THE SMITH AND CRITICAL GROUPS OF PALEY GRAPHS

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Abstract. There is a Paley graph for each prime power \( q \) such that \( q \equiv 1 \pmod{4} \). The vertex set is the field \( \mathbb{F}_q \) and two vertices \( x \) and \( y \) are joined by an edge if and only if \( x - y \) is a nonzero square of \( \mathbb{F}_q \). We compute the Smith normal forms of the adjacency matrix and Laplacian matrix of a Paley graph.

1. Introduction

Let \( \Gamma \) be a finite, simple, undirected and connected graph and let \( A \) be the adjacency matrix of \( \Gamma \) with respect to some fixed but arbitrary ordering of the vertex set of \( \Gamma \). Let \( D \) be the diagonal matrix whose \((i,i)\)-entry is the degree of the \( i \)th vertex. Then \( L = D - A \) is called the Laplacian matrix of \( \Gamma \). The matrices \( A \) and \( L \) represent endomorphisms (which will also be denoted by \( A \) and \( L \)) of the free abelian group on the vertex set. The structure of their cokernels as abelian groups is independent of the above ordering. The cokernel of \( A \) is called the Smith group \( S(\Gamma) \), since its computation is equivalent to finding the Smith normal form of the matrix \( A \). The endomorphism \( L \) maps the sum of all vertices to zero, so the cokernel of \( L \) is not a torsion group. The torsion subgroup \( K(\Gamma) \) of the cokernel of \( L \) is called the critical group of \( \Gamma \). It is known by Kirchhoff’s matrix-tree theorem that the order of \( K(\Gamma) \) is equal to the number of spanning trees of \( \Gamma \).

The critical group of a graph arises in several contexts, for example in arithmetic geometry [10], in statistical physics [6] and in combinatorics [2]. There are also interpretations of the critical group in discrete dynamics (chip-firing games and abelian sandpile models, cf. [9]). We refer the reader to [11] for a discussion of these and other connections.

So far, there are very few families of graphs for which the critical groups have been found, so it is of some interest to compute the Smith and critical groups for some well-known families of graphs.

In this note we treat the Paley graphs. Let \( q = p^t \) be a fixed prime power with \( q \equiv 1 \pmod{4} \). The Paley graph \( \text{Paley}(q) \) is defined by taking the field \( \mathbb{F}_q \) as vertex set, with two vertices \( x \) and \( y \) joined by an edge if and only if

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$x - y$ is a nonzero square in $\mathbb{F}_q$. The degree of each vertex is $k = \frac{q-1}{2}$. Let $A$ denote the adjacency matrix and $L = kI - A$ the Laplacian matrix. Our main result is the computation of $S(\text{Paley}(q))$ and $K(\text{Paley}(q))$. There has been some earlier work in this direction. The structure of the $S(\text{Paley}(q))$ was correctly conjectured in [13, Ex. 4–8]. In [11] the critical group of a conference graph on a square-free number of vertices was calculated, and Paley$(q)$ is such a graph when $q$ is a prime. The $p$-rank of the Laplacian of Paley$(q)$ was first computed in [4].

Here is a brief outline of our method. We view Paley$(q)$ as a Cayley graph, with the regular action of the additive group of $\mathbb{F}_q$. Then, in §2 we follow a standard method, applying the discrete Fourier transform while keeping track of coefficient rings, to compute the Smith group and also the $p$-complementary part of the critical group.

A different approach is needed to compute the $p$-part of $K(\text{Paley}(q))$. In §3 we study the permutation action of the group $S$ of nonzero squares on $\mathbb{F}_q$ by multiplication. The free module with basis $\mathbb{F}_q$ over a suitable ring extension decomposes into $S$-isotypic components of rank 2 (except for one of rank 3). Since $S$ preserves adjacency, these isotypic components are $A$-invariant. In the computation of the restriction of $A$ to each $S$-isotypic component, certain Jacobi sums arise naturally and the main problem is reduced to determining the $p$-adic valuations of these Jacobi sums. The classical theorem of Stickelberger on Gauss sums gives the valuation for individual sums, but there remains the problem of counting the number of sums with a given valuation. This counting problem is solved by the transfer matrix method in §4. It is also possible to count directly, but the chosen method has the advantages of being systematic and of yielding immediately the rationality of the generating function.

2. Eigenvalues and $p'$-torsion

It is well known and easily checked that Paley$(q)$ is a strongly regular graph and that its eigenvalues are $k = \frac{q-1}{2}$, $r = \frac{-1+\sqrt{q}}{2}$, and $s = \frac{-1-\sqrt{q}}{2}$, with multiplicities 1, $\frac{q-1}{2}$, and $\frac{q-1}{2}$, respectively. (See, for example, [8.1.1][3]). Hence,

$$|S(\text{Paley}(q))| = \det(A) = k \left( \frac{k}{2} \right)^k,$$

where $A$ is the adjacency matrix of Paley$(q)$. It follows that $\gcd(|S(\text{Paley}(q))|, q) = 1$. Therefore we can use the diagonalization of $A$ by the character table of $(\mathbb{F}_q, +)$ to find the Smith normal form of $A$.

Let $S$ be the set of nonzero squares in $\mathbb{F}_q$. We can view Paley$(q)$ as a Cayley graph with connecting set $S$. Let $X$ be the complex character table of the additive group of $\mathbb{F}_q$ where the elements are ordered in the same way as for
the rows and columns of $A$. The entries of $X$ lie in the ring $\mathbb{Z}[\zeta]$, where $\zeta$ is a complex primitive $p$-th root of unity. As was first observed in [12], we have the character orthogonality relation $\frac{1}{q}XAX^t = I$ and

\[
\frac{1}{q}XAX^t = \text{diag}(\psi(S))_\psi,
\]

where $\psi$ runs over the additive characters of $\mathbb{F}_q$ and $\psi(S) = \sum_{y \in S} \psi(y)$. Thus, the $\psi(S)$ are the eigenvalues of $A$. Since the eigenvalues of $A$ are all prime to $p$, the structure of $S(Paley(q))$ can be completely determined from (1). (See [14, 3.2], or [5, §2], where the same argument is used for difference sets.) It suffices to determine the structure of the $\ell$-Sylow subgroup for each prime $\ell$ different from $p$. Such an $\ell$ is unramified in $\mathbb{Z}[\zeta]$, and (1) can be interpreted as expressing the equivalence of matrices with entries in the localized ring $\mathbb{Z}[\zeta](\ell)$.

This latter ring is a principal ideal domain and the list of exact powers of $\ell$ dividing the $\psi(S)$ is precisely the list of $\ell$-elementary divisors of $A$.

By applying the above to each $\ell \neq p$, and noting that $r$ and $s$ are coprime, with $rs = \frac{q-1}{4}$, we obtain the following result.

**Theorem 2.1.** The Smith group of $Paley(q)$ is isomorphic to $\mathbb{Z}/2\mu\mathbb{Z} \oplus (\mathbb{Z}/\mu\mathbb{Z})^{2\mu}$, where $\mu = \frac{q-1}{4}$.

From the eigenvalues of $A$, we easily obtain those of $L = kI - A$ (the Laplacian matrix of $Paley(q)$), namely $0$, with multiplicity $1$, and $\left(\frac{q+\sqrt{q}}{2}\right)^k$ and $\left(\frac{q-\sqrt{q}}{2}\right)^k$, each with multiplicity $\frac{q-1}{2}$. It follows from Kirchhoff’s matrix-tree theorem that

\[
|K(Paley(q))| = \frac{1}{q} \left(\frac{q+\sqrt{q}}{2}\right)^k \left(\frac{q-\sqrt{q}}{2}\right)^k = q^{\frac{q-3}{2}} \mu^k,
\]

where $\mu = \frac{q-1}{4}$.

The $\ell$-elementary divisors of $L$ for primes $\ell \neq p$ can be found in exactly the same way as we found the elementary divisors of $A$. We can therefore determine the subgroup $K(Paley(q))_{p'}$ which is complementary to the Sylow $p$-subgroup of $K(Paley(q))$.

**Theorem 2.2.** Let $K(Paley(q)) = K(Paley(q))_p \oplus K(Paley(q))_{p'}$ be the decomposition of the critical group of $Paley(q)$ into its Sylow $p$-subgroup and $p$-complement. Then $K(Paley(q))_{p'} \cong (\mathbb{Z}/\mu\mathbb{Z})^{2\mu}$, where $\mu = \frac{q-1}{4}$.

The $p$-elementary divisors of $L$ remain to be computed and the rest of the paper is devoted to this task.

\section{Character sums and invariants}

We will adopt the same notation as in [22] In order to find the $p$-elementary divisors of $L$, we will view the entries of $L$ as coming from some $p$-adic local
ring. Let $q = p^l$ and $K = \mathbb{Q}_p(\xi_{q-1})$ be the unique unramified extension of degree $t$ over $\mathbb{Q}_p$, the field of $p$-adic numbers, where $\xi_{q-1}$ is a primitive $(q-1)^{\text{th}}$ root of unity in $K$. Let $R = \mathbb{Z}_p[\xi_{q-1}]$ be the ring of integers in $K$. Then $pR$ is the unique maximal ideal of $R$ and $R/pR \cong \mathbb{F}_q$. Let $T : \mathbb{F}_q^\times \to R^\times$ be the Teichmüller character of $\mathbb{F}_q$. Then $T$ is an $R$-valued multiplicative character of $\mathbb{F}_q$ of order $q - 1$. Hence $T$ generates the cyclic group $\text{Hom}(\mathbb{F}_q^\times, R^\times)$.

Let $R^\mathbb{F}_q$ be the free $R$-module with basis indexed by the elements of $\mathbb{F}_q$. For clarity, we write the basis element corresponding to $x \in \mathbb{F}_q$ as $[x]$. Then $\mathbb{F}_q^\times$ acts on $R^\mathbb{F}_q$, permuting the basis by field multiplication, so that $R^\mathbb{F}_q$ decomposes as the direct sum $R[0] \oplus R^\mathbb{F}_q^\times$ of a trivial module with the regular module for $\mathbb{F}_q^\times$. The regular module $R^\mathbb{F}_q^\times$ decomposes further into the direct sum of 1-dimensional $\mathbb{F}_q^\times$-isotypic components, affording the characters $T_i$, $i = 0, 1, \ldots, q - 2$. A basis element for the component affording $T_i$ is

$$e_i = \sum_{x \in \mathbb{F}_q^\times} T_i(x^{-1})[x].$$

Here the subscript $i$ is read modulo $q - 1$. So $R^\mathbb{F}_q$ has basis $\{e_i \mid i = 1, \ldots, q - 2\} \cup \{e_0, [0]\}$, where we have separated out the basis for the $\mathbb{F}_q^\times$-fixed points.

Next consider the action of the subgroup $S$ of squares in $\mathbb{F}_q^\times$ on $R^\mathbb{F}_q^\times$. Then for $0 \leq i \leq q - 2$, $T_i$ and $T^{i+k}$ are equal on $S$, and the $S$-isotypic components on $R^\mathbb{F}_q^\times$ are each 2-dimensional, with $\{e_i, e_{i+k}\}$ as basis of the component affording $T_i|_S$. For $0 < i \leq k - 1$, let $M_i$ denote this 2-dimensional space. The subspace of $S$-fixed points on $R^\mathbb{F}_q$ has basis $\{[0], e_0, e_k\}$, but since $e_0 + [0] = 1 = \sum_{x \in \mathbb{F}_q}[x]$, we will use the basis $\{1, [0], e_k\}$ instead. Let $M_0$ denote this 3-dimensional space.

We can view $A$ and $L$ as endomorphisms of $R^\mathbb{F}_q$, with

$$A([x]) = \sum_{s \in S} [x + s], \quad x \in \mathbb{F}_q$$

and

$$L([x]) = k[x] - \sum_{s \in S} [x + s], \quad x \in \mathbb{F}_q.$$ 

Both maps $A$ and $L$ are $S$-equivariant; hence they map the $S$-isotypic component $M_i$ to itself for each $0 \leq i \leq k - 1$. In other words, with respect the the decomposition of $R^\mathbb{F}_q$ into the $M_i$ ($0 \leq i \leq k - 1$), both $A$ and $L$ have block diagonal form, with a $(2 \times 2)$-block for each $M_i$, for $1 \leq i \leq k - 1$, and a $(3 \times 3)$-block for $M_0$. We are therefore reduced to computing the elementary divisors of $L|_{M_i}$ and determining for a given $p$-power its total multiplicity, as $i$ varies.
The next two lemmas compute $L$ on each of the $M_i$. The character $T^k = T^{-k}$ of $\mathbb{F}_q^\times$ is the quadratic character and we denote it by $\chi$. Following the convention of Ax [1], $T^0$ is the character that maps all elements of $\mathbb{F}_q$ to 1, while $T^{q-1}$ maps 0 to 0 and all other elements to 1. Moreover nonprincipal characters take the value 0 at 0. With these conventions the characteristic function of $S$ is

$$\frac{1}{2}(\chi + T^0 - \delta_0),$$

where $\delta_0$ is 1 at 0 and zero elsewhere. Also we will need Jacobi sums, which we define below. For any two integers $a, b$, we define the Jacobi sum $J(T^a, T^b)$ by

$$J(T^a, T^b) = \sum_{x \in \mathbb{F}_q^\times} T^a(x)T^b(1-x).$$

From the above definition and our convention on $T^0$ and $T^{q-1}$, we see that if $a \not\equiv 0 \pmod{q-1}$, then

$$J(T^a, T^0) = 0, \quad J(T^a, T^{q-1}) = -1.$$

**Lemma 3.1.** Suppose $0 \leq i \leq q-2$ and $i \neq 0, k$. Then

$$L(e_i) = \frac{1}{2}(qe_i - J(T^{-i}, \chi)e_{i+k})$$

**Proof.** We have

$$A(e_i) = \sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1}) \sum_{y \in S} [x + y]$$

$$= \frac{1}{2} \sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1}) \sum_{y \in \mathbb{F}_q^\times} (\chi(y) + T^0(y) - \delta_0(y))[x + y]$$

$$= \frac{1}{2} \sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1}) \sum_{y \in \mathbb{F}_q^\times} \chi(y)[x + y]$$

$$+ \frac{1}{2} \sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1}) \sum_{y \in \mathbb{F}_q^\times} [x + y] - \frac{1}{2} \sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1})[x].$$

The second sum is zero and the third is $-\frac{1}{2}e_i$. For the first sum, we have

$$\sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1}) \sum_{y \in \mathbb{F}_q^\times} \chi(y)[x + y] = \sum_{z \in \mathbb{F}_q^\times} \sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1})\chi(z - x)[z].$$

Then if $z \neq 0$, we have $T^i(x^{-1})\chi(z-x) = T^i(z^{-1})\chi(z)T^{-i}((x/z))\chi(1-(x/z))$. The sum over $x$ of these terms is the same over $\mathbb{F}_q^\times$ or $\mathbb{F}_q$ and is equal to $T^i(z^{-1})\chi(z)J(T^{-i}, \chi) = T^{i+k}(z^{-1})J(T^{-i}, \chi).$
If $z = 0$ then $\sum_{x \in \mathbb{F}_q^*} T^i(x^{-1}) \chi(-x) = 0$. Thus, the outer sum sum over all $z \in \mathbb{F}_q$ can be taken over $\mathbb{F}_q^*$ and is equal to $J(T^{-i}, \chi)e_{i+k}$. The lemma now follows since $L = kI - A$. \hfill \Box

**Lemma 3.2.** (i) $L(1) = 0$.
(ii) $L(e_k) = \frac{1}{2}(1 - q([0] - e_k))$.
(iii) $L([0]) = \frac{1}{2}(q[0] - e_k - 1)$.

**Proof.** (i) is obvious and (iii) is straightforward. (ii) is proved by the same calculation as in the previous lemma, using the fact that $\chi^k \chi = -\chi(-1) = -1$. The only difference in the calculation is that in equation (3) the $z = 0$ term is $(q - 1)[0]$ instead of zero. \hfill \Box

**Corollary 3.3.** The Laplacian matrix $L$ is equivalent over $R$ to the diagonal matrix with diagonal entries $J(T^{-i}, T^k)$, for $i = 1, \ldots, q - 2$ and $i \neq k$, two 1s and one zero.

In view of Corollary 3.3 to compute the $p$-elementary divisors of $L$, we will need to know the $p$-adic valuations of Jacobi sums. Using Stickelberger’s theorem on Gauss sums [16] (see [7, p. 636] for further reference) and the well-known relation between Gauss and Jacobi sums, we have.

**Theorem 3.4.** Let $a$ and $b$ be integers such that $a \neq 0 \pmod{q - 1}$, $b \neq 0 \pmod{q - 1}$, and $a + b \neq 0 \pmod{q - 1}$. For any integer $x$, we use $s(x)$ to denote the sum of digits in the expansion of the least nonnegative residue of $x$ modulo $(q - 1)$ as a base $p$ number. Then

$$\nu_p(J(T^{-a}, T^{-b})) = \frac{s(a) + s(b) - s(a + b)}{p - 1},$$

where $\nu_p(J(T^{-a}, T^{-b}))$ is the $p$-adic valuation of $J(T^{-a}, T^{-b})$. In other words, the number of times that $p$ divides $J(T^{-a}, T^{-b})$ is equal to the number of carries in the addition $a + b \pmod{q - 1}$.

By Theorem 3.4 the $p$-adic valuation of $J(T^{-i}, T^k)$ for $i \neq k$ is known to be equal to $c(i) := \frac{1}{p - 1}(s(i) + \frac{t(p - 1)}{2} - s(i + k))$. This valuation is the number of carries, when adding the $p$-expansions of $i$ and $k$, modulo $q - 1$. It is now clear that there are no elementary divisors of $L$ greater than $p^t$. It is also easy to see that the $p$-rank of $L$ (i.e., the number of times that $p^0$ appears as an elementary divisor of $L$) equals $(\frac{p + 1}{2})^t$, since a necessary and sufficient condition for there to be no carries is that each of the $p$-digits of $i$ be in the range from 0 to $\frac{p - 1}{2}$. This was already shown by Brouwer and van Eijl [4]. Also, since $c(i) + c(q - 1 - i) = t$, it follows from Corollary 3.3 that the multiplicity of $p^t$ as an elementary divisor is $((\frac{p + 1}{2})^t - 2$. It remains to find the multiplicity of
\(p^\lambda\) for \(1 \leq \lambda \leq t - 1\). In order to find the multiplicity of \(p^\lambda\) as an elementary divisor of \(L\), we have to count the number of \(i\) ranging from 1 to \(q - 2\), \(i \neq k\), such that adding \(i\) to \(k\) involves exactly \(\lambda\) carries.

4. The Counting Problem

In order to finish our computations of the critical groups of Paley(\(q\)), we will solve the following counting problem using the transfer matrix method. For a discussion of the transfer matrix method and its various applications, we refer the reader to [15, Section 4.7].

The Counting Problem. Let \(q = p^t\) be an odd prime power and \(k = \frac{q - 1}{2}\). For \(1 \leq \lambda \leq t - 1\), what is the number of \(i\), \(1 \leq i \leq q - 2\), \(i \neq k\) such that adding \(i\) to \(\frac{q - 1}{2}\) modulo \(q - 1\) involves exactly \(\lambda\) carries?

We express integers \(a\), \(0 \leq a \leq q - 1\), in base \(p\). That is, we write

\[a = a_{t-1}p^{t-1} + a_{t-2}p^{t-2} + \cdots + a_1p + a_0,\]

where \(0 \leq a_i \leq p - 1\) for all \(i\). In what follows, we will simply write \(a = a_{t-1}a_{t-2}\cdots a_1a_0\), and call the \(a_i\)’s the digits of \(a\). Since we are going to add \(a\) with \(\frac{q - 1}{2}\) modulo \(q - 1\), we will use the modular \(p\)-ary add-with-carry algorithm described in [8, Theorem 4.1].

Let \(a = a_{t-1}a_{t-2}\cdots a_1a_0\) and \(b = b_{t-1}b_{t-2}\cdots b_1b_0\) be integers in \(\{1, 2, \ldots, q - 2\}\) such that \(a + \frac{q - 1}{2} = b\) modulo \(q - 1\). By the modular \(p\)-ary add-with-carry algorithm (cf. [8, Theorem 4.1]) there is a unique carry sequence \(c = c_{t-1}c_{t-2}\cdots c_1c_0\) with \(c_i \in \{0, 1\}\) and \(c_t = c_0\) such that for all \(0 \leq i \leq t - 1\),

\[a_i + \frac{p - 1}{2} + c_i = b_i + pc_{i+1}.\]  

(4)

We will use the transfer matrix method to solve the above counting problem. This approach involves constructing a weighted digraph \(G\), and changing the counting problem to that of counting closed walks in \(G\) of certain length and weight. The above equations motivate us to construct the digraph \(G\) on \([p] \times [2]\) (here \([p] = \{0, 1, \ldots, p - 1\}\) and \([2] = \{0, 1\}\) as follows: The vertices of \(G\) are \((\alpha, \gamma) \in [p] \times [2]\). There is an arc from \((\alpha, \gamma)\) to \((\alpha', \gamma')\) if and only if

\[\alpha + \frac{p - 1}{2} + \gamma = \beta + p\gamma'\]  

for some \(\beta \in [p]\). Furthermore if there is an arc \(e\) from \((\alpha, \gamma)\) to \((\alpha', \gamma')\) we give the arc label \(\alpha\) and weight \(\gamma' := w(e)\). By the definition of \(G\), we see that with one exception there is a one-to-one correspondence between the set of closed walks in \(G\) of length \(t\) and weight \(\lambda\) and the set of \(a \in [q]\), such that when adding \(a\) with \(\frac{q - 1}{2}\) modulo \(q - 1\), there will be exactly \(\lambda\) carries. The
exception occurs when \( a = \frac{q - 1}{2} \) and is due to the fact that 0 and \( q - 1 \) are distinct elements of \([q]\) but equal modulo \( q - 1 \). In this case, there are two closed walks, one with \( b_i = p - 1 \) and \( c_i = 0 \) for all \( i \), and the other with \( b_i = 0 \) and \( c_i = 1 \) for all \( i \). (By Corollary 3.3 the exceptional case does not enter into our computation of the \( p \)-adic elementary divisors of \( L \).)

Let \( B \) be the adjacency matrix of the digraph \( G \). More precisely, the rows and columns of \( B \) are both indexed by \((\alpha, \gamma) \in [p] \times [2] \), and the entry \((\alpha, \gamma), (\alpha', \gamma')\) of \( B \) is 0 if there is no arc from \((\alpha, \gamma)\) to \((\alpha', \gamma')\); is 1 if there is an arc \( e \) from \((\alpha, \gamma)\) to \((\alpha', \gamma')\) and \( w(e) = 0 \); is \( x \) if there is an arc \( e \) from \((\alpha, \gamma)\) to \((\alpha', \gamma')\) and \( w(e) = 1 \) (here \( x \) is an indeterminate). Note that since (5) does not involve \( \alpha' \), the adjacency matrix \( B \) has only two distinct rows, each repeated \( p \) times. More explicitly,

\[
B = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots \\
1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & x & \cdots & x \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & x & \cdots & x \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & x & \cdots & x \\
\end{pmatrix}
\]

If \( \Psi = e_1 e_2 \cdots e_n \) is a walk in \( G \), we define \( \text{wt}(\Psi) = x^{w(e_1) + w(e_2) + \cdots + w(e_n)} \). Let

\[
C_G(n) = \sum_{\Psi} \text{wt}(\Psi),
\]

where the sum is extended over all closed walks of length \( n \) in \( G \). Then \( C_G(n) = \sum_{m \geq 0} f(n, m) x^m \), where \( f(n, m) \) is the number of closed walks of length \( n \) and weight \( m \). Let

\[
F(z, x) = \sum_{n \geq 1} C_G(n) z^n.
\]

By Corollary 4.7.3 of [15] p. 501], we have

\[
F(z, x) = -\frac{z \partial Q(z, x)}{y(z, x)},
\]
where $Q(z, x) = \det(I - zB)$. Using the above definition of $B$, we find that 
\[
\det(I - zB) = 1 - \frac{p+1}{2}(1 + x)z + pzx^2.
\]
It follows that 
\[
F(z, x) = \frac{U - 2V}{1 - U + V} = \frac{(U - V) - V}{1 - (U - V)},
\]
with $U = \frac{p+1}{2}(1 + x)$ and $V = pzx^2$. Extracting the coefficient of $x^mz^n$ in $F(z, x)$, we obtain 
\[
f(n, m) = \sum_{i=0}^{\min\{\lambda t - \lambda\}} \frac{n}{t-i} \binom{n-i}{i} \binom{n-2i}{m-i} (-p)^i \left(\frac{p+1}{2}\right)^{n-2i}.
\]
Notice that $f(n, m) = f(n, n-m)$.

Summing up we have the following:

**Theorem 4.1.** Let $q = p^t$ be a prime power congruent to 1 modulo 4. Then the number of $p$-adic elementary divisors of $L($Paley$(q))$ which are equal to $p^\lambda$, $0 \leq \lambda < t$, is
\[
f(t, \lambda) = \sum_{i=0}^{\min\{\lambda t - \lambda\}} \frac{t}{t-i} \binom{t-i}{i} \binom{t-2i}{\lambda - i} (-p)^i \left(\frac{p+1}{2}\right)^{t-2i}.
\]
The number of $p$-adic elementary divisors of $L($Paley$(q))$ which are equal to $p^t$ is $(\frac{p+1}{2})^t - 2$.

**Example 4.2.** In [11], $K($Paley$(25))$ is calculated directly by a computer. Here as an illustration of Theorem 4.1 we use the above formula to compute $K($Paley$(5^3))$ and $K($Paley$(5^4))$:
\[
f(3, 0) = 3^3 = 27.
\]
\[
f(3, 1) = \binom{3}{1} \cdot 3^3 - \frac{3}{2} \binom{2}{1} \binom{1}{0} \cdot 5 \cdot 3 = 36.
\]
Therefore
\[
K($Paley$(5^3)) \cong (\mathbb{Z}/31\mathbb{Z})^{62} \oplus (\mathbb{Z}/5\mathbb{Z})^{36} \oplus (\mathbb{Z}/25\mathbb{Z})^{36} \oplus (\mathbb{Z}/125\mathbb{Z})^{25}.
\]
\[
f(4, 0) = 3^4 = 81.
\]
\[
f(4, 1) = \binom{4}{1} \cdot 3^4 - \frac{4}{3} \binom{3}{1} \binom{2}{0} \cdot 5 \cdot 3^2 = 144.
\]
\[
f(4, 2) = \binom{4}{2} \cdot 3^4 - \frac{4}{3} \binom{3}{1} \binom{2}{1} \cdot 5 \cdot 3^2 + \frac{4}{2} \binom{2}{0} \binom{0}{0} \cdot 5^2 = 176.
\]
Therefore
\[
K($Paley$(5^4)) \cong (\mathbb{Z}/156\mathbb{Z})^{312} \oplus (\mathbb{Z}/5\mathbb{Z})^{144} \oplus (\mathbb{Z}/25\mathbb{Z})^{176} \oplus (\mathbb{Z}/125\mathbb{Z})^{144} \oplus (\mathbb{Z}/625\mathbb{Z})^{79}.
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