K-THEORY FOR THE SIMPLE $C^*$-ALGEBRA
OF THE FIBONACCHI DYCK SYSTEM

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Abstract. Let $F$ be the Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The Fibonacci Dyck shift is a subsystem of the Dyck shift $D_2$ constrained by the matrix $F$. Let $\mathcal{L}^{Ch}(D_F)$ be a $\lambda$-graph system presenting the subshift $D_F$, that is called the Cantor horizon $\lambda$-graph system for $D_F$. We will study the $C^*$-algebra $O_{\mathcal{L}^{Ch}(D_F)}$ associated with $\mathcal{L}^{Ch}(D_F)$. It is simple purely infinite and generated by four partial isometries with some operator relations. We will compute the K-theory of the $C^*$-algebra. As a result, the $C^*$-algebra is simple purely infinite and not semiprojective. Hence it is not stably isomorphic to any Cuntz-Krieger algebra.

Keywords: $C^*$-algebra, Cuntz-Krieger algebra, subshift, $\lambda$-graph system, Dyck shift, K-theory.

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1. Introduction

Let $\Sigma$ be a finite set with its discrete topology, that is called an alphabet. Each element of $\Sigma$ is called a symbol. Let $\Sigma^\mathbb{Z}$ be the infinite product space $\prod_{i=-\infty}^\infty \Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation $\sigma$ on $\Sigma^\mathbb{Z}$ given by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$ is called the full shift over $\Sigma$. Let $\Lambda$ be a closed subset of $\Sigma^\mathbb{Z}$ such that $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift or a symbolic dynamical system. It is written as $\Lambda$ for brevity.

In [17], the author has introduced a notion of $\lambda$-graph system as a presentation of subshifts. A $\lambda$-graph system $\Sigma = (V, E, \lambda, i)$ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \cdots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots$, a labeling map $\lambda: E \rightarrow \Sigma$ and a surjective map $i_{l,l+1}: V_{l+1} \rightarrow V_l$ for each $l \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of all nonnegative integers. An edge $e \in E_{l,l+1}$ has its source vertex $s(e)$ in $V_l$, its terminal vertex $t(e)$ in $V_{l+1}$ and its label $\lambda(e)$ in $\Sigma$ ([17]).

The theory of symbolic dynamical system has a close relationship to automata theory and language theory. In the theory of language, there is a class of universal languages due to W. Dyck. The symbolic dynamics generated by the languages are called the Dyck shifts $D_N$ (cf. [3], [10],[11],[12]). Its alphabet consists of the $2N$ brackets: $(1, \ldots, (N,)_1, \ldots, )_N$. The forbidden words consist of words that do not obey the standard bracket rules. In [14], a $\lambda$-graph system $\mathcal{L}^{Ch}(D_N)$ that presents the subshift $D_N$ has been introduced. The $\lambda$-graph system is called the Cantor horizon $\lambda$-graph system for the Dyck shift $D_N$. The K-groups for $\mathcal{L}^{Ch}(D_N)$, that are invariant under topological conjugacy of the subshift $D_N$, have been computed ([14]).
In [22] (cf. [14]), the $C^*$-algebra $\mathcal{O}_{\mathcal{L}^{Ch(D_N)}}$ associated with the Cantor horizon $\lambda$-graph system $\mathcal{L}^{Ch(D_N)}$ has been studied. In the paper, it has been proved that the algebra $\mathcal{O}_{\mathcal{L}^{Ch(D_N)}}$ is simple and purely infinite and generated by $N$ partial isometries and $N$ isometries satisfying some operator relations. Its K-groups are

$$K_0(\mathcal{O}_{\mathcal{L}^{Ch(D_N)}}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathbb{R}, \mathbb{Z}), \quad K_1(\mathcal{O}_{\mathcal{L}^{Ch(D_N)}}) \cong 0$$

where $C(\mathbb{R}, \mathbb{Z})$ denotes the abelian group of all integer valued continuous functions on a Cantor discontinuum $\mathbb{R}$ (cf. [14]).

Let $u_1, \ldots, u_N$ be the canonical generating isometries of the Cuntz algebra $\mathcal{O}_N$ that satisfy the relations: $\sum_{j=1}^N u_j u_j^* = 1$, $u_i^* u_i = 1$ for $i = 1, \ldots, N$. Then the bracket rule of the symbols $(1, \ldots, (N_1), \ldots, )$ of the Dyck shift $D_N$ may be interpreted as the relations $u_i^* u_i = 1$, $u_i^* u_j = 0$ for $i \neq j$ of the partial isometries $u_1^*, \ldots, u_N^*, u_1, \ldots, u_N$ in the $C^*$-algebra $\mathcal{O}_N$ (cf. (2.1)).

In [23], we have considered a generalization of Dyck shifts $D_N$ by using the canonical generators of the Cuntz-Krieger algebras $\mathcal{O}_A$ for $N \times N$ matrices $A$ with entries in $\{0, 1\}$. The generalized Dyck shift is denoted by $D_A$ and called the topological Markov Dyck shift for $A$ (cf. [7], [15]). Let $\alpha_1, \ldots, \alpha_N$, $\beta_1, \ldots, \beta_N$ be the alphabet of $D_A$. They correspond to the brackets $(1, \ldots, (N_1), \ldots, )_N$ respectively. Let $t_1, \ldots, t_N$ be the canonical generating partial isometries of the Cuntz-Krieger algebra $\mathcal{O}_A$ that satisfy the relations: $\sum_{j=1}^N t_j t_j^* = 1$, $t_i^* t_i = \sum_{j=1}^N A(i, j) t_j t_j^*$ for $i = 1, \ldots, N$. Consider the correspondence $\varphi(\alpha_i) = t_i^*, \varphi(\beta_i) = t_i, i = 1, \ldots, N$. Then a word $w$ of $\{\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N\}$ is defined to be admissible for the subshift $D_A$ precisely if the corresponding element to $w$ through $\varphi$ in $\mathcal{O}_A$ is not zero. Hence we may recognize $D_A$ to be the subshift defined by the canonical generators of the Cuntz-Krieger algebra $\mathcal{O}_A$. The subshifts $D_A$ are not sofic in general and reduced to the Dyck shifts if all entries of $A$ are 1.

The Cantor horizon $\lambda$-graph system $\mathcal{L}^{Ch(D_A)}$ for the topological Markov Dyck shift $D_A$ has been also studied in [23]. It has been proved to be $\lambda$-irreducible with $\lambda$-condition (I) in the sense of [21] if the matrix is irreducible with condition (I) in the sense of Cuntz-Krieger [5]. Hence the associated $C^*$-algebra $\mathcal{O}_{\mathcal{L}^{Ch(D_A)}}$ is simple and purely infinite. It is the unique $C^*$-algebra generated by $2N$ partial isometries subject to some operator relations.

In this paper we study the $C^*$-algebra $\mathcal{O}_{\mathcal{L}^{Ch(D_F)}}$ for the Fibonacci matrix $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. It is the smallest matrix in the irreducible square matrices with condition (I) such that the associated topological Markov shift $\Lambda_F$ is not conjugate to any full shift. The topological entropy of $\Lambda_F$ is $\log \frac{\sqrt{5} + 1}{2}$ the logarithm of the Perron eigenvalue of $F$. We call the subshift $D_F$ the Fibonacci Dyck shift. As the matrix is irreducible with condition (I), the associated $C^*$-algebra $\mathcal{O}_{\mathcal{L}^{Ch(D_F)}}$ is simple and purely infinite. We will compute the K-groups $K_i(\mathcal{O}_{\mathcal{L}^{Ch(D_F)}}), i = 0, 1$ of the algebra so that we have

**Theorem 1.1.** The $C^*$-algebra $\mathcal{O}_{\mathcal{L}^{Ch(D_F)}}$ associated with the $\lambda$-graph system $\mathcal{L}^{Ch(D_F)}$ is unital, separable, nuclear, simple and purely infinite. It is the unique $C^*$-algebra generated by one isometry $T_1$ and three partial isometries $S_1, S_2, T_2$ subject to the following operator relations:

$$\sum_{j=1}^2 (S_j S_j^* + T_j T_j^*) = \sum_{j=1}^2 S_j^* S_j = 1, \quad T_2^* T_2 = S_1^* S_1,$$

$$j = 1, 2,$$
where $E_{\mu_1 \cdots \mu_k} = S_{\mu_k}^* \cdots S_{\mu_1}^* T_{\mu_1} \cdots T_{\mu_k}^*$, \( k > 1 \)

\begin{equation}
E_{\mu_1 \cdots \mu_k} = \sum_{j=1}^{2} F(j, \mu_1)S_j S_j^* E_{\mu_1 \cdots \mu_k} S_j S_j^* + T_{\mu_1} E_{\mu_2 \cdots \mu_k} T_{\mu_1}^*, \quad k > 1
\end{equation}

This paper is a continuation of [23].

2. THE SUBSHIFT $D_A$ AND THE $\lambda$-GRAPH SYSTEM $L^{Ch(D_A)}$

We will briefly review the topological Markov Dyck shift $D_A$ and its Cantor horizon $\lambda$-graph system $L^{Ch(D_A)}$.

Consider a pair of $N$ symbols where $\Sigma^- = \{\alpha_1, \cdots, \alpha_N\}$, $\Sigma^+ = \{\beta_1, \cdots, \beta_N\}$. We set $\Sigma = \Sigma^- \cup \Sigma^+$. Let $A = [A(i,j)]_{i,j=1,\ldots,N}$ be an $N \times N$ matrix with entries in $\{0,1\}$. Throughout this paper, $A$ is assumed to have no zero rows or columns. Consider the Cuntz-Krieger algebra $\mathcal{O}_A$ for the matrix $A$ that is the universal $C^*$-algebra generated by $N$ partial isometries $t_1, \ldots, t_N$ subject to the following relations:

$$\sum_{j=1}^{N} t_j t_j^* = 1,$$

$$t_i^* t_i = \sum_{j=1}^{N} A(i,j) t_j t_j^* \quad \text{for } i = 1, \ldots, N$$

[[5]]. Define a correspondence $\varphi_A : \Sigma \rightarrow \{t_1^*, \ldots, t_N^*, t_1, \ldots, t_N\}$ by setting

$$\varphi_A(\alpha_i) = t_i^*,$$

$$\varphi_A(\beta_i) = t_i \quad \text{for } i = 1, \ldots, N.$$

We denote by $\Sigma^*$ the set of all words $\gamma_1 \cdots \gamma_n$ of elements of $\Sigma$. Define the set

$$\mathfrak{A}_A = \{\gamma_1 \cdots \gamma_n \in \Sigma^* | \varphi_A(\gamma_1) \cdots \varphi_A(\gamma_n) = 0\}.$$

Let $D_A$ be the subshift over $\Sigma$ whose forbidden words are $\mathfrak{A}_A$. The subshift is called the topological Markov Dyck shift defined by $A$ (cf. [7], [15]). If all entries of $A$ are 1, the subshift $D_A$ becomes the Dyck shift $D_N$ with $2N$ bracket (cf. [11],[12], [14], [22],[23]). We note the fact that $\alpha_i \beta_j \in \mathfrak{A}_A$ if $i \neq j$, and $\alpha_{i_1} \cdots \alpha_{i_t} \in \mathfrak{A}_A$ if and only if $\beta_{i_1} \cdots \beta_{i_t} \in \mathfrak{A}_A$. Consider the following subsystem of $D_A$

$$D_A^+ = \{(\gamma_i)_{i \in \mathbb{Z}} \in D_A | \gamma_i \in \Sigma^+ \text{ for all } i \in \mathbb{Z}\}.$$

The subshift $D_A^+$ is identified with the topological Markov shift

$$\Lambda_A = \{(x_i)_{i \in \mathbb{Z}} \in \{1, \ldots, N\}^\mathbb{Z} | A(x_i, x_{i+1}) = 1, i \in \mathbb{Z}\}$$

defined by the matrix $A$. Hence the subshift $D_A$ is recognized to contain the topological Markov shift $\Lambda_A$.

We denote by $B_l(D_A)$ and $B_l(\Lambda_A)$ the set of admissible words of length $l$ of $D_A$ and that of $\Lambda_A$ respectively. Let $m(l)$ be the cardinal number of $B_l(\Lambda_A)$. We use lexicographic order from the left on the words of $B_l(\Lambda_A)$, so that we may assign to a word $\mu_1 \cdots \mu_l \in B_l(\Lambda_A)$ the number $N(\mu_1 \cdots \mu_l)$ from 1 to $m(l)$. For example, if $A = F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then

$$B_1(\Lambda_F) = \{1, 2\}, \quad N(1) = 1, N(2) = 2,$$

$$B_2(\Lambda_F) = \{11, 12, 21\}, \quad N(11) = 1, N(12) = 2, N(21) = 3,$$
and so on. Hence the set $B_l(\Lambda_A)$ bijectively corresponds to the set of natural numbers less than or equal to $m(l)$. Let us now describe the Cantor horizon $\lambda$-graph system $\mathcal{L}^{Ch(D_A)}$ of $D_A$. The vertices $V_l$ at level $l$ for $l \in \mathbb{Z}_+$ are given by the admissible words of length $l$ consisting of the symbols of $\Sigma^+$. We regard $V_0$ as a one point set of the empty word $\emptyset$. Since $V_l$ is identified with $B_l(\Lambda_A)$, we may write $V_l$ as

$$V_l = \{v_{N(\mu_1, \ldots, \mu_l)}^l \mid \mu_1 \cdots \mu_l \in B_l(\Lambda_A) \}.$$ 

The mapping $\iota = \iota_{l,l+1} : V_{l+1} \to V_l$ is defined by deleting the rightmost symbol of a corresponding word such as

$$\iota(v_{N(\mu_1, \ldots, \mu_l)}^{l+1}) = v_{N(\mu_1, \ldots, \mu_l)}^l \quad \text{for} \quad v_{N(\mu_1, \ldots, \mu_l)}^{l+1} \in V_{l+1}.$$ 

We define an edge labeled $\alpha_j$ from $v_{N(\mu_1, \ldots, \mu_l)}^l \in V_l$ to $v_{N(\mu_0, \mu_1, \ldots, \mu_l)}^{l+1} \in V_{l+1}$ precisely if $\mu_0 = j$, and an edge labeled $\beta_j$ from $v_{N(j, \mu_1, \ldots, \mu_l)}^l \in V_l$ to $v_{N(\mu_0, \mu_1, \ldots, \mu_l)}^{l+1} \in V_{l+1}$. For $l = 0$, we define an edge labeled $\alpha_j$ form $v_0^1$ to $v_{N(j)}^1$, and an edge labeled $\beta_j$ form $v_1^0$ to $v_{N(j)}^1$ if $A(j,i) = 1$. We denote by $E_{l,l+1}$ the set of edges from $V_l$ to $V_{l+1}$. Set $E = \cup_{l=0}^\infty E_{l,l+1}$. It is easy to see that the resulting labeled Bratteli diagram with $\iota$-map becomes a $\lambda$-graph system over $\Sigma$, that is called the Cantor horizon $\lambda$-graph system and is denoted by $\mathcal{L}^{Ch(D_A)}$.

A $\lambda$-graph system $\mathcal{L}$ is said to present a subshift $\Lambda$ if the set of all admissible words of $\Lambda$ coincides with the set of all finite labeled sequences appearing in concatenating edges of $\mathcal{L}$. In [23], the following propositions have been proved.

**Proposition 2.1.**

(i) If $A$ satisfies condition (I) in the sense of Cuntz-Krieger [5], the subshift $D_A$ is not sofic.

(ii) The $\lambda$-graph system $\mathcal{L}^{Ch(D_A)}$ presents the subshift $D_A$.

(iii) If $A$ is an irreducible matrix with condition (I), then the $\lambda$-graph system $\mathcal{L}^{Ch(D_A)}$ is $\lambda$-irreducible with $\lambda$-condition (I) in the sense of [21].

**Proposition 2.2.** The $C^*$-algebra $O_{\mathcal{L}^{Ch(D_A)}}$ associated with the $\lambda$-graph system $\mathcal{L}^{Ch(D_A)}$ is unital, separable, nuclear, simple and purely infinite. It is the unique $C^*$-algebra generated by $2N$ partial isometries $S_i, T_i, i = 1, \ldots, N$ subject to the following operator relations:

$$\sum_{j=1}^N (S_j S_j^* + T_j T_j^*) = \sum_{j=1}^N S_j^* S_j = 1,$$

$$T_i^* T_i = \sum_{j=1}^N A(i,j) S_j^* S_j, \quad i = 1, 2, \ldots, N,$$

$$E_{\mu_1, \ldots, \mu_k} = \sum_{j=1}^N A(j, \mu_1) S_j T_{\mu_1} \cdots T_{\mu_k} S_j^* + T_{\mu_1} \cdots T_{\mu_k}^* E_{\mu_1, \ldots, \mu_k}, \quad k > 1$$

where $E_{\mu_1, \ldots, \mu_k} = S_{\mu_1}^* \cdots S_{\mu_k}^* S_{\mu_k} \cdots S_{\mu_1}$, $(\mu_1, \cdots, \mu_k) \in \Lambda_A^*$ the set of admissible words of the topological Markov shift $\Lambda_A$ defined by the matrix $A$.

3. **K-theory for $O_{\mathcal{L}^{Ch(D_A)}}$**

We will prove Theorem 1.1. The operator relations (1.1) and (1.2) are direct from the operator relations in Proposition 2.2. By Proposition 2.2, it remains to
prove the $K$-group formulae. This section is devoted to computing the $K$-groups $K_i(O_{Ch(D_F)})$, $i = 0, 1$ for the $C^*$-algebra $O_{Ch(D_F)}$. The symbols $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ of the subshift $D_F$ correspond to the brackets $(1, (2, )_1, )_2$ respectively. Let $V_l, l \in \mathbb{Z}$ be the vertex set of the $\lambda$-graph system $\mathcal{G}^{Ch(D_F)}$. They are identified with the admissible words consisting of the symbols $\beta_1, \beta_2$ in $\Sigma^+$. Since the word $\beta_2\beta_2$ is forbidden, the following is a list of the vertex sets $V_l$ for $l = 0, 1, 2, 3, 4, \ldots$:

- $V_0 : *$
- $V_1 : (\beta_1), (\beta_2)$
- $V_2 : (\beta_1\beta_1), (\beta_1\beta_2), (\beta_2\beta_1), (\beta_2\beta_2)$
- $V_3 : (\beta_1\beta_1\beta_1), (\beta_1\beta_1\beta_2), (\beta_1\beta_2\beta_1), (\beta_1\beta_2\beta_2), (\beta_2\beta_1\beta_1), (\beta_2\beta_1\beta_2), (\beta_2\beta_2\beta_1), (\beta_2\beta_2\beta_2)$, \ldots

Let $f_l$ be the $l$-th Fibonacci number for $l \in \mathbb{N}$. They are inductively defined by

$$f_1 = f_2 = 1, \quad f_{l+2} = f_{l+1} + f_l \quad \text{for } l \in \mathbb{N}.$$

By the structure of the $\lambda$-graph system $\mathcal{G}^{Ch(D_F)}$, the number $m(l)$ of the vertex set $V_l$ is $f_{l+2}$. We denote by $(\mathcal{M}_{l+1}, I_{l+1})_{l \in \mathbb{Z}_+}$ the symbolic matrix system of the Cantor horizon $\lambda$-graph system $\mathcal{G}^{Ch(D_F)}$. We write the vertex set $V_l$ as $\{v^l_1, \ldots, v^l_{m(l)}\}$.

Both the matrices $\mathcal{M}_{l+1}$ and $I_{l+1}$ are the $m(l) \times m(l+1)$ matrices for each $l \in \mathbb{Z}_+$. For $i = 1, \ldots, m(l)$, $j = 1, \ldots, m(l+1)$, the component $\mathcal{M}_{l+1}(i,j)$ denotes the formal sum of labels of edges starting at the vertex $v_i^l$ and terminating at the vertex $v_j^{l+1}$, and the component $I_{l+1}(i,j)$ denotes 1 if $l(v_{i+1}^l = v_j^l$, otherwise 0. They satisfy the relations $I_{l+1}\mathcal{M}_{l+1} = \mathcal{M}_{l+1}I_{l+1}$ for $l \in \mathbb{Z}_+$ as symbolic matrices.

The orderings of the rows and columns of the matrices are arranged lexicographically on indices $i_1, \ldots, i_n$ of the words $\beta_{i_1}, \ldots, \beta_{i_n}$ from the left. Let us denote by $0_{p,q}$ the $m(p) \times m(q)$ matrix all of whose entries are 0's.

**Lemma 3.1.** The $(m(l) \times m(l+1))$ matrix $I_{l+1}$ is given by:

$$I_{0,1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 \end{bmatrix}, \quad I_{1,2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_{l+2,l+3} = \begin{bmatrix} I_{l+1,l+2} & 0_{l+1,l+1} \\ 0_{l+1,l+2} & I_{l+1,l+1} \end{bmatrix}, \quad l \in \mathbb{Z}_+.$$

In what follows, blanks at components of matrices denote 0's. For $l \in \mathbb{Z}_+$ and $a \in \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$, let $I_l(a)$ be the $m(l) \times m(l)$ diagonal matrix with diagonal entries $a$, and $S_l(a)$ the $(m(l-1) \times m(l+1))$ matrix defined by

$$S_0(a) = \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \quad S_1(a) = \begin{bmatrix} a & a \\ a & a \\ a & a \end{bmatrix}, \quad S_{l+2}(a) = \begin{bmatrix} S_{l+1}(a) & 0_{l+1,l+1} \\ 0_{l+1,l+2} & S_l(a) \end{bmatrix}$$

where $m(-1)$ denotes 1. For $l = 2, 3, 4$ and $a \in \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$, one sees that

- $S_2(a) = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$ : $2 \times 5$ matrix,
- $S_3(a) = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$ : $3 \times 8$ matrix,
and
\[
S_1(a) = \begin{bmatrix}
  a & a & a \\
  a & a & a \\
  a & a & a \\
  a & a & a \\
  a & a & a
\end{bmatrix} : 5 \times 13 \text{ matrix}
\]

**Lemma 3.2.** The \( m(l) \times m(l + 1) \) matrix \( M_{l+1} \) is given by:
\[
M_{0,1} = [\alpha_1 + \beta_1 + \beta_2, \alpha_2 + \beta_1],
M_{l+1} = \begin{bmatrix}
  S_1(\beta_1) \\
  S_1(\beta_2)
\end{bmatrix} + \binom{0}{0, l+1}. \]

**Proof.** In the right hand side in the second equation above, the first summand describes the transitions that arise when a vertex accepts a symbol in \( \Sigma^+ \). The second summand describes the transitions that arise when a vertex accepts a symbol in \( \Sigma^- \).

We present the above matrices for \( l = 1, 2, 3, 4 \):
\[
I_{1,2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_{1,2} = \begin{bmatrix}
  \alpha_1 + \beta_1 & \beta_1 & \alpha_2 + \beta_1 \\
  \beta_2 & \alpha_1 + \beta_2 & \alpha_2 + \beta_1
\end{bmatrix},
I_{2,3} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_{2,3} = \begin{bmatrix}
  \alpha_1 + \beta_1 & \beta_1 & \beta_1 & \alpha_2 + \beta_1 \\
  \beta_2 & \alpha_1 + \beta_2 & \beta_1 & \alpha_2 + \beta_1
\end{bmatrix},
I_{3,4} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_{3,4} = \begin{bmatrix}
  \alpha_1 + \beta_1 & \beta_1 & \beta_1 & \beta_1 & \alpha_2 + \beta_1 \\
  \beta_2 & \beta_2 & \alpha_1 & \beta_1 & \alpha_2 + \beta_1
\end{bmatrix},
I_{4,5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad M_{4,5} = \begin{bmatrix}
  \alpha_1 + \beta_1 & \beta_1 & \beta_1 & \beta_1 & \beta_1 & \alpha_2 + \beta_1 & \alpha_2 + \beta_1 \\
  \beta_2 & \beta_2 & \beta_2 & \alpha_1 & \beta_1 & \beta_1 & \alpha_2 + \beta_1
\end{bmatrix}.
\]

Let \( (M_{l+1}, \bar{J}_{l+1}) \in \mathbb{Z}_+ \) be the nonnegative matrix system for \( (M, I) \). The matrix \( M_{l+1} \) for each \( l \in \mathbb{Z}_+ \) is obtained from \( M_{l+1} \) by setting all the symbols of \( M_{l+1} \) equal to 1. That is, the \((i, j)\)-component \( M_{l+1}(i, j) \) of the matrix \( M_{l+1} \) denotes the number of the symbols in \( \Sigma \) that appear in \( M_{l+1}(i, j) \). The groups \( K_0(\mathcal{O}_{\Sigma^H(D)}, K_1(\mathcal{O}_{\Sigma^H(D)}) \) are realized as the \( K \)-groups \( K_0(\mathcal{M}, I) \) and \( K_1(\mathcal{M}, I) \) for the nonnegative matrix system \( (M, I) \) respectively (cf. [18]). They are calculated by the following formulae.

**Lemma 3.3** ([18], cf. [17]).

(i) \( K_0(\mathcal{O}_{\Sigma^H(D)}) = \lim_{l \to \infty} \{ Z^{m(l+1)}/(M_{l+1} - I_{l+1})Z^{m(l)}, \bar{J}_{l+1} \} \) where the inductive limit is taken along the natural induced homomorphisms \( \bar{H}_{l+1}, l \in \mathbb{Z}_+ \), by the matrices \( I_{l+1} \).
(ii) $K_1(O_{\mathcal{CH}(D_F)}) = \lim_{l \to} \{\text{Ker}(M^t_{l,l+1} - I^t_{l,l+1}) \text{ in } \mathbb{Z}^{m(l)}, I^t_{l,l+1}\}$, where the inductive limit is taken along the homomorphisms of the restrictions of $I^t_{l,l+1}$ to $\text{Ker}(M^t_{l,l+1} - I^t_{l,l+1})$.

By the formulae of $M^t_{l,l+1}$ in Lemma 3.2, the matrices $M^t_{l,l+1} - I^t_{l,l+1}$ for $l = 1, 2, 3, 4$ are presented as in the following way:

$$M^t_{1,2} - I^t_{1,2} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 \\ 1 & -1 \end{bmatrix},$$

$$M^t_{2,3} - I^t_{2,3} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix},$$

$$M^t_{3,4} - I^t_{3,4} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix},$$

and

$$M^t_{4,5} - I^t_{4,5} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix}.$$

and
Lemma 3.4. For \( l = 3, 4, \ldots \), we have

\[
\begin{align*}
K_{l+1,l+1}^{UL} & = \begin{bmatrix} K_{l+1,l+1}^{UL} & 0_{l-1,l-2}^T \\ 0_{l-1,l-2} & I_{l-1,l-2} \end{bmatrix}, \\
K_{l+1,l+1}^{UR} & = \begin{bmatrix} S_{l-1}^T & 0_{l,l-3} \\ 0_{l,l-3} & K_{l+1,l+1}^{LR} \end{bmatrix}, \\
K_{l+1,l+1}^{LL} & = \begin{bmatrix} I_{l-1} & K_{l+1,l+1}^{UR} \\ 0_{l-2,l-1} & K_{l+1,l+1}^{LR} \end{bmatrix}, \\
K_{l+1,l+1}^{LR} & = \begin{bmatrix} K_{l+1,l+1}^{LR} & 0_{l-1,l-3}^T \\ 0_{l-2,l-2} & K_{l+1,l+1}^{LR} \end{bmatrix}.
\end{align*}
\]

Hence the sequence \( K_{l+1,l+1}, l \in \mathbb{N} \) of the matrices are inductively determined.
We set the \(m(l) \times m(l)\) square matrix \(B_t\) by setting

\[
B_t = \begin{bmatrix} A_{t+1,l}^{UL} & A_{t+1,l}^{UR} \end{bmatrix}
\]

the upper half of the matrix \(A_{t+1,l}\). We next provide a sequence \(C_{t+1,l}, l \in \mathbb{N}\) of \(m(l+1) \times m(l)\) matrix such as:

\[
C_{2,1} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad C_{3,2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad C_{4,3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},
\]

\[
C_{5,4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad C_{6,5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.
\]

To define the matrices \(C_{t+1,l}\) for \(l \geq 6\), divide \(C_{t+1,l}\) into 6 subblock matrices
\(C_{t+1,l}^{UL}, C_{t+1,l}^{UM}, C_{t+1,l}^{UR}, C_{t+1,l}^{LL}, C_{t+1,l}^{LM}, C_{t+1,l}^{LR}\) as in the following way:

\(C_{t+1,l}^{UL}(i,j) = C_{t+1,l-1}(i,j)\) for \(1 \leq i \leq m(l), 1 \leq j \leq m(l-2)\),

\(C_{t+1,l}^{UM}(i,j) = C_{t+1,l-1}(i,j + m(l - 2))\) for \(1 \leq i \leq m(l), 1 \leq j \leq m(l - 3)\),

\(C_{t+1,l}^{UR}(i,j) = C_{t+1,l-1}(i, j + m(l - 2) + m(l - 3))\) for \(1 \leq i \leq m(l), 1 \leq j \leq m(l - 2)\),

\(C_{t+1,l}^{LL}(i,j) = C_{t+1,l-1}(i + m(l), j)\) for \(1 \leq i \leq m(l - 1), 1 \leq j \leq m(l - 2)\),

\(C_{t+1,l}^{LM}(i,j) = C_{t+1,l-1}(i + m(l), j + m(l - 2))\) for \(1 \leq i \leq m(l - 1), 1 \leq j \leq m(l - 3)\),

\(C_{t+1,l}^{LR}(i,j) = C_{t+1,l-1}(i + m(l), j + m(l - 2) + m(l - 3))\) for \(1 \leq i \leq m(l - 1), 1 \leq j \leq m(l - 2)\).
They are an \(m(l) \times m(l - 2)\) matrix, an \(m(l) \times m(l - 3)\) matrix, an \(m(l) \times m(l - 2)\) matrix, an \(m(l - 1) \times m(l - 2)\) matrix, an \(m(l - 1) \times m(l - 3)\) matrix and an \(m(l - 1) \times m(l - 2)\) matrix respectively such that

\[
C_{l+1,l} = \begin{bmatrix}
C_{UL,l+1,l}^{UL} & C_{UM,l+1,l}^{UM} & C_{UR,l+1,l}^{UR} \\
C_{LL,l+1,l}^{UL} & C_{LM,l+1,l}^{LM} & C_{LR,l+1,l}^{LR}
\end{bmatrix}.
\]

These block matrices are defined inductively as in the following way:

\[
C_{UL,l+1,l}^{UL} = \begin{bmatrix}
C_{UL,l,l-1}^{UL}
0_{l,l-4}
\end{bmatrix},
C_{UM,l+1,l}^{UM} = \begin{bmatrix}
C_{UM,l,l-1}^{UL}
0_{l,l-5}
\end{bmatrix},
C_{UL,l+1,l}^{UL} = \begin{bmatrix}
0_{l,l-2}
\end{bmatrix},
C_{LL,l+1,l}^{LL} = \begin{bmatrix}
C_{UL,l-1,l-2}^{UL}
C_{LL,l-1,l-2}
\end{bmatrix},
C_{LM,l+1,l}^{LM} = \begin{bmatrix}
C_{UL,l-1,l-2}^{UL}
C_{LM,l-1,l-2}
\end{bmatrix},
C_{LR,l+1,l}^{LR} = \begin{bmatrix}
0_{l,l-2}
\end{bmatrix}.
\]

Let \(L_{l+1,l}\) be the \(m(l+1) \times m(l)\) matrix defined by the block matrix:

\[
L_{l+1,l} = \begin{bmatrix}
L_{UL,l+1,l}^{UL} & L_{UR,l+1,l}^{UR} \\
L_{LL,l+1,l}^{LL} & L_{LR,l+1,l}^{LR}
\end{bmatrix}
\]

where

\[
L_{UL,l+1,l}^{UL} = A_{l+1,l}^{UL} : m(l) \times m(l - 1) \text{ matrix},
L_{UR,l+1,l}^{UR} = \begin{bmatrix}
0_{l,l-2}
\end{bmatrix} : m(l) \times m(l - 2) \text{ matrix},
L_{LL,l+1,l}^{LL} = A_{l+1,l}^{LL} : m(l - 1) \times m(l - 1) \text{ matrix},
L_{LR,l+1,l}^{LR} = -I_{l-2,l-1} - C_{l-1,l-2} : m(l - 1) \times m(l - 2) \text{ matrix}.
\]

We write down the above matrices for \(l = 1, 2, 3, 4\).

\[
L_{2,1} = \begin{bmatrix}
1 & 2 \\
2 & -3
\end{bmatrix},
L_{3,2} = \begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -3
\end{bmatrix},
L_{4,3} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -3 & 1 \\
1 & 1 & -3 & 2
\end{bmatrix},
\]
We define the elementary column operations on integer matrices to be:

1. Multiply a column by $-1$,
2. Add an integer multiple of one column to another column.

The elementary row operations are similarly defined. We know that the matrices $L_{l+1,l}$ is obtained from $A_{l+1,l}$ by elementary column operations, that operation is denoted by $\Gamma_l$. The operation $\Gamma_l$ is an $m(l) \times m(l)$ matrix corresponding to the column operation such that

$$L_{l+1,l} = A_{l+1,l} \Gamma_l.$$ 

Since

$$L_{l+1,l} = \begin{bmatrix} 1 & 0_{l-1,l-2} \\ A_{l+1,l}^{UL} & B_{l-2} \\ A_{l+1,l}^{LL} & -I_{l-2,l-1} - C_{l-1,l-2} \end{bmatrix},$$

we may apply the elementary column operation $I_{l-1} \oplus \Gamma_{l-2}$ to $L_{l+1,l}$ so that the matrix $B_{l-2}$ in $L_{l+1,l}$ goes to

$$\begin{bmatrix} A_{l-1,l-2}^{UL} & 0_{l-3,l-1} \\ & B_{l-4} \end{bmatrix}.$$ 

The new matrix $L_{l+1,l}(I_{l-1} \oplus \Gamma_{l-2})$ is

$$L_{l+1,l}(I_{l-1} \oplus \Gamma_{l-2}) = \begin{bmatrix} A_{l+1,l}^{UL} & 0_{l-1,l-2} \\ & B_{l-2} \\ & -I_{l-2,l-1} - C_{l-1,l-2} \end{bmatrix}.$$ 

As

$$B_{l-2n} \Gamma_{l-2n} = \begin{bmatrix} A_{l-2n+1,l-2n}^{UL} & 0_{l-2n-1,l-2n-2} \\ & B_{l-2n-2} \end{bmatrix}.$$
for \( n = 1, 2, \ldots \) with \( 2n < l \), by continuing these procedures \( k \)-times for \( l = 2k, 2k + 1 \) we finally get

\[
B_2 \Gamma_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & -1 & 2
\end{bmatrix}
\]

for \( l = 2k \) and

\[
B_1 \Gamma_1 = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\]

for \( l = 2k + 1 \).

For \( l = 2k, 2k + 1 \), let \( M_{l+1,l} \) be the \( m(l + 1) \times m(l) \) matrix obtained from \( L_{l+1,l} \) after the \( k \) times procedures above. Then we have

\[
M_{l+1,l}(i, j) = \begin{cases}
0 & \text{if } i < j, \ 1 \leq i, j \leq m(l) \\
1 & \text{if } i = j, \ 1 \leq i < m(l) \\
2 & \text{if } i = j = m(l).
\end{cases}
\]

Let \( v_l = [v_l(i)]_{i=1}^{m(l-1)} \) be the column vector of length \( m(l - 1) \) defined by

\[
v_l(i) = M_{l+1,l}(m(l) + i, m(l)), \quad i = 1, 2, \ldots, m(l - 1)
\]

so that the matrix \( M_{l+1,l} \) is of the form

\[
M_{l+1,l} = \begin{bmatrix}
1 & 1 & & & \\
& \ddots & \ddots & & \\
& & 2 & & \\
\text{v}_l(1) & \text{v}_l(2) & \cdots & \text{v}_l(m(l-1))
\end{bmatrix}
\]

For \( l = 1, 2, 3, 4, 5, 6 \), we see

\[
v_1 = \begin{bmatrix}
-3
\end{bmatrix}, \quad v_2 = \begin{bmatrix}
-3
\end{bmatrix}, \quad v_3 = \begin{bmatrix}
3 \\
3
\end{bmatrix}, \quad v_4 = \begin{bmatrix}
3 \\
1
\end{bmatrix}, \quad v_5 = \begin{bmatrix}
-3 \\
-3 \\
-3
\end{bmatrix}, \quad v_6 = \begin{bmatrix}
-3 \\
-3 \\
-3 \\
-3
\end{bmatrix}
\]

By induction, one has:

**Lemma 3.5.**

1. \( v_l(i) = \begin{cases}
-3 & \text{if } l = 4k + 1, 4k + 2, k \in \mathbb{Z}_+, \text{ and } 1 \leq i \leq m(l - 2), \\
3 & \text{if } l = 4k + 3, 4k + 4, k \in \mathbb{Z}_+, \text{ and } 1 \leq i \leq m(l - 2),
\end{cases} \)

2. \( v_l(m(l - 2) + i) = \overline{v}_{l-2}(i) \) for \( i = 1, 2, \ldots, m(l - 3) \)

where for \( u = \pm 3, \pm 1 \), the integer \( \hat{u} \) is defined by

\[
\hat{u} = \begin{cases}
 u - 4 & \text{if } u = 3, 1, \\
 u + 4 & \text{if } u = -3, -1.
\end{cases}
\]
and the relations

\[
\mathbb{H}_{2,1} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad \mathbb{H}_{3,2} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbb{H}_{4,3} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbb{H}_{5,4} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}.
\]

By elementary row operations compatible to \( I_{l,t+1}^T \), one gets the matrix \( N_{l+1,t} \) from the matrix \( M_{l+1,t} \). In the matrix \( N_{l+1,t} \), for \( i = 1, 2, \ldots, m(l) \), if \( v_l(i) = -3 \), then add the \( m(l) \)-th row to the \( i + m(l) \)-th row at the \( i + m(l) \)-th row, if \( v_l(i) = 3 \), then subtract the twice of \( m(l) \)-th row from the \( i + m(l) \)-th row at the \( i + m(l) \)-th row, if \( v_l(i) = -3 \), then subtract the \( m(l) \)-th row from the \( i + m(l) \)-th row at the \( i + m(l) \)-th row, then one gets the matrix \( \mathbb{H}_{l+1,t} \). These row operations are compatible to the map \( I_{l,t+1}^T \) and the relations

\[
I_{l,t+1}^T N_{l,t-1} = N_{l+1,t} I_{l-1,t}^T, \quad I_{l,t+1}^T \mathbb{H}_{l,t-1} = \mathbb{H}_{l+1,t} I_{l-1,t}^T,
\]

for \( l = 2, 3, \ldots \) hold. As

\[
(M_{l,t+1}^T - I_{l,t+1}^T)Z^{m(l)} = A_{l+1,t}Z^{m(l)} = L_{l+1,t}Z^{m(l)} = M_{l+1,t}Z^{m(l)}, \quad l \in \mathbb{N}
\]

we see that \( Z^{m(l+1)}/(M_{l,t+1}^T - I_{l,t+1}^T)Z^{m(l)} \) coincides with the group \( Z^{m(l+1)}/M_{l+1,t}Z^{m(l)} \) for all \( l \in \mathbb{N} \). We then have

**Proposition 3.6.** There exist isomorphisms

\[
\xi_l : Z^{m(l)}/M_{l,t-1}Z^{m(l-1)} \rightarrow Z^{m(l)}/N_{l,t-1}Z^{m(l-1)},
\]

\[
\eta_l : Z^{m(l)}/N_{l,t-1}Z^{m(l-1)} \rightarrow Z^{m(l)}/\mathbb{H}_{l,t-1}Z^{m(l-1)}
\]
of abelian groups such that the following diagrams are commutative:

\[
\begin{array}{ccc}
\mathbb{Z}^{m(l)} / (M_{l-1,l}^t - I_{l-1,l}) \mathbb{Z}^{m(l-1)} & \xrightarrow{\bar{P}_{l+1}} & \mathbb{Z}^{m(l+1)} / (M_{l+1,l}^t - I_{l+1,l}) \mathbb{Z}^{m(l)} \\
\mathbb{Z}^{m(l)} / N_{l-1} \mathbb{Z}^{m(l)} & \downarrow \xi_l & \mathbb{Z}^{m(l+1)} / N_{l+1} \mathbb{Z}^{m(l)} \\
\mathbb{Z}^{m(l)} / \mathbb{H}_{l-1,l-1} \mathbb{Z}^{m(l)} & \xrightarrow{\bar{P}_{l+1}} & \mathbb{Z}^{m(l+1)} / \mathbb{H}_{l+1,l} \mathbb{Z}^{m(l)}
\end{array}
\]

where \( \bar{P}_{l+1} : \mathbb{Z}^{m(l)} / \mathbb{H}_{l-1,l-1} \mathbb{Z}^{m(l-1)} \to \mathbb{Z}^{m(l+1)} / \mathbb{H}_{l+1,l} \mathbb{Z}^{m(l)} \) is the homomorphism induced by the matrix \( I_{l+1,l}^R \). Hence we have an isomorphism

\[
K_0(\mathcal{O}_{E^{ch}(D_F)}) \cong \lim_{\mathbb{N}} \{ \bar{P}_{l+1} : \mathbb{Z}^{m(l)} / \mathbb{H}_{l-1,l-1} \mathbb{Z}^{m(l-1)} \to \mathbb{Z}^{m(l+1)} / \mathbb{H}_{l+1,l} \mathbb{Z}^{m(l)} \}.
\]

We fix \( l \geq 3 \). Define the \((m(l-1)+1) \times 1\) matrix \( R_{l-1} \) and the \((m(l-1)+1) \times (m(l-2)+1)\) matrix \( I_{l-1,l-2}^R \) by setting:

\[
R_{l-1} = \begin{bmatrix}
2 \\
\vdots \\
-1
\end{bmatrix}, \quad I_{l-1,l-2}^R = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \ddots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & I_{l-2,l-1}
\end{bmatrix}.
\]

Then the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{Z}^{m(l)} / \mathbb{H}_{l-1,l-1} \mathbb{Z}^{m(l-1)} & \xrightarrow{\bar{P}_{l+1}} & \mathbb{Z}^{m(l+1)} / \mathbb{H}_{l+1,l} \mathbb{Z}^{m(l)} \\
\mathbb{Z}^{m(l-2)+1} / R_{l-2} \mathbb{Z} & \xrightarrow{\bar{P}_{l+1}^R} & \mathbb{Z}^{m(l-1)+1} / R_{l-1} \mathbb{Z}
\end{array}
\]

where \( \bar{P}_{l+1}^R \) is the homomorphism induced by the matrix \( I_{l-1,l}^R \). Let \( \varphi_{l-2} : \mathbb{Z}^{m(l-2)+1} \to \mathbb{Z}^{m(l-2)+1} \) be an isomorphism defined by the operations on the row vectors of \( \mathbb{Z}^{m(l-2)+1} \) to add the 2-times multiplication of the second row to the first row, and subtract the second row from the \( k \)-th rows for \( k = 3, 4, \ldots, m(l-2)+1 \). It is given by the matrix:

\[
Q_{l-2} = \begin{bmatrix}
1 & 2 & \cdots & 0 \\
-1 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
-1 & \cdots & -1 & 1
\end{bmatrix}.
\]

Since \( Q_{l-2} R_{l-2} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix} \), \( \varphi_{l-2} \) yields an isomorphism

\[
\varphi_{l-2} : \mathbb{Z}^{m(l-2)+1} / R_{l-2} \mathbb{Z} \to \mathbb{Z} \oplus 0 \oplus \mathbb{Z}^{m(l-2)-1} = \mathbb{Z}^{m(l-2)}.
\]
Let $J_{l-1, l-2} : \mathbb{Z}^{m(l-2)-1} \rightarrow \mathbb{Z}^{m(l-1)-1}$ be a homomorphism defined by the $(m(l-1)-1) \times (m(l-2)-1)$ matrix

$$J_{l-1, l-2}(i, j) = \begin{cases} 0 & \text{if } i = 0, \\ I_{l-1, l-1}(i+1, j+1) & \text{if } i = 2, \ldots, m(l-2) - 1 \\ \end{cases}$$

for $i = 1, 2, \ldots, m(l-1) - 1$, $j = 1, 2, \ldots, m(l-2) - 1$. We set $\tilde{I}_{l-1, l-2} : \mathbb{Z}^{m(l-2)} \rightarrow \mathbb{Z}^{m(l-1)}$ a homomorphism defined by the $m(l-1) \times m(l-2)$ matrix

$$\tilde{I}_{l-1, l-2}(i, j) = \begin{cases} 1 & \text{if } i = j = 1, \\ 0 & \text{if } i = 1, j \geq 2, \\ 0 & \text{if } i = 2, \\ I_{l-2, l-1}(i, j) & \text{if } i = 3, 4, \ldots, m(l-2) - 1 \\ \end{cases}$$

for $i = 1, 2, \ldots, m(l-1)$, $j = 1, 2, \ldots, m(l-2)$. That is,

$$\tilde{I}_{l-1, l-2} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ 0 & \cdots & \ddots & \cdots \\ 0 & \cdots & \cdots & J_{l-1, l-2} \end{bmatrix}.$$ 

Lemma 3.7. The diagram

$$\begin{array}{ccc} \mathbb{Z}^{m(l-2)+1}/R_{l-2}\mathbb{Z} & \xrightarrow{\varphi_{l-2}} & \mathbb{Z}^{m(l-1)+1}/R_{l-1}\mathbb{Z} \\ \downarrow \varphi_{l-2} & & \downarrow \varphi_{l-1} \\ \mathbb{Z}^{m(l-2)} & \xrightarrow{\tilde{I}_{l-1, l-2}} & \mathbb{Z}^{m(l-1)} \end{array}$$

is commutative. Hence we have an isomorphism

$$K_0(\mathcal{O}_{\mathcal{O}_{\mathbb{C}P^1(D_F)}}) \cong \mathbb{Z} \oplus \lim_{\text{lim}} \{ J_{l-1, l-2} : \mathbb{Z}^{m(l-2)-1} \rightarrow \mathbb{Z}^{m(l-1)-1} \}.$$ 

Proof. Since the commutativity $\varphi_{l-1} \circ \tilde{I}_{l-1, l-2} = \varphi_{l-2} \circ \tilde{I}_{l-1, l-2}$ is immediate, one has

$$K_0(\mathcal{O}_{\mathcal{O}_{\mathbb{C}P^1(D_F)}}) \cong \lim_{\text{lim}} \{ \tilde{I}_{l-1, l-2} : \mathbb{Z}^{m(l-2)} \rightarrow \mathbb{Z}^{m(l-1)} \}.$$ 

As $\tilde{I}_{l-1, l-2} = 1 \oplus J_{l-1, l-2}$, the assertion is clear. \qed

We will compute the group of the inductive limit $\lim_{\text{lim}} \{ J_{l+1, l} : \mathbb{Z}^{m(l)-1} \rightarrow \mathbb{Z}^{m(l+1)-1} \}$, that we denote by $G$. Let $I_{l+1, l}$ be the $(m(l) - 2) \times (m(l) - 1)$ matrix defined by

$$I_{l+1, l}(i, j) = I_{l, l+1}(i+2, j+1) \quad \text{for} \quad i = 1, \ldots, m(l) - 2, \quad j = 1, \ldots, m(l) - 1.$$ 

Hence $J_{l+1, l} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$. It gives rise to a homomorphism:

$I_{l+1, l} : \mathbb{Z}^{m(l)-1} \rightarrow \mathbb{Z} \oplus I_{l+1, l} \mathbb{Z}^{m(l)-1} \subset \mathbb{Z}^{m(l)-1}$.

Put

$$\mathbb{Z}(l) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} = \mathbb{Z}^{m(l)-1}.$$
For $k \in \mathbb{N}$, take $l \in \mathbb{Z}_+$ such that $k \leq m(l)$. Define a sequence of positive integers

$$g_k = \sum_{j=1}^{k} \sum_{i=2}^{m(l+1)} I_{l,t+1}(i,j), \quad k = 1, 2, \ldots$$

that is independent of the choice of $l$, so that

$$g_1 = 1, \quad g_2 = 2, \quad g_3 = 4, \quad g_4 = 6, \quad g_5 = 7, \quad g_6 = 9, \ldots$$

Define for $l \geq k$,

$$Z(l; k) = \bigoplus_{n=0}^{g_k} \bigoplus_{n=0}^{m(l)-1-g_k} \mathbb{Z} \subset \mathbb{Z}^{m(l)-1} = \mathbb{Z}(l)$$

so that we have

$$I_{l+1,l}(Z(l; k)) \subset Z(l+1; k+1).$$

Set the group of the inductive limit

$$G_k = \varinjlim_n \{ I_{k+n+1,k+n} : \mathbb{Z}(k+n) \rightarrow \mathbb{Z}(k+n+1) \}.$$ 

Since the following diagram is commutative:

\[
\begin{array}{cccccccc}
Z(1) & \xrightarrow{I_{2,1}} & Z(2; 1) & \xrightarrow{I_{3,2}} & Z(3; 2) & \xrightarrow{I_{4,3}} & Z(4; 3) & \xrightarrow{I_{5,4}} & \cdots & \longrightarrow & G_1 \\
\downarrow{\iota} & & \downarrow{\iota} & & \downarrow{\iota} & & \downarrow{\iota} & & \cdots & & \\
Z(2) & \xrightarrow{I_{3,2}} & Z(3; 1) & \xrightarrow{I_{4,3}} & Z(4; 2) & \xrightarrow{I_{5,4}} & \cdots & \longrightarrow & G_2 \\
\downarrow{\iota} & & \downarrow{\iota} & & \downarrow{\iota} & & \downarrow{\iota} & & \cdots & & \\
Z(4) & \xrightarrow{I_{5,4}} & Z(4; 1) & \xrightarrow{I_{6,5}} & \cdots & \longrightarrow & G_3 \\
\downarrow{\iota} & & & & \downarrow{\iota} & & \cdots & & \longrightarrow & G_4 \\
\vdots & & & & & & & & & \ddots & &
\end{array}
\]

where the vertical arrows $\iota$ mean the natural inclusion maps, one sees the next lemma:

**Lemma 3.8.**

(i) For each $k = 1, 2, \ldots$, the group $G_k$ is isomorphic to the abelian group $C(\mathbb{R}_k, \mathbb{Z})$ of all integer valued continuous functions on a Cantor discontinuum $\mathbb{R}_k$.

(ii) The sequence $G_k, k = 1, 2, \ldots$ are increasing whose union generate $G$.

Hence one has

**Lemma 3.9.** The group $G$ is isomorphic to the countable direct sum of the group $C(\mathbb{R}, \mathbb{Z})$ of all integer valued continuous functions on a Cantor discontinuum $\mathbb{R}$. 

16
Proof. It is easy to see that $G_k$ is isomorphic to the direct sum $C(\mathcal{R}_{k,k-1}, \mathbb{Z}) \oplus G_{k-1}$ of all integer valued continuous functions on a Cantor discontinuum $\mathcal{R}_{k,k-1}$ and $G_{k-1}$ for each $k$. Hence we have

$$G_k \cong C(\mathcal{R}_{k,k-1}, \mathbb{Z}) \oplus G_{k-1} \cong C(\mathcal{R}_{k,k-1}, \mathbb{Z}) \oplus C(\mathcal{R}_{k-1,k-2}, \mathbb{Z}) \oplus \cdots \oplus C(\mathcal{R}_{2,1}, \mathbb{Z}) \oplus G_1.$$ 

Since both $G_1$ and $C(\mathcal{R}_{i,i-1}, \mathbb{Z})$ are isomorphic to the group $C(\mathcal{R}, \mathbb{Z})$ of all integer valued continuous functions on a Cantor discontinuum $\mathcal{R}$, we have

$$G \cong \lim_{k \to \infty} G_k \cong C(\mathcal{R}, \mathbb{Z})^\infty.$$ 

Therefore we conclude

**Theorem 3.10.**

$$K_0(\mathcal{O}_{\mathbb{C}^{Ch}(D_F)}) \cong \mathbb{Z} \oplus C(\mathcal{R}, \mathbb{Z})^\infty, \quad K_1(\mathcal{O}_{\mathbb{C}^{Ch}(D_F)}) \cong 0.$$

**Proof.** Since $K_0(\mathcal{O}_{\mathbb{C}^{Ch}(D_F)})$ is isomorphic to

$$\mathbb{Z} \oplus \lim_{l \to \infty} \{J_{l+1,l} : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}\},$$

and the second summand above denoted by $G$ is isomorphic to $C(\mathcal{R}, \mathbb{Z})^\infty$, one gets $K_0(\mathcal{O}_{\mathbb{C}^{Ch}(D_F)}) \cong \mathbb{Z} \oplus C(\mathcal{R}, \mathbb{Z})^\infty$. We have already seen the formula $K_1(\mathcal{O}_{\mathbb{C}^{Ch}(D_F)}) \equiv 0$. Therefore Theorem 1.1 holds.

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