Uniqueness of solving the problem of transport and sedimentation of multicomponent suspensions in coastal systems

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Abstract. The present work is devoted to the study of three-dimensional model of transport and sedimentation of suspended matter in the coastal zone. The model takes into account the following processes: advective transport due to the movement of the aquatic environment, microturbulent diffusion and gravitational sedimentation of particles of the suspension, and changes in the geometry of the bottom caused by sedimentation of the suspension or the rise of bottom sediments. The aim of the work was to conduct an analytical study of the uniqueness of the initial-boundary-value problem corresponding to the constructed model. In accordance with the stated aim, an initial-boundary-value problem is considered for a parabolic type equation with lower derivatives in the domain for which a quadratic functional is constructed and the uniqueness of the solution is proved by the energy method.

1. Introduction

It is necessary to use a set of models of different spatial and temporal scales of averaging to solve practical problems associated with assessing the ecological state of a reservoir. Studies of mathematical models of hydrophysical processes of various levels of complexity have been actively developed in recent decades [1–5]. The use of three-dimensional models is justified in situations with significant vertical variability of the hydrophysical fields of the object of study [6–8].

Three-dimensional mathematical model for calculating transport and sedimentation of suspended matter in relation to shallow water areas is considered in the work. It takes into account the processes of diffusion-convection, the movement of the aquatic environment, the processes of lifting, transport and sedimentation of bottom material, turbulent exchange in the vertical and horizontal directions [9–11].

The problems of parallelizing the processes of numerical solution of these problems on massive parallel systems were investigated previously, in work [12–14], in connection with the use of explicit-implicit difference schemes that ensure high efficiency of algorithms for systems containing many tens of thousands of cores. In this paper, an analytical study of the uniqueness of the solution of the initial-boundary-value problem corresponding to the constructed model is carried out, the results of which are formulated as a theorem.
2. Continuous 3D model of transport and sedimentation of suspended matter and the corresponding initial-boundary-value problem

Rectangular Cartesian coordinate system $Oxyz$ was used, where the axis $Ox$ runs along the surface of an undisturbed water surface and is directed towards the sea. Let $h=H+\eta$ is the total depth of the water area, [m]; $H$ is the depth with an unperturbed surface of the reservoir, [m]; $\eta$ is the elevation of the free surface relative to sea level, [m].

Let in the water volume $V=\{0 \leq x \leq L_x, 0 \leq y \leq L_y, 0 \leq z \leq H(x,y,t)\}$, are located $R$ types of suspended particles that are at a point $(x,y,z)$ and at time $t$ have a concentration $c_r(x,y,z,t)$, [mg / l]; $t$ is the time variable, [sec].

Consider the region $V$ as a parallelepiped sloping to the shore, for which the upper base lies on the free surface, and the lower base is part of the bottom surface (Fig. 1).

![Figure 1. The region $V$ of solutions to the problem of transport and sedimentation of suspended matter sediment.](image)

Border area $V$ is the piecewise smooth surface of form $S=S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$, where

$$S_1 = \{x = 0, 0 \leq y \leq L_y, 0 \leq z \leq H(0,y,t)\},$$

$$S_2 = \{0 \leq x \leq L_x, y = L_y, 0 \leq z \leq H(x,L_y,t)\},$$

$$S_3 = \{0 \leq x \leq L_x, y = 0, 0 \leq z \leq H(x,0,t)\},$$

$$S_4 = \{x = L_x, 0 \leq y \leq L_y, 0 \leq z \leq H(L_x,y,t)\},$$

$$S_5 = \{0 \leq x \leq L_x, 0 \leq y \leq L_y, z = 0\},$$

$$S_6 = \{0 \leq x \leq L_x, 0 \leq y \leq L_y, z = H(x,y,t)\}.$$

The system of equations describing the behavior of suspended particles will look like this:

$$\frac{\partial c_r}{\partial t} + \frac{\partial (uc_r)}{\partial x} + \frac{\partial (vc_r)}{\partial y} + \frac{\partial ((w+w_y)c_r)}{\partial z} = \mu_r \left( \frac{\partial^2 c_r}{\partial x^2} + \frac{\partial^2 c_r}{\partial y^2} + \frac{\partial^2 c_r}{\partial z^2} \right) + F_r,$$

$$F_1 = (\alpha_{c_r} c_r - \beta_{c_r}) + \Phi_{c_r}(x,y,z),$$

$$F_r = (\beta_{c_r} c_r - \alpha_{c_r}) + \alpha_{c_r} c_r - (\alpha_{c_r} c_r - \beta_{c_r}) + \Phi_{c_r}(x,y,z), \quad r = 2, ..., R-1,$$

$$F_R = (\beta_{c_R} c_R - \alpha_{c_R}) + \Phi_{c_R}(x,y,z).$$

(1)
where \( u, v, w \) are vector components, \( \vec{U} \) is fluid velocity, [m/s]; \( w_{r} \) is the hydraulic size or the rate of deposition of particles of the \( r \)-th type, [m/s]; \( \mu, \mu_r \) are the coefficients of horizontal and vertical diffusion of particles of the \( r \)-th type, [m\(^2\)/s]; \( \alpha, \beta \) are particle conversion rates of the \( r \)-th type into \((r-1)\)-th and \((r+1)\)-th type, \([m/s]\); \( \Phi \) is the power of sources of particles of the \( r \)-th type, [mg/l sec].

The terms on the left side of the first equation of system (1) (except for the time derivative) describe the convection of particles: their transport under the influence of fluid flow and gravity. The terms on the right side describe the diffusion of suspensions and their conversion from one type to another.

It is assumed that the rate of lowering of particles in an aqueous medium with the equality of gravity and resistance is constant and equal to \( w_{r} \), where \( r \) is the fraction number determined by density and particle size [15-17].

A solution to system (1) is found in some given region of continuous change of arguments \((x, y, z, t)\), representing a four-dimensional cylinder with generators parallel to the time axis \(Ot\) heights \( T \).

We add the initial conditions to equation (1)

\[ c_r(x, y, z, 0) = c_{r0}(x, y, z) \]  

and conditions on the border of the region \( C_r \)

- on the lateral border

\[ c_r = c^*, \text{ where } c^* \equiv c^*(x, y, z, t) \; ; \]  

- on the surface of the water

\[ c_r = 0; \]  

- on the bottom

\[ \frac{\partial c_r}{\partial n_m} = -\frac{w}{\mu} c_r \; \text{ or } \; \frac{\partial c_r}{\partial z} = -\frac{w}{\mu} c_r; \]  

where \( n_m \) is the external normal to the boundary of the region.

Boundary condition (5) holds for relatively small bottom slopes:

\[ \max_{s_i} \sqrt{\left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2} \leq 1. \]  

Assume that the following condition for the non-degeneracy of the solution domain for all \((x, y, z, t)\) for the stated initial-boundary value problem:

\[ H(x, y, t) \geq h_0 = \text{const} > 0, \; 0 \leq t \leq T. \]  

3. Uniqueness of the solution of the initial-regional problem of transport and sedimentation of suspensions

The existence of solutions to initial-boundary value problems for parabolic equations with lower derivatives (diffusion-convection equations) were considered in [18-19].

Assume that there is classical solution to problem (1) – (5), \( c_r(x, y, z, t) \in C^1(C_r) \cap C(\bar{C}_r) \).
Let prove the uniqueness of the solution. Consider the initial-boundary-value problem (1) – (5) formulated for three types of particles:

\[
\begin{aligned}
&\frac{\partial c_i}{\partial t} + \frac{\partial (uc_i)}{\partial x} + \frac{\partial (vc_i)}{\partial y} + \frac{\partial ((w+w_{x_i})c_i)}{\partial z} = \mu_i \left( \frac{\partial^2 c_i}{\partial x^2} + \frac{\partial^2 c_i}{\partial y^2} + \frac{\partial^2 c_i}{\partial z^2} \right) + F_i, \quad r=1,2,3, \quad (8)
\end{aligned}
\]

Assume that there are two different solutions: \( c_i^{(1)} \neq c_i^{(2)} \), \( r=1,2,3 \). When stand \( c_i^{(1)} \) и \( c_i^{(2)} \) into the first equation of system (8), we obtain the equations:

\[
\begin{aligned}
&\frac{\partial c_i^{(1)}}{\partial t} + \frac{\partial (uc_i^{(1)})}{\partial x} + \frac{\partial (vc_i^{(1)})}{\partial y} + \frac{\partial ((w+w_{x_i})c_i^{(1)})}{\partial z} = \mu_i \left( \frac{\partial^2 c_i^{(1)}}{\partial x^2} + \frac{\partial^2 c_i^{(1)}}{\partial y^2} + \frac{\partial^2 c_i^{(1)}}{\partial z^2} \right) + F_i, \quad r=1,2,3, \quad (9)
\end{aligned}
\]

\[
\begin{aligned}
&\frac{\partial c_i^{(2)}}{\partial t} + \frac{\partial (uc_i^{(2)})}{\partial x} + \frac{\partial (vc_i^{(2)})}{\partial y} + \frac{\partial ((w+w_{x_i})c_i^{(2)})}{\partial z} = \mu_i \left( \frac{\partial^2 c_i^{(2)}}{\partial x^2} + \frac{\partial^2 c_i^{(2)}}{\partial y^2} + \frac{\partial^2 c_i^{(2)}}{\partial z^2} \right) + F_i, \quad r=1,2,3. \quad (10)
\end{aligned}
\]

Subtract equation (10) from equation (9) and, introducing the notation \( \tilde{c}_i = c_i^{(1)} - c_i^{(2)} \neq 0, \quad r=1,2,3 \) we get:

\[
\begin{aligned}
&\frac{\partial \tilde{c}_i}{\partial t} + \frac{\partial (u\tilde{c}_i)}{\partial x} + \frac{\partial (v\tilde{c}_i)}{\partial y} + \frac{\partial ((w+w_{x_i})\tilde{c}_i)}{\partial z} = \mu_i \left( \frac{\partial^2 \tilde{c}_i}{\partial x^2} + \frac{\partial^2 \tilde{c}_i}{\partial y^2} + \frac{\partial^2 \tilde{c}_i}{\partial z^2} \right) + F_i, \quad r=1,2,3, \quad (11)
\end{aligned}
\]

Let supplement the equations of system (11) with the initial condition

\( \tilde{c}_i(x,y,z,0)=0 \) \quad (12)

and conditions on the border of the region \( C_T \):

– on the lateral border

\( \tilde{c}_i = 0 \) \quad (13)

– on the surface of the water

\( \tilde{c}_i = 0 \) \quad (14)

– on the bottom

\[
\frac{\partial \tilde{c}_i}{\partial n_{in}} = -\frac{w_{x_i}}{\mu_i} \quad \text{or} \quad \frac{\partial \tilde{c}_i}{\partial z} = -\frac{w_{x_i}}{\mu_i}. \quad (15)
\]

Let multiply the left and right sides of the first equation of system (11) by \( \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \) and add all three equations:
\[
\sum_{i=1}^{3} \dot{c}_i \frac{\partial c_i}{\partial t} + \sum_{i=1}^{3} \left( \frac{\partial (uc_i)}{\partial x} + \frac{\partial (vc_i)}{\partial y} + \frac{\partial (w+w_{ref}c_i)}{\partial z} \right) = 0
\]

(16)

Then integrate both sides of equation (16) over time on the interval \(0 \leq t \leq T\) and, after that, with respect to spatial variables in the region \(V\). In the first term, the integration order is changed by the Fubini theorem. Get

\[
\sum_{i=1}^{3} \int_{0}^{T} \left( \int_{V} \dot{c}_i \frac{\partial c_i}{\partial t} \, dV \right) \, dt + \sum_{i=1}^{3} \int_{0}^{T} \left( \int_{V} \left( \frac{\partial (uc_i)}{\partial x} + \frac{\partial (vc_i)}{\partial y} + \frac{\partial (w+w_{ref}c_i)}{\partial z} \right) \, dV \right) \, dt = 0
\]

(17)

The first term from the left side of equality (17) is obviously equal to

\[
\frac{1}{2} \sum_{i=1}^{3} \int_{0}^{T} \left( \int_{V} \left( \dot{c}_i \frac{\partial c_i}{\partial t} \right) \, dV \right) \, dt = \frac{1}{2} \sum_{i=1}^{3} \int_{V} \left( c_i^2(x,y,z,T) - c_i^2(x,y,z,0) \right) \, dV.
\]

(18)

The second term from the left side of equality (17), taking into account the continuity equation \(div \vec{U} = 0\) and conditions for a constant particle deposition rate \(\frac{\partial w_{ref}}{\partial z} = 0\) can be converted as follows:

\[
\sum_{i=1}^{3} \int_{0}^{T} \left( \int_{V} \left( \frac{\partial (uc_i)}{\partial x} + \frac{\partial (vc_i)}{\partial y} + \frac{\partial (w+w_{ref}c_i)}{\partial z} \right) \, dV \right) \, dt = 0
\]

(19)

where \(\vec{U}_{ref} = (u,v,w+w_{ref})\).

The equality (19) can be written using the Ostrogradsky-Gauss formula, taking into account the boundary conditions (13)–(15) and the conditions \(w_{ref} = 0\) (the suspension cannot fall below the bottom surface) in the form:

\[
\sum_{i=1}^{3} \int_{0}^{T} \left( \int_{V} \left( \frac{\partial (uc_i)}{\partial x} + \frac{\partial (vc_i)}{\partial y} + \frac{\partial (w+w_{ref}c_i)}{\partial z} \right) \, dV \right) \, dt = \frac{1}{2} \sum_{i=1}^{3} \int_{0}^{T} \left( \int_{V} \left( \partial (uc_i) \right) \, dV \right) \, dt =
\]

\[
= \int_{0}^{T} \left( \int_{V} u_{ref} c_i \, dV \right) \, dt.
\]

(20)
Note that on the boundary surface $S_3$ suspension concentration is equal to 0, therefore, flows through it are equal to 0.

We turn to the transformation of the right-hand side of equality (17). For the first term on the right-hand side of equality (17), we have

\[
\sum_{r=1}^{3} \iint_{V} \int_{0}^{T} \left( \mu_h \left( \frac{\partial \overline{c}_r}{\partial x} \right) + \mu_i \left( \frac{\partial \overline{c}_r}{\partial y} \right) + \mu_w \left( \frac{\partial \overline{c}_r}{\partial z} \right) \right) \, dt \, dV = \sum_{r=1}^{3} \iint_{V} \int_{0}^{T} \left( \frac{\partial \overline{c}_r}{\partial x} \right)^2 \, dt \, dV
\]

\[
= \sum_{r=1}^{3} \iint_{V} \left( \frac{\partial \overline{c}_r}{\partial x} \right)^2 \, dt \, dV - \sum_{r=1}^{3} \iint_{V} \left( \frac{\partial \overline{c}_r}{\partial y} \right)^2 \, dt \, dV + \sum_{r=1}^{3} \iint_{V} \left( \frac{\partial \overline{c}_r}{\partial z} \right)^2 \, dt \, dV.
\]

The second term of the right-hand side of equality (17) can be transformed as follows:

\[
\int_{0}^{T} \left( \iint_{V} \left( \overline{c}_1 F_1 + \overline{c}_2 F_2 + \overline{c}_3 F_3 \right) \, dV \right) \, dt = \int_{0}^{T} \left( \iint_{V} \left( \alpha \overline{c}_1 - \beta \overline{c}_2 + \beta \overline{c}_3 \right)^2 \right) \, dV \, dt.
\]

In view of (18), (20) – (22), equality (17) takes the form

\[
\frac{1}{2} \sum_{r=1}^{3} \iint_{V} \int_{0}^{T} \left( \frac{\partial \overline{c}_r}{\partial t} \right) \, dt \, dV + \frac{1}{2} \sum_{r=1}^{3} \iint_{V} \overline{c}_r \overline{c}_r \, dV \, dt = \frac{1}{2} \sum_{r=1}^{3} \iint_{V} \int_{0}^{T} \left( \alpha \overline{c}_1 - \beta \overline{c}_2 + \beta \overline{c}_3 \right)^2 \, dV \, dt.
\]

or

\[
\frac{1}{2} \sum_{r=1}^{3} \iint_{V} \int_{0}^{T} \overline{c}_r \, dt \, dV + \frac{3}{2} \sum_{r=1}^{3} \iint_{V} \overline{c}_r \, dV \, dt + \frac{1}{2} \sum_{r=1}^{3} \iint_{V} \int_{0}^{T} \left( \alpha \overline{c}_1 - \beta \overline{c}_2 + \beta \overline{c}_3 \right)^2 \, dV \, dt = 0.
\]

Identity (23) is fundamental in the study of the uniqueness of the solution of the initial-boundary-value problem (11) – (15).
Since each of the first two terms of equation (23) will be non-negative ($w_{grw} \geq 0$ in accordance with the physical meaning of the problem), then, assuming the sum of the third and fourth terms is non-negative, we can obtain $c_i(x,y,z,t) = 0$, $(x,y,z) \in V$, $0 \leq t \leq T$. This will contradict the requirement $c_i^{(r)} \neq c_i^{(r)}$, $r=1,2,3$ and sufficient conditions for the uniqueness of the solution of the initial-boundary-value problem are obtained.

Let

\[
\sum_{i=1}^{3} \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \frac{\partial c_i}{\partial \tau} \right)^2 + \mu_1 \left( \frac{\partial c_i}{\partial \tau} \right)^2 + \mu_2 \left( \frac{\partial c_i}{\partial \tau} \right)^2 \right] \, dt + \int_{0}^{T} \int_{V} \left[ (\beta_i \alpha_i - \alpha_i)^2 \right. - \tilde{c}_i \tilde{c}_i (\beta_i \alpha_i - \alpha_i)^2 - \tilde{c}_i \tilde{c}_i (\beta_i \alpha_i - \alpha_i)^2 \right] \, dt \geq 0.
\]

Using the Poincare inequality, write:

\[
\sum_{i=1}^{3} \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \frac{\partial c_i}{\partial \tau} \right)^2 + \mu_1 \left( \frac{\partial c_i}{\partial \tau} \right)^2 + \mu_2 \left( \frac{\partial c_i}{\partial \tau} \right)^2 \right] \, dt \geq \left( \frac{2}{L_x^2 + L_y^2} \right) \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 \right)^2 \right] \, dt + \frac{2}{L_x^2} \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 \right)^2 \right] \, dt,
\]

where $L_x = \max_{0 \leq x \leq L_x, 0 \leq y \leq L_y, 0 \leq t \leq T} H(x,y,t)$.

We transform inequality (24), replacing the first term with one that does not exceed it in accordance with inequality (25) to obtain a sufficient condition for inequality (24) to hold

\[
\left( \frac{2}{L_x^2 + L_y^2} \right) \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 \right)^2 \right] \, dt + \frac{2}{L_x^2} \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 \right)^2 \right] \, dt \geq \left( \frac{2}{L_x^2 + L_y^2} \right) \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 \right)^2 \right] \, dt + \frac{2}{L_x^2} \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 \right)^2 \right] \, dt.
\]

From elementary inequality $ab \leq \frac{a^2 + b^2}{2}$, follows that

\[
\tilde{c}_1 \tilde{c}_1 (\beta_i \alpha_i - \alpha_i)^2 \leq \frac{\tilde{c}_1^2 + \tilde{c}_3^2}{2} (\beta_i \alpha_i - \alpha_i)^2,
\]

\[
\tilde{c}_2 \tilde{c}_2 (\beta_i \alpha_i - \alpha_i)^2 \leq \frac{\tilde{c}_2^2 + \tilde{c}_3^2}{2} (\beta_i \alpha_i - \alpha_i)^2.
\]

Taking into account (27), inequality (26) will hold if inequality

\[
\left( \frac{2}{L_x^2 + L_y^2} \right) \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 \right)^2 \right] \, dt + \frac{2}{L_x^2} \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 \right)^2 \right] \, dt \geq \left( \frac{2}{L_x^2 + L_y^2} \right) \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 \right)^2 \right] \, dt + \frac{2}{L_x^2} \int_{0}^{T} \int_{V} \left[ \mu_0 \left( \tilde{c}_1 + \tilde{c}_2 + \tilde{c}_3 \right)^2 \right] \, dt.
\]
Let \( \mu_0 \equiv \min_{(x,y,z,t)} \mu,(x,y,z,t) \). Then, introducing the notation \( M = \left( \frac{4}{L_x^2} + \frac{4}{L_y^2} \right) \mu_0 + \frac{4}{L_z^2} \mu^*_0 \) and, grouping the terms with factors in inequality (28) \( c_1, c_2, c_3 \), get

\[
\int_0^T \int_V \int_V \int_V [M - \frac{1}{2}(\beta - \sqrt{\alpha})^2] dV dt + \int_0^T \int_V \int_V \int_V [M - \frac{1}{2}(\beta - \sqrt{\alpha})^2] dV dt + \int_0^T \int_V \int_V \int_V [M - \frac{1}{2}(\beta - \sqrt{\alpha})^2] dV dt \geq 0. 
\]

Inequality (29) obviously holds if the inequalities

\[
(\sqrt{\beta} - \sqrt{\alpha})^2 \leq M, \\
(\sqrt{\beta} - \sqrt{\alpha})^2 + (\sqrt{\beta} - \sqrt{\alpha})^2 \leq M, \\
(\sqrt{\beta} - \sqrt{\alpha})^2 \leq M
\]

which implies

\[
(\sqrt{\beta} - \sqrt{\alpha})^2 + (\sqrt{\beta} - \sqrt{\alpha})^2 \leq M. 
\]

Inequality (30) is the sufficient condition for the uniqueness of a solution to the investigated initial-boundary value problem.

Theorem. Let system (1) be given

\[
\begin{aligned}
&\frac{\partial c}{\partial t} + \frac{\partial (uc)}{\partial x} + \frac{\partial (vc)}{\partial y} + \frac{\partial (w+c)}{\partial z} = \mu_0 \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right) + \frac{\partial}{\partial c} \left( \mu_0 \frac{\partial c}{\partial z} \right) + F, \\
&F_r = (\beta_r c_r - \beta c_r) + \Phi_r, \\
&F_r = (\beta_r c_r - \beta c_r) + \Phi_r, r = 2,..., R - 1, \\
&F_r = (\beta_r c_r - \beta c_r) + \Phi_r,
\end{aligned}
\]

in cylinder \( C_r = V \times (0, T), V = \{0 \leq x \leq L_x, 0 \leq y \leq L_y, 0 \leq z \leq L_z \}, L_z = \max_{0 < x < L_x, 0 < y < L_y, 0 < L_z} H(x,y,z), \)

\( H(x,y,z) \equiv h = const > 0, 0 \leq t \leq T \), supplemented by initial condition (2) and boundary conditions (3) – (5). Let further \( c(x,y,z,t) \in C^2(C_r) \cap C(C_r), gruad_{(x,y,z)} \in C(C_r) \) there is a solution to problem (1) – (5) in the cylinder \( C_r \). Then problem (1)-(5) for any \( (x,y,z,t) \in C_r \) and fulfillment of the condition

\[
(\sqrt{\beta} - \sqrt{\alpha})^2 + (\sqrt{\beta} - \sqrt{\alpha})^2 \leq M,
\]

\[
M = \left( \frac{4}{L_x^2} + \frac{4}{L_y^2} \right) \mu_0 + \frac{4}{L_z^2} \mu^*_0, \mu^*_0 = \min_{(x,y,z,t)} \mu_0, \mu_0 \}
\]

has no more than one solution, i.e., any two of its solutions coincide in the general domain of definition.
4. Conclusion
The model describes 3D processes in coastal systems associated with the formation, transport, and gravitational sedimentation of suspensions having a complex particle size distribution. The model takes into account the processes of transformation of some fractions into others, as well as their possible destruction (decomposition). Analytical study of the uniqueness of the solution of the initial-boundary-value problem of transport of multicomponent suspension was carried out. The uniqueness conditions for the solution of the initial-boundary value problem are formulated in the form of corresponding theorem. This model is the basis for the description of processes in the extraction of minerals from the ocean floor, during the propagation of suspended matter in offshore areas and can be generalized to more than three types of particles (fractions).

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