Tighter monogamy and polygamy relations for a superposition of the generalized W-class state and vacuum

Le-Min Lai1, Shao-Ming Fei1,2,* and Zhi-Xi Wang1,*

1 School of Mathematical Sciences, Capital Normal University, Beijing 100048, People’s Republic of China
2 Max Planck Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

E-mail: feishm@cnu.edu.cn and wangzhx@cnu.edu.cn

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Abstract
Monogamy and polygamy relations characterize the distributions of entanglement in multipartite systems. We investigate the monogamy and polygamy relations with respect to any partitions for a superposition of the generalized W-class state and vacuum in terms of the Tsallis-q entanglement and the Rényi-α entanglement. By using the Hamming weight of the binary vectors related to the partitions of the subsystems, new classes of monogamy and polygamy inequalities are derived, which are shown to be tighter than the existing ones. Detailed examples are presented to illustrate the finer characterization of entanglement distributions.

Keywords: monogamy relation, polygamy relation, Tsallis-q entanglement, Rényi-α entanglement, generalized W-class state

1. Introduction
Quantum entanglement [1–3] is a quintessential feature of quantum mechanics which distinguishes the quantum from the classical world and plays an important role in quantum information processing. One distinguished property of quantum entanglement without any classical counterpart is its limited shareability in multipartite quantum systems, known as the monogamy of entanglement (MoE) [4, 5]. MoE is the fundamental ingredient in many quantum information processing tasks such as the security proof in quantum cryptographic scheme [6] and the security analysis of quantum key distribution [7].

*Authors to whom any correspondence should be addressed.
For a tripartite quantum state $\rho_{ABC}$ with its reduced density matrices $\rho_{AB} = \text{Tr}_C \rho_{ABC}$ and $\rho_{AC} = \text{Tr}_B \rho_{ABC}$, mathematically MoE can be characterized in terms of some bipartite entanglement measure $\varepsilon$ as $\varepsilon(\rho_{A|BC}) \geq \varepsilon(\rho_{AB}) + \varepsilon(\rho_{AC})$, where $\varepsilon(\rho_{A|BC})$ denotes the shared entanglement between subsystems $A$ and $BC$, which measures the degree of entanglement between $A$ and $BC$, and $\varepsilon(\rho_{AB})$ ($\varepsilon(\rho_{AC})$) is the bipartite entanglement between $A$ and $B$ ($A$ and $C$). This inequality conveys the MoE principle that the amount of entanglement shared between $A$ and $B$ restricts the possible amount of entanglement between $A$ and $C$ so that their sum does not exceed the total bipartite entanglement between $A$ and the composite $BC$ system. Note that the monogamy inequalities provide an upper bound for bipartite shareability of entanglement in a multipartite system. It is also known that the assisted entanglement $\varepsilon^a$ [8, 9], which is a dual amount to bipartite entanglement measures, has a dually monogamous property in multipartite systems. This dually monogamous property of entanglement is also characterized as a polygamy inequality, which is quantitatively displayed as $\varepsilon^a(\rho_{A|BC}) \leq \varepsilon^a(\rho_{AB}) + \varepsilon^a(\rho_{AC})$ for a tripartite system, where $\varepsilon^a(\cdot)$ is the corresponding entanglement measure of assistance associated to $\varepsilon$. Similarly, a polygamy inequality sets a lower bound for the distribution of bipartite entanglement in multipartite systems.

The first monogamy relation was proven by Coffman et al [10] based on the squared concurrence for arbitrary three-qubit states, known as the CKW inequality. Later, it was generalized to multipartite systems [11, 12]. Besides concurrence, the monogamy relations are also given by various entanglement measures for multipartite systems [13–24]. The polygamy relation was first obtained in terms of the tangle of assistance for three-qubit systems [8], and then generalized to multiqubit systems and arbitrary dimensional multipartite systems [24–31]. However, it was found that the CKW inequality is invalid for higher-dimensional systems [32]. In [33] the authors discovered that in some higher-dimensional systems there is no non-trivial monogamy relation satisfied by any additive entanglement measures. It seems that only the squashed entanglement satisfies the monogamy relation for arbitrary dimensional systems [34]. Therefore, the MoE for high dimensional systems has attracted much attention.

In [35] Kim et al proved that the $n$-qudit generalized $W$-class (GW) states satisfy the monogamy inequality in terms of the squared concurrence. In [36] Choi and Kim showed that the superposition of the generalized $W$-class states and vacuum (GWV) states satisfy the strong monogamy inequality based on the squared convex roof extended negativity. In [37] Kim focused on a large class of mixed states that are in a partially coherent superposition of a generalized $W$-class state and the vacuum, and showed that those states obey the strong monogamy inequality by using the squared convex roof extended negativity. Very recently, Shi et al presented in [38] new monogamy and polygamy relations with respect to any partition for $n$-qudit GWV states by using the analytical formula of the Tsallis-$q$ entanglement ($T_qE$) [25]. Moreover, Liang et al presented in [39] the monogamy and polygamy relations for GWV states in terms of the Rényi-$\alpha$ entanglement ($R_{\alpha}E$) [40, 41]. Inspired by these developments, we investigate further the monogamy and polygamy relations for the GWV states in high dimensional quantum systems.

In this paper, by using the Hamming weight of the binary vector associated with the distribution of subsystems, we establish a class of monogamy and polygamy relations for the GWV states based on $T_qE$ and $R_{\alpha}E$. We derive monogamy and polygamy inequalities which are tighter than those given in [38, 39], thus giving rise to finer characterizations of the entanglement distributions among the high dimensional quantum subsystems for the GWV states.
2. Tighter monogamy and polygamy relations based on $T_qE$ and $T_qEoA$ for GWV states

A class of $n$-qubit $W$-class states and $n$-qudit generalized $W$-class states are, respectively, defined by

$$|\psi\rangle_{A_1A_2...A_n} = a_1|0...0\rangle + a_2|01...0\rangle + ... + a_n|00...1\rangle$$  \hspace{1cm} (1)

and

$$|W_n^{(d)}\rangle_{A_1...A_n} = \sum_{i=1}^{d-1} (a_{1i}|0...0\rangle + a_{2i}|0\ 0i...0\rangle + ... + a_{ni}|00...0i\rangle),$$  \hspace{1cm} (2)

where $\sum_{i=1}^{n}|a_i|^2 = 1$ and $\sum_{i=1}^{d-1}\sum_{i=1}^{d-1}|a_{ii}|^2 = 1$. When $d = 2$, (2) reduces to the $n$-qubit $W$-class states.

Choi and Kim introduced in [36] the GWV state $|\psi\rangle_{A_1...A_n}$, defined by

$$|\psi\rangle_{A_1A_2...A_n} = \sqrt{p}|W_n^{(d)}\rangle_{A_1...A_n} + \sqrt{1-p}|00...0\rangle_{A_1...A_n}$$  \hspace{1cm} (3)

for $0 \leq p \leq 1$. Let $\rho_{A_1...A_m}$ denotes the reduced density matrix with respect to $|\psi\rangle_{A_1...A_n}$ in $m$-qudit subsystems $A_1...A_m$ with $2 \leq m \leq n - 1$. It has been shown that for any pure state decomposition of $\rho_{A_1...A_m}$,

$$\rho_{A_1...A_m} = \sum_k q_k |\phi_k\rangle_{A_1...A_m}\langle\phi_k|,$$  \hspace{1cm} (4)

$|\phi_k\rangle_{A_1...A_m}$ is a superposition of an $m$-qudit generalized $W$-class state and vacuum [36]. Moreover, for an $n$-qudit GWV state $|\psi\rangle_{A_1A_2...A_n}$ and an arbitrary partition $P = \{P_1, ..., P_r\}$ of the set $S = \{A_1, ..., A_n\}$, $r \leq n$, $P_i \cup P_j = \emptyset$ (i $\neq$ j) and $\bigcup_i P_i = S$, the state $|\psi\rangle_{P_1P_2...P_r}$ is also a GWV state [37].

The $T_qE$ of a bipartite pure state $|\psi\rangle_{AB}$ is defined as [25]

$$T_q(|\psi\rangle_{AB}) = S_q(\rho_A) = \frac{1}{q-1}(1 - \text{tr} \rho_A^q),$$  \hspace{1cm} (5)

where $q > 0$ and $q \neq 1$. When $q$ tends to 1, $T_q(\rho)$ converges to the von Neumann entropy, i.e. $\lim_{q \to 1} T_q(\rho) = -\text{tr} \rho \log_2 \rho = S(\rho)$. The $T_qE$ of a bipartite mixed state $\rho_{AB}$ is given by

$$T_q(\rho_{AB}) = \min_{\{\rho_i, \pi_i\}} \sum_i p_i T_q(|\psi_i\rangle)$$  \hspace{1cm} (6)

with the minimum taken over all possible pure state decompositions of $\rho_{AB}$. As a dual concept of $T_qE$, its Tsallis-$q$ entanglement of assistance ($T_qEoA$) is defined as

$$T_q^d(\rho_{AB}) = \max_{\{\rho_i, \pi_i\}} \sum_i p_i T_q(|\psi_i\rangle)$$  \hspace{1cm} (7)

with the maximum taken over all possible pure state decompositions of $\rho_{AB}$. There is an analytic relationship between the $T_qE$ and concurrence [43, 44] for $q \in [\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}]$ [42].

$$T_q(|\psi\rangle_{AB}) = g_q(C^2(|\psi\rangle_{AB})).$$  \hspace{1cm} (8)
where

\[ g_q(x) = \frac{1}{q - 1} \left[ 1 - \left( \frac{1 + \sqrt{1 - x}}{2} \right)^q - \left( \frac{1 - \sqrt{1 - x}}{2} \right)^q \right]. \tag{9} \]

It has also been shown that \( T_q(\langle \psi | \psi \rangle) = g_q(C^2(\langle \psi | \psi \rangle)) \) for any \( 2 \otimes m(m \geq 2) \) pure state \( |\psi \rangle \), and \( T_q(\rho) = g_q(C^2(\rho)) \) for two-qubit mixed state \( \rho \) [25]. Therefore, (8) holds for any \( q \) such that \( g_q(x) \) in (9) is monotonically increasing and convex.

Let \( \rho_{A_1 A_2 ... A_m} \) be the reduced density matrix of \( |\psi\rangle_{A_1 A_2 ... A_m} \) in (3). Denote by \( \{P_1, \ldots, P_r\} \) a partition of the set \( \{A_1, A_2, \ldots, A_m\} \), \( r \leq m \leq n \). In [38] Shi et al proved that

\[ T_q(\rho_{A_1 | A_2 ... A_m}) = g_q(C^2(\rho_{A_1 | A_2 ... A_m})) \tag{10} \]

for \( q \in \left[ \frac{5 - \sqrt{13}}{2}, \frac{5 + \sqrt{13}}{2} \right] \), and

\[ T_q^0(\rho_{A_1 | A_2 ... A_m}) = T_q(\rho_{A_1 | A_2 ... A_m}) = g_q(C^2(\rho_{A_1 | A_2 ... A_m})) \tag{11} \]

for \( q \in \left[ \frac{5 - \sqrt{13}}{2}, 2 \right] \cup [3, \frac{5 + \sqrt{13}}{2}] \).

Furthermore, for GWV states the authors in [38] established the monogamy relation

\[ T_q^\mu(\rho_{P_i | P_{j+1} ... P_r}) \geq \sum_{j=2}^r T_q^\mu(\rho_{P_i P_j}) \tag{12} \]

for \( q \in \left[ \frac{5 - \sqrt{13}}{2}, \frac{5 + \sqrt{13}}{2} \right] \) and \( \mu \in [2, \infty) \), and the general monogamy relation,

\[ T_q^\gamma(\rho_{P_i | P_{j+1} ... P_r}) \geq \sum_{j=2}^r \left( 2^\gamma - 1 \right)^{j-2} T_q^0(\rho_{P_i P_j}) + \left( 2^\gamma - 1 \right)^{r-1} T_q^0(\rho_{P_i P_r}) \]

\[ + \left( 2^\gamma - 1 \right)^{r-2} T_q^0(\rho_{P_i P_{j+1}}). \tag{13} \]

conditioned that \( T_q(\rho_{P_i | P_i}) \leq T_q(\rho_{P_i | P_{i+1} ... P_r}) \) for \( i = 2, 3, \ldots, r \) and \( T_q(\rho_{P_i P_j}) \geq T_q(\rho_{P_i | P_{j+1} ... P_r}) \) for \( j = i + 1, \ldots, r-1 \). In [38], for \( q \in \left[ \frac{5 - \sqrt{13}}{2}, 2 \right] \cup [3, \frac{5 + \sqrt{13}}{2}] \), the following polygamy relation based on the TqEOA has been obtained [38],

\[ (T_q^\mu(\rho_{P_i | P_{j+1} ... P_r}))^{\mu} \leq \sum_{j=2}^r (T_q^\mu(\rho_{P_i P_j}))^{\mu} \tag{14} \]

with \( \mu \in (0, 1] \).

Next, we provide a class of monogamy and polygamy inequalities which are tighter than inequalities (12)–(14), respectively, by using the following lemma [45].

**Lemma 1.** For any real numbers \( x, k \) and \( t \), we have

(a) \( (1 + x)^t \geq 1 + \frac{(1 + k)^{t-1}}{k} x \) for \( 0 \leq x \leq k \leq 1, t \geq 1 \);  
(b) \( (1 + x)^t \geq 1 + \frac{(1 + k)^{t-1}}{k} x \) for \( x \geq k \geq 1, 0 \leq t \leq 1 \);  
(c) \( (1 + x)^t \leq 1 + \frac{(1 + k)^{t-1}}{k} x \) for \( 0 \leq x \leq k \leq 1, 0 \leq t \leq 1 \).
2.1. Tighter monogamy relations in terms of TqE

For any nonnegative integer \( j \) and its binary expansion \( j = \sum_{i=0}^{s-1} j_i 2^i \), with \( \log_2 j \leq s \) and \( j_i \in \{0, 1\} \) for \( i = 0, \ldots, s - 1 \), one can define a unique binary vector \( \vec{j} = (j_0, j_1, \ldots, j_{s-1}) \). The Hamming weight \( \omega_H(\vec{j}) \) of \( \vec{j} \) is defined as the number of 1’s in \( \{j_0, j_1, \ldots, j_{s-1}\} \). The Hamming weight \( \omega_H(\vec{j}) \) is bounded above by \( \log_2 j \),

\[
\omega_H(\vec{j}) \leq \log_2 j \leq j. \tag{15}
\]

In the following we denote by \( \rho_{A_1A_2\ldots A_m} \) the reduced density matrices of a GWV state \( |\psi\rangle_{A_1A_2\ldots A_m} \) given in (3), and \( \{P, P_0, P_1, \ldots, P_{r-1}\} \) a partition of the set \( \{A_1, A_2, \ldots, A_m\} \), \( r \leq m - 1 \leq n - 1 \).

**Theorem 1.** If

\[
kT_q^2(\rho_{PP}) \geq T_q^2(\rho_{PP,1}) \geq 0 \tag{16}
\]

for \( j = 0, 1, \ldots, r - 2 \) and \( 0 < k \leq 1 \), we have

\[
T_q^2(\rho_{PPP_0\ldots P_{r-1}}) \geq \sum_{j=0}^{r-1} (K_{\beta})^{-\omega_H(\vec{j})} T_q^2(\rho_{PP}), \tag{17}
\]

where \( \beta \in [2, \infty) \), \( q \in [\frac{5 - \sqrt{13}}{2}, \frac{5 + \sqrt{13}}{2}] \), \( K_{\beta} = \frac{(1 + q)^{\frac{2}{q} - 1}}{q^2} \).

**Proof.** From inequality (12), one has \( T_q^2(\rho_{PPP_0\ldots P_{r-1}}) \geq \sum_{j=0}^{r-1} T_q^2(\rho_{PP}). \) Thus, it is sufficient to show that

\[
\left( \sum_{j=0}^{r-1} T_q^2(\rho_{PP}) \right)^{\frac{2}{q}} \geq \sum_{j=0}^{r-1} (K_{\beta})^{\omega_H(\vec{j})} T_q^2(\rho_{PP}). \tag{18}
\]

First, we prove that the inequality (18) holds for the case of \( r = 2^s \) by using mathematical induction on \( s \). For \( s = 1 \), using lemma 1 (a), we have

\[
(T_q^2(\rho_{PPP_0}) + T_q^2(\rho_{PPP_1}))^{\frac{2}{q}} = T_q^2(\rho_{PPP_0}) \left(1 + \frac{T_q^2(\rho_{PPP_1})}{T_q^2(\rho_{PPP_0})}\right)^{\frac{2}{q}} \geq T_q^2(\rho_{PPP_0}) \left[1 + K_{\beta} \left(\frac{T_q^2(\rho_{PPP_1})}{T_q^2(\rho_{PPP_0})}\right)\right] = T_q^2(\rho_{PPP_0}) + K_{\beta} T_q^2(\rho_{PPP_1}). \tag{19}
\]

Thus, the inequality (18) holds for \( s = 1 \).

Assume that the inequality (18) holds for \( r = 2^{s-1} \) with \( s \geq 2 \). Consider the case of \( r = 2^s \). From (16) we have \( T_q^2(\rho_{PP\ldots PP_0}) \leq k^{2^{s-1}} T_q^2(\rho_{PP}), \) for \( j = 0, 1, \ldots, 2^{s-1} - 1 \). Therefore,

\[
\frac{\sum_{j=0}^{2^{s-1}-1} T_q^2(\rho_{PP}))}{\sum_{j=0}^{2^{s-1}-1} T_q^2(\rho_{PP})} \leq k^{2^{s-1}} \leq k \leq 1.
\]
Again using lemma 1 (a), we have
\[
\left( \sum_{j=0}^{2^r-1} T_q^2(\rho_{PPj}) \right)^{\frac{\beta}{2}} = \left( \sum_{j=0}^{2^r-1} T_q^2(\rho_{PPj}) \right)^{\frac{\beta}{2}} \left( 1 + \frac{\sum_{j=2^r-1}^{2^r-1} T_q^2(\rho_{PPj})}{\sum_{j=0}^{2^r-1} T_q^2(\rho_{PPj})} \right)^{\frac{\beta}{2}}
\geq \left( \sum_{j=0}^{2^r-1} T_q^2(\rho_{PPj}) \right)^{\frac{\beta}{2}} \left[ 1 + \kappa_{\beta} \frac{\sum_{j=2^r-1}^{2^r-1} T_q^2(\rho_{PPj})}{\sum_{j=0}^{2^r-1} T_q^2(\rho_{PPj})} \right]^{\frac{\beta}{2}}
= \left( \sum_{j=0}^{2^r-1} T_q^2(\rho_{PPj}) \right)^{\frac{\beta}{2}} + \kappa_{\beta} \left( \sum_{j=2^r-1}^{2^r-1} T_q^2(\rho_{PPj}) \right)^{\frac{\beta}{2}}. \tag{20}
\]

From the induction hypothesis, we have
\[
\left( \sum_{j=0}^{2^r-1} T_q^2(\rho_{PPj}) \right)^{\frac{\beta}{2}} \geq \sum_{j=0}^{2^r-1} (\kappa_{\beta})^{\frac{\beta^2}{2}} T_q^2(\rho_{PPj}). \tag{21}
\]

By relabeling the subsystems, we can easily get
\[
\left( \sum_{j=0}^{2^r-1} T_q^2(\rho_{PPj}) \right)^{\frac{\beta}{2}} \geq \sum_{j=2^r-1}^{2^r-1} (\kappa_{\beta})^{\frac{\beta^2}{2}} T_q^2(\rho_{PPj}). \tag{22}
\]

From inequality (20) together with inequalities (21) and (22), we have
\[
\left( \sum_{j=0}^{2^r-1} T_q^2(\rho_{PPj}) \right)^{\frac{\beta}{2}} \geq \sum_{j=0}^{2^r-1} (\kappa_{\beta})^{\frac{\beta^2}{2}} T_q^2(\rho_{PPj}). \tag{23}
\]

Now we extend the above conclusion to arbitrary integer \( r \). Note that there always exists some \( s \) such that \( 0 < r \leq 2^s \). Let us consider a \((2^s + 1)\)-partite quantum state,
\[
\gamma_{PP_0P_1...P_{2^r-1}} = \rho_{PP_0P_1...P_{2^r-1}} \otimes \sigma_{P_{2^r}...P_{2^{s+1}}}, \tag{24}
\]

which is the tensor product of \( \rho_{PP_0P_1...P_{2^r-1}} \) and an arbitrary \((2^s - r)\)-partite state \( \sigma_{P_{2^r}...P_{2^{s+1}}} \). As just proved above for this state, we have
\[
T_q^j(\gamma_{PP_0P_1...P_{2^r-1}}) \geq \sum_{j=0}^{2^r-1} (\kappa_{\beta})^{\frac{\beta^2}{2}} T_q^j(\gamma_{PPj}), \tag{25}
\]

where \( \gamma_{PPj} \) is the reduced density matrix of \( \gamma_{PP_0P_1...P_{2^r-1}} \), \( j = 0, 1, \ldots, 2^r - 1 \).

Taking into account the following obvious facts: \( T_q(\gamma_{PP_0P_1...P_{2^r-1}}) = T_q(\rho_{PP_0P_1...P_{2^r-1}}) \), \( T_q(\gamma_{PPj}) = 0 \) for \( j = r, \ldots, 2^r - 1 \), and \( T_q(\gamma_{PPj}) = T_q(\rho_{PPj}) \) for each \( j = 0, \ldots, r - 1 \), we get
\[ T_q^β(ρ_{P_0P_1...P_{r-1}}) = T_q^β(γ_{P_0P_1...P_{r-1}}) \]
\[ \geq \sum_{j=0}^{2^r-1} (K_β)^{α(β)} T_q^j(γ_{PP_j}) \]
\[ = \sum_{j=0}^{r-1} (K_β)^{α(β)} T_q^j(ρ_{PP_j}). \] (26)

This completes the proof. \( \Box \)

**Remark 1.** Since \((K_β)^{α(β)} \geq 1\) for any \(β \geq 2\), we have
\[ T_q^β(ρ_{P_0P_1...P_{r-1}}) \geq \sum_{j=0}^{r-1} (K_β)^{α(β)} \sum_{j=0}^{r-1} T_q^j(ρ_{PP_j}). \] (27)

Therefore, we provide a monogamy relation based on \(T_qE\) with larger lower bound than (12) in reference [38].

In the theorem 1 when \(κ = 1\), for any GWV states in the order of the partitions \(P_0, P_1, ..., P_{r-1}\) satisfying
\[ T_q(ρ_{PP_j}) \geq T_q(ρ_{PP_{j+1}}) \geq 0, \quad j = 0, 1, ..., r - 2, \]
we get
\[ T_q^β(ρ_{P_0P_1...P_{r-1}}) \geq \sum_{j=0}^{r-1} \left( 2^{\frac{q}{2}} - 1 \right)^{α(β)} T_q^j(ρ_{PP_j}), \] (28)
which is tighter than the inequality (12) in [38]. When \(0 < κ < 1\), for the GWV states satisfying certain conditions (16), we can also improve the monogamy relations of reference [38] from (27).

Furthermore, as \((\frac{1+k}{k})^{\frac{q}{2}} - 1\) is a decreasing function of \(k\) for \(k \in (0, 1]\), \(β \geq 2\), the inequality (17) gets tighter as \(k\) decreases.

**Theorem 2.** When \(q \in \left[ \frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2} \right]\), we have
\[ T_q^β(ρ_{P_0P_1...P_{r-1}}) \geq \sum_{j=0}^{r-1} (K_β)^{α(β)} T_q^j(ρ_{PP_j}) \] (29)
conditioned that
\[ kT_q^β(ρ_{PP_j}) \geq \sum_{j=1}^{r-1} T_q^j(ρ_{PP_j}) \] (30)
for \(l = 0, 1, ..., r - 2, 0 < k \leq 1, \) where \(β \in [2, \infty)\) and \(K_β = \frac{(\frac{1+k}{k})^{\frac{q}{2}} - 1}{k}.\)

**Proof.** From inequality (12), we only need to prove
\[ \left( \sum_{j=0}^{r-1} T_q^j(ρ_{PP_j}) \right)^{\frac{α}{β}} \geq \sum_{j=0}^{r-1} (K_β)^{α(β)} T_q^j(ρ_{PP_j}). \] (31)
We use mathematical induction on $r$ here. It is obvious that inequality (31) holds for $r = 2$ from (19). Assume that it also holds for any positive integer less than $r$. Since $\frac{\sum_{j=1}^{r-1} T_q^3(\rho_{PP_j})}{T_q^3(\rho_{PP_0})} \leq k$, we have

$$
\left( \sum_{j=0}^{r-1} T_q^3(\rho_{PP_j}) \right)^{\frac{2}{3}} = T_q^3(\rho_{PP_0}) \left( 1 + \frac{\sum_{j=1}^{r-1} T_q^3(\rho_{PP_j})}{T_q^3(\rho_{PP_0})} \right)^{\frac{2}{3}} \\
\geq T_q^3(\rho_{PP_0}) \left[ 1 + \mathcal{K}_\beta \left( \frac{\sum_{j=1}^{r-1} T_q^3(\rho_{PP_j})}{T_q^3(\rho_{PP_0})} \right)^{\frac{2}{3}} \right] \\
= T_q^3(\rho_{PP_0}) + \mathcal{K}_\beta \left( \sum_{j=1}^{r-1} T_q^3(\rho_{PP_j}) \right)^{\frac{2}{3}} \\
\geq T_q^3(\rho_{PP_0}) + \mathcal{K}_\beta \sum_{j=1}^{r-1} (\mathcal{K}_\beta)^{j-1} T_q^3(\rho_{PP_j}) \\
= \sum_{j=0}^{r-1} (\mathcal{K}_\beta)^j T_q^3(\rho_{PP_j}),
$$

where the first inequality is due to lemma 1 (a) and the second inequality is due to the induction hypothesis. \hfill \square

According to inequality (15), we obtain

$$T_q^3(\rho_{PP_0}) \geq \sum_{j=0}^{r-1} (\mathcal{K}_\beta)^j T_q^3(\rho_{PP_j}) \geq \sum_{j=0}^{r-1} (\mathcal{K}_\beta)^{j-1} T_q^3(\rho_{PP_j})$$

for $\beta \geq 2$. Therefore the inequality (29) of theorem 2 is tighter than the inequality (17) of theorem 1 under certain conditions.

In general, the conditions (30) is not always satisfied. We derive the following monogamy inequality with different conditions.

**Theorem 3.** When $q \in [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}]$, we have

$$T_q^3(\rho_{PP_0}) \geq \sum_{j=0}^{t} (\mathcal{K}_\beta)^j T_q^3(\rho_{PP_j}) + (\mathcal{K}_\beta)^{t+1} \sum_{j=t+1}^{r-2} T_q^3(\rho_{PP_j}) + (\mathcal{K}_\beta)^{t+1} T_q^3(\rho_{PP_{r-1}})$$

conditioned that $kT_q^3(\rho_{PP_j}) \geq T_q^3(\rho_{PP_{j+1}})$ for $i = 0, 1, \ldots, t$ and $T_q^3(\rho_{PP_j}) \leq kT_q^3(\rho_{PP_{j+1}})$ for $j = t + 1, \ldots, r - 2$, \( \forall 0 \leq k \leq 1, \ 0 \leq t \leq r - 3, \ r \geq 3, \) where $\beta \in [2, \infty) \) and $\mathcal{K}_{\beta} \frac{(1+\beta)^{\frac{2}{3}} - 1}{\beta^{\frac{2}{3}}}$.
Proof. From theorem 1 for the case $r = 2$, we have

$$T_q^j(\rho_{P_0P_1...P_{r-1}}) \geq T_q^j(\rho_{PP_0}) + \mathcal{K}_\beta T_q^j(\rho_{PP_1...P_{r-1}})$$

$$\geq \ldots$$

$$\geq \sum_{j=0}^{r-1} (\mathcal{K}_\beta)^j T_q^j(\rho_{PP_j}) + (\mathcal{K}_\beta)^{j+1} T_q^j(\rho_{PP_{j+1}...P_{r-1}}). \quad (34)$$

Since $T_q^2(\rho_{PP_j}) \leq kT_q^j(\rho_{PP_{j+1}...P_{r-1}})$ for $j = t + 1, \ldots, r - 2$, using theorem 1 again we have

$$T_q^2(\rho_{PP_{j+1}...P_{r-1}}) \geq \mathcal{K}_\beta T_q^2(\rho_{PP_j}) + T_q^j(\rho_{PP_{j+2}...P_{r-1}})$$

$$\geq \ldots$$

$$\geq \mathcal{K}_\beta \left( \sum_{j=t+1}^{r-2} T_q^j(\rho_{PP_j}) \right) + T_q^j(\rho_{PP_{r-1}}). \quad (35)$$

Combining (34) and (35), we get the inequality (33). $\square$

Remark 2. From theorem 3, if $kT_q^j(\rho_{PP_j}) \geq T_q^j(\rho_{PP_{j+1}...P_{r-1}})$ for all $j = 0, 1, \ldots, r - 2$, one has

$$T_q^j(\rho_{PP_{0}...P_{r-1}}) \geq \sum_{j=0}^{r-1} (\mathcal{K}_\beta)^j T_q^j(\rho_{PP_j}). \quad (36)$$

Next, using lemma 1 (b), we further improve the monogamy inequality (13) provided in [38].

Lemma 2. If $T_q^j(\rho_{PP_1P_2}) \geq kT_q^j(\rho_{PP_1P_2})$, we have for $q \in \left[ \frac{1+\sqrt{5}}{2}, \frac{5+\sqrt{11}}{2} \right]$, $T_q^j(\rho_{PP_1P_2}) \geq T_q^j(\rho_{PP_1P_2}) + \mathcal{K}_\gamma T_q^j(\rho_{PP_1P_2})$, \quad (37)

where $\gamma \in [0, \mu], \mu \in [2, \infty], \mathcal{K}_\gamma = \frac{(1+\sqrt{\gamma})^2}{2\gamma^2}$ and $k \in [1, \infty]$.

Proof. From (12), we have $T_q^j(\rho_{PP_1P_2}) \geq T_q^j(\rho_{PP_1P_2}) + T_q^j(\rho_{PP_1P_2})$ for $\mu \in [2, \infty]$. Hence, we get

$$T_q^j(\rho_{PP_1P_2}) = (T_q^j(\rho_{PP_1P_2}))^\gamma$$

$$\geq (T_q^j(\rho_{PP_1P_2}) + T_q^j(\rho_{PP_1P_2}))^\gamma$$

$$= T_q^j(\rho_{PP_1P_2}) \left( 1 + \frac{T_q^j(\rho_{PP_1P_2})}{T_q^j(\rho_{PP_1P_2})} \right)^\gamma$$

$$\geq T_q^j(\rho_{PP_1P_2}) \left( 1 + \mathcal{K}_\gamma \left( \frac{T_q^j(\rho_{PP_1P_2})}{T_q^j(\rho_{PP_1P_2})} \right) \right)$$

$$= T_q^j(\rho_{PP_1P_2}) + \mathcal{K}_\gamma T_q^j(\rho_{PP_1P_2}). \quad (38)$$

Here the second inequality is due to lemma 1 (b). $\square$
Now we generalize our results to multipartite GWV states. The proof is similar to the proof of theorem 3 by using lemma 1 (b) and lemma 2.

**Theorem 4.** If \( kT^q_i(\rho_{PP}) \leq T^q_i(\rho_{PP_{i+1...r}}) \) for \( i = 0, 1, \ldots, t \) and \( T^q_i(\rho_{PP}) \geq kT^q_i(\rho_{PP_{i+1...r}}) \) for \( j = t + 1, \ldots, r - 2 \), \( \forall k \geq 1 \), \( 0 \leq t \leq r - 3 \) and \( r \geq 3 \), we have for \( q \in \left[ \frac{3+\sqrt{13}}{2}, \frac{9+3\sqrt{13}}{4} \right] \).

\[
T^q_i(\rho_{PP_{0...r-1}}) \geq \sum_{j=0}^{r-t} (K_\gamma)^j T^q_i(\rho_{PP}) + (K_\gamma)^{r-t+2} \sum_{j=r-t+1}^{r-2} T^q_i(\rho_{PP})
+ (K_\gamma)^{r-t+1} T^q_i(\rho_{PP_{r-1}})
\]

with \( \gamma \in [0, \mu], \mu \in [2, \infty) \) and \( K^\gamma = \frac{1+4\mu^2-1}{k\pi} \).

**Remark 3.** Since \( \frac{1+4\mu^2}{\pi} \geq \frac{7}{2} \) for \( \frac{7}{2} \in [0, 1] \) and \( k \in [1, \infty) \), our new monogamy relation (39) for \( TqE \) is better than (13) given in [38] which is just a special case of ours for \( k = 1 \). Moreover, the larger the \( k \), the tighter the inequality (39).

**Example 1.** Consider the three-qubit GW state

\[
|\psi\rangle_{A_1A_2A_3} = \frac{1}{\sqrt{6}} |100\rangle + \frac{1}{\sqrt{6}} |010\rangle + \frac{2}{\sqrt{6}} |001\rangle.
\]

From the definition of concurrence [43], we get \( C(\rho_{A_1|A_2A_3}) = \frac{\sqrt{3}}{2} \), \( C(\rho_{A_2|A_1A_3}) = \frac{1}{2} \), and \( C(\rho_{A_3|A_1A_2}) = \frac{1}{2} \). When \( q = 2 \), using (10) we have \( T_2(|\psi\rangle_{A_1A_2A_3}) = \frac{8}{27} \), \( T_2(\rho_{A_1A_2}) = \frac{1}{18} \), and \( T_2(\rho_{A_1A_3}) = \frac{1}{8} \). Choosing \( \mu = 3 \), we have \( 1 \leq k \leq 64 \) from lemma 2. Thus, \( T_2^q(|\psi\rangle_{A_1A_2A_3}) \geq (\frac{8}{27})^\gamma + (\frac{1+4\mu^2-1}{k\pi})^{\gamma} \) from our result (37), and \( T_2^q(|\psi\rangle_{A_1A_2A_3}) \geq (\frac{1}{2})^\gamma + (\frac{2}{7} - 1)(\frac{3}{2})^\gamma \) from the result given in [38]. One can see that the result is better than the result in [38] for \( \gamma \in [0, 3] \), and the inequality is tighter as \( k \) increases, see figure 1.

### 2.2. Tighter polygamy relations in terms of TqEoA

In this section, we present the polygamy inequalities for GWV states based on TqEoA, which improves the inequality (14).

**Theorem 5.** If the subsystems \( P_0, P_1, \ldots, P_{r-1} \) satisfy

\[
kT^q_i(\rho_{PP}) \geq T^q_i(\rho_{PP_{i+1...r}}) \geq 0
\]

with \( j = 0, 1, \ldots, r - 2 \), \( 0 < k \leq 1 \), then

\[
[T^q_i(\rho_{PP_{0...r-1}})]^\mu \leq \sum_{j=0}^{r-1} (K_\mu)^j T^q_i(\rho_{PP})]^\mu,
\]

where \( \mu \in (0, 1), q \in \left[ \frac{3+\sqrt{13}}{2}, 2 \right] \cup \left[ 3, \frac{9+3\sqrt{13}}{4} \right] \) and \( K_\mu = \frac{1+4\mu^2-1}{k\pi} \).
Figure 1. The vertical axis is the lower bound of the Tsallis-$q$ entanglement $T_2(|\psi\rangle_{A_1A_2A_3})$. The black solid line is the exact values of $T_2(|\psi\rangle_{A_1A_2A_3})$. The red dashed (green dot-dashed) line represents the lower bound from our results for the case of $k = 64$ ($k = 10$). The blue dotted line represents the lower bound from the result in [38].

Proof. Since $T_a^q(\rho_{PP_{0..r-1}}) \leq \sum_{j=0}^{r-1} T_a^q(\rho_{PP_j})$ from inequality (14), it is sufficient to prove that

$$\left( \sum_{j=0}^{r-1} T_a^q(\rho_{PP_j}) \right)^\mu \leq \sum_{j=0}^{r-1} (K_\mu)^{\mu(j)} [T_a^q(\rho_{PP_j})]^\mu. \quad (43)$$

First, we prove that the inequality (43) holds for the case of $r = 2^s$ by using mathematical induction on $s$. For $s = 1$, using lemma 1 (c) we have

$$(T_a^q(\rho_{PP_0}) + T_a^q(\rho_{PP_1}))^\mu = (T_a^q(\rho_{PP_0}))^\mu \left( 1 + \frac{T_a^q(\rho_{PP_1})}{T_a^q(\rho_{PP_0})} \right)^\mu$$

$$\leq (T_a^q(\rho_{PP_0}))^\mu \left[ 1 + K_\mu \left( \frac{T_a^q(\rho_{PP_1})}{T_a^q(\rho_{PP_0})} \right)^\mu \right]$$

$$= (T_a^q(\rho_{PP_0}))^\mu + K_\mu (T_a^q(\rho_{PP_1}))^\mu. \quad (44)$$

Assume that the inequality (43) holds for $r = 2^{s-1}$ with $s \geq 2$. Consider the case of $r = 2^s$. From (41) we have $T_a^q(\rho_{PP_{j+2^{s-1}-1}}) \leq k^{2^{s-1}} T_a^q(\rho_{PPP})$ for $j = 0, 1, \ldots, 2^{s-1} - 1$. Therefore,

$$0 \leq \frac{\sum_{j=0}^{2^{s-1}-1} T_a^q(\rho_{PPP})}{\sum_{j=0}^{2^{s-1}-1} T_a^q(\rho_{PPP})} \leq k^{2^{s-1}} \leq k \leq 1.$$
Again using lemma 1 (c), we have

\[
\left( \sum_{j=0}^{2^r-1} T_q^j(\rho_{PP}) \right)^\mu = \left( \sum_{j=0}^{2^r-1} T_q^j(\rho_{PP}) \right)^\mu \left( 1 + \sum_{j=0}^{2^r-1} \frac{T_q^j(\rho_{PP})}{\sum_{j=0}^{2^r-1} T_q^j(\rho_{PP})} \right) \mu \\
\leq \left( \sum_{j=0}^{2^r-1} T_q^j(\rho_{PP}) \right)^\mu \left[ 1 + \kappa_\mu \left( \frac{\sum_{j=0}^{2^r-1} T_q^j(\rho_{PP})}{\sum_{j=0}^{2^r-1} T_q^j(\rho_{PP})} \right)^\mu \right] \\
= \left( \sum_{j=0}^{2^r-1} T_q^j(\rho_{PP}) \right)^\mu + \kappa_\mu \left( \sum_{j=0}^{2^r-1} T_q^j(\rho_{PP}) \right)^\mu. \tag{45}
\]

From the induction hypothesis, we have

\[
\left( \sum_{j=0}^{2^r-1} T_q^j(\rho_{PP}) \right)^\mu \leq \sum_{j=0}^{2^r-1} (\kappa_\mu)^{\mu j}[T_q^j(\rho_{PP})]^\mu. \tag{46}
\]

By relabeling the subsystems, we get

\[
\left( \sum_{j=0}^{2^r-1} T_q^j(\rho_{PP}) \right)^\mu \leq \sum_{j=0}^{2^r-1} (\kappa_\mu)^{\mu j}[T_q^j(\rho_{PP})]^\mu. \tag{47}
\]

Hence we have

\[
\left( \sum_{j=0}^{2^r-1} T_q^j(\rho_{PP}) \right)^\mu \leq \sum_{j=0}^{2^r-1} (\kappa_\mu)^{\mu j}[T_q^j(\rho_{PP})]^\mu. \tag{48}
\]

Now we extend the above conclusion to arbitrary integer \( r \). Note that there always exists some \( s \) such that \( 0 < r \leq 2^s \). Let us consider a \((2^s + 1)\)-partite quantum state

\[
\gamma_{PP_0P_1...P_{2^s-1}} = \rho_{PP_0P_1...P_{2^s-1}} \otimes \sigma_{P_{2^s}...P_{2^s-1}}, \tag{49}
\]

which is the tensor product of \((pp_{0}p_{1}...p_{2^r-1})\) and an arbitrary \((2^s - r)\)-partite state \(\sigma_{P_{2^s}...P_{2^s-1}}\).

Similar to the proof above, we have

\[
[T_q^j(\gamma_{PP_0P_1...P_{2^s-1}})]^\mu \leq \sum_{j=0}^{2^s-1} (\kappa_\mu)^{\mu j}[T_q^j(\gamma_{PP})]^\mu, \tag{50}
\]

where \(\gamma_{PP}\) is the reduced density matrix of \(\gamma_{PP_0P_1...P_{2^s-1}}\), \( j = 0, 1, \ldots, 2^s - 1 \).

Taking into account the following obvious facts: \(T_q^j(\gamma_{PP_0P_1...P_{2^s-1}}) = T_q^j(\rho_{PP_0P_1...P_{2^s-1}}), T_q^j(\gamma_{PP}) = 0\) for \( j = r, \ldots, 2^s - 1, \) and \(T_q^j(\gamma_{PP}) = T_q^j(\rho_{PP})\) for each \( j = 0, \ldots, r - 1 \), we get
\[ [T_q^a(\rho_{PP_{p_1\ldots p_{r-1}}})]^{\mu} = [T_q^a(\gamma_{PP_{p_1\ldots p_{r-1}}})]^{\mu} \leq \sum_{j=0}^{2^r-1} (\mathcal{K}_\mu)^{2^r} [T_q^a(\gamma_{PP})]^{\mu} = \sum_{j=0}^{r-1} (\mathcal{K}_\mu)^{2^r} [T_q^a(\rho_{PP})]^{\mu}. \] (51)

This completes the proof. \(\square\)

**Remark 4.** Since \((1+\frac{1}{\sqrt{2}})^\mu \leq 1\) for \(\mu \in (0, 1)\) and \(k \in (0, 1)\), our new polygamy relation for \(T_q\)EoA is tighter than inequality (14) in [38] under certain assumptions for the GWV states. In particular, when \(k = 1\), we get a tighter polygamy inequality
\[ [T_q^a(\rho_{PP_{p_1\ldots p_{r-1}}})]^{\mu} \leq \sum_{j=0}^{r-1} (2^\mu - 1)^{2^r} [T_q^a(\rho_{PP})]^{\mu} \] (52)
for any GWV states without the assumptions. If one takes \(q = 2\), our polygamy inequality (52) gives rise to the one in [31]. Furthermore, the inequality (42) gets tighter as \(k\) decreases.

**Example 2.** Let us consider the four-qubit GW state,
\[ |\psi\rangle_{A_1A_2A_3A_4} = 0.3 |0001\rangle + 0.4 |0010\rangle + 0.5 |0100\rangle + \sqrt{0.5} |1000\rangle. \] (53)
We have \(\rho_{A_1A_2A_3} = 0.09 |000\rangle \langle 000| + |\phi\rangle \langle \phi|\) with |\(\phi\rangle = 0.4 |001\rangle + 0.5 |010\rangle + \sqrt{0.5} |100\rangle\), \(C(\rho_{A_1A_2}) = \frac{\sqrt{3}}{4}\) and \(C(\rho_{A_1A_3}) = \frac{2\sqrt{3}}{4}\). Set \(q = 2\). We obtain
\[ T_2^2(\rho_{A_1A_2}) = g_2 (C^2(\rho_{A_1A_2})) = \frac{1}{4}, T_2^2(\rho_{A_1A_3}) = g_2 (C^2(\rho_{A_1A_3})) = \frac{4}{25}. \]
Therefore, \([T_2^2(\rho_{A_1A_2})]^{\mu} \leq (\frac{1}{4})^{\mu} + (\frac{1+\frac{1}{\sqrt{2}}}{6})^{\mu} \) from (42), \([T_2^2(\rho_{A_1A_2})]^{\mu} \leq (\frac{1}{4})^{\mu} + (2^\mu - 1)(\frac{1}{2})^{\mu} \) from (52) and \([T_2^2(\rho_{A_1A_3})]^{\mu} \leq (\frac{1}{4})^{\mu} + (\frac{1}{25})^{\mu} \) from (14), where \(k \in [0.64, 1]\) from the condition (41). One can see that our result is better than the ones in [31, 38], and the smaller the \(k\) is, the tighter relation is, see figure 2.

Similar to the improvement from the inequality (17) to the inequality (29), we can analogously improve the polygamy inequality of theorem 5 under certain conditions.

**Theorem 6.** For \(q \in [\frac{2\sqrt{5}+1}{4}, 2] \cup [3, \frac{2\sqrt{5}+1}{2}]\) we have
\[ [T_q^a(\rho_{PP_{p_1\ldots p_{r-1}}})]^{\mu} \leq \sum_{j=0}^{r-1} (\mathcal{K}_\mu)^{2^r} [T_q^a(\rho_{PP})]^{\mu} \] (54)
conditioned that
\[ kT_q^a(\rho_{PP}) \geq \sum_{j=l+1}^{r-1} T_q^a(\rho_{PP}) \] (55)
for \(l = 0, 1, \ldots, r-2\) and \(0 \leq k \leq 1\), where \(\mu \in (0, 1]\) and \(\mathcal{K}_\mu = \frac{1+\frac{1}{\sqrt{2}}}{6}.\)
Figure 2. The vertical axis is the upper bound of the Tsallis-$q$ entanglement of assistance $T^2_{q}(\rho_{A_1|A_2A_3})$. The blue dashed line represents the upper bound from our result (42) for $k = 0.64$, the red dot-dashed line represents the upper bound from the result in [31], and the black solid line represents the upper bound from the result in [38].

Proof. From the inequality (14), we only need to prove
\[
\left( \sum_{j=0}^{r-1} T^a_q(\rho_{PPj}) \right)^\mu \leq \sum_{j=0}^{r-1} (K^\mu_j) [T^a_q(\rho_{PPj})]^\mu. \tag{56}
\]

We use mathematical induction on $r$ here. It is obvious that the inequality (56) holds for $r = 2$ from (44). Assume that it also holds for any positive integer less than $r$. Since $0 \leq \sum_{j=1}^{r-1} T^a_q(\rho_{PPj}) / T^a_q(\rho_{PP0}) \leq k$, we have
\[
\left( \sum_{j=0}^{r-1} T^a_q(\rho_{PPj}) \right)^\mu = [T^a_q(\rho_{PP0})]^\mu \left( 1 + \sum_{j=0}^{r-1} T^a_q(\rho_{PPj}) / T^a_q(\rho_{PP0}) \right)^\mu
\leq [T^a_q(\rho_{PP0})]^\mu \left[ 1 + K^\mu \left( \sum_{j=1}^{r-1} T^a_q(\rho_{PPj}) / T^a_q(\rho_{PP0}) \right)^\mu \right]
= [T^a_q(\rho_{PP0})]^\mu + K^\mu \left( \sum_{j=1}^{r-1} T^a_q(\rho_{PPj}) \right)^\mu
\leq [T^a_q(\rho_{PP0})]^\mu + K^\mu \sum_{j=1}^{r-1} (K^\mu_j)^{r-1} [T^a_q(\rho_{PPj})]^\mu
= \sum_{j=0}^{r-1} (K^\mu_j) [T^a_q(\rho_{PPj})]^\mu, \tag{57}
\]
where the first inequality is due to lemma 1 (c) and the second inequality is due to the induction hypothesis. □

Since \( [T_q^\mu(\rho_{PP})]^\mu \leq \sum_{j=0}^{r-1} (K_\mu)^j [T_q^\mu(\rho_{PP})]^\mu \leq \sum_{j=0}^{r-1} (K_\mu)^j [T_q^\mu(\rho_{PP})]^\mu \) for \( \mu \in (0, 1) \), the inequality (54) of theorem 6 is tighter than the inequality (42) of theorem 5 under the conditions. Similarly, we provide a more general result by changing the conditions of the theorem 6.

**Theorem 7.** For \( q \in \left[ \frac{5}{2}, 2 \right] \cup \left[ 3, \frac{5+\sqrt{13}}{2} \right] \) we have

\[
[T_q^\mu(\rho_{PP})]_{\mu}^\mu \leq \sum_{j=0}^{r-1} (K_\mu)^j [T_q^\mu(\rho_{PP})]_{\mu}^\mu + (K_\mu)^{r+1} \sum_{j=1}^{r-2} [T_q^\mu(\rho_{PP})]_{\mu}^\mu
\]

conditioned that \( kT_q^\mu(\rho_{PP}) \geq T_q^\mu(\rho_{PP+i-1}) \) for \( i = 1, 2, \ldots, r-2 \) and \( T_q^\mu(\rho_{PP}) \leq kT_q^\mu(\rho_{PP+i-1}) \) for \( j = 1, 2, \ldots, r-2 \), \( \forall 0 < k \leq 1, 0 \leq t \leq r-3, r \geq 3 \), where \( \mu \in (0, 1] \) and \( K_\mu = \frac{1+\sqrt{13}}{2k} \).

The proof is similar to the one of theorem 3, by using the inequality (42) for the case \( r = 2 \) and lemma 1 (c).

**Remark 5.** Note that if \( kT_q^\mu(\rho_{PP}) \geq T_q^\mu(\rho_{PP+i-1}) \) for all \( j = 1, 2, \ldots, r-2 \), one has

\[
[T_q^\mu(\rho_{PP+i-1})]_{\mu}^\mu \leq \sum_{j=0}^{r-1} (K_\mu)^j [T_q^\mu(\rho_{PP})]_{\mu}^\mu.
\]

Due to that \( T_q(\rho_{PP+i-1}, \rho_{PP+i-1}) = T_q(\rho_{PP+i-1}, \rho_{PP+i-1}) \) for \( q \in \left( \frac{5}{2}, 2 \right) \cup \left( 3, \frac{5+\sqrt{13}}{2} \right] \), the above inequalities (42), (54) and (58) also give the upper bounds of \( T_q(\rho_{PP+i-1}, \rho_{PP+i-1}) \) for GWV states \( |\psi\rangle_{A_1-A_n} \).

### 3. Tighter monogamy and polygamy relations based on R\(\alpha\)E and R\(\alpha\)EoA for GWV states

For a bipartite pure state \( |\psi\rangle_{AB} \), the R\(\alpha\)E is defined as [41] \( E_\alpha(|\psi\rangle_{AB}) = S_\alpha(\rho_A) \), where \( S_\alpha(\rho) = \frac{1}{1-\alpha} \log\{\text{tr}^\alpha(\rho)^\alpha\} \) with \( \alpha > 0, \alpha \neq 1 \). The \( S_\alpha(\rho) \) converges to the von Neumann entropy when \( \alpha \) tends to 1. For a bipartite mixed state \( \rho_{AB} \), the R\(\alpha\)E is given by

\[
E_\alpha(\rho_{AB}) = \min_{|\psi\rangle_{AB}} \sum_i p_i E_\alpha(|\psi\rangle_{AB}),
\]

where the minimum is taken over all possible pure state decompositions of \( \rho_{AB} \). As a dual concept to R\(\alpha\)E, the Rényi-\(\alpha\) entanglement of assistance (R\(\alpha\)EoA) is given by

\[
E_\alpha(\rho_{AB}) = \max_{|\psi\rangle_{AB}} \sum_i p_i E_\alpha(|\psi\rangle_{AB}),
\]

where the maximum is taken over all possible pure state decompositions of \( \rho_{AB} \).

In [41] the authors have derived an analytical relation between the R\(\alpha\)E and concurrence for any two-qubit mixed state \( \rho_{AB} \),

\[
E_\alpha(\rho_{AB}) = f_\alpha \left[ C^2(\rho_{AB}) \right],
\]

where

\[
f_\alpha = \frac{\alpha}{(\alpha-1) log(2)} - \frac{2}{(\alpha-1) log(4)}.
\]
where $\alpha \in [1, \infty)$ and $f_{\alpha}(x)$ has the form

$$f_{\alpha}(x) = \frac{1}{1 - \alpha} \log_2 \left[ \left( \frac{1 - \sqrt{1 - x}}{2} \right)^{\alpha} + \left( \frac{1 + \sqrt{1 - x}}{2} \right)^{\alpha} \right]. \quad (62)$$

Set $x = y^2$ and denote $f_{\alpha}(y) = f_{\alpha}(y^2)$. Then the function $f_{\alpha}(y)$ is monotonically increasing and convex for $y \in [0, 1]$. Later, Wang et al [46] showed that (61) also holds for $\alpha \in \left[ \frac{\sqrt{7} - 1}{2}, \infty \right)$.

Quite recently, Liang et al [39] provided the following analytic formulas for $R_{aE}$ and $R_{oE}$

$$E_{\alpha}(\rho_{A_1|A_2...A_m}) = f_{\alpha}(C^2(\rho_{A_1|A_2...A_m})) \quad (63)$$

for $\alpha \in \left[ \frac{\sqrt{7} - 1}{2}, \infty \right)$, and

$$E_{\alpha}^q(\rho_{A_1|A_2...A_m}) = E_{\alpha}(\rho_{A_1|A_2...A_m}) = f_{\alpha}(C^2(\rho_{A_1|A_2...A_m})) \quad (64)$$

for $\alpha \in \left[ \frac{\sqrt{7} - 1}{2}, \frac{\sqrt{7} - 3}{2} \right]$, together with the following monogamy relation based on $R_{aE}$ for GWV states,

$$E_{\alpha}^q(\rho_{P_iP_2...P_r}) \geq \sum_{j=2}^{r} E_{\alpha}(\rho_{P_iP_j}) \quad (65)$$

with $\mu \in [2, \infty)$ and $\alpha \in \left[ \frac{\sqrt{7} - 1}{2}, \infty \right)$, as well as the following polygamy inequalities based on $R_{oE}$,

$$(E_{\alpha}^q(\rho_{P_iP_2...P_r}))^\mu \leq \sum_{j=2}^{r} (E_{\alpha}^q(\rho_{P_iP_j}))^\mu \quad (66)$$

with $\mu \in (0, 1]$ and $\alpha \in \left[ \frac{\sqrt{7} - 1}{2}, \frac{\sqrt{7} - 3}{2} \right]$.

Instead of the $T_{qE}$ and $T_{qO}E$ used in theorems of section 2, next we consider the $R_{aE}$ and $R_{oE}$. The proofs of the theorems given below are similar to the cases for $T_{qE}$ and $T_{qO}E$.

### 3.1 Tighter monogamy relations in terms of $R_{aE}$

With a similar approach to $T_{qE}$, we first present the following tighter weighted monogamy relations based on $R_{aE}$ for GWV states.

**Theorem 8.** If the subsystems $P_0, P_1, \ldots, P_{r-1}$ satisfy

$$k E_{\alpha}^q(\rho_{PP}) \geq E_{\alpha}^q(\rho_{PP_{j+1}}) \geq 0 \quad (67)$$

for $j = 0, 1, \ldots, r-2$ and $0 < k \leq 1$, we have

$$E_{\alpha}^q(\rho_{PP_{j}P_{j+1}}) \geq \sum_{j=0}^{r-1} (K_{\beta})^j (\rho_{PP_{j+1}}) E_{\alpha}^q(\rho_{PP_{j}}) \quad (68)$$

for $\alpha \in \left[ \frac{\sqrt{7} - 1}{2}, \infty \right)$, where $\beta \in [2, \infty)$ and $K_{\beta} = \frac{(1+\beta)\frac{\sqrt{7} - 1}{k^2}}{\beta}$. 
Theorem 9. \( (K_\beta)^{-1/20} \geq 1 \) for \( \beta \geq 2 \) and \( 0 < k \leq 1 \), our new polygamy relation for RoE is tighter than the inequality (65) in [39] under certain conditions for the GWV states. Moreover, one can find that the inequality (68) gets tighter as \( k \) decreases.

Example 3. Consider the three-qubit GW state

\[
|\psi\rangle_{A_1A_2A_3} = \frac{1}{\sqrt{6}} |100\rangle + \frac{2}{\sqrt{6}} |010\rangle + \frac{1}{\sqrt{6}} |001\rangle .
\]  

(69)

We have

\[
C(|\psi\rangle_{A_1A_2A_3}) = \frac{\sqrt{5}}{3}, \quad C(\rho_{A_1A_2}) = \frac{2}{3}, \quad C(\rho_{A_1A_3}) = \frac{1}{3}.
\]

Choosing \( \alpha = 2 \), from (63) one has

\[
E_2(|\psi\rangle_{A_1A_2A_3}) = \log_2 \left( \frac{18}{13} \right),
\]

\[
E_2(\rho_{A_1A_2}) = \log_2 \left( \frac{9}{7} \right),
\]

\[
E_2(\rho_{A_1A_3}) = \log_2 \left( \frac{18}{17} \right).
\]

Then \( E^3_2(|\psi\rangle_{A_1A_2A_3}) \geq \left[ \log_2 \left( \frac{9}{7} \right) \right]^\beta + \left[ \log_2 \left( \frac{18}{17} \right) \right]^\beta \) from our result (68), and

\[
E^3_2(|\psi\rangle_{A_1A_2A_3}) \geq \left[ \log_2 \left( \frac{9}{7} \right) \right]^\beta + \left[ \log_2 \left( \frac{18}{17} \right) \right]^\beta \]

from the result (65), where \( 0.52 \leq k \leq 1 \). One can see that our result is better than the result (65) in [39] for \( \beta \geq 2 \), see figure 3.

The inequality (68) can be further improved under certain conditions.

Theorem 10. When \( \alpha \in \left[ \frac{7 + 1}{2}, \infty \right) \), we have

\[
E^3_\alpha(\rho_{PP\ldots P_{r-1}}) \geq \sum_{j=0}^{r-1} (K_\beta)^{j} E^3_\alpha(\rho_{PP}) \quad (70)
\]

conditioned that

\[
k E^3_\alpha(\rho_{PP}) \geq \sum_{j=1}^{r-1} E^3_\alpha(\rho_{PP}) \quad (71)
\]

for \( l = 0, 1, \ldots, r - 2 \) and \( 0 < k \leq 1 \), where \( \beta \in [2, \infty) \) and \( K_\beta = \frac{(\lambda + \kappa)^{\beta - 1}}{\kappa} \).

In general, we have the following monogamy inequality.

Theorem 11. When \( \alpha \in \left[ \frac{7 - 1}{2}, \infty \right) \), we have

\[
E^3_\alpha(\rho_{PP\ldots P_{r-1}}) \geq \sum_{j=0}^{r} (K_\beta)^{j} E^3_\alpha(\rho_{PP}) + (K_\beta)^{r+1} \sum_{j=1}^{r-1} E^3_\alpha(\rho_{PP})
\]

\[
+ (K_\beta)^{r+1} E^3_\alpha(\rho_{PP\ldots P_{r-1}}) \quad (72)
\]
Figure 3. The vertical axis is the lower bound of the Rényi-$\alpha$ entanglement $E_2(|\psi\rangle_{A_1 A_2 A_3})$. The solid black line represents the exact values of $E_2(|\psi\rangle_{A_1 A_2 A_3})$, the red dot-dashed line represents the lower bound from our results for $k=0.52$, and the dashed blue line represents the lower bound from the result (65) in [39].

conditioned that $kE_2^a(\rho_{PPj}) \geq E_2^a(\rho_{P|Pj+1...Pr-1})$ for $j=0,1,...,r-2$ and $0<k \leq 1$, for all $t=0,1,...,r-3$ and $r \geq 3$, where $\beta \in [2, \infty)$ and $K_{\beta} = \frac{(1+k)^\beta}{k^2}$.

Remark 6. If $kE_2^a(\rho_{PP}) \geq E_2^a(\rho_{PPj+1...Pr-1})$ for all $j=0,1,...,r-2$, we have

$$E_2^a(\rho_{PP|P0...Pr-1}) \leq \sum_{j=0}^{r-1} (K_{\beta})^j E_2^a(\rho_{PPj})$$

(73)

3.2. Tighter polygamy relations in terms of $R_{0>E0A}$

Now we establish the tighter polygamy relations for $R_{0>E0A}$ by using a similar approach to $Tq_{E0A}$.

Theorem 11. If the subsystems $P_0, P_1, \ldots, P_{r-1}$ satisfy

$$kE_2^a(\rho_{PP}) \geq E_2^a(\rho_{PPj+1}) \geq 0$$

(74)

for $j=0,1,...,r-2$ and $0<k \leq 1$, we have

$$[E_2^a(\rho_{PP|P0...Pr-1})]^\mu \leq \sum_{j=0}^{r-1} (K_{\mu})^j [E_2^a(\rho_{PPj})]^\mu$$

(75)

where $\mu \in (0,1], \alpha \in [\frac{1}{2}, \frac{1}{2}]$ and $K_{\mu} = \frac{(1+k)^\mu-1}{k^\mu}$. 

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Figure 4. The vertical axis is the upper bound of the Renyi-\(\alpha\) entanglement of assistance \(E_{1,2}^\alpha(\rho_{A_1|A_2A_3})\). The red dashed (green dot-dashed, blue dotted) line represents the upper bound from our result (75) for \(k = 1\) (\(k = 0.8, k = 0.7\)), and the black solid line represents the upper bound from (66) in [39].

Since \((K_\mu)_{\mu(\vec{\beta})} \leq 1\) for \(\mu \in (0,1]\) and \(k \in (0,1]\), our new polygamy inequality for ROEoA is tighter than the inequality (66) in [39] under certain conditions for the GWV states. Also, one finds that the smaller the \(k\) is, the tighter the inequality (75) is.

**Example 4.** Let us again consider the four-qubit GW state presented in example 2. Choosing \(\alpha = 1.2\) we have

\[
E_{1,2}^{\alpha}(\rho_{A_1A_2}) = f_{1.2} \left[ \left( \frac{\sqrt{2}}{2} \right)^2 \right] \approx 0.549339,
\]

\[
E_{1,2}^{\alpha}(\rho_{A_1A_3}) = f_{1.2} \left[ \left( \frac{2\sqrt{2}}{5} \right)^2 \right] \approx 0.372954.
\]

From (74), we get \(k \in [0.68,1]\). Then our inequality (75) yields that \([E_{1,2}^{\alpha}(\rho_{A_1|A_2A_3})]_\mu \leq 0.549339\mu + \frac{(1+k)^{\mu-1}}{\mu}0.372954\mu\), and \([E_{1,2}^{\alpha}(\rho_{A_1|A_2A_3})]_\mu \leq 0.549339\mu + (2\mu - 1)0.372954\mu\) for \(k = 1\). While, the inequality (66) yields \([E_{1,2}^{\alpha}(\rho_{A_1|A_2A_3})]_\mu \leq 0.549339\mu + 0.372954\mu\). Hence, our results are better than one in [39], and the inequality gets tighter as \(k\) decreases, see figure 4.
Analogously we can improve the polygamy inequality of theorem 11 under certain conditions.

**Theorem 12.** For \( \alpha \in [\frac{\sqrt{7}-1}{2}, \frac{\sqrt{13}+1}{2}] \) we have

\[
[E^\alpha_\alpha(\rho_{PP_{0}}_{0, \ldots, p_{r-1}})]^\mu \leq \sum_{j=0}^{r-1} (K_\mu)^j [E^\alpha_{\alpha}(\rho_{PP})]^\mu
\]

conditioned that

\[
kE^\alpha_{\alpha}(\rho_{PP}) \geq \sum_{j=l+1}^{r-1} E^\alpha_{\alpha}(\rho_{PP})_j
\]

for \( l = 0, 1, \ldots, r-2, 0 < k \leq 1 \), where \( \mu \in (0, 1) \) and \( K_\mu = \frac{1+\mu^2}{2} \).

Due to \( \omega_{\mu}(j) \leq j \), one has \( \sum_{j=0}^{r-1} (K_\mu)^j [E^\alpha_{\alpha}(\rho_{PP})]^\mu \leq \sum_{j=0}^{r-1} (K_\mu)^{\omega_{\mu}(j)} [E^\alpha_{\alpha}(\rho_{PP})]^\mu \) for \( \mu \in (0, 1) \) and \( k \in (0, 1) \). Therefore, the inequality (76) of theorem 12 is tighter than the inequality (75) of theorem 11.

Similar to the case of \( T_qEoA \), we also have the following polygamy relation for \( R_nEoA \) under certain conditions.

**Theorem 13.** For \( \alpha \in [\frac{\sqrt{7}+1}{2}, \frac{\sqrt{13}-1}{2}] \) we have

\[
[E^\alpha_\alpha(\rho_{PP_{0}}_{0, \ldots, p_{r-1}})]^\mu \leq \sum_{j=0}^{t} (K_\mu)^{j} [E^\alpha_{\alpha}(\rho_{PP})]^\mu + (K_\mu)^{j+2} \sum_{j=t+1}^{r-2} [E^\alpha_{\alpha}(\rho_{PP})]^\mu
\]

conditioned that \( kE^\alpha_{\alpha}(\rho_{PP}) \geq E^\alpha_{\alpha}(\rho_{PP_{j+1}}_{j+1, \ldots, p_{r-1}}) \) for \( i = 0, 1, \ldots, t \) and \( E^\alpha_{\alpha}(\rho_{PP}) \leq kE^\alpha_{\alpha}(\rho_{PP_{j+1}}_{j+1, \ldots, p_{r-1}}) \) for \( j = t+1, \ldots, r-2, 0 < k \leq 1, 0 \leq t \leq r-3, r \geq 3, \) where \( \mu \in (0, 1) \) and \( K_\mu = \frac{1+\mu^2}{2} \).

**Remark 7.** If \( kE^\alpha_{\alpha}(\rho_{PP}) \geq E^\alpha_{\alpha}(\rho_{PP_{j+1}}_{j+1, \ldots, p_{r-1}}) \) for all \( j = 0, 1, \ldots, r-2 \), then

\[
[E^\alpha_\alpha(\rho_{PP_{0}}_{0, \ldots, p_{r-1}})]^\mu \leq \sum_{j=0}^{r-1} (K_\mu)^j [E^\alpha_{\alpha}(\rho_{PP})]^\mu
\]

Since \( E^\alpha_{\alpha}(\rho_{PP_{0}}_{0, \ldots, p_{r-1}}) = E^\alpha_{\alpha}(\rho_{PP_{0}}_{0, \ldots, p_{r-1}}) \) for \( \alpha \in [\frac{\sqrt{7}-1}{2}, \frac{\sqrt{13}-1}{2}] \), the above inequalities (75), (76) and (78) are also upper bounds of \( E_{\alpha}(\rho_{PP_{0}}_{0, \ldots, p_{r-1}}) \) for GWV states \( |\psi\rangle_{A_1 \ldots A_n} \).

4. **Conclusion**

Both monogamy and polygamy relations of quantum entanglement are the fundamental properties of multipartite entangled states. We have investigated the monogamy and polygamy relations of multipartite entanglement for the arbitrary \( n \)-qudit GWV states with respect to different partitions. By using the Hamming weight of the binary vectors related to the partition of the subsystems, we have established a class of monogamy inequalities in terms of the \( \beta \)th power of \( T_qE \) for the GWV states when \( \beta \geq 2 \), as well as the polygamy inequalities in terms of the \( \mu \)th power of \( T_qEoA \) when \( 0 < \mu \leq 1 \). Similarly, we have also provided the monogamy
and polygamy relations based on EoE and EoEoA for the GWV states. We have further shown that our monogamy and polygamy inequalities hold under some conditions for the GWV states in a tighter way than the existing ones and can also recover the previous relations, thus they give rise to better restrictions on entanglement distribution among the subsystems of the GWV states.

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Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Le-Min Lai https://orcid.org/0000-0003-4744-4365
Zhi-Xi Wang https://orcid.org/0000-0002-8341-5142

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