Image Processing Variations with Analytic Kernels *†

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Abstract

Let \(f \in L^1(\mathbb{R}^d)\) be real. The Rudin-Osher-Fatemi model is to minimize \(\|u\|_{\dot{BV}} + \lambda\|f - u\|^2_{L^2}\), in which one thinks of \(f\) as a given image, \(\lambda > 0\) as a “tuning parameter”, \(u\) as an optimal “cartoon” approximation to \(f\), and \(f - u\) as “noise” or “texture”. Here we study variations of the R-O-F model having the form \(\inf_{u} \{\|u\|_{\dot{BV}} + \lambda\|K * (f - u)\|_{L^p}^q\}\) where \(K\) is a real analytic kernel such as a Gaussian. For these functionals we characterize the minimizers \(u\) and establish several of their properties, including especially their smoothness properties. In particular we prove that on any open set on which \(u \in W^{1,1}\) and \(\nabla u \neq 0\) almost every level set \(\{u = c\}\) is a real analytic surface. We also prove that if \(f\) and \(K\) are radial functions then every minimizer \(u\) is a radial step function.

1 Introduction

Several \(BV\) variational models have been proposed as image decomposition models (see Section 2 for the definition of \(BV\)). First, Rudin-Osher-Fatemi [27] proposed the minimization

\[
\inf_{u \in BV} \{\|u\|_{\dot{BV}} + \lambda\|f - u\|_{L^2}^2\} .
\]

In (1), \(f \in L^1(\mathbb{R}^d)\) is a real function and one thinks of \(u\) as the “cartoon” component of \(f\) and \(f - u\) as the “noise+texture” component of \(f\). By the strict convexity of the functional \(\|f - u\|_{L^2}^2\), problem (1) has a unique minimizer \(u\). However, one limitation of model (1) is

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which suggests that \( \|K\) and for that reason we choose \( u \) to be very simple, for example, to be piecewise constant or to have real analytic level sets, \( K \) that for many choices of the kernel \( \|K\) precise results about the minimizers for (2). In comparison with Allard's paper \([2]\) we note \( u \) of (2). However, because of the analyticity of \( K \) so that the regularity results from section 1.5 of that paper hold for the minimizers \( u \) of (2). Moreover, the functional in (2) is not local in the sense of \([2]\), so that the conclusions of section 1.6 of \([2]\) need not hold for the minimizers of (2).

Chan and Esedoglu \([13]\) considered the minimization

\[
\inf_{u \in BV} \left\{ \|u\|_{BV} + \lambda \int |f - u| dx \right\}
\]

(see also Alliney \([5]\) for the one-dimensional discrete case). For this problem minimizers always exist but they may not be unique. For the example \( d = 2 \) and \( f = \chi_{B(0,R)} \), \([13]\) gives \( u = f \) if \( R > \frac{2}{\lambda} \) and \( u = 0 \) if \( R < \frac{2}{\lambda} \). W. Allard \([2, 3, 4]\) analyzed extremals for the problem

\[
\inf_{u \in BV} \left\{ \|u\|_{BV} + \lambda \int \gamma(u - f) dx \right\}
\]

where \( \gamma(0) = 0 \), \( \gamma \geq 0 \), and \( \gamma \) is locally Lipschitz. Then minimizers \( u \) exist although they may not be unique. Moreover, the minimizers \( u \) satisfy the smoothness condition

\( \partial^* (\{u > t\}) \in C^{1+\alpha}, \ \alpha \in (0, 1) \)

where \( \partial^* \) denotes “measure theoretic boundary”. Allard also gave mean curvature estimates on \( \partial^* (\{u > t\}) \).

In this paper we study a cartoon+texture decomposition model defined with a positive, real analytic convolution kernel \( K \):

\[
\inf_{u \in BV} \left\{ \|u\|_{BV} + \lambda \|K \ast (f - u)\|_{L^p}^p \right\} \quad (2)
\]

where \( 1 \leq p, q < \infty \). We choose the kernel \( K \) in \([2]\) so that the Fourier transform \( \widehat{K}(\xi) \) decays rapidly as \( |\xi| \to \infty \). The motivation is that we expect \( v = f - u \) to be oscillatory, so that \( \widehat{v}(\xi) \) is large when \( |\xi| \) is large. Thus, \( \widehat{K} \cdot \widehat{v} = (\widehat{K} \ast v) \) dampens high frequencies of \( v \), which suggests that \( \|K \ast v\|_{L^p} \) is small for oscillatory \( v \). We also want the cartoon component \( u \) to be very simple, for example, to be piecewise constant or to have real analytic level sets, and for that reason we choose \( K \) to be real analytic. Examples of such \( K \) are the Gaussian kernel where \( \widehat{K}(\xi) = e^{-\pi t|\xi|^2} \) or the Poisson kernel where \( \widehat{K}(\xi) = e^{-\pi t|\xi|} \), for some \( t > 0 \).

By comparison \([13]\) takes \( p = q = 1 \) and \( K = \text{identity} \) and our choices of \( K \) yield more precise results about the minimizers for (2). In comparison with Allard's paper \([2]\) we note that for many choices of the kernel \( K \) our functional \( \|K \ast (f - u)\|_{L^p} \) is admissible in the sense of \([2]\) so that the regularity results from section 1.5 of that paper hold for the minimizers \( u \) of (2). However, because of the analyticity of \( K \) our minimizers have greater smoothness than those from \([2]\). Moreover the functional in (2) is not local in the sense of \([2]\), so that the conclusions of section 1.6 of \([2]\) need not hold for the minimizers of (2).
2 The Variational Problems

To begin we recall the definition of $BV = BV(\mathbb{R}^d)$.

**Definition 1.** Let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ be real. We say $u \in BV$ if

$$\sup \left\{ \int \text{div}\varphi dx : \varphi \in C_0^1(\mathbb{R}^d), \sup |\varphi(x)| \leq 1 \right\} = \|u\|_{BV} < \infty.$$  

If $u \in BV$ there is an $\mathbb{R}^d$-valued measure $\vec{\mu}$ such that $\frac{\partial u}{\partial x_j} = (\vec{\mu})_j$ as distributions and we write

$$Du = \vec{\mu}.$$  

The vector measure $\mu$ has a polar decomposition

$$\vec{\mu} = \vec{\rho}\mu$$  

where $\mu$ is a finite positive Borel measure and $\vec{\rho} : \mathbb{R}^d \to S^{d-1}$ is a Borel function, and

$$\|u\|_{BV} = \int d\mu.$$  

(see for example Evans-Gariepy [17]).

We assume $K$ is a positive, even, bounded and real analytic kernel on $\mathbb{R}^d$ such that $\int Kdx = 1$ and such that $K * u$ determines $u$ (i.e. the map $L^p \ni u \to K * u$ is injective). For example we may take $K$ to be a Gaussian or a Poisson kernel. We fix $\lambda > 0$, $1 \leq p < \infty$ and $1 \leq q < \infty$. For real $f(x) \in L^1$ we consider the extremal problem:

$$m_{p,q,\lambda} = \inf \{\|u\|_{BV} + \mathcal{F}_{p,q,\lambda}(f - u) : u \in BV\}$$  

(3)

where

$$\mathcal{F}_{p,q,\lambda}(h) = \lambda \|K * h\|_L^q.$$  

(4)

Since $BV \subset L^{\frac{d}{d-1}}$ and $K \in L^\infty$, a weak-star compactness argument shows that (3) has at least one minimizer $u$. Our objective is to describe, given $f$, the set $\mathcal{M}_{p,q,\lambda}(f)$ of minimizers $u$ of (3).

2.1 Convexity

Since the functional in (3) is convex, the set of minimizers $\mathcal{M}_{p,q,\lambda}(f)$ is a convex subset of $BV$. If $p > 1$ or if $q > 1$, then the functional (4) is strictly convex and the problem (3) has a unique minimizer because $K * u$ determines $u$. When $p = q = 1$ minimizers may not be unique, but they satisfy the relations given in (5) and (6) below.
Lemma 1. Let $p = q = 1$ and assume $u_1 \in \mathcal{M}_{p,q,\lambda}(f)$ and $u_2 \in \mathcal{M}_{p,q,\lambda}(f)$. For $j = 1, 2$ write

$$Du_j = \bar{\mu}_j = \bar{\rho}_j \mu_j$$

with $|\bar{\rho}_j| = 1$ and $\mu_j \geq 0$ and write $\frac{d\bar{\mu}_j}{d\mu_k}$ for the Radon-Nikodym derivative of (the absolutely continuous part of) $\bar{\mu}_j$ with respect to $\mu_k$. Then

$$\frac{K^*(f - u_1)}{|K^*(f - u_1)|} = \frac{K^*(f - u_2)}{|K^*(f - u_2)|} \quad \text{almost everywhere}$$  \hspace{1cm} (5)

on $\{ |K^*(f - u_j)| > 0 \}, j = 1, 2$; and

$$\bar{\rho}_k \cdot \frac{d\bar{\mu}_j}{d\mu_k} = \left| \frac{d\bar{\mu}_j}{d\mu_k} \right|, \ j \neq k.$$  \hspace{1cm} (6)

Proof: Since $\mathcal{M}_{p,q,\lambda}(f)$ is a convex subset of $BV$, $\frac{u_1 + u_2}{2}$ is also a minimizer. This implies

$$\left\| \frac{u_1 + u_2}{2} \right\|_{BV} + \lambda \left\| K^* \left( f - \frac{u_1 + u_2}{2} \right) \right\|_1 = \frac{1}{2} \left( \|u_1\|_{BV} + \|u_2\|_{BV} \right)$$

$$+ \frac{\lambda}{2} \left( \|K^*(f - u_1)\|_1 + \|K^*(f - u_2)\|_1 \right).$$  \hspace{1cm} (7)

On the other hand, using the convexity of $\| \cdot \|_{BV}$ and $\| \cdot \|_{L^1}$ we have

$$\left\| \frac{u_1 + u_2}{2} \right\|_{BV} \leq \frac{1}{2} \left( \|u_1\|_{BV} + \|u_2\|_{BV} \right)$$  \hspace{1cm} (8)

and

$$\left\| K^* \left( f - \frac{u_1 + u_2}{2} \right) \right\|_1 \leq \frac{1}{2} \left( \|K^*(f - u_1)\|_1 + \|K^*(f - u_2)\|_1 \right).$$  \hspace{1cm} (9)

Combining (7), (8), and (9) we obtain the equality

$$\left\| K^* \left( f - \frac{u_1 + u_2}{2} \right) \right\|_1 = \frac{1}{2} \left( \|K^*(f - u_1)\|_1 + \|K^*(f - u_2)\|_1 \right),$$  \hspace{1cm} (10)

which implies (5). We also obtain

$$\|u_1 + u_2\|_{BV} = \|u_1\|_{BV} + \|u_2\|_{BV}$$  \hspace{1cm} (11)

and for $k \neq j$ equation (11) implies

$$\int \left| \bar{\rho}_k \cdot \frac{d\bar{\mu}_j}{d\mu_k} \right| d\mu_k = \int d\mu_k + \int \left| \frac{d\bar{\mu}_j}{d\mu_k} \right| d\mu_k,$$

which yields (6). \hspace{1cm} $\Box$
2.2 Properties of $u \in \mathcal{M}_{p,q,\lambda}(f)$

Lemma 2. Let $f \in L^1$, let $u \in BV$ be a minimizer of (3) with $\|u - f\|_1 \neq 0$ and write

$$Du = \vec{\mu} = \vec{\rho} \cdot \mu.$$ 

Then whenever $h \in BV$ is real, $Dh = \vec{\nu}$ and $\vec{\nu} = \frac{d\vec{\nu}}{d\mu} \mu + \vec{\nu}_s$ is the Lebesgue decomposition of $\vec{\nu}$ with respect to $\mu$ (so that $\vec{\nu}_s$ is singular to $\mu$), we have

$$\left| \int \vec{\rho} \cdot \frac{d\vec{\nu}}{d\mu} d\mu - \lambda \int h (K \ast J_{p,q}) dx \right| \leq \|\vec{\nu}_s\|,$$

where

$$J_{p,q} = q \frac{F |F|^{p-2}}{q F^{p-q}}, \quad F = K \ast (f - u)$$

and $\|\vec{\nu}_s\|$ denotes the norm of the vector measure $\vec{\nu}_s$. Conversely, if $u \in BV$, $\|u - f\|_1 \neq 0$ and if (12) and (13) hold for all $h$, then $u \in \mathcal{M}_{p,q,\lambda}(f)$.

Note that because $\|u - f\|_1 \neq 0$ and $K \ast (f - u)$ is real analytic and bounded, $J_{p,q}$ is defined almost everywhere, and that by Lemma 1, $J_{p,q}$ is independent of $u \in \mathcal{M}_{p,q,\lambda}$ in the case $p = q = 1$.

Proof: Let $|\epsilon|$ be sufficiently small. Since $u$ is extremal, we have

$$\|u + \epsilon h\|_{BV} - \|u\|_{BV} + \mathcal{F}_{p,q,\lambda}(f - u - \epsilon h) - \mathcal{F}_{p,q,\lambda}(f - u) \geq 0. \ (14)$$

On the other hand we have

$$\left| \tilde{\rho} + \epsilon \frac{d\vec{\nu}}{d\mu} \right| = \left( 1 + 2 \epsilon \frac{d\vec{\nu}}{d\mu} + \epsilon^2 \left\| \frac{d\vec{\nu}}{d\mu} \right\|^2 \right)^{1/2} = 1 + \epsilon \frac{d\vec{\nu}}{d\mu} + o(|\epsilon|),$$

where in the last equality, we use the estimate $(1 + \alpha)^{1/2} = 1 + \frac{\alpha}{2} + o(|\alpha|)$. This implies,

$$\|u + \epsilon h\|_{BV} - \|u\|_{BV} = |\epsilon| \|\vec{\nu}_s\| + \int \left( \left| \tilde{\rho} + \epsilon \frac{d\vec{\nu}}{d\mu} \right| - 1 \right) d\mu = |\epsilon| \|\vec{\nu}_s\| + \epsilon \int \tilde{\rho} \cdot \frac{d\vec{\nu}}{d\mu} d\mu + o(|\epsilon|).$$

Moreover $K \ast (f - u)$ is bounded and non-zero almost everywhere, since $K$ is real analytic. Hence we also have

$$\mathcal{F}_{p,q,\lambda}(f - u - \epsilon h) - \mathcal{F}_{p,q,\lambda}(f - u) = -\lambda \epsilon \int (K \ast h) J_{p,q} dx + o(|\epsilon|)$$

$$= -\lambda \epsilon \int h (K \ast J_{p,q}) dx + o(|\epsilon|)$$

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since $K(-x) = K(x)$. Thus by (14), we have

$$-\epsilon \left[ \int \tilde{\rho} \cdot \frac{d\tilde{\nu}}{d\mu} d\mu - \lambda \int h(K * J_{p,q}) dx \right] \leq |\epsilon||\tilde{\nu}_s|| + o(|\epsilon|).$$

Taking $\pm \epsilon$ and noting that the right side of the above inequality does not depend on the sign of $\epsilon$, we see that (12) holds. The converse statement holds because the functional (4) is convex. □

Lemma 2 does not hold for the Chan-Esedoglu [13] functional because in that case one can have $f - u = 0$ on a set of positive measure, and this yields the additional term $\int_{\{|f - u| = 0\}} |h| dx$ on the right side of (12).

Later we will need the following alternate characterization of minimizers, due to Meyer [23] in the case of the Rudin-Osher-Fatemi model. Define

$$\|v\|_* = \inf \left\{ \|u\|_{L^\infty} : v = \sum_{j=1}^d \frac{\partial u_j}{\partial x_j}, |u|^2 = \sum_{i=1}^d |u_j|^2 \right\}$$

so that $\|v\|_*$ is (isometrically) the norm of the dual of $W^{1,1} \subset BV$ when $W^{1,1}$ is given the norm of $BV$. By the weak-star density of $W^{1,1}$ in $BV$,

$$\left| \int hv dx \right| \leq \|h\|_{BV} \|v\|_*$$

whenever $v \in L^2$. The lemma characterizes minimizers in terms of $\| \cdot \|_*$.

**Lemma 3.** Let $u \in BV$ such that $u \neq f$, and let $J_{p,q}$ be defined as in Lemma 2. Then $u$ is a minimizer for the problem (3) if and only if

$$\|K * J_{p,q}\|_* = \frac{1}{\lambda}$$

and

$$\int u(K * J_{p,q}) dx = \frac{1}{\lambda} \|u\|_{BV}.$$  \hspace{1cm} (17)

**Proof:** The short proof is the same as in [23], but we include it for the reader’s convenience. Let $u$ be a minimizer for (3). Then for any $h \in W^{1,1}$, (12) yields

$$\left| \int h(K * J_{p,q}) dx \right| \leq \frac{\|h\|_{BV}}{\lambda}$$

by the definition of $\tilde{\nu}_s$. Hence by the definition of $\| \cdot \|_*$,

$$\|K * J_{p,q}\|_* \leq \frac{1}{\lambda}.$$
But setting \( h = u \) in (12) gives (17), so that (16) follows.

Conversely, assume \( u \in BV \) satisfies (16) and (17) and note that \( u \) determines \( J_{p,q} \). Still following Meyer [23], we let \( h \in BV \) be real. Then for small \( \epsilon > 0 \), (15), (16) and (17) give

\[
\| u + \epsilon h \|_{BV} + \lambda \| K \ast (f - u - \epsilon h) \|_1 \geq \lambda \int (u + \epsilon h)(K \ast J_{p,q}) \, dx + \lambda \| K \ast (f - u) \|_1 \\
- \epsilon \lambda \int h(K \ast J_{p,q}) \, dx + o(\epsilon) \\
= \| u \|_{BV} + \epsilon \lambda \int h(K \ast J_{p,q}) \, dx \\
- \epsilon \lambda \int h(K \ast J_{p,q}) \, dx + o(\epsilon) \\
\geq 0.
\]

Therefore \( u \) is a local minimizer for the functional (3), and by convexity that means \( u \) is a global minimizer. \( \square \)

**Lemma 4.** Assume \( f \in L^1 \), \( u \in M_{p,q,\lambda}(f) \), and \( \| u - f \|_1 \neq 0 \). Let \( U \) be an open set on which \( Du = \bar{\mu} \) is absolutely continuous to Lebesgue measure and has Radon-Nikodym derivative \( \frac{d\bar{\mu}}{dx} \neq 0 \) almost everywhere. Then as distributions on \( U \)

\[
\text{div} \left( \frac{\nabla u}{|\nabla u|} \right) = -\lambda K \ast J_{p,q},
\]

and \( u \in W^{1,1}(U) \). In particular, if \( u \in C^2(U) \) then the level set \( \{ u = c \} \) is locally a \( C^2 \) surface having mean curvature \( -\lambda K \ast J_{p,q}(x) \) at \( x \in U \).

**Proof:** Since \( Du \) is absolutely continuous on \( U \) we have \( u \in W^{1,1}(U) \) and \( \bar{\mu} = \nabla u \, dx \) there. Let \( h \in C^\infty \) have compact support contained in \( U \). Then by the hypotheses, \( \bar{\nu} = Dh = \nabla h \, dx \) is absolutely continuous to \( Du \) so that by (12)

\[
\int_U \nabla h \cdot \frac{\nabla u}{|\nabla u|} \, dx = \lambda \int_U h(K \ast J_{p,q}) \, dx.
\]

This implies (13). Also, if \( u \in C^2(U) \) then (19) holds pointwise and gives the mean curvature of \( \{ u = c \} \) inside \( U \). \( \square \)

Known results on mean curvature equations can now be used to show that almost every level set \( U \cap \{ u = c \} \) is a real analytic surface, even without the assumption \( u \in C^2(U) \). Below we write \( \Lambda_{d-1} \) for \( d-1 \) dimensional Hausdorff measure.

**Theorem 1.** Assume \( f \in L^1 \), \( u \in M_{p,q,\lambda}(f) \), and \( \| u - f \|_1 \neq 0 \). Let \( U \) be an open set on which \( Du = \bar{\mu} \) is absolutely continuous to Lebesgue measure and on which the Radon-Nikodym derivative \( \frac{d\bar{\mu}}{dx} \neq 0 \) almost everywhere. Then for almost all \( c \in \mathbb{R} \) and for \( \Lambda_{d-1} \)
almost every } x_0 \in U \cap \{ u = c \} \text{ there exists a } C^1\text{-hypersurface } S \text{ with continuous unit normal } \vec{n}(x) = \frac{\sum u}{|\sum u|} \text{ and a neighborhood } V \text{ of } x \text{ such that } \Lambda_{d-1}( (V \cap \{ u = c \}) \Delta S ) = 0. \text{ After a rotation } S = \{ x_d = \varphi(y) : y = (x_1, \ldots, x_{d-1}) \in V_0 \}, \text{ where } V_0 \subset \mathbb{R}^{d-1} \text{ is open, } \varphi \in C^1(V_0), \text{ and } \vec{n}(y, \varphi(y)) = (1 + |\nabla \varphi|^2)^{-1/2}(\nabla \varphi, -1). \text{ Moreover, as a distribution on } V_0
\begin{equation}
div\left(\frac{\nabla \varphi}{(1 + |\nabla \varphi|^2)^{1/2}}\right) = -\lambda K * J_{p,q}(y, \varphi(y))dy,
\end{equation}
and the function } \varphi \text{ and the surface } S \text{ are real analytic.}

\textbf{Proof:} That } S \text{ and } \varphi \text{ exist almost everywhere follows from standard properties of BV functions and the hypothesis that } |\nabla u| > 0 \text{ a.e. on } U. \text{ See the proof of Theorem 4 below and Chapter 5 of [17]. To prove (20) we may assume } c = 0. \text{ Let } h \in C_0(\hat{(V_0)}, \text{ let } \chi(t) = \frac{1}{\varepsilon} \chi(\frac{t}{\varepsilon}) \text{ where } \chi(t) = \chi(-t) \geq 0 \text{ is } C^\infty(-1, 1) \text{ and } \int \chi dt = 1, \text{ and define }
\begin{equation}
H_\varepsilon(x) = \chi_\varepsilon(h(x_1, \ldots, x_{d-1}) - x_d)h(x_1, \ldots, x_{d-1}).
\end{equation}
Then by (18),
\begin{equation}
\int \left( \sum_{j=1}^{d-1} (\chi_\varepsilon(h(x_1, \ldots, x_{d-1}) - x_d) + \chi'_\varepsilon(h(x_1, \ldots, x_{d-1}) - x_d)h(x_1, \ldots, x_{d-1})) \right)
\begin{equation}
\frac{\partial h}{\partial x_j} \frac{1}{|\nabla u|} \frac{\partial u}{\partial x_j} - \left( \chi'_\varepsilon(h(x_1, \ldots, x_{d-1}) - x_d)h(x_1, \ldots, x_{d-1}) \right) \frac{1}{|\nabla u|} \frac{\partial u}{\partial x_d} \right) dx
\end{equation}
= \lambda \int_{V_0} H_\varepsilon(x) K * J_{p,q}(x) dx.
\end{equation}
Now for almost every } c \text{ the right side of this equation tends to } \lambda \int_{V_0} h(K * J_{p,q})(y)dy \text{ and, by the fine properties of BV functions in Chapter 5 of [17] or Chapter 3 of [6], the left side tends to }
\begin{equation}
\int_{V_0} \nabla h \cdot \frac{\nabla \varphi}{(1 + |\nabla \varphi|^2)^{1/2}} dy.
\end{equation}
That proves (20).

To prove the real analyticity of } \varphi, \text{ and hence of } S, \text{ we invoke three theorems. First, since } \varphi \in C^1, \text{ the results on mean curvature equations in Section 7.7 of [6] show that } \varphi \in W^{2,2} \cap C^{1+\alpha} \text{ whenever } 0 < \alpha < 1. \text{ Next, since } \varphi \in W^{2,2} \text{ we can rewrite (20) as }
\begin{equation}
\sum_{j,k} \frac{\delta_{j,k} - \varphi_{j} \varphi_{k}}{(1 + |\nabla \varphi|^2)^{3/2}} \varphi_{j,k} = \lambda K * J_{p,q}(y, \varphi(y)).
\end{equation}
Indeed, (21) is clear if } \varphi \in C^2, \text{ and if we set } \varphi_\varepsilon = \chi_\varepsilon \varphi \in C^2 \text{ then in the norms of } C^{1+\alpha} \text{ and } W^{2,2}, \varphi_\varepsilon \to \varphi \text{ as } \varepsilon \to 0. \text{ Hence for each } j
\begin{equation}
\int_{V_0} h_j \sum_k \frac{\delta_{j,k} - \varphi_{j} \varphi_{k}}{(1 + |\nabla \varphi|^2)^{3/2}} \varphi_{j,k} dy \to \int_{V_0} h_j \sum_k \frac{\delta_{j,k} - \varphi_{j} \varphi_{k}}{(1 + |\nabla \varphi|^2)^{3/2}} \varphi_{j,k} dy.
\end{equation}
as $\epsilon \to 0$, and consequently (21) also holds with $\varphi \in W^{2,2}$. We may assume $|\nabla \varphi| \leq 1/2$ because $\varphi$ locally parametrizes a $C^1$ surface, and then (21) becomes an elliptic equation with $C^\alpha$ coefficients (which depend on $\varphi$). It then follows by Schauder’s theorem (see [11]) that $\varphi \in C^{2+\alpha}(V_0)$ for some $\alpha > 0$. Finally, by the analyticity of the right side of (21), the function $\varphi$, and hence the surface $S$, is real analytic by a theorem of Hopf [20] (see also [24]). □

See Theorem 5 below for a related result for the case $q = 1$.

### 2.3 Radial Functions

Assume $K$ is radial, $K(x) = K(|x|)$ and assume $f$ is radial and $f \notin \mathcal{M}_{p,q,\lambda}(f)$. Then averaging over rotations shows that every $u \in \mathcal{M}_{p,q,\lambda}(f)$ is radial and

$$Du = \rho(|x|) \frac{x}{|x|} \mu,$$

where $\mu$ is invariant under rotations and where $\rho(|x|) = \pm 1$ a.e. $d\mu$. Let $H \in L^1(\mu)$ be radial and satisfy $\int H d\mu = 0$ and $H = 0$ on $|x| < \epsilon$, and define

$$h(x) = \int_{B(0,|x|)} H(|y|) \frac{1}{|y|^{d-1}} d\mu.$$ 

Then $h \in BV$ is radial and

$$Dh = \tilde{\nu} = H(|x|) \frac{x}{|x|} \mu.$$

Consequently $\tilde{\nu}_s = 0$ and (12) gives

$$\int \rho H d\mu = \lambda \int K \ast J_{p,q}(x) \int_{B(0,|x|)} \frac{H(y)}{|y|^{d-1}} d\mu(y) dx$$

$$= \lambda \int \left( \int_{|x|>|y|} K \ast J_{p,q}(x) dx \right) \frac{H(|y|)}{|y|^{d-1}} d\mu(y),$$

so that a.e. $d\mu$,

$$\rho(|y|) = \frac{\lambda}{|y|^{d-1}} \int_{|x|>|y|} K \ast J_{p,q}(x) dx.$$  

But the right side of (22) is real analytic in $|y|$, with a possible pole at $|y| = 0$, and $\rho(|y|) = \pm 1$ almost everywhere $\mu$. Therefore there is a finite set

$$\{r_1 < r_2 < \cdots < r_n\}$$

of radii such that

$$Du = \frac{x}{|x|} \sum_{j=1}^n c_j \Lambda_{d-1} \{ |x| = r_j \}.$$
for real constants \( c_1, \ldots, c_n \). By Lemma 1, \( J_{p,q} \) is uniquely determined by \( f \), and hence the set (23) is also unique. Moreover, it follows from Lemma 1 that for each \( j \), either \( c_j \geq 0 \) for all \( u \in \mathcal{M}_{p,1,\lambda}(f) \) or \( c_j \leq 0 \) for all \( u \in \mathcal{M}_{p,1,\lambda}(f) \). We have proved:

**Theorem 2.** Suppose \( K \) and \( f \) are both radial. If \( f \notin \mathcal{M}_{p,q,\lambda}(f) \), then there is a finite set (23) such that all \( u \in \mathcal{M}_{p,q,\lambda}(f) \) have the form

\[
\sum_{j=1}^{n} c_j \chi_{B(0,r_j)}.
\]

Moreover, there is \( X^+ \subset \{1, 2, \ldots, n\} \) such that \( c_j \geq 0 \) if \( j \in X^+ \) while \( c_j \leq 0 \) if \( j \notin X^+ \).

Note that by convexity \( \mathcal{M}_{p,q,\lambda}(f) \) consists of a single function unless \( p = q = 1 \). In Section 3.3 we will say more about the solutions of the form (24).

### 2.4 Example

Unfortunately, Theorem 2 does not hold more generally. The reason is that when \( u \) is not radial it is difficult to produce \( BV \) functions satisfying \( Dh = \mathring{\nu} << \mu \). For simplicity we take \( d = 2 \) and \( p = q = 1 \) and define

\[
J(x, y) = \begin{cases} 
1 & \text{if } 0 < x \leq 1 \\
-1 & \text{if } -1 < x \leq 0
\end{cases}
\]

and

\[ J(x + 2, y) = J(x, y). \]

Choose \( \lambda > 0 \) so that \( U = \lambda K * J \) satisfies \( \|U\|_* = 1 \), and note that \( \frac{U}{|U|} = J \). Also notice that \( u \in C^2 \) solves the curvature equation

\[
\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) = U
\]

if and only if the level sets \( \{u = a\} \) are curves \( y = y(x) \) that satisfy the simple ODE

\[
y'' = U(x, 0)(1 + (y')^2)^{3/2}
\]

on the line. Consequently (25) has infinitely many solutions \( u \) and both \( u \) and \( J \) satisfy (16) and (17). Hence by Lemma 3, \( u \) is a minimizer for \( f \) provided that

\[
J = \frac{K * (f - u)}{|K * (f - u)|},
\]

and there are many \( f \) that satisfy (26). For example, one can choose \( u \) and \( f \) so that \( f - u = J \). Note that in this example \( u \) can be real analytic except on \( U^{-1}(0) \) and not piecewise constant. Similar examples can be made when \( (p, q) \neq (1, 1) \).
3 Further Properties of Minimizers when $q = 1$

When $q = 1$ the minimizers $u \in M_{p,1,\lambda}(f)$ have several additional properties. The results of the next two sections do not depend on the real analyticity of the kernel $K$. They also hold when $K = I$, i.e. when $F_{p,q,\lambda}(h) = \lambda \|h\|_p$, and in the case $K = I$ somewhat stronger results have already been proved by Allard in [2]. However, since the arguments in [2] do not apply to the case $K \neq I$ we include complete but brief proofs.

3.1 Layer Cake Decomposition

Here we have been inspired by the paper of Strang [29].

**Lemma 5.** If $q = 1$ and $u \in M_{p,1,\lambda}(f)$, then $u \in M_{p,1,\lambda}(u)$.

**Proof:** If
\[
\|h\|_{BV} + \lambda \|K \ast (u - h)\|_p < \|u\|_{BV},
\]
then by the triangle inequality
\[
\|h\|_{BV} + \lambda \|K \ast (f - h)\|_p < \|u\|_{BV} + \lambda \|K \ast (f - u)\|_p
\]
so that $u$ is not a minimizer for $f$. \qed

We write
\[
M = M_{p,1,\lambda} = \bigcup_f M_{p,1,\lambda}(f).
\]

**Lemma 6.** Let $u \in BV$. Then $u \in M$ if and only if
\[
\left| \int \rho \cdot \frac{d\tilde{\nu}}{d\mu} d\mu \right| \leq \|\tilde{\nu}_s\| + \lambda \|K \ast h\|_p
\]
for all $h \in BV$, where $Dh = \tilde{\nu}$ and $\tilde{\nu}_s$ is the part of $\tilde{\nu}$ singular to $\mu$.

**Proof:** By Lemma 5 we may take $f = u$. Then for $|\epsilon|$ small we have
\[
0 \leq \|u + \epsilon h\|_{BV} - \|u\|_{BV} + \lambda \|\epsilon K \ast h\|_p
\]
\[
= |\epsilon|\|\tilde{\nu}_s\| + \epsilon \int \rho \cdot \frac{d\tilde{\nu}}{d\mu} d\mu + |\epsilon|\lambda \|K \ast h\|_p + o(|\epsilon|)
\]
and the Lemma follows from the proof of Lemma 2. \qed

Let $a < b$ be such that
\[
\mu(\{u = a\} \cup \{u = b\}) = 0.
\]

(28)

Then $u_{a,b} = \text{Min}\{(u - a)^+, (b - a)\} \in BV$ and $D(u_{a,b}) = \chi_{a < u < b} \tilde{\nu}_s$. 

Lemma 7. Assume $q = 1$.

(a) If $u \in \mathcal{M}$, then $u_{a,b} \in \mathcal{M}$.
(b) More generally, if $u \in \mathcal{M}$ and if $v \in BV$ satisfies $\mu_v \ll \mu_u$ and $\rho_v = \rho_u$ a.e. $d\mu_v$, then $v \in \mathcal{M}$.

Proof: To prove (a) we verify (27). Write $\mu_{a,b} = \chi_{(a,b)} \mu$ so that $D(u_{a,b}) = \vec{\rho} \mu_{a,b}$. Let $h \in BV$ and write $Dh = \vec{\nu}$. Then by (28)

$$\vec{\nu} = \chi_{a<u<b} \frac{d\vec{\nu}}{d\mu} \mu + \left( (\vec{\nu})_s + \chi_{u(x) \notin [a,b]} \frac{d\vec{\nu}}{d\mu} \mu \right)$$

is the Lebesgue decomposition of $\vec{\nu}$ with respect to $\mu_{a,b}$, and

$$\int \vec{\rho} \cdot \frac{d\vec{\nu}}{d\mu_{a,b}} d\mu_{a,b} = \int \vec{\rho} \cdot \frac{d\vec{\nu}}{d\mu} d\mu - \int_{g(x) \notin [a,b]} \vec{\rho} \cdot \frac{d\vec{\nu}}{d\mu} d\mu.$$

Then (27) for $\nu$ and $\mu_{a,b}$ follows from (27) for $\mu$ and $\nu$. The proof of (b) is similar. □

For simplicity we assume $u \geq 0$. Write $E_t = \{ x : u(x) > t \}$. Then by Evans-Gariepy [17], $E_t$ has finite perimeter for almost every $t$,

$$\|u\|_{BV} = \int_0^\infty \|\chi_{E_t}\|_{BV} dt,$$

and

$$u(x) = \int_0^\infty \chi_{E_t}(x) dt.$$

Moreover, almost every set $E_t$ has a measure theoretic boundary $\partial_* E_t$ such that

$$\Lambda_{d-1}(\partial_* E_t) = \|\chi_{E_t}\|_{BV}$$

and a measure theoretic outer normal $\vec{n}_t : \partial_* E_t \to S^{d-1}$ so that

$$D(\chi_{E_t}) = \vec{n}_t \Lambda_{d-1} |\partial_* E_t|.$$

Theorem 3. Assume $q = 1$.

(a) If $u \in \mathcal{M}$, then for almost every $t$, $\chi_{E_t} \in \mathcal{M}$.
(b) If $u \in \mathcal{M}$ and $u \geq 0$, then for all nonnegative $c_1, ..., c_n$ and for almost all $t_1 < ... < t_n$, $\sum c_j \chi_{E_{t_j}} \in \mathcal{M}$.

Proof: Suppose (a) is false. Then there is $\beta < 1$, and a compact set $A \subset (0, \infty)$ with $|A| > 0$ such that for all $t \in A$ (31) and (32) hold and there exists $h_t \in BV$ such that

$$\|\chi_{E_t} - h_t\|_{BV} + \lambda \| K * h_t \|_p \leq \beta \|\chi_{E_t}\|_{BV}.$$

(33)
Choose an interval \( I = (a, b) \) such that (28) holds and \( |I \cap A| \geq \frac{|I|}{2} \). Define \( h_t = 0 \) for \( t \in I \setminus A \), and take finite sums such that

\[
\sum_{j=1}^{N_n} \chi_{E_{t_j^{(n)}}} \Delta t_j^{(n)} \to u_{a,b} \quad (n \to \infty),
\]

and \( t_j^{(n)} \in A \) whenever possible. Write \( h^{(n)} = \sum_{j=1}^{N_n} h_{t_j^{(n)}} \Delta t_j^{(n)} \). Then by (30) and (33) \( \{h^{(n)}\} \) has a weak-star limit \( h \in BV \), and by (33), (34) and (35),

\[
\|u_{a,b} - h\|_{BV} + \lambda\|K \ast h\|_p \leq \frac{1 + \beta}{2} \|u_{a,b}\|_{BV},
\]

contradicting Lemma 7. The proof of (b) is similar. \( \square \)

We believe that the converse of Theorem 3 is false, but we have no counterexample. In the case \( K = I \) and \( p = 1 \) the converse of this Theorem is true. See [2], Theorem 5.3.

### 3.2 Characteristic Functions

Still assuming \( q = 1 \) we let \( E \) be such that \( \chi_E \in \mathcal{M} \). Then by Evans-Gariepy [17] \( \partial^* E = N \cup \bigcup K_j \), where \( D(\chi_E)(N) = \Lambda_{n-1}(N) = 0 \), \( K_j \) is compact and \( K_j \subset S_j \), where \( S_j \) is a \( C^1 \)–hypersurface with continuous unit normal \( \vec{n}_j(x) \), \( x \in S_j \), and \( \vec{n}_j \) is the measure theoretic outer normal of \( E \). After a coordinate change write \( S_j = \{x_d = \varphi_j(y)\}, y = (x_1, \ldots, x_{d-1}) \) with \( \nabla \varphi_j \) continuous and \( \vec{n}_j(y, \varphi_j(y)) = (1 + |\nabla \varphi_j|^2)^{-1/2}(\nabla \varphi_j, -1) \). Assume \( y = 0 \) is a point of Lebesgue density of \( (y, \varphi_j)^{-1}(K_j) \), let \( V \subset \mathbb{R}^{d-1} \) be a neighborhood of \( y = 0 \), let \( g \in C_0^\infty(V) \) with \( g \geq 0 \), and consider the variation \( u_\epsilon = \chi_{E_\epsilon} \) where \( \epsilon > 0 \) and

\[
E_\epsilon = E \cup \{0 \leq x_d \leq \epsilon g(y), y \in V\}.
\]

Then \( E \subset E_\epsilon \), and writing \( u_0 = \chi_E \), we have

\[
\|u_\epsilon\|_{BV} - \|u_0\|_{BV} = \int_V \sqrt{1 + |\nabla (\varphi_j + \epsilon g)|^2} - \sqrt{1 + |\nabla f \varphi_j|^2} dy + o(\epsilon)
\]

because by [17] page 203

\[
\Lambda_{d-1}(\partial^* (E_\epsilon \cup (E_\epsilon \setminus E))) = o(\epsilon)
\]

\( \Lambda_{d-1} \) a.e. on \( K_j \). Hence

\[
\|u_\epsilon\|_{BV} - \|u_0\|_{BV} = \epsilon \int_V \nabla g \cdot \frac{\nabla \varphi_j}{\sqrt{1 + |\nabla \varphi_j|^2}} dy + o(\epsilon).
\]
Also, a careful calculation gives

\[ \lambda \| K \ast (u_\epsilon - u_0) \|_p = \lambda |\epsilon| \left\| \int_V K(x - (y, \varphi_j(y))) g(y) dy \right\|_{L^p(dx)} + o(\epsilon). \]  

(38)

Together (37) and (38) show

\[ - \int_V \nabla g : \left( \frac{\nabla \varphi_j}{\sqrt{1 + |\nabla \varphi_j|^2}} \right) dy \leq \lambda \| K \|_p \int_V g dy. \]  

(39)

Repeating this argument with \( \epsilon < 0 \) and with \( g \leq 0 \) we obtain:

**Theorem 4.** On the hypersurface \( S_j \subset \partial \ast E \)

\[ \left| \text{div} \left( \frac{\nabla \varphi_j}{\sqrt{1 + |\nabla \varphi_j|^2}} \right) \right| \leq \lambda \| K \|_p. \]  

(40)

when viewed as a distribution on \( (y, \varphi_j)^{-1}(S_j) \).

By (40) and Section 7.7 of [6] we see that \( \varphi_j \in W^{2,2}_{\text{loc}} \cap C^{1+\alpha} \) for any \( \alpha < 1 \). Combining Theorem 4 with Theorem 3 we obtain:

**Theorem 5.** Assume \( q = 1 \) and \( u \in \mathcal{M} \). Then for almost every \( t \), \( E_t = \{ u > t \} \) has finite perimeter and \( \Lambda_{d-1} \) almost every point of the measure theoretic boundary \( \partial \ast E_t \) lies on a \( C^{1+\alpha}, \alpha < 1 \) surface having distributional mean curvature at most \( \lambda \| K \|_p \).

We note that the “distributional mean curvature” defined by (40) is the same as the generalized mean curvature defined by Allard in [2], and thus Theorem 5 complements Theorem 1.2 and Theorem 1.6 of [2]. However, unlike the situation in Theorem 1, we cannot conclude that the \( C^{1+\alpha} \) surface meeting \( \partial \ast E_t \) is real analytic because the left side of (40) may not be Hölder continuous.

### 3.3 Radial Minimizers

In this section we assume \( q = 1 \) and \( p = 1 \). For convenience we assume the kernel \( K(x) = e^{-\pi |x|^2} \), so that \( K_t \) has the form

\[ K_t(x) = t^{-d/2} K \left( \frac{x}{\sqrt{t}} \right). \]  

(41)

and

\[ K_s \ast K_t = K_{s+t}. \]  

(42)

Note that (41) and (42) imply that

\[ \| K_t \ast f \|_1 \text{ decreases in } t \]  

(43)

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and for \( f \in L^1 \) with compact support
\[
\lim_{t \to \infty} \|K_t * f\|_1 = \left| \int f \, dx \right| .
\] (44)

For fixed \( \lambda \) and \( t \) we set
\[
R(\lambda, t) = \{ r > 0 : \chi_{B(0,r)} \in \mathcal{M} \}.
\]
By Theorem 2 and Theorem 3, we have \( R(\lambda, t) \neq \emptyset \). For \( t = 0 \) and \( K = I \) our problem (2) becomes the problem
\[
\inf \{ \|u\|_{BV} + \lambda \|f - u\|_{L^1} \}
\]
studied by Chan and Esedoglu in [13], and in that case Chan and Esedoglu showed \( R(\lambda, 0) = [\frac{2}{\lambda}, \infty) \).

**Theorem 6.** There exists \( r_0 = r_0(\lambda, t) \) such that
\[
R(\lambda, t) = [r_0, \infty).
\] (45)
Moreover
\[
[0, \infty) \ni t \to r_0(t) \text{ is nondecreasing}
\] (46)
and
\[
\lim_{t \to \infty} r_0(t) = \infty.
\] (47)

**Proof:** Assume \( r \notin R(\lambda, t) \) and \( 0 < s < r \). Write \( \alpha = \frac{s}{r} > 1 \) and \( f = \chi_{B(0,r)} \). By hypothesis there is \( g \in BV \) such that
\[
\|g\|_{BV} + \lambda \|K_t * (f - g)\|_1 < \|f\|_{BV}.
\] (48)
We write \( \tilde{g}(x) = g(\alpha x), \tilde{f}(x) = f(\alpha x) = \chi_{B(0,s)}(x) \), and change variables carefully in (48) to get
\[
\alpha \|\tilde{g}\|_{BV} + \lambda \left\| \frac{1}{4s^2} \int K\left( \frac{x - y}{\sqrt{t}} \right) (\tilde{f} - \tilde{g}) \left( \frac{y}{\alpha} \right) dy \right\|_{L^1(x)} < \alpha \|\tilde{f}\|_{BV}
\]
so that
\[
\alpha \|\tilde{g}\|_{BV} + \lambda \left\| \int K\left( \frac{\alpha x' - \alpha y'}{\sqrt{t}} \right) (\tilde{f} - \tilde{g}) (y') dy' \right\|_{L^1(\alpha x')} < \alpha \|\tilde{f}\|_{BV}
\]
and
\[
\alpha \|\tilde{g}\|_{BV} + \lambda \alpha^d \left\| \int K\left( \frac{\alpha x' - \alpha y'}{\sqrt{t}} \right) (\tilde{f} - \tilde{g}) (x') dx' \right\|_{L^1(\alpha x')} < \alpha \|\tilde{f}\|_{BV}.
\]
Since \( \alpha > 1 \), this and (43) show
\[
\|\tilde{g}\|_{BV} + \lambda \|K_t * (\tilde{f} - \tilde{g})\|_1 < \|\tilde{f}\|_{BV}
\]
so that \( s \notin R(\lambda, t) \). That proves (45), and (46) now follows easily from (43). To prove (47) take \( g = \frac{s}{4s^2} \chi_{B(0,s)} \), \( s > r \) and use (44). \( \square \)

We note that not all radial minimizers have the form \( \chi_{B(0,r)} \). This is seen by considering separately, for large fixed \( t \) and \( \lambda \), the function \( \chi_{B(0,r_2)} + \chi_{B(0,r_1)} \) with \( r_1 \) and \( r_2 - r_1 \) large.
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