Decay properties of the $Z_c(3900)$ through the Fierz rearrangement

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We systematically construct all the tetraquark currents/operators of $J^{PC}=1^{+-}$ with the quark configurations $[cq][ar{c}ar{q}]$, $[cq][ar{q}c]$, and $[ar{c}c][ar{q}q]$ ($q = u/d$), and derive their relations through the Fierz rearrangement of the Dirac and color indices. Using the transformations of $[cq][ar{q}c] \rightarrow [ar{c}c][ar{q}q]$ and $[cq][ar{q}c]$, we study decay properties of the $Z_c(3900)$ as a compact tetraquark state; while using the transformation of $[cq][ar{q}c] \rightarrow [cc][qq]$, we study its decay properties as a hadronic molecular state.

Keywords: Fierz transformation, exotic hadron, compact tetraquark, hadronic molecule

I. INTRODUCTION

In the past twenty years many charmonium-like $XYZ$ states were discovered in particle experiments [1]. All of them are good multiquark candidates, and their relevant experimental and theoretical studies have significantly improved our understanding of the strong interaction at the low energy region. Especially, in 2013 BESIII reported the $Z_c(3900)^+$ in the $Y(4260) \to J/\psi \pi^+ \pi^-$ process [2], which was later confirmed by Belle [3] and CLEO [4]. Since it couples strongly to the charmonium and yet it is charged, the $Z_c(3900)^+$ is not a conventional charmonium state and contains at least four quarks. It is quite interesting to understand how it is composed of these four quarks, and there have been various models developed to explain this, such as a compact tetraquark state composed of a diquark and an antidiquark [5, 6], a loosely-bound hadronic molecular state composed of two charmed mesons [7–10], a hadro-quarkonium [11], or due to the kinematical threshold effect [12, 13], etc. We refer to reviews [14–18] for detailed discussions.

The charged charmonium-like state $Z_c(3900)$ of $J^{PC}=1^{+-}$ [19] has been observed in the $J/\psi \pi$ and $DD^*$ channels [2, 3, 20, 21], and there was some events in the $h_c \pi$ channel [22]. In a recent BESIII experiment [23], evidence for the $Z_c(3900) \to \eta_c \rho$ decay was reported with a statistical significance of $3.9\sigma$ at $\sqrt{s}=4.226$ GeV, and the relative branching ratio

$$\mathcal{R}_{Z_c} = \frac{B(Z_c(3900) \to \eta_c \rho)}{B(Z_c(3900) \to J/\psi \pi)},$$

was evaluated to be $2.2 \pm 0.9$ at the same center-of-mass energy. This ratio has been studied by many theoretical methods/models [24–32], and was suggested in Ref. [33] to be useful to discriminate between the compact tetraquark and hadronic molecule scenarios. As summarized in Table 1, this ratio was calculated in many molecular models, but the extracted values are highly model dependent. Hence, it would be useful to derive a model independent result, and it would be even better if one could do this within the same framework for both the tetraquark and molecule scenarios.

In this paper we shall study decay properties of the $Z_c(3900)$ under both the compact tetraquark and hadronic molecule interpretations. The present study is based on our previous finding that the diquark-antidiquark currents ([qq][¯qq]) and the meson-meson currents ([qq][¯qq]) are related to each other through the Fierz rearrangement of the Dirac and color indices [34–44]. See also our studies on light baryon operators [45–47]. In the present case the $Z_c(3900)$ contains four quarks, that is the $c, \bar{c}, q, \bar{q}$ quarks ($q = u/d$), so there are three configurations:

$$[cq][\bar{c}\bar{q}], \ [cq][\bar{q}c], \ \text{and} \ [\bar{c}c][qq].$$

Again, the Fierz rearrangement can be applied to relate them. Based on these relations, we shall extract some decay properties of the $Z_c(3900)$ in this paper.

There are eight independent $[cq][\bar{c}\bar{q}]$ currents of $J^{PC}=1^{+-}$, which have been systematically constructed in Ref. [48]. Here we choose one of them,

$$\eta^Z_\mu = \epsilon^{abc} c^{cde} \bar{q}^a_6 C \gamma_\mu e b \bar{q}^c_5 C e d - \{ \gamma_\mu \leftrightarrow \gamma_5 \},$$

where $C$ is the charge conjugation matrix, the subscripts $a \cdots e$ are color indices, and the sum over repeated indices is taken. This current would strongly couple to the $Z_c(3900)$, if it has the same internal structure (internal symmetry) as that state.

The above current is useful from the viewpoints of both effective field theory and QCD sum rules. Note that there are various quark-based effective field theories, which have been successfully applied to describe the meson and baryon systems, such as the Non-Relativistic QCD for the heavy quarkonium system [49, 50]:

$$\mathcal{L}_{\text{NRQCD}} = \psi^\dagger \{ iD_0 + \cdots \} \psi + \chi^\dagger \{ iD_0 + \cdots \} \chi + \frac{f_1(1S_0)}{m_1m_2} \psi^\dagger \chi \chi^\dagger \psi + \frac{f_1(3S_0)}{m_1m_2} \psi^\dagger T^a \chi \chi^\dagger T^a \psi + \frac{f_3(1S_0)}{m_1m_2} \psi^\dagger T^a \sigma \chi \chi^\dagger T^a \sigma \psi + \cdots .$$

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TABLE I: The relative branching ratio $\mathcal{R}_{Z_c} \equiv B(Z_c(3900) \to \eta_c \rho)/B(Z_c(3900) \to J/\psi \pi)$, calculated by various theoretical methods/models.

| Interpretations | $\mathcal{R}_{Z_c}$ | Methods/Models |
|-----------------|----------------------|----------------|
| compact tetraquark | $(2.3^{+3.3}_{-1.4}) \times 10^2$ | Type-I diquark-antidiquark model [33] |
| hadronic molecule | $(4.6^{+2.7}_{-1.5}) \times 10^{-2}$ | Non-Relativistic effective field theory [33] |
| QCD sum rules | 0.12 | light front model [28] |
| QCD sum rules | 0.57 | effective field theory [29] |
| QCD sum rules | 1.1 | covariant quark model [27] |
| QCD sum rules | 1.28 | covariant quark model [27] |

We refer to Ref. [51] for detailed review of this method. The above Lagrangian contains four four-fermion operators, which can be used to study the annihilation width of a heavy quarkonium into light particles. In this method the Fierz rearrangement is applied to decouple the Dirac tors, which can be used to study the annihilation width of a heavy quarkonium into light particles. In this method the Fierz rearrangement is applied to decouple the Dirac tors, which can be used to study the annihilation width of a heavy quarkonium into light particles. In this method the Fierz rearrangement is applied to decouple the Dirac tors, which can be used to study the annihilation width of a heavy quarkonium into light particles. In this method the Fierz rearrangement is applied to decouple the Dirac tors, which can be used to study the annihilation width of a heavy quarkonium into light particles. In this method the Fierz rearrangement is applied to decouple the Dirac tors, which can be used to study the annihilation width of a heavy quarkonium into light particles.

For instance (see Table II below), the above eight-quark operator can describe the fall-apart decays of the $Z_c(3900)$ into the $\eta_c \rho$ and $J/\psi \pi$ final states simultaneously, together with some other possible decay channels. In order to extract the widths of these decays, one still needs to do further calculations, which we shall not study any more. However, their relative branching ratios can be extracted much more easily, which are also useful and important to understand the nature of the $Z_c(3900)$ [52].

Consider that the meson operators, $\bar{q} \gamma _5 q$, $\bar{q} \gamma _\mu q$, $\bar{c} \gamma _5 c$, and $\bar{c} \gamma _\mu c$ couple to the $\pi$, $\rho$, $\eta_c$, and $J/\psi$ mesons respectively (see Table II below), the above eight-quark operator can describe the fall-apart decays of the $Z_c(3900)$ into the $\eta_c \rho$ and $J/\psi \pi$ final states simultaneously, together with some other possible decay channels. In order to extract the widths of these decays, one still needs to do further calculations, which we shall not study any more. However, their relative branching ratios can be extracted much more easily, which are also useful and important to understand the nature of the $Z_c(3900)$ [52].

The current $\eta_c^Z$ can also be investigated from the viewpoint of QCD sum rules [53, 54]. We assume it couples to the $Z_c(3900)$ through

$$\langle 0|\eta_c^Z|Z_c \rangle = f_{Z_c} \epsilon_\mu .$$

After the Fierz rearrangement, $\eta_c^Z$ transforms to the long expression inside Eq. (5). Through the first and second terms, it couples to the $\eta_c \rho$ and $J/\psi \pi$ channels simultaneously:

$$\langle 0|\eta_c^Z|\eta_c \rho \rangle = \frac{1}{3} \langle 0|\bar{c} c \gamma_5 c a c b \gamma_\mu q_{\bar{q} \gamma_\mu q_{\bar{q}}} + \cdots \rangle,$$

$$\langle 0|\eta_c^Z|J/\psi \pi \rangle = -\frac{1}{3} \langle 0|\bar{c} c c a c b c q_{\bar{q} \gamma_5 q_{\bar{q}}} \pi \rangle + \cdots .$$

Again, these two equations can be easily used to calculate the relative branching ratio $\mathcal{R}_{Z_c}$. Detailed discussions on this will be given below.

In the above equations we have followed the idea of the QCD factorization method [55–57], which has been
widely and successfully applied to study weak decay properties of (heavy) hadrons. In the present study we just need to replace the weak-interaction Lagrangian by some interpolating current, and the similar techniques can apply here, together with the Fierz rearrangement, to study strong decay properties of the $Z_c(3900)$. Note that a similar arrangement of the spin and color indices in the nonrelativistic case was used to study strong decay properties of the $Z_c(3900)$ in Refs. [8, 58, 59].

This paper is organized as follows. In Sec. II we systematically construct all the tetraquark currents of $J^{PC} = 1^{++}$ with the quark content $ccqq$. There are three configurations, $[cq][cq]$, $[cq][qar{c}]$, and $[cc][qar{q}]$, and their relations are also derived in this section by using the Fierz rearrangement of the Dirac and color indices. In Sec. III we discuss the couplings of meson operators to meson states, and list those which are needed in the present study. In Sec. IV and Sec. V we extract some decay properties of the $Z_c(3900)$, separately for the compact tetraquark interpretation and the hadronic molecule interpretation. The obtained results are discussed and summarized in Sec. VI.

II. TETRAQUARK CURRENTS OF $J^{PC} = 1^{++}$ AND THEIR RELATIONS

By using the $c$, $c$, $q$, $q$ quarks ($q = u/d$), one can construct three types of tetraquark currents, as illustrated in Fig. 1:

$$\gamma_\mu(x,y) = [q_a^T(x) \Gamma_1 c_b(x)] \times [q_\bar{c}(y) \Gamma_2 \bar{c}_\bar{c}^T(y)],$$

$$\xi_\mu(x,y) = [\bar{c}_a(x) \Gamma_3 q_b(x)] \times [\bar{c}_\bar{c}(y) \Gamma_4 c_\bar{c}(y)],$$

$$\theta_\mu(x,y) = [\bar{c}_a(x) \Gamma_5 c_b(x)] \times [\bar{c}_\bar{c}(y) \Gamma_6 q_\bar{c}(y)],$$

where $\Gamma_i$ are Dirac matrices, $C$ is the charge-conjugation matrix, the subscripts $a, b, c, d$ are color indices, and the sum over repeated indices is taken. One usually call $\gamma_\mu(x,y)$ the diquark-antidiquark current, and $\xi_\mu(x,y)$ and $\theta_\mu(x,y)$ the mesonic-meson currents. We separately treat them as follows.

A. $[cq][q\bar{c}]$ currents $\gamma_\mu(x,y)$

There are altogether eight independent $[cq][q\bar{c}]$ currents of $J^{PC} = 1^{++}$ [48]:

$$\gamma_\mu^1 = \frac{1}{2} \left( q_a^T C \gamma_\mu c_b - q_b^T C \gamma_\mu c_a \right),$$

$$\gamma_\mu^2 = \frac{1}{2} \left( q_a^T C \gamma_\mu c_b - q_b^T C \gamma_\mu c_a \right),$$

$$\gamma_\mu^3 = \frac{1}{2} \left( q_a^T C \gamma_\mu c_b - q_b^T C \gamma_\mu c_a \right),$$

$$\gamma_\mu^4 = \frac{1}{2} \left( q_a^T C \gamma_\mu c_b - q_b^T C \gamma_\mu c_a \right),$$

$$\gamma_\mu^5 = \frac{1}{2} \left( q_a^T C \gamma_\mu c_b - q_b^T C \gamma_\mu c_a \right),$$

$$\gamma_\mu^6 = \frac{1}{2} \left( q_a^T C \gamma_\mu c_b - q_b^T C \gamma_\mu c_a \right),$$

$$\gamma_\mu^7 = \frac{1}{2} \left( q_a^T C \gamma_\mu c_b - q_b^T C \gamma_\mu c_a \right),$$

$$\gamma_\mu^8 = \frac{1}{2} \left( q_a^T C \gamma_\mu c_b - q_b^T C \gamma_\mu c_a \right).$$

Here we have omitted the coordinates $x$ and $y$ for simplicity. Their combinations, $\eta_\mu^1 - \eta_\mu^2$, $\eta_\mu^3 - \eta_\mu^4$, $\eta_\mu^5 - \eta_\mu^6$, and $\eta_\mu^7 - \eta_\mu^8$ have the antisymmetric color structure $[cq][cq] \rightarrow [cq][cq]$, and $\eta_\mu^1 + \eta_\mu^2, \eta_\mu^3 + \eta_\mu^4, \eta_\mu^5 + \eta_\mu^6$, and $\eta_\mu^7 + \eta_\mu^8$ have the symmetric color structure $[cq][cq] \rightarrow [cq][cq]$. In the “type-II” diquark-antidiquark model proposed in Ref. [6], the ground-state tetraquarks can be written in the spin basis as $|s_{qc}, s_{q\bar{c}}\rangle_J$, where $s_{qc}$ and $s_{q\bar{c}}$ are the charmquark and antidiquark spins, respectively. There are two ground-state diquarks: the “good” one of $J^P = 0^+$ and the “bad” one of $J^P = 1^+$ [60]. By combining them, the $Z_c(3900)$ was interpreted as a diquark-antidiquark state of $J^{PC} = 1^{++}$ in Ref. [6]:

$$|0_{qc1q\bar{c}}; 1^{++}\rangle = \frac{1}{\sqrt{2}} \left( |0_{qc}, 1q\bar{c}\rangle_{J=1} - |1_{qc}, 0q\bar{c}\rangle_{J=1} \right).$$

The interpolating current having the identical internal structure is just the current $\eta_\mu^Z$ given in Eq. (2), which has been well studied in Ref. [61]:

$$\eta_\mu^Z(x,y) = \frac{1}{2} \left( [u_{c\bar{c}}^T(x)] \delta_{\bar{c}c} - [u_{c\bar{c}}^T(y)] \delta_{\bar{c}c} \right),$$

$$\gamma_\mu^Z(x,y) = \frac{1}{2} \left( [u_{c\bar{c}}^T(x)] \delta_{\bar{c}c} - [u_{c\bar{c}}^T(y)] \delta_{\bar{c}c} \right).$$

Here we have explicitly chosen the quark content $[uc][\bar{c}c]$ for the positive-charged one $Z_c(3900)^+$. In Ref. [62, 63]:

$$\xi_\mu^Z(x,y) = \frac{1}{2} \left( [\bar{c}_a(x) \gamma_\mu u_a(x)] \tilde{d}_b(y) \gamma_5 c_b(y) + \{\gamma_\mu \leftrightarrow \gamma_5\} \right).$$

Again, we have chosen the quark content $[\bar{c}u][c\bar{c}]$. 
Among them, \( J^{PC} = 1^+ \).

There are altogether eight independent \([\bar{c}c][\bar{q}q]\) currents of \( J^{PC} = 1^+ \):

\[
\theta_{\mu}^1(x, y) = \bar{c}_a(x) \gamma_5 c_a(x) \bar{q}_b(y) \gamma_\mu q_b(y),
\theta_{\mu}^2(x, y) = \bar{c}_a(x) \gamma_\mu c_a(x) \bar{q}_b(y) \gamma_5 q_b(y),
\theta_{\mu}^3(x, y) = \bar{c}_a(x) \gamma^{\nu} \gamma_5 c_a(x) \bar{q}_b(y) \sigma_{\mu\nu} q_b(y),
\theta_{\mu}^4(x, y) = \bar{c}_a(x) \sigma_{\mu\nu} c_a(x) \bar{q}_b(y) \gamma^{\nu} \gamma_5 q_b(y),
\theta_{\mu}^5(x, y) = \lambda_{ab}^{\mu} \lambda_\nu^{ab} \bar{c}_a(x) \gamma_5 c_b(x) \bar{q}_c(y) \gamma_\mu q_d(y),
\theta_{\mu}^6(x, y) = \lambda_{ab}^{\mu} \lambda_\nu^{ab} \bar{c}_a(x) \gamma_\mu c_b(x) \bar{q}_c(y) \gamma_5 q_d(y),
\theta_{\mu}^7(x, y) = \lambda_{ab}^{\mu} \lambda_\nu^{ab} \bar{c}_a(x) \gamma^{\nu} \gamma_5 c_b(x) \bar{q}_c(y) \sigma_{\mu\nu} q_d(y),
\theta_{\mu}^8(x, y) = \lambda_{ab}^{\mu} \lambda_\nu^{ab} \bar{c}_a(x) \sigma_{\mu\nu} c_b(x) \bar{q}_c(y) \gamma^{\nu} \gamma_5 q_d(y).
\]

Among them, \( \theta_{\mu}^{1,2,3,4} \) have the color structure \([\bar{c}c]_{1s}[\bar{q}q]_{1s} \rightarrow [\bar{c}c][\bar{q}q]_{1s} \), and \( \theta_{\mu}^{5,6,7,8} \) have the color structure \([\bar{c}c]_{s}[\bar{q}q]_{s} \rightarrow [\bar{c}c][\bar{q}q]_{1s} \). We will discuss their corresponding hadron states in Sec. III.

D. Fierz rearrangement

We have applied the Fierz rearrangement of the Dirac and color indices to systematically study light baryon and tetraquark operators/currents in Refs. [34–47]. It can also be used to relate the above three types of tetraquark currents. To do this, we need to use a) the Fierz transformation [64] in the Lorentz space to rearrange the Dirac indices, and b) the color rearrangement in the color space to rearrange the color indices. All the necessary equations can be found in Sec. 3.3.2 of Ref. [65].

We obtain the following relation between the local currents \( \eta_{\mu}^i(x, x) \) and \( \theta_{\mu}^i(x, x) \):

\[
\begin{bmatrix}
\eta_{\mu}^1 \\
\eta_{\mu}^2 \\
\eta_{\mu}^3 \\
\eta_{\mu}^4 \\
\eta_{\mu}^5 \\
\eta_{\mu}^6 \\
\eta_{\mu}^7 \\
\eta_{\mu}^8 \\
\end{bmatrix} =
\begin{bmatrix}
1/2 & -1/2 & i/2 & -i/2 & 0 & 0 & 0 & 0 \\
1/6 & -1/6 & i/6 & -i/6 & 1/4 & -1/4 & 1/4 & -1/4 \\
3i/2 & 3i/2 & -1/2 & -1/2 & 0 & 0 & 0 & 0 \\
i/2 & i/2 & -1/6 & -1/6 & 3i/4 & 3i/4 & -1/4 & -1/4 \\
1/2 & 1/2 & -i/2 & -i/2 & 0 & 0 & 0 & 0 \\
1/6 & 1/6 & -i/6 & -i/6 & 1/4 & 1/4 & -i/4 & -i/4 \\
3i/2 & -3i/2 & 1/2 & -1/2 & 0 & 0 & 0 & 0 \\
i/2 & -i/2 & 1/6 & -1/6 & 3i/4 & -3i/4 & 1/4 & -1/4 \\
\end{bmatrix}
\times
\begin{bmatrix}
\theta_{\mu}^1 \\
\theta_{\mu}^2 \\
\theta_{\mu}^3 \\
\theta_{\mu}^4 \\
\theta_{\mu}^5 \\
\theta_{\mu}^6 \\
\theta_{\mu}^7 \\
\theta_{\mu}^8 \\
\end{bmatrix},
\]

FIG. 1: Three types of tetraquark currents. Quarks are shown in red/green/blue color, and antiquarks are shown in cyan/magenta/yellow color.
the following relation between \( \eta_i^{x,x} (x,x) \) and \( \xi_\mu^1 (x,x) \):

\[
\left( \begin{array}{c}
\eta^1
\eta^2
\eta^3
\eta^4
\eta^5
\eta^6
\eta^7
\eta^8
\end{array} \right) = 
\left( \begin{array}{cccccccc}
0 & i/6 & -1/6 & 0 & 0 & i/4 & -1/4 & 0 \\
i/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3i/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
1/6 & 0 & 0 & -i/6 & 1/4 & 0 & 0 & -i/4 \\
1/2 & 0 & 0 & -i/2 & 0 & 0 & 0 & 0 \\
0 & -1/6 & i/2 & 0 & 0 & -1/4 & 3i/4 & 0 \\
0 & -1/2 & 3i/2 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right) 
\times 
\left( \begin{array}{c}
\xi^1
\xi^2
\xi^3
\xi^4
\xi^5
\xi^6
\xi^7
\xi^8
\end{array} \right),
\tag{17}
\]

the following relation among \( \eta^i_\mu (x,x) \), \( \epsilon^{1,2,3,4}_\mu (x,x) \), and \( \theta^{1,2,3,4}_\mu (x,x) \):

\[
\left( \begin{array}{c}
\eta^1_\mu \\
\eta^2_\mu \\
\eta^3_\mu \\
\eta^4_\mu \\
\eta^5_\mu \\
\eta^6_\mu \\
\eta^7_\mu \\
\eta^8_\mu \\
\end{array} \right) = 
\left( \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 1/2 & -1/2 & i/2 & -i/2 \\
i/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3i/2 & 3i/2 & -1/2 & -1/2 \\
-3i/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 1/2 & -i/2 & -i/2 \\
1/2 & 0 & 0 & -i/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3i/2 & -3i/2 & 1/2 & -1/2 \\
0 & -1/2 & 3i/2 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right) 
\times 
\left( \begin{array}{c}
\xi^1_\mu \\
\xi^2_\mu \\
\xi^3_\mu \\
\xi^4_\mu \\
\xi^5_\mu \\
\xi^6_\mu \\
\xi^7_\mu \\
\xi^8_\mu \\
\end{array} \right),
\tag{18}
\]

and the following relation between \( \xi^i_\mu (x,x) \) and \( \theta^i_\mu (x,x) \):

\[
\left( \begin{array}{c}
\epsilon^1_\mu \\
\epsilon^2_\mu \\
\epsilon^3_\mu \\
\epsilon^4_\mu \\
\epsilon^5_\mu \\
\epsilon^6_\mu \\
\epsilon^7_\mu \\
\epsilon^8_\mu \\
\end{array} \right) = 
\left( \begin{array}{cccccccc}
-1/6 & -1/6 & -i/6 & -i/6 & -1/4 & -1/4 & -i/4 & -i/4 \\
i/2 & i/2 & 1/6 & -1/6 & -3i/4 & 3i/4 & 1/4 & -1/4 \\
1/6 & -1/6 & -i/6 & i/6 & 1/4 & -1/4 & -i/4 & i/4 \\
i/2 & i/2 & 1/6 & 1/6 & 3i/4 & 3i/4 & 1/4 & 1/4 \\
-8/9 & -8/9 & -8i/9 & -8i/9 & 1/6 & 1/6 & i/6 & i/6 \\
-8i/3 & 8i/3 & 8/9 & -8/9 & i/2 & -i/2 & -1/6 & 1/6 \\
8/9 & -8/9 & -8i/9 & 8i/9 & -1/6 & 1/6 & i/6 & -i/6 \\
8i/3 & 8i/3 & 8/9 & 8/9 & -i/2 & -i/2 & -1/6 & -1/6 \\
\end{array} \right) 
\times 
\left( \begin{array}{c}
\theta^1_\mu \\
\theta^2_\mu \\
\theta^3_\mu \\
\theta^4_\mu \\
\theta^5_\mu \\
\theta^6_\mu \\
\theta^7_\mu \\
\theta^8_\mu \\
\end{array} \right),
\tag{19}
\]

### III. MESON OPERATORS

There are altogether six types of meson operators: \( \bar{q}_a q_a \), \( \bar{q}_a \gamma^5 q_a \), \( \bar{q}_a \gamma^\mu q_a \), \( \bar{q}_a \gamma^\mu \gamma^5 q_a \), \( \bar{q}_a \sigma_{\mu\nu} q_a \), and \( \bar{q}_a \sigma_{\mu\nu} \gamma^5 q_a \). The last two can be related to each other through

\[
\sigma_{\mu\nu} \gamma^5 = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \sigma^{\rho\sigma}.
\tag{20}
\]

The couplings of these operators to meson states are already well understood, i.e., some of them have been measured in particle experiments, and some of them have been studied and calculated by various theoretical methods, such as Lattice QCD and QCD sum rules, etc.

In the present study we need the following couplings, as summarized in Table II:

1. The scalar operators \( J^S = \bar{q}_a q_a \) and \( I^S = \bar{c}_a c_a \) of \( J^{PC} = 0^{++} \) couple to scalar mesons. In Ref. [66] the authors used the method of QCD sum rules and extracted the coupling of \( I^S \) to \( \chi_{c0}(1P) \) to be

\[
\langle 0 | \bar{c}_a c_a | \chi_{c0}(p) \rangle = m_{\chi_{c0}} \tilde{f}_{\chi_{c0}},
\tag{21}
\]

where

\[
\tilde{f}_{\chi_{c0}} = 343 \text{ MeV}.
\tag{22}
\]

See also discussions in Refs. [67–69]. The light scalar mesons have a complicated nature [70], so we shall not investigate their relevant decay channels in the present study.

2. The pseudoscalar operators \( J^P = \bar{q}_a i \gamma^5 q_a \) and \( I^P = \bar{c}_a i \gamma^5 c_a \) of \( J^{PC} = 0^{-+} \) couple to the pseudoscalar mesons \( \pi \) and \( \eta_c \), respectively. We can
TABLE II: Couplings of meson operators to meson states. Color indices are omitted for simplicity.

| Operators | $J^{PC}$ | Mesons | $J^{PC}$ | Couplings | Decay Constants |
|-----------|----------|--------|----------|------------|-----------------|
| $J^S = du$ | $0^{++}$ | – | $0^{++}$ | – | – |
| $J^P = d\gamma_5 u$ | $0^{-+}$ | $\pi^+$ | $0^{++}$ | \[0\langle J^P | \pi^+ \rangle = \lambda_\pi \]
| $f_{\rho^+} = 208$ MeV [72] |
| $J^\rho_\mu = d\gamma_\mu u$ | $1^{-+}$ | $\rho^+$ | $1^{--}$ | \[0\langle J^\rho_\mu | \rho^+ \rangle = m_\rho f_{\rho^+} \epsilon_\mu \]
| $f_{\rho^+} = 208$ MeV [72] |
| $J^A_\mu = d\gamma_\mu \gamma_5 u$ | $1^{++}$ | $\eta^+$ | $0^{--}$ | \[0\langle J^A_\mu | \eta^+ \rangle = ip_\mu f_{\eta^+} \]
| $f_{\eta^+} = 130.2$ MeV [1] |
| $J^F_\mu = d\sigma_{\mu\nu} u$ | $1^{+-}$ | $\rho^0$ | $1^{--}$ | \[0\langle J^F_\mu | \rho^0 \rangle = i f^T_\rho (p_\mu \epsilon_\nu - p_\nu \epsilon_\mu) \]
| $f^T_\rho = 159$ MeV [72] |
| $I^S = \bar{c}c$ | $0^{++}$ | $\chi_{c0}(1P)$ | $0^{++}$ | \[0\langle I^S | \chi_{c0} \rangle = m_{\chi_{c0}} f_{\chi_{c0}} \]
| $f_{\chi_{c0}} = 343$ MeV [66] |
| $I^P = \bar{c}\gamma_5 c$ | $0^{-+}$ | $\eta_c$ | $0^{-+}$ | \[0\langle I^P | \eta_c \rangle = \lambda_{\eta_c} \]
| $\lambda_{\eta_c} = \frac{f_{\eta_c} m_{\eta_c}}{\sqrt{2} m_c}$ |
| $I^V = \bar{c}\gamma_\mu c$ | $1^{--}$ | $J/\psi$ | $1^{--}$ | \[0\langle I^V_\mu | J/\psi \rangle = m_{J/\psi} f_{J/\psi} \epsilon_\mu \]
| $f_{J/\psi} = 418$ MeV [73] |
| $I^A_\mu = \bar{c}\gamma_\mu \gamma_5 c$ | $1^{++}$ | $\eta_c$ | $0^{-+}$ | \[0\langle I^A_\mu | \eta_c \rangle = ip_\mu f_{\eta_c} \]
| $f_{\eta_c} = 387$ MeV [73] |
| $I^F_\mu = \bar{c}\sigma_{\mu\nu} c$ | $1^{+-}$ | $J/\psi$ | $1^{--}$ | \[0\langle I^F_\mu | J/\psi \rangle = i f^T_{J/\psi} (p_\mu \epsilon_\nu - p_\nu \epsilon_\mu) \]
| $f^T_{J/\psi} = 410$ MeV [73] |
| $O^S = d\bar{c}$ | $0^+$ | $D^{*0}_{0}^{+}$ | $0^+$ | \[0\langle O^S | D^{*0}_{0}^{+} \rangle = m_{D^{*0}_{0}^{+}} f_{D^{*0}_{0}^{+}} \]
| $f_{D^{*0}_{0}^{+}} = 410$ MeV [98] |
| $O^P = d\gamma_5 c$ | $0^-$ | $D^+$ | $0^-$ | \[0\langle O^P | D^+ \rangle = \lambda_D \]
| $\lambda_D = \frac{f_{\eta_c} m_{\eta_c}}{2 m_c}$ |
| $O^V = \bar{c}\gamma_\mu u$ | $1^-$ | $D^a$ | $1^-$ | \[0\langle O^V_\mu | D^a \rangle = m_{D^a} f_{D^a} \epsilon_\mu \]
| $f_{D^a} = 253$ MeV [95] |
| $O^A_\mu = \bar{c}\gamma_\mu \gamma_5 u$ | $1^+$ | $D^0$ | $0^-$ | \[0\langle O^A_\mu | D^0 \rangle = ip_\mu f_D \]
| $f_D = 211.9$ MeV [1] |
| $O^F_\mu = \bar{c}\sigma_{\mu\nu} c$ | $1^+$ | $D^{*+}$ | $1^-$ | \[0\langle O^F_\mu | D^{*+} \rangle = i f^T_D (p_\mu \epsilon_\nu - p_\nu \epsilon_\mu) \]
| $f^T_D \approx 220$ MeV |
| $= 1^+$ | – | – | – | – |

3. The vector operators $J^V_\mu = \bar{q}_a \gamma_\mu q_a$ and $J^V = \bar{c}_a \gamma_\mu c_a$ of $J^{PC} = 1^{--}$ couple to the vector mesons $\rho$ and $J/\psi$, respectively. In Refs. [72, 73] the authors used the method of Lattice QCD to obtain

\[ \langle 0|\bar{d}_a \gamma_\mu u_a |\pi^+(p)\rangle = \lambda_\pi = \frac{f_{\pi^+} m_{\pi^+}^2}{m_u + m_d} \]

(23)

\[ \langle 0|\bar{c}_a \gamma_\mu c_a |\eta_c(p)\rangle = \lambda_{\eta_c} = \frac{f_{\eta_c} m_{\eta_c}^2}{2 m_c} \]

The coupling of $J^A_\mu$ to $\eta$ and $\chi_{c1}(1P)$ was evaluated by using Lattice QCD [73] and QCD sum rules [67]:

\[ \langle 0|\bar{c}_a \gamma_\mu \gamma_5 c_a |\eta_c(p)\rangle = ip_\mu f_{\eta_c} \]

(29)

where

\[ f_{\eta_c} = 254 \text{ MeV} \]

(28)

4. The axialvector operators $J^A_\mu = \bar{q}_a \gamma_\mu \gamma_5 q_a$ and $I^A_\mu = \bar{c}_a \gamma_\mu \gamma_5 c_a$ of $J^{PC} = 1^{++}$ couple to both pseudoscalar mesons ($\pi$ and $\eta$ of $J^{PC} = 0^{++}$) and axialvector mesons ($a_1(1260)$ and $\chi_{c1}(1P)$ of $J^{PC} = 1^{++}$). The coupling of $J^A_\mu$ to $\pi$ has been well measured in particle experiments [1]:

\[ \langle 0|\bar{d}_a \gamma_\mu \gamma_5 u_a |\pi^+(p)\rangle = i p_\mu f_{\pi^+} \]

(26)

while its coupling to $a_1(1260)$ was evaluated by using Lattice QCD [77]:

\[ \langle 0|\bar{d}_a \gamma_\mu \gamma_5 u_a |a_1(p, \epsilon)\rangle = m_{a_1} f_{a_1} \epsilon_\mu \]

(27)

where

\[ f_{\pi^+} = 130.2 \text{ MeV} \]

(25)

\[ f_{a_1} = 254 \text{ MeV} \]

(28)

\[ f_{J/\psi} = 418 \text{ MeV} \]

(29)

See also discussions in Refs. [74–76].
where
\[ f_{\pi c} = 387 \text{ MeV}, \]
\[ f_{\chi_{c1}} = 335 \text{ MeV}. \]

See also discussions in Refs. [67, 75, 78-84].

5. The tensor operators \( J_{T \mu}^{PC} = \bar{q}_a \sigma_{\mu\nu} q_a \) and \( I_{T \mu}^{PC} = \bar{c}_a \sigma_{\mu\nu} c_a \) of \( J^{PC} = 1^{+/-} \) couple to both vector mesons (\( \rho \) and \( J/\psi \)) and axialvector mesons (\( h_c(1235) \) and \( h_c(1P) \) of \( J^{PC} = 1^{+/-} \)). The coupling of \( J_{T \mu}^{PC} \) to \( \rho \) and \( b_1(1235) \) was calculated through Lattice QCD [72] and QCD sum rules [85]:
\[ \langle 0 | d_\mu \sigma_{\mu\nu} u_\nu | \rho^+(p, \epsilon) \rangle = i f_\rho^T (m_\rho \epsilon \nu - p_\nu \epsilon_\mu), \]
\[ \langle 0 | d_\mu \sigma_{\mu\nu} u_\nu | b_1(p, \epsilon) \rangle = i f_{b_1}^T \epsilon_{\mu\nu\alpha\beta} \epsilon^\alpha \epsilon^\beta, \]
where
\[ f_\rho^T = 159 \text{ MeV}, \]
\[ f_{b_1}^T = 180 \text{ MeV}. \]

The coupling of \( I_{T \mu}^{PC} \) to \( J/\psi \) and \( h_c(1P) \) was calculated through Lattice QCD [73]:
\[ \langle 0 | \bar{c}_a \sigma_{\mu\nu} c_a | J/\psi(p, \epsilon) \rangle = i f_{J/\psi}^T (m_{J/\psi} \epsilon \nu - p_\nu \epsilon_\mu), \]
\[ \langle 0 | \bar{c}_a \sigma_{\mu\nu} c_a | h_c(p, \epsilon) \rangle = i f_{h_c}^T \epsilon_{\mu\nu\alpha\beta} \epsilon^\alpha \epsilon^\beta, \]
where
\[ f_{J/\psi}^T = 410 \text{ MeV}, \]
\[ f_{h_c}^T = 235 \text{ MeV}. \]

See also discussions in Refs. [86-94].

6. The \( Z_c(3900) \) is above the \( D \bar{D}^* \) threshold, so we need the couplings of \( O^{O} = q_\gamma a \gamma_5 q_a \) and \( O_{\mu}^{D} = \bar{c}_a a \gamma_\mu q_a \) to the \( D^0 \) meson [1]:
\[ \langle 0 | \bar{d}_a a \gamma_5 c_a | D^+(p) \rangle = \lambda_D, \]
\[ \langle 0 | \bar{c}_a a \gamma_\mu q_a | D^0(p) \rangle = i \mu_D I_D, \]
and the couplings of \( O_{\mu}^{O} = \bar{c}_a a \gamma_\mu q_a \) and \( O_{\mu}^{V} = \bar{c}_a \gamma_\mu c_a \) to the \( D^* \) meson [95]:
\[ \langle 0 | \bar{d}_a a \gamma_\mu u_a | D^{*0}(p, \epsilon) \rangle = m_{D^*} f_{D^*} \epsilon_{\mu}, \]
\[ \langle 0 | \bar{d}_a a \gamma_\mu u_a | D^{*+}(p, \epsilon) \rangle = i f_{D^*}^T (m_\rho \epsilon \nu - p_\nu \epsilon_\mu), \]
where
\[ \lambda_D = \frac{f_{D^0} m_{D^+}^2}{m_\rho + m_\pi}, \]
\[ f_{D^*} = 211.9 \text{ MeV}, \]
\[ f_{D^{*+}} = 253 \text{ MeV}. \]

We do not find any theoretical study on the transverse decay constant \( f_{D^{*+}} \), so we simply fit among the decay constants, \( f_{\pi^0}, f_{\rho^0}, f_{\rho^+}, f_{\pi^0}, f_{J/\psi}, f_{J/\psi}, f_{D^*}, \) and \( f_{D^*}, \) to obtain
\[ f_{D^{*+}} = 220 \text{ MeV}. \]
See also discussions in Refs. [96, 97].

7. The \( Z_c(3900) \rightarrow D\bar{D}^* \rightarrow D\bar{D} \pi \) decay is kinematically allowed, so we need the coupling of \( O^S = \bar{q}_a c_a \) to the \( D^0 \) meson [98]:
\[ \langle 0 | d_\mu \sigma_{\mu\nu} u_\nu | p \rangle = m_{D^0} f_{D^0}, \]
where
\[ f_{D^0} = 410 \text{ MeV}. \]

See also discussions in Refs. [99, 100].

IV. DECAY PROPERTIES OF THE \( Z_c(3900) \) AS A COMPACT TETRAQUARK STATE

In this section and the next we shall use Eqs. (16-19) derived in Sec. II to extract some decay properties of the \( Z_c(3900) \). The two possible interpretations of the \( Z_c(3900) \) are: a) the compact tetraquark state of \( J^{PC} = 1^{+/-} \) composed of a \( J^{PC} = 1^{+/-} \) diquark/antidiquark and a \( J^{PC} = 1^{+/-} \) antidiquark/diquark [5, 6], i.e., \( \langle 0_{gc} | 1_{\bar{g}c} ; 1^{+/-} \rangle \) defined in Eq. (10); and b) the \( D^* \) hadronic molecular state of \( J^{PC} = 1^{+/-} \) [7-10], i.e., \( | D^* ; 1^{+/-} \rangle \) defined in Eq. (13). Moreover, we shall study their mixing with the \( | 1_{gc} | 1_{\bar{g}c} ; 1^{+/-} \rangle \) and \( | D^* ; 1^{+/-} \rangle \) states, whose definitions will be given below.

In this section we shall investigate the former compact tetraquark interpretation, whose relevant current \( \eta_{\mu}^Z(x, y) \) has been given in Eq. (11). This current can be transformed to \( \theta_{\mu}^Z(x, y) \) and \( \zeta_{\mu}^Z(x, y) \) according to Eqs. (16-18), through which we shall extract some decay properties of the \( Z_c(3900) \) as a compact tetraquark state in the following subsections:

A. \( \eta_{\mu}^Z([uc][\bar{d}\bar{e}]) \rightarrow \theta_{\mu}^Z([\bar{e}c] + [\bar{d}u]) \)

As depicted in Fig. 2, when the \( c \) and \( \bar{c} \) quarks meet each other and the \( u \) and \( \bar{d} \) quarks meet each other at the same time, a compact tetraquark state can decay into one charmonium meson and one light meson. This process for \( \langle 0_{gc} | 1_{\bar{g}c} ; 1^{+/-} \rangle \) can be described by the transformation (16):
\[ \eta_{\mu}^Z(x, y) \]
\[ = + \frac{i}{3} \theta_{\mu}^Z(x', y') - \frac{1}{3} \theta_{\mu}^Z(x', y') + \cdots \]
\[ = - \frac{i}{3} I^P(x') J^P_{\mu}(y') + \frac{i}{3} I^P_{\mu}(x') J^P(y') + \cdots \]
\[ = - \frac{i}{3} I^{A,\nu}(x') J^{A,\nu}_{\mu}(y') + \frac{i}{3} I^{A,\nu}_{\mu}(x') J^{A,\nu}(y') + \cdots \]
where we have only kept the direct full-apt part described by \( \theta_{\mu}^{Z,2,3,4} \), but neglected the \( \mathcal{O}(\alpha_s) \) corrections described by \( \theta_{\mu}^{Z,5,6,7,8} \).

Together with Table II, we extract the following decay channels from the above transformation:
1. The decay of $|0_{qc1\bar{q}c};1^{-+}\rangle$ into $\eta_c\rho$ is contributed by both $I^\mu \times J^\rho$ and $I^A,\nu \times J^\rho_{\mu\nu}$:

$$\langle Z_c^+(p,\epsilon)|\eta_c(p_1)\rho^+(p_2,\epsilon_2)\rangle \approx -i\frac{c_1}{3} \lambda_{\eta_c\rho}\rho_{\mu\nu} \epsilon \cdot \epsilon_2$$

(42)

2. The decay of $|0_{qc1\bar{q}c};1^{-+}\rangle$ into $J/\psi\pi$ is contributed by both $I^\mu_\mu \times J^\rho$ and $I^A,\nu \times J^\rho_{\mu\nu}$:

$$\langle Z_c^+(p,\epsilon)|J/\psi(p_1,\epsilon_1)\pi^+(p_2)\rangle \approx \frac{i\epsilon_{\pi}}{3} \lambda_{J/\psi}\rho_{\mu\nu} \epsilon \cdot \epsilon_1$$

(43)

3. The decay of $|0_{qc1\bar{q}c};1^{-+}\rangle$ into $\eta_c b_1$ is contributed by $I^A,\nu \times J^\rho_{\mu\nu}$:

$$\langle Z_c^+(p,\epsilon)|\eta_c(p_1) b_1^+(p_2,\epsilon_2)\rangle \approx -i\frac{c_1}{3} f_{J/\psi}\eta_{\eta_c\rho} b_1^+(p_2,\epsilon_2)$$

(44)

This process is kinematically forbidden, but the $|0_{qc1\bar{q}c};1^{-+}\rangle \rightarrow \eta_c b_1 \rightarrow \eta_c\omega\pi \rightarrow \eta_c + 4\pi$ decay is kinematically allowed.

4. The decay of $|0_{qc1\bar{q}c};1^{-+}\rangle$ into $\chi_{c1}\rho$ is contributed by $I^A,\nu \times J^\rho_{\mu\nu}$:

$$\langle Z_c^+(p,\epsilon)|\chi_{c1}(p_1,\epsilon_1)\rho^+(p_2,\epsilon_2)\rangle \approx -i\frac{c_1}{3} m_{\chi_{c1}} f_{\chi_{c1}} f_{J/\psi} \epsilon_1 \epsilon_2$$

(45)

This process is kinematically forbidden, but the $|0_{qc1\bar{q}c};1^{-+}\rangle \rightarrow \chi_{c1}\rho \rightarrow \chi_{c1}\pi\pi$ decay is kinematically allowed.

5. The decay of $|0_{qc1\bar{q}c};1^{-+}\rangle$ into $\chi_{c1} b_1$ is contributed by $I^A,\nu \times J^\rho_{\mu\nu}$:

$$\langle Z_c^+(p,\epsilon)|\chi_{c1}(p_1,\epsilon_1) b_1^+(p_2,\epsilon_2)\rangle \approx -i\frac{c_1}{3} m_{\chi_{c1}} f_{\chi_{c1}} f_{J/\psi} \epsilon_1 \epsilon_2$$

(46)

This process is kinematically forbidden.

6. The decay of $|0_{qc1\bar{q}c};1^{-+}\rangle$ into $h_c\pi$ is contributed by $I^A,\nu \times J^\rho_{\mu\nu}$:

$$\langle Z_c^+(p,\epsilon)|h_c(p_1,\epsilon_1)\pi^+(p_2)\rangle \approx i\frac{c_1}{3} f_{J/\psi} f_{h_c} \epsilon_\mu \epsilon_\nu$$

(47)

This process is kinematically allowed.
7. The decay of $|0_{q_c1\bar{q}_c;1^{+}-}\rangle$ into $J/\psi a_1$ is contributed by $I^T_{\mu,\nu} \times J_{A,\nu}$:

$$\langle Z^+_c(p,\epsilon)|J/\psi(p_1,\epsilon_1) a_1^+(p_2,\epsilon_2)\rangle \approx \frac{c_1}{3} f_{J/\psi} m_{a_1} f_{a_1} (\epsilon_1 \cdot \epsilon_2 \cdot p_1 - \epsilon_2 \cdot p_1 \epsilon_1)$$

$$g_{\psi a_1} (\epsilon_1 \cdot \epsilon_2 \cdot p_1 - \epsilon_2 \cdot p_1 \epsilon_1).$$

This process is kinematically forbidden, but the $|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow J/\psi a_1 \rightarrow J/\psi \rho \pi \rightarrow J/\psi + 3\pi$ decay is kinematically allowed.

8. The decay of $|0_{q_c1\bar{q}_c;1^{+}-}\rangle$ into $h_c a_1$ is contributed by $I^T_{\mu,\nu} \times J^{A,\nu}$:

$$\langle Z^+_c(p,\epsilon)|h_c(p_1,\epsilon_1) a_1^+(p_2,\epsilon_2)\rangle \approx \frac{c_1}{3} f_{h_c} m_{a_1} f_{a_1} \epsilon_{\mu\nu\rho\sigma} e^\nu e^\rho p_1^\sigma$$

$$g_{h_c a_1} \epsilon_{\mu\nu\rho\sigma} e^\nu e^\rho p_1^\sigma.$$ This process is kinematically forbidden.

Summarizing the above results, we obtain numerically

$$g_{\eta_c \rho} = -ic_1 7.29 \times 10^{10} \text{ MeV}^4, \quad g_{D^\ast \rho} = -ic_1 2.05 \times 10^{4} \text{ MeV}^2, \quad g_{\psi \pi} = ic_1 11.87 \times 10^{10} \text{ MeV}^4, \quad g_{D^\ast \pi} = ic_1 1.78 \times 10^{4} \text{ MeV}^2, \quad g_{\eta_c b_1} = -ic_2 2.32 \times 10^{4} \text{ MeV}^2, \quad g_{\chi_{c1} \rho} = -ic_1 6.23 \times 10^{7} \text{ MeV}^3, \quad g_{\chi_{c1} \pi} = -ic_1 7.06 \times 10^{7} \text{ MeV}^3, \quad g_{h_c \rho} = ic_1 1.02 \times 10^{4} \text{ MeV}^2, \quad g_{\psi a_1} = ic_1 4.27 \times 10^{7} \text{ MeV}^3, \quad g_{h_c a_1} = ic_1 2.45 \times 10^{7} \text{ MeV}^3.$$

From these coupling constants, we further obtain the following relative branching ratios, which are kinematically allowed:

$$\frac{B(|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow \eta_c \rho)}{B(|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow J/\psi \pi)} = 0.059, \quad \frac{B(|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow h_c \pi)}{B(|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow J/\psi \pi)} = 0.0088, \quad \frac{B(|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow \chi_{c1} \rho)}{B(|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow J/\psi \pi)} = 1.4 \times 10^{-6}.$$ Besides them, the following decay chains are also possible but with quite small partial decay widths:

$$|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow \eta_c b_1 \rightarrow \eta_c \omega \pi, \quad |0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow J/\psi \rho \alpha \rightarrow J/\psi + 4\pi.$$ 

B. \ $\eta_c^{u}(\overline{\langle wc|\overline{d}c\rangle}) \rightarrow \xi_{\mu}^{u}(\langle \overline{iuc} + [\overline{d}c]\rangle)$

As depicted in Fig. 3, when the $c$ and $\overline{d}$ quarks meet each other and the $u$ and $\overline{c}$ quarks meet each other at the same time, a compact tetraquark state can decay into two charmed mesons. This process for $|0_{q_c1\bar{q}_c;1^{+}-}\rangle$ can be described by the transformation (17):

$$\eta_{\mu}(x, y) \Longrightarrow -\frac{i}{3} \epsilon^2 (x', y') + \frac{1}{3} \xi_3 (x', y') + \cdots.$$ This term only keeps the direct fall-off part-process described by $\xi_2^{u,\beta}$, but neglected the $O(\alpha_s)$ corrections described by $\xi_3^{u,\beta}$.

Again, we have only kept the direct fall-off part-process described by $\xi_2^{u,\beta}$, but neglected the $O(\alpha_s)$ corrections described by $\xi_3^{u,\beta}$.

The term $\xi_2^{u,\beta}$ couples to the $D^\ast D^\ast$ and $D^\ast D_\pi$ final states, and the term $\xi_3^{u,\beta}$ couples to the $DD_\rho$ and $D_\rho D_\pi$ final states. Among them, only the $|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow DD_\rho \rightarrow DD\pi$ decay is kinematically allowed, contributed by $\xi_3^{u,\beta} = O(\rho \times O\pi)$ to be:

$$\langle Z^+_c(p,\epsilon)|DD_\rho(p_1) D_\rho^+(p_2)\rangle \approx \frac{ic_2}{3} f_{D\rho} m_{D_\rho} f_{D_\rho^\ast} \epsilon \cdot p_1$$

$$g_{DD_\rho} \epsilon \cdot p_1, \quad \langle Z^+_c(p,\epsilon)|D^+(p_1) D_\rho^0(p_2)\rangle \approx -\frac{ic_2}{3} f_{D\rho} m_{D_\rho} f_{D_\rho^\ast} \epsilon \cdot p_1$$

$$g_{DD_\rho} \epsilon \cdot p_1,$$

where $c_2$ is an overall factor.

Numerically, we obtain

$$g_{DD_\rho} = ic_2 6.80 \times 10^{7} \text{ MeV}^3.$$ Comparing the $|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow DD_\rho \rightarrow DD\pi$ decay in the present subsection with the $|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow J/\psi \pi$ and $|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow \eta_c \rho$ decays studied in the previous subsection, we obtain

$$\frac{B(|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow DD_\rho + DD_\pi)}{B(|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow J/\psi \pi + \eta_c \rho)} = 9.3 \times 10^{-8} \times \frac{c_2^2}{c_1^2}.$$ The current $\eta_c^{u}(x, y)$ does not correlate with the two terms $\xi_1^u = -iO^\nu_{\rho} \times O^\nu$ and $\xi_4^u = O^\nu_{\rho} \times O^T_{\nu,\rho}$, both of which can couple to the $DD^\ast$ final state. This suggests that $|0_{q_c1\bar{q}_c;1^{+}-}\rangle$ does not decay to the $DD^\ast$ final state with a large branching ratio,

$$g_{DD^\ast} \approx 0,$$ so that

$$\frac{B(|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow DD^\ast + DD^\ast)}{B(|0_{q_c1\bar{q}_c;1^{+}-}\rangle \rightarrow J/\psi \pi + \eta_c \rho)} \approx 0.$$ Eqs. (57) and (59) together suggest that $|0_{q_c1\bar{q}_c;1^{+}-}\rangle$ mainly decays into one charmonium meson and one light meson, other than two charmed mesons.

C. \ $\eta_{\mu}^{u}(\overline{\langle wc|\overline{d}c\rangle}) \rightarrow \theta_{1,2,3,4}^{u,\beta}(\overline{[iuc] + [\overline{d}c]} + \xi_{1,2,3,4}^{u,\beta}(\overline{[iuc] + [\overline{d}c]})$ If the above two processes investigated in Sec. IV A and Sec. IV B happen at the same time, we can use the
transformation (18), i.e., \(|q_c \bar{q_c}; 1^{+}\rangle\) can decay into one charmonium meson and one light meson as well as two charmed mesons at the same time, which process is described by the color-singlet-color-singlet currents \(\theta^{1,2,3,4}_\mu\) and \(\xi^{1,2,3,4}_\mu\) together:

\[
\eta^{\bar{Z}}(x, y) = \frac{1}{2} \theta^1_\mu(x', y') - \frac{1}{2} \theta^2_\mu(x', y') + \frac{i}{2} \theta^3_\mu(x', y') - \frac{i}{2} \theta^4_\mu(x', y') - \frac{i}{2} \xi^1_\mu(x', y') + \frac{1}{2} \xi^2_\mu(x', y') + \frac{1}{2} \xi^3_\mu(x', y') + \frac{1}{2} \xi^4_\mu(x', y').
\]

Here we have kept all the terms, and there is no \(\cdots\) in this equation.

Comparing the above equation with Eqs. (41) and (53), we arrive at the same conclusions as Sec. IV A and Sec. IV B, just with the overall factors \(c_1\) and \(c_2\) replaced by others.

\[\text{D. Mixing with } |q_c \bar{q_c}; 1^{+}\rangle\]

The relative branching ratio \(R_{Z_c}\) calculated in Sec. IV A is just 0.059, significantly smaller than the BESIII measurement \(R_{Z_c} = 2.2 \pm 0.9\) at \(\sqrt{s} = 4.226\) GeV [23]. In this subsection we slightly modify the internal structure of the \(Z_c(3900)\) to reevaluate this ratio.

Actually, in the Type-II diquark-antidiquark model [6], the \(Z_c(3900)\) was interpreted as

\[|q_c \bar{q_c}; 1^{+}\rangle = \frac{1}{\sqrt{2}} (|0_{qc} \bar{q_c}; 1^{+}\rangle_J=1 - |1_{qc} \bar{q_c}; 0\rangle_J=1),\]

and the ratio \(R_{Z_c}\) was predicted to be \(0.27_{-0.17}^{+0.40}\) [23], while in the Type-I diquark-antidiquark model [5], the \(Z_c(3900)\) was interpreted as the mixing state

\[|x_{qc} \bar{q_c}; 1^{+}\rangle = \cos \theta_1 |0_{qc} \bar{q_c}; 1^{+}\rangle + \sin \theta_1 |1_{qc} \bar{q_c}; 1^{+}\rangle,\]

where

\[|1_{qc} \bar{q_c}; 1^{+}\rangle = |1_{qc} \bar{q_c}; 1^{+}\rangle_J=1,\]

and a small \(|1_{qc} \bar{q_c}; 1^{+}\rangle\) component is able to increase this ratio to be \((2.3_{-1.4}^{+3.3}) \times 10^2\) [33], that is almost one thousand times larger.

We try to add this \(|1_{qc} \bar{q_c}; 1^{+}\rangle\) component in this subsection. The interpolating current having the identical internal structure is

\[\eta^{\bar{Z}}(x, y) = \eta^{\bar{Z}'}_{\mu}(x, y) + \eta^{\bar{Z}}_{\mu}(y^c) \cdot \cdot \cdot,\]

so that \(|x_{qc} \bar{q_c}; 1^{+}\rangle\) can be described by

\[\eta^{\text{mix}}_{\mu}(x, y) = \cos \theta_1' \eta^{\bar{Z}}_{\mu}(x, y) + \sin \theta_1' \eta^{\bar{Z}'}_{\mu}(x, y),\]

which transforms according to Eq. (16) as:

\[\eta^{\text{mix}}_{\mu}(x, y) \rightarrow + \left( -\frac{i}{3} \sin \theta_1' - \frac{i}{3} \sin \theta_1' \right) J^P(x') J_\mu^P(y') + \left( \frac{i}{3} \cos \theta_1' - \frac{i}{3} \sin \theta_1' \right) J^V(x') J_\mu^V(y') + \left( \frac{i}{3} \cos \theta_1' - \frac{i}{3} \sin \theta_1' \right) J^{A,\nu}(x') J_\mu^{A,\nu}(y') + \cdot \cdot \cdot.\]

Note that the two mixing angles \(\theta_1\) and \(\theta_1'\) are not necessarily the same (probably not the same), but they can be related to each other, i.e.,

\[\theta_1 = f(\theta_1').\]

To solve this relation, we need to know the couplings of \(\eta^{\bar{Z}}_{\mu}\) and \(\eta^{\bar{Z}'}_{\mu}\) to \(|0_{qc} \bar{q_c}; 1^{+}\rangle\) and \(|1_{qc} \bar{q_c}; 1^{+}\rangle\), which we
shall not investigate in the present study. Anyway, we can plot the three ratios:

\[
\mathcal{R}_{\psi \pi} \equiv \frac{\Gamma(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi)}{\Gamma(|0_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi)},
\]

\[
\mathcal{R}_{\eta_\rho} \equiv \frac{\Gamma(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow \eta_\rho)}{\Gamma(|0_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow \eta_\rho)},
\]

\[
\mathcal{R} \equiv \frac{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow \eta_{c}\rho)}{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi)},
\]

as functions of the mixing angle \(\theta'_1\), which are shown in Fig. 4. We find that \(\mathcal{R}_{\psi \pi}\) decreases and \(\mathcal{R}_{\eta_\rho}\) increases, so that the ratio \(\mathcal{R}\) increases rapidly, as the mixing angle \(\theta'_1\) decreasing from 0 to \(-10^\circ\).

Moreover, if fine-tuning \(\theta'_1 = -8.8^\circ\), we obtain

\[
\mathcal{R} = \frac{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow \eta_{c}\rho)}{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi)} = 2.2,
\]

\[
\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow \eta_{c}\rho) = 0.052,
\]

\[
\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi) = 1.5 \times 10^{-5}.
\]

The first ratio \(\mathcal{R}\) is 2.2, which is the same as the BESIII measurement \(\mathcal{R}_{Z_c} = 2.2 \pm 0.9\) \[23\].

The decay of \(|x_{qc}1\bar{q}_{c};1^{+}\rangle\) into two charm mesons can be described by the current \(\eta_{\mu \text{mix}}(x,y)\) together with the transformation (17):

\[
\eta_{\mu \text{mix}}(x,y) = \frac{\bar{q}'_1 \xi_{\mu}'(x',y') + \frac{1}{3} \bar{c}'_1 \xi_{\mu}'(x',y')}{\bar{q}'_1 \xi_{\mu}'(x',y') - \frac{1}{3} \bar{c}'_1 \xi_{\mu}'(x',y')} + \cdots,
\]

so that

\[
\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow D\bar{D}^* + \bar{D}D^*) \frac{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi + \eta_{c}\rho)}{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi)} = 0.26 \times \frac{c_2^3}{c_1^2},
\]

\[
\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow D\bar{D}^* + \bar{D}D^*) \frac{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi + \eta_{c}\rho)}{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi)} = 0.26 \times \frac{c_2^3}{c_1^2},
\]

\[
\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow D\bar{D}^* + \bar{D}D^*) \frac{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi + \eta_{c}\rho)}{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi)} = 0.26 \times \frac{c_2^3}{c_1^2},
\]

\[
\mathcal{R} = \frac{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow \eta_{c}\rho)}{\mathcal{B}(|x_{qc}1\bar{q}_{c};1^{+}\rangle \rightarrow J/\psi\pi)} = 2.5 \times 10^{-7} \times \frac{c_2^2}{c_1^2}.
\]

Hence, \(|x_{qc}1\bar{q}_{c};1^{+}\rangle\) can decay into the \(D\bar{D}^*\) final state, which is consistent with the BESIII observations [20, 21]. Moreover, it was proposed in Ref. [58] that: to enable the decay of the \(Z_c(3900)\), a constituent of a diquark must tunnel through the barrier of the diquark-antidiquark potential, but this tunnelling for heavy quarks is exponentially suppressed compared to that for light quarks, so the compact tetraquark couplings are expected to favour the open charm modes with respect to charmonium ones. According to this, \(c_2\) may be significantly larger than \(c_1\), so that \(|x_{qc}1\bar{q}_{c};1^{+}\rangle\) may mainly decay into two charm mesons.

V. DECAY PROPERTIES OF THE \(Z_c(3900)\) AS A HADRONIC MOLECULAR STATE  

Another possible interpretation of the \(Z_c(3900)\) is the \(D\bar{D}^*\) hadronic molecular state of \(J^{PC} = 1^{--}\) [7–10], i.e., \(|D\bar{D}^*;1^{+-}\rangle\) defined in Eq. (13). Its relevant current \(\xi_{\mu}^Z(x,y)\) has been given in Eq. (14). We can transform this current to \(\theta_\mu(x,y)\) according to the transformation (19), through which we shall extract some decay properties of the \(Z_c(3900)\) as a hadronic molecular state in the following subsections.

A. \(\xi_{\mu}^Z([\bar{c}q][\bar{q}c]) \rightarrow \theta_\mu^Z([\bar{c}c] + [q\bar{q}])\)

As depicted in Fig. 5, when the \(c\) and \(\bar{c}\) quarks meet each other and the \(u\) and \(\bar{u}\) quarks meet each other at the same time, a hadronic molecular state can decay into one charmonium meson and one light meson. This process for \(|D\bar{D}^*;1^{+-}\rangle\) can be described by the transformation (19):

\[
\xi_{\mu}^Z(x,y) \rightarrow \frac{1}{6} \theta_1^1(x',y') - \frac{1}{6} \theta_2^2(x',y').
\]
The above coupling constants are related to the $S$- and $D$-wave $|D\bar{D}^*; 1^{-+}\rangle \rightarrow \eta_c \rho$ decays, the $S$- and $D$-wave $|D\bar{D}^*; 1^{-+}\rangle \rightarrow J/\psi \pi$ decays, and the $|D\bar{D}^*; 1^{-+}\rangle \rightarrow \eta_c b_1, \chi_c b_1, \chi_c \pi, J/\psi a_1, h_c a_1$ decays, respectively. All of them contain an overall factor $c_4$.

Using the above coupling constants, we further obtain:

\[
\begin{align*}
B(|DD^*; 1^{-+}\rangle \rightarrow \eta_c \rho) &= 0.059, \\
B(|DD^*; 1^{-+}\rangle \rightarrow J/\psi \pi) &= 0.0088, \\
B(|DD^*; 1^{-+}\rangle \rightarrow \chi_c \rho &\rightarrow \chi_c \pi) = 1.4 \times 10^{-6}.
\end{align*}
\]

These values are surprisingly the same as Eqs. (51), obtained in Sec. IV A for the compact tetraquark state $|0_{q\bar{q}}; 1^{-+}\rangle$.

B. $\xi^\pm_{\mu}(\{\bar{q}q\}) \rightarrow \xi^\pm_{\mu}(\{\bar{q}q\})$

Assuming the $Z_c(3900)$ to be the $D\bar{D}^*$ hadronic molecular state of $J^{PC} = 1^{-+}$, it can naturally decay to the $D\bar{D}^*$ final state, which fall-apart process can be described by itself:

\[
\xi^\pm_{\mu}(x,y) \equiv \xi^\pm_{\mu}(x',y') = -i O^\pm_{\mu}(x') O^P(y') + \{\gamma_\mu \leftrightarrow \gamma_5\}.
\]

The decay of $|D\bar{D}^*; 1^{-+}\rangle$ into the $D\bar{D}^*$ final state is contributed by this term to be

\[
\begin{align*}
(Z_c^+(p,\epsilon)|D\bar{D}^*(p_1)\bar{D}^*\rangle(p_2,\epsilon_2)) &\approx -i c_5 \lambda_{DM} m_{D^*} f_{D^*} \epsilon \cdot \epsilon_2 \\
&\equiv h_{DD^*} \epsilon \cdot \epsilon_2,
\end{align*}
\]

\[
\begin{align*}
(Z_c^+(p,\epsilon)|\bar{D}^0(p_1)D^{**}(p_2,\epsilon_2) &\approx -i c_5 \lambda_{DM} m_{D^*} f_{D^*} \epsilon \cdot \epsilon_2 \\
&\equiv h_{DD^*} \epsilon \cdot \epsilon_2,
\end{align*}
\]

where $c_5$ is an overall factor, and it is probably significantly larger than $c_4$.

Numerically, we obtain:

\[
h_{DD^*} = -i c_5 2.95 \times 10^{11} \text{ MeV}^4.
\]
Comparing the $| D D^*; 1^{+-} \rangle \to D D^*$ decay studied in the present subsection with the $| D^* D^*; 1^{+-} \rangle \to J/\psi \pi$ and $| D D^*; 1^{+-} \rangle \to \eta_c \rho$ decays studied in the previous subsection, we obtain

$$\frac{B(|D D^*; 1^{+-}\rangle \to D D^* + \bar{D} D^*)}{B(|D D^*; 1^{+-}\rangle \to J/\psi \pi + \eta_c \rho)} = 25 \times \frac{c_3^2}{c_4^2}. \quad (78)$$

The current $\xi_\mu^3(x,y)$ does not correlate with the term $\xi_\mu^3 \equiv O_\mu^3 \times O_S^3$, so that $| D D^*; 1^{+-} \rangle$ does not decay into the $D \bar{D}_0^*$ final state:

$$B(|D D^*; 1^{+-}\rangle \to D \bar{D}_0^* + \bar{D} D_0^* \to D \bar{D} \pi) \approx 0. \quad (79)$$

Eqs. (78) and (79) suggest that $| D D^*; 1^{+-} \rangle$ mainly decays into two charmed mesons, other than one charmonium meson and one light meson. This conclusion is opposite to the one obtained in Sec. IV B for the compact tetraquark state $| 0_{q_1 \bar{q}_1; 1^{+-}} \rangle$.

### C. Mixing with the $| D^* D^*; 1^{+-} \rangle$

Similarly to Sec. IV D, we add a small $| D^* D^*; 1^{+-} \rangle$ component

$$| D^* D^*; 1^{+-} \rangle = | D^* D^* \rangle_{J=1}, \quad (80)$$

to $| D \bar{D}^*; 1^{+-} \rangle$ in this subsection to reevaluate the ratio $R_{Z_q}$. The interpolating current having the same internal structure as $| D^* D^*; 1^{+-} \rangle$ is

$$\xi_\mu^Z(x,y) = \xi_\mu^Z([\bar{c} u][\bar{d} c]), \quad (81)$$

so that we can use

$$A_{\mu}^{\text{mix}}(x,y) = \cos \theta_2 \, \xi_\mu^Z(x,y) + i \sin \theta_2 \, \xi_\mu^Z(x,y), \quad (82)$$

to describe the mixed molecular state

$$| D^{(*)} \bar{D}^*; 1^{+-} \rangle = \cos \theta_2 \, | D \bar{D}^*; 1^{+-} \rangle + \sin \theta_2 \, | D^* \bar{D}^*; 1^{+-} \rangle. \quad (83)$$

The current $A_{\mu}^{\text{mix}}(x,y)$ transforms according to Eq. (19) to be:

$$A_{\mu}^{\text{mix}}(x,y) \quad \Rightarrow \quad \left( \begin{array}{c} i \frac{1}{2} \cos \theta_2' \xi_\mu^Z(x,y) + i \frac{i}{6} \sin \theta_2' \xi_\mu^1(x,y) \\
+ i \frac{1}{2} \cos \theta_2' \xi_\mu^Z(x,y) + i \frac{i}{6} \sin \theta_2' \xi_\mu^1(x,y) \\
+ i \frac{1}{2} \cos \theta_2' \xi_\mu^Z(x,y) + i \frac{i}{6} \sin \theta_2' \xi_\mu^1(x,y) \\
+ i \frac{1}{2} \cos \theta_2' \xi_\mu^Z(x,y) + i \frac{i}{6} \sin \theta_2' \xi_\mu^1(x,y) \end{array} \right) \quad (84)$$

After fine-tuning $\theta_2' = -8.8^\circ$, we obtain

$$R' = \frac{B(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to \eta_c \rho)}{B(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to J/\psi \pi)} = 2.2, \quad (85)$$

$$R' = \frac{B(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to \chi_{c1} \rho)}{B(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to J/\psi \pi)} = 0.052, \quad (86)$$

$$R' = \frac{B(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to \chi_{c1} \pi)}{B(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to J/\psi \pi)} = 1.5 \times 10^{-5},$$

which values are the same as Eqs. (73), obtained in Sec. IV A for the mixed compact tetraquark state $| x_{q1} \bar{q}_1; 1^{+-} \rangle$. Actually, we can also plot the following three ratios

$$R_{\psi\pi} = \frac{\Gamma(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to J/\psi \pi)}{\Gamma(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to J/\psi \pi)},$$

$$R_{\eta_c \rho} = \frac{\Gamma(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to \eta_c \rho)}{\Gamma(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to \eta_c \rho)}, \quad (87)$$

$$R' = \frac{B(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to \eta_c \rho)}{B(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to J/\psi \pi)}, \quad (88)$$

as functions of the mixing angle $\theta_2'$, and the obtained figures are just identical to Fig. 4, where $R_{\psi\pi}$, $R_{\eta_c \rho}$, and $R'$ are shown as functions of $\theta_1$.

We also obtain

$$B(|D^{(*)} \bar{D}^*; 1^{+-} \rangle \to D D^* + \bar{D} D^*) \approx 0, \quad (87)$$

suggesting that $| D^{(*)} \bar{D}^*; 1^{+-} \rangle$ mainly decays into two charmed mesons.

### VI. SUMMARY AND DISCUSSIONS

In this paper we systematically construct all the tetraquark currents/operators of $J^{PC} = 1^{+-}$ with the quark content $c\bar{c}q\bar{q}$. There are three configurations: $[cq][\bar{c} q]$, $[cq][\bar{q} c]$, and $[\bar{c} c][\bar{q} q]$ ($q = u/d$), and for each configuration we construct eight independent currents. We use the Fierz rearrangement of the Dirac and color indices to derive their relations, through which we study strong decay properties of the $Z_q(3900)$:

- Using the transformation of $[cq][\bar{c} q] \to [\bar{c} c][\bar{q} q]$, we study decay properties of the $Z_q(3900)$ as a compact tetraquark state into one charmonium meson and one light meson.

- Using the transformation of $[cq][\bar{c} q] \to [cq][\bar{q} c]$, we study decay properties of the $Z_q(3900)$ as a compact tetraquark state into two charmed mesons.

- We also use the transformation of the $[cq][\bar{c} q]$ currents to the color-singlet-color-singlet $[\bar{c} c][\bar{q} q]$ and $[c q][\bar{c} q]$ currents to check the above results.
Using the transformation of $|cq⟩|q̄⟩ → [c,c]|q̄q⟩$, we study decay properties of the $Z_c(3900)$ as a hadronic molecular state into one charmonium meson and one light meson.

Through the $[c,q]|q̄⟩$ currents themselves, we study decay properties of the $Z_c(3900)$ as a hadronic molecular state into two charmed mesons.

Altogether we have investigated the two-body decays of the $Z_c(3900)$ into $J/ψπ$, $η_cρ$, $h_cπ$, and $DD^*$. We have also investigated the three-body decays $Z_c(3900) → χ_{c1}ρ → χ_{c1}ππ$ and $Z_c(3900) → DD^0_0 + DD^0_0 → DDπ$. Their relative branching ratios are calculated and summarized in Table III, where we have investigated the following interpretations of the $Z_c(3900)$:

- In the second and third columns of Table III, $|0_{qc}1_q̄q;1^{++}⟩$ and $|x_{qc}1_q̄q;1^{++}⟩$ denote compact tetraquark states of $J^P_C = 1^{++}$, defined in Eq. (10) and Eq. (61), respectively.

- In the fourth and fifth columns of Table III, $|DD^*;1^{++}⟩$ and $|D^{(*)}D^*;1^{++}⟩$ denote the hadronic molecular states of $J^P_C = 1^{++}$, defined in Eq. (13) and Eq. (83), respectively.

From Table III, we obtain:

- The relative branching ratios $B(|0_{qc}1_q̄q;1^{++}⟩ → η_cρ) / B(|0_{qc}1_q̄q;1^{++}⟩ → J/ψπ)$ and $B(|DD^*;1^{++}⟩ → η_cρ) / B(|DD^*;1^{++}⟩ → J/ψπ)$ are both around 0.059, significantly smaller than the BESIII measurement $R_{Z_c} = 2.2 ± 0.9$ at $\sqrt{s} = 4.226$ GeV [23]. However, we can add a small $|1_q̄q;1^{++}⟩$ component to $|0_{qc}1_q̄q;1^{++}⟩$ to obtain $B(|x_{qc}1_q̄q;1^{++}⟩ → η_cρ) / B(|x_{qc}1_q̄q;1^{++}⟩ → J/ψπ) = 2.2$; we can also add a small $|DD^*;1^{++}⟩$ component to $|DD^*;1^{++}⟩$ to obtain $B(|D^{(*)}D^*;1^{++}⟩ → η_cρ) / B(|D^{(*)}D^*;1^{++}⟩ → J/ψπ) = 2.2$.

Note that if the relevant mixing angles change dynamically, the ratio $R_{Z_c}$ would also change dynamically.

- Relative branching ratios of the $|DD^*;1^{++}⟩$ decays into one charmonium meson and one light meson are the same as those of the $|0_{qc}1_q̄q;1^{++}⟩$ decays. After taking proper mixing angles, relative branching ratios of the $|D^{(*)}D^*;1^{++}⟩$ decays into one charmonium meson and one light meson are also the same as those of the $|x_{qc}1_q̄q;1^{++}⟩$ decays. This suggests that one may not discriminate between the compact tetraquark and hadronic molecule scenarios by only investigating relative branching ratios of the $Z_c(3900)$ decays into one charmonium meson and one light meson.

- $|0_{qc}1_q̄q;1^{++}⟩$ mainly decays into one charmonium meson and one light meson, but $|x_{qc}1_q̄q;1^{++}⟩$ may mainly decay into two charmed mesons after taking into account the barrier of the diquark-antidiquark potential (see detailed discussions in Ref. [58] proposing $c_2 ≫ c_1$). Both $|DD^*;1^{++}⟩$ and $|D^{(*)}D^*;1^{++}⟩$ mainly decay into two charmed mesons, because $c_5$ is probably significantly larger than $c_4$.

- The decays of the $Z_c(3900)$ into $J/ψπ$ and $η_cρ$ can happen through both $S$-wave and $D$-wave. As an example, we separate them and obtain the following ratios for $|0_{qc}1_q̄q;1^{++}⟩$:

$$
\frac{B(|0_{qc}1_q̄q;1^{++}⟩ → η_cρ)}{B(|0_{qc}1_q̄q;1^{++}⟩ → J/ψπ)} \approx 0.51, \quad (88)
$$

$$
\frac{B(|0_{qc}1_q̄q;1^{++}⟩ → J/ψπ)}{B(|0_{qc}1_q̄q;1^{++}⟩ → η_cρ)} \approx 0.15. \quad (89)
$$

Hence, the $D$-wave decays are important and so can not be neglected. We also obtain:

$$
\frac{B(|0_{qc}1_q̄q;1^{++}⟩ → J/ψπ)}{B(|0_{qc}1_q̄q;1^{++}⟩ → η_cρ)} = 2.4, \quad (90)
$$

$$
\frac{B(|0_{qc}1_q̄q;1^{++}⟩ → J/ψπ)}{B(|0_{qc}1_q̄q;1^{++}⟩ → η_cρ)} = 0.82. \quad (91)
$$

| Channels | $|0_{qc}1_q̄q;1^{++}⟩$ | $|x_{qc}1_q̄q;1^{++}⟩$ | $|DD^*;1^{++}⟩$ | $|D^{(*)}D^*;1^{++}⟩$ |
|----------|-----------------|-----------------|-----------------|-----------------|
| $B(Z_c → η_cρ)$ | $B(Z_c → J/ψπ)$ | $B(Z_c → h_cπ)$ | $B(Z_c → χ_{c1}ππ)$ | $B(Z_c → DD^0_0 + DD^0_0)$ |
| $B(Z_c → J/ψπ)$ | $0.059$ | $2.2$ | $0.059$ | $2.2$ |
| $B(Z_c → h_cπ)$ | $0.0088$ | $0.052$ | $0.0088$ | $0.052$ |
| $B(Z_c → χ_{c1}ππ)$ | $1.4 \times 10^{-6}$ | $1.5 \times 10^{-5}$ | $1.4 \times 10^{-6}$ | $1.5 \times 10^{-5}$ |
| $B(Z_c → DD^0_0 + DD^0_0)$ | $\approx 0$ | $0.26 \times \frac{x_5}{c_5}$ | $25 \times \frac{x_5}{c_5}$ | $67 \times \frac{x_5}{c_5}$ |
| $B(Z_c → DD^0_0 + DD^0_0)$ | $9.3 \times 10^{-8} \times \frac{x_5}{c_5}$ | $2.5 \times 10^{-7} \times \frac{x_5}{c_5}$ | $\approx 0$ | $\approx 0$ |
The first one is consistent with Ref. [33], and the second one is consistent with Ref. [25].

- Actually, there is still one parameter not considered in above analyses, that is the phase angle $\theta$ between $S$- and $D$-wave coupling constants, for example, between $g^S_{n,p}$ and $g^D_{n,p}$:

$$g^S_{n,p} / |g^S_{n,p}| = e^{i\theta} \times (g^D_{n,p} / |g^D_{n,p}|).$$

This parameter is unknown and so not fixed, because in QCD sum rules one can only calculate the modular square of the decay constant, such as $|f_{n_c}|^2$. This might also be the case for Lattice QCD and light front model, for example, see the different definitions of $f_{n_c}$ in Refs. [73, 81].

We fix the phase angle between all the $S$- and $D$-wave coupling constants to be $\theta = \pi$, and redo the previous calculations. The results are summarized in Table IV.

Based on Tables III and IV, we conclude this paper. In this paper we systematically construct all the tetraquark currents/operators of $J^{PC} = 1^{-+}$ with the quark configurations $[cq][\bar{c}\bar{q}]$, $[cq][\bar{q}\bar{q}]$, and $[\bar{c}c][\bar{q}q]$ ($q = u/d$), and derive their relations through the Fierz rearrangement of the Dirac and color indices. Using these relations, we study decay properties of the $Z_c(3900)$ under both the compact tetraquark and hadronic molecule interpretations, within the same framework.

During the calculations, we have considered the mixing between the compact tetraquarks states

$$|0_{qc}\bar{1}_{\bar{q}c}; 1^{+-}\rangle \oplus |1_{qc}\bar{1}_{\bar{q}c}; 1^{+-}\rangle \rightarrow |x_{qc}\bar{1}_{\bar{q}c}; 1^{+-}\rangle, \quad (91)$$

and the mixing between the hadronic molecule states

$$|D\bar{D}^*; 1^{+-}\rangle \oplus |D^*\bar{D}^*; 1^{+-}\rangle \rightarrow |D^{(c)}\bar{D}^*; 1^{+-}\rangle. \quad (92)$$

We have also considered the phase angle $\theta$ between $S$- and $D$-wave coupling constants. Hence, there are indeed many degrees of freedom when investigating decay properties of the $Z_c(3900)$.

In both the compact tetraquark and hadronic molecule scenarios we obtain the same relative branching ratios of the $Z_c(3900)$ decays into one charmonium meson and one light meson, such as $R_{Z_c} \equiv \frac{\mathcal{B}(Z_c(3900) \rightarrow j^+\pi^-)}{\mathcal{B}(Z_c(3900) \rightarrow j^+\pi^-)}$. Especially, the BESIII measurement $R_{Z_c} = 2.2 \pm 0.9$ at $\sqrt{s} = 4.226$ GeV [23] can be explained in the compact tetraquark picture after considering the mixing shown in Eq. (91), while it can also be explained in the hadronic molecule picture after considering the similar mixing shown in Eq. (92). Both the hadronic molecule states $|D\bar{D}^*; 1^{+-}\rangle$ and $|D^{(c)}\bar{D}^*; 1^{+-}\rangle$ mainly decay into two charmed mesons, while the mixed compact tetraquark state $|x_{qc}\bar{1}_{\bar{q}c}; 1^{+-}\rangle$ may also mainly decay into two charmed mesons as discussed in Ref. [58].

Our result suggests that the possible decay channels of the $Z_c(3900)$ are: a) the two-body decays $Z_c(3900) \rightarrow J/\psi \pi$, $Z_c(3900) \rightarrow \eta_c \rho$, $Z_c(3900) \rightarrow h_c \pi$, and $Z_c(3900) \rightarrow D\bar{D}^*$, b) the three-body decays $Z_c(3900) \rightarrow \chi_{c1}\rho \rightarrow \chi_{c1}\pi \pi$ and $Z_c(3900) \rightarrow DD_0^+ + DD_0^0 \rightarrow DD\pi$, and c) the many-body decay chains $Z_c(3900) \rightarrow J/\psi a_1 \rightarrow J/\psi \rho \pi \rightarrow J/\psi + 3\pi$ and $Z_c(3900) \rightarrow \eta_c b_1 \rightarrow \eta_c \omega \pi \rightarrow \eta_c + 4\pi$. Their relative branching ratios calculated in the present study, under both the compact tetraquark and hadronic molecule interpretations, are very much different:

$$\mathcal{B}(Z_c(3900) \rightarrow D\bar{D}^* + \bar{D}D^*) : J/\psi \pi : \eta_c \rho : h_c \pi : \chi_{c1}\rho \rightarrow \eta_c \rho) : \chi_{c1}\rho \rightarrow \pi\pi) : \bar{D}D_0^* \rightarrow D\pi) : J/\psi a_1 \rightarrow 3\pi) : \eta_c b_1 \rightarrow 4\pi)\right) \sim 10^{+1} \sim 10^{+2} : 1 : 1 : 10^{-2} : 10^{-6} : < 10^{-6} : \cdots : \cdots \cdots (93)$$
This might be one of the reasons why many multiquark states are only observed in a few decay channels [65]. We note that in order to extract the above results, we have only considered the leading-order fall-apart decays described by color-singlet-color-singlet meson-meson currents/operators, but neglected the $\mathcal{O}(a_s)$ corrections described by color-octet-color-octet meson-meson currents/operators. To end this paper, we propose the BESIII, Belle, Belle-II, and LHCb Collaborations to search for those decay channels not observed yet, in order to better understand the nature of the $Z_c(3900)$.

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