On the width of lattice-free simplices

Jean-Michel KANTOR

I Introduction

Integral polytopes (see [Z] for the basic definitions) are of interest in combinatorics, linear programming, algebraic geometry-toric varieties [D, O], number theory [K-L]. We study here lattice-free simplices, that is simplices intersecting the lattice only at their vertices.

A natural question is to measure the “flatness” of these polytopes, with respect to integral dual vectors. This (arithmetical) notion plays a crucial role:

- in the classification (up to affine unimodular maps) of lattice-free simplices in dimension 3 (see [O, MMM].
- in the construction of a polynomial-time algorithm for integral linear programming (flatness permits induction on the dimension, [K-L]).

Unfortunately there were no known examples (in any dimension) of lattice-free polytopes with width bigger than 2. We prove here the following

**Theorem:**
Given any positive number $\alpha$ strictly inferior to $\frac{1}{e}$, for $d$ large enough there exists a lattice-free simplex of dimension $d$ and width superior to $\alpha d$.

The proof is non-constructive and uses replacing the search for lattice-free simplices in $\mathbb{Z}^d$ by the search for ”lattice-free lattices ” containing $\mathbb{Z}^d”$ (“turning the problem inside out”, see par. II), specializing in the next step to lattices of a simple kind, depending on a prime number $p$. The existence of lattice-free simplices with big width is then deduced by elementary computations, through a sufficient inequality involving the dimension $d$, the width $k$ and the prime $p$ (see (14)).

The author thanks with pleasure H. Lenstra for crucial suggestions, V. Guillemin and I. Bernstein for comments.

**Notations**

$\mathcal{P}_d$: The set of integral polytopes in $\mathbb{R}^d$; if $P$ is such a polytope, $P$ is a convex compact set, the set Vert($P$) of vertices of $P$ is a subset of $\mathbb{Z}^d$.

$\mathcal{S}_d$: The set of integral simplices in $\mathbb{R}^d$. In particular $\sigma_d$ will denote the canonical simplex with vertices at the origin and

$e_i = (0, \ldots, 0, 1, 0, \ldots, 0) — 1$ at the $i$-th coordinate—

$G_d$: the group of affine unimodular maps:

$$G_d = \mathbb{Z}^d \rtimes GL(d, \mathbb{Z})$$

acts on $\mathbb{R}^d$ (preserving $\mathbb{Z}^d$), $\mathcal{P}_d$, and $\mathcal{S}_d$.

A $d$-lattice $M$ is a lattice with

$$\mathbb{Z}^d \subset M \subset \frac{1}{m}\mathbb{Z}^d$$

for some $m \in \mathbb{N}^*$.

II. Turning simplices inside out.
II.1. Let $\sigma$ be an integral simplex of dimension $d$ in $\mathbb{R}^d$, and $L$ the sublattice of $\mathbb{Z}^d$ it generates:

$$L = \left\{ \sum_{i=1}^r m_i a_i, \ a_i \in \text{Vert}(\sigma) \ m_i \in \mathbb{Z} \right\}.$$ 

We assume for simplicity one vertex of $\sigma$ at the origin. The following is obvious:

**Proposition 1.** i) There exists a linear isomorphism

$$\varphi : \mathbb{R}^d \to \mathbb{R}^d \quad \varphi(x) = y = (y_j)_{j=1,\ldots,n}$$

such that

$$\varphi(\sigma) = \sigma_d.$$ 

It is unique up to permutation of the $y_i$'s, and

$$\varphi(L) = \mathbb{Z}^d, \quad \varphi(\mathbb{Z}^d) = M$$

where $M$ is a $d$-lattice.

ii) Conversely, given a $d$-lattice $M$, there exists a linear isomorphism

$$\psi : \mathbb{R}^d \to \mathbb{R}^d$$

such that

$$\psi(M) = \mathbb{Z}^d,$$

and the image of $\sigma_d$ by $\psi$ is an integral simplex $\sigma$ of dimension $d$ corresponding to $M$ as in i).

All $d$-lattices generate $\mathbb{R}^d$ as vector space over $\mathbb{R}$, and the Proposition is an easy consequence of this.

**Remarks.**

- Because $\varphi$ is an isomorphism, the following indices are equal

$$[\mathbb{Z}^d : L] = [M : \mathbb{Z}^d].$$

The determinant of the lattice $L$ is classically the volume of the parallelotope built on a basis of $L$. If $\sigma$ generates $L$ as above,

(1) \hspace{1cm} \text{vol}(\sigma) = \frac{1}{d!} \det L

(2) \hspace{1cm} \det M = \frac{1}{d! \text{vol} \sigma}.$$

- Proposition 1 has a straightforward extension to the case of two lattices $L$ and $M$ with

$$L \subset M \subset \frac{1}{p} L$$

II.2. Lattice-free simplices and their width

Recall the following [K, K-L].
**Definition 1.** An integral polytope $P$ in $\mathbb{R}^d$ is *lattice-free* if

$$P \cap \mathbb{Z}^d = \text{Vert}(P)$$

**Definition 2.** Given an integral non-zero vector $u$ in $(\mathbb{Z}^d)^*$, the $u$-width of the polytope $P$ of $\mathcal{P}_d$ is defined by

$$w_u(P) = \max_{x,y \in P} <u, x - y>.$$ 

The *width* of $P$ is

(4) $$w(P) = \inf_{u \in (\mathbb{Z}^d)^*, \, u \neq 0} w_u(P).$$

Remark: The width is the minimal length of all integral projections $u(P)$ for non-zero $u$.

**II.3 Known results on the width of lattice-free polytopes in dimension $d$:**

$d = 2$:
Lattice-free simplices are all integral triangles of area $1/2$; they are equivalent to $\sigma_2$. This is elementary.

$d = 3$:
Lattice-free polytopes have width one; in the case of simplices, this result has various proofs and applications (it is known sometimes as the “terminal lemma”, see [F, M-S, O, Wh]).

$d = 4$:
All lattice-free simplices have at least one basic facet (face with codimension one) [W]
-this fact is not true in higher dimensions.

Examples: There exist some interesting examples:
-L. Schläfli’s polytopes, studied by Coxeter [C];
-A recent example given by H. Scarf [private communication]; the simplex in dimension 5 with vertices the origin, the first four vectors $e_i$ and for last vertex $(23, 39, 31, 43, 57)$, has width 3.
- We have found with the help of a computer, some examples of width 2, 3 and 4 in dimension 4 and 5 [F-K].

No other result seems to be known, apart from the following asymptotic result:

**Proposition 2.** There exists a universal constant $C$ such that for any lattice-free polytope of dimension $d$

(5) $$w(P) \leq Cd^2.$$ 

*Proof.* The “Flatness Theorem” of [K-L] asserts that there exists $C$ such that any convex compact set $K$ in $\mathbb{R}^d$ with

$$K \cap \mathbb{Z}^d = \phi$$

satisfies

(6) $$w(K) \leq C d^2$$

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where \( w \) is defined as in I.2.

If \( P \) is any lattice-free polytope, take a point \( a \) in the relative interior of \( P \) and apply the previous Flatness Theorem to the homothetic \( \tilde{P} \) of \( P \) with respect to \( a \) and fixed ratio \( \alpha \) strictly less than one. Then formula (4) shows that the width of \( P \), which is proportional to the width of \( \tilde{P} \), is also bounded by a function of type (6).

Remark: `Recent results of [B] show that (5) is true with a right hand side proportional to \( d \log d \).

II.4. Turning the width inside out

Let us first define a new norm on \( \mathbb{R}^d \): If \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \)

\[
\|x\| = \sup_i |x_i|
\]

Define

(7) \[
\|x\| = \sup_i (0, x_i) - \inf_i (0, x_i)
\]

It is the support function of the following symmetric convex compact set in \((\mathbb{R}^d)^*\):

\[
K = \sigma_d - \sigma_d
\]

(see [O], p.182).

**Lemma 1.** \( \| \| \) is a norm, and

\[
\|x\|_\infty \leq \|x\| \leq 2\|x\|_\infty
\]

From now on \((\sigma, M)\) are as in Proposition 1. We can identify the dual of the lattice \( M \) with a subgroup of \((\mathbb{Z}^d)^*\):

\[
\mathbb{Z}^d \subset M, \quad M^* \subset (\mathbb{Z}^d)^* \\
\xi \in M^*: \xi = (\xi_1, \ldots, \xi_d) \in (\mathbb{Z}^d)^*
\]

**Definition 3.** Let

(8) \[
w(M) = \inf_{\xi \in M^*} \|\xi\|
\]

Then we have

**Proposition 3.**

\[
w(\sigma) = w(M)
\]

**Proof.** The isomorphism \( \varphi \) changes \( \sigma \) into \( \sigma_d \), linear forms \( u \) on \( \mathbb{Z}^d \) into linear forms on \( M \), and

\[
w_\xi(\sigma_d) = \sup_{x \in \sigma_d} (\xi, x) - \inf_{y \in \sigma_d} <\xi, y>
\]

\[
= \sup_i (0, \xi_i) - \inf_i (0, \xi_i)
\]

\[
= \|\xi\|.
\]
II.5. With notations as in II.1, we have

\[ \sigma \cap \mathbb{Z}^d = \text{Vert} \sigma \iff M \cap \sigma_d = \text{Vert} (\sigma_d) \]

We can conclude this part by asserting that the existence of an integral lattice-free simplex of dimension \( d \), volume \( v/d! \) and width at least \( k \) is equivalent with the existence of a \( d \)-lattice \( M \), containing \( \mathbb{Z}^d \), with

\[
\begin{cases}
 M \cap \sigma_d = \text{Vert} \sigma_d \\
w(M) \geq k \\
det(M) = \frac{1}{v}.
\end{cases}
\]

III. In search of lattice-free simplices (asymptotically)

III.1. We restrict our study to \( d \)-lattices of type:

\[ M(y) = \mathbb{Z}^d + \frac{1}{p} \mathbb{Z} \ y \quad y \in \mathbb{Z}^d \quad M \neq \mathbb{Z}^d \]

where \( p \) is a prime number; clearly this lattice depends only on the class of \( y \) in \((\mathbb{Z}/p\mathbb{Z})^d\).

**Lemma 2.** The set of lattices \( M \) (for a fixed \( p \)) can be identified with the space of lines in \((\mathbb{Z}/p\mathbb{Z})^d\).

In particular the number of such lattices is

\[ m(d, p) = \frac{p^d - 1}{p - 1}. \]

Let \( f(d, p) \) be the number of lattices \( M \) such as (10) satisfying

\[ M \cap \tilde{\sigma}_d \neq \phi \]

where

\[ \tilde{\sigma}_d = \sigma_d \setminus \text{Vert}(\sigma_d). \]

(The lattice \( M \) intersects \( \sigma_d \) in other points than the vertices).

**Lemma 3.**

\[ f(d, p) \leq \frac{(p + 1) \ldots (p + d)}{d!} - (d + 1). \]

**Proof.**

\[ x \in M(y) \cap \tilde{\sigma}_d \quad \{ x = z + \frac{my}{p} \} \quad \implies (m, p) = 1. \]
Writing \( \frac{mu}{p} \) as the sum of an integral vector and a remainder we get

\[
x = z + z' + \frac{\tilde{y}}{p}, \quad 0 \leq \tilde{y}_i < p, \quad \tilde{y}_i \in \mathbb{N}
\]

\[
x \in \tilde{\sigma}_d
\]

\[
\implies z + z' = 0.
\]

\[
x = \frac{\tilde{y}}{p},
\]

\[
\tilde{y} \in p\tilde{\sigma}_d \cap \mathbb{Z}^d
\]

But the vectors \( y, my, \tilde{y} \) define the same line in \((\mathbb{Z}/p\mathbb{Z})^d\). This shows that the number of lattices \( M(y) \) satisfying (12) is less than the number of points in \( p\tilde{\sigma}_d \cap \mathbb{Z}^d \), given by the right hand side of the lemma \([ E]\).

Let now \( g(d, p, k) \) be the number of lattices \( M(y) \) as in (10) with

\[
w(M(y)) \leq k.
\]

**Lemma 4.**

(13)

\[
g(p, d, k) \leq 2[(k + 1)^{d+1} - k^{d+1}]p^{d-2}.
\]

**Proof:**

The assumption on the lattice means the existence of a vector \( \xi \) in \( \mathbb{Z}^d \)

\[
\xi \neq 0 \quad y = (y_1, \ldots, y_d), \quad \xi = (\xi_1, \ldots, \xi_d)
\]

\[
\sum \xi_i y_i \in p\mathbb{Z}
\]

\[
\|\xi\| \leq k \implies \|\xi\|_{\infty} \leq k.
\]

The number of integral points \( \xi \) of norm less or equal to \( k \) is

\[
n(k, d) = (k + 1)^{d+1} - k^{d+1}
\]

[Proof :]

Let

\[
m = \inf_{i}(0, \xi_i)M = \sup_{i}(0, \xi_i)
\]

The possible values of \( m \) are

\[
m = -k, \ldots, -1, 0
\]

a/ For all values except 0 one of the \( x_i \) has value \( m \), and the others can take any value between \( m \) and \( m + k \). For each \( m \) the number of possibilities is equal to

\[
S_1 = [k + 1]^d - k^d
\]

b/ When

\[
m = 0
\]

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all $x'_i$s are non-negative, and the contribution is

$$S_2 = [k + 1]^d$$

Adding up the contributions we get

$$n(k, d) = k[(k + 1)^d - k^d] + (k + 1)^d = (k + 1)^{d+1} - k^{d+1}.$$  

Going back to the proof of Lemma 4, choose a vector $\xi$ with norm smaller than $k$ (strictly less than $p$): this implies that the linear form defined by $\xi$:

$$\hat{\xi} : (\mathbb{Z}/p\mathbb{Z})^d \to \mathbb{Z}/p\mathbb{Z}$$

is surjective, and its kernel has $p^{d-1}$ elements; the number of corresponding lattices is

$$r(p, d) = \frac{p^{d-1} - 1}{p - 1} \leq 2p^{d-2}.$$  

We can choose at most $n(k, d)$ vectors $\xi$.

Hence, by the mean value theorem,

$$g(p, d, k) \leq 2[(k + 1)^{d+1} - k^{d+1}]p^{d-2} \leq 2(d + 1)(k + 1)^dp^{d-2}. \quad (13)$$  

III.3. From Lemmas 3 and 4 we conclude that for large $d$ and $k$ the following condition

$$2(d + 1)(k + 1)^d p^{d-2} + \frac{(p + d)^d}{d!} < p^{d-1} \quad (14)$$

ensures the existence of a lattice $M(y)$ of width greater than $k$, dimension $d$, and

$$M(y) \subset \frac{1}{p}\mathbb{Z}^d.$$  

The following is well-known:

**Lemma 5.**

Given any sequence of numbers $(a_d)$ going to infinity, there exists an equivalent sequence $(p_d)$ of prime numbers.

Proof: Given $\varepsilon$ strictly positive we know from the prime number theorem that for $d$ large enough there exists a prime number $p_d$ in the interval $[(1 - \varepsilon)a_d, (1 + \varepsilon)a_d]$. This implies

$$|p_d - a_d| < \varepsilon a_d$$

for $d$ large enough.

Choose now $\alpha$ arbitrary - we will soon fix it - and a sequence $(p_d)$ of primes with

$$p_d \sim \alpha d!$$

and let us find $\alpha$ and a sequence $(k_d)$ such that

$$2(d + 1)(k_d + 1)^d p_d^{d-2} < \frac{1}{2}p_d^{d-1} \quad (15)$$
These two conditions imply (14)

The condition (16) is satisfied for large enough $d$ if

$$\alpha < \frac{1}{2}.$$ 

Indeed

$$p_d + d \sim \alpha d!;$$

since

$$\alpha < \frac{1}{2}$$

(16) follows if we can show that

$$(1 + d/p)^{d-1} \to 1$$

$$d \to \infty$$

where

$$p = p_d$$

But

$$\log(1 + d/p)^{d-1} \leq (d - 1)d/p \sim d^2/\alpha d! \to 0$$

Then (15) becomes

$$k_d + 1 < \left[ \frac{1}{4(d + 1)} p_d \right]^{\frac{1}{2}}$$

This last expression is equivalent, because of Stirling’s formula, to $\frac{d}{e}$. Hence if we choose any sequence of integral numbers $(k_d)$ with

$$k_d < \beta d$$

with

$$0 < \beta < \frac{1}{e}$$

then (15) and (16) are satisfied for large $d$.

**Theorem.** For any $\beta$ strictly less than $1/e$, there exists for sufficiently large $d$ a sequence of lattice-free simplices of dimension $d$ and width $w_d$,

$$w_d > \beta d.$$ 

Defining

$$w(d) = \sup_{\sigma} w(\sigma)$$

supremum taken over all lattice-free simplices of dimension $d$, then the previous Theorem amounts to:
$$\lim_{d \to \infty} \frac{w(d)}{d} \geq \frac{1}{e}$$

**Final Remark.** The study above raises the hope of improving the bounds on the maximal width, by introducing more general lattices generated by a finite number of rational vectors, and replacing the prime $p$ by powers in (10) (Note the study of general lattices of such type in [Sh]). Unfortunately -and rather mysteriously- our computations in these new cases give the same bounds.

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