Improved finite-time stability and instability theorems for stochastic nonlinear systems

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Abstract

This paper studies finite-time stability and instability theorems in probability sense for stochastic nonlinear systems. Firstly, a new sufficient condition is proposed to guarantee that the considered system has a global solution. Secondly, we propose improved finite-time stability and instability criteria that relax the constraints on $\mathcal{L}V$ (the infinitesimal operator of Lyapunov function $V$) by the uniformly asymptotically stable function (UASF). The improved finite-time stability theorems allow $\mathcal{L}V$ to be indefinite (negative or positive) rather than just only allow $\mathcal{L}V$ to be negative. Most existing finite-time stability and instability results can be viewed as special cases of the obtained theorems. Finally, some simulation examples verify the validity of the theoretical results.

Keywords. Stochastic nonlinear systems; finite-time stability; finite-time instability; uniformly asymptotically stable function (UASF).

I. Introduction

Stability plays a central role in systems theory and engineering applications, and is always the most fundamental consideration in system analysis and synthesis. The two most commonly used concepts in stability theory are asymptotic stability [1-5] and finite-time stability [6-9]. The asymptotic stability describes the asymptotic behavior of the system state when time tends to infinity, however, the finite-time stability illustrates the transient performance of the system state trajectory within a finite time interval. For many engineering problems, one often pays more attention to finite time stability instead of asymptotic stability; see, e.g., the tracking control of robotic manipulators [10], the position and orientation control of underwater vehicles [11]. Specially, the finite-time stable system usually shows better robustness and faster convergence rate. So finite-time stability and its related control problems have aroused great interest during the past two decades; see [12-16].

Many practical systems are subject to stochastic perturbations such as environmental noise. Actual systems in random environments are often modeled as stochastic systems, which can be studied by stochastic analysis method. It is natural that the finite-time stability and stabilization of stochastic systems have attracted a great deal of attention. For example, [17] presented the definition of finite-time stability and established finite-time stability and instability criteria in probability sense under the

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hypothesis that the considered system has a unique strong solution. Generally speaking, it is difficult to ensure that there exists a unique strong solution to stochastic nonlinear systems without Lipschitz condition. So researchers have to study finite-time stability and stabilization problems for stochastic nonlinear systems under the framework of strong solution or weak solution. [18] generalized the finite-time stability criteria of [17] and relaxed the constraint conditions on Lyapunov function, this paper proposes improved finite-time stability and instability criteria as special cases.

In this paper, we focus on the existence of the system solution and its finite-time stability and instability. A new sufficient condition is given to guarantee that the considered system has a unique strong solution. In Section 5, an example is given to show the application of our improved finite-time stability criteria. In Section 6, conclusions are drawn with some remarks.

Notations: The set of all natural numbers is denoted by \( \mathbb{N} \). \( \mathcal{R}_+ \) denotes the family of all nonnegative real numbers and \( \mathcal{R}^r \) denotes the real \( r \)-dimensional space. \( |x| \) is the Euclidean norm of a vector \( x \). The Frobenius norm of a real matrix \( X \) is \( \|X\| = [\text{Tr}(X^T X)]^{1/2} \). \( C^{1,2}([t_0, \infty) \times \mathcal{R}^r; \mathcal{R}_+) \) stands for the set of all nonnegative functions \( w(t, x) \) that are \( C^1 \) in \( t \) and \( C^2 \) in \( x \). \( C^{1,2}_0([t_0, \infty) \times \mathcal{R}^r; \mathcal{R}_+) \) denotes the set of all nonnegative functions \( w(t, x) \in C^{1,2}([t_0, \infty) \times \mathcal{R}^r; \mathcal{R}_+) \) except possibly at the point \( x = 0 \). \( \mathcal{K} \) represents the set of all strictly increasing, continuous functions \( \gamma(t) \) with \( \gamma(0) = 0 \). \( \mathcal{K}_\infty \) denotes the set of all unbounded functions \( \gamma(t) \) with \( \gamma(t) \in \mathcal{K} \). \( \mathcal{K}_\mathcal{L} \) denotes the set of all functions \( \beta(s, t) \in \mathcal{K} \) with \( t \) being fixed and \( \lim_{t \to \infty} \beta(s, t) = 0 \) with \( s \) being fixed.

II. Preliminaries

Consider the stochastic nonlinear Itô system

\[
\frac{dx(t)}{dt} = f(t, x(t))dt + g(t, x(t))dW(t),
\]

in which system state \( x(t) \in \mathcal{R}^r \) and \( x(t_0) = x_0, W(t) \in \mathcal{R}^d \), defined on the filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathcal{P}) \), is a standard Wiener process. The continuous functions \( f : [t_0, \infty) \times \mathcal{R}^r \to \mathcal{R}^r \) and \( g : [t_0, \infty) \times \mathcal{R}^r \to \mathcal{R}^{r \times d} \) satisfy \( f(t, 0) = 0 \) and \( g(t, 0) = 0 \) with \( t \in [t_0, \infty) \).

According to Lemma 3.2 in [Chapter 4, 3], the trivial solution of system (1) is impossible to reach the origin in a finite time, if \( f(t, x) \) and \( g(t, x) \) satisfy local Lipschitz condition in \( x \). Thus some conditions
should be imposed on system (1) such that system (1) has a solution, which may arrive at the origin in a finite time. The following Lemma 1 ensures that system (1) has a unique strong solution and it can be viewed as a special case of Theorem 170 in [28].

**Lemma 1** For every $N = 1, 2, \ldots$, and any $T \in [t_0, \infty)$, if system (1) satisfies that

H1: $|f(t,x)| \leq (1 + |x|)h(t)$, $|g(t,x)|^2 \leq (1 + |x|^2)h(t)$ and

H2: $2(|x_0 - x_2| f(t,x_1) - f(t,x_2)) + ||g(t,x_1) - g(t,x_2)||^2 \leq \psi_N^h(|x_0 - x_2|^2)h_N^h(t)$, $|x_1| \vee |x_2| \leq N$ for any $t \in [t_0, T]$, where the nonnegative functions $h_N^h(t)$ and $h(t)$ satisfy $\int_{t_0}^{T} h_N^h(t)dt < \infty$ and $\int_{t_0}^{T} h(t)dt < \infty$. Moreover, for any $v \geq 0$, $\psi_N^h(v) \geq 0$ is a continuous, strictly increasing, non-random and concave function satisfying $\int_{0}^{\infty} dv/\psi_N^h(v) = \infty$. Then, system (1) has a unique solution for any given initial value $x_0 \in \mathbb{R}^r$.

**Remark 1** Note that H2 in Lemma 1 is very restrictive for system (1), which limits the application scope of finite-time stability theorems. In Lemma 2, H2 is used to show the uniqueness of solution to system (1).

The following Lemma 2 and Lemma 3 give some sufficient conditions, which ensure that system (1) has a continuous strong solution.

**Lemma 2** If system (1) satisfies that

H3: $|f(t,x)|^2 + |g(t,x)|^2 \leq H(1 + |x|^2)$ where $H > 0$ is a constant, then system (1) has a continuous solution with probability 1.

**Lemma 3** For system (1), assume that $f(t,x)$ and $g(t,x)$ are locally bounded in $x$ and are uniformly bounded in $t$. If there exist a function $U(t,x) \in C^{1,2}([t_0, \infty) \times \mathbb{R}^r; \mathbb{R}_+)$, a $K$ function $\gamma(\cdot)$, a continuous function $l(t)$ and a constant $d_U \geq 0$ such that $\gamma(|x|) \leq U(t,x)$ and $\mathcal{L}U(t,x) \leq l(t)U(t,x) + d_U$ hold for any $x \in \mathbb{R}^r$, then system (1) has a continuous solution with probability 1.

Proof: For each integer $k \geq 1$, let

$$f_k(t,x) = \begin{cases} f(t, \frac{kx}{|x|}) & \text{if } |x| \geq k, \\ f(t,x) & \text{if } |x| < k, \end{cases} \quad g_k(t,x) = \begin{cases} g(t, \frac{kx}{|x|}) & \text{if } |x| \geq k, \\ g(t,x) & \text{if } |x| < k, \end{cases}$$

and define $x_k(t)$ by

$$dx_k(t) = f_k(t,x_k(t))dt + g_k(t,x_k(t))dW(t)$$

for all $t \geq t_0$, where $\psi_N(t) = \sup_{|y| \leq k \geq t_0} |f(t,y)| \leq c_1 < \infty$, $|g(t,x)| \leq \sup_{|y| \leq k \geq t_0} |g(t,y)| \leq c_2 < \infty$. According to Theorem 5.2 in [29], we can rewrite (2) as

$$dx_k(t) = f(t,x_k(t))dt + g(t,x_k(t))dW(t)$$

for any $t \in [t_0, \tau_k)$. Further, define $\tau_k = \inf\{t \geq t_0 : |x_k(t)| \geq k\}$, then (2) can be rewritten as

$$dx_k(t) = f(t,x_k(t))dt + g(t,x_k(t))dW(t)$$

for any $t \in [t_0, \tau_k)$. For any $T \in [t_0, \infty)$, let $\tilde{U}(t,x) = U(t,x)\exp\{-\int_{t_0}^{t} l(s)ds\}$, then we have

$$\mathcal{L}\tilde{U}(t,x_k(t)) = (\mathcal{L}U(t,x_k(t))) - l(t)U(t,x_k(t))e^{-\int_{t_0}^{t} l(s)ds} \leq d_U e^{-\int_{t_0}^{t} l(s)ds}$$

from which we can further obtain that

$$E[U(\tau_k \wedge T, x(\tau_k \wedge T))] \leq E[U(t_0,x_0)e^{\int_{t_0}^{\tau_k \wedge T} l(s)ds}] + d_U E[\int_{t_0}^{\tau_k \wedge T} e^{\int_{t_0}^{\tau_k \wedge T} l(s)ds} ds]$$

$$\leq U(t_0,x_0)e^{\int_{t_0}^{T} l(s)ds} + d_U \int_{t_0}^{T} e^{\int_{t_0}^{T} l(s)ds} ds$$

$$\leq U(t_0,x_0)e^{M_T(T-t_0)} + d_U T e^{M_T(T-t_0)}$$

(4)
where \( M_T = \max_{0 \leq t \leq T} |l(t)| \). Note that at
\[ \inf_{|z|= k, t \geq t_0} U(t, x)P\{ \tau_k \leq T \} \leq E[U(\tau_k \wedge T, x(\tau_k \wedge T))] \],
then (4) can be turned into
\[
P\{ \tau_k \leq T \} \leq \frac{U(t_0, x_0) + d_U T}{\inf_{|z|= k, t \geq t_0} U(t, x)} e^{M_T(T-t_0)}.
\] (5)

Let \( k \to \infty \) in (5), then we get that \( \lim_{k \to \infty} P\{ \tau_k \leq T \} = 0 \) for any \( T \in [t_0, \infty) \), where
\[
\lim_{k \to \infty} \inf_{|z|= k, t \geq t_0} U(t, x) = \infty
\]
is used. This means that \( \lim_{k \to \infty} \tau_k = \infty \) a.s.. Hence, for any \( t \in [t_0, \tau_k) \), we define \( x(t) = x_k(t) \) in (3),
then \( x(t) \) is the global solution of system (1).

**Remark 2** (i) Lemma 3 comes from Theorem 5.2 in [29]. Lemma 3 is an improved version of Lemma 1 in [19]. In Lemma 1 of [19], \( l(t) \) is a nonnegative function with \( \int_0^\infty |l(t)| dt < \infty \) and \( d_U = 0 \). However,
Lemma 3 removes these strict constraints and allow \( l(t) \) to be a continuous function and \( d_U \geq 0 \).

**Definition 1** ([19]) If system (1) has a solution \( x(t) \) and satisfies that
(i) For any \( x_0 \in \mathbb{R}^r \setminus \{ 0 \} \), the stochastic settling time \( \rho_{x_0} = \inf\{ t \geq t_0 : x(t) = 0 \} \) is a finite time a.s.,
i.e., \( P\{ \rho_{x_0} < \infty \} = 1 \).
(ii) There is a positive constant \( \delta(\varepsilon, R) \) such that \( P\{ |x(t)| < R, \quad t \geq t_0 \} \geq 1 - \varepsilon \) with \( |x_0| < \delta(\varepsilon, R) \) for
any \( 0 < \varepsilon < 1 \) and \( R > 0 \).

Then, the trivial solution of system (1) is called finite-time stable in probability.

**Definition 2** ([19]) In Definition 1, if (i) or (ii) cannot be satisfied, then the trivial solution of system (1) is finite-time instable in probability.

**Definition 3** ([19]) For system \( \dot{z}(t) = \mu(t)z(t) \) with \( t \in [t_0, \infty) \), if there exists a function \( \beta(\cdot, \cdot) \in \mathcal{KL} \)
such that \( |z(t)| \leq \beta(|z(t_0)|, t-t_0) \), then this system is globally uniformly asymptotically stable and \( \mu(t) \)
is called a UASF.

**Remark 3** According to [27], \( \mu(t) \) is a UASF if and only if there exist constants \( c_\mu > 0 \) and \( d_\mu \geq 0 \) such that \( \int_{t_0}^t \mu(s) ds \leq d_\mu - c_\mu (t-t_0) \).

For any given \( V(t, x) \in C^{1,2}([t_0, \infty) \times \mathbb{R}^r; \mathbb{R}_+) \) associated with system (1), the infinitesimal operator \( \mathcal{L} \) is defined as \( \mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2} \text{Tr} \{ g^T(t, x)V_{xx}(t, x)g(t, x) \} \).

The main goal of this paper is to present the improved finite-time stability and instability theorems under the condition that system (1) has a solution, which is in weak solution or strong solution sense. By stochastic analysis technique and UASF, the established finite-time stability and instability theorems can break through some strict constraints imposed on the existing finite-time stability and instability results.

### III. Finite-time stability theorems

In this section, our purpose is to build some improved finite-time stability theorems of stochastic nonlinear systems. Compared with some existing relative results, the improved finite-time stability theorems weaken the condition on \( \mathcal{L}V \) and permit \( \mathcal{L}V \) to be indefinite. For this end, we need the following Lemma 4 which plays an important role in the proof of Theorem 4.
Lemma 4 For system \( [1] \), if there exist a function \( V^*(t,x) \in C_0^{1,2}([t_0,\infty) \times U_k; \mathbb{R}_+) \) and a UASF \( \mu^*(t) \) such that for any \( \varepsilon \in (0,k) \),

\[
\mathcal{L}V^*(t,x) \leq \mu^*(t), \forall x \in U_{k,\varepsilon},
\]

where \( U_k = \{ x \in \mathbb{R}^r : |x| < k \} \) and \( U_{k,\varepsilon} = \{ x \in \mathbb{R}^r : \varepsilon < |x| < k \} \). Then, the trivial solution of system \( [1] \) with \( x_0 \in U_{k,\varepsilon} \) first reaches the boundary of \( U_{k,\varepsilon} \) in finite time a.s.

**Proof:** Let \( \varpi(t) = -\int_0^t \mu^*(s)ds \), then \( \varpi(t) \geq c_\mu^*(t-t_0) - d_\mu^* \) and \( \lim_{t \to \infty} \varpi(t) = +\infty \) by Remark 3. Moreover, if we let \( T_0 = \max\{ T \geq t_0 : \int_{t_0}^T \mu^*(t)dt = 0 \} \), then \( \varpi(T_0) = 0 \) with \( T_0 < \infty \) and \( \varpi(t) > 0 \) as \( t > T_0 \).

Let \( \varrho_{k,\varepsilon} = \inf\{ t \geq t_0 : |x(t)| \notin U_{k,\varepsilon} \} \), then we need to prove that \( P\{ \varrho_{k,\varepsilon}^* < \infty \} = 1 \), i.e., \( P\{ \varrho_{k,\varepsilon} = \varrho_{k,\varepsilon}^* I(\varrho_{k,\varepsilon}^* \leq T_0) + \varrho_{k,\varepsilon} I(\varrho_{k,\varepsilon}^* > T_0) < \infty \} = 1 \) with \( \varrho_{k,\varepsilon} = \inf\{ t_0 \leq t \leq T_0 : |x(t)| \notin U_{k,\varepsilon} \} \) and \( \varrho_{k,\varepsilon} = \inf\{ t > T_0 : |x(t)| \notin U_{k,\varepsilon} \} \).

Firstly, we prove the case that \( P\{ \varrho_{k,\varepsilon}^* = \varrho_{k,\varepsilon} < \infty \} = 1 \). From Dynkin’s formula and \( [9] \), we can get that for any \( x \in U_{k,\varepsilon} \),

\[
EV^*(\varrho_{k,\varepsilon} \wedge t, x(\varrho_{k,\varepsilon} \wedge t)) = EV^*(T_0, x(T_0)) + E \int_{T_0}^{\varrho_{k,\varepsilon} \wedge t} \mathcal{L}V^*(s, x(s))ds
\]

\[
\leq EV^*(T_0, x(T_0)) + E \int_{T_0}^{\varrho_{k,\varepsilon} \wedge t} \mu^*(s)ds,
\]

where \( t > T_0 \). Define \( \varpi_1(t) = -\int_0^t \mu^*(s)ds \), then \( \varpi_1(t) = \varpi(t) > 0 \) as \( t > T_0 \) and \( \lim_{t \to \infty} \varpi_1(t) = \lim_{t \to \infty} \varpi(t) = +\infty \). Further, \( [7] \) can be turned into \( EV^*(\varrho_{k,\varepsilon} \wedge t, x(\varrho_{k,\varepsilon} \wedge t)) \leq EV^*(T_0, x(T_0)) - E\varpi_1(\varrho_{k,\varepsilon} \wedge t) \), which means that

\[
E\varpi_1(\varrho_{k,\varepsilon} \wedge t) \leq EV^*(T_0, x(T_0)), \quad t > T_0.
\]

Note that for any \( t > T_0 \),

\[
\varpi_1(t)P\{ \varrho_{k,\varepsilon} \geq t \} = \int_{\{ \varrho_{k,\varepsilon} \geq t \}} \varpi_1(t)dP \leq E\varpi_1(\varrho_{k,\varepsilon} \wedge t).
\]

Substituting \( [9] \) into \( [8] \) arrives at

\[
P\{ \varrho_{k,\varepsilon} \geq t \} \leq \frac{1}{\varpi_1(t)} E\varpi_1(\varrho_{k,\varepsilon} \wedge t),
\]

where \( t > T_0 \). In \( [10] \), let \( t \to \infty \), then \( P\{ \varrho_{k,\varepsilon} < \infty \} = 1 \), that is to say, \( P\{ \varrho_{k,\varepsilon}^* = \varrho_{k,\varepsilon} < \infty \} = 1 \) can be obtained.

Secondly, we prove another case that \( P\{ \varrho_{k,\varepsilon}^* = \varrho_{k,\varepsilon} < \infty \} = 1 \). In fact, it is natural because \( P\{ \varrho_{k,\varepsilon}^* = \varrho_{k,\varepsilon} \leq T_0 \} = 1 \).

Hence, we have \( P\{ \varrho_{k,\varepsilon}^* = \varrho_{k,\varepsilon} I(\varrho_{k,\varepsilon}^* \leq T_0) + \varrho_{k,\varepsilon} I(\varrho_{k,\varepsilon}^* > T_0) < \infty \} = 1 \), i.e., \( \varrho_{k,\varepsilon}^* < \infty \) = 1. This means that the solution of system \( [1] \) with \( x_0 \in U_{k,\varepsilon} \) firstly arrives at the boundary of \( U_{k,\varepsilon} \) in finite time almost surely.

**Theorem 1** Suppose that there exists a solution to system \( [7] \). If there are functions \( \gamma^*(\cdot) \in K_\infty, \gamma^*(\cdot) \in K_\infty, V(t,x) \in C^{1,2}([t_0,\infty) \times \mathbb{R}^r; \mathbb{R}_+) \) and a UASF \( \mu(t) \), and a constant \( \kappa \in (0,1) \) such that

\[
\gamma(|x|) \leq V(t,x) \leq \gamma(|x|),
\]

\[
\mathcal{L}V(t,x) \leq \mu(t)|V(t,x)|^\kappa,
\]

then the solution of system \( [7] \) is finite-time stable in probability.
Proof: If system initial value $x_0 = 0$, then $x(t) \equiv 0$ is the solution of system \([\text{1}]\), where $f(0, t) \equiv 0$ and $g(0, t) \equiv 0$ are considered. In the following, we only discuss the case that $x(t_0) = x_0 \in \mathcal{R}^r \setminus \{0\}$.

Defining $W(V) = \int_0^V s^{-\kappa} ds$ with $V \in (0, +\infty)$ along with system \([\text{1}]\) and using Lemma 3.1 in \([\text{30}]\), then

$$\mathcal{L} W(V(t, x)) = \frac{\mathcal{L} V}{V^\kappa} - \frac{\kappa}{2} \frac{\text{Tr}((V_x g)^T V_x g)}{V^{\kappa+1}} \leq V^{-\kappa} \mathcal{L} V \leq \mu(t), \ x \in \mathcal{R}^r \setminus \{0\}. \quad (13)$$

Let $\rho_k = \rho_{k, \frac{1}{k}} = \inf\{t \geq t_0 : |x(t)| \notin U_{k, \frac{1}{k}}\}$ with $U_{k, \frac{1}{k}} = \{x \in \mathcal{R}^r : \frac{1}{k} < |x| < k\}$ and $k \in \{2, 3, \cdots\}$, then by \([\text{13}]\), we must have

$$\mathcal{L} W(V(t, x)) \leq \mu(t), \ x \in U_{k, \frac{1}{k}}. \quad (14)$$

By Dynkin’s formula, it follows from \([\text{14}]\) that

$$-W(V(t_0, x_0)) \leq EW(V(t \wedge \rho_k, x(t \wedge \rho_k))) - W(V(t_0, x_0)) \leq E \int_{t_0}^{t \wedge \rho_k} \mu(s) ds. \quad (15)$$

Because $\mu(t)$ is a UASF, there exist constants $c_\mu > 0$ and $d_\mu > 0$ such that $\int_{t_0}^{t \wedge \rho_k} \mu(s) ds \leq d_\mu - c_\mu(t - t_0)$. Hence, \([\text{15}]\) can be turned into

$$-\frac{1}{1 - \kappa} V^{1 - \kappa}(t_0, x_0) = -\int_0^{V(t_0, x_0)} s^{-\kappa} ds \leq -c_\mu E(t \wedge \rho_k) + d_\mu + d_\mu,$$

that is to say,

$$E(t \wedge \rho_k) \leq t_0 + \frac{d_\mu}{c_\mu} + \frac{\bar{\gamma}^{1 - \kappa}(|x_0|)}{c_\mu(1 - \kappa)}, \quad (16)$$

where \([\text{11}]\) is considered.

Let $W = V^*$ and $\mu(t) = \mu^*(t)$ in \([\text{13}]\), then we have $\rho_k \to \rho_\infty = \rho_{x_0}$ a.s. as $k \to \infty$ and $P\{\rho_{x_0} < \infty\} = 1$ by Lemma \([\text{3}]\) and Definition \([\text{1}]\). Further, let $t = k$ and $k \to \infty$, then $k \wedge \rho_k \to \rho_{x_0}$ a.s. Accordingly, \([\text{16}]\) can be changed into

$$E(\rho_{x_0}) \leq t_0 + \frac{d_\mu}{c_\mu} + \frac{\bar{\gamma}^{1 - \kappa}(|x_0|)}{c_\mu(1 - \kappa)}. \quad (17)$$

This implies that the trivial solution of system \([\text{1}]\) is finite-time attractive in probability.

Applying Dynkin’s formula for \([\text{13}]\), together with \([\text{11}]\), leads to

$$E[\bar{\gamma}^{1 - \kappa}(|x(t)|) I_{\{t \in [t_0, \rho_{x_0}]\}}] \leq E[V^{1 - \kappa}(t, x(t)) I_{\{t \in [t_0, \rho_{x_0}]\}}] \leq \bar{\gamma}^{1 - \kappa}(|x_0|) + d_\mu(1 - \kappa). \quad (18)$$

Meanwhile,

$$E[\bar{\gamma}^{1 - \kappa}(|x(t)|) I_{\{t \in [\rho_{x_0}, \infty]\}}] \equiv 0. \quad (19)$$

Hence, we have

$$E[\bar{\gamma}^{1 - \kappa}(|x(t)|)] = E[\bar{\gamma}^{1 - \kappa}(|x(t)|) I_{\{t \in [t_0, \rho_{x_0}]\}} + \bar{\gamma}^{1 - \kappa}(|x(t)|) I_{\{t \in [\rho_{x_0}, \infty]\}}]$$

$$= E[\bar{\gamma}^{1 - \kappa}(|x(t)|) I_{\{t \in [t_0, \rho_{x_0}]\}}] + E[\bar{\gamma}^{1 - \kappa}(|x(t)|) I_{\{t \in [\rho_{x_0}, \infty]\}}]$$

$$\leq \bar{\gamma}^{1 - \kappa}(|x_0|) + d_\mu(1 - \kappa), \ t \geq t_0 \quad (20)$$

from \([\text{18}]\) and \([\text{19}]\).

For any $\varepsilon \in (0, 1)$, let $\bar{R} > \varepsilon^{-1} \bar{\gamma}^{1 - \kappa}(|x_0|) + \varepsilon^{-1}d_\mu(1 - \kappa)$, then from \([\text{20}]\) and Chebyshev’s inequality, we can get that

$$P(\bar{\gamma}^{1 - \kappa}(|x(t)|) \geq \bar{R}, \ t \geq t_0) \leq E[\bar{\gamma}^{1 - \kappa}(|x(t)|)]\bar{R}^{-1} \varepsilon,$$
which means that

\[ P[|x(t)| < R, \ t \geq t_0] \geq 1 - \varepsilon, \]  

(21)

where \( R = \gamma^{-1}(\delta, R) \) and \( \delta(\varepsilon, R) = \gamma^{-1}([\gamma^{-1}(\varepsilon) - d_\mu(1 - \kappa)]^{1/(1 - \kappa)}) . \) This signifies that the trivial solution of system (1) is stable in probability.

**Remark 5**  
In some existing results, \( \kappa \) for any \( (i) \) Theorem 2 signifies that Theorem 1 also holds for \( \kappa \); \( \kappa \) means that \( \kappa \) signifies that \( P \) on \( L \) definite such as \( [18] \). Theorem 1 and Theorem 2 generalize the existing results and relax some constraints on \( \kappa \) according to Example 3.3 in \( [17] \). Therefore, Theorem 1 also does not hold for \( \kappa = 1 \) according to Example 3.3 in \( [17] \). Therefore, Theorem 1 also does not hold for \( \kappa = 1 \).

**Theorem 2** Suppose that there is a unique solution to system (1). If there are functions \( \gamma \in K_\infty, \) \( \bar{\gamma} \in K_\infty, \) a UASF \( \mu(t) \) and \( V(t, x) \in C^{1,2}([t_0, \infty) \times \mathbb{R}^n; \mathbb{R}_+) \) such that

\[ \underline{\gamma}(|x|) \leq V(t, x) \leq \bar{\gamma}(|x|), \]  

(22)

\[ \mathcal{L}V(t, x) \leq \mu(t), \]  

(23)

then the trivial solution of system (1) is finite-time stable in probability.

**Proof:** Similar to Theorem 1, we also only consider the case that \( x(t_0) = x_0 \in \mathbb{R}^n \setminus \{0\} \) in the following. Applying Dynkin’s formula for (23) and the property of the UASF \( \mu(t) \), we have

\[ EV(t \wedge \sigma_n \wedge \rho_{x_0}, x(t \wedge \sigma_n \wedge \rho_{x_0})) \leq V(t_0, x_0) - c_\mu E(t \wedge \sigma_n \wedge \rho_{x_0} - t_0) + d_\mu, \]  

(24)

where \( \sigma_n = \inf\{t \geq t_0 : |x(t)| > n\} \) and \( \rho_{x_0} \) represents the stochastic settling time. Note that \( \sigma_n \to \infty \) as \( n \to \infty \). So by Fatou’s lemma, (22) and (24), we have

\[ c_\mu E(t \wedge \rho_{x_0}) \leq EV(t \wedge \rho_{x_0}, x(t \wedge \rho_{x_0})) + c_\mu E(t \wedge \rho_{x_0}) \leq \bar{\gamma}(|x_0|) + c_\mu t_0 + d_\mu, \]

which indicates that

\[ E(t \wedge \rho_{x_0}) \leq \frac{\bar{\gamma}(|x_0|)}{c_\mu} + \frac{d_\mu}{c_\mu} + t_0. \]  

(25)

Applying Fatou’s lemma for (25), we have

\[ E(\rho_{x_0}) \leq \frac{\bar{\gamma}(|x_0|)}{c_\mu} + \frac{d_\mu}{c_\mu} + t_0, \]  

(26)

which signifies that \( P\{\rho_{x_0} < \infty\} = 1 \).

The proof of stability in probability is similar to that of Theorem 1 and is thus omitted here.

**Remark 4**  
(i) Theorem 3 signifies that Theorem 1 also holds for \( \kappa = 0, \) that is to say, Theorem 3 holds for any \( \kappa \in [0, 1) \). In the proof of Theorem 4 (17) also holds for \( \kappa \in [0, 1) \) since (26) is consistent with (17) with \( \kappa = 0. \) (ii) If \( \mu(t) = -c(c > 0) \) that is a UASF, then Theorem 1 and Theorem 3 are in line with Theorem 3.1 and Theorem 3.2 in (17), respectively. Note that Theorem 3.1 in (17) does not hold for \( \kappa = 1 \) according to Example 3.3 in (17). Therefore, Theorem 1 also does not hold for \( \kappa = 1. \)

**Remark 5** In some existing results, \( \mathcal{L}V \) must be negative definite such as (17) or negative semi-definite such as (18). Theorem 1 and Theorem 3 generalize the existing results and relax some constraints on \( \mathcal{L}V \) such that \( \mathcal{L}V \) can be negative or positive.

**Example 1.** For stochastic nonlinear system

\[ dx(t) = 0.5\mu_1(t)x^+(t)dt - 0.5x(t)dt + x(t)\cos(x(t))dW(t), \]  

(27)
where $\mu_1(t) = 2/(1 + t) - |\sin 2t|$ which is a UASF by [31] and $W(t) \in \mathcal{R}$ is a standard Wiener process. Note that

$$
|f(t, x)|^2 + |g(t, x)|^2 = |0.5\mu_1(t)x^5(t) - 0.5x(t)|^2 + |x(t)\cos(x(t))|^2 \\
\leq 0.25\mu_1^2(t)x^5(t) + 1.25|x(t)|^2 + 0.5|\mu_1(t)|x^5(t) \\
\leq x^2(t) + \frac{5}{4}|x(t)|^2 + x^4(t) \\
\leq \frac{1}{3}|x(t)|^2 + \frac{2}{3} + \frac{5}{4}|x(t)|^2 + \frac{2}{3}|x(t)|^2 + \frac{1}{3} \\
\leq H(|x(t)|^2 + 1),
$$

(28)

where $H = 9/4$, Lemma 3 in [24] and $\mu_1(t) = 2/(1 + t) - |\sin 2t| \leq 2$ are used. Hence, system (27) has a continuous solution by Lemma 2. Let $V_1(x) = x^2$, then we have

$$
\mathcal{L}V_1(x(t)) = 2x(t)(0.5\mu_1(t)x^5(t) - 0.5x(t)) + x^2(t)\cos^2(x(t)) \\
\leq \mu_1(t)x^5(t) \\
= \mu_1(t)V_1^2(x(t)).
$$

(29)

This implies that the trivial solution of system (27) is finite-time stable in probability by Theorem 1.

In simulation, let $x_0 = 0.6$, then the state curve of system (27) is shown in Figure 1, which shows that system state trajectory beginning from non-zero initial value converges to the origin in finite time.

**Example 2.** For stochastic nonlinear system

$$
\begin{align*}
\begin{cases}
    dx_1(t) &= f_1(t, x)dt + g_1(t, x)dW_1(t), \\
    dx_2(t) &= f_2(t, x)dt + g_2(t, x)dW_2(t),
\end{cases}
\end{align*}
$$

(30)

where $f_1(t, x) = -x_1(t) + (\psi(t) - 0.5)x^5_1(t)$, $g_1(t, x) = \sqrt{2}x_1(t)\cos(x_1(t))$, $f_2(t, x) = -x_2(t) + (\psi(t) - 0.5)x^5_2(t)$, $g_2(t, x) = \sqrt{2}x_2(t)\sin(x_2(t))$, $\psi(t) = t\sin t/(1 + t)$, $W_i(t) \in \mathcal{R}(i = 1, 2)$ are mutually independent standard Wiener processes. System (30) satisfies the condition in Lemma 2 so there is a continuous solution to system (30).

Let $V_2(x) = x^2_1 + x^2_2$, then

$$
\mathcal{L}V_2(x) \leq 2(\psi(t) - 0.5)(x^2_1 + x^2_2) = 2\mu_2(t)[(x^2_1)^\frac{m}{2} + (x^2_2)^\frac{m}{2}] \leq \mu_2(t)V_2^\frac{m}{2}(x),
$$
where Lemma 4 in [24] is used, \( \tilde{\mu}_2(t) = \psi(t) - 0.5 \) and

\[
\mu_2(t) = \begin{cases} 
2\tilde{\mu}_2(t) & \text{if } \psi(t) \geq 0.5 \\
2\mu_2(t) & \text{if } \psi(t) < 0.5
\end{cases}
\]

Since \( \tilde{\mu}_2(t) = \psi(t) - 0.5 \) is a UASF and \( \int_{t_0}^{t} \tilde{\mu}_2(s) ds \leq 5 - 0.5(t - t_0) \), it is easy to verify that \( \mu_2(t) \) is also a UASF. Therefore, the solution of system (30) with \( x_0 \in \mathbb{R}^2 \backslash \{0\} \) is finite-time stable in probability by Theorem 1 with \( \kappa = 9/10 \). For simulation, let \( x_0 = (0.4, -0.6)^T \), then the trajectories of the state \( x_1(t) \) and \( x_2(t) \) are described in Figure 2. This signifies that the trivial solution of system (30) is finite-time stable in probability.

**Remark 6**

(i) Since system (27) and (30) satisfy the condition of Lemma 2, system (27) and (30) have a continuous solution, respectively. We can also prove that Example 1 and Example 2 satisfy Lemma 3. For example, we know that \( f(x,t) \) and \( g(x,t) \) in Example 1 are locally bounded in \( x \) and are uniformly bounded in \( t \) by (28). Furthermore, let \( U_{1}(x) = V_{1}(x) = x^2 = |x|^2 \) in Example 1, then \( LU_{1}(x(t)) = \mu_1(t)U_{2}^2(x(t)) \leq 2U_{1}^2(x(t)) \leq 4/3U_{1} + 2/3 \triangleq l(t)U_{1} + 2/3 \). (ii) Example 1 and Example 2 show that the finite-time stability of stochastic nonlinear systems can be analyzed by Theorem 1. (iii) In order to further show the validity of Theorem 1, we focus on finite-time stabilization problem for a stochastic nonlinear system, which is given in Section V.

**IV. Finite-time instability theorem**

In this section, we discuss some sufficient conditions which ensure that stochastic nonlinear system (4) is finite-time instable in probability. Compared with some existing stochastic finite-time instability results (such as [17] and [19]), the given finite-time instability criterion relaxes the condition imposed on \( LV \) and allows \( LV \) to be indefinite.

**Theorem 3**

Assuming that there exists a solution to system (4) with any given non-zero initial value. If there exist a function \( V(t,x) \in C^{1,2}([t_0, \infty) \times \mathbb{R}^n; \mathbb{R}_+) \), \( K_{\infty} \) functions \( \gamma \) and \( \bar{\gamma} \), and a UASF \( \mu(t) \) such that

\[
\gamma(|x|) \leq V(t,x) \leq \bar{\gamma}(|x|),
\]

\[
\mathcal{L}V(t,x) = \mu(t)V(t,x),
\]

\[
|V_x(t,x)g(t,x)|^2 \leq a(t)V^2(t,x),
\]
where \( a(t) \geq 0 \) and \( \int_{t_0}^{\infty} a(t)dt < \infty \), then the trivial solution of system (3) is finite-time instable in probability.

**Proof:** Firstly, let \( \bar{V}(t, x) = e^{-\int_{t_0}^{t} \mu(s)ds} V(t, x) \) along system (1) and apply Itô’s formula for \( \bar{V} \), then

\[
d\bar{V} = \mathcal{L}\bar{V} dt + \bar{V}_x g(t, x(t))dW(t),
\]

where \( \bar{V}_x = e^{-\int_{t_0}^{t} \mu(s)ds} V_x(t, x) \) and \( \mathcal{L}\bar{V} = e^{-\int_{t_0}^{t} \mu(s)ds} (\mathcal{L}V(t, x) - \mu(t)V(t, x)) = 0 \). It follows from (34) that

\[
e^{-\int_{t_0}^{t} \mu(s)ds} V(t, x(t)) - V(t_0, x_0) = \int_{t_0}^{t} e^{-\int_{t}^{\tau} \mu(s)ds} V_x(t, x(t)) dt = V(t_0, x_0).
\]

From (33), (35) and Burkholder-Davis-Gundy inequality (Theorem 7.3 in [Chapter 1, 3]), we can obtain that

\[
e^{-\int_{t_0}^{t} \mu(s)ds} V(t, x(t)) - V(t_0, x_0) = \int_{t_0}^{t} e^{-\int_{t}^{\tau} \mu(s)ds} V_x(t, x(t)) dt = V(t_0, x_0).
\]

Further, by Gronwall’s inequality (Theorem 8.1 in [Chapter 1, 3]), it can be derived from (36) that

\[
E\left[\sup_{t_0 \leq t \leq T} \left( e^{-\int_{t_0}^{t} \mu(s)ds} V(t, x(t)) \right)^2 \right] \leq 2e^{\int_{t_0}^{T} \gamma(s)ds} V^2(t_0, x_0) \leq 2e^{\int_{t_0}^{\infty} \gamma(t)dt} \gamma^2(|x_0|) \leq H_0.
\]

Let \( T \to \infty \), then by Fatou’s lemma, we have

\[
E\left[\sup_{t \in [t_0, \infty)} \left( e^{-\int_{t_0}^{t} \mu(s)ds} V(t, x(t)) \right)^2 \right] \leq H_0 < \infty,
\]

which means that

\[
E\left[\sup_{t \in [t_0, \infty)} \left( e^{-\int_{t_0}^{t} \mu(s)ds} V(t, x(t)) \right) \right] < \infty.
\]

For any given \( x_0 \in \mathbb{R}^r \setminus \{0\} \), let \( \sigma_n = \inf\{t \geq t_0 : |x(t)| > n\} \). It follows from (35) that

\[
E\left\{\left. e^{-\int_{t_0}^{t} \mu(s)ds} V(t, x(t)) \right|_{t = \sigma_n \wedge \sigma_b} \right\} = V(t_0, x(t_0)),
\]

where \( \sigma_b \) represents any bounded stopping time. Moreover,

\[
0 \leq e^{-\int_{t_0}^{t_0 \wedge \sigma_b} \mu(s)ds} V(\sigma_n \wedge \sigma_b, x(\sigma_n \wedge \sigma_b)) \leq \sup_{t \in [t_0, \infty)} \left( e^{-\int_{t_0}^{t} \mu(s)ds} V(t, x(t)) \right).
\]

From (38) and (40), apply Lebesgue’s dominated convergence theorem for (39), then we have

\[
E\left[e^{-\int_{t_0}^{t_0 \wedge \sigma_b} \mu(s)ds} V(\sigma_b, x(\sigma_b)) \right] = V(t_0, x(t_0)),
\]

where \( \sigma_n \to \infty \) as \( n \to \infty \) is used. This signifies that \( e^{-\int_{t_0}^{t_0 \wedge \sigma_b} \mu(s)ds} V(t, x(t)) \) is a uniformly integrable martingale. Hence, by martingale convergence theorem (Theorem 7.11 in [32]), we have

\[
0 \leq \lim_{t \to \infty} e^{-\int_{t_0}^{t} \mu(s)ds} V(t, x(t)) = \vartheta(\omega) < \infty, \quad a.s.,
\]

where \( \vartheta(\omega) \) is a random variable.
Secondly, the conditions (31) and (32) ensure that there is a stable solution to system (1) in probability. In fact, for every given $x_0 \in \mathbb{R} \setminus \{0\}$, let $T^* = \max\{T \geq t_0 : \int_{t_0}^{T} \mu(t)dt = 0\}$. If $t \in [t_0, T^*)$, from (31), (35) and Remark 3 we obtain that
\[
E[\gamma(|x(t)|)] \leq EV(t, x(t)) \leq \gamma(|x_0|)e^{d\epsilon}.
\] (43)

If $t \in (T^*, \infty)$, by (35), we have
\[
E[\gamma(|x(t)|)] \leq EV(t, x(t)) = V(t_0, x_0)e^{\int_{t_0}^{t} \mu(s)ds} \leq \gamma(|x_0|),
\] (44)

where $\int_{t_0}^{t} \mu(s)ds = \int_{t_0}^{T^*} \mu(s)ds + \int_{T^*}^{t} \mu(s)ds < 0$ is used.

Hence, for any $t \geq t_0$, we have
\[
E[\gamma(|x(t)|)] \leq \gamma(|x_0|)e^{d\epsilon}.
\]

From Chebyshev’s inequality, for any $\varepsilon \in (0, 1)$ and $R > 0$, we have
\[
P\{|x(t)| \geq R, t \geq t_0\} = P\{\gamma(|x(t)|) \geq \gamma(R), t \geq t_0\} \leq \frac{1}{\gamma^{-1}(R)}E[\gamma(|x(t)|)] \leq \frac{1}{\gamma^{-1}(R)}\gamma(|x_0|)e^{d\epsilon} \leq \varepsilon
\] (45)
as $|x_0| < \delta(\varepsilon, R) = \gamma^{-1}(\varepsilon R) e^{-d\epsilon}$. This implies that $P\{|x(t)| < R, t \geq t_0\} \geq 1 - \varepsilon$ as $|x_0| < \delta(\varepsilon, R)$.

Finally, consider (42), we can prove that it is not finite-time attractive in probability for the solution of system (1) by the same method as Theorem 4.1 in [17].

**Remark 7** In Theorem 4.1 of [17], $\mathcal{L}V(t, x) = -c_3V(t, x)$ with $c_3 > 0$ is used. It is clear that $\mu(t) = -c_3(c_3 > 0)$ is a UASF. In Theorem 3, the constraint condition of $\mathcal{L}V(t, x) = -c_3V(t, x)$ is replaced by $\mathcal{L}V(t, x) = \mu(t)V(t, x)$. Hence, Theorem 3 improves Theorem 4.1 of [17].

V. A simulation example

**Example 3.** Consider the stochastic nonlinear system with stochastic inverse dynamics ($\Sigma$)
\[
\begin{align*}
\dot{x}(t) & = \varphi(t)x(t)\cos(x(t))x(t)dB(t), \\
\dot{x}_1(t) & = x_2(t)dt, \\
\dot{x}_2(t) & = u(t)dt + x_2(t)\sin(x(t))dB(t),
\end{align*}
\] (46)

where $\beta_1 = (2l - 1)/(2l + 1) \in (0, 1)$, $\beta_2 = 2l/(2l + 1) \in (0, 1)$ and $\beta_3 = (2l - 2)/(2l - 1) \in (0, 1)$ with $l \in \mathbb{N}$ and $l \geq 2$. $B_0(t) \in \mathcal{R}$ and $B(t) \in \mathcal{R}$ are mutually independent standard Brownian motions(Wiener processes). $\varphi(t) = 0.5(t \cos t)/(1 + t - 1.5)$.

For $x$-subsystem, we select $V_0 = x^2$, then $\mathcal{L}V_0 = 2\varphi(t)x^{\beta_1+1} + x^{\beta_2}\cos^2(x_1) \leq (2\varphi(t) + 1)x^{\beta_2} = \mu_0(t)V_0^{\beta_2}$, where $\beta_1 + 1 = \beta_2$ and $\mu_0(t) = 2\varphi(t) + 1$ is a UASF since $\int_{t_0}^{t} \mu_0(s)ds \leq -0.5(t - t_0) + 5$.

For system $\Sigma$, we introduce $V_1 = V_0 + W_0$ with $W_0 = 0.5x_1^2$ and $z_1 = x_1$, then
\[
\mathcal{L}V_1 = \mathcal{L}V_0 + \mathcal{L}W_0 = \mathcal{L}V_0 + z_1x_2 \leq \mu_0(t)V_0^{\beta_2} + z_1(x_2 - \alpha) + z_1\alpha.
\] (47)

Let the stabilizing function $\alpha = -c_1z_1$ with $\lambda = \beta_1$ and $c_1 > 0$ being a design constant, then (47) can be changed into
\[
\mathcal{L}V_1 \leq \mu_0(t)V_0^{\beta_2} + z_1(x_2 - \alpha) - c_1z_1^{1+\lambda}.
\] (48)
Further, we introduce $V_2 = V_1 + W_1, W_1 = \int_{x_2}^{x_1} (v^\gamma - \alpha^\gamma)^{2-\lambda} dv$, and $z_2 = x_2^\gamma - \alpha^\gamma$, then it follows from Proposition B.1-B.2 in [33] that $V_2$ is a positive definite function and

$$
W_0 + W_1 \leq 2(z_1^2 + z_2^2). \quad (49)
$$

Meanwhile, we can deduce that

$$
\mathcal{L}V_2 = \mathcal{L}V_1 + \mathcal{L}W_1 = \mathcal{L}V_1 + \frac{\partial W_1}{\partial x_1} x_2 + \frac{\partial W_1}{\partial x_2} \frac{\partial V}{\partial x_2} + \frac{1}{2} \frac{\partial^2 W_1}{\partial x_2^2} x_2^{2\beta_3} \sin^2(\lambda). \quad (50)
$$

Note that

$$
\frac{\partial W_1}{\partial x_1} x_2 = -(2 - \lambda) \frac{\partial}{\partial x_1} x_2 \int_{x_2}^{x_1} (v^\gamma - \alpha^\gamma)^{1-\lambda} dv
\leq c_1^\lambda (2 - \lambda) |z_2|^{1-\lambda} |x_2 - \alpha| |x_2|
\leq c_2^\lambda (2 - \lambda) 2^{1-\lambda} |z_2|^{1-\lambda} |x_2|
\leq c_2^\lambda (2 - \lambda) 2^{1-\lambda} |x_2 - \alpha| + |\alpha|
\leq c_2^\lambda (2 - \lambda) 2^{1-\lambda} (2^{1-\lambda} + c_1 |z_2| |z_2|)
\leq c_2^\lambda (2 - \lambda) 2^{1-\lambda} \left(2^{1-\lambda} z_2^{1+\lambda} + \frac{c_1 h_1^{\lambda} - \lambda^2}{1 + \lambda} z_1^{1+\lambda} + \frac{2^{1-\lambda} c_1^{1+\lambda} h_1^{\lambda}}{1 + \lambda} (2 - \lambda) z_2^{1+\lambda},
\right)

(51)

$$
\frac{1}{2} \frac{\partial^2 W_1}{\partial x_2^2} x_2^{2\beta_3} \sin^2(\lambda) = \frac{2 - \lambda}{2} \frac{1}{x_2^{1-\lambda} x_2^{1-\lambda} x_2^{2\beta_3} \sin^2(\lambda)}
\leq \frac{2 - \lambda}{2 \lambda} z_2^{1-\lambda} x_2^{2\lambda}
\leq \frac{2 - \lambda}{2 \lambda} z_2^{1-\lambda} (z_2^{2\lambda} + \alpha^2)
\leq \frac{2 - \lambda}{\lambda} z_2^{1+\lambda} + \frac{2 - \lambda}{\lambda} c_1^{1+\lambda} z_1^{1+\lambda}
\leq \frac{2 - \lambda}{\lambda} z_2^{1+\lambda} + \frac{2 - \lambda}{\lambda} - \frac{2 \lambda}{1 + \lambda} (z_2^{1+\lambda} + \frac{1 - \lambda}{1 + \lambda} h_2 z_2^{1+\lambda})
\leq \frac{2 - \lambda}{\lambda} z_2^{1+\lambda} + \frac{2}\lambda \frac{c_2 h_2^{1+\lambda}}{1 + \lambda} z_2^{1+\lambda} + \frac{2(2 - \lambda)}{1 + \lambda} c_1^{1+\lambda} z_2^{1+\lambda},
\right)

(52)

$$
\frac{\partial W_1}{\partial x_2} u = z_2^{2-\lambda} u, \quad (53)
$$

where $1 - \lambda = 2/(2l + 1)$ and $1 + \lambda = 4l/(2l + 1)$, $h_1 > 0$ and $h_2 > 0$ are to be designed constants and Lemma 4 in [21] is used.

Substituting (48), (51)–(53) into (50), we can obtain that

$$
\mathcal{L}V_2 \leq \mu_0(t) V_0^{\beta_3} + z_1 (x_2 - \alpha) - c_1 z_1^{1+\lambda} + d_2 z_2^{1+\lambda} + d_1 z_1^{1+\lambda} + z_2^{2-\lambda} u, \quad (54)
$$
where \( \tilde{d}_2 = (2 - \lambda)2^{1-\lambda}c_1^{1+\frac{1}{\lambda}}h_1^{-1}/(1 + \lambda) + (2 - \lambda)c_1^{1+\frac{1}{\lambda}}2^{2(1-\lambda)} + (1 - \lambda)(2 - \lambda)c_1^{1+\frac{1}{\lambda}}h_2^{-\frac{1}{2\lambda}}/\lambda(1 + \lambda) + (2 - \lambda)/\lambda \),
\[ \tilde{d}_1 = 2^{1-\lambda}(2 - \lambda)c_1^{1+\frac{1}{\lambda}}h_1^{-1}/(1 + \lambda) + 2(2 - \lambda)c_1^{1+\frac{1}{\lambda}}h_2^{-1}/(1 + \lambda). \]
In addition,
\[
z_1(x_2 - \alpha) \leq |z_1||z_2|^{\lambda} - (\alpha^{\frac{2}{\lambda}}) \leq 2^{1-\lambda}|z_1||z_2|^{\lambda} \leq \frac{2^{1-\lambda}}{1 + \lambda}[h_3z_1^{1+\lambda} + \lambda h_3^{1-\frac{1}{2\lambda}}z_2^{1+\lambda}],
\]
where \( h_3 > 0 \) is a constant that will be designed, and Lemma 2.3 in [34] is applied. Hence, (54) and (55) mean that
\[
\mathcal{L}V_2 \leq \mu_0(t)V_0^{\beta_2} - c_1z_1^{1+\lambda} + d_2z_2^{1+\lambda} + d_1z_1^{1+\lambda} + z_2^{-\lambda}u,
\]
where \( d_1 = \tilde{d}_1 + 2^{1-\lambda}h_3^{-1}/(1 + \lambda) \) and \( d_2 = \tilde{d}_2 + 2^{1-\lambda}h_3^{-\frac{1}{2}}/(1 + \lambda) \).

Let \( h_1 = (1 + \lambda)2^{\lambda-1}c_1^{1+\frac{1}{\lambda}}/6\lambda(2 - \lambda) \), \( h_2 = (1 + \lambda)c_1^{-1}/12(2 - \lambda) \) and \( h_3 = 2^{\lambda}c_1(1 + \lambda)/12 \), then \( d_1 = c_1/2 \) and (56) can be rewritten as
\[
\mathcal{L}V_2 \leq \mu_0(t)V_0^{\beta_2} - 0.5c_1z_1^{1+\lambda} + d_2z_2^{1+\lambda} + z_2^{-\lambda}u.
\]
Further, we choose the controller
\[
u = -d_2z_2^{2\lambda-1} - 0.5c_2z_2^{2\lambda-1},
\]
where \( c_2 > 0 \) is a design constant. Then, together with \( 1 + \lambda = 1 + \beta_1 = 2\beta_2 \), (57) and (58) lead to
\[
\mathcal{L}V_2 \leq \mu_0(t)V_0^{\beta_2} - 0.5c_1(z_1^{1+\frac{1}{\lambda}})^{\beta_2} - 0.5c_2(z_2^{1+\frac{1}{\lambda}})^{\beta_2} \\
\leq \mu_0(t)V_0^{\beta_2} - 0.5c_0((z_1^{1+\frac{1}{\lambda}})^{\beta_2} + (z_2^{1+\frac{1}{\lambda}})^{\beta_2}) \\
\leq \mu_0(t)V_0^{\beta_2} - 0.5c_0z_1^{1+\frac{1}{\lambda}} + z_2^{1+\frac{1}{\lambda}} \\
\leq \mu_0(t)V_0^{\beta_2} - 0.25c_0(W_0 + W_1)^{\beta_2} \\
\leq \mu(t)(V_0^{\beta_2} + (W_0 + W_1)^{\beta_2}) \\
\leq \mu(t)V_2^{\beta_2},
\]
where
\[
\mu(t) = \begin{cases} 2^{1-\beta_2}\tilde{\mu}(t) & \text{if } \tilde{\mu}(t) \geq 0 \\ \tilde{\mu}(t) & \text{if } \tilde{\mu}(t) < 0 \end{cases}, \quad \tilde{\mu}(t) = \begin{cases} \mu_0(t) & \text{if } \mu_0(t) \geq -0.25c_0 \\ -0.25c_0 & \text{if } \mu_0(t) < -0.25c_0 \end{cases},
\]
\( c_0 = \min\{c_1, c_2\} \). Moreover, Lemma 4 in [24, 49] and 0.5^{\beta_2} > 0.5 are used in (59). Note that \( \mu_0(t) \) is a UASF, so \( \mu(t) \) and \( \mu(t) \) are UASF's by Remark 3.

Therefore, let \( \xi = (\chi, z_1, z_2)^T \) and Lyapunov function \( V = V_0 + W_0 + W_1 \) for system \( \Sigma \), then the positive function \( V = V_0 + W_0 + W_1 \leq \chi^2 + 2(z_1^2 + z_2^2) \leq 2|\xi|^2 \) and \( \mathcal{L}V \leq \mu(t)V^{\beta_2} \), where (49) is considered. By Theorem 1, stochastic nonlinear system (16) is finite-time stabilizable in probability with the controller (58).

In simulation, we choose design parameter \( l = 4, c_1 = 0.3 \) and \( c_2 = 0.3 \), system initial value \( \chi(0) = 0.2 \), \( x_1(0) = 0.1 \) and \( x_1(0) = -0.2 \), then system control input trajectory and system state curves are given in Figure 3.

**VI. Conclusions**

This paper has further studied the finite-time stability and instability in probability for stochastic nonlinear systems. A weaker sufficient condition, which ensures that the considered system has a global
solution, has been presented. Some improved finite-time stability and instability theorems have been
given by the UASF. The obtained stability and instability results relax the strict constraint conditions
on $LV$ existing in previous references. Some examples are given to illustrate that the obtained stability
theorems can be used to analysis and synthesis of stochastic nonlinear systems including autonomous
systems and forced systems.

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