The Decay of Multiqudit Entanglement

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We investigate the decay of entanglement of a generalized $N$-qudit GHZ state with each qudit passing through independently in a quantum noisy channel. By studying the time at which the entanglement completely vanishes and the time at which the entanglement becomes arbitrarily small, we try to find how the robustness of entanglement is influenced by dimension $d$ and the number of particles $N$.

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I. INTRODUCTION

Quantum entanglement, as the most non-classical phenomenon in quantum mechanics, lies in the central position of quantum information theory and has been identified as a key resource in many applications such as quantum teleportation, quantum key distribution and quantum computation [1,2]. Aolita et al. [8] proposed that the time at which the entanglement becomes arbitrarily small is a better quantity characterizing the robustness of entanglement than the ESD time and they found that this time is inversely proportional to $N^2$. A more accurate quantification of the bipartite entanglement of $(N - n)|n$ bipartition which first disappears is that corresponding to sudden death (ESD) in Refs. [2, 6, 10, 11]. The entanglement which disappears in finite time which is named as entanglement sudden death (ESD) in Refs. [2, 4].

II. QUANTUM CHANNELS

First we would like to introduce two $d \times d$ matrices extremely useful in constructing our quantum channels. They are defined as $X|i⟩ = |i + 1⟩(mod d)$ and $Z|i⟩ = ω^i|i⟩$, where $i = 0, ..., d - 1$ and $ω = \exp(2πi/d)$. One can easily see that when $d = 2$, they are just Pauli-sigma $x$ and Pauli-sigma $z$ matrix respectively. Now we will construct three operator transformations through $X$ and $Z$:

$$\mathcal{E}_1(A) = (1-p)A + \frac{p}{d^2} \sum_{ij=0}^{d-1} X^i Z^j A Z^j X^i, \quad \mathcal{E}_2(A) = (1-p)A + \frac{p}{d} \sum_{i=0}^{d-1} Z_i^i A Z_i^i.$$  

where $p \in [0, 1]$ and $A$ is an arbitrary operator on $d$-dimensional Hilbert spaces. When $A$ is a density matrix, we can see $\mathcal{E}_i, i = 1, 2$ as three quantum channels. Through calculation, it can be shown that for an input density matrix $ρ$, $\mathcal{E}_1(ρ) = (1-p)ρ + \frac{p}{d} I$ and $\mathcal{E}_2(ρ) = (1-p)ρ + \frac{p}{d} \sum_{k=0}^{d-1} ρ_{kk} |k⟩⟨k|$, where $I$ is a $d \times d$ identity matrix. It’s obvious to see that in fact $\mathcal{E}_1$ is a depolarizing channel and $\mathcal{E}_2$ is a phase damping channel.

III. THE EVOLUTION OF ENTANGLEMENT

A generalized $N$-qudit GHZ state can be written as $|Ψ_d⟩ = \sum_{i=0}^{d-1} α_i |i⟩^⊗N$, where $α_i$ is a complex number.

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and $\sum_{i=1}^{d-1} |\alpha_i|^2 = 1$. Here we want to show how does the bipartite entanglement corresponding to $(N-n)/n$ bipartition of it evolve under the influence of the three quantum channels constructed above. In this article we adopt negativity as the measure of entanglement [3].

When each qudit is exposed to a depolarizing channel, we can calculate $\left( \bigotimes_{i=1}^{N} E_{1,i} \right) (|\Psi_d\rangle\langle\Psi_d|)$ and denote it as $\rho_1(p) = \frac{1}{N!} \sum_{i,j=0}^{N!} \sum_{k=0}^{N} \rho_{i,j} \left( \frac{1}{d} \sum_{k=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \otimes \left( \sum_{j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \right) \right) + (1-p)^N \sum_{i,j=0}^{N!} \sum_{k=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \otimes \left( \sum_{j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \right) \right) \right) \right)$, where $E_{1,i}$ means all possible permutations. Partial transposing the part of $n$ particles of $\rho_1(p)$ and noting that the first term of $\rho_1(p)$ is diagonal so it will not be changed after partial transpose, we have $\rho_1(p) = \sum_{i=0}^{d-1} \sum_{k=0}^{N} \sum_{i,j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \otimes \left( \sum_{j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \right) \right) + (1-p)^N \sum_{i,j=0}^{N} \sum_{k=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \otimes \left( \sum_{j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \right) \right) \right) \right) \right) \right)$.

We note that for any $\alpha_i$ and $\alpha_j$, which straightforwardly leads to $\rho_1(p) = \sum_{i,j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \otimes \left( \sum_{j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right)$.

There are $\frac{d(d-1)}{2}$ eigenvalues of $\rho_1(p)$ that can be negative and are determined by the smaller eigenvalues of the following $\frac{d(d-1)}{2} \times 2$ matrices

$$
\begin{pmatrix}
\lambda_{ij}^d & \alpha_i\alpha_j^*(1-p)N \\
\alpha_i^*\alpha_j(1-p)N & \lambda_{ij}^d
\end{pmatrix},
$$

where $i < j$ and $\lambda_{ij}^d = |\alpha_i|^2 \left( \frac{d}{d-1} \right) (1-\frac{d}{d-1}p)^N + |\alpha_j|^2 \left( \frac{d}{d-1} \right) (1-\frac{d}{d-1}p)^N$.

One can easily derive $\mu_{ij}^d = \xi_{ij}^d - \sqrt{(\xi_{ij}^d)^2 - \eta_{ij}^d}$, where $\xi_{ij}^d = \frac{1}{2}(\lambda^d_{ij} N - |\alpha_i|^2 N) - |\alpha_i|^2 (1-p)N$. Then we can define $\mathcal{N}_{n} = \max\{-\mu_{ij}^d, 0\}$ and the negativity can be obtained by summation $\mathcal{N}_{n} = \sum_{i<j=0}^{N} \mathcal{N}_n$.

Similar to the case of qubit [8], $\mathcal{N}_{1} \leq \mathcal{N}_{2} \leq \ldots \leq \mathcal{N}_{N/2}$ for a given pair of $\alpha_i$ and $\alpha_j$, which straightforwardly leads to $\mathcal{N}_1 \leq \mathcal{N}_2 \leq \ldots \leq \mathcal{N}_{N/2}$. So the bipartite entanglement corresponding to the most balanced partition still disappears last whereas the one corresponding to the least balanced partition disappears first. Let $\rho_n = \max_{i<j} \mu_{ij}^d$, where $\mu_{ij}^d$ is the solution of the equation $\mu_{ij}^d = 0$, then $\mathcal{N}_n(\rho_n) = 0$. Now we investigate dynamical property of $\mathcal{N}_{N/2}$, which disappears last. First we want to know when does it completely vanish. By solving $\mu_{ij}^d = 0$, it’s easy to find that

$$
p_{ij}^d = \frac{2|\alpha_i\alpha_j|^\frac{d}{d-1}}{2|\alpha_i|^2 + |\alpha_j|^2 + 2\sqrt{4|\alpha_i|^2 + |\alpha_j|^2}}
$$

which coincides with Eq.(6) of Ref.[3] when $d = 2$. After $p$ reaches the value $p_{N/2} = \max_{i<j} \mu_{ij}^d$, $N_{N/2} = 0$. It’s obvious that $p_{ij}^d < 1$ so $p_{N/2} < 1$ and so ESD happens. Before $p = p_{N/2}$, $N_{N/2} > 0$ and $\rho_1(p)$ must be still entangled. Second we want to know when does $\mathcal{N}_{N/2}$ become arbitrarily small, which is practically important because before the entanglement is zero, it can be so small that it’s useless as a resource. Like Ref.[3], we suppose $\epsilon$ is an arbitrarily small positive number and define a critical probability $p_{ij}^d$ such that $\mu_{ij}^d(p_{ij}^d) = \epsilon \mu_{ij}^d(0)$, which leads to an equation

$$
(\alpha_i^2 + \alpha_j^2) \left( \frac{p_{ij}^d}{d} \right) = (1 - \frac{d}{d-1}p_{ij}^d)^{\frac{d}{d-1}} - |\alpha_i\alpha_j|(1 - p_{ij}^d)^N = -\epsilon|\alpha_i\alpha_j|.
$$

When $p \geq p_e = \max_{i<j} p_{ij}^d$, we can think $\mathcal{N}_{N/2}$ is too small to be used as a resource.

As the second case let’s consider the situation where each qudit is exposed to a phase damping channel. Similar to the case of depolarizing channel, it can be obtained that $\rho_2(p) = \left( \bigotimes_{i=1}^{N} E_2 \right) (|\Psi_d\rangle\langle\Psi_d|) = \sum_{i=0}^{d-1} \sum_{j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \otimes \left( \sum_{j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \right) \right) \right) \right) \right) \right)$, where $\rho_2(p)$ is the partial transposing of it is $\rho_2(p) = \sum_{i=0}^{d-1} \sum_{j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \otimes \left( \sum_{j=0}^{N} \rho_{i,j} \left( (i)\langle j|^{(N-k)} \right) \right) \right) \right) \right)$.

IV. ROBUSTNESS OF ENTANGLEMENT

First we fix $d$ to study the relation between the entanglement robustness and $N$. For the depolarizing channel, Eq.(2) tells us that the ESD time of $\mathcal{N}_{N/2}$ grows with $N$ (FIG.1). Noting that $\lim_{N\to\infty} p_{ij}^d = 2d/(2d+1+\sqrt{5})$ which is independent of $i$ and $j$, we find $p_{N/2} = \max_{i<j} p_{ij}^d$ becomes closer to this value while $N$
for a fixed value of $d$.

Here we choose $\alpha = 1/\sqrt{d}$ for convenience and $N = 4$ (black cubic), $N = 6$ (red circle) and $N = 8$ (blue triangle) respectively. (a): the behavior of $p_{N/2}$ for large $d$. (b): the behavior of $p_{N/2}$ for small $d$. It’s easy to see that $p_{N/2}$ grows with $N$ for a fixed value of $d$ saturating to $2d/(2d + 1 + \sqrt{5})$ and it also grows with $d$ and saturates to 1 for any fixed $N$.

grows (FIG 1). Moreover, when $N$ is big enough, considering $p^{ij}_N$ is very small, the first term in the LHS of Eq. (4) can be omitted, leading to $p_\epsilon \sim -(1/N) \ln \epsilon$ (FIG.2). For the phase damping channel, no matter what $N$ is, no ESD happens. However, $p_\epsilon \sim -(1/N) \ln \epsilon$ still holds. We can see that so long as $d$ is fixed, the scaling relation between $p_\epsilon$ and $N$ is always the same with that in Ref. [8], where $d = 2$.

Then we fix $N$ to study the relation between the entanglement robustness and $d$. For the depolarizing channel, from Eqs. (2) and (3) one can find that when the coefficients $\alpha_i$s are given (it’s not necessary to set all $\alpha_i$s the same), both $p^{ij}_{N/2}$ (also $p_{N/2}$) and $p^{ij}_N$ (also $p_\epsilon$) grow with $d$, meaning the entanglement is more robust (FIG.1 and 2). Moreover, when $d$ is large enough, we have $\lim_{d \to \infty} p_{N/2} = 1$ (FIG.1) and $\lim_{d \to \infty} p_\epsilon = 1 - \epsilon^{1/N}$ (FIG.2). For the phase damping channel, when $N$ is fixed, it’s obvious that both the ESD time (infinity) and $p^{ij} = 1 - \epsilon^{1/N}$ are independent of $d$.

One question arises that for the depolarizing channel, what’s the dynamical behavior of the bipartite entanglement corresponding to the least balanced partition that vanishes first? As we know, in qubit case ($d=2$) it vanishes earlier when $N$ grows. Now our numerical calculation shows its relation with $N$ and $d$ (FIG.3).

FIG. 1: (color online) For the depolarizing channel, we demonstrate how can $p_{N/2}$ be influenced by both $N$ and $d$. Here we choose $\alpha = 1/\sqrt{d}$ for convenience and $N = 4$ (black cubic), $N = 6$ (red circle) and $N = 8$ (blue triangle) respectively. (a): the behavior of $p_{N/2}$ for large $d$. (b): the behavior of $p_{N/2}$ for small $d$. It can be seen that when $d$ is fixed, $p_\epsilon$ decreases with $N$ while when $N$ is fixed, it grows with $d$ to a saturated value $1 - \epsilon^{1/N}$.

FIG. 2: (color online) For the depolarizing channel, we show the relation of $p_\epsilon$ with $N$ and $d$. Here we choose $\alpha = 1/\sqrt{d}$ for convenience, $\epsilon = 0.01$ and $N = 4$ (black cubic), $N = 6$ (red circle) and $N = 8$ (blue triangle) respectively. (a): the behavior of $p_\epsilon$ for large $d$. (b): the behavior of $p_\epsilon$ for small $d$. It can be seen that when $d$ is fixed, $p_\epsilon$ decreases with $N$ while when $N$ is fixed, it grows with $d$.

FIG. 3: (color online) For the depolarizing channel, we show the relation of $p_\epsilon$ with $d$ and $N$. Here we choose $\alpha = 1/\sqrt{d}$ for convenience and $N = 4$ (black cubic), $N = 6$ (red circle) and $N = 8$ (blue triangle) respectively. It can be seen that when $d$ is fixed, $p_\epsilon$ decreases with $N$ while when $N$ is fixed, it grows with $d$. 

One could expect that the dependence of entanglement robustness on $N$ and $d$ would be similar, since in both cases we are increasing the dimension of the Hilbert space of each of the bi-partitions. But in fact their influences on the entanglement robustness are different. Roughly speaking, entanglement increases with $d$ and thus it is more robust, while entanglement becomes more fragile with $N$ since more particle are entangled together and it becomes easier to be destroyed. We take the depolarizing channel as an example, for which according to the discussion above, $p_c$ increases with $d$ while it decreases with $N$. This can be explained as follows. If $d$ is fixed, when we increase $N$, we increase the components of the state. Considering the depolarizing channel acts locally on every component, the growth of $N$ will make the entanglement more fragile. In the depolarizing channel $E_1(\rho) = (1-p)\rho + \frac{p}{N}1$, $p$ can be regarded as the probability with which $\rho$ is broken by the channel. The probability of the $N$-particle state to be broken by the collective local action of $N$ channels on each particle must grow with $N$ (about $Np$ as a rough estimation), leading that the entanglement becomes less robust. The probability with which the $N$-particle state is not broken can be estimated roughly as $(1-p)^N$ and this probability can also be expressed as $\epsilon$ considering the entanglement decay. Therefore we have $(1-p)^N = \epsilon$, leading to our familiar results. If $N$ is fixed, with the increase of $d$, the influence of the channel on $\rho$ will be more and more negligible, which leads the entanglement becomes robust.

V. SUMMARY

In this brief report, we mainly investigate the dynamical property of entanglement of a generalized $N$-qudit GHZ state under the influence of the depolarizing channel and the phase damping channel. We study the relation of the entanglement robustness with $N$ and $d$. First we consider the ESD time $t_1$ and $t_2$ respectively of the bipartite entanglement corresponding to the most balanced partition ($t_1$) and the bipartite entanglement corresponding to the least balanced partition ($t_2$). When $d$ is fixed, for the depolarizing channel $t_1$ delays when $N$ grows whereas $t_2$ becomes earlier. For the phase damping channel, no ESD happens for any $N$. These results are qualitatively the same with that of $d = 2$. When $N$ is fixed, for the depolarizing channel both $t_1$ and $t_2$ grow with $d$ whereas still no ESD happens for the phase damping channel for any $d$. Next we consider the time at which the bipartite entanglement becomes arbitrarily small ($t_3$). When $d$ is fixed the scaling relation between $p_c$ and $N$ is totally independent of $d$ therefore the same with the result in Ref. [3] for both channels. When $N$ is fixed, for the depolarizing channel $t_3$ grows with $d$ whereas it’s independent of $d$ for the phase damping channel. There are many other multiqudit entangled states such as generalized $N$-qudit W-state and other quantum channels. It is not clear how the behaviors of the entanglement differ in those situations. This is worth being studied further.

VI. ACKNOWLEDGEMENTS

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