Beyond Basins of Attraction: Evaluating Robustness of Natural Dynamics

Steve Heim, Alexander Spröwitz
Dynamic Locomotion Group
Max Planck Institute for Intelligent Systems, Germany

Abstract—It is commonly accepted that properly designing a system to exhibit favorable natural dynamics can greatly simplify designing or learning the control policy. It is however still unclear what constitutes favorable natural dynamics, and how to quantify its effect. Most studies of simple walking and running models have focused on the basins of attraction of passive limit-cycles, and the notion of self-stability. We emphasize instead the importance of stepping beyond basins of attraction. We show an approach based on viability theory to quantify robustness, valid for the family of all robust control policies. This allows us to evaluate the robustness inherent to the natural dynamics before designing the control policy or specifying a control objective. We illustrate this approach on simple spring mass models of running and show previously unexplored advantages to using a nonlinear leg stiffness. We believe designing robots with robust natural dynamics is particularly important for enabling learning control policies directly in hardware.

I. INTRODUCTION

Animals are not only agile and efficient, but also remarkably adaptable and robust [1], [2], with arguably simple control and morphology [3]–[5]. It has been difficult to reproduce this performance in legged robots. Most robots use sophisticated algorithms [6]–[9] which rely on accurate models and state-estimation at a substantial computational cost. This reliance tends to make this approach brittle.

Recently there have been attempts to combine these approaches with machine learning to improve robustness and adaptability [10]–[12], however it is notoriously difficult to apply learning directly in hardware. We are largely motivated by the question ‘how should a legged robot be designed, so that it is easier to apply model-free learning directly in hardware?’ A key component to answering this is the inherent robustness of the natural dynamics of the system.

Indeed, designing a system with favorable natural dynamics can simplify the control problem [13]–[15] as well as enable quick learning directly in hardware [16], [17]. It is however still unclear how to quantify and evaluate the effects of mechanical design choices on the control problem, especially in terms of robustness and ease of designing or learning the control policy.

Many studies focus around the concept of self-stability, and the basins of attraction of passively stable limit-cycles [18]–[21]. In this study, we advocate the importance of stepping away from thinking in terms of basins of attraction and limit-cycles. Our results are related to the results on robustness of simple walking models by Zaytsev et al. [22]. Aside from the minor difference of studying running instead of walking models, our work emphasizes an evaluation of the robustness to noise in action space. This allows us to quantify robustness for the family of all possible robust control policies, without specifying a control policy or even a control objective aside from ‘avoid failure’. In other words we quantify the robustness inherent to the natural dynamics due to mechanical design. We then use this quantification to compute the set of states that can be robustly stabilized. As a case study, we apply it to two idealized running models and expose previously unexplored advantages of using a nonlinear compared to a linear leg stiffness.

A. Natural Dynamics and Spring Mass Models

Perhaps the clearest example of natural dynamics is Tad McGeer’s passive dynamic walker [23]: this purely mechanical system with no sensors or actuators (and hence no control) exhibits passively stable limit-cycles for downhill walking, which look remarkably human-like. This idea has been extended in several robots, adding a little actuation and control in order to allow it to walk on level ground [24], [25], and more importantly to increase the basin of attraction of the passively stable limit-cycle. A key concept is to exploit the natural dynamics. The intuition behind this concept is that the control can be ‘lazy’: if a perturbation ever pushes the system out of the basin of attraction, the control should guide it back in. Once the state is inside the basin of attraction, the control policy can essentially let go and allow the system to naturally evolve to the attracting limit cycle.

Simulation studies of idealized walking models such as the rimless wheel [26] and compass walker [27] have provided more understanding of McGeer’s empirical results. These models also have passively stable limit cycles, albeit with rather small basins of attraction.

For running we turn to a different idealized model, the spring mass model, which is the focus of this work. This model is arguably the simplest and most studied model for running [28]–[32]. Numerical studies show that it also has passively stable limit cycles, with a fractal basin of attraction [33]. Rummel and Seyfarth [21], [24], as well as others [35], [36], have studied bifurcation diagrams for different spring
configurations, showing that nonlinear leg compliance improves robustness to state perturbations for passive running systems. The leg compliance can be achieved either through a different mechanical spring design or through control, simulating a virtual spring. With their numerical studies, Rummel et al. show that a nonlinear leg compliance has a larger range of parameters that exhibit passively stable limit-cycles, compared to the standard linear spring mass model. These limit-cycles also tend to have larger basins of attraction. However, at higher velocities the model with nonlinear spring stiffness no longer exhibits passively stable limit cycles, whereas with a linear spring this property is retained. These results suggested nonlinear compliance is only beneficial at lower running speeds [21].

Another important result from these models is that the continuous-time natural dynamics during the stance phase allow for a simple discrete controller to choose a new landing angle of attack once per step and achieve open-loop deadbeat control [37], [38]. There has been a lot of effort to translate control policies designed around these simple models to real-world robots. Many robots are built to mimic the natural dynamics exhibited by the spring mass model to some degree [14], [39]–[41]. While achieving the same behavior of an idealized model in real hardware can pose its own challenges, we emphasize that behaving exactly like a spring mass model is not necessary to exploit the advantages exposed with these models, especially if we drop the requirement of achieving deadbeat control and rather focus on robustness or ease of learning.

B. Some Notes on Terminology

We use terminology common to the reinforcement learning community, such as actions instead of control inputs, and control policies instead of controllers. We will speak of control policies sampling an action, or the system sampling a state-action pair, to indicate the policy may be stochastic.

Much of the mathematics in the paper revolves around sets in different spaces. Capital letters such as $S$ denote spaces (in this case state space). Capital letters with an underscore such as $S_i$ denote a set in the corresponding space, the meaning of the underscore being explained in the text (in this case the set of failure states).

C. Structure

In Section II we cover the details of the two spring mass models we examine, their dynamics, and a typical bifurcation diagram for the SLIP model. In Section III we compute the viability kernel as well as the transition map in state-action space. We illustrate how this encompasses the bifurcation diagram, and why it is limiting once we introduce control. In Section IV we introduce our definitions of robustness, and how to use this to evaluate two different designs of leg compliance prior to designing a control policy. In Section V we summarize the key contributions of the paper, the challenge of scalability and how we intend to leverage these results for designing legged robots that are amenable to model-free learning.

II. SPRING MASS MODELS

We use two well studied spring mass models to illustrate our concepts: the spring-loaded inverted pendulum (SLIP) model and a nonlinear spring mass (NSLIP) model as first studied by Rummel and Seyfarth [21] (see Fig. 1). Both models have hybrid dynamics, with the governing equations of motion switching between flight and stance phases.

During flight phase the body follows a ballistic trajectory, whereas during the stance phase it follows a spring mass motion, which depends on the modeled spring. The details of the equations of motion have been derived in [21], [28], and can be found in the appendix. For convenient comparison we use the same parameters as in [21], which are similar to human averages.

A. Discrete Analysis via Poincaré Sections

The continuous motion of the point-mass body is fully described in cartesian coordinates by the state vector $[x, y, \dot{x}, \dot{y}]^\top$. We simplify analysis by evaluating the state only on a Poincaré section at flight apex, a common approach for running motion. At flight apex, potential and kinetic energy are conveniently contained in the vertical position and forward velocity respectively. Thus, the continuous state vector $[x, y, \dot{x}, \dot{y}]^\top$ can be reduced to $[y, \dot{x}]^\top$. Taking advantage of the constant energy constraint, we can further reduce the system to a single state, the normalized apex height $s$, which defines our state space:
\[
\begin{align*}
  s &= \frac{E_{\text{pot}}}{E_{\text{kin}} + E_{\text{pot}}} = \frac{g y}{\frac{1}{2} + g y} \\
  \text{State Space: } s &\in S = [0, 1]
\end{align*}
\]

where \( E_{\text{pot}} \) and \( E_{\text{kin}} \) are potential and kinetic energy, respectively, and \( g \) is the gravitational constant.

Starting from any state at apex \( s \), we can numerically integrate the continuous time dynamics until the system either transitions to a second apex height or to a failure state. We thus obtain the Poincaré map, also called a transition map, for our discrete dynamics:

\[
s_{k+1} = P(s_k, \alpha)
\]

where the landing angle of attack \( \alpha \) is a model parameter of interest, which we shall use as our control action in section III.

We will consider as failures all states in which the body hits the ground with \( y = 0 \), as well as when the system reverses direction with \( \dot{x} < 0 \). More formally,

\[
\text{Failure Set } S_F := \{ s : y = 0 \text{ or } \dot{x} < 0 \}
\]

B. Bifurcation Diagram of the SLIP Model

A bifurcation diagram allows the study of the existence and stability of fix-points and limit-cycles, as a dependence of the model parameters.

We show in Fig. 2 the bifurcation diagram of the SLIP model, in this case with the angle of attack \( \alpha \) chosen as the parameter of interest. Similar bifurcation diagrams for spring mass models of running can be found in [42], and bifurcation diagrams for spring stiffness of these models can be found in [21], [30].

![Bifurcation Diagram](image)

**Fig. 2:** The bifurcation diagram of the passive SLIP model highlights the small range of parameters for which stable limit-cycles exist. The basins of attraction are bounded by infeasibility and unstable limit-cycles. Beyond these basins of attraction however, is a lot of structure that can be exploited through control.

We only evaluate period-1 limit-cycles, that is when \( s_{k+1} = s_k \), and do not consider limit-cycles which require multiple iterations to return to periodicity. Stable limit-cycles are marked with a solid red line and unstable limit-cycles with a dashed red line. The basins of attraction of the stable limit-cycles are highlighted by the shaded area.

These basins of attraction are bounded from below by an infeasibility constraint: below this line the foot would begin underground. The unstable limit-cycles bound the basins of attraction from above: being perturbed onto an unstable limit-cycle will keep the system at that new state, and beyond this threshold it will diverge till the model fails.

Since the basins of attraction for spring mass models are bounded by either infeasibility or unstable limit-cycles, many previous studies have been limited to identifying these bounds. The relevant range of parameters and states for studying basins of attraction tends to be narrow, as illustrated in Fig. 2. We will show in the next section that there is a lot of structure outside the basins of attraction of these passively stable limit-cycles. Once we allow parameters such as the angle of attack \( \alpha \) to be chosen as a control decision, the relevant bounds are no longer the bounds of the basins of attraction, but those of failure and viability.

III. NATURAL DYNAMICS AND VIALBE CONTROL

We begin the section by introducing control, then evaluate the effect the natural dynamics has on the set of possible control policies. A key concept is the link between the viability kernel, in state space, and the set of viable state-action pairs, in state-action space, also called “q-space”.

A. Control Policies and State-Action Space

We will now allow the system to choose the landing angle of attack \( \alpha \) freely at each flight apex. This defines our action space \( A \):

\[
a = \alpha
\]

where \( a \) is any action in \( A \). In our figures we only show the relevant range, and crop out the range which only contains failures or infeasible state-action pairs.

A control policy \( \pi \) is any function that maps a state to an action, \( a = \pi(s) \). As such, a policy spans the combined state and action spaces, which we term \( Q \)-space, after q-learning [43] in reinforcement learning.

B. Transition Map

We compute detailed grids of various state and action pairs. We thus obtain a lookup table of the transition map \( P(s_k, a_k) \), visualized in the state-action space \( Q \) in Fig. 3 for the SLIP model, and in Fig. 4 for the NSLIP model.

To highlight the limit-cycles, we use a color-map centered around \( s_k - s_{k+1} \). The warm and cold colored regions correspond to state-action pairs that result in a higher or lower state, respectively. The gray regions are state-action pairs which result in a failure state \( P(s_k, a_k) \in S_F \), and the black regions are infeasible, where the foot starts underground.
The lookup table of the SLIP model’s transition map shows all possible combinations of state (height at apex) and action (landing angle of attack), and their transition to either a second apex or a failure. State-action pairs in the grey region result in failure. State-actions in the warm and cold colored regions result in hopping higher and lower respectively, with the color indicating the change in state (vertical axis) at the next apex. Also marked are passively stable (solid red) and unstable (dashed red) limit-cycles, where the state does not change. The basin of attraction of a passively stable limit-cycle is bounded from below by an infeasibility constraint (below which the foot would start underground), and from above by an unstable limit-cycle.

We call the grey region the set of failing state-action pairs $Q_F$. Its complement, the colored region, is the non-failing set of state-action pairs $Q_N$. More formally,

$$Q_N := \{(s_k, a_k) : P(s_k, a_k) \notin S_F\} \quad (5)$$

We denote the projection of $Q_N$ onto the state space $S$ as the set $S_N = \text{proj}_S(Q_N)$. Throughout the paper, we always use orthogonal projections, that is

$$\text{proj}_S(s, a) = s. \quad (6)$$

$S_N$ is the set of controllable states, from which actions that avoid failure can be selected. More formally,

$$S_N := \{s_k : \exists a_k \text{ subject to } P(s_k, a_k) \notin S_F\} \quad (7)$$

C. Viable Sets

A viability kernel is the set of all states for which there is at least one time-evolution of the system which remains in the set for all time [44]. Since all state-action pairs $(s, a) \in Q_N$ result in at least a second step, all $s \in S_N$ have at least one failure-preventing action available. However, it is possible for a non-failing state-action pair to reach a state from which all solutions eventually reach a failed state, as was previously examined in [45]. In other words, there can be states from which immediate failure is avoidable, but from which the system is nonetheless doomed to fail within some finite time. Thus the viability kernel, which we will call $S_V$, is a subset of $S_N$.

We can compute the discretized set of viable state-action pairs $Q_V$ and its projection $S_V$ iteratively:

Algorithm 1 Compute Viable Sets

```plaintext
procedure VIABLE SETS($P, Q_N$)
    $Q_V \leftarrow Q_N$
    $S_V \leftarrow \{ \}$
    while $S_V \neq \text{proj}_S(Q_V)$ do
        $S_V \leftarrow \text{proj}_S(Q_V)$
        for all $s_{k+1} = P(s_k, a_k) \in Q_V$ do
            if $s_{k+1} \notin S_V$ then
                Remove $(s_k, a_k)$ from $Q_V$
        end for
    end while
    return $Q_V$, $S_V$
end procedure
```

For the spring-mass models we examine, $Q_V$ is equal or almost equal to $Q_N$ except in unusual corner cases. We can now compare the resulting $Q_V$ and $S_V$ for the SLIP and the NSLIP models (Fig. 5). Although the set of viable states $S_V$ is the same in both models, the set of viable state-action pairs $Q_V$ is much larger for the NSLIP model. This suggests unexplored benefits of nonlinear leg compliance, which is the focus of this paper.
D. Family of Viable Control Policies

A control policy \( \pi(s) \) must be able to sample from \( Q_V \) with non-zero probability, otherwise it will always fail in a single step. All meaningful policies must sample from \( Q_V \) with non-zero probability, or it will always fail in finite time. In order to avoid failure from every viable state for all time, a policy must sample exclusively from \( Q_V \). We call the set of all such policies the family of viable control policies. More formally, if the set \( Q_V \) is non-empty, we also have a non-empty set \( \Pi_V \) of viable policies, where

\[
\forall s_k \in S_V \exists \pi(s_k) \in \Pi_V, \ a_k = \pi(s_k) : (s_k, a_k) \in Q_V \text{ and } P(s_k, a_k) \in S_V \forall k
\]  

(8)

Both the shape and size of \( Q_V \) poses constraints on the control policies \( \pi(s) \in \Pi_V \) that we can design. The size of the projection of \( Q_V \) onto \( S \) determines the viability kernel \( S_V \). The width of \( Q_V \) in the dimensions of action space allows more flexibility in designing a viable control policy. Imagine for example a set \( Q_V \) defined by a single line covering all of \( S \): defined by a surjective function \( f(s) \). While the viability kernel \( S_V = S \) is maximal, there is exactly one control policy \( \pi(s) = f(s) \) that avoids failure. This can make it not only difficult to design or learn the control policy, but is also very brittle, as we will discuss in the next section.

IV. Robust Natural Dynamics

We define robustness as the ability of a system to avoid failure in the face of uncertainty. A key objective of this work is to evaluate the robustness inherent to the natural dynamics: we care about the robustness resulting from the mechanical design, before specifying the control policy, policy parameterization or even the exact control objective (such as converging to a specific limit cycle).

To this end, we focus on uncertainty in action-space, in other words the effect of noise on the control policy output. We will use this basis to also examine robustness to perturbations in state-space for the family of all robust controllers. We briefly discuss the link of action noise to state-estimation noise. We do not discuss model uncertainty, and leave this to future work.

A. Computing Robust Sets

Noise in the action space causes the system to sample a state-action pair with a different action than chosen by the policy:

\[
a = \pi(s_k) + \eta_a
\]

(9)

\[
s_{k+1} = P(s_k, \pi(s_k) + \eta_a)
\]

(10)

where \( \eta_a \) is some form of noise. A robust control policy needs to ensure that the chosen output never causes the system to fail despite this noise, for all time. More formally,

\[
\text{If } \pi(s_k) \in \Pi_R \text{ and } \eta_a \in U
\]

Then \( s_{k+1} = P(s_k, \pi(s_k) + \eta_a) \notin S_F \forall k
\]  

(11)

A hypersurface for arbitrary dimensional state-action space.
where $\Pi_R$ is the family of all robust control policies. For simplicity we will consider noise sampled from a symmetrical bounded set $\eta_\alpha \in U = [-\eta, \eta]$, where $\eta$ is some finite scalar. As our only control output is the landing angle of attack $\alpha$, $\eta$ is limited to $[0, 180^\circ]$). Our policy essentially chooses the axis of a cone with aperture $2\theta$, and $\alpha$ will be sampled from inside this cone.

When considering unbounded noise (such as Gaussian noise), similar arguments hold in a probabilistic sense: instead of being able to guarantee that certain state-action pairs allow the system to never fail, we can only guarantee that it will not fail with a certain probability.

The effect of action noise reduces the space available for controller design in two ways. First, the output of the control policy $a = \pi(s_k)$ must be sufficiently distant from failing state-action pairs, such that the added noise never causes an immediate failure. The second requirement is similar to that for viability: the state must always land in a state from which it can continue to sample robustly, for all time. More formally, we want that

\[ s_k \in S_R, \; \pi(s_k) \in \Pi_R, \; \eta_\alpha \in U : \]
\[ P(s_k, \pi(s_k) + \eta_\alpha) \in S_R \; \forall k \]  \hspace{1cm} (12)

We call $Q_R$ the robust control policy design set. Similar to the relation between $\Pi_V$ and $Q_V$, policies in the set $\Pi_R$ must sample exclusively from $Q_R$ in order to avoid failure for any state $s_k \in S_R$ where $S_R = \text{proj}_S(Q_R)$. Such sets are shown in Fig. 5 for various amounts of noise $\eta$. Each of these sets is computed with the following iterative process:

**Algorithm 2 Compute Robust Sets**

```plaintext
procedure ROBUST SETS($P, Q_V, U$
\[ Q_R \leftarrow Q_V \]
\[ S_R \leftarrow \{ \} \]
while $S_R \neq \text{proj}_S(Q_R)$ do
\[ S_R \leftarrow \text{proj}_S(Q_R) \]
for all $(s_k, a_k) \in Q_R$ do
\[ \text{for all } \eta_\alpha \in U \text{ do} \]
\[ \text{if } (s_k, a_k + \eta_\alpha) \notin Q_R \text{ then} \]
\[ \text{Remove } (s_k, a_k) \text{ from } Q_R \]
\[ \text{Break} \]
\[ \text{if } (s_{k+1} = P(s_k, a_k + \eta_\alpha) \notin S_R \text{ then} \]
\[ \text{Remove } (s_k, a_k) \text{ from } Q_R \]
\[ \text{Break} \]
end for
end for
return $Q_R, S_R$
end procedure
```

An important feature is that the computation of $Q_R$ depends only on the set $Q_V$, the transition map $P$ and the noise set $U$, and is valid for the family of all robust control policies $\Pi_R$. It does not depend on the exact choice of policy $\pi(s_k)$. In other words, we can evaluate the robustness inherent to the natural dynamics, before we design the control policy or define a control objective other than ‘avoid failure’.

**B. Evaluating Robustness of Different Legs**

We compare the robustness of the SLIP and NSLIP models for varying amounts of noise, as shown in Fig. 5.

With the SLIP model, $Q_R$ and $S_R$ become empty sets for noise greater than $\pm 10.75^\circ$, whereas in the NSLIP model the upper threshold is almost twice as large, at $\pm 20.0^\circ$.

For any given amount of noise, the size of the set $Q_R$ is also much greater for the NSLIP than for the regular SLIP model. The larger size of $Q_R$ means there is more flexibility to fulfill robustness requirements while also designing a control policy around different criteria.

Furthermore, action noise is one of the most common methods of introducing exploration in learning, for example with Gaussian policies [43], [46]. The amount of noise needs to be carefully balanced: a lot of noise allows for more aggressive exploration, but it can also keep the agent from converging to the true optimum, as well as lead to unstable behaviors ending in failed states. This can be particularly troublesome for learning in hardware, requiring more samples as well as potentially damaging the robot. Robustness to action uncertainty allows for more aggressive and effective exploration during learning. This is particularly important for applying model-free learning directly in hardware.

**C. Robustness to State Perturbations**

The projection of the robust policy design set onto state-space, $S_R = \text{proj}_S(Q_R)$, is the set of robust states, from which any robust policy $\pi \in \Pi_R$ can always recover. Interestingly, with small amounts of noise up to $\eta < 5$ [deg], $S_R$ remains the same for both the SLIP and NSLIP models (see Fig. 6). For greater amounts of noise, it shrinks more rapidly for the SLIP model.

The set $S_R$ is particularly useful for choosing the specific control objective. For example, if we expect perturbations in state-space to have a symmetrical distribution, we would want to stabilize a limit-cycle near the center of $S_R$. If on the other hand we expect a specific type of perturbation to occur more frequently, we can choose a limit cycle that gives a larger buffer in that specific direction.

As a specific example, a well-studied state perturbation is a change of ground height between steps [1], [13], [37]. This type of perturbation involves a change in total energy: the forward velocity at apex changes, though the effective height (and thus potential energy) changes. We can compute $S_R$ at different energy levels to then pick out operating points that remain robustly controllable across different energy levels, as shown in Fig. 7. Assuming symmetric distribution of perturbations, the control objectives should be chosen to maximize the distance from the edge of the viability kernel in each direction. For a given desired forward velocity, we can thus choose a total energy that centers the normalized height to perturbation along the vertical axis (constant energy perturbation) and along the forward velocity isolines (ground height change).
Fig. 5: The NSLIP benefits from much larger robust sets $Q_R$ for any amount of noise, which makes it easier to design or learn a robust control policy. Also, the set of robust states $S_R$ are not only larger for the NSLIP, but remain relatively large even for rather imprecise control. See Fig. 6 for the exact sizes of $S_R$. Note the dent in the upper left bound of several $Q_R$ of the NSLIP model. This is due to state-action pairs mapping to a state outside of $S_R$, the dynamic condition in eq. 12.

D. Robustness to State Estimation Uncertainty

Sensory noise causes the control policy to sample an action based on a noisy estimate of the state:

$$a = \pi(s + \eta_s)$$ (13)

where $\eta_s$ is the noise in state space. There is an equivalence between $\eta_s$ and $\eta_a$: the action used deviates from what a control policy would determine under perfect conditions, whether this is due to noise in action space or state estimation. This equivalence can be directly calculated using eq. 10 and 13:

$$\eta_a = \pi(s + \eta_s) - \pi(s)$$ (14)

If the control policy $\pi$ is affine, the equivalence is trivially $\eta_a = \pi(\eta_s)$, and for bounded estimation noise $\eta_s$ and a known policy $\pi$, the equivalent action noise $\eta_a$ is also bounded. Otherwise, we cannot guarantee bounds are available. Since this equivalence is dependent on the specific control policy, we do not investigate it further here. Suffice it to say, increasing robustness to action uncertainty can only improve robustness to state-estimation uncertainty as well.
Motivated by the need to better understand robustness in natural dynamics, we have developed our analysis based on concepts from viability theory and q-learning. To illustrate key concepts, we have applied our approach on two simple spring-mass models commonly used to model running dynamics, and elucidate previously unexplored benefits of using a nonlinear leg compliance. Similar to Zaytsev et al., a key step has been to extend the concept of a viability kernel to viable sets of state-action pairs $Q_V$, inspired in our case largely by the use of q-values in the reinforcement learning community [43]. These viable sets in state-action space define the space available for designing control policies, and we use this to evaluate models with different natural dynamics to understand how mechanical design influences control policy design.

We then examine robustness. To the best of our knowledge, other studies on reachability and viability [22], [47], [48] consider robustness to state perturbations only, and usually focus on control policy design. Instead, we focus on uncertainty in the action space. This allows us to examine the robustness of different natural dynamics for the family of all robust control policies, including stochastic policies, without specifying which control policy or even a specific control objective, aside from avoiding failure.

This allows us to evaluate the advantages and disadvantages of specific mechanical designs, before moving to the control policy design phase. For example, depending on the amount of noise the natural dynamics can tolerate, we can make an informed decision whether we need highly precise actuators or can make do with less precise and cheaper actuators and sensors.

The shape of $Q_V$ also provides valuable insight into how the control policy should be designed. The shape of $Q_V$ and its robust subsets $Q_R$ tell us a lot on the shape of a good robust controller, and the parameterization that should be chosen. The amount of noise that can be tolerated is particularly useful when designing stochastic controllers, which are commonplace when applying learning algorithms.

Throughout the paper, we have made reference to the natural dynamics as if a result purely of mechanical design. We have done this for clarity of exposition, as the intuitive real-world analogs to the springs and massless leg-segmentation in our idealized models are physical springs and a physically segmented leg design. However, the key aspect that we have exploited is simply the uncontrolled dynamics during stance, which depend on the effective leg compliance. This can be shaped both by mechanical design as well as low-level control [49], [50]. The important separation is between the level of control that is currently being considered, and the behavior of the rest of the system which might or might not include internal control loops. This is essentially the same separation in reinforcement learning between the agent and environment, where the environment consists of everything the agent cannot directly specify, including lower-level control loops, the mechanical dynamics of the robot, and the world the robot interacts with.

The possibility to apply this approach for hierarchical control strategies alleviates one of the biggest challenges to using viability theory as a design tool: scalability. We have relied on brute force computation, since our two models are simple and low dimensional. While there is ongoing work on computing or estimating reachable sets and viability kernels efficiently [51], [52], these are still limited to relatively small systems. In future work we aim to apply these tools to guide the design of mechanics and low-level controllers (which live in a high-dimensional space), and quantify the robustness of different designs in the low-dimensional state-action space exposed to the learning agent.

APPENDIX

The SLIP and NSLIP models with all their parameters are shown in Fig. 8. Integration between two apex events is split into three phases: a flight phase which terminates with a touchdown event, a stance phase which terminates with a liftoff event, and another flight phase which terminates with an apex event. The flight phase equations of motion are
Fig. 8: a) shows the parameters of the SLIP and NSLIP models. b) shows the states used in the equations of motion. Note the reference frame is reset to the foot position at each touchdown. c) shows a qualitative trajectory over an entire cycle, with the relevant phases and events.

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
0 \\
-g
\end{bmatrix}
\]

where \(x\) and \(y\) are the position of the body and \(g\) is the gravitational constant. The stance phase equations of motion are

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \frac{F_{\text{leg}}}{m} \begin{bmatrix}
\sin(\theta) \\
\cos(\theta)
\end{bmatrix} - \begin{bmatrix}
0 \\
g
\end{bmatrix}
\]

\[
\theta = \tan^{-1}\left(\frac{y}{x}\right) - \frac{\pi}{2}
\]

where \(\theta\) is the incident angle between the body and the foot (the rotation by \(\frac{\pi}{2}\) serves to keep it consistent with the landing angle of attack) and \(F_{\text{leg}}\) is the force acting on the body due to the leg spring. In the SLIP model,

\[
\text{SLIP: } F_{\text{leg}} = k(l_0 - l)
\]

\[
l = \sqrt{x^2 + y^2}
\]

where \(k\) is the prismatic spring coefficient, \(l_0\) is the resting length of the spring and \(l\) is the leg length. In the NSLIP model,

\[
\text{NSLIP: } F_{\text{leg}} = \frac{4c(\beta_0 - \beta)}{l_0^2 \sin(\beta)}
\]

\[
\beta = \arccos\left(1 - \frac{2l^2}{l_0^2}\right)
\]

where \(c\) is the torsional spring coefficient, \(\beta_0\) is the torsional spring resting angle and \(\beta\) is the knee angle.

The three events are

- touchdown: \(l = l_0\)
- liftoff: \(\theta = \arctan(y/x) - \frac{\pi}{2}\)
- apex: \(\dot{y} = 0\)

At each touchdown, the reference frame is reset to the foot position, which allows the equations of motion to be written more compactly. In the simulation, we also keep track of the foot position in an auxiliary variable, though it is not needed for the results of this paper.

For convenient comparison we use the same parameters as in [21], which are similar to human averages:

| Parameter                              | Value        |
|----------------------------------------|--------------|
| gravitational constant \(g\)          | 9.81 [m/s²]  |
| body mass \(m\)                        | 80 [kg]      |
| prismatic spring resting length \(l_0\)| 1 [m]        |
| prismatic spring coefficient \(k\)     | 8200 [N/m]   |
| torsional spring resting angle \(\beta_0\)| 170 [deg]   |
| torsional spring coefficient \(c\)     | 704 [Nm/rad] |

For the simulations shown, except in Fig. 7 the system energy simulated is 18600 Joule.

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