A sequence of potentials lying between the U(5) and X(5) symmetries

Dennis Bonatsos\#1, D. Lenis\#2, N. Minkov\textsuperscript{†}3, P. P. Raychev\textsuperscript{†}4, P. A. Terziev\textsuperscript{†}5

\# Institute of Nuclear Physics, N.C.S.R. “Demokritos”
GR-15310 Aghia Paraskevi, Attiki, Greece
\textsuperscript{†} Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences
72 Tzarigrad Road, BG-1784 Sofia, Bulgaria

Abstract

Starting from the original collective Hamiltonian of Bohr and separating the $\beta$ and $\gamma$ variables as in the X(5) model of Iachello, an exactly soluble model corresponding to a harmonic oscillator potential in the $\beta$-variable (to be called X(5)-$\beta^2$) is constructed. Furthermore, it is proved that the potentials of the form $\beta^{2n}$ (with $n$ being integer) provide a “bridge” between this new X(5)-$\beta^2$ model (occurring for $n = 1$) and the X(5) model (corresponding to an infinite well potential in the $\beta$-variable, materialized for $n \to \infty$). Parameter-free (up to overall scale factors) predictions for spectra and B(E2) transition rates are given for the potentials $\beta^2$, $\beta^4$, $\beta^6$, $\beta^8$, corresponding to $R_4 = E(4)/E(2)$ ratios of 2.646, 2.769, 2.824, and 2.852 respectively, compared to the $R_4$ ratios of 2.000 for U(5) and 2.904 for X(5). Hints about nuclei showing this behaviour, as well as about potentials “bridging” the X(5) symmetry with SU(3) are briefly discussed.

PACS numbers: 21.60.Ev, 21.60.Fw, 21.10.Re

1. Introduction

Models providing parameter-independent predictions for nuclear spectra and electromagnetic transition rates serve as useful benchmarks in nuclear theory. The recently introduced E(5) [1] and X(5) [2] models belong to this category, since their predictions for nuclear spectra (normalized to the excitation energy of the first excited state) and B(E2) transition rates (normalized to the B(E2) transition rate connecting the first excited state to the ground state) do not contain any free parameters. The E(5) model appears to be related to a phase transition from U(5) (vibrational) to O(6) ($\gamma$-unstable) nuclei [1], while X(5) is related to a phase transition from U(5) (vibrational) to SU(3) (prolate deformed) nuclei [2]. Both models originate (under certain simplifying assumptions) from the Bohr collective Hamiltonian [3], which is known to possess the U(5) symmetry of the five-dimensional (5-D) harmonic oscillator [4].

In the present paper we study a sequence of potentials lying between the U(5) symmetry of the Bohr Hamiltonian and the X(5) model. The potentials, which are of the form

\textsuperscript{1}e-mail: bonat@inp.demokritos.gr
\textsuperscript{2}e-mail: lenis@inp.demokritos.gr
\textsuperscript{3}e-mail: nminkov@inrne.bas.bg
\textsuperscript{4}e-mail: raychev@phys.uni-sofia.bg, raychev@inrne.bas.bg
\textsuperscript{5}e-mail: terziev@inrne.bas.bg
\[ u_{2n}(\beta) = \beta^{2n}/2, \] with \( n \) being integer, are depicted in Fig. 1. For \( n = 1 \) an exactly soluble model with \( R_4 = E(4)/E(2) \) ratio equal to 2.646 is obtained, while \( X(5) \) (which corresponds to an infinite well potential) occurs for \( n \to \infty \) (in practice \( n = 4 \) is already quite close to \( X(5) \)). Parameter-independent predictions for the spectra and \( B(E2) \) values (up to the overall scales mentioned above) are obtained for the potentials \( \beta^2, \beta^4, \beta^6, \beta^8 \). In addition to providing a number of models giving predictions directly comparable to experiment, the present sequence of potentials shows the way for approaching the \( X(5) \) symmetry from the direction of \( U(5) \) and gives a hint on how to approach the \( X(5) \) symmetry starting from \( SU(3) \).

In Section 2 of the present paper the exactly soluble model obtained with the \( \beta^2 \) potential, to be called \( X(5) - \beta^2 \), is introduced and compared to \( X(5) \), while in Section 3 a sequence of potentials lying between the \( X(5) - \beta^2 \) and \( X(5) \) models is considered. Numerical results for spectra and \( B(E2) \) transition rates are given for all these potentials, which lie between the \( U(5) \) symmetry of the Bohr Hamiltonian [3, 4] and the \( X(5) \) model. A brief comparison to experimental data is given in Section 4, while in Section 5 perspectives for further theoretical work are discussed and the conclusions are summarized.

2. \( X(5) - \beta^2 \): A new exactly soluble model

2.1 The \( \beta \)-part of the spectrum

The original Bohr Hamiltonian [3] is

\[ H = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4\beta^2} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2 (\gamma - \frac{2\pi}{3} k)} \right] + V(\beta, \gamma), \] (1)

where \( \beta \) and \( \gamma \) are the usual collective coordinates, while \( Q_k \) \( (k = 1, 2, 3) \) are the components of angular momentum and \( B \) is the mass parameter.

One seeks solutions of the relevant Schrödinger equation having the form \( \Psi(\beta, \gamma, \theta_i) = \phi_K(\beta, \gamma)D_{M,K}(\theta_i) \), where \( \theta_i \) \( (i = 1, 2, 3) \) are the Euler angles, \( D(\theta_i) \) denote Wigner functions of them, \( L \) are the eigenvalues of angular momentum, while \( M \) and \( K \) are the eigenvalues of the projections of angular momentum on the laboratory-fixed \( z \)-axis and the body-fixed \( z' \)-axis respectively.

As pointed out in Ref. [2], in the case in which the potential has a minimum around \( \gamma = 0 \) one can write the last term of Eq. (1) in the form

\[ \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2 (\gamma - \frac{2\pi}{3} k)} \approx \frac{4}{3} \left( Q_1^2 + Q_2^2 + Q_3^2 \right) + Q_3^2 \left( \frac{1}{\sin^2 \gamma} - \frac{4}{3} \right). \] (2)

Using this result in the Schrödinger equation corresponding to the Hamiltonian of Eq. (1), introducing reduced energies \( \epsilon = 2BE/\hbar^2 \) and reduced potentials \( u = 2BV/\hbar^2 \), and assuming that the reduced potential can be separated into two terms, one depending on \( \beta \) and the other depending on \( \gamma \), i.e. \( u(\beta, \gamma) = u(\beta) + u(\gamma) \), the Schrödinger equation can be separated into two equations [2]

\[ \left[ -\frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{4\beta^2} \frac{4}{3} L(L+1) + u(\beta) \right] \xi_L(\beta) = \epsilon_\beta \xi_L(\beta), \] (3)
\[
\left[ -\frac{1}{\langle \beta^2 \rangle} \sin 3\gamma \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} + \frac{1}{4\langle \beta^2 \rangle} K^2 \left( \frac{1}{\sin^2 \gamma} - \frac{4}{3} \right) + u(\gamma) \right] \eta_K(\gamma) = \epsilon(\gamma) \eta_K(\gamma),
\]

where \( \langle \beta^2 \rangle \) is the average of \( \beta^2 \) over \( \xi(\beta) \) and \( \epsilon = \epsilon_\beta + \epsilon_\gamma \).

In Ref. [2] Eq. (3) is solved exactly for the case in which \( u(\beta) \) is an infinite well potential

\[
u(\beta) = \begin{cases}
0 & \text{if } \beta \leq \beta_W \\
\infty & \text{for } \beta > \beta_W
\end{cases}
\]

The relevant exactly soluble model is labeled as X(5) (which is not meant as a group label, although there is relation to projective representations of E(5), the Euclidean group in 5 dimensions [2]). In particular, Eq. (3) in the case of \( u(\beta) \) being an infinite well potential is transformed into a Bessel equation, the relevant eigenvalues being

\[\epsilon_{\beta,s,L} = (k_{s,L})^2, \quad k_{s,L} = \frac{x_{s,L}}{\beta_W},\]

where \( x_{s,L} \) is the \( s \)-th zero of the Bessel function \( J_\nu(k_{s,L} \beta) \), with

\[\nu = \left( \frac{L(L+1)}{3} + \frac{9}{4} \right)^{1/2},\]

while the relevant eigenfunctions are

\[\xi_{s,L}(\beta) = c_{s,L} \beta^{-3/2} J_\nu(k_{s,L} \beta),\]

where \( c_{s,L} \) are normalization constants.

Eq. (3) is exactly soluble also in the case in which \( u(\beta) = \beta^2/2 \). In this case, to which we are going to refer as the X(5)-\( \beta^2 \) model, the eigenfunctions are [5]

\[F_n^L(\beta) = \left[ \frac{2n!}{\Gamma(n + a + \frac{5}{2})} \right]^{1/2} \beta^a L_n^{a+\frac{5}{2}}(\beta^2) e^{-\beta^2/2},\]

where \( \Gamma(n) \) stands for the \( \Gamma \)-function, \( L_n^a(z) \) denotes the Laguerre polynomials [6], and

\[a = \frac{1}{2} \left( -3 + \sqrt{9 + \frac{4}{3} L(L+1)} \right),\]

while the energy eigenvalues are

\[E_{n,L} = 2n + a + \frac{5}{2} = 2n + 1 + \sqrt{\frac{9}{4} + \frac{L(L+1)}{3}}, \quad n = 0, 1, 2, \ldots\]

In the above, \( n \) is the usual oscillator quantum number. One can see that a formal correspondence between the energy levels of the X(5) model and the present X(5)-\( \beta^2 \) model, can be established through the relation

\[n = s - 1.\]
It should be emphasized that Eq. (12) expresses just a formal one-to-one correspondence between the states in the two spectra, while the origin of the two quantum numbers is different, \( s \) labeling the order of a zero of a Bessel function and \( n \) labeling the number of zeros of a Laguerre polynomial. In the present notation, the ground state band corresponds to \( s = 1 \) (\( n = 0 \)). For the energy states the notation \( E_{s,L} = E_{n+1,L} \) of Ref. [2] will be kept.

### 2.2 The \( \gamma \)-part of the spectrum

In the original version of the \( X(5) \) model [2] the potential \( u(\gamma) \) in Eq. (4) is considered as a harmonic oscillator potential. The energy eigenvalues turn out to be

\[
E(s, L, n_\gamma, K, M) = E_0 + B(x_{s,L})^2 + A n_\gamma + C K^2, \tag{13}
\]

where \( n_\gamma \) and \( K \) come from solving Eq. (4) for \( u(\gamma) \) being a harmonic oscillator potential

\[
n_\gamma = 0, K = 0; \quad n_\gamma = 1, K = \pm 2; \quad n_\gamma = 2, K = 0, \pm 4; \quad \ldots \tag{14}
\]

For \( K = 0 \) one has \( L = 0, 2, 4, \ldots \), while for \( K \neq 0 \) one obtains \( L = K, K + 1, K + 2, \ldots \).

A variation of the \( X(5) \) model is considered in Ref. [7], in which \( u(\gamma) \) is considered not as a harmonic oscillator, but as an infinite well

\[
u' = \frac{K}{2}, \quad k_{s',K} = \frac{x_{s',K}}{\gamma_W}, \quad (k_{s',K})^2 = \epsilon_{s',K}. \tag{17}
\]

In this case the energy eigenvalues are given by

\[
E(n, L, n_\gamma, K, M) = E'_0 + B' \left( 2n + 1 + \sqrt{\frac{L(L+1)}{3}} + \frac{9}{4} \right) + A'n_\gamma + C'K^2, \tag{18}
\]

which is an analogue of Eq. (13). Eq. (14) and the discussion following it remain unchanged.

Yet another variation of the \( X(5) \) model is considered in Ref. [8]. In this case, when performing the separation of variables in Eq. (1) by using Eq. (2), one keeps the \( 4K^2/3 \) term in Eq. (3) instead of Eq. (4). As a result, in Eq. (3) the term \( L(L+1) - K^2 \) appears in the place of \( L(L+1) \), and the same substitution occurs as a consequence in Eqs. (7), (10), (11), and (18). In addition, the term \( 4K^2/3 \) disappears from Eq. (4) and, as a consequence, the term containing \( K^2 \) is eliminated in Eqs. (13), (16), and (18).

### 2.3 Numerical spectra

Numerical results for the \( \beta \)-parts of the energy spectra (which correspond to no excitations in the \( \gamma \)-variable, i.e. \( n_\gamma = 0 \)) of the \( X(5)-\beta^2 \) and \( X(5) \) models are shown in Tables
1 and 2. All levels are normalized to the energy of the first excited state, $E_{1,2} - E_{1,0} = 1.0$, where the notation $E_{s,L} = E_{n+1,L}$ is used. The model predictions for these bands are parameter independent, up to an overall scale, as seen from Eqs. (6), (11). This is not the case for bands with $n_\gamma \neq 0$, since in this case, as seen from Eqs. (13), (18) the extra parameters $A$, $C$ and $A'$, $C'$ enter respectively. Therefore, in the case of the $(n_\gamma = 1, K = 2)$-band, the energies are listed in Table 1 after subtracting from them the relevant $L = 2$ bandhead, using the same normalization as above. In the case of the $(n_\gamma = 1, K = 2)$-band, the conventions of Ref. [8], described at the end of the previous subsection, have been used. The $K = 0$ bands are not affected by these conventions, anyway.

A comparison between the spectra of the X(5)-$\beta^2$ and X(5) models, given in Tables 1 and 2, leads to the following observations:

a) The members of the ground state band are characterized by the ratios

$$R_L = \frac{E_{1,L} - E_{1,0}}{E_{1,2} - E_{1,0}} \quad (19)$$

The $R_4$ ratio within the ground state band is 2.646 in the case of X(5)-$\beta^2$, as compared to 2.904 in the case of X(5). Furthermore, all normalized energy levels within the ground state band of X(5)-$\beta^2$ are lower than the corresponding X(5) normalized energy levels. The same holds within the $n_\gamma = 1$ bands. Therefore X(5)-$\beta^2$ corresponds to nuclei “less rotational” than the ones corresponding to X(5).

b) The location of the bandheads of the various $s$-families is described by the ratios

$$\bar{R}_s = \frac{E_{s,0} - E_{1,0}}{E_{1,2} - E_{1,0}} \quad (20)$$

The $\bar{R}_2$ ratio, related to the position of the bandhead of the $s = 2$ band, is 3.562 in X(5)-$\beta^2$, while it is 5.649 in X(5). In other words, the $s = 2$ bandhead in X(5)-$\beta^2$ lies much lower than in X(5). The same holds for all bandheads of $s$-families, as seen in Table 2.

c) The $s = 2$ bandhead in X(5)-$\beta^2$ lies almost midway between the $4_1^+$ state and the $6_1^+$ state of the ground state band ($E_{1,4}$ and $E_{1,6}$ respectively), while in X(5) the $s = 2$ bandhead is almost degenerate with the $6_1^+$ state ($E_{1,6}$) of the ground state band. Indeed, in the case of X(5)-$\beta^2$ one has from Eq. (19) that $R_4 = 2.646$ and $R_6 = 4.507$, their midway being 3.577, as compared to 3.562, which is the position of the $s = 2$ bandhead.

A difference between the X(5)-$\beta^2$ and X(5) models can be seen by considering the ratios

$$R_s' = \frac{E_{s,0} - E_{s,4}}{E_{s,2} - E_{s,0}} \quad (21)$$

In the $X(5)$ case one obtains the series

$$R_{s=1,2,3,...}' = 2.904, 2.798, 2.754, 2.730, 2.714, \ldots \quad (22)$$

In addition the following limit holds

$$\lim_{s \to \infty} R_s' = 2.646. \quad (23)$$
In contrast, in the framework of the X(5)-$\beta^2$ model the $R'_s$ ratios are independent of $s = n+1$

$$R'_s^\text{osc} = \frac{\sqrt{107} - 3}{\sqrt{17} - 3} \approx 2.646.$$  

(24)

In the case of a simple 5-D harmonic oscillator this ratio would have been equal to 2.

The various ratios are shown in Fig. 2. We remark that in the X(5) model the rotational collectivity of the bands decreases with increasing $n = s - 1$. Furthermore, the X(5)-$\beta^2$ constant value of the $R'_s^\text{osc}$ ratio is the limiting value of the X(5) $R'_s$ ratio for $s \to \infty$.

2.4. B(E2) transition rates

In nuclear structure it is well known that electromagnetic transition rates are quantities sensitive to the details of the underlying microscopic structure, as well as to details of the theoretical models, much more than the corresponding spectra. It is therefore a must to calculate B(E2) ratios (normalized to $B(E2; 2^+_1 \to 0^+_1) = 100$) for the X(5) and X(5)-$\beta^2$ models.

The quadrupole operator has the form [9]

$$T^{(E2)}_\mu = t\beta \left[ D^{(2)}_{\mu,0}(\theta_i) \cos \gamma + \frac{1}{\sqrt{2}} (D^{(2)}_{\mu,2}(\theta_i) + D^{(2)}_{\mu,-2}(\theta_i)) \sin \gamma \right],$$  

(25)

where $t$ is a scale factor, while the B(E2) transition rates are given by

$$B(E2; L_s \to L'_s) = \frac{|\langle L_s | T^{(E2)} | L'_s \rangle|^2}{2L_s + 1}.$$  

(26)

The matrix elements of the quadrupole operator involve an integral over the Euler angles, which is the same as in Ref. [2] and is performed by using the properties of the Wigner $D$ functions, of which only $D^{(2)}_{\mu,0}$ participates, since $\gamma \simeq 0$ in Eq. (25) (as mentioned before Eq. (2)), as well as an integral over $\beta$. After performing the integrations over the angles one is left with

$$B(E2; L_s \to L'_s) = (L_s 2L'_s | 000 \rangle 2I_{L_s,L'_s,L}^2 2L_s + 1),$$  

(27)

where the Clebsch–Gordan coefficient $(L_s 2L'_s | 000 \rangle)$ appears, which determines the relevant selection rules. In the case of X(5) the integral over $\beta$ is

$$I_{s,L;L',L'} = \int \beta \xi_s,\ell(\beta) \xi_{s',\ell'}(\beta) \beta^4 d\beta,$$  

(28)

which, as seen from Eq. (8), involves Bessel functions, while in the case of X(5)-$\beta^2$ the integral has the form

$$I_{s,L;L',L'} = \int \beta F_n^L(\beta) F_{n'}^{L'} \beta^4 d\beta,$$  

(29)

with $n = s - 1$ and $n' = s' - 1$, which involves Laguerre polynomials, as seen from Eq. (9).

The results for intraband transitions are reported in Table 3, while interband transitions are listed in Table 4. All transitions are normalized to $B(E2 : 2^+_1 \to 0^+_1) = 100$. The following observations can be made:
a) The ratio of the lowest B(E2)s within the ground state band

\[ R_{4\rightarrow 2} = \frac{B(E2; 4^+_1 \rightarrow 2^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1')} \]  

(30)

is 1.7790 in \(X(5)-\beta^2\), while it is 1.5989 in \(X(5)\). In general, the normalized intraband B(E2)s in \(X(5)-\beta^2\) are higher than the corresponding normalized B(E2)s in \(X(5)\). This is consistent with the fact that the various bands in \(X(5)-\beta^2\) appear to be “less rotational” than the corresponding bands in \(X(5)\), as remarked above. It is well known from experimental data that in near-rotational nuclei the B(E2)s within the ground state band have high values which increase relatively slowly with increasing initial angular momentum, while in near-vibrational nuclei the B(E2)s within the ground state band have low values which increase rapidly with increasing initial angular momentum (in the absence of band crossings). This experimental picture is consistent with the intraband B(E2)s listed in Table 3.

b) As far as interband transitions are concerned, it is seen in Table 4 that transitions which are strong in \(X(5)\) appear also to be strong in \(X(5)-\beta^2\), while transitions weak in \(X(5)\) are weak in \(X(5)-\beta^2\) as well.

3. A sequence of potentials lying between \(U(5)\) and \(X(5)\)

3.1 General

The two cases mentioned in the previous section are the only ones in which Eq. (3) is exactly soluble, giving spectra characterized by \(R_4\) ratios 2.646 and 2.904 for \(X(5)-\beta^2\) and \(X(5)\) respectively. However, the numerical solution of Eq. (3) for other potentials is a straightforward task. The potentials to be used in Eq. (3) have to obey the restrictions imposed by the 24 transformations mentioned in [3] and listed explicitly in [10].

A particularly interesting sequence of potentials is given by

\[ u_{2n}(\beta) = \frac{\beta^{2n}}{2}, \]  

(31)

with \(n\) being an integer. For \(n = 1\) the \(X(5)-\beta^2\) case is obtained, while for \(n \to \infty\) the infinite well of \(X(5)\) is obtained [11], as shown in Fig. 1. Therefore this sequence of potentials interpolates between the \(X(5)-\beta^2\) model and the \(X(5)\) model, in the region lying between \(U(5)\) and \(X(5)\).

3.2 Spectra

Numerical results for the spectra of the \(\beta^4\), \(\beta^6\), and \(\beta^8\) potentials have been obtained through two different methods. In one approach, the representation of the position and momentum operators in matrix form [12] has been used, while in the other the direct integration method [13] has been applied. In the latter, the differential equation is solved for each value of \(L\) separately, the successive eigenvalues for each value of \(L\) labeled by \(s = 1, 2, 3, \ldots\) (or, equivalently, by \(n = 0, 1, 2, \ldots\)). The two methods give results mutually consistent, the second one appearing of more general applicability. The results are shown in Tables 1 and 2, where excitation energies relative to the ground state, normalized to the excitation energy of the first excited state, are exhibited.

In Tables 1 and 2 the model labels \(X(5)-\beta^4\), \(X(5)-\beta^6\), \(X(5)-\beta^8\) have been used for the above-mentioned potentials, their meaning being that the \(X(5)-\beta^{2n}\) model corresponds to the potential \(\beta^{2n}/2\) plugged in the differential equation of Eq. (3) obtained in the framework
of the X(5) model. In this notation X(5)-β^{2n} with n → ∞ is simply the original X(5) model [2].

From Tables 1 and 2 it is clear that in all bands and for all values of the angular momentum, L, the potentials β^4, β^6, β^8 gradually lead from the X(5)-β^2 case to the X(5) results in a smooth way. The same conclusion is drawn from Fig. 3(a), where several levels of the ground state band of each model are shown vs. the angular momentum L, as well as from Fig. 3(b), where the bandheads of several excited bands are shown for each model as a function of the index s.

3.3. B(E2) transition rates
The calculation of the B(E2)s follows the steps described in subsection 2.4. Eq. (27) is still valid, the only difference being that in the integral over β the wave functions in the present cases are known only in numerical form and not in analytic form as in Eqs. (28), (29).

The results of the calculations for intraband transitions are shown in Table 3, while interband transitions are shown in Table 4. In addition, the normalized B(E2) transition rates within the ground state band of each model are shown in Fig. 3(c). In all cases a smooth evolution from X(5)-β^2 to X(5) is seen. Furthermore, the results are in agreement to general qualitative expectations: the more rotational the nucleus, the less rapid the increase (with increasing initial angular momentum) of the B(E2) ratios within the ground state band should be. Indeed the most rapid increase is seen in the case of X(5)-β^2, while the slowest increase is observed in the case of X(5).

4. Comparisons to experimental data
From the above observations, we conclude that a few key features of the X(5)-β^2 model, which can serve as benchmarks in the search for nuclei exhibiting such behavior, are the following:

a) The R_4 ratio (defined in Eq. (19)) should be close to 2.646.

b) The position of the s = 2 bandhead should be almost midway between the 4_1^+ and 6_1^+(E_{1,4} and E_{1,6}) states of the ground state band, the ̄R_2 ratio (defined in Eq. (20) being 3.562.

c) The ratio of the lowest B(E2)s within the ground state band, R_{4→2} (defined in Eq. (30)) should be around 1.7790.

Analogous remarks can be made in the cases of the X(5)-β^4, X(5)-β^6, and X(5)-β^8 models.

It is clear that the first place to look for nuclei exhibiting X(5)-β^{2n} behaviour is the region close to nuclei showing X(5) structure. The best examples of nuclei corresponding to the X(5) structure are so far the N = 90 isotones ^{152}Sm [14], ^{150}Nd [15], ^{156}Dy [16]. A preliminary search in the rare earths with N < 90 shows that ^{148}Nd [17] can be a candidate for X(5)-β^2, ^{158}Er [18] can be a candidate for X(5)-β^6, while ^{160}Yb [19, 20] can be a candidate for X(5)-β^4. Existing data for the ground state bands and the β_{1-bandheads} of these nuclei are compared to the corresponding model predictions in Table 5. However, much more detailed information on spectra and B(E2) transitions is needed before final conclusions can be reached.

5. Conclusion
An exactly soluble model, labeled as X(5)-β^2, has been constructed starting from the original Bohr collective Hamiltonian, separating the β and γ variables as in the X(5) model of Iachello, and using a harmonic oscillator potential for the β-variable. Furthermore it has
been proved that the potentials $\beta^{2n}$ (with $n$ being integer) provide a “bridge” between this new X(5)-$\beta^2$ model (occurring for $n = 1$) and the X(5) model of Iachello (which is obtained by putting in the Bohr Hamiltonian an infinite well potential in the $\beta$-variable, materialized for $n \rightarrow \infty$). Parameter-free (up to overall scale factors) predictions for spectra and B(E2) transition rates have been given for the potentials $\beta^2$, $\beta^4$, $\beta^6$, $\beta^8$, called the X(5)-$\beta^2$, X(5)-$\beta^4$, X(5)-$\beta^6$, and X(5)-$\beta^8$ models, respectively, lying between the U(5) symmetry of the original Bohr Hamiltonian and the X(5) model. Hints about nuclei showing this behaviour have been given.

A sequence of potentials interpolating between the U(5) and E(5) symmetries should also be worked out. Furthermore, one should try to find a sequence of potentials interpolating between SU(3) and X(5), as well as between O(6) and E(5). In other words, one should try to approach E(5) and X(5) “from the other side”. From the classical limit of the O(6) and SU(3) symmetries of the Interacting Boson Model [21] it is clear that for this purpose potentials with a minimum at $\beta \neq 0$ should be considered, the Davidson-like potentials [22]

$$u_{2n}^D(\beta) = \beta^{2n} + \frac{\beta_0^{4n}}{\beta^{2n}}$$  \hspace{1cm} (32)

being strong candidates. The Davidson potential, corresponding to $n = 1$, is known to be exactly soluble [22, 23].

Work in these directions is in progress.

Acknowledgements

The authors are thankful to Rick Casten (Yale), Jean Libert (Orsay), and Werner Scheid (Giessen) for illuminating discussions. Support through the NATO Collaborative Linkage Grant PST.CLG 978799 is gratefully acknowledged.

References

[1] F. Iachello, Phys. Rev. Lett. 85, 3580 (2000).
[2] F. Iachello, Phys. Rev. Lett. 87, 052502 (2001).
[3] A. Bohr, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. 26, no. 14 (1952).
[4] E. Chacón and M. Moshinsky, J. Math. Phys. 18, 870 (1977).
[5] M. Moshinsky, J. Math. Phys. 25, 1555 (1984).
[6] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1965).
[7] F. Iachello, in Mapping the Triangle: International Conference on Nuclear Structure, edited by A. Aprahamian, J. A. Cizewski, S. Pittel, and N. V. Zamfir (American Institute of Physics, 2002), Vol. CP638, p. 1.
[8] R. Bijker, R. F. Casten, N. V. Zamfir, and E. A. McCutchan, Phys. Rev. C, in press.
[9] L. Wilets and M. Jean, Phys. Rev. 102, 788 (1956).
[10] T. M. Corrigan, F. J. Margetan, and S. A. Williams, Phys. Rev. C 14, 2279 (1976).

[11] C. M. Bender, S. Boettcher, H. F. Jones, and V. M. Savage, J. Phys. A 32, 6771 (1999).

[12] H. J. Korsch and M. Glück, Eur. J. Phys. 23, 413 (2002).

[13] N. Minkov and W. Scheid, INRNE Sofia preprint (2003).

[14] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. 87, 052503 (2001).

[15] R. Krücke, B. Albanna, C. Bialik, R. F. Casten, J. R. Cooper, A. Dewald, N. V. Zamfir, C. J. Barton, C. W. Beausang, M. A. Caprio, A. A. Hecht, T. Klug, J. R. Novak, N. Pietralla, and P. von Brentano, Phys. Rev. Lett. 88, 232501 (2002).

[16] M. A. Caprio, N. V. Zamfir, R. F. Casten, C. J. Barton, C. W. Beausang, J. R. Cooper, A. A. Hecht, R. Krücke, H. Newman, J. R. Novak, N. Pietralla, A. Wolf, and K. E. Zyromski, Phys. Rev. C 66, 054310 (2002).

[17] M. R. Bhat, Nucl. Data Sheets 89, 797 (2000).

[18] R. G. Helmer, Nucl. Data Sheets 77, 471 (1996).

[19] C. W. Reich, Nucl. Data Sheets 78, 547 (1996).

[20] M. Sakai, At. Data Nucl. Data Tables 31, 399 (1984).

[21] F. Iachello and A. Arima, The Interacting Boson Model (Cambridge University Press, Cambridge, 1987).

[22] P. M. Davidson, Proc. R. Soc. 135, 459 (1932).

[23] D. J. Rowe and C. Bahri, J. Phys. A 31, 4947 (1998).
Figure captions

**Fig. 1.** The potentials $\beta^{2n}$, with $n = 1$ (harmonic oscillator, solid line), $n = 2$ (dash line), $n = 3$ (dash dot), $n = 4$ (dot), $n = 8$ (das dot dot), $n = 16$ (short dash dot), $n = 32$ (short dot), gradually approaching (with increasing $n$) the infinite well potential.

**Fig. 2** (Color online) The energy ratio $R'_s$, defined in Eq. (21), for the X(5) and X(5)-$\beta^2$ models. See subsection 2.3 for further discussion.

**Fig. 3** (Color online) (a) Levels of the ground state bands of the models X(5)-$\beta^{2n}$ with $n = 1$-4 and of the X(5) model, vs. the angular momentum $L$. In each model all levels are normalized to the energy of the first excited state. See subsection 3.2 for further discussion. (b) Bandhead energies of excited bands of the same models and with the same normalization, vs. the band index $s$. See subsection 3.2 for further discussion. (c) $B(E2:L_f + 2 \rightarrow L_f)$ transition rates within the ground state bands of the same models, vs. the angular momentum of the final state, $L_f$. In each model all rates are normalized to the one between the lowest states, $B(E2:2 \rightarrow 0)$. See subsection 3.3 for further discussion.
Table 1: Spectra of the X(5)-$\beta^4$, X(5)-$\beta^6$, and X(5)-$\beta^8$ models, compared to the predictions of the X(5) (Eq. (6)) and X(5)-$\beta^2$ (Eq. (11)) models, for some $s = 1$ bands. See subsections 2.3 and 3.2 for details. For the $(n_\gamma = 1, K = 2)$-band the conventions of Ref. [8] have been used, as mentioned in subsection 2.2.

| band   | L   | X(5)-$\beta^2$ | X(5)-$\beta^4$ | X(5)-$\beta^6$ | X(5)-$\beta^8$ | X(5)   |
|--------|-----|----------------|----------------|----------------|----------------|--------|
| $s = 1, n_\gamma = 0, K = 0$ |
| 0      | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 2      | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 4      | 2.646 | 2.769 | 2.824 | 2.852 | 2.904 |        |
| 6      | 4.507 | 4.929 | 5.125 | 5.230 | 5.430 |        |
| 8      | 6.453 | 7.343 | 7.777 | 8.015 | 8.483 |        |
| 10     | 8.438 | 9.954 | 10.721 | 11.151 | 12.027 |        |
| 12     | 10.445 | 12.729 | 13.922 | 14.605 | 16.041 |        |
| 14     | 12.465 | 15.647 | 17.359 | 18.355 | 20.514 |        |
| 16     | 14.494 | 18.694 | 21.013 | 22.383 | 25.437 |        |
| 18     | 16.529 | 21.858 | 24.871 | 26.677 | 30.804 |        |
| 20     | 18.568 | 25.132 | 28.923 | 31.225 | 36.611 |        |
| 22     | 20.610 | 28.506 | 33.159 | 36.017 | 42.853 |        |
| 24     | 22.654 | 31.976 | 37.571 | 41.046 | 49.528 |        |
| 26     | 24.700 | 35.536 | 42.151 | 46.302 | 56.633 |        |
| 28     | 26.748 | 39.182 | 46.895 | 51.781 | 64.166 |        |
| 30     | 28.796 | 42.909 | 51.795 | 57.475 | 72.124 |        |
| $s = 1, n_\gamma = 1, K = 2$ |
| 2      | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 3      | 0.907 | 0.925 | 0.932 | 0.936 | 0.943 |        |
| 4      | 1.863 | 1.948 | 1.986 | 2.005 | 2.040 |        |
| 5      | 2.842 | 3.046 | 3.138 | 3.186 | 3.274 |        |
| 6      | 3.836 | 4.206 | 4.377 | 4.468 | 4.639 |        |
| 7      | 4.839 | 5.420 | 5.694 | 5.842 | 6.127 |        |
| 8      | 5.848 | 6.681 | 7.084 | 7.305 | 7.737 |        |
| 9      | 6.860 | 7.986 | 8.543 | 8.850 | 9.465 |        |
| 10     | 7.876 | 9.333 | 10.066 | 10.476 | 11.310 |        |
Table 2: Same as Table 1, but for some $s > 1$ bands.

| band | L | $X(5) - \beta^2$ | $X(5) - \beta^4$ | $X(5) - \beta^6$ | $X(5) - \beta^8$ | $X(5)$ |
|------|---|----------------|----------------|----------------|----------------|--------|
| $s = 2, n_\gamma = 0, K = 0$ | | | | | | |
| 0    | 3.562 | 4.352 | 4.816 | 5.091 | 5.649 | |
| 2    | 4.562 | 5.602 | 6.232 | 6.619 | 7.450 | |
| 4    | 6.208 | 7.733 | 8.684 | 9.288 | 10.689 | |
| 6    | 8.069 | 10.248 | 11.629 | 12.527 | 14.751 | |
| 8    | 10.014 | 12.990 | 14.896 | 16.154 | 19.441 | |
| 10   | 11.999 | 15.901 | 18.419 | 20.100 | 24.687 | |
| 12   | 14.007 | 18.951 | 22.168 | 24.331 | 30.454 | |
| 14   | 16.027 | 22.125 | 26.121 | 28.827 | 36.723 | |
| 16   | 18.056 | 25.409 | 30.267 | 33.573 | 43.481 | |
| 18   | 20.091 | 28.796 | 34.594 | 38.559 | 50.719 | |
| 20   | 22.129 | 32.278 | 39.094 | 43.777 | 58.429 | |
| $s = 3, n_\gamma = 0, K = 0$ | | | | | | |
| 0    | 7.123 | 9.384 | 10.823 | 11.758 | 14.119 | |
| 2    | 8.123 | 10.817 | 12.562 | 13.710 | 16.716 | |
| 4    | 9.769 | 13.228 | 15.520 | 17.054 | 21.271 | |
| 6    | 11.630 | 16.032 | 19.004 | 21.025 | 26.832 | |
| 8    | 13.576 | 19.050 | 22.802 | 25.385 | 33.103 | |
| 10   | 15.561 | 22.221 | 26.838 | 30.051 | 39.979 | |
| 12   | 17.568 | 25.514 | 31.079 | 34.983 | 47.413 | |
| 14   | 19.589 | 28.916 | 35.504 | 40.161 | 55.377 | |
| 16   | 21.617 | 32.416 | 40.103 | 45.571 | 63.856 | |
| 18   | 23.652 | 36.007 | 44.866 | 51.202 | 72.838 | |
| 20   | 25.691 | 39.683 | 49.786 | 57.047 | 82.315 | |
| $s = 4, n_\gamma = 0, K = 0$ | | | | | | |
| 0    | 10.685 | 14.956 | 17.831 | 19.781 | 25.414 | |
| 2    | 11.685 | 16.536 | 19.842 | 22.105 | 28.805 | |
| 4    | 13.331 | 19.177 | 23.235 | 26.044 | 34.669 | |
| 6    | 15.192 | 22.225 | 27.189 | 30.667 | 41.717 | |
| 8    | 17.137 | 25.483 | 31.458 | 35.689 | 49.551 | |
| 10   | 19.123 | 28.882 | 35.955 | 41.009 | 58.033 | |
| 12   | 21.130 | 32.394 | 40.643 | 46.584 | 67.100 | |
| 14   | 23.150 | 36.002 | 45.501 | 52.392 | 76.721 | |
| 16   | 25.179 | 39.699 | 50.519 | 58.419 | 86.876 | |
| 18   | 27.214 | 43.478 | 55.689 | 64.653 | 97.552 | |
| 20   | 29.253 | 47.334 | 61.003 | 71.089 | 108.739 | |
Table 3: Intraband B(E2) transition rates for the X(5)-$\beta^4$, X(5)-$\beta^6$, and X(5)-$\beta^8$ models, compared to the predictions of the X(5) and X(5)-$\beta^2$ models. See subsections 2.4 and 3.3 for details.

| Band | $(L_s)_i$ | $(L_s)_f$ | X(5)-$\beta^2$ | X(5)-$\beta^4$ | X(5)-$\beta^6$ | X(5)-$\beta^8$ | X(5) |
|------|-----------|-----------|----------------|----------------|----------------|----------------|------|
| $(s = 1) \rightarrow (s = 1)$ |
| 2 | 0 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| 4 | 2 | 177.90 | 169.03 | 165.31 | 163.41 | 159.89 |
| 6 | 4 | 255.18 | 226.15 | 214.62 | 208.83 | 198.22 |
| 8 | 6 | 337.06 | 279.88 | 258.09 | 247.31 | 227.60 |
| 10 | 8 | 421.32 | 330.45 | 297.02 | 280.71 | 250.85 |
| 12 | 10 | 506.85 | 378.25 | 332.37 | 310.24 | 269.73 |
| 14 | 12 | 593.11 | 423.67 | 364.85 | 336.77 | 285.42 |
| 16 | 14 | 679.84 | 467.07 | 395.01 | 360.94 | 298.69 |
| 18 | 16 | 766.88 | 508.74 | 423.25 | 383.18 | 310.11 |
| 20 | 18 | 854.13 | 548.89 | 449.86 | 403.84 | 320.04 |
| 22 | 20 | 941.54 | 587.72 | 475.10 | 423.16 | 328.79 |
| 24 | 22 | 1029.06 | 625.37 | 499.14 | 441.35 | 336.57 |
| 26 | 24 | 1116.68 | 661.98 | 522.13 | 458.56 | 343.54 |
| 28 | 26 | 1204.36 | 697.64 | 544.19 | 474.91 | 349.84 |
| 30 | 28 | 1292.10 | 732.44 | 565.43 | 490.49 | 355.55 |
| $(s = 2) \rightarrow (s = 2)$ |
| 2 | 0 | 155.69 | 121.99 | 106.03 | 97.23 | 79.52 |
| 4 | 2 | 240.30 | 187.73 | 162.89 | 149.05 | 120.02 |
| 6 | 4 | 316.27 | 239.86 | 205.80 | 187.08 | 146.75 |
| 8 | 6 | 397.68 | 290.57 | 245.80 | 221.73 | 169.31 |
| 10 | 8 | 481.90 | 339.23 | 282.84 | 253.23 | 188.55 |
| 12 | 10 | 567.55 | 385.73 | 317.15 | 281.93 | 205.12 |
| 14 | 12 | 653.98 | 430.22 | 349.09 | 308.22 | 219.55 |
| 16 | 14 | 740.88 | 472.91 | 379.00 | 332.49 | 232.24 |
| 18 | 16 | 828.08 | 514.03 | 407.16 | 355.03 | 243.52 |
| 20 | 18 | 915.48 | 553.74 | 433.81 | 376.11 | 253.63 |
| $(s = 3) \rightarrow (s = 3)$ |
| 2 | 0 | 211.85 | 144.41 | 116.82 | 102.55 | 72.52 |
| 4 | 2 | 302.74 | 208.42 | 169.03 | 149.05 | 104.36 |
| 6 | 4 | 377.38 | 256.28 | 206.61 | 180.79 | 124.81 |
| 8 | 6 | 458.35 | 304.07 | 242.92 | 211.42 | 142.94 |
| 10 | 8 | 542.55 | 350.70 | 277.41 | 240.11 | 159.02 |
| 12 | 10 | 628.33 | 395.71 | 309.93 | 266.81 | 173.30 |
| 14 | 12 | 714.93 | 439.06 | 349.09 | 308.22 | 219.55 |
| 16 | 14 | 802.00 | 480.86 | 379.00 | 332.49 | 232.24 |
| 18 | 16 | 889.36 | 521.25 | 407.16 | 355.03 | 243.52 |
| 20 | 18 | 976.92 | 560.37 | 433.81 | 376.11 | 253.63 |
Table 3: (continued)

| band | \((L_s)_i\) | \((L_s)_f\) | X(5)-\(\beta^2\) | X(5)-\(\beta^4\) | X(5)-\(\beta^6\) | X(5)-\(\beta^8\) | X(5) |
|------|------------|------------|-----------------|-----------------|-----------------|-----------------|------|
| \((s = 4) \rightarrow (s = 4)\) |
| 2_4  | 0_4        | 268.23     | 165.90          | 127.76          | 108.86          | 69.06           |
| 4_4  | 2_4        | 365.19     | 229.20          | 177.27          | 151.33          | 95.96           |
| 6_4  | 4_4        | 438.49     | 273.60          | 210.90          | 179.61          | 112.50          |
| 8_4  | 6_4        | 519.04     | 318.87          | 244.18          | 207.11          | 127.62          |
| 10_4 | 8_4        | 603.25     | 363.62          | 276.37          | 233.38          | 141.39          |
| 12_4 | 10_4       | 689.16     | 407.17          | 307.10          | 258.20          | 153.88          |
| 14_4 | 12_4       | 775.94     | 449.36          | 336.35          | 281.61          | 165.22          |
| 16_4 | 14_4       | 863.19     | 490.22          | 364.21          | 303.71          | 175.57          |
| 18_4 | 16_4       | 950.72     | 529.83          | 390.81          | 324.64          | 185.07          |
| 20_4 | 18_4       | 1038.42    | 568.29          | 416.27          | 344.52          | 193.82          |
Table 4: Same as Table 3, but for interband transitions.

| band  | $(L_s)_i$ | $(L_s)_f$ | $X(5)-\beta^2$ | $X(5)-\beta^4$ | $X(5)-\beta^6$ | $X(5)-\beta^8$ | $X(5)$ |
|-------|-----------|-----------|----------------|----------------|----------------|----------------|--------|
|       |           |           | $(s = 2) \rightarrow (s = 1)$ |                |                |                |        |
| 0_2   | 2_1       | 121.92    | 93.21          | 81.03          | 74.66          | 62.41          |        |
| 2_2   | 0_1       | 1.57      | 2.04           | 2.18           | 2.21           | 2.12           |        |
| 2_2   | 2_1       | 13.40     | 11.34          | 10.28          | 9.66           | 8.22           |        |
| 2_2   | 4_1       | 96.85     | 65.53          | 53.55          | 47.59          | 36.56          |        |
| 4_2   | 2_1       | 0.06      | 0.48           | 0.72           | 0.84           | 0.94           |        |
| 4_2   | 4_1       | 12.41     | 9.63           | 8.37           | 7.68           | 6.10           |        |
| 4_2   | 6_1       | 96.68     | 59.53          | 46.23          | 39.78          | 27.87          |        |
| 6_2   | 4_1       | 0.03      | 0.16           | 0.37           | 0.49           | 0.64           |        |
| 6_2   | 6_1       | 12.32     | 8.84           | 7.41           | 6.64           | 4.92           |        |
| 6_2   | 8_1       | 95.89     | 54.68          | 40.71          | 34.09          | 21.85          |        |
| 8_2   | 6_1       | 0.12      | 0.08           | 0.27           | 0.39           | 0.56           |        |
| 8_2   | 8_1       | 12.34     | 8.29           | 6.72           | 5.90           | 4.09           |        |
| 8_2   | 10_1      | 95.03     | 50.85          | 36.56          | 29.91          | 17.64          |        |
| 10_2  | 8_1       | 0.19      | 0.05           | 0.23           | 0.35           | 0.52           |        |
| 10_2  | 10_1      | 12.37     | 7.86           | 6.19           | 5.35           | 3.49           |        |
| 10_2  | 12_1      | 94.30     | 47.79          | 33.36          | 26.76          | 14.61          |        |
|       |           |           | $(s = 3) \rightarrow (s = 2)$ |                |                |                |        |
| 0_3   | 2_2       | 241.37    | 166.55         | 136.53         | 120.61         | 86.33          |        |
| 2_3   | 0_2       | 2.74      | 3.20           | 3.19           | 3.11           | 2.66           |        |
| 2_3   | 2_2       | 25.45     | 19.61          | 16.82          | 15.19          | 11.25          |        |
| 2_3   | 4_2       | 193.64    | 120.83         | 94.54          | 81.36          | 54.01          |        |
| 4_3   | 2_2       | 0.11      | 0.70           | 0.97           | 1.08           | 1.12           |        |
| 4_3   | 4_2       | 23.75     | 17.14          | 14.27          | 12.67          | 8.83           |        |
| 4_3   | 6_2       | 193.35    | 111.85         | 84.29          | 70.99          | 43.76          |        |
| 6_3   | 4_2       | 0.04      | 0.22           | 0.47           | 0.59           | 0.71           |        |
| 6_3   | 6_2       | 23.73     | 16.09          | 13.01          | 11.37          | 7.46           |        |
| 6_3   | 8_2       | 191.71    | 104.04         | 75.89          | 62.68          | 36.03          |        |
| 8_3   | 6_2       | 0.20      | 0.11           | 0.33           | 0.46           | 0.60           |        |
| 8_3   | 8_2       | 23.89     | 15.32          | 12.07          | 10.39          | 6.44           |        |
| 8_3   | 10_2      | 189.99    | 97.61          | 69.18          | 56.16          | 30.26          |        |
| 10_3  | 8_2       | 0.33      | 0.07           | 0.28           | 0.41           | 0.56           |        |
| 10_3  | 10_2      | 24.05     | 14.69          | 11.31          | 9.61           | 5.65           |        |
| 10_3  | 12_2      | 188.51    | 92.33          | 63.80          | 50.99          | 25.87          |        |
Table 4: (continued)

| band  | \((L_s)_i\) | \((L_s)_f\) | \(X(5)-\beta^2\) | \(X(5)-\beta^4\) | \(X(5)-\beta^6\) | \(X(5)-\beta^8\) | \(X(5)\) |
|-------|-------------|-------------|-----------------|-----------------|-----------------|-----------------|--------|
| \((s = 3) \rightarrow (s = 1)\) | | | | | | | |
| 0\(_3\) | 2\(_1\) | 0.8371 | 0.0300 | 0.0461 | 0.1835 | 0.5852 |
| 2\(_3\) | 0\(_1\) | 0.1178 | 0.0311 | 0.0036 | 0.0002 | 0.0213 |
| 2\(_3\) | 2\(_1\) | 0.4123 | 0.0770 | 0.0063 | 0.0012 | 0.0546 |
| 2\(_3\) | 4\(_1\) | 0.0170 | 0.0716 | 0.2433 | 0.3876 | 0.6769 |
| 4\(_3\) | 2\(_1\) | 0.0059 | 0.0005 | 0.0033 | 0.0139 | 0.0605 |
| 4\(_3\) | 4\(_1\) | 0.3111 | 0.0471 | 0.0012 | 0.0051 | 0.0733 |
| 4\(_3\) | 6\(_1\) | 0.0049 | 0.1241 | 0.2795 | 0.4046 | 0.6616 |
| 6\(_3\) | 4\(_1\) | 0.0022 | 0.0020 | 0.0107 | 0.0240 | 0.0790 |
| 6\(_3\) | 6\(_1\) | 0.2554 | 0.0323 | 0.0001 | 0.0085 | 0.0866 |
| 8\(_3\) | 8\(_1\) | 0.0169 | 0.1235 | 0.2503 | 0.3548 | 0.5907 |
| 8\(_3\) | 8\(_1\) | 0.0090 | 0.0043 | 0.0129 | 0.0255 | 0.0833 |
| 8\(_3\) | 10\(_1\) | 0.0220 | 0.1102 | 0.2134 | 0.3011 | 0.5207 |
| 10\(_3\) | 8\(_1\) | 0.0130 | 0.0051 | 0.0126 | 0.0240 | 0.0824 |
| 10\(_3\) | 10\(_1\) | 0.1877 | 0.0181 | 0.0002 | 0.0112 | 0.0949 |
| 10\(_3\) | 12\(_1\) | 0.0231 | 0.0960 | 0.1813 | 0.2555 | 0.4610 |
| \((s = 4) \rightarrow (s = 3)\) | | | | | | | |
| 0\(_4\) | 2\(_3\) | 359.75 | 228.92 | 179.59 | 154.38 | 99.18 |
| 2\(_4\) | 0\(_3\) | 3.77 | 4.05 | 3.86 | 3.66 | 2.85 |
| 2\(_4\) | 2\(_3\) | 36.92 | 26.34 | 21.65 | 19.06 | 12.76 |
| 2\(_4\) | 4\(_3\) | 290.41 | 169.37 | 127.74 | 107.41 | 64.60 |
| 4\(_4\) | 2\(_3\) | 0.14 | 0.84 | 1.12 | 1.22 | 1.17 |
| 4\(_4\) | 4\(_3\) | 34.54 | 23.40 | 18.80 | 16.32 | 10.39 |
| 4\(_4\) | 6\(_3\) | 290.00 | 158.78 | 116.23 | 96.05 | 54.31 |
| 6\(_4\) | 4\(_3\) | 0.06 | 0.26 | 0.52 | 0.64 | 0.73 |
| 6\(_4\) | 6\(_3\) | 34.61 | 22.27 | 17.49 | 14.98 | 9.05 |
| 6\(_4\) | 8\(_3\) | 287.51 | 149.11 | 106.29 | 86.50 | 46.15 |
| 8\(_4\) | 6\(_3\) | 0.27 | 0.12 | 0.36 | 0.49 | 0.60 |
| 8\(_4\) | 8\(_3\) | 34.93 | 21.45 | 16.48 | 13.95 | 8.02 |
| 8\(_4\) | 10\(_3\) | 284.90 | 140.88 | 98.06 | 78.69 | 39.76 |
| 10\(_4\) | 8\(_3\) | 0.45 | 0.08 | 0.31 | 0.43 | 0.56 |
| 10\(_4\) | 10\(_3\) | 35.24 | 20.74 | 15.63 | 13.08 | 7.19 |
| 10\(_4\) | 12\(_3\) | 282.67 | 133.99 | 91.26 | 72.30 | 34.73 |
Table 5: Experimental spectra of the ground state (g.s.) and $\beta_1$ bands of $^{148}$Nd [17], $^{160}$Yb [19, 20], and $^{158}$Er [18], compared to the predictions of the X(5)-$\beta^2$, X(5)-$\beta^4$, and X(5)-$\beta^6$ models respectively.

| band | $L$ | $^{148}$Nd $X(5)-\beta^2$ | $^{160}$Yb $X(5)-\beta^4$ | $^{158}$Er $X(5)-\beta^6$ |
|------|-----|----------------|----------------|----------------|
| g.s. | 2   | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
|      | 4   | 2.493 | 2.646 | 2.626 | 2.769 | 2.744 | 2.824 |
|      | 6   | 4.242 | 4.507 | 4.718 | 4.929 | 5.050 | 5.125 |
|      | 8   | 6.153 | 6.453 | 7.142 | 7.343 | 7.772 | 7.777 |
|      | 10  | 8.194 | 8.438 | 9.761 | 9.954 | 10.786 | 10.721 |
|      | 12  | 10.298 | 10.445 | 12.903 | 12.729 | 13.952 | 13.922 |
|      | 14  | 17.561 | 17.359 | 17.561 | 17.359 | 17.561 | 17.359 |
| $\beta_1$ | 0   | 3.039 | 3.562 | 4.463 | 4.352 | 4.197 | 4.816 |
The diagram shows the ratio $R'_s$ plotted against the index $s$. The data points are labeled as $X(5)$ and $X(5) - \beta^2$. The ratio $R'_s$ decreases as the index $s$ increases.
(a) Energy $E$ vs. angular momentum $L$ for different models: $X(5)-\beta^2$, $X(5)-\beta^4$, $X(5)-\beta^6$, $X(5)-\beta^8$, and $X(5)$. The graphs show a linear relationship between energy and angular momentum for each model.
energy $E$ vs index $s$
