A deformation of Sasakian structure in the presence of torsion and supergravity solutions

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Abstract
A deformation of Sasakian structure in the presence of totally skew-symmetric torsion is discussed on odd-dimensional manifolds whose metric cones are Kähler with torsion. It is shown that such a geometry inherits similar properties to those of Sasakian geometry. As their example, we present an explicit expression of local metrics. It is also demonstrated that our example of the metrics admits the existence of hidden symmetry described by non-trivial odd-rank generalized closed conformal Killing–Yano tensors. Furthermore, using these metrics as an ansatz, we construct exact solutions in five-dimensional minimal gauged/ungauged supergravity and 11-dimensional supergravity. Finally, the global structures of the solutions are discussed. We obtain regular metrics on compact manifolds in five dimensions, which give natural generalizations of Sasaki–Einstein manifolds $Y^{p,q}$ and $L_{p,b,c}$. We also briefly discuss regular metrics on non-compact manifolds in 11 dimensions.

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1. Introduction
Sasakian geometry [1] has attracted intense interest in theoretical and mathematical physics since its applications were found in higher dimensional supergravity theories, string theories and M-theory. Arguably, the most important examples are Sasaki–Einstein manifolds which have been discussed in the context of the AdS/CFT correspondence, especially in the physically interesting dimensions 5 and 7. In five dimensions, the simplest example of the Sasaki–Einstein manifold is the standard round 5-sphere, denoted by $S^5$. It provides
a supersymmetric background $\text{AdS}_5 \times S^5$ of type-IIB supergravity theory, on which D3-brane physics is conjectured to be dual of an $\mathcal{N} = 4$ four-dimensional superconformal field theory [2]. More general five-dimensional Sasaki–Einstein manifolds $M_5$ provide a variety of supersymmetric backgrounds $\text{AdS}_5 \times M_5$, which are in general dual of $\mathcal{N} = 1$ superconformal field theories. Recently, it was proposed in [3] that $\mathcal{N} = 6$ three-dimensional Chern–Simons matter theory is related to M2-brane physics on a background $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ of M-theory. This motivates us to extend $S^7$ to general seven-dimensional Sasaki–Einstein manifolds $M_7$ and study the backgrounds $\text{AdS}_4 \times M_7$ corresponding to $\mathcal{N} = 2$ Chern–Simons theories. Owing to such proposals, we now have a number of concrete examples of Sasaki–Einstein manifolds. Until recent years, the only explicit examples of Sasaki–Einstein manifolds were $S^5$ and $T^{1,1}$ in five dimensions and $M^{5,2}$, $Q^{1,1,1}$ and $V^{5,2}$ in seven dimensions. However, thanks to Gauntlett, Martelli, Sparks and Waldram, the infinite families of inhomogeneous Sasaki–Einstein manifolds were constructed in five [4, 5] and higher [6] dimensions. Further generalizations were constructed in various odd dimensions [7, 8], in connection with vacuum rotating black hole spacetimes [9–11].

In the familiar story of type-IIB supergravity theory, $\text{AdS}_5 \times M_5$ backgrounds are given as supersymmetric solutions of the ten-dimensional Einstein’s equation with the only self-dual 5-form flux. Supersymmetry then requires $M_5$ to admit the existence of Killing spinors so that $M_5$ is Sasaki–Einstein. However, in general, there can be other supersymmetric solutions which provide dual-field theories still having $\mathcal{N} = 1$ supersymmetry. Since it is expected that on these backgrounds one has some non-trivial fluxes which contribute to the energy–momentum tensor, the Sasakian structure should deform. Therefore, in order to discover such deformed backgrounds which, if they exist, give generalizations of the Sasaki–Einstein manifolds, it might be useful to think how we can deform the Sasakian structure. Pilch and Warner [12] have in fact constructed a non-trivial supersymmetric background $\text{AdS}_5 \times M_5$, where $M_5$ is deformed from $S^5$ because a non-trivial 3-form is present. In [13], a non-trivial supersymmetric background $\text{AdS}_4 \times M_7$, where $M_7$ is deformed from $S^7$, has been also constructed in M-theory. One interesting approach in this direction is the so-called Hitchin’s generalized geometry [14]. By exploiting it, the notion of ‘generalized Sasaki–Einstein geometry’ which provides general supersymmetric $\text{AdS}_5$ solutions of type-IIB supergravity theory with non-trivial fluxes was introduced, which enables us to study the general structure of the $\text{AdS}_5$/CFT$_4$ correspondence [15, 16]. Unfortunately, however, few explicit examples have been realized.

Our aim is to deform the Sasakian structure by introducing a totally skew-symmetric torsion. It is well known that pseudo-Riemannian manifolds with totally skew-symmetric torsions appear naturally in supergravity theories, where the torsions can be identified with 3-form or other-form fluxes occurring in the theories [17]. On the other hand, many kinds of torsion connections have been studied for many years (e.g., see [18]). In particular in Sasakian geometry, the torsion connection which preserves the Sasakian structure has been studied for a long time [19–22]. It is known that such a torsion is totally skew-symmetric and is written in terms of the contact 1-form: $T = \eta \wedge d\eta$. The uniqueness of the torsion was proven in [21]. However, since this kind of torsion connection does not deform the Sasakian structure, we need to explore other possibilities.

In this paper, we propose one possible deformation of the Sasakian structure in the presence of totally skew-symmetric torsion. The idea is the following: a Sasakian manifold is defined as a manifold whose metric cone in one higher dimension is Kähler. Analogously, we demand that the cone one dimension up be Kähler with torsion (KT). On a Kähler with a torsion manifold, there exists a unique torsion connection preserving the Hermitian structure, called a Bismut connection [23], and the presence of the torsion deforms the Kählerian structure. Thus, the Sasakian structure one dimension down is also deformed. To our knowledge, this attempt
to deform the Sasakian geometry has not been previously conceived and, we believe, also differs from both the ‘generalized Sasakian geometry’ discussed in [15, 16] and the Sasakian geometry with torsion connection studied in [21, 22]. We thus study the general properties of the deformed Sasakian structure.

We have another motivation to study such manifolds with torsion. It has been clarified by many authors [7–11] that a certain scaling limit of higher dimensional vacuum rotating black hole solutions [24–26] leads to toric Kähler metrics in even dimensions and Sasaki in odd ones. We thus expect a similar scaling limit for charged, rotating black hole solutions of various supergravity theories: this leads to metrics on manifolds with torsion. For example, it was demonstrated [27] that in Abelian heterotic supergravity, the Kerr–Sen black hole solutions [28–30] give rise to KT metrics. It can be also shown that the five-dimensional gauged supergravity black hole solution discovered in [31] gives rise to a metric with the deformed Sasakian structure, as we will see in section 4.

Sasakian geometry is relevant to Killing–Yano (KY) symmetry, as exemplified by KY tensors [32] and conformal Killing–Yano (CKY) tensors [33–35]. It was shown in [19, 20] that a Sasakian manifold of $2n + 1$ dimensions has rank-$(2p + 1)$ special Killing forms in the form $\eta \wedge (d\eta)^p$ ($0 \leq p \leq n$). In our case, although the Sasakian structure is deformed by torsions, the deformed Sasakian structure admits generalized special Killing forms $\eta \wedge (d^T \eta)^p$. KY symmetry has also played an important role in the study of black hole physics. One of the features is that general metrics admitting a rank-2 closed CKY tensor were obtained in four [36, 37] and higher [38–44] dimensions. Such metrics allow remarkable properties in mathematical physics, in particular separations of variables for the Hamilton–Jacobi, Klein–Gordon and Dirac equations. In this paper, we present an example of the deformed Sasakian metrics explicitly. We see that, for example, there exists a generalized Killing–Yano (GKY) symmetry providing separability of the Hamilton–Jacobi equation for geodesics.

This paper is organized as follows. In section 2, we begin with a brief review of torsion connections. After we define a notion of Sasaki with a torsion structure in the presence of totally skew-symmetric torsion (see definition 2.1), we look into general properties of the deformed Sasakian structure while clarifying differences from the standard Sasakian structure and introducing some new notions (see definition 2.5). In section 3, we present an example of local metrics admitting the deformed Sasakian structure introduced in section 2 in all odd dimensions, and elaborate on curvature properties with respect to the metrics and the cone metrics in one higher dimension. Hidden symmetry for the metrics is also discussed in this section. In section 4, the solutions of five-dimensional minimal (un)gauged supergravity and 11-dimensional supergravity are obtained. In section 5, we discuss the global structure of these solutions briefly. The condition to obtain regular metrics on compact manifolds is argued in the context of five-dimensional minimal gauged supergravity solutions. We study more on the global properties of five-dimensional solutions in the special case. In this case, the metric has enhanced isometry and can be regarded as the generalization of $Y^{p,q}$. Section 6 is devoted to summary and discussions. In appendix A, we give some calculations which are relevant to the notions introduced in section 2. In appendix B, the Riemann, Ricci and scalar curvatures for our example of the metrics are computed. We obtain them with respect to not only the Levi-Civita connection but also to the connection with the torsion. In appendix C, Calabi–Yau with torsion metrics on the cone is obtained.

2. Deformation of Sasakian structure

In the context of supergravity theories, it seems to be natural to introduce a totally skew-symmetric torsion because it can be identified with 3-form fields occurring in the theories
The Sasakian structure in the presence of torsion has been previously considered. Friedrich and Ivanov [21, 22] have used connections with totally skew-symmetric torsion preserving the Sasakian structure, which are uniquely determined by the contact 1-form η as $T = η \wedge dη$. On the other hand, what we expect now is that the presence of torsion no longer preserves the Sasakian structure because of the effect of the energy–momentum tensor which changes Einstein’s equation. We thus discuss one possible deformation of the Sasakian structure in the presence of totally skew-symmetric torsion.

Let $(M, g)$ be a Riemannian manifold, $T$ be a 3-form on $M$ and $\{e_a\}$ be an orthonormal frame on $TM$. A connection with totally skew-symmetric torsion $∇^T$ is defined by

$$g(∇^T_1 Y, Z) = g(∇^T_2 Y, Z) + \frac{1}{2} T(X, Y, Z),$$  

(1)

for any vector fields $X$, $Y$, and $Z$, where $∇^g$ is the Levi-Civita connection of $g$. The connection satisfies a metricity condition, $∇^g g = 0$, and has the same geodesics as $∇^g$. $∇^T_γ Y = ∇^T_γ γ = 0$ for a geodesic $γ$. The commutation relations are linked to the Lie brackets by

$$∇_X Y - ∇_Y X = [X, Y] + T(X, Y),$$  

(2)

where $T(X, Y, Z) = g(T(X, Y), Z)$. For a $p$-form $Ψ$, the covariant derivative is calculated as

$$∇^T_Χ Ψ = ∇^g_Χ Ψ - \frac{1}{2} \sum_a (X.\lhd e_a \rhd T) \wedge (e_a \lhd Ψ),$$  

(3)

where $\lhd$ represents the inner product. Then, we have

$$d^TΨ = \sum_a e^a \wedge ∇^T_{e_a}Ψ = dΨ - \sum_a (e_a \lhd T) \wedge (e_a \lhd Ψ),$$  

(4)

$$\delta^TΨ = - \sum_a e_{a\lhd} \cdot ∇^T_{e_a}Ψ = \deltaΨ - \frac{1}{2} \sum_{a,b} (e_{a\lhd} \cdot e_{b\lhd} \cdot T) \wedge (e_{a\lhd} \cdot e_{b\lhd} \cdot Ψ),$$  

(5)

where $\{e^a\}$ is the dual 1-form of $\{e_a\}$, $e_{a\lhd} \cdot e^b = δ^b_a$.

Suppose $(M, g, J)$ is a Hermitian manifold equipped with a complex structure $J$ and a Hermitian metric $g$ obeying $g(X, Y) = g(J(X), J(Y))$ for any vector field $X$ and $Y$. Then, it is known that there exists a unique Hermitian connection $∇^B$ with totally skew-symmetric torsion $B$, i.e., $∇^B g = 0$, $∇^B J = 0$. This connection $∇^B$ is known as a Bismut connection and the corresponding totally skew-symmetric torsion $B$ is called a Bismut torsion [23], which is written in the form

$$B(X, Y, Z) = dΩ(J(X), J(Y), J(Z)),$$  

(6)

where $Ω$ is the fundamental 2-form $Ω(X, Y) = g(J(X), Y)$. A Hermitian manifold $(M, g, J)$ equipped with the Bismut torsion $B$ is called a Kähler with the torsion manifold.

A Riemannian manifold $(M, g)$ is said to be Sasakian if its metric cone $(C(M), \tilde{g}) = (M \times R_+, \tilde{g} = dr^2 + r^2 g)$ is Kähler and its Sasakian structure is derived from the Kähler cone structure (see, e.g., reviews [45–47] and references therein). In analogy with this, we generalize the Sasakian structure to the case when torsion is present as follows:

**Definition 2.1.** Let $(M, g)$ be a Riemannian manifold and $T$ be a 3-form on $M$. Then, we call $(M, g, T)$ a Sasaki with torsion (ST) manifold if its metric cone $(C(M), \tilde{g})$ is a KT manifold whose Bismut torsion $B$ is given by $B = r^2 T$.

Propositions 2.2 and 2.3 provide three equivalent characterizations of the ST structure.
Proposition 2.2. Let \((M, g)\) be a Riemannian manifold and \(\nabla^T\) be a connection with skew-symmetric torsion \(T\). Then, the following conditions are equivalent:

(a) There exists a Killing vector field \(\xi\) of unit length on \(M\) so that the dual 1-form \(\eta\) satisfies
\[
\nabla^T_X (d^T \eta) = -2X^\circ \wedge \eta
\]
for any vector field \(X\), where \(X^\circ = g(X, -)\).
(b) There exists a Killing vector field \(\xi\) of unit length on \(M\) so that the tensor field \(\Phi\) of type \((1,1)\) defined by \(\Phi(X) = \nabla^T_X \xi\) satisfies
\[
(\nabla^T_X \Phi)(Y) = g(\xi, Y)X - g(X, Y)\xi
\]
for any pair of vector fields \(X\) and \(Y\).
(c) There exists a Killing vector field \(\xi\) of unit length on \(M\) so that the curvature satisfies
\[
R^T(X, Y)\xi = g(\xi, Y)X - g(\xi, X)Y + \Phi(T(X, Y))
\]
for any pair of vector fields \(X\) and \(Y\), where the curvature \(R^T(X, Y)\) is defined by
\[
R^T(X, Y)Z = \nabla^T_X \nabla^T_Y Z - \nabla^T_Y \nabla^T_X Z - \nabla^T_{[X,Y]} Z.
\]

Proof. (a) \iff (b). If \(\xi\) is a Killing vector field, the dual 1-form \(\eta\) satisfies
\[
\nabla^T_X \eta = \frac{1}{2} Y^\circ \cdot d^T \eta
\]
for any connection with totally skew-symmetric torsion \(\nabla^T\). Since \(\eta(Y) = g(\xi, Y)\), this is also written as
\[
g(\nabla^T_X \xi, Z) = \frac{1}{2} (d^T \eta)(Y, Z)
\]
for all vector fields \(Y\) and \(Z\). Thus, taking the covariant derivative of (12), we have
\[
g(\nabla^T_X \nabla^T_Y \xi, Z) = \frac{1}{2} \left( (\nabla^T_Y d^T \eta)(Y, Z) + \frac{1}{2} (d^T \eta)(\nabla^T_X Y, Z) \right) = \frac{1}{2} \left( (\nabla^T_X d^T \eta)(Y, Z) + g(\nabla^T_{XY} \xi, Z) \right).
\]
On the other hand, the covariant derivative of the equation \(\nabla^T_X \xi = \Phi(Y)\) yields
\[
g(\nabla^T_X \nabla^T_Y \xi, Z) = g((\nabla^T_X \Phi)(Y), Z) + g(\nabla^T_Y \xi, Z).
\]
By comparing (13) and (14), it follows that
\[
g((\nabla^T_X d^T \eta)(Y, Z) + g((\nabla^T_X \Phi)(Y), Z).
\]
which gives the equivalence of conditions (a) and (b).

(b) \implies (c). It is noted from (2) and (14) that
\[
R^T(X, Y)\xi = \nabla^T_X \nabla^T_Y \xi - \nabla^T_Y \nabla^T_X \xi - \nabla^T_{[X,Y]} \xi = (\nabla^T_X \Phi)(Y) - (\nabla^T_Y \Phi)(X) + \nabla^T_{[X,Y]} \xi.
\]
Since \(\nabla^T_{[X,Y]} \xi = \Phi(T(X, Y))\) by definition, it is easy to find that condition (b) leads to condition (c).

(b) \iff (c). Using (9) and (16), we have
\[
(\nabla^T_X \Phi)(Y) - (\nabla^T_Y \Phi)(X) = g(\xi, Y)X - g(\xi, X)Y,
\]
and also from (15), we obtain
\[
g((\nabla^T_X \Phi)(Y), Z) + g((\nabla^T_X \Phi)(Z), Y) = 0.
\]
Combining these two equations, we obtain (8).
Proposition 2.3. \( (M, g, T) \) is an ST manifold if and only if there exists a Killing vector field \( \xi \) of unit length satisfying one of the conditions given in proposition 2.2 and the torsion \( T \) obeys
\[ T(X, Y, Z) = T(X, \Phi(Y), \Phi(Z)) + T(\Phi(X), Y, \Phi(Z)) + T(\Phi(X), \Phi(Y), Z). \]
(19)

Proof. We first derive condition (b) in proposition 2.2 from the definition of the ST manifold, and later show the torsion condition (19) using the integrability of the complex structure of the metric cone. Let \( (M, g, T) \) be an ST manifold, \( X \) and \( Y \) be vector fields on \( M \), which can be also viewed as vector fields on the metric cone \( C(M) \) and \( \bar{\nabla}^B \) be the Bismut connection associated with \( C(M) \). Then, we have the following formulas:
\[
\bar{\nabla}^B_\alpha \partial_r = 0, \quad \bar{\nabla}^B_X = \bar{\nabla}^B_{\partial_r} = \frac{1}{r} X,
\]
(20)
where \( \bar{\nabla} \) is the connection on \( M \) with totally skew-symmetric torsion \( T \). Making use of the complex structure \( J \) on \( C(M) \), we define a vector field \( \xi \) on \( C(M) \) by
\[
\xi = J(\partial_r),
\]
(21)
whose length is given by \( \bar{g}(\xi, \xi) = r^2 \). Since \( \bar{\nabla}^B_X J = 0 \), we have
\[
\bar{g}(\bar{\nabla}^B_X \xi, Y) = \bar{g}(J(\bar{\nabla}^B_X (r \partial_r)), Y) = \bar{g}(J(X), Y),
\]
(22)
which is anti-symmetric under exchange of \( X \) and \( Y \). Identifying \( M \) with \( M \times \{1\} \subset C(M) \) leads us to the fact that \( \xi \) is a Killing vector field of unit length on \( M \). Let us define a tensor field \( \Phi \) of type \((1,1)\) by
\[
\Phi(X) = J(X) - \bar{g}(J(X), \partial_r) \partial_r = J(X) + r \eta(X) \partial_r,
\]
(23)
where \( \eta \) is the dual 1-form of the Killing vector field \( \xi, \eta(X) = g(\xi, X) \). Then, (8) in condition (b) follows from the covariant derivative of (23) and \( \bar{\nabla}^B J = 0 \). In fact, by virtue of formulas (20), we obtain
\[
\bar{\nabla}^B_X (\Phi(Y)) = \nabla_X^T (\Phi(Y)) - r g(X, \Phi(Y)) \partial_r = (\nabla_X^T \Phi)(Y) + \Phi(\nabla_X^T Y) - r g(X, \Phi(Y)) \partial_r.
\]
(24)
Hence, the covariant derivative of \( J(Y) \) is calculated as
\[
\bar{\nabla}^B_X (J(Y)) = \bar{\nabla}^B_X (\Phi(Y) - r \eta(X) \partial_r) = (\nabla_X^T \Phi)(Y) + \Phi(\nabla_X^T Y) - \eta(\nabla_X^T Y) - g(\xi, Y) X.
\]
(25)
On the other hand, we see
\[
J(\bar{\nabla}^B_X Y) = J(\nabla_X^T Y - r g(X, Y) \partial_r) = \Phi(\nabla_X^T Y) - \eta(\nabla_X^T Y) - g(X, Y) \xi.
\]
(26)
Since \( \bar{\nabla}^B J = 0 \), we have \( \bar{\nabla}^B_X (J(Y)) = J(\bar{\nabla}^B_X (Y)) \) and hence equating (25) and (26) shows (8).

Note that \( \Phi(\xi) = 0 \) and \( \Phi^2(X) = -X + \eta(X) \xi \) by the definition (23). Then, (8) implies that
\[
X - g(X, \xi) \xi = (\nabla_X^T \Phi)(\xi) = -\Phi(\nabla_X^T \xi).
\]
(27)
Since \( \Phi^2(\nabla_X^T \xi) = -\nabla_X^T \xi, \) the above equation yields
\[
\Phi(X) = \nabla_X^T \xi.
\]
(28)
Thus, we have obtained condition (b) in proposition 2.2. The torsion condition (19) is derived from the integrability of the complex structure \( J \) on \( C(M) \). As well known, the vanishing of
After a simple computation, the vanishing of \( NJ \) is a necessary and sufficient condition for the integrability, so we use \( NJ = 0 \). From \( \bar{\nabla}^g J = 0 \) and

\[
[X, Y] = \bar{\nabla}^g_X Y - \bar{\nabla}^g_Y X - T(X, Y), \quad [X, \partial_r] = 0,
\]
the Nijenhuis tensor is computed as

\[
NJ(X, Y) = [J(X), J(Y)] - [X, J(Y)] - J([X, Y]) - J(J(X), Y)
\]

\[
= -T(\Phi(X), \Phi(Y)) + T(X, Y) + J(T(X, \Phi(Y)) + J(T(\Phi(X), Y))
\]

and

\[
NJ(X, r\partial_r) = -T(\Phi(X), \xi) + J(T(X, \xi)).
\]

After a simple computation, the vanishing of \( NJ (NJ = 0) \) derives the condition (19).

Conversely, we can construct a KT structure on \( C(M) \) by using condition (a) in proposition 2.2 as follows. For the 1-form \( \eta \), we introduce a 2-form \( \Omega \) on \( C(M) \):

\[
\Omega = r \, dr \wedge \eta + \frac{r^2}{2} \, d^T \eta = \frac{1}{2} \, d^T (r^2 \eta).
\]

Then, the covariant derivative of \( \Omega \) in a radial direction always vanishes, while the derivative in the direction of a vector field \( X \) on \( M \) yields

\[
\bar{\nabla}^g_X \Omega = r^2 (X^\eta \wedge \eta + \frac{1}{2} \nabla^g_X d^T \eta) + r \, dr \wedge (\nabla^g_X \eta - \frac{1}{2} X \wedge d^T \eta).
\]

Condition (a) implies the vanishing of the two brackets (for the second bracket, see (11)), i.e., \( \bar{\nabla}^g \Omega = 0 \). Let us define an almost complex structure \( J \) on \( C(M) \) by

\[
J(X) = \Phi(X) - r \eta(X) \partial_r, \quad J(r \partial_r) = \xi.
\]

It is easy to see that \( g(J(X), J(Y)) = \bar{g}(\hat{X}, \hat{Y}) \) and \( \Omega(\hat{X}, \hat{Y}) = \bar{g}(J(\hat{X}), \hat{Y}) \) for all vector fields \( \hat{X}, \hat{Y} \) on \( C(M) \). Note that we have \( \bar{\nabla}^g J = 0 \) by \( \bar{\nabla}^g \bar{g} = 0 \) and \( \bar{\nabla}^g \Omega = 0 \). For \( C(M) \) to be KT, it is sufficient to show that the almost complex structure \( J \) is integrable. It follows immediately from (30) and (31) together with the torsion condition (19).

As a consequence from propositions 2.2 and 2.3, we obtain the following relations among \( \xi, \eta \) and \( \Phi \).

**Proposition 2.4.** Let \( (M, g, T) \) be an ST manifold and \( (\xi, \eta, \Phi) \) be a triple of its ST structure on \( M \) given in proposition 2.2. Then, we have

\[
\eta(\xi) = 1,
\]

\[
\Phi(\Phi(X)) = -X + \eta(X) \xi,
\]

\[
g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X) \eta(Y),
\]

\[
\Phi(\xi) = 0, \quad \eta(\Phi(X)) = 0,
\]

\[
NJ(X, Y) + d\eta(X, Y) \xi = 0,
\]

\[
d^T \eta = 2\omega, \quad \xi \wedge d\omega = 0,
\]

where \( \omega \) is the fundamental 2-form defined by \( \omega(X, Y) = g(\Phi(X), Y) \), and \( NJ \) is the Nijenhuis tensor of type \( (1, 2) \) with respect to \( \Phi \) defined by

\[
NJ(X, Y) = \left[ \Phi(X), \Phi(Y) \right] + \Phi(\Phi([X, Y])) - \Phi([\Phi(X), Y]) - \Phi([\Phi(X), Y]).
\]
A Riemannian manifold \((M, g)\) equipped with a structure \((\xi, \eta, \Phi)\) satisfying (35)–(37) is known as an almost contact metric manifold. Equation (38) is derived from such a structure, especially (35) and (36). An almost contact metric manifold \((M, g, \xi, \eta, \Phi)\) is called normal if it satisfies (39) and a contact metric manifold if it satisfies \(d\eta = 2\omega\), respectively (e.g., see [45, 46]). A Sasakian manifold is known as a normal contact metric manifold. On the other hand, the ST manifold is a normal almost contact metric manifold as the contact metric structure is deformed by the presence of torsion as seen in (40).

**Definition 2.5.** Let \((M, g)\) be a Riemannian manifold. We call an almost contact metric structure \((g, \xi, \eta, \Phi)\) satisfying \(\Delta^T \eta = 2\omega\) and \(\xi \cdot d\omega = 0\) together with a 3-form \(\omega\) satisfying (19) a T-contact metric structure, and call \((M, g, \xi, \eta, \Phi, T)\) a T-contact metric manifold. We further call a T-contact metric manifold a TK-contact metric manifold if \(\xi\) is a Killing vector field.

An almost Cauchy–Riemann (CR) structure, which is a subbundle \(E\) of \(TM\) with an almost complex structure \(J\), is said to be integrable if for any sections \(X, Y\) of \(E\) the vector field \([JX, Y] + [X, JY]\) is a section of \(E\) and the Nijenhuis tensor of \(J\) vanishes. The subbundle \(D = \ker \eta \subset TM\) has an almost complex structure defined by \(J_D = \Phi|_D\). Hence, \(D\) together with the endomorphism \(J_D\) provides \(M\) with an almost CR structure of codimension 1. The normality condition yields that the almost CR structure \((D, J_D)\) is integrable.

**Proposition 2.6.** An ST manifold is a normal T-contact metric manifold whose torsion \(T_D = T|_D\) is given by a Bismut torsion

\[ T_D(X, Y, Z) = d\omega(J_D(X), J_D(Y), J_D(Z)) \]  

for all \(X, Y, Z \in D\).

**Proof.** Let \((M, g, T)\) be an ST manifold. Since we find from proposition 2.4 that \(M\) is a normal \(T\)-contact metric manifold, we have \(N^{(i)} = 0\) \((i = 1, 2)\), where \(N^{(i)}\) are defined by (A.2) and (A.3). Then, (A.1) reduces to

\[ 2g(\nabla_X \Phi)(Y, Z) = -d\omega(X, \Phi(Y), \Phi(Z)) + d\omega(X, Y, Z) + M(X, Y, Z) \]

\[ + \Delta^T \eta(X, \Phi(Z))\eta(Y) - \Delta^T \eta(X, \Phi(Y))\eta(Z), \]  

Using (8) and

\[ d^T \eta(X, \Phi(Z))\eta(Y) - d^T \eta(X, \Phi(Y))\eta(Z) = 2\omega(X, \Phi(Z))\eta(Y) - 2\omega(X, \Phi(Y))\eta(Z) \]

\[ = 2g(X, Z)g(\xi, Y) - 2g(X, Y)g(\xi, Z), \]  

we obtain

\[ d\omega(X, \Phi(Y), \Phi(Z)) - d\omega(X, Y, Z) - M(X, Y, Z) = 0. \]  

From (40) and (A.5), it holds trivially if we take \(X = \xi, Y = \xi\) or \(Z = \xi\). Otherwise, (45) is equivalent to (42) for \(X, Y, Z \in D\).

Conversely, the normality condition \(N^{(1)} = 0\) leads to \(\mathcal{L}_\xi \Phi = 0\) (see [46]), so that \(\xi\) is a Killing vector field (see (A.9)). Following the same calculation as (43)–(45) inversely, we obtain condition (b) in proposition 2.2.

Since an almost contact metric structure is normal if and only if the almost CR structure is integrable and \(\mathcal{L}_\xi \Phi = 0\) (see [46]), we are able to restate proposition 2.6 in the following proposition.

**Proposition 2.7.** An ST manifold is a TK-contact metric manifold whose almost CR structure is integrable and torsion \(T_D = T|_D\) is given by a Bismut torsion.
Let us close this section by mentioning about some other properties of the ST manifolds. A \( p \)-form \( \phi \) is called a \textit{special Killing \( p \)-form with torsion} if it satisfies for any vector field \( X \)

\[
\nabla^T_X \phi = \frac{1}{p+1} X \wedge d^T \phi, \quad \nabla^T_X (d^T \phi) = k X \wedge \phi
\]  

(46)

with a constant \( k \). For \( \phi = \eta \), the first equation implies that its dual vector field \( \xi \) is a Killing vector field. Hence, the 1-form \( \eta \) in proposition 2.2 is a special Killing 1-form with torsion. Furthermore, it can be shown \([27]\) that the \((2\ell + 1)\)-forms

\[
\eta^{(\ell)} = \eta \wedge (d^T \eta)^\ell
\]

(47)

for \( \ell = 0, \ldots, n \), are also special Killing forms with torsion. For a special Killing \( p \)-form with torsion \( \phi \) on \( M \):

\[
\hat{\phi} = r^p \, dr \wedge \phi + \frac{r^{p+1}}{p+1} \, d^p \phi
\]

(48)

is a parallel \((p + 1)\)-form on \( C(M) \), i.e., \( \tilde{\nabla}^B \hat{\phi} = 0 \) (see [35]). In particular, for \( p = 1 \), the 1-form \( \eta \) on an ST manifold provides a parallel 2-form \( \Omega \) on \( C(M) \), which is precisely a fundamental 2-form on \( C(M) \) (cf (32)).

It is known that the Ricci tensor of a Sasakian manifold of dimension \( 2n + 1 \) is given by \( \text{Ric}(X, \xi) = 2n \eta(X) \). In the ST manifold case, the Ricci curvature follows from (9) that

\[
\text{Ric}^T (X, \xi) = - \sum_a g(R^T (X, e_a) \xi, e_a)
\]

\[
= 2n \eta(X) - \sum_a T(X, e_a, \Phi(e_a)).
\]

(49)

3. ST metrics

It would be useful to give some examples of the ST manifolds explicitly, as many examples of the Sasakian manifolds have been used for tests of AdS/CFT correspondence. In what follows, we shall discuss a concrete example of the ST metric which possesses the general properties of the ST structure we have already seen in section 2. The metric contains some unknown functions of single variable, which are determined by equations of motion of supergravity theories in section 4 and further restricted by regularity conditions in section 5. We proceed the calculation in this section while keeping the single variable functions unknown. In section 3.1, we give a physical motivation to consider our example especially in supergravity theories. In section 3.2, we confirm that the cone metric of our example is KT and then give the relation between the torsion of the ST and the Bismut torsion of the cone. In our case, it is also found that the metric possesses KY symmetry which is described by a generalized closed conformal Killing–Yano (GCCKY) 3-form. To our knowledge, it is the first example of the metric admitting such a 3-form. Therefore, we investigate in section 3.3 some properties of the ST metric from the viewpoint of KY symmetry.

3.1. Local metrics in all odd dimensions

It has been realized [7–9, 11] that the well-known examples of the toric Sasakian manifolds such as \( Y^{p,q} \) and \( L^{a,b,c} \), originally constructed by [4–6], can be obtained by taking the BPS limit of the Euclidean vacuum rotating black hole solutions in five and higher dimensions. The general Sasakian metric in \( 2n + 1 \) dimensions is locally written as an \( S^1 \)-bundle over \( 2n \)-dimensional Kähler space \((B, g_B)\):

\[
g = g_B + 4(d \psi_0 + A)^2.
\]

(50)
Since a lot of charged black hole solutions of the equations of motion in supergravity theories have been discovered, it naturally motivates us to ask what happens when we start with charged black holes in supergravity theories.

We shall explicitly present an example of local metrics admitting the deformed Sasakian structure introduced in section 2, which we call ST metrics. The ST metric in $2n+1$ dimensions we found is given in local coordinates $(x^\mu, \psi^k)$, where $\mu = 1, \ldots, n$ and $k = 0, \ldots, n$, by

$$
g = \sum_{\mu=1}^{n} \frac{dx^\mu}{Q^\mu} + \sum_{\mu=1}^{n} Q_\mu \left( \sum_{k=1}^{n} \sigma^{(k-1)} \mu d\psi_k \right)^2 + 4 \left( \sum_{k=0}^{n} \sigma^{(k)} d\psi_k + A \right)^2.
$$

where

$$
A = \sum_{\mu=1}^{n} N_\mu \sum_{k=1}^{n} \sigma^{(k-1)} \mu d\psi_k,
Q_\mu = \frac{X_\mu}{U^\mu},
U^\mu = \prod_{v=1, v \neq \mu}^{n} (x^\mu - x^v)
$$

and $\sigma^{(k)} \mu$ and $\sigma^{(k)} \psi$ are the $k$th elementary symmetric polynomials in $x^\mu$ generated by

$$
\prod_{v=1, v \neq \mu}^{n} (\lambda + x^v) = \sum_{k=0}^{n-1} \sigma^{(k)} \mu \lambda^{n-k-1},
\prod_{v=1}^{n} (\lambda + x^v) = \sum_{k=0}^{n} \sigma^{(k)} \lambda^{n-k}.
$$

The metric contains $2n$ unknown functions $X_\mu(x^\mu)$ and $N_\mu(x^\mu)$ depending only on single variable $x^\mu$. Although the unknown functions are determined by the equations of motion of various supergravity theories and the regularity of the metric as we will see in sections 4 and 5, we proceed the calculation keeping them arbitrary in this section. It is known [7–11] that the metric (51) with $A = 0$ is obtained as an ‘off-shell’ metric of the BPS limit of the odd-dimensional Kerr–NUT–(A)dS metric and leads to the toric Sasaki–Einstein metrics $Y_p$ and $L_{a,b,c}$ discovered by [4–6]. According to proposition 2.6 (or 2.7), it implies that the metric $g_B$ on $2n$-dimensional base space $(B, g_B)$ is locally KT. The present metric is known as an orthotoric Kähler metric established in [48, 49].

For later calculation, it is convenient to introduce an orthonormal frame $\{e^\mu\} = \{e^\mu, e_\hat{\mu} = \sqrt{Q^\mu}, e^0 = 2\left( \sum_{k=0}^{n} \sigma^{(k)} d\psi_k + A \right)\}$. We choose an orthonormal frame for the metric (51) as

$$
e^\mu = \frac{dx^\mu}{\sqrt{Q^\mu}} \quad e_\hat{\mu} = \sqrt{Q^\mu} \sum_{k=1}^{n} \sigma^{(k-1)} \mu d\psi_k,
\quad e^0 = 2 \left( \sum_{k=0}^{n} \sigma^{(k)} d\psi_k + A \right).
$$

From the first structure equation

$$
de^\mu + \sum_b \omega^{a}_{\mu b} \wedge e^b = 0
$$

(55)
and $\omega_{ab} = -\omega_{ba}$, we compute the connection 1-forms $\omega^a_b$ as follows:

$$\omega^a_v = -\frac{\sqrt{Q_v}}{2(x_\mu - x_v)} e^\mu - \frac{\sqrt{Q_v}}{2(x_\mu - x_v)} e^v, \quad (\mu \neq v)$$

$$\omega^a_\mu = -\partial_\mu \sqrt{Q_\mu} e^\mu + \sum_{v \neq \mu} \frac{\sqrt{Q_v}}{2(x_\mu - x_v)} e^v - (1 + \partial_\mu H) e^0,$$

$$\omega^a_\mu = \frac{\sqrt{Q_\mu}}{2(x_\mu - x_v)} e^\mu - \frac{\sqrt{Q_v}}{2(x_\mu - x_v)} e^v, \quad (\mu \neq v)$$

$$\omega^a_0 = -(1 + \partial_\mu H) e^0,$$

$$\omega^a_0 = (1 + \partial_\mu H) e^\mu,$$

where $H$ is defined by

$$H = \sum_{\mu=1}^n \frac{N_\mu}{U_\mu}. \quad (57)$$

Firstly, we shall see the conditions in proposition 2.4. We introduce a 1-form $\eta$, vector field $\xi$ and endomorphism $\Phi$ as

$$\eta = e^0, \quad \xi = e_0,$$

$$\Phi(e_\mu) = e_\mu, \quad \Phi(e_0) = 0. \quad (58)$$

For the triple $(\xi, \eta, \Phi)$ together with the metric $g$, conditions (35)–(37) in proposition 2.4 hold clearly, so that $(g, \xi, \eta, \Phi)$ is an almost contact metric structure. Using $e_{\alpha} \nabla e_{\beta} e^{\beta} = -\omega^{\alpha}_{\beta}(e_{\beta})$, we compute the covariant derivatives with respect to the Levi-Civita connection $\nabla$ as (B.1) in appendix B, and its commutation relations $[e_{\alpha}, e_{\beta}]$ are obtained. From the obtained commutation relations, we are able to confirm condition (39), which means that the almost contact metric structure is normal. However, $\eta$ is not in general a contact 1-form because we have

$$d\eta = 2 \sum_{\mu=1}^n (1 + \partial_\mu H) e^\mu \wedge e^0,$$

and hence there is a possibility that $\eta \wedge (d\eta)^n = 0$ at some points. If $H$ is constant, we have $d\eta = 2\omega$, where $\omega$ is the fundamental form, so that $\eta$ is a contact 1-form. It is also found that the present metric is a quasi-Sasakian metric [50], whose fundamental form satisfies $d\omega = 0$.

In fact, we have

$$\omega = \sum_{\mu=1}^n e^\mu \wedge e^0 = d \left[ \sum_{k=0}^n \psi_k \right]. \quad (60)$$

Next, let us see the conditions in proposition 2.3. We introduce the torsion $T$ and compute the covariant derivatives with respect to the torsion connection $\nabla_T$. Since the torsion $T$ satisfying (40) is given by

$$T = 2 \sum_{\mu=1}^n \partial_\mu H e^\mu \wedge e^0, \quad (61)$$
we can check that (19) holds. We emphasize again that the torsion (61) differs from the torsion preserving the Sasakian structure, $\eta \wedge d\eta$, discussed in [21]. Namely $\nabla^T \xi \neq 0$, $\nabla^T \eta \neq 0$ and $\nabla^T \Phi \neq 0$. The covariant derivatives with respect to $\nabla^T$ are calculated as (B.9) in appendix B. Using these expressions, we find that

$$\nabla^T \xi = \Phi(X).$$

(62)

It is also shown that for any vector field $X$,

$$\nabla^T X \xi \neq 0, \quad \nabla^T X \eta \neq 0 \quad \text{and} \quad \nabla^T /\Phi X \neq 0.$$  

The covariant derivatives with respect to $\nabla^T$ are calculated as (B.9) in appendix B. Using these expressions, we find that

$$\nabla^T X \xi = /\Phi 1(X).$$  

(62)

It is also shown that for any vector field $X$,

$$\nabla^T X \eta = 1/2 X \wedge \eta, \quad \nabla^T X (d^T \eta) = -2 X \wedge \eta,$$

(63)

which proves (46) with $k = -2$ so that $\eta$ is a special Killing 1-form with torsion.

3.2. The cone metric

Going back to the definition (2.1), we confirm that the Riemannian cone metric of our example is KT. For later calculation, we introduce an orthonormal frame $\{\bar{e}^\alpha\}$ ($\alpha = r, 1, \ldots, n$)

$$\bar{e}^r = dr, \quad \bar{e}^a = re^a$$

(64)

with respect to the cone metric $\bar{g} = dr^2 + r^2 g$

(65)

where $g$ is given by (51).

The connection 1-forms $\bar{\omega}^{a}_{\beta}$ with respect to $\bar{g}$ are calculated as

$$\bar{\omega}^{r}_{\alpha} = -1/r \bar{e}^\alpha, \quad \bar{\omega}^{a}_{b} = \omega^{a}_{b},$$

(66)

where $\omega^{a}_{b}$ is given by (55), and the commutation relations $[\bar{e}^a, \bar{e}^\beta]$ are calculated in the similar manner to previous section. We introduce an almost complex structure $J$

$$J(\bar{e}^r) = \bar{e}^0, \quad J(\bar{e}^0) = -\bar{e}^r, \quad J(\bar{e}^\mu) = \bar{e}^\mu, \quad J(\bar{e}^\hat{\mu}) = -\bar{e}_{\mu}.$$  

(67)

Then, it is directly checked that for the almost complex structure $J$, the Nijenhuis tensor vanishes so that $J$ is integrable, and the cone metric $\bar{g}$ is Hermitian:

$$\bar{g}(X, Y) = \bar{g}(J(X), J(Y)).$$

(68)

The fundamental form $\Omega(X, Y) = \bar{g}(J(X), Y)$ can be written as

$$\Omega = \bar{e}^r \wedge e^0 + \sum_{\mu=1}^n \bar{e}^\mu \wedge \bar{e}^\hat{\mu} = 1/2 d^T (r^2 e^0).$$

(69)

Since $(M, g, J)$ is a Hermitian manifold, there exists the Bismut connection, a unique Hermitian connection $\bar{\nabla}^B$ with totally skew-symmetric torsion $B$. From (6), the Bismut torsion is explicitly obtained as

$$B = \sum_{\mu=1}^n 2/P \partial_\mu H \bar{e}^\mu \wedge \bar{e}^\hat{\mu} \wedge \bar{e}^0 = r^2 T,$$

(70)

where $T$ is given by (61). We finally note that the Killing vector fields $\partial/\partial \psi_k (k = 0, 1, \ldots, n)$ preserve the KT structure on the cone:

$$L_{\psi_k} \Omega = 0, \quad L_{\psi_k} B = 0.$$  

(71)

12
3.3. Hidden symmetry

It is known that a GKY symmetry in the presence of totally skew-symmetric torsion, which were introduced in [51], appears for the black hole solutions of the five-dimensional minimal gauged supergravity [51] and of the Abelian heterotic supergravity in four [28], five [29] and higher dimensions [30]. Moreover, it has been realized later that the GKY symmetry is related to Kähler manifolds established in [48, 49] and toric Sasakian manifolds which are obtained as the BPS limit of Euclideanized higher dimensional black hole spacetimes [27, 55, 57, 58]. As we have seen in previous sections, the ST metric (51) can be regarded as a natural generalization of Sasakian metrics in the presence of torsion. Since the ordinary Sasakian metric obtained from vacuum black holes admits the GKY symmetry, it is natural to expect that the ST metric (51) also admits the GKY symmetry.

A generalized conformal Killing–Yano (GCKY) tensor $k$ was introduced in [51] as a p-form satisfying for any vector field $X$ and a totally skew-symmetric torsion $\mathbf{T}$:

$$\nabla^T_X k = \frac{1}{p+1} X \cdot d^T k - \frac{1}{D-p+1} X^\flat \wedge \delta^T k,$$

(72)

where $X^\flat$ is the dual 1-form of $X$. In particular, a GCKY tensor $f$ obeying $\delta^T f = 0$ is called a GKY tensor, and a GCKY tensor $h$ obeying $d^T h = 0$ is called a GCCKY tensor. From general properties [52–54], any GKY tensors $f$ of rank-$(p-1)$ always provide rank-2 Killing tensors $K$ obeying

$$K_{ab} = f_{ac_1} e_{c_2} \cdots e_{c_{p-1}}$$

(73)

When a Hamilton–Jacobi equation for geodesics can be solved by the separation of variables, the separation constants $\kappa^{(i)}$ are given as the eigenvalues of rank-2 Killing tensors $K^{(i)}$, $\kappa^{(i)} = K^{(i)ab} p_ap_b$. Hence, the separability of Hamilton–Jacobi equations for geodesics provides rank-2 Killing tensors. On the other hand, not all the rank-2 Killing tensors can be decomposed into the square of KY tensors as (73). Nevertheless, it is easy to demonstrate that for the metric (51), the Hamilton–Jacobi equation for geodesics separates, and we obtain rank-2 Killing tensors. Therefore, it is an interesting problem to investigate whether the Killing tensors are given by GKY tensors or not.

To explore the GKY tensors for the metric (51), we have to determine a torsion connection first. The natural torsion is the 3-form $\mathbf{T}$ related to the ST structure, given by (61). Since the first equation in (46) is the same as the GKY equation, a special Killing $p$-form with torsion is alternatively said to be a rank-$p$ special GKY tensor. As already seen in (47), $\eta^{(\ell)} \equiv \eta \wedge (\delta^T \eta)^\ell$ for $\ell = 0, \ldots, n$ are rank-$(2\ell + 1)$ special GKY tensors with respect to torsion $\mathbf{T}$. Thus, we have $n+1$ GKY tensors. However, these GKY tensors $\eta^{(\ell)}$ do not give rise to non-trivial rank-2 Killing tensors. In fact, every GKY tensor generates the only metric essentially.

Introducing another torsion, we find other GKY tensors $f^{(j)}$ for the metric (51), which are not special GKY. We introduce a 2-form $\hat{h}$ and 3-form $G$ as

$$\hat{h} = \sum_{\mu=1}^{n} \sqrt{x_\mu} e^\mu \wedge e^0,$$

(74)

$$G = \sum_{\mu \neq 0} \frac{1}{\sqrt{x_\mu} + \sqrt{x_0}} \sqrt{Q_{x_0}} e^\mu \wedge e^0 \wedge e^0 + \sum_{\mu=1}^{n} 2 (1 + \partial_\mu H) e^\mu \wedge e^0 \wedge e^0.$$

(75)

Then, it is demonstrated that for the metric (51), the $(2j+1)$-forms

$$h^{(\ell)} \equiv e^0 \wedge (\hat{h})^\ell = e^0 \wedge \hat{h} \wedge \cdots \wedge \hat{h}$$

(76)
for \( j = 1, \ldots, n \), are rank-\((2j + 1)\) GCCKY tensors with respect to torsion \( G \), obeying for any vector field \( X \)

\[
\nabla^G_X h^{(j)} = -\frac{1}{D - 2j} X^\alpha \delta^{G h^{(j)}}.
\]

From general properties of GCKY tensors (e.g., see [52], GCCKY tensors \( h^{(j)} \) generate GKY tensors \( f^{(j)} \) by \( f^{(j)} = \ast h^{(j)} \). These GKY tensors \( f^{(j)} \) generate rank-2 Killing tensors \( K^{(j)} \) by

\[
K^{(j)}_{ab} = \sum_{\mu=1}^n \sigma^{(j)}_{\mu} \left( e^\mu \otimes e^a + e^a \otimes e^\mu \right).
\]

4. Supergravity solutions

In the context of supergravity theories, it seems to be natural to introduce a totally skew-symmetric torsion because it can be identified with 3-form fields occurring in the theories [17, 18]. In this section, we investigate Euclidean solutions of two particular supergravity theories: the five-dimensional gauged minimal supergravity in section 4.1 and the 11-dimensional supergravity in section 4.2. As mentioned before, there is a correspondence between Kerr–dS black holes and toric Sasakian manifolds, which can be seen through a Wick rotation and a certain scaling limit. Analogously, it is expected to obtain an Euclidean solution corresponding to the charged Kerr–dS black hole solution [31] in the five-dimensional gauged minimal supergravity. By making use of the canonical form (51) for the ST metric in section 3, we attempt to solve equations of motion of the theory. Similar to five dimensions, we also explore a Euclidean solution of the 11-dimensional supergravity under the same ansatz because it is suggested that there are a lot of similarities between the 5- and 11-dimensional supergravities. Since any charged, rotating black hole solution is not known, if exists, the Euclidean solution might give us a clue for finding new black hole solution in 11-dimensional supergravity.

4.1. Five-dimensional minimal gauged supergravity

The five-dimensional minimal gauged supergravity is given by the (Lorentzian) action

\[
S_5 = \int (* (\mathcal{R} - \Lambda) - \frac{1}{2} F_{(2)} \wedge * F_{(2)} + \frac{1}{3 \sqrt{3}} F_{(2)} \wedge F_{(2)} \wedge A_{(1)}),
\]

where \( F_{(2)} = d A_{(1)} \) is a 2-form field strength of a Maxwell field \( A_{(1)} \), \( \mathcal{R} \) is the Ricci curvature of a gravitational field \( g_5 \) and \( \Lambda \) is the cosmological constant. The equations of motion are given by

\[
R_{ab} = -4 g_{ab} + \frac{1}{2} \left( F_{(2)cde} F_{(2)}^{cde} - \frac{1}{6} g_{ab} F_{(2)cde} F_{(2)}^{cde} \right),
\]

\[
d * F_{(2)} - \frac{1}{\sqrt{3}} F_{(2)} \wedge F_{(2)} = 0,
\]

where the cosmological constant has been normalized as \( \Lambda = -12 \).

It should be noted here that for Euclidean solutions, we must consider the Euclidean action which is obtained by the Wick rotation. Since it corresponds to change the sign of the whole right-hand side of (80), the cosmological constant can be interpreted as positive. The Wick rotation we take transforms the only fiber direction from spacelike into timelike, so as
to satisfy the original Einstein equation (80), and does not break the reality of the matter flux. Therefore, we investigate the Einstein equation for the Euclidean signature:

\[ R_{ab} = 4g_{ab} - \frac{1}{2} (F_{(2)ac}F_{(2)b}^c - \frac{1}{8}g_{ab}F_{(2)cd}F^{(2)}_{(2)}) . \] (82)

As for the gauge potential \( A_{(1)} \) and the functions \( N_{\mu} \), we assume the following form so as to solve the Maxwell–Chern–Simons equation (81):

\[ A_{(1)} = c_F \sum_{\mu=1}^{2} \frac{q_{\mu}}{U_{\mu}} \sum_{k=1}^{2} \sigma_{\mu}^{(k-1)} \, d\psi_k , \] (83)

\[ N_{\mu} = a_1 x_{\mu} + q_{\mu} , \] (84)

with constant parameters \( c_F \), \( a_1 \) and \( q_{\mu} \). Since \( a_1 \) is a gauge parameter, we set \( a_1 = 0 \). In the form, the field strength is given by

\[ F_{(2)} = c_F (\partial_1 H \, e^1 \wedge e^1 + \partial_2 H \, e^2 \wedge e^2) , \] (85)

where \( H \) is, as before, given by (57). This immediately shows that \( \partial_1 H = - \partial_2 H \) and hence

\[ \ast F_{(2)} = - F_{(2)} \wedge \eta , \] (86)

where \( \eta \) is the contact 1-form and \( \omega \) is the fundamental 2-form given by (60). Thus, the Maxwell equation (81) can be solved easily as

\[ d \ast F_{(2)} = - F_{(2)} \wedge d\eta = - \frac{2}{c_F} F_{(2)} \wedge F_{(2)} , \] (87)

where the constant \( c_F \) is determined as \( c_F = -2\sqrt{3} \). The Einstein equation (82) requires that \( X_{\mu}(x_{\mu}) \) takes the form

\[ X_{\mu} = - 4x_{\mu}^3 + \sum_{i=1}^{2} c_i x_{\mu}^i + b_{\mu} - 8q_{\mu} x_{\mu} , \] (88)

where \( c_i \), \( b_{\mu} \) and \( q_{\mu} \) are constants.

Finally, let us comment a solution of the five-dimensional ungauged minimal supergravity. We obtain an ungauged minimal supergravity solution in the similar way. The solution is provided when (51) and (83) take the form

\[ X_{\mu} = \sum_{i=1}^{2} c_i x_{\mu}^i + b_{\mu} , \quad N_{\mu} = - x_{\mu}^2 + a_1 x_{\mu} + q_{\mu} . \] (89)

The solutions can be changed into the Lorentzian signature as in the case of the gauged supergravity solutions. In the ungauged case, the Wick rotation changes only the metric in the form

\[ g_L = \sum_{\mu=1}^{2} \frac{dx_{\mu}^2}{Q_{\mu}} + \sum_{\mu=1}^{2} Q_{\mu} \left( \sum_{k=1}^{2} \sigma_{\mu}^{(k-1)} \, d\psi_k \right)^2 - 4 \left( \sum_{k=0}^{2} \sigma^{(k)} \, d\psi_k + A \right)^2 . \] (90)

The gauged minimal supergravity solutions need to correct \( X_{\mu} \) as

\[ X_{\mu} = 4x_{\mu}^3 + \sum_{i=1}^{2} c_i x_{\mu}^i + b_{\mu} + 8q_{\mu} x_{\mu} . \] (91)

This arises from the negativity of the cosmological constant. In both cases, the vector potential remains the form as (83).
4.2. Eleven-dimensional supergravity

We consider the 11-dimensional supergravity. The action is given by
\[ L_{11} = \ast R - \tfrac{1}{2} F(4) \wedge \ast F(4) + \tfrac{1}{12} F(4) \wedge F(4) \wedge A(3) \] (92)
where \( F(4) = dA(3) \) is a 4-form flux of a 3-form gauge potential \( A(3) \). The equations of motion are
\[ R_{ab} = \tfrac{1}{12} (F(4)_{acde} F(4)_{b}^{\ cde} - \tfrac{1}{12} g_{ab} F(4)_{abcd} F(4)^{abcd}) \] (93)
\[ d \ast F(4) - \tfrac{1}{2} F(4) \wedge F(4) = 0. \] (94)

As the five-dimensional case, we examine the Euclidean solutions satisfying the Einstein equation which are obtained by changing the sign of the right-hand side in (93).

We assume that the field strength \( F(4) \) takes the form
\[ F(4) = \sqrt{Q} \sum_{\mu \neq \nu} F_{\mu \nu} e^{\mu} \wedge e^{\hat{\mu}} \wedge e^{\nu} \wedge e^{\hat{\nu}}, \] (95)
where
\[ F_{\mu \nu} = 2 \ell_1 + \ell_2 (\partial_\mu H + \partial_\nu H), \] (96)
and \( H \) is still given by (57) and \( \ell_1, \ell_2 \) are constant. Under this assumption, the field strength becomes closed, \( dF(4) = 0 \) and the co-derivative is given by
\[ \delta F(4) = - \sum_{\mu \neq \nu} \sqrt{Q} \left( \partial_\mu F_{\mu \nu} + \sum_{\rho \neq \mu, \nu} \frac{F_{\mu \nu} - F_{\rho \nu}}{x_\mu - x_\rho} \right) e^{\hat{\mu}} \wedge e^{\nu} \wedge e^{\hat{\nu}} \left( 1 + \partial_\mu H \right) e^{\nu} \wedge e^{\hat{\nu}} \wedge e^{\mu}. \] (97)

Substituting expressions (95) and (97) into (94), we obtain \( \ell_1 = \ell_2 = -2 \) and
\[ N_{\mu} = -x_\mu^2 + \sum_{i=1}^{4} a_i x_\mu^i + q_\mu. \] (98)

Then, we have
\[ F(4) = -2 \sum_{\mu \neq \nu} \left( 1 + \partial_\mu H \right) e^{\mu} \wedge e^{\hat{\mu}} \wedge e^{\nu} \wedge e^{\hat{\nu}}. \] (99)

The Einstein equation (93) reduces to
\[ \partial_\nu^2 Q_T - 4 \sum_{\nu \neq \mu} K_{\mu \nu} = 0, \] (100)
where
\[ K_{\mu \nu} \equiv -\frac{1}{4} \frac{\partial_\mu Q_T}{x_\mu - x_\nu} + \frac{1}{4} \frac{\partial_\nu Q_T}{x_\mu - x_\nu}, \quad Q_T \equiv \sum_{\mu=1}^{n} Q_\mu. \] (101)

This equation can be solved by
\[ X_\mu = -\sum_{i=1}^{5} c_i x_\mu^i + b_\mu \] (102)
with free parameters \( c_i \) and \( b_\mu \).
5. Global analysis

5.1. Compact manifolds in five dimensions

In this section, we discuss the global structure of the five-dimensional minimal gauged supergravity solution obtained in section 4 and construct regular metrics on compact manifolds.

5.1.1. Generalization of $L^{a,b,c}$. The metric is written in the form

$$g_5 = \frac{x-y}{X} \, dx^2 + \frac{y-x}{Y} \, dy^2 + \frac{X}{x-y} (d\psi_1 + y \, d\psi_2)^2 + \frac{Y}{y-x} (d\psi_1 + x \, d\psi_2)^2$$

$$+ 4 \left( d\psi_0 + (x+y) \, d\psi_1 + xy \, d\psi_2 + \frac{q_1 - q_2}{x-y} \, d\psi_1 + \frac{q_1 y - q_2 x}{x-y} \, d\psi_2 \right)^2,$$

(103)

where

$$X = -4x(x - \alpha_1)(x - \alpha_2) + b_1 - 8q_1 x,$$

(104)

$$Y = -4y(y - \alpha_1)(y - \alpha_2) + b_2 - 8q_2 y,$$

and $\alpha_i$ ($i = 1, 2$), $b_\mu$ and $q_\mu$ ($\mu = 1, 2$) are free parameters. However, not all the parameters are non-trivial. There is a scaling symmetry of the metric, under which we take

$$x_\mu \rightarrow \lambda x_\mu, \quad \lambda \rightarrow 1,$$

$$\psi_k \rightarrow \lambda^2 \psi_k, \quad \alpha_i \rightarrow \lambda \alpha_i, \quad b_\mu \rightarrow \lambda^3 b_\mu, \quad q_\mu \rightarrow \lambda^2 q_\mu.$$

(105)

The metric also has a shift symmetry which is taken by

$$x_\mu \rightarrow x_\mu + \lambda, \quad \psi_0 \rightarrow \psi_0 - \lambda^2 \psi_2, \quad \psi_1 \rightarrow \psi_1 - \lambda \psi_2,$$

$$\alpha_1 + \alpha_2 \rightarrow \alpha_1 + \alpha_2 - 3\lambda,$$

$$\alpha_1 \alpha_2 + 2q_\mu \rightarrow \alpha_1 \alpha_2 + 2q_\mu - 2(\alpha_1 + \alpha_2)\lambda + 3\lambda^2,$$

$$b_\mu \rightarrow b_\mu - 4(\alpha_1 \alpha_2 + 2q_\mu)\lambda + 4\lambda^2 - 4\lambda^3.$$

(106)

In order to obtain regular metrics on compact manifolds, we must impose appropriate regions of the coordinates. This corresponds to making an appropriate choice of the parameters. Suppose that $x_i$ and $y_j$ ($i = 1, 2, 3$) are real roots of the equations $X(x) = 0$ and $Y(y) = 0$ and satisfy the inequalities $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$. If we choose the region of the coordinates as $x_1 \leq x \leq x_2 < y_2 < y_3$, then the metric is positive definite, except for the boundaries $x = x_1$ and $x_2$ as well as $y = y_2$ and $y_3$. From the relationship between the coefficients and solutions, we have

$$\alpha_1 + \alpha_2 = x_1 + x_2 + x_3 = y_1 + y_2 + y_3,$$

$$\alpha_1 \alpha_2 + 2q_1 = x_1 x_2 + x_1 x_3 + x_2 x_3,$$

$$\alpha_1 \alpha_2 + 2q_2 = y_1 y_2 + y_1 y_3 + y_2 y_3,$$

$$b_1 = 4x_1 x_2 x_3, \quad b_2 = 4y_1 y_2 y_3.$$

(107)

Following [7, 8], we can extend the metric smoothly onto the boundaries. Since $\partial/\partial \psi_0$, $\partial/\partial \psi_1$ and $\partial/\partial \psi_2$ are linearly independent Killing vector fields, the general Killing vector field is written as

$$v = \sum_{k=0}^{2} \omega_k \frac{\partial}{\partial \psi_k},$$

(108)
where \( \omega_k \) are constants. The length of \( v \) is given by
\[
v^2 = \frac{X(x)}{x-y} (\omega_1 + \omega_2)^2 + \frac{Y(y)}{y-x} (\omega_1 + \omega_2)^2
+ 4 \left( \omega_0 (x+y) \omega_1 + xy \omega_2 + \frac{q_1 q_2}{x-y} \omega_1 + \frac{q_1 y - q_2 x}{x-y} \omega_2 \right)^2.
\] (109)

Using this expression, we construct the associated normalized Killing vector fields \( v_i \) \( (i = 1, 2) \) and \( \ell_j \) \( (j = 2, 3) \) such that their lengths are vanishing at the corresponding boundaries \( x = x_i \) and \( y = y_j \). Namely we have
\[
v_i = \frac{2}{X'(x_i)} \left( (q_1 + x_i^2) \frac{\partial}{\partial \psi_0} - x_i \frac{\partial}{\partial \psi_1} + \frac{\partial}{\partial \psi_2} \right),
\]
\[
\ell_j = \frac{2}{Y'(y_j)} \left( (q_2 + y_j^2) \frac{\partial}{\partial \psi_0} - y_j \frac{\partial}{\partial \psi_1} + \frac{\partial}{\partial \psi_2} \right),
\]
where their normalizations are taken so that the surface gravity is equal to unity:
\[
g^{ab} \left( \partial_a v^2 \right) \left( \partial_b v^2 \right) \mid_{x=x_i} = 1.
\] (111)

The metric extends smoothly onto the boundaries if the Killing vector fields \( v_i \) and \( \ell_j \) have the period \( 2\pi \).

Since we have four vector fields \( v_i \) and \( \ell_j \), they must satisfy a linear relation
\[
n_1 v_1 + n_2 v_2 + m_1 \ell_2 + m_2 \ell_3 = 0
\] (112)
for integral coefficients \((n_1, n_2, m_1, m_2)\), which are assumed to be coprime. To avoid conical singularities, any three of the integers must be also coprime. Substituting (110) into (112), it can be solved as
\[
\frac{n_1}{(x_3 - x_1)(q + (x_2 - y_2)(x_2 - y_3))} = \frac{n_2}{(x_3 - x_2)(q + (x_1 - y_2)(x_1 - y_3))}
= \frac{m_1}{(y_2 - y_1)(q - (x_1 - y_3)(x_2 - y_3))}
= \frac{m_2}{(y_3 - y_1)(q - (x_1 - y_2)(x_2 - y_2))},
\] (113)
where
\[
q = q_1 - q_2 = \frac{x_1 x_2 + x_1 x_3 + x_2 x_3 - y_1 y_2 - y_1 y_3 - y_2 y_3}{2}.
\] (114)

Since we have degrees of freedom under the scaling symmetry (105) and the shift symmetry (106), the value of (113) can be set to 1 and we can take \( b_2 = 0 \) without loss of generality. Then, we have \( y_1 = 0 \) and (113) leads to
\[
n_1 = (x_3 - x_1)(q + (x_2 - y_2)(x_2 - y_3)),
\] (115)
\[
n_2 = (x_3 - x_2)(q + (x_1 - y_2)(x_1 - y_3)),
\] (116)
\[
m_1 = y_2(q - (x_1 - y_3)(x_2 - y_3)),
\] (117)
\[
m_2 = y_3(q - (x_1 - y_2)(x_2 - y_2)),
\] (118)
where
\[
q = \frac{x_1 x_2 + x_1 x_3 + x_2 x_3 - y_2 y_3}{2}.
\] (119)

Thus, the problem of constructing regular metrics on compact manifolds results in solving four coupled algebraic equations (115)–(118) for a set of coprime integers \((n_1, n_2, m_1, m_2)\),
supergravity solution (103), we have discussed the global metrics on compact manifolds (see table 1).

\[
\begin{array}{cccccccccc}
 n_1 & n_2 & m_1 & m_2 & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & q \\
-4 & -3 & -1 & -2 & -1.32023 & 1.25127 & 3.11486 & 0.167499 & 0.375858 & -3.21042 & \\
-4 & -2 & -1 & -3 & -1.27727 & 1.4007 & 3.17205 & 0.155888 & 0.59382 & -3.15254 & \\
-4 & -1 & -2 & -3 & -1.25653 & 1.04966 & 3.20068 & 0.329791 & 0.564966 & -3.12434 & \\
-4 & 1 & -2 & -3 & -1.18938 & -0.78852 & 3.0232 & 0.372652 & 0.672647 & -2.6462 & \\
-4 & 2 & -1 & -3 & -1.11468 & -0.543466 & 2.82041 & 0.188869 & 0.9734 & -2.12735 & \\
-4 & 3 & -1 & -2 & -1.11385 & -0.202506 & 2.46249 & 0.263875 & 0.882254 & -1.62438 & \\
-3 & -2 & -1 & -4 & -1.15876 & -1.09101 & 3.24621 & 0.142997 & 0.853444 & -3.08052 & \\
-3 & -1 & -2 & -4 & -1.14654 & -1.00989 & 3.26472 & 0.302705 & 0.805588 & -3.06305 & \\
-3 & 1 & -2 & -4 & -1.05629 & -0.741916 & 3.11358 & 0.3276 & 0.987783 & -2.56939 & \\
-3 & 2 & -1 & -4 & -0.899549 & -0.461233 & 3.04032 & 0.146529 & 1.53301 & -1.97347 & \\
-2 & -1 & -3 & -4 & -1.06256 & -0.9939 & 3.29155 & 0.483516 & 0.751555 & -3.0381 & \\
-2 & 1 & -3 & -4 & -0.969754 & -0.731764 & 3.13442 & 0.536657 & 0.896244 & -2.55231 & \\
-1 & 2 & -3 & -4 & -0.761789 & -0.47932 & 2.987 & 0.578562 & 1.16732 & -2.00871 & \\
-1 & 4 & -3 & -2 & -0.358631 & 0.309102 & 2.83766 & 0.50795 & 2.28018 & -0.704807 & \\
1 & 4 & -3 & -2 & 0.167966 & 1.36545 & 2.52543 & 1.90031 & 2.15854 & 0 & \\
2 & 1 & -3 & -4 & 1.58023 & 2.19861 & 2.46249 & 2.66499 & 3.57634 & 1.62438 & \\
2 & 1 & 3 & 4 & 2.739 & 2.94736 & 3.11486 & 4.36613 & 4.43508 & 3.12042 & \\
2 & 3 & -4 & -1 & 0.36689 & 0.625118 & 2.52543 & 1.15997 & 2.35746 & 0 & \\
2 & 3 & 4 & 1 & 0.557479 & 2.32971 & 2.83766 & 2.52855 & 3.19629 & 0.704807 & \\
3 & 1 & -2 & 4 & 1.84701 & 2.63154 & 2.82041 & 3.36388 & 3.93509 & 2.12735 & \\
3 & 1 & 2 & 4 & 2.57323 & 3.01616 & 3.17205 & 4.31211 & 4.44931 & 3.15254 & \\
3 & 2 & -1 & 4 & 2.35055 & 2.65055 & 3.0232 & 3.81172 & 4.21258 & 2.6462 & \\
3 & 2 & 1 & 4 & 2.63598 & 2.87089 & 3.20068 & 4.25033 & 4.45721 & 3.12434 & \\
4 & 1 & -2 & 3 & 1.50731 & 2.8938 & 3.04032 & 3.50156 & 3.93987 & 1.97347 & \\
4 & 1 & 2 & 3 & 2.39276 & 3.10321 & 3.24621 & 4.33722 & 4.40497 & 3.08052 & \\
4 & 2 & -1 & 3 & 2.1258 & 2.78598 & 3.11358 & 3.8555 & 4.16987 & 2.56939 & \\
4 & 2 & 1 & 3 & 2.45913 & 2.96201 & 3.26472 & 4.27446 & 4.41126 & 3.06305 & \\
4 & 3 & -2 & 1 & 1.81967 & 2.40843 & 2.987 & 3.46631 & 3.74878 & 2.00871 & \\
4 & 3 & 1 & 2 & 2.23817 & 2.59776 & 3.13442 & 3.86618 & 4.10417 & 2.55231 & \\
4 & 3 & 1 & 2 & 2.53997 & 2.80801 & 3.29155 & 4.28543 & 4.35409 & 3.0381 & \\
\end{array}
\]

together with the conditions for the real roots \(x_1 \) and \(y_1 \), \(x_1 = y_3 + x_3 - x_1 - x_2, x_1 < x_2 < x_3, 0 < y_2 < y_3 \) and \(x_2 < y_2 \). In particular, when we take \(q = 0 \), (115)–(118) give rise to the condition discussed in [5, 7, 8]:

\[
n_1 + n_2 + m_1 + m_2 = 0,
\]

which leads to the toric Sasaki–Einstein metrics \(L^{n_1,n_2,m_1} \) on \(S^2 \times S^3 \). When \(q \) is non-zero, the present metric is parameterized by independent four integers, which we denote by \(L^{n_1,n_2,m_1,m_2} \) (see table 1).

5.1.2. Generalization of \(Y^{p,q} \). Making use of the five-dimensional minimal gauged supergravity solution (103), we have discussed the global metrics on compact manifolds \(M_3 \) and it has been seen that they can be regarded as a generalization of \(L^{a,b,c} \). In the special case, we can also and this time rather precisely discuss the global properties of the metrics which can be regarded as a generalization of \(Y^{p,q} \) [4]. Taking a certain limit of the solution (103), we obtain the metric locally given by

\[
g = (\xi - x)(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dx^2}{Q(x)} + Q(x)(d\psi_1 + \cos \theta d\phi)^2 + 4\left(d\psi_0 + \left(x + \frac{q}{x - \xi}\right) d\psi_1 + \left(x - \xi + \frac{q}{x - \xi}\right) \cos \theta d\phi\right)^2,
\]

(121)
where
\[
Q(x) = \frac{4x^3 + (1 - 12\xi)x^2 + (8q - 2\xi + 12\xi^2)x + k}{\xi - x}
\]  
(122)
and \(q, \xi\) and \(k\) are free parameters. The metric is again an ST metric and satisfies the equations of motion of five-dimensional minimal gauged supergravity with the Maxwell potential:
\[
A_{(1)} = -\frac{2\sqrt{3}q}{x - \xi} (d\psi_1 + \cos \theta \, d\phi).
\]  
(123)
The torsion 3-form is given by \(T = *F_{2\ell}/\sqrt{3}\).

Following [4, 6], we study global properties of the metric (121). Before starting the analysis, we perform the following coordinate transformation:
\[
\psi_1 = -\psi + \alpha, \quad \psi_0 = \xi \psi.
\]  
(124)
Then, the metric is
\[
g_5 = (\xi - x)(d\theta^2 + \sin^2 \theta \, d\phi^2) + \frac{dx^2}{Q(x)} + \frac{4\xi^2 Q(x)}{F(x)} (d\psi - \cos \theta \, d\phi)^2
\]
\[
+ F(x)(d\alpha - f(x)(d\psi - \cos \theta \, d\phi))^2,
\]  
(125)
where
\[
F(x) = Q(x) + 4 \left( x + \frac{q}{x - \xi} \right)^2,
\]  
(126)
\[
f(x) = \frac{Q(x) + 4 (x + \frac{q}{x - \xi}) (x - \xi + \frac{q}{x - \xi})}{F(x)}.
\]  
(127)
It should be noted that when \(q = 0\) and \(\xi = 1/6\), the metric is the local form of the Sasaki–Einstein metric \(Y^{p,q}\). In addition, we obtain the homogeneous Sasaki–Einstein metric \(T^{1,1}\) if we take the coordinate transformation \(x = c/6y\) and send \(c \to 0\). We also find that when \(k = -4\xi^3 + \xi^2 - 8q\xi\), the function \(Q(x)\) degenerates to a polynomial of degree 2 and we have \(Q = -4x^3 + (8\xi - 1)x - 4\xi^2 + \xi - 8q\). Then, the metric is the standard \(S^3\) metric when \(q = 0\). Otherwise, \(Q(x)\) is a rational function and henceforth we will focus on the case.

The metric \(g_5\) is positive definite when there exist three distinct real roots \(x_1, x_2, x_3\) of the equation \(Q(x) = 0\) such that
\[
x_1 < x_2 < x_3, \quad x_2 < \xi,
\]  
(128)
and the coordinate \(x\) takes the range \(x_1 \leq x \leq x_2\). Although we will show later that the five-dimensional space \((M_5, g_5)\) is an \(S^1\)-bundle over four-dimensional space \(B\) given by the metric
\[
g_B = (\xi - x)(d\theta^2 + \sin^2 \theta \, d\phi^2) + \frac{dx^2}{Q(x)} + \frac{4\xi^2 Q(x)}{F(x)} (d\psi - \cos \theta \, d\phi)^2,
\]  
(129)
we shall see first that \(g_B\) can extend globally on \(S^2\)-bundle over \(S^3\). Fixing the coordinates \((\theta, \phi)\) and introducing a new coordinate \(r = 2|(x - x_i)/Q'(x_i)|^{1/2}\), we can evaluate the behavior near \(x = x_i\) of the fiber metric as
\[
dr^2 + \left( \frac{\xi(x_i - \xi)Q'(x_i)}{2(x_i(x_i - \xi) + q)} \right)^2 r^2 d\psi^2.
\]  
(130)
Hence, avoiding conical singularities at \(x = x_i\) requires both the condition
\[
\frac{\xi(x_i - \xi)Q'(x_i)}{x_i(x_i - \xi) + q} = \pm n
\]  
(131)
and the range of $\psi$ given by $0 \leq \psi \leq 4\pi/n$ with a constant $n \neq 0$. Equation (131) is explicitly written as
\begin{equation}
(12\xi - q_{i} \xi_{i})^{2} + \xi (2 - 24\xi + n_{i}), x_{i} + 2\xi (4q - \xi + 6\xi^{2}) - qn_{i} = 0, \quad (i = 1, 2), 
\end{equation}
where $n_{i}$ take $\pm n$, respectively. Thus, two of the three parameters $q$, $k$ and $\xi$ are fixed by the regular condition (132). Since the Chern number is calculated as 
\begin{equation}
S_{i}(132). \text{ Since the Chern number is calculated as}
\end{equation}
\begin{equation}
Then, the roots of the function $Q(x)$, $x_{1}$, $x_{2}$ and $x_{3}$ are given by
\begin{equation}x_{1,2} = \frac{2\xi + n\xi - 24\xi^{2} \pm \sqrt{(n - 2)(-n + 5n + 10n - 48\xi^{2})}}{2(n - 12\xi)}, \end{equation}
\begin{equation}x_{3} = \frac{-n + 4\xi + 8n\xi - 48\xi^{2}}{4(n - 12\xi)}, \end{equation}
where the choice of the sign in (137) depends on the sign of $n - 12\xi$. The reality condition of $x_{1}$ and $x_{2}$ and the inequalities (128) require the following ranges of $\xi$ for each integer $n$:
\begin{itemize}
\item [(a)] $n \geq 4$, \quad $\xi_{1} < \xi < \frac{n}{4(n + 1)}$, \quad (139)
\item [(b)] $n = 1$, \quad $\frac{15 - \sqrt{33}}{96} < \xi < \frac{1}{8}$, \quad (140)
\item [(c)] $n \leq -1$, \quad $\frac{n}{8} < \xi < \xi_{1} \quad \text{or} \quad \xi_{2} < \xi < \xi_{3}$, \quad (141)
\end{itemize}
where the quantities $\xi_{1}$, $\xi_{2}$ and $\xi_{3}$ are defined by
\begin{equation}
\xi_{1} = \frac{1}{96}(10 + 5n - \sqrt{100 - 92n + 25n^{2}}), \quad (142)
\end{equation}
\begin{equation}
\xi_{2} = \frac{1}{96}(10 + 5n + \sqrt{100 - 92n + 25n^{2}}), \quad (143)
\end{equation}
\begin{equation}
\xi_{3} = \frac{1}{48}(5 + n + \sqrt{25 - 14n + n^{2}}), \quad (144)
\end{equation}
the four-dimensional space $B$ is a trivial bundle $S^{2} \times S^{2}$ for an even integer $n$ and a twisted $S^{2}$-bundle for an odd integer $n$, respectively. For simplicity, we deal with the case $n_{1} = n_{2} = n$. We note that (132) becomes trivial when $q = 0, \xi = 1/6$ and $n = 2$, which reproduces the Sasaki–Einstein metric $Y^{p,q}$. In the case $\xi \neq n/12$ nor $n/16$, we obtain more general solutions of (132).
\begin{equation}q = \frac{(2 - n)\xi (-n + 4\xi + 4n\xi)}{4(n - 16\xi)(n - 12\xi)}, \quad (134)
\end{equation}
\begin{equation}k = \frac{\xi (-n + 4\xi + 8n\xi - 48\xi^{2})L(\xi)}{4(n - 16\xi)(n - 12\xi)^{3}}, \quad (135)
\end{equation}
where
\begin{equation}L(\xi) = 2n^{2} - n^{3} + 4(n^{3} - n^{2} - 6n)\xi + 16(n^{2} + 16n + 4)\xi^{2} - 192(7n + 8)\xi^{3} + 9216\xi^{4}. \quad (136)
\end{equation}
The regular condition for the five-dimensional metric $g_5$ gives rise to a further constraint, under which we must choose the period of the fiber direction $\alpha$ in (125) so as to describe a principal $S^1$-bundle over $B$. Since the connection 1-form is given by
\[
A = f(x)(d\psi - \cos \theta \, d\phi),
\]
the periods $P_i (i = 1, 2)$ are calculated as [6]
\[
P_1 = \frac{1}{2\pi} \int_{C_i} dA = \frac{2}{n} (f(x_2) - f(x_1)),
\]
\[
P_2 = \frac{1}{2\pi} \int_{C_i} dA = 2f(x_2),
\]
where $n$ is the Chern number given by (133). $C_1$ and $C_2$ represent the basis for $H_2(B, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Note that the cycle $C_1$ is the $S^2$-fiber of $B$ at some fixed point $(\theta, \phi)$ on the base space, while $C_2$ is the sub-manifold $S^2$ of $B$ at $x = x_2$, where the length of $\partial / \partial \psi$ vanishes. Now we require
\[
\frac{f(x_1)}{f(x_2)} = \frac{\ell}{m}
\]
where $\ell, m \in \mathbb{Z}$. Then, $\kappa^{-1} \frac{dA}{2\pi}$ has integral periods if we set $\kappa = 2hf(x_2)/(nm)$ with $h = \gcd(\ell - m, nn)$. Thus, we take the range $0 \leq \alpha \leq 2\pi \kappa$. A numerical calculation shows that our solution (134) admits the parameter $\xi$ satisfying condition (148), and hence the five-dimensional space $M_5$ becomes an $S^1$-bundle over $B$ parameterized by three integers $\ell, m$ and $n$.

It is straightforward to verify that the following four Killing vectors
\[
v_1 = \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi}, \quad v_2 = -\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi},
\]
\[
\ell_1 = \frac{2}{n} \left( \frac{\partial}{\partial \psi} + f(x_1) \frac{\partial}{\partial \alpha} \right), \quad \ell_2 = \frac{2}{n} \left( \frac{\partial}{\partial \psi} + f(x_2) \frac{\partial}{\partial \alpha} \right)
\]
vanish with the surface gravity 1 on the sub-manifolds given by $\theta = 0, \theta = \pi, x = x_1$ and $x = x_2$, respectively, and they have a linear relation
\[
(N_1 - N_2)(v_1 + v_2) + nN_2 \ell_1 - nN_1 \ell_2 = 0
\]
with $N_1 = n\ell / h \in \mathbb{Z}, N_2 = nm / h \in \mathbb{Z}$ (cf (112)).

The volume is given by
\[
\text{Vol}(M_5) = \pi^3 \left| \frac{32\kappa \kappa (x_2 - x_1)(2\xi - x_1 - x_2)}{n} \right|
\]
Moreover, since $B$ is a simply connected manifold, it follows that $M_5$ is also simply connected. Note also that $M_5$ is a spin manifold [6]. Smale’s theorem states that any simply connected compact 5-manifold which is spin and has no torsion in the second homology group is diffeomorphic to $S^2 \sharp k(S^2 \times S^3)$ for some non-negative integer $k$. Thus, together with the analysis in [4, appendix A], we see that $M_5$ is topologically $S^2 \times S^3$.

5.2. Non-compact manifolds in 11 dimensions

Next, we turn to discussing the global structure of the 11-dimensional supergravity solution. We assume that the functions $X_{n}(x_{i})$ take the form
\[
X(x) \equiv X_1(x_1) = c(x - \alpha)P(x),
\]
\[
Y_k(y_k) \equiv X_{k+1}(x_{k+1}) = c \prod_{i=1}^{5} (y_k - \beta_i), \quad k = 1, \ldots, 4,
\]
where \( P(x) \) is a positive-definite polynomial of degree 4 and \( a, c, \beta_i \) are real constants satisfying
\[
c > 0, \quad \beta_1 < \beta_2 < \cdots < \beta_5 < a.
\] (153)

Then, we choose the region of the coordinates \( x, y_i \) as
\[
\beta_1 \leq y_1 \leq \beta_2 \leq \cdots \leq y_4 \leq \beta_5 < a < x < \infty.
\] (154)

The fact that the region of \( x \) is infinite corresponds to the non-compactness of manifold. Then, the metric is positive definite except for the boundaries \( y_k = \beta_k, y_k = \beta_{k+1} \) and \( x = a \). From appendix B, we see that the curvature is finite at the points \( y_k = y_{k+1} = \beta_k \). Some calculations analogous to the five-dimensional case yield that the following vector fields are Killing vector fields vanishing at the boundaries \( x = a, y_k = \beta_k \) and \( y_k = \beta_{k+1} \) \((k = 1, 2, 3, 4)\), respectively:

\[
v_0 = \frac{2}{X^2(a)} \left( (N_1(a) + a^2) \frac{\partial}{\partial \psi_0} + \sum_{\ell=1}^{5} (1)^{\ell} \xi a^{5-\ell} \frac{\partial}{\partial \psi_\ell} \right),
\]

\[
v_k = \frac{2}{Y_k^2(\beta_k)} \left( (N_{k+1}(\beta_k) + \beta_k^2) \frac{\partial}{\partial \psi_0} + \sum_{\ell=1}^{5} (1)^{\ell} \beta_k^{5-\ell} \frac{\partial}{\partial \psi_\ell} \right),
\]

\[
w_k = \frac{2}{Y_k^2(\beta_{k+1})} \left( (N_{k+1}(\beta_{k+1}) + \beta_{k+1}^2) \frac{\partial}{\partial \psi_0} + \sum_{\ell=1}^{5} (1)^{\ell} \beta_{k+1}^{5-\ell} \frac{\partial}{\partial \psi_\ell} \right).
\] (155)

These Killing vector fields have a unit surface gravity. If we impose the condition \( q_2 = q_3 = q_4 = q_5 \), then we have \( N_2 = N_3 = N_4 = N_5 \), which implies the relation \( v_k = w_{k-1} \) \((k = 2, 3, 4)\). Hence, we can use them as the new Killing coordinates \( \phi_i \) with the period \( 2\pi \) representing the canonical coordinate of torus \( T^6 \):

\[
\frac{\partial}{\partial \phi_0} = v_0, \quad \frac{\partial}{\partial \phi_1} = v_1, \quad \frac{\partial}{\partial \phi_k} = v_k = w_{k-1} \quad (k = 2, 3, 4), \quad \frac{\partial}{\partial \phi_5} = \omega_4.
\] (156)

6. Summary and discussion

Motivated by supergravity theories, we have introduced an ST manifold, which is defined as a Riemannian manifold whose metric cone is KT. In terms of almost contact metric structure, the ST manifold is a normal almost contact metric manifold on which the vector field \( \xi \) is a Killing vector field of unit length. The dual 1-form \( \eta \) of \( \xi \) is a special Killing 1-form. Furthermore, we also find special Killing forms \( \eta \wedge (d^2 \eta)^p \) of higher degrees. These are all known examples of special Killing forms at least in ordinary Sasakian manifolds except for round spheres.

In section 3, we have presented an example of the ST metric in \( 2n + 1 \) dimensions. The metric is quasi-Sasakian and further admits \( n + 1 \) Killing vector fields preserving the KT structure. We also have demonstrated that there exist two kinds of hidden symmetries: one is given by special Killing forms mentioned above and another by GKY tensors which are related to non-trivial rank-2 Killing tensors. Although the former exists in the general ST manifold, the existence of the GKY tensors could not always be expected. In our case, the GKY tensors are given by the Hodge duals of GCCKY tensors of odd ranks: the ST metric we presented is the first example admitting such odd-rank GCCKY tensors. The GKY tensors lead to the separation of variables in the Hamilton–Jacobi equation for geodesics. It would be interesting to examine in this geometry whether the GKY tensors generate the separation of variables for the Klein–Gordon and Dirac equations.

Using the ST metric (51) as an ansatz, we have constructed exact solutions in 5-dimensional minimal gauged supergravity and 11-dimensional supergravity in section 4,
and discussed the global structures of the solutions in section 5. The ST metrics on the five-dimensional compact manifolds provide a natural generalization of the toric Sasaki–Einstein metrics $Y^{p,q}$ and $L^{a,b,c}$. Indeed, there exists a toric action preserving the KT structure and the Einstein condition is replaced by the equations of motion of the minimal gauged supergravity. In 11 dimensions, we have briefly analyzed the ST metrics on non-compact manifolds. Further global analysis in 11 dimensions remains as a future problem. We also find that deformed $S^5$ [12] and $S^7$ [13] describing non-trivial supersymmetric solutions of supergravity theories are ST manifolds. Therefore, it is expected that the notion of ST manifolds works well for finding other supersymmetric solutions which play an important role in the AdS/CFT correspondence.

For the ST manifolds in section 2, we have three kinds of connections with totally skew-symmetric torsion. The first one is the Bismut connection which preserves the KT structure of the cone. For the five-dimensional solution of minimal gauged supergravity, the corresponding torsion (61) can be identified with the Maxwell field as

$$T = \frac{1}{\sqrt{3}} \ast F_{(2)}. \quad \text{(157)}$$

The second is the connection preserving the almost contact metric structure [21]. The associated torsion $T_c$ is given by $T_c = T + 2\eta \wedge \omega$. It was pointed out in the recent paper [59] that this relation holds in general ST manifolds. A supersymmetric solution in five-dimensional heterotic supergravity was discovered in [60], where the 3-form flux is identified with the torsion $T_c$ preserving the almost contact metric structure. The last connection appears in the hidden symmetry of our metrics. However, the relation between the torsion $G$ of hidden symmetry and $T_c$ (or $T$) is not yet fully understood since not all the torsion $G$ given by (75) can be expressed using the Maxwell field or the almost contact metric structure. It would be interesting to clarify the physical meaning of these three connections and the relationship between them.

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Appendix A. Some properties of T-contact metric manifolds

In this section, we show some useful formulas on T-contact metric manifolds in order to prove proposition 2.6 in section 2. Let $(M, g, T, \xi, \eta, \Phi)$ be an almost contact metric manifold equipped with a 3-form $T$ satisfying (19) and define a fundamental 2-form $\omega$ by $\omega(X, Y) = g(\Phi(X), Y)$. Then, a straightforward calculation leads us to

$$2g((\nabla^T \Phi)(Y), Z) = -d\omega(X, \Phi(Y), \Phi(Z)) + d\omega(X, Y, Z) + M(X, Y, Z)$$
$$+ g(N^{(1)}(Y, Z), \Phi(X)) + \eta(X)N^{(2)}(Y, Z)$$
$$+ d^T \eta(X, \Phi(Z))\eta(Y) - d^T \eta(X, \Phi(Y))\eta(Z), \quad \text{(A.1)}$$
where \( N^{(i)}(i = 1, 2) \) are tensor fields defined in [46, section 6] by

\[
N^{(1)}(X, Y) = N_\Phi(X, Y) + d\eta(X, Y)\xi, \tag{A.2}
\]

\[
N^{(2)}(X, Y) = (\mathcal{L}_\Phi(Y)\eta)(Y) - (\mathcal{L}_\Phi(X)\eta)(X), \tag{A.3}
\]

and \( M \) is a tensor field defined by

\[
M(X, Y, Z) = T(X, /\Phi(Y), Z) + T(X, Y, /\Phi(Z))
- T(\xi, X, /\Phi(Y)\eta)(Z) + T(\xi, X, /\Phi(Z))\eta(Y). \tag{A.4}
\]

Note that

\[
M(\xi, X, Y) = M(\xi, X, Y) = M(\xi, X, Y) = 0. \tag{A.5}
\]

On a T-contact metric manifold, (A.1) simplifies. In fact, we have

\[
N^{(2)}(X, Y) = d\eta(X, /\Phi(Y)) + d\eta(\Phi(X), Y)
= d^T\eta(X, Y) + d\eta(Y, X) + T(\xi, X, \Phi(Y)) + T(\xi, \Phi(X), Y)
= 0, \tag{A.6}
\]

where we have used (19), (38) and (40) at the last equality. Furthermore, we obtain

\[
\xi \cdot d^T\eta = \xi \cdot d\eta = 0. \tag{A.7}
\]

This implies that \( \mathcal{L}_\xi \eta = 0 \) and \( \mathcal{L}_\xi d\eta = 0 \). It is also obtained that

\[
\mathcal{L}_\xi d^T\eta = d\xi \cdot d^T\eta + \xi \cdot dd^T\eta = 2\xi \cdot d\omega = 0, \tag{A.8}
\]

which leads to

\[
2(\mathcal{L}_\xi g)(X, Y) = d^T\eta(X, (\mathcal{L}_\xi \Phi)(Y)). \tag{A.9}
\]

Thus, \( \xi \) is a Killing vector field if and only if \( \mathcal{L}_\xi \Phi = 0 \).

Appendix B. Some technical results

In this section, we collect some technical results. For the metric (51), we compute the covariant derivatives with respect to the Levi-Civita connection \( \nabla \) in section B.1 and with respect to the connection with skew-symmetric torsion \( \nabla^T \) (given in section 3.1) in section B.2, respectively. In section B.1, we also compute the curvature quantities with respect to \( \nabla \). The resulting curvatures have been used for solving Einstein equations of the supergravity theories considered in section 4.

B.1. The Levi-Civita connection

We have chosen the orthonormal frame (54) and obtained the connection 1-forms (56) for the metric (51) in section 3.1. Then, using the relation \( \nabla_{e_\mu} e^\nu = -\omega^{\nu}_{\mu}(e_0) \), we can compute the covariant derivatives as follows:

\[
\nabla_{e_\mu} e_\nu = \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_\rho, \quad \mu \neq \nu,
\]

\[
\nabla_{e_\mu} e_\nu = -\frac{\sqrt{Q_v}}{2(x_\mu - x_v)} e_\mu, \quad \mu \neq v,
\]

\[
\nabla_{e_\mu} e_\rho = \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_\rho - (1 + \partial_\mu H) e_0,
\]
\[ \nabla_{e_{\mu}} e^{\nu} = -\frac{\sqrt{Q_v}}{2(x_\mu - x_\nu)} e_\mu, \quad \mu \neq \nu \]

\[ \nabla_{e_{\nu}} e_{\mu} = \partial_\mu \sqrt{Q_v} e_\mu - \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\rho)} e_\rho + (1 + \partial_\mu H) e_0, \]

\[ \nabla_{e_{\nu}} e_\nu = -\frac{\sqrt{Q_v}}{2(x_\mu - x_\nu)} e_\mu + \frac{\sqrt{Q_\mu}}{2(x_\mu - x_\nu)} e_\nu, \quad \mu \neq \nu \]

\[ \nabla_{e_{\rho}} e_{\mu} = -\partial_\mu \sqrt{Q_\rho} e_\mu + \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_\mu - x_\nu)} e_\rho, \]

\[ \nabla_{e_{\rho}} e_\nu = \frac{\sqrt{Q_v}}{2(x_\mu - x_\nu)} e_\mu - \frac{\sqrt{Q_\mu}}{2(x_\mu - x_\nu)} e_\nu, \quad \mu \neq \nu \]

\[ \nabla_{e_0} e_0 = (1 + \partial_\mu H) e_\mu, \]

\[ \nabla_{e_\mu} e_\mu = -(1 + \partial_\mu H) e_\mu, \]

\[ \nabla_{e_\mu} e_\nu = (1 + \partial_\mu H) e_\nu, \]

\[ \nabla_{e_\nu} e_\mu = -(1 + \partial_\mu H) e_\mu, \]

\[ \nabla_{e_\nu} e_\nu = 0, \quad (B.1) \]

where the function $H$ is given by (57).

From the second structure equation

\[ R^a_{\ b} = d\omega^a_{\ b} + \sum_{c} \omega^a_{\ c} \wedge \omega^c_{\ b}, \quad (B.2) \]

the curvature 2-forms $R^a_{\ b}$ are obtained as follows:

\[ R^\mu_{\ \nu} = K_{\mu\nu} e^\mu \wedge e^\nu + (K_{\mu\nu} - (1 + \partial_\mu H)(1 + \partial_\mu H)) e^{\mu} \wedge e^{\nu} \]

\[ -\frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_v} e^{\mu} \wedge e^{\nu} + \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\mu} e^{\mu} \wedge e^{\nu}, \quad (\mu \neq \nu) \]

\[ R^\mu_{\ \hat{\nu}} = -\frac{1}{2} \left( \partial^2 \mu \bar{Q}_v + 6(1 + \partial_\mu H)^2 \right) e^{\mu} \wedge e^{\hat{\nu}} \]

\[ + 2 \sum_{\nu \neq \mu} (K_{\mu\nu} - (1 + \partial_\mu H)(1 + \partial_\nu H)) e^{\nu} \wedge e^{\hat{\nu}} \]

\[ -\sqrt{Q_v} \partial^2 \mu \bar{Q}_v e^{\mu} \wedge e^{\nu} - \sum_{\nu \neq \mu} \frac{\partial_\mu H - \partial_\nu H}{x_\mu - x_\nu} \sqrt{Q_v} e^{\nu} \wedge e^{\nu} \]

\[ R^\mu_{\ \hat{\nu}} = K_{\mu\nu} e^{\mu} \wedge e^{\hat{\nu}} + (K_{\mu\nu} - (1 + \partial_\mu H)(1 + \partial_\nu H)) e^{\nu} \wedge e^{\mu} \]

\[ -\frac{\partial_\nu H - \partial_\mu H}{2(x_\mu - x_\nu)} \sqrt{Q_v} e^{\mu} \wedge e^{\nu} + \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\mu} e^{\nu} \wedge e^{\nu}, \quad (\mu \neq \nu) \]

\[ R^\mu_{\ \hat{\nu}} = K_{\mu\nu} e^{\hat{\nu}} \wedge e^{\mu} + (K_{\mu\nu} - (1 + \partial_\mu H)(1 + \partial_\nu H)) e^{\nu} \wedge e^{\nu} \]

\[ -\frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_v} e^{\mu} \wedge e^{\nu} + \frac{\partial_\mu H - \partial_\nu H}{2(x_\mu - x_\nu)} \sqrt{Q_\mu} e^{\mu} \wedge e^{\nu}, \quad (\mu \neq \nu) \]

\[ R^\mu_{\ \hat{\nu}} = -\sqrt{Q_v} \partial^2 \mu \bar{Q}_v e^{\mu} \wedge e^{\hat{\nu}} - \sum_{\nu \neq \mu} \frac{\partial_\mu H - \partial_\nu H}{x_\mu - x_\nu} \sqrt{Q_v} e^{\mu} \wedge e^{\hat{\nu}} \]

\[ + \sum_{\nu \neq \mu} \frac{\partial_\mu H - \partial_\nu H}{x_\mu - x_\nu} \sqrt{Q_\mu} e^{\nu} \wedge e^{\hat{\nu}} \]

\[ + (1 + \partial_\mu H)^2 e^{\mu} \wedge e^{\nu}, \quad (B.2) \]
\[
\mathcal{R}^{\hat{\mu}}_{\hat{0}} = - \sum_{\nu \neq \mu} \frac{\partial_{\mu} H - \partial_{\nu} H}{2(x_{\mu} - x_{\nu})} \sqrt{Q_{\nu}} e^{\mu} \wedge e^{\nu} - \sum_{\nu \neq \mu} \frac{\partial_{\mu} H - \partial_{\nu} H}{2(x_{\mu} - x_{\nu})} \sqrt{Q_{\nu}} e^{\mu} \wedge e^{\nu} + (1 + \partial_{\mu} H)^2 e^{\hat{\mu}} \wedge e^{\hat{0}},
\]

(B.3)

where \(K_{\mu \nu}\) and \(Q_T\) are given by (101). The Ricci curvature is defined by
\[
\text{Ric}(e_a, e_b) = \sum_{\epsilon} \mathcal{R}^{\epsilon}_{\epsilon}(e_a, e_b).
\]

(B.4)

Thus, non-zero components of the Ricci curvature are
\[
\text{Ric}(e_\mu, e_\mu) = \text{Ric}(e_\mu, e_\mu) = - \frac{1}{2} \partial_{\mu}^2 Q_T + 2 \sum_{\nu \neq \mu} K_{\mu \nu} - 2(1 + \partial_{\mu} H)^2,
\]

(B.5)

\[
\text{Ric}(e_0, e_\mu) = - \sqrt{Q_\mu} \left( \partial_{\mu}^2 H + \sum_{\nu \neq \mu} \frac{\partial_{\nu} H - \partial_{\mu} H}{x_{\mu} - x_{\nu}} \right).
\]

The scalar curvature is defined by
\[
\text{scal} = \sum_{a} \text{Ric}(e_a, e_a).
\]

(B.6)

Thus, we obtain
\[
\text{scal} = - \sum_{\mu = 1}^{n} \partial_{\mu}^2 Q_T + 4 \sum_{\mu \neq \nu} K_{\mu \nu} - 2 \sum_{\mu = 1}^{n} (1 + \partial_{\mu} H)^2.
\]

(B.7)

**B.2. The connection with totally skew-symmetric torsion**

We next compute the covariant derivatives with respect to the torsion \(\nabla^T\), with respect to the orthonormal frame (54) of the metric (51). Since we have obtained the covariant derivatives with respect to the Levi-Civita connection \(\nabla\), (B.1), we can compute from (1) the covariant derivatives with respect to the torsion connection \(\nabla^T\) as
\[
\nabla^T_{e_a} e_b = \nabla_{e_a} e_b + \frac{1}{2} T(e_a, e_b).
\]

(B.8)

Thus, we obtain
\[
\nabla^T_{e_\mu} e_\mu = \frac{\sqrt{Q_\mu}}{2(x_{\mu} - x_\rho)} e_\rho,
\]
\[
\nabla^T_{e_\mu} e_\nu = - \frac{\sqrt{Q_\nu}}{2(x_{\mu} - x_\nu)} e_\mu,
\]
\[
\nabla^T_{e_\mu} e_\hat{\mu} = \frac{\sqrt{Q_\rho}}{2(x_{\mu} - x_\rho)} e_\hat{\rho} - e_0,
\]
\[
\nabla^T_{e_\mu} e_\hat{\nu} = - \frac{\sqrt{Q_\nu}}{2(x_{\mu} - x_\nu)} e_\hat{\mu},
\]
\[
\nabla^T_{e_\mu} e_\mu = \partial_{\mu} \sqrt{Q_\mu} e_\mu - \sum_{\rho \neq \mu} \frac{\sqrt{Q_\rho}}{2(x_{\mu} - x_\rho)} e_\rho + e_0.
\]

(27)
Note that if we restrict the connection 1-forms \(\omega\), the Bismut torsion \(T\), and the curvature 2-form \(R\) to the hyperplane of \(r = 1\), then we obtain the connection 1-form with torsion \(\tilde{\omega}^{Ra}_{\beta}\) and the Ricci form \(\rho^{\beta}(X, Y)\) are given [56] as

\[
\tilde{R}^{Ra}_{\beta}(X, Y) = g^{(2n+1)}(\tilde{\rho}^{\beta}(X, Y)\tilde{e}_a, \tilde{e}_\beta),
\]

\[
\rho^{\beta}(X, Y) = \frac{1}{2} \sum_a \tilde{R}^{Ra}_{\beta}(X, Y, \tilde{e}_a, J(\tilde{e}_a)),
\]

where the function \(H\) is again given by (57).

### Appendix C. Calabi–Yau with the torsion metric on a cone

We begin with the metric (65) and choose the same orthonormal frame as (64); then, the connection 1-forms are calculated as (66). For the Hermitian connection \(\nabla^\mathbb{H}\) with respect to the Bismut torsion (70), the connection 1-form with torsion \(\tilde{\omega}^{Ra}_{\beta}\) are calculated as

\[
\tilde{\omega}^{Ra}_{\beta} = \tilde{\omega}^a_{\beta} - \frac{1}{2} \sum_\gamma B^a_{\beta\gamma} \tilde{e}^\gamma.
\]  

That is, we have

\[
\tilde{\omega}^{Ra}_{\alpha} = -\tilde{e}^a, \\
\tilde{\omega}^{Ra}_{\mu\nu} = -\sqrt{\Omega_\mu} \tilde{e}^\mu \sqrt{\Omega_\nu} \tilde{e}^\nu - \sqrt{\Omega_\mu} \tilde{e}^\nu \sqrt{\Omega_\nu} \tilde{e}^\mu, \quad \mu \neq \nu, \\
\tilde{\omega}^{Ra}_{\mu\hat{\nu}} = -\tilde{e}^\mu + \sum_{\nu \neq \mu} \sqrt{\Omega_\nu} \tilde{e}^\nu - (1 + 2\delta_{\mu} H) \tilde{e}^\nu, \\
\tilde{\omega}^{Ra}_{\mu\hat{\nu}} = -\sqrt{\Omega_\nu} \tilde{e}^\nu - \sqrt{\Omega_\mu} \tilde{e}^\mu, \quad \mu \neq \nu, \\
\tilde{\omega}^{Ra}_{00} = -\tilde{e}^0, \\
\tilde{\omega}^{Ra}_{0\hat{\nu}} = -\tilde{e}^0 - \tilde{e}^\nu.
\]  

Note that if we restrict the connection 1-forms \(\tilde{\omega}^{Ra}_{\beta}\) on the hyperplane of \(r = 1\), then we obtain the connection 1-form \(\alpha^{(2n+1)}\) with respect to the original metric \(g^{(2n+1)}\) and the torsion \(T\). Since the curvature 2-form \(\tilde{R}^{Ra}_{\beta}\) and the Ricci form \(\rho^{\beta}(X, Y)\) are given [56] as

\[
\tilde{R}^{Ra}_{\beta}(X, Y) = g^{(2n+1)}(\tilde{\rho}^{\beta}(X, Y)\tilde{e}_a, \tilde{e}_\beta),
\]

\[
\rho^{\beta}(X, Y) = \frac{1}{2} \sum_a \tilde{R}^{Ra}_{\beta}(X, Y, \tilde{e}_a, J(\tilde{e}_a)),
\]
where $\bar{R}^B(X, Y)$ is the curvature defined by (10) with respect to $\bar{\nabla}^B$, we have the curvature 2-form as

$$\bar{R}^B_{\mu \nu} = -2 \sum_{\mu=1}^n \partial_\mu H \, e^\mu \wedge e^\nu,$$

and the non-zero components of the Ricci form as

$$\rho^B(e_\mu, e_\nu) = -\frac{1}{2} \partial_\mu \mathcal{Q}_T + \frac{1}{2} \sum_{\nu \neq \mu} \left( -\frac{\partial_\mu \mathcal{Q}_T}{x_\mu - x_\nu} + \frac{\partial_\nu \mathcal{Q}_T}{x_\mu - x_\nu} \right) e^\nu \wedge e^\mu,$$

Thus, we find that $\rho^B(X, Y) = 0$ for all vector fields $X, Y$, when provided that the functions $X_\mu$ and $N_\mu$ take the form

$$X_\mu(x_\mu) = -4x_\mu + \sum_{j=1}^n c_j x_j^2 + b_\mu - 4(n + 1)q_\mu x_\mu,$$

and

$$N_\mu(x_\mu) = \sum_{i=1}^{n-1} a_i x_i^2 + q_\mu,$$

where $a_i, b_\mu, m_\mu$ and $q_\mu$ are constant parameters. This gives a Calabi–Yau with the torsion metric on a cone. The function (C.7) in five dimensions is different from (88).

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