NEUTRALLY EXPANDABLE MODELS OF ARITHMETIC

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Abstract. A subset of a model of PA is called neutral if it does not change the dcl relation. A model with undefinable neutral classes is called neutrally expandable. We study the existence and non-existence of neutral sets in various models of PA. We show that cofinal extensions of prime models are neutrally expandable, and \( \omega_1 \)-like neutrally expandable models exist, while no recursively saturated model is neutrally expandable. We also show that neutrality is not a first-order property. In the last section, we study a local version of neutral expandability.

1. Introduction

All models in this note are models of PA and their expansions. We will use \( M, N, K \), etc. for models of PA, and \( M, N, K \), etc. for their respective domains. We will write \( M \prec_{\text{end}} N \) if \( N \) is an elementary end extension of \( M \), \( M \prec_{\text{cof}} N \) if \( N \) is a cofinal extension of \( M \), and \( M \prec_{\text{cons}} N \) if \( N \) is an elementary conservative extension of \( M \).

A subset \( X \) of a model \( M \) is a class if for each \( a \in M \), \( \{ x \in X : M \models x < a \} \) is definable in \( M \). A subset \( X \) of \( M \) is inductive if \( (M, X) \models PA^* \), i.e. the induction schema holds in \( (M, X) \) for all formulas of the language of PA with a unary predicate symbol interpreted as \( X \). All inductive sets are classes.

Simpson proved that every countable model \( M \) has an inductive undefinable subset \( X \) such that \( (M, X) \) is pointwise definable \( \mathfrak{g} \). Enayat showed that there are non-prime models \( M \), such that for every undefinable class \( X \), \( (M, X) \) is pointwise definable \( \mathfrak{g} \). The first author called such models Enayat, and studied them in detail in \([1]\).

In search for a general notion of a generic subset of a model of PA, Alf Dolich suggested the definition below. Since the term “generic” has been already used in arithmetic context, we will use another name.

Definition 1. A subset \( X \) of \( M \) is neutral if for all \( a \) in \( M \), the definable closure of \( a \) in \( M \), \( \text{dcl}^M(a) \), and the definable closure of \( a \) in \( (M, X) \), \( \text{dcl}^{(M,X)}(a) \), are the same. We will call a model neutrally expandable if it has an undefinable neutral class.

For trivial reasons, prime models are neutrally expandable. Every subset of a prime model is neutral. In every model, 0-definable sets are neutral. It is less trivial that the standard cut is neutral in every model; this result follows from a generalization of a theorem of Kanovei [4], [6, Theorem 8.4.7]. Constructing neutral undefinable classes in models that are not prime is a harder task.

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2. Neutral Inductive Sets

If \( X \) is a neutral inductive subset of \( M \), then for each \( K \prec M \), \( K \) is closed under the Skolem terms of \( \mathcal{L}_{\text{PA}} \cup \{ X \} \), i.e. the language of \( \text{PA} \) with the unary relation symbol interpreted in \( M \) as \( X \). Hence, by the Tarski-Vaught test, we have the following characterization of inductive neutral sets.

**Proposition 2.** If \( X \) is an inductive subset of \( M \), then \( X \) is neutral iff for every \( K \prec M \), \( (K, X \cap K) \prec (M, X) \).

**Proposition 2** has an immediate corollary.

**Corollary 3.** If \( X \) is a neutral inductive subset of \( M \), and \( K \prec M \), then \( X \cap K \) is an inductive neutral subset of \( K \).

**Proposition 4.** If \( M \) is prime and \( M \prec_{\text{col}} N \), then \( N \) is neutrally expandable.

**Proof.** Let \( X \) be an undefinable inductive subset of \( M \). Such sets always exist; for example we can take \( X \) to be a generic subset of \( M \), see \cite{9} or \cite{13, Chapter 6}.

By the Kotlarski-Schmerl lemma \cite[Theorem 1.3.7]{6}, there is a unique \( Y \subseteq N \) such that \( (M, X) \prec (N, Y) \), and it follows that \( Y \) is an inductive subset of \( N \).

Suppose that for \( a \) and \( b \) in \( N \), \( a \in \text{dcl}((N, Y))(b) \). Then, there is a Skolem term \( t \) such that \( a = t(b, X) \). Fix \( c \in M \) such that \( a, b < c \). There is \( d \in M \) such that

\[
(M, X) \models \forall x, y < c [x = t(y, X) \iff x = (d)_y].
\]

Since \( (M, X) \prec (N, Y) \), it follows that \( a = (d)_b \), proving that \( a \in \text{dcl}(b) \).

Let us note that in the proof of **Proposition 4**, the witness \( Y \) to neutral expandability of \( N \) is a generic.

If \( M \) is countable, and \( X \subseteq M \) is coded in a countable elementary end extension \( N \), then \( X \) can be extended to a generic subset of \( N \). If \( X \) is generic, and \( Y \) is a generic subset of \( N \) extending \( X \), then \( (M, X) \prec (N, Y) \) (see \cite[Corollary 6.2.8]{6}).

An extension \( N \) of a model \( M \) is **superminimal** if for each \( a \in N \setminus M \), \( N = \text{dcl}(a) \).

It follows directly from definitions that if \( X \) is a neutral subset of \( M \), \( N \) is a superminimal elementary extension of \( M \), and \( (M, X) \prec (N, Y) \), then \( Y \) is a neutral subset of \( N \).

**Theorem 5** of \cite{8} implies that any inductive subset of a countable model can be coded in an elementary superminimal end extension. Hence, we have the following lemma.

**Lemma 5.** If \( X \) is a generic neutral subset of a countable model \( M \), then there is \( (N, Y) \) such that \( (M, X) \prec_{\text{end}} (N, Y) \), \( Y \) is neutral in \( N \), and \( N \) is a superminimal extension of \( M \).

**Theorem 6.** Every completion of \( \text{PA} \) has \( \omega_1 \)-like neutrally expandable models.

**Proof.** Start with any generic subset \( X_0 \) of a prime model \( M_0 \) and iterate **Lemma 5** along \( \omega_1 \). That is, let \( \{ (M_\alpha, X_\alpha) \}_{\alpha \in \omega_1} \) be a continuous elementary chain such that \( M_{\alpha + 1} \) is a superminimal elementary end extension of \( M_\alpha \), and \( X_\alpha \) is undefinable and neutral in \( M_\alpha \) that is coded in \( M_{\alpha + 1} \). Let \( N = \bigcup_{\alpha \in \omega_1} M_\alpha \) and \( X = \bigcup_{\alpha \in \omega_1} X_\alpha \).

Since each \( M_{\alpha + 1} \) is a superminimal extension of \( M_\alpha \), every elementary submodel of \( N \) is \( M_\alpha \) for some \( \alpha < \omega_1 \). Further, for each \( \alpha < \omega_1 \), \( X_\alpha = X \cap M_\alpha \), so
Neutral Classes

If \( X \subseteq N \) is inductive and neutral, and \( M \) is such that \( M \prec N \), then \( X \cap M \) is definable in \( M \) and it follows from Proposition 2 that \( X \) is definable in \( N \). We will show that the same result holds under the assumption that \( X \) is a class.

**Lemma 8.** If \( M \prec N \) and \( X \) is a neutral class of \( N \), then \( X \cap M \) is a class of \( M \).

**Proof.** Let \( X \) be a neutral class of \( N \). Let

\[
m(x) = \min \{ y : \forall z < x (z \in y \iff z \in X) \}.
\]

For each \( c \in N \), \( m(c) \) is well-defined in \( N \) because \( X \) is a class, and for \( c \in M \), \( m(c) \in M \) because \( X \) is a neutral class. \( \square \)

**Theorem 9.** Let \( X \) be a class of a model \( N \). If there is \( M \prec N \) such that \( M \) is a conservative extension of its prime submodel, then \( X \) is a neutral class iff \( X \) is 0-definable in \( N \).

**Proof.** Since all 0-definable sets are neutral, we only need to prove one direction of the theorem. Let \( M_0 \) be the prime elementary submodel of \( N \), and suppose that \( X \) is a neutral class of \( N \). By Lemma 8 \( X \cap M \) is a class of \( M \), and, since \( M_0 \prec \text{cons} \ M_0 \), it follows that \( X \cap M_0 \) is 0-definable. Let \( \varphi(x) \) be a formula defining \( X \cap M_0 \) in \( M_0 \). If \( \varphi(N) \neq X \), then

\[
e = \min \{ y : \varphi(y) \iff y \notin X \}
\]

is well-defined in \( N \) because \( X \) is a class, and \( e > M_0 \), which contradicts neutrality of \( X \). \( \square \)

For the proofs of the next result and the results in Section 4 we will use Gaifman’s minimal types. See Chapter 3 of [6], in particular Theorem 3.2.10, for all necessary background.

Since every recursively saturated model realizes minimal types and elementary end extensions generated by an element realizing a minimal type are conservative, we have the following corollary.

**Corollary 10.** If \( N \) has a recursively saturated elementary submodel, and \( X \) is a class of \( N \), then \( X \) is a neutral class iff \( X \) is 0-definable.

By a result of Kotlarski, Krajewski, and Lachlan, each countable recursively saturated model of \( \text{PA} \) has a full satisfaction class [7]. Satisfaction classes are never definable, and they may not be classes; however, Smith showed [10, Theorem 2.10]
that if $S$ is a full satisfaction class for a model $M$, then $M$ has an undefined class that is definable in $(M, S)$. This result and Theorem 9 imply that no full satisfaction class is neutral.

The following terminology was introduced by Smoryński [11]. A model $M$ is short if for some $a \in M$, $dcl(a)$ is cofinal in $M$. A model that is not short is tall. A model $M$ is short recursively saturated if it realizes all finitely realizable recursive types that contain a formula $x < b$ for some $b \in M$. It is easy to see that every tall short recursively saturated model is recursively saturated.

Let $N$ be recursively saturated. For $a \in N$ let $I(N; a)$ be the elementary submodel of $N$ whose domain is $\{x : \exists b \in dcl(a) N \models x < b\}$. All models of the form $I(N; a)$ are short recursively saturated. Every short, short recursively saturated model is of the form $I(N; a)$ for a recursively saturated $N$. This was proved for countable models in [11, Theorem C], and in full generality in [5, Theorem 5.1].

For every parameter free complete type $p(x)$ that is realized in $N$ by some $a > I(N; 0)$, the set of its realizations of $p(x)$ in $N$ is coinitial with $I(N; 0)$. Hence, Corollary 10 applies to all models $I(N; a)$ for $a > I(N; 0)$. Models of the form $I(N; 0)$ are exceptional. We know from Proposition 11 that $I(N; 0)$ is neutrally expandable. In fact, we can show more: if $I(N; 0)$ is nonstandard, it is an example of a model that has a neutral class that is not inductive.

**Proposition 11.** Let $N$ be short recursively saturated and let $N_0$ be the prime elementary submodel of $N$. Then, every subset of $N_0$ is neutral in $N$.

**Proof.** Let $X$ be a subset of $N_0$ and let $a, b \in N \setminus N_0$ be such that $a \notin dcl(b)$. If $N$ is uncountable, then instead of $N$ we can take a countable short recursively saturated model $N'$ such that $a, b \in N'$ and $(N', X) \prec (N, X)$. So let us assume that $N$ is countable.

If $N$ is tall, and hence recursively saturated, then there is an automorphism $f$ of $N$ that fixes $b$ and moves $a$. Since $f$ fixes $N_0$ pointwise and fixes $I(N; 0)$ setwise, $a \notin dcl(N, X)(b)$. If $N$ is short, then it has a countable recursively saturated elementary end extension $K$, and we can repeat the same argument working in $K$. \hfill \Box

In Proposition 11 instead of requiring that $N$ is short recursively saturated we could assume that for all $a, b \in N$ such that $a \notin dcl(b)$, there is an automorphism of $N$ fixing $b$ and moving $a$. Models with that property do not have to be short recursively saturated.

For an example of a non-prime model that has a neutral class that is not inductive, we can apply Proposition 11 to a countable short recursively saturated model $N$ that is a cofinal extension of $N_0$ and any unbounded $X \subseteq N_0$ of order type $\omega$.

**Problem 12.** Are there neutral non-inductive classes other than those given by Proposition 11?

Let $K$ and $N$ be recursively saturated models such that $K \prec_{cof} N$, and such that the standard cut is strong in $K$ but not in $N$. Let $K_0$ be the prime elementary submodel of $K$, and let $X$ be an unbounded subset of $K_0$ of order type $\omega$. Then, $X$ is a neutral class of $I(K; 0)$ and $I(N; 0)$, and since $\omega$ is definable (by the same definition) in both models, it follows that $(I(N; 0), X)$ is not an elementary extension of $(I(K; 0), X)$. This shows that Proposition 2 does not hold for classes, but still we have a weaker version of it that is a corollary of Lemma 3 and its proof.
Proposition 13. If $X$ is a neutral unbounded class of $\mathcal{N}$, then for each $K \prec \mathcal{N}$, $X \cap K$ is an unbounded class of $K$.

Proposition 13 may be prove to be useful, but it may also turn out to hold trivially if the answer to Problem 12 is negative.

Let us finish this part with an obvious question.

Problem 14. Are there neutrally expandable models other than those shown in this section?

3.1. Neutrality is not first-order. If $X$ is a neutral subset of a model $\mathcal{M}$, and $(\mathcal{M}, X) \prec (\mathcal{N}, Y)$, then $Y$ may not be neutral. An easy example is given by the standard model $\mathbb{N}$, an undefinable set of natural numbers $X$, and $(\mathbb{N}, Y)$ that is a recursively saturated elementary extension of $(\mathbb{N}, X)$. $Y$ is an inductive undefinable subset of $\mathbb{N}$, but it is not neutral, because $\mathbb{N}$, being recursively saturated, does not have neutral classes.

Another set of examples is given by the tall neutrally expandable models constructed in the proof of Theorem 6. It is not difficult to see that every recursively saturated model $\mathcal{M}$ (of any first-order theory), and any set $X$, $X$ is neutral iff $(\mathcal{M}, X) \models T$. We provide a (partial) negative answer here.

Corollary 15. There is no theory $T$ extending $\text{PA}$ such that for any recursively saturated $\mathcal{M}$ and any set $X$, $X$ is neutral iff $(\mathcal{M}, X) \models T$.

Proof. Suppose $T$ is such a theory. Corollary 10 implies that if $\mathcal{M}$ is a countable recursively saturated model of $\text{PA}$, then for any subset $X$ of $\mathcal{M}$,

$$X \text{ is 0-definable iff } (\mathcal{M}, X) \models T + X \text{ is a class.}$$

Let $\mathcal{M}$ be a recursively saturated model such that $T$ is coded by a set in the standard system of $\mathcal{M}$. Let $S$ be the theory $T$ extended by “$X$ is a class” and the sentences of the form $\exists x \neg\{x \in X \iff \varphi(x)\}$, for all formulas $\varphi(x)$ of the language of $\text{PA}$. Since $\mathcal{M}$ is resplendent and $S$ is in the standard system of $\mathcal{M}$, one can find an expansion $(\mathcal{M}, Y) \models S$ (see [6, Theorem 1.9.3]). Then, $Y$ is a neutral class and is not 0-definable, which is a contradiction.

4. A-Neutral Sets

Definition 16. Let $A \subseteq M$. A subset $X$ of $\mathcal{M}$ is called $A$-neutral if, for all $a, b \in A$, $a \in \text{dcl}(\mathcal{M})(b)$ iff $a \in \text{dcl}((\mathcal{M}, X))(b)$.

The two theorems below are a strengthening of the results we previously obtained for finite $A$, and for coded $\omega$-sequences. The proofs are due to Jim Schmerl, and they are included here with his kind permission.

Theorem 17. Let $\mathcal{M}$ be countable and recursively saturated and let $A$ be a bounded subset of $M$. Then $\mathcal{M}$ has an inductive, undefinable $A$-neutral subset $X$. 
Proof. Without loss of generality, we can assume that $A$ is an elementary cut of $\mathcal{M}$. Let $p(x)$ be a minimal type realized in $\mathcal{M}$, and let $C \subseteq M \setminus A$ be a cofinal set of realizations of $p(x)$. Let $N = \text{dcl}(A \cup C)$. Since minimal types are definable, we have $A \prec_{\text{cons}} N \prec_{\text{cof}} \mathcal{M}$. First we notice that if

$$Y = \{ x : N \models \varphi(x, \bar{a}) \},$$

where $a \in A$ and $\bar{e}$ is a tuple of elements of $C$, then $Y \cap A$ is definable using only $a$ as a parameter.

Let $G \subseteq \text{dcl}(C)$ be generic. By the Kotlarski-Schmerl lemma, there is $X \subseteq M$ such that $(\text{dcl}(C), G) \prec (\mathcal{M}, X)$. We notice that $G$ is cofinal in $X$ and, since $G$ is generic, $X$ is also generic. Let $a, b \in A$ be such that $a \in \text{dcl}^{(\mathcal{M}, X)}(b)$. That is, there is a formula $\varphi(x, y)$ in the language $L_{PA} \cup \{ X \}$ such that

$$(\mathcal{M}, X) \models \varphi(a, b) \land \exists x \varphi(x, b).$$

By the forcing lemma for arithmetic [4, Lemma 6.2.6], there is a relation $\models$ definable in $L_{PA}$ such that for each formula $\sigma(x, y)$ of $L_{PA} \cup \{ X \}$, and for all parameters $m, n \in M$, there is $p \in X$ such that

$$(\mathcal{M}, X) \models \sigma(m, n) \text{ iff } M \models p \models \sigma(m, n).$$

Let $p \in X$ be such that $M \models p \models [\varphi(a, b) \land \exists x \varphi(x, b)]$. Since $G$ is cofinal in $X$, there is $q \in G$ extending $p$, so $M \models q \models [\varphi(a, b) \land \exists x \varphi(x, b)]$. Let

$$Y = \{ \langle x, y \rangle : N \models q \models [\varphi(x, y) \land \exists z \varphi(z, y)] \}$$

Notice that $Y$ is definable using only the parameter $q \in G \subseteq C$, so $Y \cap A$ is definable in $A$ without parameters. Since $a$ is unique such that $\langle a, b \rangle \in Y \cap A$, $a \in \text{dcl}(b)$. □

We can also find cofinal subsets $A$ of countable, recursively saturated models for which there exist $A$-neutral sets.

**Proposition 18.** If $\mathcal{M}$ is countable and recursively saturated, then there are a cofinal $A \subseteq M$ and an undefinable inductive $X \subseteq M$ that is $A$-neutral.

**Proof.** Let $X$ be an inductive, undefinable subset of $M$ such that $(\mathcal{M}, X)$ is recursively saturated. Such $X$ exists by chronic resplendence of $\mathcal{M}$. Let $p(x)$ be a minimal type in the language $L_{PA} \cup \{ X \}$ realized in $(\mathcal{M}, X)$. Let $A$ be the set of realizations of $p(x)$ in $\mathcal{M}$. Then $A$ is cofinal in $X$. We will show that $X$ is $A$-neutral. To see this, let $a, b \in A$ and suppose $a \in \text{dcl}^{(\mathcal{M}, X)}(b)$. By Ehrenfeucht’s Lemma [2] (see also [6, Theorem 1.7.2]), if $a \neq b$, then $\text{tp}^{(\mathcal{M}, X)}(a) \neq \text{tp}^{(\mathcal{M}, X)}(b)$. Since $a$ and $b$ realize the same type, this means $a = b$. □

In the previous result, we found a particular cofinal subset $A$ of a recursively saturated model $\mathcal{M}$ for which there exists an $A$-neutral inductive subset of $\mathcal{M}$. Our last question asks if there are cofinal elementary submodels $\mathcal{A}$ of $\mathcal{M}$ for which we have $A$-neutral inductive subsets.

**Problem 19.** If $\mathcal{M}$ is recursively saturated, is there $\mathcal{A} \prec_{\text{cof}} \mathcal{M}$ such that $\mathcal{M}$ has an undefinable $A$-neutral inductive subset?
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