A principle of corresponding states for two-component, self-gravitating fluids

R. Caimmi

May 4, 2010

Abstract

Macrogases are defined as two-component, large-scale celestial objects where the subsystems interact only via gravitation. The macrogas equation of state is formulated and compared to the van der Waals (VDW) equation of state for ordinary gases. By analogy, it is assumed that real macroisothermal curves in macrogases occur as real isothermal curves in ordinary gases, where a phase transition (vapour-liquid observed in ordinary gases and gas-stars assumed in macrogases) takes place along a horizontal line in the macrovolume-macropressure ($OXV_XP$) plane. The intersections between real and theoretical (deduced from the equation of state) macroisothermal curves, makes two regions of equal surface as for ordinary gases obeying the VDW equation of state. A numerical algorithm is developed for determining the following points of a selected theoretical macroisothermal curve on the ($OXV_XP$) plane: the three intersections with the related real macroisothermal curve, and the two extremum points (one maximum and one minimum). Different kinds of macrogases are studied in detail: UU, where U density profiles are flat, to be conceived as a simple guidance case; HH, where H density profiles obey the Hernquist (1990) law, which satisfactorily fits to observed spheroidal components of galaxies; HN/NH, where N density profiles obey the Navarro-Frenk-White (1995, 1996, 1997) law, which satisfactorily fits to simulated...
nonbaryonic dark matter haloes. A different trend is shown by theoretical macroisothermal curves on the \((O \times V, X_p)\) plane, according if density profiles are sufficiently mild (UU) or sufficiently steep (HH, HN/NH). In the former alternative, no critical macroisothermal curve exists, below or above which the trend is monotonic. In the latter alternative, a critical macroisothermal curve exists as shown by VDW gases, where the critical point may be defined as the horizontal inflexion point. In any case, by analogy with VDW gases, the first quadrant of the \((O \times V, X_p)\) plane may be divided into three parts, namely (i) the G region, where only gas exists; (ii) the S region, where only stars exist; (iii) the GS region, where both gas and stars exist. With regard to HH and HN/NH macrogases, an application is made to a subsample \((N = 16)\) of elliptical galaxies extracted from larger samples \((N = 25, N = 48)\) of early type galaxies investigated within the SAURON project (Cappellari et al. 2006, 2007). Under the simplifying assumption of universal mass ratio of the two subsystems, \(m\), different models characterized by different scaled truncation radii i.e. concentrations, \(\Xi_i, \Xi_j\), are considered and the related position of sample objects on the \((O \times V, X_p)\) plane is determined. Macrogases fitting to elliptical galaxies are expected to lie within the S region or slightly outside the boundary between the S and the GS region at most. Accordingly, models where sample objects lie outside the S region and far from its boundary, or cannot be positioned on the \((O \times V, X_p)\) plane, are rejected. For each macrogas, twenty models are considered for different values of \((\Xi_i, \Xi_j, m)\), namely \(\Xi_i, \Xi_j = 5, 10, 20, +\infty (\Xi_i, \Xi_j, both \ either \ finite \ or \ infinite)\), and \(m = 10, 20\). Acceptable models are \((10, 10, 20)\), \((10, 20, 20)\), \((20, 10, 20)\), \((20, 20, 20)\), for HH macrogases, and \((10, 5, 10)\), \((10, 10, 20)\), \((20, 10, 20)\), \((20, 20, 20)\), for HN/NH macrogases. Typically, fast rotators are found to lie within the S region, while slow rotators are close (from both sides) to the boundary between the S and the GS region. The net effect of the uncertainty affecting observed quantities, on the position of sample objects on the \((O \times V, X_p)\) plane, is also investigated. Finally, a principle of corresponding states is formulated for macrogases with assigned density profiles and scaled truncation radii.

**keywords - galaxies: evolution - dark matter: haloes.**

# 1 Introduction

Tidal interactions between neighbouring objects span across the whole admissible range of lengths in nature: from, say, atoms to cluster of galaxies i.e. from micro to macrocosmos. The role of tidal interactions is of basic
importance in driving a wide variety of physical phenomena. In dealing with microcosmos, tidal forces between molecules are responsible for the occurrence of the liquid and solid phase, and the presence of a triple point where the gas, liquid, and solid phase coexist, for an assigned homogeneous substance (e.g., Landau & Lifchitz, 1967, Chaps. VII-VIII, hereafter quoted as LL67). In dealing with ordinary cosmos, the tidal action of a white dwarf star on a sufficiently close (filling the whole volume enclosed by the Roche equipotential surface) red giant companion, makes mass transfer into the white dwarf until a critical mass is attained and the star ends its life into a catastrophic SnIa supernova explosion (e.g., Burrows, 2000). In dealing with macrocosmos, the tidal action induced by massive haloes on hosted galaxies affects their formation and evolution process, due to a larger depth of the potential well, resulting in a different correlation of observables with respect to galaxies in absence of massive halos (e.g., D’Onofrio et al., 2006).

Ordinary fluids are collisional, which makes the stress tensor be isotropic and the velocity distribution obey the Maxwell law. Tidal interactions therein act between colliding particles (e.g., LL67, Chap. VII, §74). Astrophysical fluids (leaving aside extremely dense environments such as galactic nuclei) are collisionless, which makes the stress tensor be anisotropic and the velocity distribution do not obey the Maxwell law. Tidal interactions therein act between a single particle and the system as a whole.

Given that tidal interactions are at work in both collisional and collisionless fluids, the existence of an analogy between the two may be the subject of a legitimate question. To this respect, an investigation must necessarily be restricted to theoretical considerations, as astrophysical fluids (conceived as macrogases) cannot be tested in laboratory. More specifically, a macrogas equation of state has to be formulated in terms of three variables (macrovolume, macropressure, macrotemperature), and the related macroisothermal curves (i.e. the macropressure as a function of the macrovolume for selected constant macrotemperatures) has to be compared with their counterparts deduced from the van der Waals (hereafter quoted as VDW) equation of state for ordinary gases. If some analogy exists, it can be extended to (undetectable) real macroisothermal curves as a working hypothesis. Finally, an application can be made to galaxies or clusters of galaxies.

The VDW equation of state can be expressed in (dimensionless) reduced volume, reduced pressure, and reduced temperature. Similarly to the Lane-Emden equation for polytropes (e.g., Chandrasekhar 1939, Chap.IV, §4; Caimmi, 1986), the reduced VDW equation holds for a class of fluids instead of a single fluid. In general, the states of two systems with equal values of the reduced variables, are defined as corresponding states. According to the principle of corresponding states, two fluids which obey the reduced VDW
equation of state and exhibit equal values of two among three reduced variables, necessarily exhibit equal values of the remaining reduced variable. For further details refer to classical textbooks (e.g., LL67, Chap. VIII, §85).

In the light of an analogy between ordinary gases and macrogases, the formulation of a principle of corresponding states in the latter case could be highly rewarding. A macrogas equation of state was formulated in earlier attempts (Caimmi and Secco, 1990, hereafter quoted as CS90; Caimmi and Valentinuzzi, 2008, hereafter quoted as CV08), where the analogy between macrogases and VDW gases was only mentioned, and iso fractional mass \( m = \text{const} \) curves were plotted for a few selected density profiles. The current paper aims to establish a closer analogy, where the macrovolume is related to the fractional radius, \( y \), the macropressure to the fractional mass, \( m \), and the macrotemperature to the fractional energy, \( \phi \). The basic assumptions and the formalism remain unchanged with respect to the last parent paper (CV08).

The present investigation is mainly devoted to the following points: (i) expression of an equation of state for two-component astrophysical fluids, conceived as macrogases; (ii) comparison between macroisothermal curves and isothermal curves related to VDW gases, with regard to a simple guidance case and two cases which satisfactorily fit to observations or simulations; (iii) application to a subsample (\( N = 16 \)) of elliptical galaxies (CV08), extracted from larger samples (\( N = 25, N = 48 \)) of early-type galaxies investigated within the SAURON project (Cappellari et al., 2006, 2007, hereafter quoted as S IV, S X, respectively).

The work is organized as follows. The equation of state of ideal and VDW gases are reviewed, and related isothermal curves are shown, in Section 2. A macrogas equation of state is formulated in terms of macrovolume, macropressure, macrotemperature, and related macroisothermal curves are shown for flat and steep density profiles, in Section 3. An application to elliptical galaxies for which masses, radii, and rms velocities can be determined, is performed in Section 4, where the selection of acceptable models is made and an interpretation of the results is outlined. The conclusion is drawn in Section 5. Further details on two specific points are reported in the Appendix.

2 Ordinary fluids

Ordinary fluids are conceived as fluids where the effects of gravitation on the equation of state may safely be neglected e.g., on the surface of the Earth. The simplest description is provided by the theory of ideal gas. Ideal gases are collisional fluids defined by the following properties: (i)
particles are identical spheres; (ii) the number of particles is extremely large; (iii) the motion of particles is random; (iv) collisions between particles or with the wall of the box are perfectly elastic; (v) interactions between particles or with the wall of the box are null.

The equation of state of ideal gases may be written under the form (e.g., LL67, Chap. IV, §42):

\[ pV = kNT \]  \hspace{1cm} (1)

where \( p \) is the pressure, \( V \) the volume, \( T \) the temperature, \( N \) the particle number, and \( k \) the Boltzmann constant. The product, \( pV \), has the dimensions of an energy and, in fact, the mean kinetic energy per degree of freedom equals the product, \((1/2)kNT\), and the mean kinetic energy of motions along the \( X_p \) axis reads:

\[ \langle E_{\text{kin}} \rangle_{pp} = \frac{1}{2}Nm\sigma_{pp}^2 = \frac{1}{2}kNT \]  \hspace{1cm} (2a)

\[ m\sigma_{pp}^2 = kT \]  \hspace{1cm} (2b)

where \( m \) is the mean particle mass and \( \sigma_{pp} \) the rms velocity component along the \( X_p \) axis. In the light of the theory of ideal gases, Eqs. (1) and (2) disclose the meaning of the Boltzmann constant: for fixed pressure, volume, and particle number, the mean kinetic energy remains unchanged, regardless of the nature of the gas.

In getting a better description of real gases, the above assumption (v) is relaxed and interactions between particles are taken into consideration. The VDW generalization of the equation of state of ideal gases, Eq. (1), reads (van der Waals, 1873):

\[ \left( p + A \frac{N^2}{V^2} \right) (V - NB) = kNT \]  \hspace{1cm} (3)

where \( A \) and \( B \) are constants which depend on the nature of the particles. More specifically, the presence of an attractive interaction between particles reduces both the force and the frequency of particle-wall collisions: the net effect is a reduction of the pressure, proportional to the square numerical density, expressed as \( A(N/V)^2 \). On the other hand, the whole volume of the box, \( V \), is not accessible to particles, in that they are conceived as identical spheres: the free volume within the box is \( V - NB \), where \( B \) is the volume of a single sphere. For further details refer to specific textbooks (e.g., LL67, Chap. VII, §74).

The isothermal \( (T = \text{const}) \) curves for ideal gases are hyperbolas with axes, \( p = \mp V \), conformly to Eq. (1). In VDW theory of real gases, the
isothermal curves exhibit two extremum points, which reduce to a single horizontal inflexion point when a critical temperature is attained, as shown in Fig. 1. Well above the critical isothermal curve, $T \gg T_c$, the trends exhibited by ideal and VDW gases look very similar. Below the critical isothermal curve, $T < T_c$, the behaviour of VDW gases is different with respect to ideal gases and, in addition, the related isothermal curves provide a wrong description within a specific region where saturated vapour and liquid phases coexist. Further details are shown in Fig. 2. Above the critical isothermal curve ($T = T_c$) the trend is similar with respect to ideal gases. Below the critical isothermal curve and on the right of the dashed curve, the supersaturated vapour still behaves as an ideal gas. Below the critical isothermal curve and on the left of the dashed curve, the liquid shows little change in volume as the pressure rises. Within the bell-shaped area bounded by the dashed
curve, the liquid phase is in equilibrium with the saturated vapour phase. A reduced volume implies smaller saturated vapour fraction and larger liquid fraction at constant pressure, and vice versa. The VDW equation of state is no longer valid in this region. The dashed curve (including the central branch) is the locus of intersections between VDW and real isothermal curves, the latter being related to constant pressure where liquid and vapour phases coexist. The dotted curve is the locus of VDW isothermal extremum points.

A specific \((T/T_c = 0.85)\) VDW and corresponding real isothermal curve, are represented in Fig.\([3]\). The VDW isothermal curve and the real isothermal curve coincide within the range, \(V \leq V_A\) and \(V \geq V_E\). The VDW isothermal curve exhibits two extremum points: a minimum, \(B\), and a maximum, \(D\), while the real isothermal curve is flat, within the range, \(V_A \leq V \leq V_E\). Configurations related to the VDW isothermal curve within the range, \(V_A \leq V \leq V_B\) (due to tension forces acting on the particles yielding superheated liquid), and \(V_D \leq V \leq V_E\) (due to the occurrence of undercooled vapour), may be obtained under special conditions, while configurations within the range, \(V_B \leq V \leq V_D\), are always unstable. The volumes, \(V_A\) and \(V_E\), correspond to the maximum value in presence of the sole liquid phase and the minimum value in presence of the sole vapour phase, respectively.

The surfaces, ABC and CDE, are equal, as first inferred by Maxwell \(\text{e.g., Rostagni, 1957, Chap. XII, §19}\). The VDW and real isothermal curves represented in Fig.\([3]\) being related to the same temperature, \(T\), the cycle, ABCDECA, is completely both isothermal and reversible, and the work, \(W\), performed therein cannot be positive to avoid violation of the second law of the thermodynamics. The cycles, ABCA and CDEC, occurring in counter-clockwise and clockwise sense, respectively, are also completely both isothermal and reversible. Accordingly, \(W_{ABDECA} = W_{ABC} - W_{CDEC} \leq 0\). A similar procedure, related to the reversed cycle, ACEDCBA, yields \(W_{ACEDCBA} = W_{CDEC} - W_{CBAC} \leq 0\). Then \(W_{ABDECA} = W_{ACEDCBA} = 0\), which implies \(W_{ABCA} = W_{CDEC} = W_{CEDC} = W_{CBAC}\) and, in turn, the equality between the related surfaces. For further details refer to specific textbooks \(\text{e.g., LL67, Chap. VIII, §85}\).

In order to simplify both notation and calculations, it is convenient to deal with (dimensionless) reduced variables \(\text{e.g., Rostagni, 1957, Chap. XII, §16; LL67, Chap. VIII, §85}\). To this aim, the first step is the knowledge of the parameters related to the critical point, \(V_c, p_c, T_c\). Using the VDW equation of state, Eq.\([3]\), the pressure and its first and second partial derivatives, with
respect to the volume, read:

\[ p = \frac{kNT}{V - NB} - AN^2 V^2 ; \quad N = \text{const} ; \quad (4) \]

\[ \left( \frac{\partial p}{\partial V} \right)_{V,T} = -\frac{kNT}{(V - NB)^2} + 2AN^2 V^3 ; \quad (5) \]

\[ \left( \frac{\partial^2 p}{\partial V^2} \right)_{V,T} = \frac{2kNT}{(V - NB)^3} - 6AN^2 V^4 ; \quad (6) \]

where the domain is \( V > NB \), \( V = NB \) is a vertical asymptote, and \( p = 0 \) is a horizontal asymptote. The critical isothermal corresponds to the highest temperature allowing a liquid phase, which occurs therein only at the critical point. The critical isothermal curve exhibits neither a minimum nor a maximum, which are replaced by a horizontal inflexion point coinciding with the critical point. Accordingly, \( (\partial p/\partial V)_{V_c,T_c} = 0 \), \( (\partial^2 p/\partial V^2)_{V_c,T_c} = 0 \), and \( p_c = kNT_c/(V_c - NB) - AN^2/V_c^2 \). The solution of the related system is:

\[ V_c = 3NB ; \quad (7) \]
\[ T_c = \frac{8A}{27Bk} ; \quad (8) \]
\[ p_c = \frac{1}{27} \frac{A}{B^2} ; \quad (9) \]
\[ Z_c = \frac{p_cV_c}{kNT_c} = \frac{3}{8} ; \quad (10) \]

where, in general, the compressibility factor, \( Z = pV/(kNT) \), defines the degree of departure from the behaviour of ideal gases, for which \( Z = 1 \), according to Eq. (1). For further details refer to specific textbooks (e.g., Rostagni, 1957, Chap. XII, §20; LL67, Chap. VIII, §85).

With regard to the reduced variables:

\[ \frac{\vartheta}{V_c} = \frac{V}{V_c} ; \quad \frac{\psi}{p_c} = \frac{p}{p_c} ; \quad \frac{T}{T_c} = \frac{T}{T_c} ; \quad (11) \]

the ideal gas equation of state, Eq. (1), and the VDW equation of state, Eq. (3), reduce to:

\[ \frac{\psi}{\vartheta} = \frac{8}{3} T ; \quad (12) \]

\[ \left( \frac{\psi}{\vartheta} + \frac{3}{\vartheta^2} \right) \left( \vartheta - \frac{1}{3} \right) = \frac{8}{3} T ; \quad \vartheta > \frac{1}{3} ; \quad (13) \]
and Eqs. (4), (5), and (6), reduce to:

\[ \dot{\varphi} = \frac{8\mathcal{T}}{3\mathcal{V}^2} - \frac{3}{\mathcal{V}^2} ; \quad (14) \]

\[ \left( \frac{\partial \dot{\varphi}}{\partial \mathcal{V}} \right) = -\frac{24\mathcal{T}}{(3\mathcal{V} - 1)^2} + \frac{6}{\mathcal{V}^3} ; \quad (15) \]

\[ \left( \frac{\partial^2 \dot{\varphi}}{\partial \mathcal{V}^2} \right) = \frac{144\mathcal{T}}{(3\mathcal{V} - 1)^3} - \frac{18}{\mathcal{V}^4} ; \quad (16) \]

where, for assigned \( \mathcal{T} \), the domain of the function, \( \dot{\varphi}(\mathcal{V}) \), is \( \mathcal{V} > \frac{1}{3} \), \( \mathcal{V} = \frac{1}{3} \) is a vertical asymptote, and \( \dot{\varphi} = 0 \) is a horizontal asymptote. In the special case of the critical point, \( \mathcal{V} = 1, \mathcal{T} = 1, \dot{\varphi} = 1 \), the partial derivatives are null, as expected.

The extremum points, via Eq. (15), are defined by the relation:

\[ f(\mathcal{V}) = \frac{(3\mathcal{V} - 1)^2}{4\mathcal{V}^3} = \mathcal{T} ; \quad (17) \]

which is satisfied at the critical point, as expected. The function on the left-hand side of Eq. (17) has two extremum points: a minimum at \( \mathcal{V} = 1/3 \) (outside the physical domain) and a maximum at \( \mathcal{V} = 1 \), where \( \mathcal{T} = 1 \). Accordingly, Eq. (17) is never satisfied for \( \mathcal{T} > 1 \), which implies no extremum point for related isothermal curves, as expected. The contrary holds for \( \mathcal{T} < 1 \), where it can be seen that the third-degree equation associated to Eq. (15) has three real solutions, related to extremum points. One lies outside the physical domain, which implies \( \mathcal{V} \leq 1/3 \). The remaining two are obtained as the intersections between the curve, \( f(\mathcal{V}) \), expressed by Eq. (17), and the straight line, \( y = \mathcal{T} \), keeping in mind that \( f(1/3) = 0, f(1) = 1, \) and \( \lim_{\mathcal{V} \to +\infty} f(\mathcal{V}) = 0 \).

The third-degree equation associated to Eq. (15), may be ordered as:

\[ \mathcal{V}^3 - 9a\mathcal{V}^2 + 6a\mathcal{V} - a = 0 ; \quad (18a) \]

\[ a = \frac{1}{4\mathcal{V}} ; \quad (18b) \]

with regard to the standard formulation (e.g., Spiegel, 1968, Chap. 9):

\[ x^3 + a_1x^2 + a_2x + a_3 = 0 ; \quad (19) \]

the discriminants of Eq. (18a) are:

\[ Q = \frac{3a_2 - a_1^2}{9} = a(2 - 9a) ; \quad (20) \]
\[
R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54} = \frac{a(1 - 18a + 54a^2)}{2}; \quad (21)
\]

\[
D = Q^3 + R^2 = \frac{a^2(1 - 4a)}{4}; \quad (22)
\]

where \( D = 0 \) in the special case of the critical isothermal curve (\( T = 1, a = 1/4 \)), \( D < 0 \) for \( T < 1 \), and \( D > 0 \) for \( T > 1 \). Accordingly, three coincident real solutions exist if \( D = 0 \), three (at least two) different real solutions if \( D < 0 \), one real (outside the physical domain) and two complex conjugate if \( D > 0 \).

The three real solutions \( (D \leq 0) \) may be expressed as (e.g., Spiegel, 1968, Chap. 9):

\[
V_1 = 2\sqrt{-Q} \cos \left( \pi + \frac{\theta}{3} \right) - \frac{1}{3}a_1 ; \quad (23a)
\]

\[
V_2 = 2\sqrt{-Q} \cos \left( \pi + \frac{\theta}{3} + \frac{2\pi}{3} \right) - \frac{1}{3}a_1 ; \quad (23b)
\]

\[
V_3 = 2\sqrt{-Q} \cos \left( \pi + \frac{\theta}{3} + \frac{4\pi}{3} \right) - \frac{1}{3}a_1 ; \quad (23c)
\]

\[
\theta = \arctan \frac{\sqrt{-D}}{R} ; \quad (23d)
\]

where \( a_1 = -9a \) and, in the special case of the critical isothermal curve, \( a = 1/4, Q = -1/16, D = 0 \), which implies \( V_0 = \min(V_1, V_2, V_3), V_A = V_B = V_C = V_D = V_E = \max(V_1, V_2, V_3) \). In the special case, \( T \to 0 \), Eq. (13a) reduces to a second-degree equation whose solutions are \( V_{01} = V_{02} = 1/3 \), while the related function is otherwise divergent as \( a \to +\infty \). In general, the extremum points of VDW isothermal curves (\( T \leq 1 \)) occur at \( V = V_B \) (minimum) and \( V = V_D \) (maximum), \( V_B \leq V_D \). As \( T \to 0 \), \( V_B \to 1/3, V_D \to +\infty \), where, in all cases, \( 1/3 < V_B \leq 1 \leq V_D \).

The two areas defined by the intersection of a generic VDW isothermal curve (\( T \leq 1 \)) and related real isothermal curves (see Fig. 3), are expressed as:

\[
W_1 = \int_{V_A}^{V_C} p_C dV - \int_{V_A}^{V_C} p dV = pcV_C \left[ \varphi_C(V_C - V_A) - \int_{V_A}^{V_C} \varphi dV \right]; \quad (24a)
\]

\[
W_2 = \int_{V_C}^{V_E} p dV - \int_{V_C}^{V_E} pc dV = pcV_C \left[ \int_{V_C}^{V_E} \varphi dV - \varphi_C(V_E - V_C) \right]; \quad (24b)
\]

and the substitution of Eq. (14) into (24) allows explicit expressions for the integrals. The result is:

\[
\frac{W_i}{pcV_C} = \varphi_C(V_C - V_A) - \frac{8}{3} T \ln \frac{3V_C - 1}{3V_A - 1} + \frac{3(V_C - V_A)}{V_A V_C}; \quad (25a)
\]
\[
\frac{W_2}{p_C V_C} = \frac{8}{3} T \ln \left( \frac{3 V_E - 1}{3 V_C - 1} \right) - \frac{3 (V_E - V_C)}{V_C V_E} - \phi_C (V_E - V_C) \quad ; \quad (25b)
\]

and the condition, \( W_1 = W_2 \), after some algebra reads:
\[
\phi_C = \frac{8}{3} \frac{T}{V_E - V_A} \ln \left( \frac{3 V_E - 1}{3 V_A - 1} \right) - \frac{3}{V_A V_E} \quad ; \quad (26)
\]

where, for a selected isothermal curve, the unknowns are \( \phi_C = \phi_A = \phi_E, V_A, \) and \( V_E \).

The reduced volumes, \( V_A, V_C, V_E \), see Fig. 3, may be considered as intersections between a VDW isothermal curve \( (T < 1) \) and a horizontal straight line, \( \phi = \phi_C \), in the \((O \bar{V} \rho)\) plane. In other words, \( V_A, V_C, V_E \), are the real solutions of the third-degree equation:
\[
V^3 - \left( \frac{1}{3} + \frac{8}{3} \frac{T}{\phi_C} \right) V^2 + \frac{3}{\phi_C} V - \frac{1}{\phi_C} = 0 \quad ; \quad (27)
\]

which has been deduced from Eq. (14), particularized to \( \phi = \phi_C \). The related solutions may be calculated using Eqs. (23). The last unknown, \( \phi_C \), is determined from Eq. (26).

An inspection of Fig. 3 shows that the points, A and E, are located on the left of the minimum, B, and on the right of the maximum, D, respectively. Keeping in mind the above results, the following inequality holds: \( V_A \leq V_B \leq 1 \leq V_D \leq V_E \), which implies further investigation on the special case, \( V_C = 1 \). The particularization of the VDW equation of state, Eq. (14), to the point, \( C = C_1 \), assuming \( V_{C_1} = 1 \), yields:
\[
T = \frac{\phi_{C_1} + 3}{4} \quad ; \quad (28)
\]

and Eq. (27) reduces to:
\[
V^3 - (1 + 2b) V^2 + 3b V - b = 0 \quad ; \quad (29a)
\]
\[
b = \frac{1}{\phi_{C_1}} \quad ; \quad (29b)
\]

with regard to the generic third-degree equation, Eq. (19), the three solutions, \( x_1, x_2, x_3 \), satisfy the relations (e.g., Spiegel, 1968, Chap. 9):
\[
x_1 + x_2 + x_3 = -a_1 \quad ; \quad (30a)
\]
\[
x_1 x_2 + x_2 x_3 + x_3 x_1 = a_2 \quad ; \quad (30b)
\]
\[
x_1 x_2 x_3 = -a_3 \quad ; \quad (30c)
\]
where, in the case under discussion:

\[ a_1 = -1 - 2b ; \quad a_2 = 3b ; \quad a_3 = -b ; \quad (31a) \]
\[ x_1 = \mathcal{V}_A ; \quad x_2 = \mathcal{V}_{C_1} = 1 ; \quad x_3 = \mathcal{V}_E ; \quad (31b) \]

and the substitution of Eqs. (31) into two among (30) yields:

\[ \mathcal{V}_A = b - \sqrt{b^2 - b} ; \quad (32a) \]
\[ \mathcal{V}_E = b + \sqrt{b^2 - b} ; \quad (32b) \]

and the combination of Eqs. (28), (29b), and (32) produces:

\[ \mathcal{V}_A = \frac{1 - 2\sqrt{1 - T}}{4T - 3} ; \quad T \leq 1 ; \quad (33a) \]
\[ \mathcal{V}_E = \frac{1 + 2\sqrt{1 - T}}{4T - 3} ; \quad T \leq 1 ; \quad (33b) \]

which, together with \( \mathcal{V}_{C_1} = 1 \), are the abscissae of the intersection points between a selected VDW isothermal curve in the \((\mathcal{O} \mathcal{V} \hat{\mathcal{p}})\) plane and the straight line, \( \hat{\mathcal{p}} = \hat{\mathcal{p}}_{C_1} \), in the special case under discussion.

The substitution of Eqs. (33) into (26), the last being related to the real isothermal curve, yields:

\[ \frac{T}{\sqrt{1 - T}} \ln \frac{3 - 2T + 3\sqrt{1 - T}}{3 - 2T - 3\sqrt{1 - T}} = 6 ; \quad (34) \]

which holds only for the critical isothermal curve, \( T = 1 \). Accordingly, the abscissa of the intersection point, \( C \), between a selected VDW isothermal curve and related real isothermal curve, see Fig. 3, cannot occur at \( \mathcal{V}_C = 1 \) unless the critical isothermal curve is considered. Then the third-degree equation, Eq. (27), must be solved in the general case by use of Eqs. (23). The results are shown in Tab. 1 where the following parameters (in reduced variables) are listed for each VDW isothermal curve, see Fig. 3: the temperature, \( T \), the lower volume limit, \( \mathcal{V}_A \), for which the liquid and vapour phase coexist; the extremum point (minimum) volume, \( \mathcal{V}_B \); the intermediate volume, \( \mathcal{V}_C \); for which the pressure equals its counterpart related to the corresponding lower and upper volume limit, for which the liquid and vapour phase coexist; the extremum point (maximum) volume, \( \mathcal{V}_D \); the upper volume limit, \( \mathcal{V}_E \), for which the liquid and vapour phase coexist; the extremum point (minimum) pressure, \( \hat{p}_B \); the pressure, \( \hat{p}_A = \hat{p}_C = \hat{p}_E \), related to the horizontal real isothermal curve; the extremum point (maximum) pressure, \( \hat{p}_D \). The locus
Table 1: Values of parameters, $T$, $V_A$, $V_B$, $V_C$, $V_D$, $V_E$, $p_B$, $p_C$, $p_D$, within the range, $0.85 \leq T \leq 0.99$, using a step, $\Delta T = 0.01$. All values equal unity at the critical point. Index captions: A, C, E - intersections between VDW and real isothermal curves; B - extremum point of minimum; D - extremum point of maximum. Extremum points are related to VDW isothermal curves, while their real counterparts are flat in presence of both liquid and vapour phase. For aesthetical reasons, 01 on head columns stands for unity.

| $T$  | $10V_A$ | $10V_B$ | $10V_C$ | $10V_D$ | $10V_E$ | $10p_B$ | $10p_C$ | $10p_D$ |
|------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.85 | 5.5336  | 6.7168  | 1.1453  | 1.7209  | 3.1276  | 0.4963  | 5.0449  | 6.2055  |
| 0.86 | 5.6195  | 6.8003  | 1.1337  | 1.6821  | 2.9545  | 1.2750  | 5.3125  | 6.4005  |
| 0.87 | 5.7116  | 6.8883  | 1.1225  | 1.6436  | 2.7909  | 2.0346  | 5.5887  | 6.6011  |
| 0.88 | 5.8106  | 6.9814  | 1.1116  | 1.6052  | 2.6360  | 2.7752  | 5.8736  | 6.8076  |
| 0.89 | 5.9176  | 7.0804  | 1.1009  | 1.5669  | 2.4889  | 3.4965  | 6.1674  | 7.0205  |
| 0.90 | 6.0340  | 7.1860  | 1.0905  | 1.5285  | 2.3488  | 4.1984  | 6.4700  | 7.2401  |
| 0.91 | 6.1615  | 7.2994  | 1.0804  | 1.4900  | 2.2151  | 4.8807  | 6.7816  | 7.4669  |
| 0.92 | 6.3022  | 7.4221  | 1.0706  | 1.4511  | 2.0869  | 5.5430  | 7.1021  | 7.7014  |
| 0.93 | 6.4593  | 7.5561  | 1.0610  | 1.4117  | 1.9634  | 6.1849  | 7.4318  | 7.9443  |
| 0.94 | 6.6369  | 7.7040  | 1.0516  | 1.3715  | 1.8438  | 6.8058  | 7.7707  | 8.1963  |
| 0.95 | 6.8412  | 7.8697  | 1.0425  | 1.3300  | 1.7271  | 7.4049  | 8.1188  | 8.4584  |
| 0.96 | 7.0819  | 8.0593  | 1.0336  | 1.2867  | 1.6118  | 7.9811  | 8.4762  | 8.7319  |
| 0.97 | 7.3756  | 8.2830  | 1.0249  | 1.2404  | 1.4960  | 8.5328  | 8.8429  | 9.0185  |
| 0.98 | 7.7554  | 8.5611  | 1.0164  | 1.1892  | 1.3761  | 9.0576  | 9.2191  | 9.3209  |
| 0.99 | 8.3091  | 8.9461  | 1.0081  | 1.1278  | 1.2430  | 9.5510  | 9.6048  | 9.6437  |
of the intersections between VDW and real isothermal curves is presented in Fig. 2 as a trifid curve, where the left, the right, and the middle branch correspond to $V_A$, $V_E$, and $V_C$, respectively. The common starting point coincides with the critical point. The locus of the VDW isothermal curve extremum points is represented in Fig. 2 as a dotted curve starting from the critical point, where the left and the right branch corresponds to minimum and maximum points, respectively.

A fluid state can be represented in reduced variables as $(\bar{V}, \bar{\rho}, \bar{T})$, where one variable may be expressed as a function of the remaining two, by use of the reduced ideal gas equation of state, Eq. (12), or the reduced VDW equation of state, Eq. (13). The formulation in terms of reduced variables, Eqs. (11), makes the related equation of state universal i.e. it holds for any fluid. Similarly, the Lane-Emden equation expressed in polytropic (dimensionless) variables, describes the whole class of polytropic gas spheres with assigned polytropic index, in hydrostatic equilibrium (e.g., Chandrasekhar 1939, Chap. IV, §4).

The states of two fluids with equal $(\bar{V}, \bar{\rho}, \bar{T})$, are defined as corresponding states. The mere existence of an equation of state yields the following result. **Law of corresponding states.** Given two fluids, the equality between two among three reduced variables, $\bar{V}$, $\bar{\rho}$, $\bar{T}$, implies the equality between the remaining related reduced variables i.e. the two fluids are in corresponding states.

The law was first formulated by van der Waals in 1880. For further details refer to specific textbooks (e.g., LL67, Chap. VIII, §85).

### 3 Astrophysical fluids

Let macrogases be defined as two-component fluids which interact only gravitationally. The virial theorem for subsystems reads (Caimmi et al., 1984; Caimmi and Secco, 1992; CV08):

$$
2(E_u)_{\text{kin}} + (E_{uv})_{\text{vir}} = 0 \quad ; \quad u = i, j \quad ; \quad v = j, i \quad ; \quad (35a)
$$

$$
(E_{uv})_{\text{vir}} = (E_u)_{\text{sel}} + (E_{uv})_{\text{tid}} \quad ; \quad (35b)
$$

where $i$ and $j$ denote the inner and outer subsystem, respectively, $E_{\text{kin}}$ is the kinetic energy, $E_{\text{sel}}$, $E_{\text{tid}}$, and $E_{\text{vir}}$, are the self, tidal, and virial potential energy, respectively. The related definitions are:

$$
(E_u)_{\text{kin}} = \frac{1}{2} \int_{S_u} \rho_u(x_1, x_2, x_3) \sum_{s=1}^{3} (v_u)_s^2 \, d^3S_u \quad ;
$$

$$
(E_{uv})_{\text{vir}} = \int_{S_u} \rho_u(x_1, x_2, x_3) \sum_{s=1}^{3} (v_u)_s^2 \, d^3S_u \quad ;
$$

$$
(E_u)_{\text{sel}} = \int_{S_u} \rho_u(x_1, x_2, x_3) \sum_{s=1}^{3} (v_u)_s^2 \, d^3S_u \quad ;
$$

$$
(E_{uv})_{\text{tid}} = \int_{S_u} \rho_u(x_1, x_2, x_3) \sum_{s=1}^{3} (v_u)_s^2 \, d^3S_u \quad ;
$$

$$
(E_{uv})_{\text{vir}} = \int_{S_u} \rho_u(x_1, x_2, x_3) \sum_{s=1}^{3} (v_u)_s^2 \, d^3S_u \quad ;
$$

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\[(E_u)_{\text{sel}} = \int_{S_u} \rho_u(x_1, x_2, x_3) \sum_{s=1}^{3} x_s \frac{\partial V_u}{\partial x_s} d^3S_u \]
\[= -\frac{1}{2} \int_{S_u} \rho_u(x_1, x_2, x_3) V_u(x_1, x_2, x_3) d^3S_u ; \quad (37)\]
\[(E_{uv})_{\text{tid}} = \int_{S_u} \rho_u(x_1, x_2, x_3) \sum_{s=1}^{3} x_s \frac{\partial V_v}{\partial x_s} d^3S_u ; \quad (38)\]

where \(\rho\) is the density, \(v_s\) the velocity component, \(S\) the volume, and \(V\) the gravitational potential.

The virial theorem makes a necessary (but not sufficient) condition for dynamical or hydrostatic equilibrium, which implies the parameters of the virialized configuration must be considered as averaged on a sufficiently long time. For further details refer to specific textbooks (e.g., Landau and Lifshitz, 1966, Chap. II, §10) and to an earlier paper (Caimmi, 2007). On the other hand, general trends exhibited by virialized configurations hold, in particular, for self-consistent density profiles implying nonnegative distribution functions. Accordingly, density profiles shall be chosen regardless from their self-consistency, aiming to investigate general trends instead of local properties. To avoid the determination of the gravitational potential, which is the most difficult step towards an explicit formulation of potential energies, future considerations shall be restricted to homeoidally striated ellipsoids (Roberts, 1962). The following results are taken from earlier attempts (CV08, and further references therein), to which an interested reader is addressed.

The isopycnic (i.e. constant density) surfaces are defined by the following law:
\[\rho_u = \rho_u^* f_u(\xi_u) ; \quad f_u(1) = 1 ; \quad u = i, j ; \quad (39a)\]
\[\xi_u = \frac{r_u}{r_u^*} ; \quad 0 \leq \xi_u \leq \Xi_u ; \quad \Xi_u = \frac{R_u}{r_u^*} ; \quad (39b)\]

where the dagger denotes a selected reference isopycnic surface, \(r\) is the radial coordinate along a selected direction, \(R\) the related truncation radius, \(\xi\) a related scaled radial coordinate, and \(\Xi\) the related scaled truncation radius.

The mass and the self potential energy read:
\[M_u = (\nu_u)_{\text{mas}} M_u^\dagger ; \quad M_u^\dagger = \frac{4\pi}{3} \rho_u^* (a_u^*)_1^1 (a_u^*)_2^1 (a_u^*)_3^1 ; \quad (40a)\]
\[(\nu_u)_{\text{mas}} = \frac{3}{2} \int_{0}^{\Xi_u} F_u(\xi_u) d\xi_u ; \quad F_u(\xi_u) = 2 \int_{\xi_u}^{\Xi_u} f_u(\xi_u) \xi_u d\xi_u ; \quad (40b)\]
\[\nu_u(\xi)_{\text{sel}} = -\frac{G(M_u^\dagger)^2}{(a_u^*)_1^1} B_u ; \quad (\nu_u)_{\text{sel}} = \frac{9}{16} \int_{0}^{\Xi_u} F_u^2(\xi_u) d\xi_u ; \quad (41)\]
where $a_i^\dagger$ are semiaxes of the reference isopycnic surface, $\nu_{\text{mas}}$ and $\nu_{\text{sel}}$ are profile factors which depend only on the mass distribution, and $B$ is a shape factor. For homogeneous configurations, $\nu_{\text{mas}} = \Xi^3$ and $\nu_{\text{sel}} = (3/10) \Xi^5$. For spherical shapes, $B = 2$.

Under the further restriction of similar and similarly placed boundaries, the following relations hold:

$$
\xi_i = y^\dagger \xi_j ; \quad \Xi_j \Xi_i = y y^\dagger ; \quad \frac{(\nu_j)}{(\nu_i)}_{\text{mas}} = \frac{m}{m^\dagger} ; \quad (42a)
$$

$$
y = \frac{R_j}{R_i} ; \quad y^\dagger = \frac{r_j^\dagger}{r_i^\dagger} ; \quad m = \frac{M_j}{M_i} ; \quad m^\dagger = \frac{M_j^\dagger}{M_i^\dagger} ; \quad (42b)
$$

which makes tidal and virial potential energy reduce to:

$$
(E_{uv})_{xxx} = -\frac{G (M_u^\dagger)^2}{(a_u)_{\text{1}}} (\nu_{uv})_{xxx} B ; \quad (43a)
$$

$$
u = i, j ; \quad v = j, i ; \quad xxx = \text{tid, vir} ; \quad (43b)
$$

and the explicit expression of the profile factors reads:

$$
(\nu_{ij})_{\text{tid}} = -\frac{9}{8} m^\dagger w^{(\text{ext})}(\eta) ; \quad (\nu_{ji})_{\text{tid}} = -\frac{9}{8} y^\dagger w^{(\text{int})}(\eta) ; \quad (44a)
$$

$$
(\nu_{uv})_{\text{vir}} = (\nu_{u})_{\text{sel}} + (\nu_{uv})_{\text{tid}} ; \quad u = i, j ; \quad v = j, i ; \quad (44b)
$$

$$
\eta = \frac{\Xi_i}{y^\dagger} = \frac{\Xi_j}{y} ; \quad y \geq 1 ; \quad (44c)
$$

where the functions, $w^{(\text{int})}$ and $w^{(\text{ext})}$, are defined as:

$$
w^{(\text{int})}(\eta) = \int_0^\eta \xi_j F_j(\xi_j) \frac{dF_j}{d\xi_j} d\xi_j ; \quad (45a)
$$

$$
w^{(\text{ext})}(\eta) = \int_0^\eta \xi_j F_i(\xi_i) \frac{dF_i}{d\xi_i} d\xi_j ; \quad (45b)
$$

for further details refer to Appendix [A]. In conclusion, Eqs. (39)-(45) allow the calculation of the virial potential energy for homeoidally striated ellipsoids related to similar and similarly placed boundaries.

The fractional virial potential energy, reads:

$$
\phi = \frac{(E_{ij})_{\text{vir}}}{(E_{ij})_{\text{vir}}} = \frac{(m^\dagger)^2}{y^\dagger} \frac{(\nu_{ij})_{\text{vir}}}{(\nu_{ij})_{\text{vir}}} = \frac{m^2 \Xi_j}{\Xi_i} \left[ \frac{(\nu_i)}{(\nu_j)}_{\text{mas}} \right]^2 \frac{(\nu_{ij})_{\text{vir}}}{(\nu_{ij})_{\text{vir}}} ; \quad (46)
$$

which, for assigned density profiles, depends on either the reference fractional mass, $m^\dagger$, and the fractional scaling radius, $y^\dagger$, or the fractional mass, $m$, and the fractional truncation radius, $y$. 

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Strictly speaking, Eq. (46) is valid provided the indices, \( i \) and \( j \), denote the embedded and the embedding subsystem, respectively, which implies \( y \geq 1 \). If the role of the two subsystems is reversed, \( 0 \leq y \leq 1 \), it has to be kept in mind that the inner and the outer component are denoted by the indices, \( j \) and \( i \), respectively. Then the quantities of interest must be calculated according to the changes, \( m \rightarrow m^{-1} \), \( m^\dagger \rightarrow (m^\dagger)^{-1} \), \( y \rightarrow y^{-1} \), \( y^\dagger \rightarrow (y^\dagger)^{-1} \), \( i \leftrightarrow j \), which yields \( \phi = (E_{ij})_{\text{vir}}/(E_{ji})_{\text{vir}} \) in the domain, \( y \geq 1 \). Following the above mentioned procedure where, in addition, \( \phi \rightarrow \phi^{-1} \), allows the explicit expression of the fractional virial energy, \( \phi = (E_{ji})_{\text{vir}}/(E_{ij})_{\text{vir}} \) in the domain, \( 0 \leq y \leq 1 \), which extends the whole domain to \( 0 \leq y < +\infty \).

In absence of truncation radius, \( \Xi \rightarrow +\infty \), \( \eta \rightarrow +\infty \), the reversion occurs when the density drops to zero and nothing changes except in infinitesimal terms of higher order and infinite terms of lower order. Accordingly, there is no need to perform the reversion in this case.

The combination of Eqs. (44) and (46) yields:

\[
\phi = \frac{(m^\dagger)^2 (\nu_{j})_{\text{sel}} - \frac{9}{8} m^\dagger w^{(\text{int})}(\eta)}{y^\dagger (\nu_i)_{\text{sel}} - \frac{9}{8} m^\dagger w^{(\text{ext})}(\eta)} ; \quad (47a)
\]

\[
\phi = \frac{\Xi_j}{\Xi_i} \left[ \frac{(\nu_j)_{\text{mas}}}{(\nu_i)_{\text{mas}}} \right]^2 \frac{m^\dagger^2 (\nu_{j})_{\text{sel}} - \frac{9}{8} \Xi_i (\nu_i)_{\text{mas}} \frac{m}{y} w^{(\text{int})}(\eta)}{(\nu_i)_{\text{sel}} - \frac{9}{8} (\nu_i)_{\text{mas}} m w^{(\text{ext})}(\eta)} ; \quad (47b)
\]

where, for assigned scaled truncation radii, \( \Xi_i \) and \( \Xi_j \), the independent variables are \( m^\dagger, y^\dagger, \) or \( m, y \). In terms of a second-degree equation in \( m^\dagger \) or \( m \), Eqs. (47) read:

\[
\hat{A}x^2 + \hat{B}x + \hat{C} = 0 ; \quad (48a)
\]

\[
\hat{A} = k_A(\nu_{j})_{\text{sel}} ; \quad k_A = 1, \frac{\Xi_j}{\Xi_i} \left[ \frac{(\nu_j)_{\text{mas}}}{(\nu_i)_{\text{mas}}} \right]^2 ; \quad (48b)
\]

\[
\hat{B} = -\frac{9}{8} k_B \left[ w^{(\text{int})}(\eta) - \phi w^{(\text{ext})}(\eta) \right] ; \quad k_B = \frac{1}{y^\dagger}, (\nu_i)_{\text{mas}} y \quad (48c)
\]

\[
\hat{C} = -k_C(\nu_i)_{\text{sel}} \phi ; \quad k_C = y^\dagger, y \quad (48d)
\]

where the positive solution is:

\[
x = \frac{\sqrt{\hat{B}^2 - 4\hat{A}\hat{C}} - \hat{B}}{2\hat{A}} ; \quad x = m^\dagger, m \quad (49)
\]
and the negative solution has been disregarded due to the lack of physical meaning.

In the special case of coinciding density profiles, \( f_i = f_j, F_i = F_j \), and scaled truncation radii, \( \Xi_i = \Xi_j \), fractional masses and truncation radii also coincide, \( m^\dagger = m, y^\dagger = y \), via Eqs. (42), and the same holds for the profile factors, \( (\nu_i)_{\text{mas}} = (\nu_j)_{\text{mas}}, (\nu_i)_{\text{sel}} = (\nu_j)_{\text{sel}} \), which depend on the scaled truncation radii. Accordingly, Eqs. (47a), (47b), also coincide.

To get a closer analogy with real gases, let Eqs. (47) be rewritten after a change of variables, as:

\[
1 - \frac{9}{8} \frac{w^{(\text{int})}(\eta)}{(\nu_i)_{\text{sel}} (X^\dagger p)^{1/2} X^\dagger V} = K^\dagger(\Xi_i, \Xi_j) X^\dagger T ; \quad (50a)
\]

\[
X^\dagger p X^\dagger V = \frac{1}{\eta X^\dagger V} = (X^\dagger p)^{1/2} X^\dagger V \quad (50b)
\]

\[
K(\Xi_i, \Xi_j) = \Xi_i \Xi_j \left[ \frac{(\nu_j)_{\text{mas}}}{(\nu_i)_{\text{mas}}} \right]^2 \frac{(\nu_i)_{\text{sel}}}{(\nu_j)_{\text{sel}}} ; \quad (50c)
\]

\[
X^\dagger p = (m^\dagger)^2 ; \quad X^\dagger V = \frac{1}{y^\dagger} ; \quad X^\dagger T = \phi ; \quad K^\dagger(\Xi_i, \Xi_j) = \frac{(\nu_i)_{\text{sel}}}{(\nu_j)_{\text{sel}}} ; \quad (50d)
\]

which can be conceived as an equation of state for macrogases. The variables, \( X_V, X_p, X_T \), play a similar role as the volume, the pressure, and the temperature, for ordinary gases. Accordingly, \( X_V, X_p, X_T \), shall be defined as macrovolume, macropressure, and macrotemperature, respectively.

The combination of Eqs. (42), (50d), and (50e) yields:

\[
X^\dagger V = \frac{\Xi_i}{\Xi_j} X_V ; \quad X^\dagger p = \left[ \frac{(\nu_j)_{\text{mas}}}{(\nu_i)_{\text{mas}}} \right]^2 X_p ; \quad X^\dagger T = X_T ; \quad (51)
\]

which links the variables, \( (X^\dagger V, X^\dagger p, X^\dagger T) \), to \( (X_V, X_p, X_T) \), and vice versa.

Strictly speaking, the macrogas equation of state should be deduced from dimensional (instead of dimensionless) virial equations for an assigned subsystem, as outlined in Appendix B. On the other hand, a description in terms of dimensionless variables turns out to be more useful.

If the interaction terms are omitted, \( w^{(\text{int})} = w^{(\text{ext})} = 0 \), Eqs. (50a) and (50b) reduce to:

\[
X^\dagger p X^\dagger V = K^\dagger(\Xi_i, \Xi_j) X^\dagger T ; \quad (52a)
\]
\[ X_p X_v = K(\Xi_i, \Xi_j) X_T \; \]  

which may be considered as equation of state of ideal macrogases, where “ideal” means “the interaction terms are omitted”.

The parameters, \( K^\dagger \) and \( K \), appearing in either macrogas equation of state, depend on the scaled truncation radii, \( \Xi_i \) and \( \Xi_j \), and on the selected density profiles. In other words, the macrogas equation of state is not universal, but takes a different form for different density profiles. A restricted number of special cases shall be studied below, grounding on earlier results (CV08), to which an interested reader is addressed for further details. In any case, the following method shall be used: (i) select two density profiles; (ii) fix related scaled truncation radii, \( \Xi_i \) and \( \Xi_j \); (iii) choose a macrotemperature, \( \phi \); (iv) plot related macroisothermal curves, by solving Eq. (50a) or (50b).

### 3.1 UU macrogases

The related density profiles maintain uniform, which is equivalent to polytropes with index, \( n = 0 \) (e.g., Chandrasekhar, 1939, Chap. IV, §4; Caimmi, 1986), but implies negative distribution functions for stellar fluids (Vandervoort, 1980). The particularization of the general expressions to the case under discussion, yields for the quantities of interest (CV08):

\[
\begin{align*}
  f_u(\xi_u) &= 1 \ ; \quad 0 \leq \xi_u \leq \Xi_u \ ; \quad u = i, j \ ; \\
  F_u(\xi_u) &= \Xi_u^2 - \xi_u^2 \ ; \quad u = i, j \ ; \\
  (\nu_u)_{\text{mas}} &= \Xi_u^3 \ ; \quad u = i, j \ ; \\
  (\nu_u)_{\text{sel}} &= \frac{3}{10} \Xi_u^5 \ ; \quad u = i, j \ ; \\
  w^{(\text{int})}(\eta) &= -\frac{4}{15} \Xi_u^2 \eta^3 \left( \frac{5}{2} \eta^2 - \frac{3}{2} \right) \ ; \\
  w^{(\text{ext})}(\eta) &= -\frac{4}{15} \Xi_u^2 \eta^3 \ ;
\end{align*}
\]

where Eqs. (57) and (58) hold under the additional restriction of similar and similarly placed boundaries (CV08).

In the case under consideration of uniform density profiles, without loss of generality, it can be assumed a scaled truncation radius, \( \Xi_u = R_u/r_u^\dagger = 1 \), which implies \( y = y^\dagger \), \( m = m^\dagger \), due to Eqs. (12) and (55). Accordingly,
Eq. (46) reduces to (CV08):

$$\phi = \frac{(m')^2}{y'} \left( \frac{y}{y'} \right)^5 \frac{1 + \frac{(y')^3}{m'^2} \left( \frac{5}{2} y'^2 - 3 \right)}{1 + \frac{m'}{(y')^2}} ; \quad y \geq 1 ; \quad (59a)$$

$$\phi = \frac{m'(y')^2}{1 + m'(\frac{y}{y'})^5} \left[ 1 + \frac{m'}{(y')^2} \right] \left( \frac{5}{2} - 3 y'^2 \right) ; \quad 0 \leq y \leq 1 ; \quad (59b)$$

where $y$ is the outer to inner ellipsoid axis ratio, $R_j / R_i$, according to Eq. (42b).

In addition, Eqs. (48) and (49) reduce to:

$$m = \sqrt{\beta^2 + y \phi - \beta} ; \quad (60a)$$

$$\beta = \frac{1}{2} \frac{1}{y^2} \left( \frac{5}{2} y^2 - \frac{3}{2} - \phi \right) ; \quad y \geq 1 ; \quad (60b)$$

$$\beta = \frac{1}{2} y^3 \left[ 1 - \left( \frac{5}{2} \phi^2 - \frac{3}{2} \right) \phi \right] ; \quad 0 \leq y \leq 1 ; \quad (60c)$$

$$A = \frac{3}{10} ; \quad C = -\frac{3}{10} y \phi ; \quad B = \frac{3}{5} \beta ; \quad (60d)$$

where the negative solution is not considered due to the lack of physical meaning.

The explicit expression of the square fractional mass, $m^2$, extracted from Eq. (60a) in dimensionless variables, $X_p = X'_p = m^2 = (m')^2$, $X_V = X'_V = 1/y = 1/y'$, $X_T = X'_T = \phi$, Eqs. (50d), (50e), reads:

$$X_p = 2 \beta^2 + \frac{X_T}{X_V} - 2 \beta \sqrt{\beta^2 + \frac{X_T}{X_V}} ; \quad (61a)$$

$$2 \beta = X_V^2 \left( \frac{5}{2} \phi^2 - \frac{3}{2} - X_T \right) ; \quad 0 < X_V \leq 1 ; \quad (61b)$$

$$2 \beta = \frac{1}{X_V^3} \left[ 1 - \left( \frac{5}{2} \phi^2 - \frac{3}{2} \right) X_T \right] ; \quad X_V \geq 1 ; \quad (61c)$$

or, more explicitly:

$$X_p = 2 \frac{1}{4} X_V^4 \left( \frac{5}{2} \frac{1}{X_V^2} - \frac{3}{2} - X_T \right)^2 + \frac{X_T}{X_V} - X_V^2 \left( \frac{5}{2} \frac{1}{X_V^2} - \frac{3}{2} - X_T \right)$$

$$\cdot \left[ \frac{1}{4} X_V^4 \left( \frac{5}{2} \frac{1}{X_V^2} - \frac{3}{2} - X_T \right)^2 + \frac{X_T}{X_V} \right]^{1/2} ; \quad 0 < X_V \leq 1 ; \quad (62a)$$
\[
X_p = 2^{1/4} \frac{1}{X_V^4} \left[ 1 - \left( \frac{5}{2} X_V^2 - \frac{3}{2} \right) X_T \right]^2 + \frac{X_T}{X_V} - \frac{1}{X_V^4} \left[ 1 - \left( \frac{5}{2} X_V^2 - \frac{3}{2} \right) X_T \right] \]
\[
\cdot \left\{ \frac{1}{4} \frac{1}{X_V^4} \left[ 1 - \left( \frac{5}{2} X_V^2 - \frac{3}{2} \right) X_T \right]^2 + \frac{X_T}{X_V} \right\}^{1/2}; \quad X_V \geq 1; \quad (62b)
\]

which is the actual UU (AUU) macrogas equation of state.

The ideal UU (IUU) macrogas equation of state is obtained by the combination of Eqs. (50e), (52b), (55), and (56). The result is:

\[
X_p = \frac{X_T}{X_V}; \quad (63)
\]

which represents a hyperbola with equal axes, for fixed \(X_T\).

Macroisothermal curves related to IUU (tidal potential energy excluded) and AUU (tidal potential energy included) macrogases, are plotted in Fig. 4 left and right panel, respectively, for values of the macrotemperature, \(X_T = 0.85, 0.90, 0.95, 1.00, 1.05, 1.10\), from bottom to top. The coordinates, \(X_V = X_V^\dagger, X_p = X_p^\dagger, X_T = X_T^\dagger\), may be conceived as normalized to their fictitious critical counterparts, \(X_V^c = X_V^\dagger = 1, X_p^c = X_p^\dagger = 1, X_T^c = X_T^\dagger = 1\), as \(\phi = m = m^\dagger\) for \(y = y^\dagger = 1\), according to Eqs. (59) or (123), which implies \(\phi = 1\) for \(m = m^\dagger = 1\). The comparison with ideal and VDW gases, plotted in Fig. 1 shows a similar trend, except the absence of a critical macroisothermal curve, above which the extremum points disappear.

Contrary to ordinary gases, no experiment can be performed on macrogases to ascertain the existence of a phase transition moving along a selected macroisothermal curve, where the path is a horizontal line instead of a curve including the extremum points. Then the existence of the above mentioned phase transition and flat real macroisothermal curves, must necessarily be assumed as a working hypothesis, by analogy with VDW isothermal curves (below the critical one). The loci of extremum points of AUU macroisothermal curves plotted in Fig. 4 (right panel), are represented as dotted lines in Fig. 5.

Unlike the VDW equation of state, Eq. (14), the AUU macrogas equation of state, Eq. (62), is not analytically integrable. Then the procedure used for determining a selected macroisothermal curve, must be numerically performed. The main steps are (i) calculate the intersections, \(X_{V_A}, X_{V_C}; X_{V_k}, X_{V_A} < X_{V_C} < X_{V_k}\), between the generic horizontal line in the \((0X_VX_P)\) plane, \(X_p = \text{const}\), and the AUU macrogas equation of state, within the range, \(X_{P_B} < X_p < X_{P_D}\), where \(B\) and \(D\) denote the extremum points of minimum and maximum, respectively; (ii) calculate the area of the regions, \(ABC\) and \(CDE\); (iii) find the special value, \(X_p = X_{p_0}\), which makes the
two areas equal; (iv) trace the real UU (RUU) macroisothermal curve as a horizontal line connecting the points, \((X_{V_A}, X_{p_A}), (X_{V_C}, X_{p_C}), (X_{V_E}, X_{p_E})\), \(X_{p_A} = X_{p_C} = X_{p_E} = X_{p_0}\).

The loci of the intersections between AUU and RUU macroisothermal curves, are represented as dashed lines in Fig.5. The absence of a critical macroisothermal curve makes a band-like instead of a bell-shaped region exist on the \((OX_Xp)\) plane, as shown by comparison with Fig.2. The AUU and RUU macroisothermal curves are plotted in Fig.6, with regard to the special case, \(X_T = 0\).

Values of parameters, \(X_T, X_{V_A}, X_{V_B}, X_{V_C}, X_{V_D}, X_{V_E}, X_{p_B}, X_{p_C}, X_{p_D}\), are listed in Tab.2 within the range, \(0.85 \leq X_T \leq 1.10\), using a step, \(\Delta X_T = 0.01\).

3.2 HH macrogases

The related density profiles exhibit a central cusp and null value at infinite distances (Hernquist, 1990), and have been proved to be consistent with nonnegative distribution functions, in an acceptable parameter range (Ciotti, 1996). The particularization of the general expressions to the case under discussion, yields for the quantities of interest (CV08):

\[
f_u(\xi_u) = \frac{8}{\xi_u(1 + \xi_u)^3} ; \quad 0 \leq \xi_u \leq \Xi_u ; \quad u = i, j ; \quad (64)
\]

\[
F_u(\xi_u) = \frac{8}{(1 + \xi_u)^2} - \frac{8}{(1 + \Xi_u)^2} ; \quad u = i, j ; \quad (65)
\]

\[
(\nu_u)_{\text{mas}} = \frac{12 \Xi_u^3}{(1 + \Xi_u)^2} ; \quad u = i, j ; \quad (66)
\]

\[
(\nu_u)_{\text{sel}} = \frac{12 \Xi_u^3(4 + \Xi_u)}{(1 + \Xi_u)^4} ; \quad u = i, j ; \quad (67)
\]

\[
w^{\text{(int)}}(\eta) = -128y^\dagger \left\{ \frac{1}{2} \left( \frac{1}{(y^\dagger - 1)^4} \right) \left[ \frac{(y^\dagger - 1)^2 y^\dagger \eta}{(y^\dagger \eta + 1)^2} + \frac{2(y^\dagger - 1)\eta}{1 + \eta} \right] + \frac{(y^\dagger - 1)(y^\dagger + 3)y^\dagger \eta}{y^\dagger \eta + 1} \right\} + 2(2y^\dagger + 1) \ln \left( \frac{\eta + 1}{y^\dagger \eta + 1} \right) - \frac{1}{2} \frac{1}{(1 + \Xi_u)^2} \left( \frac{1}{y^\dagger} \right)^2 \left[ 1 - \frac{2y^\dagger \eta + 1}{(y^\dagger \eta + 1)^2} \right] \right\} ; \quad y^\dagger \neq 1 ; \quad (68a)
\]

\[
w^{\text{(int)}}(\eta) = -128 \left\{ \frac{1}{12} \left[ -\frac{4\eta + 1}{(\eta + 1)^4} + 1 \right] \right\} - \frac{1}{2} \frac{1}{(1 + \Xi_u)^2} \left( \frac{\eta^2}{(\eta + 1)^2} \right) ; \quad y^\dagger = 1 ; \quad (68b)
\]
Table 2: Values of parameters, $X_T$, $X_{Va}$, $X_{Vb}$, $X_{Vc}$, $X_{Vd}$, $X_{Ve}$, $X_{pB}$, $X_{pC}$, $X_{pD}$ (to be conceived as normalized to their fictitious critical counterparts, $X_{Vc} = 1$, $X_{pC} = 1$, $X_{Te} = 1$), within the range, $0.85 \leq X_T \leq 1.10$, using a step, $\Delta X_T = 0.01$. Index captions: A, C, E - intersections between AUU and RUU macroisothermal curves; B - extremum point of minimum; D - extremum point of maximum. Extremum points are related to AUU macroisothermal curves, while their RUU counterparts are flat within the range, $X_{Va} \leq X_V \leq X_{Ve}$. For aesthetical reasons, 01 on head columns stands for unity.

| $X_T$ | $10X_{Va}$ | $10X_{Vb}$ | $01X_{Vc}$ | $01X_{Vd}$ | $01X_{Ve}$ | $10X_{pB}$ | $01X_{pC}$ | $01X_{pD}$ |
|-------|------------|------------|------------|------------|------------|------------|------------|------------|
| 0.85  | 2.4980     | 6.6478     | 1.1115     | 1.5541     | 2.5993     | 3.7895     | 1.0228     | 1.4699     |
| 0.86  | 2.4939     | 6.6407     | 1.1105     | 1.5521     | 2.5952     | 3.8711     | 1.0440     | 1.5008     |
| 0.87  | 2.4899     | 6.6337     | 1.1094     | 1.5501     | 2.5912     | 3.9534     | 1.0654     | 1.5320     |
| 0.88  | 2.4858     | 6.6267     | 1.1084     | 1.5482     | 2.5873     | 4.0364     | 1.0870     | 1.5635     |
| 0.89  | 2.4818     | 6.6197     | 1.1073     | 1.5463     | 2.5836     | 4.1201     | 1.1087     | 1.5953     |
| 0.90  | 2.4776     | 6.6127     | 1.1064     | 1.5445     | 2.5798     | 4.2046     | 1.1307     | 1.6275     |
| 0.91  | 2.4735     | 6.6059     | 1.1054     | 1.5427     | 2.5762     | 4.2897     | 1.1529     | 1.6599     |
| 0.92  | 2.4693     | 6.5991     | 1.1044     | 1.5409     | 2.5725     | 4.3755     | 1.1753     | 1.6927     |
| 0.93  | 2.4653     | 6.5922     | 1.1034     | 1.5392     | 2.5692     | 4.4620     | 1.1979     | 1.7258     |
| 0.94  | 2.4610     | 6.5854     | 1.1025     | 1.5374     | 2.5657     | 4.5492     | 1.2207     | 1.7591     |
| 0.95  | 2.4569     | 6.5786     | 1.1016     | 1.5358     | 2.5623     | 4.6371     | 1.2437     | 1.7928     |
| 0.96  | 2.4526     | 6.5719     | 1.1007     | 1.5341     | 2.5590     | 4.7256     | 1.2669     | 1.8268     |
| 0.97  | 2.4483     | 6.5653     | 1.0998     | 1.5325     | 2.5557     | 4.8149     | 1.2903     | 1.8611     |
| 0.98  | 2.4442     | 6.5587     | 1.0989     | 1.5309     | 2.5527     | 4.9048     | 1.3138     | 1.8958     |
| 0.99  | 2.4398     | 6.5520     | 1.0981     | 1.5293     | 2.5495     | 4.9954     | 1.3377     | 1.9307     |
| 1.00  | 2.4356     | 6.5455     | 1.0972     | 1.5278     | 2.5465     | 5.0866     | 1.3616     | 1.9659     |
| 1.01  | 2.4313     | 6.5389     | 1.0964     | 1.5263     | 2.5435     | 5.1785     | 1.3858     | 2.0015     |
| 1.02  | 2.4271     | 6.5328     | 1.0955     | 1.5248     | 2.5407     | 5.2711     | 1.4101     | 2.0374     |
| 1.03  | 2.4228     | 6.5260     | 1.0947     | 1.5233     | 2.5378     | 5.3643     | 1.4347     | 2.0735     |
| 1.04  | 2.4184     | 6.5195     | 1.0939     | 1.5219     | 2.5349     | 5.4582     | 1.4595     | 2.1100     |
| 1.05  | 2.4142     | 6.5131     | 1.0931     | 1.5204     | 2.5323     | 5.5528     | 1.4844     | 2.1468     |
| 1.06  | 2.4098     | 6.5068     | 1.0924     | 1.5191     | 2.5295     | 5.6480     | 1.5097     | 2.1839     |
| 1.07  | 2.4055     | 6.5004     | 1.0916     | 1.5177     | 2.5269     | 5.7438     | 1.5350     | 2.2213     |
| 1.08  | 2.4011     | 6.4942     | 1.0908     | 1.5163     | 2.5242     | 5.8403     | 1.5606     | 2.2590     |
| 1.09  | 2.3969     | 6.4879     | 1.0901     | 1.5150     | 2.5217     | 5.9374     | 1.5863     | 2.2971     |
| 1.10  | 2.3925     | 6.4816     | 1.0894     | 1.5137     | 2.5192     | 6.0351     | 1.6123     | 2.3354     |

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\[ w(\text{ext})(\eta) = -128 \left\{ -\frac{1}{2} \frac{1}{(y^\dagger - 1)^4} \left[ (y^\dagger - 1)^2 \eta - (y^\dagger)^2 (y^\dagger - 1) \eta \right] + \frac{1}{2} \frac{1}{(1 + \Xi i)^2 (\eta + 1)^2} \right\} ; \quad y^\dagger \neq 1 ; \] 
\[ w(\text{ext})(\eta) = 128 \left\{ \frac{1}{12} \left[ \frac{4\eta + 1}{(\eta + 1)^2} - 1 \right] + \frac{1}{2} \frac{1}{(1 + \Xi i)^2 (\eta + 1)^2} \right\} ; \quad y^\dagger = 1 ; \] 

using Eqs. (44) and (66)-(69), the actual HH (AHH) macrogas equation of state is obtained from the particularization of Eqs. (50) to the case of interest for the domain, \( y \geq 1 \). The extension to the domain, \( 0 \leq y \leq 1 \), can be done following the procedure outlined above in dealing with Eq. (46).

In absence of truncation radius, \( \Xi \to +\infty, \eta \to +\infty \), and Eqs. (65)-(69) reduce to (CV08):

\[ \lim_{\Xi \to +\infty} F_u(\xi_u) = \frac{8}{(1 + \xi_u)^2} ; \quad u = i, j ; \] 
\[ \lim_{\Xi \to +\infty} (\nu_u)_{\text{mas}} = 12 ; \quad u = i, j ; \] 
\[ \lim_{\Xi \to +\infty} (\nu_u)_{\text{sel}} = 12 ; \quad u = i, j ; \] 

\[ \lim_{\eta \to +\infty} w(\text{int})(\eta) = \frac{64 y^\dagger}{(y^\dagger - 1)^5} \left[ -2(2y^\dagger + 1) \ln y^\dagger + (y^\dagger - 1)(y^\dagger + 5) \right] ; \quad y^\dagger \neq 1 ; \] 
\[ \lim_{\eta \to +\infty} w(\text{int})(\eta) = -\frac{32}{3} ; \quad y^\dagger = 1 ; \] 

\[ \lim_{\eta \to +\infty} w(\text{ext})(\eta) = -\frac{64}{(y^\dagger - 1)^5} \left[ 2y^\dagger (y^\dagger + 2) \ln y^\dagger - (y^\dagger - 1)(5y^\dagger + 1) \right] ; \quad y^\dagger \neq 1 ; \] 
\[ \lim_{\eta \to +\infty} w(\text{ext})(\eta) = -\frac{32}{3} ; \quad y^\dagger = 1 ; \]
using Eqs. (44) and (71)-(74), the AHH macrogas equation of state in the special situation under discussion, is obtained from the particularization of Eqs. (50) to the case of interest.

The ideal situation, where the interaction terms are omitted, is obtained using Eqs. (52) instead of Eqs. (50). More specifically, the ideal HH (IHH) macrogas equation of state is derived by the combination of Eqs. (50a), (52b), (66), and (67). The result is:

$$X_p = \frac{4 + \Xi_i}{4 + \Xi_j} X_T$$

which represents a hyperbola with different axes (unless $\Xi_i = \Xi_j$), for fixed $X_T$.

Macroisothermal curves ($X_p = X_p/X_{pe}$ vs. $X_V = X_V/X_{Ve}$) related to IHH (tidal potential energy excluded) and AHH (tidal potential energy included) macrogases, are plotted in Fig. 7 left and right panels, for different values of scaled truncation radii, ($\Xi_i$, $\Xi_j$), labelled on each panel, and same values of the reduced macrotemperature, $X_T = X_T/X_{T_c} = 0.90, 0.95, 1.00, 1.05, 1.10, 1.15$, from bottom to top. The limit, $(\Xi_i, \Xi_j) \to (+\infty, +\infty)$, makes only little changes. The comparison with ideal and VDW gases, plotted in Fig. 1 shows a similar but reversed trend. More specifically, extremum points occur above, instead of below, the critical macroisothermal curve. A complete analogy can be obtained using the transformations, $X_V \to 1/X_V$, $X_p \to 1/X_p$, $X_T \to 1/X_T$.

The existence of a phase transition moving along a selected macroisothermal curve, where the path is a horizontal line instead of a curve including the extremum points, must necessarily be assumed as a working hypothesis, due to the analogy between VDW isothermal curves and AHH macroisothermal curves. As in the case of UU macrogases, AHH macroisothermal curves must be numerically determined, following the same procedure outlined in Subsection 3.1. Characteristic loci of AHH macroisothermal curves plotted in Fig. 8 (right panels), are represented in Fig. 8. The loci of intersections between AHH and real HH (RHH) macroisothermal curves, are represented in Fig. 8 as trifid curves, where the left branch corresponds to $X_{VA}$, the right branch to $X_{VE}$, and the middle branch to $X_{VC}$. The critical point is the common starting point. The loci of AHH macroisothermal curve extremum points are represented in Fig. 8 as dotted curves starting from the critical point, where the left branch corresponds to minimum points and the right branch to maximum points. The RHH macroisothermal curves, when different from their AHH counterparts, lie within the larger bell-shaped regions. The limit, $(\Xi_i, \Xi_j) \to (+\infty, +\infty)$, makes only little changes. Values of parameters, $X_T$, $X_{VA}$, $X_{VB}$, $X_{VC}$, $X_{VD}$, $X_{VE}$, $X_{pB}$, $X_{pC}$, $X_{pD}$, are listed in Tab. 3.
within the range, $1.00 < X_T \leq 1.15$, using a step, $\Delta X_T = 0.01$, in the limit, $(\Xi_i, \Xi_j) \to (+\infty, +\infty)$. The AHH macroisothermal curves are plotted in Fig. 9 for scaled truncation radii, $(\Xi_i, \Xi_j)$, as in Fig. 8 with regard to the special case, $X_T = 1.15$. The limit, $(\Xi_i, \Xi_j) \to (+\infty, +\infty)$, makes only little changes, as shown by comparison with the dashed curve.

Values of parameters, $m^\dagger, m, y^\dagger, y, \phi$, related to the critical macroisothermal curve, for selected scaled truncation radii, $(\Xi_i, \Xi_j)$, are listed in Tab. 4.

A better method has been used with respect to an earlier attempt (CV08), yielding improved results.

### 3.3 HN/NH macrogases

A description of H density profiles has been provided in Subsection 3.2. The remaining N density profiles also exhibit a central cusp and null values at infinite distances (Navarro et al., 1995, 1996, 1997). Mass distributions defined by an inner H and outer N density profile were found to be self-consistent, in an acceptable parameter range, with regard to the non negativity of the distribution function (Lowenstein and White, 1999), using a theorem stated in an earlier attempt (Ciotti and Pellegrini, 1992). The particularization of the general expressions to N density profiles and HN macrogases, yields for the quantities of interest (CV08):

\[ f_u(\xi_u) = \frac{4}{\xi_u(1 + \xi_u)^2} ; \quad 0 \leq \xi_u \leq \Xi_u ; \quad u = N ; \quad (76) \]

\[ F_u(\xi_u) = \frac{8}{1 + \xi_u} - \frac{8}{1 + \Xi_u} ; \quad u = N ; \quad (77) \]

\[ (\nu_u)_{mas} = 12 \left[ \ln(1 + \Xi_u) - \frac{\Xi_u}{1 + \Xi_u} \right] ; \quad u = N ; \quad (78) \]

\[ (\nu_u)_{sel} = 36 \frac{\Xi_u(2 + \Xi_u) - 2(1 + \Xi_u) \ln(1 + \Xi_u)}{(1 + \Xi_u)^2} ; \quad (79) \]

\[ w^{(int)}(\eta) = -\frac{64y^\dagger}{(y^\dagger - 1)^3} \left[ -\frac{(y^\dagger - 1)^2 y^\dagger \eta(y^\dagger \eta + 2)}{(y^\dagger \eta + 1)^2} + \frac{2y^\dagger \eta(y^\dagger - 1)}{y^\dagger \eta + 1} \right. \]

\[ \left. + 2 \ln \frac{\eta + 1}{y^\dagger \eta + 1} - \frac{(y^\dagger - 1)^3 \eta^2}{1 + \Xi_j (y^\dagger \eta + 1)^2} \right] ; \quad y^\dagger \neq 1 ; \quad i = H ; \quad j = N ; \quad (80a) \]

\[ w^{(int)}(\eta) = -\frac{64\eta^2}{(\eta + 1)^2} \left[ \frac{\eta + 3}{3(\eta + 1)} - \frac{1}{1 + \Xi_j} \right] ; \quad y^\dagger = 1 ; \quad i = H ; \quad j = N ; \quad (80b) \]
Table 3: Values of parameters, $X_T$, $X_{V_A}$, $X_{V_B}$, $X_{V_C}$, $X_{V_D}$, $X_{V_E}$, $X_{p_B}$, $X_{p_C}$, $X_{p_D}$, within the range, $1.00 < X_T \leq 1.15$, using a step, $\Delta X_T = 0.01$, except near the critical point, where convergence was not attained. All values equal unity at the critical point. Results are related to infinitely extended configurations, $(\Xi_i, \Xi_j) \to (+\infty, +\infty)$. Index captions: A, C, E - intersections between AHH and RHH macroisothermal curves; B - extremum point of minimum; D - extremum point of maximum. Extremum points are related to AHH macroisothermal curves, while their RHH counterparts are flat within the range, $X_{V_A} \leq X_V \leq X_{V_E}$. For aesthetical reasons, 01 on head columns stands for unity.

| $X_T$ | 10$X_{V_A}$ | 10$X_{V_B}$ | 10$X_{V_C}$ | 01$X_{V_D}$ | 01$X_{V_E}$ | 01$X_{p_B}$ | 01$X_{p_C}$ | 01$X_{p_D}$ |
|-------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1.014 | 8.3150      | 8.9774      | 9.9858      | 1.1016      | 1.1854      | 1.0264      | 1.0266      | 1.0268      |
| 1.015 | 8.2595      | 8.9420      | 9.9849      | 1.1051      | 1.1921      | 1.0283      | 1.0285      | 1.0287      |
| 1.02  | 8.0099      | 8.7810      | 9.9803      | 1.1211      | 1.2231      | 1.0378      | 1.0381      | 1.0384      |
| 1.03  | 7.6038      | 8.5136      | 9.9711      | 1.1475      | 1.2756      | 1.0568      | 1.0575      | 1.0580      |
| 1.04  | 7.2739      | 8.2912      | 9.9620      | 1.1694      | 1.3203      | 1.0759      | 1.0770      | 1.0778      |
| 1.05  | 6.9927      | 8.0979      | 9.9533      | 1.1884      | 1.3600      | 1.0951      | 1.0966      | 1.0978      |
| 1.06  | 6.7463      | 7.9255      | 9.9446      | 1.2054      | 1.3961      | 1.1144      | 1.1165      | 1.1181      |
| 1.07  | 6.5262      | 7.7690      | 9.9359      | 1.2207      | 1.4293      | 1.1338      | 1.1365      | 1.1385      |
| 1.08  | 6.3269      | 7.6252      | 9.9277      | 1.2348      | 1.4604      | 1.1533      | 1.1566      | 1.1591      |
| 1.09  | 6.1447      | 7.4920      | 9.9192      | 1.2478      | 1.4895      | 1.1730      | 1.1770      | 1.1799      |
| 1.10  | 5.9767      | 7.3676      | 9.9112      | 1.2600      | 1.5171      | 1.1927      | 1.1975      | 1.2009      |
| 1.11  | 5.8209      | 7.2508      | 9.9031      | 1.2714      | 1.5434      | 1.2126      | 1.2182      | 1.2222      |
| 1.12  | 5.6755      | 7.1406      | 9.8950      | 1.2821      | 1.5685      | 1.2325      | 1.2391      | 1.2436      |
| 1.13  | 5.5393      | 7.0362      | 9.8872      | 1.2923      | 1.5925      | 1.2526      | 1.2601      | 1.2652      |
| 1.14  | 5.4111      | 6.9370      | 9.8796      | 1.3019      | 1.6155      | 1.2728      | 1.2813      | 1.2870      |
| 1.15  | 5.2902      | 6.8425      | 9.8720      | 1.3111      | 1.6378      | 1.2931      | 1.3027      | 1.3090      |
Table 4: Values of the scaling fractional mass, $m^\ddagger$, the fractional mass, $m$, the scaling fractional radius, $y^\ddagger$, the fractional truncation radius, $y$, and the fractional energy, $\phi$, related to the critical point i.e. the horizontal inflexion point on the critical macroisothermal curve, for selected scaled truncation radii, $(\Xi_i, \Xi_j)$, of HH density profiles. In absence of truncation radius, $(\Xi_i, \Xi_j) \to (+\infty, +\infty)$, the results are independent of $y/y^\ddagger = \Xi_j/\Xi_i$.

| $(\Xi_i, \Xi_j)$ | $m^\ddagger$ | $m$ | $y^\ddagger$ | $y$ | $\phi$ |
|-----------------|--------------|-----|--------------|-----|-------|
| (05,05)         | 07.10        | 07.10 | 2.32         | 2.32 | 06.86 |
| (05,10)         | 07.72        | 09.18 | 2.43         | 4.86 | 08.17 |
| (05,20)         | 07.90        | 10.31 | 2.46         | 9.83 | 08.62 |
| (10,05)         | 11.15        | 09.37 | 3.18         | 1.59 | 09.44 |
| (10,10)         | 11.88        | 11.88 | 3.30         | 3.30 | 11.03 |
| (10,20)         | 12.10        | 13.28 | 3.33         | 6.67 | 11.57 |
| (20,05)         | 15.06        | 11.53 | 4.00         | 1.00 | 12.09 |
| (20,10)         | 15.83        | 14.42 | 3.88         | 1.94 | 13.94 |
| (20,20)         | 16.08        | 16.08 | 3.92         | 3.92 | 14.59 |
| $(\infty, \infty)$ | 20.22    | 20.22 | 4.26         |      | 18.15 |

\[ w^{(\text{ext})}(\eta) = -\frac{64}{(y^\ddagger - 1)^2} \left\{ -\frac{\eta}{\eta + 1} - \frac{y^\dagger \eta}{y^\dagger + 1} - \frac{y^\ddagger + 1}{y^\ddagger - 1} \ln \frac{\eta + 1}{y^\ddagger + 1} \right\} \]

\[ w^{(\text{ext})}(\eta) = -\frac{64}{(y^\ddagger - 1)^2} \left\{ -\frac{\eta}{\eta + 1} + \ln(\eta + 1) \right\} \]

\[ y^\ddagger \neq 1 ; \quad i = \text{H} ; \quad j = \text{N} ; \quad (81a) \]

\[ w^{(\text{int})}(\eta) = -\frac{1}{y^\ddagger} \left\{ \frac{\eta^2(\eta + 3)}{6(\eta + 1)^3} - \frac{1}{(1 + \Xi)^2} \left[ -\frac{\eta}{\eta + 1} + \ln(\eta + 1) \right] \right\} \]

\[ y^\dagger = 1 ; \quad i = \text{H} ; \quad j = \text{N} ; \quad (81b) \]

using Eqs. (44), (66), (67), and (78)-(81), the HN macrogas equation of state is obtained from the particularization of Eqs. (50) to the case of interest for the domain, $y \geq 1$. The extension to the domain, $0 \leq y \leq 1$, can be done following the procedure outlined above in dealing with Eq. (46).

To this aim, the counterparts of Eqs. (80) and (81), related to NH macrogases ($i = N, j = H$), are needed. The particularization of Eqs. (15) to the case under discussion, yields:

\[ w^{(\text{int})}(\eta) = -\frac{64y^\dagger}{(y^\ddagger - 1)^2} \left\{ -\frac{\eta}{1 + \eta} - \frac{\Xi_i}{1 + \Xi_i} - \frac{y^\ddagger + 1}{y^\ddagger - 1} \ln \frac{1 + \eta}{1 + \Xi_i} \right\} \]
\[-\frac{(y^\dagger - 1)^2}{(y^\dagger)^2} \frac{1}{(1 + \Xi_j)^2} \left\{ \ln(1 + \Xi_j) - \frac{\Xi_j}{1 + \Xi_j} \right\} \right\};
\]
\[y^\dagger \neq 1 ; \quad i = N ; \quad j = H ; \quad (82a)\]

\[w^{(\text{int})}(\eta) = -64 \left\{ \frac{1}{6 (1 + \eta)^3} - \frac{1}{(1 + \Xi_i)^2} \left[ \ln(1 + \eta) - \frac{\eta}{1 + \eta} \right] \right\} ;
\]
\[y^\dagger = 1 ; \quad i = N ; \quad j = H ; \quad (82b)\]

\[w^{(\text{ext})}(\eta) = -\frac{64}{(y^\dagger - 1)^3} \left\{ \frac{2(y^\dagger - 1)\eta}{1 + \eta} + \frac{(y^\dagger - 1)^2\eta(2 + \eta)}{(1 + \eta)^2} \right\} + 2y^\dagger \ln \frac{1 + \eta}{1 + \Xi_i} - \frac{(y^\dagger - 1)^3\eta^2}{1 + \Xi_i (1 + \eta)^2} \right\};
\]
\[y^\dagger \neq 1 ; \quad i = N ; \quad j = H ; \quad (83a)\]

\[w^{(\text{ext})}(\eta) = -64 \left\{ \frac{1}{3 (1 + \eta)^3} - \frac{1}{1 + \Xi (1 + \eta)^2} \right\} ;
\]
\[y^\dagger = 1 ; \quad i = N ; \quad j = H ; \quad (83b)\]

where \(\Xi_i = \Xi_N, \Xi_j = \Xi_H,\) while the contrary holds with regard to Eqs. (80) and (81). Using Eqs. (11), (66), (67), (78), (79), (82) and (83), the NH macrogas equation of state is obtained from the particularization of Eqs. (50) to the case of interest for the domain, \(y \geq 1,\) which corresponds to \(0 \leq y \leq 1\) for HN macrogases and vice versa.

In absence of truncation radius, \(\Xi \to +\infty, \eta \to +\infty,\) and Eqs. (77)-(81) reduce to:

\[\lim_{\Xi_u \to +\infty} F_u(\xi_u) = \frac{8}{1 + \xi_u} ; \quad u = N ; \quad (84)\]

\[\lim_{\Xi_u \to +\infty} (\nu_u)_{\text{mas}} = +\infty ; \quad u = N ; \quad (85)\]

\[\lim_{\Xi_u \to +\infty} (\nu_u)_{\text{sel}} = 36 ; \quad u = N ; \quad (86)\]

\[\lim_{\eta \to +\infty} w^{(\text{int})}(\eta) = -\frac{64}{(y^\dagger - 1)^3} \left[ (y^\dagger)^2 - 1 - 2y^\dagger \ln y^\dagger \right] ; \quad y^\dagger \neq 1 ; (87a)\]

\[\lim_{\eta \to +\infty} w^{(\text{int})}(\eta) = -\frac{64}{3} ; \quad y^\dagger = 1 ; \quad (87b)\]

\[\lim_{\eta \to +\infty} w^{(\text{ext})}(\eta) = -\frac{64}{(y^\dagger - 1)^2} \left[ \frac{y^\dagger + 1}{y^\dagger - 1} \ln y^\dagger - 2 \right] ; \quad y^\dagger \neq 1 ; \quad (88a)\]

\[\lim_{\eta \to +\infty} w^{(\text{ext})}(\eta) = -\frac{32}{3} ; \quad y^\dagger = 1 ; \quad (88b)\]
where the self potential-energy profile factor remains finite, although the mass profile factor undergoes a logarithmic divergence. Similarly, Eqs. (82) and (83) reduce to:

\[
\lim_{\eta \to +\infty} w^{(\text{int})}(\eta) = \frac{-64 y^{\dagger}}{(y^{\dagger} - 1)^2} \left[ \frac{y^{\dagger} + 1}{y^{\dagger} - 1} \ln y^{\dagger} - 2 \right] ; \quad y^{\dagger} \neq 1 ; \quad (89a)
\]

\[
\lim_{\eta \to +\infty} w^{(\text{int})}(\eta) = \frac{-32}{3} ; \quad y^{\dagger} = 1 ; \quad (89b)
\]

\[
\lim_{\eta \to +\infty} w^{(\text{ext})}(\eta) = \frac{-64}{3} \left[ (y^{\dagger})^2 - 1 - 2y^{\dagger} \ln y^{\dagger} \right] ; \quad y^{\dagger} \neq 1 ; \quad (90a)
\]

\[
\lim_{\eta \to +\infty} w^{(\text{ext})}(\eta) = \frac{-64}{3} ; \quad y^{\dagger} = 1 ; \quad (90b)
\]

where \( \Xi_i = \Xi_N, \) \( \Xi_j = \Xi_H, \) while the contrary holds with regard to Eqs. (87) and (88). Using Eqs. (44), (66), (67), (71), (72), and (78)-(90), the actual HN (AHN) and NH (ANH) macrogas equation of state in the special situation under discussion, is obtained from the particularization of Eqs. (50) to the case of interest.

The ideal situation, where the interaction terms are omitted, is obtained using Eqs. (52) instead of (50). More specifically, the ideal HN (IHN) and NH (INH) macrogas equation of state is obtained from the combination of Eqs. (50c), (52b), (66), (67), (78), and (79). The result is:

\[
X_p = \frac{14 + \Xi_H}{3} \frac{(1 + \Xi_N) \ln(1 + \Xi_N) - \Xi_N^2}{\Xi_N(2 + \Xi_N) - 2(1 + \Xi_N) \ln(1 + \Xi_N)} \frac{X_T}{X_V} ; \quad (91)
\]

which represents a hyperbola with different axes for fixed \( X_T. \) In the limit of infinite extension, \( \Xi \to +\infty, \) both the left and right-hand side of Eq. (91) diverge, but a different equation of state may be derived starting from Eqs. (50d) and (52a), following a similar procedure. The result is:

\[
X_p^\dagger = \frac{13\Xi_H^4(4 + \Xi_H)}{3} \frac{(1 + \Xi_N)^2}{\Xi_N(2 + \Xi_N) - 2(1 + \Xi_N) \ln(1 + \Xi_N)} \frac{X_T^\dagger}{X_V^\dagger} ; \quad (92)
\]

which also represents a hyperbola with different axes for fixed \( X_T^\dagger. \)

Macroisothermal curves (\( X_p = X_p/X_p c \) vs. \( X_V = X_V/X_V c \)) related to IHN/INH (tidal potential energy excluded) and AHN/ANH (tidal potential energy included) macrogases, are plotted in Fig. 10 left and right panels, for different values of scaled truncation radii, (\( \Xi_i, \Xi_j, \)) labelled on each panel, and same values of the reduced macrotemperature, \( X_T = X_T/X_T c = 0.90, 0.95, \)
The existence of a phase transition moving along a selected macroisothermal curve, where the path is a horizontal line instead of a curve including the extremum points, must necessarily be assumed as a working hypothesis, due to analogy between VDW isothermal curves and AHN/ANH macroisothermal curves. As in the case of UU macrogases, AHN/ANH macroisothermal curves must be numerically determined, following the same procedure outlined in Subsection 3.1. The loci of AHN/ANH macroisothermal curves plotted in Fig. 10 (right panels), are represented in Fig. 11. The loci of intersections between AHN/ANH and real HN/NH (RHN/RNH) macroisothermal curves, are represented in Fig. 11 as trifid curves, where the left branch corresponds to $X_{V_A}$, the right branch to $X_{V_E}$, and the middle branch to $X_{V_C}$. The critical point is the common starting point. The loci of AHN/ANH macroisothermal curve extremum points are represented in Fig. 11 as dotted curves starting from the critical point, where the left branch corresponds to minimum points and the right branch to maximum points. The RHN/RNH macroisothermal curves, when different from their AHN/ANH counterparts, lie within the larger bell-shaped regions. The limit, $(\Xi_i, \Xi_j) \rightarrow (+\infty, +\infty)$, makes only little changes, as shown by comparison with the dashed curve.

Values of parameters, $m^i, m, y^i, y, \phi$, related to the critical macroisothermal curve, for selected scaled truncation radii, $(\Xi_i, \Xi_j)$, are listed in Tab. 6. A better method has been used with respect to an earlier attempt (CV08), yielding improved results with the occurrence of the critical macroisothermal curve in all cases. In absence of truncation radius, $(\Xi_i, \Xi_j) \rightarrow (+\infty, +\infty)$, small differences appear in the results, probably due to (i) the divergence of the fractional mass, $m \rightarrow +\infty$, as $\Xi_N \rightarrow +\infty$, and (ii) numerical calculations have been performed, taking $\Xi \sim 10^{10}$. Related results listed in Table 6 may be considered as weighted means.
Table 5: Values of parameters, $X_T$, $X_{VA}$, $X_{Vb}$, $X_{VC}$, $X_{VD}$, $X_{VE}$, $X_{PB}$, $X_{PC}$, $X_{PD}$, within the range, $1.01 \leq X_T \leq 1.15$, using a step, $\Delta X_T = 0.01$. All values equal unity at the critical point. Results are related to infinitely extended configurations, $(\Xi_i, \Xi_j) \to (+\infty, +\infty)$. Index captions: A, C, E - intersections between AHN/ANH and RHN/RNH macroisothermal curves; B - extremum point of minimum; D - extremum point of maximum. Extremum points are related to AHN/ANH macroisothermal curves, while their RHN/RNH counterparts are flat within the range, $X_{VA} \leq X_V \leq X_{VE}$. For aesthetical reasons, 01 on head columns stands for unity.

| $X_T$ | 10$X_{VA}$ | 10$X_{Vb}$ | 10$X_{VC}$ | 01$X_{VD}$ | 01$X_{VE}$ | 01$X_{PB}$ | 01$X_{PC}$ | 01$X_{PD}$ |
|-------|------------|------------|------------|------------|------------|------------|------------|------------|
| 1.01  | 8.4036     | 9.0357     | 9.9949     | 1.0976     | 1.1768     | 1.1085     | 1.0186     | 1.0187     |
| 1.02  | 7.8103     | 8.6548     | 9.9935     | 1.1372     | 1.2535     | 1.0370     | 1.0374     | 1.0377     |
| 1.03  | 7.3771     | 8.3678     | 9.9918     | 1.1675     | 1.3139     | 1.0555     | 1.0563     | 1.0570     |
| 1.04  | 7.0273     | 8.1303     | 9.9903     | 1.1929     | 1.3656     | 1.0742     | 1.0754     | 1.0764     |
| 1.05  | 6.7309     | 7.9247     | 9.9888     | 1.2151     | 1.4118     | 1.0929     | 1.0947     | 1.0961     |
| 1.06  | 6.4724     | 7.7420     | 9.9874     | 1.2347     | 1.4540     | 1.1118     | 1.1142     | 1.1159     |
| 1.07  | 6.2427     | 7.5768     | 9.9859     | 1.2527     | 1.4930     | 1.1307     | 1.1338     | 1.1360     |
| 1.08  | 6.0356     | 7.4256     | 9.9844     | 1.2694     | 1.5295     | 1.1498     | 1.1535     | 1.1562     |
| 1.09  | 5.8488     | 7.2872     | 9.9830     | 1.2845     | 1.5637     | 1.1687     | 1.1733     | 1.1765     |
| 1.10  | 5.6738     | 7.1558     | 9.9815     | 1.2991     | 1.5967     | 1.1881     | 1.1936     | 1.1974     |
| 1.11  | 5.5137     | 7.0340     | 9.9797     | 1.3126     | 1.6279     | 1.2074     | 1.2139     | 1.2182     |
| 1.12  | 5.3647     | 6.9194     | 9.9785     | 1.3254     | 1.6577     | 1.2269     | 1.2343     | 1.2392     |
| 1.13  | 5.2256     | 6.8111     | 9.9770     | 1.3376     | 1.6864     | 1.2464     | 1.2549     | 1.2605     |
| 1.14  | 5.0952     | 6.7084     | 9.9754     | 1.3491     | 1.7139     | 1.2661     | 1.2757     | 1.2819     |
| 1.15  | 4.9726     | 6.6108     | 9.9737     | 1.3601     | 1.7406     | 1.2858     | 1.2967     | 1.3035     |
Table 6: Values of the scaling fractional mass, $m^\dagger$, the fractional mass, $m$, the scaling fractional radius, $y^\dagger$, the fractional truncation radius, $y$, and the fractional energy, $\phi$, related to the critical point i.e. the horizontal inflexion point on the critical macroisothermal curve, for selected scaled truncation radii, $(\Xi_i, \Xi_j)$, of HN/NH density profiles. In absence of truncation radius, $(\Xi_i, \Xi_j) \to (+\infty, +\infty)$, the results are independent of $y/y^\dagger = \Xi_j/\Xi_i$.

| $(\Xi_i, \Xi_j)$ | $m^\dagger$ | $m$  | $y^\dagger$ | $y$  | $\phi$ |
|------------------|-------------|------|-------------|------|-------|
| (05, 05)         | 02.57       | 03.54| 1.01        | 1.01 | 03.66 |
| (05, 10)         | 04.41       | 09.45| 1.04        | 2.08 | 09.16 |
| (05, 20)         | 05.14       | 15.47| 1.14        | 4.57 | 13.42 |
| (10, 05)         | 05.84       | 06.77| 1.88        | 0.94 | 07.33 |
| (10, 10)         | 06.36       | 11.45| 1.38        | 1.38 | 11.47 |
| (10, 20)         | 07.24       | 18.32| 1.50        | 3.00 | 16.46 |
| (20, 05)         | 09.40       | 09.93| 2.13        | 0.53 | 11.30 |
| (20, 10)         | 08.44       | 13.85| 1.92        | 0.96 | 14.35 |
| (20, 20)         | 09.10       | 20.99| 1.75        | 1.75 | 19.49 |
| $(\infty, \infty)$ | 12.40       | $+\infty$ | 2.06 | $\ldots$ | 35.82 |

4 Application to elliptical galaxies

The luminosity-weighted second moment of the line-of-sight velocity distribution within the half-light radius has recently been determined for samples of early-type galaxies, using integral-field spectroscopy such as SAURON (S IV; S X). Compared to the central velocity dispersion, which was sometimes used before, the above mentioned quantity has the advantage that it is only weakly dependent on the details of the aperture used. In addition, it is an approximation to the second velocity moment which appears in the virial equations (e.g., Binney and Tremaine, 1987, Chap. 4, §4.3). In other words, it may be conceived as a rms mass-weighted velocity, with a weak dependence on the features of the orbital distribution. For further details refer to the parent papers (S IV; S X).

An application of the current model to SAURON sample objects can be performed along the following steps: (i) select SAURON data of interest; (ii) calculate parameters appearing in the virial equations; (iii) make a correspondence between model galaxies and sample objects; (iv) represent model galaxies as points on the $(O_X V X_p)$ plane.
4.1 Data selection

The sample used (CV08, \(N = 16\)) is made of elliptical galaxies, extracted from richer samples of early-type galaxies investigated within the SAURON project (SIV, \(N = 25\); SX, \(N = 48\)). The selection has been restricted to elliptical galaxies common to the above mentioned samples for two main reasons, namely (i) the current model best applies to elliptical galaxies and their hosting haloes, and (ii) some physical parameters of interest are listed in either SIV or SX. The whole data set needed for the application is shown in Tab. 7. For further details refer to the parent papers (SIV, columns 2-6; SX, columns 7-11).

Values to be actually used in the application are related to: the effective (half-light) radius, \(R_e\) (SIV); the total observed \(I\) band galaxy magnitude, \(I_T\) (SIV); the mass-luminosity ratio of the stellar population, \(M_i/L\) (SIV); the galaxy distance modulus, \(\hat{m} - \hat{M}\) (hats avoid confusion with the fractional mass, \(m = M_j/M_i\), and the total mass, \(M = M_i + M_j\)) (SIV); the luminosity-weighted average ellipticity, \(\hat{e}_\perp\), on a plane perpendicular to the line of sight, within either an isophote enclosing an area, \(\hat{A} = \pi R_e^2\), or the largest isophote fully contained within the SAURON field, whichever is smaller (SX); the luminosity-weighted squared mean velocity component, parallel to the line of sight, within either an ellipse of area, \(\hat{A}\), ellipticity, \(\hat{e}_\perp\), and related position angle, or the largest similar ellipse fully contained within the SAURON field, whichever is smaller (SX); the luminosity-weighted squared velocity dispersion, parallel to the line of sight, within either an ellipse of area, \(\hat{A}\), ellipticity, \(\hat{e}_\perp\), and related position angle, or the largest similar ellipse fully contained within the SAURON field, whichever is smaller (SX).

Parameters listed to gain further insight even if not used in the application are: the ratio between maximum radius, \(R_{\text{max}}\), sampled by the kinematical observations, and effective radius, \(R_e\) (SIV); the ratio between square root luminosity-weighted squared mean velocity component and dispersion velocity component, parallel to the line of sight (SX); the kinematic classification (F - fast rotator; S - slow rotator) (SX).

For further details refer to the parent papers (SIV; SX) and an earlier attempt (Binney, 2005).

4.2 Determination of model parameters

With regard to the stellar subsystem, two model parameters can directly be inferred from the data (CV08). More specifically, stellar masses are deduced from luminosities and mass-luminosity ratios (in \(I\) band), as \(M_i/M_{10} = (L/L_\odot)[(M_i/L)/(10^{10}m_\odot/L_\odot)]; L/L_\odot = \exp_{10}\{-0.4[I_T - (\hat{m} - \hat{M}) - 4.11]\};\)
Table 7: Data related to a sample \((N = 16)\) of elliptical galaxies, extracted from larger samples of early-type galaxies investigated within the SAURON project (S IV, \(N = 25\); S X, \(N = 48\)), which are used in the current paper. Column captions: (1) NGC number; (2) effective (half-light) radius, \(R_e\), measured in the \(I\) band (S IV); (3) ratio between maximum radius, \(R_{\text{max}}\), and effective radius, \(R_e\) (S IV); (4) total observed \(I\) band galaxy magnitude (S IV); (5) mass-luminosity ratio of the stellar population (S IV); (6) galaxy distance modulus (hats avoid confusion with the fractional mass, \(m\), and the total mass, \(M\)) (S IV); (7) luminosity-weighted average ellipticity, \(\hat{e}_\perp\), on a plane perpendicular to the line of sight, within either an isophote enclosing an area, \(\hat{A} = \pi R_e^2\), or the largest isophote fully contained within the SAURON field, whichever is smaller (S X); (8) luminosity-weighted squared mean velocity component, parallel to the line of sight, within either an ellipse of area, \(\hat{A}\), ellipticity, \(\hat{e}_\perp\), and related position angle, or the largest similar ellipse fully contained within the SAURON field, whichever is smaller (S X); (9) luminosity-weighted squared velocity dispersion, parallel to the line of sight, within either an ellipse of area, \(\hat{A}\), ellipticity, \(\hat{e}_\perp\), and related position angle, or the largest similar ellipse fully contained with the SAURON field, whichever is smaller (S X); (10) ratio between the square root luminosity-weighted squared mean velocity component and dispersion velocity component, parallel to the line of sight (S X); (11) kinematic classification (F - fast rotator; S - slow rotator) (S X). For further details refer to the parent papers (S IV; S X) and an earlier attempt (Binney, 2005).

| NGC  | \(R_e\)  | \(\frac{R_{\text{max}}}{R_e}\) | \(I_T\)    | \(\frac{M_l}{L}\) | \(\hat{m} - \hat{M}\) | \(\frac{<\hat{e}_{\parallel}>}{<\hat{e}_{\perp}>}\) | \(\frac{<\hat{v}_{\parallel}^2>^\frac{1}{2}}{<\hat{v}_{\perp}^2>^\frac{1}{2}}\) | \(\frac{<\sigma_{\parallel}^2>^\frac{1}{2}}{<\sigma_{\perp}^2>^\frac{1}{2}}\) | KC    |
|------|---------|-------------------|-------------|----------------|-----------------|-----------------|-----------------|-----------------|-----|
| 0821 | 039.0   | 0.62              | 09.47       | 2.60           | 31.85           | 0.40            | 048             | 182             | 0.26 F         |
| 2974 | 024.0   | 1.04              | 09.43       | 2.34           | 31.60           | 0.37            | 127             | 180             | 0.70 F         |
| 3377 | 038.0   | 0.53              | 08.98       | 1.75           | 30.19           | 0.46            | 057             | 117             | 0.49 F         |
| 3379 | 042.0   | 0.67              | 08.03       | 3.08           | 30.06           | 0.08            | 028             | 198             | 0.14 F         |
| 3608 | 041.0   | 0.49              | 09.40       | 2.57           | 31.74           | 0.18            | 008             | 179             | 0.05 S         |
| 4278 | 032.0   | 0.82              | 08.83       | 3.05           | 30.97           | 0.12            | 044             | 228             | 0.19 F         |
| 4374 | 071.0   | 0.43              | 07.69       | 3.08           | 31.26           | 0.15            | 007             | 282             | 0.03 S         |
| 4458 | 027.0   | 0.74              | 10.68       | 2.27           | 31.12           | 0.12            | 010             | 084             | 0.12 S         |
| 4473 | 027.0   | 0.79              | 08.94       | 2.88           | 30.92           | 0.41            | 041             | 188             | 0.22 F         |
| 4486 | 105.0   | 0.29              | 07.23       | 3.33           | 30.97           | 0.04            | 007             | 306             | 0.02 S         |
| 4552 | 032.0   | 0.63              | 08.54       | 3.35           | 30.87           | 0.04            | 013             | 257             | 0.05 S         |
| 4621 | 046.0   | 0.56              | 08.41       | 3.12           | 31.25           | 0.34            | 052             | 207             | 0.23 F         |
| 4660 | 011.0   | 1.83              | 09.96       | 2.96           | 30.48           | 0.44            | 079             | 163             | 0.49 F         |
| 5813 | 052.0   | 0.53              | 09.12       | 2.97           | 32.48           | 0.15            | 032             | 223             | 0.14 S         |
| 5845 | 004.6   | 4.45              | 11.10       | 2.96           | 32.01           | 0.35            | 081             | 226             | 0.36 F         |
| 5846 | 081.0   | 0.29              | 08.41       | 3.33           | 31.92           | 0.07            | 007             | 240             | 0.03 S         |
and scaling radii are calculated from effective radii (in arcsec) and distances, as $r_i^{1/kpc} = (R_e/kpc)/1.81; R_e/kpc = [(R_e/arcsec)(d/Mpc)]/206.265; d/Mpc = \exp_{10}[(\hat{m} - M)/5 - 5]$; where the factor, 1.81, is related to an assumed H density profile for the inner subsystem, and the factor, 206.265, is related to the choice of measure units (CV08). For further details refer to the parent paper (SIV).

Two additional parameters can be inferred by fitting the data with dynamical models. More specifically, the inclination angle, $i$, is deduced from the best fitting two-integral Jeans model (SIV), and the anisotropy parameter, $\delta$, is determined from the solution of the dynamical models, supposed to be axisymmetric (SX). In fact, fast rotators show evidence of large anisotropy and axial symmetry, while slow rotators appear to be nearly isotropic and moderately triaxial. For further details refer to the parent paper (SX).

The intrinsic axis ratio, $\epsilon$, is deduced from the computed inclination, under the assumption of axisymmetric configurations (SX), using the relation (Binney and Tremaine, 1987, Chap. 4, §4.3):

$$1 - \epsilon^2 = \frac{1 - \epsilon_{\text{obs}}^2}{\sin^2 i}; \quad (93)$$

where $\epsilon_{\text{obs}}$ is the observed axis ratio related to an inclination angle, $i$, between the symmetry axis and the line of sight ($i = 90^\circ$ for edge-on configurations).

The mass-weighted mean square velocity component and dispersion velocity component, parallel to the line of sight, for a galaxy observed (obs) at an inclination angle, $i$, under the assumption of axisymmetric ($a_1 = a_2$) and isotropic on the equatorial plane ($\sigma_{11} = \sigma_{22}$) configurations (SX), are related to their edge-on (edo) counterparts as (Binney and Tremaine, 1987, Chap. 4, §4.3):

$$[< \tilde{v}_\parallel^2 >^{1/2}]_{\text{obs}} = [< \tilde{v}_\parallel^2 >^{1/2}]_{\text{edo}} \sin i; \quad (94)$$

$$[< \sigma_\parallel^2 >^{1/2}]_{\text{obs}} = [< \sigma_\parallel^2 >^{1/2}]_{\text{edo}} (1 - \delta \cos^2 i)^{1/2}; \quad (95)$$

and the intrinsic mean rotational velocity and velocity dispersion are expressed as:

$$< \tilde{v}_{\phi\phi} >^{1/2} = \sqrt{2[< \tilde{v}_\parallel^2 >^{1/2}]_{\text{edo}}} = \frac{\sqrt{2}}{\sin i} [< \tilde{v}_\parallel^2 >^{1/2}]_{\text{obs}} ; \quad (96)$$

$$< \sigma^2 >^{1/2} = \{2[< \sigma_\parallel^2 >]_{\text{edo}} + (1 - \delta)[< \sigma_\parallel^2 >]_{\text{edo}}\}^{1/2}$$

$$= (3 - \delta)^{1/2}[< \sigma_\parallel^2 >^{1/2}]_{\text{edo}} = \left(\frac{3 - \delta}{1 - \delta \cos^2 i}\right)^{1/2} [< \sigma_\parallel^2 >^{1/2}]_{\text{obs}}; \quad (97)$$

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in terms of observed quantities, where \( \delta = 1 - \frac{\sigma_{33}^2}{\sigma_{11}^2} = 1 - \frac{\sigma_{33}^2}{\sigma_{22}^2} \) by definition (e.g., Binney, 2005).

### 4.3 Model galaxies vs. sample objects

The kinetic energy of the stellar subsystem is:

\[
(E_i)_{\text{kin}} = \frac{1}{2} M_i \left\{ \langle (v_{\phi\phi})_i^2 \rangle + \langle \sigma_i^2 \rangle \right\} ; \quad (98)
\]

and the combination of Eqs. (35), (10), (11), (43), (44), and (98) yields:

\[
M_i \left\{ \langle (v_{\phi\phi})_i^2 \rangle + \langle \sigma_i^2 \rangle \right\} = \frac{(\nu_i)_\text{sel}}{(\nu_i)_\text{mas}^2} G(M_i)^2 (a_i^1) B + \frac{(\nu_{ij})_\text{tid}}{(\nu_i)_\text{mas}^2} G(M_i)^2 \ ; \quad (99)
\]

which, after some algebra, takes the form:

\[
\frac{\langle \sigma_i^2 \rangle (a_i^1) B}{GM_i} \left\{ \langle (v_{\phi\phi})_i^2 \rangle + 1 \right\} = \frac{(\nu_i)_\text{sel} + (\nu_{ij})_\text{tid}}{(\nu_i)_\text{mas}^2} ; \quad (100)
\]

where, for an inner H density profile, the scaling radius, \( r_i^* = (a_i^1)_1 \), may be chosen as:

\[
r_i^* = \frac{R_e}{1.81} \quad ; \quad M_i(r_i^*) = \frac{1}{4} M_i \quad ; \quad M_i(R_e) = \frac{1}{2} M_i \ ; \quad (101)
\]

and the shape factor, \( B \), reads (e.g., Chandrasekhar, 1969, Chap. 3 §§17, 22; Caimmi, 2009):

\[
B = 2 \arcsin \sqrt{1 - \epsilon^2} \quad = 2 \frac{\arcsin \sqrt{1 - (1 - \langle \hat{e}_\perp \rangle)^2}}{\sin i} \frac{\sin i}{\sin i} ; \quad (102)
\]

for axisymmetric configurations, where the last equality is owing to Eq. (93) and the definition of ellipticity, \( \hat{e} = 1 - \epsilon \).

The combination of Eqs. (96), (97), (98), (100), and (101) yields:

\[
\frac{\langle \sigma_\parallel^2 \rangle_{\text{obs}} R_e}{2GM_i(R_e) 1.81 B} \left\{ \frac{2}{\sin^2 i} \langle \sigma_\parallel^2 \rangle_{\text{obs}} + \frac{3 - \delta}{1 - \delta \cos^2 i} \right\} = c_{ij} ; \quad (103a)
\]

\[
c_{ij} = \frac{(\nu_i)_\text{sel} + (\nu_{ij})_\text{tid}}{(\nu_i)_\text{mas}^2} ; \quad (103b)
\]

37
where the left-hand side of Eq. (103a) is expressed in terms of quantities which are deduced from either observations or fitting with dynamic models, and then may be determined for an assigned sample object. Conversely, the right-hand side of Eq. (103a) depends on the selected density profiles, and then may be determined for an assigned model galaxy. In this view, Eq. (103a) may be read as a correspondence between sample objects (left) and model galaxies (right).

The dimensionless energy, \(c_{ij}\), defined by Eq. (103b), depends on four variables via Eqs. (40)-(45): the scaled truncation radii, \(\Xi_i, \Xi_j\), the fractional mass, \(m^\dagger\) (or \(m\)), and the fractional radius, \(y^\dagger\) (or \(y\)).

### 4.4 Model galaxies on the \((O_X V_X p)\) plane

For assigned density profiles and scaled truncation radii, \(\Xi_i, \Xi_j\), two unknowns remain: the fractional mass, \(m^\dagger\) (or \(m\)), and the fractional radius, \(y^\dagger\) (or \(y\)). The combination of Eqs. (44a) and (103b) yields:

\[
w^{(\text{ext})}(\eta) = \frac{8}{9m^\dagger} \left\{ (\nu_i)_{\text{sel}} - c_{ij}[(\nu_i)_{\text{mas}}]^2 \right\}; \tag{104}
\]

where the function, \(w^{(\text{ext})}\), depends on \(\Xi_i\) and \(y^\dagger\), conformably to Eqs. (40b), (44c), and (45b). In the case under discussion, Eq. (104) may be conceived as a link between the unknowns, \(m^\dagger\) and \(y^\dagger\). At this stage, one additional relation is needed (CV08).

The mere existence of a fundamental plane (Djorgovski and Davis, 1987; Dressler et al., 1987) indicates that structural properties in elliptical galaxies span a narrow range, suggesting that some self-regulating mechanism must be at work during formation and evolution. In particular, projected light profiles from elliptical galaxies exhibit large degree of homogeneity and may well be fitted by the \(r^{1/4}\) de Vaucouleurs law. Accordingly, a narrow range may safely be expected also for fractional masses of elliptical galaxies and the assumption, \(m = \text{const}\), appears to be a viable approximation. This is the reason for which the sample used (\(N = 16\)) is made of only elliptical galaxies, extracted from larger samples (\(N = 25; N = 48\)) of early-type galaxies investigated within the SAURON project (SIV; SX).

Then the fractional radius, \(y^\dagger\) (or \(y\)), remains as the sole unknown, which can be determined by solving Eq. (104) with numerical techniques. If no solution exists, no model galaxy corresponds to the selected sample object or, in other words, the related density profiles provide no fit to the data, and some input value has to be changed.

Values of parameters needed for representing model galaxies on the \((O_X V_X p)\) plane, are listed in Tab. 8: the galaxy stellar mass, \(M_i\); the galaxy scaling
radius, $r_i^1$; the inclination angle, $i$, of the best fitting two-integral Jeans model (SIV); the anisotropy parameter, $\delta$, determined from the solution of the dynamic models, supposed to be axisymmetric ($a_1 = a_2$) and isotropic on the equatorial plane ($\sigma_{11} = \sigma_{22}$) (SX); the intrinsic axis ratio, $\epsilon$, deduced from the computed inclination, under the assumption of axisymmetric configurations (SX); the dimensionless energy, $c_{ij}$, defined by Eqs. (103). The kinematic classification is listed again to get more insight. The galaxy stellar mass within the effective radius is calculated as:

$$\frac{M_i}{M_{10}} = \frac{L}{L_\odot} \frac{M_i/L}{10^{10} m_\odot/L_\odot}; \quad (105a)$$

$$\frac{L}{L_\odot} = \exp_{10}\{ -0.4[I_T - (\hat{m} - \hat{M}) - 4.11] \} ; \quad (105b)$$

and the galaxy effective radius is calculated as:

$$\frac{R_e}{\text{kpc}} = \frac{R_e}{\text{arcsec}} \frac{1}{206.265 \text{Mpc}}; \quad (106a)$$

$$\frac{d}{\text{Mpc}} = \exp_{10}\left[ \frac{\hat{m} - \hat{M}}{5} - 5 \right] ; \quad (106b)$$

where the factor, 206.265, is related to the choice of measure units. For further details refer to an earlier attempt (CV08).

The values of the reduced variables, $X_V$, $X_p$, $X_T$, are determined via Eqs. (47) and (50e). Both HH and HN macrogases have been considered, for the following values of parameters. Scaled truncation radii (both finite or infinite): $\Xi = (k_i, k_j); k_i = 5, 10, 20, +\infty; k_j = 5, 10, 20, +\infty$. Fractional masses: $m = 10, 20$. Under the working hypothesis of an analogy between VDW gases and macrogases, the $(O_{XV}Xp)$ plane may be divided into three parts: (i) a reversed bell-shaped region where two phases, gas and stars, coexist and the lower point coincides with the critical point (hereafter quoted as the GS region); (ii) a region limited by the left boundary of the reversed bell-shaped region and the rising side of the critical macroisothermal curve, both branching off from the critical point, where only stars are present (hereafter quoted as the S region); (iii) a region limited by the right boundary of the reversed bell-shaped region and the rising side of the critical macroisothermal curve, both branching off from the critical point, and the coordinate axes, where only gas is present (hereafter quoted as the G region).

If the density profiles are only slightly affected in time, the evolution of a galaxy on the $(O_{XV}Xp)$ plane is represented by a track, starting from the G region and ending within the GS or the S region. The critical point is the sole which is common to the three regions.
Table 8: Parameters calculated from the data listed in Tab. 7 for a sample \((N = 16)\) of elliptical galaxies, extracted from larger samples of early-type galaxies investigated within the SAURON project (S IV, \(N = 25\); SX, \(N = 48\)), which are used in the current paper. Column captions: (1) NGC number; (2) galaxy stellar mass deduced from luminosities and mass-luminosity ratios (in \(I\) band), conformably to Eqs. (105); (3) galaxy scaling radius, calculated using Eqs. (101) and (106); (4) inclination angle of the best fitting two-integral Jeans model (S IV); (5) anisotropy parameter, determined from the solution of the dynamic models, supposed to be axisymmetric (SX); (6) intrinsic axis ratio, deduced from the computed inclination, under the assumption of axisymmetric configurations (SX); (7) dimensionless energy, defined by Eqs. (103); (8) kinematic classification (F - fast rotator; S - slow rotator) (SX). For further details refer to the parent papers (S IV; SX) and an earlier attempt (Binney, 2005).

| NGC   | \(M_i\)  | \(r_i^1\) | \(i\) | \(\delta_i\) | \(\epsilon_i\) | \(c_{ij}\) | KC  |
|-------|----------|-----------|------|------------|----------------|-----------|-----|
| 0821  | 10.26    | 2.45      | 90   | 0.20       | 0.60           | 0.23      | F   |
| 2974  | 07.61    | 1.34      | 57   | 0.24       | 0.38           | 0.23      | F   |
| 3377  | 02.35    | 1.11      | 90   | 0.25       | 0.54           | 0.20      | F   |
| 3379  | 08.80    | 1.16      | 90   | 0.03       | 0.92           | 0.18      | F   |
| 3608  | 09.77    | 2.45      | 90   | 0.13       | 0.82           | 0.25      | S   |
| 4278  | 09.64    | 1.34      | 45   | 0.18       | 0.74           | 0.25      | F   |
| 4374  | 36.35    | 3.40      | 90   | 0.08       | 0.85           | 0.24      | S   |
| 4458  | 01.50    | 1.21      | 90   | 0.09       | 0.88           | 0.19      | S   |
| 4473  | 07.86    | 1.10      | 73   | 0.34       | 0.54           | 0.14      | F   |
| 4486  | 45.97    | 4.40      | 90   | 0.00       | 0.96           | 0.31      | S   |
| 4552  | 12.62    | 1.28      | 90   | 0.02       | 0.96           | 0.23      | S   |
| 4621  | 18.80    | 2.19      | 90   | 0.18       | 0.66           | 0.15      | F   |
| 4660  | 02.11    | 0.37      | 70   | 0.30       | 0.47           | 0.15      | F   |
| 5813  | 28.89    | 4.36      | 90   | 0.08       | 0.85           | 0.25      | S   |
| 5845  | 03.02    | 0.31      | 90   | 0.15       | 0.65           | 0.17      | F   |
| 5846  | 37.19    | 5.25      | 90   | 0.01       | 0.93           | 0.28      | S   |
The position of a model galaxy on the \((O X V X_p)\) plane is affected by errors of different kind, due to: (1) scatter around mean values listed in Table \(7\); (2) scatter in fitting observed to model density profiles; (3) uncertainty on the determination of the critical point. The third contribution may safely be neglected with respect to the other ones. Values related to the first contribution are not completely found in literature (to the knowledge of the author). The second contribution could be determined using fitting procedures, provided light distributions are available and light traces stellar mass in elliptical galaxies. In summary, error calculation on the position of sample objects on the \((O X V X_p)\) plane would be cumbersome, and perhaps of little meaning.

A notable simplification can be attained if model galaxies instead of sample objects are considered, in dealing with only errors of the first kind mentioned above. For fixed fractional mass, \(m\), the uncertainty on \(X_p = (m/m_c)^2\), is negligible with respect to \(X_V = y_c/y\), which is determined by solving Eq. (103a) for assigned density profiles. The combination of Eqs. (101), (102), and (103b) yields an expression of the dimensionless energy, \(c_{ij}\), in terms of observables listed in Table 7 as:

\[
c_{ij} = \frac{1}{1163.335} \frac{1}{G \zeta_5} \exp_{10}[0.4\zeta_6 - 0.2\zeta_7 - 6.644] \cdot \left[ \frac{1 - (1 - \zeta_4^{2})^{1/2}}{\sin i} \right]^{1/2} \left\{ \frac{2\zeta_5^2}{\sin^2 i} + \frac{(3 - \delta)\zeta_5^2}{1 - \delta \cos^2 i} \right\} ; \quad (107a)
\]

\[
\zeta_4 = \langle \dot{\varepsilon}_\perp \rangle ; \quad \zeta_2 = \left[ \langle \dot{v}_\parallel^2 \rangle^{1/2} \right]_{\text{obs}} ; \quad \zeta_3 = \left[ \langle \sigma_\parallel^2 \rangle^{1/2} \right]_{\text{obs}} ; \quad (107b)
\]

\[
\zeta_4 = \frac{R_e}{\text{arcsec}} ; \quad \zeta_5 = \frac{M_i/L_i}{10^{10} m_\odot/L_\odot} ; \quad \zeta_6 = I_T ; \quad \zeta_7 = \hat{m} - \hat{M} ; \quad (107c)
\]

where some symbols have been changed to gain simplicity. An inspection of Eq. (107a) shows that the dimensionless energy, \(c_{ij}\), is monotonically increasing with increasing \(\zeta_4, \zeta_6, \zeta_2, \zeta_3\), and decreasing \(\zeta_5, \zeta_7\); and vice versa. Establishing the trend with the remaining variables, \(\zeta_4, i, \delta\), demands further considerations.

The function, \(B(\epsilon)\), defined by Eq. (102), is monotonically decreasing in the domain, \(0 \leq \epsilon \leq 1\), where \(\pi = B(0) \geq B(\epsilon) \geq B(1) = 2\). Accordingly, \(B(\epsilon)\) is monotonically decreasing with decreasing \(\zeta_4 = \langle \dot{\varepsilon}_\perp \rangle\) and/or decreasing \(i\), and vice versa.

With regard to the anisotropy parameter, \(\delta\), the following identity holds:

\[
\frac{3 - \delta}{1 - \delta \cos^2 i} = \left[ \cos^2 i + \frac{\sin^2 i - 2 \cos^2 i}{3 - \delta} \right]^{-1} ; \quad (108)
\]

and the special inclination angle, \(i_0\), which makes null the fraction within
brackets, is the solution of the equation:
\[
\sin^2 i_0 - 2 \cos^2 i_0 - 3 \sin^2 i_0 - 2 = 0 \quad ; \quad (109)
\]
the result is:
\[
\sin i_0 = \sqrt{\frac{2}{3}} \quad ; \quad i_0 = 0.955 \, 316 \, 6 = 54.735 \, 61^\circ \quad ; \quad (110)
\]
accordingly, the fraction on the left-hand side of Eq. (108) is monotonically
decreasing for increasing \( \delta \) in the range, \( 0 < i < i_0 \), and is monotonically
increasing for increasing \( \delta \) in the range, \( 0 \leq i < i_0 \), while no dependence on \( \delta \)
occurs in the special case, \( i = i_0 \). Finally, Eqs. (103a), (108), and (109) show
that the dimensionless energy, \( c_{ij} \), is monotonically increasing or decreasing
for increasing \( \delta \) according if \( 0 < i < i_0 \) or \( i_0 < i \leq \pi/2 \), respectively, and vice
versa, while no dependence on \( \delta \) occurs in the special case, \( i = i_0 \).

The above results may be reduced to a single relation, as:
\[
c_{ij} \mp \Delta c_{ij} = \frac{1}{1163.335 G \zeta_5 \pm \Delta \zeta_5} \cdot \exp_{10}[0.4 (\zeta_6 \mp \Delta \zeta_6) - 0.2 (\zeta_7 \mp \Delta \zeta_7) - 6.644]
\cdot \{1 - [1 - (\zeta_1 \pm \Delta \zeta_1)^2]^{1/2} / \sin(i \pm \Delta i)
\cdot \arcsin\{1 - [1 - (\zeta_1 \pm \Delta \zeta_1)^2]^{1/2} / \sin(i \pm \Delta i)\}
\cdot \{2(\zeta_4 \mp \Delta \zeta_4)^2 \pm \{3 - [\delta \pm \sgn(i - i_0) \Delta \delta] (\zeta_3 \mp \Delta \zeta_3)^2\}
\cdot \sin^2(i \pm \Delta i) + 1 - [\delta \pm \sgn(i - i_0) \Delta \delta] \cos^2(i \pm \Delta i)\} \}; (111)
\]
where upper and lower signs correspond to the lower and upper \( c_{ij} \) value,
respectively, and \( \sgn \) is the sign function, defined as \( \sgn(x) = x/|x|, \; x \neq 0; \)
\( \sgn(0) = 0 \). It is worth emphasizing that Eq. (111) makes an exact for-
mulation of the uncertainty on the dimensionless energy, \( c_{ij} \), as defined by
Eq. (103a). On the contrary, standard linear and quadratic error propagation
formulae apply to any kind of functions allowing Taylor series development,
but are approximate instead of being exact.

To the knowledge of the author, part of the errors on the right-hand
side of Eq. (111) are not available in literature. For the inclination angle,
\( i \), and the anisotropy parameter, \( \delta \), the reason is in that they depend on a
reference dynamic model (SIV; SX) and cannot be specified. The following
values are found: \( \Delta \zeta_4/\zeta_4 = 0.17 \) (SIV); \( \Delta \zeta_6/\zeta_6 = 0.20 \) (SX); \( \Delta \zeta_8/\zeta_8 = 0.13 \) (SIV); \( \Delta \zeta_5/\zeta_5 = 0.10 \), strongly dependent on the assumptions made (SIV); \( \Delta \zeta_7 = 0.09 - 0.33 \), with a value listed for each sample object (SIV). In this view,
it seems better starting from assigned values of the dimensionless energy
relative error, \( \Delta c_{ij}/c_{ij} \), and numerically evaluate the related uncertainty on
the position of a selected model galaxy on the \((O|X_Y X_p)\) plane, to visualize
the trend.
4.5 Results

With regard to HH macrogases, model galaxies corresponding to sample objects listed in Tables 7 and 8 are represented on the \((O X_v X_p)\) plane of Fig. 13 for different choices of scaled truncation radii, \(\Xi_i, \Xi_j\), and fractional mass, \(m\). The critical macroisothermal curve (left) and the boundary of the GS region (right) are also plotted for each case. The critical point is marked by a composite symbol. Two sample objects cannot be modelled for low inner scaled truncation radii \((\Xi_i = 5)\) and, for this reason, related cases are not considered. Lower and upper symbols of the same kind correspond to \(m = 10, 20\), respectively.

Under the working hypothesis of an analogy between VDW gases and macrogases, modelled elliptical galaxies are expected to lie in the S region or slightly outside the S region within the GS region at most. An inspection of Fig. 13 shows the following: (1) model galaxies with low fractional mass \((m = 10)\) and/or no truncation radii \((\Xi \rightarrow +\infty)\) lie below the critical macroisothermal curve, in the G region, and for this reason cannot be accepted; (2) about one half of model galaxies with low outer scaled radii \((\Xi_j = 5)\) lie well inside the GS region, and for this reason cannot be accepted; (3) more than one half of model galaxies with larger scaled radii \((\Xi_i = 10, 20; \Xi_j = 10, 20)\) lie within the S region, and for this reason are accepted.

With regard to viable cases, the plot of Fig. 13 is repeated in Fig. 14 where model galaxies are distinguished according if their parent sample object is a fast (squares) or a slow (diamonds) rotator. The related scaled truncation radii (from top to bottom) are \((\Xi_i, \Xi_j) = (10, 10), (10, 20), (20, 10), (20, 20),\) and the fractional mass is \(m = 20\). The curves are as in Fig. 13.

Restricting to viable cases, the plot of Fig. 13 is repeated in Fig. 15 where the effect of assigned errors in dimensionless energy, \(\Delta c_{ij}/c_{ij} = 5\%, 10\%, 15\%, 20\%\), labelled on each panel, on model galaxies, is represented. More specifically, the position of model galaxies is marked by diamonds, and the change due to lowered \((c_{ij} - \Delta c_{ij})\) and increased \((c_{ij} + \Delta c_{ij})\) dimensionless energy, is marked by Greek and St. Andrew’s crosses, respectively. Scaled truncation radii are, from top to bottom, \((\Xi_i, \Xi_j) = (10, 10), (10,20), (20,10), (20,20),\) and the fractional mass is \(m = 20\) in all cases. In the special cases, \((\Xi_i, \Xi_j) = (10, 10), (10,20),\) and \(\Delta c_{ij}/c_{ij} = 20\%,\) a sample object (NGC 4473) cannot be modelled for lowered dimensionless energies, \(c_{ij} - \Delta c_{ij}\). In general, lowered and increased dimensionless energies, \(c_{ij},\) make model galaxies shift on the left and on the right, respectively, from their position on the \((O X_v X_p)\) plane. Accordingly, a fraction of model galaxies enter the G region and the GS region, respectively, and the fit could be improved by changing the input parameters, \(\Xi_i, \Xi_j,\) and \(m\).
With regard to HN/NH macrogases, model galaxies corresponding to sample objects listed in Tables 7 and 8 are represented on the \((O_X, V_x)\) plane of Fig. 16 for different choices of scaled truncation radii, \(\Xi_i, \Xi_j\), and fractional mass, \(m\). The critical macroisothermal curve (left) and the boundary of the GS region (right) are also plotted for each case. The critical point is marked by a composite symbol. Two sample objects cannot be modelled for low inner scaled truncation radii \((\Xi_i = 5)\) and, for this reason, related cases are not considered. The same holds, to a larger extent, for cases \((\Xi_i, \Xi_j) \rightarrow (+\infty, +\infty)\), due to an infinite mass of the NFW density profile. Lower and upper symbols of the same kind correspond to \(m = 10, 20\), respectively.

Under the working hypothesis of an analogy between VDW gases and macrogases, modelled elliptical galaxies are expected to lie in the S region or slightly outside the GS region at most. An inspection of Fig. 16 shows the following: (1) model galaxies with low fractional mass \((m = 10)\) and/or large outer scaled truncation radius \((\Xi_j = 20)\) lie (at least partially) below the critical macroisothermal curve in the G region, and for this reason the related cases cannot be accepted; (2) more than one half of model galaxies with large fractional mass \((m = 20)\) and/or low outer scaled truncation radius \((\Xi_j = 5)\) lie well inside the GS region, and for this reason the related cases cannot be accepted; (3) more than one half of model galaxies with large fractional mass \((m = 20)\) and outer scaled radius \((\Xi_j = 10)\), or low fractional mass \((m = 10)\) and outer scaled radius \((\Xi_j = 5)\) lie within the S region, and for this reason the related cases are accepted.

With regard to viable cases, the plot of Fig. 16 is repeated in Fig. 17, where model galaxies are distinguished according if their parent sample object is a fast (squares) or a slow (diamonds) rotator. The related scaled truncation radii (from top to bottom) are \((\Xi_i, \Xi_j) = (10, 10), (10, 5), (20, 10)\), and the fractional mass is \(m = 20\) for \(\Xi_j = 10\), and \(m = 10\) for \(\Xi_j = 5\). The curves are as in Fig. 16.

Restricting to viable cases, the plot of Fig. 16 is repeated in Fig. 18, where the effect of assigned errors in dimensionless energy, \(\Delta c_{ij}/c_{ij} = 5\%, 10\%, 15\%, 20\%\), labelled on each panel, on model galaxies, is represented. More specifically, the position of model galaxies is marked by diamonds, and the change due to lowered \((c_{ij} - \Delta c_{ij})\) and increased \((c_{ij} + \Delta c_{ij})\) dimensionless energy, is marked by Greek and St. Andrew’s crosses, respectively. Scaled truncation radii and fractional masses are \((\Xi_i, \Xi_j, m) = (10, 10, 20), (10, 5, 10), (20, 10, 20)\), from top to bottom.

In the special case, \((\Xi_i, \Xi_j, m) = (10, 5, 10)\), and \(\Delta c_{ij}/c_{ij} = 20\%\), a sample object (NGC 4486) is out of scale on the right for increased dimensionless energies, \(c_{ij} + \Delta c_{ij}\). In the special cases, \((\Xi_i, \Xi_j, m) = (10, 5, 10), (10, 10, 20)\),
and $\Delta c_{ij}/c_{ij} = 20\%$, a sample object (NGC 4473) cannot be modelled for lowered dimensionless energies, $c_{ij} - \Delta c_{ij}$. In general, lowered and increased dimensionless energies, $c_{ij}$, make model galaxies shift on the left and on the right, respectively, from their position on the $(O, X_V, X_P)$ plane. Accordingly, a fraction of model galaxies enter the G region and the GS region, respectively, and the fit could be improved by changing the input parameters, $\Xi_i, \Xi_j$, and $m$.

### 4.6 Discussion

Current cosmological models imply large-scale celestial bodies, such as galaxies and clusters of galaxies, are embedded within nonbaryonic dark haloes, where the two subsystems interact only via gravitation. Accordingly, large-scale celestial bodies can be modelled as macrogases where macrovolume, macropressure, and macrotemperature can be defined, and a counterpart of the VDW theory for ordinary gases can be developed. Sufficiently steep density profiles, as in HH and HN/NH macrogases, show a similar trend with respect to VDW gases: the macroisothermal curves are nonmonotonic with two extremum points (one maximum and one minimum) above a critical macrotemperature, and are monotonic below a critical macrotemperature. The critical macroisothermal curve is characterized by a single extremum point, where the maximum and the minimum coincide yielding a horizontal inflexion point. On the other hand, sufficiently mild density profiles, such as in UU and PP (CV08) macrogases, show only nonmonotonic macroisothermal curves with two extremum points (one maximum and one minimum) where no critical macroisothermal curve exists.

A generic macrogas equation of state is formulated in terms of dimensionless variables normalized to critical values (or conveniently chosen in absence of the critical point). Similar to what has been done in dealing with the reduced VDW equation for ordinary gases (e.g., LL67, Chap. VIII, §85), the states of two large-scale celestial bodies with equal $X_V, X_P, X_T$, or $y, m, \phi$, for assigned scaled truncation radii, $\Xi_i, \Xi_j$, and belonging to the same family of macrogases, may be defined as corresponding states. The mere existence of a macrogas equation of state yields the following result.

**Law of corresponding states.** Given two large-scale celestial bodies belonging to the same family of macrogases with assigned scaled truncation radii, $\Xi_i, \Xi_j$, the equality between two among three reduced variables, $X_V, X_P, X_T$, or $y, m, \phi$, implies the equality between the remaining related reduced variables i.e. the two macrogases are in corresponding states.

The law of corresponding states cannot be extended to macrogases with
different scaled truncation radii, as shown in Figs. 9 and 12. A possible explanation may be the following. Contrary to ordinary gases, bounded by rigid walls which have no influence on the equation of state, macrogases are confined by “gravitational” walls appearing in the equation of state via the potential energy terms which, in turn, depend on the scaled truncation radii.

Ordinary gases exhibit monotonic isothermal curves where the central part of the related VDW isothermal curve, including the extremum points, is replaced by a horizontal line and a phase transition occurs therein. With regard to macrogases, the existence of a phase transition and monotonic macroisothermal curves of the kind considered, must necessarily be assumed as a working hypothesis by analogy with VDW gases. The phase transition must be conceived between gas and stars, and the \((O_X V \times p)\) plane may be divided into three parts, namely (i) the G region, where only gas exists; (ii) the S region, where only stars exist; (iii) the GS region, where both gas and stars exist. In this view, model elliptical galaxies are expected to lie within the S region or slightly outside the boundary between the S and the GS region at most.

It is the case for different models related to both HH and HN/NH macrogases, where acceptable values of scaled truncation radii and fractional masses are used, as shown in Figs. 13 and 16. The assumption of universal fractional mass for sample objects is not a limit of the model, which equally holds assigning different fractional masses to different sample objects.

It can be seen from Figs. 14 and 17 that fast rotators lie within the S region, while slow rotators are close (from both sides) to the boundary between the S and the GS region. This dichotomy could be interpreted by the different nature of the two classes of sample objects. More specifically, fast rotators seem consistent with elliptical galaxies with disky isophotes, which experienced minor mergers and accreted a significant amount of gas \((S_X)\), suddenly turned into stars. On the other hand, systematically more massive (with the exception of NGC 4458 and, marginally, NGC 3608, see Table 8) slow rotators may be related to elliptical galaxies with boxy isophotes, which experienced major gas-rich mergers, or sequences of mergers, and regulation by the feedback of a powerful central active galactic nucleus \((S_X)\), in some cases allowing gas survival as e.g., diffuse, still undetected interstellar medium.

Accordingly, in the \((O_X V \times p)\) plane fast rotators and a fraction of slow rotators are expected to lie within the S region, and the remaining part of slow rotators to be placed in the GS region, as shown in Figs. 14 and 17. The following trend is also exhibited: fast rotators are systematically on the left with respect to slow rotators, with the exception of NGC 4458, which is the sole sample slow rotator with low mass, possibly due to having
experienced minor instead of major mergers. In this view, NGC 4458 should be considered as a peculiar fast rotator, where low rotation could be due to special configurations e.g., still undetected counter-rotating components, as observed in the disk of NGC 4450 (e.g., SX).

The reduced variable, $\mathcal{X}_V = X_V / X_{Vc}$, via Eqs. (42b) and (50c) is proportional to the fractional truncation radius, $1/y = R_i / R_j$. The above mentioned dichotomy, exhibited by fast and slow rotators in the $(O \mathcal{X}_V \mathcal{X}_p)$ plane, implies a larger fractional truncation radius, $y = R_j / R_i$, for fast rotators with respect to slow rotators. This result could be interpreted as due to different formation mechanisms: minor mergers would produce larger contraction of the baryonic matter, while major mergers would make (possibly via active galactic nuclei) smaller contraction of the baryonic matter, yielding a larger or smaller fractional radius, respectively.

5 Conclusion

In the current attempt, two-component large-scale celestial bodies where the subsystems interact only via gravitation, are conceived as macrogases bounded by “gravitational” walls. The macrogas equation of state is formulated in terms of macrovolume, macropressure, and macrotemperature, which are dimensionless variables. For sufficiently steep density profiles, which fit to observed elliptical galaxies (or more generally, spheroid components) and to simulated nonbaryonic dark matter haloes, macroisothermal curves on the $(O \mathcal{X}_V \mathcal{X}_p)$ plane show an analogy with VDW isothermal curves exhibited by VDW gases. More specifically, a critical macroisothermal curve exists, below and above which the macroisothermal curves are monotonic and non-monotonic (with two extremum points, one maximum and one minimum), respectively. The critical macroisothermal curve is characterized by a single extremum point (a horizontal inflexion point), which is the critical point.

Contrary to ordinary gases, macrogases cannot be tested in laboratory, and for this reason a working hypothesis is inescapable. By analogy with ordinary gases, real macroisothermal curves are supposed to occur instead of their theoretical counterparts (deduced from the macrogas equation of state), where the central part containing the extremum points is replaced by a horizontal line along which a phase transition takes place. The intersection between a selected theoretical macroisothermal curve and its real counterpart, yields two regions of equal area. The phase transition is assumed to be gas-stars instead of vapour-liquid as in ordinary gases. Accordingly, the first quadrant of the $(O \mathcal{X}_V \mathcal{X}_p)$ plane is divided into three parts, namely (i) the G region, where only gas exists; (ii) the S region, where only stars exist; (iii)
the GS region, where both gas and stars exist.

For selected density profiles and scaled truncation radii, $\Xi_i, \Xi_j$, the macro-gas equation of state depends on three parameters, $X_V, X_p, X_T$, or the fractional truncation radius, $y$, the fractional mass, $m$, and the fractional energy, $\phi$. If elliptical galaxies and their hosting nonbaryonic dark haloes are conceived as macrogases, a selected model is accepted only if a whole set of sample objects (from which radii, masses, and rsm velocities can be determined) lies within the S region or slightly outside the boundary between the S and the GS region at most. The sample used (CV08, $N = 16$) is extracted from larger samples of early-type galaxies investigated within the SAURON project (S IV, $N = 25$; S X, $N = 48$). The position of model galaxies on the $(O, X_V, X_p)$ plane is determined through the following steps: (i) select SAURON data of interest; (ii) calculate the parameters appearing in the virial equations; (iii) make a correspondence between model galaxies and sample objects; (iv) represent model galaxies on the $(O, X_V, X_p)$ plane.

The main results found in the present investigation may be summarized as follows.

(1) A new numerical algorithm has been used for determining the critical point of selected HH and HN/NH macrogases, improving earlier results (CV08). In particular, the critical point exists for all the cases considered.

(2) A principle of corresponding states rigorously holds for selected density profiles and scaled truncation radii, and to a first extent only for selected density profiles.

(3) The following models (on a total of 20) can be accepted in the above mentioned sense: $(\Xi_i, \Xi_j, m) = (10, 10, 20), (10, 20, 20), (20, 10, 20), (20, 20, 20)$, with regard to HH macrogases, and $(\Xi_i, \Xi_j, m) = (10, 5, 10), (10, 10, 20), (20, 10, 20)$, for HN/NH macrogases. The values of model parameters may be changed by the occurrence of systematic errors.

(4) Fast rotators exhibit larger fractional truncation radii with respect to slow rotators, which makes the former lie within the S region and the latter close (from both sides) to the boundary between the S and the GS region, with regard to acceptable models. This dichotomy could be interpreted in terms of a different evolution related to fast and slow rotators, where gas is currently absent in the former and can be present (even if still undetected) in the latter.
6 Acknowledgements

The author is indebted to an anonymous referee for helpful comments which improved an earlier version of the manuscript. Thanks are due to T. Valentinuzzi for fruitful discussions.

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Appendix

A Tidal potential energy profile factors

The tidal potential energy for homeoidally striated ellipsoids related to similar and similarly placed boundaries, depends on the reference fractional mass, $m^\dagger$, the fractional scaling radius, $y^\dagger$, and the functions, $w^{(int)}(\eta)$, $w^{(ext)}(\eta)$, $\eta = \Xi_i/y^\dagger = \Xi_j/y$, expressed by Eqs. \[15\]. Let the two subsystems be denoted as A and B regardless of what is the outer and what is the inner.
Conformingly, Eqs. (42) read:

$$\Xi_B = \frac{y_{BA}}{y_{BA}^\dagger} \quad ; \quad \eta_{BA} = \Xi_A = \frac{\Xi_B}{y_{BA}} \quad ; \quad y_{BA} = \frac{r_B}{r_A}^\dagger \quad ; \quad y_{BA} = \frac{R_B}{R_A} \geq 1 ;$$

$$\xi_A = y_{BA}^\dagger \xi_B \quad ; \quad m_{BA} = \frac{M_B}{M_A} ; \quad m_{BA} = \frac{M_B}{M_A} ;$$

$$\Xi_A = \frac{y_{AB}}{y_{AB}^\dagger} \quad ; \quad \eta_{AB} = \Xi_B = \frac{\Xi_A}{y_{AB}} \quad ; \quad y_{AB} = \frac{r_A}{r_B}^\dagger \quad ; \quad y_{AB} = \frac{R_A}{R_B} \geq 1 ;$$

$$\xi_B = y_{AB}^\dagger \xi_A \quad ; \quad m_{AB} = \frac{M_A}{M_B} ; \quad m_{AB} = \frac{M_A}{M_B} ;$$

(112a)

according if $R_B \geq R_A$ or $R_A \geq R_B$, respectively.

In dealing with sequences of configurations where the scaled truncation radii, $\Xi_A$ and $\Xi_B$, are kept unchanged, the combination of Eqs. (112a) and (112b) yields:

$$y_{BA}^\dagger y_{AB}^\dagger = y_{BA} y_{AB} ;$$

(113)

for any pair of configurations belonging to opposite sides of the sequence, with respect to $y_{BA} = y_{AB} = 1$.

In the special case of equal scaled density profiles, $F_A = F_B = F$, and equal scaled truncation radii, $\Xi_A = \Xi_B = \Xi$, Eqs. (45) reduce to:

$$w^{\text{int}}(\Xi, y_{BA}^\dagger) = \int_{0}^{\Xi/y_{BA}} F(\xi) \frac{dF(y_{BA}^\dagger \xi)}{d\xi} d\xi \quad ; \quad y_{BA} = y_{BA} \geq 1 \quad (114a)$$

$$w^{\text{ext}}(\Xi, y_{BA}^\dagger) = \int_{0}^{\Xi/y_{BA}} F(y_{BA}^\dagger \xi) \frac{dF(\xi)}{d\xi} d\xi \quad ; \quad y_{BA} = y_{BA} \geq 1 \quad (114b)$$

$$w^{\text{int}}(\Xi, y_{AB}^\dagger) = \int_{0}^{\Xi/y_{AB}} F(\xi) \frac{dF(y_{AB}^\dagger \xi)}{d\xi} d\xi \quad ; \quad y_{AB} = y_{AB} \geq 1 \quad (114c)$$

$$w^{\text{ext}}(\Xi, y_{AB}^\dagger) = \int_{0}^{\Xi/y_{AB}} F(y_{AB}^\dagger \xi) \frac{dF(\xi)}{d\xi} d\xi \quad ; \quad y_{AB} = y_{AB} \geq 1 \quad (114d)$$

in the special case where the fractional scaling radius coincides for both configurations, $y_{BA}^\dagger = y_{AB}^\dagger$, the combination of Eqs. (114a) and (114c); (114b) and (114d); yields:

$$w^{\text{int}}(\Xi, y_{BA}^\dagger) = w^{\text{int}}(\Xi, y_{AB}^\dagger) \quad ; \quad y_{BA} = y_{AB}^\dagger \quad ; \quad (115a)$$

$$w^{\text{ext}}(\Xi, y_{BA}^\dagger) = w^{\text{ext}}(\Xi, y_{AB}^\dagger) \quad ; \quad y_{BA} = y_{AB}^\dagger \quad ; \quad (115b)$$

regardless of the scaled truncation radius, $\Xi$.

If, on the other hand, $\Xi_A \neq \Xi_B$, the upper integration limits are $\Xi_A/y_{BA}^\dagger$ and $\Xi_B/y_{AB}^\dagger$ for Eqs. (114a) and (114b); (114c) and (114d); respectively.
Then \( y_{BA}^\dagger = y_{AB}^\dagger \) implies equal integrands but different upper integration limits, while \( \Xi_A/y_{BA}^\dagger = \Xi_B/y_{AB}^\dagger \) implies equal upper integration limits but different integrands, with regard to Eqs. (114a) and (114c); (114b) and (114d); respectively. Accordingly, Eqs. (115a) and (115b) no longer hold in the case under discussion, unless \( \Xi_A \to +\infty, \Xi_B \to +\infty \), which erases the dependence on \( \Xi_A \) or \( \Xi_B \), regardless of the value of \( \lim_{(\Xi_A,\Xi_B) \to +\infty}(\Xi_B/\Xi_A) \).

More specifically, Eqs. (114) where \( \Xi = \Xi_A, \Xi_B; y^\dagger = y_{BA}^\dagger, y_{AB}^\dagger \); reduce to:

\[
w_{(\text{int})}^{\infty}(y^\dagger) = \lim_{\Xi \to +\infty} w_{(\text{int})}^{\infty}(\Xi, y^\dagger) = \int_0^{+\infty} F(\xi) \frac{dF(y^\dagger \xi)}{d\xi} \xi d\xi ; \quad y^\dagger = y \geq 1 ; \quad (116a)
\]

\[
w_{(\text{ext})}^{\infty}(y^\dagger) = \lim_{\Xi \to +\infty} w_{(\text{ext})}^{\infty}(\Xi, y^\dagger) = \int_0^{+\infty} F(y^\dagger \xi) \frac{dF(\xi)}{d\xi} \xi d\xi ; \quad y^\dagger = y \geq 1 ; \quad (116b)
\]

replacing \( y^\dagger \) with \( 1/y^\dagger \) yields:

\[
w_{(\text{int})}^{\infty}\left(\frac{1}{y^\dagger}\right) = \int_0^{+\infty} F(\xi) \frac{dF(\xi/y^\dagger)}{d\xi} \xi d\xi ; \quad y^\dagger = y \geq 1 ; \quad (117a)
\]

\[
w_{(\text{ext})}^{\infty}\left(\frac{1}{y^\dagger}\right) = \int_0^{+\infty} F(\xi y^\dagger) \frac{dF(\xi)}{d\xi} \xi d\xi ; \quad y^\dagger = y \geq 1 ; \quad (117b)
\]

which, choosing \( \xi/y^\dagger \) as integration variable, and keeping in mind that integrals are independent of integration variables, is equivalent to:

\[
w_{(\text{int})}^{\infty}\left(\frac{1}{y^\dagger}\right) = y^\dagger \int_0^{+\infty} F(y^\dagger \xi) \frac{dF(\xi)}{d\xi} \xi d\xi ; \quad y^\dagger = y \geq 1 ; \quad (118a)
\]

\[
w_{(\text{ext})}^{\infty}\left(\frac{1}{y^\dagger}\right) = y^\dagger \int_0^{+\infty} F(\xi y^\dagger) \frac{dF(\xi)}{d\xi} \xi d\xi ; \quad y^\dagger = y \geq 1 ; \quad (118b)
\]

and the combination of Eqs. (116a) and (118b); (116b) and (118a); produces:

\[
w_{(\text{int})}^{\infty}\left(\frac{1}{y^\dagger}\right) = y^\dagger w_{(\text{ext})}^{\infty}(y^\dagger) ; \quad (119a)
\]

\[
w_{(\text{ext})}^{\infty}\left(\frac{1}{y^\dagger}\right) = y^\dagger w_{(\text{int})}^{\infty}(y^\dagger) ; \quad (119b)
\]

where, on the other hand, \( 1/y^\dagger = 1/y \leq 1 \) is outside the domain, and the role of the two components should be interchanged therein, according to
Eqs. (114c) and (114d), which makes the above result only mathematically relevant.

Turning to the special case, \( F_A = F_B = F, \Xi_A = \Xi_B = \Xi \), the last implying \( (\nu_A)_{\text{sel}} = (\nu_B)_{\text{sel}} = \nu_{\text{sel}} \), and taking, in addition, \( y_{BA}^\dagger = y_{AB}^\dagger = 1 \), Eqs. (114) reduce to:

\[
\begin{align*}
E_{\text{kin}}(\Xi, 1) &= E_{\text{ext}}(\Xi, 1) = E(\Xi, 1) ; \\
\text{and the fractional virial potential energy, expressed by Eq. (47a), reduces to:}
\end{align*}
\]

\[
\begin{align*}
\phi &= m^\dagger \frac{m^\dagger \nu_{\text{sel}} - (9/8)w(\Xi, 1)}{\nu_{\text{sel}} - (9/8)m^\dagger w(\Xi, 1)} ; \\
y^\dagger &= 1 ; \\
\nu_{\text{sel}} &= \frac{9}{16} \int_0^\Xi F^2(\xi) \, d\xi ; \\
w(\Xi, 1) &= \int_0^\Xi F(\xi) \frac{dF}{d\xi} \xi \, d\xi ; \\
F(\Xi) &= 0 ;
\end{align*}
\]

where Eqs. (121b) and (121d) follow from the definition of \( \nu_{\text{sel}} \) and \( F(\xi) \), respectively. For further details refer to earlier attempts (e.g., Roberts, 1962; Caimmi and Secco, 1992; CV08).

Integrating by parts Eq. (121c) and combining with (121b) and (121d), yields:

\[
w(\Xi, 1) = -\frac{8}{9} \nu_{\text{sel}} ;
\]

finally, substituting Eq. (122) into (121a) produces:

\[
\phi = m^\dagger ; \\
y^\dagger = 1 ;
\]

which is a general result for subsystems with equal scaled density profiles, \( F_A(\xi) = F_B(\xi) = F(\xi), 0 \leq \xi \leq \Xi \).

**B Dimensional macrogas equation of state**

Let macrogases be defined as large-scale collisionless fluids with the following properties: (i) particles are identical mass points; (ii) particle number is extremely large; (iii) particle motions obey Newton laws of mechanics; (iv) particle collisions are absent; (v) particle interactions obey Newton law of gravitation.

Under the assumption of homeoidally striated density profiles, by use of Eq. (41), the virial theorem reads:

\[
2E_{\text{kin}} - \nu_{\text{sel}} \frac{G(M^\dagger)^2}{a_1^\dagger} B = 0 ;
\]
let the macrovolume, \( V_M \), the macropressure, \( p_M \), the macrotemperature, \( T_M \), and the mass weighted rms velocity, \( \sigma_M \), be defined as:

\[
V_M = \frac{4\pi}{3} a_1 a_2 a_3 ; \quad (125)
\]

\[
p_M = \frac{GM^2}{a_1^2 a_2 a_3} ; \quad (126)
\]

\[
kT_M = \frac{2}{3} N E_{\text{kin}} ; \quad (127)
\]

\[
\sigma_M = \left( \frac{2E_{\text{kin}}}{M} \right)^{1/2} ; \quad (128)
\]

where \( N \) is the total number of particles, \( k \) the Boltzmann’s constant, and the index, \( M \), means macrogas. In particular, Eq. (127) discloses that the macrotemperature, \( T_M \), coincides with the temperature of an ideal gas with particle number, \( N \), and translational kinetic energy, \( E_{\text{kin}} \).

Typical values for galaxies are \( M = 3 \cdot 10^{11} \text{m}_{\odot} \), \( N = 3 \cdot 10^{11} \), \( \sigma_M = 100\sqrt{3} \text{ km s}^{-1} \), which yields via Eqs. (127) and (128):

\[
kT_M = \frac{M \sigma_M^2}{3N} = \left( \frac{3}{M} \right) \left( \frac{2E_{\text{kin}}}{M} \right)^{1/2} \cdot (128);
\]

\[
T_M = \frac{10^{14} \cdot 1.99 \cdot 10^{33}}{1.44 \cdot 10^{36}} \text{ K} = 1440 \text{ K},
\]

where \( K_M = 10^{60} \text{ K} \) is the macrodegree, assumed as macrotemperature unit.

The combination of Eqs. (39)-(41) and (124)-(128) yields:

\[
\frac{p_M V_M}{N k T_M} = 4\pi \frac{GM}{a_1 \sigma_M^2} = 4\pi \frac{(\nu_{\text{mas}})^2}{\Xi_{\text{sel}} B} ; \quad (129)
\]

which may be conceived as a compressibility factor.

In presence of a similar, similarly placed, homeoidally striated density profile, by use of Eqs. (41)-(43), the virial theorem for the subsystem under consideration reads:

\[
2(E_u)_{\text{kin}} - \frac{G(M_u)^2}{(a_u)_1} \left( \nu_u \right)_{\text{sel}} \left[ 1 + \frac{(\nu_{uv})_{\text{tid}}}{(\nu_u)_{\text{sel}}} \right] B ; \quad (130)
\]

and the macrogas equation of state is:

\[
(p_u)_M(V_u)_M = \frac{4\pi [(\nu_u)_{\text{mas}}]^2}{\Xi_u (\nu_u)_{\text{sel}} B} \left[ 1 + \frac{(\nu_{uv})_{\text{tid}}}{(\nu_u)_{\text{sel}}} \right]^{-1} N_u k(T_u)_M ; \quad (131a)
\]

\[
u = j, i ; \quad u = i, j ; \quad (131b)
\]

where \( i \) and \( j \) denote the inner and the outer subsystem, respectively. The compressibility factor is:

\[
\frac{(p_u)_M(V_u)_M}{N_u k(T_u)_M} = 4\pi \frac{GM_u}{(a_u)_1[(\sigma_u)_M]^2} = 4\pi \frac{[(\nu_u)_{\text{mas}}]^2}{\Xi_u (\nu_u)_{\text{sel}} B} \left[ 1 + \frac{(\nu_{uv})_{\text{tid}}}{(\nu_u)_{\text{sel}}} \right]^{-1} ; \quad (132a)
\]

\[
u = j, i ; \quad u = i, j ; \quad (132b)
\]
in all cases, the effect of the tidal potential is expressed by the sum within square brackets. If the two subsystems were infinitely distant one from the other, \((\nu_{uv})_{\text{tid}} = 0\), and the above results reduce to their one-component counterparts.

In this view, one-component macrogases should be conceived as “ideal” and two-component macrogases as “VDW”, the related equations of state resembling ideal and VDW gas equation of state, respectively. In fact, the mass ratio, \(m = M_j/M_i\), and the axis ratio, \(y = (a_j)/(a_i) = (a_j)/(a_i) = (a_j)/(a_i)\), appear in the explicit expression of \((\nu_{uv})_{\text{tid}}\). Owing to Eqs. (125) and (126), the following relations hold:

\[
\begin{align*}
\frac{(V_j)_M}{(V_i)_M} &= \frac{(a_j)(a_j)(a_j)}{(a_i)(a_i)(a_i)} = y^3 \quad ; (133) \\
\frac{(p_j)_M}{(p_i)_M} &= \frac{(M_j)(a_j)(a_j)(a_j)}{(M_i)(a_i)(a_i)(a_i)} = \frac{m^2}{y^4} \quad ; (134) \\
\frac{N_j(T_j)_M}{N_i(T_i)_M} &= \frac{(E_j)_\text{kin}}{(E_i)_\text{kin}} = \phi \quad ; (135)
\end{align*}
\]

which show the profile factor, \((\nu_{uv})_{\text{tid}}\), depends on the fractional macrovolume and the fractional macropressure, via \(m\) and \(y\).

The combination of Eqs. (127), (131), (133), and (134) yields:

\[
m^2 \frac{1}{y} = \frac{\Xi_i}{\Xi_j} \left[ \frac{\nu_j}{\nu_i} \right]_{\text{mas}}^2 \left( \nu_i \right)_{\text{sel}} + \left( \nu_{ij} \right)_{\text{tid}} \left( \nu_j \right)_{\text{sel}} + \left( \nu_{ji} \right)_{\text{tid}} \phi \quad ; (136)
\]

which, using Eqs. (15), may be cast under the equivalent form:

\[
X_pX_V \frac{1}{1 - 9\frac{\Xi_i}{\Xi_j} \left( \frac{\nu_j}{\nu_i} \right)_{\text{mas}} \frac{1}{8} \left( \frac{\nu_j}{\nu_i} \right)_{\text{sel}} w^{(\text{int})}(X_V, \Xi_j) \frac{1}{X_p^{1/2} X_V^{1/3}}} = K(\Xi_i, \Xi_j)X_T \quad ; (137a)
\]

\[
K(\Xi_i, \Xi_j) = \frac{\Xi_i}{\Xi_j} \left[ \left( \frac{\nu_j}{\nu_i} \right)_{\text{mas}} \right]^{2} \left( \frac{\nu_j}{\nu_i} \right)_{\text{sel}} \left( \frac{\nu_j}{\nu_i} \right)_{\text{sel}} ; (137b)
\]

\[
X_p = \frac{(p_j)_M}{(p_i)_M} = \frac{m^2}{y^4} \quad ; \quad X_V = \frac{(V_j)_M}{(V_i)_M} = y^3 \quad ; \quad X_T = \frac{N_j(T_j)_M}{N_i(T_i)_M} = \phi \quad ; (137c)
\]

that is the macrogas fractional equation of state. A simpler and more intuitive choice, adopted in the text, is \(X_p = m^2\), \(X_V = 1/y\), but the connection with the fractional macropressure and fractional macrovolume is lost in this case.
Figure 2: Same as in Fig.1 (right panel), where the occurrence (within the bell-shaped area bounded by the dashed curve) of saturated vapour is considered. Above the critical isothermal curve ($T = T_c$) the trend is similar with respect to ideal gases. Below the critical isothermal curve and on the right of the dashed curve, the gas still behaves as an ideal gas. Below the critical isothermal curve and on the left of the dashed curve, the liquid shows little change in volume as the pressure rises. Within the bell-shaped area bounded by the dashed curve, the liquid phase is in equilibrium with the saturated vapour phase. A reduced volume implies smaller saturated vapour fraction and larger liquid fraction at constant pressure, and vice versa. The VDW equation of state is no longer valid in this region. The dashed curve (including the central branch) is the locus of intersection between VDW and real isothermal curves, the latter being related to constant pressure where liquid and vapour phases coexist. The dotted curve is the locus of VDW isothermal extremum points.
Figure 3: A specific \( T/T_e = 0.85 \) VDW and corresponding real isothermal curve. The above mentioned curves coincide within the range, \( V \leq V_A \) and \( V \geq V_E \). The VDW isothermal curve exhibits two extremum points: a minimum, \( B \), and a maximum, \( D \), while the real isothermal curve is flat within the range, \( V_A \leq V \leq V_E \). Configurations related to the VDW isothermal curve within the range, \( V_A \leq V \leq V_B \) (due to tension forces acting on the particles yielding superheated liquid), and \( V_D \leq V \leq V_E \) (due to the occurrence of undercooled vapour), may be obtained under special conditions, while configurations within the range, \( V_B \leq V \leq V_D \), are always unstable. The volumes, \( V_A \) and \( V_E \), correspond to the maximum value in presence of the sole liquid phase and the minimum value in presence of the sole vapour phase, respectively. The regions, \( ABC \) and \( CDE \), have equal area. For further details refer to the text.
Figure 4: Macroisothermal curves related to IUU (left panel) and AUU (right panel) macrogases, respectively. Macroisothermal curves (from bottom to top) correspond to $X_T = 0.85, 0.90, 0.95, 1.00, 1.05, 1.10$. No critical macroisothermal curve exists, above or below or above which the extremum points disappear. The coordinates, $X_V = X_V^\dagger$, $X_p = X_p^\dagger$, $X_T = X_T^\dagger$, may be conceived as normalized to their fictitious critical counterparts, $X_{V_c} = X_{V_c}^\dagger = 1$, $X_{p_c} = X_{p_c}^\dagger = 1$, $X_{T_c} = X_{T_c}^\dagger = 1$. 
Figure 5: Same as in Fig. 4 (right panel). The loci of the extremum points are represented as dotted lines. The loci of the intersections between actual and real macroisothermal curves are represented as dashed lines. The absence of the critical macroisothermal curve makes a band-like instead of a bell-shaped region exist on the plane. The coordinates, $X_V = X_V^\dagger$, $X_p = X_p^\dagger$, $X_T = X_T^\dagger$, may be conceived as normalized to their fictitious critical counterparts, $X_{Vc} = X_{Vc}^\dagger = 1$, $X_{pc} = X_{pc}^\dagger = 1$, $X_{Tc} = X_{Tc}^\dagger = 1$. 
Figure 6: A specific ($X_T = 0.85$) AUU and corresponding RUU macroisothermal curve. The above mentioned curves coincide within the range, $X_V \leq X_{V_A}$, $X_V \geq X_{V_E}$. The AUU macroisothermal curve exhibits two extremum points: a minimum, $B$, and a maximum, $D$, while the RUU macroisothermal curve is flat within the range, $X_{V_A} \leq X_V \leq X_{V_E}$. The regions, $ABC$ and $CDE$, have equal area. The coordinates, $X_V = X^\dagger_V$, $X_p = X^\dagger_p$, $X_T = X^\dagger_T$, may be conceived as normalized to their fictitious critical counterparts, $X_{V_c} = X^\dagger_{V_c} = 1$, $X_{p_c} = X^\dagger_{p_c} = 1$, $X_{T_c} = X^\dagger_{T_c} = 1$. 

$\text{Figure 6: A specific ($X_T = 0.85$) AUU and corresponding RUU macroisothermal curve. The above mentioned curves coincide within the range, } X_V \leq X_{V_A}, X_V \geq X_{V_E}. \text{ The AUU macroisothermal curve exhibits two extremum points: a minimum, } B, \text{ and a maximum, } D, \text{ while the RUU macroisothermal curve is flat within the range, } X_{V_A} \leq X_V \leq X_{V_E}. \text{ The regions, } ABC \text{ and } CDE, \text{ have equal area. The coordinates, } X_V = X^\dagger_V, X_p = X^\dagger_p, X_T = X^\dagger_T, \text{ may be conceived as normalized to their fictitious critical counterparts, } X_{V_c} = X^\dagger_{V_c} = 1, X_{p_c} = X^\dagger_{p_c} = 1, X_{T_c} = X^\dagger_{T_c} = 1. \text{ }$
Figure 7: Macroisothermal curves (\( X_p = X_p/X_p^c \) vs. \( X_V = X_V/X_V^c \)) related to IHH (left panels) and AHH (right panels) macrogases, respectively, for different values of scaled truncation radii, \((\Xi_i, \Xi_j)\), labelled on each panel. Macroisothermal curves (from bottom to top) correspond to \( X_T = X_T/X_T^c = 0.90, 0.95, 1.00, 1.05, 1.10, 1.15 \). The limit, \((\Xi_i, \Xi_j) \to (+\infty, +\infty)\), makes only little changes. Owing to Eqs. (42a) and (51), \( X_V^\dagger = X_Vy^\dagger/y \) and \( X_p^\dagger = X_p(m^\dagger/m)^2 \), which implies \( X_V^\dagger = X_V \) and \( X_p^\dagger = X_p \).
Figure 8: AHH macroisothermal curves (\(X_p = X_p/X_{p_c}\) vs. \(X_V = X_V/X_{V_c}\)) for different choices of scaled truncation radii, \((\Xi_i, \Xi_j)\), labelled on each panel. Macroisothermal curves (from bottom to top) correspond to \(X_T = X_T/X_{T_c} = 0.90, 0.95, 1.00, 1.05, 1.10, 1.15\). The limit, \((\Xi_i, \Xi_j) \to (+\infty, +\infty)\), makes only little changes. RHH macroisothermal curves, when different from their AHH counterparts, lie within the larger reversed bell-shaped region in each panel. The loci of intersections between AHH and RHH macroisothermal curves are represented as trifid curves, where the left branch corresponds to \(X_{\lambda V}\), the right branch to \(X_{\mu V}\), and the middle branch to \(X_{\nu V}\). The critical point is the common origin. The loci of AHH macroisothermal curve extremum points are represented as dotted curves starting from the critical point, where the left branch corresponds to minimum points and the right branch to maximum points. Owing to Eqs. (42a) and (51), \(X_V^\dagger = X_V y^\dagger/y\) and \(X_p^\dagger = X_p (m^\dagger/m)^2\), which implies \(X_{\lambda V}^\dagger = X_{\lambda V}\) and \(X_p^\dagger = X_p\).
Figure 9: AHH macroisothermal curves for scaled truncation radii, \((\Xi_i, \Xi_j)\), as in Fig. 8, with regard to the special case, \(X_T = X_{Tc} = 1.15\). The limit, \((\Xi_i, \Xi_j) \rightarrow (+\infty, +\infty)\), makes only little changes, as shown by comparison with the dashed curve. Owing to Eqs. (42a) and (51), \(X_V^\dagger = X_{Vc}^{\dagger} = U_T^{\dagger} / y\) and \(X_p^\dagger = X_p (m^\dagger / m)^2\), which implies \(X_V^\dagger = X_v\) and \(X_p^\dagger = X_p\).
Figure 10: Macroisothermal curves \( \bar{X}_p = X_p/X_{pc} \) vs. \( \bar{X}_V = X_V/X_{vc} \) related to IHN/INH (left panels) and AHN/ANH (right panels) macrogases, respectively, for different values of scaled truncation radii, \((\Xi_i, \Xi_j)\), labelled on each panel. Macroisothermal curves (from bottom to top) correspond to \( \bar{X}_T = X_T/X_{Tc} = 0.90, 0.95, 1.00, 1.05, 1.10, 1.15 \). The limit, \((\Xi_i, \Xi_j) \to (+\infty, +\infty)\), makes only little changes. Owing to Eqs. (42a) and (51), \( \bar{X}_V = X_V y^\dagger/y \) and \( \bar{X}_p = X_p (m^\dagger/m)^2 \), which implies \( \bar{X}_V = \bar{X}_V \) and \( \bar{X}_p = \bar{X}_p \).
Figure 11: AHN/ANH macroisothermal curves ($X_p = X_p/X_{pc}$ vs. $X_V = X_V/X_{vc}$) for different choices of scaled truncation radii, $(\Xi_i, \Xi_j)$, labelled on each panel. Macroisothermal curves (from bottom to top) correspond to $X_T = X_T/X_{Tc} = 0.90, 0.95, 1.00, 1.05, 1.10, 1.15$. The limit, $(\Xi_i, \Xi_j) \rightarrow (+\infty, +\infty)$, makes only little changes. RHN/RNH macroisothermal curves, when different from their AHN/ANH counterparts, lie within the large reversed bell-shaped region in each panel. The loci of intersections between AHN/ANH and RHN/RNH macroisothermal curves are represented as trifid curves, where the left branch corresponds to $X_{Va}$, the right branch to $X_{Ve}$, and the middle branch to $X_{Vc}$. The critical point is the common origin. The loci of AHN/ANH macroisothermal curve extremum points are represented as dotted curves starting from the critical point, where the left branch corresponds to minimum points and the right branch to maximum points. Owing to Eqs. (42a) and (51), $X_{Va} = X_V y^/y$ and $X_{p} = X_p (m^/m)^2$, which implies $X_{Va} = X_V$ and $X_{p} = X_p$. 

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Figure 12: AHN/ANH macroisothermal curves ($\xi_p = X_p/X_{pc}$ vs. $\xi_V = X_V/X_{vc}$) for scaled truncation radii, $(\Xi_i, \Xi_j)$, as in Fig.11 with regard to the special case, $\xi_T = X_T/X_{Tc} = 1.15$. The limit, $(\Xi_i, \Xi_j) \to (+\infty, +\infty)$, makes only little changes, as shown by comparison with the dashed curve. Owing to Eqs. (42a) and (51), $X_V^\dagger = X_V y^\dagger/y$ and $X_p^\dagger = X_p (m^\dagger/m)^2$, which implies $\xi_V^\dagger = \xi_V$ and $\xi_p^\dagger = \xi_p$. 

$x_p/x_{pc}=1.15$

Figure 12: AHN/ANH macroisothermal curves ($\xi_p = X_p/X_{pc}$ vs. $\xi_V = X_V/X_{vc}$) for scaled truncation radii, $(\Xi_i, \Xi_j)$, as in Fig.11 with regard to the special case, $\xi_T = X_T/X_{Tc} = 1.15$. The limit, $(\Xi_i, \Xi_j) \to (+\infty, +\infty)$, makes only little changes, as shown by comparison with the dashed curve. Owing to Eqs. (42a) and (51), $X_V^\dagger = X_V y^\dagger/y$ and $X_p^\dagger = X_p (m^\dagger/m)^2$, which implies $\xi_V^\dagger = \xi_V$ and $\xi_p^\dagger = \xi_p$. 

$x_p/x_{pc}=1.15$
Figure 13: Elliptical galaxies listed in Tables 7 and 8 modelled as HH macrogases for different choices of scaled truncation radii, $\Xi_i$, $\Xi_j$, and fractional mass, $m$. The critical macroisothermal curve (left) and the boundary of the GS region (right) are also plotted for each case. Symbol caption and line style: $(\Xi_i, \Xi_j) = (10, 5)$ - crosses, full; $(10, 10)$ - asterisks, dotted; $(10, 20)$ - diamonds, dashed; $(20, 5)$ - triangles, dot-dashed; $(20, 10)$ - squares, long-short-dashed; $(20, 20)$ - St. Andrew's crosses, long-dashed; $(+\infty, +\infty)$ - dots, full. Lower and upper symbols of the same kind are related to $m = 10, 20$, respectively. The composite symbol marks the critical point. Cases where $\Xi_i = 5$ make two galaxies unable to be modelled, and for this reason are not considered.
Figure 14: Same as in Fig. 13 for scaled truncation radii, $(\Xi_i, \Xi_j) = (10, 10)$, $(10, 20)$, $(20, 10)$, $(20, 20)$, from top to bottom, and fractional mass, $m = 20$, where model galaxies are distinguished according if their parent sample object is classified as fast (squares) or slow (diamonds) rotator.
Figure 15: Same as in Fig. 13 for different choices of scaled truncation radii, $(\Xi_i, \Xi_j) = (10,10), (10,20), (20,10), (20,20)$, from top to bottom, and fractional masses, $m = 20$, where dimensionless energies, $c_{ij}$, defined by Eq. (103a), are lowered to $c_{ij} - \Delta c_{ij}$ (Greek crosses) and increased to $c_{ij} + \Delta c_{ij}$ (St. Andrew’s crosses) with respect to their original values (diamonds), by a factor equal to 5%, 10%, 15%, 20%, respectively. In the last case, a sample object (NGC 4473) cannot be modelled for lowered values, $c_{ij} - \Delta c_{ij}$, and $(\Xi_i, \Xi_j) = (10,10), (10,20)$. 
Figure 16: Elliptical galaxies listed in Tables 7 and 8 modelled as HN/NH macrogases for different choices of scaled truncation radii, Ξ_i, Ξ_j, and fractional mass, m. The critical macroisothermal curve (left) and the boundary of the GS region (right) are also plotted for each case. Symbol caption and line style: (Ξ_i, Ξ_j) = (10,5) - crosses, full; (10,10) - asterisks, dotted; (10,20) - diamonds, dashed; (20,5) - triangles, dot-dashed; (20,10) - squares, long-short-dashed; (20,20) - St. Andrew’s crosses, long-dashed. Lower and upper symbols of the same kind are related to m = 10, 20, respectively. The composite symbol marks the critical point. Cases where Ξ_i = 5 make two galaxies unable to be modelled, and for this reason are not considered. The same holds, to a larger extent, for cases (Ξ_i, Ξ_j) → (+∞, +∞), due to an infinite mass of the NFW density profile.
Figure 17: Same as in Fig. 16 for scaled truncation radii, \((\Xi_i, \Xi_j) = (10, 10), (10, 5), (20, 10)\), from top to bottom, and fractional mass, \(m = 20, \Xi_j = 10\), and \(m = 10, \Xi_j = 5\), where model galaxies are distinguished according if their parent sample object is classified as fast (squares) or slow (diamonds) rotator.
Figure 18: Same as in Fig. 16 for different choices of scaled truncation radii and fractional masses, \((\Xi_i, \Xi_j, m) = (10, 10, 20), (10, 5, 10), (20, 10, 20)\), from top to bottom, where dimensionless energies, \(c_{ij}\), defined by Eq. (103a), are lowered to \(c_{ij} - \Delta c_{ij}\) (Greek crosses) and increased to \(c_{ij} + \Delta c_{ij}\) (St. Andrew’s crosses) with respect to their original values (diamonds), by a factor equal to 5%, 10%, 15%, 20%, respectively. In the last case, a sample object (NGC 4473) cannot be modelled for lowered values, \(c_{ij} - \Delta c_{ij}\), and \((\Xi_i, \Xi_j, m) = (10, 5, 10)\), (10,10,20), and a sample object (NGC 4486) is out of scale on the right for increased values, \(c_{ij} + \Delta c_{ij}\), and \((\Xi_i, \Xi_j, m) = (10, 5, 10)\).