CUCKER-SMALE FLOCKING PARTICLES WITH MULTIPlicative NOiSES: STOCHASTic MEAN-FIELD LIMIT AND PHASE TRANSITION

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ABSTRACT. In this paper, we consider the Cucker-Smale flocking particles which are subject to the same velocity-dependent noise, which exhibits a phase change phenomenon occurs bringing the system from a “non flocking” to a “flocking” state as the strength of noises decreases. We rigorously show the stochastic mean-field limit from the many-particle Cucker-Smale system with multiplicative noises to the Vlasov-type stochastic partial differential equation as the number of particles goes to infinity. More precisely, we provide a quantitative error estimate between solutions to the stochastic particle system and measure-valued solutions to the expected limiting stochastic partial differential equation by using the Wasserstein distance. For the limiting equation, we construct global-in-time measure-valued solutions and study the stability and large-time behavior showing the convergence of velocities to their mean exponentially fast almost surely.

1. Introduction. In the current work, we are interested in stochastic flocking systems with multiplicative noises in the Stratonovich sense. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{F}_t)_{t \geq 0}$. Here $\Omega$ is the random set, $\mathbb{P}$ and $\mathcal{F}$ are measure and $\sigma$-algebra on the set, respectively. On that probability space, $(B_t)_{t \geq 0}$ denotes a real-valued Brownian motion. Let $X^i_t \in \mathbb{R}^d$ and $V^i_t \in \mathbb{R}^d$ be position and velocity of $i$-th particle at time $t \geq 0$, respectively, then our main stochastic differential equations read as follows:

$$dX^i_t = V^i_t \, dt, \quad i = 1, \ldots, N, \quad t > 0,$$

$$dV^i_t = F[\mu^N_t](X^i_t, V^i_t) \, dt + \sqrt{2\sigma} \big( V^i_t - V^i_t \big) \circ dB_t, \quad \mu^N_t := \frac{1}{N} \sum_{i=1}^N \delta_{(X^i_t, V^i_t)}, \quad (1.1)$$

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or, equivalently, in the Itô form,
\[
dX_t^i = V_t^i \, dt, \quad 1, \ldots, N, \quad t > 0,
\]
\[
dV_t^i = F[\mu^N_t](X_t^i, V_t^i) \, dt - \sigma (\bar{V}_t - V_t^i) \, dt + \sqrt{2\sigma} (\bar{V}_t - V_t^i) \, dB_t,
\]
subject to the deterministic initial data \((X_0^i, V_0^i)\), for \(i = 1, \ldots, N\). Here \(\bar{V}_t\) is an averaged particle velocity, i.e., \(\bar{V}_t := \frac{1}{N} \sum_{j=1}^N V_t^j\) and \(F[\mu]\) represents a velocity alignment force given by
\[
F[\mu](x, v) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|x - y|)(w - v) \, \mu(dy, dw) \quad \text{for} \quad \mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d),
\]
where \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) called a communication weight, which is in general non-increasing function. Strong existence and pathwise uniqueness for the particle system (1.1) holds up to any finite time \(T > 0\) thanks to standard results (see [11, Theorems 3.1 and 3.2] or [15, Proposition 3.28]). When there is no noise, i.e., \(\sigma = 0\), the stochastic particle system (1.1) is reduced to the Cucker-Smale model [9]. We refer to [5, 7] for a recent overview of Cucker-Smale and its variants. The system (1.1) is proposed in [1] by taking into account uniform randomness in the communication weight function. In [1], a flocking estimate showing the relative positions are uniformly bounded in time and relative velocities converge to zero as time goes to infinity almost surely is obtained. Later, in [19], the phase change phenomenon from non flocking to flocking states in (1.1) is observed by considering the convergence of relative velocities in the \(L^2\)-norm. We also refer to [6] for the study of Cucker-Smale type models with different types of stochastic noises. For a flocking estimate of the Cucker-Smale model and its variants with \(N\)-independent noise of uniform strength, we refer to [13, 16].

Formal passage to the mean-field limit \(N \to \infty\) for the particle system (1.1) yields the following stochastic partial differential equation:
\[
d\mu_t + (v \cdot \nabla_x \mu_t) dt + (\nabla_v \cdot (F[\mu_t] \mu_t)) dt + \sqrt{2\sigma} \nabla_v \cdot ((\bar{v}_t - v) \mu_t) \circ dB_t = 0,
\]
or again equivalently, in the Itô form:
\[
d\mu_t + (v \cdot \nabla_x \mu_t) dt + (\nabla_v \cdot (F[\mu_t] \mu_t)) dt + \sqrt{2\sigma} \nabla_v \cdot ((\bar{v}_t - v) \mu_t) dB_t
\]
\[
= \sigma \nabla_v \cdot ((\bar{v}_t - v) \mu_t) \cdot dB_t + \sigma \nabla_v \cdot ((\bar{v}_t - v) \mu_t) dt,
\]
where \(\bar{v}_t := \int_{\mathbb{R}^d \times \mathbb{R}^d} v \, \mu_t(dx, dv)\). The limiting equation (1.3) is indeed stochastic in this case as shown for instance in [8], where a system of interacting particles are subject to the same space-dependent noise is discussed. This is due to the fact that the noise which drives the motion of each particle in (1.2) is the same. In classical McKean-Vlasov particle system, the noise seen by each particle are independent from each other [18, Theorem 1.1], and the limiting equation becomes a deterministic diffusion equation. This result can be classically proved by coupling the \(N\)-particle system with independent initial condition to the \(N\) independent copies of the nonlinear particle. It is worth emphasizing that the independence of noises in the system is important in that coupling method, see [18] for more details on that.

The first purpose of this paper is to establish the global existence and uniqueness of measure-valued solutions to the stochastic partial differential equation (1.3), and the rigorous analysis of the stochastic mean-field limit of the system (1.1). As pointed out in [8], the equation (1.3) can be understood as a standard transport PDE as the random \(\omega \in \Omega\) is fixed. The empirical measure \((\mu^N_t)_{t \geq 0}\) associated to
the stochastic particle system (1.1) solves the stochastic partial differential equation (1.3) for any finite \( N \), see Section 2.1 below for details. This enables us to take a strategy based on weak-weak/weak-strong stability estimates used for deterministic transport type equations [3, 4, 10] for fixed \( \omega \in \Omega \). More precisely, if \((\mu_t)_{t \geq 0}\) and \((\nu_t)_{t \geq 0}\) are two solutions to (1.3) for the respective initial data \( \mu_0 \) and \( \nu_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d) \) in the space \( C([0,T]; \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)) \), we need to establish some (local in time) inequalities of the type:

\[
\sup_{t \in [0,T]} \mathbb{E}[W_2(\mu_t, \nu_t)] \leq C_T \mathbb{E}[W_2(\mu_0, \nu_0)], \tag{1.5}
\]

where \( C_T \) is a nonnegative constant depending on the time and other parameters of the problem, or a weaker version

\[
\sup_{t \in [0,T]} W_2(\mu_t, \nu_t) \leq C_T W_2(\mu_0, \nu_0) \quad \text{almost surely.} \tag{1.6}
\]

Here \( C_T \) is a nonnegative almost surely finite random variable depending on the time and other parameters of the problem. Here \( W_2 \) denotes the Wasserstein distance of order 2 defined by

\[
W_2^2(\mu, \nu) := \inf_{\xi \in \Gamma(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \xi(dx, dv) \right) = \inf_{(X \sim \mu, Y \sim \nu)} \mathbb{E}[|X - Y|^2],
\]

where \( \Gamma(\mu, \nu) \) is the set of all probability measures on \( \mathbb{R}^d \times \mathbb{R}^d \) with first and second marginals \( \mu \) and \( \nu \), respectively, and \((X,Y)\) are all possible couples of random variables with \( \mu \) and \( \nu \) as respective laws. Compared to the classical case of globally Lipschitz and bounded potentials, the force fields in (1.1) are only locally Lipschitz and bounded in velocity and the result by Dobrushin [10] cannot be directly applied. Note that a classical feature of the Cucker-Smale equation with nonnegative communication weight is to keep the speed of particle velocity bounded by the maximal speed at initial state. On the other hand, in the presence of diffusion, that maximum principle does not hold since the Brownian motion can make the velocities as high as wanted with some non-zero probability. In [2], similar Newtonian types of equations with independent standard Brownian motions, which have locally Lipschitz potentials, are considered, and the high speed of particle velocities are controlled by imposing the exponential moments bound, see also [16], where the propagation of chaos for a stochastic flocking model is studied by using a different strategy based on probability theory. However, in our case for the stochastic transport PDE, we can obtain a \( \mathbb{P} \)-almost sure propagation of the compact support in velocity if the initial data is compactly supported in velocity. This only gives that the force fields are Lipschitz and bounded \( \mathbb{P} \)-almost surely, thus we can have a similar inequality as (1.6), but not the type of (1.5) since the Lipschitz constant of force fields is a random variable which does not have any exponential moments, see Proposition 1. This stability estimate enables us to approximate a solution \( \mu_t \) to the equation (1.3) by the empirical measure \( \mu_N^t \) associated to the particle system (1.1), and in fact, this provides the stochastic mean-field limit. We remark that the mean-field limit of the particle system (1.1) is studied in [12], and a Fokker-Planck type equation is derived as the corresponding mean-field equation. However, that corresponds to the Cucker-Smale model with \( N \)-independent Brownian motions, i.e., adding \( \sqrt{2\sigma} (\bar{V}_t - V^i_t) dB^i_t \), not the dependent Brownian motion appeared in (1.1).
Our second goal in this paper is to discuss the phase change phenomenon in the limiting stochastic kinetic equation (1.3) showing the transition from non flocking to flocking states as the strength of noises decreases. We notice that flocking behavior of solutions implies the concentration of velocities of particles, i.e., formation of a Dirac delta in velocity, see Remark 4. Thus it is natural to consider measure-valued solutions in our notion of solutions for the time-asymptotic behavior of solutions. Since the empirical measures associated to the particle system (1.1) well approximate the measure-valued solutions to the stochastic partial differential equation (1.3), see Proposition 1, we can easily extend the result of phase change phenomenon at particle level to the infinite-dimensional one.

Before stating our main results, we first introduce a notion of measure-valued solutions to the kinetic system (1.3). For this, we use a standard notation:

$$\langle \nu, \phi \rangle := \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x, v) \nu(dx, dv), \quad \text{for} \quad \nu \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d).$$

**Definition 1.1.** A family $\{\mu_t(w) : t \geq 0, w \in \Omega\}$ of random probability measures taking value in $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ is a measure-valued solution of the equation (1.3) if

(i) $\mu_t$ is weakly continuous: for all $\phi \in C^2_0(\mathbb{R}^d \times \mathbb{R}^d)$, $\langle \mu_t, \phi \rangle$ is an adapted process with a continuous version.

(ii) $\mu_t$ satisfies the stochastic integral equation: for all $\phi \in C^2_0(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\langle \mu_t, \phi \rangle = \langle \mu_0, \phi \rangle + \int_0^t \langle \mu_s, v \cdot \nabla_x \phi + F[\mu_s] \cdot \nabla_v \phi \rangle ds + \sqrt{2\sigma} \int_0^t \langle \mu_s, (\bar{v}_s - v) \cdot \nabla_v \phi \rangle dB_s.$$

**Remark 1.** The weak formulation in Definition 1.1 can be rewritten as

$$\langle \mu_t, \phi \rangle = \langle \mu_0, \phi \rangle + \int_0^t \langle \mu_s, v \cdot \nabla_x \phi + (F[\mu_s] - \sigma(\bar{v}_s - v)) \cdot \nabla_v \phi \rangle ds$$

$$+ \sqrt{2\sigma} \int_0^t \langle \mu_s, (\bar{v}_s - v) \cdot \nabla_v \phi \rangle dB_s$$

$$+ \sigma \int_0^t \langle \mu_s, (\bar{v}_s - v) \otimes (\bar{v}_s - v) : \nabla_v^2 \phi \rangle ds.$$  

Here $\cdot \otimes$ denotes the outer product of two vectors, i.e., $(u \otimes v)_{ij} = u_i v_j$ for $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$ with $n, m \in \mathbb{N}$, and $\cdot$ represents the standard inner product between matrices, i.e., $A : B = \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij}$ for $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$.

We now state our first result on the global existence and uniqueness of measure-valued solutions to the stochastic partial differential equation (1.3).

**Theorem 1.2.** Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ be compactly supported in velocity and $T > 0$. Suppose that the communication weight $\psi \in C^1_b(\mathbb{R}_+)$.

Then there exists at most one measure-valued solution to equation (1.3) $\mu_t \in C([0, T], \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d))$ in the sense of Definition 1.1, which is almost surely compactly supported in velocity. Moreover, $\mu_t$ is determined as the push-forward of the initial density through the stochastic flow map generated by the local Lipschitz field $(v, F[\mu_t] + \sigma(v - \bar{v}_t) - \sqrt{2\sigma}(v - \bar{v}_t) dB_t/dt)$ in phase space. Furthermore, if $\mu$ and $\overline{\mu}$ are two such solutions to the equation (1.3) with compactly supported initial data $\mu_0$ and $\overline{\mu}_0$ in velocity, we have

$$W_2(\mu_t, \overline{\mu}_t) \leq C W_2(\mu_0, \overline{\mu}_0) e^{C(t + W_2(\mu_0, \overline{\mu}_0))},$$

for $t \in [0, T]$ almost surely, where the constant $C$ depends only on $\psi, T, \sigma$, $\sup_{t \in [0, T]} |B_t|$, and the support in velocity of $\mu_0$ and $\overline{\mu}_0$. 

Remark 2. As mentioned before, the empirical measure \( \mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_t^i, v_t^i)} \) associated to the particle system (1.1) is the solution to the stochastic partial differential equation (1.3) in the sense of Definition 1.1 starting from \( \mu_0^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_0^i, v_0^i)} \), see Section 2.1. Thus it follows from the stability estimate in Theorem 1.2 that

\[
\sup_{0 \leq t \leq T} W_2(\mu_t, \mu_t^N) \leq CW_2(\mu_0, \mu_0^N) e^{C(1+ W_2(\mu_0, \mu_0^N))},
\]

where \( C \) is a random variable independent of \( N \). Thus if \( W_2(\mu_0, \mu_0^N) \to 0 \) as \( N \to \infty \), we have

\[
\sup_{0 \leq t \leq T} W_2(\mu_t, \mu_t^N) \to 0 \quad \text{as} \quad N \to \infty, \quad \text{almost surely.}
\]

Note that we can construct the initial atomic measures \( \mu_0^N \) approximating the initial data \( \mu_0 \) such that \( W_2(\mu_0, \mu_0^N) \to 0 \) as \( N \to \infty \) in the standard way: \( \mu_0^N \) can be chosen as a dirac masses centered at i.i.d. random variables of law \( \mu_0 \).

Our second result on the phase change phenomenon from a “non flocking” to a “flocking” state depending on the strength of noises is presented below. In order to state our theorem, we need to introduce a variance functional of the stochastic particle velocity fluctuation around \( \bar{v}_t \):

\[
E[\mu_t] := \int_{\mathbb{R}^d \times \mathbb{R}^d} |\bar{v}_t - v|^2 \mu_t(dx, dv).
\]

For the sake of notational simplicity, we denote by \( E_t := E[\mu_t] \).

**Theorem 1.3.** Let \( \mu_t \) be a measure-valued solution to equation (1.3). Suppose that the communication weight function \( \psi \) satisfies \( 0 < \psi_m \leq \psi(s) \leq \psi_M \) for \( s \in \mathbb{R}_+ \). Then we have

\[
E[\mu_0] e^{-2(\psi_M - 2\sigma)t} \leq E[\mu_t] \leq E[\mu_0] e^{-2(\psi_m - 2\sigma)t} \quad t \geq 0.
\]

This subsequently implies

\[
\lim_{t \to \infty} E[\mu_t] = \begin{cases} 
0 & \text{if } \psi_m > 2\sigma, \\
\infty & \text{if } \psi_m < 2\sigma.
\end{cases}
\]

**Remark 3.** We can obtain the convergence of the variance functional \( E_t \) without taking the expectation. More precisely, we find the following almost surely convergence when \( \psi(s) \geq \psi_m > 0 \):

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |\bar{v}_t - v|^2 \mu_t(dx, dv) \to 0 \quad \text{as} \quad t \to \infty, \quad a.s.,
\]

at least exponentially fast. Find the details of the proof in Proposition 2.

**Remark 4.** It follows from Theorem 1.3 that

\[
W_1(\mu_t(x, v), \rho_t(x) \delta_{\bar{v}_0}(v)) \to 0 \quad \text{as} \quad t \to \infty \quad \text{in probability},
\]

where \( W_1 \) denotes the Wasserstein distance of order 1 and \( \rho_t \) is the random spatial probability, i.e., \( \rho_t(dx) := \int_{\mathbb{R}^d} \mu_t(dx) \). Indeed, for any bounded Lipschitz function \( \phi \), we find

\[
\begin{align*}
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x, v) \mu_t(dx, dv) - \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x, v) \rho_t(dx) \delta_{\bar{v}_0}(dv) \right| \\
= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} (\phi(x, v) - \phi(x, \bar{v}_0)) \mu_t(dx, dv) \right|
\end{align*}
\]
\[ \leq \| \phi \|_{L^p} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{v}_0| \mu_t(dx, dv) \]
\[ \leq \| \phi \|_{L^p} E_t^{1/2} \to 0, \]
as \( t \to \infty \) in probability, due to Theorem 1.3. This, together with the fact that \( W_1 \) is equivalent to the bounded Lipschitz distance, concludes the desired result.

The rest of the paper is organized as follows. In Section 2, we show that the empirical measures associated to the particle system (1.1) are solutions to the equation (1.3) in the sense of Definition 1.1. We also present a stochastic Gronwall type inequality which will be used later for the almost surely bound estimate of compact support of solutions in velocity. Using that support bound estimate in velocity together with the weak stability estimate, we provide details on the proof of Theorem 1.2 in Section 3. Finally, we show the phase change phenomenon of the stochastic partial differential equation (1.3), which proves Theorem 1.3, in Section 4.

2. Preliminaries.

2.1. Itô’s formula. In this part, we show that the empirical measures associated to the stochastic particle system (1.2) are weak solutions to the stochastic partial differential equation by employing the Itô’s formula. For the sake of mathematical simplicity, we work in the corresponding Itô form (1.2). We want to emphasize that this observation implies the limiting system cannot be deterministic since the empirical measures are stochastic for any \( N \).

For \( \phi \in C_b^2(\mathbb{R}^d \times \mathbb{R}^d) \), if we apply Itô’s formula to the system (1.2), then we obtain
\[
\phi(X_t^i, V_t^i) = \phi(X_0^i, V_0^i) + \int_0^t \nabla_x \phi(X_s^i, V_s^i) \cdot V_s^i \, ds + \int_0^t \nabla_v \phi(X_s^i, V_s^i) \cdot F[\mu^N_s](X_s^i, V_s^i) \, ds \\
- \sigma \int_0^t \nabla_v \phi(X_s^i, V_s^i) \cdot (\bar{V}_s - V_s^i) \, ds + \sqrt{2\sigma} \int_0^t \nabla_v \phi(X_s^i, V_s^i) \cdot (\bar{V}_s - V_s^i) \, dB_s \\
+ \sigma \int_0^t (\bar{V}_s - V_s^i) \odot (\bar{V}_s - V_s^i) : \nabla^2_v \phi(X_s^i, V_s^i) \, ds.
\]
Averaging the above equation over \( i = 1, \cdots, N \) deduces
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x, v) \mu^N_t(dx, dv) \\
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x, v) \mu^N_0(dx, dv) + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x \phi(x, v) \cdot v \, \mu^N_s(dx, dv) \, ds \\
+ \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot F[\mu^N_s](x, v) \, \mu^N_s(dx, dv) \, ds \\
- \sigma \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot (\bar{V}_s - v) \, \mu^N_s(dx, dv) \, ds \\
+ \sqrt{2\sigma} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot (\bar{V}_s - v) \, \mu^N_s(dx, dv) \, dB_s \\
+ \sigma \int_0^t ((\bar{V}_s - v) \odot (\bar{V}_s - v)) : \nabla^2_v \phi(x, v) \mu^N_s(dx, dv) \, ds.
\]
Note that \( \bar{V}_t = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mu^N_t(dx, dv) \). We also easily check that
\[
\int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x, v) \nabla_v \cdot ( (\bar{v}_s - v) \nabla_v \cdot ( (\bar{v}_s - v) \mu_s(dx, dv) )) \, ds
\]
\[
= - \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot ( (\bar{v}_s - v) \nabla_v \cdot ( (\bar{v}_s - v) \mu_s(dx, dv) ) ) \, ds
\]
\[
= \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot ( (\bar{v}_s - v) \mu_s(dx, dv) ) \, ds
\]
\[
- \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot \nabla_v \cdot ( (\bar{v}_s - v) \otimes (\bar{v}_s - v) ) \mu_s(dx, dv) \, ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla^2_v \phi(x, v) : ( (\bar{v}_s - v) \otimes (\bar{v}_s - v) ) \mu_s(dx, dv) \, ds,
\]
where we used
\[
\nabla_v \cdot ( (\bar{v}_s - v) \otimes (\bar{v}_s - v) ) \mu_s ) = -(\bar{v}_s - v) \mu_s + (\bar{v}_s - v) \nabla_v \cdot ( (\bar{v}_s - v) \mu_s).
\]
This yields that \( \mu^N_t \) associated to (1.2) satisfies the weak formulation in Definition 1.1. Furthermore, \( (\mu^N_t, \phi) \) with \( \phi \in C_b^2(\mathbb{R}^d \times \mathbb{R}^d) \) is \( \mathcal{F}_t \)-adapted since the processes \( (X^i_t, V^i_t)_{i=1, \ldots, N} \) are solutions to the system (1.1). Thus they are adapted and continuous and this gives that \( \mu^N_t \) is a weak solution to the system (1.4) in the sense of Definition 1.1.

2.2. Stochastic Gronwall type inequality. In this subsection, we provide a stochastic Gronwall type inequality which will be crucially used in this paper.

**Lemma 2.1.** Let \( (X_t)_{t \geq 0}, (\beta_t)_{t \geq 0}, (A_t)_{t \geq 0} \) be some real valued processes satisfying
\[
X_t = X_0 + \int_0^t \beta_s \, ds - c_2 \int_0^t X_s \, dB_s \quad \text{for} \quad t \geq 0,
\]
\[
\beta_s \leq c_1 X_s + A_s \quad \text{for} \quad s \geq 0,
\]
then it holds
\[
X_t \leq X_0 e^{\left( c_1 - c_2^2/2 \right) t} e^{-c_2 B_t} \left( \int_0^t e^{-\left( c_1 - c_2^2/2 \right) s} e^{c_2 B_s} A_s \, ds + 1 \right),
\]
with probability one.

**Proof.** Let us denote
\[
\tilde{Y}_t := X_0 e^{\left( c_1 - c_2^2/2 \right) t} e^{-c_2 B_t}.
\]
Using Itô’s rule, we find that \( \tilde{Y}_t \) solves
\[
d\tilde{Y}_t = \left( c_1 - \frac{c_2^2}{2} \right) X_0 e^{\left( c_1 - c_2^2/2 \right) t} e^{-c_2 B_t} dt - c_2 X_0 e^{\left( c_1 - c_2^2/2 \right) t} e^{-c_2 B_t} dB_t
\]
\[
+ \frac{c_2^2}{2} X_0 e^{\left( c_1 - c_2^2/2 \right) t} e^{-c_2 B_t} \, dt
\]
\[
= (c_1 dt - c_2 dB_t) \tilde{Y}_t.
\]
We next consider
\[
Y_t := X_0 \int_0^t e^{\left( c_1 - c_2^2/2 \right) t - c_2 B_t} A_s \, ds =: X_0 \int_0^t \frac{Z_t}{Z_s} A_s \, ds.
\]
On the other hand, since $Z_t$ satisfies
\[
dZ_t = \left( c_1 - \frac{c_2^2}{2} \right) e^{(c_1-c_2^2/2)t} e^{-c_2 B_t} dt + e^{(c_1-c_2^2/2)t} \left( -c_2 e^{-c_2 B_t} dB_t + \frac{c_2^2}{2} e^{-c_2 B_t} dt \right)
\]
\[
= (c_1 dt - c_2 dB_t)Z_t.
\]
Itô’s rule yields
\[
dY_t = X_0 \left( \int_0^t \frac{1}{Z_s} \, dA_s \right) dZ_t + X_0 Z_t \frac{1}{Z_t} A_t \, dt.
\]
This yields that $H_t = Y_t + \tilde{Y}_t$ solves
\[
dH_t = c_1 H_t dt - c_2 H_t dB_t + X_0 A_t dt,
\]
with the initial condition $H_0 = X_0$. This together with [14, Theorem 1.1, p 437] concludes
\[
X_t \leq H_t \quad \text{for} \quad t \geq 0,
\]
almost surely. This completes the proof. \qed

3. Well-posedness of stochastic PDE: Proof of Theorem 1.2. In this section we establish the global well-posedness of the equation (1.3). Note that we already observed that the empirical measure associated to the particle system (1.1) is a weak solution to (1.3) in the previous section.

Let us consider $(\mu_t)_{t \geq 0}$ solution to the stochastic partial differential equation (1.3) in the sense of Definition 1.1 with the initial data $\mu_0 \in P_2(\mathbb{R}^d \times \mathbb{R}^d)$. Throughout this section, we may assume that the particle velocity average is zero, i.e., $\tilde{v}_t = 0$ for $t \geq 0$, almost surely. Indeed, since the velocity average is conserved in time, if we consider a measure $\tilde{\mu}_t(dx, dv)$ with new variables $\tilde{x} := x - \tilde{v}_0 t$ and $\tilde{v} := v - \tilde{v}_0$, then $\tilde{\mu}_t$ satisfies the equation (1.3) with $\tilde{v}_t \equiv 0$. Moreover, we find
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{\mu}_t(dx, dv) = \int_{\mathbb{R}^d \times \mathbb{R}^d} (v - \tilde{v}_0) \mu_t(dx, dv) = 0 \quad \text{for} \quad t \geq 0, \quad \text{almost surely.}
\]

In order to give the proof of Theorem 1.1, we first need to write $\mu_t$ as the push-forward of $\mu_0$ by some suitable stochastic transport application. Define $(Z_t)_{t \geq 0} = (\mathcal{X}_t, \mathcal{V}_t)_{t \geq 0}$, a family of stochastic transport maps from $\mathbb{R}^{2d}$ into $\mathbb{R}^{2d}$ such that
\[
\mu_t = Z_t \# \mu_0.
\]
Due to Definition 1.1 (i), one can choose this family almost surely continuous in time. Then Definition 1.1 (ii) rewrites from any $\phi \in C^2(\mathbb{R}^{2d})$
\[
\langle \mu_0, \phi(Z_t) \rangle = \langle \mu_0, \phi \rangle + \int_0^t \langle \mu_0, \mathcal{V}_s \cdot \nabla_x \phi(Z_s) + F[\mu_s](Z_s) \cdot \nabla_v \phi(Z_s) \rangle \, ds
\]
\[
- \sqrt{2\sigma} \int_0^t \langle \mu_0, \mathcal{V}_s \cdot \nabla_v \phi(Z_s) \rangle \circ dB_s,
\]
from which we infer that $(Z_t(x, v))_{t \geq 0} = (\mathcal{X}_t(x, v), \mathcal{V}_t(x, v))_{t \geq 0}$ solve
\[
\mathcal{X}_t(x, v) = x + \int_0^t \mathcal{V}_s(x, v) \, ds, \quad t \geq 0,
\]
\[
\mathcal{V}_t(x, v) = v + \int_0^t (F[\mu_s](Z_s(x, v)) + \sigma \mathcal{V}_s(x, v)) \, ds - \sqrt{2\sigma} \int_0^t \mathcal{V}_s(x, v) \, dB_s,
\]
(3.7)
for \( \mu_0 \)-almost all \((x,v) \in \mathbb{R}^d \). Since \((\mu_t)_{t \geq 0} \) satisfies Definition 1.1 (i), \( F[\mu_t](x,v) : \Omega \times [0,T] \to \mathbb{R}^d \) is measurable for every \((x,v) \in \mathbb{R}^{2d} \), w.r.t. the algebra of progressively measurable subsets of \( \Omega \times [0,T] \) associated to \((\mathcal{F}_t)_{t \geq 0} \) (see [15, Definition 1.25]). Then using [15, Proposition 3.28], the well-posedness for equation (3.7) holds, and \((Z_t)_{t \geq 0} \) is uniquely determined.

In the lemma below, we provide a priori kinetic energy and velocity support estimates.

**Lemma 3.1.** Let \((\mu_t)_{t \geq 0}\) be a solution of (1.3), and \((Z_t)_{t \geq 0}\) be the associated stochastic characteristic. Then, it holds

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 \mu_t(dx, dv) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathcal{V}_t(x,v)|^2 \mu_0(dx, dv)
\]

\[
\leq \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 \mu_0(dx, dv) \right) e^{-2\sqrt{\sigma_B}t},
\]

with probability one. Moreover, still with probability one, it holds

\[
|\mathcal{V}_t(x,v)|^2 \leq |v|^2 e^{\psi_M t - 2\sqrt{\sigma_B} t} \left( \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |w|^2 \mu_0(dy, dw) \right) \frac{1}{\psi_M} + 1 \right),
\]

for \((x,v) \in \mathbb{R}^d \times \mathbb{R}^d\).

**Proof.** It follows from (3.7) that

\[
|\mathcal{V}_t(x,v)|^2 = |v|^2 + 2 \int_0^t \langle \mathcal{V}_s(x,v), F[\mu_s](Z_s(x,v)) \rangle \, ds + 4\sigma \int_0^t |\mathcal{V}_s(x,v)|^2 \, ds
\]

\[
- 2\sqrt{2\sigma} \int_0^t |\mathcal{V}_s(x,v)|^2 \, dB_s.
\]

Integrating (3.8) with respect to \( \mu_0(dx, dv) \), we find

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathcal{V}_t(x,v)|^2 \, \mu_0(dx, dv)
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 \mu_0(dx, dv)
\]

\[
- 2 \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(\mathcal{X}_t(x,v) - \mathcal{X}_t(y,w)) |\mathcal{V}_t(x,v) - \mathcal{V}_t(y,w)|^2 \times \mu_0(dy, dw) \mu_0(dx, dv) \, ds
\]

\[
+ 4\sigma \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathcal{V}_s(x,v)|^2 \mu_0(dx, dv) \, ds
\]

\[
- 2\sqrt{2\sigma} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathcal{V}_s(x,v)|^2 \mu_0(dx, dv) \, dB_s
\]

\[
\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 \mu_0(dx, dv) + 4\sigma \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathcal{V}_s(x,v)|^2 \mu_0(dx, dv) \, ds
\]

\[
- 2\sqrt{2\sigma} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathcal{V}_s(x,v)|^2 \mu_0(dx, dv) \, dB_s.
\]

This together with Lemma 2.1 gives

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathcal{V}_t(x,v)|^2 \mu_0(dx, dv) \leq \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 \mu_0(dx, dv) \right) e^{-2\sqrt{\sigma_B}t}.
\]
Coming back to (3.8), we obtain
\[
|\psi_t(x,v)|^2 = |v|^2 + 2 \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(\mathcal{X}_t(x,v) - \mathcal{X}_t(y,w)) \times (\mathcal{V}_s(x,v), \mathcal{V}_s(y,w) - \mathcal{V}_s(x,v)) \mu_0(dy, dw) ds \\
+ 4\sigma \int_0^t |\mathcal{V}_s(x,v)|^2 ds - 2\sqrt{2\sigma} \int_0^t |\mathcal{V}_s(x,v)|^2 dB_s \\
\leq |v|^2 + 2 \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi_M |(\mathcal{V}_s(x,v), \mathcal{V}_s(y,w))| \mu_0(dy, dw) ds \\
+ 4\sigma \int_0^t |\mathcal{V}_s(x,v)|^2 ds - 2\sqrt{2\sigma} \int_0^t |\mathcal{V}_s(x,v)|^2 dB_s \\
\leq |v|^2 + (4\sigma + \psi_M) \int_0^t |\mathcal{V}_s(x,v)|^2 ds - 2\sqrt{2\sigma} \int_0^t |\mathcal{V}_s(x,v)|^2 dB_s \\
+ \int_0^t \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |w|^2 \mu_0(dy, dw) \right) e^{-2\sqrt{2\sigma} B_s} ds.
\]
Hence, Lemma 2.1, we have
\[
|\psi_t(x,v)|^2 \leq |v|^2 e^{\psi_M t - 2\sqrt{2\sigma} B_t} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |w|^2 \mu_0(dy, dw) \right) \frac{1 - e^{-\psi_M t}}{\psi_M} + 1,
\]
and this concludes the desired result. \(\square\)

**Remark 5.** If the initial data \(\mu_0\) is compactly supported in velocity, i.e., \(\text{supp}_v(\mu_0) := \{v \in \mathbb{R}^d : \mu_0(x,v) \neq 0\} \subseteq B(0,R)\) for some \(R > 0\), then we have
\[
|\psi_t(x,v)|^2 \leq R^2 e^{\psi_M t - 2\sqrt{2\sigma} B_t} \left(\frac{R^2}{\psi_M} \frac{1 - e^{-\psi_M t}}{\psi_M} + 1\right) \text{ for } (x,v) \in \mathbb{R}^d \times \mathbb{R}^d.
\]
This yields that the support of \(\mu_t\) in velocity, \(\text{supp}_v(\mu_t) := \{v \in \mathbb{R}^d : \mu_t(x,v) \neq 0\}\), is almost surely bounded for \(t \in [0,T]\).

We now provide the stability estimate of solutions in 2-Wasserstein distance in the proposition below.

**Proposition 1.** Let \(\mu_t, \tilde{\mu}_t\) be solutions to the equation (1.3) with compactly supported initial data \(\mu_0, \tilde{\mu}_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)\) in velocity, respectively. Then there exists an almost surely finite random variable \(C_T\) depends only on \(\psi, T, \sigma, \sup_{t \in [0,T]} |B_t|\), and the support in velocity of \(\mu_0\) and \(\tilde{\mu}_0\), such that
\[
W_2(\mu_t, \tilde{\mu}_t) \leq C_T W_2(\mu_0, \tilde{\mu}_0) e^{2T + C_T W_2(f_0, g_0)},
\]
for \(t \in [0,T]\), \(\mathbb{P}\)-almost surely.

**Proof.** We first choose an optimal transport map \(\mathcal{T}^0(x,v) = (\mathcal{T}^0_1(x,v), \mathcal{T}^0_2(x,v))\) between the initial datum \(\mu_0\) and \(\tilde{\mu}_0\) with respect to 2-Wasserstein distance \(W_2\), i.e., \(\tilde{\mu}_0 = \mathcal{T}^0 \# \mu_0\) and
\[
W_2(\mu_0, \tilde{\mu}_0) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |(x,v) - \mathcal{T}^0(x,v)|^2 \mu_0(dx, dv).
\]
We now consider stochastic characteristics $\mathcal{Z}_t = (\mathcal{X}_t, \mathcal{V}_t)$ and $\tilde{\mathcal{Z}}_t = (\tilde{\mathcal{X}}_t, \tilde{\mathcal{V}}_t)$ defined as in (3.7) associated to the solutions $\mu_t$ and $\tilde{\mu}_t$, respectively. Then defining $T^t := \tilde{\mathcal{Z}}_t \circ T^0 \circ \mathcal{Z}_{-t}$ for $t \in [0, T]$, we find $T^t \# \mu_t = \tilde{\mu}_t$ and

$$W_2^2(\mu_t, \tilde{\mu}_t) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |(x, v) - T^1(x, v)|^2 \mu_t(dx, dv)$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \mathcal{Z}_t(x, v) - \tilde{\mathcal{Z}}_t(T^0(x, v)) \right|^2 \mu_0(dx, dv),$$

due to $T^t \circ \mathcal{Z}_t = \tilde{\mathcal{Z}}_t \circ T^0$. Applying Itô’s lemma yields

$$\left| \mathcal{V}_t(x, v) - \tilde{\mathcal{V}}_t(T^0(x, v)) \right|^2$$

$$= |v - T^0_2(x, v)|^2$$

$$+ 2 \int_0^t \left< \mathcal{V}_s(x, v) - \tilde{\mathcal{V}}_s(T^0(x, v)), F[\mu_s](\mathcal{Z}_s(x, v)) - F[	ilde{\mu}_s](\tilde{\mathcal{Z}}_s(T^0(x, v))) \right> ds$$

$$+ 4\sigma \int_0^t \left| \mathcal{V}_s(x, v) - \tilde{\mathcal{V}}_s(T^0(x, v)) \right|^2 ds$$

$$- 2\sqrt{2\sigma} \int_0^t \left| \mathcal{V}_s(x, v) - \tilde{\mathcal{V}}_s(T^0(x, v)) \right|^2 dB_s.$$

(3.9)

Let us denote

$$P_t := \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \mathcal{X}_t(x, v) - \tilde{\mathcal{X}}_t(T^0(x, v)) \right|^2 \mu_0(dx, dv)$$

and

$$Q_t := \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \mathcal{V}_t(x, v) - \tilde{\mathcal{V}}_t(T^0(x, v)) \right|^2 \mu_0(dx, dv).$$

Then it follows from (3.9) that

$$Q_t = Q_0 + 4\sigma \int_0^t Q_s ds - 2\sqrt{2\sigma} \int_0^t Q_s dB_s + 2 \int_0^t I_s ds,$$

where

$$I_s := \int_{\mathbb{R}^d \times \mathbb{R}^d} \left< \mathcal{V}_s(x, v) - \tilde{\mathcal{V}}_s(T^0(x, v)), F[\mu_s](\mathcal{Z}_s(x, v)) - F[\tilde{\mu}_s](\tilde{\mathcal{Z}}_s(T^0(x, v))) \right> \mu_0(dx, dv).$$

Note that

$$F[\mu_s](\mathcal{Z}_s(x, v)) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|\mathcal{X}_s(x, v) - y|)(w - \mathcal{V}_s(x, v)) \mu_s(dy, dw)$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|\mathcal{X}_s(x, v) - \mathcal{X}_s(y, w)|)(\mathcal{V}_s(y, w) - \mathcal{V}_s(x, v)) \mu_0(dy, dw),$$

due to $\mu_s = \mathcal{Z}_s \# \mu_0$. Similarly, we also get

$$F[\tilde{\mu}_s](\tilde{\mathcal{Z}}_s(T^0(x, v)))$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|\tilde{\mathcal{X}}_s(T^0(x, v)) - \mathcal{X}_s(T^0(y, w))|)$$

$$\times (\tilde{\mathcal{V}}_s(T^0(y, w)) - \mathcal{V}_s(T^0(x, v))) \mu_0(dy, dw).$$
This gives

\[ F[\mu_s](Z_s(x,v)) - F[\hat{\mu}_s](\tilde{Z}_s(T^0(x,v))) \]

\[ = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \psi(\mathcal{X}_s(x,v) - \mathcal{X}_s(y,w)) - \psi(\hat{\mathcal{X}}_s(T^0(x,v)) - \hat{\mathcal{X}}_s(T^0(y,w))) \right) \]

\[ \times (\mathcal{V}_s(y,w) - \mathcal{V}_s(x,v)) \mu_0(dy,dw) \]

\[ + \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(\hat{\mathcal{X}}_s(T^0(x,v)) - \hat{\mathcal{X}}_s(T^0(y,w))) \]

\[ \times (\mathcal{V}_s(y,w) - \mathcal{V}_s(x,v) - \hat{\mathcal{V}}_s(T^0(y,w)) + \hat{\mathcal{V}}_s(T^0(x,v))) \mu_0(dy,dw) \]

\[ =: I_s^1 + I_s^2. \]

Using these newly defined terms, we split \( I_s \) into two terms:

\[ I_s = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \mathcal{V}_s(x,v) - \hat{\mathcal{V}}_s(T^0(x,v)), J_s^1 + J_s^2 \right) \mu_0(dx,dv) =: I_s^1 + I_s^2, \]

and here \( I_s^1 \) can be estimated as follows.

\[ I_s^1 \leq \| \psi \|_{Lip} H_s \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \mathcal{V}_s(x,v) - \hat{\mathcal{V}}_s(T^0(x,v)) \right| \left| \mathcal{X}_s(x,v) - \hat{\mathcal{X}}_s(T^0(x,v)) \right| \mu_0(dx,dv) \]

\[ + \| \psi \|_{Lip} H_s \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \mathcal{V}_s(x,v) - \hat{\mathcal{V}}_s(T^0(x,v)) \right| \left| \mathcal{X}_s(y,w) - \hat{\mathcal{X}}_s(T^0(y,w)) \right| \]

\[ \times \mu_0(dx,dv) \mu_0(dy,dw) \]

\[ \leq 2\| \psi \|_{Lip} H_s P_s^{1/2} Q_s^{1/2}, \]

where we used the fact

\[ J_s^1 \leq \| \psi \|_{Lip} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |\mathcal{X}_s(x,v) - \hat{\mathcal{X}}_s(T^0(x,v))| + |\mathcal{X}_s(y,w) - \hat{\mathcal{X}}_s(T^0(y,w))| \right) \]

\[ \times |\mathcal{V}_s(y,w) - \mathcal{V}_s(x,v)| \mu_0(dy,dw) \]

\[ \leq \| \psi \|_{Lip} H_s \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |\mathcal{X}_s(x,v) - \hat{\mathcal{X}}_s(T^0(x,v))| + |\mathcal{X}_s(y,w) - \hat{\mathcal{X}}_s(T^0(y,w))| \right) \]

\[ \times \mu_0(dy,dw). \]

Here \( H_s \) is given by

\[ H_s := 2R^2 e^{\psi_M t - 2\sqrt{2\sigma}} \left( \frac{R^2}{\psi_M} + 1 \right), \]

and we used Remark 5, and \( R > 0 \) is chosen such that \( \text{supp}(\mu_0) \subseteq B(0,R) \) since \( \mu_0 \) is compactly supported in velocity. We next estimate \( I_s^2 \) as

\[ I_s^2 \leq \psi_M \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \mathcal{V}_s(x,v) - \hat{\mathcal{V}}_s(T^0(x,v)) \right| \left| \mathcal{V}_s(y,w) - \hat{\mathcal{V}}_s(T^0(y,w)) \right| \]

\[ \times \mu_0(dx,dv) \mu_0(dy,dw) \]

\[ \leq \psi_M Q_s. \]

Combining all the above estimates, we obtain

\[ Q_t \leq Q_0 + (4\sigma + 2\psi_M) \int_0^t Q_s \, ds - 2\sqrt{2\sigma} \int_0^t Q_s \, ds + 2\| \psi \|_{Lip} H_T \int_0^t (P_s + Q_s) \, ds, \]
for $0 \leq t \leq T$ almost surely, where $\tilde{H}_T := \sup_{0 \leq t \leq T} H_t$. We now apply Lemma 2.1 with $c_1 = 4\sigma + 2\psi_M + 2\|\psi\|_{\text{Lip}}\tilde{H}_T$, $c_2 = 2\sqrt{2\sigma}$, and $A_\sigma = 2\|\psi\|_{\text{Lip}}\tilde{H}_T P_\sigma$ to get

$$Q_t \leq Q_0 e^{c_1 t} e^{-2\sqrt{2\sigma} + 2\|\psi\|_{\text{Lip}}\tilde{H}_T P_\sigma} \left( \int_0^t e^{-c_1 s} e^{2\sqrt{2\sigma} B_s} 2\|\psi\|_{\text{Lip}} \tilde{H}_T P_\sigma \, ds + 1 \right),$$

where $C_T > 0$ depends only on $\psi, \sigma$, $\sup_{t \in [0,T]} |B_t|$, and $R$. On the other hand, we easily find

$$P_t \leq P_0 + 2 \int_0^t P_\sigma \, ds + 2 \int_0^t Q_\sigma \, ds.$$

This together with (3.10) yields

$$P_t + Q_t \leq (1 + C_T)(P_0 + Q_0) + (2 + C_T Q_0) \int_0^t (P_\sigma + Q_\sigma) \, ds,$$

and applying Gronwall’s inequality gives

$$P_t + Q_t \leq (1 + C_T)(P_0 + Q_0)e^{2T + C_T Q_0 T} \quad \text{for} \quad 0 \leq t \leq T \quad \text{almost surely}.$$

This completes the proof. \(\square\)

We now define the stochastic empirical measure $\mu^N_t$ associated to the solution to the stochastic particle system (1.1) as

$$\mu^N_t = \frac{1}{N} \sum_{i=1}^N \delta_{(X^i_t, Y^i_t)}$$

with the initial data satisfying

$$W_2(\mu^N_0, \mu_0) \to 0 \quad \text{as} \quad N \to \infty,$$

where $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ is compactly supported in velocity. On the other hand, $(\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d), W_2)$ is a complete metric space, thus the sequence $\mu^N_0$ is Cauchy. As discussed before, $\mu^N_t$ is the solution to the equation (1.3) starting from $\mu^N_0$ and this together with the stability estimate in Proposition 1 yields

$$\sup_{t \in [0,T]} W_2(\mu^N_t, \mu^N_t) \leq C_T W_2(\mu^N_0, \mu^N_0) e^{2T + C_T W_2(\mu^N_0, \mu^N_0)}.$$

This implies that the sequence $(\mu^N)_{N \in \mathbb{N}}$ converges to some $\mu_\ast \in \mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d))$ almost surely. Then it remains to prove that $(\mu^N_{\ast})_{t \in [0,T]}$ solves our main equation (1.3). Since the $\mu^N_0$ is compactly supported in velocity uniformly in $N$, it follows from Remark 5 that the solution $\mu^N_\ast$ is also compactly supported in velocity uniformly in $N$.

For $t \in [0,T]$ and $\phi \in C^2_0(\mathbb{R}^d \times \mathbb{R}^d)$, we define

$$I_{\phi, \ast}(\mu) := \langle \mu_\ast, \phi \rangle - \langle \mu_0, \phi \rangle - \int_0^t \langle \mu_\ast, v \cdot \nabla_x \phi \rangle + (F[\mu_\ast] + \sigma v) \cdot \nabla_v \phi \rangle \, ds$$

$$+ \sqrt{2\sigma} \int_0^t \langle \mu_\ast, v \cdot \nabla_v \phi \rangle \, dB_v - \sigma \int_0^t \langle \mu_\ast, v \otimes v : \nabla_v \phi \rangle \, ds.$$

(3.11)
We notice that $I_{\phi,t}(\mu^N) = 0$ for any $N \in \mathbb{N}$ almost surely. Thus we find
\[
I_{\phi,t}(\mu) = I_{\phi,t}(\mu) - I_{\phi,t}(\mu^N)
\]
\[
= (\mu_s - \mu^N_s, \phi) - (\mu_s - \mu^N_s, \phi) - \int_0^t (\mu_s - \mu^N_s, v \cdot \nabla_x \phi + \sigma v \cdot \nabla_v \phi) \, ds
\]
\[
- \sigma \int_0^t (\mu_s - \mu^N_s, v \cdot \nabla_v \phi) \, ds
\]
\[
- \int_0^t \langle \mu_s, F[\mu_s] \cdot \nabla_v \phi \rangle - \langle \mu^N_s, F[\mu^N_s] \cdot \nabla_v \phi \rangle \, ds
\]
\[
=: \sum_{i=1}^6 I^N_i.
\]
We then claim that $\sum_{i=1}^6 I^N_i \to 0$ as $N \to \infty$ in probability to conclude that $\mu_t$ is the solution to equation (1.3) in the sense of Definition 1.1.

\( \diamond \) Estimate of $\sum_{i=1}^4 I^N_i$: Note that the terms $I^N_i, 1 \leq i \leq 4$ are linear. Since $\phi \in C^2_b(\mathbb{R}^d)$, and $\mu^N_t$ and $\mu_t$ are compactly supported in velocity uniformly in $N$, we easily obtain that
\[
\left| \sum_{i=1}^4 I^N_i \right| \leq C \sup_{0 \leq t \leq T} W_2(\mu^N_t, \mu_t) \to 0 \quad as \quad N \to \infty, \quad almost \ surely.
\]

\( \diamond \) Estimate of $I^N_5$: Note that $I^N_5$ is a linear stochastic integral term. Let us denote by
\[
H^N_s := \int_{\mathbb{R}^d \times \mathbb{R}^d} v \cdot \nabla_v \phi(x,v) \left( \mu^N_s(dx, dv) - \mu_s(dx, dv) \right).
\]
Since $(x,v) \mapsto v \cdot \nabla_v \phi(x,v)$ is locally bounded and Lipschitz, and $\mu^N_s$ converges to $\mu_s$ almost surely (uniformly in time), $H^N_s$ converges to $0$ almost surely (uniformly in time) as $N$ goes to infinity. Note also that $\|H^N_s\| \leq C \|\nabla_v \phi\|_{L^\infty}$ almost surely. Thus, by the stochastic dominated convergence theorem [17, Theorem 2.12, Chapter IV], we have
\[
\left| \int_0^t H^N_s dB_s \right| \to 0 \quad as \quad N \to \infty \quad in \ probability.
\]

(\h) Estimate of $I^N_6$: We now estimate the nonlinear term. We rewrite
\[
I^N_6 = \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x,v) \cdot (F[\mu^N_s](x,v) - F[\mu_s](x,v)) \mu^N_s(dx, dv) \, ds
\]
\[
\leq \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x,v) \cdot (F[\mu^N_s](x,v) - F[\mu_s](x,v)) \mu^N_s(dx, dv) \, ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x,v) \cdot F[\mu_s](x,v) \left( \mu^N_s(dx, dv) - \mu_s(dx, dv) \right) \, ds,
\]
where the second term on the right hand side goes to $0$ as $N$ goes to infinity almost surely since $\nabla_v \phi \cdot F[\mu_s]$ is locally bounded and Lipschitz, and $\mu^N_s$ converges to $\mu_t$ as $N$ goes to infinity uniformly in time. Indeed, we get
\[
\left| \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x,v) \cdot F[\mu_s](x,v) \left( \mu^N_s(dx, dv) - \mu_s(dx, dv) \right) \, ds \right|
\]
\[
\leq C \sup_{0 \leq t \leq T} W_2(\mu^N_t, \mu_t) \to 0,
\]
as \( N \to \infty \), almost surely. For the estimate of the first term, we rewrite it that as
\[
\left| \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot (F[\mu_s^N](x, v) - F[\mu_s](x, v)) \mu_s^N(dx, dv) \, ds \right|
\]
\[
= \left| \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot (|x - y|(w - v) - \psi(|x - y|)(w - v)) \mu_s^N(dx, dv) \right|
\]
\[
\times \left( \mu_s(dy, dw) - \mu_s(dy, dw) \right) ds.
\]
In order to treat this last term we use the following lemma.

**Lemma 3.2.** Define the functional \( G(\mu_t^N) \) on \( \mathbb{R}^d \times \mathbb{R}^d \) as
\[
G(\mu_t^N) : (y, w) \mapsto \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot \psi(|x - y|)(w - v) \mu_t^N(dx, dv) \right).
\]
Then \( \mathbb{P} \)-almost surely, for any \( t \in [0, T] \), the vector field \( G(\mu_t^N) \) is bounded and Lipschitz locally in velocity.

**Proof.** For any \( \mu \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d) \) we first easily find that
\[
|G(\mu)|_{L^\infty} \leq C|\nabla_v \phi|_{L^\infty} (1 + |w|)|\psi_M|,
\]
for some \( C > 0 \). Next for \( (y, w), (y', w') \in \mathbb{R}^d \times \mathbb{R}^d \), we obtain
\[
|G(\mu)(y, w) - G(\mu)(y', w')| = \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot (\psi(|x - y|)(w - v) - \psi(|x - y'|)(w' - v)) \mu(dx, dv)
\]
\[
\leq \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|x - y|) (\nabla_v \phi(x, v) \cdot (w - w')) \mu(dx, dv) \right|
\]
\[
+ \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} (\psi(|x - y|) - \psi(|x - y'|)) (\nabla_v \phi(x, v) \cdot (w' - v)) \mu(dx, dv) \right|
\]
\[
=: I + J.
\]
where \( I \) is easily estimated by
\[
I \leq \psi_M \|\nabla_v \phi\|_{L^\infty} |w - w'|.
\]
For the estimate of \( J \), we also easily get
\[
J \leq \|\psi\|_{Lip} \|\nabla_v \phi\|_{L^\infty} \left( |w'| + \left( \int |v|^2 \mu(dx, dv) \right)^{1/2} \right) |y - y'|.
\]
Hence it holds
\[
|G(\mu_t^N)(y, w) - G(\mu_t^N)(y', w')| \leq C_{\psi, \phi} \left( 1 + |w'| + \left( \int |v|^2 \mu_t^N(dx, dv) \right)^{1/2} \right) |(y, w) - (y', w')|.
\]
This together with Lemma 3.1 completes the proof. \( \square \)

Using Lemma 3.2, since \( \mu_t^N \) and \( \mu_s \) are compactly supported in velocity and \( \mu_t^N \) converges to \( \mu_t \) almost surely uniformly in time, we have
\[
\left| \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \phi(x, v) \cdot (F[\mu_s^N](x, v)\mu_s^N(dx, dv) - F[\mu_s](x, v)\mu_s(dx, dv)) \, ds \right| \to 0,
\]
as \( N \to \infty \), almost surely.
Putting all those estimates together deduces that $I_{\phi, \tau}(\mu)$ defined in (3.11) is equal to sequences which go to 0 as $N \to \infty$ in probability, thus that is equal to 0 almost surely. This concludes that $(\mu_t)_{t \in [0,T]}$ satisfies the weak formulation in Definition 1.1 (ii). We now show that $\mu_t$ is weakly continuous, i.e., it satisfies Definition 1.1 (i). Note that we can always assume that the process $(\langle \mu_t, \phi \rangle)_{t \in [0,T]}$ is adapted by changing the notion of the filtration $(\mathcal{F}_t)_{t \in [0,T]}$. Indeed, that can be obtained as the limit of the sequence of the adapted processes $(\langle \mu^N_t, \phi \rangle)_{t \in [0,T]}$ in law. Thus it only remains to establish the existence of a continuous version of the process $(\langle \mu_t, \phi \rangle)_{t \in [0,T]}$ to complete the proof. Recall that $\mu_t$ obtained above satisfies

$$\langle \mu_t, \phi \rangle = \langle \mu_0, \phi \rangle + \int_0^t \langle \mu_s, v \cdot \nabla x \phi + (F[\mu_s] + \sigma v) \cdot \nabla_v \phi \rangle \, ds$$

$$+ \sigma \int_0^t \langle \mu_s, v \otimes v : \nabla^2_v \phi \rangle \, ds - \sqrt{2\sigma} \int_0^t \langle \mu_s, v \cdot \nabla_v \phi \rangle \, dB_s$$

$$=: \langle \mu_0, \phi \rangle + \int_0^t \mathcal{L}_s \, ds + \sqrt{2\sigma} \int_0^t \mathcal{I}_s \, dB_s.$$

In the rest of this section, we estimate the Lebesgue and Itô integrals as follows.

- **Estimate of the Lebesgue integral**: Since $\phi \in C^2_c(\mathbb{R}^d \times \mathbb{R}^d)$ is compactly supported in both position and velocity, we get

$$\langle \nu, v \cdot \nabla x \phi + \sigma v \cdot \nabla_v \phi + v \otimes v : \nabla^2_v \phi \rangle \leq C_{\sigma, \phi},$$

for any $\nu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, where $C_{\sigma, \phi}$ is a constant which depends only on $\sigma, \phi$. This gives

$$|\mathcal{L}_s| \leq C_{\sigma, \phi} + |\langle \mu_s, F[\mu_s] \cdot \nabla_v \phi \rangle|.$$

On the other hand, it follows from Remark 5 that $\mu_s$ is compactly supported in velocity. Thus we obtain that the alignment force field is almost surely bounded from above by

$$|F[\mu_s]| \leq 2\psi_M R^2 e^{\psi_M s - 2\sqrt{2\sigma B_s}} \left( R^2 \frac{1 - e^{-\psi_M s}}{\psi_M} + 1 \right).$$

Hence we have that $(\int_0^t \mathcal{L}_s \, ds)_{t \in [0,T]}$ is almost surely Lipschitz in time.

- **Estimate of the Itô integral**: Applying Burkholder-Davis-Gundy’s inequality yields

$$\mathbb{E} \left[ \sup_{s \leq u < t} \left| \int_s^u \mathcal{I}_s \, dB_s \right|^{2p} \right] \leq C_{p} \mathbb{E} \left[ \left| \int_0^t \mathcal{I}_s^2 \, ds \right|^{p} \right] = C_{p} \mathbb{E} \left[ \left| \int_s^t \langle \mu_s, v \cdot \nabla_v \phi \rangle^2 \, ds \right|^{p} \right]$$

$$\leq C_{p, \phi} \mathbb{E} \left[ \left| \int_s^t \langle \mu_s, |v|^2 \rangle \, ds \right|^{p} \right]$$

$$\leq C_{p, \phi} \mathbb{E} \left[ \left| \int_s^t \langle \mu_0, |v|^2 \rangle e^{-2\sqrt{2\sigma B_t}} \, ds \right|^{p} \right]$$

$$\leq C_{p, \phi, \mu_0} \left| t - s \right|^{p} \mathbb{E} \left[ e^{2p \sqrt{2\sigma} \sup_{t \in [0,T]} |B_t|} \right],$$

for any $p > 1$ and $s, t \in [0, T]$, where we used Jensen’s inequality with $\mu_s \in \mathcal{P}(\mathbb{R}^{2d})$ in the second line, the large-time behavior estimate in Lemma 3.1 in the third line, and the fact that $\mu_0$ is compactly supported in velocity in the fourth line. On the
other hand, classical properties of one dimensional Brownian motion provide
\[ \mathbb{E} \left[ 2^{p \sqrt{2T}} \sup_{r \in [0,T]} |B_r| \right] \leq 2 \mathbb{E} \left[ 2^{p \sqrt{2T}} |B_T| \right] \leq C_{\sigma,T,p}, \]
putting all those estimates together we find that there is a constant \( C_{p,\mu_0,\sigma,T} \) such that
\[ \mathbb{E} \left[ \sup_{s < u < t} \left| \int_s^u \mathcal{I}_s dB_s \right|^{2p} \right] \leq C_{p,\mu_0,\sigma,T} |t - s|^p. \]
Finally, we use Kolmogorov’s continuity theorem to conclude that there exists a continuous version of the process \((\int_0^t \mathcal{I}_s dB_s)_{t \in [0,T]}\).

4. Phase change phenomenon: Proof of Theorem 1.3. In this section, we provide details of the proof of Theorem 1.3 on the flocking and non flocking estimates for the stochastic kinetic equation (1.3). As mentioned in Introduction, we employ a similar strategy used for the particle system (1.1) proposed in [1, 19].

Recall the variance functions of stochastic particle velocity fluctuation around \( \bar{v}_0 \):
\[ E_t = \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \bar{v}_0 - v \rangle^2 \mu_t(dx, dv). \]
Note that
\[ \bar{v}_t = \bar{v}_0 + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} F[\mu_s](x, v) \mu_s(dx, dv) ds + \sqrt{2\sigma} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} (\bar{v}_s - v) \mu_s(dx, dv) \circ dB_s = \bar{v}_0, \]
where we used
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} F[\mu_s](x, v) \mu_s(dx, dv) = \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \psi(|x - y|)(w - v) \mu_s(dx, dv)\mu_s(dy, dw) = 0. \]
This, together with a straightforward computation, yields
\[
\begin{align*}
\frac{1}{2} dE_t &= \int_{\mathbb{R}^d \times \mathbb{R}^d} (w - \bar{v}_t) \cdot F[\mu_t](x, v) \mu_t(dx, dv) dt \\
&\quad + 2\sigma \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{v}_t|^2 \mu_t(dx, dv) dt \\
&\quad - \sqrt{2\sigma} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{v}_t|^2 dB_t \\
&\quad = -\frac{1}{2} \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \psi(|x - y|)|w - v|^2 \mu_t(dx, dv)\mu_t(dy, dw) dt \\
&\quad + 2\sigma E_t dt - \sqrt{2\sigma} E_t dB_t. 
\end{align*}
\]
Taking the expectation to the above gives
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \mathbb{E}[E_t] + \frac{1}{2} \mathbb{E} \left[ \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \psi(|x - y|)|w - v|^2 \mu_t(dx, dv)\mu_t(dy, dw) \right] \\
&= 2\sigma \mathbb{E}[E_t]. 
\end{align*}
\]
Note that
\[ \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} |w - v|^2 \mu_t(dx, dv)\mu_t(dy, dw) = 2E_t. \]
Thus we get
\[ \psi_mE[E_t] \leq \frac{1}{2} \mathbb{E} \left[ \int_{\mathbb{R}^2 \times \mathbb{R}^d} \psi(|x - y|)|w - v|^2 \mu_t(dx, dv) \mu_t(dy, dw) \right] \leq \psi_M E[E_t], \]
and this together with (4.13) provides
\[ -2(\psi_M - 2\sigma)E[E_t] \leq \frac{d}{dt} E[E_t] \leq -2(\psi_m - 2\sigma)E[E_t]. \]
Applying Gronwall’s inequality, we have
\[ E[E_t]e^{-2(\psi_M - 2\sigma)t} \leq E[E_t] \leq E[E_0]e^{-2(\psi_m - 2\sigma)t} \quad t \geq 0. \]
This completes the proof of Theorem 1.3.

As mentioned in Remark 3, we can also obtain the convergence of the variance functional $E_t$ without taking the expectaiton even though it does not provide the phase change phenomenon. Let us go back to equation (4.12). Using the similar strategy as the above, we estimate the drift term in (4.12) as
\[ -2(\psi_m - 2\sigma)E_t dt - 2\sqrt{2}\sigma E_t dB_t \leq dE_t \leq -2(\psi_m - 2\sigma)E_t dt - 2\sqrt{2}\sigma E_t dB_t, \quad \text{a.s.} \]
Applying Itô’s formula gives
\[ d\log E_t = \frac{dE_t}{E_t} - \frac{|dE_t|^2}{2E_t^2} = \frac{dE_t}{E_t} - 4\sigma dt \leq -2\psi_m dt - 2\sqrt{2}\sigma dB_t, \quad \text{a.s.} \]
Taking the time integration to the above inequality gives
\[ \log E_t \leq \log E_0 - 2\psi_m t - 2\sqrt{2}\sigma \int_0^t dB_s = \log E_0 - 2\psi_m t - 2\sqrt{2}\sigma B_t, \]
i.e.,
\[ E_t \leq E_0 \exp \left( -2\psi_m t - 2\sqrt{2}\sigma B_t \right). \] (4.14)
On the other hand, by using the fact from the law of the iterated logarithm that for any $\epsilon > 0$, almost surely, there exists $t_0 > 0$ such that $|B_t| \leq \epsilon t$ for all $t \geq t_0$, we can further estimate
\[ E_t \leq E_0 \exp \left( -2(\psi_m - \epsilon_0)t \right), \quad \text{a.s.,} \]
for some $0 < \epsilon_0 < \psi_m$ and $t \geq t_0 > 0$.

Summarizing the above discussion, we have the following result.

**Proposition 2.** Let $\mu_t$ be a measure-valued solution for the equation (1.3). Suppose that there exists a positive constant $\psi_m$ such that $0 < \psi_m \leq \psi(s)$ for $s \in \mathbb{R}_+$. Then we have the following time asymptotic flocking estimate.
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} |\bar{v}_0 - v|^2 \mu_t(dx, dv) \to 0 \quad \text{as} \quad t \to \infty, \quad \text{a.s.,} \]
at least exponentially fast.

**Remark 6.** The stochastic process $E_t$ in (4.12) resembles a geometric Brownian motion. Note that the inequality (4.14) can be rewritten as
\[ E_t \leq E_0 e^{-2\sqrt{2}\sigma B_t^{(\psi_m)/\sqrt{2}\sigma}}, \]
where $B_t^{\nu}$ denotes the one dimensional Brownian motion with constant drift $\nu \in \mathbb{R}$, i.e., $B_t^{\nu} = B_t + \nu t$. 
Remark 7. When there is no multiplicative noise $\sigma = 0$, then the system (1.1) becomes the original Cucker-Smale model. For the Cucker-Smale model, the unconditional flocking estimate can be obtained if the communication weight $\psi$ is not integrable, see [7]. If not, suitable assumptions for the initial configurations are needed for the flocking estimate. In [1], the multiplicative noise is considered for the stochastic particle system (1.1) in the Itô sense, and in that case, we can get the flocking behavior for any nonnegative communication weight $\psi$, i.e., taking into account the multiplicative noises in the Cucker-Smale model enable us to have the unconditional flocking estimate. This implies that the multiplicative noise in the Itô sense plays a role as a (stochastic) control which enhances the flocking behavior of particles.

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