A GENERALIZED GÄVRUTA STABILITY OF
COHOMOLOGICAL EQUATIONS IN NONQUASIANALYTIC
CARLEMAN CLASSES

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This modest work is dedicated to the memory of our beloved master Ahmed Intissar
(1951-2017), a distinguished professor, a brilliant mathematician, a man with a golden heart.

ABSTRACT. In this paper we introduce the notion of generalized Gavruta stability of functional equations in order to study, in the framework of a nonquasi-analytic Carleman class, the stability of a class of cohomological equations.

1. Introduction

The important concept of stability of a functional equation was first introduced by Ulam in 1940 when he asked in a talk before the Mathematics Club of the University of Wisconsin ([46]) the following question:

"Let $G_1$ be a group and let $(G_2, d)$ be a metric group. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality:

$$d(h(xy), h(x)h(y)) \leq \delta$$

for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with:

$$d(H(x), h(x)) \leq \varepsilon$$

for all $x \in G_1$?"

Hyers ([22]) was the first to answer partially this question when he showed in 1941 the following result:

"If $E_1$, $E_2$ are Banach spaces and $f : E_1 \to E_2$ is a mapping which satisfy, for some constant $\delta > 0$ and for all $x, y \in E_1$, the condition:

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

then there exists a unique mapping $T : E_1 \to E_2$ such that:

$$T(x + y) = T(x) + T(y)$$

for all $x, y \in E_1$ and:

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E_1$.”

In 1978 Rassias ([43]) has generalized the result of Hyers in the following way:

"Let $f : E_1 \to E_2$ be a mapping between Banach spaces and let $p < 1$ be fixed. If $f$ satisfies, the inequality:

$$\|f(x + y) - f(x) - f(y)\| \leq \theta (\|x\|^p + \|y\|^p)$$

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holds for each \( x, y \in E_1 \) (resp. all \( x, y \in E_1 \setminus \{0\} \)) and for some constant \( \theta > 0 \).

Then there exists a unique mapping \( T : E_1 \to E_2 \) such that:

\[
T(x + y) = T(x) + T(y)
\]

for all \( x, y \in E_1 \) and:

\[
\|T(x) - f(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p
\]

for all \( x \in E_1 \) (resp. all \( x \in E_1 \setminus \{0\} \)). If in addition, \( t \mapsto f(tx) \) is continuous for each fixed \( x \in E_1 \), then \( T \) is linear.

In 1994 Gavruta ([15]) has given a new generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. In fact he showed that:

"Let \( G \) be an abelian group and \((X, \|\cdot\|)\) a Banach space. Let \( \varphi : G \times G \to \mathbb{R}^+ \) a mapping satisfying, for all \( x, y \in G \), the condition:

\[
\tilde{\varphi}(x, y) := \sum_{k=0}^{+\infty} 2^{-k} \varphi(2^k x, 2^k y) < +\infty
\]

Let \( f : G \to X \) be a mapping which fullfiles, for each \( x, y \in G \), the condition:

\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)
\]

Then there exists a unique mapping \( T : G \to X \) such that:

\[
T(x + y) = T(x) + T(y)
\]

for all \( x, y \in G \) and:

\[
\|f(x) - T(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x)
\]

for all \( x \in G \)."

In this paper we introduce the notion of generalized Gavruta stability of functional equations in order to study, in the framework of a nonquasianalytic Carleman class \( C_M\{R\} \), the stability of the so-called cohomological equation \((E_{\psi, \chi})\):

\[
(E_{\psi, \chi}) : f - (f \circ \psi) = \chi
\]

where \( f \) is the unknown function and \( \chi : \mathbb{R} \to \mathbb{C}, \psi : \mathbb{R} \to \mathbb{R} \) are a given functions belonging to \( C_M\{\mathbb{R}\} \). Let us recall that cohomological equations play a fundamental role in the study of dynamical systems. Indeed, the study of certain forms of invariance, rigidity and stability of dynamical systems can be reduced to the investigation of the solvability in certain regularity classes of some cohomological equations ([11], [7]-[10], [14]-[30], [32]-[47]). However, despite the great interest devoted to these functional equations, there is at our knowledge a lack of works on their solvability and their stability in the setting of Carleman classes. Finally let us pointwise that we were mainly motivated in the preparation of this paper, by the works ([1], [5]) of G. Belitskii, E. M. Dyn’kin and V. Tkachenko. Finally to illustrate our main result, we will consider the cohomological equations of the form:

\[
(E_{\chi}) : f(x) - f(x + 1) = \chi(x)
\]

which are a particular case of a functional equations called traditionally difference equations. Let us also recall that, such a functional equations were studied by numerous authors ([12], [41], [18], [19], [39], [37], [28], [13], [6], [20], [29], [45], [40], [23], [3], etc.) because of their great importance in applied and fundamental sciences.
2. Preliminary notes and statement of the main result

2.1. Basic notations and main definitions. For all \( x \in \mathbb{R} \) we set:

\[
\begin{align*}
\lfloor x \rfloor & := \max \{ \{ p \in \mathbb{Z} : p \leq x \} \\
\{ x \} & := x - \lfloor x \rfloor \\
\lceil x \rceil & := \min \{ \{ p \in \mathbb{Z} : x \leq p \} \\
x^+ & := \max (x, 0)
\end{align*}
\]

We denote by \( B_r \), for each \( r \in \mathbb{N} \), the Bernoulli number of order \( r \) ([11], page 297, 299).

Let \( f : S \rightarrow \mathbb{C} \) be a function. \( \| f \|_{\infty, S} \) denotes the quantity:

\[
\| f \|_{\infty, S} := \sup_{z \in S} |f(z)|
\]

Let \( X \) be a nonempty set and \( F : X \rightarrow X \) a mapping. We denote by \( F^{\langle n \rangle} \) for each \( n \in \mathbb{N} \) the iterate of order \( n \) of the mapping \( F \). If \( F \) is a bijection, then we will denote by \( F^{\langle -1 \rangle} \) the compositional inverse of the mapping \( F \) and for each \( n \in \mathbb{N} \), by \( F^{\langle -n \rangle} \) the iterate of order \( n \) of the mapping \( F^{\langle -1 \rangle} \).

Definition 2.1. Let \( E \) be a nonempty set, \( F \) a nonempty subset of the set of mappings from \( E \) to a metric space \((V,d)\), \( \Phi : F \rightarrow F \) a given mapping and \( g \) a given element of \( F \). We say that the functional equation:

\[
\Phi(y)(x) = g(x)
\]

has the generalized Găvruta stability (GGS) in \( F \) if the following condition is fulfilled:

For every mapping \( \delta : E \rightarrow \mathbb{R}^+ \) there exists a mapping \( \mu : E \rightarrow \mathbb{R}^+ \) depending only on \( \delta \) and \( \Phi \) such that for each mapping \( y \in F \) satisfying the inequality:

\[
d(\Phi(y)(x), g(x)) \leq \delta(x), \ x \in E
\]

there exists a solution \( z \in F \) of the functional equation (2.1) such that the following condition holds:

\[
d(y(x), z(x)) \leq \mu(x), \ x \in E
\]

Definition 2.2. Let \( A := (A_n)_{n \geq 0} \) be a sequence of strictly positive real numbers.

i. The Carleman class \( C_A \{ \mathbb{R} \} \) is then the set of all functions \( f : \mathbb{R} \rightarrow \mathbb{C} \) of class \( C^\infty \) such that:

\[
\| f^{(n)} \|_{\infty, K} \leq C_K \rho_K^n A_n, \ n \in \mathbb{N}
\]

for every compact interval \( K \) of \( \mathbb{R} \) with some constants \( C_K, \rho_K > 0 \).

ii. The Carleman class \( C_A \{ \mathbb{R} \} \) is said to be nonquasianalytic if there exists a nonidentically vanishing function \( f_0 \in C_A \{ \mathbb{R} \} \) such that:

\[
f_0^{(n)}(x_0) = 0, \ n \in \mathbb{N}
\]

for some \( x_0 \in \mathbb{R} \).

iii. The sequence \( A \) is said to be almost increasing if there exists a constant \( C > 0 \) such that:

\[
A_p \leq CA_q \text{ if } p \leq q
\]
2.2. Assumptions and related notations. Along this paper we make the following assumptions:

- $a < b$ are a fixed real numbers.
- $M := (M_n)_{n \geq 0}$ is a fixed sequence of strictly positive real numbers such that $C_M(\mathbb{R})$ is nonquasianalytic and the following conditions hold:

\begin{enumerate}
\item[(2.2)] the sequence $\left( \frac{M_n}{n!} \right)_{n \in \mathbb{N}}$ is almost increasing
\item[(2.3)] $1 = M_0 \leq M_1$
\item[(2.4)] $\left( \frac{M_{n+1}}{(n+1)!} \right)^2 \leq \frac{M_n}{n!} \frac{M_{n+2}}{(n+2)!}, n \in \mathbb{N}$
\item[(2.5)] $\sup_{n \in \mathbb{N}} \left( \frac{M_{n+1}}{(n+1)!} \frac{M_n}{n!} \right)^{\frac{1}{2}} < +\infty$
\item[(2.6)] $\liminf_{n \to +\infty} \left( \frac{M_n}{n!} \right)^{\frac{1}{2}} > 0$
\end{enumerate}

- $\psi, \chi: \mathbb{R} \to \mathbb{R}$ are fixed functions belonging to the Carleman class $C_M(\mathbb{R})$ such that the following conditions hold:

\begin{enumerate}
\item[(2.7)] $\psi(s) > s, s \in \mathbb{R}$
\item[(2.8)] $\lim_{t \to -\infty} \psi(t) = -\infty$
\item[(2.9)] $\psi'(s) > 0, s \in \mathbb{R}$
\end{enumerate}

**Remark 2.3.** It follows from the assumptions (2.7-2.9) that $\psi$ is a diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and that the following relations hold for each $s \in \mathbb{R}$:

\begin{align*}
\lim_{n \to +\infty} \psi^{(n)}(s) &= +\infty \\
\lim_{n \to +\infty} \psi^{(-n)}(s) &= -\infty
\end{align*}

Let us set for every $s, t \in \mathbb{R}$:

\begin{align*}
\mathcal{N}_{\psi, t}^+(s) &:= \min \left\{ p \in \mathbb{N} : \psi^{(p)}(s) \geq t \right\} \\
\mathcal{N}_{\psi, t}^-(s) &:= \min \left\{ p \in \mathbb{N} : \psi^{(-p)}(s) \leq t \right\}
\end{align*}

It is clear that the integer valued function:

\begin{align*}
\mathcal{N}_{\psi, t}^+: \mathbb{R} &\to \mathbb{R}^+ \\
s &\mapsto \mathcal{N}_{\psi, t}^+(s)
\end{align*}

is increasing while the integer valued function:

\begin{align*}
\mathcal{N}_{\psi, t}^-: \mathbb{R} &\to \mathbb{R}^+ \\
s &\mapsto \mathcal{N}_{\psi, t}^-(s)
\end{align*}

is decreasing. We can easily prove that the following inequality holds for each $s \in \mathbb{R}$ and all real numbers $t_1 < t_2$:

$$-\mathcal{N}_{\psi, t_1}^-(s) \leq \mathcal{N}_{t_2}^+(s) - 1$$
2.3. Statement of the main result. Our main result in this paper is the following theorem.

**Theorem 1.**

The cohomological equation:

\[(E_{\psi, \chi}) : f - (f \circ \psi) = \chi\]

has, under the above assumptions, the GGS in the Carleman class \( C_M(\mathbb{R}) \). More precisely if a function \( y \in C_M(\mathbb{R}) \) satisfies the condition:

\[|y(s) - y(\psi(s)) - \chi(s)| \leq \delta(s), \; s \in \mathbb{R}\]

where \( \delta : \mathbb{R} \to \mathbb{R}^+ \), then there exists a solution \( z \in C_M(\mathbb{R}) \) of the CE \( (E_{\psi, \chi}) \) such that:

\[|y(s) - z(s)| \leq \sum_{-\psi,a(s) \leq n \leq \psi,b(s) - 1} \delta(\psi(n))(s), \; s \in \mathbb{R}\]

3. Proof of the main result

3.1. A key result. We prove first the following result.

**Proposition 2.**

The cohomological equation:

\[(E_{\psi, \chi}) : f - (f \circ \psi) = \chi\]

has a solution \( g \) in the Carleman class \( C_M(\mathbb{R}) \) such that the following inequality holds for every \( s \in \mathbb{R} \):

\[|g(s)| \leq \sum_{-\psi,a(s) \leq n \leq \psi,b(s) - 1} |\chi(\psi(n))(s)|\]

**Proof.** Since the Carleman class \( C_M(\mathbb{R}) \) is nonquasianalytic there exists, thanks to a result due to S. Mandelbrojt (1921), a function \( \kappa \in C_M(\mathbb{R}) \) such that:

\[
\begin{cases}
0 \leq \kappa(s) \leq 1, \; s \in \mathbb{R} \\
\kappa(s) = 0, \; s \leq a \\
\kappa(s) = 1, \; s \geq b
\end{cases}
\]

Then let us set for every \( s \in \mathbb{R} \):

\[\chi_-(s) = \kappa(s) \chi(s), \; \chi_+(s) = (1 - \kappa(s)) \chi(s)\]

The functions \( \chi_+ \) and \( \chi_- \) belong to \( C_M(\mathbb{R}) \) and satisfy the following conditions:

\[
\begin{cases}
\chi_+(s) = 0, \; s \geq b \\
\chi_-(s) = 0, \; s \leq a
\end{cases}
\]

Since \( \psi \) is a diffeomorphism from \( \mathbb{R} \) onto \( \mathbb{R} \) and belongs to the Carleman class \( C_M(\mathbb{R}) \) it follows from the assumptions (2.2) - (2.6), according to (2), that \( f \circ \psi, f \circ \psi^{(-1)} \) belong to the Carleman class \( C_M(\mathbb{R}) \) for each \( f \in C_M(\mathbb{R}) \). Let us then define the operators:

\[
\begin{align*}
\mathcal{L}_+ : & \quad C_M(\mathbb{R}) \to C_M(\mathbb{R}) \\
& f \mapsto f \circ \psi \\
\mathcal{L}_- : & \quad C_M(\mathbb{R}) \to C_M(\mathbb{R}) \\
& f \mapsto f \circ \psi^{(-1)}
\end{align*}
\]
On the other hand it follows from the assumptions on the function $\psi$ that the sequences of intervals $([\psi^{(n)}(b), +\infty])_{n \in \mathbb{N}}$ and $([-\infty, \psi^{(n)}(a)])_{n \in \mathbb{N}}$ are both increasing coverings of $\mathbb{R}$. Thence the following inclusion holds for each compact interval $K := [\alpha, \beta]$ of $\mathbb{R}$:

$$K \subset [\psi^{(-N^-_{\psi,a}(\alpha))}(a), +\infty[ \cap [-\infty, \psi^{(N^+_{\psi,b}(\beta))}(b)]$$

Furthermore we have for every $s \in K$ and $n \in \mathbb{N}$:

$$\left\{ \begin{array}{l}
\chi_+(\psi^{(n)}(s)) = 0 \text{ if } n \geq N^+_{\psi,b}(\beta) \\
(-\chi_-) \circ \psi^{(-1)}((\psi^{(-1)})^{(n)}(s)) = 0 \text{ if } n \geq N^-_{\psi,a}(\alpha)
\end{array} \right.$$

Thence the series $\sum L_+^{(n)}(\chi_+(s))$ and $\sum L_-^{(n)}((-\chi_-) \circ \psi^{(-1)})(s)$ contain finitely many non-vanishing terms. Consequently the functions $g_+, g_- : \mathbb{R} \to \mathbb{C}$ defined by the relations:

$$g_+(s) := \sum_{0 \leq n \leq N^+_{\psi,a}(s)-1} L_+^{(n)}(\chi_+(s))$$
$$g_-(s) := \sum_{0 \leq n \leq N^-_{\psi,a}(s)-1} L_-^{(n)}((-\chi_-) \circ \psi^{(-1)})(s)$$

belong to $C_M(\mathbb{R})$ according to (2). Furthermore easy computations show that the following estimates hold for each $s \in \mathbb{R}$:

$$\left\{ \begin{array}{l}
|g_+(s)| \leq \sum_{0 \leq n \leq N^+_{\psi,a}(s)-1} |\chi(\psi^{(n)}(s))| \\
|g_-(s)| \leq \sum_{-N^-_{\psi,a}(s) \leq n \leq -1} |\chi(\psi^{(n)}(s))|
\end{array} \right.$$

(3.1)

It is also clear that we have for every $s \in \mathbb{R}$:

$$\left\{ \begin{array}{l}
g_+(s) - g_+(\psi(s)) = \chi_+(s) \\
g_-(s) - g_-(\psi(s)) = \chi_-(s)
\end{array} \right.$$

(3.2)

It follows from (3.1) and (3.2) that the function $g := g_+ + g_-$ belongs to $C_M(\mathbb{R})$ and is a solution of the cohomological equation $(E_{\psi, \chi})$ such that:

$$|g(s)| \leq \sum_{-N^-_{\psi,a}(s) \leq n \leq N^+_{\psi,a}(s)-1} |\chi(\psi^{(n)}(s))|, \ s \in \mathbb{R}$$

The proof of the proposition is then complete.

3.2. End of the proof of the main result. Let $y \in C_M(\mathbb{R})$ and $\delta : \mathbb{R} \to \mathbb{R}^+$. We assume that the following inequality holds for every $s \in \mathbb{R}$:

$$|y(s) - y(\psi(s)) - \chi(s)| \leq \delta(s)$$

Let us then consider the function:

$$\varphi : \mathbb{R} \quad \rightarrow \quad \mathbb{C}$$
$$s \quad \mapsto \quad y(s) - y(\psi(s)) - \chi(s)$$
Then thanks to (2), the function \( \varphi \) belongs to the Carleman class \( C_M(\mathbb{R}) \). According to the proposition 3, there exists a function \( h \in C_M(\mathbb{R}) \) such that:

\[
h(s) - h(\psi(s)) = \varphi(s), \quad s \in \mathbb{R}
\]

Then the function \( z := y - h \in C_M(\mathbb{R}) \) is a solution of the cohomological equation \((E_{\psi,\chi})\). Furthermore we have for each \( s \in \mathbb{R} \):

\[
|y(s) - z(s)| = |h(s)| \\
\leq \sum_{-N^-_\psi(s) \leq n \leq N^+_\psi(s) - 1} |\varphi(\psi^{(n)}(s))|, \quad s \in \mathbb{R}
\]

It follows that the cohomological equation \((E_{\psi,\chi})\) has the GGS in the Carleman class \( C_M(\mathbb{R}) \).

We have then achieved the proof of our main result.

\[ \square \]

4. Example

The function \( \psi_0 : s \mapsto s + 1 \) satisfies the conditions (2.7-2.9). Furthermore the following relations hold for every \( s, t \in \mathbb{R} \):

\[
N^+_{\psi_0,t}(s) = \lceil (t - s)^+ \rceil, \quad N^-_{\psi_0,t}(s) = \lceil (s - t)^+ \rceil
\]

Then, according to the above main result, the cohomological equation:

\[
(E_{\psi_0,\chi}) : f(s) - f(s + 1) = \chi(s)
\]

is satisfied if \( \chi \in C_M(\mathbb{R}) \) and the function \( \delta : \mathbb{R} \to \mathbb{R}^+ \) satisfies the condition

\[
|y(s) - y(s + 1) - \chi(s)| \leq \delta(s), \quad s \in \mathbb{R}
\]

then there exists a solution \( z \in C_M(\mathbb{R}) \) of the CE \((E_{\psi_0,\chi})\) such that

\[
|y(s) - z(s)| \leq \sum_{n=-(s-a)^+}^{(b-s)^+ - 1} \delta(s + n), \quad s \in \mathbb{R}
\]

1. If the function \( \delta \) is periodic with period 1, then the estimate (4.1) becomes:

\[
|y(s) - z(s)| \leq \left( \lceil (b - s)^+ \rceil + \lceil (s - a)^+ \rceil \right) \delta(s)
\]

2. If the function \( \delta \) is of class \( C^1 \) on \( \mathbb{R} \) then we can improve the estimate (4.1) by means of a special case of the Euler Mac-Laurin formula (11), page 302-303).
Indeed we have for each \( s \in \mathbb{R} \):

\[
\begin{align*}
\left\lfloor (b-s)^+ \right\rfloor - 1 & \sum_{n=-\left\lfloor (s-a)^+ \right\rfloor}^{\left\lfloor (b-s)^+ \right\rfloor - 1} \delta(s+n) \\
= & \int_{-(s-a)^+}^{-(s-a)^+} \delta(s+t)dt + \\
& + \frac{\delta \left( s - \left\lfloor (s-a)^+ \right\rfloor \right) + \delta \left( s + \left\lfloor (b-s)^+ \right\rfloor - 1 \right)}{2} \\
& + \int_{-(s-a)^+}^{s+(b-s)^+} \left( \{t\} - \frac{1}{2} \right) \delta'(s+t)dt \\
& + \int_{s-(s-a)^+]}^{s+(b-s)^+]_{\sigma}} \left( \{u\} - \frac{1}{2} \right) \delta'(u)du \\
\leq & \left\lfloor (b-s)^+ \right\rfloor + \left\lfloor (s-a)^+ \right\rfloor \|\delta\|_{\infty, x_s} + \\
& + \frac{\left\lfloor (b-s)^+ \right\rfloor + \left\lfloor (s-a)^+ \right\rfloor - 1}{2} \|\delta'\|_{\infty, x_s}
\end{align*}
\]

where \( I_s \) denotes the interval \([s - \left\lfloor (s-a)^+ \right\rfloor, s + \left\lfloor (b-s)^+ \right\rfloor - 1]\). Thence the estimate (4.1) entails that :

\[
\begin{align*}
|y(s) - z(s)| & \leq \left\lfloor (b-s)^+ \right\rfloor + \left\lfloor (s-a)^+ \right\rfloor \|\delta\|_{\infty, x_s} + \\
& + \frac{\left\lfloor (b-s)^+ \right\rfloor + \left\lfloor (s-a)^+ \right\rfloor - 1}{2} \|\delta'\|_{\infty, x_s}
\end{align*}
\]

3. If the function \( \delta \) is of class \( C^{2r+1} \) on \( \mathbb{R} (r \in \mathbb{N}^*) \) then we can improve the estimate (4.1) by means of a the general Euler Mac-Laurin formula (II), page
Indeed we have for each $s \in \mathbb{R}$:

$$|y(s) - z(s)| \leq \sum_{n=-[(s-a)^+] \ldots [(s-a)^+ - 1]}^{[(b-s)^+] - 1} \delta(s + n)$$

Finally the estimate (4.1) becomes:

$$|y(s) - z(s)| \leq \sum_{n=-[(s-a)^+] \ldots [(s-a)^+ - 1]}^{[(b-s)^+] - 1} \delta(s + n)$$

Finally the estimate (4.4) becomes:

$$|y(s) - z(s)| \leq \left( \sum_{n=-[(s-a)^+] \ldots [(s-a)^+ - 1]}^{[(b-s)^+] - 1} \delta(s + n) \right) \|\delta\|_{\infty, J_1} +$$

$$+ \sum_{j=1}^{r} B_j \left( \sum_{n=-[(s-a)^+] \ldots [(s-a)^+ - 1]}^{[(b-s)^+] - 1} \delta(s + n) \right) \|\delta^{(2j)}\|_{\infty, J_1} +$$

$$+ \sum_{j=1}^{r} B_j \left( \sum_{n=-[(s-a)^+] \ldots [(s-a)^+ - 1]}^{[(b-s)^+] - 1} \delta(s + n) \right) \|\delta^{(2j+1)}\|_{\infty, J_1} +$$

Finally the estimate (4.4) becomes:

$$|y(s) - z(s)| \leq \left( \sum_{n=-[(s-a)^+] \ldots [(s-a)^+ - 1]}^{[(b-s)^+] - 1} \delta(s + n) \right) \|\delta\|_{\infty, J_1} +$$

$$+ \sum_{j=1}^{r} B_j \left( \sum_{n=-[(s-a)^+] \ldots [(s-a)^+ - 1]}^{[(b-s)^+] - 1} \delta(s + n) \right) \|\delta^{(2j)}\|_{\infty, J_1} +$$

$$+ \sum_{j=1}^{r} B_j \left( \sum_{n=-[(s-a)^+] \ldots [(s-a)^+ - 1]}^{[(b-s)^+] - 1} \delta(s + n) \right) \|\delta^{(2j+1)}\|_{\infty, J_1} +$$

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