Asymptotic analysis for the eikonal equation with the dynamical boundary conditions

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Received 11 August 2013, revised 22 January 2014, accepted 27 January 2014
Published online 4 April 2014

Key words Asymptotic behavior, eikonal equations, dynamical boundary conditions, Hamilton-Jacobi equations

MSC (2010) 35F21, 35B40, 35D40, 35F31, 49L25

We study the dynamical boundary value problem for Hamilton-Jacobi equations of the eikonal type with a small parameter. We establish two results concerning the asymptotic behavior of solutions of the Hamilton-Jacobi equations: one concerns with the convergence of solutions as the parameter goes to zero and the other with the large-time asymptotics of solutions of the limit equation.

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1 Introduction and the main results

We consider the initial-boundary value problem for the eikonal equation

\[
\begin{aligned}
\varepsilon u_{\varepsilon}^t(x, t) + |Du_{\varepsilon}(x, t)| &= 1 & \text{in } Q, \\
u(x) \cdot Du_{\varepsilon}(x, t) &= 0 & \text{on } \partial \Omega \times (0, \infty), \\
\lim_{\varepsilon \to 0} u_{\varepsilon}(x, 0) &= u_0(x) & \text{for } x \in \overline{\Omega}.
\end{aligned}
\]  

(1.1)

Here \( \varepsilon \in (0, 1) \) is a parameter, \( \Omega \) is a bounded open connected subset of \( \mathbb{R}^n \), with \( C^1 \) boundary, \( Q := \Omega \times (0, \infty) \), \( \nu(x) \) denotes the outer unit normal of \( \Omega \) at \( x \in \partial \Omega \), and \( u_0 \) represents the initial data. We adapt the notion of viscosity solution as the notion of solution of eikonal equations in this article. Throughout this article we assume for simplicity that \( u_0 \in \text{Lip}(\overline{\Omega}) \), i.e., \( u_0 \) is Lipschitz continuous on \( \overline{\Omega} \).

The boundary condition in the above problem is called the dynamical boundary condition. If we set

\[
\tilde{\nu}(x, t) = (\nu(x), 0) \quad \text{and} \quad \gamma(x, t) = (\nu(x), 1) \quad \text{for } (x, t) \in \partial Q \times (0, \infty),
\]

then \( \tilde{\nu}(x, t) \) is the outer unit normal vector of \( Q \) at \( (x, t) \in \partial Q \times (0, \infty) \),

\[
\tilde{\nu}(x, t) \cdot \gamma(x, t) = 1 \quad \text{for all } (x, t) \in \partial Q \times (0, \infty)
\]

and

\[
u(x) \cdot Du(x, t) = \gamma(x, t) \cdot D\tilde{u}(x, t).
\]

From this observation, we see that the above dynamical boundary condition is a kind of Neumann type boundary condition posed on the portion \( \partial Q \times (0, \infty) \) of the boundary \( \partial Q \) of the domain \( Q \).

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Motivated with applications to superconductivity and surface evolution, Elliott-Giga-Goto [7] have studied the well-posedness of a Hamilton-Jacobi equation with a dynamical boundary condition, where the boundary condition is “tangential” to the lateral boundary and has the form \( u_t(x, t) + g(x, t) = 0 \) (see also (1.4) below). As far as the authors know, a general study of Hamilton-Jacobi equations with dynamical boundary conditions goes back to Barles [3], where the well-posedness of dynamical boundary problems has been established (see for instance [3, Théorème 4.11]).

We are also motivated by the recent studies on the Laplace equation

\[ \Delta_v u(x, t) = 0 \quad \text{in } \Omega \times (0, \infty) \]

with the nonlinear dynamical boundary condition of the type

\[ u_t(x, t) + v(x) \cdot D_u u(x, t) = |u(x, t)|^q, \quad \text{with a constant } q > 1, \]

due to Amann-Fila [1], Fila-Ishige-Kawakami [10] and others, where the blow-up phenomena and large time behavior of solutions are investigated. The Laplace equation above is, of course, the limit equation of the heat equations \( \varepsilon u_t(x, t) - \Delta u(x, t) = 0 \) in \( \Omega \times (0, \infty) \) as \( \varepsilon \to 0+ \). Here we replace these heat equations by the eikonal equations and the nonlinear dynamical boundary condition by the linear one as in (1.1).

We are thus concerned with the asymptotic behavior of the solution \( u^\varepsilon \) of (1.1) as \( \varepsilon \to 0+ \). Roughly speaking, if there is a limit function of \( u^\varepsilon \) as \( \varepsilon \to 0+ \), the limit function \( u \) should satisfy

\[
\begin{align*}
|D_v u(x, t)| &= 1 \quad \text{in } Q, \\
u_t(x, t) + v(x) \cdot D_v u(x, t) &= 0 \quad \text{on } \partial \Omega \times (0, \infty).
\end{align*}
\]

(1.2)

Regarding the initial condition for the limit function, as we will see in our main results, the solutions \( u^\varepsilon \) develop an initial layer and the original initial condition \( u(\cdot, 0) = u_0 \) does not make sense for the limit function \( u \) in general.

To overcome the difficulty of initial layer, we introduce a new (slower) time scale and, for the solutions \( u^\varepsilon \) of (1.1), we set \( v^\varepsilon(x, t) = u^\varepsilon(x, \varepsilon t) \) for \( (x, t) \in \overline{Q} \). Note that the function \( v^\varepsilon \) satisfies

\[
\begin{align*}
v^\varepsilon_t(x, t) + |D_v v^\varepsilon(x, t)| &= 1 \quad \text{in } Q, \\
v^\varepsilon_t(x, t) + \varepsilon v(x) \cdot D_v v^\varepsilon(x, t) &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
v^\varepsilon(x, 0) &= u_0(x) \quad \text{for } x \in \overline{\Omega}.
\end{align*}
\]

(1.3)

In the informal level, by setting \( \varepsilon = 0 \) we get the problem for the limit function \( v \) of the \( v^\varepsilon \) as \( \varepsilon \to 0+ \):

\[
\begin{align*}
v_t(x, t) + |D_v v(x, t)| &= 1 \quad \text{in } Q, \\
v_t(x, t) &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
v(x, 0) &= u_0(x) \quad \text{for } x \in \overline{\Omega}.
\end{align*}
\]

(1.4)

The initial condition for the limit function \( u \) of the \( u^\varepsilon \) is then given as the limit function \( v_{\infty}(x) \) of the solution \( v(x, t) \) of (1.4) as \( t \to \infty \). The recent developments concerning the large time asymptotics for solutions of Hamilton-Jacobi equations (see [4], [12]) suggest that the limit function \( v_{\infty} \) should be described as follows: define first the function \( v_0^\infty \) on \( \overline{\Omega} \) as the maximal subsolution of the stationary eikonal equation

\[ |D_v v(x)| = 1 \quad \text{in } \Omega \]

(1.5)

among those \( v \) which satisfy \( v \leq u_0 \) on \( \overline{\Omega} \), and then \( v_{\infty} \) as the minimal solution of (1.5) among those \( v \) which satisfy \( v \geq v_0^\infty \) on \( \overline{\Omega} \). It is well-known (see the end of this section) that \( v_0^\infty \) and \( v_{\infty} \) are Lipschitz continuous on \( \overline{\Omega} \) with a Lipschitz bound depending only on the domain \( \Omega \). See [12, Lemma 2.2] for this Lipschitz continuity.

The main purpose of this paper is twofold. First, we consider the convergence of \( u^\varepsilon \) as \( \varepsilon \to 0+ \) and, second, we study the large time asymptotics for solutions of (1.2). Our result on the convergence of \( u^\varepsilon \) is stated as follows:

**Theorem 1.1** Let \( u^\varepsilon \in C(\overline{Q}) \) be a solution of (1.1), with \( \varepsilon \in (0, 1) \), and \( u \in C(\overline{Q}) \) a solution of (1.2) satisfying the initial condition \( u(\cdot, 0) = v_{\infty} \) on \( \overline{\Omega} \). Then

\[
\lim_{\varepsilon \to 0^+} u^\varepsilon(x, t) = u(x, t) \quad \text{uniformly on } \overline{\Omega} \times [T^{-1}, T]
\]

for all \( T > 1 \).
The stationary problem corresponding to (1.2) is the following.

\[
\begin{cases}
|Du(x)| = 1 & \text{in } \Omega, \\
1 + v(x) \cdot Du(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\] (1.6)

As we will see, this problem has a solution in \(C(\Omega)\) and \(v_\infty\) is a supersolution of this problem. We define the function \(u_\infty\) as the maximal solution of (1.6) among those \(u\) which satisfy \(u \leq v_\infty\) on \(\Omega\). Our result concerning the large time behavior of solutions of (1.2) is as follows.

**Theorem 1.2** Let \(u \in C(\Omega)\) be a solution of (1.2) satisfying the initial condition \(u(\cdot, 0) = v_\infty\). Then

\[
\lim_{t \to \infty} (u(x, t) - t) = u_\infty(x) \quad \text{uniformly on } \overline{\Omega}.
\]

We use the following notation: as above let \(\Omega \subset \mathbb{R}^n\) be a bounded domain with \(C^1\) boundary. By the implicit function theorem, there exists a function \(\rho \in C^1(\mathbb{R}^n)\) such that

\[
\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\},
\]
\[
D\rho(x) \neq 0 \quad \text{for all } x \in \partial \Omega.
\]

Note that \(D\rho(x) = |D\rho(x)| v(x)\) for all \(x \in \partial \Omega\). We call such a function \(\rho\) a defining function of \(\Omega\). Let \(\phi \in C(\Omega)\) be a (viscosity) supersolution of \([D\phi(x)] \leq 1\) in \(\Omega\). It is well-known (see [12, Proposition 1.14] and [2], [3], [14]) that this property is equivalent to the following Lipschitz property: \(|\phi(x) - \phi(y)| \leq |x - y|\) for all \(x, y \in B\) and all ball \(B \subset \Omega\). Due to the \(C^1\) regularity, connectedness and boundedness of \(\Omega\), any such function \(\phi\) is Lipschitz continuous on \(\overline{\Omega}\), with a uniform Lipschitz bound. (See [12, Lemma 2.2] for this.) In the following, the minimum of such uniform bounds will be denoted by \(L_\Omega\). It is obvious that \(L_\Omega \geq 1\). For any bounded function \(f\) on a set \(A\), \(\sup_{x \in \overline{A}} |f(x)|\) represents the sup norm \(\sup_{x \in A} |f(x)|\). For any \(T > 0\), \(Q_T\) denotes the domain \(\Omega \times (0, T)\).

## 2 Preliminaries

We begin with the following theorem.

**Theorem 2.1** Let \(\varepsilon > 0\). There exists a unique solution \(u^\varepsilon \in \text{Lip}(\overline{\Omega})\) of (1.1).

Recall that \(u_0 \in \text{Lip}(\overline{\Omega})\) is assumed here, which is crucial to conclude the Lipschitz continuity of \(u^\varepsilon\) in the above theorem. On the other hand, for any continuous \(u_0\), one can show the unique existence of a uniformly continuous solution of (1.1). The above result is known in the literature (see for instance [3], [5]), but we give a proof here for the reader's convenience.

**Proof.** We first note that the uniqueness of solution of (1.1) is a direct consequence of Theorem A.1 (comparison theorem) in the appendix.

We next show that there exists a solution of (1.1) which is continuous on \(\overline{\Omega}\). Let \(L > 0\) be a Lipschitz bound of \(u_0\). Define the functions \(U^\pm \in \text{Lip}(\overline{\Omega})\) by \(U^\pm(x, t) = u_0(x) \pm \max\{\varepsilon^{-1}, \varepsilon^{-1}L, L\} t\). It is easily checked that \(U^+\) and \(U^-\) are, respectively, a supersolution and a subsolution of

\[
\begin{cases}
\varepsilon u_{\tau}(x, t) + |D_u u(x, t)| = 1 & \text{in } \Omega, \\
u_t(x, t) + v(x) \cdot D_u u(x, t) = 0 & \text{on } \partial \Omega \times (0, \infty).
\end{cases}
\] (2.1)

According to Perron’s method (see [6], [2], [3], [12] for instance), if we denote by \(S\) the set of all subsolutions \(\phi\) of (2.1) such that \(U^- \leq \phi \leq U^+\) on \(\overline{\Omega}\) and set

\[
u^\varepsilon(x, t) = \sup\{\phi(x, t) : \phi \in S\} \quad \text{for } (x, t) \in \overline{\Omega},
\]

then \(\nu^\varepsilon\) is a solution of (2.1) and \(\nu^\varepsilon \in S\). More precisely, \(\nu^\varepsilon\) is a solution of (2.1) in the sense that \(\nu^\varepsilon \in \text{USC}(\overline{\Omega})\), \(\nu^\varepsilon\) is a subsolution of (2.1) and the lower semicontinuous envelope \(\nu^\varepsilon\) of \(\nu^\varepsilon\) is a supersolution of (2.1). It is obvious that \(U^- \leq \nu^\varepsilon \leq U^+\) on \(\overline{\Omega}\). We apply the comparison theorem (Theorem A.1 in the Appendix) to \(\nu^\varepsilon\) and \(\nu^\varepsilon\), to obtain \(\nu^\varepsilon \leq \nu^\varepsilon\) on \(\overline{\Omega}\). Thus we see that \(\nu^\varepsilon = \nu^\varepsilon\) is continuous on \(\overline{\Omega}\). It is now obvious that \(\nu^\varepsilon(x, 0) = u_0(x)\) for all \(x \in \overline{\Omega}\) and that \(\nu^\varepsilon\) is a solution of (1.1).
Finally we show that $u^r \in \text{Lip}(\Omega)$. We set $M_r = \max \{ \varepsilon^{-1}, \varepsilon^{-1}L, L \}$ and observe that $u^r(x, h) - M_r h \leq u^r(x, h) - M_r h \leq u_0(x)$ for all $x \in \overline{\Omega}$ and $h > 0$. For any $h > 0$, by comparison between the solutions $u^r(x, t + h) - M_r h$ and $u^r(x, t)$ of (1.1), we get $u^r(x, t + h) - M_r h \leq u^r(x, t)$ for all $(x, t) \in \overline{\Omega}$. Similarly, we get $u^r(x, t + h) + M_r h \geq u^r(x, t)$ for all $(x, t) \in \overline{\Omega}$. Hence, we get

$$|u^r(x, t) - u^r(x, t)| \leq M_r |t - s|$$

for all $t, s \geq 0, x \in \overline{\Omega}$.

This Lipschitz estimate together with Lemma A.2 in the Appendix guarantees that $u^r \in \text{Lip}(\overline{\Omega} \times (0, \infty))$. But, since $u^r \in C(\overline{\Omega})$, we conclude that $u^r \in \text{Lip}(\overline{\Omega})$.

Given a function $u^r \in \text{Lip}(\overline{\Omega})$, we define the function $v^r \in \text{Lip}(\overline{\Omega})$ by

$$v^r(x, t) := u^r(x, t).$$

It is easy to check that $u^r$ is a solution of (1.1) if and only if $v^r$ is a solution of (1.3). Hence, Theorem 2.1 implies the following proposition.

**Corollary 2.2** There exists a unique solution $v^r \in \text{Lip}(\overline{\Omega})$ of (1.3).

We remark here on the definition of $v_0^-, v_\infty$ and $u_\infty$. By definition, the function $v_0^- : \overline{\Omega} \to \mathbb{R}$ is given by

$$v_0^-(x) = \sup \{ \phi(x) : \phi \in S_0 \},$$

where $S_0$ denotes the set of all subsolutions $\phi \in \text{Lip}(\overline{\Omega})$ of (1.5) satisfying the inequality $\phi \leq u_0$ on $\overline{\Omega}$. It is a classical observation that $S_0 \neq \emptyset$ and the above formula gives a Lipschitz continuous subsolution of (1.5). The function $v_\infty : \overline{\Omega} \to \mathbb{R}$ is defined by

$$v_\infty(x) = \inf \{ \phi(x) : \phi \in S \},$$

where $S$ denotes the set of all solutions $\phi \in \text{Lip}(\overline{\Omega})$ of (1.5) satisfying $\phi \geq v_0^-$ on $\overline{\Omega}$. It is well-known (see also Proposition A.4 in the appendix or Perron’s method as well) that $S \neq \emptyset$ and the above formula gives a solution in $\text{Lip}(\overline{\Omega})$ of (1.5).

The definition of $u_\infty$ is related to the additive eigenvalue problem (or, ergodic problem): consider the problem of finding a pair $(c, v) \in \mathbb{R} \times \text{Lip}(\overline{\Omega})$ such that $v$ is a solution of

$$\begin{cases}
|Dv(x)| = 1 & \text{in } \Omega, \\
\epsilon + v \cdot Dv(x) = 0 & \text{on } \partial \Omega.
\end{cases}$$

(2.2)

It is clear that if $(c, v)$ is a solution of the above additive eigenvalue problem, so is the pair $(c, v + A)$, with any constant $A \in \mathbb{R}$. On the other hand, the following theorem assures that the additive eigenvalue $c$ is unique and, indeed, $c = 1$.

**Theorem 2.3** There exists a solution $v \in C(\overline{\Omega})$ of (1.6). Moreover, for any $c \neq 1$, there exists no solution of (2.2).

**Proof.** 1. To prove the existence of a solution of (1.6), we show that the function $v(x) := \text{dist}(x, \partial \Omega)$ is a solution of (1.6). It is well-known and easily checked that the function $v$ is a solution of (1.5).

Next let $\phi \in C^1(\overline{\Omega})$ and $x \in \partial \Omega$. We first assume that $v - \phi$ has a maximum at $x \in \partial \Omega$. We set $y = x - \varepsilon v$, where $v = v(x)$ and $\varepsilon > 0$. Note that if $\varepsilon > 0$ is sufficiently small, then $y \in \Omega$. The $C^1$ regularity of $\Omega$ ensures that $v(y) - v(x) = v(x - \varepsilon v) = \varepsilon + o(\varepsilon)$ as $\varepsilon \to 0+$. Hence, as $\varepsilon \to 0+$, we get

$$\varepsilon + o(\varepsilon) = v(y) - v(x) \leq \phi(y) - \phi(x) = -\varepsilon v \cdot D\phi(x) + o(\varepsilon),$$

which yields $1 + v \cdot D\phi(x) \leq 0$.

Now, we assume that $v - \phi$ has a minimum at $x \in \partial \Omega$. Set $y = x - \varepsilon v$, with $\varepsilon > 0$. Observe that as $\varepsilon \to 0+$,

$$\varepsilon + o(\varepsilon) = v(y) - v(x) \geq \phi(y) - \phi(x)$$

$$= D\phi(x) \cdot (y - x) + o(\varepsilon) = -\varepsilon v D\phi(x) + o(\varepsilon),$$

from which we get $1 + v \cdot D\phi(x) \geq 0$. We thus conclude that $v$ is a supersolution of (1.6) and moreover that $v$ is a solution of (1.6).
2. We next show that \( e = 1 \) is the only possible choice for which (2.2) has a solution. We actually show that if there exist solutions \((c_1, u), (c_2, v) \in \mathbb{R} \times \text{Lip}(\bar{\Omega})\) of (2.2), then \( c_1 = c_2 \). By symmetry, we only need to show that \( c_1 \leq c_2 \). To this end, we argue by contradiction. Thus, we assume that the inequality \( c_1 > c_2 \) holds. Let \( A > 0 \), and define the functions \( V, W \in \text{Lip}(\bar{\Omega}) \) by

\[
\begin{align*}
V(x, t) &= v(x) + c_1 t, \\
W(x, t) &= w(x) + c_2 t + A.
\end{align*}
\]

It is easily seen that \( V \) and \( W \) are both solutions of (1.2). We select \( A \) sufficiently large so that \( W(x, 0) \geq V(x, 0) \) for all \( x \in \bar{\Omega} \). By the comparison principle (see Theorem A.1), we obtain \( W(x, t) \geq V(x, t) \) for all \( (x, t) \in \bar{\Omega} \). But this is a contradiction since \( c_1 > c_2 \). Thus we must have \( c_1 \leq c_2 \).

\[\square\]

3 Proof of the main results

**Theorem 3.1** For each \( \varepsilon > 0 \) let \( v^\varepsilon \in \text{Lip}(\bar{\Omega}) \) be the solution of (1.3). Then there exists a function \( v \in \text{Lip}(\bar{\Omega}) \) such that

\[
\lim_{\varepsilon \to 0^+} v^\varepsilon(x, t) = v(x, t) \quad \text{uniformly on } \bar{\Omega} \times [0, T)
\]

for all \( 0 < T < \infty \). The function \( v \) is a solution of (1.4).

The existence and uniqueness of solution of (1.3) have been shown in Corollary 2.2. The following lemma is needed in our proof of the above theorem.

**Lemma 3.2 (Comparison)** Let \( v \in \text{Lip}(\bar{\Omega}) \) and \( w \in \text{Lip}(\bar{\Omega}) \) be a subsolution and a supersolution of

\[
\begin{align*}
& u_t(x, t) + |D_x u(x, t)| = 1 & \text{in } \Omega, \\
& u_t(x, t) = 0 & \text{on } \partial \Omega \times (0, \infty),
\end{align*}
\]

respectively. Assume that \( v(x, 0) \leq w(x, 0) \) for all \( x \in \bar{\Omega} \). Then \( v \leq w \) on \( \bar{\Omega} \).

The above comparison principle does not hold in general if the Lipschitz regularity of the functions \( v, w \) is removed. For this see Example A.5 in the Appendix.

**Proof.** Fix any \( \varepsilon > 0 \). Let \( M > 0 \) be a Lipschitz bound of the functions \( v \) and \( w \). It is easily checked that the functions \( v_\varepsilon(x, t) := v(x, t) - \varepsilon M t \) and \( w_\varepsilon(x, t) := w(x, t) + \varepsilon M t \) are, respectively, a subsolution and a supersolution of

\[
\begin{align*}
& u_t(x, t) + |D_x u(x, t)| = 1 & \text{in } \Omega, \\
& u_t(x, t) + \varepsilon v(x) \cdot D_x u(x, t) = 0 & \text{on } \partial \Omega \times (0, \infty),
\end{align*}
\]

and that \( v_\varepsilon(x, 0) = v(x, 0) \leq w(x, 0) = w_\varepsilon(x, 0) \) for all \( x \in \bar{\Omega} \). Applying a standard comparison theorem (for instance, Theorem A.1), we get

\[
v_\varepsilon(x, t) = v(x, t) - \varepsilon M t \leq w_\varepsilon(x, t) = w(x, t) + \varepsilon M t \quad \text{for all } (x, t) \in \bar{\Omega}.
\]

Sending \( \varepsilon \to 0 \) yields the desired inequality. \[\square\]

The following proposition is an immediate consequence of Theorem 3.1 and Lemma 3.2.

**Corollary 3.3** There exists a unique solution of (1.4) in the class \( \text{Lip}(\bar{\Omega}) \).

**Proof of Theorem 3.1.** We show first that the family \( \{v^\varepsilon\}_{\varepsilon > 0, \varepsilon < 1} \) is equi-Lipschitz continuous on \( \bar{\Omega} \). The argument is similar to the last part of the proof of Theorem 2.1.
Let $0 < \varepsilon < 1$. Let $M \geq 1$ be a Lipschitz bound of the function $u_0$. It is easily checked that the functions $U^+, U^- \in \text{Lip}(\mathcal{Q})$, given by $U^\pm(x, t) = u_0(x) \pm Mt$, are a supersolution and a subsolution of (1.3), respectively.

By comparison (Theorem A.1), we get

$$U^-(x, t) \leq v^e(x, t) \leq U^+(x, t) \quad \text{for all } (x, t) \in \overline{Q}.$$  

Consequently, for any $h > 0$, we have

$$v^e(x, h) - Mh \leq u_0(x) = v^e(x, 0) \leq v^e(x, h) + Mh \quad \text{for all } x \in \overline{Q}.$$  

Again, by comparison, we get

$$v^e(x, t + h) - Mh \leq v^e(x, t) \leq v^e(x, t + h) + Mh \quad \text{for all } (x, t) \in \overline{Q}.$$  

Hence, using Lemma A.2, we deduce that the collection $\{v^\varepsilon\}_{0 < \varepsilon < 1}$ is equi-Lipschitz continuous on $\overline{Q}$.

Thanks to the Ascoli-Arzela theorem, there are a sequence ${\varepsilon_j} \subset (0, 1)$ converging to zero and a function $v \in \text{Lip}(\overline{Q})$ such that

$$\lim_{j \to \infty} \|v^\varepsilon_j - v\|_{\infty, \mathcal{Q} \times [0, T]} = 0 \quad \text{for every } T > 0. \quad (3.2)$$

By the well-known stability property of viscosity solutions, we see that $v$ is a solution of (1.4).

To complete the proof, we need to show that for any $T > 0$,

$$\lim_{\varepsilon \to 0^+} \|v^\varepsilon - v\|_{\infty, \mathcal{Q} \times [0, T]} = 0.$$  

For this, we argue by contradiction and suppose that there were a sequence $\{\varepsilon_j\} \subset (0, 1)$ converging to zero and a constant $0 < S < \infty$ such that

$$\limsup_{j \to \infty} \|v^{\varepsilon_j} - v\|_{\infty, \mathcal{Q} \times [0, S]} > 0. \quad (3.3)$$

Passing to a subsequence and arguing as in the case of the sequence $\{\varepsilon_j\}$, we may assume that there is a solution $w \in \text{Lip}(\overline{Q})$ of (1.4) such that for all $T > 0$,

$$\lim_{j \to \infty} \|v^{\varepsilon_j} - w\|_{\infty, \mathcal{Q} \times [0, T]} = 0. \quad (3.4)$$

But, by Lemma 3.2, we must have $v = w$ on $\overline{Q}$, and (3.3) contradicts (3.4).

**Theorem 3.4** Let $v \in \text{Lip}(\overline{Q})$ be the solution of (1.4). Then

$$\lim_{t \to \infty} v(x, t) = v_{\infty}(x) \quad \text{uniformly on } \mathcal{Q}.$$  

Indeed, one can prove that there exists a constant $T > 0$ such that

$$v(x, t) = v_{\infty}(x) \quad \text{for all } (x, t) \in \mathcal{Q} \times [T, \infty).$$

**Proof of Theorem 3.4.** 1. We show first that $v$ is bounded on $\overline{Q}$. Fix an $e \in \mathbb{R}^n$ so that $|e| = 1$. For any $C \in \mathbb{R}$ the function $w_C(x, t) := e \cdot x + C$ is a solution of (3.1). Hence, choosing $C > 0$ so large that $|u_0(x) - e \cdot x| \leq C$ for all $x \in \mathcal{Q}$, by comparison (Lemma 3.2), we get $w_{-C}(x, t) \leq v(x, t) \leq w_C(x, t)$ for all $(x, t) \in \overline{Q}$, which shows that $v$ is bounded on $\overline{Q}$.

2. We next show that for each $x \in \partial \Omega$ the function $t \mapsto v(x, t)$ is nonincreasing on $[0, \infty)$. We fix any $\hat{x} \in \partial \Omega$ and show that the function $t \mapsto v(\hat{x}, t)$ is nonincreasing on $[0, \infty)$. To this end, we assume, by contradiction, that there were two positive numbers $t_0 < t_1$ such that $v(\hat{x}, t_0) < v(\hat{x}, t_1)$.

We may choose an increasing function $\psi \in C^1([t_0, t_1])$ such that $v(\hat{x}, \cdot) - \psi$ attains a strict maximum at some point $\hat{t} \in (t_0, t_1)$ and $\inf_{r \geq 0} \psi'(r) > 0$.

For $\alpha \geq 0$ we introduce the function

$$\Phi_\alpha(x, t) := v(x, t) - \psi(t) - \alpha|x - \hat{x}|^2 + \alpha^2 \rho(x)$$

on $\mathcal{Q} \times [t_0, t_1]$, where $\rho$ is a defining function of $\Omega$. Note that $\Phi_\alpha(x, t) \leq \Phi_0(x, t)$ for all $(x, t) \in \mathcal{Q} \times [t_0, t_1]$ and all $\alpha > 0$. Let $\alpha > 0$ and let $(x_\alpha, t_\alpha)$ be a maximum point of the function $\Phi_\alpha$ over $\mathcal{Q} \times [t_0, t_1]$. Note that

$$\Phi_0(\hat{x}, \hat{t}) = \Phi_\alpha(\hat{x}, \hat{t}) \leq \Phi_\alpha(x_\alpha, t_\alpha) \leq \Phi_0(x_\alpha, t_\alpha) - \alpha|x_\alpha - \hat{x}|^2.$$ 

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This sequence of inequalities guarantees that \( \lim_{\alpha \to \infty} (x_\alpha, t_\alpha) = (\hat{x}, \hat{t}) \). In particular, if \( \alpha \) is sufficiently large, then \( t_0 < t_\alpha < t_1 \). For such a large \( \alpha \), by the viscosity property of \( v \), we have either
\[
\psi'(t_\alpha) \leq 0 \text{ or } \psi'(t_\alpha) + |2\alpha(x_\alpha - \hat{x}) - \alpha^2 D\rho(x_\alpha)| \leq 1.
\]
Noting that \( \lim_{\alpha \to \infty} |2\alpha(x_\alpha - \hat{x}) - \alpha^2 D\rho(x_\alpha)| = \infty \) and sending \( \alpha \to \infty \), we get \( \psi'(<) \leq 0 \), which contradicts our choice of \( \psi \). Thus we see that for each \( x \in \partial \Omega \) the function \( t \mapsto v(x, t) \) is nonincreasing on \([0, \infty)\) and, therefore, the limit
\[
\lim_{t \to \infty} v(x, t) = v_b(x) \quad \text{exists for any } x \in \partial \Omega,
\]
where \( v_b \) is a function on \( \partial \Omega \). Noting that \( v \in \text{Lip}(\overline{\Omega}) \), we see that \( v_b \in \text{Lip}(\partial \Omega) \).

3. We define \( v^\pm \in \text{Lip}(\overline{\Omega}) \) by
\[
\begin{cases}
v^+(x, t) = \sup_{s \geq 0} v(x, t + s), \\
v^-(x, t) = \inf_{s \geq 0} v(x, t + s).
\end{cases}
\]
By the monotonicity of the function \( t \mapsto v(x, t) \) for \( x \in \partial \Omega \), we see that
\[
v^-(x, t) = v_b(x) \quad \text{for all } (x, t) \in \partial \Omega \times [0, \infty).
\]

It is a standard observation (see [12, Proposition 1.10], [6], [2], [3], [14] for instance) that \( v^+ \) and \( v^- \) are a subsolution and a supersolution of (3.1), respectively. Because of the Lipschitz continuity and boundedness of \( v^\pm \), we find that
\[
\begin{align*}
\lim_{s \to \infty} v^+(x, t + s) &= V^+(x, t), \\
\lim_{s \to \infty} v^-(x, t + s) &= V^-(x, t)
\end{align*}
\]
for some functions \( V^\pm \in \text{Lip}(\overline{\Omega}) \), where the convergence is uniform for \( (x, t) \in \overline{\Omega} \times [0, T] \), with every \( T > 0 \). By the stability of the viscosity property under uniform convergence, we see that \( V^+ \) and \( V^- \) are a subsolution and a supersolution of (3.1), respectively. It is easily seen that the functions \( V^\pm(x, t) \) in fact do not depend on \( t \). We may thus denote them respectively by \( V^\pm(x) \), and we have for all \( x \in \overline{\Omega} \),
\[
\begin{align*}
V^+(x) &= \lim_{t \to \infty} v(x, t), \\
V^-(x) &= \lim_{t \to \infty} v(x, t).
\end{align*}
\]
Obviously we have
\[
\begin{align*}
V^+(x) &= V^-(x) = v_b(x) \quad \text{for all } x \in \partial \Omega, \\
V^+(x) &\geq V^-(x) \quad \text{for all } x \in \overline{\Omega},
\end{align*}
\]
and the functions \( V^+ \) and \( V^- \) are a subsolution and a supersolution of the eikonal equation (1.5), respectively. By the standard comparison result (see Lemma A.3 in the Appendix and also [2], [3], [14]), we get \( V^+(x) \leq V^-(x) \) for all \( x \in \overline{\Omega} \). It is now clear that \( \lim_{\alpha \to \infty} v(x, t) = V^+(x) = V^-(x) \) uniformly on \( \overline{\Omega} \).

4. Set \( V := V^+ = V^- \). We intend to identify \( V \) with \( v_\infty \). By the definition of \( v_\infty^- \), we have \( v_\infty^- (x) \leq u_0(x) \) for all \( x \in \overline{\Omega} \), and the function \( v_\infty^- (x) \) as a function on \( \overline{\Omega} \) is a subsolution of (3.1). By comparison, we get \( v_\infty^- (x) \leq v(x, t) \) for all \( (x, t) \in \overline{\Omega} \). Consequently, we have \( v_\infty^- (x) \leq V(x) \) for all \( x \in \overline{\Omega} \). Since \( V \) is a solution of (1.5), we see that
\[
\lim_{t \to \infty} v(x, t) \leq V(x) \quad \text{for all } x \in \overline{\Omega}.
\]

It is immediate to see by the definition of \( v^- \) that for each \( x \in \overline{\Omega} \), the function \( t \mapsto v^-(x, t) \) is nondecreasing on \([0, \infty)\). Since (3.1) is a convex Hamilton-Jacobi equation, we deduce (see Proposition A.4) that \( v^- \) is a solution of \( u_t(x, t) + |D_x u(x, t)| = 1 \) in \( \overline{\Omega} \). From these observations, we infer that for each \( t > 0 \) the function \( x \mapsto v^-(x, t) \) is a subsolution of (1.5). Noting that \( u_0(x) \geq v^-(x, 0) \) for all \( x \in \overline{\Omega} \) and that
\[
v^-(x, 0) = \lim_{t \to 0^+} v^-(x, t) \quad \text{uniformly for all } x \in \overline{\Omega},
\]
which shows that the function $x \mapsto v^-(x, 0)$ is a subsolution of (1.5), we see that $v^-(x, 0) \leq v_0^-(x) \leq v_\infty(x)$ for all $x \in \overline{Q}$. By the constancy (3.5), it is now easy to check that $v^-$ is a subsolution of (3.1). Also, the function $v_\infty(x)$, regarded as a function of $(x, t)$, is a solution of (3.1). Hence, by comparison, we get $v^-(x, t) \leq v_\infty(x)$ for all $(x, t) \in \overline{Q}$. Consequently, we get $V(x) \leq v_\infty(x)$ for all $x \in \overline{Q}$. This together with (3.6) guarantees that $V = v_\infty$.

Proof of Theorem 1.1. We set $v^+(x, t) := u^+(x, \epsilon t)$ for $(x, t) \in \overline{Q}$. Let $v \in \text{Lip}(\overline{Q})$ be the solution of (1.4). Fix any $\delta > 0$. By Theorem 3.4, there is a constant $0 < T < \infty$ such that $|v(x, T) - v_\infty(x)| < \delta$ for all $x \in \overline{Q}$. By Theorem 3.1, there exists a constant $\epsilon_0 \in (0, 1)$ such that $|v^+(x, T) - v(x, T)| < \delta$ for all $x \in \overline{Q}$ and all $0 < \epsilon < \epsilon_0$. Thus, we have $|v^+(x, t) - v_\infty(x)| < 2\delta$ for all $x \in \overline{Q}$ and all $0 < \epsilon < \epsilon_0$, which reads

$$|u^+(x, \epsilon t) - v_\infty(x)| < 2\delta \quad \text{for all } x \in \overline{Q} \text{ and } 0 < \epsilon < \epsilon_0. \tag{3.7}$$

Since $v_\infty$ is a solution of (1.5), the function $v_\infty$ is Lipschitz continuous on $\overline{Q}$ with a Lipschitz bound $L_\Omega$. Hence, the functions $v_\infty(x) \pm L_\Omega \cdot t$ are a supersolution and a subsolution of (1.1), respectively. Using (3.7), by comparison (Theorem A.1), we get

$$|u^+(x, t + \epsilon T) - v_\infty(x)| < 2\delta + L_\Omega \cdot t \quad \text{for all } (x, t) \in \overline{Q}, \quad 0 < \epsilon < \epsilon_0. \tag{3.8}$$

We define the functions $u^\pm$ on $\overline{Q} \times (0, \infty)$ by

$$
\begin{aligned}
&u_+^+(x, t) = \lim_{r \to 0^+} \sup \{u^+(y, s) : (y, s) \in \overline{Q} \times (0, \infty), \quad 0 < \epsilon < r, \quad |y - x| + |s - t| < r\}, \\
&u_-^+(x, t) = \lim_{r \to 0^+} \inf \{u^+(y, s) : (y, s) \in \overline{Q} \times (0, \infty), \quad 0 < \epsilon < r, \quad |y - x| + |s - t| < r\}.
\end{aligned}
$$

The functions $u_+^+$ and $u_-^+$ are called the half-relaxed limits of the functions $u^+$, and it is well-known (see [12, Theorem 1.3], [6], [2], [3]) that $u^+ \in \text{USC}(\overline{Q} \times (0, \infty))$, $u^- \in \text{LSC}(\overline{Q} \times (0, \infty))$, $u^- \leq u^+$ in $\overline{Q} \times (0, \infty)$, and $u^+ \text{ and } u^-$ are a subsolution and a supersolution of (1.2), respectively. Due to estimate (3.8), we see that if we set $u^+(x, 0) = v_\infty(x)$ for $x \in \overline{Q}$, then $u^+ \in \text{USC}(\overline{Q})$ and $u^- \in \text{LSC}(\overline{Q})$. By comparison (Theorem A.1), we get $u^+ \leq u \leq u^-$ on $\overline{Q}$, which shows that $u^+ = u^- = u$ on $\overline{Q}$ and moreover that as $\epsilon \to 0^+$, $u^+(x, t) \to u(x, t)$ uniformly for $(x, t) \in \overline{Q} \setminus [S^-1, S]$ for every $S > 1$.

Proof of Theorem 1.2. We define the function $w \in \text{Lip}(\overline{Q})$ by $w(x, t) = u(x, t) - t$, and observe that $w$ is a solution of

$$
\begin{aligned}
&D_xw(x, t) = 1 \quad \text{in } \overline{Q}, \\
&w_t(x, t) + 1 + v(x) \cdot D_xw(x, t) = 0 \quad \text{on } \partial \overline{Q} \times (0, \infty).
\end{aligned} \tag{3.9}
$$

The function $v_\infty$ is a solution of (1.5) and hence it is a supersolution of (1.6). (Indeed, for any $x \in \partial \overline{Q}$ and $p \in -\nabla v_\infty(x)$, we have two possibilities: either $|p| \geq 1$ or $|p| < 1$, and if $|p| < 1$, then $1 + |v(x)| \cdot p \geq 1 - |v(x)||p| > 0$.) Accordingly, the function $v_\infty(x)$, as a function of $(x, t)$, is a supersolution of (3.9). By comparison (Theorem A.1), we get $w(x, t) \leq v_\infty(x)$ for all $(x, t) \in \overline{Q}$. Again, by the comparison between $w(x, t)$ and $w(x, t + h)$, with $h > 0$, we get $w(x, t + h) \leq w(x, t)$ for all $(x, t) \in \overline{Q}$ and $h > 0$. That is, for each $x \in \overline{Q}$, the function $t \mapsto w(x, t)$ is nonincreasing on $[0, \infty)$. Note that the function $u_\infty(x)$ is, as a function of $(x, t)$, a solution of (3.9) and that $u_\infty(x) \leq v_\infty(x)$ for all $x \in \overline{Q}$. By comparison, we get $w(x, t) \geq u_\infty(x)$ for all $(x, t) \in \overline{Q}$. Noting also that $w$ is bounded and Lipschitz continuous on $\overline{Q}$, we see that, as $t \to \infty$, $w(x, t)$ converges to $w_\infty(x)$ uniformly on $\overline{Q}$ for some function $w_\infty \in \text{Lip}(\overline{Q})$. Clearly, $w_\infty$ is a solution of (1.6) and satisfies $u_\infty(x) \leq w_\infty(x) \leq v_\infty(x)$ for all $x \in \overline{Q}$. Because of the maximality of $u_\infty$, we conclude that $w_\infty = u_\infty$ and that, as $t \to \infty$, $w(x, t)$ converges to $u_\infty(x)$ uniformly on $\overline{Q}$.

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4 Initial value problem for (1.2)

We discuss here the well-posedness of the initial value problem for (1.2) and consider first the initial value problem

\[
\begin{cases}
|D_1u(x, t)| = 1 & \text{in } Q, \\
u_t(x, t) + v(x) \cdot D_xu(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\
u(x, 0) = u_0(x) & \text{for } x \in \Omega.
\end{cases}
\]

(4.1)

This problem has been studied in the previous sections, but it is overdetermined in its initial condition. Indeed, if \(u\) is a solution of (4.1) and continuous on \(\overline{Q}\), then the function \(u_0\) should be a solution of \(|D_1u(x)| = 1\) in \(\Omega\) and therefore it should be given by the boundary data \(u_0|_{\partial\Omega}\). This suggests another formulation: let \(u_0 \in C(\partial\Omega)\) and consider the initial value problem

\[
\begin{cases}
|D_1u(x, t)| = 1 & \text{in } Q, \\
u_t(x, t) + v(x) \cdot D_xu(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\
u(x, 0) = u_0(x) & \text{for } x \in \partial\Omega.
\end{cases}
\]

(4.2)

Suppose that there is a solution \(u \in C(Q \cup (\partial\Omega \times [0, \infty)))\) of (4.2). Observe that for each \(t > 0\) the function \(v(x) := u(x, t)\) is a solution of \(|Dv(x)| = 1\) in \(\Omega\), which shows that the collection \(\{u(\cdot, t) : t > 0\}\) is equi-Lipschitz continuous on \(\overline{\Omega}\). Hence, \(\Omega\), we may choose a sequence \(t_j \to 0^+\) such that the limit

\[\pi_0(x) = \lim_{j \to \infty} u(x, t_j)\]

exists for all \(x \in \overline{\Omega}\) and the convergence is uniform on \(\overline{\Omega}\). It is a standard observation that the limit function \(\pi_0\) is a unique solution of the Dirichlet problem

\[
\begin{cases}
|Dv(x)| = 1 & \text{in } \Omega, \\
v(x) = u_0(x) & \text{for } x \in \partial\Omega.
\end{cases}
\]

(4.3)

Moreover, it follows that, as \(t \to 0^+\), \(u(x, t)\) converges to \(\pi_0(x)\) uniformly for \(x \in \overline{\Omega}\). We thus see that if \(u \in C(Q \cup (\partial\Omega \times [0, \infty)))\) is a solution of (4.2), then \(u\) is extended uniquely to a function on \(\overline{Q}\) and the resulting function solves (4.1), with \(u_0\) replaced by the unique solution \(v\) of the Dirichlet problem (4.3). It is well-known that there exists a solution \(v \in \text{Lip}(\overline{\Omega})\) of (4.3) if and only if there exists a subsolution \(w \in \text{Lip}(\overline{\Omega})\) of (4.3).

Now, an existence result for (4.2) is stated as follows.

**Theorem 4.1** Let \(u_0 \in \text{Lip}(\partial\Omega)\). Assume that there exists a subsolution \(w \in \text{Lip}(\overline{\Omega})\) of (4.3). Then there exists a unique solution \(u \in \text{Lip}(\overline{\Omega})\) of (4.2).

Before giving a proof of the above theorem, we present a comparison principle.

**Lemma 4.2** Let \(v \in \text{USC}(Q \cup (\partial\Omega \times [0, \infty)))\) and \(w \in \text{LSC}(Q \cup (\partial\Omega \times [0, \infty)))\) be a subsolution and a supersolution of (1.2), respectively. Assume that \(v(x, 0) \leq w(x, 0)\) for all \(x \in \partial\Omega\). Then \(v \leq w\) on \(Q \cup (\partial\Omega \times [0, \infty))\).

**Proof.** Fix any \(\varepsilon > 0\). By the semicontinuity of \(v\), \(w\) and the inequality \(v(\cdot, 0) \leq w(\cdot, 0)\) on \(\partial\Omega\), we find that there exists a constant \(\delta > 0\) such that \(v(x, t) \leq w(x, t) + \varepsilon\) for all \((x, t) \in \partial\Omega \times (0, \delta)\). For each \(0 < t < \delta\), since \(|D_1v(x, t)| \leq 1\) and \(|D_1w(x, t)| \geq 1\) in \(\Omega\) hold in the viscosity sense, by a standard comparison result (see Lemma A.3 or [2], [3]), we see that \(v(x, t) \leq w(x, t) + \varepsilon\) for all \((x, t) \in \overline{\Omega} \times (0, \delta)\). We now apply Theorem A.1, with the interval \([t, \infty)\) in place of \([0, \infty)\), in the Appendix, to conclude that

\[v(x, t + s) \leq w(x, t + s) + \varepsilon\]

for all \((x, s) \in \overline{Q}, t \in (0, \delta)\),

which implies that \(v(x, t) \leq w(x, t)\) for all \((x, t) \in Q \cup (\partial\Omega \times [0, \infty))\).
of the function \( w \). We define the functions \( \overline{U} \) and \( \overline{V} \) by \( \overline{U}(x, t) = \overline{w}(x) \pm \overline{L} \), and observe that \( \overline{U} \) and \( \overline{V} \) are a supersolution and a subsolution of (1.2), respectively. By Perron’s method, we may find a function \( u \) such that \( u \in \text{USC}(\overline{Q}) \) is the outer unit normal vector of \( \overline{Q} \). Also, by the comparison (see Theorem A.1) between \( u \) and \( u_* \), we see that \( u = u_* \in C(\overline{Q}) \). Consequently, we obtain for all \( x \in \overline{Q} \) and \( t, s \in [0, \infty) \).

For each \( t > 0 \) the function \( x \mapsto u(x, t) \) is a solution of \( |Du(x, t)| = 1 \) in \( \Omega \), which shows that \( |u(x, t) - u(y, t)| \leq L \) for all \( x, y \in \overline{Q} \) and \( t > 0 \). Thus we see that \( u \in \text{Lip}(\overline{Q}) \) and \( u \) is a solution of (4.1), with \( w \) in place of \( u_0 \).

### 5 Variational formulas

We have studied several Hamilton-Jacobi equations of the eikonal type. We discuss in this section the variational (or optimal control) formulas for solutions of such Hamilton-Jacobi equations.

We recall first the general principle (the Bellman principle). Let \( U \) be an open subset of \( \mathbb{R}^m \) and \( \Gamma \) a closed subset of \( \partial U \). Let \( \gamma : \partial U \setminus \Gamma \to \mathbb{R}^m \) be a continuous vector field such that \( \gamma(x) \cdot v(x) > 0 \) (or \( \gamma(x) \cdot v(x) \geq 0 \)) for all \( x \in \partial U \setminus \Gamma \), where \( v(x) \) denotes the outer unit normal vector of \( U \) at \( x \) and \( p \cdot q \) denotes the inner product in \( \mathbb{R}^m \). Given a function \( v \in L^\infty([0, \infty), \mathbb{R}^m) \), the Skorokhod problem is then to seek for \( \tau \in [0, \infty) \) and \( (X, l) \in \text{Lip}([0, \tau], \mathbb{R}^m) \times L^\infty([0, \tau], \mathbb{R}) \) such that

\[
\begin{align*}
\dot{X}(t) + l(t)\gamma(X(t)) &= v(t) \quad \text{a.e. } t \in [0, \tau], \\
X(t) &\in U \quad \text{for all } t \in [0, \tau], \\
l(t) &\geq 0 \quad \text{a.e. } t \in [0, \tau], \\
l(t) &= 0 \text{ if } X(t) \in U \quad \text{a.e. } t \in [0, \tau], \\
X(\tau) &\in \Gamma.
\end{align*}
\]

The Skorokhod problem has been investigated extensively in the literature (see [15]), and we refer to [12, Theorem 5.2], [13] for the existence results convenient for our discussion here. We denote by \( \text{SP} \) the set of all quadruples \((X, l, \tau, v)\) which satisfy (5.1). We consider the function

\[
V(x) = \inf \int_0^\tau \{ f(X(t), v(t)) + l(t)g(X(t)) \} \, dt + V_0(X(\tau))
\]

on \( \overline{U} \) where \( V_0, f \) and \( g \) are given functions on \( \Gamma, \overline{U} \times \mathbb{R}^m \) and \( \partial U \), respectively, and the infimum is taken over all \((X, l, \tau, v) \in \text{SP} \) such that \( X(0) = x \). This is an optimal control problem, where the function \( v \) plays the role of control and where the function \( V \) is called the value function. The dynamic programming principle leads to the boundary-value problem for the value function \( V \):

\[
\begin{align*}
\sup_{\nu \in \mathbb{R}^m} \{ -\nu \cdot DV(x) - f(x, \nu) \} &= 0 \quad \text{in } U, \\
\gamma(x) \cdot DV(x) &= g(x) \quad \text{on } \partial U \setminus \Gamma, \\
V(x) &= V_0(x) \quad \text{for } x \in \Gamma,
\end{align*}
\]

where \( V_0 \) is a given function representing the Dirichlet (or initial) data on \( \Gamma \).

We apply the above principle to find correct variational formulas for solutions of the Hamilton-Jacobi equations discussed in the previous sections.

We treat first Equation (1.1), where the Hamilton-Jacobi equation can be written as

\[
\sup \{ - (\eta, \sigma) \cdot Du(x, t) - \delta_{\overline{Q}(0)}(\eta, -\sigma) - 1 : (\eta, \sigma) \in \mathbb{R}^m \times \mathbb{R} \} = 0,
\]

where \( \delta_A \) denotes the indicator function of the set \( A \), i.e., the function \( \delta_A \) is defined by \( \delta_A(z) = 0 \) if \( z \in A \) and \( \delta_A(z) = \infty \) otherwise. The sets \( Q \) and \( \overline{Q} \) in (1.1) correspond to \( U \) and \( \Gamma \) in the above, respectively, and the
vector field \((v(x), 1)\) on \(\partial \Omega \times (0, \infty)\), where \(v(x)\) and \(\varepsilon\) are from (1.1), corresponds to the \(\gamma(x, t)\) in the above. Given functions \((v, w) \in L^\infty([0, \infty), \mathbb{R}^n \times \mathbb{R})\), our current Skorokhod problem is stated as

\[
\begin{aligned}
\dot{X}(t) + l(t)\gamma(X(t)) &= v(t) & \text{a.e. } t \in [0, \tau], \\
T(t) + l(t) &= w(t) & \text{a.e. } t \in [0, \tau], \\
X(t) \in \overline{\mathcal{D}} \text{ and } T(t) &\geq 0 & \text{for all } t \in [0, \tau], \\
l(t) &\geq 0 & \text{a.e. } t \in [0, \tau], \\
l(t) &= 0 \text{ if } (X(t), T(t)) \in \mathcal{Q} & \text{a.e. } t \in [0, \tau], \\
T(\tau) &= 0,
\end{aligned}
\]

(5.3)

where \(\tau \in [0, \infty)\) and \((X, T, l) \in \text{Lip}([0, \tau], \mathbb{R}^n \times \mathbb{R}) \times L^\infty([0, \tau], \mathbb{R})\) are to be looked for. Accordingly, \(\text{SP}\) denotes the set of all sextuples \((X, T, l, \tau, v, w)\) satisfying (5.3) and the minimization problem at \((x, t) \in \overline{Q}\), i.e., the value at \((x, t)\) of the optimal control problem associated with (5.3) or (5.2) is stated as

\[
\inf \left\{ \int_0^\tau (\delta_{\Pi(0)}(x,t) + (v(t), -w(t)) + 1) \, dt + u_0(X(\tau)) : (X, T, l, \tau, v, w) \in \text{SP}, X(0) = x, T(0) = t \right\}.
\]

For any \((X, T, l, \tau, v, w) \in \text{SP}\), with \((X(0), T(0)) = (x, t) \in \overline{Q}\), if the integral is finite in the above minimization formula, then \(|v(t)| \leq 1\) and \(w(t) = -\varepsilon\) for a.e. \(t \in [0, \tau]\) and \(\tau\) is characterized by

\[
-t + \int_0^\tau l(t) \, dt = -\varepsilon \tau.
\]

These observations suggest a modification of \(\text{SP}\) and we introduce \(\text{SP}(1.1; x, t)\) as the set of all triples \((X, l, \tau)\) of \(\tau \in [0, \infty), X \in \text{Lip}([0, \tau], \mathbb{R}^n)\) and \(l \in L^\infty([0, \tau], \mathbb{R})\) such that

\[
\begin{aligned}
\dot{X}(s) + l(s)v(X(s)) &\in \overline{\mathcal{D}}(0) & \text{a.e. } s \in [0, \tau], \\
l = \int_0^\tau l(s) \, ds + \varepsilon \tau, \\
X(0) = x, &X(s) \in \overline{\mathcal{D}} & \text{for all } s \in [0, \tau], \\
l(s) = 0 \text{ if } X(s) \in \Omega, &l(s) \geq 0 & \text{a.e. } s \in [0, \tau].
\end{aligned}
\]

(5.4)

**Theorem 5.1** The solution \(u^\varepsilon \in \text{Lip}(\overline{Q})\) of (1.1) is represented as

\[
u^\varepsilon(x, t) = \inf\{\tau + u_0(X(\tau)) : (X, l, \tau) \in \text{SP}(1.1; x, t)\} \text{ for all } (x, t) \in \overline{Q}.\]

(5.5)

**Proof.** We write \(V(x, t)\) for the right-hand side of (5.5).

1. It is a standard observation that the dynamic programming principle holds: for any \((x, t) \in \overline{\mathcal{D}} \times (0, \infty)\) and \(\delta \in (0, t)\),

\[
\nu(x, t) = \inf\{\tau + V(X(\tau), t - \delta) : (X, l, \tau) \in \text{SP}(1.1; x, \delta)\}.
\]

(5.6)

Here we have for any \((X, l, \tau) \in \text{SP}(1.1; x, \delta)\),

\[
\delta = \varepsilon \tau + \int_0^\tau l(r) \, dr,
\]

and hence,

\[
\max_{x \in [0, 1]} |X(s) - x| \leq \int_0^\tau (l(r) + 1) \, dr \leq (\varepsilon^{-1} + 1) \delta.
\]

(5.7)

2. We have shown in the proof of Theorem 2.1 that, for some constant \(A > 0\), the function \(U(x, t) := u_0(x) - At\) is a subsolution of (1.1). Fix such a function \(U\) on \(\overline{Q}\). Fix any \((x, t) \in \overline{Q}\) and \((X, l, \tau) \in \text{SP}(1.1; x, t)\).

We set

\[
T(s) := t - \varepsilon s - \int_0^s l(r) \, dr \text{ and } v(s) := \dot{X}(s) + l(s)v(X(s)) \text{ for } s \in [0, \tau],
\]
and compute informally that
\[
U(x, t) = U(X(0), T(0)) = U(X(\tau), T(\tau)) - \int_0^\tau \frac{d}{ds} U(X(s), T(s)) ds
\]
\[
= u_0(X(\tau)) + \int_0^\tau \left[ -U_t(X(s), T(s)) \dot{T}(s) - D_x U(X(s), T(s)) \cdot \dot{X}(s) \right] ds
\]
\[
= u_0(X(\tau)) + \int_0^\tau \left[ \varepsilon U_t(X(s), T(s)) - D_x U(X(s), T(s)) \cdot v(s) \right] ds
\]
\[
+ \int_0^\tau I(s) \left[ U_t(X(s), T(s)) + D_x U(X(s), T(s)) \cdot v(X(s)) \right] ds
\]
\[
\leq u_0(X(\tau)) + \int_0^\tau \left[ \varepsilon U_t(X(s), T(s)) + |D_x U(X(s), T(s))| \right] ds
\]
\[
+ \int_0^\tau I(s) \left[ U_t(X(s), T(s)) + D_x U(X(s), T(s)) \cdot v(X(s)) \right] ds
\]
\[
\leq u_0(X(\tau)) + \varepsilon t.
\]

The above computation is easily justified by approximating \( u_0 \) by smooth functions, and we conclude that
\[
V(x, t) \geq u_0(x) - \varepsilon t \quad \text{for all} \quad (x, t) \in \overline{Q}, \quad \text{which yields}
\]
\[
V_+(x, 0) \geq u_0(x) \quad \text{for all} \quad x \in \overline{\Omega},
\] (5.8)
where \( V_+ \) denotes the lower semicontinuous envelope of the function \( V \).

3. Next, we show that
\[
V^+(x, 0) \leq u_0(x) \quad \text{for all} \quad x \in \overline{\Omega},
\] (5.9)
where \( V^+ \) denotes the upper semicontinuous envelope of the function \( V \). To see this, we fix any \((x, t) \in \overline{Q}, \) define the triple \((X, l, \tau)\) by
\[
\tau := \varepsilon^{-1} t \quad \text{and} \quad X(s) := x, \quad l(s) := 0 \quad \text{for} \quad s \in [0, \tau],
\]
and observe that \((X, l, \tau) \in SP(1; 1; x, t)\) and
\[
V(x, t) \leq \tau + u_0(x) = u_0(x) + \varepsilon^{-1} t.
\]
This clearly shows that (5.9) holds.

4. We prove that \( V^+ \) is a subsolution of (1.1). Let \( \phi \in C^1(\overline{Q}) \) and \((\hat{x}, \hat{t}) \in \overline{\Omega} \times (0, \infty), \) and assume that \( V^+ - \phi \) has a strict maximum at \((\hat{x}, \hat{t})\). We treat here only the case where \( \hat{x} \in \partial \Omega \). The other case can be handled similarly and more easily. We argue by contradiction and hence assume that
\[
\varepsilon \phi_t(\hat{x}, \hat{t}) + |D_x \phi(\hat{x}, \hat{t})| > 1 \quad \text{and} \quad \phi_t(\hat{x}, \hat{t}) + v(\hat{x}) \cdot D_x \phi(\hat{x}, \hat{t}) > 0.
\] (5.10)

We choose a unit vector \( e \in \mathbb{R}^n \) so that \(|D_x \phi(\hat{x}, \hat{t})| = -e \cdot D_x \phi(\hat{x}, \hat{t})\), and then a constant \( R \in (0, \hat{t}) \) so that for all \((x, t) \in \overline{Q} \cap (B_R(\hat{x}) \times [\hat{t} - R, \hat{t} + R])\),
\[
\varepsilon \phi_t(x, t) - e \cdot D_x \phi(x, t) > 1 \quad \text{and} \quad \phi_t(x, t) + v(x) \cdot D_x \phi(x, t) > 0.
\]
In view of (5.7), we fix an \( r \in (0, R/2) \) so that \((\varepsilon^{-1} + 1)r < R/2\), and choose a point \((\bar{x}, \bar{t}) \in \overline{Q} \cap (B_{R/2}(\hat{x}) \times (\hat{t} - r/2, \hat{t} + r/2))\) so that
\[
(V - \phi)(\bar{x}, \bar{t}) > \max\{(V^+ - \phi)(x, \hat{t} - r/2) : x \in \overline{\Omega}\}.
\]
Such a choice is possible since
\[
(V^+ - \phi)(\hat{x}, \hat{t}) > \max\{(V^+ - \phi)(x, \hat{t} - r/2) : x \in \overline{\Omega}\}.
\]
We set \( \delta := \hat{t} - (\hat{t} - r/2), \) and note that \( 0 < \delta < (\hat{t} - r/2) < r \) and \((\varepsilon^{-1} + 1)\delta < R/2\).

According to the existence results in [12], [13], there exists a solution \((X, l, \tau) \in SP(1; 1; \bar{x}, \delta)\) such that
\[
\dot{X}(s) + l(s)v(X(s)) = e \quad \text{a.e.} \quad s \in [0, \tau].
\]
We set
\[ T(s) := \tilde{t} - \varepsilon s - \int_0^s l(r)dr \quad \text{for } s \in [0, \tau]. \]

By (5.6), we get
\[ V(\tilde{x}, \tilde{t}) \leq \tau + V(X(\tau), \tilde{t} - \delta) = \tau + V(X(\tau), T(\tau)). \]

We may assume by adding a constant to the function \( \phi \) that \((V - \phi)(\tilde{x}, \tilde{t}) = 0\), which implies that
\[ (V - \phi)(x, T(\tau)) = (V - \phi)(x, \tilde{t} - r/2) < 0 \quad \text{for all } x \in \bar{\mathcal{D}}. \]

Hence, we get
\[
0 \leq \tau + V(X(\tau), T(\tau)) - V(\tilde{x}, \tilde{t}) < \tau + \phi(X(\tau), T(\tau)) - \phi(\tilde{x}, \tilde{t})
\]
\[
= \int_0^\tau [1 + \phi_1(X(s), T(s))\tilde{T}(s) + D_x \phi(X(s), T(s)) \cdot \tilde{X}(s)] ds
\]
\[
= \int_0^\tau [1 + e \cdot D_x \phi(X(s), T(s)) - \varepsilon \phi_1(X(s), T(s))] ds
\]
\[
+ \int_0^\tau l(s) [-\phi_1(X(s), T(s)) - \nu(X(s)) \cdot D_x \phi(X(s), T(s))] ds. \tag{5.11}
\]

By estimate (5.7) and our choice of \( \delta \), we have \( X(s) \in \mathcal{B}_{R/2}(\tilde{x}) \subset \mathcal{B}_R(\tilde{x}) \). Now, using (5.10), we get
\[
0 < \int_0^\tau [1 + e \cdot D_x \phi(X(s), T(s)) - \varepsilon \phi_1(X(s), T(s))] ds
\]
\[
+ \int_0^\tau l(s) [-\phi_1(X(s), T(s)) - \nu(X(s)) \cdot D_x \phi(X(s), T(s))] ds < 0,
\]
which is a contradiction. Thus, \( V^* \) is a subsolution of (1.1).

5. We next prove that \( V_* \) is a supersolution of (1.1). The argument here is similar to that in the previous step. Let \( \phi \in C^1(\mathcal{Q}) \) and \((\hat{x}, \hat{t}) \in \mathcal{Q} \times (0, \infty)\), and assume that \( V_* - \phi \) has a strict minimum at \((\hat{x}, \hat{t})\). Here again we treat only the case where \( \hat{x} \in \partial \Omega \). We assume by contradiction that
\[ \varepsilon \phi_1(\hat{x}, \hat{t}) + |D_x \phi(\hat{x}, \hat{t})| < 1 \quad \text{and} \quad \phi_1(\hat{x}, \hat{t}) + \nu(\hat{x}) \cdot D_x \phi(\hat{x}, \hat{t}) < 0. \]

We choose a constant \( R \) so that for all \((x, t) \in \mathcal{Q} \cap (\bar{B}_R(\hat{x}) \times [\hat{t} - R, \hat{t} + R]),\)
\[ \varepsilon \phi_1(x, t) + |D_x \phi(x, t)| < 1 \quad \text{and} \quad \phi_1(x, t) + \nu(x) \cdot D_x \phi(x, t) < 0. \]

As in the previous step, we may choose a point \((\hat{x}, \hat{t}) \in \mathcal{Q}\) and a constant \( \delta > 0 \) such that
\[
\begin{cases}
(V - \phi)(\hat{x}, \hat{t}) < \min\{(V_* - \phi)(x, \hat{t} - \delta) : x \in \mathcal{D}\}, \\
\bar{B}_{R/2}(\hat{x}) \times [\hat{t} - \delta, \hat{t}] \subset \bar{B}_R(\hat{x}) \times [\hat{t} - R, \hat{t} + R], \\
(\varepsilon^{-1} + 1)\delta < R/2.
\end{cases}
\]

We may assume as before that \((V - \phi)(\hat{x}, \hat{t}) = 0\) and, hence, \((V_* - \phi)(x, \hat{t} - \delta) > 0\) for all \( x \in \mathcal{D} \). Setting
\[ \gamma := \min\{(V_* - \phi)(x, \hat{t} - \delta) : x \in \mathcal{D}\}, \]
by (5.6), we may choose a triple \((X, l, \tau) \in \text{SP}(1.1; \hat{x}, \hat{t})\) so that \( V(\tilde{x}, \tilde{t}) + \gamma > \tau + V(X(\tau), \tilde{t} - \delta) \), which yields \( \phi(\tilde{x}, \tilde{t}) > \tau + \phi(X(\tau), \tilde{t} - \delta) \). Noting that \( X(s) \in \mathcal{B}_R(\hat{x}) \) for all \( s \in [0, \tau] \), and setting
\[
T(s) := \tilde{t} - \varepsilon s - \int_0^s l(r)dr \quad \text{and} \quad \nu(s) := \dot{X}(s) + l(s)\nu(X(s)) \quad \text{for } s \in [0, \tau],
\]
we compute that
\[
0 > \tau + \phi(X(\tau), T(\tau)) - \phi(\bar{x}, \bar{r}) = \int_0^\tau [1 + \phi_t(X(s), T(s)) T(s) + D_x\phi(X(s), T(s)) \cdot \dot{X}(s)] ds
\]
\[
= \int_0^\tau [1 + v(s) \cdot D_x\phi(X(s), T(s)) - \epsilon \phi_t(X(s), T(s))] ds + \int_0^\tau l(s) [-\phi_t(X(s), T(s)) - v(X(s)) \cdot D_x\phi(X(s), T(s))] ds
\]
\[
\geq \int_0^\tau [1 - |D_x\phi(X(s), T(s))| - \epsilon \phi_t(X(s), T(s))] ds \geq 0,
\]
which is a contradiction. We have thus shown that \( V_* \) is a supersolution of \((1.1)\).

6. We now apply a comparison result (for instance, Theorem A.1), to conclude that \( V_* \leq u^c \leq V_* \) on \( \bar{Q} \), which obviously shows that \( u^c = V \) on \( \bar{Q} \).

Replacing \((l, t)\) by \((\epsilon l, \epsilon t)\) in \((5.4)\) is a simple modification to get the right Skorokhod problem for \((1.3)\). We thus denote by \( SP(1.3; x, t) \) the set of all triples \((X, l, \tau)\) of \( \tau \in [0, \infty) \), \( X \in \text{Lip}(\Omega, \mathbb{R}^n) \) and \( l \in L^\infty([0, \tau], \mathbb{R}) \) such that

\[
\dot{X}(s) + \epsilon l(s) v(X(s)) \in \bar{B}_1(0) \quad \text{a.e. } s \in [0, \tau],
\]
\[
t = \int_0^\tau l(s) ds + \tau,
\]
\[
X(0) = x, \quad X(s) \in \bar{Q} \quad \text{for all } s \in [0, \tau],
\]
\[
l(s) = 0 \text{ if } X(s) \in \Omega, \quad l(s) \geq 0 \quad \text{a.e. } s \in [0, \tau].
\]

The following proposition is an immediate consequence of the previous theorem.

**Corollary 5.2** The solution \( u^c \in \text{Lip}(\bar{Q}) \) of \((1.3)\) is represented as

\[
u^c(x, t) = \inf \{ \tau + u_0(X(\tau)) : (X, l, \tau) \in SP(1.3; x, t) \} \quad \text{for all } (x, t) \in \bar{Q}.
\]

The Skorokhod problem associated with problem \((1.4)\) or \((3.1)\) is given by the collection \( SP(1.4; x, t) \) of all triples \((X, l, \tau)\) of \( \tau \in [0, \infty) \), \( X \in \text{Lip}(\Omega, \mathbb{R}^n) \) and \( l \in L^\infty([0, \tau], \mathbb{R}) \) such that

\[
|\dot{X}(s)| \leq 1 \quad \text{a.e. } s \in [0, \tau],
\]
\[
t = \int_0^\tau l(s) ds + \tau,
\]
\[
X(0) = x, \quad X(s) \in \bar{Q} \quad \text{for all } s \in [0, \tau],
\]
\[
l(s) = 0 \text{ if } X(s) \in \Omega, \quad l(s) \geq 0 \quad \text{a.e. } s \in [0, \tau].
\]

The variational formula for the solution of \((1.4)\) is stated as follows.

**Theorem 5.3** Let \( v \in \text{Lip}(\bar{Q}) \) be the solution of \((1.4)\). Then

\[
v(x, t) = \inf \{ \tau + u_0(X(\tau)) : (X, l, \tau) \in SP(1.4; x, t) \} \quad \text{for all } (x, t) \in \bar{Q}.
\]

Because of the lack of a “good” comparison theorem (see Lemma 3.2 and Example A.5), our strategy for the proof of the above theorem differs substantially from that of Theorem 5.1.

**Proof.** We write \( V(x, t) \) for the right-hand side of \((5.14)\). The dynamic programming principle holds, which is stated as

\[
V(x, t + h) = \inf \{ \tau + V(X(\tau), t) : (X, l, \tau) \in SP(1.4; x, h) \} \quad \text{for all } (x, t) \in \bar{Q}, \ h \geq 0.
\]

1. Let \((x, t) \in \bar{Q}\) and \( h > 0 \). We set \( X(t) = x \) and \( l(t) = 0 \) for \( t \in [0, h] \), and note that \((X, l, h) \in SP(1.4; x, h)\). By \((5.15)\), we have \( V(x, t + h) \leq h + V(x, t) \). That is, we have

\[
V(x, t + h) \leq V(x, t) + h \quad \text{for all } (x, t) \in \bar{Q}, \ h \geq 0.
\]
2. Let $A > 0$ be a Lipschitz bound of the function $u_0$. Following Step 2 of the proof of Theorem 5.1, we obtain $V(x, t) \geq u_0(x) - At$ for all $(x, t) \in \widehat{Q}$. Hence, using (5.15), we get

$$V(x, t + h) = \inf \{ \tau + V(X(\tau), h) : (X, l, \tau) \in \text{SP}(1.4; x, t) \} \geq \inf \{ \tau + u_0(X(\tau)) - Ah : (X, l, \tau) \in \text{SP}(1.4; x, t) \}$$

$$= V(x, t) - Ah \quad \text{for all } (x, t) \in \widehat{Q}, h \geq 0.$$  

Thus, combining (5.16) and (5.17), we see that the functions $t \mapsto V(x, t)$, with $x \in \widehat{Q}$, are equi-Lipschitz continuous on $[0, \infty)$ with a Lipschitz bound $\max \{A, 1\}$.

3. Let $t \geq 0$, $B \subset \Omega$ be a ball and choose $x, y \in B$ so that $x \neq y$. Set

$$h = |x - y|, \quad l(s) = 0 \quad \text{and} \quad X(s) = x + sh^{-1}(y - x) \quad \text{for } s \in [0, h].$$

Noting that $(X, l, h) \in \text{SP}(1.4; x, h)$, by (5.15), we get $V(x, t + h) \leq h + V(y, t)$. Combining this with (5.17) yields $V(x, t) \leq V(y, t) + (A + 1)|x - y|$. Hence, by the symmetry in $x$ and $y$, we get $|V(x, t) - V(y, t)| \leq (A + 1)|x - y|$. This shows that the functions $x \mapsto V(x, t)$, with $t \geq 0$, are equi-Lipschitz continuous on $\Omega$ with a Lipschitz bound $L_\Omega (A + 1)$. This combined with the result in Step 2 assures that $V \in \text{Lip}(\widehat{Q})$.

4. Let $x \in \partial \Omega$ and $t \geq 0$. Observe that if $\varepsilon > 0$ is small enough, then the line segment connecting the points $x$ and $x - \varepsilon v(x)$ lies in $\widehat{Q}$. Arguing as in Step 3, we deduce that $|V(x, t) - V(x - \varepsilon v(x), t)| \leq (A + 1)\varepsilon$. This shows together with the observation in Step 3 that the functions $x \mapsto V(x, t)$, with $t \geq 0$, are continuous on $\widehat{Q}$, which moreover implies that the functions $x \mapsto V(x, t)$, with $t \geq 0$, are equi-Lipschitz continuous on $\widehat{Q}$ with a Lipschitz bound $L_\Omega (A + 1)$. This combined with the result in Step 2 assures that $V \in \text{Lip}(\widehat{Q})$.

5. Following the argument of the proof of Theorem 5.1, we see easily that $V$ is a solution of (1.4). Lemma 3.2 guarantees that $v = V \in \widehat{Q}$.

For the problem (1.2) with initial condition, a natural choice is the Skorokhod problem $\text{SP}(1.2; x, t)$, defined as the set of of all triples $(X, l, \tau)$ of $\tau \in [0, \infty), X \in \text{Lip}([0, \tau], \mathbb{R}^n)$ and $l \in L^\infty([0, \tau], \mathbb{R})$ such that

$$\begin{align*}
\dot{X}(s) + l(s) v(X(s)) &\in \overline{B}_1(0) \quad \text{a.e. } s \in [0, \tau], \\
\tau &= \int_0^\tau l(s) ds, \\
X(0) &= x, \quad X(s) \in \overline{Q} \quad \text{for all } s \in [0, \tau], \\
l(s) &= 0 \text{ if } X(s) \in \Omega, \quad l(s) \geq 0 \quad \text{a.e. } s \in [0, \tau].
\end{align*}$$

We remark that the first condition in (5.18) is equivalent to the inequality

$$|\dot{X}(s)|^2 + l(s)^2 \leq 1 \quad \text{a.e.}$$

To see this, let $x, y \in \overline{Q}$ and $(X, l, \tau) \in \text{SP}(x)$ be such that $X(\tau) = y$. Let $\rho$ be a defining function of $\Omega$ and note that for any $t \in [0, \tau]$, if $X(t) \in \partial \Omega$, then the function $s \mapsto \rho(X(s))$ attains the maximum value $0$ at $t$. Hence, if $t \in (0, \tau)$ is a point where $X(t) \in \partial \Omega$ and the function $X$ is differentiable at $t$, then

$$0 = \frac{d}{ds} \rho(X(s)) \bigg|_{s=t} = D\rho(X(t)) \cdot \dot{X}(t).$$

that is, two vectors $v(X(t))$ and $\dot{X}(t)$ are perpendicular. Accordingly, we have

$$v(X(s)) \cdot \dot{X}(s) = 0 \quad \text{if } X(s) \in \partial \Omega \quad \text{a.e.}$$

Thus, the first condition in (5.18) is equivalent to condition (5.19).

**Theorem 5.4** Let $u \in \text{Lip}(\overline{Q})$ be a (unique) solution of (1.2) satisfying the initial condition $u(\cdot, 0) = u_0$. Then

$$u(x, t) = \inf \{ \tau + u_0(X(\tau)) : (X, l, \tau) \in \text{SP}(1.2; x, t) \} \quad \text{for all } (x, t) \in \overline{Q}.$$
which there exists a function $X \in \text{Lip}([0, \tau], \overline{\Omega})$ such that $X(t) \in \overline{\Omega}$ for all $t \in [0, \tau]$, $X(0) = x$ and $X(\tau) = y$, and $\lambda_{\Omega}(x, y)$ is defined by

$$\lambda_{\Omega}(x, y) = \inf \left\{ \int_0^\tau (1 - l(s))ds : (X, l, \tau) \in \text{SP}(x), \ X(\tau) = y \right\},$$

where $\text{SP}(x)$ denotes the set of all triples $(X, l, \tau)$ of $\tau > 0, l \in L^\infty([0, \tau], \mathbb{R})$ and $X \in \text{Lip}([0, \tau], \mathbb{R}^n)$ such that

$$\begin{cases}
\dot{X}(s) + l(s)v(X(s)) \in \overline{B}_1(0) & \text{a.e. } s \in [0, \tau], \\
x(0) = x, \ X(s) \in \overline{\Omega} & \text{for all } s \in [0, \tau], \\
l(s) = 0 \text{ if } X(s) \in \Omega, \ l(s) \geq 0 & \text{a.e. } s \in [0, \tau].
\end{cases}$$

Note that

$$\text{SP}(x) = \bigcup_{t \geq 0} \text{SP}(1.1; x, t) = \bigcup_{t \geq 0} \text{SP}(1.2; x, t),$$

by (5.19) that $\lambda_{\Omega}(x, y) \geq 0$ for all $x, y \in \overline{\Omega}$, and that

$$d_{\Omega}(x, y) = \inf \left\{ \int_0^\tau (1 - l(s))ds : (X, l, \tau) \in \text{SP}(x), \ X(\tau) = y, \ l(s) \equiv 0 \right\} \geq \lambda_{\Omega}(x, y).$$

We note as well that $d_{\Omega}(x, x) = \lambda_{\Omega}(x, x) = 0$ for all $x \in \overline{\Omega}$.

**Theorem 5.5** The functions $v_0^-, v_\infty$ and $u_\infty$ are represented as

$$v_0^-(x) = \inf \{d_{\Omega}(x, y) + u_0(y) : y \in \overline{\Omega} \},$$

$$v_\infty(x) = \inf \{d_{\Omega}(x, y) + v_0^-(y) : y \in \partial \Omega \},$$

$$u_\infty(x) = \inf \{\lambda_{\Omega}(x, y) + v_0^-(y) : y \in \partial \Omega \}.$$

**Lemma 5.6** Let $y \in \overline{\Omega}$. (i) The function $x \mapsto d_{\Omega}(x, y)$ is a solution of (1.5) in $\Omega \setminus \{y\}$ and a subsolution of (1.5) in $\Omega$. (ii) The function $x \mapsto \lambda_{\Omega}(x, y)$ is a solution of

$$\begin{cases}
|Du(x)| = 1 & \text{in } \Omega \setminus \{y\}, \\
1 + v(x) \cdot Du(x) = 0 & \text{on } \partial \Omega,
\end{cases}$$

and a subsolution of (1.6).

**Proof.** 1. Note that the function $v(x) = |x - y|$ in $\mathbb{R}^n$, with $y \in \mathbb{R}^n$, is a solution of $|Du(x)| = 1$ in $\mathbb{R}^n \setminus \{y\}$ and is a subsolution of $|Du(x)| = 1$ in $\mathbb{R}^n$.

2. Let $y \in \overline{\Omega}$ and let $B \subset \Omega$ be an open ball such that $y \notin B$. According to the dynamic programming principle, we deduce that for any $x \in B$

$$d_{\Omega}(x, y) = \inf \{d_{\Omega}(x, z) + d_{\Omega}(z, y) : z \in \partial B \} = \inf \{|x - z| + d_{\Omega}(z, y) : z \in \partial B \},$$

and

$$\lambda_{\Omega}(x, y) = \inf \{|x - z| + \lambda_{\Omega}(z, y) : z \in \partial B \}.$$

By the observation in Step 1, using Proposition A.4, we see that both the functions $x \mapsto d_{\Omega}(x, y)$ and $x \mapsto \lambda_{\Omega}(x, y)$ are Lipschitz continuous in $B$ and are solutions of $|Du(x)| = 1$ in $B$, which implies that they are both solutions of $|Du(x)| = 1$ in $\Omega \setminus \{y\}$.

Next let $y \in \Omega$ and let $B \subset \Omega$ be an open ball such that $y \in B$. For any $x \in B$, we have

$$d_{\Omega}(x, y) = \lambda_{\Omega}(x, y) = |x - y|,$$

from which we see that both the functions $x \mapsto d_{\Omega}(x, y)$ and $x \mapsto \lambda_{\Omega}(x, y)$ are subsolutions of $|Du(x)| = 1$ in $B$. This together with the previous observation, we conclude that the functions $x \mapsto d_{\Omega}(x, y)$ and $x \mapsto \lambda_{\Omega}(x, y)$
are subsolutions of $|Du(x)| = 1$ in $\Omega$. This ensures that these functions are Lipschitz continuous in $\Omega$ with a Lipschitz bound $L_{\Omega}$.

A consideration based on the dynamic programming principle similar to the above shows that if $\varepsilon > 0$ is sufficiently small, then, for all $x \in \partial \Omega$ and $y \in \Omega$,

$$\max\{|d_{\Omega}(x - \varepsilon u(x), y) - d_{\Omega}(x, y)|, |\lambda_{\Omega}(x - \varepsilon u(x), y) - \lambda_{\Omega}(x, y)|\} \leq \varepsilon.$$ 

This and the Lipschitz continuity of the functions $x \mapsto d_{\Omega}(x, y)$ and $x \mapsto \lambda_{\Omega}(x, y)$ in $\Omega$ guarantee that these functions are Lipschitz continuous on $\Omega$.

3. By following the argument (Steps 4 and 5) of the proof of Theorem 5.1, it is now not hard to check that the function $x \mapsto \lambda_{\Omega}(x, y)$ on $\Omega$ is a solution of (5.23).

$\square$

Lemma 5.7

(i) If $u \in \text{Lip}(\overline{\Omega})$ is a subsolution of (1.5), then

$$u(x) - u(y) \leq d_{\Omega}(x, y) \quad \text{for all } x, y \in \Omega.$$ 

(ii) If $u \in \text{Lip}(\overline{\Omega})$ is a subsolution of (1.6), then

$$u(x) - u(y) \leq \lambda_{\Omega}(x, y) \quad \text{for all } x, y \in \Omega.$$

Proof. (i) Let $u \in \text{Lip}(\overline{\Omega})$ be a subsolution of (1.5). We approximate $u$ by a smooth function $u_{\varepsilon} \in C^{1}(\overline{\Omega})$ with $\varepsilon > 0$ such that $| Du_{\varepsilon}(x) | \leq 1 + \varepsilon$ in $\Omega$, observe that for any $x, y \in \Omega$,

$$u_{\varepsilon}(x) = u_{\varepsilon}(y) - \int_{l}^{r} Du_{\varepsilon}(X(s)) \cdot \dot{X}(s) ds \leq u_{\varepsilon}(y) + (1 + \varepsilon)\tau,$$

where $(X, l, \tau) \in \text{SP}(\Omega)$ satisfies $X(\tau) = y$ and $l(s) \equiv 0$, and conclude that $u(x) \leq u(y) + d_{\Omega}(x, y)$ for all $x, y \in \Omega$.

(ii) Let $u \in \text{Lip}(\overline{\Omega})$ be a subsolution of (1.6). For each $\varepsilon > 0$, there is a function $u_{\varepsilon} \in C^{1}(\overline{\Omega})$ which satisfies

$$\begin{cases}
|Du_{\varepsilon}(x)| \leq 1 + \varepsilon & \text{for all } x \in \Omega, \\
1 + v(x) \cdot Du_{\varepsilon}(x) \leq 0 & \text{for all } x \in \partial \Omega, \\
||u_{\varepsilon} - u||_{Lip, \Omega} < \varepsilon.
\end{cases}$$

(See [12, Theorem 4.2] for this.) Then, arguing as in the proof of (i) above, we easily conclude that $u(x) \leq u(y) + \lambda_{\Omega}(x, y)$ for all $x, y \in \Omega$.

$\square$

Lemma 5.8

(i) If $u \in \text{Lip}(\overline{\Omega})$ is a solution of (1.5), then

$$u(x) = \min\{u(y) + d_{\Omega}(x, y) : y \in \partial \Omega\} \quad \text{for all } x \in \Omega.$$ 

(ii) If $u \in \text{Lip}(\overline{\Omega})$ is a solution of (1.6), then

$$u(x) = \min\{u(y) + \lambda_{\Omega}(x, y) : y \in \partial \Omega\} \quad \text{for all } x \in \Omega.$$ 

Proof. (i) We set

$$V(x) = \min\{u(y) + d_{\Omega}(x, y) : y \in \partial \Omega\} \quad \text{for } x \in \Omega.$$ 

By Lemma 5.7, we have $u(x) \leq V(x)$ for all $x \in \overline{\Omega}$. By the definition of $V$, we see that $V(x) \leq u(x)$ for all $x \in \partial \Omega$. Hence, we have $u(x) = V(x)$ for all $x \in \partial \Omega$. According to Proposition A.4, the function $V$ is a solution of (1.5). Hence, by Lemma A.3, we conclude that $u = V$ on $\overline{\Omega}$.

The proof of (ii) is similar to the above, and we skip it here.

$\square$

Proof of Theorem 5.4. 1. We set

$$V(x, t) = \inf\{\tau + u_{0}(X(\tau)) : (X, l, \tau) \in \text{SP}(1.2; x, t)\} \quad \text{for } (x, t) \in \overline{\Omega}.$$
We show that
\[ V^*(x, 0) \leq u_0(x) \leq V_u(x, 0) \quad \text{for all } x \in \overline{\Omega} \]
as well as the locally boundedness of the function \( V \). Once this is done, we just need to follow Steps 4, 5 and 6 of the proof of Theorem 5.1.

2. It is a standard observation that for each \( t > 0 \) the function \( w : x \mapsto u(x, t) \) is a solution of the eikonal equation \( |Dw(x)| = 1 \) in \( \Omega \). By assumption, we have \( u \in \text{Lip}(\overline{\Omega}) \). Hence, by the stability of the viscosity property, we see that \( u_0 \) is a solution of \( |Dw(x)| = 1 \) in \( \Omega \). As in Step 2 of the proof of Theorem 5.1, we easily find a constant \( A > 0 \) such that \( V(x, t) \geq u_0(x) - At \) for all \( (x, t) \in \overline{\Omega} \), which proves that \( V \) is locally bounded below in \( \overline{\Omega} \) and that \( V_u(x, 0) \geq u_0(x) \) for all \( x \in \overline{\Omega} \).

3. Next, fix any \( (x, t) \in \partial \Omega \times (0, \infty) \) and set
\[ \tau = t, \quad l(s) = 1 \quad \text{and} \quad X(s) = x \quad \text{for } s \in [0, \tau]. \]
Observe that \( (X, l, \tau) \in \text{SP}(1.2; x, t) \) and that
\[ V(x, t) \leq \tau + u_0(X(\tau)) = u_0(x) + t. \]
(5.24)

Now fix any \( (x, t) \in \overline{\Omega} \times (0, \infty) \). By (i) of Lemma 5.8, there exists a point \( y \in \partial \Omega \) such that
\[ u_0(x) = u_0(y) + d_\Omega(x, y). \]
(5.25)

By the dynamic programming principle, we have \( V(x, t) \leq d_\Omega(x, y) + V(y, t) \). Combining this with (5.24) and using (5.25), we get \( V(x, t) \leq d_\Omega(x, y) + u_0(y) + t = u_0(x) + t \), which shows that \( V \) is locally bounded above on \( \overline{\Omega} \) and that \( V^*(x, t) \leq u_0(x) \) for all \( x \in \overline{\Omega} \).

\textbf{Proof of Theorem 5.1.} 1. We write \( V(x) \) for the right-hand side of (5.20). Since \( v_0 \) is a subsolution of (1.5), by Lemma 5.7 we have \( v_0(x) \leq V(x) \) for all \( x \in \overline{\Omega} \). On the other hand, in view of Proposition A.4 we see that \( V \) is a subsolution of (1.5). Also, we have \( V(x) \leq v_0(x) + d_\Omega(x, x) \leq u_0(x) \) for all \( x \in \overline{\Omega} \). Now, the maximality of \( v_0 \) ensures that \( V \leq v_0 \) on \( \overline{\Omega} \). Thus, we conclude that \( V = v_0 \) on \( \overline{\Omega} \).

2. Let \( V(x) \) denote the right-hand side of (5.21). By Lemma 5.7, we have
\[ v_0(x) \leq v_0(y) + d_\Omega(x, y) \quad \text{for all } x, y \in \overline{\Omega}, \]
and hence, \( v_0 \) is a subsolution of (1.5). By the minimality of \( v_\infty \), we see that \( v_\infty \leq V \) on \( \overline{\Omega} \). Note that \( v_\infty \geq v_0 \) on \( \overline{\Omega} \) and \( V(x) \leq v_0(x) \) for all \( x \in \partial \Omega \). Hence, we have \( v_\infty(x) = v_0(x) \) for all \( x \in \partial \Omega \). Now, by comparison (Lemma A.3), we get \( v_\infty = V \) on \( \overline{\Omega} \).

3. Let \( V(x) \) denote the right-hand side of (5.22). As noted before, the function \( v_\infty \) is a supersolution of (1.6). In Step 2 above, we have observed that \( v_\infty = v_0 \) on \( \partial \Omega \). According to Lemma 5.6, the function \( V \) is a solution of (1.6). Since \( V \leq u_\infty \) on \( \partial \Omega \), by comparison, we get \( V \leq u_\infty \) on \( \overline{\Omega} \). Hence, by the maximality of \( u_\infty \), we see that \( V \leq u_\infty \) on \( \overline{\Omega} \). On the other hand, by (ii) of Lemma 5.8, we find that
\[ V(x) = \min\{\lambda_\Omega(x, y) + v_\infty(y) : y \in \partial \Omega\} \geq \min\{\lambda_\Omega(x, y) + u_\infty(y) : y \in \partial \Omega\} = u_\infty(x) \quad \text{for all } x \in \overline{\Omega}. \]
Thus, we have \( u_\infty = V \) on \( \overline{\Omega} \).

6 More on the function \( \lambda_\Omega \)

By the assumption that \( \Omega \) is a bounded, open connected subset of \( \mathbb{R}^n \) and is of class \( C^1 \), we deduce that \( \partial \Omega \) consists of a finite number of connected components \( \Omega_i \), with \( i = 1, 2, \ldots, N \).

We have shown in the proof of Theorem 2.3 that the function \( x \mapsto \text{dist}(x, \partial \Omega) \) on \( \overline{\Omega} \) is a solution of (1.6). The same proof shows that for each \( i = 1, \ldots, N \), the function \( u(x) := \text{dist}(x, \Omega_i) \) on \( \overline{\Omega}_i \) is a solution of
\[
\begin{cases}
|Du(x)| = 1 & \text{in } \Omega, \\
1 + v(x) \cdot Du(x) = 0 & \text{on } \Omega_i.
\end{cases}
\]
(6.1)
For \( y \in \overline{\Omega} \) and \( i, j = 1, \ldots, N \) we define
\[
\gamma(y, i) := \text{dist}(y, \Gamma_i) = \min\{|y - z| : z \in \Gamma_i\},
\]
\[
\gamma(i, j) := \text{dist}(\Gamma_i, \Gamma_j) = \min\{|x - y| : x \in \Gamma_i, \ y \in \Gamma_j\}.
\]
Let \( I \) denote the set of all finite sequences \((i_1, \ldots, i_m)\) such that \( i_j \in \{1, \ldots, N\} \) for all \( j = 1, \ldots, m \) and \( i_j \neq i_k \) if \( j \neq k \). For \( y \in \overline{\Omega} \) and \( i = 1, \ldots, N \) we set
\[
a_i(y) = \min\left\{\gamma(y, i_1) + \sum_{j=1}^{m-1} \gamma(i_j, i_{j+1}) : (i_1, \ldots, i_m) \in I, \ i_m = i\right\}.
\]

**Theorem 6.1** We have
\[
\lambda(y, x) = \min\{|x - y|, \ a_i(y) + \text{dist}(x, \Gamma_i) : i = 1, \ldots, N\} \quad \text{for all } x, y \in \overline{\Omega}.
\] (6.2)

**Lemma 6.2** For each \( i = 1, \ldots, N \) we have
\[
\lambda(y, x) = 0 \quad \text{for all } x, y \in \Gamma_i.
\]

Before going into the proof of the above lemma, we remark that \( \lambda(y, x) \) is symmetric in \( x \) and \( y \), that is,
\[
\lambda(y, x) = \lambda(x, y) \quad \text{for all } x, y \in \overline{\Omega}.
\]
To see this, let \( x, y \in \overline{\Omega} \) and \((X, l, \tau) \in \text{SP}(x)\) be such that \( X(\tau) = y \). We set
\[
Y(s) = X(\tau - s) \quad \text{and} \quad m(s) = l(\tau - s) \quad \text{for } s \in [0, \tau],
\]
then \((Y, m, \tau) \in \text{SP}(y)\) and \( Y(\tau) = x \). Moreover, we have
\[
\int_0^\tau (1 - l(s))ds = \int_0^\tau (1 - m(s))ds,
\]
and find that \( \lambda(y, x) = \lambda(x, y) \) for all \( x, y \in \overline{\Omega} \). From this symmetry, \( \lambda(y, x) = \lambda(x, y) \), it follows that \( \lambda(y, x) \) is Lipschitz continuous on \( \overline{\Omega} \times \overline{\Omega} \).

We remark also that the triangle inequality holds for \( \lambda(y, x) \):
\[
\lambda(y, x) \leq \lambda(y, z) + \lambda(z, x) \quad \text{for all } x, y, z \in \overline{\Omega}.
\]
Indeed, for any \((X, l, \tau) \in \text{SP}(x)\) and \((Z, m, \sigma) \in \text{SP}(z)\) such that \( X(\tau) = z \) and \( Z(\sigma) = y \), we define \((\xi, p, \tau + \sigma) \in \text{SP}(x)\) by concatenating \((X, l)\) and \((Z, m)\), i.e., by setting
\[
(\xi(s), p(s)) = \begin{cases} (X(s), l(s)) & \text{for } s \in [0, \tau), \\ (Z(s), m(s)) & \text{for } s \in [\tau, \tau + \sigma], \end{cases}
\]
and observe that \( \xi(\tau + \sigma) = y \) and
\[
\lambda(y, x) \leq \int_0^{\tau + \sigma} (1 - p(s))ds = \int_0^\tau (1 - l(s))ds + \int_0^\sigma (1 - m(s))ds,
\]
which implies that \( \lambda(y, x) \leq \lambda(y, x) + \lambda(x, z) \).

**Proof of Lemma 6.2.** Let \( i = 1, \ldots, N \) and \( x, y \in \Gamma_i \). By the connectedness and \( C^1 \) regularity of \( \Gamma_i \), there exists a curve \( X \in \text{Lip}([0, \tau], \mathbb{R}^p) \) starting at \( x \) and ending at \( y \) such that \( X(s) \in \Gamma_i \) for all \( s \in [0, \tau] \). We may assume by an appropriate scaling if needed that \( |X(s)| \leq 1 \) a.e. in \([0, \tau]\). Fix any \( \varepsilon \in (0, 1) \) and set
\[
X_\varepsilon(s) = X(\varepsilon s) \quad \text{and} \quad l_\varepsilon(s) = 1 - \varepsilon^2 \quad \text{for } s \in [0, \varepsilon^{-1} \tau].
\]
Observe that
\[
|X_\varepsilon(s)|^2 + l_\varepsilon(s)^2 \leq \varepsilon^2 + (1 - \varepsilon^2)^2 < 1 \quad \text{a.e. in } [0, \varepsilon^{-1} \tau],
\]
which assures that \( (X, l, \varepsilon^{-1} \tau) \in \text{SP}(x) \). Also, we have
\[
\int_0^{\tau} (1 - l(s)) ds = \varepsilon \tau.
\]
Sending \( \varepsilon \to 0 \), we conclude that \( \lambda_{\Omega}(x, y) = 0 \).

We divide the proof of Theorem 6.1 into two parts.

**Proof of Theorem 6.1. Part 1.** We fix any \( y \in \overline{\Omega} \) and write \( v(x) \) for the right-hand side of formula (6.2). Here we prove that
\[
v(x) \leq \lambda_{\Omega}(x, y) \quad \text{for all } x \in \overline{\Omega}.
\]
We first prove that \( v \) is a solution of (1.6). Let \( i = 1, \ldots, N \). By the definition of \( a_i(y) \), for any sequence \((i_1, \ldots, i_m) \in I \) such that \( i_m = i \), we have
\[
a_i(y) \leq \gamma(y, i_1) + \sum_{k=1}^{m-1} \gamma(i_k, i_{k+1}).
\]
In particular, if \( m = 1 \), then we get
\[
a_i(y) \leq \gamma(y, i) \leq |y - x| \quad \text{for all } x \in \Gamma_i.
\]
Also, for any \( j = 1, \ldots, N \), if we choose \((i_1, \ldots, i_m) \in I \) with \( i_{m-1} = j \), optimally so that
\[
a_j(y) = \gamma(y, i_1) + \sum_{k=1}^{m-2} \gamma(i_k, i_{k+1}).
\]
then we get
\[
a_i(y) \leq a_j(y) + \gamma(j, i) \leq a_j(y) + \text{dist}(x, \Gamma_j) \quad \text{for all } x \in \Gamma_i.
\]
Hence, by the definition of \( v \), we see that \( v(x) = a_i(y) \) for all \( x \in \Gamma_i \). Note as well by the definition of \( v \) that
\[
v(x) \leq a_i(y) + \text{dist}(x, \Gamma_i) \quad \text{for all } x \in \overline{\Omega}.
\]
Let \( \varepsilon > 0 \) and set
\[
v_\varepsilon(x) = \min\{v(x), a_i(y) + \text{dist}(x, \Gamma_i) - \varepsilon\} \quad \text{for } x \in \overline{\Omega}.
\]
There exists an open neighborhood \( V_\varepsilon \), relative to \( \mathbb{R}^n \), of \( \Gamma_i \) such that
\[
v_\varepsilon(x) = a_i(y) + \text{dist}(x, \Gamma_i) - \varepsilon \quad \text{for all } x \in V_\varepsilon \cap \overline{\Omega}.
\]
It is now a standard observation that \( v_\varepsilon \) is a solution of (6.1). It is clear that
\[
\lim_{\varepsilon \to 0} v_\varepsilon(x) = v(x) \quad \text{uniformly on } \overline{\Omega}.
\]
Hence, by the stability of the viscosity property under uniform convergence, we see that \( v \) is a solution of (6.1). Since our choice of \( i \) is arbitrary, we may conclude that \( v \) is a solution of (1.6). Noting that \( v(y) = 0 \), by Lemma 5.7, we conclude that (6.3) holds.

Recalling the Jordan-Brouwer separation theorem (see for instance [11]), since the \( \Gamma_i \) are compact, connected \( C^1 \) hypersurfaces, we see that for each \( i = 1, 2, \ldots, N \) the open subset \( \mathbb{R}^n \setminus \Gamma_i \) of \( \mathbb{R}^n \) has exactly two connected components \( O_i^+ \) and \( O_i^- \). Since \( \Omega \) is connected and does not intersect \( \partial \Omega = \cup_i \Gamma_i \), for each \( i \) we have either \( \Omega \subset O_i^+ \) or \( \Omega \subset O_i^- \). We choose our notation so that \( \Omega \subset O_i^- \) for all \( i = 1, \ldots, N \).

**Lemma 6.3** Let \( i, j \in \{1, \ldots, N\} \). If \( i \neq j \), then \( \Gamma_j \subset O_i^- \).

**Proof.** Since \( \Gamma_j \subset \overline{\Omega} \subset \overline{O_i^-} = \Gamma_i \cup O_i^- \) and \( \Gamma_j \cap \Gamma_i = \emptyset \), we have \( \Gamma_j \subset O_i^- \).

**Lemma 6.4** We have \( \Omega = \bigcap_{i=1}^N O_i^- \).

**Proof.** 1. We first show that the set \( \bigcap_{i=1}^N O_i^- \) is connected. To do this, fix \( i = 1, \ldots, N \) and an open connected subset \( O \) of \( \mathbb{R}^n \) such that \( \Gamma_i \subset O \) and prove that \( O \cap O_i^- \) is connected. Fix \( x, y \in O \cap O_i^- \). Since \( O \) is arc-wise
connected, there exists a curve $\xi \in C([0, 1], \mathbb{R}^n)$ such that $\xi(0) = x$, $\xi(1) = y$ and $\xi(t) \in O$ for all $t \in [0, 1]$. If $\xi(t) \in O^{-}_{\tau}$ for all $t \in [0, 1]$, then we are done. Otherwise, we may choose two numbers $0 < \sigma \leq \tau < 1$ so that $\xi(\sigma), \xi(\tau) \in \Gamma_{\tau}$ and $\xi(t) \in O^{-}_{\tau}$ for all $t \in [\sigma, \tau] \cup (\tau, 1]$. Now, since $\Gamma_{\tau}$ is locally diffeomorphic to a hyperplane, it is not hard to find a small constant $\epsilon > 0$ and a continuous curve $\eta \in C([\sigma - \epsilon, \tau + \epsilon], \mathbb{R}^n)$ such that $\eta(\sigma - \epsilon) = \xi(\sigma - \epsilon), \eta(\tau + \epsilon) = \xi(\tau + \epsilon)$ and $\eta(t) \in O^{-}_{\tau}$ for all $t \in [\sigma - \epsilon, \tau + \epsilon]$. Here it is assumed that $0 < \sigma - \epsilon < \tau + \epsilon < 1$. Moreover, we may select the curve $\eta$ so that the distance of the curve $\eta$ to the hypersurface $\Gamma_{\tau}$,

$$
\max_{t \in [\sigma - \epsilon, \tau + \epsilon]} \text{dist}(\eta(t), \Gamma_{\tau}),
$$

is as small as required. Consequently, we may assume that $\eta(t) \in O$ for all $t \in [\sigma - \epsilon, \tau + \epsilon]$. Concatenating three curves $\xi|_{[\sigma - \epsilon, \sigma - \epsilon]}$ (the restriction of $\xi$ to $[\sigma - \epsilon, \sigma - \epsilon]$), $\eta$ and $\xi|_{[\tau + \epsilon, 1]}$, we get a continuous curve in $O \cap O^{-}_{\tau}$ connecting $x$ and $y$. Hence, $O \cap O^{-}_{\tau}$ is arc-wise connected, which shows that it is connected.

We assume that $N \geq 2$, note by Lemma 6.3 that $I_{\tau} \subset O^{-}_{\tau}$ and apply the above observation to $O^{-}_{\tau}$ and $O^{-}_{\tau}$, to see that $O^{-}_{\tau} \cap O^{-}_{\tau}$ is connected. If $N \geq 3$, then we note by Lemma 6.3 that $I_{\tau} \subset O^{-}_{\tau} \cap O^{-}_{\tau}$ and use the above observation, to see that $O^{-}_{\tau} \cap O^{-}_{\tau} \cap O^{-}_{\tau}$ is connected. In general, by induction, we conclude that the set $\bigcap_{i=1}^{N} O^{-}_{\tau}$ is connected.

2. We know now that $\bigcap_{i=1}^{N} O^{-}_{\tau}$ is connected and includes the set $\Omega$. To show the identity $\bigcap_{i=1}^{N} O^{-}_{\tau} = \Omega$, we suppose that there exists a point $x \in \bigcap_{i=1}^{N} O^{-}_{\tau} \setminus \Omega$ and will get a contradiction. Fix a point $x_{0} \in \Omega$ and select a curve in $\bigcap_{i=1}^{N} O^{-}_{\tau}$ connecting $x_{0}$ and $x$. Since $x \notin \Omega$, the curve intersects $\partial \Omega$ at a point $x_{1}$. Since, for each $i$, $\Gamma_{i}$ does not intersects $O^{-}_{\tau}$, the set $\partial \Omega = \bigcup_{i=1}^{N} \Gamma_{i}$ does not intersects $\bigcap_{i=1}^{N} O^{-}_{\tau}$. These together yield a contradiction:

$$
x_{1} \in \partial \Omega \cap \bigcap_{i=1}^{N} O^{-}_{\tau} = \emptyset.
$$

**Lemma 6.5** For any $x, y \in \Omega$ we have

$$
\lambda_{\Omega}(x, y) \leq |x - y| \text{ for all } x, y \in \overline{\Omega}.
$$

**Proof.** By continuity, it is enough to show inequality (6.4) only for $x, y \in \overline{\Omega}$. Fix any $x, y \in \overline{\Omega}$ and consider the curve $\phi$ given by $\phi(t) := (1 - t)x + ty$ for $t \in [0, 1]$. Indeed, $\phi$ represents the line segment between $x$ and $y$.

1. Assume first that $\phi(t) \in \Omega$ for all $t \in [0, 1]$. Fix any $\epsilon > 0$, set $\tau_{\epsilon} := |x - y| + \epsilon$ and $\phi_{i}(t) := \phi_{i}(t_{i+1})$ for $t \in [0, \tau_{\epsilon}]$, and note that $(\phi_{t},(t_{i}, 0) \in SP(\phi))(\text{that is, } l(t) \equiv 0 \text{ in the usual notation})$ and $\phi_{i}(\tau_{\epsilon}) = y$. By the definition of $\lambda_{\Omega}$, we get $\lambda_{\Omega}(x, y) \leq \tau_{\epsilon} = |x - y| + \epsilon$, which shows that (6.4) holds in this case.

2. Next assume that the curve $\phi$ intersects the complement of $\Omega$. We show that there are sequences $\{s_{i}\}_{i=1}^{m} \subset (0, 1)$, $\{t_{i}\}_{i=1}^{m} \subset (0, 1)$ and $\{i_{k}\}_{i=1}^{m} \subset \{1, \ldots, N\}$ such that

$$
\begin{align*}
0 < s_{1} & \leq t_{1} \leq s_{2} \leq t_{2} \leq \cdots \leq s_{m} \leq t_{m} < 1, \\
i_{k} & \neq i_{j} \quad \text{if } k \neq j, \\
\phi(s_{k}) & \in \Gamma_{i_{k}}, \quad \phi(t_{k}) \in \Gamma_{i_{k}} \quad \text{for all } k = 1, \ldots, m, \\
\phi(t) & \in \Omega \quad \text{for all } t \in [0, s_{1}) \cup \bigcup_{k=1}^{m-1} (t_{k}, s_{k+1}) \cup (t_{m}, 1].
\end{align*}
$$

(6.5)

Here, since the $i_{k}$ are mutually different, $m$ is not more than $N$.

It is obvious that $\phi([0, 1]) \cap \partial \Omega \neq \emptyset$. We set

$$
\begin{align*}
s_{1} & = \min\{t \in [0, 1] : \phi(t) \in \partial \Omega\}, \\
t_{1} & = \max\{t \in [0, 1] : \phi(t) \in \Gamma_{i_{1}}\},
\end{align*}
$$

where $i_{1} \in \{1, \ldots, N\}$ is chosen so that $\phi(s_{1}) \in \Gamma_{i_{1}}$. Note that such an $i_{1}$ is uniquely determined.

Since both $\phi(0)$ and $\phi(1)$ are in $\Omega$, it is clear that $0 < s_{1} \leq t_{1} < 1$ and also that $\phi(t) \in \Omega$ for all $t \in [0, s_{1})$. Note that $\phi(1) \in \Omega \subset O^{-}_{\tau}$ and that $\phi([t_{1}, 1]) \cap \Gamma_{i_{1}} = \emptyset$. Hence, the connected set $\phi([t_{1}, 1])$ is included in $\mathbb{R}^{n} \setminus \Gamma_{i_{1}}$ and intersects $O^{-}_{\tau}$, which implies that

$$
\phi([t_{1}, 1]) \subset O^{-}_{\tau}.
$$

(6.6)
By Lemma 6.3, we have \( \phi(t_1) \in \Gamma_i \subset \bigcap_{j \neq i} O_j^- \), which implies that \( \phi((t_1, \tau_1)) \subset \bigcap_{j \neq i} O_j^- \) for some \( \tau_1 \in (t_1, 1] \). Combining this with (6.6) and using Lemma 6.4, we see that

\[
\phi((t_1, \tau_1)) \subset \bigcap_{i=1}^{N} O_i^- = \Omega.
\]

If \( \phi((t_1, 1)) \subset \Omega \), then we set \( m = 1 \) and we are done. Otherwise, we repeat the previous argument, with the interval \([0, 1]\) replaced by \([t_1, 1]\). (Note that \( t_1 < \tau_1 < 1 \).) That is, we set

\[
\begin{align*}
S_2 &= \min\{t \in [t_1, 1] : \phi(t) \in \partial \Omega\}, \\
t_2 &= \max\{t \in [t_1, 1] : \phi(t) \in \Gamma_i\},
\end{align*}
\]

where \( i_2 \in \{1, \ldots, N\} \) is the integer such that \( \phi(s_2) \in \Gamma_{i_2} \). By the choice of \( t_1 \), it is clear that \( i_2 \neq i_1 \). As in the first step of this iteration, we see that \( \phi((t_2, \tau_2)) \subset \Omega \) for some \( \tau_2 \in (t_2, 1] \).

We repeat this procedure of finding \( (s_k, t_k, i_k) \) at most \( N \) times before arriving the situation that \( \phi((t_k, 1)) \subset \Omega \), to conclude that there exist sequences \( \{s_k\}_{k=1}^{m} \subset (0, 1) \), \( \{t_k\}_{k=1}^{m} \subset (0, 1) \) and \( \{i_k\}_{k=1}^{m} \subset \{1, \ldots, N\} \) such that all the conditions of (6.5) hold.

3. By the triangle inequality, we get

\[
\lambda_{\Omega}(x, y) \leq \lambda_{\Omega}(\phi(0), \phi(s_1)) + \sum_{k=1}^{m} \lambda_{\Omega}(\phi(s_k), \phi(t_k))
\]

\[
+ \sum_{k=1}^{m-1} \lambda_{\Omega}(\phi(t_k), \phi(s_{k+1})) + \lambda_{\Omega}(\phi(t_m), \phi(1)).
\]

According to Lemma 6.2, we have \( \lambda_{\Omega}(\phi(s_k), \phi(t_k)) = 0 \) for all \( k = 1, \ldots, m \). Noting that

\[
\phi(t) \in \Omega \quad \text{for all } t \in [0, s_1) \cup \bigcup_{k=1}^{m-1} (t_k, s_{k+1}) \cup (t_m, 1]
\]

and arguing as in Step 1, we get

\[
\begin{align*}
\lambda_{\Omega}(\phi(0), \phi(s_1)) &\leq |\phi(0) - \phi(s_1)|, \\
\lambda_{\Omega}(\phi(t_k), \phi(s_{k+1})) &\leq |\phi(t_k) - \phi(s_{k+1})| \quad \text{for all } k = 1, \ldots, m-1, \\
\lambda_{\Omega}(\phi(t_m), \phi(1)) &\leq |\phi(t_m) - \phi(1)|.
\end{align*}
\]

Adding these all together, we obtain

\[
\lambda_{\Omega}(x, y) \leq |\phi(0) - \phi(s_1)| + \sum_{k=1}^{m-1} |\phi(t_k) - \phi(s_{k+1})| + |\phi(t_m) - \phi(1)| \leq |x - y|.
\]

The proof is complete. \( \square \)

Proof of Theorem 6.1. Part 2. As in Part 1, we fix any \( y \in \overline{\Omega} \) and write \( v(x) \) for the right-hand side of formula (6.2). We show that

\[
\lambda_{\Omega}(x, y) \leq v(x) \quad \text{for all } x \in \overline{\Omega},
\]

(6.7)

which will complete the proof of the theorem.

Fix any \( i \in \{1, \ldots, N\} \). There exists a sequence \((i_1, \ldots, i_m) \in I\) such that \( i_m = i \) and

\[
a_i(y) = \gamma(y, i_1) + \sum_{k=1}^{m-1} \gamma(i_k, i_{k+1}).
\]

We may choose sequences \((x_1, \ldots, x_m) \in (\partial \Omega)^m\) and \((y_1, \ldots, y_{m-1}) \in (\partial \Omega)^{m-1}\) so that

\[
\begin{align*}
\gamma(y, i_1) &= |y - x_1|, \quad x_1 \in \Gamma_i, \\
\gamma(i_k, i_{k+1}) &= |y_k - x_{k+1}|, \quad y_k \in \Gamma_i, \ x_{k+1} \in \Gamma_{i_{k+1}} \quad \text{for all } k = 1, \ldots, m-1.
\end{align*}
\]
By the triangle inequality, we get
\[ \lambda_\Omega(y, x_m) \leq \lambda_\Omega(y, x_1) + \sum_{k=1}^{m} \lambda_\Omega(x_k, y_k) + \sum_{k=1}^{m-1} \lambda_\Omega(y_k, x_{k+1}). \]

Hence, using Lemmas 6.2 and 6.5, we obtain
\[ \lambda_\Omega(y, x_m) \leq |y - x_1| + \sum_{k=1}^{m-1} |y_k - x_{k+1}| = \gamma(y, l_1) + \sum_{k=1}^{m-1} \gamma(i_k, i_{k+1}) = a_i(y). \]

Thus we get
\[ \lambda_\Omega(y, x) \leq \lambda_\Omega(y, x_m) + \lambda_\Omega(x_m, x) = \lambda_\Omega(y, x_m) \leq a_i(y) \quad \text{for all } x \in \Gamma_i. \tag{6.8} \]

By the definition of \( a_i(y) \), it is obvious that \( v(x) = a_i(y) \) for all \( x \in \Gamma_i \) and \( i = 1, \ldots, N \). This together with (6.8) assures that \( \lambda_\Omega(x, y) \leq v(x) \) for all \( x \in \partial \Omega \). By the standard comparison result, we conclude that \( \lambda_\Omega(x, y) \leq v(x) \) for all \( x \in \overline{\Omega} \).

\[ \square \]

A Appendix

We collect basic results for the eikonal equation. They are known in the literature (see, for instance, [3], [4], [8], [9], [12]).

We first consider the problem
\[
\begin{aligned}
au_t(x, t) + |D_x u(x, t)| &= f(x) & \text{in } Q_T := \Omega \times (0, T), \\
bv_t(x, t) + cu(x) \cdot D_x u(x, t) &= g(x) & \text{on } \partial \Omega \times (0, T),
\end{aligned}
\]

where \( T > 0, a \geq 0, b \geq 0 \) and \( c > 0 \) are constants and \( f \) and \( g \) are continuous functions on \( \overline{\Omega} \) and \( \partial \Omega \), respectively.

**Theorem A.1** Assume that \( a + b > 0 \) and that \( \min_{\overline{\Omega}} f > 0 \) if \( a = 0 \). Let \( u \in \text{USC}(\overline{\Omega} \times [0, T]) \) and \( v \in \text{LSC}(\overline{\Omega} \times [0, T]) \) be a subsolution and a supersolution of (A.1), respectively. Assume that \( u(x, 0) \leq v(x, 0) \) for all \( x \in \overline{\Omega} \) and that \( u \) and \( v \) are bounded on \( \overline{\Omega} \times [0, T] \). Then \( u \leq v \) on \( \overline{\Omega} \times (0, T) \).

In what follows we set
\[ H(x, p) = |p| - f(x) \quad \text{for } (x, t) \in \overline{\Omega} \times \mathbb{R}^n, \]
so that our equation reads
\[ au_t + H(x, D_x u) = 0 \quad \text{in } Q_T. \]

**Lemma A.2** Let \( u \in \text{USC}(\overline{\Omega} \times (0, T)) \) be a subsolution of (A.1). Assume that the family \( \{u(x, \cdot)\}_{x \in \overline{\Omega}} \) of functions in \( (0, T) \) is equi-Lipschitz continuous. Then \( u \) is Lipschitz continuous in \( \overline{\Omega} \times (0, T) \). Moreover, if \( L \) is a Lipschitz bound of the family \( \{u(x, \cdot)\}_{x \in \overline{\Omega}} \) in \( (0, T) \), then the constant
\[ 2L_{\Omega}(aL + \| f \|_{\infty, \Omega}) \]
is a Lipschitz bound of the function \( u \) in \( Q_T \).

Notice that \( L_{\Omega} \) indicates the Lipschitz constant introduced at the end of Section 1.

We outline the proofs of the above lemma and Theorem A.1 for the reader’s convenience.

**Outline of proof.** Let \( L > 0 \) be a Lipschitz bound for the family \( \{u(x, \cdot)\}_{x \in \overline{\Omega}} \). Let \((x, t) \in Q_T \) and \((p, q) \in D^+u(x, t) \). Since \( u \) is a subsolution of (A.1), we have
\[ aq + H(x, p) \leq 0. \tag{A.2} \]
Also, by the choice of \( L \), we have \( |q| \leq L \). Hence, we get \( |p| \leq aL + \|f\|_{\infty, \Omega} \), which implies that the family \( \{u(\cdot, t)\}_{t \in (0, T)} \) of functions in \( \Omega \) is equi-Lipschitz continuous with \( L_{\Omega}(aL + \|f\|_{\infty, \Omega}) \) as a Lipschitz bound. Hence, the function \( u \) is Lipschitz continuous on \( QT \) and satisfies

\[
|u(x, t) - u(y, s)| \leq 2L_{\Omega}(aL + \|f\|_{\infty, \Omega})(|x - y|^2 + |t - s|^2)^{1/2} \quad \text{for all} \quad (x, t), (y, s) \in QT.
\]

It is now enough to show that \( u \in C(\overline{\Omega} \times (0, T)) \). Since \( u \) is upper semicontinuous on \( \overline{\Omega} \times (0, T) \), we need only to show that for any \((y, s) \in \partial \Omega \times (0, T) \),

\[
\lim_{Q_T \ni (x, t) \to (y, s)} u(x, t) \geq u(y, s),
\]

where the limit exists thanks to the Lipschitz continuity of \( u \) in \( QT \). If we assume by contradiction that for some \((y, s) \in \partial \Omega \times (0, T) \),

\[
\lim_{Q_T \ni (x, t) \to (y, s)} u(x, t) < u(y, s),
\]

then there exist sequences \( \{\phi_j\} \subset C^1(\overline{\Omega} \times (0, T)) \) and \( \{(y_j, s_j)\} \subset \partial \Omega \times (0, T) \) such that for \((p_j, q_j) := D\phi_j(y_j, s_j) \),

\[
\begin{cases}
\max_{\Omega \times (0, T)} (u - \phi_j) = (u - \phi_j)(y_j, s_j) \quad \text{for all} \ j, \\
\lim_{j \to \infty} \min_{\partial \Omega \times (0, T)} (aq_j + |p_j|, \ b q_j + cv(y_j) \cdot p_j) = \infty.
\end{cases}
\]

These two properties contradict the fact that \( u \) is a subsolution of (A.1). We leave it to the interested reader to check the existence of such sequences. See [12, Lemma 3.3] for a related proposition.

**Outline of proof of Theorem A.1.** We argue by contradiction and assume that \( \sup_{\Omega \times (0, T)} (u - v) > 0 \) and will get a contradiction.

We may replace \( T \) by a smaller number so that \( u \in \text{USC}(\overline{\Omega}_T) \) and \( v \in \text{LSC}(\overline{\Omega}_T) \). We choose \( \varepsilon \in (0, 1) \) small enough. If \( a > 0 \), then, by setting temporarily

\[
U^\varepsilon(x, t) = u(x, t) - \frac{\varepsilon}{T - t + \varepsilon^2} \quad \text{for} \quad (x, t) \in \overline{\Omega}_T,
\]

and replacing \( u \) by \( U^\varepsilon \), we may assume that there is a subsolution of

\[
\begin{cases}
a u_t(x, t) + H(x, D_x u(x, t)) \leq -\delta & \text{in} \quad QT, \\
b u_t(x, t) + cv(x) \cdot D_x u(x, t) \leq g(x) & \text{on} \quad \partial \Omega \times (0, T),
\end{cases}
\]

where \( \delta \) is a positive constant. If \( a = 0 \) instead, then, choosing \( M > 0 \) so that \( \|g\|_{\infty, \Omega} \leq M \), putting

\[
U^\varepsilon(x, t) := (1 - \varepsilon)u(x, t) - \frac{\varepsilon b^{-1} M (T + \varepsilon^2)^2}{T - t + \varepsilon^2}
\]

for \((x, t) \in \overline{\Omega}_T \),

and compute informally that

\[
a U_t^\varepsilon(x, t) + H(x, D_x U^\varepsilon(x, t)) = (1 - \varepsilon)|D_x u(x, t)| - f(x) \leq -\varepsilon f(x),
\]

and for any \((x, t) \in \partial \Omega \times (0, T) \),

\[
b U_t^\varepsilon(x, t) + cv(x) \cdot D_x U^\varepsilon(x, t) = (1 - \varepsilon)|b u_t(x, t) + cv(x) \cdot D_x u(x, t)| - \frac{\varepsilon b M (T + \varepsilon^2)^2}{(T - t + \varepsilon^2)^2}
\]

\[
\leq (1 - \varepsilon)g(x) - \varepsilon M \leq g(x),
\]

we conclude that we may assume, after replacing \( u \) and \( \delta > 0 \) by \( U^\varepsilon \) and a smaller number, respectively, that \( u \) satisfies (A.3). Moreover, noting that

\[
\lim_{\varepsilon \to 0+} \frac{\varepsilon}{T - t + \varepsilon^2} \bigg|_{t=T} = \infty \quad \text{and} \quad \lim_{\varepsilon \to 0+} \frac{\varepsilon b^{-1} M (T + \varepsilon^2)^2}{T - t + \varepsilon^2} \bigg|_{t=T} = \infty,
\]
we may assume that \( \max_{\Omega \times (0, T)} (u - v) \leq 0 \). We now adapt the classical comparison argument for the Neumann type boundary value problem (see [13, Theorem 3.1]) for Hamilton-Jacobi equations to the present situation, to obtain a contradiction, which completes the proof.

The stationary eikonal equation (1.5) is of a special importance in this article and the following is a well-known comparison result (see [2], [3], [14] for instance) for (1.5).

**Lemma A.3** Let \( v \in \text{USC}(\overline{\Omega}) \) and \( w \in \text{LSC}(\overline{\Omega}) \) be a subsolution and a supersolution of (1.5), respectively. Assume that \( v(x) \leq w(x) \) for all \( x \in \partial \Omega \). Then \( v \leq w \) on \( \overline{\Omega} \).

The following proposition is a well-known result for convex Hamilton-Jacobi equations (see, for instance, [8], [9], [12] for the proof).

**Proposition A.4** Let \( U \) be an open subset of \( \mathbb{R}^n \) and \( H \in C(U \times \mathbb{R}^n) \). Assume that for each \( x \in U \), the function \( p \mapsto H(x, p) \) is convex in \( \mathbb{R}^n \). Let \( \mathcal{F} \) be a nonempty collection of subsolutions of

\[
H(x, Du(x)) = 0 \quad \text{in} \quad U.
\]

Assume that \( \mathcal{F} \) is uniformly bounded and equi-Lipschitz in \( U \). Set

\[
u(x) = \inf \{ v(x) : v \in \mathcal{F} \}
\]

for \( x \in U \).

Then the function \( u \) is a subsolution of (A.4).

As is well-known, if we replace “inf” by “sup” in the above definition of \( u \), the same conclusion as above holds without the convexity of \( H \).

**Example A.5** Let \( n = 1 \) and \( \Omega = (0, 1) \). Consider the problem

\[
\begin{align*}
u_t + |D_u| & = 1 \quad \text{in} \quad \Omega \times (0, \infty), \\
u_t & = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty),
\end{align*}
\]

Together with the initial condition

\[
u(x, 0) = 2x \quad \text{for} \quad x \in \overline{\Omega}.
\]

As Lemma 3.2 states, the comparison principle holds for Lipschitz continuous subsolutions and supersolution of the above problem. In what follows we show that the comparison principle does not hold for semicontinuous subsolutions and supersolutions of (3.1).

We set

\[
u(x) = x, \quad w(x, t) = 2x - t \quad \text{and} \quad u(x, t) = \max\{\nu(x), w(x, t)\}.
\]

Note that \( \nu \) is a classical solution of (3.1) and that \( w \) is a classical solution of \( \nu_t + |D_u| = 1 \) in \( \Omega \times (0, \infty) \) and a classical subsolution of (3.1). Accordingly, \( u \) is a viscosity subsolution of (3.1).

Our claim here is that \( u \) is a solution of (3.1) satisfying the initial condition (A.5). It is clear that \( u(x, 0) = 2x \) for all \( x \in \overline{\Omega} \). We have already checked that \( u \) is a subsolution of (3.1). Note that \( u(x, t) = \nu(x) \) if \( t > x \), which shows that \( u \) is a classical solution of

\[
\begin{align*}
u_t + |D_u| & = 1 \quad \text{in} \quad Q^+, \\
u_t & = 0 \quad \text{on} \quad \{0\} \times (0, \infty),
\end{align*}
\]

where \( Q^+ := \{(x, t) : t > x\} \). Similarly, noting that \( u = w \) if \( t < x \), we see that \( u \) is a classical solution of \( u(x, t) + |D_u(x, t)| = 1 \) in \( Q^- \), where \( Q^- := \{(x, t) \in Q : t < x\} \).

Fix any \((\hat{x}, \hat{t}) \in \overline{\Omega} \times (0, \infty)\) such that \( \hat{t} \leq \hat{t} \). Let \( \phi \in C^1_{+}(Q) \) and assume that \( u - \phi \) has a minimum at \((\hat{x}, \hat{t})\). The function \( r \mapsto (u - \phi)(r + \hat{x} - \hat{t}, r) = r + 2(\hat{x} - \hat{t}) - \phi(r + \hat{x} - \hat{t}, r) \) on \((0, \hat{t})\) has a minimum at \( r = \hat{t} \) and we get \( 0 \geq 1 - \phi_x(\hat{x}, \hat{t}) - \phi_x(\hat{x}, \hat{t}) \), that is, \( \phi_x(\hat{x}, \hat{t}) + \phi_x(\hat{x}, \hat{t}) \geq 1 \), which shows that \( \phi_x(\hat{x}, \hat{t}) + |\phi_x(\hat{x}, \hat{t})| \geq 1 \). This assures that \( u \) is a supersolution of (3.1). Thus, we conclude that \( u \) is a viscosity solution of (3.1).

We set

\[
U(x, t) = \begin{cases} u(x, t) & \text{for} \ (x, t) \in \{0, 1\} \times [0, \infty), \\
2 & \text{for} \ (x, t) \in \{1\} \times [0, \infty). \end{cases}
\]
The functions $u$ and $U$ differ only on the set $\{1\} \times (0, \infty)$ and the function $U$ is upper semicontinuous on $\overline{Q}$. It is obvious to see that $U$ is a viscosity subsolution of (3.1) and that $U(x, 0) = 2x = u(x, 0)$ for all $x \in [0, 1] = \overline{I}$. Moreover, the inequality $U \leq u$ on $\overline{Q}$ does not hold. That is, in the framework of semicontinuous viscosity solutions, the comparison principle does not hold.

Acknowledgements This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under grant No. (2-130/1433 HiCi). The authors, therefore, acknowledge technical and financial support of KAU. The work of HI was supported in part by KAKENHI #21224001, #23340028 and #23244015, JSPS.

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