On the $\mu$-admissible set in the extended affine Weyl groups of $E_6$ and $E_7$

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1 Introduction

In [6], Kottwitz and Rapoport introduce the notions of the $\mu$-admissible set $\text{Adm}(\mu)$ and $\mu$-permissible set $\text{Perm}(\mu)$ for a cocharacter $\mu$ (we will recall these notions in section 2) and in [6] §11 they prove that $\text{Adm}(\mu) \subset \text{Perm}(\mu)$. Haines and Ngo show in [3, Theorem 3], that the equality $\text{Adm}(\mu) = \text{Perm}(\mu)$ does not hold for general $\mu \in X^*$. On the other hand, Kottwitz and Rapoport prove in [6] that $\text{Adm}(\mu) = \text{Perm}(\mu)$ holds for minuscule cocharacters for the root systems $A_n$ and $C_n$. They raise the question whether the $\mu$-permissible set $\text{Perm}(\mu)$ and the $\mu$-admissible set $\text{Adm}(\mu)$ agree for any minuscule cocharacter. Smithling proves $\text{Adm}(\mu) = \text{Perm}(\mu)$ for minuscule cocharacters for the root systems $B_n$ and $D_n$ in [7]. Thus, this question has a positive answer for all classical root systems.

The aim of the present paper is to show that the question has a negative answer in the cases $E_6$ and $E_7$.

We will consider a root datum for the root system $E_6$ and a minuscule cocharacter $\mu$. We exhibit a certain element $x$ of its extended affine Weyl group (everything will be defined in subsection 3.1), and investigate, whether $x$ is $\mu$-permissible, resp. $\mu$-admissible. In subsection 3.2 we show that $x$ is $\mu$-permissible using only simple matrix calculations. Afterwards, in subsection 3.3 we find that $x$ is not $\mu$-admissible, using computer software and relying on a result of He and Lam which characterizes the $\mu$-admissible set. As a double-check, we prove in subsection 3.4 that $x$ is not $\mu$-admissible using computer software again, but this time relying on a result of Haines, instead of the characterization by He and Lam. Thus, the existence of $x$ gives a negative answer to the question of Kottwitz and Rapoport.

In section 4 we repeat the same considerations for a certain element of the extended affine Weyl group attached to some root datum for the root system $E_7$, again yielding a counterexample to the question of Kottwitz and Rapoport.

The counterexamples presented in this paper were found using the CHEVIE-package of the software GAP. The author wrote a program that calculated the size of the permissible and the admissible set in the above settings. This gave 20159 for the cardinality of the admissible set and 20303 for the cardinality of the permissible set for $E_6$; for $E_7$, the admissible set has 1227151 elements and the permissible set has 1298607 elements.

As the conjecture of Kottwitz and Rapoport does not depend on the actual choice of the root datum, our calculations show that $\text{Adm}(\mu) \neq \text{Perm}(\mu)$ holds in general for the considered minuscule coweights of $E_6$ and $E_7$. Note that $E_7$ has only one non-trivial dominant minuscule coweight, and in $E_6$ the two non-trivial dominant minuscule coweights are interchanged by an automorphism of the root system of $E_6$. Hence we get $\text{Adm}(\mu) \neq \text{Perm}(\mu)$ for all the non-trivial minuscule coweights $\mu$ of $E_6$ and $E_7$. As $\text{Adm}(\mu) = \text{Perm}(\mu)$ has been proved for minuscule coweights for classical root systems, this finally gives a complete answer to the question on the relation between the admissible and the permissible set for minuscule coweights (note that in the other cases $E_8$, $F_4$ and $G_2$ there are no non-trivial minuscule coweights).

Of course, it is not clear whether computer calculations can be considered as a proof. We need the computer to check $x \notin \text{Adm}(\mu)$. We do this in several different ways and with different software (the
CHEVIE-package of GAP as well as Sage. We only apply standard functions of these softwares. Thus, a mistake in these functions in all the different approaches seems improbable.

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2 Notions

Let us recall the notions of \( \mu \)-admissibility and \( \mu \)-permissibility introduced by Kottwitz and Rapoport in [6].

Let \((X^*, X_*, R, R')\) be a root datum with reduced root system \(R\). Let \(W\) be its (finite) Weyl group, \(W_a = Q(R') \times W = ZR' \rtimes W\) its affine Weyl group and \(\tilde{W} = X_* \times W\) its extended affine Weyl group.

Set \(V := X_* \otimes \mathbb{R}\). We choose a base \(B = \{\alpha_1, \ldots, \alpha_l\}\) of \(R\) and denote by \(A\) the base alcove

\[ A = \{ v \in V | (\alpha_i, v) > 0 \text{ for } i = 1, \ldots, l \text{ and } (\tilde{\alpha}, v) < 1 \} \]

in \(V\), where

\[ \tilde{\alpha} = \sum_{i=1}^l n_i \alpha_i \]

is the highest root of \(R\). We consider the stabilizer

\[ \Gamma := \{ x \in \tilde{W} | x(\mathfrak{A}) = \mathfrak{A} \} \]

of \(A\) under the \(\tilde{W}\)-operation on the alcoves in \(V\). Every element \(x \in \tilde{W}\) can be written uniquely in the form \(x = y\gamma\) with \(y \in W_a\) and \(\gamma \in \Gamma\), since \(W_a\) acts freely and transitively on the set of alcoves.

Now we can extend the Bruhat order on \(W_a\) to \(\tilde{W}\) as follows: For elements \(x = y\gamma\) and \(x' = y'\gamma'\) of the extended affine Weyl group \(\tilde{W}\) with \(y, y' \in W_a\) and \(\gamma, \gamma' \in \Gamma\), we set \(x \leq x'\), if \(\gamma = \gamma'\) and \(y \leq y'\) in the Bruhat order on \(W_a\).

If we consider some \(\lambda \in X_*\) as an element of \(\tilde{W}\), we denote it by \(t_\lambda\). Thus, \(t_\lambda\) is the translation \(V \to V, v \mapsto v + \lambda\).

Let \(\mu \in X_*\) be a fixed cocharacter.

An element \(x \in \tilde{W}\) of the extended affine Weyl group is called \(\mu\)-admissible, if \(x \leq t_{w(\mu)}\) for some \(w \in W\). The set of all \(\mu\)-admissible \(x \in \tilde{W}\) is denoted by \(\text{Adm}(\mu)\).

Let \(P_\mu\) be the convex hull in \(V\) of the \(W\)-orbit \(W\mu = \{w(\mu)\mid w \in W\}\). An element \(x \in \tilde{W}\) of the extended affine Weyl group is called \(\mu\)-permissible, if it satisfies both of the following conditions:

(i) If \(x = y_1\gamma_1\) and \(t_\mu = y_2\gamma_2\) with \(y_1, y_2 \in W_a\) and \(\gamma_1, \gamma_2 \in \Gamma\), we have \(\gamma_1 = \gamma_2\).

(ii) For every element \(v \in \mathfrak{A}\) in the closure of the base alcove \(\mathfrak{A}\) we have \(x(v) - v \in P_\mu\).

The first condition (i) is equivalent to \(x \in W_\mu t_\mu\). The set of all \(\mu\)-permissible \(x \in \tilde{W}\) is denoted by \(\text{Perm}(\mu)\).

Let \(\{\gamma_1, \ldots, \gamma_l\}\) be the dual basis in \(Q(R') \otimes \mathbb{R}\) of the basis \(\{\alpha_1, \ldots, \alpha_l\}\) of \(ZR' \otimes \mathbb{R}\). Furthermore, we set

\[ a_1 := \frac{\gamma_1}{n_1}, \ldots, a_l := \frac{\gamma_l}{n_l} \text{ and } a_{l+1} := 0. \]

Then \(a_1, \ldots, a_{l+1}\) are elements of the minimal facets of \(\mathfrak{A}\). Therefore, condition (ii) in the definition of \(\mu\)-permissibility is equivalent to \(x(a_i) - a_i \in P_\mu\) for \(i = 1, \ldots, l + 1\). In the following we will consider root data, where \(a_1, \ldots, a_{l+1}\) are in fact the vertices of \(\mathfrak{A}\) and every element of \(\mathfrak{A}\) is a convex combination of \(a_1, \ldots, a_{l+1}\). Here, the equivalence of condition (ii) and \(x(a_i) - a_i \in P_\mu\) for \(i = 1, \ldots, l + 1\) is easy to see.

Recall that \(\mu \in X_*\) is called minuscule, if \(\langle \alpha, \mu \rangle \in \{-1, 0, 1\}\) for all \(\alpha \in R\).
3 The case of $E_6$

3.1 Setting

Let $R$ denote the root system $E_6$. Furthermore, let $R^\vee$ be the dual root system, $X^* := Q(R)$ the root lattice and $X_* := P(R^\vee)$ the coweight lattice. Then $(X^*, X_*, R, R^\vee)$ is a root datum with (finite) Weyl group $W$, affine Weyl group $W_0 = Q(R^\vee) \rtimes W$ and extended affine Weyl group $\tilde{W} = X_* \rtimes W = P(R^\vee) \rtimes W$. Observe that by definition the cocharacters $X_*$ agree with the coweights $P(R^\vee)$.

For our calculations we consider the explicit construction of $R$ given in [1, Plate V(I)]: Let

$$V^* := \{(x_1, \ldots, x_8) \in \mathbb{R}^8 \mid x_6 = x_7 = -x_8\}.$$  

Furthermore, let $e_1, \ldots, e_8$ the standard basis vectors of $\mathbb{R}^8$. Then the 72 vectors in $V^*$

$$\pm e_i \pm e_j$$

for $1 \leq i < j \leq 5$ and

$$\pm \frac{1}{2} \left( e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\delta(i)} e_i \right)$$

with $\sum_{i=1}^{5} \delta(i)$ even form a root system of type $E_6$, which we identify with $R$.

Let $e'_1, \ldots, e'_8$ the dual basis of $e_1, \ldots, e_8$ in the dual space of $\mathbb{R}^8$. The dual space of $V^*$ can be identified with

$$V := \{x'_1 e'_1 + \cdots + x'_8 e'_8 \in \mathbb{R}^8 \mid x'_6 = x'_7 = -x'_8\}$$

and the dual root system $R^\vee$ consists of

$$\pm e'_i \pm e'_j$$

for $1 \leq i < j \leq 5$ and

$$\pm \frac{1}{2} \left( e'_8 - e'_7 - e'_6 + \sum_{i=1}^{5} (-1)^{\delta(i)} e'_i \right)$$

with $\sum_{i=1}^{5} \delta(i)$ even, see [1, Plate V(V)].

Then $V^* = Q(R) \otimes_{\mathbb{Z}} \mathbb{R} = X^* \otimes_{\mathbb{Z}} \mathbb{R}$ and $V = Q(R^\vee) \otimes_{\mathbb{Z}} \mathbb{R} = P(R^\vee) \otimes_{\mathbb{Z}} \mathbb{R} = X_* \otimes_{\mathbb{Z}} \mathbb{R}$, which agrees with the notation from section 2.

As in [1, Plate V(II)] we consider the base of $R$ given by

$$\begin{align*}
\alpha_1 &= \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + e_4 + e_5 + e_6 + e_7), \\
\alpha_2 &= e_1 + e_2, \\
\alpha_3 &= e_2 - e_1, \\
\alpha_4 &= e_3 - e_2, \\
\alpha_5 &= e_4 - e_3, \\
\alpha_6 &= e_5 - e_4.
\end{align*}$$

The simple reflections belonging to $\alpha_1, \ldots, \alpha_6$ are as usual denoted by $s_1, \ldots, s_6$, respectively. Thus, $s_i : V \to V$ is given by

$$s_i(v) = v - (\alpha_i, v)\alpha_i^\vee.$$  

According to [1, Plate V(IV)] the highest root of $R$ is $\bar{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. Thus,

$$\mu := g_1 \frac{2}{3}(e'_8 - e'_7 - e'_6)$$

is a dominant minuscule coweight (see [1, Plate V(VI)]).
We set $x = w_2t_\mu w_1^{-1} \in \hat{W}$ with
\[ w_1 = s_2s_4s_5s_6s_3s_4s_5s_2s_4s_3s_1 \]
and
\[ w_2 = s_4s_5s_6s_2s_4s_5. \]

In the following subsections we will check that $x \in \text{Perm}(\mu)$, but $x \notin \text{Adm}(\mu)$. Hence $\text{Perm}(\mu) \neq \text{Adm}(\mu)$ for $\mu = \varrho_1$. As there is an automorphism of the root system $E_6$ interchanging $\alpha_1$ and $\alpha_6$, we also get $\text{Perm}(\varrho_0) \neq \text{Adm}(\varrho_0)$. Note that $\varrho_1$ and $\varrho_0$ are the only dominant minuscule coweights for $E_6$.

### 3.2 $x \in \text{Perm}(\mu)$

In this subsection we prove that $x$ is $\mu$-permissible. The orbit $W\mu$ consists of the following 27 elements:

\[ \mu = \frac{2}{3}(e_8' - e_7' - e_6'), \]
\[ -\frac{1}{3}(e_8' - e_7' - e_6') + e_i' \]

with $\sum_{i=1}^{5} \delta(i)$ even and

with $1 \leq i \leq 5$. This has been calculated by hand, and afterwards checked using the CHEVIE-package of GAP. Now $P_\mu$ is the convex hull of these 27 points and we have to show $x(a_i) - a_i \in P_\mu$ for $i = 1, \ldots, 7$. Here, the $a_i$ are given by (see [1, Plate V(IV)])

\[ a_1 = \frac{\varrho_1}{1} = \mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{2}{3} \\ -\frac{3}{2} \end{pmatrix}, \quad a_2 = \frac{\varrho_2}{2} = \begin{pmatrix} 1 \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}, \quad a_3 = \frac{\varrho_3}{3} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}, \quad a_4 = \frac{\varrho_4}{3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{6} \\ -\frac{1}{6} \end{pmatrix}, \quad a_5 = \frac{\varrho_5}{2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -\frac{3}{2} \end{pmatrix}, \quad a_6 = \frac{\varrho_6}{1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad a_7 = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \]

Since $\mu$ is fixed by $s_2$, $s_3$, $s_4$, $s_5$ and $s_6$, we have

\[ x = w_2t_\mu w_1^{-1} = t_{w_2(\mu)}w_2w_1^{-1} = t_\mu w_2w_1^{-1} = t_\mu s_4s_5s_6s_2s_4s_5s_1s_3s_4s_2s_4s_3s_8s_6s_5s_4s_2. \]

Multiplying the matrices for the simple reflections $s_i$ we get that $w_2w_1^{-1}$ is represented by the matrix

\[ M := \frac{1}{4} \begin{pmatrix} -1 & 3 & -1 & -1 & -1 & 1 & 1 & -1 \\ -3 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -3 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -3 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 3 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 3 \end{pmatrix}. \]
This product was calculated using Octave.

Now \( x(a_i) - a_i = M a_i + \mu - a_i \) for \( i = 1, \ldots, 7 \) yields the following results (also calculated with Octave)

\[
x(a_1) - a_1 = \begin{pmatrix}
-\frac{1}{2} \\
0 \\
-\frac{1}{6}
\end{pmatrix},
\]

\[
x(a_2) - a_2 = \begin{pmatrix}
-\frac{1}{2} \\
0 \\
-\frac{1}{6}
\end{pmatrix} + \begin{pmatrix}
\frac{1}{2} \\
0 \\
\frac{1}{6}
\end{pmatrix}.
\]

\[
x(a_3) - a_3 = \begin{pmatrix}
0 \\
\frac{1}{6} \\
0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-\frac{1}{4} \\
\frac{1}{2} \\
-\frac{1}{6}
\end{pmatrix} + \frac{1}{2} \begin{pmatrix}
\frac{1}{4} \\
0 \\
\frac{1}{6}
\end{pmatrix}.
\]

\[
x(a_4) - a_4 = \begin{pmatrix}
0 \\
\frac{1}{6} \\
0
\end{pmatrix} = \frac{1}{3} \begin{pmatrix}
-\frac{1}{4} \\
\frac{1}{2} \\
-\frac{1}{6}
\end{pmatrix} + \frac{1}{3} \begin{pmatrix}
\frac{1}{4} \\
0 \\
\frac{1}{6}
\end{pmatrix} + \frac{1}{3} \begin{pmatrix}
0 \\
\frac{1}{6} \\
0
\end{pmatrix}.
\]

\[
x(a_5) - a_5 = \begin{pmatrix}
-\frac{1}{2} \\
0 \\
-\frac{1}{6}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-\frac{1}{4} \\
\frac{1}{2} \\
-\frac{1}{6}
\end{pmatrix} + \frac{1}{2} \begin{pmatrix}
\frac{1}{4} \\
0 \\
\frac{1}{6}
\end{pmatrix}.
\]

\[
x(a_6) - a_6 = \begin{pmatrix}
-\frac{1}{2} \\
0 \\
-\frac{1}{6}
\end{pmatrix},
\]

\[
x(a_7) - a_7 = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

The values \( x(a_i) - a_i \) have been calculated independently using CHEVIE, which yielded the same results. Alternatively, all the calculations in this subsection could have also been done by hand.

As given above, every \( x(a_i) - a_i \) for \( i = 1, \ldots, 7 \) is a convex combination of points in \( W \mu \) and therefore \( x(a_i) - a_i \in P_\mu \). Hence, condition (ii) for \( x \in \text{Perm}(\mu) \) is fulfilled. On the other hand condition (i) is trivial:

\[
x = w_2 t_\mu w_1^{-1} = w_2 w_1^{-1} t_{w_1(\mu)} = W_\alpha t_{w_1(\mu)} = W_\alpha t_\mu.
\]

**3.3 First test of \( x \not\in \text{Adm}(\mu) \)**

Let us denote by

\[
I(\mu) := \{ s_i \mid i \in \{ 1, \ldots, l \}, (\alpha_i, \mu) = 0 \}
\]
the set of simple reflections fixing $\mu$. Since $\mu = e_1$, we have $I(\mu) = \{s_2, s_3, s_4, s_5, s_6\}$ in our case.

Now let $W_{I(\mu)}$ be the subgroup of $W$ generated by $I(\mu)$. We denote the set of minimal length representatives in $W/W_{I(\mu)}$ by $W^{I(\mu)}$.

According to He and Lam [5, Theorem 2.2(1)] the map 

$$W^{I(\mu)} \times W \to W_{t_\mu}, \quad (z_1, z_2) \mapsto z_2t_\mu z_1^{-1}.$$ 

is a bijection (see also [4, Proposition 3.2(1)]). Furthermore [5, Theorem 2.2(2)] states that an element $z_2t_\mu z_1^{-1} \in W$ with $z_1 \in W^{I(\mu)}$ and $z_2 \in W$ is $\mu$-admissible iff $z_2 \leq z_1$ (see also [4, Proposition 3.2(2)]).

Thus, for proving $x = w_2t_\mu w_1^{-1} \not\in \text{Adm}(\mu)$ it is sufficient to show $w_1 \in W^{I(\mu)}$ and $w_2 \not\leq w_1$.

Unfortunately, it is laborious to check these statements with calculations by hand. However, there are standard routines to check them via computer. The statements were checked with Sage and with the CHEVIE-package of GAP independently.

### 3.3.1 Calculations with Sage

```python
sage: W=WeylGroup("E6")
sage: [s1,s2,s3,s4,s5,s6]=W.simple_reflections()
sage: w1=s2*s4*s5*s6*s3*s4*s5*s2*s4*s3*s1
sage: w2=s4*s5*s6*s2*s4*s5
sage: w2.bruhat_le(w1)
False
sage: w1.coset_representative([2,3,4,5,6])==w1
True
```

In the first line, $W$ is defined to be the Weyl group of $E_6$. In the second line the simple reflections of $W$ are given names. In the third and the fourth line, the element $w_1$ and $w_2$ are defined as products of simple reflections as in subsection 3.1. The fifth line asks whether $w_2 \leq w_1$. Sage’s answer is ‘False’. Finally, the sixth line asks whether $w_1$ equals the minimal length representative of the coset $w_1 W^{I(\mu)}$, where as above $W_{I(\mu)}$ denotes the subgroup of $W$ generated by $\{s_2, s_3, s_4, s_5, s_6\}$. The answer is ‘True’.

### 3.3.2 Calculations with CHEVIE

In CHEVIE there is only a function for minimal length elements of right cosets and not for left cosets. Since the length of every element equals the length of its inverse and also the Bruhat order inequalities $w_2 \leq w_1$ and $w_2^{-1} \leq w_1^{-1}$ are equivalent, we can solve this issue just by considering the inverses of $w_1$ and $w_2$ instead. Then the following calculations in GAP using the CHEVIE-package yield the desired statements:

```gap
gap> W:=CoxeterGroup("E",6);  CoxeterGroup("E",6)
gap> W_Imu:=ReflectionSubgroup(W,[2,3,4,5,6]);  ReflectionSubgroup(CoxeterGroup("E",6), [ 2, 3, 4, 5, 6 ] )
gap> w1inv:=EltWord(W,[1,3,4,2,5,4,3,6,5,4,2]);  ( 1,72,59,20,21,29)( 2, 6, 3,17,60,69)( 4, 5)( 7,67,71,54,25,32)
< 8,11, 9,22,55,66)(12,64,70,48,28,34)(13,27,50,52,51,62)(14,16,15,26,49,63)
(18,61,68,43,31,35)(19,30,44,47,45,58)(23,56,57,65,37,36)(24,33,38,42,39,53)
(40,41)
gap> w2inv:=EltWord(W,[5,4,2,6,5,4]);  ( 2, 5,56, 4, 6)( 3,24,21,13,31)( 7,26,23,17,33)( 8,11,52,10,50)
( 9,28,25,19,15)(12,30,27,22,18)(14,44,47,16,46)(20,40,42,38,41)
(29,35,32,36,34)(39,60,57,49,67)(43,62,59,53,69)(45,64,61,55,51)
(48,66,63,58,54)(65,71,68,72,70)
gap> Bruhat(W,w2inv,w1inv);
```
In the first line, \( W \) is defined to be the Weyl group of \( E_6 \), GAP repeats this definition. In the next line, \( W_1(\mu) \) is defined as the subgroup of \( W \) generated by \( s_2, s_3, s_4, s_5, s_6 \) as above (which GAP repeats, again). In the third and the fourth line, the elements \( w_1^{-1} \) and \( w_2^{-1} \) of \( W \) are defined (the inverses of the elements \( w_1 \) and \( w_2 \) considered above). GAP answers by printing the corresponding permutations of the roots by those elements. In the fifth line GAP is asked, whether \( w_2^{-1} \leq w_1^{-1} \) in the Bruhat order on \( W \), and it returns ‘false’. In the sixth line GAP is asked, whether \( w_1^{-1} \) equals the minimal length representative of the coset \( W_1(\mu)w_1^{-1} \), and it returns ‘true’.

Thus, both Sage and CHEVIE computed the statements \( w_1 \in W_1(\mu) \) and \( w_2 \not\leq w_1 \) to be true.

### 3.4 Second test of \( x \not\in \text{Adm}(\mu) \)

In this subsection \( x \not\in \text{Adm}(\mu) \) is checked again, this time relying on a result of Haines instead of the result by He and Lam used in the last subsection. In \cite{haines2011representations} proof of Proposition 4.6 Haines showed \( w \leq t_\mu(0) \) for every \( \mu \)-admissible element \( w \in \tilde{W} \). Thus, for proving \( x \not\in \text{Adm}(\mu) \) it is enough to check \( x \not\leq t_\mu(0) \) (as \( x(0) = w_2t_\mu w_1^{-1}(0) = w_2(\mu) = \mu \)).

Again, the software Sage was used, which provides a function `reduced_word_of_alcove_morphism`, that calculates for any \( x' \in \tilde{W} \) the corresponding element \( y' \in W_a \) such that \( x' = y'\gamma \) for some \( \gamma \in \Gamma \).

The following calculation works with weights instead of coweights, but this does not make a difference since the root system \( E_6 \) is self-dual.

```python
sage: R = RootSystem(["E",6,1]).weight_lattice()
sage: Lambda = R.fundamental_weights()
sage: W = WeylGroup(R)
sage: s = W.simple_reflections()
sage: R.reduced_word_of_alcove_morphism((Lambda[1]-Lambda[0]).translation)
[0, 2, 4, 3, 5, 4, 2, 0, 6, 5, 4, 2, 3, 4, 5, 6]
sage: y1=s[0]*s[2]*s[4]*s[3]*s[5]*s[4]*s[2]*s[0]*s[6]*s[5]*s[4]*s[2]*s[3]*s[4]*s[5]*s[6]
sage: w1=s[2]*s[4]*s[5]*s[6]*s[3]*s[4]*s[5]*s[2]*s[4]*s[3]*s[1]
sage: w1mu=w1.action(Lambda[1]-Lambda[0])
sage: w2=s[4]*s[5]*s[6]*s[2]*s[4]*s[5]
sage: w1mu=w1.action(Lambda[1]-Lambda[0])
sage: R.reduced_word_of_alcove_morphism(w1mu.translation)
[2, 4, 3, 5, 4, 2, 0, 6, 5, 4, 2, 3, 1, 4, 5, 6]
sage: y2=w2*w1inv*s[2]*s[4]*s[3]*s[5]*s[4]*s[2]*s[0]*s[6]*s[5]*s[4]*s[2]*s[3]*s[1]*s[4]*s[5]*s[6]
sage: y2.bruhat_le(y1)
False
```

Here \( R \) is defined to be the affine root system \( E_6 \) (together with its weight lattice) in the first line. The fundamental weights are denoted by ‘\( \Lambda \)’ in the second line and the affine Weyl group by \( W \) in the third line. In the fourth line, ‘\( s \)’ is set as a notion for the simple reflections. The function `reduced_word_of_alcove_morphism` computes in the fifth line a reduced word for \( y_1 \) in \( W_a \) such that \( t_\mu = y_1\gamma_1 \) for some \( \gamma_1 \in \Gamma \) (recall that \( \mu = \rho_1 \)) and in the sixth line \( y_1 \) is defined as the corresponding element of \( W_a \).

In the 7th, 8th and 9th line the elements \( w_1, w_1^{-1} \) and \( w_2 \) of the finite Weyl group are defined as defined in subsection 3.3. Then in the 10th line ‘\( w_1 \mu \)’ is defined to be \( w_1(\mu) \). In the 11th line the function `reduced_word_of_alcove_morphism` computes a reduced word for \( y' \) in \( W_a \) such that \( t_{w_1(\mu)} = y'\gamma_2 \) for some \( \gamma_2 \in \Gamma \). The 12th line defines \( y_2 = w_2w_1^{-1}y' \). In fact, we have

\[
x = w_2t_\mu w_1^{-1} = w_2w_1^{-1}t_{w_1(\mu)} = w_2w_1^{-1}y'\gamma_2 = y_2\gamma_2
\]
with \( y_2 \in W_a \) and \( \gamma_2 \in \Gamma \).

For checking \( x \not\leq t_\mu \) it is therefore enough to ask whether \( y_2 \leq y_1 \). This is done in the 13th line. Indeed, Sage returns ‘False’.

4 The case of \( E_7 \)

This section is basically a repetition of the last section, this time working with \( E_7 \) instead of \( E_6 \). All considerations and calculations are fully analogous, therefore the computer calculations for \( x \not\in \text{Adm}(\mu) \) will just be displayed, but not explained again.

4.1 Setting

Let \( R \) now denote the root system \( E_7 \). As before, let \( R^\vee \) be the dual root system, \( X^* := Q(R) \) the root lattice and \( X_* := P(R^\vee) \) the coweight lattice. Then \( (X^*, X_*, R, R^\vee) \) is a root datum with (finite) Weyl group \( W \), affine Weyl group \( W_a := Q(R^\vee) \rtimes W \) and extended affine Weyl group \( \tilde{W} = X_* \rtimes W = P(R^\vee) \rtimes W \). Observe that by definition the cocharacters \( X_* \) agree with the coweights \( P(R^\vee) \).

For our calculations we consider the explicit construction of \( R \) given in [1, Plate VI(I)]: Let

\[
V^* := \{(x_1, \ldots, x_8) \in \mathbb{R}^8 \mid x_7 = -x_8\}.
\]

Furthermore, let \( e_1, \ldots, e_8 \) the standard basis vectors of \( \mathbb{R}^8 \). Then the 126 vectors in \( V^* \)

\[
\pm e_i \pm e_j
\]

for \( 1 \leq i < j \leq 6 \),

\[
\pm (e_7 - e_8)
\]

and

\[
\pm \frac{1}{2} \left( e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\delta(i)} e_i \right)
\]

with \( \sum_{i=1}^{6} \delta(i) \) odd form a root system of type \( E_7 \), which we identify with \( R \).

Let \( e_1', \ldots, e_8' \) be the dual basis of \( e_1, \ldots, e_8 \) in the dual space of \( \mathbb{R}^8 \). The dual space of \( V^* \) can be identified with

\[
V := \{x_1' e_1' + \cdots + x_8' e_8' \in \mathbb{R}^8 \mid x_7' = -x_8'\}
\]

and the dual root system \( R^\vee \) consists of

\[
\pm e_i' \pm e_j'
\]

for \( 1 \leq i < j \leq 6 \),

\[
\pm (e_7' - e_8')
\]

and

\[
\pm \frac{1}{2} \left( e_7' - e_8' + \sum_{i=1}^{6} (-1)^{\delta(i)} e_i' \right)
\]

with \( \sum_{i=1}^{6} \delta(i) \) odd, see [1, Plate VI(V)].

Then \( V^* = Q(R) \otimes_{\mathbb{Z}} \mathbb{R} = X^* \otimes_{\mathbb{Z}} \mathbb{R} \) and \( V = Q(R^\vee) \otimes_{\mathbb{Z}} \mathbb{R} = P(R^\vee) \otimes_{\mathbb{Z}} \mathbb{R} = X_* \otimes_{\mathbb{Z}} \mathbb{R} \), which agrees with the notation from section 2.
As in [1] Plate VI(II)] we consider the base of $R$ given by

$$\begin{align*}
\alpha_1 &= \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + e_4 + e_5 + e_6 + e_7) \\
\alpha_2 &= e_1 + e_2 \\
\alpha_3 &= e_2 - e_1 \\
\alpha_4 &= e_3 - e_2 \\
\alpha_5 &= e_4 - e_3 \\
\alpha_6 &= e_5 - e_4 \\
\alpha_7 &= e_6 - e_5
\end{align*}$$

The simple reflections belonging to $\alpha_1, \ldots, \alpha_7$ are as usual denoted by $s_1, \ldots, s_7$, respectively. Thus, $s_i : V \to V$ is given by

$$s_i(v) = v - \langle \alpha_i, v \rangle \alpha_i^\vee.$$

According to [1] Plate VI(IV)], the highest root of $R$ is $\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. Thus,

$$\mu := \varpi_7 = \frac{1}{2}(\varepsilon'_4 - \varepsilon'_7) + \varepsilon'_6$$

is the only dominant minuscule coweight (see [1] Plate VI(VI)]).

We set $x = w_2\mu w_1^{-1} \in \tilde{W}$ with

$$w_1 = s_2s_4s_5s_8s_1s_3s_2s_4s_5s_6s_7$$

and

$$w_2 = s_4s_3s_2s_4s_1.$$

In the following subsections we will check that $x \in \text{Perm}(\mu)$, but $x \notin \text{Adm}(\mu)$. This implies that $\text{Perm}(\mu) \neq \text{Adm}(\mu)$ for $\mu = \varpi_7$.

### 4.2 $x \in \text{Perm}(\mu)$

In this subsection we prove that $x$ is $\mu$-permissible. The orbit $W\mu$ consists of the following 56 elements:

$$\pm \frac{1}{2}(\varepsilon'_4 - \varepsilon'_7) \pm \varepsilon'_i$$

with $1 \leq i \leq 6$ and

$$\frac{1}{2} \left( \sum_{i=1}^{6} (-1)^{\delta(i)} \varepsilon'_i \right)$$

with $\sum_{i=1}^{6} \delta(i)$ even. This was calculated by hand and independently using the CHEVIE-package of GAP. Now $P_\mu$ is the convex hull of these 56 points and we have to show $x(a_i) - a_i \in P_\mu$ for $i = 1, \ldots, 8$. Here, the $a_i$ are given by (see [1] Plate V(IV])

$$a_1 = \frac{\varpi_1}{2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \quad a_2 = \frac{\varpi_2}{2} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad a_3 = \frac{\varpi_3}{3} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}, \quad a_4 = \frac{\varpi_4}{4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}.$$
\[
a_5 = \frac{\theta_5}{3} = \left(\begin{array}{ccc}
0 \\
0 \\
0 \\
\frac{1}{2} \\
-\frac{1}{4}
\end{array}\right), \quad a_6 = \frac{\theta_6}{2} = \left(\begin{array}{ccc}
0 \\
0 \\
\frac{1}{2} \\
-\frac{1}{4}
\end{array}\right), \quad a_7 = \frac{\theta_7}{4} = \mu = \left(\begin{array}{ccc}
0 \\
0 \\
\frac{1}{2} \\
-\frac{1}{2}
\end{array}\right), \quad a_8 = 0 = \left(\begin{array}{ccc}
0 \\
0 \\
0 \\
0
\end{array}\right).
\]

Since \( \mu \) is fixed by \( s_1, s_2, s_3, s_4 \) (and \( s_5, s_6 \)), we have
\[
x = w_2 t_\mu w_4^{-1} = t_{w_2(\mu)} w_2 w_4^{-1} = t_\mu s_4 s_3 s_2 s_4 s_1 s_3 s_7 s_6 s_5 s_4 s_2 s_3 s_1 s_4 s_3 s_5 s_4 s_2.
\]

Multiplying the matrices for the simple reflections \( s_i \) we get that \( w_2 w_4^{-1} \) is represented by the matrix
\[
M := \frac{1}{4} \left(\begin{array}{cccc}
-1 & 1 & -1 & 3 \\
1 & -1 & -3 & 1 \\
3 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 -1 1 -1 1 \\
1 & -1 & 1 & -1 & 3 -1 -1 1 \\
1 & -1 & 1 & -1 & -1 3 1 \\
-1 & 1 & -1 & -1 & 1 1 1 3
\end{array}\right),
\]
which has been calculated using Octave.

Now \( x(a_i) - a_i = M a_i + \mu - a_i \) for \( i = 1, \ldots, 7 \) yields the following results (also calculated with Octave)

\[
x(a_1) - a_1 = \left(\begin{array}{ccc}
0 \\
\frac{1}{4} \\
\frac{1}{4} \\
\frac{1}{4}
\end{array}\right) + \frac{1}{2} \left(\begin{array}{ccc}
0 \\
0 \\
0 \\
1
\end{array}\right), \quad x(a_2) - a_2 = \left(\begin{array}{ccc}
\frac{1}{4} \\
\frac{1}{4} \\
\frac{1}{4} \\
\frac{1}{4}
\end{array}\right) + \frac{1}{2} \left(\begin{array}{ccc}
0 \\
0 \\
0 \\
\frac{1}{2}
\end{array}\right),
\]

\[
x(a_3) - a_3 = \left(\begin{array}{ccc}
0 \\
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3}
\end{array}\right) + \frac{1}{3} \left(\begin{array}{ccc}
-1 \\
0 \\
0 \\
\frac{1}{2}
\end{array}\right), \quad x(a_4) - a_4 = \left(\begin{array}{ccc}
0 \\
\frac{1}{4} \\
\frac{1}{4} \\
\frac{1}{4}
\end{array}\right) + \frac{1}{4} \left(\begin{array}{ccc}
0 \\
0 \\
0 \\
\frac{1}{2}
\end{array}\right),
\]

\[
x(a_5) - a_5 = \left(\begin{array}{ccc}
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3}
\end{array}\right) + \frac{1}{3} \left(\begin{array}{ccc}
0 \\
0 \\
0 \\
\frac{1}{2}
\end{array}\right), \quad x(a_6) - a_6 = \left(\begin{array}{ccc}
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2}
\end{array}\right) + \frac{1}{2} \left(\begin{array}{ccc}
0 \\
0 \\
0 \\
\frac{1}{2}
\end{array}\right), \quad x(a_7) - a_7 = \left(\begin{array}{ccc}
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2}
\end{array}\right) + \frac{1}{2} \left(\begin{array}{ccc}
0 \\
0 \\
0 \\
\frac{1}{2}
\end{array}\right),
\]

\[
x(a_8) - a_8 = \left(\begin{array}{ccc}
0 \\
0 \\
0 \\
0
\end{array}\right).
\]
\[
x(a_6) - a_6 = \begin{pmatrix} 0 \\ 0 \\ -1/2 \\ 0 \\ 0 \\ -1/2 \\ 0 \\ -1/2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ -1/2 \end{pmatrix}, \quad x(a_7) - a_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1/2 \end{pmatrix}, \quad x(a_8) - a_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1/2 \end{pmatrix}.
\]

These results for \(x(a_i) - a_i\) agree with independent calculations using CHEVIE. Alternatively, all the calculations in this subsection could have also been done by hand.

As given above, every \(x(a_i) - a_i\) for \(i = 1, \ldots, 8\) is a convex combination of points in \(W_{\mu}\) and therefore \(x(a_i) - a_i \in P_{\mu}\). Hence, condition (ii) for \(x \in \text{Perm}(\mu)\) is fulfilled. On the other hand, condition (i) is trivial:

\[
x = w_2t_\mu w_1^{-1} = w_2w_1^{-1}t_{\mu} \in W_\mu t_{\mu} = W_\mu.
\]

### 4.3 First test of \(x \not\in \text{Adm}(\mu)\)

As mentioned before, all the calculations are completely analogous to the case of \(E_6\). We will just display them here. For explanations and comments the reader is referred to the section about \(E_6\). Note that, as \(\mu = \rho_7\), we now have \(I(\mu) = \{s_1, s_2, s_3, s_4, s_5, s_6\}\).

#### 4.3.1 Calculations with Sage

```python
sage: W=WeylGroup("E7")
sage: [s1,s2,s3,s4,s5,s6,s7]=W.simple_reflections()
sage: w1=s2*s3*s4*s5*s6*s1*s3*s2*s4*s5*s6*s7
sage: w2=s4*s3*s2*s4*s1*s3
sage: w2.bruhat_le(w1)
False
sage: w1.coset_representative([1,2,3,4,5,6])=w1
True
```

#### 4.3.2 Calculations with CHEVIE

```plaintext
gap> W:=CoxeterGroup("E",7);
CoxeterGroup("E",7)
gap> W:=ReflectionSubgroup(W,[1,2,3,4,5,6]);
ReflectionSubgroup(CoxeterGroup("E",7), [ 1, 2, 3, 4, 5, 6 ])
gap> w1inv:=ElWord(W,[7,6,5,4,2,3,1,4,3,5,4,2]);
( 1, 5, 23, 39, 78, 77, 80, 98,110, 2)( 3, 4)( 6, 7,119, 90,108, 69,
70, 56, 27, 45)( 8, 11, 29, 44, 72, 71, 74, 92,107, 9)( 12, 30, 91,105,
117, 87,104, 16, 33, 48)( 13,116,126,122,112, 76, 53, 63, 59, 49,
14, 17, 35, 47, 65, 64, 68, 86,102, 15)( 18, 36, 85,101,114, 81, 99, 22, 38,
51)( 19,100,113,125,120, 94, 26, 43, 60, 52)( 20, 21, 40, 58, 34, 83, 84,103,
121, 97)( 24, 41, 79, 96,111, 75, 93, 28, 42, 54)( 25, 95,109,124,118, 88,
32, 46, 61, 55)( 31, 89,106,123,115, 82, 37, 50, 62, 57)( 66, 67
gap> w2inv:=ElWord(W,[3,1,4,2,3,4]);
( 1, 2, 3, 83, 4)( 5, 28, 21, 16, 37)( 8, 77,10, 78, 9,
11, 32, 26, 22, 17)( 12, 35, 27, 23, 42)( 14, 73, 15, 72, 71,
18, 38, 33, 29, 24)( 19, 41, 34, 30, 47)( 20, 67, 64, 65, 66,
25, 44, 39, 36, 31)( 40, 50, 43, 53, 46)( 45, 54, 48, 56, 51,
49, 57, 52, 58, 55)( 59, 61, 63, 60, 62)( 68, 91, 84, 79,100,
74, 95, 89, 85, 80)( 75, 98, 90, 86,105)( 81,101, 96, 92, 87,
82,104, 97, 93,110)( 88,107,102, 99, 94)(103,113,106,116,109,
108,117,111,119,114)(112,120,115,121,118)(122,124,126,123,125)
```

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4.4 Second test of \( x \not\in \text{Adm}(\mu) \)

Again, everything is completely analogous to the section about \( E_6 \). Just note that this time we have \( \mu = \varrho_7 \) and, of course, different \( w_1 \) and \( w_2 \).

```python
sage: R = RootSystem(['E',7,1]).weight_lattice()
sage: Lambda = R.fundamental_weights()
sage: W = WeylGroup(R)
sage: s = W.simple_reflections()
sage: R.reduced_word_of_alcove_morphism((Lambda[7]-Lambda[0]).translation)
[0, 1, 3, 4, 2, 5, 4, 3, 1, 0, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 5, 4, 3, 1, 0]
sage: y1=s[0]*s[1]*s[3]*s[4]*s[2]*s[5]*s[4]*s[3]*s[1]*s[0]*s[6]*s[5]*s[4]*s[2]*s[3]
sage: w1mu=w1.action(Lambda[7]-Lambda[0])
sage: R.reduced_word_of_alcove_morphism(w1mu.translation)
[2, 4, 3, 1, 0, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 5, 4, 3, 1, 0, 7, 6, 5, 4, 3, 1]
sage: y2=y2*bruhat_le(y1)
False
```

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