REMARK ON POLARIZED K3 SURFACES OF GENUS 36

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ABSTRACT. Smooth primitively polarized K3 surfaces of genus 36 are studied. It is proved that all such surfaces $S$, for which there exists an embedding $R \hookrightarrow \text{Pic}(S)$ of some special lattice $R$ of rank 2, are parameterized up to an isomorphism by some 18-dimensional unirational algebraic variety. More precisely, it is shown that a general $S$ is an anticanonical section of a (unique) Fano 3-fold with canonical Gorenstein singularities.

1. INTRODUCTION

Let $K_g$ be the moduli space of all smooth primitively polarized K3 surfaces of genus $g$. $K_g$ is known to be a quasi-projective algebraic variety (see for example [25]). This makes it possible to consider the fundamental questions of birational geometry about $K_g$ such as its rationality, unirationality, rational connectedness, Kodaira dimension, and etc.

S. Mukai’s vector bundle method, developed to classify higher dimensional Fano manifolds of Picard number 1 and coindex 3 (see [15], [18]), allowed to prove unirationality of $K_g$ for $g \in \{2, \ldots, 10, 12, 13, 18, 20\}$ (see [17], [20], [16], [21]). At the same time, $K_g$ turns out to be non-unirational for general $g \geq 43$ (see [1], [13], [14]). In principle, the proof of unirationality of $K_g$ is based on the observation that general K3 surface $S_g$ with primitive polarization $L_g$ and “not very big” $g$ is an anticanonical section of a smooth Fano 3-fold $X_g$ of genus $g$ so that $L_g = -K_{X_g}|_{S_g}$ (see [17], [16], [19]). The latter gives a rational dominant map from the moduli space $\mathcal{F}_g$ of pairs $(X_g, S_g)$, where $S_g \in |-K_{X_g}|$ is smooth, to $K_g$ by sending $(X_g, S_g)$ to $S_g$, with $\mathcal{F}_g$ typically being a rational algebraic variety. However, this construction has the restriction that $X_g$ must have Picard number 1, which does not hold for most $g$ (see [7]).

In order to generalize the above arguments for every possible $g$, to a given smooth Fano 3-fold $V$ of genus $g$ one associates the Picard lattice $R_V := \text{Pic}(V)$, equipped with the pairing $(D_1, D_2) := D_1 \cdot D_2 \cdot (-K_V)$ for $D_1, D_2 \in \text{Pic}(V)$, and considers the moduli space $K^R_V$ of all smooth K3 surfaces $S_g$, equipped with a primitive embedding $R_V \hookrightarrow \text{Pic}(S_g)$ which maps $-K_V$ to an ample class on $S_g$ of square $g$ (let us call such $S_g$ a K3 surface of type $R_V$). A beautiful result due to A. Beauville states that a general K3 surface of type $R_V$ is the anticanonical section of a smooth Fano 3-fold $X_g$ of genus $g$ such that $R_{X_g} \cong R_V$ (see [1]). The proof employs the same idea as above, but instead of $\mathcal{F}_g$ the moduli space $\mathcal{F}^R_g$ of pairs $(X_g, S_g)$, where $S_g \in |-K_{X_g}|$ is smooth and $X_g$ is equipped with the lattice isomorphism $R_{X_g} \cong R_V$, is considered. Again the forgetful map $(X_g, S_g) \mapsto S_g$ from $\mathcal{F}^R_g$ to $K^R_g$ turns out to be generically surjective. However, these arguments can be applied only to some $g \leq 33$ (see [7]).

In the present paper, we study primitively polarized smooth K3 surfaces of genus 36 and consider the following

Conjecture 1.1. The moduli space $K_{36}$ is unirational.

To develop an approach to prove Conjecture 1.1 we employ the above ideas to realize a general smooth primitively polarized K3 surface of genus 36 as an anticanonical section of some Fano 3-fold, which must be singular in this case (see [7]). The natural candidate for the latter is the Fano 3-fold $X$ with canonical Gorenstein singularities and genus 36, constructed and studied in [9], [8]. This $X$ has only one singular point (see Corollary 3.10) and the anticanonical linear system $|-K_X|$ gives an embedding $X \hookrightarrow \mathbb{P}^{37}$ (see Remark 3.12), which implies that a general surface $S \in |-K_X|$ is smooth. Also the Picard group of $X$ is generated by $K_X$ (see Corollary 3.11).

Unfortunately, the divisor class group of $X$ has two generators, $K_X$ and some surface $E$ (see Corollary 3.11), so that the restrictions $K_X|_S$ and $E|_S$ generate a primitive sublattice $R_S \subset \text{Pic}(S)$. In particular, the Picard number of $S$ must be at least 2, and hence $S$ cannot be general. However, all lattices $R_S$, $S \in |-K_X|$, are isomorphic to the lattice $R \cong \mathbb{Z}^2$ with the associated quadratic form $70x^2 + 4xy - 2y^2$ (see the end of Section 3), and, as above,
we can consider the moduli space $\mathcal{K}^{36}_{36}$ of K3 surfaces of type R. On the other hand, we may also consider the moduli space $\mathcal{F}$ of pairs $(X^{\sharp}, S^{\sharp})$, where $X^{\sharp}$ is a Fano 3-fold of genus 36 with canonical Gorenstein singularities and $S^{\sharp} \in |-K_{X^{\sharp}}|$ is smooth (see Remark 1.3 below for the precise description of $\mathcal{F}$). Let us state the main result of the present paper:

**Theorem 1.2.** The forgetful map $s : \mathcal{F} \rightarrow \mathcal{K}^{36}_{36}$ is generically surjective.

**Remark 1.3.** In the proof of Theorem 1.2 we do not appeal to Akizuki–Nakano Vanishing Theorem, used in [11] to show that $\mathcal{F}_{g}$ (or $\mathcal{F}^{g}_{RV}$) is a smooth stack, since it is not clear how to apply this theorem in the singular case. Instead, we note that $X$ is unique up to an isomorphism (see Proposition 3.7), and, moreover, it admits a crepant resolution $f : Y \rightarrow X$, with $Y$ being also unique up to an isomorphism (see Proposition 3.8). Then one can prove (see Proposition 3.11) that $\mathcal{F}$ carries the structure of a normal scheme, being the geometric quotient $\mathcal{U}/\text{Aut}(\mathcal{Y})$ of an open subset $\mathcal{U}$ in $\mathbb{P}^{37}$ by the group $\text{Aut}(\mathcal{Y})$ of regular automorphisms of $\mathcal{Y}$. The proof of Theorem 1.2 then goes along the same lines as in [1] (see Lemma 4.10 below).

**Remark 1.4.** Taking $X = \mathbb{P}(1,1,1,3)$ in the above considerations, one might apply the arguments from [11] directly (cf. Remark 1.3) to prove that the moduli space $\mathcal{K}^{10}_{10}$ is unirational (see [9], [8] for geometric properties of $\mathbb{P}(1,1,1,3)$).

Furthermore, since the forgetful map $\mathcal{K}^{36}_{36} \rightarrow \mathcal{K}_{36}$ is finite and representable (see [11, (2.5)]), from Theorem 1.2 the construction of $\mathcal{F}$ and quasi-projectivity of $\mathcal{K}_{36}$ we deduce the following

**Corollary 1.5.** There exists a 18-dimensional unirational algebraic variety which parameterizes up to an isomorphism all smooth K3 surfaces of type R. For general such surface $S$, $S \in |-K_X|$ and the Picard lattice of $S$ is isomorphic to $\mathbb{R}$.

**Remark 1.6.** On the opposite, it follows from the proof of Theorem 1.2 and [2], [3], [23] that no general smooth primitively polarized K3 surface $S$ of genus 36 can be an ample anticanonical section of a normal algebraic 3-fold, except for the cone over $S$.

**Remark 1.7.** Corollary 1.5 gives only unirational hypersurface in $\mathcal{K}_{36}$ but not the whole $\mathcal{K}_{36}$, and hence the proof of Conjecture 1.1 is still to go. It would be also interesting to know whether the map $s$ from Theorem 1.2 is 1-to-1 and $\mathcal{K}^{36}_{36}$ is rational (it follows from the proof of Theorem 1.2 that $s$ is generically étale).

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### 2. Notation and conventions

We use standard notions and facts from the theory of minimal models (see [12], [11]). We also use standard notions and facts from the theory of algebraic varieties and schemes (see [3]). All algebraic varieties are assumed to be defined over $\mathbb{C}$. Throughout the paper we use standard notions and notation from [12], [11], [5]. However, let us introduce some:

- We denote by $\text{Sing}(V)$ the singular locus of an algebraic variety $V$. For $P \in \text{Sing}(V)$, we denote by $(O \in V)$ the analytic germ of $V$ at $P$.
- For a $\mathbb{Q}$-Cartier divisor $M$ and an algebraic cycle $Z$ on a normal algebraic variety $V$, we denote by $M|_{Z}$ the restriction of $M$ to $Z$. We denote by $Z_1 \cdot \ldots \cdot Z_k$ the intersection of algebraic cycles $Z_1, \ldots, Z_k$, $k \in \mathbb{N}$, in the Chow ring of $V$.
- $M_1 \equiv M_2$ (respectively, $Z_1 \equiv Z_2$) stands for the numerical equivalence of two $\mathbb{Q}$-Cartier divisors $M_1$, $M_2$ (respectively, of two algebraic 1-cycles $Z_1$, $Z_2$) on a normal algebraic variety $V$. We denote by $\rho(V)$ the Picard number of $V$. $D_1 \sim D_2$ stands for the the linear equivalence of two Weil divisors $D_1$, $D_2$ on $V$. We denote by $N_1(V)$ the group of classes of algebraic cycles on $V$ modulo numerical equivalence. We denote by $\text{Cl}(V)$ (respectively, $\text{Pic}(V)$) the group of Weil (respectively, Cartier) divisors on $V$ modulo linear equivalence.
A normal algebraic three-dimensional variety is called **Fano threefold** if it has at worst canonical Gorenstein singularities and the anticanonical divisor $-K_V$ is ample. A normal algebraic three-dimensional variety is called **weak Fano threefold** if it has at worst canonical singularities and the anticanonical divisor $-K_V$ is nef and big. The number $(-K_V)^3$ (respectively, $\frac{2}{3}(-K_V)^3 + 1$) is called (anticanonical) **degree** (respectively, **genus**) of $V$.

- For a Weil divisor $D$ on a normal algebraic variety $V$, we denote by $O_V(D)$ the corresponding divisorial sheaf on $V$ (sometimes we denote both by $O_V(D)$ (or by $D$)).
- For a vector bundle $E$ on smooth projective variety $V$, we denote by $c_i(E)$ the $i$-th Chern class of $E$.
- We denote by $T_P(V)$ the Zariski tangent space to an algebraic variety $V$ at a point $P \in V$. For $V$ smooth and a smooth hypersurface $D \subset V$, we denote by $T_V(D)$ the subsheaf of the tangent sheaf on $V$ which consists of all vector fields tangent to $D$.
- For a Cartier divisor $M$ on a normal projective variety $V$, we denote by $|M|$ the corresponding complete linear system on $V$. For an algebraic cycle $Z$ on $V$, we denote by $|M - Z|$ the linear subsystem in $|M|$ which consists of all divisors passing through $Z$. For a linear system $\mathcal{M}$ on $V$ without base components, we denote by $\Phi_{\mathcal{M}}$ the corresponding rational map.
- For a birational map $\psi : V' \rightarrow V$ between normal projective varieties and an algebraic cycle $Z$ (respectively, a linear system $\mathcal{M}$) on $V$, we denote by $\psi_*^{-1}(Z)$ (respectively, by $\psi_*^{-1}(\mathcal{M})$) the proper transform of $Z$ (respectively, of $\mathcal{M}$) on $V'$.
- We denote by $\mathbb{F}_n$ the Hirzebruch surface with the class of a fiber $l$ and the minimal section $h$ of the natural projection $\mathbb{F}_n \rightarrow \mathbb{P}^1$ such that $(h^2) = -n, n \in \mathbb{Z}_{\geq 0}$.

### 3. Preliminaries

In what follows, $X$ is a Fano 3-fold of genus 36 (or degree 70). Let us present the construction and some properties of $X$ (see [9] for more details).

Consider the weighted projective space $\mathbb{P} := \mathbb{P}(1, 1, 4, 6)$ with weighted homogeneous coordinates $x_0, x_1, x_2, x_3$ of weights 1, 1, 4, 6, respectively. $\mathbb{P}$ is a Fano 3-fold of degree 72. Furthermore, the linear system $| -K_\mathbb{P}|$ gives an embedding of $\mathbb{P}$ in $\mathbb{P}^{38}$ such that the image $\Phi_{-K_\mathbb{P}}(\mathbb{P})$ is an intersection of quadrics. In what follows, we assume that $\mathbb{P} \subset \mathbb{P}^{38}$ is anticanonically embedded. Then $L := \text{Sing}(\mathbb{P})$ is a line on $\mathbb{P}$ with respect to this embedding. Moreover, there are two points $P$ and $Q$ on $L$ such that the singularities $P \in \mathbb{P}, Q \in \mathbb{P}$ are of types $\frac{1}{6}(4, 1, 1), \frac{1}{4}(2, 1, 1)$, respectively, and for every point $O \in L \setminus \{P, Q\}$ the singularity $O \in \mathbb{P}$ is analytically isomorphic to $(0, o) \in \mathbb{C} \times W$, where $o \in W$ is the singularity of type $\frac{1}{2}(1, 1)$ (see [9] Example 2.13).

**Proposition 3.1.** $L$ is the unique line on $\mathbb{P}$.

**Proof.** Let $L_0 \neq L$ be another line on $\mathbb{P}$. Since $-K_\mathbb{P} \sim O_\mathbb{P}(12)$, we have

\begin{equation}
O_\mathbb{P}(1) \cdot L_0 = \frac{1}{12}.
\end{equation}

which implies that $L \cap L_0 \neq \emptyset$. Consider the crepant resolution $\phi : T \rightarrow \mathbb{P}$ of $\mathbb{P}$. Set $L'_0 := \phi^{-1}(L_0), E_0 := \phi^{-1}(Q), E_P := \phi^{-1}(P)$ and $E_L := \phi^{-1}(L \setminus \{P, Q\})$, the Zariski closure in $T$ of $\phi^{-1}(L \setminus \{P, Q\})$. These are all the components of the $\phi$-exceptional locus. Furthermore, we have $E_P = E_P^{(1)} \cup E_P^{(2)}$, where $E_P^{(i)}$ are irreducible components of the divisor $E_P$ such that $E_P^{(1)} \cap E_L = \emptyset$ and $E_P^{(2)} \cap E_L \neq \emptyset$ (see [9] Example 2.13 for the explicit construction of $\phi$).

Since $\rho(\mathbb{P}) = 1$, the group $N_1(T)$ is generated by the classes of $\phi$-exceptional curves and some curve $Z$ on $T$ such that $R := \mathbb{R}_+[Z]$ is the $K_T$-negative extremal ray (see [24] Lemmas 4.2, 4.3). In particular, since $-K_T \cdot L'_0 = 1$, [24] Lemmas 4.2, 4.3] implies that

\begin{equation}
L'_0 = Z + E^*,
\end{equation}

where $E^*$ is a linear combination with nonnegative coefficients of irreducible $\phi$-exceptional curves. Further, the linear projection $\pi_L$ of $\mathbb{P}$ from $L$ is given by the linear system $\mathcal{H} \subset | -K_\mathbb{P}|$ of all hyperplane sections of $\mathbb{P}$ containing $L$. In addition, $\pi_L$ maps $L_0$ to the point because $L \cap L_0 \neq \emptyset$ and $\mathbb{P}$ is the intersection of quadrics. On the other hand, $\phi$ factors through the blow up of $\mathbb{P}$ at $L$ (see [9, 8]). Hence the linear system $\phi_*^{-1}\mathcal{H}$ is basepoint-free on $T$ and $H \cdot L'_0 = 0$ for $H \in \phi_*^{-1}\mathcal{H}$. In particular, $H \in \mathcal{H}$.
Lemma 3.4. In (3.3), the support $\text{Supp}(E^*)$ of $E^*$ is either $\emptyset$ or $e_P$, where $e_P \subset E_P^{(1)}$.

Proof. As we saw, the face of the Mori cone $\overline{NE}(T)$, which corresponds to the nef divisor $H$, contains the class of the curve $L_0'$. Then from (3.3) we get

$$H \cdot Z = H \cdot E^* = 0.$$ 

In particular, $H$ intersects trivially every curve in $\text{Supp}(E^*)$. On the other hand, we have $\text{Supp}(E^*) \subseteq \{e_P, e_Q, e_L\}$, where $e_P, e_Q, e_L$ are the curves in $E_P, E_Q, E_L$, respectively. But for $e_P \subset E_P^{(2)}$ intersections $H \cdot e_P, H \cdot e_Q, H \cdot e_L$ are all non-zero. Thus, $\text{Supp}(E^*)$ is either $\emptyset$ or $e_P$, where $e_P \subset E_P^{(1)}$.

Consider the extremal contraction $f_R : T \to T'$ of $R$. The morphism $f_R$ is birational with the exceptional divisor $E_R$ (see [9], [8]).

Lemma 3.5. The divisor $-K_{T'}$ is nef.

Proof. Suppose that $-K_{T'}$ is nef, i.e., $T'$ is a weak Fano 3-fold (with possibly non-Gorenstein singularities). If $T'$ has only terminal factorial singularities, then since $(-K_{T'})^3 \geq (-K_T)^3 = 72$ (see [24] Proposition-definition 4.5), $T'$ is a terminal $\mathbb{Q}$-factorial modification either of $\mathbb{P}(1, 1, 1, 3)$ or of $\mathbb{P}(1, 1, 4, 6)$. In particular, either $\rho(Y') = 5$ or $\rho(Y') = 2$ (see [9], [8]). On the other hand, $\rho(T') = \rho(T) - 1 = 4$, a contradiction.

Thus, the singularities of $T'$ are worse than factorial. In this case, $f_R(E_R)$ is a point (see [24] Proposition-definition 4.5)) and we get

$$(3.6) \quad E_P \cap E_R = E_Q \cap E_R = \emptyset.$$ 

On the other hand, it follows from (3.3) that $-K_T \cdot \phi_s(Z) = 1$, i.e., $\phi(Z)$ is a line on $\mathbb{P}$. In particular, as for $L_0$ above, we have $\phi_s(Z) \cap L \neq \emptyset$. But then (3.6) implies that $0 = K_T \cdot Z = -1$, a contradiction. \qed

It follows from Lemma 3.5 that $E_R = \mathbb{F}_1$ or $\mathbb{F}^1 \times \mathbb{F}^1$ (see [24] Proposition-definition 4.5)). But if $E_R = \mathbb{F}_1$, then $\phi(E_R)$ is a plane on $\mathbb{P}$ such that $L \not\subset \phi(E_R)$ (see [24] Proposition-definition 4.5)). This implies that there is a line on $\mathbb{P}$ not intersecting $L$, a contradiction (see [3.2]). Finally, in the case when $E_R = \mathbb{F}^1 \times \mathbb{F}^1$, we have $Z \subset E_R = E_L$ (see [24] Proposition-definition 4.5)), and if $\text{Supp}(E^*) = \emptyset$ in (3.3), then $L_0 = L$, a contradiction. Hence, by Lemma 3.4 we get $\text{Supp}(E^*) = e_P$, where $e_P \subset E_P^{(1)}$. Further, on $E_R$ we have:

$$Z \sim l, \quad E_P|_{E_R} = E_P^{(2)}|_{E_R} \sim h \sim E_Q|_{E_R},$$

which implies that $E_P^{(2)} \cdot Z = E_Q \cdot Z = 1$. On the other hand, since $L_0 \neq L$, we have either $E_P^{(2)} \cdot L_0 = 0$ or $E_Q \cdot L_0 = 0$. Then, intersecting (3.3) with $E_P^{(2)}$ and $E_Q$, we get a contradiction because $E_P^{(2)} \cdot e_P$ and $E_Q \cdot e_P \geq 0$.

Thus, we get $L_0 = L$, a contradiction. Proposition 3.1 is completely proved. \qed

Coming back to the construction of $X$, take any point $O$ in $L \setminus \{P, Q\}$ and consider the linear projection $\pi : \mathbb{P} \to \mathbb{P}^{37}$ from $O$. Then the image of $\pi$ is a Fano 3-fold $X_O$ of degree 70 (see [9], [8]).

Proposition 3.7. For any point $O'$ in $L \setminus \{P, Q, O\}$, the image of the linear projection $\mathbb{P} \to \mathbb{P}^{37}$ from $O'$ is a Fano 3-fold $X_{O'}$ isomorphic to $X_O$.

Proof. In the above notation, $L$ is given by equations $x_0 = x_1 = 0$ on $\mathbb{P}$, with equations of $P$ and $Q$ being $x_0 = x_1 = x_2 = 0$ and $x_0 = x_1 = x_3 = 0$, respectively (see [6] 5.15). Then the torus $(\mathbb{C}^*)^3$, acting on $\mathbb{P}$, acts transitively on the set $L \setminus \{P, Q\}$, which induces an isomorphism $X_{O'} \simeq X_O$. \qed

In what follows, because of Proposition 3.7 we fix the point $O \in L \setminus \{P, Q\}$, the linear projection $\pi : \mathbb{P} \to \mathbb{P}^{37}$ from $O$, and denote the image of $\pi$ by $X$. Let us construct a terminal $\mathbb{Q}$-factorial modification of $X$. Consider the blow up $\sigma : W \to \mathbb{P}$ of $\mathbb{P}$ at $O$, and the following commutative diagram:

$$\begin{array}{c}
\sigma \\
\downarrow \\
W \\
\downarrow \\
\mathbb{P} \\
\downarrow \pi \\
X.
\end{array}$$

The type of the singularity $O \in \mathbb{P}$ implies that $W$ has at most canonical Gorenstein singularities. Moreover, we have $\text{Sing}(W) = \sigma^{-1}_*(L)$ and the singularities of $W$ are exactly of the same kind as of $\mathbb{P}$, i.e., locally near every point
in \( \text{Sing}(W) \), \( W \) is isomorphic to \( \mathbb{P} \). Then, resolving the singularities of \( W \) in the same way as for \( \mathbb{P} \), we arrive at the birational morphism \( \tau : Y \to W \), with \( Y \) being smooth and \( K_Y = \tau^*(K_W) \) (see [9, 8]). Set \( f := \tau \circ \mu : Y \to X \).

**Proposition 3.8.** \( f : Y \to X \) is a terminal \( \mathbb{Q} \)-factorial modification of \( X \). Moreover, \( Y \) is unique up to isomorphism, i.e., every smooth weak Fano 3-fold of degree 70 is isomorphic to \( Y \).

**Proof.** The linear projection \( \pi \) is given by the linear system \( H \subset | - K_P | \) of all hyperplane sections of \( \mathbb{P} \) passing through \( O \). For a general \( H \in H \), we have
\[
\sigma^{-1}_\pi(H) = \sigma^*(H) - E_\sigma,
\]
where \( E_\sigma \) is the \( \sigma \)-exceptional divisor. On the other hand, from the adjunction formula we get
\[
K_W = \sigma^*(K_P) + E_\sigma.
\]
Thus, the morphism \( \mu : W \to X \) is given by the linear system \( \sigma^{-1}_\pi(H) \subseteq | - K_W | \). Furthermore, since \( \mathbb{P} \) is an intersection of quadrics, \( \pi \) is a birational map, which implies that \( \mu \) and \( f \) are also birational with \( K_Y = f^*(K_X) \).

In particular, \( ( - K_Y )^3 = ( - K_X )^3 = 70 \).

Thus, it remains to prove that every smooth weak Fano 3-fold of degree 70 is isomorphic to \( Y \). Let \( Y' \) be another smooth weak Fano 3-fold of degree 70. Then its image under the morphism \( f' := \Phi_{( - K_Y )} \), \( n \in \mathbb{N} \), is a Fano threefold \( X' \) such that \( K_{X'} = f'^*(K_X) \) (see [10]). Since \( ( - K_Y )^3 = ( - K_X )^3 = 70 \), we get \( X' \simeq X \) and \( Y' \) is a terminal \( \mathbb{Q} \)-factorial modification of \( X \). Then, since \( Y' \) and \( Y \) are relative minimal models over \( X \), the induced birational map \( Y \to Y' \) is either an isomorphism or a sequence of \( K_Y \)-flops over \( X \) (see [12]).

**Lemma 3.9.** Every \( K_Y \)-trivial extremal birational contraction \( f_1 : Y \to Y_1 \) is divisorial.

**Proof.** Suppose that \( f_1 \) is small. In the notation from the proof of Proposition 3.8 denote by \( E_{Y,L} \), \( E_1^{(i)} \), \( E_Q \), \( L \) the proper transforms on \( Y \) of \( E_L \), \( E_1 \), \( E_Q \), \( L \) respectively. The resolution \( \tau : Y \to W \) (or \( \phi : T \to \mathbb{P} \)) is locally toric near \( \text{Sing}(W) \). In particular, we have \( E_1^{(1)} \simeq F_1 \), \( E_1^{(2)} \simeq F_2 \), \( E_Q \simeq F_2 \), \( E_{Y,L} \simeq F_m \) for some \( m \in \mathbb{N} \) (see [9] Example 2.13), and hence the only possibility for \( f_1 \) is to contract the curve \( Z = h \) on \( E_{Y,L} \) such that \( \tau(Z) = \sigma^{-1}(L) \).

On the other hand, \( Y \) is obtained by the blow up of the 3-fold \( T \) at the curve \( \phi^{-1}(O) \simeq \mathbb{P}^1 \) (see [9, 8]). Furthermore, since \( \mathbb{P} \) is singular along the line, we have \( E_L \simeq \mathbb{P}^1 \times \mathbb{P}^1 \) (see [24] Proposition-definition 4.5), and hence \( E_{Y,L} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \), a contradiction.

It follows from Lemma 3.9 that \( Y' \simeq Y \). Proposition 3.8 is completely proved. \( \square \)

**Corollary 3.10.** \( \text{Sing}(X) \) consists of a unique point.

**Proof.** Since the morphism \( \mu : W \to X \) is given by the linear system \( \sigma^{-1}_\pi(H) \subseteq | - K_W | = | \sigma^*(-K_P) - E_\sigma | \) (see the proof of Proposition 3.8), it contracts only \( \sigma^{-1}_\pi(L) = \text{Sing}(W) \) to the unique singular point on \( X \) (see Proposition 3.11). \( \square \)

**Corollary 3.11.** We have \( \text{Pic}(X) = \mathbb{Z} \cdot K_X \) and \( \text{Cl}(X) = \mathbb{Z} \cdot K_X \oplus \mathbb{Z} \cdot E \), where \( E := \mu_*(E_\sigma) \).

**Proof.** This follows from the construction of \( X \) and equalities \( \rho(\mathbb{P}) = 1 \), \( ( - K_X )^3 = 70 \). \( \square \)

**Remark 3.12.** It follows from the construction of \( X \) that \( f = \Phi_{( - K_Y )} \) and \( X \subseteq \mathbb{P}^{37} \) is anticanonically embedded.

**Remark 3.13.** Since \( Y \) is a smooth weak Fano 3-fold, we have \( \text{Pic}(Y) \simeq H^2(Y, \mathbb{Z}) \) (see [7] Proposition 2.1.2) and \( H^2(Y, \mathcal{O}_Y) = 0 \) by Kawamata–Viehweg Vanishing Theorem.

It follows from Corollary 3.10 that a general surface \( S \in | - K_X | \) is smooth. Furthermore, Corollary 3.11 implies that the cycles \( K_X |_S \) and \( E |_S \) are not divisible in \( \text{Pic}(S) \), linearly independent in \( H^2(S, \mathbb{Q}) \), and hence they generate a primitive sublattice \( R_S \) in \( \text{Pic}(S) \). It follows from the construction of \( X \) that all lattices \( R_S \), \( S \in | - K_X | \), are isomorphic to the lattice \( \mathbb{R} \simeq \mathbb{Z}^2 \) with the associated quadratic form \( 70x^2 + 4xy - 2y^2 \), and we can consider the moduli stack \( \mathcal{K} := R_{\mathbb{R}^2} \) of \( K3 \) surfaces of type \( R \) (see [1] (2.3)]. \( \mathcal{K} \) is actually an algebraic space because the forgetful map \( \mathcal{K} \to \mathcal{K}_{36} \) is representable and 1-to-1 in our case (see [1] (2.5)]).

**Proposition 3.14** (see [1]). Let \( S \) be the \( K3 \) surface of type \( R \). Then

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1) It can be also easily seen that the class of a \((-2)\)-curve in \( \text{Pic}(S) \) is unique and generated by the conic \( E |_S \).
the first order deformations of $(S, R)$ are parameterized by the orthogonal of $c_1(R) \subset H^1(S,\Omega^1_S)$ in $H^1(S, T_S)$;
the space $K$ is smooth, irreducible, of dimension 18.

4. Proof of Theorem 1.2

We use the notation and conventions of Section 3. Since $f : Y \rightarrow X$ is the crepant resolution (see Proposition 3.8), it follows from Corollary 3.10 that we can assume a general $S \in \mathcal{G}_{X}$ to be a surface in $\mathcal{G}_{Y}$ on $Y$. We can also assume that $S \cap \text{Exc}(f) = \emptyset$ for the $f$-exceptional locus $\text{Exc}(f)$. Further, it follows from Remark 3.12 that the points in $(\mathbb{P}^3)^*$, corresponding to such $S$’s, form an open subset $U \subset (\mathbb{P}^3)^*$. Consider the natural (faithful) action of the group $G := \text{Aut}(Y)$ on $U$. Shrinking $U$ if necessary, we obtain the following

**Proposition 4.1.** The geometric quotient $U/G$ exists as a smooth scheme.

*Proof.* Let us calculate the group $G$ first. Take $g \in \text{Aut}(\mathbb{P})$ to be an automorphism of $\mathbb{P}$ which fixes the point $O$. Then $g$ lifts to the automorphism of $Y$ (see the construction of $X$ and $Y$ in Section 3). Conversely, take any $g \in G$.

**Lemma 4.2.** The morphism $\tau : Y \rightarrow W$ is $g$-equivariant.

*Proof.* Since the morphism $f = \Phi_{-K_Y} : Y \rightarrow X$ is $g$-equivariant (see Remark 3.12), it follows from the construction of $Y$ in Section 3 that the irreducible components of $\text{Exc}(f)$ are all $g$-invariant. Thus, since $\text{Pic}(Y)$ is generated by $K_Y$, the irreducible components of $E_f$ and $E_{Y,\sigma} := \tau_f^{-1}(E_{\sigma})$, it is enough to prove that $g(E_{Y,\sigma}) = E_{Y,\sigma}$. Suppose that $g(E_{Y,\sigma}) \neq E_{Y,\sigma}$. Then, since all the curves in $E_{\sigma}$ (respectively, in $\tau_f(g(E_{Y,\sigma}))$) are numerically proportional and $\tau$ is divisorial, we must have $E_{\sigma} \cap \tau_f(g(E_{Y,\sigma})) = \emptyset$. The latter implies that there exists a curve $C \equiv \sigma_x(-K_W \cdot \tau_f(g(E_{Y,\sigma})))$ on $W$ with $K_W \cdot C = 4$ and $C \cap L = \emptyset$. On the other hand, since $K_W \sim O(W)$, we get $O_W(1) \cdot C = 1$, a contradiction. \hfill \Box

It follows from Lemma 4.2 that $g$ acts on $W$. Further, considering the induced $g$-action on the cone $\overline{NE}(W)$, we obtain, since $\text{Pic}(W) = \mathbb{Z} \cdot K_W \oplus \mathbb{Z} \cdot E_{\sigma}$, that $\sigma : W \rightarrow \mathbb{P}$ is $g$-equivariant. The latter gives a $g$-action on $\mathbb{P}$ with the fixed point $O$.

Thus, $G$ is isomorphic to the stabilizer in $\text{Aut}(\mathbb{P})$ of the point $O$, and to describe the $G$-action on $U$ we may consider the action of the corresponding subgroup in $\text{Aut}(\mathbb{P})$ on the linear system $| - K_{\mathbb{P}} - O|$. Note that, since $P \in \mathbb{P}$, $Q \in \mathbb{P}$, $O \in \mathbb{P}$ are the pairwise non-isomorphic singularities, every $g \in G$ fixes every point on $L$. Finally, since $O_{\mathbb{P}}(1)$, $O_{\mathbb{P}}(4)$, $O_{\mathbb{P}}(6)$ are $G$-invariant, the $g$-action on $\mathbb{P}$ can be described as follows:

$$
(4.3) \quad x_0 \mapsto ax_0 + bx_1, \\
\quad x_1 \mapsto cx_0 + dx_1, \\
\quad x_2 \mapsto \lambda^4 x_2 + f_4(x_0, x_1), \\
\quad x_3 \mapsto \lambda^6 x_3 + x_2^2 x_2 f_2(x_0, x_1) + f_6(x_0, x_1),
$$

where $\lambda \in \mathbb{C}^*$, $(a \ b \ c \ d) \in GL(2, \mathbb{C})/\{\pm 1\}$, $f_i := f_i(x_0, x_1)$ are arbitrary homogeneous polynomials of degree $i$ in $x_0, x_1$. On the other hand, since $-K_{\mathbb{P}} \sim O(W)$, a general element in $| - K_{\mathbb{P}} - O|$ can be given by the equation

$$
(4.4) \quad \alpha x_3^2 + x_2^3 + a_6(x_0, x_1)x_3 + a_2(x_0, x_1)x_2x_3 + a_4(x_0, x_1)x_2^2 + a_6(x_0, x_1)x_2 + a_{12}(x_0, x_1) = 0
$$

on $\mathbb{P}$, where $a_i := a_i(x_0, x_1)$ are arbitrary general homogeneous polynomials in $x_0, x_1$ of degree $i$, and $\alpha \in \mathbb{C}^*$ is fixed.

Take a general surface $S_0$ on $\mathbb{P}$ with the equation (4.3) such that $a_2 = a_4 = a_6 = 0$.

**Lemma 4.5.** If $S_0$ is $g$-invariant for some $g \neq id$ from (4.3), then $f_2 = f_4 = f_6 = 0$, $c = b = 0$, $a = d = \sqrt{-1}$, $\lambda^4 = 1$.

*Proof.* $g$-invariance of $S_0$ implies that $f_2 = f_4 = f_6 = 0$ and

$$
(4.6) \quad a_8(x_0, x_1) = a_8(ax_0 + bx_1, cx_0 + dx_1), \quad a_{12}(x_0, x_1) = a_{12}(ax_0 + bx_1, cx_0 + dx_1).
$$

Without loss of generality we may assume that $a_8(x_0, x_1) b_6$ for some $b_6 := b_6(x_0, x_1)$ coprime to $x_0$ and $x_1$. Then (4.3) and generality of $S_0$ imply that $(a \ b \ c \ d) = (\sqrt{-1} \ 0 \ 0 \ d)$, and we get:

$$
\begin{align*}
a^{12} &= 1, & a^{i+1} a^{d-i} &= 1, & a^i d^{6-i} &= a^i d^{6-j}.
\end{align*}
$$

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for all $0 \leq i, j \leq 6$. In particular, $a = d, a^8 = a^{12} = 1$, i.e., $a = d = \sqrt{-1}$. Finally, since $x_2 \mapsto \lambda^4 x_2$ (see 4.3) and hence $a_8(x_0, x_1) = \lambda^4 a_8(x_0, x_1)$ (see 4.4), we get $\lambda^4 = 1$.

\textbf{Lemma 4.7.} Let $g \in G$, given by 4.3, be such that $f_2 = f_4 = f_6 = 0$, $c = b = 0$, $a = d = \sqrt{-1}$, $\lambda = \pm \sqrt{-1}$. Then $g = \text{id}$.

\textit{Proof.} Let $\pi$ be the geometric quotient $U/G$ exists as a smooth scheme. Proposition 4.1 is completely proved.

It follows from Lemmas 4.3 and 4.7 since $\lambda^4 = 1$ implies $\lambda^2 = \pm 1$, that the stabilizer of $S_0$ in $G$ is a group of order 2, generated by some $g_0 \in G$ with $\lambda^2 = 1$ (see 4.3). Consider the normal algebraic subgroup $G' \subset G$ generated by $g^{-1}g_0g$ for all $g \in G$, i.e., generators of $G'$ are all the elements in $G$ for which $f_4 = 0$, $c = b = 0$, $a = d = \sqrt{-1}$ and $\lambda = 1$ in 4.3. Then the $G'$-action on $U$ is proper, and we can consider the geometric quotient $U' := U/G'$, which exists as a normal scheme (see 22). Further, take the $G'' := G/G'$-equivariant factorization map $\pi_G : U \to U'$ and consider the induced $G''$-action on $U'$. Shrinking $U$ if necessary, we obtain

\textbf{Lemma 4.8.} The $G''$-action on $U'$ is free.

\textit{Proof.} Let $S_0'$ be the image on $U'$ of $S_0$ under $\pi_G$. Then we have $G'' \cdot S_0' \simeq G''$ for the $G''$-orbit of $S_0'$, and, by the dimension count, there exists a Zariski open subset in $U'$ with a free $G''$-action. Lemma 4.8 and 24 imply that the geometric quotient $U/G \simeq U'/G''$ exists as a smooth scheme. Proposition 1.1 is completely proved.

Set $\mathcal{F} := U/G$ to be the scheme from Proposition 4.1. It follows from Proposition 4.8 and Remark 4.12 that $\mathcal{F}$ is a (coarse) moduli space which parameterizes the pairs $(Y^2, S^2)$ consisting of smooth weak Fano 3-fold $Y^2$ of degree 70 and smooth surface $S^2 \subset -K_{Y^2}$ (see (2.2)). These give the following

\textbf{Lemma 4.9.} For $o := (Y, S) \in \mathcal{F}$, we have $H^1(Y, T_Y(S)) = T_o \mathcal{F}$.

\textit{Proof.} This follows from the fact that $\mathcal{F}$ is smooth and $H^1(Y, T_Y(S))$ parameterizes the first order deformations of $(Y, S)$ (see [1] Proposition 1.1).

Consider the forgetful morphism $s : \mathcal{F} \to \mathcal{K}$, which sends $(Y, S)$ to $S$.

\textbf{Lemma 4.10.} $s$ is generically surjective.

\textit{Proof.} Consider the restriction map $r : T_Y(S) \to T_S$. It fits into the exact sequence

\begin{equation}
0 \to \Omega^2_Y \to T_Y(S) \to T_S \to 0,
\end{equation}

since $\text{Ker}(r) = T_Y(-S)$ is a subsheaf of $T_Y(S)$ consisting of the vector fields vanishing along $S$, for which we have $T_Y(-S) \simeq \Omega^2_Y$. From (4.11) we get the exact sequence

$$H^1(Y, T_Y(S)) \xrightarrow{H^1(r)} H^1(S, T_S) \xrightarrow{\partial} H^2(Y, \Omega^2_Y).$$

The map $\partial$ is dual to the restriction map $i : H^1(Y, \Omega^2_Y) \to H^1(S, \Omega^1_S)$ (see [1]). In particular, $\text{Ker}(\partial)$ is the orthogonal of $\text{Im}(i)$. On the other hand, we have $\text{Im}(i) = Z \cdot c_1(K_Y \mid_S) \oplus Z \cdot c_1(\tau^{-1}E_s \mid_S) \simeq Z \cdot K_X \mid_S \oplus Z \cdot E \mid_S$ (see Corollary 3.11 and Remark 3.13), and hence $H^1(r)$ coincides with the tangent map to $s$ at $(Y, S)$, with $\text{Im}(H^1(r)) = \text{Ker}(\partial)$ being the tangent space to $\mathcal{K}$ at $S$ (see Lemma 3.9 and Proposition 3.11). Thus, since $\mathcal{K}$ is irreducible (see Proposition 3.14), we get that $s$ is generically surjective.

\textbf{Theorem 1.2} is completely proved.
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