Coupled quintessence with double exponential potentials

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March 27, 2014

Abstract

We study flat Friedmann-Robertson-Walker (FRW) models with a perfect fluid matter source and a scalar field non minimally coupled to matter having a double exponential potential. It is shown that the scalar field almost always diverges to infinity. Under conditions on the parameter space, we show that the model is able to give an acceptable cosmological history of our universe, that is, a transient matter era followed by an accelerating future attractor. It is found that only a very weak coupling can lead to viable cosmology. We study in the Einstein frame, the cosmological viability of the asymptotic form of a class of $f(R)$ theories predicting acceleration. The role of the coupling constant is briefly discussed.

1 Introduction

The standard inflationary idea requires that there be a period of slow-roll evolution of a scalar field (the inflaton) during which, its potential energy drives the universe in a quasi-exponential expansion. Besides a cosmological constant, a nearly massless scalar field (quintessence) provides the simplest mechanism to obtain accelerated expansion of the universe within General Relativity. Therefore, scalar fields play a prominent role in the construction of cosmological scenarios aiming to describe the evolution of the early and the present universe.

Earlier investigations in scalar-field cosmology assumed a minimal coupling of the scalar field, (see for example [1] and [2] for models containing both a perfect fluid of ordinary matter and a scalar field with an exponential potential, the so-called “scaling” cosmologies; [3] and references therein for

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scalar-tensor theories with exponential potential,\cite{4} for a phase-space analysis of the qualitative evolution of cosmological models with a scalar field with positive or negative exponential potentials). Inclusion of non minimal coupling increases the mathematical difficulty of the analysis, yet it is important to consider non minimal coupling in scalar field cosmology\cite{5}. As is stressed in\cite{6}, the introduction of non minimal coupling is not a matter of taste; a large number of physical theories predict the presence of a scalar field coupled to matter and we mention a few important examples:

In the string effective action, the dilaton field is generally coupled to matter in the Einstein frame\cite{7}. In scalar-tensor theories of gravity\cite{5}, the action in the Einstein frame takes the form

\[ S = \int d^4x \sqrt{-g} \left\{ R - \left( (\partial \phi)^2 + 2V(\phi) \right) + 2\chi^{-2}L_m(\tilde{g}_{\mu\nu}, \Psi) \right\}, \quad (1) \]

with

\[ \tilde{g}_{\mu\nu} = \chi^{-1}g_{\mu\nu}, \]

where \( \chi = \chi(\phi) \) is the coupling function and matter fields are collectively denoted by \( \Psi \). In particular, for higher order gravity (HOG) theories derived from Lagrangians of the form

\[ f \left( \tilde{R} \right) + 2L_m(\tilde{g}_{\mu\nu}, \Psi), \quad (2) \]

it is well known that under the conformal transformation, \( g_{\mu\nu} = f''(\tilde{R}) \tilde{g}_{\mu\nu} \), the field equations reduce to the Einstein field equations with a scalar field \( \phi \) as an additional matter source. The conformal equivalence can be formally obtained by conformally transforming the Lagrangian (2) and the resulting action becomes\cite{8},

\[ S = \int d^4x \sqrt{-g} \left\{ R - \left( (\partial \phi)^2 + 2V(\phi) \right) + 2e^{-\frac{2}{\beta\phi}}L_m \left( e^{-\frac{2}{\beta\phi}}g_{\mu\nu}, \Psi \right) \right\}. \]

Therefore the Lagrangian of HOG theories is a particular case of the general scalar-tensor Lagrangian with \( \chi(\phi) = e^{\frac{2}{\beta\phi}} \), in equation (1). Non minimally coupling occurs also in models of the so called coupled quintessence\cite{9},\cite{10},

\[ S = \int d^4x \sqrt{-g} \left\{ R - \left( (\partial \phi)^2 + 2V(\phi) \right) + 2L_m(\tilde{g}_{\mu\nu}, \Psi) \right\}, \]

with

\[ \tilde{g}_{\mu\nu} = e^{2\beta\phi}g_{\mu\nu}, \]

where \( \beta \) is a coupling constant. The same form of coupling has been proposed in models of the so called coupled quintessence\cite{11} (see also\cite{12} for a generalization involving a scalar field coupled both to matter and a vector field).

Variation of the action (1) with respect to the metric \( g \) yields the field equations,

\[ G_{\mu\nu} = T_{\mu\nu}(g, \phi) + T_{\mu\nu}^m(g, \Psi), \quad (3) \]
where $T^m_{\mu\nu}$ is the matter energy momentum tensor. The Bianchi identities imply that the total energy-momentum tensor is conserved and therefore there is an energy exchange between the scalar field and ordinary matter. In all the above examples, the conservation of their sum is provided by the equations (compare to [11]),

$$\nabla^\mu T^m_{\mu\nu}(g, \Psi) = QT^m\nabla_\nu \phi, \quad \nabla^\mu T_{\mu\nu}(g, \phi) = -QT^m\nabla_\nu \phi,$$

where $Q := d \ln \chi / d\phi$, depends in general on $\phi$ and $T^m$ is the trace of the matter energy-momentum tensor, i.e., $T^m = g^{\mu\nu}T^m_{\mu\nu}(g, \Psi)$. Variation of $S$ with respect to $\phi$ yields the equation of motion of the scalar field,

$$\Box \phi - \frac{dV}{d\phi} = -QT^m.$$ (4)

In this paper we study the late time evolution of initially expanding flat FRW models, with a scalar field coupled to matter and having a potential of the form

$$V(\phi) = V_1 e^{-\alpha \phi} + V_2 e^{-\beta \phi},$$ (5)

where $\alpha, \beta$ are positive constants and $V_1, V_2$ are constants of arbitrary sign. Without loss of generality, we assume $0 < \alpha < \beta$. For $0 < \alpha = \beta$ the case reduces to a single exponential potential. We also assume that the coupling coefficient is a constant, of order $Q \lesssim 1$. The double exponential potential is usually the asymptotic form of other potentials. For example in Kaluza-Klein theories with $d$ extra dimensions reformulated in the Einstein frame, $\alpha$ and $\beta$ are $\sqrt{2d/(d+2)}$ and $\sqrt{2(d+2)/d}$ respectively, [5]. The physical reason for the choice (5), is that in quintessence models, the dark energy is the energy of a slowly varying scalar field $\phi$ with equation of state $p_\phi = w \rho_\phi$, $w \simeq -1$. In most of the models of dark energy, it is assumed that the cosmological constant is zero and the potential energy, $V(\phi)$, of the scalar field driving the present stage of acceleration, slowly decreases and eventually vanishes as the field approaches the value $\phi = \infty$, [13]. In this case, after a transient accelerating stage, the speed of expansion of the universe decreases and the universe reaches Minkowski regime. Double exponential potentials of the form (5) were investigated in [14, 15]; solutions were obtained in [16] with the ansatz $\dot{\phi} = \lambda H$. A scalar field with a double exponential potential without coupling to matter was investigated in [17]. For exact solutions of a scalar field non coupled to dust with single and double exponential potentials see [18]. Quintessence cosmologies of double exponential potentials in the absence of matter were studied in [19] with the techniques of phase space analysis.

The plan of the paper is as follows. In the next Section we write the field equations for flat FRW models as a constrained four-dimensional dynamical system. Assuming an initially expanding universe, we show that for potentials (5) the scalar field almost always diverges to plus or minus infinity as $t \to \infty$, depending on the signs of $V_1, V_2$. Using expansion-normalized variables, the system is written as a polynomial three-dimensional system. In
Section 3 we study the equilibrium points and analyze the structure of the solutions. It is shown that under conditions on the parameter space, the model is able to give an acceptable cosmological history of our universe: a transient matter era followed by an accelerating future attractor. In particular, if we assume that ordinary matter satisfies plausible energy conditions, i.e., $\gamma \gtrsim 1$, the scale factor during the matter era evolves approximately as $a \sim t^{2/3}$, provided that the coupling constant, $Q$, takes very small values. In Section 4 we examine the asymptotic form of a popular class of $f(R)$ theories predicting acceleration; in the Einstein frame this theory is equivalent to a scalar field with a double exponential potential and we discuss its cosmological viability. Section 5 is a brief discussion on the acceptable range of the coupling constant.

2 Coupled scalar field model

For homogeneous and isotropic flat spacetimes the field equations (3) and (4), (ordinary matter is described by a perfect fluid with equation of state $p = (\gamma - 1)\rho$, where $0 < \gamma < 2$), reduce to the Friedmann equation,

$$3H^2 = \rho + \frac{1}{2}\phi^2 + V(\phi),$$

(6)

the Raychaudhuri equation,

$$\dot{H} = -\frac{1}{2}\phi^2 - \frac{\gamma}{2}\rho,$$

(7)

the equation of motion of the scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \frac{4 - 3\gamma}{2}Q\rho,$$

(8)

and the conservation equation,

$$\dot{\rho} + 3\gamma\rho H = -\frac{4 - 3\gamma}{2}Q\rho\dot{\phi}.$$

(9)

Although there is an energy exchange between the two matter components, it is easy to see that the set, $\rho > 0$, is invariant under the flow of (7)-(9), therefore $\rho$ is nonzero if initially $\rho(t_0)$ is nonzero. We adopt the metric and curvature conventions of [20]. $a(t)$ is the scale factor, an overdot denotes differentiation with respect to time $t$, $H = \dot{a}/a$ and units have been chosen so that $c = 1 = 8\pi G$. Here $V(\phi)$ is the potential energy of the scalar field and $V'(\phi) = dV/d\phi$. As is explained in the last paragraph of the Appendix, the physically interesting cases are $V_1, V_2 > 0$ or $V_1 > 0, V_2 < 0$.

The dynamical system (7)-(9) has for $V_1 > 0, V_2 < 0$, only one finite equilibrium point, $(\phi = \phi_m, \dot{\phi} = 0, \rho = 0, H = \sqrt{V_{\text{max}}/3})$, see Figure 1. It represents de Sitter solutions and is easy to see that it is unstable. The
other asymptotic states of the system correspond to the points at infinity, \( \phi \to \pm \infty \).

For potentials (5) with \( V_1, V_2 > 0 \), it can be shown the global result that, for expanding flat models, \( \phi \to \infty \) as \( t \to \infty \). In fact, the following slightly stronger result holds, which generalizes Proposition 4 in [21].

**Proposition 1** Let \( V \) be a potential function with the following properties:
1. \( V \) is non-negative. 2. \( V' (\phi) < 0 \). 3. If \( A \subseteq \mathbb{R} \) is such that \( V \) is bounded on \( A \), then \( V' \) is bounded on \( A \). Then \( \lim_{t \to +\infty} \dot{\phi} = 0 = \lim_{t \to +\infty} \rho \), and \( \lim_{t \to +\infty} \phi = +\infty \).

**Proof.** Since \( V (\phi) \geq 0 \), it follows from (6) that \( H \) is never zero, thus it cannot change sign. Hence, \( H \) is always non-negative if \( H (t_0) > 0 \). Furthermore, \( H \) is decreasing in view of (7), thus \( H (t) \leq H (t_0) \), for all \( t \geq t_0 \).

We then deduce from (6) that each of the terms \( \rho, \frac{1}{2} \dot{\phi}^2 \) and \( V \) is bounded by \( 3H (t_0)^2 \). Since \( H \) is decreasing, \( \exists \lim_{t \to +\infty} H = \eta \geq 0 \), therefore (7) implies that

\[
\frac{1}{2} \int_{t_0}^{+\infty} \left( \dot{\phi}^2 + \gamma \rho \right) dt = H (t_0) - \eta < +\infty. \tag{10}
\]

In general, if \( f \) is a non-negative function, the convergence of \( \int_{t_0}^{\infty} f (t) \) does not imply that \( \lim_{t \to \infty} f (t) = 0 \), unless the derivative of \( f \) is bounded. In our case and setting \( \lambda = (4 - 3\gamma) Q \),

\[
\frac{d}{dt} \left( \dot{\phi}^2 + \gamma \rho \right) = -6H \dot{\phi}^2 - 2\dot{\phi}V' (\phi) - 3\gamma^2 \rho H + \lambda \left( 1 - \frac{\gamma}{2} \right) \rho \dot{\phi} \\
\leq -2\dot{\phi}V' (\phi) + \lambda \left( 1 - \frac{\gamma}{2} \right) \rho \dot{\phi}.
\]

As we already remarked, \( \dot{\phi} \) and \( \rho \) are bounded; also, by our assumption on \( V, V' (\phi) \) is bounded. We conclude that the derivative of the function

![Diagram](image_url)
Suppose on the contrary that, \( \lim_{\phi \to +\infty} \dot{\phi} = +\infty \), then from (6), \( V \) remains less than \( V_{\max} \) since \( H \) is decreasing. We conclude that \( V(\phi(t)) < V_{\max} \) for all \( t \geq t_0 \), thus \( \dot{\phi} \) cannot pass to the left of \( \phi_m \). In the interval \( (\phi_m, +\infty) \) the potential satisfies the assumptions of the above Proposition and therefore, \( \dot{\phi} \to \infty \) as \( t \to \infty \).

The case \( V_1 > 0, V_2 < 0 \), is more delicate and the asymptotic state depends on the initial conditions. (i) If initially \( \phi_0 > \phi_m \), and \( 3H(t_0)^2 < V_{\max} \), then \( V(\phi) \) remains less than \( V_{\max} \) since \( H \) is decreasing. We conclude that \( V(\phi(t)) < V_{\max} \) for all \( t \geq t_0 \), thus \( \dot{\phi} \) cannot pass to the left of \( \phi_m \). (ii) If initially \( \phi_0 < \phi_m \), and \( \phi_0 \) is larger than the critical value \( \phi_{\text{crit}} > 0 \), which allows for \( \phi \) to pass on the right of \( \phi_m \), then the conclusions of case (i) hold. (iii) Finally, suppose that initially \( \phi_0 < \phi_m \), and \( \phi_0 \) is less than the critical value \( \phi_{\text{crit}} > 0 \), i.e., \( -\infty < \phi_0 < \phi_{\text{crit}} \). From (7), \( H \) is monotonically decreasing and not bounded below from zero, hence eventually \( H \) may change sign. We cannot use the same argument as in Proposition 1 concerning the asymptotic behavior of \( \dot{\phi}(t)^2 \) and \( \rho(t) \), since \( V \) and \( V' \) are not bounded. Suppose, firstly, that \( \lim_{t \to +\infty} H = \eta \), where \( \eta \) is finite. But, an asymptotic state of the form, \( p = \left( H = \eta, \rho = \rho_*, \dot{\phi} = \dot{\phi}_*, \phi = \phi_* \right) \), is impossible, i.e., the point \( p \) cannot be an equilibrium point of the dynamical system (7)-(9) for \( \phi_* < \phi_m \). Although we cannot exclude periodic orbits, or strange attractors as \( \omega \)-limit sets for our system, numerical experiments suggest that, \( H \) diverges to \( -\infty \). If this is the case, it can be shown that \( H \) diverges to \( -\infty \), in a finite time.

Suppose on the contrary that, \( \lim_{t \to +\infty} H = -\infty \). Since \( \gamma < 2 \),

\[
3H^2 = \frac{\dot{\phi}^2}{2} + \rho + V(\phi) < \frac{\dot{\phi}^2 + \gamma \rho}{\gamma} + V(\phi) = \frac{-2\dot{H}}{\gamma} + V(\phi),
\]

hence,

\[
3 < -\frac{2\dot{H}}{\gamma H^2} + \frac{V(\phi)}{H^2}. \tag{11}
\]

Taking limits as \( t \to +\infty \), and since \( V(\phi) \) is bounded from above, \( \lim_{t \to +\infty} V(\phi)/H^2 \leq 0 \). Inequality (11) implies that \( \lim_{t \to +\infty} \left( -\frac{\dot{H}}{H^2} \right) \geq \frac{3\gamma}{2}, \) which is impossible, since \( -\dot{H}/H^2 = d/dt (1/H) \) and \( 1/H \to 0 \). In view of (3), \( \dot{\phi}^2 + \gamma \rho \) also diverges to infinity. Again, an asymptotic state of the form, \( H = -\infty, \dot{\phi}^2 + \gamma \rho = \infty \) and \( \phi = \text{finite} \) is impossible, therefore \( \phi \) diverges to \( -\infty \) in a finite time. The above arguments, supported by numerical investigation, establish the following result, although we were unable to prove it rigorously:

**Proposition 2** Let \( V \) be a \( C^1 \) potential function with the following properties: 1. \( V \) is negative and monotonically increasing for \( \phi < 0 \), with \( \lim_{\phi \to -\infty} V(\phi) = -\infty \). 2. \( V \) has a global maximum at some \( \phi_m > 0 \).

Suppose that the following initial conditions hold: \( H(t_0) > 0, \phi(t_0) < \phi_m, \)
and $-\infty < \dot{\phi}(t_0) < \dot{\phi}_{\text{crit}}$, where $\dot{\phi}_{\text{crit}} > 0$, is the critical value which allows for $\phi$ to pass to the right of $\phi_m$. Then $H$ and $\phi$ diverge to $-\infty$ in a finite time.

This result generalizes previous investigations indicating that negative potentials may drive a flat initially expanding universe to recollapse, see [4, 22, 23]. Negative potentials appear also in ekpyrotic models (see for example [24] and references therein and [25] with multiple fields).

The function (5) belongs to the class of multi-exponential potentials of the form

$$V(\phi) = \sum_{i=1}^{N} V_i e^{-k_i \phi},$$

which arise as a special case of generalized models with multiple fields studied in the context of assisted inflation (see for example [26]; for an elegant mathematical generalization see [27]). There exists a well established mathematical procedure for the investigation of scalar field cosmologies with exponential potentials in the context of dynamical systems theory [1, 20]. It consists in the introduction of the so called, expansion normalized variables by defining

$$x = \frac{\dot{\phi}}{\sqrt{6}H}, \quad y = \sqrt{\frac{V_1 e^{-\alpha \phi}}{3H^2}}, \quad z = \sqrt{\frac{V_2 e^{-\beta \phi}}{3H^2}}, \quad \Omega = \frac{\rho}{3H^2},$$

and a new time variable $\tau = \ln a$. The Friedmann equation (6) imposes the constraint

$$\Omega = 1 - \left(x^2 + y^2 + z^2\right),$$

(13)

to the state vector $(x, y, z, \Omega)$. This equation can be used to eliminate $\Omega$ from the evolution equations and we end up with a three-dimensional dynamical system,

$$x' = \sqrt{6}Q - \frac{3}{2}\sqrt{\frac{3}{2}} \gamma Q + \left(\frac{3\gamma}{2} - 3\right) x + \left(\frac{3}{2} \sqrt{\frac{3}{2}} \gamma - \sqrt{6}\right) Q x^2 +$$

$$+ \left(3 - \frac{3\gamma}{2}\right) x^3 + \left(\sqrt{\frac{3}{2}} \alpha - \sqrt{6}Q + \frac{3}{2} \sqrt{3/2} \gamma Q\right) y^2 +$$

$$+ \left( \sqrt{\frac{3}{2}} \beta - \sqrt{6}Q + \frac{3}{2} \sqrt{3/2} \gamma Q\right) z^2 - \frac{3}{2} \gamma xy^2 - \frac{3}{2} \gamma xz^2,$$

$$y' = y \left(\frac{3\gamma}{2} - \sqrt{\frac{3}{2}} \alpha x + \left(3 - \frac{3\gamma}{2}\right) x^2 - \frac{3\gamma}{2} y^2 - \frac{3\gamma}{2} z^2\right),$$

$$z' = z \left(\frac{3\gamma}{2} - \sqrt{\frac{3}{2}} \beta x + \left(3 - \frac{3\gamma}{2}\right) x^2 - \frac{3\gamma}{2} y^2 - \frac{3\gamma}{2} z^2\right),$$

(14)

where

$$x^2 + y^2 + z^2 \leq 1,$$

(15)
| Label | \( (x, y, z) \) | \( \Omega \) | Stability | \( a(t) \) |
|-------|----------------|---------|-----------|-----------|
| \( A_\pm \) | \((\pm 1, 0, 0)\) | 0 | Unstable | \( t^{1/3} \) |
| \( B \) | \( \left( \frac{(4-3\gamma)Q}{\sqrt{6}(2-\gamma)}, 0, 0 \right) \) | 1 - \( \frac{(4-3\gamma)^2 Q^2}{6(2-\gamma)} \) | Saddle | \( t^{4(2-\gamma)/(6\gamma(2-\gamma)+(4-3\gamma)^2 Q^2)} \) |
| \( C_\pm \) | \( \left( \frac{\alpha}{\sqrt{6}}, \pm \sqrt{1 - \frac{\alpha^2}{6}}, 0 \right) \) | 0 | Stable | \( t^{2/\alpha^2} \) |
| \( D_\pm \) | \( \left( 0, \pm \sqrt{\frac{\beta}{\beta - \alpha}}, \pm \sqrt{\frac{\alpha}{\alpha - \beta}} \right) \) | 0 | Saddle | \( e^t \) |

and a prime denotes derivative with respect to \( \tau \). Note that \( y \) and \( z \) can take both real and pure imaginary values, depending on the signs of \( V_i \). With this choice we avoid to have four different dynamical systems (see however [4] where real normalized variables are used). For \( V_1, V_2 > 0 \), the phase space is the closed unit ball in \( \mathbb{R}^3 \). For \( V_1 > 0 \) and \( V_2 < 0 \), the phase space is the one sheet hyperboloid \( x^2 + y^2 - (\text{Im} z)^2 = 1 \) and its interior. The resulting dynamical system depends on four parameters \((\gamma, \alpha, \beta, Q)\). Using \( (7) \), the effective equation of state,

\[
\frac{w_{\text{eff}}}{3H^2} = -1 - \frac{2\dot{H}}{3H^2},
\]

is written in terms of the new variables as,

\[
\frac{w_{\text{eff}}}{3H^2} = -1 + 2x^2 + \gamma \Omega.
\]

### 3 Cosmologically acceptable solutions

By inspection, system \((14)\) is symmetric under reflection, with respect to the planes \( x - z \) and \( x - y \). The planes \( y = 0 \) and \( z = 0 \) are invariant sets for the system \((14)\). The full list and analysis of the critical points of our system is presented in the Appendix. In this section, we discuss only these equilibria which allow for a viable cosmological history of the universe. In Table 1 are shown the equilibria for \( V_1 > 0 \) and

\[
\alpha < \sqrt{2}, \ \gamma \leq 1, \ (4 - 3\gamma) Q \in \left( \max \left\{ 0, 2 \left( \frac{\alpha^2 - 3\gamma}{\alpha} \right) \right\}, \sqrt{6}(2-\gamma) \right).
\]

The two critical points \( A_\pm \) correspond to kinetic dominated solutions which are unstable and are only expected to be relevant at early times. Point \( B \) represents a type of scaling solution, i.e., the kinetic energy density of the scalar field remains proportional to that of the perfect fluid. Points \( C_\pm \) are accelerated only for \( V_1 > 0 \). They correspond to scalar field dominated solutions which exist for sufficiently flat potentials, \( \alpha < \sqrt{6} \). These are the same conclusions as in [28] for an exponential potential and \( Q = \sqrt{2/3} \), and
also in [1, 4] and [17] and in the case of a scalar field non coupled to matter, although the ranges of the parameters \((\alpha, \gamma)\) are different. Points \(D_{\pm}\) exist only in models with \(V_1 > 0, V_2 < 0\). They correspond to the unstable state \(\left(\phi = \phi_m, \dot{\phi} = 0, \rho = 0, H = \sqrt{V_{\text{max}}/3}\right)\) and represent de Sitter solutions.

A successful cosmological model should comprise an accelerating solution as a future attractor. It is evident that points \(C_{\pm}\), could satisfy the condition for acceleration, \(w_{\text{eff}} < -1/3\), provided that \(\alpha < \sqrt{2}\), (compare with the conclusions in [1]). From now on we assume this range for the parameter \(\alpha\). Moreover, the equilibria \(C_{\pm}\), are stable for all physically interesting values of \(\gamma\). For a cosmological theory to be acceptable, it has to possess a matter dominated epoch followed by a late time accelerated attractor. The saddle character of point \(B\), implies that it represents a transient phase and therefore, it is a good candidate for a matter point, provided that \(\Omega\) is close to one. This happens only for very small values of the coupling parameter \(Q\) and for \(\gamma\) close to one. Another way to see this, is the following. During the matter era, the scale factor has to expand approximately as \(a \sim t^{2/3}\). The scale factor near \(B\) evolves as \(a \sim t^{2/3 \left( w_{\text{eff}} + 1\right)}\), therefore, \(w_{\text{eff}}\), has to be close to zero. As seen in Table 1, \(a(t)\) at \(B\), evolves as \(t^{2/3}\) when \(Q\) takes the values

\[
Q = \frac{\sqrt{6} (2 - \gamma) (1 - \gamma)}{(4 - 3 \gamma)}, \quad \gamma \leq 1. \tag{16}
\]

Therefore, the realistic value \(\gamma = 1\), corresponding to dust, is incompatible to scalar field coupled to matter, i.e., the coupling parameter \(Q\) must be zero. On the other hand, (5) and (9) imply that for \(\gamma = 4/3\), the value of \(Q\) is undetermined. Below we summarize our results for the particular values \(\gamma = 1, 4/3, 2/3\).

A. Dust \((\gamma = 1)\). The critical points of our system are those of Table 1 for \(\alpha < \sqrt{2}, \beta > \alpha, Q = 0\). Note that the future attractors \(C_{\pm}\) have non phantom acceleration for every value of \(\alpha\) in the interval \((0, \sqrt{2})\). A cosmologically acceptable trajectory should pass near \(B\) and finally land on one of the points \(C_{\pm}\), depending on the initial conditions. Note that \(A_{\pm}, B\) and \(C_{\pm}\) lie on the invariant plane \(z = 0\) and \(C_{\pm}\) exist only in potentials with \(V_1 > 0\). We consider the projection of the system (14) on that plane. The phase portrait is shown in Figure 2 and is the same in both cases where the phase space is a sphere \((V_2 > 0)\), or a one sheet hyperboloid \((V_2 < 0)\).

B. Radiation \((\gamma = 4/3)\). The case of \(\gamma = 4/3\) corresponds to radiation, and therefore there is no matter point with a scale factor \(a \sim t^{2/3}\). Instead, point \(B\), which coincides with the origin \((0, 0, 0)\), now represents the well-known radiation dominated solution, \(a \sim t^{1/2}\), as a transient phase. \(C_{\pm}\) are future attractors for \(\alpha < \sqrt{2}\).

C. The value \(\gamma = 2/3\) corresponds to ordinary matter marginally satisfying the strong energy condition. Eq. (10) implies \(Q = \sqrt{2}/3\). An acceptable trajectory exists for \(\alpha < \sqrt{2}\). For these values of \(\alpha\) and \(Q\), points \(A_{\pm}\) are always unstable. Point \(B \equiv (1/2, 0, 0)\), corresponds to the transient matter era, with \(\Omega = 3/4\). The accelerated points \(C_{\pm}\) are future attractors.
Figure 2: Phase portrait of the projected three-dimensional system on the invariant set $z = 0$.

Throughout this paper we do not consider the case, $\alpha \beta < 0$, for the potentials \cite{5}. The reason is that for $\alpha \beta < 0$, and $V_1, V_2 > 0$, the function $V(\phi)$ in \cite{5} has a strictly positive minimum, say $V_{\text{min}}$, and the de Sitter solution with $H = \sqrt{V_{\text{min}}/3}$, is the future attractor for the system, \cite{28}. This follows directly either from the original equations (7)-(9), or from the system (14) written in the new variables. Moreover, it is easy to see that a matter era represented by a saddle equilibrium $\mathcal{B}$, precedes the final accelerated epoch.

4 Asymptotic form of some $f(R)$ theories predicting acceleration

A large class of dynamical dark energy models is based on the large-distance modification of gravity (see \cite{29} for recent reviews). For example, in the context of $f(R)$ gravity theories the models $f(R) = R - \mu^{2(n+1)} / R^n$, where $\mu > 0, n > 1$, were proposed to explain the late-time cosmic acceleration \cite{30,31}. The obvious idea is the introduction of modifications to the Einstein-Hilbert Lagrangian which become important at low curvatures. For these models the potential functions in the Einstein frame have the form,

$$V_n(\phi) = \frac{\mu^2 (n+1) n^{1/(n+1)} (e^{\sqrt{2/3} \phi} - 1)^{n/(n+1)}}{2 \eta e^{2 \sqrt{2/3} \phi}}. \tag{17}$$

These functions are defined only for $\phi \geq 0$, and their behavior is similar to that indicated in Figure 1, i.e., they have a local maximum at some $\phi_m$ depending on $n$, and for large $\phi$ they approach zero exponentially. As $n \rightarrow \infty$
the potentials (17) approach the function,

\[ V(\phi) = \frac{\mu^2}{2} \left( e^{-\sqrt{2/3}\phi} - e^{-2\sqrt{2/3}\phi} \right), \]

(18)
corresponding to the asymptotic form of these theories, [31]. Thus, (18) is a particular case of (5) with \( \beta = 2\alpha = 2\sqrt{2/3} \), \( V_1 = -V_2 = \mu^2/2 > 0 \), cf. Figure 1. Note that for large \( \phi \), \( V \) in (18) behaves similarly to \( V_n \) in (17). In contrast to the family (17), \( V \) in (18) has the nice property that it is defined for all \( \phi \in \mathbb{R} \). As mentioned in the introduction, the coupling coefficient takes the value \( Q = \sqrt{2/3} \), regardless of the form of \( f(R) \), [32].

The constraint (15) implies that the phase space is the set \( x^2 + y^2 - (\text{Im } z)^2 \leq 1 \). There are up to seven critical points for that system, depending on the value of \( \gamma \).

| Label | \((x, y, z)\) | \(\Omega\) | Existence | Stability | \(a(t)\) |
|-------|---------------|----------|-----------|-----------|--------|
| \(A_{\pm}\) | \((\pm 1, 0, 0)\) | 0 | always unstable | \(t^{1/3}\) |
| \(B\) | \(\left(\frac{4-3\gamma}{3(2-\gamma)}, 0, 0\right)\) | \(\frac{4(5-3\gamma)}{9(2-\gamma)^2}\) | \(\gamma \leq 5/3\) saddle | \(t^{3(2-\gamma)/(8-3\gamma)}\) |
| \(C_{\pm}\) | \(\left(\frac{1}{3}, \pm \frac{\sqrt{2}}{\sqrt{3}}, 0\right)\) | 0 | always stable | \(t^3\) |
| \(D_{\pm}\) | \((0, \pm \sqrt{2}/\sqrt{3}, \pm i)\) | 0 | always saddle | \(e^t\) |

Points \(C_{\pm}\) are future attractors and have non phantom acceleration with \( w_{\text{eff}} = -7/9 \). However, in the case of dust, \( \gamma = 1 \), the scale factor at matter point \(B\) evolves as \( a \sim t^{3/5} \), rather than the usual \( a \sim t^{2/3} \). The scale factor evolves “correctly” only for \( \gamma = 2/3 \). The absence of the standard matter epoch is associated with the fact that matter is strongly coupled to gravity. This result is in agreement with the general conclusions in [32], [33], [34], that these \( f(R) \) dark energy models are not cosmologically viable.

5 Conclusion

In this paper we have focused on a general treatment of a scalar field with a double exponential potential non minimally coupled to a perfect fluid. A full analysis of the equilibrium points of the resulted dynamical system is quite complicated, yet it revealed that the model predicts a late accelerated phase of the universe for a wide range of the parameters, \( \alpha, \beta, \gamma \) and \( Q \). Moreover, there exists transient solutions representing a matter era, preceding the accelerating attractor. However, in most cases the scale factor near these transient phases evolves as \( a(t) \sim t^{q(Q)} \), where the exponent \( q \) is in general different from the usual 2/3. The “wrong” matter epoch is associated with the fact that for values of \( Q \) of order unity, matter is strongly coupled to gravity. A coupling constant of order unity means that matter feels an additional scalar force as strong as gravity itself, cf [32]. Assuming that ordinary matter satisfies plausible energy conditions, i.e., \( \gamma \gtrsim 1 \), the coupling constant, \( Q \), has to be very small; more precisely, \( q(Q) \rightarrow 2/3 \), only for \( Q \rightarrow 0 \).
Table 2: Critical Points

| Label | $(x, y, z)$ | $\Omega$ | $\omega_{\text{eff}}$ |
|-------|------------|----------|---------------------|
| $A_{\pm}$ | $(\pm 1, 0, 0)$ | 0 | 1 |
| $B$ | $\left(\frac{(4-3\gamma)Q}{\sqrt{6(2-\gamma)}}, 0, 0\right)$ | $1 - \frac{(4-3\gamma)^2Q^2}{6(2-\gamma)^2}$ | $-1 + \gamma + \frac{(4-3\gamma)^2Q^2}{6(2-\gamma)}$ |
| $C_{\pm}$ | $\left(\frac{\alpha}{\sqrt{6}}, \pm \sqrt{1 - \frac{\alpha^2}{6}}, 0\right)$ | 0 | $-1 + \frac{\alpha^2}{3}$ |
| $D_{\pm}$ | $\left(0, \pm \sqrt{\frac{\beta - \alpha}{\beta - \alpha}}, \pm \sqrt{\frac{\alpha}{\alpha - \beta}}\right)$ | 0 | $-1$ |
| $D'_{\pm}$ | $\left(0, \pm \sqrt{\frac{\beta}{\beta - \alpha}}, \mp \sqrt{\frac{\alpha}{\alpha - \beta}}\right)$ | 0 | $-1$ |
| $E_{\pm}$ | $(u_\alpha, \pm v_\alpha, 0)$ | $\omega_\alpha$ | $-1 + \sqrt{\frac{2}{3}}\alpha u_\alpha$ |
| $F_{\pm}$ | $\left(\frac{\beta}{\sqrt{6}}, 0, \pm \sqrt{1 - \frac{\beta^2}{6}}\right)$ | 0 | $-1 + \frac{\beta^2}{3}$ |
| $G_{\pm}$ | $(u_\beta, 0, \pm v_\beta)$ | $\omega_\beta$ | $-1 + \sqrt{\frac{2}{3}}\beta u_\beta$ |

where $u_\alpha = \sqrt{\frac{6\gamma}{2\alpha - (4-3\gamma)Q}}$, $v_\alpha = \sqrt{\frac{(4-3\gamma)^2Q^2 - 2\alpha(4-3\gamma)Q + 6\gamma(2-\gamma)}{2\alpha - (4-3\gamma)Q^2}}$, $\omega_\alpha = 2\frac{2\alpha^2 \gamma - \alpha(4-3\gamma)Q}{(2\alpha - (4-3\gamma)Q)^2}$, and similarly for $u_\beta, v_\beta, \omega_\beta$.

Therefore, only a very weak coupling of the scalar field to ordinary matter can lead to acceptable cosmological histories of the universe. This surprising result, indicates that cosmological evolution imposes strict constraints on the choice of the correct Lagrangian of a gravity theory.

Acknowledgements

We thank N. Hadjisavvas and S. Cotsakis for useful comments.

Appendix

We present here the full analysis of the stability of the system \(^{(14)}\). The critical points are listed in Table 2.

We assume that $0 < \alpha < \beta$. The case $0 < \beta < \alpha$, is a mere renaming of some of the equilibrium points. According to the definition \(^{(12)}\), the modulus of $z$ lies between 0 and the absolute value of $y$. Therefore, points $D'_{\pm}, F_{\pm}$ and $G_{\pm}$ are not acceptable. The eigenvalues of the remaining equilibria are presented in the Table 3.
exist at the same time at least one matter point with
by at least one accelerated future attractor.

determine under which conditions on the parameters
$\alpha, \beta, \gamma$ and $B$
$\Omega > 0. (ii) the “right” scale factor condition, $a \sim t^{2/3}$, (or equivalently, $w_{\text{eff}}$
close to zero), and (iii) is a saddle point, i.e., represents a transient phase.
On the other hand, an acceptable late attractor has to be (iv) accelerated,
close to zero), and (iii) is a saddle point, i.e., represents a transient phase. We are going to
determine under which conditions on the parameters $\alpha, \beta, \gamma$ and $Q$
there exist at the same time at least one matter point with $w_{\text{eff}}$ close to 1, followed
by at least one accelerated future attractor.

As mentioned in the main text, a cosmologically acceptable trajectory
passes near a matter point and lands to an accelerated point. A critical point
is a good candidate for a matter point if it satisfies (i) the matter condition,
$\Omega > 0$, (ii) the “right” scale factor condition, $a \sim t^{2/3}$, (or equivalently, $w_{\text{eff}}$
close to zero), and (iii) is a saddle point, i.e., represents a transient phase. We are going to
determine under which conditions on the parameters $\alpha, \beta, \gamma$ and $Q$
there exist at the same time at least one matter point with $w_{\text{eff}}$ close to 1, followed
by at least one accelerated future attractor.

$C_\pm$ Following the terminology of [4], these are kinetic-potential scaling sol-
solutions and exist in potentials with $V_1 > 0$ for $\alpha < \sqrt{6}$ and in potentials
with $V_1 < 0$ for $\alpha > \sqrt{6}$. They are stable and accelerated whenever

\[
(4 - 3\gamma)Q > \frac{2(\alpha^2 - 3\gamma)}{\alpha} \quad \text{and} \quad \alpha < \sqrt{2}. \quad \text{(A.1)}
\]

Hence, they are good candidates as accelerated late attractors only in
potentials with $V_1 > 0$.

$E_\pm$ Points $E_\pm$ are fluid-kinetic-potential scaling solutions (see also Ref. [4]
for the uncoupled case). They enter the phase space when

\[
(4 - 3\gamma)Q \leq \frac{2(\alpha^2 - 3\gamma)}{\alpha}, \quad \text{(A.2)}
\]

Points $E_\pm$ may be used for the matter epoch if they satisfy conditions
(i), (ii) and (iii), i.e., if

\[
Q = 2\alpha \frac{1 - \gamma}{4 - 3\gamma}, \gamma \leq 1, \quad \alpha > \sqrt{\frac{3}{2}} \frac{2 - \gamma}{1 - \gamma}. \quad \text{(A.3)}
\]
In that case, points $E_{\pm}$ exist only for potentials with $V_1 < 0$. Hence, when $E_{\pm}$ are used as matter points, points $C_{\pm}$ cannot be used as the accelerated attractors. The only candidate left for the accelerated epoch is $B$, but as we will see, $B$ cannot be accelerated for $Q$ given in (A.3).

In order for points $E_{\pm}$ to be used for the accelerated epoch they have to satisfy conditions (iv) and (v). This happens for

$$(4 - 3\gamma)Q < (2 - 3\gamma)\alpha \text{ and } (4 - 3\gamma)^2Q^2 - 2\alpha (4 - 3\gamma)Q + 6\gamma(2 - \gamma) > 0.$$  

(A.4)

Whenever $E_{\pm}$ are accelerated attractors, the only remaining candidate for the matter epoch is point $B$, but as we shall see right below, $B$ does not satisfy the matter point conditions for the range of the parameters given in (A.4).

$B$ This is a fluid-kinetic scaled solution. Point $B$ enters the phase space when

$$Q \leq \sqrt{6} \frac{2 - \gamma}{|4 - 3\gamma|},$$

(A.5)

for $\gamma \neq 4/3$ and lies always in the phase space for $\gamma = 4/3$, irrespective of the nature of the potential. For $\gamma < 4/3$, condition (A.5) is always satisfied for sufficiently small values of $Q$, e.g., $Q \lesssim 1$. Matter point conditions (i), (ii) and (iii) are satisfied whenever

$$Q = \frac{\sqrt{6}(2 - \gamma)(1 - \gamma)}{4 - 3\gamma}, \quad \gamma \leq 1, \quad \alpha < \sqrt{\frac{3}{2}}\sqrt{\frac{2 - \gamma}{1 - \gamma}}.$$  

(A.6)

On the other hand, point $B$ may be an accelerated attractor if (iv) and (v) hold, provided that (A.5) is satisfied. The condition for acceleration (iv) gives

$$Q < \sqrt{\frac{2(2 - \gamma)(2 - 3\gamma)}{4 - 3\gamma}},$$

(A.7)

with $\gamma < 2/3$. Assuming (A.7), the stability condition, (v), gives

$$(4 - 3\gamma)^2Q^2 - 2\alpha (4 - 3\gamma)Q + 6\gamma(2 - \gamma) < 0.$$  

Nevertheless, $Q$ given in (A.3) do not satisfy (A.7). Hence, matter points $E_{\pm}$ cannot be combined with accelerated attractors $B$.

We conclude that there is only one case in which we have at the same time at least one matter point and at least one accelerated attractor. This happens whenever $B$ represents the matter solution and $C_{\pm}$ stand for attractors. In that case, potential has $V_1 > 0$ and the parameters take the values

$$\alpha < \sqrt{2}, \quad \gamma \leq 1, \quad Q = \frac{\sqrt{6}(2 - \gamma)(1 - \gamma)}{4 - 3\gamma},$$

(A.8)

leading to Table 1 in the main text.
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