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Bifurcation phenomena of mean-field coupled self-sustained oscillators with two different time scales under the influence of external white noise

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Abstract. We report stochastic phenomena of inter-cluster synchronization in ensembles of coupled self-sustained oscillators with different time scales. A mean-field model of two inter-connected clusters of coupled limit cycle oscillators having same trajectories with two native frequencies is considered to analytically derive the time evolution of the order parameters in the thermodynamic limit. A set of the order parameters obtained for the system exhibits noise induced bifurcations, including those from torus type attractors to fixed point ones for the case with two different heat baths in accordance with the introduction of the time scale.

1. Introduction
Synchronization phenomena are ubiquitous ones observed in many fields of natural sciences [1]. While effects of noise on synchronization phenomena in oscillatory systems are breakdown of synchronization, they are widely reported to exhibit opposite effects on synchronization [2, 3]. As far as we know, there are few papers which study the outcome of considering same trajectories with different native frequencies within the frame work of noise induced synchronization. In this paper, we study the relationships between noise influence and synchronization phenomena in ensembles of two clusters of globally coupled limit cycle oscillators with different time scales. A nonlinear Fokker-Planck equation (NFPE) approach to our model, which turns out to be noise level-free analysis, is applied to derive the time evolution equation of the order parameters in the thermodynamic limit. Noting that our basic oscillator model includes an excitable element model with changes in the parameters, observations of noise induced synchronization in our system will be of practical interest.

2. Model and nonlinear Fokker-Planck equation approach
We present a simple model where the ensemble consists of two inter-connected clusters of mean-field coupled limit cycle oscillators having same trajectories with two native frequencies [4, 5, 6]:

\[
\frac{d\phi_{i}^{(x)}}{dt} = -a^{(x)}\phi_{i}^{(x)} + \frac{J^{(x)}}{N} \sum_{j=1}^{N} F_{i}^{(x)}(y)\phi_{j}^{(y)} + I
\]
where \( z_{\alpha i}^{(\mu)} (\mu = x, y) (\alpha = 1, 2) \) are the dynamical variables of the two dimensional oscillators at site \( i (i = 1, \cdots, N) \) of cluster \( \alpha \), \( a^{(\mu)} \) and \( b^{(\mu, \nu)} \) are constants, \( J^{(\mu)} \) and \( K^{(\mu)} \) are intra- and inter-cluster coupling strengths, \( F^{(\mu)}_\alpha (\cdot) \) are nonlinear coupling functions, and \( \eta^{(\mu)}_{\alpha i} (t) \) are independent external noise, respectively. Due to the introduction of the time scale parameter \( \tau \), the native frequency of each oscillator in cluster 2 can be different from that of each oscillator in cluster 1, in spite of their same trajectories. The inter-cluster coupling strength \( \epsilon \) may take any real constant value, not limited to weak connections. The external noise are of independent white Gaussian type, \( \langle \eta^{(\mu)}_{\alpha i} (t) \rangle = 0, \langle \eta^{(\mu)}_{\alpha i} (t) \eta^{(\nu)}_{\beta j} (t') \rangle = 2D^{(\mu)}_{\alpha \beta} \delta_{ij} \delta_{\alpha \beta} \delta(t - t'). \)

Applying an NFPE approach to this model [4, 5, 6], we derive the time evolution of each moment and obtain a set of closed ordinary differential equations involving at most second moments of the Gaussian probability density ensured for large times by the H theorem:

\[
\frac{d\langle z^{(x)}_{1i} \rangle}{dt} = -a^{(x)} \langle z^{(x)}_{1i} \rangle + J^{(x)} \langle F^{(x)}_2 \rangle + I + \epsilon K^{(x)} \langle F^{(x)}_2 \rangle,
\]

\[
\frac{d\langle z^{(y)}_{1i} \rangle}{dt} = \kappa(-a^{(y)} \langle z^{(y)}_{1i} \rangle + J^{(y)} \langle F^{(y)}_1 \rangle + \epsilon K^{(y)} \langle F^{(y)}_1 \rangle),
\]

\[
\frac{d\langle z^{(x)}_{2i} \rangle}{dt} = \tau(-a^{(x)} \langle z^{(x)}_{2i} \rangle + J^{(x)} \langle F^{(x)}_1 \rangle + I + \epsilon K^{(x)} \langle F^{(x)}_1 \rangle),
\]

\[
\frac{d\langle z^{(y)}_{2i} \rangle}{dt} = \kappa(-a^{(y)} \langle z^{(y)}_{2i} \rangle + J^{(y)} \langle F^{(y)}_1 \rangle + \epsilon K^{(y)} \langle F^{(y)}_1 \rangle).
\]

We note that \( \langle u^{(\mu)}_{\alpha i} \rangle \rightarrow D^{(\mu)}_{\alpha i} / \{ a^{(\mu)} \left[ (\tau - 1) \delta_{2i} + 1 \right] \left[ (\kappa - 1) \delta_{y} + 1 \right] \} \) and \( \langle u^{(\mu)}_{\alpha i} u^{(\nu)}_{\beta j} \rangle \rightarrow 0 \) (\( t \rightarrow \infty \)).

To investigate qualitative dynamical behavior of the system, we specify the coupling functions as \( F^{(x)}_\alpha (z) = z \exp \left( -z^2 / 2 \right) \) and \( F^{(y)}_\alpha (z) = z \), which reflects excitable features of neural oscillators [5]. Then, one has

\[
\langle F^{(x)}_\alpha \rangle = \frac{m^{(x)}_{\alpha}}{(2\sigma^2_{\alpha} + 1)^{3/2}} \exp \left[ -\frac{m^{(x)}_{\alpha}^2}{2(2\sigma^2_{\alpha} + 1)} \right], \langle F^{(y)}_\alpha \rangle = m^{(y)}_{\alpha},
\]
Figure 1. Dependence of the first and second Lyapunov exponents $\lambda_1$, $\lambda_2$ on the Langevin noise intensity $D^{(x)}_z = D^{(x)}_y / \tau = D^{(x)}$. With an increase of noise intensity, the qualitative changes in the attractors formed in the order parameter system are observed, involving the appearance and disappearance of the inter-cluster synchronization phenomena. In this system, the occurrence of the bifurcations from torus type attractors to fixed point ones is worth noting. The model parameter values are $a^{(x)} = 1.6875$, $a^{(y)} = 2.0925$, $b^{(x,x)} = 0.87750$, $b^{(x,y)} = -0.33750$, $b^{(y,x)} = 2.7000$, $b^{(y,y)} = -0.67500$, $J^{(x)} = K^{(x)} = 2.5650$, $J^{(y)} = K^{(y)} = 2.835$, $\kappa = 0.0068000$, $I = -2.0$, $D^{(y)} = 0$, $\epsilon = 0.0012$, $\tau = 0.9$, $D^{(y)} = 0$.

where $m^{(\mu)}_a = b^{(u,x)} \langle z^{(x)}_a \rangle_a + b^{(u,y)} \langle z^{(y)}_a \rangle_a$, $\sigma^2_a = b^{(x,x)} \langle u^{(x)}_a \rangle_a + b^{(x,y)} \langle u^{(y)}_a \rangle_a$.

3. Nonequilibrium phase transitions including stochastic inter-cluster synchronization

Using an NFPE approach, we have reduced the system of two clusters of mean-field coupled limit cycle oscillators from the set of the Langevin equations (1) - (4) to the moment equations (5) - (8). Investigating nonlinear aspects of the obtained reduced system, we can reveal the effects of noise, frequency differences, and coupling strengths on inter-cluster synchronization.

We performed numerical calculations with the fourth-order Runge-Kutta method. Stochastic phenomena of inter-cluster synchronization associated with nonequilibrium phase transitions are systematically investigated with changes in the inter-cluster coupling strength $\epsilon$, the time scale $\tau$, and the Langevin noise intensity $D^{(x)}_z = D^{(x)}_y / \tau = D^{(x)}$ which corresponds to the introduction of the two different heat baths based on the time scale $\tau$.

The most interesting result of stochastic phenomena in this system is the noise induced bifurcations from torus type attractors to fixed point ones as shown in Figs. 1 and 2. Conducting a linear stability analysis of fixed points near the bifurcation point $D^{(x)} = 0.017378$, we show that eigenvalues of the Jacobian matrix of the order parameter equations (5) - (8) are two pairs of purely imaginary numbers. Since our model setting of the mean-field coupled oscillators having same trajectories with two native frequencies and two heat baths in accordance with the time scale contribute to this kind of bifurcation structure, it may not be frequently observed in other systems. The typical case of noise induced inter-cluster synchronization phenomena is also observed, as were shown by previously reported work [6].
Figure 2. The behavior of a torus type oscillation in the order parameter system corresponding to Fig. 1 with $D^{(x)} = 0.017060$. a), b), c) The trajectories in phase space projected to the $\langle z^{(x)}_1 \rangle_G - \langle z^{(y)}_1 \rangle_G$, $\langle z^{(x)}_2 \rangle_G - \langle z^{(y)}_2 \rangle_G$, and $\langle z^{(x)}_1 \rangle_G - \langle z^{(x)}_2 \rangle_G$ planes, respectively. d) The time evolution of the order parameters of the system. Sizes of the torus type attractors are very small and shapes of those are almost circular near the bifurcation point.

4. Concluding remarks
We have shown the effects of the independent white Gaussian noise on synchronization phenomena in ensembles of limit cycle oscillators having same trajectories with different native frequencies. We have considered a mean-field model of two clusters of coupled limit cycle oscillators with different time scales. Using an NFPE approach, we have analytically reduced the system from a set of the Langevin equations to a set of ordinary differential equations of the order parameters. This noise level-free analysis has allowed us to reveal various interesting bifurcations including the inter-cluster synchronization and torus type attractors by external noise.

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