Lagrange’s Theorem For Hom-Groups

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Abstract
Hom-groups are nonassociative generalizations of groups where the unitality and associativity are twisted by a map. We show that a Hom-group \((G, \alpha)\) is a pointed idempotent quasigroup (pique). We use Cayley table of quasigroups to introduce some examples of Hom-groups. Introducing the notions of Hom-subgroups and cosets we prove Lagrange’s theorem for finite Hom-groups. This states that the order of any Hom-subgroup \(H\) of a finite Hom-group \(G\) divides the order of \(G\). We linearize Hom-groups to obtain a class of nonassociative Hopf algebras called Hom-Hopf algebras. As an application of our results, we show that the dimension of a Hom-sub-Hopf algebra of the finite dimensional Hom-group Hopf algebra \(\mathbb{K}G\) divides the order of \(G\). The new tools introduced in this paper could potentially have applications in theories of quasigroups, nonassociative Hopf algebras, Hom-type objects, combinatorics, and cryptography.

1 Introduction

Nonassociative objects such as quasigroups, loops, non-associative algebras, and Hopf algebras have many applications in several contexts. Among all of these, Hom-type objects have been under intensive research in the last decade. Hom-Lie algebras have appeared in quantum deformations of Witt and Virasoro algebras [AS], [CKL], [CZ]. Hom-Lie algebras, [HLS], are generalizations of Lie algebras where Jacobi identity is twisted by a linear map. The Witt algebra is the complexification of the Lie algebra of polynomial vector fields on a circle with a basis \(L_n = -z^{n+1}\frac{\partial}{\partial z}\) and the Lie bracket which is given by \([L_m, L_n] = (m - n)L_{m+n}\). This Lie algebra can also be viewed as the Lie algebra of derivations \(D\) of the ring \(\mathbb{C}[z, z^{-1}]\), where \(D(ab) = D(a)b + aD(b)\). The Lie bracket of two derivations \(D\) and \(D'\) is given by \([D, D'] = D \circ D' - D' \circ D\). This algebra has a central extension, called the Virasoro algebra, which appears in two-dimensional conformal field theory and string theory. One can define the quantum deformation of \(D\) given by \(D_q(f)(z) = \frac{f(qz) - f(z)}{qz - z}\). These linear operators are different from the usual derivations and they satisfy \(\sigma\)-derivation property \(D_q(fg) = gD_q(f) + \sigma(f)D_q(g)\) where \(\sigma(f)(z) = f(qz)\). An example of a \(\sigma\)-derivation is the Jackson derivative on polynomials in one variable.
The set of $\sigma$-derivations with the classical bracket is a new type of algebra so called $\sigma$-deformations of the Witt algebra. This algebra does not satisfy the Jacobi identity. Instead, it satisfies Hom-Jacobi identity and it is called Hom-Lie algebra. The corresponding associative algebras, called Hom-associative algebras, were introduced in [MS1]. Any Hom-associative algebra with the bracket $[a, b] = ab - ba$ is a Hom-Lie algebra. Later, other nonassociative objects such as Hom-coalgebras [MS2], Hom-bialgebras [MS2], [MS3], [Ya2], [GMMP], and Hom-Hopf algebras [MS2], [Ya3], [Ya4], were introduced and studied.

Hom-groups are nonassociative objects which were recently appeared in the study of group-like elements of Hom-Hopf algebras [LMT]. Studying Hom-groups gives us more information about Hom-Hopf algebras. One knows that groups and Lie algebras have important rules to develop many concepts related to Hopf algebras. During the last years, Hom-Lie algebras had played important rules to understand the structures of Hom-Hopf algebras. However, a lack of the notion of Hom-groups can affect to miss some concepts which potentially can be extended from Hom-groups to Hom-group Hopf algebras and therefore possibly to all Hom-Hopf algebras. A Hom-group $(G, \alpha)$ is a set $G$ with a bijective map $\alpha : G \rightarrow G$ which is endowed with a multiplication that satisfies the Hom-associativity property $\alpha(a)(bc) = (ab)\alpha(c)$. Furthermore $G$ has the Hom-unit element 1 which satisfies $\alpha 1 = 1 \alpha = \alpha(a)$. Every element $g \in G$ has an inverse $g^{-1}$ satisfying $gg^{-1} = g^{-1}g = 1$. If $\alpha = \text{Id}$, then $G$ is a group. Although the twisting map $\alpha$ of a Hom-group $(G, \alpha)$ does not need to be invertible in the original works [LMT], [H1], however, more interesting results, including the main results of this paper, are obtained if $\alpha$ is invertible. As a result through the paper, we assume that the twisting map $\alpha$ is bijective. Since any Hom-group gives rise to a Hom-Hopf algebra, called Hom-group Hopf algebra [H2], it is interesting to know what properties will be enforced by the invertibility of $\alpha$ on Hom-Hopf algebras. It is shown in [CG] that the category of modules over a Hom-Hopf algebra with invertible twisting map is monoidal. For this reason, they called them monoidal Hom-Hopf algebras. Also many interesting properties of Hom-Hopf algebras, such as integrals, modules, comodules, and Hopf representations are obtained when $\alpha$ is bijective, see [CWZ], [H2], [PSS] [ZZ]. The author in [H1] introduced some basics of Hom-groups, their representations, and Hom-group (co)homology. They showed that the Hom-group (co)homology is related to the Hochschild (co)homology, [HSS], of Hom-group algebras.

Lagrange-type’s theorem for nonassociative objects is a nontrivial problem. For instance, whether Lagrange’s theorem holds for Moufang loops was an open problem in the theory of Moufang loops for more than four decades [CKRV]. In fact, not every loop satisfies the Lagrange property and the problem was finally answered in [GZ]. The authors in [BS] proved a version of Lagrange’s theorem for Bruck loops. The strong Lagrange property was shown for left Bol loops of odd order in [FKP]. However, it is still an open problem whether Bol loops satisfy the Lagrange property. The authors in [SW] proved Lagrange’s theorem for gyrogroups which are a class of Bol loops. In this paper, we prove Lagrange’s theorem for Hom-groups which are an interesting class of quasigroups. More precisely, in Section 2, we introduce some fundamental concepts of Hom-groups such as Hom-subgroups, cosets,
center, and the centralizer of an element. In Theorem 2.11 we show that any Hom-group is a quasigroup. This means division is always possible to solve the equations $ax = b$ and $ya = b$. The unit element 1 is an idempotent element and in fact, this class of interesting quasigroups is known as pointed idempotent quasigroups (piques) [BH]. Indeed, a Hom-group is a special case of piques which satisfies certain twisted associativity condition given by the idempotent element 1. If a Hom-group $G$ is a loop then $1x = \alpha(x) = x$ which means $\alpha = \text{Id}$, and therefore $G$ should be a group. We use properties of Cayley table of Hom-groups to present some examples. The Cayley tables of quasigroups have been used in combinatorics and cryptography, see [CPS], [DK], [SC], [BBW]. In Section 3, we use cosets to partition a Hom-group $G$. Then we prove Lagrange’s theorem for Hom-groups which states that for any finite Hom-group, the order of any Hom-subgroup divides the order of the Hom-group. In Section 4, we apply Lagrange’s theorem to show that for a Hom-sub-Hopf algebra $A$ of $KG$, $\text{dim}(A)$ divides $|G|$. We finish the paper by raising some conjectures about Hom-groups.

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2 Hom-groups

In this section, we introduce basic notions for a Hom-group $(G, \alpha)$ such as Hom-subgroups, cosets, and the center. Our definition of a Hom-group is a special case of the one discussed in [LMT] and [H1]. Through the paper, we assume the map $\alpha$ is invertible. Therefore our axioms will be different from the ones in the original definition. We show that some of the axioms can be obtained by Hom-associativity when $\alpha$ is invertible.

Definition 2.1. A Hom group consists of a set $G$ together with a distinguished member $1 \in G$, a bijective set map: $\alpha : G \to G$, a binary operation $\mu : G \times G \to G$, where these pieces of structure are subject to the following axioms:

i) The product map $\mu : G \times G \to G$ is satisfying the Hom-associativity property

$$\mu(\alpha(g), \mu(h, k)) = \mu(\mu(g, h), \alpha(k)).$$

For simplicity when there is no confusion we omit the multiplication sign $\mu$.

ii) The map $\alpha$ is multiplicative, i.e, $\alpha(gk) = \alpha(g)\alpha(k)$. 

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iii) The element 1 is called unit and it satisfies the Hom-unitality conditions
\[ g1 = 1g = \alpha(g), \quad \alpha(1) = 1. \]

v) For every element \( g \in G \), there exists an element \( g^{-1} \in G \) which
\[ gg^{-1} = g^{-1}g = 1. \]

Based on the definition of a Hom-group in [LMT, H1, H2], for any element \( g \in G \) there exists a natural number \( n \) satisfying the Hom-invertibility condition \( \alpha^n(gg^{-1}) = \alpha^n(g^{-1}g) = 1 \), where the smallest such \( n \) is called the invertibility index of \( g \). Clearly if \( \alpha \) is invertible this condition will be simplified to our condition (v). One also notes that the condition (iii) implies \( \mu(1, 1) = 1 \). This shows that 1 is an idempotent element. The following lemma is crucial for our studies of Hom-groups.

**Lemma 2.2.** Let \((G, \alpha)\) be a Hom-group. Then

i) The inverse of any element is unique.

ii) \((ab)^{-1} = b^{-1}a^{-1}\) for all \( a, b \in G \).

**Proof.** i) First we show that the right inverse is unique. Let us assume an element \( g \in G \) has right inverses \( a, b \in G \). So \( ga = 1 \) and \( gb = 1 \). Since there exists \( g^{-1} \in G \) where \( g^{-1}g = 1 \), then
\[ \alpha(g^{-1})(ga) = \alpha(g^{-1})1. \]

By Hom-associativity we have
\[ (g^{-1}g)\alpha(a) = \alpha^2(g^{-1}) \]

By Hom-unitality we obtain \( \alpha^2(a) = \alpha^2(g^{-1}) \). Since \( \alpha \) is invertible then \( a = g^{-1} \). Similarly \( \alpha \) is invertible and therefore \( a = b \). Likewise we can prove that left inverse is unique. Now we show that left and right inverses are the same. Let \( g \in G \), \( ag = 1 \) and \( gb = 1 \). Therefore
\[ (ag)\alpha(b) = 1\alpha(b). \]

By Hom-associativity we have \( \alpha(a)(gb) = \alpha^2(b) \). Then \( \alpha(a)1 = \alpha^2(b) \). So \( \alpha^2(a) = \alpha^2(b) \).

By invertibility of \( \alpha \) we obtain \( a = b \). Therefore the inverse element is unique.

ii) The following computations shows that \( b^{-1}a^{-1} \) is the inverse of \( ab \).

\[
\begin{align*}
(ab)(b^{-1}a^{-1}) & = \alpha(\alpha^{-1}(ab))[b^{-1}a^{-1}] \\
& = [\alpha^{-1}(ab)b^{-1}]\alpha(a^{-1}) \\
& = ([\alpha^{-1}(a)\alpha^{-1}(b)b^{-1}] \alpha(a^{-1}) \\
& = ([\alpha^{-1}(a)\alpha^{-1}(b)]\alpha^{-1}(b^{-1})) \alpha(a^{-1}) \\
& = [a (\alpha^{-1}(b)\alpha^{-1}(b^{-1}))] \alpha(a^{-1}) \\
& = (a1)\alpha(a^{-1}) \\
& = \alpha(a)\alpha(a^{-1}) = 1.
\end{align*}
\]
We used the Hom-associativity in the third equality, multiplicity of $\alpha$ in the fourth equality, and Hom-associativity in the fifth equality. 

The following proposition which was introduced in [H1] provides a source of examples for Hom-groups.

**Proposition 2.3.** Let $(G, \mu)$ be a group and $\alpha : G \to G$ a group automorphism. Then $(G, \alpha \circ \mu, \alpha)$ is a Hom-group.

**Definition 2.4.** A subset $H$ of a Hom-group $(G, \alpha)$ is called a Hom-subgroup [H1] of $G$ if $(H, \alpha)$ is itself a Hom-group. We denote a Hom-subgroup $H$ of $G$ by $H \leq G$.

**Definition 2.5.** The set $gH = \{gh, \ h \in H\}$ is called the left coset of the Hom-subgroup $H$ in $G$ with respect to the element $g$. Similarly the set $Hg = \{hg, \ h \in H\}$ is called the right coset of $H$ in $G$.

We denote the number of elements of a Hom-group $G$ by $|G|$.

**Definition 2.6.** The Center $Z(G)$ of a Hom-group $(G, \alpha)$ is the set of all $x \in G$ where $xy = yx$ for all $y \in G$.

**Proposition 2.7.** Let $(G, \alpha)$ be a Hom-group. Then $Z(G) \simeq G$.

**Proof.** Let $x, y \in Z(G)$. Then for all $a \in G$ we have

\[(xy)a = (xy)(\alpha^{-1}(a)) = \alpha(x)(y\alpha^{-1}(a)) = \alpha(x)[\alpha^{-1}(a)y] = [x\alpha^{-1}(a)]\alpha(y) = [\alpha^{-1}(a)x]\alpha(y) = a(xy).\]

Thus $Z(G)$ is closed under multiplication of $G$. Also if $x \in Z(G)$ and $a \in G$ then $xa^{-1} = a^{-1}x$. Then $(xa^{-1})^{-1} = (a^{-1}x)^{-1}$. Therefore $ax^{-1} = x^{-1}a$. So $x^{-1} \in Z(G)$. Therefore $Z(G)$ is a Hom-subgroup of $G$.

**Definition 2.8.** The centralizer of an element $x \in G$ is the set of all elements $g \in G$ where $gx = xg$ and it is denoted by $C_G(x)$.

**Proposition 2.9.** Let $(G, \alpha)$ be a Hom-group. Then $C_G(x) \simeq G$ for all $x \in G$.

**Proof.** The proof is similar to previous Proposition.

Let $(G, \alpha)$ and $(H, \beta)$ be two Hom-groups. The morphism $f : G \to H$ is called a morphism of Hom-groups [H1], if $\beta(f(g)) = f(\alpha(g))$ and $f(gk) = f(g)f(k)$ for all $g, k \in G$. Two Hom-groups $G$ and $H$ are called isomorphic if there exist a bijective morphism of Hom-groups $f : G \to H$.

**Example 2.10.** Let $(G, \alpha)$ and $(G', \alpha')$ be two Hom-groups. Then $(G \times G', \alpha \times \alpha')$ is a Hom-group by the multiplication given by $(g, h)(g', h') = (gg', hh')$. 

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Here we recall the definition of a quasigroup. A quasigroup \((Q, \ast)\) is a set \(Q\) with a multiplication \(\ast : Q \times Q \rightarrow Q\), where for all \(a, b \in Q\), there exist unique elements \(x, y \in Q\) such that
\[
a \ast x = b, \quad y \ast a = b.
\]
In the following theorem we show that Hom-groups are a class of quasigroups.

**Theorem 2.11.** Every Hom-group \((G, \alpha)\) is a quasigroup.

**Proof.** First we show that the \(x = \alpha^{-1}(a^{-1})\alpha^{-2}(b)\) satisfies the equation \(ax = b\).

\[
a x = a (\alpha^{-1}(a^{-1})\alpha^{-2}(b)) = \alpha(\alpha^{-1}(a)) (\alpha^{-1}(a^{-1})\alpha^{-2}(b)) = (\alpha^{-1}(a)\alpha^{-1}(a^{-1})) \alpha^{-1}(b) = \alpha^{-1}(aa^{-1})\alpha^{-1}(b) = \alpha^{-1}(1)\alpha^{-1}(b) = 1\alpha^{-1}(b) = b.
\]

We used the Hom-associativity in the third equality, invertibility of \(a\) in the fifth equality and the Hom-unitality in the last equality. Similarly \(y = \alpha^{-2}(b)\alpha^{-1}(a^{-1})\) satisfies the equation \(ya = b\). Therefore \(G\) is a quasigroup.

Since the element \(1 \in G\) is an idempotent element then any Hom-group is a pointed idempotent quasigroup (piques). This is an interesting class of quasigroups which have been under intensive research [S], [CPS].

**Remark 2.12.** (Cayley table of finite Hom-groups)

Since every Hom-group is a quasigroup, then the Cayley table of a Hom-group has all the properties of the one for quasigroup. However having invertibility and Hom-unitarity conditions, one obtains more properties. We put all the different elements of \(G\) in the first row and column such that the Hom-unit 1 is in the first place. For simplicity we use the matrix notation \([c_{ij}]\) for the Cayley matrix. Some of the properties of Hom-groups are as follows:

i) Every row and column of the Cayley table of a Hom-group \((G, \alpha)\) is a permutation of the set \(G\). This is because Hom-groups have the cancelation property. In fact the Cayley table of a Hom-group is an example of a Latin square.

ii) Rows and columns can not be the identity permutation of \(G\) except \(\alpha = \text{Id}\), which means \(G\) is a group.
iii) If the element in the $i^{th}$ row and $j^{th}$ column is 1 then the element in the $j^{th}$ row and $i^{th}$ column also should be 1. This is because of the invertibility condition.

iv) The Cayley table is symmetric if and only if $(G, \alpha)$ is abelian.

v) The first row and first column are the same, because $1a = a1 = \alpha(a)$.

**Example 2.13. (Classification of Hom-groups of order 3).**

In this example we show that there is only one Hom-group of order 3. We use the Cayley table to classify all the Hom-groups of order 3. Let $G = \{1, a, b\}$. To fill out the Cayley table, we start from the first row. Since by property (ii), the first row can not be the identity permutation of $G$, therefore the only possibility is $c_{12} = b$ and $c_{13} = a$. Since the first column is the same as the first row therefore there will be only 4 spots $c_{22}, c_{23}, c_{32}, c_{33}$ to find. Now we argue on the place of the unit 1 in the second row. We note that $c_{22} \neq 1$ because otherwise $c_{23} = a$ which is a contradiction as the third column will have two copies of $a$. Therefore $c_{22} = a$ and $c_{23} = 1$. Now since the second and the third columns should be a permutation of $G$ then there is only one case left which is the following Cayley table:

| $G$ | 1 | a | b |
|-----|---|---|---|
| 1   | 1 | b | a |
| a   | b | a | 1 |
| b   | a | 1 | b |

This defines the twisting map by $\alpha(a) = b$ and $\alpha(b) = a$. One can check that $(G, \alpha)$ with above Cayley table satisfies the Hom-associativity and therefore it is an abelian Hom-group. This Hom-group is isomorphic to $Z_3^2$ where the multiplication is obtained by twisting the multiplication of the additive cyclic group $Z_3$ by the group automorphism $\alpha : Z_3 \rightarrow Z_3$ given by $\alpha(1) = 2$. See Proposition 2.3.

We finish this section by introducing a non-abelian Hom-group of order 6.

**Example 2.14. (Hom-Dihedral group $D_3$)**

We recall that the Dihedral group $D_3$ is the smallest non-abelian group which is given by $\{r, s \mid r^3 = s^2 = 1, \ srs = r^{-1}\}$. In fact the elements of $D_3$ are $\{1, r, r^2, s, sr, rs\}$. We consider the conjugation automorphism $\varphi_s(x) = sx^{-1}s^{-1}$. Now we twist the multiplication of $D_3$ by $\varphi_s$, as explained in Proposition 2.3, to obtain a Hom-group $D_3^a$ where $a = \varphi_s$.

The Cayley table is given by

| $D_3^a$ | 1 | r | $r^2$ | s | $rs$ | $sr$ |
|---------|---|---|-------|---|------|------|
| 1       | 1 | r | $r^2$ | s | $rs$ | $sr$ |
| r       | r | 1 | r     | s | s    | rs   |
| $r^2$   | $r^2$ | 1 | $r^2$ | s | s    | $sr$ |
| s       | s | s | $rs$  | s | r    | $r^2$ |
| $rs$    | $rs$ | s | $rs$  | r | $r^2$ | 1    |
| $sr$    | $sr$ | s | $rs$  | r | $r^2$ | 1    |
Lagrange’s theorem for a class of quasigroups

Lagrange-type’s theorem for nonassociative structures (magmas) is a challenging problem due to nonassociativity. In this section we focus on finite Hom-groups. We prove Lagrange’s theorem for this interesting class of quasigroups. This in fact generalizes the theorem for groups. The Hom-associativity condition plays an important rule in our proof. First we need the following lemma which shows that the number of elements of a Hom-subgroup and its cosets are the same.

Lemma 3.1. Let \((G, \alpha)\) be a finite Hom-group. If \(H \trianglelefteq G\), then \(|gH| = |H|\) for all \(g \in G\).

Proof. It is enough to show that for \(h_i \neq h_j\), the elements \(gh_i\) and \(gh_j\) are different in \(gH\). Suppose \(gh_i = gh_j = b\). By Theorem 2.11 we have

\[ h_i = \alpha^{-1}(g^{-1})\alpha^{-2}(b) = h_j. \]

Lemma 3.2. Let \((G, \alpha)\) be a finite Hom-group and \(H \trianglelefteq G\). Then \(gH = H\) if and only if \(g \in H\).

Proof. If \(g \in H\) then \(gH \subseteq H\). However by Lemma 3.1 we have \(|gH| = |H|\). Therefore \(gH = H\). Conversely if \(gH = H\) then \(g1 \in H\). So \(\alpha(g) \in H\). Since \(\alpha\) is invertible and \(\alpha(H) = H\) then \(g \in H\).

Lemma 3.3. Let \((G, \alpha)\) be a finite Hom-group and \(H \trianglelefteq G\). For all \(x, y \in G\) if \(xH \cap yH \neq \emptyset\) then \(xH = yH\).

Proof. Since \(xH \cap yH \neq \emptyset\), there exists \(h_1, h_2 \in H\) such that \(xh_1 = yh_2\). Then by Theorem 2.11 we have

\[ x = \alpha^{-2}(yh_2)\alpha^{-1}(h_1^{-1}). \]

By invertibility of \(\alpha\) we obtain

\[ x = [\alpha^{-2}(y)\alpha^{-2}(h_2)]\alpha(\alpha^{-2}(h_1^{-1})). \]

By Hom-associativity we have

\[ x = \alpha^{-1}(y)[\alpha^{-2}(h_2)\alpha^{-2}(h_1^{-1})] = \alpha^{-1}(y)[\alpha^{-2}(h_2h_1^{-1})]. \]

Now we show that \(xH \subseteq yH\). Let \(xh \in xH\). Then

\[ xh = [\alpha^{-1}(y)[\alpha^{-2}(h_2h_1^{-1})]] h = [\alpha^{-1}(y)[\alpha^{-2}(h_2h_1^{-1})]] \alpha(\alpha^{-1}(h)) = y[\alpha^{-2}(h_2h_1^{-1})] \alpha^{-1}(h). \]
We used invertibility of $\alpha$ in the second equality, and Hom-associativity in the third equality. Since $\alpha(H) = H$ then

$$x = y[\alpha^{-2}(h_2h_1^{-1})\alpha^{-1}(h)] \in yH.$$  

Therefore $xH \subseteq yH$. By Lemma 3.1, we have $|xH| = |H| = |yH|$. Therefore $xH = yH$. □

As a consequence of the previous results we obtain the following proposition.

**Proposition 3.4.** Let $(G, \alpha)$ be a finite Hom-group and $H \leq G$. Then the set of all cosets of $H$ in $G$ gives a partition of the set $G$.

**Theorem 3.5.** (Lagrange’s theorem for Hom-groups)

Let $(G, \alpha)$ be a finite Hom-group and $H \leq G$. Then $|H|$ divides $|G|$.

**Proof.** By the previous Proposition the cosets of $H$ in $G$ gives a partition of $G$. By Lemma 3.1 the size of all cosets are the same as the size of $H$. Since $G = \cup_{x \in G} xH$, then $|G|$ is a multiplication of $|H|$. □

**Example 3.6.** Let $G$ be a group, $\alpha : G \to G$ a group automorphism and $H \leq G$ which is preserved by $\alpha$, i.e, $\alpha(H) = H$. We twist the multiplication of $G$ by $\alpha$ to obtain the Hom-group $G_\alpha$ as we explained in Proposition 2.3. Clearly we have $H \leq G_\alpha$. One notes that if $\alpha$ does not preserve $H$ then $H$ will not be a Hom-subgroup of $G_\alpha$ because $\alpha(h) = 1h \in H$. Therefore studying group of $Aut(G)$ has an important rule to have examples of Hom-(sub)groups of $G_\alpha$. As an example if $\alpha \in Inn(G)$ and $N \triangleleft G$ then $\alpha(N) = N$ and therefore $N \leq G_\alpha$.

**Example 3.7.** All cyclic groups of order 6 are isomorphic to $Z_6$. We define the group automorphism $\alpha : Z_6 \to Z_6$ given by

$$\alpha(1) = 5, \ \alpha(2) = 4, \ \alpha(3) = 3, \ \alpha(4) = 2, \ \alpha(5) = 1, \ \alpha(0) = 0$$

Now we twist the multiplication of $Z_6$ by $\alpha$ as we explained in Proposition 2.3 to obtain a Hom-group $Z_6^\alpha$ given by the following Cayley table of multiplication,

| $Z_6^\alpha$   | 0 | 1 | 2 | 3 | 4 | 5 |
|----------------|---|---|---|---|---|---|
| 0              | 0 | 1 | 2 | 3 | 4 | 5 |
| 1              | 5 | 4 | 3 | 2 | 1 | 0 |
| 2              | 4 | 3 | 2 | 1 | 0 | 5 |
| 3              | 3 | 2 | 1 | 0 | 5 | 4 |
| 4              | 2 | 1 | 0 | 5 | 4 | 3 |
| 5              | 1 | 0 | 5 | 4 | 3 | 2 |

It can be verified that $Z_2^\alpha = \{0, 3\}$ and $Z_3^\alpha = \{0, 2, 4\}$ are the only non-trivial Hom-subgroups of the Hom-group $Z_6^\alpha$. They are of orders 2 and 3 which both divides the order of $Z_3^\alpha$. One notes that the Hom-subgroup $Z_3^\alpha$ is not cyclic in the usual sense. In fact $2 + 2$ in $Z_3^\alpha$ is 2 and $4 + 4$ is 4. Therefore 2 and 4 can not be the generators of $Z_3^\alpha$ in the usual sense. Therefore the proper notion of power of an element in Hom-groups is not clear to us.
4 Linearization of Hom-groups

In this section first we recall the linearization of Hom-groups from [H1], [H2] to obtain some examples of an interesting class of nonassociative Hopf algebras called Hom-Hopf algebras. This linearization is called Hom-group Hopf algebras. Then we apply Lagrange’s theorem for finite Hom-groups to find out about dimensions of Hom-sub-Hopf algebras of Hom-group Hopf algebras. First we recall the definitions of Hom-algebras, Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras. By [MS1], a Hom-associative algebra $A$ over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space with a bilinear map $\mu : A \otimes A \rightarrow A$, called multiplication, and a linear homomorphism $\alpha : A \rightarrow A$ satisfying the Hom-associativity condition

$$\mu(\alpha(a), \mu(b, c)) = \mu(\mu(a, b), \alpha(c)),$$

for all elements $a, b, c \in A$. A Hom-associative algebra $A$ is called unital with unit 1 if $\alpha(1) = 1$, and $a1 = 1a = \alpha(a)$. By [MS2], [MS3], a Hom-coalgebra is a triple $(C, \Delta, \varepsilon, \beta)$, where $C$ is a $\mathbb{K}$-vector space, $\Delta : C \rightarrow C \otimes C$ a linear map, called comultiplication, with a Sweedler notation $\Delta(c) = c^{(1)} \otimes c^{(2)}$, counit $\varepsilon : \mathbb{K} \rightarrow C$, and $\beta : C \rightarrow C$ a linear map satisfying the Hom-coassociativity condition,

$$\beta(c^{(1)}) \otimes c^{(2)(1)} \otimes c^{(2)(2)} = c^{(1)(1)} \otimes c^{(1)(2)} \otimes \beta(c^{(2)}),$$

and

$$c^{(1)} \varepsilon(c^{(2)}) = \varepsilon(c^{(1)})c^{(2)} = \beta(c), \quad \varepsilon(\beta(c)) = \varepsilon(c).$$

A $(\alpha, \beta)$-Hom-bialgebra is a tuple $(B, m, 1, \alpha, \Delta, \varepsilon, \beta)$ where $(B, m, 1, \alpha)$ is a unital Hom-algebra and $(B, \varepsilon, \Delta, \beta)$ is a counital Hom-coalgebra where $\Delta$ and $\varepsilon$ are morphisms of Hom-algebras, that is

i) $\Delta(hk) = \Delta(h)\Delta(k)$.
ii) $\Delta(1) = 1 \otimes 1$.
iii) $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$.
iv) $\varepsilon(1) = 1$.
v) $\varepsilon(\alpha(x)) = \varepsilon(x)$.

Here we recall the definition of Hom-Hopf algebras from [MS2] and [MS3]. A Hom-bialgebra $(B, m, \eta, \alpha, \Delta, \varepsilon, \beta)$ is called a $(\alpha, \beta)$-Hom-Hopf algebra if it is endowed with a morphism $S : B \rightarrow B$, called antipode, satisfying

a) $S \circ \eta = \eta$ and $\varepsilon \circ S = \varepsilon$.
b) $S$ is an inverse of the identity map $\text{Id} : B \rightarrow B$ for the convolution product, i.e., for any $x \in B$,

$$S(x^{(1)})x^{(2)} = x^{(1)}S(x^{(2)}) = \varepsilon(x)1_B. \quad (4.1)$$

This definition of a Hom-Hopf algebra and specially antipodes is different from the one in [LMT]. For more details see [H2]. However if $\alpha$ is invertible, both definitions will be equivalent.
**Example 4.1.** For any Hom-group \((G, \alpha)\), the Hom-group algebra \(\mathbb{K}G\) is a \((\alpha, \text{Id})\)-Hom-Hopf algebra. It is a free algebra on \(G\) where the coproduct is given by \(\Delta(g) = g \otimes g\), counit by \(\varepsilon(g) = 1\), the antipode by \(S(g) = g^{-1}\), and \(\beta = \text{Id}\) with \(\alpha\) which is linearly extended from \(G\) to \(\mathbb{K}G\). One notes that elements \(g \in \mathbb{K}G\) are group-like elements. Also \(\mathbb{K}G\) is a cocommutative Hom-Hopf algebra. If \(G\) is an abelian Hom-group then \(\mathbb{K}G\) is a commutative Hom-Hopf algebra.

**Lemma 4.2.** Let \((G, \alpha)\) be a Hom-group. The vector space \(A\) is a Hom-sub Hopf algebra of \(\mathbb{K}G\) if and only if there exists \(H \preceq G\) where \(A = \mathbb{K}H\).

**Proof.** Let \(A\) be a Hom-sub-Hopf algebra of \(\mathbb{K}G\). Then the set of group-like elements of \(A\) forms a Hom-group \(H\), see [LMT], and clearly \(A = \mathbb{K}H\). Conversely if \(H \preceq G\) then by the structure of the product, coproduct and the antipode explained in the previous example, \(\mathbb{K}H\) is a Hom-sub Hopf algebra of \(\mathbb{K}G\). \(\square\)

**Theorem 4.3.** Let \((G, \alpha)\) be a Hom-group and \(\mathbb{K}G\) be the Hom-group Hopf algebra. If \(A\) is a Hom-sub-Hopf algebra of \(\mathbb{K}G\), then \(\dim(A)\) divides \(|G|\).

**Proof.** By previous Lemma there exists \(H \preceq G\) where \(A = \mathbb{K}H\). Since \(\dim(A) = |H|\), then by Lagrange's theorem \(\dim(A)\) divides \(|G|\). \(\square\)

**Example 4.4.** Let us consider the cyclic group \(Z_5\). We define a group automorphism \(\alpha : Z_5 \to Z_5\) given by

\[
\begin{align*}
\alpha(1) = 2, & \quad \alpha(2) = 4, \quad \alpha(3) = 1, \quad \alpha(4) = 3, \quad \alpha(0) = 0.
\end{align*}
\]

One twists the multiplication of \(Z_5\) by \(\alpha\) as explained in Proposition 2.3 to obtain a Hom-group \(Z_5^\alpha\) given by the following table of multiplication

\[
\begin{array}{c|cccc}
\hline
Z_5^\alpha & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 2 & 4 & 1 & 3 \\
1 & 2 & 4 & 1 & 3 & 0 \\
2 & 4 & 1 & 3 & 0 & 2 \\
3 & 1 & 3 & 0 & 2 & 4 \\
4 & 3 & 0 & 2 & 4 & 1 \\
\hline
\end{array}
\]

Since the order of \(\mathbb{K}Z_5^\alpha\) is prime, by previous theorem it does not have any non-trivial Hom-sub Hopf algebra.

**Example 4.5.** Consider the Hom-group Hopf algebra \(\mathbb{K}G\). Since \(Z(G) \preceq G\) then the center of the Hom-Hopf algebra \(\mathbb{K}G\) is the same as \(\mathbb{K}Z(G)\) and its dimension divides \(|G|\).

**Remark 4.6.** Conjectures

A challenge in studying a Hom-group \((G, \alpha)\) is defining a proper notion of power of an element. Since \(G\) is not associative we can define two different types of powers called left
and right powers. Following the contexts of nonassociative objects such as quasigroups, an approach to define a right power of an element $x$ in a Hom-group $(G, \alpha)$ is as follows. We set $x^1 = x$, $x^2 = xx$. Now $x^3 = (x^2)x$ and inductively we can define other right powers. In fact one can define the right multiplication function $R_a(x) = xa$. So $x^2 = R_a(x)$, $x^3 = R_a(x^2)$ and generally $x^n = R_a(x^{n-1})$. Similarly if $L_a(x) = x$ then the left powers of $x$ can inductively be defined by $x^n = L_a(x^{n-1})$. However, this method has some problems such as defining cyclic Hom-subgroups. The notions of power and order of an element of $G$ are not clear for Hom-groups. Consequently, some fundamental theorems of group theory such as Cauchy’s theorem will be left as a conjecture for Hom-groups; if $(G, \alpha)$ is a finite Hom-group and $p$ is a prime number dividing the order of $G$, then $G$ contains an element, and therefore a Hom-subgroup of order $p$.

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