From Benjamin-Feir instability to focusing dam breaks in water waves

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We report water wave experiments performed in a long tank where we consider the evolution of nonlinear deep-water surface gravity waves with the envelope in the form of a large-scale rectangular barrier. Our experiments reveal that, for a range of initial parameters, the nonlinear wave packet is not disintegrated by the Benjamin-Feir instability but exhibits a specific, strongly nonlinear modulation, which propagates from the edges of the wave packet towards the center with finite speed. Using numerical tools of nonlinear spectral analysis of experimental data we identify the observed envelope wave structures with focusing dispersive dam break flows, a peculiar type of dispersive shock waves recently described in the framework of the semi-classical limit of the 1D focusing nonlinear Schrödinger equation (1D-NLSE). Our experimental results are shown to be in a good quantitative agreement with the predictions of the semi-classical 1D-NLSE theory. This is the first observation of the persisting dispersive shock wave dynamics in a modulationally unstable water wave system.

I. INTRODUCTION

Following the pioneering works by Whitham and Lighthill [1, 2], Benjamin and Feir reported in 1967 the fundamental experimental and theoretical investigations of the time evolution of nonlinear deep-water surface gravity waves [3, 4]. They demonstrated that a uniform continuous wave train is unstable with respect to small long-wave perturbations of its envelope, which may eventually lead to its disintegration after some evolution time [3–9]. In 1968 Zakharov showed that, for narrowband perturbations, the governing hydrodynamic equations can be reduced to a single equation for the complex wave envelope: the focusing one-dimensional nonlinear Schrödinger equation (1D-NLSE) [10, 11]. It was then understood that the instability, first observed in water waves, represents a ubiquitous phenomenon in focusing nonlinear media. Nowadays this phenomenon is called modulational instability (MI) and it has been observed and studied in many physical situations including plasma waves, matter waves, electromagnetic and optical waves [12–22].

According to the conventional picture, the early (linear) stage of MI is manifested in the exponential growth of all the perturbations of a plane wave background that fall in the region of the Fourier spectrum below a certain cut-off wavenumber. A particular scenario of the MI development strongly depends on the type of the initial perturbation considered. For periodic modulations of a constant background, the nonlinear stage of the MI development is dominated by breather-like structures such as the Akhmediev, Kuznetsov-Ma, Peregrine breathers and their generalizations [23–28]. On the other hand, localized perturbations of a constant plane wave background produce a very different evolution pattern that depends on the soliton content of the localized perturbation. For localized “solitonless” initial perturbations, the nonlinear dynamics of MI is characterized by the development of a universal nonlinear oscillatory structure that expands in time with finite speed [29–32]. Localized perturbations having pure solitonic content may induce the generation of pairs of breathers with opposite velocities, termed super-regular breathers in ref. [33–35]. It was recently shown in ref. [36] that these two scenarios of the evolution of localized perturbations may coexist. Finally, the MI of an infinite plane wave induced by a random perturbation leads to the emergence of a complex nonlinear wave structure associated with the so-called integrable turbulence [37] requiring statistical approaches to the description of the evolution of the nonlinear wave system [38–47].

Another type of the evolution in a modulationally unstable systems occurs when the initial field represents a “slowly modulated” plane wave (e.g. a broad localized wave packet with a smooth envelope). In this case, the initial evolution is dominated by nonlinear effects, and the classical MI (understood as an exponential growth of small long-wave initial perturbations) plays a secondary role. The dynamical evolution of such wave fields in nonlinear focusing dispersive media gives rise to generic dynamical features which are in sharp contrast with the conventional MI scenarios [48]. As shown in the optical fiber experiments reported in ref. [49], the nonlinear focusing of such wave packets (light pulses) results in a
gradient catastrophe that is regularized by dispersive effects through the universal mechanism yielding the local Peregrine soliton structure [48]. Note that these dynamical features have been observed in the nonlinear evolution of deep-water wave packets [50, 51] even though they had not been connected to the universal semi-classical mechanism of the generation of the Peregrine soliton discussed in ref. [48].

There is a third class of problems where the initial profile is neither a plane wave nor a broad wavepacket but is rather characterized by a sharp and significant change of the amplitude and/or the phase. The arising dynamics are most acutely captured by the so-called Riemann problems when the evolution is initiated by a jump discontinuity between two uniform values of the initial field [52]. In nonlinear focusing media described by the 1D-NLSE, Riemann problems can give rise to dispersive shock waves (DSWs), a phenomenon that has attracted considerable attention in recent years but considered predominantly for stable media [53]. A particular type of the Riemann problems in modulationally unstable systems, the so-called dam break problem has been studied in detail at the theoretical level in ref. [54, 55].

While the three aforementioned evolution scenarios have been under a very active theoretical investigation over several decades, their experimental realization in a “pure” form can be quite challenging, especially for longer evolution times, due e.g. to the inevitable presence of dissipation or of higher-order nonlinear effects. The experimental observation of the space-time evolutions associated with Riemann problems is especially challenging because of the co-existence of strongly nonlinear oscillatory DSWs (which have enhanced stability properties [54]) and a plane wave (which is subject to classical modulational instability). As a result, at large enough times the two types of the evolution cannot be meaningfully separated, and the eventual long time dynamics are attributed to the developed (nonlinear) stage of spontaneous modulational instability. As a result, at large enough times the two types of the evolution cannot be meaningfully separated, and the eventual long time dynamics are attributed to the developed (nonlinear) stage of spontaneous modulational instability with no regular oscillatory structure present. Indeed, despite some recent progress in the observation of qualitative DSW dynamics in focusing nonlinear optical media [56], there has been no clear experimental demonstration of such dynamics that could be compared with the existing theory.

In this paper, we present water wave experiments in which we demonstrate the persistent “focusing DSW” dynamics in the evolution of water wave packets of large extent and constant amplitude. Performing experiments in a long tank, we consider the evolution of nonlinear deep-water surface gravity waves having their envelope in the form of a large-scale rectangular barrier (a “box”) of finite height. Our experimental observations reveal that, for a range of input parameters, the nonlinear wave train does not get disintegrated by the spontaneous Benjamin-Feir instability but instead, exhibits a regular DSW type behavior that dominates the dynamics of the nonlinear wave at intermediate times.

More specifically, we observe that the initial sharp transitions between the uniform plane wave and the zero background undergo a very special nonlinear evolution leading to the emergence of two counter-propagating dispersive dam break flows having the characteristic DSW structure, in good agreement with the scenario studied theoretically in ref. [54, 55] using semi-classical analysis of the focusing 1D-NLSE. We show that there exist ranges of parameters for which the dynamics observed in the experiment are nearly integrable and quantitatively agree with the theoretical predictions of [54, 55]. We show also that the observed behaviors exhibit significant degree of robustness to perturbative higher-order effects.

We would like to stress that DSWs in shallow water waves (often termed undular bores) are a classical subject of fluid dynamics [57] with numerous contributions over the last 60 or so years [58], starting from the pioneering paper by Benjamin and Lighthill [59]. In particular some recent optical fiber experiments have demonstrated that the light may evolve as a fluid mimicking the features of undular bores [60] and of the dam break in shallow water [61, 62]. In contrast, there has been no experimental demonstration of the DSW dynamics on deep water so far. Our paper presents the first observation of DSWs in deep water waves, supported by the previously developed semi-classical theory [54, 55].

The paper is organized as follows. In Sec. II, we describe our experimental results obtained in a one-dimensional water tank. In Sec. III, we introduce the semi-classical formalism in which the observed dynamical behaviors can be interpreted. In Sec. IV, we perform quantitative comparison between experimental results and the semi-classical theory. In Sec. V, we show that the observed dynamics exhibits some degree of robustness to perturbative higher-order effects. A brief summary of our work is presented in Sec. VI together with a short discussion about possible perspectives.

II. WATER WAVE EXPERIMENT

The experiment was performed in a wave flume at the Hydrodynamics, Energetics and Atmospheric Environment Lab (LHEEA) in Ecole Centrale de Nantes (France). The flume which is 148 m long, 5 m wide and 3 m deep is equipped with a parabolic shaped absorbing beach that is approximately 8 m long. With the addition of pool lanes arranged in a W pattern in front of the beach the measured amplitude reflection coefficient is as low as 1%. Unidirectional waves are generated with a computer assisted flap-type wavemaker. The setup comprises 20 equally spaced resistive wave gauges that are installed along the basin at distances $z_j = 6 + (j - 1)6$ m, $j = 1, 2, \ldots, 20$ from the wavemaker located at $z = 0$ m. This provides an effective measuring range of 114 m.

In the first experimental run presented in Fig. 1(a), the wavemaker produced one single large-scale wavepacket having a near rectangular envelope. The duration $\Delta T_0$ of the wave packet is $\sim 160$ s. The water wave has a carrier
FIG. 1. Experimental results showing the nonlinear evolution of several rectangular wave trains along the 1D water tank. The evolution is plotted in the frame of reference moving at the group velocity $\omega_0/(2k_0)$ of the wave packets. (a) A large scale wave packet of constant amplitude unstable to small perturbations of its envelope is disintegrated by the Benjamin Feir instability (The wave steepness is $k_0a_0 \simeq 0.19$, the carrier period is $T_0 = 0.87$ s and the wave amplitude is $a_0 = 3.7$ cm (see text)). (b) Three “boxes” of constant identical amplitudes, which are not disintegrated by the Benjamin-Feir instability, undergo some strongly nonlinear modulation which propagates with finite speed from the edges of the wavepacket towards the center under the form of counter-propagating dispersive dam break flows with DSW structure (The wave steepness is $0.082$, the carrier period is $T_0 = 0.99$ s and the wave amplitude is $a = 2$ cm (see text)). The thick black dashed lines represent the theoretical breaking lines separating the genus 1 region from the genus 0 region, see calculation details in Sec. III B and Sec. IV A.

In the second experimental run presented in Fig. 1(b), the computer controlled wavemaker produced a sequence of three large-scale wave packets having rectangular envelopes. The three rectangular wave trains are individually generated over a global time interval of $\sim 220$ s where they have increasing durations of $\Delta T_1 = 30$ s, $\Delta T_2 = 45$ s, $\Delta T_3 = 60$ s. The period of the carrier wave has been changed to $T_0 = 0.99$ s ($k_0 = 4.10$ m$^{-1}$, $\lambda_0 = 2\pi/k_0 \simeq 1.51$ m) and the amplitude of the generated envelope has been reduced to $a = 2$ cm which implies that the wave steepness $k_0a_0 = 0.082$ is 2.3 times smaller than in Fig. 1(a).

Fig. 1(a) shows that disintegration of the large rectangular wavepacket occurs as a result of Benjamin-Feir instability, in good qualitative agreement with experimental results previously reported in ref. [4, 5, 8, 9, 63]. Note that some dispersive breaking starts to occur near the edges of the box at a propagation distance of $\sim 42$ m, simultaneously with the growth of the Benjamin-Feir
Nonlinear diffraction of light beams in focusing media has been considered in a few optical experiments. The experiment reported in ref. [71] has investigated diffraction from an edge in a self-focusing nonlinear photorefractive medium using a spatially incoherent light beam. In the very recent experimental work [56] the evolution of a 1D optical beam having a square profile was observed in a focusing photorefractive medium. While some of the robust qualitative features of the DSW dynamics predicted by the semi-classical 1D-NLSE theory [53] have been observed and interpreted in the context of the “topological control of extreme waves” [56], the quantitative comparison with the theory was limited because of the significant competition between the DSW dynamics and noise amplification in the modulationally unstable photorefractive medium.

Recent optical fiber experiments reported in ref. [72] have also evidenced a spatio-temporal evolution very similar to the one that we observe with the rectangular wave train of the smallest width ($\Delta T = 30\,\text{s}$), compare Fig. 1(b) with Fig. 3(a) of ref. [72]. However the work reported in ref. [72] was concentrated on the emergence of Peregrine-like events and did not allow for a meaningful quantitative, or even qualitative, identification of the observed wave patterns with DSWs due to very few oscillations observed.

### III. DISPERSIVE FOCUSING DAM BREAK FLOWS: SEMI-CLASSICAL THEORY

The experimental results shown in Fig. 1(b) clearly indicate that the envelope of the wavepacket develops oscillations with the typical period significantly smaller than the temporal extent of the wavepacket. This separation of scales suggests the usefulness of an asymptotic WKB-type approach to the theoretical understanding of the arising dynamics. In this section, we show that the mathematical framework of dispersive hydrodynamics [73], a semi-classical theory of nonlinear dispersive waves, provides some insightful interpretation of the experimentally observed multi-scale coherent structures.

#### A. The semi-classical framework

The experimental results reported in Fig. 1(b) can be interpreted within the framework of the focusing 1D-NLSE (2), written in a dimensional form as a spatial evolution equation

$$
\frac{\partial A}{\partial z} + \frac{k_0}{\omega_0} \frac{\partial^2 A}{\partial t^2} + \alpha k_0^2 |A|^2 A = 0,
$$

(1)

where $A(z,t)$ represents the complex envelope of the water wave that changes in space $z$ and in time $t$ [74]. $\alpha = 0.91$ is a corrective term to the cubic nonlinear term. It has been introduced in Eq. (1) in order to take into
account finite depth effects. In our experiment where
the water depth \( h \) is 3 m, the numerical value of \( k_0 h \) is
\( \sim 12.3 \). This is large enough to consider that the condition
of propagation in deep water regime is well verified but not large enough not to include some small corrective
term in the nonlinear coefficient. A comprehensive
discussion about the influence of finite depth effects on
the values of linear and nonlinear coefficients is given in
Appendix A.

In the experimental evolution reported in Fig. 1(b),
the dynamics of the nonlinear wave is ruled by the interplay of two characteristic length scales associated
with the temporal duration \( \Delta T_j \) \( (j = 1, 2, 3) \) of the
rectangular envelopes, namely, the nonlinear length
\( L_{NL} = 1/(\alpha k_0^3 a^2) \) and the linear dispersion length
\( L_D = (\omega_0 \Delta T_j)^2/(2 k_0) = g \Delta T_j^2/2 \). Normalizing the propagation distance \( z \) along the flume as \( \xi = z/\sqrt{L_{NL} L_D} \), the physical time as \( \tau = t/\Delta T_j \), the complex field envelope
as \( \psi = A/a \), Eq. (1) takes the following dimensionless
"semi-classical" form

\[
  i \epsilon \frac{\partial \psi}{\partial \xi} + \frac{\epsilon^2}{2} \frac{\partial^2 \psi}{\partial \tau^2} + |\psi|^2 \psi = 0
\]  

(2)

where \( \epsilon = \sqrt{L_{NL}/L_D} \ll 1 \) is a small dispersion parameter.

The numerical values of the physical and dimensionless parameters describing our experiment are reported
in Table I for the three rectangular wave trains with temporal widths \( \Delta T_j \) \( (j = 1, 2, 3) \). It can be easily seen
that our experiments are always placed in a regime where
\( L_{NL} \ll L_D \) which implies that the experimental values
of the \( \epsilon \) parameter are much smaller than 1. Therefore
our experimental observations can be interpreted within
the mathematical framework of dispersive hydrodynamics [73], a semi-classical theory of nonlinear dispersive
waves suitable for such multi-scale coherent structures.

FIG. 2. Numerical simulation of Eq. (2) showing (a) the space time evolution of the wave field having a profile specified by
Eq. (3) at \( \xi = 0 \) \( (q = 1, T = 1, \epsilon = 0.04) \). The space-time evolution is separated into three regions of increasing genus \( g \)
(see text). The genus \( g = 0 \) region corresponds locally to the plane wave solution. The genus \( g = 1 \) region is associated to
DSWs that are generated from the edges of the box. The genus \( g = 2 \) region emerges from the collision of the two focusing
dam break flows, see also the Supplemental Material Video S2. Curves plotted in blue lines in (b), (c), (d) represent the wave amplitude profiles at \( \xi = 0.25, \xi = 0.35, \xi = 0.477 \), respectively. Red dashed lines in (b) and (c) represent the amplitudes of the modulated cnoidal waves that are determined from Eqs. (5), (6), (7). (e), (f) Spectral (IST) portraits of isolated structures
made at \( \xi = 0.477 \). The spectral portrait in (e) is mostly composed of two complex conjugate bands demonstrating that the
analyzed structure has a genus \( g = 1 \) (soliton-like structure, see text). The spectral portrait in (f) is composed of three bands
demonstrating that the genus of the analyzed structure is \( g = 2 \) (breather-like structure).
Here is the processed text:

In this section, we summarize some important theoretical results about the 1D-NLSE box problem. The quantitative comparison between these theoretical results and the experimental results will be presented in Sec. IV.

First, it is instructive to use the Madelung transform $\psi(\tau, \xi) = \sqrt{\rho(\tau, \xi)} \exp(i e^{-1} \int^\tau u(\tau', \xi) d\tau')$ to represent the 1D-NLSE in the dispersive hydrodynamic form

$$u_\xi + uu_\tau - \rho_\tau = 0,$$

where $\rho$ and $u$ are analogues of the fluid depth and velocity respectively. Within the hydrodynamic interpretation [Eq. (4)], the box initial data [Eq. (3)] can be viewed as a combination of two hydrodynamic dam breaks (i.e. step transitions from finite depth $\rho = q^2$ to “dry bottom” $\rho = 0$) of opposite polarities, placed at the distance $2T$ from each other. It is important to stress that, the 1D-NLSE “fluid” here has nothing to do with the underlying water wave context of the original problem; moreover, due to the focusing nature of the 1D-NLSE (2), the classical “pressure” term in the hydrodynamic representation (4) is negative.

The dispersive hydrodynamic representation (4) provides an important insight into the 1D-NLSE evolution of different types of initial data. Linearising system (4) about a constant equilibrium flow $\rho = \rho_0$, $u = 0$ (a plane wave of the 1D-NLSE) one obtains the usual 1D-NLSE dispersion relation $\omega = \pm k \sqrt{(e k)^2 - 4 \rho_0}$ implying modulational instability of plane waves for long enough waves with $ek < 2\sqrt{\rho_0}$. This is the classical Benjamin-Feir instability, which is manifested as a dispersion-dominated, linear wave phenomenon within 1D-NLSE. The initial exponential growth of harmonic, long-wave perturbations is mediated by nonlinearity leading to the formation of Akhmediev breathers or more complicated breather structures associated with integrable turbulence [75, 76].

The focusing 1D-NLSE dam break problem (2), (3) is rather special in the sense that it triggers both nonlinearity and dispersion in (4) from the early time of the evolution. As a result, it leads to the formation of a coherent, unsteady nonlinear wave structure that is very different from those arising in the development of the BF instability or in the evolution of smooth humps. This structure can be viewed as a focusing counterpart of the well-known dispersive-hydrodynamic phenomenon, called a dispersive shock wave (DSW) [53], which represents an expanding, nonlinear wave train connecting two disparate constant fluid states. DSW is described by a slowly modulated, locally periodic wave solution of a dispersive equation (1D-NLSE in our case) gradually transforming from a soliton at one edge to a vanishing amplitude, harmonic wave at the opposite edge. The special modulation providing such a transition has been found in [77] as a self-similar solution of the Whitham modulation equations [57] associated with the 1D-NLSE. Typically, DSWs are the features of stable media, described by such equations as the KdV or defocusing NLS equations (see [53] and references therein) but for a special Riemann data (dam break) the DSWs can be generated in unstable (focusing) media ([78], [54], [79]). The persistence of DSW dynamics in focusing dam break problem is due to a special "hyperbolic" modulation as explained below.

The periodic solutions of the focusing 1D-NLSE are known to be modulationally unstable with respect to small initial perturbations (see e.g. [80]), but this modulational instability is more subtle than the BF instability of a plane wave. It turns out that the instability of non-linear periodic solution can be "inhibited" by a special modulation yielding a "hyperbolic" wave behavior characterized by finite speeds of propagation. This modulation is described by a similarity solution of the Whitham modulation equations associated with the 1D-NLSE [81], [82] and it is exactly the modulation that is realised in the dispersive regularization of the dam break flow in the focusing 1D-NLSE and enables the persistent DSW structure that can be observed in an experiment.
The box problem (2), (3) has been studied analytically in [54, 55] using a combination of the Whitham modulation theory and an IST-based Riemann-Hilbert problem approach [83]. The theoretical developments of [54] important for the interpretation of our experimental results in water waves can be conveniently explained by considering Fig. 2 where the numerical simulation of the focusing dam break problem for the 1D-NLSE equation is presented along with the results of the so-called “local IST” analysis [76] of the emerging wave structures (Fig. 2(c),(f)). The plots in Fig. 2(c),(f) show the qualitative changes of the nonlinear (IST) spectra occurring in the course of the wave propagation. These spectra and the associated nonlinear waves are characterised by a fundamental integer index \( g \) called genus which enables classification of the emerging wave structures in terms of the number \( N = g + 1 \) of “nonlinear Fourier modes” involved.

The genus itself characterises topology of the hyperelliptic Riemann surfaces associated with the special class of the 1D-NLSE solutions, called finite-gap potentials (see e.g. [74, 84]). As shown in [54] the solutions of the semi-classical 1D-NLSE box problem can be asymptotically described by slowly modulated finite-gap 1D-NLSE solutions with the genus changing across certain lines in \( \tau-\xi \) plane called breaking curves. In particular, the wave structures regularising the initial dam breaks at \( \tau = \pm T \) in the box problem have genus \( g = 1 \) while the genus 2 structures emerge as a result of the interaction of two counter-propagating dispersive dam break flows having the signature structure of dispersive shock waves (DSWs) [53].

The asymptotic solution of the box problem for the small dispersion 1D-NLSE (2) has different form in different regions of \( \tau-\xi \) plane (see Fig. 2). For \( \xi < \xi^* \), where \( \xi^* = \frac{T}{2\sqrt{2q}} \) the solution represents two counter-propagating focusing DSWs—seen as the genus one regions in Fig. 1—connecting two disparate genus zero states: the “dry bottom” state \( \psi = 0 \) at \( |\tau| > T \) and the constant state \( \psi = q \) for \( (2\sqrt{2q}\xi - T) < \tau < (-2\sqrt{2q}\xi + T) \). The local structure of both DSWs is described by the elliptic (“cnoidal”) solution of the 1D-NLSE

\[
\rho = (q + b)^2 - 4qb \cdot \frac{1}{m} \left( 2\sqrt{q/m} (\tau - a_0) e^{-1}; m \right),
\]

where \( m \) is a Jacobi elliptic function with the modulus \( m \in [0, 1] \) given by

\[
m = \frac{4qb}{a^2 + (q + b)^2}.
\]

The modulation parameters \( a(\tau, \xi), b(\tau, \xi) \) are found from equations

\[
a = \pm \frac{2q}{m\mu(m)} \sqrt{1 - m} \mu^2(m) + m - 1],
\]

\[
b = \frac{q}{m\mu(m)} \left[ 2 - m \right] \mu(m) - 2(1 - m)],
\]

where \( \mu(m) = E(m)/K(m) \). \( K(m) \) and \( E(m) \) are the complete elliptic integrals of the first and second kind respectively. The signs \( \pm \) in (6), (7) correspond to right- and left-propagating waves. The initial position \( \tau_0 \) in (5) is given by \( \tau_0 = \pm T \). In practice, \( \tau_0 \) depends on the way the sharp edges of the “box” are smoothed in the experimental signal or in numerical simulations so for a practical comparison with the theory, one chooses \( \tau_0 \) by fitting to the experimental/numerical data.

Solution (5), (6), (7) describes two symmetric oscillatory structures exhibiting the fundamental 1D-NLSE solitons \( m = 1 \) with the amplitude \( |\psi_m| = 2q \), located at \( \tau = \pm T \). The structures degenerate, via the modulated elliptic regime, into the vanishing amplitude linear wave \( m = 0 \) at the internal moving edges propagating towards the box centre with constant velocities \( \pm 2\sqrt{2q} \). The solutions computed from Eq. (5), (6), (7) are plotted with red dashed lines in Fig. 2(b),(c). Very good quantitative agreement is found between these theoretical solutions and numerical simulations of the 1D-NLSE (blue lines in Fig. 2(b),(c)).

The equation of the first breaking curve \( \Gamma_1 \) separating the genus \( g = 0 \) region from the genus \( g = 1 \) region in the diagram in Fig. 2(a) is

\[
\Gamma_1 : \quad \xi = \frac{T - |\tau|}{2\sqrt{2q}}
\]

Equation (8) yields the DSW collision time \( \xi^* = \frac{T}{2\sqrt{2q}} \) corresponding to Fig. 2(c). For \( \xi > \xi^* \) the region with \( g = 2 \) is formed, confined to another breaking curve (not shown in Fig. 2(a)). One of the prominent features of the genus 2 region is the occurrence of a large-amplitude breather at the center with the characteristics close to those of the Peregrine soliton (see Fig. 2(d)) as predicted in [54] and experimentally observed in fiber optics in [72].

We now demonstrate that deep water waves, while providing the classical example of the Benjamin-Feir instability, present also a medium supporting the “hyperbolic” dispersive dam break (DSW) scenario of the wavepacket evolution for a range of input parameters. This is done in Sec. IV by a quantitative comparison of the water wave experiment with the modulated 1D-NLSE solution [54] and the “local IST” analysis of the experimentally observed wave patterns [76, 85], confirming the spectral topological index (genus) of generated waves.
IV. DATA ANALYSIS AND COMPARISON WITH THE THEORY

A. Numerical simulations of the 1D-NLSE, breaking lines, collision points and modulated cnoidal waves

In this section, we focus on the quantitative comparison between experimental results and the semi-classical theory. First we have performed numerical simulations of Eq. (1) by taking as initial condition the complex envelope \( A(z_1,t) \) of the signal measured by the gauge closest to the wavemaker \((z_1 = 6 \text{ m})\). The complex envelope has been computed from the experimentally-recorded signals by using standard techniques based on the Hilbert transform, as discussed e.g. in ref. [74]. Fig. 3 shows the modulus \(|A(z_{20},t)|\) of the complex envelope that is computed at \( z_{20} = 120 \text{ m} \), the position where is located the gauge furthest from the wavemaker. The agreement between the experimental results and the numerical simulations is quantitatively good for each of the three generated rectangular wave trains.

![Graph](image)

**Fig. 3.** Modulus of the water wave envelopes with durations (a) \( \Delta T_1 = 30 \text{ s} \), (b) \( \Delta T_2 = 45 \text{ s} \), (c) \( \Delta T_3 = 60 \text{ s} \). The red (resp. blue) lines represent the experimental envelopes of the signal recorded at \( z_1 = 6 \text{ m} \) (resp. \( z_{20} = 120 \text{ m} \), close to (resp. far from) the wavemaker. The magenta lines represent the envelopes computed from the numerical simulation of Eq. (1) \((k_0 = 4.1 \text{ m}^{-1}, \omega_0 = 6.34 \text{ s}^{-1}, \alpha = 0.91)\) by taking as initial condition the complex envelope measured by the gauge closest to the wavemaker (red lines, \( z_1 = 6 \text{ m} \)).

As a first valuable test of the theory introduced in Sec. III, we plot the linear breaking curves separating the genus 0 (plane wave) regions from the genus 1 (DSW) regions. Rephrasing Eq. (8) in physical units, we easily find that the slopes \( s_{\pm} \) of the breaking lines in the \( z-t \) plane read \( s_{\pm} = \pm \omega_0/\left(4a\sqrt{\alpha k_0^3}\right) \) and that the collision between the two counterpropagating dam break flows occurs at the position \( z^* = \omega_0 \Delta T_j/(8a\sqrt{\alpha k_0^3}) \).

The numerical values of the positions at which the collisions between the counterpropagating dam break flows occur are summarized in Table I for the three boxes generated in our experiment. The breaking lines separating the genus 0 region from the genus 1 region are plotted in Fig. 1(b). It can be readily seen that there is a good quantitative agreement between theoretical and experimental results. In particular, the distance at which the collision is predicted to occur for the largest box is larger than the physical length of the water tank and it is clear that the collision between the dam break flows is not experimentally observed in this situation, see Fig. 1(b) and also Fig. 4.

To go one step further in the analysis of our experiment, we now compare experimental data with the modulated cnoidal solution that has been discussed in Sec. III. To this end Eqs. (5), (6), (7) are solved and rephrased to physical variables according to the transformations introduced in Sec. IIIA. Considering only the box of largest size where the counterpropagating dam break flows are the most developed near the end of the water tank \((z \sim 120 \text{ m})\), Fig. 4 shows that the modulated cnoidal wave envelope determined from the semi-classical theory matches quantitatively well the experimental results over the whole range of evolution of the dam break flows (i.e. from \( z = 6 \text{ m} \) to \( z = 120 \text{ m} \)). The numerical value of \( \tau_0 \) has been determined from the signal measured at the last gauge, at \( z_{20} = 120 \text{ m} \) \((\tau_0 = -0.48)\). Note that good quantitative agreement between experimental and theoretical results has been obtained for the two boxes of smaller sizes, though it is not presented here for the sake
of simplicity.

B. Inverse scattering transform analysis of the experimental data

Some other insights into our experimental results can be obtained from the perspective of the inverse scattering transform (IST) method. The configuration considered in our experiments corresponds to the initial value problem specified by Eq. (2) and Eq. (3). As shown by Zakharov and Shabat [86], the nonlinear dynamics in this kind of problem is determined by the IST spectrum that is composed of two components: a discrete part related to the soliton content of the box data and of a continuous part related to the dispersive radiation. In particular, it is known that the number \( N \) of solitons embedded inside the initial box is given by \( N = \text{int}(1/2 + 1/(\pi \epsilon)) \), where \( \text{int}(x) \) denotes the integer part of \( x \) [69, 70, 87–89].

As shown in Table I, the number of solitons that are embedded inside the rectangular wave trains is predicted to grow from \( N = 3 \) for the smallest box (\( \epsilon = 0.095 \)) to \( N = 7 \) for the largest box (\( \epsilon = 0.047 \)). To check this result from experimental signals and to investigate more in depth the integrable nature of the features experimentally observed, we now consider the non-self-adjoint Zakharov-Shabat eigenvalue problem

\[
\epsilon \frac{dY}{d\tau} = \begin{pmatrix} -i\lambda & \psi_0 \\ -\psi_0^* & i\lambda \end{pmatrix} Y
\]  

(9)

that is associated with Eq. (2). \( Y(\tau; \lambda, \epsilon) \) is a vector where \( \lambda \in \mathbb{C} \) represent the eigenvalues composing the discrete spectrum associated with the soliton content of the complex envelope \( \psi_0 \) measured at some given propagation distance. Note that the linear spectral problem (9) can be identified as one half of the Lax pair for Eq. (2) [90].

Fig. 5 shows the complex eigenvalues \( \lambda \) that are computed from the numerical resolution of Eq. (9) made by using the Fourier collocation method described and used e.g. in ref. [76, 85, 90]. For the sake of clarity only the upper part of the complex plane is represented but complex conjugate eigenvalues are obviously obtained from the numerical resolution of Eq. (9). Fig. 5 (left column) shows the complex eigenvalues computed for the three experimental envelopes (blue lines in Fig. 3) measured at \( z_1 = 6 \) m, close to the wavemaker. Remarkably, nearly all of the non-zero eigenvalues numerically computed are distributed close to the vertical imaginary axis, demonstrating that the solitons embedded inside the three rectangular wave trains have negligible velocity at the initial time. In fact, the rigorous semi-classical IST analysis of the box problem [55] shows that the discrete spectrum is located on the imaginary axis as \( \epsilon \to 0 \). The number of discrete eigenvalues found from numerical IST analysis and reported in Fig. 5(a), (c), (e) is in good agreement with results reported in Table I.

Fig. 5 (right column) shows the complex eigenvalues that are computed for the three experimental envelopes (blue lines in Fig. 3) measured at \( z_2 = 120 \) m, far from the wavemaker. In the IST theory of the 1D-NLSE, these discrete eigenvalues do not change in the evolution time. In the experiment, we find that this isospectrality condition is not perfectly verified because of the unavoidable occurrence of small perturbative effects. It is however clear that the number of eigenvalues is preserved over the propagation distance characterizing our experiment, i.e. between \( z_1 = 6 \) m and \( z_2 = 120 \) m. Moreover the global shape of the IST spectra is well preserved (compare left and right columns in Fig. 5), thus confirming the nearly integrable nature of the features observed in the experiment.

V. WATER WAVE EXPERIMENT: ROBUSTNESS OF THE OBSERVED DYNAMICS TO HIGHER-ORDER EFFECTS

A. Space-time evolution

In this section, we demonstrate that the observed dynamics exhibits some degree of robustness to higher-order effects that unavoidably perturb the wave evolution when experimental parameters are changed in such a way that the strength of nonlinearity increases. To do so, we have simply increased the frequency of the wavemaker from \( f_0 = 1/T_0 = 1.01 \) Hz to \( f_0 = 1.28 \) Hz while also decreas-
ing the amplitude of the wave envelope from 2 cm to 1.4 cm. With these changes, the nonlinear length decreases from $L_{NL} = 39.86$ m to $L_{NL} = 18.77$ m while the linear dispersive lengths $L_D$ remain unchanged and identical to those summarized in Table 1.

Fig. 6 shows the space-time evolutions of the three rectangular envelopes that are observed in this situation where the nonlinearity strength is increased, see also the Supplemental Material Video S3. Contrary to the experimental space-time evolutions considered in Sec. II and in Sec. IV, there is now a marked asymmetry in the space evolution of the three wave packets. This represents a clear qualitative indication of the presence of higher-order effects associated with some red-shift of the wave Fourier spectrum which cannot be described by the 1D-NLSE but rather by other models like the unidirectional Zakharov equation or the Dysthe equation [50]. Note such higher-order effects are already noticeable in the details of Fig. 4(a) where the envelope of the modulated cnoidal wave fits better the left part than the right part of the wave packet at large distances from the wavemaker.

Despite the undisputable presence of higher-order nonlinear effects in water wave experiments reported in Fig. 6, it is clear that the scenario of emergence of counter-propagating dispersive dam break flows remains qualitatively well observed. White lines plotted in Fig. 6 represent the breaking lines that are computed from the semi-classical theory presented in Sec. III (see Eq. 8). At a qualitative level, the breaking lines still clearly separate regions where DSWs (genus 1) are found from regions where the (unmodulated) plane waves (genus 0) are found. Therefore these breaking lines retain some relevance to the description of the dynamics, even in the presence of perturbative higher-order effects.

### B. Nonlinear spectral analysis

In the regime where higher-order nonlinear effects influence the dynamics, the wave system is no longer described by the 1D-NLSE and rigorously speaking, the dynamics is no longer of an integrable nature. However mathematical tools of nonlinear spectral analysis can still be used to advantage for getting relevant information about the wave system. For instance, it has been shown in ref. [85] that dissipative effects occurring in a water tank produce some slow modulation of the spectral (IST) portrait of the Peregrine soliton recorded in water wave experiments reported in ref. [91]. More recently the IST has been applied to characterize coherent structures in dissipative nonlinear systems described by the cubic Ginzburg-Landau equation [92].

Here, we apply nonlinear spectral analysis to examine the soliton content of the rectangular wave packets in the propagation regime displayed in Fig. 6. For the sake of simplicity, we only present here the numerical results that are associated with the box of duration $\Delta T_2 = 45$ s (central box in Fig. 6). As described in Sec. IV B and also more in details in ref. [76, 85], the determination of the discrete IST eigenvalues relies on the numerical resolution of Eq. (9) for the potentials $\psi_0$ that are measured in the experiment.

Fig. 7(a) shows the rectangular envelope of the central box of Fig. 6 that has been measured at $z_1 = 6$ m, close to the wavemaker. Fig. 7(b) shows the corresponding discrete IST spectrum which is composed of 7 eigenvalues located well above the real axis. Let us recall that the discrete IST spectrum of the same box was only composed of 5 eigenvalues in the regime where the dynamics was described by the integrable focusing 1D-NLSE, see Sec. IV B. The result of an increased nonlinearity is therefore that the number of solitons embedded within the box has increased, which is not that surprising but which is here substantiated and quantified with the IST.

Fig. 7(c) shows the envelope of the central box of Fig. 6 that has been measured at $z_2 = 120$ m, far from the wavemaker. Fig. 7(d) shows the corresponding discrete IST spectrum. Comparing Fig. 7(d) and Fig. 5(d), we obtain the clear signature that higher-order effects significantly perturbate the discrete IST spectrum (i.e. the soliton content of the rectangular wave packet). Six eigenvalues well above the real axis are observed instead of seven near the wavemaker, see Fig. 7(a). Moreover the real parts of most of these eigenvalues become nonzero which means that the solitons embedded within the box have acquired some velocity, a feature that is fully compatible with the fact that the rectangular box exhibits some slow drift in the space-time plot, see Fig. 6.

To investigate the change of the genus [54] of the co-

![Figure 6](image-url)
amplitude (m)

FIG. 7. (a) Envelope of the central wavepacket of Fig. 6 measured at \( z_1 = 6 \) m, close to the wavemaker. (b) Discrete IST spectrum of the wavefield plotted in (a). (c) Envelope of the central wavepacket of Fig. 6 measured at \( z_20 = 120 \) m, far from the wavemaker. (d) Discrete IST spectrum of the wavefield plotted in (c). (e) Local IST spectrum of the coherent structure highlighted in red in (c). (f) Local IST spectrum of the coherent structure highlighted in magenta in (c).
the DSW dynamics plays the dominating role and the
effects of the Benjamin-Feir instability can be neglected,
but it would be interesting to examine in detail how the
Benjamin-Feir instability affects the DSW structure at
longer propagation times. Finally, our experiments have
shown that the generated DSWs exhibit certain robust-
ness to higher-order nonlinear effects. It is another in-
teresting and challenging question to investigate these
higher order effects more in detail from the theoretical
perspective.

Appendix A: Finite depth effects

As shown in ref. [95] the weakly nonlinear, narrow-
banded approximation of the fully nonlinear irrotational
inviscid water wave equations is the 1D-NLSE under
the following form

$$\frac{\partial A}{\partial t} + \frac{1}{2} \frac{\partial A}{\partial z} + \frac{1}{8} \left( \frac{\partial A}{\partial z} \right)^2 + \gamma A = 0,$$

(A1)

where $A$ is the complex envelope of the water wave. $\nu$
is the correction to the group velocity for finite depth.
$\kappa$ and $\gamma$ are coefficients that in general depend on
the water depth $h$ at the dominant wave number $k_0$ and
at the corresponding angular frequency $\omega_0$.

The general expressions of $\nu$, $\kappa$ and $\gamma$ are given by (see
e.g. ref. [66] and see ref. [95] for the derivation)

$$\nu = 1 + \frac{2k_0h}{\sinh(2k_0h)} \quad (A2)$$

$$\kappa = -\nu^2 + 2 + 8(k_0h)^2 \frac{\cosh(2k_0h)}{\sinh^2(2k_0h)} \quad (A3)$$

$$\gamma = \frac{\cosh(4k_0h) + 8 - 2 \tanh^2(k_0h)}{8 \sinh^2(k_0h)}$$

$$\left( \frac{2k_0h + 0.5 \nu^2}{\sinh^2(2k_0h)} - \frac{\nu^2}{4} \right) \quad (A4)$$

For a hydrodynamic wavemaker problem, it is conven-
te to use the 1D-NLSE in the form of an evolution
equation in space. Using changes of variables described
in ref. [74, 96], one obtains the following evolution equa-
tion

$$\frac{\partial A}{\partial z} + \kappa \frac{k_0}{\nu^2} \frac{\partial^2 A}{\partial \omega_0^2} + \gamma k_0^3 |A|^2 A = 0, \quad (A5)$$

in the frame of reference moving with the group velocity
of the wave packets.

In the experiments reported in Fig. 1(b), the numerical
value of $k_0h$ is 12.3 and the numerical values of the
corrective terms $\nu$ and $\kappa$ given by Eqs. (A2) and (A3)
are very close to unity. However the numerical value of $\gamma$
is $\sim 0.91$ which means that the finite-depth correction to
the cubic nonlinearity is small but not negligible. There-
fore our experiments are described by Eq. (A5) in which
the values of the corrective terms are set to $\nu = 1$
and $\gamma = \alpha = 0.91$.

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