A COMBINATORIAL RECIPROCITY THEOREM FOR HYPERPLANE ARRANGEMENTS

CHRISTOS A. ATHANASIADIS

Abstract. Given a nonnegative integer $m$ and a finite collection $A$ of linear forms on $\mathbb{Q}^d$, the arrangement of affine hyperplanes in $\mathbb{Q}^d$ defined by the equations $\alpha(x) = k$ for $\alpha \in A$ and integers $k \in [-m, m]$ is denoted by $A^m$. It is proved that the coefficients of the characteristic polynomial of $A^m$ are quasi-polynomials in $m$ and that they satisfy a simple combinatorial reciprocity law.

1. Introduction

Let $V$ be a $d$-dimensional vector space over the field $\mathbb{Q}$ of rational numbers and $A$ be a finite collection of linear forms on $V$ which spans the dual vector space $V^*$. We denote by $A^m$ the essential arrangement of affine hyperplanes in $V$ defined by the equations $\alpha(x) = k$ for $\alpha \in A$ and integers $k \in [-m, m]$ (we refer to [8,12] for background on hyperplane arrangements). Thus $A^0$ consists of the linear hyperplanes which are the kernels of the forms in $A$ and $A^m$ is a deformation of $A^0$, in the sense of [1,9].

The characteristic polynomial [8, Section 2.3] [12, Section 1.3] of $A^m$, denoted $\chi_A(q,m)$, is a fundamental combinatorial and topological invariant which can be expressed as

$$\chi_A(q,m) = \sum_{i=0}^{d} c_i(m) q^i.$$  

We will be concerned with the behavior of $\chi_A(q,m)$ as a function of $m$. Let $\mathbb{N} := \{0,1,\ldots\}$ and recall that a function $f : \mathbb{N} \to \mathbb{R}$ is called a quasi-polynomial with period $N$ if there exist polynomials $f_1,f_2,\ldots,f_N : \mathbb{N} \to \mathbb{R}$ such that $f(m) = f_i(m)$ for all $m \in \mathbb{N}$ with $m \equiv i \ (\text{mod} \ N)$. The degree of $f$ is the maximum of the degrees of the $f_i$. Our main result is the following theorem.

**Theorem 1.1.** Under the previous assumptions on $A$, the coefficient $c_i(m)$ of $q^i$ in $\chi_A(q,m)$ is a quasi-polynomial in $m$ of degree at most $d-i$. Moreover, the degree of $c_0(m)$ is equal to $d$ and

$$\chi_A(q,-m) = (-1)^d \chi_A(-q, m - 1).$$

In particular we have $\chi_A(q,-1) = (-1)^d \chi_A(-q)$, where $\chi_A(q)$ is the characteristic polynomial of $A^0$. Let $A^m_\mathbb{R}$ denote the arrangement of affine hyperplanes in the real $d$-dimensional vector space $V_\mathbb{R} = V \otimes_{\mathbb{Q}} \mathbb{R}$ defined by the same equations defining

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the hyperplanes of $A^m$. Let $r_A(m) = (-1)^d \chi_A(-1, m)$ and $b_A(m) = (-1)^d \chi_A(1, m)$ so that, for $m \in \mathbb{N}$, $r_A(m)$ and $b_A(m)$ count the number of regions and bounded regions, respectively, into which $V_R$ is dissected by the hyperplanes of $A^m$. \cite{22} [12, Section 2.2] \cite{13}.

Corollary 1.2. Under the previous assumptions on $A$, the function $r_A(m)$ is a quasi-polynomial in $m$ of degree $d$ and, for all positive integers $m$, $(-1)^d r_A(-m)$ is equal to the number $b_A(m-1)$ of bounded regions of $A^m_{V_R}$. 

Theorem 1.1 and its corollary belong to a family of results demonstrating some kind of combinatorial reciprocity law; see \cite{10} for a systematic treatment of such phenomena. Not surprisingly, the proof given in Section 2 is a simple application of the main results of Ehrhart theory \cite{11} Section 4.6. More specifically, equation (2) will follow from the reciprocity theorem \cite{11} Theorem 4.6.26 for the Ehrhart quasi-polynomial of a rational polytope. An expression for the coefficient of the leading term $m^d$ of either $c_0(m)$ or $r_A(m)$ is also derived in that section. Some examples, including the motivating example in which $A_0$ is the arrangement of reflecting hyperplanes of a Weyl group, and remarks are discussed in Section 3. In the remainder of this section we give some background on characteristic and Ehrhart (quasi-)polynomials needed in Section 2. We will denote by $\# S$ or $|S|$ the cardinality of a finite set $S$.

Arrangements of hyperplanes. Let $V$ be a $d$-dimensional vector space over a field $K$. An arrangement of hyperplanes in $V$ is a finite collection $\mathcal{H}$ of affine subspaces of $V$ of codimension one (we will allow this collection to be a multiset). The intersection poset of $\mathcal{H}$ is the set $L_{\mathcal{H}} = \{ \cap F : F \subseteq \mathcal{H} \}$ of all intersections of subcollections of $\mathcal{H}$, partially ordered by reverse inclusion. It has a unique minimal element $\hat{0} = V$, corresponding to the subcollection $\mathcal{F} = \emptyset$. The characteristic polynomial of $\mathcal{H}$ is defined by

$$\chi_{\mathcal{H}}(q) = \sum_{x \in L_{\mathcal{H}}} \mu(x) q^{\dim x}$$

where $\mu$ stands for the Möbius function on $L_{\mathcal{H}}$ defined by

$$\mu(x) = \begin{cases} 1, & \text{if } x = \hat{0} \\ -\sum_{y < x} \mu(y), & \text{otherwise.} \end{cases}$$

Equivalently \cite{8} Lemma 2.55] we have

$$\chi_{\mathcal{H}}(q) = \sum_{\mathcal{G} \subseteq \mathcal{H}} (-1)^{\# \mathcal{G}} q^{\dim(\cap \mathcal{G})}$$

where the sum is over all $\mathcal{G} \subseteq \mathcal{H}$ with $\cap \mathcal{G} \neq \emptyset$.

In the case $K = \mathbb{R}$, the connected components of the space obtained from $V$ by removing the hyperplanes of $\mathcal{H}$ are called regions of $\mathcal{H}$. A region is bounded if it is a bounded subset of $V$ with respect to a usual Euclidean metric.

Ehrhart quasi-polynomials. A convex polytope $P \subseteq \mathbb{R}^n$ is said to be a rational or integral polytope if all its vertices have rational or integral coordinates, respectively. If $P$ is rational and $P^o$ is its relative interior then the functions defined for
nonnegative integers $m$ by the formulas

\begin{align*}
i(P, m) &= \# (mP \cap \mathbb{Z}^n) \\
i(P, m) &= \# (mP^o \cap \mathbb{Z}^n)
\end{align*}

are quasi-polynomials in $m$ of degree $d = \dim(P)$, related by the Ehrhart reciprocity theorem [11, Theorem 4.6.26]

\begin{equation}
i(P, -m) = (-1)^d \bar{i}(P, m).
\end{equation}

The function $i(P, m)$ is called the Ehrhart quasi-polynomial of $P$. The coefficient of the leading term $m^d$ in either $i(P, m)$ or $\bar{i}(P, m)$ is a constant equal to the normalized $d$-dimensional volume of $P$ (meaning the $d$-dimensional volume of $P$ normalized with respect to the affine lattice $V_P \cap \mathbb{Z}^n$, where $V_P$ is the affine span of $P$ in $\mathbb{R}^n$). If $P$ is an integral polytope then $i(P, m)$ is a polynomial in $m$ of degree $d$, called the Ehrhart polynomial of $P$.

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1 and Corollary 1.2 and derive a formula for the coefficient of the leading term $m^d$ of $r_A(m)$. In what follows $A$ is as in the beginning of Section 1. We use the notation $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ and $[a, b]_\mathbb{Z} = [a, b] \cap \mathbb{Z}$ for $a, b \in \mathbb{Z}$ with $a \leq b$.

**Proof of Theorem 1.1 and Corollary 1.2.** Using formula (4) we get

\begin{equation}
\chi_A(q, m) = \sum_{G \subseteq A} (-1)^{\# G} q^{\dim(\cap G)}
\end{equation}

where the sum is over all $G \subseteq A^m$ with $\cap G \neq \emptyset$. Clearly for this to happen $G$ must contain at most one hyperplane of the form $\alpha(x) = k$ for each $\alpha \in A$. In other words we must have $G = F_b$ for some $F \subseteq A$ and map $b : F \rightarrow [-m, m]_\mathbb{Z}$ sending $\alpha$ to $b_\alpha$, where $F_b$ consists of the hyperplanes $\alpha(x) = b_\alpha$ for $\alpha \in F$. Let us denote by $\dim F$ the dimension of the linear span of $F$ in $V^*$ and observe that $\dim(\cap F_b) = d - \dim F$ whenever $\cap F_b$ is nonempty. From the previous observations and (7) we get

\[
\chi_A(q, m) = \sum_{F \subseteq A} (-1)^{\dim F} \sum_{b : F \rightarrow [-m, m]_\mathbb{Z}, \cap F_b \neq \emptyset} (-1)^{\# F_b} q^{\dim(\cap F_b)}
\]

where the sum is over all $F \subseteq A^m$ with $\cap F \neq \emptyset$. Clearly for this to happen $F$ must contain at most one hyperplane of the form $\alpha(x) = k$ for each $\alpha \in A$. In other words we must have $F = F_b$ for some $F \subseteq A$ and map $b : F \rightarrow [-m, m]_\mathbb{Z}$ sending $\alpha$ to $b_\alpha$, where $F_b$ consists of the hyperplanes $\alpha(x) = b_\alpha$ for $\alpha \in F$. Let us denote by $\dim F$ the dimension of the linear span of $F$ in $V^*$ and observe that $\dim(\cap F_b) = d - \dim F$ whenever $\cap F_b$ is nonempty. From the previous observations and (7) we get

\[
\chi_A(q, m) = \sum_{F \subseteq A} (-1)^{\dim F} \sum_{b : F \rightarrow [-m, m]_\mathbb{Z}, \cap F_b \neq \emptyset} (-1)^{\# F_b} q^{\dim(\cap F_b)}
\]

Let us write $F = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $b_i = b_{\alpha_i}$, so that $b$ can be identified with a column vector in $\mathbb{Q}^n$. Then $\cap F_b$ is nonempty if and only if the linear system $\alpha_i(x) = b_i$, $1 \leq i \leq n$, has a solution in $\mathbb{Q}^d$ or, equivalently, if and only if $b$ lies in the image $\text{Im} T_F$ of the linear transformation $T_F : \mathbb{Q}^d \rightarrow \mathbb{Q}^n$ mapping $x \in \mathbb{Q}^d$ to the column vector in $\mathbb{Q}^n$ with coordinates $\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x)$. It follows that
Equivalently we have
\[
\# \{ b : F \to [-m, m]_\mathbb{Z}, \cap F_b \neq \emptyset \} = \# (\text{Im}T_{\mathcal{F}} \cap [-m, m]_\mathbb{Z})^n
\]
\[
= \# \text{Im}T_{\mathcal{F}} \cap [-m, m]^n \cap \mathbb{Z}^n
\]
\[
= \# (m (\text{Im}T_{\mathcal{F}} \cap [-1, 1]^n) \cap \mathbb{Z}^n)
\]
\[
= \# (m P_{\mathcal{F}} \cap \mathbb{Z}^n)
\]
\[
i(P_{\mathcal{F}}, m)
\]
where \( P_{\mathcal{F}} = (\text{Im}T_{\mathcal{F}} \otimes_{\mathbb{Q}} \mathbb{R}) \cap [-1, 1]^n \), and hence that
\[
\chi_{\mathcal{A}}(q, m) = \sum_{\mathcal{F} \subseteq \mathcal{A}} (-1)^{\dim \mathcal{F}} q^{d - \dim \mathcal{F}} i(P_{\mathcal{F}}, m).
\]
Equivalently we have
\[
c_i(m) = \sum_{\dim \mathcal{F} = d - i} (-1)^{\dim \mathcal{F}} i(P_{\mathcal{F}}, m)
\]
for \( 0 \leq i \leq d \), where the \( c_i(m) \) are as in (1). Clearly \( P_{\mathcal{F}} \) is a rational convex polytope of dimension \( \dim (\text{Im}T_{\mathcal{F}}) = \dim \mathcal{F} \) and hence \( i(P_{\mathcal{F}}, m) \) is a quasi-polynomial in \( m \) of degree \( \dim \mathcal{F} \). It follows from (2) that \( c_i(m) \) is a quasi-polynomial in \( m \) of degree at most \( d - i \) and that \( r_{\mathcal{A}}(m) = \sum_{i=0}^{d} (-1)^{d-i} c_i(m) \) is a quasi-polynomial in \( m \) of degree at most \( d \). Moreover we have \( r_{\mathcal{A}}(m) \geq (2m+2)^d \) for \( m \geq 0 \) since \( \mathcal{A} \) contains \( d \) linearly independent forms and the corresponding hyperplanes of \( \mathcal{A}^m \) dissect \( V_{\mathbb{R}} \) into \( (2m+2)^d \) regions. It follows that the degree of \( r_{\mathcal{A}}(m) \) is no less than \( d \), which implies that the degrees of \( r_{\mathcal{A}}(m) \) and \( c_0(m) \) are, in fact, equal to \( d \).

It remains to prove the reciprocity relation (2). For \( \mathcal{F} \subseteq \mathcal{A} \) with \( \# \mathcal{F} = n \) let \( W_{\mathcal{F}} \) be the real linear subspace \( \text{Im}T_{\mathcal{F}} \otimes_{\mathbb{Q}} \mathbb{R} \) of \( \mathbb{R}^n \), so that \( P_{\mathcal{F}} = W_{\mathcal{F}} \cap [-1, 1]^n \). We have
\[
m P_{\mathcal{F}}^\circ \cap \mathbb{Z}^n = (W_{\mathcal{F}} \cap [-m, m]^n)^\circ \cap \mathbb{Z}^n
\]
\[
= W_{\mathcal{F}} \cap [-(m-1), m-1]^n \cap \mathbb{Z}^n
\]
\[
= (m-1) P_{\mathcal{F}} \cap \mathbb{Z}^n
\]
and hence \( i(P_{\mathcal{F}}, m) = i(P_{\mathcal{F}}, m-1) \). The Ehrhart reciprocity theorem (3) implies that
\[
i(P_{\mathcal{F}}, -m) = (-1)^{\dim \mathcal{F}} i(P_{\mathcal{F}}, m-1).
\]
Equation (2) follows from (3) and (10).

The following corollary is an immediate consequence of the case \( i = 0 \) of (9), the equation \( r_{\mathcal{A}}(m) = \sum_{i=0}^{d} (-1)^{d-i} c_i(m) \) and the fact that the degree of \( c_i(m) \) is less than \( d \) for \( 1 \leq i \leq d \).

**Corollary 2.1.** The coefficient of the leading term \( m^d \) in \( r_{\mathcal{A}}(m) \) is equal to the expression
\[
\sum_{\substack{\mathcal{F} \subseteq \mathcal{A} \\ \dim \mathcal{F} = d}} (-1)^{\dim \mathcal{F} - d} \text{vol}_d(P_{\mathcal{F}}),
\]
where \( P_{\mathcal{F}} \) is as in the proof of Theorem 1.1 and \( \text{vol}_d(P_{\mathcal{F}}) \) is the normalized \( d \)-dimensional volume of \( P_{\mathcal{F}} \).
3. Examples and remarks

In this section we list a few examples, questions and remarks.

Example 3.1. If $V = \mathbb{Q}$ and $A$ consists of two forms $\alpha_1, \alpha_2 : V \to \mathbb{Q}$ with $\alpha_1(x) = x$ and $\alpha_2(x) = 2x$ for $x \in V$ then $A^m$ consists of the affine hyperplanes (points) in $V$ defined by the equations $x = k$ and $x = k/2$ for $k \in [-m, m]$. One can check that

$$
\chi_A(q, m) = \begin{cases} 
q - 3m - 1, & \text{if } m \text{ is even} \\
q - 3m - 2, & \text{if } m \text{ is odd}
\end{cases}
$$

and that (2) holds. Moreover we have

$$
r_A(m) = \begin{cases} 
3m + 2, & \text{if } m \text{ is even} \\
3m + 3, & \text{if } m \text{ is odd}.
\end{cases}
$$

Note that $\text{vol}_d(P_F)$ takes the values 2, 2 and 1 when $F = \{\alpha_1\}, \{\alpha_2\}$ and $\{\alpha_1, \alpha_2\}$, respectively.

Example 3.2. If $V = \mathbb{Q}^d$ and $A$ consists of the coordinate functions $\alpha_i(x) = x_i$ for $1 \leq i \leq d$ then $A^m$ consists of the affine hyperplanes in $V$ given by the equations $x_i = k$ with $1 \leq i \leq d$, $k \in [-m, m]$, and $\chi_A(q, m) = (q - 2m - 1)^d$, which is a polynomial in $q$ and $m$ satisfying (2).

Example 3.3. Let $\Phi$ be a finite, irreducible, crystallographic root system spanning the Euclidean space $\mathbb{R}^d$, endowed with the standard inner product $(\ , \ )$ (we refer to [4, 5, 7] for background on root systems). Fix a positive system $\Phi^+$ and let $Q_{\Phi}$ and $W$ be the coroot lattice and Weyl group, respectively, corresponding to $\Phi$. Let also $A^m_\Phi$ denote the $m$th generalized Catalan arrangement associated to $\Phi$ [1, 2, 9], consisting of the affine hyperplanes in $\mathbb{R}^d$ defined by the equations $(\alpha, x) = k$ for $\alpha \in \Phi^+$ and $k \in [-m, m]$ (so that $A^0_\Phi$ is the real reflection arrangement associated to $\Phi$). If $V$ is the $\mathbb{Q}$-span of $Q_{\Phi}$ then there exists a finite collection $A$ of linear forms on $V$ (one for each root in $\Phi^+$) such that, in the notation of Section 1, $A^m_\Phi$ coincides with $A^m_\Phi$. In [2, Theorem 1.2] a uniform proof was given of the formula

$$
\chi_A(q, m) = \prod_{i=1}^{d} (q - mh - e_i)
$$

for the characteristic polynomial of $A^m_\Phi$, where $e_1, e_2, \ldots, e_d$ are the exponents and $h$ is the Coxeter number of $\Phi$. Thus the reciprocity law (2) in this case is equivalent to the well-known fact [5, Section V.6.2] [7, Lemma 3.16] that the numbers $h - e_i$ are a permutation of the $e_i$. As was already deduced in [2, Corollary 1.3], it follows from (11) that

$$
r_A(m) = \prod_{i=1}^{d} (mh + e_i + 1)
$$

and

$$
b_A(m) = \prod_{i=1}^{d} (mh + e_i - 1)
$$

are polynomials in $m$ of degree $d$ (a fact which was the main motivation behind this paper). Setting $N(\Phi, m) = \frac{1}{|W|} r_A(m)$ and $N^+(\Phi, m) = \frac{1}{|W|} b_A(m)$, as in [3, 6], our
Corollary 1.2 implies that

\[(12) \quad (-1)^d N(\Phi, -m) = N^+(\Phi, m - 1).\]

It was suggested in [4 Remark 12.5] that this equality, first observed in [6 (2.12)], may be an instance of Ehrhart reciprocity. This was confirmed in [3 Section 7] using an approach which is different from the one followed in this paper. Finally we note that Corollary 2.1 specializes to the curious identity

\[(13) \quad h^d = \sum_F (-1)^#F - d \, \text{vol}_d(P_F)\]

where in the sum on the right hand side \(F\) runs through all subsets \(\{\alpha_1, \alpha_2, \ldots, \alpha_n\}\) of \(\Phi^+\) spanning \(\mathbb{R}^d\), \(P_F\) is the intersection of the cube \([-1, 1]^n\) with the image of the linear transformation \(T_F: \mathbb{R}^d \to \mathbb{R}^n\) mapping \(x \in \mathbb{R}^d\) to the column vector in \(\mathbb{R}^n\) with coordinates \((\alpha_1, x), (\alpha_2, x), \ldots, (\alpha_n, x)\) and \(\text{vol}_d(P_F)\) is the normalized \(d\)-dimensional volume of \(P_F\). If \(\Phi\) has type \(A_d\) in the Cartan-Killing classification then \((13)\) translates to the equation

\[(14) \quad (d + 1)^d = \sum_G (-1)^{e(G) - d} \, \text{vol}_d(Q_G)\]

where in the sum on the right hand side \(G\) runs through all connected simple graphs on the vertex set \(\{1, 2, \ldots, d + 1\}\), \(e(G)\) is the number of edges of \(G\) and \(Q_G\) is the \(d\)-dimensional polytope in \(\mathbb{R}^d\) defined in the following way. Let \(T\) be a spanning tree of \(G\) with edges labeled in a one to one fashion with the variables \(x_1, x_2, \ldots, x_d\). For any edge \(e\) of \(G\) which is not an edge of \(T\) let \(R_e\) be the region of \(\mathbb{R}^d\) defined by the inequalities \(-1 \leq x_i + x_j \pm \cdots + x_k \leq 1\), where \(x_1, x_2, \ldots, x_k\) are the labels of the edges (other than \(e\)) of the fundamental cycle of the graph obtained from \(T\) by adding the edge \(e\). The polytope \(Q_G\) is the intersection of the cube \([-1, 1]^d\) and the regions \(R_e\).

**Remark 3.4.** It is well-known [12 Corollary 3.5] that the coefficients of the characteristic polynomial of a hyperplane arrangement strictly alternate in sign. As a consequence, in the notation of [11], we have \((-1)^{d-i} c_i(m) > 0\) for all \(m \in \mathbb{N}\) and \(0 \leq i \leq d\). We do not know of an example of a collection \(\mathcal{A}\) of forms for which a negative number appears among the coefficients of the quasi-polynomials \((-1)^{d-i} r_i(m)\).

**Remark 3.5.** If the matrix defined by the forms in \(\mathcal{A}\) with respect to some basis of \(V\) is integral and totally unimodular, meaning that all its minors are \(-1, 0\) or \(1\), then the polytopes \(P_F\) in the proof of Theorem 1.1 are integral and, as a consequence, the functions \(c_i(m)\) and \(r_i(m)\) are polynomials in \(m\). This assumption on \(\mathcal{A}\) is satisfied in the case of graphical arrangements, that is when \(\mathcal{A}\) consists of the forms \(x_i - x_j\) on \(Q^r\), where \(1 \leq i < j \leq r\), corresponding to the edges \(\{i, j\}\) of a simple graph \(G\) on the vertex set \(\{1, 2, \ldots, r\}\). The degree of the polynomial \(r_G(m) := r_{\mathcal{A}}(m)\) is equal to the dimension of the linear span of \(\mathcal{A}\), in other words to the rank of the cycle matroid of \(G\).

**Remark 3.6.** Let \(\mathcal{A}\) be finite collections of linear forms on a \(d\)-dimensional \(\mathbb{Q}\)-vector space \(V\) spanning \(V^*\). Using the notation of Section 1 let \(\mathcal{H}_m\) denote the union of \(\mathcal{A}_m^*\) with the linear arrangement \(\mathcal{H}_m^0\). It follows from Theorem 1.1 the Deletion-Restriction theorem [8 Theorem 2.56] and induction on the cardinality of \(\mathcal{H}\) that the function \(r(\mathcal{H}_m)\) is a quasi-polynomial in \(m\) of degree \(d\). Given a region
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R of $\mathcal{H}_0^R$, let $r_R(m)$ denote the number of regions of $\mathcal{H}_m$ which are contained in $R$, so that

$$r(\mathcal{H}_m) = \sum_R r_R(m)$$

where $R$ runs through the set of all regions of $\mathcal{H}_0^R$. Is the function $r_R(m)$ always a quasi-polynomial in $m$?

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DEPARTMENT OF MATHEMATICS (DIVISION OF ALGEBRA-GEOMETRY), UNIVERSITY OF ATHENS, PANEPISTIMIOPOULIS, 15784 ATHENS, GREECE
E-mail address: caath@math.uoa.gr