SMALL EIGENVALUES OF LARGE HANKEL MATRICES

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In this note we shall determine the asymptotic behavior as \(N \to \infty\) of the smallest eigenvalue of the Hankel matrix

\[
H_N = (c_{m+n}) \quad m, n = 0, \ldots, N.
\]

It is assumed that the \(c_n\) are the moments of a distribution function \(\alpha(x)\) on the finite interval \([a, b]\),

\[
c_n = \int_a^b x^n \, d\alpha(x),
\]

where \(w(x) = \alpha'(x)\) satisfies

\[
\int_a^b \frac{\log w(x)}{(x-a)^{1/2}(b-x)^{1/2}} \, dx > -\infty.
\]

We shall see that for the smallest eigenvalue \(\lambda_N\) of \(H_N\) there is an asymptotic formula of the form

\[
\lambda_N \sim \rho N^{1/2} \sigma^{-2N}
\]

where \(\rho\) and \(\sigma\) are constants which will be explicitly determined. In the case of the Hilbert matrix \((c_m = 1/(m+1))\) a partial result was obtained by Todd in [3]. (In certain exceptional cases the exponent \(\frac{1}{2}\) must be replaced by \(\frac{1}{4}\).) It will be found that \(\sigma\) depends only on the interval \([a, b]\).

It will be assumed throughout that \(a+b \geq 0\). This entails no loss of generality since the Hankel matrix corresponding to the distribution function \(-\alpha(-x)\) on \([-b, -a]\) has exactly the same eigenvalues as \(H_N\).

**Lemma 1.** Let \(P_n(x)\) \((n=0, 1, \ldots)\) denote the orthogonal polynomials associated with \(\alpha(x)\). Then \(H_N^{-1}\) is similar to the matrix whose \(m, n\) entry is

\[
a_{m,n} = \frac{1}{2\pi} \int_0^{2\pi} P_m(e^{i\theta}) P_n(e^{i\theta})^* \, d\theta, \quad m, n = 0, \ldots, N.
\]

**Proof.** Write \(P_n(x) = \sum_{i=0}^n b_{n,i} x^i\). Then

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\[ \delta_{m,n} = \int_{a}^{b} P_m(x)P_n(x) \, d\alpha(x) = \sum_{i,j=0}^{N} b_{m,i}c_{i+j}b_{n,j} \]

and so if \( K_N \) denotes the matrix

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & b_{1,0} & \cdots & 0 \\
0 & b_{2,0} & b_{2,1} & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
0 & b_{N,0} & b_{N,1} & \cdots & b_{N,N}
\end{bmatrix}
\]

we have \( I = K_NH_NK_N^T \). Thus \( H_N^{-1} = K_N^T(K_NK_N^T)(K_N)^{-1} \). But the \( m, n \) entry of \( K_NK_N^T \) is

\[
\sum_{i=0}^{N} b_{m,i}b_{n,i} = \frac{1}{2\pi} \int_{0}^{2\pi} P_m(e^{i\theta})P_n(e^{i\theta})^* \, d\theta,
\]

which proves the lemma.

We shall be concerned now with the asymptotic behavior of \( a_{m,n} \) as \( m, n \to \infty \). This will turn out to be simple enough to enable us to deduce the asymptotic behavior of the largest eigenvalue of \( (a_{m,n}) \).

Lemma 2. We have, uniformly for \( z \) bounded away from the interval \([a, b]\),

\[ P_n(z) \sim (b - a)^{-1/2}\pi^{-1/2}z^nA(\xi), \]

where

\[ \xi = \frac{2}{b - a} z - \frac{b + a}{b - a} + \left[ \frac{2}{b - a} z - \frac{b + a}{b - a} \right]^2 - 1 \]

(the square root denoting that branch which is positive for large positive \( z \)), \( A(\xi) \) is analytic in \(|\xi| > 1\) and

\[
\log |A(\rho e^{i\phi})| = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[ w \left( \frac{b - a}{2} \cos t + \frac{b + a}{2} \right) \left| \sin t \right| \right] \frac{\rho^2 - 1}{1 - 2\rho \cos(\phi - t) + \rho^2} \, dt.
\]

Proof. If \( a = -1, b = 1 \) this is Theorem 12.1.2 of [2] if \( \alpha(x) \) is absolutely continuous and is Theorem 9.3 of [1] for general \( \alpha \). The case of the interval \([a, b]\) may be reduced to this by a linear change of variable since if \( q_n(x) \) are the orthogonal polynomials associated
with the distribution function
\[ \alpha \left( \frac{b - a}{2} x + \frac{b + a}{2} \right) \]
on \([-1, 1]\) then
\[ P_n(x) = q_n \left( \frac{2}{b - a} x - \frac{b + a}{b - a} \right). \]

We omit the details.

In view of Lemma 2 we expect that the asymptotic behavior of \( a_{m,n} \) depends on the maximum of \(|\xi(z)|\) as \( z \) runs over the unit circle. The next lemma will describe this maximum. It is convenient at this point to distinguish three cases:

Case 1. \( a > -\frac{b}{1 + 2b} \).
Case 2. \( a = -\frac{b}{1 + 2b} \).
Case 3. \( a < -\frac{b}{1 + 2b} \).

Lemma 3. The maximum value of \( g(\theta) = |\xi(e^{i\theta})| \) is given by
\[
\sigma = \begin{cases} 
\frac{b + a + 2}{b - a} + \left[ \left( \frac{b + a + 2}{b - a} \right)^2 - 1 \right]^{1/2} & \text{Cases 1 and 2}, \\
\left( \frac{1}{|a| \cdot b} + 1 \right)^{1/2} + \left( \frac{1}{|a| \cdot b} \right)^{1/2} & \text{Cases 2 and 3}.
\end{cases}
\]

In Cases 1 and 2 the maximum occurs at \( \theta = \pi \) (and only there mod \( 2\pi \)) and in Case 3 at \( \theta = \pm \theta_0 \) (and only there mod \( 2\pi \)) where
\[ \cos \theta_0 = \frac{b + a}{2ab}. \]

Moreover in Case 1 we have \( g''(\pi) \neq 0 \), in Case 2 we have \( g''(\pi) = 0 \) but \( g'''(\pi) \neq 0 \), and in Case 3 we have \( g''(\theta_0) \neq 0 \).

The proof of the lemma is completely elementary and need not be reproduced here.

Lemma 4. There is a constant \( A \), depending only on the distribution function \( \alpha(x) \), such that for all \( m, n \)
\[
|a_{m,n}| \leq \begin{cases} 
A(m + n + 1)^{-1/2} \sigma^{m+n} & \text{Cases 1 and 3}, \\
A(m + n + 1)^{-1/4} \sigma^{m+n} & \text{Case 2}.
\end{cases}
\]

Proof. It follows from Lemma 2 that as long as the unit circle
does not intersect the interval \([a, b]\) we have

\[ |a_{m,n}| \leq \text{const} \int_0^{2\pi} g(\theta)^{m+n} d\theta \]

and the desired conclusions follow readily from Lemma 3 using standard techniques.

To show that the same estimates hold even if the unit circle does intersect \([a, b]\) let us assume that 1 belongs to the interval but \(-1\) does not. (The case in which they both belong to the interval is more complicated in only a trivial way.) We can write, for any \(\epsilon > 0\)

\[ |a_{m,n}| \leq \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} |P_m(e^{i\theta})P_n(e^{i\theta})| d\theta + \frac{1}{2\pi} \int_{\epsilon}^{2\pi-\epsilon} |P_m(e^{i\theta})P_n(e^{i\theta})| d\theta. \]

Since the asymptotic formula of Lemma 2 holds uniformly for \(\epsilon \leq \theta \leq 2\pi - \epsilon\), the last integral will satisfy the estimate in the statement of the lemma. To estimate the first integral, denote by \(R_\epsilon\) the rectangle with vertices \(e^{\pm i\epsilon}, 1 \pm i\tan \epsilon\). This rectangle contains the arc of the unit circle given by \(|\theta| \leq \epsilon\). Since the polynomial \(P_m(z)P_n(z)\) has only real zeros (Theorem 3.3.1 of \([2]\)) its maximum absolute value on \(R_\epsilon\) is attained on the horizontal sides of \(R_\epsilon\). On these sides we may apply the asymptotic formula of Lemma 2, and so

\[ \limsup_{m+n \to \infty} \max_{R_\epsilon} |P_m(z)P_n(z)|^{1/(m+n)} = g(\epsilon + O(\epsilon^2)). \]

Therefore we have as \(m+n \to \infty\)

\[ \int_{-\epsilon}^{\epsilon} |P_m(e^{i\theta})P_n(e^{i\theta})| d\theta = O(\epsilon^{m+n}) \]

for any \(t > g(\epsilon + O(\epsilon^2))\). A little computation shows that \(g(2\epsilon) > g(\epsilon + O(\epsilon^2))\) if \(\epsilon\) is small enough. Thus

\[ \int_{-\epsilon}^{\epsilon} |P_m(e^{i\theta})P_n(e^{i\theta})| d\theta = O(g(2\epsilon)^{m+n}). \]

But \(\sigma > g(2\epsilon)\), again for sufficiently small \(\epsilon\) (recall that \(g(\theta)\) does not attain its maximum \(\sigma\) at \(\theta = 0\)), and so certainly

\[ \int_{-\epsilon}^{\epsilon} |P_m(e^{i\theta})P_n(e^{i\theta})| d\theta = o((m+n)^{-1/2}\sigma^{m+n}). \]

This completes the proof of the lemma.

The next lemma gives the asymptotic behavior of \(a_{m,n}\) as \(m, n \to \infty\). First some more notation. We write
\[ \gamma = \begin{cases} 
\frac{A(\xi(-1))}{2^{1/2} \pi^{3/2}} & \text{Case 1,} \\
\frac{1}{2^{1/4} \pi^{1/2}} & \text{Case 2,} \\
\frac{1}{\pi^{3/2}} & \text{Case 3,} 
\end{cases} \]

where \( |A(\xi)| \) is given in Lemma 2 and \( \theta_0 \) in Lemma 3. We shall write, in Case 3,

\[ \text{sgn} \xi(e^{i\theta_0}) = e^{i\phi_0}. \]

(In Cases 1 and 2, \( \text{sgn} \xi(-1) = -1. \))

Lemma 5. The following hold as \( m, n \to \infty \) with \( m - n \) bounded:

\[ a_{m,n} \sim \gamma(-1)^{m-n}(m+n)^{-1/2} \sigma^{m+n} \] Case 1,
\[ a_{m,n} \sim \gamma(-1)^{m-n}(m+n)^{-1/4} \sigma^{m+n} \] Case 2,
\[ a_{m,n} = \gamma \cos (m-n) \phi_0 (m+n)^{-1/2} \sigma^{m+n} + o((m+n)^{-1/2} \sigma^{m+n}) \] Case 3.

Proof. Suppose the unit circle does not intersect \([a, b]\). (The case in which it does can be handled just as in the proof of Lemma 4.) Then by Lemma 2,

\[ a_{m,n} = \frac{1}{2\pi^2 (b-a)} \int_0^{2\pi} \{ g(\theta)^{m+n}[\text{sgn} \xi(e^{i\theta})]^{m-n} |A(\xi(e^{i\theta}))| \} d\theta + o(g(\theta)^{m+n}) \]

In Cases 1 and 2 the maximum of \( g(\theta) \) occurs at \( \theta = \pi \) (and nowhere else) and the result follows from Lemma 3 using standard techniques. In Case 3 the maximum occurs at \( \pm \theta_0 \). Since

\[ \xi(e^{-i\theta_0}) = (\xi(e^{i\theta_0}))^*, \quad |A(\xi)| = |A(\xi)| \]

the conclusion in this case also follows easily from Lemma 3.

Theorem. If \( \lambda_N \) is the smallest eigenvalue of \( H_N \), then as \( N \to \infty \),

\[ \lambda_N \sim \gamma^{-1}(\sigma^2 - 1)(2N)^{1/2} \sigma^{-2(N+1)} \] Case 1,
\[ \lambda_N \sim \gamma^{-1}(\sigma^2 - 1)(2N)^{1/4} \sigma^{-2(N+1)} \] Case 2,
\[ \lambda_N \sim 2\gamma^{-1} \left[ \frac{1}{\sigma^2 - 1} + \left( \frac{1}{\sigma^4 - 2\sigma^2 \cos 2\phi_0 + 1} \right)^{1/2} \right]^{-1} (2N)^{1/2} \sigma^{-2(N+1)} \] Case 3.
Proof. We shall consider in detail only Case 3; the others are easier. Let us write
\[ b_{m,n} = \cos (m - n)\phi_0 e^{m+n} , \]
\[ c_{m,n} = a_{m,n} - \gamma (2N)^{-1/2} b_{m,n} . \]
Fix \( N_0 \) and \( \epsilon \). It follows from Lemma 5 that if \( m \) and \( n \) are sufficiently large, but \( |m - n| \leq N_0 \), we shall have
\[ |a_{m,n} - \gamma \cos (m - n)\phi_0 (m + n)^{-1/2} e^{m+n}| \leq \epsilon (m + n)^{-1/2} e^{m+n}. \]
Therefore if both \( m \) and \( n \) exceed \( N - N_0 \) and \( N \) is sufficiently large we shall have
\[ |c_{m,n}| = |a_{m,n} - \gamma \cos (m - n)\phi_0 (2N)^{-1/2} e^{m+n}| \]
\[ \leq \epsilon (m + n)^{-1/2} e^{m+n} + \gamma e^{m+n} [(2N - 2N_0)^{1/2} - (2N)^{1/2}] \]
\[ \leq \epsilon N^{-1/2} e^{m+n}. \]
It follows from Lemma 4 that for all \( m, n \)
\[ |c_{m,n}| \leq A_1 (m + n + 1)^{-1/2} e^{m+n} \]
where \( A_1 \) is a constant depending only on the distribution function \( \alpha(x) \). Denote by \( \mu_N \) the eigenvalue of largest absolute value of the matrix \( (c_{m,n}) \) \( (m, n = 0, \ldots, N) \). Then from (2) and (3) we obtain
\[ \frac{2}{\mu_N} \leq \sum_{m=n=0}^{N} c_{m,n} \leq \epsilon N \sum_{m=n=0}^{N} e^{2(m+n)} + 2A_1 \sum_{m=0}^{N-N_0} \sum_{n=0}^{N} \frac{\sigma^2(m+n)}{m+n+1} \]
\[ \leq \frac{\epsilon \sigma^4 (N+1)}{(\sigma^2 - 1)^2 N} + A_2 \frac{\sigma^2 (2N-N_0)}{2N-N_0} , \]
where \( A_2 \) is another constant. If now \( N_0 \) is taken sufficiently large in comparison to \( \epsilon \), this will imply for sufficiently large \( N \)
\[ |\mu_N| \leq \frac{2\epsilon \sigma^2 (N+1)}{(\sigma^2 - 1) N^{1/2}} . \]
Now Lemma 1 implies that \( \lambda_N^{-1} \) is the largest eigenvalue of \( (a_{m,n}) \) \( (m, n = 0, \ldots, N) \). It follows therefore from (1) and (4) that if \( \nu_N \) is the largest eigenvalue of \( (b_{m,n}) \) \( (m, n = 0, \ldots, N) \), we have
\[ \gamma (2N)^{-1/2} \nu_N - \frac{2\epsilon \sigma^2 (N+1)}{(\sigma^2 - 1)^{N^{1/2}}} \leq \lambda_N^{-1} \leq \gamma (2N)^{-1/2} \nu_N + \frac{2\epsilon \sigma^2 (N+1)}{(\sigma^2 - 1)^{N^{1/2}}} \]
for sufficiently large \( N \). Since the eigenvectors of \( (b_{m,n}) \) must be linear combinations \( \alpha \cos n\phi_0 \theta^n + \beta \sin n\phi_0 \theta^n \) it is easy to see that
\( \nu_N \) is the largest eigenvalue of

\[
\begin{bmatrix}
A & B \\
B & C
\end{bmatrix}
= \begin{bmatrix}
\sum_0^N \cos^2 n\phi_0 \sigma^{2n} & \sum_0^N \sin n\phi_0 \cos n\phi_0 \sigma^{2n} \\
\sum_0^N \sin n\phi_0 \cos n\phi_0 \sigma^{2n} & \sum_0^N \sin^2 n\phi_0 \sigma^{2n}
\end{bmatrix}.
\]

We find that as \( N \to \infty \)

\[
A = \frac{1}{2} \left[ \frac{1}{\sigma^2 - 1} + \frac{\sigma^2 \cos 2N\phi_0 - \cos 2(N + 1)\phi_0}{\sigma^4 - 2\sigma^2 \cos 2\phi_0 + 1} \right] \sigma^{2(N+1)} + O(1),
\]

\[
C = \frac{1}{2} \left[ \frac{1}{\sigma^2 - 1} - \frac{\sigma^2 \cos 2N\phi_0 - \cos 2(N + 1)\phi_0}{\sigma^4 - 2\sigma^2 \cos^2 \phi_0 + 1} \right] \sigma^{2(N+1)} + O(1),
\]

\[
B = \frac{1}{2} \frac{\sigma^2 \sin 2N\phi_0 - \sin 2(N + 1)\phi_0}{\sigma^4 - 2\sigma^2 \cos^2 \phi_0 + 1} \sigma^{2(N+1)} + O(1),
\]

and from these there follows easily

\[(6) \quad \nu_N \sim \frac{1}{2} \left[ \frac{1}{\sigma^2 - 1} + \left( \frac{1}{\sigma^4 - 2\sigma^2 \cos 2\phi_0 + 1} \right)^{1/2} \right] \sigma^{2(N+1)} + O(1).\]

The theorem follows from (6) and (5) if we observe that \( \epsilon \) was arbitrarily small.

We regret to announce that in the case of the Hilbert matrix

\[
\left( \frac{1}{m + n + 1} \right) \quad (m, n = 0, 1, \ldots, N)
\]

our result takes the form

\[
\lambda_N \sim 2^{9/8} \pi^{3/2} (73 - 48(2)^{1/2})^{-1} N^{1/2} (3 + 2(2)^{1/2})^{-2N - 3/4} \quad (N \to \infty).
\]

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