TWO KINDS OF HOOK LENGTH FORMULAS FOR COMPLETE $m$-ARY TREES

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Abstract. In this paper, we define two kinds of hook length for internal vertices of complete $m$-ary trees, and deduce their corresponding hook length formulas, which generalize the main results obtained by Du and Liu.

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1. Introduction

Postnikov’s hook length formula [3] states that

$$n! \sum_{T} \prod_v \left(1 + \frac{1}{h_v}\right) = (n + 1)^{n-1},$$

where the sum is over all unlabeled complete binary trees $T$ with $n$ internal vertices, the product is over all internal vertices $v$ of $T$, and $h_v$ is the “hook length” of $v$ in $T$, namely, the number of internal vertices in the subtree of $T$ rooted at $v$. Postnikov derived the formula indirectly and asked for a combinatorial proof which was provided by Seo [4], Chen and Yang [1]. Later, Lascoux conjectured that

$$\sum_{T} \prod_v \left(x + \frac{1}{h_v}\right) = \frac{1}{(n+1)!} \prod_{i=0}^{n-1} \left((n+1+i)x + n + 1 - i\right).$$

This is equivalent to the more suggestive form

$$(1.1) \quad \sum_{T} \prod_v \left(\frac{(h_v+1)x-h_v+1}{2h_v}\right) = \frac{1}{n+1} \binom{(n+1)x}{n},$$

which was proved by Du and Liu [2]. Moreover, they generalized (1.1) from counting complete binary trees to counting complete $(m+1)$-ary trees and obtained the following formula for $(m+1)$-ary trees:

$$(1.2) \quad \sum_{T \in T_{n,m+1}} \prod_v \left(\frac{mh_v+1)x-h_v+1}{(m+1)h_v}\right) = \frac{1}{mn+1} \binom{(mn+1)x}{n},$$

or equivalently

$$(1.3) \quad \sum_{T \in T_{n,m+1}} \prod_v \left(x + \frac{1}{h_v}\right) = \frac{x+1}{n!} \prod_{i=1}^{n-1} \left((mn+i+1)(x+1) - (m+1)i\right).$$

where $T_{n,m+1}$ denotes the set of complete $(m+1)$-ary trees with $n$ internal vertices, the product is over all internal vertices $v$ of $T$. 
Recall that a plane forest is a forest of plane trees that are linearly ordered. Let \( \mathcal{F}(n) \) denote the set of plane forests with \( n \) vertices. For any vertex \( v \) of \( F \in \mathcal{F}(n) \), the hook length \( H_v \) of \( v \) is defined as the number of vertices in the subtree rooted at \( v \). Note that this definition is slightly different to that of hook length defined above for \((m+1)\)-ary trees. Du and Liu [2] investigated the hook length polynomials for plane forests and obtained that

\[
\sum_{F\in \mathcal{F}(n)} \prod_{v\in V(F)} \left( x + \frac{1}{H_v} \right) = \frac{(x+1)^n}{n!} \prod_{i=1}^{n-1} \left( (2n+1-i)(x+1) - i \right),
\]

or equivalently,

\[
F_n(x) = \sum_{F\in \mathcal{F}(n)} \prod_{v\in V(F)} \frac{(2h_v-1)x - H_v + 1}{H_v} = \frac{1}{2n+1} \left( \frac{(2n+1)x}{n} \right),
\]

where \( V(F) \) is the set of vertices of \( F \).

It is well known that there exists a simple bijection between plane forests and complete binary trees. For the sake of completeness, we present it here. Given any plane forest \( F \in \mathcal{F}(n) \), we pick the first plane tree \( T \) of \( F \) with root \( u \). Let \( T' \) denote the plane forest deduced from \( T \) by removing the root \( u \). Then the bijection can be defined recursively as follows: \( \psi(F) \) is the complete binary tree with root \( u \) such that it has the left subtree \( \psi(T') \) and the right subtree \( \psi(T'') \).

It is clear that the bijection maps the hook length of \( v \) in \( V(F) \) to the number of internal vertices of the left component of \( v \) of \( \psi(F) \). This motivates us to define the first kind of hook length \( \mathcal{H}_v \) for an internal vertex \( v \) of \( m \)-ary trees \( T \). Let \( T_v \) denote the \( m \)-ary subtree of \( T \) rooted at \( v \) and let \( T' \) denote the reduced tree from \( T_v \) by removing the rightmost subtree of \( v \). Define \( \mathcal{H}_v \) to be the number of internal vertices of the subtree \( T' \).

Moreover, the definition of the first hook length inspires us defining the second kind of hook length. Let \( S \) be a subset of \([m] = \{1, 2, \ldots, m\}\), for an internal vertex \( v \) of \((m+1)\)-ary trees \( T \), let \( T'_v \) denote the \((m+1)\)-ary subtree of \( T \) rooted at \( v \), and let \( v_1, v_2, \ldots, v_{m+1} \) be the children of \( v \), first delete the subtree rooted at \( v \), then delete \( r \)th subtree of \( v \) for all \( r \in S \); then delete the \( r \)th subtree of \( v_j \) for all \( r \in S \) and \( j \in [m+1] \setminus S \), and then continue this process; one can obtain an \((m+1-|S|)\)-ary tree \( T'_v^S \). Define \( \mathcal{H}_v^S \) to be the number of internal vertices of \( T'_v^S \). See Figure 1 for example. Then we have the second main result which, in the case \( m=2 \), the hook length \( \mathcal{H}_v \) reduces to \( H_v \) up to the bijection \( \psi \). Then we have the first main result which is a generalization of \((1.4)\) and \((1.5)\).

\[\text{Theorem 1.1.} \] For any integer \( m \geq 2 \),

\[
\sum_{T\in T_{n,m}} \prod_{v\in \mathcal{I}(T)} \left( x + \frac{1}{\mathcal{H}_v} \right) = \frac{(x+1)^n}{n!} \prod_{i=1}^{n-1} \left( (mn+1-i)(x+1) - (m-1)i \right),
\]

or equivalently,

\[
\sum_{T\in T_{n,m}} \prod_{v\in \mathcal{I}(T)} \frac{(m\mathcal{H}_v-1)x - \mathcal{H}_v + 1}{(m-1)\mathcal{H}_v} = \frac{1}{mn+1} \left( \frac{(mn+1)x}{n} \right),
\]

where \( \mathcal{I}(T) \) is the set of internal vertices of \( T \in T_{n,m} \).

\[\text{Theorem 1.2.} \] For any integer \( m \geq 1 \), \( S \subseteq [m] \) and \( s = |S| \),

\[
\sum_{T\in T_{n,m+1}} \prod_{v\in \mathcal{I}(T)} \left( x + \frac{1}{\mathcal{H}_v^S} \right) = \frac{(x+1)^n}{n!} \prod_{i=1}^{n-1} \left( (mn+i+1)(x+1) - (m-s+1)i \right),
\]
or equivalently,

\[(1.9) \quad \sum_{T \in \mathcal{T}_{n,m}} \prod_{v \in \mathcal{I}(T)} \left( \frac{(m-s)H_v^S + 1)x - H_v^S + 1}{(m-s+1)H_v^S} \right) = \frac{1}{mn+1} \binom{mn+1}{n} x.\]

In the next two Sections, we present the proofs of Theorem 1.1 and 1.2 respectively.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemma obtained by Seo [4].

Lemma 2.1. Fix positive integers \(a\) and \(b\). Let \(\Omega := \Omega(t) = 1 + \sum_{n \geq 1} \Omega_n t^n\) be a formal power series in \(t\) satisfying

\[\Omega' = x\Omega^{b+1} + at\Omega^b\Omega',\]

where the prime denotes the derivative of \(\Omega\) with respect to \(t\). Then \(\Omega_n\) can be given by

\[\Omega_n = \frac{x}{m} \prod_{i=1}^{n-1} \left( ai + bx(n-i) + x \right).\]

Proof of Theorem 1.1: Define

\[\mathcal{H}_{n,m}(x) = \sum_{T \in \mathcal{T}_{n,m}} \prod_{v \in \mathcal{I}(T)} \left( \frac{(mH_v - 1)x - H_v + 1}{(m-1)H_v} \right).\]

Given any \(m\)-ary tree \(T \in \mathcal{T}_{n,m}\) with root \(u\) for \(n \geq 1\), let \(T_1, T_2, \ldots, T_m\) be the \(m\) subtrees of \(u\) from left to right with \(i_1, i_2, \ldots, i_m\) internal vertices respectively. Then \(\mathcal{H}_u = i_1 + i_2 + \cdots + i_{m-1} + 1\). Therefore, we can deduce the recurrence relation for \(\mathcal{H}_{n,m}(x)\),

\[\mathcal{H}_{n,m}(x) = \sum_{i_1+i_2+\cdots+i_m=n-1} \prod_{j=1}^{m} \mathcal{H}_{i_j,m}(x)\]

\begin{align*}
\mathcal{H}_{n,m}(x) &= \sum_{i_1+i_2+\cdots+i_m=n-1} \prod_{j=1}^{m} \mathcal{H}_{i_j,m}(x) \\
&= \sum_{i_1+i_2+\cdots+i_m=n-1} \left( \frac{mx}{m-1} + \frac{1-x}{(m-1)(i_1+i_2+\cdots+i_{m-1}+1)} \right) \prod_{j=1}^{m} \mathcal{H}_{i_j,m}(x)\
\end{align*}
Define the generating function for $\mathcal{H}_{n,m}(x)$ by

$$\mathcal{H}_{n,m}(x) = 1 + \sum_{n \geq 1} \mathcal{H}_{n,m}(x)t^n.$$ 

Then by the above relation and the following series expansion

$$\mathcal{H}_m(x; t) = 1 + \sum_{n \geq 1} t^n \sum_{i_1 + i_2 + \cdots + i_k = n} \prod_{j=1}^{k} \mathcal{H}_{i_j,m}(x),$$

one can get

$$\mathcal{H}_m(x; t) = 1 + \frac{mx - 1}{m - 1} t \mathcal{H}_{m+1}(x; t) + \frac{1 - x}{m - 1} \mathcal{H}_m(x; t) \int_0^t \mathcal{H}_m^{m-1}(x; y) dy,$$

from which, one can derive that

$$\mathcal{H}_m'(x; t) = x \mathcal{H}_{m+1}(x; t) + (mx - 1)t \mathcal{H}_m(x; t) \mathcal{H}_m'(x; t),$$

where the prime denotes the derivative of $\mathcal{H}_m(x; t)$ with respect to $t$. Using Lemma 2.1, we have

$$\mathcal{H}_{n,m}(x) = \frac{x}{n!} \prod_{i=1}^{n-1} \left( (m(n - i) + 1)x + (mx - 1)i \right)$$

which proves (1.7).

Dividing by $\left(\frac{1 - x}{m - 1}\right)^n$ on both sides of (1.7), and then replacing $\frac{mx - 1}{1 - x}$ by $x$, one can get (1.6) immediately.

If choose the special values 0 or $-m$ for $x$ in (1.6), we get the following identities

Corollary 2.2. For any integer $m \geq 2$,

$$\sum_{T \in \mathcal{T}_{n,m}} \prod_{v \in \mathcal{I}(T)} \frac{1}{\mathcal{H}_v} = \frac{m^n}{mn + 1} \left( \frac{mn + 1}{m} \right),$$

$$\sum_{T \in \mathcal{T}_{n,m}} \prod_{v \in \mathcal{I}(T)} \left( m - \frac{1}{\mathcal{H}_v} \right) = \frac{(m - 1)^n}{n!} (mn + 1)^{n-1}.$$

3. Proof of Theorem 1.2

Define

$$s \mathcal{H}_{n,m}(x) = \sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} \left( x + \frac{1}{\mathcal{H}_v} \right),$$

and define the generating function for $s \mathcal{H}_{n,m}(x)$ by

$$s \mathcal{H}_m(x; t) = 1 + \sum_{n \geq 1} s \mathcal{H}_{n,m}(x)t^n.$$ 

First, we consider the case when $S$ is the empty set $\emptyset$. Note that in this case, (1.8) and (1.9) reduce to the results (1.3) and (1.2) obtained by Du and Liu [2]. Given any $(m+1)$-ary tree $T \in \mathcal{T}_{n,m+1}$ with root $u$ for $n \geq 1$, let $T_1, T_2, \ldots, T_{m+1}$ be the $m+1$ subtrees of $u$ from left to
right with $i_1, i_2, \ldots, i_{m+1}$ internal vertices respectively. Then $H_n^0 = i_1 + i_2 + \cdots + i_{m+1} + 1$. Therefore, we can deduce a recurrence relation for $\emptyset H_{n,m}(x)$,

\[
\emptyset H_{n,m}(x) = \sum_{T \in T_{n,m+1}} \prod_{v \in I(T)} \left( x + \frac{1}{H_n^u} \right)
\]

\[
= \sum_{i_1+i_2+\cdots+i_{m+1}=n-1} \left( x + \frac{1}{H_n^u} \right) \prod_{j=1}^{m+1} \sum_{T_j \in T_{i_j,m+1}} \prod_{v \in I(T_j)} \left( x + \frac{1}{H_n^t} \right)
\]

\[
= \sum_{i_1+i_2+\cdots+i_{m+1}=n-1} \left( x + \frac{1}{i_1 + i_2 + \cdots + i_{m+1} + 1} \right) \prod_{j=1}^{m+1} \emptyset H_{i_j,m+1}(x).
\]

Similar to the proof of Theorem 1.1, an equation for $\emptyset H_m(x;t)$ can be derived as

\[
\emptyset H_m(x;t) = 1 + xt \emptyset H_{m+1}(x;t) + \int_0^t \emptyset H_{m+1}(x;y)dy,
\]

from which, one can get

\[
\emptyset H'_m(x;t) = (x+1) \emptyset H_{m+1}(x;t) + (m+1)xt \emptyset H'_m(x;t),
\]

where the prime denotes the derivative of $\emptyset H_m(x;t)$ with respect to $t$.

For any complete $(m+1)$-ary tree $T$ with $k \geq 1$ internal vertices and an $s$-subset $S \in [m]$, according to the definition of the second kind of hook length, $T$ can be uniquely partitioned into a complete $(m-s+1)$-ary tree with $n$ internal vertices for some $n \geq 1$ and an ordered forest of $ns$ complete $(m+1)$-ary trees. Hence we get a recurrence relation for $S H_m(x;t)$, namely

\[
S H_m(x;t) = 1 + \sum_{n \geq 1} \emptyset H_{n,m-s}(x)t^n S H^n_m(x;t) = \emptyset H_{m-s}(x;S H^n_m(x;t)).
\]

Taking the derivative on both side of (3.2) with respect to $t$, using (3.1), we have

\[
S H'_m(x;t) = (x+1) S H_{m+1}(x;t) + \left((m-s+1)x + s(x+1)\right)t S H'_m(x;t) S H_m(x;t).
\]

Applying Lemma 2.1 to (3.3), one can obtain that

\[
S H_{n,m}(x) = \frac{x+1}{n!} \prod_{i=1}^{n-1} \left((m-s+1)x + s(x+1)i + m(x+1)(n-i) + x + 1\right)
\]

\[
= \frac{(x+1)}{n!} \prod_{i=1}^{n-1} \left((mn + i + 1)(x+1) - (m-s+1)i\right),
\]

which proves (1.8).

Dividing by $((m-s+1)-(x+1))^n$ on both sides of (1.8), and then replacing $\frac{x+1}{(m-s+1)-(x+1)}$ by $x$, one can get (1.9) immediately.

If choose the special values $m-s-1$ or $m-s$ for $x$ in (1.8), or choose $s=m$ in (1.9), we get the following identities
Corollary 3.1. For any integer $m \geq 0$, $S \subset [m]$ and $s = |S|$, 
\[
\sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} (m - s - 1 + \frac{1}{H^S_v}) = \frac{1}{mn + 1} \binom{(mn + 1)(m - s)}{n},
\]
\[
\sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} (m - s + 1 + \frac{1}{H^S_v}) = \frac{(m - s + 1)^n (mn + 1)^{n-1}}{n!},
\]
\[
\sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} \frac{x - H^S_v}{H^S_v} + 1 = \frac{1}{mn + 1} \binom{(mn + 1)x}{n}.
\]

Remark 3.2. Motivated by Lemma 2.1 and the proof of Theorem 1.2, we can consider the function
\[
\Phi := \Phi(t) = \Omega(t^s(t)),
\]
where $\Omega(t)$ is defined in Lemma 2.1 and $\Phi = 1 + \sum_{n \geq 1} \Phi_n t^n$. Then it is easy to derive that $\Phi(t)$ satisfies the following differential equation
\[
\Phi' = x\Phi^{b+s+1} + (a + sx)t\Phi^{b+s}\Phi',
\]
from which, by Lemma 2.1, we can deduce the explicit expression for $\Phi_n$,
\[
\Phi_n = \frac{x}{n!} \prod_{i=1}^{n-1} \left( ai + bx(n - i) + (sn + 1)x \right).
\]

We wonder if there is any combinatorial explanation for the relation.

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