Non-equilibrium DC noise in a Luttinger liquid with impurity

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We compute exactly the non-equilibrium DC noise in a Luttinger liquid with an impurity and an applied voltage. By generalizing Landauer transport theory for Fermi liquids to interacting, integrable systems, we relate this noise to the density fluctuations of quasiparticles. We then show how to compute these fluctuations using the Bethe ansatz. The non-trivial density correlations from the interactions result in a substantial part of the non-equilibrium noise. The final result for the noise is a scaling function of the voltage, temperature and impurity coupling. It may eventually be observable in tunneling between edges of a fractional quantum Hall effect device.

I. INTRODUCTION

The study of fluctuations in systems of non-interacting particles is a well-understood problem. Of particular interest is the noise due to free electrons scattering off an impurity, which has been discussed carefully in [1–3]. In these calculations, the only interactions are those by individual particles with a barrier. The fact that the particles do not interact with each other is of course crucial. When interactions are present and the system is out of equilibrium, it is not possible in general to go beyond a perturbative analysis of the fluctuations. In particular, the issue of possible singularities in the noise is very difficult to resolve.

In this paper, we present an exact calculation of the non-equilibrium noise in a one-dimensional (interacting) Luttinger liquid with a single impurity. This problem has attracted a great deal of attention because the edge of a fractional quantum Hall device should be a Luttinger liquid [4–7]. A point contact pinching the sample plays the role of the impurity, with the gate voltage allowing the impurity coupling to be varied. The noise here is especially important because DC transport measurements are not sensitive to the charge of the carriers, while non-equilibrium noise measurements do depend on this charge [8]. We obtain the general current (and voltage) noise as a function of the applied voltage, temperature and impurity coupling. We check in particular that the current noise corresponds to tunneling of uncorrelated electrons in the weak-backscattering limit, and to tunneling of uncorrelated Laughlin quasiparticles of fractional charge in the strong-backscattering limit.

In a critical theory, conformal invariance usually yields the correlators, from which the fluctuations follow. The situation changes dramatically when an impurity is introduced. This adds interactions to the model and destroys conformal invariance. One can use the Keldysh formalism to relate various types of fluctuations to each other, and to obtain results to lowest order in perturbation theory in the interaction strength [9,10]. Perturbed conformal field theory yields other information [11] (in particular it predicts a singularity at finite frequency), but in neither approach has it been possible to compute exactly the non-equilibrium noise in the presence of interactions.

In a series of papers with A. Ludwig, we have developed a description of a Luttinger liquid in terms of its interacting quasiparticles. As in our earlier work, the results rely crucially on the fact that the Luttinger model with impurity is integrable [12–14]. This formalism makes it possible to do exact non-perturbative calculations in the presence of the impurity. We have computed the current and conductance through the impurity, and the DC noise at zero temperature. At zero temperature, the fluctuations are pure shot noise coming from the impurity, so finding them does not require understanding the non-trivial quantum fluctuations of the interacting quasiparticles. However, at non-zero temperature, the thermal noise and the shot noise can not be separated, so the full noise requires understanding these fluctuations. The effect of these fluctuations is substantial; without including them one does not even obtain the equilibrium Johnson-Nyquist formula when the impurity is not present. In this paper, we calculate exactly the quasiparticle fluctuations, and use them to find the exact DC noise at any temperature, voltage and impurity coupling.

In sect. 2, we show how to compute density fluctuations of the quasiparticles in an integrable theory, and apply the results to find the current noise in the Luttinger model without an impurity. In sect. 3, we couple an impurity to the Luttinger liquid, and find the exact scaling curve. This involves a reformulation of Landauer’s approach suitable for interacting quasiparticles. In sect. 4, we consider various limits, including the weak and strong backscattering limit, and also derive some generalized fluctuation-dissipation results.

II. DENSITY AND CURRENT FLUCTUATIONS IN AN INTEGRABLE THEORY

In this section we first review known results for the quasiparticle densities in an integrable theory. We extend these calculations to compute the density fluctuations. We then specialize to the case of interest, the Luttinger
liquid (in this section, without an impurity). For this model, we find the current and charge fluctuations.

A. General considerations

We consider an integrable theory whose space of states is described entirely in terms of quasiparticles. In an integrable theory, the scattering is completely elastic and factorizable, which means that the energy and momentum of the quasiparticles are individually conserved, and that the multi-particle S matrix decomposes into a product of two-particle elements. For simplicity we require that the scattering is diagonal as well, which means that the types of the particles do not change in the collision. This does not mean the scattering is trivial, because there can be a phase shift; the S matrix element \( S_{ij}^{bulk}(p_1, p_2) \) for scattering a particle of type \( i \) with one of type \( j \) is a non-trivial function of the momenta. It will be convenient to parametrize the momenta in terms of the rapidity \( \theta \) defined for gapless theories as \( p \propto \exp \theta \). Then scale invariance requires that S matrix element is a function of \( p_1/p_2 \), or equivalently, \( \theta_1 - \theta_2 \). (In a massive theory one has \( p \propto \sinh \theta \) and relativistic invariance requires also that \( S \) is a function of \( \theta_1 - \theta_2 \); the results of the section apply to this case as well.)

We study the system on a circle of large length \( L \) and at temperature \( T \). Each quasiparticle has the energy \( \epsilon_i \); in a gapless theory \( \epsilon_i = \pm p_i \propto \exp \theta \). We introduce the level densities \( n_i(\theta) \) and the filling fractions \( f_i(\theta) \), defined so that \( \int n_i(\theta) d\theta \) is the number of allowed states for a quasiparticle of type \( i \) in the rapidity range between \( \theta \) and \( \theta + d\theta \) and \( f_i(\theta) \) is the fraction of these which are occupied. Thus the density of occupied states per unit length \( P_i(\theta) \) is given by

\[
P_i(\theta) = n_i(\theta) f_i(\theta).
\]

Requiring that the wave functions be periodic in space gives rise to the Bethe equations (setting \( \hbar = 1 \), not the usual \( \hbar = 1 \))

\[
n_i(\theta) = \frac{dp_i}{d\theta} + \sum_j \int d\theta' \Phi_{ij}(\theta - \theta') P_j(\theta'),
\]

where the kernel is defined by

\[
\Phi_{ij}(\theta) = \frac{1}{2\pi} \frac{\partial}{\partial \theta} \ln S_{ij}^{bulk}(\theta),
\]

and all rapidity integrals in this paper run from \( -\infty \) to \( \infty \). These Bethe equations hold for any allowed configuration, whether it is the equilibrium configuration or not. Notice that in a free theory the kernel vanishes and we obtain the free-particle relation \( n_i(\theta) = dp_i/d\theta \).

We generalize the usual analysis by allowing for a chemical potential which depends on \( \theta \), thus contributing

\[
\exp \left[ L \sum_i \int d\theta \mu_i(\theta) P_i(\theta) \right]
\]

to the Boltzmann weight of a configuration. This generalization may seem a bit odd, but in an integrable theory with diagonal scattering, collisions change neither the energy nor type of a particle. Therefore \( P_i(\theta) \) is independent of time, even when a current is flowing. In the next section we discuss how the densities change with time, but in the very special situation where the change results from a single impurity.

To determine the equilibrium values of the densities, we find the configuration which minimizes the free energy. This procedure is known as the thermodynamic Bethe ansatz (TBA) and is discussed in the appendix. We define the “pseudoenergies” \( \epsilon_i \) by

\[
f_i = \frac{P_i}{n_i} = \frac{1}{1 + e^{\epsilon_i - \mu_i}},
\]

where \( \epsilon_i \) and the chemical potentials \( \mu_i \) are scaled by the temperature to make them dimensionless. The result is the thermodynamic Bethe ansatz equations

\[
\tau_j(\theta) = -\sum_k \int d\theta' \Phi_{jk}(\theta - \theta') \ln \left[ 1 + e^{-\tau_j(\theta') + \mu_k(\theta')} \right],
\]

\[
+ c_j(\theta) \frac{\mu_j(\theta)}{T}
\]

The bar indicates thermal equilibrium values. The equilibrium value \( \overline{F} \) of the free energy per unit length is

\[
\overline{F} = -T \sum_i \int d\theta \epsilon_i(\theta) \ln \left[ 1 + e^{-\tau_i(\theta) + \mu_i(\theta)} \right].
\]

The relations (4) and (5) still hold when the \( \mu_i \) depends on \( \theta \). This makes it possible to compute correlators by taking functional derivatives of the free energy with respect to \( \mu_i(\theta) \), and then setting \( \mu_i \) to the appropriate value. This is because the partition function can be written as the functional integral

\[
Z = \int \prod_i D P_i(\theta) e^{-F_0 L/T + L \sum_i \int d\theta \mu_i(\theta) P_i(\theta)},
\]

where the explicit form of \( F_0 \) is given in the appendix (here, the exact form of the integration measure is somewhat unclear, but this will not affect the following analysis). The equilibrium value \( -\overline{F} L/T \) is the saddle-point value of the exponent. Taking derivatives with respect to \( \mu_i(\theta) \) brings down powers of \( P_i(\theta) \) and gives expectation values. For example, the equilibrium particle densities are given by

\[
\overline{P}_i(\theta) = \frac{1}{ZL} \frac{\delta Z}{\delta \mu_i(\theta)} = \frac{1}{T} \frac{\delta F}{\delta \mu_i(\theta)}.
\]

Of course doing this functional derivation is not necessary for determining \( \overline{P}_i \); a direct way is to relate it to \( \tau \) using the definition (3) and the Bethe equations (1).
To find the current noise we will need the density-density correlation function, which is given by second-order functional derivatives:

\[
\frac{\delta^2 Z}{\delta \mu_i(\theta) \delta \mu_j(\theta')} \frac{\delta \mu_i(\theta)}{\delta \mu_j(\theta')}
\]

It is convenient to define the difference of an quantity and its equilibrium value, e.g. \(\Delta P_i \equiv P_i - \overline{P}_i\). We then have

\[
\frac{\Delta P_i(\theta) \Delta P_j(\theta')}{\Delta P_i(\theta) \Delta P_j(\theta')} = -\frac{1}{LT} \frac{\delta^2 \mathcal{F}}{\delta \mu_i(\theta) \delta \mu_j(\theta')}
\]

(7)

In general, this equilibrium value is not diagonal: it can be non-zero even when \(\theta \neq \theta'\) and \(i \neq j\). The diagonal correlations come from the Fermi repulsion of the particles, while the non-diagonal part arises from the interactions. However, we show in the appendix that in integrable systems there is at least one kind of diagonal fluctuation, even if interacting:

\[
\frac{\Delta f_i(\theta) \Delta f_j(\theta')}{\Delta f_i(\theta) \Delta f_j(\theta')} \propto \delta_{ij} \delta(\theta - \theta')
\]

It would be interesting to find a deep reason for this relation.

There is a simple alternative formulation of (6) in terms of the \(\overline{\tau}_i\). We discuss this in the appendix. The result is written in terms of a function \(K_{ij}\) defined by

\[
K_{ij}(\theta, \theta') = \Phi_{ij}(\theta - \theta') - (1 + e^{\overline{\tau}_i(\theta) - \mu_i(\theta)}) \delta_{ij} \delta(\theta - \theta').
\]

(8)

Then we find that

\[
\frac{\Delta P_i(\theta) \Delta P_j(\theta')}{\Delta P_i(\theta) \Delta P_j(\theta')} = \frac{T}{L} \left( \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta'} \right) K_{ij}^{-1}(\theta, \theta'),
\]

(9)

where the inverse is defined via

\[
\sum_k \int d\theta'' K_{ik}(\theta, \theta'') K_{kj}^{-1}(\theta'', \theta') = \delta_{ij} \delta(\theta - \theta').
\]

Unfortunately, this reformulation, while rather elegant, is also rather useless because we cannot find a closed-form expression for \(K^{-1}\) explicitly. The problem in inverting (5) is that \(K\) is not a function of the difference \(\theta - \theta'\) but of both the variables individually. Observe from (6) that, as expected from general considerations, fluctuations of densities per unit length are of order \(O(1/L)\).

**B. Current noise in a Luttinger liquid**

We now use these general results to compute the current noise in a particular system, the Luttinger liquid. We first find the noise using conformal field theory, and then rederive it using the quasiparticle picture. This leads to a result which will be crucial to our analysis in the next section including an impurity. We find that even though the current operator creates multi-particle states, only one-particle states contribute to the DC fluctuations.

The bulk Hamiltonian of the Luttinger model can be written in terms of the charge current \(j_{\ell,L}\):

\[
H_0 = \frac{\pi}{g} \int_0^L dx \left[ j_{L}^2 + j_{R}^2 \right].
\]

(10)

The overall constant \(g\) follows from the normalization of the current; in the fractional quantum Hall problem this is fixed by the charge of the electron and we have \(g = \nu\) when \(1/\nu\) is an odd integer. Since in this system the left and right movers are decoupled, we here treat only the right movers. We study the model at non-zero temperature \(T\), which corresponds to imaginary times on a circle of circumference \(1/T\).

Without an impurity, the charge and current fluctuations in a Luttinger liquid (or any other conformal field theory) can be found very straightforwardly from conformal field theory. The left- and right-moving current operators \(j_{L}(x + t)\) and \(j_{R}(x - t)\) are individually conserved. To find the noise, we need to integrate the two-point function of the currents. Conformal invariance fixes it to be

\[
\langle j_{R}(x, t) j_{R}(0) \rangle = \frac{\pi^2 T^2 g}{[\sin(2\pi^2 T(x - t))]^2}
\]

(11)

and likewise for the left movers (The strange normalization in (11) arises because of our choice \(h = 1\) instead of the more common \(h = 1\). Also, we are ignoring subtleties coming from the chiral anomaly which vanish in the infinite-volume limit). The DC current fluctuations are then given simply by

\[
\langle \Delta I \rangle^2 \equiv \lim_{\omega \to 0} \int dt e^{i\omega t} \frac{1}{2} \langle \{ j_{R}(x, t), j_{R}(0, 0) \} \rangle + (L \to R) = 2 g T,
\]

(12)

where we added the contribution of left movers. This is the well-known Johnson-Nykqvist formula (for spinless electrons), because the conductance \(G\) of this Luttinger system is equal to \(g\) (see [14] for example).

We now rederive this result for the current noise from the quasiparticles. We have discussed the quasiparticles for this system at length in [12]. The values of \(g\) where the scattering is diagonal are \(g = 1/t\), \(t\) an integer. We therefore restrict to these values, although we expect that the final results will extend straightforwardly to general coupling. There are \(t\) different types of quasiparticles: the kink (labeled by +), the antikink (−), and the breathers (1 . . . \(t - 2\)). Since there is no scale in the theory, these right-moving particles are massless, and have the dispersion relation \(e_i = p_i = m_i c_i^2\), where \(m_+ = m_- = m/2\) and \(m_j = m j \sin[\pi j/(2(t - 1))]\). The overall scale \(m\) is arbitrary and cancels out of any physical quantities. The explicit form of the kernels \(\Phi_{jk}\) in the Luttinger liquid is
discussed for example in [12], where we also give a simpler form of the thermodynamic Bethe ansatz equations ([12]).

The general fluctuations depend on space and time. To compute them using the quasiparticles, one would like to be able to recover expressions such as ([13]) based on the action of current operators on multiparticle states. Unfortunately the theory, although it appears “almost free” from the Bethe-ansatz point of view, is truly interacting, and physical observables act on the quasiparticle basis in a very complicated fashion. Typically, operators like the current or the energy density have non-vanishing matrix elements (so-called form factors) between all pairs of states with identical number of kinks minus number of antikinks; for instance, \( j_R(x,t) \) acting on the vacuum can create an arbitrary number of breathers and pairs of kinks and antikinks. (Operators that are simple in the quasiparticle basis, such as the density of kinks, are highly non-local in terms of the original physical current \( j_R \), and presumably of no physical interest.) This makes the computation of the space- and time-dependent fluctuations rather difficult, although not impossible.

While local operators are quite complicated, their integral over space can become much simpler. A typical example is the stress-energy tensor, whose integral over space can become much simpler. A typical fluctuation is the stress-energy tensor, whose integral over space is

\[
\langle \Delta Q \rangle^2 = -4LT \frac{\partial^2 \mathcal{F}}{\partial V^2} = 2T \frac{\partial Q}{\partial V}.
\]

We can find \( \mathcal{Q}(V) \) directly from the TBA. By comparing the derivative of \( \mathcal{Q}(V) \) with \( \mu \), one finds that when \( \mu \) is independent of \( \theta \)

\[
\hat{\pi}_i(\theta, V) = T \frac{\partial \mathcal{F}_i(\theta, V)}{\partial \theta}.
\]

Using the definitions of \( n_i \) and \( f_i \) gives

\[
\int d\theta \mathcal{F}_\pm(\theta) = T \int d\theta \frac{\partial \mathcal{F}_\pm(\theta, V)}{\partial \theta} \frac{1}{1 + e^{\pm(\theta, V) - \mu_\pm}}.
\]

The integrand is a total derivative, and from \([13]\) we see that

\[
\mathcal{Q}(V) = LT \left[ \ln \frac{1 + e^{-\tau_+(V/2T)}}{1 + e^{-\tau_-(V/2T)}} \right]^{\tau(\infty) - \tau(-\infty)}.
\]

We see from \([12]\) that \( \tau(\infty) = \infty \), and it is straightforward to show that

\[
\exp(\tau_\pm(-\infty)) = \frac{\sinh[(t-1)V/(2T)]}{\sinh[V/(2T)]}.
\]

This gives \( \mathcal{Q}(V) = VL/t \) and using \([13]\) gives

\[
\langle \Delta Q \rangle^2 = 2LT g
\]

for any \( V \). Using \( \langle \Delta I \rangle^2 = \frac{1}{2} \langle \Delta Q \rangle^2 \) reproduces the desired result \([12]\).

For completeness, we can use this computation to give a quick proof of the diagonal action of \( Q \). For any multiparticle state which we denote by \( |\mathcal{M}\rangle \), characterized densities \( P_i \), one has

\[
\langle \mathcal{M} | \int dx j_R(x,t) \int dx j_R(x',t') |\mathcal{M}\rangle = X + L^2 \int d\theta d\theta' \langle P_+(\theta) - P_-(\theta) | P_+(\theta') - P_-(\theta') \rangle.
\]

Here \( X \) represents the other contributions to the two-point function:

\[
X = \sum_{M'} \left| \langle M | \int dx j_R(x) | M' \rangle \right|^2 e^{i(E_M - E_{M'}) t},
\]

and \( E_M \) is the energy of the multiparticle state \( |M\rangle \). It is convenient to define
\[
D(\theta, \theta') = \Delta(P_+ - P_-)(\theta)\Delta(P_+ - P_-)(\theta')
\]  

(18)

We then have
\[
\langle \Delta T \rangle^2 = L \int \int D(\theta, \theta') d\theta d\theta' + \frac{1}{L} \chi
\]
\[= \frac{1}{L} \langle \Delta Q \rangle^2 + \frac{1}{L} \chi
\]
\[= 2gT + \frac{1}{L} \chi,
\]

(19)

where the second equality follows from the definition of \(Q\) in \([13]\), the last equality follows from \([14]\), and \(\chi\) has the same expression as \([11]\) but with a sum on intermediate states restricted to adding a term \(\phi_j\). An impurity at \(x\) allows backscattering. An impurity at \(x\) says that \(\phi_j\) must vanish individually. Therefore, the complicated matrix elements of the local current all disappear after spatial integration, and the charge operator, which is all we need to compute DC current fluctuations, acts diagonally on multiparticle states, and is a simple sum of one particle operators. This fact will prove crucial in the next section, where we include an impurity.

We also note that if one were to neglect the non-diagonal density fluctuations, one does not obtain the Johnson-Nyquist formula even in this simple situation with no impurity. This can easily be checked, because if we were to treat the particles as free but with a non-Fermi distribution function, then \(K^{-1}_{ij}(\theta, \theta') = -f_i\delta_{ij}(\theta - \theta')\). Using this to find \(D(\theta, \theta')\) yields
\[
\langle \Delta T \rangle^2_{\text{approx}} = 2T \sinh(gV/2T) \cosh((1 - g)V/2T) \sinh(V/2T).
\]

(20)

Except at \(g = 1/2\) (where the Luttinger liquid is equivalent to a free fermion), this contradicts not only the Johnson-Nyquist formula but also the simple physical result that when there is no impurity, the left movers and right movers are not coupled, and the noise should not depend on the voltage.

III. THE EFFECT OF AN IMPURITY ON THE NOISE

In this section we add a single impurity to the Luttinger liquid. This couples the left and right movers and allows backscattering. An impurity at \(x = 0\) corresponds to adding a term
\[
H_B = \lambda \cos[\phi_L(0) - \phi_R(0)]
\]
to the Hamiltonian; we have written the theory in terms of a boson so that \(j_L = -\partial_t \phi_L(x, t)/2\pi\) and \(j_R = \partial_x \phi_R(x, t)/2\pi\). Basically, the idea is that we express the noise in terms of properties of the system far from the impurity combined with the \(S\) matrix elements for tunneling through the impurity. The former are given in the preceding section, while the \(S\) matrix elements are known \([14]\). The analysis is similar to one given for free electrons: it is possible because of the simple action of \(Q\) on multiparticle states explained the previous section.

As previously discussed \([14]\), the transport properties of a Luttinger liquid with a backscattering impurity can be mapped to a Luttinger system of right movers alone. The backscattered charge \(Q_L - Q_R\) due to left-right tunneling in the original problem is proportional to the charge \(Q\) in the new problem. It is not conserved because of the impurity scattering.

To start, we discuss the one-point function of the current in the presence of the impurity, quickly rederiving earlier results \([14]\). There are two types of asymptotic states: the in states injected by the reservoir from the left, and the out states, after scattering through the impurity. The out state must have the same momentum and energy because the integrability allows only elastic scattering, so a kink can scatter only into a kink or antikink of the same rapidity. The \(S\) matrix element for a particle of rapidity \(\theta\) and type \(i\) to scatter off the impurity into a particle of type \(j\) is denoted by \(S_{ij}(\theta)\). These are written out in \([14]\). All we will need for the noise are the unitarity relation
\[
\sum_k S^\dagger_{ik} S_{kj} = \delta_{ij}
\]

(21)

and the magnitude
\[
T(\theta) \equiv |S_{++}(\theta)|^2 = \frac{1}{1 + e^{2(g - 1)(\theta - \theta_B)/g}}
\]

(22)

where \(\theta_B\) parametrizes the impurity strength and is related to \(\lambda\). In particular, it follows from the unitarity relation that \(|S_{+-}|^2 = 1 - T\).

The effect of the impurity is that the asymptotic states \(|\theta\>_+\) and \(|\theta\>_-\) are not individual eigenstates of the Hamiltonian any more. The single-particle eigenstates are now
\[
|\theta\>_+ = \frac{1}{\sqrt{2}} \left( |\theta\>^\text{in}_+ + S_{++}(\theta)|\theta\>^\text{out}_+ + S_{+-}(\theta)|\theta\>^\text{out}_- \right)
\]
\[
|\theta\>_- = \frac{1}{\sqrt{2}} \left( |\theta\>^\text{in}_- + S_{--}(\theta)|\theta\>^\text{out}_- + S_{-+}(\theta)|\theta\>^\text{out}_+ \right).
\]

(23)

The unitarity of the \(S\) matrix yields \(\langle \theta| |\theta\rangle_j = \delta_{ij}\). Describing multiparticle eigenstates is a little more complicated; for our purpose it will be sufficient to think of them as tensor products of the states \([14]\).

Although \(Q\) acts diagonally on the asymptotic states, it acts in a more complicated fashion on the eigenstates of the problem with impurity: this time, \(Q\) and the Hamiltonian do not commute and cannot be simultaneously diagonalized. First one has
\[
+ \langle |Q| \theta\rangle_+ = \frac{1}{2} \left( 1 + |S_{++}(\theta)|^2 - |S_{+-}(\theta)|^2 \right) = T(\theta)
\]

(24)
The charge acts on multi-particle states by multiplying by the appropriate densities. Since we will be interested in the leading contributions to the current, we can neglect the effect of the boundary when quantizing the allowed rapidities. Therefore the densities are exactly those discussed in the previous section. Since the impurity problem is again mapped on purely right movers, the relation between DC fluctuations of the current and fluctuations of the charge holds as before. It follows that

\[ T(V) = \int [\mathcal{T}_+(\theta, V) - \mathcal{T}_-(\theta, V)] T(\theta)d\theta. \]  

This is the central result of \cite{13, 22}, where it was however derived quite differently.

The current noise follows in a similar fashion. The central result of this paper is that

\[ \langle \Delta I \rangle^2 = L \int \int D(\theta, \theta') T(\theta) T(\theta') d\theta d\theta' + \int [\mathcal{T}_+(\theta)(1 - \mathcal{T}_-(\theta)) + \mathcal{T}_-(\theta)(1 - \mathcal{T}_+(\theta)) ] \times T(\theta)[1 - T(\theta)]d\theta, \]

where we remind the reader that \( D(\theta, \theta') \) is the density-density correlator defined in \cite{13}. Again, fluctuations are computed without the impurity, its effect being negligible at leading order in \( L \). The first term in (27) is the equivalent of (18), with the \( T \) terms arising from (24) and (25). The second term is an additional contribution at coincident rapidities. It occurs because \( Q \) is not diagonal on the eigenstates. Since \( Q \) is diagonal on the asymptotic states, the only additional contribution due to intermediate states is proportional to

\[ + |Q| |\theta \rangle - - |Q| |\theta \rangle = T(\theta)(1 - T(\theta)), \]

and another with \( + \leftrightarrow - \). This correction does not occur if the multiparticle state has a double occupancy at a particular rapidity \( \theta \) (a state of the form \( |\ldots, \theta, \theta, \ldots, -\rangle \)), because then the intermediate state \( |\ldots, \theta, \theta, \ldots, -\rangle \) has two particles in the same state, which is not allowed by the Bethe equations. This gives rise to the additional densities in the second term of (27), whose equilibrium value is

\[ \mathcal{P}_+(\theta)(1 - f_-(\theta)) + \mathcal{P}_-(\theta)(1 - f_+(\theta)). \]

Of course, using the usual arguments (see the appendix) the fluctuations in this term are suppressed by a factor of \( 1/L \), so we can replace it with the product of the individual equilibrium values, as in (27). Since fluctuations of densities per unit length are of order \( O(1/L) \), both terms in (27) are of order \( O(1) \).

\( Q \) acts diagonally on multi-particle states as a sum of one particle operators. In the presence of the impurity, \( Q \) acts again as a sum of one-particle operators, albeit non-diagonally. For each rapidity, there are two possible states \( |\theta \rangle \) and \( |\theta \rangle \), and in this subspace \( Q \) acts as a two-by-two matrix. This matrix can alternatively be represented by introducing fictitious fermion creation and annihilation operators as

\[ \begin{bmatrix} a_+(\theta)a_+(\theta) - a_-(\theta)a_-\theta \rangle & a_\pm(\theta)S_{++}\rangle \rangle \end{bmatrix} \]

\[ a_+(\theta)a_\pm(\theta)S_{++}\rangle \rangle = a_-\theta a_\pm(\theta)S_{++}\rangle \rangle - a_\pm(\theta)a_\pm(\theta)S_{++}\rangle \rangle \]

We can thus represent the DC current as an integral over rapidities with (29) as the integrand. This formally coincides with the free-electron calculation in \cite{13}. There are however major differences since populations at different rapidities are correlated. For completeness, let us briefly rederive (27) in this approach. Since we neglect the \( (O(1/L)) \) effect of the impurity on the densities, averages have to be computed using the asymptotic states and the Bethe equations for the bulk system. One then checks that for any such states, \( \langle a_\pm(\theta)a_\pm(\theta)\rangle = 0 \) and \( \langle a_\pm(\theta)a_\pm(\theta)a_\pm(\theta)a_\pm(\theta)\rangle = 0 \) unless \( \theta = \theta' \). The first term of (27) follows right away. For the second term, we need also to observe that \( \langle a_\pm(\theta)a_\pm(\theta)a_\pm(\theta)a_\pm(\theta)\rangle = \langle a_\pm(\theta)a_\pm(\theta)\rangle \langle a_\pm(\theta)a_\pm(\theta)\rangle + O(1/L) \), and the \( P_+(1 - f_-) \) and \( P_-(1 - f_+) \) follow for the coincident term.

We finally discuss how to compute (27) numerically. The second term is straightforward; the densities follow from the TBA equations (4) and the relation (15). There is a trick to make the numerics even easier; as indicated in sect. 4.6 below, this piece can be simply written as a derivative of the current. The first term is much more difficult. Extracting \( D(\theta, \theta') \) from (4) is difficult because it would require inverting large matrices and taking their derivatives numerically. Taking functional derivatives to determine \( D(\theta, \theta') \) directly from (7) is also difficult. However, if we choose the rapidity-dependent chemical potentials to be

\[ \mu_+(\theta) = -\mu_-(\theta) = \frac{V}{2T} + x|S_{++}(\theta)|^2, \]

this defines a partition function \( Z(x) \). Then

\[ L \int \int D(\theta, \theta') T(\theta) T(\theta') d\theta d\theta' = -\frac{1}{T} \left. \frac{\partial^2 F}{\partial x^2} \right|_{x=0} \]

Using this method, we calculated the noise explicitly at \( g = 1/3 \). Some curves are given in fig. 1. A more physical way to express the impurity coupling is as \( \theta_T = \ln[T_B/T] \), where \( T_B \) is a crossover parameter analogous to the Kondo temperature. (This definition corresponds to setting the arbitrary scale \( m = T_0 \).) The noise \( \langle \Delta I \rangle^2 \) is then a function of three variables \( V, T \) and \( T_B/T \). Since \( \langle \Delta I \rangle^2/T \) is dimensionless, it is a function of two variables, say \( V/T \) and \( T_B/T \). In fig. 1 each curve is a function of \( T_B/T \) at a fixed value of \( V/T \). We notice that as \( V/T \) is increased, a peak develops, similarly to the conductance. However,
IV. VARIOUS LIMITS

We now consider various limits of (27) to check our expressions and to investigate further physical aspects of the noise in the interacting system. A first trivial limit is of course the case with no impurity, where $T = 1$ and the results of section 3 reduce to those of section 2.

A. $T = 0$

At vanishing temperature, there are no fluctuations, so $D(\theta, \theta') = 0$: the noise is pure shot noise. For positive voltage, $f_+ = 0$, so the noise reads simply

$$\langle (\Delta I)^2 / T \rangle = \int \overline{\mathbf{T}}_+(\theta) [\mathbf{T}(\theta) - \mathbf{T}(\theta')^2] d\theta,$$

in agreement with (25). There it is also shown that a simple fluctuation-dissipation-type relation relates the noise to the current. Unfortunately, we have not been able to find a finite-temperature analog of this relation.

B. $V = 0$

When the voltage $V = 0$, we show that the charge fluctuations $D(\theta, \theta')$ are diagonal and find that the Johnson-Nyquist formula holds. We first observe that due to the symmetry of the kernels $\Phi_{ij} = \Phi_{ji}$, the Bethe equations (4) and the TBA equations (5) yield

$$\overline{\pi}_+ = \overline{\pi}_- \equiv \overline{\pi} \quad \overline{T}_+ = \overline{T}_- \equiv \overline{T}$$

at any voltage. Similarly, when $\mu_+(\theta) = -\mu_-(\theta) \equiv \mu(\theta)$, we have

$$\overline{T}_+ [\mu(\theta)] = \overline{T}_- [-\mu(\theta)],$$

from which it follows that

$$\overline{\pi}[\mu(\theta)] = \overline{\pi}[-\mu(\theta)] \quad \overline{T}[\mu(\theta)] = \overline{T}[-\mu(\theta)].$$

(31)

We see from section 2 that $D(\theta, \theta')$ can be written as the functional derivative

$$D(\theta, \theta')|_{V=0} = \frac{2}{L} \int_{\mathbb{R}} \overline{\pi}_i \overline{f}_j (1 - \overline{T}) \delta(\theta - \theta').$$

(32)

Therefore, the charge fluctuations are diagonal and free in the absence of an applied voltage. This fact also follows easily from the symmetry of the TBA equations under $+ \leftrightarrow -$ when $V = 0$, combined with (5). One then finds

$$\langle (\Delta I)^2 / T \rangle |_{V=0} = 2 \int \overline{\pi}(1 - \overline{T}) d\theta = 2TG,$$

(33)

where we used the linear-response result for the conductance $G$. Thus in limit $V = 0$ we recover the Johnson-Nyquist formula. This is an extremely non-trivial check on our result, because the conductance here depends on the impurity coupling and temperature.

C. Free fermions

When $g = 1/2$, the Luttinger liquid is equivalent to free fermions. The bulk $S$ matrices are unity, so the kernels $\Phi_{ij} = 0$ and the pseudoenergy $T \epsilon_i(\theta) = m_i e^\theta$. Then $K_{ij}(\theta, \theta') = -(1/f_i) \delta_{ij} \delta(\theta - \theta')$ is easy to invert, yielding

$$D(\theta, \theta') = \frac{1}{L} [\overline{T}_+ (1 - \overline{T}_+) + \overline{T}_- (1 - \overline{T}_-)] \delta(\theta - \theta')$$

and using $n_+ = n_- = n$, we have

$$\langle (\Delta I)^2 / T \rangle = \int \overline{\pi} \left\{ [\overline{T}_+ (1 - \overline{T}_-) + \overline{T}_- (1 - \overline{T}_+)] Tight. $$

$$\left. - (\overline{T}_+ - \overline{T}_-)^2 T^2 \right\} d\theta$$
in agreement with the known results.

This limit illustrates the importance of the fluctuations. We note that if one were to use this formula for $g \neq 1/2$ as an approximation for the correct formula (27), one obtains an answer larger than the correct one, even if we use the appropriate interacting distribution functions derived from the TBA. For example, as $\mathcal{T} \to 1$, one obtains (20) instead of the correct $2g\mathcal{T}$. The difference between the two is substantial for $V \neq 0$, especially when $g$ is small.

D. Strong-backscattering limit

From (22) we see that for $\theta \ll \theta_B$, the transmission amplitude $\mathcal{T} \to 0$. Thus the limit $\theta_B \to \infty$ is the strong backscattering limit, where the impurity splits the system in two. In the expression (27) for the noise, in this limit we can neglect $\mathcal{T}^2$ compared to $\mathcal{T}$, except for very energetic particles with $\theta \approx \theta_B$. However, for $\theta \to \infty$ we have $\mathcal{T} \to 0$ and $\mathcal{F} \to 0$, and the contribution of this region to the integral is negligible. We can therefore approximate (27) in this limit by neglecting the $\mathcal{T}^2$ terms entirely:

$$\langle \Delta I^2 \rangle \approx \int \pi [\mathcal{F}_+(1 - \mathcal{F}_-) + \mathcal{F}_-(1 - \mathcal{F}_+)] \mathcal{F} d\theta.$$ 

By simple algebra one finds

$$f_+(1 - f_-) + f_+(1 - f_-) = \coth \left( \frac{V}{2\mathcal{T}} \right) (f_+ - f_-)$$

Hence

$$\langle \Delta I^2 \rangle \approx \coth \left( \frac{V}{2\mathcal{T}} \right) \int \pi (\mathcal{F}_+ - \mathcal{F}_-) \mathcal{T} = \coth \left( \frac{V}{2\mathcal{T}} \right) \mathcal{T}.$$ 

This is the noise coming from the tunneling of uncorrelated electrons, a result expected on physical grounds.

E. Generalized-fluctuation dissipation results

By the same arguments as in section 2, the conductance $G = d\mathcal{I}/dV$ can be written as

$$G = \frac{L}{2T} \int D(\theta, \theta') \mathcal{T}(\theta)d\theta d\theta'.$$ 

(35)

The voltage difference $v$ between the two channels is given by

$$v = \frac{1}{g}(I_{\text{max}} - I),$$

where $I_{\text{max}}$ is current with no impurity. Since the noise in $v$ is proportional to the noise in the reflected (as opposed to transmitted) current, it is therefore given by the same arguments of section 3 with the transmission amplitude $\mathcal{T}$ replaced with the reflection amplitude $1 - \mathcal{T}$. Therefore

$$\langle \Delta v^2 \rangle = L \int \mathcal{T}(\theta, \theta') \left(1 - \mathcal{T}(\theta)(1 - \mathcal{T}(\theta'))d\theta d\theta' \right.$$ 

$$+ \int \left[\mathcal{T}_{-}(\theta)(1 - \mathcal{T}_{-}(\theta)) + \mathcal{T}_{+}(\theta)(1 - \mathcal{T}_{+}(\theta)) \right] \mathcal{T}(\theta)(1 - \mathcal{T}(\theta))d\theta,$$ 

(36)

By simple algebra, and using (35), we derive the following identity

$$\langle \Delta I^2 \rangle = g^2\langle \Delta v^2 \rangle + 2T(2G - g).$$ 

(37)

This is the zero-frequency limit of the general relation between current and voltage fluctuations, equation (3) of [1].

F. Weak-backscattering limit

The weak-backscattering limit is given by $\theta_B \to \infty$. This is not as easy to treat as the strong backscattering limit, because the fluctuations for very-low-energy particles are not negligible, preventing a naive expansion in $1 - \mathcal{T}$. We therefore have not succeeded in checking the weak backscattering limit analytically. Some steps can be accomplished however. We can evaluate the behavior of the second term in (27) using the identity

$$\mathcal{T}(1 - \mathcal{T}) = \frac{g}{2(g - 1)} \frac{\partial}{\partial \theta_B} \mathcal{T}.$$ 

Therefore

$$\int \pi [\mathcal{F}_+(1 - \mathcal{F}_-) + \mathcal{F}_-(1 - \mathcal{F}_+)] \mathcal{T}(1 - \mathcal{T})d\theta$$ 

$$= \frac{g}{2(g - 1)} \coth \left( \frac{V}{2T} \right) \int \pi (\mathcal{F}_+ - \mathcal{F}_-) \frac{\partial}{\partial \theta_B} \mathcal{T}$$ 

$$= \frac{g}{2(g - 1)} \coth \left( \frac{V}{2T} \right) \frac{\partial}{\partial \theta_B}. \quad (38)$$

In the weak backscattering limit, the current can be expanded in a power series $\mathcal{T} = \mathcal{T}_{\text{max}} + c(V, g, T)e^{2(1 - g)\theta_B}$ (where $\mathcal{T}_{\text{max}} = gV$), so

$$\frac{\partial \mathcal{T}}{\partial \theta_B} = 2(1 - g)(\mathcal{T} - \mathcal{T}_{\text{max}})$$

and

$$\langle \Delta I^2 \rangle \approx g \coth(V/(2T))(\mathcal{T}_{\text{max}} - \mathcal{T}) + 2T(2G - g)$$ 

$$+ \int \int D(\theta, \theta')(1 - \mathcal{T}(\theta))(1 - \mathcal{T}(\theta'))d\theta d\theta'.$$

On the other hand, in this limit physical arguments say that the noise comes from the uncorrelated tunneling of Laughlin quasiparticles of charge $g$, yielding

$$\langle \Delta I^2 \rangle \approx g \coth(gV/(2T))\mathcal{T}_{\text{max}} - \mathcal{T} + 2T(2G - g).$$
This will hold if and only if

\[ L \int \int D(\theta, \theta')(1 - T(\theta))(1 - T(\theta'))d\theta d\theta' \approx g \left[ \coth\left(\frac{gV}{2T}\right) - \coth\left(\frac{V}{2T}\right) \right] (T_{\text{max}} - T). \]  

(39)

This can be checked explicitly in the case \( g = 1/2 \), but in general we need to resort to a numerical check. For \( g = 1/3 \) we computed this integral by expressing it as the derivative of \( Z(x) \), as discussed at the end of section 3. We found \( Z(x) \) by numerically solving the TBA equations, and took the derivatives numerically. We find that [29] is indeed satisfied for any voltage as one takes \( \theta_B \to \infty \).

V. CONCLUSION

We have presented in this paper an exact computation of the non-equilibrium noise in a Luttinger liquid with a single impurity. The approach is somewhat similar to the one for free electrons because the model is integrable. However, the quasiparticles of this integrable model are not free, and the computation requires as a key ingredient the equilibrium fluctuations of densities, which are not diagonal and are quite complicated.

The computation of AC properties appears more difficult. In addition to the correlations between densities at different rapidities induced by the Bethe equations, another key ingredient to take into account is that local physical operators are infinite sums of multiparticle operators with complicated matrix elements (form factors). We hope however that a computation similar to [28] is possible.

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APPENDIX A: DENSITY FLUCTUATIONS FROM THE THERMODYNAMIC BETHE ANSÄTZ

The energy per unit length is given simply by summing up the individual energies \( e_i(\theta) \) of the particles:

\[ E = \int e_i(\theta) P_i(\theta) d\theta. \]

In order to compute the entropy, we utilize the fact that solutions of the Bethe equations allow only one quasiparticle per level, so that the filling fractions obey \( 0 \leq f_i \leq 1 \). The number of states in the interval between \( \theta \) and \( \theta + d\theta \) is then

\[ \frac{(n_i(\theta)d\theta)!}{(P_i(\theta)d\theta)!(n_i(\theta) - P_i(\theta))d\theta)!}. \]

Stirling’s formula gives the entropy

\[ S = \sum_i \int d\theta \left[ n_i \ln n_i - P_i \ln P_i - (n_i - P_i) \ln(n_i - P_i) \right]. \]

A given configuration is completely characterized by the knowledge of the densities \( \{P_i(\theta)\} \), or by the knowledge of the pseudoenergies \( \{\epsilon_i(\theta)\} \) defined in [30]. These sets are individually complete because the Bethe equation [31] gives the \( n_i \) in terms of the \( P_i \), and so knowing the \( \epsilon_i \) determines the \( P_i \) and vice versa. Either set can be used as convenient variables for the functional integration. In what follows we keep dependent variables in intermediate computations for notational simplicity.

The various quantities are obtained by a functional integration with the Boltzmann weight \( e^{-FL/T} \), where \( F = E - TS \). As discussed in sect. 2, the full action includes a chemical potential, and it can be written as

\[ F = \sum_i \int (e_i - \mu_i) P_i \; d\theta - T \sum_i \int P_i \left[ \ln(1 + e^{\epsilon_i - \mu_i}) + e^{\epsilon_i - \mu_i} \ln(1 + e^{-\epsilon_i + \mu_i}) \right] d\theta, \]

where we generalize the customary situation by allowing rapidity-dependent chemical potentials \( \mu_i(\theta) \).

The equilibrium values \( \overline{\epsilon_i} \) or \( \overline{P_i} \) minimize the action [32] subject to the constraint of the Bethe equations [33]. The first-order variation gives

\[ \Delta F = \sum_i \int \epsilon_i \Delta P_i - \sum_i \int P_i e^{\epsilon_i - \mu_i} \ln(1 + e^{-\epsilon_i + \mu_i}) \Delta \epsilon_i \]

\[ - T \sum_i \int \left[ \ln(1 + e^{\epsilon_i - \mu_i}) + e^{\epsilon_i - \mu_i} \ln(1 + e^{-\epsilon_i + \mu_i}) \right] \Delta P_i. \]

Recall that the \( \Delta P_i \) are given in terms of \( \Delta \epsilon_i \) via the Bethe equations. Requiring \( \Delta F = 0 \) gives the TBA equations [34] with the equilibrium free energy [35].

To compute the fluctuations, we expand this action to second order around the saddle point values (denoted with a bar). Writing \( \Delta^{(2)} P_i \) for the variation of \( P_i \) induced by first and second order variations in \( \epsilon \) we get

\[ \Delta^{(2)} F/T = \sum_i \int e_i \Delta^{(2)} P_i / T \]

\[ - \sum_i \int P_i e^{\epsilon_i - \mu_i} \ln(1 + e^{-\epsilon_i + \mu_i}) \Delta \epsilon_i \]

\[ - \sum_i \int \Delta^{(2)} P_i \left[ \ln(1 + e^{\epsilon_i - \mu_i}) + e^{\epsilon_i - \mu_i} \ln(1 + e^{-\epsilon_i + \mu_i}) \right] \]

\[ - \sum_i \int P_i e^{\epsilon_i - \mu_i} \left[ \ln(1 + e^{-\epsilon_i + \mu_i}) - \frac{1}{1 + e^{\epsilon_i - \mu_i}} \right] \left( \Delta \epsilon_i \right)^2. \]
The Bethe ansatz equations (1) yield 
\[ e^{-\tau_i + \mu_i} \Delta_i^{(2)} (n_i - P_i) = \Delta_i^{(2)} P_i + \overline{\Phi}_i \left[ \Delta_i + \frac{1}{2} (\Delta_i)^2 \right]. \]

Also, from the Bethe equations (2), we have
\[ \Delta_i^{(2)} n_i = \sum_j \Phi_{ij} * \Delta_i^{(2)} P_j. \]

It follows that \( \Delta_i^{(2)} P_i \) depends only on the combination \( \Delta_i + \frac{1}{2} (\Delta_i)^2 \). Because the first-order variation \( \Delta F = 0 \), the linear terms in \( \Delta \) and many others cancel out, leaving only
\[ \Delta_i^{(2)} F = \frac{1}{2} \sum_i \int \frac{\overline{\Phi}_i}{1 + e^{-\tau_i + \mu_i}} (\Delta_i)^2. \]

(A similar result appears in the original paper by Yang and Yang\textsuperscript{14} on bosons with repulsive delta interactions). Hence the fluctuations of \( \epsilon \) are diagonal:
\[ \langle \Delta \epsilon_i(\theta) \Delta \epsilon_j(\theta') \rangle = \frac{1}{L} \frac{e^{-\tau_i + \mu_i}}{1 + e^{-\tau_i + \mu_i}} \delta_{ij} \delta(\theta - \theta'). \] (A1)

In terms of the filling fractions \( f_i \), we can recast this as
\[ \langle \pi_i(\theta) \pi_j(\theta') \rangle = \frac{1}{L} \frac{e^{-\tau_i + \mu_i}}{1 + e^{-\tau_i + \mu_i}} \delta_{ij} \delta(\theta - \theta'). \]

This is the same result as one would have for a free theory: fluctuations restricted to the filling fractions are free. The difference with a free theory is that we have correlated fluctuations \( \langle \Delta n_i \Delta n_j \rangle \) as well as \( \langle \Delta n_i \Delta f_j \rangle \). The density-density fluctuation \( D(\theta, \theta') \) can be obtained from (A1) by expressing \( \Delta \epsilon_i \) in terms of the \( \Delta P_i \). The Bethe ansatz equations (1) yield
\[ (1 + e^{-\tau_i - \mu_i}) \Delta P_i + e^{-\tau_i + \mu_i} \overline{\Phi}_i \Delta \epsilon_i = \sum_j \Phi_{ij} * \Delta P_j, \]

which can be recast as
\[ e^{-\tau_i + \mu_i} \overline{\Phi}_i \Delta \epsilon_i = \sum_j K_{ij}(\theta, \alpha) \Delta P_j(\alpha) d\alpha, \]

where \( K_{ij} \) is defined in (1). Plugging this into (A1) gives an explicit form of \( D(\theta, \theta') \) in terms of an integral of \( K^{-1} \):
\[ \langle \Delta P_i(\theta) \Delta P_j(\theta') \rangle = \frac{1}{L} \sum_k \int d\alpha K_{ik}^{-1}(\theta, \alpha) K_{jk}^{-1}(\theta', \alpha) e^{\tau_i(\alpha) - \mu_i} \pi_k(\alpha). \]

This equation can be simplified. First note that when \( \mu \) is independent of \( \alpha \), (1) yields
\[ e^{\tau_i(\alpha)} \pi_k(\alpha) = T \frac{\partial e^{\tau_i(\alpha)}}{\partial \alpha}. \]

Then it follows from the definition of \( K \) that
\[ \frac{\partial e^{\tau_i(\alpha)}}{\partial \alpha} \delta(\alpha - \alpha') \delta_{km} = - \left( \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha'} \right) K_{km}(\alpha, \alpha'). \]

Using these two identities and the definition of the inverse of \( K \) yields the relation (1) relating the density fluctuations to \( K^{-1} \).

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