Explanation of the Stern-Gerlach splitting of spinor condensates based on symmetry

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The Stern-Gerlach splitting of spinor condensates is explained based on the total spin-states with specified SU(2) and permutation symmetries.

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The experimental realization of the spinor Bose-Einstein condensation in optical traps [1,2,3] is a great step in probing the microscopic world. In the experiment by Stenger, et al.,[2] after the atoms had been trapped sufficiently long by the optical trap, the trap was suddenly switched off and the atoms are allow to expand, and a magnetic field gradient was applied to yield a Stern-Gerlach splitting. Then the expanding cloud was observed via the absorption imaging where the atoms are condensing into the same spatial state, the spatially normalized total spin-states are good quantum numbers for all the eigenstates. Thus, to understand the orientation of the hyperfine state can be evaluated. We found that there is a strong symmetry background governing the splitting, this is reported as follows.

After a sufficient long time of evolution, the system of condensed atoms would arrive at a state of equilibrium, and would be distributed among the low-lying eigen-states, the probability \( P(E_i) \) of staying at a particular eigen-state \( \Psi_i \) is determined by thermodynamics, namely, \( P(E_i) \propto e^{-E_i/kT} \), where \( T \) is the temperature. Let \( M \) be the \( Z \)-component of the total spin. During the evolution \( M \) remains unchanged. Furthermore, when all \( N \) atoms condense into the same spatial state, the spatial wave function must be completely symmetric with respect to particle interchanges, accordingly the total spin-state must also be completely symmetric. Furthermore, due to the property of the imaging, the total spin \( S \) together with \( M \) are good quantum numbers for all the eigenstates. Thus, to understand the orientation of the spins, it is crucial to understand the completely symmetric normalized total spin-states \( \vartheta_{S,M}^{[N]} \), where \( S \) is ranged from \( N, N-2, \ldots \) to \( M \) (or \( M+1 \)) if \( N-M \) is even (or odd) [4,3].

When \( N \) is small, \( \vartheta_{S,M}^{[N]} \) is simple, e.g., for \( N=3 \),

\[
\vartheta_{1,M}^{[3]} = \frac{\sqrt{5}}{3} [(\chi(1)\chi(2))_{0}\chi(3)]_{1,M} + \frac{2}{3}[(\chi(1)\chi(2))_{2}\chi(3)]_{1,M}
\]

(1)

where \( \chi(i) \) is the spin-state of the \( i \)-th particle, particles 1 and 2 are first coupled to spin zero and two, respectively, then all three particles are coupled to \( S=1 \). However, when \( N \) is larger, \( \vartheta_{S,M}^{[N]} \) becomes very complicated. Fortunately, the expression of \( \vartheta_{S,M}^{[N]} \) itself is not really necessary. Making use of the fractional parentage coefficients \( a_S^{[N]} \) and \( b_S^{[N]} \) derived in our previous papers [6], we can extract anyone of the particles (say, particle 1) from \( \vartheta_{S,M}^{[N]} \) as

\[
\vartheta_{S,M}^{[N]} = a_S^{[N]} \chi(1)\vartheta_{S+1,M}^{[N-1]} + b_S^{[N]} \chi(1)\vartheta_{S-1,M-1}^{[N-1]}
\]

(2)

\[
= a_S^{[N]} \sum_{\mu} C_{1\mu,S+1,M-\mu}^{S,M} \chi(1)\vartheta_{S+1,M-\mu}^{[N-1]} + b_S^{[N]} \sum_{\mu} C_{1\mu,S-1,M-\mu}^{S,M} \chi(1)\vartheta_{S-1,M-\mu}^{[N-1]}
\]

where

\[
a_S^{[N]} = \frac{1}{(2N(2S+1))^1/2} (1 + (-1)^{N-S}) (N-S)(S+1)
\]

(3)

\[
b_S^{[N]} = \frac{1}{(2N(2S+1))^1/2} (1 + (-1)^{N-S}) S(S+1)
\]

(4)

and \( C_{1\mu,S+1,M-\mu}^{S,M} \) are the Clebesh-Gorden coefficients. It is clear from (3) and (4) that \( N-S \) must be even.

For the state \( \vartheta_{S,M}^{[N]} \) from (2), the probability of a spin at \( \mu \) is

\[
P_{\mu} = a_S^{[N]} C_{1\mu,S+1,M-\mu}^{S,M} + b_S^{[N]} C_{1\mu,S-1,M-\mu}^{S,M}
\]

(5)

Not only the fractional parentage coefficients, the related Clebesh-Gorden coefficients in [5] have also analytical

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Thus $P_{S,M}^\mu$ has an analytical form as

$$P_{1}^{S,M} = \frac{1}{2(2S+1)} \left( \frac{1}{2S+3} + \frac{1}{2S-1} \right)$$

$$P_{0}^{S,M} = \frac{1}{(2S+1)} \left( \frac{1}{2S+3} + \frac{1}{2S-1} \right)$$

and in general

$$P_{S,-M}^{\mu} = P_{-\mu}^{S,M}$$

It is recalled that $\vartheta_i^{N,S,M}$ are completely symmetric, therefore the spin of each particle has exactly the same probability $P_{S,M}^\mu$. From (12-14) we have

$$P_{1}^{S,M} + P_{0}^{S,M} + P_{-1}^{S,M} \equiv 1$$

$$N (P_{1}^{S,M} - P_{-1}^{S,M}) \equiv M$$

and

$$\sum_{M=-S}^{S} p_{S,M}^{\mu} = \frac{1}{3}$$

Eq. (16) is a basic requirement because $\mu$ has only three choices. (17) implies that $N$ $P_{S,M}^{\mu}$ is the number of bosons at $\mu$. Eq. (18) implies that, for a nonpolarized system, the probability of a particle staying at a given $\mu$ is $1/3$. In particular, when $M = 0$, we have

$$P_{1}^{S,0} = P_{0}^{S,0} = \frac{(1-1/2N)S(S+1)-1}{(2S+3)(2S-1)}$$

$$P_{-1}^{S,0} = \frac{(1+1/2N)S(S+1)-1}{(2S+3)(2S-1)}$$

Since $N$ is usually large, we can neglect the term $1/N$. Then we found both $P_{1}^{S,0}$ and $P_{-1}^{S,0}$ are close to $1/4$ and

$$P_{0}^{S,0}$$

is close to $1/2$ (unless $S$ is very small). This is a crucial point to explain the splitting experiment in Ref. [2].

When $M = N$ (in this case $S = N$ is the only choice) we have $P_{1}^{N,N} = 1$, $P_{0}^{N,N} = P_{-1}^{N,N} = 0$ as expected.

The $P_{S,M}^{\mu}$ with $N = 10000$, $M = N/4$, $N/2$, and $3N/4$ are plotted in Fig. 1; the curves are not sensitive to $N$, as it appears in (12-14). For excited states, the total spin-states are not necessary to be completely symmetric (e.g., for the first excited band, both the spatial states and total spin-states have the $\{N - 1, 1\}$ symmetry. However, if the temperature is sufficiently low, only low-lying states are concerned, where only a very small part of particles are excited. It implies that most particles are condensed, and the spin-states of these majority must be completely symmetric, while the effect of the excited particles on the average spin-orientation is very small.

This probability $P_{S,M}^{\mu}$ holds, in good approximation, for all low-lying states.

When the final state is a thermodynamical distribution over the eigen-states with the same $M$, the probability of a particle at $\mu$ is

$$P(M, \mu) = \Theta \sum_i e^{-E_i/T} p_{S_i, M}^{\mu}$$

where $i$ is the label of the levels, $E_i$ and $S_i$ are the corresponding energy and total spin, $\Theta$ is a constant just for the normalization.

From our previous study [5, 6, 8], the low-lying states of spinor condensates are divided into bands, the states in a band have similar spatial wave functions but they are different in $S$. The energy splitting inside the band is caused by the spin-dependent atom-atom interaction. However, for realistic case, the spin-dependence is weak. It was found from our calculation that the splitting of energy levels in a band is very small. Hence, for the levels of a band the factor $e^{-E_i/kT}$ can be roughly considered.
µ condensate starts with the account, the normalized probability appears as \( P_{\text{equilibrium}} \). It is found that \( \Theta_j \) as a constant, \([9]\) thus the contribution of the curve.

\[
(\gamma) = 0 \quad (1) \quad \text{if } N - M \text{ is even (odd).} \quad P(M, \mu) \text{ is plotted in Fig.2, which is the probability of a boson at } \mu \text{ if the condensate starts with } M \text{ and finally arrives at thermal-equilibrium. It is found that } P(M, \mu) \text{ depends only on } M/N \text{ as shown in Fig.2.}
\]

Obviously, due to \( [12,13,14] \), \( P(-M, \mu) = P(M, -\mu) \). There are the following features.

(i) The Stern-Gerlach splitting of spinor condensates is described by \( P(M, \mu) \), which is system-independent, i.e., it does not depend on the species and the details of interactions, but is essentially determined by symmetry.

(ii) Let \( N_\mu \) be the number of particles at \( \mu \) in the initial state. \( P(M, \mu) \) also does not depend on the details of \( N_{mn} \) but only on \( N_1 - N_{-1} = M \). This coincides with the experiment by Stenger, et al (Fig.2 of Ref. [2]).

(iii) \( P(M, \mu) \) depends on \( M/N \) nearly linearly.

(iv) When \( M = 0 \), \( P(M, \mu) = 1/4 \), and \( 1/4 \) when \( \mu = 1, 0, \) and \( -1 \). This is also supported by the above experiment.

(v) When \( M = N \), \( P(M, \mu) = 1/0,0 \) and 0 if \( \mu = 1, 0, \) and \( -1 \) as expected.

(vi) When \( M = N/2 \), \( P(M, 1) = 5/8 \), \( P(M, 0) = 2/8 \), and \( P(M, -1) = 1/8 \). This is supported by the experiment as shown in Fig.5 of Ref. [10].

In fact, disregarding any set of initial \( N_1, N_0, \) and \( N_{-1} \), the Stern-Gerlach splitting can be predicted based on (22) or Fig.2.

In summary, it is recalled that, in explaining the Stern-Gerlach splitting, the details of dynamics is not involved. Instead, the assumption of arriving at thermal-equilibrium together with a strict symmetry consideration play the role. Nonetheless, in the above derivation, the total spin-state is assumed to be completely symmetric. This is exactly true for the ground band and therefore the above theory is rigorously valid at the low-temperature limit. When the temperature is nonzero but is still low (say, \( T \approx 100 \nu k \)) so that the number of excited particles is still small, the above theory remains qualitatively valid.

Symmetry is well known to be crucial for various few-body systems. This paper gives an example that many-body systems are also governed by symmetry.

\[
P(M, \mu) = \frac{2}{N - M + 2 - \gamma} \sum_{S} P_{\mu}^{S, M}
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where \( \gamma = 0 \) (1) if \( N - M \) is even (odd). \( P(M, \mu) \) is plotted in Fig.2, which is the probability of a boson at \( \mu \) if the condensate starts with \( M \) and finally arrives at thermal-equilibrium. It is found that \( P(M, \mu) \) depends only on \( M/N \) as shown in Fig.2. Obviously, due to \([12,13,14]\), \( P(-M, \mu) = P(M, -\mu) \). There are the following features.

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