A Multi-Quantile Regression Time Series Model with Interquantile Lipschitz Regularization for Wind Power Probabilistic Forecasting

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Abstract

Producing probabilistic forecasts for renewable generation (RG) has become an important topic in power systems applications. This is due to the significant growth of RG participation in power systems worldwide. Additionally, it is well-known that decision making under uncertainty generally benefits from stochastic aware models, specially those relying on non symmetric costs and risk-averse assessments. Therefore, objective of this work is to propose an adaptive non-parametric time-series model driven by a regularized multiple-quantile-regression (MQR) framework. The goal is to derive a dynamic model for the conditional distribution function (CDF) describing renewable generation time series. To accomplish that, the quantile space is discretized within a user-defined granularity and an interpolation method is used to derive the full predictive CDF. Instead of estimating each quantile model separately, all models are jointly estimated through a single linear optimization problem. Thus, the estimation process converges to the global optimal parameters in polynomial time. Parsimony is imposed to the coefficients across quantiles and covariates. An innovative feature of our work is the consideration of a penalization term based on a Lipschitz regularization of the first derivative of coefficients in the quantile space. The proposed regularization imposes a smooth coupling effect among quantiles creating a single non-parametric CDF model with improved out-of-sample performance. A case study with realistic wind-power generation data from the Brazilian system shows: 1) the regularization model is capable to improve the performance of MQR probabilistic forecasts, and 2) our MQR model outperforms five relevant benchmarks: two based on the MQR framework, and three based on parametric models, namely, SARIMA, and GAS with Beta and Weibull CDF.

Keywords: Least absolute shrinkage and selection operator, Lipschitz regularization, multi-quantile regression, non parametric time-series, probabilistic forecast, renewable generation forecast.

1. Introduction

Renewable generation (RG) forecasting is a growing topic among power-systems community due to the number of applications that benefit from it. The intermittent nature of renewable energy sources and the complexity of power-system applications require specific and challenging developments, as for example, the replacement of usual point forecasting methods by more sophisticated probabilistic forecasting approaches. Such a probabilistic forecasts are in general used to improve decisions with risk-analysis-based information relying on the description of extreme quantiles [1].

Hence, new statistical models capable of addressing such technicalities have evolved into an emerging field in the power systems literature. See, for example, [2] [3] [4] [5] [6] [7] [8]. The main objective in such literature is to propose models capable of generating scenarios from a joint probability distribution of RG time series. This is particularly important in applications in power system based on stochastic optimization models [9]. For instance, energy trading [10], unit commitment [11] [12], grid expansion planning and investment decisions [13] [14] [15] are relevant and timely examples where scenarios play a crucial role.

1.1. Literature review on wind-power forecast

In [16], the commonly used methodologies regarding wind power probabilistic forecasting models are reviewed, and classified into parametric and nonparametric classes. The main characteristics of parametric models are (i) assuming a distribution shape and (ii) low computational costs. The ARIMA-GARCH model, for example, fits the RG series by assuming a Gaussian distribution a priori. On the other hand, nonparametric models have the following characteristics: (i) do not require a specified distribution for data description, (ii) require more data to fit a model and (iii) have higher computational costs. Popular methods are quantile regression (QR) [3] [5], kernel density estimation [17] [2], artificial intelligence [18] [19], or a combination of them [20] [3].

Although many of the familiar time series models rely on the assumption of Gaussian errors, RG time series, such as wind and solar, are reported as non-Gaussian (see [2] [17] and [21]). For instance, in [21], a recent publication proposing a Generalized Auto Regressive Score (GAS) [22] parametric model was proposed and tested to address a wind power time series in monthly basis. In such work, the non-Gaussian model has shown better results in comparison to the traditional based Gaussian-based models such as SARIMA. Still, GAS models rely on the a priori choice of the parametric distribution and the
estimation process is based on non-convex optimization problems for which global optimality can not be ensured. To circumvent this problem, the usage of nonparametric methods - which do not rely on a distribution assumption - appears as a promising alternative.

The importance of characterizing the whole distribution becomes even more relevant in applications with asymmetric costs such as those found in power systems applications [23]. For instance, load-shed costs are in general much higher than the cost of spilling renewable energy; the price and quantity risk due to the need of purchasing in high spot prices when RG falls short of meeting contract amounts is, in general, higher than the risk of clearing RG surplus in low spot prices [10]; just to mention a few. Hence, having a good estimate of the conditional distribution function (CDF) is essential for meeting accurate estimates of the risk involved in operational and planning decisions.

The seminal work [24] defines Quantile Regression (QR) as the solution of an optimization problem where the sum of the “check” function $\rho$ (defined formally in the next session), a piecewise linear convex function, is minimized. Conditional quantiles are the result of the above optimization problem, and this approach is employed in many works [25, 26, 27, 4, 5, 6, 7, 8]. Many fields benefit from such applications, ranging from risk measures in asset managements (Value at Risk) to a central measure robust to outliers, but in what follows we will focus on its usage in wind power time series.

In [4, 5, 6, 7, 8], QR is employed to model the conditional distribution of wind power time series. An updating quantile regression model is presented in [5]. The authors presented a modified version of the simplex algorithm to incorporate new observations without restarting the optimization procedure. In [6], the authors build a quantile model from an already existent independent wind power forecasts.

The authors in [4] individually estimates multiple quantile regressions, where each quantile model is conceived as a linear regression on a basis of functions. The quantile regression uses regularization through a penalty on the norm in a Reproducing Kernel Hilbert Space (RKHS), which is equivalent to a regularization one the explanatory variables coefficients. This work also develops an on-line learning technique, where the model is updated after each new observation arrives. In [8], wind power probabilistic forecasts are made by using QR with a special type of neural network with one hidden layer, called an extreme learning machine. In this setup, each quantile is a different linear combination of the hidden layer features. The authors of [28] use a weighted Nadaraya-Watson kernel estimator to estimate a conditional density function from a given time series.

Variable selection is another topic already explored in previous QR works. The work by [29] defines the properties and convergence rates of QR when adding a regularization term to select covariates according to the Least Absolute Shrinkage and Selection Operator (LASSO) [30]. The adaptive LASSO (adaLASSO) used in a QR model was investigated in [31]. In this variant, the penalty for each explanatory variable has a different weight, and this modification ensures that the oracle property is being respected. In [32, 33], the adaLASSO is applied in QR with MQR.

Until now, the main benefit of using a single multi-quantile regression (MQR) model is the guarantee that the quantiles will not overlap, thereby enabling simultaneously estimated quantile that give rise to a coherent forecasting model for the conditional distribution [8]. However, it is well-known that non-parsimonious models, in general, overfit in-sample data sets, and don’t provide a good generalization for out-of-sample data. One would also expect a similar behavior in conditional quantile model for the purpose of obtaining a probabilistic forecast. Therefore, the benefits of parsimonious models motivate a step forward in terms of developments in coupled MQR models.

1.2. Objective and contributions

The objective of this work is to propose an adaptive nonparametric CDF-time-series model driven by an MQR model, i.e., a linked array of QR simultaneously estimated. The goal is to approximately derive a dynamic model for the CDF describing renewable energy time-series. To accomplish that, the quantile space is discretized within a user-defined granularity and an interpolation method is used to derive the full CDF. However, instead of estimating each quantile model separately, all QR models are estimated through a single mathematical optimization problem. In this framework, parsimony is imposed to the coefficient estimates across quantiles and covariates.

Based on the parsimony principle, we expect that the same coefficient should not drastically change within small variations of the quantile probabilities. The first approach aiming to mitigate this issue was [33]. In this paper, the absolute value of the discrete first derivative of coefficients across quantiles were penalized within a MQR model applied to cross-sectional data. However, the aforementioned penalization is known to produce stepwise-shaped filtered signals [34], contradicting the idea of a single continuous interquantile model for the CDF. Hence, an innovative feature of our work is the consideration of a penalization term based on a Lipschitz regularization for the first derivative of the estimated coefficients across quantiles. The proposed regularization avoids the stepwise-shaped issue while imposing a smooth coupling effect among quantiles. This smoothed coupling effect creates the idea of a single nonparametric CDF model, which, for the tested data, has shown an improved generalization capability in a large rolling horizon out-of-sample test.

Therefore, in our work, a novel Multi-Quantile Regression with Lipschitz Regularization (MQR-LR) model is proposed for the time-series framework. To ensure a continuous (piecewise linear) shape for the parameters across quantiles, we include a Lipschitz regularization term of the first derivative of coefficients in the optimization problem used to estimate the MQR-LR model. Inspired in the $\ell_1$-filter [35], this term penalizes the absolute value of the second-discrete derivative of the QR coefficients across the quantile probabilities. As a result, a smooth link among the multiple QR models is induced. Finally, a second regularization term is used to select the best covariates among a set of many candidates. In this regard, we use the adaLASSO [35].
Interestingly, in general, statistical estimation procedures focus on minimizing point forecasting errors. However, power-system applications heavily rely on multi-step-ahead probabilistic forecasts \[ \text{1, 2].\] Unfortunately, universal evaluation metrics summarizing all the characteristics of forecast errors of all qualities are not available. In this context, the evaluation metric should reflect the objective of the user \[ \text{36].\] Therefore, to determine the best regularization parameters leading to an accurate probabilistic forecast up to \( K \) steps ahead, we use a rolling-horizon out-of-sample evaluation procedure. Within this procedure, we use a norm-1 distance between the empirical distribution and the predictive CDF is minimized, which in this work is chosen as the objective loss metric.

Summarizing, the contributions of this work are twofold:

- An adaptive non-parametric CDF-based time-series model for RG. The proposed model is conceived based on a MQR model with two regularization terms. The first term uses an \( \ell_1 \)-filter \[ \text{35] to consider a Lipschitz regularization to the first derivative of the coefficients with respect to the quantile probability. The proposed Lipschitz regularization term introduces parsimony in the estimation process through a smoothed link among the coefficients of the different QR models. The second regularization term considers the adaLASSO method to select the best explanatory variables (auto-regressive terms, exogenous variables or any function basis) \[ \text{35].\]

- A linear optimization problem to estimate the proposed MQR-LR model ensuring the aforementioned properties for the predictive non-parametric CDF. The model ensures global optimality to the joint estimation of all parameters.

The features previously described in the two contribution items significantly differentiates our model from the state-of-the-art reported works. As a minor contribution, we perform long-term out-of-sample rolling horizon test to show the improvement of our proposed inter-quantile regularization scheme. For a realistic wind-power generation data from the Brazilian system, results show: 1) the regularization model is capable to improve the performance of MQR probabilistic forecasts in out-of-sample assessments, and 2) our MQR model outperforms five relevant benchmarks, namely, MQR models without any regularization scheme, MQR models without cross-quantile regularization scheme, SARIMA models, and GAS models with Beta and Weibull CDF.

The remainder of this work is organized as follows. In Section II, we present the quantile regression framework and the proposed model, the MQR-LR model. In Section III we discuss how to estimate the MQR-LR model in order to obtain a full CDF. Section IV shows how to estimate, evaluate and simulate the proposed model. In Section V, we present two computational experiments to evaluate our model: i) a controlled experiment where data is generated through an auto-regressive model and ii) a case study based on real data from a Brazilian wind farm. Section VI concludes this study.

2. Quantile regression based time series model

Let \( Q_{y|x} : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R} \) be the conditional quantile function of a dependent random variable \( Y \) for a given value \( x \) of a \( d \)-dimensional explanatory random variable \( X \) (also known as vector of covariates). The \( \alpha \)-quantile function can be defined as follows:

\[
Q_{y|x}(\alpha) = F_{y|x=a}^{-1} \left( \alpha \right) = \inf \{ y : F_{y|x=a}(y) \geq \alpha \}. \quad (1)
\]

The function \( Q \) is the inverse of the conditional distribution function \( F \), and represents the smallest value for which the conditional distribution function is greater than a given probability \( \alpha \).

Let \( \rho \) be the “check” function

\[
\rho_\alpha(x) = \begin{cases} \alpha x & \text{if } x \geq 0 \\ (\alpha - 1)x & \text{if } x < 0 \end{cases}. \quad (2)
\]

The \( \alpha \)-quantile function for a given finite sample \( \{ y_t, x_t \}_{t \in T} \), where \( T \) is the set of time indexes, can be estimated through the solution of the following convex optimization problem:

\[
\hat{Q}_{y|x}(\alpha, \cdot) = \arg \min_{\theta \in \mathcal{Q}} \sum_{t \in T} \rho_\alpha(y_t - q_\alpha(x_t)). \quad (3)
\]

For inference on QR and finite sample properties, see Chapter 3 in \[ \text{37].\} The \( \alpha \)-quantile function \( q_\alpha(\cdot) \) belongs to a function space \( \mathcal{Q} \). We might have different assumptions for space \( \mathcal{Q} \), depending on the type of function we want to find for \( q \). A few properties, however, must be part of our choice of space, such as being continuous and having a limited first derivative. However, a general linear regression model,

\[
q_\alpha(x) = \beta_0 + \sum_{p} \beta_{p,\alpha} x_{p,t}, \quad (4)
\]

is capable of approximating any well-behaved non-linear function. This can be achieved by expanding the dimension \( |P| \) of vectors \( \beta_p := [\beta_{1,p}, ..., \beta_{|P|,p}]^T \) and \( x_t := [x_{1,t}, ..., x_{|P|,t}]^T \) to consider as many components as needed of a non-linear functional basis (see \[ \text{41} \] for an example where a trigonometric basis is used).

When dealing with high-dimensional vector of covariates, with many candidates to explain a given quantile, one has to properly select the relevant ones. In practice, this means that some coefficients from the vector \( \beta_p \) might assume a value of zero, for each quantile \( \alpha \). There are many ways of selecting a subset of variables among the available options. A classical approach for this problem is the stepwise algorithm \[ \text{38, 39, 30] \], which includes new variables iteratively.

Newly advocated variable selection methods that fits on a linear programming context are the LASSO/adaLASSO techniques, which consist of penalizing the \( \ell_1 \)-norm of the coefficient’s size. In addition to shrinking the coefficients towards zero, it has also the capability of effectively pushing the coefficients to zero (an effect that ridge regression cannot achieve \[ \text{30] \}). The usage of LASSO/adaLASSO in the QR context is the topic of study in \[ \text{40, 41, 29, 32, 35] \]. We refer the reader to
the work from [29], where it is possible to find specific properties and convergence rates when using the LASSO to perform model selection in a QR framework.

Regarding the penalization parameter $\lambda$, which dictates the shrinkage magnitude of the linear coefficients, the level of parsimony of the model can be defined by the user through such quantity. This is because higher values of $\lambda$ means less variables selected to be nonzero.

The single $\alpha$-quantile adaLASSO is estimated by the following optimization problem:

$$
\min_{\beta_0, \beta} \sum_{t \in T} \rho_\alpha(y_t - q_\alpha(x_t)) + \sum_{p \in P} w_p |\beta_{p,\alpha}|.
$$

(5)

What makes the adaLASSO different from the LASSO is the inclusion of the term $w_p$. If the model (5) is estimated with all $w_p = 1$, the output of the optimization problem are coefficients of LASSO $\beta_{p,\alpha}$. The adaLASSO coefficients $\beta_{p,\alpha}$ are obtained when solving the same optimization problem by letting $w_p = 1/|\beta_{p,\alpha}|$. The main advantage of AdaLASSO over the LASSO is the oracle property for variable selection, which is attended by the former but not by the latter. We refer the interested reader to [31].

3. Conditional distribution based on MQR-LR for time series

In the previous section, we presented a linear model for estimating a single $\alpha$-quantile using QR with adaLASSO as a regularization strategy to select the best covariates. However, to build a CDF from an array of quantiles, we propose estimating them at once by a single model in order to explore the coupling effect, i.e., parsimony and generalization capability, across different quantiles.

Let the finite discretization of the interval $[0, 1]$ be composed of a sequence of probabilities $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_J | J < 1$ and denote $A$ as the set $A = \{\alpha_j | j \in J\}$, where $J$ is an index set for the probabilities $\alpha_j$. The $\alpha$-quantiles are, from this point forward, indexed by $j$, to account for the different models that are simultaneously estimated. A property that must be respected is the monotonicity of the quantile function $Q$, such that $q_{\alpha_1} \leq q_{\alpha_2} \cdots \leq q_{\alpha_J}$. The sequence of quantiles defines a continuous quantile function after interpolation, from which a CDF can be obtained after inverting the estimated quantile function.

To produce a proper distribution function via the estimation of several quantile functions, the output of the problem must respect certain properties, such as being monotonically increasing. If a sequence of quantiles do not respect such a property, the issue is known as crossing quantiles. In addition to monotonicity, parsimony is a modeling virtue as it mitigates well-known side effects of over-fitting. In our case, where multiple quantile regressions are being jointly estimated to form a single distribution model, parsimony can be understood as coefficients that do not drastically change through quantiles. If the coefficient of a given $p$ covariate changes too abruptly with respect to a change in the quantile probability $\alpha$, it is an evidence that the estimated model has captured too much noise from the in-sample data and might not generalize well the true process. As a consequence, poor out-of-sample results are expected. To account for all these issues, all quantiles must be estimated at once.

To ensure parsimonious (smooth) transitions on the estimation of coefficients $\beta_{p,\alpha}$ through the quantiles $\alpha \in A$, we use a $\ell_1$-norm to consider a Lipschitz regularization on the first derivative of coefficients across quantiles. Inspired in second derivative filter [34], we define the discrete second derivative of $[\beta_{p,\alpha}]_{\alpha \in A}$ as follows:

$$
D^2_\alpha \beta_{p,j} := \frac{\beta_{p,\alpha+1,j} - 2 \beta_{p,\alpha,j} + \beta_{p,\alpha-1,j}}{\alpha_{j+1} - \alpha_j}.
$$

(6)

where for the sake of notation simplicity, hereinafter we assume $\beta_{p,j} := \beta_{p,\alpha,j}$. Therefore, the proposed MQR-LR model is defined by the vector of coefficients $\beta_{j} := [\beta_{1,j}, ..., \beta_{p,j}]^T$ and intercept $\beta_{0,j}$ that solve the following minimization problem:

$$
\begin{align*}
\text{Minimize} & \sum_{j \in J} \sum_{t \in T} \rho_\alpha(y_t - (\beta_{0,j} + \beta_{j}^T x_t)) \\
& + \lambda \sum_{p, j \in J} w_p | \beta_{p,j} | + \gamma \sum_{p \in J} \sum_{j \in J} | D^2_\alpha \beta_{p,j} |,
\end{align*}
$$

(7)

subject to:

$$
|\beta_{0,j} + \beta_{j}^T x_t| \leq |\beta_{0,j+1} + \beta_{j+1}^T x_t|, \forall t \in T, \forall j \in J(-1),
$$

(8)

where weights $w_{p,j} = 1/|\beta_{p,j}|$ are defined based on the values of coefficients $\beta_{p,j}$ estimated with the same model disregarding the AdaLASSO penalty. As in [31], in this work we set $\delta$ equal to one. The sum of the absolute values that compose the second derivative filter, $\sum_{p \in J} \sum_{j \in J} |D^2_\alpha \beta_{p,j}|$, is added on the objective function multiplied by a tuning parameter $\gamma$. Finally, $J(-1) = \{2, ..., |J|\}$ is the set containing all indexes but the first and $J’ = \{1, ..., |J| - 1\}$ is the set containing all indexes but the first and the last. These two auxiliary sets are used to implement the constraint (6), which ensures a monotonic quantile function by forcing that, for every $x_t$ and $\alpha_j$-quantile, $q_{\alpha,j}(x) \leq q_{\alpha_{j+1}}(x)$.

With this approach, one can keep track of the aforementioned crossing quantiles issue as well as using an interquantile coherence structure as a strategy to produce a parsimonious (regularized) non-parametric CDF-based time series models. It is worth mentioning that if we consider the supremum norm instead, this penalization term would reflect the minimization of the Lipschitz constant of the derivative of the coefficients across quantiles.

4. Estimation and Simulation procedures

This section presents computational aspects of the estimation of our proposed model, such as the mathematical programming formulation that accounts for the two regularization terms introduced in previous sections. We also present the algorithm to perform a Monte Carlo simulation with the estimated model and an out-of-sample procedure to determine the best regularization parameters. The methodology is implemented in R [42] and Julia [43] languages, using packages JuMP [44], Gurobi, RCall and Dierckx.
4.1. Estimation of the MQR-LR model

We assume that all variables are normalized. If they are not in the same scale, the shrinkage feature of the adaLASSO will fail, as different variables may have different weights according to their relative size. Thus, let \( \tilde{x}_{t,p} \) be an input observation at time \( t \) of covariate variable \( p \). The normalization process is a linear transformation to each covariate \( p \), such that all have a mean of 0 and a variance of 1. We apply the transformation \( x_{t,p} = (\tilde{x}_{t,p} - \bar{x}_p)/\tilde{x}_p \), where \( \bar{x}_p \) and \( \tilde{x}_p \) are the covariate \( p \)'s unconditional mean and standard deviation, respectively. The response variable \( Y \) does not need to be transformed. More information about this process is available in [45].

The MQR-LR model, as described in problem (7)-(8), can be implemented as a linear programming problem as shown below:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{p \in P} \sum_{j \in J} (\alpha_{p,j} + (1 - \alpha_{p,j})\epsilon_j^+ - \epsilon_j^-) + \lambda \sum_{p \in P} \sum_{j \in J} \gamma \max(D_{p,j}^+, D_{p,j}^-), \\
\text{subject to:} & \quad \epsilon_j^+ - \epsilon_j^- = y_j - \beta_0 - \beta_j^+x_j, \quad \forall t \in T, \forall j \in J, \\
& \quad \epsilon_j^+ - \epsilon_j^- = \beta_j, \quad \forall p \in P, \forall j \in J, \\
& \quad D_{p,j}^+ - D_{p,j}^- = \frac{\beta_{p,j+1} - \beta_{p,j}}{\alpha_{p,j+1} - \alpha_{p,j}}, \quad \forall p \in P, \forall j \in J'.
\end{align*}
\]

(9)

Variables \( \epsilon_j^+ \) and \( \epsilon_j^- \) represent the quantities \( |y - q(\cdot)|^+ \) and \( |y - q(\cdot)|^- \), respectively. The first line on the objective function in (9) represents the sum of the function \( \rho \) over all \( j \), i.e., \( \rho_0(y - q(\cdot)) = \alpha \epsilon_j^+ + (1 - \alpha)\epsilon_j^- \). The second derivative term \( D_{p,j}^+ - D_{p,j}^- \) is implemented on the optimization problem by adding a penalty on the objective function to penalize its absolute value, modeled as the sum of auxiliary variables \( D_{p,j}^+ - D_{p,j}^- \). The tuning parameter \( \gamma \) controls how rough the sequence \( \{\beta_{p,j}\}_{p,j} \) can be, for a given \( p \).

4.2. Estimating the regularization parameters

Most statistical models are designed to provide a good fit on point forecasts. In this work, however, the objective is to produce a predictive CDF. To evaluate the statistical quality of a given pair of regularization parameters \((\lambda, \gamma)\), the estimated CDF is compared with the actual observed data one-step ahead within a rolling horizon out-of-sample test. To do that, the difference between the quantile probability forecast \( \alpha_j \) and its actual cumulative frequency of occurrence \( F_j \) is computed for an out-of-sample set of data. While \( \alpha_j \) is given by the model specification (or, in case of assessing the performance \( K \) steps ahead, by simulation using Algorithm [1]), \( F_j \) is computed based on the ratio of observations belonging to the quantile interval \((-\infty, q_{\alpha_j}(x_j))\) along a user-defined estimation horizon.

The mean absolute error (MAE) among conditional-quantile forecasts and observed ones for a given out-of-sample horizon \( H \) is defined by

\[
\text{MAE}_H = \frac{1}{|J|} \sum_{j \in J} |\alpha_j - F_j|, 
\]

(17)

where \( F_j \) depends on the vector of parameters \( \theta \).

The MAE applied to the quantiles emphasizes the estimated CDF accuracy for data not seen before. Depending on the application, it might be interesting to put different weights on different quantiles. In this work, however, we will treat every quantile as equal concerning the error measure.

For the sake of completeness, we also test another relevant metric largely used in time-series analysis, namely, the Information Criteria (IC). It is also employed in other multiple quantile model studies [32, 33] to tune parameters. An IC summarizes two desirable characteristics in model selection: in-sample goodness of fit and penalization of complexity (or lack of parsimony) as given by the model size. Hence, in order for a covariate to be included in the model, it must supply a sufficient goodness of fit. The expression for SIC appropriate for quantile autoregression is presented below:

\[
SIC = \sum_{j \in J} \log \left( \sum_{p \in P} \rho_0(y_j - \beta_0 - \beta_j x_j) + \frac{\log(|T|)\epsilon_0}{2|T|} \right),
\]

(18)

where \( \theta = [\gamma \lambda]^T \) and \( \epsilon_0 \) is the elbow set, defined as \( \epsilon_0 = |(l, j) : y_l - q_{\theta}(x_l) = 0| \). The authors in [40] show that the quantity \( |\epsilon_0| \) is the effective degrees of freedom in the quantile regression.

4.3. Monte Carlo scenario generation

We use a Monte Carlo (MC) approach to produce \( S \) different future scenarios \( \{y_{\tau,j}\}_{\tau=1}^{T+K} \) of length \( K \). It consists of first estimating the model for a given time \( \tau \) and scenario \( s \). Then, for each of these we take the input vector \( x_{\tau,s} \) and calculate the discrete quantile function \( Q_{\tau,j|x} \), which is an intermediate step to estimate the continuous quantile function \( Q_{\tau,j|x} \). This same MC procedure is employed to produce scenario simulations for a variety of MQR models. This procedure is detailed in Algorithm [1] that can be found in the Appendix.

5. Case Studies

In the following two case studies, our model MQR-LR is tested against benchmark models. We study the performance of our model through two case studies. In the first, a controlled study is performed to check the capability of the model to capture well-known patterns of Gaussian models. In the second, a real wind-power generation time series from the Brazilian power system is used and the five benchmarks are compared.

5.1. Benchmark models

In the next two subsections, different comparisons are performed to test the modeling capacity and performance of our proposed MQR-LR model. To do that, five relevant benchmarks are used. The first two benchmarks are conceived to isolate the
effect of the proposed cross-quantile regularization scheme. In this sense, the first benchmark model is exactly our proposed model but disregarding the two regularization terms in (9), i.e., with $\lambda = 0$ and $\gamma = 0$. Hereinafter, this benchmark model is referred to as $\textit{MQR-B1}$. To test the effect of the proposed cross-quantile Lipschitz regularization in our model, we devise a benchmark model keeping the adalASSO regularization but disregard the second-derivative penalty term (last term) in (9), i.e., similar to our model but with $\gamma = 0$. Hereinafter, this benchmark model is referred to as $\textit{MQR-B2}$.

To cover the family of models relying on parametric CDF’s, three other relevant benchmark models are selected. Thus, the third benchmark model is based on SARIMA models implemented in [46]. Finally, to provide a more interesting benchmark based on state-of-the-art non-Gaussian parametric models, two instances of the Generalized Auto-regressive with Score (GAS) model implemented in [47] are selected. Therefore, the forth and fifth benchmark are the GAS using the Beta CDF, referred to as $\textit{GAS (BETA)}$, and the GAS using Weibull CDF, referred to as $\textit{GAS (WEIBULL)}$.

5.2. Controlled Studies I - Auto regressive Process

In our first simulation study, the capability of the proposed MQR-LR methodology to recover a non-order auto-regressive model, AR(1), is tested against MQR-B1. In this case, as we know the true model is an AR(1), we define $\lambda = 0$ for all MQR models and no variable selection approach is used. Hence, MQR-B1 and MQR-B2 are equivalent and the difference between the MQR-B1 and our MQR-LR relies solely on the cross-quantile regularization.

The used AR(1) model is the following:

$$ y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, 1), \quad t = 1, \ldots, 400, \quad (19) $$

with $\beta_0 = 0$, $\beta_1 = 0.3$ and $y_0 = 0$. The interquantile regularization parameter $\gamma$ (see equations (9)-(16)) is estimated using cross-validation, which is a popular technique for selecting optimal parameters values in cross-sectional data. After simulating 1000 different time series given by equation (19), the three models are estimated.

Since the main objective of this controlled experiment is to assess the capability of our non-order model to recover a given AR(1) CDF, its performance can be evaluated by examining how close the estimated quantiles are from the population ones. The results for each model are depicted in Figure 1, where a box-plot containing the results for the 1000 simulations are shown. The conclusions from this experiment are: (i) coefficient estimation errors for the central quantiles deviate very little from those estimated by the AR model; (ii) extreme quantiles are usually harder to estimate due to fewer observations available, consequently the estimation error increases on the extremities; (iii) MQR-LR has an advantage over MQR-B1 because it shows smaller variance of estimators.

5.3. Case study with realistic data

In this section, the MQR-LR methodology is tested in generating probabilistic forecasts for a real wind power unit generation. The wind power time series, measured in megawatts, is composed of 2 years (from June-2011 to May-2013) of hourly power generation observations from a wind farm located in the Northeast of Brazil.

As previously mentioned in Section 4, the case study resorts to a rolling horizon scheme. At each step, the model is estimated using a window of size 720 (corresponding to a month’s worth of hourly observations) and the quantiles of the next $K$ periods are forecasted. The rolling horizon estimation is repeated for 500 times, each containing the forecast of $K$ hours ahead. The parameters $\lambda$ and $\gamma$ are constant across all data set. Then, the regularization parameters are selected among a thin grid of values according to two metrics: MAE and SIC. For the sake of clarity and comparison purposes, we selected the last 48-hour lags as covariates for all benchmark models.

5.3.1. Analysis of Results

We tested our model against the five benchmarks for the one-step-ahead forecast, as is typical in the literature on the theme, and for the four-step-ahead horizon to illustrate the forecast capability of the proposed methodology. The estimation of parametric models remained identical for both one- and four-step-ahead tests. The parameters $\lambda$ and $\gamma$ may differ for each horizon. This guarantees that the model will have the best performance according to the criteria and horizon for which it is being built. The forecasted quantiles from the MQR-LR model are directly evaluated against the historic data on the one-step-ahead case. On the other hand, when forecasting four steps ahead the CDF is not directly available and need to be empirically obtained. To do that, we use Algorithm 1 (Appendix) to simulate scenarios...
to generate the empirical K-step-ahead CDF for all MQR-based models tested in this study. The MAE metric is evaluated over the set of quantiles \( \Lambda = \{0.05, 0.1, \ldots, 0.9, 0.95\} \), totaling 19 measurements, for each time \( t \). In this work, the out-of-sample performance is assessed through a rolling-horizon scheme. Although we focus on the quantile MAE metric in this work, we also assess the performances of MQR-based models using the SIC.

Results for our MQR-LR model calibrated with both MAE and SIC are presented in Table 1. From the results of this table, it can be seen that the two metrics are effective in improving out-of-sample performance. For instance, note that MQR-B1, where no calibration scheme is needed, out-of-sample evaluation produced a value for MAE equal to 3.5. When calibrating our model via SIC, the out-of-sample evaluation via MAE has decreased to 2.04, which represents a 41.7% improvement in out-of-sample performance. Notwithstanding, when calibrating our model via MAE, this gain was even higher, as expected. In this case, MAE dropped to 0.98, which represents a 72% improvement. The same pattern of out-of-sample improvement via MAE is also observed for the 4-hour horizon.

One has to bear in mind that the selection of a particular calibration metric is a modeling choice of the user. In this forecasting study, the model calibrated via MAE is outperformed by the model calibrated via SIC when SIC is used as the evaluation metric. Thus, although the metric selection is a modeling choice, we suggest the quantile-MAE metric as more appropriate for specifying a robust CDF in terms of performance against unseen data.

Additionally, note that the benefit of the two regularization metrics defining the MQR-LR model can be decomposed. Given the MAE drop from MQR-B1 to MQR-B2 (2.09 for the one-hour ahead forecast and 0.12 for the 4h case), we see the benefit of the AdaLASSO regularization, and from MQR-B2 to MQR-LR (0.43 for the one-hour ahead forecast and 0.34 for the 4h case) we can assess the additional benefit of the interquantile regularization.

It is worth mentioning that the three parametric benchmarks, SARIMA, GAS (WEIBULL), and GAS (BETA), are outperformed by the proposed MQR-LR calibrated under both metrics (SIC and MAE) in both time horizons.

In the sequel, we investigate the forecasting performance of our proposed model for 4 hours ahead. Figure 2 presents a heatmap of the MAE metric for the MQR-LR model considering a combination of regularization parameters. We can see there is a region of optimal regularization levels according to this criteria. The worst performances occur when \( \lambda = 0 \), such that all covariates are included in the model.

Since coefficients are estimated using a rolling-horizon scheme, they are updated at each step. As the regularization parameters are kept constant, the figures of a given period are sufficient to understand the coefficients behaviour in the experiment as a whole. Figures 3 and 4 present the estimated coefficients on the first period of the experiment for the MQR-LR (MAE) and the MQR-B1, respectively. For each model, \( \beta_0(\alpha) \) is shown on the left side of the figure, while \( \beta(\alpha) \), for each lag, is on the right side. The comparison of coefficients across MQR-LR (MAE) and MQR-B1 illustrates the effect of regularizations on the first model. One advantage of the proposed model is that quantiles are all estimated by a single model, which helps to decrease the variance of the estimators. As a consequence, only a handful of coefficients are selected to be nonzero, and its \( \beta \) coefficients follow regularized piecewise linear functions, in contrast to MQR-B1’s noisier and higher variability coefficients (as seen in the experiment in section 5.2).

An advantage of MQR models is the fact that they are able to capture an asymmetric non-Gaussian distribution - which a SARIMA model cannot - as illustrated by Figure 5 where we present a cumulated probability error function (CE) to compare the distribution fit across quantiles of MQR-LR (MAE) and SARIMA. A consequence of a better CDF estimation is that simulated scenarios are more accurate in relation to real data, as corroborated by the results presented on Table 1. In Figure 6 we compare the median (50%), extreme quantiles (5% and 95%), and the 1st and 3rd quartiles (25% and 75%), obtained from the MQR-LR (MAE) model via simulation, with the asso-

Table 1: Cumulated statistics across all quantiles

| Model (tuning criteria) | Horizon | \( \lambda \) | \( \gamma \) | 0 | 0.17 | 0.33 | 0.5 | 1 | 2 | 3 | 5 | 7 | 10 |
|-------------------------|---------|--------------|-------------|---|-------------------------------|------------------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| MQR-LR (SIC)            | 1h      | 0            | 2.10        | 2.54 | 2.66 | 2.66 | 2.65 | 2.61 | 2.57 | 2.58 | 2.60 | 2.56 |
|                         | 1h      | 0.25         | 2.11        | 2.47 | 2.41 | 2.46 | 2.42 | 2.47 | 2.41 | 2.45 | 2.45 | 2.40 |
|                         | 1h      | 0.5          | 2.11        | 2.61 | 2.44 | 2.30 | 2.25 | 2.23 | 2.24 | 2.29 | 2.22 | 2.27 |
|                         | 1h      | 1            | 2.30        | 2.48 | 2.34 | 2.25 | 2.12 | 2.14 | 2.17 | 2.24 | 2.16 | 2.11 |
|                         | 1h      | 1.75         | 2.11        | 2.45 | 2.28 | 2.01 | 2.10 | 2.04 | 2.11 | 2.01 | 2.04 | 2.08 |
|                         | 1h      | 2.5          | 2.14        | 2.33 | 2.26 | 2.06 | 1.98 | 2.02 | 2.04 | 2.04 | 2.04 | 2.03 |
|                         | 1h      | 3.25         | 2.13        | 2.31 | 2.18 | 2.00 | 1.93 | 1.92 | 1.96 | 2.02 | 1.97 | 1.93 |
|                         | 1h      | 4            | 2.07        | 2.19 | 1.98 | 1.86 | 1.89 | 1.79 | 1.84 | 1.80 | 1.79 | 1.81 |
|                         | 1h      | 5.25         | 2.15        | 2.14 | 1.88 | 1.91 | 1.77 | 1.74 | 1.81 | 1.78 | 1.69 | 1.68 |
|                         | 1h      | 6            | 2.13        | 2.12 | 1.95 | 1.82 | 1.68 | 1.74 | 1.72 | 1.68 | 1.68 | 1.71 |
|                         | 1h      | 6.75         | 2.08        | 2.08 | 1.93 | 1.82 | 1.71 | 1.72 | 1.76 | 1.66 | 1.64 | 1.74 |
|                         | 1h      | 8            | 2.02        | 2.04 | 1.99 | 1.89 | 1.74 | 1.68 | 1.68 | 1.68 | 1.69 | 1.68 |
|                         | 1h      | 10           | 1.96        | 2.10 | 2.01 | 1.82 | 1.78 | 1.67 | 1.72 | 1.70 | 1.75 | 1.83 |
|                         | 1h      | 15           | 2.02        | 2.09 | 2.04 | 1.97 | 1.87 | 1.87 | 1.94 | 2.00 | 1.97 | 1.93 |
|                         | 1h      | 20           | 2.15        | 2.19 | 2.17 | 2.19 | 1.98 | 1.99 | 2.07 | 2.05 | 2.11 | 2.05 |

* MAE values scaled by a factor of 100.

Figure 2: Calculated MAE of forecasting quantiles in a four-hours window. Lower values have a lighter tone, while higher ones are darker. The MAE values are scaled by a factor of 100.
Finally, to illustrate the benefit of interquantile regularization as adopted in our model, in Fig. 7 we present the MAE metric restricted to some extreme quantiles, i.e., accounting only for the lower and higher quantiles, \{0.05, 0.10, 0.15, 0.85, 0.90, 0.95\}. By connecting quantiles, the MQR-LR model uses information from other quantiles when estimated across several quantiles, resulting in better accuracy than the MQR benchmark model, as regarding tails behavior, where fewer observations are available.

6. Concluding Remarks

In this work we propose an adaptive non-parametric conditional distribution function (CDF) time-series model driven by an array of quantile regressions. The estimation process for the proposed CDF model simultaneously estimates all quantiles through a single linear programming problem with two regularization terms. The regularization terms used account for 1) explanatory variable selection via adaLASSO, and 2) the link of all the quantile models aiming to create a single CDF-model structure. Based on the proposed CDF-time-series associated historic time series.
model, we developed an algorithm that can be used to generate synthetic scenarios for wind power generation, featuring the relevant properties of the empirical distribution and its dynamics. Such a simulation procedure can be used to feed the various applications in power systems relying on simulated scenarios, namely, risk analysis in energy trading, expansion planning, unit commitment, and economic dispatch. Our results show that the scenarios generated through the proposed model outperforms other benchmarks such as other Multiple Quantile Regression models and the classical SARIMA models and state-of-the-art GAS models. Our proposed CDF-model also exhibits a relevant tradeoff between parsimony and flexibility, which results in better evaluation metrics for the extreme quantiles in an out-of-sample test. As interesting future research topics we highlight the consideration of a kernel density estimation process on top of the estimated quantiles and the study of a nonlinear basis of functions.

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Appendix A. Monte Carlo simulation algorithm

Algorithm 1 MC procedure for simulating $S$ scenarios $K$ steps ahead

1. Estimate an MQR model (for the MQR-LR solve the optimization problem defined in equation (9)-(16)). A set of coefficients $\hat{\beta}_0$, $\hat{\beta}_j$ is the output from this optimization, using time series $y_t$ and $x_{pt}$ from the period $t = 1, \ldots, T$.

2. Initialize time index $\tau = |T| + 1$.

3. For each scenario $s \in S$, do:
   a. Let $x_{\tau,s} = [y_{\tau-1,s}, \ldots, y_{\tau-24,s}]$ be the vector of explanatory variables used as the input to predict the conditional distribution function in time $\tau$ and scenario $s$.
   b. Let $\hat{Q}_{y|X} : A \times \mathbb{R}^d \to \mathbb{R}$ be the discrete quantile function. Its values are mapped according to the estimated quantile $\hat{Q}_{y|X}(\alpha, x_{\tau,s}) \leftarrow \hat{\beta}_0 + \hat{\beta}_j x_{\tau,s}$, for all $j \in J$.
   c. To define the continuous function $\hat{Q}_{y|X} : [0, 1] \times \mathbb{R}^d \to \mathbb{R}$ from $\hat{Q}_{y|X}$, use a linear interpolation to connect the points. As $0 < \alpha_1 < \cdots < \alpha_J < 1$, there are no quantile estimates for the intervals $[0, \alpha_1]$ and $[\alpha_J, 1]$. These gaps are filled by linearly extending the line that connects $\alpha_1$ to $\alpha_2$ on the left hand side and extending the line that connects $\alpha_{J-1}$ to $\alpha_J$ on the right hand side until the support $[0, 1]$ is fully mapped.
   d. Let $U$ be a random variable with a uniform distribution over the interval $[0, 1]$. As $Q_{y_i}(U)$ has the same distribution as $y_\tau$, by taking
      $$ y_{\tau,s} \leftarrow \hat{Q}_{y|X}(u), \quad u \sim U[0, 1], $$
      we simulate scenarios next values.

4. Let $\tau = \tau + 1$. If $\tau > K$, then stop. Else, go back to step 3).