Surface Segmentation Using Implicit Divergence Constraint Between Adjacent Minimal Paths

Abstract

We introduce a novel approach for object segmentation from 3D images using modified minimal path Eikonal equation. The proposed method utilizes an implicit constraint - a second order correction to the inhomogeneous minimal path Eikonal - preventing the adjacent minimal path trajectories to diverge uncontrollably. The proposed modification greatly reduces the surface area uncovered by minimal paths allowing the use of the calculated minimal path set as “parameter lines” of an approximate surface. It also has a loose connection with the true minimal surface Eikonal equations that are also deduced.

Keywords: image segmentation, 3D object extraction, minimal surface Eikonal equations, minimal path Eikonal with divergence constraint

1 Introduction

The active contour models have proven record of success in various image segmentation problems. Since its inception [10] the method evolved in many directions. From theoretical point of view, the most prominent changes were: a) the switch from the parametric models to the geometric [15, 5, 4], where the functional energy does not depend on the parameterization, only on the image content and the intrinsic geometric properties of the contour and b) the transition from the Lagrangian approach to the Hamilton-Jacobi. In the former case, the solution is obtained via the Euler-Lagrange equation associated with the functional, whereas in the latter, the Eikonal (the “time independent” Hamilton-Jacobi) equation needs to be solved for the minimal action map [9, 11]. The Hamilton-Jacobi approach has also turned the view from the ad hoc interpretation of the segmentation functionals to the geodesic calculation in a Riemannian manifold wrt the metrics the functional represent, placing the segmentation problems into one consistent theoretical framework. Beyond the theoretical aspects, the biggest advantage of the Hamilton-Jacobi formulation compared to the Lagrangian - is that it always provides unique, globally minimal segmentation avoiding local minima traps, the Lagrangian approach often suffers from; and this is without significant - if any - increase in run-time. It is

In the minimal path framework, the minimal action map can be identified with distance map wrt a distance function induced by metric. For this reason, in this paper we use the minimal action map and the distance map interchangeably.
also worth to mention that despite the fact that the Eikonal equations can only be derived from functionals having only the first order quantity of the path geometry - for the price of additional dimension(s) - higher order quantities, like the Euler’s elastica [13] can also be modeled [7] as well as problems, formulated with region energy [5]-[6][8]. An extensive summary of the geodesic methods can be found in [14].

However, despite the great success in the field of standard 2D image segmentation, the Hamilton-Jacobi approach cannot be directly transferred into three dimensional problems due to the essentially different nature of the minimal surface equations. The imparity is the consequence of the fact that the minimal surface is determined by the two dimensional sub-determinant, whilst the minimal path functionals rely only on the one-dimensional minor of the space-metric tensor - an obvious loss in the information that would be needed for a theoretically correct 3D object segmentation with minimal surfaces. This information hiatus is manifested in the discontinuity in the minimal path network [12]. Attempts have been carried out to mitigate the - often devastating - effect of the metric-information loss, one of the most promising one being the exploitation of the transport equations [2]. In this paper we follow a different approach that attempts to expand the Eikonal equation with an implicit constraint preventing the development of high divergence bulges on the momentary distance map boundary during its construction. This in turn penalizes the distances growing uncontrollably high between the adjacent minimal paths. It will also be shown, that the constraint we used, has loose connection with the minimal surface Eikonal equation derived in the Appendix: The minimal surface problem

2 Segmentation with Minimal Paths

Segmenting images in the variational framework is the task of finding extremal curves of a specifically designed functional that takes its extrema at the object boundaries corresponding to a logical partitioning of the image. In the image processing, two major approaches are used to find these extremals: a) solving the associated Euler-Lagrange ODE system or b) using the Hamilton-Jacobi PDEs. Both have advantages and drawbacks. Euler-Lagrange equations can relatively easy be derived for functionals incorporating higher order derivatives of the sought contours or for multiple integrals i.e. for boundary surface determination in 3D voxel images. On the other hand, using the Euler-Lagrange equations requires initial contour (2D) or surface (3D) definition close to the solution otherwise the evolving contour can be trapped in local minima. Defining the whole initial contour, let alone a surface is a slow, tedious task.

In contrast, wherever applicable the use of the Hamilton-Jacobi equations is preferred. It always provides globally optimal solution that is the shortest distance minimal path between any two points wrt the enforced metric (see Appendix A1-A2). Minimal user input, a few - often only two - points in 2D or few “zero set” contours in 3D are sufficient. On the other hand, the direct use of the Hamilton-Jacobi equation is limited to functionals of one-variable with Lagrangian composed from the zeroth and first - as the highest - order derivatives. However, incorporating higher-order derivatives is not impossible for the price of additional dimension(s), more complex Lagrangian and approximate nature. An efficient such extension is found in [7] that uses Euler’s elastica modeling “turning inertia”. The method can be used to segment tubular structures (blood vessels) from X-ray images correctly handling crossings, bifurcations. Unfortunately, the extension of the Hamilton-Jacobi theory to the double integral minimal surface problem leads to a completely different problem: minimal action (i.e. distance) map creation in the space of the function-triplets (see Appendix [A.3]).

The Minimal Surface problem. In this paper we do not pursue the solution in this infinite dimensional space. Rather we propose an extension of the classical minimal path Eikonal with an implicit constraint that has some (albeit loose) connection with the true minimal surface problem.

In three dimensions, functionals, realizing isotropic inhomogeneous metric

$$\int \phi(r) \| r \| dt, \ r(t) \in \mathbb{R}^3$$  \hspace{1cm} (2.1)

are almost exclusively used. This metric is the special form of the Riemannian metrics $\int \sqrt{(r \cdot G \cdot r)} \ dt$ where the metric tensor is proportional to the identity tensor $G = \phi^2(r) I$. The $\phi$ is a carefully defined function usually of the image intensity gradient $\nabla I$, representing edges. The typical choices are $\frac{1}{2} \chi_{\sqrt{\nabla I} = \beta} \exp(-\beta |\nabla I|)$ that take small values at high gradient magnitudes. Isotropic metrics have the useful property that their minimal paths (the geodesics wrt the metric) are always perpendicular to the level sets of the distance function the metric induce. The proposed divergence constraint relies on this property.

3 The Minimal Path Network and the Minimal Surface Problem

The inspiration of this paper is as follows.

To emphasize the connection with the minimal surface problem, in this section we use the following notations: let $s_1, s_2$ be two closed curves (hereinafter contours) of $\mathbb{R}^3$. The path network $N_{s_2}^{s_1}$ introduced in [1] contains paths $S_{s_2}^{s_1}(u)$
between the points of \( s_2 \) and \( s_1 \):

\[
\mathcal{N}^2_{s_1} = \{ S^q_{s_1} \}_{q \in s_2}.
\]  

(3.1)

An individual path’s Riemannian length wrt an isotropic inhomogeneous metric is defined by the functional

\[
\int \phi ( S^q_{s_1} ) \frac{dS^q_{s_1}}{du}, \quad S^q_{s_1} = \frac{ds^q_{s_1}}{du} \tag{6.12}
\]

(see Appendix A.2 Specialization). The network “energy” \( S_{net} \) is defined to be the sum of these lengths:

\[
S_{net} := \int_{q \in s_2} u(q) \int \phi ( S^q_{s_1} ) \frac{dS^q_{s_1}}{du} \, dvdu.
\]

(3.2)

Minimizing \( S_{net} \) is the equivalent of the task of finding all minimal paths between the points \( q = s_2 (v) \) and the \( s_1 \) as zero set. (Note that the endpoints are - by definition - dense on \( s_2 \), but usually discontinuous on \( s_1 \).) Compared (3.2) to the isotropic minimal surface action A.3 The Minimal Surface problem/formula (6.27)

\[
S = \int \int \tilde{\Phi} ( S ) \frac{|S_v|}{|S_u|} \, dvdu
\]

(3.3)

one can notice striking resemblance with the only real difference being the presence of the multiplicator \( |S_v|, \frac{dS}{dv} \) in the latter. This quantity represents the infinitesimal distances \(|dS_{s_1}| = |S_v| \, dv\) between the adjacent \( u = const \) parameter lines. This slight difference in the Lagrangians has huge impact in the result insomuch it inhibits the parameter lines to diverge without control.

Despite the semblance shown above, the nature of the real minimal surface problem is fundamentally different as it requires minimal path determination in the infinite dimensional function (triplet) space: A.3 The Minimal Surface problem. In this paper we do not attempt to search the solution in the infinite dimensional space. Rather we incorporate divergence constraint into the minimal path Eikonal equation that - in some degree - imitates the effect of the missing factor \( |S_v| \).

### 4 Object Extraction with Modified Minimal Paths

In this section first we collect some mathematical results that are used in the proposed model then introduce the divergence constraint.

#### 4.1 Metric in the Vicinity of a Reference Surface

In this section we turn our attention to the distance map solution of the usual three dimensional Eikonal problem. The level sets of this map are the locus of the points having constant distance defined by a metric-based distance function, see A.2 Specialization. We assume that these points - except isolated singularities - constitute a regular surface in the space.

Spatial points \( \mathbf{R} \in \mathbb{R}^3 \) in the vicinity of a reference surface \( \mathbf{Q} (p, r) \) may be parameterized as

\[
\mathbf{R} (p, r, s) = \mathbf{Q} (p, r) + s \mathbf{m} (p, r)
\]

(4.1)

where \( \mathbf{m} (p, r) \) is the unit normal vector of the surface at parameter values \( (p, r) \); then \( s \) is the Euclidean distance measured from the surface. Spatial points \( \mathbf{R} (p, r, s = const) \) constitute equidistant surfaces to \( \mathbf{Q} (p, r) \). The normal vectors of the equidistant surfaces are obtained by the cross product of the coordinate (covariant) basis vectors

\[
\mathbf{R}_p = \frac{\partial \mathbf{R}}{\partial p}, \quad \mathbf{R}_r = \frac{\partial \mathbf{R}}{\partial r}, \quad \mathbf{M} (p, r, s) = \mathbf{R}_p (p, r, s) \times \mathbf{R}_r (p, r, s), \quad \text{their magnitude can be calculated as} \ |\mathbf{M}| (p, r, s) = \mathbf{m} (p, r) \cdot \mathbf{M} (p, r, s), \ i.e.
\]

\[
|M| = |\mathbf{m} \cdot (\mathbf{Q}_p \times \mathbf{Q}_r + s (\mathbf{Q}_p \times \mathbf{m}_r + \mathbf{m}_p \times \mathbf{Q}_r) + s^2 \mathbf{m}_p \times \mathbf{m}_r|.
\]

(4.2)

The coefficient of \( s \) can be expressed with the inverse (contravariant) basis \( \mathbf{Q}^c, \mathbf{Q}^r \) as

\[
\mathbf{m} \cdot (\mathbf{Q}_c \times \mathbf{m}_r) = \mathbf{m}_r \cdot (\mathbf{Q}_p \times \mathbf{Q}_c) = |\mathbf{Q}_p \times \mathbf{Q}_c| \, \mathbf{m}_r \cdot \mathbf{Q}^c
\]

\[
\mathbf{m} \cdot (\mathbf{m}_p \times \mathbf{Q}_r) = \mathbf{m}_p \cdot (\mathbf{Q}_c \times \mathbf{m}) = |\mathbf{Q}_p \times \mathbf{Q}_c| \, \mathbf{m}_p \cdot \mathbf{Q}^p
\]

(easy to check that indeed: \( \mathbf{Q}_c \cdot \mathbf{Q}^k = \delta^k_i, i, k \in \{p, r\} \) where \( \delta^k_i \) is the Kronecker delta). Denoting the reference surface normal by \( \mathbf{M}_0 (\mathbf{Q} (r, s = 0)) = \mathbf{Q}_p \times \mathbf{Q}_r \), expression (4.2) becomes:

\[
|M| = |\mathbf{M}_0| \left[ 1 + s (\mathbf{m}_p \cdot \mathbf{Q}^p + \mathbf{m}_r \cdot \mathbf{Q}^r) \right] + s^2 \mathbf{m} \cdot (\mathbf{m}_p \times \mathbf{m}_r).
\]

(4.3)
In (4.3) \( m_p \cdot Q^p + m_r \cdot Q^r \) is the divergence of the unit normal vector \( \nabla \cdot m \), known to be the negative of the sum curvature \( K_S \). (Also, by simple calculation, the last term is \( s^2 m \cdot (m_p \times m_r) = s^2 |M_0| K_G \) with \( K_G \) being the Gaussian curvature.)

Considering non-equidistant surfaces, the direction of the unit normal vector is the function of \( s \): \( m(p,r,s) \neq m(p,r) \). In this case, equation (4.3) is applicable only to the immediate neighbors with infinitesimal distance \( ds \) from \( Q \), then the last term \( ds^2 K_G \) becomes insignificant, hence omitted:

\[
|M|(p,r,ds) = |M_0|(1 - K_S ds).
\]

(4.4)

Same holds for the invariant elementary surface area \( dA = |M| dpdr \).

### 4.2 Alteration of the Area Element of an Evolving Surface

Now, we track the change of the magnitude of a surface patch evolving in the normal direction (that is \( p, r \) are constant). Here we identify the reference surface with the \( S \) level set of an evolving surface and assume that the rate of change of the Euclidean dilation wrt the level set value \( \frac{ds}{dS}(S) \) is known. From (4.4) we obtain

\[
|M|(S + dS) = |M|(S) \left( 1 - K_S(S) \frac{ds(S)}{dS} dS \right).
\]

(4.5)

This leads to the differential equation

\[
\frac{d|M|}{dS}(S) = -\frac{d|M|}{dS}(S) K_S(S) \frac{ds(S)}{dS}
\]

with the solution

\[
|M|(S) = |M|(0) \exp \left( - \int_0^S K_S(S) \frac{ds(S)}{dS} dS \right); \tag{4.6}
\]

however, we will not need this cumulative form, for the reason that we always consider unit area around each point of the ever growing minimal action map boundary.

![Figure 1: A path with the evolving unit area around.](image)

The area of an elementary surface patch also changes with the rate of (4.5). The alteration of the unit area is (Fig. 1):

\[
\frac{|M|(S + dS)}{|M|(S)} = \left( 1 - K_S(S) \frac{ds(S)}{dS} dS \right).
\]

(4.7)

The length of the one-dimensional structures inscribed in an elementary surface patch is - in some average sense - proportional with the square root of the area of that patch. The real rate of change of the elementary lengths (vector magnitudes) is the function of the direction in the tangent plane that the one-dimensional structure passes. If we wish to incorporate some elementary length/local distance information into our equations, but also want to maintain their isotropic nature we are left with this “in average sense” possibility. Therefore, the change rate of the “mean elementary distances” can be expressed as

\[
\sqrt{\frac{|M|(S + dS)}{|M|(S)}} = \sqrt{1 - K_S(S) \frac{ds(S)}{dS} dS}
\]

that for small sum curvatures can further be approximated with \( 1 - H \frac{ds(S)}{dS} dS \), where \( H = \frac{1}{2} K_S \) is the mean curvature.

Based on these reasonings, now we can turn our attention to the divergence constraint.

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2 In the case, the evolving surface is the front of a distance map under construction, the level set values are the geodesic distances measured from the zero set.

3 Level set value 0 is assumed on the boundary of the initial action map.
4.3 The Divergence Constraint

The distance map solutions for the minimal path Eikonal problems can be obtained by the updating scheme \[ \delta S = \mathbf{p} \cdot \delta \mathbf{r} \] (4.8) gradually expanding the initial zero set - where the action value \( S \) is zero - in the normal direction. The boundary of the momentary map (the “front” surface) always represents a constant level set \( S = \text{const} \) of the distance map. Raising the action value from \( S \) to \( S + \delta S \), the dilation of the map \( |\delta \mathbf{r}| \) in the normal direction is obtained for known \( |\mathbf{p}| \). The momentum \( \mathbf{p} \) then can be retrieved from the constructed distance map in the gradient descent direction. In general, the tangent of the minimal path is not the homogeneous function of \( \mathbf{p} \), but it is in the isotropic case.

Summary of the isotropic case is as follows. The functional is defined as \[ S = \int_a^t \phi(\mathbf{r}) |\dot{\mathbf{r}}| \, dt \] (4.9) and the corresponding Eikonal equation \( \nabla S = \mathbf{p} \left( \mathbf{p} = \frac{\partial \phi}{\partial \dot{\mathbf{r}}} \right) \) becomes \[ \nabla S = \phi(\mathbf{r}) \mathbf{e}, \] (4.10) where \( \mathbf{e} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} \) is the unit tangent of the minimal paths, coinciding with the unit normal \( \mathbf{m} \) of the level sets of the distance map at the same point in \( \mathbb{R}^3 \). The updation scheme in the normal direction of the frontal \( S = \text{const} \) level set is \[ \delta S = |\mathbf{p}| |\delta \mathbf{r}| \] where dilation \( |\delta \mathbf{r}| \) is (also) the elongation of the minimal paths. Hereinafter we denote it by \( \delta s := |\delta \mathbf{r}| \) (also index 0 is attached to the variation of the action value for later referencing):

\[ \delta s = \frac{\delta S_0}{\phi(\mathbf{r})}. \] (4.11)

We wish to incorporate a quantity into (4.11) that reduces the unit area/mean distance between the adjacent minimal paths. By (4.4)(4.7) this task is equivalent to the reduction of the divergence of the unit normal \( \mathbf{m} (= \mathbf{e}) \) of the level sets[5]. Let’s assume, that the front surface is the level set of the action map at some value \( S \). We consider the modified action increment, corrected by either the sum curvature (4.7) or the mean curvature (hereinafter \( K(S) \) stands for either one of them) examined in the previous section Alteration of the Area Element of an Evolving Surface

\[ \delta S := \delta S_0 \left( 1 - \lambda K(S) \frac{d\phi}{dS} \delta S_0 \right), \quad \lambda \in [0, 1]. \] (4.12)

From (4.9) \( (ds = |\dot{r}| \, dt \rightarrow \frac{ds}{d\phi} = \frac{1}{\phi} ) \). Also substituting the variation of the action value (4.11) \( \delta S_0 = \phi \delta s \) and choosing \( \lambda = 1 \) to maximize the effect of our divergence constraint, we get the following quadratic equation

\[ \delta S = \phi \delta s \left( 1 - K \delta s \right) \rightarrow K \delta s^2 - \delta s + \frac{\delta S}{\phi} = 0. \] (4.13)

This is a second-order correction to (4.11) as it includes the square of the sought normal dilation \( \delta s \) - the value equivalent to the elongation of the minimal path. The solution for this value is:

\[ \delta s = \frac{1 - \sqrt{1 - 4K\frac{\delta S}{\phi}}}{2K}. \] (4.14)

The sign of the square root is determined by the sign convention chosen for the curvatures and the limit value \[ \lim_{K \to 0} \delta s, \] since in this limit the solution of (4.13) needs to be reduced to the original (4.11).

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4 In the isotropic case, the value of \( |\mathbf{p}| \) is completely determined by the image content.

5 The reduction of the divergence can be achieved in many ways. We look for a direct modification of the updation scheme (4.11).
4.4 Discussing the Divergence Constraint

In this section we refer formula (4.14) as $\delta_{s,corr}$ and assume same $\delta S$ elevation in the in the action value for both update schemes 4.11-4.14. First we check the limit of $\delta_{s,corr}$ for zero curvatures. By using L’Hopital’s rule, it is

$$\lim_{K \to 0} \frac{1 - \sqrt{1 - 4K \frac{\delta S}{\phi}}}{2K} = \lim_{K \to 0} -\frac{4 \frac{\delta S}{\phi}}{4\sqrt{1 - 4K \frac{\delta S}{\phi}}} = \delta S$$

as expected.

Formula (4.14) defines the distance map dilation $\delta s$ in the normal direction as the function of the geodesic distance elevation $\delta S$ and the curvature $K$ values. We used the sign convention $K_S = -\nabla \cdot m$. Hence, the divergence of the outward unit vectors are a) positive for the points where the action map stretches forward and b) negative for the points lagging behind their neighbors. For the same elevation of the action values the corresponding advances: $\delta_{s,corr} < \delta s$ in case a) and $\delta_{s,corr} > \delta s$ in case b), causing more uniform evolution of the action map. This in turn reduces the divergence of the normal vectors on the map boundary - or equivalently the divergence of the minimal paths (recall, in the isotropic case, $\dot{r}$ is the homogeneous function of the surface normal: $\dot{r} \propto m$).

For extremely high - positive, case b) - curvatures, the discriminant $1 - 4K \frac{\delta S}{\phi}$ can become negative. To avoid this situation, the maximum elevation of the action value needs to be limited to

$$\delta S \leq \frac{\phi}{4K},$$

where the right hand side is the worst case among the points on the momentary distance map boundary.

Another updation strategy can be to choose the constant action value elevation $\delta S := \phi_{min} \delta s_{max}$. Here $\phi_{min}$ is the smallest possible value - determined by the image and $\delta s_{max}$ is a constant parameter - the maximum enabled normal-dilation increment by 4.11. Then wherever the discriminant would become negative, we have to limit the dilation to the maximum value $\delta s = 2 \frac{\delta S}{\phi}$.

5 Results

Figure 2: Columns: The top-side-front views of the path networks calculated using the original (left) and the divergence-corrected (right) distance map updation schemes. Rows: results obtained from increasingly asymmetrical plane setup. As the asymmetry increased more surface area becomes deserted.

The method was tested on a synthetic voxel intensity image. The test environment and the user inputs are as follows.

The test object is composed from two inhomogeneous ellipsoids. Artificial noise is added. The user input is a point and a plane. The initial distance map (the zero set) is an $\epsilon$-ball around the point with the radius of 3-voxels. The map evolves until all pixels of the plane are reached. Then the object contour on the plane ($s_2$ in The Minimal Path Network and the Minimal Surface Problem) is automatically detected. Finally, the path networks are retrieved starting from the contour of the object-plane intersection. The comparative results, using update schemes a) (4.11):

$$\delta s = \frac{\delta S}{\phi}$$
\[ \delta s = \frac{1 - \sqrt{1 - 4K \delta S \phi}}{2K}, \]

for increasingly asymmetrical plane setup, are shown in Fig. The enforced space metric is an isotropic inhomogeneous metric extracted from the image

\[ \phi(r) = \alpha + (1 - \alpha \exp(-\beta |\nabla I(r)|)), \quad (5.1) \]

with the choices \( \alpha = 0.01, \beta = 14. \) \(|\nabla I(r)|\) is the magnitude of the gradient of the normed (\( I \in [0, 1] \)) image intensity function.

### 6 Conclusion

In this paper we presented a second order correction to the inhomogeneous minimal path Eikonal equations that prevents the adjacent minimal path trajectories to diverge uncontrollably. We showed that the minimal path network obtained from the modified equations greatly reduces the surface area left uncovered with paths. This denser covering greatly improves the accuracy of the approximate surface that can be retrieved directly from the path network. It can also be used to initialize the transport equation enhancing its robustness.

**Appendix: The minimal surface problem**

In the appendix we 1) summarize the relevant aspects of the Hamilton-Jacobi Theorem; 2) Its specialization to the Minimal Path Eikonal equation; 2) The generalization of the Minimal Path Eikonal equation to the Minimal Surface problem.

#### A.1 The Hamilton-Jacobi equation

The functional of variable endpoints \( S(q_i, t) \) of \( n \) function \( q_i(t), i = 1 \ldots n \):

\[ S(q_i, t) = \int_a^t L(q_i, \dot{q}_i, \dot{t}) \, d\dot{t} \quad (6.1) \]

can always be decomposed into the sum

\[ S(q_i, t) = \sum_{i=1}^{n} \int_{q_i}^{q_i} p_i dq_i - \int_a^t H(q_i, p_i, t) \, d\dot{t} \quad (6.2) \]

where \( p_i = \frac{\partial L}{\partial \dot{q}_i}, i = 1 \ldots n \) are the generalized momenta, \( H(q_i, p_i, t) = \sum_{i=1}^{n} p_i q_i - L \) is the Hamiltonian of the problem. Decomposition (6.2) arises from the analysis of the partial alterations of the action value \( S \) as the function of the change in the boundary conditions wrt a) the end coordinates (the 1\(^{st}\) terms on the right) and b) the parameter value at the end coordinates (the 2\(^{nd}\) term on the right). The corresponding local equations are:

a) \( p_i = \frac{\partial S}{\partial q_i}, i = 1 \ldots n \)

b) \( \frac{\partial S}{\partial t} = -H, \quad H = H(q_i, p_i, t). \quad (6.3) \)

Combining a) and b), one can get the Hamilton-Jacobi PDE:

\[ \frac{\partial S}{\partial t} + H(q_i, \frac{\partial S}{\partial q_i}, t) = 0 \quad (6.4) \]

equating the relation between partial variations a) and b).
A.2 Specialization

We restrict our further examination to the geometric functionals. By definition, a functional is geometric, if it is immune to the reparameterization, that is, for any admissible $t \rightarrow \gamma(t)$ we require:

$$\int L(q_i, \dot{q}_i, t) \, dt = \int L\left(q_i, \frac{dq_i}{d\gamma}, t(\gamma)\right) \frac{1}{\gamma} \, d\gamma := \int L\left(q_i, \frac{dq_i}{d\gamma}, \gamma\right) \, d\gamma. \quad (6.5)$$

Because of this independence criterion, in the geometric case, only the change of coordinates – encoded in the $1^{st}$ terms on the right of (6.2) – has effect, hence requirement (6.5) implies identically zero Hamiltonian.

Then formula (6.2) is reduced to the abbreviated functional

$$S(q_i) = \sum_{i=1}^{n} q_i \partial L / \partial q_i,$$

(6.6)

describing static action map.

Henceforth the general coordinates $q_i$ are identified with the coordinate functions of a "real" position vector $r(t) \in \mathbb{R}^3$, or for planar problems $r(t) \in \mathbb{R}^2$, the Hamilton-Jacobi equation (6.4) becomes the Eikonal equation that can concisely be written

$$\frac{dS}{dr} = \nabla S = p, \quad p = \frac{\partial L}{\partial r}. \quad (6.7)$$

(From 6.3 b) $\frac{dS}{dt} = 0 \rightarrow \frac{dS}{dr} = \frac{ds}{dt} = p$. The equivalent formulation using differentials

$$\delta S = p \cdot \delta r \quad (6.8)$$

is used to construct the solution by gradually extending the action map starting from a zero set (often chosen to be an "ε-ball").

The static action map solution of the Hamilton-Jacobi equation for the geometric functionals can be identified with distance map wrt a distance function defined by metric, where the metric of the space is encoded in the Lagrangian. The parameterization-independent Finsler metric:

$$L = \psi(r, e) |\dot{r}|$$

$$e = \frac{\dot{r}}{|\dot{r}|}, \quad (6.9)$$

realizes the most general form of such functionals. Its specialization sequence includes the asymmetric Randers metric:

$$L = \left(\sqrt{e \cdot G \cdot e + \omega \cdot e}\right) |\dot{r}|, \quad (6.10)$$

the anisotropic Riemannian metric ($G$ stands for the positive definite symmetric metric tensor):

$$L = \sqrt{e \cdot G \cdot e} |\dot{r}| = \sqrt{\dot{r} \cdot G \cdot \dot{r}}, \quad (6.11)$$

the inhomogeneous, isotropic metric:

$$L = \phi(r) |\dot{r}| = \left(G = \phi^2(r) I\right), \quad (6.12)$$

and the Euclidean metric ($\phi(r) = const$). The ratio of the Riemannian and Euclidean lengths $\frac{\sqrt{\dot{r} \cdot G \cdot \dot{r}}}{|\dot{r}|} = \sqrt{e \cdot G \cdot e}$, $|e| = 1$ is encoded in the one-dimensional minor of the metric tensor in the direction of the path tangent.

In the isotropic case (and only in the isotropic case), the minimal paths are perpendicular to the level sets of the of the distance map: $p = \frac{\partial L}{\partial r} = \phi(r) e$, hence the abbreviated functional is obtained by rearrangement: $\int \phi(r) |\dot{r}| \, dt = \int \phi(r) (e \cdot \dot{r}) \, dt = \int \phi(r) e \cdot (\dot{r} dt) = \int \phi(r) e \cdot dr$.

A.3 The Minimal Surface problem

Summarizing the discussion of the previous sections, the functionals $\int L \, dt$ for the one-parameter minimal path problems can be reformulated as $\int p \cdot dr$, where $p = \frac{\partial L}{\partial r}$ is the new (momentum) variable and the invariant integration variable is

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Footnote: This further implies scleronomic ($\frac{\partial L}{\partial t} = 0$) functional.
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Figure 3: One particular path. $S = \text{const}$ contours are boundary points on the consecutive action (hyper-)surface.

the tangent times the path parameter $dr = \dot{r} dt$. We seek analogous formulation – including the corresponding Eikonal equations – for the two-parameter minimal surface problem. The minimal path is the shortest distance path among all possible paths joining point pairs in $\mathbb{R}^3$. Similarly, the “shortest distance between two contours” can be identified by the surface having the the minimal surface area among all possible surfaces connecting them.

Notation: The two-parameter position vector to the surface points is denoted by $S(u,v)$. Let the directed invariant surface element be denoted by $dA = N |dudv|$ – where $N(u,v)$ is the normal of the surface at parameter values $(u,v)$ – it can be expressed by the local covariant basis vectors $S_u = \frac{\partial S}{\partial u}$, $S_v = \frac{\partial S}{\partial v}$ as $N = S_u \times S_v$; let the unit normal vector be $n = \frac{N}{|N|}$. The absolute value (magnitude) of the directed invariant surface element expresses the elementary Euclidean surface area: $|N| |dudv| = dA$.

The general form of the functional given by double integrals with Lagrangian having only the first order partial derivatives is:

$$\iint_A L(S, S_u, S_v, u, v) |dudv|. \quad (6.13)$$

We restrict our further examination to the geometric case. The two dimensional equivalent of the geometric Finsler functional (6.9) is:

$$S = \int_A \Psi(S, n) dA. \quad (6.14)$$

with Lagrangian $L = \Psi(S, n) |N|$. Note that it does not include the covariant basis vectors (neither the parameters) explicitly. Hereinafter we assume that the Lagrangian of the minimal surface problem must not include more explicitness and work by analogy. One may define the surface momentum

$$P := \frac{\partial L}{\partial N} \quad (6.15)$$

and the abbreviated functional as

$$S = \int_A P \cdot dA. \quad (6.16)$$

In the case of the Finsler functional (6.14), $P = \Psi_n \cdot (I - nn) + \Psi n = \Psi_n|_T + \Psi n$ - where $I$ is the identity tensor ($I - nn$ is the projector to the tangent space, $\cdot|_T$ stands for the projection to the tangent space) and $\Psi_n = \frac{\partial \Psi}{\partial n}$. Then (6.16) indeed gives (6.14) back.

In agreement with our explicitness assumption, now we consider the effect of the varying boundary condition for the particular form of the double double integral

$$S = \iint_A L(S, N) |dudv|. \quad (6.17)$$
This form is more general than (6.14) but does not depend explicitly on the covariant basis $S_u$, $S_v$. First, we need the variation of the surface normal in the direction of $h_N$:

$$\mathbf{N} + \varepsilon h_N = (S_u + \varepsilon h_u) \times (S_v + \varepsilon h_v)$$

$$= \mathbf{N} + \varepsilon (h_u \times S_v + S_u \times h_v) + \varepsilon^2 h_u \times h_v \quad (6.18)$$

then the variation of $\delta S$ is

$$\delta S = \int \int L (\mathbf{S} + \varepsilon \mathbf{h}, \mathbf{N} + \varepsilon h_N) - L (\mathbf{S}, \mathbf{N}) \, du \, dv$$

$$= \int \int \frac{\partial L}{\partial \mathbf{S}} \cdot \varepsilon \mathbf{h} + \left( \frac{\partial L}{\partial \mathbf{N}} \cdot (\varepsilon h_u \times S_v + S_u \times h_v) \right) \, du \, dv$$

$$= \int \int \varepsilon \mathbf{h} \cdot \frac{\partial L}{\partial \mathbf{S}} + \varepsilon h_u \cdot \left( S_v \times \frac{\partial L}{\partial \mathbf{N}} \right) + \varepsilon h_v \cdot \left( \frac{\partial L}{\partial \mathbf{N}} \times S_u \right) \, du \, dv \quad (6.19)$$

After applying the integration by parts step, we take the usual assumption that at the extremal surface the Euler-Lagrange term vanishes. Also, the boundary terms are reduced to boundary integrals by the Green’s theorem, then we are left with (here the notations $dS(u) = S_u du$, $dS(v) = S_v dv$ and the “cross tensor” $\varepsilon \cdot \varepsilon \cdot \varepsilon$):

$$(6.21)$$

Note that in the case of geometric functionals this is the only non-zero term, arising alone from the boundary variation. Finally, using the usual notation $\varepsilon \mathbf{h} = \delta S$, also the fact that $dS(v) = dS(u)$ is tangent to the boundary curve (hence denoted by $dS_B$) we arrived to the differential form of the Eikonal equation - an equivalent to (6.8) - for minimal surfaces:

$$\delta S = \int dS_B \cdot \left( \frac{\partial L}{\partial \mathbf{N}} \right) \times \delta \mathbf{S} \quad (6.22)$$

where $\delta A$ stands for the varying end of the surface. Now it is tempting to choose special parameterization to get deeper insight into the minimal surface problem. Adapting the $u = const$ parameter lines to the boundary of the momentary surface – that is with identifications $dS_B \rightarrow dS(v)$, $dS(v) = S_v dv$ and using the surface moment (6.15):

$$\delta S = \int dS(v) \cdot S_v \times P \cdot \delta \mathbf{S}$$

(Note: integrating (6.22): $\int \delta S du$ we get back the abbreviated Finsler functional (6.16) parameterized specially: $\int \delta S du$)

$$\int \delta L \cdot N \, du \, dv$$. By analogy (see (6.7), (6.8)) formula (6.22) can be interpreted as the inner product of the action gradient $S_v (v) \times P (v)$ and the endpoint variation $\delta S (v)$ both are the elements of some function (-triplet) space:

$$\delta S = \langle \nabla S, \delta \mathbf{S} \rangle, \quad \nabla S (v) = S_v (v) \times P (v) \quad (6.23)$$

Summarizing the result of our examination on the minimal surface problem: we need to find minimal paths in the space of boundary contours, i.e. in the space of coordinate function triplets – or alternative contour representation with the same information content (see Fig.3). The gradient $S_v (v) \times P (v)$ designates the normal at the momentary contour (at a particular point on the momentary action surface). Again, by analogy, if $\delta S (v)$ is parallel with this direction, that is $S_v (v) \times P (v) \parallel \delta S (v)$ (for any parameter value $v$ along the momentary contour) then the endpoint variation formula (6.22) can be written so that it contains the point-wise magnitudes of the gradient and the endpoint variation functions:

$$\delta S = \int dS(v) \cdot |S_v \times P_{|\delta \mathbf{S}}|$$

(6.24)
the formula suitable for the action map construction.

Author’s warning: the equations derived for minimal surfaces - despite it provides insight to many aspects of the minimal surface problem - require rigorous mathematical foundation of the Banach space of endcontours including but not limited to: the coherent definitions for the basis set, norm, level sets, the normal spaces of the level sets etc. All of these required for the construction of the static action map - i.e. the distance function - in the space of endcontours, where the shortest distance between two contours is identified with the surface area of the minimal surface spanned between them.

A.4 The Isotropic case

First we consider the relation between the Euclidean and Riemannian elementary area that is \( |N| \, du \, dv \leftrightarrow \sqrt{\det^* [G_T]} |N| \, du \, dv \), where \( \det^* [G_T] \) is the determinant of the two-dimensional minor of the space metric expressed in the tangent plane in any normalized local coordinate system \( s_u, s_v \) with the normalization being \( s_u \times s_v = n \), \( |n| = 1 \). (Without normalization the determinant expresses the space metric with the parameterization dependent surface metric blended in: \( \det [G_T] = \det^* [G_T] \, \det [S_{ik}] \), \( i, k \in \{ u, v \} \), where \( \det [S_{ik}] = |N|^2 \), similarly to the length expression (6.11)).

If the space metric is isotropic, the matrix of \( G_T \) can be given in orthonormal basis as \( \begin{bmatrix} \Phi (S) & 0 \\ 0 & \Phi (S) \end{bmatrix} \) then the Riemannian area expression becomes \( \Phi (S) |N| \, du \, dv \). We consider this form as the simplest version of the geometric minimal surface functionals, still suitable for segmentation. The Lagrangian of the isotropic, inhomogeneous space is

\[
L = \Phi (S) |N| .
\] (6.25)

This form is the direct equivalent to (6.12). Similarly to the minimal path problem, the associated abbreviated functional can be obtained by rearrangement:

\[
S = \int_A \Phi (S) |N| \, du \, dv = \int_A \Phi (S) \, n \cdot dA \\
\rightarrow P = \Phi (S) \, n
\] (6.26)

or using the special parameterization introduced above (including the \( S_v \times n \parallel S_u \) from (6.24))

\[
S = \int_a^u \int_v \Phi (S) |S_v| \, |S_u| \, dv \, du .
\] (6.27)

Its particular Eikonal’s differential form is:

\[
\delta S = \int_v \Phi (S) |S_v| \, |\delta S| .
\] (6.28)

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