1 Introduction.

The experimental discovery of quasicrystals by D Shechtman, D Gratias, I Blech, and J W Cahn in 1984 [43] provided the paradigm for a new type of long-range order of solid matter in nature. This discovery stimulated an explosion of new experimental and theoretical research. In years prior to the discovery, there was a very active development of various gateways to quasicrystals in theoretical and mathematical physics. Without this conceptual basis, it would have been impossible to grasp and explore efficiently the structure and physical properties of quasicrystals. The aim in what follows is to give a non-technical and condensed account of the conceptual gateways to quasicrystals prior to their discovery.

2 A Bravais, J B J Fourier, A M Schönflies and E S Fedorov: Classical periodic crystallography.

Crystals in the natural world caught the attention by their regular geometrical polyhedral form. It was A Bravais [4] who found the fundamental insight into their internal structure by introducing the idea of an underlying periodic lattice Λ, see Fig. 1. Bravais was able to explain the regularity of crystal faces by associating them with planes uniformly occupied by lattice points. The finite translations that connect lattice points form the translation group of the lattice, denoted also for short by Λ. If the next distances and directions in a lattice are tuned in particular ways, finite rotations with respect to a fixed lattice point may turn lattice points into lattice points. The lattice then is compatible with a point group. The combined symmetry under lattice translations and rotations led to the concept of space group symmetries. The question then arose: what are the possible crystal structures in 3-dimensional space? This question was explored in the 19th century and culminated in the systematic classification of all possible crystal structures in terms of space group theory by Schönflies [11] and Fedorov [8]. The emergence of an atomic structure of solid matter...
in the 19th century offered the possibility of viewing a crystal lattice as being formed by atoms. This structure was verified in 1912 in diffraction experiments with X rays following von Laue \[30\] and Bragg \[3\].

The concept of a periodic lattice implies other basic notions for crystals: The periodic lattice symmetry requires that the long-range distribution of atoms is completely determined once it is known inside a unit cell. The analysis of periodic systems was fundamentally advanced by Fourier’s \[9\] concept of the series expansion of a periodic function into elementary periodic functions. For a complex-valued periodic function $f^p(x)$ on the real line, this expansion in a condensed complex version takes the form

$$f^p(x) = \sum_{\nu} a(k_\nu) \exp(ik_\nu x), \quad k_\mu = \frac{1}{2\pi} \mu, \mu = 0, \pm 1, \pm 2, ...$$

The Fourier coefficients $a(k_\mu)$ in this expansion, given as integrals over the function $f^p(x)$ inside the unit interval, may be considered as functions defined on the points $k_\nu$ of a lattice $\Lambda^R$ in a Fourier $k$-space. Then the Fourier series represents the function $f^p(x)$ with domain the unit interval as a function $f^p(k)$ on the points of the so-called reciprocal lattice $\Lambda^R$ in

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**Figure 1:** **Hexagonal and dual triangular periodic lattices.** The centers (white and black circles) of the dual triangular cells are located at the vertices of the hexagonal lattice cells (centers black squares). The hexagons and triangles are examples of the Voronoi and dual Delone cells in general periodic lattices.
$k$-space. For crystals with lattices in 3-dimensional space $E^3$, the Fourier coefficients live on a $E^3$ $k$-space equipped with a 3-dimensional reciprocal lattice $\Lambda^R$.

With the advent of scattering theory by quantum wave mechanics, von Laue [30] and Bragg [3] related the magnitude of the Fourier coefficients directly to the observed intensity of waves scattered from crystals. The intensity in scattering from crystals is characterized by sharp peaks in selected directions. In mathematical terms one speaks of a Fourier point spectrum. The determination of the atomic structure of matter up to date is based on the interpretation of scattering data by Fourier series analysis.

The three related notions of a periodic lattice $\Lambda$, an atomic unit cell, and a Fourier series analysis characterize crystals as periodic atomic long-range structures.

3 Point symmetry: Das Pentagramma macht Dir Pein?

Another geometric aspect of crystals observed in nature were the systematic angles between their outer faces. With respect to the center of the crystal, these faces often displayed 2fold, 4fold or 6fold point symmetry as part of their polyhedral symmetry. These properties found an explanation in terms of the Bravais periodic lattice theory: It was shown that all the observed point symmetries could be related to what became known as the seven Bravais lattices. The compatibility of point and periodic lattice symmetry in the framework of space groups formed the basis of the classification by Schoenflies and Fedorov.

There remained an enigma expressed by J W Goethe in his drama Faust [11]: Das Pentagramma macht Dir Pein? Certain well-known point symmetries did not fit into lattice theory. Among them are the 5fold and the icosahedral symmetry, associated with the cyclic group $C_5$ and the icosahedral group $J$ of rotations in 2- and 3-dimensional space. Already Plato [40] in his study of regular polyhedra had noted the regular dodecahedron and icosahedron with icosahedral symmetry. For him, four regular polyhedra were geometric building blocks of the four elements whereas the dodecahedron he associated with the overall symmetry.

J Kepler [20] was impressed by the Platonic catalogue. In a first attempt he tried to use them for the determination of the radii of spheres of the planets. Later, after his discovery of the elliptic orbits for the planets, he studied [21] regular polygons, see Fig. 2, and polyhedra in order to deduce rational relations between astronomical data for the orbits of the planets. In his studies he also looked at tilings of the plane by regular pentagons, and enlarged the list of polyhedra by the half-regular triacontahedron.

With the success of crystallographic lattice theory in the 19th century and thereafter its atomic setting by quantum theory, 5fold and icosahedral point symmetry, being incompatible with any 2- or 3-dimensional lattice, were stigmatized as being non-crystallographic. Of course, the 5fold and icosahedral point symmetry can and does appear in molecules. But all the known paradigms of long-range order, thought to be periodic and so lattice-based, excluded these point symmetries.

In view of their geometrical possibility in 3-dimensional space the enigma remained: Are these point symmetries simply forbidden in nature, since they are not compatible with any
 Kepler’s planar tiling with decagons and pentagons. He found that, to leave no gaps, he needed pentagons and pentagonal stars in addition to decagons.

lattice, or can they be the gateways towards a new type of long-range order in nature?

4 H Zassenhaus and C Hermann 1948,49: Mathematical crystallography in $n$ dimensions for $n > 3$.

The determination of all space groups in 3-dimensional space was a clear classification problem in mathematical physics. This classification obviously had a counterpart in Euclidean spaces of higher dimension. The systematic analysis of these symmetries and lattices was advanced in particular by H Zassenhaus [17] and by C Hermann [13]. This work, reviewed by Schwarzenberger [32], showed that the counterparts of all essential findings of classical crystallography in $E^3$ can be found in $nD$ lattices. In $E^4$ the classification of space groups was completed in the work of Brown et al. [5]. Of course the lattices in $E^4$ also include 5fold point symmetry.

The work on high dimensional crystallography gained new weight in physics with the advent of quasicrystals.
Figure 3: The Penrose pattern: The tiles have thin or thick rhombus shape.

5 R Penrose 1974: Aperiodic tilings of the plane.

In mathematical crystallography, the Euclidean space $E^n$ is tiled without gaps or overlaps by repeated copies of the unit cell of the lattice. The position of the centers of these copies is given by all the lattice translations. A natural generalization of periodicity are tilings into copies of a finite number of cells. If such a tiling cannot be organized by a lattice, one has to find new ways to introduce a long-range order. R Penrose [38] proposed such a tiling of the plane with two rhombus tiles, known as the Penrose pattern, Fig. 3. The edges of the two tiles have the same length. The angles between the edges of these tiles are multiples of $2\pi/5$ and so are adapted to 5fold symmetry. It follows that all the edges in the tile point in only five directions. This already suggests an average 5fold symmetry. Of course the tiles could be arranged into periodic tilings, but Penrose wanted to avoid a lattice periodicity. As a local rule for the long-range order he introduced the concept of matching rules. The matching rules demand that the marked directed edges of adjacent tiles must correspond to one another. Penrose demonstrated a number of interesting properties of his patterns. The paradigm of the Penrose pattern was very appealing to scientists as a template for a generalizations of classical crystallography. One could imagine to fix atoms to positions on the rhombus tiles and study the properties of the resulting generalized crystals. Of particular interest was the question what other properties of classical crystallography extend to Penrose patterns. A first question about the Penrose patterns was the long-range order implied. Can any patch of a tilings, built according to the local rules, be extended to cover the full plane? This question has a negative answer: There are finite patches of tilings obeying the matching
rules which cannot be extended in some parts without violating the matching rules. Penrose later in [39] called this the non-locality of the pattern. Another problem were the Fourier and diffraction properties of the Penrose pattern.

6 A L Mackay 1981/2: Cells and diffraction properties from the Penrose pattern.

A L Mackay [32] presented the Penrose pattern as a paradigm for crystallography with 5fold point symmetry. He discussed the planar Penrose rhombus pattern, suggested its two cells as non-periodic generalizations of crystallographic cells, and proposed the name quasi-lattice for the pattern. He also pointed out a 3D generalization to two rhombohedra whose edges point in the six directions perpendicular to the faces of the regular dodecahedron. He demonstrated in [32] Figure 8 that these rhombohedra can build Kepler’s triacontahedron. In [33] he posed the question what diffraction would result if one placed scatterers to the vertices of the Penrose pattern. By an optical transform of circles, placed at vertex positions of a portion of a Penrose pattern, he arrived at a diffraction pattern governed by sharp peaks of intensity whose distribution exhibited 10-fold point symmetry. Mackays result strongly suggested that the Penrose generalization of crystals shared with classical crystals the discrete point spectrum in diffraction which was the classical basis of structure determination by Fourier series analysis, section 2.

7 H Bohr 1925: Quasiperiodicity and Fourier module from n-dimensional lattice embedding.

H Bohr back in the year 1925, in part II of two papers [2], devoted to a careful mathematical analysis of almost periodicity, had on pp. 111-117, pp. 137-140, pp. 160-162 explored the notion of quasiperiodicity. He considered a lattice Λ in a Euclidean space $E^n$ of dimension $n > 3$ and functions $f^p$ periodic on this space. His approach can be described as follows: He introduced a decomposition of Euclidean space $E^n = E^m \parallel + E^{(n-m)} \perp$, with $E^m \parallel$ irrational. Irrationality meant: If $E^m \parallel$ is parallel shifted by a vector $t$ so as to intersect with a lattice point $P$, then the intersection of the shifted subspace with the set of all lattice point contains only $P$, $(E^m \parallel + t) \cap \Lambda = P$. The Fibonacci tiling, see section 8 provides the simplest example of an irrational section.

Bohr then analyzed the restriction of a periodic function $f^p$ on $E^n$, with domain restricted to the subspace $E^m \parallel$. He showed that this restriction has quasiperiodic properties. Moreover he considered the Fourier series of $f^p$. He analyzed the Fourier transform of a function restricted to the irrational parallel subspace. His finding can be expressed in terms of the lattice $\Lambda^R$ reciprocal to the original lattice $\Lambda$ in $E^n$ in Fourier $k$-space: If he projects the points of the reciprocal lattice to the parallel $k$-subspace $E^m \parallel$, the discrete set of these
projections carries the Fourier coefficients of the quasiperiodic function. The discrete set of projected reciprocal lattice points forms what in mathematical terminology is called a Z-module. Its points by construction can be related in $E^m$ by integer linear combinations of basis vectors of the reciprocal lattice, projected to this subspace. In contrast to the reciprocal lattice points of periodic crystals, the points belonging the quasiperiodic module are discrete but become dense, that is come arbitrarily close, to any other point of the module.

The ingredients of Bohr’s description of quasiperiodic functions were then a periodic lattice $\Lambda$ in $E^n, n > 3$, and an irrational subspace $E^m$. On this basis Bohr provided a discrete Fourier module whose points carried the Fourier coefficients for quasiperiodic functions. The Fibonacci paradigm, discussed in the next section, provides a simple example of Bohr’s theory. For general applications of Bohr’s ideas there remained a problem: The points of a lattice form only a countable subset in $E^n$ leaving ample gaps for irrational subspaces. Among the infinite set of irrational subspaces, what guideline can lead to a significant choice?

8 Fibonacci 1202, M Lothaire 1983: Scaling and the square lattice.

Leonardo de Pisa published in 1202 in Pisa the hand-written monograph Liber abaci. In it he presented his famous series of the integer Fibonacci numbers defined recursively by

$$f_{n+1} = f_n + f_{n-1}, \quad f_1 = f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, ...$$

(2)

The Fibonacci numbers appear in mathematical combinatorics, M Lothaire [31] p. 10, as follows: Consider an alphabet $A = \{a, b\}$ and words formed recursively by the concatenation of letters

$$f_{i_{n+1}} = f_{i_n}f_{i_{n-1}}, \quad n \geq 2,$$

$$f_{i_1} = b, f_{i_2} = a, f_{i_3} = ab, f_{i_4} = aba, f_{i_5} = abaaab, ....$$

(3)

Counting the number of letters in successive words, called the word length $|f_{i_n}|$, one finds

$$|f_{i_1}| = |f_{i_2}| = 1, |f_{i_{n+1}}| = |f_{i_n}| + |f_{i_{n-1}}| = f_{n+1}.$$ 

(4)

So the word length is a Fibonacci number. From the relative frequency of the letters $\{a, b\}$ in the words one can easily proof that the Fibonacci words cannot be periodic. The Fibonacci words can be converted into an aperiodic tiling by interpreting the letters $(a, b)$ as intervals on the line of length $(1, \tau)$ respectively.

The recursive construction in eq. 3 is the first approach to the Fibonacci tiling. In a second step we now relate the Fibonacci tiling to Bohr’s theory of quasiperiodic functions. For
Figure 4: The Fibonacci tiling from the square lattice. The black squares denote the points of the square lattice. The two squares $A, B$ give a periodic tiling of the plane. The Fibonacci matrix $g$ determines scalings in two perpendicular directions $e_\parallel, e_\perp$. A line parallel to $e_\parallel$ intersects the two squares in two intervals of length in proportion $\tau$. The sequence of intervals generates on the line the Fibonacci tiling, beginning with $abaab$, eq. 3.

this we follow [28] pp. 311-12 and define

$$g = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(5)

This matrix belongs to the group $GL(2, \mathbb{Z})$ with integer matrix elements and determinant $\pm 1$. Computation of the powers of this matrix relates them to the Fibonacci numbers since

$$g^n = \begin{bmatrix} f_n & f_{n-1} \\ f_{n-1} & f_{n-2} \end{bmatrix}, \quad \det(g^n) = (-1)^n.$$  

(6)

We determine the two eigenvalues of $g$ as

$$\lambda_1 = -\tau^{-1} = -\tau + 1, \quad \lambda_2 = \tau = (1 + \sqrt{5})/2.$$  

(7)

and get the eigenvectors from

$$BgB^{-1} = \begin{bmatrix} -\tau^{-1} & 0 \\ 0 & \tau \end{bmatrix}, \quad B = \begin{bmatrix} -\sqrt{\frac{-\tau + 3}{5}} & \sqrt{\frac{\tau + 2}{5}} \\ \sqrt{\frac{\tau + 2}{5}} & -\sqrt{\frac{-\tau + 3}{5}} \end{bmatrix}.$$  

(8)
as the two orthonormal column vectors $B = (b^1, b^2)$ of the matrix $B$. These two vectors are obtained from an initial orthogonal basis by application of the matrix $B$. Writing eq. 8 as

$$\begin{bmatrix} -\tau^{-1} & 0 \\ 0 & \tau \end{bmatrix} B = B \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(9)
gives the following interpretation: The basis vectors $(b^1, b^2)$ span a square lattice. The integer linear combinations of the basis vectors $(b^1, b^2)$, given on the right-hand side by the right action of $g$ on $B$, transform lattice points into lattice points. The left-hand side shows that the two vectors $(b^1, b^2)$ are scaled respectively by the factors $(\lambda_1, \lambda_2)$. We can combine this result with the Bohr theory of quasiperiodic functions: The orthogonal basis vectors $e_\parallel = (1, 0)$, $e_\perp = (0, 1)$ determine two irrational orthogonal directions through the square lattice and provide one-dimensional irrational subspaces $E_\parallel, E_\perp$. With respect to these vectors, the original matrix $g$ becomes diagonal. One can construct [28] a new periodic tiling of $E^2$ by two squares whose boundaries run in the directions of these subspaces, see Fig 4.

Now let a line parallel to $e_\parallel$ intersect these two squares in two intervals. These intervals belong to a module on the line. Their length is in the golden ratio $\tau$. The tiling on the parallel line is the Fibonacci tiling. A parallel scaling with factor $\tau$ transforms end points of intervals into end points. So the Fibonacci tiling has a scaling symmetry and displays a cell structure. The scaling symmetry selects a particular one-dimensional irrational subspace in the square lattice, and so by Bohr’s theory becomes a source of quasiperiodicity. Similar lattice scalings by powers of $\tau$ appear in lattices of $E^4, E^6$ with 5fold and icosahedral symmetry. These scalings underly the notions of inflation and self-similarity.

It follows from Bohr’s theory that the Fibonacci tiling is quasiperiodic and has a discrete Fourier module.

9 Y Meyer 1970, 1972: Harmonious sets.

In [31], [35], Y Meyer, starting from a mathematical study of harmonic analysis on locally compact abelian groups, introduced certain discrete point sets he called harmonious sets. After the discovery of quasicrystals it was realized that these harmonious or Meyer sets generalize the notion of lattices to in general aperiodic structures, and so provide a mathematical frame including and generalizing quasicrystals.

Of the seven equivalent characterizations of Meyer sets given by R V Moody [36] pp. 403-41, we mention here only a geometric one: It starts from a Delone set $\Lambda \in R^k$, defined as a relatively dense and uniformly discrete set. This Delone set becomes a Meyer set if there is a finite set $F$ such that the set of differences obeys $\Lambda - \Lambda = \Lambda + F$. Clearly one sees the generalization from the notion of a lattice, whose set of differences would result in $F = 0$.

R V Moody [36] pp. 403-41 gives a detailed mathematical account of Meyer’s harmonious sets in the light of our present knowledge of aperiodic structures. The broader field of mathematics for aperiodic structures is the subject of the volume [36], edited in 1997 by R V Moody.
A first and successful application in line with Bohr’s theory of quasiperiodic functions to crystallography was made by A Janner and T Janssen [17], [18], based on previous work of P M de Wolff [7]. The idea was to describe so-called incommensurate and modulated structures, found in certain classes of crystals, by the extension of 3D space to a superspace equipped with a superlattice. The extra dimensions then are used to describe incommensuration and modulation.

Introduction of a reciprocal Fourier superspace, and projection to the usual Fourier space then provided, beyond the usual diffraction pattern, a pattern of satellite diffraction peaks whose structure encodes the specific nature of the incommensuration or modulation. Here the Fourier analysis beyond periodicity was developed, extended and applied in the spirit of Bohr’s frame of quasiperiodicity.

From the mathematical side, de Bruijn [6] presented the first analysis of the Penrose pattern by use of a lattice embedding into 4-dimensional space $E^4$. His choice of $E^4$ was guided by the wish to incorporate 5fold point symmetry. The Euclidean space $E^4$ has the lowest dimension to allow for a lattice embedding with 5fold point symmetry. If one examines the action of the cyclic group $C_5$ on $E^4$, one finds two orthogonal subspaces of dimension 2. The unique subspace $E^2_\parallel$ of $E^4$ in which the cyclic group generator acts as a rotation by an angle $2\pi/5$ is the natural choice of $E^2_\parallel$ for a quasiperiodic function. This subspace has the property of being irrational with respect to the chosen lattice in $E^4$.

The next task of de Bruijn was to identify the Penrose rhombus tiles as projections. We take the liberty to describe his finding in the terminology of a later analysis of the same lattice in $E^4$ given in [1]. The lattice provides two tilings of $E^4$ by 4-polytopes: One is a tiling by Voronoi polytopes, which are the Wigner-Seitz cells of the lattice centered at the lattice points. The second, dual tiling is given by so-called Delone polytopes, centered at the vertices of the Voronoi domains. His geometric view allowed de Bruijn to identify the Penrose rhombus tiles with what is denoted in [1] as the projections of 2-dimensional boundaries of the so-called Delone cells. De Bruijn introduced a so-called pentagrid for the construction by projection of a Penrose rhombus tiling.

De Bruijn’s contribution to the theory of quasicrystal presented major advances: His construction of an irrational lattice embedding into 4 dimensional space related the planar Penrose tiling construction to the theory of Bohr. This construction showed that the requirement of 5fold point symmetry uniquely determines the irrational subspace required by Bohr’s theory. So indeed the 5fold point symmetry promised to be the gateway to a new type of long-range order. When combined with Bohr’s theory, it followed from de
Bruijn’s construction that the Fourier transform of a Penrose pattern can be described by a module of sharp diffraction points, with positions the projections of the reciprocal lattice to a 2-dimensional Fourier k-subspace. This consequence confirmed that Mackay’s conjecture of sharp diffraction peaks from a Penrose pattern had a strict mathematical basis.

12 P Kramer 1982/84: Icosahedral tilings in 3 dimensions.

The pentagram had led to the Penrose paradigm of a quasiperiodic planar tiling in 2 dimensions. Crystals in physics are phenomena in 3 dimensions. The counterpart of the pentagram is the icosahedron, whose point symmetry is forbidden in 3-dimensional lattices. There arose now in 3 dimensional space the question of tiles and tilings with forbidden icosahedral point symmetry. The first recursive and the second lattice approach discussed in section [8] for the Fibonacci tiling looked promising. Kramer in [24] constructed a first set of seven elementary convex polyhedral tiles with two properties:

(i) Copies of them could be packed into a regular dodecahedron.

(ii) Copies of the seven tiles could be packed into polyhedra of the same seven shapes, but scaled by a factor which was the third power of the golden section number \( \tau = \frac{1}{2}(1 + \sqrt{5}) \).

It was clear then that, by repeated application of this self-similar scaling, any region of 3-dimensional space could be covered by a tiling of the seven elementary tiles. Mosseri and Sadoc [37] managed to reduce the number of these tiles from seven to four.

The findings by de Bruijn [6] and by Bohr [2] suggested the following lattice construction for icosahedral quasicrystals:

(1) One had to find a lattice in \( \mathbb{E}^n \) which under the action of the icosahedral group \( \mathcal{J} \) is transformed into itself, (2) Moreover one should find a subspace \( E_3^3 \in E^n \) of dimension 3, invariant under the action of \( \mathcal{J} \).

Kramer and Neri [25] showed that the hypercubic lattice in 6-dimensional space was compatible with icosahedral point symmetry, and moreover provided a unique 3-dimensional subspace invariant under the icosahedral rotation group \( \mathcal{J} \).

By considering the Voronoi 6-polytopes of the hypercubic lattice in \( \mathbb{E}^6 \) and their 3-dimensional boundaries, both projected to the parallel space \( E_3^3 \) with icosahedral symmetry, there emerged Kepler’s triacontahedron from the Voronoi polytope, and rhombohedra in two shapes from the 3-dimensional boundaries. So this icosahedral tiling is organized exactly by the tiles considered by Mackay [33]. As was found out later, Kowalewski [23] in 1938 in a book on recreational mathematics had already described Kepler’s triacontahedron and the two rhombohedra as icosahedral projections of the hypercube in 6 dimensions and its boundaries.

This work generated in three dimensions the first paradigm of a quasicrystal with icosahedral point symmetry. Combined with Bohr’s general theory, a diffraction analysis on an appropriate icosahedral Fourier module could be devised. A generalization of de Bruijn’s
The enigma of the pentagram was finally solved, 5fold and icosahedral symmetry were back on their way into physics.

13 D Shechtman, D Gratias, I Blech and J W Cahn
1984: Discovery of icosahedral quasicrystals.

In the previous sections we surveyed the theoretical approaches to quasicrystals prior to their experimental discovery.

In 1984 D Shechtman, D Gratias, I Blech and J W Cahn announced the discovery of quasicrystals exhibiting a diffraction pattern with sharp peaks of icosahedral point symmetry. This discovery implied that atomic matter could organize itself in the new paradigm
of quasiperiodic long-range order. An international workshop at Les Houches in 1986 \[14\] brought together many protagonists of quasicrystal theory with D Shechtman and his colleagues. A brief review along similar lines as given here can be found in the epilogue by J W Cahn \[16\], pp. 807-10 to the 5th International Conference on Quasicrystals, Avignon 1995.

14 Postscriptum: D Levine and P J Steinhardt 1984, A Katz and M Duneau 1986, B Grünbaum and G C Shepard 1987, H Q Ye and K H Kuo et al. 1984, Ishimasa et al. 1985.

The extraordinary development of quasicrystals after 1984, both on the experimental and the theoretical level, is a new story. Here it remains to briefly postscribe theoretical and experimental work by authors that was published shortly after the experimental discovery of quasicrystals.

D Levine and P J Steinhardt \[29\] in 1984 devised a construction method based on the Fibonacci sequence, and proposed the name quasicrystals for the new ordered structures. A Katz and M Duneau \[19\] in 1986 developed projection methods for the construction of icosahedral tilings by rhombohedra. Tilings were well described in a monograph written in 1987 by B Grünbaum and G C Shepard \[12\].

Enlarging the field of quasicrystals on the experimental side, H Q Ye and K H Kuo \[46\] in 1984 studied quasicrystals with layers of forbidden 10fold point symmetry. T Ishimasa, H U Nissen, and Y Fukano \[15\] in 1985 prepared structures with (Ni, Cr) atomic composition and non-crystallographic 12fold point symmetry.

After 1984, the broad development of quasicrystal preparation, structure analysis and new physical properties became manifest in the Proceedings of International Conferences on Quasicrystals, 1986 \[14\] in Les Houches (France), 1989 \[45\] in Vista Hermosa (Mexico), 1992 \[22\] in St Louis (USA), 1995 \[16\] in Avignon (France), 1997 \[44\] in Tokyo (Japan), and 1999 \[10\] in Stuttgart (Germany).
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