BOUNDARY LAYER SOLUTION OF THE BOLTZMANN EQUATION FOR DIFFUSIVE REFLECTION BOUNDARY CONDITIONS IN HALF-SPACE

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Abstract. We study steady Boltzmann equation in half-space, which arises in the Knudsen boundary layer problem, with diffusive reflection boundary conditions. Under certain admissible conditions and the source term decaying exponentially, we establish the existence of boundary layer solution for both linear and nonlinear Boltzmann equation in half-space with diffusive reflection boundary condition in $L^\infty_{x,v}$ when the far-field Mach number of the Maxwellian is zero. The continuity and the spacial decay of the solution are obtained. The uniqueness is established under some constraint conditions.

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1. INTRODUCTION

In the present paper, we consider the steady Boltzmann equation
\[ v_3 \cdot \partial_x F = Q(F,F) + S, \quad (x,v) \in \mathbb{R}_+ \times \mathbb{R}^3, \] (1.1)
with $\mathbb{R}_+ = (0, +\infty)$. The Boltzmann collision term $Q(F_1,F_2)$ on the right is defined in terms of the following bilinear form
\[ Q(F_1,F_2) \equiv \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) F_1(u') F_2(v') \, d\omega du - \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) F_1(u) F_2(v) \, d\omega du \]
\[ := Q_+ (F_1,F_2) - Q_-(F_1,F_2), \] (1.2)
where the relationship between the post-collision velocity $(v', u')$ of two particles with the pre-collision velocity $(v, u)$ is given by
\[ u' = u + [(v - u) \cdot \omega] \omega, \quad v' = v - [(v - u) \cdot \omega] \omega, \]
for $\omega \in S^2$, which can be determined by conservation laws of momentum and energy
\[ u' + v' = u + v, \quad |u'|^2 + |v'|^2 = |u|^2 + |v|^2. \]

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The Boltzmann collision kernel \( B = B(v - u, \theta) \) in (1.2) depends only on \(|v - u|\) and \( \theta \) with \( \cos \theta = (v - u) \cdot \omega / |v - u| \). Throughout this paper, we consider the hard sphere model, i.e.,
\[
B(v - u, \theta) = |(v - u) \cdot \omega|
\]

There have been many studies on the half-space problem of the steady Boltzmann equation in the literature. The existence, uniqueness and properties of asymptotic behavior were proved in [1] for the linearized Boltzmann equation of a hard sphere gas for the Dirichlet type boundary condition, see [4] for a classification of well-posed kinetic boundary layer problem. Later, the existence of nonlinear boundary layers with small magnitudes and Dirichlet boundary condition for the hard sphere model was established in [14], it was shown that the existence of a solution depends on the Mach number of the far field Maxwellian, and an implicit solvability conditions yielding the co-dimensions of the boundary data (see also [3]), and we refer [15, 6] for the time-asymptotic stability of such boundary layer solution. Wu [16] established the existence of unique solution for a modified boundary layer solution with Dirichlet boundary condition in \( L^\infty_{x,v} \) space which is used to prove the Hilbert expansion (diffusive scaling) in a disk. Recently, Bernhoff-Golse [2] offered the existence and uniqueness of a uniformly decaying boundary layer type solution in the situation that gas is in contact with its condensed phase. For the specular reflection condition and the solution tends to a global Maxwellian with zero Mach number at the far field, Golse-Perthame-Sulem [10] and Huang-Jiang-Wang [13] proved the existence, uniqueness and asymptotic behavior in different functional space, respectively.

For the diffusive reflection boundary condition, the existence of steady nonlinear Boltzmann solution is proved [5, 7] in weighted \( L^\infty_{x,v} \) in smooth bounded domain, and the time-asymptotic stability of such steady solutions is also obtained. For half-space problem, Coron-Golse-Sulem [4] proved the existence of solution for linear Boltzmann equation in half-space with zero source term in the functional space \( L^\infty(e^{\gamma x} dx; L^2(\{|v_3|dv\})) \). To the best of our knowledge, there is no result on the existence of solution to the nonlinear boundary layer problem (1.1) under diffusive reflection boundary condition in \( L^\infty_{x,v} \) when the Mach number of far-field Maxwellian is zero. In fact, the continuity of boundary layer solution is very important to close the Hilbert expansion of Boltzmann equation in the initial boundary value problem, see [12] for instance.

We supplement the Boltzmann equation (1.1) with the perturbed diffusive reflection boundary conditions
\[
F(0, v)|_{v_3 > 0} = \sqrt{2\pi} \mu(v) \int_{u_3 < 0} F(0, u)|u_3|du + R(v)
\]
and
\[
\lim_{x \to \infty} F(x, v) = p_E^0 \mu(v),
\]
where \( p_E^0 > 0 \) is some given positive constant, and \( \mu(v) \) is the normalized global Maxwellian
\[
\mu(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v|^2}{2}}.
\]
The boundary function \( R(v) \) is defined only for \( v_3 > 0 \), and one can extend its definition to \( \mathbb{R}^3 \) by taking \( R(v) = 0 \) when \( v_3 \leq 0 \). The positive constant \( p_E^0 \) appear in general when we derive the Knudsen boundary layers in the process of Hilbert expansion of Boltzmann equation (compressible Euler scaling) with physical boundary. In fact, the \( p_E^0 \) involves with the boundary pressure of Euler solution.

In general, for given source term \( S(x, v) \) and boundary perturbed term \( R(v) \), the boundary value problem (BVP) (1.1) (1.3)-(1.4) may be not solvable. In fact, for given \( S \) and \( R \), we have to replace the far-field condition (1.4) by a new one. Hence the main difficulty is to construct the
new far-field state and obtain uniform estimates for both the new far-field state and the solution. To achieve our goal, we shall use the $L^2_{x,v} - L^\infty_{x,v}$-framework initiated by Guo [11].

We search the solution of (1.1) in the following form

$$F(x, v) = p^0_B \mu(v) + \sqrt{\mu(v)} f(x, v).$$

Then (1.1), (1.3), (1.4) are rewritten as

$$
\begin{aligned}
&v_3 \cdot \partial_x f + p^0_B \mathbf{L} f = \Gamma(f, f) + \mathcal{G}, \\
&f(0, v)|_{v_3 > 0} = P_\gamma f(0, v) + \mathcal{R}(v), \\
&\lim_{x \to \infty} f(x, v) = 0,
\end{aligned}
$$

(1.7)

where we have used the notations

$$
\mathcal{G} := \frac{\mathcal{S}}{\sqrt{\mu(v)}}, \quad \mathcal{R} := \frac{\mathcal{R}}{\sqrt{\mu(v)}},
$$

(1.8)

and the linearized operator $\mathbf{L}$ is given by $\mathbf{L} = \nu(v) - K$ with $K = K_2 - K_1$ and

$$
\begin{aligned}
(K_1 f)(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - u) \cdot \omega| \sqrt{\mu(u)} \mu(v) f(u) \, d\omega \, du, \\
(K_2 f)(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - u) \cdot \omega| \sqrt{\mu(u)} \mu(v') f(v') \, d\omega \, du, \\
&\quad + \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \sqrt{\mu(u)} \mu(u') f(u') \, d\omega \, du,
\end{aligned}
$$

and

$$
\nu(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \mu(u) \, d\omega \, du \equiv 1 + |v|,
$$

$$
\Gamma(f, f) = \frac{1}{\sqrt{\mu(v)}} Q(\sqrt{\mu} f, \sqrt{\mu} f).
$$

The null space of the operator $\mathbf{L}$ is the 5-dimensional space of collision invariants:

$$
\mathcal{N} = \ker \mathbf{L} = \text{span} \left\{ \sqrt{\mu}, \ v \sqrt{\mu}, \ \frac{\sqrt{\mu}}{2} |v|^2, \ \frac{1}{2} \left( |v|^2 - 3 \right) \sqrt{\mu} \right\}.
$$

And let $\mathbf{P}$ denote the projection operator from $L^2(\mathbb{R}^3)$ to $\mathcal{N}$. We list some useful properties of $\mathbf{L}$ and $K, \Gamma$ in Appendix A.

For later use we define the velocity weight function

$$
w(v) = \left( 1 + |v|^2 \right)^{\beta/2} e^{\varpi |v|^2}, \quad 0 \leq \varpi < \frac{1}{4},
$$

(1.10)

In general, for given $\mathcal{G}$ and $\mathcal{R}$, the boundary value problem (1.7) may be not solvable. In fact, our main result is

**Theorem 1.1.** Recall the weight function $w(v)$ in (1.10), and let $\beta \geq 3$ and $0 \leq \varpi < \frac{1}{8}$. Let $\sigma_0 \in (0, 1)$ be suitably small. Assume $\mathcal{G} \in \mathcal{N}_1^1$ and $\int_{v_3 > 0} v_3 \sqrt{\mu(v)} \mathcal{R}(v) \, dv = 0$. There exist a small $\delta_0 > 0$ such that if

$$
\|e^{\sigma_0 x} \nu^{-1} w \mathcal{G}\|_{L^2_{x,v}} + |w \mathcal{R}|_{L^\infty_{x,v}(\mathbb{R}^3_+)} := \delta \leq \delta_0,
$$

(1.11)

then there is a unique function $f^\infty := \mathcal{G}(\mathcal{G}, \mathcal{R})$ with

$$
\begin{aligned}
f^\infty(v) &= \left\{ b_1^\infty v_1 + b_2^\infty v_2 + c^\infty \left( \frac{|v|^2}{2} - \frac{3}{2} \right) \sqrt{\mu(v)} \right\} \in \mathcal{N}, \\
|\{b_1^\infty, b_2^\infty, c^\infty\}| &\leq C \left\{ \|e^{\sigma_0 x} \nu^{-1} w \mathcal{G}\|_{L^2_{x,v}} + |w \mathcal{R}|_{L^\infty_{x,v}(\mathbb{R}^3_+)} \right\},
\end{aligned}
$$

(1.12)
 such that the following nonlinear boundary layer problem of Boltzmann equation

\[
\begin{align*}
\left\{ 
\begin{array}{l}
v_3 \cdot \partial_x f + p_3^0 L f = \Gamma(f, f) + \mathcal{S}, 
(x, v) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
f(0, v)|_{v_3 > 0} = P_2 f(0, v) - (I - P_2) f^\infty(v) + \mathcal{R}(v), \\
\lim_{x \to \infty} f(x, v) = 0, 
\end{array}
\right.
\end{align*}
\]

has a unique mild solution \( f(x, v) \) satisfying

\[
\|e^{\sigma x} w f\|_{L^\infty_{x,v}(\mathbb{R}^3)} + \|w f(0)\|_{L^\infty_{x,v}(\mathbb{R}^3)} \leq \frac{C}{\sigma_0 - \sigma} \left\{ \|e^{\sigma_0 x} \nu^{-1} w \mathcal{S}\|_{L^\infty_{x,v}} + \|w \mathcal{R}\|_{L^\infty_{x,v}(\mathbb{R}^3_+)} \right\},
\]

where \( \sigma > 0 \) is a constant such \( \sigma \in (0, \sigma_0) \), and \( C > 0 \) is a constant independent of \( \sigma \). Moreover, if \( \mathcal{S} \) is continuous in \( \mathbb{R}_+ \times \mathbb{R}^3 \) and \( \mathcal{R}(v) \) is continuous in \( \{v \in \mathbb{R}^3_+\} \), then \( f(x, v) \) is continuous away from the grazing set \( \{0, v \) : \( v \in \mathbb{R}^3, v_3 = 0\} \).

Furthermore, let \( \mathcal{S}_i \in \mathcal{N}^\perp \) and \( \int_{v_3 > 0} v_3 \sqrt{\mu(v)} \mathcal{R}_i(v) dv = 0, i = 1, 2 \) satisfying (1.11). Let \( f_i, f_i^\infty \) be the solutions obtain above by replacing \( \mathcal{S}, \mathcal{R} \) by \( \mathcal{S}_i, \mathcal{R}_i \), and denote

\[
f_i^\infty(v) = \mathcal{G}(\mathcal{S}_i, \mathcal{R}_i) = \left\{ \begin{array}{l}
b_{i,1}^\infty v_1 + b_{i,2}^\infty v_2 + c_i^\infty \left( \frac{|v|^2}{2} - \frac{3}{2} \right) \sqrt{\mu(v)} \end{array} \right\} i = 1, 2,
\]

then it holds that \( \mathcal{G}(0, v) = 0 \) and

\[
\begin{aligned}
\|e^{\sigma x} w (f_1 - f_2)\|_{L^\infty_{x,v}(\mathbb{R}^3)} + \|w (f_1 - f_2)(0)\|_{L^\infty_{x,v}(\mathbb{R}^3)} &+ \|(b_{1,1}^\infty - b_{2,1}^\infty, b_{1,2}^\infty - b_{2,2}^\infty, c_1^\infty - c_2^\infty)\|
\leq \frac{C}{\sigma_0 - \sigma} \left\{ \|e^{\sigma_0 x} \nu^{-1} w (\mathcal{S}_1 - \mathcal{S}_2)\|_{L^\infty_{x,v}} + \|w (\mathcal{R}_1 - \mathcal{R}_2)\|_{L^\infty_{x,v}(\mathbb{R}^3_+)} \right\}.
\end{aligned}
\]

That means the solution of \( f, f^\infty \) of (1.13) depend continuously on \( \mathcal{S}, \mathcal{R} \) in the sense of (1.14).

Remark 1.2. We note that \( (I - P_\gamma)(a^\infty \sqrt{\mu} + f^\infty) \equiv (I - P_\gamma)(f^\infty) \) for all \( a^\infty \in \mathbb{R} \), so the uniqueness of \( f^\infty \) is in the sense that we normalize \( a^\infty = 0 \). In fact, the uniqueness of \( f, f^\infty \) is under the constraints conditions (1.12) and (1.14).

Corollary 1.3. Recall the weight function \( w(v) \) in (1.10), and let \( \beta \geq 3 \) and \( 0 \leq \omega < \frac{1}{3} \). Let \( \sigma_0 \in (0, 1) \) be suitably small. Assume \( \mathcal{S} \in \mathcal{N}^\perp \) and \( \int_{v_3 > 0} v_3 \sqrt{\mu(v)} \mathcal{R}(v) dv = 0 \). If there is a function \( \mathbf{r}(v) \) with \( w \mathbf{r} \in L^\infty_{x,v}(\mathbb{R}^3_+) \) such that

\[
\mathcal{R}(v) = -(I - P_\gamma) \mathcal{G}(\mathcal{S}, \mathbf{r})(v) + \mathbf{r}(v),
\]

and \( \|e^{\sigma_0 x} \nu^{-1} w \mathcal{S}\|_{L^\infty_{x,v}(\mathbb{R}^3)} + \|w \mathbf{r}\|_{L^\infty_{x,v}(\mathbb{R}^3_+)} \leq \delta_0 \), then there is a unique mild solution \( f(x, v) \) to the nonlinear boundary layer problem (1.17), and satisfies

\[
\begin{aligned}
\|\mathcal{G}(\mathcal{S}, \mathbf{r})\|_{L^\infty_{x,v}(\mathbb{R}^3)} &\leq C \left\{ \|e^{\sigma_0 x} \nu^{-1} w \mathcal{S}\|_{L^\infty_{x,v}} + \|w \mathbf{r}\|_{L^\infty_{x,v}(\mathbb{R}^3_+)} \right\}; \\
\|e^{\sigma x} w f\|_{L^\infty_{x,v}(\mathbb{R}^3)} + \|w f(0)\|_{L^\infty_{x,v}(\mathbb{R}^3)} &\leq \frac{C}{\sigma_0 - \sigma} \left\{ \|e^{\sigma_0 x} \nu^{-1} w \mathcal{S}\|_{L^\infty_{x,v}} + \|w \mathbf{r}\|_{L^\infty_{x,v}(\mathbb{R}^3_+)} \right\}.
\end{aligned}
\]

Moreover, if \( \mathcal{S} \) is continuous in \( \mathbb{R}_+ \times \mathbb{R}^3 \) and \( \mathcal{R}(v) \) is continuous in \( \{v \in \mathbb{R}^3_+\} \), then \( f(x, v) \) is continuous away from the grazing set \( \{0, v \) : \( v \in \mathbb{R}^3, v_3 = 0\} \).

Remark 1.4. Corollary (1.3) means that the nonlinear boundary layer problem (1.17) are solvable if the boundary perturbation \( \mathcal{R} \) is on the locally continuous manifold \( \mathcal{M} := \{\mathcal{R} : \mathcal{R} = -(I - P_\gamma) \mathcal{G}(\mathcal{S}, \mathbf{r}) + \mathbf{r}, \|e^{\sigma_0 x} \nu^{-1} w \mathcal{S}\|_{L^\infty_{x,v}} + \|w \mathbf{r}\|_{L^\infty_{x,v}(\mathbb{R}^3_+)} \leq \delta_0 \} \).
To study the well-posedness of nonlinear boundary layer problem, we need first to consider the existence of solution for the following linearized boundary layer problem with a source term

$$\begin{cases} v_3 \partial_x f + p_{E}^0 L f = g, & (x, v) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ f(0, v)|_{v_3>0} = P_{\gamma} f(0, v) + r(v), \\ \lim_{x \to \infty} f(x, v) = 0. \end{cases}$$

(1.19)

**Theorem 1.5.** Recall the weight function $w(v)$ in (1.10), and let $\beta \geq 3$ and $0 \leq \alpha < \frac{1}{8}$. Let $\sigma_0 \in (0, 1)$ be suitably small. Assume $g \in \mathcal{N}^1$ and $\int_{v_3>0} v_3 \sqrt{\mu(v)} r(v) dv = 0$ with

$$\|e^{\sigma_0 \nu^{-1}} w g\|_{L^\infty_{v_3}} + |w r|_{L^\infty_{(\mathbb{R}_+^3)}} < +\infty.$$  

(1.20)

Then there exists a unique function $f^\infty(v) := G(g, r)(v)$ with

$$f^\infty(v) = \begin{cases} b_1^\infty v_1 + b_2^\infty v_2 + c^\infty \left( \frac{|v|^2}{2} - \frac{3}{2} \right) \sqrt{\mu(v)}, \\ (b_1^\infty, b_2^\infty, c^\infty) \leq C \left\{ \|e^{\sigma_0 \nu^{-1}} w g\|_{L^\infty_{v_3}} + |w r|_{L^\infty_{(\mathbb{R}_+^3)}} \right\}, \end{cases}$$

(1.21)

such that the following linearized boundary layer problem of Boltzmann equation

$$\begin{cases} v_3 \partial_x f + p_{E}^0 L f = g, & (x, v) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ f(0, v)|_{v_3>0} = P_{\gamma} f(0, v) - (I - P_{\gamma}) f^\infty + r(v), \\ \lim_{x \to \infty} f(x, v) = 0, \end{cases}$$

(1.22)

has a unique solution $f(x, v)$ satisfying

$$\|e^{\sigma x} w f\|_{L^\infty_{v_3}} + |w f(0)|_{L^\infty(\mathbb{R}^3)} \leq C \left\{ \frac{1}{\sigma_0 - \sigma} \|e^{\sigma_0 \nu^{-1}} w g\|_{L^\infty_{v_3}} + |w r|_{L^\infty_{(\mathbb{R}_+^3)}} \right\}.$$  

(1.23)

The uniqueness of $f^\infty(v)$ is under the constraints (1.21). Moreover, if $g$ is continuous in $\mathbb{R}_+ \times \mathbb{R}^3$ and $r(v)$ is continuous in $\{ v \in \mathbb{R}_+^3 \}$, then $f(x, v)$ is continuous away from the grazing set $\{ (0, v) : v \in \mathbb{R}^3, v_3 = 0 \}$.

**Remark 1.6.** Let $g_i \in \mathcal{N}^1$ and $\int_{v_3>0} v_3 \sqrt{\mu(v)} r_i(v) dv = 0$, $i = 1, 2$ satisfying (1.20). Let $f_i, f_i^\infty$ be the solutions obtain in (1.21), (1.22) by replacing $g, r$ by $g_i, r_i$, and denote

$$f_i^\infty(v) = G(g_i, r_i) = \begin{cases} b_1^\infty v_1 + b_2^\infty v_2 + c^\infty \left( \frac{|v|^2}{2} - \frac{3}{2} \right) \sqrt{\mu(v)} & i = 1, 2, \end{cases}$$

(1.24)

then it follows from the uniqueness, (1.21), (1.22) and (1.23) that

$$G(g_1, r_1) + G(g_2, r_2) = G(g_1 + g_2, r_1 + r_2), \quad G(0, 0) = 0$$

(1.25)

and

$$| (b_1^\infty - b_2^\infty, b_1^\infty - b_2^\infty, c_1^\infty - c_2^\infty) | \leq C \left\{ \|e^{\sigma_0 \nu^{-1}} w (g_1 - g_2)\|_{L^\infty_{v_3}} + |w (r_1 - r_2)|_{L^\infty_{(\mathbb{R}_+^3)}} \right\}.$$  

(1.26)

Then means $f, f^\infty$ depend continuously on $g, r$ in the sense of (1.20).

**Remark 1.7.** Theorem (1.5) can be used to determine the boundary conditions of higher order viscous boundary layers and Knudsen boundary layers in the Hilbert expansion of Boltzmann equation (compressible Euler scaling) with diffusive reflection boundary conditions.
Corollary 1.8. Assume \( g, r \) satisfy the conditions in Theorem 1.7. If there exists a function \( r(v) \) with \( w \in L^\infty(\mathbb{R}^3_+) \) such that
\[
  r(v) = -(I - P_\gamma)G(g, r)(v) + r(v), \quad \forall \ v \in \mathbb{R}^3_+, \tag{1.27}
\]
then there is a unique mild solution \( f(x, v) \) to the linearized steady boundary layer problem (1.19), and satisfies
\[
  |G(g, r)|_{L^\infty(\mathbb{R}^3)} \leq C\left\{ \left| e^{\sigma_0} \nu^{-1} w g \right|_{L^\infty, \nu} + |w|_{L^\infty(\mathbb{R}^3_+)} \right\},
\]
\[
  \left| e^{\sigma x} w f \right|_{L^\infty, \nu} + |w f(0)|_{L^\infty(\mathbb{R}^3)} \leq C\left\{ \frac{1}{\sigma_0 - \sigma} \left| e^{\sigma_0} \nu^{-1} w g \right|_{L^\infty, \nu} + |w|_{L^\infty(\mathbb{R}^3_+)} \right\}. \tag{1.28}
\]
Moreover, if \( g \) is continuous in \( \mathbb{R}_+ \times \mathbb{R}^3 \) and \( r(v) \) is continuous in \( \{ v \in \mathbb{R}^3_+ \} \), then \( f(x, v) \) is continuous away from the grazing set \( \{ (0, v) : v \in \mathbb{R}^3, v_3 = 0 \} \).

Remark 1.9. Corollary 1.8 means that the steady boundary value problem (1.19) are solvable if the boundary perturbation \( r \) is on the manifold \( M := \{ r : r = -(I - P_\gamma)G(g, r) + r \text{ with } e^{\sigma_0} \nu^{-1} w g \in L^\infty_{x,v}, w \in L^\infty(\mathbb{R}^3_+), \} \).

We now briefly comment on the analysis of the present paper. The main part of this paper is to prove Theorem 1.5 that is to prove the existence of solution for the linear boundary layer problem of Boltzmann equation with diffusive reflection boundary conditions. We start with the construction of approximate solutions of the truncated problem (2.12) with penalization. Firstly we establish a priori uniform \( L^\infty_{x,v} \)-estimate which is independent of both truncation parameter \( d \geq 0 \) and penalization parameter \( \varepsilon \in [0, 1] \), see Lemma 2.4 for details. With the help of the uniform \( L^\infty_{x,v} \)-estimate, and by similar arguments as in [5], we can obtain the solution \( \bar{f} \) of approximate problem (2.12), with
\[
  \left| w f \right|_{L^\infty, \nu} + |w f(0)|_{L^\infty(\gamma)} \leq C\left\{ \left| \nu^{-1} w g \right|_{L^\infty, \nu} + |w|_{L^\infty(\gamma, \gamma)} \right\}, \tag{1.29}
\]
see Lemmas 2.4 2.4 for details.

We note that the bound on the right hand side (RHS) of (1.29) depends on \( d \). However, to construct the boundary layer solution of Boltzmann equation in half-space, we need some uniform estimates independent of \( d \) so that we can take the limit \( d \to +\infty \). We define
\[
  \tilde{f}(x, v) := f(x, v) - \sqrt{2\pi \mu(v)} z_{\gamma_+}(f),
\]
where \( z_{\gamma_+}(f) := \int_{v_3 < 0} |v_3|\sqrt{\mu(v)} f(0, v) dv \). Clearly, \( z_{\gamma_+}(\tilde{f}) = \int_{v_3 < 0} |v_3|\sqrt{\mu(v)} f(0, v) dv = 0 \) and the equation of \( \tilde{f} \) has the same form as \( f \), see [5, 24]. Then, by energy estimate, we can prove
\[
  \left| (I - P_\gamma) \tilde{f}(0) \right|_{L^2(\gamma_+)}^2 + \int_0^d e^{2\sigma_1 x} \| (I - P) \tilde{f}(x) \|_{L^2}^2 dx \leq C\left\{ \left| |r|_{L^2(\gamma_+)} \right| + \int_0^d e^{2\sigma_1 x} \left| g(x) \right|_{L^2(\gamma_+)}^2 dx \right\} \quad \text{for} \quad \sigma_1 \in [0, \sigma_0]. \tag{1.30}
\]
It is very important that the term on RHS of (1.30) is independent of \( d \). To obtain the uniform in \( d \) estimate for macroscopic term, motivated by [10], we define
\[
  \tilde{f}(x, v) := \tilde{f}(x, v) + \Phi(v)
\]
with
\[
  \Phi(v) := \left[ \phi_0 + \phi_1 v_1 + \phi_2 v_2 + \phi_3 \left( \frac{|v|^2}{2} - \frac{3}{2} \right) \sqrt{\mu(v)} \right],
\]
where \((\phi_0, \phi_1, \phi_2, \phi_3)(d)\) are four constants determined in Lemma 2.11. And \(\tilde{f}\) satisfies

\[
\begin{aligned}
\begin{cases}
\nu \partial_x \tilde{f} + P_E \mathbf{L} \tilde{f} = g, & (x, v) \in \Omega_d \times \mathbb{R}^3, \\
\tilde{f}(0, v)|_{v_3 > 0} = P_x \tilde{f}(0, v) + (I - P_\gamma) \Phi + r(v), \\
\tilde{f}(d, v)|_{v_3 < 0} = \tilde{f}(x, \mathcal{R} v).
\end{cases}
\end{aligned}
\]

Then we can get the uniform \(L^2_{x,v}\)-estimate for \(\tilde{f}\), i.e.,

\[
\|e^{\sigma x} \tilde{f}\|_{L^2_{x,v}} \leq C \left\{ |r|_{L^2(\gamma^-)} + \frac{1}{\sigma_1 - \sigma} \|e^{\sigma_1 x} g\|_{L^2_{x,v}} \right\},
\]

with \(0 < \sigma < \sigma_1 \leq \sigma_0\), and the constant \(C > 0\) is independent of \(d\), see Lemma 2.12. Fortunately, the right hand side of (1.31) is independent of \(\Phi\). Applying the \(L^2_{x,v}\) estimate to \(e^{\sigma x} \tilde{f}\), we can prove

\[
\|e^{\sigma x} w \tilde{f}\|_{L^\infty_{x,v}} + \|e^{\sigma x} w \tilde{f}\|_{L^\infty_{x,v}} \leq C \left\{ \frac{1}{\sigma_0 - \sigma} \|e^{\sigma_0 x} \nu^{-1} w g\|_{L^\infty_{x,v}} + |w((I - P_\gamma) \Phi + r)|_{L^\infty(\gamma^-)} \right\},
\]

for details. From the proof of Lemma 2.11, we know that the constants \((\phi_0, \phi_1, \phi_2, \phi_3)(d)\) should depend on \(d\), hence the RHS of (1.32) still depends on \(d\). Hence, to obtain the uniform estimate for \(\tilde{f}\), we have to establish the uniform in \(d\) estimate for \((\phi_0, \phi_1, \phi_2, \phi_3)(d)\). This is the key part of the present paper. In fact, we express the macroscopic part \(P \tilde{f}(x, v) = [\bar{a}(x) + \bar{b}(x) \cdot v + \bar{c}(x)(|v|^2 - \frac{3}{2})]\sqrt{\nu} \bar{g} by using the boundary value (see Lemma 2.15) and get

\[
|\tilde{f}(d, \bar{v})| \leq C \left\{ |(I - P_\gamma) \tilde{f}(d)|_{L^2_{x,v}} + \|g\|_{L^2_{x,v}} + |(I - P_\gamma) \tilde{f}(0)|_{L^2_{x,v}} + |r|_{L^2(\gamma^-)} \right\},
\]

which, together with (1.32), (1.33) and (1.35), yields that

\[
|\tilde{f}(d, \bar{v})| \leq C e^{-\sigma d} |e^{\sigma d} w \tilde{f}(d)|_{L^\infty_{x,v}} + C \left\{ \|g\|_{L^\infty_{x,v}} + |r|_{L^2(\gamma^-)} \right\}
\]

\[
\leq C e^{-\sigma d} |\tilde{f}(d, \bar{v})| \leq C \left\{ \frac{1}{\sigma_0 - \sigma} \|e^{\sigma_0 x} \nu^{-1} w g\|_{L^\infty_{x,v}} + |w|_{L^\infty(\gamma^-)} \right\}.
\]

Then we have

\[
|\tilde{f}(d, \bar{v})| \leq C \left\{ \|e^{\sigma_0 x} \nu^{-1} w g\|_{L^\infty_{x,v}} + |w|_{L^\infty(\gamma^-)} \right\},
\]

and hence

\[
\|e^{\sigma x} w \tilde{f}\|_{L^\infty_{x,v}} + \|e^{\sigma x} w \tilde{f}\|_{L^\infty_{x,v}} \leq C \left\{ \frac{1}{\sigma_0 - \sigma} \|e^{\sigma_0 x} \nu^{-1} w g\|_{L^\infty_{x,v}} + |w|_{L^\infty(\gamma^-)} \right\},
\]

where the constants \(C > 0\) are independent of \(d\), see Lemma 2.10 for details. Therefore we have established the uniform in \(d\) estimates for both \(\tilde{f}\) and \((\phi_0, \phi_1, \phi_2, \phi_3)(d)\).

To take the limit \(d \to +\infty\), we still need to obtain the asymptotic behavior of \((\phi_0, \phi_1, \phi_2, \phi_3)(d)\). Using (1.34) and energy estimate, we can prove

\[
|\tilde{f}_d - \tilde{f}_d(0)|_{L^2(\gamma^+)} \leq C e^{-\sigma_1 d} \left\{ |r|_{L^2(\gamma^+)} + \int_0^{\sigma_1 d} e^{2\sigma_1 x} |g(x)|^2_{L^2_{x,v}} dx \right\}^{\frac{1}{2}}.
\]

where we have denoted \(\tilde{f}\) to be \(\tilde{f}_d\) to emphasize the dependent on \(d\), see Lemma 2.17 for details. Then using (2.13), (1.35) and Lemma 2.15 we can show that there exist constants \((\phi_0^\infty, \phi_1^\infty, \phi_2^\infty, \phi_3^\infty)\) such that \(\lim_{d \to +\infty}(\phi_0(0), \phi_1(0), \phi_2(0), \phi_3(0)) = (\phi_0^\infty, \phi_1^\infty, \phi_2^\infty, \phi_3^\infty)\) and

\[
|\tilde{f}(d, \bar{v})| \leq C \left\{ \|e^{\sigma_0 x} \nu^{-1} w g\|_{L^\infty_{x,v}} + |w|_{L^\infty(\gamma^-)} \right\}.
\]

With above uniform estimates, we can finally prove Theorem 2.5.
The paper is organized as follows: In section 2, we prove Theorem 1.5 by a series of approximations and some uniform estimates. In section 3, we prove Theorem 1.6. Some useful known results are presented in Appendix A.

We list some notations that will be used in this paper. Throughout this paper, \( C \) denotes a generic positive constant which may vary from line to line. And \( C_a, C_b, \ldots \) denote the generic positive constants depending on \( a, b, \ldots \), respectively, which also may vary from line to line. We denote \( \Omega = \mathbb{R}_+ \) or \( (0, d) \), and the phase boundary \( \gamma = \partial \Omega \times \mathbb{R}_+^3 \). We denote \( \| \cdot \|_{L^p} \) the standard \( L^p(\Omega \times \mathbb{R}_+^3) \)-norm or \( L^p(\mathbb{R}_+^3) \)-norm, and \( \| \nu \|_\nu = \| \sqrt{\nu} \|_{L^2} \). When the norms need to be distinguished from each other, we write \( \| \cdot \|_{L^p}\), \( \| \cdot \|_{L^p} \) and \( \| \cdot \|_{L^p,\nu} \), respectively. For the phase boundary integration, we define \( | \cdot |_{L^\infty(\gamma)} \) denotes the \( L^\infty(\gamma) \)-norm, \( | \cdot |_{L^2(\gamma)} \) denotes the \( L^2(\gamma, |v_3|dv) \)-norm.

2. Existence for the Linearized Problem

We denote \( \Omega_d := (0, d) \) with \( d \geq 1 \) and denote the phase boundary of \( \Omega_d \times \mathbb{R}^3 \) as \( \gamma = \partial \Omega_d \times \mathbb{R}^3 \). We split \( \gamma \) into three disjoint parts, outgoing boundary \( \gamma_+ \), the incoming boundary \( \gamma_- \), and the singular boundary \( \gamma_0 \) for grazing velocities:

\[
\gamma_+ = \{(x, v) \in \partial \Omega_d \times \mathbb{R}^3 : n(x) \cdot v > 0 \},
\gamma_- = \{(x, v) \in \partial \Omega_d \times \mathbb{R}^3 : n(x) \cdot v < 0 \},
\gamma_0 = \{(x, v) \in \partial \Omega_d \times \mathbb{R}^3 : n(x) \cdot v = 0 \}.
\]

where \( \tilde{n}(x) \) is the outward unit normal. It is direct to know that \( \partial \Omega_d = \{0, d\} \), \( \tilde{n}(0) = (0, 0, -1) \) and \( \tilde{n}(d) = (0, 0, 1) \).

To construct a solution of the linearized boundary layer problem in half-space, we first consider the truncated problem with penalized term

\[
\left\{
\begin{array}{l}
\varepsilon f^\varepsilon + v_3 \cdot \partial_x f^\varepsilon + p^\varepsilon_0 \nu f^\varepsilon = g, \\
f^\varepsilon(0, v)|_{v_3 > 0} = (P_\gamma f)(0, v) + r(v), \\
f^\varepsilon(d, v)|_{v_3 < 0} = f^\varepsilon(d, \mathcal{R} v),
\end{array}
\right.
\tag{2.1}
\]

where \( \varepsilon \in (0, 1] \) and \( \mathcal{R} v := (v_1, -v_3) \) with \( v_0 := (v_1, v_2) \). We define

\[
h^\varepsilon(x, v) := w(v) f^\varepsilon(x, v),
\]

then (2.1) can be rewritten as

\[
\left\{
\begin{array}{l}
\varepsilon h^\varepsilon + v_3 \cdot \partial_x h^\varepsilon + p^\varepsilon_0 \nu(v) h^\varepsilon = p^\varepsilon_0 K_w h^\varepsilon + wg, \\
h^\varepsilon(0, v)|_{v_3 > 0} = \frac{1}{\tilde{w}(v)} \int_{v_3 < 0} h^\varepsilon(0, u) \tilde{w}(u) d\sigma + (wr)(v), \\
h^\varepsilon(d, v)|_{v_3 < 0} = h^\varepsilon(d, \mathcal{R} v),
\end{array}
\right.
\tag{2.2}
\]

where

\[
\tilde{w}(v) = \frac{1}{\sqrt{2\pi\mu(v)|v_3|dv}} \quad \text{and} \quad d\sigma = \sqrt{2\pi\mu(v)|v_3|dv}.
\tag{2.3}
\]

It is easy to check that

\[
\int_{v_3 < 0} d\sigma = \int_{v_3 < 0} \sqrt{2\pi\mu(v)|v_3|dv} = 1.
\]

Hereafter \( K_w h = w K_\frac{h}{\varepsilon^3} \) and

\[
K_w h(v) = \int_{\mathbb{R}^3} k_w(v, u) h(u) du \quad \text{with} \quad k_w(v, u) = w(v) k(v, u) w(u)^{-1}.
\tag{2.4}
\]
2.1. A priori \( L_{x,v}^\infty \) estimate. For the approximate problem \((2.2)\), the most difficult part is to obtain the \( a \ priori \) \( L_{x,v}^\infty \)-bound uniform in \( \varepsilon \in [0,1] \) and \( d \in [1, \infty) \).

**Definition 2.1.** Given \((t,x,v)\), let \([X(s),V(s)]\) be the backward characteristics for \((2.2)\), which is determined by

\[
\begin{align*}
\frac{dX(s)}{ds} &= V_3(s), \\
\frac{dV(s)}{ds} &= 0,
\end{align*}
\]

\([X(t),V(t)] = [x,v] \). The solution is then given by

\([X(s),V(s)] = [X(s; t,x,v),V(s; t,x,v)] = [x-(t-s)v_3,v] \).

Now for each \((x,v)\) with \( x \in \Omega_d \) and \( v_3 \neq 0 \), we define its backward exit time \( t_b(x,v) \geq 0 \) to be the last moment at which the back-time straight line \([X(-\tau; 0,x,v),V(-\tau; 0,x,v)]\) remains in \( \Omega_d \):

\( t_b(x,v) = \sup\{s \geq 0 : x-\tau v_3 \in \Omega_d \text{ for } 0 \leq \tau \leq s\} \).

We also define

\( x_b(x,v) = x(t_b) = x-t_b(x,v) v_3 \in \partial \Omega_d \).

We point out that \( X(s), t_b(x,v) \) and \( x_b(x,v) \) are independent of the horizontal velocity.

Let \( x \in \Omega_d, (x,v) \notin \gamma_0 \cup \gamma_- \) and denote \((t_0,x_0,v_0) := (t,x,v) \) hereafter. We firstly define

\( v_k \in V_k := \{v_k \in \mathbb{R}^3 : v_k \cdot \vec{n}(0) = -v_{k,3} > 0\}, \text{ for } k \geq 1, \) \hspace{1cm} (2.5)

which is defined only at the lower boundary \( x = 0 \). Then the back-time cycle is defined as

\[
\begin{align*}
X_d(s; t,x,v) &= \sum_k 1_{[t_{k+1}, t_k]}(s) \{x_k - v_{k,3} \cdot (t_k-s)\}, \\
V_d(s; t,x,v) &= \sum_k 1_{[t_{k+1}, t_k]}(s) v_k,
\end{align*}
\]

with

\( (t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_b(x_k,v_k), x_b(x_k,v_k), v_{k+1}) \), \hspace{1cm} (2.6)

where

\[
\begin{align*}
v_{k+1} &= \mathcal{R} v_k = (v_k,-v_{k,3}) \text{ if } x_{k+1} = d, \\
v_{k+1} &\in V_{k+1} \text{ if } x_{k+1} = 0.\)
\] \hspace{1cm} (2.7)

Clearly, for \( k \geq 1 \) and \((x,v) \notin \gamma_0 \cup \gamma_-\), it holds that

\[
\begin{align*}
x_k &= \frac{1-(-1)^k}{2} x_1 + \frac{1+(-1)^k}{2} x_2, \\
t_k - t_{k+1} &= \frac{d}{|v_{k,3}|} > 0. \)
\]

\hspace{1cm} (2.8)

We also define the iterated integral

\[
\int_{\Pi_{j=1}^{k-1} V_j} \Pi_{j=1}^{k-1} d\sigma_j = \int_{V_1} d\sigma_1 \int_{V_2} d\sigma_2 \cdots \int_{V_{k-1}} d\sigma_{k-1}, \int_{V_k} d\sigma_k,
\]

\hspace{1cm} (2.9)

where

\[ d\sigma_j = \sqrt{2\pi \mu(v_j)} |v_{j,3}| dv_j, \text{ } j = 1,2,\ldots,k-1. \]

It is direct to know that \( d\sigma_j, j = 1,2,\ldots,k-1 \) are probability measure.

The following lemma will be used in the proof of Lemma \((2.4)\) and the proof is very similar to the one in Guo \((11)\), so we omit the proof here.
Lemma 2.2. Let \((t, x, v)\) with \(t \in [0, T_0]\), \(x \in \bar{Q}_d\), \(v \in \mathbb{R}^3\) and \(v_3 > 0\), there exist constants \(C_1, C_2 > 0\), independent of \(T_0\), such that for \(k \geq k_0 := C_1 T_0^4\), it holds

\[
\int_{\Pi_{k}^{k-1} \mathcal{V}_{2l-1}} 1_{\{(t_{2(l-1)+1}, t_{2l-1}), ..., (t_{2(k-1)+1}, t_{2k-1})\} > 0} \Pi_{l=1}^{k-1} d\sigma_{2l-1} \leq \left( \frac{1}{2} \right)^{C_2 T_0^4}.
\] (2.10)

We point out that all the constants above are independent of \(d \geq 1\).

We can represent the solution of (2.2) in a mild formulation which enables us to get the \(L_\infty\) bound of solutions. Indeed, for later use, we consider the following iterative linear problems involving a parameter \(\lambda \in [0, 1]::

\[
\begin{cases}
\varepsilon h^{i+1} + v_3 \partial_x h^{i+1} + v_0^0 \nu(v) h^{i+1} = \lambda p_0^0 K_w h^i + w g, \\
\left. h^{i+1}(0, v) \right|_{v_3 > 0} = \frac{1}{\bar{w}(v)} \int_{v_3 < 0} h^i(0, v') \bar{w}(v') d\sigma + (wr)(v), \\
\left. h^{i+1}(d, v) \right|_{v_3 < 0} = h^i(d, Rv) + (wr')(v)
\end{cases}
\] (2.11)

for \(i = 0, 1, 2, \ldots\), where \(h^0 \equiv 0\) and \((wr)(v), (wr')(v) \in L_\infty(\gamma)\) are some given functions. For the mild formulation of (2.11), we have the following lemma whose proof can be given by induction arguments, but omitted for brevity here.

Lemma 2.3. Let \(\varepsilon \in [0, 1]\) and \(\lambda \in [0, 1]\). For each \((x, v) \in \bar{Q}_d \times \mathbb{R}^3 \setminus (\gamma_0 \cup \gamma_-)\), we have:

Case 1. If \(v_{0,3} > 0\), it holds

\[
h^{i+1}(x, v) = \sum_{n=1}^{3} J_n + \sum_{n=4}^{16} I_{\{t_1 > 0\}} J_n,
\] (2.12)

with \(\nu_\varepsilon(v) := \varepsilon + \nu_0^0 \nu(v)\), and

\[
J_1 = \int_{\{t_1 \leq 0\}} e^{-\nu_\varepsilon(v)t} h^{i+1}(x - v_{0,3} t, v),
J_2 + J_3 = \int_{\max\{t_1, 0\}}^{t_1} e^{-\nu_\varepsilon(v)(t-s)} \left( \lambda K_w h^i + w g \right)(x - v_{0,3}(t-s), v) ds,
\]

\[
J_4 = e^{-\nu_\varepsilon(v)(t-t_1)} (wr)(x_1, v),
J_5 = \frac{e^{-\nu_\varepsilon(v)(t-t_1)}}{\bar{w}(v)} \left\{ \sum_{l=1}^{k-2} \sum_{l=1}^{k-2} 1_{\{v_{2l-1} > 0\}} (wr)(v_{2l-1}) d\Sigma^{k-1}_{l} (t_{2l+1}) \right\}.
\]
and

\[ J_6 = \frac{e^{-\nu_1(t-t_1)}}{\tilde{w}(v)} \int_{\Pi_j=1}^{k-1} v_{2j-1} \sum_{l=1}^{k-1} 1 \{ t_{2l} \leq t_{2l-1} \} h^{i+2-2l} (x_{2l-1} - v_{2l-1,3} t_{2l-1} - v_{2l-1}) d\Sigma^{k-1}_{l-1}(0), \]

\[ J_7 + J_8 = \frac{e^{-\nu_1(t-t_1)}}{\tilde{w}(v)} \int_{\Pi_j=1}^{k-1} v_{2j-1} \sum_{l=1}^{k-1} 1 \{ t_{2l} \leq t_{2l-1} \} \int_0^{t_{2l-1}} \lambda K_w h^{i+1-2l} + w \gamma (x_{2l-1} - v_{2l-1,3} (t_{2l-1} - s), v_{2l-1}) ds d\Sigma^{k-1}_{l-1}(s), \]

\[ J_9 + J_{10} = \frac{e^{-\nu_1(t-t_1)}}{\tilde{w}(v)} \int_{\Pi_j=1}^{k-1} v_{2j-1} \sum_{l=1}^{k-1} 1 \{ t_{2l} > 0 \} \int_0^{t_{2l}} \lambda K_w h^{i+1-2l} + w \gamma (x_{2l-1} - v_{2l-1,3} (t_{2l-1} - s), v_{2l-1}) ds d\Sigma^{k-1}_{l-1}(s), \]

\[ J_{11} = \frac{e^{-\nu_1(t-t_1)}}{\tilde{w}(v)} \int_{\Pi_j=1}^{k-1} v_{2j-1} \sum_{l=1}^{k-1} 1 \{ t_{2l+1} \leq t_{2l} \} h^{i+1-2l}(x_{2l} + v_{2l-1,3} t_{2l}), \mathcal{R} v_{2l-1} d\Sigma^{k-1}_{l-1}(0), \]

\[ J_{12} + J_{13} = \frac{e^{-\nu_1(t-t_1)}}{\tilde{w}(v)} \int_{\Pi_j=1}^{k-1} v_{2j-1} \sum_{l=1}^{k-1} 1 \{ t_{2l+1} > 0 \} \int_0^{t_{2l+1}} \lambda K_w h^{i+2-2l} + w \gamma (x_{2l} + v_{2l-1,3} (t_{2l} - s), v_{2l-1}) ds d\Sigma^{k-1}_{l-1}(s), \]

\[ J_{14} + J_{15} = \frac{e^{-\nu_1(t-t_1)}}{\tilde{w}(v)} \int_{\Pi_j=1}^{k-1} v_{2j-1} \sum_{l=1}^{k-1} 1 \{ t_{2l+1} > 0 \} \int_0^{t_{2l+1}} \lambda K_w h^{i+2-2l} + w \gamma (x_{2l} + v_{2l-1,3} (t_{2l} - s), v_{2l-1}) ds d\Sigma^{k-1}_{l-1}(s), \]

\[ J_{16} = \frac{e^{-\nu_1(t-t_1)}}{\tilde{w}(v)} \int_{\Pi_j=1}^{k-1} v_{2j-1} 1 \{ t_{2(k+1)+1} > 0 \} h^{i+1-2(k-1)}(x_{2(k+1)}, v_{2(k-1)}) ds d\Sigma^{k-1}_{l-1}(t_{2(k+1)+1}), \]

where we have used the notation

\[ d\Sigma^{k-1}_{l-1}(s) := \{ \Pi_j=1 d\sigma_{2j-1} \} \cdot \{ \tilde{w}(v_{2l-1}) e^{-\nu_1(v_{2l-1}) (t_{2l-1} - s)} ds \} \cdot \{ \Pi_j=1 e^{-\nu_1(v_{2l-1}) (t_{2l-1} - t_{2l+1})} d\sigma_{2j-1} \}. \]  

**Case 2.** If \( v_{0,3} < 0 \), it holds

\[ h^{i+1}(x, v) = 1 \{ t_1 \leq 0 \} \cdot e^{-\nu_1(t-s)} h^{i+1}(x - v_{0,3} t, v) \]

\[ + \int_{\max\{ t_1, 0 \}}^{t} e^{-\nu_1(t-s)} \left( \lambda K_w h^{i} + w \gamma \right)(x - v_{0,3} (t - s), v) ds \]

\[ + 1 \{ t_1 > 0 \} \cdot e^{-\nu_1(t-t_1)} \cdot (\tilde{w}) \gamma (v_1) + 1 \{ t_1 > 0 \} \cdot e^{-\nu_1(t-t_1)} h^i (d, v_1). \]

**Lemma 2.4.** Let \( h^i \), \( i = 0, 1, 2, \ldots \), be the solutions to (2.11), satisfying

\[ \| h^i \|_{L^\infty_v} + | h^i |_{L^\infty_\gamma} < \infty. \]

Then there exists \( T_0 > 1 \) large enough such that for \( i \geq k_0 := C_T T_0^3 \), it holds that

\[ \| h^{i+1} \|_{L^\infty_v} + \| h^{i+1} \|_{L^\infty_\gamma} \leq \frac{1}{8} \sup_{0 \leq t \leq 2k_0} \left\{ \| h^{i-t} \|_{L^\infty_v} + | h^{i-t} |_{L^\infty_\gamma} \right\} \]

\[ + C \left\{ \| \nu^{-1} w \|_{L^\infty_v} + | w \|_{L^\infty_\gamma} + | \tilde{w} \|_{L^\infty_\gamma} \right\} + C \sup_{0 \leq t \leq 2k_0} \left\{ \left\| \frac{h^{i-t}}{w} \right\|_{L^\infty_v} \right\}. \]
Moreover, if \( h^i \equiv h \) for \( i = 1, 2, \ldots, \) i.e., \( h \) is a solution, then (2.16) is reduced to the following estimate

\[
\| h \|_{L^\infty_{t,x}} + \| \frac{\partial}{\partial t} h \|_{L^\infty_{t,x}} \leq C\{ \| \nu^{-1} w \|_{L^\infty_{t,x}} + \| \nu_{\gamma} \|_{L^\infty_{t,x}} + \| \nu_{\gamma} \|_{L^\infty_{t,x}} \} + C \left\| \frac{\partial}{\partial t} h \right\|_{L^\infty_{t,x}} .
\] (2.16)

We emphasize that the positive constant \( C > 0 \) depends on \( k_0 \), and is independent of \( d, \lambda \) and \( \epsilon > 0 \).

**Proof.** By the definition of \( \nu_\epsilon(v) \), we first note that

\[
\nu_\epsilon(v) \geq p_0(v) \geq \nu_0 > 0,
\] (2.17)

where \( \nu_0 \) is a positive constant independent of \( \epsilon \) and \( v \in \mathbb{R}^3 \). Since the proof is complicate, we divide the proof into four steps.

**Step 1.** In this step, we assume \( v_{0,3} > 0 \). Hence, \( h^{i+1}(x, v) \) can be expressed by (2.12). Then, for \( J_1 \), it follows from (2.17) that

\[
|J_1| \leq e^{-\nu_0 t} \| h^{i+1} \|_{L^\infty_{t,x}}.
\] (2.18)

For terms involving the source \( g \), we notice that

\[
\frac{1}{w(v)} \leq w(v) e^{-\frac{|v|^2}{2}} \leq Ce^{-\frac{|v|^2}{4}}, \quad \tilde{w}(v) \mu(v) \| v_3 \| \leq C v \sqrt{\mu(v)} \leq Ce^{-\frac{|v|^2}{4}}
\] (2.19)

which immediately yields that

\[
\int_{\Omega_{j+1}^{k-1}} \tilde{w}(v_i) \Pi_{j=1}^{k-1} d\sigma_{j-1} \leq C < \infty, \quad \text{for} \quad 1 \leq l \leq k - 1,
\] (2.20)

Then it follows from (2.19) and (2.20) that

\[
|J_3| + |J_8| + |J_{10}| + |J_{13}| + |J_{15}| \leq C k \| \nu^{-1} w \|_{L^\infty_{t,x}},
\] (2.21)

\[
|J_4| + |J_5| \leq C k \{ \| \nu_\gamma \|_{L^\infty_{t,x}} + \| \nu_{\gamma} \|_{L^\infty_{t,x}} \},
\] (2.22)

and

\[
|J_6| \leq \frac{C e^{-\frac{|v|^2}{4}}}{d} \frac{1}{w(v)} e^{-\nu_0(t-t_1)} \int_{\Omega_{j+1}^{k-1}} \sum_{l=1}^{k-1} \{ 1 \{ t_{2l} \leq t_{2l-2} \} \} \| h^{i+2-2l} \|_{L^\infty_{t,x}} d\Sigma_l^{k-1}(0)
\]

\[
\leq C k e^{-\nu_0 t} e^{-\frac{|v|^2}{4}} \sup_{1 \leq l \leq k - 1} \{ \| h^{i+2-2l} \|_{L^\infty_{t,x}} \}.
\] (2.23)

Similarly we have

\[
|J_{11}| \leq C k e^{-\nu_0 t} e^{-\frac{|v|^2}{4}} \sup_{1 \leq l \leq k - 1} \{ \| h^{i+1-2l} \|_{L^\infty_{t,x}} \}.
\] (2.24)

From the boundary condition (2.11), it holds that

\[
|h^{i+1-2(k-1)}(0)|_{L^\infty_{t,x}} \leq C \| h^{i+2(k-1)}(0) \|_{L^\infty_{t,x}} + |wr|_{L^\infty_{t,x}}.
\] (2.25)

Hence, for \( J_{16} \), it follows from (2.19), (2.25) and Lemma 2.2 that

\[
|J_{16}| \leq C e^{-\frac{|v|^2}{4}} \left\{ \frac{1}{2} \right\} \sup_{1 \leq l \leq k - 1} \{ \| h^{i+1-2l} \|_{L^\infty_{t,x}} \}
\]

\[
\leq C e^{-\frac{|v|^2}{4}} \left\{ \frac{1}{2} \right\} \sup_{1 \leq l \leq k - 1} \{ \| h^{i+2(k-1)}(0) \|_{L^\infty_{t,x}} + |wr|_{L^\infty_{t,x}} \}
\] (2.26)

where we have taken \( k = C_1 T_0^\frac{1}{4} \) and \( T_0 > 1 \) is a large constant to be chosen later.
For $J_7$, it holds that

$$
|J_7| \leq Ce^{-\frac{1}{2}|v|^2} \sum_{l=1}^{k-1} \int_{\mathbb{R}^3} \Pi_{j=1}^{l-1} d\sigma_{2j-1} \int_0^{t_{2j-1}} e^{-t_{2j-1}^0(t-s)} ds \int_{V_{2j-1}} \int \mathbf{1}_{\{v_{2j-1} \leq t_{2j-1} \}} |k_w(v_{2j-1}, v')| h^{i+1-2l}(x_{2j-1} - v_{2j-1, 3}(t_{2j-1} - s), v') dv' d\sigma_{2j-1} \leq Cke^{-\frac{1}{2}|v|^2} \cdot e^{-\frac{1}{2}|v|^2} \sum_{1 \leq l \leq k-1} \|h^{i+1-2l}\|_{L^\infty_{x,v}} \tag{2.27}
$$

We shall estimate the right-hand terms of (2.27) as follows. A direct calculation shows

$$
\sum_{l=1}^{k-1} J_{7l} \leq Ce^{-\frac{1}{2}|v|^2} \sum_{l=1}^{k-1} \int_{\mathbb{R}^3} \Pi_{j=1}^{l-1} d\sigma_{2j-1} \int_0^{t_{2j-1}} e^{-t_{2j-1}^0(t-s)} ds \int_{V_{2j-1}} \int \mathbf{1}_{\{v_{2j-1} \leq t_{2j-1} \}} |k_w(v_{2j-1}, v')| e^{-\frac{1}{2}|v_{2j-1}|^2} dv_{2j-1} \cdot \sup_{1 \leq l \leq k-1} \|h^{i+1-2l}\|_{L^\infty_{x,v}} \tag{2.28}
$$

For each term $J_{72l}$, we have

$$
J_{72l} \leq Ce^{-\frac{1}{2}|v|^2} \int_{\mathbb{R}^3} \Pi_{j=1}^{l-1} d\sigma_{2j-1} \int_0^{t_{2j-1}} e^{-t_{2j-1}^0(t-s)} ds \int_{V_{2j-1}} \int \mathbf{1}_{\{v_{2j-1} \leq t_{2j-1} \}} |k_w(v_{2j-1}, v')| e^{-\frac{1}{2}|v_{2j-1}|^2} dv' e^{-\frac{1}{2}|v'|^2} \cdot \sup_{1 \leq l \leq k-1} \|h^{i+1-2l}\|_{L^\infty_{x,v}} \tag{2.29}
$$
To estimate the first term on the right-hand side of (2.29), a direct calculation shows
\[
\int_{V_{2l-1} \cap \{|v_{2l-1}| \leq N\}} \int_{\{|v'| \leq 2N\}} 1_{\{t_{2l} \leq 0 \leq t_{2l-1}\}} e^{-\frac{1}{N} |v_{2l-1}|^2} \\
\times |k_w(v_{2l-1}, v')h^{i+1-2l}(x_{2l-1} - v_{2l-1,3}(t_{2l-1} - s), v')| dv' dv_{2l-1}
\]
\[
\leq C_N \left\{ \int_{V_{2l-1} \cap \{|v_{2l-1}| \leq N\}} \int_{\{|v'| \leq 2N\}} 1_{\{t_{2l} \leq 0 \leq t_{2l-1}\}} e^{-\frac{1}{N} |v_{2l-1}|^2} |k_w(v_{2l-1}, v')|^2 dv' dv_{2l-1} \right\}^{\frac{1}{2}}
\times \left\{ \int_{V_{2l-1} \cap \{|v_{2l-1}| \leq N\}} \int_{\{|v'| \leq 2N\}} 1_{\{t_{2l} \leq 0 \leq t_{2l-1}\}} \frac{|h^{i+1-2l}(y_{2l-1}, v')|^2}{w(v')} dv' dv_{2l-1} \right\}^{\frac{1}{2}}.
\]
(2.30)
where we have used the notation \(y_{2l-1} = x_{2l-1} - v_{2l-1,3}(t_{2l-1} - s)\). It is direct to know that \(y_{2l-1} \in \Omega_d\) for \(s \in [0, t_{2l-1} - \frac{1}{N}]\). A direct computation shows that
\[
\frac{\partial y_{2l-1}}{\partial v_{2l-1,3}} = |t_{2l-1} - s| \geq \frac{1}{N}, \quad \text{for} \quad s \in [0, t_{2l-1} - \frac{1}{N}].
\]
(2.31)
Thus, by making change of variable \(v_{2l-1,3} \rightarrow y_{2l-1}\) and using (2.31), one obtains that
\[
\left\{ \int_{V_{2l-1} \cap \{|v_{2l-1}| \leq N\}} \int_{\{|v'| \leq 2N\}} 1_{\{t_{2l} \leq 0 \leq t_{2l-1}\}} \frac{|h^{i+1-2l}(y_{2l-1}, v')|^2}{w(v')} dv' dv_{2l-1} \right\}^{\frac{1}{2}}
\leq C_N \left\{ \int_{\Omega_d} \int_{|v'| \leq 2N} \frac{|h^{i+1-2l}(y_{2l-1}, v')|^2}{w(v')} dv' dy_{2l-1} \right\}^{\frac{1}{2}} \leq C_N \left\| \frac{|h^{i+1-2l}|}{w} \right\|_{L_{2,v}^1},
\]
which together with (2.30) and (2.29) yield that
\[
J_{2l} \leq \frac{C}{N} e^{-\frac{1}{N} |v|^2} \cdot \left\| h^{i+1-2l} \right\|_{L_{\infty,v}^\infty} + C_N e^{-\frac{1}{N} |v|^2} \left\| \frac{|h^{i+1-2l}|}{w} \right\|_{L_{2,v}^1}.
\]
(2.32)
Thus it follows from (2.32), (2.28) and (2.27) that
\[
|J_{7l}| \leq \frac{Ck}{N} e^{-\frac{1}{N} |v|^2} \cdot \sup_{1 \leq l \leq k-1} \left\{ \left\| h^{i+1-2l} \right\|_{L_{\infty,v}^\infty} \right\} + C_N ke^{-\frac{1}{N} |v|^2} \sup_{1 \leq l \leq k-1} \left\{ \left\| \frac{|h^{i+1-2l}|}{w} \right\|_{L_{2,v}^1} \right\}.
\]
(2.33)
By similar arguments as in (2.27)-(2.33), one can obtain
\[
|J_0| + |J_{12}| + |J_{14}| \leq \frac{Ck}{N} e^{-\frac{1}{N} |v|^2} \cdot \sup_{1 \leq l \leq k-1} \left\{ \left\| h^{i+1-2l} \right\|_{L_{\infty,v}^\infty} + \left\| \frac{|h^{i-2l}|}{w} \right\|_{L_{\infty,v}^\infty} \right\} + C_N ke^{-\frac{1}{N} |v|^2} \sup_{1 \leq l \leq k-1} \left\{ \left\| \frac{|h^{i+1-2l}|}{w} \right\|_{L_{2,v}^1} + \left\| \frac{|h^{i-2l}|}{w} \right\|_{L_{2,v}^1} \right\}.
\]
(2.34)
Now substituting (2.33)-(2.34), (2.20), (2.21)-(2.24) and (2.18) into (2.12), we get, for \((t, x) \in [0, T_0] \times \Omega_d\) and \(v_{0,3} > 0\), that
\[
|h^{i+1}(x, v)| \leq \int_{t_{1,0}}^t e^{-\nu_0^v(x-s)} \int_{\mathbb{R}^3} |k_w(v, v')h^i(x - v_{0,3}(t - s), v')| dv' ds + A_{1,1}(t, v),
\]
(2.35)
where we have denoted

\[ A_{1,i}(t,v) := Ce^{-\frac{1}{2} |v|^2} \left\{ k e^{-\nu_0^2 t} + \left( \frac{1}{2} \right) C_2 T_0^\frac{k}{N} \right\} \cdot \sup_{0 \leq t \leq (k-1)} \left\{ \| h^{i-l} \|_{L_{x,v}^\infty} + \| h^{i-l} \|_{L_{x,v}^\infty(\gamma_+)} \right\} \\
+ e^{-\nu_0^2 t} \| h^{i+1} \|_{L_{x,v}^\infty} + Ck \left\{ \| \nu^{-1} g \|_{L_{x,v}^\infty} + \| wr \|_{L_{x,v}^\infty(\gamma_-)} + \| w \|_{L_{x,v}^\infty(\gamma_-)} \right\} \\
+ C_{N,k} e^{-\frac{1}{2} |v|^2} \cdot \sup_{1 \leq t \leq (k-1)} \left\{ \| h^{i-l} \|_{L_{x,v}^2} \right\}. \]

**Step 2.** We consider the case \( v_{0,3} < 0 \). In fact, it follows from (2.14) that

\[
\| h^{i+1}(x,v) \| \leq e^{-\nu_0^2 t} \| h^{i+1} \|_{L_{x,v}^\infty} + C \| \nu^{-1} g \|_{L_{x,v}^\infty} \\
+ \int_{\max \{ t_1, 0 \}}^{t_1} e^{-\nu_0^2 (t-s)} \int_{\mathbb{R}^3} |k_x(v,v') h^i(d - v_{0,3}(t-s), v')|dv'|ds \\
+ \mathbf{1}_{\{ t_1 > 0 \}} \cdot e^{-\nu_{v_0}(t-t_1)} \cdot |h^i(d, v_1)|. \tag{2.36}
\]

Noting \( v_{1,3} = -v_{0,3} > 0 \) and \( |v_1| = |\mathcal{R}v| = |v| \), then we apply (2.35) to \( h^i(d, v_1) \) to obtain

\[
\| h^i(d, v_1) \| \leq \mathbf{1}_{\{ t_1 > 0 \}} \int_{\max \{ t_2, 0 \}}^{t_1} e^{-\nu_0^2 (t-s)} \int_{\mathbb{R}^3} |k_x(v_1,v') h^i(d - v_{0,3}(t-1-s), v')|dv'|ds + A_{2,i}(t,v) \\
\leq \mathbf{1}_{\{ t_1 > 0 \}} \int_{\max \{ t_2, 0 \}}^{t_1} e^{-\nu_0^2 (t-s)} \int_{\mathbb{R}^3} |k_x(Rv, v') h^i(2d - x + v_{0,3}(t-s), v')|dv'|ds + A_{2,i}(t,v), \tag{2.37}
\]

where we have used the fact \( d - v_{0,3}(t-1-s) = 2d - x + v_{0,3}(t-s) \) and denoted

\[ A_{2,i}(t,v) := Ce^{-\frac{1}{2} |v|^2} \left\{ k e^{-\nu_0^2 t} + \left( \frac{1}{2} \right) C_2 T_0^\frac{k}{N} \right\} \cdot \sup_{1 \leq t \leq (k-1)+1} \left\{ \| h^{i-l} \|_{L_{x,v}^\infty} + \| h^{i-l} \|_{L_{x,v}^\infty(\gamma_+)} \right\} \\
+ e^{-\nu_0^2 t} \| h^i \|_{L_{x,v}^\infty} + Ck \left\{ \| g \|_{L_{x,v}^\infty} + \| wr \|_{L_{x,v}^\infty(\gamma_-)} + \| w \|_{L_{x,v}^\infty(\gamma_-)} \right\} \\
+ C_{N,k} e^{-\frac{1}{2} |v|^2} \cdot \sup_{2 \leq t \leq (k-1)+1} \left\{ \| h^{i-l} \|_{L_{x,v}^2} \right\}. \]

Substituting (2.37) into (2.36), one can get

\[
\| h^{i+1}(x,v) \| \leq \int_{\max \{ t_1, 0 \}}^{t_1} e^{-\nu_0^2 (t-s)} \int_{\mathbb{R}^3} |k_x(v,v') h^i(x - v_{0,3}(t-s), v')|dv'|ds \\
+ \mathbf{1}_{\{ v_{0,3} < 0 \}} \cdot \| h^{i+1} \|_{L_{x,v}^\infty} + A_{2,i}(t,v). \tag{2.38}
\]

**Step 3.** Combining (2.35) and (2.38), for \( t \in [0, T_0], (x,v) \in (\bar{\Omega}_d \times \mathbb{R}^3) \setminus (\gamma_0 \cup \gamma_-) \), we obtain

\[
\| h^{i+1}(x,v) \| \leq \int_{\max \{ t_1, 0 \}}^{t_1} e^{-\nu_0^2 (t-s)} \int_{\mathbb{R}^3} |k_x(v,v') h^i(x - v_{0,3}(t-s), v')|dv'|ds \\
+ \mathbf{1}_{\{ v_{0,3} < 0 \}} \cdot \| h^{i+1} \|_{L_{x,v}^\infty} + A_i(t), \tag{2.39}
\]
where we have denoted

\[
A_i := C \left\{ k e^{-\nu_0 t} + \left( \frac{1}{7} \right)^{C_2 T_0^4} + \frac{k}{N} \right\} \sup_{0 \leq l \leq 2(k-1)+1} \left\{ \| h^{i-l} \|_{L^\infty_{x,v}} + \| h^{i-l} \|_{L^\infty_{v}} \right\} \\
+ C e^{-\theta_v^0 t} \| h^{i+1} \|_{L^\infty_{x,v}} + Ck \left\{ \| v^{-1} \|_{L^\infty_{x,v}} + \| w \|_{L^\infty_{x,v}} \right\} \\
+ C \sup_{1 \leq l \leq 2(k-1)+1} \left\{ \| h^{i-l} \|_{L^\infty_{x,v}} \right\}.
\]  

(2.40)

Step 4. For later use, we denote \( x' = x - v_{0,3}(t - s) \in \bar{\Omega}_d \) and \( t_i = t(s, x', v') \) for \( s \in (\max\{t_1, 0\}, t) \). We also denote \( \tilde{x}' = 2d - x + v_{0,3}(t - s) \in \bar{\Omega}_d \) and \( \tilde{t}_i = t_i(s, \tilde{x}, v') \). Noting (2.39), one has that

\[
|h^i(x', v')| \leq \left| \int_{\max\{t_i', 0\}}^{s} e^{-\nu_0^0 \tau} \int_{\mathbb{R}^3} |k_w(v', v'') h^{i-1}(x' - v_{0,3}(s - \tau), v'')| dv' d\tau \right| + A_{i-1}(s),
\]

(2.41)

and

\[
|h^i(\tilde{x}', v')| \leq \left| \int_{\max\{\tilde{t}_i', 0\}}^{s} e^{-\nu_0^0 \tau} \int_{\mathbb{R}^3} |k_w(v', v'') h^{i-1}(\tilde{x}' - v_{0,3}(s - \tau), v'')| dv' d\tau \right| + A_{i-1}(s).
\]

(2.42)

Substituting (2.41) and (2.42) into (2.39), we can obtain

\[
|h^{i+1}(x, v)| \leq A_1(t) + B_1 + B_2 + B_3 + B_4 + B_5,
\]

(2.43)

where we have used the following notations

\[
B_1 := C \int_{\max\{t_i', 0\}}^{s} ds \left\{ |k_w(v, v')| + |k_w(\mathcal{R}v, v')| \right\} A_{i-1}(s) dv' ds,
\]

(2.44)

\[
B_2 := \int_{\max\{t_i', 0\}}^{s} ds \int_{\mathbb{R}^3} e^{-\nu_0^0 \tau} \left( \int_{\mathbb{R}^3} \left| k_w(v, v') \right| k_w(v', v'') \right) dv' dv' \left| h^{i-1}(x' - v_{0,3}(s - \tau), v'') \right| dv'' dv',
\]

(2.45)

\[
B_3 := \int_{\max\{t_i', 0\}}^{s} ds \int_{\mathbb{R}^3} e^{-\nu_0^0 \tau} \left( \int_{\mathbb{R}^3} \left| k_w(v, v') \right| k_w(\mathcal{R}v', v'') \right) \left( \int_{\max\{t_i', 0\}}^{s} ds \int_{\mathbb{R}^3} \left| k_w(\mathcal{R}v', v'') \right| \right) dv' dv' \left| h^{i-1}(2d - x' + v_{0,3}(s - \tau), v'') \right| dv'' dv',
\]

(2.46)

\[
B_4 := \int_{\max\{t_i', 0\}}^{s} ds \int_{\mathbb{R}^3} e^{-\nu_0^0 \tau} \left( \int_{\max\{t_i', 0\}}^{s} ds \int_{\mathbb{R}^3} \left| k_w(\mathcal{R}v, v') \right| k_w(v', v'') \right) \left( \int_{\max\{t_i', 0\}}^{s} ds \int_{\mathbb{R}^3} \left| k_w(v', v'') \right| \right) \left| h^{i-1}(\tilde{x}' - v_{0,3}(s - \tau), v'') \right| dv'' dv',
\]

(2.47)
and
\[
B_5 := 1_{\{v_0 < 0\}} 1_{\{t_1 > 0\}} \int_0^t ds \int_0^s 1_{\{\max(t_2, 0) < s < t_1\}} e^{-U_k(t-\tau)} d\tau \\
\times \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} |k_w(Rv, v')e^{-U_k(Rv', v''h^{-1}(2d-\tilde{d'} + v_0 \cdot (s-\tau), v'')}| \\
\times 1_{\{v_0 < 0\}} 1_{\{t_1 > 0\}} 1_{\{\max(t_2, 0) < t < t_1\}} d\nu'' d\nu'.
\]

For the term $B_1$, a direct calculation shows that
\[
B_1 \leq C \left\{ k e^{-\frac{4}{3}v \cdot t} + \left( \frac{1}{2} \right) C_2 T_0^2 + \frac{k}{N} \right\} \cdot \sup_{0 \leq t \leq 2k} \left\{ \| h^{i-l} \|_{L^\infty_{x,v}} + |h^{i-l}|_{L^\infty_{x,v}} \right\} \\
+ Ck \left\{ \| v^{-1} \|_{L^\infty_{x,v}} + |w|_{L^\infty_{x,v}} + |w|_{L^\infty_{x,v}} \right\} + C_{N,k} \sup_{1 \leq t \leq 2k} \left\{ \| h^{i-l} \|_{L^\infty_{x,v}} \right\} .
\]

For the term $B_2$, we split the estimate into several cases.
Case 1. For $|v| \geq N$, we have
\[
B_2 \leq \frac{C}{N} \| h^{-1} \|_{L^\infty_{x,v}}.
\]

Case 2. For $|v| \leq N, |v'| \geq 2N$ or $|v'| \leq 2N, |v''| \geq 3N$. In this case, we note from (A.1) that
\[
\left\{ \int_{|v| \leq N, |v'| \geq 2N} \left| k_w(v, v') e^{-\frac{|v^\prime|}{2}} \right| d\nu' \leq C(1 + |v|)^{-1}, \\
\int_{|v'| \leq 2N, |v''| \geq 3N} \left| k_w(v', v'') e^{-\frac{|v^\prime|}{2}} \right| d\nu'' \leq C(1 + |v'|)^{-1}.
\]

This yields that
\[
\int_0^t ds \int_0^s e^{-U_k(t-\tau)} d\tau \left\{ \int_{|v| \leq N, |v'| \geq 2N} + \int_{|v'| \leq 2N, |v''| \geq 3N} \right\} (\cdots) d\nu'' d\nu' \\
\leq e^{-\frac{N^2 t^2}{2}} \| h^{-1} \|_{L^\infty_{x,v}} \int_{|v| \leq N, |v'| \geq 2N} |k_w(v, v') e^{-\frac{|v^\prime|}{2}}| \cdot |k_w(v', v'')| d\nu'' d\nu' \\
+ e^{-\frac{N^2 t^2}{2}} \| h^{-1} \|_{L^\infty_{x,v}} \int_{|v'| \leq 2N, |v''| \geq 3N} |k_w(v, v')| \cdot |k_w(v', v'') e^{-\frac{|v^\prime|}{2}}| d\nu'' d\nu' \\
\leq C e^{-\frac{N^2 t^2}{2}} \| h^{-1} \|_{L^\infty_{x,v}} .
\]
Case 3. For \( |v| \leq N, |v'| \leq 2N, \) and \( |v''| \leq 3N, \) we first note that

\[
\int_0^t ds \int_0^s e^{-\nu(t-\tau)} d\tau \int_{|v'| \leq 2N, |v''| \leq 3N} (\cdots) dv'' dv' \\
\leq C \frac{N}{N} \|h^{i-1}\|_{L^\infty_{x,v}} + \int_0^t ds \int_0^s e^{-\nu(t-\tau)} d\tau \int_{|v'| \leq 2N, |v''| \leq 3N} (\cdots) dv'' dv' \\
\leq C \frac{N}{N} \|h^{i-1}\|_{L^\infty_{x,v}} + C_{N,k} \int_0^t ds \int_0^s e^{-\nu(t-\tau)} d\tau \left\{ \int_{|v'| \leq 2N, |v''| \leq 3N} (\cdots) dv'' dv' \right\}^\frac{1}{2} \\
\times \left\{ \int_{|s| \leq 2N, |s''| \leq 3N} \mathbf{1}_{\{\max(t_1,0) < s < t\}} \mathbf{1}_{\{\max(t_1',0) < \tau < s\}} \frac{|h^{i-1}(y', v'')|}{w(v''')} \right\}^\frac{1}{2} \frac{1}{\sqrt{w}} d\tau,
\]

(2.52)

where we have denoted \( y' = x' - \nu_{0,3}(s - \tau) \in \Omega_4 \) for \( s \in (\max\{t_1,0\}, t) \) and \( \tau \in (\max\{t_1',0\}, s) \). Similar to (2.31), we make change of variable \( v'_{0,3} \mapsto y' \), so that the second term on the right-hand side of (2.52) is bounded as

\[
\int_0^t ds \int_0^s e^{-\nu(t-\tau)} d\tau \left\{ \int_{|v'| \leq 2N, |v''| \leq 3N} (\cdots) dv'' dv' \right\} \leq CN^\frac{1}{2} \|h^{i-1}\|_{L^\infty_{x,v}}.
\]

which together with (2.52) yield that

\[
\int_0^t ds \int_0^s e^{-\nu(t-\tau)} d\tau \int_{|v'| \leq 2N, |v''| \leq 3N} (\cdots) dv'' dv' \leq C \frac{N}{N} \|h^{i-1}\|_{L^\infty_{x,v}} + C_{N,k} \left\| \frac{\sqrt{v}h^{i-1}}{w} \right\|_{L^2_{x,v}}. \tag{2.53}
\]

Combining (2.40), (2.41) and (2.43), we have

\[
B_2 \leq C \frac{N}{N} \|h^{i-1}\|_{L^\infty_{x,v}} + C_{N,k} \left\| \frac{\sqrt{v}h^{i-1}}{w} \right\|_{L^2_{x,v}}. \tag{2.44}
\]

By similar arguments as above, we can obtain

\[
B_3 + B_4 + B_5 \leq C \frac{N}{N} \|h^{i-1}\|_{L^\infty_{x,v}} + C_{N,k} \left\| \frac{\sqrt{v}h^{i-1}}{w} \right\|_{L^2_{x,v}}. \tag{2.45}
\]

Hence substituting (2.53), (2.44), (2.40) into (2.43), we have for \( t \in [0, T_0], \)

\[
|h^{i+1}(x,v)| \leq C \left\{ ke^{-\nu_{0,3}(t)} + \left( \frac{1}{2} \right)^{C_2 T_0^\frac{1}{2}} + \frac{k}{N} \right\} \sup_{0 \leq l \leq 2k} \left\{ \|h^{i-1}\|_{L^\infty_{x,v}} + \|h^{i-1}\|_{L^\infty_{x,v}} + \|w\|_{L^\infty_{x,v}} + \|w\|_{L^\infty_{x,v}} + \|wT\|_{L^\infty_{x,v}} \right\} \\
+ C_{N,k} \sup_{0 \leq l \leq 2k} \left\{ \|h^{i-1}\|_{L^\infty_{x,v}} \right\}.
\]
Now we take $k = C_1 T_0^{\frac{3}{4}}$ and choose $t = T_0$. We first fix $T_0$ large enough, and then choose $N$ large enough, so that one has $e^{-\nu E(t)} \leq \frac{1}{2}$ and

$$C \left\{ k e^{-\frac{3}{4} \nu_0 t} + \left( \frac{1}{2} \right)^{C_2 T_0^{\frac{3}{4}}} + \frac{k}{N} \right\} \leq \frac{1}{16},$$

Therefore (2.15) follows. This completes the proof of Lemma 2.4.

### 2.2. Approximate solutions and uniform estimates

Now we are in a position to construct solutions of (2.1) or equivalently (2.2). First of all, we consider the following approximate problem

$$\begin{aligned}
\varepsilon f^n + v_3 \partial_x f^n + p_0^0 \nu(v) f^n - p_0^0 K f^n &= g, \quad (x, v) \in \Omega \times \mathbb{R}^3, \\
|f^n(0, v)|_{v_3 > 0} &= (1 - \frac{1}{n}) P_i f^n(0, v) + r(v), \\
|f^n(d, v)|_{v_3 < 0} &= (1 - \frac{1}{n}) f^n(d, \mathbb{R}v)
\end{aligned}
$$

(2.56)

where $\varepsilon \in (0, 1]$ is arbitrary and $n > 1$ is an integer. For later use, we choose $n_0 > 1$ large enough such that

$$\frac{1}{8} (1 - \frac{2}{n} + \frac{3}{2n^2})^{-2k_0 - 1} \leq \frac{1}{2},$$

for any $n \geq n_0$, where $k_0 > 0$ is the one fixed in Lemma 2.4.

**Lemma 2.5.** Let $\varepsilon > 0, d \geq 1, n \geq n_0$, and $\beta \geq 3$. Assume $\|\nu^{-1} w g\|_{L_{\infty}^\infty} + \|w r\|_{L_{\infty}(\gamma_\cdot)} < \infty$. Then there exists a unique solution $f^n$ to (2.56) satisfying

$$\|w f^n\|_{L_{\infty}^\infty} + \|w f^n\|_{L_{\infty}(\gamma_\cdot)} \leq C_{\varepsilon, n} \left\{ \|\nu^{-1} w g\|_{L_{\infty}^\infty} + \|w r\|_{L_{\infty}(\gamma_\cdot)} \right\},$$

where the positive constant $C_{\varepsilon, n} > 0$ depends only on $\varepsilon$ and $n$. Moreover, if $g$ is continuous in $\Omega_d \times \mathbb{R}^3$ and $r(v)$ is continuous in $\mathbb{R}_d^3$, then $f^n$ is continuous away from grazing set $\gamma_0$.

**Proof.** We consider the solvability of the following boundary value problem

$$\begin{aligned}
\mathcal{L}_\lambda f := \varepsilon f + v_3 \partial_x f + p_0^0 \nu(v) f - p_0^0 \lambda K f &= g, \\
|f(0, v)|_{v_3 > 0} &= (1 - \frac{1}{n}) P_i f + r(v), \\
|f(d, v)|_{v_3 < 0} &= (1 - \frac{1}{n}) f(d, \mathbb{R}v)
\end{aligned}
$$

(2.57)

for $\lambda \in [0, 1]$. For brevity we denote $\mathcal{L}_0^{-1}$ to be the solution operator associated to the problem, meaning that $f := \mathcal{L}_0^{-1} g$ is a solution to the BVP (2.57). Our idea is to prove the existence of $\mathcal{L}_0^{-1}$, and then extend to obtain the existence of $\mathcal{L}_1^{-1}$ by a continuous argument on $\lambda$. Since the proof is very long, we split it into four steps.

**Step 1.** In this step, we prove the existence of $\mathcal{L}_0^{-1}$. We consider the following approximate sequence

$$\begin{aligned}
\mathcal{L}_0 f^{i+1} := \varepsilon f^{i+1} + v_3 \partial_x f^{i+1} + p_0^0 \nu(v) f^{i+1} &= g, \\
|f^{i+1}(0, v)|_{v_3 > 0} &= (1 - \frac{1}{n}) P_i f^{i} + r(v), \\
|f^{i+1}(d, v)|_{v_3 < 0} &= (1 - \frac{1}{n}) f^{i}(d, \mathbb{R}v)
\end{aligned}
$$

(2.58)

for $i = 0, 1, 2, \cdots$, where we have set $f^0 \equiv 0$. We will construct $L_{x,v}^\infty$ solutions to (2.58) for $i = 0, 1, 2, \cdots$, and establish $L_{x,v}^\infty$-estimates.
Firstly, we will solve inductively the linear equation (2.58) by the method of characteristics. Let
\[ h^{i+1}(x, v) = w(v)f^{i+1}(x, v). \]
For almost every \((x, v) \in \Omega_d \times \mathbb{R}^3 \setminus (\gamma_0 \cup \gamma_-)\), one can write
\[
\begin{align*}
    h^{i+1}(x, v) &= 1_{\{(0, 0) > 0\}}e^{-(\varepsilon + p^0_w^0u(v))t_b} \cdot w(v) \left[1 - \frac{1}{n}\right]P_{\gamma}f^i(0, v) + r(v) \\
    &+ 1_{\{(0, 0) < 0\}}e^{-(\varepsilon + p^0_w^0u(v))t_b} \cdot w(v)\left[1 - \frac{1}{n}\right]f^i(d, Rv) \\
    &+ \int_{t - t_1}^t e^{-(\varepsilon + p^0_w^0u(v))(t - s)}(wg)(x - v_{0, 3}(t - s), v)ds,
\end{align*}
\]
where \(t_1 = t - t_b(x, v)\). Noting the definition of \(P_{\gamma}f\), we have
\[
|wP_{\gamma}f(0)|_{L^\infty} \leq C|f(0)|_{L^\infty(\gamma_+)}.
\]
We consider (2.59) with \(i = 0\). Since \(f^0 \equiv 0\), it is straightforward to see that
\[
\|h^1\|_{L^\infty} + |h^1|_{L^\infty(\gamma_+)} \leq C \left\{ \|\nu^{-1}wg\|_{L^\infty} + |w\nu|_{L^\infty(\gamma_-)} \right\} < \infty.
\]
Therefore we have obtained the solution to (2.58) with \(i = 0\). Assume that we have already solved (2.58) for \(i \leq l\) and obtained
\[
\|h^{l+1}\|_{L^\infty} + |h^{l+1}|_{L^\infty(\gamma_+)} \leq C_{l+1} \left\{ \|\nu^{-1}wg\|_{L^\infty} + |w\nu|_{L^\infty(\gamma_-)} \right\} < \infty.
\]
We now consider (2.58) for \(i = l + 1\). Noting (2.61), then we can solve (2.58) by using (2.59) with \(i = l + 1\). We still need to prove \(h^{l+2} \in L^\infty\). Indeed, it follows from (2.59) that
\[
\begin{align*}
    \|h^{l+2}\|_{L^\infty} + |h^{l+2}|_{L^\infty(\gamma_+)} &\leq C|h^{l+1}|_{L^\infty(\gamma_+)} + C|w\nu|_{L^\infty(\gamma_-)} + C\|\nu^{-1}wg\|_{L^\infty} \\
    &\leq C_{l+2} \left\{ \|\nu^{-1}wg\|_{L^\infty} + |w\nu|_{L^\infty(\gamma_-)} \right\} < \infty.
\end{align*}
\]
Therefore, inductively, we have solved (2.58) for \(i = 0, 1, 2, \ldots\) and obtained
\[
\|h^i\|_{L^\infty} + |h^i|_{L^\infty(\gamma_+)} \leq C_i \left\{ \|\nu^{-1}wg\|_{L^\infty} + |w\nu|_{L^\infty(\gamma_-)} \right\} < \infty.
\]
The positive constant \(C_i\) may increase to infinity as \(i \to \infty\). Here, we emphasize that we first need to know the sequence \(\{h^i\}_{i=0}^{\infty}\) is in \(L^\infty\)-space, otherwise one can not use Lemma 2.4 to get uniform \(L^\infty\) estimates.

Let \((x, v) \in (\Omega_d \times \mathbb{R}^3) \setminus \gamma_0\), then it is easy to check that \(t_b(x, v)\) and \(x_b(x, v)\) are continuous for \(v_{0, 3} \neq 0\). Therefore if \(g\) and \(r\) are continuous, by using (2.59), we can check that \(f^i(x, v)\) is continuous away from grazing set \(\gamma_0\).

Secondly, in order to take the limit \(i \to \infty\), one has to get some uniform estimates. Multiplying (2.58) by \(f^{i+1}\) and integrating it over \(\Omega_d \times \mathbb{R}^3\), one obtains that
\[
\begin{align*}
    \varepsilon\|f^{i+1}\|_{L^2}^2 + \frac{1}{2}\left\{\|f^{i+1}(0)\|_{L^2(\gamma_+)}^2 + \|f^{i+1}(d)\|_{L^2(\gamma_-)}^2\right\} + \|f^{i+1}\|_{\nu}^2 \\
    \leq \frac{1}{2}(1 - \frac{2}{n} + \frac{3}{2n^2})\|f^i(0)\|_{L^2(\gamma_+)}^2 + \frac{1}{2}(1 - \frac{1}{n})\|f^i(d)\|_{L^2(\gamma_-)}^2 \\
    + C_n\|r\|_{L^2}^2 + \frac{C}{\varepsilon}\|g\|_{L^2}^2 + \frac{C}{4}\|f^{i+1}\|_{L^2}^2 \\
    \leq \frac{1}{2}(1 - \frac{2}{n} + \frac{3}{2n^2})\{\|f^i(0)\|_{L^2(\gamma_+)}^2 + \|f^i(d)\|_{L^2(\gamma_-)}^2\} \\
    + C_n\|r\|_{L^2}^2 + \frac{C}{\varepsilon}\|g\|_{L^2}^2 + \frac{C}{4}\|f^{i+1}\|_{L^2}^2
\end{align*}
\]
where we have used the fact $|P_\gamma f(0)|_{L^2(\gamma_-)} = |P_\gamma f(0)|_{L^2(\gamma_+)} \leq |f(0)|_{L^2(\gamma_+)}$. Then it follows from (2.68) that
\[ \frac{3}{2} \varepsilon \| f^{i+1} \|^2_{L^2_{\nu,v}} + \frac{3}{2n} \| f^{i+1} \|^2_{L^2_{\nu,v}} + 2 \| f^{i+1} \|^2_{\| f^{i+1} \|^2_{L^2_{\nu,v}}} \leq (1 - \frac{2}{n} + \frac{3}{2n^2}) \left\{ \| f^{i+1} \|^2_{L^2_{\nu,v}} + \frac{3}{2n^2} \right\} \left\{ \| f^{i+1} \|^2_{L^2_{\nu,v}} + \frac{3}{2n^2} \right\} \leq C_{\varepsilon,n} \left\{ \| g \|^2_{L^2_{\nu,v}} + |r|_{L^2_{\nu,v}} \right\}. \] (2.64)

Now we take the difference $f^{i+1} - f^i$ in (2.68), then by similar energy estimate as above, we obtain
\[ \frac{3}{2} \varepsilon \| f^{i+1} - f^i \|^2_{L^2_{\nu,v}} + \frac{3}{2n} \| f^{i+1} - f^i \|^2_{L^2_{\nu,v}} + 2 \| f^{i+1} - f^i \|^2_{\| f^{i+1} - f^i \|^2_{L^2_{\nu,v}}} \leq (1 - \frac{2}{n} + \frac{3}{2n^2}) \left\{ \| f^{i+1} - f^i \|^2_{L^2_{\nu,v}} + \frac{3}{2n^2} \right\} \left\{ \| f^{i+1} - f^i \|^2_{L^2_{\nu,v}} + \frac{3}{2n^2} \right\} \leq C_{\varepsilon,n} \cdot \left\{ (1 - \frac{2}{n} + \frac{3}{2n^2}) \frac{3}{2n^2} \right\} \left\{ \| \nu^{-1} w g \|_{L^\infty_{\nu,v}} + |wr|_{L^\infty_{\nu,v}} \right\} < \infty. \] (2.65)

Noting $1 - \frac{2}{n} + \frac{3}{2n^2} < 1$, thus $\{ f^i \}_{i=0}^\infty$ is a Cauchy sequence in $L^2_{\nu,v}$, i.e.,
\[ \| f^i - f^j \|^2_{L^2_{\nu,v}} + \| f^i - f^j \|^2_{\| f^i - f^j \|^2_{L^2_{\nu,v}}} \to 0, \quad \text{as } i,j \to \infty. \]

And we also have, for $i = 0, 1, 2, \ldots$, that
\[ \| f^i \|^2_{L^2_{\nu,v}} + \| f^i \|^2_{\| f^i \|^2_{L^2_{\nu,v}}} \leq C_{\varepsilon,n} \left\{ \| g \|^2_{L^2_{\nu,v}} + |r|_{L^2_{\nu,v}} \right\}. \] (2.66)

Next we consider the uniform $L^\infty_{\nu,v}$ estimate. Here we point out that Lemma 2.4 still holds by replacing 1 by $1 - \frac{1}{n}$ in the boundary condition of (2.11), and the constants in Lemma 2.4 do not depend on $n \geq 1$. Thus we apply Lemma 2.4 to obtain that
\[ \| h^{i+1} \|_{L^\infty_{\nu,v}} + |h^{i+1}|_{L^\infty_{\nu,v}} \leq \frac{1}{8} \sup_{0 \leq t \leq 2k_0} \left\{ \| h^{i+1} - |h^{i+1}|_{L^\infty_{\nu,v}} \right\} \right\} \leq C_{\varepsilon,n} \left\{ \| \nu^{-1} w g \|_{L^\infty_{\nu,v}} + |wr|_{L^\infty_{\nu,v}} \right\}. \] (2.67)

where we have used (2.66) in the second inequality. Now we apply Lemma A.1 to obtain that for $i \geq 2k_0 + 1$,
\[ \| h^i \|^2_{L^2_{\nu,v}} + |h^i|_{L^\infty_{\nu,v}} \leq \left( \frac{1}{8} \right) \max_{1 \leq t \leq 4k_0} \left\{ \| h^i \|^2_{L^2_{\nu,v}} + |h^i|_{L^\infty_{\nu,v}} \right\} \leq C_{k_0,\varepsilon,n,d} \left\{ \| \nu^{-1} w g \|_{L^\infty_{\nu,v}} + |wr|_{L^\infty_{\nu,v}} \right\}. \] (2.68)
Taking the difference $h^{i+1} - h^i$ and then applying Lemma 2.4 to $h^{i+1} - h^i$, we have that for $i \geq 2k_0$,
$$
\|h^{i+2} - h^{i+1}\|_{L_{x,v}} + |h^{i+2} - h^{i+1}|_{L^\infty(\gamma_+)} \\
\leq \frac{1}{8} \max_{0 \leq l \leq 2k_0} \left\{ \|h^{i+1-l} - h_l^{i-l}\|_{L_{x,v}} + |h^{i+1-l} - h_l^{i-l}|_{L^\infty(\gamma_+)} \right\} \\
+ C \sup_{0 \leq l \leq 2k_0} \left\{ \|f^{i+1-l} - f_l^{i-l}\|_{L_{x,v}^2} \right\},
$$
where we have used (2.69) and denoted $\eta_n := 1 - \frac{2}{n} + \frac{3}{2n^2} < 1$. Here we choose $n$ large enough so that $\frac{1}{8} \eta_n^{-2k_0-1} \leq \frac{1}{4}$, then it follows from (2.69) and Lemma 2.1 that
$$
\|h^{i+2} - h^{i+1}\|_{L_{x,v}^\infty} + |h^{i+2} - h^{i+1}|_{L^\infty(\gamma_+)} \\
\leq \left( \frac{1}{8} \right) \left( \frac{1}{n^2} \right) \max_{1 \leq l \leq 2k_0+1} \{ \|h_l^{i}\|_{L_{x,v}^\infty} + |h_l^{i}|_{L^\infty(\gamma_+)} \} \\
+ C_{\varepsilon,n,d} \{ \|\nu^{-1} w g\|_{L_{x,v}^\infty} + |w r|_{L^\infty(\gamma_-)} \} \cdot \eta_n \\
\leq C_{\varepsilon,n,k_0,d} \{ \|\nu^{-1} w g\|_{L_{x,v}^\infty} + |w r|_{L^\infty(\gamma_-)} \} \cdot \left( \frac{1}{8} \right) \left( \frac{1}{n^2} \right) \eta_n + \eta_n^i,
$$
for $i \geq 2k_0 + 1$. Then (2.70) implies immediately that $\{h_i\}_{i=0}^\infty$ is a Cauchy sequence in $L_{x,v}^\infty$, i.e., there exists a limit function $h \in L_{x,v}^\infty$ so that $\|h^i - h\|_{L_{x,v}^\infty} + |h^i - h|_{L^\infty(\gamma_+)} \to 0$ as $i \to \infty$. Thus we obtained a function $f := \frac{h}{n}$ solves
$$
\begin{align*}
L_0 f &= \varepsilon f + v_3 \partial_x f + \nu(v) f = g, \\
f(0, v)|_{v_3 > 0} &= (1 - \frac{1}{n}) P_x f(0, v) + r(v), \\
f(d, v)|_{v_3 < 0} &= (1 - \frac{1}{n}) f(d, R v),
\end{align*}
$$
with $n \geq n_0$ large enough. Moreover, from (2.69), there exists a constant $C_{\varepsilon,n,k_0,d} > 0$ such that
$$
\|h\|_{L_{x,v}^\infty} + |h|_{L^\infty(\gamma_+)} \leq C_{\varepsilon,n,k_0,d} \{ \|\nu^{-1} w g\|_{L_{x,v}^\infty} + |w r|_{L^\infty(\gamma_-)} \}.
$$

Step 2. A priori uniform estimates. For any given $\lambda \in [0,1]$, let $f^n$ be the solution of (2.57), i.e.,
$$
\begin{align*}
L \lambda f^n &= \varepsilon f^n + v_3 \partial_x f^n + P^0_E \nu(v) f^n - P^0_E \lambda K f^n = g, \\
f^n(0, v)|_{v_3 > 0} &= (1 - \frac{1}{n}) P_x f^n(0, v) + r(v), \\
f^n(d, v)|_{v_3 < 0} &= (1 - \frac{1}{n}) f^n(d, R v),
\end{align*}
$$
Moreover we also assume that $\|w f^n\|_{L_{x,v}^\infty} + |w f^n|_{L^\infty(\gamma_-)} < \infty$. We note that $(L f^n, f^n) \geq 0$, which implies that
$$
(K f^n, f^n) \leq \|f^n\|_{\nu}.
$$
On the other hand, a direct calculation shows that

\[ |f^n(0)|^2_{L^2(\gamma_-)} = \left| (1 - \frac{1}{n})P_\gamma f^n(0, v) + r(v) \right|^2_{L^2(\gamma_-)} \leq (1 - \frac{2}{n} + \frac{3}{2n^2})|f^n|_{L^2(\gamma_+)}^2 + C_n |r|^2_{L^2(\gamma_-)}, \]  

(2.73)

and

\[ |f^n(d)|^2_{L^2(\gamma_-)} = (1 - \frac{1}{n})^2 |f^n(d)|_{L^2(\gamma_-)}^2. \]  

(2.74)

Multiplying (2.71) by \( f^n \), one has that

\[ \varepsilon \|f^n\|^2_{L^2_{x,v}} + \frac{1}{2} \|f^n\|^2_{L^2(\gamma_+)} - \frac{1}{2} \|f^n\|^2_{L^2(\gamma_-)} + p_E^0 \|\|f^n\|_v^2 - \lambda (K f^n, f^n)\| \leq \frac{\varepsilon}{4} \|f^n\|_{L^2_{x,v}} + C \varepsilon \|g\|^2_{L^2_{x,v}}, \]

which, together with (2.72), (2.73) and (2.74), yields that

\[ \|\mathcal{L}_\lambda^{-1} g\|^2_{L^2_{x,v}} + \|\mathcal{L}_\lambda^{-1} g\|^2_{L^2(\gamma_+)} = \|f^n\|^2_{L^2_{x,v}} + |f^n|_{L^2(\gamma_-)} \leq C_{\varepsilon,n} \{ \|g\|^2_{L^2_{x,v}} + |r|^2_{L^2(\gamma_-)} \}. \]  

(2.75)

Let \( h^n := w f^n \). Then, by using (2.71) and (2.75), we obtain

\[ \|w \mathcal{L}_\lambda^{-1} g\|_{L^\infty_{x,v}} + |w \mathcal{L}_\lambda^{-1} g|_{L^\infty(\gamma)} = \|h^n\|_{L^\infty_{x,v}} + |h^n|_{L^\infty(\gamma)} \leq C_{\varepsilon,n,k_0,d} \left\{ \|\nu^{-1} w g\|_{L^\infty_{x,v}} + |w r|_{L^\infty(\gamma)} \right\}. \]  

(2.76)

On the other hand, let \( w g_1 \in L^\infty_{x,v} \) and \( w g_2 \in L^\infty_{x,v} \). Let \( f^n_1 = \mathcal{L}_\lambda^{-1} g_1 \) and \( f^n_2 = \mathcal{L}_\lambda^{-1} g_2 \) be the solutions to (2.71) with \( g \) replaced by \( g_1 \) and \( g_2 \), respectively. Then we have that

\[ \begin{cases}
  \varepsilon (f^n_2 - f^n_1) + \nu_3 \partial_x (f^n_2 - f^n_1) + p_E^0 \nu(v)(f^n_2 - f^n_1) - p_E^0 \lambda K(f^n_2 - f^n_1) = g_2 - g_1, \\
  (f^n_2 - f^n_1)(0, v)|_{v_3 > 0} = (1 - \frac{1}{n}) P_\gamma (f^n_2 - f^n_1)(0, v), \\
  (f^n_2 - f^n_1)(d, v)|_{v_3 < 0} = (1 - \frac{1}{n}) (f^n_2 - f^n_1)(d, \mathcal{R} v)
\end{cases} \]

By similar arguments as in (2.71)-(2.76), we can obtain

\[ \|\mathcal{L}_\lambda^{-1} g_2 - \mathcal{L}_\lambda^{-1} g_1\|^2_{L^2_{x,v}} + \|\mathcal{L}_\lambda^{-1} g_2 - \mathcal{L}_\lambda^{-1} g_1\|^2_{L^2(\gamma_+)} \leq C_{\varepsilon,n,k_0} \|g_2 - g_1\|^2_{L^2_{x,v}}, \]  

(2.77)

and

\[ \|w(\mathcal{L}_\lambda^{-1} g_2 - \mathcal{L}_\lambda^{-1} g_1)\|_{L^\infty_{x,v}} + |w(\mathcal{L}_\lambda^{-1} g_2 - \mathcal{L}_\lambda^{-1} g_1)|_{L^\infty(\gamma)} \leq C_{\varepsilon,n,k_0,d} \|\nu^{-1} w (g_2 - g_1)\|_{L^\infty_{x,v}}. \]  

(2.78)

The uniqueness of solution to (2.71) also follows from (2.77). We point out that the constants \( C_{\varepsilon,n,k_0,d} \) in (2.77)-(2.78) do not depend on \( \lambda \in [0, 1] \). This property is crucial for us to extend \( \mathcal{L}_0^{-1} \) to \( \mathcal{L}_1^{-1} \) by a bootstrap argument.

**Step 3.** In this step, we shall prove the existence of solution \( f^n \) to (2.57) for sufficiently small \( 0 < \lambda \ll 1 \), i.e., to prove the existence of operator \( \mathcal{L}_\lambda^{-1} \). Firstly, we define the Banach space

\[ \mathbf{X} := \left\{ f = f(x, v) : w f \in L^\infty(\Omega_d \times \mathbb{R}^3), \ w f \in L^\infty(\gamma), \right. \\
\left. f(0, v)|_{v_3 > 0} = (1 - \frac{1}{n}) P_\gamma f(0, v) + r(v), \ f(d, v)|_{v_3 < 0} = (1 - \frac{1}{n}) f(d, \mathcal{R} v) \right\}. \]

Now we define

\[ T_\lambda f = \mathcal{L}_0^{-1} \left( p_E^0 \lambda K f + g \right). \]
For any \( f_1, f_2 \in X \), by using (2.78), we have that
\[
\|w(T_{\lambda}f_1 - T_{\lambda}f_2)\|_{L^\infty_x} + |w(T_{\lambda}f_1 - T_{\lambda}f_2)|_{L^\infty(\gamma)} \\
= \|w\{L^{-1}_0(p_0^0 E)Kf_1 + g - L^{-1}_0(p_0^0 E)Kf_2 + g)\|_{L^\infty_v} \\
+ |w\{L^{-1}_0(p_0^0 E)Kf_1 + g - L^{-1}_0(p_0^0 E)Kf_2 + g)\|_{L^\infty(\gamma)} \\
\leq C_{\epsilon, n, \alpha, d, n}\|\nu^{-1}w(p_0^0 E)Kf_1 + g - (p_0^0 E)Kf_2 + g)\|_{L^\infty_v} \\
\leq \lambda p_0^0 E C_{\epsilon, n, \alpha, d, n}\|\nu^{-1}w(Kf_1 - Kf_2)\|_{L^\infty_v} \\
\leq \lambda p_0^0 E C_{\epsilon, n, \alpha, d, n}d\|w(f_1 - f_2)\|_{L^\infty_v}.
\]

We take \( \lambda_0 > 0 \) sufficiently small such that \( \lambda_0 p_0^0 E C_{\epsilon, n, \alpha, d, n} \leq 1/2 \), then \( T_{\lambda} : X \rightarrow X \) is a contraction mapping for \( \lambda \in [0, \lambda_0] \). Thus \( T_{\lambda} \) has a fixed point, i.e., \( \exists f^\lambda \in X \) such that
\[
f^\lambda = T_{\lambda} f^\lambda = L^{-1}_0(p_0^0 E)Kf^\lambda + g,
\]
which yields immediately that
\[
L_{\lambda} f^\lambda = \epsilon f^\lambda + v_3 \partial_x f^\lambda + p_0^0 E \nu(v)f^\lambda - p_0^0 E \lambda K f^\lambda = g.
\]

Hence, for any \( \lambda \in [0, \lambda_0] \), we have solved (2.57) with \( f^\lambda = L^{-1}_0 g \in X \). Therefore we have obtained the existence of \( L^{-1}_0 \) for \( \lambda \in [0, \lambda_0] \). Moreover the operator \( L^{-1}_0 \) has the properties (2.73)-(2.78).

Step 4. Finally we define
\[
T_{\lambda_0 + \lambda} f = L^{-1}_0 \lambda Kf + g.
\]

Noting the estimates for \( L^{-1}_0 \) are independent of \( \lambda_0 \). By similar arguments, we can prove \( T_{\lambda_0 + \lambda} : X \rightarrow X \) is a contraction mapping for \( \lambda \in [0, \lambda_0] \). Then we obtain the existence of operator \( L^{-1}_0 \) and (2.75)-(2.78). Step by step, we can finally obtain the existence of operator \( L^{-1}_0 \), and \( \lambda K \) satisfies the estimates in (2.73)-(2.78). The continuity is easy to obtain since the convergence of sequence under consideration is always in \( L^\infty_{x, v} \). Therefore we complete the proof of Lemma 2.5.”

Before taking the limit \( n \rightarrow + \infty \), we first introduce a useful lemma which will be used later.

**Lemma 2.6.** Define the near grazing set \( \gamma^\epsilon_\pm \) as
\[
\gamma^\epsilon_\pm := \{ (x, v) \in \gamma \ : |v| \leq \epsilon \text{ or } |v| \geq \frac{1}{\epsilon} \}. \quad (2.79)
\]
Let \( \epsilon > 0 \) be a small positive constant, then it holds that
\[
|f(0)|_{L^1 \gamma} \leq \frac{d}{\epsilon^2} \left\{ \|f\|_{L^1_{x, v}} + \|v_3 \partial_x f\|_{L^1_{x, v}} \right\}. \quad (2.80)
\]
More precisely, we have
\[
|f(0)|_{L^1 \gamma} \leq \frac{4}{\epsilon^2} \left\{ \|f\|_{L^1(\Omega \times \mathbb{R}^3)} + \|v_3 \partial_x f\|_{L^1(\Omega \times \mathbb{R}^3)} \right\}, \quad (2.81)
\]
where \( \Omega = (0, \frac{1}{2}) \).

**Lemma 2.7.** Let \( \epsilon > 0 \), \( d \geq 1 \) and \( \beta \geq 3 \), and assume \( \|\nu^{-1}wg\|_{L^\infty_{x, v}} + |wr|_{L^\infty(\gamma_\pm)} < \infty \). Then there exists a unique solution \( f^\epsilon \) to solve the approximate linearized steady Boltzmann equation (2.1). Moreover, it satisfies
\[
\|w f^\epsilon\|_{L^\infty_{x, v}} + |wf^\epsilon|_{L^\infty(\gamma)} \leq C_{\epsilon, d} \cdot \left\{ \|\nu^{-1}wg\|_{L^\infty_{x, v}} + |wr|_{L^\infty(\gamma_\pm)} \right\}, \quad (2.82)
\]
where the positive constant \( C_{\epsilon, d} > 0 \) depends only on \( \epsilon \) and \( d \). Moreover, if \( g(x, v) \) is continuous in \( \Omega_d \times \mathbb{R}^3 \) and \( r(v) \) is continuous in \( \mathbb{R}^3 \), then \( f^\epsilon \) is continuous away from the grazing set \( \gamma_0 \).
Proof. Let $f^n$ be the solution of (2.56) constructed in Lemma 2.5 for $n \geq n_0$ with $n_0$ large enough. Multiplying (2.56) by $f^n$, one obtains that
\[
\varepsilon \|f^n\|^2_{L^2(\gamma_+)} + \|f^n(0)\|^2_{L^2(\gamma_-)} + 2c_0\|\big(I - P\big)f^n\|^2 \leq C_{\varepsilon} \|g\|^2_{L^2_{\gamma_+}} + \left| \left( 1 - \frac{1}{n} \right) P_\gamma f^n(0) + r \right|^2_{L^2(\gamma_-)}.
\] (2.83)

A direct calculation shows that
\[
\left| \left( 1 - \frac{1}{n} \right) P_\gamma f^n(0) + r \right|^2_{L^2(\gamma_-)} \leq |P_\gamma f^n(0)|^2_{L^2(\gamma_-)} + 2|P_\gamma f^n(0)|_{L^2(\gamma_-)} \cdot |r|_{L^2(\gamma_-)} + |r|^2_{L^2(\gamma_-)}.
\]
which, together with (2.83), yields that
\[
\varepsilon \|f^n\|^2_{L^2(\gamma_+)} + \|f^n(0)\|^2_{L^2(\gamma_-)} + 2c_0\|\big(I - P\big)f^n\|^2 \leq \delta |P_\gamma f^n(0)|^2_{L^2(\gamma_-)} + C_{\varepsilon, \delta} \left\{ \|g\|^2_{L^2_{\gamma_+}} + |r|^2_{L^2(\gamma_-)} \right\}.
\] (2.84)

where $\delta > 0$ is a small constant to be chosen later.

We still need to bound the first term on RHS of (2.84). Firstly, a direct calculation shows that
\[
\frac{1}{2} |P_\gamma f^n(0)|^2_{L^2(\gamma_-)} \leq |P_\gamma f^n(0)|I_{\gamma_+} |f^n|_{L^2(\gamma_\gamma_+)}^2,
\] (2.85)
provided that $\varepsilon \ll 1$. We notice that
\[
f^n(0, v) = (I - P_\gamma)f^n(0, v) + P_\gamma f^n(0, v), \quad \forall (0, v) \in \gamma_+,
\]
which yields that
\[
|P_\gamma f^n(0, v)I_{\gamma_+ \gamma_\gamma_+} |^2_{L^2(\gamma_-)} \leq 2|f^n(0, v)I_{\gamma_+ \gamma_\gamma_+} |^2_{L^2(\gamma_-)} + 2|(I - P_\gamma)f^n(0, v)I_{\gamma_+ \gamma_\gamma_+} |^2_{L^2(\gamma_-)}.
\] (2.86)

On the other hand, it follows from (2.56) that
\[
\frac{1}{2} v_3 \partial_x (|f^n|^2) = -\varepsilon |f^n|^2 - p_0^n f^n L f^n + g f^n,
\]
which immediately implies that
\[
\|v_3 \partial_x (|f^n|^2)\|_{L^1_{\gamma_+}} \leq C \left\{ \|f^n\|^2_{L^2_{\gamma_+}} + \|\big(I - P\big)f^n\|^2_{L^2_{\gamma_+}} + \|g\|^2_{L^2_{\gamma_+}} \right\}.
\] (2.87)

Thus, using Lemma 2.6 and (2.84) to obtain
\[
|f^n(0)|I_{\gamma_+ \gamma_\gamma_+} |^2_{L^2(\gamma_-)} = \|f^n(0)\|^2I_{\gamma_+ \gamma_\gamma_+} |^2_{L^1(\gamma_-)}
\leq \frac{4}{c_\varepsilon} \left\{ \|f^n\|^2_{L^2_{\gamma_+}} + \|v_3 \cdot \partial_x (f^n)\|^2_{L^2_{\gamma_+}} \right\}
\leq \frac{C}{c_\varepsilon} \left\{ \|f^n\|^2_{L^2_{\gamma_+}} + \|\big(I - P\big)f^n\|^2_{L^2_{\gamma_+}} + \|g\|^2_{L^2_{\gamma_+}} \right\}
\leq C_{\varepsilon, \delta} |P_\gamma f^n(0)|^2_{L^2(\gamma_-)} + C_{\varepsilon, \delta} \left\{ |g|^2_{L^2_{\gamma_+}} + |r|^2_{L^2(\gamma_-)} \right\},
\] (2.88)

which, together with (2.84)-(2.86), yields that
\[
|P_\gamma f^n(0)|^2_{L^2(\gamma_-)} \leq C_{\varepsilon, \delta} |P_\gamma f^n(0)|^2_{L^2(\gamma_-)} + C_{\varepsilon, \delta} \left\{ |g|^2_{L^2_{\gamma_+}} + |r|^2_{L^2(\gamma_-)} \right\}.
\]

Now taking $0 < \delta \ll 1$ so that $C_{\varepsilon, \delta} \cdot \delta \leq \frac{1}{2}$, one obtains that
\[
|P_\gamma f^n(0)|^2_{L^2(\gamma_-)} \leq C_{\varepsilon, \delta} \left\{ |g|^2_{L^2_{\gamma_+}} + |r|^2_{L^2(\gamma_-)} \right\},
\] (2.89)

which, together with (2.84), shows that
\[
\|f^n\|^2_{L^2_{\gamma_+}} + |f^n(0)|^2_{L^2(\gamma_-)} + 2c_0\|\big(I - P\big)f^n\|^2 \leq C_{\varepsilon, \delta} \left\{ |g|^2_{L^2_{\gamma_+}} + |r|^2_{L^2(\gamma_-)} \right\}.
\] (2.90)
We apply (2.16) and use (2.90) to obtain

\[ \|wf^n\|_{L^\infty_v} + |wf^n|_{L^\infty(\gamma)} \leq C \left\{ \|\nu^{-1}wg\|_{L^\infty_v} + |wr|_{L^\infty(\gamma_-)} + \|f^n\|_{L^2_v} \right\} \]
\[ \leq C_{\varepsilon, \alpha, d} \left\{ \|\nu^{-1}wg\|_{L^\infty_v} + |wr|_{L^\infty(\gamma_-)} \right\}. \]  

(2.91)

Taking the difference \( f^{n_1} - f^{n_2} \) with \( n_1, n_2 \geq n_0 \), we know that

\[ \begin{cases} 
\varepsilon(f^{n_1} - f^{n_2}) + v_3 \partial_z (f^{n_1} - f^{n_2}) + p^0_b L(f^{n_1} - f^{n_2}) = 0, \\
(f^{n_1} - f^{n_2})(0, v)_{|v_3 > 0} = (1 - \frac{1}{n_1}) P(f^{n_1} - f^{n_2})(0, v) + \frac{1}{n_1} P \gamma f^{n_2}(0, v), \\
(f^{n_1} - f^{n_2})(d, v)_{|v_3 < 0} = (1 - \frac{1}{n_1}) (f^{n_1} - f^{n_2})(d, Rv) + \frac{1}{n_1} f^{n_2}(d, Rv) 
\end{cases} \]  

(2.92)

Multiplying (2.92) by \( f^{n_1} - f^{n_2} \) and integrating it over \( \Omega_d \times \mathbb{R}^3 \), by similar arguments as in (2.83)-(2.90), we can obtain

\[ \|(f^{n_1} - f^{n_2})\|_{L^2_v}^2 + \|(f^{n_1} - f^{n_2})(0)\|_{L^2(\gamma_+)} + 2c_0 \|\{I - P\} (f^{n_1} - f^{n_2})\|_{L^2_v}^2 \\ \leq C_{\varepsilon, \alpha} \left( \frac{1}{n_2} - \frac{1}{n_1} \right) P(f^{n_2}(0)) + \frac{1}{n_2} \left\{ \|f^{n_1}\|_{L^2(\gamma_+)} + \|f^{n_2}\|_{L^2(\gamma_+)} \right\} \]  
\[ \leq C_{\varepsilon, \alpha} \cdot \left\{ \frac{1}{n_1} + \frac{1}{n_2} \right\} \cdot \left\{ \|f^{n_1}\|_{L^2(\gamma_+)} + \|f^{n_2}\|_{L^2(\gamma_+)} \right\} \]  
\[ \leq C_{d, \varepsilon, \alpha} \cdot \left\{ \frac{1}{n_1} + \frac{1}{n_2} \right\} \cdot \left\{ \|\nu^{-1}wg\|_{L^\infty_v} + |wr|_{L^\infty(\gamma_-)} \right\} \]  
\[ \rightarrow 0, \]  

(2.93)

as \( n_1, n_2 \rightarrow \infty \), where we have used the uniform estimate (2.91) in the last inequality. Applying (2.16) to \( f^{n_1} - f^{n_2} \) and using (2.93), then one has

\[ \|w(f^{n_1} - f^{n_2})\|_{L^\infty_v} + |w(f^{n_1} - f^{n_2})|_{L^\infty(\gamma)} \]  
\[ \leq C \left( \frac{1}{n_2} + \frac{1}{n_1} \right) |wf^{n_2}|_{L^\infty(\gamma_+)} + C \|f^{n_1} - f^{n_2}\|_{L^2_v} \]  
\[ \leq C_{\varepsilon, \alpha, d} \cdot \left( \frac{1}{n_1} + \frac{1}{n_2} \right) \cdot \left\{ \|\nu^{-1}wg\|_{L^\infty_v} + |wr|_{L^\infty(\gamma_-)} \right\} \rightarrow 0, \]  

as \( n_1, n_2 \rightarrow \infty \), which yields that \( w f^n \) is a Cauchy sequence in \( L^\infty_v \). We denote \( f^\varepsilon = \lim_{n \rightarrow \infty} f^n \), then it is direct to check that \( f^\varepsilon \) is a solution to (2.41), and (2.82) holds. The continuity of \( f^\varepsilon \) is easy to obtain since the convergence of sequences is always in \( L^\infty_v \) and \( f^n \) is continuous away from the grazing set. Therefore we have completed the proof of Lemma 2.7. \( \Box \)

From now on, we assume

\[ \int_0^d \int_{\mathbb{R}^3} \sqrt{\mu} g(x, v) dv dx = 0 \quad \text{and} \quad \int_{v_3 > 0} r(v) \sqrt{\mu} |v_3| dv = 0. \]  

(2.94)

Let \( f^\varepsilon \) be the solution constructed in Lemma 2.7 and we denote

\[ P f^\varepsilon(x, v) = \{ a^\varepsilon(x) + b^\varepsilon \cdot v + \frac{1}{2} c^\varepsilon(x) (|v|^2 - 3) \} \sqrt{\mu}. \]

Multiplying (2.41) by \( \sqrt{\mu} \) and integrating over \([0, d] \times \mathbb{R}^3 \) to obtain

\[ \varepsilon \int_0^d \int_{\mathbb{R}^3} \sqrt{\mu} f^\varepsilon(x, v) dv dx = \varepsilon \int_0^d a^\varepsilon(x) dx = 0, \]  

(2.95)

where we have used (2.94).
Lemma 2.8. Let $d \geq 1$. Assume (2.7.4) and let $f^\varepsilon$ be the solution of (2.1) constructed in Lemma 2.7 then it holds that

$$\|P f^\varepsilon\|^2_{L^2_{\varepsilon}} \leq C d \left\{ \| (I - P) f^\varepsilon \|_{L^2_{\varepsilon}}^2 + \| g \|^2_{L^2_{\varepsilon}} + \| (I - P_{\gamma}) f^\varepsilon \|_{L^2_{(\gamma_+)}}^2 + \| r \|^2_{L^2_{(\gamma_-)}} \right\}. \tag{2.96}$$

Proof. The weak formulation of (2.1) is

$$\varepsilon \int_0^d \int_{\mathbb{R}^3} f^\varepsilon(x,v) \psi(x,v) dv dx - \int_0^d \int_{\mathbb{R}^3} v_3 f^\varepsilon(x,v) \partial_x \psi(x,v) dv dx$$

$$= - \int_{\mathbb{R}^3} v_3 f^\varepsilon(d,v) \psi(d,v) dv + \int_{\mathbb{R}^3} v_3 f^\varepsilon(0,v) \psi(0,v) dv$$

$$- \int_0^d \int_{\mathbb{R}^3} \psi(x,v) L f^\varepsilon(x,v) dv dx + \int_0^d \int_{\mathbb{R}^3} g(x,v) \psi(x,v) dv dx. \tag{2.97}$$

Motivated by [7, 8], we choose some special test function $\psi$ to calculate the macroscopic part of $f^\varepsilon$.

Step 1. Estimate on $c^\varepsilon$. Define

$$c^\varepsilon_c(x) = \int_x^d c^\varepsilon(z) dz.$$

It is easy to check that

$$c^\varepsilon_c(d) = 0, \quad |c^\varepsilon_c(0)| \leq d^{\frac{1}{2}} \| c^\varepsilon \|_{L^2_{\varepsilon}} \quad \text{and} \quad \| c^\varepsilon_c \|_{L^2_{\varepsilon}} \leq d \| c^\varepsilon \|_{L^2_{\varepsilon}}. \tag{2.98}$$

We define the test function $\psi$ in (2.97) to be

$$\psi = \psi_c(x,v) = v_3(|v|^2 - 5) \sqrt{\mu} c^\varepsilon_c(x).$$

Then the second term on LHS of (2.97) is estimated as

$$- \int_0^d \int_{\mathbb{R}^3} v_3 f^\varepsilon(x,v) \partial_x \psi_c^\varepsilon(x,v) dv dx$$

$$= \int_0^d \int_{\mathbb{R}^3} [a^\varepsilon + b^\varepsilon \cdot v + \frac{1}{2} c^\varepsilon(x)(|v|^2 - 3)] v_3^2(|v|^2 - 5) \mu(v) c^\varepsilon(x) dv dx$$

$$+ \int_0^d \int_{\mathbb{R}^3} (I - P)f^\varepsilon(x,v) v_3^2(|v|^2 - 5) \sqrt{\mu} c^\varepsilon(x) dv dx$$

$$\geq 5 \| c^\varepsilon \|^2_{L^2_{\varepsilon}} - C \| (I - P)f^\varepsilon \|_{L^2_{\varepsilon}} \| c^\varepsilon \|_{L^2_{\varepsilon}} \geq 4 \| c^\varepsilon \|^2_{L^2_{\varepsilon}} - C \| (I - P)f^\varepsilon \|^2_{L^2_{\varepsilon}}, \tag{2.99}$$

where we have used

$$\int_{\mathbb{R}^3} v_3^2(|v|^2 - 3)(|v|^2 - 5) \mu(v) dv = 10, \quad \int_{\mathbb{R}^3} v_3^2(|v|^2 - 5) \mu(v) dv = 0. \tag{2.100}$$

By using (2.100), the first term on LHS of (2.97) is bounded as

$$\varepsilon \left| \int_0^d \int_{\mathbb{R}^3} f^\varepsilon(x,v) \psi_c^\varepsilon(x,v) dv dx \right| \leq C \varepsilon \| (I - P)f^\varepsilon \|_{L^2_{\varepsilon}} \| c^\varepsilon_c \|_{L^2_{\varepsilon}} \leq C \varepsilon d \| (I - P)f^\varepsilon \|_{L^2_{\varepsilon}} \| c^\varepsilon \|_{L^2_{\varepsilon}}.$$

For the boundary terms, we note

$$f^\varepsilon(0,v) = P_{\gamma} f^\varepsilon(0,v) + I_{\gamma_+} \cdot (I - P_{\gamma}) f^\varepsilon(0,v) + I_{\gamma_-} \cdot r(v) \quad \text{for } (0,v) \in \gamma, \tag{2.101}$$

where

$$P_{\gamma} f^\varepsilon(0,v) = \sqrt{2\pi \mu(v)} z_\gamma(f^\varepsilon) \quad \text{with} \quad z_\gamma(f^\varepsilon) := \int_{u_3 < 0} f^\varepsilon(0,u) \sqrt{\mu(u)} |u_3| du. \tag{2.102}$$
Noting \((2.98)\), one has that
\[
\left| \int_{\mathbb{R}^3} v_3 f^\varepsilon (d, v) \psi^\varepsilon_c (d, v) dv + \int_{\mathbb{R}^3} v_3 f^\varepsilon (0, v) \psi^\varepsilon_c (0, v) dv \right|
= \left| \int_{\mathbb{R}^3} v_3^2 (|v|^2 - 5) \sqrt{\mu (v)} \zeta^\varepsilon (0) \left[ \sqrt{2\pi \mu (v)} z_{\gamma} (f^\varepsilon) + I_{\gamma^+} \cdot (I - P_{\gamma^+}) f^\varepsilon (0, v) + I_{\gamma^-} \cdot r (v) \right] dv \right|
= |\zeta^\varepsilon (0)| \cdot \left| \int_{\mathbb{R}^3} v_3^2 (|v|^2 - 5) \sqrt{\mu (v)} \left[ I_{\gamma^+} \cdot (I - P_{\gamma^+}) f^\varepsilon (0, v) + I_{\gamma^-} \cdot r (v) \right] dv \right|
\leq C d^2 \| \varepsilon \|_{L^2} \left[ \|(I - P_{\gamma^+}) f^\varepsilon (0)\|_{L^2 (\gamma^+)} + \|r\|_{L^2 (\gamma^-)} \right].
\] (2.103)
Hence the RHS of \((2.97)\) is bounded as
\[
\text{RHS of } (2.97) \leq C d \left( \|(I - \mathbf{P}) f^\varepsilon\|_\nu + \|g\|_{L^2, \omega} + \|(I - P_{\gamma^+}) f^\varepsilon (0)\|_{L^2 (\gamma^+)} + \|r\|_{L^2 (\gamma^-)} \right) \| \varepsilon \|_{L^2}. \] (2.104)
Combining \((2.99)\), \((2.101)\) and \((2.104)\), one obtains
\[
\| \varepsilon \|_{L^2} \leq C d^2 \left( \|(I - \mathbf{P}) f^\varepsilon\|_\nu^2 + \|g\|_{L^2, \omega}^2 + \|(I - P_{\gamma^+}) f^\varepsilon (0)\|_{L^2 (\gamma^+)}^2 + \|r\|_{L^2 (\gamma^-)}^2 \right).
\] (2.105)

Step 2. Estimate on \(b^\varepsilon\). We define
\[
\zeta^\varepsilon_{\delta, i} (x) = \int_x^d b^\varepsilon_i (z) dz, \quad i = 1, 2, 3.
\]
It holds that
\[
\zeta^\varepsilon_{\delta, i} (d) = 0, \quad |\zeta^\varepsilon_{\delta, i} (0)| \leq d^2 \| b^\varepsilon_i \|_{L^2} \quad \text{and} \quad \| \zeta^\varepsilon_{\delta, i} \|_{L^2} \leq d \| b^\varepsilon_i \|_{L^2}, \quad i = 1, 2, 3.
\] (2.106)
Now we take the test function \(\psi\) in \((2.97)\) to be
\[
\psi = \psi^\varepsilon_{\delta, 3} (x, v) = (v_3^2 - 1) \sqrt{\mu (v)} \zeta^\varepsilon_{\delta, 3} (x).
\]
Then the second term on LHS of \((2.97)\) is controlled as
\[
- \int_0^d \int_{\mathbb{R}^3} v_3 f^\varepsilon (x, v) \partial_x \psi^\varepsilon_{\delta, 3} (x, v) dv dx
= \int_0^d \int_{\mathbb{R}^3} \left[ a^\varepsilon + b^\varepsilon \cdot v + \frac{1}{2} c^\varepsilon (x) (|v|^2 - 3) \right] v_3 (v_3^2 - 1) \mu (v) b^\varepsilon_i (x) dv dx
+ \int_0^d \int_{\mathbb{R}^3} (I - \mathbf{P}) f^\varepsilon (x, v) \cdot v_3 (v_3^2 - 1) \sqrt{\mu (v)} b^\varepsilon_i (x) dv dx
= 2 \| b^\varepsilon_i \|_{L^2}^2 + \int_0^d \int_{\mathbb{R}^3} (I - \mathbf{P}) f^\varepsilon (x, v) \cdot v_3 (v_3^2 - 1) \sqrt{\mu (v)} b^\varepsilon_i (x) dv dx
\geq \frac{7}{4} \| b^\varepsilon_i \|_{L^2}^2 - C \|(I - \mathbf{P}) f^\varepsilon\|_\nu^2,
\] (2.107)
where we have used
\[
\int_{\mathbb{R}^3} v_3^2 (v_3^2 - 1) \mu (v) dv = 2.
\]
The first term on LHS of \((2.97)\) is bounded as
\[
\varepsilon \left| \int_0^d \int_{\mathbb{R}^3} f^\varepsilon (x, v) \psi^\varepsilon_{\delta, 3} (x, v) dv dx \right| \leq C \varepsilon d \| \varepsilon \|_{L^2} \| b^\varepsilon_3 \|_{L^2} + C \varepsilon \|(I - \mathbf{P}) f^\varepsilon\|_\nu \| b^\varepsilon_3 \|_{L^2}
\leq \frac{1}{4} \| b^\varepsilon_3 \|_{L^2}^2 + C \varepsilon^2 d^2 \left( \|(I - \mathbf{P}) f^\varepsilon\|_\nu^2 + \| \varepsilon \|_{L^2}^2 \right).
For the boundary terms on RHS of (2.97), it follows from (2.100) that
\[
\left| - \int_{\mathbb{R}^3} v_3 f^\varepsilon (d, v) \psi_{b,i}^\varepsilon (d, v) dv + \int_{\mathbb{R}^3} v_3 f^\varepsilon (0, v) \psi_{b,i}^\varepsilon (0, v) dv \right|
\]
\[
= \left| \int_{\mathbb{R}^3} v_3 (v_3^2 - 1) \sqrt{\mu(v) \zeta_{b,i}^\varepsilon (0)} \left[ \sqrt{2\pi \mu(v)} z_\gamma^\varepsilon (0) + I_{\gamma_+} \cdot (I - P_\gamma) f^\varepsilon (0, v) + I_{\gamma_-} \cdot r(v) \right] dv \right|
\]
\[
= \left| \int_{\mathbb{R}^3} v_3 (v_3^2 - 1) \sqrt{\mu(v) \zeta_{b,i}^\varepsilon (0)} \left[ I_{\gamma_+} \cdot (I - P_\gamma) f^\varepsilon (0, v) + I_{\gamma_-} \cdot r(v) \right] dv \right|
\]
\[
\leq C d^2 \| b_i^\varepsilon \|_{L^2_y} \left[ |(I - P_\gamma) f^\varepsilon (0)|_{L^2(\gamma_+)} + |r|_{L^2(\gamma_-)} \right]. \quad (2.108)
\]

Thus we have
\[
\text{RHS of } (2.97) \leq C d \| b_i^\varepsilon \|_{L^2_y} \left\{ \| (I - P) f^\varepsilon \|_\nu + \| g \|_{L^2_{\gamma,nu}} + |(I - P_\gamma) f^\varepsilon (0)|_{L^2(\gamma_+)} + |r|_{L^2(\gamma_-)} \right\}. \quad (2.109)
\]

Then combining (2.107)–(2.109) and using (2.105), one obtains that
\[
\| b_i^\varepsilon \|_{L^2_y} \leq C d^2 \left\{ \| (I - P) f^\varepsilon \|_\nu + \| g \|_{L^2_{\gamma,nu}} + |(I - P_\gamma) f^\varepsilon (0)|_{L^2(\gamma_+)} + |r|_{L^2(\gamma_-)} \right\}. \quad (2.110)
\]

For the estimate of $b_i^\varepsilon (x)$, $i = 1, 2$, we define
\[
\psi = \psi_{b,i} (x, v) = |v|^2 v_i v_3 \sqrt{\mu(v) \zeta_{b,i}^\varepsilon (x)}, \quad i = 1, 2. \quad (2.111)
\]

By a tedious calculation, one has
\[
- \int_0^d \int_{\mathbb{R}^3} v_3 f^\varepsilon (x, v) \partial_x \psi_{b,i}^\varepsilon (x, v) dv dx
\]
\[
= \int_0^d \int_{\mathbb{R}^3} \left[ a^\varepsilon + b^\varepsilon \cdot v + \frac{1}{2} c^\varepsilon (x)(|v|^2 - 3) \right] |v|^2 v_i v_3 \mu(v) b_i^\varepsilon (x) dv dx
\]
\[
+ \int_0^d \int_{\mathbb{R}^3} (I - P) f^\varepsilon (x, v) \cdot |v|^2 v_i v_3 \mu(v) b_i^\varepsilon (x) dv dx
\]
\[
= 7 |b_i^\varepsilon \|_{L^2_y}^2 \int_{\mathbb{R}^3} |v|^2 v_i v_3 \mu(v) dv + \int_0^d \int_{\mathbb{R}^3} (I - P) f^\varepsilon (x, v) \cdot |v|^2 v_i v_3 \mu(v) b_i^\varepsilon (x) dv dx
\]
\[
\geq 6 |b_i^\varepsilon \|_{L^2_y}^2 - C \| (I - P) f^\varepsilon \|_\nu^2, \quad (2.112)
\]

where we have used
\[
\int_{\mathbb{R}^3} |v|^2 v_i^2 v_3 \mu(v) dv = 7, \quad i = 1, 2.
\]

The first term on LHS of (2.97) is bounded as
\[
\varepsilon \left| \int_0^d \int_{\mathbb{R}^3} f^\varepsilon (x, v) \psi_{b,i}^\varepsilon (x, v) dv dx \right| \leq C d \varepsilon \| (I - P) f^\varepsilon \|_\nu \| b_i^\varepsilon \|_{L^2_y}
\]
\[
\leq \frac{1}{4} |b_i^\varepsilon \|_{L^2_y}^2 + C d^2 d^2 \| (I - P) f^\varepsilon \|_\nu^2. \quad (2.113)
\]

For the boundary term, we have
\[
- \int_{\mathbb{R}^3} v_3 f^\varepsilon (d, v) \psi_{b,i}^\varepsilon (d, v) dv + \int_{\mathbb{R}^3} v_3 f^\varepsilon (0, v) \psi_{b,i}^\varepsilon (0, v) dv
\]
\[
= \int_{\mathbb{R}^3} |v|^2 v_i v_3 \sqrt{\mu(v) \zeta_{b,i}^\varepsilon (0)} \left[ \sqrt{2\pi \mu(v)} z_\gamma^\varepsilon (0) + I_{\gamma_+} \cdot (I - P_\gamma) f^\varepsilon (0, v) + I_{\gamma_-} \cdot r(v) \right] dv
\]
\[
= \int_{\mathbb{R}^3} |v|^2 v_i v_3 \sqrt{\mu(v) \zeta_{b,i}^\varepsilon (0)} \left[ I_{\gamma_+} \cdot (I - P_\gamma) f^\varepsilon (0, v) + I_{\gamma_-} \cdot r(v) \right] dv
\]
\[
\leq C d^2 \| b_i^\varepsilon \|_{L^2_y} \left[ |(I - P_\gamma) f^\varepsilon (0)|_{L^2(\gamma_+)} + |r|_{L^2(\gamma_-)} \right].
\]
Thus the terms on RHS of (2.97) is bounded as
\[
\text{RHS of (2.97)} \leq C_d \|b_\varepsilon\|_{L^2_x} \cdot \left\{ \|s_{I - P}f\|_{L^2_v} + \|g\|_{L^2_{x,v}} + |(I - P_\gamma)f^{(0)}|_{L^2(\gamma^+)} + |r|_{L^2(\gamma^-)} \right\}.
\]
(2.114)
Substituting (2.112), (2.113) and (2.114) into (2.97), it holds that
\[
\|b_\varepsilon\|^2_{L^2_x} \leq C_d^2 \left\{ \|s_{I - P}f\|^2_{L^2_v} + \|g\|^2_{L^2_{x,v}} + |(I - P_\gamma)f^{(0)}|^2_{L^2(\gamma^+)} + |r|^2_{L^2(\gamma^-)} \right\}, \quad i = 1, 2.
\]
(2.115)

**Step 3. Estimate on \(a^\varepsilon\).** Define
\[
\zeta_\alpha^\varepsilon(x) = -\int_0^x a^\varepsilon(z)dz.
\]
From (2.95), it is easy to check that
\[
\zeta_\alpha^\varepsilon(x)|_{x=0,d} = 0, \quad \text{and} \quad \|\zeta_\alpha^\varepsilon\|_{L^2_{x,v}} \leq d \|a^\varepsilon\|_{L^2_{x,v}}.
\]
(2.116)
We define the test function \(\psi\) in (2.97) to be
\[
\psi = \psi_\alpha^\varepsilon(x, v) = (|v|^2 - 10)\sqrt{v_3} \zeta_\alpha^\varepsilon(x).
\]
Then the second term on RHS of (2.97) is estimated as
\[
- \int_{\mathbb{R}^3} v_3 f^{(0)}(x, v) \partial_x \psi_\alpha(x, v)dvdx
\]
\[
= \int_{\mathbb{R}^3} \left[ a^\varepsilon + b^\varepsilon \cdot v + \frac{1}{2} c^\varepsilon(x)(|v|^2 - 3) \right] \cdot (|v|^2 - 10) v_3^2 \mu(v)a^\varepsilon(x)dx dv
\]
\[
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v_3 (I - P) f^{(0)}(x, v)(|v|^2 - 10) v_3^2 \mu(v)a^\varepsilon(x) dx dv
\]
\[
\geq 5 \|a^\varepsilon\|^2_{L^2_v} - C\|s_{I - P}f\|_{L^2_v}\|a^\varepsilon\|_{L^2_{x,v}} \geq 4 \|a^\varepsilon\|^2_{L^2_v} - C\|s_{I - P}f\|^2_{L^2_v},
\]
(2.117)
where we have used
\[
\int_{\mathbb{R}^3} v_3^2 \cdot (|v|^2 - 10) \mu(v)dv = 5 \quad \text{and} \quad \int_{\mathbb{R}^3} (|v|^2 - 3) v_3^2 \cdot (|v|^2 - 10) \mu(v)dv = 0.
\]
A direct calculation shows that
\[
\varepsilon \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{(0)}(x, v) \psi_\alpha(x, v) dv dx \right|
\]
\[
\leq \varepsilon \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} b_\varepsilon^\gamma(x) \zeta_\alpha^\varepsilon(x) v_3^2(|v|^2 - 10) \mu(v) dv dx + C \varepsilon \|s_{I - P}f\|_{L^2_v}\|a^\varepsilon\|_{L^2_{x,v}}
\]
\[
\leq \|a^\varepsilon\|^2_{L^2_{x,v}} + C d^2 \varepsilon^2 (\|s_{I - P}f\|^2_{L^2_v} + \|b_\varepsilon^\gamma\|^2_{L^2_x}).
\]
(2.118)
It follows from (2.111) that
\[- \int_{\mathbb{R}^3} v_3 f^{(0)}(d, v) \psi_\alpha^\varepsilon(d, v) dv dx + \int_{\mathbb{R}^3} v_3 f^{(0)}(0, v) \psi_\alpha^\varepsilon(0, v) dt dv = 0.
\]
Hence, for the RHS of (2.97), it holds that
\[
\text{RHS of (2.97)} \leq C_d \|a^\varepsilon\|_{L^2_x} \left\{ \|s_{I - P}f\|^2_{L^2_v} + \|g\|_{L^2_{x,v}} \right\}.
\]
(2.119)
Combining (2.117)-(2.118) and using (2.110), we obtain
\[
\|a^\varepsilon\|^2_{L^2_x} \leq C_d^2 \left\{ \|s_{I - P}f\|^2_{L^2_v} + \|g\|^2_{L^2_{x,v}} + \|b_\varepsilon^\gamma\|^2_{L^2_x} \right\}
\]
\[
\leq C_d^6 \left\{ \|s_{I - P}f\|^2_{L^2_v} + \|g\|^2_{L^2_{x,v}} + |(I - P_\gamma)f^{(0)}|^2_{L^2(\gamma^+)} + |r|^2_{L^2(\gamma^-)} \right\}.
\]
(2.119)
Therefore, (2.90) follows directly from (2.119), (2.111), (2.110) and (2.106). The proof of Lemma 2.8 is complete.

Lemma 2.9. Let \( d \geq 1, \beta \geq 3 \). Assume (2.114) and \( \|v^{-1}wg\|_{L^\infty_{x,v}} + |wr|_{L^\infty(\gamma_-)} < \infty \). Under the condition \( \int_0^d \int_{\mathbb{R}^3} f(x,v)\sqrt{\mu dvdx} = 0 \), there exists a unique solution \( f = f(x,v) \) to the linearized steady Boltzmann equation

\[
\begin{aligned}
\nu_3 \partial_v f + \nu \mathbf{e}_1 \cdot \mathbf{L} f &= g, \quad (x,v) \in \Omega_d \times \mathbb{R}^3, \\
f(0,v)|_{v_3 > 0} &= P_y f(0,v) + r(v), \\
f(d,v)|_{v_3 < 0} &= f(x,\mathcal{R}v),
\end{aligned}
\]

with

\[
\|w f\|_{L^\infty_{x,v}} + |w f|_{L^\infty(\gamma)} \leq C_d \{ \|v^{-1}wg\|_{L^\infty_{x,v}} + |wr|_{L^\infty(\gamma_-)} \}.
\]

Moreover, if \( g \) is continuous in \( \Omega_d \times \mathbb{R} \) and \( r(v) \) is continuous in \( \mathbb{R}_+^3 \), then \( f \) is continuous away from the grazing set \( \gamma_0 \).

Proof. Let \( f^\varepsilon \) be the solution of (2.11) constructed in Lemma 2.7 for \( \varepsilon \in (0,1] \). Multiplying (2.11) by \( f^\varepsilon \) and integrating it over \( \Omega_d \times \mathbb{R}^3 \), we have

\[
\begin{aligned}
\varepsilon \|f^\varepsilon\|_{L^2_{x,v}}^2 &+ \frac{1}{2}(I - P_\gamma)(f^\varepsilon(0))_{L^2(\gamma_+)}^2 + C_0 \|v_0\|_{L^\infty_{x,v}} = \delta |P_\gamma f^\varepsilon(0)|_{L^2(\gamma_+)}^2 + C_\delta |r|_{L^2(\gamma_-)}^2 + g \|f^\varepsilon\|_{L^2_{x,v}},
\end{aligned}
\]

which, together with Lemma 2.8, yields that

\[
\begin{aligned}
\frac{d}{d\varepsilon}\|f^\varepsilon\|_{L^2_{x,v}}^2 + \|v_0\|_{L^\infty_{x,v}} = \delta |P_\gamma f^\varepsilon(0)|_{L^2(\gamma_+)}^2 + C_\delta |r|_{L^2(\gamma_-)}^2.
\end{aligned}
\]

By similar arguments as (2.86)-(2.89), one can obtain

\[
|P_\gamma f^\varepsilon(0)|_{L^2(\gamma_+)}^2 \leq C \left\{ \|f^\varepsilon\|_{L^2_{x,v}}^2 + \|(I - P_\gamma) f^\varepsilon\|_{L^2(\gamma_+)}^2 + |r|_{L^2(\gamma_-)}^2 \right\}.
\]

Substituting (2.123) into (2.123), then taking \( \delta \) suitably small, one has

\[
\|f^\varepsilon\|_{L^2_{x,v}}^2 + \|(I - P_\gamma) f^\varepsilon\|_{L^2(\gamma_+)}^2 + |r|_{L^2(\gamma_-)}^2 \leq C_{\varepsilon,d} \left\{ \|g\|_{L^2_{x,v}}^2 + |r|_{L^2(\gamma_-)}^2 \right\}.
\]

Now applying (2.10) to \( f^\varepsilon \) and using (2.123), then we obtain

\[
\|w f\|_{L^\infty_{x,v}} + |w f|_{L^\infty(\gamma)} \leq C_{\varepsilon,d} \left\{ \|v^{-1}wg\|_{L^\infty_{x,v}} + |wr|_{L^\infty(\gamma_-)} \right\}.
\]

Next we consider the convergence of \( f^\varepsilon \) as \( \varepsilon \to 0^+ \). For any \( \varepsilon_1, \varepsilon_2 > 0 \), we consider the difference \( f^{\varepsilon_2} - f^{\varepsilon_1} \) satisfying

\[
\begin{aligned}
\nu_3 \partial_v (f^{\varepsilon_2} - f^{\varepsilon_1}) + \mathbf{L}(f^{\varepsilon_2} - f^{\varepsilon_1}) &= -\varepsilon_2 f^{\varepsilon_2} + \varepsilon_1 f^{\varepsilon_1}, \\
(f^{\varepsilon_2} - f^{\varepsilon_1})(0,v)|_{v_3 > 0} &= P_y (f^{\varepsilon_2} - f^{\varepsilon_1})(0,v), \\
(f^{\varepsilon_2} - f^{\varepsilon_1})(d,v)|_{v_3 < 0} &= (f^{\varepsilon_2} - f^{\varepsilon_1})(d,\mathcal{R}v).
\end{aligned}
\]

Multiplying (2.127) by \( f^{\varepsilon_2} - f^{\varepsilon_1} \), by similar arguments as in (2.122)-(2.125), and using (2.124), one gets

\[
\begin{aligned}
\|f^{\varepsilon_2} - f^{\varepsilon_1}\|_{L^2_{x,v}}^2 + \|(I - P_\gamma)(f^{\varepsilon_2} - f^{\varepsilon_1})\|_{L^2(\gamma_+)}^2 + |r|_{L^2(\gamma_-)}^2 \leq C_{\varepsilon,d} \left\{ \|v^{-1}wg\|_{L^\infty_{x,v}} + |wr|_{L^\infty(\gamma_-)} \right\} \to 0,
\end{aligned}
\]
as $\varepsilon_1, \varepsilon_2 \to 0^+$. Finally, applying (2.10) to $f^{\varepsilon_2} - f^{\varepsilon_1}$ and using (2.128), then we obtain

$$
\|w(f^{\varepsilon_2} - f^{\varepsilon_1})\|_{L^\infty(\Omega)} + |w(f^{\varepsilon_2} - f^{\varepsilon_1})|_{L^\infty(\gamma)} \\
\leq C \{\|\nu^{-1}w(\varepsilon_2 f^{\varepsilon_2} - \varepsilon_1 f^{\varepsilon_1})\|_{L^\infty(\Omega)} + \|f^{\varepsilon_2} - f^{\varepsilon_1}\|_{L^2(\Omega)} \}
$$

$$
\leq C_{c,d} \cdot (\varepsilon_1 + \varepsilon_2) \{\|\nu^{-1}w\|_{L^\infty(\Omega)} + |wr|_{L^\infty(\Omega^-)} \} \to 0, \quad (2.129)
$$
as $\varepsilon_1, \varepsilon_2 \to 0^+$. With (2.129), we know that there exists a function $f$ so that $\|w(f^\varepsilon - f)\|_{L^\infty(\Omega)} \to 0$ as $\varepsilon \to 0^+$. And it is direct to see that $f$ solves (2.120). Also, (2.121) follows immediately from (2.120). The continuity of $f$ follows directly from the $L^\infty$-convergence and the continuity of $f^\varepsilon$. Therefore the proof of Lemma 2.9 is complete.

To obtain the solution for half-space problem, we need some uniform estimates independent of $d$, then we can take the limit $d \to \infty$. Let $f(x,v)$ be the solution of (2.120), we denote

$$
P \mathbf{f}(x,v) = [a(x) + b(x) \cdot v + c(x)(\frac{|v|^2}{2} - \frac{3}{2})] \sqrt{\mu}.
$$

It follows from (2.99) that

$$
\int_0^d \int_{\mathbb{R}^3} \sqrt{\mu(v)} f(x,v) dv dx = \int_0^d a(x) dx = 0. \quad (2.130)
$$

Noting $z_{\gamma^+}(f) := \int_{v_3 < 0} |v_3| \sqrt{\mu(v)} f(0,v) dv$, we define

$$
\tilde{f}(x,v) = f(x,v) - \sqrt{2\pi \mu(v)} z_{\gamma^+}(f), \quad (2.131)
$$

then it is easy to check that $\tilde{f}$ satisfies

$$
\begin{aligned}
\{v_3 \partial_x \tilde{f} + p_E^d \mathbf{L} \tilde{f} = g, \quad (x,v) \in \Omega_d \times \mathbb{R}^3, \\
\tilde{f}(0,v)|_{v_3 > 0} = P_d \tilde{f}(0,v) + r(v), \\
\tilde{f}(d,v)|_{v_3 < 0} = \tilde{f}(d,v). \nonumber
\end{aligned}
$$

(2.132)

It is direct to check that

$$
z_{\gamma^+}(\tilde{f}) = \int_{v_3 < 0} |v_3| \sqrt{\mu(v)} \tilde{f}(0,v) dv = 0. \quad (2.133)
$$

Moreover it follows from (2.121) and (2.131) that

$$
\|\nu \tilde{f}\|_{L^\infty(\Omega)} + |\nu \tilde{f}|_{L^\infty(\Omega^-)} \leq C_d \{\|\nu^{-1}w\|_{L^\infty(\Omega)} + |wr|_{L^\infty(\Omega^-)} \}. \quad (2.134)
$$

For later use, we denote

$$
P \mathbf{f}(x,v) = [a(x) + \tilde{b}(x) \cdot v + \tilde{c}(x)(\frac{|v|^2}{2} - \frac{3}{2})] \sqrt{\mu}.
$$

From now on, we assume $g \in \mathcal{N}^\perp$, then multiplying (2.132) by $\sqrt{\mu}$, one has that

$$
\frac{d}{dx} \int_{\mathbb{R}^3} v_3 \sqrt{\mu} \tilde{f}(x,v) dv = \int_{\mathbb{R}^3} g(x,v) \sqrt{\mu} dv \equiv 0. \quad (2.135)
$$

It follows from the specular reflection boundary condition that

$$
\int_{\mathbb{R}^3} v_3 \sqrt{\mu} \tilde{f}(d,v) dv = 0,
$$

which, together with (2.135), yields

$$
\tilde{b}_3(x) = 0, \quad \text{for} \quad x \in [0,d]. \quad (2.136)
$$
Similarly, multiplying (2.132) by \( v_i \sqrt{\mu} \), \( i = 1, 2 \), and \( (|v|^2 - 5) \sqrt{\mu} \), we can obtain, for \( x \in [0, d] \), that

\[
0 = \int_{\mathbb{R}^3} v_3 v_i \sqrt{\mu} \bar{f}(x,v) \, dv = \int_{\mathbb{R}^3} v_3 \sqrt{\mu} (I - P) \bar{f}(x,v) \, dv,
\]

\[
0 = \int_{\mathbb{R}^3} v_3 (|v|^2 - 5) \sqrt{\mu} \bar{f}(x,v) \, dv = \int_{\mathbb{R}^3} v_3 (|v|^2 - 5) \sqrt{\mu} (I - P) \bar{f}(x,v) \, dv,
\]

Utilizing (2.136) and (2.137), a direct calculation shows that

\[
\int_{\mathbb{R}^3} v_3 |P \bar{f}(x,v)|^2 \, dv = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} v_3 P \bar{f}(x,v) (I - P) \bar{f}(x,v) \, dv = 0, \quad \forall x \in [0, d],
\]

which implies that

\[
\int_{\mathbb{R}^3} v_3 |\bar{f}(x,v)|^2 \, dv = \int_{\mathbb{R}^3} v_3 |(I - P) \bar{f}(x,v)|^2 \, dv, \quad \forall x \in [0, d].
\]

Lemma 2.10. For the solution of (2.132), it holds that

\[
||I - P_\gamma||^2_{L^2(\gamma_+)} + \int_0^d e^{2\sigma_1 x} ||(I - P) \bar{f}(x)||^2_{L^2} \, dx \\
\leq C \left\{ |r|^2_{L^2(\gamma_-)} + \int_0^d e^{2\sigma_1 x} ||g(x)||^2_{L^2} \, dx \right\} \quad \text{for} \quad \sigma_1 \in [0, \sigma_0].
\]

Proof. Multiplying (2.132) by \( e^{2\sigma_1 x} \bar{f}(x,v) \), one has

\[
\frac{d}{dx} \left\{ e^{2\sigma_1 x} \int_{\mathbb{R}^3} v_3 |\bar{f}(x,v)|^2 \, dv \right\} + (c_0 P_E^0 - C\sigma_1) e^{2\sigma_1 x} ||(I - P) \bar{f}(x)||^2_{L^2} \leq C e^{2\sigma_1 x} ||g(x)||^2_{L^2},
\]

where we have used (2.139). Integrating above inequality over \([0, d]\), using (2.139) and taking \( \sigma_1 > 0 \) suitably small, one obtains

\[
- \int_{\mathbb{R}^3} v_3 |\bar{f}(0,v)|^2 \, dv + \frac{1}{2} c_0 P_E^0 \int_0^d e^{2\sigma_1 x} ||(I - P) \bar{f}(x)||^2_{L^2} \, dx \leq C \int_0^d e^{2\sigma_1 x} ||g(x)||^2_{L^2} \, dx,
\]

where we have used the fact \( \int_{\mathbb{R}^3} v_3 |\bar{f}(d,v)|^2 \, dv = 0 \) due to the specular boundary condition.

Using the diffuse reflection boundary condition, a direct calculation shows

\[
- \int_{\mathbb{R}^3} v_3 |\bar{f}(0,v)|^2 \, dv = ||I - P_\gamma||^2_{L^2(\gamma_+)} - |r|^2_{L^2(\gamma_-)} - 2 \int_{v_3 > 0} z_{\gamma_+}(\bar{f}) \sqrt{2\pi \mu(v)} v_3 r(v) \, dv
\]

\[
= ||I - P_\gamma||^2_{L^2(\gamma_+)} - |r|^2_{L^2(\gamma_-)},
\]

which, together with (2.141), yields (2.140). Therefore the proof is complete. \( \square \)

To obtain the uniform in \( d \) estimate for macroscopic term, motivated by [10, 13], we define

\[
\tilde{f}(x,v) := \bar{f}(x,v) + \Phi(v)
\]

with

\[
\Phi(v) := [\phi_0 + \phi_1 v_1 + \phi_2 v_2 + \phi_3 \frac{|v|^2}{2} - \frac{3}{2}] \sqrt{\mu(v)},
\]

where \( \phi_0, \phi_1, \phi_2, \phi_3 \) are four constants determined later. Clearly, \( \tilde{f} \) satisfies

\[
\begin{aligned}
&v_3 \partial_x \tilde{f} + p_E^0 L \tilde{f} = g, \quad (x,v) \in \Omega_d \times \mathbb{R}^3, \\
&\tilde{f}(0,v)|_{v_3 > 0} = P_\gamma \bar{f}(0,v) + (I - P_\gamma) \Phi + r(v), \\
&\tilde{f}(d,v)|_{v_3 < 0} = \tilde{f}(x,\mathcal{R}v),
\end{aligned}
\]

(2.143)
Also we have
\[
\int_{v_3>0} \{(I - P_7)\Phi + r\} v_3 \sqrt{\mu(v)} dv = 0, 
\]
(2.144)
and
\[
P \tilde{f}(x,v) = \left[ \bar{a}(x) + \bar{b}_1(x)v_1 + \bar{b}_2(x)v_2 + \bar{c}(x)\left(\frac{|v|^2}{2} - \frac{3}{2}\right) \right] \sqrt{\mu(v)},
\]
(2.145)

\[
(I - P) \tilde{f}(x,v) = (I - P)\tilde{f}(x,v) = (I - P)f(x,v),
\]

with
\[
\begin{align*}
\bar{a}(x) &= \bar{a}(x) + \phi_0 = a(x) + \phi_0 - \sqrt{2}\pi z_i(f), \\
\bar{b}_i(x) &= \bar{b}_i(x) + \phi_i = b_i(x) + \phi_i, \ i = 1, 2, \\
\bar{c}(x) &= \bar{c}(x) + \phi_3 = c(x) + \phi_3.
\end{align*}
\]
(2.146)

For later use, we denote
\[
A_{ij}(v) = \{v_i v_j - \delta_{ij}\frac{|v|^2}{3}\} \sqrt{\mu(v)},
\]
\[
B_i(v) = \frac{1}{2} v_i (|v|^2 - 5) \sqrt{\mu(v)}.
\]

It is obvious to know $A_{ij}, B_i \in \mathcal{N}^\perp$. We define
\[
\kappa_1 := \langle A_{ij}, L^{-1} A_{ij} \rangle > 0, \ i \neq j, \ i, j = 1, 2, 3.
\]
(2.147)

**Lemma 2.11.** *There exist constants $\phi_0, \phi_1, \phi_2, \phi_3$ such that*
\[
\int_{\mathbb{R}^3} v_3 \tilde{f}(d,v) \cdot v_3 \sqrt{\mu} dv = 0,
\]
\[
\int_{\mathbb{R}^3} v_3 \tilde{f}(d,v) \cdot L^{-1}(A_{33}) dv = 0, \ i = 1, 2,
\]
\[
\int_{\mathbb{R}^3} v_3 \tilde{f}(d,v) \cdot L^{-1}(B_3) dv = 0.
\]
(2.148)

**Proof.** Motivated by [10], a direct calculation shows that
\[
\int_{\mathbb{R}^3} v_3 \tilde{f}(x,v) \cdot v_3 \sqrt{\mu} dv = \bar{a}(x) + \bar{c}(x) + \int_{\mathbb{R}^3} A_{33}(v) \cdot (I - P)\tilde{f}(x,v) dv
\]
\[
= \phi_0 + \phi_3 + \bar{a}(x) + \bar{c}(x) + \int_{\mathbb{R}^3} A_{33}(v) \cdot (I - P)\tilde{f}(x,v) dv,
\]
(2.149)

\[
\int_{\mathbb{R}^3} v_3 \tilde{f}(x,v) \cdot L^{-1}(A_{31}) dv = \kappa_1 \bar{b}_1(x) + \int_{\mathbb{R}^3} v_3(I - P)\tilde{f}(x,v) \cdot L^{-1}(A_{31}) dv
\]
\[
= \kappa_1 \phi_1 + \kappa_1 \bar{b}_1(x) + \int_{\mathbb{R}^3} v_3(I - P)\tilde{f}(x,v) \cdot L^{-1}(A_{31}) dv,
\]
(2.150)

\[
\int_{\mathbb{R}^3} v_3 \tilde{f}(x,v) \cdot L^{-1}(A_{32}) dv = \kappa_1 \bar{b}_2(x) + \int_{\mathbb{R}^3} v_3(I - P)\tilde{f}(x,v) \cdot L^{-1}(A_{32}) dv
\]
\[
= \kappa_1 \phi_2 + \kappa_1 \bar{b}_2(x) + \int_{\mathbb{R}^3} v_3(I - P)\tilde{f}(x,v) \cdot L^{-1}(A_{32}) dv,
\]
(2.151)

\[
\int_{\mathbb{R}^3} v_3 \tilde{f}(x,v) \cdot L^{-1}(B_3) dv = \kappa_2 \bar{c}(x) + \int_{\mathbb{R}^3} v_3(I - P)\tilde{f}(x,v) \cdot L^{-1}(B_3) dv
\]
\[
= \kappa_2 \phi_3 + \kappa_2 \bar{c}(x) + \int_{\mathbb{R}^3} v_3(I - P)\tilde{f}(x,v) \cdot L^{-1}(B_3) dv,
\]
(2.152)
where we have used (2.144). Using (2.149)-(2.152), then (2.148) is equivalent as

\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & \kappa_1 & 0 & 0 \\
0 & 0 & \kappa_1 & 0 \\
0 & 0 & 0 & \kappa_2
\end{pmatrix}
\begin{pmatrix}
\phi_0 \\
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}
= -
\begin{pmatrix}
\bar{a}(d) + \bar{c}(d) + \int_{\mathbb{R}^3} (I - P) \ddot{f}(d, v) \cdot A_{33}(v) dv \\
\kappa_1 \check{b}_1(d) + \int_{\mathbb{R}^3} v_3 (I - P) \ddot{f}(d, v) \cdot L^{-1}(A_{31}) dv \\
\kappa_1 \check{b}_2(d) + \int_{\mathbb{R}^3} v_3 (I - P) \ddot{f}(d, v) \cdot L^{-1}(A_{32}) dv \\
\kappa_2 \check{c}(d) + \int_{\mathbb{R}^3} v_3 (I - P) \ddot{f}(d, v) \cdot L^{-1}(B_3) dv
\end{pmatrix}.
\] (2.153)

Noting the matrix is non-singular, hence \((\phi_0, \phi_1, \phi_2, \phi_3)\) is found. Therefore the proof of Lemma 2.11 is complete. \(\square\)

**Lemma 2.12.** Let \(\phi_0, \phi_1, \phi_2, \phi_3\) be the constants determined in Lemma 2.11, then it holds that

\[
\|e^{\sigma x} \ddot{f}\|_{L^2_{\gamma,v}} \leq C \left\{ |r|_{L^2(\gamma, -)} + \frac{1}{\sigma_1 - \sigma} \|e^{\sigma x} g\|_{L^2_{\gamma,v}} \right\},
\] (2.154)

with \(0 < \sigma < \sigma_1 \leq \sigma_0\), and the constant \(C > 0\) is independent of \(d\). It is important that the right hand side of (2.154) is independent of \(\Phi\).

**Proof.** It follows from (2.137) that

\[
\int_{\mathbb{R}^3} L(I - P) \ddot{f} \cdot L^{-1}(A_{3i}) dv = \int_{\mathbb{R}^3} (I - P) \ddot{f} \cdot A_{3i} dv = 0, \quad i = 1, 2,
\] (2.155)

\[
\int_{\mathbb{R}^3} L(I - P) \ddot{f} \cdot L^{-1}(B_3) dv = \int_{\mathbb{R}^3} (I - P) \ddot{f} \cdot B_3 dv = 0.
\]

Multiplying (2.143) by \(L^{-1}(A_{31}), L^{-1}(A_{32})\) and \(L^{-1}(B_3)\), respectively, and using (2.155), we can obtain

\[
\frac{d}{dx} \begin{pmatrix}
\int_{\mathbb{R}^3} v_3 \ddot{f}(x, v) \cdot L^{-1}(A_{31}) dv \\
\int_{\mathbb{R}^3} v_3 \ddot{f}(x, v) \cdot L^{-1}(A_{32}) dv \\
\int_{\mathbb{R}^3} v_3 \ddot{f}(x, v) \cdot L^{-1}(A_{33}) dv
\end{pmatrix}
= \begin{pmatrix}
\int_{\mathbb{R}^3} [g - p^{0}_E L(I - P) \ddot{f}] \cdot L^{-1}(A_{31}) dv \\
\int_{\mathbb{R}^3} [g - p^{0}_E L(I - P) \ddot{f}] \cdot L^{-1}(A_{32}) dv \\
\int_{\mathbb{R}^3} [g - p^{0}_E L(I - P) \ddot{f}] \cdot L^{-1}(A_{33}) dv
\end{pmatrix}
= \begin{pmatrix}
\int_{\mathbb{R}^3} g L^{-1}(A_{31}) dv \\
\int_{\mathbb{R}^3} g L^{-1}(A_{32}) dv \\
\int_{\mathbb{R}^3} g L^{-1}(A_{33}) dv
\end{pmatrix}.
\] (2.156)

Integrating above system over \([x, d]\) and using (2.148), we get

\[
\begin{pmatrix}
\int_{\mathbb{R}^3} v_3 \ddot{f}(x, v) \cdot L^{-1}(A_{31}) dv \\
\int_{\mathbb{R}^3} v_3 \ddot{f}(x, v) \cdot L^{-1}(A_{32}) dv \\
\int_{\mathbb{R}^3} v_3 \ddot{f}(x, v) \cdot L^{-1}(A_{33}) dv
\end{pmatrix}
= - \int_x^d \begin{pmatrix}
\int_{\mathbb{R}^3} g L^{-1}(A_{31}) dv \\
\int_{\mathbb{R}^3} g L^{-1}(A_{32}) dv \\
\int_{\mathbb{R}^3} g L^{-1}(A_{33}) dv
\end{pmatrix} (z) dz,
\] (2.157)

which, together with (2.150)-(2.152), yields that

\[
| (\kappa_1 \check{b}_1, \kappa_1 \check{b}_2, \kappa_2 \check{c})(x) | \leq C \| (I - P) \ddot{f}(x) \|_{\nu} + C \int_x^d \| g(z) \|_{L^2} dz.
\] (2.158)
Multiplying (2.153) by $e^{\sigma x}$ with $0 < \sigma < \sigma_1 \leq \sigma_0$ and using (2.140), one has
\[
\int_0^d e^{2\sigma x} |(\tilde{h}_1, \tilde{b}_2, \tilde{c})| dx \leq C \int_0^d e^{2\sigma x} \| (I - P) \tilde{f} (x) \|_{L^2}^2 dx \\
+ C \int_0^d e^{2\sigma x} \left( \int_x^d \| g(z) \|_{L^2}^2 dz \right) dx \\
\leq C \left\{ \left| r \right|_{L^2(\gamma)}^2 + \frac{1}{\sigma_1 - \sigma} \| e^{\sigma x} g \|_{L^2_w}^2 \right\}. \tag{2.159}
\]

Finally, we consider the case for $\tilde{a}$. In fact, multiplying (2.143) by $v_3 \sqrt{\mu}$, we get
\[
\frac{d}{dx} \int_{\mathbb{R}^3} \tilde{f}(x,v) \cdot v_3 \sqrt{\mu} dv = \int_{\mathbb{R}^3} g \cdot v_3 \sqrt{\mu} dv = 0. \tag{2.160}
\]
Integrating above equation over $[x, d]$, and using (2.143)–(2.149), one obtains
\[
\tilde{a}(x) = -\tilde{c}(x) + \int_{\mathbb{R}^3} (I - P) \tilde{f}(x,v) \cdot v_3 \sqrt{\mu} dv. \tag{2.161}
\]
Multiplying (2.161) by $e^{\sigma x}$ with $0 < \sigma < \sigma_1 \leq \sigma_0$ and using (2.140), (2.159), it holds that
\[
\int_0^d e^{2\sigma x} |\tilde{a}(x)|^2 dx \leq C \left\{ \left| r \right|_{L^2(\gamma)}^2 + \frac{1}{\sigma_1 - \sigma} \| e^{\sigma x} g \|_{L^2_w}^2 \right\}. \tag{2.162}
\]
Combining (2.162), (2.159) and (2.144), we prove (2.154). Therefore the proof of Lemma 2.12 is complete.

**Lemma 2.13.** Let $\tilde{f}$ the solution of (2.143), then it holds that
\[
\| e^{\sigma x} w \tilde{f} \|_{L^\infty_w} + \| e^{\sigma x} w \tilde{f} \|_{L^\infty(\gamma)} \leq C \left\{ \frac{1}{\sigma_0 - \sigma} \| e^{\sigma_0 x} \nu \|_{L^\infty_w} + \| \nu \|_{L^\infty(\gamma)} \right\}. \tag{2.163}
\]

**Proof.** Let $\tilde{h} := e^{\sigma x} w \tilde{f}$. Multiplying (2.143) by $e^{\sigma x} w$ to have
\[
\begin{cases}
 v_3 \partial_x \tilde{h} + P_0 \nu \sigma(w) \tilde{h} = P_0 K_w \tilde{h} + e^{\sigma x} w g, \\
 \tilde{h}(0,v)_{v_3 > 0} = w(v) P_\gamma \tilde{f}(0,v) + w(v) \{ (I - P_\gamma) \Phi + r(v) \}, \\
 \tilde{h}(d,v)_{v_3 < 0} = \tilde{h}(d,v) \Phi_w,
\end{cases}
\]
where $\nu(v) := \nu(v) - \nu v_3$. We further take $\sigma_0 > 0$ small such that $\nu(v) \geq \frac{1}{2} \nu(v) > 0$. By the same arguments as in Lemma 2.4 and using (2.151), we can obtain
\[
\begin{align*}
\| \tilde{h} \|_{L^\infty_w} + \| \tilde{h} \|_{L^\infty(\gamma)} & \leq C \left\{ \frac{1}{\sigma_0 - \sigma} \| e^{\sigma_0 x} \nu \|_{L^\infty_w} + \| \nu \|_{L^\infty(\gamma)} \right\} \\
& \leq C \left\{ \frac{C}{\sigma_1 - \sigma} \| e^{\sigma x} g \|_{L^2_w} + \| e^{\sigma x} \nu \|_{L^2_w} \right\} \right\}
\end{align*}
\]
where we have chosen $\sigma_1 = \sigma + \frac{\sigma_0 - \sigma}{2}$ such that $0 < \sigma < \sigma_1 < \sigma_0$. Hence the proof of Lemma 2.13 is completed.

**Remark 2.14.** From (2.159), we know that the constants $(\phi_0, \phi_1, \phi_2, \phi_3)$ depend on $d$, hence the right hand side of (2.163) still depends on $d$. In the following, we shall write down them as $(\phi_0, \phi_1, \phi_2, \phi_3)(d)$ to emphasize the dependence of $d$. To obtain the uniform estimate for $\tilde{f}$, we need...
to establish the uniform in $d$ estimate for $(\phi_0, \phi_1, \phi_2, \phi_3)(d)$. This is the key part of the present paper.

Hereafter we also denote the solutions of (2.120), (2.132) and (2.143) as $\tilde{f}_d$, $\tilde{f}_d$ and $\tilde{f}_d$, respectively, to emphasize the dependence of $d$. For later use, we denote

$$\mathbf{P} \tilde{f}_d(x, v) = \{ \tilde{a}_d(x) + \tilde{b}_{d,1}(x)v_1 + \tilde{b}_{d,2}v_2 + \tilde{c}_d(x)(|v|^2 - \frac{3}{2}) \} \sqrt{\mu(v)}.$$  

(2.164)

Firstly we give a formula of $(\tilde{a}_d, \tilde{b}_{d,1}, \tilde{b}_{d,2}, \tilde{c}_d)(x)$ by using the boundary data at $x = 0$.

**Lemma 2.15.** It holds that

$$\kappa_1 \tilde{b}_{d,1}(x) = -\left\langle (\mathbf{I} - \mathbf{P}) \tilde{f}_d(x, v), v_3 \mathbf{L}^{-1}(A_{31}) \right\rangle + \int_0^x \int_{\mathbb{R}^3} g(z) \mathbf{L}^{-1}(A_{31}) dv dz$$

$$+ \int_{\gamma_+} (I - P_\gamma) \tilde{f}_d(0, v) \cdot v_3 \mathbf{L}^{-1}(A_{31}) dv + \int_{\gamma_-} v_3 \mathbf{L}^{-1}(A_{31}) \cdot r(v) dv, \quad i = 1, 2, \quad (2.165)$$

$$\kappa_1 \tilde{c}_d(x) = -\left\langle (\mathbf{I} - \mathbf{P}) \tilde{f}_d(x, v), v_3 \mathbf{L}^{-1}(A_{32}) \right\rangle + \int_0^x \int_{\mathbb{R}^3} g(z) \mathbf{L}^{-1}(A_{32}) dv dz$$

$$+ \int_{\gamma_+} (I - P_\gamma) \tilde{f}_d(0, v) \cdot v_3 \mathbf{L}^{-1}(A_{32}) dv + \int_{\gamma_-} v_3 \mathbf{L}^{-1}(A_{32}) \cdot r(v) dv, \quad (2.166)$$

$$\tilde{a}_d(x) = -\left\langle (\mathbf{I} - \mathbf{P}) \tilde{f}_d(x, v), A_{33} - \frac{1}{\kappa_2} v_3 \mathbf{L}^{-1}(B_3) \right\rangle - \frac{1}{\kappa_2} \int_0^x \int_{\mathbb{R}^3} g(z) \mathbf{L}^{-1}(B_3) dv dz$$

$$+ \int_{\gamma_+} (I - P_\gamma) \tilde{f}_d(0, v) \cdot v_3 \left\{ v_3 \sqrt{\mu(v)} - \frac{1}{\kappa_2} \mathbf{L}^{-1}(B_3) \right\} dv$$

$$+ \int_{\gamma_-} r(v) \cdot v_3 \left\{ v_3 \sqrt{\mu(v)} - \frac{1}{\kappa_2} \mathbf{L}^{-1}(B_3) \right\} dv. \quad (2.167)$$

**Proof.** Multiplying (2.132) by $\mathbf{L}^{-1}(A_{31}), \mathbf{L}^{-1}(A_{32})$ and $\mathbf{L}^{-1}(B_3)$, respectively, and integrating over $[0, x]$ and using (2.134), one gets

$$\int_{\mathbb{R}^3} v_3 \tilde{f}_d(x, v) \cdot \mathbf{L}^{-1}(A_{31}) dv = \int_{\mathbb{R}^3} v_3 \tilde{f}_d(0, v) \cdot \mathbf{L}^{-1}(A_{31}) dv + \int_0^x \int_{\mathbb{R}^3} g(z) \mathbf{L}^{-1}(A_{31}) dv dz,$$

$$\int_{\mathbb{R}^3} v_3 \tilde{f}_d(x, v) \cdot \mathbf{L}^{-1}(A_{32}) dv = \int_{\mathbb{R}^3} v_3 \tilde{f}_d(0, v) \cdot \mathbf{L}^{-1}(A_{32}) dv + \int_0^x \int_{\mathbb{R}^3} g(z) \mathbf{L}^{-1}(A_{32}) dv dz,$$

$$\int_{\mathbb{R}^3} v_3 \tilde{f}_d(x, v) \cdot \mathbf{L}^{-1}(B_3) dv = \int_{\mathbb{R}^3} v_3 \tilde{f}_d(0, v) \cdot \mathbf{L}^{-1}(B_3) dv + \int_0^x \int_{\mathbb{R}^3} g(z) \mathbf{L}^{-1}(B_3) dv dz,$$

which, together with a tedious calculation, gives

$$\kappa_1 \tilde{b}_{d,1}(x) = -\left\langle (\mathbf{I} - \mathbf{P}) \tilde{f}_d(x, v), v_3 \mathbf{L}^{-1}(A_{31}) \right\rangle + \int_{\mathbb{R}^3} v_3 \tilde{f}_d(0, v) \cdot \mathbf{L}^{-1}(A_{31}) dv$$

$$+ \int_0^x \int_{\mathbb{R}^3} g(z) \mathbf{L}^{-1}(A_{31}) dv dz, \quad (2.168)$$

$$\kappa_1 \tilde{b}_{d,2}(x) = -\left\langle (\mathbf{I} - \mathbf{P}) \tilde{f}_d(x, v), v_3 \mathbf{L}^{-1}(A_{32}) \right\rangle + \int_{\mathbb{R}^3} v_3 \tilde{f}_d(0, v) \cdot \mathbf{L}^{-1}(A_{32}) dv$$

$$+ \int_0^x \int_{\mathbb{R}^3} g(z) \mathbf{L}^{-1}(A_{32}) dv dz, \quad (2.169)$$
and
\[ \kappa_2 \tilde{\epsilon}_d(x) = -\left( (I - P) \tilde{f}_d(x, v), v_3 L^{-1}(B_3) \right) + \int_{\mathbb{R}^3} v_3 \tilde{f}_d(0, v) \cdot L^{-1}(B_3) dv + \int_0^x \int_{\mathbb{R}^3} g(z) L^{-1}(B_3) dv dz. \] (2.170)

For the expression of $\tilde{a}(x)$, multiplying (2.132) by $v_3 \sqrt{\mu}$, and integrating over $[0, x]$, one obtains
\[ \tilde{a}_d(x) = -\tilde{\epsilon}_d(x) - \left( (I - P) \tilde{f}_d(x, v), A_{33} \right) + \int_{\mathbb{R}^3} \tilde{f}_d(0, v) v_3^2 \sqrt{\mu(v)} dv. \] (2.171)

It follows from (2.133) that
\[ \tilde{f}_d(0, v) = I_{x^+} \cdot (I - P_\gamma) \tilde{f}_d(0, v) + I_{x^-} \cdot r(v). \] (2.172)

Noting (2.133) and (2.172), a direct calculation shows
\[ \int_{\mathbb{R}^3} v_3 \tilde{f}_d(0, v) \cdot L^{-1}(A_{3i}) dv = \int_{\gamma^+} (I - P_\gamma) \tilde{f}_d(0, v) \cdot v_3 L^{-1}(A_{3i}) dv + \int_{\gamma^-} v_3 L^{-1}(A_{3i}) \cdot r(v) dv, \] (2.173)
\[ \int_{\mathbb{R}^3} v_3 \tilde{f}_d(0, v) \cdot L^{-1}(B_3) dv = \int_{\gamma^+} (I - P_\gamma) \tilde{f}_d(0, v) \cdot v_3 L^{-1}(B_3) dv \]
\[ + \int_{\gamma^-} v_3 L^{-1}(B_3) \cdot r(v) dv, \] and
\[ \int_{\mathbb{R}^3} \tilde{f}_d(0, v) v_3^2 \sqrt{\mu(v)} dv = \int_{\gamma^+} (I - P_\gamma) \tilde{f}_d(0, v) v_3^2 \sqrt{\mu(v)} dv + \int_{\gamma^-} r(v) \cdot v_3^2 \sqrt{\mu(v)} dv, \] (2.174)

which, together with (2.168)-(2.171), yields (2.165)-(2.167). Therefore the proof of Lemma 2.15 is complete. \( \square \)

**Lemma 2.16.** For $d \geq 1$, it holds that
\[ |(\phi_0, \phi_1, \phi_2, \phi_3)(d)| \leq C \left\{ \| e^{\gamma v} \nu^{-1} w g \|_{L^\infty_{\gamma v}} + |r|_{L^\infty(\gamma^-)} \right\}, \] (2.175)
and
\[ \| e^{\alpha x} w \tilde{f} \|_{L^\infty_{\gamma v}} + |e^{\alpha x} w \tilde{f} \|_{L^\infty(\gamma)} \leq C \left\{ \frac{1}{\sigma_0 - \sigma} \| e^{\gamma v} \nu^{-1} w g \|_{L^\infty_{\gamma v}} + |r|_{L^\infty(\gamma^-)} \right\}. \] (2.176)

where the constants $C > 0$ are independent of $d$. Hence we have obtained the uniform in $d$ estimates for both $\tilde{f}$ and $(\phi_0, \phi_1, \phi_2, \phi_3)(d)$.

**Proof.** Noting that (2.176) follows directly from (2.169) and (2.175), hence we need only to prove (2.175).

Noting $(I - P) \tilde{f}_d(x, v) = (I - P) \tilde{f}_d(x, v)$, it follows from (2.165)-(2.167) and (2.140) that
\[ |(\tilde{a}_d, \tilde{b}_{d, 1}, \tilde{b}_{d, 2}, \tilde{\epsilon}_d)(d)| \leq C \left\{ \| (I - P) \tilde{f}_d(d) \|_{L^\infty} + \| g \|_{L^2_{x, v}} + |(I - P_\gamma) \tilde{f}_d(0) \|_{L^2(\gamma^-)} + |r|_{L^2(\gamma^-)} \right\} \]
\[ \leq C \left\{ \| (I - P) \tilde{f}_d(d) \|_{L^\infty} + \| g \|_{L^2_{x, v}} + |r|_{L^2(\gamma^-)} \right\}. \] (2.177)
On the other hand, it follows from (2.158) and (2.171) that
\[
\frac{1}{|\sigma_0 - \sigma|} e^\sigma w_f d_t |_{L^\infty} + C \left\{ \| g \|_{L^2_{\gamma +}} + |r|_{L^2_{\gamma +}} \right\}
\]
\[
\leq Ce^{-\sigma d}|e^\sigma w_f d_t |_{L^\infty} + C \left\{ \| g \|_{L^2_{\gamma +}} + |r|_{L^2_{\gamma +}} \right\}
\]
\[
\leq Ce^{-\sigma d} \left( \frac{1}{|\sigma_0 - \sigma|} e^\sigma w_f d_t |_{L^\infty} + |r|_{L^\infty_{\gamma +}} \right),
\]
where we have used (2.169) in the last inequality.

Taking \( \sigma = \frac{1}{2} \sigma_0 \), and \( d_0 \) suitably large so that \( Ce^{-\sigma d_0} \leq \frac{1}{2} \), then (2.178) is reduced to
\[
\frac{1}{|\sigma_0 - \sigma|} e^\sigma w_f d_t |_{L^\infty} + |r|_{L^\infty_{\gamma +}} \leq C_{d_0} \left\{ \| g \|_{L^2_{\gamma +}} + |r|_{L^\infty_{\gamma +}} \right\},
\]
for \( d \geq d_0 \geq 1. \) (2.179)

For \( d \in [1, d_0] \), using (2.150) and (2.134), one has
\[
\left( \frac{1}{|\sigma_0 - \sigma|} e^\sigma w_f d_t |_{L^\infty} + |r|_{L^\infty_{\gamma +}} \right),
\]
which together with (2.179), yields (2.175). Hence the proof of Lemma 2.16 is complete. \( \square \)

In the following, we study the asymptotic behavior of \((\phi_t, \phi_2, \phi_3)(0) \) as \( d \to \infty \). Let \( 1 \leq d_1 \leq d_2 < \infty \), then it holds that
\[
\left\{ \begin{array}{l}
v_3 \partial_x (f_{d_2} - \tilde{f}_{d_1}) + P_E^0 V f_{d_2} - \tilde{f}_{d_1} = 0, \quad (x, v) \in (0, d_1) \times \mathbb{R}^3, \\
(f_{d_2} - \tilde{f}_{d_1})(0, v)|_{v_3 > 0} = P_{\gamma}(f_{d_2} - \tilde{f}_{d_1})(0, v).
\end{array} \right.
\]

**Lemma 2.17.** It holds that
\[
|I - P_{\gamma} (f_{d_2} - \tilde{f}_{d_1})(0)|_{L^2_{\gamma +}} \leq Ce^{-\sigma d_1} \left\{ r_{2 L^2_{\gamma +}} + \int_0^{d_1} e^{2\sigma_1 x} \| g(x) \|_{L^2_{\gamma +}} dx \right\}^{\frac{1}{2}}.
\]

**Proof.** Multiplying (2.180) by \( f_{d_2} - \tilde{f}_{d_1} \), one can obtain
\[
\frac{1}{2} \int_{\mathbb{R}^3} v_3 |(f_{d_2} - \tilde{f}_{d_1})(x, v)|^2 dv + c^0 P_E^0 \int_0^x \|I - P\| \tilde{f}_{d_2} - \tilde{f}_{d_1}(z)\|_{L^2_{\gamma +}}^2 dz
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^3} v_3 |(f_{d_2} - \tilde{f}_{d_1})(0, v)|^2 dv, \quad \forall x \in [0, d_1].
\]

Using (2.180)₂, we get that
\[
\int_{\mathbb{R}^3} v_3 |(f_{d_2} - \tilde{f}_{d_1})(0, v)|^2 dv = -|I - P_{\gamma} (f_{d_2} - \tilde{f}_{d_1})(0)|_{L^2_{\gamma +}}^2.
\]

Using (2.138) and (2.139), one has
\[
\int_{\mathbb{R}^3} v_3 |(f_{d_2} - \tilde{f}_{d_1})(x, v)|^2 dv
\]
\[
= \int_{\mathbb{R}^3} v_3 \tilde{f}_{d_1}(x, v)^2 dv - 2 \int_{\mathbb{R}^3} v_3 \tilde{f}_{d_1}(x, v) \tilde{f}_{d_1}(x, v) dv + \int_{\mathbb{R}^3} v_3 |\tilde{f}_{d_1}(x, v)|^2 dv
\]
\[
= \int_{\mathbb{R}^3} v_3 |I - P\| \tilde{f}_{d_1}(x, v)|^2 dv - 2 \int_{\mathbb{R}^3} v_3 \|I - P\| \tilde{f}_{d_1}(x, v) \cdot (I - P) \tilde{f}_{d_1}(x, v) dv
\]
\[
+ \int_{\mathbb{R}^3} v_3 |I - P\| \tilde{f}_{d_1}(x, v)|^2 dv.
\]

Substituting (2.138) and (2.139) into (2.182), one gets
\[
|I - P_{\gamma} (f_{d_2} - \tilde{f}_{d_1})(0)|_{L^2_{\gamma +}} \leq C \left\{ \|I - P\| \tilde{f}_{d_1}(x)|^2 + \|I - P\| \tilde{f}_{d_1}(x)|^2 \right\}, \quad \forall x \in [0, d_1].
\]
Integrating (2.185) over \( x \in [d_1 - 1, d_1] \), and using (2.140), then we obtain
\[
| (I - P_\gamma)(\tilde{f}_{d_2} - \tilde{f}_{d_1})(0) |^2_{L^2(\gamma_+)} \\
\leq C \left\{ \int_{d_1 - 1}^{d_1} \| (I - P) \tilde{f}_{d_1}(x) \|_{L^2}^2 dx + \int_{d_1 - 1}^{d_1} \| (I - P) \tilde{f}_{d_2}(x) \|_{L^2}^2 dx \right\} \\
\leq C e^{-2\sigma d_1} \left\{ \int_{d_1 - 1}^{d_1} e^{2\sigma \tau x} \| (I - P) \tilde{f}_{d_1}(x) \|_{L^2}^2 dx + \int_{d_1 - 1}^{d_1} e^{2\sigma \tau x} \| (I - P) \tilde{f}_{d_2}(x) \|_{L^2}^2 dx \right\} \\
\leq C e^{-2\sigma d_1} \left\{ |r|_{L^2(\gamma_+)}^2 + \int_0^{d_1} e^{2\sigma \tau x} \| g(x) \|_{L^2_x}^2 dx \right\},
\]
which immediately yields (2.181). Therefore the proof of Lemma 2.17 is complete. \( \square \)

**Lemma 2.18.** There exist constants \((\phi_0^\infty, \phi_1^\infty, \phi_2^\infty, \phi_3^\infty)\) such that
\[
\lim_{d \to +\infty} \left( \phi_0, \phi_1, \phi_2, \phi_3 \right)(d) = (\phi_0^\infty, \phi_1^\infty, \phi_2^\infty, \phi_3^\infty),
\]
with
\[
\| (\phi_0^\infty, \phi_1^\infty, \phi_2^\infty, \phi_3^\infty) \| \leq C \left\{ \| e^{\sigma \nu x} v^{-1} w g \|_{L^\infty_x} + |w r|_{L^\infty_x(\gamma_-)} \right\}.
\]

**Proof.** Let \( 1 \leq d_1 \leq d_2 < \infty \), it follows from (2.153), (2.165)-(2.167), (2.181), (2.145) and (2.170) that
\[
\left| \left( \phi_0(d_2) - \phi_0(d_1), \phi_1(d_2) - \phi_1(d_1), \phi_2(d_2) - \phi_2(d_1), \phi_3(d_2) - \phi_3(d_1) \right) \right| \\
\leq \left| \left( \tilde{a}_{d_2}, \tilde{a}_{d_1}, \tilde{b}_{d_2,1} - \tilde{b}_{d_1,1}, \tilde{b}_{d_2,2} - \tilde{b}_{d_1,2}, \tilde{c}_{d_2} - \tilde{c}_{d_1} \right) \right| \\
+ C \left| (I - P) \tilde{f}_{d_1}(d_1) \right|_{L^\infty_x} + C \left| (I - P) \tilde{f}_{d_2}(d_2) \right|_{L^\infty_x} \\
\leq |(I - P_\gamma)(\tilde{f}_{d_2} - \tilde{f}_{d_1})(0) |_{L^2(\gamma_+)} + \int_{d_1}^{d_2} \| g(z) \|_{L^2_x} dz \\
+ C e^{-\sigma d_1} \left\{ \frac{1}{\sigma_0 - \sigma} \| e^{\sigma \nu x} v^{-1} w g \|_{L^\infty_x} + |w r|_{L^\infty_x(\gamma_-)} \right\} \\
\leq C e^{-\sigma d_1} \left\{ \frac{1}{\sigma_0 - \sigma} \| e^{\sigma \nu x} v^{-1} w g \|_{L^\infty_x} + |w r|_{L^\infty_x(\gamma_-)} \right\} \to 0, \quad \text{as} \quad d_1 \to +\infty,
\]
which immediately gives (2.180). And (2.187) follows directly from (2.186) and (2.175). Therefore the proof of Lemma 2.18 is complete. \( \square \)

**2.3. Proof of Theorem 1.5.** Let \( 1 \leq d_1 \leq d_2 < \infty \), we denote
\[
f(x, v) := (\tilde{f}_{d_2} - \tilde{f}_{d_1})(x, v), \quad \text{and} \quad h(x, v) = w(v) f(x, v), \quad \forall (x, v) \in [0, d_1] \times \mathbb{R}^3,
\]
then it follows from (2.143) that
\[
\begin{cases}
\nu_3 \partial_v \tilde{f} + P_\gamma^2 \mathbf{L} f = 0, \quad (x, v) \in (0, d_1) \times \mathbb{R}^3, \\
f(0, 0) = P_\gamma f(0, v) + (I - P_\gamma)(\Phi(d_2) - \Phi(d_1)).
\end{cases}
\]
We divide the proof into three steps.
Step 1. Convergence in $L^2$-norm. Multiplying (2.190) by $\mathfrak{f}$ to obtain

$$\notag |(I - P_\gamma)\mathfrak{f}(0)|_{L^2(\gamma^+)}^2 + p_E^0 c_0 \int_0^{d_1} \int_{\mathbb{R}^3} (1 + |v|)|I - P|\mathfrak{f}(x,v)|^2 dvdx$$

$$\notag \leq C \int_{\mathbb{R}^3} |v_3| \cdot |\mathfrak{f}(d_1, v)|^2 dv + C(|\Phi(d_2) - \Phi(d_1)|)_{L^2(\gamma)}^2$$

$$\notag \leq C \left| \left( \phi_0(d_2) - \phi_0(d_1), \phi_1(d_2) - \phi_1(d_1), \phi_2(d_2) - \phi_2(d_1), \phi_3(d_2) - \phi_3(d_1) \right) \right|^2$$

$$\notag + |wf(d_1)|_{L^\infty}^2$$

$$\notag \leq C e^{-2\sigma d_1} \left\{ \frac{1}{\sigma_0 - \sigma} \| e^{\sigma_0} \nu^{-1} w_g \|_{L^\infty_{x,v}} + |wr|_{L^\infty_{\gamma_-}} \right\}^2, \quad (2.191)$$

where we have used (2.170) and (2.188) in the last inequality.

We still need to control the macroscopic part of $\mathfrak{f}$. We denote

$$\notag \mathcal{P} \mathfrak{f}(x,v) = \left[ a(x) + b_1(x)v_1 + b_2(x)v_2 + c(x)\left( \frac{1}{2} |v|^2 - \frac{3}{2} \right) \right] \sqrt{\mu(v).} \quad (2.192)$$

Noting from (2.159), one has

$$\notag \left( \int_{\mathbb{R}^3} v_3 \mathfrak{f}(x,v) \cdot \mathbf{L}^{-1}(A_{31})dv \right) = \left( \int_{\mathbb{R}^3} v_3 \mathfrak{f}(d_1,v) \cdot \mathbf{L}^{-1}(A_{31})dv \right)$$

$$\notag \left( \int_{\mathbb{R}^3} v_3 \mathfrak{f}(x,v) \cdot \mathbf{L}^{-1}(A_{32})dv \right) = \left( \int_{\mathbb{R}^3} v_3 \mathfrak{f}(d_1,v) \cdot \mathbf{L}^{-1}(A_{32})dv \right),$$

which, together with (2.150) - (2.152), yields

$$\notag |(b_1(x), b_2(x), c(x))| \leq C|f(d_1)|_{L^\infty} + \left( \int_{\mathbb{R}^3} |(I - P)\mathfrak{f}(x,v)|^2 dv \right)^{\frac{1}{2}}.$$ 

Hence it follows from (2.176) and (2.191) that

$$\notag \int_0^{d_1} |(b_1, b_2, c)(x)|^2 dx \leq Cd_1 |f(d_1)|^2_{L^\infty} + \int_0^{d_1} \int_{\mathbb{R}^3} |(I - P)|\mathfrak{f}(x,v)|^2 dvdx$$

$$\notag \leq Cd_1 e^{-2\sigma d_1} \left\{ \frac{1}{\sigma_0 - \sigma} \| e^{\sigma_0} \nu^{-1} w_g \|_{L^\infty_{x,v}} + |wr|_{L^\infty_{\gamma_-}} \right\}^2. \quad (2.193)$$

Noting (2.161) and using (2.193), (2.191), one can obtain

$$\notag \int_0^{d_1} |a(x)|^2 dx \leq \int_0^{d_1} |c(x)|^2 dx + \int_0^{d_1} \int_{\mathbb{R}^3} |(I - P)|\mathfrak{f}(x,v)|^2 dvdx$$

$$\notag \leq Cd_1 e^{-2\sigma d_1} \left\{ \frac{1}{\sigma_0 - \sigma} \| e^{\sigma_0} \nu^{-1} w_g \|_{L^\infty_{x,v}} + |wr|_{L^\infty_{\gamma_-}} \right\}^2,$$ 

which, together with (2.191) and (2.193), gives

$$\notag \int_0^{d_1} \int_{\mathbb{R}^3} |f(x,v)|^2 dvdx \leq Cd_1 e^{-2\sigma d_1} \left\{ \frac{1}{\sigma_0 - \sigma} \| e^{\sigma_0} \nu^{-1} w_g \|_{L^\infty_{x,v}} + |wr|_{L^\infty_{\gamma_-}} \right\}^2. \quad (2.194)$$

Step 2. Convergence in $L^\infty_{x,v}$-norm. In the following, we use $t_k = t_k(t,x,v), X_{cl}(s; t, x, v), x_k = x_k(v, x)$ to denote the back-time cycles defined for domain $[0, d_1] \times \mathbb{R}^3$. Let $(x,v) \in [0, d_1] \times \mathbb{R}^3 \setminus (\gamma_0 \cup$
\(\gamma_{-}\), it follows from (2.199) that
\[
\mathbf{h}(x, v) = 1_{\{\nu_{0,3} > 0\}} \frac{e^{-\nu_{E}(v)(t-t_1)}}{\bar{w}(v)} \int_{t_1}^{t} \int_{t_2}^{t_1} e^{-\nu_{E}(v)(t_1-s)} \left( K_{w} \mathbf{h}(x-v_{1,3}(t_1-s), v_1) \right) \bar{w}(v_1) dsd\sigma_1
\]
\[
+ \int_{t}^{t_1} e^{-\nu_{E}(v)(t-s)} \left( K_{w} \mathbf{h}(x-v_{0,3}(t-s), v) \right) ds + \mathfrak{B},
\]  
(2.195)
where we have used the notations \(\nu_{E}(v) := \nu_{E}^{\nu}(v)\) and
\[
\mathfrak{B} := 1_{\{\nu_{0,3} > 0\}} \left\{ e^{-\nu_{E}(v)(t-t_1)} w(v) \cdot (I - P_{\gamma})(\Phi(d_2) - \Phi(d_1))(v) \right. \\
+ \frac{e^{-\nu_{E}(v)(t-t_1)}}{\bar{w}(v)} \int_{t_1}^{t} e^{-\nu_{E}(v)(1-t-t_2)} \mathbf{h}(d_1, v_1) \bar{w}(v_1) d\sigma_1 \left. \right\}
\]
\[
+ 1_{\{\nu_{0,3} \leq 0\}} \cdot e^{-\nu_{E}(v)(t-t_1)} \cdot \mathbf{h}(d_1, v).
\]  
(2.196)
Using (2.188), (2.176), it holds that
\[
|\mathfrak{B}| \leq C\left| (\Phi(d_2) - \Phi(d_1)) \right| + |\mathbf{h}(d_1)|_{L^\infty} \\
\leq Ce^{-\sigma d_1} \frac{1}{\sigma_0 - \sigma} \left\{ \left\| e^{\sigma_\nu x} \nu^{-1}w \right\|_{L^\infty_{x,v}} + |w_r|_{L^\infty(\gamma_{-})} \right\}.
\]  
(2.197)
By similar arguments as in (2.217)-(2.233), one can obtain
\[
1_{\{\nu_{0,3} > 0\}} \frac{e^{-\nu_{E}(v)(t-t_1)}}{\bar{w}(v)} \left| \int_{t_1}^{t} \int_{t_2}^{t_1} e^{-\nu_{E}(v)(t_1-s)} \left( K_{w} \mathbf{h}(x-v_{1,3}(t_1-s), v_1) \right) \bar{w}(v_1) dsd\sigma_1 \right|
\leq \frac{C}{N} \|\mathbf{h}\|_{L^\infty([0,d_1] \times \mathbb{R}^3)} + C_N \|f\|_{L^2([0,d_1] \times \mathbb{R}^3)}.
\]  
(2.198)
Combining (2.195), (2.197) and (2.198), one gets, for \((x, v) \in [0, d_1] \times \mathbb{R}^3\)
\[
|\mathbf{h}(x, v)| \leq \int_{t_1}^{t} e^{-\nu_{E}(v)(t-s)} ds \int_{\mathbb{R}^3} |k_w(v, v') \mathbf{h}(x', v')| dv' + \frac{C}{N} \|\mathbf{h}\|_{L^\infty([0,d_1] \times \mathbb{R}^3)} \\
+ C_N \|f\|_{L^2([0,d_1] \times \mathbb{R}^3)} + Ce^{-\sigma d_1} \frac{1}{\sigma_0 - \sigma} \left\{ \left\| e^{\sigma_\nu x} \nu^{-1}w \right\|_{L^\infty_{x,v}} + |w_r|_{L^\infty(\gamma_{-})} \right\}
\]  
(2.199)
where we have denoted \(x' := x - v_{0,3}(t-s) \in [0, d_1]\).

For the first term on right hand side of (2.199), we use (2.199) again to obtain
\[
\int_{t_1}^{t} e^{-\nu_{E}(v)(t-s)} ds \int_{\mathbb{R}^3} |k_w(v, v') \mathbf{h}(x', v')| dv' \\
\leq \int_{t_1}^{t} e^{-\nu_{E}(v)(t-s)} ds \int_{t_1}^{t} e^{-\nu_{E}(v)(s)} ds \int_{t_1}^{t} e^{-\nu_{E}(v)(s)} ds \int_{\mathbb{R}^3} |k_w(v, v')| k_w(v', v'') \mathbf{h}(x'', v'') dv'' dv' \\
+ \frac{C}{N} \|\mathbf{h}\|_{L^\infty([0,d_1] \times \mathbb{R}^3)} + C_N \|f\|_{L^2([0,d_1] \times \mathbb{R}^3)} \\
+ Ce^{-\sigma d_1} \frac{1}{\sigma_0 - \sigma} \left\{ \left\| e^{\sigma_\nu x} \nu^{-1}w \right\|_{L^\infty_{x,v}} + |w_r|_{L^\infty(\gamma_{-})} \right\}
\]  
(2.200)
where \(t'_{1} = t_1 (s, x', v'')\) and \(x'' = x' - v_{0,3}(s - \tau)\). Applying similar arguments as (2.200)-(2.233) to the first term on right hand side of (2.200), then we can have
\[
\int_{t_1}^{t} e^{-\nu_{E}(v)(t-s)} ds \int_{\mathbb{R}^3} |k_w(v, v') \mathbf{h}(x', v')| dv' \leq \frac{C}{N} \|\mathbf{h}\|_{L^\infty([0,d_1] \times \mathbb{R}^3)} + C_N \|f\|_{L^2([0,d_1] \times \mathbb{R}^3)} \\
+ Ce^{-\sigma d_1} \frac{1}{\sigma_0 - \sigma} \left\{ \left\| e^{\sigma_\nu x} \nu^{-1}w \right\|_{L^\infty_{x,v}} + |w_r|_{L^\infty(\gamma_{-})} \right\},
\]
which, together with (2.199), yields that, for \((x, v) \in ([0, d_1] \times \mathbb{R}^3) \setminus (\gamma_0 \cup \gamma_-)\)

\[
|b(x, v)| \leq \frac{C |b|}{N} L^\infty([0, d_1] \times \mathbb{R}^3) + C_N \|f\| L^2([0, d_1] \times \mathbb{R}^3)
\]

\[+ Ce^{-\sigma_1\frac{1}{\sigma_0 - \sigma}} \|e^{\sigma_1 x} \nu^{-1} w g\| L^\infty_{x, v} + |w r| L^\infty_{\gamma_-}.\]

Hence it holds

\[
\|b\|_{L^\infty([0, d_1] \times \mathbb{R}^3)} + |b(0)| L^\infty_{\gamma_-} \leq \frac{C |b|}{N} L^\infty([0, d_1] \times \mathbb{R}^3) + C_N \|f\| L^2([0, d_1] \times \mathbb{R}^3)
\]

\[+ Ce^{-\sigma_1\frac{1}{\sigma_0 - \sigma}} \|e^{\sigma_1 x} \nu^{-1} w g\| L^\infty_{x, v} + |w r| L^\infty_{\gamma_-}.\]

Taking \(N > 0\) suitably large so that \(\frac{C}{N} \leq \frac{1}{4}\) and using (2.176), (2.194), then one obtains

\[
\|b\|_{L^\infty([0, d_1] \times \mathbb{R}^3)} + |b(0)| L^\infty_{\gamma_-} \leq C \|f\| L^2([0, d_1] \times \mathbb{R}^3) + Ce^{-\sigma_1\frac{1}{\sigma_0 - \sigma}} \|e^{\sigma_1 x} \nu^{-1} w g\| L^\infty_{x, v} + |w r| L^\infty_{\gamma_-} \]

\[\leq Cd_1 e^{-\frac{1}{\sigma_0 - \sigma} \frac{1}{\sigma_0 - \sigma}} \|e^{\sigma_1 x} \nu^{-1} w g\| L^\infty_{x, v} + |w r| L^\infty_{\gamma_-} \] for \(d_1 \to \infty.\) (2.201)

With the help of (2.201), there exists a function \(f(x, v)\) with \((x, v) \in \mathbb{R}_+ \times \mathbb{R}^3\) so that

\[\|w(\tilde{f}_d - f)\| L^\infty([0, d_1] \times \mathbb{R}^3) + |w(\tilde{f}_d - f)(0)| L^\infty_{\gamma_+} \to 0,\] as \(d \to \infty,\)

It follows from (2.201) and the strong convergence in \(L^\infty_{x, v}\) that

\[
\|e^{\sigma_1 x} w f\|_{L^\infty_{x, v}} + |e^{\sigma_1 x} w f(0)| L^\infty_{\gamma_-} \leq C \left\{ \frac{1}{\sigma_0 - \sigma} \|e^{\sigma_1 x} \nu^{-1} w g\| L^\infty_{x, v} + |w r| L^\infty_{\gamma_-} \right\}.\] (2.202)

The continuity of \(f\) follows directly from the \(L^\infty_{x, v}\)-convergence and the continuity of \(\tilde{f}_d\). It is direct to see that \(f(x, v)\) solves

\[
\begin{aligned}
v_3 \partial_v f + p^0_L f + g, & \quad (x, v) \in (0, \infty) \times \mathbb{R}^3, \\
(0, v)|_{v_3 > 0} = P_\gamma f(0, v) + (I - P_\gamma)(\Phi^\infty)(v) + r(v), & \quad v_3 \to \infty f(x, v) = 0,
\end{aligned}
\] (2.203)

with

\[
\Phi^\infty(v) = \left\{ \phi^\infty_0 + \phi^\infty_1 v_1 + \phi^\infty_2 v_2 + \phi^\infty_3 (\frac{|v|^2}{2} - \frac{3}{2}) \right\} \sqrt{\mu(v)},\] (2.204)

\[
||\phi^\infty_0, \phi^\infty_1, \phi^\infty_2, \phi^\infty_3 || \leq C \left\{ \|e^{\sigma_0 x} \nu^{-1} w g\| L^\infty_{x, v} + |w r| L^\infty_{\gamma_-} \right\}.\] (2.205)

where the constants \((\phi^\infty_0, \phi^\infty_1, \phi^\infty_2, \phi^\infty_3)\) are the ones determined in Lemma 2.18.

We note that

\[(I - P_\gamma)(\Phi^\infty) \equiv (I - P_\gamma)(\alpha \sqrt{\mu} + \Phi^\infty), \quad \forall \alpha \in \mathbb{R},\]

that means the system (2.203) is invariant under such transformation. So we always normalized \(\Phi^\infty\) such that

\[
\phi^\infty_0 = 0.\] (2.206)

**Step 3. Uniqueness.** For any given \(g \in \mathcal{N}^\perp\) and \(r\) with

\[
\|e^{\sigma_1 x} \nu^{-1} w g\| L^\infty_{x, v} + |w r| L^\infty_{\gamma_-} < \infty, \quad \text{and} \quad \int_{v_3 > 0} v_3 \sqrt{\mu(v)} r(v) dv = 0.\] (2.207)
Let \( \hat{f}_1, \Phi_1^\infty \) and \( \hat{f}_2, \Phi_2^\infty \) be two solutions of (2.203), respectively, with \( 2.202), 2.204)-(2.206). Noting (2.204) and (2.206), we denote

\[
\Phi_1^\infty := \left\{ \phi_{1,1}^\infty v_1 + \phi_{1,2}^\infty v_2 + \phi_{1,3}^\infty \left( \frac{|v|^2}{2} - \frac{3}{2} \right) \right\} \sqrt{\mu(v)}, \\
\Phi_2^\infty := \left\{ \phi_{2,1}^\infty v_1 + \phi_{2,2}^\infty v_2 + \phi_{2,3}^\infty \left( \frac{|v|^2}{2} - \frac{3}{2} \right) \right\} \sqrt{\mu(v)}. \tag{2.208}
\]

Moreover, for the solutions \( \hat{f}_i, i = 1, 2 \), it is direct to check, for \( x \in [0, +\infty) \), that

\[
\int_{\mathbb{R}^3} v_3\sqrt{\mu} \hat{f}_i(x, v) dv = 0, \\
\int_{\mathbb{R}^3} \mathcal{A}_3(v) \hat{f}_i(x, v) dv = \int_{\mathbb{R}^3} \mathcal{A}_3(v) (I - \mathcal{P}) \hat{f}_i(x, v) dv = 0, \quad \forall i, j = 1, 2. \tag{2.209}
\]

We define

\[ \hat{f}(x, v) := [\hat{f}_1(x, v) - \Phi_1^\infty] - [\hat{f}_2(x, v) - \Phi_2^\infty], \tag{2.210} \]

then it follows from (2.203) that

\[
\begin{align*}
&v_3 \partial_x \hat{f} + p_3^L \hat{f} = 0, \quad (x, v) \in (0, \infty) \times \mathbb{R}^3, \\
&\hat{f}(0, v)|_{v_3 > 0} = P_\gamma \hat{f}(0, v), \\
&\lim_{x \to \infty} e^{\sigma x} \| w(v) [\hat{f} - (-\Phi_1^\infty + \Phi_2^\infty)] \|_{L_2^\infty} = 0.
\end{align*} \tag{2.211}
\]

Multiplying (2.211) by \( \hat{f} \) and integrating over \( (x, v) \in \mathbb{R}^+ \times \mathbb{R}^3 \), one can obtain

\[
\frac{1}{2} \int_{\mathbb{R}^3} v_3 ([\Phi_1^\infty - \Phi_2^\infty](v))^2 dv + c_0 p_E^0 \int_0^\infty \| (I - \mathcal{P}) \hat{f}(x) \|^2_v \leq \frac{1}{2} \int_{\mathbb{R}^3} v_3 |\hat{f}(0, v)|^2 dv, \tag{2.212}
\]

where we have used (2.211).

Noting (2.208), one can check that

\[
\int_{\mathbb{R}^3} v_3 |(\Phi_1^\infty - \Phi_2^\infty)(v)|^2 dv = 0. \tag{2.213}
\]

It follows from (2.211) that

\[
\int_{\mathbb{R}^3} v_3 |\hat{f}(0, v)|^2 dv = -\| (I - P_\gamma) \hat{f}(0) \|^2_{L_2^\infty}. \tag{2.214}
\]

Substituting (2.213) and (2.214) into (2.212), one has

\[
\| (I - P_\gamma) \hat{f}(0) \|^2_{L_2^\infty} + c_0 p_E^0 \int_0^\infty \| (I - \mathcal{P}) \hat{f}(x) \|^2_v \leq 0, \tag{2.215}
\]

which immediately yields that

\[
0 \equiv (I - \mathcal{P}) \hat{f}(x, v), \quad \forall (x, v) \in [0, \infty) \times \mathbb{R}^3, \tag{2.216}
\]

and

\[
\hat{f}(0, v) = P_\gamma \hat{f}(0, v), \quad \forall v \in \mathbb{R}^3. \tag{2.217}
\]

We still need to prove \( \mathcal{P} \hat{f}(x, v) \equiv 0 \). We denote

\[
\mathcal{P} \hat{f}(x, v) = \left[ \alpha(x) + \hat{b}_1(x)v_1 + \hat{b}_2(x)v_2 + \hat{c}(x)\left( \frac{1}{2} |v|^2 - \frac{3}{2} \right) \right] \sqrt{\mu(v)}.
\]
Using (2.217), one can check that
\[
\int_{\mathbb{R}^3} \hat{f}(0, v) \cdot v_3^2 \sqrt{\mu(v)} dv = \sqrt{2\pi z_{\gamma_+}}(\hat{f}),
\]
\[
\int_{\mathbb{R}^3} \hat{f}(0, v) \cdot v_3 L^{-1}(A_{31}) dv = 0, \quad i = 1, 2,
\]
\[
\int_{\mathbb{R}^3} \hat{f}(0, v) \cdot v_3 L^{-1}(B_3) dv = 0,
\]
where \(z_{\gamma_+}(\hat{f}) = \int_{v_3 < 0} |v_3| \sqrt{\mu(v)} \hat{f}(0, v) dv\).

Noting (2.216), multiplying (2.211) by \(v_3 \sqrt{\mu} L^{-1}(A_{31}), L^{-1}(A_{32})\) and \(L^{-1}(B_3)\), respectively, and integrating over \([0, x]\) and using (2.218), one gets
\[
\int_{\mathbb{R}^3} \hat{f}(x, v) \cdot v_3^2 \sqrt{\mu(v)} dv = \int_{\mathbb{R}^3} \hat{f}(0, v) \cdot v_3^2 \sqrt{\mu(v)} dv = \sqrt{2\pi z_{\gamma_+}}(\hat{f}),
\]
\[
\int_{\mathbb{R}^3} \hat{f}(x, v) \cdot v_3 L^{-1}(A_{31}) dv = \int_{\mathbb{R}^3} \hat{f}(0, v) \cdot v_3 L^{-1}(A_{31}) dv = 0, \quad i = 1, 2,
\]
\[
\int_{\mathbb{R}^3} \hat{f}(x, v) \cdot v_3 L^{-1}(B_3) dv = \int_{\mathbb{R}^3} \hat{f}(0, v) \cdot v_3 L^{-1}(B_3) dv = 0.
\]
Noting (2.216), by similar arguments as in (2.149)-(2.152), then one has
\[
\sqrt{2\pi z_{\gamma_+}}(\hat{f}) = \int_{\mathbb{R}^3} \hat{f}(x, v) \cdot v_3^2 \sqrt{\mu(v)} dv = \hat{a}(x) + \hat{\epsilon}(x),
\]
\[
0 = \int_{\mathbb{R}^3} \hat{f}(x, v) \cdot v_3 L^{-1}(A_{31}) dv = \kappa_1 \hat{b}_i(x), \quad i = 1, 2,
\]
\[
0 = \int_{\mathbb{R}^3} \hat{f}(x, v) \cdot v_3 L^{-1}(B_3) dv = \kappa_2 \hat{\epsilon}(x),
\]
that is
\[
\hat{a}(x) = \sqrt{2\pi z_{\gamma_+}}(\hat{f}), \quad \hat{b}_i(x) = 0, \quad i = 1, 2, \quad \hat{\epsilon}(x) = 0, \quad \forall \ x \in [0, +\infty).
\]
(2.220)

It follows from (2.220) and (2.216) that
\[
\hat{f}(x, v) \equiv \sqrt{2\pi z_{\gamma_+}}(\hat{f}) \sqrt{\mu(v)},
\]
which, together with (2.211)³, yields
\[
\hat{\epsilon}(0) = 0 \quad \text{and} \quad \Phi_1^\infty = \Phi_2^\infty.
\]

Therefore we have proved
\[
\hat{f}_1(x, v) \equiv \hat{f}_2(x, v) \quad \text{and} \quad \Phi_1^\infty = \Phi_2^\infty.
\]
That is the solution \(f, \Phi^\infty\) of (2.203) is unique under the constraints (2.202), (2.204)-(2.206).

Finally we define
\[
G(g, r) \equiv f^\infty(v) := \left\{ b_1^\infty v_1 + b_2^\infty v_2 + c^\infty \left( \frac{|v|^2}{2} - \frac{3}{2} \right) \sqrt{\mu(v)} \right\},
\]
with
\[
b_1^\infty = -\phi_1^\infty, \quad b_2^\infty = -\phi_2^\infty, \quad c^\infty = -\phi_3^\infty.
\]
Then it is direct to know that \(f, f^\infty\) constructed above satisfies (1.22). Therefore the proof of Theorem 1.3 is complete. \(\square\)
3. Proof of Theorem 1.1

We consider the following iterative sequence
\[
\begin{align*}
\nu_j \cdot \partial_t f_{j+1} + p_{j} b_j \mathbb{L}_{f_{j+1}} &= \Gamma(f_j, f_{j}) + \mathcal{G}, \\
 f_{j+1}(0, v)|_{\nu_j > 0} &= P_{\gamma} f_{j+1}(0, v) - (I - P_{\gamma}) f_{j+1}^\infty + \mathcal{R}, \quad (3.1)
\end{align*}
\]
for \(j = 0, 1, 2, \cdots\) with \(f_0 \equiv 0\). It follows from (A.2) that \(\Gamma(f_{j}, f_{j}) \in \mathcal{N} \) and
\[
\|\nu^{-1} w \Gamma(f_{j}, f_{j})\|_{L^\infty} \leq C \|w f_{j}\|_{L^\infty}^2. \quad (3.2)
\]
Taking \(\frac{1}{2} \sigma_0 \leq \sigma < \sigma_0\), using (A.2) and Theorem 1.6, we can find solutions \((f_{j+1}, f_{j+1}^\infty)\) of (3.1) inductively for \(j = 0, 1, 2, \cdots\), where
\[
f_{j+1}^\infty = \mathcal{G} \left( \Gamma(f_j, f_j) + \mathcal{G}, \mathcal{R} \right) := \left\{ b_{j+1, 1}^\infty v_1 + b_{j+1, 2}^\infty \, c_{j+1}^\infty \left( \frac{|v|^2}{2} - \frac{3}{2} \right) \right\} \sqrt{\mu(v)},
\]
It follows from (A.2), (1.21) and (1.23) that
\[
|\hat{b}_{j+1, 1}^\infty, \hat{b}_{j+1, 2}^\infty, \hat{c}_{j+1}^\infty| \leq \hat{C}_1 \left( |w \mathcal{R}|_{L^\infty} + \|e^{\sigma_0 x} \nu^{-1} w \mathcal{G}\|_{L^\infty_{x,v}} + \|e^{\frac{1}{2} \sigma_0 x} w f_{j}\|_{L^\infty_{x,v}} \right). \quad (3.3)
\]
and
\[
\|e^{\sigma x} w f_{j+1}\|_{L^\infty_{x,v}} + |w f_j(0)|_{L^\infty} \leq \frac{\hat{C}_1 \delta}{\sigma_0 - \sigma}, \quad (3.4)
\]
We denote
\[
\delta := |w \mathcal{R}|_{L^\infty} + \|e^{\sigma_0 x \nu^{-1} w \mathcal{G}}\|_{L^\infty_{x,v}}.
\]
By induction, we shall prove that
\[
|\hat{b}_{j, 1}^\infty, \hat{b}_{j, 2}^\infty, \hat{c}_{j}^\infty| \leq 2 \hat{C}_1 \delta, \quad (3.5)
\]
Indeed, for \(j = 0\), it follows from \(f_0 \equiv 0\) and (3.3), (3.4) that
\[
|\hat{b}_{0, 1}^\infty, \hat{b}_{0, 2}^\infty, \hat{c}_{0}^\infty| \leq \hat{C}_1 \delta,
\]
\[
\|e^{\sigma x} w f_{1}\|_{L^\infty_{x,v}} + |w f_1(0)|_{L^\infty} \leq \frac{\hat{C}_1 \delta}{\sigma_0 - \sigma}.
\]
Now we assume that (3.5) holds for \(j = 1, 2, \cdots, l\), then we consider the case for \(j = l + 1\). Indeed it follows from (3.3), (3.4) that
\[
|\hat{b}_{l+1, 1}^\infty, \hat{b}_{l+1, 2}^\infty, \hat{c}_{l+1}^\infty| \leq \hat{C}_1 \delta + \hat{C}_1 \|e^{\frac{1}{2} \sigma_0 x} w f_{j}\|_{L^\infty_{x,v}}^2 \leq \hat{C}_1 \delta \left( 1 + \delta \left( \frac{4 \hat{C}_1}{\sigma_0} \right)^2 \right) \leq \frac{3}{2} \hat{C}_1 \delta,
\]
and
\[
\|e^{\sigma x} w f_{l+1}\|_{L^\infty_{x,v}} + |w f_{l+1}(0)|_{L^\infty} \leq \frac{\hat{C}_1 \delta}{\sigma_0 - \sigma} + \frac{\hat{C}_1}{\sigma_0 - \sigma} \|e^{\frac{1}{2} \sigma_0 x} w f_{l+1}\|_{L^\infty_{x,v}}^2 \leq \frac{3}{2} \hat{C}_1 \delta,
\]
where we have used (3.5) with \(j = l\), and chosen \(\delta \leq \delta_0\) with \(\delta_0\) small enough such that \(\left( \frac{4 \hat{C}_1}{\sigma_0} \right)^2 \delta_0 \leq 1/2\). Therefore we have proved (3.5) by induction.
Finally we consider the convergence of sequence $f_j$. For the difference $f_{j+1} - f_j$, we apply (3.26) to have

$$
\frac{4C_1}{\sigma_0}\left(\|e^{\frac{1}{2} \sigma_0 x} w(f_{j+1} - f_j)\|_{L^\infty_{x,v}} + \|e^{\frac{1}{2} \sigma_0 x} w(f_{j+1} - f_j)(0)\|_{L^\infty(\gamma)} + \|b_{j+1,1} - b_{j,1}, b_{j+1,2} - b_{j,2}, c_{j+1} - c_{j}\|\right)
\leq 4C_1\left(\|e^{\frac{1}{2} \sigma_0 x} w \Gamma(f_j - f_{j-1}, f_j)\|_{L^\infty_{x,v}} + \|e^{\frac{1}{2} \sigma_0 x} w \Gamma(f_j - f_{j-1})\|_{L^\infty_{x,v}}\right)
\leq \frac{4C_1}{\sigma_0}\left(\|e^{\frac{1}{2} \sigma_0 x} w f_j\|_{L^\infty_{x,v}} + \|e^{\frac{1}{2} \sigma_0 x} w f_{j-1}\|_{L^\infty_{x,v}}\right)
\leq \frac{4C_1}{\sigma_0}\left(\|e^{\frac{1}{2} \sigma_0 x} w f_j - f_{j-1}\|_{L^\infty_{x,v}} \leq \frac{1}{2}\|e^{\frac{1}{2} \sigma_0 x} w f_j - f_{j-1}\|_{L^\infty_{x,v}}\right),\tag{3.6}
$$

where we have used that fact $2\frac{4C_1}{\sigma_0}^2 \delta_0 \leq 1/2$. Hence both $f_j$ and $f_j^\infty$ are Cauchy sequences, then we obtain the solution by taking the limits

$$
f = \lim_{j \to \infty} f_j \quad \text{and} \quad G(\mathcal{S}, \mathcal{R}) = \lim_{j \to \infty} f_j^\infty. \tag{3.7}
$$

The estimates (3.12) and (3.14) follow from (3.24) and (3.25). The uniqueness can also be obtained by using the inequality as (3.10). The continuity of $f$ is a direct consequence of $L^\infty_{x,v}$-convergence. It follows from the uniqueness that $G(0, 0) = 0$.

Finally we prove that the solution $f, f^\infty$ depend continuously on $\mathcal{S}, \mathcal{R}$. Let $\mathcal{S}_1 \in \mathcal{N}_1$ and \(\int_{v_3 > 0} v_3 \sqrt{\mu(v)} \mathcal{R}_1(v)dv = 0, i = 1, 2\) satisfying (1.14). Let $f_1, f_2^\infty$ be the solutions obtained above by replacing $\mathcal{S}, \mathcal{R}$ by $\mathcal{S}_1, \mathcal{R}_1$, and denote

$$
f_i^\infty(v) = G(\mathcal{S}_1, \mathcal{R}_1) = \left\{b_{i,1}^\infty v_1 + b_{i,2}^\infty v_2 + c_i^\infty \left(\frac{|v|^2}{2} - \frac{3}{2}\right)\right\} \sqrt{\mu(v)} \quad i = 1, 2.
$$

It is direct to check that $f_1 - f_2$ satisfies

$$
\begin{align*}
\left\{v_3 \cdot \partial_x (f_1 - f_2) + p_i^{(2)} L(f_1 - f_2) = \Gamma(f_1 - f_2, f_1) + \Gamma(f_2, f_1 - f_2) + \mathcal{S}_1 - \mathcal{S}_2, \\
(f_1 - f_2)(0,v)|_{v_3 > 0} = P_i(f_2, f_1 - f_2)(0,v) - (I - P_i) (f_2^\infty - f_1^\infty) + \mathcal{R}_1 - \mathcal{R}_2, \\
\lim_{x \to \infty} (f_1 - f_2)(x,v) = 0.
\end{align*}\tag{3.8}
$$

We can look at (3.8) as a linear problem. Taking $\frac{1}{2} \sigma_0 \leq \sigma < \sigma_0$, then we apply (3.26) to (3.3) to have

$$
\begin{align*}
\|e^{\sigma x} w(f_1 - f_2)\|_{L^\infty_{x,v}} + \|e^{\sigma x}(f_1 - f_2)(0)\|_{L^\infty(\gamma)} + \|b_{1,1}^\infty - b_{2,1}^\infty, b_{1,2}^\infty - b_{2,2}^\infty, c_1^\infty - c_2^\infty\|_1
\leq & \frac{2C_1}{\sigma_0} \left\{\|e^{\sigma x} w \Gamma(f_1 - f_2, f_1)\|_{L^\infty_{x,v}} + \|e^{\sigma x} w \Gamma(f_2, f_1 - f_2)\|_{L^\infty_{x,v}} + \|e^{\sigma x} w (\mathcal{S}_1 - \mathcal{S}_2)\|_{L^\infty_{x,v}} + \|w(\mathcal{R}_1 - \mathcal{R}_2)\|_{L^\infty(\gamma_\gamma)}\right\}
\leq & \frac{2C_1}{\sigma_0} \left\{\|e^{\sigma x} w f_1\|_{L^\infty_{x,v}} + \|e^{\sigma x} w f_2\|_{L^\infty_{x,v}}\right\} \cdot \left\{\|e^{\sigma x} w (f_1 - f_2)\|_{L^\infty_{x,v}} + \|w(\mathcal{R}_1 - \mathcal{R}_2)\|_{L^\infty(\gamma_\gamma)}\right\}. \tag{3.9}
\end{align*}
$$

Taking $\sigma = \frac{1}{2} \sigma_0$, by a direct calculation, we have from (3.9) that

$$
\|e^{\frac{1}{2} \sigma_0 x} w(f_1 - f_2)\|_{L^\infty_{x,v}} \leq \frac{8C_1}{\sigma_0} \left\{\|e^{\sigma x} w (\mathcal{S}_1 - \mathcal{S}_2)\|_{L^\infty_{x,v}} + \|w(\mathcal{R}_1 - \mathcal{R}_2)\|_{L^\infty(\gamma_\gamma)}\right\}. \tag{3.10}
$$
Substituting (3.10) into (3.9), one obtains
\[
\|e^{\sigma \nu} w(f_1 - f_2)\|_{L_\infty} + \|e^{\sigma \nu} (f_1 - f_2)(0)\|_{L_\infty} + \|A(\xi_1, \xi_2)\|_{L_\infty} + \|A(\xi_1, \xi_2)\|_{L_\infty} \leq C \int_{\mathbb{R}^3} |k(v, \eta)| \, d\eta
\]
which proves (1.16). Therefore we complete the proof of Theorem 1.1.

APPENDIX A. SOME USEFUL KNOWN RESULTS

The operator $K$ satisfies the following Grad’s estimates
\[
K f(v) = \int_{\mathbb{R}^3} k(v, \eta) f(\eta) \, d\eta,
\]
where $k(v, \eta)$
\[
0 \leq |k(v, \eta)| \leq \frac{C}{|v - \eta|} e^{-\frac{|v - \eta|^2}{2}} + C |v - \eta| e^{-\frac{|v|^2}{4}},
\]
where $C > 0$ is a given constant. Following (A.1), it is direct to have
\[
\int_{\mathbb{R}^3} |k(v, \eta)| \cdot \frac{(1 + |v|)^\alpha}{(1 + |\eta|)^\alpha} \, d\eta \leq C_\alpha (1 + |v|)^{-1}.
\]
It is known [1] that $L$ satisfies
\[
\int_{\mathbb{R}^3} g L g d\nu \geq c_0 \|g\|^2_{H^1},
\]
From Guo [11], it holds
\[
\Gamma(f, f) \in \mathcal{N}^1, \quad \|v^{-1} w \Gamma(f, f)\|_{L_\infty} \leq C \|w f\|_{L_\infty}^2.
\]

We introduce a lemma which will be used to obtain the uniform $L_\infty^\nu$ of approximate solutions.

Lemma A.1. [5] Consider a sequence $\{a_i\}_{i=0}^\infty$ with each $a_i \geq 0$. For any fixed $k \in \mathbb{N}_+$, we denote
\[
A_k = \max\{a_i, a_{i+1}, \cdots, a_{i+k}\}.
\]
(1) Assume $D \geq 0$. If $a_{i+1+k} \leq \frac{1}{8} A_k^i + D$ for $i = 0, 1, \cdots$, then it holds that
\[
A_k^i \leq \left(\frac{1}{8}\right)^{\left\lceil \frac{i}{k+1} \right\rceil} \cdot \max\{A_k^0, A_k^1, \cdots, A_k^k\} + \frac{8 + k}{7} D, \quad \text{for} \quad i \geq k + 1.
\]
(2) Let $0 \leq \eta < 1$ with $\eta^{k+1} \geq \frac{1}{2}$. If $a_{i+1+k} \leq \frac{1}{8} A_k^i + C_k \cdot \eta^{k+1}$ for $i = 0, 1, \cdots$, then it holds that
\[
A_k^i \leq \left(\frac{1}{8}\right)^{\left\lceil \frac{i}{k+1} \right\rceil} \cdot \max\{A_k^0, A_k^1, \cdots, A_k^k\} + 2C_k \frac{8 + k}{7} \eta^{k+1}, \quad \text{for} \quad i \geq k + 1.
\]

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