Supersymmetry in the boundary tricritical Ising field theory

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Abstract

We argue that it is possible to maintain both supersymmetry and integrability in the boundary tricritical Ising field theory. Indeed, we find two sets of boundary conditions and corresponding boundary perturbations which are both supersymmetric and integrable. The first set corresponds to a "direct sum" of two non-supersymmetric theories studied earlier by Chim. The second set corresponds to a one-parameter deformation of another theory studied by Chim. For both cases, the conserved supersymmetry charges are linear combinations of $Q$, $\bar{Q}$ and the spin-reversal operator $\Gamma$. 
1 Introduction

Consider a 1+1-dimensional quantum field theory which is integrable in the bulk. Demanding that this field theory remain integrable in the presence of a boundary places severe restrictions on the possible boundary interactions \[ \text{[1]} \]. Similarly, if a field theory has bulk supersymmetry, then demanding that supersymmetry be preserved in the presence of a boundary evidently also restricts the possible boundary interactions. These considerations immediately raise the question: to what extent can both integrability and supersymmetry be maintained in the presence of a boundary? We address this question here in the context of the tricritical Ising field theory \[ \text{[2]} \] – i.e., the tricritical Ising conformal field theory (CFT) \[ \text{[3, 4, 5]} \] perturbed by the \( \Phi_{(1,3)} \) operator \[ \text{[6]} \]. Several authors have already investigated this field theory in the presence of a boundary \[ \text{[7]–[12]} \]. Although the question of whether supersymmetry can be maintained in the boundary theory was raised in the seminal work of Chim \[ \text{[7]} \], it has not been addressed until now. \[ \text{[1]} \]

We have recently investigated the issue of integrability and supersymmetry in the presence of a boundary for other models: the scaling supersymmetric Yang-Lee model \[ \text{[15]} \], and the \( N = 1 \) and \( N = 2 \) sine-Gordon models \[ \text{[16]} \]. However, in contrast to those models whose particles have vertex-type scattering matrices, the tricritical Ising field theory contains kinks which have RSOS-type scattering matrices. Also, the tricritical Ising field theory is an example of a perturbed minimal model which, unlike the supersymmetric Yang-Lee model, is unitary.

To the above question, we find an affirmative answer: it is possible to maintain both supersymmetry and integrability in the boundary tricritical Ising field theory. Indeed, we find two sets of boundary conditions and corresponding boundary perturbations which are both supersymmetric and integrable. The first boundary condition involves a superposition of two pure “Cardy” boundary conditions \[ \text{[17]} \]. Hence, the corresponding field theory is in fact a “direct sum” of two non-supersymmetric theories studied in \[ \text{[7]} \]. We explicitly construct the conserved supersymmetry charge, and find that it contains a term proportional to the spin-reversal operator. The field theory corresponding to the second set of boundary conditions is a one-parameter deformation of another theory studied in \[ \text{[7]} \].

In Section 2 we briefly review the pertinent results from \[ \text{[2]} \] on the bulk tricritical Ising field theory. We also introduce the spin-reversal operator \( \Gamma \), which – as already noted – plays an important role in the boundary theory. In Section 3 we recall \[ \text{[7]} \] the conformal boundary conditions and corresponding conformal boundary states of the tricritical Ising CFT, and we

\[ \text{[1]} \] An alternative field theory for perturbed tricritical Ising has been proposed \[ \text{[13, 14]} \], which we shall not discuss here.
argue that certain boundary states and combinations thereof have superconformal symmetry. Our main results are contained in Sections 4 and 5 where we study supersymmetric perturbations of these superconformal boundary conditions. In particular, we propose specific boundary perturbing operators and the corresponding conserved supersymmetry charges and boundary $S$ matrices. We conclude in Section 6 with a brief discussion of our results.

2 Bulk TIM

Zamolodchikov has described the bulk massive integrable quantum field theory obtained as a perturbation of the tricritical Ising CFT by the $\Phi_{(1,3)}$ operator. This model, to which we shall refer as the (bulk) tricritical Ising field theory, or simply (bulk) TIM, can be described by the “action”

$$A = A_{\mathcal{M}(4/5)} + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \, \Phi_{\left(\frac{2}{5}, \frac{2}{5}\right)}(x, y), \quad \lambda < 0,$$ (2.1)

where $A_{\mathcal{M}(4/5)}$ is the action of the tricritical Ising CFT (i.e., the minimal unitary model $\mathcal{M}(4/5)$ with central charge $c = \frac{7}{10}$), and $\Phi_{\left(\frac{2}{5}, \frac{2}{5}\right)}$ is the spinless $(1, 3)$ primary field of this CFT with dimensions $(\frac{2}{5}, \frac{2}{5})$. The $\mathcal{M}(4/5)$ Kac table is given in Table 1. Moreover, $\lambda$ is a bulk parameter with dimension length $-\frac{4}{5}$.

Following [2], we restrict our attention to the case $\lambda < 0$, for which there is a three-fold vacuum degeneracy, and the spectrum consists of massive kinks $K_{a,b}(\theta)$ that separate neighboring vacua, $a, b \in \{-1, 0, 1\}$ with $|a - b| = 1$. Multi-kink states

$$|K_{a_1,b_1}(\theta_1) K_{a_2,b_2}(\theta_2) \ldots\rangle$$ (2.2)

must obey the adjacency conditions $b_1 = a_2$, etc.

The two-kink $S$ matrix has four distinct amplitudes defined by [2]

$$|K_{0,a}(\theta_1) K_{a,0}(\theta_2)\rangle_{in} = A_0(\theta_{12})|K_{0,a}(\theta_2) K_{a,0}(\theta_1)\rangle_{out} + A_1(\theta_{12})|K_{0,-a}(\theta_2) K_{-a,0}(\theta_1)\rangle_{out},$$

$$|K_{a,0}(\theta_1) K_{0,a}(\theta_2)\rangle_{in} = B_0(\theta_{12})|K_{a,0}(\theta_2) K_{0,a}(\theta_1)\rangle_{out},$$

$$|K_{a,0}(\theta_1) K_{0,-a}(\theta_2)\rangle_{in} = B_1(\theta_{12})|K_{a,0}(\theta_2) K_{0,-a}(\theta_1)\rangle_{out},$$ (2.3)
where \( \theta_{12} = \theta_1 - \theta_2 \), and \( a = \pm 1 \). These amplitudes are given by [2]

\[
A_0(\theta) = \cosh \frac{\theta}{4} A(\theta), \quad A_1(\theta) = -i \sinh \frac{\theta}{4} A(\theta),
\]
\[
B_0(\theta) = \cosh \frac{1}{4}(\theta - i\pi) B(\theta), \quad B_1(\theta) = \cosh \frac{1}{4}(\theta + i\pi) B(\theta),
\]

(2.4)

where

\[
A(\theta) = e^{\gamma \theta} S(\theta), \quad B(\theta) = \sqrt{2} e^{-\gamma \theta} S(\theta),
\]

(2.5)

with \( \gamma = \frac{1}{2\pi} \ln 2 \), and

\[
S(\theta) = \frac{1}{\sqrt{\pi}} \prod_{k=1}^{\infty} \frac{\Gamma(k - \frac{\theta}{2\pi i}) \Gamma(-\frac{1}{2} + k + \frac{\theta}{2\pi i})}{\Gamma(\frac{1}{2} + k - \frac{\theta}{2\pi i}) \Gamma(k + \frac{\theta}{2\pi i})}.
\]

(2.6)

As remarked in [2], these amplitudes do not have poles in the physical strip, corresponding to the fact that there are no bulk bound states. Also, these amplitudes are essentially the Boltzmann weights of the critical Ising lattice model [18, 19].

The tricritical Ising CFT contains a conserved supercurrent, and in fact has superconformal symmetry [3]. The holomorphic component \( G(z) \) of the supercurrent corresponds to the primary field \( \Phi_{(\frac{3}{2}, 0)} \) with spin \( \frac{3}{2} \), and similarly, the antiholomorphic component \( \bar{G}(\bar{z}) \) is the primary field \( \Phi_{(0, \frac{3}{2})} \) with spin \( -\frac{3}{2} \). Using the methods developed in [6], Zamolodchikov has shown that in the perturbed theory (2.1) the supercurrent remains conserved,

\[
\partial_z \bar{G} = \partial_{\bar{z}} \Psi, \quad \partial_{\bar{z}} G = \partial_z \bar{\Psi},
\]

(2.7)

where \( \bar{\Psi} = 4\lambda \bar{G} \Phi_{(\frac{3}{2}, \frac{1}{2})} \) and \( \Psi = 4\lambda G \Phi_{(-\frac{3}{2}, \frac{1}{2})} \). Hence, the model has fermionic integrals of motion \( Q \) and \( \bar{Q} \) of spin \( \pm \frac{1}{2} \), respectively,

\[
Q = \int G \, dz + \bar{\Psi} \, d\bar{z}, \quad \bar{Q} = \int \bar{G} \, d\bar{z} + \Psi \, dz,
\]

(2.8)

that obey the \( N = 1 \) supersymmetry algebra

\[
Q^2 = \hat{P}, \quad \bar{Q}^2 = \bar{\hat{P}}, \quad \{Q, \bar{Q}\} = 2t,
\]

(2.9)

where \( t \) is the topological charge. The action of the supersymmetry charges on multi-kink states (both “in” and “out”) is given by [2]

\[
Q|K_{a_1, a_2}(\theta_1) \ldots K_{a_N, a_{N+1}}(\theta_N)\rangle = \sum_{j=1}^{N} \left\{ \sqrt{m} \, \beta(a_j, a_{j+1}) \, e^{2\gamma} \right\} \times |K_{-a_1, -a_2}(\theta_1) \ldots K_{-a_{j-1}, -a_j}(\theta_{j-1}) K_{-a_j, a_{j+1}}(\theta_j) K_{a_N, a_{N+1}}(\theta_N)\rangle,
\]

(2.10)
\[ Q|K_{a_1,a_2}(\theta_1) K_{a_2,a_3}(\theta_2) \ldots K_{a_N,a_{N+1}}(\theta_N)\rangle = \sum_{j=1}^{N} \left\{ \sqrt{m} \beta(a_j,a_{j+1}) e^{-\frac{\theta_j}{m}} \times |K_{\bar{a}_1,-a_2}(\theta_1) \ldots K_{-a_j,-a_{j+1}}(\theta_j) \ldots K_{\bar{a}_N,a_{N+1}}(\theta_N)\rangle \right\}, \quad (2.11) \]

where
\[ \beta(a,b) = i(a + ib), \quad \bar{\beta}(a,b) = -i(a - ib), \quad (2.12) \]

and \( m \) is the kink mass. Moreover, the topological charge acts according to
\[ t|K_{a_1,a_2}(\theta_1) \ldots K_{a_N,a_{N+1}}(\theta_N)\rangle = -(a_1^2 - a_{N+1}^2)|K_{a_1,a_2}(\theta_1) \ldots K_{a_N,a_{N+1}}(\theta_N)\rangle. \quad (2.13) \]

One can show that these charges commute with the above \( S \) matrix. Indeed, this is how the \( S \) matrix is determined in [2].

We now define the spin-reversal operator \( \Gamma \) by the following action on multi-kink states (both “in” and “out”):
\[ \Gamma |K_{a_1,a_2}(\theta_1) K_{a_2,a_3}(\theta_2) \ldots K_{a_N,a_{N+1}}(\theta_N)\rangle = |K_{\bar{a}_1,-a_2}(\theta_1) K_{-a_2,-a_3}(\theta_2) \ldots K_{-a_N,-a_{N+1}}(\theta_N)\rangle. \quad (2.14) \]

Evidently, the spin-reversal operator satisfies
\[ \Gamma^2 = 1, \quad (2.15) \]

and it commutes with the bulk \( S \) matrix [2,3]. Moreover, \( \Gamma \) anticommutes with the supersymmetry charges,
\[ \{ \Gamma, Q \} = 0, \quad \{ \Gamma, \bar{Q} \} = 0, \quad (2.16) \]

as follows from the fact \( \beta(-a,-b) = -\beta(a,b) \), and similarly for \( \bar{\beta}(a,b) \). These properties suggest that \( \Gamma \) corresponds in the (perturbed) CFT to the operator \((-1)^F\), where \( F \) is the Fermion-number operator.

### 3 Conformal boundary conditions

The Cardy states [17] for the tricritical Ising boundary CFT are given in terms of Ishibashi states [20] by [7]
\[ (-) : \quad \tilde{0} = C \left[ \left| 0 \right\rangle + \frac{3}{2} \right\rangle + \eta \left( \left| \frac{1}{10} \right\rangle + \frac{3}{5} \right\rangle + \sqrt{2} \left| \frac{7}{16} \right\rangle + \sqrt{2} \left| \frac{3}{80} \right\rangle \right], \]
\( (-0) : \langle \frac{1}{10} | = C \left[ \eta^2 \left( |0\rangle + |\frac{3}{2}\rangle \right) - \eta^{-1} \left( |\frac{1}{10}\rangle + |\frac{3}{5}\rangle \right) - \sqrt{2} \eta^2 \left| \frac{7}{16} \rightangle + \sqrt{2} \eta^{-1} \left| \frac{3}{80} \rightangle \right] , \)

\( (0+) : \langle \frac{3}{5} | = C \left[ \eta^2 \left( |0\rangle + |\frac{3}{2}\rangle \right) - \eta^{-1} \left( |\frac{1}{10}\rangle + |\frac{3}{5}\rangle \right) + \sqrt{2} \eta^2 \left| \frac{7}{16} \rightangle - \sqrt{2} \eta^{-1} \left| \frac{3}{80} \rightangle \right] , \)

\( (+) : \langle \frac{3}{2} | = C \left[ \left( |0\rangle + |\frac{3}{2}\rangle \right) + \eta \left( |\frac{1}{10}\rangle + |\frac{3}{5}\rangle \right) - \sqrt{2} \left| \frac{7}{16} \rightangle - \sqrt{2} \left| \frac{3}{80} \rightangle \right] , \)

\( (0) : \langle \frac{7}{16} | = \sqrt{2} C \left[ \left( |0\rangle - |\frac{3}{2}\rangle \right) - \eta \left( |\frac{1}{10}\rangle - |\frac{3}{5}\rangle \right) \right] , \)

\( (d) : \langle \frac{3}{80} | = \sqrt{2} C \left[ \eta^2 \left( |0\rangle - |\frac{3}{2}\rangle \right) + \eta^{-1} \left( |\frac{1}{10}\rangle - |\frac{3}{5}\rangle \right) \right] , \)

where

\[ C = \sqrt{\frac{\sin \frac{\pi}{5}}{\sqrt{5}}}, \quad \eta = \sqrt{\frac{2 \sin \frac{2\pi}{5}}{5 \sin \frac{\pi}{5}}} . \] 

In \( \text{(3.1)} \) are also given the corresponding conformal boundary conditions (CBC) which Chim has identified. Let us recall that, in the bulk, the three vacua \(-1, 0, +1\) are degenerate. However, these vacua do not necessarily remain degenerate at the boundary. Indeed, for the boundary conditions \((-), (0), (+)\), the order parameter is fixed at the boundary to the vacua \(-1, 0, +1\), respectively. For the boundary condition \((-0)\), the vacua \(-1\) and 0 are degenerate at the boundary; hence, the order parameter at the boundary may be in either of these two vacua. Similarly, for the boundary condition \((0+)\), the 0 and +1 vacua are degenerate at the boundary. Finally, for the boundary condition \((d)\), all three vacua \(-1, 0, +1\) are degenerate at the boundary (as well as in the bulk); i.e., the order parameter at the boundary may be in any of the three vacua.

In the remainder of this Section, we argue that the boundary states corresponding to the conformal boundary conditions \((-)\&(+), (-0)\&(0+), (o)\) and \((d)\) have superconformal symmetry. We observe that the boundary states corresponding to these conformal boundary conditions are given by

\( (-)\&(+) : \langle 0 | + \langle \frac{3}{2} | = 2C \left( |0_+\rangle + \eta \left| \frac{1}{10_+} \right\rangle \right) , \)

\( (-0)\&(0+) : \langle \frac{1}{10} | + \langle \frac{3}{5} | = 2C \left( \eta^2 |0_+\rangle - \eta^{-1} \left| \frac{1}{10_+} \right\rangle \right) , \)

\( (o) : \langle \frac{7}{16} | = \sqrt{2} C \left( |0_-\rangle - \eta \left| \frac{1}{10_-} \right\rangle \right) , \)

\( (d) : \langle \frac{3}{80} | = \sqrt{2} C \left( \eta^2 |0_-\rangle + \eta^{-1} \left| \frac{1}{10_-} \right\rangle \right) , \) 

(3.3)
where
\[ |0_\pm\rangle = |0\rangle \pm |\frac{3}{2}\rangle, \quad |\frac{1}{10}_\pm\rangle = |\frac{1}{10}\rangle \pm |\frac{3}{5}\rangle. \] (3.4)

Notice that the first two states in (3.3) are superpositions of “pure” Cardy states. Also, these states are related by duality [7]. Indeed, under the duality transformation \( \mathcal{D} \), the Ishibashi states \(|0\rangle\) and \(|\frac{3}{5}\rangle\) remain invariant, while the states \(|\frac{1}{10}\rangle\) and \(|\frac{3}{2}\rangle\) pick up a minus sign. (See Appendix A.) Thus,
\[ \mathcal{D}|0_\pm\rangle = |0_\mp\rangle, \quad \mathcal{D}|\frac{1}{10}_\pm\rangle = -|\frac{1}{10}_\mp\rangle. \] (3.5)

It follows that
\[ \mathcal{D}\left(|\tilde{0}\rangle + |\tilde{3}\rangle\right) = \sqrt{2}\left|\frac{7}{16}\right\rangle, \quad \mathcal{D}\left(|\tilde{1}\rangle + |\tilde{3}\rangle\right) = \sqrt{2}\left|\frac{3}{80}\right\rangle. \] (3.6)

Let us briefly review some basic facts about superconformal field theory [5]. The \( N = 1 \) superconformal algebra is defined by the (anti) commutation relations
\[ \left[L_m, L_n\right] = (m - n)L_{m+n} + \frac{1}{12}c(m^3 - m)\delta_{m+n,0}, \]
\[ \left[L_m, G_r\right] = \left(\frac{1}{2}m - r\right)G_{m+r}, \]
\[ \left\{G_r, G_s\right\} = 2L_{r+s} + \frac{1}{3}c(r^2 - \frac{1}{4})\delta_{r+s,0}, \] (3.7)

where \( r, s \in \mathbb{Z} \) for the Ramond (R) sector and \( r, s \in \mathbb{Z} + \frac{1}{2} \) for the Neveu-Schwarz (NS) sector. The operators \( \bar{L}_n, \bar{G}_r \) corresponding to the antiholomorphic components of the energy-momentum tensor and supercurrent obey similar relations. Highest weight irreducible representations of the holomorphic algebra are generated from highest weight states \(|\Delta\rangle\) satisfying
\[ L_0|\Delta\rangle = \Delta|\Delta\rangle, \quad L_n|\Delta\rangle = G_r|\Delta\rangle = 0, \quad n > 0, \quad r > 0. \] (3.8)

The corresponding highest weight states of the antiholomorphic algebra are denoted by \(|\bar{\Delta}\rangle\). The tricritical Ising model, which has \( c = \frac{7}{10} \), can be regarded either as the conformal minimal model \( \mathcal{M}(4/5) \) or the superconformal minimal model \( S\mathcal{M}(3/5) \).

Since the Ishibashi states \(|j\rangle\) are annihilated by \((L_n - \bar{L}_{-n})\), then so are the linear combinations (3.3). We wish to show that \(^2\)
\[ (G_r - \bar{G}_{-r})\left(|\frac{1}{10}\rangle + |\frac{3}{5}\rangle\right) = 0, \quad (G_r - \bar{G}_{-r})\left(|\tilde{0}\rangle + |\frac{3}{2}\rangle\right) = 0, \]

\(^2\)For recent discussions of superpositions of Cardy states, see e.g. [11, 21, 22].

\(^3\)If \( G_r \) and \( \bar{G}_r \) are assumed to be operators in the “open” channel, then the rotation to the “closed” channel produces certain \( \pm i \) factors. For convenience, we shall assume here instead that the operators \( G_r \) and \( \bar{G}_r \) obeying [57] are in the closed channel.
\[
(G_r + \bar{G}_{-r}) \left| \frac{7}{16} \right> = 0,
\]
\[
(G_r + \bar{G}_{-r}) \left| \frac{3}{80} \right> = 0,
\]
for \( r \in \mathbb{Z} + \frac{1}{2} \). In view of (3.3), it suffices to show that the boundary states \( |0\pm\rangle, \left| \frac{1}{10} \pm \right> \) obey
\[
(G_r + \bar{G}_{-r}) |0\pm\rangle = 0, \quad (G_r + \bar{G}_{-r}) \left| \frac{1}{10} \pm \right> = 0.
\]
(3.10)

(This result is stated without proof in [23].) To this end, we observe that the vacuum state \( |0\rangle \) of \( \mathcal{M}(4/5) \) and \( \mathcal{SM}(3/5) \) is the same, since it is unique. Thus, it satisfies
\[
L_n |0\rangle = 0, \quad n \geq -1,
\]
\[
G_r |0\rangle = 0, \quad r \geq -\frac{1}{2}.
\]
(3.11)
The state \( |\frac{1}{10}\rangle \), which is a highest weight in both \( \mathcal{M}(4/5) \) and \( \mathcal{SM}(3/5) \), satisfies
\[
G_r |\frac{1}{10}\rangle = 0, \quad r > 0.
\]
(3.12)
The state \( |\frac{3}{2}\rangle \), which is a highest weight in \( \mathcal{M}(4/5) \), is a descendant in \( \mathcal{SM}(3/5) \),
\[
|\frac{3}{2}\rangle = \sqrt{\frac{15}{7}} G_{-\frac{1}{2}} |0\rangle,
\]
(3.13)
where the coefficient is fixed by the normalization condition \( \langle \frac{3}{2} |\frac{3}{2}\rangle = 1 \). Similarly,
\[
|\frac{3}{5}\rangle = \sqrt{5} G_{-\frac{1}{2}} |\frac{1}{10}\rangle.
\]
(3.14)
With the help of the representation for the conformal Ishibashi states given in [24]
\[
|j\rangle = \left( 1 + \frac{L_{-1} \bar{L}_{-1}}{2\Delta_j} + \ldots \right) |j\rangle \otimes |\bar{j}\rangle, \quad j \neq 0,
\]
\[
|0\rangle = \left( 1 + \frac{L_{-2} \bar{L}_{-2}}{c/2} + \ldots \right) |0\rangle \otimes |\bar{0}\rangle,
\]
(3.15)
one can proceed to verify (3.10). Indeed, for the case \( r = \frac{1}{2} \),
\[
(G_{\frac{1}{2}} - \bar{G}_{-\frac{1}{2}}) |0, r\rangle = \left( G_{\frac{1}{2}} - \bar{G}_{-\frac{1}{2}} \right) \left( |0\rangle \otimes |\bar{0}\rangle + |\frac{3}{2}\rangle \otimes |\frac{3}{2}\rangle \right)
\]
\[
+ \frac{20}{7} L_{-2} |0\rangle \otimes \bar{L}_{-2} |\bar{0}\rangle + \frac{1}{3} L_{-1} |\frac{3}{2}\rangle \otimes \bar{L}_{-1} |\frac{3}{2}\rangle + \ldots \).
\]
(3.16)
The operator \( (G_{\frac{1}{2}} - \bar{G}_{-\frac{1}{2}}) \) annihilates the first term due to (3.11); and acting on the second term, this operator gives
\[
-2 \sqrt{\frac{15}{7}} |\frac{3}{2}\rangle \otimes \bar{L}_{-2} |\bar{0}\rangle,
\]
(3.17)
as follows from (3.13), (3.7) and (3.11). There is an equal and opposite contribution from the action of $G_{\frac{1}{2}}$ on the third term in (3.16). The actions of $G_{\frac{1}{2}}$ on the third term, and of \(G_{\frac{1}{2}} - G_{\frac{1}{2}}\) on the fourth term, produce contributions which presumably are cancelled by corresponding contributions from higher-order terms represented by ellipses in (3.16). Similarly,

\[
\left( G_{\frac{1}{2}} - G_{\frac{1}{2}} \right) |\frac{1}{10}\rangle = \left( G_{\frac{1}{2}} - G_{\frac{1}{2}} \right) \left( |\frac{1}{10}\rangle \otimes |\frac{3}{5}\rangle + |\frac{3}{5}\rangle \otimes |\frac{3}{5}\rangle + \ldots \right)
\]

The first term vanishes due to (3.12). The second term is equal to

\[
- \frac{1}{\sqrt{5}} |\frac{1}{10}\rangle \otimes |\frac{3}{5}\rangle,
\]

as follows from (3.14); and it is cancelled by the third term. Other values of $r$ can presumably be treated in a similar manner.

## 4 Boundary TIM: NS case

We shall consider supersymmetric perturbations of the tricritical Ising boundary CFT with two different (super)conformal boundary conditions. In this Section we consider the CBC $(-0)\&(0+)$; and in Section 5 we consider the CBC $(d)$. We refer to these two cases as NS and R, respectively, since these are the sectors to which the corresponding boundary states belong.

### 4.1 Definition of the model as a perturbed CFT

We now consider the boundary tricritical Ising field theory, with the action \(^4\)

\[
A = A_{M(4/5)+(-0)\&(0+)} + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx \Phi_{(\frac{2}{5}, \frac{3}{5})}(x, y)
- h \int_{-\infty}^{\infty} dy \left( \phi_{(\frac{3}{5}), (-0)}(y) - \phi_{(\frac{3}{5}), (0+)}(y) \right),
\]

\(^4\)This boundary action differs in two important respects from a similar one considered by Chim [7]: (1) While he considers the CBC $(-0)\&(0+)$ corresponding to a pure Cardy state, we consider the CBC $(-0)\&(0+)$ corresponding to a superposition state. As we have already argued, the latter is supersymmetric, while the former is not. (2) Chim considers a single perturbing boundary operator ($\phi_{(\frac{3}{5}), (-0)}$ in our notation), whereas we consider the difference of two such operators. The latter generates an RG boundary flow which is supersymmetric, while the former does not.
where again we restrict to the case $\lambda < 0$. We now explain each of the terms in turn.

The first term in (4.1) is the action for the tricritical Ising boundary CFT $\mathcal{M}(4/5)$ with the conformal boundary condition $(-0)\&(0+).$ We have argued in the previous Section that the conformal boundary state corresponding to this CBC is annihilated not only by $(L_n - \bar{L}_{-n})$, but also by $(G_r - \bar{G}_r).$ Hence, it is in fact a superconformal boundary state. The corresponding boundary condition is superconformal,

$$
(T(y + ix) - \bar{T}(y - ix)) \bigg|_{x=0} = 0, \quad (G(y + ix) - \bar{G}(y - ix)) \bigg|_{x=0} = 0. \quad (4.2)
$$

The second term in the action (4.1) is the bulk perturbation, which is the same as in the bulk action (2.1), except that the $x$ integral is now restricted to the half-line $x \leq 0.$

The last term in the action (4.1) is the boundary perturbation. It involves the boundary primary fields $\phi_{(\frac{3}{5}),(-0)}$ and $\phi_{(\frac{3}{5}),(0+)}$ with dimension $\Delta_{(1,3)} = \frac{3}{5}$ which act on $(-0)$ and $(0+)$, respectively. The reason for the relative minus sign between the two boundary primary fields will be given below, when we discuss boundary flows. Moreover, $h$ is a boundary parameter which has dimensions length $^{-\frac{2}{5}}$.

Since the boundary perturbation has the same dimension $\Delta_{(1,3)}$ as the bulk perturbation, the analysis of [1] suggests that this boundary perturbation is integrable. One can also use the arguments of [1] to infer that the boundary perturbation preserves supersymmetry. Indeed, consider the bulk conformal limit $\lambda = 0.$ In view of the boundary condition (1.2), by computing the operator product $[G(y + ix) - \bar{G}(y - ix)] \left( \phi_{(\frac{3}{5}),(-0)}(y') - \phi_{(\frac{3}{5}),(0+)}(y') \right)$, one can conclude to first order in perturbation theory that the quantity

$$
\hat{Q} = \int_{-\infty}^{0} dx \left[ G(x, y) + \bar{G}(x, y) \right] + \Theta(y), \quad (4.3)
$$

with $\Theta(y) \propto h(1 - 2\Delta_{(1,3)}) \left( \phi_{(\frac{3}{5}),(-0)}(y) - \phi_{(\frac{3}{5}),(0+)}(y) \right)$ is an integral of motion. It is plausible that, for the general massive case $\lambda \neq 0$, this becomes

$$
\hat{Q} = Q + \bar{Q} + \Theta, \quad (4.4)
$$

where $Q$ and $\bar{Q}$ are given by (2.8), with the $x$ integrals restricted to the half-line.

Various arguments [7, 9, 10, 11, 12, 21] support the following pattern of renormalization group boundary flows for the tricritical Ising boundary CFT:

$$
(-0) + \phi_{(\frac{3}{5})} \rightarrow (0), \quad (0-) + \phi_{(\frac{3}{5})} \rightarrow (-), \quad (4.5)
$$

$$
(0+) + \phi_{(\frac{3}{5})} \rightarrow (+), \quad (0+) - \phi_{(\frac{3}{5})} \rightarrow (0). \quad (4.6)
$$

\[\text{In general, boundary operators } \phi_a \text{ and } \phi_b \text{ which act on conformal boundary conditions } a \text{ and } b \text{ commute; i.e., their operator product expansion with each other is zero. Such operators have recently been studied in [22].}\]
This would imply that the boundary perturbation in (4.1) generates the boundary flows
\[
\begin{align*}
(0)&(0+) & \rightarrow & \ 2(0) & \text{for} \ h < 0 , \\
(0)&(0+) & \rightarrow & \ (-)&(+) & \text{for} \ h > 0 .
\end{align*}
\]

We have argued in Section 3 that both the pure Cardy state \(|\tilde{\tau}_{7/16}\rangle\) which corresponds to the CBC \(0\), and also the superposition state \(|\tilde{\theta}_{0} + \frac{1}{2}\rangle\) which corresponds to the CBC \((-)&(+)\), are superconformal boundary states. Hence, the boundary flows (4.7) connect superconformal boundary conditions, and we refer to such flows as “supersymmetric flows.” Indeed, we arrived at the particular boundary perturbation in (4.1) (specifically, the relative minus sign between the boundary primary fields \(\phi_{(0+)}\)) by requiring that it produce this supersymmetric flow. Curiously, since the \(g\) factors (4.8)
\[
\begin{align*}
g_{(-0)} &= g_{(0+)} , & g_{(-)} &= g_{(+)} ,
\end{align*}
\]
the ratio \(g_{UV}/g_{IR}\) of \(g\) factors corresponding to the ultraviolet and infrared fixed points is the same for the non-supersymmetric flows (4.5),(4.6) and for the corresponding supersymmetric flows (4.7). That is,
\[
\begin{align*}
\frac{g_{(-0)}}{g_{(0)}} &= \frac{g_{(0+)}}{g_{(0)}} = \frac{g_{(-0)&(0+)}}{g_{2(0)}} , \\
\frac{g_{(-0)}}{g_{(-)}} &= \frac{g_{(0+)}}{g_{(+)}} = \frac{g_{(-0)&(0+)}}{g_{(-)&(+)}} .
\end{align*}
\]

It is important to notice that the boundary perturbation in (4.1) breaks spin-reversal symmetry. Indeed, under spin reversal, \((-0) \leftrightarrow (0+)\). Hence, although the CBC \((-0)&(0+)\) remains invariant, the perturbation \((\phi_{(-0)} - \phi_{(0+)})\) picks up a minus sign.

### 4.2 Boundary scattering theory

We now turn to the boundary scattering theory. Following Chim [7], we assume that the boundary can have (at most) three possible states, corresponding to the three different vacua. We therefore define the boundary operator \(B_{a}\) with \(a \in \{-1, 0, 1\}\), in terms of which the possible boundary states are \(|B_{a}\rangle\), with corresponding energies \(e_{a}\). Multi-kink states have the form
\[
|K_{a_{1},a_{2}}(\theta_{1}) K_{a_{2},a_{3}}(\theta_{2}) \ldots K_{a_{N},a}(\theta_{N}) B_{a}\rangle
\]

We extend the action (2.10), (2.11) of the supercharges \(Q\) and \(\bar{Q}\) on such states in the obvious way (namely, \(B_{a}\) remains invariant); and we extend the action (2.14) of the spin-reversal operator \(\Gamma\) such that \(\Gamma B_{a} = B_{-a}\).
The kink boundary $S$ matrix has six amplitudes defined by (7)

$$\begin{align*}
|K_{1,0}(\theta) B_0\rangle_{in} & = R_+(\theta)|K_{1,0}(-\theta) B_0\rangle_{out}, \\
|K_{-1,0}(\theta) B_0\rangle_{in} & = R_-(\theta)|K_{-1,0}(-\theta) B_0\rangle_{out}, \\
|K_{0,1}(\theta) B_1\rangle_{in} & = P_+(\theta)|K_{0,1}(-\theta) B_1\rangle_{out} + V_+(\theta)|K_{0,-1}(-\theta) B_{-1}\rangle_{out}, \\
|K_{0,-1}(\theta) B_{-1}\rangle_{in} & = P_-(\theta)|K_{0,-1}(-\theta) B_{-1}\rangle_{out} + V_-(\theta)|K_{0,1}(-\theta) B_1\rangle_{out}. 
\end{align*} \tag{4.11}$$

The unitarity constraints are given by (7)

$$\begin{align*}
R_+(\theta)R_+(-\theta) & = 1, \\
P_+(\theta)V_+(-\theta) + V_+(\theta)P_-(\theta) & = 0, \\
P_+(\theta)P_+(-\theta) + V_+(\theta)V_-(-\theta) & = 1, 
\end{align*} \tag{4.12}$$

The cross-unitarity constraints (1) are given by (7)

$$R_+(\theta) = A_0(2\theta) + A_1(2\theta) + i\tilde{\alpha} \Gamma, \tag{4.13}$$

Motivated by the plausibility argument given above that the boundary TIM (4.1) has a conserved supersymmetry charge (4.4) and by the observation that the boundary perturbation breaks spin-reversal symmetry (see also (15), (16)), we consider the following operator

$$\hat{Q} = Q + \bar{Q} + \frac{2i\sqrt{m}}{\alpha} \Gamma, \tag{4.14}$$

where $\alpha$ is a parameter which is yet to be determined. Requiring that this operator commute with the boundary $S$ matrix (4.11) yields the following constraints on the amplitudes:

$$\begin{align*}
\frac{R_+(\theta)}{R_-(\theta)} & = \frac{1 + \alpha \sinh \frac{\theta}{2}}{1 - \alpha \sinh \frac{\theta}{2}}, \\
V_+(\theta) & = V_-(\theta) \equiv V(\theta), \\
V(\theta) & = \frac{i}{2\alpha \cosh \frac{\theta}{2}} (P_-(\theta) - P_+(\theta)). \tag{4.15}
\end{align*}$$
These constraints are a special case of those obtained in \cite{7} from the boundary Yang-Baxter equations.

Our goal is to determine the boundary $S$ matrix for the perturbation of the CBC $(−0)$ & $(0+)$. To this end, it is important to first recall \cite{7} some results for the simpler (non-supersymmetric) case of the perturbation of the CBC $(−0)$. For that case, the boundary can exist in either the states (vacua) $−1$ or $0$. The energies $e_{−1}$ and $e_{0}$ of these states depend on the value of the boundary parameter $h$: for $h = 0$, the two energies are equal; while for nonzero $h$, the energies are no longer degenerate. Consider the situation (say, positive $h$) that $e_{−1} < e_{0}$; that is, the state $−1$ is the ground state of the boundary, and the state $0$ is an excited state of the boundary. Since the boundary cannot exist in the state $+1$, the amplitudes $P_{±}(θ)$ and $V_{±}(θ)$ vanish. \footnote{Such amplitudes evidently violate the third unitarity constraint in \cite{11}. However, since the boundary cannot exist in the state $+1$, this constraint should not be imposed.} The amplitude $P_{−}(θ)$ is given by $P_{−}(θ) = P(θ)$, where

$$P(θ) = P_{ξ}^{CDD}(θ)P_{min}(θ), \quad (4.16)$$

where $P_{min}(θ)$ is the minimal solution of the equations

$$P_{min}(θ)P_{min}(−θ) = 1, \quad P_{min}(\frac{iπ}{2} − θ) = B_{0}(2θ)P_{min}(\frac{iπ}{2} + θ), \quad (4.17)$$

with no poles in the physical strip, and is given by \cite{7}

$$P_{min}(θ) = e^{γθ} \prod_{k=1}^{∞} \frac{Γ(k - \frac{θ}{2πi})Γ(k - \frac{1}{4} + \frac{θ}{2πi})Γ(k + \frac{1}{4} + \frac{θ}{2πi})}{Γ(k + \frac{θ}{2πi})Γ(k + \frac{1}{4} − \frac{θ}{2πi})Γ(k - \frac{1}{4} − \frac{θ}{2πi})}. \quad (4.18)$$

Moreover, $P_{ξ}^{CDD}(θ)$ is the CDD factor

$$P_{ξ}^{CDD}(θ) = \frac{\sin ξ - i \sinh θ}{\sin ξ + i \sinh θ}, \quad (4.19)$$

which has a pole at $θ = iξ$. The parameter $ξ$ is related in some way to the boundary parameter $h$. The state $0$, which is an excited state of the boundary, can be regarded as a boundary bound state, which is associated with this pole when it lies in the physical strip $0 ≤ ξ ≤ \frac{π}{2}$. The energies of the states $−1$ and $0$ are related by

$$e_{0} = e_{−1} + m \cos ξ. \quad (4.20)$$

In particular, $h = 0$ corresponds to $ξ = \frac{π}{2}$. Using the boundary bound state bootstrap equations \cite{11}, one can determine \cite{7} the amplitudes $R_{±}(θ)$,

$$R_{±}(θ) = \frac{1}{2}(\cos \frac{ξ}{2} ± i \sinh \frac{θ}{2})B(θ - iξ)B(θ + iξ)P(θ). \quad (4.21)$$
For the opposite sign of $h$, the situation is reversed: $e_0 < e_{-1}$, and so the state 0 is the ground state of the boundary, and the state $-1$ is an excited state of the boundary. Chim has explained in detail how the above boundary $S$ matrix can give rise to the RG boundary flows in Eq. (4.5).

For the case of the perturbation of the CBC $(0+)$, the results are parallel: the boundary can exist in either the states 0 or $+1$. Hence, the amplitudes $P_-(\theta)$ and $V_{\pm}(\theta)$ vanish, $P_+(\theta) = P(\theta)$, and $R_{\pm}(\theta)$ is given by (4.21). The corresponding RG boundary flows are given in Eq. (4.6).

Finally, let us return to the case of the perturbation of the CBC $(0)\&(0+)$. Since the corresponding boundary state is a superposition of two “pure” boundary states, the vacua $-1$ and $+1$ are not states of the same “irreducible” theory. Indeed, we have a “direct sum” of two “irreducible” theories: one theory with only boundary states $-1$ and 0, and another theory with only boundary states 0 and $+1$. In particular, the unitarity, crossing and bootstrap constraints involving both $-1$ and $+1$ boundary states should not be imposed. Thus, the boundary $S$ matrix is the “direct sum” of the boundary $S$ matrices given above for the perturbations of $(0)$ and $(0+)$. That is, we propose for the boundary TIM (4.1) the following boundary $S$ matrix:

$$P_+(\theta) = P_-(\theta) = P(\theta), \quad V_{\pm}(\theta) = 0,$$

$$R_{\pm}(\theta) = \frac{1}{2}(\cos \frac{\xi}{2} \pm i \sinh \frac{\theta}{2})B(\theta - i\xi)B(\theta + i\xi)P(\theta), \quad (4.22)$$

where $P(\theta)$ is given by (4.16). This set of amplitudes satisfies the supersymmetry constraints (4.15), with the parameter $\alpha$ which appears in the supersymmetry charge (4.14) given by

$$\alpha = \frac{i}{\cos \frac{\xi}{2}}. \quad (4.23)$$

We expect that the corresponding RG boundary flows should be given by Eq. (4.7).

## 5 Boundary TIM: R case

We now wish to consider the perturbation of the tricritical Ising boundary CFT with the CBC $(d)$. We have argued in Section 3 that the corresponding boundary state has superconformal symmetry. Hence, this CBC is in fact superconformal,

$$(T(y + ix) - \bar{T}(y - ix)) \bigg|_{x=0} = 0, \quad (G(y + ix) + \bar{G}(y - ix)) \bigg|_{x=0} = 0. \quad (5.1)$$

Since the CBC $(d)$ is related to the CBC $(0)\&(0+)$ by a duality transformation (see Eq. (3.6)), we expect that the action should be given by the image of the action (4.1) under duality. The bulk perturbing operator is invariant under duality. (See Appendix A) However,
the individual conformal boundary conditions \((-0)\) and \((0+)\) transform in a complicated manner under duality, so that only their sum transforms simply. Hence, the boundary perturbing operator \(D\left(\phi_{\left(\frac{\theta}{2}\right),(0)}(y) - \phi_{\left(\frac{\theta}{2}\right),(0+)}(y)\right)\) is also complicated; in particular, it breaks spin-reversal symmetry, and is different from the operator \(\phi_{\left(\frac{\theta}{2}\right),(0)}(y)\) considered in [7]. The expected RG boundary flow is dual to (4.7), namely,

\[
(d) \quad \rightarrow \quad (-0)&(+) \quad \text{for} \quad h < 0, \\
(d) \quad \rightarrow \quad (0) \quad \text{for} \quad h > 0. \tag{5.2}
\]

By duality, we expect that the supersymmetry charge

\[
\hat{Q} = Q - \bar{Q} + \frac{2i\sqrt{m}}{\alpha} \Gamma
\]

should be conserved. This charge differs from the one considered previously (4.14) by the sign in front of \(\bar{Q}\). Requiring that this operator commute with the boundary \(S\) matrix (4.11) yields the following constraints on the amplitudes:

\[
R_+ = R_-(\theta) \equiv R(\theta), \\
P_+ = P_-(\theta) \equiv P(\theta), \\
P(\theta) = \frac{i}{2\alpha \sinh \frac{\theta}{2}} \left(V_-(\theta) - V_+(\theta)\right). \tag{5.4}
\]

These constraints are also a special case of those obtained in [7] from the boundary Yang-Baxter equations.

We now proceed to determine the boundary \(S\) matrix. For \(h = 0\), all three vacua are degenerate at the boundary, \(e_{-1} = e_0 = e_1\). We assume that for positive \(h\), \(e_0 < e_{-1} = e_{+1}\); that is, the state 0 is the ground state of the boundary, and the states \(-1\) and \(+1\) are degenerate excited states. We further assume that these excited states can be regarded as boundary bound states associated with a pole of \(R_{\pm}(\theta)\) at \(\theta = i\xi\) which lies in the physical strip. The energies of the three states are therefore related by

\[
e_{\pm 1} = e_0 + m \cos \xi, \tag{5.5}
\]

implying that \(h = 0\) corresponds to \(\xi = \frac{\pi}{2}\). Moreover, for \(\theta \sim i\xi\),

\[
R_{\pm}(\theta) \sim \frac{g_{0\pm}g_{\pm 0}}{\theta - i\xi}, \tag{5.6}
\]

where \(g_{0\pm}\) and \(g_{0}\) are particle-boundary coupling constants [1]. From the constraint \(R_+(\theta) = R_-(\theta) \equiv R(\theta)\) in Eq. (5.3), it follows that

\[
r \equiv \frac{g_{0+}}{g_{0-}} = \frac{g_{+0}}{g_{-0}}. \tag{5.7}
\]
The boundary bound state bootstrap equations imply that

\[ P(\theta) = [A_0(\theta - i\xi)A_0(\theta + i\xi) + A_1(\theta - i\xi)A_1(\theta + i\xi)] R(\theta) \]  

(5.8)

and

\[ V_\pm(\theta) = r [A_0(\theta - i\xi)A_1(\theta + i\xi) + A_1(\theta - i\xi)A_0(\theta + i\xi)] R(\theta) \]

(5.9)

Apart from the factors of \( r \) in \( V_\pm(\theta) \), these results coincide with those obtained in [7]. It follows that

\[ R(\theta) = e^{-2\gamma \theta} P^{CDD}_\xi P_{\min}(\theta), \]

\[ P(\theta) = \cos \frac{\xi}{2} A(\theta - i\xi)A(\theta + i\xi) R(\theta), \]

\[ V_+(\theta) = -i r \sinh 2 A(\theta - i\xi)A(\theta + i\xi) R(\theta), \]

\[ V_-(\theta) = -i r \sinh 2 A(\theta - i\xi)A(\theta + i\xi) R(\theta). \]  

(5.10)

This set of amplitudes satisfies the supersymmetry constraints (5.4), with the parameter \( \alpha \) given by

\[ \alpha = \frac{1 - r^2}{2r \cos \frac{\xi}{2}}. \]  

(5.11)

For the opposite sign of \( h \), the situation is reversed: \( e_{-1} = e_{+1} < e_0 \), and so the ground state of the boundary is two-fold degenerate, consisting of the vacua \(-1\) and \(+1\), and the vacuum \(0\) is an excited state. Indeed, the above boundary \( S \) matrix seems to be consistent with the boundary flows proposed in Eq. (5.2).

We have seen that the parameter \( r \) is not determined by the constraints of \( S \)-matrix theory (unitarity, crossing, etc.), integrability, or supersymmetry. It is clear that for \( h = 0 \), the action has spin-reversal symmetry (i.e., it commutes with \( \Gamma \)), which implies \( V_+ = V_- \). That is, \( r = 1 \) for \( h = 0 \) (i.e., for \( \xi = 2 \)). However, this constraint is rather mild, as it can be satisfied in infinitely many ways, e.g. \( r = \sin \xi \). We expect that through a more detailed analysis of the boundary perturbation it should be possible to completely determine \( r \), as well as \( \xi \), in terms of \( h \). However, we shall not pursue this problem here.

Finally, it should be noted that the parameter \( r \) can be set to unity by an appropriate gauge transformation [8] of the kink operators,

\[ K_{0,1}(\theta) \rightarrow e^{i\phi} K_{0,1}(\theta), \quad K_{0,-1}(\theta) \rightarrow e^{-i\phi} K_{0,-1}(\theta), \]  

(5.12)
which transforms the amplitudes \( V_\pm(\theta) \) as

\[
V_+(\theta) \to e^{-2i\varphi}V_+(\theta), \quad V_-(\theta) \to e^{2i\varphi}V_-(\theta),
\]

and leaves the other amplitudes unchanged. Ghoshal and Zamolodchikov have argued that performing a gauge transformation corresponds to adding a total derivative term to the boundary action density. Thus, setting \( r \to 1 \) corresponds to adding a total derivative term to the boundary action which restores spin-reversal symmetry, in which case the supersymmetry charge (5.3) reduces to \( \Gamma \), since \( \alpha \to 0 \) (5.11). It is this limiting case which was considered in [7].

## 6 Discussion

We have seen that it is possible to maintain both supersymmetry and integrability in the boundary tricritical Ising model. The NS case (Section 4) corresponds to a “direct sum” of two non-supersymmetric theories studied in [7]. The R case (Section 5) corresponds to a one-parameter (\( r \)) deformation of another theory studied in [7]. For both cases, the conserved supersymmetry charges (4.14), (5.3) are linear combinations of \( Q, \bar{Q} \) and the spin-reversal operator \( \Gamma \). For the other boundary supersymmetric integrable models which we have studied [15, 16], the conserved supersymmetry charges have a similar structure. We expect that the phenomenon of forming superconformal boundary states from superpositions of pure Cardy states, which we have witnessed in the tricritical Ising model, may occur in other boundary supersymmetric models as well.

An important check on the picture presented here would be to verify directly using the TBA that the proposed boundary \( S \) matrices (4.22), (5.10) describe the corresponding proposed boundary flows (4.7), (5.2). This work is now in progress [26]. It would also be interesting to clarify the relation between the operator \( \Theta \) in the perturbed CFT expression (4.4) for the conserved supersymmetry charge, and the operator \((-1)^F\), to which the spin-reversal operator \( \Gamma \) seems to correspond.

Our results should help to precisely determine the boundary \( S \) matrix for the solitons of the boundary \( N = 1 \) supersymmetric sine-Gordon model. Indeed, this model is known to have supersymmetry, and the breather boundary \( S \) matrix has also been proposed [16]. However, the soliton boundary \( S \) matrix [8], which contains as one of its factors the tricritical Ising model boundary \( S \) matrix, has not yet been completely determined.

We expect that “higher” perturbed boundary minimal models should have generalizations of the boundary supersymmetry which we have considered here. For instance, the self-dual perturbation of the tricritical 3-state Potts model (\( \mathcal{M}(6/7) \)) has [2] bulk spin 1/3 integrals.
of motion, certain combinations of which (plus boundary terms) should be conserved in the corresponding integrable boundary theory. In general, perturbed minimal models have bulk quantum group symmetries, which can survive in the presence of a boundary. It would be very interesting to work out the corresponding conserved charges, boundary $S$ matrices and boundary flows.

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**A Duality**

The operator product algebra of the tricritical Ising CFT has a discrete Kramers-Wannier-like symmetry known as duality [2, 5]. We list here the duality transformation properties of some of the primary fields. The following primary fields are invariant under duality:

\[
\begin{align*}
\Phi_{(0,0)} &\to \Phi_{(0,0)}, & \Phi_{(\frac{1}{2},\frac{3}{2})} &\to \Phi_{(\frac{3}{2},\frac{1}{2})}, \\
\Phi_{(\frac{1}{10},\frac{1}{9})} &\to \Phi_{(\frac{1}{9},\frac{1}{10})}, & \Phi_{(\frac{1}{2},0)} &\to \Phi_{(0,\frac{1}{2})}.
\end{align*}
\]  

(A.1)

The following primary fields pick up a minus sign under duality:

\[
\begin{align*}
\Phi_{(\frac{1}{10},\frac{1}{9})} &\to -\Phi_{(\frac{1}{9},\frac{1}{10})}, & \Phi_{(\frac{3}{2},\frac{1}{2})} &\to -\Phi_{(\frac{1}{2},\frac{3}{2})}, \\
\Phi_{(\frac{1}{2},\frac{1}{2})} &\to -\Phi_{(\frac{3}{2},\frac{1}{2})}, & \Phi_{(0,\frac{1}{2})} &\to -\Phi_{(\frac{1}{2},0)}.
\end{align*}
\]  

(A.2)

The following primary fields pick up a factor of $G_0$ under duality:

\[
\begin{align*}
\Phi_{(\frac{1}{10},\frac{1}{9})} &\to G_0\Phi_{(\frac{1}{9},\frac{1}{10})}, & \Phi_{(\frac{3}{2},\frac{1}{2})} &\to G_0\Phi_{(\frac{1}{2},\frac{3}{2})}, \\
\Phi_{(\frac{1}{2},0)} &\to G_0\Phi_{(0,\frac{1}{2})}.
\end{align*}
\]  

(A.3)

**Note added:**

It was pointed out in [27] that the bulk $S$ matrix (2.4) should be rescaled by a minus sign. (See also the correction of the scalar factor (2.6).) Moreover, it was pointed out in [28] that the amplitude $P(\theta)$ (4.16) should be rescaled by the factor $i\tanh(\frac{i\pi}{4} - \frac{i\theta}{2})$, in order that
it have a simple (rather than double) pole at $\theta = \frac{i\pi}{2}$ for $\xi = \frac{\pi}{2}$. Similarly, the amplitudes (5.10) should also be rescaled by this factor. I am grateful to L. Chim for bringing \cite{28} to my attention.

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ERRATUM to “Supersymmetry in the boundary tricritical Ising field theory”

Section 4: NS case

We argued in [1] that the supersymmetric perturbation of the “superposition” of Cardy conformal boundary conditions \((-0)\&(0+)\) is given by the boundary operator \(\phi_{(\frac{3}{5})}(-0)\) \(-\phi_{(\frac{3}{5})}(0+)\), i.e., with a relative minus sign. (See Eq. (4.1).) The argument is based on the results (see, e.g., [2]) for the “pure” boundary flows summarized in Eqs. (4.5), (4.6). However, Graham has argued [3] that the signs of the perturbing operators in (4.6) should be reversed; i.e., the boundary flows are given by

\[
(-0) + \phi_{(\frac{3}{5})} \rightarrow (0), \quad (-0) - \phi_{(\frac{3}{5})} \rightarrow (-),
\]

\[
(0+) - \phi_{(\frac{3}{5})} \rightarrow (+), \quad (0+) + \phi_{(\frac{3}{5})} \rightarrow (0).
\]

Therefore, in order to obtain the supersymmetric flows (4.7) of the superposition of conformal boundary conditions \((-0)\&(0+)\), the perturbing boundary operator should instead be

\[
\phi_{(\frac{3}{5})}(-0)(y) + \phi_{(\frac{3}{5})}(0+)(y),
\]

i.e., with a relative plus sign. It follows that the boundary perturbation is even under spin reversal, instead of odd. (The arguments given in [1] that the perturbation is integrable and supersymmetric (4.3), (4.4) also hold if the relative sign is plus.)

We also argued in [1] that the boundary S matrix for the NS case, corresponding (e.g., for boundary parameter \(h > 0\)) to the boundary flow \((-0)\&(0+) \rightarrow (-)\&(+)\), should be given by the “direct sum” of boundary S matrices corresponding to the two pure boundary flows \((-0) \rightarrow (-)\) and \((0+) \rightarrow (+)\), which were found by Chim [4]. However, the explicit expression for the boundary S matrix which we proposed there cannot be strictly correct, since it is not consistent with the boundary bound-state bootstrap equations [5].\(^1\) Moreover, it does not have spin-reversal symmetry.

In order to formulate the “direct sum” of two boundary S matrices, we consider two “copies” of the boundary states. Hence, instead of the boundary operator \(B_a\) which has a single index \(a \in \{-1, 0, 1\}\) (corresponding to the three possible vacua or “spins”), we now have the boundary operator \(B_{(a,\epsilon)}\) where \(a \in \{-1, 0, 1\}\) and \(\epsilon \in \{-1, 1\}\). Boundary scattering is described by the boundary S matrix \(R_a^{(c,\epsilon)}_{(b,\epsilon')}(\theta)\), which is defined by

\[
|K_{a,b}(\theta)\ B_{(b,\epsilon)}\rangle_{in} = \sum_{c,\epsilon'} R_a^{(c,\epsilon')}_{(b,\epsilon)}(\theta) \ |K_{a,c}(-\theta)\ B_{(c,\epsilon')}\rangle_{out}.
\]

\(^1\)We were aware of this difficulty, and we suggested in [1] that this constraint should not be imposed. The new boundary S matrix which we propose here does not involve such unphysical assumptions.
In particular, for the boundary flow \((-0)\&(0+) \rightarrow (-)\&(+),\) we assume that the boundary can exist only in the following four states, labeled by \((a, \epsilon)\): the two degenerate ground states \((-1, -1), (1, 1)\) and the two degenerate excited states \((0, -1), (0, 1)\). Moreover, we propose that the boundary \(S\) matrix has the following nonvanishing amplitudes

\[
R^0 \begin{pmatrix} (-1, -1) \\ (-1, -1) \end{pmatrix}(\theta) = R^0 \begin{pmatrix} (1, 1) \\ (1, 1) \end{pmatrix}(\theta) = P(\theta),
\]

\[
R^1 \begin{pmatrix} (0, -1) \\ (0, -1) \end{pmatrix}(\theta) = R^{-1} \begin{pmatrix} (0, 1) \\ (0, 1) \end{pmatrix}(\theta) = R_+(\theta),
\]

\[
R^{-1} \begin{pmatrix} (0, -1) \\ (0, -1) \end{pmatrix}(\theta) = R^1 \begin{pmatrix} (0, 1) \\ (0, 1) \end{pmatrix}(\theta) = R_-(\theta),
\]

where \(P(\theta)\) is given by Eqs. (4.16)-(4.19) in [1]; and \(R_{\pm}(\theta)\) are given by (4.21) when the boundary parameter \(\xi\) is in the range \(0 < \xi < \pi/2\) (in which case the simple pole of the amplitude \(P(\theta)\) at \(\theta = i\xi\) corresponds to a boundary bound state), and are otherwise zero.

The amplitudes with \(\epsilon = -1\) correspond to the boundary flow \((-0) \rightarrow (-)\); while the amplitudes with \(\epsilon = 1\) correspond to the boundary flow \((0+) \rightarrow (+)\). One can verify that this boundary \(S\) matrix satisfies the boundary Yang-Baxter equation, boundary unitarity, boundary cross-unitarity, as well as the boundary bound-state bootstrap equations.

We now argue that this boundary \(S\) matrix has both supersymmetry and spin-reversal symmetry. To this end, we assume that the supersymmetry operators \(Q\) and \(\bar{Q}\) and the spin-reversal operator \(\Gamma\) have the following action on the boundary operators for the ground states:

\[
Q B_{(\pm, \pm)} = 0 = \bar{Q} B_{(\pm, \pm)}, \quad \Gamma B_{(\pm, \pm)} = B_{(\mp, \mp)}. \tag{6}
\]

Following Bajnok et al. [6], we determine the action of these operators on the boundary operators for the excited states by using a bootstrap construction for the latter, namely,

\[
B_{(0, \mp)} = \frac{1}{g_{\mp 0}} K_{0, \mp 1}(i\xi) B_{(\mp 1, \mp 1)}, \tag{7}
\]

where \(g_{\mp 0}\) are particle-boundary coupling constants. Indeed, it follows that

\[
Q B_{(0, \mp)} = \pm \sqrt{me^{\mp i\xi}} B_{(0, \mp)}, \quad \bar{Q} B_{(0, \mp)} = \pm \sqrt{me^{-\mp i\xi}} B_{(0, \mp)}, \tag{8}
\]

and

\[
\Gamma B_{(0, -1)} = r B_{(0, 1)}, \quad \Gamma B_{(0, 1)} = \frac{1}{r} B_{(0, -1)}, \tag{9}
\]

where \(r = g_{+0}/g_{-0}\). One can now verify that both \(Q + \bar{Q}\) and \(\Gamma\) commute with the boundary \(S\) matrix [6]. These properties are consistent with those of the boundary perturbation [3]. In particular, the operator

\[
\hat{Q} = Q + \bar{Q} + \frac{2i\sqrt{m}}{\alpha} \Gamma, \tag{10}
\]
(see (4.14) in [1]) commutes with the boundary $S$ matrix for any value of $\alpha$. (Hence, the restriction (4.23) is not necessary.) Following the arguments of [1], the value of $\alpha$ is related to the energy of the ground state, $E_0 = -2m/\alpha^2$.

Section 5: R case

For the R case, the boundary perturbing operator is the dual of the operator (3), which is given by $\phi_{(d)}(y)$. This operator is even under spin-reversal. Precisely this case was considered by Chim [1]. The corresponding boundary flows are given by (5.2).

For the boundary flow $(d) \to (0)$, the boundary has ground state 0, and degenerate excited states $\pm 1$. As in [1], we assume that the supersymmetry operators $Q$ and $\bar{Q}$ and the spin-reversal operator $\Gamma$ act on the boundary operator for the ground state as follows,

$$ Q B_0 = 0 = \bar{Q} B_0, \quad \Gamma B_0 = B_0. \quad (11) $$

However, following Bajnok et al. [6], let us now use a bootstrap construction for the boundary operators for the excited states

$$ B_{\mp 1} = \frac{1}{g_{0\mp}} K_{\mp 1,0}(i\xi) B_0 \quad (12) $$

to determine how the symmetry operators act on them:

$$ Q B_1 = i r \sqrt{m} e^{i\xi} B_{-1}, \quad Q B_{-1} = -i \frac{r}{e} \sqrt{m} e^{i\xi} B_1, $$
$$ \bar{Q} B_1 = -ir \sqrt{m} e^{-i\xi} B_{-1}, \quad \bar{Q} B_{-1} = i \frac{r}{e} \sqrt{m} e^{-i\xi} B_1, $$
$$ \Gamma B_1 = r B_{-1}, \quad \Gamma B_{-1} = \frac{1}{r} B_1, \quad (13) $$

where $r = g_{0-}/g_{0+}$. (In contrast, we assumed in [1] that $Q$ and $\bar{Q}$ annihilate all the boundary operators $B_a$.) Demanding that both $Q - \bar{Q}$ and $\Gamma$ commute with the boundary $S$ matrix now leads to the following constraints on the amplitudes:

$$ R_+ (\theta) = R_- (\theta) \equiv R(\theta), \quad P_+ (\theta) = P_- (\theta) \equiv P(\theta), $$
$$ rV_- (\theta) = \frac{1}{r} V_+ (\theta), \quad P(\theta) = \frac{ir \cos \frac{\theta}{2}}{\sinh \frac{\theta}{2}} V_-(\theta). \quad (14) $$

Comparing with Eq. (5.4) in [1], we see that the first two lines are the same, but the third line is different. The boundary $S$ matrix (5.10) satisfies all of these constraints. In particular, the operator (5.3)

$$ \hat{Q} = Q - \bar{Q} + \frac{2i \sqrt{m}}{\alpha} \Gamma \quad (15) $$

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commutes with the boundary \( S \) matrix for any value of \( \alpha \). (Hence, the restriction (5.11) is not necessary.) Following the arguments of [6], the value of \( \alpha \) is related to the energy of the ground state, \( E_0 = -2m/\alpha^2 \).

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