CURRENTS OF ARBITRARY SPIN IN $\text{AdS}_3$

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Abstract

We study conserved currents of any integer or half integer spin built from massless scalar and spinor fields in $\text{AdS}_3$. 2-forms dual to the conserved currents in $\text{AdS}_3$ are shown to be exact in the class of infinite expansions in higher derivatives of the matter fields with the coefficients containing inverse powers of the cosmological constant. This property has no analog in the flat space and may be related to the holography of the AdS spaces.

1 Introduction

The role of anti-de Sitter (AdS) geometry in the high energy physics increased greatly due to the Maldacena conjecture [1] on the duality between the theory of gravity in the AdS space and conformal theory on the boundary of the AdS space [2, 3]. The holography hypothesis suggests that the two types of theories are equivalent. The same time, AdS geometry plays very important role in the theory of higher spin (HS) gauge fields (for a brief review see [4]) because interactions of HS gauge fields contain negative powers of the cosmological constant [4]. The theory of HS gauge fields may be considered [4] as a candidate for a most symmetric phase of string theory.

The group manifold case of $\text{AdS}_3$ is special and interesting in many respects. HS gauge fields are not propagating in analogy with the usual Chern-Simons gravitational and Yang-Mills fields, although the HS gauge symmetries remain nontrivial. The HS
currents can be constructed from the matter fields of spin 0 and spin 1/2. Their couplings to HS gauge potentials describe interactions of the matter via HS gauge fields.

Schematically, the equations of motion in the gauge field sector have a form $R = J(C; W)$, where $R = dW - W \wedge W$ denotes all spin $s \geq 1$ curvatures built from the HS potential $W$, while $C$ denotes the matter fields (precise definitions are given in the sect. 2). To analyze the problem perturbatively, one fixes a vacuum solution $W_0$ that solves $R_0 = 0$, assuming that $W = W_0 + W_1$, while $C$ starts from the first-order part. When gravity is included, as is the case in the HS gauge theories, $W_0$ is different from zero and describes background geometry. In the lowest nontrivial order one gets

$$R_1 \equiv D_0 W_1 = J_2(C^2),$$

(1.1)

where $D_0$ is built from $W_0$ and the 2-form $J_2(C^2)$ dual to the 3d conserved current vector field obeys the conservation law

$$D_0 J_2(C^2) = 0$$

(1.2)

on the free equations of motion of the matter fields.

A nonlinear system of equations of motion describing HS gauge interactions for the spin 0 and spin 1/2 matter fields in $AdS_3$ in all orders in interactions has been formulated both for massless \[6\] and massive \[7\] matter fields. An interesting property of the proposed equations discovered in \[7\] is that there exists a mapping of the full nonlinear system to the free one. This mapping is a nonlinear field redefinition having a form of infinite power series in higher derivatives of the matter fields and is therefore generically nonlocal. The coefficients of such expansions contain inverse powers of the cosmological constant and therefore do not admit a flat limit. We call such expansions in higher derivatives pseudolocal. Comparison of the results of \[7\] with (1.1) implies that such a field redefinition exists in a nontrivial model if

$$J_2(C^2) = D_0 U(C^2),$$

(1.3)

where $U$ is some pseudolocal functional of the matter fields. The cohomological interpretation with $D_0$ as de Rahm differential is straightforward because $D_0^2 = R_0 = 0$. Indeed, from (1.2) it follows that the current $J_2(C^2)$ should be closed on the free equations for matter fields, while (1.3) implies that it is exact in the class of pseudolocal functionals. This fact has been already demonstrated for the spin 2 current in \[4\], where we have found a pseudolocal $U$ for the stress tensor constructed from a massless scalar field. In this paper, we generalize this result to the currents of an arbitrary integer or half integer spin.

Exact currents with local $U$ containing at most a finite number of derivatives of the matter fields reproduce “improvements”, i.e., modifications of the currents which are trivially conserved. The new result about AdS space established in this paper is that the true currents can also be treated as “improvements” in the class of pseudolocal expansions. This sounds very suggestive in the context of the holography hypothesis.

The paper is designed as follows. In sect. 2 we collect some facts about the equations of motion of the Chern-Simons HS gauge fields and the “unfolded” formulation of the equations of motion for the massless spin 0 and 1/2 matter fields in $AdS_3$. In sect. 3 we propose a formalism of generating functions to describe differential forms bilinear in
2 Higher Spin and Matter Fields in $AdS_3$

The 3d HS gauge fields are described by a spacetime 1-form $W = dx^\mu W_\mu(y, \psi|x)$ depending on the spacetime coordinates $x^\mu$ ($\mu = 0, 1, 2$), auxiliary commuting spinor variables $y_\alpha$ (indices $\alpha, \beta, \gamma = 1, 2$ are lowered and raised by the symplectic form $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon_{12} = \epsilon_{21} = 1$, $A^{\alpha} = A^{\alpha\beta}A_\beta$, $A_\alpha = A^{\beta}\epsilon_{\beta\alpha}$), and the central involutive element $\psi$,

$$W_\mu(y, \psi|x) = \sum_{n=0}^{\infty} \frac{1}{2in!} \left[ \omega_{\mu,\alpha(n)}(x) + \lambda \psi h_{\mu,\alpha(n)}(x) \right] y^{\alpha_1} \ldots y^{\alpha_n}. \quad (2.1)$$

A constant parameter $\lambda$ is to be identified with the inverse radius of $AdS_3$.

The HS gauge algebra is a Lie superalgebra built via (anti)commutators from the associative algebra spanned by the elements $a(y, \psi)$ with the product law “$*$” defined via the generating relations $y_\alpha * y_\beta - y_\beta * y_\alpha = 2i\epsilon_{\alpha\beta}, y_\alpha * \psi = \psi * y_\alpha, \psi * \psi = 1$. (The boson-fermion parity $\pi$ is defined in a usual way as $a(-y, \psi) = (-)^{\pi(a)}a(y, \psi)$.) The field strength is $R(y, \psi|x) = dW(y, \psi|x) - W(y, \psi|x) \wedge W(y, \psi|x)$. The role of the element $\psi$ is to make the 3d HS superalgebra semisimple ($hs(2) \oplus hs(2)$, where, in notation of $[\mathcal{H}], hs(2)$ is a superalgebra spanned by the $\psi$-independent elements $a(y)$), with simple components singled out by the projectors $P_{\pm} = \frac{1}{2}(1 \pm \psi)$. This is similar to the $AdS_3$ isometry algebra $o(2, 2) \sim sp(2) \oplus sp(2)$. The latter is identified with a subalgebra of $hs(2) \oplus hs(2)$ spanned by $L_{\alpha\beta} = \frac{1}{2} y_\alpha y_\beta$ and $P_{\alpha\beta} = \frac{1}{2} y_\alpha y_\beta \psi$. We therefore identify the $o(2, 2)$ components of $W(y, \psi|x)$ (2.1) with the gravitational Lorentz connection 1-form $\omega^{\alpha\beta}(x) = dx^\mu \omega_{\mu,\alpha\beta}(x)$ and the dreibein 1-form $h^{\alpha\beta}(x) = dx^\mu h_{\mu,\alpha\beta}(x)$. Since $AdS_3$ algebra $o(2, 2)$ is a proper subalgebra of the d3 HS algebra it is a consistent ansatz to require the vacuum value of $W(y, \psi|x)$ to be non-zero only in the spin 2 sector. Then the equation $R_0 = 0$ is equivalent to the $o(2, 2)$ zero-curvature conditions

$$d\omega_{\alpha\beta} = \omega_{\alpha\gamma} \wedge \omega_{\beta\gamma} + \lambda^2 h_{\alpha\gamma} \wedge h_{\beta\gamma}, \quad dh_{\alpha\beta} = \omega_{\alpha\gamma} \wedge h_{\beta\gamma} + \omega_{\beta\gamma} \wedge h_{\alpha\gamma}. \quad (2.2)$$

For the metric interpretation, the dreibein $h_{\mu,\alpha\beta}$ should be non-degenerate, thus admitting the inverse dreibein $h^{\nu,\alpha\beta}$ defined via $h_{\mu,\alpha\beta} h^{\nu,\alpha\beta} = \frac{1}{2} \delta^{\alpha\beta}_{\gamma\delta} + \delta^{\beta\gamma}_{\alpha\delta} + \delta^{\gamma\delta}_{\beta\alpha}$. Then, the second equation in (2.2) reduces to the zero-torsion condition which expresses Lorentz connection $\omega_{\mu,\alpha\beta}$ via dreibein $h_{\mu,\alpha\beta}$, while the first one implies that $R_{\alpha\beta} = -\lambda^2 h_{\alpha\gamma} \wedge h_{\beta\gamma}$, where $R_{\alpha\beta}$ is the Riemann tensor 2-form. Therefore, the equations (2.2) describe $AdS_3$ with radius $\lambda^{-1}$.

The massless Klein-Gordon and Dirac equations in $AdS_3$ read

$$\Box C = \frac{3}{2} \lambda^2 C \quad \text{and} \quad h^{\mu,\alpha\beta} \nabla_\mu C_{\beta} = 0 \quad (2.3)$$

for the spin 0 boson field $C(x)$ and spin $\frac{1}{2}$ fermion field $C_\alpha(x)$. Here $\Box = \nabla^\mu \nabla_\mu$, where $\nabla_\mu$ is the full covariant derivative with the symmetric Christoffel connection defined via the metric postulate $\nabla_\mu h_{\nu,\alpha\beta} = 0$. 

\[3\]
The “unfolded” formulation \cite{10} of the equations (2.3) in the form of some covariant constancy conditions is most convenient for the analysis of cohomology of currents. To this end one introduces an infinite set of symmetric multispinors $C_{\alpha_1...\alpha_n}$ for all $n \geq 0$. (We will assume total symmetrization of indices denoted by the same letter and will use the notation $C_{\alpha(n)} = C_{\alpha_1...\alpha_n}$ when only a number of indices is important.) As shown in \cite{10}, the infinite chain of equations

$$D^L C_{\alpha(n)} = \frac{i}{2} \left[ h^{\beta\gamma} C_{\beta\gamma\alpha(n)} - \lambda^2 n(n-1) h_{\alpha\alpha} C_{\alpha(n-2)} \right], \quad (2.4)$$

where $D^L$ is the background Lorentz covariant differential, $D^L C_{\alpha(n)} = dC_{\alpha(n)} + n \omega_\alpha \gamma C_{\gamma\alpha(n-1)}$, is equivalent to the equations (2.3) for the lowest rank components $C$ and $C_\alpha$ along with some constraints expressing highest multispinors via highest spacetime derivatives of $C$ and $C_\alpha$. For example, for bosons

$$C_{\alpha(2n)}(x) = (-2i)^n h^{\nu_1,..,\nu_2} h^{\nu_3,..,\nu_6} \ldots h^{\nu_n,..,\nu_n} \nabla_{\nu_1} \nabla_{\nu_2} \ldots \nabla_{\nu_n} C(x), \quad (2.5)$$

where $\nabla_\mu$ is a full background derivative (for multispinors $\nabla_\mu C_{\alpha(n)} = D^L_\mu C_{\alpha(n)}$).

Following \cite{10}, let us introduce the generating function

$$C(y, \psi|x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \lambda^{-1} \psi \right)^n C_{\alpha_1...\alpha_n}(x) y^{\alpha_1} \ldots y^{\alpha_n} = \lambda^2 \pi(C) \tilde{C}(\lambda^{-1} y, \psi|x), \quad (2.6)$$

where $[n+a] = n$, $\forall n \in \mathbb{Z}$ and $0 \leq a < 1$, and the boson-fermion parity $\pi(C) = 0(1)$ for even (odd) functions $C(y)$. The equations (2.4) can be rewritten in the form \cite{10},

$$D^L C(y, \psi) = \frac{i\lambda}{2} \psi h^{\alpha\beta} \left[ \frac{\partial}{\partial y^{\alpha}} \frac{\partial}{\partial y^{\beta}} - y_\alpha y_\beta \right] C(y, \psi), \quad D^L = d - \omega_\alpha y_\alpha \frac{\partial}{\partial y^{\beta}}. \quad (2.7)$$

The fields $C_{\alpha_1...\alpha_n}$ are identified with all on-mass-shell nontrivial derivatives of the matter fields according to (2.3). The condition that the system is on–mass–shell is encoded in the fact that the multispinors $C_{\alpha_1...\alpha_n}$ are totally symmetric. This allows us to work with $C_{\alpha_1...\alpha_n}$ instead of explicit derivatives of the matter fields. Consider a function $F[C_{\alpha(n)}(x)]$ of all components of $C_{\alpha_1...\alpha_n}(x)$ at some fixed point $x$. $F$ is not supposed to contain any derivatives with respect to the spacetime coordinates $x$ and therefore looks like a local function of matter fields. One has to be careful however because, when the equations (2.3) hold, (2.3) is true. We will therefore call a function $F[C_{\alpha(n)}]$ pseudolocal if it is an infinite expansion in the field variables $C_{\alpha(n)}(x)$ and local if $F$ is a polynomial. In terms of the generating functions $C(y, \psi|x)$ this can be reformulated as follows. Let $F(C|x)$ be some functional of the generating function $C(y, \psi|x)$ at some fixed point of spacetime $x$. According to (2.5) its spacetime locality is equivalent on–mass–shell to the locality in the $y$ space. Indeed, from (2.7) it follows that the derivatives in the spinor variables form in a certain sense a square root of the spacetime derivatives.

The equation (1.1) for the d3 HS system reads (in the rest of the paper we use the symbol $D$ instead of $D_0$)

$$DW_1(y, \psi|x) = J(C^2)(y, \psi|x) \quad (2.8)$$
with the background AdS covariant differential
\[ D = D^L - \lambda \psi \ h^{\alpha \beta} \ y_\alpha \frac{\partial}{\partial y^\beta} = d - (\omega^{\alpha \beta} + \lambda \psi \ h^{\alpha \beta}) \ y_\alpha \frac{\partial}{\partial y^\beta}. \tag{2.9} \]

That \( \omega^{\alpha \beta}(x) \) and \( h^{\alpha \beta}(x) \) obey the equations (2.2) guarantees \( D^2 = 0 \). Thus, our problem is to study the cohomology of \( D \) (2.9). Clearly, \( D \) commutes with the Euler operator \( N = y_\alpha \frac{\partial}{\partial y^\alpha} \). Its eigenvalues are identified with spin \( s \) via \( N = 2(s-1) \). The problem therefore is to be analyzed for different spins independently.

Conserved currents of an arbitrary integer spin in d4 Minkowski spacetime were considered in [11]. For \( d = 2 \), HS conserved currents were constructed in [12].

In the case of \( AdS_4 \), conserved currents of any integer spin \( s \geq 1 \) built from two massless scalar fields \( C, C' \) have a form
\[ J_{\mu, \alpha(2s-2)}(C, C') = \sum_{k=0}^{s-2} \frac{2(-1)^k}{(2k+1)!(2s-2k-3)!} h_{\mu, \gamma \gamma} C_\gamma \alpha(2k+1) C_\gamma \alpha(2s-2k-3) + \sum_{k=0}^{s-1} \frac{(-1)^k}{(2k)!(2s-2k-2)!} h_{\mu, \gamma \gamma} [C_\gamma \gamma \alpha(2k) C'_\alpha(2s-2k-2) - C_\alpha(2k) C'_\gamma \gamma \alpha(2s-2k-2)]. \tag{2.10} \]

Analogously, one can write down the currents of any integer spin \( s \geq 1 \) built from two massless spinors, and the “supercurrents” of any half integer spin \( s \geq 3/2 \) built from one scalar and one spinor [13]. The lowest spin conserved currents read
\[ J_{\mu, \gamma \gamma}^{(1)}(C, C') = h_{\mu, \gamma \gamma} (C_\gamma \gamma \alpha \gamma - C C'_\gamma \gamma), \quad J_{\mu, \gamma \gamma}^{(1)}(C_\alpha, C'_\alpha) = h_{\mu, \gamma \gamma} C_\gamma \gamma C'_\gamma \gamma, \tag{2.11} \]
\[ J_{\mu, \gamma \gamma}^{(3/2)}(C, C') = h_{\mu, \gamma \gamma} (C_\gamma \gamma C'_\alpha + 2 C_\gamma \gamma C'_\gamma \gamma), \tag{2.12} \]
\[ J_{\mu, \alpha \alpha}^{(2)}(C, C') = \frac{1}{2} h_{\mu, \gamma \gamma} (C_\gamma \gamma C'_\alpha - C C'_\gamma \gamma + 4 C_\gamma \gamma C'_\gamma \gamma). \tag{2.13} \]

These currents are all local, containing a finite number of terms (i.e., higher derivatives (2.3)). The same expressions remain valid in the flat limit with \( \nabla_\mu \rightarrow \partial_\mu \) in (2.3).

3 Generating Functions

To analyze the cohomology problem for currents of an arbitrary spin we first elaborate a technique operating with the generating functions (2.6) rather than with the individual multispinors. A generic Lorentz covariant spacetime 1-form of spin \( s = n/2 + 1 \) bilinear in two different matter fields \( C \) and \( C' \) and their on–mass–shell nontrivial derivatives is
\[ \Phi_{\alpha(n)}(C, C')(x) = \sum_{k+l=n-2}^{\infty} \sum_{m=0}^{\infty} a^{klm} \ h_{\alpha \alpha} C_{\alpha(k)}^{\beta(m)}(x) \ C'_{\alpha(l)}^{\theta(m)}(x) \]
\[ + \sum_{k+l=n-1}^{\infty} \sum_{m=0}^{\infty} \left[ b^{klm} C_{\gamma \gamma \alpha(k)}^{\beta(m)}(x) \ C'_{\gamma \gamma \alpha(l)}^{\theta(m)}(x) + c^{klm} \ h_{\gamma \gamma} C_{\gamma \alpha(k)}^{\beta(m)}(x) \ C'_{\gamma \alpha(l)}^{\theta(m)}(x) \right] \]
\[ + \sum_{k+l=n}^{\infty} \sum_{m=0}^{\infty} \left[ d^{klm} \ h_{\gamma \gamma} C_{\gamma \gamma \alpha(k)}^{\beta(m)}(x) \ C'_{\gamma \gamma \alpha(l)}^{\theta(m)}(x) + e^{klm} \ h_{\gamma \gamma} C_{\gamma \alpha(k)}^{\beta(m)}(x) \ C'_{\gamma \alpha(l)}^{\theta(m)}(x) \right], \tag{3.1} \]
where \(a_{klm}^{\alpha}, b_{1,2}^{klm}\), and \(e_{1,2,3}^{klm}\) are arbitrary constants and \(h_{a\alpha}\) is the dreibein 1-form. Introducing \(\Phi(y, \psi|x) = \Phi_{a1...\alpha_n}(\psi|x) y^{a_1} \ldots y^{a_n}\), one can equivalently rewrite this formula as

\[
\Phi(y, \psi|x) = h_{a\alpha} \frac{1}{(2\pi i)^2} \oint dr \oint ds \oint \tau^{-2} d\tau \oint d^2u \exp \left\{ \frac{i}{\tau}(u, \tau) \right\} \times C(u - ry, \psi|x) \left[ \frac{1}{2}(q + \bar{q}) - \frac{1}{2\bar{z}}(z - \bar{z})y, \psi \right| x] C' \left[ \frac{1}{2\bar{z}}(q + \bar{q}) + \frac{1}{2}(z + \bar{z})y, \psi \right| x] H_{a\alpha},
\]

where \(r, s, \tau\) are complex variables, \(u_\alpha\) and \(v_\alpha\) (\(\alpha = 1, 2\)) are spinor variables. The quantities \(f_i(r, s, \tau), i = 1, \ldots, 6\) are polynomials in \(r^{-1}\) and \(s^{-1}\) and formal series in \(\tau^{-1}\),

\[
f_i(r, s, \tau) = \sum_{a<k, l<p, m=1}^{\infty} f_i(k, l, m) r^{-l} s^{-m} \tau^{-m}.
\]

The contour integrations are normalized as \(\oint \tau^{-n} d\tau = \delta_n^1\). The Gaussian integrations with respect to \(u_\alpha\) and \(v_\alpha\) should be completed prior the contour integrations.

Inserting (2.6) into (3.2) and completing elementary integrations one arrives at (3.1) with the coefficients \(a_{klm}^{\alpha}, b_{1,2}^{klm},\) and \(e_{1,2,3}^{klm}\) expressed via \(f_i(k, l, m)\). For example,

\[
a_{klm}^{\alpha} = (-)^{k+m} \frac{i^m}{k! l! m!} (\lambda^{-1})^{\left\{ \frac{k+m}{2} \right\} + \left\{ \frac{\pm m}{2} \right\}} f_i(k + 1, l + 1, m + 1).
\]

Therefore (3.2) indeed describes a general Lorentz covariant 1-form bilinear in the matter fields. Note that the formula (3.2) produces a spacetime local expression if all the coefficients \(f_i\) contain a finite number of terms in (3.3) and pseudolocality if some of the expansions in negative powers of \(\tau\) are infinite.

In practice, the following representations of rank \(n = 0, 1, 2, 3\) differential forms \(\Phi_n(x)\) are most convenient,

\[
\Phi_{0,3}(y, \psi|x) = H_{0,3} \frac{1}{(2\pi i)^2} \oint dz \oint \bar{z} \oint d\tau \oint d^2q d^2\bar{q} \exp \left\{ -\frac{1}{2\tau}(q, \bar{q}) \right\} \times C \left[ \frac{1}{2}(q + \bar{q}) - \frac{1}{2\bar{z}}(z - \bar{z})y, \psi \right| x] C' \left[ \frac{1}{2\bar{z}}(q + \bar{q}) + \frac{1}{2}(z + \bar{z})y, \psi \right| x] H_{a\alpha},
\]

\[
\Phi_{1,2}(y, \psi|x) = H_{1,2\alpha} \frac{1}{(2\pi i)^2} \oint dz \oint \bar{z} \oint \tau^{-2} d\tau \oint d^2q d^2\bar{q} \exp \left\{ -\frac{1}{2\tau}(q, \bar{q}) \right\} \times C \left[ \frac{1}{2}(q + \bar{q}) - \frac{1}{2\bar{z}}(z - \bar{z})y, \psi \right| x] C' \left[ \frac{1}{2\bar{z}}(q + \bar{q}) + \frac{1}{2}(z + \bar{z})y, \psi \right| x]
\]

\[
\times \left\{ R_{1,2}(z, \bar{z}, \tau) y^\alpha q^\alpha + \frac{1}{2\tau z} W_{1,2}(z, \bar{z}, \tau) y^\alpha \bar{q}^\alpha + \frac{1}{2\tau \bar{z}} \bar{W}_{1,2}(z, \bar{z}, \tau) q^\alpha \bar{q}^\alpha + \frac{1}{2\tau^2 z^2} Y_{1,2}(z, \bar{z}, \tau) q^\alpha \bar{q}^\alpha + \frac{1}{2\tau^2 \bar{z}^2} \bar{Y}_{1,2}(z, \bar{z}, \tau) q^\alpha \bar{q}^\alpha \right\}.
\]

Here \(H_0 = \psi, H_3 = -\frac{\lambda^2}{12} h_{\alpha\beta} \wedge h^\beta \gamma \wedge h^\gamma \alpha, H_{1\alpha} = h_{a\alpha},\) and \(H_{2a\alpha} = -\frac{\lambda}{2} \psi h_{\alpha\beta} \wedge h^\beta \alpha,\) where the factors of \(\psi, -\frac{\lambda}{2} \psi,\) and \(-\frac{\lambda^2}{12}\) are introduced for future convenience. It is not hard to see that the expressions (3.3) and (3.5) reproduce arbitrary Lorentz covariant forms bilinear in the matter fields and their on-mass-shell nontrivial derivatives.
Let \( n, \bar{n}, \) and \( n_\tau \) be the following operators,
\[
n = z \frac{\partial}{\partial z}, \quad \bar{n} = \bar{z} \frac{\partial}{\partial \bar{z}}, \quad n_\tau = \tau \frac{\partial}{\partial \tau}
\] (using the same notations for their eigenvalues). The quantities \( R_{1,2}(z, \bar{z}, \tau), W_{1,2}(z, \bar{z}, \tau), \)
\( \bar{W}_{1,2}(z, \bar{z}, \tau), Y_{1,2}(z, \bar{z}, \tau), \bar{Y}_{1,2}(z, \bar{z}, \tau), \)
\( V_{1,2}(z, \bar{z}, \tau), \) and \( E_{0,3}(z, \bar{z}, \tau) \) give a non-zero contribution to \((3.3)\)
and \((3.6)\) when \( n, \bar{n}, \) and \( n_\tau \) satisfy the following restrictions:

\[
\begin{array}{ccc}
R_1, R_2 & n \leq -1 & \bar{n} \leq -1 & n_\tau \leq -1 \\
W_1, W_2 & n \leq -1 & \bar{n} \leq 0 & n_\tau \leq -1 \\
\bar{W}_1, \bar{W}_2 & n \leq 0 & \bar{n} \leq -1 & n_\tau \leq -1 \\
Y_1, Y_2 & n \leq -1 & \bar{n} \leq 1 & n_\tau \leq -1 \\
\bar{Y}_1, \bar{Y}_2 & n \leq 1 & \bar{n} \leq -1 & n_\tau \leq -1 \\
V_1, V_2, E_0, E_3 & n \leq 0 & \bar{n} \leq 0 & n_\tau \leq -1 \\
\end{array}
\] (3.8)

Beyond these regions, the coefficients do not contribute and therefore their values can
be fixed arbitrarily. As a result, the quantities \( R_{1,2}, W_{1,2}, \bar{W}_{1,2}, \ldots \) are defined modulo
arbitrary polynomials \( P(\tau) = \sum_{k=0}^{b_0} P_k \tau^k. \)

For the two-component spinors, antisymmetrization over any three two-component
spinor indices gives zero. This is expressed by the identity
\( a_\alpha(b_\beta c_\delta) + b_\alpha(c_\beta a_\delta) + c_\alpha(a_\beta b_\delta) = 0 \)
valid for any three commuting two-component spinors \( a_\alpha, b_\alpha \), and \( c_\alpha. \) As a result, the
forms discussed so far are not all independent. The ambiguity in adding any terms which
vanish as a consequence of this identity can be expressed in a form of some equivalence
(gauge) transformations of the coefficients in \((3.2)\) and \((3.6)\). We call these equivalence
transformations Fierz transformations. Using the partial integrations w.r.t. \( \tau, z, \) and \( \bar{z}, \)
one can check that the transformations
\[
\delta R_{1,2} = -\partial_\tau \chi_{1,2}, \quad \delta W_{1,2} = -\partial_\tau \xi_{1,2} + 2i\bar{n}\chi_{1,2}, \quad \delta \bar{W}_{1,2} = -\partial_\tau \bar{\xi}_{1,2} - 2in\chi_{1,2}, \\
\delta V_{1,2} = -i(n\xi_{1,2} - \bar{n}\bar{\xi}_{1,2}), \quad \delta Y_{1,2} = i(\bar{n} - 1)\xi_{1,2}, \quad \delta \bar{Y}_{1,2} = -i(n - 1)\bar{\xi}_{1,2}
\] (3.9)
with arbitrary parameters \( \chi_{1,2}(z, \bar{z}, \tau), \xi_{1,2}(z, \bar{z}, \tau), \) and \( \bar{\xi}_{1,2}(z, \bar{z}, \tau) \) describe all possible
Fierz transformations of the 1- and 2-forms \((3.6)\).

4 On-Mass-Shell Current Complex

In this section we study the on–mass-shell action of the operator \( D(2.3) \) on the differential
forms defined in sect. 3. The advantage of the formulation of the dynamical equations
in the unfolded form \((2.7)\) is that it expresses the spacetime derivative of \( C \) via some
operators acting in the auxiliary spinor space. As a result, on–mass–shell action of \( D \)
reduces to some mapping \( D \) acting on the coefficients in the formulae \((3.3)-(3.6).\)

Let us consider the example of a 0-form. Using the Leibnitz rule for \( D^\alpha \) and taking into
account the equations of motion \((2.7)\) and the zero torsion condition \( D^L h_{\alpha \beta} = 0 \)
\((2.2)\), one arrives at the 1-form \( \Phi_1 = D\Phi_0 \) with the coefficients \( R^D_1(E_0), W^D_1(E_0), \bar{W}^D_1(E_0), Y^D_1(E_0), \)
\( \bar{Y}_1^P(E_0) \), and \( V_1^P(E_0) \) of the form

\[
\begin{align*}
R_1^P(E_0) &= -\frac{i\lambda}{2}E_0, \\
W_1^P(E_0) &= i\lambda(1-i\tau)E_0, \\
\bar{W}_1^P(E_0) &= i\lambda(1+i\tau)E_0, \\
V_1^P(E_0) &= -i\lambda(1+\tau^2)E_0, \\
Y_1^P(E_0) &= \bar{Y}_1^P(E_0) = 0.
\end{align*}
\] (4.1)

Analogously one derives the mapping \( \mathcal{D} : \Phi_i(y) \rightarrow \Phi_{i+1}(y) = D\Phi_i(y)|_{\text{on-shell}}, i = 1,2 \) on the coefficients of the differential forms \((3.5), (3.6)\), \( \mathcal{D} \{R_1, W_1, \ldots\} = \{R_2^P, W_2^P, \ldots\} \), \( \mathcal{D} \{R_2, W_2, \ldots\} = E_3^P \) with

\[
\begin{align*}
R_2^P &= -(1-i\tau)nR_1 - (1+i\tau)nR_1 + 2R_1 + \frac{i}{4}(1+i\tau)\partial_\tau W_1 - \frac{i}{4}(1-i\tau)\partial_\tau \bar{W}_1 \\
W_2^P &= -\frac{i}{2}\partial_\tau[(1+\tau^2)W_1] + \frac{3}{2}(1+i\tau)W_1 + 2(1+\tau^2)nR_1 + \frac{1}{2}(1-i\tau)n\bar{W}_1 + (1-2n)Y_1 \\
&- \frac{1}{2}(1+i\tau)(\bar{n}-1)W_1 + i(1+i\tau)\partial_\tau Y_1 + \left(\frac{3}{2} - \bar{n}\right)Y_1 - \frac{i}{2}(1-i\tau)\partial_\tau V_1, \\
V_2^P &= \frac{1}{2}(1+\tau^2)(nW_1 + n\bar{W}_1) + \frac{1}{2}(1+i\tau)(\bar{n}-1)V_1 + \frac{1}{2}(1-i\tau)(n-1)V_1 + V_1 \\
&+ (1+i\tau)nY_1 + (1-i\tau)n\bar{Y}_1, \\
Y_2^P &= \frac{1}{2}(1+\tau^2)(\bar{n}-1)W_1 - i(1+\tau^2)\partial_\tau Y_1 + (1+i\tau)Y_1 + (1-i\tau)nY_1 \\
&+ \frac{1}{2}(1-i\tau)(\bar{n}-1)V_1, \\
E_3^P &= 4i n\bar{n}(1+\tau^2)R_2 + i n\bar{n}(W_2 + \bar{W}_2) - 3\tau(nW_2 - n\bar{W}_2) - \tau n\bar{n}(W_2 - \bar{W}_2) \\
&+ \partial_\tau[(1+\tau^2)(nW_2 - n\bar{W}_2)] + 2\partial_\tau[(nY_2 - n\bar{Y}_2) - (n - \bar{n})\partial_\tau V_2 \\
&+ 2i(n\bar{n} - n - \bar{n} + 1)V_2 + i(n + \bar{n} - 2)\tau \partial_\tau V_2 + i(1+\tau^2)\partial_\tau \partial_\tau V_2 \\
&- 3i(nW_2 + n\bar{W}_2) + (i + \tau)n(n+1)W_2 + (i - \tau)\bar{n}(\bar{n} + 1)\bar{W}_2 \\
&- 4i(nY_2 + \bar{n}\bar{Y}_2) + 2i\tau \partial_\tau(nY_2 + \bar{n}\bar{Y}_2) + 2i[n(n+1)Y_2 + \bar{n}(n+1)\bar{Y}_2].
\end{align*}
\] (4.2)

The corresponding formulae for the parameters \( \bar{W}_2^P \) and \( \bar{Y}_2^P \) are given by \((4.3)\) and \((4.4)\) respectively with the replacements \( i \rightarrow -i, n \leftrightarrow \bar{n}, W_1 \leftrightarrow \bar{W}_1, \) and \( Y_1 \leftrightarrow \bar{Y}_1 \) on r.h.s. As expected, \( \mathcal{D}^2 = 0 \) and therefore the mapping \( \mathcal{D} \) defines a complex \((T, \mathcal{D})\) with \( T = \bigoplus_{i=0,1,2,3} T_i \), where \( T_{0,3} = \{E_{0,3}\} \) and \( T_{1,2} = \{R_{1,2}, W_{1,2}, \bar{W}_{1,2}, V_{1,2}, Y_{1,2}, \bar{Y}_{1,2}\} \). The reformulation of the problem in terms of \((T, \mathcal{D})\) effectively accounts the fact that the fields are on–mass–shell. We identify the cohomology of currents with the cohomology of \( \mathcal{D} \).

The remarkable property of the mapping \( \mathcal{D} \) is that it contains \( z, \bar{z}, \frac{\partial}{\partial z}, \) and \( \frac{\partial}{\partial \bar{z}} \) only via \( n \) and \( \bar{n} \) \((3.7)\), thus implying the separation of variables: the differential \( \mathcal{D} \) leaves invariant eigensubspaces of \( n \) and \( \bar{n} \). In fact, this is the main reason for using the particular representation \((3.3)-(3.6)\).

As expected, the system \((4.2)-(4.6)\) is consistent with the Fierz transformations \((3.9)\). Namely, any Fierz transformation of the quantities \( R_1, W_1, \ldots \) leads to some Fierz transformation of the quantities \( R_2^P, W_2^P, \ldots \), and any Fierz transformation of \( R_2, W_2, \ldots \) does not affect the parameter \( E_3^P \) \((4.6)\).
Following [1] we study the currents containing the minimal possible number of space-time derivatives for a given spin s. From (2.3) it is clear that this is the case if the number of the contracted indices \( \beta \) in (3.3) is zero. Since the number of contractions is \(-(n_\tau + 1)\) (see sect. 3) we consider 2-forms \( \Phi_{2n,1}^{\alpha} \) with \( n_\tau = -1 \). Thus we set in (3.6)

\[
R_2 = \alpha_R(n, \bar{n}) z^n \bar{z}^{\bar{n}} \tau^{-1}, \quad W_2 = \alpha_W(n, \bar{n}) z^n \bar{z}^{\bar{n}} \tau^{-1}, \quad Y_2 = \alpha_Y(n, \bar{n}) z^n \bar{z}^{\bar{n}} \tau^{-1}, \quad V_2 = \alpha_V(n, \bar{n}) z^n \bar{z}^{\bar{n}} \tau^{-1}
\]

with some constant parameters \( \alpha_R(n, \bar{n}), \alpha_W(n, \bar{n}), \ldots \sim \lambda^s \), where \( s = 1 - \frac{1}{2}(n + \bar{n}) \). The conservation condition means that \( \Phi_{2n,1}^{\alpha} \) should be \( \mathcal{D} \)-closed. The requirement \( E_3^D = 0 \) modulo terms that do not contribute to (3.5) imposes the following conditions

\[
4n \bar{n} \alpha_R + (n + \bar{n} - 2)(n \alpha_W + \bar{n} \alpha_W) + 2(n - 2)\alpha_Y + 2n(\bar{n} - 2)\alpha_Y = 0, \\
n \alpha_W - \bar{n} \alpha_W + 2(n \alpha_Y - \bar{n} \alpha_Y) = 0, \\
\alpha_Y = 0,
\]

for \( n \neq 1, \bar{n} \neq 1 \). For \( n = 1 \) or \( \bar{n} = 1 \) \( \Phi_{2n,1}^{\alpha} \) is closed as a consequence of (3.8).

Our problem is to investigate whether there exist coefficients \( R_1, W_1, \ldots \) such that \( R_1^P, W_1^P, \ldots \) have a form (4.7). To this end one has to solve the system (4.2)-(4.5) in terms of the formal series \( f(\tau) = \sum_{k=0}^{\infty} f_k \tau^k \).

The Fierz transformations (3.9) for \( \Phi_1 \) together the exact shifts of \( R_1, W_1, \ldots \) by any \( R_1^P, W_1^P, \ldots \) (4.4) produce the following equivalence transformations

\[
\delta R_1 = -\partial_\tau \xi_1 + \varepsilon, \quad \delta W_1 = -\partial_\tau \xi_1 + 2i\bar{n} \chi_1 - 2(1 - i\tau)\varepsilon, \\
\delta Y_1 = -\partial_\tau \xi_1 - 2in \chi_1 - 2(1 + i\tau)\varepsilon, \quad \delta V_1 = 2(1 + \tau^2)\varepsilon - i(n \xi_1 - \bar{n} \bar{\xi}_1), \\
\delta Y_1 = i(\bar{n} - 1) \xi_1, \quad \delta Y_1 = -i(n - 1) \bar{\xi}_1.
\]

We consider separately two cases: (i) with \( n = 1, \bar{n} = 1 - 2s \) or \( \bar{n} = 1, n = 1 - 2s \) and (ii) with \( n < 1 \) and \( \bar{n} < 1 \). As shown below, the case (i) corresponds to the nontrivial physical conserved currents, whereas the case (ii) describes all possible “improvements”.

Let us start with the case (i) setting for definiteness \( \bar{n} = 1 \). According to (3.8), \( Y_2 \) is the only coefficient giving a non-zero contribution to \( \Phi_{2n,1}^{\alpha}(y) \). Obviously, a 2-form with \( \bar{n} = 1 \) is invariant under the transformations (3.9). The only non-trivial equation is (4.5).

With \( Y_2^D \) (4.7) it takes the form

\[
(1 + \tau^2) \partial_\tau Y_1 = -i(1 + i\tau) Y_1 + i(2s - 1)(1 - i\tau) Y_1 + i\alpha_Y(1 - 2s, 1) z^{1-2s} \bar{z} \tau^{-1}
\]

modulo terms polynomial in \( \tau \). It is not hard to see that a generic solution of (4.10) is

\[
Y_1(z, \bar{z}, \tau) = -\frac{i}{2} \alpha_Y(1 - 2s, 1) z^{1-2s} \bar{z} (1 - i\tau)(1 + i\tau)^{2s-1} \ln (1 + \tau^{-2}) + \sigma z^{1-2s} \bar{z} (1 - i\tau)(1 + i\tau)^{2s-1} \ln \frac{1 + i\tau^{-1}}{1 - i\tau^{-1}} + Q(\tau),
\]

where \( \sigma \) is an arbitrary constant and \( Q(\tau) \) is some inessential polynomial. The logarithms are treated as power series in \( \tau^{-1} \).

At any \( \sigma \), the solution (4.11) is an infinite series in \( \tau^{-1} \), thus corresponding to some pseudolocal 1-form. Thus, the 2-forms \( \Phi_2^2(y|x) \) constructed with the polynomials \( Y_2 \) at
\( \tilde{n} = 1 \) and with \( \tilde{Y}_2 \) at \( n = 1 \) are \( D \)-closed and cannot be represented as \( D\Phi_s^i(y|x) \) with a spacetime local \( \Phi_s^i(y|x) \). We therefore argue that the 2-form \( \Phi_s^2(y) \) describes a physical conserved current of spin \( s \). The currents \((4.10)\) as well as the currents containing fermions are reproduced via \( Y_2 \) \((4.7)\) with \( \alpha_Y(1-2s,1) = 2^{2s-1}(\lambda\psi)_s^{|s} \). The formula \((4.11)\) solves the problem of reformulation of the physical currents as pseudolocally exact 2-forms. Note that the currents generated by \( Y_2(n = 1 -2s, \tilde{n} = 1) \) and \( \tilde{Y}_2(n = 1, \tilde{n} = 1 -2s) \) are equivalent by the interchange \( C \leftrightarrow C' \).

Note that the solution \((4.11)\) is not unique, containing an arbitrary parameter \( \sigma \). Since the transformations \((4.9)\) are trivial for \( Y_1 \) at \( \tilde{n} = 1 \), this one-parametric ambiguity cannot be compensated this way. This means that we have found a pseudolocal 1-form that is \( D \)-closed but not \( D \)-exact, i.e., the cohomology group \( H^1(T, D) \) is nontrivial. This fact is in agreement with the one-parametric ambiguity found in [7] for the spin 2 case.

Let us now consider the case of \( n < 1, \tilde{n} < 1 \). Substituting \((4.7)\) into the system \((1.12)-(1.15)\) one can see that, if the conditions \((4.8)\) guaranteeing that \( \Phi_s^2 \) is \( D \)-closed are satisfied, then it admits the following solution,

\[
R_1(z, \bar{z}, \tau) = \frac{1}{4n\tilde{n}} \left( n\alpha_w + \tilde{n}\alpha_w - \frac{2n}{\tilde{n}-1} \alpha_Y - \frac{2\tilde{n}}{n-1} \alpha_Y \right) z^n \bar{z}^{\tilde{n}} \tau^{-1}, \quad n, \tilde{n} < 0, \quad (4.12)
\]

\[
W_1(z, \bar{z}, \tau) = \frac{\alpha_Y}{\tilde{n}-1} z^n \bar{z}^{\tilde{n}} \tau^{-1}, \quad \tilde{W}_1(z, \bar{z}, \tau) = \frac{\alpha_Y}{n-1} z^n \bar{z}^{\tilde{n}} \tau^{-1}, \quad V_1 = Y_1 = \tilde{Y}_1 = 0. \quad (4.13)
\]

We observe that the 1-form \( \Phi_s^{n,\tilde{n}}(y) \) leads to a spacetime local expression since \( R_1, W_1 \), and \( \tilde{W}_1 \) \((1.12), (1.13)\) are linear in \( \tau^{-1} \). Therefore, \( \Phi_s^{n,\tilde{n}}(y|x) = D\Phi_s^{n,\tilde{n}}(y|x) \) with some local \( \Phi_s^{n,\tilde{n}}(y|x) \). Thus, it is an “improvement” of the physical current 2-form \( J(C^2) \) on the r.h.s. of \((2.8)\), which can be compensated by a local field redefinition of the (HS) gauge fields.

To investigate what happens in the flat limit \( \lambda \to 0 \) one should use the generating function \( \tilde{C}(y) \) \((2.7)\) instead of \( C(y) \). A simple analysis show that in terms of \( \tilde{C}(y) \) the solution \((4.11)\) contain an inverse power of \( \lambda \) together with each power of \( \tau^{-1} \). Hence, a representation of physical current 2-forms \( \Phi_s^2(y) \) as some differentials \( D\Phi_s^i(y) \) becomes meaningless in the flat limit.

**Conclusion**

It is shown that local conserved currents of an arbitrary spin in \( AdS_3 \) can be treated as “improvements” within the class of infinite power expansions in higher derivatives, i.e., 2-forms \( J \) dual to the physical conserved currents are shown to be exact in this class, \( J = DU \). The 1-forms \( U \) are constructed explicitly what allows us to write down nonlocal field redefinitions compensating matter sources in the equations of motion for the Chern-Simons gauge fields of all spins. The coefficients in the expansion of \( U \) in derivatives of the matter fields contain negative powers of the cosmological constant (i.e. positive powers of the AdS radius) and therefore do not admit a flat limit. The existence of \( U \) may be related to the holography in the AdS/CFT correspondence since it indicates that local current interactions in \( AdS_3 \) are in a certain sense trivial and can, up to some surface terms, be compensated by a field redefinition. We expect similar phenomenon to take place for \( AdS_d \) with any \( d \).
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