Gravitational collapse of thick domain walls

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Abstract
Numerical simulations are performed of the gravitational collapse of a scalar field with a $\lambda \phi^4$ potential. Comparisons for the wall motion and stress-energy are made with the thin shell approximation. It is found that the thin shell approximation generally works well, especially for the motion of the wall. However, the approximation is not as good at late times as it is at early times.

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1. Introduction

Domain walls in general relativity [1] are usually treated in the thin shell approximation [2]. However, there is also a description of a domain wall as a soliton of a field theory. In a certain limit one expects the field theory description to reduce to the thin shell description. Indeed, one can formally expand the Einstein-scalar equations in powers of the thickness and obtain the thin shell equations at lowest order in this expansion [3, 4]. Nonetheless, one cannot simply assume the validity of such an expansion. Instead in a field theory one gets to choose initial data and then the field equations give the results of the evolution of those data. Thus the thin shell approximation is a good approximation for a thick domain wall to the extent that the evolution of initial data that are well approximated by the thin shell treatment continue under evolution to be well approximated by this treatment. To test this approximation, we choose initial data for a spherically symmetric thick domain wall, perform a numerical evolution, and compare to the corresponding thin shell approximation. The initial data are chosen so that the spherical wall locally approximates as closely as possible a planar domain wall in flat spacetime. A comparison is made between the motion of the wall in the simulation and the corresponding motion in the thin shell approximation. Comparisons are also made between the stress-energy of the wall and the stress-energy of the thin shell approximation. By choice of initial data, the thin shell approximation works well at the initial time. However, we also find that it works extremely well for the motion of the wall throughout the evolution. Furthermore, the approximations for the stress-energy work well throughout the evolution, though less well...
at late times than at early times. Methods are described in section 2, results in section 3 and conclusions in section 4.

2. Methods

The Lagrangian for a thick wall takes the form

$$L = -\frac{1}{2} \nabla^a \phi \nabla_a \phi - V(\phi),$$

where $V(\phi)$ is a potential with two minima. The equation of motion associated with this Lagrangian is

$$\nabla^a \nabla_a \phi - \frac{dV}{d\phi} = 0.$$  (2)

The shell also satisfies the Einstein field equation

$$G_{ab} = \kappa T_{ab}.$$  (3)

Here $G_{ab}$ is the Einstein tensor, $\kappa = 8\pi G$ where $G$ is Newton’s gravitational constant, and the stress-energy of the scalar field is given by

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab}(\nabla^c \phi \nabla_c \phi + 2V).$$  (4)

Simulations of such thick walls in the case where their self-gravity can be neglected were done by [5] and [6]. The dynamics of self-gravitating vacuum bubbles have also been studied numerically [7, 8]. We will use the standard $\lambda \phi^4$ potential

$$V = \lambda (\phi^2 - \eta^2)^2.$$  (5)

For simulations in spherical symmetry, one often takes the area radius as one of the spatial coordinates and chooses time to be orthogonal to this radial coordinate [9]. However, this coordinate system breaks down when a trapped surface forms and thus cannot follow the evolution after black hole formation. Instead we will use the method of [10] and use maximal slicing with radial length as the radial coordinate. Maximal slicing allows us to simulate part of the region inside the black hole without encountering the singularity. The metric takes the form

$$ds^2 = -\alpha^2 dt^2 + (dr + \beta^r dt)^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$  (6)

Note that the usual area radius $R$ is not one of the coordinates and is instead a function of the coordinates $t$ and $r$. The relevant evolution and constraint equations for the scalar field and the metric components are analogous to those of [10] and are given in the appendix. These equations have the following structure: as is usual for spherical symmetry, the only physical degrees of freedom are the scalar field $\phi$ and its normal derivative $P$. The Einstein field equation, the maximal slicing condition, and the form of the metric allow one to compute the metric components $\alpha$, $\beta$, and $R$ as well as the components of the extrinsic curvature $K_{ab}$ once $\phi$ and $P$ are known. Thus the only real evolution is the evolution of $\phi$ and $P$ which takes place through equation (2).

We now consider the choice of initial data. We can think of the initial configuration as that of the wall itself plus some additional scalar field radiation, which in analogy to the corresponding effect in binary black hole simulations we will refer to as ‘junk radiation’. Just as in the binary black hole simulations, we would like to minimize the amount of junk radiation present in our initial data. This is easiest to do when the wall is at its maximum radius and is therefore momentarily static. The initial configuration is then a moment of time symmetry so that $P$ and $K_{ab}$ vanish. To minimize junk radiation, we will choose the initial wall radius
$r_0$ sufficiently large so that the wall can be approximated as planar, and will then choose the scalar field profile to match as closely as possible that of a planar flat spacetime domain wall. More precisely, define the quantities $\epsilon$ and $\sigma$ by

$$\epsilon = \frac{1}{\eta \sqrt{2 \lambda}},$$

$$\sigma = \frac{4}{3} \sqrt{2 \lambda \eta^3},$$

where $\eta$ and $\lambda$ are the parameters of the potential. For planar walls in flat spacetime $\sigma$ is the energy per unit area of the wall and $\epsilon$ is an effective wall thickness. Formally the thin shell limit of the solution is the limit as $\epsilon \to 0$ at constant $\sigma$. Correspondingly, one can specify $\epsilon$ and $\sigma$; then $\lambda$ and $\eta$ are determined by

$$\lambda = \frac{2}{3 \sigma \epsilon^3}$$

$$\eta = \sqrt{\frac{3}{4} \sigma \epsilon}.$$ (10)

In flat spacetime a static planar domain wall solution is given by

$$\phi = \eta \tanh(z/\epsilon).$$

(11)

We choose for initial data

$$\phi = \eta \tanh((r - r_0)/\epsilon),$$

(12)

where the constant $r_0$ can be chosen arbitrarily but should be chosen to be much larger than $\epsilon$. The quantity $S$ is set equal to $\partial_t \phi$. Equation (A.13) is integrated for $R$ using the fact that at the origin $R = 0$ and $\partial_t R = 1$. Then at each time step, the evolution proceeds as follows: first equations (A.11) and (A.13) are integrated to find $K'_r$ and $R$. And then equation (A.15) is solved for the lapse $\alpha$ (using a tridiagonal method and the fact that $\partial_r \alpha = 0$ at the origin and $\alpha \to 1$ at infinity). Then equation (A.7) is integrated to find the shift $\beta_r$. Finally, the quantities $\phi$, $S$ and $P$ are evolved to the next time step using equations (A.16), (A.17) and (A.18) respectively. The evolution is done using the iterated Crank–Nicholson method, and all spatial derivatives are found using standard centered differences. We use units where $\kappa = 1$.

As the evolution proceeds we can check for black hole formation by looking for the presence of a marginally outer trapped surface. In spherical symmetry, such a surface is given by the condition

$$\nabla^a R \nabla_a R = 0$$

which in our coordinate system is equivalent to

$$\partial_r R + \frac{1}{2} R K'_r = 0.$$ (13)

3. Results

The simulations are run with $n + 1$ points evenly spaced between $r = 0$ and a maximum value $r_{\text{max}}$. We choose $r_{\text{max}} = 20$. We would like to know how the initial scalar field profile changes under evolution. Figure 1 shows the result of a simulation with $r_0 = 5$, $\epsilon = 0.25$ and $\sigma = 0.15$. This simulation was run with $n = 9600$ and results are shown for the scalar field $\phi$ at times 0, 2, 4, 6, and 8. As the evolution proceeds, the scalar field profile moves inward. Furthermore, the profile becomes steeper, and departures from the simple tanh form become more pronounced. A marginally outer trapped surface forms at $t = 5.09$.

In order to be sure that these results are not numerical artifacts, we need to know that the code is convergent and that these results are within the convergent regime. Note
that equation (A.8) is not used in the evolution, but should nonetheless be satisfied to within numerical error due to finite differencing. Thus this equation provides a check on the performance of the simulation. More precisely, define the constraint quantity $C$ by

$$C = \frac{\alpha}{2} K''_r + R^{-1} (\beta' \partial_t R - \partial_t R)$$

(14)

and let $\|C\|$ be the $L_2$ norm of $C$. Then in an exact solution $C$ should vanish, while in a numerical treatment $C$ should converge to zero in the limit of zero step size. Figure 2 shows $\ln \|C\|$ as a function of time. Here the parameters are as in the previous simulation, except that one simulation is run with $n = 9600$ and one with $n = 19200$. The results demonstrate second-order convergence: that is, halving the step size reduces $C$ by a factor of 4.
In the thin shell formalism one is mostly concerned with the motion of the wall. The motion of a spherical wall is described by giving its area radius as a function of proper time. That is,

$$R = R_0(\tau). \quad (15)$$

Using the results of [1, 2] one can show that the equation of motion of the thin shell is

$$\ddot{R}_0 = \frac{3}{4} \kappa \sigma \left( 1 + \dot{R}_0^2 \right)^{1/2} - 2\dot{R}_0^{-1}(1 + \dot{R}_0^2). \quad (16)$$

We would like to know how well the motion given by solving equation (16) models the behavior of the scalar field domain wall. Here we can do a direct comparison: at any given time, the position of the wall will be taken to be the place where $\phi = 0$ and the proper time $\tau$ will be that of an observer who is always at the position of the wall. At each time step of the simulation, we can find the position of the wall, so it remains to evaluate $\tau$. Let $u^a$ be the four-velocity of the observer who remains at the position of the wall. Then it follows that $u^a$ is a unit timelike vector for which

$$u^a \nabla_a \phi = 0.$$ 

It then follows using equations (A.1) and (A.10) that

$$u^a = A^{-1} \left[ n^a - \frac{P}{S} \left( \frac{\partial}{\partial r} \right)^a \right], \quad (17)$$

where the quantity $A$ is defined by

$$A = \sqrt{1 - \frac{P^2}{S^2}} \quad (18)$$

evaluated at the position of the wall. The relation between the normal vector and the time coordinate is

$$n_a = -\alpha \nabla_a t \quad (19)$$

so using equation (17) we have

$$\frac{dt}{d\tau} = u^a \nabla_a t = -\alpha^{-1} u^a n_a = \alpha^{-1} A^{-1} \quad (20)$$

and therefore we find

$$d\tau = \alpha A dt. \quad (21)$$

Choosing $\tau = 0$ at the beginning of the simulation, we then integrate equation (21) to find $\tau$ at all times of the simulation. Since we also have $R_0$ at all points of the simulation, we can produce the thick wall $R_0(\tau)$ for comparison with the $R_0(\tau)$ of the thin wall. Figure 3 shows such a comparison. Here the parameters of the wall are the same as in the previous simulations, with the solid line being that of the simulation and the dashed line the solution of equation (16).

We now consider a comparison of the stress-energy of the simulation to that of the thin shell approximation. From equations (4) and (A.10) it follows that the trace of the stress-energy tensor is

$$T = P^2 - S^2 - 4V. \quad (22)$$

However, in the treatment of [3] to lowest order in thickness of the wall we have

$$T = \frac{-9\sigma}{4\epsilon \cosh^4(z/\epsilon)}. \quad (23)$$

where $z$ is geodesic distance from the center of the wall. Calculating geodesic distance from a simulation can be complicated; but luckily because the expression for $T$ in equation (23) falls off so rapidly, we only need $z$ in the vicinity of the wall’s center and thus can approximate $z$
using a Taylor series. In particular, note that at the center of the wall $\nabla_\alpha z$ must be a unit vector orthogonal to $u^\alpha$. It then follows that
\[\nabla_\alpha z = A^{-1} \left[ \left( \frac{\partial}{\partial r} \right)^\alpha - \frac{P}{S} n^\alpha \right].\] (24)

Taking the inner product of equation (24) with $(\partial/\partial r)^\alpha$ we find that at the center of the wall
\[\frac{\partial z}{\partial r} = A^{-1}\] and we therefore find that near the wall $z$ is well approximated by
\[z = (r - R_0)/A.\] (26)

Thus the thin wall expression for $T$ is given by equation (23) with $z$ given by equation (26).

The thin wall approximation also predicts that the component of the stress-energy perpendicular to the wall vanishes. Since the wall is at $\phi = 0$, we can define a general ‘direction perpendicular to the wall’ to be the unit direction parallel to $\nabla^\alpha \phi$. We can then define $T_\perp$ to be the component of the stress-energy in this direction. It then follows from equation (4) that
\[T_\perp = \frac{1}{2} (S^2 - P^2) - V.\] (27)

Thus if the thin wall approximation is a good approximation to the behavior of thick walls, then we would expect that $T$ of the simulation should closely agree with $T$ of the thin wall approximation, and that in the simulation $T_\perp$ should be $\ll T$. Such a comparison is given in figure 4 for $t = 2$ and in figure 5 for $t = 6$. The parameters of the wall are the same as in the previous simulations. Here the trace of the stress-energy $T$ is plotted for the simulation (solid line representing the expression of equation (22)) and for the thin shell approximation (dashed line representing the expression of equation (23)). The $T_\perp$ of the simulation (equation (27)) is represented by a dotted line. Note that at both times the simulation results for $T$ are well approximated by those of the thin shell expression. Also note that at both times $T_\perp \ll T$. And finally note that both the thin shell approximation for $T$ and the approximation that $T_\perp$ vanishes become less good as the evolution proceeds.
4. Conclusions

Our simulations indicate that the thin wall approximation is an excellent approximation for thick wall gravitational collapse. Perhaps more importantly, we have developed a robust numerical method for thick wall collapse that could be used on other projects. In particular, we plan to simulate the collapse of a charged thick domain wall. Thin charged walls collapse to charged black holes and can even be used to form an extreme (charge equals mass) black
hole [11]. In contrast, simulations of the collapse of a free charged scalar field yield black holes that are always far from extreme [12]. It will be interesting to see whether an extreme black hole can be formed by the collapse of a charged thick domain wall.

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Appendix. Equations of motion

The metric takes the form
\[ ds^2 = -\alpha^2 \, dt^2 + (dr + \beta^r \, dr)^2 + R^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \]  
(A.1)

Thus the spatial metric \( \gamma_{ab} \) has components
\[
\begin{align*}
\gamma_{rr} &= 1 \\
\gamma_{\theta\theta} &= R^2 \\
\gamma_{\phi\phi} &= R^2 \sin^2 \theta .
\end{align*}
\]  
(A.2)

The extrinsic curvature, \( K_{ab} \) is defined by
\[ K_{ab} = -\gamma_a^c \nabla_c n_b , \]  
(A.3)

where \( n^a \) is the unit normal to the surfaces of constant time \( t \). However, due to spherical symmetry and maximal slicing, there is only one independent component of the extrinsic curvature. Specifically we have
\[ K^\theta_\theta = K^\phi_\phi = -\frac{1}{2} K^r_r . \]  
(A.4)

Equation (A.3) is equivalent to
\[ \frac{\partial}{\partial t} \gamma_{ij} = -2 \alpha K_{ij} + D_i \beta_j + D_j \beta_i , \]  
(A.5)

where \( D_i \) is the covariant derivative of the spatial metric \( \gamma_{ij} \). The \( rr \) component of equation (A.5) yields
\[ \frac{\partial}{\partial t} \beta^r = \alpha K^r_r , \]  
(A.6)

whose solution is
\[ \beta^r = \int_0^r \alpha K^r_r \, dr . \]  
(A.7)

The \( \theta \theta \) component of equation (A.5) yields
\[ \frac{\partial}{\partial t} R = \beta^r \partial_r R + \frac{\alpha}{2} R K^r_r . \]  
(A.8)

We now use the momentum constraint of the Einstein field equation to determine the extrinsic curvature. For maximal slicing (\( K = 0 \)) this constraint is
\[ D_a K^{ab} = -\kappa \gamma^{bc} n^d T_{cd} . \]  
(A.9)

Define the quantities \( P \) and \( S \) by
\[ P = n^a \nabla_a \phi , \quad S = \partial_r \phi . \]  
(A.10)

Then equation (A.9) becomes
\[ \frac{\partial}{\partial t} K^r_r + 3 R^{-1} K^r_r = -\kappa PS . \]  
(A.11)
Note that there is also a Hamiltonian constraint associated with the Einstein field equation. In the case of maximal slicing, this constraint is

\[ (3)R - K_{ab}K^{ab} = 2\kappa T_{ab}n^a n^b, \]  

(3.12)

where \( (3)R \) is the spatial scalar curvature. This equation yields

\[ \partial_t \partial_t R = 1 - \left( \partial_r R \right)^2, \]  

(3.13)

We now determine the lapse \( \alpha \). It follows from the maximal slicing condition that

\[ D_a D^a \alpha = \alpha \left[ K_{ab}K^{ab} + \frac{\kappa}{2} T_{ab}(n^a n^b + \gamma^{ab}) \right] \]  

(3.14)

which yields

\[ \partial_t \partial_t \alpha = \alpha \left[ \frac{3}{2} (K^r_r)^2 + \kappa (P^2 + S^2 + 2V) \right]. \]  

(3.15)

We now consider the evolution of the scalar field. From the definitions of \( P \) and \( S \) it follows that

\[ \partial_t \phi = \alpha P + \beta S \]  

(3.16)

\[ \partial_t S = \alpha (\partial_t P + K^r_r S) + \beta \partial_t \alpha + \beta \partial_t S. \]  

(3.17)

The equation of motion, equation (2) becomes after some straightforward but tedious algebra

\[ \partial_t P = \beta^2 \partial_t \phi + \partial_t \alpha + \alpha \left[ R^{-2} \partial_t (R^2 S) - \frac{dV}{d\phi} \right]. \]  

(3.18)

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