Existence and uniqueness of mean-ratio quasi-stationary distribution for one-dimensional diffusions

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Abstract
In this paper, we study mean-ratio quasi-stationary distribution (MRQSD) for one-dimensional diffusion $X$ killed at 0, when $+\infty$ is an entrance boundary and 0 is an exit boundary. More precisely, we not only show that the process is $R$-positive, but also prove the existence and uniqueness of MRQSD by using the spectral theory tool. As a consequence, this unique MRQSD is just the stationary distribution of $Q$-process.

Keywords: One-dimensional diffusion; Mean-ratio quasi-stationary distribution; $R$-positive; Quasi-stationary distribution; Entrance boundary; Exit boundary

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1. Introduction

We are interested in the long-term behavior of killed Markov processes. Quasi-ergodicity, is one of the most interesting topics in this direction. The main motivation of this work is the existence and uniqueness of mean-ratio quasi-stationary distribution (MRQSD) for one-dimensional diffusion $X$ killed at 0, when $+\infty$ is an entrance boundary and 0 is an exit boundary.

The notion of MRQSD for diffusions, which may be defined as the limiting distribution of the expected empirical distributions of the process conditioned on absorption not occurred, is a kind of phase transition and also quite different from quasi-stationary distribution (QSD), but coincide with a certain double limiting distribution. To the best of our knowledge, the study of MRQSD may first appeared in Darroch and Seneta (1965) and Seneta and Vere-Jones (1966) for Markov processes with countable state spaces with the name of limiting conditional mean ratio. For Markov processes with general state spaces under the condition that the process is a positive Harris $\lambda$-recurrent process with $\lambda \leq 0$, Breyer and Roberts (1999) proved the existence and uniqueness of MRQSD and the corresponding quasi-ergodicity for time average of the process. Recently, for absorbing Markov processes under the assumption on $\lambda$-positivity, Chen et al. (2012) studied some problems related to MRQSD, attempting to interpretate MRQSD from different perspectives. Soon afterwards, Chen and Jian (2013) proved the existence and uniqueness of both QSD and MRQSD for killed Brownian motion by using an eigenfunction expansion for the transition density.

In this paper, by using the spectral theory tool, we will prove the existence and uniqueness of MRQSD for one-dimensional diffusion $X$ killed at 0, when $+\infty$ is an entrance boundary and 0 is an exit boundary. Our proof is completely different from the previous papers. This paper is related to Cattiaux et al. (2009) and Littin (2012). Under the same conditions, Littin (2012) have proved the existence of a unique Yaglom limit and that of a unique QSD for the process $X$.

The remainder of this paper is organized as follows. In Section 2 we present some preliminaries that will be needed in the sequel. In Section 3 we mainly show that the process is $R$-positive. In Section 4 we will study some problems related to MRQSD and prove the existence and uniqueness of MRQSD. Finally, in Section 5 we show that this unique MRQSD is just the stationary distribution of $Q$-process.

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2. Preliminaries

We consider the generator $Lu := \frac{1}{2}\partial_{xx}u - \alpha\partial_x u$. Denote by $X$ the diffusion whose infinitesimal generator is $L$, or in other words the solution of the stochastic differential equation (SDE)

$$
    dX_t = dB_t - \alpha(X_t)dt,
$$

where $(B_t; t \geq 0)$ is a standard one-dimensional Brownian motion and $\alpha \in C^1(0, \infty)$. In this paper, $\alpha$ is allowed to explode at the origin. There exists a pathwise unique solution to the SDE (1), with initial condition $x > 0$, up to the explosion time $\tau$. Thus, $-\alpha$ is the drift of $X$.

Associated to $\alpha$, we consider the following two functions

$$
    \Lambda_1(x) = \int_1^x e^{Q(y)}dy \quad \text{and} \quad \kappa(x) = \int_1^\infty e^{Q(y)} \left( \int_1^y e^{-Q(z)}dz \right)dy,
$$

where $Q(y) = \int_0^y 2\alpha(x)dx$. Notice that $\Lambda_1$ is the scale function for $X$.

The other important piece of information is the following measure defined on $(0, \infty)$:

$$
    \mu(dy) := e^{-Q(y)}dy.
$$

Notice that $\mu$ is the speed measure for $X$.

Let $T_a := \inf\{0 \leq t < \tau : X_t = a\}$ be the hitting time of $a \in (0, \infty)$ for $X$. We denote by $T_\infty = \lim_{a \to \infty} T_a$ and $T_0 = \lim_{a \to 0} T_a$ be the hitting time of $X$ at $0$. Because $\alpha$ is regular in $(0, \infty)$, $\tau = \min\{T_0, T_\infty\}$. Let $\mathbb{P}_x$ and $\mathbb{E}_x$ stand for the probability and the expectation, respectively, associated with $X$ when initiated from $x$. For any distribution $\nu$ on $(0, \infty)$, we define $\mathbb{P}_\nu(\cdot) := \int_0^\infty \mathbb{P}_x(\cdot)\nu(dx)$. We denote by $\mathcal{B}(0, \infty)$ the Borel $\sigma$-algebra on $(0, \infty)$, $\mathcal{P}(0, \infty)$ the set of all probability measures on $(0, \infty)$ and $\langle f, g \rangle_\mu = \int_0^\infty f(\mu)g(\mu)d\mu$.

For all the results in this paper we will use the following hypothesis (H), that is,

**Definition 1.** We say that hypothesis (H) holds if the following explicit conditions on $\alpha$, all together, are satisfied:

(H1) for all $x > 0$, $\mathbb{P}_x(\tau = T_0 < T_\infty) = 1$.

(H2) for any $\varepsilon > 0$, $\mu(0, \varepsilon) = \infty$.

(H3) $S = \int_0^\infty e^{Q(y)} \left( \int_1^\infty e^{-Q(z)}dz \right)dy < \infty$.

If (H1) holds, then it is equivalent to $\Lambda_1(\infty) = \infty$ and $\kappa(\infty) < \infty$ (see, e.g., Ikeda and Watanabe (1989), Chapter VI, Theorem 3.2). If (H1) and (H2) are satisfied, we say that 0 is an exit boundary in the sense of Feller (see Karlin and Taylor (1981), Chapter 15). If (H1) and (H3) are satisfied, then $+\infty$ is an entrance boundary in the sense of Feller (also see Karlin and Taylor (1981), Chapter 15).

We know from Cattiaux et al. (2009) and Littin (2012) that $L$ is the generator of a strongly continuous symmetric semigroup of contractions on $L^2(\mu)$ denoted by $(P_t)_{t \geq 0}$. This semigroup is sub-Markovian, that is, if $0 \leq f \leq 1$, then $0 \leq P_tf \leq 1$ a.e.. Also from Cattiaux et al. (2009) and Littin (2012) we get that when absorption is sure, that is (H1) holds, the semigroup of $X$ killed at 0 can be given by $P_tf(x) = \mathbb{E}_x[f(X_\tau), T_\tau > t]$.

In this paper, we need some results obtained by Littin (2012) or required in the proof of Littin (2012), which are summarized in the following proposition.

**Proposition 1.** ([Littin(2012)]) Assume (H) holds. Then we have

(i) $-L$ has purely discrete spectrum. The eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ are simple, $\lim_{n \to \infty} \lambda_n = +\infty$, and the eigenfunction $\eta_\lambda$ associated to $\lambda_\lambda$ has exactly $n$ roots in $(0, \infty)$ and an orthonormal basis of $L^2(\mu)$. In particular, $\eta_1$ can be chosen to be strictly positive.

For $g \in L^2(\mu)$,

$$
    P_tg = \sum_{i \geq 1} e^{-\lambda_i t} \langle \eta_i, g \rangle_\mu \eta_i \quad \text{in} \quad L^2(\mu),
$$

then for $f, g \in L^2(\mu)$,

$$
    \lim_{t \to \infty} e^{\lambda_1 t} \langle g, P_tf \rangle_\mu = \langle \eta_1, f \rangle_\mu \langle \eta_1, g \rangle_\mu.
$$
Definition 2. said to be and it takes values on \((0, \infty)\), that is, \(P\).

The SDE

\[
\text{for all bounded Borel function } f. \text{ Moreover, the density } r(t, x, \cdot) \in \mathbb{L}^2(\mu) \text{ for all } x > 0, t > 0. \text{ In particular, there exists a function } B(t) \geq 0, \lim_{t \to \infty} B(t) = 0 \text{ such that}
\]

\[
\int_0^\infty r^2(t, x, y) \mu(dy) < r(t, x, x)B(t) < \infty.
\]

(iv) For all \(x > 0\), there exists a locally bounded function \(\Theta(x)\) such that, for all \(y > 0\) and all \(t > 1\),

\[
r(t, x, y) \leq \Theta(x)e^{-\lambda t} \eta_1(y).
\]

3. \(R\)-positive

In this section, we will show that, under (H), the one-dimensional diffusion \(X\) killed at \(0\) is \(R\)-positive. This means that the processes \(Y\), whose law is the conditional law of \(X\) to never hit the origin, is positive recurrent.

Next, as in Collet et al. (1995) (Theorem B) or Cattiaux et al. (2009) (Corollary 6.1), we can also describe the law of the process \(X\) conditioned to never be absorbed, usually referred to as the Q-process (see Lambert (2007)).

Proposition 2. Assume (H) holds. Then for all \(x > 0\), \(s \geq 0\) and all \(A \in \mathcal{B}(C[0, s])\), we have

\[
\lim_{t \to \infty} \mathbb{P}_x(X \in A | T_0 > t) = Q_x(A),
\]

where \(Q_x\) is the law of a diffusion process on \((0, \infty)\) with transition probability densities (w.r.t. the Lebesgue measure) given by

\[
h(s, x, y) = e^{\lambda s} \frac{\eta_1(y)}{\eta_1(x)} r(s, x, y)e^{-\lambda y},
\]

that is, \(Q_x\) is locally absolutely continuous w.r.t. \(\mathbb{P}_x\) and

\[
Q_x(X \in A) = \mathbb{E}_x \left( \mathbf{1}_A(X)e^{\lambda s} \frac{\eta_1(X_s)}{\eta_1(x)}, T_0 > s \right).
\]

Proof. The same proof as in Cattiaux et al. (2009) works. \(\square\)

As we see that, for \(x > 0\) fixed, the following limit exists and defines a diffusion \(Y\):

\[
\lim_{t \to \infty} \mathbb{P}_x(X \in A | T_0 > t) = e^{\lambda s} \mathbb{E}_x \left( \frac{\eta_1(X_s)}{\eta_1(x)}, X \in A, T_0 > s \right) = \mathbb{P}_x(Y \in A),
\]

where we put \(\mathbb{E}_x(f(B), A) = \mathbb{E}_x(f(B) \mathbf{1}_A)\), for a measurable set \(A\) and an integrable function \(f\). The diffusion \(Y\) satisfies the SDE

\[
dY_t = dB_t - \phi(Y_t)dt \quad \text{where } \phi(y) = \alpha(y) - \frac{\eta'_1(y)}{\eta_1(y)},
\]

and it takes values on \((0, \infty)\). In fact, since its drift is of order \(1/x\) for \(x\) near \(0\), so it never reaches \(0\).

The connection between the classification of \(Y\) and the \(R\)-classification of the killed diffusion \(X^{T_0}\) is given in the following definition, where as usual \(X^{T_0}\) corresponds to \(X\) killed at \(0\).

Definition 2. If the process \(Y\) is positive recurrent (resp. recurrent, null recurrent, transient), then the process \(X^{T_0}\) is said to be \(R\)-positive (resp. \(R\)-recurrent, \(R\)-null, \(R\)-transient).
We may now state the following result.

**Theorem 1.** Assume (H) holds. Then $X^{T_2}$ is $R$-positive.

**Proof.** We consider the functions

$$Q^Y(y) = \int_1^\infty 2\phi(x)dx = Q(y) - 2\log(\eta_1(y)/\eta_1(1))$$

and

$$\Lambda^Y(y) = \int_1^\infty e^{Q^x(\cdot)}dz = \eta_1^2(1) \int_1^\infty \eta_1(z)e^{Q^x(\cdot)}dz.$$

Because $\eta_1(x) = x + O(x^2)$ for $x$ near 0, we first obtain that $\Lambda^Y(0^+) = -\infty$.

The speed measure $m$ of $Y$ is given by

$$m(dx) = \frac{2dx}{(\Lambda^Y(x))^2}$$

(see Karatzas and Shreve (1988), formula (5.51)). So we obtain

$$m(dx) = \frac{2\eta_1^2(x)e^{-Q^x(\cdot)}dx}{\eta_1^2(1)} = \frac{2\eta_1^2(x)\mu(dx)}{\eta_1^2(1)}.$$  \hspace{1cm} (10)

If (H) is satisfied, we know from Proposition 1 that $\eta_1 \in L^2(\mu)$, i.e. $\int_0^\infty \eta_1^2(z)e^{Q^x(\cdot)}dz < \infty$, which implies $\int_0^\infty \eta_1^2(z)e^{Q^x(\cdot)}dz = \infty$. Then $\Lambda^Y(\infty) = \infty$ and from (10) we obtain $m(0,\infty) < \infty$.

Let $T^Y_a := \inf\{t > 0 : Y_t = a\}$ be the hitting time of $a$ for the process $Y$. For any $x, a \in (0,\infty)$, we know that the process $Y$ is positive recurrent when $\mathbb{E}_x(T^Y_a) < \infty$. By using the formulas on page 353 in Karatzas and Shreve (1988), we deduce that $Y$ is positive recurrent. Therefore, $X^{T_2}$ is $R$-positive. \hfill \Box

4. Mean-ratio quasi-stationary distribution

In this section, we study the existence and uniqueness of mean-ratio quasi-stationary distribution for one-dimensional diffusion $X$ killed at 0, when $+\infty$ is an entrance boundary and 0 is an exit boundary. More formally, the following definition captures the main object of interest of this work.

**Definition 3.** We say that $v \in \mathcal{B}(0,\infty)$ is a mean-ratio quasi-stationary distribution (MRQSD) if, for all $t > 0$ and any $A \in \mathcal{B}(0,\infty)$, there exists a $\pi \in \mathcal{B}(0,\infty)$ such that the following limit exists in the weak sense

$$\lim_{t \to +\infty} \mathbb{E}_\pi\left(\frac{1}{t} \int_0^t 1_A(X_s)ds\big|T_0 > t\right) = v(A).$$

Define

$$v_1(A) := \int_A \eta_1^2(x)\mu(dx), \quad A \in \mathcal{B}(0,\infty).$$

We know from Proposition 1 that $\|\eta_1\|_{L^2(\mu)} = 1$, then $v_1$ is a distribution on $(0,\infty)$.

We may now state the following result from the spectral point of view.

**Theorem 2.** Assume (H) holds. Then for any $0 < q < 1$, any $v \in \mathcal{B}(0,\infty)$ and any $f \in \mathbb{L}^1(\mu)$, we have

$$\lim_{t \to +\infty} \mathbb{E}_\pi(f(X_{T_0})|T_0 > t) = \int_0^\infty f(y)v_1(dy).$$
Proof. We know from Proposition that \( r(1, x, \cdot) \in L^2(\mu) \). By using Proposition writing \( r(t, x, \cdot) = P_{\tau^{-1}} r(1, x, \cdot) \) \( \mu \)-a.s. and noticing that
\[
\int_0^\infty r(1, x, y) \eta_1(y) d\mu(dy) = (P_1 \eta_1)(x) = e^{-t} \eta_1(x),
\]
we deduce that
\[
\lim_{t \to \infty} e^{t} r(t, x, \cdot) = e^{t}(r(1, x, \cdot), \eta_1)_\mu \eta_1(\cdot) = \eta_1(x) \eta_1(\cdot) \tag{11}
\]
Similarly, for \( t > \min\left( \frac{1}{\eta_1}, \frac{1}{\lambda q} \right) \), writing \( r(qt, x, \cdot) = P_{\tau^{-1}} r(1, x, \cdot) \mu \)-a.s. and \( r(t - qt, z, \cdot) = P_{\tau^{-q}} r(1, z, \cdot) \mu \)-a.s., we deduce that
\[
\lim_{t \to \infty} e^{t} r(qt, x, \cdot) r(t - qt, z, \\cdot) = e^{t}(r(1, x, \cdot), \eta_1)_\mu \eta_1(\cdot)(r(1, z, \cdot), \eta_1)_\mu \eta_1(\cdot) = \eta_1(x) \eta_1(\cdot) \eta_1(z) \eta_1(\cdot) \tag{12}
\]
By the semigroup property, for any \( (t, x, y) \in (0, \infty) \times (0, \infty) \times (0, \infty) \), we have
\[
\int_0^\infty r(t, x, z) r(t - qt, z, y) d\mu(dz).
\]
It suffices to prove for \( q = \delta_z \), \( x \in (0, \infty) \). Moreover, we know from Proposition that the equality holds, then for \( t > 1, e^{t} r(t, x, \cdot) \in L^1(\mu) \) and is dominated by \( \Theta(t) \eta_1(\cdot) \). Similarly, the equality holds, then for \( t > \min\left( \frac{1}{q}, \frac{1}{\lambda q} \right), e^{t} r(qt, x, \cdot) r(t - qt, z, \cdot) \) is dominated by \( \Theta^q(t) \eta_1(\cdot) \eta_1(\cdot) \). Then it now follows from the dominated convergence theorem and Fubini’s theorem that for any \( 0 < q < 1 \) and any \( f \in L^1(\mu) \),
\[
\lim_{t \to \infty} E_x(f(X_0) | T_0 > t) = \lim_{t \to \infty} \frac{E_x(f(X_0), T_0 > t)}{P_x(T_0 > t)}
\]
\[
= \lim_{t \to \infty} \frac{\int_0^\infty \int_0^\infty r(qt, x, z) f(z) r(t - qt, z, y) d\mu(dz) d\mu(dy)}{\int_0^\infty r(t, x, y) d\mu(dy)}
\]
\[
= \lim_{t \to \infty} \frac{e^{t} \int_0^\infty \int_0^\infty r(qt, x, z) f(z) r(t - qt, z, y) d\mu(dz) d\mu(dy)}{e^{t} \int_0^\infty r(t, x, y) d\mu(dy)}
\]
\[
= \int_0^\infty \int_0^\infty \eta_1(x) \eta_1(z) f(z) d\mu(dz) d\mu(dy)
\]
\[
= \int_0^\infty \eta_1(z) f(z) d\mu(dz)
\]
\[
= \int_0^\infty f(y) \nu_1(dy).
\]
Note that when \( q = 1 \), we know from Theorem 4.2 of Litvin (2012) that for \( A \in \mathcal{B}(0, \infty), \lim_{t \to \infty} P_x(X_0 \in A | T_0 > t) = \nu_2(A) \), where \( \nu_2(A) = \frac{\int_A \eta_1(x) d\mu(x)}{\eta_1(\mu(\cdot))} \) is a quasi-stationary distribution. As we have seen that the above result describes a mixing property and exhibits a certain phase transition. A further investigation of \( \nu_1 \) shows that \( \nu_1 \) can also be described as the following double limit.

**Theorem 3.** Assume (H) holds. Then for any \( \nu \in \mathcal{M}(0, \infty) \) and any bounded Borel function \( f \) on \( (0, \infty) \), we have
\[
\lim_{t \to \infty} \lim_{T \to \infty} E_{\nu}(f(X_0) | T_0 > T) = \int_0^\infty f(y) \nu_1(dy).
\]

Proof. For \( t > 1, T - t > 1 \), writing \( r(T - t, z, \cdot) = P_{\tau^{-1}} r(1, z, \cdot) \mu \)-a.s., we deduce that
\[
\lim_{t \to \infty} e^{t} r(T - t, z, \cdot) = e^{t(1+1)} r(1, z, \cdot), \eta_1)_\mu \eta_1(\cdot) = e^{t} \eta_1(z) \eta_1(\cdot) \tag{13}
\]
Moreover, we know from Proposition 1 that the equality (11) holds, then for $t > 1$, $T - t > 1$, $e^{1/t}r(t, x, \cdot)$, $e^{1/T}r(T, x, \cdot)$ are dominated by $\Theta(x)\eta_1(\cdot)$ and $e^{1/T}r(T - t, x, \cdot)$ is dominated by $\Theta^*(x)\eta_1(\cdot)$. Then by the dominated convergence theorem, the semigroup property, the equality (11) and Fubini’s theorem, we have

$$
\lim_{t \to \infty} \lim_{T \to \infty} \mathbb{E}_\nu(f(X_t)T_0 > T) = \lim_{t \to \infty} \lim_{T \to \infty} \int_0^\infty \mathbb{E}_\nu(f(X_t), T_0 > T)\nu(dx) \\
= \lim_{t \to \infty} \lim_{T \to \infty} \int_0^\infty \int_0^\infty \int_0^\infty r(t, x, z)f(z)r(T - t, z, y)\mu(dy)\nu(dx) \\
= \lim_{t \to \infty} \lim_{T \to \infty} e^{1/T} \int_0^\infty \int_0^\infty r(T, x, y)\mu(dy)\nu(dx) \\
= \lim_{t \to \infty} \lim_{T \to \infty} \int_0^\infty e^{1/t}r(t, x, z)f(z)\eta_1(z)\eta_1(y)\mu(dy)\nu(dx) \\
= \int_0^\infty \int_0^\infty e^{1/t}r(t, x, z)f(z)\eta_1(z)\eta_1(y)\mu(dy)\nu(dx) \\
= \int_0^\infty \eta_1(z)\nu(dx) \\
= \int_0^\infty \eta_1(z)f(z)\mu(dx) = \int_0^\infty \int_0^\infty f(y)\nu_1(dy).
$$

We now describe the quasi-ergodic behavior of one-dimensional diffusion $X$ killed at 0, when $+\infty$ is an entrance boundary and 0 is an exit boundary. The following result implies that $\nu_1$ is the unique MRQSD of the process.

**Theorem 4.** Assume (H) holds. Then for any $\nu \in \mathcal{P}(0, \infty)$ and any bounded Borel function $f$ on $(0, \infty)$, we have

$$
\lim_{t \to \infty} \mathbb{E}_\nu \left( \frac{1}{t} \int_0^t f(X_s)ds | T_0 > t \right) = \int_0^\infty f(y)\nu_1(dy).
$$

**Proof.** We set $s = qt$. Then by change of variable in the Lebesgue integral, the dominated convergence theorem and Theorem 2 we obtain

$$
\lim_{t \to \infty} \mathbb{E}_\nu \left( \frac{1}{t} \int_0^t f(X_s)ds | T_0 > t \right) = \lim_{t \to \infty} \mathbb{E}_\nu \left( \int_0^1 f(X_{qs})dq | T_0 > t \right) \\
= \lim_{t \to \infty} \int_0^1 \mathbb{E}_\nu(f(X_{qs})| T_0 > t)dq \\
= \int_0^\infty f(y)\nu_1(dy).
$$

5. The stationary distribution of $Q$-process

In this section, we study the existence and uniqueness of the stationary distribution for the conditional limit process $Y$, namely, the $Q$-process, which has been introduced in the section 3.

The following result implies that $\nu_1$ is just the unique stationary distribution of $Q$-process.
**Theorem 5.** Assume (H) holds. Then for any $\nu \in \mathcal{P}(0, \infty)$ and any bounded Borel function $f$ on $(0, \infty)$, we have

$$\lim_{t \to \infty} \mathbb{E}_\nu(f(Y_t)) = \int_0^\infty f(y)\nu_1(dy).$$

**Proof.** By the dominated convergence theorem, Proposition 2, the equality 11 and Fubini’s theorem, we have

$$\lim_{t \to \infty} \mathbb{E}_\nu(f(Y_t)) = \lim_{t \to \infty} \int_0^\infty \int_0^\infty h(t, x, y)f(y)d\nu(dy)dx$$

$$= \lim_{t \to \infty} \int_0^\infty \int_0^\infty e^{\lambda_1 t} \frac{\eta_1(x)}{\eta_1(y)} r(t, x, y)f(y)e^{-Q(y)}dyd\nu(dx)$$

$$= \int_0^\infty \int_0^\infty \eta_1(y)f(y)\mu(dy)d\nu(dx)$$

$$= \int_0^\infty \eta_1(y)f(y)\mu(dy) = \int_0^\infty f(y)\nu_1(dy).$$

□

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