Strong disorder effects of a Dirac fermion with a random vector field

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We study a Dirac fermion model with a random vector field, especially paying attention to a strong disorder regime. Applying the bosonization techniques, we first derive an equivalent sine-Gordon model, and next average over the random vector field using the replica trick. The operator product expansion based on the replica action leads to scaling equations of the coupling constants (“fugacities”) with nonlinear terms, if we take into account the fusion of the vertex operators. These equations are converted into a nonlinear diffusion equation known as the KPP equation. By the use of the asymptotic solution of the equation, we calculate the density of state, the generalized inverse participation ratios, and their spatial correlations. We show that results known so far are all derived in a unified way from the point of view of the renormalization group. Remarkably, it turns out that the scaling exponent obtained in this paper reproduces the recent numerical calculations of the density correlation function. This implies that the freezing transition has actually revealed itself in such calculations.

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I. INTRODUCTION

A Dirac fermion model has proved to be quite useful, even in condensed matter physics, as an effective theory for various kinds of phenomena, e.g. the integer quantum Hall transition, d-wave superconductors, etc. It is also involved with statistical mechanical models such as the Ising model or the XY-model. Especially, including randomness, this model has clarified remarkable aspects of disordered systems, for example, In ln $\tau$ behavior of the specific heat of the random bond Ising model and multifractal scaling dimensions.

Many problems, however, still remain to be explored. Especially, the critical theory of a Dirac fermion with generic disorder, which is believed to be an effective theory for the integer quantum Hall transition, is still missing. In this case, coupling constants describing disorder strength flow to a strong coupling regime, which we cannot reach by weak coupling approaches. Even simpler model with a random vector field only, whose zero energy state can be obtained exactly for any realization of randomness, yields non-trivial strong disorder phase, recognized through the calculation of the generalized inverse participation ratios (IPR).

On the other hand, Carpentier and Le Doussal have recently developed a theory for the XY-model with a random gauge field. It has been shown that a naive replica theory for this model concludes a reentrant transition which is not consistent to the numerical calculations, and to exact arguments by the use of the Nishimori-line. Although this inconsistency has been a long standing problem, underlying physics has recently been clarified. In particular, Scheidl has pointed out the importance of a role which vortices with higher charges play. Carpentier and Le Doussal have derived new scaling equations for the Coulomb gas model taking into account “fusion” of vortices, from which higher charge vortices are generated consistently. The renormalization group (RG) equations they have derived have intimate relationship with the freezing transition of spin glasses (the random energy model).

By the use of the fact that the exact zero energy wave function of a Dirac fermion model with a random vector field is equivalent to the vertex operator of the boson field theory, Carpentier and Le Doussal have applied their RG method to a Dirac fermion model, calculating the IPR. Variational method has also been applied in order to calculate the density of state (DOS). It has turned out that these quantities shows peculiar behavior in a strong coupling regime due to the freezing transition.

Motivated by these developments, we rederive, in this paper, scaling equations for a replicated sine-Gordon model with a random vector field, which is the bosonized model of a Dirac fermion with a random vector field. The main point is that we take into account the fusion of the vertex operators. In order to derive scaling equations at one-loop order, we use the operator product expansion (OPE) techniques. Based on these, we calculate the DOS, the IPR, and their spatial correlations of a Dirac fermion model. We show that it is possible to derive all results known so far in a unified way from the RG point of view. Moreover, it turns out that the scaling exponent obtained in this paper reproduces the recent numerical calculations by Ryu and Hatsugai of the density correlation function. This implies that the freezing transition has manifestly revealed itself in their calculations.

This paper is organized as follows: In the next section, a Dirac fermion model is introduced and a corresponding sine-Gordon model is derived. In Sec. the OPEs are calculated, which lead to scaling equations, by taking account of the fusion of the vertex operators, and in Sec. they are converted into more convenient form, known as the KPP equation. By the use of the asymptotic solution of the equation, the DOS and the IPR are cal-
Field-theoretically, the ensemble-averaged DOS \( \rho_n \) where \( \Psi \) is a numerical constant, is a numerical constant. \( A \text{ summary and discussions are given in Sec. VI.} \)

II. DIRAC FERMION WITH RANDOM VECTOR FIELD

In this section, we first introduce a Dirac fermion model including a random vector field and give formulae to calculate the DOS, etc. Next, using the bosonization techniques, we convert the Dirac action into an equivalent sine-Gordon action. The latter is the model we directly investigate in this paper.

A. Dirac fermion model

The action functional is defined by

\[
S = \int d^2x \left[ \overline{\psi} i\gamma_{\mu} (\partial_{\mu} - iA_{\mu}) \psi - ig \overline{\psi} \right],
\]

where \( y = \omega - iE \), \( \gamma_{\mu} = \sigma_{\mu} \) (the Pauli matrices), and \( A_{\mu} \) denotes a quenched random vector field with the following gaussian probability distribution

\[
P[A_{\mu}] \propto \exp \left( -\frac{1}{2\pi g} \int d^2x A_{\mu}^2 \right).
\]

Here we include an extra factor \( \pi \) into the definition of the distribution width \( g \) for later convenience. The Green function is computed as

\[
\text{Im tr}(\psi(x)\overline{\psi}(x)) = \frac{\omega}{\Gamma(x)} \left[ \gamma_{\mu} (\partial_{\mu} - iA_{\mu}) - E \right]^2 + \omega^2 |x|,
\]

where \( \text{tr} \) is the trace with respect to the two-component spinor. Let us further define the \( q \)-th power of the Green function

\[
\Gamma^{(q)}(x) = \left( \text{Im tr}(\psi(x)\overline{\psi}(x)) \right)^q.
\]

Then, the following relation is valid

\[
\omega^{q-1} \Gamma^{(q)}(x) \rightarrow C_q \sum_n |\Psi_n(x)|^{2q} \delta(E_n - E), \quad (\omega \rightarrow +0),
\]

where \( \Psi_n(x) \) is the eigenstate of the Hamiltonian with energy \( E = E_n \), namely, \( i\gamma_{\mu} (\partial_{\mu} - iA_{\mu}) \Psi_n = E_n \Psi_n \), and \( C_q \) is a numerical constant \( C_q = \pi/(2q - 3)!!/(2q - 2)!! \).

Field-theoretically, the ensemble-averaged DOS \( \rho(E) \), the generalized IPR \( P^{(q)}(E) \), and their spatial correlations \( Q^{(q_1,q_2)}(x,y,E) \) are defined by, in the limit \( \omega \rightarrow +0 \),

\[
\rho(E) = \frac{\Gamma(1)(x)}{C_1},
\]

\[
P^{(q)}(E) = \omega^{q-1} \frac{\Gamma(q)(x)}{[C_q \rho(E)]},
\]

\[
Q^{(q_1,q_2)}(x,y,E) = \omega^{q_1+q_2-1} \frac{\Gamma(q_1)(x)\Gamma(q_2)(y)}{[C_{q_1}C_{q_2}\rho(E)]}.
\]

Here we have assumed that the ensemble-average recovers the translational invariance and hence \( \Gamma^{(q)}(x) \) does not depend on \( x \) and \( \Gamma^{(q_1)}(x)\Gamma^{(q_2)}(y) \) depends on the separation \( x - y \) only.

B. Sine-Gordon model

Bosonization of (2.1) yields the following sine-Gordon action

\[
S = \int \frac{d^2x}{4\pi} \left[ \frac{1}{2} \left( \partial_{\mu} \phi \right)^2 + 2iA_{\mu} \epsilon_{\mu\nu} \partial_{\nu} \phi - y \cos \phi \right].
\]

By the use of the replica trick, ensemble-average over the vector field \( A_{\mu} \) gives

\[
\overline{Z^m} = \int D\phi e^{-S^{(m)}},
\]

\[
= \int \frac{d^2x}{4\pi} \left[ \frac{1}{2} \sum_{a,b=1}^m \partial_{\mu} \phi_a G_{ab} \partial_{\nu} \phi_b - y \sum_{a=1}^m \cos \phi_a \right],
\]

where \( a, b \) denote \( m \) replicas and

\[
G_{ab} = \frac{1}{K} \delta_{ab} + g.
\]

The bare coupling constant is given by \( K = 1 \), since the present system is a free fermion model.

So far we have derived a replicated sine-Gordon model by the bosonization of the Dirac fermion model. We also reach the same model from the XY-model with a random gauge field as a result of applying the Villain approximation, “integrating” over vortices, and performing the quenched average with a replica trick. In this case, the coupling constant \( K = J/T \), where \( J \) and \( T \) denotes, respectively, the spin coupling constant and the temperature. The problem had been that the RG analysis based on this action predicts a reentrant transition, which contradicts numerical studies and exact arguments by the use of the Nishimori-line.

Recent developments, however, have made it possible to describe the random XY model within a framework of the replica method, without resort to the replica symmetry breaking. In the language of the sine-Gordon model, the main point is that higher charge vertex operators, which are not included in the above action but are generated in the process of the renormalization, are taken into account. The next section is devoted to the reformulation of the method.
III. SCALING EQUATIONS

Carpentier and Le Doussal have recently proposed improved scaling equations for the Coulomb gas model. They have taken into account the fusion of vortices, which results in scaling equations with nonlinear terms. Without these terms, the flow of the coupling constant $g$ (also called “fugacity” in the context of the Coulomb gas model) is governed by the exponent $2 - x$, where $x$ is the dimension of $\sum_a \cos \phi_a$ in Eq. (2.3). The nonlinearity of the scaling equations, however, has turned out to play a crucial role in the RG flow. In what follows, we rederive the scaling equations for the present sine-Gordon model by the use of the OPE techniques.

We start with the same sine-Gordon action but with generalized fugacities,

$$S = \int \frac{d^2x}{4\pi} \left[ \frac{1}{2} \partial_\mu \phi_a G_{ab} \partial_\mu \phi_b - \sum_{n \neq 0} Y(n) e^{i n_a \phi_a} \right],$$

where $a, b = 1, 2, \ldots, m$, $n' = (n_1, n_2, \ldots, n_m)$ is a $m$ component vector with integer elements $n_a$ which will be specified momentarily. The vector $n$ denotes a “charge” of the vertex operator, so that we define a total charge $n = \sum_n n_a$. Initially, the bare fugacity is $Y(n) = y_1 (\delta_{nn_1} + \delta_{nn_{-n_1}})$, where the vector $n_1$ is such that only one of its elements is 1 and others are zero. In this case, $\sum_{n \neq 0} Y(n) e^{i n_a \phi_a} = 2 y_1 \sum_n \cos(\phi_a(n))$, and the action (3.1) reduces to the normal replicated sine-Gordon action (2.9).

The unperturbed correlation function reads

$$\langle \phi_a(x) \phi_b(y) \rangle = \left( \frac{G}{4\pi} \right)_a^{b-1} (-\partial^2)^{-1}(x - y) = G_{ab}^{-1} \ln |x - y|^{-2}. \quad (3.2)$$

The inverse of the matrix $G$ is easy to calculate,

$$G_{ab}^{-1} = K \delta_{ab} - \frac{gK^2}{1 + mgK} \rightarrow K \delta_{ab} - gK^2, \quad (3.3)$$

where we have taken the replica limit $m \to 0$ in the last equation. By using the correlation function (3.2), we readily find,

$$\langle e^{i n_a \phi_a(z)} e^{-i n_a \phi_a(0)} \rangle = \frac{1}{|z|^{2x(n)}}, \quad (3.4)$$

where the dimension $x(n)$ of $e^{i n_a \phi_a}$ is

$$x(n) = n' G^{-1} n' = K |n|^2 - gK^2 n^2, \quad (3.5)$$

with $|n|^2 = n \cdot n$ and $n \cdot n' = \sum_n n_n'$. The OPE of the vertex operators in the case $n + n' = n' \neq 0$ is given by

$$e^{i n_a \phi_a(z)} e^{-i n_a \phi_a(0)} = \frac{1}{|z|^{2x(n, n')}} e^{i n_a' \phi_a(0)}, \quad (3.6)$$

with

$$x(n, n') = x(n) + x(n') - x(n''')$$

$$= -2 n' G^{-1} n'$$

$$= 2 (gK^2 n_{n'} - K n \cdot n'), \quad (3.7)$$

where we have used the fact that the matrix $G$ is symmetric. On the other hand, in the case $n + n' = 0$, we find

$$e^{i n_a \phi_a(z)} e^{-i n_a \phi_a(0)} \sim \frac{1}{|z|^{2x(n)}} e^{i n_a (z \partial \bar{z} + \bar{z} \partial) \phi_a(z)}$$

$$\sim -\frac{1}{|z|^{2x(n)}} \partial \phi_a(n n') \partial \bar{z} \phi_a, \quad (3.8)$$

where $z, \bar{z} = x_1 \pm i x_2$, $\partial = \partial _z$ and $\bar{\partial} = \partial _\bar{z}$.

First, the OPE (3.6) leads us directly to the scaling equations,

$$\frac{dY(n)}{dl} = [2 - x(n)] Y(n) + \frac{1}{4} \sum_{n', n''} Y(n') Y(n''), \quad (3.9)$$

where $l = \ln L$ with the system size $L$.

So far we have not specified the vector charges $n$. The initial condition at $l = 0$ is such that there exist $n_1$-type vectors only. However, the “fusion” of the vertex operators (3.6) yields higher charge vectors in the process of the renormalization. Let us fix the charge $n$, and compute the scaling dimensions of the vertex operators. Consider the case with $n = 1$ for simplicity. Then, we have $n' = (1, 0, 0, \cdots), (1, 1, -1, 0, \cdots)$, and $(2, -1, 0, \cdots)$, for example, whose dimensions are, respectively, $K - gK^2$, $3K - gK^2$, and $5K - gK^2$. It is readily seen that the vectors of the first type have the most relevant dimension. In a similar way, given a charge $n$, we find that the most relevant vectors are the ones that have $|n|$ 1’s (−1’s) for a positive (negative) $n$.

In what follows, we restrict ourselves to these most relevant ones, $n$ with $n_q = 0, 1$ and with $n_q = 0, -1$, and therefore, vectors whose charges lie between $\pm m$ are taken into account. Let us define a set of vectors $N_q$ which includes vectors $n$ with $n_q = 0, 1$ and $\sum_n n_q = q$. The scaling dimensions depend, in this case, only on $|n|$, so that let us define, for $n > 0$

$$x(n) = x_n = K n - gK^2 n^2. \quad (3.10)$$

Then the scaling equations (3.9) decouple into positive and negative charge sectors, which obey the same scaling equations with the same initial condition. Thus, it turns out that $Y(−n) = Y(n)$. Moreover, $Y(n)$ depends on $n$ only. Denoting $Y(n) = Y(−n) = y_n$ and counting
the multiplicity in the summation over \( \mathbf{n}' \) and \( \mathbf{n}'' \) in Eq. (3.9), we end up with a closed set of scaling equations
\[
\frac{dy_n}{dt} = (2 - Kn + gK^2n^2) y_n + \frac{1}{4} \sum_{n'=1}^{n} \binom{n}{n'} y_{n'} y_{n-n'}.
\]
(3.11)

Next consider the OPE (3.8). The exponent in this equation is \( 2x_n - 2 = 2(n - gn^2 - 1) \), where we have used the free fermion condition \( K = 1 \). It should be noted that in a weak disorder regime, a naively replicated model including \( y_1 \) only could describe the critical behavior correctly. The exponent of this fusion is 2, which is definitely negative, and therefore, gives rise to no ultraviolet singularity. Hence, the kinetic term should not be renormalized. This may be valid in a strong coupling regime, since large \( g \) tends to make the exponent negative, although in some range of \( n \), positive exponents appear if one includes higher charge vectors. Taking these into account, we postulate that the kinetic term is not renormalized in any regime of interest \( K \lesssim 1 \), in what follows.

The Green function can be described by the replicated boson fields as \( \Gamma(q) \sim \langle \cos(\phi_q) \rangle \). If one keeps the most relevant field, one finds
\[
\frac{\Gamma(q)(x)}{\Gamma(n)(x)\Gamma(q)(y)} \sim \langle \cos(\mathbf{n}'\phi_q(x)) \cos(\mathbf{n}''\phi_q(y)) \rangle,
\]
(3.12)
after the ensemble average, where \( \mathbf{n} \in \mathbb{N}_q, \mathbf{n}' \in \mathbb{N}_{q_1}, \) and \( \mathbf{n}'' \in \mathbb{N}_{q_2} \).

IV. THE KPP EQUATION

So far we have derived the scaling equations with nonlinear terms due to the fusion of the vertex operators. Remarkably, it has been shown that the nonlinear terms are involved with the freezing transition of the random energy model [2]. This section is devoted to the review of converting the scaling equations (3.11) into the KPP equation, according to Carpentier and Le Doussal [4]. First, make the Mellin transformation \( y_n(l) = \int_{0}^{\infty} dze^{zn-1}P(z, l) \), which is converted, by the use of the variable \( z = e^u \), into a more convenient form for the present purpose as
\[
y_n(l) = \int_{-\infty}^{\infty} du e^{nu}P(u, l).
\]
(4.1)

Integration over \( u \) of the both sides of the equation yields
\[
\partial_t y_0 = 2y_0 - \frac{1}{4} y_0^2.
\]
(4.3)

It turns out that \( y_0 \), which is just the normalization of the function \( P(u, l) \), flows as \( y_0 \rightarrow 8 \). Therefore, setting \( y_0 = 8 \) gives
\[
\partial_t P(u, l) = \left[ K \partial_u + gK^2\partial_u^2 \right] P(u, l) - 2P(u, l) + \frac{1}{4} \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2 \delta(u - \ln(e^{u_1} + e^{u_2})) \times P(u_1, l)P(u_2, l).
\]
(4.4)

Although it may be difficult to solve this equation analytically, a crucial observation was made by Carpentier and Le Doussal as follows: Define a generating functional of \( y_n \) by
\[
G(x, l) = 1 - \langle \exp(-ze^{-x+K}) \rangle_P = \frac{1}{8} \sum_{n=1}^{\infty} \frac{(z-1)^{n-1}}{n!} y_n(l) e^{-n(x-K)},
\]
(4.5)

where \( z = e^u \), and we have introduced the notation \( \langle A \rangle_P = \int_{-\infty}^{\infty} \frac{dA}{A} P(u, l)A \). Since the function \( P(u, l) \) can be interpreted, after normalized to unity, as the probability distribution of the fugacity \( z \). What is remarkable is that the function \( G(x, l) \) obeys the following equation
\[
\frac{1}{2} \partial_t G = DG^2 + G(1 - G),
\]
(4.6)

where \( D = gK^2/2 \). This equation is known as the KPP equation [1]. Since \( y_0 = 8 \), the boundary condition is \( G(-\infty, l) = 1 \) and \( G(\infty, l) = 0 \).

V. THE DOS AND THE IPR

In order to calculate the dominant exponents of the DOS, etc in Eq. (2.6), we need to know the scaling exponents of the local composite operators \( \Gamma(a) \). Basically they are given by (3.10) through the relation (3.12). However, it will turn out that the nonlinear terms of the scaling equations induce nontrivial dynamical scaling exponents in a strong disorder regime. Therefore, we first have to read the dynamical scaling exponents from the solution of the KPP equation, and next calculate various exponents of the DOS, etc.
A. Dynamical scaling exponents of the vertex operators

First of all, let us summarize the remarkable selection rule of the front velocity of the KPP equation \( \{4.4, \} \). Provided that \( G(x, l) \) has a boundary condition \( G(\infty, l) = 1 \) and \( G(\infty, l) = 0 \) as well as an initial condition \( G(x, l = 0) \sim e^{-\mu x} \) at large \( x \). In this case, the KPP equation admits the following traveling solution \( \{5.1\} \)

\[
G(x, l) = g(x - m(l)), \quad l \to \infty. \tag{5.1}
\]

The solution tells that the wavefront (a kink) travels with a speed \( \partial_l m \) if we interpret \( l \) as time. Neglecting the nonlinear term \( G^2 \) and assuming the form of the solution as \( \exp[-\mu(x - vl)] \), we expect that the velocity \( v \), defined by \( m(l) = vl \), is given by \( v = 2(D\mu + 1/\mu) \). Actually, this is valid, as long as \( \mu \) is small enough. However, it has been shown \( \{4.4, \} \) that if \( \mu \) is larger than \( \mu_{cr} = 1/\sqrt{D} \), the relation does not hold any longer, and the velocity is a constant (the minimum value) independent of \( \mu \). To be precise, \( m(l) \) is given by \( \{5.2\} \)

\[
m(l) = \begin{cases} 2(D\mu + 1/\mu)l + O(1) & \text{for } \mu < 1/\sqrt{D}, \\ \sqrt{D}l & \text{for } \mu = 1/\sqrt{D}, \\ \sqrt{D}(4l - 3/2 \ln l) + O(1) & \text{for } \mu > 1/\sqrt{D}. \end{cases} \tag{5.2}
\]

Next task is to obtain the behavior of the fugacities \( y_n \), which are related to the above solution of \( G(x, l) \) via Eqs. \( \{4.4, \} \) and \( \{4.3\} \). To this end, notice that the front velocity above is related to a typical value of \( u \) via the relation \( \{4.4, \} \)

\[
u_{typ} \equiv \ln z |_P
= \int_{-\infty}^{\infty} dx \left[ \exp(-e^{-x+Kl}) - \exp(-ze^{-x+Kl}) \right]_P
= \int_{-\infty}^{\infty} dx \left[ G(x, l) - \{1 - \exp(-e^{-x+Kl})\} \right]
\sim m(l) - Kl. \tag{5.3}
\]

Here, in the last line, we have used the fact that the wavefront of \( G(x, l) \) is located at \( x \sim m(l) \), whereas the wavefront of \( 1 - \exp(-e^{-x+Kl}) \) is \( x \sim Kl \). The distribution function \( P(u, l) \) is broad especially in the case of a pulled front (\( \mu > 1/\sqrt{D} \)), and the typical value corresponds to the maximum of the distribution function. The typical \( \nu_{typ} \) may define typical values of \( y_n, \nu_{typ} \), through the relation \( \{4.4\} \)

\[
y_n, \nu_{typ} \sim e^{[m(l) - Kl]q}, \quad \nu_{typ} \sim e^{[m(l) - Kl]q}. \tag{5.4}
\]

Now we can define the dynamical scaling exponents \( z_n \) from the \( l \)-dependence of the typical values \( y_n, \nu_{typ} \). To this end, let us recall the following fact. In a weak disorder regime, the dynamical exponent of \( y_1 \) is, as expected, just \( 2 - x_1 \), where \( x_1 \) is the scaling dimension of \( \cos n_q \phi_a \) with \( n \in \mathbb{N}_+ \). In a strong disorder regime, however, it has been shown \( \{4.4\} \) that the interaction terms in Eq. \( \{3.1\} \) with the initial condition \( y_n(0) = y_1\delta_{n,1} \) plays a crucial role in the RG flow, and therefore it is quite important to keep the operator \( y_n \sum \cos n_q \phi_a \) in the action and to derive the scaling equation of \( y_n \) coupled together with those of higher \( y_n \) generated by the fusion of the vertex operators.

Similarly, if one wants to know the dynamical scaling exponent of the higher \( y_n \), one has to perturb the action by adding the corresponding operator \( y_q \sum \cos n_q \phi_a \) (\( n \in \mathbb{N}_q \)) as a source term. Namely, one has to consider the initial condition \( y_n(0) = y_q \delta_{n,q} \), and hence \( \mu = q \) in Eq. \( \{5.2\} \). Thus, we find \( y_q, \nu_{typ}(l) \sim e^{zl} \), where \( z_q \) is defined by

\[
z_q = \begin{cases} 2 - Kq + gK^2 q^2 & \text{for } gK^2 < 2/q^2, \\ ((\sqrt{8q} - 1)Kq) & \text{for } gK^2 \geq 2/q^2. \end{cases} \tag{5.5}
\]

Here we have neglected the logarithmic corrections appearing in Eq. \( \{5.2\} \). The vertex operator \( \cos(n_q \phi_a) \) with total charge \( q \) i.e., \( n \in \mathbb{N}_q \), therefore, obeys the scaling law with scaling exponent \( 2 - z_q \).

In passing, we mention that the logarithmic corrections in Eq. \( \{5.2\} \) are universal and give corrections to the correlation functions like marginal perturbations, although we have neglected them in this section. We will briefly discuss them in the next Sec. \( \{6\} \) calculating logarithmic corrections to the DOS.

B. Calculation of the DOS and the IPR

By using the dynamical scaling exponents, we first calculate the DOS defined in Eq. \( \{2.4\} \). Although we are basically interested in the Dirac fermion model, \( K = 1 \), we will derive formulas below using generic \( K \), because they are in fact valid for \( K \ll 1 \). Notice that the energy \( E \) has the same dimension as \( y_1 \). Then, we find

\[
\frac{\Lambda}{E} \sim \begin{cases} e^{(2 - K + gK^2)l} & \text{for } gK^2 < 2, \\ e^{(\sqrt{8g} - 1)Kl} & \text{for } gK^2 \geq 2. \end{cases} \tag{5.6}
\]

where \( \Lambda \) is a renormalized energy, and in what follows, we set \( \Lambda = 1 \) for simplicity. On the other hand, since the scaling exponent of \( \Gamma(l) \) is \( 2 - z_1 \), we have

\[
\rho(l) \sim \begin{cases} e^{-(K - gK^2)l} & \text{for } gK^2 < 2, \\ e^{-(2 - K + K\sqrt{8g})l} & \text{for } gK^2 \geq 2. \end{cases} \tag{5.7}
\]

This equation, together with Eq. \( \{5.6\} \), yields

\[
\rho(E) \sim \begin{cases} \frac{E^{K - gK^2}}{2 - 2\sqrt{gK} + K} & \text{for } gK^2 < 2, \\ \frac{E^{K - gK^2}}{(\sqrt{8g} - 1)K} & \text{for } gK^2 \geq 2. \end{cases} \tag{5.8}
\]

This result coincides with the one obtained by a variational method \( \{4.4\} \) in the weak coupling regime of the Dirac fermion, \( g < 2 \) with \( K = 1 \), this DOS is just the same as that obtained via usual weak coupling approaches \( \{4.4\} \).
Next, let us calculate the IPR according to the definition (2.6). The IPR are expected to show a power law behavior $P(q) \sim L^{-\tau(q)}$. Since the energy $E$ scales as $E \sim |\omega|$, the IPR obey the scaling law $P(q)(E) \sim E^{\tau(q)/z}$ as a function of the energy. One can read in this way the exponent $\tau(q)$ from $P(q)(E)$ in what follows. In the case of $gK^2 < 2$, the $q$th moment scales as

$$\omega^{q-1}\Gamma(q) \sim \begin{cases} e^{-(q-1)(2-K+gK^2)l}-(Kq-gK^2q^2)l & \text{for } q < \sqrt{2/gK^2}, \\ e^{-(q-1)(2-K+gK^2)l}-(2-\sqrt{8g}Kq+Kq)l & \text{for } q \geq \sqrt{2/gK^2}, \end{cases}$$

(5.9)

where we have used the fact that $\omega$ obeys the same scaling law as $E$ in Eq. (5.6). In the same way as the DOS, we reach

$$P(q)(E) \sim \begin{cases} E^{2-\sqrt{8g}Kl}/(\sqrt{8g}-1) & \text{for } q < \sqrt{2/gK^2}, \\ E^0 & \text{for } q \geq \sqrt{2/gK^2}. \end{cases}$$

(5.10)

Contrary to this, in the case $gK^2 \geq 2$, we have

$$\omega^{q-1}\Gamma(q) \sim \begin{cases} e^{-(q-1)(\sqrt{8g}-1)Kl-(Kq-gK^2q^2)l} & \text{for } q < \sqrt{2/gK^2}, \\ e^{-(q-1)(\sqrt{8g}-1)Kl-(2-\sqrt{8g}Kq+Kq)l} & \text{for } q \geq \sqrt{2/gK^2}. \end{cases}$$

(5.11)

Therefore, we arrive at

$$P(q)(E) \sim \begin{cases} E^{2-\sqrt{8g}Kl}/(\sqrt{8g}-1) & \text{for } q < \sqrt{2/gK^2}, \\ E^0 & \text{for } q \geq \sqrt{2/gK^2}. \end{cases}$$

(5.12)

These results completely reproduce those obtained so far by using various methods. The merit of the present approach lies in the fact that spatial correlations are quite easy to calculate. We assume that $q_1 \geq q_2$ without loss of generality. Noticing Eq. (5.13) leads to

$$Q^{(q_1,q_2)}(x-y,E) \sim \omega^{q_1+q_2-1}\Gamma(q_1)(x)\Gamma(q_2)(y)/\rho(E)$$

$$\sim \frac{1}{|x-y|^{\Delta(q_1,q_2)}}\omega^{q_1+q_2-1}\Gamma(q_1+q_2)(y)/\rho(E)$$

$$\sim \frac{1}{|x-y|^{\Delta(q_1,q_2)}}P^{(q_1+q_2)}(E),$$

(5.13)

where we have kept the most relevant operator for the OPE in the second line. Some comments are in order: First, the OPE used here is valid in the energy scale $|x-y| \ll E^{-z_1}$, where $z_1$ is the dynamical exponent in Eq. (5.3). Next, the formula (5.10) or (5.12) applies to $P^{(q_1+q_2)}(E)$ in the last line of (5.13). Finally, the exponent $\Delta(q_1,q_2)$ is calculated as

$$\Delta(q_1,q_2) = (2-z_{q_1}) + (2-z_{q_2}) - (2-z_{q_1+q_2})$$

$$= \begin{cases} \frac{2gK^2q_1q_2}{\sqrt{8g}K(q_1+q_2) - gK^2(q_1^2 + q_2^2)} & \text{for } gK^2 \leq 2/(q_1+q_2)^2, \\ \frac{2gK^2q_1q_2}{\sqrt{8g}Kq_2 - gK^2q_2^2} & \text{for } 2/(q_1+q_2)^2 \leq gK^2 \leq 2/q_1^2, \\ \frac{2gK^2q_1q_2}{2q_2^2} & \text{for } 2/q_2^2 \leq gK^2. \end{cases}$$

(5.14)

The equation shows that if disorder is strong enough, the exponent $\Delta(q_1,q_2)$ saturates at 2 for any $q_1$ and $q_2$. This saturation implies that the model is in the fully frozen phase.

Ryu and Hatsugai have recently calculated the density correlation function of the zero energy state of the Dirac fermion model, which corresponds to $q_1 = q_2 = 1$ and $K = 1$ case. Substituting these values into Eq. (5.14), we have

$$\Delta = \begin{cases} \frac{2g}{2} & \text{for } g \leq 1/2, \\ \frac{2}{\sqrt{\frac{g}{2}} - g - 1} & \text{for } 1/2 \leq g \leq 2, \\ \frac{2}{g} & \text{for } 2 \leq g. \end{cases}$$

(5.15)

It should be noted that if one compares this formula directly with the calculation of Ryu and Hatsugai, one has to replace $g \rightarrow g/\pi$. The above formula seems to fit their calculation, and therefore we can claim that an evidence
VI. SUMMARY AND DISCUSSIONS

In this paper, we have studied the Dirac fermion model with the random vector field in order to calculate the DOS, the IPR, and their spatial correlations both in the strong and the weak disorder regime. We have derived scaling equations of one-loop order, using the OPE techniques as well as taking account of the fusion of the vertex operators. We have been able to reproduce the results known so far. Especially, we have shown that the numerical calculation of the density correlation function can be explained by the RG method in this paper. It turns out that the saturation of the exponent at 2 is manifestly due to the freezing transition.

So far we have studied the random Dirac fermion in the continuum limit. In this case, the fermion has of course chiral symmetry at any realization of disorder. On the other hand, starting from the lattice model, e.g., the random hopping fermion on a square lattice with \( \pi \)-flux per plaquette\(^3\) or on a honeycomb lattice\(^4\) we find, in a continuum limit, a Dirac fermion with two flavors due to the species doubling. These lattice models could be described by such a Dirac fermion with a random imaginary vector field as well as a random mass term\(^5\) which has indeed chiral symmetry. The pioneering work by Gade\(^6\) of the disorder systems with chiral symmetry has predicted that the DOS diverges toward the zero energy as 

\[
E^{-1} \exp(-c|\ln E|^{\frac{2}{3}}),
\]

with \( x = 1/2 \). Recently, Motrunich, Damle, and Huse\(^7\) have pointed out that the DOS of such models should obey the same scaling law but with \( x = 2/3 \) due to strong disorder effects. Surprisingly, Mudry, Ryu and Furusaki\(^8\) have quite recently shown that such a behavior is also due to the freezing transition. Their method is based on the RG equation of the DOS, by taking account of all relevant perturbations. The KPP equation also plays a crucial role in their analysis.

In this sense, the KPP equation, which was proposed originally as a biological problem of the gene\(^9\) is a key equation to understand various kinds of random systems such as spin glasses\(^2\), directed polymers on a Cayley tree\(^10\) and a random hopping fermion on a lattice\(^11\).

Although we claimed in Sec. \( \ref{sec:logcorr} \) that the logarithmic corrections in Eq. (5.2) are universal, we have neglected them for simplicity. It is, however, readily seen that they give the logarithmic corrections, for example, to the DOS as

\[
\rho(E) \sim E^{\frac{2-(\sqrt{gK^2}-1)}{\sqrt{gK^2}-1}} |\ln E|^{-2\alpha \sqrt{gK^2}/\sqrt{gK^2}-1},
\]

where

\[
\alpha = \begin{cases} 
\frac{1}{8} & \text{for } gK^2 = 2, \\
\frac{1}{8} & \text{for } gK^2 > 2.
\end{cases}
\]

In deriving the equation, we have included the logarithmic corrections into the dynamical scaling exponents via the relation

\[
z_q = (\partial^2 u_{\text{typ}}) q = |\partial m(l) - K| q.
\]

In the weak disorder regime, however, no logarithmic corrections should exist in the DOS, as can be seen from Eq. (5.2).

In pure systems without disorder, it is well-known that marginal perturbations give rise to such logarithmic corrections. In random systems, however, logarithmic corrections are allowed even in the fixed-point theories themselves due to the presence of so-called logarithmic operators\(^3\). In the present case, it may be quite interesting to address the question on the origin of the logarithmic corrections appearing in the KPP equation.

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