Poincaré inequalities on graphs

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Abstract

We prove local $L^p$-Poincaré inequalities, $p \in [1, \infty]$, on quasiconvex sets in infinite graphs endowed with a family of locally doubling measures, and global $L^p$-Poincaré inequalities on connected sets for flow measures on trees. We also discuss the optimality of our results.

1 Introduction and notation

Poincaré inequalities have played an important role in analysis on noncompact manifolds, Lie groups, infinite graphs, and more generally on metric spaces. A complete overview of the literature on this subject would go out of the scope of this paper, so we refer the reader to [6, 8] and the references therein.

While the key role of these inequalities is largely understood, and their validity is considered a natural geometric assumption in the context of analysis in metric spaces, on the other hand, quite surprisingly, in the discrete setting there is a lack of concrete examples of graphs fulfilling these inequalities in the literature. In this note we focus on infinite graphs and we prove that some concrete common families of measures and graphs actually fulfil the desired property of supporting Poincaré-type inequalities.

(Local) Poincaré inequalities combined with the (local) doubling condition, are a standard tool to obtain (local) Harnack inequalities both in continuous and discrete settings (see [4, 5, 14, 16]). Applications of the Poincaré inequalities discussed in this note in such direction will be object of further investigation in [11].

Let $X$ be an infinite, locally finite, connected, and undirected graph. We identify $X$ with its set of vertices and we write $x \sim y$ whenever $x, y \in X$ are neighbors, namely, when they are connected by an edge. We denote by $\deg(x)$ the number of neighbors of $x$. We say that the graph $X$ has bounded degree $b + 1$ if $\deg(x) \leq b + 1$, for some $b \geq 1$ and every $x \in X$. A path of length $n \in \mathbb{N}$ connecting two vertices $x$ and $y$ is a sequence $\{x_0, x_1, \ldots, x_n\} \subset X$, with no repeated vertices, such that $x_0 = x$, $x_n = y$, and $x_i \sim x_{i+1}$ for every $i = 0, \ldots, n - 1$. The distance $d(x, y)$ is defined as the minimum of the lengths of the paths connecting $x$ and $y$. For any $x \in X$ and $r \geq 0$, the ball of radius $r$ and center $x$ is $B_r(x) = \{y \in X : d(x, y) \leq r\}$. For every subset $E$ of $X$ the diameter of $E$ is $\operatorname{diam}(E) = \sup\{d(x, y) : x, y \in E\}$.

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We say that a subset $E$ of $X$ is quasiconvex if for every couple of vertices $x, y$ in $E$ there exists a path contained in $E$ connecting $x$ and $y$ of length $\leq 2\text{diam}(E)$. This is the same as asking that $E$ is quasiconvex as a metric space (see for instance [6, 9]) with the metric induced by the ambient space $X$. Examples of sets which are quasiconvex in any graph are all the geodesically convex sets, as well as balls (which, on a general graph, might not be geodesically convex). Observe that on trees, which are connected graphs without cycles, the notion of quasiconvex set coincides with the notion of connected set; indeed, each connected set in a tree is geodesically convex, since every path is a geodesic.

Any nonnegative function $\mu$ on $X$ induces a measure on $X$; with slight abuse of notation, for any subset $E \subseteq X$, we set $\mu(E) = \sum_{x \in E} \mu(x)$. In particular, we denote by $|\cdot|$ the counting measure, i.e., $|E|$ is the cardinality of $E$.

A measure $\mu$ is said to be locally doubling if for any $R > 0$ there exists a constant $D(R)$ such that for any $0 \leq r \leq R$

$$\mu(B_{2r}(x)) \leq D(R)\mu(B_r(x)), \quad \text{for every } x \in X. \quad (1)$$

For any $1 \leq p < \infty$, we denote by $L^p(X, \mu)$ the space of functions $f : X \to \mathbb{C}$ such that the norm $\|f\|_{L^p(X, \mu)} = \left(\sum_{x \in X} |f(x)|^p \mu(x)\right)^{1/p}$ is finite, and by $L^\infty(X, \mu)$ the space of functions such that $\|f\|_{L^\infty(X, \mu)} = \sup_{x \in X} |f(x)| < \infty$.

For every function $f : X \to \mathbb{C}$ we define the length of the gradient of $f$ as the function $\nabla f : X \to \mathbb{R}$ defined by

$$\nabla f(x) = \sum_{y \sim x} |f(x) - f(y)|, \quad x \in X.$$  

This is a standard notion for analysis on graphs and it plays the role that the upper gradients defined in [6] play in analysis on metric spaces.

We say that $(X, \mu)$ satisfies a local $L^p$-Poincaré inequality, $p \in [1, \infty]$, if for any $R > 0$ there exists a positive constant $P_p(R)$ such that for any function $f : X \to \mathbb{C}$ and any quasiconvex set $E$ of diameter $0 \leq 2r \leq R$ it holds

$$\|f - f_E\|_{L^p(E, \mu)} \leq P_p(R)r\|\nabla f\|_{L^p(E, \mu)},$$  

where $f_E = \frac{1}{\mu(E)} \sum_{x \in E} f(x) \mu(x)$.

In case the constant $P_p(R)$ may be made independent of $R$, we say that $(X, \mu)$ satisfies a global $L^p$-Poincaré inequality. More precisely, $(X, \mu)$ satisfies a global $L^p$-Poincaré inequality, $p \in [1, \infty]$, if there exists a positive constant $P_p$ such that for any function $f : X \to \mathbb{C}$ and any quasiconvex set $E$ of diameter $2r$ it holds

$$\|f - f_E\|_{L^p(E, \mu)} \leq P_p r\|\nabla f\|_{L^p(E, \mu)}.$$

Notice that when $E$ is a ball, [2] and [3] are the standard local and global $L^p$-Poincaré inequalities studied in the literature [2, 3, 11, 16].

In Section 2 we prove an $L^p$-estimate for $f - f_E$ expressed in terms of the $L^p$-norm of $\nabla f$ for every function $f$ and every quasiconvex set $E$ on a graph $X$ endowed with measures positively bounded from below (see Theorem 2.1). As a consequence, we prove a local $L^p$-Poincaré inequality for quasiconvex sets on every infinite graph endowed with a measure positively bounded from below and above (see Corollary 2.3). The counting measure is obviously included in this class of measures.
In Section 3 we discuss the optimality of the results of Section 2. First, we prove that the assumption on the quasiconvexity of the set $E$ in Theorem 2.1 cannot be weakened by simply assuming that $E$ is connected. Next, we show that also the assumption on the boundedness from below of the measure cannot in general be dropped by exhibiting an appropriate example (see Example 3.2). Moreover, we prove that the growth of the constant involved in the local $L^p$-Poincaré inequality, which may be exponential with respect to the radius of the balls, is optimal in the case when $p = 1, \infty$ for a suitable class of trees which includes the homogeneous tree. It is worth mentioning that a similar discussion on the exponential growth of the constant was carried out on the so called $ax + b$ groups in [1].

Surprisingly, in the last section we are able to prove a global $L^p$-Poincaré inequality for quasiconvex sets and for flow measures on infinite trees, a class of measures introduced in [12] (see (8) for the precise definition). This represents a further evidence that these measures, despite being nondoubling, of exponential growth and not positively bounded from below nor from above, are very well behaved with respect to analysis on trees, see also [13].

We remark that the Poincaré inequalities that we prove, and to which we address simply as $L^p$-Poincaré inequalities, are indeed $(p, p)$-strong Poincaré inequalities. The term strong here refers to the fact that the integral on the right-hand side of (2) is taken on the same set than the integral on the left-hand side, and not on an enlarged set. The specification $(p, p)$, instead, denotes a difference with another class of inequalities, the $(1, p)$-Poincaré inequalities (see for example [10, Equation (8.1.1)]). These are, for instance, the Poincaré inequalities treated in [6, 9] in the generality of metric measure spaces. We point out that the $L^1$-Poincaré inequalities that we prove imply the corresponding $(1, p)$-Poincaré inequalities for every $p > 1$, while in general they are not enough to imply $L^p$-Poincaré inequalities for any $p > 1$. Nevertheless, as previously described, for the cases under study, we obtain $L^p$-Poincaré inequalities for every $p \in [1, \infty]$.

Along the paper, we use the standard notation $f_1(x) \lesssim f_2(x)$ to indicate that there exists a positive constant $C$, independent of the variable $x$ but possibly depending on some involved parameters, such that $f_1(x) \leq Cf_2(x)$ for every $x$. When both $f_1(x) \lesssim f_2(x)$ and $f_2(x) \lesssim f_1(x)$ are valid, we will write $f_1(x) \approx f_2(x)$.

## 2 Bounded measures on graphs

Let $X$ be an infinite, locally finite, connected, and undirected graph. For every $\alpha > 0$ we denote by $M_\alpha$ the class of measures positively bounded from below by $\alpha$, namely, $\mu \in M_\alpha$ if $\mu(x) \geq \alpha$ for every $x \in X$. We underline that the counting measure belongs to $M_1$.

**Theorem 2.1.** Let $E \subset X$ be a finite quasiconvex set with $\text{diam}(E) = 2r$, $p \in [1, \infty]$, and $f$ be any function on $X$. Then, for any $\alpha > 0$ and for any $\mu \in M_\alpha$,

$$
\|f - f_E\|_{L^p(E, \mu)} \leq \left( \frac{\mu(E)}{\alpha} \right)^{1/p} (4r)^{1-1/p} \|\nabla f\|_{L^p(E, \mu)}.
$$

**Proof.** For $x, y \in E$, let $[x, y] = \{x_i\}_{i=0}^n \subset E$ be a path of length smaller or equal than $2 \text{diam}(E)$ connecting $x$ to $y$, where $x_0 = x$, $x_n = y$ and $x_i \sim x_{i+1}$, $i = 0, ..., n - 1$. Then,

$$
|f(x) - f(y)| \leq \sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})| \leq \sum_{z \in [x, y]} |\nabla f(z)|.
$$
For $x \in E$ then one has
\[
|f(x) - f_E| \leq \frac{1}{\mu(E)} \sum_{y \in E} |f(x) - f(y)| \mu(y) \leq \frac{1}{\mu(E)} \sum_{y \in E} \sum_{z \in [x, y]} |\nabla f(z)| \mu(y),
\]
from which immediately follows,
\[
\|f - f_E\|_{L^\infty(E, \mu)} \leq 4r \|\nabla f\|_{L^\infty(E, \mu)}.
\]
For $p \in [1, \infty)$, by Jensen’s and Hölder’s inequalities we get
\[
|f(x) - f_E|^p \leq \frac{1}{\mu(E)} \sum_{y \in E} \left( \sum_{z \in [x, y]} |\nabla f(z)| \right)^p \mu(y)
\leq \frac{1}{\mu(E)} \sum_{y \in E} [x, y]^{p-1} \sum_{z \in [x, y]} |\nabla f(z)|^p \mu(y)
\leq (4r)^{p-1} \frac{\mu(E)}{\alpha} \sum_{y \in E} \sum_{z \in E} |\nabla f(z)|^p \mu(y) = (4r)^{p-1} \sum_{z \in E} |\nabla f(z)|^p.
\]
Since $\mu(z) \geq \alpha$ for any $z \in X$, we obtain
\[
\sum_{x \in E} |f(x) - f_E|^p \mu(x) \leq \frac{\mu(E)}{\alpha} (4r)^{p-1} \sum_{z \in E} |\nabla f(z)|^p \mu(z).
\]
\[\Box\]

**Remark 2.2.** Analyzing the proof of Theorem 2.1, it is not difficult to observe that for $p = 1$ the assumption on the quasiconvexity of the set $E$ may be replaced by the weaker request that $E$ is connected. On the other hand, we will show in Section 3 that for $p = \infty$, it is not necessary to require the positive boundedness from below of the measure. Nevertheless, in the next section we will prove that such request cannot be dropped when $p < \infty$. \[\Box\]

For every finite $\beta > 0$, we denote by $M^\beta$ the class of positive measures bounded from above by $\beta$, namely, $\mu \in M^\beta$ if $\mu(x) \leq \beta$ for every $x \in X$. For every $0 < \alpha \leq \beta$, we set $M^\beta_\alpha = M_\alpha \cap M^\beta$. It is worth mentioning that $M^\beta_1$ uniquely consists of the counting measure.

We remark that a measure $\mu \in M^\beta_\alpha$ is locally doubling if and only if the graph $X$ has bounded degree. Indeed, if $\mu$ is locally doubling, then for every $x \in X$
\[
\deg(x) + 1 = |B_1(x)| \leq \frac{\mu(B_1(x))}{\alpha} \leq D(1/2) \frac{\mu(B_{\frac{1}{2}}(x))}{\alpha} = D(1/2) \frac{\mu(x)}{\alpha} \leq D(1/2) \frac{\beta}{\alpha},
\]
where $D(1/2)$ is as in (1). Hence the degree is bounded. On the other hand, if $X$ has bounded degree $b + 1$, then for every $x \in X$ and $r \geq 0$, $|B_r(x)| \leq 3b^r$ and therefore, for every $R \geq r$
\[
\frac{\mu(B_{2r}(x))}{\mu(B_r(x))} \leq \frac{\beta |B_{2r}(x)|}{\alpha |B_r(x)|} \leq \frac{\beta}{\alpha} |B_{2r}(x)| \leq \frac{\beta}{\alpha} 3b^{2R} =: D(R).
\]

The following corollary shows that for measures in the class $M^\beta_\alpha$ on graphs of bounded degree, we have a local $L^p$-Poincaré inequality for quasiconvex sets.
Finally, we prove that for \( L \) measures which are not positively bounded from below the local \( k \geq 2 \).

**Corollary 2.3.** Suppose that \( X \) has bounded degree \( b + 1 \). Fix \( R > 0 \) and let \( E \subset X \) be a quasiconvex set with \( \text{diam}(E) = 2r \leq R, \) \( p \in [1, \infty], \) and \( f \) be any function on \( X \). Then, for every \( 0 < \alpha \leq \beta < \infty \) and \( \mu \in \mathcal{M}_\alpha^\beta \), \((X, \mu)\) satisfies the \( L^p \)-Poincaré inequality \((2)\), i.e.,

\[
\|f - f_E\|_{L^p(E, \mu)} \leq P_p(R) r \|\nabla f\|_{L^p(E, \mu)},
\]

with \( P_p(R) = 4 \left( \frac{3\beta R}{4\alpha} \right)^{1/p} \).

**Proof.** If \( x \in E \), then \( E \subseteq B_R(x) \), so that

\[
\mu(E) \leq \mu(B_R(x)) \leq 3\beta R.
\]

If \( r < 1 \) the result is trivial, so we can suppose \( r \geq 1 \). Then the result directly follows from Theorem 2.1.

**Remark 2.4.** We remark that, for \( p = \infty \), the conclusion of Corollary 2.3 coincides with that of Theorem 2.1. In particular, in this case, there is no actual need to assume any boundedness of the measure, nor bounded degree of the graph; the local \( L^\infty \)-Poincaré inequality for quasiconvex sets holds for any graph and any measure.

We mention that a global inequality related to \((2)\) was obtained in [3] for \( \alpha = \beta = 1 \) in the case \( p = 1, 2 \), under the additional assumption that the counting measure is globally doubling.

### 3 Optimality of Theorem 2.1

The scope of this section is to discuss the optimality of Theorem 2.1 under different aspects. First we will prove by means of an example that for \( p > 1 \), in general, the assumption of quasiconvexity on \( E \) cannot be replaced by the weaker assumption of being connected. Next, we prove that for measures which are not positively bounded from below the local \( L^p \)-Poincaré inequality may fail. Finally, we prove that for \( p = 1, \infty \) the constant in formula \((4)\) is optimal when \( E \) is a ball, \( X \) is a tree and \( \mu \in \mathcal{M}_\alpha^\beta \).

**Example 3.1.** Referring to Figure 1 consider the infinite connected graph \( X \) with vertex set labelled by \( \mathbb{N}^2 \) and the following proximity rule: \((j_1, k_1) \sim (j_2, k_2)\) if and only if \((i) \ |j_1 - j_2| + |k_1 - k_2| = 1 \) or \((ii)\) \( k_1 = k_2 \) odd, \( j_1, j_2 \leq k_1 \) and \( |j_1 - j_2| \geq 4 \).

For any \( E \subset X \) and \( p \geq 1 \), let \( \| \cdot \|_{L^p(E)} \) denote \( \| \cdot \|_{L^p(|\cdot|)} \). Define the sequence of sets \( E_k = \{(j, k) \in X : j \leq k\}, k \in 2\mathbb{N} \). It is clear that \( E_k \) is connected, \( 1 \leq \text{diam}(E_k) \leq 3 \) and \( |E_k| = k + 1 \).

Consider the function \( f : X \rightarrow \mathbb{C} \) such that \( f(j, k) = j \), for every \((j, k) \in X\).

The average of \( f \) on \( E_k \), with respect to the counting measure, is equal to

\[
f_{E_k} = \frac{1}{k + 1} \sum_{(j, k) \in E_k} f(j, k) = \frac{1}{k + 1} \sum_{j=0}^{k} j = \frac{k}{2}.
\]

It follows that, for \( 1 < p < \infty \),

\[
\|f - f_{E_k}\|_{L^p(E_k)}^p = \sum_{j=0}^{k} \left| j - \frac{k}{2} \right|^p = 2 \sum_{j=0}^{k/2} j^p \approx k^{p+1},
\]

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Figure 1: A portion of the graph $X$. Proximity rule $(i)$ defines the grid, while $(ii)$ introduces the curved edges.

and

$$\|f - f_{E_k}\|_{L^\infty(E_k)} = \max_{j=0,\ldots,k} \left| j - \frac{k}{2} \right| = \frac{k}{2}.$$  

Moreover, for $(j, k) \in E_k$, $\nabla f(j, k) = |j - (j - 1)| + |j - (j + 1)| = 2$ if $j \neq 0$, and $\nabla f(0, k) = 1$. Therefore, $\|\nabla f\|_{L^\infty(E_k)} = 2$ and for $1 < p < \infty$,

$$\|\nabla f\|_{L^p(E_k)}^p \approx \sum_{j=0}^k 1 = k + 1.$$  

Altogether, for any $1 < p \leq \infty$,

$$\frac{\|f - f_{E_k}\|_{L^p(E_k)}}{|E_k|^{1/p} \text{diam}(E_k)^{1-1/p} \|\nabla f\|_{L^p(E_k)}} \approx k^{1-1/p} \to \infty, \quad \text{as } k \to \infty,$$

contradicting (4). \hfill \Box

We now exhibit a graph endowed with a measure which is not positively bounded from below on which the $L^p$-Poincaré inequality fails for $1 \leq p < \infty$.

**Example 3.2.** Let $X = \mathbb{Z}$ endowed with the measure

$$\mu(j) = \begin{cases} \frac{1}{|j|} & \text{if } j \neq 0, \\
1 & \text{otherwise}. \end{cases}$$
Consider the sequence of sets $E_k = [-k, k] \cap \mathbb{Z}$ and define $f(j) = j$. It is clear that $f_{E_k} = 0$ for any $k \geq 1$. Moreover, $\mu(E_k) \approx \log k$ and diam$(E_k) = 2k + 1$ for any $k \geq 2$. A simple computation shows that for any $1 \leq p < \infty$,

$$\|f - f_{E_k}\|_{L^p(E_k, \mu)} = \|f\|_{L^p(E_k, \mu)} = \left(\sum_{j=-k}^{k} |j|^{p-1}\right)^{1/p} \approx k.$$  

Moreover,

$$\|\nabla f\|_{L^p(E_k, \mu)} \approx \left(\sum_{j=1}^{k} \frac{1}{|j|}\right)^{1/p} \approx (\log k)^{1/p},$$

for any $k \geq 2$. It follows that

$$\frac{\|f - f_{E_k}\|_{L^p(E_k, \mu)}}{\mu(E_k)^{1/p} \text{diam}(E_k)^{1-1/p} \|\nabla f\|_{L^p(E_k, \mu)}} \approx \frac{k^{1/p}}{(\log k)^{2/p}} \to \infty, \quad \text{as } k \to \infty. \qedhere$$

We now discuss the optimality of the constant in Theorem 2.1 on trees. Let $T$ be a tree such that $\text{deg}(x) \geq 2$ for every $x \in T$. For any couple of points $x_0 \sim y_0 \in T$, we define the triangle of height $r \in \mathbb{N}$ and root $[x_0, y_0]$ to be the set $T_0 = \{x \in T : d(x, x_0) = d(x, y_0) - 1 \leq r\}$. The base of $T_0$ is intended to be the set of points of $T_0$ at distance $r$ from $x_0$.

**Proposition 3.3.** Let $T$ be a tree such that $2 \leq \text{deg}(x) \leq b+1$ for every $x \in T$, $0 < \alpha \leq \beta < \infty$ and $\mu \in \mathcal{M}_b^\beta$. Then, for every ball $B \subset T$ with diam$(B) = 2r$ and $p \in [1, \infty]$, there exists a function $f$ such that

$$\frac{\|f - f_{B}\|_{L^p(B, \mu)}}{\|\nabla f\|_{L^p(B, \mu)}} \geq C_p(B, \mu),$$

where $C_p(B, \mu) = \mu(B)^{1/p}$ for every $p \in [1, \infty)$ and $C_\infty(B, \mu) = r$.

**Proof.** Let $B = B_r(x_0) \subset T$. Consider two points $x_1, x_2 \sim x_0$ and let $T_1, T_2$ be the two disjoint triangles of height $r - 1$ and roots, respectively, $[x_1, x_0], [x_2, x_0]$. Clearly $T_1, T_2 \subset B$. Define $f$ on $B$ as

$$f(x) = \begin{cases} 
  d(x, x_0) & \text{if } x \in T_1, \\
  -C d(x, x_0) & \text{if } x \in T_2, \\
  0 & \text{otherwise},
\end{cases}$$

and extend $f$ on $T$ by imposing $f(x) = f(y)$ if $x \sim y$ on the remaining vertices. We choose

$$C = \frac{\sum_{x \in T_1} d(x, x_0) \mu(x)}{\sum_{x \in T_2} d(x, x_0) \mu(x)},$$

so that $f_B = 0$. Observe that $\|f\|_{L^\infty(B, \mu)} = r \max\{1, C\}$ and $\|\nabla f\|_{L^\infty(B, \mu)} \leq (b+1) \max\{1, C\}$. Thus,

$$\frac{\|f\|_{L^\infty(B, \mu)}}{\|\nabla f\|_{L^\infty(B, \mu)}} \geq \frac{r}{b+1}.$$
which is (5) for \( p = \infty \).

We now focus on the case \( p \in [1, \infty) \). We claim that for every triangle \( T_0 \) there exists a triangle \( T' \subset T_0 \), whose base is contained in the base of \( T_0 \), such that \( \mu(T') \approx \mu(T_0 \setminus T') \), with constants depending only on \( \alpha, \beta \) and \( b \). If the above claim holds, we are done.

Indeed, it is easy to see that \( B \setminus \{x_0\} \) can be decomposed as the union of at most \( b + 1 \) disjoint triangles \( \{T_i\}_{i=1}^b \) with roots \( [x_i, x_0] \), where \( x_i \sim x_0 \), and height \( r - 1 \). Since \( \mu(x_0) \leq \beta \) and \( n \leq b + 1 \), it is clear that there exists at least one triangle \( T_j \) among them with measure \( \mu(T_j) \gtrsim \mu(B) \). Now, by the aforementioned claim, we can choose a triangle \( T' \subset T_j \) with root \([x', y']\) such that \( \mu(T') \approx \mu(T_j \setminus T') \). Clearly \( \mu(T') \approx \mu(T_j) \approx \mu(B) \). We conclude by defining \( f \) on \( B \) by

\[
f(x) = \begin{cases} 1 & x \in T', \\ -\mu(T')/\mu(T_j \setminus T') & x \in T_j \setminus T', \\ 0 & \text{otherwise}; \end{cases}
\]

and we extend \( f \) on \( T \) by defining \( f(x) = f(y) \) if \( x \sim y \) on the remaining vertices. It is obvious that \( f_B = 0 \), and

\[
\sum_{x \in B} |f(x)|^p \mu(x) \approx \mu(B).
\]

Moreover, \( \nabla f(x) = 0 \) unless \( x = x_j, x_0, x', y' \), in which case

\[
\nabla f(x) \approx 1 + \frac{\mu(T')}{\mu(T_j \setminus T')} \approx 1.
\]

It follows that for every \( p \in [1, \infty) \),

\[
\frac{\|f - f_B\|_{L^p(B, \mu)}}{\|\nabla f\|_{L^p(B, \mu)}} \gtrsim \mu(B)^{1/p},
\]

which is inequality (5).

It remains to prove the claim. Let \( T_0 \) be a triangle of height \( r \) and, for every integer \( n \in [1, r] \), let \( T_n \) be the triangle of height \( r - n \) of maximal measure among those contained in \( T_{n-1} \). Let \( n \) be the minimum integer for which

\[
\frac{\mu(T_n)}{\mu(T_0 \setminus T_n)} \leq 2.
\]

We can assume that \( r \) is large enough so that the above inequality is actually satisfied for some \( n \in [1, r] \) (indeed, if \( r \) is small there is nothing to prove). We have \( \mu(T_n) \leq 2\mu(T_0 \setminus T_n) \) and, on the other hand,

\[
\mu(T_0 \setminus T_n) \leq \mu(T_0 \setminus T_{n-1}) + b\mu(T_n) + \beta
\]
\[
< \frac{1}{2} \mu(T_{n-1}) + b\mu(T_n) + \beta
\]
\[
\leq \frac{1}{2} (b\mu(T_n) + \beta) + b\mu(T_n) + \beta \leq \frac{3}{2} \left( b + \frac{\beta}{\alpha} \right) \mu(T_n).
\]

The claim is proved and the proof is completed.
We underline that when \( \deg(x) \geq 3 \) the term \( C_p(B, \mu) = \mu(B)^{1/p} \) has exponential growth with respect to the radius of \( B \) since \( \mu(B) \geq \alpha 2^r \) if \( B = B_r(x_0) \) for some \( x_0 \in T \) and \( r \in \mathbb{N} \).

We now apply the previous proposition in order to deduce an optimal Poincaré inequality for \( p = 1 \) and \( p = \infty \) on a suitable class of trees, which includes the homogeneous tree endowed with the counting measure.

**Theorem 3.4.** Let \( T \) be a tree such that \( 2 \leq \deg(x) \leq b + 1 \) for every \( x \in T \). Fix \( 0 < \alpha \leq \beta < \infty \) and let \( \mu \in \mathcal{M}_p^\beta \) be a measure on \( T \) such that, for every ball \( B \subset T \) with \( \text{diam}(B) = R \), \( \mu(B) \approx h(R) \) where \( h : \mathbb{N} \to \mathbb{R} \) is a given function. Then, if \( p \in [1, \infty] \), the following inequalities hold

\[
\|f - f_B\|_{L^p(B, \mu)} \lesssim h(R)^{1/p} R^{1-1/p} \|\nabla f\|_{L^p(B, \mu)}. \tag{6}
\]

Moreover, if \( p = 1, \infty \), the previous inequalities are optimal, i.e., there exists a function \( g \) such that

\[
\|g - g_B\|_{L^p(B, \mu)} \gtrsim h(R)^{1/p} R^{1-1/p}, \quad p = 1, \infty. \tag{7}
\]

**Proof.** Since \( \mu(B) \approx h(R) \), Theorem 2.1 implies (6) and, in the case \( p = 1, \infty \), Proposition 3.3 yields (7).

\[\Box\]

## 4 Flow measures on trees

Let \( T \) be a tree such that \( \deg(x) \geq 2 \) for every \( x \in T \). Fix a point \( o \in T \), which we call the origin, and a half-infinite geodesic \( o = x_0, x_1, x_2, \ldots \). We denote by \( \ell(x) \) the level of the vertex \( x \), which is defined by \( \ell(x) = \lim_{n \to \infty} (n - d(x, x_n)) \). For each vertex \( x \), let \( p(x) \) be its only neighbor such that \( \ell(p(x)) > \ell(x) \) and let \( s(x) \) be the set of the remaining neighbors. We define a partial order relation on \( T \) according to which \( x \geq y \) if and only if \( y \) is closer to \( x \) than to \( p(x) \).

In this context, we say that \( \mu : T \to \mathbb{R}^+ \) is a flow measure if

\[
\mu(x) = \sum_{y \in s(x)} \mu(y), \quad \text{for every } x \in T, \tag{8}
\]

and we denote by \( \mathcal{F} \) the family of flow measures on \( T \). The conservation property (8) characterizing flows is equivalent to Kirchhoff’s current law: the total current received by a vertex must equal the total current released by the vertex. Flows have remarkable properties from the harmonic analysis point of view. Indeed, in [7] the authors develop a Calderón–Zygmund theory on a homogeneous tree endowed with a particular flow and, in [12], this theory is adapted to general trees endowed with any locally doubling flow measure.

We define the difference operator acting on functions \( f : T \to \mathbb{C} \) as

\[
d f(x) = f(x) - f(p(x)).
\]

Observe that for any \( f : T \to \mathbb{C} \) and \( x \in T \), \( |d f(x)| \leq \nabla f(x) \).

We now prove a global \( L^p \)-Poincaré inequality on connected sets for trees endowed with flow measures. What is remarkable here, with respect to the results of Section 2, is that flow measures are not bounded above nor below and the tree is not required to have bounded degree. The latter statement implies (see [12] Corollary 2.3) that the upcoming result applies also to flow measures which are not even locally doubling.
\textbf{Theorem 4.1.} Let $E \subset T$ be a connected set with $\text{diam}(E) = 2r$, $p \in [1, \infty]$ and $f$ any function on $X$. Then, for every $\mu \in P$, $(T, \mu)$ satisfies the $L^p$-Poincaré inequality \(3\) with $P_p = 4$, i.e.,

\[
\|f - f_E\|_{L^p(E, \mu)} \leq 4r \|\nabla f\|_{L^p(E, \mu)}.
\]

\textbf{Proof.} Let $E \subset T$ be a finite connected set with $\text{diam}(E) = 2r$. It is easy to see that

\[
\sup_{x \in E} \{|z \in E : z \geq x\} \leq 2r.
\]

Denote by $x_E$ the vertex with maximum level in $E$. Then, we have that

\[
|f(x) - f_E| \leq \sum_{y \in E} \left( \sum_{x_E \geq z \geq x} |df(z)| + \sum_{x_E \geq z \geq y} |df(z)| \right) \frac{\mu(y)}{\mu(E)}
\]

\[
\leq 2\|df\|_{L^\infty(E, \mu)} \sup_{z \in E} \{|z \in E : z \geq x\}
\]

\[
\leq 4r \|df\|_{L^\infty(E, \mu)}.
\]

Passing to the supremum and using that $|df| \leq \nabla f$, we get the desired inequality when $p = \infty$.

Assume now $p \in [1, \infty)$. By applying Jensen’s inequality, we get that

\[
\sum_{x \in E} |f(x) - f_E|^p \mu(x) = \sum_{x \in E} \left( \sum_{y \in E} (f(x) - f(y))^p \frac{\mu(y)}{\mu(E)} \right) \mu(x)
\]

\[
\leq \sum_{x \in E} \sum_{y \in E} |f(x) - f(y)|^p \frac{\mu(y)}{\mu(E)} \mu(x)
\]

\[
\leq \sum_{x \in E} \sum_{z \in E} (\sum_{x \geq z \geq x} |df(z)| + \sum_{x \geq z \geq y} |df(z)|) \frac{\mu(y)}{\mu(E)} \mu(x).
\]

Then, since $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for any $a, b \geq 0$, by Hölder’s inequality, \(4\) and Fubini’s Theorem we obtain

\[
\sum_{x \in E, y \in E} \left( \sum_{x \geq z \geq x} |df(z)| + \sum_{x \geq z \geq y} |df(z)| \right) \frac{\mu(y)}{\mu(E)} \mu(x)
\]

\[
\leq 2^p(2r)^{p/p'} \sum_{x \in E} \sum_{x \geq z \geq x} |df(z)|^p \mu(x)
\]

\[
= 2^p(2r)^{p/p'} \sum_{x \in E} |df(z)|^p \sum_{E \ni x \leq x} \mu(x)
\]

\[
\leq 2^p(2r)^{p/p'+1} \sum_{x \in E} |df(z)|^p \mu(x).
\]

In the last line we have used that, for a flow measure, $\sum_{E \ni x \leq x} \mu(x) \leq \mu(x) \text{diam}(E)$. Since $|df| \leq \nabla f$, the above inequalities imply the desired result. \( \square \)

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