THE FIBERWISE INTERSECTION THEORY

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Abstract. We define a bordism invariant for the fiberwise intersection theory. Under some certain conditions, this invariant is an obstruction for the theory.

1. Introduction

We start with the following assumptions for the intersection theory;

(i) Let $P^p$ and $M^m$ be smooth manifolds. Suppose $Q^q \subseteq M^m$ is a closed submanifold and $f : P \to M$ smooth map such that $f \pitchfork Q$ in $M$.

(ii) Let $E(f,i_Q) :=$ the homotopy pullback of $[P \xrightarrow{f} M \xleftarrow{i_Q} Q]$. We have a smooth map $f^{-1}(Q) \to E(f,i_Q)$ and the bundle data determines an element in $\Omega^{fr}_{p+q-m}(E(f,i_Q))$.

(iii) Let $N^{p+q-m}$ be a submanifold of $P$ and let $N \to E(f,i_Q)$ be a map which represents the same such an element in $\Omega^{fr}_{p+q-m}(E(f,i_Q))$.

In 1974, Allan Hatcher and Frank Quinn [H-Q] showed in their work that if $f$ is an immersion and assume $m > q + \frac{p}{2} + 1, m > p + \frac{q}{2} + 1$, then we can homotope a map $f$ the the new map $g$ so that $g^{-1}(Q) = N$. We develop their result to the case where $f$ is any smooth map and also weaken the dimension condition as follows; (See [Sun] for more details.)

**Theorem 1.** (Classical version) Given a smooth map $f : P^p \to M^m$ and $Q^q$ is a closed submanifold of $M$. Assume $m > q + \frac{p}{2} + 1$. Then there is a map $g$ homotopic to $f$ such that $g^{-1}(Q) = N$.

Our proof is using the bundle data to construct the required homotopy step by step. In this paper, we proceed along the same lines as the proof of the classical case to get the result for the fiberwise case which we will describe it in the next section.

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2. Intersection Theory. (Fiberwise version)

(I) Suppose that $E^{p+k}_P, E^{q+k}_Q$ and $E^{m+k}_M$ are smooth fiber bundles over a compact manifold $B^k$. Let $f : E^{p+k}_P \to E^{m+k}_M$ be a bundle map and $E_Q$ be a subbundle of $E_M$ with the inclusion bundle map $i_Q : E_Q \hookrightarrow E_M$.

We have a commutative diagram

\begin{align}
& P^p \quad M^m \quad Q^q \\
\downarrow & \quad \downarrow & \quad \downarrow \\
E_P \xrightarrow{f} E_M \xleftarrow{i_Q} E_Q \\
\downarrow{pr_P} & \quad \downarrow{pr_M} & \quad \downarrow{pr_Q} \\
B
\end{align}

(2.1)

where $P, Q$ and $M$ are the fibers of $pr_P, pr_Q$ and $pr_M$, respectively.

We may assume that $f \cap E_Q$ in $E_M$ (See [Koz]).

The homotopy pullback is

$E(f, i_Q) := \{(x, \lambda, y) \in E_P \times E'_M \times E_Q \mid \lambda(0) = f(x), \lambda(1) = y\}$.

We have a diagram which commutes up to homotopy

\begin{align}
E(f, i_Q) \xrightarrow{\pi_Q} E_Q \\
\downarrow{\pi_P} & \quad \downarrow{\pi_I} \\
E_P \xrightarrow{f} E_M \\
\downarrow{f^{-1}(E_Q)} & \quad \downarrow{E_Q} \\
f^{-1}(E_Q)
\end{align}

(2.2)

where $\pi_P$ and $\pi_Q$ are the trivial projections, i.e. we have a homotpy

$E(f, i_Q) \times I \xrightarrow{K} E_M$ defined by $K(x, \lambda, y, t) = \lambda(t)$.

We also have a map $c : f^{-1}(E_Q) \to E(f, i_Q)$ defined by $x \mapsto (x, c_{f(x)}, f(x))$

where $c_{f(x)} = \text{constant path in } E_M$ at $f(x)$. Note that $E(f, i_Q)$ and $f^{-1}(E_Q)$ are not necessarily the fiber bundles over $B$.

Transversality yields a bundle map

\begin{align}
\nu_{f^{-1}(E_Q)} \subseteq E_P \xrightarrow{\nu_{E_Q} \subseteq E_M} \\
\downarrow & \quad \downarrow{f_{f^{-1}(E_Q)}} \\
\nu_{f^{-1}(E_Q)} \subseteq E_P & \quad \downarrow{f^{-1}(E_Q)} \xrightarrow{E_Q}
\end{align}

(2.3)
Choose an embedding $E_P^{p+k} \subseteq S^{p+k+l}$, for sufficiently large $l$. Then we get $f^{-1}(E_Q) \hookrightarrow E_P^{p+k} \subseteq S^{p+k+l}$. So $\nu_{f^{-1}(E_Q) \subseteq S^{p+k+l}} \cong \nu_{f^{-1}(E_Q) \subseteq E_P} \oplus i^* \nu_{E_P \subseteq S^{p+k+l}}$.

The commutative diagram

\[
\begin{array}{ccc}
f^{-1}(E_Q) & \xrightarrow{f_{f^{-1}(E_Q)}} & E(f, i_Q) \\
\downarrow c & & \downarrow \pi_Q \\
E(f, i_Q) & \xrightarrow{\pi_P} & E_P \\
\end{array}
\]

yields a bundle map

\[
\nu_{f^{-1}(E_Q) \subseteq S^{p+k+l}} \xrightarrow{\hat{c}} \pi^*(\nu_{E_M \subseteq E_M}) \oplus \pi^*(\nu_{E_P \subseteq S^{p+k+l}}) := \xi
\]

(2.5)

\[
f^{-1}(E_Q) \xrightarrow{c} E(f, i_Q)
\]

Thus $(c, \hat{c})$ determines an element $[c, \hat{c}] \in \Omega_{p+q+k-m}^f(E(f, i_Q); \xi)$.

(II) Suppose that $(N \xrightarrow{c} E(f, i_Q), \nu_{N \subseteq S^{p+k+l}} \xrightarrow{\hat{c}} \xi)$ is another representative of $[c, \hat{c}]$, where $N^{p+q+k-m} \subseteq E_P^{p+k} \subseteq S^{p+k+l}$. This means we have a normal bordism $(W \xrightarrow{\hat{c}} E(f, i_Q), \nu_{W} \xrightarrow{\hat{c}_1} \xi)$ between $(c, \hat{c})$ and $(c_1, \hat{c}_1)$, i.e.

(\(i\)) $W^{p+q+k-m+1} \subseteq (S^{p+k+l} \times I)$,
(\(ii\)) $\partial W \subseteq (S^{p+k+l} \times \partial I)$,
(\(iii\)) $W \cap (S^{p+k+l} \times \partial I)$,
(\(iv\)) $W \cap (S^{p+k+l} \times 0) = f^{-1}(E_Q)$ and $W \cap (S^{p+k+l} \times 1) = N$

such that

$C_{f^{-1}(E_Q)} = c : f^{-1}(E_Q) \to E(f, i_Q)$, $C_N = c_1 : N \to E(f, i_Q)$

$\hat{c}_{\nu_{f^{-1}(E_Q) \subseteq S^{p+k+l}}} = \hat{c}$, $\hat{c}_{\nu_N \subseteq S^{p+k+l}} = \hat{c}_1$.

**Theorem 2.** Let $M^n$ and $N^n$ be smooth manifolds and $f : M \to N$ be a smooth map. If $n > 2m$, then $f$ is homotopic to an embedding $g : M \to N$.

**Proof.** See [Muk].
Lemma 1. Let $f : M^m \to N^n$ be a map between two smooth manifolds. Let $A$ be a closed submanifold of $M$. Assume that $f|_A$ is an embedding.

If $n > 2m$, then $f$ is homotopic to an embedding $g$ relative to $A$.

Proof. Let $T$ be a tubular neighborhood of $A$ in $M$.

Step I Extend the embedding $f|_A : A \to N$ to an embedding $f_T : T \to N$.

Let $\nu(A, M)$ be the normal bundle of $A$ in $M$ and $D(\nu)$ denote the disc bundle of $\nu$. Then the tubular neighborhood theorem implies $D(\nu(A, M)) \cong T$.

Claim: For any given an embedding $A \xrightarrow{g} N$ and a vector bundle $\eta$ over $A$. Then

\[
\left( g \text{ extends to an embedding of } D(\eta) \text{ into } N \right) \iff \left( \text{There exists a bundle monomorphism } \phi : \eta \to \nu(g) \right)
\]

where $\nu(g)$ is the normal bundle of $A$ in $N$ via $g$.

Proof Claim

$(\Leftarrow)$ We have a diagram

\[
\begin{array}{ccc}
D(\eta) & \xrightarrow{\phi} & D(\nu(g)) \\
\downarrow{\exp} & & \downarrow{\cong} \\
N & \cong & A
\end{array}
\]

where $\exp$ is the exponential map.

Note that $\exp(D(\nu(g))) \cong$ tubular neighborhood of $A$ in $N$ via $g$. Then $\exp \circ \phi : D(\eta) \to N$ is a desired embedding.

$(\Rightarrow)$ Assume there exists an embedding $g_T$ so that the following diagram commutes

\[
\begin{array}{ccc}
D(\eta) & \xrightarrow{g_T} & N \\
\downarrow{i} & & \downarrow{g} \\
A & \xrightarrow{g} & A
\end{array}
\]

Then $\nu(g) \cong \eta \oplus i^*\nu(g_T)$. 

We are in the situation that we have a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f|_{A}=g} & N \\
\downarrow^{i} & & \downarrow^{f} \\
M & \xrightarrow{f} & N
\end{array}
\]  

(2.8)

Let \( \nu(f) := f^{*}\tau_{N} - \tau_{M} \). Then \( i^{*}(\nu(f)) \oplus \nu(A, M) \overset{\text{stable}}{\cong} \nu(g) \).

If \( n - a > a \), then \( i^{*}(\nu(f)) \oplus \nu(A, M) \cong \nu(g) \), so there exists a bundle monomorphism

\[
\begin{array}{ccc}
\nu(A, M) & \xrightarrow{\cdot} & \nu(g) \\
\downarrow & & \downarrow \\
A
\end{array}
\]  

(2.9)

Apply Claim when \( g = f|_{A} \) and \( \eta = \nu(A, M) \), then we have an extension embedding of \( g \) from \( D(\nu(A, M)) \cong T \overset{f_{T}}{\hookrightarrow} N \).

**Step 2** We have a map \( f|_{M - \text{int}(T)} : M \setminus \text{int}(T) \to N \) and \( \partial(M \setminus \text{int}(T)) = \partial T \).

\( b > (c + \frac{a}{2} + 1) \) and Theorem 2 \( \Rightarrow f|_{M - \text{int}(T)} \) is homotopic to an embedding \( g_{M - \text{int}(T)} \).

Define \( g : M \to N \) by

\[
g(x) = \begin{cases} 
g_{M - \text{int}(T)}(x) & \text{if } x \in M \setminus \text{int}(T) \\
g_{T}(x) & \text{if } x \in T. \end{cases}
\]

Then \( f \) is homotopic to \( g \) relative to \( A \).  

\( \square \)
**Theorem 3.** Assume \( m > q + \left(\frac{p+k}{2}\right) + 1 \). Then there exists a smooth map over \( B \)

\[
\Psi : E_P \times I \to E_M
\]

such that \( \Psi_{|E_P \times \{0\}} = f \), \( \Psi \cap E_Q \) and \( \Psi_{|E_P \times \{1\}}^{-1}(E_Q) = N \).

Note that if we let \( g = \Psi_{|E_P \times \{1\}} \). Then \( g \) is fiber-preserving homotopic to \( f \) and \( g^{-1}(E_Q) = N \).

**Proof.** We divide the proof into 3 steps,

**Step 1:** Goal: Homotope the map \( W \xrightarrow{a:= \pi_P \circ c} E_P \) to an embedding over \( B \).

By assumption, we have

\[
W \xrightarrow{c} E(f,i_Q) \subseteq E_P \times E_M^I \times E_Q.
\]

and we also have maps

\[
W \xrightarrow{a:= \pi_P \circ c} E_P, \quad W \xrightarrow{b:= \pi_Q \circ c} E_Q, \quad W \times I \xrightarrow{H} E_M
\]

where \( H := K \circ (c \times id_I) \), so \( H_{|W \times 0} = f \circ a, H_{|W \times 1} = b \).

Recall that \( \partial W = f^{-1}(E_Q) \cup N \), \( a_{|\partial W} \) is just the inclusion of \( f^{-1}(E_Q) \) and \( N \) into \( E_P \).

Apply the condition \( m > q + \frac{p+k}{2} + 1 \) to Lemma 1, there exists an embedding \( A \simeq a \) (rel \( \partial W \)), i.e. we have a commutative diagram

\[
\begin{array}{ccc}
W \times 1 & \xrightarrow{A} & A \\
\downarrow & & \downarrow \\
W \times I & \xrightarrow{L} & E_P \\
\downarrow & & \downarrow \\
W \times 0 & \xrightarrow{a} & E_P \\
\end{array}
\]

We have a map \( W \xrightarrow{b:= \pi_Q \circ c} E_Q \). By concatenating the homotopy \( H \) and \( f \circ L \) together, we get a commutative diagram
Thus the following diagram commutes up to homotopy

\[
\begin{array}{c}
\begin{array}{c}
W \times 2 \xrightarrow{b} E_Q \\
\downarrow i_Q \\
W \times [1, 2] \xrightarrow{V} E_M \\
\downarrow f \\
W \times 1 \xrightarrow{A} E_P \\
\end{array}
\end{array}
\]

Next, we want to modify the homotopy \( V \) such that it is fiber preserving with respect to \( pr_M \).

Note that we have a commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
W \xrightarrow{b} E_Q \\
\downarrow A \\
E_P \xrightarrow{f} E_M \\
\end{array}
\end{array}
\]

We can apply the homotopy lifting property for \( pr_Q \) to get a homotopy of \( b \) to \( b' \) through \( V' \) such that the following diagram commute

\[
\begin{array}{c}
\begin{array}{c}
W \times 2 \xrightarrow{b'} E_Q \\
\downarrow V' \\
W \times [1, 2] \xrightarrow{pr_M \circ V} B \\
\end{array}
\end{array}
\]

Let \( \Psi_W := b' : W \to E_Q \). Then \( \Psi_W \) is a bundle map over \( B \) through the lifting \( V' \).

**Step 2** Goal: Construct a bundle isomorphism

\[
\nu(A) \oplus \epsilon^1 \simeq b'^* (\nu_{E_Q \subseteq E_M})
\]
where $\epsilon^1$ is the trivial bundle. Since $\text{dim} W < \text{rank } \nu(A)$, it is enough to give a stable equivalence between such bundles.

Now, we have
\[ W \xhookrightarrow{A} E_P \subseteq S^{p+k+I} \implies \nu_{W \subseteq S^{p+k+I}} \cong \nu(A) \oplus A^* (\nu_{E_P \subseteq S^{p+k+I}}). \tag{2.13} \]

We also have a commutative diagram
\[
\begin{array}{ccc}
W & \xrightarrow{\nu} & E(f, i_Q) \\
\downarrow{C} & & \downarrow{\pi_Q} \\
A & \xrightarrow{\pi_P} & E_P
\end{array}
\]
\tag{2.14}

Thus the bundle map $\hat{C} : \nu_{W \subseteq S^{p+k+I} \times I} \to \xi = \pi_P^* (\nu_{E_P \subseteq S^{p+k+I}}) \oplus \pi_Q^* (\nu_{E_Q \subseteq E_M})$ yields the following stable isomorphism
\[ \nu_{W \subseteq S^{p+k+I}} \oplus \epsilon^1 \cong stable (\pi_P \circ C)^* (\nu_{E_P \subseteq S^{p+k+I}}) \oplus (\pi_Q \circ C)^* (\nu_{E_Q \subseteq E_M}). \tag{2.17} \]

Putting (2.13), (2.15) (2.16)and (2.17) together, we get
\[ \nu(A) \oplus (\pi_P \circ C)^* (\nu_{E_P \subseteq S^{p+k+I}}) \oplus \epsilon^1 \cong stable (\pi_P \circ C)^* (\nu_{E_P \subseteq S^{p+k+I}}) \oplus b'^* (\nu_{E_Q \subseteq E_M}). \tag{2.18} \]

Consequently, we have
\[ \nu(A) \oplus \epsilon^1 \cong b'^* (\nu_{E_Q \subseteq E_M}). \tag{2.19} \]

This implies that we did construct a bundle map
\[
\begin{array}{ccc}
\nu(A) \oplus \epsilon^1 & \xrightarrow{b'} & \nu_{E_Q \subseteq E_M} \\
W & \xrightarrow{b'} & E_Q
\end{array}
\]
\tag{2.20}

which give us the extension of map $b'$ to the tubular neighborhood of $W$ in $E_P$. More precisely,
\[ \Psi_T : D(\nu(A)) \hookrightarrow D(\nu(A) \oplus \epsilon^1) \xrightarrow{\hat{\psi}_T} D(\nu_{EQ \subseteq EM}) \]

where \( D \) denotes the disc bundle. Note that \( \Psi_T \upharpoonright E_Q \) and \( \Psi_T(\partial D(\eta_1)) \subseteq E_M \setminus E_Q \).

Since \( \nu(A) \oplus \epsilon^1 \cong b^*(\nu_{EQ \subseteq EM}) \), we can find a subbundle \( \eta_2 \) of \( b^*(\nu_{EQ \subseteq EM}) \) such that \( \eta_2 \cong \nu(A) \). For simplicity, let \( \eta_1 := \nu(A) \).

**Step 3.** Goal: Construct the smooth map \( \Psi : E_P \times I \rightarrow E_M \) over \( B \).

Recall that we have

\[ W \hookrightarrow D(\nu(A)) \cong D(\nu(A) \oplus \epsilon^1) \cong D(b^*(\nu_{EQ \subseteq EM})). \]

Then there exists a neighborhood \( \tilde{D} \) of \( W \) in \( D(b^*(\nu_{EQ \subseteq EM})) \) such that \( \tilde{D} \cong D(b^*(\nu_{EQ \subseteq EM})) \) and \( \tilde{D} \cong D(\nu(A)) \).

According to (2.20), there exists a bundle \( \eta \) over \( W \times I \) such that \( \eta|_{W \times i} = \eta_i \) for \( i = 1, 2 \).

Let \( \eta_1 = \nu(A \oplus \epsilon^1) \), \( \eta_2 = b^*(\nu_{EQ \subseteq EM}) \).

According to (2.20), there exists a bundle \( \eta \) over \( W \times I \) such that \( \eta|_{W \times i} = \eta_i \) for \( i = 1, 2 \).

Let \( D_1 := D(\nu(A)) \), \( D_2 := \tilde{D} \).

Then \( D_i \hookrightarrow D(\eta_i) \) is a homotopy equivalence for \( i = 1, 2 \) and also \( D_1 \cong D_2 \).

Since \( D_1 \cup W \times [1, 2] \cup D_2 \hookrightarrow D_1 \times [1, 2] \) is a cofibration and a homotopy equivalence, there exist an extension \( D_1 \times [1, 2] \xrightarrow{\hat{\psi}_1} E_M \times [1, 2] \) such that the following diagram commutes

\[
\begin{array}{ccc}
W \times 2 & \xrightarrow{V'_{|W \times 2}} & E_M \times 2 \\
\downarrow & & \downarrow \\
W \times [1, 2] & \xrightarrow{\Psi_1} & E_M \times [1, 2] \\
\downarrow & & \downarrow \\
W \times 1 & \xrightarrow{V'_{|W \times 1}} & E_M \times 1
\end{array}
\]
Next we want to construct an embedding $W \xhookrightarrow{A'} D_2 \times [2, 3]$ such that the following hold:

1. $A'(W) \cap \{D_2 \times 2\} = f^{-1}(E_Q)$
2. $A'(W) \cap \{D_2 \times 3\} = N$
3. $A' \cap D_2 \times \partial[2, 3]$

We start by letting $\alpha : W \to [2, 3]$ be a smooth map such that $\alpha \cap \partial[2, 3], \alpha^{-1}(2) = f^{-1}(E_Q)$ and $\alpha^{-1}(3) = N$, we also have an inclusion $W \xhookrightarrow{i_W} D_2$.

Let $A' := i_W \times \alpha$. Then $A'$ is such a required map.

By the construction, we have

$$D(\nu(A')) \cong D_2 \times [2, 3].$$

Let $\psi_2$ be the composition of the maps

$$W \xhookrightarrow{A'} D_2 \times [2, 3] \xrightarrow{\Psi_T|_{D_2 \times [2, 3]}} M \times [2, 3] \xrightarrow{\text{proj}} M$$

Define a map $\Psi_2 := \psi_2 \times \alpha : W \to E_M \times [2, 3]$.

Using the fact that $D(\nu(A')) \cong D_2 \times [2, 3]$, then $D_2 \times \partial[2, 3] \cup A'(W) \hookrightarrow D_2 \times [2, 3]$ is a cofibration and homotopy equivalence. Hence there exist an extension $D_2 \times [2, 3] \xrightarrow{\Psi_T} E_M \times [2, 3]$.

Note that for $(x, 3) \in D_2 \times 3$ such that $\Psi_2(x, 3) \in E_Q \times 3$, the map $\Psi_2|_{D_2 \times 3} = \Psi_T$ forces that $x$ has to be in $W$, so by the definition of $\Psi_2$ implies $x \in N$. Thus $\hat{\Psi}_2^{-1}(E_Q) = N$.

We define a map $\tilde{\Psi} : \{E_P \times I\} \cup \{D_1 \times [1, 2]\} \cup \{D_2 \times [2, 3]\} \to E_M \times [0, 3]$ by

$$\tilde{\Psi}(p, t) = \begin{cases} (f(p), t) & \text{if } t \in [0, 1] \\ (\Psi_1(p), t) & \text{if } t \in [1, 2] \\ (\Psi_2(p), t) & \text{if } t \in [2, 3]. \end{cases} \quad (2.23)$$
Then \( \tilde{\Psi} \) is well-defined map over \( B \) by the construction.

It’s not hard to see that \( \{ E_P \times I \} \cup \{ D_1 \times [1, 2] \} \cup \{ D_2 \times [2, 3] \} \) is diffeomorphic to \( E_P \times I \). Define the map \( \Psi \) to be the composition of maps

\[
E_P \times I \xrightarrow{\tilde{\Psi}} \{ E_P \times I \} \cup \{ D_1 \times [1, 2] \} \cup \{ D_2 \times [2, 3] \} \xrightarrow{\Psi} E_M \times [0, 3] \xrightarrow{\text{proj}} E_M
\]

where \( \text{proj} \) is the projection to the first factor.

Thus, we get a map \( \Psi : E_P \times I \rightarrow E_M \) over \( B \) so that \( \Psi|_{E_P \times 0} = f \). By construction, \( \Psi \cap E_Q \) and \( \Psi^{-1}|_{E_P \times 1}(E_Q) = N \) as required.

\[ \square \]

**Corollary 1.** Assume \( m > q + \left( \frac{p+k}{2} \right) + 1 \). Then we can fiber-preserving homotope a map \( f \) to the map that its image does not intersect \( E_Q \) if and only if \( [c, \hat{c}] = 0 \in \Omega^{fr}_{p+q+k-m}(E(f,i_Q); \xi) \).

### 3. Application to fixed point theory.

Let \( p : M^{m+k} \rightarrow B^k \) be a smooth fiber bundle with compact fibers and \( k > 2 \). Assume that \( B \) is a closed manifold. Let \( f : M \rightarrow M \) be a smooth map over \( B \), i.e. \( p \circ f = p \).

The fixed point set of \( f \) is

\[
\text{Fix}(f) := \{ x \in M \mid f(x) = x \}.
\]

We have a homotopy pull-back diagram

\[
\begin{array}{ccc}
L_fM & \xrightarrow{\text{ev}_0} & M \\
\downarrow \text{ev}_1 & & \downarrow \Delta \\
M & \xrightarrow{\Delta_f} & M \times_B M
\end{array}
\] (3.24)

where

(i) \( L_fM := \{ \alpha \in M^I \mid f(\alpha(0)) = \alpha(1) \} \),

(ii) \( \text{ev}_0 \) and \( \text{ev}_1 \) are the evaluation map at 0 and 1 respectively,

(iii) \( \Delta := \) the diagonal map, defined by \( x \mapsto (x, x) \),

(iv) \( \Delta_f := \) the twisted diagonal map, defined by \( x \mapsto (x, f(x)) \),

(v) \( M \times_B M := \) the fiber bundle over \( B \) with fiber over \( b \in B \), given by \( F_b \times F_b \) where \( F_b \) is the fiber of \( p \) over \( b \).
**Proposition 1.** There exists a homotopy from $f$ to $f_1$ such that $\triangle f_1 \pitchfork \triangle$.

The proof relies on the work of Kozniowski [Koz], relating to $B$-manifolds. Let $B$ be a smooth manifold. A $B$-manifold is a manifold $X$ together with a locally trivial submersion $p : X \to B$. A $B$-map is a smooth fiber-preserving map.

**Lemma 2.** Let $X$ and $Y$ be $B$-manifolds and $Z$ be a $B$-submanifold of $Y$. Let $g : X \to Y$ be a $B$-map. Then there is a fiber-preserving smooth $B$-homotopy $H : X \to Y$ such that $H_0 = g$ and $H_1 \pitchfork Z$.

**Proof.** See [Cou] for the proof. □

We have a transversal (pullback) square

$$
\begin{array}{ccc}
Fix(f_1) & \xrightarrow{i} & M \\
\downarrow i & & \downarrow \triangle \\
M & \xrightarrow{\triangle f_1} & M \times M
\end{array}
$$

(3.25)

where $i$ is the inclusion. Transversality yields that $\nu(i) \cong i^*(\nu(\triangle)) \cong i^*(\tau M)$.

Choose an embedding $M \hookrightarrow S^{m+k}$ for sufficiently large $k$. Then we have

$$
\nu_{Fix(f_1) \subseteq S^{m+k}} \cong \nu(i) \oplus i^*(\nu_{M \subseteq S^{m+k}}) \cong i^*(\tau M) \oplus i^*(\nu_{M \subseteq S^p}) \cong \epsilon.
$$

We denote this bundle isomorphism by $\nu_{Fix(f_1) \subseteq S^{m+k}} \xrightarrow{\hat{g}} \epsilon$. We also have a map $Fix(f_1) \xrightarrow{\hat{g}} \mathcal{L}_f M$ defined by $x \mapsto c_x$, where $c_x$ is the constant map at $x$. Thus $L^{bord}(f) := [Fix(f_1), g, \hat{g}]$ determines the element in $\Omega^fr_k(\mathcal{L}_f M; \epsilon)$.

Applying Theorem 3, we obtain the following corollary;

**Corollary 2.** (Converse of fiberwise Lefschetz fixed point theorem)

Let $f : M^{m+k} \to M^{m+k}$ be a smooth bundle map over the closed manifold $B^k$. Assume that $m \geq k + 3$. Then $f$ is fiber homotopic to a fixed point free map if and only if $L^{bord}(f) = 0 \in \Omega^fr_k(\mathcal{L}_f M; \epsilon)$.

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