Partial Sum of Analytic Odd Function Defined by Salagean Differential Operator

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Abstract. This paper is emphasizing on the new analytic odd function defined using Salagean differential operator $H(m, z)$ in the open unit disc. The main objective is to study the ratio of the function $H(m, z)$ to its sequence of partial sum. Also, to determine the sharp upper bound for $\text{Re}\left\{H(m, z)/H_n(m, z)\right\}$, $\text{Re}\left\{H_n(m, z)/H(m, z)\right\}$, $\text{Re}\left\{H'(m, z)/H'_n(m, z)\right\}$ and $\text{Re}\left\{H'_n(m, z)/H'(m, z)\right\}$.

1. Introduction
Let $\mathcal{A}$ denote the class of normalized analytic functions $f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$  \hspace{1cm} (1)

where $z \in \mathcal{U} = \{z : |z| < 1\}$, and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. We denote $S$ be the subclass of $\mathcal{A}$ consisting of analytic univalent functions $f(z)$ in $\mathcal{U}$.

For each function $f \in S$, the square-root transform is an odd univalent function given by

$$h(z) = \sqrt{h(z)} = z + c_3 z^3 + c_5 z^5 + \cdots = z + \sum_{k=2}^{\infty} c_{2k-1} z^{2k-1}.$$  \hspace{1cm} (2)

The set of all odd functions in $S$ is denoted by $S^{(2)}$. It is easy to see the square-root transform of Koebe function is

$$\frac{z}{1-z^2} = z + z^3 + c_5 z^5 + \cdots.$$  \hspace{1cm} (3)

It is to be expected that this function will play the role of the Koebe function in the class $S^{(2)}$ (see Duren \cite{3}).

Now, we denote by $S^*(\alpha)$, $K(\alpha)$, $0 \leq \alpha \leq 1$, the class of univalently starlike functions of order $\alpha$ and the class of univalently convex of order $\alpha$ respectively, where

$$S^*(\alpha) = \left\{ h \in S : \text{Re} \left\{ \frac{zh'(z)}{h(z)} \right\} > \alpha, \ z \in \mathcal{U} \right\}.$$  \hspace{1cm} (4)
and
\[ K(\alpha) = \left\{ h \in S : \text{Re} \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} > \alpha, \quad z \in U \right\}. \quad (5) \]

For functions \( h \in A \), by using the approach introduced by Salagean [4], we introduced the following operator:
\[ D^0 h(z) = h(z) = z + c_3 z^3 + c_5 z^5 + \cdots = z + \sum_{k=2}^{\infty} c_{2k-1} z^{2k-1} \]
\[ D^1 h(z) = Dh(z) = zh'(z) = z + 3c_3 z^3 + 5c_5 z^5 + \cdots = z + \sum_{k=2}^{\infty} (2k-1)c_{2k-1} z^{2k-1} \]
\[ \vdots \]
\[ D^m h(z) = D(D^{m-1} h(z)) = z + \sum_{k=2}^{\infty} (2k-1)^m c_{2k-1} z^{2k-1}. \]

Thus, we write univalent odd function defined by Salagean derivative operator as
\[ \mathcal{H}(m, z) = z + \sum_{k=2}^{\infty} (2k-1)^m c_{2k-1} z^{2k-1}. \quad (6) \]

We note that, \( S_0(\alpha) = S^*(\alpha) \) and \( S_1(\alpha) = K(\alpha) \), since
\[ \frac{Dh'(z)}{h(z)} = \frac{zh'(z)}{h(z)} = \mathcal{H}(0, z) \quad \text{and} \quad \frac{D^2h'(z)}{Dh(z)} = \frac{z(h'(z))'}{zh'(z)} = \mathcal{H}(1, z), \quad (7) \]

respectively.

2. Coefficient Inequalities

The necessary and sufficient condition for functions \( h \) to be in the subclass \( S_n(\alpha) \) given as follow

**Theorem 2.1** Let \( h(z) \) given by (2), then \( h \in S_n(\alpha) \) if and only if
\[ \sum_{k=2}^{\infty} |(2k-1)^m - (2k-1)^m c_{2k-1}| \leq 1. \quad (8) \]

**Proof.** It is suffices to show that
\[ \left| \frac{D^{m+1}h(z)}{D^m h(z)} - 1 \right| \leq 1 - \alpha. \]

We have
\[
\begin{align*}
\left| \frac{D^{m+1}h(z)}{D^m h(z)} - 1 \right| &= \left| \frac{D^{m+1}h(z) - D^m h(z)}{D^m h(z)} \right| \\
&= \left| \frac{\sum_{k=2}^{\infty} (2k-1)^m c_{2k-1} z^{2k-1} - 1 (z + \sum_{k=2}^{\infty} (2k-1)^m c_{2k-1} z^{2k-1})}{z + \sum_{k=2}^{\infty} (2k-1)^m c_{2k-1} z^{2k-1}} \right| \\
&\leq \sum_{k=2}^{\infty} |(2k-1)^m - (2k-1)^m c_{2k-1}| |z^{2k-1}|, \quad |z| = 1 \\
&\leq \sum_{k=2}^{\infty} |(2k-1)^m - (2k-1)^m c_{2k-1}| |z^{2k-1}| \\
&= \sum_{k=2}^{\infty} |(2k-1)^m - (2k-1)^m c_{2k-1}|. 
\end{align*}
\]
On the other hand,
\[ \sum_{k=2}^{\infty} \left| (2k-1)^{m+1} - \alpha(2k-1)^m \right| c_{2k-1} \leq 1 - \alpha \]
\[ \sum_{k=2}^{\infty} \left| (2k-1)^{m+1} - (2k-1)^m + (2k-1)^m - \alpha(2k-1)^m \right| c_{2k-1} \leq 1 - \alpha. \]

Hence,
\[ \sum_{k=2}^{\infty} \left| (2k-1)^{m+1} - (2k-1)^m \right| c_{2k-1} \leq 1 - \alpha - \sum_{k=2}^{\infty} \left| (2k-1)^m - \alpha(2k-1)^m \right| c_{2k-1} \]
\[ \sum_{k=2}^{\infty} \left| (2k-1)^{m+1} - (2k-1)^m \right| c_{2k-1} \leq (1 - \alpha) \left[ 1 - \sum_{k=2}^{\infty} (2k-1)^m |c_{2k-1}| \right] \]
\[ \frac{\sum_{k=2}^{\infty} \left| (2k-1)^{m+1} - (2k-1)^m \right| c_{2k-1}}{1 - \sum_{k=2}^{\infty} |(2k-1)^m| c_{2k-1}} \leq 1 - \alpha. \]

Thus, we obtain
\[ \left| \frac{D^{m+1}h(z)}{D^m h(z)} - 1 \right| \leq 1 - \alpha, \]
and the proof is complete.

In the present paper and by following the earlier work by Silverman [1] and Frasin [2], we will examine the ratio of univalent odd functions (6) to its sequence of partial sums when the coefficients of the \( H(m, z) \) are sufficiently small to satisfy either condition (8). We will determine sharp lower bound for \( \text{Re} \{ H(m, z)/H_n(m, z) \}, \ \text{Re} \{ H_n(m, z)/H(m, z) \}, \ \text{Re} \{ H'(m, z)/H'_n(m, z) \} \) and \( \text{Re} \{ H'_n(m, z)/H'(m, z) \} \).

3. Partial Sums

In this section the partial sums of odd function defined by Salagean differential operator \( S_n(\alpha) \) are given.

**Theorem 3.1** If \( H \) of the form (6) satisfies condition (8), then
\[ \text{Re} \left\{ \frac{H(m, z)}{H_n(m, z)} \right\} \geq \frac{(2n+1)^m(2n+1-\alpha) - 1 + \alpha}{(2n+1)^m(2n+1-\alpha)}, \quad (z \in U) \quad (9) \]
\[ \text{and} \]
\[ \text{Re} \left\{ \frac{H_n(m, z)}{H(m, z)} \right\} \geq \frac{(2n+1)^m(2n+1-\alpha)}{(2n+1)^m(2n+1-\alpha) + 1 - \alpha}, \quad (z \in U). \quad (10) \]

The result (9) and (10) is sharp with the function given by
\[ H(m, z) = z + \frac{1 - \alpha}{(2n+1)^m(2n+1-\alpha)} z^{2n+1}. \]
Proof. To prove (9), we may write

\[
\frac{1 + w(z)}{1 - w(z)} = \frac{(2n+1)^m(2n+1 - \alpha)}{1 - \alpha} \left[ \mathcal{H}(m, z) - (2n+1)^m(2n+1 - \alpha) - 1 + \alpha \right]
\]

\[
= 1 + \sum_{k=2}^{\infty} \frac{(2k-1)^m c_{2k-1} z^{2k-2}}{1 - \alpha} + \sum_{k=2}^{\infty} \frac{(2k-1)^m(2n+1) \alpha}{1 - \alpha} \sum_{k=n+1}^{\infty} (2k-1)^m c_{2k-1} z^{2k-2}
\]

\[
= 1 + A(z) + B(z). \tag{10}
\]

Define the function \( w(z) \) by setting \( \frac{1+A(z)}{1-B(z)} = \frac{1+w(z)}{1-w(z)} \), so that \( w(z) = \frac{A(z)-B(z)}{2+A(z)+B(z)} \). Then

\[
w(z) = \frac{\left[ \frac{(2n+1)^m(2n+1 - \alpha)}{1 - \alpha} \sum_{k=n+1}^{\infty} (2k-1)^m c_{2k-1} z^{2k-2} \right]}{2 + 2 \sum_{k=2}^{\infty} (2k-1)^m c_{2k-1} z^{2k-2} + \left[ \frac{(2n+1)^m(2n+1 - \alpha)}{1 - \alpha} \sum_{k=n+1}^{\infty} (2k-1)^m c_{2k-1} z^{2k-2} \right]}
\]

and

\[
|w(z)| \leq \frac{\left[ \frac{(2n+1)^m(2n+1 - \alpha)}{1 - \alpha} \sum_{k=n+1}^{\infty} (2k-1)^m c_{2k-1} \right]}{2 + 2 \sum_{k=2}^{\infty} (2k-1)^m c_{2k-1} + \left[ \frac{(2n+1)^m(2n+1 - \alpha)}{1 - \alpha} \sum_{k=n+1}^{\infty} (2k-1)^m c_{2k-1} \right]}. \tag{11}
\]

Now \( |w(z)| \leq 1 \) if and only if

\[
2 \left[ \frac{(2n+1)^m(2n+1 - \alpha)}{1 - \alpha} \right] \sum_{k=n+1}^{\infty} (2k-1)^m c_{2k-1} \leq 2 - 2 \sum_{k=2}^{n} (2k-1)^m c_{2k-1}
\]

which is equivalent to

\[
\sum_{k=2}^{n} (2k-1)^m c_{2k-1} + \sum_{k=n+1}^{\infty} \left[ \frac{(2n+1)^m(2n+1 - \alpha)}{1 - \alpha} \right] (2k-1)^m c_{2k-1} \leq 1. \tag{12}
\]

It is sufficient to show the LHS of (11) is bounded above by

\[
\sum_{k=2}^{\infty} \left[ \frac{(2k-1)^m(2k-1 - \alpha)}{1 - \alpha} \right] c_{2k-1} \tag{13}
\]

or, equivalently

\[
\sum_{k=2}^{n} (2k-1)^m c_{2k-1} + \sum_{k=n+1}^{\infty} \left[ \frac{(2n+1)^m(2n+1 - \alpha)}{1 - \alpha} \right] (2k-1)^m c_{2k-1} \leq \sum_{k=2}^{\infty} \left[ \frac{(2k-1)^m(2k-1 - \alpha)}{1 - \alpha} \right] c_{2k-1} \]

\[
\Rightarrow \sum_{k=2}^{n} c_{2k-1} + \sum_{k=n+1}^{\infty} \left[ \frac{(2n+1)^m(2n+1 - \alpha)}{1 - \alpha} \right] c_{2k-1} \leq \sum_{k=2}^{\infty} \left[ \frac{(2k-1)^m(2k-1 - \alpha)}{1 - \alpha} \right] c_{2k-1} \]
which is equivalent to
\[
\sum_{k=2}^{\infty} \left| \frac{(2k-1)^m(2k-1-\alpha) - 1 + \alpha}{1-\alpha} \right| c_{2k-1} \left| c_{2k-1} \right|^{2n+1} 
+ \sum_{k=n+1}^{\infty} \left| \frac{(2k-1)^m(2k-1-\alpha) - (2n+1)^m(2n+1-\alpha)}{1-\alpha} \right| c_{2k-1} \left| c_{2k-1} \right|^{2n+1} \geq 0.
\]

To see that the function given by
\[
H(m, z) = z + \frac{1-\alpha}{(2n+1)^m(2n+1-\alpha) - 1 + \alpha} z^{2n+1}
\]
gives the sharp result, we observe that for \( z = re^{i\pi} \)
\[
\frac{H(m, z)}{H_n(m, z)} = 1 + \frac{1-\alpha}{(2n+1)^m(2n+1-\alpha)} z^{2n+1} \rightarrow 1 - \frac{1-\alpha}{(2n+1)^m(2n+1-\alpha)} \text{ when } r \rightarrow 1^{-}.
\]
The proof is complete. The proof of (10) is similar to that of (9), and this is omitted.

Letting \( m = 0 \) in Theorem 3.1 respectively, we have the following corollary

**Corollary 3.1** If \( H \) of the form (6) satisfies condition (8), then
\[
\text{Re} \left\{ \frac{H(0, z)}{H_n(0, z)} \right\} \geq \frac{2n}{2n+1-\alpha}, \quad (z \in \mathcal{U}) \quad (13)
\]
and
\[
\text{Re} \left\{ \frac{H_n(0, z)}{H(0, z)} \right\} \geq \frac{(2n+1-\alpha)}{2(n+1-\alpha)}, \quad (z \in \mathcal{U}). \quad (14)
\]
The results (13) and (14) is sharp with the function given by
\[
H(0, z) = z + \frac{1-\alpha}{2n+1-\alpha} z^{2n+1}.
\]

Letting \( m = 1 \) in Theorem 3.1 respectively, we have the following corollary

**Corollary 3.2** If \( H \) of the form (6) satisfies condition (8), then
\[
\text{Re} \left\{ \frac{H(1, z)}{H_n(1, z)} \right\} \geq \frac{2n(2n+2-\alpha)}{(2n+1)(2n+1-\alpha)}, \quad (z \in \mathcal{U}) \quad (15)
\]
and
\[
\text{Re} \left\{ \frac{H_n(1, z)}{H(1, z)} \right\} \geq \frac{(2n+1)(2n+1-\alpha)}{(2n+1)(2n+1-\alpha) + 1 - \alpha}, \quad (z \in \mathcal{U}). \quad (16)
\]
The results (15) and (16) is sharp with the function given by
\[
H(1, z) = z + \frac{1-\alpha}{(2n+1)(2n+1-\alpha)} z^{2n+1}.
\]

Similarly, we can prove the following theorem.
Theorem 3.2 If $H$ of the form (6) satisfies condition (8), then
\[
\text{Re} \left\{ \frac{H^\prime(m, z)}{H_n^\prime(m, z)} \right\} \geq \frac{(2n+1)^m(2n+1-\alpha)}{(2n+1)^m(2n+1-\alpha) + 1 - \alpha}, \quad (z \in \mathcal{U})
\] (17)
and
\[
\text{Re} \left\{ \frac{H_n^\prime(m, z)}{H^\prime(m, z)} \right\} \geq \frac{(2n+1)^m(2n+1-\alpha)}{(2n+1)^m(2n+1-\alpha) + 1 - \alpha}, \quad (z \in \mathcal{U}).
\] (18)

The results (17) and (18) is sharp with the function given by
\[H(m, z) = z + \frac{1 - \alpha}{(2n+1)^m(2n+1-\alpha)} z^{2n+1}.
\]

If we let $m = 0$ and $m = 1$ in (3.2), we have the following corollaries

Corollary 3.3 If $H$ of the form (6) satisfies condition (8), then
\[
\text{Re} \left\{ \frac{H^\prime(0, z)}{H_n^\prime(0, z)} \right\} \geq \frac{2n+1 - \alpha}{(2n+1 - \alpha) + (n+1)(1-\alpha)} = \frac{n(1+\alpha)}{2n+1-\alpha}, \quad (z \in \mathcal{U})
\] (19)
and
\[
\text{Re} \left\{ \frac{H_n^\prime(0, z)}{H^\prime(0, z)} \right\} \geq \frac{2n+1 - \alpha}{(2n+1 - \alpha) + (n+1)(1-\alpha)} = \frac{2n+1 - \alpha}{n(3-\alpha) + 2(1-\alpha)}, \quad (z \in \mathcal{U}).
\] (20)

The results (19) and (20) is sharp with the function given by
\[H(0, z) = z + \frac{1 - \alpha}{(2n+1-\alpha)} z^{2n+1}.
\]

Corollary 3.4 If $H$ of the form (6) satisfies condition (8), then
\[
\text{Re} \left\{ \frac{H^\prime(1, z)}{H_n^\prime(1, z)} \right\} \geq \frac{(2n+1)(2n+1-\alpha)}{(2n+1)(2n+1-\alpha) + 1 - \alpha}, \quad (z \in \mathcal{U})
\] (21)
and
\[
\text{Re} \left\{ \frac{H_n^\prime(1, z)}{H^\prime(1, z)} \right\} \geq \frac{(2n+1)(2n+1-\alpha)}{(2n+1)(2n+1-\alpha) + 1 - \alpha}, \quad (z \in \mathcal{U}).
\] (22)

The results (21) and (22) is sharp with the function given by
\[H(1, z) = z + \frac{1 - \alpha}{(2n+1)(2n+1-\alpha)} z^{2n+1}.
\]

Acknowledgement. The work is supported by research Grant: GUP-2017-064. The authors would like to thank the referee for valuable suggestions.

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