Bounds on the Lyapunov exponent via crude estimates on the density of states

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Abstract

We study the Chirikov (standard) map at large coupling $\lambda \gg 1$, and prove that the Lyapunov exponent of the associated Schrödinger operator is of order $\log \lambda$ except for a set of energies of measure $\exp(-c\lambda^\beta)$ for some $1 < \beta < 2$. We also prove a similar (sharp) lower bound on the Lyapunov exponent (outside a small exceptional set of energies) for a large family of ergodic Schrödinger operators, the prime example being the $d$-dimensional skew shift.

1 Introduction

1.1 Standard map

The Chirikov or Standard map is the two-dimensional area preserving dynamical system $T$ of the torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ to itself defined by

$$T(x_1, x_2) = (x_2, 2x_2 + \lambda \sin x_2 - x_1).$$

For small $\lambda$ the Chirikov map is known to have a set of invariant curves on which the motion is quasi-periodic. This follows from theory of Kolmogorov,

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Arnold and Moser (KAM). In fact by a theorem of Duarte \[8, 9\], elliptic islands are known to appear for an open dense set of \(\lambda\).

One of the major problems in dynamical systems is to prove that there is a set of initial conditions of positive measure for which the dynamics is chaotic, i.e. the Lyapunov exponent is positive. Equivalently, the Kolmogorov–Sinai metric entropy \(h(T)\) is conjectured to be positive, and to be of order \(\ln \lambda\) when \(\lambda \gg 1\). The conjecture remains unproved for any value of \(\lambda\).

The orbit of the Chirikov map is determined by its initial condition \((x_{-1}, x_0)\) and may be expressed in the form

\[
\cdots, (x_{-1}, x_0), (x_0, x_1), (x_1, x_2), \cdots,
\]

where \(x_j\) satisfy the equation for the discrete time pendulum

\[
(\triangle x)_n = x_{n+1} + x_{n-1} - 2x_n = \lambda \sin x_n.
\]

Setting

\[
\psi(n) = \frac{\partial x_n}{\partial x_0},
\]

we obtain an equation for the linearization about an orbit:

\[
H_\omega \psi = \left( -\lambda^2 \triangle - \cos x_n \right) \psi = 0.
\]

Here \(H_\omega\) is the discrete Schrödinger operator associated with the map \(T\). The potential \(v = -\cos x_n\) is evaluated along the orbit above and will depend on the initial condition \(\omega = (x_{-1}, x_0)\). By Pesin’s entropy formula \[11\], the metric entropy of \(T\) is equal to the integral over \((x_{-1}, x_0)\) of the Lyapunov exponent of \(H\) at energy 0 (which corresponds to energy \(E = 2\lambda^{-1}\) after the diagonal part of the Laplacian is incorporated in the potential). The precise definition of the Lyapunov exponent (in the non-ergodic setting) is given in Section 2 below. We can also define the average Lyapunov exponent \(\gamma(E)\) at energy \(E\) by studying the related equation \((H - E)\psi = 0\).

Our main theorem roughly states that \(\gamma(E) \approx \log \lambda\) except for a set of \(E\) of Lebesgue measure less than \(e^{-c\lambda^\beta}\) for \(1 < \beta < 2\).

**Theorem 1.** Fix \(\epsilon > 0\). For any \(\beta < 4/3\) and sufficiently large \(\lambda\),

\[
\text{meas} \{ -1/2 \leq E \leq 1/2 \mid \gamma(E) < (1 - \epsilon) \ln \lambda \} < C_\epsilon \exp \left[ -C_\epsilon^{-1} \lambda^{\beta} \right].
\]
For any $\beta < 2$, and any $E_0 \in (-1, 1)$, there exists a set $\Lambda(E_0)$ of $\lambda$ so that for every $\lambda \in \Lambda(E_0)$

$$\text{meas}\{E_0 - \lambda^{-2} \leq E \leq E_0 + \lambda^{-2} \mid \gamma(E) < (1 - \epsilon) \ln \lambda\} < C_{\beta, \epsilon} \exp\left[-C_{\beta, \epsilon}^{-1} \lambda^\beta\right].$$

and

$$\text{meas}\{\Lambda(E_0) \cap [2\pi \ell, 2\pi(\ell + 1)]\} \to 2\pi \quad \text{as} \quad \ell \to \infty.$$ 

Here and further meas denotes Lebesgue measure.

**Remarks.**

1. We are mostly interested in energy $E_0 = 0$, since the metric entropy is equal to the average Lyapunov exponent at $E = 0$.

2. The interval $(-1/2, 1/2)$ in the first estimate can be extended to $(-1 + \xi, 1 - \xi)$, for an arbitrary $\xi > 0$.

3. The set $\Lambda(E_0)$ may be chosen to be the complement of the Minkowski (element-wise) sum

$$A + \bigcup_{\ell=1}^\infty (2\pi \ell - \ell^{-\delta}, 2\pi \ell + \ell^{-\delta})$$

for some finite set $A \subset (-\pi, \pi]$ depending on $E_0$ and some small $\delta > 0$ (see the proof of Lemma 8). For $E_0 = 0$, one may choose $A = \{0, \pi\}$.

4. Instead of eliminating a set of energies of small measure we could have added a random potential (or noise) to the Schrödinger operator with variance $\approx e^{-\lambda^\beta}$.

5. For any $E$, one has the complementary inequality $\gamma(E) \leq \ln \lambda + C$; see Section 2. This remark also applies to the setting of Theorem 2 below.

Our proof relies on a formula of Jones and Thouless relating the density of states $\rho(E)$ to the Lyapunov exponent $\gamma(E)$. To get a lower bound on $\gamma(0)$ it is sufficient get some mild Hölder regularity of $\rho$ near $E = 0$ for large $\lambda$. This observation, in more general context, goes back to the work of Avron, Craig, and Simon [1]. Although Hölder regularity of the density of states is not known for the standard map, we obtain our theorem by bounding $\rho(-\delta, \delta)$ for small $\delta$ depending on $\lambda$. 

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In [14], the second-named author proved an inequality similar to that of Theorem 1 with the bound \( \approx e^{-C - \frac{1}{\lambda \ln \lambda}} \) on the size of the exceptional set. While our argument is based on the strategy of [14], the latter only uses the regularity of the distribution of \( f \), whereas our result requires an analysis of resonances.

There is also partial unpublished work of Carelson and the second-named author [5] which tried to extend the ideas of Benedicks and Carleson [2] on the Hénon map to the standard map. See also the paper by Ledrappier et al. [13] for other approaches to metric entropy.

In [4], Bourgain showed that, for sufficiently large \( \lambda \), \( \gamma(E) \) is positive outside a set of energies of zero measure. While Bourgain’s result has a much smaller exceptional set than Theorem 1, his method does not yield an explicit lower bound on the Lyapunov exponent.

A recent paper by Gorodetski [10] proves that there is a set of Hausdorff dimension 2 for which the Lyapunov exponent is positive for all \( \lambda > 0 \). This paper also gives an up to date overview of the dynamics of the standard map.

1.2 General bounds on the Lyapunov exponent

The second result of this paper pertains to a large family of Schrödinger operators. The setting is as follows. \((\Omega, \mathcal{B}, \mu)\) is an arbitrary probability space, \( T: \Omega \to \Omega \) is a measure-preserving map, and the one-dimensional Schrödinger operators \( H_\omega, \omega \in \Omega \), are constructed via

\[
(H\psi)(n) = \lambda^{-1}(\psi(n-1) + \psi(n+1)) + V(n)\psi(n),
\]

where

\[
V(n) = V_\omega(n) = f(T^n \omega)
\]

for some bounded measurable \( f: \Omega \to \mathbb{R} \). The definition of the Lyapunov exponent associated with \( H \) and \( E \in \mathbb{R} \) is discussed in Section 2.

**Theorem 2.** Let \( \ell \geq 1 \) be a natural number, and assume that \( 2\ell - 1 \) consecutive values \( V(-\ell+1), V(-\ell+2), \ldots, V(\ell-1) \) of the potential are independent bounded random variables, and that the restriction of their common distribution to an interval \([a,b]\) has bounded density. Then, for any closed interval \( I \subset (a,b) \) and large \( \lambda \gg 1 \),

\[
|\{ E \in I \mid \gamma(E) \leq \ln \lambda - C \ln \ln \lambda \}| \leq C \exp \left( -\lambda^\ell \frac{\ln \ln \lambda}{(C \ln \lambda)^{2\ell-1}} \right).
\]

For \( \ell = 1 \), this result can be derived from the arguments in [14].
Example. Let $\Omega = \mathbb{T}^d = (\mathbb{R}/(2\pi \mathbb{Z}))^d$ be the $d$-dimensional torus equipped with the normalized Lebesgue measure. Fix $\alpha \notin \mathbb{Q}$, and consider the skew shift
\[ T(\omega_1, \cdots, \omega_d) = (\omega_1 + \alpha, \omega_2 + \omega_1, \cdots, \omega_d + \omega_{d-1}) . \]

Let $h : \mathbb{T} \to \mathbb{R}$ be a function such that
1. $h \in C^2(\mathbb{T})$
2. $h$ has one non-degenerate maximum $M$ and one non-degenerate minimum $m$ (without loss of generality $M = h(0) = h(2\pi)$, $m = h(c)$ for some $0 < c < 2\pi$)
3. $h$ is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 2\pi]$
4. $|h''(c)|, |h''(0)| > 0$.

Let $f(\omega) = h(\omega_d)$ and let $H = H_\omega$ be the corresponding Schrödinger operator. Set $\ell = \lfloor \frac{d-1}{2} \rfloor$. 

**Corollary 1.** For any $\xi > 0$,
\[
\text{meas} \left\{ m + \xi \leq E \leq M - \xi \mid \gamma(E) \leq \ln \lambda - C \ln \ln \lambda \right\} \leq C_\xi \exp \left[ -\lambda^{\ell} \frac{\ln \ln \lambda}{(C \ln \lambda)^{2\ell-1}} \right]
\]

The corollary can be extended to more general functions $h$. For example, if $h$ is a Morse function, the bound holds in any interval away from the critical values of $h$.

We remark that a related result for the skew shift was obtained by Chan, Goldstein, and Schlag \[6\]. There, the assumptions on the function $h$ are weaker, and the lower bounds on the Lyapunov exponent are complemented by a proof of Anderson localization; on the other hand, the exceptional set of energies of \[6\] is much larger than ours (a power of $\lambda^{-1}$).

Stronger results have been obtained for the skew shift in the case when the function $h$ is analytic; see the book of Bourgain \[3\].

2 Preliminaries

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space, and let $T : \Omega \to \Omega$ be a measure-preserving invertible transformation. We shall consider one-dimensional discrete Schrödinger operators $H = H_\omega$, $\omega \in \Omega$, acting on $\ell^2(\mathbb{Z})$ by
\[
(H\psi)(n) = \lambda^{-1}(\psi(n-1) + \psi(n+1)) + V(n)\psi(n) ,
\]
where $\lambda \gg 1$ is the coupling constant, and $V = V_\omega$ is constructed from $T$ and a bounded measurable function $f : \Omega \to \mathbb{R}$ by the formula

$$V_\omega(n) = f(T^n \omega).$$

If the transformation $T$ is ergodic, the operator $H$ is called ergodic. In this case the Lyapunov exponent $\gamma(E; \omega)$ is defined for every $E \in \mathbb{R}$ and $\mu$-almost every $\omega$; moreover, $\gamma(E; \omega)$ is equal to an $\omega$-independent number $\gamma(E)$ on a set of $\omega$ of full measure. In addition, the density of states $\rho$ is defined (as an $\omega$-independent probability measure on $\mathbb{R}$). The Lyapunov exponent is related to the density of states by the Thouless formula

$$\gamma(E) = \ln \lambda + \int \ln |E - E'| d\rho(E').$$  \hspace{1cm} (1)

These facts may be found for example in the book [7] of Cycon, Froese, Kirsch, and Simon.

If $T$ is not ergodic, the Lyapunov exponent and the density of states admit a following generalization, based on the theorem on ergodic decomposition (first proved by von Neumann [15] and Krylov–Bogolyubov [12]). Let us briefly recall the definitions.

A $T$-invariant probability measure $\eta$ on $(\Omega, \mathcal{B})$ is called $T$-ergodic if $T$ is ergodic on $(\Omega, \mathcal{B}, \eta)$. The space of all $T$-ergodic measures is denoted $\text{Erg}(T)$. The ergodic decomposition theorem (see Walters [16, pp. 27–28]) states that there exists a probability measure $\eta$ on $\text{Erg}(T)$ so that

$$\mu = \int_{\text{Erg}(T)} \nu \, d\eta(\nu).$$  \hspace{1cm} (2)

The representation (2) is called the ergodic decomposition of $\mu$.

By Fubini’s theorem, one can prove the following:

**Lemma 1.** The Lyapunov exponent $\gamma(E; \omega)$ is defined for every $E \in \mathbb{R}$ and $\mu$-almost every $\omega \in \Omega$. The density of states $\rho_\omega$ is defined for $\mu$-almost every $\omega$.

**Proof.** Let $B$ be the set of $\omega$ for which $\gamma(E; \omega)$ is not defined. Then, for any $\eta \in \text{Erg}T$, $\eta(B) = 0$ by the ergodic case. Hence by (2), $\mu(B) = 0$. The second statement is proved in a similar way. \hfill $\square$
We set
\[ \gamma(E) = \int \gamma(E, \omega) d\mu(\omega) \] (3)
and
\[ \rho = \int \rho_\omega d\mu(\omega) . \] (4)

Fubini's theorem also yields the following lemmata:

**Lemma 2.** For every \( z = E_0 + i\delta \in \mathbb{C} \setminus \mathbb{R} \),
\[ \int \frac{d\rho(E)}{E - z} = \int (H_\omega - z)^{-1}(0,0)d\mu(\omega) . \]

In particular, for any \( 0 < \alpha \leq 1 \),
\[ \rho(E_0 - \delta, E_0 + \delta) \leq (2\delta)^\alpha \int [\Im(H_\omega - E_0 - i\delta)^{-1}(0,0)]^\alpha d\mu(\omega) \leq (2\delta)^\alpha \int |(H_\omega - E_0 - i\delta)^{-1}(0,0)|^\alpha d\mu(\omega) . \]

**Lemma 3.** The Thouless-Jones formula (1) remains valid with the definitions (3), (4).

Lemma 3 yields the following upper bound on the Lyapunov exponent (which can be also easily obtained by other means):
\[ \gamma(E) \leq \ln \lambda + \max_{E' \in \sigma(H)} \ln |E - E'| \leq \ln \lambda + \ln \text{diam} \sigma(H) \leq \ln \lambda + \ln(\max f - \min f + 4\lambda^{-1}) \leq \ln \lambda + C . \]

### 3 Application of the Thouless–Jones formula

Fix an energy \( E_0 \in \mathbb{R}, t > 0, \) and \( 1 \geq \xi > \delta > 0 \). Denote
\[ g(\delta) = g(\delta; E_0) = \max \left( \delta, \sup_{|E - E_0| \leq \xi} \rho[E - \delta, E + \delta] \right) \]
and
\[ Z_t = Z_t(E_0) = \left\{ E \left| |E - E_0| \leq \delta \quad \text{and} \quad \gamma(E) \leq t \right. \} . \]
Proposition 1. In the notation above, for any $\delta > 0$ we have:

$$\text{meas } Z_t(E_0) \leq 2e \exp \left\{ -\frac{\ln \lambda - t - 6\xi \ln e^{2g(\delta; E_0)}}{2g(\delta; E_0)} \right\}.$$  

Remark. The main goal of this paper is to make the right-hand side as small as possible by getting good estimates on $\rho(E - \delta, E + \delta)$ for small $\delta$.

Proof of Proposition[7]. From the Thouless–Jones formula (see[1] and Lemma[3]),

$$\gamma(E) = \ln \lambda + \int \ln |E - E'| d\rho(E').$$

Therefore by definition of $Z_t = Z_t(E_0)$

$$t \text{ meas } Z_t \geq \int_{Z_t} \gamma(E) dE$$

$$= \int_{Z_t} \left\{ \ln \lambda + \int \ln |E - E'| d\rho(E') \right\} dE$$

$$\geq \text{meas } Z_t \ln \lambda - \int_{Z_t} dE \int_{|E' - E| \leq 1} \ln |E - E'|^{-1} d\rho(E').$$

Hence

$$(\ln \lambda - t) \text{ meas } Z_t \leq \int_{Z_t} dE \int_{|E' - E| \leq 1} \ln |E - E'|^{-1} d\rho(E')$$

$$= \int_{E_0 - 1 - \delta}^{E_0 + 1 + \delta} d\rho(E') \int_{Z_t \cap \{|E - E'| \leq 1\}} \ln |E - E'|^{-1} dE. \quad (5)$$

Denote

$$J(E') = \int_{Z_t \cap \{|E - E'| \leq 1\}} \ln |E - E'|^{-1} dE$$

and decompose

$$\int_{E_0 - 1 - \delta}^{E_0 + 1 + \delta} J(E') d\rho(E')$$

$$= \left\{ \int_{|E' - E_0| \leq 2\delta} + \sum_{k=1}^{k_0 - 1} \int_{2k\delta \leq |E' - E_0| \leq (2k + 2)\delta} + \int_{|E' - E_0| \geq 2k_0\delta} \right\} J(E') d\rho(E').$$
where we shall later choose \( k_0 = \left\lfloor \frac{\xi}{2g(\delta)} \right\rfloor + 1. \)

Consider the following cases:

i) \( |E' - E_0| \leq 2\delta \); then by the rearrangement inequality

\[
J(E') \leq \int_{Z_t} \ln |E - E'|^{-1} dE \\
\leq \int_{E' + \text{meas}_{Z_t/2}}^{E' + \text{meas}_{Z_t/2}} \ln |E - E'|^{-1} dE \\
= \text{meas}_{Z_t} \ln \frac{2e}{\text{meas}_{Z_t}}.
\]

Therefore

\[
\int_{|E' - E_0| \leq 2\delta} J(E') d\rho(E') \leq 2g(\delta) \text{meas}_{Z_t} \ln \frac{2e}{\text{meas}_{Z_t}}.
\]

ii) \( 2k\delta \leq |E' - E_0| < (2k + 2)\delta \) for some \( 1 \leq k \leq \xi\delta^{-1} - 2 \). Then for \( E \in Z_t \) we have \( |E - E_0| \leq \delta \) and hence \( |E - E'| \geq (2k - 1)\delta \). Therefore

\[
\int_{2k\delta \leq |E' - E_0| \leq (2k+2)\delta} J(E') d\rho(E') \leq 2g(\delta) \text{meas}_{Z_t} \ln \frac{1}{(2k - 1)\delta}.
\]

iii) By the same reasoning,

\[
\int_{2k_0\delta \leq |E' - E_0|} J(E') d\rho(E') \leq \text{meas}_{Z_t} \ln \frac{1}{(2k_0 - 1)\delta}.
\]

Combining these estimates, we obtain:

\[
\int_{E_0 + 1 + \delta}^{E_0 - 1 - \delta} J(E') d\rho(E') \\
\leq \text{meas}_{Z_t} \left\{ 2g(\delta) \ln \frac{2e}{\text{meas}_{Z_t}} + 2g(\delta)k_0 \ln \frac{e^2}{(2k_0 - 1)\delta} + \ln \frac{1}{(2k_0 - 1)\delta} \right\}.
\]

Taking \( k_0 = \left\lfloor \frac{\xi}{2g(\delta)} \right\rfloor + 1 \) and plugging into (5) yields:

\[
\ln \lambda - t \leq 2g(\delta) \ln \frac{2e}{\text{meas}_{Z_t}} + 6\xi \ln \frac{e^2 g(\delta)}{\xi \delta},
\]

whence

\[
\text{meas}_{Z_t} \leq 2e \exp \left\{ -\ln \lambda - t - 6\xi \ln \frac{e^2 g(\delta)}{\xi \delta} \right\}.
\]
4 Theorem 2: proof

Let $H_m = H_{m, \omega}$ be the restriction of $H = H_\omega$ to $\{0, \ldots, m - 1\}$. Denote

$$\Delta_m(\omega) = \det(H_{m, \omega} - E - i\delta) .$$

For $m = 0$, set $\Delta_0(\omega) = 1$. The proof of Theorem 2 uses the following auxiliary proposition, which we prove after we prove the theorem.

**Proposition 2.** Let $m \geq 1$ be a fixed integer. Suppose $V(0), \ldots, V(m - 1)$ are independent, and that their common distribution has bounded density $\leq A$ in $[E - \xi, E + \xi]$. Then for any $a$ with $\Im a \geq 0$ and $|a| \leq \xi/2$,

$$\int d\omega |\Delta_m(\omega) - a\Delta_{m-1}(\omega)|^{-1} \leq (3A \ln(1 + 1/(A\delta)) + 2\xi^{-1})^m .$$

**Remark.** Only the case $a = 0$ is needed to prove the theorem; however, the stronger statement is more suited for an inductive proof and may have other applications.

**Proof of Theorem 2.** Let $G_\omega(z) = (H_\omega - z)^{-1}$, and $G_{\omega,2\ell-1} = (H_{\omega,2\ell-1} - z)^{-1}$, where $H_{\omega,2\ell-1}$ is the restriction of $H_\omega$ to $\{0, 1, \ldots, 2\ell - 2\}$. According to the resolvent identity,

$$G_\omega(E + i\delta; \ell - 1, \ell - 1) = G_{\omega,2\ell-1}(E + i\delta; \ell - 1, \ell - 1)
+ \frac{1}{\lambda} \left[ G_{\omega,2\ell-1}(E + i\delta; \ell - 1, 0)G_\omega(E + i\delta; -1, \ell - 1)
+ G_{\omega,2\ell-1}(E + i\delta; \ell - 1, 2\ell - 2)G_\omega(E + i\delta; 2\ell - 1, \ell - 1) \right] . \quad (6)$$

Let $M = \|V - E - i\delta\|_\infty$. By Cramer’s rule we can express the matrix elements of $G_{\omega,2\ell-1}$ as ratios of two determinants. Using the inequality

$$|\Delta_m(\omega)| \leq \left( M + \frac{1}{\lambda} \right)^m , \quad m = 0, 1, 2, \ldots , \quad (7)$$

(cf. (8) below), and recalling that our matrix is of size $(2\ell - 1) \times (2\ell - 1)$ with indices numbered from 0 to $2\ell - 2$, we obtain:

$$|G_{\omega,2\ell-1}(E + i\delta; \ell - 1, \ell - 1)| \leq \left( M + \frac{1}{\lambda} \right)^{2(\ell-1)} |\Delta_{2\ell-1}(\omega)|^{-1} ,$$

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whereas
\[ |G_{\omega,2\ell-1}(E + i\delta; \ell - 1, 0)| \leq \lambda^{-(\ell-1)} \left( M + \frac{1}{\lambda} \right)^{\ell-1} |\Delta_{2\ell-1}(\omega)|^{-1} \]
and
\[ |G_{\omega,2\ell-1}(E + i\delta; \ell - 1, 2\ell - 2)| \leq \lambda^{-(\ell-1)} \left( M + \frac{1}{\lambda} \right)^{\ell-1} |\Delta_{2\ell-1}(\omega)|^{-1}. \]
Combining these inequalities with the bound \( \|G_{\omega}(E + i\delta)\| \leq \delta^{-1} \), we obtain from (6) with \( \delta = \lambda^{-\ell} \) and \( \lambda \geq 2 \):
\[ |G_{\omega}(E + i\delta; \ell - 1, \ell - 1)| \leq \left[ (M + \frac{1}{\lambda})^{2(\ell-1)} + \lambda^{-\ell} \frac{2}{\delta} (M + \frac{1}{\lambda})^{\ell-1} \right] |\Delta_{2\ell-1}(\omega)|^{-1} \]
\[ \leq (M + 3)^{2(\ell-1)} |\Delta_{2\ell-1}(\omega)|^{-1}. \]
By Proposition 2
\[ \int d\mu(\omega) |G_{\omega}(E + i\delta; \ell - 1, \ell - 1)| \]
\[ \leq (M + 3)^{2(\ell-1)} (3A \ln(1 + 1/(A\delta)) + 2\xi^{-1})^{2\ell-1} \]
\[ \leq (C \ln \lambda)^{2\ell-1}, \]
where \( A \) is a bound on the density of \( V \) in \([a,b]\), and \( \xi \) is the distance from \( I \) to \([a,b]\). By Proposition 1 one can choose \( \tilde{C} \) so that
\[ \text{meas } Z_{\ln \lambda - C \ln \lambda}(E) \leq 2e^{-\lambda^{\frac{\ln \ln \lambda}{(C \ln \lambda)^{2\ell-1}}}}. \]

The proof of Proposition 2 is based on two estimates (the first one follows from a rearrangement inequality, the second one follows from the first one).

**Lemma 4.** Let \( \tau \) be a sub-probability measure (i.e. \( \tau(\mathbb{R}) \leq 1 \)) with density bounded by \( A \). Then
\[ \int \frac{d\tau(v)}{\sqrt{|v - E|^2 + \delta^2}} \leq 3A \ln(1 + 1/(A\delta)) \]
for any \( E \in \mathbb{R} \) and \( \delta > 0 \). 

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Lemma 5. Let $\sigma$ be a probability measure on $\mathbb{R}$ so that the restriction of $\sigma$ to $[-\xi, \xi]$ has density $\leq A$. Then for any real $a$ with $|a| \leq |\xi|/2$,
\[
\int \frac{d\sigma(v)}{\sqrt{(v-a)^2 + \delta^2}} \leq 3A \ln(1 + 1/(A\delta)) + 2\xi^{-1}.
\]

Proof of Proposition 2. Without loss of generality we may set $E = 0$. Denote
\[
F_m(a) = \int \frac{d\sigma(V(0))d\sigma(V(1))\cdots d\sigma(V(m-1))}{|\Delta_m - a\Delta_{m-1}|}, \quad M_m = \max F_m.
\]

The proof is by induction on $m$. For $m = 1$ the statement follows directly from Lemma 5. For the induction step, note that
\[
\Delta_m = V(m-1)\Delta_{m-1} - \lambda^{-2}\Delta_{m-2}.
\]

Represent $\sigma = \sigma_1 + \sigma_2$, where $\sigma_1 = \sigma|_{-\xi,\xi}$; then
\[
\text{supp} \sigma_1 \subset [-\xi, \xi], \quad \text{supp} \sigma_2 \cap (-\xi, \xi) = \emptyset,
\]
and the density of $\sigma_1$ is bounded by $A$. Then
\[
F_m(a) = \int \frac{d\sigma(V(0))\cdots d\sigma(V(m-1))}{|\Delta_m - a\Delta_{m-1}|}
\]
\[
= \left[ \int d\sigma(V(0))\cdots d\sigma_1(V(m-1)) + \int d\sigma(V(0))\cdots d\sigma_2(V(m-1)) \right]
\]
\[
= I_1 + I_2.
\]

The first integral is equal to
\[
I_1 = \int \frac{d\sigma(V(0))\cdots d\sigma(V(m-2))d\sigma_1(V(m-1))}{|\Delta_{m-1}|} \int \frac{d\sigma_1(V(m-1))}{|V(m-1) - a_1|},
\]
where $a_1 = a + \lambda^{-2}\Delta_{m-2}\Delta_{m-1}^{-1}$ has non-negative imaginary part. Hence by Lemma 4
\[
I_1 \leq 3A \ln(1 + 1/(A\delta))F_{m-1}(0) \leq 3A \ln(1 + 1/(A\delta))M_{m-1}.
\]
On the other hand,

\[ I_2 = \int \frac{d\sigma_2(V(m-1))}{|V(m-1) - a|} \int \frac{d\sigma(V(0)) \cdots d\sigma(V(m-2))}{|\Delta_{m-1} - a_2 \Delta_{m-2}|}, \]

where \( a_2 = \lambda^{-2}(V(m-1) - a) \). Then \( \Im a_2 \geq 0 \) and

\[ |a_2| \leq \frac{\lambda^{-2}}{\xi/2} \leq \xi/2 \]

for sufficiently large \( \lambda \). Therefore

\[ I_2 \leq \int \frac{d\sigma_2(V(m-1))}{|V(m-1) - a|} F_{m-1}(a_2) \leq \frac{2}{\xi} M_{m-1}. \] (10)

Combining (9) with (10), we obtain

\[ M_m \leq (3A \ln(1 + 1/(A\delta)) + 2\xi^{-1}) M_{m-1}. \]

\[ \square \]

5 Standard map: proofs

Let \( z = E + i\delta \), and set

\[ \Delta_3(z; x_0, x_1) = \Delta_3(z; \omega) = \det \begin{pmatrix} -\cos x_{-1} - z & \frac{1}{\lambda} \\ \frac{1}{\lambda} & -\cos x_0 - z & \frac{1}{\lambda} \\ 0 & \frac{1}{\lambda} & -\cos x_1 - z \end{pmatrix} \]

\[ = -\left[ c_{-1}(c_0 c_1 - \frac{1}{\lambda^2}) - \frac{1}{\lambda^2} c_1 \right] \]

\[ = -c_0 \left[ c_1 c_{-1} - \frac{1}{\lambda^2 c_0} (c_1 + c_{-1}) \right], \]

where

\[ c_1 = \cos x_1 + z \]

\[ c_0 = \cos x_0 + z \]

\[ c_{-1} = \cos x_{-1} + z = \cos(2x_0 + \lambda \sin x_0 - x_1) + z. \]

Let \( a > 0 \) be a large number independent of \( \lambda \) and define

\[ A_a = \{(x_0, x_1) \mid |c_0(x_0, x_1)| \geq a\lambda^{-2}\} = A_0^a \times [-\pi, \pi]. \]
Let
\[ I = I_\lambda(z; a) = \int_{A_a} |\Delta_3(z; x_0, x_1)|^{-\alpha} dx_0 dx_1. \]

The next lemma is the main result of this section.

**Lemma 6.** For sufficiently large \( a \) and \( \lambda \), the following estimates hold:
1. \( I_\lambda(E, a) \leq C_\alpha \) for any \( \alpha < 2/3 \);
2. \( I_\lambda(E, a) \leq C_\alpha \) if \( \lambda \) is outside a small exceptional set (in the sense of Theorem 1 and the subsequent remark) and \( \alpha < 1 \).

Here \( C_\alpha > 0 \) depends only on \( \alpha \).

**Proof of Theorem 1 using Lemma 6.** We apply Proposition 1 with \( \delta = \lambda^{-2} \). The density of states is bounded as follows. Let \( G_\omega(z) = (H_\omega - z)^{-1} \); to bound \( G_\omega(z; 0, 0) \), we introduce the restriction \( H_\omega(3) \) of \( H_\omega \) to \( \{-1, 0, 1\} \), and denote its Green function by \( G_\omega(3)(z) = (H_\omega(3) - z)^{-1} \). We have:
\[
G_\omega(z; 0, 0) = G_\omega(3)(z; 0, 0) + \lambda^{-1} G_\omega(3)(z; 0, 1) G_\omega(z; 2, 0) + \lambda^{-1} G_\omega(3)(z; 0, -1) G_\omega(z; -2, 0).
\]

By Cramer’s rule,
\[
|G_\omega(3)(z; 0, 0)| \leq C |\Delta_3(z; \omega)|^{-1}; \quad |G_\omega(3)(z; 0, \pm 1)| \leq C \lambda^{-1} |\Delta_3(z; \omega)|^{-1}.
\]

For \( \Im z = \delta \), we also have the trivial bound \( \|G_\omega\| \leq \delta^{-1} = \lambda^2 \). Thus
\[
|G_\omega(z; 0, 0)| \leq |G_\omega(3)(z; 0, 0)| + \frac{1}{\lambda \delta} (|G_\omega(3)(z; 0, 1)| + |G_\omega(3)(z; 0, -1)|) \leq C |\Delta_3(z; \omega)|^{-1}.
\]

Therefore
\[
\int |G_\omega(z; 0, 0)|^\alpha d\mu(\omega) \leq C_\alpha \int_{A_a} |\Delta_3(z; \omega)|^{-\alpha} d\mu(\omega) + (1 - \mu(A_a)) \lambda^2 = 4\pi^2 C_\alpha I_\lambda(z; a) + (1 - \mu(A_a)) \lambda^2.
\]

It is easy to see that \( \mu(A_a) \leq C_a \lambda^{-2} \) (since \( E \) is away from \( \pm 1 \)). Also, \( |\Delta_3(z; x_0, x_1)| \) is monotone in \( \Im z \), since \( \Delta_3(\cdot; \omega) \) has real roots. By Lemma 6
\[
\int |G_\omega(z; 0, 0)|^\alpha d\mu(\omega) \leq C'.
\]
for sufficiently large \(a\), under the assumptions of Theorem 1. By Lemma 2

\[ \rho(E - \delta, E + \delta) \leq C'' \delta^\alpha. \]

By Proposition 1 we obtain the statement. \(\square\)

**Proof of Lemma 6.** First, we write \(I\) as

\[
I = \int_{A_0} \frac{dx_0}{\cos x_0 + E} \int_{-\pi}^{\pi} \frac{dx_1}{|c_1 c_{-1} - \frac{1}{\lambda^2 c_0} (c_1 + c_{-1})|^\alpha} \\
= \int_{A_0} \frac{dx_0}{\cos x_0 + E} J(E, \theta(x_0), \alpha, \frac{1}{\lambda^2 (\cos x_0 + E)}),
\]

where

\[ \theta(x_0) = 2x_0 + \lambda \sin x_0, \]

\[ g(x_1) = g(E, \epsilon, \theta; x_1) = (\cos x_1 + E)(\cos(x_1 - \theta) + E) - \epsilon((\cos x_1 + E) + (\cos(x_1 - \theta) + E)), \]

and

\[ J(E, \theta, \alpha; \epsilon) = \int_{-\pi}^{\pi} \frac{dx_1}{|g(E, \epsilon, \theta, x_1)|^\alpha}. \]

**Lemma 7.** There exists \(\epsilon_0 > 0\) such that for any \(\frac{1}{2} < \alpha < 1\) and \(|\epsilon| \leq \epsilon_0\),

\[ J(E, \theta, \alpha, \epsilon) \leq C \text{dist}^{-2\alpha - 1}(\bar{\theta}, \{0, \pm 2 \arccos(-E)\}) , \]

where \(\bar{\theta} = \theta \mod 2\pi\) and the constant \(C\) depends only on \(\alpha\).

Note that \(J\) does not depend on \(\lambda\), hence neither does \(\epsilon_0\). We will later apply this lemma with \(\epsilon = (\lambda^2 (\cos x_0 + E))^{-1}\).

**Proof.** Without loss of generality we may assume that \(|\theta| \leq \pi\). As a function of \(x\), \(g(x)\) is a trigonometric polynomial of degree 2; hence it has 4 zeros. For \(\epsilon = 0\), the zeros are \(\pm x^*, \pm x^* + \theta\), where \(x^* = \arccos(-E)\). Since \(E \in (-1/2, 1/2)\) is bounded away from \(\pm 1\), \(x^*\) is bounded away from \(-x^*\), and \(x^* + \theta\) is bounded away from \(-x^* + \theta\). Consider four cases.

**Case 1:** \(\theta\) is bounded away from 0, 2\(x^*\). Then the 4 zeros of \(g(E, 0, \theta; \cdot)\) are separated; hence the same is true for \(g(E, \epsilon, \theta; \cdot)\) when \(\epsilon\) is sufficiently small, \(|\epsilon| \leq \epsilon_0(\theta)\). Therefore

\[ J(E, \theta, \alpha, \epsilon) \leq C. \]
**Case 2:** $\theta$ is close to 0. Then $x^* + \theta$ is close to $x^*$ and $-x^* + \theta$ is close to $-x^*$. The corresponding zeros $x_{1/2}$ and $x_{3/4}$ of $g(E, \epsilon, \theta; \cdot)$ satisfy

$$|x_1 - x_2| \geq C^{-1} |\theta|, \quad |x_3 - x_4| \geq C^{-1} |\theta|,$$

as one can see, for example, plugging the linear approximation

$$\cos x + E \simeq -\sin x^* (x - x^*), \quad \cos(x - \theta) + E \simeq -\sin x^* (x - \theta - x^*)$$

near $x^*$ into the definition of $g$. Therefore

$$J(E, \theta, \alpha, \epsilon) \leq C |\theta|^{(2\alpha - 1)},$$

where we have used the following observation: for any $d$ such that $0 < |d| < 1$ ($d$ may be complex) and $1/2 < \alpha < 1$,

$$\int |x(x - d)|^{-\alpha} dx \leq C_\alpha |d|^{-(2\alpha - 1)}. \quad (11)$$

**Cases 3 and 4:** $\theta$ is close to $\pm 2x^*$. Then by similar reasoning

$$J(E, \theta, \alpha, \epsilon) \leq C |\theta \mp 2x^*|^{-(2\alpha - 1)}.$$ 

To complete the proof of Lemma 6, we need to estimate

$$\int_{-\pi}^{\pi} \frac{dx_0}{|\cos x_0 + E|^{\alpha}} \text{dist}^{-(2\alpha - 1)}(\hat{\theta}(x_0), \{0, \pm 2x^*\})$$

$$\leq \int_{-\pi}^{\pi} \frac{dx_0}{|\cos x_0 + E|^{\alpha}} \text{dist}^{-(2\alpha - 1)}(\hat{\theta}(x_0), 0)$$

$$+ \int_{-\pi}^{\pi} \frac{dx_0}{|\cos x_0 + E|^{\alpha}} \text{dist}^{-(2\alpha - 1)}(\hat{\theta}(x_0), 2x^*)$$

$$+ \int_{-\pi}^{\pi} \frac{dx_0}{|\cos x_0 + E|^{\alpha}} \text{dist}^{-(2\alpha - 1)}(\hat{\theta}(x_0), -2x^*) .$$

Let

$$K(\lambda, b, E, \alpha) = \int_{-\pi}^{\pi} \frac{dx_0}{|\cos x_0 + E|^{\alpha}} \text{dist}^{-(2\alpha - 1)}(\hat{\theta}(x_0), b).$$
Lemma 8. If $E$ is bounded away from $\pm 1$ and $\alpha < 2/3$,

$$K(\lambda, b, E, \alpha) < C_\alpha .$$

For $\lambda$ outside a small exceptional set (satisfying the measure estimate in Theorem 1), the same estimate holds for any $\alpha < 1$.

Proof. For simplicity of presentation, we first discuss in detail the case $E = 0$ and $b = 0$, and then comment on the modifications needed for the general case. Setting $x = \pi/2 - x_0$,

$$K(\lambda, \alpha) = K(\lambda, 0, 0, \alpha) = \int \frac{dx}{|\sin x|^\alpha \text{dist}^{-(2\alpha - 1)}(\bar{h}(x), 0)}$$

where

$$\bar{h}(x) = \lambda \cos x + 2x ,$$

and as before $\bullet$ denotes reduction modulo $2\pi$. Now we need to estimate the integral over $[-\pi/2, \pi/2]$ and over $[-\pi, \pi] \setminus [-\pi/2, \pi/2]$. We show the argument for the former, since the argument for the latter is identical.

Decompose

$$[-\pi/2, \pi/2] = \bigcup_{|\ell| \leq \left\lfloor \frac{\lambda + \pi}{2\pi} \right\rfloor} \{(2\ell - 1)\pi \leq h(x) \leq (2\ell + 1)\pi \} .$$

All but one set in this decomposition are unions of two intervals which do not contain the origin (0) with a zero (root) of $\bar{h}$ is each. One is a single interval (a neighborhood of 0) with two zeros of $\bar{h}$.

First consider the intervals

$$I_\ell^+ \cup I_\ell^- = \{(2\ell - 1)\pi \leq h(x) \leq (2\ell + 1)\pi \}$$

which do not contain the origin. If $\ell$ is such that $|\lambda - 2\pi\ell| \geq c\lambda$, where $c > 0$ is a small constant, we have the following estimates for $x \in I_\ell^\pm$:

$$|x| \geq c_1 , \quad |h'(x)| \geq c_1\lambda ,$$

which imply that

$$\sum_{|\lambda - 2\pi\ell| \geq c\lambda} \int_{I_\ell^+ \cup I_\ell^-} \frac{dx}{|\sin x|^\alpha |h(x)|^{2\alpha - 1}}$$
is bounded for any $\alpha < 1$.

For the intervals with $|\lambda - 2\pi \ell| < c\lambda$ which do not contain the origin, we use the Taylor expansion

$$h(x) = \lambda(1 - \frac{x^2}{2} + O(x^4)) + 2x,$$

which implies that the zeros $x^\pm_\ell \in I^\pm_\ell$ of $\bar{h}$ are given by

$$x^\pm_\ell = \frac{2}{\lambda} \pm \sqrt{\frac{4}{\lambda^2} + 2 \frac{\lambda - 2\pi \ell}{\lambda} + \epsilon^\pm_\ell}, \quad |\epsilon^\pm_\ell| \leq C|x^\pm_\ell|^4.$$  

(14)

This relation yields the estimates

$$|x| \geq \frac{c_2|\lambda - 2\pi \ell|^{1/2}}{\lambda^{1/2}}, \quad |h'(x)| \geq c_2|\lambda - 2\pi \ell|^{1/2}\lambda^{1/2}$$

for $x \in I^\pm_\ell$, as well as the bound

$$\text{meas } I^\pm_\ell \leq 1/(c_2|\lambda - 2\pi \ell|^{1/2}\lambda^{1/2}).$$

Hence

$$\int_{I^\pm_\ell} \frac{dx}{\sin x^\alpha |h(x)|^{2\alpha-1}} \leq \frac{C}{|\lambda - 2\pi \ell|^{1+\alpha}\lambda^{1+\alpha}}.$$}

Since $2\pi \ell$ is bounded away from $\lambda$ for intervals not containing the origin, the sum of all these integrals is also bounded for any $\alpha < 1$. (This estimate is valid for any large $\lambda$, and the properties of $\lambda$ play a role only in the estimate for the interval with two zeros, which we consider next.)

Finally, consider the interval $I_{\ell_0}$ containing two zeros $x^\pm_{\ell_0}$ of $\bar{h}$ (observe that $\lambda - 2\pi \ell_0 = \bar{\lambda}$). From the Lagrange interpolation formula

$$h(x) = h(x^+_{\ell_0})\frac{x - x^-_{\ell_0}}{x^+_{\ell_0} - x^-_{\ell_0}} + h(x^-_{\ell_0})\frac{x - x^+_{\ell_0}}{x^-_{\ell_0} - x^+_{\ell_0}} + \frac{1}{2}h''(\bar{x})(x - x^+_{\ell_0})(x - x^-_{\ell_0})$$

($\bar{x}$ lies in the interval spanned by $x$ and $x^\pm_{\ell_0}$) we have:

$$|\bar{h}(x)| \geq \frac{1}{3}\lambda|x - x^+_{\ell_0}||x - x^-_{\ell_0}|$$

(15)

for $x \in I_{\ell_0}$. Also the relation (14) is still valid. If $|\bar{\lambda}| \geq \lambda^{-\delta}$, we have:

$$|x^\pm_{\ell_0}| \geq c_3/\lambda^{1+\delta},$$

for $\lambda \geq 1/\ell_0^\delta$. This completes the proof of the lower bounds.
whence the integral
\[ \int_{I_{\ell_0}} \frac{dx}{|\sin x|^{\alpha}|h(x)|^{2\alpha-1}} \]  
(16)
is bounded for any \( \alpha < 1 \), provided that \( \delta = \delta(\alpha) > 0 \) is sufficiently small.

This establishes Theorem 1 with \( \beta < 2 \) for \( E = 0 \) for \( \lambda \in \Lambda = \{ |\tilde{\lambda}| \geq \lambda^{-\delta} \} \). Note that \( \tilde{\lambda} \approx 0 \) corresponds to the presence of an elliptic island.

Without restrictions on \( \lambda \), (13) implies that one has either
\[ |x_{\ell_0}^+| \geq c/\lambda \quad \text{and} \quad -|x_{\ell_0}^-| \geq c/\lambda \]
or
\[ |x_{\ell_0}^+ - x_{\ell_0}^-| \geq c/\lambda \, . \]

In both cases, the integral (16) is bounded for \( \alpha < 2/3 \). For example, in the first case (15) and the Cauchy–Schwarz inequality imply
\[ \int_{I_{\ell_0}} \frac{dx}{|\sin x|^\alpha |h(x)|^{2\alpha-1}} \leq \frac{C}{\lambda^{2\alpha-1}} \int_{I_{\ell_0}} \frac{dx}{|x|^\alpha |x - x_{\ell_0}^+|^{|2\alpha-1|} |x - x_{\ell_0}^-|^{|2\alpha-1|}} \]
\[ \leq \frac{C}{\lambda^{2\alpha-1}} \int_{I_{\ell_0}} \frac{dx}{|x|^\alpha |x - c/\lambda|^{4\alpha-2}} ; \]
for \( \alpha \leq 2/3 \), the right-hand side does not exceed
\[ \frac{C}{\lambda^{2\alpha-1}} \int_{I_{\ell_0}} \frac{dx}{|x|^\alpha |x - c/\lambda|^\alpha} ; \]
and this expression is bounded by a constant due to the estimate (11). This concludes the proof of the lemma for the case \( E = b = 0 \).

In the general case, set \( \tilde{x}^* = \pi - x^* \); then (for \( E \) away from \( \pm 1 \))
\[ |x_0 + E| \geq \frac{1}{C} \min(|x_0 - \tilde{x}^*|, |x_0 + \tilde{x}^*|) , \]
hence we may replace the term \( \cos x_0 + E \) in the denominator with \( |x_0 \pm \tilde{x}^*| \).

Letting \( x = x_0 \pm \tilde{x}^* \), we obtain an integral similar to (12), with the function
\[ 2(x_0 \mp \tilde{x}^*) + \lambda \sin(x_0 \mp \tilde{x}^*) \]
in place of \( h \). The form of the Taylor expansion (13) and its corollary (14) is similar to that in the case \( b = E = 0 \). The final step of the argument requires no modification. \( \square \)
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