Subset Synchronization in Monotonic Automata

Andrew Ryzhikov\textsuperscript{1,2}, Anton Shemyakov\textsuperscript{3}

\textsuperscript{1} Université Grenoble Alpes, Laboratoire G-SCOP, 38031 Grenoble, France
\textsuperscript{2} United Institute of Informatics Problems of NASB, 220012 Minsk, Belarus
\textsuperscript{3} Belarusian State University, 220030 Minsk, Belarus
ryzhikov.andrew@gmail.com, shemyanton@gmail.com

Abstract

We study extremal and algorithmic questions of subset synchronization in monotonic automata. We show that several synchronization problems that are hard in general automata can be solved in polynomial time in monotonic automata, even without knowing a linear order of the states preserved by the transitions. We provide asymptotically tight lower and upper bounds on the maximum length of a shortest word synchronizing a subset of states in monotonic automata. We prove that the Finite Automata Intersection problem is NP-hard for monotonic weakly acyclic automata over a three-letter alphabet, and the problem of computing the rank of a subset of states is NP-hard to approximate within a factor of \( \frac{9}{8} - \epsilon \) for any \( \epsilon > 0 \) for the same class of automata. Finally, we give a simple necessary and sufficient condition when a strongly connected digraph with a selected subset of vertices can be transformed into a deterministic automaton where the corresponding subset of states is synchronizing.

1 Introduction

Let \( A = (Q, \Sigma, \delta) \) be a deterministic finite automaton (which we further simply call an automaton), where \( Q \) is the set of its states, \( \Sigma \) is a finite alphabet and \( \delta : Q \times \Sigma \to Q \) is a transition function. Note that our definition of automata does not include initial and accepting states. An automaton is called synchronizing if there exists a word that maps every its state to some fixed state \( q \in Q \). Synchronizing automata play an important role in manufacturing, coding theory and biocomputing and model systems that can be controlled without knowing their actual state [Vol08].

A set \( S \subseteq Q \) of states of an automaton \( A = (Q, \Sigma, \delta) \) is called synchronizing if there exists a word \( w \in \Sigma^* \) and a state \( q \in Q \) such that the word \( w \) maps each state \( s \in S \) to the state \( q \). The word \( w \) is said to synchronize the set \( S \). It follows from the definition that an automaton is synchronizing if and only if the set \( Q \) of all its states is synchronizing.

An automaton \( A = (Q, \Sigma, \delta) \) is monotonic if there is a linear order \( \leq \) of its states such that for each \( x \in \Sigma \) if \( q_1 \leq q_2 \) then \( \delta(q_1, x) \leq \delta(q_2, x) \). In this case we say that the transitions of the automaton preserve, or respect this order. Monotonic automata play an important role in the part-orienting process in manufacturing [AV04]. Once an order
\[ q_1, \ldots, q_n \] of the states is fixed, we denote \([q_i, q_j] = \{q_\ell \mid i \leq \ell \leq j\}\), and \(\min S, \max S\) as the minimum and maximum states of \(S \subseteq Q\) with respect to the order.

An automaton \(A = (Q, \Sigma, \delta)\) is called weakly acyclic if there exists an order of its states \(q_1, \ldots, q_n\) such that if \(\delta(q_i, x) = q_j\) for some \(x \in \Sigma\), then \(i \leq j\). Note that a monotonic automaton does not have to be weakly acyclic, and vice versa. Both weakly acyclic and monotonic automata present proper subclasses of a widely studied class of aperiodic automata [Vol08]. An automaton is called strongly connected if any its state can be mapped to any other state by some word. An automaton is called orientable, if there exists a cyclic order of its states that is preserved by all transitions of the automaton (see [Vol08] for the discussion of this definition). Each monotonic automaton is obviously orientable.

Synchronizing automata model devices that can be reset to some particular state without having any information about their current state. Automata with a synchronizing set of states model devices that can be reset to a particular state with some partial information about the current state, namely when it is known that the current state belongs to a synchronizing subset of states. Checking whether an automaton is synchronizing can be performed in polynomial time [Vol08], but checking whether a given subset of states in an automaton is synchronizing (the Sync Set problem) is a PSPACE-complete problem in binary strongly connected automata [Vor16], and a NP-complete problem in binary weakly acyclic automata [Ryz17].

Eppstein [Epp90] provides a polynomial algorithm for the Sync Set problem, as well as for some other problems, in orientable automata. However, proposed algorithms assume that a cyclic order of the states preserved by transitions is known. Since the problem of recognizing monotonic (respectively, orientable) automata is NP-complete [Szy15], a linear (respectively, cyclic) order preserved by the transitions of an automaton cannot be computed in polynomial time unless \(P = NP\). Thus, we should avoid using this order explicitly in algorithms, so we have to investigate other structural properties of monotonic automata. As shown in this paper, several synchronization problems are still solvable in polynomial time in monotonic automata without knowing an order of states preserved by the transitions.

If an automaton (or a subset of its states) is synchronizing, it is reasonable to find a shortest word synchronizing it. A lot of effort was put into the investigation of this problem, both from extremal and algorithmic point of view. It is known that any synchronizing \(n\)-state automaton can be synchronized by a word of length \(\frac{n^3-6n}{6}\) [Pin83], and the famous Černý conjecture states that the length of such word is at most \((n-1)^2\) [Vol08]. Approximating the length of a shortest word synchronizing a \(n\)-state automaton within a factor of \(O(n^{1-\epsilon})\) for any \(\epsilon > 0\) in polynomial time is impossible unless \(P = NP\) [GS15].

For words synchronizing a subset of states, the situation is quite different. It is known that the length of a shortest word synchronizing a subset of states in a binary strongly connected automaton can be exponential in the number of states of the automaton [Vor16]. In weakly acyclic automata, there is a quadratic upper bound on the length of such words, but approximating this length is still a hard problem [Ryz17]. For orientable \(n\)-state automata, a tight \(n^2 - 2n + 1\) bound on the length of a shortest word synchronizing a subset of states is known [Epp90].
Given an automaton \( A = (Q, \Sigma, \delta) \), the rank of a word \( w \in \Sigma^* \) with respect to a set \( S \subseteq Q \) is the size of the image of \( S \) under the mapping defined by \( w \) in \( A \), i.e. the number \(|\{\delta(s, w) \mid s \in S\}|\). The rank of an automaton (respectively, of a subset of states) is the minimum among the ranks of all words \( w \in \Sigma^* \) with respect to the whole set \( Q \) of states of the automaton (respectively, to the subset of states). It follows from the definition that a set of states has rank 1 if and only if it is synchronizing. A state in an automaton is a sink state if all letters map this state to itself.

In this paper, we study both extremal and algorithmic questions of subset synchronization in monotonic automata. In Section 2, we provide structural results about synchronizing sets of states in monotonic automata and give algorithmic consequences of this results. In Section 3, we provide lower and upper bounds on the maximum length of shortest words synchronizing a subset of states in monotonic automata. In Section 4, we provide NP-hardness and inapproximability of several problems related to subset synchronization in weakly acyclic monotonic automata over a three-letter alphabet. In Section 5 we give necessary and sufficient conditions when a strongly connected digraph can be colored resulting in an automaton with a pre-defined synchronizing set.

2 Structure of Synchronizing Sets

Let \( A \) be an automaton, and \( S \) be a subset of its states. In general, if any two states in \( S \) can be synchronized (i.e., form a synchronizing set), \( S \) does not necessarily have to be synchronizing, as it is shown by the following theorem.

**Theorem 1.** There exists a binary weakly acyclic automaton \( A \) and a subset \( S \) of its states such that each pair of states in \( S \) in synchronizing, but the rank of \( S \) equals \(|S| - 1\).

**Proof.** Consider the following automaton \( A = (Q, \{0, 1\}, \delta) \). Let \( k = 2^\ell \) for an integer number \( \ell \), and let \( \text{bin}(i) \) be a word which is equal to the binary representation of \( i \) of length \( \ell \) (possibly with zeros at the beginning). We introduce new states \( t_i, p_i \) for \( 1 \leq i \leq k \), a state \( f \), and new intermediate states in \( Q \) as following. For each \( s_i, 1 \leq i \leq k \), consider a construction sending \( s_i \) to \( f \) for a word \( \text{bin}(i) \), and to \( t_i \) otherwise (such a construction has \( k+1 \) state). For each \( t_i \), consider the same construction sending \( t_i \) to \( f \) for a word \( \text{bin}(i) \), and to \( p_i \) otherwise. For each \( i \), define both transitions from \( p_i \) as self-loops. Define both transitions from \( f \) as self-loops.

In this construction, each word applied after a word of length \( 2\ell \) obviously has no effect. Consider a word \( w \) of length \( 2\ell \), \( w = w_1w_2 \), where both \( w_1 \) and \( w_2 \) have length \( \ell \). If \( w_1 = w_2 \), then the image of \( S \) under the mapping defined by \( w \) has size \( k \). Otherwise, \( w \) synchronizes two states \( s_i \) and \( s_j \) with \( \text{bin}(i) = w_1 \) and \( \text{bin}(j) = w_2 \) and maps all other states to different states. Thus, the rank of \( S \) equals \( k - 1 \). \( \square \)

The size of the whole automaton is \( O(|S|(\log |S|)^2) \), thus \( S \) can be large comparing to the size of the whole set of states in the automaton.

Since the Sync Set problem is PSPACE-complete in strongly connected automata, pairwise synchronization of states in a subset does not imply that this subset is synchronizing for this class of automata unless \( P = \text{PSPACE} \). Thus, it is reasonable to ask the following question.
Question 1. How large can be the rank of a subset of states in a strongly connected automaton such that each pair of states in this subset can be synchronized?

For the rest of section, fix a monotonic automaton $A = (Q, \Sigma, \delta)$, and let $q_1, \ldots, q_n$ be an order of its states preserved by all transitions.

Theorem 2. Let $S \subseteq Q$ be a subset of states of $A$. Then $S$ is synchronizing if and only if any two states in $S$ can be synchronized.

Proof. Obviously, any subset of a synchronizing set is synchronizing.

In the other direction, if any two states in $S$ can be synchronized, then the minimal state $q_\ell = \min S$ and the maximal state $q_r = \max S$ in $S$ can be synchronized by a word $w \in \Sigma^*$. Let $q = \delta(q_\ell, w) = \delta(q_r, w)$. Then the interval $[q_\ell, q_r] = \{q_\ell, \ldots, q_r\}$ is synchronized by $w$, because each state of $[q_\ell, q_r]$ is mapped to the interval $[\delta(q_\ell, w), \delta(q_r, w)] = \{q\}$, since $A$ is monotonic. Thus, $S \subseteq [q_\ell, q_r]$ is synchronizing. \hfill \Box

Theorem 3. The problem of checking whether a given set $S$ is synchronizing can be solved in polynomial time for monotonic automata.

Proof. By Theorem 2 it is enough to check that each pair of states in $S$ can be synchronized, which can be done by solving the reachability problem in the subautomaton of the power automaton, build on all 2-element and 1-element subsets of $Q$ [Vol08]. \hfill \Box

Theorem 4. A shortest word synchronizing a given subset $S$ of states can be found in polynomial time for monotonic automata.

Proof. Consider the following algorithm. For each pair of states, find a shortest word synchronizing this pair. This can be done by solving the shortest path problem in the subautomaton of the power automaton, build on all 2-element and 1-element subsets of $Q$ [Vol08]. Let $W$ be the set of all such words that synchronize $S$. Output the shortest word in $W$.

By an argument similar to the proof of Theorem 2, any shortest word synchronizing $\{\min S, \max S\}$ is a shortest word synchronizing $S$, thus the algorithm is optimal. Since there are $\binom{|S|}{2}$ pairs of states, the algorithm is polynomial. \hfill \Box

Define the synchronizing graph $G(A)$ of an automaton $A = (Q, \Sigma, \delta)$ as following. The set of vertices of $G(A)$ is $Q$. Two vertices are adjacent in $G$ if and only if the two corresponding states can be synchronized.

The following fact follows from the proof of Theorem 3 in [Ryz16] and shows that in general the structure of synchronization graphs can be very complicated.

Theorem 5. Each graph is an induced subgraph of a synchronization graph of some binary weakly acyclic automaton.

However, synchronizing graphs of monotonic automata are very special. An interval graph is the intersection graph of a set of intervals of a line.

Theorem 6. Synchronization graphs of monotonic automata are interval graphs.
Proof. For each state $q \in Q$ in $A$, consider the set $S_q$ of all such states $q'$ that $q$ and $q'$ can be synchronized. It follows from the proof of Theorem 2 that each maximal (by inclusion) synchronizing set is an interval $[q_i, q_j]$ for some $q_i, q_j$. Thus, each set $S_q$ form an interval, which is the union of all intervals that are maximal synchronizing sets containing $q$.

Observe that two states can be synchronized if and only if the corresponding intervals intersect. Thus, we obtain that the synchronization graph of $A$ is the intersection graph of the constructed set of intervals.

Theorem 6 implies the following.

**Theorem 7.** A synchronizing subset of states of maximum size can be found in polynomial time in monotonic automata.

**Proof.** By Theorem 2, synchronizing sets in monotonic automata correspond to cliques (complete subgraphs) in their synchronization graph. By Theorem 6, synchronization graphs of monotonic automata are interval. A clique of maximum size can be found in polynomial time in interval graphs [Gol04].

The problem of finding a synchronizing subset of states of maximum size in general automata is PSPACE-complete [Ryz16]. Türker and Yenigün [TY15] study a variation of this problem, which is to find a set of states of maximum size that can be mapped by some word to a subset of a given set of states in a given monotonic automaton. They reduce the N-QUEENS PUZZLE problem [BS09] to this problem to prove its NP-hardness. However, their proof is unclear, since in the presented reduction the input has size $O(\log N)$, and the output size is polynomial in $N$.

## 3 Shortest Words Synchronizing a Subset of States

The length of a shortest word synchronizing a $n$-state monotonic automaton is at most $n-1$ [AV04]. In this section we investigate a more general question of bounding the length of a shortest word synchronizing a subset of states in a $n$-state synchronizing automaton.

**Theorem 8.** Let $S$ be a synchronizing set of states in a monotonic $n$-state automaton $A$. Then for $n \geq 3$ the length of a shortest word synchronizing $S$ is at most $\frac{(n-1)^2}{4}$.

**Proof.** Let $A = (Q, \Sigma, \delta)$, and $\{q_1, \ldots, q_n\}$ be an order of the states preserved by all transitions of $A$. Define $q_\ell = \min S$, $q_r = \max S$. We can assume that $S$ can be mapped only to states in $[q_\ell, q_r]$. Indeed, assume without loss of generality that $q_i, i < \ell$, is a state such that $S$ can be mapped to $q_i$. Note that for the states $q_i \leq q_j \leq q_k \leq q_\ell$, if $q_k$ can be mapped to $q_i$, and $q_\ell$ can be mapped to $q_j$, then $q_\ell$ can be mapped to a state $q'_\ell \leq q_i$, because $A$ is monotonic. Thus, $S$ can be synchronized by applying only letters that map consecutive images of its states to states with smaller indexes and a shortest word synchronizing $S$ has length at most $n-1$.

Now we can assume that $S$ can be mapped only to states in $[q_\ell, q_r]$. This means that $S$ can be mapped to a subinterval $[q_i, q_j]$ of $[q_\ell, q_r]$, and no state outside $[q_i, q_j]$ is reachable from any state of $[q_i, q_j]$. If both $q_\ell$ and $q_r$ are mapped to states inside $[q_i, q_j]$, they can
be synchronized by applying a word of length at most \( j - i \), for example by applying only letters mapping the consecutive images of \( q_r \) to states with smaller indexes.

Suppose now that \( w = w_1 \ldots w_m \) is a shortest word synchronizing \( S \). Consider the sequence of pairs \((t_i, s_i) = (\delta(q_r, w_1 \ldots w_i)\), \( \delta(q_r, w_1 \ldots w_i)\))\), \( i = 1, 2, \ldots, m \).

As \( w \) is a shortest word synchronizing \( S \), and synchronization of \( S \) is equivalent to synchronization of \( \{q_i, q_r\} \), no pair appears in this sequence twice, and the only pair with equal components is \((s_m, t_m)\). Further, for each \( k \), \( 1 \leq k \leq m \), \( \delta(q, w_1 \ldots w_k) \leq q_i \leq q_j \leq \delta(q, w_1 \ldots w_k) \). Thus, the maximum length of \( w \) is reached when \( q_i = q_j \) (since after both images are in \([q_i, q_j]\) the remaining length of a synchronizing word is at most \( j - i \)) and is at most \((i - 1)(n - i) \leq \frac{(n-1)^2}{4} \).

The bound is almost tight for monotonic automata over a three-letter alphabet as shown by the following example.

**Theorem 9.** For each \( m \geq 1 \), there exist a \((2m + 3)\)-state monotonic automaton \( A \) over a three-letter alphabet, which has a subset \( S \) of states, such that the length of a shortest word synchronizing \( S \) is \( m^2 + 1 \).

**Proof.** Consider the following monotonic automaton \( A = (Q, \Sigma, \delta) \), \( Q = \{q_1, \ldots, q_{2m+3}\} \). Let \( \Sigma = \{0, 1, 2\} \). Let states \( q_1, q_{m+2} \) and \( q_{2m+3} \) be sink states. For every state \( q_i \), \( 2 \leq i \leq m + 1 \), we set \( \delta(q_i, 0) = q_{i+1}, \delta(q_i, 1) = q_i, \delta(q_i, 2) = q_1 \). For every state \( q_i \), \( m + 4 \leq i \leq 2m + 2 \), we set \( \delta(q_i, 0) = q_{2m+3}, \delta(q_i, 1) = q_{i-1}, \delta(q_i, 2) = q_i \). Finally we define \( \delta(q_{m+3}, 0) = q_{m+2}, \delta(q_{m+3}, 1) = q_{m+3}, \delta(q_{m+3}, 2) = q_m+2 \). See Figure 1 for an illustration of the construction.

![Figure 1: The automaton providing a lower bound for subset synchronization in monotonic automata over a three-letter alphabet.](image)

All transitions of \( A \) respect the order \( q_1, \ldots, q_{2m+3} \), so \( A \) is monotonic. Let us show that the shortest word synchronizing the set \( S = \{q_2, q_{m+3}\} \) is \( w = (01^{m-1})m2 \). Let \( S' \) be a set of states such that \( q_i, q_j \in S', 2 \leq i \leq m + 1, m + 3 \leq j \leq 2m + 2 \). The set \( S' \) can be mapped only to \( q_{m+2} \), because \( A \) is monotonic. Hence if any state of \( S' \) is mapped by a word to \( q_1 \) or to \( q_{2m+3} \), then this word cannot synchronize \( S' \).

We start with the set \( S = \{q_2, q_{m+3}\} \). There is only one letter 0 that does not map the state \( q_2 \) to \( q_1 \) or the state \( q_{m+3} \) to \( q_{2m+3} \), and maps \( S \) not to itself. Indeed, 1 maps \( q_2 \) to \( q_2 \) and \( q_{m+3} \) to \( q_{m+3} \), and 2 maps \( q_2 \) to \( q_1 \). Thus, any shortest synchronizing word can start only with 0. Consider now the set \( \{\delta(q_2, 0), \delta(q_{m+3}, 0)\} = \{q_3, q_{2m+2}\} \). There is only letter 1 that does not map the state \( q_3 \) to \( q_1 \), or \( q_{2m+2} \) to \( q_{2m+3} \) and maps this set not to itself. Indeed, 0 maps \( q_{2m+2} \) to \( q_{2m+3} \) and 2 maps \( q_3 \) to \( q_1 \). So the second letter of
the shortest synchronizing word can only be 1. By a similar reasoning (at each step there is exactly one letter that maps a pair of states not to itself and does not map the states to the sink states \(q_1\) and \(q_{2m+3}\), we deduce that any shortest synchronizing word has to begin with \((01^{m-1})^m\) and it is easy too see that \((01^{m-1})^m 2\) synchronizes \(S\). Thus, \(w\) is a shortest word synchronizing \(S\), and its length is \(m^2 1\).

For a \(n\)-state automaton, the lower bound on the length of a shortest word in this theorem is \((\frac{n-3}{4})^2 1\), which is very close to the lower bound \((\frac{n-1}{4})^2\) from Theorem 8.

By taking \(q_2\) and \(q_{m+3}\) as initial states in two equal copies of the automaton in the proof of Theorem 9, and taking \(q_{m+2}\) as the only accepting state in both copies, we obtain the following result.

**Corollary 1.** A shortest word accepted by two \((2m + 3)\)-state monotonic automata can have length \(m^2 1\).

For binary monotonic automata, our lower bound is slightly smaller, but still quadratic.

**Theorem 10.** For each \(m \geq 1\), there exist a \((4m + 3)\)-state binary monotonic automaton \(A\), which has a subset \(S\) of states such that the length of a shortest word synchronizing \(S\) is at least \(m^2\).

**Proof.** Consider the following automaton \(A = (Q, \Sigma, \delta)\) with \(Q = \{q_1, \ldots, q_{4m+3}\}\), \(\Sigma = \{0, 1\}\). Define \(\delta\) as following. Set \(q_1, q_{2m+2}, q_{4m+3}\) to be sink states. Define \(\delta(q_i, 1) = q_{i-1}\) for all \(i \neq 1, 2m + 2, 4m + 3\). For each \(i, 2 \leq i \leq m + 1\), define \(\delta(q_i, 0) = q_{i+m}\), and for each \(i, m + 2 \leq i \leq 2m + 1\), define \(\delta(q_i, 0) = q_{2m+2}\). For each \(i, 2m + 3 \leq i \leq 3m + 3\), define \(\delta(q_i, 0) = q_{m+i-1}\), and for each \(i, 3m + 4 \leq i \leq 4m + 2\), define \(\delta(q_i, 0) = q_{4m+3}\). The defined binary automaton is monotonic, since all its transitions respect the order \(q_1, \ldots, q_{4m+3}\). See Figure 2 for an example of the construction.

![Figure 2](image-url)  

**Figure 2:** The automaton providing a lower bound for subset synchronization in binary monotonic automata. Dashed arrows represent transitions for the letter 0, solid – for the letter 1. The states \(q_1, q_{2m+2}, q_{4m+3}\) are sink states. The picture is divided into two parts because of its width.

Define \(S = \{q_{m+2}, q_{4m+3}\}\). Let us prove that a shortest word synchronizing \(S\) has length at least \(m^2\).

The set \(S\) can only be mapped to \(q_{2m+2}\), since it is a sink state between \(\min S\) and \(\max S\). Thus, no synchronizing word maps any state of \(S\) to \(q_1\) or \(q_{4m+3}\). Consider now an
interval \([q_i, q_j]\) for \(2 \leq i \leq m + 1, 2m + 3 \leq j \leq 4m + 2\) and note that applying 0 reduces its length by 1 (or maps its right end to \(q_{4m+3}\)), and applying 1 maps its ends to the ends of another interval of this form with the same length (or maps its left end to \(q_1\)). The maximal length of a segment of this form that allows its left end to be mapped to \(q_{2m+2}\) is \(2m + 1\), so before any end of the interval is mapped to \(q_{2m+2}\), the letter 0 has to be applied at least \(m\) times. Each application of 0 moves the right end of the intervals \(m - 1\) states to the right, so each application of 0 requires \(m - 1\) applications of 1 so that 0 can be applied one more time. Thus, the word mapping \(S\) to \(q_{2m+2}\) has length at least \(m^2\). Note that \(S\) can be synchronized by a word \(w = (1^{m-1}0)^m 1^2\) of length \(|w| = m^2 + 2m\). \(\square\)

For a \(n\)-state binary monotonic automaton we get a lower bound of \(\frac{(n-3)^2}{16}\) from this theorem.

### 4 Complexity Results

In this section, we obtain complexity results for several problems related to subset synchronization in monotonic automata. We improve Eppstein’s construction [Epp90] to make it suitable for monotonic automata. We shall need the following NP-complete SAT problem [Sip12].

**SAT**

**Input:** A set \(X\) of \(n\) boolean variables and a set \(C\) of \(m\) clauses;  

**Output:** Yes if there exists an assignment of values to the variables in \(X\) such that all clauses in \(C\) are satisfied, No otherwise.

Provided a set \(X\) of boolean variables \(x_1, \ldots, x_n\) and a clause \(c_j\), construct the following automaton \(A_j = (Q, \Sigma, \delta)\). Take

\[
Q = \{q_1, \ldots, q_{n+1}\} \cup \{q'_2, \ldots, q'_n\} \cup \{s, t\}.
\]

Let \(\Sigma = \{0, 1\}\). Define the transition function \(\delta\) as following. For each \(i, 1 \leq i \leq n\), map a state \(q_i\) to \(q'_{i+1}\) (or to \(t\) if \(i = n\)) by a letter \(x \in \{0, 1\}\) if the assignment \(x_i = x\) satisfies \(c_j\), and to \(q_{i+1}\) otherwise. For each \(i, 2 \leq i \leq n - 1\), set \(\delta(q'_i, x) = \delta(q'_i, 1)\) for \(x \in \{0, 1\}\). Set \(\delta(q'_n, x) = t\) for \(x \in \{0, 1\}\). Define transitions from \(t\) for letters 0, 1 as self-loops. Finally, define \(\delta(q, r) = s\) for \(q \in Q \setminus \{t\}\), \(\delta(t, r) = t\). See Figure 3 for an example.

Note that \(A_j\) is monotonic, since it respects the order

\[s, q_1, q_2, q'_2, q_3, q'_3, \ldots, q_n, q'_n, q_{n+1}, t.\]

It is also weakly acyclic, since its underlying digraph has no simple cycles of length at least 2.

Also, provided the number of variables \(n\), construct an automaton \(T = (Q_T, \Sigma, \delta_T)\) as following. Take \(Q_T = \{a, p_1, \ldots, p_{n+1}, b\}\), \(\Sigma = \{0, 1\}\). Define \(\delta(p_i, x) = p_{i+1}\) for each \(i, 1 \leq i \leq n\), and \(x \in \{0, 1\}\), and \(\delta(p_{n+1}, x) = b\) for \(x \in \{0, 1\}\). Define also \(\delta(a, x) = a\) and \(\delta(b, x) = b\) for each \(x \in \Sigma\), and \(\delta(p_i, r) = a\) for \(1 \leq i \leq n + 1\). See Figure 4 for an
Example. This automaton is monotonic, since it respects the order $a, p_1, \ldots, p_{n+1}, b$, and it is obviously weakly acyclic.

First, we prove NP-completeness of the following problem.

**FINITE AUTOMATA INTERSECTION**

*Input:* Automata $A_1, \ldots, A_k$ (with input and accepting states);
*Output:* Yes if there is a word which is accepted by all automata, No otherwise.

This problem is PSPACE-complete for general automata [Koz77], and NP-complete for binary weakly acyclic automata [Ryz17].

**Theorem 11.** The Finite Automata Intersection problem is NP-complete for monotonic weakly acyclic automata over a three-letter alphabet.

**Proof.** The fact that the problem is in NP follows from the fact that Finite Automata Intersection for weakly acyclic automata is in NP [Ryz17].

To prove hardness, we reduce the SAT problem. For each clause $c_j \in C$, construct an automaton $A_j$, and set $q_1$ as its initial state and $t$ as its only accepting state. Construct also the automaton $T$ with initial state $p_1$ and accepting state $a$.

We claim that $C$ is satisfiable if and only if all automata $\{A_j \mid c_j \in C\} \cup \{T\}$ accept a common word $w$. Indeed, assume that there is a common word accepted by all these automata. Then none of the first $n$ letters of this word can be $r$, otherwise all automata $A_j$ are mapped to $s$, which is a non-accepting sink state. The next letter has to be $r$, otherwise $T$ is mapped to $b$, which is a non-accepting sink state. But that means that in
each $A_j$, the set $q_1$ is mapped by a $n$-letter word $z_1 \ldots z_n$ to the accepting state $t$. Thus, by construction, the assignment $x_i = z_i$ satisfies all clauses in $C$. By the same reasoning, if the assignment $x_i = z_i$, $1 \leq i \leq n$, satisfies all clauses in $C$, then $z_1 \ldots z_nr$ is a word accepted by all automata.

Now we switch to a related Set Rank problem.

**Set Rank**

*Input:* An automaton $A$ and a set $S$ of its states;

*Output:* The rank of $S$.

This problem is hard to approximate for binary weakly acyclic automata $\text{[Ryz17]}$. To get inapproximability results for monotonic automata, we use the following problem.

**Max-3SAT**

*Input:* A set $X$ of $n$ boolean variables and a set $C$ of $m$ 3-term clauses;

*Output:* The maximum number of clauses that can be simultaneously satisfied by some assignment of values to the variables.

This problem cannot be approximated in polynomial time with in a factor of $\frac{7}{8} - \epsilon$ for any $\epsilon > 0$ unless P = NP $\text{[Hås01]}$.

**Theorem 12.** The Set Rank problem cannot be approximated in polynomial time within a factor of $\frac{9}{8} - \epsilon$ for any $\epsilon > 0$ in monotonic weakly acyclic automata over a three-letter alphabet unless P = NP.

**Proof.** We reduce the Max-3SAT problem. For each clause $c_j \in C$, construct an automaton $A_j$. Construct also $m$ copies of the automaton $T$, denoted $T_j$, $1 \leq j \leq m$. Define an automaton $A$ with the set of states which is the union of all sets of states of $\{A_j, T_j | 1 \leq j \leq m\}$, alphabet $\Sigma$ and transition functions defined in all constructed automata. For each $j$, identify the state $t$ in $A_j$ with the state $a$ in $T_j$. Take $S$ to be the set of states $q_1$ from each automaton $A_j$. The constructed automaton is monotonic and weakly acyclic.

If $h$ is the minimum number of clauses in $C$ that are not satisfied by an assignment, the set $S$ has rank $m+h$. Indeed, consider an assignment $x_i = z_i$, $1 \leq i \leq n$, not satisfying exactly $h$ clauses in $C$. Then the word $z_1 \ldots z_nr$ has rank $m+h$ with respect to $S$.

In the other direction, let $w$ be a word of minimum rank with respect to the set $S$. If any of the first $n$ letters of $w$ is $r$, then $q_1$ in each $A_i$ is mapped to $s$ in the corresponding automaton, and thus $w$ has rank $2m$ with respect to $S$. The same is true if $(n+1)$th letter of $w$ is not $r$, because then $p_1$ in each $T_i$ is mapped to $b$ in the corresponding automaton. If first $n$ letters $z_1, \ldots, z_n$ of $w$ are not $r$, and the next letter is $r$, then the assignment $x_i = z_i$ does not satisfy exactly $h'$ clauses, where $m + h'$ is the rank of the word $w$ with respect to $S$. For the word of minimum rank, we get the required equality.

It is NP-hard to decide between (i) all clauses in $C$ are satisfiable and (ii) at most $\frac{7}{8}m$ clauses in $C$ can be satisfied by an assignment $\text{[Hås01]}$. In the case (i), the rank of $S$ is $m$, in the case (ii) it is at least $m + \frac{1}{8}m$. Since it is NP-hard to decide between this two options, we get $\left(\frac{9}{8} - \epsilon\right)$-inapproximability for any $\epsilon > 0$.

By using an argument similar to the proof of Theorem 12, we can show inapproximability of the maximization version of Finite Automata Intersection (where we are
asked to find a maximum number of automata accepting a common word). Indeed, take \( m \) copies of \( T \) together with the set \( \{ A_j \mid c_j \in C \} \) as the input of Finite Automata Intersection and reduce Max-3SAT to it (input and accepting states are assigned according to the construction in Theorem 11). Then the maximum number of automata accepting a common word is \( m + g \), where \( g \) is the maximum number of simultaneously satisfied clauses in \( C \), since all copies of \( T \) have to accept this word. Thus it is NP-hard to decide between (i) all \( 2m \) automata accept a common word and (ii) at most \( m + \frac{7}{8}m \) automata accept a common word, and we get the following result.

**Theorem 13.** The maximization version of the Finite Automata Intersection problem cannot be approximated in polynomial time within a factor of \( \frac{16}{15} - \epsilon \) for any \( \epsilon > 0 \) in monotonic weakly acyclic automata over a three-letter alphabet unless \( P = NP \).

**Question 2.** What is the complexity of the mentioned problems for binary monotonic automata? Are the problems discussed in this section in NP for monotonic automata?

We note that it does not matter in the provided reductions whether a linear order preserved by all transitions is known or not.

A set \( S \subseteq Q \) in an automaton \( A = (Q, \Sigma, \delta) \) is said to be saturated by a word \( w \) if \( S \) is a pre-image of a subset of \( \{ \delta(q, w) \mid q \in Q \} \) under the mapping defined by \( w \). In other words, no state in \( Q \setminus S \) is mapped to the set \( \{ \delta(s, w) \mid s \in S \} \) by \( w \). An automaton \( A \) (with initial and accepting states) is called biconnected if \( A \) and the determinization of the reversal of \( A \) are both strongly connected. In [DPR+] it is proved that a strongly connected automaton \( A \) is biconnected if and only if the set of its accepting states is saturated by a word \( w \) such that the rank of \( w \) with respect to \( A \) is equal to the rank of \( A \) (i.e., \( w \) is a word of minimum rank with respect to \( A \)). Thus, the problem of recognizing a subset of states saturated by a word of minimum rank arises naturally. We show that this problem is NP-hard.

**Theorem 14.** Given a monotonic weakly acyclic automaton \( A \) over a three-letter alphabet and a subset \( S \) of its states, it is NP-hard to decide whether \( S \) is saturated by a word of minimum rank with respect to \( A \).

**Proof.** We reduce the SAT problem. Construct the automaton \( A \) as described in the proof of Theorem 12. Take \( S \) to be the set of all states except \( s \) in each \( A_j \) (including the state \( t = a \)), together with \( p_1 \) in each \( T_j \).

We claim that the set \( S \) is saturated by a word of minimum rank (with respect to \( A \)) if and only if all clauses in \( C \) can be satisfied by some assignment. Indeed, note that the rank of \( A \) is \( 3m \). If \( C \) can be satisfied by an assignment \( x_i = z_i, 1 \leq i \leq n \), then the word \( z_1 \ldots z_n r \) is a word of minimum rank saturating \( S \).

In the other direction, assume that \( S \) is saturated by a word \( w \) of minimum rank. If \( w \) maps some state to a sink state not included in \( S \), this sink state will be mapped by \( w \) to \( \{ \delta(s, w) \mid s \in S \} \) as well. Hence \( w \) maps all states of \( S \) in each \( A_j \) and \( T_j \) to \( t = a \). The length of \( w \) is at least \( n + 1 \), since in each \( A_j \) the state \( q_1 \) must be mapped to \( t \) (which can be done only by a word of length at least \( n \)), and thus \( p_1 \) can be mapped to \( t \) only by a word of length \( n + 1 \). Now, to map all this states to \( t \), \( w \) needs to have first \( n \) letters equal to 0 or 1, and next letter to be equal to \( r \) (otherwise some state of \( S \) will be mapped to
s or b in some gadget). By construction, that means that in each gadget $A_j$ the state $q_1$ is mapped to $t$ by a word of length $n$, and thus if $w = z_1 \ldots z_n r$, the assignment $x_i = z_i$, $1 \leq i \leq n$, satisfies all clauses.

Hence, for general automata the considered problem is NP-hard as well. However, biconnected automata are strongly connected, and checking whether an automaton is strongly connected can be performed in polynomial time. Thus, the complexity of recognizing biconnected automata is equivalent to the following question.

**Question 3.** What is the complexity of deciding whether a subset of states in a strongly connected automaton is saturated by a word of minimum rank?

## 5 Subset Road Coloring

The famous Road Coloring problem is formulated as following. Given a strongly connected digraph with all vertices of equal out-degree $k$, is it possible to find a coloring of its arcs with letters of alphabet $\Sigma$, $|\Sigma| = k$, resulting in a synchronizing deterministic automaton. This problem was stated in 1977 by Adler, Goodwyn and Weiss [AGW77] and solved in 2007 by Trahtman [Tra09]. A natural generalization of this problem is to find a coloring of a strongly connected digraph turning it into a deterministic automaton where a given subset of states is synchronizing. We introduce the problem formally and show that its solution is a consequence of [BP14]. In particular, the problem of deciding whether such a coloring exists is solvable in polynomial time.

Let $G = (V, E)$ be a strongly connected digraph such that each its vertex has out-degree $k$. A coloring of $G$ with letters from alphabet $\Sigma$, $|\Sigma| = k$, is a function assigning each arc of $G$ a letter from $\Sigma$, such that for each vertex, each pair of arcs outgoing from it achieves different letters. We say that a coloring synchronizes $S \subseteq V$ in $G$ if $S$ is a synchronizing set in the resulting automaton.

If the greatest common divisor of the lengths of all cycles of $G$ is $\ell$, the set $V$ can be partitioned into sets $V_1, \ldots, V_\ell$ in such a way that if $(v, u)$ is an arc of $G$, then $v \in V_i, u \in V_{i+1}$ or $v \in V_\ell, u \in V_1$ [Fri90]. Moreover, such partition is unique.

**Theorem 15.** An strongly connected digraph $G$ with vertices of equal out-degree $k$ has a coloring synchronizing a set $S \subseteq V$ if and only if $S \subseteq V_i$ for some $i$.

**Proof.** Obviously, if two vertices of $S$ belong to distinct sets $V_i$ and $V_j$, $S$ can not be synchronized. Assume that $S \subseteq V_i$ for some $i$. As proved in [BP14], there exists a coloring of $G$ such that the resulting automaton $A$ has rank $\ell$. In this coloring each $V_j$, $1 \leq j \leq \ell$, is a synchronizing set, since no two states from two different sets $V_p, V_t$, $p \neq t$, can be synchronized and $A$ has rank $\ell$. Hence, $S \subseteq V_i$ is also a synchronizing set. 

According to this theorem, checking whether there exists such a coloring can be performed in polynomial time. Construction of this coloring can be done in polynomial time using the algorithm from [BP14].
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