Singular Vectors of the Virasoro Algebra

Adrian Kent

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Silver Street,
Cambridge, CB3 9EW, U.K.

ABSTRACT

We give expressions for the singular vectors in the highest weight representations of the Virasoro algebra. We verify that the expressions — which take the form of a product of operators applied to the highest weight vector — do indeed define singular vectors. These results explain the patterns of embeddings amongst Virasoro algebra highest weight representations.
Conformal field theory relies on a description of the Virasoro algebra’s highest weight representations, and in particular on the classifications of the levels at which representations have singular vectors\(^{1,2}\) and the embedding relations amongst these vectors\(^{3,4}\). These embedding relations are encoded in the irreducible Virasoro characters\(^{5}\), from which the partition functions of conformal field theories are built\(^{6,7,8}\).

While this information is enough for most conformal field theoretic purposes, there are several applications\(^{9,10,11}\) in which explicit expressions for the singular vectors are needed. The first relevant work was the beautiful paper of Malikov, Feigin and Fuchs\(^{12}\) which gives expressions for the general singular vectors in finite dimensional Lie algebra Verma modules. Feigin and Fuchs\(^{4}\) have also presented some partial results describing projections, and asymptotic properties, of the Virasoro algebra singular vectors. Benoit and Saint-Aubin\(^{13}\) (BSA) found remarkable explicit expressions for the sub-class of the singular vectors \(v_{p,q}\) in which either \(p\) or \(q\) is 1. Recently, Bauer et al.\(^{14,15}\) have rewritten the BSA expressions in a compact form in which their singularity is manifest, and in which very interesting connections to integrable systems and to W-algebra theory appear. Bauer et al. have also given a new algorithm by which any vector \(v_{p,q}\) can in principle be calculated: we shall discuss this later. In this letter, we give expressions for all the singular vectors \(v_{p,q}\), show how these expressions explain the embeddings of the Virasoro algebra’s highest weight representations, and sketch proofs of these results.

First let us recall some basic facts and describe the results of Benoit and Saint-Aubin. The Virasoro algebra has commutation relations

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}C, \\
[L_m, C] &= 0.
\end{align*}
\]

The Verma module \(V(h, c)\) is the representation which contains a vector \(|h\rangle\) such
that
\[ L_m |h\rangle = 0 \quad \text{if} \ m > 0, \]
\[ L_0 |h\rangle = h |h\rangle, \]
\[ C |h\rangle = c |h\rangle, \tag{2} \]
and which has a basis comprising the states \( L_{-i_1} \ldots L_{-i_r} |h\rangle \) with \( i_1 \geq \ldots \geq i_r > 0 \).

We have the decomposition
\[ V(h, c) = \bigoplus_{n=0,1,2\ldots} V_n(h, c), \tag{3} \]
where the level \( n \) space \( V_n(h, c) \) is the eigenspace of \( L_0 \) with eigenvalue \( (h + n) \).

Define a singular vector in \( V(h, c) \) to be a vector \( v \), lying in some \( V_n(h, c) \) for \( n > 0 \), with the property that
\[ L_m v = 0 \quad \text{if} \ m > 0. \tag{4} \]

It is known\[1,2,3,4\] that there is a singular vector at level \( N \) in \( V(h, c) \) if and only if, for some positive integers \( p \) and \( q \) and complex number \( t \), we have \( N = pq \), and
\[\begin{align*}
c &= c(t) &= 13 - 6t - 6t^{-1}, \\
h &= h_{p,q}(t) &= \frac{p^2 - 1}{4}t - \frac{pq - 1}{2} + \frac{q^2 - 1}{4}t^{-1}. \tag{5}\end{align*}\]

It is also known that the singular vector at level \( N \), when it exists, is unique up to scalar multiplication. Thus, given \( p \) and \( q \), the singular vector \( v_{p,q} \) is a function of \( t \). In fact, one can write \( v_{p,q}(t) = O_{p,q}(t)|h_{p,q}(t)\rangle \), where
\[ O_{p,q}(t) = \sum_{|I|=pq} a_{I}^{p,q}(t)L_{-I} \tag{6} \]
and the \( a_{I}^{p,q}(t) \) depend polynomially on \( t \) and \( t^{-1} \). Here the sum is over sequences \( I = \{i_1, \ldots, i_n\} \) of positive integers ordered so that \( i_1 \geq \ldots \geq i_n \), we write \( L_{-I} = L_{-i_1} \ldots L_{-i_n} \) and \( |I| = i_1 + \ldots + i_n \), and take the coefficient of \( (L_{-1})^{pq} \) to be 1.

However, this turns out not to be the most convenient form in which to describe the operators \( O_{p,q}(t) \). In the cases when \( p = 1 \) or \( q = 1 \), BSA obtained
remarkably simple expressions for the operators:

\[ O_p(t) = \sum_{I = \{i_1, \ldots, i_n\} \atop |I| = p} c_p(i_1, \ldots, i_n) (-t)^{p-n} L_{-I}, \]

\[ O_{q,t}(t) = \sum_{I = \{i_1, \ldots, i_n\} \atop |I| = q} c_q(i_1, \ldots, i_n) (-t)^{-q+n} L_{-I}. \]

These sums are over all sequences of positive integers summing to \( p \) or \( q \), without any ordering restriction. The coefficients are defined by

\[ c_r(i_1, \ldots, i_n) = \prod_{1 \leq k < r \atop k \neq \sum_{j=1}^s i_j} k(r-k). \]

Our first step in generalising these results follows the ideas of Malikov-Feigin-Fuchs\[12\] by extending the enveloping algebra of the Virasoro algebra to include operators of the form \((L_{-1})^a\) for arbitrary complex values of \( a \). Thus, as well as the relations (1), we have

\[ [L_m, (L_{-1})^a] = \sum_{n=1}^{m+1} \prod_{r=1}^{n} \frac{(m+2-r)(a+1-r)}{r} (L_{-1})^{a-n} L_{m-n} \quad \text{if} \ m \geq 0, \]

\[ [(L_{-1})^a, L_m] = \sum_{n=1}^{\infty} \prod_{r=1}^{n} \frac{-(m+2-r)(a+1-r)}{r} L_{m-n} (L_{-1})^{a-n} \quad \text{if} \ m < 0, \]

and

\[ (L_{-1})^a L_{-1} = L_{-1} (L_{-1})^a = (L_{-1})^{a+1}, \quad (L_{-1})^a (L_{-1})^b = (L_{-1})^{a+b}. \]

(That is, we are considering the central extension of the algebra generated by differential operators \( z^n d \) and generalised pseudodifferential operators \( d^a \).) Denote the algebra generated by \( L_m, C \) and the \( (L_{-1})^a \) by \( \tilde{V} \). Define the \( \tilde{V} \) representation \( \tilde{V}(h,c) \) to be a generalised Verma module with a vacuum vector \( |h\rangle \) and on which \( C \) acts as the scalar \( c \). That is, equations (2) hold and the vectors \( L_{-n_1} \ldots L_{-n_r} (L_{-1})^a |h\rangle \), with \( n_1 \geq \ldots n_r \geq 2 \) and \( a \) unrestricted, form a basis for \( \tilde{V}(h,c) \). Note that \((L_{-1})^a |h\rangle \) is not zero, even when \( a \) is negative, so that \( |h\rangle \) is neither highest nor lowest weight in \( \tilde{V}(h,c) \). We shall be interested in the Virasoro
singular vectors in $\tilde{V}(h,c)$ — that is, those vectors $v$ lying in some $L_0$ eigenspace and obeying equation (11).

We say an infinite sum in $\tilde{V}$ is a well-defined operator if it has the form

$$a_0(L_{-1})^a + \sum_I a_1 L_{-1}(L_{-1})^{a-|I|},$$

where the coefficients $a_0$ and $a_I$ are finite and the sum is over sets $I = \{i_1, \ldots, i_n\}$ with $n$ finite and the integers $i_j \geq 2$; we say that the operator is of level $a$ and that a term $a_1 L_{-1}(L_{-1})^{a-|I|}$ is of order $|I|$. We also say an operator is well-defined if it can be reduced to the form (11) by commuting all $(L_{-1})^a$ terms (of any power $a$) through to the right and if moreover, for any $n$, the operator can be reduced to the form

$$a_0(L_{-1})^a + \sum_{|I| \leq n} a_1 L_{-1}(L_{-1})^{a-|I|} + O((L_{-1})^{a-n-1})$$

by a finite number of these reordering operations. Here $O((L_{-1})^m)$ means a sum of monomials of the form

$$AM_1(L_{-1})^{a_1}M_2(L_{-1})^{a_2} \ldots M_r(L_{-1})^{a_r}M_{r+1}$$

where $A$ is a scalar, the $M_i$ are either 1 or products of $L_{-i}$ with $i \geq 2$, and $a_1 + \ldots + a_r \leq m$. This means in particular that if two operators are well-defined then their product is also well-defined.

The naive generalisation of expressions (11) to $\tilde{V}$ is not well-defined. However, a well-defined generalisation can be obtained in the following way. First, rewrite the expressions (11) by commuting all $L_{-1}$ operators to the right, with no other reordering. This gives

$$O_{p,1}(t) = \sum_{k_1, \ldots, k_r} P_{k_1, \ldots, k_r}(p,t)L_{-k_1} \ldots L_{-k_r}(L_{-1})^{p-\sum k_i},$$

$$O_{1,q}(t) = \sum_{k_1, \ldots, k_r} P_{k_1, \ldots, k_r}(q,t^{-1})L_{-k_1} \ldots L_{-k_r}(L_{-1})^{q-\sum k_i},$$

(14)
where \( P_{k_1, \ldots, k_r}(p, t) \) is defined for \( p \geq \sum_i k_i \), in which range it is a polynomial function in \( p \) and \( t \). We analytically extend \( P_{k_1, \ldots, k_r}(p, t) \) to arbitrary \( p \) and define operators in \( \tilde{V} \) by

\[
\mathcal{O}_{a,1}(t) = (L_{-1})^a + \sum_{n=2}^{\infty} \sum_{\substack{k_1 + \ldots + k_r = n \\ k_i \geq 2}} P_{k_1, \ldots, k_r}(a, t) L_{-k_1} \ldots L_{-k_r}(L_{-1})^{a-n}
\]

\[
\mathcal{O}_{1,b}(t) = (L_{-1})^b + \sum_{n=2}^{\infty} \sum_{\substack{k_1 + \ldots + k_r = n \\ k_i \geq 2}} P_{k_1, \ldots, k_r}(b, t^{-1}) L_{-k_1} \ldots L_{-k_r}(L_{-1})^{b-n},
\] (15)

for any complex numbers \( a \) and \( b \). Now, when applied to the vacua, these operators create Virasoro singular vectors in the modules \( \tilde{V}(h_{a,1}(t), c(t)) \) and \( \tilde{V}(h_{1,b}(t), c(t)) \).

To see this, consider the vector

\[
L_1 \mathcal{O}_{a,1}(t)|h_{a,1}(t)\rangle
\] (16)

expressed as a sum of the form

\[
\sum_{k_1 \geq \ldots \geq k_r \geq 2} Q_{k_1, \ldots, k_r}(a, t, t^{-1}) L_{-k_1} \ldots L_{-k_r}(L_{-1})^{a-k_1-\ldots-k_r-1}|h_{a,1}(t)\rangle.
\] (17)

Each \( Q_{k_1, \ldots, k_r}(a, t, t^{-1}) \) is a polynomial, obtained as a sum of multiples of polynomials \( P_{k_1', \ldots, k_r'}(a, t) \) with \( \sum_{i=1}^{a} k_i' \leq (\sum_{i=1}^{r} k_i) + 2 \). Thus if \( N \) is an integer larger than \( (\sum_{i=1}^{r} k_i) + 1 \), we have that \( Q_{k_1, \ldots, k_r}(N, t, t^{-1}) \) is the coefficient of

\[
L_{-k_1} \ldots L_{-k_r}(L_{-1})^{N-k_1-\ldots-k_r-1}|h_{N,1}(t)\rangle
\] (18)

in the expression \( L_1 \mathcal{O}_{N,1}|h_{N,1}(t)\rangle \). That is, \( Q_{k_1, \ldots, k_r}(N, t, t^{-1}) = 0 \) for all integers \( N \geq (\sum_{i=1}^{r} k_i) + 2 \), and so the polynomial \( Q \) must be identically zero.

Similarly, \( L_2 \mathcal{O}_{a,1}(t)|h_{a,1}(t)\rangle \), \( L_1 \mathcal{O}_{1,b}(t)|h_{1,b}(t)\rangle \), and \( L_2 \mathcal{O}_{1,b}(t)|h_{1,b}(t)\rangle \) are all zero; hence \( \mathcal{O}_{a,1}(t)|h_{a,1}(t)\rangle \) and \( \mathcal{O}_{1,b}(t)|h_{1,b}(t)\rangle \) are Virasoro singular vectors.

It is now easy to see that a vector of the form

\[
X_{a_n} \ldots X_{a_1}|h\rangle,
\] (19)
where each \( X_a = \mathcal{O}_{a,1}(t) \) or \( \mathcal{O}_{1,a}(t) \), will be singular provided that, for each \( r \) from 1 to \( n \), if \( X_{ar} = \mathcal{O}_{ar,1}(t) \) then \( h + \sum_{i=1}^{r-1} a_i = h_{ar,1} \) and if \( X_{ar} = \mathcal{O}_{1,ar}(t) \) then \( h + \sum_{i=1}^{r-1} a_i = h_{1,ar} \).

To make use of these expressions, we first need to show that if \( X \) is a well-defined operator of level 0, and if \( X|\hbar\rangle \) is a Virasoro singular vector in \( \tilde{V}(h,c) \), then \( X \) is a scalar multiple of the identity. Suppose this is not so, and expand \( X \) as a sum over canonically ordered partitions:

\[
X = a_0 + \sum_{I=\{i_r, \ldots, i_1\}} \sum_{I_r \geq \cdots \geq i_1 \geq 2} a_I L_{-I}(L_{-1})^{|I|}.
\]

(20)

If \( I = \{i_r, \ldots, i_2, i_1\} \) and \( J = \{j_s, \ldots, j_2, j_1\} \) are two canonically ordered partitions, let \( I > J \) if \( |I| > |J| \), and if \( |I| = |J| \) let \( I > J \) if for some \( k \) we have \( i_k > j_k \) and \( i_l = j_l \) for all \( l < k \). Then let \( I' = \{i'_r, \ldots, i'_1\} \) be the lowest partition with non-zero coefficient in \( X \) (that is, \( a_{I'} \neq 0 \) and if \( I < I' \) then \( a_I = 0 \)). Then, letting \( I'' = \{i''_r, \ldots, i''_2\} \), we have that the coefficient of \( L_{-I''}(L_{-1})^{|I''|-i''_1+1}|\hbar\rangle \) in \( L_{i_1-1}X|\hbar\rangle \) is non-zero, contrary to our original assumption.

Now it follows from equation (3) that

\[
\begin{align*}
h_{a,1} + a &= h_{a',1} \quad \Rightarrow \quad a' = -a \text{ or } a' = a + 2t^{-1}, \\
h_{1,b} + b &= h_{1,b'} \quad \Rightarrow \quad b' = -b \text{ or } b' = b + 2t, \\
h_{a,1} &= h_{1,b} \quad \Rightarrow \quad t(a \pm 1) = (1 \pm b).
\end{align*}
\]

(21)

Hence \( \mathcal{O}_{a,1}(t)\mathcal{O}_{a,1}(t)|\hbar\rangle \) is a singular vector at level 0. Since

\[
\mathcal{O}_{a,1}(t)\mathcal{O}_{a,1}(t) = (1 + O((L_{-1})^{-2})),
\]

(22)

and since similar observations apply to \( \mathcal{O}_{1,b}(t)\mathcal{O}_{1,b}(t) \), we have that

\[
\begin{align*}
\mathcal{O}_{a,1}(t)\mathcal{O}_{a,1}(t) &= 1, \\
\mathcal{O}_{1,b}(t)\mathcal{O}_{1,b}(t) &= 1.
\end{align*}
\]

(23)
A similar argument shows that $O_{-p,1}(t)O_{p,1}(t) = 1 = O_{1,-q}(t)O_{1,q}(t)$ for positive integers $p$ and $q$, and so

$$
O_{p,1}(t) = O_{p,1}(t),
O_{1,q}(t) = O_{1,q}(t). 
$$

That is, the operators (15) are analytic extensions of the BSA operators (7). (This is not obvious from their definition.) It also follows that if $h = h_{a,1}(t) = h_{1,b}(t)$, with $t(1+a) = (1+b)$, then

$$
O_{-a,1}(t)O_{1,-b-2}(t)O_{a+2,1}(t)O_{1,1}(t)|h\rangle
$$

is a singular vector at level 0, and so we have the identity

$$
O_{1,t(1+a)+1}(t)O_{a,1}(t) = O_{a+2,1}(t)O_{1,t(1+a)-1}(t).
$$

Thus, for generic $(h, c)$, the vector $|h\rangle$ in $\tilde{V}(h, c)$ lies at one vertex of a commutative diagram which takes the form of an infinite rectangular lattice whose points correspond to Virasoro singular vectors and whose edges correspond to operators of the form $O_{a,1}$ and $O_{1,b}$. (See Figure 1.)

Next we consider $\tilde{V}(h, c)$ when $c = c(t)$ and $h = h_{p,q}(t)$ for some positive integers $p$ and $q$. We have

$$
a = p - (q - 1)t^{-1} \Rightarrow h_{p,q}(t) = h_{a,1}(t),
b = q - (p - 1)t \Rightarrow h_{p,q}(t) = h_{1,b}(t). \quad (27)
$$

Hence the vectors

$$
O_{p+(q-1)t^{-1},1}(t)O_{p+(q-3)t^{-1},1}(t)\ldots O_{p-(q-1)t^{-1},1}(t)|h_{p,q}(t)\rangle
$$

and

$$
O_{1,q+(p-1)t}(t)O_{1,q+(p-3)t}(t)\ldots O_{1,q-(p-1)t}(t)|h_{p,q}(t)\rangle
$$

are Virasoro singular vectors at level $pq$ in $\tilde{V}(h_{p,q}(t), c(t))$. But, since

$$
O_{-p+(q-1)t^{-1},1}(t)\ldots O_{-p-(q-1)t^{-1},1}(t)O_{p,q}(t)|h_{p,q}(t)\rangle
$$

and

\[ O_{1,-q+(p-1)t}(t) \ldots O_{1,-q-(p-1)t}(t)O_{p,q}(t)|h_{p,q}(t)\]  \hspace{1cm} (31)

are singular vectors at level 0, we have that

\[
O_{p,q}(t) = O_{p+(q-1)t-1,1}(t)O_{p+(q-3)t-1,1}(t) \ldots O_{p-(q-1)t-1,1}(t) \\
= O_{1,q+(p-1)t}(t)O_{1,q+(p-3)t}(t) \ldots O_{1,q-(p-1)t}(t).
\]  \hspace{1cm} (32)

Note that, although the individual operators on the right hand side of equations (32) do not generally belong to the Virasoro enveloping algebra, the two products do: the equations describe operator identities, not merely relations that hold to \(O((L_{-1})^{-1})\). This means that (28) and (29) are in fact two equivalent expressions for the singular vector \(v_{p,q}(t)\) in the Virasoro algebra Verma module \(V(h_{p,q}(t), c(t))\). These general formulae for the singular vectors of the Virasoro algebra are our main results.

An immediate application of these results is an explanation of the regularity of the Virasoro algebra’s Verma module embeddings, and hence of its character formulae. For example, take a discrete series representation \(V(h_{p,q}(t), c(t))\) with \(t = (m+1)/m\), for some integers \(m, p, q\) with \(m \geq 2\) and \(1 \leq q \leq p \leq (m-1)\). Its singular vectors fall into an embedding pattern of type \(III_\ast\) in Feigin-Fuchs’ classification. The regularity of this pattern derives from two types of identities, of which the simplest examples are

\[
O_{m+p,m+1+q}(t) = O_{2m+p,q}(t)O_{m+p,m+1-q}(t)O_{p,q}(t)  \hspace{1cm} (33)
\]

and

\[
O_{m+p,m+1-q}(t)O_{p,q}(t) = O_{p,2(m+1)-q}(t)O_{m-p,m+1-q}(t).  \hspace{1cm} (34)
\]

(See Figure 2.) The first of these follows directly from equation (32); both sides are equal to

\[
\prod_{r=0}^{m+q} O_{(m+p)+(m+q-2r)t^{-1},1}(t), \hspace{1cm} (35)
\]
where for non-commuting operators we set $\prod_{i=1}^r A_i = A_1 \ldots A_r$. The second identity follows from equations (32) and (26). We have

$$O_{m+p,m+1-q}(t)O_{p,q}(t) = \prod_{r=0}^{m-q} O_{(m+p)+(m-q-2r)t^{-1},1}(t) \prod_{r=0}^{p-1} O_{1,q+(p-1-2r)t}(t)$$

and

$$O_{p,2(m+1)-q}(t)O_{m-p,m+1-q}(t) = \prod_{r=0}^{p-1} O_{1,(2(m+1)-q)+(p-1-2r)t}(t) \prod_{r=0}^{m-q} O_{(m-p)+(m-q-2r)t^{-1},1}(t).$$

Then equation (26) shows that the relevant quadrilateral in figure 2 can be refined to an $O$ operator commutative diagram in the form of a $p \times (m+1-q)$ rectangular lattice, with the right hand sides of (36) and (37) each represented by two of the outer edges.

The literature now contains quite a few results about Virasoro singular vectors, and so we conclude with a brief comparative discussion. Perhaps it is worth remarking that the Kac determinant formula already provides not only an existence theorem, but also an algorithm for calculating each singular vector $v_{p,q}(t)$: the singular vector is the zero eigenvector of the inner product matrix $M_{pq}(h_{p,q}(t), c(t))$ at generic $t$, and the inner product matrix can be obtained from the Virasoro algebra’s commutation relations. Bauer et al. gave recursion relations which define a chain of vectors of levels $0, 1, 2, \ldots, pq$ in the Verma module $V(h_{p,q}(t), c(t))$: the last of these vectors is the singular vector $v_{p,q}(t)$. While these results haven’t yet led to explicit expressions for the singular vectors, they might well eventually do so; they might also — as Bauer et al. suggest — lead to a new proof of the Kac determinant formula, and perhaps also of the Feigin-Fuchs classification. This would certainly be valuable: an intrinsic method of proving determinant and character formulae may well be necessary in understanding the representation theory of some of the Virasoro algebra’s extensions. In any case, Bauer et al. have described an interesting algebraic structure within the Verma module, which needs to be better understood; the connections with W-algebra theory and integrable models are particularly fascinating.
Our expressions for the Virasoro singular vectors in the modules \( \tilde{V}(h_{p,q}(t), c(t)) \) are simple and completely explicit. One can easily calculate the projection of one of these vectors to the Verma module \( V(h_{p,q}(t), c(t)) \). However, we still do not have general explicit expressions — that is, expressions given solely in terms of elementary functions and enveloping algebra elements — for the singular vectors in the Verma modules. Such expressions would certainly be of some intellectual interest. On the other hand, we expect that the product formulae (32), together with the relations (23) and (26), will actually prove much more useful in any theoretical applications. For example, it seems unlikely that the Verma module embeddings can be understood so readily without use of the product formulae (32). (Compare Malikov-Feigin-Fuchs’ formulae\(^{12}\) for the \( sl(n) \) singular vectors: the Verma module embeddings can easily be read off from the complex exponent formula (19), but are thoroughly obscured in equation (20).)

To summarise: the Virasoro singular vectors derive from a simple algebraic structure within the modules \( \tilde{V}(h, c) \), in which the analytically continued BSA operators are the analogues of the powers of simple roots used by Malikov-Feigin-Fuchs in the Kac-Moody case.\(^{12}\) We suspect that similar structures underlie the highest weight representation theory of the Virasoro algebra’s extensions.

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Figure 1
\[ |h_{p,q}(m)\rangle \]

Figure 2
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