Memory Constraint Online Multitask Classification

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Abstract

We investigate online kernel algorithms which simultaneously process multiple classification tasks while a fixed constraint is imposed on the size of their active sets. We focus in particular on the design of algorithms that can efficiently deal with problems where the number of tasks is extremely high and the task data are large scale. Two new projection-based algorithms are introduced to efficiently tackle those issues while presenting different trade-offs on how the available memory is managed with respect to the prior information about the learning tasks. Theoretically sound budget algorithms are devised by coupling the Randomized Budget Perceptron and the Forgetron algorithms with the multitask kernel. We show how the two seemingly contrasting properties of learning from multiple tasks and keeping a constant memory footprint can be balanced, and how the sharing of the available space among different tasks is automatically taken care of. We propose and discuss new insights on the multitask kernel. Experiments show that online kernel multitask algorithms running on a budget can efficiently tackle real world learning problems involving multiple tasks.

1 Introduction

In recent years there has been a growing interest in online learning algorithms processing data from multiple and related sources. Many interesting multitask problems involve large scale data sets or pose memory and real-time restrictions. For example, massive personalized spam detectors and ad serving systems need real-time, scalable, and continuously adaptable learning methods. Multi-sensors, memory limited handheld devices deployed on the field are often required to process and classify readings without relying on a centralized, dedicated mainframe, and therefore face severe memory restrictions. Online multitask algorithms are also a natural choice for a growing number of applications that do not necessarily involve online data processing, but where data sets are so large that the computationally expensive batch algorithms can not be used (see, for example, [1]).

In this paper we leverage on known results ([3], [8], [11]) to design online kernel algorithms that effectively address these problems. First, we cast new light on how the multitask kernel of [9] acts as a proxy to wire prior information about task relations into learning algorithms. Second, we build upon the Projectron algorithm [11] to design two new and highly efficient

1
multitask budget online algorithms that aggressively retain learned information by sharing the global budget across different tasks. This is achieved using two multitask mediated projection steps. We also show how existing budget online kernel algorithms can be combined with the multitask paradigm in order to obtain scalable and accurate solutions that retain strong theoretical bounds. In doing so we show how the available space is automatically shared by and assigned to different tasks. Third, we provide an empirical evaluation of the proposed algorithms in a variety of different experimental setups. We stress the fact that our algorithms are particularly apt to be applied to problems where both the data per task and the tasks themselves are great in number. This goal is achieved by combining the mild dependence on the number of tasks with the enforced budget size.

As of recently, several papers have considered large scale learning of multitask problems. For example, [15] introduces a highly scalable linear algorithm for spam classification based on hashing techniques. Our work focuses on kernel algorithms, and therefore it is not directly comparable to [15]. Moreover, by relying on the multitask kernel, the algorithms discussed here can be easily applied to model situations where tasks exhibit a specific pattern of relationships. A different approach is the one outlined in [12], where the focus is on learning and tracking continuously changing relations among tasks. The nature of the problem considered there makes their algorithms not suitable to large scale applications, since the active set can easily grow unbounded and the dependence on the number of tasks tends to be quadratic. In Section 4 we show that under certain assumptions our techniques can effectively deal with shifting tasks.

2 Preliminaries and notation

We consider the usual online classification protocol, where learning proceeds in trials, with an additional complication due to the presence of multiple, possibly related, tasks. At each time step $t$ an instance vector $x_t \in \mathbb{R}^d$ for a given task $i_t \in \{1, \ldots, k\}$, chosen among a fixed set of $k$ different tasks, is disclosed, and the algorithm outputs a corresponding binary prediction $\hat{y}_t \in \{-1, +1\}$. The true label $y_t$ is then revealed and the algorithm acts accordingly, choosing if and how to update its internal state. We follow [4] and define the $t$-th multitask instance as the pair $[x_t, i_t]$. Similarly, the multitask example at time $t$ is defined to be the pair comprised of the $t$-th multitask instance together with the label $y_t$. We do not assume any specific generative model for the sequence of the multitask examples, i.e., no assumptions are made on the instance vectors $x_1, x_2, \ldots$, the task markers $i_1, i_2, \ldots$ and the labels $y_1, y_2, \ldots$. In this work our main interest focuses on classification algorithms that are rotationally invariant and allow for the adoption of the so-called kernel trick. Let $K : (\mathbb{R}^d \times \{1, \ldots, K\}) \times (\mathbb{R}^d \times \{1, \ldots, K\}) \to \mathbb{R}$ be a symmetric, positive semidefinite kernel operator between multitask instances and denote with $\mathcal{H}_K$ its associated Reproducing Kernel Hilbert Space (RKHS). Following the online literature, theoretical results are stated in the form of relative mistake bounds, where the number of mistakes made by the algorithm is compared against a measure of performance obtained by the best reference classifier in a given comparison class. We assume the standard hinge loss function as measure of performance, which is defined, for any $\tilde{y} \in \mathcal{H}_K$, by $\ell_t(\tilde{y}) = \max\{0, 1 - y_t\tilde{y}(x_t, i_t)\}$. Since theoretical results are given with respect to arbitrary sequences, we also introduce the cumulative hinge loss $L(\tilde{y}) = \sum_{t} \ell_t(\tilde{y})$.

With the intent of properly modeling a number of scenarios that are frequently occurring in practice, we also consider the so-called shifting model in which the reference classifier is allowed to change, or shift, throughout the ongoing learning phase. Under this assumption the single reference classifier $\tilde{y}$ is replaced by the sequence $\tilde{y}_1, \tilde{y}_2, \ldots$ and the cumulative hinge loss becomes $L(\{\tilde{y}_t\}_{1,2,\ldots}) = \sum_t \ell_t(\tilde{y}_t)$. We omit the argument of the cumulative loss whenever no ambiguity arises. Unsurprisingly, the presence of shifting reference classifiers will be reflected in theoretical bounds through a term that takes into account the overall amount of shifting that the reference sequence undergoes. Such term, known as the total shift and denoted with
\[ S_A(\{\bar{g}_t\}), \text{ or } S_A \text{ whenever } \{\bar{g}_t\} \text{ is understood from the context, is defined as a sum of the distances computed with respect to a given positive definite operator } A \text{ between consecutive classifiers in the sequence, or, formally, as } S_A(\{\bar{g}_t\}) = \sum_t \|A(\bar{g}_t - \bar{g}_{t-1})\|]. \]

As previously mentioned, this paper concentrates on kernel-based online algorithms. The prediction function \( \bar{f} : \mathbb{R}^d \times \{1, \ldots, K\} \to \mathbb{R} \) of such algorithms can be encoded in the so-called dual form \( \bar{f}(\cdot) = \sum_j \beta_j \mathcal{K}(\bar{x}_j, \cdot) \), that is, as a linear combination of terms \( \beta_j \mathcal{K}(\bar{x}_j, \cdot) \) where the weights \( \beta_j \)'s are real coefficients. We refer to the (multi)set of those instances that appear in the linear combination for \( \bar{f} \) as to the active set \( \mathcal{S} \) and to the instances themselves as to active instances. For convenience we also adopt the shorthand \( J = \{ j : [x_j, i_j] \in \mathcal{S} \} \). A common drawback of the dual form expansion is the tendency of the active set to grow unbounded. Online algorithms such that the expanded or dual form is limited to \( B \) terms are known as budget algorithms. The budget requirement effectively imposes a memory constraint and forces the online algorithm to throw away an active example whenever a new one has to be added to the active set. In the rest of the paper we use the expressions “budget” and “active set” interchangeably when we refer to budget algorithms.

### 3 From single to multiple tasks

We first provide a brief description of the multitask kernel as introduced in \cite{9} and the rationale behind it. Denote with \( \mathcal{K}' : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) the kernel operator between single task instances and with \( \mathcal{H}_{\mathcal{K}'} \) its associated RKHS. In order to streamline the presentation as much as possible we assume \( \|x_t\| = \sqrt{\mathcal{K}'(x_t, x_t)} = 1 \) for all \( t \).

For our purposes the multitask kernel can be seen as a meta-kernel in that it is responsible for properly balancing the impact that a given instance \( x_t \) has on the learning of task \( i_t \) as well as of the possibly related, remaining tasks \( 1, \ldots, i_t - 1, i_t + 1, \ldots, k \). Ideally, in order to meet this goal the multitask kernel should be defined according to a priori information encoded in a graph that establishes mutual relations among tasks. Let \( G = (V_G, E_G) \) be such a graph, with \( V_G \) representing the tasks and \( E_G \) being their relations, and define the associated Laplacian matrix \( L_G \) as

\[
(L_G)_{i,j} = \begin{cases} 
  d_i & \text{if } i = j, \\
  -1 & \text{if } (i, j) \in E_G, \\
  0 & \text{otherwise},
\end{cases}
\]

where \( d_i \) denotes the number of tasks \( i \) is related to. Let \( A_{G} = I + L_G \) be the so-called interaction matrix for the graph \( G \). The graph induced kernel product is defined by \( \mathcal{K}(x_s, i_s), [x_t, i_t]) = (A_{G}^{-1})_{i_s, i_t} \mathcal{K}'(x_s, x_t) \). Given a sequence of multitask examples, it easily follows that \( \sqrt{\mathcal{K}(x_s, i_s), [x_t, i_t])} \leq \max_t (I + L_G)_{i_t, i_t}^{-1/2} = c_G \) for all time steps \( s \) and \( t \) in the sequence. The magnitude of \( c_G \) scales according to the connectivity of the multitask kernel inducing graph \( G \). In particular, \( c_G \) ranges from \( \sqrt{2}/(k + 1) \) if \( G \) is complete to up to 1 when \( G \) has at least one isolated task. Note that, since the model considered here involves multiple tasks, it is often found to be more natural to actually describe any reference classifier as set of classifiers, each one representing a classifier for a subsequence \( x_{t_1}, x_{t_2}, \ldots \), where \( t_1, t_2, \ldots \) are such that \( i_{t_1} = i_{t_2} = \ldots \). For the sake of convenience we adopt an extended, vector-based notation and denote with \( \bar{g} = [g_1, \ldots, g_k] \) the multitask reference classifier made by the single task classifiers \( g_t \in \mathcal{H}_{\mathcal{K}'} \), \( g_k \in \mathcal{H}_{\mathcal{K}'} \). Moreover, by denoting with \( \Theta(\cdot) \) the null operator in \( \mathcal{H}_{\mathcal{K}'} \) and keeping in with the above notation, it turns out that the kernel inducing feature map \( \psi_\mathcal{K} \) is such that \( \psi_\mathcal{K}(x_t, i_t) = \mathcal{K}(x_t, i_t, \cdot) = [\Theta(\cdot), \ldots, \Theta(\cdot), \mathcal{K}'(x_t, \cdot), \Theta(\cdot), \ldots, \Theta(\cdot)] A_{G}^{-1/2} \), therefore mapping multitask instances to the space of vector valued functions endowed with the inner product \( \langle \bar{f}, \bar{g} \rangle = \text{trace}(\mathcal{K}_{\bar{f}, \bar{g}}) \), where \( (K_{\bar{f}, \bar{g}})_{i,j} = \langle f_i, g_j \rangle \) for any \( f, g \in \mathcal{H}_K \) is the standard inner

\footnote{The identity matrix ensures the positive definiteness of \( A_{G} \) and allows for an easier treatment. It is however possible to define \( A_{G} = L_G \) and then use the pseudoinverse \( A_{G}^+ \) in place of the inverse \( A_{G}^{-1} \).}
product for the space $H_K$. It is also worthwhile to observe that, when the multitask kernel is employed, and the foregoing interpretation on the structure of $\tilde{g} \in H_K$ holds, the hinge loss incurred at time $t$ by $\tilde{g}$ reduces to the hinge loss incurred on the example $(x_t, y_t)$ by the $i_t$-th single task reference classifier, i.e., $\ell_t(\tilde{g}) = \max \{0, 1 - y_t g_i(x_t)\}$. The multitask kernel is therefore a meta-kernel since it is a scalar multiple of the underlying kernel computed on the single task instances, disregarding task relations. This latter information, indeed a pair of coordinates pointing to an entry in the inverse of the interaction matrix $A_G$, is instead taken into account to determine the actual value of the scaling factor. In particular, the following statement holds (proof deferred to the Appendix).

**Proposition 3.1** Let $G$ be a graph of $k$ nodes and $G'$ be the augmented graph obtained from $G$ by adding a dummy node that is connected to every nodes in $G$. Then $(A_G^{-1})_{i,j}$ is equal to

$$-\frac{1}{2}(R_{G'})_{i,j} + \frac{1}{2(k+1)} \sum_{l=1}^{k+1} (R_{G'})_{i,l} + \frac{1}{2(k+1)} \sum_{l=1}^{k+1} (R_{G'})_{j,l} - \frac{1}{(k+1)^2} \|R_{G'}\|_1 + \frac{k+2}{(k+1)^2}$$

where $\| \cdot \|_1$ denotes the entrywise 1-norm and $R_{G'}$ is the resistance matrix whose entries $(R_{G'})_{i,j}$ are the resistance distances between tasks $i$ and $j$ in the augmented graph $G'$.

The augmented graph is needed due to the fact that $A_G = I + L_G$ and therefore it is not a Laplacian matrix. Note that $(A_G^{-1})_{i,i}$ is 1 if task $i$ is isolated and gets smaller and smaller as the number of its related tasks grows. On the other hand, $(A_G^{-1})_{i,j}$, with $i \neq j$, decreases as the resistance distance between tasks $i$ and $j$ grows, and as the average resistance distances between each of the two tasks $i$ and $j$ and the rest of the tasks in the graph increase. This amounts to say that two identical single task instances from loosely connected tasks may be considered more “different” than slightly different single task instances from tightly connected tasks. The magnitude of $(A_G^{-1})_{i,i}$ taken in isolation has more a balancing role than anything else. In particular, note that $(A_G^{-1})_{i,j}$ for $i = j$ is always greater than $i \neq j$ for all tasks $i,j$ belonging to the same connected component. Indeed, as we expect, $K([x_t,i],[x_t,i]) \geq K([x_t,i],[x_t,j])$ for any $j \neq i$. Finally, it is worth mentioning that, while some of the more compelling properties of the multitask kernel arise from its interpretation in terms of the graph $G$, nothing prevents one to replace the Laplacian $L_G$ with an arbitrary positive semidefinite matrix.

## 4 Dealing with multiple tasks with a memory budget constraint

Motivated by the elegant and general theoretical guarantees and by the easiness of their implementation, we now introduce several budget algorithms for the multitask setup.

### 4.1 The Multitask Budget Projectron

The first multitask algorithm (Algorithm [MTBPRJ] $\text{mtbprj}$) considered here is a modification of the Projectron algorithm [11] where a hard constraint is imposed on the size of the active set. Similarly to [14], the general idea of the algorithm aims at optimizing the allotted space by capitalizing on the examples already stored in the active set.

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2It is enough to observe that the entries on $i$-th row and the $i$-th column are zeros.
Algorithm 1 MTBPRJ

Require: Graph $G$, Budget size $B > 0$, Projection threshold $\eta > 0$
1: $S \leftarrow \emptyset$
2: for all $t = 1, 2, \ldots$ do
3: Get $(x_t, i_t, y_t)$ and let $f_t(x_t) = \sum_{j \in J} \beta_j (A_G)_{i_t, i_t}^{-1} K_j(x_t, x_t)$
4: if $y_t f_t(x_t) \leq 0$ then
5: if $\|P^b_j K(x_t, i_t)\| \leq \eta$ then
6: $\beta_j \leftarrow \beta_j + y_t \alpha_j$, $\forall j \in J$ {see text}
7: else
8: $\beta_i \leftarrow y_t$
9: if $|S| \leq B$ then
10: $S \leftarrow S \cup [x_t, i_t]$
11: else
12: $r \leftarrow \arg\min_{j \in J} \beta_j \|P^b_{\{j\}, \cup(t)} K(x_t, i_t, \cdot)\|
13: $S \leftarrow S \cup [x_t, i_t]$
14: $\beta_j \leftarrow \beta_j + \beta_r \gamma_j$, $\forall j \in J$ {see text}

Denote with $P^b_j(\cdot)$ the projection operator on the space spanned by $\{K([x_t, i_t], \cdot)\}_{j \in J}$ and with $P^b_j(\cdot)$ the corresponding orthogonal projection operator. When a mistake occurs at time step $t$, the expression $\|P^b_j (K([x_t, i_t], \cdot) - K([x_t, i_t], \cdot))\|$ is evaluated as how to assess how much of $K([x_t, i_t], \cdot)$ can not be written as a linear combination of the current active multitask instances. In particular, if $\|P^b_j (K([x_t, i_t], \cdot))\|$ is smaller than the user supplied threshold $\eta$ (i.e., a portion of size no bigger than $\eta$ will not be preserved after projection), then the budget is left untouched and the weights $\beta_j$’s are updated to reflect that we are actually storing $P^b_j (K([x_t, i_t], \cdot))$ in place of $K([x_t, i_t], \cdot)$. In fact, $P^b_j (K([x_t, i_t], \cdot)) = \sum_{j \in J} \alpha_j K([x_t, i_t], \cdot)$ where the $\alpha_j$’s are the entries of the vector $H^{-1}[\cdots K([x_t, i_t], [x_t, i_t]) \cdots]^T$, $\forall l \in J$, and $H$ denotes the Gram matrix of the current active multitask instances. The projection step depends on task relations in such a way that the condition on line 6 is unlikely to be true if $i_t$ is loosely connected to tasks $i_l$, even if $K’([x_t, \cdot])$ is in the space spanned by $\{K’([x_t, \cdot])\}_{j \in J}$. Otherwise, either $|S| \leq B$ and $K([x_t, i_t], \cdot)$ is simply loaded into the budget or $|S| = B$ in which case a projection-based budget maintenance policy is triggered. As result, MTBPRJ singles out for eviction the multitask instance $[x_t, i_t]$ that can be removed and projected back onto the remaining active instances with little overall damage as measured by $\beta_r \|P^b_{\cup(t)} K([x_t, i_t], \cdot)\|$. Here $\|P^b_{\cup(t)} K([x_t, i_t], \cdot)\|$ is the amount of $K([x_t, i_t], \cdot)$ that is lost after projecting it back. The weights $\beta_j$’s are then updated pretty much in the same way as on line 7 with $\gamma_j$’s being the coefficients of the expansion of $K([x_t, i_t], \cdot)$ as a linear combination of $\{K([x_t, i_t], \cdot)\}_{j \in J}$.

Unfortunately, MTBPRJ suffers a major drawback in that instances from an isolated task tend to undermine the projection mechanism. In fact, if tasks $i_t$ and $i_j$’s are unrelated, $K([x_t, i_t], [x_t, i_j]) = 0$ which implies $\|P^b_j K([x_t, i_t], \cdot)\| > \eta$ almost surely. This in turn prevents the efficient storage of $K([x_t, i_t], \cdot)$ in terms of its projection. For the same reason active multitask instances from isolated tasks are seldom chosen for eviction since no part of them can be retained as a linear combination of the remaining instances. To overcome these limits we designed a new projection-based budget multitask algorithm (Algorithm 2 MTBPRJ-2). We denote with $P^b_j(\cdot)$ the projection operator on the space spanned by $\{K([x_t, \cdot])\}_{j \in J}$ and with $P^b_J(\cdot)$ the corresponding orthogonal projection operator. The algorithm maintains $k$ sets of weights $(\beta_i)_j$’s for $i = 1, \ldots, k$ and in doing so it trades off space (used to store the weights $(\beta_i)_j$) to increase retention and ultimately accuracy. More specifically, the projection applied to the instance observed at time $t$ is equivalent to the one employed by MTBPRJ except that the task markers are now ignored, effectively increasing the chance of
Algorithm 2 MTBPRJ-2

Require: Graph $G$, Budget size $B > 0$, Projection threshold $\eta > 0$
1: $S \leftarrow \emptyset$
2: for all $t = 1, 2, \ldots$ do
3:   Get $(x_t, i_t, y_t)$ and let $f_{i_t}(x_t)$ be $\sum_{j \in J}(\beta_{i_t,j})K'(x_j, x_t)$
4: if $y_t f_{i_t}(x_t) \leq 0$ then
5:   if $\| (P')_{\gamma_{i_t,j} \cdot 1} K'(x_j, \cdot) \| \leq \eta$ then
6:     $(\beta_{i_t,j}) \leftarrow (\beta_{i_t,j}) + \alpha_j y_t (A_G^{-1})_{i_t,i_t}, \forall j \in J, l \in \{1, \ldots, k\}$
7: else
8:   if $|S| \leq B$ then
9:     $S \leftarrow S \cup [x_t, i_t]$
10:    $(\beta_{i_t,j}) \leftarrow y_t (A_G^{-1})_{i_t,i_t}, \forall l \in \{1, \ldots, k\}$
11: else
12:     $r \leftarrow \text{argmin}_{j \in J} ||d_j||$ \{see text\}
13:     $S \leftarrow S \cup [x_t, i_t] \setminus [x_r, i_r]$
14:    $(\beta_{i_t,j}) \leftarrow (\beta_{i_t,j}) + \gamma_j (\beta_{i_t,j}) r (A_G^{-1})_{i_t,i_t}, \forall j \in J, l \in \{1, \ldots, k\}$

the instance to be efficiently stored through its projection. The projection based maintenance policy now prescribes to remove an active instance $K'(x_j, \cdot)$ so that the $l_2$-norm of the vector $d_j = [(\beta_{i_t,j}) \| (P')_{\gamma_{i_t,j} \cdot 1} K'(x_j, \cdot) \| \cdots (\beta_{i_t,j}) \| (P')_{\gamma_{i_t,j} \cdot 1} K'(x_j, \cdot) \|]_2$ is as small as possible. As one can easily observe, each of the $k$ entries of this vector gauges to what extent the removal of $K'(x_j, \cdot)$ affects one of the $k$ prediction functions that the algorithms maintains.

4.2 The Multitask Randomized Budget Perceptron

The next algorithm we consider (Algorithm 3 MTRBP) is the multitask version of the Randomized Budget Perceptron [3] algorithm.

Algorithm 3 MTRBP

Require: Graph $G$, Budget size $B > 0$
1: $S \leftarrow \emptyset$
2: for all $t = 1, 2, \ldots$ do
3:   Get $(x_t, i_t, y_t)$ and let $f_{i_t}(x_t)$ be $\sum_{j \in J} \beta_j (A_G)_{i_j,i_t}^{-1} K'(x_j, x_t)$
4: if $y_t f_{i_t}(x_t) \leq 0$ then
5:   $\beta_{i_t} \leftarrow y_t$
6: if $|S| < B$ then
7:   $S \leftarrow S \cup [x_t, i_t]$
8: else
9:   $S \leftarrow S \cup [x_t, i_t] \setminus \text{RND}(S)$

MTRBP resembles the Perceptron algorithm and relies on a very simple scheme to deal with the memory budget constraint. If a mistake occurs on time step $t$, the instance $[x_t, i_t]$ is added to the budget with the weight set to $y_t$ and a random-based space management policy is triggered whenever the budget size grows beyond $B$. In the pseudocode of Algorithm 3 the primitive $\text{RND}(\cdot)$ samples a random element from its set argument. The following theorem is a fairly straightforward multitask version of [3] Theorem 5]. Since the algorithm is randomized, the theoretical guarantee provided in Theorem 1 bounds the expected value of the (random) number of mistakes, rather than the number of mistakes itself.

Theorem 1 The expected value of the number of mistakes $M$ made by the MTRBP algorithm, run with a graph $G$ and with a budget size $B > 0$, on any finite sequence of $n$ multitask examples
The variable $\epsilon$ in Theorem 1 trades off the size of the comparison class for the tightness of the bound, the larger the former (which is the case when $\epsilon$ leans towards 1), the looser the latter. Since Theorem 1 holds for arbitrary choices of $\epsilon$, MTRBP effectively competes against the best multitask classifier in the best traded off comparison class. Let us observe how the multitask kernel affects bound (1). The role of the kernel appears evident in the shifting term, both explicitly through the presence of the interaction matrix $A_G$ and implicitly through the weighting constant factor $c_G$. Consider the expansion of $S_{A_{G}^{1/2}}$ as $\sum_{t=1}^{n-1} \langle \langle \bar{g}_t - \bar{g}_{t-1} \rangle A_G, \langle \bar{g}_t - \bar{g}_{t-1} \rangle \rangle$. Each term under the square root has the form

$$\sum_{i=1}^{k} \|g_t,i - g_{t-1,i}\|^2 + \sum_{(i,j) \in E} \|g_t,i - g_{t,j}\| - (g_{t-1,i} - g_{t-1,j})\|^2 \tag{2}$$

where the first summand summarizes how much each of the $k$ reference classifiers shifts from time step $t-1$ to time step $t$ and therefore does not take into account the relations among tasks. The second summand of (2) is where those relations come into play. Specifically, for each pair of related tasks the difference of their relative positions after consecutive time steps is evaluated.

This expression is closely related to the reference classifiers of related tasks shift in similar ways.\footnote{Think of a multi language spam classifier. Different trends may arise over time but the language(task) relations stay the same.} In particular, when this shifting pattern holds and $G$ is a complete graph the shifting term $c_G S_{A_{G}^{1/2}}$ becomes, excluding constant factors, $\sum_{t=1}^{n-1} \sum_{i=1}^{k} \|g_t,i - g_{t-1,i}\|^2$, and it is therefore similar to the one we would get if there were only one task. Aside from the shifting term, the impact of the multitask kernel is then largely confined in the comparison class inequality that binds $B$ to $\text{TRACE}(K_{\bar{g}_t, \bar{g}_t} A_G)$. In fact, setting a given value of $B$ amounts to define the shape of the class of multitask reference classifiers the algorithm competes against. After rewriting $\text{TRACE}(K_{\bar{g}_t, \bar{g}_t} A_G)$ as $\sum_{i=1}^{K} \|g_t,i\|^2 + \sum_{(i,j) \in E} \|g_t,i - g_t,j\|^2$ it is easy to see that a given choice of $B$ imposes a constraint on the norms of the task-specific reference classifiers and on the spatial relations they entertain with each other. To better illustrate how the memory constraint works in this respect, consider the following two opposite situations where we again set $G$ to be the complete graph. First assume that the worst-case multitask reference classifiers are stationary, i.e., $\bar{g}_t = \bar{g}$ for all $t$, and their single task classifiers are overlapped, i.e., $g_1 = g_2 = \cdots = g_k$. In this case $\|g_t\|^2 \leq \frac{k+1}{k} B$ for all $i = 1, \ldots, k$, excluding constant factors. In other words, the algorithm can compete against longer reference classifiers whose norm is nearly $B$ instead of $B/k$ as a naive, non multitask approach would imply. Implementation-wise, this means that the whole allotted space can be devoted to learn a single, more complex unique task rather than inefficiently fragmented to track $k$ equal reference classifiers. On the other hand, if the $k$ single task reference classifiers are distant from each other,\footnote{When the reference classifiers have the same norm and $G$ is a complete graph this amounts to say that the $(g_t)_i$’s are the vertices of a $k$-simplex centered at the origin.} as measured by the metrics defined by $A_{G}^{1/2}$, then the average norm of the single task reference vectors the algorithm can compete against is reduced by an amount proportional to how much they are spread apart. In this case, since the tasks we are learning are different from each other, MTRBP is forced to reserve a portion of the available space to each task.
4.3 The Multitask Self-Tuned Forgetron

Adapting the Forgetron algorithm of [8] to run within the multitask protocol is relatively straightforward. Before investigating the details of the algorithm (MTFORG) recall that its theoretical behavior is sub-optimal, since the size of the comparison class is of the order $O(\sqrt{B/\log(B)})$, a factor $\sqrt{\log(B)}$ worse than the optimal $O(\sqrt{B})$ achieved by MTRBP. Nonetheless, the algorithm may prove to be more effective in a number of real world scenarios where the sequence of examples is not adversarial and the policy of always dropping the oldest instance may turn up to be beneficial.

We consider the self-tuned version of the algorithm which performs Perceptron-like updates whenever there is still room in the active set and operates as follows otherwise. If a mistake occurs at time step $t$ and the current budget size $|S|$ is already $B$, then the oldest active multitask instance $[x_r,i_r]$ with $r = \min(J)$ is singled out for eviction and the incoming instance $[x_t,i_t]$ is loaded into the budget with the weight $\beta_t$ set to $y_t$ (removal step). As a way to control the detrimental effect of the removal of $[x_r,i_r]$ from the budget, MTFORG also reduces the weights $\beta_j$’s by an adaptive factor $\phi$ (shrinking step) so that older instances have smaller weights. Because it is likely that both steps negatively affect the overall performance of the algorithm, the rescaling factor $\phi$ is adaptively set to moderate the impact of the shrinking step.

We take advantage of the fact that the norm of multitask instances is bounded by $c_G$, which may be much less then 1 for several non trivial graphs, and slightly fine tune the algorithm presented in [8] by setting

$$\phi = \max_{\chi \in [0,1]} \left( \Psi_G(\beta_r y_r \chi, \beta_r \chi f_{i_r}(x_r)) + Q \leq \frac{15c_G^2 M}{32} \right)$$

(3)

where $\Psi_G(\lambda,\mu) = c_G^2 \lambda^2 + 2c_G \lambda - 2\lambda \mu$. The resulting algorithm is similar to Algorithm 3 where line 9 is replaced by

9: $r \leftarrow \min(J)$
10: $S \leftarrow S \cup [x_t,i_t] \setminus [x_r,i_r]$
11: $\beta_j \rightarrow \phi \beta_j, \forall j \in J\{\phi \text{ computed as in (3)}\}$
12: $Q \leftarrow Q + \Psi_G(\beta_r y_r \phi, \beta_r \phi f_{i_r}(x_r))$

The following theorem, which is an easy consequence of [8, Theorem 3], provides a deterministic worst-case upper bound on the number of mistakes made by MTFORG.

**Theorem 2** The number of mistakes $m$ made by the MTFORG algorithm, run with a graph $G$ and with a budget size $B > 83$, on any finite sequence of $n$ multitask examples $([x_1,i_1], y_1), \ldots, ([x_n,i_n], y_n)$ satisfies

$$m \leq 4L + \frac{B + 1}{2\log(B + 1)}$$

for any multitask reference classifier $\bar{g}$ such that $\sqrt{\text{trace}(K_{\bar{g}\bar{g}} G)} \leq \frac{1}{4c_G} \sqrt{\frac{B + 1}{\log(B + 1)}}$ and $K_{\bar{g}\bar{g}}$ is the Gram matrix computed among the single task classifiers $g_1, \ldots, g_k$.

First, observe that the above bound only applies to a stationary comparison multitask classifier. As far as we know no shifting analysis is known for the Self-Tuned version of the Forgetron algorithm. Nonetheless, a shifting bound can be obtained for an experimentally less appealing variant by following the argument and the analysis given in [13]. Second, as for MTRBP, the role of the multitask kernel is mainly reflected in the bound through the comparison class inequality

$$\sqrt{\text{trace}(K_{\bar{g}\bar{g}} G)} \leq \frac{1}{4c_G} \sqrt{\frac{B + 1}{\log(B + 1)}}.$$ 

In this respect, note that it is the presence of $c_G$ in (3) that

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Footnote: For details on why $O(\sqrt{B})$ is the largest norm an algorithm with a budget size $B$ can compete against see [8].
allows the size of the comparison class to scale as a function of the tasks through the constant factor $1/4c \Gamma$ (see subsection 4.2 for a detailed discussion on the comparison class inequality within the multitask framework). Third, observe that when a mistake occurs and $|S|$ is $B$, then the removal step only affects those tasks that are related to $i_r$, since $K([x_r, i_r], [x_j, i_j]) = 0$ if tasks $i_r$ and $i_j$'s are unrelated. As a result, tasks belonging to connected components that seldom need processing can be quickly forgotten altogether. Combining the Forgetron budget maintenance policy with the multitask kernel has thus the important effect that the allocation of the available space to the most frequent tasks is automatically taken care of.

### 4.4 Implementation details

While the implementation of the budget algorithms discussed in this paper is straightforward, it is still useful to point out a few remarks. Except for MTBPRJ-2, all the algorithms discussed in this paper only maintain $B$ real weights and $B$ multitask instance vectors regardless of the number $k$ of the tasks at hand. Of course the matrix $A \Gamma$ employed by the multitask kernel may still require $O(k^2)$ space. Note, however, that if $G$ is not overly complex and exhibits a certain regularity, the required space may be much smaller, and of course it may be more efficient to opt for a programmatic implementation of $A \Gamma$ over the naive table-based approach. Both MTRBP and MTFORG require $O(1)$ operations to update their internal state on mistaken rounds, whereas it is not hard to show that MTBPRJ take $O(B^2)$ operations. As for MTBPRJ-2 we should note that the algorithm only needs to store $B$ real weights for each connected component of $G$ which results in a much milder dependence on $G$ than a naive implementation would imply.

### 5 Experiments

The experimental performance of budget multitask algorithms on non-synthetic data sets is of key importance because the constraint on the size of the budget and the favorable dependence on the number of tasks make them suited even for large scale applications. In this section we evaluate the multitask budget algorithms discussed in Section 4 over three data sets, the PKDD 2006 Spam Task A [5] data set ($k = 3, d = 106780, n = 7500$), the School [6] data set ($k = 139, d = 28, n = 15362$) and the Sentiment [7] data set ($k = 4, d = 473856, n = 8000$).

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**Table 1:** Online training F-measures achieved by the multitask budget algorithms run with $G$ set to C (complete graph) or D (totally disconnected graph) and $B$ set to 25%, 10% or 5% of the size of the active set obtained by a battery of $k$ Perceptrons on the PKDD 2006 spam task A, School binary, and Sentiment data sets. As a reference, the F-measures achieved by the $k$ independent Perceptrons are shown below the name of each data set.

| Algorithm | C(25%) | D(25%) | C(10%) | D(10%) | C(5%) | D(5%) |
|-----------|--------|--------|--------|--------|--------|--------|
| **Pkdd06 Spam A** (92.0%) | | | | | | |
| MTRBP     | 84.8%  | 81.4%  | 78.0%  | 73.1%  | 72.8%  | 66.7%  |
| MTFORG    | 85.1%  | 81.7%  | 77.9%  | 73.0%  | 72.3%  | 66.7%  |
| MTBPRJ    | 91.6%  | 89.4%  | 87.8%  | 84.0%  | 83.4%  | 76.0%  |
| MTBPRJ-2  | 91.6%  | 90.3%  | 89.3%  | 87.3%  | 85.4%  | 82.3%  |
| **School binary** (39.1%) | | | | | | |
| MTRBP     | 40.4%  | 35.0%  | 38.6%  | 30.9%  | 37.3%  | 26.0%  |
| MTFORG    | 39.7%  | 35.1%  | 38.0%  | 31.5%  | 36.9%  | 25.9%  |
| MTBPRJ    | 40.6%  | 37.4%  | 40.2%  | 32.6%  | 39.4%  | 23.8%  |
| MTBPRJ-2  | 41.2%  | 39.1%  | 40.9%  | 39.0%  | 39.6%  | 37.9%  |
| **Sentiment** (71.5) | | | | | | |
| MTRBP     | 66.8%  | 63.3%  | 62.0%  | 58.3%  | 59.4%  | 56.1%  |
| MTFORG    | 66.7%  | 63.4%  | 62.3%  | 58.8%  | 58.8%  | 56.5%  |
| MTBPRJ    | 71.4%  | 67.7%  | 66.6%  | 63.6%  | 64.0%  | 60.6%  |
| MTBPRJ-2  | 71.7%  | 68.5%  | 67.9%  | 64.9%  | 64.5%  | 61.7%  |
Since examples in the School data set have real valued labels, a preprocessing was required to turn those into binary values. Therefore, we assigned a positive label to those instances whose original score was above the 75-th percentile and a negative label to those instances whose original score was below that threshold. Moreover we rescaled non binary features in the range $[0,1]$. We used a Gaussian kernel for the School data set and a polynomial kernel for the PKDD 2006 Spam Task A and Sentiment data sets.

The simple yardstick for our algorithms is the online classifier that runs a battery of $k$ Perceptrons in parallel with no constraints on the size of their active sets. We evaluated the budget multitask algorithms for different sizes of their budget. In particular, we set $B$ to 25%, 10% and 5% of the size of the active set obtained by the baseline algorithm after a single training epoch. For mtbprj and mtbprj-2 we set $\eta = 0.01$.

Imposing a budget restriction should be detrimental to the overall performance and it should be even more so when $k$ rather than a single task are to be processed. On the other hand, a proper multitask formulation should lessen and ideally negate, this impact. The F-measure values achieved after a single pass over the training set are reported in Table 1. The numbers show that disregarding multitask information (i.e., $G = D$), and imposing a constraint on the size of the active set, really negatively affects the performance of budget multitask algorithms, and this behavior is unsurprisingly shared by all algorithms. Even in this case, however, the projection based algorithms tend to present a relatively better behavior confirming that the projection schemes are a first step towards a better retention. Specifically, the global projection step employed by mtbprj-2 turns out to be particularly effective, as evidenced by comparing the F-measures obtained on the School data set, and to a lesser extent on the PKDD 2006 Spam Task A data set, when $B = 5\%$.

It is of course more interesting to see how this decrease in the overall performance can be offset by taking into account multitask relations. In fact, setting $G = C$ results in better F-measure values for all three data sets and for all algorithms. Moreover the differences in the F-measures obtained for the different choices of $G$ grow larger as the budget size is shrunk to smaller values. This should not come to a surprise, since when the available memory is scarce an efficient management policy, which is the main benefit that the multitask kernel brings to budget algorithms, becomes crucial. It is particularly surprising that the multitask kernel can be so effective that for $B = 25\%$ all four algorithms match the performance of the baseline and, for mtbprj and mtbprj-2 this holds true even when $B$ goes down to 5\%.

6 Appendix

Proof of Proposition 3.1: First, observe that the Laplacian matrix for the augmented graph $G'$ is

$$L_{G'} = \begin{bmatrix} A_G & -1 \\ -1^\top & k \end{bmatrix}$$

where 1 is the vector of all ones. We use [2, Theorem 3.3.2] to compute the pseudoinverse of $A_{G'}^+$

$$A_{G'}^+ = \begin{bmatrix} A_G^{-1} - (1 + k)^{-1} A_G^{-1} 1 1^\top A_G^{-2} - (1 + k)^{-1} A_G^{-2} 1 1^\top A_G^{-1} - (1 + k)^{-1} A_G^{-2} 1 \\ - (1 + k)^{-1} A_G^{-1} - 1^\top A_G^{-1} \\ + \frac{1}{(1 + k)^2} A_G^{-3} 1 \\ - A_G^{-1} 1^\top A_G^{-1} \\ - 1^\top A_G^{-1} \\ + \frac{1}{(1 + k)^2} \end{bmatrix}$$
Denoting with $e_i$ the $i$-th standard vector of size $k$ and observing that $A_G \mathbf{1} = (I + L_G) \mathbf{1} = 1 + 0$ since $L_G$ is a Laplacian matrix, we have, for all $i = 1, \ldots, k$ and $j = 1, \ldots, k$

\[
(A^+_G)_{i,j} = e_i^\top A^+_G e_j = e_i^\top A_G^{-1} e_j - \frac{2}{1+k} + \frac{k}{(1+k)^2} = (A_G^{-1})_{i,j} - \frac{2 + k}{(1+k)^2} \tag{4}
\]

Finally, by [10, Theorem 7] it holds that

\[
A_G' = -\frac{1}{2} \left[ R_{G'} - \frac{1}{1+k} \left( R_{G'} \mathbf{11}^\top + \mathbf{11}^\top R_{G'} \right) + \frac{1}{(1+k)^2} \mathbf{11}^\top R_{G'} \mathbf{11}^\top \right] \tag{5}
\]

Substituting (5) back into (4) yields the desired result. \[\square\]

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