/3 + \varepsilon/3 = \varepsilon
\]

for sufficiently large \( x \), as required. This concludes the proof of Proposition 1.
Remark 2. Note that we actually did not need to use the PVT structure or the asymptotic essential connectedness of the random support \( S \) here. We only used the fact that \( S \) is a stabilizing random tessellation and the complete independence properties of the point processes of users \( X^\lambda \) and of relays \( Y \) given their random support \( S \). As a matter of fact, Proposition 1 can be generalized to any stabilizing random tessellation \( S \) in \( \mathbb{R}^d \).

3.4. Proof of Theorem 1

As mentioned earlier, proving Theorem 1 is equivalent to proving Reformulation 1. This in turn is equivalent to showing that \( G \) does not percolate when \( p = 1 \) and \( \lambda, r \) are sufficiently small but positive. We will use a coarse-graining argument and introduce a discrete site percolation model on the integer lattice \( \mathbb{Z}^d \), constructed in such a way that if it does not percolate, then neither does \( G \). Proving the absence of percolation of the integer lattice model will then be done by appealing to its local dependence.

To this end, for \( n \geq 1 \), we say a site \( z \in \mathbb{Z}^d \) is \( n \)-good if the following conditions are satisfied:

1. \( R(Q_n(nz)) < n \);
2. for any \( e \in E \), if \( s_{z,e} := e \cap Q_n(nz) \neq \emptyset \), then \( s_{z,e} \) is closed.

We say a site \( z \in \mathbb{Z}^d \) is \( n \)-bad if it is not \( n \)-good.

Our first claim is the following.

Lemma 1. Percolation of \( G \) implies percolation of the process of \( n \)-bad sites.

Proof. Assume \( G \) percolates and denote by \( C \) an unbounded (connected) component of \( G \). Define \( Z = Z_n := \{ z \in \mathbb{Z}^d : C \cap Q_n(nz) \neq \emptyset \} \). Since \( C \) is unbounded we have \( \#(Z) = \infty \). Observe that for all \( z \in Z \), \( z \) is \( n \)-bad, since Condition 2 of \( n \)-goodness is not satisfied (there exists an open street segment intersecting \( Q_n(nz) \)). Also, \( Z \) is almost surely connected in \( \mathbb{Z}^d \) (in the following sense: \( z, z' \in \mathbb{Z}^d, z \neq z' \) are connected in \( \mathbb{Z}^d \) if \( \|z - z'\|_1 = 1 \)). This follows from the fact that the probability that some edge \( e \in E \) of the PVT intersects \( \mathbb{Z}^d \) is equal to zero. (This is actually also true for the Voronoi tessellation generated by any stationary point process, as a consequence of the fact that such a process does not have points which are equidistant to a given, fixed location; see e.g. [8, Lemma 11.2.3].) Hence, the process of \( n \)-bad sites percolates.

By Lemma 1, it suffices to prove that the process of \( n \)-bad sites does not percolate (for some \( n \)) when \( \lambda \) and \( r \) are sufficiently small but positive. This will be done using the fact that it is a 3-dependent percolation model on the integer lattice \( \mathbb{Z}^d \).

Definition 5. Let \( X = (X_z)_{z \in \mathbb{Z}^d} \) be a discrete random field. Let \( k \geq 1 \). Then \( X \) is said to be \( k \)-dependent if for all \( q \geq 1 \) and all \( \{s_1, \ldots, s_q\} \subset \mathbb{Z}^d \) finite with the property that \( \forall i \neq j, \|s_i - s_j\|_\infty > k \), the random variables \( (X_{s_i})_{1 \leq i \leq q} \) are independent.

As previously stated, we have the following.

Lemma 2. For \( z \in \mathbb{Z}^d \), set \( \zeta_z := 1 \{ z \text{ is } n \text{-bad} \} \). Then \( (\zeta_z)_{z \in \mathbb{Z}^d} \) is a 3-dependent random field.

Proof. As a starting point, note that for every \( z \in \mathbb{Z}^d \), \( \zeta_z = 1 - 1 \{ z \text{ is } n \text{-good} \} \). It is therefore equivalent to prove that the process of \( n \)-good sites is 3-dependent.
For $z \in \mathbb{Z}^d$, set $\xi_z = \mathbb{1} \{z \text{ is } n\text{-good}\}$. Let $\{z_1, \ldots, z_q\} \subset \mathbb{Z}^d$ be such that for any $i \neq j$, $\|z_i - z_j\|_\infty > 3$. We want to show that the random variables $(\xi_z)_{1 \leq i \leq q}$ are independent. Since we are dealing with indicator functions, this is equivalent to showing that

$$E \left( \prod_{i=1}^{q} \xi_{z_i} \right) = \prod_{i=1}^{q} E(\xi_{z_i}).$$

Now, we have

$$E \left( \prod_{i=1}^{q} \xi_{z_i} \right) = E \left[ E \left( \prod_{i=1}^{q} \xi_{z_i} \mid \Lambda \right) \right]$$

$$= E \left[ E \left( \prod_{i=1}^{q} \mathbb{1}\{R(Q_n(nz_i)) < n\} \prod_{i=1}^{q} \mathbb{1}\{\forall e \in E, s_{z_i,e} \text{ is closed}\} \mid \Lambda \right) \right]$$

$$= E \left[ \prod_{i=1}^{q} \mathbb{1}\{R(Q_n(nz_i)) < n\} \prod_{i=1}^{q} E(\mathbb{1}\{\forall e \in E, s_{z_i,e} \text{ is closed}\} \mid \Lambda) \right], \quad (9)$$

where we have used $\Lambda$-measurability of the random variables $\{R_x\}_{x \in \mathbb{R}^d}$ in (9).

For $1 \leq i \leq q$, set $A_{z_i} := \{\forall e \in E, s_{z_i,e} \text{ is closed}\}$. According to Definition 4, for a given $1 \leq i \leq q$, the event $A_{z_i}$ depends only on the configuration of the random measure $\Lambda$ and of the Cox point process $X^\lambda$ inside the $d$-dimensional cube $Q_n(nz_i)$. Therefore, given $\Lambda$, the events $\{A_{z_i} : 1 \leq i \leq q\}$ depend only on $X^\lambda \cap Q_n(nz_i)$, $1 \leq i \leq q$. Since we have $\|z_i - z_j\|_\infty > 3$ for all $i \neq j$, the $d$-dimensional cubes $Q_n(nz_i)$ are disjoint. Moreover, given $\Lambda$, $X^\lambda$ has the distribution of a Poisson point process. Thus, by the Poisson independence property, the events $(A_{z_i})_{1 \leq i \leq q}$ are conditionally independent given $\Lambda$. Hence (9) yields

$$E \left( \prod_{i=1}^{q} \xi_{z_i} \right) = E \left[ \prod_{i=1}^{q} \mathbb{1}\{R(Q_n(nz_i)) < n\} \prod_{i=1}^{q} E(\mathbb{1}\{\forall e \in E, s_{z_i,e} \text{ is closed}\} \mid \Lambda) \right]. \quad (10)$$

Recall that the restriction of the random measure $\Lambda$ to the $d$-dimensional cube $Q_n(x)$ is denoted by $\Lambda_{Q_n(x)}(\cdot) := \Lambda(Q_n(x) \cap \cdot)$, and set

$$f(\Lambda_{Q_n(x)}) := E \left( \mathbb{1}\{\forall e \in E, s_{x,e} \text{ is closed}\} \mid \Lambda \right).$$

Then $f$ is a deterministic, bounded, and measurable function of $\Lambda_{Q_n(x)}$. Moreover, the set $\varphi := \{nz_1, \ldots, nz_q\} \subset \mathbb{R}^d$ is a finite subset of $\mathbb{R}^d$ satisfying

$$\forall i \neq j, \quad \|nz_i - nz_j\|_\infty > 3n.$$

Since the infinite norm is always bounded above by the Euclidean norm, we have $\|nz_i - nz_j\|_2 > 3n$ for all $i \neq j$, and so $\varphi$ satisfies

$$\forall x \in \varphi, \quad \text{dist}(x, \varphi \setminus \{x\}) > 3n.$$


Hence, by Condition 3 in the definition of stabilization (Definition 1), the random variables appearing in the right-hand side of (10) are independent. This yields

$$\mathbb{E}\left(\prod_{i=1}^{q} \xi_{z_i}\right) = \prod_{i=1}^{q} \mathbb{E}\left(1\{R(Q_{n}(nz_i)) < n\}\prod_{i=1}^{q} \mathbb{E}\left(1\{\forall e \in E, s_{z_i,e} \text{ is closed}\}\right| \Lambda\right)$$

$$= \prod_{i=1}^{q} \mathbb{E}(\xi_{z_i}) ,$$

thus concluding the proof of the lemma. □

Now we prove that the probability that an arbitrary site, which by stationarity can be chosen to be the origin $0 \in \mathbb{Z}^d$, is $n$-bad can be made arbitrarily small by first taking some large enough finite $n$ and then small enough positive $\lambda, r$, as stated in the following lemma.

**Lemma 3.**

$$\lim_{n \uparrow \infty} \lim_{\lambda, r \downarrow 0} \mathbb{P}(0 \text{ is } n\text{-bad}) = 0.$$

**Proof.** Note that we have

$$\mathbb{P}(0 \text{ is } n \text{-bad}) = \mathbb{P}\left(\left\{R(Q_{n}) \geq n\right\} \cup \left\{\exists e \in E: e \cap Q_{n} \neq \emptyset \text{ and open}\right\}\right) \leq \mathbb{P}(R(Q_{n}) \geq n) + \mathbb{P}(\exists e \in E: e \cap Q_{n} \neq \emptyset \text{ and open})$$

$$\leq \mathbb{P}(R(Q_{n}) \geq n) + \mathbb{P}(\exists e \in E: 0 < |e \cap Q_{n}| \leq r) + \mathbb{P}(\exists e \in E: e \cap Q_{n} \text{ satisfies Cond. } 2 \text{ in Definition } 4).$$

Take any $\epsilon > 0$. By the stabilization property of the PVT (Proposition 3) we have $\lim_{n \uparrow \infty} \mathbb{P}(R(Q_{n}) \geq n) = 0$, and so we can fix $n$ large enough to make the probability in (a) smaller than $\epsilon/3$. Then $Q_{n}$ intersects almost surely zero or a finite number of edges $e \in E$. Hence the probability in (b) converges to 0 when $r \to 0$, and consequently we can take $r$ small enough to make the probability in (b) smaller than $\epsilon/3$. Finally, for given $n$ (and independently of $r$), we can take $\lambda$ small enough to make the probability in (c) smaller than $\epsilon/3$. Indeed, this latter probability is dominated by the probability that $X^{\lambda}(Q_{n}) \geq 1$ and thus converges to 0 when $\lambda \to 0$ for any finite $n$. This concludes the proof of Lemma 3. □

By Lemmas 2 and 3, using [31, Theorem 0.0], for large enough $n \leq \infty$ and small enough $r > 0$, $\lambda > 0$, the process of $n$-bad sites is stochastically dominated from above by an independent site percolation model on the integer lattice where the probability of having an open site is arbitrarily small. Hence this independent site percolation model is subcritical. Consequently, we can make the process of $n$-bad sites non-percolating. By Lemma 1 the same is true for $\mathcal{G}$, which concludes the proof of Theorem 1.

### 3.5. Proof of Theorem 2

We shall prove that for large enough $p < 1$ the model $\mathcal{G}$ percolates with positive probability for all $r > 0$ and large enough $\lambda < \infty$ (depending on $r$).

As in the proof of Theorem 1, we will use a coarse-graining argument. To this end, consider the following percolation model on the integer lattice $\mathbb{Z}^d$. For $n \geq 1$, we say a site $z \in \mathbb{Z}^d$ is $n$-good if the following conditions are satisfied:
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Lemma 5.

For z ∈ Zd,

set ξz := 1{z is n-good}. Then (ξz)z∈Zd is an 18-dependent random field.

Proof. In the same way as in the proof of Lemma 2, it suffices to prove that for all finite

ψ = {z1, . . . , zd} ⊂ Zd such that ∀i ̸= j, ||zi − zj||∞ > 18, we have

\[
E\left(\prod_{i=1}^{q} \xi_{zi}\right) = \prod_{i=1}^{q} E(\xi_{zi}).
\]

Denote respectively by Azi, Bzi, Czi, Dzi, Fzi the events that Conditions 1, 2, 3, 4, 5 in the definition of n-goodness hold for z ∈ Zd. We thus have

\[
\forall z ∈ Zd, \quad ξz = 1\{Azi\} 1\{Bzi\} 1\{Czi\} 1\{Dzi\} 1\{Fzi\}.
\]
Note first that whenever \( z \in \mathbb{Z}^d \), the indicators \( \mathbb{1}\{ A_z \} \) and \( \mathbb{1}\{ B_z \} \) are \( \Lambda \)-measurable. Thus, we have
\[
\mathbb{E}\left( \prod_{i=1}^{q} \xi_{zi} \right) = \mathbb{E}\left[ \mathbb{E}\left( \prod_{i=1}^{q} \xi_{zi} \mid \Lambda \right) \right]
\]
\[
= \mathbb{E}\left[ \prod_{i=1}^{q} \mathbb{1}\{ A_{zi} \} \mathbb{1}\{ B_{zi} \} \mathbb{E}\left( \prod_{i=1}^{q} \mathbb{1}\{ C_{zi} \cap D_{zi} \cap F_{zi} \} \mid \Lambda \right) \right].
\]
Now note that conditioned on \( \Lambda \), for each \( 1 \leq i \leq q \), the event \( C_{zi} \cap D_{zi} \cap F_{zi} \) depends only on the configuration of \( X^\lambda \) and \( Y \) inside of the \( d \)-dimensional cube \( Q_{6n}(nz_i) \). Since \( \psi \) satisfies \( \|z_i - z_j\|_\infty > 18 \) for all \( i \neq j \), we have \( \|nz_i - nz_j\|_\infty > 18n \) for all \( i \neq j \). As a matter of fact, the cubes \( \{Q_{6n}(nz_i) : 1 \leq i \leq q\} \) are disjoint, i.e.
\[
\forall i \neq j, \quad Q_{6n}(nz_i) \cap Q_{6n}(nz_j) = \emptyset.
\]
By the complete independence of Poisson and Bernoulli processes (recall that, given \( \Lambda \), \( X^\lambda \) has the distribution of a Poisson point process and \( Y \) the distribution of a Bernoulli point process), we have
\[
\mathbb{E}\left( \prod_{i=1}^{q} \xi_{zi} \right) = \mathbb{E}\left[ \prod_{i=1}^{q} \mathbb{1}\{ A_{zi} \} \mathbb{1}\{ B_{zi} \} \mathbb{E}\left( \prod_{i=1}^{q} \mathbb{1}\{ C_{zi} \cap D_{zi} \cap F_{zi} \} \mid \Lambda \right) \right]
\]
\[
= \mathbb{E}\left[ \prod_{i=1}^{q} \mathbb{1}\{ A_{zi} \} \mathbb{1}\{ B_{zi} \} \prod_{i=1}^{q} \mathbb{E}\left( \mathbb{1}\{ C_{zi} \cap D_{zi} \cap F_{zi} \} \mid \Lambda \right) \right]
\]
\[
= \mathbb{E}\left[ \prod_{i=1}^{q} \mathbb{1}\{ A_{zi} \} \prod_{i=1}^{q} \mathbb{E}\left( \mathbb{1}\{ B_{zi} \cap C_{zi} \cap D_{zi} \cap F_{zi} \} \mid \Lambda \right) \right]
\]
\[
= \mathbb{E}\left[ \prod_{i=1}^{q} \mathbb{1}\{ R(Q_{6n}(nz_i)) < 6n \mid f(\Lambda_{Q_{6n}(nz_i)}) \} \right],
\]
where \( f(\Lambda_{Q_{6n}(x)}) := \mathbb{E}\left( \mathbb{1}\{ B_x \cap C_x \cap D_x \cap F_x \} \mid \Lambda \right) \), a bounded measurable deterministic function of \( \Lambda_{Q_{6n}(x)} \), and where by \( \Lambda \)-measurability of the events \( \{ B_{zi} : 1 \leq i \leq q \} \), we put their indicators into the conditional expectation given \( \Lambda \). Now, the set \( \varphi := \{ nz_1, \ldots , nz_p \} \subset \mathbb{R}^d \) is finite and satisfies
\[
\forall i \neq j, \quad \|nz_i - nz_j\|_\infty > 18n.
\]
Since the infinite norm is always bounded above by the Euclidean norm, we have \( \|nz_i - nz_j\|_2 > 18n \) for all \( i \neq j \), and so \( \varphi \) satisfies
\[
\forall x \in \varphi, \quad \text{dist}(x, \varphi \setminus \{ x \}) > 18n = 3 \times 6n.
\]
We can therefore apply Condition 3 in Definition 1 (with \( n \) replaced by \( 6n \)) to get that the random variables appearing in the right-hand side of (11) are independent. Hence
\[
\mathbb{E}\left( \prod_{i=1}^{q} \xi_{zi} \right) = \prod_{i=1}^{q} \mathbb{E}\left[ \mathbb{1}\{ R(Q_{6n}(nz_i)) < 6n \mid f(\Lambda_{Q_{6n}(nz_i)}) \} \right]
\]
\[
= \prod_{i=1}^{q} \mathbb{E}(\xi_{zi}),
\]
which concludes the proof of Lemma 5. \( \square \)
Lemma 6. For any $r > 0$ we have

$$\lim_{n \to \infty} \lim_{p \to 1, \lambda \to \infty} \mathbb{P}(0 \text{ is } n\text{-good}) = 1.$$ 

Proof. We shall prove that

$$\lim_{n \to \infty} \lim_{p \to 1, \lambda \to \infty} \mathbb{P}(0 \text{ is } n\text{-bad}) = 0.$$ 

Take any $\epsilon > 0$. Denote respectively by $A$, $B$, $C$, $D$, $F$ the events that Conditions 1, 2, 3, 4, 5 in the definition of $n$-goodness hold for $z = 0$. Denote also by $\tilde{A}$ the event that $R(Q_{6n}) < n/2$. Note that $\tilde{A} \subset A$ and thus we have

$$\mathbb{P}(0 \text{ is } n\text{-bad}) = \mathbb{P}(A^c \cup B^c \cup C^c \cup D^c \cup F^c) \leq \mathbb{P}(\tilde{A}^c) + \mathbb{P}(B^c) + \mathbb{P}(B \cap C^c) + \mathbb{P}(D^c) + \mathbb{P}(\tilde{A} \cap D \cap F^c).$$

First, partitioning the cube $Q_{6n}$ into $12^d$ subcubes $(K_i)_{1 \leq i \leq 12^d}$ of side length $n/2$, we get

$$\mathbb{P}(\tilde{A}^c) = \mathbb{P}(R(Q_{6n}) \geq n/2) \leq 12^d \mathbb{P}(R(Q_{n/2}) \geq n/2) \leq \mathbb{P}(E \cap Q_n = \emptyset) \quad \text{by stationarity of the Rs.}$$

Therefore, by Condition 2 of Definition 1, we get $\lim_{n \to \infty} \mathbb{P}(\tilde{A}^c) = 0$. Also,

$$\mathbb{P}(B^c) = \mathbb{P}(E \cap Q_n = \emptyset),$$

and thus $\lim_{n \to \infty} \mathbb{P}(B^c) = 0$. Fix $n$ large enough so that $\mathbb{P}(\tilde{A}^c) \leq \epsilon/5$ and $\mathbb{P}(B^c) \leq \epsilon/5$. For such $n$, $Q_n$, $Q_{3n}$, and $Q_{6n}$ intersect almost surely zero or a finite number of edges and vertices.

Let us now deal with the quantity $\mathbb{P}(B \cap C^c)$. We have

$$\mathbb{P}(B \cap C^c) = \mathbb{P}(E \cap Q_n \neq \emptyset \quad \forall \ e \in E \cap Q_n : e \text{ is closed}).$$

This latter probability converges to 0 when $\lambda \to \infty$ (for fixed $n$ and $r > 0$). Hence, for large enough $\lambda < \infty$ (depending on $n$, $r$) we have $\mathbb{P}(B \cap C^c) \leq \epsilon/5$. Similarly,

$$\mathbb{P}(D^c) = \mathbb{P}(\exists v \in V \cap Q_{6n} : v \text{ is closed}) \leq \epsilon/5,$$

converges to 0 when $p \to 1$ (for fixed $n$ and $r > 0$); hence, for large enough $p < 1$, we have $\mathbb{P}(D^c) \leq \epsilon/5$.

Regarding the event $\tilde{A} \cap D \cap F^c$, note that under the event $\tilde{A}$, we have

$$R(Q_{6n}) < n/2 < 3n/2.$$ 

Hence, by asymptotic essential connectedness (see Definition 2), we have that $\text{supp}(A_{Q_{6n}}) \neq \emptyset$, and moreover, there exists a connected component $\Delta$ of $\text{supp}(A_{Q_{6n}})$ such that $\text{supp}(A_{Q_{6n}}) \subset \Delta \subset \text{supp}(A_{Q_{6n}})$. Therefore

$$\tilde{A} \cap D \cap F^c \subset (\exists e \in E \cap Q_{6n} : e \text{ is closed}).$$
Clearly, for fixed \( n, r \) and independently of \( p \),
\[
\lim_{\lambda \to \infty} \mathbb{P}(\exists e \in E \cap Q_{6n} : e \text{ is closed}) = 0.
\]
Hence, we can find \( \lambda < \infty \) large enough (depending on \( n, r \)) so that \( \mathbb{P}(\tilde{\Lambda} \cap D \cap F^c) \leq \epsilon/5 \). Since \( \epsilon > 0 \) was arbitrary, this concludes the proof of Lemma 6. \( \square \)

By Lemmas 5 and 6, using [31, Theorem 0.0], the process of \( n \)-good sites is stochastically dominated from below by a supercritical Bernoulli process for large enough \( n < \infty \), \( \lambda < \infty \), \( p < 1 \). Thus, we can make the process of \( n \)-good sites percolating. By Lemma 4, the connectivity graph \( G \) with these values of \( \lambda \), \( p \) percolates, which concludes the proof of Theorem 2.

3.6. Proof of Proposition 2

We first prove that \( p^* < 1 \) as a consequence of Theorem 2.3. Then, using an adapted path-count argument, very much as in classical i.i.d. percolation on the square grid (see e.g. [32, Theorem 1.1]), we show that \( p^* > 0 \).

\( p^* < 1: \) By Theorem 2, we can find large enough \( p \in (0, 1) \) so that \( P(p, U, H) > 0 \) for any \( H \in [0, \infty) \) and large enough \( U < \infty \) depending on the chosen \( H \). Choosing \( H = 0 \), we thus obtain the existence of some \( p \in (0, 1) \) such that \( P(p, U, 0) > 0 \) for some \( U < \infty \). Now, as noted in Section 2.3.1, the case \( H = 0 \) corresponds to PVT site percolation, and the percolation probability \( P(p, U, 0) =: P_{\text{PVT}}(p) \) does not depend on \( U \). Thus, \( P_{\text{PVT}}(p) > 0 \) for some \( p < 1 \), and so \( p^* < 1 \).

\( p^* > 0: \) It is known that the degree of all vertices (i.e. 0-facets) of a \( d \)-dimensional PVT generated by a homogeneous Poisson point process in \( \mathbb{R}^d \) is almost surely equal to \( d + 1 \), as a consequence of the following facts:

- Such a PVT is almost surely regular and normal (see [37, Proposition PV2]).
- The cells of such a PVT are almost surely polytopes, i.e. convex and bounded polyhedra (see [37, Proposition PV1]).
- Each \( s \)-facet of a Voronoi cell in \( \mathbb{R}^d \) lies on an \( s \)-dimensional hyperplane whose points are equidistant of \( d - s + 1 \) atoms of the Poisson point process having generated the PVT (this is a consequence of [37, Property IV3] and [37, Property IV4]).

Moreover, \( S \), being a regular and normal tessellation, is locally finite. This observation combined with the degree bound allows us to use an adapted path-count argument, as follows.

First, denote by \( \tilde{G}_p =: G_{p,0,0} \) the random graph arising from the PVT site percolation process with parameter \( p \), and note that the percolation of \( \tilde{G}_p \) is equivalent to the percolation of \( G_{p,U,0} \) whenever \( p \in [0, 1] \) and \( U \geq 0 \) (indeed, the fact that \( H = 0 \) makes the percolation independent of the Cox points in our model). Hence

\[
P_{\text{PVT}}(p) = \mathbb{P} \left( \tilde{G}_p \text{ has an infinite connected component} \right).
\]

Denote by \( \Phi \) the point process of crossroads (i.e. vertices of the PVT \( S \)), and denote by \( \mathbb{P}^0 \) its Palm probability. Since \( Y \) is a doubly stochastic Bernoulli point process supported by the crossroads, the conditional distribution of \( Y \) given \( S \) is the same under the stationary probability \( \mathbb{P} \) and under the Palm probability \( \mathbb{P}^0 \). As a matter of fact, for every crossroad \( v \in V \),
we have $\mathbb{P}^0(Y(v)) > 0 \mid S) = \mathbb{P}(Y(v)) > 0 \mid S) = p$. Moreover, conditionally to the realization of the PVT $S$, the states of distinct crossroads (i.e. open or closed) remain independent.

Fix some $n \geq 1$. A self-avoiding path $\gamma$ of length $n$ starting from the typical crossroad $0$ is a sequence of crossroads $0 = v_1, \ldots, v_n \in V$ with $v_i \neq v_j$ for $i \neq j$ and such that $v_i$ and $v_{i+1}$ are adjacent in $S$ whenever $1 \leq i \leq n - 1$. If the typical crossroad $0$ belongs to an infinite connected component in $\tilde{G}_p$ (which we denote by $0 \xrightarrow{\infty} \infty$), there must exist such a path with a Bernoulli point present at all crossroads of the path. Denote this event by $A_n$.

Then we have

$$
\mathbb{P}^0(0 \xrightarrow{\infty}) \leq \mathbb{P}^0(A_n).
$$

Let $SAP_n$ denote the set of self-avoiding paths of length $n$ starting from the typical crossroad $0$. By the union bound, we have

$$
\mathbb{P}^0(A_n) \leq \mathbb{E}^0 \left[ \sum_{(v_1, \ldots, v_n) = \gamma \in SAP_n} \mathbb{P}^0 \left( \cap_{i=1}^n \{Y(v_i) > 0\} \mid S \right) \right]
$$

$$
= \mathbb{E}^0 \left[ \sum_{(v_1, \ldots, v_n) = \gamma \in SAP_n} p^n \right]
$$

$$
= \mathbb{E}^0 \left[ \#(SAP_n) p^n \right],
$$

where $\#(SAP_n)$ denotes the cardinality of $SAP_n$, and where we have used the conditional independence of the states of the vertices as well as the conditional distribution of $Y$ given $S$ to get the first equality. Now, using the fact that for every $v \in V$, $\deg v = d + 1$ almost surely, we get that $\#(SAP_n) \leq (d + 1) \times d^{n-1}$. Hence

$$
\mathbb{P}^0(0 \xrightarrow{\infty}) \leq (d + 1) \times d^{n-1} p^n = \frac{d + 1}{d}(dp)^n.
$$

When $p < 1/d$, the quantity in the right-hand side converges to $0$ as $n \uparrow \infty$. Hence, for $p < 1/d$, we have $\mathbb{P}^0(0 \xrightarrow{\infty}) = 0$.

To conclude that $\tilde{G}_p$ does not percolate, we proceed as follows. For a crossroad $v \in V$, denote by $\{v \xrightarrow{\infty}\}$ the event that $v$ belongs to an infinite connected component of the PVT site percolation graph $\tilde{G}_p$. By Markov’s inequality, we have

$$
\mathbb{P} \left( \tilde{G}_p \text{ has an infinite connected component} \right) = \mathbb{P}(\exists v \in V : v \xrightarrow{\infty})
$$

$$
\leq \mathbb{E} \left[ \# \{v \in V : v \xrightarrow{\infty}\} \right],
$$

and so, by (12), we get

$$
P_{PVT}(p) \leq \mathbb{E} \left[ \# \{v \in V : v \xrightarrow{\infty}\} \right].
$$

(13)

Denote by $\lambda_0$ the intensity of the point process $\Phi$ of crossroads of $S$ and fix some $p < 1/d$. By the Campbell–Little–Mecke–Matthes theorem (see [2, Theorem 6.1.28]), we have

$$
\mathbb{E} \left[ \# \{v \in V : v \xrightarrow{\infty}\} \right] = \mathbb{E} \left[ \int_{\mathbb{R}^d} 1\{x \xrightarrow{\infty}\} \Phi(dx) \right]
$$

$$
= \lambda_0 \int_{\mathbb{R}^d} \mathbb{E}^0 \left[ 1\{0 \xrightarrow{\infty}\} \right] dx
$$

$$
= \lambda_0 \int_{\mathbb{R}^d} \mathbb{P}^0(0 \xrightarrow{\infty}) dx
$$

$$
= 0,
$$

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where we have used the fact that $p < 1/d \Rightarrow P^0(0 \sim \infty) = 0$ to get the last equality. By (13), we have $P_{pV}(p) = 0$ whenever $p < 1/d$, and thus $p^* \geq 1/d > 0$.

### 3.7. Proof of Theorem 3

Proving Theorem 3 amounts to showing that $\mathcal{G}$ percolates with positive probability when $\lambda = 0$, $p < 1$ is sufficiently large, and $r < \infty$ is sufficiently large. We thus assume throughout the rest of this subsection that $\lambda = 0$ and that $r$ and $p$ are the varying parameters, and we still refer to $\mathcal{G}$ for the associated connectivity graph.

Say a street $e \in E$ is hard-geometric-open if its length is smaller than the connectivity threshold $r$:

$$|e| \leq r.$$ 

If not, say $e$ is hard-geometric-closed.

Once again, we will use a coarse-graining argument. Since the development is very similar to the one presented in the previous subsection, we only give details on what modifications should be applied to the proof of Theorem 2 to prove Theorem 3.

To this end, we consider now the following percolation model on the integer lattice $\mathbb{Z}^d$. For $n \geq 1$, say as it $z \in \mathbb{Z}^d$ is $n$-good if it satisfies the following conditions:

1. $R(Q_{6n}(nz)) < 6n$.
2. $E \cap Q_n(nz) \neq \emptyset$; i.e. the cube $Q_n(nz)$ contains a full street.
3. There exists $e \in E \cap Q_n(nz)$ such that $|e| \leq r$. In other words, there exists a hard-geometric-open street that is fully included in the cube $Q_n(nz)$.
4. All crossroads in $Q_{6n}(nz)$ are open, in the sense of Definition 3.
5. Every two hard-geometric-open streets $e, e' \in E \cap Q_{3n}(nz)$ (i.e. such that $|e| \leq r$ and $|e'| \leq r$) are connected by a path in $\mathcal{G} \cap Q_{6n}(nz)$.

We say a site $z \in \mathbb{Z}^d$ is $n$-bad if it is not $n$-good.

Note that this new definition of $n$-goodness is exactly the same as the one given in the proof of Theorem 2 but with Conditions 3 and 5 replaced by $\hat{3}$ and $\hat{5}$. In other words, openness is replaced by hard-geometric-openness.

Since we are now dealing with hard-geometric openness and all the other conditions are unchanged, it is straightforward to prove the following by adapting the proof of Lemma 4.

**Lemma 7.** Percolation of the process of $n$-good sites implies percolation of the connectivity graph $\mathcal{G}$.

In the same way, we get the result below.

**Lemma 8.** For $z \in \mathbb{Z}^d$, set $\xi_z := 1\{z \text{ is } n\text{-good}\}$. Then $(\xi_z)_{z \in \mathbb{Z}^d}$ is an $18$-dependent random field.

**Proof.** It suffices to adapt the proof of Lemma 5 as follows.

Denote respectively by $A_z, B_z, \hat{C}_z, D_z, \hat{F}_z$ the events that Conditions 1, 2, $\hat{3}$, 4, and $\hat{5}$ in the definition of $n$-goodness hold for $z \in \mathbb{Z}^d$.

Note first that whenever $z \in \mathbb{Z}^d$, the indicators $1\{A_z\}, 1\{B_z\}, 1\{\hat{C}_z\}$ are all $\Lambda$-measurable. Doing exactly the same calculations as in the proof of Lemma 5, we arrive at dealing with the quantity

$$E \left( \prod_{i=1}^q \xi'_{z_i} \right) = E \left[ \prod_{i=1}^q 1\{A_{z_i}\} 1\{B_{z_i}\} 1\{\hat{C}_{z_i}\} E \left( \prod_{i=1}^q 1\{D_{z_i} \cap \hat{F}_{z_i}\} \mid \Lambda \right) \right].$$
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Now note that conditioned on $\Lambda$, for each $1 \leq i \leq q$, the event $D_{z_i} \cap \hat{F}_{z_i}$ depends only on the configuration of $Y$ inside of the cube $Q_{6n}(nz_i)$. We can thus proceed as in the aforementioned proof by using the complete independence of $Y$ (recall that, given $\Lambda$, $Y$ has the distribution of a Bernoulli point process). Finally, it is clear that $\mathbb{1}\{\hat{C}_z\}$ is a bounded deterministic function of $\Lambda_{Q_n(nz)}$ and that

$$\mathbb{E}\left(\mathbb{1}\{D_z \cap \hat{F}_z\} \mid \Lambda\right)$$

is a bounded deterministic function of $\Lambda_{Q_{6n}(nz)}$. This allows us to proceed exactly as in the proof of Lemma 5 and conclude.

Finally, for the hard-geometric model, we still have the following.

**Lemma 9.**

$$\lim_{n \uparrow \infty} \lim_{p \uparrow 1} \lim_{r \uparrow \infty} \mathbb{P}(0 \text{ is } n\text{-good}) = 1.$$  

**Proof.** Again, we shall prove that

$$\lim_{n \uparrow \infty} \lim_{p \uparrow 1} \lim_{r \uparrow \infty} \mathbb{P}(0 \text{ is } n\text{-bad}) = 0.$$  

Take any $\epsilon > 0$. We adapt the proof of Lemma 6 as follows. Denote respectively by $A$, $B$, $\hat{C}$, $D$, $\hat{F}$ the events that Conditions 1, 2, $\hat{3}$, 4, and $\hat{5}$ in the definition of $n$-goodness hold for $z = 0$. Also denote by $\hat{A}$ the event that $R(Q_{6n}) < n/2$. As in the aforementioned proof, we have

$$\mathbb{P}(0 \text{ is } n\text{-bad}) \leq \mathbb{P}(\hat{A}^c) + \mathbb{P}(B^c) + \mathbb{P}(B \cap \hat{C}^c) + \mathbb{P}(D^c) + \mathbb{P}(\hat{A} \cap D \cap \hat{F}^c).$$

In the above inequality, we deal with the first, second, and fourth quantities as before, and so we can fix $n$ large enough such that $\mathbb{P}(\hat{A}^c) \leq \epsilon/5$ and $\mathbb{P}(B^c) \leq \epsilon/5$. For such $n$, $Q_n$, $Q_{3n}$, and $Q_{6n}$ intersect almost surely zero or finitely many edges and vertices. We can then fix $p < 1$ large enough so that $\mathbb{P}(D^c) \leq \epsilon/5$.

Let us now deal with the quantity $\mathbb{P}(B \cap \hat{C}^c)$. We have

$$\mathbb{P}(B \cap \hat{C}^c) = \mathbb{P}(E \cap Q_n \neq \emptyset \text{ and } \forall e \in E \cap Q_n \mid e \mid > r)$$

$$= \mathbb{E}\left(\mathbb{1}\{E \cap Q_n \neq \emptyset\} \prod_{e \in E \cap Q_n} \mathbb{1}\{|e| > r\}\right).$$

Note first that on the event $\{E \cap Q_n \neq \emptyset\}$, the latter product is non-empty. Moreover, since $E \cap Q_n$ contains finitely many edges (recall that $n$ is fixed) and since we have

$$\forall e \in E, \lim_{r \uparrow \infty} \mathbb{1}\{|e| > r\} = 0 \text{ almost surely,}$$

by dominated convergence we have that the latter expectation converges to 0 when $r \to \infty$ (for fixed $n$). Therefore, $\lim_{r \uparrow \infty} \mathbb{P}(B \cap \hat{C}^c) = 0$ (for fixed $n$).

Regarding the event $\hat{A} \cap D \cap \hat{F}^c$, we proceed as before and use asymptotic essential connectedness to get

$$\hat{A} \cap D \cap \hat{F}^c \subset (\exists e \in E \cap Q_{6n} : e \text{ is hard-geometric-closed}).$$
Clearly, for fixed $n$,
\[ \lim_{r \to \infty} \mathbb{P}(\exists e \in E \cap Q_{6n} : e \text{ is hard-geometric-closed}) = \lim_{r \to \infty} \mathbb{P}(\exists e \in E \cap Q_{6n} : |e| > r) = 0. \]

Hence, we can find $r < \infty$ large enough (depending on $n$) so that $\mathbb{P}(B \cap \hat{C}^c) \leq \epsilon/5$ and $\mathbb{P}(\hat{A} \cap D \cap \hat{F}^c) \leq \epsilon/5$. Since $\epsilon > 0$ was arbitrary, this concludes the proof of Lemma 9. \(\Box\)

By Lemmas 8 and 9, using [31, Theorem 0.0], the process of $n$-good sites is stochastically dominated from below by a supercritical Bernoulli process for large enough $n < \infty$, $p < 1$, $r < \infty$. Thus, we can make the process of $n$-good sites percolating. By Lemma 7, the connectivity graph $\mathcal{G}$ with these values of $p$ and $r$ percolates, which concludes the proof of Theorem 3.

4. Model extensions and concluding remarks

In this paper, we have introduced and studied a new model for continuum line-of-sight percolation in a random environment. This mathematical model can be seen as a good candidate for the modeling of telecommunications networks in an urban scenario with regular obstructions: the random support (equivalent to the street system of the city) is modeled by a PVT. Users are dropped on the edges of the PVT according to a Cox point process with linear intensity $\lambda$. Obstructive connectivity conditions require the presence of an additional Bernoulli point process (representing relays which could be either real users or physical antennas) at the vertices of the PVT (crossroads of the city) to ensure connectivity between adjacent streets. We have proven via coarse-graining arguments that a minimal relay proportion $p > p^*$ is necessary to allow for percolation of the connectivity graph, that a nontrivial subcritical phase exists whenever the connectivity threshold $r$ is not too large, and that a supercritical phase exists for all $r > 0$. Moreover, we also performed Monte Carlo simulations to get numerical estimates of critical parameters of our model and estimate the frontiers between the different connectivity regimes and particular cases of our model (PVT site percolation, PVT hard-geometric bond percolation, and PVT soft-geometric bond percolation).

Our line-of-sight percolation model in a random environment has many possible generalizations. The most obvious one is the dual tessellation of the PVT: a Poisson–Delaunay tessellation [18, Section 9.2], which is known to be stabilizing and essentially asymptotically connected (see [27, Exercise 3.2.7] and [26, Section 3.1]).

More interestingly, one can try to prove these stabilization and asymptotic essential connectedness properties (which are fundamental for our approach) for the generalized Poisson–Voronoi weighted tessellation [21], including, as special cases, Laguerre and Johnson–Mehl tessellations. Note that in this latter paper, a different stabilization property is used to prove expectation and variance asymptotics, as well as central limit theorems for unbiased and asymptotically consistent estimators of geometric statistics of the typical cell.

Finally, one may ask whether the Poisson process, which underlies all the above models, can be replaced by a more general point process, sufficiently mixing [38] or having a sufficiently fast decay of correlations [12]. These latter properties (the mixing property and fast decay of correlations) were originally used or developed for the aforementioned context of the limit theory for the typical cell, and it is not clear whether they can be used in our percolation context.
Concluding, we believe our model paves the way to the study of a new class of random connection models in a random environment that not only conditions the locations of points but also the connection function. As a natural generalization of the line-of-sight connection function, one can consider connectivity conditions based on two connection radii: one for the nodes of the network being located on the same edge of the random support and another (typically smaller) for non-line-of-sight (NLOS) connections, i.e. with nodes of the network being located on different edges of the random support. This could be of particular interest for more realistic models of telecommunications networks.

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