The Bernstein–Gelfand Tensor Product Functor and the Weight-2 Eisenstein Series

Martin Raum *

Abstract: The Bernstein–Gelfand tensor product functors are endofunctors of the category of Harish-Chandra modules provided by tensor products with finite dimensional modules. We provide an automorphic analogue of these tensor product functors, implemented by vector-valued automorphic representations that are trivial at all finite places. They naturally explain the role of vector-valued modular forms in recent work by Bringmann–Kudla on Harish-Chandra modules associated with harmonic weak Maaß forms. We give a detailed account of the image $\text{sym}^1 \otimes \varpi(E_2)$ of the automorphic representation $\varpi(E_2)$ generated by the Eisenstein series of weight 2 under one of those tensor product functors. This builds upon work by Roy–Schmidt–Yi, who recently determined the structure of $\varpi(E_2)$. They found that $\varpi(E_2)$ does not decompose as a restricted tensor product over all places of $\mathbb{Q}$, while we discover that $\text{sym}^1 \otimes \varpi(E_2)$ has a direct summand that does. This summand corresponds to a holomorphic and modular, vector-valued analogue of $E_2$. The complement in $\text{sym}^1 \otimes \varpi(E_2)$ arises from one of the vector-valued examples in the work of Bringmann–Kudla. Our approach allows us to determine its structure at the finite places.

mock modular forms ■ harmonic weak Maaß forms ■ Harish-Chandra modules ■ vector-valued modular forms

MSC Primary: 11F11 ■ MSC Secondary: 11F12, 11F70

*The author was partially supported by Vetenskapsrådet Grant 2019-03551.
VECTOR-VALUED modular forms already appear in classical work by Kuga and Shimura [7], who describe a relation to scalar-valued modular forms, which was recast in the light of quasi-modular forms by various authors [5, 13]. They play a crucial role in the recent classification of Harish-Chandra modules associated with harmonic weak Maaß forms [3]. Without them the classification provided by Bringmann–Kudla would require several special cases related to the exceptional role of the Eisenstein series of weight 2 and level 1. With the help of vector-valued modular forms, its special behavior can be extended to other weights, yielding a full case in the classification that is on equal footing with the other ones. Vector-valued modular forms and vector-valued Maaß forms are thus firmly set up to endure in the theory of harmonic weak Maaß forms. The goal of the present paper is to highlight a connection between them and tensor product functors for Harish-Chandra modules that were studied by Bernstein and Gelfand [1] in the 80ies, who build upon earlier work of Kostant [6]. As applications, we relate the construction of Bringmann–Kudla to these functors and examine a holomorphic and modular analogue $E_{2}^{\text{vec}}$ of the weight-2 Eisenstein series.

Throughout, we work with the algebraic group $G = \text{SL}_2$ and the associated complexified Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$. Specialized to this case, Bernstein–Gelfand investigated the functors $M \rightarrow V \otimes M$ assigning to a $\mathfrak{g}$-module $M$ its tensor product with a finite dimensional $\mathfrak{g}$-module $V$. They showed that the functors $V \otimes \cdot$ provide equivalences between categories of Harish-Chandra modules of specific Harish-Chandra parameters. In our situation, possible indecomposable $V$ are the symmetric power representations $\text{sym}^m$ of the standard representation for nonnegative integers $m$.

For us, the key observation is that the equivalences of categories provided by tensor product functors requires a projection to specific blocks of the category. In other words, the tensor products themselves, without these projections, may have constituents in further blocks. We write $\partial(E_2)$ for the Harish-Chandra module associated with $E_2$. Then Bringmann–Kudla’s construction is an instance where the nontrivial extension class that is manifested in $\partial(E_2)$ is sent to another nontrivial one realized by a submodule of $\text{sym}^m \otimes \partial(E_2)$. This one is related to the equivalence of categories found by Bernstein–Gelfand. However, if $m \geq 1$ then $\text{sym}^m \otimes \partial(E_2)$ also splits off an irreducible direct summand, which is a limit of discrete series if $m = 1$. This is the representation theoretic essence of our construction of $E_{2}^{\text{vec}}$, and does not participate in the equivalence studied by Bernstein–Gelfand.

The space $\mathcal{S}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ of automorphic forms for $G$ is a $(\mathfrak{g}, K_\infty)$-module and a smooth $G(\mathbb{A}_f)$-representation, where $K_\infty = \text{SO}_2(\mathbb{R}) \subset G(\mathbb{R})$ and $\mathbb{A}_f$ are the finite adeles. Refining the definition given by Bringmann–Kudla, we provide spaces of vector-
valued automorphic forms $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}),\text{sym}^m)$. We say that a smooth $(g, K_\infty) \times G(\mathbb{A}_f)$-representation is vector-valued automorphic if it occurs as a subquotient of $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}),\text{sym}^m)$. Our first result, instrumental to make the connection with the functors by Bernstein–Gelfand, provides vector-valued automorphic representations that are trivial at the finite places. We thus have a natural candidate for the modules $V$ in the formalism of Bernstein–Gelfand. The next theorem also confirms that it is compatible with the notion of vector-valued automorphic representations. The trivial representation of $G(\mathbb{Q}_v)$ for a place $v$ of $\mathbb{Q}$ is denoted $1_v$. Restricted tensor products in this paper are always taken with respect to spherical vectors.

**Theorem A.** There are vector-valued automorphic representations $\varpi(\varepsilon_{m,0}) \subset \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}),\text{sym}^m)$ that admit the following restricted tensor product decomposition:

$$\varpi(\varepsilon_{m,0}) \cong \text{sym}^m \otimes \bigotimes_{v \neq \infty} 1_v.$$ 

Given an automorphic representation $\varpi$, that is a subquotient of $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$, the tensor product $\varpi(\varepsilon_{m,0}) \otimes \varpi$ is vector-valued automorphic.

The vector-valued automorphic representations $\varpi(\varepsilon_{m,0})$ are associated with the almost holomorphic vector-valued modular forms $\epsilon_{j,m-j}$ that appear in the work of Bringmann–Kudla [3], and previously in work of Zemel [13]. It is well-known how to associate an automorphic form $\tilde{f}$ to a modular form $f$, and thus the automorphic representation $\varpi(f)$ generated by $\tilde{f}$. This procedure extends to vector-valued modular forms and we find that the representations $\varpi(\varepsilon_{j,m-j})$ for $0 \leq j \leq m$ are identical.

For cuspidal newforms $f$, it is known that $\varpi(f)$ decomposes as a restricted tensor product $\bigotimes' \varpi(f)_v$ over the place $v$ of $\mathbb{Q}$ where $\varpi(f)_v$ is irreducible. When giving up the assumption that $f$ is cuspidal, this no longer holds in general. For example, Roy–Schmidt–Yi [11] determined the structure of $\varpi(E_2)$ for the modular, nonholomorphic Eisenstein series $E_2$ of weight 2 and level 1. They found that it fits into an exact sequence

$$1 \longleftarrow \varpi(E_2) \longrightarrow \mathcal{D}(1) \otimes \bigotimes'_{v \neq \infty} \Pi_{1/2,v},$$

where $\mathcal{D}(k-1)$ for positive integers $k$ is the (limit of) holomorphic discrete series of Harish-Chandra parameter $k-1$ and the representations $\Pi_{1/2,v}$ are degenerate principal series for $G(\mathbb{Q}_v)$. Our second theorem exhibits the structure of the tensor product $\varpi(\varepsilon_{1,0}) \otimes \varpi(E_2)$.
Theorem B. We have inclusions of internal direct sums

$$\varpi(\epsilon_{1,0}) \oplus \varpi(E_{\text{vec}}^2) \subset \varpi(\epsilon_{1,0}) \cdot \varpi(E_2) \subset \mathcal{A}(G(Q) \backslash G(A), \text{sym}^1)$$

and a short exact sequence

$$\varpi(\epsilon_{1,0}) \oplus \varpi(E_{\text{vec}}^2) \longrightarrow \varpi(\epsilon_{1,0}) \otimes \varpi(E_2) \longrightarrow \varpi$$

for a vector-valued automorphic representation

$$\varpi \cong \mathcal{D}(2) \otimes \bigotimes_{v \neq \infty}^\prime \Pi_{\frac{1}{2},v}.$$ 

In particular,

$$\varpi(E_{\text{vec}}^2) \cong \mathcal{D}(0) \otimes \bigotimes_{v \neq \infty}^\prime \Pi_{\frac{1}{2},v}.$$ 

The irreducible direct summand that splits off $\varpi(\epsilon_{1,0}) \otimes \varpi(E_2)$ in Theorem B is a natural source for our construction of $E_{\text{vec}}^2$. The remaining direct summand corresponds to a vector-valued harmonic Maaß form in weight 3 that Bringmann–Kudla encounter in their case III(b).

In Section 1, we revisit some of the classical theory of modular forms and recall the functions $\epsilon_{j,m-j}$ from the work of Bringmann–Kudla including their images under the Maaß operators. In Section 2, we define the vector-valued analogue $E_{\text{vec}}^2$ of the weight-2 Eisenstein series. We give two expressions for it: One as a linear combination of $\epsilon_{j,m-j}$ with coefficients in modular forms, and another one in terms of vector-valued Eisenstein series and their residues. In Section 3, we revisit some of the connection between modular forms and automorphic forms. We also recall the definition of principal series, restate one of the results by Roy–Schmidt–Yi, and establish a related statement on subrepresentations of the space of automorphic forms. In the final Section 4, we introduce vector-valued automorphic forms and corresponding automorphic representations. Theorem A is a combination of Theorem 4.1 and Theorem 4.3, which we state and prove in this section. We conclude the paper with Theorem 4.4, which encompasses the statement of Theorem B, and its Corollaries 4.5 and 4.6, which present the consequences for $E_{\text{vec}}^2$ and the example of Bringmann–Kudla.

Acknowledgement The author is grateful to Claudia Alfes-Neumann for fruitful discussions and comments. He thanks the Institut Mittag-Leffler, where parts of this work was conducted during the program on Moduli and Algebraic Cycles.
1 Modular forms

In this section, we revisit some basic notions of the theory of scalar- and vector-valued modular forms. We also recall some of the special vector-valued modular forms that occurred in the work of Bringmann–Kudla [3] and Zemel [13].

1.1 Preliminaries  Given a complex number $z$, we set $e(z) = \exp(2\pi i z)$. Throughout the text, we set $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which are generators for $\text{SL}_2(\mathbb{Z})$. We let $\Gamma_\infty = \{ \pm T^n : n \in \mathbb{Z} \}$. The Poincaré upper half plane will be written as $\mathbb{H}$. Its elements will be commonly denoted $\tau = x + i y$, $x, y \in \mathbb{R}$. It carries an action of $\text{SL}_2(\mathbb{R})$ by Möbius transformations: $g \tau = \frac{a \tau + b}{c \tau + d}$, $\tau \in \mathbb{H}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$.

We have a surjection $\text{SL}_2(\mathbb{R}) \rightarrow \mathbb{H}$ of $\text{SL}_2(\mathbb{R})$-sets, which maps $g$ to $g i$ and whose kernel is $\text{SO}_2(\mathbb{R})$.

For integers $k$, we have slash actions on functions from $\mathbb{H}$ to $\mathbb{C}$, defined by $$(f|k \gamma)(\tau) = (c \tau + d)^{-k} f(g \tau), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}).$$

The compatibility between these actions and the action of $\text{SL}_2(\mathbb{R})$ by right shift on functions from $\text{SL}_2(\mathbb{R})$ to $\mathbb{C}$, which is standard in automorphic representation theory, will be explained in Section 3.

We obtain the usual notion of holomorphic modular forms of weight $k \in \mathbb{Z}$. Using the Maaß lowering and raising operators and the Laplace operator

$$L_k := -2 i y^2 \partial_\tau, \quad R_k := 2 i \partial_\tau + k y^{-1}, \quad \Delta_k := -R_k - L_k,$$

we define an almost holomorphic modular form of weight $k$ as a smooth function $f : \mathbb{H} \rightarrow \mathbb{C}$ subject to the condition that (i) $f$ is almost holomorphic, i.e., $L_k^{d+1} f = 0$ for some nonnegative integer $d$. (ii) $f|k \gamma = f$ for every $\gamma \in \text{SL}_2(\mathbb{Z})$, (iii) $|f(x + i y)| \ll y^a$ for some $a > 0$ uniformly in $x$ as $y \rightarrow \infty$.

1.2 Vector-valued modular forms  We define an arithmetic type as a finite dimensional, complex representation of $\text{SL}_2(\mathbb{Z})$. We write $V(\rho)$ for the representation space of such an arithmetic type $\rho$. We have a vector-valued slash action on functions $f : \mathbb{H} \rightarrow V(\rho)$ defined by

$$f|_{k, \rho} \gamma := \rho(\gamma)^{-1} (f|_{k} \gamma).$$
We say that a function \( f : \mathbb{H} \to V(\rho) \) is a holomorphic modular form of weight \( k \) and (arithmetic) type \( \rho \) if (i) \( f \) is holomorphic, (ii) \( f|_{k,\gamma} = f \) for all \( \gamma \in \text{SL}_2(\mathbb{Z}) \), (iii) \( \| f(x + iy) \| \ll y^a \) for some \( a > 0 \) locally uniformly in \( x \) as \( y \to \infty \) for any norm \( \| \cdot \| \) on \( V(\rho) \). The space of such forms is written as \( M_k(\rho) \).

There are analogous notions of real-analytic modular forms. In particular, we have a space \( M^{ahol}_k(\rho) \) of almost holomorphic modular forms, which vanish under a power of the lowering operator defined in (1.1), and the notion of harmonic Maaß forms, which vanish under the Laplace operator \( \Delta_k \).

**Symmetric powers**

Given a nonnegative integer \( m \), we let \( \text{sym}^m \) be the symmetric power of the standard representation of \( \text{SL}_2 \) over \( \mathbb{C} \), which is an arithmetic type.

When performing calculations, we will employ the realization on the space \( \mathbb{C}[X]_m \) of polynomials in a formal variable \( X \) of degree at most \( m \) with the usual \( \| \cdot \| \) action.

In order to differentiate between the action on \( \tau \) and the action on \( X \), we will use two indices for the slash action and suppress the negative sign for \( m \), writing \( |_{k,m} \).

Specifically, given a function \( f : \mathbb{H} \to \mathbb{C}[X]_m \), we set

\[ (f|_{k,m} g)(\tau)(X) :=(cX+d)^m(c\tau+d)^{-k}f(g\tau)(gX) \in \mathbb{C}[X]_m, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}). \]

We record for clarity and for later use that

\[ X^j|_{0,m} S = (-1)^j X^{m-j}, \quad 1|_{0,1} S = X, \quad X|_{0,1} S = -1. \]

We write \( M_{k,m} \) and \( M^{ahol}_{k,m} \) for \( M_k(\text{sym}^m) \) and \( M^{ahol}_k(\text{sym}^m) \), that is, the associated spaces of holomorphic and almost holomorphic modular forms.

We have a vector-valued holomorphic modular form \( (X - \tau)^m \) of weight \( -m \) and arithmetic type \( \text{sym}^m \); See for example [9]. More generally, Zemel [13] and later Bringmann-Kudla [3] provided a family of almost holomorphic modular forms, which goes back to at least as early as Verdier [12], that will be convenient in the context of raising and lowering operators. For an integer \( 0 \leq j \leq m \), we adapt the notation of Bringmann–Kudla and set

\[ \epsilon_{j,m-j}(\tau) := \frac{(-1)^{m-j}}{j!} y^{j-m}(X - \tau)^j (X - \overline{\tau})^{m-j} \in M^{ahol}_{m-2j,m}. \quad (1.3) \]

By a direct calculation, we can verify the stronger modular invariance condition with respect to the real Lie group:

\[ \forall g \in \text{SL}_2(\mathbb{R}) : \epsilon_{j,m-j}|_{m-2j,m} g = \epsilon_{j,m-j}. \quad (1.4) \]
As an almost special case of Proposition 3.1 of [13] in the spirit of Kuga–Shimura [7], we record that
\[
M^{\text{hol}}_{k,d} = \bigoplus_{j=0}^{m} \epsilon_{j,m-j} \cdot M^{\text{hol}}_{k+m-2j}.
\] (1.5)

There is no simple, analogous statement for holomorphic modular forms that holds for all weights \(k\); see Theorem 3.4 of [13].

1.3 Maaß operators on vector-valued modular forms We revisit the covariance properties of the classical Maaß operators in (1.1) in the vector-valued setting, including a sketch of a proof that circumvents the explicit verification of covariance. We claim that for all smooth functions \(f : \mathbb{H} \to \mathbb{C}[(X)]\) and all \(g \in \SL_2(\mathbb{R})\), we have
\[
L_k(f|_{k,m} g) = (L_k f)|_{k-2,m} g \quad \text{and} \quad R_k(f|_{k,m} g) = (L_R f)|_{k+2,m} g.
\] (1.6)

Since the lowering and raising operators are graded derivations, we have, for a smooth function \(f : \mathbb{H} \to \mathbb{C}\) and nonnegative integers \(i\) and \(j\), the relations
\[
L_{k+j-i}(\epsilon_{i,j} f) = (L_{j-i} \epsilon_{i,j}) f + \epsilon_{i,j}(L_k f),
\]
\[
R_{k+j-i}(\epsilon_{i,j} f) = (R_{j-i} \epsilon_{i,j}) f + \epsilon_{i,j}(R_k f).
\]

Further, Bringmann-Kudla [3] calculate the action of lowering and raising operators on \(\epsilon_{i,j}\):
\[
L_{j-i} \epsilon_{i,j} = (i+1)j \epsilon_{i+1,j-1}, \quad R_{j-i} \epsilon_{i,j} = \epsilon_{i-1,j+1}.
\] (1.7)

Now, the modular invariance of \(\epsilon_{i,j}\) under \(\SL_2(\mathbb{R})\) implies the covariance in (1.6), since we have for all \(g \in \SL_2(\mathbb{R})\)
\[
L_{j-i}(\epsilon_{i,j}|_{j-i,j+i+g}) = L_{j-i} \epsilon_{i,j} = (i+1)j \epsilon_{i+1,j-1} = (i+1)j \epsilon_{i+1,j-1}|_{j-i-2,j+i+g}
\]
\[
R_{j-i}(\epsilon_{i,j}|_{j-i,j+i+g}) = R_{j-i} \epsilon_{i,j} = \epsilon_{i-1,j+1} = \epsilon_{i-1,j+1}|_{j-i-2,j+i+g}.
\]

1.4 Real-analytic Eisenstein series For \(k \in \mathbb{Z}\) and \(s \in \mathbb{C}\), \(2\Re(s) + k > 2\), we define
\[
E_k(\tau, s) := \sum_{\gamma \in \Gamma_{\infty} \setminus \SL_2(\mathbb{Z})} \gamma^s|_{k}\gamma.
\] (1.8)

If \(k > 2\), we set \(E_k(\tau) = E_k(\tau,0)\).
Employing Maaß’s formula on pages 210 of [8], which provides the Fourier expansion of the modified Eisenstein series \( y^{s+k-1} E_k(\tau, s) \), and Maaß’s modified Whittaker function on page 181 of [8], we find that

\[
E_k(\tau, s) = y^s + (-1)^{\frac{k}{2}} 2^{2-k-2s} \pi \Gamma(1+2s) \Gamma(s) \frac{\Gamma(k-1+2s)}{\Gamma(k+2s)} \frac{\zeta(k-1+2s)}{\zeta(k+2s)} y^{1-k-s} \\
+ \frac{(-1)^{\frac{k}{2}} 2^{s+\frac{k}{2}} \pi^{k+2s}}{\zeta(k+2s)} \sum_{n \in \mathbb{Z}, n \neq 0} \frac{\sigma_{k-1+2s}(|n|)}{\Gamma(s+(1+\text{sgn}(n))\frac{k}{2})} y^{\frac{k}{2}} W_{\text{sgn}(n)} \frac{k}{2}, s \frac{k-1}{2} (4\pi |n| y) e(nx). \tag{1.9}
\]

Analytic continuation via the Fourier expansion allows us to evaluate at \( s = 0 \) even if \( k = 2 \). We set \( E_2(\tau) = E_2(\tau, 0) \) and find (see also p. 19 of [4])

\[
E_2(\tau) = E_2^{\text{hol}}(\tau) - \frac{3}{\pi} y^{-1}, \quad E_2^{\text{hol}}(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e(n\tau).
\]

Observe that \( E_2^{\text{hol}} \) is not a modular form. Instead, it is both a quasi-modular form and a mock modular form; One of the few instances where these two notions overlap. Correspondingly, \( E_2 \) is both an almost holomorphic modular form and a harmonic Maaß form.

### 2 Vector-valued Eisenstein series

In this section, we define the vector-valued analogue \( E_2^{\text{vec}} \) of the quasi-modular holomorphic Eisenstein series in weight 2 and provide two constructions that reveal its modularity. In (2.2), we express it as a linear combination of Bringmann–Kudla’s \( \epsilon_{j,1-j} \) with coefficients in almost holomorphic modular forms. In Sections 2.2, we define real-analytic vector-valued Eisenstein series. This allows us to express \( E_2^{\text{vec}} \) in terms of a residue of Eisenstein series and an Eisenstein series.

#### 2.1 An analogue of the weight-2 Eisenstein series

We define a vector-valued modular form that combines features of \( E_2 \) and \( E_2^{\text{hol}} \). On the one hand it is modular, on the other hand it is holomorphic. It has weight 1, but nevertheless we write \( E_2^{\text{vec}} \) in order to emphasize the connection to \( E_2 \) and \( E_2^{\text{hol}} \).

\[
E_2^{\text{vec}}(\tau) := (X - \tau) E_2^{\text{hol}}(\tau) - \frac{6}{\pi i} \in \mathbb{C}[X]_1. \tag{2.1}
\]

We can recover \( E_2^{\text{hol}} \) as the \( X \)-component of the vector-valued holomorphic modular form \( E_2^{\text{vec}} \).
It is also helpful to express $E_{2}^{\text{vec}}$ in terms of the basis $\epsilon_{j,1-j}(\tau)$. We have

$$E_{2}^{\text{vec}}(\tau) = \epsilon_{1,0}(\tau)E_{2}(\tau) - \epsilon_{0,1}(\tau)\frac{3}{\pi} = (X - \tau)(E_{2}^{\text{hol}}(\tau) - \frac{3}{\pi}y^{-1}) - (X - \tau)^{-1/3} - y^{-1}(X - \tau)^{-1/3}. \quad (2.2)$$

This manifests the connection to the modular Eisenstein series $E_{2}$.

**Proposition 2.1.** We have

$$E_{2}^{\text{vec}} \in M_{1,1}.$$

**Proof.** We have $E_{2}^{\text{vec}} \in M_{1,1}^{\text{hol}}$ by (2.2). Further, $E_{2}^{\text{vec}}$ is holomorphic by its defining expression (2.1).

**Remark 2.2.** It is instructive in the context of Section 4.3 to verify that $E_{2}^{\text{vec}}$ is holomorphic from the expression in (2.2):

$$L_{1}E_{2}^{\text{vec}} = L_{1}(\epsilon_{1,0}E_{2} - \epsilon_{0,1}\frac{3}{\pi}) = \epsilon_{1,0}\frac{3}{\pi} - \epsilon_{1,0}\frac{3}{\pi} = 0.$$

### 2.2 Real-analytic Eisenstein series

We extend the real-analytic Eisenstein series in Section 1.4 to the vector-valued setting and relate it to the former. For a weight $k \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq 0}$ with $k \equiv m \pmod{2}$, $j \in \mathbb{Z}$ with $0 \leq j \leq m$, and $s \in \mathbb{C}$ with $2\text{Re}(s) + k > 2 + m$, we define

$$E_{k,m}(\tau, j, s) := \sum_{\gamma \in \Gamma_{\infty}\backslash \SL_{2}(\mathbb{Z})} (X - \tau)^{j}y^{s}|_{k,m}\gamma. \quad (2.3)$$

Obverse that $X - \tau$ is invariant under the action of $\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$, $b \in \mathbb{R}$, for any $k$ and $m$. The parity condition on $k$ and $m$ ensures invariance under $\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence the action of $\gamma$ in the summand only depends on $\Gamma_{\infty}\gamma$. The condition on $\text{Re}(s)$ ensures absolute convergence by usual estimates.

To provide an analytic continuation of $E_{k,d}(\cdot, j, s)$ we contrast it with a type of Eisenstein series that directly emerges from (1.5). We set

$$E_{k,m}^{\text{hol}}(\tau, j, s) := \sum_{\gamma \in \Gamma_{\infty}\backslash \SL_{2}(\mathbb{Z})} \epsilon_{j,m-j}y^{s}|_{k,m}\gamma. \quad (2.4)$$

As opposed to the Eisenstein series in (2.3), the one in (2.4) is in general not holomorphic even for $s = 0$, as $\epsilon_{j,m-j}$ is merely almost holomorphic. We can, however, readily identify them as products of $\epsilon_{j,m-j}$ with classical Eisenstein series.
Proposition 2.3. Given integers $k, m \geq 0$, and $0 \leq j \leq m$, and a complex number $s$ with $2\text{Re}(s) + k > 2 + m$, we have

$$E_{k,m}^{\text{ahol}}(\tau, j, s) = \epsilon_{j,m-j} \cdot E_{k-m+2j}(\tau, s).$$

In particular, the Eisenstein series in (2.4) admits an analytic continuation to $s \in \mathbb{C}$ with at most simple poles.

Proof. We use the modular $\text{SL}_2(\mathbb{R})$-invariance of $\epsilon_{j,m-j}$ in (1.4). Then after inserting the definition of the Eisenstein series, we find that

$$E_{k,m}^{\text{ahol}}(\tau, j, s) = \sum_{\gamma \in \Gamma_{\infty}} \epsilon_{j,m-j} \gamma^s \big|_{k,m} = \sum_{\gamma \in \Gamma_{\infty}} \epsilon_{j,m-j} \gamma^s \big|_{k-m+2j,m} = \epsilon_{j,m-j} \sum_{\gamma \in \Gamma_{\infty}} \gamma^s \big|_{k-m+2j} = \epsilon_{j,m-j} \cdot E_{k-m+2j}(\tau, s).$$

The Eisenstein series $E_{k,m}^{\text{ahol}}(\tau, j, s)$ and $E_{k,m}(\tau, j, s)$ can be related to each other via an explicit formula, which we make precise in the next statement.

Proposition 2.4. Given integers $k, m \geq 0$, and $0 \leq j \leq m$, and a complex number $s$ with $2\text{Re}(s) + k > 2 + m$, we have

$$E_{k,m}(\tau, j, s) = \left(\frac{i}{2}\right)^{m-j} \sum_{r=j}^{m} \binom{m-j}{r-j} r! \epsilon_{r,m-r} E_{k-m+2r-2j}(\tau, s-r+j)$$

$$= \left(\frac{i}{2}\right)^{m-j} \sum_{r=j}^{m} \binom{m-j}{r-j} r! E_{k,m}^{\text{ahol}}(\tau, r, s-r+j).$$

In particular, the Eisenstein series in (2.3) admits an analytic continuation to $s \in \mathbb{C}$ with at most simple poles.

Proof. This follows when expanding $(X-\tau)^{m-j} 1^{m-j}$ according to the following equations:

$$1 = \frac{-1}{2i} \left((X-\tau) - (X-\overline{\tau})\right),$$

$\cdots$
and hence
\[(X - \tau)^j \frac{1}{m-j} = \left(\frac{i}{2y}\right)^m \left(\frac{1}{\tau} - \left(X - \frac{1}{\tau}\right)\right)^m \]
\[= \left(\frac{i}{2y}\right)^m \sum_{r=0}^{m-j} \binom{m-j}{r} (-1)^{m-j-r} (X - \tau)^j r (X - \bar{\tau})^{m-j-r} \]
\[= \left(\frac{i}{2y}\right)^m \sum_{r=0}^{m-j} \binom{m-j}{r} (j + r)! y^{-r} \epsilon_{j+r,m-j-r}. \]

We are now in position, to express $E_{2\text{vec}}$ via vector-valued Eisenstein series.

**Proposition 2.5.** We have

\[E_{2\text{vec}}(\tau) = E_{1,1}(\tau, 1, 0) + 2i \text{Res}_{s=1} E_{1,1}(\tau, 0, s). \quad (2.7)\]

**Proof.** Note that Proposition 2.4 immediately implies that

\[E_{1,1}(\tau, 1, 0) = \epsilon_{1,0}(\tau) E_2(\tau). \]

We also obtain the expression

\[E_{1,1}(\tau, 0, s) = \frac{i}{2} \left( -\epsilon_{1,0} E_2(\tau, s - 1) + \epsilon_{0,1} E_0(\tau, s) \right), \]

which reveals a simple pole at $s = 1$. Recall from Miyake’s Corollary 7.2.10 in [10] (or employ (1.9)) that

\[\text{Res}_{s=1} E_0(\tau, s) = \frac{\pi}{2\xi(2)} = \frac{3}{\pi}, \]

where care must be taken that he defines modified real-analytic Eisenstein series in (7.2.1), which lack the factor $y^s$ and are normalized by an additional factor $\zeta(2s+k)$. This yields

\[\text{Res}_{s=1} E_{1,1}(\tau, 0, s) = \frac{i}{2} \epsilon_{0,1} \text{Res}_{s=1} E_0(\tau, s) = \frac{3i}{2\pi} \epsilon_{0,1}. \]

Combining this with (2.2), we conclude that

\[E_{2\text{vec}}(\tau) = E_2(\tau) \epsilon_{1,0}(\tau) - \frac{3}{\pi} \epsilon_{0,1}(\tau) = E_{1,1}(\tau, 1, 0) + 2i \text{Res}_{s=1} E_{1,1}(\tau, 0, s). \]
3 Automorphic forms

In this section, we revisit some of the theory of automorphic forms, principal series, and automorphic Eisenstein series. We base this section on recent work by Roy–Schmidt–Yi [11], who examined the automorphic representation associated with $E_2$. The content of this section is mostly preparatory for Section 4. The only statement not contained in the literature is given in Section 3.3.

3.1 Preliminaries  We write $\mathbb{Q}_p$ and $\mathbb{Z}_p$ for the rings of $p$-adic rationals and integers. We write $\mathbb{A}$ for the adeles of $\mathbb{Q}$, and $\mathbb{A}_f \subset \mathbb{A}$ for the finite adeles. To shorten notation and make it slightly more compatible with common use in automorphic representation theory, we will write $G$ for the algebraic group $\text{SL}_2$, $P$ for its parabolic subgroup of upper triangular matrices, $K_\infty = \text{SO}_2(\mathbb{R})$, $K_p = G(\mathbb{Q}_p)$ for primes $p$, $K_f = \prod_p K_p$, and $K = K_f \times K_\infty$. We write $\mathfrak{g}$ for the complexified Lie algebra of $G(\mathbb{R})$ and $\mathfrak{z}$ for the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

To emphasize the connection between modular and automorphic forms, we usually employ the following notation

$$p = \begin{pmatrix} \tilde{y} & x & \tilde{y}^{-1} \\ 0 & 1 & \tilde{y}^{-1} \end{pmatrix} \in P(\mathbb{Q}_v),$$

where at the infinite place we assume that $\tilde{y} > 0$ and we write $p(\tau)$ with $\tau = x + iy^2$, if we need to emphasize the coordinates.

We recall the definition of automorphic forms, adopted in a slightly different form from [2]. An automorphic form on $G$ is a function $\tilde{f} : G(\mathbb{A}) \to \mathbb{C}$ such that $\tilde{f}(\gamma g) = \tilde{f}(g)$ for all $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbb{A})$, $\tilde{f}$ is $K$-finite for the action by right shifts, $\tilde{f}$ is $\mathfrak{z}$-finite, and for every norm $\| \cdot \|$ on $G(\mathbb{A})$ in the sense of [2], there is $a > 0$ such that $|\tilde{f}(g)| \ll \|g\|^a$ for all $g \in G(\mathbb{A})$. The space of automorphic forms $\mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ is a module for $(g, K_\infty) \times G(\mathbb{A}_f)$. By construction this is a smooth representation. Automorphic representations are defined as constituents of $\mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$.

We write $\mathcal{O}(k-1)$, $k \geq 1$ an integer, for the (limit of) holomorphic discrete series of Harish-Chandra parameter $k-1$. Its lowest weight has index $k-1 + \frac{1+1}{2} = k$. Similarly, $\mathcal{O}(k-1)$ denotes the (limit of) anti-holomorphic discrete series. The Steinberg representation for $\text{SL}_2(\mathbb{Q}_v)$ will written as $\text{St}_v$. We write the trivial representation of $G(\mathbb{Q}_v)$ as $1_v$.

Modular forms  We next recall the procedure by which one associates an automorphic form $\tilde{f}$ to an almost holomorphic modular form $f$ of weight $k$, or more generally
a real-analytic one that vanishes under a polynomial in the Laplace operator. We define

\[ \tilde{f}(\gamma g_\infty k_t) := (f|_k g_\infty)(i), \quad \gamma \in G(\mathbb{Q}), \, g_\infty \in G(\mathbb{R}), \, k_t \in K_t. \] (3.2)

Since every \( g \in G(\mathbb{A}) \) can be decomposed as a product \( \gamma g_\infty k_t \), this provides all values of \( \tilde{f} \). Observe that \( \tilde{f} \) depends on the weight \( k \), which is only implicitly given via \( f \).

While \( \tilde{f} \) is a \( K_f \)-fixed vector by construction, its \( K_\infty \)-type is readily calculated. For all \( k = (d - c) \in K_\infty \), we have

\[ \tilde{f}(g k_\infty) := (f|_k g k_\infty)(i) = (ci + d)^{-k}(f|_k g)(k_\infty i) = \tilde{f}(g)(ci + d)^{-k}. \]

We write \( \varpi(f) \) for the representation in \( \mathcal{S}(G(\mathbb{Q}) \backslash G(\mathbb{A})) \) generated by \( \tilde{f} \).

3.2 Principle series  

Langlands proved in the appendix to work by Borel–Jacquet [2] that automorphic representations are exactly the constituents of global principal series, which we define in this section. Specifically, we recall the normalized induction from \( P \) to \( G \), for which we will need the modular character \( \delta_P \) of \( P(\mathbb{A}) \).

Given a place \( v \) of \( \mathbb{Q} \), let \( \mu_{P,v} \) be the normalized right Haar measure on \( P(\mathbb{Q}_v) \). Then for \( p \in P(\mathbb{Q}_v) \), we define \( \delta_{P,v}(p) \) by

\[ \delta_{P,v}(p) = p^*(\mu_{P,v}), \]

where \( p^* \) denotes the pullback under inverse multiplication from the left. More concretely, for \( p \) as in (3.1), we have \( \delta_{P}(p) = |y|^2 \). At the finite places \( v \), we can verify this by comparing the measure of \( P(\mathbb{Z}_v) \) and its image under \( p^{-1} \):

\[ \mu(p^{-1}P(\mathbb{Z}_v)) = \mu\left( \left\{ \begin{array}{c} y \left( \begin{array}{cc} a & b \\ 0 & \tilde{y}^{-1} \end{array} \right) \end{array} : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p \right\} \right) = (y|_v)^2. \]

At the infinite place, a similar computation with, say, \( a \in [\frac{1}{2}, 2] \) and \( b \in [-1, 1] \) works. The global modular function \( \delta_{P,v} \), defined in an analogous way, equals the product of its local contributions \( \delta_{P,v} \) over all places of \( \mathbb{Q} \).

Note that \( G(\mathbb{Q}_p) \) is unimodular, i.e., its modular character is trivial. Local normalized induction of a representation \( \sigma \) of \( P(\mathbb{Q}_p) \) can then be defined by the action via right shifts on the space

\[ V(\text{Ind}_{P(\mathbb{Q}_p)}^G \sigma) := \{ f : G(\mathbb{Q}_v) \to V(\sigma) : f \text{ smooth}, f(pg) = \delta_{P,v}(p)^{\frac{1}{2}} \sigma(p)f(g) \}. \] (3.3)
The representation $\sigma$ in this work is merely a character on the Levi factor. In order to accommodate the relation to $\text{PGL}_2$, we write $| \cdot |_p^s \times | \cdot |_p^{-s}$ for the character that maps $p$ as in (3.1) to $|y|^s = |\tilde{y}|^{2s}$. We write $\Pi_{s,v}$ for its induction if $v \neq \infty$ and $\Pi_{1/2,\infty}$ for the $K_\infty$-finite vectors of the induction at $v = \infty$.

The important case for us will be $s = \frac{1}{2}$. If $v = \infty$, the principal series fits into the following exact sequence of Harish-Chandra modules:

$$
\mathcal{D}(1) \oplus \mathcal{D}(1) \longrightarrow \Pi_{1/2,\infty} \longrightarrow \text{sym}^1. \quad (3.4)
$$

If $v$ is finite, we have an analogous sequence of $G(\mathbb{Q}_v)$-modules

$$
\text{St}_v \longrightarrow \Pi_{1/2,v} \longrightarrow \mathbb{1}_v. \quad (3.5)
$$

### 3.3 The Eisenstein series of weight 2

Roy–Schmidt–Yi [11] determined the exact structure of $\varpi(E_2)$ in the context of automorphic forms for $\text{GL}_2$ in their Theorem 5.11. Since the restriction of the Steinberg representation from $\text{GL}_2(\mathbb{Q}_v)$ to $\text{SL}_2(\mathbb{Q}_v)$ does not decompose further, we can adopt their result and find that $\varpi(E_2)$ sits in the short exact sequence

$$
\mathbb{1} \longrightarrow \varpi(E_2) \longrightarrow \mathcal{D}(1) \otimes \bigotimes_{v \neq \infty} \Pi_{1/2,v}^\prime. \quad (3.6)
$$

Note that the discrete series $\mathcal{D}(1)$ in this sequence is the one for $\text{SL}_2(\mathbb{R})$ and thus different from the one in [11] for $\text{GL}_2(\mathbb{R})$.

The maximal irreducible quotient $\varpi(E_2)$ of $\varpi(E_2)$ can be reasonably viewed as the automorphic representation associated with $E_2^{\text{hol}}$. We want to verify that $\varpi(E_2)$ is not a subrepresentation of $\mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. While this is folklore, to our knowledge it is not stated in the required form in the available literature.

**Proposition 3.1.** There is no intertwining inclusion of $\varpi(E_2)$ into $\mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$.

**Proof.** Assuming the contrary, we have a map

$$
\varpi(E_2) \hookrightarrow \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A})). \quad (3.7)
$$

Let $\phi = \bigotimes_v \phi_v \in \varpi(E_2)$ be the spherical vector at all finite places $v$ and a generator of the lowest $K_\infty$-type at the infinite place. We write $\tilde{f}$ for the image of $\phi$ under (3.7) and define a function $f : \mathbb{H} \rightarrow \mathbb{C}$ by

$$(f |_2 g_\infty)(i) := \tilde{f}(g_\infty), \quad g_\infty \in G(\mathbb{R}).$$
This is well-defined since \( \tilde{f} \) has \( K_\infty \)-type corresponding to modular weight 2. It is invariant under \( \Gamma \), since \( \tilde{f} \) is automorphic and spherical at the finite places. Since \( \phi_\infty \) is a lowest weight vector, \( f \) is holomorphic. At the finite places \( \phi_v \) has the same Hecke eigenvalue \( 1+p \) as \( E_2 \). The growth condition for \( \tilde{f} \) implies that \( f \) has moderate growth. From the Hecke eigenvalues we conclude that \( f \) is a multiple of \( E_2^{\text{hol}} \) up to an additive constant, which is not modular; A contradiction.

4 Vector-valued automorphic forms

In this section we define the space \( \mathcal{A} (G(Q) \backslash G(A), \rho) \) of vector-valued automorphic forms. We introduce finite dimensional, vector-valued automorphic representations in Section 4.2. Then in Section 4.3 we relate them to the tensor product functors studied by Bernstein–Gelfand. This allows us to reinterpret the vector-valued automorphic representation \( \varpi(E_{2}^{\text{vec}}) \) generated by \( E_{2}^{\text{vec}} \). The tensor product functors partially split the extension class of \( \varpi(E_{2}) \).

4.1 Preliminaries

Recall the notation set up in Section 3.1. We start by defining vector-valued automorphic forms following Bringmann–Kudla [3]. We translate their notion to the adelic setting and amend the condition that vector-valued automorphic forms are \( z \)-finite, which appears in the definition of usual automorphic forms.

We let \( \rho \) be an arithmetic type, that is, a representation of \( SL_2(Z) = G(Z) \), that extends to \( G(Q) \). For \( \gamma \in G(Q) \) embedded diagonally into \( G(A) \), we set \( \rho(\gamma) = \rho(\gamma_\infty) \), where \( \gamma = \gamma_f \gamma_\infty \) with \( \gamma_f \in G(A_f) \) and \( \gamma_\infty \in G(R) \). A vector-valued automorphic form on \( G \) of arithmetic type \( \rho \) is a function \( \tilde{f} : G(A) \to \mathcal{V}(\rho) \) such that \( \tilde{f}(\gamma g) = \rho(\gamma) \tilde{f}(g) \) for all \( \gamma \in G(Q) \), \( g \in G(A) \), \( \tilde{f} \) is \( K \)-finite for the action by right shifts, \( \tilde{f} \) is \( z \)-finite, and for every norm \( \| \cdot \| \) on \( G(A) \) and every norm \( \| \cdot \| \) on \( V(\rho) \) there is \( a > 0 \) such that \( \| \tilde{f}(g) \| \ll \| g \|^a \) for all \( g \in G(A) \). The corresponding space of vector-valued automorphic forms \( \mathcal{A} (G(Q) \backslash G(A), \rho) \) is a module for \( (g, K_\infty) \times G(A_f) \). As in the automorphic case, it is a smooth representation by construction.

Modular forms

Similar to classical modular forms, one can pass from vector-valued modular forms to vector-valued automorphic forms. In the real-analytic case we need to assume that it vanishes under a polynomial in the Laplace operator. Given a vector-valued modular form \( f \) of weight \( k \) and type \( \rho \) where \( \rho \) extends to \( SL_2(Q) \) we can associate a vector-valued automorphic form as follows:

\[
\tilde{f}(\gamma g_\infty k_f) := \rho(\gamma) \left( f \mid_k g_\infty \right) (i), \quad \gamma \in G(Q), \; g_\infty \in G(R), \; k_f \in K_f.
\]
The defining properties of vector-valued modular forms can all be directly verified provided that the right hand side is well-defined. For clarity, we emphasize that the right hand side does not agree with the vector-valued slash action that appears in the definition of vector-valued modular forms.

To verify that the right hand side is well-defined, we consider $\gamma \in G(Q) \cap G(\mathbb{R}) K_t$ and write $\gamma = \gamma_f \gamma_{\infty}$ with $\gamma_f \in K_t$ and $\gamma_{\infty} \in G(\mathbb{R})$. Then for $\gamma' \in G(Q)$, $g_{\infty} \in G(\mathbb{R})$, and $k_t \in K_t$, we have

$$\tilde{f}(\gamma' g_{\infty}k_t) = \tilde{f}(\gamma' \gamma_f^{-1} \gamma_{\infty} k_t) = \rho(\gamma' \gamma_f^{-1}) (f|_{K} \gamma_{\infty} g_{\infty})(i) = \rho(\gamma' \gamma_f^{-1}) \rho(\gamma_{\infty}) (f|_{K} g_{\infty})(i) = \rho(\gamma')(f|_{K} g_{\infty})(i).$$

We write $\omega(f)$ for the representation in $\mathcal{A}(G(Q)/G(A), \rho)$ generated by $\tilde{f}$.

### 4.2 Finite dimensional representations

Recall $\varepsilon_{j,m-j}$ from Section 1.3, which is a vector-valued modular form of weight $m-2j$ and type $\text{sym}^m$. The sole purpose of this section is to determine the associated vector-valued automorphic representations.

**Theorem 4.1.** Given a nonnegative integer $m$, we have $\omega(\varepsilon_{j,m-j}) = \omega(\varepsilon_{m,0})$ for every integer $0 \leq j \leq m$. Further, we have the restricted tensor product decomposition

$$\omega(\varepsilon_{m,0}) \cong \text{sym}^m \otimes \bigotimes_{v \neq \infty} 1_v,$$

where $\text{sym}^m$ is the $m+1$-dimensional irreducible $(g, K_{\infty})$-module and $1_v$ is the trivial representation of $G(Q_v)$.

**Remark 4.2.** The isomorphism can be concretely realized by the evaluation map that sends $f \in \omega(\varepsilon_{j,m-j})$ to $f(1) \in \mathbb{C}[X]_m$. To see that it is injective, note that if $f(1) = 0$ for $f \in \omega(\varepsilon_{j,m-j})$, then for all $g_{\infty} \in G(\mathbb{R})$, we have $f(g) = \text{sym}^m(g_{\infty}) f(1) = 0$ by (4.3) in the next proof.

**Proof.** Using the $\text{SL}_2(\mathbb{R})$-covariance in (1.4), we find that the vector-valued automorphic form associated to $\varepsilon_{j,m-j}$ is

$$\tilde{\varepsilon}_{j,m-j}(\gamma g_{\infty}k_t) = \text{sym}^m(\gamma_{\infty} g_{\infty}) \varepsilon_{j,m-j}(t),$$

where $\gamma = \gamma_{\infty} \gamma_f \in G(Q)$, $\gamma_{\infty}, g_{\infty} \in G(\mathbb{R})$, $\gamma_f \in G(A_f)$, and $k_t \in K_t$. In particular, we discover that $\tilde{\varepsilon}_{j,m-j}(g) = \tilde{\varepsilon}_{j,m-j}(g_{\infty})$ for $g = g_{\infty} \gamma_f \in G(A)$ with $g_{\infty} \in G(\mathbb{R})$ and $g_f \in G(A_f)$. 

Since $\omega(\epsilon_{j, m-j})$ is defined as the representation generated by $\tilde{\epsilon}_{j, m-j}$, we conclude that $f(g_{\infty}g) = f(g_{\infty})$ for all $f \in \omega(\epsilon_{j, m-j})$. This shows that $\omega(\epsilon_{j, m-j})$ is trivial as a representation of $G(\mathbb{Q}_v)$ for all places $v \neq \infty$ of $\mathbb{Q}$.

Since the $\epsilon_{j, m-j}(i)$ form a basis of $\mathbb{C}[X]_m$ as $j$ runs through integers $0 \leq j \leq m$, we find as we claimed that

$$\omega(\epsilon_{j, m-j}) = \omega(\epsilon_{m,0}) = \text{span} \{ \tilde{\epsilon}_{j', m-j'} : 0 \leq j' \leq m \} \quad \text{and} \quad \omega(\epsilon_{m,0}) \cong \text{sym}^m. \quad \blacksquare$$

### 4.3 Automorphic tensor products

In general, constituents of tensor products of automorphic forms are not automorphic. In this section, we show that the tensor product with $\omega(\epsilon_{m,0})$ yields maps from automorphic forms to vector-valued automorphic forms. This yields an automorphic version of the tensor product functors by Bernstein–Gelfand. We examine the structure of $\omega(\epsilon_{j, m-j}) \otimes \omega(E_2)$ and relate it to one of the cases in the classification by Bringmann–Kudla [3] and to $\omega(E_2^{\text{vec}})$.

**Theorem 4.3.** For all nonnegative integers $m$ and $0 \leq j \leq m$, we have an embedding of $G(\mathbb{A}_d)$-representations

$$\mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A})) \hookrightarrow \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}), \text{sym}^m), f \longmapsto \tilde{\epsilon}_{j, m-j} \cdot f. \quad (4.4)$$

In particular, if $\omega$ is an automorphic representation that appears as a subrepresentation, i.e., $\omega \subseteq \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$, then

$$\omega(\epsilon_{m,0}) \otimes \omega \cong \omega(\epsilon_{m,0}) \cdot \omega \subseteq \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}), \text{sym}^m). \quad (4.5)$$

More generally, if $\omega$ is an automorphic representation for $G(\mathbb{Q})$, then $\omega(\epsilon_{m,0}) \otimes \omega$ is vector-valued automorphic of type $\text{sym}^m$.

**Proof.** The embedding in (4.4) follows when verifying the definition of vector-valued automorphic forms for $\tilde{\epsilon}_{j, m-j} \cdot f$. For the isomorphism in (4.5), it suffices to note that the $\tilde{\epsilon}_{j, m-j}(1)$ form a basis of $\mathbb{C}[X]_m$.

To establish the final statement of the theorem, choose automorphic representations $\tilde{\omega}, \tilde{\omega}_0 \subseteq \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ with

$$\tilde{\omega}_0 \hookrightarrow \tilde{\omega} \longrightarrow \omega.$$

Since $\omega(\epsilon_{m,0})$ is finite dimensional and trivial as a representation of $G(\mathbb{A}_d)$, the functor $\omega(\epsilon_{m,0}) \otimes \cdot$ is exact. We obtain a short exact sequence

$$\omega(\epsilon_{m,0}) \otimes \tilde{\omega}_0 \cong \omega(\epsilon_{m,0}) \cdot \tilde{\omega}_0 \hookrightarrow \omega(\epsilon_{m,0}) \otimes \tilde{\omega} \cong \omega(\epsilon_{m,0}) \cdot \tilde{\omega} \longrightarrow \omega(\epsilon_{m,0}) \otimes \omega.$$

Since the first two representation are contained in $\mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}), \text{sym}^m)$, we finish the proof. \quad \blacksquare
The prime example of Theorem 4.3 in this paper is given in the next statement.

**Theorem 4.4.** We have inclusions of internal direct sums

\[ \varpi(e_{1,0}) \oplus \varpi(E^\text{vec}_{2}) \subset \varpi(e_{1,0}) \cdot \varpi(E_2) \subset \mathcal{S}(G(Q) \backslash G(A), \text{sym}^1) \]

and a short exact sequence

\[ \varpi(e_{1,0}) \oplus \varpi(E^\text{vec}_{2}) \longrightarrow \varpi(e_{1,0}) \otimes \varpi(E_2) \longrightarrow \varpi \]

for a vector-valued automorphic representation

\[ \varpi \cong \mathcal{D}(2) \otimes \bigotimes_{v \neq \infty} \Pi^1_{2,v}. \]

**Corollary 4.5.** We have the restricted tensor product decomposition

\[ \varpi(E^\text{vec}_{2}) \cong \mathcal{D}(0) \otimes \bigotimes_{v \neq \infty} \Pi_{1,2,v}. \]

**Proof of Theorem 4.4 and Corollary 4.5.** Observe that the constant function \( L_2 E_2 \) lifts to a constant function in \( \varpi(E_2) \). That is, we have \( \varpi(1) \subset \varpi(E_2) \), and thus find that

\[ \varpi(e_{1,0}) = \varpi(e_{1,0}) \cdot \varpi(1) \subset \varpi(e_{1,0}) \cdot \varpi(E_2). \]

In a similar vein, we conclude from (2.2) that

\[ \tilde{E}^\text{vec}_{2} = \tilde{e}_{1,0} \tilde{E}_2 - \tilde{e}_{0,1} \frac{3}{\pi} \in \varpi(e_{1,0}) \cdot \varpi(E_2). \quad (4.6) \]

To see that the sum representation \( \varpi(e_{1,0}) + \varpi(E^\text{vec}_{2}) \) is a direct sum, we inspect the summands as Harish-Chandra modules that are isotypical for \( \text{sym}^1 \) and \( \mathcal{D}(0) \). Since these have different Harish-Chandra parameters, we have

\[ \text{Ext}_{(g, \mathcal{K}_\infty)}(\text{sym}^1, \mathcal{D}(0)) = \text{Ext}_{(g, \mathcal{K}_\infty)}(\mathcal{D}(0), \text{sym}^1) = 0. \]

This shows the first part of Theorem 4.4:

\[ \varpi(e_{1,0}) + \varpi(E^\text{vec}_{2}) = \varpi(e_{1,0}) \oplus \varpi(E^\text{vec}_{2}) \subset \varpi(e_{1,0}) \cdot \varpi(E_2). \]

From the first part of Theorem 4.3 and the short exact sequence (3.6) for \( \varpi(E_2) \), we obtain the exact sequence

\[ \varpi(e_{1,0}) \longrightarrow \varpi(e_{1,0}) \cdot \varpi(E_2) \longrightarrow \varpi(e_{1,0}) \otimes \left( \mathcal{D}(1) \otimes \bigotimes_{v \neq \infty} \Pi_{1,2,v} \right). \]
Theorem 4.1 allows us to determine the right module. It is isomorphic to
\[
\left( \mathcal{D}(0) \oplus \mathcal{D}(2) \right) \otimes \bigotimes_{v \neq \infty} \Pi_{\frac{1}{2}, v}^v.
\]

We thus finish the proof when we establish Corollary 4.5.

To this end we inspect the \( K_{\infty} \times G(\mathbb{A}_f) \)-module given by the lowest \( K_{\infty} \)-type. We write \( \pi_k \) for the irreducible \( K_{\infty} \)-representation corresponding to modular weight \( k \), and denote corresponding \( K_{\infty} \)-isotypical components by square brackets \( \varpi \left[ k \right] \). Further, we write \( \tilde{\varpi}_f(E_2) \) and \( \varpi_f(1) \) for the \( K_{\infty} \times G(\mathbb{A}_f) \)-modules generated by \( \tilde{E}_2 \) and the constants. As a consequence of (3.6), we find that
\[
\varpi(E_2)[0] = \tilde{\varpi}_f(1)[0] \cong 1 \quad \text{and} \quad \varpi(E_2)[2] \cong \pi_2 \otimes \bigotimes_{v \neq \infty} \Pi_{\frac{1}{2}, v}^v.
\]

Theorem 4.1 then shows that
\[
\varpi(e_1, 0) \cdot \varpi(E_2)[1] = \tilde{\varpi}_f(0, 1) \cdot \varpi(E_2)[0] \cong \varpi_f(1)[0] \oplus \varpi_f(1)[2] \cong \pi_1 \otimes \bigotimes_{v \neq \infty} \Pi_{\frac{1}{2}, v}^v,
\]
where \( 1_f \) is the trivial \( G(\mathbb{A}_f) \)-representation.

From (4.6) and Theorem 4.1, we conclude that
\[
\tilde{\varpi}_f(E_{2}^{\text{vec}}) + \varpi_f(e_0, 1) = \tilde{\varpi}_f(1) \oplus \varpi_f(E_2),
\]
which we have identified in the previous equation. Taking the quotient by \( \tilde{\varpi}_f(1)[0] \cong \pi_1 \otimes 1_f \), we obtain Corollary 4.5. 

The sequence in Theorem 4.4 does not split. In our final corollary, we identify the complement to \( \varpi_f(E_{2}^{\text{vec}}) \). It matches one of the Examples of case III(b) in Bringmann–Kudla’s classification [3].

**Corollary 4.6.** For
\[
f = e_1, 0 E_2 + \frac{1}{2} e_0, 1 R_2 E_2 \quad (4.7)
\]
we have an exact sequence
\[
\varpi(e_1, 0) \longrightarrow \varpi(f) \longrightarrow \mathcal{D}(2) \otimes \bigotimes_{v \neq \infty} \Pi_{\frac{1}{2}, v}^v.
\]

**Remark 4.7.** As a special case of the calculation by Bringmann–Kudla in (6.10) of [3], one can use the relation \( LR E_2 = 2 E_2 \) to directly verify that
\[
L_3 (e_1, 0 E_2 + \frac{1}{2} e_0, 1 R_2 E_2) = e_0, 1 E_2 + e_1, 0 \frac{3}{\pi} + \frac{1}{2} e_0, 1 L_4 R_2 E_2 = e_1, 0 \frac{3}{\pi}.
\]
Proof. It suffices to show that \( \varpi(f) + \varpi(E_2^{\text{vec}}) = \varpi(\epsilon_{1,0}) \cdot \varpi(E_2). \) Write \( \varpi \) for the left hand side. From the remark, we have \( \tilde{e}_{j,1-j} \in \varpi \) for \( j \in \{0,1\} \). Then the expression for \( E_2^{\text{vec}} \) in (2.2) shows that \( \tilde{e}_{1,0} \tilde{E}_2 \in \varpi \). The action of \( g \) implies that

\[
\tilde{e}_{0,1} \tilde{E}_2 + \tilde{e}_{1,0} \tilde{R} \tilde{E}_2 \in \varpi,
\]

which together with the expression for \( f \) finishes the proof. 

\[ \square \]

[1] J. N. Bernstein and S. I. Gelfand. Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. *Compos. Math.* 41 (1980).

[2] A. Borel and H. Jacquet. Automorphic forms and automorphic representations. *Automorphic forms, representations and \( L \)-functions*, Proc. Symp. Pure Math. Am. Math. Soc., Corvallis/Oregon 1977, Proc. Symp. Pure Math. 33, 1. 1979.

[3] K. Bringmann and S. Kudla. A classification of harmonic Maass forms. *Math. Ann.* 370.3-4 (2018).

[4] J. H. Bruinier, G. van der Geer, G. Harder, and D. B. Zagier. *The 1-2-3 of modular forms*. Ed. by K. Ranestad. Universitext. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004. Springer-Verlag, Berlin, 2008.

[5] Y. Choie and M. H. Lee. Symmetric tensor representations, quasimodular forms, and weak Jacobi forms. *Adv. Math.* 287 (2016).

[6] B. Kostant. On the tensor product of a finite and an infinite dimensional representation. *J. Funct. Anal.* 20 (1975).

[7] M. Kuga and G. Shimura. On vector differential forms attached to automorphic forms. *J. Math. Soc. Japan* 12 (1960).

[8] H. Maass. *Lectures on modular functions of one complex variable*. Notes by Sunder Lal. Tata Institute of Fundamental Research Lectures on Mathematics, No. 29. Bombay: Tata Institute of Fundamental Research, 1964.

[9] M. H. Mertens and M. Raum. Modular forms of virtually real-arithmetic type I: Mixed mock modular forms yield vector-valued modular forms. *Math. Res. Lett.* 28.2 (2021).

[10] T. Miyake. *Modular forms*. Translated from the Japanese by Yoshitaka Maeda. Springer, Berlin, 1989.

[11] M. Roy, R. Schmidt, and S. Yi. *Classical and adelic Eisenstein series*. arXiv:2109.07649. 2021.

[12] J.-L. Verdier. Sur les intégrales attachées aux formes automorphes (d’après Goro Shimura). *Séminaire Bourbaki*, Vol. 6. Soc. Math. France, Paris, 1995.

[13] S. Zemel. On quasi-modular forms, almost holomorphic modular forms, and the vector-valued modular forms of Shimura. *Ramanujan J.* 37.1 (2015).