Generalization of the Glasser - Manna - Oloa integral and some new integrals of similar type

N.M. Vildanov

Abstract As was shown in the previous works by other authors, Glasser - Manna - Oloa integral arise in the study of the Laplace transform of the dilogarithm function and can be evaluated in a closed form. In this article, we give a one parametric generalization of the Glasser - Manna - Oloa integral. The method employed in the course of derivation allows to obtain some new integrals of similar type. These include also integral representation of the Hurwitz zeta function and some beautiful formulae involving the logarithm of the gamma function of the argument $-ix + \ln(2\cos x)$.

Keywords Definite integrals · Dilogarithm · Hurwitz zeta function · Laplace transform

Mathematics Subject Classification (2000) MSC 30E20

1 Motivation and results

Recently, there has been some interest in the integrals of the form

$$M(a) = \frac{4}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{x^2}{x^2 + \ln^2(2e^{-a\cos x})} \, dx.$$ \hspace{1cm} (1)

[1,2,3,4,5]. In [1], it was shown that $M(a)$ is related to the Laplace transform of the dilogarithm function $\psi(s+1)$

$$L(a) = \int_{0}^{\infty} e^{-as} \psi(s+1) \, ds.$$ \hspace{1cm} (2)

N.M. Vildanov
I.E.Tamm Department of Theoretical Physics, P.N.Lebedev Physics Institute - 119991 Moscow, Russia
Tel.: none
Fax: none
E-mail: niyazvil@list.ru
through

\[ L(a) = M(a) - \frac{\gamma}{a} - \frac{\ln(e^a - 1)}{1 - e^{-a}}H(\ln2 - a), \quad (3) \]

where \( H \) is the unit step function and \( \gamma \) is the Euler’s constant. The value \( M(0) \) can be evaluated in a closed form and reads

\[ M(0) = \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2}{x^2 + \ln^2(2\cos x)} \, dx = \frac{1}{2} (1 - \gamma + \ln2\pi). \quad (4) \]

In this article, we give a proof of the following formula

\[ \int_{-\pi/2}^{\pi/2} x (1 + e^{-2ix})^\beta \ln \left( 1 + e^{-2ix} \right) \, dx = \frac{\pi}{8} \left( 1 + \ln2\pi - \gamma(2\beta + 1) - 2\ln\Gamma(\beta + 1) \right), \quad (5) \]

provided \( \text{Re} \beta > -1 \). This is a one-parametric generalization of the integral \( (4) \), which directly follows from \( (5) \) when \( \beta = 0 \).\(^1\)

Using the method of derivation of \( (5) \) the following integral representation of the Hurwitz zeta function

\[ I(\alpha, \beta) = \int_{-\pi/2}^{\pi/2} (1 + e^{-2ix})^\beta \ln^{\alpha} (1 + e^{-2ix}) \, dx = -\frac{\pi}{\Gamma(-\alpha)} \zeta(\alpha + 1, \beta + 1) \quad (6) \]

can be obtained. \( (6) \) is valid for \( \beta > -1 \) and arbitrary complex \( \alpha \); if \( \text{Re} \beta = -1 \), then \( \text{Re} \alpha < -1 \). For \( \alpha = -1 \), \( (6) \) reduces to

\[ \int_{-\pi/2}^{\pi/2} (1 + e^{-2ix})^\beta \ln (1 + e^{-2ix}) \, dx = \frac{\pi}{2} \left( 1 + 2\beta \right) \quad (7) \]

and for \( \beta = 0 \) it is equivalent to the integral representation of the Riemann zeta function

\[ \int_{-\pi/2}^{\pi/2} \ln^{\alpha-1} (1 + e^{-2ix}) \, dx = -\frac{\pi \zeta(\alpha)}{\Gamma(1 - \alpha)} \quad (8) \]

Integral in \( (8) \) converges for arbitrary complex \( \alpha \).

The following integrals can be obtained from \( (7) \) and \( (8) \):

\[ \int_{0}^{\pi/2} \frac{\ln(2\cos x)}{x^2 + \ln^2(2\cos x)} \, dx = \frac{\pi}{4}, \quad (9) \]

\[ \int_{0}^{\pi/2} \ln \left[ x^2 + \ln^2(2\cos x) \right] \, dx = 0, \quad (10) \]

\[ \int_{0}^{\pi/2} \frac{1}{x^2 + \ln^2(2\cos x)} \, dx - 2 \int_{0}^{\pi/2} \frac{x^2}{[x^2 + \ln^2(2\cos x)]^2} \, dx = \frac{\pi}{24}. \quad (11) \]

Putting \( \beta = 0 \) in \( (7) \), we obtain \( (9) \). \( (10) \) is obtained from \( (8) \) differentiating by \( \alpha \) at \( \alpha = 1 \) (see also \( (12) \)). Assuming \( \alpha = -1 \) in \( (8) \) and taking into account the value of the Riemann zeta function \( \zeta(-1) = -\frac{1}{12} \) yields \( (11) \).\(^1\)

\(^1\) It is easy to check that \( \frac{1}{2i} \int_{-\pi/2}^{\pi/2} \frac{x}{\ln(1 + e^{-2ix})} \, dx = \int_{0}^{\pi/2} \frac{x}{x^2 + \ln^2(2\cos x)} \, dx \).
In the light of formula (10) it is natural to study the integrals

\[ f(a) = \int_0^\infty \ln \left( 1 + x^2 \right) \, dx = \frac{\pi a}{e^b - 1}, \quad (12) \]

\[ g(a) = \int_0^\infty \ln \left( \frac{1}{1 - e^{-2x}} \right) \, dx = \pi \left( 1 - \frac{1}{a} \right), \quad (13) \]

where \( b = \min \{a, \ln 2\} \). Surprisingly they can be evaluated for any real \( a \).

From (12), the following identities can be deduced

\[ \int_0^\infty \frac{\ln \Gamma(1 - e^{-2x})}{\ln \left( 1 - e^{-2x} \right)} \, dx = \frac{\pi}{2} \ln \prod_{n=1}^\infty (1 - r^n), \quad 0 < r < \frac{1}{2} \quad (14) \]

\[ \int_0^\infty \ln \left| \Gamma(c + ix - \ln(2 \cos x)) \right| \, dx = \frac{\pi}{2} \ln \Gamma(c), \quad c \geq \ln 2 \quad (15) \]

In the rest of the article we give the proofs of the formulas listed above, namely, (5), (6), (12), (15).

2 Proofs of the new results presented in section 1

Proof of formula (5). Let us make the substitution \( y = \ln (1 + e^{-2ix}) \) in the integral

\[ I(\beta) = \frac{1}{2i} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x (1 + e^{-2ix})^\beta \ln (1 + e^{-2ix}) \, dx. \]

The result is

\[ I(\beta) = \frac{1}{8i} \int_{-\infty}^{0+} \ln \left( e^\nu - 1 \right) \frac{e^{\nu(\beta+1)}}{e^\nu - 1} \, dy. \quad (16) \]

This integral should be understood in the following sense: there is a cut along the line \((-\infty, 0]\) in the complex \( y \) plane; the path extends from \(-\infty\), circumvents the origin in the anticlockwise direction and goes back to \(-\infty\) on the opposite side of the cut (Hankel contour; see, e.g., (6)). It is also assumed that the path does not contain the points \( y = \pm 2\pi in, \ n = 1, 2, 3, \ldots \), which are the poles of \( \frac{1}{e^\nu - 1} \). To be more specific, we will assume that the contour is composed of three parts: \( L_1 \) – the line \((-\infty, \epsilon)\) in the lower side of the cut, the circle \( I \) of radius \( \epsilon \to 0 \) traversed in the anticlockwise direction and \( L_2 \) – the line \((\infty, \epsilon)\) on the upper side of the cut. Such deformation of the path is allowed, since no poles are intercepted. The logarithm is \( \ln \left( e^\nu - 1 \right) = \ln |e^\nu - 1| + \pi i \) on the lower and upper sides of the cut, respectively.

Let us calculate the following two integrals

\[ I_{L_1}(\beta) = \frac{1}{8i} \int_{L_1+L_2} \frac{\ln \left( e^\nu - 1 \right) e^{\nu(\beta+1)}}{y} \frac{e^{\nu(\beta+1)}}{e^\nu - 1} \, dy, \quad (17) \]

\[ I_{L_2}(\beta) = \frac{1}{8i} \int_{L_2} \frac{\ln \left( e^\nu - 1 \right) e^{\nu(\beta+1)}}{y} \frac{e^{\nu(\beta+1)}}{e^\nu - 1} \, dy \quad (18) \]
separately. First, we rewrite \( I_L(\beta) \) as follows

\[
I_L(\beta) = -\frac{\pi}{4} \int_{\epsilon}^{\infty} \frac{e^{-\beta y}}{y^{(\epsilon - 1)}} \, dy = -\frac{\pi}{4} \int_{\epsilon}^{\infty} \frac{e^{-\beta y}}{y^{\epsilon - 1}} \left( 1 + \frac{1}{y \left[ 1 + y \left( \frac{1}{2} + \beta \right) \right]} \right) \frac{\pi}{4} \int_{\epsilon}^{\infty} \frac{dy}{y^{2 \left[ 1 + y \left( \frac{1}{2} + \beta \right) \right]}}. \tag{19}
\]

The first integral in the second line of (19) converges and therefore is a constant up to terms \( O(\epsilon) \). Asymptotic form of the second integral for small \( \epsilon \) can be easily found and we obtain

\[
I_L(\beta) = -\frac{\pi}{4} \int_{0}^{\infty} \frac{dy}{y} \left\{ e^{-\beta y} \left( 1 + \frac{1}{y \left[ 1 + y \left( \frac{1}{2} + \beta \right) \right]} \right) \right\} - \frac{\pi}{4} \left( \beta + \frac{1}{2} \right) \ln \frac{\beta + 1}{2} + O(\epsilon). \tag{20}
\]

Then we rewrite \( I_G(\beta) \) making the substitution \( y = \epsilon e^{i\phi} \) as

\[
I_G(\beta) = \frac{1}{8} \int_{-\pi}^{\pi} d\phi \, e^{-i\phi} \frac{1 + \left( \beta + \frac{1}{2} \right) \epsilon e^{i\phi}}{\epsilon} \left( \ln \epsilon + i\phi + \frac{1}{2} \epsilon e^{i\phi} \right) + O(\epsilon)
= \frac{\pi}{4} \left( \frac{1}{\epsilon} + \left( \beta + \frac{1}{2} \right) \ln \frac{\beta + 1}{2} \right) + O(\epsilon). \tag{21}
\]

Summing up \( I_L(\beta) \) and \( I_G(\beta) \) and assuming \( \epsilon \to 0 \), we obtain

\[
I(\beta) = -\frac{\pi}{4} \int_{0}^{\infty} \frac{dy}{y} \left\{ e^{-\beta y} \left( 1 + \frac{1}{y \left[ 1 + y \left( \frac{1}{2} + \beta \right) \right]} \right) \right\} + \frac{\pi}{8} \left( \beta + \frac{1}{2} \right) \ln \left( \beta + \frac{1}{2} \right). \tag{22}
\]

Thus, we reduced the task of calculation of the initial integral to a much easier task of calculation of the integral

\[
J(\beta) = \int_{0}^{\infty} \frac{dy}{y^{\epsilon - 1}} \left\{ e^{-\beta y} \left( 1 + \frac{1}{y \left[ 1 + y \left( \frac{1}{2} + \beta \right) \right]} \right) \right\}, \tag{23}
\]

which we seek as the limit

\[
J(\beta) = \lim_{\alpha \to 0} \int_{0}^{\infty} \frac{dy}{y^{\epsilon - \alpha}} \left\{ e^{-\beta y} \left( 1 + \frac{1}{y \left[ 1 + y \left( \frac{1}{2} + \beta \right) \right]} \right) \right\}, \quad \alpha \to 0. \tag{24}
\]

Since [6]

\[
\int_{0}^{\infty} \frac{dy}{y^{\epsilon - \alpha} \epsilon^{\epsilon - 1}} = \Gamma(\alpha) \zeta(\alpha, \beta + 1) \tag{25}
\]
and [6]
\[ \int_{0}^{\infty} \frac{dy}{y^{2-\alpha} + 1 + \frac{1}{y^2 + \beta}} = - \left( \beta + \frac{1}{2} \right)^{1-\alpha} \frac{\pi}{\sin \pi \alpha}. \] (26)
we obtain
\[ J(\beta) = \lim_{\alpha \to 0} \left\{ \Gamma(\alpha) \zeta(\alpha, \beta + 1) + \left( \beta + \frac{1}{2} \right)^{1-\alpha} \frac{\pi}{\sin \pi \alpha} \right\}, \] (27)
Using [6]
\[ \zeta(0, \beta + 1) = - \frac{1}{2} - \beta, \quad \frac{d}{d\alpha} \zeta(\alpha, \beta + 1) \bigg|_{\alpha=0} = \ln \Gamma(\beta + 1) - \frac{1}{2} \ln 2 \pi, \] (28)
we come to
\[ J(\beta) = \gamma \left( \beta + \frac{1}{2} \right) + \ln \Gamma(\beta + 1) - \frac{1}{2} \ln 2 \pi - \left( \beta + \frac{1}{2} \right) \ln \left( \beta + \frac{1}{2} \right). \] (29)
Substitution of (29) into (22) yields the desired result (5).

**Corollary: proof of the formula** [6]. After substitution \( y = \ln \left( 1 + e^{-2i \alpha} \right) \) in (6), we obtain
\[ I(\alpha, \beta) = \frac{1}{2i} \int_{-\infty}^{0+} \frac{e^{(\beta + 1) y} - 1}{y} \, dy. \] (30)
The path is the same as before. One can recognize in (30) the integral representation of the Hurwitz zeta function \( \zeta(\alpha + 1, \beta + 1) \) [6] (up to the factor \( -\pi / \Gamma(-\alpha) \)).

**Proof of formulae** (12) and (13). First, we note that (12) can be presented as
\[ f(a) = \int_{-\pi}^{\pi} \ln \left[ -a + \ln \left( 1 + e^{-2i \alpha} \right) \right] \, dx \] (31)
and after making the substitution \( y = \ln \left( 1 + e^{-2i \alpha} \right) \) as
\[ \frac{1}{2i} \oint_{C} \ln \left( -a + y \right) \frac{e^{y}}{e^{y} - 1} \, dy. \] (32)
The path \( C \) is different for two cases i) \( a < \ln 2 \) and ii) \( a > \ln 2 \). The difference comes from the fact that the point \( y = \ln 2 \) should belong to the path \( C \). However, there is also a cut along the line \( (-\infty, a] \). \( C \) can be arbitrarily deformed such that the pole \( y = 0 \) is not intercepted in the course of deformation. From this requirements, we find that in the case i) \( C \) extends from \( -\infty \), circumvents the the point \( y = \max \{a, 0\} \), and extends back to \( -\infty \); in the case ii) it is composed of two branches: first extends from \( -\infty \) and terminates at \( y = \ln 2 \) at the lower side of the cut, and the second emerges at \( y = \ln 2 \) at the upper side of the cut and extends to \( -\infty \). The logarithm is \( \ln \left( -a + y \right) = \ln \left| a - y \right| + \pi i \) at the lower and upper sides of the cut, respectively. We also assume that \( C \) does not contain the points \( y = \pm 2\pi in, \ n = 1, 2, 3, ... \).
Now we can write (in the case $a > 0$)

$$f(a) = \pi \int_{e^\epsilon}^\infty \frac{dy}{y^\epsilon - 1} - \pi \int_e^{e^\epsilon} \frac{e^\epsilon dy}{y^\epsilon - 1}$$

$$+ \frac{1}{2i} \int_{-\pi}^0 i \, d\phi \ln|a| + \pi i + \frac{1}{2i} \int_{0}^{\pi} i \, d\phi \ln|a| - \pi i) + O(\epsilon). \quad (33)$$

Terms containing $\epsilon$ emerge from the integration around the pole $y = 0$. Calculating the elementary integrals and proceeding to the limit $\epsilon \to 0$, we obtain (12). If $a < 0$ is the case, calculations are analogous to the presented above and lead to the same result, therefore we omit them.

To prove (13), we present it as a sum

$$g(a) = g_1 + g_2, \quad (34)$$

$$g_1 = \frac{1}{2} \, \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \left[-a + \ln(1 + e^{2ix})\right] e^{-2iks} \, dx, \quad (35)$$

$$g_2 = \frac{1}{2} \, \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \left[-a + \ln(1 + e^{-2ix})\right] e^{2iks} \, dx. \quad (36)$$

(35) and (36) can be evaluated in the same manner as (31). The only complication is the additional factors $e^\epsilon - 1$ and $\frac{1}{e^\epsilon - 1}$ that should be embedded in (32). We present the final result only:

$$g_1 = -\frac{\pi}{2} e^b$$

$$g_2 = \frac{\pi}{2} \left(1 - \frac{1}{a} + \frac{1}{e^b - 1}\right)$$

Proof of formulae (14) and (15). Although it is almost obvious how to obtain (14) and (15) from (12), they are not so obvious by themselves. Assuming $a = \pm n \ln \frac{1}{r}, 0 < r < \frac{1}{2}, n = 1, 2, 3, \ldots$ in (31) and (12), we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \left[-ix - n \ln \frac{1}{r} + \ln(2 \cos x)\right] \, dx = \pi \ln \left[n \ln \frac{1}{r}\right], \quad (37)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \left[-ix + n \ln \frac{1}{r} + \ln(2 \cos x)\right] \, dx = \pi \ln \frac{n \ln \frac{1}{r}}{1 - r^n} \quad (38)$$

and subtracting (38) from (37)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \left[-ix - n \ln \frac{1}{r} + \ln(2 \cos x)\right] \, dx = \pi \ln(1 - r^n). \quad (39)$$

Using the following formula for the gamma function

$$\Gamma(x) = \lim_{n \to \infty} \frac{n! n^x}{x(x + 1) \ldots (x + n)}, \quad n \to \infty \quad (40)$$
and the well known fact that
\[ \int_{0}^{\frac{\pi}{2}} \ln(2 \cos x) \, dx = 0, \quad (41) \]

one finally comes to (14).

(15) is obtained from (see (31) and (12))
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln[-ix - n - c + \ln(2 \cos x)] \, dx = \pi \ln(n + c), \quad c > \ln 2, \quad n = 0, 1, 2, \ldots \]

with the aid of identities (40) and (41).

### 3 Some other results

We end this article by listing some integrals without proofs, which are either the direct consequence of the integrals in the main text or new integrals, which can be evaluated exactly in the same manner.

\[ \int_{0}^{\frac{\pi}{2}} \ln[x^2 + \ln^2 \cos x] \, dx = \frac{\pi}{2} \ln 2 \quad (42) \]
\[ \int_{0}^{\frac{\pi}{2}} \ln[x^2 + \ln^2 \cos x] \cos 2x \, dx = -\frac{\pi}{\ln 2} \quad (43) \]
\[ \int_{0}^{\frac{\pi}{2}} \ln[x^2 + \ln^2(2 \cos x)] \cos 2x \, dx = -\frac{\pi}{4} \quad (44) \]
\[ \int_{0}^{\frac{\pi}{2}} \frac{\ln \cos x}{x^2 + \ln^2 \cos x} \, dx = \frac{\pi}{2} \left( 1 - \frac{1}{\ln 2} \right) \quad (45) \]
\[ \int_{0}^{\frac{\pi}{2}} \frac{x \sin 2x}{x^2 + \ln^2 \cos x} \, dx = \frac{\pi}{4 \ln^2 2} \quad (46) \]
\[ \int_{0}^{\frac{\pi}{2}} \frac{x \sin 2x}{x^2 + \ln^2(2 \cos x)} \, dx = \frac{13\pi}{48} \quad (47) \]
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1 + e^{-2ax})^\beta}{\beta \ln(1 + e^{-2ax}) - a} \, dx = -\frac{\pi}{a} + \frac{e^{\beta+1} - 1}{e^a - 1} H(\ln 2 - a) \quad (48) \]
\[ \int_{0}^{\frac{\pi}{2}} \frac{x \sin 2x}{x^2 + \ln^2(2e^{-a} \cos x)} \, dx = \frac{\pi}{4a^2} + \frac{\pi e^a}{4} \left( 1 - \frac{1}{(e^a - 1)^2} \right) H(\ln 2 - a) \quad (49) \]

---

2 Consequence of Lobachevskii’s integral. The proof is elementary:
\[ \int_{0}^{\frac{\pi}{2}} \ln \cos x \, dx = \int_{0}^{\frac{\pi}{2}} \ln \sin x \, dx = \frac{1}{2} \int_{0}^{\pi} \ln \sin x \, dx = \frac{\pi}{2} \ln 2 + 2 \int_{0}^{\frac{\pi}{2}} \ln \cos x \, dx \]
References

1. M.L. Glasser and D. Manna, On the Laplace transform of the psi function, Tapas in Experimental Mathematics (T. Amdeberhan and V. Moll, eds.), Contemporary Mathematics, vol. 457, Amer. Math. Soc., Providence, RI, 205-214 (2008)

2. O. Oloa, Some Euler-type integrals and a new rational series for Euler's constant, Tapas in Experimental Mathematics (T. Amdeberhan and V. Moll, eds.), Contemporary Mathematics, vol. 457, Amer. Math. Soc., Providence, RI, 253-264 (2008)

3. T. Amdeberhan, O. Espinosa and V.H. Moll, The Laplace transform of the digamma function: An integral due to Glasser, Manna and Oloa, Proc. Amer. Math. Soc., 136, 3211-3221 (2008)

4. D. Bailey and J. Borwein, Computer-Assisted Discovery and Proof, Tapas in Experimental Mathematics (T. Amdeberhan and V. H. Moll, eds), Providence, RI: Amer. Math. Soc., pp. 21–52 (2008).

5. A. Dixit, The Laplace transform of the psi function, Proc. Amer. Math. Soc., Volume 138, 593-603 (2010)

6. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge University Press, Cambridge, (1927).