Matching of $\lambda_B$ onto HQET

Volker Pilipp

Arnold Sommerfeld Center, Department für Physik
Ludwig-Maximilians-Universität München
Theresienstrasse 37, 80333 München, Germany

Abstract

The quantity $1/\lambda_B$, the inverse moment of the $B$-meson light-cone wave function, plays an important role in exclusive $B$-decays. I calculate the matching of $\lambda_B$ defined in QCD onto $\lambda_B$ defined in HQET. This is useful for comparing results that have been obtained in QCD to results obtained in HQET.

1 Introduction

The $B$-meson wave function can be defined with two different types of fields. Either we use the $b$-quark field, which occurs in the ordinary QCD Lagrangian, or the heavy quark field, which occurs in the Lagrangian of HQET [1, 2]. In any case we get different wave functions, which differ at subleading order in $\alpha_s$. If we want to compare a QCD calculation to the corresponding calculation, made in an effective theory like HQET or SCET, we need the connection between the QCD and the HQET wave functions. In this paper I will compute the matching of $\lambda_B$, the inverse moment of the $B$-meson wave function. This parameter often appears in exclusive $B$-decays like $B \to \pi\pi$ or $B \to \gamma\ell\nu$. Usually the LO result only depends on $\lambda_B$, while the higher logarithmic moments appear first at NLO. It is then sufficient to know the $\alpha_s$-corrections of the matching coefficient of $\lambda_B$. I have tested my result by calculating the NLO of the hard spectator scattering amplitude in $B \to \pi\pi$ in QCD [3]. The same calculation has been performed before by [4, 5] within the framework of SCET. The results agree, if the matching of $\lambda_B$ is properly taken into account.

2 Definitions

For the $B$-meson wave function I use the definition of [6, 7]:

$$i\hat{f}_B(\mu)m_B\Phi^{HQET}_+(\omega, \mu) = \frac{1}{2\pi} \int dt e^{i\omega t}\langle 0|\bar{q}(z)[z, 0]\not\!\not\!\gamma_{\nu}(0)|\bar{B}\rangle$$

(1)
Figure 1: NLO contributions to $\lambda_B$. The double line stands for the $b$-quark field.

and analogously for the QCD fields

$$i f_B m_B \phi^\text{QCD}_+ (\omega, \mu) = \frac{1}{2\pi} \int dt \, e^{i\omega t} \langle 0 | \bar{q}(z) [z, 0] \gamma_5 b(0) | B \rangle. \quad (2)$$

Here $n$ is an arbitrary Lorentz vector with $n^2 = 0$ and $n \cdot v = 1$, where $v$ is the four-velocity of the $B$-meson. We assume $z \parallel n$. The integration in (1) and (2) goes over $t = v \cdot z$. The path-ordered gauge factor is given by $[z, 0] = \text{Pexp}[ig s \int_0^1 dt \, z \cdot A(tz)]$.

We define the $B$-meson decay constant by

$$i f_B m_B = \langle 0 | \bar{q}(0) \gamma_5 b(0) | B \rangle \quad (3)$$

and analogously the HQET decay constant, which depends on the renormalisation scale $\mu$, by

$$i f_B(\mu) m_B = \langle 0 | \bar{q}(0) \gamma_5 h_v(0) | B \rangle. \quad (4)$$

The definition of $\lambda_B$ reads

$$\frac{1}{\lambda^\text{QCD}_B (\mu)} = \int_0^\infty d\omega \, \frac{\phi^\text{QCD}_+ (\omega, \mu)}{\omega} \quad (5)$$

and

$$\frac{1}{\lambda^\text{HQET}_B (\mu)} = \int_0^\infty d\omega \, \frac{\phi^\text{HQET}_+ (\omega, \mu)}{\omega}. \quad (6)$$

Because $\lambda_B$ usually appears in the combination $f_B/\lambda_B$ we define our matching coefficient $C_{\lambda_B}$ in the following way:

$$\frac{f_B}{\lambda^\text{QCD}_B (\mu)} = C_{\lambda_B}(\mu) \frac{\hat{f}_B(\mu)}{\lambda^\text{HQET}_B (\mu)}. \quad (7)$$

3 Matching calculation

It is obvious that at LO in $\alpha_s$ and in leading power in $\Lambda_{\text{QCD}}/m_b$ we get $C_{\lambda_B} = 1$. We get the NLO correction of $C_{\lambda_B}$ by calculating the convolution integrals over $\omega$ in (5)
and up to $O(\alpha_s)$. The corresponding diagrams are shown in fig. 1. Because $C_{\lambda\rho}$ does not depend on the hadronic physics, we use wave functions that are defined by free on-shell quark states, i.e. we replace $|B\rangle$ in (11) and (2) by $|b(p)\bar{q}(l)\rangle$. We assign to the $b$-quark the momentum $p = v(m_b - \tilde{\omega}) (-v\tilde{\omega}$ resp.) in the case of pure QCD (HQET resp.) and $l = v\tilde{\omega}$ to the soft constituent quark, where $v$ is the four-velocity of the $B$-meson. We assume that $\tilde{\omega} \ll m_b$ and calculate the diagrams only in leading power in $\tilde{\omega}/m_b$.

The diagram in fig. 1(b) is trivially identical in QCD and HQET, as the heavy quark field does not occur in the loop integral. It turns out that also fig. 1(c) does not contribute to (7) in leading power. This is due to the fact that fig. 1(c) only contributes in leading power in the region where the exchanged gluon is soft. In this region however QCD and HQET coincide. We can see this explicitly by writing down this diagram for QCD, which reads up to constant factors:

$$\int \frac{d^4k}{(2\pi)^d} \frac{\bar{q}\gamma^\nu \not{k} \not{\gamma}_5 (m_b \not{\gamma} + m_b - \not{k}) \gamma^\nu b}{k^2(k + \tilde{\omega}v)^2(k^2 + 2k \cdot v m_b)k \cdot n}. \quad (8)$$

In the region, where $k$ is soft, (8) simplifies to

$$\int \frac{d^4k}{(2\pi)^d} \frac{\bar{q} \gamma^\nu \not{\gamma}_5 b}{k^2(k + \tilde{\omega}v)^2 k \cdot v k \cdot n}. \quad (9)$$

where we used the equation of motion $(1 - \not{v})b = 0$. Eq. (9) is actually the contribution of fig. 1(c) in HQET.

For the diagram in fig. 1(a) we obtain in QCD:

$$\frac{\alpha_s}{4\pi} C_F A_0 \left( \frac{2 + 2 \ln \frac{\tilde{\omega}}{m_b}}{\epsilon} + 4 \ln \frac{\mu}{m_b} + 4 - \frac{\pi^2}{6} - 2 \ln^2 \frac{\tilde{\omega}}{m_b} + 4 \ln \frac{\tilde{\omega}}{m_b} \ln \frac{\mu}{m_b} \right) \quad (10)$$

and in HQET

$$\frac{\alpha_s}{4\pi} C_F A_0 \left( -\frac{1}{\epsilon^2} + \frac{2 \ln \frac{\tilde{\omega}}{\mu}}{\epsilon} - 2 \ln^2 \frac{\tilde{\omega}}{\mu} - \frac{\pi^2}{4} \right). \quad (11)$$

Here

$$A_0 = i f_B \int_0^\infty d\omega \frac{\phi_0^{(0)}(\omega)}{\omega} \quad (12)$$

denotes the LO matrix element, which is the same for QCD and HQET. In (10) and (11) we have set the dimension to $d = 4 - 2\epsilon$ and redefined $\mu^2 \to \mu^2 \frac{2^{\epsilon}\pi}{4\pi}$, which corresponds to the \text{MS}-scheme.

The wave function renormalisation constants of the heavy quark fields are given in the on-shell scheme for the QCD $b$-field:

$$Z_{2b}^b = 1 + \frac{\alpha_s}{4\pi} C_F \left( -\frac{1}{2\epsilon} - \frac{1}{\epsilon_{\text{IR}}} - 3 \ln \frac{\mu}{m_b} - 2 \right) \quad (13)$$

and for the HQET field $h_v$:

$$Z_{2h_v}^h = 1 + \frac{\alpha_s}{4\pi} C_F \left( \frac{1}{\epsilon} - \frac{1}{\epsilon_{\text{IR}}} \right). \quad (14)$$
The renormalisation of the $q$-field drops out in the matching. Diagrammatically the matching equation (7) reads:

$$Z_{2b}^{\lambda} \left( \begin{array}{c} \lambda \\ \lambda \\ \lambda \\ \lambda \end{array} \right)_{\text{QCD}} = C_{\lambda b} Z_{2b}^{\lambda} \left( \begin{array}{c} \lambda \\ \lambda \\ \lambda \end{array} \right)_{\text{HQET}}.$$ (15)

Finally we obtain

$$C_{\lambda b}(\mu) = 1 + \frac{\alpha_s}{4\pi} C_F \left( 2 \ln^2 \frac{\mu}{m_b} + \ln \frac{\mu}{m_b} + 2 + \frac{\pi^2}{12} \right)$$ (16)

where we have renormalised the UV-divergences in the $\overline{\text{MS}}$-scheme.

The matching coefficient for $f_B$, which has been calculated in [2], [8]-[12], reads:

$$\hat{f}_B(\mu) = \left( 1 + \frac{\alpha_s}{4\pi} C_F \left( 3 \ln \frac{\mu}{m_b} + 2 \right) \right) f_B.$$ (17)

This leads to:

$$\lambda_{\text{HQET}}^B = \left( 1 + \frac{\alpha_s}{4\pi} C_F \left( 2 \ln^2 \frac{\mu}{m_b} + 4 \ln \frac{\mu}{m_b} + 2 + \frac{\pi^2}{12} \right) \right) \lambda_{\text{QCD}}^B.$$ (18)

4 Discussion

Eq. (18) allows us to express the dependence of $\lambda_{b}^{\text{QCD}}$ on $\mu$ by the first logarithmic moment. From [7] we get:

$$\frac{d}{d \ln \mu} \int_0^\infty d \omega \frac{\phi_{\text{HQET}}(\omega)}{\omega} = C_F \frac{\alpha_s}{4\pi} \int_0^\infty d \omega \frac{\phi_{\text{HQET}}(\omega)}{\omega} \left( -4 \ln \frac{\mu}{\omega} + 2 \right).$$ (19)

This leads immediately to

$$\frac{d}{d \ln \mu} \int_0^\infty d \omega \frac{\phi_{\text{QCD}}(\omega)}{\omega} = C_F \frac{\alpha_s}{4\pi} \int_0^\infty d \omega \frac{\phi_{\text{QCD}}(\omega)}{\omega} \left( 4 \ln \frac{\omega}{m_b} + 6 \right).$$ (20)

The $\ln \mu$-term on the right hand side of (19) has disappeared in (20). This term has been removed by the double logarithm $\ln^2 \mu$ in (18). As already stated in the introduction, I calculated the hard spectator scattering amplitude of $B \to \pi \pi$ in QCD, which has been calculated before in [4, 5] in the framework of SCET. Beside the fact that using (16) makes our results coincide, it turned out that (20) leads to the right dependence of the amplitude on $\mu$.

There are crude approximations of $\lambda_B$ from sum rules [13, 14, 15]. In order to get an impression of the numerical implications of (18) we use the value from [13]:

$$\lambda_{b}^{\text{QCD}}(1\text{GeV}) = 460 \pm 160\text{MeV}.$$ (21)
This leads to the numerical value of $\lambda_{B}^\text{HQET}$:

$$\lambda_{B}^\text{HQET}(1\text{GeV}) = 560 \pm 200\text{MeV}, \quad (22)$$

where $\alpha_s$ is defined by four active flavours and $\Lambda_{\text{QCD}}^{\text{MS}(4)} = 325\text{MeV}$. We see that numerically the value of $\lambda_{B}^\text{HQET}$ is slightly enhanced. However this enhancement is within the error range of (21).

Instead of using sum rules $\lambda_{B}$ might be obtained experimentally from radiative decays $B \to \gamma l\nu, \gamma ll, \gamma\gamma$. These decays have been calculated in [16, 17, 18] at order $\alpha_s$, where the results are given in terms of $\lambda_{B}^\text{HQET}$. The corresponding $\lambda_{B}^\text{QCD}$ can be obtained by [18].

Acknowledgement

I would like to thank Gerhard Buchalla for useful discussions and for comments on the manuscript.

References

[1] H. Georgi, Phys. Lett. B 240 (1990) 447
[2] M. Neubert, Phys. Rept. 245 (1994) 259
[3] V. Pilipp, PhD-thesis, in preparation
[4] M. Beneke and S. Jäger, Nucl. Phys. B 751 (2006) 160 [arXiv:hep-ph/0512351]
[5] N. Kivel (2007) [arXiv:hep-ph/0608291]
[6] A.G. Grozin, M. Neubert, Phys. Rev. D 55 (1997) 272, [arXiv:hep-ph/9607366]
[7] B.O. Lange and M. Neubert, Phys. Rev. Lett. 91 102001 (2003) [arXiv:hep-ph/0303082]
[8] M. Voloshin and M. Shifman, Sov. J. Nucl. Phys. 45 (1987) 292
[9] H. Politzer and M. Wise, Phys. Lett. B 206 (1988) 681
[10] H. Politzer and M. Wise, Phys. Lett. B 208 (1988) 504
[11] X. Ji and M. Musolf, Phys. Lett. B 257 (1991) 409
[12] D. Broadhurst and A. Grozin, Phys. Lett. B 267 (1991) 105
[13] A. Khodjamirian, T. Mannel, N. Offen, Phys. Lett. B 620 (2005) 52, [arXiv:hep-ph/0504091]
[14] V. Braun, D. Ivanov and G. Korchemsky, Phys. Rev. D69 (2004) 034014 [arXiv:hep-ph/0309330]
[15] P. Ball and E. Kou, JHEP 0304 (2003) 029 [arXiv:hep-ph/0301135]

[16] S. Descotes-Genon and C. Sachrajda, Nucl. Phys. B 650 (2003) 356 [arXiv:hep-ph/0209216]

[17] S. Descotes-Genon and C. Sachrajda, Phys. Lett. B 557 (2003) 213 [arXiv:hep-ph/0212162]

[18] E. Lunghi, D. Pirjol and D. Wyler, Nucl. Phys. B 649 (2003) 349 [arXiv:hep-ph/0210091]