The $q$-Riccati Algebra

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Abstract: For $q \in (0, 1)$, we introduce the $q$-Riccati Lie algebra. Using the $q$-derivative (or Jackson derivative), we give a representation of this Lie algebra.

Keywords: $q$-Derivative, $q$-Riccati Lie Algebra

Introduction

In the mathematical field of representation theory, the representation of a Lie algebra is a way of writing a Lie algebra as a set of matrices (or endomorphisms of a vector space) in such a way that the Lie bracket is given by the commutator. More precisely, a representation of a Lie algebra $g$ is a linear transformation:

$$\psi : g \to M(V)$$

where, $M(V)$ is the set of all linear transformations of a vector space $V$. In particular, if $V = \mathbb{R}^n$, then $M(V)$ is the set of $n \times n$ square matrices. The map $\psi$ is required to be a map of Lie algebras so that:

$$\psi[(A,B)] = \psi(A)\psi(B) - \psi(B)\psi(A)$$

for all $A,B \in g$. Note that the expression $AB$ only makes sense as a matrix product in a representation. For example, if $A$ and $B$ are antisymmetric matrices, then $AB-BA$ is skew-symmetric, but $AB$ may not be antisymmetric. The possible irreducible representations of complex Lie algebras are determined by the classification of the semi simple Lie algebras. Any irreducible representation $V$ of a complex Lie algebra $g$ is the tensor product $V = V_0 \otimes L$, where $V_0$ is an irreducible representation of the quotient $g_{\text{Rad}}(g)$ of the algebra $g$ and its Lie algebra radical and $L$ is a one-dimensional representation. In the study of representations of a Lie algebra, a particular ring, called the universal enveloping algebra, associated with the Lie algebra plays an important role. The Riccati algebra is a finite-dimensional linear space that is closed under commutator, that is $R$ is a Lie algebra.

In recent years the $q$-deformation of the Heisemburg commutation relation has drawn attention. Leeuwen and Maassen (1995) and many of other researcher like (Altoum, 2018a; 2018b; Rguigui, 2015a; 2015b; 2016a; 2016b; 2018a; 2018b; Altoum et al., 2017), the purpose is to study the probability distribution of a non-commutative random variable $a + a^*$, where $a$ is a bounded operator on some Hilbert space satisfying:

$$aa^* - qa^* = 1,$$

for some $q \in [-1, 1)$. The calculation is inspired by the case, $q = 0$, where $a$ and $a^*$ turn out to be the left and right shift on $l^2(\mathbb{N})$: In this case $a$ and $a^*$ can be quite nicely represented as operators on the Hardy class $H^2$ of all analytic functions on the unit disk with $L^2$ limits toward the boundary. Subsequently, they find a measure $\mu_q$, $q \in [0, 1)$, on the complex plane that replaces the Lebesgue measure on the unit circle in the above: $\mu_q$ is concentrated on a family of concentric circle, the largest of which has the radius $\frac{1}{\sqrt{1-q}}$. Their representation space (Leeuwen and Maassen, 1995) will be $S^2(\mathcal{D}_q, \mu_q)$, the completion of the analytic functions on
\[ D_q = \left\{ z \in \mathbb{C} \mid z^2 < \frac{1}{1-q} \right\} \] with respect to the inner product defined by \( \mu_q \). In this space annihilation operator \( a \) is represented by a \( q \) difference operator \( D_q \). As \( q \) tends to 1, \( \mu_q \) will tend to the Gauss measure on \( \mathbb{C} \) and \( D_q \) becomes differentiation. We recall some basic notations of the language of \( q \)-calculus (Abdi, 1962; Adams, 1929; Gasper and Rahman, 1990; Jackson, 1910; Leeuwen and Maassen, 1995). For \( q \in (0, 1) \) and analytic \( f: \mathbb{C} \to \mathbb{C} \) define operators \( Z \) and \( D_q \) as follows (Gasper and Rahman, 1990; Jackson, 1910; Leeuwen and Maassen, 1995):

\[
\begin{align*}
(Zf)(z) &= f(z), \\
(D_q f)(z) &= \begin{cases} 
  f(z) - f(qz) & , z \neq 0 \\
  f'(0) &
\end{cases}
\end{align*}
\]

In this paper, we introduce the \( q \)-Riccati Algebra. This paper is organized as follows: In Section 1, we present preliminaries include \( q \)-calculus. In Section 2, we introduce the \( q \)-Riccati algebra. In section 3, we give a representation of this algebra.

**Representation of the \( q \)-Riccati Algebra**

Let \( q \in (0, 1) \). Then, we define the \( q \)-Riccati Lie algebra as follows:

\[ R_q = \langle A, B, C, D \rangle \]

such that:

1. \([A, B] = AD\).
2. \([A, C] = [2]_q CD\).
3. \([B, C] = qCD\).
4. \([A, D] = 0\).
5. \([B, D] = (1-q)BD\).
6. \([C, D] = (1-q)[2]_q CD\).

**Representation of the \( q \)-Riccati Algebra**

Let \( M_{0,q}, M_{1,q} \) and \( M_{2,q} \) given by:

\[
\begin{align*}
M_{0,q} &= D_q \\
M_{1,q} &= XD_q \\
M_{2,q} &= X^2 D_q
\end{align*}
\]

where, \( D_q \) and \( X \) are defined as follows:

\[
\begin{align*}
D_q f(x) &= \frac{f(x) - f(qx)}{x(1-q)} \\
X f(x) &= x f(x).
\end{align*}
\]

**Proposition 3.1**

For \( q \in (0, 1) \) we have:

i) \([M_{0,q}, M_{1,q}] = M_{0,q} H_q\)

ii) \([M_{0,q}, M_{2,q}] = [2]_q M_{1,q} H_q\)

iii) \([M_{1,q}, M_{2,q}] = q M_{2,q} H_q\)

where, \( H_q \) is given by \( H_q f(x) = f(qx) \)

**Proof**

We have:

\[
\begin{align*}
[M_{0,q}, M_{1,q}] &= \left[D_q, XD_q\right] \\
&= D_q X D_q - XD_q D_q
\end{align*}
\]

But:

\[
\begin{align*}
D_q X D_q f(x) &= D_q \left( x \frac{f(x) - f(qx)}{x(1-q)} \right) \\
&= \frac{1}{1-q} D_q \left( f(x) - f(qx) \right) \\
&= \frac{1}{1-q} \frac{f(x) - f(qx) - f(qx) + f(q^2 x)}{x(1-q)} \\
&= \frac{1}{1-q} \frac{f(x) - 2f(qx) + f(q^2 x)}{x(1-q)}
\end{align*}
\]

and:

\[
\begin{align*}
X D_q D_q f(x) &= x D_q \left( \frac{f(x) - f(qx)}{x(1-q)} \right) \\
&= x \left( \frac{f(x) - f(qx) + f(q^2 x)}{x(1-q)} \right) \\
&= x \left( \frac{x f(x) - qf(qx) - f(qx) + f(q^2 x)}{x(1-q)} \right) \\
&= \frac{1}{1-q} \frac{qf(x) - qf(qx) - f(qx) + f(q^2 x)}{qx(1-q)}
\end{align*}
\]

Then, we obtain:

\[
\begin{align*}
[M_{0,q}, M_{1,q}] f(x) &= \frac{f(qx)(1-q)-(1-q)f(q^2 x)}{qx(1-q)^2} \\
&= \frac{f(qx) - f(q^2 x)}{qx(1-q)} \\
&= D_q f(qx) \\
&= D_q H_q f(x).
\end{align*}
\]

But:
\begin{align*}
D_x X^2 D_x f(x) &= x D_x \left( \frac{x^2 f(x) - f(qx)}{x(1-q)} \right) \\
&= \frac{1}{1-q} D_x \left( sf(x) - sf(qx) \right) \\
&= \frac{1}{1-q} \left( \frac{sf(x) - sf(qx)}{x(1-q)} - \frac{xqf(qx) - xqf(q^2x)}{x(1-q)} \right) \\
&= \frac{1}{(1-q)^2} \left( f(x) - (1+q) f(qx) + qf(q^2x) \right)
\end{align*}

Similarly, we get:
\begin{align*}
X^2 D_x f(x) &= x^2 D_x \left( \frac{f(x) - f(qx)}{x(1-q)} \right) \\
&= \frac{1}{1-q} \left( \frac{qf(x) - qf(qx)}{q(x(1-q))} + \frac{f(qx) + f(q^2x)}{q(x(1-q))} \right) \\
&= \frac{1}{q(1-q)} \left( qf(x) - (1+q) f(qx) + f(q^2x) \right)
\end{align*}

Which gives:
\begin{align*}
\left[ M_{0,q}, M_{z,q} \right] &= \frac{1}{q(1-q)} \left( (1+q)(-q+1)f(qx) + (q-1)f(q^2x) \right) \\
&= x(1+q) \left( \frac{f(qx) - f(q^2x)}{q} \right) \\
&= x[2] q D_x f(qx) \\
&= [2] q X D_x f(qx)
\end{align*}

We have:
\begin{align*}
\left[ M_{0,q}, M_{z,q} \right] f(x) &= \left[ XD_x, X^2 D_x \right] \\
&= XD_x X^2 D_x - X^2 D_x XD_x
\end{align*}

\begin{align*}
XD_x X^2 D_x f(x) &= x D_x \left( \frac{xf(x) - xf(qx)}{1-q} \right) \\
&= \frac{x}{1-q} \left( \frac{xf(x) - xf(qx) - qsf(qx) + qsf(q^2x)}{x(1-q)} \right) \\
&= \frac{x}{(1-q)^2} \left( f(x) - (1+q) f(qx) + qf(q^2x) \right)
\end{align*}

Similarly, we have:
\begin{align*}
X^2 D_x XD_x f(x) &= x^2 D_x \left( \frac{f(x) - f(qx)}{1-q} \right) \\
&= \frac{x}{1-q} \left( \frac{f(x) - f(qx) - f(q^2x) - f(q^3x)}{x(1-q)} \right) \\
&= \frac{x}{q(1-q)^2} \left( f(x) - 2f(qx) + f(q^2x) \right)
\end{align*}

Then, we get:
\begin{align*}
\left[ M_{1,q}, M_{z,q} \right] f(x) &= \frac{x}{(1-q)^2} \left( (1-q)f(qx) - (q-1)f(q^2x) \right) \\
&= \frac{x}{1-q} \left( f(qx) - f(q^2x) \right) \\
&= qx^2 D_x f(qx) \\
&= qx^2 D_x H_q f(x)
\end{align*}

**Proposition 3.2**

For $q \in (0, 1)$ we have:

i) $[M_{0,q}, H_q] = 0.$

ii) $[M_{1,q}, H_q] = (1-q)M_{1,q}H_q.$

iii) $[M_{z,q}, H_q] = (1-q)[2]_q M_{z,q}H_q.$

**Proof**

We have:
\begin{align*}
\left[ D_q, H_q \right] f(x) &= D_q H_q f(x) - H_q D_q f(x) \\
&= D_q f(qx) - H_q \left( \frac{f(x) - f(qx)}{x(1-q)} \right) \\
&= \frac{f(qx) - f(q^2x)}{q(x(1-q))} - \frac{f(qx) - f(q^3x)}{q(x(1-q))} \\
&= 0.
\end{align*}

Then, we get:
\begin{align*}
\left[ M_{0,q}, H_q \right] &= 0.
\end{align*}

We have:
\begin{align*}
\left[ XD_q, H_q \right] f(x) &= XD_q H_q f(x) - H_q XD_q f(x) \\
&= x D_q f(qx) - H_q \left( x D_q f(x) \right) \\
&= x D_q f(qx) - x D_q f(qx) \\
&= 0.
\end{align*}
Then, we get:

\[
\left[ M_{\lambda,q}, H_q \right] = (1-q)M_{\lambda,q}H_q.
\]

We have:

\[
\begin{align*}
& \left[ X^2D_q, H_q \right] f(x) = X^2D_qH_qf(x) - H_q\left( X^2D_qf(x) \right) \\
& = x^2D_qf(qx) - (qx)^2D_qf(x) \\
& = (1-q)^2X^2D_qH_qf(x) \\
& = (1-q)[2_q]X^2D_qH_qf(x).
\end{align*}
\]

Then, we obtain:

\[
\left[ M_{\lambda,q}, H_q \right] = (1-q)[2_q]M_{\lambda,q}H_q,
\]

which complete the proof.

Now, we give the representation theorem of the q-Riccati algebra.

**Theorem 3.3**

Let \( \varphi: R_q \rightarrow g(\mathcal{D}_q, \mu_q) \) a linear mapping such that:

\[
\begin{align*}
\varphi(A) &= M_{\lambda,q} \\
\varphi(B) &= M_{\lambda,q} \\
\varphi(C) &= M_{\lambda,q} \\
\varphi(D) &= H_q.
\end{align*}
\]

Then, \( (\mathcal{D}_q, \mu_q, \varphi) \) is a representation of \( R_q \).

**Proof**

The proof follows from Proposition 3.1 and Proposition 3.2.

**Author’s Contributions**

All authors equally contributed in this work.

**Ethics**

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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