STEALTH BRANES

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Abstract

We discuss the brane world model of Dvali, Gabadadze and Porrati in which branes evolve in an infinite bulk and the brane curvature term is added to the action. If $Z_2$ symmetry between the two sides of the brane is not imposed, we show that the model admits the existence of “stealth branes” which follow the standard 4D internal evolution and have no gravitational effect on the bulk space. Stealth branes can nucleate spontaneously in the bulk spacetime. This process is described by the standard 4D quantum cosmology formalism with tunneling boundary conditions for the brane world wave function. The notorious ambiguity in the choice of boundary conditions is fixed in this case due to the presence of the embedding spacetime. We also point to some problematic aspects of models admitting stealth brane solutions.

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I. INTRODUCTION

The idea that our Universe could be a (3+1)-dimensional surface (brane) floating in a higher dimensional embedding (bulk) space dates back to Regge and Teitelboim [1]. They considered pure gravity with a conventional looking action

\[ S(4)_G = \frac{M_{(4)}^2}{2} \int_b d^4x \sqrt{-g} R \]  

where \( M_{(4)} \) is the Planck mass and \( R \) is the scalar curvature. However, the integration in (1) is performed over a (3+1)-dimensional surface

\[ y^A = y^A(x) \]

in an N-dimensional embedding spacetime and the metric \( g_{\mu\nu} \) is the induced metric on the surface,

\[ g_{\mu\nu} = G_{AB} \frac{\partial y^A}{\partial x^\mu} \frac{\partial y^B}{\partial x^\nu}. \]

Here, \( y^A (A = 0, 1, ..., N - 1) \) are the coordinates in the embedding space and the surface is parametrized by the coordinates \( x^\mu (\mu = 0, 1, 2, 3) \). The independent dynamical variables of the theory are not the metric components \( g_{\mu\nu} \) but the embedding functions \( y^A(x) \), with the bulk metric \( G_{AB}(y) \) being fixed. Generalizations have later been considered [2] with a matter term added to the action

\[ S_m^{(4)} = \int_b d^4x \sqrt{-g} L_m. \]

The simplest form of the matter term is a 4D cosmological constant, \( L_m = \rho_v = \text{const} \), which correspond to the brane tension term from the bulk point of view.

In most of the recent work on brane cosmology, the bulk geometry is assumed to be dynamical, while the curvature term on the brane is omitted. The corresponding action includes the bulk curvature term.
\[ S_{G}^{(N)} = \frac{M_{(N)}^{N-2}}{2} \int d^{N} y \sqrt{-G} \mathcal{R}, \]  

(5)

the brane matter term (4), and the bulk matter term

\[ S_{m}^{(N)} = \int d^{N} y \sqrt{-G} \mathcal{L}_{m}. \]  

(6)

The latter term is usually taken in the form of a bulk cosmological constant, which may take different values on the two sides of the brane. The 4D gravity on the brane is then recovered either by compactifying the extra dimensions [3] or by introducing a large negative cosmological constant in the bulk, which causes the bulk space to warp, confining low-energy gravitons to the brane [4].

Quite recently, Dvali, Gabadadze and Porrati (DGP) [5] have pointed out that 4D gravity can be recovered even in an asymptotically Minkowski bulk, provided that one includes the brane curvature action (1),

\[ S = S_{G}^{(N)} + S_{m}^{(N)} + S_{G}^{(4)} + S_{m}^{(4)}. \]  

(7)

Assuming a 5-dimensional bulk and a \( Z_2 \) symmetry of reflections with respect to the brane, they found that gravity on the brane is effectively 4D on scales \( r \ll r_0 \), with

\[ r_0 = \frac{M_{(4)}^{2}}{2M_{(5)}^{3}}, \]  

(8)

and becomes 5D on larger scales. Analysis of cosmological solutions with a Robertson-Walker metric on the brane [6] indicates the same crossover scale (8) in this class of models.

The DGP model can be extended in several directions. First, one can consider \( N > 5 \). One finds that the brane gravity is always 4D for zero-thickness branes [7]. However, the crossover to a higher-dimensional behaviour is recovered when the finite thickness of the brane is taken into account.
Another possible extension is to lift the requirement of $Z_2$ symmetry. This symmetry is certainly not mandated by the action \( (7) \) and actually cannot be enforced in 5D models where the brane is coupled to a 5-form field, so that the 5D cosmological constant is different on the two sides of the brane. Brane world cosmology without $Z_2$ symmetry has been discussed by a number of authors [8–18], but in most of this work the brane curvature term \( (1) \) has not been included in the action. Somewhat surprisingly, effective 4D gravity on the brane could still be obtained for weak sources [13,14], but this behaviour does not extend to strong gravitational fields or to the early Universe cosmology. Some effects of including the brane curvature term have been discussed in [17,18].

In this paper we are going to discuss brane world cosmology in an infinite bulk with the brane curvature included in the action and without the assumption of $Z_2$ symmetry. We will show that inclusion of the curvature term results in some qualitatively new, interesting, but potentially problematic features of the model. In particular, we show that the brane produces no gravitational effect in the bulk, provided that 4D Einstein’s equations are satisfied on the brane. The brane can then float without disturbing the embedding space, and its internal evolution will be identical to that in the genuine 4D case.

Closed branes in this model can be spontaneously created in the bulk. The corresponding instanton is a trivial embedding of the usual $S_4$ instanton of 4D quantum cosmology. An important difference, however, is that nucleation of brane worlds occurs in an embedding spacetime. As a result, the brane world wave function has a clear interpretation, and the issue of boundary conditions in quantum brane cosmology can be definitively addressed.

The paper is organized as follows. In the next Section we demonstrate the existence of branes that satisfy 4D Einstein’s equations and do not disturb the bulk (we refer
to them as “stealth” branes). Quantum nucleations of spherical branes is discussed in Section III. In Section IV we consider the dynamics of more general (not necessary stealth) spherical branes in a 5-dimensional embedding space. Here we also allow for a nonzero bulk cosmological constant which can differ on the two sides of the brane. In Section V we discuss small-scale quantum fluctuations of the branes and argue that these fluctuations would be unacceptably large if we lived on a stealth brane. Finally, in Section VI we summarize our conclusions and discuss the potential problems of models admitting stealth brane solutions.

II. STEALTH BRANES

We consider an action of the following form

\[ S = \int_M d^N Y \sqrt{-G} \left( \frac{1}{2k} R + \mathcal{L}_m \right) \]
\[ + \int_b d^4 x \sqrt{-g} \left( \frac{1}{2k'} R + L_m \right), \]

(9)

where \( k = M^{2-N}_{(N)} \), \( k' = M^{-2}_{(4)} \), \( M_{(N)} \) and \( M_{(4)} \) being the bulk and the brane Planck masses, respectively. The 4D integration is over the brane \( y^A(y) = y^A(x) \), and the induced metric on the brane can be expressed as

\[ g_{\mu\nu} = \int_M d^N y G_{AB}(y) \partial_\mu y^A(x) \partial_\nu y^B(x) \delta^N (y^A - y^A(x)). \]

(10)

Variation of the action (8) with respect to \( G_{AB} \) gives

\[ \delta S = -\frac{1}{2k} \int_M d^N y \sqrt{-G} \left( R^{AB} - \frac{1}{2} G^{AB} R - k T_{\text{bulk}}^{AB} \right) \delta G_{AB} \]
\[ -\frac{1}{2k'} \int_b d^4 x \sqrt{-g} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - k' T^{\mu\nu} \right) \delta g_{\mu\nu}, \]

(11)

where
\[
\begin{align*}
\mathcal{T}_{\text{bulk}}^{AB} &= \frac{2}{\sqrt{-G}} \frac{\delta \mathcal{L}_M}{\delta G_{AB}}, \\
T^{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g_{\mu\nu}},
\end{align*}
\]

are the bulk and the brane energy-momentum tensors, respectively, and the variation \(\delta g_{\mu\nu}\) is to be expressed in terms of \(\delta G_{AB}\) using the relation (10). We thus obtain the N-dimensional Einstein’s equations

\[
\mathcal{R}^{AB} - \frac{1}{2} G^{AB} \mathcal{R} = k (\mathcal{T}_{\text{bulk}}^{AB} + T_{\text{brane}}^{AB}),
\]

where

\[
T_{\text{brane}}^{AB} = \frac{1}{\sqrt{-G}} \int_b d^4x \sqrt{-g} \left( T^{\mu\nu} - \frac{1}{k'} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \right) \partial_\mu y^A(x) \partial_\nu y^B(x) \delta^N (Y^A - Y^A(x)).
\]

(14)

From equations (13), (14) we see immediately that if 4D Einstein’s equations are satisfied on the brane,

\[
\tilde{T}^{\mu\nu} \equiv T^{\mu\nu} - \frac{1}{k'} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0,
\]

then the brane has no gravitational effect on the N-dimensional bulk. The brane can then be treated as evolving in a fixed background geometry. We shall refer to such branes as “stealth” branes.

The brane equations of motion can be obtained by varying the action with respect to the embedding functions \(y^A(x)\). These equations can be expressed as

\[
\tilde{T}^{\mu\nu} K^i_{\mu\nu} = 0
\]

(16)

where \(\tilde{T}^{\mu\nu}\) is from equation (13),

\[
K^i_{\mu\nu} = -n^i_A D_\mu e^A_\nu
\]

(17)
is the extrinsic curvature, \( n^i \) are the (N-4) unit normal vectors to the worldsheet with tangent vectors \( e_A^A = y_A^A \), and \( D_\mu = e_A^A \nabla_\mu \), with \( \nabla_A \) being the covariant derivative in the metric \( G_{AB} \).

Suppose now that we have a solution of 4D Einstein’s equations (15) which can be embedded into a N-dimensional Minkowski space with some functions \( y^A(x) \). It is then clear that such an embedding gives a solution both of the N-dimensional Einstein’s equations (23) (N-dimensional Minkowski space with \( T^{AB} = 0 \)) and of the brane equations of motions (16). Thus any 4D Universe which is embeddable into Minkowski bulk represents a possible internal evolution of a brane world.\(^\dagger\)

For \( N \geq 10 \), any 4D Universe can be embedded at least locally [21], and for \( N \geq 91 \), a global embedding is also possible [22]. Any solution of 4D Einstein’s equations can then be realized as a brane world evolution.

In most of the recent work on brane worlds it is assumed that the bulk is 5-dimensional, in which case there are significant restrictions on the possible embeddings. For example, the Schwarzschild solution can only be embedded for \( N \geq 6 \) [23]. However, closed and flat Friedman-Robertson-Walker cosmologies are all embeddable in 5D Minkowski space.

We assumed so far that stealth branes evolve in a Minkowski bulk. However, it is clear from equations (13)-(16) that one can use practically any fixed background. In particular, we shall later consider stealth branes in de Sitter and anti-de Sitter spaces.

\(^\dagger\)A special case of this general result was noted by Dick [20], who pointed out that a spatially flat 3-brane in a 5D bulk without \( Z_2 \) symmetry may follow the standard Friedman evolution.
III. NUCLEATION OF STEALTH BRANES

From the point of view of the bulk, stealth branes have vanishing energy and momentum, and their nucleation is not forbidden by any conservation laws. One can imagine small, nearly spherical branes filled with a high-energy false vacuum nucleating in Minkowski space. The branes then go through a period of inflation, thermalize, and evolve (locally) along the lines of the standard hot cosmological model. This picture is identical to that usually adopted in quantum cosmology [24–28], except for the presence of the embedding space‡. One can expect that the nucleation rate is described by the same 4-sphere instanton, which is now embedded in a Minkowski bulk. The bulk does not contribute to the instanton action, so the action is also the same,

\[ S^{(E)} = \int d^4x \sqrt{g} \left( -\frac{1}{2k'} R + \rho_V \right) \]
\[ = -24\pi^2 M_4^4 / \rho_V, \]  

where \( \rho_V \) is the 4D false vacuum energy density.

The role of the vacuum energy in inflationary models is usually played by a scalar field potential \( V(\varphi) \). This potential is assumed to be very flat, at least in some range of \( \varphi \), so that \( \varphi \) evolves very slowly and \( V(\varphi) \) acts as a vacuum energy. Then, the instanton action

\[ S^{(E)}(\varphi) = -24\pi^2 M_4^4 / V(\varphi) \]  

should determine the relative probability of nucleation for branes with different values of \( \varphi \).

‡Note that our discussion here differs from [30] who considered nucleation of 5D Universes consisting of AdS regions joined along a brane.
The situation is complicated, however, by the ongoing debate about how the instanton action appears in the nucleation rate \( P \). Different results are obtained depending on one’s choice of the boundary conditions for the wave function of the Universe. The Hartle-Hawking wave function gives

\[
P_{HH} \propto \exp(-S^{(E)}) ,
\]

while the tunneling and Linde wave functions give

\[
P_T \propto \exp(-|S^{(E)}|) .
\]

Since the instanton action is negative, the difference between and is quite dramatic. This issue has not found a clear resolution in 4D quantum cosmology, mainly due to the conceptual problems with the interpretation of the wave function of the Universe. We believe the brane world picture can shed some new light on this problem, since the wave function of a brane nucleating in a Minkowski bulk has a clear physical interpretation.

Quantum cosmology of spherical branes in a 5D Minkowski space has been studied by Davidson et. al. in the framework of Regge-Teitelboim theory. For stealth branes in our model, the bulk gravitational field is absent and the model of is an appropriate minisuperspace approximation. The 5D metric can be written as

\[
ds_5^2 = \sigma^2(-d\tau^2 + da^2 + a^2d\Omega_3^2) ,
\]

where \( d\Omega_3^2 \) is the metric on a unit 3-sphere and \( \sigma^2 \) is a normalization factor. The evolution of the brane is described by specifying the functions \( a(t) \) and \( \tau(t) \), where \( t \) is a time parameter on the brane. We shall choose \( t \) to be the proper time, which introduces the gauge condition

\[
\tau^2 - \dot{a}^2 = 1,
\]
where dots stand for derivatives with respect to $t$.

With a suitable choice of the normalization, $\sigma^2 = (12\pi^2 M_{(4)}^2)^{-1}$, the conserved energy of the brane is given by

$$E = \frac{1}{2}(\dot{a}^2 + 1 - H^2 a^2) a \sqrt{1 + \dot{a}^2}, \quad (24)$$

where

$$H^2 = \rho_V (6\pi M_{(4)}^2)^{-2}. \quad (25)$$

In the case of a stealth brane $E = 0$,

$$\dot{a}^2 + 1 - H^2 a^2 = 0, \quad (26)$$

and one recovers the usual 4D Wheeler-DeWitt equation

$$[p_a^2 + a^2 (1 - H^2 a^2)] \psi = 0 \quad (27)$$

where

$$p_a = -\dot{a} \quad (28)$$

is the momentum conjugate to $a$. (We disregard the factor ordering ambiguities in this discussion). In the “coordinate” representation, the wave function is $\psi = \psi(a)$ and

$$p_a = -i \frac{\partial}{\partial a}.$$

Davidson et. al. were not specifically concerned with the problem of brane nucleation, so they considered branes of arbitrary energy $E$ and a variety of possible boundary conditions at $a \to \infty$. They also required that the wave function should vanish at $a = 0$, which in our view is not justified.

Let us now address the problem of boundary conditions for the nucleation of stealth branes ($E = 0$). The Wheeler-DeWitt equation has the form of a one-dimensional
Schroedinger equation for a “particle” described by a coordinate $a(t)$, having zero energy, and moving in a potential

$$U(a) = a^2(1 - H^2a^2)$$

(29)

The classically allowed region is $a \geq H^{-1}$, and the WKB solutions of equation (27) in this region are

$$\psi_{\pm}(a) = [p(a)]^{-1/2} \exp[\pm i \int_{H^{-1}}^{a} p(a') da' + \frac{i\pi}{4}],$$

(30)

where $p(a) = [-U(a)]^{1/2}$. The three widely discussed choices of boundary conditions correspond to

$$\psi_T(a > H^{-1}) = \psi_-(a)$$

(31)

for the tunneling wave function,

$$\psi_{HH}(a > H^{-1}) = \psi_+(a) - \psi_-(a)$$

(32)

for the Hartle-Hawking wave function, and

$$\psi_L(a > H^{-1}) = \psi_+(a) + \psi_-(a)$$

(33)

for the Linde wave function.

For the two semiclassical wave functions (30) we have

$$\hat{p}_a \psi_{\pm}(a) \approx \pm p(a) \psi_{\pm}(a).$$

(34)

Combined with equation (28), this indicates that $\psi_-(a)$ and $\psi_+(a)$ describe an expanding and a contracting Universe, respectively. Hence, the tunneling wave function includes only the expanding component, while the Hartle-Hawking and Linde wave functions include both components with equal weight and appear to describe contracting and re-expanding Universes.
This interpretation has been questioned by Rubakov [33] who notes that the time coordinate \( t \) is just an arbitrary label, so changing expansion to contraction does not do anything, as long as the directions of all other physical processes are also reversed. In our simple minisuperspace model \( a(t) \) is the only variable, and thus \( \psi_+(a) \) and \( \psi_-(a) \) may both correspond to an expanding Universe.

Rubakov’s objection does not, however, apply to the case of nucleating branes. In this case we have

\[
\dot{a} = \frac{da}{d\tau} \sqrt{1 + \dot{a}^2},
\]

and thus the brane expanding in terms of its proper time \( t \) is also expanding in terms of the Minkowski time \( \tau \). We could of course reverse the internal time coordinate, which would introduce a minus sign on the right-hand side of equation (35). But still, one of the wave functions (30) would correspond to an expanding and the other to a contracting brane, in terms of the Minkowski time \( \tau \). In the brane nucleation process, the newly created branes expand and no contracting branes are present. Thus the tunneling wave function (31) (or its complex conjugate) is the only correct choice in this case.

Stealth brane nucleation can also occur in curved spacetime. Important examples here are de Sitter and anti-de Sitter spaces. The instanton describing stealth brane nucleation in de Sitter space is the same 4-sphere instanton as in Minkowski space, except now it is embedded into an \( N \)-sphere (Euclideanized \( N \)-dimensional de Sitter space). The difference between the instanton action and the \( N \)-sphere action without a brane is still given by Eq. (18), so the rate of brane nucleation is also unchanged.

Turning now to anti-de Sitter space, we consider the case of a single extra dimension for simplicity. The Euclideanized anti-de Sitter space metric can be written as (see, e.g., [30])
\[ ds^2_E = dr^2 + l^2 \sinh^2(r/l)[d\chi^2 + \sin^2 \chi d\Omega_3^2], \] (36)

where \( l = (-6/\Lambda)^{1/2} \) and \( \Lambda < 0 \) is the 5D cosmological constant. The brane worldsheet is a sphere \( r = r_0 \), where \( r_0 \) is determined from

\[ H^{-1} = l \sinh(r_0/l). \] (37)

The Lorentzian evolution is obtained by the analytic continuation \( \chi \to iHt + \pi/2 \),

\[ ds^2 = dr^2 + (lH)^2 \sinh^2(r/l)[-dt^2 + H^{-2} \cosh^2(Ht)d\Omega_3^2], \] (38)

with the brane worldsheet still given by \( r = r_0 \).

**IV. SPHERICAL BRANES IN 5D**

We now turn to a more general situation, when 4D Einstein’s equations on the brane are not necessarily satisfied. Here, we shall specialize to the case of spherical branes and of a 5D bulk. We shall also assume a non-zero bulk cosmological constant and allow for the possibility that it can take different values on the two sides of the brane.

Following the same steps as in \[3\], the equations of motion of the brane can be expressed as

\[ [K]g_{\mu\nu} - [K_{\mu\nu}] = k\bar{T}_{\mu\nu}, \] (39)

\[ \bar{T}^{\mu\nu} < K_{\mu\nu} > = [T_{\mu\nu}], \] (40)

\[ \nabla_\nu(T^\nu_{\mu}) = -[T_{\mu\nu}] \] (41)

Here, \( K_{\mu\nu} \) is the extrinsic curvature of the brane, \( T_{\mu\nu} = (T_{bulk})_{AB}n^An^B, T_{\mu\nu} = (T_{bulk})_{AB}e^A_{\mu}n^B \), the square and angular brackets stand, respectively, for the difference and the average of the corresponding quantity on the two sides of the brane, e.g.,
\[ [K_{\mu\nu}] = K^+_{\mu\nu} - K^-_{\mu\nu}, \quad <K_{\mu\nu}> = \frac{1}{2}(K^+_{\mu\nu} + K^-_{\mu\nu}), \] where “+” and “−” correspond, respectively, to the brane exterior and interior.

We shall assume that the bulk energy-momentum tensor has the vacuum form,

\[ T^\pm_{AB} = -k^{-1} \Lambda^\pm g_{AB}. \] (42)

Then, using the generalized Birkhoff theorem, the 5D metric can be expressed as

\[ ds_5^2 = -A_\pm d\tau^2 + A_\pm^{-1} da^2 + a^2 d\Omega_3^2 \] (43)

with

\[ A_\pm = 1 - \frac{\Lambda^\pm}{6} a^2 - \frac{2M^\pm}{M_5^3} a^2. \] (44)

In the proper time gauge, the metric on the brane is

\[ ds_4^2 = -dt^2 + a^2(t) d\Omega_3^2. \] (45)

The brane embedding is described by specifying the functions \( a(t), \tau(t) \).

The components of the extrinsic brane curvature of the worldsheet are

\[ K^\pm_{\tau\tau} = -\left( \dot{a} + \frac{1}{2} \frac{\partial A_\pm}{\partial a} \right) \left( \dot{a}^2 + A_\pm \right)^{1/2} \] (46)

\[ K^\pm_{\chi\chi} = K^\pm_{\theta\theta} = K^\pm_{\phi\phi} = \frac{(\dot{a}^2 + A_\pm)^{1/2}}{a}. \] (47)

Substituting this in the junction condition equations and assuming that the brane energy-momentum tensor is of the form

\[ \text{§The exterior and interior are identical for } Z_2 \text{ symmetric branes, in which case we either have two spherical 4D regions joined at the brane, or two possibly infinite regions connected by a wormhole. Here we are interested in the case of trivial topology, with spherical interior region and an infinite region outside.} \]
\[ T^\nu_\mu = \text{diag}(\rho, -P, -P, -P) , \quad (48) \]

one obtains \[ \dot{a}^2 + A_-^{1/2} - (\dot{a}^2 + A_+^{1/2}) = \frac{ka}{3} \left( \rho - \frac{3(\dot{a}^2 + 1)}{k'a^2} \right) . \quad (49) \]

Equation (49) expresses the energy-momentum conservation on the brane,

\[ \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \quad (50) \]

while equation (49) gives a relation which is obtainable from (49) and (50).

A stealth brane corresponds to \( M^\pm = 0 \) and \( \Lambda^+ = \Lambda^- \equiv \Lambda \). As expected, in this case Eqs. (49), (50) reduce to the standard FRW evolution equations. Even if the bulk cosmological constant is non-zero, it has no effect on the evolution of the brane.

It should be noted that, in the case of a positive \( \Lambda \), the static coordinates (43) and (44) cannot be used beyond the de Sitter horizon, \( a < H^{-1} \), where \( H = (\Lambda / 6)^{1/2} \).

However, since no singularities are developed at \( a = H^{-1} \), one can expect that the solution for \( a(t) \) will be analytically continued, so that the standard evolution will continue at \( a > H^{-1} \).

We now briefly consider some examples of deviations from stealth brane evolution.

Suppose first that the bulk cosmological constant vanishes, \( \Lambda^\pm = 0 \), but the brane has a nonzero mass, \( M^+ \equiv M^\neq 0 \), with \( M^- = 0 \). For sufficiently large brane radii, \( a^2 \gg M/M(t)^3(\)\), the left-hand side of (49) can be expanded to yield

\[ \frac{\dot{a}^2 + 1}{a^2} = \frac{1}{3M(t)^2} \left( \rho - \frac{M}{a^3\sqrt{a^2 + 1}} \right) . \quad (51) \]

The second term in the brackets represents a correction to the stealth evolution due to a non-zero brane mass. (Note that the mass \( M \) can be both positive or negative.)

The same equation (51) is obtained in the Regge-Teitelboim limit of vanishing bulk gravity, \( M(t) \rightarrow \infty \). The brane dynamics in this regime has been investigated by Davidson [2].
Suppose now that $\mathcal{M}^\pm = 0$, while $\Lambda^+$ and $\Lambda^-$ are both non-zero, and that the brane has a vacuum equation of state, $\rho = \text{const}$. Then it is easily understood that Eq. (19) has a de Sitter solution,

$$a(t) = H^{-1} \cosh(Ht),$$

where $H$ can be found from the equation

$$\left( H^2 - \frac{\Lambda^-}{6} \right)^{1/2} - \left( H^2 - \frac{\Lambda^+}{6} \right)^{1/2} = \frac{1}{M_{(3)}^3} \left( \frac{\rho}{3} - M_{(3)}^2 H^2 \right).$$

The instanton describing nucleation of such branes consists of de Sitter 5-spheres of radii $H_+^{-1} = (6/\Lambda^+/2)$ and $H_-^{-1} = (6/\Lambda^-/2)$ joint at a 4-sphere of radius $H^{-1}$ with $H$ from Eq. (53).

The corresponding Euclidean action can be expressed as

$$S^{(E)} = -\frac{8\pi^2}{H^2} M_{(4)}^2 - 4\pi^2 M_{(5)}^3 \left( \frac{H^2 - H_+^2}{H_+^2} - \frac{H^2 - H_-^2}{H_-^2} \right)$$

$$- 4\pi^2 M_{(5)}^3 \left( \frac{\phi_-}{H_-^3} - \frac{\phi_+}{H_+^3} \right),$$

where $\sin \phi_+ = \frac{H_+^2}{H}$ and $\sin \phi_- = \frac{H_-^2}{H}$. This cumbersome expression can be simplified in the limit when $H_+ \ll H$,

$$S^{(E)} \approx -\frac{24\pi^2 M_{(4)}^4}{\rho} - \frac{16\pi^2 M_{(5)}^3}{5H^5} M_{(4)}^2 (H_+^2 - H_-^2)$$

with $H^2 \approx \rho/3M_{(4)}^2$.

\textbf{V. DO WE LIVE ON A STEALTH BRANE?}

Our discussion in sections II and III naturally leads to the question: Is it possible that we live on a stealth brane? From the classical point of view, this is quite possible, since the classical interior evolution of stealth branes is identical to that of genuine 4D
universes. We shall now argue, however, that quantum-mechanically this picture leads to physically unacceptable consequences.

If we live on a stealth brane, then the tension of the brane is equal to the 4D cosmological constant. This is observationally constrained by

\[ \rho_v \lesssim (10^{-3} \text{eV})^4. \]  

(56)

We are going to argue that with such extremely small tension, the quantum fluctuations of the brane on small length scales would be unacceptable large. Moreover, the scalar particles corresponding to these fluctuations would be copiously produced in high-energy particle collisions in accelerators.

Small fluctuations on a 3-brane in an N-dimensional bulk spacetime can be described by a set of \((N - 4)\) scalar fields with the Lagrangian \[34\]

\[ -\frac{1}{2\rho_v}(Y_a^\alpha Y_a^\alpha + RY_a^a). \]  

(57)

Here, \(Y_a^\alpha(x)\) have the meaning of brane displacements in \((N - 4)\) orthogonal directions and we assume summation over \(a = 1, \ldots, N - 4\). For a nearly flat brane, the curvature term in (57) is negligible, and we have a set of massless scalar fields.

The characteristic amplitude of zero-point brane fluctuations on a length scale \(L\) can be estimated by requiring that the corresponding action in a spacetime volume \(\sim L^4\) is \(\sim 1\),

\[ \rho_v(Y_a^a/L)^2L^4 \sim 1. \]  

(58)

This gives

\[ Y_a^a \sim \rho_v^{-1/2}L^{-1} \]  

(59)

The brane can be treated as a well-localized classical object as long as \(Y_a^a \ll L\), that is, on scales \(L \gg \rho_v^{-1/4}\). On smaller scales, the brane inhabitants would find that the
classical picture of spacetime no longer applies. With $\rho_v$ satisfying the bound (56), deviations from the classical picture would be seen on scales as large as 1 mm.

A related problem is that, for small values of $\rho_v$, quanta of the fields $Y^a$ can be copiously created in particle collisions.\footnote{We are grateful to Gia Dvali for pointing this out to us.} Consider for example the coupling of $Y^a$ to the electromagnetic field $F_{\mu\nu}$. The canonically normalized fields corresponding to $Y^a$ are $\psi^a = \rho_v^{-1/2} Y^a$. In terms of these fields, the 4D metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + \rho_v^{-1} \psi^a_{,\mu} \psi^a_{,\nu},$$

and the interaction Lagrangian is

$$\rho_v^{-1} \left( \psi^a_{,\mu} \psi^a_{,\nu} - \frac{1}{4} \eta_{\mu\nu} \psi^a_{,\sigma} \psi^a_{,\sigma} \right) F^{\mu\rho} F^\nu_{\rho}.$$  (61)

The effective coupling constant at energy $\epsilon$ is $\lambda_{\text{eff}} \sim \epsilon^4 / \rho_v$, so the theory becomes strongly interacting at energies $\epsilon \gtrsim \rho_v^{1/4}$. This is in conflict with observations, unless

$$\rho_v \gtrsim (1\text{TeV})^4.$$  (62)

The conclusion is that in brane world models without $Z_2$ symmetry the brane tension must satisfy the bound (62). These models include the Randall-Sundrum-type 5D model with a positive brane tension $\rho_v$, a negative bulk cosmological constant $\Lambda$, and a brane curvature term in the action. The values of $\rho_v$ and $\Lambda$ are tuned to allow (nearly) flat brane solutions. The original Randall-Sundrum model assumed $Z_2$ symmetry, but it can be lifted, as long as (62) is satisfied. Then, we may be living on a flat brane separating two anti-de Sitter regions, but stealth branes will nucleate and expand in the anti-de Sitter bulk. Their internal geometry will be de Sitter, with the vacuum energy given by the brane tension. The expanding stealth branes will occasionally run
into our brane, and one can derive constraints on the model parameters by requiring that such brane collisions should be sufficiently rare. This has been done in [35] for a somewhat different model.

Another possibility is a model with a flat 4D brane world in two extra dimensions. The brane evolution equation (16) is trivially satisfied for $K_{\mu\nu} = 0$, and the bulk Einstein’s equations are solved by a conical metric in the extra dimensions. The bulk metric is then locally flat, and stealth brane nucleation will proceed as in flat spacetime. To avoid an excessive rate of collisions of our brane with stealth branes, one can introduce a small cosmological constant in the bulk, and brane collisions will be rare, provided that the brane nucleation rate is sufficiently low.

VI. CONCLUSIONS AND DISCUSSION

We discussed the classical and quantum cosmology of 3-branes with an Einstein curvature term added to the brane action. If $Z_2$ symmetry between the two sides of the brane is not imposed, the model admits the existence of stealth branes which follow the standard 4D internal evolution and have no gravitational effect on the bulk space. Stealth branes have vanishing energy in the bulk space, $E = 0$, and can therefore nucleate spontaneously. This process is described by the standard 4D quantum cosmology formalism with tunneling boundary condition for the brane world wave function. The notorious ambiguity in the choice of the boundary conditions is fixed in this case due to the presence of the embedding spacetime.

Apart from stealth branes, the model also allows the existence of branes with negative bulk energy. So one could have spontaneous creation of brane pairs, one with energy $E_1 > 0$ and the other with $E_2 = -E_1$. Alternatively, a positive energy brane can increase its energy by chopping off a negative energy brane.
We have argued that we are not likely to live on a stealth brane, since otherwise the tension of our brane would be extremely small, \( \rho_0 \lesssim (10^{-3} \text{eV})^4 \). As a result, the brane would be subject to large quantum fluctuations, and the classical picture of spacetime would break down for the brane inhabitants on unacceptably large length scales. Quanta of brane fluctuations would also be copiously produced in high-energy collisions in particle accelerators. But even though we do not live on a stealth brane, such branes can still nucleate and expand in the bulk spacetime.

The existence of stealth branes indicates that the bulk space is unstable, and there is a danger that it may fill up with nucleated branes, so that the brane inhabitants will be constantly bombarded by other branes hitting them from extra dimensions. One way around this problem is to enforce the \( Z_2 \) symmetry, thus disallowing stealth branes. Alternatively, one can try to impose constraints on the parameters of the model, e.g., an upper bound on stealth brane nucleation rate, to reduce the rate of brane collisions to an acceptable value. One could also consider an expanding de Sitter bulk. The branes will then be driven apart by the exponential expansion of the bulk, and brane collisions will be rare, provided that the brane nucleation rate is sufficiently low.

We finally point to a disturbing property of stealth brane solutions which is manifested in cases when the brane spacetime allows a continuum of embeddings. Consider for example a flat brane

\[
ds^2_1 = -dt^2 + dx^2 + dy^2 + dz^2
\]

(63)

embedded into a flat 5D Minkowski space

\[
ds^2_5 = -d\tau^2 + dx^2 + dy^2 + dz^2 + dw^2.
\]

(64)

A possible embedding is given by any functions \( \tau(t) \) and \( w(t) \) satisfying

\[
\dot{\tau}^2 - \dot{w}^2 = 1.
\]

(65)
In particular, we could have $\tau = t$, $w = 0$ at $t < 0$ and an arbitrary function $w(t)$ at $t > 0$, with $\tau(t)$ determined from (35). For example, the brane could suddenly start moving with an acceleration, $w(t) = at^2/2$. This shows that the evolution of the brane is degenerate, in the sense that it is not uniquely predicted by the initial data.††

One could try to resolve this problem by considering small perturbations about stealth brane solutions. The degeneracy of brane evolution may be removed if only perturbatively stable stealth solutions are allowed (that is, solutions which admit a full spectrum of infinitesimal perturbations). Consider, for example, a brane which suddenly starts to accelerate in a flat $5D$ embedding space. For perturbations violating Einstein’s equations on the brane, the $5D$ energy of the brane is generally non-zero, and its accelerated motion in flat space is inconsistent with energy and momentum conservation. An accelerating brane solution can be obtained by introducing an appropriate gravitational field in the bulk. However, an infinitesimal perturbation of the bulk Minkowski geometry can result only in an infinitesimal acceleration of the brane. Thus, a brane which suddenly starts moving with a finite acceleration is not perturbatively stable.

A detailed analysis of this issue is beyond the scope of the present paper.

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