Brace algebras and the cohomology comparison theorem\textsuperscript{(*)}.

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**Abstract.** The Gerstenhaber and Schack cohomology comparison theorem asserts that there is a cochain equivalence between the Hochschild complex of a certain algebra and the usual singular cochain complex of a space. We show that this comparison theorem preserves the brace algebra structures. This result gives a structural reason for the recent results establishing fine topological structures on the Hochschild cohomology, and a simple way to derive them from the corresponding properties of cochain complexes.

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Introduction

A theorem of Gerstenhaber and Schack (the cohomology comparison theorem, CCT) asserts that, for a given triangulated topological space, there exists an associative algebra $A$ and a quasi-isomorphism of cochain complexes between the cohomological Hochschild complex of $A$ and the singular cochain complex of the space $[5,6]$. Besides, there are brace differential graded algebra (BDGA) structures on the cohomological Hochschild complexes of associative algebras and on the singular cochain complexes. We prove that the Gerstenhaber-Schack quasi-isomorphism preserves these algebraic structures. This result should make clear the origin of the fine topological structures appearing on the Hochschild cohomology according to the Deligne conjecture [10,12,11], and why cochain algebras and Hochschild complexes share many algebraic properties, the CCT providing a systematic tool for "structure transportation" between the two theories.

1 Brace differential graded algebras

Let us introduce first BDGAs. These algebras first appeared in the work of Getzler-Jones on algebras up to homotopy (without a specific name) as a particular case of $B_\infty$-algebras, associated in particular to Hochschild complexes of associative algebras, see [4, Sect. 5.2]. When Gerstenhaber and Voronov studied them more in detail [7,14,13], they decided to call these algebras homotopy $G$-algebras. However, this terminology appeared to be a misleading one after Tamarkin had shown that the name $G$(erstenhaber)-algebra up to homotopy should be naturally given to another class of algebras [12]. We therefore call them by a name that reflects their properties and should not create confusion, namely: brace differential graded algebras.

The basic idea is that BDGAs are associative differential graded algebras together with extra (brace) operations that behave exactly as the Kadeishvili-Getzler brace operations on the Hochschild cohomological complex of an associative algebra $[5,5]$. We write, as usual, $B(A)$ for the cobar coalgebra over a differential graded algebra (DGA) $A$, where the product is written $\cdot$ and the differential (of degree +1) $\delta$. That is, $B(A)$ is the cofree graded coalgebra $T(A[1]) := \bigoplus_{n \in \mathbb{N}} A[n]^{\otimes n}$ over the desuspension $A[1]$ of $A$ ($A[1]_n := A_{n+1}$). We use the bar notation and write $[a_1|...|a_n]$ for $a_1 \otimes ... \otimes a_n \in A[1]^{\otimes n}$. In particular, the coproduct on $T(A[1])$ is given by:

$$\Delta[a_1|...|a_n] := \sum_{i=0}^n [a_1|...|a_i] \otimes [a_{i+1}|...|a_n].$$

There is a differential coalgebra structure on $B(A)$ induced by the DGA structure on $A$. In fact, since $B(A)$ is cofree as a graded coalgebra, the properties of the cofree coalgebra functor imply that, in general, a coderivation $D \in Coder(B(A))$ is entirely determined by the composition (written as a degree 0 morphism):

$$\tilde{D} : B(A) \xrightarrow{D} B(A)[1] \xrightarrow{P} A[2],$$

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where \( p \) is the natural projection. In particular, the differential \( d \) on \( B(A) \) is induced by the maps:

\[
\delta : A[1] \to A[2],
\]

and

\[
\mu : A[1] \otimes A[1] \to A[2],
\]

where \( \mu(a, b) := (-1)^{|a|} a \cdot b \). The algebra \( A \) is a BDGA if it is provided with a set of extra-operations called the braces:

\[
B_k : A[1] \otimes A[1]^{\otimes k} \to A[1], \quad k \geq 1,
\]

satisfying certain relations. These relations express exactly the fact that the braces have to induce a differential Hopf algebra structure on \( B(A) \). Explicitly, the relations satisfied by the braces are then [4, Sect. 5.2] and [9, 13] (we use Getzler’s notation: \( v\{v_1, ..., v_n\} := B_n(v \otimes (v_1 \otimes ... \otimes v_n)) \)):

1. The brace relations (the associativity relations for the product on \( B(A) \)).

\[
(v\{v_1, ..., v_n\} w_1, ..., w_n) = \sum_{0 \leq i_1 \leq i_2 \leq ... \leq i_m \leq n} (-1)^{\sum_{i=1}^{k} (|v_i|-1)} v\{v_{i_1}, ..., v_{i_m}, w_{i_{m+1}}, ..., w_{i_2}, w_{i_1}, ..., w_{i_1} w_{i_{m+1}}, ..., w_n\},
\]

with the usual conventions on indices: for example, an expression such as \( v_5\{w_7, ..., w_6\} \) has to be read \( v_5\{\emptyset\} = v_5 \).

2. The distributivity relations of the product w.r. to the braces.

\[
(v \cdot w)\{v_1, ..., v_n\} = \sum_{k=0}^{n} (-1)^{|w|} \sum_{p=1}^{k} (|v_p|-1) v\{v_{1}, ..., v_k\} \cdot w\{v_{k+1}, ..., v_n\},
\]

3. The boundary relations.

\[
\delta(v\{v_1, ..., v_n\}) - \delta v\{v_1, ..., v_n\} + \sum_{i=1}^{n} (-1)^{|v|+|v_1|+...+|v_i-1|-i+1} v\{v_1, ..., v_i, \delta v_{i+1}, ..., v_n\} = (-1)^{|v|(|v_1|-1)} v_1 \cdot (v\{v_2, ..., v_n\})
\]

\[
- \sum_{i=1}^{n-1} (-1)^{|v|+|v_1|+...+|v_{i-1}|-i-1} v\{v_1, ..., v_i \cdot v_{i+1}, ..., v_n\}
\]

\[
+ (-1)^{|v|+|v_1|+...+|v_{n-1}|-n} (v\{v_1, ..., v_{n-1}\}) \cdot v_n.
\]
Let us write down explicit formulas for the BDGA structure on the Hochschild cochain complex $C^*(A,A)$ of an associative algebra $A$ over a commutative unital ring $k$. Recall that $C^n(A,A) = \text{Hom}_k(A^\otimes n, A)$ and that the brace operations on $C^*(A,A)$ are the multilinear operators defined for $x, x_1, \ldots, x_n$ homogeneous elements in $C^*(A,A)$ and $a_1, \ldots, a_m$ elements of $A$ by:

$$\{x\} \{x_1, \ldots, x_n\}(a_1, \ldots, a_m) := \sum_{0 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq |x|} (-1)^{\sum_i (|x_i| - 1)} x(a_1, a_{i_1+1}, x_1(x_{i_1+2} + \ldots + x_{i_2}), \ldots, a_{i_n+1}, \ldots, a_{i_n+|x_i|}, \ldots a_m).$$

The other operations defining the BDGA structure, $\delta$ and $\cdot$, are, respectively, the Hochschild coboundary and the cup product.

There is also a BDGA structure on the cochain complex of a simplicial set $Σ$. Recall Gerstenhaber and Schack’s cohomology comparison theorem for finite simplicial cohomology. The first one, written $\hat{Σ}$, is the usual barycentric subdivision of $Σ$. It is the simplicial set whose $n$-simplices are the ordered morphisms (weakly...
increasing maps) from the ordered set \( \{0, ..., n\} \) to \( \Sigma \) or, equivalently, the increasing sequences in \( \Sigma \), written \( \sigma_0 \leq ... \leq \sigma_n \). The cohomology of \( \hat{\Sigma} \) is isomorphic to the simplicial cohomology of \( \Sigma \).

The other object is the incidence algebra \( I_{\Sigma} \) of the poset \( \Sigma \): it is the algebra generated linearly (over a commutative ring \( k \)) by the pairs of simplices \( (\sigma, \sigma') \) with \( \sigma \leq \sigma' \). The product of two pairs \( (\sigma, \sigma') \) and \( (\beta, \beta') \) is \( (\sigma, \beta') \) if \( \sigma' = \beta \) and 0 else. This algebra is a triangular algebra. The Hochschild cohomology of such algebras can be computed explicitly by means of a spectral sequence, introduced recently by S. Dourlens [4]. We refer from now on to [3] and [4] for the general properties of the Hochschild cohomology of triangular and incidence algebras that are recalled below.

The incidence algebra \( I_{\Sigma} \) has a separable subalgebra \( S_{\Sigma} \) generated as a \( k \)-algebra by the pairs \( (\sigma, \sigma) \). The \( n \)-cochains of the Hochschild complex of \( I_{\Sigma} \) relative to \( S_{\Sigma} \) are the elements of \( \text{Hom}_{S_{\Sigma}}(I_{\hat{\Sigma}}^{\otimes n}, I_{\Sigma}) \), with the usual formula for the Hochschild coboundary. This relative Hochschild complex, written \( C^*_{S_{\Sigma}}(I_{\Sigma}, I_{\Sigma}) \), computes the usual Hochschild cohomology of \( I_{\Sigma} \). A direct inspection shows that \( \text{Hom}_{S_{\Sigma}}(I_{\hat{\Sigma}}^{\otimes n}, I_{\Sigma}) \) is generated linearly (over \( k \)) by the maps sending a given tensor product \( ((\sigma_0, \sigma_1), (\sigma_1, \sigma_2), ..., (\sigma_{n-1}, \sigma_n)) \) to \( (\sigma_0, \sigma_n) \), where the \( (\sigma_i, \sigma_{i+1}) \)s belong to the set of generators of \( I_{\Sigma} \), and all the other tensor products of generators of \( I_{\Sigma} \) to 0.

The Gerstenhaber and Schack cohomology comparison theorem states that there is a canonical cochain isomorphism between the singular cohomology of the barycentric subdivision of \( \Sigma \) and this relative Hochschild complex. See [3] (in particular sect. 23) for details, generalizations, and a survey of the history of this theorem.

**Theorem 1** There is a cochain algebra isomorphism \( \iota \) between the singular complex of \( \Sigma \) and the relative Hochschild complex of \( I_{\Sigma} \) given by: for \( f \in \text{Hom}_k(\hat{\Sigma}, k) \)

\[
\iota(f)((\sigma_0, \sigma_1), (\sigma_1, \sigma_2), ..., (\sigma_{n-1}, \sigma_n)) := f(\sigma_0 \leq \sigma_1 \leq ... \leq \sigma_n) \cdot (\sigma_0, \sigma_n).
\]

In particular:

\[
HH^*(I_{\Sigma}, I_{\Sigma}) \cong H^*(\Sigma, k).
\]

**Proposition 1** The isomorphism \( \iota \) commutes with the action of the brace operations on \( C^*(\Sigma, k) \) and \( C^*_{S_{\Sigma}}(I_{\Sigma}, I_{\Sigma}) \).

Indeed, let \( f, f_1, ..., f_k \) belong respectively to \( C^n(\hat{\Sigma}), C^{n_1}(\hat{\Sigma}), ..., C^{n_k}(\hat{\Sigma}) \). Let \( (\sigma_0 \leq \sigma_1 \leq ... \leq \sigma_{m-1} \leq \sigma_m) \in \hat{\Sigma}_m \), where \( m := n + n_1 + ... + n_k - k \). Let us also introduce the following useful convention. Let e.g. \( (\sigma_{i_0}, \sigma_{i_1}, k_1, k_2, \sigma_{i_3}, ..., \sigma_{i_q}, k_p) \) be any sequence, the elements of which are either scalars, either simplices of \( \Sigma \), and assume that \( (\sigma_{i_0} \leq ... \leq \sigma_{i_q}) \) is a simplex of \( \Sigma \). Then, we write \( f(\sigma_{i_0}, \sigma_{i_1}, k_1, k_2, \sigma_{i_3}, ..., \sigma_{i_q}, k_p) \) for \( (\prod_{i=1}^q k_i) \cdot f(\sigma_{i_0} \leq ... \leq \sigma_{i_q}) \).

Then, we have, according to the definition of the braces:
\[ f\{f_1, ..., f_k\}(\sigma_0 \leq \sigma_1 \leq ... \leq \sigma_m) \]
\[ = \sum \pm f(\sigma_0, ..., \sigma_{i_1}, f_1(\sigma_{i_1} \leq ... \leq \sigma_{i_1+n_1}), ..., f_k(\sigma_{i_k} \leq ... \leq \sigma_{i_k+n_k}), \sigma_{i_k+n_k}, ..., \sigma_m). \]

Therefore:
\[ \iota(f\{f_1, ..., f_k\})((\sigma_0, \sigma_1), (\sigma_1, \sigma_2), ..., (\sigma_{m-1}, \sigma_m)) \]
\[ = \{\sum \pm f(\sigma_0, ..., \sigma_{i_1}, f_1(\sigma_{i_1} \leq ... \leq \sigma_{i_1+n_1}), ..., f_k(\sigma_{i_k} \leq ... \leq \sigma_{i_k+n_k}), \sigma_{i_k+n_k}, ..., \sigma_m)\} \cdot (\sigma_0, \sigma_m). \]
\[ = \sum \pm \iota(f)((\sigma_0, \sigma_1), ..., (\sigma_{i_1-1}, \sigma_{i_1}), f_1(\sigma_1 \leq ... \leq \sigma_{i_1+n_1}) \cdot (\sigma_{i_1}, \sigma_{i_1+n_1}), \sigma_{i_1+n_1+1}, ..., (\sigma_{i_k-1}, \sigma_{i_k}), \sigma_{i_k} \leq ... \leq \sigma_{i_k+n_k}, ..., (\sigma_{m-1}, \sigma_m)) \]
\[ = \sum \pm \iota(f)\iota(f_1), ..., \iota(f_k))(((\sigma_0, \sigma_1), (\sigma_1, \sigma_2), ..., (\sigma_{m-1}, \sigma_m)), \]

and the proof of the proposition follows.

Notice that the signs in the definition of the brace operations on cochains have been chosen in such a way that the last identity holds.

**Theorem 2** The morphism \( \iota \) is an isomorphism of BDGAs between the singular cochain complex of the barycentric subdivision of a finite simplicial complex \( \Sigma \) and the \( S_\Sigma \)-relative Hochschild cochain complex of the incidence algebra of \( \Sigma \).

In particular, the cohomology comparison theorem of Gerstenhaber and Schack relating singular cohomology and Hochschild cohomology can be realized, at the cochain level, as a quasi-isomorphism of BDGAs.

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