A COMPLEXITY THEOREM FOR THE NOVELLI–PAK–STOYANOVSKII ALGORITHM

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Abstract. We describe two aspects of the behaviour of entries of tabloids during the application of the Novelli–Pak–Stoyanovskii algorithm. We derive two theorems which both contain a generalized version of a conjecture by Krattenthaler and Müller concerning the complexity of the Novelli–Pak–Stoyanovskii algorithm as corollary.

1. Introduction

The Novelli–Pak–Stoyanovskii bijection [3] is known as an elegant way to prove the hook-lengths formula [2] which counts the standard Young tableaux of a given shape $\lambda$. The bijection contains a sorting algorithm. If $\lambda$ contains $n$ cells then this sorting algorithm transforms a permutation of $\{1, 2, \ldots, n\}$ into a standard Young tableau of shape $\lambda$, and each standard Young tableau is hit by the same number of permutations. Thus, this sorting algorithm can be used as a random generation algorithm of standard Young tableaux of shape $\lambda$. Motivated by this observation, Krattenthaler and Müller defined the complexity of this sorting algorithm as the average runtime of the sorting algorithm. They conjectured that the Novelli–Pak–Stoyanovskii algorithm has the same complexity no matter whether it is applied row-wise or column-wise.

We consider a generalized version of the Novelli–Pak–Stoyanovskii algorithm, where the sorting order is given by an arbitrary standard Young tableau. We find that any two algorithms have the same complexity whenever they produce each standard Young tableau the same number of times. Since the row-wise and the column-wise Novelli–Pak–Stoyanovskii algorithms satisfy this condition, the conjecture follows. The proof relies on a recursion for exchange numbers from which one can calculate the complexity.

Motivated by earlier attempts to prove the conjecture of Krattenthaler and Müller, we use this knowledge to derive another surprising result. Namely, that the function that describes the positions of the entries, when they reach their maximal

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distance to the top left corner during the application of the sorting algorithm, fulfills the same property as the complexity.

In Section 2 we review the basic definitions. We present the complexity and the conjecture in Section 3. In Section 4 we derive the recursion for the exchange numbers and the complexity theorem that includes the conjecture. In Section 5 we analyse the extreme positions of the entries during the application of the sorting algorithm, and find a different recursion, which also includes the conjecture but is interesting by itself. Finally, we make a few remarks and give an example in Section 6.

2. THE NOVELLI–PAK–STOYANOVSKII ALGORITHM

In this section we recap the basic definitions and present the Novelli–Pak–Stoyanovskii algorithm.

Let \(n \in \mathbb{N}\), and \(\lambda\) be a partition of \(n\), i.e. a weakly decreasing sequence \((\lambda_1, \lambda_2, \ldots)\) of nonnegative integers such that \(\sum_{i=1}^{\infty} \lambda_i = n\). We identify \(\lambda\) with its Young diagram \(\{(i,j) : i, j \in \mathbb{N}, 1 \leq i, 1 \leq j \leq \lambda_i\}\). We adopt the English convention of visualising a Young diagram by arranging cells like entries of a matrix such that \(\lambda\) appears as a left justified array of rows with the \(i\)-th row containing \(\lambda_i\) cells (see Figure 1). The conjugated partition \(\lambda'\) of \(\lambda\) corresponds to the Young diagram \(\{(j,i) : (i,j) \in \lambda\}\). Alternatively, \(\lambda' = (\lambda'_1, \lambda'_2, \ldots)\) where \(\lambda'_i = \max\{j : \lambda_j \geq i\}\).

A tabloid of shape \(\lambda\) is a map \(T : \lambda \to \mathbb{N}\). We will only consider the case where \(T : \lambda \to \{1, \ldots, n\}\) is a bijection, and we denote the set of all such tabloids by \(\text{SYT}(\lambda)\). We call \(T(x)\) the entry of the cell \(x \in \lambda\). Note that if \(T \in \text{SYT}(\lambda)\) is a tabloid and \(\sigma \in \mathfrak{S}_n\) is a permutation, then \(\sigma \circ T\) is again a tabloid in \(\text{SYT}(\lambda)\). For convenience, we denote \(T(i,j) := T((i,j))\). Furthermore, a tabloid \(T \in \text{SYT}(\lambda)\) is called standard Young tableau if \(T\) is increasing along rows from left to right, and along columns from top to bottom. That is, \(T(i,j) < T(i+1,j)\) and \(T(i,j) < T(i,j+1)\) whenever \((i,j)\) and \((i+1,j)\) respectively \((i,j+1)\) are cells of \(\lambda\). We denote the set of standard Young tableaux by \(\text{SYT}(\lambda)\).

Let \(x = (i,j)\) be a cell in the Young diagram \(\lambda\). We recall the usual definitions of the arm \(a_\lambda(x) := \lambda_i - j\), the leg \(l_\lambda(x) := \lambda'_j - i\), the coarm \(a'_\lambda(x) := i - 1\), and the coleg \(l'_\lambda(x) := j - 1\) as the numbers of cells strictly to the right, below, to the left, and above \(x\). Moreover, we define the hook length as \(h_\lambda(x) := a_\lambda(x) + l_\lambda(x) + 1\) and the height as \(h'_\lambda(x) := a'_\lambda(x) + l'_\lambda(x)\). We denote by \(N^-_\lambda(x) := \{(i-1,j), (i,j-1)\} \cap \lambda\) and \(N^+_\lambda(x) := \{(i+1,j), (i,j+1)\} \cap \lambda\) the sets of left and top respectively lower and right neighbours of \(x\) in \(\lambda\). Lastly, given an entry \(1 \leq a \leq n\) and a tabloid \(T \in \text{SYT}(\lambda)\) we will use the notation \(h'(a, T) := h'_\lambda(T^{-1}(a))\).

Since we asked \(T\) to be a bijection we have \(|\text{SYT}(\lambda)| = |\mathfrak{S}_n| = n!\). On the other hand, it is a classical result that \(f_\lambda := |\text{SYT}(\lambda)|\) is given by the hook-lengths
We have a partition \( \lambda = (5, 4, 2, 1, 1, 1) \), a cell \( x = (2, 3) \in \lambda \) with \( a_\lambda(x) = 1 \), \( l_\lambda(x) = 0 \), \( a'_\lambda(x) = 2 \), \( l'_\lambda(x) = 1 \), \( N^-_\lambda(x) = \{(1, 3), (2, 2)\} \), \( N^+_\lambda(x) = \{(2, 4)\} \), and \( \lambda' = (6, 3, 2, 2, 1) \).

Novelli, Pak and Stoyanovskii prove this formula bijectively in [3]. They define a hook function of shape \( \lambda \) to be a map \( H : \lambda \to \mathbb{Z} \) such that \(-l_\lambda(x) \leq H(x) \leq a_\lambda(x)\) for every \( x \in \lambda \). Given a tabloid \( T \in T(\lambda) \) Novelli, Pak and Stoyanovskii construct a standard Young tableau \( W \in \text{SYT}(\lambda) \) and a hook function \( H \) on \( \lambda \) such that \( T \) can be regained from the pair \( (W, H) \). Since the number of hook functions of shape \( \lambda \) equals \( \prod_{x \in \lambda} h_\lambda(x) \), the hook-lengths formula follows.

Let \( \varphi \) denote the map \( T \mapsto W \) used by Novelli, Pak and Stoyanovskii. This map is given by a simple sorting algorithm which is a variation of the jeu de taquin. At each step the algorithm exchanges the entries of two adjacent cells in \( \lambda \). We will be interested in the average number of steps needed to transform a tabloid into a standard Young tableau. We briefly recap the algorithm, which is illustrated in Figure 2, in the following paragraphs. First however, we note that for all \( W \in \text{SYT}(\lambda) \) we have

\[ |\varphi^{-1}(W)| = \frac{n!}{f_\lambda}. \]

We impose the lexicographic order on the cells of \( \lambda \) by letting \((i, j) \prec (k, l)\) if \( j < l \) or \( j = l \) and \( i < k \). Now, set \( T_0 = T \).

If \( T_s \) is not already a standard Young tableau, the algorithm turns to the maximal cell \( x \) of \( \lambda \) with respect to \( \prec \) such that \( N^+_\lambda(x) \neq \emptyset \) and \( T_s(x) > \min\{T_s(y) : y \in N^+_\lambda(x)\} \). We define a new tabloid \( T_{s+1} := \sigma \circ T_s \), where

\[ \sigma = (T_s(x), \min\{T_s(y) : y \in N^+_\lambda(x)\}) \in \mathfrak{S}_n \]

is a transposition. That is, we exchange the entry of \( x \) with the minimal entry among its lower and right neighbours.

It is clear that the algorithm terminates after yielding a finite sequence \((T_0, \ldots, T_r)\) of \( r + 1 \) tabloids such that \( W = T_r \) is a standard Young tableau.

Before we make a few observations about the nature of the Novelli–Pak–Stoyanovskii algorithm, let us consider a slightly more general setting. Clearly,
We have $W = T_b = (3,4)(1,4)(1,7)(6,5)(2,6)(2,5) \circ T$.

for any linear order on the cells of $\lambda$ an analogous sorting algorithm can be defined. The set of linear orders on $\lambda$ can be identified with the set $T(\lambda)$ in the following way: For each $U \in T(\lambda)$ let

$$x \prec_U y : \iff U(x) < U(y).$$

By this definition $U$ is a standard Young tableau if and only if the corresponding order $\prec_U$ refines both the partial row-wise order (given by $(i,j) \prec (k,l)$ if $i = k$ and $j < l$), and the partial column-wise order (given by $(i,j) \prec (k,l)$ if $i < k$ and $j = l$). The left hand side of Figure 3 shows the standard Young tableau which gives the order used in Figure 2.

From now on we will only consider orders $\prec_U$ with $U \in \text{SYT}(\lambda)$.

Note that there is a unique standard Young tableau $U \in \text{SYT}(\lambda)$ such that $U(i+1,j) = U(i,j) + 1$ whenever $(i,j)$ and $(i+1,j)$ are cells of $\lambda$. We call the induced order the linear column-wise order. This is precisely the ordering used by Novelli, Pak and Stoyanovskii. Analogously, there is a unique $U \in \text{SYT}(\lambda)$ such that $U(i,j+1) = U(i,j) + 1$ whenever $(i,j)$ and $(i,j+1)$ are cells of $\lambda$. We call the induced order the linear row-wise order.

We conclude the following definitions.

**Definition 2.1** (Novelli–Pak–Stoyanovskii algorithm). Let $n \in \mathbb{N}$, $\lambda$ be a partition of $n$, and $U \in \text{SYT}(\lambda)$. For each $T \in T(\lambda)$ let $\varphi_U(T) := (T_0, T_1, \ldots, T_r)$ where the tabloids $T_i$ arise from the algorithm with respect to $\prec_U$ as described above. We call the map $\varphi_U$ the Novelli–Pak–Stoyanovskii algorithm corresponding to $U$.

Accordingly, we denote $\varphi_U : T(\lambda) \to \text{SYT}(\lambda)$, $T \mapsto T_r$. Note that $r$ is the number of steps the Novelli–Pak–Stoyanovskii algorithm needs to sort $T$ and thus depends on the tabloid $T$. To make this explicit, we denote this number by $r_U(T)$. 

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**Figure 2.** We have $W = T_b = (3,4)(1,4)(1,7)(6,5)(2,6)(2,5) \circ T$.

**Figure 3.** The linear column-wise (left) and row-wise (middle) orders on $\lambda = (3,3,1)$, and a mixed order on $\lambda = (4,3,2)$ (right).
Furthermore, we call the algorithm with respect to the linear column-wise order the column-wise Novelli–Pak–Stoyanovskii algorithm (see Figure 2). Analogously, we define the row-wise Novelli–Pak–Stoyanovskii algorithm to be the algorithm with respect to the linear row-wise order.

Remark 2.2. Let $U \in \text{SYT}(\lambda)$, $T_0 \in T(\lambda)$, and $\varphi_U(T) = (T_0, \ldots, T_{r_U(T)})$ given by the corresponding Novelli–Pak–Stoyanovskii algorithm.

Two tabloids $T_{i-1}$ and $T_i$ differ in exactly two neighbouring cells $x$ and $y$. The transition $T_{i-1} \rightarrow T_i$ is given by the transposition $\tau_U(i,T) = (T_i(x), T_i(y))$. It follows that

\begin{equation}
\varphi_U(T) = \tau_U(r_U(T), T) \cdots \tau_U(1,T) \circ T.
\end{equation}

Suppose $\tau_U(i,T) = (a,b)$ for some $1 \leq a < b \leq n$. Then we have

\begin{equation}
h'(a, T_i) = h'(a, T_{i-1}) - 1,
\end{equation}

and equivalently, $h'(b, T_i) = h'(b, T_{i-1}) + 1$.

Next, we want to have a closer look at the behaviour of the entries. To do so we introduce one more notation.

**Definition 2.3.** For every $x = (i, j) \in \lambda$ we refer to the area weakly to the right of $x$ and weakly below $x$ as the **dropping zone** of $x$ in $\lambda$, and denote it by

$$J(x) = \{(k, l) \in \lambda : k \geq i, l \geq j\}.$$ 

For any tabloid $T \in T(\lambda)$ and a subset $I \subseteq \lambda$ we say $T$ is ordered on $I$ if

$$T(x) \leq \min\{T(y) : y \in N^I_\lambda(x)\}$$

for all $x \in I$.

Since $U$ is a standard Young tableau and thus $\prec_U$ refines the partial column-wise and row-wise orders, we have $x \prec_U y$ for all $y \in J(x) - \{x\}$. Hence, if $T_i(x) \neq T_{i-1}(x)$ for a cell $x$, then $T_{i-1}$ must be ordered on $J(x) - \{x\}$.

**Remark 2.4.** Let $x_1 \prec_U x_2 \prec_U \cdots \prec_U x_n$ be the cells of $\lambda$, and $b = T(x_s)$ be the entry of the cell $x_s$. Choose $0 \leq i \leq r_U(T)$ minimal such that $T_i$ is ordered on $\{x_{s+1}, \ldots, x_n\}$, and thus in particular on $(J(x_s) - \{x_s\})$. We distinguish two cases.

Firstly, assume that $T_i$ is ordered on $\{x_s, \ldots, x_n\}$. Then we have $T_i(y) > b$ for all $y \in J(x_s) - \{x_s\}$. Clearly, no entry less than $b$ can be exchanged into $J(x_s)$ thereafter. Moreover, suppose that $b$ is at some later point exchanged with the entry $T_j(x')$, then also $T_j(y) > b$ for all $y \in J(x') - \{x'\}$. It follows that $b$ cannot be exchanged with an entry less than $b$ for the rest of the sorting procedure.

Secondly, assume that $T_i$ is not ordered on $\{x_s\}$. Then $\tau_U(i+1,T) = (a_1,b)$ for some $1 \leq a_1 < b$, where $a_1$ is the entry of a lower or right neighbour of $x_s$. Now again, there are two possibilities. Either $T_{i+1}$ is ordered on $\{x_1, \ldots, x_s\}$, and we are in the situation of the first case, or $\tau_U(i+2,T) = (a_2,b)$ for some $a_1 < a_2 < b$. Note that $a_1 < a_2$ because $T_i$ is ordered on $J(x_s) - \{x_s\}$.

Iterating this argument, we see that all transpositions that exchange the entry $b$ with an entry less than $b$ are processed consecutively, and in increasing order with
respect to the entry less than \( b \). Informally, we also say the entry \( b \) drops, since each such exchange moves \( b \) away from the top left corner of \( \lambda \).

Now, let \( b_s := T(x_s) \). Applying the above observation to the sorting algorithm as a whole, we can divide \((T_1, \ldots, T_r)\) into successive (possibly empty) subsequences \( \varphi_U(x_s, T) \), such that each \( T_j \) belonging to \( \varphi_U(x_s, T) \) differs from \( T_{j-1} \) only by a transposition of \( b_s \) and an entry less than \( b_s \). That is, each subsequence \( \varphi_U(x_s, T) \) describes the dropping of the entry \( b_s \).

Moreover, the length of the sequence \( \varphi_U(x_s, T) \) is given by \( \mu_U(s, T) - \mu_U(s + 1, T) \), where \( \mu_U(s, T) \) denotes the minimal integer \( i \) such that \( T_i \) is ordered on \( \{x_s, \ldots, x_n\} \). Thus, the sequence \( \varphi_U(x_s, T) \) is non-empty if and only if \( \mu_U(s + 1, T) < \mu_U(s, T) \).

Looking back to Figure 2 for an example, we find that \( \mu_U(7, T) = 0, \mu_U(6, T) = 1, \mu_U(5, T) = 1, \mu_U(4, T) = 3, \mu_U(3, T) = 3, \mu_U(2, T) = 4, \) and \( \mu_U(1, T) = 6 = r_U(T) \). Thus, the non-empty sequences \( \varphi_U(x_s, T) \) correspond to the dropping of the fillings \( T(x_2) = 5, T(x_4) = 6, T(x_6) = 7, \) and \( T(x_7) = 4 \).

3. Complexity and the Conjecture of Krattenthaler and Müller

In this section we are going to present the conjecture that was the motivation for the present work. To do so we first have to define the complexity of a Novelli–Pak–Stoyanovskii algorithm.

**Definition 3.1 (Complexity).** Let \( n \in \mathbb{N}, \lambda \) be a partition of \( n \), \( U \in \text{SYT}(\lambda) \) and \( \varphi_U \) the corresponding Novelli–Pak–Stoyanovskii algorithm. The complexity of \( \varphi_U \), denoted by \( C(U) \), is defined to be the average number of transitions in the sequences \( \varphi_U(T) \), where \( T \) ranges over \( T(\lambda) \). That is,

\[
C(U) := \frac{1}{n!} \sum_{T \in T(\lambda)} r_U(T).
\]

**Conjecture 3.2 (Krattenthaler, Müller).** Let \( n \in \mathbb{N}, \lambda \) be a partition of \( n \), and \( U, V \in \text{SYT}(\lambda) \) be the standard Young tableaux defining the linear column-wise and linear row-wise orders on \( \lambda \) respectively. Then we have

\[
C(U) = C(V).
\]

In other words, the row-wise and the column-wise Novelli–Pak–Stoyanovskii algorithms have the same complexity.

**Remark 3.3.** Given a standard Young tableau \( U \in \text{SYT}(\lambda) \) we may define the conjugated standard Young tableau as \( U'(j,i) := U(i,j) \) for all \( (i,j) \in \lambda \). We obtain \( U' \in \text{SYT}(\lambda') \). More precisely, this correspondence defines a bijection between \( \text{SYT}(\lambda) \) and \( \text{SYT}(\lambda') \).

Let \( V \in \text{SYT}(\lambda) \) denote the standard Young tableau defining the linear row-wise order on \( \lambda \), then \( V' \) induces the linear column-wise order on \( \lambda' \). Thereby, also the row-wise algorithm has the property \(|\varphi_V^{-1}(W)| = n!/f_\lambda \) for all \( W \in \text{SYT}(\lambda) \).
Additionally we want to consider orders defined by any standard Young tableau $U$ of the following form. At each step choose column or row. Depending on your choice, assign to the leftmost unlabelled cell of each row or to the topmost unlabelled cell of each column the minimal possible entry (see Figure 3). Each such order induces a sorting algorithm $\varphi_U$ such that $|\varphi_U^{-1}(W)| = n!/f_\lambda$ for all $W \in \text{SYT}(\lambda)$. This can be seen from the fact that during the construction of the hook function after Novelli, Pak and Stoyanovskii a column of the hook function is not altered after the corresponding column of the tabloid has been sorted (see [4, Section 3.10]).

4. The proof

In this section we are going to prove that a fixed entry is exchanged equally often with every greater entry and derive the recursion for the exchange numbers. The conjecture of Krattenthaler and Müller will follow.

From now on let $\varphi_U$ be the Novelli–Pak–Stoyanovskii algorithm corresponding to an arbitrary $U \in \text{SYT}(\lambda)$. First, we observe that the transposition $(a,b)$ may occur at most once while sorting any fixed $T \in T(\lambda)$.

We introduce a function that decides if there is an exchange of two entries at a given position. For $1 \leq a,b \leq n$, $x,y \in \lambda$ and $T \in T(\lambda)$ define

$$m_U(a,b,x,y,T) := \begin{cases} 1 & \text{if } T_{i-1}(x) = a, T_{i-1}(y) = b, T_i(x) = b, \text{ and } T_i(y) = a \text{ for some } 1 \leq i \leq r_U(T), \\ 0 & \text{else}, \end{cases}$$

where $\varphi_U(T) = (T_0, \ldots, T_{r_U(T)})$, and

$$m_U(a,b,x,y) := \sum_{T \in T(\lambda)} m_U(a,b,x,y,T).$$

Obviously $m_U(a,b,x,y,T)$ and $m_U(a,b,x,y)$ both vanish unless $x$ and $y$ are neighbours. Next we define similar functions which just count whether $a$ and $b$ are exchanged during the algorithm without any condition on the involved cells. Let

$$m_U(a,b,T) := \begin{cases} 1 & \text{if there is an } 1 \leq i \leq r_U(T) \text{ such that } \tau_U(i,T) = (a,b), \\ 0 & \text{else}, \end{cases}$$

and

$$m_U(a,b) := \sum_{T \in T(\lambda)} m_U(a,b,T).$$

Furthermore, we define the exchange matrix $M_U := (m_{a,b})_{a,b}$ to be the $n \times n$-matrix with entries $m_{a,b} = m_U(a,b)$ when $1 \leq a < b \leq n$ and $m_{a,b} = 0$ otherwise. The essential insight of our proof is the fact that if one exchanges $a$ and $a+1$ in $T$, then up to the point when both entries $a$ and $a+1$ have dropped, the
tabloids $T_i$ arising in the Novelli–Pak–Stoyanovskii algorithm differ at most by the transposition $(a, a+1)$. We use this fact to prove the following central proposition.

**Proposition 4.1.** Let $n \in \mathbb{N}$, $\lambda$ be a partition of $n$, $U \in \text{SYT}(\lambda)$ a standard Young tableau, and $\varphi_U$ the Novelli–Pak–Stoyanovskii algorithm corresponding to $U$. For all $a, b, c \in \mathbb{N}$ with $1 \leq a < b \leq n$ and $1 \leq a < c \leq n$, and all $x, y \in \lambda$ we have

$$m_U(a, b, x, y) = m_U(a, c, x, y). \quad (4.1)$$

Furthermore, we have the symmetry

$$m_U(a, b, x, y) = m_U(b, a, y, x). \quad (4.2)$$

Finally, we have

$$m_U(a, b) = m_U(a, c). \quad (4.3)$$

Hence, for $1 \leq a < b \leq n$ we denote the exchange numbers by

$$m_U(a) = m_U(a, b),$$

and the local exchange numbers by

$$m_U(a, x, y) = m_U(a, b, x, y).$$

**Proof.** The symmetry in (4.2) is evident. Moreover, (4.3) follows from (4.1) by summation over all pairs of cells $x, y \in \lambda$. To show (4.1) it suffices to consider the case $c = b + 1$. Let $T \in \text{T}(\lambda)$ and $x_1 \prec_U x_2 \prec_U \cdots \prec_U x_n$ be the cells of $\lambda$. Now, choose $1 \leq i, j \leq r_U(T)$ such that $T(x_i) = b$ and $T(x_j) = b + 1$. Set $\sigma = (b, b + 1)$ and $T^* = \sigma \circ T$. Without loss of generality we may assume that $i < j$. For convenience we denote the tabloids that appear during the application of the Novelli–Pak–Stoyanovskii algorithm to $T^*$ by $T_k^*$ and the corresponding transposition by $\tau_k^*$.

Obviously, for $0 \leq k \leq \mu_U(j + 1, T)$ we have $T_k = \sigma \circ T_k^*$, since none of the involved transitions are influenced by the entries of $x_i$ or $x_j$.

Now, we consider the dropping of the entry of $x_j$. Since all entries different from $b$ and $b + 1$ are either less than both $b$ and $b + 1$ or greater than both $b$ and $b + 1$, the dropping path of the entry of $x_j$ does not depend on whether it is $b$ or $b + 1$. Hence, also for $\mu_U(j + 1, T) < k \leq \mu_U(j, T)$ we have $T_k = \sigma \circ T_k^*$.

For the same reason the dropping paths in $T$ and $T^*$ are the same for the initial entries of $x_i$, for all $x_i \prec_U x_1 \prec_U x_j$. Hence, also for $\mu_U(j, T) < k \leq \mu_U(i + 1, T)$ we have $T_k = \sigma \circ T_k^*$.

Finally, we need to treat the dropping of the initial entry of $x_i$. Since $T_{\mu_U(i + 1, T)} = \sigma \circ T_{\mu_U(i + 1, T)}$ the dropping paths will again agree, unless $b$ and $b + 1$ are exchanged at some point (i.e., if $\sigma$ occurs as transition). The only situation where this may happen, is if $\tau_{\mu_U(i, T)+1}^* = \tau_{\mu_U(i, T^*)}^* = \sigma$ (i.e., the very last transition of the dropping of $b + 1$ in $T^*$ may be $\sigma$). Hence, for $\mu_U(i + 1, T) < k \leq \mu_U(i, T)$ we have once more $T_k = \sigma \circ T_k^*$. 

Summarizing, $T_k = \sigma \circ T_k^*$ for all $0 \leq k \leq \mu_U(i, T)$. The rest of the sequences $\varphi_U(T)$ and $\varphi_U(T^*)$ may differ heavily. However, we know that all transitions $\tau_k$ and $\tau_k^*$ that exchange $b$ or $b+1$ with an entry $a < b$ happen solely up to the index $\mu_U(i, T)$. Hence, the dropping path of $b$ in $T$ agrees exactly with the dropping path of $b+1$ in $T^*$ and the dropping path of $b+1$ in $T$ agrees exactly with the dropping path of $b$ in $T^*$. Therefore, we have

$$m_U(a, b, x, y; T) = m_U(a, b+1, x, y; T^*)$$

and

$$m_U(a, b+1, x, y; T) = m_U(a, b, x, y; T^*).$$

Since $T \mapsto \sigma \circ T$ is an involution, summation over $T \in T(\lambda)$ yields (4.1), and the proof is complete. \hfill \Box

To state and prove the recursion for the exchange numbers we need some more notation.

**Definition 4.2.** Let $n \in \mathbb{N}$, $\lambda$ be a partition of $n$, $U \in \text{SYT}(\lambda)$ and $\varphi_U$ the Novelli–Pak–Stoyanovskii algorithm corresponding to $U$. For $W \in \text{SYT}(\lambda)$ we define the **multiplicity** of $W$ with respect to $U$ as

$$z_U(W) := |\{T \in T(\lambda) : \phi_U(T) = W\}|$$

and the **distribution vector** of $U$ as

$$Z_U := (z_U(W))_{W \in \text{SYT}(\lambda)}.$$

Moreover, we call $\varphi_U$ **uniformly distributed** if all entries of $Z_U$ agree. That is, for all $W \in \text{SYT}(\lambda)$ we have

$$z_U(W) = \frac{n!}{f_\lambda}.$$

Before the application of the Novelli–Pak–Stoyanovskii algorithm every entry has a distance to the top left corner. Summing up these distances over all tabloids we define the **total initial height** of the entry $b$ as

$$\alpha_\lambda(b) := \sum_{x \in \lambda} h'(b, T).$$

After the application the entry has taken its terminal position in a standard Young tableau with a (different) distance to the top left corner. Summing up these distances over all initial tabloids we define the **total terminal height** of the entry $b$ as

$$\omega_U(b) := \sum_{x \in \lambda} h'(b, \varphi_U \circ T).$$
Remark 4.3. The above parameter $\alpha_\lambda(b)$ does not depend on $b$, i.e.

$$\alpha_\lambda(b) = \sum_{T \in T(\lambda)} h'(b, T) = \sum_{T \in T(\lambda)} \left( \sum_{x \in \lambda, T(x) = b} h'_\lambda(x) \right) = \sum_{x \in \lambda} \left( \sum_{T \in T(\lambda), T(x) = b} h'_\lambda(x) \right) = (n - 1)! \sum_{x \in \lambda} h'_\lambda(x).$$

Hence, we denote it by $\alpha_\lambda$. Note that we could calculate $\alpha_\lambda$ also as a sum of hook lengths $\alpha_\lambda = (n - 1)! \left( -n + \sum_{x \in \lambda} h_\lambda(x) \right)$ or even in terms of $\lambda_i$ as $\alpha_\lambda = (n - 1)! \sum_{i \in \mathbb{N}} \left( \binom{\lambda_i}{2} + (i - 1)\lambda_i \right)$.

Furthermore, $\omega_U(b)$ does not depend on $\varphi_U$ but rather on $Z_U$, i.e.

$$\omega_U(b) = \sum_{T \in T(\lambda)} h'(b, \varphi_U \circ T) = \sum_{W \in \text{SYT}(\lambda)} \left( \sum_{T \in T(\lambda), \varphi_U(T) = W} h'(b, W) \right) = \sum_{W \in \text{SYT}(\lambda)} z_U(W) h'(b, W).$$

We are now in good shape to derive the predicted recursion.

**Theorem 4.4 (Exchange numbers).** Let $n \in \mathbb{N}$, $\lambda$ be a partition of $n$, $U \in \text{SYT}(\lambda)$ and $\varphi_U$ the Novelli–Pak–Stoyanovskii algorithm corresponding to $U$. Then for $1 \leq b \leq n$ we have the recursion

$$(n - b) m_U(b) = \alpha_\lambda - \omega_U(b) + \sum_{a=1}^{b-1} m_U(a).$$

**Proof.** From (2.2) we conclude

$$\alpha_\lambda + \sum_{a=1}^{b-1} m_U(a, b) - \sum_{c=b+1}^{n} m_U(b, c) = \omega_U(a),$$

which says that the starting height plus the steps away from the top left corner minus the steps towards the top left corner equals the total terminal height.
we get
\[ \alpha \lambda + \sum_{a=1}^{b-1} m_U(a) - \sum_{c=b+1}^{n} m_U(b) = \omega_U(a), \]
hence,
\[ \alpha \lambda + \sum_{a=1}^{b-1} m_U(a) - (n - b)m_U(b) = \omega_U(a). \]
This finishes the proof. \( \square \)

Note that for \( b = 1 \) the sum on the right hand side of (4.4) is empty. Moreover, \( \omega_U(1) = 0 \) since the entry 1 will always end up in the top left corner of the standard Young tableau. Hence, the recursion yields its own initial condition
\[ m_U(1) = \frac{\alpha \lambda}{n - 1}. \]
Therefore, we can recursively compute the exchange matrix. We observe that the exchange matrix depends only on \( \lambda \) and \( Z_U \) rather than on \( U \). We obtain the following corollaries.

**Corollary 4.5** (Complexity Theorem). In the situation of Theorem 4.4 we can calculate the complexity of \( \varphi_U \) as
\[ C(U) = \frac{1}{n!} \sum_{a=1}^{n} (n - a)m_U(a). \]
Moreover, let \( V \in \text{SYT} \) be such that \( Z_U = Z_V \). Then we have
\[ C(U) = C(V). \]

**Corollary 4.6.** In the situation of Theorem 4.4 consider \( \varphi_U \) to be uniformly distributed. Then we obtain the same recursion with the specialization
\[ \omega_U(b) = \frac{n!}{f^\lambda} \sum_{W \in \text{SYT}(\lambda)} h'(b, W). \]

Since the row-wise and column-wise Novelli–Pak–Stoyanovskii algorithm are both uniformly distributed, we have in particular that the conjecture of Krattenthaler and Müller holds.

5. **Intermediate Targets of Entries**

In this section we are going to define the drop function counting the tabloids for which a certain cell is the one farthest from the top left corner to contain a certain entry during the application of the Novelli–Pak–Stoyanovskii algorithm for a given \( U \). We derive another recursion implying that also the drop function does not depend on \( U \) but only on \( Z_U \).
As pointed out before, during the application of a Novelli–Pak–Stoyanovskii algorithm each entry first raises its height to a maximum and then lowers it to its final height. Therefore, for each entry \(1 \leq b \leq n\) and tabloid \(T \in T(\lambda)\) there is a unique cell of \(\lambda\) with maximal height which contains \(b\) at some point during the sorting of \(T\). We denote this maximal height by \(\beta_U(b,T)\).

Let \(\varphi_U(T) = (T_0, \ldots, T_{r_U(T)})\), and suppose \(b = T(x_s)\) for some \(1 \leq s \leq n\), where \(x_1 \prec_U \cdots \prec_U x_n\) are the cells of \(\lambda\) ordered with respect to \(U\), then the maximal height is given by \(\beta_U(b,T) = h'(b, T_{\mu_U(s,T)})\).

Summation over \(T\) yields the statistic \(\beta_U(b) := \sum_{T \in T(\lambda)} \beta_U(b,T)\).

Using the notation from Theorem 4.4 we can immediately derive the relation

\[
C(U) = \frac{1}{n!} \sum_{b=1}^{n} (\beta_U(b) - \alpha_\lambda).
\]

Let \(U, V \in \text{SYT}(\lambda)\) be the standard Young tableaux defining the row-wise and the column-wise Novelli–Pak–Stoyanovskii algorithms, then Conjecture 3.2 would follow from \(\beta_U(b) = \beta_V(b)\) for all entries \(1 \leq b \leq n\). This approach naturally raises the question to which cells a given entry actually drops. The rest of this section is devoted to answering this question.

**Definition 5.1 (Drop function).** Let \(n \in \mathbb{N}\), \(\lambda\) be a partition of \(n\) and \(U \in \text{SYT}(\lambda)\) define a Novelli–Pak–Stoyanovskii algorithm \(\varphi_U\). Given an entry \(1 \leq b \leq n\), a cell \(x \in \lambda\) and a tabloid \(T \in T(\lambda)\) we define

\[
d_U(b,x,T) := \begin{cases} 
1 & \text{if } h'_\lambda(x) = \beta_U(b,T) \text{ and } T_i(x) = b \\
& \text{for some } 1 \leq i \leq r_U(T), \\
0 & \text{else},
\end{cases}
\]

where \(\varphi_U(T) = (T_0, \ldots, T_{r_U(T)})\). The **drop function** \(d_U(b,x)\) counts how often the entry \(b\) drops to the cell \(x\), when all tabloids in \(T(\lambda)\) are considered, i.e.,

\[
d_U(b,x) := \sum_{T \in T(\lambda)} d_U(b,x,T).
\]

Note that if \(b\) is never exchanged with a label less than \(b\), then it drops to its starting position. Thus, \(\sum_{x \in \lambda} d_U(b,x) = n!\) for all \(b\), and in particular \(d_U(1,x) = (n-1)!\) for all \(x \in \lambda\).

In order to calculate the drop function we need to define intermediate quantities which are suitable to construct a recursion.
Definition 5.2 (Signed exit number). With the notation of Theorem 4.4 the signed exit number of the entry $b$ at the cell $x$ is defined as

$$\Delta_U(b, x) := \sum_{y \in N^-(x)} m_U(b, x, y) - \sum_{y \in N^+(x)} m_U(b, y, x).$$

Furthermore, let

$$\omega_U(b, x, T) := \begin{cases} 1 & \text{if } \varphi_U(T)(x) = b, \\ 0 & \text{else.} \end{cases}$$

Then

$$\omega_U(b, x) := \sum_{T \in \Pi(x)} \omega_U(b, x, T)$$

counts the number of tabloids in which the terminal position of $b$ after the application of the sorting algorithm is $x$.

Theorem 5.3 (Signed exit numbers). Let $n \in \mathbb{N}$, $\lambda$ be a partition of $n$ and $U \in \text{SYT}(\lambda)$. For all cells $x \in \lambda$ and for all entries $1 \leq b \leq n$ we have the recursion

$$(n - b) \Delta_U(b, x) = (n - 1)! - \omega_U(b, x) + \sum_{a=1}^{b-1} \Delta_U(a, x).$$

Proof. Let $N(x) = N^-_\lambda(x) \cup N^+_\lambda(x)$ be the set of adjacent cells of $x$ in $\lambda$. Fix a label $b$ and a cell $x$. The number of tabloids $T$ such that $\varphi_U(T)(x) = b$ is obtained by adding the number of tabloids in which $b$ starts in $x$ and the number of times $b$ is exchanged to $x$, and subtracting the number of times $b$ is exchanged away from $x$. That is,

$$\omega_U(b, x) = (n - 1)! + \sum_{b \neq a} \sum_{y \in N(x)} \left( m_U(b, a, y, x) - m_U(b, a, x, y) \right).$$

Using Propostion 4.1 the double sum in the above equation becomes
\[
\sum_{a=1}^{b-1} \left( \sum_{y \in N^-_\lambda(x)} m_U(a, b, x, y) - \sum_{y \in N^+_\lambda(x)} m_U(a, b, y, x) \right) + \\
+ \sum_{c=b+1}^{n} \left( \sum_{y \in N^+_\lambda(x)} m_U(a, b, x, y) - \sum_{y \in N^-_\lambda(x)} m_U(a, b, y, x) \right) = \\
= \sum_{a=1}^{b-1} \left( \sum_{y \in N^-_\lambda(x)} m_U(a, x, y) - \sum_{y \in N^+_\lambda(x)} m_U(a, x, y) \right) + \\
+ \sum_{c=b+1}^{n} \left( \sum_{y \in N^+_\lambda(x)} m_U(b, x, y) - \sum_{y \in N^-_\lambda(x)} m_U(b, x, y) \right) = \\
= \sum_{a=1}^{b-1} \Delta_U(a, x) - (n - b) \Delta_U(b, x).
\]

This completes the proof. \(\square\)

As before, this recursion generates its own initial condition. Hence, we can recursively compute \(\Delta_U(a, x)\) and use it to determine the drop function.

**Corollary 5.4 (Drop Theorem).** With the notation of Theorem 5.3, the drop function can be derived from the signed exit numbers as

\[
d_U(b, x) = (n - 1)! + \sum_{a=1}^{b-1} \Delta_U(a, x).
\]

Furthermore, if \(U, V \in \text{SYT}(\lambda)\) define equidistributed Novelli–Pak–Stoyanovskii algorithms, i.e. \(Z_U = Z_V\), then for all \(x \in \lambda\) and \(1 \leq b \leq n\) we have \(d_U(b, x) = d_V(b, x)\).

**Proof.** The entry \(b\) drops to a cell \(x\) either if it enters \(x\) from \(N^-_\lambda(x)\) or starts there, and does not leave it towards \(N^+_\lambda(x)\). Summing over \(T \in \text{T}(\lambda)\) we find that \(b\) starts at \(x\) exactly \((n - 1)!\) times, it enters \(x\)

\[
\sum_{T \in \text{T}(\lambda)} \sum_{a=1}^{b-1} \sum_{y \in N^-_\lambda(x)} m_U(b, a, y, x; T)
\]
times from the left or from above, and leaves it

\[
\sum_{T \in \text{T}(\lambda)} \sum_{a=1}^{b-1} \sum_{y \in N^+_\lambda(x)} m_U(b, a, x, y; T)
\]
times to the right or below. Hence, we get

\[ d_U(b, x) = (n - 1)! + \sum_{a=1}^{b-1} \left( \sum_{y \in N^-_\lambda(x)} m_U(a, b, x, y) - \sum_{y \in N^+_\lambda(x)} m_U(a, b, y, x) \right) \]

\[ = (n - 1)! + \sum_{a=1}^{b-1} \left( \sum_{y \in N^-_\lambda(x)} m_U(a, x, y) - \sum_{y \in N^+_\lambda(x)} m_U(a, y, x) \right) \]

\[ = (n - 1)! + \sum_{a=1}^{b-1} \Delta_U(a, x). \]

The second claim follows from the fact that \( \omega_U(b, x) \) depends only on \( Z_U \) rather than on \( U \). Thus, by Theorem 5.3, also the signed exit numbers and the drop function depend only on \( Z_U \). \( \square \)

6. Remarks

The above two sections both come to the conclusion that the objects of study (the complexity, the signed exit number and the drop function) depend on \( Z_U \) rather than on \( U \). The distribution vector implicitly appeared earlier in a result by Fischer [1] which we want to quote here as a remark.

Remark 6.1 (Fischer 2002). Let \( n \in \mathbb{N}, \lambda \) be a partition of \( n \) and \( (z_U(W))_{U,W \in \text{SYT}(\lambda)} \) be the matrix of multiplicities of \( W \) with respect to \( U \). Then \( (z_U(W))_{U,W \in \text{SYT}(\lambda)} \) is symmetric.

Secondly, we want to consider possible generalisations.

Remark 6.2. Similar as in [1], our arguments can be generalised to the skew and shifted case. But since the row-wise and column-wise orders may yield different distribution vectors, the conjecture of Krattenthaler and Müller does not apply.

Thirdly, we mention a property of the signed exit number.

Remark 6.3. The signed exit number \( \Delta_U(b, x) \) measures the difference between how often \( b \) leaves \( x \) towards \( N^-_\lambda(x) \) and how often it enters \( x \) from \( N^+_\lambda(x) \). Thus is measures how much stronger (or weaker if it is negative) it is as source for pushing \( b \) towards the top left corner, as it is as sink for \( b \) on its way there. For our purposes (namely computing the dropping function) it would be enough to consider it as a formal quantity. Nevertheless, we want to remark, that summing over all \( x \in \lambda \) it counts every exchange of \( b \) once with positive and once with negative sign. Hence, we have for all \( 1 \leq b \leq n \)

\[ \sum_{x \in \lambda} \Delta_U(b, x) = 0. \]

Finally, we want to compute the values of the drop function in a special case.
Example 6.4 (The drop function for a single lined Young diagram). Let \( n \in \mathbb{N} \) and consider the partition \( \lambda = (n) \). We can treat a cell \( x \) as a single index \( 1 \leq x \leq n \). Moreover, there is only one standard Young tableau \( U \in \text{SYT}(\lambda) \). Hence, we shall index all functions by \( n \) instead of \( U \). To calculate the drop function explicitly without using the signed exit numbers we introduce the **partial drop function** counting the number of tabloids in which the entry \( a \) drops from the starting position \( x \) to \( y \),

\[
d_n(a, x, y) = |\{ T \in T(\lambda) : T(x) = a, d_n(a, y, T) = 1 \}|.
\]

First, note that we may give the partial drop functions explicitly as

\[
d_n(a, x, y) = \binom{x-1}{y-a} \binom{n-x}{n-y} (a-1)! (n-a)!.
\]

Thus, the desired result can be obtained using the Chu–Vandermonde identity. Instead we shall derive a combinatorial recursion

\[
d_n(a+1, x, y) = \frac{y-a}{n-a} d_n(a, x, y) + \frac{n-y+1}{n-a} d_n(a, x, y-1).
\]

Exchanging \( a \) and \( a+1 \) defines a bijection between the tabloids in which \( a+1 \) drops from \( x \) to \( y \) and the tabloids in which either \( a \) drops from \( x \) to \( y \) and \( a+1 \) is to the left of \( a \), or \( a \) drops from \( x \) to \( y-1 \) and \( a+1 \) starts to its right.

The left summand corresponds to the first case since here \( y-a \) of the \( n-a \) entries larger than \( a \) must start to its left. Analogously, the right summand corresponds to the latter case since there \( n-y+1 \) entries larger than \( a \) start to its right.

Summation over \( x \) yields the recursion

\[
d_n(a+1, y) = \frac{y-a}{n-a} d_n(a, y) + \frac{n-y+1}{n-a} d_n(a, y-1).
\]

Since the entry 1 always drops to its initial position, we have the initial condition

\[
d_n(1, x) = (n-1)!
\]

for all \( x \in \lambda \). A straightforward calculation yields

\[
d_n(a, x) = \begin{cases} 
\frac{n!}{n-a+1} & \text{if } x \geq a, \\
0 & \text{else.}
\end{cases}
\]

Remark 6.5. The above example might raise the hope of finding a recursion directly for the drop function for a general \( \lambda \) that uses partial drop functions rather than signed exit numbers. For at least two reasons this seems unlikely.

- The partial drop functions for a general \( \lambda \) do depend on \( U \) and not only on \( Z_U \).
- The above calculation is based on the fact that \( a+1 \) drops to \( y+1 \) if it the exchange with \( a \) moves \( a+1 \) to a smaller cell (with respect to some standard Young tableau). In the general case it could not just drop to
Nevertheless, we observe that for $\lambda = (n)$ we have
\[
\gcd\{d_n(a, x) : 1 \leq a, x \leq n\} = \gcd\left\{\frac{n!}{n - a + 1} : 1 \leq a \leq n\right\} = \frac{n!}{\operatorname{lcm}\{1, \ldots, n\}}.
\]
Surprisingly, in a computer experiment for the row-wise Novelli–Pak–Stoyanovskii algorithm applied to some tabloids of different small partitions this equality held as well. For some others the greatest common divisor was still a factor of the right hand side. Hence, we close with the following conjecture.

**Conjecture 6.6.** Let $n \in \mathbb{N}$, $\lambda$ be a partition of $n$ and $U \in \operatorname{SYT}(\lambda)$ such that the corresponding Novelli–Pak–Stoyanovskii algorithm is uniformly distributed. Then
\[
\frac{n!}{\operatorname{lcm}\{1, \ldots, n\}} \cdot \gcd\{d_U(a, x) : 1 \leq a \leq n, x \in \lambda\} \in \mathbb{N}.
\]

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