THE SPACE OF MARGINALLY TRAPPED SURFACES IN PERTURBED SCHWARZSCHILD SPACETIMES

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ABSTRACT. Marginally trapped surfaces are an important concept in Lorentzian geometry. They play a similar role like minimal surfaces do in the Riemannian setting. Penrose used closed trapped surfaces to prove his famous incompleteness theorem in general relativity; stating that under physical assumptions the existence of such a surface implies future geodesic incompleteness of the spacetime. In the Schwarzschild black hole spacetime, the event horizon is foliated by marginally trapped surfaces. In this paper, we study the moduli space of marginally trapped surfaces in a perturbed Schwarzschild spacetime. More generally, we develop a method to study the geometry of spacelike surfaces and apply it to characterise the moduli space of marginally trapped surfaces in such a spacetime.

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1. Introduction

The concept of a closed trapped surface was introduced by Penrose in [12], which is a closed spacelike surface whose future null expansions are both negative. Based on this concept, Penrose proved his famous incompleteness theorem which says that a globally hyperbolic vacuum spacetime cannot be future null geodesically complete if it contains a closed trapped surface. This implies that a vacuum spacetime with a complete future null infinity and a closed trapped surface must exhibit a black hole region containing the closed trapped surface. In general, a black hole region is defined as the complement of the past of future null infinity.

The concept of a closed marginally trapped surface is closely related to the one of a closed trapped surface. A closed spacelike surface is called marginally trapped, if one of its future null expansions is negative and the other one is constantly zero. Like closed trapped surfaces, closed marginally trapped surfaces are also related to black holes in the following way: the event horizon in any Schwarzschild or Kerr black hole spacetime is foliated by closed marginally trapped surfaces. In particular, a closed spacelike surface embedded into the event horizon of a Schwarzschild or Kerr black hole spacetime is marginally trapped. Hence the moduli space of closed marginally trapped surfaces is the space of closed embedded spacelike surfaces inside the event horizon.
In this paper, we consider the moduli space of marginally trapped surfaces in perturbed Schwarzschild black hole spacetimes \((M, g)\). The nice symmetry of the Schwarzschild case breaks down by perturbations, making the moduli space much more complicated. Within spacetimes close to Schwarzschild, we consider a neighborhood of a closed marginally trapped surface and study the moduli space of marginally trapped surfaces nearby.

In order to investigate this moduli space, we develop a general method to study the geometry of spacelike surfaces. We suppose that a double null coordinate system is given on the perturbed Schwarzschild spacetime. With the help of this coordinate system, we can parametrize any spacelike surface by two functions on the sphere. Then we can calculate all the geometric quantities of the spacelike surface by these two functions parametrizing the surface and the background geometric quantities of the spacetime. In particular, we can calculate the future null expansions in this way.

A closed spacelike surface being marginally trapped is equivalent to its outgoing future null expansion being zero. We apply the above mentioned method to a marginally trapped surface to derive an equation for the two functions parametrizing it, requiring that the outgoing future null expansion is zero. In the following, we will call this equation the zero null expansion equation. From the point of view of analysis, the moduli space of marginally trapped surfaces is identified with the space of solutions of the zero null expansion equation.

Assuming that we are given a marginally trapped surface in \((M, g)\), we have a trivial solution of the zero null expansion equation. We will use a perturbation method to find all the solutions near the trivial solution. By the equivalence of marginally trapped surfaces and solutions of the zero null expansion equation, we actually find all the marginally trapped surfaces near the a priori given marginally trapped surface. One of the main results obtained in this paper is that there exists a unique marginally trapped surface in any incoming null hypersurface in the perturbed Schwarzschild spacetime.

This paper is organised as follows.

Section 2 gives a short review of the Schwarzschild spacetime and introduces the perturbed Schwarzschild spacetime.

In section 3 we introduce two methods to parametrize spacelike surfaces by two functions. These will be crucial geometric-analytic tools to investigate the geometry of spacelike surfaces. We denote the two functions in the first method of parametrization by \((\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}})\), and the two functions in the second method by \((s=0)\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}})\). We will use both parametrizations throughout this paper. Moreover, we calculate the geometric quantities on spacelike surfaces by the first parametrization \((\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}})\). In the later sections, we will always assume the second parametrization \((s=0)\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}})\) is given a priori. In this section, we also present the method to obtain the first parametrization from the second one.

In section 4 we estimate the first parametrization \((\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}})\) by the a priori given second parametrization \((s=0)\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}})\). From the point of view of analysis, we study the mapping from the second parametrization to the first parametrization: \((s=0)\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}}) \mapsto \tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}}\).

In section 5 we estimate the geometric quantities of spacelike surfaces, combining the results obtained in section 3 and the estimates of \((\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}})\) in section 4.
In sections 6 and 7, we study the perturbations of spacelike surfaces.
In section 8, we construct the linearization of the mapping \( \tilde{s} \mapsto (\tilde{f}, \tilde{\tilde{f}}) \).
In section 9, we construct the linearized perturbations of the geometric quantities of spacelike surfaces.
In section 10, we construct the moduli space of marginally trapped surfaces near the a priori given one in \((M, g)\) by solving the zero null expansion equation \( \tilde{\text{tr}} \tilde{\tilde{\chi}} = 0 \).

We prove that for any \( \tilde{s} \) close to zero, there exists a unique solution \( \tilde{f} \) solving the zero null expansion equation. This implies that there exists a unique closed marginally trapped surface embedded in any incoming null hypersurface near the a priori given closed marginally trapped surface in \((M, g)\).

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2. Schwarzschild spacetime and its perturbation

The Schwarzschild spacetime is the spherically symmetric vacuum spacetime. In the polar coordinate system \( \{t, r, \theta, \phi\} \), the Schwarzschild metric \( g_{\text{Sch}} \) can be expressed as

\[
g_{\text{Sch}} = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( m \) is a positive constant which is the total mass of the Schwarzschild spacetime. The metric is singular at \( r = 2m \) and \( r = 0 \). The singularity of the metric at \( r = 2m \) is due to the choice of coordinate system, but the singularity at \( r = 0 \) is a real singularity, i.e., it cannot be resolved by any change of coordinate systems.

In another coordinate system \( \{\nu, \upsilon, \theta, \phi\} \), the Schwarzschild metric takes the form

\[
g_{\text{Sch}} = -\frac{8m^2}{r} \exp \left(-\frac{r}{2m}\right) (d\nu \otimes d\upsilon + d\upsilon \otimes d\nu) + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( r \) is given by

\[
\nu \upsilon = -(r - 2m) \exp \frac{r}{2m}.
\]
The region \( \{0 < \nu v < 2m, \nu > 0\} \) is called the black hole, since it is the set of points which cannot send signals to future null infinity. The event horizon is defined to be the past boundary of the black hole. It consists of two null hypersurfaces \( \{\nu = 0, v \geq 0\} \) and \( \{\nu \geq 0, v = 0\} \) which intersect at the surface \( \Sigma_{\nu=0,v=0} \). The event horizon is foliated by future marginally trapped surfaces. The past lightcone \( \{\nu = \nu_0, v \leq 0\} \) issuing from the future marginally trapped surface \( \Sigma_{\nu=\nu_0,v=0} \) in the event horizon intersects past null infinity.

The coordinate system \( \{\nu, v, \theta, \phi\} \) is called a double coordinate system and the level sets \( \{C_{\nu}\} \cup \{C_{v}\} \) of \( \nu, v \) is a double null foliation.

In the double null coordinate \( \{\nu, v, \theta, \phi\} \), we define

\[
L_{\nu} = -\frac{\partial}{\partial \nu}, \quad L_{v} = \frac{\partial}{\partial v},
\]

and

\[
2\Omega^2_{\nu v} = g_{Sch}(L_{\nu}, L_{v}) = \frac{8m^2}{r} \exp\left(-\frac{r}{2m}\right).
\]

Then the null geodesic vector fields \( L_{\nu}, L_{v} \) tangential to \( C_{\nu} \) and \( C_{v} \) respectively are

\[
\begin{align*}
L_{\nu} &= \Omega^{-2} L_{\nu} = -\frac{r}{4m^2} \exp\left(-\frac{r}{2m}\right) \frac{\partial}{\partial \nu}, \\
L_{v} &= \Omega^{-2} L_{v} = \frac{r}{4m^2} \exp\left(-\frac{r}{2m}\right) \frac{\partial}{\partial \nu}.
\end{align*}
\]

From equation (3) of \( r \), we can calculate its differential in \( \{\nu, v\} \) coordinates,

\[
dr = -\frac{2m}{r} \exp\left(-\frac{r}{2m}\right) (\nu dv + v d\nu) = \frac{2m(r-2m)}{r} (d \log \nu + d \log v),
\]

so

\[
\begin{align*}
\frac{\partial r}{\partial v} &= -\frac{2mv}{r} \exp\left(-\frac{r}{2m}\right), \\
\frac{\partial r}{\partial \nu} &= -\frac{2mv}{r} \exp\left(-\frac{r}{2m}\right).
\end{align*}
\]

On the null hypersurface \( C_{\nu=0} \),

\[
\frac{\partial r}{\partial v} = 0
\]

which means that \( r = 2m \) along \( C_{\nu=0} \) and \( \Sigma_{\nu=0,v} \) is marginally trapped for all values of \( v \). We define

\[
r_0 = 2m.
\]

We choose the sphere \( \Sigma_{\nu=0,v=\nu_0} \) and the null hypersurface \( C_{\nu=\nu_0} \), and introduce a function \( s = s(\nu) \) such that

\[
s(\nu) = r(\nu, \nu_0) - 2m = r(\nu, \nu_0) - r_0,
\]

and since

\[
\frac{ds}{d\nu} = \frac{\partial r}{\partial v}(\nu, \nu_0) = -\frac{2mv_0}{r(\nu, \nu_0)} \exp\left(-\frac{r(\nu, \nu_0)}{2m}\right) = -\frac{r_0 v_0}{r(\nu, \nu_0)} \exp\left(-\frac{r(\nu, \nu_0)}{r_0}\right) \neq 0,
\]
s is a foliation on \( C_{\nu=v_0} \). We set \( \Sigma_{0,0} = \Sigma_{\nu=0,v=v_0} \) and \( C_0 = C_{\nu=v_0} \). In following we will construct a double null foliation \( \{ C_s \} \cup \{ \mathcal{C}_s \} \) and the corresponding double null coordinate system \( \{ s, \underline{s}, \theta, \phi \} \) We will find the transformation between \( \{ \nu, \upsilon \} \) and \( \{ s, \underline{s} \} \) to define \( \{ s, \underline{s} \} \).

The associated outgoing null vector field \( L_s^\nu \) on \( \mathcal{C}_0 \) is
\[
L_s^\nu = \frac{\partial}{\partial s} = \frac{d\nu}{ds} \frac{\partial}{\partial \nu} = \left( \frac{\partial r}{\partial \nu}(\nu, \upsilon_0) \right)^{-1} \frac{\partial r}{\partial \nu} = \frac{r(\nu, \upsilon_0)}{r_0 \upsilon_0} \exp \frac{r(\nu, \upsilon_0)}{r_0} L^\nu,
\]
then we define its conjugate null vector field as \( L_s^\upsilon \) on \( \Sigma_{0,0} \), which is given by
\[
g(L_s^\nu, L_s^\upsilon) = 2 \Rightarrow L_s^\upsilon = \frac{\upsilon_0}{r_0} L_\upsilon.
\]
We extend \( L_s^\upsilon \) to a null geodesic vector field on \( C_0 = C_{\nu=0} \) and notice that \( L_s^\upsilon \) is geodesic, hence the extension of \( L_s^\upsilon \) on \( C_0 \) is
\[
L_s^\upsilon = \frac{\upsilon_0}{r_0} L_\upsilon = \frac{\upsilon_0}{r_0} L^\upsilon.
\]
Then by the equation \( L_s^s \mathcal{C}_0 = 1 \) on \( C_0 \), we get
\[
\frac{\upsilon_0}{r_0} \frac{\partial \mathcal{C}_0}{\partial \upsilon} = 1,
\]
hence
\[
\mathcal{C}_0(\nu) = \frac{r_0}{\upsilon_0}(\nu - \upsilon_0),
\]
and
\[
L_s^\underline{s} = \frac{\partial}{\partial \underline{s}} = \frac{d\nu}{d\underline{s}} \frac{\partial}{\partial \nu} = \frac{\upsilon_0}{r_0} L^\nu.
\]
We obtained the transformation from \( \{ \nu, \upsilon \} \) coordinates to \( \{ s, \underline{s} \} \) coordinates
\[
\begin{cases}
\mathcal{C}_0(\nu) = r(\nu, \upsilon_0) - r_0, \\
\underline{s}(\nu) = \frac{r_0}{\upsilon_0}(\nu - \upsilon_0),
\end{cases}
\]
we can reverse the transformation to get
\[
\begin{cases}
\nu(s) = -\frac{s}{\upsilon_0} \exp \frac{s + r_0}{r_0}, \\
\underline{s}(\nu) = \frac{\upsilon_0}{r_0}(\nu - r_0).
\end{cases}
\]
Substituting to equation (3) of \( r \), we get
\[
(r - r_0) \exp \frac{r}{r_0} = (\underline{s} + r_0) \frac{s}{r_0} \exp \frac{s + r_0}{r_0}.
\]
We calculate the metric component \( \Omega \) in \( \{ s, \underline{s} \} \) coordinates,
\[
2\Omega^2 = g_{\text{Sch}}(L_s^\nu, L_s^\upsilon) = \frac{r(\nu, \upsilon_0)}{r_0} \exp \frac{r(\nu, \upsilon_0)}{r_0} g_{\text{Sch}}(L_\nu^\nu, L_\upsilon^\upsilon) = \frac{2}{r_0} \exp \frac{s + r_0}{r_0} \exp \frac{s + r_0}{r_0}.
then we substitute equation (21) to get
\begin{equation}
\Omega^2 = \frac{s + r_0}{r} \exp \frac{s + r_0}{r_0} \exp - \frac{r}{r_0} = \frac{r_0(r - r_0)(s + r_0)}{rs(s + r_0)}
\end{equation}

So in the double null coordinate system \(\{s, s, \theta, \phi\}\), the Schwarzschild metric takes the form
\begin{equation}
g_{\text{Sch}} = \frac{2(s + r_0)}{r} \exp \frac{s + r_0}{r_0} - \frac{r}{r_0} (ds \otimes ds + ds \otimes ds) + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\end{equation}

and \(r\) is determined by equation (21). In the following, we use \(\hat{g}\) to denote the round metric \(d\theta^2 + \sin^2 \theta d\phi^2\) on the sphere.

In the following, we calculate the structure coefficients and curvature components of the double null foliation \(\{s, s\}\). First, we calculate the differential of \(r\) in \(\{s, s\}\) coordinates by taking the differential of equation (21)
\begin{equation}
r \exp \frac{r}{r_0} dr = s \exp \frac{s + r_0}{r_0} ds + (s + r_0) \frac{s + r_0}{r_0} \exp \frac{s + r_0}{r_0} ds,
\end{equation}
which implies
\begin{equation}
\begin{align*}
Dr &= \frac{\partial r}{\partial s} = \frac{s + r_0}{r_0} \exp \frac{s + r_0}{r_0} \exp - \frac{r}{r_0}, \\
Ds &= \frac{\partial r}{\partial s} = \frac{s + r_0}{r_0} \exp - \frac{r}{r_0}.
\end{align*}
\end{equation}

The null second fundamental forms \(\chi, \chi\) are
\begin{equation}
\chi = \frac{1}{2} D \left(r^2 \hat{g}\right) = r D r \hat{g} = s \exp \frac{s + r_0}{r_0} \exp - \frac{r}{r_0} \hat{g},
\end{equation}
\begin{equation}
\chi = \frac{1}{2} D \left(r^2 \hat{g}\right) = r D r \hat{g} = \frac{s + r_0}{r_0} (s + r_0) \exp \frac{s + r_0}{r_0} \exp - \frac{r}{r_0} \hat{g}.
\end{equation}

The shears \(\hat{\chi}, \hat{\chi}\), which are the trace free parts of \(\chi, \chi\), and the null expansions \(\text{tr} \chi, \text{tr} \chi\) are
\begin{equation}
\hat{\chi} = 0, \quad \text{tr} \chi = \frac{2s}{r^2} \exp \frac{s + r_0}{r_0} \exp - \frac{r}{r_0} = \frac{r_0}{s + r_0} \frac{2(r - r_0)}{r^2},
\end{equation}
\begin{equation}
\hat{\chi} = 0, \quad \text{tr} \chi = \frac{2(s + r_0)}{r^2} \frac{s + r_0}{r_0} \exp - \frac{r}{r_0} = \frac{2(r - r_0)(s + r_0)}{s}.\end{equation}

Since the spacetime is spherically symmetric, the torsions \(\eta, \eta\) vanishes, i.e.
\begin{equation}
\eta = \frac{1}{2} g(\nabla L^s, L^s) = 0, \quad \eta = \frac{1}{2} g(\nabla L^s, L^s) = 0.
\end{equation}

The accelerations \(\omega, \omega\) are
\begin{equation}
\omega = D \log \Omega = D \frac{1}{2} \left\{ \log (s + r_0) - \log r + \frac{s + r_0}{r_0} - \frac{r}{r_0} \right\} = - \left( \frac{1}{2r} + \frac{1}{2r_0} \right) \frac{r}{r_0},
\end{equation}
\begin{equation}
\omega = D \log \Omega = D \frac{1}{2} \left\{ \log (s + r_0) - \log r + \frac{s + r_0}{r_0} - \frac{r}{r_0} \right\} = \frac{1}{2(s + r_0)} + \frac{1}{2r_0} - \left( \frac{1}{2r} + \frac{1}{2r_0} \right) \frac{r}{r_0} \hat{r}.
\end{equation}
then substituting (26), we have

\[ \omega = - \left( \frac{1}{2r} + \frac{1}{2r_0} \right) \frac{s + r_0}{r_0} \exp \frac{s + r_0}{r_0} \exp \frac{-r}{r_0}, \]

\[ \omega = \frac{1}{2(s + r_0)} + \frac{1}{2r_0} - \left( \frac{1}{2r} + \frac{1}{2r_0} \right) \frac{s + r_0}{r_0} \frac{s + r_0}{r_0} \exp \frac{s + r_0}{r_0} \exp \frac{-r}{r_0} \]

From the above structure coefficients of the double null foliation \( \{s, s\} \), then we can calculate the curvature components by the Gauss equations, the Codazzi equations and the propagation equations in [C], which takes the following forms in \( \{s, s\} \) coordinate system.

\[ \rho = -K + \frac{1}{2} \Omega^2 \frac{\text{tr} \chi}{\text{tr} \chi} - \frac{1}{2 \Omega^2} = -\frac{r_0}{\beta^2}, \quad \sigma = 0, \quad \beta = \beta = 0, \quad \alpha = \alpha = 0, \]

where \( K \) is the Gauss curvature of the surface \( \Sigma_{s,s} \).

\( \{s, s\} \) is a double null foliation with nice properties on \( C_{s=0} \), since

\[ \Omega = 1, \quad \omega = 0 \]

on \( C_{s=0} \), however the shortcoming of this double null foliation is that the structure coefficients have no good decay approaching the past null infinity \( I^- \), for example \( \omega \). So we consider the following renormalised null function

\[ s^r = r_0 \log \frac{s + r_0}{r_0}, \quad \frac{s}{r_0} = r_0 \exp \frac{s}{r_0}, \quad \frac{ds^r}{d\bar{s}} = \frac{r_0}{s + r_0}, \]

which implies

\[ L^r_{\bar{s}} = \left( \frac{ds^r}{d\bar{s}} \right)^{-1} L_{\bar{s}} = \frac{s + r_0}{r_0} \]

Now for the double null foliation \( \{s, s^r\} \), we calculate the metric components and the structure coefficients.

\[ (\Omega^r)^2 = \frac{s + r_0}{r_0} \Omega^2 = \frac{s + r_0}{r_0} \frac{s + r_0}{r_0} \exp \frac{s + r_0}{r_0} \exp \frac{-r}{r} = \frac{s + r_0}{r} \frac{r - r_0}{r} \]

\[ \text{tr} \chi^r = \frac{s + r_0}{r_0} \text{tr} \chi = \frac{s + r_0}{r_0} \frac{2s}{r_0} \exp \frac{s + r_0}{r_0} \exp \frac{-r}{r_0} = \frac{2(r - r_0)}{r^2} \]

For the acceleration \( \omega^r \), we have

\[ \nabla_{L^r_{\bar{s}}} s^r = 2 \omega^r L^r_{\bar{s}} \Rightarrow 2 \omega^r = \frac{1}{r_0} + 2 \omega \frac{s + r_0}{r_0}, \]

then substituting \( \omega \) from (34),

\[ \omega^r = \frac{r_0}{2r^2}. \]

Notice that at the event horizon \( r = r_0 \), \( \omega^r(r = r_0) = \frac{1}{2r_0} = \frac{1}{4m} \) which is the surface gravity at the event horizon.

From now on we replace the double null foliation \( \{s, s\} \) by \( \{s, s^r\} \). In the above, we construct a double null coordinate system in the Schwarzschild spacetime and study the geometry of the spacetime using this coordinate system. In the following, we introduces a class of spacetimes which are close to the Schwarzschild spacetime.
Definition 2.1 \((\kappa, \tau)\)-neighbourhood \(M_{\kappa, \tau}\) of \(\Sigma_{0,0}\). Let \((S, g_{Sch})\) be the Schwarzschild spacetime of mass \(m\). Let \(\{s, \overline{s}\}\) be the double null foliation constructed as above. We define the \(\kappa\)-neighbourhood \(M_{\kappa, \tau}\) of \(\Sigma_{0,0}\) to be the portion of \(S\) such that for any point \(p \in M_{\kappa}\),

\[
|s(p)| \leq \kappa r_0, \quad |\overline{s}(p)| < \tau r_0,
\]

i.e.

\[
M_{\kappa} = \{p \in S : |s(p)| \leq \kappa r_0, |\overline{s}(p)| < \tau r_0\}.
\]

The \((\kappa, \tau)\)-neighbourhood \(M_{\kappa, \tau}\) is simply a differential manifold. In the coordinate system \(\{s, \overline{s}, \theta^1, \theta^2\}\), the Schwarzschild metric takes the form

\[
g_{Sch} = 2\left(\frac{s + r_0}{r}\right) (ds \otimes d\overline{s} + d\overline{s} \otimes ds) + r^2 g_{ab} d\theta^a \otimes d\theta^b,
\]

where \(r\) is given by

\[
(r - r_0) \exp \frac{r}{r_0} = s \exp \frac{s + \overline{s} + r_0}{r_0}.
\]

In the following we consider more general metrics on \(M_{\kappa, \tau}\) which have the following form:

Let \(\Omega\) be a positive function on \(M_{\kappa, \tau}\), \(\hat{g}\) be a positive definite symmetric \(\{\Sigma_{s, \overline{s}}\}\) tangential covariant 2-tensor field and \(\vec{b}\) be a \(\{\Sigma_{s, \overline{s}}\}\) tangential vector field on. In the coordinate system,

\[
\hat{g} = \hat{g}_{ab} d\theta^a \otimes d\theta^b, \quad \vec{b} = b^1 \frac{\partial}{\partial \theta^1} + b^2 \frac{\partial}{\partial \theta^2},
\]

then we can construct a Lorentzian metric \(g\) as follows

\[
g = 2\Omega^2 \left(ds \otimes d\overline{s} + d\overline{s} \otimes ds\right) + \hat{g}_{ab} (d\theta^a - b^a \overline{ds}) \otimes (d\theta^b - b^b \overline{ds}).
\]

It is easy to see that \(\{s, \overline{s}, \theta^1, \theta^2\}\) is also a double null coordinate system of the Lorentzian manifold \((M_{\kappa, \tau}, g)\), and the associated null vector fields \(L^s, L^{\overline{s}}\) satisfy

\[
L^s = \frac{\partial}{\partial s}, \quad L^{\overline{s}} = \frac{\partial}{\partial \overline{s}} + \vec{b}.
\]

Definition 2.2 (Lorentzian metric \(g_{\epsilon}\) on \((\kappa, \tau)\)-neighbourhood \(M_{\kappa, \tau}\)). Let \(g_{\epsilon}\) be a Ricci-flat Lorentzian metric on the \((\kappa, \tau)\)-neighbourhood \(M_{\kappa, \tau}\) and assume that in coordinates

\[
g_{\epsilon} = 2\Omega^2 \left(ds \otimes d\overline{s} + d\overline{s} \otimes ds\right) + (\hat{g}_{\epsilon})_{ab} (d\theta^a - b^a \overline{ds}) \otimes (d\theta^b - b^b \overline{ds}).
\]

Define the area radius function \(r_\epsilon(s, \overline{s})\) by

\[
4\pi r_\epsilon^2(s, \overline{s}) = \int_{\Sigma_{s, \overline{s}}} 1 \cdot d\text{vol}_{\hat{g}_{\epsilon}}.
\]

Then we have the structure coefficients and curvature components of \((M_{\kappa, \tau}, g_{\epsilon})\) associated to the double null foliation \(\{s, \overline{s}\}\). We call \(g_{\epsilon}\) \(\epsilon\)-close to the Schwarzschild metric on \(M_{\kappa, \tau}\).
if the following assumptions are true: $\Sigma_{0,0}$ is marginally outer trapped and

\[
1 - \epsilon < \frac{r_\epsilon}{r_{Sch}} < 1 + \epsilon,
\]

\[
|\log \Omega - \log \Omega_{Sch}| < \frac{\epsilon r_0}{r_\epsilon},
\]

\[
\partial_k \phi (\log \Omega - \log \Omega_{Sch}) < \frac{\epsilon r_0}{r_\epsilon^{1+n}},
\]

\[
\partial_k \phi (\log \Omega - \log \Omega_{Sch}) < \frac{\epsilon r_0}{r_\epsilon^2},
\]

\[
\partial_k \partial_{\chi} (\log \Omega - \log \Omega_{Sch}) < \frac{\epsilon r_0}{r_\epsilon^{2+n}},
\]

\[
\partial_k \partial_{\chi} (\log \Omega - \log \Omega_{Sch}) < \frac{\epsilon r_0}{r_\epsilon^m},
\]

\[
|b_\epsilon| \frac{\epsilon}{r_\epsilon^3},
\]

\[
\partial_k \phi b_\chi \frac{\epsilon r_0}{r_\epsilon^{3+n}},
\]

\[
\partial_k \partial_{\chi} b_\chi \frac{\epsilon r_0}{r_\epsilon^{3+n+m}},
\]

\[
\partial_k \partial_{\chi} \partial_{\chi} b_\chi \frac{\epsilon}{r_\epsilon^{3+n+m-2}},
\]
\begin{align}
\left| \phi_\epsilon - \phi_{Sch} \right|_g &< \epsilon r_\epsilon^2, \\
\left| \nabla^k \phi_\epsilon - \phi_{Sch} \right|_g &< \epsilon r_\epsilon^{1-n}, \\
\left| \nabla^k \partial^n_s (\phi_\epsilon - \phi_{Sch}) \right|_g &< \epsilon r_\epsilon^{1-3n}, \\
\left| \nabla^k \partial^n_s \partial^m_s (\phi_\epsilon - \phi_{Sch}) \right|_g &< \epsilon r_\epsilon^{1-3n-1}.
\end{align}

\begin{align}
\left| \omega_\epsilon - \omega_{Sch} \right|_g &< \frac{\epsilon r_\epsilon}{r_\epsilon^2}, \\
\left| \nabla^k (\omega_\epsilon - \omega_{Sch}) \right|_g &< \frac{\epsilon r_\epsilon}{r_\epsilon^2}, \\
\left| \nabla^k \partial^n_s (\omega_\epsilon - \omega_{Sch}) \right|_g &< \frac{\epsilon r_\epsilon^{1-n}}{r_\epsilon^{1+n}}, \\
\left| \nabla^k \partial^n_s \partial^m_s (\omega_\epsilon - \omega_{Sch}) \right|_g &< \frac{\epsilon r_\epsilon^{1-3n-1}}{r_\epsilon^{1+n-m-1}}.
\end{align}

\begin{align}
\left| \eta_\epsilon \right|_g &< \frac{\epsilon r_\epsilon}{r_\epsilon}, \\
\left| \nabla^k \eta_\epsilon \right|_g &< \frac{\epsilon r_\epsilon}{r_\epsilon}, \\
\left| \nabla^k \partial^n_s \eta_\epsilon \right|_g &< \frac{\epsilon r_\epsilon^{1-n}}{r_\epsilon^{1+n}}, \\
\left| \nabla^k \partial^n_s \partial^m_s \eta_\epsilon \right|_g &< \frac{\epsilon}{r_\epsilon^{1+n-m-1}}.
\end{align}

\begin{align}
\left| \eta_\epsilon \right|_g &< \frac{\epsilon r_\epsilon}{r_\epsilon}, \\
\left| \nabla^k \eta_\epsilon \right|_g &< \frac{\epsilon r_\epsilon}{r_\epsilon}, \\
\left| \nabla^k \partial^n_s \eta_\epsilon \right|_g &< \frac{\epsilon r_\epsilon^{1-n}}{r_\epsilon^{1+n}}, \\
\left| \nabla^k \partial^n_s \partial^m_s \eta_\epsilon \right|_g &< \frac{\epsilon}{r_\epsilon^{1+n-m-1}}.
\end{align}

\begin{align}
\left| \text{tr}_\epsilon - \text{tr}_\epsilon \chi_{Sch} \right|_g &< \frac{\epsilon r_\epsilon}{r_\epsilon} + \frac{\epsilon r_\epsilon^2}{r_\epsilon^2} \left( \frac{r_\epsilon}{r_\epsilon} \right), \\
\left| \nabla^k (\text{tr}_\epsilon - \text{tr}_\epsilon \chi_{Sch}) \right|_g &< \frac{\epsilon r_\epsilon}{r_\epsilon} + \frac{\epsilon r_\epsilon^2}{r_\epsilon^2} \left( \frac{r_\epsilon}{r_\epsilon} \right),
\end{align}

\text{THE SPACE OF MARGINALLY TRAPPED SURFACES}
In our setting, by construction, the second parametrization of the spacelike surface. The second one is given by the embedding of the spacelike surface. Compared to the first method, we will characterize a spacelike surface by two functions to parametrize the spacelike surface. The second method is more indirect compared to the first method.

Remark 2.3. In the definitions of \((M, g, \epsilon)\) for some \(\kappa, \tau\) and \(\epsilon\). To simplify the notations, we use \((M, g)\) to denote \((M, g, \epsilon)\).

3. Parametrization of Spacelike Surfaces and Incoming Null Hypersurfaces

In this section, we study two methods to parametrize spacelike surfaces in \((M, g)\). The first method consists of representing a spacelike surface as the graph of two functions \(f, \bar{f}\) on \(S^2\) in the double null coordinate system \(\{s, \bar{s}, \vartheta\}\) as in formula (63). We use these two functions to parametrize the spacelike surface. The second method is more indirect compared to the first method. We will characterize a spacelike surface by two functions. The first function is given by the parametrization of the incoming null hypersurface containing the spacelike surface. The second one is given by the embedding of the spacelike surface into the null hypersurface. Then these two functions together parametrize the spacelike surface. In our setting, by construction, the second parametrization of the surface.
spacelike surface is given. Then the first parametrization of the spacelike surface is obtained from the given (i.e. the second) parametrization by transformation.

In section 3.1 we introduce the first method to parametrize spacelike surfaces and give its application to the geometry of spacelike surfaces. In section 3.2 we introduce the method to parametrize incoming null hypersurfaces. In section 3.3 we introduce the second method to parametrize spacelike surfaces and in section 3.4, we study how to obtain the first parametrization of spacelike surfaces from the second parametrization.

3.1. First method to parametrize spacelike surfaces. Let Σ be a spacelike surface in \((M, g)\) and we assume that Σ is the graph of two functions \(f, f\) on \(S^2\) in the double null coordinate system \(\{s, \bar{s}, \theta\}\), i.e.

\[
\Sigma = \{(s, \bar{s}, \vartheta) = (f(\vartheta), f(\vartheta), \vartheta) \in S^2\}.
\]

We can also parametrize an open domain of Σ in the double null coordinate system \(\{s, s, \theta_1, \theta_2\}\)

\[
\{(s, s, \theta_1, \theta_2) = (f(\theta_1, \theta_2), f(\theta_1, \theta_2), \theta_1, \theta_2) \in \Sigma\}.
\]

We call the pair of functions \((f, f)\) the first parametrization of Σ.

Let \(T_\Sigma\) be the tangent space of Σ, we have the coordinate frame \(\{\tilde{\partial}_1, \tilde{\partial}_2\}\) where

\[
\tilde{\partial}_i = \partial_i + f_i \partial_s + f_{,i} \partial_s,
\]

and by \(L = \partial_s, L = \partial_s + b^i \partial_i\), the above is equivalent to

\[
\tilde{\partial}_i = \left(\delta^j_i - f_i b^j\right) \partial_j + f_i L + f_{,i} L = B^j_i \partial_j + f_i L + f_{,i} L,
\]

where

\[
B^j_i = \delta^j_i - f_i b^j.
\]

The intrinsic metric \(\tilde{g} = g|_\Sigma\) is

\[
\tilde{g}_{ij} = \tilde{g} \left(\tilde{\partial}_i, \tilde{\partial}_j\right) = g \left(\tilde{\partial}_i, \tilde{\partial}_j\right) = B^k_i B^l_j \tilde{g}_{kl} + 2 \Omega^2 \left(f_i f_j + f_j f_i\right).
\]

Let \(N_\Sigma\) be the normal bundle of Σ in \((M, g)\). We can find a null frame \(\{\tilde{L}_\Sigma, \tilde{L}_\Sigma\}\) of \(N_\Sigma\), where

\[
\begin{align*}
\tilde{L}_\Sigma &= L + \varepsilon L + \varepsilon^i \partial_i, \\
\tilde{L}_\Sigma &= L + \varepsilon L + \varepsilon^i \partial_i.
\end{align*}
\]

\(\varepsilon, \varepsilon^i, \varepsilon, \varepsilon^i\) are given by

\[
\varepsilon = \frac{-|e|^2}{(2 \Omega^2 + e \cdot e + \sqrt{(2 \Omega^2 + e \cdot e)^2 - |e|^2 |e|^2})},
\]

and

\[
\varepsilon^k = e^k + \varepsilon e^k, \quad e^k = e^k + \varepsilon e^k,
\]

\[
\varepsilon = \frac{-|e|^2}{(2 \Omega^2 + e \cdot e + \sqrt{(2 \Omega^2 + e \cdot e)^2 - |e|^2 |e|^2})}.
\]
where \( e^k, \underline{e}^k \) are given by
\[
e^k = -2\Omega^2 f_i (B^{-1})_j^i (\bar{g}^{-1})_{jk}^j, \quad \underline{e}^k = -2\Omega^2 f_i (B^{-1})_j^i (\bar{g}^{-1})_{jk}^j,
\]
and
\[
|e|^2 = \phi_{ij} e^i e^j, \quad |\underline{e}|^2 = \phi_{ij} \underline{e}^i \underline{e}^j, \quad e \cdot \underline{e} = \phi_{ij} e^i \underline{e}^j.
\]

To simplify the notation, we will omit the script \( \Sigma \) in \( \tilde{L}_\Sigma, \tilde{L}_\Sigma \). We have the function \( \tilde{\Omega} \) given by
\[
(76) \quad 2\tilde{\Omega}^2 = g (\tilde{\xi}, \tilde{\xi}) = 2\Omega^2 (1 + \varepsilon \cdot \bar{\varepsilon}) + \bar{\varepsilon} \cdot \bar{\varepsilon}
\]
where
\[
(77) \quad \bar{\varepsilon} \cdot \bar{\varepsilon} = \phi_{ij} \varepsilon^i \varepsilon^j.
\]
then we define another null frame \( \{ \tilde{L}', \tilde{L}' \} \) by
\[
(78) \quad \tilde{L}' = \tilde{\Omega}^{-2} \tilde{L}, \quad \tilde{L}' = \tilde{\Omega}^{-2} \tilde{L}.
\]
We can also define the conjugate null frames \( \{ \tilde{L}, \tilde{L}' \} \) and \( \{ \tilde{L}', \tilde{L} \} \), i.e.
\[
(79) \quad g(\tilde{L}, \tilde{L}') = 2, \quad g(\tilde{L}', \tilde{L}) = 2
\]

We recall the structure coefficients associated with the null frame \( \{ \tilde{L}, \tilde{L} \} \) on \( \Sigma \),
\[
(80) \quad \tilde{\chi}_{ij} = g \left( \nabla_{\tilde{\partial}_i} \tilde{L}, \tilde{\partial}_j \right), \quad \tilde{\chi}_{ij} = g \left( \nabla_{\tilde{\partial}_i} \tilde{L}, \tilde{\partial}_j \right),
\]
\[
\tilde{\eta}_i = \frac{1}{2} g \left( \nabla_{\tilde{\partial}_i} \tilde{L}, \tilde{\partial}_j \right), \quad \tilde{\eta}_i = \frac{1}{2} g \left( \nabla_{\tilde{\partial}_i} \tilde{L}, \tilde{\partial}_j \right).
\]
We have the following formulas for these structure coefficients.

**Proposition 3.1.** Let \( \Sigma \) be a given spacelike surface in \((M, g)\). Assume that \( \Sigma \) is represented by the first parametrization \((f, \underline{f})\). Then we have
\[
(81) \quad \tilde{\chi}_{ij} = \chi_{ij} + \varepsilon \chi_{ij} + (\varepsilon \cdot b - 2\Omega^2 \varepsilon)^p \nabla^2_{ij} f - 2\Omega^2 \nabla^2_{ij} \underline{f} - 2\text{sym} \left\{ \left[ \chi(\underline{\varepsilon}) + 2\Omega^2 \eta \right] \otimes \tilde{\partial}_f \right\}_{ij}
\]
\[
+ 2\text{sym} \left\{ \left[ \nabla b \cdot \varepsilon - \chi(\varepsilon) - \varepsilon \chi(b) - 2\Omega^2 \varepsilon \eta \right] \otimes \tilde{\partial}_f \right\}_{ij}
\]
\[
+ \left[ 2\chi(b, \varepsilon) + \chi(b, b) + \varepsilon \chi(b, b) + 4\Omega^2 \varepsilon b \cdot \eta - 
\frac{\partial b}{\partial s} \cdot \varepsilon + 4\Omega^2 \varepsilon \omega \right]
\]
\[
\cdot \left( \tilde{\partial}_f \otimes \tilde{\partial}_f \right)_{ij}
\]
\[
+ 2\text{sym} (\tilde{\partial}_f \otimes \tilde{\partial}_f)_{ij} \left[ 2\Omega^2 \eta \cdot \varepsilon + \chi(b, \varepsilon) + 2\Omega^2 b \cdot \eta \right] - 4\omega \Omega^2 (\tilde{\partial}_f \otimes \tilde{\partial}_f)_{ij},
\]
(82) 
\[ \tilde{\chi}_{ij} = \chi_{ij} + \varepsilon \chi_{ij} + (\varepsilon' \cdot b - 2 \Omega^2) p \nabla^2_{ij} f - 2 \Omega^2 \varepsilon \ p \nabla^2_{ij} f - 2 \text{sym} \left\{ \left[ \chi (\varepsilon) + 2 \Omega^2 \varepsilon \eta \right] \cdot \tilde{d} f \right\}_{ij} + 2 \text{sym} \left\{ \left[ \nabla b \cdot \varepsilon - \chi (\varepsilon) - \varepsilon \chi (b) - 2 \Omega^2 \eta \right] \cdot \tilde{d} f \right\}_{ij} + \left[ 2 \chi (b, \varepsilon) + \chi (b, b) + \varepsilon \chi (b, b) + 4 \Omega^2 b \cdot \eta - \nabla b b \cdot \varepsilon + \frac{\partial b}{\partial s} \cdot \varepsilon + 4 \Omega^2 \omega \right] \right. \\
 \left. \cdot \left( \tilde{d} f \otimes \tilde{d} f \right)_{ij} + 2 \text{sym}(\tilde{d} f \otimes \tilde{d} f)_{ij} \left[ 2 \Omega^2 \eta \cdot \varepsilon + \chi (b, \varepsilon) + 2 \Omega^2 \varepsilon b \cdot \eta \right] - 4 \varepsilon \omega \Omega^2 (\tilde{d} f \otimes \tilde{d} f)_{ij}, \right. \\
\]

(83) \[ \tilde{\eta} = \left[ 2 \Omega^2 (1 + \varepsilon \overline{\varepsilon}) + \varepsilon' \cdot \varepsilon \right]^{-1}. \]

\[ \left\{ 2 \Omega^2 \eta + 2 \Omega^2 \varepsilon \eta - \chi (\varepsilon) \cdot i + \varepsilon \chi (\varepsilon) \cdot i + \chi (\varepsilon) \cdot i - \varepsilon \chi (\varepsilon) \cdot i + f_1 [4 \Omega^2 \omega - 2 \Omega^2 b \cdot \eta - 2 \Omega^2 \varepsilon b \cdot \eta + 2 \Omega^2 \varepsilon \eta - 2 \Omega^2 \varepsilon \eta - \varepsilon \right. \\\n\left. \chi (b, \varepsilon) - \varepsilon \chi (b, \varepsilon) + \chi (b, \varepsilon) - \chi (b, \varepsilon) + \chi (b, \varepsilon) - \varepsilon \chi (b, \varepsilon) - (\nabla b) \cdot \varepsilon \right] + f_1 \left[ -2 \Omega^2 \eta \cdot \varepsilon + 2 \Omega^2 \eta \cdot \varepsilon + 4 \Omega^2 \varepsilon \omega + \chi (b, \varepsilon) \right] + 2 \Omega^2 \varepsilon (\tilde{d} \varepsilon) \cdot i + (\nabla \varepsilon) \cdot \varepsilon, \right\}, \]

where \( \tilde{d} \) is the differential operator on \( \Sigma \), \( p \nabla \) on \( \Sigma \) is the pull back of \( \nabla \) on \( (M, g) \) via the embedding of \( \Sigma \), i.e., for any tensor field \( T \) on \( \Sigma \),

\[ p \nabla_i T_{i_1 \cdots i_k} = \partial_i T_{i_1 \cdots i_k} - T_{j_1 \cdots j_i} \cdot i_{i_1 \cdots i_k} \cdot i_{i_1 \cdots i_k} \cdot i_{i_1 \cdots i_k} + T_{j_1 \cdots j_i} \cdot i_{i_1 \cdots i_k} \cdot i_{i_1 \cdots i_k}, \]

and the product \( \cdot \) are taken with respect to the metric \( g \).

**Remark 3.2.** Proposition 3.1 holds for general Lorentzian manifolds, not only Ricci flat Lorentzian manifolds. So are propositions 3.3, 3.4.

There are two important special case of proposition 3.1 \( f \equiv s_0 \) and \( f \equiv 2 \Omega \).

**Proposition 3.3.** Let \( \Sigma \) be a given spacelike surface in \( (M, g) \). Assume that \( \Sigma \) is represented by the first parametrization \( (f, f) \). Moreover, we assume that \( \Sigma \) is embedded in the outgoing null hypersurface \( C_{s=s_0} \). Then \( f \equiv s_0 \) in the first parametrization of \( \Sigma \) and we have

(85) \[ \tilde{\chi}_{ij} = \chi_{ij} - \Omega^2 \left| d f \right|_g^2 \chi_{ij} - 2 \Omega^2 p \nabla_{ij} f - 4 \Omega^2 \text{sym} \left\{ \eta \otimes \tilde{d} f \right\}_{ij} - 4 \omega \Omega^2 (\tilde{d} f \otimes \tilde{d} f)_{ij} + 4 \Omega^2 \text{sym} \left\{ \tilde{d} f \otimes \chi (p \nabla f) \right\}_{ij}, \right. \\
\]

(86) \[ \tilde{\nabla} \tilde{\chi} = \text{tr} \chi - 2 \Omega^2 p \Delta f - \Omega^2 \left| d f \right|_g^2 \chi - 4 \Omega^2 \eta \cdot \tilde{d} f - 4 \Omega^2 \omega \left| d f \right|_g^2 + 4 \Omega^2 \chi (p \nabla f, p \nabla f), \]

(87) \[ \tilde{\chi}_{ij} = \chi_{ij}, \quad \tilde{\nabla} \chi = \text{tr} \chi, \quad \tilde{\eta} = \eta_0 + \chi (p \nabla f)_{ij}. \]
where \( p\nabla^i f = (\hat{g}^{-1})^{ij} f_j \), \( p\Delta f = (\hat{g}^{-1})^{ij} p\nabla^i \nabla_j f \), the inner product \( \cdot \) is taken with respect to \( \hat{g} \) and \( \hat{\nabla} \) is the trace operator with respect to \( \hat{g} \).

**Proposition 3.4.** Let \( \Sigma \) be a given spacelike surface in \((M, g)\). Assume that \( \Sigma \) is represented by the first parametrization \((f, f)\). Moreover, we assume that \( \Sigma \) is embedded in the incoming null hypersurface \( \Sigma_{\infty} = \Sigma_0 \). Then \( f \equiv \Sigma_0 \) in the first parametrization of \( \Sigma \) and we have

\[
\tilde{\chi}_{ij} = \chi_{ij} + 2\text{sym}\{(-\chi(b)) \otimes \hat{\nabla} f \}_{ij} + \chi(b, b) f_i f_j,
\]

\[
\tilde{\epsilon}_{ij} = \chi_{ij} + \epsilon' \tilde{\chi}_{ij} + (b \cdot \tilde{\epsilon} - 2) \ n^2 ij f.
\]

\[
\tilde{\eta}_i = \eta_i + \frac{1}{2} \tilde{\chi}(\tilde{\epsilon})_i + f_i \left[ 2\omega - b \cdot \eta - \frac{1}{2} \chi(b, \tilde{\epsilon}) \right].
\]

**Remark 3.5.** In propositions 3.1, 3.2, \( p\nabla \) (the pull back of \( \nabla \) via the embedding of \( \Sigma \) into \((M, g)\)) only applies to the function \( f_i f \) of the first parametrization of \( \Sigma \). To simplify the notation, we will drop the script \( p \) when it applies to \( f_i f \). For example, \( \nabla f \) means \( p\nabla f \), and \( \nabla^2 f \) means \( p\nabla^2 f \). Similarly, we drop the script \( p \) in the operator \( p\Delta \) when it applies to \( f_i f \).

**3.2. Parametrization of incoming null hypersurfaces.** Let \( \tilde{C} \) be an incoming null hypersurface embedded in \((M, g)\) which is regular and extends to past null infinity \( \mathcal{I}^\pm \). We define \( \Sigma_s \) as the intersection of \( \tilde{C} \) with \( C_s \), i.e.

\[
\Sigma_s = \tilde{C} \cap C_s,
\]

then the family of spacelike surfaces \( \{ \Sigma_s \} \) is a foliation of \( \tilde{C} \). In particular, \( \Sigma_0 \) is the intersection of \( \tilde{C} \) with \( C_{s=0} \). Geometrically, we can see that \( \tilde{C} \) is determined by \( \hat{\Sigma}_0 \), so we can use \( \Sigma_0 \) to characterize \( \tilde{C} \). We assume that the first parametrization of \( \Sigma_0 \) is \((0, \Sigma_0)\), then we will use the function \( s = 0 \) to parametrize the null hypersurface \( \tilde{C} \).

Associated with the foliation \( \{ \Sigma_s \} \), we can define the tangential null vector fields \( \tilde{L} \) on \( \tilde{C} \) by the condition

\[
\tilde{L}_s = 1,
\]

and define the null vector field \( \tilde{L}' \) by the condition that \( \{ \tilde{L}, \tilde{L}' \} \) is a conjugate null frame of the normal space \( N\Sigma_s \) for each \( \Sigma_s \). We can restrict coordinate functions \( \{ s, \theta \} \) or \( \{ s, \theta^1, \theta^2 \} \) to \( \tilde{C} \) to get a coordinate system of \( \tilde{C} \). \( \{ \tilde{\partial}_s = \partial / \partial s, \tilde{\partial}_1 = \partial / \partial \theta^1, \tilde{\partial}_2 = \partial / \partial \theta^2 \} \) is the coordinate frames obtained from the coordinate system \( \{ s, \theta^1, \theta^2 \} \) on \( \tilde{C} \). Note that \( \tilde{\partial}_s \)
may not be null, which implies that \( \vec{L} \) and \( \vec{\partial}_s \) may not coincide, so we can assume that there exists a \( \Sigma_s \)-tangential vector field \( \vec{b} \) such that

\[
\vec{L} = \vec{\partial}_s + \vec{b}, \quad \vec{b} = b^1 \vec{\partial}_1 + b^2 \vec{\partial}_2.
\]

We can apply section 3.1 to \( \{\Sigma_s\} \). We assume that the first parametrization of \( \Sigma_s \) is \( (\vec{s}, \vec{f}) \), and since \( \Sigma_s \) is contained in \( C_s \), we have \( \vec{s} \equiv s \), then the coordinate frames are

\[
(93) \quad \vec{L} = \vec{\partial}_s + \vec{b}, \quad \vec{b} = b^1 \vec{\partial}_1 + b^2 \vec{\partial}_2.
\]

and the associated null vector field \( \vec{L} \) is

\[
(94) \quad \vec{L} = \vec{\partial}_s + \vec{\partial}_s \vec{f} \vec{\partial}_2, \quad \vec{\partial}_i = \vec{\partial}_i + \vec{\partial}_i \vec{f} \vec{\partial}_2,
\]

and substituting from equation (96), we get

\[
(100) \quad \vec{\partial}_s \vec{f} = -b^i \vec{\partial}_i \vec{f} + \Omega^2 (\vec{g}^{-1})_{ij} \vec{\partial}_i \vec{f} \vec{\partial}_j \vec{f},
\]

The above equation is a first order nonlinear equation of \( \vec{f} \). If \( s = 0 \) is known, then we can solve \( \vec{f} \) for any \( s \) via this equation. Thus (cccc So, Thus, Therefore) the above equation confirms again that \( \vec{C} \) is uniquely determined by \( \vec{C}_0 \) or \( s = 0 \). Therefore we use \( \vec{C}_0 \) or \( \vec{C}_{s=0} \) to denote the incoming null hypersurface \( \vec{C} \) when we need to emphasize. Sometimes in order to simplify the notations, we will use the following conventions,

\[
(101) \quad \vec{s} \vec{f} = \vec{\partial}_s \vec{f}, \quad \vec{s} \vec{L} = \vec{\partial}_s \vec{L}.
\]

The structure coefficients and curvature components on \( \Sigma \) associated with the foliation \( \{\Sigma_s\} \) can be calculated by propositions 3.3. The acceleration \( \vec{\omega} \) is defined by

\[
(102) \quad \nabla_{\vec{f}} \vec{L} = 2 \vec{\omega} \vec{L}.
\]

It is not covered in proposition 3.3, so we give the formula here.

**Proposition 3.6.** The acceleration \( \vec{\omega} \) can be written as

\[
(103) \quad \vec{\omega} = \vec{\omega} - 2 \Omega^2 (\vec{g}^{-1})_{ij} \eta^{ij} \vec{f} \vec{L} - \Omega^2 \chi^{ij} \vec{f} \vec{L},
\]

where

\[
(104) \quad \chi^{ij} = (\vec{g}^{-1})_{ik} (\vec{g}^{-1})_{jl} \chi_{kl}.
\]
Remark 3.7. We put the upper script $s$ to the structure coefficients and curvature components on their left, to emphasis they are coefficients on the surface $\Sigma_s$. Whenever it is clear from the context which surface we are working on, we omit this script $s$.

3.3. Second method to parametrize spacelike surfaces. Let $\tilde{\Sigma}$ be a spacelike surface embedded in $\tilde{\mathcal{C}}$, we can parametrize it as the graph of a function $\tilde{f}$ on $\mathbb{S}^2$ using the coordinate system $\{s, \vartheta\}$ on $\tilde{\mathcal{C}}$, i.e.

$$\tilde{\Sigma} = \{(s, \vartheta) = (\tilde{f}(\vartheta), \vartheta), \vartheta \in \mathbb{S}^2\} \subset \tilde{\mathcal{C}}.$$  \hspace{1cm} (105)

The same applies to the coordinate system $\{s, \theta^1, \theta^2\}$ on $\tilde{\mathcal{C}}$. We denote by $\{\tilde{\partial}_1, \tilde{\partial}_2\}$ the frame on $T\tilde{\Sigma}$ with

$$\tilde{\partial}_i = \partial_i + \tilde{\partial}_i \tilde{f} \tilde{n}_s = \tilde{B}_j^i \tilde{\partial}_j + \tilde{\partial}_i \tilde{n}, \quad \tilde{B}_i^j = \delta_i^j - \tilde{\partial}_i \tilde{n}.$$  \hspace{1cm} (106)

The intrinsic metric $\tilde{g}$ on $T\tilde{\Sigma}$ is

$$\tilde{g}_{ij} = \tilde{B}_k^i \tilde{B}_l^j \tilde{g}_{kl} = \tilde{g}_{ij} - \left( \tilde{g}_{ij} \tilde{f}_i \tilde{f}_j + \tilde{g}_{ij} \tilde{n} \tilde{f}_i \right) + \tilde{f}_i \tilde{f}_j \tilde{n}^2.$$  \hspace{1cm} (107)

The set of null vector fields $\{\tilde{L}, \tilde{L}'\}$ form a conjugate null frame of $N\tilde{\Sigma}$, where

$$\begin{cases}
\tilde{L}^{\tilde{\Sigma}} = \tilde{L} \\
\tilde{L}'^{\tilde{\Sigma}} = \tilde{L}' + \tilde{\varepsilon}' \tilde{\partial}_i,
\end{cases}$$  \hspace{1cm} (108)

and here

$$\tilde{\varepsilon}' = - \left( \tilde{\phi}^{-1} \right)^{kl} (\tilde{B}^{-1})^i_j (\tilde{B}^{-1})^j_i \tilde{f}_i \tilde{f}_j, \quad \tilde{\varepsilon}'^{jk} = -2 \left( \tilde{\phi}^{-1} \right)^{kl} (\tilde{B}^{-1})^i_j \tilde{f}_i.$$  \hspace{1cm} (109)

Then we can calculate the structure coefficients

$$\begin{aligned}
\tilde{\chi}^{\tilde{\Sigma}}, & \quad \tilde{\chi}'^{\tilde{\Sigma}}, & \quad \tilde{\eta}^{\tilde{\Sigma}} \\
\end{aligned}$$

on $\tilde{\Sigma}$ associated with the null frame $\{\tilde{L}^{\tilde{\Sigma}}, \tilde{L}'^{\tilde{\Sigma}}\}$, by proposition 3.4 from the structure coefficients on $\tilde{\mathcal{C}}$ associated with the foliation $\{\Sigma_s\}$.

The above construction actually provides another method to parametrize a spacelike surface in $(M, g)$, i.e. a spacelike surface $\tilde{\Sigma}$ can be determined by two sets of data:

1. the incoming null hypersurface $\tilde{\mathcal{C}}$ which $\tilde{\Sigma}$ is embedded in,
2. the function $\tilde{f}$ whose graph in the coordinate system $\{s, \vartheta\}$ on $\tilde{\mathcal{C}}$ is $\tilde{\Sigma}$.

Since $\tilde{\mathcal{C}}$ can be parametrized by the initial surface $\tilde{\Sigma}_0$ or the function $s=\tilde{f}_0$, we get another method to parametrize $\tilde{\Sigma}$ by two functions $s=\tilde{f}_0$ and $\tilde{f}$. We call the pair of functions $\left(\tilde{f}, s=\tilde{f}_0\right)$ the second parametrization of $\tilde{\Sigma}$. 

3.4. Transformation from the second to the first parametrization. Let \( \tilde{\Sigma} \) be a spacelike surface with the second parametrization \((s = \tilde{f}, \tilde{g})\). We want to obtain the first parametrization of \( \tilde{\Sigma} \) from the second parametrization. Assume that the first parametrization of \( \tilde{\Sigma} \) is \((\tilde{g}, \tilde{f})\). Note that the first and second parametrizations share the function \( \tilde{f} \). This is due to the coordinate function \( s \) being part of the double null system \( \{s, \tilde{g}, \vartheta\} \) and the system \( \{s, \vartheta\} \) on \( \tilde{C} \).

We present three ways to obtain the first parametrization of \( \tilde{\Sigma} \) from the second one. In (I), we see the most simple situation. However it is not practical for calculation, since the function \( s \) in (I) is not known a priori. Therefore we introduce (II) and (III), which yield an important partial differential equation ((121), (128)) each, that will be crucial for our further investigations.

(I): since we can calculate the embedding of \( \tilde{C} \) into \( M \), which is given by
\[
\tilde{C} \hookrightarrow M : (s, \vartheta) \mapsto (s, \tilde{g} = \tilde{f}(\vartheta), \theta),
\]
and we know the embedding of \( \tilde{\Sigma} \) into \( \tilde{C} \)
\[
\tilde{\Sigma} \hookrightarrow \tilde{C} : \vartheta \mapsto (s = \tilde{f}(\vartheta), \theta),
\]
then we just compose these two embeddings to get the embedding of \( \tilde{\Sigma} \) into \( M \), which is
\[
\tilde{\Sigma} \hookrightarrow M : \vartheta \mapsto (s, \tilde{g}, \theta) = (\tilde{f}(\vartheta), s = \tilde{f}(\vartheta) \tilde{f}, \theta),
\]
hence
\[
\tilde{f}(\vartheta) = s = \tilde{f}(\vartheta) \tilde{f}(\vartheta).
\]

(II): we define \( \tilde{f} \) by
\[
\tilde{f} = t \tilde{f}
\]
and \( \tilde{\Sigma}_t \) is the spacelike surface embedded into \( \tilde{C} \) by
\[
\tilde{\Sigma}_t \hookrightarrow \tilde{C} : \vartheta \mapsto (s, \tilde{g}) = (t \tilde{f}(\vartheta), \theta),
\]
then we assume the first parametrization of \( \tilde{\Sigma}_t \) is \( (\tilde{f}, \tilde{f}) \). We derive an equation for \( \tilde{f} \). On one hand, from the first solution we know
\[
\tilde{f} = s = t \tilde{f},
\]
and \( \tilde{f} \) satisfies equation (100), on the other hand we have the following lemma.

**Lemma 3.8.** Let \( \{u\} \) be a family of functions on \( \{\tilde{\Sigma}_t\} \) and \( u \) satisfies the equation
\[
\tilde{\partial}_s u = F(s, u, \tilde{\partial}_t u).
\]
We define \( \{\tilde{u}\} \), a family of functions on \( \{\tilde{\Sigma}_t\} \) by
\[
\tilde{u}(\theta) = s = t \tilde{f}(\theta) u(\theta),
\]
then $\tilde{u}$ satisfies the equation

\[
\tilde{\partial}_t \tilde{u} = \tilde{f} F(t, \tilde{u}, -t f \tilde{f}^{-1} \tilde{\partial}_t \tilde{u} + \tilde{\partial}_t \tilde{u}).
\]

We can apply lemma 3.8 to $\tilde{f}$ and obtain

\[
\tilde{\partial}_t \tilde{f}_L = -f b' \left( -t f \tilde{f}^{-1} \tilde{\partial}_t \tilde{f}_L + \tilde{\partial}_t \tilde{f} \right) + f \Omega^2 (g^{-1})^{ij} \left( -t f \tilde{f}^{-1} \tilde{\partial}_t \tilde{f}_L + \tilde{\partial}_j \tilde{f}_L \right),
\]

through which in principle, we can solve $\tilde{f}_L$ with the initial condition $t=0, L = s=0$ and in the end we set $t=1$ to get $\tilde{f}_L$, which is equal to $t=1$ $\tilde{f}_L$ by definition.

\( (III): \) we derive another equation for $\tilde{f}_L$ in (II). We have that the first parametrization of $\tilde{\Sigma}_t$ is $(\tilde{f}_L, \tilde{f}_L)$, then by equation (121) we have

\[
\tilde{\Sigma}_t = L + \tilde{\xi} \partial_t \partial_s + \tilde{\xi} \partial_s \partial_s + (b' + \tilde{\xi}) \partial_t
\]
and the tangential frame $\{ \tilde{\partial}_1, \tilde{\partial}_2 \}$ is given by

\[
\tilde{\partial}_1 = \partial_t + \tilde{f}_L \partial_s + \tilde{f}_L \partial_s,
\]
hence substituting equation (123) in (122), we obtain

\[
\tilde{\Sigma}_t = \partial_s + \tilde{\xi} \partial_s + (b' + \tilde{\xi}) \partial_t \tilde{\partial}_1 - \tilde{f}_L \partial_s = (b' + \tilde{\xi}) \partial_t + \left[ \tilde{\xi} - (b' + \tilde{\xi}) \tilde{f}_L \right] \partial_s + \left[ 1 - (b' + \tilde{\xi}) \tilde{f}_L \right] \partial_s.
\]

Thus on $\tilde{\Sigma}_t$ we have

\[
\tilde{\partial}_s = \partial_s + \left[ 1 - (b' + \tilde{\xi}) \tilde{f}_L \right]^{-1} \left[ \tilde{\xi} - (b' + \tilde{\xi}) \tilde{f}_L \right] \partial_s,
\]

\[
\tilde{\partial}_t = \left( \tilde{\partial}_t \tilde{f}_L \right) \partial_s + \tilde{f} \left[ 1 - (b' + \tilde{\xi}) \tilde{f}_L \right]^{-1} \left[ \tilde{\xi} - (b' + \tilde{\xi}) \tilde{f}_L \right] \partial_s.
\]

On the other hand, we can write

\[
\tilde{\partial}_t = \left( \tilde{\partial}_t \tilde{f}_L \right) \partial_s + \left( \tilde{\partial}_t \tilde{f}_L \right) \partial_s.
\]

Then we get the equation for $\tilde{f}_L$

\[
\tilde{\partial}_t \tilde{f}_L = \tilde{f} \cdot \left[ 1 - (b' + \tilde{\xi}) \tilde{f}_L \right]^{-1} \cdot \left[ \tilde{\xi} - (b' + \tilde{\xi}) \tilde{f}_L \right],
\]

which is a nonlinear first order equation, and the initial condition of this equation is $t=0, L = s=0$.

\textbf{Remark 3.9} (Comparison of the two parametrizations of spacelike surfaces). The first parametrization is convenient when studying the geometry of spacelike surfaces, because of proposition 3.1. The second parametrization has the advantage that it tells which null hypersurface the spacelike surface lies in.

When we study the geometry of a null hypersurface via a foliation of spacelike surfaces, we can easily parametrize these spacelike surfaces via the second method, then find the first parametrization by the solutions given above, and eventually calculate the structure coefficients of these spacelike surfaces through proposition 3.1.
Remark 3.10 (Regularity of the transformation between two parametrizations). Given \((s=\tilde{\varphi}, \tilde{\hat{f}})\), we can solve equation \((128)\). If we examine the regularity of the solution \(\tilde{f}\), its differentiability will be one order less than \(\tilde{\hat{f}}\), since on the right hand side, the differential of \(\tilde{f}\) shows up. Thus by (III), the transformation from the second parametrization \((s=\hat{\varphi}, \hat{f})\) to the first parametrization \((\tilde{\varphi}, \tilde{f})\) will lose one derivative. For example, if \(s=\tilde{\varphi}, \tilde{\hat{f}} \in W^{5,p}\), then by solving equation \((128)\), we can only show that \(\tilde{f} \in W^{4,p}\). Then if we calculate the structure coefficients of the surface, they are only in the Sobolev space \(W^{2,p}\) by proposition \(3.1\). However, in section 4, we can actually show that the structure coefficients will be in the Sobolev space \(W^{3,p}\) if \(s=\tilde{\varphi}, \tilde{\hat{f}} \in W^{3,p}\).

4. Estimating \(\tilde{f}\) of the first parametrization of spacelike surfaces

We estimate the function \(\tilde{f}\) of the first parametrization of spacelike surfaces when given the second parametrization. Assume that \(\tilde{\Sigma}\) is given with the second parametrization \((s=\hat{\varphi}, \hat{f})\), where \(s=\hat{\varphi}\) parametrizes the null hypersurface \(\tilde{\Sigma}\) containing \(\tilde{\Sigma}\). In the previous section, we give three ways to find the first parametrization of \(\tilde{\Sigma}\). From the point of view of analysis, we are interested in the map: \((s=\hat{\varphi}, \hat{f}) \to (\tilde{\varphi}, \tilde{f})\).

We show that the size of the first parametrization \((\tilde{\varphi}, \tilde{f})\) can be bounded by the size of the second parametrization \((\hat{\varphi}, \hat{f})\). We state the main theorems of this section.

**Theorem 4.1.** Assume that for the second parametrization of \(\tilde{\Sigma}\) we have

\[
\left\|d^{s=\hat{\varphi}} f^{n+1,p} \right\| \leq \delta_0 r_0, \quad \left\|s=\hat{\varphi} f^{n+1,p} \right\| \leq \delta_m r_0, \quad \tilde{f} \equiv s_0,
\]

where \(n \geq 1, p > 2 \) or \(n \geq 2, p \geq 1\).

Then for \(c, \delta_0, \delta_m\) sufficiently small depending on \(n, p\), where \(c\) is the one in the definition of \(c\)-close Schwarzschild metric \(g\), there exist constants \(c_0(n,p), c_{m,m}(n,p), c_{m,o}(n,p)\), such that for the first parametrization of \(\tilde{\Sigma}\) we obtain

\[
\left\|d^{s=\tilde{\varphi}} f^{n+1,p} \right\| \leq c_0(n,p) \left\|d^{s=\hat{\varphi}} f^{n+1,p} \right\|, \\
\left\|s=\tilde{\varphi} f^{n+1,p} \right\| \leq c_{m,m}(n,p) \left\|s=\hat{\varphi} f^{n+1,p} \right\| + c_{m,o}(n,p) \frac{1}{r_0} \left(\left\|d^{s=\tilde{\varphi}} f^{n+1,p}\right\| \right)^2.
\]

**Theorem 4.2.** Assume that in the second parametrization of \(\tilde{\Sigma}\),

\[
\left\|d^{s=\tilde{\varphi}} f^{n+1,p} \right\| \leq \delta_0 r_0, \quad \left\|s=\tilde{\varphi} f^{n+1,p} \right\| \leq \delta_m r_0, \quad \tilde{f} = s_0, \quad \left\|d^{s=\tilde{\varphi}} f^{n+1,p} \right\| \leq \delta_o (r_0 + s_0),
\]

where \(n \geq 1, p > 2\) or \(n \geq 2, p \geq 1\).
Then for \( c_0, a, \delta_0 \) sufficiently small depending on \( n, p \), where \( c_0 \) is the one in the definition of \( \epsilon \)-close Schwarzschild metric \( g_\epsilon \), there exist constants \( c_0(n, p), c_{m,m}(n, p), c_{m,0}(n, p) \), such that in the first parameterization of \( \bar{\Sigma} \),

\[
\begin{align*}
\| \hat{d} \hat{\tilde{f}} \|^{n,p} &\leq c_0(n, p) \| \hat{d} s=0 \hat{\tilde{f}} \|^{n,p}, \\
\| \hat{\tilde{f}} \| &\leq c_{m,m}(n, p) \left( \| \hat{d} s=0 \hat{\tilde{f}} \|^{n,1} + c_{m,0}(n, p) \right) \frac{1}{r_0} \left( \| \hat{d} s=0 \hat{\tilde{f}} \|^{n+1,p} \right)^2,
\end{align*}
\]

Remark 4.3. Notice that in theorem 4.2, we do not have the highest order estimate of \( \hat{\tilde{f}} \) which is the estimate of the \((n+1)\)-order derivatives, therefore theorem 4.2 does not imply theorem 4.1.

Remark 4.4. Both theorems will be proven using bootstrap arguments. Let us consider the case in theorem 4.2 to explain the idea of the bootstrap arguments. Let \( \{\bar{\Sigma}_s\} \) be a family of surfaces with the second parametrization \((s=0, f, s)\). We will assume that the theorem holds for \( \bar{\Sigma}_s \), where \( s \in [0, s_0] \), which is called the bootstrap assumption. Then we will use this bootstrap assumption to show that at the end point \( s = s_0 \), we can actually get strict inequalities for \( \| \hat{d} \hat{\tilde{f}} \|^{n+1,p}, \| \hat{\tilde{f}} \| \). Hence we can actually extend the bootstrap assumption beyond the end point \( s = s_0 \), so eventually the bootstrap assumption holds for all \( s \), which proves theorem 4.1.

4.1. Proof of theorem 4.1. We have that \( \bar{\Sigma} = \bar{\Sigma}_{s=s_0} \) which is a leaf of the foliation \( \{\bar{\Sigma}_s = C_0 \cap \bar{\mathcal{C}}_s\} \) and \( \hat{\tilde{f}} = s=s_0 \). \( \hat{\tilde{f}} \) satisfies equation (100) which we cite here again

\[
\hat{\partial}_s \hat{\tilde{f}} = -b^i \hat{\partial}_i \hat{\tilde{f}} + \Omega^2 (g^{-1})^{ij} \hat{\partial}_i \hat{\tilde{f}} \hat{\partial}_j \hat{\tilde{f}}.
\]

This equation is nonlinear in the first as well as the highest derivatives. In order to estimate the solution, we need to differentiate the equation. In our case, we take the Laplacian of the equation.

\[
\begin{align*}
\hat{\partial}_s \Delta \hat{\tilde{f}} &= -b^i (\hat{\Delta} \hat{\tilde{f}})_i + 2 \Omega^2 g^{-1} b^j \tilde{\tilde{f}}_j \left( \hat{\Delta} \hat{\tilde{f}} \right)_i - b^i \tilde{\tilde{f}}_j - (\hat{\Delta} b)_j \tilde{\tilde{f}}_j - 2 (\hat{\nabla} b) \nabla k \tilde{\tilde{f}}_j \\
&\quad - \hat{\partial}_k (\hat{\nabla} b) \tilde{\tilde{f}}_j + \hat{\partial}_k b^j \tilde{\tilde{f}}_j \hat{\Delta} \hat{\tilde{f}} - 2 \hat{\partial}_k b^j \tilde{\tilde{f}}_j \nabla i \tilde{\tilde{f}} - \partial_k^2 b^j \tilde{\tilde{f}}_j \nabla i \tilde{\tilde{f}} - \partial_k^2 b^j \tilde{\tilde{f}}_j \nabla i \tilde{\tilde{f}} \\
&\quad + 2 \Omega^2 g^{-1} \nabla k \tilde{\tilde{f}}_j + \hat{\Delta} (\Omega^2 g^{-1})^{ij} \tilde{\tilde{f}}_i \tilde{\tilde{f}}_j + 4 \nabla k (\Omega^2 g^{-1})^{ij} \tilde{\tilde{f}}_i \tilde{\tilde{f}}_j \\
&\quad + 2 \Omega^2 g^{-1} \nabla k \tilde{\tilde{f}}_j + \hat{\Delta} (\Omega^2 g^{-1})^{ij} \tilde{\tilde{f}}_i \tilde{\tilde{f}}_j + 4 \nabla k (\Omega^2 g^{-1})^{ij} \tilde{\tilde{f}}_i \tilde{\tilde{f}}_j \\
&\quad + \hat{\Delta} \tilde{\tilde{f}}_j + (\Omega^2 g^{-1})^{ij} \tilde{\tilde{f}}_i \tilde{\tilde{f}}_j + \hat{\partial}_k (\Omega^2 g^{-1})^{ij} \tilde{\tilde{f}}_i \tilde{\tilde{f}}_j.
\end{align*}
\]
where \( * \) is a polynomial of type \( \Omega \).

Then we write these two terms as follows:

For example \( \tilde{s}_f \) derivatives and \( \nabla \) is the covariant derivative on \( (\tilde{\Sigma}, \tilde{g}) \), i.e. for any tensor field of any type \( T \) on \( (M, g) \) for example \( \tilde{b}, \tilde{g}, \tilde{g}^{-1} \),

\[
\tilde{\nabla}_i T_{j_1 \cdots j_l} = \partial_i T_{j_1 \cdots j_l} - T_{j_1 \cdots j_l} \Gamma^r_i_{j_m} + T_{j_1 \cdots j_l} \Gamma^r_i_{j_m} \tilde{g}_{k_s}.
\]

\( \tilde{\nabla} \) is the covariant derivative on \( (\tilde{\Sigma}, \tilde{g}) \), i.e. for any tensor field on \( \tilde{\Sigma} \) for example \( \tilde{f}, \tilde{d} \tilde{f} \)

\[
P_{m,m'}(\tilde{\nabla}, \tilde{s}_f) \text{ represent polynomials of } \tilde{\nabla} \tilde{s}_f \text{ where } m \text{ is the sum of the order of covariant derivatives and } m' \text{ is the degree of the polynomial, i.e.}
\]

\[
P_{m,m'}(\tilde{\nabla}, \tilde{s}_f) = \sum_{k_1, \ldots, k_{m'} m_k = m, k_1, \ldots, k_{m'} > 1} c_{k_1 \cdots k_{m'}} \left( \tilde{\nabla}^{k_1} \tilde{s}_f \right) \cdots \left( \tilde{\nabla}^{k_{m'}} \tilde{s}_f \right).
\]

For example \( \tilde{\nabla}_i \tilde{\nabla}_j \tilde{s}_f \) is a polynomial of type \( P_{3,2}(\tilde{\nabla}, \tilde{s}_f) \). * means the contraction of tensor fields with respect to the metric \( \tilde{g} \). For example we write

\[
-2\tilde{\partial}_i \tilde{b} \tilde{s}_f \tilde{\nabla}_i \tilde{s}_f \tilde{\nabla}_j \tilde{s}_f \text{ as } \partial_i \tilde{b} * \tilde{\nabla} \tilde{s}_f * \tilde{\nabla} \tilde{s}_f \text{ whenever things do not depend on the specific contraction.}
\]

In equation (134), for the first and second terms on the right hand side, we define the vector field \( \tilde{s}_X \)

\[
\tilde{s}_X = b^i \tilde{\partial}_i - 2\Omega^2 (\tilde{g}^{-1})^{ij} \tilde{s}_f \tilde{\nabla}_j \tilde{\partial}_i.
\]

Then we write these two terms as follows:

\[
-\tilde{s}_X^i (\tilde{\nabla} \tilde{s}_f)_i.
\]

We prove theorem 1.1 using the bootstrap argument with the help of equation (133). We state the bootstrap assumption in the following.
Assumption 4.5 (Bootstrap assumption of $\tilde{s}^m_f$ in theorem 4.1). Assume that for all $s \in [0, s_a]$ where $s_a \in (0, \infty)$ or $s \in [s_a, 0]$ where $s_a \in (-\kappa r_0, 0)$, we have

\begin{equation}
\left\| \tilde{\mathbf{f}}^{(n+1),p} \right\|_{m,m+1} \leq c(n,p) \left\| \tilde{s}^{(n+1),p} \right\|_{n+1,m+1},
\end{equation}

where $\left\| \tilde{\mathbf{f}}^{(n+1),p} \right\|_{m,m+1} \leq c(n,p) \left\| \tilde{s}^{(n+1),p} \right\|_{n+1,m+1}$.

The goal is to prove that at $s = s_a$, we can actually get strict inequalities in the estimate (140) of $s = s_a$. Then by continuity, we can find a slightly larger $s'$ such that the bootstrap assumption is also valid for all $s \in [0, s'_a]$, hence we can conclude that the bootstrap assumption is valid for all $s \geq 0$. We carry out these steps in the following.

We use the bootstrap assumption of $\tilde{s}^m_f$ to estimate the vector fields $\tilde{\mathbf{X}}$, $\sum_{m+n \leq 3} P_{m,m+1}(\tilde{\mathbf{X}})$, and $\sum_{m+n \leq 4} P_{m,m+2}(\tilde{\mathbf{X}})$, which are the results of lemmas 4.6 and 4.7. Then we integrate equation (138) to get the estimates on $\left\| \tilde{\mathbf{f}}^{(n+1),p} \right\|_{m,m+1}$ and $\left\| \tilde{s}^{(n+1),p} \right\|_{m,m+1}$, which are contained in lemmas 1.8 and 1.9.

Finally, by choosing the suitable constants $c(n,p), c_{m,m'}, c_{m,o}$, we can strengthen the inequalities in the bootstrap assumption to strict inequalities.

Lemma 4.6 (Estimate of the vector field $\tilde{\mathbf{X}}$ on $\tilde{\Sigma}$). Let the vector field $\tilde{\mathbf{X}}$ be given by equation (138). Then under the bootstrap assumption for $\tilde{s}^m_f$, we have the following estimate of the vector field $\tilde{\mathbf{X}}$

\begin{equation}
\left\| \tilde{\mathbf{X}} \right\|_{m,m+1} \leq c(n,p) \left( c_{m,m'} + c_{m,o} \right) c_{n,m} \frac{c_{o,r_0} r_0}{(r_0 + s)^3} + c(n,p) c_{o,r_0} r_0.
\end{equation}

Proof. By the bootstrap assumption and proposition 1.3. \qed

Lemma 4.7. Under the bootstrap assumption for $\tilde{s}^m_f$, we have the following estimates.

\begin{equation}
\left\| \sum_{m+n \leq 3} P_{m,m+1}(\tilde{\mathbf{X}}) \right\|_{m,m+1} \leq c(n,p) \left( c_{m,m'} + c_{m,o} \right) c_{n,m} \frac{c_{o,r_0} r_0}{(r_0 + s)^3} + c(n,p) c_{o,r_0} r_0,
\end{equation}

and

\begin{equation}
\left\| \sum_{m+n \leq 4} P_{m,m+2}(\tilde{\mathbf{X}}) \right\|_{m,m+2} \leq c(n,p) \left( c_{m,m'} + c_{m,o} \right) c_{n,m} \frac{c_{o,r_0} r_0}{(r_0 + s)^3} + c(n,p) c_{o,r_0} r_0.
\end{equation}
By the elliptic theory on the sphere, we get that
\[
(145) \quad \left\| \sum_{m_1+2m_2\leq 2} P_{m_1,1}(\nabla, \tilde{s}^f) * P_{m_2,1}(\nabla, \tilde{s}^g) * (\Omega^2 g^{-1}) \right\|_{\mathcal{O}_s, \tilde{g}}^{n,p} \leq \frac{c(n,p)}{(r_0 + s)^2} c_0^2 e^{-\tilde{r}^2}.
\]

**Proof.** By the bootstrap assumption and proposition 13.3.

Then by the above lemmas, we conclude the following lemma for the propagation equation for $\tilde{\Delta} \tilde{s}^f$.

In order to state the lemma, we introduce the following simplified notation. In equation (134), we denote by $\tilde{\mathcal{C}}$ the low order terms from the point of view of the derivatives of $\tilde{s}^f$. Then equation (134) takes the following form

\[
(145) \quad \left( \partial_s + sX^i \partial_i \right) \tilde{\Delta} \tilde{s}^f = \tilde{\mathcal{C}}.
\]

**Lemma 4.8.** Under the bootstrap assumptions 13.7 for $\tilde{s}^g$, the Laplacian $\tilde{\Delta} \tilde{s}^f$ satisfies the propagation equation

\[
(146) \quad \left( \partial_s + sX^i \partial_i \right) \tilde{\Delta} \tilde{s}^f = \tilde{\mathcal{C}},
\]

where

\[
(147) \quad \left\| sX \right\|_{\mathcal{O}_s, \tilde{g}}^{n+1,p} \leq \frac{c(n,p)c_0 \delta \partial \delta r}{(r_0 + s)^2} + \frac{c(n,p)c_m \delta \partial \delta r_0^2}{(r_0 + s)^3},
\]

\[
(148) \quad \left\| \tilde{\mathcal{C}} \right\|_{\mathcal{O}_s, \tilde{g}}^{n,p} \leq \frac{c(n,p)c_0^2 \tilde{\delta} \tilde{\partial} \delta \tilde{r}_0^2}{(r_0 + s)^2} + \frac{c(n,p)c_m \delta \tilde{\delta} \tilde{\partial} \delta \tilde{r}_0^3}{(r_0 + s)^3}.
\]

Then for $\delta_n$ and $\tilde{\delta}_n$ chosen suitably small, we have

\[
(149) \quad \left\| \tilde{\Delta} \tilde{s}^f \right\|_{\mathcal{O}_s, \tilde{g}}^{n,p} \leq \left\{ \left\| \tilde{s}^f \right\|_{\mathcal{O}_s, \tilde{g}}^{n,p} + \int_0^s \left\| \tilde{\mathcal{C}} \right\|_{\mathcal{O}_s, \tilde{g}}^{n,p} \right\}.
\]

By the elliptic theory on the sphere, we get that

\[
(150) \quad \left\| d\tilde{s}^f \right\|_{\mathcal{O}_s, \tilde{g}}^{n+1,p} \leq c(n,p) \delta \partial \delta r_0 + c(n,p) \left( c_0 \delta_n + c_m \delta_n \right) \delta c \partial \delta r_0.
\]

**Proof.** The lemma follows from propositions 4.8, 13.7 and lemma A.14.

The above lemma enables us to extend the bootstrap assumption 13.5 for $d \tilde{s}^g$ to a larger interval for $s$. We still need to consider extending the bootstrap assumption 13.5 for $\tilde{s}^g$. We have the following lemma.
Lemma 4.9. Under the bootstrap assumption 4.4 of $\tilde{f}$, we have
\begin{align}
-\beta^i \partial_i \tilde{s}^j &\leq (n, p) c(c_{m,m} \delta^i_m + c_{m,o} \delta^i_o + \epsilon \delta^i_o) \frac{c_r o_r}{r_0^3}, \\
\Omega^2 \left( \theta^{-1} \right)^j \partial_i \tilde{s}^i \tilde{\partial}_j \tilde{s}^k &\leq (n, p) c^2 \delta^2 o_r^2 \frac{r_0^2}{(r_0 + s)^2}, \\
\tilde{\partial}_i \tilde{s}^j &\leq (n, p) c(c_{m,m} \delta^i_m + c_{m,o} \delta^i_o + \epsilon \delta^i_o) \frac{c_r o_r}{r_0^3} + (n, p) c^2 \delta^2 o_r^2 \frac{r_0^2}{(r_0 + s)^2}.
\end{align}

Then we have the following estimate of $\tilde{s}^j$ and $\tilde{s}^j$:
\begin{align}
\left| \tilde{s}^j \right|^2 &\leq \left| s^j - \tilde{s}^j \right|^2 + \int_s^\infty \left| \tilde{\partial}_i s - \tilde{\partial}_i \tilde{s} \right|^2 ds' \\
&\leq \delta^i_n r_0 + (n, p) (c_{m,m} c_{m,o} \delta^i_n \delta^i_o + c_{m,o} c_{m,o}^2 + (1 + \epsilon) c^2 \delta^2 o_r^2) r_0.
\end{align}

Now we can extend the bootstrap assumption 4.5 for $\tilde{f}$ using lemmas 4.8, 4.9 for suitable $c_0, c_{m,m} c_{m,o}$ and sufficiently small $\epsilon, \delta^i_n, \delta^i_o$. We assume that $c_0, c_{m,m} c_{m,o}$ satisfy the following inequalities:
\begin{align}
\epsilon &> c_0, \\
1 + c(n, p) c_{m,m} c_{m,o} < c_{m,m}, \\
c(n, p) c_{m,m} c_{m,o} + c(n, p) (1 + \epsilon) c_0^2 < c_{m,o}.
\end{align}

There exist $c_0, c_{m,m} c_{m,o}$ such that this system of inequalities holds if $c_0, \delta^i_n, \delta^i_o$ sufficiently small. For example,
\begin{align}
(c_0, c_{m,m}, c_{m,o}) = (2c(n, p), 2, 8c(n, p)^3)
\end{align}
is a solution of the system sufficiently small $c_0, \delta^i_n, \delta^i_o$. We conclude the proof of theorem 4.1.1

4.2. Proof of theorem 4.2. Now we turn to theorem 4.2. In order to estimate $\left\| \tilde{d} f \right\|^{n,p}$ and $\tilde{f}$, we recall the third solution to transform the second parametrization $\tilde{f}$ of $\tilde{f}$ to the first parametrization $\tilde{f}$. We cite equation 128 here.

\begin{align}
\tilde{\partial}_i \tilde{f}^j &\equiv \tilde{f} \cdot \left[ 1 - \beta^i_0 \tilde{f}_i \right]^{-1} \cdot \left[ \tilde{f}_i - \beta^i_0 \tilde{f}_i \right], \\
\tilde{\partial}_i \tilde{f}^j &\equiv \tilde{f} \cdot \left[ 1 - t_0 \tilde{f}_i - t_0 \tilde{f}_i - t_0 \tilde{f}_i \right]^{-1} \cdot \left[ \tilde{f}_i - t_0 \tilde{f}_i - t_0 \tilde{f}_i - \tilde{f}_i \right]
\end{align}

where we recall that $\tilde{f}^i, \tilde{f}_i$ are given by equations 124, 125, which we cite here,
\begin{align}
\tilde{f}^j = -2\Omega^2 \tilde{f}_i (B^{-1})^i_j (\theta^{-1})^j_k, \\
\tilde{f}^k = -2\Omega^2 \tilde{f}_j (B^{-1})^i_j (\theta^{-1})^j_k, \\
B^i_j = \delta^i_j - t_0 \tilde{f}_i \tilde{f}_j
\end{align}
and
\begin{align}
\xi = \frac{-|\epsilon|^2}{(2 \Omega^2 + e \cdot \epsilon) + \sqrt{(2 \Omega^2 + e \cdot \epsilon)^2 - |\epsilon|^2 |\epsilon|^2}}.
\end{align}
The equation for $\tilde{f}$ is a first order nonlinear equation and we take derivatives of the equation to estimate its solution. We write equation (128) as follows

$$\tilde{\partial}_t \tilde{f} = F \left( \tilde{f}, \tilde{t}' \tilde{f}, \tilde{t}^2 \tilde{f}, \tilde{t}' \tilde{f}, \tilde{\epsilon} \tilde{t}' \tilde{f}, \tilde{\epsilon}^2 \tilde{f}, \tilde{\epsilon} \tilde{f} \tilde{f}, \tilde{t}' \tilde{f} \tilde{f}, b \tilde{t}' \tilde{f} \right),$$

(159)

$$F = \tilde{f} \cdot \left[ 1 - \tilde{t}' \tilde{f} - \tilde{t}^2 \tilde{f} - \tilde{t}^2 \tilde{f} \right]^{-1} \cdot \left[ \tilde{\epsilon} - b \tilde{t}' \tilde{f} - \tilde{\epsilon} \tilde{t}' \tilde{f} - \tilde{\epsilon} \tilde{t}' \tilde{f} \right],$$

(160)

then we take the Laplacian of the equation

$$\tilde{\partial}_t \tilde{\Delta} \tilde{f} = \tilde{X}^i \left( \tilde{\Delta} \tilde{f} \right)_i + \tilde{\epsilon}^k,$$

(161)

where

$$\tilde{t}' \tilde{X}^i = \partial_{\tilde{t}'} \tilde{f} \cdot \tilde{f}_k \left[ -2\Omega^2 (B^{-1})^i_j (\tilde{f}^{-1})^j_k \right] + \partial_{\tilde{t}'} \tilde{f} \cdot \left\{ \tilde{\epsilon} - 2\Omega^2 (B^{-1})^i_j (\tilde{f}^{-1})^j_k \right\}$$

(162)

$$\tilde{F} = \tilde{f} \cdot \left[ 1 - \tilde{t}' \tilde{f} - \tilde{t}^2 \tilde{f} - \tilde{t}^2 \tilde{f} \right]^{-1} \cdot \left[ \tilde{\epsilon} - b \tilde{t}' \tilde{f} - \tilde{\epsilon} \tilde{t}' \tilde{f} - \tilde{\epsilon} \tilde{t}' \tilde{f} \right],$$

and

$$\tilde{\epsilon}^k = \partial_{\tilde{t}'} \tilde{f} \cdot \tilde{f}_k \left[ -2\Omega^2 (B^{-1})^i_j (\tilde{f}^{-1})^j_k \right] + \partial_{\tilde{t}'} \tilde{f} \cdot \left\{ \tilde{\epsilon} - 2\Omega^2 (B^{-1})^i_j (\tilde{f}^{-1})^j_k \right\}$$

(163)

where $a, b$ denote the terms $\tilde{f}, \tilde{t}' \tilde{f}, \tilde{t}^2 \tilde{f}, \tilde{t}' \tilde{f}, \tilde{\epsilon} \tilde{t}' \tilde{f}, \tilde{\epsilon}^2 \tilde{f}, \tilde{\epsilon} \tilde{f} \tilde{f}, \tilde{t}' \tilde{f} \tilde{f}, b \tilde{t}' \tilde{f}.$

The strategy to prove theorem 4.2 is via bootstrap arguments. We introduce the following bootstrap assumption of $\tilde{f}.$
Assumption 4.10 (Bootstrap assumption of $\tilde{f}$ for theorem 4.2). Assume that for all $t \in [0, t_a]$ where $t_a \in (0, 1]$, we have

$$
\left\| \tilde{d}^\circ \tilde{f} \right\|_{\Sigma_t, \tilde{g}}^{n,p} \leq c_o \left\| \tilde{d}^{s=0} \tilde{f} \right\|_{n,p}, \quad \left\| \tilde{d} \tilde{f} \right\| \leq c_{m,m} \left\| s=0 \tilde{f} \right\| + c_{m,o} \frac{1}{r_0} \left( \left\| s=0 \tilde{f} \right\|_{n,p} \right)^2.
$$

The goal is to prove that at $t = t_a$, we get that

$$
\text{Assumption is valid for all } t = t_a,
$$

Then we have

$$
\text{Assumption is also valid for all } t \in [0, t_a'],
$$

Finally, by choosing the suitable constants $c_o, c_{m,m}, c_{m,o}$, we can strengthen the inequalities in the bootstrap assumption to strict inequalities.

Lemma 4.11. Under the bootstrap assumption of $\tilde{f}$, for $\delta_o, \delta_m, \delta_o$ chosen suitably small, $\tilde{\Delta} \tilde{f}$ satisfies the propagation equation

$$
\tilde{\Delta} \tilde{f} = \tilde{X} \tilde{f} + \tilde{\tau} f,
$$

where

$$
\left\| \tilde{\Delta} \tilde{f} \right\|_{\Sigma_t, \tilde{g}}^{n,p} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n,p)c_o r_0}{(r_0 + t s_0)^2} \sum_o + \frac{c(n,p)r_0^2}{(r_0 + t s_0)^2} \sum_o \right\},
$$

Then we have

$$
\left\| \tilde{\Delta} \tilde{f} \right\|_{\Sigma_t, \tilde{g}}^{n-1,p} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n,p)c_o r_0^2}{(r_0 + t s_0)^2} \sum_o + \frac{c(n,p)c_o r_0^2}{(r_0 + t s_0)^2} \sum_o \right\}.
$$

By the elliptic theory on the sphere, we get that

$$
\left\| \tilde{\Delta} \tilde{f} \right\|_{\Sigma_t, \tilde{g}}^{n-1,p} \leq c(n,p) \delta_o r_0 + c(n,p) \left\{ \frac{c_o^2 \delta_o^2}{r_0} + c_o e(c_m, m, \delta_o^2) \right\} r_0 t_a.
$$

Proof. The proof follows from the estimates of $\tilde{X}$, $\tilde{\tau} f$, and lemma [A.14]
Lemma 4.12. Under the bootstrap assumption of \( \tilde{\Sigma} \), we have

\[
\left| \tilde{\Sigma} \right| \leq (s_0 + r_0\delta_0) \cdot \left\{ \frac{c(n,p)c_r r_0^2}{(r_0 + ts_0)^2} \delta_m + \frac{c(n,p)c_o r_0^2}{(r_0 + ts_0)^2} \delta_o \right\} \cdot c_0 \delta_m r_0.
\]

By the propagation equation of \( \tilde{\Sigma} \), we have

\[
\left| \tilde{\Sigma} \right| \leq (s_0 + r_0\delta_0) \cdot \left\{ \frac{c(n,p)c_r r_0^2}{(r_0 + ts_0)^2} \delta_m + \frac{c(n,p)c_o r_0^2}{(r_0 + ts_0)^2} \delta_o \right\},
\]

\[
\left| \tilde{\Sigma} \right| \leq \delta_m r_0 + c(n,p)c_o^2 \delta_m^2 r_0 + (n,p)c_o(\delta_m \delta_m + \delta_o \delta_o)\delta_o r_0.
\]

Combining lemmas 4.11 and 4.12, we can extend the bootstrap assumption of \( \tilde{\Sigma} \) to a larger interval of \( t \) for suitable \( c_o, \delta_m, \delta_m \) and sufficient small \( \epsilon, \delta_m \). We require that the following inequalities hold:

\[
c(n,p) + c(n,p)c_o^2 \delta_m + c(n,p)c_o(\delta_m \delta_m + \delta_o \delta_o) < c_o,
\]

\[
1 + c(n,p)\epsilon \delta_o \delta_m c_o < c_m,
\]

\[
c(n,p)c_o^2 + c(n,p)\epsilon \delta_o c_o < c_m.\]

The above system of inequalities has solution for sufficient small \( \epsilon, \delta_m, \delta_m \), for example,

\[
(c_o, \delta_m, \delta_m) = (2c(n,p), 2, 8c(n,p)^3)
\]

is a solution of the system for sufficient small \( \epsilon, \delta_m, \delta_m \). We conclude the proof of theorem 4.2.

5. Estimate for the Structure Coefficients of Spacelike Surfaces

In this section, we estimate the structure coefficients on a spacelike surface \( \tilde{\Sigma} \) in \( (M, g) \). Recall that in section 5.1 we give the formulas to express the structure coefficients in terms of the background quantities and the first parametrization \( (\tilde{f}, \tilde{\gamma}) \) of the surface \( \tilde{\Sigma} \). In order to explain the strategy, we consider equation (82),

\[
\tilde{\Sigma} = \nabla \chi + \varepsilon \nu \cdot b - 2(\Omega^2 \varepsilon) \nabla^2 \tilde{f} - 2\Omega^2 \tilde{\gamma} \tilde{f}
\]

\[
+ 2\text{sym} \left\{ \left[ \nabla b \cdot \tilde{\zeta} - \chi(b) - \varepsilon \nabla \tilde{f} \right] \tilde{f} \right\}_{ij} - 2\text{sym} \left\{ \left[ \nabla \varepsilon \tilde{\zeta} \right] \nabla \tilde{f} \right\}_{ij}
\]

\[
+ 2\left[ 2\chi(b, \tilde{\zeta}) + \chi(b, b) + \varepsilon \chi(b, b) + 4\Omega^2 \varepsilon b \cdot \eta - \nabla b \cdot \tilde{\zeta} + \frac{\partial b}{\partial s} \cdot \tilde{\zeta} + 4\Omega^2 \varepsilon \omega \right] \left[ \tilde{f} \right]_{ij}
\]

\[
+ 2\text{sym} \left[ \tilde{f} \cdot \nabla \tilde{f} \right] \left[ 2\Omega^2 \varepsilon \nabla \tilde{f} + \chi(b, \tilde{\zeta}) + 2\Omega^2 b \cdot \eta \right] - 4\Omega^2 \left[ \tilde{f} \cdot \nabla \tilde{f} \right]_{ij}.
\]
We want to estimate the above quantity under the assumption that

\[(178) \quad f, \tilde{d} f\]

are small, for example of magnitude \(\delta\). Note that the quadratic terms like \(\omega \Omega^2 (\tilde{d} f \otimes \tilde{d} f)_{ij}\) are of magnitude \(\delta^2\) and higher degree terms like \(\text{sym}(\tilde{d} f \otimes \tilde{d} f)_{ij} \Omega^2 \eta \cdot \tilde{\varepsilon}\) are of magnitude \(\delta^n, n \geq 3\). Thus it is a natural idea to divide the terms in \(\tilde{\Sigma}^{ij}\) into low and high degree terms w.r.t. the powers of \(f, \tilde{d} f\).

When we try to estimate the structure coefficient \(\tilde{\chi}^{ij}\) using equation (82), we will encounter the issue of regularity as we described in remark 3.10. That is, if \(s = 0\), the second parametrization of \(\tilde{\Sigma}\), are both \((n + 2)\) times differentiable, then \(\tilde{f}\) in the first parametrization of \(\tilde{\Sigma}\) is \((n + 1)\) times differentiable. Hence the structure coefficient is \((n - 1)\) times differentiable. Our wish is that the structure coefficient \(\tilde{\chi}^{ij}\) is two orders less differentiable than the parametrization. However it seems that the second parametrization does not fulfil this, which indicates that the second parametrization might not be a good method to parametrize the spacelike surface.

Fortunately, we can fill this gap by choosing another method to estimate the structure coefficients which is more coherent with the second parametrization. Recall that the idea of the second parametrization is to use two sets of data to parametrize the spacelike surface \(\tilde{\Sigma}\):

1. the incoming null hypersurface \(\tilde{C}\) into which \(\tilde{\Sigma}\) is embedded.
2. the function \(\tilde{f}\) whose graph is \(\Sigma\) in the coordinate system \(\{s, \vartheta\}\) on \(\tilde{C}\).

So when we estimate the structure coefficients, we proceed as follows:

1. Study the structure coefficients associated with the foliation \(\{\tilde{\Sigma}_s = \tilde{C} \cap C_s\}\) on the null hypersurface \(\tilde{C}\), which can be calculated by proposition 3.3.
2. Use the structure coefficients of the foliation \(s\) on \(\tilde{C}\) and the function \(\tilde{f}\) to study the structure coefficients of \(\tilde{\Sigma}\) by proposition 3.4.

By proposition 3.4, we have

\[(179) \quad \tilde{\chi}^{ij} = \Sigma^{ij} + 2\text{sym}\left\{(-\tilde{\chi}(\tilde{b})) \otimes \tilde{d} f\right\}_{ij} + \tilde{\chi}(\tilde{b}, \tilde{f}) \tilde{f}_i \tilde{f}_j,\]

where \(\tilde{\chi}\) is the structure coefficient of \(\tilde{\Sigma}_s\) in \(\tilde{C}\), which is calculated from the first parametrization \((\tilde{f}, s)\) of \(\tilde{\Sigma}_s\) by proposition 3.3 and \(\tilde{b}\) is the difference of \(\tilde{L}\) and \(\tilde{\partial}\) on \(\tilde{C}\), which is given by formula (98). Adapting equation (85), \(\tilde{\chi}\) takes the form

\[(180) \quad \tilde{\Sigma}^{ij} = \Sigma^{ij} - \Omega^2 \left|\tilde{d} \tilde{f}\right|^2 \chi^{ij} - 2\Omega^2 \nabla^2 \tilde{f}^2 - 4\Omega^2 \text{sym}\left\{\eta \otimes \tilde{d} \tilde{f}\right\}_{ij} - 4\Omega^2 (\tilde{d} \tilde{f} \otimes \tilde{d} \tilde{f})_{ij} + 4\Omega^2 \text{sym}\left\{\tilde{d} \tilde{f} \otimes \chi(\nabla \tilde{f})\right\}_{ij}.\]
So in order to estimate \( \tilde{\Sigma}_{ij} \), we need to estimate \( \tilde{\Sigma}_{ij} \) on \( \tilde{\Sigma} \), hence we need to estimate \( \nabla_{ij} \tilde{f} \) on \( \tilde{\Sigma} \). Notice that on \( \tilde{\Sigma} \), \( \tilde{f} \) is equal to \( \tilde{f} \), however we can show that \( \left( \nabla_{ij} \tilde{f} \right) \big|_{\tilde{\Sigma}} \) is \( n \)th order differentiable, which is one order higher than \( \nabla^2 \tilde{f} \) in equation (82). The distinction between \( \left( \nabla_{ij} \tilde{f} \right) \big|_{\tilde{\Sigma}} \) and \( \nabla^2 \tilde{f} \) is that the covariant derivatives in \( \left( \nabla_{ij} \tilde{f} \right) \big|_{\tilde{\Sigma}} \) are \( \tilde{\Sigma} \) tangential, but the covariant derivatives in \( \nabla^2 \tilde{f} \) is \( \tilde{\Sigma} \) tangential. This is the crucial point why we can actually show that the structure coefficients are two orders less differentiable than the second parametrization. The more detailed discussion on the differentiabilities of \( \tilde{\Sigma}_{ij} \) are contained in section 5.2.

### 5.1. Decomposition of structure coefficients.
Throughout this section, we will use the Sobolev norms defined in appendix A.1. Assume that the second parametrization of \( \tilde{\Sigma} \) is \( (s=0, \tilde{f}) \) and the first parametrization of \( \tilde{\Sigma} \) is \( (\tilde{f}, \tilde{f}) \). \( \tilde{\Sigma} \) is the surface having the second parametrization \( (s=0, \tilde{f}) \) and the first parametrization \( (\tilde{f}, \tilde{f}) \). \( \tilde{\Sigma} \) is the incoming null hypersurface foliated by \( \left\{ \tilde{\Sigma}_{ij} \right\} \). We assume that

\[
\left\| \tilde{d} s - \tilde{0} \right\|_{\tilde{\Sigma}_{ij}}^{n+1,p} \leq \delta_{m} r_{0}, \quad \left\| \tilde{d} \tilde{f} \right\|_{\tilde{\Sigma}_{ij}}^{n+1,p} \leq \delta_{o}(r_{0} + s_{0}), \quad \left\| \tilde{f} \right\| = s_{0},
\]

then by theorem 4.1 we have

\[
\left\| \tilde{d} \tilde{f} \right\|_{\tilde{\Sigma}_{ij}}^{n+1,p} \leq \delta_{o} r_{0}, \quad \left\| \tilde{f} \right\| \leq \delta_{m} r_{0}, \quad \left\| \tilde{\Sigma}_{ij} \right\| \leq \delta_{m,0} m, + c_{m,0} n^{2},
\]

\[
\left\| \tilde{d} \tilde{f} \right\|_{\tilde{\Sigma}_{ij}}^{n+1,p} \leq \delta_{m} r_{0}, \quad \left\| \tilde{f} \right\| \leq \delta_{m} r_{0}, \quad \left\| \tilde{\Sigma}_{ij} \right\| \leq \delta_{m,0} m, + c_{m,0} n^{2},
\]

**Definition 5.1.** We decompose the structure coefficients on \( \tilde{\Sigma} \) into low and high degree terms.

\[
\begin{align*}
\text{a. } \left\{ \tilde{\Sigma}_{ij} \right\} & := \chi_{ij} - 2\Omega^{2} \nabla_{ij} \tilde{f} - 4\Omega^{2} \tilde{f} \nabla_{ij} + 4\Omega^{2} \tilde{f} \nabla_{ij} + 4\Omega^{2} \tilde{f} \nabla_{ij} \nabla_{ij}, \\
\text{b. } \left\{ \tilde{\Sigma}_{ij} \right\} & := -4\Omega\tilde{f} \tilde{f} \tilde{f} \tilde{f}, \\
\text{c. } \left\{ \tilde{\Sigma}_{ij} \right\} & := \chi_{ij}, \\
\text{d. } \left\{ \tilde{\Sigma}_{ij} \right\} & := 0, \\
\text{e. } \left\{ \tilde{\Sigma}_{ij} \right\} & := \chi_{ij}, \\
\text{f. } \left\{ \tilde{\Sigma}_{ij} \right\} & := 0,
\end{align*}
\]

\[
\begin{align*}
\text{g. } \left\{ \tilde{\Sigma}_{ij} \right\} & := \chi_{ij}, \\
\text{h. } \left\{ \tilde{\Sigma}_{ij} \right\} & := 0,
\end{align*}
\]

\[
\begin{align*}
\text{i. } \left\{ \tilde{\Sigma}_{ij} \right\} & := \chi_{ij}, \\
\text{j. } \left\{ \tilde{\Sigma}_{ij} \right\} & := 0,
\end{align*}
\]

\[
\begin{align*}
\text{k. } \left\{ \tilde{\Sigma}_{ij} \right\} & := \omega - 2\Omega^{2} \tilde{f} \tilde{f}, \\
\text{l. } \left\{ \tilde{\Sigma}_{ij} \right\} & := -2\Omega^{2} \tilde{f} \tilde{f} \tilde{f} \tilde{f}.
\end{align*}
\]
The above decomposition is based on the power of \( \bar{f} \) whose Sobolev norm we assume to be small. On the other hand, we have another smallness requirement which is contained in the assumption for the metric \( g_{i} \). Therefore we introduce another decomposition of the structure coefficients of \( \tilde{\Sigma}_{s} \). Heuristically, we may think of \( 4\Omega^{2}\text{sym} \left\{ \eta \otimes \partial_{f} \bar{f} \right\} _{ij} \) as being order \( \epsilon \cdot \delta \). This is a second order term. Recall that \( \eta = O(\epsilon) \) and \( \partial_{f} \bar{f} = O(\delta) \).

**Definition 5.2.** We decompose the structure coefficients on \( \tilde{\Sigma}_{s} \) into low and high degree terms. The decomposition is according to the powers of \( \bar{f} \) and the orders in \( \epsilon \) in definition 2.2.

\[
\begin{align*}
\eta_{ij} & := \chi_{ij}, & \eta_{ij} & := 0, \\
\bar{\epsilon}_{ij} & := -\Omega^{2} \left| \partial_{f} \bar{f} \right|_{g}^{2} \chi_{ij} - 4\Omega^{2} \text{sym} \left\{ \eta \otimes \partial_{f} \bar{f} \right\}_{ij}, & \eta_{ij} & := 0, \\
\bar{\epsilon}_{ij} & := -\Omega^{2} \left| \partial_{f} \bar{f} \right|_{g}^{2} \text{tr} \chi - 4\Omega^{2} \chi \left( \partial_{f} \bar{f} \right)_{ij} - 4\Omega^{2} \omega \left| \partial_{f} \bar{f} \right|_{g}^{2} - 4\Omega^{2} \chi \left( \nabla_{\bar{f}} \bar{f}, \nabla_{\bar{f}} \bar{f} \right), & \eta_{ij} & := 0, \\
\bar{\epsilon}_{ij} & := \chi_{ij}, & \eta_{ij} & := 0, \\
\bar{\epsilon}_{ij} & := \text{tr} \chi, & \eta_{ij} & := 0, \\
\bar{\epsilon}_{ij} & := \eta_{i} + \chi \left( \nabla_{\bar{f}} \bar{f} \right)_{i}, & \eta_{ij} & := 0, \\
\bar{\epsilon}_{ij} & := \omega, & \eta_{ij} & := 0.
\end{align*}
\]

Before proceeding to the decomposition of the structure coefficients on \( \tilde{\Sigma} \), we compare the covariant derivative \( \tilde{\nabla} \) of \( \left( \tilde{\Sigma}_{s}, \tilde{g} \right) \) with the covariant derivative \( \nabla \) restricted on \( \tilde{\Sigma}_{s} \).

**Proposition 5.3.** Let \( \tilde{\nabla}_{ij}^{k} \) be the Christoffel symbol of \( \tilde{\nabla} \) and \( \nabla_{ij}^{k} \) be the Christoffel symbol of \( \nabla \), then we define \( \tilde{\Delta}_{ij}^{k} := \tilde{\nabla}_{ij}^{k} - \nabla_{ij}^{k} \).

We have

\[
\tilde{\Delta}_{ij}^{k} = \frac{1}{2} \left( g^{-1} \right)^{kl} \left( \tilde{\nabla}_{j} \circ \partial_{g} \bar{g} + \tilde{\nabla}_{i} \circ \partial_{g} \bar{g} \right) - \tilde{\Delta}_{ij}^{k} = \left( g^{-1} \right)^{kl} \left( \tilde{\nabla}_{j} \chi_{il} + \tilde{\nabla}_{i} \chi_{jl} - \tilde{\nabla}_{ij} \chi_{l} \right).
\]

For any tensor field \( T \) on \( \tilde{\Sigma} \), we have

\[
\left( \tilde{\nabla} - \nabla \right)_{i} T_{j_{1} \ldots j_{l}}^{j_{1} \ldots j_{l}} = \tilde{\Delta}_{i}^{j_{1} \ldots j_{l}} T_{j_{1} \ldots j_{l}}^{j_{1} \ldots j_{l}} - \tilde{\Delta}_{i}^{j_{1} \ldots j_{l}} T_{j_{1} \ldots j_{l}}^{j_{1} \ldots j_{l}}.
\]

Now we turn to the decomposition of the structure coefficients on \( \tilde{\Sigma} \), which has the second parametrization \( \left( \tau = \partial_{f} \bar{f}, \bar{f} \right) \). We have the foliation \( \{ \tilde{\Sigma}_{s} \} \) on \( \tilde{\Sigma} \) and the metric on
\( \tilde{C} \) is

\[
g_{\tilde{C}} = \tilde{g}_{ij}(d\theta^i - \tilde{\beta}^i ds) \otimes (d\theta^j - \tilde{\beta}^j ds),
\]

where

\[
\tilde{g} = g, \quad \tilde{\beta}^i = b^i - 2\Omega^2 (g^{-1})^{ik} \tilde{f}_k.
\]

The structure coefficients on \( \tilde{\Sigma} \) can be calculated by proposition 3.4 in terms of \( \tilde{f} \) and the structure coefficients of \( \tilde{\Sigma}_s \) on \( \tilde{C} \). Then we can decompose the structure coefficients on \( \tilde{\Sigma} \) into low and high degree terms.

**Definition 5.4.** We decompose the metric \( \tilde{g} \) on \( \tilde{\Sigma} \) into low and high degree terms.

\[
\begin{align*}
\text{l.} \left\{ \tilde{g}_{ij} \right\} &:= g_{ij}, \\
\text{h.} \left\{ \tilde{g}_{ij} \right\} &:= -(\tilde{g}_{il} \tilde{f}_j + \tilde{g}_{jl} \tilde{f}_i) + \tilde{f}_i \tilde{f}_j \mid \tilde{g} \mid^2, \\
\text{l.} \left\{ \left( \tilde{g}^{-1} \right)^{ij} \right\} &:= (g^{-1})^{ij}, \\
\text{h.} \left\{ \left( \tilde{g}^{-1} \right)^{ij} \right\} &:= (g^{-1})^{ik} \text{h.} \left\{ \tilde{g}_{kl} \right\} \left( \tilde{g}^{-1} \right)^{jl}.
\end{align*}
\]
We decompose the structure coefficients on $\tilde{\Sigma}$ into low and high degree terms according to the degrees of $\tilde{f}$ and $\tilde{d} f$ and the orders in $\epsilon$ in definition 2.2.

\[(209) \quad \mathcal{H}_k \{ \tilde{\chi}_{ij} \} := \mathcal{H}_k \{ \chi_{ij} \}, \]
\[(210) \quad \mathcal{H}_k \{ \tilde{\nu}_{ij} \} := \mathcal{H}_k \{ \nu_{ij} \} + 2\text{sym} \left\{ (-\chi' (\tilde{b})) \otimes \tilde{d} f \right\}_{ij} + \chi (\tilde{\nu} \delta b) f_i f_j, \]
\[(211) \quad \mathcal{H}_k \{ \tilde{\chi}_{ij} \} := \mathcal{H}_k \{ \chi_{ij} \} - (\tilde{g}^{-1})^{ij} \mathcal{H}_k \{ \tilde{\chi}_{ij} \}, \]
\[(212) \quad \mathcal{H}_k \{ \tilde{\nu}_{ij} \} := \mathcal{H}_k \{ (\tilde{g}^{-1})^{ij} \} \mathcal{H}_k \{ \tilde{\nu}_{ij} \} + (\tilde{g}^{-1})^{ij} \mathcal{H}_k \{ \tilde{\nu}_{ij} \}, \]
\[(213) \quad \mathcal{H}_k \{ \tilde{\nu}_{ij} \} := \chi_{ij} - 2 \nabla_{ij}^2 \tilde{f}, \]
\[(214) \quad \mathcal{H}_k \{ \tilde{\chi}_{ij} \} := \epsilon' \chi_{ij} + \tilde{\nu} \cdot \tilde{\nu} \nabla_{ij}^2 \tilde{f} + 2 \nabla_{ij} \tilde{f} \tilde{f} + 2 \nabla_{ij} \tilde{f} \cdot \tilde{f} \]
\[+ 2\text{sym} \left\{ \tilde{d} f \otimes \left[ \nabla \tilde{\nu} \cdot \varepsilon - \nabla (\tilde{\nu})^2 \right] \right\}_{ij} \]
\[+ \left[ \nabla \left( \tilde{\nu} \cdot \varepsilon \right) \right]_{ij} + \left[ \nabla \chi (\tilde{b}, \tilde{b}) \right]_{ij} + \chi (\tilde{\nu} \cdot \tilde{b}) + 4 \tilde{\nu} \cdot \tilde{\nu} \]
\[- \nabla \left( \tilde{\nu} \cdot \tilde{\nu} \right) + \tilde{\nu} \cdot \tilde{\nu}^2 - \delta_{ij} \tilde{f} \tilde{f} - 4 \tilde{\nu} \]
\[(215) \quad \mathcal{H}_k \{ \tilde{\chi}_{ij} \} := \mathcal{H}_k \{ (\tilde{g}^{-1})^{ij} \} \mathcal{H}_k \{ \tilde{\chi}_{ij} \} = \text{tr} (\tilde{\chi} - 2 \nabla \tilde{f}), \]
\[(216) \quad \mathcal{H}_k \{ \tilde{\nu}_{ij} \} := \mathcal{H}_k \{ (\tilde{g}^{-1})^{ij} \} \mathcal{H}_k \{ \tilde{\nu}_{ij} \} + (\tilde{g}^{-1})^{ij} \mathcal{H}_k \{ \tilde{\nu}_{ij} \}, \]
\[(217) \quad \mathcal{H}_k \{ \tilde{\nu}_{ij} \} := \tilde{\nu} + \frac{1}{2} \chi (\tilde{\nu}), \]
\[(218) \quad \mathcal{H}_k \{ \tilde{\nu}_{ij} \} := \frac{1}{2} \left\{ -2 \Omega^2 \nabla_{ij}^2 \tilde{f} - 4 \Omega^2 \text{sym} \left[ \tilde{\nu} \otimes \tilde{d} f \right]_{ij} + \mathcal{H}_k \{ \chi_{ij} \} \right\} \varepsilon_{ij}
\[+ \tilde{f} \left[ \nabla \left( \tilde{\nu} \cdot \varepsilon \right) \right]_{ij} - \frac{1}{2} \nabla \chi (\tilde{b}, \tilde{b}) \right\}. \]

5.2. **Estimate the Differential $\tilde{\Phi} f$.** In order to obtain the estimate of the low and high degree terms of the structure coefficients on $\tilde{\Sigma}$, we need to know the estimate of $\tilde{d} f$ on $\tilde{\Sigma}$. We state the result of this section on the estimate of $\tilde{d} f$ on $\tilde{\Sigma}$.

**Theorem 5.5.** Assume that in the second parametrization of $\tilde{\Sigma}$,

\[(219) \quad \left\| \tilde{d} \left|_{s=0} f \right| \right\|_{n+1,p} \leq \delta_{0} r_{0}, \quad \left\| \tilde{d} \left|_{s=0} f \right| \right\|_{n+1,p} \leq \delta_{0} r_{0}, \quad \left\| \tilde{f} \right\|_{n+1,p} \leq \delta_{0} (r_{0} + s_{0}), \]

where $n \geq 1, p > 2$ or $n \geq 2, p > 1$.

Then for $\epsilon, \delta_{0,}, \delta_{0,}$ sufficiently small depending on $n, p$, where $\epsilon$ is given in definition 2.2 of $g_{e}$, there exist constants $\epsilon_{0,}$, such that for the restriction of the differential $\tilde{d} f$ on
we have the following estimate,

\begin{equation}
\left\| \tilde{d} \tilde{s} \tilde{f} \right\|^{n+1,p} \leq c_0 \left\| \tilde{d} s^{-\partial_s} \tilde{f} \right\|^{n+1,p}.
\end{equation}

**Remark 5.6.** Notice that the above theorem is optimal according to the differentiability, i.e. no loss of derivative appears here, which is not the case for \( \tilde{d} t \tilde{f} \) in theorem 4.2. This theorem is not a corollary of theorem 4.7 since the spacelike surface in theorem 4.7 is \( \tilde{\Sigma}_s \), which is different from here.

The proof of the above theorem follows the similar pattern as in the proofs of theorems 4.1 and 4.2. We consider a family of surfaces \( \tilde{\Sigma}_t \) whose second parametrization is \( \left( s=0, t \tilde{f} \right) \), and the differential \( \tilde{d} t \tilde{f} \) restricted on \( \tilde{\Sigma}_t \). Then we derive a propagation equation for \( \left( \tilde{d} t \tilde{f} \right) \big|_{\tilde{\Sigma}_t} \). In the end, we prove theorem 5.5 by bootstrap arguments.

Estimating \( \tilde{d} t \tilde{f} \) on \( \tilde{\Sigma} \) is equivalent to estimating \( \tilde{R}_a \tilde{s} \tilde{f}, a=1, 2, 3 \) on \( \tilde{\Sigma} \). Here \( \{ R_a \}_{a=1,2,3} \) are the rotational vector fields as defined in appendix A.1. So in the following, we will consider the estimate of \( \tilde{R}_a \tilde{s} \tilde{f} \). We derive the propagation equation of \( \tilde{R}_a \tilde{s} \tilde{f} \) from the propagation equation (100) of \( \tilde{f} \) in the following,

\[ \partial_s \tilde{R}_a = -b^i \partial_i \tilde{R}_a + \Omega^2 (g^{-1})^{ij} \partial_j \tilde{R}_a \]

then take the derivative of the above equation in the direction of \( \tilde{R}_a \) to obtain

\begin{equation}
\partial_s \tilde{R}_a \tilde{s} \tilde{f} = -b^i \tilde{R}_a \tilde{s} \tilde{f}, \quad + 2 \Omega^2 (g^{-1})^{ij} \partial_j \tilde{R}_a \tilde{s} \tilde{f}, \quad \quad - [R_a, b]^{ij} \tilde{s} \tilde{f}, \quad \quad - \left( \tilde{R}_a \tilde{s} \tilde{f} \right) \partial_s b^{ij} \tilde{s} \tilde{f},
\end{equation}

From the above equation, we can derive the propagation equation of \( \tilde{R}_a \tilde{s} \tilde{f} \) on \( \tilde{\Sigma}_t \). First, we introduce the following abbreviation: let \( T \) be any tensor field on \( S^2 \), then

\begin{equation}
T^{b_1 \cdots b_k}_{a_1 \cdots a_k} := g \left( T(R_{a_1}, \cdots, R_{a_k}), R_{b_1} \otimes \cdots \otimes R_{b_k} \right).
\end{equation}

By lemma A.3 we have that for the contraction of tensors, for example, for a \((1,1)\)-type tensor field \( T \) on \( S^2 \),

\begin{equation}
\text{tr} T = T^i_i = \sum_{a=1,2,3} T^a_a = \sum_{a=1,2,3} g \left( T(R_a), R_a \right),
\end{equation}

where the first equality is the normal notation of tensor calculus, and the second equality follows from equation (222) and lemma A.3. Because of the above property of equation (222), we can use it as if doing tensor calculus. We also use the Einstein summation convention when taking contractions. We can write \( \tilde{R}_a \tilde{s} \tilde{f} \) as \( \tilde{s} \tilde{f}_{ia} \). And we will use \( t^{ia} \tilde{s} \tilde{f} \) to denote \( \tilde{R}_a \tilde{s} \tilde{f} \) on \( \tilde{\Sigma}_t \).
Now we apply lemma 3.8 to equation (221) to get the propagation equation of $t, s f^\rightarrow_\alpha$.

$$\partial_t t, s f^\rightarrow_\alpha = -\tilde{f} b \left( -t_\tilde{f} f^\rightarrow_\alpha + \partial_t t, s f^\rightarrow_\alpha + 2\tilde{f} f^\rightarrow_\alpha \right) + 2\tilde{f} f^\rightarrow_\alpha (g^{-1})^{ij} t, s f^\rightarrow_j - [R_{\alpha}, b]^t, s f^\rightarrow_\alpha - t, s f^\rightarrow_\alpha \partial_\alpha b^{ij, s f^\rightarrow_j} + [R_{\alpha}, 2\Omega g^{-1})^{ij} t, s f^\rightarrow_j t, s f^\rightarrow_j + t, s f^\rightarrow_\alpha \partial_\alpha (\Omega^2 g^{-1})^{ij} t, s f^\rightarrow_j t, s f^\rightarrow_j,$$

where

$$t, s f^\rightarrow_\alpha = s = t f^\rightarrow_\alpha = R_{\alpha} \left( s = t f^\rightarrow_\alpha \right).$$

We rewrite the propagation equation of $t, s f^\rightarrow_\alpha$ as follows:

$$\partial_t t, s f^\rightarrow_\alpha = t, s \tilde{X} \cdot t, s f^\rightarrow_\alpha + t, s \tilde{\kappa},$$

$$t, s \tilde{X} = f \left( 1 - b f^\rightarrow_\alpha + 2\Omega g^{-1})^{ij} t, s f^\rightarrow_j \right)^{-1} \left( -b f^\rightarrow_\alpha + 2\Omega g^{-1})^{ij} t, s f^\rightarrow_j \right) \partial_t,$$

$$t, s \tilde{\kappa} = f \left( 1 - b f^\rightarrow_\alpha + 2\Omega g^{-1})^{ij} t, s f^\rightarrow_j \right)^{-1} \left\{ - [R_{\alpha}, b]^t, s f^\rightarrow_\alpha - t, s f^\rightarrow_\alpha \partial_\alpha b^{ij, s f^\rightarrow_j} + [R_{\alpha}, 2\Omega g^{-1})^{ij} t, s f^\rightarrow_j t, s f^\rightarrow_j + t, s f^\rightarrow_\alpha \partial_\alpha (\Omega^2 g^{-1})^{ij} t, s f^\rightarrow_j t, s f^\rightarrow_j \right\}.$$

The initial condition of the propagation equation is

$$t=0, s f^\rightarrow_\alpha = s = 0 t f^\rightarrow_\alpha.$$

The equations for $\{t, s f^\rightarrow_\alpha\}_{a=1,2,3}$ are nonlinear first order partial differential equations without quadratics in first derivatives. We estimate $t, s f^\rightarrow_\alpha$ by bootstrap arguments. We introduce the following bootstrap assumption.

**Assumption 5.7** (Bootstrap assumption of $t, s f^\rightarrow_\alpha, a = 1, 2, 3$). Assume that for $t \in [0, t_\alpha]$, we have

$$\left\| t, s f^\rightarrow_\alpha \right\|_{\tilde{X}_t, g_\alpha}^{n+1,p} \leq \tilde{c}_0 \left\| t, s f^\rightarrow_\alpha \right\|_{s = 0, f^\rightarrow_\alpha}^{n+1,p}, \quad a = 1, 2, 3.$$

The goal is to prove that at $t = t_\alpha$, we can actually get strict inequalities in the estimate of $t, s f^\rightarrow_\alpha$. Then by continuity, we can find a slightly larger $t_\alpha$ such that the bootstrap assumption is also valid for all $t \in [0, t_\alpha]$, hence we can conclude that the bootstrap assumption is valid for all $t \in [0, 1]$, which proves theorem 5.3.

We use the bootstrap assumption of $\tilde{X}$ to estimate the vector field $t, s \tilde{X}$ and $t, s \tilde{\kappa}$, then we integrate equation (226) to get the estimates on $\left\| \tilde{X}_t, g_\alpha \right\|^{n+1,p}$, which are lemmas 5.8, 5.9 and 5.10. Finally, by choosing the suitable constants $\tilde{c}_0$, we can strengthen the inequality in the bootstrap assumption to the strict inequality.
Lemma 5.8 (Estimate of the vector field $t_s \tilde{X}$). Under the assumption of theorem 5.5 and the bootstrap assumption of $t_s \tilde{F}_f$ for $t \in [0, t_a]$, we have

\begin{align}
\|t^i f_i\|^{n+1,p}_{t, a} &\leq \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o) \delta_o, \\
\|2 \Omega^2 (g^{-1}) \|^{n+1,p}_{t, g} &\leq \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o), \\
\|b^i\|^{n+1,p}_{s, t, g} &\leq \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o), \\
\|2 \Omega^2 (g^{-1}) \|^{n+1,p}_{t, g} &\leq \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o),
\end{align}

and

\begin{align}
\|t_s \tilde{X}\|^{n+1,p}_{t, a, g} &\leq (s_0 + r_0 \delta_0) \cdot \left\{ \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o) + \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o) \right\}, \\
\int_0^{t_a} \left| \text{div } t_s \tilde{X} \right| \, dt &\leq c(n, p) \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o) + c(n, p) \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o).
\end{align}

Lemma 5.9 (Estimate of the term $t_s \tilde{e}$). Under the assumption of theorem 5.5 and the bootstrap assumption of $t_s \tilde{F}_f$ for $t \in [0, t_a]$, we have

\begin{align}
\|-[R_a, b] \|^{n+1,p}_{t, a} &\leq \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o) \delta_o, \\
\|t_s \tilde{e}\|^{n+1,p}_{t, a} &\leq \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o) \delta_o, \\
\| [R_a, \Omega^2 g^{-1}] \|^{n+1,p}_{t, g} &\leq \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o), \\
\|t_s \tilde{e}\|^{n+1,p}_{t, a} &\leq \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o),
\end{align}

and

\begin{align}
\|t_s \tilde{e}\|^{n+1,p}_{t, a} &\leq (s_0 + r_0 \delta_0) \cdot \left\{ \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o) \delta_o, \\
&\quad + \frac{c(n, p)r_0^3}{(r_0 + t s_0)^3} \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o) \delta_o \right\}, \\
\int_0^{t_a} \left| t_s \tilde{e} \right| \, dt &\leq c(n, p) \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o) + c(n, p) \epsilon (c_m \Delta_m + c_{o, o} \Delta_o + c_o \Delta_o).
\end{align}
Proof. The above two lemmas follow from proposition C.3, theorem 4.2, and the bootstrap assumption of $\tilde{t,s}_{\tilde{f},a}$.

\[\square\]

**Lemma 5.10** (Integrating the propagation equation of $\tilde{t,s}_{\tilde{f},a}$). Under the bootstrap assumption of $\tilde{t,s}_{\tilde{f},a}$, we have

\[\begin{align*}
\left\| \tilde{t,s}_{\tilde{f}} \right\|_{\tilde{L}^{n+1,p}_{\tilde{\Sigma}_{t,a}}} &\leq c(n,p) \left\| \tilde{r}_{\tilde{\Sigma}_0,\tilde{g}} \right\|_{\tilde{L}^{n+1,p}_{\tilde{\Sigma}_0,\tilde{g}}} + c(n,p) \int_0^{t_a} \left\| \tilde{t,s}_{\tilde{r}} \right\|_{\tilde{L}^{n+1,\tilde{g}}_{\tilde{\Sigma}_0,\tilde{g}}} dt \\
&\leq c(n,p)\tilde{\delta}_0 r_0 + c(n,p)\epsilon (c_{m,m}\tilde{\delta}_m + c_{m,o}\tilde{\delta}_o + c_o\tilde{\delta}_o)^\epsilon c_o\tilde{\delta}_o r_0 t_a \\
&\quad + c(n,p)\epsilon c_o^2 \tilde{\delta}_o^2 r_0 t_a + c(n,p)\epsilon c_o^3 \tilde{\delta}_o^3 r_0 t_a.
\end{align*}\]

Then we can extend the bootstrap assumption of $\tilde{t,s}_{\tilde{f},a}$ to a larger interval of $t$ by the above lemma. We assume that the following inequality holds

\[\begin{align*}
c(n,p) + c(n,p)\epsilon (c_{m,m}\tilde{\delta}_m + c_{m,o}\tilde{\delta}_o + c_o\tilde{\delta}_o)^\epsilon c_o + c(n,p)\epsilon c_o^2 \tilde{\delta}_o^2 + c(n,p)\epsilon c_o^3 \tilde{\delta}_o^3 < \xi_o.
\end{align*}\]

The inequality has solutions for sufficient small $\epsilon, \tilde{\delta}_m, \tilde{\delta}_o, \delta_o$, for example,

\[\begin{align*}
\xi_o = 2c(n,p)
\end{align*}\]

is a solution for the inequality for sufficient small $\epsilon, \tilde{\delta}_m, \tilde{\delta}_o, \delta_o$. We conclude the estimate of $\tilde{t,s}_{\tilde{f},a}$, $a = 1, 2, 3$.

From now on, we will not specify the constant $c_o, c_{m,m}, c_{m,o}, \xi_o$, but denote them by the simplified notation $c(n,p)$.

5.3. **Estimate the structure coefficients.** Now we can estimate the structure coefficients on $\tilde{\Sigma}$. 
**Proposition 5.11.** For $\epsilon, \hat{\omega}, \delta, \delta_o$ sufficiently small, we have

\begin{align}
(246) & \| s_{\hat{\omega}, \delta} \|_{\Sigma, \hat{\omega}, \delta}^{n+1, p} \leq \frac{c(n, p)r_0}{r_0 + s_0} \delta_o, \\
(247) & \| \tilde{q}_{ij} \|_{\Sigma, \hat{\omega}, \delta}^{n+1, p} \leq \frac{c(n, p)r_0^2}{(r_0 + s_0)^2} \hat{\omega}_m + \frac{c(n, p)r_0}{(r_0 + s_0)^2} \delta_o, \\
(248) & \| \tilde{h}(\tilde{g}_{ij}) \|_{\Sigma, \hat{\omega}, \delta}^{n+1, p} \leq \frac{c(n, p)(r_0 + s_0)^2}{2}, \\
(249) & \| \tilde{h}(\tilde{g}_{ij}) \|_{\Sigma, \hat{\omega}, \delta}^{n+1, p} \leq \frac{c(n, p)r_0^2}{(r_0 + s_0)^2} \hat{\omega}_n \delta_o + \frac{c(n, p)r_0}{(r_0 + s_0)^2} \hat{\omega}_o \delta_o, \\
(250) & \| \tilde{h}(\tilde{g}_{ij}) \|_{\Sigma, \hat{\omega}, \delta}^{n+1, p} \leq \frac{c(n, p)}{(r_0 + s_0)^2}, \\
(251) & \| \tilde{h}(\tilde{g}_{ij}) \|_{\Sigma, \hat{\omega}, \delta}^{n+1, p} \leq \frac{c(n, p)r_0^2}{(r_0 + s_0)^2} \hat{\omega}_n \delta_o + \frac{c(n, p)r_0}{(r_0 + s_0)^2} \hat{\omega}_o \delta_o, \\
(252) & \| \tilde{h}(\tilde{g}_{ij}) \|_{\Sigma, \hat{\omega}, \delta}^{n+1, p} \leq \frac{c(n, p)}{(r_0 + s_0)^2},
\end{align}

\textit{Proof.} The proposition follows from proposition \[C.3\] theorem 4.2 and theorem 5.5. \[\square\]

**Proposition 5.12** (Estimate of the low degree terms $\tilde{h}(\cdot)$ and high degree terms $\tilde{h}(\cdot)$ of the structure coefficients on $\tilde{\Sigma}$). Assume that $\epsilon, \hat{\omega}, \delta, \delta_o$ sufficiently small, we have the following estimates for the low degree terms $\tilde{h}(\cdot)$ on $\tilde{\Sigma}$:

\begin{align}
(255) & \| \tilde{h}(\tilde{\Sigma}_{ij}) \|_{\tilde{\Sigma}, \delta}^{n, p} \leq c(n, p)(r_0 + s_0), \\
(256) & \| \tilde{h}(\tilde{\Sigma}_{ij}) \|_{\tilde{\Sigma}, \delta}^{n, p} \leq \frac{c(n, p)}{r_0 + s_0}, \\
(257) & \| \tilde{h}(\tilde{\Sigma}_{ij}) \|_{\tilde{\Sigma}, \delta}^{n, p} \leq c(n, p)s_0 + c(n, p)(r_0 + s_0)(\epsilon + \delta_o), \\
(258) & \| \tilde{h}(\tilde{\Sigma}_{ij}) \|_{\tilde{\Sigma}, \delta}^{n, p} \leq \left(\frac{c(n, p)s_0}{r_0 + s_0} + \frac{c(n, p)}{r_0 + s_0} \delta_o + \frac{c(n, p)r_0}{(r_0 + s_0)^2} \epsilon \delta_o + \frac{c(n, p)r_0}{(r_0 + s_0)^2} \delta_o \right), \\
(259) & \| \tilde{h}(\tilde{\Sigma}_{ij}) \|_{\tilde{\Sigma}, \delta}^{n+1, p} \leq \frac{c(n, p)r_0}{r_0 + s_0}(\epsilon + \delta_o) + \frac{c(n, p)r_0}{(r_0 + s_0)^2} \delta_o.
\end{align}

For the high degree terms $\tilde{h}(\cdot)$ on $\tilde{\Sigma}$, we have the following estimates

\begin{align}
(260) & \| \tilde{h}(\tilde{\Sigma}_{ij}) \|_{\tilde{\Sigma}, \delta}^{n, p} \leq \frac{c(n, p)r_0^2}{r_0 + s_0}(\delta o^2 + \epsilon \delta_o + \epsilon \delta o^2 + \epsilon \delta o \delta_\Lambda) + c(n, p)r_0 \delta o \delta o.
\end{align}
We define the parametrization of spacelike surfaces \((s=0, t, a) \rightarrow \tilde{t}, \tilde{f}, \tilde{a}\).

Proof. The proposition follows from the definition of the quantities, proposition \ref{prop:propagation} theorem \ref{thm:properties} and theorem \ref{thm:properties2}.

6. Perturbation of the parametrization of spacelike surfaces

In section \ref{sec:propagation} we study the transformation from the second parametrization of spacelike surfaces to the first parametrization, i.e. the map \((s=0, t, \tilde{a}) \rightarrow (\tilde{t}, \tilde{f}, \tilde{a})\). In this section, we study the perturbation of this map.

Let \(\tilde{\Sigma}_1\) and \(\tilde{\Sigma}_2\) be two spacelike surfaces and \((s=0, a, \tilde{a})\) be the second parametrization of the surface \(\tilde{\Sigma}_a\), \(a = 1, 2\). Assume that the first parametrization of the surface \(\tilde{\Sigma}_a\) is \((a, \tilde{f}, \tilde{a})\). We are interested in the perturbation of the first parametrization \(\tilde{2f} - \tilde{1f}\).

Following the construction in section \ref{sec:propagation} let \(\tilde{\Sigma}_{a,t}\) be the spacelike surface with the second parametrization of spacelike surfaces \((s=0, a, \tilde{a})\). We assume that the first parametrization of \(\tilde{\Sigma}_{a,t}\) is \((t, a, \tilde{a})\).

Recall in section \ref{sec:propagation} we use the propagation equations (\ref{eq:propagation}) of \(t, a, \tilde{a}\) and \(\tilde{t}, a, \tilde{a}\) to establish the estimate of \(t, a, \tilde{a}\). In this section, we will derive the propagation equations of \(\tilde{2f} - \tilde{1f}\) and establish the estimates of \(\tilde{2f} - \tilde{1f}\) by these equations. We define

\begin{align}
|\nabla{\tilde{r}}\rangle &:= \frac{c(n,p)r_0^2}{(r_0+s_0)^3}(\delta_0^2 + c\delta_0 + c\delta_0^2 + c\delta_0^2\delta_m) + \frac{c(n,p)r_0}{(r_0+s_0)^2}\delta_0\beta_0, \\
|\nabla{\tilde{z}}\rangle &:= \frac{c(n,p)r_0^2}{(r_0+s_0)^3}\delta_0^2 + c(n,p)r_0\delta_0\beta_0 + c(n,p)r_0\delta_0 + \frac{c(n,p)r_0^2}{(r_0+s_0)^2}\delta_0\beta_0, \\
|\nabla{\tilde{x}}\rangle &:= \frac{c(n,p)r_0}{(r_0+s_0)^2}\delta_0\beta_0 + \frac{c(n,p)r_0^2}{(r_0+s_0)^2}\delta_0\beta_0 + \frac{c(n,p)r_0}{r_0+s_0}\delta_0.
\end{align}
We show that the size of \( \mathbb{D}\{ \tilde{f} \}, \mathbb{D}\{ \tilde{f} \} \), the perturbation of the first parametrisation, can be bounded by the size of \( \mathbb{D}\{ \epsilon^{-0.5} \}, \mathbb{D}\{ \tilde{f} \} \), the perturbation of the second parametrisation. We state the main theorems of this section.

**Theorem 6.1.** Let \((s=0,a,\tilde{f})\) be the second parametrisation of the surface \(\tilde{\Sigma}_a, a = 1, 2\) respectively. We assume that

\[
\begin{align*}
\left\| \tilde{d}^{s=0.5} \tilde{f} \right\|^{n+1}_s &\leq \delta_{s} r_0, \\
\left\| \tilde{d}^{s=0.5} a \tilde{f} \right\|^{n+1}_s &\leq \delta_{s} a r_0, \\
\left\| \tilde{d} \mathbb{D}\{ \epsilon^{-0.5} \} \right\|^{n+1}_s &\leq \mathbb{D}_{s} r_0, \\
\left\| \tilde{d} \mathbb{D}\{ \tilde{f} \} \right\|^{n+1}_s &\leq \mathbb{D}_{s} m (r_0 + s_0, 1).
\end{align*}
\]

Then for \(\tilde{\epsilon}, \tilde{\delta}, \tilde{\delta}_m, \delta, \tilde{\delta}_n, \tilde{\delta}_n, \delta, \mathbb{D}_n, \mathbb{D}_m\) sufficiently small, there exist constants \(c_0, c_{0,m}, c_{0}, c_{m}, c_{m,o}, c_{m,o}, c^0\), such that

\[
\begin{align*}
\left\| \tilde{d} \mathbb{D}\{ \tilde{f} \} \right\|^{n+1}_{\Sigma_1, t, \tilde{g}} &\leq c_{0} \mathbb{D}_n r_0 + c_{0,m} (\tilde{\delta}_n + c_{0,m} \mathbb{D}_n r_0 + c^0 (\tilde{\delta}_n + c_{m,m} \mathbb{D}_m, \mathbb{D}_n) \mathbb{D}_m, \mathbb{D}_n) r_0, \\
\left\| \tilde{d} \mathbb{D}\{ \tilde{f} \} \right\|^{n+1}_{\Sigma_1, t, \tilde{g}} &\leq c_{m} \mathbb{D}_n r_0 + c_{m,o} (\tilde{\delta}_n + c_{m,o} \mathbb{D}_n r_0 + c^0 (\tilde{\delta}_n + c_{m,m} \mathbb{D}_m, \mathbb{D}_n) \mathbb{D}_m, \mathbb{D}_n) r_0.
\end{align*}
\]

**Theorem 6.2.** Under the assumptions of theorem 6.1 and in addition we assume that

\[
\tilde{\Delta} t, \tilde{f} = 0.
\]

Then for \(\tilde{\epsilon}, \tilde{\delta}, \tilde{\delta}_n, \delta, \tilde{\delta}_n, \tilde{\delta}_n, \delta, \mathbb{D}_n, \mathbb{D}_m\) sufficiently small, there exist constants \(c_0, c_{0,m}, c_{0}, c_{m}, c_{m,o}, c_{m,o}, c^0\), such that

\[
\begin{align*}
\left\| \tilde{d} \mathbb{D}\{ \tilde{f} \} \right\|^{n+1}_{\Sigma_1, t, \tilde{g}} &\leq c_{0} \mathbb{D}_n r_0 + c_{0,m} (\tilde{\delta}_n + c_{0,m} \mathbb{D}_n r_0 + c^0 (\tilde{\delta}_n + c_{m,m} \mathbb{D}_m, \mathbb{D}_n) \mathbb{D}_m, \mathbb{D}_n) r_0, \\
\left\| \tilde{d} \mathbb{D}\{ \tilde{f} \} \right\|^{n+1}_{\Sigma_1, t, \tilde{g}} &\leq c_{m} \mathbb{D}_n r_0 + c_{m,o} (\tilde{\delta}_n + c_{m,o} \mathbb{D}_n r_0 + c^0 (\tilde{\delta}_n + c_{m,m} \mathbb{D}_m, \mathbb{D}_n) \mathbb{D}_m, \mathbb{D}_n) r_0.
\end{align*}
\]

**Remark 6.3.** Notice that in the estimate of \(\tilde{f}^2\) in theorem 6.1, we cannot obtain the estimate of the highest order which is \(\left\| \tilde{d} \mathbb{D}\{ \tilde{f} \} \right\|^{n+1}_{\Sigma_1, t, \tilde{g}}\) as in theorem 6.2. The loss of the regularity in the estimate is due to the term \(\mathbb{D}\{ \tilde{X}^i \} \left( \tilde{\Delta} t, \tilde{f} \right) i\) in the propagation equation \((280)\) of \(\tilde{\Delta} \mathbb{D}\{ \tilde{f} \}\). If we have a higher order derivative estimate of \(t, \tilde{f}\) for
example as we assume that $\tilde{\Delta}^{t,\tilde{\alpha}} = 0$ in theorem 6.2, it will also improve the estimate of $\mathbf{d}\{\tilde{f}\}$.

6.1. Propagation equation of perturbation of the first parametrization. In this section, we derive the equations for the perturbation of the first parametrization $\mathbf{d}\{\tilde{f}\}$ and its Laplacian $\tilde{\Delta} \mathbf{d}\{\tilde{f}\}$.

The first parametrization $t, a \tilde{f}$ of the surface $\tilde{\Sigma}_{a,t}$, $a = 1, 2$ satisfies the propagation equations [128], [161], which we cite here

$$
\frac{\partial}{\partial t} t, a \tilde{f} = t, a F,
$$

$$
\frac{\partial}{\partial t} \tilde{\Delta} t, a \tilde{f} = t, a \tilde{X}^i \left( \frac{\partial}{\partial t} t, a \tilde{f} \right)_i + t, a \tilde{\epsilon},
$$

where

$$
t, a F = \tilde{F} \left( a \tilde{f}, t, a \tilde{f}, t, a \tilde{f}_i, t, a \tilde{f} \right)
$$

$$= \tilde{a} \cdot \left[ 1 - b_i t, a \tilde{f}_i - \tilde{z}^i t, a \tilde{f}_i - \tilde{z}^i t, a \tilde{f} \right] \left[ \tilde{z} - b_i t, a \tilde{f}_i - \tilde{z}^i t, a \tilde{f}_i - \tilde{z}^i t, a \tilde{f} \right]^{-1},
$$

then the perturbation $\mathbf{d}\{\tilde{f}\}$ satisfies the propagation equations

$$
\frac{\partial}{\partial t} \mathbf{d}\{\tilde{f}\} = \tilde{d}\{t F\},
$$

$$
\frac{\partial}{\partial t} \tilde{\Delta} \mathbf{d}\{\tilde{f}\} = \tilde{t}, 1 \tilde{X}^i \mathbf{d}\{\tilde{f}\}_i + \mathbf{d}\{\tilde{X}^i\}_i + \mathbf{d}\{t \tilde{\epsilon}\},
$$

where

$$
\mathbf{d}\{t F\} = t, 2 \tilde{F} - t, 1 \tilde{F},
$$

$$
\mathbf{d}\{\tilde{X}^i\} = t, 2 \tilde{X}^i - t, 1 \tilde{X}^i,
$$

$$
\mathbf{d}\{t \tilde{\epsilon}\} = t, 2 \tilde{\epsilon} - t, 1 \tilde{\epsilon},
$$

and the initial condition of the propagation equations is

$$
\mathbf{d}\{t=0 \tilde{f}\} = \mathbf{d}\{s=0 \tilde{f}\} = \tilde{s} = 0, 2 \tilde{F} = s = 0, 1 \tilde{f}.
$$

6.2. Estimate the perturbation of the first parametrization. In this section, we prove the main theorems 6.1 and 6.2 on the perturbations of the first parametrization $\mathbf{d}\{\tilde{f}\}$. The idea is to estimate the solution of the propagation equations of $\mathbf{d}\{\tilde{f}\}$ derived
in the above section. We introduce the following bounds on the perturbations

\[ \| \mathcal{D} \{ s=0 \mathcal{F} \} \|_{\Sigma_{0.2}}^{n+1, p} \leq \mathcal{D}_n r_0, \quad \| \mathcal{D} \{ s=0 \mathcal{F} \} \|_{\Sigma_{0.2}} \leq \mathcal{D}_n r_0. \]

Notice in the above bounds (285), \( \mathcal{D} \{ \hat{f} \} \) is one order less differentiable than in theorem 4.1. These bounds will be used in the proof in theorem 6.1. For theorem 6.2, we introduce the following stronger bounds on the perturbation,

\[ \| \mathcal{D} \{ s=0 \mathcal{F} \} \|_{\Sigma_{0.2}}^{n+1, p} \leq \mathcal{D}_n r_0, \quad \| \mathcal{D} \{ s=0 \mathcal{F} \} \|_{\Sigma_{0.2}} \leq \mathcal{D}_n r_0. \]

The only difference between the bounds (285) with the previous bounds (286) is that we strengthen the differentiability of \( \mathcal{D} \{ \hat{f} \} \) from \( n - 1 \)th-order to \( n \)th-order.

The question is how to estimate \( \mathcal{D}_n \) and \( \mathcal{D}_m \). We will first estimate the terms showing up in the propagation equations by \( \mathcal{D}_n \) and \( \mathcal{D}_m \), including \( \hat{t}_1 \hat{X} \) and \( \mathcal{D} \{ \hat{t} \mathcal{F} \} \), \( \mathcal{D} \{ \hat{t} \hat{X} \} \), \( \mathcal{D} \{ \hat{t} \hat{e} \} \), then derive integral inequalities of \( \mathcal{D}_n \) and \( \mathcal{D}_m \), and finally obtain their estimates from the integral inequalities.

The estimate of the vector \( \hat{t}_1 \hat{X} \) is given by lemma 4.11. The estimates of \( \mathcal{D} \{ \hat{t} \mathcal{F} \} \), \( \mathcal{D} \{ \hat{t} \hat{X} \} \) and \( \mathcal{D} \{ \hat{t} \hat{e} \} \) are the main difficulties. Their estimates will follow from the key lemma 6.4 in the following.
First, let us take a closer look at the terms $\mathcal{D}\{F\}, \mathcal{D}\{\tilde{X}_f\}$ and $\mathcal{D}\{\tilde{e}_f\}$. We can view $\mathcal{F}, \mathcal{F}, \tilde{\mathcal{F}}$ as maps from the tensor fields on $\mathbb{S}^2$ to the tensor fields on $\mathbb{S}^2$, i.e. we can understand $\mathcal{F}, \mathcal{F}, \tilde{\mathcal{F}}$ as following,

\begin{align}
\mathcal{F} : \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right) &\mapsto F \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right), \\
\mathcal{F} : \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right) &\mapsto \tilde{X} \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right), \\
\mathcal{F} : \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right) &\mapsto \tilde{e}_f \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right).
\end{align}

On the other hand, pointwisely we can also think $\mathcal{F}, \mathcal{F}, \tilde{\mathcal{F}}$ as functions composed by tensor calculations, for example, we can understand $\mathcal{F}$ as follows:

\begin{align}
\mathcal{F} : \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right) \\
\mapsto F \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right) = \tilde{f} \cdot \left[ 1 - b \, \tilde{f} \, \tilde{f} - \tilde{e} \, \tilde{f} \, \tilde{f} - \tilde{e} \tilde{e} \tilde{f} \tilde{f} \right]^{-1} \cdot \left[ \tilde{e} - b \, \tilde{f} \, \tilde{f} - \tilde{e} \, \tilde{f} \, \tilde{f} - \tilde{e} \tilde{e} \tilde{f} \tilde{f} \right],
\end{align}

For such maps like $\mathcal{F}, \mathcal{F}, \tilde{\mathcal{F}}$, we have the following lemma to estimate their perturbations.

**Lemma 6.4.** Let $A$ be a map between tensor fields on $\tilde{\Sigma}_f$, which pointwisely is defined by the background quantities of the double null foliation on $\tilde{\Sigma}_f$, $\tilde{F}, \tilde{f}, \tilde{t}$ and their derivatives through tensor calculations such as contraction and inversion. For example, $A$ could be $\mathcal{F}$

\begin{align}
A : \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right) \mapsto \tilde{f} \cdot \left[ 1 - b \, \tilde{f} \, \tilde{f} - \tilde{e} \, \tilde{f} \, \tilde{f} - \tilde{e} \tilde{e} \tilde{f} \tilde{f} \right]^{-1} \cdot \left[ \tilde{e} - b \, \tilde{f} \, \tilde{f} - \tilde{e} \, \tilde{f} \, \tilde{f} - \tilde{e} \tilde{e} \tilde{f} \tilde{f} \right],
\end{align}

or more general, $A$ could be

\begin{align}
A : \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right) \mapsto Q_A \left( \tilde{f}, \tilde{f}, \tilde{f}, \tilde{f} \right)
\end{align}

where $Q_A$ could be any rational functions and when substituting values, the products should be understood as tensor products and the inversion should be understood as the inversion of linear operators, for example,

\begin{align}
Q_A \left( \text{id}, \Omega^2, \tilde{f}, \eta, \tilde{f}, \tilde{f}, \tilde{f} \right) = \tilde{f} \left( \tilde{B}^{-1} \right)^j_i \Omega^2 \left( \tilde{g}^{-1} \right)^{jk} \tilde{f}_j \tilde{f}_{ik},
\end{align}

\begin{align}
\tilde{B}_i^j = \delta_i^j - b \, \tilde{f}_i \tilde{f}_j + 2 \Omega^2 \tilde{f}_i \tilde{f}_j \left( \tilde{g}^{-1} \right)^{jk} \tilde{f}_k,
\end{align}

\begin{align}
Q_A (q_1, q_2, q_3, q_4, q_5, q_6, q_7) = q_5 \left( q_1 - q_4 q_7 + 2 q_2 q_3 q_7 \right)^{-1} q_2 q_3 q_6 q_7.
\end{align}
We assume that \( \tilde{f}, \tilde{f}, \tilde{f} \) satisfy the following estimates

\[
\| \tilde{f} \tilde{f} \|^{n+1,p} \leq \delta_o(r_o + s_o), \quad \tilde{f} = s_0
\]

(296)

\[
\| \tilde{f} \tilde{f} \|^{n+1,p} \leq \delta_o(r_o + s_o), \quad \tilde{f} = s_0
\]

(297)

\[
\| \tilde{f} \tilde{f} \|^{n+1,p} \leq \delta_o(r_o + t s_o) \leq t \delta_o(r_o + s_o), \quad \tilde{f} = t f, \quad \tilde{f} = t \tilde{f},
\]

(298)

then in principle that we can estimate the Sobolev norms of \( A \) by the above bounds and the estimate of the background quantities on \( \tilde{\Sigma}_t \).

Assume \( A \) satisfies the following estimate

\[
\| A \|^{m,p}_{\tilde{\Sigma}_t, \tilde{\Sigma}} \leq P(\epsilon, \delta_o, \tilde{\delta}_n, r_0, r_t = r_0 + t s_0, s_0)
\]

(299)

where \( m \leq n - 1 \) and \( P(\epsilon, \delta_o, \tilde{\delta}_n, r_0, r_t = r_0 + t s_0, s_0) \) is a rational function, whose denominator only involves \( r_0, r_t \). Then if the bounds 255 are satisfied, the perturbation of \( A \) with respect to the perturbation of spacelike surfaces \( \tilde{\Sigma}_{a,t}, a = 1, 2, \) satisfies the estimate

\[
\| \partial \{ A \} \|^{m,p}_{\tilde{\Sigma}_{a,t}, \tilde{\Sigma}} \leq c(n, p) P(\epsilon, \delta_o, \tilde{\delta}_n, \tilde{\delta}_o, r_0, r_t, s_0): \\
\leq c(n, p) P(\epsilon, \delta_o, \tilde{\delta}_n, \tilde{\delta}_o, r_0, r_t, s_0) (\tilde{\delta}_m + \tilde{\delta}_o + \tilde{\delta}_m + \tilde{\delta}_o)
\]

(300)
where \( P_{\Delta} (\epsilon, \delta, \hat{\Delta}, r_0, r_t, s_0) \) is the terms in \( P(\epsilon, \delta, \hat{\Delta}, r_0, r_t, s_0) \) which involves \( \hat{\Delta} \), for example,

\[
P(\epsilon, \delta, \hat{\Delta}, t, r_0, r_t, s_0) = \frac{c(n, p) r_0}{r_0 + t s_0} (\epsilon + \hat{\Delta}, \delta) + \frac{c(n, p) r_0^2}{(r_0 + t s_0)^2} \epsilon^2 (\hat{\Delta} \delta + \hat{\Delta} \delta^2 + \hat{\Delta} \delta),
\]

\[
P_{\Delta} (\epsilon, \delta, \hat{\Delta}, t, r_0, r_t, s_0) = - \frac{c(n, p) r_0^2}{(r_0 + t s_0)^2} \epsilon^2 \hat{\Delta} \delta.
\]

Furthermore if we assume that the bounds \( \lVert B(t, s) \lVert \) are satisfied, we can improve \( m \), the order of differentiability of \( A \), to \( n \), and get the similar estimate of \( \lVert d \{ A \} \rVert_{\Sigma_{1,1}, \delta} \).

The proof of the above lemma follows from the next lemma.

**Lemma 6.5.** Let \( B \) be any structure coefficient of the background double null coordinate system. The perturbation of \( B \) on \( \Sigma_{1,1} \) is

\[
d \{ B \} = B_{\Sigma_{2,1}} - B_{\Sigma_{1,1}}
\]

\[
= \int_{\frac{1}{2} t \delta}^{\frac{1}{2} t \delta} \mathcal{L}_{\partial s} B (s, t, s) ds + \int_{\frac{1}{2} t \delta}^{\frac{1}{2} t \delta} \mathcal{L}_{\partial t} B (s, t, s) ds,
\]

satisfies the following estimate

\[
\lVert d \{ B \} \rVert_{\Sigma_{1,1}, \delta}^{m,p} \leq \sup_{\Sigma_{1,1}, \delta} \mathcal{L}_{\partial s} B \lVert \mathcal{L}_{\partial s} B \rVert_{\Sigma_{1,1}, \delta}^{m,\infty} \lVert d \{ \tilde{f} \} \rVert_{\Sigma_{1,1}, \delta}^{m,p} + \sup_{\Sigma_{1,1}, \delta} \mathcal{L}_{\partial t} B \lVert \mathcal{L}_{\partial t} B \rVert_{\Sigma_{1,1}, \delta}^{m,\infty} \lVert d \{ \tilde{f} \} \rVert_{\Sigma_{1,1}, \delta}^{m,p}
\]

where the supremum is taken in the range

\[
(s, t) \in \left[ \frac{1}{2} t \delta, \frac{1}{2} t \delta \right] \times \left[ \frac{1}{2} t \delta, \frac{1}{2} t \delta \right]
\]

\[
\subset [-\hat{\Delta} - c(n, p) \hat{\Delta} \delta, \hat{\Delta} + c(n, p) \hat{\Delta} \delta] \times [s_{0,\min} - c(n, p) (r_0 + t s_{0,\max}) \delta, s_{0,\max} + c(n, p) (r_0 + t s_{0,\max}) \delta]
\]

where

\[
s_{0,\min} = \min \left\{ \frac{1}{2} t \delta, \frac{1}{2} t \delta \right\}, \quad s_{0,\max} = \max \left\{ \frac{1}{2} t \delta, \frac{1}{2} t \delta \right\}.
\]

Remark 6.6. In the inequality \( (300) \), the term

\[
c(n, p) \mathbb{P}(\epsilon, \delta, \hat{\Delta}, r_0, r_t, s_0).
\]

\[
\left( (r_0 + s_0)^{-1} \lVert d \{ \tilde{f} \} \rVert_{\Sigma_{1,1}, \delta}^{n,p} + (r_0 + t \delta)^{-1} \lVert d \{ \tilde{f} \} \rVert_{\Sigma_{1,1}, \delta}^{n,p} + r_0^{-1} \lVert d \{ \tilde{f} \} \rVert_{\Sigma_{1,1}, \delta}^{n,p} \right)
\]

comes from the perturbations of structure coefficients and metric components, the term

\[
c(n, p) P_{\Delta} (\epsilon, \delta, \hat{\Delta}, r_0, r_t, s_0) \hat{\Delta}^{-1} r_0^{-1} \lVert d \{ \tilde{f} \} \rVert_{\Sigma_{1,1}, \delta}^{n,p}
\]
comes from the perturbation of \( b^i, \omega \) and \( \text{tr} \chi - \text{tr} \chi_{\text{Sch}} \), the term
\[
\tag{310}
c(n, p) \partial_s \delta_r P(\epsilon, \delta_0, \delta_m, \delta_n, r_0, r_t, s_0) \left\| \partial\{\tilde{f}\} \right\|^{n,p}
\]
comes from the perturbation of \( \tilde{f} \), the term
\[
\tag{311}
c(n, p) \partial_{s_0} P(\epsilon, \delta_0, \delta_m, \delta_n, r_0, r_t, s_0)(r_0 + ts_0)^{-1} \left\| \partial\{f_{\tilde{f}}\} \right\|^{n-1,p}
\]
comes from the perturbation of \( f_{\tilde{f}} \) and \( \text{tr} \chi - \text{tr} \chi_{\text{Sch}} \), the term
\[
\tag{312}
c(n, p) \partial_{r_0} P(\epsilon, \delta_0, \delta_m, \delta_n, r_0, r_t, s_0)(r_0 + ts_0)^{-1} \left\| \partial\{f_{\tilde{f}}\} \right\|^{n-1,p}
\]
comes from the perturbation of \( \tilde{f} \), \( b^i, \omega \) and \( \text{tr} \chi - \text{tr} \chi_{\text{Sch}} \). The above cases cover all the possibilities of terms in the perturbation of \( A \).

**Remark 6.7.** The estimate from lemma 6.4 sometimes is not optimal, for example, for \( \text{tr} \chi \) from proposition C.3 we have the estimate
\[
\tag{313}
\left\| \text{tr} \chi \right\|^{n+1,p}_{\Sigma_{1,1}, \tilde{g}} \leq \frac{c(n, p)}{r_0 + ts_0, 1},
\]
then by lemma 6.4
\[
\tag{314}
\left\| \partial\{\text{tr} \chi\} \right\|^{n-1,p}_{\Sigma_{1,1}, \tilde{g}} \leq \frac{c(n, p)}{(r_0 + ts_0, 1)} \left( (r_0 + s_0, 1)^{-1} \left\| \partial\{\tilde{f}\} \right\|^{n,p} + (r_0 + ts_0, 1)^{-1} \left\| \partial\{f_{\tilde{f}}\} \right\|^{n,p} + r_0^{-1} \left\| \partial\{f_{\tilde{f}}\} \right\|^{n,p} \right),
\]
hence by lemma 6.3 and proposition C.3
\[
\tag{315}
\left\| \partial\{\text{tr} \chi\} \right\|^{n,p}_{\Sigma_{1,1}, \tilde{g}} \leq \frac{c(n, p)}{(r_0 + ts_0, 1)^2} \left( \left\| \partial\{\tilde{f}\} \right\|^{n,p} + \left\| \partial\{f_{\tilde{f}}\} \right\|^{n,p} \right).
\]
The more precise estimate gives the estimate of higher order Sobolev norms and finer estimate concerning the order of \( r_0 + ts_0, 1 \). The reason is that \( \left\| \partial_s \text{tr} \chi \right\|^{n,\infty}_{\Sigma_{1,1}, \tilde{g}} \) has better estimate than \( \left\| \text{tr} \chi \right\|^{n,\infty}_{\Sigma_{1,1}, \tilde{g}} \) concerning the degree of \( r_i = r_0 + ts_0 \), but which is not always true for other background quantities, for example \( \omega, \omega, \eta \). The above shows that we may strengthen our estimate if the \( \partial_s \) derivatives of the background quantities involved in \( A \) have better estimates, or \( A \) does not involve \( \tilde{f} \).

Applying lemma 6.4 to \( t^F, t^X \) and \( t^\omega \), we get the estimates of their perturbations.
Lemma 6.8 (Estimate of $\mathcal{D}\{\tilde{X}\}$). Assuming the bounds, we have the following estimate for $\mathcal{D}\{\tilde{X}\}$,

\begin{equation}
\left\| \mathcal{D}\{\tilde{X}\} \right\|_{L_{1,t;\tilde{\varpi}}}^{n-1,p} \leq (s_0 + r_0 \delta_o) \cdot \frac{c(n,p)r_0}{(r_0 + ts_0)^2} \left\{ \delta_o + \frac{r_0}{r_0 + ts_0} \epsilon \delta_m \right\} (\mathcal{D}_m + \mathcal{D}_o + t\mathcal{D}_m + \epsilon t\mathcal{D}_o) \\
+ (s_0 + r_0 \delta_o) \cdot \frac{c(n,p)r_0^2}{(r_0 + ts_0)^3} \epsilon \left( t\mathcal{D}_m + \epsilon t\mathcal{D}_o \right) + (s_0 + r_0 \delta_o) \frac{c(n,p)r_0}{(r_0 + ts_0)^2} \cdot \mathcal{D}_o \\
+ \frac{c(n,p)r_0(r_0 + s_0)}{(r_0 + ts_0)^2} \left\{ \delta_o + \frac{c(n,p)r_0}{r_0 + ts_0} \epsilon \delta_m \right\} (\mathcal{D}_m + \mathcal{D}_o) \\
\leq \frac{c(n,p)r_0(r_0 + s_0)}{(r_0 + ts_0)^2} \left\{ \delta_o + \frac{c(n,p)r_0}{r_0 + ts_0} \epsilon \delta_m \right\} (\mathcal{D}_m + \mathcal{D}_o) \\
+ (s_0 + r_0 \delta_o) \cdot \frac{c(n,p)r_0}{(r_0 + ts_0)^2} \left\{ \delta_o + \frac{r_0}{r_0 + ts_0} \epsilon \right\} t\mathcal{D}_m \\
+ (s_0 + r_0 \delta_o) \frac{c(n,p)r_0}{(r_0 + ts_0)^2} \epsilon t\mathcal{D}_o.
\end{equation}

Then for $\mathcal{D}\{\tilde{X}^i\} \left( \tilde{\Delta} \tilde{\frac{\partial}{\partial t}} \right)_i$, we have

\begin{equation}
\left\| \mathcal{D}\{\tilde{X}^i\} \left( \tilde{\Delta} \tilde{\frac{\partial}{\partial t}} \right)_i \right\|_{L_{1,t;\tilde{\varpi}}}^{n-2,p} \leq \frac{c(n,p)r_0^2(r_0 + s_0)}{(r_0 + ts_0)^2} \left\{ \delta_o^2 + \frac{c(n,p)r_0}{r_0 + ts_0} \epsilon \delta_m \right\} (\mathcal{D}_m + \mathcal{D}_o) \\
+ (s_0 + r_0 \delta_o) \cdot \frac{c(n,p)r_0^2}{(r_0 + ts_0)^3} \left\{ \delta_o^2 + \frac{r_0}{r_0 + ts_0} \epsilon \delta_m \right\} t\mathcal{D}_m \\
+ (s_0 + r_0 \delta_o) \frac{c(n,p)r_0^2}{(r_0 + ts_0)^2} \epsilon t\mathcal{D}_o.
\end{equation}

If we assume further that $\tilde{\Delta} \tilde{\frac{\partial}{\partial t}} = 0$, which is the assumption of theorem, then the term $\mathcal{D}\{\tilde{X}^i\} \left( \tilde{\Delta} \tilde{\frac{\partial}{\partial t}} \right)_i$ vanish in the propagation equation for $\mathcal{D}\{\tilde{X}\}$.
Lemma 6.9 (Estimate of the perturbation $\mathcal{D}\{\tilde{\tau}\}$). Assuming the bounds (285), we have the following estimate for $\mathcal{D}\{\tilde{\tau}\}$,

\begin{equation}
\|\mathcal{D}\{\tilde{\tau}\}\|_{n-2,p,\xi} \leq (s_0 + r_0\delta_o) \frac{c(n,p)r_0^2}{(r_0 + t s_0)^2} \left\{ \delta_o^2 + \frac{c(n,p)r_0}{r_0 + t s_0} \epsilon \delta_m \delta_o \right\} \cdot (\mathcal{D}_m + \mathcal{D}_o + \partial_m + \partial_o) \\
+ (s_0 + r_0\delta_o) \frac{c(n,p)r_0^2}{(r_0 + t s_0)^2} \epsilon \delta_o (\partial_m + \partial_o) \\
+ c(n,p)r_0^2 (r_0 + s_0) \left\{ \delta_o^2 + \frac{r_0}{r_0 + t s_0} \epsilon \delta_m \delta_o \right\} (\mathcal{D}_m + \mathcal{D}_o) \\
+ (s_0 + r_0\delta_o) \frac{c(n,p)r_0^2}{(r_0 + t s_0)^2} \left\{ \delta_o + \frac{c(n,p)r_0}{r_0 + t s_0} \epsilon \delta_m \right\} (\mathcal{D}_o) \\
\leq \frac{c(n,p)r_0^2 (r_0 + s_0)}{(r_0 + t s_0)^2} \left\{ \delta_o^2 + \frac{r_0}{r_0 + t s_0} \epsilon \delta_m \delta_o \right\} (\mathcal{D}_m + \mathcal{D}_o) \\
+ (s_0 + r_0\delta_o) \frac{c(n,p)r_0^2}{(r_0 + t s_0)^2} \left\{ \delta_o + \frac{c(n,p)r_0}{r_0 + t s_0} \epsilon \delta_m \right\} (\mathcal{D}_o) \\
+ (s_0 + r_0\delta_o) \frac{c(n,p)r_0^2}{(r_0 + t s_0)^2} \left\{ \delta_o + \frac{c(n,p)r_0}{r_0 + t s_0} \epsilon \delta_m \right\} (\mathcal{D}_o).
\end{equation}

Moreover if we assume the stronger bounds (286), we get the same estimate for the stronger norm of $\mathcal{D}\{\tilde{\tau}\}$, $\|\mathcal{D}\{\tilde{\tau}\}\|_{n-1,p,\xi}^\perp$.

Lemma 6.10 (Estimate of $\mathcal{D}\{\tilde{F}\}$). Assuming the bounds (285), we have the following estimates for $\mathcal{D}\{\tilde{F}\}$,

\begin{equation}
|\mathcal{D}\{\tilde{F}\}|, |\mathcal{D}\{\tilde{F}\}|^\perp \leq \frac{c(n,p)r_0^2 (r_0 + s_0)}{(r_0 + t s_0)^2} \left\{ \delta_o^2 + \frac{r_0}{r_0 + t s_0} \epsilon \delta_m \delta_o \right\} (\mathcal{D}_m + \mathcal{D}_o) \\
+ (s_0 + r_0\delta_o) \frac{c(n,p)r_0^2}{(r_0 + t s_0)^2} \left\{ \delta_o + \frac{c(n,p)r_0}{r_0 + t s_0} \epsilon \delta_m \right\} (\mathcal{D}_o) \\
+ (s_0 + r_0\delta_o) \frac{c(n,p)r_0^2}{(r_0 + t s_0)^2} \left\{ \delta_o + \frac{c(n,p)r_0}{r_0 + t s_0} \epsilon \delta_m \right\} (\mathcal{D}_o).
\end{equation}

Combining the above three lemmas 6.8, 6.9, 6.10, we can integrate the propagation equations (279), (280).

Lemma 6.11 (Integrating the propagation equations of $\tilde{\mathcal{D}}\mathcal{D}\{\tilde{\tau}\}$ and $\mathcal{D}\{\tilde{F}\}$). Assuming the bounds (285), we integrate equations (279) and (280) and have the following
Proof. We prove theorem 6.1 by bootstrap argument. Assume that the estimates of $\mathfrak{d}\{t,\tilde{\mathfrak{f}}\}$ is valid for $t \in [0, t_a]$, then apply lemma 6.11 and the bootstrap assumption to obtain

\begin{align}
\mathfrak{d}\{t=0, \tilde{\mathfrak{f}}\} & \leq c(n, p)\mathfrak{d}_m + c(n, p)[1 + (\tilde{\mathfrak{d}}^2 + e\tilde{\mathfrak{d}}_n + e\tilde{\mathfrak{d}}_m)\mathfrak{c}^3](\tilde{\mathfrak{d}}^2 + e\tilde{\mathfrak{d}}_n, \tilde{\mathfrak{d}}_o)(\mathfrak{d}_m + \mathfrak{d}_o) \\
& \quad + c(n, p)(\tilde{\mathfrak{d}}^2 + e\tilde{\mathfrak{d}}_n)\mathfrak{c}^2\mathfrak{d}_m + c(n, p)(\tilde{\mathfrak{d}}^2 + e\tilde{\mathfrak{d}}_n)(\tilde{\mathfrak{d}}_o + e\tilde{\mathfrak{d}}_m)\mathfrak{c}_m, \mathfrak{d}_o \\
& \quad + c(n, p)(\tilde{\mathfrak{d}} + e\tilde{\mathfrak{d}}_m)\tilde{\mathfrak{d}}_o + c(n, p)(\tilde{\mathfrak{d}}_o + e\tilde{\mathfrak{d}}_m)(\tilde{\mathfrak{d}}_o^3 + e\tilde{\mathfrak{d}}_o^2)\mathfrak{c}_{o,m}\mathfrak{d}_m.
\end{align}

Moreover, if we assume the stronger bound (280), then we can get the same bound for the estimate of the stronger norm of $\mathfrak{d}\{t=0, \tilde{\mathfrak{f}}\}$.

Now we can prove the main theorems 6.1 and 6.2 of this section.
We require that \( \frac{\partial}{\partial t}, \frac{\partial}{\partial n}, \frac{\partial}{\partial \nu}, c^0 \) satisfy the following inequalities,
\[
\begin{align*}
(324) \quad & c(n, p) + c(n, p)(\delta^2 + \epsilon \delta_m) + c(n, p) + (\delta_m + \epsilon \delta_m) + c(n, p) \leq c^0, \\
(325) \quad & c(n, p) + c(n, p)(\delta^2 + \epsilon \delta_m) + c(n, p) + (\delta_m + \epsilon \delta_m) \leq c^0, \\
(326) \quad & c(n, p)[1 + (\delta^2 + \epsilon \delta_m) + \delta_m + \epsilon \delta_m] \leq c^0, \\
(327) \quad & 1 + c(n, p)(\delta^2 + \epsilon \delta_m) + c(n, p)(\delta^2 + \epsilon \delta_m) + \delta_m = c^0, \\
(328) \quad & c(n, p)(\delta^2 + \epsilon \delta_m) + c(n, p) \leq c^0,
\end{align*}
\]
of which
\[
(329) \quad (c_{0, m}, c_{0, m}, c_{0, m}, c^0) = (2c(n, p), 4c(n, p)^2, 2c(n, p), 4c(n, p)^2, 2c(n, p))
\]
is a solution for \( \epsilon, \delta_m, \delta_m, \delta_m \) sufficiently small. Then we can extend the bootstrap assumption beyond \([0, t_n]\), hence the theorem follows.

The proof of theorem (6.2) follows the similar pattern. \( \square \)

**Remark 6.12.** Notice that in the proof of theorem (6.1), we only need the estimate of \( \| \partial \{ s^0 \} \|_{\Sigma_{1, t_m}}^{n-2, p} \), which is one order less than the estimates we get in lemma (6.3). Hence we can release the condition of \( \partial \{ s^0 \} \) and \( \partial \{ s^0 \} \) in theorem (6.1).

**Theorem 6.13.** Let \( (s^0, a^0, a^0) \) be the second parametrization of the surface \( \tilde{\Sigma}_a, a = 1, 2 \) respectively. We assume that
\[
\begin{align*}
(330) \quad & \| \partial s^0, a^0 \|_{\Sigma_{1, t_m}}^{n+1, p} \leq \delta_{\nu} r_0, \\
(331) \quad & \| \partial a^0 \|_{\Sigma_{1, t_m}}^{n+1, p} \leq \delta_{\nu} (r_0 + s_{0, a}), \\
(332) \quad & \| \partial \partial \{ s^0 \} \|_{\Sigma_{1, t_m}}^{n-1, p} \leq \delta_{\nu} r_0, \\
(333) \quad & \| \partial \partial \{ \tilde{\Sigma} \} \|_{\Sigma_{1, t_m}}^{n+1, p} \leq \delta_{\nu} (r_0 + s_{0, 1})
\end{align*}
\]
Then for \( \epsilon, \delta_m, \delta_m, \delta_m, \delta_m, \delta_m, \delta_m, \delta_m, \delta_m, \delta_m \) sufficiently small, we have
\[
\begin{align*}
(334) \quad & \| \partial \partial \{ \tilde{\Sigma} \} \|_{\Sigma_{1, t_m}}^{n+1, p} \leq \epsilon \delta_{\nu} r_0 + c^2_{\nu} (\delta_m + \epsilon \delta_m) \delta_{\nu} r_0 + c^2 (\delta_m + \epsilon \delta_m) (\delta_{\nu} + \epsilon \delta_{\nu}) (\delta_{\nu} + \epsilon \delta_{\nu}) r_0, \\
(335) \quad & \| \partial \partial \{ \tilde{\Sigma} \} \|_{\Sigma_{1, t_m}}^{n+1, p} \leq \epsilon \delta_{\nu} \delta_{\nu} r_0 + c^2_{\nu} (\delta_m + \epsilon \delta_m) \delta_{\nu} r_0 + c^2 (\delta_m + \epsilon \delta_m) (\delta_{\nu} + \epsilon \delta_{\nu}) (\delta_{\nu} + \epsilon \delta_{\nu}) r_0.
\end{align*}
\]

**7. Perturbation of structure coefficients of spacelike surfaces**

In section (5), we estimate the structure coefficients on a spacelike surface \( \tilde{\Sigma} \) with the second parametrization \((s^0, a^0, a^0)\). In this section, we consider the perturbation of the structure coefficients with respect to the perturbations of spacelike surfaces.
More precisely, like in section 5, let \( \tilde{\Sigma}_1 \) and \( \tilde{\Sigma}_2 \) be two spacelike surfaces and \((s=0, a \tilde{f}, a \tilde{f})\) be the second parametrization of the surface \( \tilde{\Sigma}_a \), \( a = 1, 2 \). Assume that the first parametrization of the surface \( \tilde{\Sigma}_a \) is \((\tilde{a} \tilde{f}, \tilde{a} \tilde{f})\). We are interested in the difference of the structure coefficients on \( \tilde{\Sigma}_1 \) and \( \tilde{\Sigma}_2 \).

Recall that in section 5, we see that the structure coefficients on \( \tilde{\Sigma} \) can be estimated by the background structure coefficients on \( \tilde{\Sigma} \), the differentials of \( \tilde{f} \), the differentials of \( \tilde{f} \) restricted on \( \tilde{\Sigma} \), respectively, like the construction in section 6. Then we obtain the propagation equation (347) of \( \tilde{f} \), for each spacelike surface \( \tilde{\Sigma}_a \), \( a = 1, 2 \).

In section 6.2, we construct a family of spacelike surfaces \( \{ \tilde{\Sigma}_t \} \) for each spacelike surface \( \tilde{\Sigma}_a \) respectively, like the construction in section 5. Then we obtain the propagation equation (347) of \( \{ \tilde{\Sigma}_t \} \), the perturbation of \( \tilde{f} \) restricted on \( \tilde{\Sigma}_a \). We will estimate \( \{ \tilde{\Sigma}_t \} \) by integrating the propagation equation. The estimate of \( \{ \tilde{\Sigma}_t \} \) is obtained similarly.
We show that the size of $\mathcal{O}\{t^s \tilde{r}\}, \mathcal{O}\{\nabla^2_{ij} t^s \tilde{r}\}$ can be bounded by the size of $(\mathcal{O}\{s=0,\tilde{r}\}, \mathcal{O}\{\tilde{f}\})$, the perturbation of the second parametrisation. We state the main theorems of section 7.1.

**Theorem 7.1.** Let $(s=0,\tilde{r}, \tilde{s} \tilde{f})$ be the second parametrisation of the surface $\tilde{\Sigma}_a$, $a = 1, 2$ respectively. We assume that

\begin{align}
\left\| \tilde{d} \{s=0,\tilde{r}\} \right\|_{n+1,p} & \leq \delta_o r_0, \\
\left\| \tilde{z} \tilde{d} \{s=0,\tilde{r}\} \right\|_{n+1,p} & \leq \delta_o (r_0 + s_{0,a}), \\
\left\| \tilde{d} \{s=0,\tilde{r}\} \right\|_{n+1,p} & \leq \delta_o r_0, \\
\left\| \tilde{d} \{s=0,\tilde{r}\} \right\|_{n+1,p} & \leq \delta_o r_0.
\end{align}

Then for $\epsilon, \delta_t, \delta_n, \delta_o, \epsilon_o, \delta_o, \delta_m, \delta_o$, sufficiently small, there exists $c_o$, such that

\begin{equation}
\left\| \tilde{d} \{t^s \tilde{r}\} \right\|_{n+1,p} \lesssim c_o \tilde{d}_o r_0 + c_o \left(\epsilon \tilde{\Sigma}_m + \delta_o\right) \left(\delta_m + \epsilon_o\right) + c_o \left(\epsilon \tilde{\Sigma}_n + \delta_o\right) \left(\delta_m + \epsilon_o\right).
\end{equation}

**Theorem 7.2.** Under the assumption of theorem 7.1 and in addition we assume that

\begin{equation}
\Delta s=0,2\tilde{r} = 0.
\end{equation}

Then for $\epsilon, \delta_o, \delta_n, \delta_o, \epsilon_o, \delta_o, \delta_m, \delta_o$ sufficiently small, there exists $c_o$ such that

\begin{equation}
\left\| \tilde{d} \{t^s \tilde{r}\} \right\|_{n+1,p} \lesssim c_o \tilde{d}_o r_0 + c_o \left(\epsilon \tilde{\Sigma}_m + \delta_o\right) \left(\delta_m + \epsilon_o\right) + c_o \left(\epsilon \tilde{\Sigma}_n + \delta_o\right) \left(\delta_m + \epsilon_o\right).
\end{equation}

In section 7.2 we obtain the estimates of the perturbations of the structure coefficients.

7.1. **Estimate the perturbation of the differential $t^s \tilde{r}$.** As mentioned above, like in section 6, we construct a family of spacelike surfaces $\{\tilde{\Sigma}_{a,t}\}$ for each $\tilde{\Sigma}_a$, that the second parametrisation of $\tilde{\Sigma}_{a,t}$ is $(s=0,\tilde{r}, t\tilde{f}_{a,t})$. Let $t^s \tilde{r}_{a,t}$ be the restriction of $s^a \tilde{r}$ on $\tilde{\Sigma}_{a,t}$. They satisfy the propagation equations \textbf{(220)}, which we cite here

\begin{equation}
\tilde{\partial}_t t^s \tilde{r}_{a,t} = t^s \tilde{\partial}_k t^s \tilde{r}_{a,t} + t^s \tilde{r}_{a,t},
\end{equation}

where $t^s \tilde{\partial}_k t^s \tilde{r}_{a,t}$ are given by equations \textbf{(227)}, \textbf{(228)}.

The perturbation $\tilde{d} \{t^s \tilde{r}_{a,t}\}$ is the difference between $t^s \tilde{r}_{a,t}$,

\begin{equation}
\tilde{d} \{t^s \tilde{r}_{a,t}\} = t^s \tilde{r}_{a,t} - t^s \tilde{r}_{a,t}.
\end{equation}
which satisfies the propagation equations, derived from taking the difference of the propagation equations of $t,s,a$,  

\[
\tilde{\partial}_d \{ t,s, \tilde{f}_i \} = t,s,1 \tilde{X}_k \tilde{\partial}_k \{ t,s, \tilde{f}_i \} + d \{ t,s, \tilde{f}_i \},
\]

(347)  

\[
\partial \{ t,s, \tilde{X}^k \} = t,s,2 \tilde{X}^k - t,s,1 \tilde{X}^k,
\]

(348)  

\[
\partial \{ t,s, \tilde{r} \} = t,s,2 \tilde{r} - t,s,1 \tilde{r},
\]

(349)  

and the initial condition is

\[
\partial \{ t=0, s, \tilde{f}_i \} = s=0,2 \tilde{f}_i - s=0,1 \tilde{f}_i.
\]

(350)  

We integrate the above propagation equation to obtain the estimate of $\partial \{ t,s, \tilde{f}_i \}$. We need to estimate $t,s,1 \tilde{X}_k$, $\partial \{ t,s, \tilde{X}_k \} \tilde{\partial}_k \{ t,s, \tilde{f}_i \}$ and $\partial \{ t,s, \tilde{r} \}$. The estimate of $t,s,1 \tilde{X}^k$ is given by lemma \[5.8\]. For the estimates of $\partial \{ t,s, \tilde{X}^k \} \tilde{\partial}_k \{ t,s, \tilde{f}_i \}$ and $\partial \{ t,s, \tilde{r} \}$, we prove a lemma which is analogous to lemma \[5.4\].

We assume the following two different bounds of $\partial \{ t,s, \tilde{f}_i \}$ that

\[
\left\| \partial \{ t,s, \tilde{f}_i \} \right\|_{\tilde{\Sigma}_t, \tilde{g}}^{n,p} \leq t,s,2 \tilde{f}_i r_0,
\]

(351)  

and

\[
\left\| \partial \{ t,s, \tilde{f}_i \} \right\|_{\tilde{\Sigma}_t, \tilde{g}}^{n+1,p} \leq t,s,2 \tilde{f}_i r_0.
\]

(352)  

The first bound (351) is used in the proof of theorem \[7.1\] and the second bound (352) is used in the proof of theorem \[7.2\].

**Lemma 7.3.** Let $\mathcal{A}$ be a map between tensor fields on $\tilde{\Sigma}_t$, which pointwisely is defined by the background quantities of the double null foliation on $\tilde{\Sigma}_t$, $\tilde{f}_i$, and $\tilde{r}_i$ through tensor calculations such as contraction and inversion, like $t,s, \tilde{X}^k$ and $t,s, \tilde{r}$. Assume that $\mathcal{A}$ satisfies the following estimate

\[
\| \mathcal{A} \|_{\tilde{\Sigma}_t, \tilde{g}}^{m,p} \leq P(\epsilon, \delta_0, \delta_m, \delta_0, r_0, t_1, s_0),
\]

(353)  

where $m \leq n$ and $P(\epsilon, \delta_o, \tilde{\Sigma}, r_0, r_t, s_0)$ is a rational function, whose denominator only involves $r_0, r_t$, then the perturbation of $A$ satisfies the estimate

\begin{equation}
\| \mathcal{A} \|_{m,p, \Sigma_{1,t,0}}^{m,p} \leq c(n,p) P(\epsilon, \delta_o, \tilde{\Sigma}, r_0, r_t, s_0) \\
\left\{ (r_0 + s_0)^{-1} \left\| \mathcal{A} \right\|_{m,p}^{m,p} + (r_0 + t s_0)^{-1} \left\| \mathcal{A} \right\|_{m,p}^{m,p} + r_0^{-1} \left\| \mathcal{A} \right\|_{m,p}^{m,p} \right\}
\end{equation}

Remark 7.4. The proof of lemma 7.3 follows the same pattern of lemma 6.4.

Remark 7.5. Like the example in remark 6.7, the estimate of $\mathcal{A}$ in the above lemma can be improved if the $\partial_{X}$ derivatives of the background quantities involved in $A$ have better estimates.

Remark 7.6. If we have better regularity for $\partial^{(t,s)\tilde{X}}$ and $\partial^{(t,s)\tilde{f}}$, then there is an improved version of lemma 7.3. For example we assume the assumptions in theorem 7.2, we have the estimate of $\| \mathcal{A} \|_{m,p, \Sigma_{1,t,0}}^{m,p}$ by theorem 6.2 and if we assume further the bound

\begin{equation}
\| \mathcal{A} \|_{m,p, \Sigma_{1,t,0}}^{m,p} \leq t s \delta_o r_0,
\end{equation}

then $m$, the regularity of $A$, can be improved to $n + 1$.

We apply the lemma, together with lemmas 5.8 and 5.9 to $\partial^{(t,s)\tilde{X}}$, $\partial^{(t,s)\tilde{f}}$, then we get their estimates.

Lemma 7.7 (Estimates of the vector field $\partial^{(t,s)\tilde{X}}$ and $\partial^{(t,s)\tilde{f}}$). Under the assumption of theorem 7.4 and moreover we assume the bound 351, then we have the following
estimates of $\mathcal{D}\{t, \tilde{\mathcal{X}}^i\}$ and $\mathcal{D}\{t, \tilde{\mathcal{C}}\}$,

\begin{equation}
\|\mathcal{D}\{t, \tilde{\mathcal{X}}^i\}\|^{n,p}_{\Sigma_{1,1,\tilde{g},t}} \leq \frac{c(n,p)(s_0 + r_0 \delta_0) r_0^3}{(r_0 + ts_0)^2} \left\{ \frac{t^0 r_0}{r_0 + ts_0} \left( e^{\delta_m} d_0 + d_0^2 \right) \right\} (\delta_m + \delta_o + \delta_m + \delta_o + t^s \delta_0) \\
+ \frac{c(n,p)(s_0 + r_0 \delta_0) r_0^3}{(r_0 + ts_0)^2} (\delta_m + \delta_o) + \frac{c(n,p)(s_0 + r_0 \delta_0) r_0^3}{(r_0 + ts_0)^2} \left( e^{\delta_m} + e^{\delta_o} + \delta_m^2 \right) (\delta_m + \delta_o) \\
+ \frac{c(n,p)(s_0 + r_0 \delta_0) r_0^3}{(r_0 + ts_0)^2} \left( e^{\delta_m} + e^{\delta_o} + \delta_m^2 \right) (\delta_m + \delta_o) \\
+ \frac{c(n,p)(s_0 + r_0 \delta_0) r_0^3}{(r_0 + ts_0)^2} \left( e^{\delta_m} + e^{\delta_o} + \delta_m^2 \right) (\delta_m + \delta_o).
\end{equation}

Furthermore if we assume that $\Delta^{s=0} f = 0$, which is the assumption of theorem 7.2 and we assume the bound (52), then we have the same bounds as above for the stronger Sobolev norms $\|\mathcal{D}\{t, \tilde{\mathcal{X}}^i\}\|^{n+1,p}_{\Sigma_{1,1,\tilde{g},t}}$.

Proof. Under the assumption of theorem 7.1 we can get the estimate of $\|\tilde{f}\|^{n,p}_{\Sigma_{1,1,\tilde{g},t}}$ by theorem 6.1, then the estimates of $\|\mathcal{D}\{t, \tilde{\mathcal{X}}^i\}\|^{n,p}_{\Sigma_{1,1,\tilde{g},t}}$ and $\|\mathcal{D}\{t, \tilde{\mathcal{C}}\}\|^{n,p}_{\Sigma_{1,1,\tilde{g},t}}$ follow from lemma 7.3. The estimates of $\|\mathcal{D}\{t, \tilde{\mathcal{X}}^i\}\|^{n+1,p}_{\Sigma_{1,1,\tilde{g},t}}$ and $\|\mathcal{D}\{t, \tilde{\mathcal{C}}\}\|^{n+1,p}_{\Sigma_{1,1,\tilde{g},t}}$ follow from theorem 7.2 and lemma 7.7.

Then we can integrate the propagation equation using lemma 7.7.
Lemma 7.8 (Integrate the propagation equation of $\mathfrak{d}\{f, s\}$. Under the assumption of theorem 7.1, and we assume the bound (351), then we have

\[
\begin{align*}
\| &\mathfrak{d}\{t=a, s\} \|^{n,p}_{\hat{\Sigma}_{1, t_a, \bar{g}}} \\
&\leq c(n,p)\mathfrak{d}_a r_0 + c(n,p) \left( (\mathfrak{d}_{\mathfrak{d}_m} + \mathfrak{d}_m^2)(\mathfrak{d}_m + \mathfrak{d}_o) + (\mathfrak{d}_o + \mathfrak{d}_o^2)(\mathfrak{d}_m + \mathfrak{d}_o) \right) r_0 \\
&+ \int_0^{t_a} \frac{c(n,p)(s_0 + r_0 \epsilon_0)}{(r_0 + t s_0)^2} (\mathfrak{d}_{\mathfrak{d}_m} + \mathfrak{d}_o) t_s \mathfrak{d}_o r_0 dt.
\end{align*}
\]

Furthermore if we assume that \( \frac{\epsilon}{\delta} s = 0.2 \), which is the assumption of theorem 7.2, and the bound (352), then we have the same bound as above for the stronger Sobolev norms \( \| \mathfrak{d}\{t=a, s\} \|^{n+1,p}_{\hat{\Sigma}_{1, t, \bar{g}}} \).

Now we can prove the main theorems 7.1, 7.2 in this section.

Proofs of theorems 7.1, 7.2: The proofs of these two theorem follow the similar pattern of the proofs of theorems 6.1 and 6.2. For theorem 7.1, we assume the bound (351). We introduce the bootstrap assumption that for \( t \in [0, t_a] \),

\[
\mathfrak{d}_o \leq c_0 \mathfrak{d}_m + c_o (\mathfrak{d}_m + \mathfrak{d}_m^2)(\mathfrak{d}_m + \mathfrak{d}_o) + c_o (\mathfrak{d}_o + \mathfrak{d}_o^2)(\mathfrak{d}_m + \mathfrak{d}_o).
\]

We show that at \( t = t_a \), the inequality of \( t_a \mathfrak{d}_o \) can be improved to the strict inequality. Then by lemma 7.8 we have that

\[
\begin{align*}
\| &\mathfrak{d}\{t=a, s\} \|^{n,p}_{\hat{\Sigma}_{1, t, \bar{g}}} \\
&\leq \left\{ c(n,p) + c_o (\mathfrak{d}_m + \mathfrak{d}_o) \right\} \mathfrak{d}_m r_0 + \left\{ c(n,p) + c_o (\mathfrak{d}_m + \mathfrak{d}_o) \right\} \left( \mathfrak{d}_m + \mathfrak{d}_o \right) r_0 \\
&+ \left\{ c(n,p) + c_o (\mathfrak{d}_m + \mathfrak{d}_o) \right\} \left( \mathfrak{d}_m + \mathfrak{d}_o \right) r_0.
\end{align*}
\]

We choose \( c_o \) such that

\[
\left( c(n,p) + c_o (\mathfrak{d}_m + \mathfrak{d}_o) < c_o, \right.
\]

for example, for sufficient small \( \epsilon, \mathfrak{d}_m, \mathfrak{d}_o \) such that \( \epsilon \mathfrak{d}_m + \mathfrak{d}_o < \frac{1}{2} \), we can choose \( c_o = 2c(n,p) \). Then

\[
\| \mathfrak{d}\{t=a, s\} \|^{n,p}_{\hat{\Sigma}_{1, t, \bar{g}}} < c_o \mathfrak{d}_m r_0 + c_o \left( \mathfrak{d}_m + \mathfrak{d}_o \right) r_0 + c_o \left( \mathfrak{d}_m + \mathfrak{d}_o \right) r_0,
\]

so we can extend the bootstrap assumption to a slightly larger interval. Hence the bootstrap assumption can be extended to the whole interval \( t \in [0, 1] \).

Theorem 7.2 can be proved similarly as theorem 7.1. \( \square \)
7.2. Estimate the perturbation of the structure coefficients. We can estimate the perturbation of the structure coefficients on $\tilde{\Sigma}$ by proposition [C.3] lemma [F.3] and theorem [F.1] [F.2].

**Proposition 7.9** (Estimate of the perturbation of the structure coefficients). Under the assumption of theorem [F.1], we have

\[ \|d\{I, \tilde{\chi}_j\}\|_{n-1,p, \tilde{\Sigma}, \tilde{\partial}} \leq c(n, p)(r_0 + s_0)(\partial_m + \partial_o + \tilde{\partial}_m + \tilde{\partial}_o), \]

\[ \|d\{I, \tilde{\chi}_j\}\|_{n-1,p, \tilde{\Sigma}, \tilde{\partial}} \leq \frac{c(n, p)}{r_0 + s_0}(\partial_m + \partial_o + \tilde{\partial}_m + \tilde{\partial}_o), \]

\[ \|d\{I, \tilde{\chi}_j\}\|_{n-1,p, \tilde{\Sigma}, \tilde{\partial}} \leq c(n, p)(s_0 + r_0(\epsilon + \delta_o))(\tilde{\partial}_m + \tilde{\partial}_o) + c(n, p)(r_0 + s_0)(\partial_m + \partial_o), \]

\[ \|d\{I, \tilde{\chi}_j\}\|_{n-1,p, \tilde{\Sigma}, \tilde{\partial}} \leq \frac{c(n, p)(s_0 + r_0(\epsilon + \delta_o))}{(r_0 + s_0)^2}(\tilde{\partial}_m + \tilde{\partial}_o) + \frac{c(n, p)}{r_0 + s_0}(\partial_m + \partial_o), \]

\[ \|d\{I, \tilde{\chi}_j\}\|_{n-1,p, \tilde{\Sigma}, \tilde{\partial}} \leq c(n, p)\partial_o + c(n, p)\left\{ \frac{r_0}{r_0 + s_0}\epsilon + \frac{r_0}{r_0 + s_0}\delta_o + \delta_o \right\} \partial_m + c(n, p)\left\{ \frac{r_0}{r_0 + s_0}\epsilon + \frac{r_0s_0}{(r_0 + s_0)^2}\delta_o + \delta_o \right\} \tilde{\partial}_m + c(n, p)\left\{ \frac{r_0}{r_0 + s_0}\epsilon + \frac{r_0s_0}{(r_0 + s_0)^2}\delta_o + \delta_o \right\} \tilde{\partial}_o, \]

\[ \|d\{\eta, \tilde{\chi}_j\}\|_{n-1,p, \tilde{\Sigma}, \tilde{\partial}} \leq c(n, p)q_2(\epsilon, \delta_o, \epsilon\tilde{\partial}_m, \tilde{\partial}_o)(\partial_m + \tilde{\partial}_m) + c(n, p)q_1(\epsilon, \delta_o, \epsilon\tilde{\partial}_m, \tilde{\partial}_o)(\partial_o + \tilde{\partial}_o), \]

\[ \|d\{\eta, \tilde{\chi}_j\}\|_{n-1,p, \tilde{\Sigma}, \tilde{\partial}} \leq \frac{c(n, p)r_0}{(r_0 + s_0)^2}q_2(\epsilon, \delta_o, \epsilon\tilde{\partial}_m, \tilde{\partial}_o)(\partial_m + \tilde{\partial}_m) + \frac{c(n, p)r_0}{(r_0 + s_0)^2}q_1(\epsilon, \delta_o, \epsilon\tilde{\partial}_m, \tilde{\partial}_o)(\partial_o + \tilde{\partial}_o), \]

\[ \|d\{\eta, \tilde{\chi}_j\}\|_{n-1,p, \tilde{\Sigma}, \tilde{\partial}} \leq c(n, p)(r_0 + s_0)q_2(\epsilon, \delta_o, \epsilon\tilde{\partial}_m, \tilde{\partial}_o)(\partial_m + \tilde{\partial}_m) \]

\[ + c(n, p)(r_0 + s_0)q_1(\epsilon, \delta_o, \epsilon\tilde{\partial}_m, \tilde{\partial}_o)(\partial_o + \tilde{\partial}_o), \]

\[ \|d\{\eta, \tilde{\chi}_j\}\|_{n-1,p, \tilde{\Sigma}, \tilde{\partial}} \leq \frac{c(n, p)}{r_0 + s_0}q_2(\epsilon, \delta_o, \epsilon\tilde{\partial}_m, \tilde{\partial}_o)(\partial_m + \tilde{\partial}_m) + \frac{c(n, p)}{r_0 + s_0}q_1(\epsilon, \delta_o, \epsilon\tilde{\partial}_m, \tilde{\partial}_o)(\partial_o + \tilde{\partial}_o), \]
In section 6.1, we derive the propagation equations (279), (280) for the first parametrization of spacelike surfaces. Analytically, we construct the linearization of the map approximates the map first parametrization, which we cite here, Linearized perturbation of the first parametrization. We introduce the surfaces \( \tilde{\Sigma}_a, a = 1, 2 \) as in section 6 and assume that they have the first parametrization \( (\tilde{a}, \tilde{a}, \tilde{a}, \tilde{a}) \) and the second parametrization \( (s=0, \tilde{a}, \tilde{a}, \tilde{a}) \). In section 6.1 we study the perturbation of the second parametrization \( \mathcal{D}(s=0) \), which is the map \( \mathcal{D}(s=0) \rightarrow \mathcal{D}(\tilde{f}) \). In this section, we construct a linear map \( \mathcal{D}(s=0) \rightarrow \mathcal{D}(\tilde{f}) \) which approximates the map \( \mathcal{D}(s=0) \rightarrow \mathcal{D}(\tilde{f}) \).

8. Linearization of the perturbation of the parametrization of spacelike surfaces

In this section, we construct the linearization of the perturbation of the first parametrization of spacelike surfaces. Analytically, we construct the linearization of the map \( (s=0, \tilde{f}) \rightarrow (\tilde{f}, \tilde{f}) \), the map from the second parametrization to the first parametrization. We introduce the surfaces \( \tilde{\Sigma}_a, a = 1, 2 \) as in section 6 and assume that they have the first parametrization \( (a, a, a, a) \) and the second parametrization \( (s=0, a, a, a) \). In section 6.1, we derive the propagation equations (279), (280) for \( \mathcal{D}(\tilde{f}) \), the perturbation of the first parametrization, which we cite here,

\[
\begin{align*}
\tilde{\delta}_{t} \tilde{\delta} \mathcal{D}(\tilde{f}) &= \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}), \\
\tilde{\delta}_{t} \tilde{\delta} \mathcal{D}(\tilde{f}) &= \tilde{\delta} \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}) \tilde{\delta} \mathcal{D}(\tilde{f}),
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{D}(F) &= \tilde{\mathcal{D}}(F) - \tilde{\mathcal{D}}(F), \\
\mathcal{D}(X) &= \tilde{\mathcal{D}}(X) - \tilde{\mathcal{D}}(X), \\
\mathcal{D}(\tilde{e}) &= \tilde{\mathcal{D}}(\tilde{e}) - \tilde{\mathcal{D}}(\tilde{e}).
\end{align*}
\]
We construct the linearized perturbation of the second parametrization, which is denoted by $\delta \{ \tilde{f} \}$, as the solution of the following linear equations,

$$\delta \{ \tilde{f} \} = \delta \{ s=0 \tilde{f} \} \circ \tilde{g},$$

$$\tilde{\partial}_t \Delta \delta \{ \tilde{f} \} = t^{1,1} \tilde{X}^i \left( \tilde{\Delta} \delta \{ \tilde{f} \} \right)_i,$$

with the initial condition

$$\delta \{ t=0 \tilde{f} \} = \delta \{ s=0 \tilde{f} \}.$$

8.2. Estimate the error of linearized perturbation of the parametrization. We denote by $e \{ \tilde{f} \}$ the error of linearized perturbation of $\tilde{f}$, which is the difference of $\delta \{ \tilde{f} \}$ and $\delta \{ \tilde{f} \}$. We can get the propagation equations of $e \{ \tilde{f} \}$,

$$\tilde{\partial}_t e \{ \tilde{f} \} = \tilde{\partial} \{ \tilde{f} \} \circ \tilde{g},$$

$$\tilde{\partial}_t \Delta e \{ \tilde{f} \} = t^{1,1} \tilde{X}^i \left( \tilde{\Delta} e \{ \tilde{f} \} \right)_i + \delta \{ t^{1,1} \tilde{X}^i \} \left( \tilde{\Delta} t^{1,1} \tilde{X}^i \right)_i + \tilde{\partial} \{ t^{1,1} \},$$

with the initial condition

$$e \{ t=0 \tilde{f} \} = 0.$$

We integrate the propagation equations (376), (377) to estimate $e \{ \tilde{f} \}$. We state the main theorems of this section.

**Theorem 8.1.** Under the assumption of theorem 6.2, we have the following estimates of $e \{ \tilde{f} \}$,

$$\left\| \tilde{\partial}_t e \{ \tilde{f} \} \right\|_{L^1_t, L^p_x} \leq (\delta^2 + e \delta_\alpha) \mathcal{P}_m r_0 + (\delta_\alpha + e \delta_m) \mathcal{P}_\alpha r_0 + (\delta^2 + e \delta_\alpha \delta_m) (\mathcal{P}_m + \mathcal{P}_\alpha) r_0,$$

$$\left\| e \{ \tilde{f} \} \right\|_{L^2_t, L^p_x} \leq \delta^2 + e \delta_\alpha \mathcal{P}_m r_0 + (\delta_\alpha + e \delta_m) \mathcal{P}_\alpha r_0 + (\delta^2 + e \delta_\alpha \delta_m) (\mathcal{P}_m + \mathcal{P}_\alpha) r_0.$$

**Theorem 8.2.** Under the assumption of theorem 6.2, and moreover we assume that $\tilde{\Delta} t^{1,1} \tilde{X}^i 0$, which is the additional assumption in theorem 6.2, we have the following estimate of $e \{ \tilde{f} \}$,

$$\left\| \tilde{\partial}_t e \{ \tilde{f} \} \right\|_{L^1_t, L^p_x} \leq (\delta^2 + e \delta_\alpha) \mathcal{P}_m r_0 + (\delta_\alpha + e \delta_m) \mathcal{P}_\alpha r_0 + (\delta^2 + e \delta_\alpha \delta_m) (\mathcal{P}_m + \mathcal{P}_\alpha) r_0.$$

To integrate the propagation equations (376), (377), we need to estimate the terms $\tilde{\partial} \{ \tilde{f} \} \circ \tilde{g}$, $\delta \{ t^{1,1} \tilde{X}^i \} \left( \tilde{\Delta} t^{1,1} \tilde{X}^i \right)_i$ and $\tilde{\partial} \{ t^{1,1} \}$. The estimates of these terms are given in the following lemma 8.3 which is proved by lemmas 6.8, 6.9, 6.10 and theorems 6.1, 6.2.
Lemma 8.3 (Estimates of $\mathcal{D}\{F\}$, $\mathcal{D}\{\tilde{X}^i\}$ and $\mathcal{D}\{\tilde{t} e\}$). Under the assumption of theorem 8.1 we have

\[
(382) \quad \left| \mathcal{D}\{F\} \right|, \left| \mathcal{D}\{\tilde{F}\} \right| \leq \frac{c(n,p)r_0^2(r_0 + s_0)}{(r_0 + ts_0)^2} \left\{ \delta_0^2 + \frac{r_0}{r_0 + ts_0} \varepsilon \delta_0 \right\} (\mathcal{D}_m + \mathcal{D}_o) \\
+ (s_0 + r_0 \delta_0) \frac{c(n,p)r_0^2}{(r_0 + ts_0)^2} \left\{ \delta_0^2 + \frac{c(n,p)r_0}{r_0 + ts_0} \delta_0 \right\} \mathcal{D}_n \\
+ (s_0 + r_0 \delta_0) \frac{c(n,p)r_0^2}{(r_0 + ts_0)^2} \left\{ \delta_0^2 + \frac{r_0}{r_0 + ts_0} \varepsilon \delta_0 \right\} \mathcal{D}_o.
\]

\[
(383) \quad \left\| \mathcal{D}\{\tilde{X}^i\} \left( \frac{z}{\tilde{\Delta} + \lambda X} \right) i \right\|_{\hat{\Sigma}_{1,t,\tilde{g}}} \leq \frac{c(n,p)r_0^2(r_0 + s_0)}{(r_0 + ts_0)^2} \left\{ \delta_0^2 + \frac{r_0}{r_0 + ts_0} \varepsilon \delta_0 \right\} (\mathcal{D}_m + \mathcal{D}_o) \\
+ (s_0 + r_0 \delta_0) \frac{c(n,p)r_0^2}{(r_0 + ts_0)^2} \left\{ \delta_0^2 + \frac{c(n,p)r_0}{r_0 + ts_0} \delta_0 \right\} \mathcal{D}_n \\
+ (s_0 + r_0 \delta_0) \frac{c(n,p)r_0^2}{(r_0 + ts_0)^2} \left\{ \delta_0^2 + \frac{r_0}{r_0 + ts_0} \varepsilon \delta_0 \right\} \mathcal{D}_o.
\]

\[
(384) \quad \left\| \mathcal{D}\{\tilde{t} e\}\right\|_{\hat{\Sigma}_{1,t,\tilde{g}}}^{n-2,p} \leq \frac{c(n,p)r_0^2(r_0 + s_0)}{(r_0 + ts_0)^2} \left\{ \delta_0^2 + \frac{r_0}{r_0 + ts_0} \varepsilon \delta_0 \right\} (\mathcal{D}_m + \mathcal{D}_o) \\
+ (s_0 + r_0 \delta_0) \frac{c(n,p)r_0^2}{(r_0 + ts_0)^2} \left\{ \delta_0^2 + \frac{c(n,p)r_0}{r_0 + ts_0} \delta_0 \right\} \mathcal{D}_n \\
+ (s_0 + r_0 \delta_0) \frac{c(n,p)r_0^2}{(r_0 + ts_0)^2} \left\{ \delta_0^2 + \frac{r_0}{r_0 + ts_0} \varepsilon \delta_0 \right\} \mathcal{D}_o.
\]

Furthermore, under the assumption of theorem 8.3, we can improve the above estimates to the stronger Sobolev norm

\[ \mathcal{D}\{\tilde{X}^i\} \left( \frac{z}{\tilde{\Delta} + \lambda X} \right) \left\|_{\hat{\Sigma}_{1,t,\tilde{g}}}^{n-1,p}, \right. \mathcal{D}\{\tilde{t} e\}\right\|_{\hat{\Sigma}_{1,t,\tilde{g}}}^{n-1,p}. \]

We can prove theorems 8.1, 8.2 now.

Proofs of theorems 8.1, 8.2 The theorems follow immediately from integrating the propagation equations (376), (377) by substituting the estimates in lemma 8.3. □
9. Linearization of the Perturbation of the Structure Coefficients of Spacelike Surfaces

In this section, we construct the linearization of the perturbation of the structure coefficients of spacelike surfaces. As in section 7, we introduce spacelike surfaces \( \Sigma_a, a = 1, 2 \), with the first parametrization \( (s=0, a_f, a_f) \) and the second parametrization \( (s=0, a_f, a_f) \). In section 7, we study the perturbation of the structure coefficients of spacelike surfaces. We take the structure coefficients \( \tilde{\Sigma} \chi' \) as an example. We may think of \( \tilde{\Sigma} \chi' \) as a map: \( (s=0, \tilde{a}, \tilde{a}) \mapsto (\tilde{\Sigma} \chi') \). In section 7, we study the perturbation of the above map: \( \delta\{s=0, \tilde{a}, \tilde{a}\} \mapsto \delta\{\tilde{\Sigma} \chi'\} \). In this section, we construct two linear maps \( \delta\{s=0, \tilde{a}, \tilde{a}\} \mapsto \delta\{\tilde{\Sigma} \chi'\} \) and \( \delta\{s=0, \tilde{a}, \tilde{a}\} \mapsto \delta\{\tilde{\Sigma} \chi'\} \) to approximate the linearization of \( \tilde{\Sigma} \chi' \). In principle, we can construct the linearization of the perturbation of all the structure coefficients. However, for the purpose of constructing marginally trapped surfaces, we need to consider only the structure coefficient \( \tilde{\Sigma} \chi' \).

Recall that in section 7, we view the structure coefficients as functions of the background quantities of the double null foliation, \( \tilde{f}_i, \tilde{\nabla}_{ij}^2 \tilde{f}, \tilde{\nabla}_{ij}^2 \tilde{f} \) on spacelike surfaces. In order to construct the linearization, we need first to construct the linearization of these quantities.

The linearization of the background quantities of the double null foliation can be constructed by the following: taking \( \tilde{\Sigma} \chi' \) as an example, we can construct its linearization by \( L\chi' \cdot \delta\{\tilde{f}\} + L\chi' \cdot \delta\{\tilde{f}\} \).

The linearization of \( \tilde{f}_i \) can be constructed in a straightforward manner as \( \delta\{\tilde{f}\}_{i} \). \( \nabla_{ij}^2 \tilde{f} \) can be decomposed into \( \nabla_{ij}^2 \tilde{f} + \left( \nabla_{ij} \cdot \nabla_{ij} \right) \tilde{f} \). The linearization of \( \nabla_{ij}^2 \tilde{f} \) can be constructed as \( \nabla_{ij}^2 \delta\{\tilde{f}\} \). The term \( \left( \nabla_{ij} \cdot \nabla_{ij} \right) \tilde{f} \) involves the difference of the background covariant derivative with the covariant derivative on the standard sphere. So formally, the linearization of \( \left( \nabla_{ij}^2 - \nabla_{ij} \right) \tilde{f} \) can be constructed as \( \delta\left\{ \left( \nabla_{ij}^2 - \nabla_{ij} \right) \right\} \tilde{f} + \left( \nabla_{ij}^2 - \nabla_{ij} \right) \delta\{\tilde{f}\} \).

The linearizations of \( \tilde{f}_i \) and \( \tilde{f}_i \) on spacelike surfaces are more complicated to construct. For \( \tilde{f}_i \), we consider the linearization \( \delta\{\tilde{f}\}_{i} \) of the propagation equation \( \nabla_{ij}^2 \tilde{f} = 1 \) and take the solution of the linearized propagation equation as the linearization of \( \tilde{f}_i \). For \( \nabla_{ij}^2 \tilde{f} \), we can also decompose it as \( \nabla_{ij}^2 \tilde{f} + \left( \nabla_{ij} - \nabla_{ij} \right) \tilde{f} \). The linearization of \( \nabla_{ij}^2 \tilde{f} \) can be constructed similarly as the case of \( \tilde{f}_i \), and the linearization of \( \left( \nabla_{ij} - \nabla_{ij} \right) \tilde{f} \) can be constructed similarly as in the case of \( \left( \nabla_{ij}^2 - \nabla_{ij} \right) \tilde{f} \).
we construct the linearization \( \delta \{ \text{tr} \tilde{\chi}' \} \) of the perturbation of \( \text{tr} \tilde{\chi}' \).

In section 9.1 we construct the linearization of \( \check{d}_{\mathcal{I}} \), and estimate its error with the perturbation of \( \check{d}_{\mathcal{I}} \) in section 9.2 In section 9.3 we construct two linearizations of the outgoing null expansion \( \text{tr} \tilde{\chi}' \) and estimate their errors with the perturbations of \( \text{tr} \tilde{\chi}' \) in section 9.4.

9.1. Linearized perturbation of the differential \( \check{d}_{\mathcal{I}} \). In section 7.1 we derive the propagation equations (347) for \( \delta \{ t, \check{s}_{\mathcal{I}} \} \), the perturbation of the differential \( \check{d} t, \check{s}_{\mathcal{I}} \), which we cite here

\[
\begin{align*}
\tilde{\partial}_t \delta \{ t, \check{s}_{\mathcal{I}} \} & = t, s, 1 \tilde{\chi}^k \tilde{\partial}_k \delta \{ t, \check{s}_{\mathcal{I}} \} + \mathcal{O} \{ t, s, 2 \tilde{\chi}^k \} \tilde{\partial}_k \delta \{ t, \check{s}_{\mathcal{I}} \} + \mathcal{O} \{ t, s, \check{\chi} \}, \\
\delta \{ t, \check{s}_{\mathcal{I}} \} & = t, s, 2 \tilde{\chi}^k - t, s, 1 \tilde{\chi}^k, \\
\delta \{ t, \check{\chi} \} & = t, s, 2 \tilde{\chi}^k - t, s, 1 \tilde{\chi}^k.
\end{align*}
\]

We construct the linearized perturbation of the differential, which is denoted by \( \delta \{ t, \check{s}_{\mathcal{I}} \} \), as the solution of the following linear equations. Note that the last two terms in equation (387) are dropped.

\[
\begin{align*}
\tilde{\partial}_t \delta \{ t, \check{s}_{\mathcal{I}} \} & = t, s, 1 \tilde{\chi}^k \tilde{\partial}_k \delta \{ t, \check{s}_{\mathcal{I}} \}, \\
\text{with the initial condition} & \quad \delta \{ t = 0, \check{s}_{\mathcal{I}} \} = \mathcal{O} \{ s = 0 \check{\chi} \},
\end{align*}
\]

9.2. Estimate the error of the linearized perturbation of the differential \( \check{d}_{\mathcal{I}} \).

We denote by \( \mathcal{E} \{ t, \check{s}_{\mathcal{I}} \} \) the error of the linearized perturbation of \( \check{d} t, \check{s}_{\mathcal{I}} \), which is the difference of \( \delta \{ t, \check{s}_{\mathcal{I}} \} \) and \( \delta \{ t, \check{s}_{\mathcal{I}} \} \). We can get the propagation equations of \( \mathcal{E} \{ t, \check{s}_{\mathcal{I}} \} \),

\[
\begin{align*}
\tilde{\partial}_t \mathcal{E} \{ t, \check{s}_{\mathcal{I}} \} & = t, s, 1 \tilde{\chi}^k \tilde{\partial}_k \mathcal{E} \{ t, \check{s}_{\mathcal{I}} \} + \mathcal{O} \{ t, s, 2 \tilde{\chi}^k \} \tilde{\partial}_k \mathcal{E} \{ t, \check{s}_{\mathcal{I}} \} + \mathcal{O} \{ t, s, \check{\chi} \}, \\
\text{with the initial condition} & \quad \mathcal{E} \{ t = 0, \check{s}_{\mathcal{I}} \} = 0.
\end{align*}
\]

We integrate the propagation equation (387) to obtain the estimates of \( \mathcal{E} \{ t, \check{s}_{\mathcal{I}} \} \).

Theorem 9.1. Under the assumption of theorem 7.1, we have the following estimate of \( \mathcal{E} \{ t, \check{s}_{\mathcal{I}} \} \),

\[
\mathcal{E} \{ t, \check{s}_{\mathcal{I}} \} \left\|_{\Sigma_{t, \check{\chi}}} \right\| p \leq c(n, p)(e \check{\delta}_m \check{\delta}_n + \check{\delta}_m^2) \check{\delta}_m + c(n, p) \check{\delta}_m \check{\delta}_n r_0 + c(n, p) \check{\delta}_m \check{\delta}_n r_0.
\]

Theorem 9.2. Under the assumption of theorem 7.1 and the additional condition \( \Delta t, \check{s}_{\mathcal{I}} = 0 \), we have the following estimate of \( \mathcal{E} \{ t, \check{s}_{\mathcal{I}} \} \),

\[
\mathcal{E} \{ t, \check{s}_{\mathcal{I}} \} \left\|_{\Sigma_{t, \check{\chi}}} \right\| p+1 \leq c(n, p)(e \check{\delta}_m \check{\delta}_n + \check{\delta}_m^2) \check{\delta}_m + c(n, p) \check{\delta}_m \check{\delta}_n r_0 + c(n, p) \check{\delta}_m \check{\delta}_n r_0.
\]
In order to integrate equation (387), we need to obtain the estimates of $\mathcal{O}\{t^s X^{\hat{k}}\} \partial^{t^s, t^s} f$ and $\mathcal{O}\{t^s \hat{\tau}\}$, which are given in the following lemma 9.3 which follows from lemma 7.1 and theorems 7.1, 7.2.

**Lemma 9.3** (Estimates of $\mathcal{O}\{t^s X^{\hat{k}}\} \partial^{t^s, t^s} f$ and $\mathcal{O}\{t^s \hat{\tau}\}$). Under the assumption of theorem 7.1, then we have the following estimates of $\mathcal{O}\{t^s X^{\hat{k}}\} \partial^{t^s, t^s} f$ and $\mathcal{O}\{t^s \hat{\tau}\}$,

\[
\|\mathcal{O}\{t^s X^{\hat{k}}\} \partial^{t^s, t^s} f\|_{\Sigma_1, s, \partial} \leq c(n, p)(r_0 + s_0) r_0 \left\{ \frac{r_0}{r_0 + s_0} e_\alpha d_\alpha + \delta_\alpha^2 \right\} (d_m + d_0) + \frac{c(n, p)(r_0 + s_0) r_0}{(r_0 + s_0)^2} \left\{ \frac{r_0}{r_0 + s_0} e_\alpha d_\alpha + \delta_\alpha^2 \right\} d_n + \frac{c(n, p)(r_0 + s_0) r_0^3}{(r_0 + s_0)^3} (e_\alpha d_\alpha + \delta_\alpha^2) d_n.
\]

Furthermore if we assume that \( \sum_{s=0}^{2} \frac{\mathcal{O}\{t^s X^{\hat{k}}\} \partial^{t^s, t^s} f}{\partial} \) vanishes in the propagation equation (387), and we have the same bounds as above for the stronger Sobolev norm \( \|\mathcal{O}\{t^s \hat{\tau}\}\|_{\Sigma_1, s, \partial} \) of $\mathcal{O}\{t^s \hat{\tau}\}$.

Now we can prove theorems 9.1, 9.2.

**Proofs of theorems 9.1, 9.2.** The proofs of these two theorems are straightforward, just integrating the propagation equation (387) and substituting from lemma 9.3 the estimates of $\mathcal{O}\{t^s X^{\hat{k}}\} \partial^{t^s, t^s} f$ and $\mathcal{O}\{t^s \hat{\tau}\}$.

\[\square\]

### 9.3. Linearized perturbation of the structure coefficients.

To construct the linearized perturbation of the structure coefficients, we need to construct first a linearized perturbation of background quantities, including metric components and structure coefficient, of the double null coordinate system, which is introduced in the next definition.

**Definition 9.4.** Let $A$ be a background quantity of the double null coordinate system, which could be a function like $\text{tr} \chi$ or a tenor field like $\chi_{ij}$. We define $A$’s linearized perturbation with respect to the perturbation of spacelike surfaces $\Sigma_a, a = 1, 2$ as

\[
\delta\{A\} = (\mathcal{L}_a A)_{\tilde{\Sigma}_1} \cdot \delta\{\tilde{f}\} + (\mathcal{L}_\Sigma A)_{\tilde{\Sigma}_1} \cdot \delta\{\tilde{f}\},
\]

where

\[
\delta\{\tilde{f}\} = \mathcal{O}\{t^s \tilde{f}\} = 2\tilde{f} - 1\tilde{f},
\]

and
and \( \delta \{ \tilde{f} \} \) is given by the linearized perturbation constructed in section 8.7.

Let \( \mathcal{A} \) be a function or a tensor field on \( \tilde{\Sigma} \), which is constructed from the background quantities of the double null coordinate system and \( \tilde{f}_a, \tilde{\nabla}_{ab} \tilde{f}, \tilde{s}_a, \tilde{\nabla}_{ab} \tilde{s} \) on \( \tilde{\Sigma} \), by tensor calculus such as tensor product and contraction. Suppose that

\[
\mathcal{A} = A_1 \ast A_2 \ast \cdots \ast A_n,
\]

where \( \ast \) represents either tensor product or contraction, then we define \( \mathcal{A} \)'s linearized perturbation with respect to the perturbation of spacelike surfaces \( \tilde{\Sigma}_a, a = 1, 2 \) as

\[
\delta \{ \mathcal{A} \} = \sum_{i=1,2,\ldots,n} (A_1)_{\tilde{\Sigma}_1} \ast (A_2)_{\tilde{\Sigma}_1} \ast \cdots \ast \delta \{ A_i \} \ast \cdots \ast (A_n)_{\tilde{\Sigma}_1}
\]

where

\[
\delta \{ \tilde{f}_a \} = \delta \{ \tilde{f} \}_a, \quad \delta \{ \tilde{\nabla}_{ab} \tilde{f} \} = \tilde{\nabla}_{ab} \delta \{ \tilde{f} \},
\]

and \( \delta \{ \tilde{s}_a \} \) and \( \delta \{ \tilde{\nabla}_{ab} \tilde{s} \} \) are constructed as in section 9.7.

In the following, we construct the linearized perturbation of the structure coefficients. One obvious linearization would be to adapt the above definition. However, at this point we can simplify the process significantly by introducing another linearization as follows. In section 4.1, we decompose structure coefficients into low degree and high degree terms. The high degree terms do not interfere with the main structure of the linearization. Thus we omit the linearization of high degree terms. For the purpose of constructing marginally trapped surfaces, we only define the linearized perturbation of the outgoing null expansion \( \tilde{\Delta} \tilde{\chi}' \).

**Definition 9.5.** We define the following linearized perturbation of \( \tilde{\Lambda} \tilde{\chi}' \):

\[
\delta_{\text{l,Sch}} \{ \tilde{\Lambda} \tilde{\chi}' \}
\]

\[
:= (\partial_{\text{l,Sch}} \tilde{\chi}')_{\tilde{\Sigma}_{\tilde{s}}} \delta \{ \tilde{f} \} + (\partial_{\text{l,Sch}} \tilde{\chi}')_{\tilde{\Sigma}_{\tilde{s}}} \delta \{ \tilde{\nabla}_{ab} \tilde{f} \} - 2r_{\text{Sch}}^{-2} (s = \tilde{f}, \tilde{s} = \tilde{\nabla}_{ab} \tilde{f}) \Delta \tilde{\nabla} \delta \{ \tilde{f} \},
\]

where \( r_{\text{Sch}}(s, \tilde{s}) \) is the area radius of the surface \( \Sigma_{s, \tilde{s}} \) of the double null foliation in the Schwarzschild spacetime.

**9.4. Estimate the error of the linearized perturbation of structure coefficients.**

We have the following lemma to estimate the error of the linearized perturbation constructed in definition 9.7.

**Lemma 9.6.** Let \( \mathcal{A} \) be as in definition 9.4. Assume that its perturbation with respect to the perturbation of spacelike surfaces \( \tilde{\Sigma}_a, a = 1, 2 \), satisfies the following estimate

\[
\| \delta \{ \mathcal{A} \} \|_{\tilde{\Sigma}_{\tilde{f}}} \leq p_{\mathcal{A}}(\epsilon, \delta_0, \tilde{s}_m, \tilde{\nabla}_0, r_0, s_0) \delta_m + p_{\mathcal{A}}(\epsilon, \delta_0, \tilde{s}_m, \tilde{\nabla}_0, r_0, s_0) \delta_m
\]

\[
+ p_{\mathcal{A}}(\epsilon, \delta_0, \tilde{s}_m, \tilde{\nabla}_0, r_0, s_0) \delta_m + p_{\mathcal{A}}(\epsilon, \delta_0, \tilde{s}_m, \tilde{\nabla}_0, r_0, s_0) \delta_m,
\]
where $m \leq n - 1$, then for $e, \delta_m, \delta_o, \delta_p, \delta_m, \delta_o, \delta_p$ sufficiently small depending on $n, p$ and $A$, there exists a constant $c(n, p, A)$ such that the difference $e\{A\}$ between the perturbation $\delta\{A\}$ and the linearization $\delta\{A\}$ in definition 7.4 satisfies the estimate

$$\left\| e\{A\}\right\|_{\Sigma, g}^m \leq c(n, p, A) p_{\delta_m} (e, \delta_o, \delta_m, \delta_o, r_0, s_0) (r_0 + s_0)^{-1} \left\| e\{\tilde{f}\}\right\|_{\Sigma, \tilde{g}}^m$$

$$+ c(n, p, A) p_{\delta_o} (e, \delta_o, \delta_m, \delta_o, r_0, s_0) (r_0 + s_0)^{-1} \left\| e\{\tilde{f}\}\right\|_{\Sigma, \tilde{g}}^m$$

$$+ c(n, p, A) p_{\delta_p} (e, \delta_o, \delta_m, \delta_o, r_0, s_0)^{-1} \left\| e\{\tilde{f}\}\right\|_{\Sigma, \tilde{g}}^m$$

$$+ c(n, p, A) (p_{\delta_m} \delta_m + p_{\delta_o} \delta_o + p_{\delta_p} \delta_p + p_{\delta_o} \delta_o) \left( \delta_m + \delta_o + \delta_p + \delta_o \right)$$

$$+ c(n, p, A) (p_{\delta_m} \delta_m + p_{\delta_o} \delta_o + p_{\delta_p} \delta_p + p_{\delta_o} \delta_o) \left( \delta_m + \delta_o \right)$$

Furthermore, if we assume that $\tilde{\Delta} \tilde{\Delta} \tilde{\Delta} = 0$, then by theorem 7.2, we have the estimate of $\left\| e\{\tilde{f}\}\right\|_{\Sigma, \tilde{g}}^m$, so we can improve $m$, the order of differentiability of $A$, to $n$, and get the similar estimate of $\| e\{A\}\|_{\Sigma, \tilde{g}}^m$.

Applying the above lemma and proposition 7.8 together with theorems 8.1, 8.2 on the estimates of $e\{\tilde{f}\}$, and theorems 9.1, 9.2 on the estimates of $e\{\tilde{f}\}$, we can estimate the error of the linearized perturbation of the structure coefficients. Here we consider the structure coefficient $\tilde{\chi}'$.

Lemma 9.7. From section 5.7, we decompose $\tilde{\chi}'$ into $\{\tilde{\chi}'\}$ and $\{\tilde{\chi}'\}$ where

$$\{\tilde{\chi}'\} = \tilde{\chi}' - 2\Delta \tilde{f} = \tilde{\chi}' - 2 (\tilde{g}^{-1})^{ij} \left( \tilde{\nabla}_{ij} \tilde{f} - \Delta_{ij}^{*} \tilde{f} \right),$$

with

$$\Delta_{ij}^{*} = 2 \tilde{\nabla}_{ij}^{*} \tilde{f} - \tilde{g}^{-1} \left( \tilde{\nabla}_{i} \tilde{g}_{j} + \tilde{\nabla}_{j} \tilde{g}_{i} - \tilde{\nabla}_{i} \tilde{g}_{ij} \right).$$
The linearized perturbation of \( l \{ \tilde{\text{tr}} \tilde{\chi} \} \) defined in definition \( 9.4 \) is

\[
\delta \{ l \{ \tilde{\text{tr}} \tilde{\chi} \} \} = (\partial_x \text{tr} \chi)_{\tilde{\Sigma}_l} \delta \{ \tilde{f} \} + (\partial_x \text{tr} \chi)_{\tilde{\Sigma}_l} \delta \{ \tilde{f} \} - 2 (\tilde{g}^{-1})^{ij} \nabla_{ij} \delta \{ \tilde{f} \}
\]

\[
- 2 \delta \{ (\tilde{g}^{-1})^{ij} \} \nabla_{ij} \tilde{f} + 2 \delta \{ (\tilde{g}^{-1})^{ij} \Delta_{ij} \tilde{f}_k \}
\]

Under the assumption of theorem \( 7.1 \),

\[
(\partial_x \text{tr} \chi)_{\tilde{\Sigma}_l} \delta \{ \tilde{f} \}
\]

is given by proposition \( 7.9 \). We need to estimate

\[
\[4]]
\]

Moreover, if we assume that \( s = 0 \), the above estimate of \( e \{ l \{ \tilde{\text{tr}} \tilde{\chi} \} \} \) is satisfied for the \( W^{n,p} \) norm

\[
\left\| e \{ l \{ \tilde{\text{tr}} \tilde{\chi} \} \} \right\|_{\tilde{\Sigma}_l, 0}^{n,p} = 0.
\]

We are interested in the error of the linearized perturbation \( \delta_{l, \text{Sch}} \{ \tilde{\text{tr}} \tilde{\chi} \} \). We denote by \( e_{l, \text{Sch}} \{ \tilde{\text{tr}} \tilde{\chi} \} \) the difference of \( \delta \{ \tilde{\text{tr}} \tilde{\chi} \} \) with \( \delta_{l, \text{Sch}} \{ \tilde{\text{tr}} \tilde{\chi} \} \). We estimate \( e_{l, \text{Sch}} \{ \tilde{\text{tr}} \tilde{\chi} \} \) in the following.

\[
e_{l, \text{Sch}} \{ \tilde{\text{tr}} \tilde{\chi} \} = e \{ l \{ \tilde{\text{tr}} \tilde{\chi} \} \} + \delta \{ l \{ \tilde{\text{tr}} \tilde{\chi} \} \} - \delta_{l, \text{Sch}} \{ \tilde{\text{tr}} \tilde{\chi} \} + \delta \{ \tilde{\mathcal{H}} \{ \tilde{\text{tr}} \tilde{\chi} \} \}.
\]

So we estimate \( e \{ l \{ \tilde{\text{tr}} \tilde{\chi} \} \}, \delta \{ l \{ \tilde{\text{tr}} \tilde{\chi} \} \} - \delta_{l, \text{Sch}} \{ \tilde{\text{tr}} \tilde{\chi} \}, \delta \{ \tilde{\mathcal{H}} \{ \tilde{\text{tr}} \tilde{\chi} \} \} \) separately to obtain the estimate of \( e_{l, \text{Sch}} \{ \tilde{\text{tr}} \tilde{\chi} \} \) in lemma \( 9.8 \).

The estimate of \( e \{ l \{ \tilde{\text{tr}} \tilde{\chi} \} \} \) is given by lemma \( 9.7 \) and the estimate of \( \delta \{ \tilde{\mathcal{H}} \{ \tilde{\text{tr}} \tilde{\chi} \} \} \) is given by proposition \( 7.8 \). We need to estimate \( \delta \{ l \{ \tilde{\text{tr}} \tilde{\chi} \} \} - \delta_{l, \text{Sch}} \{ \tilde{\text{tr}} \tilde{\chi} \} \) which is the following.

\[
\delta \{ l \{ \tilde{\text{tr}} \tilde{\chi} \} \} - \delta_{l, \text{Sch}} \{ \tilde{\text{tr}} \tilde{\chi} \}
\]

\[
= (\partial_x \text{tr} \chi - \partial_x \text{tr} \chi_{\text{Sch}})_{\tilde{\Sigma}_l} \delta \{ \tilde{f} \} + (\partial_x \text{tr} \chi - \partial_x \text{tr} \chi_{\text{Sch}})_{\tilde{\Sigma}_l} \delta \{ \tilde{f} \}
\]

\[
- 2 \left\{ (\tilde{g}^{-1})^{ij} - r_{\text{Sch}}(s = \tilde{s}, f = \tilde{f}) (\tilde{g}^{-1})^{ij} \right\} \nabla_{ij} \delta \{ \tilde{f} \}
\]

\[
- 2 \delta \{ (\tilde{g}^{-1})^{ij} \} \nabla_{ij} \tilde{f} + 2 \delta \{ (\tilde{g}^{-1})^{ij} \Delta_{ij} \tilde{f}_k \}
\]
By lemma C.3 we have

\[ \| (\partial_s \text{tr} \chi' - \partial_s \text{tr} \chi'_{\text{Sch}})_{\Sigma_1} \delta \{ \tilde{f} \} \|_{\Sigma_1, \tilde{g}}^{n+2,p} \leq \frac{c(n, p)}{r_0 + s_0} \epsilon (\mathcal{D}_m + \mathcal{D}_o), \]

\[ \| (\partial_s \text{tr} \chi' - \partial_s \text{tr} \chi'_{\text{Sch}})_{\Sigma_1} \delta \{ \tilde{f} \} \|_{\Sigma_1, \tilde{g}}^{n+1,p} \leq \frac{c(n, p) r_0}{(r_0 + s_0)^2} \epsilon (\mathcal{D}_m + \mathcal{D}_o + (\mathcal{D}_m + \mathcal{D}_o)(\mathcal{D}_m + \mathcal{D}_o)), \]

\[ \| (\tilde{g}^{-1})^{ij} - r_{\text{Sch}} \left( \tilde{g}^{-1} \right)^{ij} \|_{\Sigma_1, \tilde{g}}^{n+1} \leq \frac{c(n, p)}{(r_0 + s_0)^2} \epsilon, \]

\[ \| \delta \{ (\tilde{g}^{-1})^{ij} \} \|_{\Sigma_1, \tilde{g}}^{n+1} \leq \frac{c(n, p) r_0}{(r_0 + s_0)^2} \left( \epsilon + \frac{s_0 + (r_0 + s_0) \delta_o}{r_0 + s_0} \right) \delta_o (\mathcal{D}_m + \mathcal{D}_o) + \frac{c(n, p)}{r_0 + s_0} \epsilon \mathcal{D}_o. \]

For \((\tilde{g}^{-1})^{ij} \triangle^k \tilde{f}_k\), it is a high order term,

\[ \| (\tilde{g}^{-1})^{ij} \triangle^k \tilde{f}_k \|_{\Sigma_1, \tilde{g}}^{n+1,p} \leq \frac{c(n, p)}{r_0 + s_0} \epsilon \mathcal{D}_o. \]

\[ \| \delta \{ (\tilde{g}^{-1})^{ij} \triangle^k \tilde{f}_k \} \|_{\Sigma_1, \tilde{g}}^{n+1,p} \leq \frac{c(n, p)}{r_0 + s_0} \epsilon \delta_o (\mathcal{D}_m + \mathcal{D}_o) + \frac{c(n, p)}{r_0 + s_0} \epsilon \mathcal{D}_o. \]

Combining the above estimates, we get the estimate for \(\delta \{ \{ \tilde{t}, \tilde{r} \} \} - \delta_{t, \text{Sch}} \{ \tilde{r} \}, \)

\[ \| \delta \{ \{ \tilde{t}, \tilde{r} \} \} - \delta_{t, \text{Sch}} \{ \tilde{r} \} \|_{\Sigma_1, \tilde{g}}^{n+1} \leq \frac{c(n, p) r_0}{(r_0 + s_0)^2} \left( \epsilon + \frac{s_0 + (r_0 + s_0) \delta_o}{r_0 + s_0} \right) \delta_o (\mathcal{D}_m + \mathcal{D}_o) + \frac{c(n, p)}{r_0 + s_0} \epsilon \mathcal{D}_o. \]

Hence we have the following estimate for the error \(e_{t, \text{Sch}} \{ \tilde{r} \} \).
Lemma 9.8. Under the assumption of theorem \(7.1\)

\[
\left\| \mathbf{e}_{l,Sch}\{\tilde{\chi} \tilde{\chi}'\} \right\|_{\tilde{\Sigma}_1}^{n-1,p} \\
\leq \frac{c(n,p)}{(r_0 + s_0)^2} \left\{ (\epsilon + \delta_o) r_0 + \delta_o^2(r_0 + s_0) + \left( \delta_o^2 + \epsilon \delta_o + \frac{r_0}{r_0 + s_0} \delta_o \right) s_0 \right\} \partial_m \\
+ \frac{c(n,p)}{r_0 + s_0} \left\{ (\epsilon + \delta_o) r_0 + \delta_o^2(r_0 + s_0) + \left( \delta_o + \epsilon \delta_o + \frac{r_0}{r_0 + s_0} \delta_o \right) s_0 \right\} \partial_o \\
+ \frac{c(n,p)}{r_0 + s_0} \left\{ (\epsilon + \delta_o) r_0 + \delta_o^2(r_0 + s_0) + \left( \delta_o + \epsilon \delta_o + \frac{r_0}{r_0 + s_0} \delta_o \right) s_0 \right\} \partial_o \\
+ \left\{ \frac{c(n,p)(s_0 + r_0)(\epsilon + \delta_o)}{(r_0 + s_0)^2}(\partial_m + \partial_o) + \frac{c(n,p)}{r_0 + s_0}(\partial_m + \partial_o) \right\} (\partial_m + \partial_o + \partial_m + \partial_o).
\]

Moreover, if we assume that \(\tilde{\chi} \tilde{\chi}' \sim 0\), the above estimate of \(\mathbf{e}_{l,Sch}\{\tilde{\chi} \tilde{\chi}'\}\) is satisfied for the \(W^{n,p}\) norm \(\left\| \mathbf{e}_{l,Sch}\{\tilde{\chi} \tilde{\chi}'\} \right\|_{\tilde{\Sigma}_1}^{n,p} \).

10. Construct marginally trapped surfaces near \(\Sigma_{0,0}\)

Recall that in the double null foliation of \((M, g)\), the surface \(\Sigma_{0,0}\) is marginally trapped. In this section, we want to find all the marginally trapped surfaces near \(\Sigma_{0,0}\). Suppose that \(\tilde{\Sigma}\) is a spacelike surface near \(\Sigma_{0,0}\). By definition, \(\tilde{\Sigma}\) being marginally trapped is equivalent to \(\tilde{\chi} \tilde{\chi}' \tilde{\Sigma} = 0\). We assume that \(\tilde{\Sigma}\) is parametrized by \((s = 0, \tilde{f})\). Then the outgoing null expansion \(\tilde{\chi} \tilde{\chi}' \tilde{\Sigma}\) can be viewed as a functional which is denoted by \(\tilde{\chi} \tilde{\chi}' (s = 0, \tilde{f})\). Finding the marginally trapped surfaces near \(\Sigma_{0,0}\) is equivalent to finding the spacelike surface \(\tilde{\Sigma}\) which solves the equation \(\tilde{\chi} \tilde{\chi}' (s = 0, \tilde{f}) = 0\).

The second parametrization of the surface \(\Sigma_{0,0}\) is \((0, 0)\) and \(\tilde{\chi} \tilde{\chi}' (0, 0) = 0\). We will use the linearization of \(\tilde{\chi} \tilde{\chi}'\) constructed in section 9.3 to solve the equation \(\tilde{\chi} \tilde{\chi}' (s = 0, \tilde{f}) = 0\). We prove that for any \(s = \tilde{f}\) sufficiently small, there exists a unique \(\tilde{f}\) which is also small and solves \(\tilde{\chi} \tilde{\chi}' (s = \tilde{f}, \tilde{f}) = 0\). The idea is that given \(s = \tilde{f}\), we use the linearization of \(\tilde{\chi} \tilde{\chi}'\) to construct a Picard iteration to approximate the solution \(\tilde{f}\) of the equation \(\tilde{\chi} \tilde{\chi}' (s = \tilde{f}, \tilde{f}) = 0\).

Geometrically speaking, in this section we prove that in any incoming null hypersurface \(\tilde{\Sigma}\) near the incoming null hypersurface \(\Sigma_{0,0}\), there exists a unique marginally trapped surface \(\tilde{\Sigma}\) embedded in \(\tilde{\Sigma}\) and close to \(\Sigma_{0,0}\).
We use the solution of $\tilde{\chi}'(s=0, \tilde{f}) = 0$ to construct a map $f$ such that for any $s=0, \tilde{f}$ sufficiently small, $f \left( s=0, \tilde{f} \right)$ is the function $\tilde{f}$ solving $\tilde{\chi}'(s=0, \tilde{f}) = 0$. With the help of this map, we can identify the space of marginally trapped surfaces near $\Sigma_{0,0}$ with the space of functions $s=0, \tilde{f}$, where the identification is given by the following: a marginally trapped surface $\tilde{\Sigma}$ near $\Sigma_{0,0}$ corresponds to the function $s=0, \tilde{f}$ in the second parametrization of $\tilde{\Sigma}$, and vice versa a function $s=0, \tilde{f}$ corresponds to the spacelike surface $\tilde{\Sigma}$ parametrized by $\left( s=0, f \left( s=0, \tilde{f} \right) \right)$. The construction of $f$ relies on the existence and uniqueness of the solution of $\tilde{\chi}'(s=0, \tilde{f}) = 0$ for $\tilde{f}$, thus $f$ is constructed in the end of this section.

In section 10.1, we introduce the map $t$ from the pair of functions $\left( s=0, \tilde{f} \right)$ to the incoming null expansion $\tilde{\chi}'$ of the spacelike surface $\tilde{\Sigma}$ parametrized by $\left( s=0, \tilde{f} \right)$. We introduce the appropriate Sobolev norms of the functions $s=0, \tilde{f}$ and $\tilde{\chi}'$. Therefore we view the map $t$ as an nonlinear map between the Sobolev spaces. We construct the linearization of $t$ using the results of section 9. In section 10.2, we construct the iteration scheme aiming to solve the equation $t \left( s=0, \tilde{f} \right) = 0$ with given $s=0, \tilde{f}$. In section 10.3 we prove boundedness of the iteration scheme. In section 10.4, we prove the convergence of the iteration scheme. In section 10.5, we show that the limit of the iteration scheme solves the equation $t \left( s=0, \tilde{f} \right) = 0$. In section 10.6, we prove the uniqueness of the solution $\tilde{f}$ of $t \left( s=0, \tilde{f} \right) = 0$ with given $s=0, \tilde{f}$. In section 10.7, we define the map $f$, construct its linearization and estimate the error of its linearization.

10.1. **Analytic description of the outgoing null expansion** $\tilde{\chi}'$. We view $\tilde{\chi}'$ as a map from the space of spacelike surfaces to a function over the surfaces. Since we know that the space of spacelike surfaces can be parametrized by two functions $\left( s=0, \tilde{f} \right)$ over $\Sigma$, then we can view $\tilde{\chi}'$ as a map from the space of pairs of functions on $\mathbb{S}^2$ to the space of functions on $\mathbb{S}^2$, which we denote by $t$,

$$
(416) \quad t : W^{n+2,p} \times W^{n+2,p} \rightarrow W^{n,p}, \quad \left( s=0, \tilde{f} \right) \mapsto t(s=0, \tilde{f}) = \tilde{\chi}'_{s=0, \tilde{f}},
$$

where

$$
(417) \quad W^{m,p}_{b_m,b_0} = \left\{ f \in W^{m,p} : \left\| f \right\|_{b_m} \leq b_m r_0, \left\| d f \right\|_{m-1,p} \leq b_0 r_0 \right\}.
$$

In section 10.3, we construct the linearized perturbation of the outgoing null expansion $\tilde{\chi}'$, which we cite here: for the following perturbation of spacelike surfaces,

$$
(418) \quad \delta(s=0, \tilde{f}) = s=0, 2\tilde{f} - s=0, 1\tilde{f}, \quad \delta(\tilde{f}) = 2\tilde{f} - 1\tilde{f}.
$$
the linearized perturbation of \( \bar{\text{tr}} \bar{x}' \) is given by

\[
\delta_{l,Sch} \{ \bar{\text{tr}} \bar{x}' \} = (\partial_s \text{tr} \chi_{Sch}', \xi) \delta \{ \bar{f} \} + (\partial_s \text{tr} \chi_{Sch}', \xi) \delta \{ \bar{f} \} - 2r_{Sch}^2 (s = \frac{\bar{f}}{r}, \bar{s} = \frac{\bar{f}}{r}) \Delta \delta \{ \bar{f} \},
\]

hence we construct the following linearization of the map \( t \),

\[
\delta_{l,Sch} \{ t \} \left[ s = 0, \frac{\bar{f}}{r} \right] \left( \delta \{ s = 0, \frac{\bar{f}}{r} \} \right) \delta \{ \bar{f} \} = \partial_s \text{tr} \chi_{Sch}' \cdot \delta \{ \bar{f} \} + \partial_s \text{tr} \chi_{Sch}' \cdot \delta \{ \bar{f} \} - 2r_{Sch}^2 (s = \frac{\bar{f}}{r}, \bar{s} = \frac{\bar{f}}{r}) \Delta \delta \{ \bar{f} \},
\]

and the partial linearizations of the map \( t \)

\[
\delta_{s=0} \left[ s = 0, \frac{\bar{f}}{r} ) \right] \left( \delta \{ s = 0, \frac{\bar{f}}{r} \} \right) \delta \{ \bar{f} \} = \delta_{l,Sch} \{ t \} \left[ s = 0, \frac{\bar{f}}{r} \right] \left( \delta \{ s = 0, \frac{\bar{f}}{r} \} \right),
\]

\[
\delta_{f} \left[ s = 0, \frac{\bar{f}}{r} \right] \left( \delta \{ \bar{f} \} \right) = \delta_{l,Sch} \{ t \} \left[ s = 0, \frac{\bar{f}}{r} \right] \left( \delta \{ \bar{f} \} \right) - 2r_{Sch}^2 (s = \frac{\bar{f}}{r}, \bar{s} = \frac{\bar{f}}{r}) \Delta \delta \{ \bar{f} \},
\]

The error of the linearization of \( t \), which we denoted by \( e_{l,Sch} \{ t \} \), is

\[
e_{l,Sch} \{ t \} \left[ s = 0, 1, \frac{\bar{f}}{r} \right] \left( \delta \{ s = 0, \frac{\bar{f}}{r} \} \right) = \theta \left[ s = 0, \frac{\bar{f}}{r} \right] \left( \delta \{ \bar{f} \} \right) - \delta_{l,Sch} \{ t \} \left[ s = 0, 1, \frac{\bar{f}}{r} \right] \left( \delta \{ s = 0, \frac{\bar{f}}{r} \} \right) \left( \delta \{ \bar{f} \} \right),
\]

\[
e_{l,Sch} \{ t \} \left[ s = 0, 1, \frac{\bar{f}}{r} \right] \left( \delta \{ s = 0, \frac{\bar{f}}{r} \} \right) \left( \delta \{ \bar{f} \} \right),
\]

By lemma 9.8 the error satisfies the estimate

\[
\left\| e_{l,Sch} \{ t \} \left[ s = 0, 1, \frac{\bar{f}}{r} \right] \left( \delta \{ s = 0, \frac{\bar{f}}{r} \} \right) \right\|_{-1,p} \leq \frac{c(n,p)}{(r_0 + s_0)^2} \left( \frac{\partial \delta_0}{r_0} + \frac{\partial_0}{r_0 + s_0} \right) + \frac{c(n,p)}{r_0 + s_0} \left( \frac{\partial \delta_0}{r_0} + \frac{\partial_0}{r_0 + s_0} \right) + \frac{c(n,p)}{r_0 + s_0} \left( \frac{\partial \delta_0}{r_0} + \frac{\partial_0}{r_0 + s_0} \right) + \frac{c(n,p)}{(r_0 + s_0)^2} \left( \partial m + \partial o \right).}
\]
To simplify the calculation in the later sections, we introduce $e_{s,m}, e_{s,n}, e_{s,o}$ in equation (426) and rewrite the above estimate as follows:

\[(425) \quad \left\| e_{s,\text{Sch}} \{ t \} \left[ s=0, \tilde{f} \right] \left( \delta \{ s=0, \tilde{f} \}, \delta \{ \tilde{f} \} \right) \right\|_{n-1,p} \leq \frac{c(n,p)}{r_0 + s_0} (e_{s,m} \circ \delta_m + e_{s,n} \circ \delta_n + e_{s,o} \circ \delta_o) \]

\[+ \frac{c(n,p)}{r_0 + s_0} \left\{ s_0 + r_0 (\epsilon + \delta_o) (e_{s,m} \circ \delta_m + e_{s,n} \circ \delta_n + (\delta_m + \delta_n)) \right\} (e_{s,m} + e_{s,n} + e_{s,o}).\]

We have

\[(426) \quad e_{s,m} = \frac{1}{r_0 + s_0} \left\{ (\epsilon + \delta_s \circ \delta_m) r_0 + \delta_s^2 (r_0 + s_0) + \left( \delta_s^2 + \epsilon_{s,\text{Sch}} + \frac{r_0}{r_0 + s_0} \delta_o \right) s_0 \right\},\]

\[(427) \quad e_{s,n} = \frac{1}{r_0 + s_0} \left\{ (\epsilon + \delta_o) r_0 + \delta_o^2 (r_0 + s_0) + \left( \delta_o + \epsilon_{s,\text{Sch}} + \frac{r_0}{r_0 + s_0} \delta_o \right) s_0 \right\},\]

\[(428) \quad e_{s,o} = \epsilon + \delta_o + \frac{s_0}{r_0 + s_0} \delta^2_o,\]

\[(429) \quad e_{s,m} = \epsilon + \delta_o + \frac{r_0}{r_0 + s_0} \delta^2_o.\]

Finding a marginally trapped surface is equivalent to solving the equation

\[(430) \quad t^{(s=0, \tilde{f})} = 0\]

In the Schwarzschild spacetime, for any $s=0, \tilde{f}$, the above equation has a unique solution for $\tilde{f}$,

\[(431) \quad t^{(s=0, \tilde{f}, \tilde{f} = 0)} = 0.\]

In theorem 10.2 we prove that for any $s=0, \tilde{f}$, there exists a unique $\tilde{f}$ such that $(s=0, \tilde{f})$ is a solution of the equation. Given this to be true, we can construct a map from the space of the function $s=0, \tilde{f}$ to the space of the function $\tilde{f}$, which we denote by $\tilde{f}$,

\[(432) \quad f : W_{s,n+2,p}^{n+2,p} \rightarrow W_{s,n+2,p}^{n+2,p}, \quad s=0, \tilde{f} \mapsto f^{(s=0, \tilde{f})}, \quad t^{(s=0, \tilde{f}, f^{(s=0, \tilde{f})})} = 0.\]

We construct the following linearization of the map $f$

\[(433) \quad \delta_{s,\text{Sch}} \{ f \} \left[ s=0, \tilde{f} \right] \left( \delta \{ s=0, \tilde{f} \} \right) = \left\{ \partial^s \{ t \} \left[ s=0, \tilde{f}, f^{(s=0, \tilde{f})} \right] \right\}^{-1} \left( \partial_0 \{ t \} \left[ s=0, \tilde{f}, f^{(s=0, \tilde{f})}, (s=0, \tilde{f}) \right] \right).\]

10.2. The iteration scheme to solve $t^{(s=0, \tilde{f}, \tilde{f})} = 0$. Assume that $(s=0, \tilde{f})$ is a solution of the equation $t^{(s=0, \tilde{f}, \tilde{f})} = 0$. Consider a new function $s=0, \tilde{f}$ which is a perturbation of $s=0, \tilde{f}$. We want to solve the equation $t^{(s=0, \tilde{f}, \tilde{f})} = 0$ for $t^{(s=0, \tilde{f})}$. We solve the equation by constructing the following iteration sequence $\{ \tilde{f} \}$ using the linearization of
t,

\begin{align}
0 & = \tilde{f} \\
\sum_{k=1}^{n+1} k\tilde{f} - k\tilde{f} &= \left(\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, p\tilde{f}, k\tilde{f}}\right)^{-1} \left(0 - t_{\tilde{s}=0, p\tilde{f}, k\tilde{f}}\right).
\end{align}

To show the above iteration scheme is well defined, we need to show that the operator is invertible, which is given by the following lemma.

**Lemma 10.1.** Assume that

\begin{align}
\left\|d_{s=0,\tilde{f}}^{s=0,\tilde{f}}\right\|_{0,0}^{n+1,p} & \leq \delta_{r_0}, \\
\left\|d_{s=0,\tilde{f}}^{s=0,\tilde{f}}\right\| & \leq \delta_{n} r_0, \\
\tilde{f} & = s_0, \\
\left\|d\tilde{f}\right\| & \leq \delta_{o}(r_0 + s_0),
\end{align}

then for sufficiently small \(s_{n,0}^{-1}, r_{0}^{-1}, \delta_{o},\)

\begin{equation}
\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, p\tilde{f}, k\tilde{f}} : W^{n+2,p} \rightarrow W^{n,p}
\end{equation}

is invertible.

**Proof.** By definition

\begin{align}
\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, \tilde{f}} & \left[\tilde{f} \right] = \partial_{s} \chi_{s}^{s_{n} \tilde{f}}(\tilde{f}, \tilde{f}) \cdot \delta\{\tilde{f}\} - 2\tilde{r}_{0,s}^{2} \tilde{r}_{0,s}^{2} \hat{\Delta} \delta\{\tilde{f}\}
\end{align}

\begin{align}
\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, \tilde{f}} & \left[0,0\right] \left[\tilde{f}\right] = 2\tilde{r}_{0}^{2} - \delta\{\tilde{f}\} - 2\tilde{r}_{0}^{2} \hat{\Delta} \delta\{\tilde{f}\}
\end{align}

\(\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, \tilde{f}}\left[0,0\right]\) is clearly invertible, and to show that \(\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, \tilde{f}}\) is invertible, we estimate its difference with \(\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, \tilde{f}}\left[0,0\right]\).

\begin{align}
\left\|\partial_{s} \chi_{s}^{s_{n} \tilde{f}}(\tilde{f}, \tilde{f}) - \partial_{s} \chi_{s}^{s_{n} \tilde{f}}(0,0)\right\|_{0,0}^{n+1,p} & \leq c(n, p)\tilde{r}_{0}^{2}(s_{0}|\tilde{r}_{0}^{-1} + \delta_{o}),
\end{align}

\begin{align}
\left|\tilde{r}_{0,s}^{2} - \tilde{r}_{0}^{2}\right| & \leq c(n, p)\tilde{r}_{0}^{-3}|s_{0}|
\end{align}

Hence

\begin{align}
\left\|\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, \tilde{f}} - \partial_{\tilde{f}}\{t\}_{\tilde{s}=0, \tilde{f}}\left[0,0\right]\right\|_{0,0}^{n+2-n,p} & \leq c(n, p)\tilde{r}_{0}^{2}(s_{0}|\tilde{r}_{0}^{-1} + \delta_{o})
\end{align}

Hence for sufficiently small \(\delta_{n}, \delta_{o}, s_{0}\tilde{r}_{0}^{-1}, \delta_{o},\)

\(\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, \tilde{f}}\) is invertible, and

\begin{align}
\left\|\left(\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, \tilde{f}}\right)^{-1} - \left(\partial_{\tilde{f}}\{t\}_{\tilde{s}=0, \tilde{f}}\left[0,0\right]\right)^{-1}\right\|_{0,0}^{n-n-2,p} & \leq c(n, p)\tilde{r}_{0}^{2}(s_{0}|\tilde{r}_{0}^{-1} + \delta_{o}).
\end{align}
10.3. Boundedness of the iteration sequence in \( W^{n+2,p} \). In this section, we will show that the iteration sequence \( \{k_f^j\}_{j=0,1,\ldots} \) is bounded in \( W^{n+2,p} \). We assume that

\[
\|d{s=0,p\tilde{f}}\|^{n+1,p} \leq p_\delta^r r_0, \quad \|s{=0,p\tilde{f}}\| \leq p_\delta^m r_0,
\]

(444)

\[
k_f^j = s_{0,k}, \quad k_\delta^m = |s_{0,k}|r_0^{-1} \left\|q^k\|^{n+1,p} \leq k_\delta^o(r_0 + s_{0,k}),
\]

(445)

and moreover we make one additional assumption that

\[
k_\delta^m \leq \frac{1}{2},
\]

(446)

which will be justified after we get the estimate for \( \{k_f^j\}_{j=0,1,\ldots} \). With this additional assumption, \( r_0 \) and \( r_0 + s_{0,k} \) are comparable. In the following, we obtain the recurrence inequality (452) of \( k_\delta^m, k_\delta^o, k = 1, 2, 3, \ldots \). Then we can prove the boundedness of \( k_\delta^m, k_\delta^o \) from (452).

\[
k_f^{j+1} = k_f^j - \{\partial_f^j(s=0,p\tilde{f}, k_f^j)\}^{-1} \left(t(s=0,p\tilde{f}, k_f^j)\right)
\]

\[
= -\{\partial_f^j(s=0,p\tilde{f}, k_f^j)\}^{-1} \left(t(s=0,p\tilde{f}, k_f^j) - \partial_f^j(s=0,p\tilde{f}, k_f^j)(k_f^j) - \partial_s(s=0,p\tilde{f}, k_f^j)(s=0,p\tilde{f})\right)
\]

\[
+ \{\partial_f^j(s=0,p\tilde{f}, k_f^j)\}^{-1} \left(m_{s=0}(s=0,p\tilde{f}, k_f^j)(s=0,p\tilde{f})\right)
\]

\[
= -\{\partial_f^j(s=0,p\tilde{f}, k_f^j)\}^{-1} \left(e_{s=0}(s=0,p\tilde{f}, k_f^j)(s=0,p\tilde{f})\right)
\]

By lemma 9.8, the following holds

\[
\left\|e_{s=0}(s=0,p\tilde{f}, k_f^j)(s=0,p\tilde{f})\right\|^{n,p}
\]

\[
\leq c(n,p)r_0^{-1} \left(e_{p\delta_m}p_\delta^m + e_{p\delta_o}p_\delta^o + e_{\delta_m}k_\delta^m + e_{\delta_o}k_\delta^o\right)
\]

(447)

\[
+ c(n,p)r_0^{-1} \left(\left(k_\delta^m + \epsilon + k_\delta^o\right) \left(p_\delta^m + p_\delta^o\right) + k_\delta^m + k_\delta^o\right)
\]

and

\[
\partial_{s=0}(s=0,p\tilde{f}, k_f^j)(s=0,p\tilde{f}) = \partial_{s=0}(s=0,p\tilde{f}, k_f^j)(s=0,p\tilde{f}),
\]

(448)

\[
\left\|\partial_{s=0}(s=0,p\tilde{f}, k_f^j)(s=0,p\tilde{f})\right\|^{n+1,p} \leq c(n,p)r_0^{-2} \left(k_\delta^m + k_\delta^o\right) \left(p_\delta^m + p_\delta^o + k_\delta^m + k_\delta^o\right),
\]

(449)
We obtain the estimate of $k+\tilde{f}$

\begin{equation}
\|k+\tilde{f}\|^{n+2,p} 
\end{equation}

\begin{align*}
\leq c(n, p) r_0 \left( e_{\mathbf{m}} \frac{p_k \delta_m + e_{\mathbf{n}} \frac{p_k \delta_o + e_{\mathbf{m}} k \delta_m + e_{\mathbf{o}} k \delta_o} {p_k \delta_m} \right) 
\end{align*}

\begin{align*}
+ c(n, p) r_0 \left( \left( k \delta_m + \epsilon + k \delta_o \right) \left( p_k \delta_m + p_k \delta_o + k \delta_m + k \delta_o \right) \right) 
\end{align*}

\begin{align*}
+ c(n, p) r_0 \left( k \delta_m \left( p_k \delta_m + p_k \delta_o + k \delta_m + k \delta_o \right) \right) 
\end{align*}

Substituting $e_{\mathbf{m}}, e_{\mathbf{n}}, e_{\mathbf{o}},$

\begin{equation}
\|k+\tilde{f}\|^{n+2,p} 
\end{equation}

\begin{align*}
\leq c(n, p) r_0 \left( \epsilon + p_k \delta_o \right) \left( p_k \delta_m \right) 
\end{align*}

\begin{align*}
+ c(n, p) r_0 \left( k \delta_o \right) 
\end{align*}

\begin{align*}
+ c(n, p) r_0 \left( k \delta_m \right) 
\end{align*}

\begin{align*}
+ c(n, p) r_0 \left( k \delta_o \right) 
\end{align*}

\begin{align*}
+ c(n, p) r_0 \left( k \delta_m \right) 
\end{align*}

\begin{align*}
\leq c(n, p) r_0 \left( p_k \delta_m + p_k \delta_o \right) 
\end{align*}

\begin{align*}
+ c(n, p) r_0 \left( p_k \delta_m + p_k \delta_o \right) 
\end{align*}

\begin{align*}
+ c(n, p) r_0 \left( k \delta_m \right) 
\end{align*}

\begin{align*}
+ c(n, p) r_0 \left( k \delta_o \right) 
\end{align*}

By the above estimate, we have that for sufficiently small $0 \delta_m, 0 \delta_o, p_k \delta_m, p_k \delta_o,$ and $k \geq 0,$

\begin{equation}
2k+1 \delta_m + 2k+1 \delta_o, 2k+2 \delta_m + 2k+2 \delta_o 
\end{equation}

\begin{align*}
\leq c(n, p) \epsilon \left( p_k \delta_m + p_k \delta_o \right) + c(n, p) \left\{ \epsilon + p_k \delta_o + p_k \delta_m + 0 \delta_m + 0 \delta_o \right\}^k \left( 0 \delta_m + 0 \delta_o \right) 
\end{align*}

The above estimate can be shown by induction. Moreover, if $s=0 \tilde{f}=0, 0 \tilde{g}=0,$ then we can choose $0 \delta_m = 0 \delta_o = 0,$ hence we get the better bound

\begin{equation}
2k+1 \delta_m + 2k+1 \delta_o, 2k+2 \delta_m + 2k+2 \delta_o \leq c(n, p) \epsilon \left( p_k \delta_m + p_k \delta_o \right) . 
\end{equation}
So we prove the boundedness of \( \{k_j^\infty\}_{k=0,1,\ldots} \) in \( W^{n+2,p} \) under the assumption \( k\delta_m \leq \frac{1}{2} \).  
This assumption is justified by the estimate (452), provided that

\[
0\delta_m + 0\delta_o \leq c(n,p) (0\delta_m + 0\delta_o) \leq \frac{1}{4},
\]

(454)

\[
\{\epsilon + p\delta_o + \delta_m + 0\delta_o\}^k \leq \left\{\epsilon + p\delta_o + \delta_m + \frac{1}{4}\right\}^k \leq 1
\]

(455)

\[
c(n,p)\epsilon (p\delta_m + p\delta_o) \leq \frac{1}{4}.
\]

(456)

In section [10.0] we will show that for sufficiently small \( \epsilon, p\delta_m, p\delta_o \), the above bounds are satisfied, hence the assumption \( k\delta_m \leq \frac{1}{2} \) is justified for sufficiently small \( \epsilon, p\delta_m, p\delta_o \).

10.4. Strong convergence of the iteration sequence in \( W^{n+1,p} \). In this section, we will show that the iteration sequence \( \{k_j^\infty\}_{k=0,1,\ldots} \) converges in \( W^{n+1,p} \). It is crucial that we need the boundedness in \( W^{n+2,p} \) to prove the strong convergence in \( W^{n+1,p} \). We do not have the strong convergence of the sequence in \( W^{n+2,p} \). Assume that

\[
\|s=0,p\overline{f} - s=0,\overline{f}\|^{n,p} \leq p\mathfrak{d}_0\mathfrak{r}_0, \quad \|s=0,p\overline{f} - s=0,\overline{f}\|^{p} = p\mathfrak{d}_0\mathfrak{r}_0.
\]

(457)

\[
\|k+1\overline{f} - k\overline{f}\|^{n,p} \leq k\mathfrak{d}_0(r_0 + s_0,k), \quad \|k+1\overline{f} - k\overline{f}\|^{p} = k\mathfrak{d}_m(r_0 + s_0,k).
\]

(458)

Recall that

\[
k^{k+2}\overline{f} - k^{k+1}\overline{f} = \left\{\partial_j\{t\}_{[s=0,p\overline{f},k^{k+1}\overline{f}]^{-1}\left(0 - t(s=0,p\overline{f},k^{k+1}\overline{f}\right),
\]

(459)

\[
k^{k+1}\overline{f} - k\overline{f} = \left\{\partial_j\{t\}_{[s=0,p\overline{f},k\overline{f}]^{-1}\left(0 - t(s=0,p\overline{f},k\overline{f}\right),
\]

(460)

then by taking the difference of the above and following the general idea proving the convergence of the Picard iteration in the implicit function theorem, we get

\[
k^{k+2}\overline{f} - k^{k+1}\overline{f} = \left\{\partial_j\{t\}_{[s=0,p\overline{f},k^{k+1}\overline{f}]^{-1}.
\]

(461)
In order to obtain the estimate of $k^2 + 2\tilde{f} - k^2 + 1\tilde{f}$ by the above equation, we need to estimate the two terms \( \left\{ \partial_j f(t) \right\}_{s=0, p_f^2, p_{\tilde{f}}} = 1/(\partial_{\tilde{f}} f(t)) \) and \( \left\{ \partial_j f(t) \right\}_{s=0, p_f^2, p_{\tilde{f}}} = 1/(\partial_{\tilde{f}} f(t)) \). The first term is estimated by the bound of the operator of lemma 9.8 on the estimate of the second term, we need to estimate the difference of the operators

\[
\left( \partial_j f(t) \right\}_{s=0, p_f^2, p_{\tilde{f}}} = 1/(\partial_{\tilde{f}} f(t)) - \left( \partial_j f(t) \right\}_{s=0, p_f^2, p_{\tilde{f}}} = 1/(\partial_{\tilde{f}} f(t))
\]

which is obtained in the estimate \( \left( \partial_j f(t) \right\}_{s=0, p_f^2, p_{\tilde{f}}} = 1/(\partial_{\tilde{f}} f(t)) \) by the following deductions. We can write the difference as the following,

\[
\left( \partial_j f(t) \right\}_{s=0, p_f^2, p_{\tilde{f}}} = 1/(\partial_{\tilde{f}} f(t)) - \left( \partial_j f(t) \right\}_{s=0, p_f^2, p_{\tilde{f}}} = 1/(\partial_{\tilde{f}} f(t))
\]

By definition,

\[
\partial_j f(t)_{s=0, p_f^2, p_{\tilde{f}}} - \partial_j f(t)_{s=0, p_f^2, p_{\tilde{f}}}
\]

\[
= (\partial_s \text{tr}_S f(k^2, k^2) - \partial_s \text{tr}_S f(k^2 + 1, k^2 + 1/2)) - 2 \left( r^{-2}(k^2, k^2) - r^{-2}(k^2 + 1, k^2 + 1/2) \right) \Delta.
\]

Since

\[
\left\| \partial_s \text{tr}_S f(k^2, k^2) - \partial_s \text{tr}_S f(k^2 + 1, k^2 + 1/2) \right\|^{n+1, p} = \left( r^{-2}(k^2, k^2) - r^{-2}(k^2 + 1, k^2 + 1/2) \right) \Delta,
\]

\[
\left( r^{-2}(k^2, k^2) - r^{-2}(k^2 + 1, k^2 + 1/2) \right)
\]

\[
\leq c(n, p) \left( r^{-2}(k^2, k^2) - r^{-2}(k^2 + 1, k^2 + 1/2) \right) \Delta
\]
where \( k, k+1 \delta_m = \max \{ k \delta_m, k+1 \delta_m \} \). Hence

\[
(467) \quad \left\| \frac{\partial f}{\partial \bar{z}} (t) [s=0, p \tilde{f}, k \tilde{f}] - \frac{\partial f}{\partial \bar{z}} (t) [s=0, p \tilde{f}, k+1 \tilde{f}] \right\|^{n+1-1} \leq c(n, p) \left( k \delta_m + k \delta_o \right),
\]

\[
(468) \quad \left\| \left( \frac{\partial f}{\partial \bar{z}} (t) [s=0, p \tilde{f}, k \tilde{f}] \right)^{-1} - \left( \frac{\partial f}{\partial \bar{z}} (t) [s=0, p \tilde{f}, k+1 \tilde{f}] \right)^{-1} \right\|^{n+1-1} \leq c(n, p) \left( k \delta_m + k \delta_o \right),
\]

We can estimate \( \| k^2 \tilde{f} - k+1 \tilde{f} \|^{n+1, p} \) now.

\[
(469) \quad \left\| \left( \frac{\partial f}{\partial \bar{z}} (t) [s=0, p \tilde{f}, k \tilde{f}] \right)^{-1} - \left( \frac{\partial f}{\partial \bar{z}} (t) [s=0, p \tilde{f}, k+1 \tilde{f}] \right)^{-1} \right\|^{n+1, p} \leq c(n, p) r_0^2 \left( k \delta_m + k \delta_o \right) \left\| t^2 = 0, p \tilde{f}, k \tilde{f} \right\|^{n+1, p} \leq c(n, p) r_0 \left( k \delta_m + k \delta_o \right) \left( k \delta_m + k \delta_o \right),
\]

\[
(470) \quad \left\| \left( \frac{\partial f}{\partial \bar{z}} (t) [s=0, p \tilde{f}, k+1 \tilde{f}] \right)^{-1} \left( e_{t, Sch} (t) [s=0, p \tilde{f}, k+1 \tilde{f}] \right) \right\|^{n+1, p} \leq c(n, p) r_0 \left\| \epsilon + k, k+1 \delta_m + k, k+1 \delta_o + p \delta_o \right\| \left( k \delta_m + k \delta_o \right) c(n, p) r_0 \left( k \delta_m + k \delta_o \right)^2,
\]

hence for \( k \geq 0 \),

\[
(471) \quad \left\| k^2 \tilde{f} - k+1 \tilde{f} \right\|^{n+1, p} \leq c(n, p) r_0 \left\| \epsilon + k, k+1 \delta_m + k, k+1 \delta_o + p \delta_o \right\| \left( k \delta_m + k \delta_o \right) c(n, p) r_0 \left( k \delta_m + k \delta_o \right)^2.
\]

To estimate \( \| k^2 \tilde{f} - k+1 \tilde{f} \|^{n+1, p} \) by the above inductive estimate, we still need to know \( \| \tilde{f} - 0 \tilde{f} \|^{n+1, p} \), which is estimated as follows,

\[
(472) \quad \left\| \tilde{f} - 0 \tilde{f} \right\|^{n+1, p} = \left\| \left( \frac{\partial f}{\partial \bar{z}} (t) [s=0, p \tilde{f}, 0 \tilde{f}] \right)^{-1} \left( t^2 = 0, p \tilde{f}, 0 \tilde{f} \right) \right\|^{n+1, p} \leq c(n, p) r_0 \left( \epsilon^2 + 0 \delta_m + 0 \delta_o \right) \left( \epsilon^2 + 0 \delta_m + 0 \delta_o \right).
\]

Now we can use the inductive estimate (471) and estimate (472) to get the estimate of \( \| k^2 \tilde{f} - k+1 \tilde{f} \|^{n+1, p} \). Here we make an assumption on \( k \delta_m, k \delta_o \), which will be justified in section 10.6

\[
(473) \quad k \delta_m + k \delta_o \leq c_0 \epsilon \left( 0, \tilde{f}, 0, \tilde{f} \right) \leq \epsilon,
\]

where \( 0, \tilde{f} = \max \left( 0 \delta, \tilde{f} \right) \). Notice if \( s = 0, \tilde{f} = 0 \tilde{f} = 0 \), then the above assumption is valid by estimate (453). Under the above assumption, from estimates (471) and (472), we
have

\[(474)\]
\[2k+1 \mathcal{D}_m + 2k+1 \mathcal{D}_o, 2k+2 \mathcal{D}_m + 2k+2 \mathcal{D}_o \leq c(n, p) \epsilon (\epsilon + c_o (0, p \mathcal{D}_m + 0, p \mathcal{D}_o)) (\epsilon + p \mathcal{D}_o)^k (p \mathcal{D}_m + p \mathcal{D}_o).
\[
\leq c(n, p) \epsilon (\epsilon + p \mathcal{D}_o)^k (p \mathcal{D}_m + p \mathcal{D}_o).
\]

Then for small \(\epsilon, \frac{p \mathcal{D}_o}{\epsilon}\), \(\{k \tilde{f}\}_{k=0,1,\ldots}\) converges in \(W^{n+1,p}\). Then we denote the limit of \(\{k \tilde{f}\}_{k=0,1,\ldots}\) by \(\tilde{f}\). For \(\epsilon \tilde{f} - \tilde{f}\), we have

\[(475)\]
\[\|\epsilon \tilde{f} - \tilde{f}\|^{n+1,p} \leq \sum_{k=0,1,\ldots} \|k+1 \tilde{f} - k \tilde{f}\|^{n+1,p} \leq c(n, p) \epsilon (\epsilon + c_o (0, p \mathcal{D}_m + 0, p \mathcal{D}_o)) (p \mathcal{D}_m + p \mathcal{D}_o).
\]

Since \(\{k \tilde{f}\}_{k=0,1,\ldots}\) is bounded in \(W^{n+2,p}\), the sequence converges to \(\epsilon \tilde{f}\) weakly, and

\[(476)\]
\[\|\epsilon \tilde{f}\|^{n+2,p} \leq \limsup_{k \to \infty} \|k \tilde{f}\|^{n+2,p} \leq c(n, p) \epsilon (p \mathcal{D}_m + p \mathcal{D}_o).
\]

Moreover, if \(s=0, \tilde{f} = 0\), the assumption \(473\) is satisfied automatically by estimate \(453\).

10.5. Solving \(t(s=0, \tilde{f}) = 0\): the limit \(\epsilon \tilde{f}\) of \(\{k \tilde{f}\}_{k=0,1,\ldots}\). In this section, we show that \(\epsilon \tilde{f}\), the limit of \(\{k \tilde{f}\}_{k=0,1,\ldots}\), solves the equation

\[(477)\]
\[t(s=0, \epsilon \tilde{f}, \epsilon \tilde{f}) = 0.
\]

We have

\[(478)\]
\[t(s=0, \epsilon \tilde{f}, \epsilon \tilde{f}) - t(s=0, \epsilon \tilde{f}, k \tilde{f}) + t(s=0, \epsilon \tilde{f}, k \tilde{f}) \]
\[= [t(s=0, \epsilon \tilde{f}, \epsilon \tilde{f}) - t(s=0, \epsilon \tilde{f}, k \tilde{f})] + \partial \{\tilde{t}\}|_{s=0, \epsilon \tilde{f}, k \tilde{f}} (k \tilde{f} - k+1 \tilde{f}).
\]

The first term on the right hand side converges to zero since \(k \tilde{f}\) converges to \(\epsilon \tilde{f}\) and the second one also converges to zero since \(k+1 \tilde{f} - k \tilde{f}\) converges to zero. We have the following arguments:

\[(479)\]
\[\|t(s=0, \epsilon \tilde{f}, \epsilon \tilde{f}) - t(s=0, \epsilon \tilde{f}, k \tilde{f})\|^{n+1,p} \leq c(n, p) r_0^{-2} \|\epsilon \tilde{f} - k \tilde{f}\|^{n+1,p}
\]

\[(480)\]
\[\|\partial \{\tilde{t}\}|_{s=0, \epsilon \tilde{f}, k \tilde{f}} (k \tilde{f} - k+1 \tilde{f})\|^{n-1,p} \leq c(n, p) r_0^{-2} \|\epsilon \tilde{f} - k+1 \tilde{f}\|^{n+1,p},
\]

by taking limits in \(W^{n-1,p}\), we get

\[(481)\]
\[t(s=0, \epsilon \tilde{f}, \epsilon \tilde{f}) = 0.
\]
There exist Theorem 10.2.

We have proven the existence of \( s = 0 \), \( \tilde{t}, \tilde{f} \) satisfy the following bound

\[
\| \tilde{d} s = 0 \|^{n+1,p} \leq \delta_n R_0, \quad \| \tilde{s} s = 0 \| \leq \delta_n R_0,
\]

\[
\| \tilde{d} \tilde{f} \|^{n+1,p} \leq \delta_n (R_0 + s_0), \quad \| \tilde{f} \|, \| \tilde{f}' \| \leq s_0.
\]

Then

\[
\tilde{f} - \tilde{f}' = \left\{ \partial_{\tilde{f}} \{ s = 0, \tilde{f} \} \right\}^{-1} \left( \left( t^{s = 0}, \tilde{f}' \right) - t^{s = 0}, \tilde{f} \right) - \partial_{\tilde{f}} \{ s = 0, \tilde{f} \} \left( \tilde{f}' - \tilde{f} \right)
\]

\[
= \left\{ \partial_{\tilde{f}} \{ s = 0, \tilde{f} \} \right\}^{-1} \left( e_{l, Sch} \{ t \}^{s = 0}, \tilde{f} \right) (\tilde{f}' - \tilde{f})
\]

hence

\[
\| \tilde{f} - \tilde{f}' \|^{n+1,p} \leq c(n, p) r_0^2 \| e_{l, Sch} \{ t \}^{s = 0}, \tilde{f} \| (0, \tilde{f}' - \tilde{f}) \|^{n-1,p}
\]

\[
\leq c(n, p) r_0 \left\{ \epsilon + \delta_o + \delta_n \right\} \| \tilde{f} - \tilde{f}' \|^{n+1,p} + c(n, p) r_0 \left( \| \tilde{f} - \tilde{f}' \|^{n+1,p} \right)^2
\]

Then for \( \epsilon, \delta_o, \delta_n \) sufficiently small, we can derive that

\[
\tilde{f} - \tilde{f}' = 0.
\]

Now we can justify the assumption (473). Suppose that \( t^{s = 0}, \tilde{f} \) = 0, then by the uniqueness of the solution, \( \tilde{f} \) can be obtained as the limit of the iteration sequence starting from \( s = 0 \), \( \tilde{f} = 0 \). Hence by the bound (476) obtained in the last section, we get the estimate for \( \tilde{f} \),

\[
\| \tilde{f} \|^{n+2,p} \leq c(n, p) \epsilon \left( \delta_n + \delta_o \right).
\]

Then the assumption (473) is justified for \( k = 0 \). For \( k \geq 1 \), the assumption can be justified by the estimate (452).

10.7. Analytic properties of the function \( f \). Recall the definition of the map \( f \),

\[
f : W^{n+2,p}_{\delta_n, \delta_o} \to W^{n+2,p}, \quad s = 0, \tilde{f} \to f^{s = 0}(\tilde{f}), \quad t^{s = 0}(\tilde{f}) = 0.
\]

We have proven the existence of \( f \), which we formulate as the following theorem.

**Theorem 10.2.** There exist \( \epsilon^*, \delta^*_n, \delta^*_o, \) depending on \( n \geq 3, p > 1 \) such that for any \( \epsilon \leq \epsilon^* \), \( \delta_n \leq \delta^*_n \), \( \delta_o \leq \delta^*_o \), there exists a map \( f \) such that

\[
f : W^{n+2,p}_{\delta_n, \delta_o} \to W^{n+2,p}, \quad s = 0, \tilde{f} \to f^{s = 0}(\tilde{f}), \quad t^{s = 0}(\tilde{f}) = 0.
\]
and the map \( f \) satisfies the following estimate

\[
\| f(s=0^{+}) \|_{n+2,p} \leq c(n,p) \epsilon \| s=0^{+} f \|_{n+2,p} .
\]

Assume \( s=0^{+} f = s=0^{+} g + \delta \{ s=0^{+} f \} \). Recall the linearization of the map \( f \),

\[
\delta_{i,Sch} \{ f \} \left[ s=0^{+} f \right] \left( \delta \{ s=0^{+} f \} \right)
= \left\{ \partial_{f}^{i} \{ f \}\left[ s=0^{+} f \right] \left( \delta \{ s=0^{+} f \} \right) \right\}^{-1} \left( -\partial_{s=0^{+}} \{ f \}\left[ s=0^{+} f \right] \right) .
\]

The error of the above linearization of \( f \) is

\[
e_{i,Sch} \{ f \} \left[ s=0^{+} f \right] \left( \delta \{ s=0^{+} f \} \right) = f(s=0^{+} f) - f(s=0^{+} f) - \delta_{i,Sch} \{ f \} \left[ s=0^{+} f \right] \left( \delta \{ s=0^{+} f \} \right).
\]

A special case is that \( s=0^{+} f = 0 \). In this case,

\[
\delta_{i,Sch} \{ f \} \left[ 0 \right] \left( \delta \{ s=0^{+} f \} \right)
= \left\{ \partial_{f}^{i} \{ f \}\left[ 0,0 \right] \right\}^{-1} \left( -\partial_{s=0^{+}} \{ f \}\left[ 0,0 \right] \right) = 0 .
\]

We have the following theorem on the error of the above constructed linearization of \( f \).

**Theorem 10.3.** Under the assumption of theorem 10.2

\[
\| e_{i,Sch} \{ f \} \left[ 0 \right] \left( \delta \{ s=0^{+} f \} \right) \|_{n+2,p} \leq c(n,p) \epsilon \| \delta \{ s=0^{+} f \} \|_{n+2,p} ,
\]

\[
\| e_{i,Sch} \{ f \} \left[ s=0^{+} f \right] \left( \delta \{ s=0^{+} f \} \right) \|_{n+1,p} \leq c(n,p) \epsilon \| \delta \{ s=0^{+} f \} \|_{n+1,p} ,
\]

\[
\| e_{i,Sch} \{ f \} \left[ s=0^{+} f \right] \left( -s=0^{+} f \right) \|_{n+2,p} \leq c(n,p) \epsilon \| s=0^{+} f \|_{n+2,p} .
\]

**Proof.** The error of the linearization of \( f \) at \( s=0^{+} f = 0 \) satisfies

\[
\| e_{i,Sch} \{ f \} \left[ 0 \right] \left( \delta \{ s=0^{+} f \} \right) \|_{n+2,p} = \| f \left( \delta \{ s=0^{+} f \} \right) \|_{n+2,p} \leq c(n,p) \epsilon \| \delta \{ s=0^{+} f \} \|_{n+2,p} .
\]

For the general case,

\[
e_{i,Sch} \{ f \} \left[ s=0^{+} f \right] \left( \delta \{ s=0^{+} f \} \right)
= \left\{ \partial_{f} \{ f \}\left[ s=0^{+} f \right] \right\}^{-1} \left( \delta_{i,Sch} \{ f \} \left[ s=0^{+} f \right] \left( f(s=0^{+} f) - f(s=0^{+} f) \right) \right)
= \left\{ \partial_{f} \{ f \}\left[ s=0^{+} f \right] \right\}^{-1} \left( \delta_{i,Sch} \{ f \} \left[ s=0^{+} f \right] \left( f(s=0^{+} f) - f(s=0^{+} f) \right) \right)\delta \{ s=0^{+} f \} \right) .
\]
Then by estimate (425), we have

\[
\left\| e_{l,Sch}\{f\} \left[ \delta^{\{s=0\}_{\mathbb{I}}} \right] \right\|_{n+1,p}^{n+1,p} \\
\leq c(n, p) r_0 \left\{ \epsilon + \delta_0 + (\delta_0 + \epsilon \delta_m) \delta_m \right\} \left\| \delta^{\{s=0\}_{\mathbb{I}}} \right\|_{n+1,p}^{n+1,p} \\
+ c(n, p) r_0 \left\{ \epsilon + \delta_0 + \delta_0 \right\} \left\| f^{\{s=0,p\}_{\mathbb{I}}} - f^{\{s=0\}_{\mathbb{I}}} \right\|_{n+1,p}^{n+1,p} \\
+ c(n, p) r_0 \left\{ (\delta_m + \epsilon) \left\| \delta^{\{s=0\}_{\mathbb{I}}} \right\|_{n+1,p}^{n+1,p} + \left\| f^{\{s=0,p\}_{\mathbb{I}}} - f^{\{s=0\}_{\mathbb{I}}} \right\|_{n+1,p}^{n+1,p} \right\}.
\]

By estimates (491) and (475),

\[
\delta_m + \delta_0 \leq c(n, p) \epsilon (\delta_m + \delta_0),
\]

(501)

\[
\left\| f^{\{s=0,p\}_{\mathbb{I}}} - f^{\{s=0\}_{\mathbb{I}}} \right\|_{n+1,p}^{n+1,p} \leq c(n, p) \epsilon (\epsilon + \delta_m + \delta_0) \left\| \delta^{\{s=0\}_{\mathbb{I}}} \right\|_{n+1,p}^{n+1,p},
\]

we get

(502)

\[
\left\| e_{l,Sch}\{f\} \left[ \delta^{\{s=0\}_{\mathbb{I}}} \right] \right\|_{n+1,p}^{n+1,p} \leq c(n, p) \epsilon \left\| \delta^{\{s=0\}_{\mathbb{I}}} \right\|_{n+1,p}^{n+1,p}.
\]

If \( s=0, p=0 \), i.e. \( \delta^{\{s=0\}_{\mathbb{I}}} = - s=0\), then we can improve all the estimates above to the stronger Sobolev space \( W^{n+2,p} \), therefore we obtain the estimate of the error in \( W^{n+2,p} \), i.e.

(503)

\[
\left\| e_{l,Sch}\{f\} \left[ s=0\right] \right\|_{n+1,p}^{n+1,p} \leq c(n, p) \epsilon \left\| s=0\right\|_{n+1,p}^{n+1,p}.
\]

We conclude the estimates for the error of the linearization of the map \( f \) in the theorem. □

**APPENDIX A. ANALYTIC TOOLS**

**A.1. Function spaces on spheres.** Let \((\mathbb{S}^2, \mathbb{g})\) be the unit sphere embedded in the Euclidean space \( \mathbb{E}^3 \) and \( \{R_1, R_2, R_3\} \) be the rotation vector fields tangential to \( \mathbb{S}^2 \) which are

\[
R_i = \epsilon_{ijk} x^j \partial_k.
\]

We define the Sobolev norms \( \|f\|^{n,p} \) and \( \|\cdot\|_R^{n,p} \) on \( \mathbb{S}^2 \)
Definition A.1. Let $f$ be a function on $S^2$, for any $n \in \mathbb{N}, p > 1$ we define $\|f\|_{R}^{n,p}$ as

\[
\|f\|_{R}^{n,p} = \sum_{k=0}^{n} \left\{ \int_{S^2} \left| \frac{\partial^k}{\partial \nu^k} f \right|^p \, d\nu \right\}^{\frac{1}{p}},
\]

\[
\|f\|_{R}^{n,p} = \sum_{k=0}^{n} \left\{ \int_{S^2} \left| \nabla f \right|^p \, d\nu \right\}^{\frac{1}{p}},
\]

\[
\|f\|_{R}^{n,\infty} = \sum_{k=0}^{n} \left\{ \sup_{S^2} \left| \nabla f \right| \right\}^{\frac{1}{p}},
\]

\[
\|f\|_{R}^{n,\infty} = \sum_{k=0}^{n} \left\{ \sup_{S^2} \left| \nabla f \right| \right\}^{\frac{1}{p}}.
\]

Let $\xi$ be a $(l, s)$ tensor field on $S^2$, we define $\|\xi\|^{n,p}$ and $\|\xi\|_{R}^{n,p}$ as

\[
\|\xi\|^{n,p} = \sum_{k=0}^{n} \left\{ \int_{S^2} \left| \frac{\partial^k}{\partial \nu^k} \xi \right|^p \, d\nu \right\}^{\frac{1}{p}},
\]

\[
\|\xi\|_{R}^{n,p} = \sum_{k=0}^{n} \left\{ \int_{S^2} \left| \nabla \xi \right|^p \, d\nu \right\}^{\frac{1}{p}},
\]

\[
\|\xi\|_{R}^{n,\infty} = \sum_{k=0}^{n} \left\{ \sup_{S^2} \left| \nabla \xi \right| \right\}^{\frac{1}{p}},
\]

\[
\|\xi\|_{R}^{n,\infty} = \sum_{k=0}^{n} \left\{ \sup_{S^2} \left| \nabla \xi \right| \right\}^{\frac{1}{p}}.
\]

The two Sobolev norms $\|\cdot\|^{n,p}$ and $\|\cdot\|_{R}^{n,p}$ are actually equivalent.

Proposition A.2. There exist constants $c(n, p)$ for any $n \in \mathbb{N}, p > 1$ such that for any function $f$ on $S^2$

\[
c(n, p)^{-1} \|f\|^{n,p} \leq \|f\|_{R}^{n,p} \leq c(n, p) \|f\|^{n,p}.
\]

There also exist constants $c(n, p, l, s)$ for any $n, l, s \in \mathbb{N}, p > 1$ such that for any $(l, s)$ tensor field $\xi$ on $S^2$

\[
c(n, p, l, s)^{-1} \|\xi\|^{n,p} \leq \|\xi\|_{R}^{n,p} \leq c(n, p, l, s) \|\xi\|^{n,p}.
\]

The above proposition follows from the lemma 11.2 in [C], of which we cite the simplest cases in the following.

Lemma A.3. Let $f$ be a function on $S^2$. Then we have

\[
\sum_{i=1,2,3} (R_i f)^2 = |d f|^2_g.
\]
Let \( \xi \) be a 1-form on \( S^2 \). Then we have
\[
\sum_{i=1,2,3} |\mathcal{L}_g \xi|^2 = |\nabla_g \xi|^2 + |\xi|^2.
\]

We can also define Sobolev norms on \( S^2 \) with respect to a different metric \( \hat{g} \).

**Definition A.4.** Let \( f \) be a function on \( S^2 \), for any \( n \in \mathbb{N}, p > 1 \) we define \( \| f \|^{n,p}_{\hat{g}} \) and \( \| f \|^{n,p}_{R,\hat{g}} \) as
\[
\| f \|^{n,p}_{\hat{g}} = \sum_{k=0}^{n} \left\{ \int_{S^2} |\nabla^k f|^{p}_{\hat{g}} \text{dvol}_{\hat{g}} \right\}^{\frac{1}{p}},
\]
\[
\| f \|^{n,p}_{R,\hat{g}} = \sum_{k=0}^{n} \left\{ \int_{S^2} \sum_{i_1,\ldots,i_k=1,2,3} |R_{i_1} \cdots R_{i_k} f|^p \text{dvol}_{\hat{g}} \right\}^{\frac{1}{p}},
\]
\[
\| f \|^{n,\infty}_{\hat{g}} = \sum_{k=0}^{n} \sup_{S^2} |\nabla^k f|_{\hat{g}},
\]
\[
\| f \|^{n,\infty}_{R,\hat{g}} = \sum_{k=0}^{n} \sum_{i_1,\ldots,i_k=1,2,3} \sup_{S^2} |R_{i_1} \cdots R_{i_k} f|.
\]

Let \( \xi \) be a \((l,s)\) tensor field on \( S^2 \), we define \( \| \xi \|^{n,p}_{\hat{g}} \) and \( \| \xi \|^{n,p}_{R,\hat{g}} \) as
\[
\| \xi \|^{n,p}_{\hat{g}} = \sum_{k=0}^{n} \left\{ \int_{S^2} |\nabla^k \xi|^{p}_{\hat{g}} \text{dvol}_{\hat{g}} \right\}^{\frac{1}{p}},
\]
\[
\| \xi \|^{n,p}_{R,\hat{g}} = \sum_{k=0}^{n} \left\{ \int_{S^2} \sum_{i_1,\ldots,i_k=1,2,3} |\mathcal{L}_{R_{i_1}} \cdots \mathcal{L}_{R_{i_k}} \xi|^p \text{dvol}_{\hat{g}} \right\}^{\frac{1}{p}},
\]
\[
\| \xi \|^{n,\infty}_{\hat{g}} = \sum_{k=0}^{n} \sup_{S^2} |\nabla^k \xi|_{\hat{g}},
\]
\[
\| \xi \|^{n,\infty}_{R,\hat{g}} = \sum_{k=0}^{n} \sum_{i_1,\ldots,i_k=1,2,3} \sup_{S^2} |\mathcal{L}_{R_{i_1}} \cdots \mathcal{L}_{R_{i_k}} \xi|.
\]

Comparing the Sobolev norms with respect to different metrics, we have proposition

**Proposition A.5.** Assume that for given \( n \geq 1, p > 1 \),
\[
\| \nabla \hat{g} \|^{n,p} \leq \delta, \quad \left| \log \frac{\det \hat{g}}{\det g} \right| \leq \delta,
\]
then there exist \( c(n,p,\delta) \) and \( c(n,p,l,s,\delta) \) such that for any \( m \leq n \)
\[
(1 + c(n,p)\delta)^{-1} \| f \|^{m+2,p}_{\hat{g}} \leq \| f \|^{m+2,p}_{\hat{g}} \leq (1 + c(n,p)\delta) \| f \|^{m+2,p}_{R,\hat{g}},
\]
\[
(1 + c(n,p)\delta)^{-1} \| f \|^{m+2,p}_{R,\hat{g}} \leq \| f \|^{m+2,p}_{R,\hat{g}} \leq (1 + c(n,p)\delta) \| f \|^{m+2,p}_{R,\hat{g}}.
\]
and

\[(529)\quad (1 + c(n, p, l, s)\delta)^{-1} \|\xi\|^{m+1,p} \leq \|\xi\|_{g}^{m+1,p} \leq (1 + c(n, p, l, s)\delta) \|\xi\|^{m+1,p},\]

\[(530)\quad (1 + c(n, p, l, s)\delta)^{-1} \|\xi\|_{R}^{m+1,p} \leq \|\xi\|_{R,g}^{m+1,p} \leq (1 + c(n, p, l, s)\delta) \|\xi\|_{R}^{m+1,p},\]

The above proposition follows from the following lemma.

**Lemma A.6.** Let \(\{\theta^1, \theta^2\}\) be any coordinate chart of \(S^2\), and denote the Christoffel symbols of \(\tilde{g}\) and \(g\) by \(\hat{\Gamma}^k_{ij}\) and \(\Gamma^k_{ij}\) respectively. Then the difference of Christoffel symbols \(\hat{\Gamma}^k_{ij} - \Gamma^k_{ij}\) is a \((2,0)\) tensor field and

\[(531)\quad \hat{\Gamma}^k_{ij} - \Gamma^k_{ij} = \frac{1}{2} (\tilde{g}^{-1})^{kl} \left( \nabla_i \tilde{g}_{jl} + \nabla_j \tilde{g}_{il} - \nabla_l \tilde{g}_{ij} \right).\]

Under the assumptions of proposition A.5, in addition we require \(n \geq 2, p > 2\), there exists \(c(n, p, \delta)\) such that

\[(532)\quad (1 + c(n, p)\delta)^{-1} \tilde{g} \leq \hat{g} \leq (1 + c(n, p)\delta) \tilde{g},\]

\[(533)\quad \left\| \hat{\Gamma}^k_{ij} - \Gamma^k_{ij} \right\|^{n-1,p} \leq c(n, p)\delta,
\]

and

\[(534)\quad (1 + c(n, p)\delta)^{-1} \leq K_{\hat{g}} \leq 1 + c(n, p)\delta,\]

\[(535)\quad \|dK\|^{n-2,p} \leq c(n, p)\delta.\]

**A.2. Elliptic theory on spheres.** In this section, we discuss the estimates for elliptic equations on \(S^2\). All the theories in this section are well known and we refer to [CK] [S] for detailed expositions. We consider the following equations on \((S^2, \hat{g})\),

\((H_0)\)

\(\hat{\Delta} \phi = f,\)

\((H_1)\)

\[\begin{cases}
\hat{\Delta} \zeta = f, \\
\hat{\text{curl}} \zeta = f, \\
\hat{\text{div}} \xi = F,
\end{cases}\]

where \(\phi, f, f_*\) are scalar functions, \(\zeta, F\) are vector fields and \(\xi\) is a 2-covariant symmetric tracefree tensor fields.

**Theorem A.7** (Solvability). Assume that the right hand sides of the above equations \(H_0 - H_2\) are in \(L^2(S^2, \hat{g})\). Then

\((H_6)\) is solvable if and only if \(\hat{T}^g = 0\). The solution \(\phi\) is unique up to a constant.

\((H_1)\) is solvable if and only if \(\hat{T}^g = \hat{T}^g_* = 0\). The solution is unique.

\((H_2)\) is solvable if and only if \(F\) is \(L^2\)-orthogonal to the set of conformal Killing fields of \((S^2, \hat{g})\). The solution \(\xi\) is unique.
Theorem A.8 ($L^2$ estimates). Assume that we are given solutions to equations $\{H_0\}$-(\textit{H}_2) with the right hand sides in $L^2(S^2, \hat{g})$. Then the following $L^2$-estimates hold.

\begin{align}
(536) & \int_{S^2} \left( |\nabla^2 \phi|_{\hat{g}} + K_{\hat{g}} |\nabla \phi|_{\hat{g}}^2 \right) d\text{vol}_{\hat{g}} = \int_{S^2} |f|^2 d\text{vol}_{\hat{g}}, \\
(537) & \int_{S^2} \left( |\nabla \zeta|_{\hat{g}} + K_{\hat{g}} |\zeta|_{\hat{g}}^2 \right) d\text{vol}_{\hat{g}} = \int_{S^2} (|f|^2 + |f_*|^2) d\text{vol}_{\hat{g}}, \\
(538) & \int_{S^2} \left( |\nabla \xi|_{\hat{g}} + 2K_{\hat{g}} |\xi|_{\hat{g}}^2 \right) d\text{vol}_{\hat{g}} = 2 \int_{S^2} |F|^2 d\text{vol}_{\hat{g}}.
\end{align}

Theorem A.9 (Uniformization theorem). Assume that

\begin{equation}
(539) (1 + \delta)^{-1} \leq K \leq 1 + \delta, \quad \|d K\|^n_{p, \delta} \leq \delta.
\end{equation}

Then there exist a conformal factor $\Omega > 0$ and a constant $c(n, p, \delta)$ such that

\begin{equation}
(540) \hat{g} = \Omega^2 \hat{\gamma}
\end{equation}

and

\begin{equation}
(541) |\log \Omega| \leq c(n, p, \delta), \quad \|d \Omega\|^n_{p, \delta} \leq c(n, p, \delta),
\end{equation}

where $\hat{\gamma}$ is a metric on $S^2$ with constant curvature 1. Moreover for $\delta$ sufficiently small, there exists a constant $c(n, p)$ such that we can choose the conformal factor satisfying

\begin{equation}
(542) |\log \Omega| \leq c(n, p) \delta, \quad \|d \Omega\|^n_{p, \delta} \leq c(n, p) \delta.
\end{equation}

Theorem A.10 ($L^p$-estimates). Assume that $\hat{g}$ satisfies the assumptions in proposition A.3, in addition we require that $n \geq 2, q > 2$ or $n \geq 3, q > 1$. Then there exist universal constants $c(n, p, \delta)$ such that the following $L^p$-estimate holds for the solutions to equations $\{H_0\}$-(\textit{H}_2) with the right hand sides in the Sobolev spaces $W^{m,p}_{\hat{g}}, m \leq n$,

\begin{align}
(543) & \left\| \phi - \phi^0 \right\|^{m+2,p}_{\hat{g}} \leq c(n, p, \delta) \|f\|^m_{p, \delta}, \\
(544) & \left\| \zeta \right\|^{m+1,p}_{\hat{g}} \leq c(n, p, \delta) \left( \|f\|^m_{p, \delta} + \|f_*\|^m_{p, \delta} \right), \\
(545) & \left\| \xi \right\|^{m+1,p}_{\hat{g}} \leq c(n, p, \delta) \|F\|^m_{p, \delta}.
\end{align}

A.3. Propagation equations and the Gronwall’s inequality. Consider the product space with a degenerate metric $(S^2 \times \mathbb{R}, \hat{g})$ and let $\{X\}$ be a family of vector fields tangential to $\{S^2 \times \{t\}\}$, and $\{f\}, \{a\}$ be families of functions on $\{S^2 \times \{t\}\}$. We consider the following equation

\begin{equation}
(546) \partial_t f + X^i \partial_i f = a.
\end{equation}

Lemma A.11. Define the one parameter family of diffeomorphisms $\{\varphi\}$ generated by $X$, i.e.

\begin{equation}
(547) \varphi : S^2 \rightarrow S^2, \quad \partial_t \varphi(\vartheta) = X(\vartheta), \quad t=0 \varphi = \text{Id}
\end{equation}

We define the push forward metric $\gamma$ of $\hat{g}$ via $\varphi$

\begin{equation}
(548) \gamma = \varphi_* \hat{g}, \quad \text{dvol}_{\gamma} = \varphi_* \text{dvol}_{\hat{g}}.
\end{equation}
Then there exists a family of functions \( \{ \phi \} \) such that
\[
\text{dvol}_g = t \text{dvol}_g,
\]
and \( \phi \) satisfies the equation
\[
\partial_t \phi + X^i \partial_i \phi = -\bar{o} \phi \div \bar{X}, \quad (\partial_t + X^i \partial_i) \log \phi = -\bar{o} \div \bar{X}.
\]
Assume that
\[
\sup_{\mathcal{S}^2} \left| \bar{o} \div \bar{X} \right| \leq \frac{\delta r_0}{(r_0 + t)^2},
\]
then
\[
|\log \phi| \leq \delta.
\]

Remark A.12. (552) is obtained by a simple integration
\[
|\log \phi| \leq \int_0^t \sup_{\mathcal{S}^2} \left| \bar{o} \div (t = t' \bar{X}) \right| \, dt' \leq \int_0^t \frac{\delta r_0}{(r_0 + t')^2} \, dt' = \frac{\delta r_0}{r_0} - \frac{\delta r_0}{r_0 + t} \leq \delta.
\]

Lemma A.13. Let \( \{ R_i, i = 1, 2, 3 \} \) be the rotation fields introduced in section A.1. Then
\[
\partial_t R_i f + \bar{X}(R_i f) + [R_i, \bar{X}] f = R_i \bar{a}.
\]
Moreover for a sequence of rotation vector fields \( \{ R_{i_1}, \ldots, R_{i_n} \} \), we have
\[
\partial_t (R_{i_1} \cdots R_{i_n} f) + \bar{X} (R_{i_1} \cdots R_{i_n} f)
= - \sum_{\{i_1, \ldots, i_n\} \cup \{k_1, \ldots, k_{n-s}\} = \{1, \ldots, n\} \atop i_1 < \ldots < i_s < k_1 < \ldots < k_{n-s}} [R_{k_1} \cdots R_{k_{n-s}}, \bar{X}] (R_{i_1} \cdots R_{i_s} f) + R_{i_1} \cdots R_{i_n} \bar{a},
\]
where
\[
[R_{k_1} \cdots R_{k_{n-s}}, \bar{X}] = \mathcal{L}_{R_{k_1}} \cdots \mathcal{L}_{R_{k_{n-s}}} \bar{X}.
\]

Lemma A.14. Consider equation (546) and assume that \( \bar{a} \in W^{n,p}_g \) and \( \bar{X} \in W^{n,p}_g \) where \( n \geq 1, p > 2 \) or \( n \geq 2, p > 1 \). Moreover we assume that
\[
\| \bar{X} \|_{n,p} \leq \frac{\delta r_0}{(r_0 + t)^2}, \quad 0 < \delta < \frac{1}{2}.
\]
Then there exists a constant \( c(n,p) \) such that
\[
\left| \frac{d}{dt} \left\| f \right\|_{R_{i_1} \cdots R_{i_n} \bar{g}}^{n,p} \right| \leq c(n,p) \frac{\delta r_0}{(r_0 + t)^2} \left\| f \right\|_{R_{i_1} \cdots R_{i_n} \bar{g}}^{n,p} + c(n,p) \left\| \bar{a} \right\|_{R_{i_1} \cdots R_{i_n} \bar{g}}^{n,p}.
\]
By the Gronwall’s inequality
\[
\left| f \right|_{R_{i_1} \cdots R_{i_n} \bar{g}}^{n,p} \leq \exp(c(n,p)\delta) \left\{ \left| f \right|_{\bar{g}}^{n,p} + \int_0^t \left| \left| f \right|_{R_{i_1} \cdots R_{i_n} \bar{g}}^{n,p} \right| \, dt' \right\},
\]
and by lemma A.11 there exists a new constant, still denoted by \( c(n,p) \), such that
\[
\left| f \right|_{R_{i_1} \cdots R_{i_n} \bar{g}}^{n,p} \leq c(n,p) \left\{ \left| f \right|_{\bar{g}}^{n,p} + \int_0^t \left| \left| f \right|_{R_{i_1} \cdots R_{i_n} \bar{g}}^{n,p} \right| \, dt' \right\}.
\]
Remark A.15. The following equality is used in proving the above lemma

\[
\frac{d}{dt} \int_{S^2} |f|^p \, d\text{vol}_{\varphi^* g} = \frac{d}{dt} \int_{S^2} |f \circ \varphi|^p \, d\text{vol}_{\varphi^* g} \\
= \int_{S^2} |f|^{p-1} \partial_t (f \circ \varphi) \, d\text{vol}_{\varphi^* g} \\
= \int_{S^2} |f \circ \varphi|^{p-1} (\partial_t f + X^f) \circ \varphi \, d\text{vol}_{\varphi^* g} \\
= \int_{S^2} |f|^{p-1} (\partial_t f + X^f) \, d\text{vol}_{\varphi^* g} \\
= \int_{S^2} |f|^{p-1} \, d\text{vol}_{\varphi^* g}.
\]

(561)

Similar equations holds for higher order derivatives of \(f\).

Now recall definition 2.2 and consider a given null hypersurface \(C_{\varphi_0}\) in \((M, g)\). We assume that the parameter \(s\) on \(C_{\varphi_0}\) is a parameter by area radius, i.e.

\[
r(s, \varphi = 0) = r(s = 0, \varphi = 0) + s.
\]

(562)

Let \(\{s, \vartheta\}\) be a coordinate system on \(C_{\varphi_0}\), hence this coordinate system introduce a diffeomorphism between \(C_{\varphi_0}\) and \(S^2 \times [-\epsilon, +\infty)\). In another coordinate system \(\{s, \vartheta^1, \vartheta^2\}\), the intrinsic metric of \(C_{\varphi_0}\) which is degenerated takes the form

\[
g|_{C_{\varphi_0}} = (\hat{g})_{ab} (d\vartheta^a - \eta^a ds) \otimes (d\vartheta^b - \eta^b ds).
\]

(563)

and the tangential null vector field \(L^s\) is

\[
L^s = \partial_s + \hat{\vartheta}.
\]

(564)

Notice we change our notation a bit by adding the index \(s\) to emphasis quantities on \(\Sigma_{s, \varphi_0}\). Assume that \(\{\varphi\}, \{\vartheta\}\) is a family of functions on \(\Sigma_{s, \varphi_0}\) and satisfies the equation

\[
L^s \varphi = \partial_s \varphi + \hat{\vartheta} \partial_s \varphi = -\lambda_0 \text{tr}_\Sigma \varphi + \vartheta.
\]

(565)

Then we have, Apply lemma A.11 to the vector field \(\hat{\vartheta}\), then we have

\[
\mathcal{L}_{L^s} (|\varphi|^p \, d\text{vol}_g) = p |\varphi|^{p-2} \varphi \mathcal{L}_{L^s} (\varphi) \, d\text{vol}_g \\
= |\varphi|^{p-2} \varphi \left\{-p\lambda_0 \text{tr}_\Sigma \varphi \varphi + p \vartheta \right\} \, d\text{vol}_g \\
= \left\{-p\lambda_0 \text{tr}_\Sigma |\varphi|^p + p \vartheta |\varphi|^{p-2} \varphi \right\} \, d\text{vol}_g,
\]

hence

\[
\frac{d}{ds} \left\{ \int_{\Sigma_{s, \varphi_0}} |\varphi|^p \, d\text{vol}_g \right\} = \int_{\Sigma_{s, \varphi_0}} \mathcal{L}_{L^s} (|\varphi|^p \, d\text{vol}_g) \\
= \int_{\Sigma_{s, \varphi_0}} \left\{-p\lambda_0 \text{tr}_\Sigma |\varphi|^p + p \vartheta |\varphi|^{p-2} \varphi \right\} \, d\text{vol}_g.
\]

(567)
which implies

\[
\frac{d}{ds} \left\{ \int_{\Sigma_{s, \Sigma_0}} |\mathbf{f}|^p \, d\text{vol}_y \right\} + p\lambda_0 \int_{\Sigma_{s, \Sigma_0}} \mathbf{tr}^X |\mathbf{f}|^p \, d\text{vol}_y \leq \int_{\Sigma_{s, \Sigma_0}} \left| p \mathbf{u} |\mathbf{f}|^{p-2} \mathbf{f} \right| \, d\text{vol}_y
\]

(568)

\[
\frac{d}{dt} \left\{ \|\mathbf{f}\|_y^p \right\} + \lambda_0 t \mathbf{tr}^X |\mathbf{f}|^p \|\mathbf{f}\|_y^p \leq c(p) \|\mathbf{u}\|_y^p + \frac{\lambda_0 \left( \mathbf{tr}^X - \mathbf{tr}^X Y \right) |\mathbf{f}|^p}{\|\mathbf{f}\|_y^{p-1}},
\]

then by the Gronwall’s inequality we have the following lemma

**Lemma A.16.** Assume \( |\mathbf{f}| \) satisfies equation (568) then for any \( p > 1 \), there exists a constant \( c(p) \) such that

(570)

\[
\frac{d}{dt} \left\{ \|\mathbf{f}\|_y^p \right\} + \lambda_0 t \mathbf{tr}^X |\mathbf{f}|^p \|\mathbf{f}\|_y^p \leq c(p) \|\mathbf{u}\|_y^p + \frac{\lambda_0 \left( \mathbf{tr}^X - \mathbf{tr}^X Y \right) |\mathbf{f}|^p}{\|\mathbf{f}\|_y^{p-1}}.
\]

Moreover if \( \mathbf{tr}^X \mathbf{f} \) satisfies

\[
|\mathbf{tr}^X - \mathbf{tr}^X Y| \leq \frac{\delta r_0}{r_s^2},
\]

then

(572)

\[
\frac{d}{dt} \left( r_s^{2\lambda_0} \|\mathbf{f}\|_y^p \right) \leq c(p) r_s^{2\lambda_0} \|\mathbf{u}\|_y^p + \frac{\delta r_0}{r_s^2} \left( r_s^{2\lambda_0} \|\mathbf{f}\|_y^p \right),
\]

and by the Gronwall’s inequality, there exists a constant \( c(p, \delta) \) such that

(573)

\[
r_s^{2\lambda_0} \|\mathbf{f}\|_y^p \leq c(p, \delta) \left\{ r_0^{2\lambda_0} \|\mathbf{f}\|_y^p + \int_0^s r_s^{2\lambda_0} \|\mathbf{x}\|_y^p \, ds \right\}
\]

For the Sobolev norms of \{\( |\mathbf{f}| \)\}, we have the corresponding lemma.

**Lemma A.17.** Let \( \{\mathbf{R}_i, i = 1, 2, 3\} \) be the rotation fields introduced in section A.1. Then

(574)

\[
\partial_t \mathbf{R}_i \mathbf{f} + \mathbf{X} (\mathbf{R}_i \mathbf{f}) + [\mathbf{R}_i, \mathbf{X}] \mathbf{f} = \mathbf{R}_i \mathbf{tr}^X Y \mathbf{f} - \lambda_0 \mathbf{tr}^X \mathbf{R}_i \mathbf{f} - \mathbf{R}_i \mathbf{u}.
\]

Moreover for a sequence of rotation vector fields \( \{\mathbf{R}_1, \cdots, \mathbf{R}_n\} \), we have

(575)

\[
\partial_t (\mathbf{R}_1 \cdots \mathbf{R}_n \mathbf{f}) + \mathbf{X} (\mathbf{R}_1 \cdots \mathbf{R}_n \mathbf{f}) - \lambda_0 \mathbf{tr}^X (\mathbf{R}_1 \cdots \mathbf{R}_n \mathbf{f}) - \sum_{\{1, \cdots, n \} \cup \{k_1, \cdots, k_{n-s} \} = \{1, \cdots, n\}} [\mathbf{R}_{k_{1}}, \cdots, \mathbf{R}_{k_{n-s}}] \mathbf{X} (\mathbf{R}_{k_{1}} \cdots \mathbf{R}_{k_{n-s}} \mathbf{f}) = - \lambda_0 \mathbf{tr}^X (\mathbf{R}_1 \cdots \mathbf{R}_n \mathbf{f}) - \sum_{\{1, \cdots, n \} \cup \{k_1, \cdots, k_{n-s} \} = \{1, \cdots, n\}} [\mathbf{R}_{k_{1}}, \cdots, \mathbf{R}_{k_{n-s}}] \mathbf{X} (\mathbf{R}_{k_{1}} \cdots \mathbf{R}_{k_{n-s}} \mathbf{f}) + \mathbf{R}_i \cdots \mathbf{R}_n \mathbf{u}.
\]
Lemma A.18. Consider equation (565) and assume that \( a, \text{tr}^{\chi} \in W^{n,p}_{\tilde{g}} \) and \( b \in W^{n,p}_{\tilde{g}} \) where \( n \geq 1, p > 2 \) or \( n \geq 2, p > 1 \). Moreover we assume that

\[
\| d \text{tr}^{\chi} \|_{n-1,p} \leq \frac{\delta r_0}{r^2}, \quad \| b \|_{n,p} \leq \frac{\delta r_0}{r^2}, \quad 0 < \delta < \frac{1}{2}.
\]

Then there exists a constant \( c(n,p) \) such that

\[
\frac{d}{dt} \| f \|_{R \cdot \tilde{g}} + \lambda_0 \| \text{tr}^{\chi} \|_{n,p} \leq c(n,p) \frac{\delta r_0}{r^2} \| f \|_{R \cdot \tilde{g}} + c(n,p) \| b \|_{n,p},
\]

which implies

\[
\frac{d}{dt} \left( r_2^{\lambda_0} \| f \|_{R \cdot \tilde{g}} \right) \leq c(n,p) \frac{\delta r_0}{r^2} \left( r_2^{\lambda_0} \| f \|_{R \cdot \tilde{g}} \right) + c(n,p) \left( r_2^{\lambda_0} \| b \|_{n,p} \right),
\]

and by the Gronwall’s inequality, there exists a constant \( c(n,p) \) such that

\[
r_2^{2\lambda_0} \| f \|_{R \cdot \tilde{g}} \leq c(n,p) \left\{ r_2^{2\lambda_0} \| f \|_{R \cdot \tilde{g}} + \int_0^s r_2^{2\lambda_0} \| s \|_{s} \| b \|_{n,p} \, ds \right\}.
\]

Appendix B. Proof of Lemmas 4.6 and 4.7

We introduce the rotation fields \( \bar{R}_i, i = 1, 2, 3 \) on \( \bar{\Sigma}_s \).

Lemma B.1. Let \( T \) be any tensor field on \( (M, g) \) and \( T|_{\bar{\Sigma}_s} \) is the restriction of \( T \) on \( \bar{\Sigma}_s \). Then the Lie derivatives of \( T|_{\bar{\Sigma}_s} \) with respect to the rotation fields \( \bar{R} \) is

\[
\mathcal{L}_{\bar{R}_i} T_{j_1 \cdots j_k} = \mathcal{L}_{\bar{R}_i} T_{j_1 \cdots j_k} + \bar{R}_i \bar{\mathcal{L}}_{\bar{\partial}_j} T_{j_1 \cdots j_k}
\]

and the covariant derivative of \( T \) restricted on \( \bar{\Sigma}_s \) is

\[
\bar{\nabla}_j T_{j_1 \cdots j_k} = \bar{\nabla}_j T_{j_1 \cdots j_k} + \bar{\nabla}_j \bar{\mathcal{L}}_{\bar{\partial}_j} T_{j_1 \cdots j_k}.
\]

Moreover, for higher order Lie derivatives, we have

\[
\mathcal{L}_{\bar{R}_{i_1}} \cdots \mathcal{L}_{\bar{R}_{i_m}} T_{j_1 \cdots j_k} = \left( \mathcal{L}_{\bar{R}_{i_1}} + \bar{R}_{i_1} \bar{\mathcal{L}}_{\bar{\partial}_j} \right) \cdots \left( \mathcal{L}_{\bar{R}_{i_m}} + \bar{R}_{i_m} \bar{\mathcal{L}}_{\bar{\partial}_j} \right) T_{j_1 \cdots j_k}
\]

where the summation is taken over the set of the following disjoint partition of \( \{1, \cdots, m\} \),

\[
\{ p_{1,1}, \cdots, p_{q_1,1} \} \cup \cdots \cup \{ p_{1,h}, \cdots, p_{q_h,h} \} \cup \{ r_1, \cdots, r_t \} = \{1, \cdots, m\},
\]

\[
p_{1,1} < \cdots < p_{q_1,1}, \cdots, p_{1,h} < \cdots < p_{q_h,h}, r_1 < \cdots < r_t,
\]

and for higher order of covariant derivatives

\[
\bar{\nabla}_{n_1} \cdots \bar{\nabla}_{n_m} T_{j_1 \cdots j_k} = \left( \bar{\nabla}_{n_1} + \bar{\nabla}_{n_1} \bar{\mathcal{L}}_{\bar{\partial}_j} \right) \cdots \left( \bar{\nabla}_{n_m} + \bar{\nabla}_{n_m} \bar{\mathcal{L}}_{\bar{\partial}_j} \right) T_{j_1 \cdots j_k}
\]

where the summation is taken over the set of the following disjoint partition of \( \{1, \cdots, m\} \),

\[
\{ p_{1,1}, \cdots, p_{q_1,1} \} \cup \cdots \cup \{ p_{1,h}, \cdots, p_{q_h,h} \} \cup \{ r_1, \cdots, r_t \} = \{1, \cdots, m\},
\]

\[
p_{1,1} < \cdots < p_{q_1,1}, \cdots, p_{1,h} < \cdots < p_{q_h,h}, r_1 < \cdots < r_t,
\]
where the summation is taken over the same set of the disjoint partition of \( \{ 1, \cdots, m \} \) as for the Lie derivatives.

**Lemma B.2.** Let \( T \) be any tensor field on \((M, g)\). Then the Sobolev norm \( \|T\|^{m,p}_{\Sigma_{s, \tilde{\eta}}} \) of \( T|_{\Sigma_{s}} \) which is \( T \) restricted on \( \Sigma_{s} \) satisfies the following inequality

\[
\|T\|^{m,p}_{\Sigma_{s, \tilde{\eta}}} \leq \left\{ \sup_{s \in [\inf, \sup]} \|T\|^{m,\infty}_{\Sigma_{s, \tilde{\eta}}} \right\} + c(m, p) \sum_{h \geq 1} \left( \|T^h\|^{m-1,p}_{\Sigma_{s, \tilde{\eta}}} \right) \sup_{t \leq h \leq m} \sup_{s \leq \tilde{\eta} \leq m} \sup_{\Sigma_{s, \tilde{\eta}}} \|T^h_{\Sigma_{s, \tilde{\eta}}}\|^{\infty}_{\Sigma_{s, \tilde{\eta}}}. 
\]

Assume that

\[
\|T^h\|^{m-1,p}_{\Sigma_{s, \tilde{\eta}}} \leq \tilde{\eta}_0 r_0 \leq r_0, \quad \|T^h\|^{\infty}_{\Sigma_{s, \tilde{\eta}}} \leq \tilde{\eta}_m r_0 \leq r_0, 
\]

then there exists a constant \( c(n, p) \) such that for any \( 0 \leq k \leq m \leq n + 2 \),

\[
\inf \left[ \inf_{s \in [\inf, \sup]} T, \sup_{s \in [\inf, \sup]} T \right] \subset [-r_0 (\tilde{\eta}_m + c(n, p)\tilde{\eta}_0), r_0 (\tilde{\eta}_m + c(n, p)\tilde{\eta}_0)].
\]

Hence by the above lemma and the assumptions of the \( \epsilon \)-close Schwarzschild metric, we have the following estimates of the background quantities of the double null coordinate system \( \{ s, \tilde{\eta}, \theta \} \) when restricted on \( \Sigma_{s} \).

**Proposition B.3** (Estimates of the background quantities of the double null coordinate system \( \{ s, \tilde{\eta}, \theta \} \) when restricted on \( \Sigma_{s} \)). Assume that

\[
\|T^h\|^{m-1,p}_{\Sigma_{s, \tilde{\eta}}} \leq \tilde{\eta}_0 r_0 \leq r_0, \quad \|T^h\|^{\infty}_{\Sigma_{s, \tilde{\eta}}} \leq \tilde{\eta}_m r_0 \leq r_0, 
\]
we have the following estimates. For any $0 \leq m \leq n+2$, odifaohfaoasndoifhosaifhiashd-goiahsodihoasidgjoiashd-goaisdogosdihoasidgjoiashdgoasdgoadosfdoighaos[idhgio[asdhgaios[hdg

\[
\| \log \Omega - \log \Omega_{Sch} \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p)r_0}{(r_0 + s)^{1+k}},
\]

\[
\| \partial_\nu (\log \Omega - \log \Omega_{Sch}) \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p,k)r_0}{(r_0 + s)^{1+k}},
\]

\[
\| \partial_\nu^2 (\log \Omega - \log \Omega_{Sch}) \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p,l)r_0}{(r_0 + s)^{2+k}},
\]

\[
\| \partial_\nu^2 (\log \Omega - \log \Omega_{Sch}) \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p,k,l)r_0}{(r_0 + s)^{3+k}},
\]

\[
\| \log \Omega_{Sch} \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p)r_0}{(r_0 + s)^{1+k}},
\]

\[
\| \partial_\nu^k \log \Omega_{Sch} \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p,k)r_0}{(r_0 + s)^{1+k}},
\]

\[
\| \nabla \log \Omega_{Sch} \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p,k)r_0}{(r_0 + s)^{1+k}},
\]

\[
\| \nabla \partial_\nu^k \log \Omega_{Sch} \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p,k)r_0}{(r_0 + s)^{2+k}},
\]

\[
\| \nabla \partial_\nu^k \log \Omega_{Sch} \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p,k)r_0}{(r_0 + s)^{3+k}},
\]

(591) \[
\| b \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p)(\delta_{m+2} + \delta_{s})r_0^2}{(r_0 + s)^{1+k}},
\]

\[
\| \partial_\nu b \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p,k)(\delta_{m+2} + \delta_{s})r_0^2}{(r_0 + s)^{2+k}},
\]

\[
\| \partial_\nu^2 b \|_{\Sigma_{\nu}, g}^{m,p} \leq \frac{c(n,p,k,l)(\delta_{m+2} + \delta_{s})r_0^2}{(r_0 + s)^{3+k}},
\]
\begin{align}
\| \phi - r_{\mathcal{S}ch}\phi \|^2_{\Sigma_{r \phi}} & \leq c(n, p)(r_0 + s)^2 \epsilon, \\
\| \partial^k_{\Sigma_{r \phi}} (\phi - r_{\mathcal{S}ch}\phi) \|^2_{\Sigma_{r \phi}} & \leq c(n, p, k)(r_0 + s)^{1-k} \epsilon, \\
\| \partial^l_{\Sigma_{r \phi}} (\phi - r_{\mathcal{S}ch}\phi) \|^2_{\Sigma_{r \phi}} & \leq c(n, p, l)(r_0 + s)^{1-l} \epsilon, \\
\| \partial^k \partial^l_{\Sigma_{r \phi}} (\phi - r_{\mathcal{S}ch}\phi) \|^2_{\Sigma_{r \phi}} & \leq c(n, p, k, l)\epsilon, \\
\| \phi - r_{\mathcal{S}ch}\phi \|^2_{\Sigma_{r \phi}} & \leq c(n, p)(r_0 + s)^2 \epsilon, \\
\| \partial^k_{\Sigma_{r \phi}} \phi \|^2_{\Sigma_{r \phi}} & \leq c(n, p, k)(r_0 + s)^{1-k} \epsilon, \\
\| \partial^l_{\Sigma_{r \phi}} \phi \|^2_{\Sigma_{r \phi}} & \leq c(n, p, l)(r_0 + s)^{1-l} \epsilon, \\
\| \partial^k \partial^l_{\Sigma_{r \phi}} \phi \|^2_{\Sigma_{r \phi}} & \leq c(n, p, k, l)\epsilon.
\end{align}

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\[
\begin{align*}
\|\bar{\nabla}\omega_{Sch}\|_{\Sigma, g}^{m-1,p} & \leq \frac{c(n,p)r_0}{(r_0 + s)^2} \bar{g} \cdot r_0, \\
\|\bar{\nabla}\bar{\partial}_{\Sigma}^k\omega_{Sch}\|_{\Sigma, g}^{m-1,p} & \leq \frac{c(n,p,k)r_0}{(r_0 + s)^{2+k}} \bar{g} \cdot r_0, \\
\int_{\bar{\Sigma}} \nabla (\omega_{Sch}) \leq & \frac{c(n,p,l)r_0}{(r_0 + s)^{2+l}} \bar{g} \cdot r_0, \\
\|\bar{\nabla}\bar{\partial}_{\Sigma}^k\omega_{Sch}\|_{\Sigma, g}^{m-1,p} & \leq \frac{c(n,p,l,k)r_0}{(r_0 + s)^{2+k+l}} \bar{g} \cdot r_0, \\
\|\omega_{Sch}\|_{\Sigma, g}^{m,p} & \leq \frac{c(n,p,r_0)}{(r_0 + s)^2} \bar{g} \cdot r_0, \\
\|\bar{\nabla}\omega_{Sch}\|_{\Sigma, g}^{m,p} & \leq \frac{c(n,p,l)r_0}{(r_0 + s)^{2+l}} \bar{g} \cdot r_0, \\
\|\bar{\nabla}\bar{\partial}_{\Sigma}^k\omega_{Sch}\|_{\Sigma, g}^{m,p} & \leq \frac{c(n,p,l,k)r_0}{(r_0 + s)^{2+k+l}} \bar{g} \cdot r_0.
\end{align*}
\]
\[ \left\| \partial^k (\text{tr} \chi - \text{tr} \chi_{\text{Sch}}) \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, k) \epsilon}{(r_0 + s)^{2} r_0^{k-1}}, \quad \left\| \partial^k (\text{tr} \chi - \text{tr} \chi_{\text{Sch}}) \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, l) \epsilon}{(r_0 + s)^{1+k}}, \]

\[ \left\| \text{tr} \chi_{\text{Sch}} \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p) s}{(r_0 + s)^2}, \quad \left\| \partial^k \text{tr} \chi_{\text{Sch}} \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, k) s}{(r_0 + s)^{2+k}}, \]

\[ \left\| \partial^k \text{tr} \chi_{\text{Sch}} \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, l)}{(r_0 + s)^{1+k}}, \quad \left\| \partial^k \partial^l \text{tr} \chi_{\text{Sch}} \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, l)}{(r_0 + s)^{1+k}}, \]

\[ \left\| \nabla (\text{tr} \chi - \text{tr} \chi_{\text{Sch}}) \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p) s}{(r_0 + s)^2}, \quad \left\| \nabla \partial^k (\text{tr} \chi - \text{tr} \chi_{\text{Sch}}) \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, k) \epsilon}{(r_0 + s)^{2+k}}, \]

\[ \left\| \nabla \partial^k (\text{tr} \chi - \text{tr} \chi_{\text{Sch}}) \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, l) \epsilon}{(r_0 + s)^{k}}, \quad \left\| \nabla \partial^k \partial^l (\text{tr} \chi - \text{tr} \chi_{\text{Sch}}) \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, l)}{(r_0 + s)^{1+k}}, \]

\[ \left\| \text{tr} \chi_{\text{Sch}} \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p) s}{(r_0 + s)^2}, \quad \left\| \partial^k \text{tr} \chi_{\text{Sch}} \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, k) s}{(r_0 + s)^{2+k}}, \]

\[ \left\| \nabla \partial^k (\text{tr} \chi - \text{tr} \chi_{\text{Sch}}) \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, l) \epsilon}{(r_0 + s)^{2+k}}, \quad \left\| \nabla \partial^k \partial^l (\text{tr} \chi - \text{tr} \chi_{\text{Sch}}) \right\|_{\Sigma_{s, \delta}}^{m, p} \leq \frac{c(n, p, l) \epsilon}{(r_0 + s)^{2+k}}, \]

(597)
\[
\gamma_{\Sigma_{S, g}} \left( \nabla^k \partial^L \right) \left( \text{tr}_x - \text{tr}_{\Sigma_{ch}} \right) \leq \frac{c(n, p, k, l)\epsilon}{(r_0 + s)^{3 + l} + k^2},
\]
\[
\gamma_{\Sigma_{S, g}} \left( \text{tr}_{\Sigma_{ch}} \right) \leq \frac{c(n - 1, p)\epsilon}{(r_0 + s)^{\frac{2}{3} + k}},
\]
\[
\gamma_{\Sigma_{S, g}} \left( \text{tr}_{\Sigma_{ch}} \right) \leq \frac{c(n, p, k)\epsilon}{(r_0 + s)^{2 + k}},
\]
\[
\gamma_{\Sigma_{S, g}} \left( \text{tr}_{\Sigma_{ch}} \right) \leq \frac{c(n, p, k)\epsilon}{(r_0 + s)^{2 + k + l}},
\]

(598)
\[
\gamma_{\Sigma_{S, g}} \leq c(n, p)\epsilon(r_0 + s),
\]
\[
\gamma_{\Sigma_{S, g}} \leq \frac{c(n, p, k)\epsilon}{(r_0 + s)^{1 - l}},
\]
\[
\gamma_{\Sigma_{S, g}} \leq \frac{c(n, p, k)\epsilon}{(r_0 + s)^{1 - l + r_0^m}},
\]

(599)
\[
\gamma_{\Sigma_{S, g}} \leq c(n, p)\epsilon r_0,
\]
\[
\gamma_{\Sigma_{S, g}} \leq \frac{c(n, p, k, l)\epsilon}{(r_0 + s)^{\frac{3}{2} + l + k}},
\]
\[
\gamma_{\Sigma_{S, g}} \leq \frac{c(n, p, k, l)\epsilon}{(r_0 + s)^{\frac{3}{2} + l + k - 3}},
\]
\[
\gamma_{\Sigma_{S, g}} \leq \frac{c(n, p, k, l)\epsilon}{(r_0 + s)^{\frac{3}{2} + l + k}},
\]
\[
\gamma_{\Sigma_{S, g}} \leq \frac{c(n, p, k, l)\epsilon}{(r_0 + s)^{\frac{3}{2} + l + k}},
\]

Proof. The proof is simply applying lemma 3.2 to those background quantities and using the assumptions on the metric \(\epsilon\)-close to Schwarzschild. For example

\[
\left\| \log \Omega - \log \Omega_{Sch} \right\|_{\Sigma_{S, g}}^{m, p} \leq \left\{ \sup_{x} \left\| \log \Omega - \log \Omega_{Sch} \right\|_{\Sigma_{S, g}}^{m, \infty} \right\} + c(n, p)\epsilon r_0 \left\{ \sum_{1 \leq h \leq n} \sup_{x} \left\| \partial^h \left( \log \Omega - \log \Omega_{Sch} \right) \right\|_{\Sigma_{S, g}}^{m, h, \infty} \right\}
\]
where \( s \in [-r_0(\bar{\delta}_m + c(n, p)\bar{\delta}_n), r_0(\bar{\delta}_m + c(n, p)\bar{\delta}_n)] \).

Now we can apply proposition \( \text{B.3} \) to estimate the vector field \( b^i \bar{\partial}_i - 2\Omega^2 (\bar{\phi}^{-1})^{ij} s^2 \tilde{\omega}^0 \partial_t \tilde{\omega}^1 \) and the terms
\[
\sum_{m+n \leq 3} P_{m, m^\prime + 1} (\nabla, \tilde{\omega}^0 \nabla) \tilde{\partial}_m b, \quad \sum_{m+n \leq 4} P_{m, m^\prime + 2} (\nabla, \tilde{\omega}^0 \nabla) \tilde{\partial}_m^\prime (\Omega^2 \bar{\phi}^{-1}),
\]
which show up in the propagation equation (133) of \( \tilde{\omega}^0 \tilde{\omega}^0 \).

**Appendix C. Proof of lemma \( \text{A.1.1} \)**

In order to estimate \( \tilde{\omega}^0 \), we need to obtain the estimates of \( \tilde{\omega}^0 \) and \( \tilde{\omega}^0 \), hence we need first to get the estimate of the background quantities of the background double null coordinate system restricted on \( \Sigma_t \).

**Lemma C.1.** Let \( T \) be any tensor field on \( (M, g) \). Then the Lie derivatives of \( T \) on \( \tilde{\Sigma}_t \) which is \( T \) restricted on \( \tilde{\Sigma}_t \) with respect to the rotation fields \( \tilde{\omega}^0 \) is
\[
\mathcal{L}_{R_{1}} T_{1i}^{\cdots jk} = \mathcal{L}_{R_{1}} T_{1i}^{\cdots jk} + \tilde{\omega}^0 \mathcal{L}_{\partial_t} T_{1i}^{\cdots jk} + \tilde{\omega}^0 \mathcal{L}_{\partial_t} T_{1i}^{\cdots jk},
\]
and the covariant derivative of \( T \) restricted on \( \tilde{\Sigma}_t \) is
\[
\tilde{\nabla}_i T_{1i}^{\cdots jk} = \tilde{\nabla}_i T_{1i}^{\cdots jk} + \tilde{\omega}^0 \mathcal{L}_{\partial_t} T_{1i}^{\cdots jk} + \tilde{\omega}^0 \mathcal{L}_{\partial_t} T_{1i}^{\cdots jk}.
\]
Moreover, for higher order Lie derivatives, we have
\[
\mathcal{L}_{R_{1} \cdots R_{m}} T_{1i}^{\cdots jk} = \mathcal{L}_{R_{1} \cdots R_{m}} T_{1i}^{\cdots jk} + \tilde{\omega}^0 \mathcal{L}_{\partial_t} T_{1i}^{\cdots jk} + \tilde{\omega}^0 \mathcal{L}_{\partial_t} T_{1i}^{\cdots jk} + \tilde{\omega}^0 \mathcal{L}_{\partial_t} T_{1i}^{\cdots jk}.
\]
where the summation is taken over the set of the following disjoint partition of \( \{1, \cdots, m\} \),
\[
\bigg\{ n_{p_1, 1}, \cdots, n_{p_1, h} \bigg\} \cup \cdots \cup \bigg\{ n_{p_{q_1}, 1}, \cdots, n_{p_{q_1, 1}} \bigg\} \cup \cdots \cup \bigg\{ n_{p_{q_h}, 1}, \cdots, n_{p_{q_h, h}} \bigg\}
\]
and \( \{r_1, \cdots, r_t\} \) is \( \{1, \cdots, m\} \).
and for higher order of covariant derivatives

\[(605)\]
\[
\tilde{\nabla}_{n_1} \cdots \tilde{\nabla}_{n_m} T^{j_1 \cdots j_k}_{i_1 \cdots i_k} \\
= \left( \tilde{\nabla}_{n_1} + \tilde{\nabla}_{n_1} \tilde{\nabla} L_{\partial_z} + \tilde{\nabla}_{n_1} \tilde{\nabla} L_{\partial_s} \right) \cdots \left( \tilde{\nabla}_{n_m} + \tilde{\nabla}_{n_m} \tilde{\nabla} L_{\partial_z} + \tilde{\nabla}_{n_m} \tilde{\nabla} L_{\partial_s} \right) T^{j_1 \cdots j_k}_{i_1 \cdots i_k} \\
= \sum \left( \tilde{\nabla}_{n_{p_1,1}} \cdots \tilde{\nabla}_{n_{p_1,1}} \tilde{\nabla} \tilde{T}_{p_1} \right) \cdots \left( \tilde{\nabla}_{n_{p_l,1}} \cdots \tilde{\nabla}_{n_{p_l,1}} \tilde{\nabla} \tilde{T}_{p_l} \right). \\
\]

where the summation is taken over the same set of the disjoint partition of \( \{1, \ldots, m\} \) as for the Lie derivatives.

**Lemma C.2.** Let \( T \) be any tensor field on \((M, g)\). Then the Sobolev norm \( \|T\|_{S_t}^{m,p} \) of \( T|_{\Sigma_t} \) which is \( T \) restricted on \( \Sigma_t \) satisfies the following inequality

\[(606)\]
\[
\|T\|_{S_t}^{m,p} \leq \left\{ \sup_{k} \|T\|_{\Sigma_t}^{m,\infty} \right\} + c(m, p) \sum_{k \geq 1} \left( \|\tilde{\nabla} \tilde{T}_{f}\|_{\Sigma_t}^{m-1,p} \right)^h \left( \|\tilde{\nabla} \tilde{T}_{f}\|_{\Sigma_t}^{m-1,p} \right)^h \left\{ \sup_{t \geq 0} \left\| \tilde{\nabla} L_{\partial_z}^{s} T_{i_1 \cdots i_k} \right\|_{\Sigma_t}^{\infty} \right\}
\]

where the supremum is taken in the range \( s \in [\inf \tilde{T}_{f}, \sup \tilde{T}_{f}] \), \( s \in [\inf \tilde{T}_{f}, \sup \tilde{T}_{f}] \). Assume that

\[(607)\]
\[
\|\tilde{\nabla} \tilde{T}_{f}\|_{\Sigma_t}^{m,p} \leq \| \tilde{\nabla} \|_{\Sigma_t}^{m,p} \leq \| \tilde{\nabla} \|_{\Sigma_t}^{m,p} \leq \| \tilde{\nabla} \|_{\Sigma_t}^{m,p} \leq \| \tilde{\nabla} \|_{\Sigma_t}^{m,p} \leq t \delta \leq (r_0 + s_0) \leq t (r_0 + s_0), \quad \tilde{T}_{f} = t s_0,
\]

then there exists a constant \( c(n, p) \) such that for any \( 0 \leq k \leq m \leq n + 1 \),

\[(608)\]
\[
\left[ \inf \tilde{T}_{f}, \sup \tilde{T}_{f} \right] \subset \left[ -r_0 \left( \| \tilde{\nabla} \|_{\Sigma_t}^{m,p} + c(n, p) \| \tilde{\nabla} \|_{\Sigma_t}^{m,p} \right) \right.
\]
\[(609)\]
\[
\left[ \inf \tilde{T}_{f}, \sup \tilde{T}_{f} \right] \subset \left[ t s_0 - c(n, p) t \delta, t s_0 + c(n, p) t \delta \right]
\]
(610) \[ \|T\|_{\tilde{\Sigma}_{t,\tilde{g}}}^{m,p} \leq \left\{ \sup_{\tilde{\Sigma}^{s}} \| T \|_{\tilde{\Sigma}^{\infty}}^{m,\infty} + c(n,p)\delta_{n}r_{0} \right\} + \left\{ \sum_{t+h \leq m, h \geq 1} \sup_{\tilde{\Sigma}^{s}} \left\| \tilde{\nabla}^{t} \mathcal{L}_{\partial_{\tilde{g}}}^{h} T \right\|_{\tilde{\Sigma}_{s}}^{\infty} \right\} \\
+ c(n,p)t\delta_{o}(r_{0} + s_{0}) \left\{ \sum_{t+h \leq m, h \geq 1} \sup_{\tilde{\Sigma}^{s}} \left\| \tilde{\nabla}^{t} \mathcal{L}_{\partial_{\tilde{g}}}^{h} T \right\|_{\tilde{\Sigma}_{s}}^{\infty} \right\} \\
+ c(n,p)t\delta_{o}\tilde{\delta}_{n}r_{0}(r_{0} + s_{0}) \left\{ \sum_{t+h \leq m, h \geq 1} \sup_{\tilde{\Sigma}^{s}} \left\| \tilde{\nabla}^{t} \mathcal{L}_{\partial_{\tilde{g}}}^{h} T \right\|_{\tilde{\Sigma}_{s}}^{\infty} \right\} \]

(611) \[ \left\| \tilde{\nabla}^{m-k,p} T \right\|_{\tilde{\Sigma}_{t,\tilde{g}}}^{m-k,p} \leq \left\{ \sup_{\tilde{\Sigma}^{s}} \left\| \tilde{\nabla}^{m-1,\infty} T \right\|_{\tilde{\Sigma}^{\infty}}^{s,\infty} + c(n,p)\delta_{n}r_{0} \right\} + \left\{ \sum_{t+h \leq m, h \geq 1} \sup_{\tilde{\Sigma}^{s}} \left\| \tilde{\nabla}^{t} \mathcal{L}_{\partial_{\tilde{g}}}^{h} T \right\|_{\tilde{\Sigma}_{s}}^{\infty} \right\} \\
+ c(n,p)t\delta_{o}(r_{0} + s_{0}) \left\{ \sum_{t+h \leq m, h \geq 1} \sup_{\tilde{\Sigma}^{s}} \left\| \tilde{\nabla}^{t} \mathcal{L}_{\partial_{\tilde{g}}}^{h} T \right\|_{\tilde{\Sigma}_{s}}^{\infty} \right\} \\
+ c(n,p)t\delta_{o}\tilde{\delta}_{n}r_{0}(r_{0} + s_{0}) \left\{ \sum_{t+h \leq m, h \geq 1} \sup_{\tilde{\Sigma}^{s}} \left\| \tilde{\nabla}^{t} \mathcal{L}_{\partial_{\tilde{g}}}^{h} T \right\|_{\tilde{\Sigma}_{s}}^{\infty} \right\} \]

where the supremum is taken in the range \( s \in [-r_{0} (\tilde{\delta}_{n} + c(n,p)\delta_{n}), r_{0} (\tilde{\delta}_{n} + c(n,p)\delta_{n})], s \in [t_{s_{0}} - c(n,p)t\delta_{o}, t_{s_{0}} + c(n,p)t\delta_{o}] \).

Then by the above lemma and the assumptions of the metric \( \epsilon \)-close to Schwarzschild, we have the following estimates of the background quantities restricted to \( \tilde{\Sigma}_{t} \).

**Proposition C.3** (Estimates of the background quantities restricted to \( \tilde{\Sigma}_{t} \)). Assume that

(612) \[ \left\| \tilde{d} \tilde{f} \right\|_{\tilde{\Sigma}_{t,\tilde{g}}}^{m,p} \leq \tilde{\delta}_{n}r_{0} \leq r_{0}, \quad \left\| \tilde{t} \tilde{f} \right\|_{\tilde{\Sigma}_{t,\tilde{g}}}^{m,p} \leq \delta_{n}r_{0} \leq r_{0}, \quad \left\| \tilde{d} \tilde{f} \right\|_{\tilde{\Sigma}_{t,\tilde{g}}}^{m,p} \leq \delta_{o}(r_{0} + t_{s_{0}}) \leq r_{0} + t_{s_{0}}, \quad \tilde{t} \tilde{f} = t_{s_{0}} \]

we have the following estimates of the background quantities of the background double null coordinate system restricted on \( \tilde{\Sigma}_{t} \). For any \( 0 \leq m \leq n + 1 \),

(613) \[ \| \log \Omega - \log \Omega_{Sch} \|_{\tilde{\Sigma}_{t,\tilde{g}}}^{m,p} \leq c(n,p)r_{0} \frac{r_{0}}{r_{0} + t_{s_{0}}} \epsilon, \quad \| \partial_{\tilde{t}} \log \Omega - \omega_{Sch} \|_{\tilde{\Sigma}_{t,\tilde{g}}}^{m,p} \leq c(n,p,k)r_{0} \frac{(r_{0} + t_{s_{0}})^{2} \epsilon}{(r_{0} + t_{s_{0}})^{2}}, \]

\[ \| \partial_{\tilde{t}} \log \Omega - \partial_{\tilde{t}}^{k-1} \omega_{Sch} \|_{\tilde{\Sigma}_{t,\tilde{g}}}^{m,p} \leq c(n,p,k) \frac{(r_{0} + t_{s_{0}})^{2} \epsilon}{(r_{0} + t_{s_{0}})^{2}}, \]

\[ \| \partial_{\tilde{t}} \left( \log \Omega - \log \Omega_{Sch} \right) \|_{\tilde{\Sigma}_{t,\tilde{g}}}^{m,p} \leq \frac{c(n,p,l)r_{0}}{(r_{0} + t_{s_{0}})^{1/2} \epsilon}, \]
\[ \| \partial_{s}^{k} \partial_{s}^{l} (\log \Omega - \log \Omega_{Sch}) \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, l)r_{0}}{(r_{0} + t_{s0})^{3 + l} r_{0}^{3 - l} \epsilon}, \]
\[ \| \partial_{s}^{k} \partial_{s}^{l} (\log \Omega - \log \Omega_{Sch}) \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, k, l)}{(r_{0} + t_{s0})^{3 + l} k_{0}^{3 - l} \epsilon}, \]
\[ \| \nabla (\log \Omega - \log \Omega_{Sch}) \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, l)r_{0}}{(r_{0} + t_{s0})^{2 + l} \epsilon}, \]
\[ \| \nabla (\partial_{s}^{k} \log \Omega - \partial_{s}^{k-1} \log \Omega_{Sch}) \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, k)}{(r_{0} + t_{s0})^{3 + l} k_{0}^{3 - l} \epsilon}, \]
\[ \| \nabla \partial_{s}^{k} \partial_{s}^{l} (\log \Omega - \log \Omega_{Sch}) \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, l)r_{0}}{(r_{0} + t_{s0})^{2 + l} \epsilon}, \]
\[ \| \log \Omega_{Sch} \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, l)r_{0}}{(r_{0} + t_{s0})^{1 + l} \epsilon}, \]
\[ \| \partial_{s}^{k} \log \Omega_{Sch} \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, l)r_{0}}{(r_{0} + t_{s0})^{1 + k + l} \epsilon}, \]
\[ \| \log \Omega_{Sch} \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, l)r_{0}}{(r_{0} + t_{s0})^{1 + l} \epsilon}, \]
\[ \| \partial_{s}^{k} \log \Omega_{Sch} \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, l)r_{0}}{(r_{0} + t_{s0})^{1 + k + l} \epsilon}, \]
\[ \| b \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, l)r_{0}}{(r_{0} + t_{s0})^{3} \epsilon}, \]
\[ \| \partial_{s}^{k} b \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, k, l)}{(r_{0} + s)^{3} r_{0}^{2} \epsilon}, \]
\[ \| \partial_{s}^{k} \partial_{s}^{l} b \|_{\hat{\Sigma}_{t}, g}^{m,p} \leq \frac{c(n, p, k, l)}{(r_{0} + s)^{3} r_{0}^{2} \epsilon}. \]
\[ \| \varphi - r_{Sch}^2 \|_{(n, p)} \leq c(n, p)(r_0 + ts_0)^2 \epsilon, \quad \| \partial^k_m \left( \varphi - r_{Sch}^2 \right) \|_{\Sigma, g}^{m, p} \leq c(n, p, k)(r_0 + ts_0)^{1-k} \epsilon, \]
\[ \| \partial^k_s \left( \varphi - r_{Sch}^2 \right) \|_{\Sigma, g}^{m, p} \leq c(n, p, l)(r_0 + ts_0)^{1-l} \epsilon, \]
\[ \| \partial^k \partial^l_s \left( \varphi - r_{Sch}^2 \right) \|_{\Sigma, g}^{m, p} \leq c(n, p, k, l)(r_0 + ts_0)^{1-k-l} \epsilon, \]
\[ \| r_{Sch}^2 \|_{\Sigma, g}^{m, p} \leq c(n, p)(r_0 + ts_0)^2, \]
\[ \| \partial^k r_{Sch}^2 \|_{\Sigma, g}^{m, p} \leq \frac{c(n, p, k)}{(r_0 + ts_0)^{k-2}} \frac{r_0 + ts_0}{r_0 + ts_0}, \]
\[ \| \partial^k_s r_{Sch}^2 \|_{\Sigma, g}^{m, p} \leq \frac{c(n, p, l)}{(r_0 + ts_0)^{l-2}}, \quad \| \partial^k_s \partial^l_s r_{Sch}^2 \|_{\Sigma, g}^{m, p} \leq \frac{c(n, p, k, l)}{(r_0 + ts_0)^{k+l-2}}, \]
\[ \| \nabla \left( \varphi - r_{Sch}^2 \right) \|_{\Sigma, g}^{m, p} \leq c(n, p)(r_0 + ts_0)^2 \epsilon, \]
\[ \| \nabla \partial^k_s \left( \varphi - r_{Sch}^2 \right) \|_{\Sigma, g}^{m, p} \leq c(n, p, k)(r_0 + ts_0)^{1-k} \epsilon, \]
\[ \| \nabla \partial^k \left( \varphi - r_{Sch}^2 \right) \|_{\Sigma, g}^{m, p} \leq c(n, p, l)(r_0 + ts_0)^{1-l} \epsilon, \]
\[ \| \nabla \partial^k \partial^l_s \left( \varphi - r_{Sch}^2 \right) \|_{\Sigma, g}^{m, p} \leq c(n, p, k, l)(r_0 + ts_0)^{1-k-l} \epsilon, \]
\[ \| \nabla r_{Sch}^2 \|_{\Sigma, g}^{m, p} \leq c(n, p)(r_0 + ts_0)^2 \left( \frac{ts_0 + \delta_o r_0}{r_0 + ts_0} \right) + \delta_o, \]
\[ \| \nabla \partial^k \partial^l_s r_{Sch}^2 \|_{\Sigma, g}^{m, p} \leq \frac{c(n, p, k)}{(r_0 + ts_0)^{k-2}} \left( \frac{ts_0 + \delta_o r_0}{(r_0 + ts_0)^2} + \delta_o \right), \]
\[ \| \nabla \partial^k \partial^l_s r_{Sch}^2 \|_{\Sigma, g}^{m, p} \leq \frac{c(n, p, l)}{(r_0 + ts_0)^{l-2}} \left( \frac{\delta_o r_0}{r_0 + ts_0} + \delta_o \right), \]
\[ \| \nabla \partial^k \partial^l_s r_{Sch}^2 \|_{\Sigma, g}^{m, p} \leq \frac{c(n, p, k, l)}{(r_0 + ts_0)^{k+l-2}} \left( \frac{\delta_o r_0}{r_0 + ts_0} + \delta_o \right), \]

(615) \[
\| \omega - \omega_{Sch} \|_{\Sigma, g}^{m, p} \leq \frac{c(n, p, 0)}{(r_0 + ts_0)^2} \epsilon, \quad \| \partial^k \left( \omega - \omega_{Sch} \right) \|_{\Sigma, g}^{m, p} \leq \frac{c(n, p, 0)}{(r_0 + ts_0)^3} \epsilon, \]
\[ \| \partial^k_s \left( \omega - \omega_{Sch} \right) \|_{\Sigma, g}^{m, p} \leq \frac{c(n, p, l, 0)}{(r_0 + ts_0)^2} \epsilon, \]
\[ \| \partial^k_m \left( \omega - \omega_{Sch} \right) \|_{\Sigma, g}^{m, p} \leq \frac{c(n, p, m)}{(r_0 + ts_0)^3} \epsilon, \]
\[
\left\| \frac{\partial^n \partial^l}{\partial \gamma \partial^k} (\omega - \omega_{Sch}) \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, k, l)}{(r_0 + t s_0)^{3+i} r_0^{-k-2} \epsilon},
\]
\[
\left\| \omega_{Sch} \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p) r_0}{(r_0 + t s_0)^2}, \quad \left\| \frac{\partial^n \omega_{Sch}}{\partial \gamma \partial^k} \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, k) r_0}{(r_0 + t s_0)^{2+k}},
\]
\[
\left\| \frac{\partial^n \omega_{Sch}}{\partial \gamma \partial^k} \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, l) r_0}{(r_0 + t s_0)^{2+i+k}}, \quad \left\| \frac{\partial^n \partial^l}{\partial \gamma \partial^k} \omega_{Sch} \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, k, l) r_0}{(r_0 + t s_0)^{3+i+k}},
\]
\[
\left\| \frac{\partial^n}{\partial \gamma} (\omega - \omega_{Sch}) \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p) r_0}{(r_0 + t s_0)^{2+i} \epsilon}, \quad \left\| \frac{\partial^n}{\partial \gamma} \frac{\partial^n}{\partial \gamma} (\omega - \omega_{Sch}) \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, k) r_0}{(r_0 + t s_0)^{3+i+k}},
\]
\[
\left\| \frac{\partial^n}{\partial \gamma} (\omega - \omega_{Sch}) \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, k) r_0}{(r_0 + t s_0)^{2+i+k} \epsilon}, \quad \left\| \frac{\partial^n}{\partial \gamma} \frac{\partial^n}{\partial \gamma} (\omega - \omega_{Sch}) \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, k, l) r_0}{(r_0 + t s_0)^{3+i+k}},
\]
\[
\left\| \frac{\partial^n}{\partial \gamma} (\omega - \omega_{Sch}) \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, k, l) r_0}{(r_0 + t s_0)^{2+i+k} \epsilon}, \quad \left\| \frac{\partial^n}{\partial \gamma} \frac{\partial^n}{\partial \gamma} (\omega - \omega_{Sch}) \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, k, l, m, p) r_0}{(r_0 + t s_0)^{3+i+k}},
\]
\[
\left\| \frac{\partial^n}{\partial \gamma} (\omega - \omega_{Sch}) \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, k, l, m, p) r_0}{(r_0 + t s_0)^{2+i+k} \epsilon}, \quad \left\| \frac{\partial^n}{\partial \gamma} \frac{\partial^n}{\partial \gamma} (\omega - \omega_{Sch}) \right\|_{m,p}^{\Sigma_r, g} \leq \frac{c(n, p, k, l, m, p) r_0}{(r_0 + t s_0)^{3+i+k}},
\]
\[
\begin{align*}
\|\omega - \omega_{Sch}\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p)\rho_0}{(r_0 + ts_0)^2} \epsilon, \\
\|\partial_s^l (\omega - \omega_{Sch})\|_{\Sigma_{t,\hat{g}}}^{m,p,l} & \leq \frac{c(n,p,l)\rho_0}{(r_0 + ts_0)^{2+l}} \epsilon, \\
\|\partial_s^k \partial_s^l (\omega - \omega_{Sch})\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p,k,l)}{(r_0 + ts_0)^{3+l} r_0^{k-2} \epsilon}, \\
\|\omega_{Sch}\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p,l)\rho_0}{(r_0 + ts_0)^2} \epsilon, \\
\|\partial_s^k \omega_{Sch}\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p,l)\rho_0}{(r_0 + ts_0)^{2+l}} \epsilon, \\
\|\partial_s^k \partial_s^l \omega_{Sch}\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p,k,l)\rho_0}{(r_0 + ts_0)^{2+k+l}} \epsilon, \\
\|\nabla_s (\omega - \omega_{Sch})\|_{\Sigma_{t,\hat{g}}}^{m-1,p} & \leq \frac{c(n,p)\rho_0}{(r_0 + ts_0)^2} \epsilon, \\
\|\nabla_s \partial_s^k (\omega - \omega_{Sch})\|_{\Sigma_{t,\hat{g}}}^{m-1,p} & \leq \frac{c(n,p,k)}{(r_0 + ts_0)^3} r_0^{k-2} \epsilon, \\
\|\nabla_s \partial_s^k \partial_s^l (\omega - \omega_{Sch})\|_{\Sigma_{t,\hat{g}}}^{m-1,p} & \leq \frac{c(n,p,k,l)}{(r_0 + ts_0)^{3+l} r_0^{k-2} \epsilon}, \\
\|\nabla_s \omega_{Sch}\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p,l)\rho_0}{(r_0 + ts_0)^2} \epsilon, \\
\|\nabla_s \partial_s^k \omega_{Sch}\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p,k)\rho_0}{(r_0 + ts_0)^{2+k} r_0^{k-2} \epsilon}, \\
\|\nabla_s \partial_s^k \partial_s^l \omega_{Sch}\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p,k,l)\rho_0}{(r_0 + ts_0)^{2+k+l}} \epsilon, \\
\|\nabla_s \omega_{Sch}\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p,l)\rho_0}{(r_0 + ts_0)^2} \epsilon, \\
\|\nabla_s \partial_s^k \omega_{Sch}\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p,k)\rho_0}{(r_0 + ts_0)^{2+k} r_0^{k-2} \epsilon}, \\
\|\nabla_s \partial_s^k \partial_s^l \omega_{Sch}\|_{\Sigma_{t,\hat{g}}}^{m,p} & \leq \frac{c(n,p,k,l)\rho_0}{(r_0 + ts_0)^{2+k+l}} \epsilon.
\end{align*}
\]
\begin{align}
(618) \quad \|q\|_{\tilde{\mathcal{S}}_{t,\tilde{g}}}^{m,p} & \leq \frac{c(n,p)r_0}{r_0 + ts_0} \epsilon, \\
\|\partial^k q\|_{\tilde{\mathcal{S}}_{t,\tilde{g}}}^{m,p} & \leq \frac{c(n,p,k)}{(r_0 + ts_0)^{1+k} \epsilon}, \\
\|\partial^k \partial^l q\|_{\tilde{\mathcal{S}}_{t,\tilde{g}}}^{m,p} & \leq \frac{c(n,p,k,l)}{(r_0 + ts_0)^{1+l+k} \epsilon}, \\
\end{align}

\begin{align}
(619) \quad \|\text{tr} \chi - \text{tr} \chi_{\text{Sch}}\|_{\tilde{\mathcal{S}}_{t,\tilde{g}}}^{m,p} & \leq \frac{c(n,p)}{r_0 + ts_0} \left( \frac{ts_0 + \epsilon(t \delta_m + \delta_o)}{r_0 + ts_0} + \epsilon \delta_o \right), \\
\|\partial^k (\text{tr} \chi - \text{tr} \chi_{\text{Sch}})\|_{\tilde{\mathcal{S}}_{t,\tilde{g}}}^{m,p} & \leq \frac{c(n,p,k)}{(r_0 + ts_0)^{2+k} \epsilon}, \\
\|\partial^k \partial^l (\text{tr} \chi - \text{tr} \chi_{\text{Sch}})\|_{\tilde{\mathcal{S}}_{t,\tilde{g}}}^{m,p} & \leq \frac{c(n,p,k,l)}{(r_0 + ts_0)^{1+l+k} \epsilon}, \\
\|\text{tr} \chi_{\text{Sch}}\|_{\tilde{\mathcal{S}}_{t,\tilde{g}}}^{m,p} & \leq \frac{c(n,p)}{r_0 + ts_0} \left( \frac{ts_0 + \epsilon(t \delta_m + \delta_o)}{r_0 + ts_0} + \epsilon \delta_o \right), \\
\|\partial^k \text{tr} \chi_{\text{Sch}}\|_{\tilde{\mathcal{S}}_{t,\tilde{g}}}^{m,p} & \leq \frac{c(n,p,k)}{(r_0 + ts_0)^{2+k} \epsilon}, \\
\|\partial^k \partial^l \text{tr} \chi_{\text{Sch}}\|_{\tilde{\mathcal{S}}_{t,\tilde{g}}}^{m,p} & \leq \frac{c(n,p,k,l)}{(r_0 + ts_0)^{1+k+l} \epsilon}, \\
\frac{\partial}{\partial t} \left( (\text{tr} \chi - \text{tr} \chi_{\text{Sch}})^{m-1,p} \right)_{\tilde{\mathcal{S}}_{t,\tilde{g}}} & \leq \frac{c(n,p)}{r_0 + ts_0} \left( \frac{ts_0 + \epsilon(t \delta_m + \delta_o)}{r_0 + ts_0} + \epsilon \delta_o \right), \\
\frac{\partial}{\partial t} \left( \partial^k (\text{tr} \chi - \text{tr} \chi_{\text{Sch}})^{m-1,p} \right)_{\tilde{\mathcal{S}}_{t,\tilde{g}}} & \leq \frac{c(n,p,k)}{(r_0 + ts_0)^{2+k} \epsilon}, \\
\frac{\partial}{\partial t} \left( \partial^k \partial^l (\text{tr} \chi - \text{tr} \chi_{\text{Sch}})^{m-1,p} \right)_{\tilde{\mathcal{S}}_{t,\tilde{g}}} & \leq \frac{c(n,p,k,l)}{(r_0 + ts_0)^{1+k+l} \epsilon}, \\
\frac{\partial}{\partial t} \left( \text{tr} \chi_{\text{Sch}}^{m-1,p} \right)_{\tilde{\mathcal{S}}_{t,\tilde{g}}} & \leq \frac{c(n,p)}{r_0 + ts_0} \left( \frac{ts_0 + \epsilon(t \delta_m + \delta_o)}{r_0 + ts_0} + \epsilon \delta_o \right), \\
\frac{\partial}{\partial t} \left( \partial^k \text{tr} \chi_{\text{Sch}}^{m-1,p} \right)_{\tilde{\mathcal{S}}_{t,\tilde{g}}} & \leq \frac{c(n,p,k)}{(r_0 + ts_0)^{2+k} \epsilon}, \\
\frac{\partial}{\partial t} \left( \partial^k \partial^l \text{tr} \chi_{\text{Sch}}^{m-1,p} \right)_{\tilde{\mathcal{S}}_{t,\tilde{g}}} & \leq \frac{c(n,p,k,l)}{(r_0 + ts_0)^{1+k+l} \epsilon}. 
\end{align}
\[ \left\| \nabla \frac{\partial_k}{\partial_t} \text{tr} \chi_{\text{Sch}} \right\|_{\hat{\Sigma}_{t, \bar{g}}}^{m-1, p} \leq \frac{c(n, p, k, l)}{(r_0 + ts_0)^{1+k+l}} \left( \frac{\delta_0 r_0}{r_0 + ts_0} + \delta_0 \right), \]
Proof. By the Sobolev embedding theorem, we have

\[ \| \nabla \hat{\chi} \|_{\Sigma_t, g}^{m,p} \leq c(n,p,r_0 + t_s_0) \epsilon, \quad \| \nabla \hat{\chi} \|_{\Sigma_t, g}^{m,p} \leq c(n,p,k)(r_0 + t_s_0) \epsilon, \]

\[ \| \partial_i \hat{\chi} \|_{\Sigma_t, g}^{m,p} \leq \frac{c(n,p,l)}{(r_0 + t_s_0)^{l-1}} \epsilon, \quad \| \partial_i \hat{\chi} \|_{\Sigma_t, g}^{m,p} \leq \frac{c(n,p,k,l)}{(r_0 + t_s_0)^{l-1}r_0^k} \epsilon, \]

\[ \| \frac{\partial}{\partial \hat{\chi}} \|_{\Sigma_t, g}^{m,p} \leq c(n-1,p)(r_0 + t_s_0) \epsilon, \quad \| \frac{\partial}{\partial \hat{\chi}} \|_{\Sigma_t, g}^{m,p} \leq \frac{c(n,p,k,l)(r_0 + t_s_0)}{r_0^k} \epsilon, \]

\[ \| \frac{\partial}{\partial \hat{\chi}} \|_{\Sigma_t, g}^{m,p} \leq \frac{c(n,p,l)}{(r_0 + t_s_0)^{l-1}} \epsilon, \quad \| \frac{\partial}{\partial \hat{\chi}} \|_{\Sigma_t, g}^{m,p} \leq \frac{c(n,p,k,l)}{(r_0 + t_s_0)^{l-1}r_0^k} \epsilon, \]

\[ \| \frac{\partial}{\partial \hat{\chi}} \|_{\Sigma_t, g}^{m,p} \leq \frac{c(n,p,l)}{(r_0 + t_s_0)^{l-1}} \epsilon, \quad \| \frac{\partial}{\partial \hat{\chi}} \|_{\Sigma_t, g}^{m,p} \leq \frac{c(n,p,k,l)}{(r_0 + t_s_0)^{l-1}r_0^k} \epsilon, \]

(623) \[ \| \hat{f} - \hat{f} \|_{\Sigma_t, g}^{n,p} \leq c(n,p) \| \hat{\chi} \|_{\Sigma_t, g}^{n,p} \leq c(n,p) \hat{\xi}_0 r_0, \]

(624) \[ \| \hat{f} - \hat{f} \|_{\Sigma_t, g}^{n,p} \leq c(n,p) \| \hat{\chi} \|_{\Sigma_t, g}^{n,p} \leq c(n,p) \hat{\xi}_0 (r_0 + t_s_0), \]

which implies that on \( \hat{\xi}_t \)

(625) \[ \| \hat{f} \|_{\Sigma_t, g} \leq \hat{\xi}_m r_0 + c(n,p) \hat{\xi}_0 r_0, \]

(626) \[ \| \hat{f} \|_{\Sigma_t, g} \leq t_s_0 + c(n,p) \hat{\xi}_0 (r_0 + t_s_0). \]
Then we get the estimates of the background quantities by lemma C.2 and the assumptions on the metric \( \epsilon \)-close to Schwarzschild. For example,

\[
(627) \quad \| \log \Omega - \log \Omega_{Sch} \|_{\Sigma_{t, g}}^{m, p} \\
\leq \sup_{\Delta^s} \| \log \Omega - \log \Omega_{Sch} \|_{\Sigma_{t, g}}^{m, \infty} \\
+ c(n, p) \delta_o r_0 \left\{ \sum_{t+h \leq m, h \geq 1} \sup_{\Delta^s} \left\| \nabla t \mathcal{L}_{\delta_o}^h (\log \Omega - \log \Omega_{Sch}) \right\|_{\Sigma_{t, g}}^{\infty} \right\} \\
+ c(n, p) \delta_o (r_0 + ts_0) \left\{ \sum_{t+h \leq m, h \geq 1} \sup_{\Delta^s} \left\| \nabla t \mathcal{L}_{\delta_0}^h (\log \Omega - \log \Omega_{Sch}) \right\|_{\Sigma_{t, g}}^{\infty} \right\} \\
+ c(n, p) \delta_o (r_0 + ts_0) \delta_o r_0 \left\{ \sum_{t+h \leq m, h \geq 1} \sup_{\Delta^s} \left\| \nabla t \mathcal{L}_{\delta_o}^h (\log \Omega - \log \Omega_{Sch}) \right\|_{\Sigma_{t, g}}^{\infty} \right\} \\
\leq \frac{c(n, p) \epsilon}{r_0 + ts_0} + \frac{c(n, p) \delta_o r_0}{(r_0 + ts_0)^2} + c(n, p) \delta_o (r_0 + ts_0) \frac{er_0}{(r_0 + ts_0)^2} \\
+ c(n, p) \delta_o (r_0 + ts_0) \delta_o r_0 \frac{er_0^2}{(r_0 + ts_0)^3} \\
\leq \frac{c(n, p) \epsilon}{r_0 + ts_0} \left( \epsilon + \frac{\epsilon \delta_o r_0}{r_0 + ts_0} + \frac{\epsilon \delta_o (r_0 + ts_0)}{r_0 + ts_0} \right) \\
\leq \frac{c(n, p) \epsilon}{r_0 + ts_0}.
\]

All the other estimates are similar. \( \square \)

Now we can estimate the vector fields \( \tilde{\tilde{X}} \) and the term \( \tilde{t} \tilde{e} \) in the propagation equation \( \tilde{\Delta} \tilde{f} \) of \( \tilde{\tilde{f}} \).

**Lemma C.4.** \( \tilde{\tilde{f}} \) satisfies the following estimates on \( \tilde{\tilde{\Sigma}}_t \),

\[
(628) \quad \left\| \tilde{\tilde{\xi}} \right\|_{\tilde{\tilde{\Sigma}}_t}^{n+1, p} \leq \delta_o (r_0 + ts_0) \leq r_0 + ts_0, \quad \left\| \tilde{\tilde{\xi}} \right\| = ts_0 = ts_0
\]

where

\[
(629) \quad \delta_o \leq \frac{t(r_0 + s_0)}{r_0 + s_0} \delta_o \leq 2 \delta_o.
\]

**Lemma C.5** (Estimates of the terms \( \tilde{\tilde{f}}, \tilde{t} \tilde{e} f_t, \tilde{\tilde{e}}^{i} f_{ij}, b^{ij} f_{\tilde{f}}, \tilde{\tilde{e}}^{i} \tilde{f} \), \( \tilde{\tilde{e}}^{i} \), \( \tilde{\tilde{e}}^{i} \), \( \tilde{\tilde{\xi}} \), \( \tilde{\tilde{\xi}} \), \( \tilde{\tilde{\xi}} \), \( \tilde{\tilde{\xi}} \)).

Under the bootstrap assumption 4.10 of \( \tilde{\tilde{f}} \), we have the following estimates of the terms

\[
\tilde{\tilde{f}}, \tilde{t} \tilde{e} f_t, \tilde{\tilde{e}}^{i} f_{ij}, b^{ij} f_{\tilde{f}}, \tilde{\tilde{e}}^{i} \tilde{f}, \tilde{\tilde{e}}^{i} \tilde{f}, \tilde{\tilde{e}}^{i} \tilde{f}, \tilde{\tilde{e}}^{i} \tilde{f}, \tilde{\tilde{e}}^{i} \tilde{f}, \tilde{\tilde{e}}^{i} \tilde{f}, |e|, e, e, |e|^2.
\]
on $\tilde{\Sigma}_t$, 

\begin{align}
\| \tilde{f} \|_{\tilde{\Sigma}_t, g}^{n+2,p} & \leq s_0 + r_0 \delta_o, \\
\| \tilde{g} \|_{\tilde{\Sigma}_t, g}^n \leq & \frac{c(n, p)}{(r_0 + ts_0)^2} t(r_0 + s_0) \delta_o \tilde{\Sigma}_0, \\
\| \tilde{f} \|_{\tilde{\Sigma}_t, g}^n & \leq c(n, p) \Delta o, \\
\| \tilde{g} \|_{\tilde{\Sigma}_t, g}^{n+1,p} & \leq \frac{c(n, p)}{(r_0 + ts_0)^2} t(r_0 + s_0) \delta_o, \\
\| \tilde{f} \|_{\tilde{\Sigma}_t, g}^{n+1,p} & \leq c(n, p) \Delta o, \\
\| \tilde{g} \|_{\tilde{\Sigma}_t, g}^n & \leq c(n, p) \Delta o, \\
\| \tilde{g} \|_{\tilde{\Sigma}_t, g}^{n+1,p} & \leq c(n, p) \Delta o, \\
\| \tilde{g} \|_{\tilde{\Sigma}_t, g}^n & \leq c(n, p) \Delta o.
\end{align}

Lemma C.6 (Estimates of the partial differentials of $F$ and $\tilde{F}$). Under the bootstrap assumption 4.10 of $\tilde{f}$, we have the following estimates of the partial differentials of $F$ and $\tilde{F}$.

\[
\partial^i F, \ \partial^i \tilde{f}_j, \ \partial^i \tilde{g}_j, \ \partial^i \tilde{g}_j, \ \partial^i \tilde{f}_j, \ \partial^i \tilde{g}_j, \ \partial^i \tilde{g}_j, \ \partial^i \tilde{f}_j, \ \partial^i \tilde{g}_j, \ \partial^i \tilde{g}_j, \ \partial^i \tilde{g}_j
\]

on $\tilde{\Sigma}_t$, 

\begin{align}
\| \partial^i F \|^2_{\tilde{\Sigma}_t, g} & \leq \frac{c(n, p) r_o^2}{(r_0 + ts_0)^2} \Delta o + \frac{c(n, p) c_o r_o^3}{(r_0 + ts_0)^3} \epsilon (c_m, \delta_o m + c_m, \delta_o^2 + c_o, \delta_o) \Delta o.
\end{align}
\[
\left\| \partial_{\tilde{\omega} f_i} F \right\|_{\tilde{\Sigma}'_0}^{n,p} \leq \left( s_0 + r_0 \delta_o \right) \cdot \left\{ \frac{c(n,p) c_o r_0^2}{s^2} + \frac{c(n,p) c_o r_0}{(r_0 + t s_0)^2} \right\} \cdot \left( c_{m,n} \delta_m + c_m \delta^2_n + c_o \delta_o \right) \cdot \delta_o \cdot \delta_o \cdot \delta_o.
\]

\[
\left\| \partial_{\tilde{t} f_i} F \right\|_{\tilde{\Sigma}'_0}^{n,p} \leq \left( s_0 + r_0 \delta_o \right) \cdot \left\{ \frac{c(n,p) c_o r_0^2}{s^2} + \frac{c(n,p) c_o r_0}{(r_0 + t s_0)^2} \right\} \cdot \left( c_{m,n} \delta_m + c_m \delta^2_n + c_o \delta_o \right) \cdot \delta_o \cdot \delta_o \cdot \delta_o.
\]

\[
\left\| \partial_{\tilde{\ell} f_i} F \right\|_{\tilde{\Sigma}'_0}^{n,p} \leq \left( s_0 + r_0 \delta_o \right) \cdot \left\{ \frac{c(n,p) c_o r_0^2}{s^2} + \frac{c(n,p) c_o r_0}{(r_0 + t s_0)^2} \right\} \cdot \left( c_{m,n} \delta_m + c_m \delta^2_n + c_o \delta_o \right) \cdot \delta_o \cdot \delta_o \cdot \delta_o.
\]
We have the following estimates of the second order partial differentials of $F$ and $\tilde{z}$ on $\tilde{\Sigma}_t$, 

\begin{align}
\|\partial^2_{f_i\partial y_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2 + \frac{c(n,p)c_{\alpha}r_0^3}{(r_0 + ts_0)^3}\epsilon(c_{m,m}\delta_m + c_{m,o}\delta_o^2 + c_o\delta_o)\delta_0, \\
\|\partial^2_{f_i\partial z_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2 + \frac{c(n,p)c_{\alpha}r_0^3}{(r_0 + ts_0)^3}\epsilon(c_{m,m}\delta_m + c_{m,o}\delta_o^2 + c_o\delta_o)\delta_0, \\
\|\partial^2_{\tilde{f}_i\tilde{z}_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2, \\
\|\partial^2_{\tilde{f}_i\tilde{y}_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2, \\
\|\partial^2_{a_i\partial y_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2, \\
\|\partial^2_{a_i\partial z_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2, \\
\|\partial^2_{\tilde{a}_i\tilde{y}_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2, \\
\|\partial^2_{\tilde{a}_i\tilde{z}_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2, \\
\|\partial^2_{\tilde{e}_i\tilde{y}_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2, \\
\|\partial^2_{\tilde{e}_i\tilde{z}_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2, \\
\|\partial^2_{\tilde{e}_i\tilde{y}_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2, \\
\|\partial^2_{\tilde{e}_i\tilde{z}_i} F\|_{\tilde{\Sigma}_t}^{n,p} & \leq \frac{c(n,p)c_{\alpha}^2\partial^2}{(r_0 + ts_0)^2}\delta_0^2.
\end{align}
We have the following estimate of Lemma C.7. Under the bootstrap assumption 4.10 of $\hat{g}$, we have the following estimates on $\hat{\Sigma}_t$.

\[
\left| \partial_{\xi, j}^{n, p} F \right|^{n, p}_{\hat{\Sigma}_t, \hat{\Sigma}_t} \leq (s_0 + r_0 \delta_o) \cdot \frac{c(n, p)c_r^2r_0^2}{(r_0 + ts_0)^2 \delta_o},
\]

\[
\left| \partial_{\xi}^{n, p} F \right|^{n, p}_{\hat{\Sigma}_t, \hat{\Sigma}_t} \leq (r_0 + c(n, p)\delta_o(r_0 + s_0)) \cdot c(n, p)\delta_o,
\]

\[
\left| \partial_{\xi}^{n, p} F \right|^{n, p}_{\hat{\Sigma}_t, \hat{\Sigma}_t} \leq c(n, p)(r_0 + ts_0)^2 \delta_o^2.
\]

\[
\left| \partial_{\xi}^{n, p} F \right|^{n, p}_{\hat{\Sigma}_t, \hat{\Sigma}_t} \leq c(n, p)c_r^4r_0^4 \delta_o^4,
\]

\[
\left| \partial_{\xi}^{n, p} F \right|^{n, p}_{\hat{\Sigma}_t, \hat{\Sigma}_t} \leq c(n, p)c_r^6r_0^6 \delta_o^6.
\]

We have the following estimate of $\hat{X}$ on $\hat{\Sigma}_t$.

\[
\left| \text{div } \hat{X} \right|^{n, p}_{\hat{\Sigma}_t, \hat{\Sigma}_t}, \quad ||\hat{X}||^{n, p}_{\hat{\Sigma}_t, \hat{\Sigma}_t}
\]
Lemma C.8 (Estimate of the term $\tilde{t} \tilde{r}$). Under the bootstrap assumption 4.10 of $\tilde{f}$, we have the following estimates

\begin{align}
\leq (s_0 + r_0 \delta_0) \cdot \\
\left\{ \frac{c(n, p)c_o^2 r_0^2}{(r_0 + t s_0)^2} \delta_o^2 + \frac{c(n, p)c_o r_0^3}{(r_0 + t s_0)^3} \epsilon(m, m \delta_m + c_m \delta_o^2 + c_o \delta_o \delta_o) \right\} \cdot \frac{c(n, p)}{r_0 + t s_0} \delta_o \\
+ (s_0 + r_0 \delta_0) \cdot \frac{c(n, p)c_0 r_0}{(r_0 + t s_0)^2} \epsilon(m, m \delta_m + c_m \delta_o^2 + c_o \delta_o) r_0 \\
+ (s_0 + r_0 \delta_0) \cdot \frac{c(n, p)c_o r_0}{(r_0 + t s_0)^2} \delta_o,
\end{align}

and

\begin{align}
(678) \\
\int_0^{t a} \left| \tilde{z} \tilde{X} \right| \text{div} \tilde{\mathbf{X}} \, dt \leq & \ c(n, p) \left\{ \frac{c_o^2 \delta_o^2 + c_o \epsilon(m, m \delta_m + c_m \delta_o^2 + c_o \delta_o \delta_o)}{(r_0 + t s_0)^2} \delta_o \right\} \cdot (r_0 + s_0) \delta_o, \\
+ & \ c(n, p) \left\{ \epsilon(m, m \delta_m + c_m \delta_o^2 + c_o \delta_o) + \epsilon(m, m \delta_m + c_m \delta_o^2 + c_o \delta_o) \delta_o \right\} t_a \\
+ & \ c(n, p) \delta_o t_a.
\end{align}

Proof. All the estimates follow from the bootstrap assumption and the previous lemmas. The last estimate of the integration is proved in two cases $s_0 \leq r_0$ and $s_0 > r_0$ by direct calculations.

Lemma C.8 (Estimate of the term $\tilde{t} \tilde{r}$). Under the bootstrap assumption 4.10 of $\tilde{f}$, we have the following estimates

\begin{align}
(679) \\
\left\| \partial j \tilde{F} \cdot \tilde{\Delta} \tilde{f} \right\|^{\tilde{z}, \tilde{\chi}, \tilde{F}} \leq & \ \left\{ \frac{c(n, p)c_o^2 r_0^2}{(r_0 + t s_0)^2} \delta_o^2 + \frac{c(n, p)c_o r_0^3}{(r_0 + t s_0)^3} \epsilon(m, m \delta_m + c_m \delta_o^2 + c_o \delta_o \delta_o) \right\} \cdot (r_0 + s_0) \delta_o, \\
(680) \\
\left\| \partial \tilde{t} \tilde{f} \tilde{F} \cdot \tilde{\Delta} \tilde{f} \right\|^{\tilde{z}, \tilde{\chi}, \tilde{F}} \leq & \ \left\{ \frac{c(n, p)c_o^2 r_0^2}{(r_0 + t s_0)^2} \delta_o^2 + \frac{c(n, p)c_o r_0^3}{(r_0 + t s_0)^3} \epsilon(m, m \delta_m + c_m \delta_o^2 + c_o \delta_o) \right\} \cdot (r_0 + s_0) \delta_o, \\
(681) \\
\left\| \partial \tilde{t} \tilde{f} \tilde{F} \cdot \tilde{\Delta} \tilde{f} \right\|^{\tilde{z}, \tilde{\chi}, \tilde{F}} \leq & \ (r_0 + t s_0) \delta_o \cdot \frac{c(n, p)c_o^2 r_0^2}{(r_0 + t s_0)^2} \delta_o^2, \\
+ & \ \left\{ \frac{c(n, p)c_o^2 r_0^2}{(r_0 + t s_0)^2} \delta_o^2 + \frac{c(n, p)c_o r_0^3}{(r_0 + t s_0)^3} \epsilon(m, m \delta_m + c_m \delta_o^2 + c_o \delta_o) \right\} \cdot (r_0 + s_0) \delta_o^2.
\end{align}
\[ (682) \left\| \partial_{t^bf^i_f} F \cdot \left( \frac{\tilde{z}_0^2}{\tilde{z}_0^2} \right) \right\|^2_{\tilde{\Sigma}_{t^0}} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n,p)c_0 r_0^2 \delta_0^2}{(r_0 + t s_0)^2} + \frac{c(n,p)c_0 r_0^3}{(r_0 + t s_0)^3} \varepsilon (c_{m,m} \delta_m + c_{m,o} \delta_o^2 + c_o \delta_o) \right\} \]

\[ (683) \left\| \partial_{\bar{z}F} F \cdot \partial^2_{\bar{z}F} \right\|_{\tilde{\Sigma}_{t^0}} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n,p)c_0 r_0^2 \delta_0^2}{(r_0 + t s_0)^2} \right\} \]

\[ (684) \left\| \partial_{\bar{z}F} F \cdot \partial^2_{\bar{z}F} \right\|_{\tilde{\Sigma}_{t^0}} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n,p)c_0 r_0^2 \delta_0^2}{(r_0 + t s_0)^2} \right\} \]

\[ (685) \left\| \partial_{w^i_w} F \cdot \partial^2_{w^i_w} \right\|_{\tilde{\Sigma}_{t^0}} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n,p)c_0 r_0^2 \delta_0^2}{(r_0 + t s_0)^2} \right\} \]

\[ (686) \left\| \partial_{w^i_w} F \cdot \partial^2_{w^i_w} \right\|_{\tilde{\Sigma}_{t^0}} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n,p)c_0 r_0^2 \delta_0^2}{(r_0 + t s_0)^2} \right\} \]

\[ (687) \left\| \partial_{w^i_w} F \cdot \partial^2_{w^i_w} \right\|_{\tilde{\Sigma}_{t^0}} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n,p)c_0 r_0^2 \delta_0^2}{(r_0 + t s_0)^2} \right\} \]

\[ (688) \left\| \partial^2_{a^j} F \cdot \partial^2_{a^j} \right\|_{\tilde{\Sigma}_{t^0}} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n,p)c_0 r_0^2 \delta_0^2}{(r_0 + t s_0)^2} \right\} \]
(689) \[ \left\| \frac{\partial^2}{\partial a \partial b} F \cdot \frac{\partial}{\partial t} \frac{\partial}{\partial \hat{\nu}} \right\|_{\hat{\Sigma}_t, g}^{n-1,p} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n, p)c_o r_0^2}{(r_0 + t_s_0)^2} \delta_o^2 + \frac{c(n, p)c_o r_0^3}{(r_0 + t_s_0)^3} (c_m, m \delta_m + c_m, o \delta_o^2 + c_o \delta_o) \right\} \cdot \left\{ \frac{c(n, p)c_o (r_0 + s_o) t}{(r_0 + t_s_0)^2} \delta_o + \frac{c(n, p) r_0^2 (r_0 + s_o) t}{(r_0 + t_s_0)^3} c_\delta (c_m, m \delta_m + c_m, o \delta_o^2 + c_o \delta_o) \right\}^2, \]

a, b = t \tilde{a} f, t \tilde{b} f, \tilde{e} i \tilde{f}, \tilde{e} i \tilde{f}, (690)

(691) \[ \left\| \frac{\partial^2}{\partial a \partial \tilde{a}} F \cdot \frac{\partial}{\partial t} \frac{\partial}{\partial \tilde{f}} \right\|_{\hat{\Sigma}_t, g}^{n-1,p} \leq c(n, p)(s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n, p)c_o r_0^2}{(r_0 + t_s_0)^2} \delta_o^2 + \frac{c(n, p)c_o r_0^3}{(r_0 + t_s_0)^3} (c_m, m \delta_m + c_m, o \delta_o^2 + c_o \delta_o) \right\}^2, \]

a = \tilde{a}, \tilde{e} i \tilde{f}, b = \tilde{b} \tilde{f}, (692)

(693) \[ \left\| \tilde{\epsilon} e \right\|_{\hat{\Sigma}_t, g}^{n-1,g} \leq (s_0 + r_0 \delta_o) \cdot \left\{ \frac{c(n, p)c_o r_0^2}{(r_0 + t_s_0)^2} \delta_o^2 + \frac{c(n, p)c_o r_0^3}{(r_0 + t_s_0)^3} (c_m, m \delta_m + c_m, o \delta_o^2 + c_o \delta_o) \right\}, \]

We have the following estimate of \( \tilde{\epsilon} e \) on \( \hat{\Sigma}_t \),
\[ \int_0^{t_a} \left\| \tilde{\tau} \right\|_n^{n-1,p} \, dt \leq c(n, p) \left\{ c_0^2 \delta_m^2 + c_0 \epsilon (c_{m,m} \delta_m + c_{m,o} \delta_o^2) \right\} r_0 t_a. \]  

Then by the above lemmas of $\tilde{\tau}$ and $\tilde{\tau}$, we conclude the following lemma for the propagation equation for $\tilde{\tau}$.

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