BERNOULLI FREE BOUNDARY PROBLEM
FOR THE INFINITY LAPLACIAN

GRAZIANO CRASTA, ILARIA FRAGALÀ

ABSTRACT. We study the interior Bernoulli free boundary for the infinity laplacian. Our results cover existence, uniqueness, and characterization of solutions (above a threshold representing the “infinity Bernoulli constant”), their regularity, and their relationship with the solutions to the interior Bernoulli problem for the p-laplacian.

1. INTRODUCTION

This paper concerns the following interior Bernoulli-type problem:

\[
(P)\lambda \quad \begin{cases}
\Delta_\infty u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\
u = 1 & \text{on } \partial\Omega, \\
|\nabla u| = \lambda & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega,
\end{cases}
\]

where \(\Omega\) is an open bounded connected domain in \(\mathbb{R}^n\) \((n \geq 2)\), and \(\Delta_\infty\) is the infinity laplacian, defined by

\[
\Delta_\infty u := \nabla^2 u \nabla u \cdot \nabla u \quad \forall u \in C^2(\Omega).
\]

Before presenting our results, we wish to put them into context by saying few words on related literature.

1.1. Bernoulli problem for the p-laplacian. The analogue of problem \((P)_\lambda\) for the p-laplacian, namely

\[
\begin{cases}
\Delta_p u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\
u = 1 & \text{on } \partial\Omega, \\
|\nabla u| = \lambda & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega,
\end{cases}
\]

corresponds to the classical Bernoulli problem when \(p = 2\), and by now it has been widely studied also in the nonlinear case of an arbitrary \(p > 1\). It is motivated by several physical and industrial applications, which comprehend fluid dynamics, optimal insulation, and electro-chemical machining (see [34] for a more precise description). The main questions are the existence and uniqueness of solutions, the geometric properties of the free boundary \(F(u)\), and especially its regularity (for an overview on these topics, we address to [15,33]). When \(\Omega\) is convex and regular, it was proved by Henrot and Shahgholian that there exists a positive constant \(\lambda_{\Omega,p}\), called the Bernoulli constant for the p-Laplacian, such that the interior \(p\)-Bernoulli problem admits a non-constant solution if and only if \(\lambda \geq \lambda_{\Omega,p}\); this solution is in general not unique, it has convex level sets, and its free boundary \(F(u)\) is of class \(C^{2,\alpha}\) (see [16,35]).

\textbf{Date:} April 23, 2018.
\textbf{2010 Mathematics Subject Classification.} Primary 49K20, Secondary 35J70, 35J40.
\textbf{Key words and phrases.} Bernoulli problem, infinity laplacian, capacitary potential, distance function.
When $Ω$ is an arbitrary domain, not necessarily convex, one way of finding solutions is to use the approach which in the linear case $p = 2$ was introduced by Alt and Caffarelli in the seminal work [2]. It amounts to minimize the integral functional

$$J_λ^p(u) := \frac{1}{p} \int_{Ω} \left( \frac{|\nabla u|}{λ} \right)^p + \frac{p-1}{p} |\{u > 0\}|$$

over the space $u ∈ W^{1,p}_1(Ω)$ of functions $u ∈ W^{1,p}_1(Ω)$ which are equal to 1 on $∂Ω$. This minimization problem admits a non-constant solution if and only if $λ ≥ Λ_{Ω,p}$, where $Λ_{Ω,p}$ is a positive constant satisfying $Λ_{Ω,p} ≥ λ_{Ω,p}$ [26]. A non-constant minimizer of $J_λ^p$ over $W^{1,p}_1(Ω)$ solves the $p$-Bernoulli problem provided the free boundary condition $|\nabla u| = λ$ is intended in a suitable weak sense. The free boundary $F(u)$ turns out to be a locally analytic hyper-surface, except for a $H^{n-1}$-negligible singular set (in the wide literature about the free boundary regularity, we limit ourselves to quote as main contributions [2,11,30] for the case $p = 2$ and [27,28,49] for general $p$).

1.2. Free boundary problems for the infinity laplacian. This highly nonlinear and strongly degenerated operator was discovered by Aronsson in the sixties [3]. However, the study of boundary value problems for the infinity Laplacian started only in the early nineties, with the advent of viscosity solutions theory. Bhattacharya, DiBenedetto and Manfredi were the first to consider the Dirichlet problem for infinity harmonic functions and to prove the existence of a solution in the viscosity sense [8]; shortly afterwards, a fundamental contribution came by Jensen [40], who obtained uniqueness and discovered the connection with the problem of finding optimal Lipschitz extensions (see also [4,6]). The last decade has seen a renewed and increasing interest around the infinity laplacian, also due to its connections with differential games. With no attempt of completeness, among the topics under investigation in this growing field let us mention: inhomogeneous equations [9,46], regularity of solutions [22,31,32,45,55], ground states [24,39,41,57], overdetermined problems [20,21,23], tug-of-war games [44,48]. In this scenario, the study of free boundary problems involving the infinity laplacian seems to be rather at its early stage. To the best of our knowledge, only the following exterior version of Bernoulli problem has been considered in the literature (see [47]):

$$\begin{align*}
\begin{cases}
Δ_∞ u = 0 & \text{in } Ω^+(u) := \{x ∈ Ω^c : u(x) > 0\}, \\
u = 1 & \text{on } ∂Ω, \\
|\nabla u| = a(x) & \text{on } F(u) := ∂Ω^+(u) ∩ Ω^c.
\end{cases}
\end{align*}$$

In particular, when $Ω$ is a regular convex set and $a(x) ≡ λ$, the situation looks relatively simple: a unique explicit solution exists, given by $1 - \frac{1}{λ}\text{dist}(x, Ω)$. It satisfies the condition $|\nabla u| = λ$ in a classical sense along its free boundary, which is a parallel set of $Ω$ (hence of class $C^1$). Further, such solution can be identified with the pointwise limit, as $p → +∞$, of the unique solutions $u_p$ to the analogue exterior Bernoulli problem for the $p$-laplacian. On the variational side, let us mention that the asymptotics as $p → +∞$ of integral energies associated with the exterior $p$-Bernoulli problem (loosely speaking, functionals of the type (1) with $Ω$ replaced by its complement) has been studied in [43]. In a somewhat close spirit, the limiting behaviour as $p → +∞$ of the minimization problems for the $p$-Dirichlet integral with a positive boundary datum and a constraint on the volume of the support, has been studied in [52]. Still in theme of free boundary problems for the infinity laplacian, see also [51,53,56].
1.3. **Notion of solutions.** A delicate point before starting the analysis of problem \((P)_\lambda\) is to establish what is meant by a solution. Clearly the PDE has to be intended in the viscosity sense. Going further we point out that, contrarily to the case of the exterior problem mentioned above, for solutions to problem \((P)_\lambda\) the free boundary will not be globally \(C^1\). Consequently, a solution is not expected to be differentiable up to the boundary (see [37,38]), so that also the free boundary condition cannot be interpreted in a pointwise, classical way. Thus, even at the boundary, a viscosity interpretation seems to be the most convenient one in order to manage both existence and uniqueness questions.

More precisely, throughout the paper we intend solutions to \((P)_\lambda\) according to the next definition, which is inspired by De Silva’s work [29, Def 2.2 and 2.3]. If \(u,v: \Omega \to \mathbb{R}\) are two functions and \(x \in \Omega\), by \(u \prec x v\) we mean that \(u(x) = v(x)\) and \(u(y) \leq v(y)\) in a neighborhood of \(x\). Moreover, we denote by \(\varphi\) a test function of class \(C^2\), and we set \(\varphi^+ := \max\{\varphi,0\}\).

**Definition 1.** A function \(u \in C(\Omega)\) is a viscosity solution to \((P)_\lambda\) if

1. \(u\) is infinity harmonic at every \(x \in \Omega^+(u)\) in the viscosity sense, i.e.
   - (a1) if \(u \prec x \varphi\), then \(-\Delta_\infty \varphi(x) \leq 0\);
   - (a2) if \(\varphi \prec x u\), then \(-\Delta_\infty \varphi(x) \geq 0\);
2. the Dirichlet condition \(u = 1\) holds pointwise on \(\partial \Omega\);
3. the free boundary condition holds at every \(y \in F(u)\) in the following viscosity sense:
   - (c1) if \(\varphi^+ \prec y u\), then \(|\nabla \varphi(y)| \leq \lambda;\)
   - (c2) if \(u \prec y \varphi^+\), with \(\nabla \varphi(y) \neq 0\), then \(|\nabla \varphi(y)| \geq \lambda\).

It is clear from the definition that \(u = 0\) on \(F(u)\), so we shall think of \(u\) as equal to 0 on \(\Omega \setminus \Omega^+(u)\).

We point out that a solution in the sense of Definition 1 is also a solution in the sense proposed by Caffarelli in [12, Def. 1] (see also [13, 14]). The converse is a priori not true, because a touching ball as in Caffarelli’s definition does not exist necessarily at all points of the free boundary. Some of our results (e.g. Proposition 2 and Proposition 3) remain true if solutions are intended in the sense of [12]. However, Definition 1 à la De Silva seems to be the one which allows to deal in the optimal way with the existence question (in particular, in the proof of Theorem 15 (b)).

1.4. **Synopsis of the results.** We carry over a detailed analysis of problem \((P)_\lambda\) which covers existence, uniqueness, and characterization of solutions, their regularity, and their relationship with the solutions to the interior Bernoulli problem for the \(p\)-laplacian. We postpone to a companion paper [19] the study of the variational problem which is naturally associated to \((P)_\lambda\) namely the minimization of the supremal functional

\[ J_\lambda(u) := ||\nabla u||_{\infty} + \lambda|\{u > 0\}| \]

over the space of functions \(u \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)\) which are equal to 1 on \(\partial \Omega\).

- **Existence.** By analogy with the case of the \(p\)-laplacian, we define the \(\infty\)-Bernoulli constant of \(\Omega\) as
  \[
  \lambda_{\infty,\Omega} := \inf \left\{ \lambda > 0 : (P)_\lambda \text{ admits a non-constant solution} \right\}. 
  \]

Then we identify \(\lambda_{\infty,\Omega}\) with the reciprocal of the inradius \(R_\Omega\) of \(\Omega\). Indeed, for \(\lambda < 1/R_\Omega\), problem \((P)_\lambda\) does not admit any non-constant solution (Theorem 15 (b)). The proof is
based on a gradient estimate obtained via the gradient flow for infinity harmonic functions (Proposition 2). On the other hand, for \( \lambda \geq 1/R_\Omega \), we get existence. More precisely, it is convenient to distinguish between trivial and non-trivial solutions, according to whether the set \( \{ u = 0 \} \) is Lebesgue negligible or not. For any \( \lambda \geq 1/R_\Omega \), it is easily seen that problem \((P)_\lambda\) admits a bunch of trivial solutions, given by the infinity harmonic potentials of suitable compact subsets with empty interior contained into the set of points \( x \in \Omega \) with \( \text{dist}(x, \partial \Omega) \geq 1/\lambda \) (Proposition 9). So the interesting feature is the existence of a non-trivial solution: if \( \lambda > 1/R_\Omega \), we show that it is given precisely by the infinity harmonic potential \( w_{1/\lambda} \) of the set \( \overline{\Omega}_{1/\lambda} \), being \( \Omega_{1/\lambda} \) the parallel set of points \( x \in \Omega \) such that \( \text{dist}(x, \partial \Omega) > 1/\lambda \) (Theorem 15 (a)). This is obtained by constructing suitable upper and lower bounds for \( w_{1/\lambda} \), and taking advantage of the behaviour of infinity harmonic potentials along rays of the distance function (see Section 2.3).

- **Uniqueness.** For \( \lambda > 1/R_\Omega \), we obtain uniqueness of non-trivial solutions under two assumptions on the parallel set \( \Omega_{1/\lambda} \): connectedness and “open regularity” (Theorem 16); moreover, we show that these assumptions are sharp (Examples 20 and 21). It turns out that they are satisfied for example when \( \Omega \) is convex. Remarkably, such uniqueness result on convex domains distinguishes the case of the \( \infty \)-laplacian from the case of the \( p \)-laplacian, when we have multiplicity of solutions also in case of the ball. We address the interesting open question of establishing whether the unique solution on a convex domain has convex level sets.

- **Characterization of solutions.** For \( \lambda \geq 1/R_\Omega \), we show that \( u \) is a solution to \((P)_\lambda\) if and only if it is the infinity harmonic potential of a set \( K \) belonging to a suitable family of compact subsets of \( \overline{\Omega}_{1/\lambda} \). This result (Theorem 26) gives a complete picture of solutions to \((P)_\lambda\) also in case \( \Omega \) is an arbitrary domain, possibly non-convex.

- **Regularity.** As a by-product of the results described so far, combined with well-known facts about the regularity of infinity harmonic functions, we obtain that, for \( \lambda \geq 1/R_\Omega \), any non-trivial solution is everywhere differentiable in \( \Omega \) (and \( C^{1,\alpha} \) in dimension \( n = 2 \)). Furthermore, the free boundary essentially shares the same regularity properties of the level set \( \{ \text{dist}(x, \partial \Omega) = 1/\lambda \} \) of the distance function. More precisely, if we denote by \( \Sigma(\Omega) \) the cut locus of \( \Omega \) (i.e., the closure of the set of points where the distance from \( \partial \Omega \) is not differentiable), then \( F(u) \setminus \Sigma(\Omega) \) is locally \( C^{1,1} \). As a particular case, if \( \lambda > 1/\text{dist}(\partial \Omega), \Sigma(\Omega) \), then \( F(u) \) is of class \( C^{1,1} \) and, if in addition \( \partial \Omega \) is of class \( C^{k,\alpha} \) for some \( k \geq 2 \), then \( F(u) \) is of class \( C^{k,\alpha} \) (see e.g. [25, Theorem 6.10]).

- **Relationship with the \( p \)-Bernoulli problem.** We show that, if \( \Omega \) is convex and regular, both the \( p \)-Bernoulli constants \( \lambda_{\Omega, p} \) and \( \Lambda_{\Omega, p} \) defined as in Section 1.1 above converge to \( \lambda_{\Omega, \infty} = 1/R_\Omega \) in the limit as \( p \to +\infty \) (Corollary 28). Moreover, if \( u_p \) are solutions to the interior \( p \)-Bernoulli problem, we prove that they converge uniformly to the solution to problem \((P)_\lambda\) provided we are in a setting when such solution is unique, and provided \( u_p \) are **variational** solutions, namely they are issued from the minimization of functionals \((\Pi)\) over \( W^{1,1}_p(\Omega) \) (Theorem 31).

1.5. **Some notation.** We shall write for brevity \( d(x) := \text{dist}(x, \partial \Omega), \ x \in \overline{\Omega} \). Moreover we set \( R_\Omega \) the inradius of \( \Omega \), and for any \( r \in [0, R_\Omega] \), we shall use the notation

\[
\Omega_r = \{ d > r \} = \{ x \in \Omega : d(x) > r \},
\]

\[
\{ d \geq r \} = \{ x \in \Omega : d(x) \geq r \},
\]

\[
D_r := \Omega \setminus \overline{\Omega_r}.
\]
2. Some preliminary results

In this section we collect some material which will be useful throughout the paper. To be at most self-contained, we start by giving a quick recall of some basic facts about infinity harmonic functions, for which we refer to [5, 17, 18].

Then we establish some general properties of (non-constant) solutions to \((P)_{\lambda}\) and of infinity harmonic potentials, which will play a crucial role in the sequel.

2.1. About infinity harmonic functions. A function \(u \in C(\Omega)\) is called infinity subharmonic (resp. infinity superharmonic) if it satisfies condition (a1) (resp. (a2)) in Definition 1. It is called infinity harmonic if it is both infinity subharmonic and superharmonic. An infinity harmonic function on \(\Omega\) is differentiable at every point \(x\). It is called infinity harmonic if \(u\) satisfies condition (a1) (resp. (a2)) in Definition 2.1.

The following facts are equivalent:

(i) \(u\) is infinity harmonic in \(\Omega\);

(ii) \(u \in AML(\Omega)\), which stands for absolutely minimizing Lipschitz, and means that

\[ \|\nabla u\|_{L^\infty(\omega)} \leq \|v\|_{L^\infty(\omega)} \]

for every open set \(\omega \subset \Omega\), and every \(v \in C(\overline{\omega})\) satisfying \(v = u\) on \(\partial\omega\);

(iii) the functions \(w = u\) and \(w = -u\) enjoy comparison with cones from above in \(\Omega\), which means that, for every open set \(\omega \subset \Omega\) and for every \(a, b \in \mathbb{R}\) and \(x_0 \in \mathbb{R}^n\), it holds

\[ w(x) \leq C(x) := a + b|x - x_0| \forall x \in \partial(\omega \setminus \{x_0\}) \Rightarrow w(x) \leq C(x) \forall x \in \omega. \]

Let \(u\) be infinity harmonic in \(\Omega\), and let \(\partial B_r(x) \subset \Omega\). Then

\[ \max_{y \in \partial B_r(x)} u(y) = \max_{y \in \partial B_r(x)} u(y), \quad \min\left(\max_{y \in \partial B_r(x)} u(y)\right) = \min_{y \in \partial B_r(x)} u(y), \]

and the following relations hold:

\[ |\nabla u(x)| \leq \max_{y \in \partial B_r(x)} \frac{u(y) - u(x)}{r} = \max_{y \in \partial B_r(x)} \frac{u(y) - u(x)}{r}, \]

\[ |\nabla u(x)| \leq -\min_{y \in \partial B_r(x)} \frac{u(y) - u(x)}{r} = -\min_{y \in \partial B_r(x)} \frac{u(y) - u(x)}{r}. \]

(see [17, Lemma 4.6]). Moreover, if the maximum and minimum at the right–hand side of (4), (5) are attained respectively at \(p, q \in \partial B_r(x)\), i.e. if

\[ p, q \in \partial B_r(x) : \quad u(p) = \max_{y \in \partial B_r(x)} u(y), \quad u(q) = \min_{y \in \partial B_r(x)} u(y), \]

then the following increasing slope estimates hold:

\[ |\nabla u(x)| \leq |\nabla u(p)|, \quad |\nabla u(x)| \leq |\nabla u(q)| \]

(see [17, Proposition 6.2]).

2.2. Properties of solutions to \((P)_{\lambda}\).

**Proposition 2** (gradient estimate). Let \(u \in C(\overline{\Omega})\) be a non-constant solution to \((P)_{\lambda}\). Then \(|\nabla u(x)| \leq \lambda\) for every \(x \in \Omega^+(u)\).
Proof. Let $x_0 \in \Omega^+(u)$ and let us prove that $|\nabla u(x_0)| \leq \lambda$. Since the statement is trivial if $\nabla u(x_0) = 0$, let us assume that $\nabla u(x_0) \neq 0$. In this case, we claim that there exists a finite family $x_0, x_1, \ldots, x_N$ of points with the following properties:

$$x_0, \ldots, x_{N-1} \in \{u \leq u(x_0)\} \cap \Omega^+(u), \quad x_N \in F(u),$$

$$|\nabla u(x_j)| \geq |\nabla u(x_{j-1})| \quad \forall j = 1, \ldots, N-1, \quad u(x_{N-1}) \geq \text{dist}(x_{N-1}, F(u))|\nabla u(x_{N-1})|.$$ 

(7) 

Since $u(x_0) < 1$ and $u$ is continuous, the sub-level $C := \{u \leq u(x_0)\}$ is a compact subset of $\Omega$. Hence we can find $\rho > 0$ such that $C \subset \Omega_\rho$. Then we fix $r \in (0, \rho)$ and we proceed as follows.

Assume we are given $x_{j-1} \in \{u \leq u(x_0)\} \cap \Omega^+(u)$, and let us construct the point $x_j$.

If $B_r(x_{j-1}) \subset \Omega^+(u)$, then we let $x_j \in B_r(x_{j-1})$ be such that

$$u(x_j) = \min_{y \in B_r(x_{j-1})} u(y).$$

By definition, we have immediately $u(x_j) \leq u(x_{j-1})$, so that $x_j \in C \cap \Omega^+(u)$. Moreover, since $u$ is infinity-harmonic in $\Omega^+(u)$, by (3) and (6) it turns out that $x_j \in \partial B_r(x_{j-1})$ and $|\nabla u(x_j)| \geq |\nabla u(x_{j-1})|$. If $\overline{B}_r(x_{j-1})$ is not contained in $\Omega^+(u)$, by our choice of $r$ we have necessarily $\overline{B}_r(x_{j-1}) \cap F(u) \neq \emptyset$. (Indeed, since $x_{j-1} \in C \subset \Omega$, and $r \in (0, \rho)$, we have $\overline{B}_r(x_{j-1}) \cap \partial \Omega = \emptyset$.) In this case, we set $N = j$, ending the construction, and we let $x_N \in F(u)$ be the projection of $x_{N-1}$ on $F(u)$. Setting $\delta := \text{dist}(x_{N-1}, F(u)) = |x_N - x_{N-1}|$ and taking into account $u(x_N) = 0 = \min_{y \in \overline{B}_r(x_{N-1})} u(y)$, by (5) we obtain

$$|\nabla u(x_{N-1})| \leq -\min_{y \in \overline{B}_r(x_{N-1})} \frac{u(y) - u(x_{N-1})}{\delta} = \frac{u(x_{N-1})}{\delta}.$$ 

It remains to show that our construction always stops in a finite number of steps. Namely, for every $j = 1, \ldots, N-1$, applying again by (5), we obtain

$$|\nabla u(x_j)| \leq -\min_{y \in B_r(x_{j-1})} \frac{u(x) - u(x_{j-1})}{r} = \frac{u(x_{j-1}) - u(x_j)}{r};$$

hence

$$u(x_j) \leq u(x_{j-1}) - r|\nabla u(x_{j-1})| \leq u(x_{j-1}) - r|\nabla u(x_0)|$$

so that in a finite number of steps we arrive at $F(u)$ thanks to the assumption $\nabla u(x_0) \neq 0$. Now, let us consider the open ball $B_\delta(x_{N-1}) \subset \Omega^+(u)$. By comparison with cones [18, Theorem 3.1], we have

$$u(x) \geq u(x_{N-1}) \left(1 - \frac{1}{\delta} |x - x_{N-1}|\right) \quad \forall x \in B_\delta(x_{N-1}).$$

(9) 

We observe that, setting $\nu := (x_{N-1} - x_N)/|x_{N-1} - x_N|$ (so that $x_{N-1} = x_N + \delta \nu$) we have

$$|x - x_{N-1}| = |x - x_N - \delta \nu| = \delta \left(1 - \frac{2}{\delta} \langle x - x_N, \nu \rangle + \frac{1}{\delta^2} |x - x_N|^2\right)^{1/2}.$$ 

Hence, for $x \in \Omega^+(u)$ near $x_N$, it holds

$$|x - x_{N-1}| \leq \delta \left(1 - \frac{1}{\delta} \langle x - x_N, \nu \rangle + \frac{1}{2\delta^2} |x - x_N|^2\right) = \delta - \langle x - x_N, \nu \rangle + o(|x - x_N|).$$

(10)
We infer that there exists a smooth function $\varphi$ such that
$$
\varphi^+ \preceq u, \quad |\nabla \varphi(x_N)| = \frac{u(x_{N-1})}{\delta}(\neq 0).
$$

Then, by applying first Definition $[\text{c1}]$ and then the inequalities $[\text{8}]$, we finally get
$$
\lambda \geq \frac{u(x_{N-1})}{\delta} \geq |\nabla u(x_{N-1})| \geq |\nabla u(x_0)|,
$$
and the proof is completed. \hfill \Box

**Proposition 3** (free boundary location). Let $u \in C(\overline{\Omega})$ be a non-constant solution to $(P)_\lambda$. Then $\text{dist}(F(u), \partial \Omega) \geq \frac{d}{\lambda}$ (or, equivalently, $\{u = 0\} \subseteq \{d \geq \frac{1}{\lambda}\}$). If, in addition, $\text{int}\{u = 0\} \neq \emptyset$, then $\text{dist}(F(u), \partial \Omega) = \frac{d}{\lambda}$.

**Proof.** Let $x \in F(u)$ and let $y \in \partial \Omega$ be its projection on $\partial \Omega$. If $\{y, x \cap F(u) \neq \emptyset$, let $x_0 \in [y, x \cap F(u)$ be the nearest point of $\{y, x \cap F(u)$ to $\partial \Omega$, otherwise let $x_0 := x$. By Proposition $[\text{2}]$ we have
$$
1 = u(y) - u(x_0) \leq \lambda d(x_0),
$$
hence
$$
d(x) \geq d(x_0) \geq \frac{1}{\lambda},
$$
i.e. $x \in \{d \geq \frac{1}{\lambda}\}$. Hence, $F(u) \subseteq \{d \geq \frac{1}{\lambda}\}$, i.e. $\text{dist}(F(u), \partial \Omega) \geq \frac{1}{\lambda}$.

Let us prove that, if $\text{int}\{u = 0\} \neq \emptyset$, then also the opposite inequality holds. Let $r := \text{dist}(\partial \Omega, F(u))$. The function $v(x) := \frac{1}{r} \text{dist}(x, F(u))$ is infinity superharmonic in $\Omega^+(u)$ (see e.g. [42, p. 212]), and satisfies $v = 0$ on $F(u)$ and $v \geq 1$ on $\partial \Omega$. Hence, by the comparison principle for infinity harmonic functions [40, Theorem 2.22], we have that
$$
v \geq u \text{ in } \Omega^+(u).
$$
Since $\text{int}\{u = 0\} \neq \emptyset$, there exists a ball $B = B_\rho(y) \subseteq \Omega^-(u)$ that is tangent to $F(u)$ at some point $x_0 \in F(u)$. Hence, if $\nu := (x_0 - y)/|x_0 - y|$, it holds
$$
u(x) \leq v(x) \leq \frac{1}{r} \langle x - x_0, \nu \rangle + o(|x - x_0|), \quad x \in \Omega^+(u).
$$
We infer that there exists a smooth function $\varphi$ such that
$$
u \prec x_0 \varphi^+, \quad |\nabla \varphi(x_0)| = \frac{1}{r}(\neq 0),
$$
and by Definition $[\text{c2}]$ we conclude that $1/r \geq \lambda$. \hfill \Box

2.3. Properties of infinity–harmonic potentials.

**Definition 4.** Given a non-empty compact set $K \subset \Omega$, the infinity–harmonic potential of $K$ relative to $\Omega$ is the unique viscosity solution $w_K$ to the problem

$$
\begin{cases}
-\Delta_\infty w_K = 0, & \text{in } \Omega \setminus K, \\
w_K = 1, & \text{on } \partial \Omega, \\
w_K = 0, & \text{on } K.
\end{cases}
$$

(11)
Remark 5. Since \( \Omega \setminus K \) may be disconnected, some words to explain the well-posedness of the above definition are in order. Let us write the open set \( \Omega \setminus K \) as the union of its connected components \( \{ A^\alpha : \alpha \in I \} \). For every \( \alpha \in I \), we have that \( \partial A^\alpha \subseteq \partial \Omega \cup K \), and the function \( f^\alpha : \partial A^\alpha \to \mathbb{R} \) defined by
\[
 f^\alpha := \begin{cases} 
 1, & \text{on } \partial A^\alpha \cap \partial \Omega, \\
 0, & \text{on } \partial A^\alpha \cap K, 
 \end{cases}
\]
is continuous on \( \partial A^\alpha \) (being constant on each connected component of \( \partial A^\alpha \)). Therefore, for every \( \alpha \in I \), there exists a unique solution \( w^\alpha \in C(\overline{\Omega}) \) to the Dirichlet problem
\[
\begin{align*}
 -\Delta_\infty w^\alpha &= 0, & \text{in } A^\alpha, \\
 w^\alpha &= f^\alpha, & \text{on } \partial A^\alpha,
\end{align*}
\]
(see [4, Theorems 3.1 and 6.1]). Consequently, problem (11) admits a unique solution, which is precisely the function \( w_K \in C(\Omega) \) defined by \( w_K = w^\alpha \) on \( \overline{A^\alpha} \), \( \alpha \in I \).

Remark 6. It is clear from Definition 4 that the set \( K \) is contained into \( \{ w_K = 0 \} \). We point out that the inclusion may be strict. For instance, this happens when \( \Omega = B_2(0) \) and \( K = \partial B_1(0) \); in this case, \( K \) is strictly contained in \( \{ w_K = 0 \} = \overline{B_1} \). In general, it is not difficult to characterize the set \( \{ w_K = 0 \} \) by looking at the behaviour of the connected components \( A^\alpha \) of \( \Omega \setminus K \) introduced in the previous remark, or equivalently of the functions \( f^\alpha \) defined in (12). Letting
\[
 I_0 := \{ \alpha \in I : \partial A^\alpha \subseteq K \} = \{ \alpha \in I : f^\alpha \equiv 0 \},
\]
we have
\[
 \{ w_K = 0 \} = K \cup \bigcup_{\alpha \in I_0} A^\alpha, \quad \{ w_K > 0 \} = \bigcup_{\alpha \in I \setminus I_0} A^\alpha.
\]
Actually, for every \( x_0 \) belonging to a set \( A^\alpha \) with \( \alpha \in I \setminus I_0 \), one can give a more precise estimate from below for the value \( w_K(x_0) \). This can be done by observing that such a point \( x_0 \) can be joined to \( \partial \Omega \) through a path in \( \Omega \setminus K \), and then exploiting the following result, which is essentially taken from [7, Lemma 3.2].

Proposition 7 (Harnack inequality). Let \( K \subset \Omega \) be a non-empty compact set, and let \( w_K \) be its infinity–harmonic potential relative to \( \Omega \). Let \( x_0 \in A^\alpha \), with \( \alpha \in I \setminus I_0 \), and let \( \gamma \) be a path in \( \Omega \setminus K \) connecting \( x_0 \) to \( \partial \Omega \). Then
\[
 w_K(x_0) \geq e^{-L/\delta},
\]
where \( L \) is the length of \( \gamma \), and \( \delta \) is the distance from \( \gamma \) to \( K \).

Proof. By possibly taking a slightly larger value of \( \delta \) (but lower than \( \text{dist}(\gamma, K) \)), it is not restrictive to assume that \( \gamma \) is a polygonal curve. Moreover, for \( m \in \mathbb{N} \) large enough, we can assume that the polygonal has exactly \( m+1 \) vertices \( x_0, x_1, \ldots, x_m = y \) with \( |x_j - x_{j-1}| = L/m \) for every \( j = 1, \ldots, m \). By possibly moving a bit the point \( y \) (shortening the curve), we can also assume that \( y \) is a projection of \( x_m-1 \) on \( \partial \Omega \). Since \( x_{j-1} \in B_\delta(x_j) \subset \Omega \setminus K \) for every \( j = 1, \ldots, m \), by comparison with cones, we have
\[
 w_K(x_{j-1}) \geq w_K(x_j) \left( 1 - \frac{|x_j - x_{j-1}|}{\delta} \right) = w_K(x_j) \left( 1 - \frac{L}{m\delta} \right),
\]
so that
\[
 w_K(x_0) \geq w_K(y) \left( 1 - \frac{L}{m\delta} \right)^m.
\]
Since $w_K(y) = 1$ and $m$ can be taken arbitrarily large, we finally get (15). □

We conclude with a useful characterization of the infinity harmonic potential $w_K$ along rays connecting $K$ with $\partial \Omega$:

**Proposition 8** (potential along rays). Let $K \subset \Omega$ be a non-empty compact set, and let $w_K$ be the infinity–harmonic potential of $K$ relative to $\Omega$. If $y \in \partial \Omega$ and $z \in K$ are two points such that $|y - z| = \text{dist}(\partial \Omega, K)$, then $w_K$ is affine on the segment $[y, z]$.

**Proof.** Set $R := \text{dist}(\partial \Omega, K)$. Since $w_K$ enjoys comparison with cones from below, we have

$$w_K(x) \geq f(x) := 1 - \frac{|x - y|}{R}, \quad \forall x \in B_R(y) \cap \Omega.$$ 

On the other hand, the function $g(x) := \frac{1}{R} \text{dist}(x, K)$ is infinity superharmonic in $\Omega \setminus K$, with $g = 0$ on $K$ and $g \geq 1$ on $\partial \Omega$, hence $g \geq w_K$ by the comparison principle for infinity harmonic functions. Since $f = g$ on the segment $[y, z]$, the statement follows. □

3. Existence

We start the analysis of existence of solutions to problem \((P)_\lambda\) by observing that, for any $\lambda \geq 1/R_\Omega$, it admits a bunch of “trivial” solutions. Inspired by the results of the previous section, they are found among infinity harmonic potentials $w_K$ of suitably chosen compact sets $K$ contained into $\{d \geq \frac{1}{\lambda}\}$. Recall that the zero set of $w_K$ can be characterized as in (14); in particular, we have $\{w_K = 0\} = K$ if and only if the set $I_0$ defined in (13) is empty.

**Proposition 9.** Let $\lambda \geq 1/R_\Omega$, and let $K \subset \{d \geq \frac{1}{\lambda}\}$ be a non-empty compact set. Assume that

\[
\text{int}(K) = \emptyset \quad \text{and} \quad I_0 = \emptyset. 
\]

Then the infinity–harmonic potential $w_K$ of $K$ relative to $\Omega$ is a solution to \((P)_\lambda\).

**Proof.** By (16) and (14), the set $\{w_K = 0\}$ agrees with $K$ and has empty interior, so that $\{w_K = 0\} = K = F(w_K)$. Thus, we have to show that the free boundary condition in Definition 1 is satisfied at every point $x_0 \in K$.

Since $K \subset \{d \geq \frac{1}{\lambda}\}$, by comparison with cones we have that $w_K(x) \leq \lambda|x - x_0|$ for every $x \in \Omega$. If $\varphi^+ \prec_{x_0} w_K$, then necessarily $|\nabla \varphi(x_0)| \leq \lambda$, hence condition (c1) in Definition 1 is satisfied.

On the other hand, if $w_K \prec_{x_0} \varphi^+$, then $\varphi \geq 0$ in $\Omega$, because $\varphi^+ \geq w_K > 0$ in $\Omega \setminus \{w_K = 0\}$ and $\Omega \setminus \{w_K = 0\} = \overline{\Omega}$. Since $\varphi(x_0) = 0$, then $x_0$ is a minimum point for the regular function $\varphi$, hence we can conclude that $\nabla \varphi(x_0) = 0$, and also condition (c2) in Definition 1 is satisfied. □

Motivated by Proposition 9, we give the following definition.

**Definition 10** (Non-trivial solutions). We say that a solution $u$ to \((P)_\lambda\) is non-trivial if the set $\{u = 0\}$ has non-empty interior (and trivial otherwise).

**Remark 11.** In the special case $\lambda = \frac{1}{R_\Omega}$, problem \((P)_\lambda\) admits only trivial solutions. Indeed, we know from Proposition 3 that, for every solution $u$ to problem \((P)_\lambda\) $F(u)$ is contained into the high ridge $\{d(x) = R_\Omega\}$ and hence the set $\{u = 0\}$ has necessarily empty interior.
We are now going to deal with the existence of non-trivial solutions to \((P)_,_\lambda\) for \(\lambda \in (0, \frac{1}{\mu_\Omega})\). To that aim, we introduce two more definitions.

**Definition 12.** Given \(r \in (0, R_\Omega)\), we define \(w_r\) as the infinity harmonic potential of \(\Omega_r\), namely the unique solution to
\[
\begin{cases}
\Delta_\infty w_r = 0 & \text{in } D_r := \Omega \setminus \overline{\Omega}_r \\
w_r = 1 & \text{on } \partial \Omega \\
w_r = 0 & \text{in } \overline{\Omega}_r.
\end{cases}
\]

**Definition 13.** Given \(r \in (0, R_\Omega]\), we set
\[
\hat{D}_r := \bigcup_{y \in \partial \Omega_r} \{y, z : z \in \Pi_{\partial \Omega}(y)\},
\]
where \(\Pi_{\partial \Omega}(y) := \{z \in \partial \Omega : d(y) = |z - y|\}\).

**Remark 14.** Notice that, by definition, \(\hat{D}_r\) is a subset of \(D_r\), with possibly strict inclusion (think for instance to the case when \(D\) is a square, see Figure 1).

**Theorem 15.** (a) For every \(\lambda > \frac{1}{\mu_\Omega}\), the function \(w_{\frac{1}{\lambda}}\) is a non-trivial solution to problem \((P)_{\lambda}\) moreover it satisfies the estimates
\[
1 - \lambda d(x) \leq w_{\frac{1}{\lambda}}(x) \leq \lambda \dist(x, \partial \Omega_{\frac{1}{\lambda}}) \quad \text{in } \overline{D}_{\frac{1}{\lambda}}, \text{ with equalities in } \overline{\hat{D}}_{\frac{1}{\lambda}}.
\]
(b) For every \(\lambda \in (0, \frac{1}{\mu_\Omega})\), problem \((P)_{\lambda}\) does not admit non-constant solutions.

**Proof.** Throughout the proof, since \(\lambda\) is fixed, we set for brevity
\[
w := w_{\frac{1}{\lambda}}, \quad D := D_{\frac{1}{\lambda}}, \quad \hat{D} := \hat{D}_{\frac{1}{\lambda}}.
\]
Let us first show that \(w\) satisfies the inequalities in (17).
The function \(v(x) := 1 - \lambda d(x)\) is infinity subharmonic (since \(d\) is infinity superharmonic), and satisfies the equality \(v = w\) on both \(\partial \Omega\) and \(\partial \Omega_{\frac{1}{\lambda}}\). By the comparison principle for infinity harmonic functions, it follows that \(w \geq v\) in \(\overline{\hat{D}}\).

Similarly, the function \(z(x) := \lambda \dist(x, \partial \Omega_{\frac{1}{\lambda}})\) is infinity superharmonic, and satisfies \(z = w = 0\) on \(\partial \Omega_{\frac{1}{\lambda}}\), \(w \leq z\) on \(\partial \Omega\). Again by the comparison principle for infinity harmonic functions, we infer that \(w \leq z\) in \(\overline{\hat{D}}\).
In order to obtain that the inequalities in (17) hold as equalities in $\hat{D}$, we firstly notice that $\|w\|_\infty = \lambda$. Indeed, the inequality $\|w\|_\infty \geq \lambda$ follows immediately from the estimate

$$\|w\|_\infty \geq \sup \left\{ \frac{|w(x) - w(y)|}{|x - y|} : x \in \partial \Omega, \ y \in \partial \Omega \right\};$$

the converse one follows from the fact that $w$ has the AML property in $D$, which entails in particular $\|w\|_\infty \leq \|v\|_\infty = \lambda$.

Now assume by contradiction that the strict inequality $w > v$ holds at some point $x \in \hat{D}$. If $x$ belongs to the segment $[y, z]$, with $y \in \partial \Omega \backslash \lambda$ and $z \in \Pi_\partial \Omega(y)$, we have

$$\|w\|_\infty \geq \frac{|w(x) - w(y)|}{|x - y|} = \frac{w(x)}{|x - y|} > \frac{v(x)}{|x - y|} = \frac{|v(x) - v(y)|}{|x - y|} = \lambda.$$

Here in the last equality we have exploited the fact that $d(x) - d(y) = |x - y|$. Indeed, if $x \in [y, z] \subset \hat{D}_\lambda$, with $y \in \partial \Omega\lambda$ and $z \in \Pi_\partial \Omega(y)$, it holds $d(x) = r - |x - y|$ and $\text{dist}(x, \partial \Omega\lambda) = r - |x - z|$, which implies in particular

$$r - d(x) = |x - y| = r - |x - z| = \text{dist}(x, \partial \Omega\lambda).$$

We have thus contradicted the equality $\|w\|_\infty = \lambda$, and we conclude that $w(x) = v(x)$. Since by (18) $v(x) = z(x)$ on $\hat{D}$, the proof of (17) is achieved.

(a) We are now in a position to prove that $w$ solves problem (P)$_\lambda$, which amounts to show that it satisfies the free boundary condition (c) of Definition 1 along the free boundary $F(w) = \partial \Omega_\lambda$.

Let $x_0 \in \partial \Omega_\lambda \backslash \lambda$, let $\varphi^+ \prec_{x_0} w$, with $p := \nabla \varphi(x_0) \neq 0$. By the upper bound inequality in (17), we have

$$\varphi(x) \leq w(x) \leq \lambda \text{dist}(x, \partial \Omega_\lambda) \quad \forall x \in D,$$

hence

$$\varphi(x_0 + tp) \leq \lambda \text{dist}(x_0 + tp, \partial \Omega_\lambda) \leq \lambda t |p|, \quad t > 0 \text{ small}.$$ 

Dividing by $t > 0$ and taking the limit as $t \to 0^+$ we get $|p|^2 \leq \lambda |p|$, hence $|p| \leq \lambda$, so that (c1) holds.

Let us now consider condition (c2) at a point $x_0 \in \partial \Omega_\lambda \backslash \lambda$. For every $y \in \Pi_\partial \Omega(x_0)$, by (17), the function $w$ is affine with slope $\lambda$ on the segment $[x_0, y] \subset \hat{D}$. Consequently, if $\Pi_\partial \Omega(x_0)$ is not a singleton, we can find no smooth function $\varphi$ such that $w \prec_{x_0} \varphi^+$, so that condition (c2) is empty. Hence, we are going to assume that $\Pi_\partial \Omega(x_0) = \{y\}$. In this case, setting $\nu := (y - x_0)/|y - x_0|$, we claim that

$$w \prec_{x_0} \varphi^+, \quad \text{with } p := \nabla \varphi(x_0) \neq 0 \quad \Rightarrow \quad p = \alpha \nu \quad \text{for some } \alpha > 0.$$ 

To prove the claim, we observe firstly that $\Omega_\lambda$ satisfies an exterior sphere condition of radius $\frac{1}{\lambda}$ at $x_0$. (Indeed, we have $B_{\frac{1}{\lambda}}(y) \cap \Omega_\lambda = \emptyset$ and $x_0 \in \partial B_{\frac{1}{\lambda}}(y) \cap \partial \Omega_\lambda$.)

Now, we assume without loss of generality that $x_0 = (x', x_n) = (0, 0)$, $\nu = e_n$, and we denote by $x_\nu = g(x')$ be a local parametrization of $\partial B_{\frac{1}{\lambda}}(y)$ near $x_0 = 0$ (so that $g$ is defined in a neighbourhood $U(0) \subset \mathbb{R}^{n-1}$ of the origin, satisfies $g(0) = 0$, and is differentiable at $x' = 0$ with $\nabla g(0) = 0$). If $\varphi$ is as in (19), we have by definition

$$w(x) \leq ((p, x) + o(x))^+ \quad \forall x \in \overline{D}.$$
We now take $x = (x', g(x'))$. Taking into account that, on $\partial B_{\frac{1}{\lambda}}(y) \setminus \{x_0\}$, we have $w > 0$ and hence $\varphi^+ = \varphi$, we get

$$0 \leq w(x', g(x')) \leq \langle (p', p_n), (x', g(x')) \rangle + o(x', g(x')) \quad \forall x' \in \mathcal{U}(0).$$

Since $g(x') = o(x')$ as $x' \to 0$, we infer that

$$0 \leq \langle p', x' \rangle + o(x') \quad \forall x' \in \mathcal{U}(0),$$

and hence $p' = 0$, yielding (19).

Now, if $\varphi$ is a test function as in condition (c2), by (19) we have $\langle \nabla \varphi(x_0), \nu \rangle = |\nabla \varphi(x_0)|$.

 Hence

$$\lambda t = w(x_0 + tv) \leq \varphi(x_0 + tv)^+ = |\nabla \varphi(x_0)|t + o(t), \quad t > 0 \text{ small.}$$

Dividing by $t$ and taking the limit as $t \to 0^+$ we get $\lambda \leq |\nabla \varphi(x_0)|$, and (c2) follows.

(b) We observe that, if $u$ is a solution to $(P)_\lambda$ (for an arbitrary $\lambda > 0$), it holds

$$\sup_{x \in \Omega^+(u)} |\nabla u(x)| \geq 1 / R_\Omega.$$  \hspace{1cm} (20)

Indeed, if we assume that $|\nabla u(x)| \leq L < 1 / R_\Omega$ for every $x \in \Omega^+(u)$, then we obtain

$$u(x) \geq 1 - L d(x) \geq 1 - L R_\Omega > 0 \quad \forall x \in \Omega^+(u),$$

a contradiction.

The proof of statement (b) now follows by combining (20) with Proposition 2 \hfill \square

4. Uniqueness

Prior to starting the analysis of the uniqueness of solutions for problem $(P)_\lambda$, we emphasize that one has to restrict attention to the class of non-trivial solutions and to choose $\lambda > 1 / R_\Omega$. Indeed, if these requirements are dropped, by applying the results of the previous section we readily get the following conclusions:

- For $\lambda > 1 / R_\Omega$, according to Proposition 6 there exist infinitely many trivial solutions to $(P)_\lambda$ corresponding the infinity harmonic potentials of any compact set $K \subseteq \{d \geq \frac{1}{\lambda}\}$ satisfying (16).

- For $\lambda = 1 / R_\Omega$, we know that all the solutions to $(P)_\lambda$ are trivial (cf. Remark 11). Moreover, it is easy to see that any compact set $K$ contained into the high ridge of $\Omega$ satisfies (16). Therefore, there exist either one or multiple non-constant solutions to $(P)_\lambda$ respectively when the high ridge is a singleton or not.

We are thus led to formulate the question as:

When uniqueness of non-trivial solutions to $(P)_\lambda$ occurs for $\lambda > 1 / R_\Omega$?

Our answer is given in the statement below.

**Theorem 16** (Uniqueness of non-trivial solutions). Let $\lambda > 1 / R_\Omega$. Assume that

(H1) $\Omega_{\frac{1}{\lambda}}$ is connected;

(H2) $\Omega_{\frac{1}{\lambda}} = \{d \geq \frac{1}{\lambda}\}$.

Then $w_1$ is the unique non-trivial solution to $(P)_\lambda$.

**Corollary 17.** Assume $\Omega$ is convex. For every $\lambda > 1 / R_\Omega$, $w_1$ is the unique non-trivial solution to problem $(P)_\lambda$. 

Remark 18. (About the connectness assumption (H1)). When $\Omega$ is convex, assumption (H1) is satisfied because also $\Omega_r$ is convex for every $r \in [0, R_{\Omega})$. For general $\Omega$, (H1) is satisfied if $\frac{1}{\lambda} < \text{dist}(\partial \Omega, \Sigma(\Omega))$, $\Sigma(\Omega)$ being the cut locus of $\Omega$, namely the closure of the set of points where the distance from $\partial \Omega$ is not differentiable. Indeed, if $r < \text{dist}(\partial \Omega, \Sigma(\Omega))$, then $\Sigma(\Omega) \subset \Omega_r$ and $\Sigma(\Omega_r) = \Sigma(\Omega)$. By Theorem 5.3 in [1], $\Omega$ and $\Omega_r$ have the same homotopy class as $\Sigma(\Omega)$. Since $\Omega$ is connected by assumption, then also $\Sigma(\Omega)$ and $\Omega_r$ are connected.

Remark 19. (About the regularity assumption (H2)). When $\Omega$ is convex, assumption (H2) is satisfied because $\Omega_r$ agrees with $\{d \geq r\}$ for every $r \in [0, R_{\Omega})$. For general $\Omega$, we have the inclusion $\Omega_r \subseteq \{d \geq r\}$, which may be possibly strict (see for instance Example 21 below). Assumption (H2) can be also rephrased by asking that the set $C := \{d \geq \frac{1}{\lambda}\}$ satisfies $C = \text{int}(C)$. In topology, sets satisfying this last condition are known as regular closed sets. It is clear from the definition that such sets are closed in the usual sense, and have a non-empty interior if they are not empty.

Assumptions (H1) and (H2) are sharp, as we can have multiple non-trivial solutions as soon as $\Omega_{\frac{1}{\lambda}}$ is not connected and/or $\Omega_{\frac{1}{\lambda}} \neq \{d \geq \frac{1}{\lambda}\}$. This fact is illustrated in Examples 20 and 21 below.

Example 20 (Multiplicity of non-trivial solutions without (H1)). If $\Omega_{\frac{1}{\lambda}}$ is not connected, then problem $(P)_1$ may have more than one non-trivial solution. Let us show this phenomenon with an explicit example. Let $\Omega \subset \mathbb{R}^2$ be the set 
$$\Omega := B_3((-4, 0)) \cup B_3((4, 0)) \cup ((-4, 4) \times (-1, 1))$$
(see Figure 2), and let $\lambda = 1$.

The set $\Omega$ is not connected, since it is the disjoint union of two connected components $\Omega_1^- := \Omega_1 \cap \{x_1 < 0\}$ and $\Omega_1^+ := \Omega_1 \cap \{x_1 > 0\}$.

We have proved in Theorem 15 that the function $w_1$ is a solution to $(P)_1$.

On the other hand, we claim that the infinity–harmonic potentials of $\Omega_1^\pm$ relative to $\Omega$ are both solutions to $(P)_1$. Let us prove this claim when $u$ is the infinity–harmonic potentials of $\Omega_{\frac{1}{\lambda}}^-$. By Proposition 8 we have that $u(x) = w_1(x)$ on the set
$$A^- := \{(x_1, x_2) \in \Omega \setminus \Omega_{\frac{1}{\lambda}}^- : x_1 < 2\sqrt{2} - 4\}.$$
Hence, we already know that $u$ satisfies the free boundary condition of Definition 11 at all points $x_0 \in F(u) = \partial \Omega_{\frac{1}{\lambda}}^-$, $x_0 \neq p := (2\sqrt{2} - 4, 0)$. It remains to prove that the free boundary condition is satisfied at $p$. Since $p$ has two projections on $\partial \Omega$, it does not exist.
a smooth function $\varphi$ such that $u \prec_p \varphi^+$. On the other hand, if $\varphi$ is a smooth function such that $\varphi^+ \prec_p u$, then necessarily $|\nabla \varphi(p)| \leq 1$, since $u(x) \leq \text{dist}(x, \Omega^{-1})$. This proves that $u$ is a solution to $(P)_1$.

One can also construct infinitely many other non-trivial solutions to $(P)_1$. Namely, let $q := -p$, let $C$ be a closed subset of $[p, q] \cup \Omega^{-1}$ with empty interior, and let $K := C \cup \Omega^{-1}$. Then the infinity–harmonic potential of $K$ relative to $\Omega$ turns out to be a solution to $(P)_1$.

Another symmetric family of non-trivial solutions can be constructed by taking $C$ a closed subset of $[p, q] \cup \Omega^{-1}$ with empty interior and $K := C \cup \Omega^{-1}$. (For both families, the free boundary condition can be checked by arguing with minor modifications as done in the proof of Proposition 9).

**Example 21 (Multiplicity of non-trivial solutions without (H2)).** More than one non-trivial solution may occur also in case $\Omega^{-1}$ is strictly contained into $\{d \geq \frac{1}{\lambda}\}$. To enlighten this fact, let us modify the above example by considering the set

$$
\Omega := B_3((-4, 0)) \cup B_1((4, 0)) \cup ((-4, 4) \times (-1, 1)).
$$

Again, we take $\lambda = 1$. In this case, $\{d \geq 1\} \neq \Omega^{-1}$. In a similar way as above, for every closed subset $C$ of the segment $[p, q]$, with $p := (2\sqrt{2} - 4, 0)$ and $q := (4, 0)$, the infinity–harmonic potential of $K := \Omega^{-1} \cup C$ relative to $\Omega$ is a solution to $(P)_\lambda$.

We now turn to the proof of Theorem 16. It is based on the characterization of the set $\text{int}\{u = 0\}$ (see Proposition 23 below). We start by proving a simple geometric lemma.

**Lemma 22.** Let $A$ be a non-empty open subset of $\Omega$ such that, for some constant $R > 0$,

$$
(21) \quad d(x) = \text{dist}(x, \partial A) + R, \quad \forall x \in A.
$$

Then $A$ is a union of connected components of $\Omega_R$. In particular, if $\Omega_R$ is connected, then $A = \Omega_R$.

**Proof.** From (21) we have that $d(x) > R$ for every $x \in A$, hence $A \subseteq \Omega_R$.

We claim that $\partial A \subseteq \partial \Omega_R$. Namely, let $y \in \partial A$. For every $\varepsilon > 0$ there exists a point $x \in A$ such that $|x - y| < \varepsilon$, so that, by (21),

$$
d(y) < d(x) + \varepsilon = \text{dist}(x, \partial A) + R + \varepsilon < R + 2\varepsilon,
$$

$$
d(y) > d(x) - \varepsilon = \text{dist}(x, \partial A) + R - \varepsilon > R - 2\varepsilon,
$$

hence $d(y) = R$, and the claim is proved.
Let $A'$ be a connected component of $A$, and let $B$ a connected component of $\Omega_R$ such $A' \cap B \neq \emptyset$. By the previous claim, $\partial A' \cap B = \emptyset$, hence $B$ can be written as the union of the two open sets $A'$ and $B \setminus A'$. On the other hand, $B$ is connected, hence necessarily $B \setminus A' = \emptyset$ and $A' = B$. \hfill \Box

**Proposition 23.** Let $\lambda > 1/R\Omega$ and let $u$ be a solution to $(P)_\lambda$. Then $\text{int}\{u = 0\}$ is a (possibly empty) union of connected components of $\Omega_\lambda$.

**Proof.** We are going to prove that, if the set $A := \text{int}\{u = 0\}$ is not empty, it satisfies the assumption (21) of Lemma 22 with $R = \frac{1}{\lambda}$.

Let $x \in A$, let $x_0 \in \Pi_{\partial A}(x)$ and let $r := |x - x_0|$, so that $B_r(x) \subset A$ and $x_0 \in F(u)$. Let us consider the function

$$
\varphi(y) := \frac{|y - x| - r}{d(x) - r}, \quad y \in \Omega.
$$

We have that $\varphi(y) \geq 0$ for every $y \in \partial A \subset F(u)$, and $\varphi(y) \geq 1$ for every $y \in \partial \Omega$. Hence, by comparison, $\varphi \geq u$ in $\Omega \setminus A$ and, in particular, $u \prec_{x_0} \varphi^+$. By Definition 1 it follows that

$$
|\nabla \varphi(x_0)| = \frac{1}{d(x) - r} \geq \lambda.
$$

Let $z \in \Pi_{\partial A}(x)$. The point $y_0 := x + r \frac{z - x}{|z - x|}$ belongs to $\overline{B}_r(x) \subset \overline{A}$ and, by Proposition 3, $\overline{B}_r(x) \subseteq \overline{A} \subseteq \overline{\Omega}_\lambda$, so that

$$
\frac{1}{\lambda} \leq d(y_0) = |y_0 - z| = |x - z| - |x - y_0| = d(x) - r.
$$

From (22) and (23) it follows that $d(x) - r = \frac{1}{\lambda}$, i.e. the assumptions of Lemma 22 hold with $R = \frac{1}{\lambda}$. \hfill \Box

We are now in a position to give:

**Proof of Theorem 16.** Let $u$ be a non-trivial solution to $(P)_\lambda$ for some $\lambda > 1/R\Omega$.

Since by assumption the interior of $\{u = 0\}$ is not empty, by Proposition 23 it is a union of connected components of $\Omega_\lambda$ and hence, by assumption (H1), it agrees with $\Omega_\frac{1}{\lambda}$.

On the other hand, by Proposition 3 the closed set $\{u = 0\}$ is contained into $\{d \geq \frac{1}{\lambda}\}$ and, by assumption (H2), we have $\{d \geq \frac{1}{\lambda}\} = \overline{\Omega}_\lambda$.

Summarizing, we have

$$
\Omega_\frac{1}{\lambda} = \text{int}\{\{u = 0\}\} \subseteq \{u = 0\} \subseteq \{d \geq \frac{1}{\lambda}\} = \overline{\Omega}_\lambda.
$$

Hence, $\{u = 0\} = \overline{\Omega}_\lambda$ and $u = w_\lambda$. \hfill \Box

5. Characterization of solutions

In the following theorem we will characterize all solution to $(P)_\lambda$ as the infinity-harmonic potentials of compact subsets of $\Omega$.

**Definition 24.** For a fixed $\lambda \geq 1/R\Omega$, let $K_\lambda$ be the family of all sets $K \subset \mathbb{R}^n$ satisfying the following properties:

(i) $K$ is a compact subset of $\{d \geq 1/\lambda\}$.

(ii) If $\tilde{K}$ is a connected component of $K$ with non-empty interior, then $\text{int}\tilde{K}$ coincides with a connected component of $\Omega_\frac{1}{\lambda}$. 
(iii) If $\Omega \setminus K$ is decomposed as in Section 2.3, then the set $I_0$ defined in (13) is empty.

**Remark 25.** In connection with property (iii), we recall that, for a compact set $K \subset \Omega$, the following properties are equivalent:

1) the set $I_0$ defined in (13) is empty;
2) $\{w_K = 0\} = K$;
3) every point $x \in \Omega \setminus K$ can be joined to $\partial \Omega$ through a path in $\overline{\Omega} \setminus K$.

**Theorem 26.** Let $\lambda \geq 1/R_\Omega$. Then a function $u \in C(\Omega)$ is a solution to $(P)_\lambda$ if and only if there exists a set $K \in K_\lambda$ such that $u = w_K$.

**Proof.** The case $\lambda = 1/R_\Omega$ is trivial (see Remark 11), so that we shall assume that $\lambda > 1/R_\Omega$.

Let $u \in C(\Omega)$ be a solution to $(P)_\lambda$. Let us prove that the set $K := \{u = 0\}$ belongs to the class $K_\lambda$ introduced in Definition 24, and that $u = w_K$.

Condition (i) is satisfied by Proposition 3.

Condition (ii) is clearly satisfied if $u$ is a trivial solution, while it follows from Proposition 23 if $u$ is a non-trivial solution.

Condition (iii) can be easily checked arguing by contradiction. Namely, assume that the set $I_0$ defined in (13) is not empty. In this case, there exists a connected component $A$ of $\Omega \setminus K$ such that $\partial A \subset K$. But then, by uniqueness, necessarily $u = 0$ on $A$, with $A \cap K = \emptyset$, against the definition of $K$.

We have thus proved that $K \in K_\lambda$. Finally we observe that, since $K$ satisfies condition (iii), we have $\{w_K = 0\} = K$ (cf. Remark 25), and hence $u = w_K$.

Viceversa, let $K \in K_\lambda$ and let us prove that $w_K$ is a solution to $(P)_\lambda$.

By property (iii) in Definition 24, we have that $F(w_K) = \partial K$, hence it is enough to prove that the free boundary condition is satisfied at any point of $\partial K$. Let $x_0 \in \partial K$.

We have two possibilities: either $x_0 \notin \text{int} K$, or $x_0 \in \partial B$, where $B$ is a connected component of $\text{int} K$ (which thanks to property (ii) in Definition 24 is also a connected component of the open set $\Omega_{1/\lambda}$).

If $x_0 \notin \text{int} K$, we are done by arguing exactly as in Proposition 9 (in particular, by exploiting property (i) in Definition 24).

If $x_0 \in \partial B$, we argue as in the proof of Theorem 15(a). More precisely, we prove firstly that the following inequalities analogous to (17) are satisfied

$$1 - \lambda d(x) \leq w_K(x) \leq \lambda \text{dist}(x, \partial B), \quad \forall x \in \overline{\Omega} \setminus K,$$

with equalities for every $x \in [x_0, y_0]$, being $y_0 \in \Pi_{\partial \Omega}(x_0)$. Then, by using (24), we obtain the free boundary condition at $x_0$ by proceeding in the same way as in the second part of the proof of Theorem 15(a).


6. Asymptotics of $p$-Bernoulli problems as $p \to +\infty$

In this section we explore the relation between problem $(P)_\lambda$ and the interior Bernoulli problem for the $p$-laplacian. For the benefit of the reader, we start by revisiting in more detail some facts which in part have been already mentioned in the Introduction (some bibliographical references already given therein are skipped below).
The interior Bernoulli free boundary problem for the $p$-Laplacian, for a given $p > 1$, consists in finding a (non-constant) solution to

$$
\begin{cases}
\Delta_p u = 0 & \text{in } \Omega^+(u) := \{ x \in \Omega : u(x) > 0 \}, \\
u = 1 & \text{on } \partial \Omega, \\
|\nabla u| = \lambda & \text{on } F(u) := \partial \Omega^+(u) \cap \Omega.
\end{cases}
$$

Then the Bernoulli constant for the $p$-Laplacian is defined by

$$
\lambda_{\Omega,p} := \inf \{ \lambda > 0 : \text{(25) admits a non-constant solution} \}.
$$

If a solution to (25) is meant as a classical one, i.e. as a function $u \in C(\Omega^+(u)) \cap C^2(\Omega^+(u))$, then the free boundary condition can be understood in the following pointwise sense:

$$
\lim_{\Omega^+(u) \ni y \to x} |\nabla u(y)| = \lambda, \quad \forall x \in F(u).
$$

Actually, when $\Omega$ is a regular convex domain, the following results due to Henrot and Shahgholian hold:

- for every $\lambda \geq \lambda_{\Omega,p}$, problem (25) admits a classical non-constant solution $u$, which has convex level sets; moreover, the free boundary $F(u)$ is of class $C^{2,\alpha}$ [36, Thm. 2.1];
- $\lambda_{\Omega,p}$ can be characterized loosely speaking as the infimum of positive $\lambda$ such that the family of sub-solutions to (25) is not empty, and it satisfies the lower bound

$$
\lambda_{\Omega,p} \geq 1/R_{\Omega}
$$

[35, Thms. 3.1 and 3.2].

When $\Omega$ is an arbitrary domain, not necessarily convex, following the celebrated work [2] by Alt and Caffarelli, in order to find solutions to problem (25) one can consider the integral functionals

$$
J^p_\lambda(u) := \frac{1}{p} \int_\Omega \left( \frac{|\nabla u|}{\lambda} \right)^p + \frac{p-1}{p} |\{ u > 0 \}|
$$

and search for minimizers to

$$
\min \left\{ J^p_\lambda(u) : u \in W^{1,p}_1(\Omega) \right\}, \quad W^{1,p}_1(\Omega) := 1 + W^{1,p}_0(\Omega).
$$

Accordingly, the constant

$$
\Lambda_{\Omega,p} := \inf \{ \lambda > 0 : \text{[27] admits a non-constant solution} \},
$$

can be regarded as a variational Bernoulli constant for the $p$-Laplacian. We have:

- for every $\lambda \geq \Lambda_{\Omega,p}$, problem (27) admits a non-constant minimizer $u$ (see [26, Thm. 1.1]); such minimizer turns out to be a solution to the Bernoulli problem (25), provided the free boundary condition $|\nabla u| = \lambda$ is intended in a suitable weak sense [27, Thm. 2.1]; moreover, the free boundary $F(u)$ is a locally analytic hyper-surface, except for a $H^{n-1}$-negligible singular set [27, Cor. 9.2].
- as consequence of the results recalled at the above item, we have

$$
\Lambda_{\Omega,p} \geq \lambda_{\Omega,p};
$$

this inequality may be strict, as the explicit computation of both constants $\Lambda_{\Omega,p}$ and $\lambda_{\Omega,p}$ in case of the ball reveals [26, Section 4].

Being this a quick picture of the state of the art, in the light of the results proved in the previous sections for problem $[P]_\lambda$ it is natural to ask:
What is the asymptotics of the Bernoulli constants \( \lambda_{\Omega,p} \) and \( \Lambda_{\Omega,p} \) as \( p \to +\infty \)? Further, if \( p \) is fixed, does \( \Lambda_{\Omega,p} \) admit a non-constant solution \( u_p \) to (25), what is the limiting behaviour of \( u_p \) as \( p \to +\infty \)?

Starting from the asymptotics of the Bernoulli constants \( \lambda_{\Omega,p} \) and their variational counterparts \( \Lambda_{\Omega,p} \), we have:

**Proposition 27.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open domain. Then

\[
\limsup_{p \to +\infty} \lambda_{\Omega,p} \leq \limsup_{p \to +\infty} \Lambda_{\Omega,p} \leq 1/R_\Omega.
\]

**Proof.** In view of the inequality (28), it is enough to prove that

\[
\limsup_{p \to +\infty} \Lambda_{\Omega,p} \leq 1/R_\Omega.
\]

To obtain this inequality we observe that, if we fix \( \lambda > 1/R_\Omega \), for \( p \) large enough problem (27) admits a non-constant minimizer. Indeed, setting \( v_\lambda := (1 - \lambda d)_+ \), for \( p \gg 1 \) we have

\[
J_p(v_\lambda) = \frac{1}{p} |D\lambda| + \frac{p - 1}{p} |D\lambda| = |D\lambda| < \frac{p - 1}{p} |\Omega| = J_p(1).
\]

**Corollary 28.** Assume that \( \Omega \) is convex with \( \partial \Omega \) of class \( C^1 \). Then

\[
\lim_{p \to +\infty} \lambda_{\Omega,p}(\Omega) = \lim_{p \to +\infty} \Lambda_{\Omega,p}(\Omega) = 1/R_\Omega.
\]

**Proof.** It follows from (26) and Proposition 27.

Now, let \( \Omega \) be convex and let \( \lambda > 1/R_\Omega \). By Proposition 27, for \( p \) large enough there exists a non-constant solution \( u_p \) to (25). Moreover, by Corollary 17, problem (P)_\lambda admits a unique solution given precisely by the infinity harmonic potential \( w_\lambda \) of \( \Omega \). Nevertheless, we cannot expect that, in general, \( u_p \) converge to \( w_\lambda \) as \( p \to +\infty \). To enlighten this fact and get a feeling of the situation, let us have a look at what happens in case of the ball.

**Example 29** (The radial case). Let \( B_R \) be the ball of center 0 and radius \( R \) in \( \mathbb{R}^n \), and let \( \lambda > 1/R \). It is well-known that for \( \lambda = \lambda_p(B_R) \) the Bernoulli problem (25) on \( B_R \) admits a unique solution, which is called parabolic, whereas for any \( \lambda > \lambda_p(B_R) \) it admits two solutions, which are called hyperbolic and elliptic (as they are respectively decreasing and increasing with respect to the parameter \( \lambda \)).

Since we want to examine the asymptotic behaviour of these solutions as \( p \to +\infty \), let us briefly recover their expressions. By a result of Reichel [50], a solution to problem (25) on \( B_R \) is necessarily radial. Hence, for \( \rho \in (0, R) \) and \( p > n \), we compute the \( p \)-harmonic function \( u_p \) in \( B_R \setminus \partial B_R \), which satisfies the Dirichlet boundary conditions \( u_p = 1 \) on \( \partial B_R \) and \( u_p = 0 \) on \( \partial B_\rho \). It is given by

\[
u_p(x) = \frac{|x|^\alpha - \rho^{\alpha}}{R^{\alpha} - \rho^{\alpha}}, \quad \rho < |x| < R, \quad \alpha := \frac{p - n}{p - 1}
\]

(observe that, for \( p > n \), the exponent \( \alpha \) belongs to \( (0, 1) \), and tends to \( 1 \) as \( p \to +\infty \)).

We are interested in finding the values of \( \rho \in (0, R) \) such that

\[
|\nabla u_p(x)| = \lambda, \quad \text{for } |x| = \rho.
\]

Since \( |\nabla u_p(x)| = \alpha \frac{|x|^{\alpha - 1}}{R^{\alpha} - \rho^{\alpha}} \), condition (30) is equivalent to

\[
f_\alpha(\rho) := \lambda \rho^\alpha + \alpha \rho^{\alpha-1} - \lambda R^\alpha = 0.
\]
It is immediate to check that $f_{\alpha}$ is strictly decreasing in $\left(0, \frac{1-\alpha}{\rho}\right)$ and strictly increasing in $\left(\frac{1-\alpha}{\rho}, R\right)$, so that

$$m_{\alpha} := \min_{(0,R)} f_{\alpha} = f_{\alpha} \left(\frac{1-\alpha}{\rho}\right) = \left(\frac{\lambda}{1-\alpha}\right)^{1-\alpha} - \lambda R^{\alpha}.$$  

Moreover,

$$\lim_{\rho \to 0^+} f_{\alpha}(\rho) = +\infty, \quad f_{\alpha}(R) = \alpha R^{\alpha-1} > 0.$$  

Hence, equation (31) has one solution if $m_{\alpha} = 0$, two solutions if $m_{\alpha} < 0$, no solutions if $m_{\alpha} > 0$. Observe that

$$m_{\alpha} \leq 0 \iff \lambda \geq \lambda_{p}(B_{R}) := \frac{1}{R} \left(1 - \alpha\right)^{1-1/\alpha} = \frac{1}{R} \left(\frac{n-1}{p-1}\right)^{-(n-1)/(p-n)}.$$  

In particular, for $p$ large enough, since $\lim_{\alpha \to 1^-} m_{\alpha} = 1 - \lambda R < 0$, equation (31) has exactly two zeros $\rho'_{\alpha}$ and $\rho''_{\alpha}$; correspondingly, the sets $\partial B_{\rho'_{\alpha}}$ and $\partial B_{\rho''_{\alpha}}$ are the free boundaries of the so-called hyperbolic and elliptic solutions to (25).

Now, let us look at what happens as $p \to +\infty$. We know from the above computations that

$$0 < \rho'_{\alpha} < \frac{1-\alpha}{\lambda} < \rho''_{\alpha} < R.$$  

This gives at once $\rho'_{\alpha} \to 0$ as $\alpha \to 1^-$. On the other hand it is easily seen that, for every $0 < \varepsilon < \min\left\{1/\lambda, R - 1/\lambda\right\}$, it holds

$$\lim_{\alpha \to 1^-} f_{\alpha}\left(R - \frac{1}{\lambda} - \varepsilon\right) = -\varepsilon \lambda < 0, \quad \lim_{\alpha \to 1^-} f_{\alpha}\left(R - \frac{1}{\lambda} + \varepsilon\right) = \varepsilon \lambda > 0,$$

so that $\rho''_{\alpha} \to R - \frac{1}{\lambda}$ as $\alpha \to 1^-$. Summarizing, the above analysis shows that the two families of $p$-harmonic functions which fit the Bernoulli free boundary condition (30) for $\rho = \rho'_{p}$ and $\rho = \rho''_{p}$ have respectively the following asymptotic behaviour: their free boundary degenerates or converge to the parallel set $\partial \Omega_{\frac{n}{p}}$, i.e.,

$$\rho'_{p} \to 0, \quad \rho''_{p} \to R - \frac{1}{\lambda},$$

and the functions $u_{p}$, as given by (29), converge to

$$w_{R}(x) = \frac{|x|}{R}, \quad x \in \overline{B}_{R}, \quad w_{\frac{n}{p}}(x) = 1 - \lambda(R - |x|), \quad x \in \overline{B}_{R} \setminus B_{R - \frac{1}{\lambda}}.$$  

In particular, only the elliptic family converges to the unique solution of (P)$_{\lambda}$. Let us remark that, for $\lambda \geq \Lambda_{p}(B_{R})$, contrary to the hyperbolic solutions, the elliptic ones are variational, namely they solve the minimization problem (27) on $B_{R}$ (see [34, Sec. 5.3], [26, Sec. 4]).

Now, as suggested by the example of the ball, we give a convergence result for variational solutions. Preliminarily, we give a useful remark about the asymptotic behaviour of the functionals $J_{p}^{\lambda}$ as $p \to +\infty$. 
Remark 30. Let $q > 1$ be fixed. It is easy to check that, for every $p \geq q$, the functional $J_p^\lambda$ is lower semicontinuous with respect to the weak topology of $W^{1,q}(\Omega)$. Moreover, as observed in [43, Proposition 1], for every fixed $u \in \text{Lip}(\Omega)$, the map $p \mapsto J_p^\lambda(u)$ is monotone nondecreasing. Namely, if $1 < p \leq q$, by applying Young’s inequality $AB \leq (A^r/r) + (B^s/s)$, with $A = |\nabla u|/\lambda$, $B = 1$, $r = q/p$ and $s = r/(r-1)$, we obtain
\[
J_p^\lambda(u) \leq \frac{1}{q} \int_\Omega \left( \frac{|\nabla u|}{\lambda} \right)^q \, dx + \left( \frac{q-p}{pq} + \frac{p-1}{p} \right) |\{u > 0\}| = J_1^\lambda(u).
\]
Thanks to this monotonicity property, by applying [10, Remark 1.40(ii)] we conclude that the sequence $(J_p^\lambda)_p$ $\Gamma$-converges, with respect to the weak topology of $W^{1,q}(\Omega)$, to its “pointwise” limit, namely the functional functional $J_\infty$ given by
\[
J_\infty(u) := \begin{cases} |\{u > 0\}|, & \text{if } \|\nabla u\|_{\infty} \leq \lambda, \\ +\infty, & \text{otherwise.} \end{cases}
\]
The reader may find a similar $\Gamma$-convergence result in the paper [43], where the Authors deal with the asymptotic behaviour of variational energies related to the exterior Bernoulli boundary problem for the $p$-laplacian as $p \to +\infty$.

**Theorem 31.** Let $\lambda > 1/R_\Omega$. For $p$ large enough, let $u_{\lambda,p}$ be a solution to the $p$-Bernoulli problem [25] which is found by solving the minimization problem [27]. Then, there exists an increasing sequence $(p_j)$, diverging to $+\infty$, such that
\[
u_{\lambda,p_j} \rightharpoonup u_\infty \text{ weakly in } W^{1,q}(\Omega) \quad \forall q > 1, \quad u_{\lambda,p_j} \to u_\infty \text{ uniformly in } \overline{\Omega},
\]
where $u_\infty$ is a solution of the $\infty$-Bernoulli problem (P)$^\lambda$ satisfying
\[
\text{int}\{u_\infty = 0\} = \{d > 1/\lambda\}.
\]

**Proof.** Thanks to the assumption $\lambda > 1/R_\Omega$ and to Proposition [27] we know that for $p$ large enough problem (27) admits a solution $u_{\lambda,p}$. As $\lambda$ is fixed, we shall write for brevity $J_p$ in place of $J_p^\lambda$ and $u_p$ in place of $u_{\lambda,p}$.

Let us first show that, for every fixed $q > 1$, the family $(u_p)$ is uniformly bounded in $W^{1,q}(\Omega)$. Since
\[
\frac{1}{p} \int_\Omega \left( \frac{|\nabla u_p|}{\lambda} \right)^p \, dx \leq J_p(u_p) \leq J_p(1) = \frac{p-1}{p} |\Omega| \leq |\Omega|,
\]
we get
\[
\|\nabla u_p\|_p \leq \lambda p^{1/p} |\Omega|^{1/p}.
\]
Then, by Hölder inequality, for every $p > q + 1$ it holds
\[
\|\nabla u_p\|_q \leq \|\nabla u_p\|_p |\Omega|^{\frac{p}{q(p-q)}} \leq \lambda p^{1/p} |\Omega|^{\frac{1}{p} + \frac{q}{q(p-q)}} \leq C,
\]
where $C > 0$ is a constant independent of $p$.

Using a diagonal argument, we can construct an increasing sequence $p_j \to +\infty$ such that $u_{p_j}$ converges weakly in $W^{1,q}(\Omega)$ for every $q > 1$ and uniformly in $\overline{\Omega}$ to a function $u_\infty$. We claim that $u_\infty$ is a solution to (P)$^\lambda$ satisfying $\text{int}\{u_\infty = 0\} = \{d > 1/\lambda\}$.

The fact that $u_\infty$ is infinity harmonic in its positivity set is a standard consequence of the fact that $u_{p_j}$ solve (27) with $p = p_j \to +\infty$, see for instance the arguments in [52, proof of Theorem 1].

Moreover, since $u_{p_j} = 1$ on $\partial \Omega$ for every $j$, from the uniform convergence it follows immediately that the same condition is satisfied by $u_\infty$. 

Next we are going to show that the set
\[ K := \{ u_\infty = 0 \} \]
satisfies \( \text{int}(K) = \{ d > 1/\lambda \} \), and that it belongs to the class introduced in Definition 24. From the second inequality in (33), we see that \( \| \nabla u_\infty \|_{L^\infty(\Omega)} \leq \lambda \). Since \( u = 1 \) on \( \partial \Omega \), we conclude that
\[ u \in \text{Lip}_1(\Omega) := \{ u \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega) : u = 1 \text{ on } \partial \Omega \} . \]
Therefore we have \( \{ u_\infty > 0 \} \supseteq D_{1/\lambda} \), and we deduce as a first information on \( K \) the inclusion
\[ K \subseteq \{ d \geq 1/\lambda \} . \tag{34} \]
To go farther, we claim that \( u_\infty \) solves the minimization problem
\[ \min \left\{ J_\infty(u) : u \in \text{Lip}_1(\Omega) \right\} , \tag{35} \]
where \( J_\infty \) is the functional defined by (32). Indeed, since \( u_{p_j} \to u_\infty \) uniformly in \( \Omega \), for every fixed \( \varepsilon > 0 \), there exists an index \( j_\varepsilon \in \mathbb{N} \) such that
\[ |\{ u_\infty > 0 \}| < |\{ u_{p_j} > 0 \}| + \varepsilon, \quad \forall j > j_\varepsilon . \]
Then, for \( j > j_\varepsilon \), it holds
\[ J_{p_j}(u_\infty) \leq \frac{1}{p_j} |\{ u_\infty > 0 \}| + \frac{p_j - 1}{p_j} |\{ u_\infty > 0 \}| \]
\[ \leq \frac{1}{p_j} |\Omega| + \frac{p_j - 1}{p_j} (|\{ u_{p_j} > 0 \}| + \varepsilon) \]
\[ \leq \frac{1}{p_j} |\Omega| + J_{p_j}(u_{p_j}) + \varepsilon . \tag{36} \]
Thanks to the monotonicity property pointed out in Remark 30, we can now pass to the limit as \( j \to +\infty \) in (36). By the arbitrariness of \( \varepsilon > 0 \), and recalling that \( u_{p_j} \) are solutions to (27) (with \( p = p_j \)), we obtain, for every \( u \in \text{Lip}_1(\Omega) \),
\[ J_\infty(u_\infty) = \lim_{j \to +\infty} J_{p_j}(u_\infty) \leq \liminf_{j \to +\infty} J_{p_j}(u_{p_j}) \leq \liminf_{j \to +\infty} J_{p_j}(u) = J_\infty(u), \]
so that \( u_\infty \) solves problem (35) as claimed. Consequently, by taking as a competitor the function \( (1 - \lambda d)_+ \), we deduce that \( |\{ u_\infty > 0 \}| \leq |D_{1/\lambda}| \), or equivalently
\[ |K| \geq |\{ d \geq 1/\lambda \}| . \tag{37} \]
Since \( \text{int}\{ d \geq 1/\lambda \} = \{ d > 1/\lambda \} \) and \( |\{ d \geq 1/\lambda \}| = |\{ d > 1/\lambda \}| \), by combining conditions (34) and (37) we obtain \( \text{int} K = \{ d > 1/\lambda \} \). As a consequence, \( K \) belongs to the family \( K_\lambda \) introduced in Definition 24, so that, by Theorem 26, \( u_\infty \) is a solution to \( (P)_\lambda \). \[ \square \]

**Corollary 32.** Let \( \lambda > 1/R_\Omega \). Then, under the assumptions (H1)–(H2) of Theorem 16 (hence, in particular, when \( \Omega \) is convex), in the limit as \( p \to +\infty \) we have
\[ u_{\lambda,p} \rightharpoonup w_\lambda \text{ weakly in } W^{1,q}(\Omega) \quad \forall q > 1, \quad u_{\lambda,p_j} \to w_\lambda, \text{ uniformly in } \overline{\Omega} , \]
where \( w_\lambda \) is the infinity harmonic potential of \( \overline{\Omega}_\lambda \), namely the unique solution to the \( \infty \)-Bernoulli problem \( (P)_\lambda \).
Proof. From Theorem 31, there exists an increasing sequence \( p_j \nearrow \infty \) such that \( u_{\lambda,p_j} \to u_\infty \), with \( u_\infty \) solution to \((P)_\lambda\). Hence, by Theorem 16 we have that \( u_\infty = \frac{w_1}{\lambda} \). By the same argument, any other converging subsequence must converge to \( \frac{w_1}{\lambda} \). \( \square \)

Acknowledgments. We thank Bozhidar Velichkov for a useful discussion about the viscosity interpretation of free boundary conditions. The authors have been supported by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

REFERENCES

[1] P. Albano, P. Cannarsa, K.T. Nguyen, and C. Sinestrari, Singular gradient flow of the distance function and homotopy equivalence, Math. Ann. 356 (2013), no. 1, 23–43. MR3038120
[2] H. W. Alt and L. A. Caffarelli, Existence and regularity for a minimum problem with free boundary, J. Reine Angew. Math. 325 (1981), 105–144.
[3] G. Aronsson, Extension of functions satisfying Lipschitz conditions, Ark. Mat. 6 (1967), 551–561 (1967).
[4] G. Aronsson, M.G. Crandall, and P. Juutinen, A tour of the theory of absolutely minimizing functions, Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 4, 439–505.
[5] Gunnar Aronsson, Michael G. Crandall, and Petri Juutinen, A tour of the theory of absolutely minimizing functions, Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 4, 439–505. MR2083637
[6] E. N. Barron, R. R. Jensen, and C. Y. Wang, The Euler equation and absolute minimizers of \( L^\infty \) functionals, Arch. Ration. Mech. Anal. 157 (2001), no. 4, 255–283.
[7] T. Bhattacharya, On the properties of \( \infty \)-harmonic functions and an application to capacitary convex rings, Electron. J. Differential Equations (2002), No. 101, 22. MR1938397
[8] T. Bhattacharya, E. DiBenedetto, and J. Manfredi, Limits as \( p \to \infty \) of \( \Delta_p u_p = f \) and related extremal problems, Rend. Sem. Mat. Univ. Politec. Torino Special Issue (1989), 15–68 (1991). Some topics in nonlinear PDEs (Turin, 1989).
[9] T. Bhattacharya and A. Mohammed, Inhomogeneous Dirichlet problems involving the infinity-Laplacian, Adv. Differential Equations 17 (2012), no. 3-4, 225–266.
[10] A. Braides, \( \Gamma \)-convergence for beginners, Oxford University Press, New York, 2002.
[11] L. A. Caffarelli, D. Jerison, and C.E. Kenig, Global energy minimizers for free boundary problems and full regularity in three dimensions, Noncompact problems at the intersection of geometry, analysis, and topology, 2004, pp. 83–97.
[12] L.A. Caffarelli, A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are \( C^{1,\alpha} \), Rev. Mat. Iberoamericana 3 (1987), no. 2, 139–162. MR990856
[13] _____, A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz, Comm. Pure Appl. Math. 42 (1989), no. 1, 55–78. MR973745
[14] _____, A Harnack inequality approach to the regularity of free boundaries. III. Existence theory, compactness, and dependence on \( X \), Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 15 (1988), no. 4, 583–602 (1989). MR1029856
[15] _____, A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz, Comm. Pure Appl. Math. 42 (1989), no. 1, 55–78. MR973745
[16] L.A. Caffarelli and S. Salsa, A geometric approach to free boundary problems, Graduate Studies in Mathematics, vol. 68, American Mathematical Society, Providence, RI, 2005. MR2145284
[17] P. Cardaliaguet and R. Tahraoui, Some uniqueness results for Bernoulli interior free-boundary problems in convex domains, Electron. J. Differential Equations (2002), No. 102, 16.
[18] M.G. Crandall, A visit with the \( \infty \)-Laplace equation, Calculus of variations and nonlinear partial differential equations, 2008, pp. 75–122.
[19] M.G. Crandall, L.C. Evans, and R.F. Gariepy, Optimal lipschitz extensions and the infinity Laplacian, Calc. Var. Partial Differential Equations 13 (2001), no. 2, 123–139.
[20] G. Crasta and I. Fragalà, A variational problem of Bernoulli–supremal type. in preparation.
[21] _____, On the Dirichlet and Serrin problems for the inhomogeneous infinity Laplacian in convex domains: regularity and geometric results, Arch. Ration. Mech. Anal. 218 (2015), no. 3, 1577–1607. MR3401015
[21] A symmetry problem for the infinity Laplacian, Int. Math. Res. Not. IMRN 18 (2015), 8411–8436. MR3417681
[22] A C1 regularity result for the inhomogeneous normalized infinity Laplacian, Proc. Amer. Math. Soc. 144 (2016), no. 6, 2547–2558. MR3477071
[23] Characterization of stadium-like domains via boundary value problems for the infinity Laplacian, Nonlinear Anal. 133 (2016), 228–249. MR3449756
[24] Rigidity results for variational infinity ground states, 2017. to appear in Indiana Univ. Math. J.
[25] G. Crasta and A. Malusa, The distance function from the boundary in a Minkowski space, Trans. Amer. Math. Soc. 359 (2007), 5725–5759.
[26] D. Daners and B. Kawohl, An isoperimetric inequality related to a Bernoulli problem, Calc. Var. Partial Differential Equations 39 (2010), no. 3-4, 547–555. MR2729312
[27] D. Danielli and A. Petrosyan, A minimum problem with free boundary for a degenerate quasilinear operator, Calc. Var. Partial Differential Equations 23 (2005), no. 1, 97–124. MR2133664
[28] Full regularity of the free boundary in a Bernoulli-type problem in two dimensions, Math. Res. Lett. 13 (2006), no. 4, 667–681.
[29] D. De Silva, Free boundary regularity for a problem with right hand side, Interfaces Free Bound. 13 (2011), no. 2, 223–238. MR2813524
[30] D. De Silva and D. Jerison, A singular energy minimizing free boundary, J. Reine Angew. Math. 635 (2009).
[31] L.C. Evans and O. Savin, C1,α regularity for infinity harmonic functions in two dimensions, Calc. Var. Partial Differential Equations 32 (2008), 325–347.
[32] L.C. Evans and C.K. Smart, Everywhere differentiability of infinity harmonic functions, Calc. Var. Partial Differential Equations 42 (2011), 289–299.
[33] A. Figalli and H. Shahgholian, An overview of unconstrained free boundary problems, Philos. Trans. Roy. Soc. A 373 (2015), no. 2050.
[34] M. Flucher and M. Rumpf, Bernoulli’s free-boundary problem, qualitative theory and numerical approximation, J. Reine Angew. Math. 486 (1997), 165–204.
[35] A. Henrot and H. Shahgholian, Existence of classical solutions to a free boundary problem for the p-Laplace operator. II. The interior convex case, Indiana Univ. Math. J. 49 (2000), no. 1, 311–323. MR1777029
[36] E. Hewitt and K. Stromberg, Real and abstract analysis, Springer-Verlag, Berlin, 1969.
[37] G. Hong, Boundary differentiability for inhomogeneous infinity Laplace equations, Electron. J. Differential Equations (2014), No. 72, 6.
[38] R. Hynd, C.K. Smart, and Y. Yu, Nonuniqueness of infinity ground states, Calc. Var. Partial Differential Equations 48 (2013), no. 3-4, 545–554.
[39] R. Jensen, Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient, Arch. Rational Mech. Anal. 123 (1993), 51–74.
[40] P. Juutinen, P. Lindqvist, and J.J. Manfredi, The ∞-eigenvalue problem, Arch. Ration. Mech. Anal. 148 (1999), no. 2, 89–105.
[41] The infinity Laplacian: examples and observations, Papers on analysis, 2001, pp. 207–217. MR1886623
[42] B. Kawohl and H. Shahgholian, Gamma limits in some Bernoulli free boundary problem, Arch. Math. (Basel) 84 (2005), no. 1, 79–87. MR2106407
[43] R. V. Kohn and S. Serfaty, A deterministic-control-based approach to motion by curvature, Comm. Pure Appl. Math. 59 (2006), no. 3, 344–407.
[44] E. Lindgren, On the regularity of solutions of the inhomogeneous infinity Laplace equation, Proc. Amer. Math. Soc. 142 (2014), no. 1, 277–288. MR319202
[45] G. Lu and P. Wang, Inhomogeneous infinity Laplace equation, Adv. Math. 217 (2008), 1838–1868.
[46] J. Manfredi, A. Petrosyan, and H. Shahgholian, A free boundary problem for ∞-Laplace equation, Calc. Var. Partial Differential Equations 14 (2002), no. 3, 359–384. MR1899452
[47] Y. Peres, O. Schramm, S. Sheffield, and D. B. Wilson, Tug-of-war and the infinity Laplacian, J. Amer. Math. Soc. 22 (2009), no. 1, 167–210.
A. Petrosyan, On the full regularity of the free boundary in a class of variational problems, Proc. Amer. Math. Soc. 136 (2008), no. 8, 2763–2769.

W. Reichel, Radial symmetry by moving planes for semilinear elliptic BVPs on annuli and other non-convex domains, Elliptic and parabolic problems (Pont-à-Mousson, 1994), 1995, pp. 164–182.

J. D. Rossi, E. V. Teixeira, and J. M. Urbano, Optimal regularity at the free boundary for the infinity obstacle problem, Interfaces Free Bound. 17 (2015), no. 3, 381–398.

J.D. Rossi and E.V. Teixeira, A limiting free boundary problem ruled by Aronsson's equation, Trans. Amer. Math. Soc. 364 (2012), no. 2, 703–719. MR2846349

J.D. Rossi and P. Wang, The limit as $p \to \infty$ in a two-phase free boundary problem for the $p$-Laplacian, Interfaces Free Bound. 18 (2016), no. 1, 115–135.

O. Savin, $C^1$ regularity for infinity harmonic functions in two dimensions, Arch. Ration. Mech. Anal. 176 (2005), no. 3, 351–361. MR2185662 (2006i:35108)

J. Siljander, C. Wang, and Y. Zhou, Everywhere differentiability of viscosity solutions to a class of Aronsson's equations, 2014. preprint arXiv:1409.6804.

R. Teymurazyan and J.M. Urbano, A free boundary optimization problem for the $\infty$-Laplacian, J. Differential Equations 263 (2017), no. 2, 1140–1159.

Y. Yu, Some properties of the ground states of the infinity Laplacian, Indiana Univ. Math. J. 56 (2007), 947–964.

(Graziano Crasta) Dipartimento di Matematica “G. Castelnuovo”, Univ. di Roma I, P.le A. Moro 2 – 00185 Roma (Italy)
E-mail address: crasta@mat.uniroma1.it

(Ilaria Fragalà) Dipartimento di Matematica, Politecnico, Piazza Leonardo da Vinci, 32 – 20133 Milano (Italy)
E-mail address: ilaria.fragala@polimi.it