Lebesgue–Stieltjes measures and the play operator

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Abstract. The play operator with variable characteristics and BV inputs is usually defined by means of an integral variational inequality which involves a suitable version of the Kurzweil integral. We propose here a new formulation which makes use of the Lebesgue-Stieltjes integral. In this way we can apply standard results of measure theory and obtain simpler proofs.

Keywords: play operator, Lebesgue-Stieltjes measures, hysteresis, variational inequalities, functions of bounded variation.

1. Introduction

The play operator is an input-output relation between functions of time that can be defined in the following way. Given \( T, r > 0, z_0 \in [-r, r] \), and an absolutely continuous function \( u : [0, T] \rightarrow \mathbb{R} \), we have to find \( w : [0, T] \rightarrow \mathbb{R} \) that is absolutely continuous and such that

\[
|u(t) - w(t)| \leq r \quad \forall t \in [0, T],
\]

\[
(u(t) - w(t) - z)w'(t) \geq 0 \quad \forall z \in [-r, r], \text{ for a.e. } t \in [0, T],
\]

\[
u(0) - w(0) = z_0.
\]

The constraint \([-r, r]\) is usually called characteristic. By standard theorems on variational inequalities, it can be proved that (1.1)-(1.3) have a unique solution \( w =: P(u) \), and that \( w \) is Lipschitz continuous if \( u \) has the same regularity. The play operator has an important role in elastoplasticity and in hysteresis phenomena. It has been widely studied in the monographs [4, 13, 1, 5], where the well-posedness of (1.1)-(1.3) is proved. In particular it is proved that the operator \( P : u \mapsto w \) is uniformly continuous with respect to the uniform convergence, and it can be extended to a unique operator which is defined on the space of functions of bounded variation, \( BV([0, T]) \). This abstract extension result does not provide a formulation that describes the output \( w \), at variance with (1.1)-(1.3). This problem was solved by Krejčí who introduced in [5] an integral formulation for the play operator with continuous inputs of bounded variation and in [6] generalized this formulation allowing discontinuous inputs and variable characteristic \([-r(t), r(t)]\) (see also [2], where the play with variable characteristic is introduced). The integral formulation of [6] reads as follows. Given \( T > 0, u \) and \( r \) right-continuous and with bounded...
variation, and \( z_0 \in [-r(0), r(0)] \), find \( w \in BV([0, T]) \) right-continuous such that
\[
|u(t) - w(t)| \leq r(t) \quad \forall t \in [0, T],
\]
\[
K \int_0^t (u(\sigma) - w(\sigma) - z) \, dw(\sigma) \geq 0 \quad \forall z \in BV([0, T]), \quad |z(\sigma)| \leq r(\sigma), \quad \forall t \in [0, T],
\]
\[
u(0) - w(0) = z_0.
\]
The integral appearing in (1.5) is the Kurzweil integral in the form presented in [12]. The original definition of the Kurzweil integral can be found in [7]. The aim of our note is to show that the play operator can be simply defined by using the standard Lebesgue integral with respect to the Stieltjes measure associated with a function of bounded variation. In this way classic measure theory tools are available and proofs are shorter and simpler. This is in particular evident in the main tool for the proof of the uniform continuity of the play operator: an integral characterization of monotone functions. The characterization in terms of the Kurzweil integral needs the long and ingenious proof of [6], whereas the analogous characterization in terms of Lebesgue integral has a proof reduced to a few lines.

2. Preliminaries

In the sequel \( \mathbb{N} \) denotes the set of strictly positive integers and \( I \) is an interval of the real line \( \mathbb{R} \). We say that a function \( f : I \to \mathbb{R} \) is increasing if \( (f(s) - f(t))(s - t) \geq 0 \) for every \( s, t \in I \), with the same convention for the term decreasing. We set \( \|f\|_\infty := \sup_{t \in I} |f(t)| \).

2.1. Step, regulated, and BV functions

A function \( f : I \to \mathbb{R} \) is called regulated if at each point \( t \in I \) there exist \( f(t-) := \lim_{s \uparrow t} f(s) \) and \( f(t+) := \lim_{s \downarrow t} f(s) \), and are finite, with the convention that \( f(t-) := f(t) \) (resp. \( f(t+) = f(t) \)) if \( t \) is the left (resp. right) endpoint of \( I \). We denote by \( Reg(I) \) the set of regulated functions on \( I \) and we define \( Reg_r(I) := \{ f \in Reg(I) : f(t+) = f(t) \forall t \in I \} \), the set of right-continuous regulated functions. Given \( f : I \to \mathbb{R} \), the (pointwise) variation of \( f \) on \( I \), denoted by \( V_p(f, I) \), is defined by
\[
V_p(f, I) := \sup \left\{ \sum_{j=1}^m |f(t_j) - f(t_{j-1})| : m \in \mathbb{N}, \; t_j \in I, \; t_0 < t_1 < \cdots < t_m \right\}.
\]
We indicate by \( BV_r(I) \) the space of right-continuous functions of bounded variation defined on \( I \), i.e. \( BV_r(I) := \{ f : I \to \mathbb{R} : V_p(f, I) < \infty, \; f(t+) = f(t) \forall t \in I \} \). We use the symbol \( St(I) \) to denote the set of step functions, that is functions \( f : I \to \mathbb{R} \) such that \( I \) can be partitioned into a finite number of (possibly degenerate) intervals \( J_1, \ldots, J_m \) and \( f \) is constant on each \( J_j \), for \( j = 1, \ldots, m \). Finally \( St_r(I) \) is the set of right-continuous regulated functions. It is well known that \( St_r(I) \subseteq BV_r(I) \subseteq Reg_r(I) \) (see, e.g. [11, Theorem 8.13, p. 161]). We have the following

**Lemma 2.1.** If \( I = [a, b] \) is compact and \( f \in Reg(I) \) then \( u \) is bounded and there is a sequence \( f_n \in St(I) \) such that \( \|f_n\|_\infty \leq \|f\|_\infty \), \( f_n(a) = f(a) \), \( V_p(f_n, I) \leq V_p(f, I) \) for every \( n \in \mathbb{N} \), and \( \|f - f_n\|_\infty \to 0 \) as \( n \to \infty \). If \( f \) is right-continuous we can take \( f_n \in St_r([0, T]) \) for every \( n \in \mathbb{N} \).

**Proof.** Given \( n \in \mathbb{N} \) by the compactness of \( I \) there exist \( a = t_0 < t_1 < \cdots < t_m = b \) such that the oscillation of \( f \) on \( [t_{j-1}, t_j] \) is less than \( 1/n \) for every \( j = 1, \ldots, m \). The function \( f_n \) is defined by \( f_n(t_j) := f(t_j) \) for \( j = 0, \ldots, m \) and \( f_n(t) := f(t_j) \) if \( t \in [t_{j-1}, t_j], \; j = 1, \ldots, m \), if \( f \) is right-continuous. Otherwise we define \( f_n(t) := f((t_{j-1} + t_j)/2) \) if \( t \in [t_{j-1}, t_j], \; j = 1, \ldots, m \). The remaining assertions are obvious. \( \square \)

It is easy to see that if \( f_n \in St_r(I) \) and \( \|f - f_n\|_\infty \to 0 \), then \( f \in Reg_r(I) \). Therefore, if \( I \) is compact, from Lemma 2.1 we infer the completeness of \( Reg_r(I) \) with respect to the norm \( \|\cdot\|_\infty \).
2.2. Signed measures on the real line

Now we recall some basic facts about finite signed measures on the real line. Proofs and details can be found, e.g., in [11, Chapter 6]. The symbol $\mathcal{B}(I)$ denotes the family of Borel sets, i.e., the smallest $\sigma$-algebra of subsets of $I$ containing all the open sets of $I$. A signed Borel measure is a map $\nu: \mathcal{B}(I) \rightarrow \mathbb{R}$ which is countably additive, i.e., $\nu(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \nu(B_j)$ whenever $(B_j)$ is a family of mutually disjoint Borel sets. If the range of $\nu$ is $[0, \infty[$ the measure $\nu$ is also called a positive finite Borel measure. In the sequel we will omit the term “Borel”. The theory of positive measures and integration with respect to positive measures is standard and can be found in [11, Chapters 1, 2]. For convenience we recall just few results, the former being usually referred as the averaging theorem (cf. [11, Theorem 1.40, p. 30]).

**Proposition 2.1.** Let $\mu: \mathcal{B}(I) \rightarrow [0, \infty[$ be a positive finite measure and assume that $f : I \rightarrow \mathbb{R}$ is $\mu$-integrable. If $S \subseteq \mathbb{R}$ is a closed subset and

$$\int_B f \, d\mu \in S \quad \forall B \in \mathcal{B}(I), \quad \mu(B) > 0,$$

then $f(t) \in S$ for $\mu$-a.e. $t \in \mathbb{R}$.

**Proposition 2.2.** If $\mu: \mathcal{B}(I) \rightarrow [0, \infty[$ is a positive finite measure then for every $B \in \mathcal{B}(I)$ we have $\mu(B) = \inf\{\mu(A) : B \subseteq A, A \text{ open}\}$.

The property in the last proposition is usually called outer regularity (see [11, Theorem 2.18, p. 48]). Many properties of a finite signed measure $\nu: \mathcal{B}(I) \rightarrow \mathbb{R}$ can be obtained from the theory of positive measures by means of the total variation of $\nu$ which is the smallest positive measure dominating the map $B \mapsto |\nu(B)|$ and will be denoted by $|\nu|$. It turns out that $|\nu|$ is finite. Setting $\nu_+ := (|\nu| + \nu)/2$ and $\nu_- := (|\nu| - \nu)/2$ we have defined two finite positive measures (the positive part and the negative part of $\nu$) such that $\nu = \nu_+ - \nu_-$ and $|\nu| = \nu_+ + \nu_-$. If $f$ is a $|\nu|$-integrable function we define $\int_I f \, d\nu := \int_I f \, d\nu_+ - \int_I f \, d\nu_-$ and $\int_B f \, d\nu := \int_I \chi_B f \, d\nu$ for $B \in \mathcal{B}(I)$, $\chi_B$ being the characteristic function of $B$: $\chi_B(t) = 1$ if $t \in B$ and $\chi_B(t) = 0$ if $t \notin B$. It is clear that if $I$ is compact, then every regulated function is $|\nu|$-integrable. From the averaging theorem and the outer regularity we immediately infer the following two facts.

**Proposition 2.3.** Let $\mu: \mathcal{B}(I) \rightarrow [0, \infty[$ be a finite positive measure and assume that $f : I \rightarrow \mathbb{R}$ is $\mu$-integrable. Let us define the signed measure $\nu := f\mu: \mathcal{B}(I) \rightarrow \mathbb{R}$ by

$$\nu(B) := f\mu(B) := \int_B f \, d\mu, \quad B \in \mathcal{B}(I). \quad (2.1)$$

Then

$$\nu \geq 0 \iff f \geq 0 \quad \mu\text{-a.e. in } I. \quad (2.2)$$

**Proof.** Assume that $\nu \geq 0$. For every $B \in \mathcal{B}(I)$ such that $\mu(B) > 0$ we have that $\frac{1}{\mu(B)} \int_B f \, d\mu \in [0, \infty[$. Therefore, by the averaging theorem, we deduce that $f(t) \geq 0$ for $\mu$-a.e. $t \in I$. The other implication is trivial. \hfill $\Box$

**Lemma 2.2.** If $\nu: \mathcal{B}(I) \rightarrow \mathbb{R}$ is a signed measure and $\nu(I \cap ]s, t]) \geq 0$ for every $s, t \in \mathbb{R}$, $s < t$, then $\nu \geq 0$.

**Proof.** Every open set in $I$ is a countable disjoint union of intervals of the form $I \cap ]s, t]$, therefore $\nu(A) \geq 0$, i.e. $\nu_+(A) \geq \nu_-(A)$, for every open set $A$. Therefore by the outer regularity we infer that $\nu_+(B) \geq \nu_-(B)$ for every $B \in \mathcal{B}(I)$, that is $\nu \geq 0$. \hfill $\Box$
Let us finally recall the so called polar representation of measures (cf. [11, Theorem 6.12, p. 126]):

**Theorem 2.1.** If \( \nu : \mathcal{B}(I) \rightarrow \mathbb{R} \) is a signed measure, then there is a measurable function \( h : I \rightarrow \mathbb{R} \) such that \( |h(t)| = 1 \) for every \( t \in I \) and

\[
\nu(B) = \int_B h \, d|\nu| \quad \forall B \in \mathcal{B}(I).
\]

With the notation of the previous theorem it is easy to see that if \( f \) is \( |\nu| \)-integrable then
\[
\left| \int_B f \, d\nu \right| \leq \int_B |f| \, d|\nu| \quad \forall B \in \mathcal{B}(I).
\] (2.3)

3. Lebesgue-Stieltjes measures

As we outlined in the Introduction we wish to formulate the play operator by using standard integration theory, rather than other types of integral. The starting point is the fact that every function of bounded variation is associated with a signed measure. Let us briefly recall this correspondence in the case of a compact interval \( [a, b] \), \( a, b \in \mathbb{R} \), \( a < b \). Details and proofs can be found in [11, Chapter 8]. If \( g \in BV, ([a, b]) \) we will automatically extend \( g \) to \( \mathbb{R} \) by setting \( g(t) = g(a) \) for \( t < a \) and \( g(t) = g(b) \) for \( t > b \). In this way it turns out that \( g \in BV, (\mathbb{R}) \). There exists a unique signed measure \( \lambda_g : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R} \) such that

\[
\lambda_g([c, d]) = g(d) - g(c) \quad \forall -\infty < c < d < \infty.
\] (3.1)

Such measure is called the Lebesgue-Stieltjes measure associated with \( g \). We have that ([11, Theorem 8.14, p. 163])

\[
|\lambda_g|([c, d]) = V_f(g, [c, d]),
\] (3.2)

for \( -\infty < c \leq d < \infty \), in particular

\[
\lambda_g(\{t\}) = g(t) - g(t-) \quad \forall t \in \mathbb{R}.
\] (3.3)

It can also be proved that \( \lambda_g \) is the distributional derivative of \( g \) on \( \mathbb{R} \), but we do not need this fact. Vice versa, if \( \nu : \mathcal{B}([a, b]) \rightarrow \mathbb{R} \) is a signed measure, we can “extend” it to \( \mathbb{R} \) by setting \( \nu(B) := \nu(B \cap [a, b]) \) for \( B \in \mathcal{B}(\mathbb{R}) \), and we can define \( g : \mathbb{R} \rightarrow \mathbb{R} \) by \( g(t) := \nu([-\infty, t]) \), \( t \in \mathbb{R} \). Then it turns out that \( \lambda_g = \nu \). This correspondence is linear in the sense that

\[
\lambda_{ag+h} = a\lambda_g + \lambda_h \quad \forall g, h \in BV, ([a, b]), \forall a \in \mathbb{R}.
\] (3.4)

**Lemma 3.1.** Assume that \( g \in BV, ([a, b]) \). Then \( g \) is increasing if and only if \( \lambda_g \) is positive.

*Proof.* The “if” part is obvious. The “only if” part is a consequence of Lemma 2.2 and the fact that \( \lambda_g([s, t]) = \lambda_g([-\infty, t]) - \lambda_g([-\infty, s]) = g(t) - g(s) \geq 0 \) whenever \( -\infty < s < t < \infty \). \( \square \)

The key tool in [6] for proving the continuity of the play operator, is a characterization of monotone functions by means of the Kurzweil integral. The proof of this property is rather long and ingenious. We now show that an analogous characterization by means of the Lebesgue-Stieltjes integral is an easy consequence of general results of measure theory.
**Proposition 3.1.** Assume $g \in BV_c([a,b])$ and let $f : [a,b] \to \mathbb{R}$ be a $\lambda_g$-integrable function such that $f > 0$. Let $\nu := f\lambda_g : \mathcal{B}([a,b]) \to \mathbb{R}$ be the signed measure defined by

$$
\nu(B) := f\lambda_g(B) := \int_B f \, d\lambda_g, \quad B \in \mathcal{B}([a,b]).
$$

Then

$$
\nu \geq 0 \iff \lambda_g \geq 0 \iff g \text{ is increasing.}
$$

**Proof.** By the polar representation there exists a function $h$ such that $\lambda_g = h|\lambda_g|$, therefore

$$
\nu(B) = \int_B f(t)h(t) \, d|\lambda_g|(t) \quad \forall B \in \mathcal{B}([a,b]).
$$

Hence by Proposition 2.3 we have that $\nu \geq 0 \iff fh \geq 0 |\lambda_g|\text{-a.e.} \iff h \geq 0 |\lambda_g|\text{-a.e.} \iff \lambda_g \geq 0$. The proposition follows from the previous lemma. \hfill \Box

We finish the section with two useful results.

**Lemma 3.2.** Let us assume that $f \in St([a,b])$ and $g \in BV_c([a,b])$. Then $|\int_{[a,b]} f \, d\lambda_g| \leq (2\|f\|_{\infty} + V_p(f,[a,b]))\|g\|_{\infty}$.

**Proof.** We can assume that $a = t_0 < t_1 < \cdots < t_m = b$ is a subdivision of $[a,b]$ such that $f$ is constant on $|t_{j-1}, t_j|$ for every $j \in \{1, \ldots, m\}$. We have

$$
\left| \int_{[a,b]} f \, d\lambda_g \right| = \left| \sum_{j=1}^{m} \int_{|t_{j-1}, t_j|} f \, d\lambda_g + \sum_{j=1}^{m} \int_{\{t_j\}} f \, d\lambda_g \right|
$$

$$
= \left| \sum_{j=1}^{m} f(t_{j-1}+)[g(t_j) - g(t_{j-1})] + \sum_{j=1}^{m-1} f(t_j)[g(t_{j+1}) - g(t_j)] \right|
$$

$$
= \left| f(b)g(b) - f(a)g(a) \right|
$$

$$
+ \sum_{j=1}^{m} g(t_{j-1})|f(t_{j-1}+) - f(t_j)| + \sum_{j=1}^{m-1} g(t_j)|f(t_{j+1}) - f(t_{j-1})| \right|
$$

$$
\leq \|g\|_{\infty} \left( |f(b)| + |f(a)| + \sum_{j=1}^{m} |f(t_{j-1}) - f(t_{j-1})| \right) \leq \|g\|_{\infty} (2\|f\|_{\infty} + V_p(f,[a,b])).
$$

\hfill \Box

**Proposition 3.2.** Assume that $f, f_n \in Reg([a,b])$, $g, g_n \in BV_c([a,b])$ for every $n \in \mathbb{N}$. If $f_n \to f$ and $g_n \to g$ uniformly on $[a,b]$, and there exists $c > 0$ such that $V_p(g_n, [a,b]) \leq c$, then

$$
\lim_{n \to \infty} \int_{[a,b]} f_n \, d\lambda_{g_n} = \int_{[a,b]} f \, d\lambda_g.
$$

**Proof.** Let $\varepsilon > 0$ be arbitrarily fixed and let $f_{\varepsilon} \in St([a,b])$ be such that $\|f - f_{\varepsilon}\|_{\infty} < \varepsilon/(V_p(g, [a,b]) + c)$. Using inequality $V_p(g - g_n, [a,b]) \leq V_p(g, [a,b]) + V_p(g_n, [a,b])$, together
with (2.3), (3.2), (3.4), and applying the previous lemma we get
\[
\left| \int_{[a,b]} f_n \, d\lambda_{g_n} - \int_{[a,b]} f \, d\lambda_g \right|
\leq \int_{[a,b]} (f_n - f) \, d\lambda_{g_n} + \int_{[a,b]} (f - f_\varepsilon) \, d\lambda_{g_n} - g + \int_{[a,b]} f_\varepsilon \, d\lambda_{g_n} - g
\leq \|f_n - f\|_\infty \|V_p(g_n, [a,b])\| + \|f - f_\varepsilon\|_\infty \|V_p(g_n - g, [a,b])\| + (2\|f_\varepsilon\|_\infty + V_p(f_\varepsilon, [a,b]))\|g_n - g\|_\infty
\leq \|f_n - f\|_\infty c + \varepsilon (2\|f_\varepsilon\|_\infty + V_p(f_\varepsilon, [a,b]))\|g_n - g\|_\infty.
\]

Now, letting \( n \to \infty \) we get that \( \lim_{n \to \infty} \left| \int_{[a,b]} f_n \, d\lambda_{g_n} - \int_{[a,b]} f \, d\lambda_g \right| \leq \varepsilon \). The proposition follows from the arbitrariness of \( \varepsilon \).

4. The play operator with discontinuous inputs

In the sequel \( T > 0 \) will be a fixed final time. Let us give the formulation of the problem that will define the play operator.

**Problem (P).** Assume that we are given \( u, r \in BV_r([0, T]) \), \( r \geq 0 \), and \( z_0 \in [-r(0), r(0)] \). Find \( w \in BV_r([0, T]) \) such that
\[
|u(t) - w(t)| \leq r(t) \quad \forall t \in [0, T], \tag{4.1}
\]
\[
\int_{[s,t]} (u - w - z) \, d\lambda_w \geq 0 \quad \forall z \in Reg([0, T]), \ |z| \leq r, \ \forall s, t \in [0, T], \ s \leq t, \tag{4.2}
\]
\[
u(0) - w(0) = z_0. \tag{4.3}
\]

In this section we will prove that Problem (P) is well-posed. Let us stress the fact that we strictly follow the ideas of [6], the main differences consist in the formulation and in the tools: in (4.2) we use the Lebesgue integral, and for the proofs we exploit measure theory tools. It is convenient to introduce the following notation. For every \( a \geq 0 \) we denote by \( \Pi_a(x) \) the projection on \([-a, a]\) defined by
\[
\Pi_a(x) := \begin{cases} 
  x & \text{if } |x| \leq a \\
  -a & \text{if } x < -a \\
  a & \text{if } x > a
\end{cases} \quad x \in \mathbb{R}.
\]

Observe that
\[
\Pi_a(x) \leq a \quad \forall x \in \mathbb{R}, \tag{4.4}
\]
\[
|x - \Pi_a(x)| = \text{dist}(x, [-a, a]) := \inf\{|x - y| : |y| \leq a\} \quad \forall x \in \mathbb{R}, \tag{4.5}
\]
\[
(\Pi_a(x) - z)(x - \Pi_a(x)) \geq 0 \quad \forall z \in [-a, a], \tag{4.6}
\]
\[
|x| \leq a_1 \implies |x - \Pi_{a_2}(x)| \leq |a_1 - a_2| \quad \forall a_1, a_2 \in \mathbb{R}. \tag{4.7}
\]

The first step is to solve problem (P) for data belonging to \( \text{St}_r([0, T]) \). The construction of the solution is based on the so-called *catching up algorithm* (cf. [8]).

**Proposition 4.1.** Assume that \( u, r \in \text{St}_r([0, T]) \) and \( z_0 \in [-r(0), r(0)] \). Then Problem (P) admits a solution \( w \in \text{St}_r([0, T]) \) such that \( V_p(w, [0, T]) \leq V_p(u, [0, T]) + V_p(r, [0, T]) \).

**Proof.** We can assume that there is a subdivision \( 0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = T \) such that both \( u \) and \( r \) are constant on \([t_{j-1}, t_j]\) for every \( j \in \{1, \ldots, m\} \). Thus there are \( u_j, r_j \in \mathbb{R} \) such that
\[
u(t) = u_j, \quad r(t) = r_j \quad \forall t \in [t_{j-1}, t_j], \quad 1 \leq j \leq m, \quad u(T) = u_m, \quad r(T) = r_m. \tag{4.8}
\]
We define the family \((w_j)\) recursively by
\[
\begin{align*}
  w_0 & := u_0 - z_0, \\
  w_j & := u_j - \Pi_{r_j}(u_j - w_{j-1}) \\
        & = w_{j-1} + (u_j - w_{j-1}) - \Pi_{r_j}(u_j - w_{j-1}) \quad \forall j = 1, \ldots, m.
\end{align*}
\]

(4.9)

(4.10)

Now we check that the step function \(w \in St_r([0, T])\) defined by
\[
w(t) = w_{j-1} \quad \forall t \in [t_{j-1}, t_j], \quad 1 \leq j \leq m, \quad w(T) = w_m, \quad (4.11)
\]
is the solution to the problem \((P)\). Indeed if \(0 \leq s \leq t \leq T\) we can assume that \(s = t_h\) and \(t = t_k\) for some \(0 \leq h \leq k \leq T\). Since \(\lambda_w([t_{j-1}, t_j]) = \lambda_w([t_j, t_{j+1}]) = w_j - w_{j-1}\), if we set \(w_{-1} := w_0\), then for every \(z \in \text{Reg}([0, T])\) with \(|z| \leq r\), we have
\[
\int_{[s,t]} (u - w - z) \, d\lambda_w = \sum_{j=h}^{k-1} \int_{[t_j, t_{j+1}]} (u_j - w_j - z(\sigma)) \, d\lambda_w(\sigma) + \int_{(t_k]} (u - w - z) \, d\lambda_w
\]
\[
= \sum_{j=h}^{k} (u_j - w_j - z(t_j))(w_j - w_{j-1}).
\]
For every \(j \in \{1, \ldots, m\}\) we have, thanks to (4.6),
\[
(u_j - w_j - z(t_j))(w_j - w_{j-1}) = [\Pi_{r_j}(u_j - w_{j-1}) - z(t_j)][(u_j - w_{j-1}) - \Pi_{r_j}(u_j - w_{j-1})] \geq 0,
\]
which implies that \(\int_{[s,t]} (u - w - z) \, d\lambda_w \geq 0\). The other conditions (4.1), (4.3) are obvious. We are left to prove the estimate for the variation of \(w\). We have
\[
\nu_p(w, [0, T]) = \sum_{j=1}^{m} |w_j - w_{j-1}| = \sum_{j=1}^{m} |(u_j - w_{j-1}) - \Pi_{r_j}(u_j - w_{j-1})|.
\]

Observe that \(u_j - w_{j-1} = u_j - u_{j-1} + \Pi_{r_{j-1}}(u_{j-1} - w_{j-2})\) for every \(j\), hence \(|u_j - w_{j-1}| \leq |u_j - u_{j-1}| + |\Pi_{r_{j-1}}(u_{j-1} - w_{j-2})| \leq |u_j - u_{j-1}| + r_{j-1}\). Therefore from (4.7) we get that
\[
|(u_j - w_{j-1}) - \Pi_{r_j}(u_j - w_{j-1})| \leq |u_j - u_{j-1} + r_{j-1} - r_j| \leq |u_j - u_{j-1}| + |r_j - r_{j-1}| \quad \text{from which we deduce that } \nu_p(w, [0, T]) \leq \sum_{j=1}^{m} (|u_j - u_{j-1}| + |r_j - r_{j-1}|) = \nu_p(u, [0, T]) + \nu_p(r, [0, T]).
\]

Now we prove the continuous dependence on the data. We argue as in the proof of [6, Theorem 2.1], but we use measure theory tools.

**Proposition 4.2.** Assume that \(u_j, r_j \in \text{Reg}_r([0, T])\), \(z_{0j} \in [-r_j(0), r_j(0)]\) for \(j = 1, 2\). Let \(w_j \in BV_r([0, T])\) satisfy (4.1)-(4.3) with \(u, r, z_0, w_0, w, u_j, w_j, j = 1, 2\). If \(u := u_1 - u_2, r := r_1 - r_2, z_0 := z_{01} - z_{02}\), and \(w := u_1 - w_2\), then we have
\[
\|w\| \leq \max\{|z_0|, \|u\| + \|r\|\}.
\]

(4.12)

**Proof.** We claim that
\[
w > ||u|| + \|r\| \quad \text{on } [c, d] \quad \implies \quad \lambda_w \leq 0 \quad \text{on } [c, d]
\]

(4.13)
whenever $0 \leq c \leq d \leq T$. Indeed take $z(\sigma) = \Pi_{r_1(\sigma)}(u_j(\sigma) - w_j(\sigma))$ in the inequality for $w_j$ with $i \neq j$. We get
\[ \int_{[s,t]} [u(\sigma) - w(\sigma) + (u(\sigma) - w(\sigma)) - \Pi_{r_1(\sigma)}(u(\sigma) - w(\sigma))] \, d\lambda_{u_1}(\sigma) \geq 0, \] (4.14)
\[ \int_{[s,t]} [w(\sigma) - u(\sigma) + (u(\sigma) - w(\sigma)) - \Pi_{r_2(\sigma)}(u(\sigma) - w(\sigma))] \, d\lambda_{w_2}(\sigma) \geq 0 \] (4.15)
for every $s \leq t$. Observe that $|u_j - w_j| \leq r_j$ hence by (4.7) we have that $|(u(\sigma) - w(\sigma)) - \Pi_{r_1(\sigma)}(u(\sigma) - w(\sigma))| \leq |r_j(\sigma)|$, therefore for every $\sigma \in [c, d]$ we get
\[ u(\sigma) - w(\sigma) + (u(\sigma) - w(\sigma)) - \Pi_{r_1(\sigma)}(u(\sigma) - w(\sigma)) < \|u\|_\infty - \|u\|_\infty - \|r\|_\infty + \|r\|_\infty = 0 \]
and similarly $-u(t) + w(t) + (u(t) - w(t)) - \Pi_{r_2(t)}(u(t) - w(t)) > 0$. Proposition 3.1 and Lemma 2.2 imply that $\lambda_{w_2} \leq 0$ and $\lambda_{w_2} \geq 0$, thus the claim follows. Now we can prove the proposition. If $\|w\|_\infty \leq \|u\|_\infty + \|r\|_\infty$ there is nothing to prove. Thus assume that there exists $t \in \{0, T\}$ such that $|w(t)| > \|u\|_\infty + \|r\|_\infty$. We deal only with the case $w(t) > \|u\|_\infty + \|r\|_\infty$, because the case $w(t) < -\|u\|_\infty + \|r\|_\infty$ is analogous. Applying (4.13) with $c = d = t$ we get that $\lambda_{w_2}(t) \leq 0$, i.e. $w(t) - w(t-0) \leq 0$ (cf. (3.3)), hence there exists $s \in [0, t]$ such that $w > \|u\|_\infty + \|r\|_\infty$ on $[s, t]$. Applying again (4.13) with $c = s, d = t$ we infer that $\lambda_{w} \leq 0$, hence $w$ is decreasing on $[s, t]$. We have that $s_0 := \inf\{s \in [0, t] : w > \|u\|_\infty + \|r\|_\infty\} = 0$, because otherwise we could apply (4.13) with $c = d = s_0$ and get a contradiction. Hence $w$ is decreasing on $[0, t]$ thus $w(t) \leq |w(0)| = |z_0|$. 

Now we can solve Problem (P) by means of a classic approximation argument.

**Theorem 4.1.** Assume that $u, r \in BV_r([0, T])$ and $z_0 \in [-r(0), r(0)]$. Then Problem (P) admits a unique solution $w = P(u, r, z_0) \in BV_r([0, T])$. If we set
\[ D := \{ (u, r, z) \in BV_r([0, T])^2 \times \mathbb{R} : |z| \leq r(0) \} \]
endowed with the topology induced by $\|\cdot\| := \max\{\|u\|_\infty, \|r\|_\infty, |z|\}$, then the resulting operator $P : D \longrightarrow BV_r([0, T])$ is Lipschitz continuous when $BV_r([0, T])$ is endowed with topology of uniform convergence.

**Proof.** Uniqueness is a consequence of Proposition 4.2. By Lemma 2.1 there exist sequences $u_n, r_n \in St_r([0, T])$ such that $\|u_n - u\|_\infty + \|r_n - r\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, $z_0 \in [-r(0), r(0)]$, and $V_r(u_n, [0, T]) + V_r(r_n, [0, T]) \leq V_r(u, [0, T]) + V_r(r, [0, T]) = c$. If $w_n \in St_r([0, T])$ is the solution to Problem (P) with data $u_n, r_n, z_0$, then by Proposition 4.2 there exists $w \in BV_r([0, T])$ such that $w_n \rightarrow w$ uniformly, moreover Proposition 4.1 yields $V_r(w_n, [0, T]) \leq c$. Hence by Proposition 3.2 we can take the limit in the inequality for $w_n$ and deduce that $w$ is the solution to Problem (P). The Lipschitz continuity of $P$ follows from Proposition 4.2.

The extension of $P$ to the space of regulated functions is standard, indeed if $\tilde{D} := \{ (u, r, z) \in Reg_r([0, T])^2 \times \mathbb{R} : |z| \leq r(0) \}$ is endowed with the topology induced by $\|\cdot\| := \max\{\|u\|_\infty, \|r\|_\infty, |z|\}$, then $\tilde{D}$ is dense in $\tilde{D}$ by Lemma 2.1. Therefore, since $Reg_r([0, T])$ is complete, $P$ can uniquely be extended to a Lipschitz continuous operator $\tilde{P} : \tilde{D} \longrightarrow Reg_r([0, T])$.

5. Comparison with the Kurzweil formulation
Of course we have to check that Problem (P) defines the same play operator determined in [6] by (1.4)-(1.6). This is straightforward. Indeed let us observe that if $u, r \in St_r([0, T])$ are defined by formula (4.8), then the step function $w$ defined by (4.9)-(4.10) is the unique solution of the Kurzweil formulation given in [6]. Since the play operator defined in [6] is Lipschitz continuous with respect to the uniform convergence, we obtain the result that the two formulations are equivalent.
6. Conclusions

As we have seen the use of Lebesgue integral makes some proofs shorter, in particular the proof of characterization of monotone functions. Another advantage of Lebesgue-Stieltjes measures lies in the fact that we can prove the existence theorem for the play operator simply by applying a suitable change of variable formula. Let us sketch the proof in the case of constant characteristic \([-r,r]\) for some \(r > 0\). If \(|z_0| \leq r\), let us consider a nonconstant input function \(u \in BV_r([0,T])\) and set \(h(t) := TV_p(u,[0,t])/V_p(u,[0,T])\). Then there exists a Lipschitz function \(\hat{u} : [0,T] \rightarrow \mathbb{R}\) such that \(u(t) = (\hat{u} \circ h)(t) = \hat{u}(h(t))\) for every \(t \in [0,T]\) (cf., e.g., [3, p. 109]). If \(P(\hat{u}) := P(\hat{u},r,z_0)\), we can apply the following formula

\[
\int_{[0,T]} f(h(t)) \; d\lambda_{goh}(t) = \int_{[0,T] \setminus A} f(s)g'(s) \; d\lambda_h(s) + \sum_{t \in A} f(h(t))(g(h(t)) - g(h(t-)))
\]

with \(f(s) = \hat{u}(s) - P(\hat{u})(s) - z(s)\) and \(g(s) = P(\hat{u})(s)\), where \(A\) is the set of discontinuity points of \(h\), and \(z \in \text{Reg}([0,T])\), \(|z| \leq r\). Since \(\hat{u}\) is Lipschitz, \(P(\hat{u})\) solves (1.1)-(1.3), therefore it is not difficult to infer that \(\int_{[0,T]} f(h(t)) \; d\lambda_{goh}(t) \geq 0\), i.e. that \(P(\hat{u}) \circ h\) is the solution to Problem (P) with \(r(t) = r\). The details will be given in a forthcoming paper. Let us only remark that this technique provides a kind of representation formula for the play, namely \(P(u) = P(\hat{u}) \circ h\), and does not require an approximation-a priori estimates-limit procedure. Moreover it seems not to be easy to use the Kurzweil integral, because we have to integrate on the set \([0,T] \setminus A\) that in general is not an interval. The above formula for \(P(u)\) was also obtained in [9, 10] with different methods.

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