THERMAL TIDES IN FLUID EXTRASOLAR PLANETS

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ABSTRACT

Asynchronous rotation and orbital eccentricity lead to time-dependent irradiation of the close-in gas giant exoplanets—the hot Jupiters. This time-dependent surface heating gives rise to fluid motions which propagate throughout the planet. We investigate the ability of this “thermal tide” to produce a quadrupole moment which can couple to the stellar gravitational tidal force. While previous investigations discussed planets with solid surfaces, here we focus on entirely fluid planets in order to understand gas giants with small cores. The Coriolis force, thermal diffusion, and self-gravity of the perturbations are ignored for simplicity. First, we examine the response to thermal forcing through analytic solutions of the fluid equations which treat the forcing frequency as a small parameter. In the “equilibrium tide” limit of zero frequency, fluid motion is present but does not induce a quadrupole moment. In the next approximation, finite frequency corrections to the equilibrium tide do lead to a nonzero quadrupole moment, the sign of which torques the planet away from synchronous rotation. We then numerically solve the boundary value problem for the thermally forced, linear response of a planet with neutrally stratified interior and a stably stratified envelope. The numerical results find quadrupole moments in agreement with the analytic non-resonant result at a sufficiently long forcing period. Surprisingly, in the range of forcing periods of 1–30 days, the induced quadrupole moments can be far larger than the analytic result due to response of internal gravity waves which propagate in the radiative envelope. We discuss the relevance of our results for the spin, eccentricity, and thermal evolution of hot Jupiters.

Key words: hydrodynamics – planetary systems – planets and satellites: atmospheres – planet–star interactions – waves

Online-only material: color figures

1. INTRODUCTION

Many of the gas giant exoplanets orbiting close to their parent stars—the hot Jupiters—are observed to have radii far larger than the radius of Jupiter, $R_J$, implying high temperatures deep in the planetary interior (Burrows et al. 2000). A powerful internal heat source must be present in order to prevent their rapid contraction. Tidal heating is a possible solution. However, the gravitational tide synchronization and circularization timescales are much shorter than the age of the system if the tidal quality factor is comparable to that of Jupiter. Tidal heating could then power the observed radii only for a brief period, far shorter than the age of observed systems. A number of possible mechanisms have been put forward to generate tidal heating over long timescales, as is required to explain the large radii of $\gtrsim$Gyr old planets: tidal dissipation due to orbital eccentricity induced by an external planet (Bodenheimer et al. 2001), dissipation of the kinetic energy of deep winds driven by insolation (Showman & Guillot 2002), tidal dissipation due to obliquity maintained in a Cassini state (Winn & Holman 2005; Fabrycky et al. 2007; Levrard et al. 2007), and Ohmic dissipation induced by currents driven by atmospheric flows (Batygin & Stevenson 2010).

We propose that time-dependent thermal forcing of hot Jupiters leads to a “thermal tide” torque pushing the planet away from synchronous rotation, and possibly circular orbits. The equilibrium spin state is set by the competition between the opposing thermal and gravitational tide torques. Gravitational tide dissipation continues to operate in this torque equilibrium, implying a steady state heat source to power the large radii. The thermal tide torque mechanism was initially proposed by Gold & Soter (1969, GS from here on) in order to explain Venus’ slow, retrograde spin. Correia et al. (2008) have applied the GS thermal tide calculation to understand the rotation rate of extrasolar planets with a solid surface. Here, we generalize their analysis for fluid planets in which the central solid core is either non-existent or dynamically unimportant.

The existence of dynamically important thermal tide torques in fluid planets has recently been cast in doubt (Goodman 2009; Gu & Ogilvie 2009). These authors argue that at forcing frequencies smaller than the planet’s dynamical frequency, the quadrupole moment will be negligible due to isostatic adjustment. Furthermore, the former author asserts that the torque will push the planet toward synchronous rotation, rather than away. A careful study of the fluid motion induced by surface thermal forcing in fluid gas giant planets is needed to address the following questions. What is the direction and depth dependence of the resulting flow? What is the relevance of the concept of “isostatic adjustment” to this inherently time-dependent problem? What sets the “dynamical frequency” below which the quadrupole moments become small? What is the sense of the torque (synchronous versus asynchronous)? What is the magnitude of the torque? Once these questions have been answered, the ability of thermal tide torques to promote asynchronous rotation and tidally inflated radii can then be assessed.

Here, we study the fluid flow and resulting quadrupole moments induced by thermal forcing at the surface of a gas giant planet. To isolate what we believe to be the relevant physical mechanisms, we ignore thermal diffusion, the Coriolis force, and self-gravity of the perturbations, while utilizing a simplified model for the planetary structure.

The plan of this paper is as follows. In Section 2, we provide motivation for performing the ensuing calculations. We then describe the problem setup and details of the background model for
the fluid planet in Section 3. Section 4 reviews the relation between the density perturbation and the quadrupole moment, and the torque on the planet and orbit in terms of the quadrupole moment. The linearized fluid equations are presented in Section 5. In Sections 5.1 and 5.2, we develop analytic solutions in the zero and small, but finite, frequency limit. Numerical results to the section 8.

In Sections 5.1 and 5.2, we develop analytic solutions in the zero and small, but finite, frequency limit. Numerical results to the linearized boundary value problem are presented in Section 5.3. In Section 6, the numerical results are explained in terms of the response of envelope gravity waves that are excited by thermal forcing. In Section 7, we discuss the major approximations that are employed in our fluid dynamical analysis. We summarize in Section 8.

2. MOTIVATION

We begin with a brief discussion of the possible role of thermal tides for the rotation rates and radii of hot Jupiters.

Thermal forcing may induce a “bulge” which is misaligned with star–planet line (see Figure 1). The misalignment is a result of thermal inertia, the tendency of maximum temperature to lag maximum heating. Since insolation is finite on the day side and the thermal time of the absorbing layer is given by

\[ t_{th} \sim \frac{M_p c_p T}{R_p^2 F_*}. \]

where \( c_p \) is the specific heat at constant pressure, \( T \) is the temperature, and \( F_* \) is the stellar flux. The frequency dependence in Equation (1) reflects that the temperature and density perturbations in the atmosphere become small when the forcing period is short. The accuracy with which Equation (1) represents a fluid atmosphere is the subject of the remainder of this paper.

The quadrupole moment induced by the stellar gravitational tide which leads to secular evolution is given by (e.g., Goldreich & Soter 1966)

\[ Q^{(th)} \sim \frac{\Delta M R_p^2}{\sigma_{th}}. \]

where \( \Delta M \) is the mass down to the photospheric layer for the starlight, \( R_p \) is the planet’s radius, \( \sigma \) is the tidal forcing frequency, and the thermal time of the absorbing layer is given by

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The ratio between the thermal and gravitational tide quadrupoles is given by

\[ \frac{Q^{(th)}}{Q^{(grav)}} \sim \frac{n}{\sigma} \left( \frac{\Delta M}{M_p} \right) \left( \frac{M_p a^3}{M_* R_p^2} \right) \left( \frac{Q_p}{n t_{th}} \right) \]

\[ \sim \frac{n}{\sigma} \left( \frac{\Delta M}{10^{-8} M_p} \right) \left( \frac{M_p}{10^{-3} M_*} \right) \]

\[ \times \left( \frac{a}{10^2 R_p} \right)^3 \left( \frac{Q_p}{10^3} \right) \left( \frac{1}{n t_{th}} \right). \]
The fiducial estimate in Equation (4) shows that thermal and gravitational tidal effects are competitive for hot Jupiters, with large departures from synchronous spin $\sigma \sim n$ expected.

Given the equilibrium spin rate, the gravitational tide will generate heat at a rate

$$\dot{E}^{(GT)} = \frac{\sigma}{m} n^2 Q^{(grav)} = -\frac{\sigma}{m} n^2 Q^{(th)}$$

$$\simeq 10^{38} \text{erg s}^{-1} \left( \frac{\sigma}{n} \right) \left( \frac{3 \text{days}}{P_{\text{orb}}} \right)^3 \left( \frac{Q^{(th)}}{10^{-8} M_J R_J^2} \right). \quad (5)$$

This fiducial estimate of the heating rate is comparable, or larger than, the core cooling rates found in Arras & Bildsten (2006) for many "problem" planets, which cannot be explained by passive cooling. If this tidal heating is deposited sufficiently deep in the core, it may then prevent these planets from contracting, explaining the inflated radii of some hot Jupiters.

3. PROBLEM SETUP

In this section, we expand the thermal and gravitational forcing terms in terms of a Fourier expansion, define the coordinate system used and discuss the background planetary structure model.

3.1. Description of Time-dependent Thermal and Gravitational Forcing

We seek solutions to the boundary value problem describing linear response of a gaseous planet to a gravitational tidal acceleration $-\nabla U(x, t)$ and an imposed entropy perturbation, $\Delta s(x, t)$, that results from time-dependent insolation.

Consider a planet of mass $M_p$ and radius $R_p$ in a circular orbit of separation $a$ around a solar-type star of mass $M_\star = M_\odot$, radius $R_\star = R_\odot$, and effective temperature $T_\star = 5800 \text{ K}$. The bolometric flux at the planet’s surface is then $F_\star = \sigma_{sb} T_\star^4 (R_\star/a)^2$, where $\sigma_{sb}$ is the Stefan–Boltzmann constant. The orbital frequency is $n = (GM_\star/a^3)^{1/2}$ and the planet’s spin frequency is $\Omega$. The orbital phase in the frame corotating with the planet will be denoted as $\Phi = (n - \Omega) t$.

It is convenient to express $U(x, t)$ and $\Delta s(x, t)$ in a Fourier series in time and spherical harmonics in angle, i.e.,

$$\Delta s(x, t) = \sum_{\ell m} \Delta s_{\ell m}(r) Y_{\ell m}(\theta, \phi) e^{-i \sigma_{\ell m} t} \quad (6)$$

$$U(x, t) = \sum_{\ell m} U_{\ell m}(r) Y_{\ell m}(\theta, \phi) e^{-i \sigma_{\ell m} t}. \quad (7)$$

The forcing frequency is then $\sigma_{\ell m} = m(n - \Omega).$ For $\ell = 2$, only $m = \pm 2$ are allowed due to the even parity of $U$ and $\Delta s$. We note that while the longitude-dependent stellar irradiation is zero on the night side and nonzero on the day side, the individual Fourier components are nonzero on both day and night sides. By adding together many Fourier components one recovers the true position-dependent flux.

Lagrangian and Eulerian variations are denoted by $\Delta$ and $\delta$, respectively. The Lagrangian entropy perturbation, $\Delta s$, is a convenient variable to use as it varies solely due to non-adiabatic processes, here the time-dependent heating is due to insolation.

3.2. Background Model

Hot Jupiters possess a convective core, blanketed by a thin radiative envelope at the surface (e.g., Burrows et al. 2000; Arras & Bildsten 2006). Rather than constructing a model

For semidiurnal forcing of an asynchronously rotating planet, the stellar insolation $\delta \varepsilon$ achieves its maximum value at noon and at midnight. The entropy perturbation $\Delta s$ then attains maxima at 3 p.m. and 3 a.m., and minima at 9 a.m. and 9 p.m.

The $\ell = 2$ coefficients for the tidal potential are given by

$$U_{2m} = \sqrt{\frac{3\pi}{10}} n^2 \gamma^2$$

for both $m = \pm 2$.

3.2. Background Model
with a detailed equation of state and radiative transfer, we use a simple parametrized equation of state which enforces neutral stratification in the core and stable stratification with roughly constant scale height in the envelope. The equation of hydrostatic balance is integrated using roughly constant scale height in the envelope. The equation of neutral stratification in the core and stable stratification with use a simple parametrized equation of state which enforces a narrow transition near the surface with and gravity waves can propagate where at the radiative–convective boundary. The core compressibility factors with and are related by where is the Lagrangian displacement vector. Mass conservation implies where the volume integral is performed over the background planet. Equation (16) may then be expanded as

The factors with define the time-dependent multipole moments of the planet

Here and are the orbital separation and phase. Integrating by parts and using the continuity equation for the perturbations

Equation (19) becomes

Hence, it is the Eulerian density perturbation which is needed for the quadrupole moment.

Since is a real quantity, and the moments must satisfy . By defining , the interaction Hamiltonian in Equation (16) may be conveniently expressed in terms of a sum over spherical harmonics as

The rate of change of orbital angular momentum, is given by (e.g., Newcomb 1962)
The secular evolution of the planet–orbit system is due to the presence of quadrupole moments that are out of phase with the tidal acceleration, i.e., the imaginary component of the quadrupole. The torque on the planet for a circular orbit is dominated by the semidiurnal term with $|n| = 2$ and forcing frequency $2(n - \Omega)$. The torque at this order is

$$N = 4 \left( \frac{3\pi}{10} \right)^{1/2} \left( \frac{M_p + M_*}{M_*} \right) n^2 \text{Im} (Q_{22}) \approx 4n^2 \text{Im} (Q_{22}).$$

The sign in Equation (24) is such that density perturbations $\delta \rho$ lead maximum heating tend to torque the planet away from synchronous rotation and vice versa.

5. SOLUTION OF THE BOUNDARY VALUE PROBLEM

Conservation of mass and momentum, the first law of thermodynamics, and two boundary conditions are employed in order to obtain the planet’s response to thermal and gravitational forcing. First, we develop equations that can be solved as a boundary value problem. We then show that planet’s response can be represented as a sum of eigenmodes. The governing equations below are standard and can be found in, for example, Unno et al. (1989).

The equations of conservation of horizontal and vertical momentum are given by

$$-\sigma^2 \xi_h = -\left( \frac{\delta p/\rho + U}{r} \right)$$

and

$$-\sigma^2 \xi_r = -\frac{1}{\rho} \frac{d\delta \rho}{dr} - g \frac{\delta \rho}{\rho} - \frac{dU}{dr},$$

respectively. Here, $\delta \rho$ and $\delta p$ are the Eulerian pressure and density perturbations. We ignore the self-gravity of the perturbations for simplicity. The continuity Equation (20) may be written as

$$\frac{\delta \rho}{\rho} = -\frac{1}{r^2 \rho} \frac{d}{dr} (r^2 \rho \xi_r) + \frac{\ell (\ell + 1)}{r} \xi_n$$

where we have used Equation (25). By substituting the relation between Lagrangian and Eulerian quantities, i.e., $\Delta = \delta + \xi \cdot \nabla$, Equation (9) becomes

$$\frac{\delta \rho}{\rho} = \frac{\delta p}{\rho c^2} + \frac{N^2}{g} \xi_r + \rho_s \frac{\Delta s}{C_p},$$

where $c^2 = \Gamma_1 p/\rho$ is the adiabatic sound speed. The imposed entropy perturbations drive vertical fluid motion through buoyancy forces $-g \delta \rho \Delta s/c_p$. The vertical fluid motion in turn drives horizontal motion.

We solve the system of three Equations (26), (28), and (29), for the three variables $\delta p$, $\delta \rho$, and $\xi_r$. While $\delta \rho$ can be eliminated as well, we find that better numerical precision is achieved by keeping $\delta \rho$ in the system of equations.

Finally, we remark on the boundary conditions. Equation (29) is algebraic, so no boundary condition is needed as this equation can be evaluated on the boundaries. At the center of the planet, we require all variables to be finite. This implies (e.g., Unno et al. 1989; take $\xi_r \propto r^{\ell - 1}$ and $\delta p/\rho \propto r^{\ell}$ in Equation (26))

$$\sigma^2 \xi_r = \frac{\ell}{r} \left( \frac{\delta p}{\rho} + U \right).$$

We present results for two different upper boundary conditions. The “standard” boundary condition requires that the Lagrangian pressure perturbation $\Delta p$ vanish (Unno et al. 1989)

$$\delta p/\rho = g \xi_r.$$  \hfill (31)

If the fluid perturbation is evanescent, this boundary condition is valid. We also present results for an “outgoing wave” boundary condition, in which wave energy propagates outward at the upper boundary (e.g., Unno et al. 1989). We implement this outgoing wave boundary condition as follows. The atmosphere above the upper boundary is idealized as isothermal, with constant $H$, $N$, $g$, $c$, and horizontal wavenumber $k_\perp = \sqrt{(\ell + 1)/R_p}$. In an isothermal atmosphere, the fluid variables $\delta p/\rho, \xi_r \propto e^{bc}$ (e.g., Goldreich & Kumar 1990), where $z$ is an altitude and the complex constant

$$b = \frac{1}{2H} \pm i \left[ k_\perp^2 N^2 - k_\parallel^2 + \frac{\sigma^2 - 1}{4H^2} \right]^{1/2}$$

determines the run with height. When the argument of the square root is positive, the wave is propagating. The sign of the second-order wave flux is

$$F_{\text{wave}} = \delta p \xi_r \propto -\left( \frac{\sigma}{N^2 - \sigma^2} \right) \text{Im}(b).$$

We choose outgoing wave flux by choosing the appropriate sign of $\text{Im}(b)$ to make $F_{\text{wave}} > 0$, given the signs of $\sigma$ and $N^2 - \sigma^2$. For the propagating case, the boundary condition is enforced as

$$\frac{d}{dr} \left( \frac{\delta p}{\rho} \right) = b \left( \frac{\delta p}{\rho} \right).$$

When the argument of the square root in Equation (32) is negative, the wave is evanescent and we enforce the “hydrostatic” boundary condition in Equation (31).

5.1. Zero-frequency Limit: The Equilibrium Tide

The equilibrium tide solution is found by setting $\sigma \to 0$ (e.g., Goldreich & Nicholson 1989). The pressure perturbation is found from Equation (25) to be

$$\delta p^{\text{eq}} = -\rho U.$$  \hfill (35)

Substitution of Equation (35) into Equation (26) gives

$$\frac{\delta p^{\text{eq}}}{\rho} = \frac{d \ln \rho}{dr} \left( \frac{U}{g} \right).$$

Inserting Equations (35) and (36) into Equation (29) gives

$$N^2 \xi_r = -\frac{N^2}{g} U - g \rho_s \frac{\Delta s}{C_p}.$$  \hfill (37)

In stably stratified regions, $N^2 > 0$ and Equation (37) can be solved for the vertical displacement

$$\xi_r^{\text{eq}} = -\frac{U}{g} - \frac{g}{N^2 \rho_s} \frac{\Delta s}{C_p}.$$  \hfill (38)
In the central convection zone, where $N^2 \simeq 0$, no solution exists for Equation (37), since the terms with $N^2$ go to zero while the $\Delta\sigma$ term is nonzero. It follows that the concept of an equilibrium tide breaks down in regions that are neutrally stratified and therefore, the forcing frequency cannot be safely set to zero. That is, the fluid response in neutrally stratified regions is inherently a “non-equilibrium tide.” Finally, the substitution of Equations (35), (36), and (38) into Equation (27) yields the equilibrium horizontal displacement

$$r\delta\xi_h^{(eq)} = \frac{1}{\ell(\ell + 1)} \left[ \frac{d}{dr} \left( r^2 \rho\xi_t^{(eq)} \right) + r^2 \delta\rho^{(eq)} \right].$$  

(39)

Hence, both vertical and horizontal motions exist in the equilibrium tide limit, driven by gravity and entropy fluctuations.

Clearly, the equilibrium tide perturbation in Equation (36), and thus the equilibrium tide quadrupole moment, has no contribution from time-dependent surface heating. Inspection of Equation (29) shows that the terms $\propto \Delta\sigma$ cancel out one another. The last term in Equation (29) represents a density decrease due to heating, which is precisely compensated for by the $N^2\xi_t/g$ term that results from moving across surfaces of constant pressure. In other words, denser fluid is brought up from below, exactly canceling the local density decrease due to heating.

Time-dependent insolation heats the surface and initiates horizontal motion. Consider subsynchronous rotation $(n - \Omega > 0)$ so that an observer at rest with respect to the planet sees the star rotate with angle $\Phi(t) = (n - \Omega)t$ in the positive $\phi$ direction (see Figure 1). For $m = 2$, the heating function $\delta\epsilon \propto \cos[2(\phi - \Phi)]$ and maxima in the heating function occur at $\phi - \Phi = 0, \pi$. The entropy perturbation takes the form (Equation (12))

$$\Delta\sigma(x, t) = \left( \frac{-1}{n-\Omega} \right) \left( \frac{k_F^*}{c_p T} \right) \left( \frac{\kappa_p}{g} \right) \sin \left[ 2(\phi - \Phi) \right],$$  

(40)

giving the hottest points ($\Delta\sigma$ a maximum) at $\phi - \Phi = -\pi/4, 3\pi/4$, and the coldest points at $\phi - \Phi = \pi/4, -3\pi/4$. This phase shift reflects the lag between maximum heating and maximum temperature. The displacement in the $\phi$ direction becomes (Equation (39))

$$\xi_\phi(x, t) = \left( -1 \right) \left( \frac{1}{n-\Omega} \right) \left( \frac{1}{3\rho} \right) \left( \frac{\kappa_p}{g} \right) \sin \left[ 2(\phi - \Phi) \right].$$  

(41)

The quantity in parentheses and its derivatives are shown in Figure 3. It varies mainly as $p \exp(-\kappa_p r/g)$, which has a sharp maximum near the base of the heating layer at $p = g/\kappa_p$. Above the base, this derivative is negative and $\xi_\phi(\phi) \propto \cos[2(\phi - \Phi)]$ describes motion away from regions of high entropy, and toward regions of low entropy. Were there no return flow, mass would accumulate at $\phi - \Phi = \pi/4, -3\pi/4$, leading the star. For mass “bumps” leading the star, the quadrupole moment would act to decrease the planet’s spin, pushing it further from synchronous spin. However, there is a return flow, since below the base the direction of $\xi_\phi$ reverses (see the sign of $df/d\tau$ in Figure 3). As a result, there is no mass accumulation, the density perturbation is identically zero, and there is no net quadrupole. What is required to drive asynchronous spin is a net flow, integrated over depth, away from the hottest points in the atmosphere.

### 5.2. Finite Frequency Correction

In this section, we derive an analytic expression for the density perturbation due to thermal forcing by using perturbation theory in powers of the frequency. While instructive, this limit is only applicable when resonances with internal waves are unimportant, i.e., at forcing periods $\ll 1$ day or $\gg 1$ month.

Let $\delta\rho$ and $\xi_\phi$ denote the complete solution including both the $\sigma = 0$ equilibrium tide and the finite frequency corrections. To first order in $\sigma^2$, conservation of horizontal momentum gives

$$\delta\rho = -\rho U + \sigma^2 r\rho\xi_h^{(eq)} = \delta\rho^{(eq)} + \sigma^2 r\rho\xi_h^{(eq)}. $$  

(42)

Substituting Equation (42) into Equation (26) allows us to write the density perturbation as

$$\delta\rho \simeq \delta\rho^{(eq)} + \frac{\sigma^2}{g} \left[ \rho\xi_t^{(eq)} - \frac{d}{dr} \left( \rho r\xi_h^{(eq)} \right) \right].$$  

(43)

For now, we ignore the tidal potential $U$ and use Equation (39) to write

$$\delta\rho = \frac{\sigma^2}{g} \left[ \rho\xi_t^{(eq)} - \frac{1}{\ell(\ell + 1)} \frac{d^2}{dr^2} \left( r^2 \rho\xi_h^{(eq)} \right) \right].$$  

(44)

This equation shows that finite fluid inertia effects, scaling as $\sigma^2$, give rise to a nonzero density perturbation and consequently, a quadrupole moment.

Attempts to directly integrate Equation (44) lead to numerical difficulties due to the second derivative term, which oscillates with depth. To the extent that $r^2\xi_h^{(eq)}$ is constant, this expression gives a perfect derivative, and the integral would depend only on the endpoints at the center and surface of the planet, where the integrand is negligible. In the Appendix, we show that
integration by parts and use of the equations of motion can be used to obtain a more monotonically integrand with significantly less cancellation error.

The expression in Equation (A3) can be estimated analytically in the low-frequency limit. If we ignore the \( \delta \sigma \propto \sigma^2 \) term, substitute Equation (38) for \( \xi \), and treat the interior mass \( m(r) \propto M_p \) as a constant, we find

\[
Q = \left( \frac{(4 + \ell)(3 + \ell)}{\ell(\ell + 1)} - 1 \right) \int_0^R dr \rho r^{2+c} \frac{\Delta s}{\rho_s c_p}. \tag{45}
\]

This formula directly gives the quadrupole moment in terms of the applied entropy perturbation. Note that the \( \ell \)-dependent prefactor is equal to \( 4 \) for \( \ell = 2 \) and vanishes as \( \ell \to \infty \).

By comparison, Gold & Soter (1969) assumed constant pressure and ignored vertical motion to obtain a density perturbation \( \delta \rho / \rho = -\delta T / T = -\Delta s / c_p \) (for \( \rho_s = -1 \)). This gives the quadrupole moment

\[
Q^{(GS)} = \int_0^R dr \rho r^{2+c} \Delta s / c_p. \tag{46}
\]

The integrand in Equation (45) has the same sign as that of Equation (46), implying a torque which generates asynchronous spin, but suffers a reduction relative to their expression by a factor

\[
\left( \frac{(4 + \ell)(3 + \ell)}{\ell(\ell + 1)} - 1 \right) \frac{\sigma^2}{N^2}. \tag{47}
\]

As we will see, our numerical solutions to the boundary value problem asymptote to the expression in Equation (45) at long forcing periods.

5.3. Numerical Results

In this section, we present numerical results for the solution of Equations (26), (28), and (29) for \( \delta \rho, \delta p, \) and \( \xi \). To solve these equations, we finite difference in radius with second-order accuracy and write the resulting inhomogeneous equation in matrix form as \( M x = B \), where \( M, x, \) and \( B \) denote the differential operators, solution vector (\( \delta \rho, \delta p, \) and \( \xi \) as a function of radius), and forcing vector (involving \( U \) and \( \Delta s \)). We find the solution of this linear system using a band-diagonal solver. The quadrupole moment is then evaluated using Equation (A3).

Figure 4 shows the quadrupole moment \( Q_{22}(\sigma) \) as a function of forcing frequency \( \sigma \) for standard boundary condition in Equation (31). Forcing is solely by \( \Delta s \), i.e., \( U = 0 \). The parameters are \( M_p = 0.7 M_J, R_p = 1.3 R_J, a = 0.05 \) AU, a solar-type star, base of the heating layer at \( g / k_s = 1 \) bar and base of the radiative zone at \( p_b = 100 \) bar. The absolute value of the imaginary part is plotted. The sign is shown by the line type, solid red line for \( \text{Im}(Q_{22}(\sigma)) = 0 \), and dashed blue for \( \text{Im}(Q_{22}(\sigma)) < 0 \). The positive forcing frequencies used imply sub-synchronous rotation. Torque has the same sign as \( \text{Im}(Q_{22}) \), so that \( \text{Im}(Q_{22}) < 0 \) drives the planet further from the synchronous state and vice versa. It is immediately apparent that the response to thermal forcing is dominated by the low-order \( g \)-modes at periods \( 2\pi / \sigma \simeq 1–30 \) days. This is in contrast to the gravitational tide, which has the largest response for the \( f \)-mode at \( 2\pi / \sigma \simeq 0.1 \) days. Consequently, the equilibrium tide limit does not apply until the frequency is well below the periods of low-order \( g \)-modes. In other words, the low-frequency limit of Section 5.2 is not a good approximation until periods \( 2\pi / \sigma > 30 \) days for the model shown, rather than \( 2\pi / \sigma \sim 0.1 \) days, as one would expect for the gravitational tide.

For comparison, Figure 4 shows the GS approximation in Equation (46) and the finite frequency correction to the equilibrium tide given by Equation (45). The amplitude at the resonances is set by the frequency spacing used to make the plot; for finer frequency spacing the peaks would be larger. The solution asymptotes to Equation (45) for forcing periods much longer than that of the low-order \( g \)-modes. In the range \( 1–30 \) days, the quadrupole moment is much larger than the value given by Equation (45), even midway between resonances. The torque alternates sign across some resonances, but not others. For short forcing periods \( \sim 0.1–1 \) day, the torque acts to synchronize the spin, reinforcing the dissipative gravitational tide torque. However, in between the two lowest order \( g \)-modes, at periods \( \simeq 1–2 \) days, the thermal tide torque is large and acts to torque the planet away from a synchronous spin state.

The sharp peaks in Figure 4 are due to resonant response. In the absence of damping, the energy of a standing wave diverges (in linear theory) as the resonance is approached. This is due to the fact that waves can reflect back and forth in the resonant cavity, continually being pumped by the forcing. A second limit one can imagine is that waves with sufficiently short wavelengths at the top of the atmosphere (above the photosphere) propagate upward, carrying away their energy. Figure 5 shows the results for the same parameters as in Figure 4, the only modification being the upper boundary condition. Again, there is no forcing by the gravitational tide (\( U = 0 \)).

\footnote{It is unclear if Gold & Soter (1969) assume \( \delta \rho = 0 \) or \( \Delta \rho = 0 \).}
is the sum of the gravitational tidal force
\[ a^G = -\nabla U \] (49)
and the force due to time-dependent insolation, \( a^\delta \). We have also added an ad hoc damping term \( \mathbf{D} \cdot \mathbf{\xi} \). To compute the form of \( a^\delta \), the pressure and buoyancy forces in Equation (26) must be expressed in terms of \( \mathbf{\xi} \) using Equations (20) and (29) using
\[
\delta p = \rho c^2 \left( \frac{\delta \rho}{\rho} - \frac{N^2}{g} \xi_r - \rho \frac{\Delta s}{c_p} \right) = -c^2 \left( \nabla \cdot (\rho \mathbf{\xi}) + \rho \frac{N^2}{g} \xi_r + \rho \rho_0 \frac{\Delta s}{c_p} \right). \tag{50}
\]
Terms involving \( \mathbf{\xi} \) combine to give \( \mathbf{C} \cdot \mathbf{\xi} \). The total pressure force contains a non-adiabatic term that results from time-dependent insolation, which we identify as
\[
a^\delta = \frac{1}{\rho} \nabla \left( \rho c^2 \frac{\Delta s}{c_p} \right). \tag{51}
\]
The entropy fluctuation \( \Delta s \) is specified by Equation (10).

The externally forced fluid displacement \( \mathbf{\xi} \) can be decomposed into the free adiabatic eigenmodes \( \mathbf{\hat{\xi}}_\alpha \) as
\[
\mathbf{\xi}(\mathbf{x}, t) = \sum_\alpha q_\alpha(t) \mathbf{\hat{\xi}}_\alpha(\mathbf{x}). \tag{52}
\]
Here, \( q_\alpha(t) \) is the time-dependent eigenmode amplitude. The free adiabatic eigenmodes \( \mathbf{\hat{\xi}}_\alpha(\mathbf{x})e^{-i\sigma_\alpha t} \) with eigenfrequency \( \sigma_\alpha \) obey
\[
-\sigma_\alpha^2 \mathbf{\hat{\xi}}_\alpha + \mathbf{C} \cdot \mathbf{\hat{\xi}}_\alpha = 0. \tag{53}
\]
The Hermitian nature of the operator \( \mathbf{C} \) implies that the eigenfunctions obey the orthogonality condition \( \int d^3 x \rho \mathbf{\hat{\xi}}_\alpha^\ast \mathbf{\hat{\xi}}_\beta = A_{\alpha\alpha} \delta_{\alpha\beta} \), where \( \delta_{\alpha\beta} \) is the delta function and \( A_{\alpha\alpha} = \int d^3 x \rho | \mathbf{\hat{\xi}}_\alpha |^2 \).

By projecting Equation (48) onto mode \( \alpha \) using the orthogonality relation, the amplitude \( q_\alpha(t) \) obeys the following forced oscillator equation:
\[
\ddot{q}_\alpha + \sigma_\alpha^2 q_\alpha = -\gamma_\alpha \dot{q}_\alpha + \frac{1}{A_{\alpha\alpha}} \int d^3 x \rho \mathbf{\hat{\xi}}_\alpha^\ast \cdot \mathbf{a}(\mathbf{x}, t), \tag{54}
\]
where \( \gamma_\alpha \) is the damping rate. In general, damping can be due to a microphysical process such as viscosity and thermal diffusion. It may also represent the loss of energy at the upper boundary due to the outgoing wave boundary condition, which can also lead to a broadening of the resonant response. By inserting in a particular harmonic \( \mathbf{a}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x}, \sigma)e^{-i\sigma t} \), the amplitude of the forced response is given by
\[
q_\alpha(\sigma) = \frac{\int d^3 x \rho \mathbf{\hat{\xi}}_\alpha^\ast \cdot \mathbf{a}(\mathbf{x}, \sigma)}{A_{\alpha\alpha}(\sigma_\alpha^2 - \sigma^2 - i \sigma \gamma_\alpha)}. \tag{55}
\]
Note that the mode amplitude \( q_\alpha(\sigma) \) can be forced by both tidal gravity, which leads to an in-phase response, and the time-dependent insolation, which leads to an out-of-phase response.

6.2. Quadrupole Moment in Terms of Overlap Integrals

The frequency-dependent quadrupole moment is found by combining the mode amplitude in Equation (55) and the
Quadrupole moments from Equation (19) to find
\[
Q_{\ell m}(\sigma) = \sum_\alpha \left[ \int d^3x \rho \xi_\alpha^\ast \cdot a(x, \sigma) \right] \left[ \int d^3x \rho \xi_\alpha \cdot \nabla (r^4 Y_{\ell m}(\theta, \phi)) \right] / A_{\alpha\alpha}(\sigma^2 - \sigma^2 - i\gamma_\alpha \sigma) \tag{56}
\]
\[
= \sum_\alpha Q_{\alpha m}(\sigma) \left( \frac{2\alpha^2}{\sigma^2 - \sigma^2 - i\gamma_\alpha \sigma} \right), \tag{57}
\]
where we identify the quadrupole moment of mode \(\alpha\)
\[
Q_{\alpha m}(\sigma) = \left[ \int d^3x \rho \xi_\alpha^\ast \cdot a(x, \sigma) \right] \left[ \int d^3x \rho \xi_\alpha \cdot \nabla (r^4 Y_{\ell m}(\theta, \phi)) \right] / 2A_{\alpha\alpha} \sigma^2. \tag{58}
\]

For forcing frequencies \(\sigma \ll \sigma_\alpha\), Equation (57) shows that \(\k Q_{\alpha m}\) can be identified with the quadrupole moment contribution from mode \(\alpha\) at low frequency. Note that \(Q_{\alpha m}\) is a physical quantity, since the normalization factors for the eigenmodes cancel out. Also note that the \(Q_{\alpha m}\) have frequency dependence solely due to \(\Delta \sigma \propto \sigma^{-1}\). This dependence is factored out by writing
\[
\text{Im}[Q_{\alpha m}(\sigma)] = \text{Im}[Q_{\alpha m}(\sigma_\alpha)] \left( \frac{\sigma_\alpha}{\sigma} \right). \tag{59}
\]

The imaginary part of the quadrupole moment, which determines the secular torque on the orbit and spin, can be written out as
\[
\text{Im}[Q_{\ell m}(\sigma)] = \sum_\alpha \left( \frac{2\alpha^2}{\sigma^2 - \sigma^2 + \gamma_\alpha^2 \sigma^2} \right) \times \left( \text{Re}[Q_{\alpha m}(\sigma)] \gamma_\alpha \sigma_\alpha + \text{Im}[Q_{\alpha m}(\sigma)] (\sigma_\alpha^2 - \sigma^2) \right). \tag{60}
\]

In the absence of dissipation (\(\gamma_\alpha = 0\)), the gravitational tide exerts no torque since the response is in phase, and \(\text{Im}[Q_{\alpha m}(\sigma)] = 0\). Including dissipation (\(\gamma_\alpha \neq 0\)), the gravitational tide gives rise to a torque due to the first term in Equation (60). By contrast, the thermal tide response is inherently out of phase. Hence even when \(\gamma_\alpha = 0\), the thermal tide causes a secular torque. Inclusion of damping for the thermal tide will act to prevent divergent response at resonances, due to the denominator of the Lorentzian in Equation (60).

Figure 6 shows the quadrupole moments \(Q_{\alpha 22}(\sigma_\alpha)\) for each mode as defined in Equations (57) and (58) for the same parameters used to find the frequency-dependent quadrupole moment in Figure 4. These quadrupole moments were computed by fitting \(Q_{\alpha 22}(\sigma)\) near resonance using Equation (57) in order to read off the coefficient \(Q_{\alpha 22}(\sigma_\alpha)\). It is immediately apparent that the dominant overlap is with the lowest order \(g\)-mode at period \(\approx 1\) day. The \(f\)- and \(p\)-mode quadrupoles are down by 1–2 orders of magnitude.

To verify the equivalence between the solution of the boundary value problem and the sum over eigenmodes, the frequency-dependent response \(Q_{\alpha 22}(\sigma)\) is compared to the sum over eigenmodes from Equation (57) in Figure 7. Damping is neglected, i.e., the standard boundary condition in Equation (31) is used. Despite the fact that only a handful of low-order \(g\)-modes are used in the sum and that the \(f\)- and \(p\)-modes are not included, there is good agreement with the solution to the full boundary value problem.

The difference resulting from the two different upper boundary conditions as shown in Figures 4 and 5 can be understood using Equation (60). Figure 4 has \(\gamma_\alpha = 0\), and hence divergences appear in the quadrupole moment near the frequencies of internal standing waves. In Figure 5, damping is introduced through the outgoing wave boundary condition. In fact this damping is so large that \(\gamma_\alpha \sim \sigma_\alpha\), and the Lorentzian factors \(\sigma_\alpha^2/((\sigma_\alpha^2 - \sigma^2)^2 + \gamma_\alpha^2 \sigma^2) = O(1)\). For Lorentzian factor of order...
unity, the induced quadrupole moment is of order of the overlap integral $Q_{\sigma \ell m}(\sigma_a)$.

Even for large damping rates $\gamma_a \sim \sigma_a$, the low-order $g$-modes still dominate the response since their overlap integrals with the thermal tide force are far larger than the $f$- and $p$-modes (see Figure 6). In the duration of one wave travel time in the radiative layer, the wave can build up enough energy to dominate the response. The overlap integrals in Figure 6 are larger than near-equilibrium tide result from Equation (45) in the period range 1 day–1 month, precisely the region where enhanced response is seen in Figure 5.

The spatial profiles of thermal and gravitational forcing differ greatly from one another. Thermal forcing is localized in physical space to the outer layers of the planet, resembling a delta function in space. By contrast, the gravitational tide most closely resembles the $f$-mode, and can be thought of as a delta function in momentum space (Reisenegger 1994). In other words, since thermal forcing is narrow in physical space, the fluid response, in terms of normal modes, is broad in momentum space and vice versa for the gravitational tide. However, the $g$-modes which propagate within the stably stratified envelope dominate since their frequencies are comparable to forcing frequencies of interest, and their eigenfunctions are confined to the radiative layer where the thermal forcing occurs. The frequencies of $f$- and $p$-modes are too high to be resonant, and their eigenfunctions extend over the entire planet.

7. ON THE NEGLECT OF THE CORIOLIS FORCE, THERMAL DIFFUSION, AND BACKGROUND FLOWS

In an attempt to elucidate the origin of quadrupole moments from thermal forcing, we have made a number of simplifying assumptions so as not to obscure the basic physics. In this section, we comment on how our results might be changed, both qualitatively and quantitatively, by relaxing these assumptions.

7.1. The Coriolis Force

Consider the effect of uniform rotation, with spin frequency $\Omega$. When $\sigma \lesssim \Omega$, the Coriolis force becomes important for the fluid response and we anticipate several additional effects due to its presence. First, the gravity wave dispersion relation and fluid motions are altered. For $\sigma \sim \Omega$, we expect the change in the quadrupole to be at the factor of a few level in the “bump” between 1 and 30 days. Second, new wave families arise, mainly restored by the Coriolis force—the Rossby waves or inertial waves. These waves can propagate throughout both the radiative envelope and the convective core. We expect that, for sufficiently rapid supersynchronous rotation, Rossby resonances may give rise to a larger quadrupole moment. In the perturbation theory calculation of Section 5.2, we expect larger quadrupole moments when the spin is nearly synchronous, since the Coriolis terms $\propto \sigma \Omega$ will be larger than the inertia terms $\propto \sigma^2$. In Figures 4 and 5, this will lead to the “analytic” line becoming flat, instead of decreasing to a longer forcing period.

For sufficiently large quadrupole moment, the equilibrium spin will be asynchronous enough that $\sigma \geq \Omega$, and we expect our results to be approximately quantitatively correct. Substitution of the quadrupole moment in Figure 5 into Equation (4), for $Q_\sigma = 10^6$ we find $\sigma \sim n$ in the period range of a few days, implying the spin frequency, orbital frequency, and forcing frequency are all comparable in torque equilibrium. In this event, the Coriolis force is important, but will only change the frequencies and eigenfunctions of the gravity waves at the order unity level.

7.2. Thermal Diffusion

The heating function in Equation (11) describes the law of exponential attenuation of stellar radiation. By setting this to occur at a pressure $\sim$1 bar, we are assuming there are no absorbers high in the atmosphere. Until recently, this assumption was standard, but there may now be evidence of an inversion layer due to such absorbers in a fraction of observed hot Jupiters (Fortney et al. 2008). In that case, incorporating thermal forcing by the boundary condition used in Gu & Ogilvie (2009) may be more appropriate.

The entropy perturbation in Equation (10) assumes that the forcing period is much shorter than the thermal time in the heated layer, so that radiation diffusion may be ignored. In fact, thermal diffusion and damping of downward propagating waves is expected, as in the solutions of Gu & Ogilvie (2009). Furthermore, a shorter diffusion time may partially “erase” temperature perturbations, reducing the buoyancy force upon which gravity waves rely. This may cause wave damping or reflection near the optical or infrared photospheres. By choosing the “perfect reflector” and “leaky” upper boundary conditions in Section 5.3, we attempted to simulate two extreme limits of the effects of radiative diffusion. While thermal diffusion may decrease the amplitude of gravity waves near the photosphere, deeper in the envelope the thermal time becomes sufficiently long so that damping can be ignored.

For forcing period of order a few days, the thermal time at the base of the heating layer is of order the forcing period. Therefore, the approximation that the layer possesses a large thermal inertia is only approximately valid. Another effect we have ignored is that nonlinear effects become important when the forcing period is longer than the thermal time.

7.3. Fluid Flow in the Background State

We now comment on the possible role of fluid motion such as zonal flows in the background state. Fluid motion in the background state does not preclude the existence of the gravity (or inertial) wave response as described in this paper. Ignoring shearing of the velocity field, such a flow would simulate uniform rotation, so that the Coriolis force becomes important. However, when there is velocity shear in the background state, we expect the wave motions to be significantly modified when the shear rate is comparable to the wave frequency. Since inertial wave response relies on angular momentum gradients, we expect these waves to persist in the presence of velocity shear. Most atmospheric circulation simulations for the hot Jupiters (e.g., Dobbs-Dixon & Lin 2008; Showman et al. 2009) assume synchronous rotation and find powerful advective flows. Relatively few simulations with asynchronous rotation have been performed, but the recent results by Showman et al. (2009) are useful in this context. They found that when the degree of asynchronous spin was increased, the flow velocities were decreased. It remains to be seen what winds are induced for rotation rates appropriate for the torque equilibrium described in Section 2. All current studies of atmospheric circulation in hot Jupiters ignore the thermal tide torques. From the analysis presented in this work, it seems likely that the qualitative and quantitative outcome of these simulations will change once thermal tidal torques are included.

5 If one considers a two-frequency, two-angle approximation to the radiative transfer, both the “greenhouse” and the “stratosphere” cases may be included.
8. SUMMARY

We considered the ability of a planet, subjected to time-dependent stellar irradiation, to develop net quadrupole moments. The existence of such quadrupole moments allows the stellar tidal field to exert torques on the planet, possibly creating asynchronous spin. Such asynchronous spin could lead to large tidal heating rates, through the dissipative gravitational tide, perhaps sufficient to power the large observed radii. Section 2 contains a brief outline of how the thermal tide-generated torques could induce gravitational tide dissipation.

In Sections 5.1 and 5.2, the fluid equations are solved treating inertia as a small parameter. At zeroth order in the forcing frequency, a circulation pattern is found from hot to cold at small depths \( p \lesssim g/\kappa_\star \), and a return flow from cold to hot at larger depth \( p \gtrsim g/\kappa_\star \). There is no density perturbation associated with this flow and therefore zero quadrupole moment. The inclusion of finite inertia as a small perturbation gives rise to a finite density perturbation and therefore quadrupole moment. In the assumed limit of small fluid inertia, the phase of the quadrupole is found to have the correct sign to induce asynchronous rotation in the planet, similar to the work of Gold & Soter (1969). However, the magnitude of this quadrupole is reduced by a factor \( \sim 4(\sigma/N)^2 \). A propagating wave-like response of any form is eliminated in this “near-equilibrium tide” approximation. The value of the resulting quadrupole moment is insensitive to boundary conditions and depends solely on the local forcing in the atmosphere. Numerical calculations presented in Figures 4 and 5 confirm that the quadrupole moment approaches this analytic limit at long forcing periods.

In Section 5.3, the fluid equations are solved as a boundary value problem. Two different upper boundary conditions are used, which effectively treat the radiative envelope as a perfect resonant cavity, or as a cavity with an open upper lid, allowing waves to be radiated upward. Using both boundary conditions, which may bracket the true result including radiative transfer effects, it is found that the quadrupole moments in the period range 1 day–1 month are far larger than the non-resonant near-equilibrium tide calculation in Equation (45) by 1–3 orders of magnitude. The reason for this enhanced response is discussed in Section 6. Unlike the gravitational tidal forcing, time-dependent thermal forcing and low-order envelope \( g \)-modes couple well due to a favorable spatial overlap.

In Section 2, we argued that the magnitude of the thermal tide torques using the formula of Gold & Soter (1969) is sufficient to generate large deviations from synchronous rotation as well as large tidal heating rates, easily sufficient to power the observed planet radii. Our numerical results in Section 5.3 show that the full numerical solutions for the torque can indeed have the correct sign to generate asynchronous spin for some ranges of forcing (rotation) frequency. The magnitude of the torques is smaller than the Gold & Soter (1969) result by a factor of a few, but still large enough to be interesting. We expect that the inclusion of the Coriolis force will increase the torque at low forcing periods, since the Coriolis force is larger than fluid inertia in that limit.

There is considerable scatter in planetary radii, even for planets of similar mass and orbital period, orbiting similar stars. Increasing the size of a putative high density core has the effect of decreasing the planetary radius. While this paper has focused on finding a sufficiently powerful heat source to explain the largest planets, comparatively smaller planets may in principle be explained through a larger core size. Hence, the range in planetary radii may then be due to a range of core sizes.

In future investigations, we plan on including the effects of the Coriolis force and thermal diffusion in the calculation of the quadrupole moments. We expect that both physical effects could lead to qualitative and quantitative changes to the non-resonant and resonant thermal tidal responses. The resulting torques will then be used to solve for the equilibrium values of the planet’s spin, radius, and eccentricity.

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APPENDIX

ACCURATE EVALUATION OF THE QUADRUPOLE MOMENT: INTEGRATION BY PARTS

As discussed in Section 5.2, direct integration of \( r^2 \delta \rho \) to find the quadrupole moment is subject to large cancellation errors. This behavior is somewhat surprising and does not occur for the gravitational tide quadrupole moment. The origin of the problem is that \( \delta \rho \) is proportional to the second derivative of a certain quantity, resulting in large cancellations during the integration. We noticed this cancellation error during numerical resolution studies in which the results did not converge rapidly with increasing resolution.

Here, we show that this cancellation error can be ameliorated through successive integration by parts and use of the equations of motion. By using the exact equations of motion, this method does not introduce approximations. Integration by parts leads to an integrand that is much more monotonically, leading to better convergence.

The quadrupole moment is given by

\[
Q = \int_0^R drr^{2+\ell}\delta \rho. \tag{A1}
\]

By substituting for \( \delta \rho \) from Equation (26) gives

\[
Q = \int_0^R drr^{2+\ell}\rho \left( \sigma^2 \xi_r - \frac{1}{\rho} \frac{d\delta p}{dr} - \frac{dU}{dr} \right). \tag{A2}
\]

Integrating the pressure gradient term by parts, discarding the (small) boundary terms, and solving Equation (28) for \( \delta \rho \), we find

\[
Q = \int_0^R drr^{2+\ell} \rho \left( \frac{\sigma^2}{g} \xi_r - \frac{dU}{dr} \right) + \frac{\sigma^2}{\ell(\ell+1)} \int_0^R \frac{d}{dr} \left( \frac{r^{2+\ell}}{g} \right) \times \left[ r^{2+\ell} + \frac{d}{dr} (r^2 \rho \xi_r) - \frac{\ell(\ell+1)}{\sigma^2} \rho U \right].
\]

Integrating the \( d/dr(r^2 \rho \xi_r) \) term by parts and simplifying gives the final result.
\[
Q = \int_0^R dr r^{2+\ell} \delta \rho^{\text{eq}} \\
+ \sigma^2 \int_0^R dr \left[ \rho \xi_r \left( \frac{r^{2+\ell}}{g} - \frac{r^2}{\ell(\ell+1)} \frac{d^2}{d r^2} \left( \frac{r^{2+\ell}}{g} \right) \right) \\
+ \frac{r^2}{\ell(\ell+1)} d \left( \frac{r^{2+\ell}}{g} \right) \delta \rho \right]. 
\]

(A3)

The first term in Equation (A3) is just the equilibrium \( \sigma^2 = 0 \) quadrupole from gravitational forcing alone—see Equation (36). Although the \( \delta \rho \) term in Equation (A3) may be subject to cancellation error, this term should be smaller by a factor \( \sigma^2 r/g \) compared to the other terms after integration. Since the equilibrium tide component of \( \xi_r \) is monotonic, the terms involving \( \xi_r \) do not suffer cancellation error. We find numerically that integrand is far more monotonic than \( r^{2+\ell} \delta \rho \) and is less subject to cancellation error. Throughout, we use Equation (A3) when numerically evaluating the quadrupole moment that results from thermal forcing.

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