ON THE REGULARITY OF CROSSED PRODUCTS

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Abstract. We study some generalizations of the notion of regular crossed products \( K \ast G \). For the case when \( K \) is an algebraically closed field, we give necessary and sufficient conditions for the twisted group ring \( K \ast G \) to be an \( n \)-weakly regular ring, a \( \xi^*N \)-ring or a ring without nilpotent elements.

1. Introduction

Let \( G \) be a group, \( U(K) \) the group of units of the associative ring \( K \) with identity and let \( \sigma : G \to \text{Aut}(K) \) be a map of \( G \) into the group \( \text{Aut}(K) \) of automorphisms of \( K \). Let \( K \ast G = K_\rho^*G = \{ \sum_{g \in G} u_g \alpha_g \mid \alpha_g \in K \} \) be the crossed product (in the sense of [1]), of the group \( G \) over the ring \( K \) with respect to the factor system

\[
\rho = \{ \rho(g,h) \in U(K) \mid g, h \in G \}
\]

and the map \( \sigma : G \to \text{Aut}(K) \). Moreover we assume that the factor system \( \rho \) is normalized, i.e. \( \rho(g,1) = \rho(1,g) = \rho(1,1) = 1 \) for any \( g \in G \).

In particular, if \( \sigma = 1 \), then the crossed product \( K \ast G \) is called a \textit{twisted group ring}, which we denote by \( K_\rho G \). If the factor system \( \rho \) is unitary, i.e. \( \rho(g, h) = 1 \) for all \( g, h \in G \), then \( K \ast G \) is called a \textit{skew group ring} and is denoted by \( K^\ast G \). In the case, when \( \rho = 1 \) and \( \sigma = 1 \), then \( K \ast G \) is the ordinary group ring \( KG \).

In the present paper we study properties of crossed products \( K \ast G \) which are generalizations of the notion of a regular ring. For the case when \( K \ast G \) is a twisted group ring over the algebraically closed field \( K \), we give necessary and sufficient conditions for \( K \ast G \) to be an \( n \)-weakly regular ring \( (n \geq 2) \), a \( \xi^*N \)-ring or a ring without nilpotent elements. Our investigation can be considered as a generalization of certain results of [2 3 4 7 11 12] earlier obtained for group rings. Note that we exclude the case when \( K \ast G \) is a skew group ring, so we do not cite any reference from that topic.

2. Twisted group algebras without nilpotent elements

Denote the \( K \)-basis of \( K \ast G \) by \( U_G = \{ u_g \mid g \in G \} \). The multiplication of \( u_g, u_h \in U_G \) is defined by \( u_g u_h = \rho(g,h) u_{gh} \), where \( \rho(g,h) \in \rho \) and \( g, h \in G \). The factor system \( \rho \) of the crossed product \( K \ast G \) is called \textit{symmetric}, if for all elements \( g, h \in G \) the condition \( gh = hg \) yields \( \rho(g,h) = \rho(h,g) \). The finite subset \( \text{Supp}(a) = \{ g \in G \mid \alpha_g \neq 0 \} \) of \( G \) is called the \textit{support} of the element \( a \in K \ast G \).

We shall freely use the following.

\textbf{Lemma 1.} Let \( K \ast G \) be a crossed product and suppose that \( axb = c \) for some \( x, a, b, c \in K \ast G \). If \( H \) is the subgroup of \( G \) generated by \( \text{Supp}(a), \text{Supp}(b) \) and \( \text{Supp}(c) \), then there exists an element \( y \in K \ast H \), such that \( ayb = c \).

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Proof. Indeed, if \( x = y + z \), then \( aby + azb = c \), where \( y = \sum_{h \in H} u_h \alpha_h \) and \( z = \sum_{g \in H} u_g \beta_g \). This shows that \( \text{Supp}(aby) \subseteq H \). Since \( fgh \not\in H \) for \( f \in \text{Supp}(a) \), \( g \in \text{Supp}(b) \) and \( h \in \text{Supp}(b) \), we conclude that \( aby = 0 \) and \( aby = c \), as it was requested. \( \square \)

**Corollary 1.** If \( g \in G \) has infinite order, then \( u_g - 1 \) is neither a one-sided zero divisor, nor a one-sided invertible element of the crossed product \( K \ast G \).

Proof. In fact, if \( u_g - 1 \) is either a one-sided zero divisor, or a one-sided invertible element of \( K \ast G \), then by Lemma \( \square \) we may assume that \( u_g = u_1 \) is also such an element of \( K \ast H \), where \( H = \langle g \rangle \) is an infinite cyclic group. But \( H \) is an ordered group, a contradiction. \( \square \)

For twisted group algebras we give a refinement of Corollary 2 and Lemma 2 of \( \square \) (see p.68) which were earlier proved for group rings.

**Theorem 1.** Let \( K \ast G \) be a twisted group algebra of a torsion group \( G \) over the algebraically closed field \( K \). The ring \( K \ast G \) does not contain nilpotent elements if and only if the following conditions hold:

(i) \( G \) is an abelian group;
(ii) the order of every elements in \( G \) is invertible in \( K \);
(iii) the factor system \( \rho \) is symmetric.

Proof. Assume that the conditions (i), (ii) and (iii) hold. Then the twisted group ring \( K \ast G \) is commutative. If \( x \in K \ast G \) is a nonzero nilpotent element and \( H = \langle \text{Supp}(x) \rangle \), we conclude that \( K \ast H \) is a commutative artinian ring with a nonzero nilpotent element \( x \). So, by Theorem 2.2 of (\( \square \), p.415), we get a contradiction.

Conversely, let \( K \ast G \) be a twisted group ring without nilpotent elements. If \( g \in G \) is of order \( n \) and \( u_g^n = u_1 \alpha_g \), where \( \alpha_g \in U(K) \), then there exists an element \( \mu_g \in U(K) \) such that \( \mu_g^n = \alpha_g^{-1} \), because \( K \) is algebraically closed. So for the element \( v_g = u_g \mu_g \) we have \( v_g^n = 1 \). Obviously,

\[
x = (v_g - 1)u_h (1 + v_g + v_g^2 + \cdots + v_g^{n-1})
\]

is a nilpotent element of \( K \ast G \) for all \( h \in G \) as far as \( x^2 = 0 \). Thus \( x = 0 \), so we conclude that

\[
(1) \quad u_h = v_g u_h v_g^i \quad (0 \leq i \leq n - 1).
\]

Examining the supports we can deduce that \( h^{-1}gh = g^{-i} \) (\( h \in G \)). Therefore all cyclic subgroups of \( G \) are normal. This implies that \( G \) is either abelian or hamiltonian. If \( g h = h g \), then \( i = n - 1 \) and by (1) it follows that \( u_h v_g = v_g u_h \), since \( v_g^n = 1 \) is the identity element of \( K \ast G \). So we conclude that \( \rho(g, h) = \rho(h, g) \), i.e. the factor system \( \rho \) is symmetric and condition (iii) holds.

If \( \text{char}(K) = p > 0 \) and \( G \) contains an element \( g \) of order \( p \), then

\[
(1 + v_g + v_g^2 + \cdots + v_g^{p-1})^p = 0
\]

and we get a contradiction. This implies that condition (ii) also follows.

Assume that \( G \) is hamiltonian and \( \langle g, h \mid g^4 = h^4 = 1, g^2 = h^2, g^b h = g^{-1} \rangle \cong Q_8 \) is the quaternion group of order 8. Then \( h^{-1}gh = g^{-1} \) and \( i = 1 \). Therefore in this case by (1) we have \( u_h = v_g u_h v_g \), i.e.

\[
(2) \quad v_h = v_g v_h v_g,
\]
where \( v_h = u_h \mu_h \) and \( v_g^4 = v_h^4 = 1 \). Since \( G \) contains 2-elements, it follows from (ii) that \( \text{char}(K) \neq 2 \).

\( K \) being an algebraically closed field, it is clear that there exist nonzero elements \( \alpha, \beta \in K \) for which \( \alpha^2 + \beta^2 = 0 \). Then by [2] it is easy to verify that

\[
w = \alpha(v_g^2 v_h - v_h) + \beta(v_g^3 v_h - v_g v_h)
\]

is a nonzero nilpotent element of \( K \).

Indeed, \( h \in \text{Supp}(\alpha(v_g^2 v_h - v_h)) \), but \( h \notin \text{Supp}(\beta(v_g^3 v_h - v_g v_h)) \). Thus we have \( w \neq 0 \). Moreover, by [2] we obtain that \( u_h^2 v_g = v_g u_h^2 \) and \( u_h v_g^2 = v_g u_h^2 \). Then \( w^2 = (v_g^2 - 1)^2(\alpha v_h + \beta v_g v_h)^2 \). Since \( (v_g^2 - 1)^2 = 2(1 - v_g^2) \) and

\[
(\alpha v_h + \beta v_g v_h)^2 = (\alpha^2 + \beta^2) v_g^2 + \alpha \beta v_g^2 (v_g^2 + 1)v_g
\]

we obtain \( w^2 = 2(1 - v_g^2)\alpha \beta v_h^2 (1 + v_g^2)v_g = 0 \), which is impossible. Hence condition (i) follows, as requested. \( \square \)

3. Regular crossed products

An associative ring \( R \) with unity is called regular (strongly regular) if for every \( a \in R \) there is an element \( b \in R \), such that \( aba = a \) (\( ba^2 = a \), respectively). A ring \( R \) is called \( \xi \)-ring (\( \xi^* N \)-ring) if for every \( a \in R \) there exists \( b \in R \) such that \( aba - a \) is a central (central nilpotent, respectively) element of \( R \). It is clear that every regular ring is a \( \xi^* N \)-ring and every \( \xi^* N \)-ring is a \( \xi \)-ring (see [7,12]).

By the theorem of Auslander, Connell and Willamayor (see [3], Theorem 3, p.660), it is well known that a group ring is regular if and only if \( K \) is regular, \( G \) is a locally finite group and the order of every element \( g \in G \) is invertible in \( K \).

Our first result for this section is the following.

**Theorem 2.** Let \( K * G \) be a crossed product of the group \( G \) over the ring \( K \) such that one of the following conditions is satisfied:

1. \( K * G \) is a \( \xi \)-ring;
2. \( K * G \) is \( \xi^* N \)-ring.

Then \( G \) is a torsion group.

**Proof.** (i) Suppose that \( g \in G \) is an element of infinite order. Then there exists a \( b \in K * G \) and a natural number \( n \geq 1 \) such that

\[
x = (u_g - 1)b(u_g - 1) - (u_g - 1)
\]

is a central element of \( K * G \) and \( x^n = 0 \). If \( n = 1 \), then \( x = 0 \) and

\[
(u_g - 1)[b(u_g - 1) - 1] = 0.
\]

Since, by Corollary [1] the element \( u_g - 1 \) is not a left zero divisor in \( K * G \), we obtain that \( b(u_g - 1) = 1 \), i.e. \( u_g - 1 \) is a left invertible element in \( K * G \), which is also impossible. Therefore \( n > 1 \) and

\[
x^n = (u_g - 1)[b(u_g - 1) - 1]x^{n-1} = 0.
\]

In the same way we obtain that \( z_1 = [b(u_g - 1) - 1]x^{n-1} = 0 \). Suppose that for some \( k \geq 1 \) we have \( z_k = [b(u_g - 1) - 1]^k x^{n-k} = 0 \). If \( 1 < k < n \), as far as \( x \) is
central,
\[
z_k = x[b(u_g - 1) - 1]^k x^{n-k-1} = (u_g - 1)[b(u_g - 1) - 1]^{k+1} x^{n-k-1} = 0.
\]
Now applying Corollary 1 we obtain that
\[
z_{k+1} = [b(u_g - 1) - 1]^{k+1} x^{n-k-1} = 0.
\]
Thus, by induction we conclude that \( z_n = [b(u_g - 1) - 1]^n = 0. \)

The last equality shows that there exists \( z \in K \ast G \) such that \( z(u_g - 1) = 1, \)
which, by Corollary 1, is impossible.

(ii) Suppose that \( g \in G \) is an element of infinite order. Then for some \( b, c \in K \ast G \)
we have \( u_g - 1 = (u_g - 1)b(u_g - 1)^n c. \) By Corollary 1 we have
\[
(u_g - 1)(1 - b(u_g - 1)^n c) = 0,
\]
we conclude that \( b(u_g - 1)^n c = 1. \) Hence it follows that \( b(u_g - 1)x = 1, \)
where \( x = (u_g - 1)^{n-1} c. \) If \( e = xb(u_g - 1), \) then
\[
e^2 = x[b(u_g - 1)x]b(u_g - 1) = xb(u_g - 1) = e,
\]
i.e. \( e \) is a central idempotent of \( K \ast G. \) Thus we have
\[
1 = b(u_g - 1)x = xb(u_g - 1),
\]
i.e. \( u_g - 1 \) has a left invertible element \( xb \in K \ast G. \) Now again by Corollary 1 we obtain a contradiction, so the proof is complete.

**Corollary 2.** If the crossed product \( K \ast G \) is a regular ring, then \( K \) is also a regular ring and \( G \) is a torsion group.

**Proof.** The claim follows from Theorem 2 and Lemma 1.

Observe that the theorem of Auslander, Connell and Willamayor (see [3], Theorem 3, p.660) does not apply for crossed products. Indeed, if \( K \) is a non-perfect field of characteristic \( p > 0 \) and \( G \) is the \( p^\infty \)-group, then there exists a twisted group ring \( K_p G, \) which must be a field (see [9], Proposition 4.2).

If \( G \) satisfies the maximum condition for finite normal subgroups and the group ring \( KG \) is a \( \xi N \)-ring, then \( G \) is locally finite (see [14], Theorem 3, p.16).

We shall prove the locally finiteness of \( G \) without the assumption of the maximum condition when \( K \) is a field. First we recall that (see [10], p.308)
\[
\Delta(G) = \{ g \in G \mid [G : C_G(g)] < \infty \}
\]
is a subgroup of \( G, \) where \( C_G(g) \) is the centralizer of \( g \) in \( G. \) Furthermore, we put
\[
\Delta^p(G) = \{ g \in \Delta(G) \mid g \text{ is a } p\text{-element } \},
\]
that is the subgroup of \( \Delta(G) \) which is generated by all \( p \)-elements of \( \Delta(G). \)

Now we are ready to prove the following.

**Theorem 3.** Let \( KG \) be the group algebra of a group \( G \) over a field \( K. \) If \( KG \) is a \( \xi N \)-ring, then \( G \) is a locally finite group. Moreover, if \( \text{char}(K) = p > 0 \) then \( \Delta^p(G) \) contains all \( p \)-elements of \( G. \)
Proof. Let $\mathfrak{N}(KG)$ be the union of all nilpotent ideals of $KG$. In particular, the central nilpotent elements of $KG$ are in $\mathfrak{N}(KG)$ and, consequently, $KG/\mathfrak{N}(KG)$ is a regular ring.

Assume $\text{char}(K) = p > 0$. By Theorem 8.19 ([10], p.309),
$$\mathfrak{N}(KG) = \mathfrak{Rad}(K[\Delta^p(G)])KG,$$
where $\mathfrak{Rad}(K[\Delta^p(G)])$ is the Jacobson radical of the group ring $K[\Delta^p(G)]$. Obviously, the augmentation ideal $\omega(K[\Delta^p(G)])$ is a maximal ideal of $K[\Delta^p(G)]$, so
$$\mathfrak{N}(KG) = \mathfrak{Rad}(K[\Delta^p(G)])KG \subseteq \omega(K[\Delta^p(G)])KG.$$

It is well-known (see [3], Theorem 3, p.660) that
$$K[G/(\Delta^p(G))] \cong KG/\omega(K[\Delta^p(G)])KG$$
and therefore the group algebra $K[G/(\Delta^p(G))]$ is regular, as a homomorphic image of $KG/\mathfrak{N}(KG)$. This implies, by the theorem of Auslander, Connell and Villamayor (see [3], Theorem 3, p.660), that $G/\Delta^p(G)$ is locally finite and has no $p$-element. Thus we obtain that $\Delta^p(G)$ contains all the $p$-elements of $G$ and the group $G$ is locally finite (see [5], Theorem 23.1.1, p.215).

If $\text{char}(K) = 0$, then $\mathfrak{N}(KG) = 0$ and $KG$ is regular. According to Auslander-Connell-Villamayor’s theorem the proof is complete. □

4. n-weakly regular twisted group algebras

Let $n \geq 2$ be a fixed natural number. A ring $R$ is called $n$-weakly regular [1] if for every $a \in R$ there exist elements $b, c \in R$ such that $a = aba^n c$.

Obviously, an $n$-weakly regular ring $R$ has no nonzero nilpotent element. Indeed, if $R$ contains a nonzero nilpotent element, then there exists a nonzero nilpotent element $a \in R$ with $a^2 = 0$. Hence $a = aba^n c = 0$, which is impossible. From this fact we can conclude that all idempotents of an $n$-weakly regular ring are central.

In [2] (Theorem 2, p.119) it was proved that the group algebra $KG$ over a field $K$ is $n$-weakly regular ($n \geq 2$) if and only if $K$ and $G$ satisfy at least one of the following two conditions:

(i) $\text{char}(K) = p > 0$ and $G$ is an abelian torsion group without $p$-elements;
(ii) $\text{char}(K) = 0$ and $G$ is either an abelian torsion group or a hamiltonian group $G = Q \times E \times A$, where $A$ is an abelian torsion group without 2-elements and the equation $x^2 + y^2 + z^2 = 0$ in $KA$ has only the trivial solution.

In the case when $K$ is an algebraically close field, this result can be extended to.

Theorem 4. A twisted group algebra $K\rho G$ of a group $G$ over the algebraically closed field $K$ is $n$-weakly regular ($n \geq 2$) if and only if the following conditions hold:

(i) $G$ is an abelian torsion group;
(ii) the order of every element of $G$ is invertible in $K$;
(iii) the factor system $\rho$ is symmetric.

Proof. Suppose that $K\rho G$ is $n$-weakly regular. Then conditions (i), (ii) and (iii) hold by Theorems [1] and [2].

Conversely, if $K$ and $G$ satisfy the conditions (i), (ii) and (iii), then $K\rho G$ is a commutative ring. Let $a \in K\rho G$ be an arbitrary element. Then $a \in K\rho H$, where $H = \langle \text{Supp}(a) \rangle$ is a finite abelian group. Since $K\rho H$ is a commutative semisimple artinian ring ([8], Theorem 2.2), we conclude that $K\rho H$ is a direct product of
fields, so $K\rho H$ is $n$-weakly regular. This implies that $K\rho G$ is $n$-weakly regular, as requested.

Analyzing the result of [2] (see Theorem 2, p.119) on $n$-weakly regular group rings and [7] (see Corollary 2, p.70) about strongly regular group rings we deduce that when $K$ is a field, then these two classes coincide.

In the case of twisted group algebras over an algebraically closed basic field we have the following.

**Corollary 3.** Let $K\rho G$ be a twisted group algebra of a group $G$ over an algebraically closed field $K$. The following statements are equivalent:

(i) $K\rho G$ is strongly regular;

(ii) $K\rho G$ is $n$-weakly regular for every natural number $n \geq 2$;

(iii) $K\rho G$ is $n$-weakly regular for some natural number $n \geq 2$;

(iv) $G$ is an abelian torsion group, the order of every element of $G$ is invertible in $K$ and the factor system $\rho$ is symmetric.

**Proof.** Suppose that $K\rho G$ is a strongly regular ring. If $a \in K\rho G$ and $a = a^2b$, then $a = aba$, because $K\rho G$ does not contain nilpotent elements. Now by induction it follows that $a = ab^nc$ for some $c \in K\rho G$ and for every natural number $n \geq 1$. So (i) implies (ii) and, obviously, (ii) implies (iii). By the preceding theorem, (iii) implies (iv). Finally, by the Auslander-Connell-Villamayor theorem and by (iv) it follows that $K\rho G$ is a commutative von Neumann ring and so (iv) implies (i). □

5. **$\xi N$-twisted group algebras**

A ring $R$ is called a $\xi N$-ring if for any $a \in R$ there exists $b \in R$ such that $a^2b - a$ is a central nilpotent element of $R$ (see [11]).

Obviously, every $\xi N$-ring is a $\xi$-ring and, therefore, (see [6], Theorem 1, p.714) we deduce that every $\xi N$-ring is a $\xi^* N$-ring. Moreover, (see [6], Lemma 2, p.715) it follows that in $\xi N$-rings all nilpotent elements are central.

$\xi N$-group rings over commutative rings are described in [11] (Theorem 2, p.15). From this description, it follows that a group ring $KG$ over a field $K$ of characteristic $p > 0$ is a $\xi N$-ring if and only if $G$ is an abelian torsion group.

Finally we prove the following.

**Theorem 5.** A twisted group algebra $K\rho G$ of a group $G$ over the algebraically closed field $K$ is a $\xi N$-ring if and only if the following conditions hold:

(i) $G$ is an abelian torsion group;

(ii) the factor system $\rho$ is symmetric.

**Proof.** Let $K\rho G$ be a $\xi N$-ring. Then ([6], Theorem 1, p.714) the ring $K\rho G$ is a $\xi^* N$-ring and, in view of Theorem 2, we conclude that $G$ is a torsion group. As far as $K$ is an algebraically closed field, for every element $g \in G$ of order $n$ there exists an $\mu_g \in U(K)$, such that $v_g = u_g \mu_g$ ($u_g \in U_G$) and $v^n_g = 1$. Then we put

$$z = (v_g - 1)v_h(1 + v_g + v^2_g + \cdots + v^{n-1}_g), \quad (h \in H).$$
Clearly, $z^2 = 0$ and therefore $z$ is a central element of $K_ρG$. Thus $zv_h = v_h z$, and, so we obtain the equality

$$2v_h v_g v_h + \sum_{i=1}^{n-1} v_h^i v_g v_h v_h + \sum_{i=2}^{n-1} v_h^i v_h$$

$$= \sum_{i=1}^{n-1} v_h^i v_h^2 + \sum_{i=0}^{n-1} v_h^i v_g v_h v_g.$$  

If char($K$) = 2, then $2v_h v_g v_h = 0$. Consequently for the product $v_h v_g^2 v_h$ and for the corresponding supports we obtain the following three cases:

(A1) $v_h v_g^2 v_h = v_h^i v_h v_g v_h$, $h g h^2 = g^i h g h$ and $h g h^{-1} = g^i (1 \leq i \leq n - 1)$;

(A2) $v_h v_g^2 v_h = v_h v_g^2 v_h v_g$, $h g h^2 = g h^2 g$ and $h g h^{-1} = g^2 - i (1 \leq i \leq n - 1)$;

(A3) $v_h v_g v_h = v_g^2 v_h$, $h g h^2 = g^2 h^2$ and $h g h^{-1} = g^i (1 \leq i \leq n - 1)$.

This shows that $\langle g^2 \rangle$ is a normal cyclic subgroup of $G$.

If $g$ is a 2-element of $G$, then $1 + v_g$ is nilpotent and by Lemma 2 of [6] we deduce that $1 + v_g$ is a central element of $K_ρG$. Therefore $v_g v_h = v_h v_g$ for every $h \in G$.

If $g$ is an element of odd order, then $\langle g^2 \rangle = \langle g \rangle$ and from (A1), (A2) and (A3) we obtain that every cyclic subgroup of $G$ is normal, i.e. $G$ is either abelian, or hamiltonian. Since the 2-elements of $G$ are central, we conclude that $G$ is an abelian torsion group, i.e. condition (i) holds. Now by (A1) and (A2) it follows that $i = 1$ and $v_g v_h = v_h v_g$. In case (A3) we have $i = 2$ and $v_g v_h^2 = v_g v_h$. But $\langle v_g \rangle = \langle v_h \rangle$, so $v_h$ commutes with $v_g$ for all $i = 1, \ldots, n - 1$. Therefore condition (ii) also holds.

Now, suppose that char($K$) $\neq 2$. Then by [3], we conclude that for the product $v_h v_g v_h$ we have the following four cases:

(B1) $v_h v_g v_h = v_h v_g^2 v_h$, $h g h = g^2 h^2$ and $h g h^{-1} = g^i (1 \leq i \leq n - 1)$;

(B2) $v_h v_g v_h = v_h v_g v_g v_h$, $h g h = g h^2 h g$ and $h g h^{-1} = g^{i-1} (0 \leq i \leq n - 1)$;

(B3) $v_h v_g v_h = -v_h v_g^2 v_h$, $h g h = g h^2 h$ and $g^{i-1} = 1$, which is impossible, because $2 \leq i \leq n - 1$ and $g$ is of order $n$;

(B4) $v_h v_g v_h = -v_h v_g v_h v_h$, $h g h = g h^2 h g$ and $g^i = 1$, which is impossible, because $1 \leq i \leq n - 1$.

Therefore $\langle g \rangle$ is a normal cyclic subgroup of $G$ for every $g \in G$. Hence $G$ is either abelian or a hamiltonian group.

Assume that $G$ is hamiltonian and $\langle g, h \mid g^4 = 1, h^2 = g^2, h g h^{-1} = g^{-1} \rangle \cong Q_8$. Then by (B1) and (B2), it follows that either $i = 3$ or $i = 2$, respectively. Hence we obtain that $v_h v_g = v_g v_h$, where $v_g^4 = v_h^4 = 1$.

Let $(\alpha, \beta)$ be a nontrivial solution of the equation $x^2 + y^2 = 0$ in $K$. Then as in the proof of Theorem [4] we establish that

$$w = \alpha(v_g^2 v_h - v_h) + \beta(v_g^3 v_h - v_g v_h)$$

is a nonzero nilpotent element of $K_ρG$ with $z^2 = 0$. Therefore $w$ is a central element of $K_ρG$. But $w v_h \neq v_h w$, so we obtain a contradiction. Thus $G$ is abelian and condition (i) holds. If $g h = h g$, then by (B1) and (B2) it follows that either $i = 1$ or $i = 2$, respectively. Hence we obtain that $v_h v_g = v_g v_h$ for all $g, h \in G$ and so condition (ii) also follows.

Conversely, if the conditions (i) and (ii) hold, then $K_ρG$ is a commutative ring. For every element $a \in K_ρG$ with $H = \langle \text{Supp}(a) \rangle$, the ring $K_ρH$ is artinian and
$R \cong K_\rho H/\text{nil}(K_\rho H)$ is a finite sum of fields. Therefore $R$ is strongly regular and hence $K_\rho H$ is a $\xi N$-ring. Since $a \in K_\rho H$, we deduce that $K_\rho G$ is a $\xi N$-ring.

Note that if $K_\rho G$ is a $\xi N$-ring, then the periodicity of $G$ can be proved directly. Indeed, if $g \in G$ is an element of infinite order and $z = (u_g - 1)x - (u_g - 1)$ is a central nilpotent element of $K_\rho G$, then $z^n = 0$ for some $n \geq 1$. By Corollary 1 we deduce that $[(u_g - 1)x - 1]z^{n-1} = 0$.

Using the fact that $z$ is central, we can prove by induction that $[(u_g - 1)x - 1]kz^{n-k} = 0$

for every $k \geq 1$. Therefore $[(u_g - 1)x - 1]^n = 0$. This equality shows that $u_g - 1$ is right invertible in $K_\rho G$, which again is impossible by Corollary 1. □

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