Viscous-like forces control the impact response of shear-thickening dense suspensions

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We experimentally and theoretically study impacts into dense cornstarch and water suspensions. We vary impact speed as well as intruder size, shape and mass, and we characterize the resulting dynamics using high-speed video and an onboard accelerometer. We numerically solve previously proposed models, most notably the added-mass model as well as a class of viscous-like models. In the viscous-like models, the intruder dynamics is dominated by large, viscous-like forces at the boundary of the jammed front where large shear rates and accompanying large viscosities are present. We find that our experimental data are consistent with this class of models and inconsistent with the added-mass model. Our results strongly suggest that the added-mass model, which is the dominant model for understanding the dynamics of impacts into shear-thickening dense suspensions, should be updated to include these viscous-like forces.

Key words: suspensions

1. Introduction

A dense suspension consists of solid particles, with sizes on the scale of 1–100 μm, placed into a Newtonian fluid such that, in the absence of external forcing or driving, the particle phase is crowded but not jammed (O’herm et al. 2003; van Hecke 2009; Torquato & Stillinger 2010). Such systems are common in a variety of engineering and geophysical contexts. If a dense suspension is sheared or compressed, there can be a dramatic increase in the viscosity, called shear thickening or, if the viscosity changes rapidly enough, discontinuous shear thickening (DST). A growing consensus suggests that this effect arises when short-range repulsive forces (e.g. lubrication, electrostatic or chemical) are overcome and particles make solid–solid contact with each other (Seto et al.)
This picture helps explain why the rheology of the suspension varies strongly with the volume fraction \( \phi \) occupied by the particles. For small particle volume fraction \( \phi \) (typically \( \phi < 0.4 \)), the suspension behaves as a Newtonian fluid, with a constant viscosity \( \eta \) that increases with \( \phi \). For \( \phi > \phi_J \) (typically \( \phi_J \approx 0.6 \)), the particles are jammed, and the material behaves as a yield-stress solid (Brown & Jaeger 2014). In between these two limits (typically \( 0.4 < \phi < 0.6 \)), \( \eta \) increases dramatically if the shear rate \( \dot{\gamma} \) exceeds some critical shear rate \( \dot{\gamma}_c \), the value of which depends on \( \phi \) (Hoffman 1972; Barnes 1989; Brown & Jaeger 2009; Fall et al. 2010; Brown & Jaeger 2014) and other microscopic features.

Impact into a shear-thickening dense suspension (e.g. cornstarch mixed with water or another liquid at \( \phi \approx 0.45–0.5 \)) by a foreign intruder can be similarly dramatic (Lee, Wetzel & Wagner 2003; Waitukaitis & Jaeger 2012; Peters & Jaeger 2014; Han, Peters & Jaeger 2016; Mukhopadhyay, Allen & Brown 2018; Han et al. 2019b; Pradipto & Hayakawa 2021). Yet, a simple application of steady-state rheology cannot explain the impact response, as the stresses predicted by DST are far too small to, e.g. support a person running across a cornstarch–water suspension; see the Introduction of Mukhopadhyay et al. (2018) for a complete discussion. This discrepancy is not entirely surprising, since impact is a highly inhomogeneous, transient process involving both compression and shear. Experiments have repeatedly shown that the impact leads to a dynamically jammed region that grows rapidly away from the point of impact (Waitukaitis & Jaeger 2012; Peters & Jaeger 2014; Han et al. 2016; Mukhopadhyay et al. 2018), which is thought to dominate the decelerating forces on the impacting object. The dynamically jammed region was originally thought to be associated with a local increase in \( \phi \) (Waitukaitis & Jaeger 2012; Peters & Jaeger 2014), but later work demonstrated that ‘solidification proceeds without a detectable increase in packing fraction’ (Han et al. 2016). Thus, the underlying physical mechanisms that control the impact-induced solidification are still debated.

Regardless of the microscopic origins of the dynamic jamming during impact, the dominant theory assumes that the intruder deceleration is dominated by momentum conservation due to the growing ‘added mass’ of the solidified region (Waitukaitis & Jaeger 2012). This model essentially divides the suspension into two regions: the solid-like region, which co-moves with the intruder and grows in time, and the rest of the suspension, which does not move. A third region, specifically the boundary layer between these two regions, is explicitly neglected by the added-mass model. Han et al. (2016) experimentally demonstrated the existence of a thin layer surrounding the moving, dynamically jammed region, with characteristic thickness of the order of millimetres. This thin layer separates the (moving) jammed region from the rest of the (static) suspension, so a very large shear rate \( \dot{\gamma} \) must be present. Thus, this layer can be characterized by having a large, nearly constant viscosity, corresponding to what is commonly observed in the shear-thickened regime (i.e. at large shear rates) in steady-state rheology studies of dense suspensions (Brown & Jaeger 2009; Fall et al. 2012). We note that added mass alone was sufficient to explain the experimental data in Peters & Jaeger (2014), but these experiments were two-dimensional (2-D), meaning that large, viscous-like forces would only act over a thin, quasi-1-D boundary. In three dimensions, the relative role of viscous-like drag acting on a surface and inertial effects from changing volumes is significantly different. We return to this discussion in our conclusions, §4.

Here, we show via theoretical analysis and impact experiments that large, viscous-like forces at the boundary of the growing jammed mass likely play a dominant role in the dynamics of the intruder. Drawing from previous work, we theoretically analyse the case of an intruder impact into a suspension, where the dynamics includes added-mass forces as well as these large, viscous-like forces at the boundary of the jammed region. We find...
that the original added-mass model as well as modified versions robustly predict that the maximum force $F_{\text{max}}$ achieved during impact and associated time scale $t_{\text{max}}$ scale with the impact velocity $v_0$ as $F_{\text{max}} \propto v_0^2$ and $t_{\text{max}} \propto v_0^{-1}$. These predictions are inconsistent with the data from our experiments as well as those from Waitukaitis & Jaeger (2012). These data show $F_{\text{max}} \propto v_0^\alpha$ and $t_{\text{max}} \propto v_0^\beta$, but with $\alpha \approx 1.5$ and $\beta \approx -0.5$ (instead of $2$ and $-1$, respectively). We find that we can better predict these observed scaling laws by assuming that viscous-like forces at the boundary of the dynamically jammed region are dominant. In addition, we consider how $F_{\text{max}}$ and $t_{\text{max}}$ depend on intruder size, mass and shape, and we again find that models dominated by viscous forces at the boundary perform better than models based on added mass. Our results suggest that the added-mass model is incomplete and may be improved by including large, viscous-like forces at the boundary of the dynamically jammed region.

2. Theoretical analysis

Prior experiments have repeatedly demonstrated that impact into a shear-thickening dense suspension results in a dynamically jammed, solid-like region. The boundary separating the jammed region from the rest of the suspension propagates away from the impact point at a characteristic speed that increases with packing fraction. Several theories have been proposed to explain this phenomenon. The original picture from Waitukaitis & Jaeger (2012) and Waitukaitis et al. (2013) posits that the formation and dynamics of the propagating front are primarily related to volume conservation, where volume swept out by the intruder must be compensated for by compaction of particle phase. However, as mentioned above, subsequent work has demonstrated that shear jamming is a more likely cause (Han et al. 2016, 2018; Han, James & Jaeger 2019a). Thus, Reynolds dilatancy, which is closely related to shear jamming (Bi et al. 2011; Wang et al. 2018), likely plays a crucial role in the front dynamics. Darcy–Reynolds theory, experimentally tested by Jerome, Vandenberghe & Forterre (2016), also demonstrated that there is a strong coupling between dilation and fluid flow through the pore structure. Thus, for very small particles like cornstarch, the Darcy pressure is proportional to the inverse of particle size squared. This may explain why the growing jammed front remains quasi-solid and relaxes slowly; we return to this point in our discussion of the force relaxation dynamics in §3.4. Our perspective in this paper is to assume that the growing, dynamically jammed region exists and to remain agnostic as much as possible regarding its microscopic origins. Waitukaitis & Jaeger (2012) also showed that the impact response was not primarily due to the dynamically jammed region reaching the boundary of the system and thus connecting the intruder to the boundary. They verified this by changing the depth of the suspension and demonstrating that the impact response did not depend on it (we also verify this for our experiments, as described below). When the dynamically jammed region reaches the boundary, there is a very large increase in the force felt by the intruding object, which has also been seen in other studies (Peters & Jaeger 2014; Maharjan et al. 2018). Here, we focus on the regime where system-spanning dynamically jammed regions do not occur.

2.1. Quasi-one-dimensional front development

If the front propagation process is assumed to be quasi-one-dimensional, meaning the compacted region grows only downward in a column with depth $z_f$ and not laterally, then the volume-conservation picture Waitukaitis et al. (2013) can be applied to obtain

$$k \equiv \frac{v_f}{v} = \frac{\phi_f}{\phi_f - \phi_0},$$

(2.1)
where \( v_f = \frac{dz_f}{dt} \) is the characteristic speed of the front and \( v = \frac{dz}{dt} \) is the intruder’s speed. For a schematic, see figure 1(a). Despite the fact that no detectable compaction of the suspension occurs, this dependence on \( \phi_0 \) and \( \phi_J \) was corroborated by Peters & Jaeger (2014) and Han et al. (2016) for 2-D and 3-D impacts, where \( \phi_J \approx 0.51 \) is the jamming packing fraction for the cornstarch particles, low due to swelling (Chen et al. 2019). This implies \( k \approx 10 \) when \( \phi_0 = 0.46 \). Since we do not vary \( \phi \) in our experiments, we will assume a constant value of \( k \).

2.2. Including lateral front growth

Experiments show that the jammed region below the intruder does not grow strictly downward in a quasi-1-D column but spreads out laterally as well, albeit at a smaller speed. Experiments with small impacting objects (Peters & Jaeger 2014; Han et al. 2016) typically find that the transverse dimension is approximately half of \( z_f \), meaning that the volume of the jammed region still scales as \( z_f^d \), where \( d \) is the dimensionality of the system (2 or 3). Additionally, experimental data show that the front slows down as it moves; see figure 4 of Peters & Jaeger (2014) and figure 4 of Han et al. (2016). The data from these papers appear consistent with \( z_f \propto z^\Gamma \) with \( \Gamma < 1 \). We discuss models below including this slowing-down behaviour, corresponding to \( z_f \propto z^\Gamma \), where \( \Gamma \) is varied to demonstrate that the results are relatively insensitive to its value. If compression-induced jamming were the sole cause, the volume of the jammed region scales as \( z_f^d \) while the volume swept out by the intruder is linear \( z \), meaning that \( z_f \) scales as \( z^{1/d} \). So, \( \Gamma = 1/d \) represents a reasonable lower bound for \( \Gamma \).

2.3. General equation of motion

To understand how the front growth, including its shape, affects the resulting dynamics, we consider a generic equation of motion that describes the dynamics of the intruder. Assuming that the growing solid-like region is rigidly connected to the intruder, then the total momentum of both is \( p = \{m + m_a(t)\}v(t) \), where \( m \) is the constant intruder mass, \( m_a(t) \) is the added mass and \( v(t) \) is the velocity of the intruder and solid-like region. The shape of the jammed region and its growth rate will set \( m_a(t) \). To complete the equation of motion, there are three external forces to consider. Two relate to gravity: the weight of the intruder, \( F_g = mg \), where \( g \) is the gravitational acceleration, and a buoyant force \( F_b \).
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from the displaced suspension (this term is typically negligible). The third, which is not included in the added-mass model, is any viscous-like forces $F_v$ that act at the boundary of the jammed region. Newton’s second law can then be written as

$$ (m + m_a) \frac{dv}{dt} + v \frac{dm_a}{dt} = F_b + F_g + F_v. \quad (2.2) $$

Before (2.2) can be solved for $z(t)$ (including the derivatives $v(t) = dz/dt$, and the acceleration $a(t) = dv/dt$), assumptions must be made about the mathematical form of $m_a, F_b$ and $F_v$. Based on our front dynamics discussion, we now consider a few scenarios, shown in figure 1, and solve (2.2) numerically or, if possible, exactly. The original added-mass model, shown in (a), assumed that a solid, cylindrical plug grows straight down, but that the total added mass is some proportion of an inverted cone-shaped region that grows downward and outward at the same rate. We note that schematic in figure 1(a), which follows Waitukaitis & Jaeger (2012), does not show a cone but a truncated cone; this difference does not affect the scaling laws shown. An alternative, shown in (b) is to consider drag force $F_v$ on the growing solid plug through shear in a boundary layer with thickness $\delta$. A version of this model was proposed in appendix D of Waitukaitis (2014). Although the existence of this thin boundary layer was documented in Han et al. (2016), the microscopic physics that set $\delta$ are not known. Some possibilities include that $\delta$ is set by $p$, similar to the front width in Waitukaitis et al. (2013), or by the particle size, perhaps through Darcy pressure. Finally, several 2-D and 3-D imaging experiments suggest that the solid-like region grows laterally. In this case, the solid region experiences a drag force that grows with its surface area, as depicted in figure 1(c). We first describe the first two cases in §§ 2.4 and 2.5. We then discuss scaling laws predicted by these models in §§ 2.6 and 2.7. Finally, we discuss the third case and its scaling laws in § 2.8.

2.4. Case 1: added-mass model, no viscous drag

First, we consider the original added-mass model (Waitukaitis & Jaeger 2012), which assumed that the solidified region is an inverted cone whose height and radius grow at the same rate $v_f = kv$, yielding $m_a = C_m \rho (1/3) \pi (D/2 + kz)^2 kz$, where $\rho$ is the suspension mass density, $C_m$ is an added-mass coefficient found experimentally to be $C_m \approx 0.37$, and $D$ is the intruder diameter. The fact that $C_m < 1$ means that the entirety of the added-mass region is not perfectly rigidly connected to the intruder. They assumed that viscous drag was negligible or absent and set $F_v = 0$ and that $F_b$ comes from displaced fluid in a conical depression near the intruder, $F_b = 1/3 \pi \rho g z (D/2 + kz)^2$. Numerical solutions to this model are qualitatively similar to experimental trajectories, as shown in figure 2 (thick black dashed line) and in Waitukaitis & Jaeger (2012).

2.5. Case 2: cylindrical jammed region with large, viscous-like drag

While the added-mass model provides qualitative features that can be matched to experiments, it is not unique in this respect: other choices for $F_v$ and $m_a$ yield similar results and can also be calibrated to match experimental trajectories, as also shown in figure 2 (thinner black dot-dashed line). In particular, models where $F_v$ is dominant yield similar results and have the advantage of matching other features of the dynamics, as discussed below. As shown in figure 2 of Han et al. (2016), the growing jammed region moves at approximately the same speed $v$ as the intruder, and it is surrounded by a thin layer of thickness $\delta \approx 5$ mm where the shear rate is $v/\delta$. Thus, on dimensional grounds,
we can approximate viscous force as

\[ F_v = -C_v \eta_s S \frac{v}{\delta}. \]  \hspace{1cm} (2.3)

Here, \( C_v \) is a dimensionless drag coefficient, \( \eta_s \) is the effective (constant) viscosity of the suspension and \( S \) is the surface area of the jammed region. We reemphasize that our use of a constant viscosity \( \eta_s \) does not mean that we consider the suspension to be a simple, viscous fluid. Instead, we refer to the large, nearly constant viscosity observed for \( \dot{\gamma} > \dot{\gamma}_c \). For example, figure 11 of Fall et al. (2012) shows a nearly constant value of \( \eta_s \approx 100 \text{ Pa s} \) for cornstarch suspensions in the fully shear-thickened regime for several different volume fractions. Additionally, the shear rates in this layer would be of the order of \( \dot{\gamma} \sim 10–100 \text{ s}^{-1} \), which is larger than \( \dot{\gamma}_c \sim 1–10 \) for cornstarch suspensions at volume fractions near \( \phi = 0.4 \) (Fall et al. 2012).

This model is a generalized form of a model appearing in Appendix D of Waitukaitis’ Ph.D. thesis (Waitukaitis 2014), which involves a columnar, solid-like front growing beneath the intruder with height \( h_F = k_z \) and thus \( S = \pi D k_z + \pi D^2 / 4 \). We note that their dimensional analysis used \( D \) in place of \( S / \delta \), which then required \( \eta_s \) to be unphysically large, \( \eta_s \approx 2000 \). This situation is sketched in figure 1(b). If the second term is dropped on the grounds that \( D \) is much smaller than \( k_z \) or that the viscous-like forces only act on the sides of the growing cylinder, then the resulting dynamics is exactly solvable. This approximation is certainly not valid during the very early stages of impact, which would be particularly important for large \( D \) and small \( m \). In this case, (2.3) becomes

\[ F_v = -C_v \pi D \eta_s k_z \frac{v}{\delta}. \]  \hspace{1cm} (2.4)

Figure 2. Experimental trajectories (red) for velocity \( v(t) \) and acceleration \( a(t) \) are compared with the solutions to the intruder equation of motions for the added-mass (thick black dashed line) and viscous (thinner black dot-dashed line) models discussed in §§2.4 and 2.5, respectively. (a,b) Show impacts with \( v_0 \approx 4 \text{ ms}^{-1} \) and \( v_0 \approx 1 \) using cylindrical intruders with \( m = 189 \text{ g} \) and \( D = 25.4 \text{ mm} \). The fit parameters in both models can be adjusted to approximately match the experiments. The added-mass model trajectories for both panels were solved using \( C_m = 0.1 \), which is much smaller than the value \( C_m = 0.37 \) used in Waitukaitis & Jaeger (2012). The viscous model trajectories are those in (2.5) and (2.6) using \( C_v = 0.4, \eta_s = 20 \text{ Pa} \) and \( \delta = 1 \text{ mm} \).
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Assuming other forces can be neglected and \( m_a \) is negligible, then (2.2) can be exactly solved, yielding

\[
v(t) = v_0 \text{sech}^2(t/\tau),
\]

\[
a(t) = -\sqrt{\frac{2C_v \pi D \eta_s m}{m \delta}} v_0^{3/2} \text{sech}^2(t/\tau) \tanh(t/\tau),
\]

where \( \tau = \sqrt{C_v \pi D \eta_s v_0 / 2m \delta} \). These functions are plotted in figure 2 and agree well with experiments. The viscous model solution in figure 2 uses \( C_v = 0.42 \), \( \delta = 0.5 \text{ cm} \) and \( \eta_s = 100 \text{ Pa s} \). This agrees well with the viscosity of cornstarch and water–CsCl suspensions in the shear-thickening regime with similar values of \( \phi \), as shown in figure 11 of Fall et al. (2012). This comparison demonstrates that reasonable parameter values can be used in matching to experiments, although there is some flexibility and therefore uncertainty in the values of these parameters.

2.6. Scaling laws for the added-mass model

Since both the added-mass and the viscous drag models can be reasonably matched to experimentally observed intruder trajectories, some further validation can come from comparing how \( F_{\text{max}} \) and \( t_{\text{max}} \) scale with \( v_0 \), \( m \) and \( D \). Such scalings have been previously used in the case of impact to connect macroscale dynamics with the microscale mechanisms that give rise to them (Uehara et al. 2003; Walsh et al. 2003; Goldman & Umbanhowar 2008; Clark, Petersen & Behringer 2014; Zhao et al. 2015; Krizou & Clark 2020). For all experiments and theoretical models, we find that these scalings can be well approximated by

\[
F_{\text{max}} = A v_0^\alpha
\]

\[
t_{\text{max}} = B v_0^\beta.
\]

The prefactors \( A \) and \( B \) can vary with intruder properties, and we will examine how they depend on \( m, D \) and, for conical intruders, cone angle \( \theta \).

In the added-mass model, \( F_{\text{max}} \) and \( t_{\text{max}} \) can depend on four parameters: \( m, \rho, v_0, D \). A total of three dimensionless quantities can be formed by these six total parameters, so \( F_{\text{max}}/\rho v_0^2 D^2 \) and \( t_{\text{max}} v_0/D \) must each be a function of a single dimensionless quantity \( m/\rho D^3 \). This suggests that, when \( m/\rho D^3 \) is fixed, \( F_{\text{max}} \propto v_0^2 \) and \( t_{\text{max}} \propto v_0^{-1} \). The added-mass model was solved numerically by Mukhopadhyay et al. (2018), finding that \( F_{\text{max}} \propto v_0^2 m^{2/3} \). We also numerically solve the added-mass model and find the same result, along with \( t_{\text{max}} \propto v_0^{-1} m^{1/3} \). We find \( F_{\text{max}} \) and \( t_{\text{max}} \) to be nearly independent of \( D \) in the range of parameters studied here, in agreement with Mukhopadhyay et al. (2018). The lack of dependence on \( D \), combined with the dimensional analysis, also requires \( F_{\text{max}} \propto \rho v_0^2 D^2 (m/\rho D^3)^{2/3} \) and \( t_{\text{max}} \propto D / v_0 (m/\rho D^3)^{1/3} \), in agreement with the numerical solutions. Figure 3 shows numerical solutions as well as these scaling laws for the added mass model.

2.7. Scaling law for viscous models

In the viscous model, there are two additional dimensionless numbers: \( Re = \rho v_0 D / \eta_s \) and \( D/\delta \). This means that the dimensional analysis no longer directly predicts a relationship between \( F_{\text{max}} \) and \( v_0 \). However, the peak force and time at which the peak acceleration occurs can be directly calculated by differentiating \( a(t) \) in (2.6), setting the result to zero,
and solving for \( t \). This time \( t_{\text{max}} \) can be substituted back into (2.6) to calculate \( a_{\text{max}} = a(t_{\text{max}}) \). By this method, the peak force \( F_{\text{max}} = ma_{\text{max}} \) and \( t_{\text{max}} \) are found to be

\[
F_{\text{max}} = \sqrt{2C_vDk\eta}m/\delta v_0^{3/2}\text{sech}^2(\beta)\tanh(\beta),
\]

\[
t_{\text{max}} = \sqrt{\frac{2m\delta}{C_vDk\eta v_0}}\beta,
\]

where \( \beta = \frac{1}{2}\log(2 + \sqrt{3}) \). We also solve the viscous model numerically and find the same result. Thus, for the viscous model where the jammed region is a cylindrical column, \( F_{\text{max}} \propto v_0^{3/2}m^{1/2}D^{1/2} \) and \( t_{\text{max}} \propto v_0^{-1/2}m^{1/2}D^{1/2} \). These scaling relations can also be written in terms of the dimensionless quantities given above, e.g. \( F_{\text{max}}/\rho v_0^2D^2 \propto Re^{-1/2}(D/\delta)^{1/2}(m/\rho D^3)^{1/2} \).

2.8. Case 3: hybrid models and scaling laws

The simple viscous model discussed above does not include several features that may make a comparison with experimental data more complicated. First, the solidified region grows in all three dimensions, not just straight down in a cylindrical column. This is particularly relevant when the contact point for the impacting object is point-like. Second, momentum is being transferred to the solidified region, so added-mass terms must be included generally. Third, the rate at which the front is growing tends to decrease with time...
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(i.e. $\Gamma < 1$). Finally, many of the physical parameters in these models have been previously studied, so there are physical bounds on, e.g. $\eta_s$ based on experimental measurements. To check how sensitive the scaling laws are for varying shapes of the growing jammed region, we study a case where the volume of the solidified region grows as $zf^3$ and the surface area grows as $zf^2$, where $zf = kz^\Gamma$. For simplicity, we approximate the growing jammed region as a hemisphere where the volume and surface area are $2\pi zf^3/3$ and $2\pi zf^2$, respectively. This situation is sketched in figure 1(c). We note that this overestimates the volume (and thus the added mass) in particular, since Han et al. (2016) showed that the mass is better approximated as a half-ellipsoid with semi-minor axes of $zf$, $zf/2$ and $zf/2$, meaning that the volume is $1/4$ of the hemisphere. This yields

$$m_a = C_m \rho \frac{2\pi}{3}(kz^\Gamma)^3,$$  \hspace{1cm} (2.11)

$$F_v = -2C_v \eta_s \pi (kz^\Gamma)^2 \frac{v}{\delta}.$$  \hspace{1cm} (2.12)

We first consider the case where the added-mass term is dominant. We set $C_m = 0.2$, $C_v = 0$, $F_b = 0$ and numerically solve (2.2). We find that $F_{\max} \propto v_0^2$ and $t_{\max} \propto v_0^{-1}$ persist for all values of $\Gamma$ we study ($1/3 < \Gamma < 1$). We emphasize that these scaling exponents are required by the dimensional analysis and inconsistent with experimental data shown below. Next, we consider the case where the viscous term is dominant, setting $C_v = 0.5$, $C_m = 0$ and $F_b = 0$. For $\Gamma = 1$, corresponding to the front moving at a constant speed in all directions, we find $F_{\max} \propto v_0^{1.63}m^{0.69}$ and $t_{\max} \propto v_0^{-0.67}m^{0.33}$. For $\Gamma = 1/3$, corresponding to the lower-bound case where compression-induced jamming is dominant (see § 2.2), we find $F_{\max} \propto v_0^{1.40}m^{0.40}$ and $t_{\max} \propto v_0^{-0.40}m^{0.60}$. The behaviour smoothly varies between these limits as $\Gamma$ is varied. For example, when $\Gamma = 0.7$, we find $F_{\max} \propto v_0^{1.58}m^{0.59}$ and $t_{\max} \propto v_0^{-0.60}m^{0.42}$.

We also find that, if added-mass and viscous terms are both present, then the values of the exponents fall in between the predictions of each model, depending on the relative strength of each term. To illustrate this, figure 4 shows numerical solutions using (2.11) and (2.12). We choose $C_m = 0.1$, $\rho = 1630 \text{ kg m}^{-3}$, $k = 4$, $\Gamma = 0.7$, $C_v = 0.5$, $\eta_s = 20 \text{ Pa s}$ and $\delta = 2 \text{ mm}$, all values that are based on previous experiments or within reasonable physical bounds. This yields $F_{\max} \propto v_0^{1.61}m^{0.59}$ and $t_{\max} \propto v_0^{-0.90}m^{0.46}$.

2.9. Summary of theoretical considerations

Table 1 shows a summary of all the theoretical predictions from these models, assuming the forms $F_{\max} \propto v_0^\alpha m^\xi D^\lambda$ and $t_{\max} \propto v_0^\beta m^\kappa D^\mu$. These scalings can be derived analytically only for case 2, as shown above and in Waitukaitis (2014), as well as case 1 in the limit of small $D$ (Mukhopadhyay et al. 2018). We also solve these models numerically to confirm the analytical scalings, including the non-zero values of $D$ we use in experiments for case 1. All results from case 3 are obtained numerically.

All added-mass models (case 1 and case 3 with $C_v = 0$) predict $\alpha = 2$ and $\beta = -1$. Viscous models (case 2 and case 3 with $C_m = 0$) predict $1.4 < \alpha < 1.63$ and $-0.67 < \beta < -0.4$. The exponents associated with $m$, $\xi$ and $\kappa$, are similar for all models. The exponents associated with $D$, $\lambda$ and $\mu$, are zero for all added-mass models, but are non-zero for the viscous model, case 2, with $\lambda = 1/2$ and $\mu = -1/2$. This is due to the fact that the size of the jammed region (and therefore the surface area experiencing viscous-like drag) scales with the intruder diameter in this model.
Figure 4. Mixed model numerical solutions using parameters described in the text. As in figure 3, each symbol represents a different mass.

Table 1. A table summarizing predictions of the models shown in figure 1, where $F_{\text{max}} \propto v_0^\alpha m^\beta D^{\lambda}$ and $t_{\text{max}} \propto v_0^\mu m^\kappa D^{\mu}$. See § 2.9 for a discussion.

| Model                  | $\alpha$ | $\zeta$ | $\lambda$ | $\beta$ | $\kappa$ | $\mu$ |
|------------------------|----------|----------|------------|----------|----------|--------|
| Added mass (case 1)    | 2        | 2/3      | 0          | -1       | 1/3      | 0      |
| Viscous (case 2)       | 3/2      | 1/2      | 1/2        | -1/2     | 1/2      | -1/2   |
| Hybrid (case 3): $1/3 < \Gamma < 1$, $C_v = 0$, $C_m = 0.2$ | 2        | —        | 0          | -1       | —        | 0      |
| Hybrid (case 3): $\Gamma = 1$, $C_v = 0.5$, $C_m = 0$ | 1.63     | 0.69     | 0          | -0.67    | 0.33     | 0      |
| Hybrid (case 3): $\Gamma = 1/3$, $C_v = 0.5$, $C_m = 0$ | 1.4      | 0.4      | 0          | -0.4     | 0.6      | 0      |
| Hybrid (case 3): $\Gamma = 0.7$, $C_v = 0.5$, $C_m = 0$ | 1.58     | 0.59     | 0          | -0.6     | 0.42     | 0      |
| Hybrid (case 3): $\Gamma = 0.7$, $C_v = 0.5$, $C_m = 0.1$ | 1.61     | 0.59     | 0          | -0.9     | 0.46     | 0      |

3. Experiments

To compare with the scaling laws summarized in § 2.9, we perform experiments of intruders falling under gravity to impact a free surface of a suspension of food-grade cornstarch particles in tap water. The dimensions of the cornstarch suspension were approximately 20 cm $\times$ 20 cm $\times$ 20 cm (length, width, height). Figure 2 of Waitukaitis & Jaeger (2012) shows that boundary effects become dominant when the fill height is less than 10 cm, meaning that our data should also be independent of boundary effects. We explicitly verified this by performing selected experiments into a suspension with larger dimensions (30 cm $\times$ 30 cm $\times$ 30 cm), and found that our results did not depend
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on system size. The density of the cornstarch was 46% by volume. We also tested impacts with 49% by volume and found only a very slight upward shift in the forces, in agreement with Waitukaitis & Jaeger (2012). The packing fraction of cornstarch was inferred by weighing both the water and cornstarch added and assuming a specific gravity of 1.6 for the cornstarch (Han, Van Ha & Jaeger 2017). Intruders of varying shapes (cylinders, spheres and cones) and diameters $D$ were attached to threaded rods and held by an electromagnet. They were held at variable heights and then released, yielding impact speeds of up to $v_0 \approx 4 \text{ m s}^{-1}$. The mass $m$ of the intruder was varied by adding additional weights on the rod. The impacts were recorded by high-speed video using a Phantom V711 at frame rates between 175 000 and 230 000 frames per second. A ball was attached to the threaded rod and tracked using MATLAB, yielding the position of the intruder at each frame. Discrete differentiation and a lowpass filter (Clark, Kondic & Behringer 2012) were used to obtain the velocity and acceleration. An accelerometer with sample rate of 5000 Hz (Sparkfun ADXL377) was connected to the rod, showing good agreement with the acceleration obtained from video tracking. The accelerometer data had better time resolution, since two lowpass filters were applied to the video acceleration data. Therefore, all velocity data shown are from video tracking and all acceleration data are from the accelerometer.

3.1. Experimental results

Figure 5 shows $F_{\text{max}} = -ma_{\text{max}}$ and $t_{\text{max}}$ plotted as a function of $v_0$ for four representative experiments of cylinders impacting cornstarch suspensions: three from our experiments (one cylinder, one sphere and one cone) as well as the experimental data from Waitukaitis & Jaeger (2012). These quantities appear to scale with $v_0$ according to power-law relations, (2.7) and (2.8). Comparison with the fit line that is shown strongly suggests that $\alpha \approx 1.5$; linear fits to the data from our experiments confirm this, returning $1.3 < \alpha < 1.6$ for all intruders we study. This is consistent with the range predicted by viscous models discussed in § 2. The data for $t_{\text{max}}$ are more scattered, making clear determination of $\beta$ more difficult. Additionally $v_0 > 3 \text{ m s}^{-1}$, the $t_{\text{max}}$ data from all intruders appear to flatten out and even curve upward slightly, which is not predicted by any of the theoretical models. However, best fits for impact velocities $0.5 < v_0 < 3 \text{ m s}^{-1}$ give $\beta \approx -0.5$. These values, $\alpha = 1.5$ and $\beta = -0.5$, agree with the viscous model well, but do not agree with the predictions of the added-mass models discussed above, $\alpha = 2$ and $\beta = -1$. This strongly suggests that viscous terms at the boundary of the dynamically jammed region play an important, and likely dominant, role in the deceleration of the intruder.

To further examine the consistency of these models with the experimental data, figure 6 shows how the prefactors $A$ and $B$ scale with $m$, $D$ and cone angle $\theta$. We measure $A$ and $B$ as the mean of $F_{\text{max}}/v_0^\alpha$ and $t_{\text{max}}/v_0^\beta$, with $\alpha = 3/2$ and $\beta = -1/2$. Given the scatter in the $t_{\text{max}}$ data, we also measure $B$ using other values of $\beta$ ranging from $-0.4$ to $-1$, and the results shown in figure 6 are insensitive to our choice of $\beta$. Error bars represent the standard error on the mean; e.g. $A$ is measured as the mean of $F_{\text{max}}/v_0^\alpha$, and the size of the error bar is given by the square root of these data divided by the number of data points. Figures 6(a) and 6(b) show $A$ and $B$ vs $m$ for three cylindrical intruders with the same diameter $D = 25 \text{ mm}$ but with varied mass, $m \approx 80$, 150 and 230 g. Power-law fit lines are shown in black for the predictions of the original added-mass model, $A \propto m^{2/3}$ and $B \propto m^{1/3}$, and the viscous model involving a quasi-1-D cylindrical dynamically jammed region, $A \propto m^{1/2}$ and $B \propto m^{1/2}$. The data appear more consistent with the viscous model predictions, especially for $B$. However, we note that the details of the shape of the added
mass, as well as how dramatically the propagating front slows down as it moves, can cause these exponents to vary somewhat.

3.2. Intruder size scaling

Figures 6(c) and 6(d) show data from cylindrical and spherical intruders of similar m but varying D. The cylinders have m ≈ 190 g and D = 12.5, 25 and 50 mm, and the spheres have m ≈ 200 g and D = 20, 30 and 50 mm. The added-mass model predicts that, for these sizes and weights, there is very little dependence of A or B on D, i.e. A ∝ D^0 and B ∝ D^0. The first viscous model predicts A ∝ D^{1/2} and B ∝ D^{-1/2}. Overall, the experimental data show that A increases with D and B decreases with D, which is inconsistent with the added-mass model. We note that increase is clearer for the cylindrical intruders (square symbols) than for the spherical intruders (circular symbols). The cylindrical intruders appear to follow the predictions of the first viscous model, A ∝ D^{1/2} and B ∝ D^{-1/2}. For spherical intruders, the hybrid model from § 2.8 may be more relevant, where the impact is more point like instead of a circular surface that makes simultaneous contact with the fluid. These models still predicted α ≈ 1.5 and β ≈ −0.5, but they had no D dependence.

3.3. Cone-shaped intruders

Intruder shape affects A and B somewhat, as can be observed from the dynamics of cone-shaped intruders. Figures 6(e) and 6(f) show conical intruder data whose mass and diameter are constant, m ≈ 195 g and diameter D = 30 mm, but with varied angles θ = 0°, 20°, 30°, 45°, 55° and 70°. Here, θ = 0° corresponds to a flat cylinder and θ = 90° is the maximum possible value. We observe F_{max} ∝ v_0^{1.5} and t_{max} ∝ v_0^{-0.5} for all cone angles, suggesting that viscous-like forces are again dominant. However, A decreases with increasing θ, while B increases with increasing θ. One hypothesis for this behaviour is that larger θ corresponds to a smaller contact area, equivalent to smaller D. Another explanation could come from the fact that increasing θ means that the dynamically jammed region transitions from being generated primarily by normal compression (for θ = 0°) to being generated primarily through shear jamming. As shown by Han et al. (2018),

![Graph showing the relationship between F_{max} and v_0 for different intruder shapes.](image-url)
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Figure 6. (a,b) Show $A$ and $B$ vs $m$ for four cylindrical intruders with $D = 2.5$ cm (for the three square symbols) and $D = 1.86$ cm (for the magenta stars, from Waitukaitis & Jaeger 2012). (c,d) Show $A$ and $B$ vs $D$ for experimental data on cylindrical (square symbols) and spherical intruders with $m = 189$ g and $m = 199$ g, respectively. (e,f) Show $A$ and $B$ for conical intruders with $m = 195$ g and $D = 2.5$ cm as a function of cone angle $\theta$. Error bars represent the standard error of the mean (see text).

the value of $k$ is smaller for fronts created by shear jamming. Our data are inconclusive on this question, except for the fact that we consistently find $\alpha \approx 1.5$ and $\beta \approx -0.5$ for all values of $\theta$, which is consistent with the class of viscous models discussed in § 2.

3.4. Relaxation after peak deceleration

Finally, we consider the intruder dynamics after the peak deceleration, which also provides information on the microscopic physics of suspension dynamics. While the dynamics
Figure 7. Acceleration curves and post-peak acceleration curves for a few representative examples of experimental data sets with (a,c) $1 < v_0 < 1.5 \text{ m s}^{-1}$ and (b,d) $3 < v_0 < 3.5 \text{ m s}^{-1}$. In (c,d), we indicate exponential decay with time scales of 2 and 3 ms with solid lines in both panels for reference. The cones have $m = 195 \text{ g}$ and $D = 2.5 \text{ cm}$; the light and heavy cylinders have $D = 2.5 \text{ cm}$ with $m = 77.8 \text{ g}$ and $m = 227 \text{ g}$, respectively; and the small and large spheres have $m = 199 \text{ g}$ and $D = 2 \text{ cm}$ and $D = 5 \text{ cm}$, respectively.

of impact before peak deceleration is highly sensitive to various experimental control parameters, the post-peak dynamics is not. This is shown in figure 7. Figures 7(a) and 7(b) show impacts with $1 < v_0 < 1.5$ and $3 < v_0 < 3.5 \text{ m s}^{-1}$, respectively, both with a wide variety of intruder properties; $t_{\text{max}}$ varies dramatically with $v_0$ and intruder properties, which has been the subject of our analysis so far. However, figures 7(c) and 7(d) show that forces decay quasi-exponentially with a time scale between 2 and 3 ms; this behaviour is largely independent of speed and intruder properties. This implies that, e.g. the time scale $\tau = \sqrt{C_1 k n_1 v_0 / 2m}$ in (2.5) and (2.6) might capture the buildup to peak force but not the decay. This suggests that the relaxation dynamics is dominated by the microscopic material composition in a way that is not sensitive to the intruder. Thus, this dynamics appears to lie outside the description of either the added-mass or viscous models. Our observations are consistent with Peters & Jaeger (2014), who found that changing the viscosity of the suspending fluid affected the dynamics of the relaxation of the jammed front but not its growth.

The fact that fluid viscosity slows down the relaxation dynamics suggests that Darcy–Reynolds theory, demonstrated by Jerome et al. (2016), may play a crucial role in holding the solidified region together. The system studied by Jerome et al. (2016) involves
much larger, inorganic glass beads with diameter of approximately 100 μm or larger, whereas (organic) cornstarch particles are approximately 1 to 10 μm. The grains were gravitationally loaded into a settled configuration – that is, the initial state of the system involves all particles making solid–solid contact – as opposed to the cornstarch particles, which are stirred between impacts and remain suspended in the fluid at relatively low volume fractions. Then, if the grains dilate during impact, there is a strong solidification that is strictly driven by a combination of Reynolds dilatancy (the packing expands) and Darcy flow (high resistance of the fluid being sucked into the expanding pores between grains); if the grains compact, there is no solidification. Our system begins at a low volume fraction and likely does not compact significantly (Han et al. 2016). Thus, it is unlikely that the Darcy–Reynolds theory plays a direct, dominant role in the present experiments, but it may play a crucial, secondary role. Relaxation of the solidified region occurs as particles rearrange and lose contact with each other, which requires fluid flow through the pore structure. Thus, Darcy flow in particular may be responsible for holding the solidified region together and may also control the relaxation dynamics shown in figure 7.

4. Conclusion
Here, we have theoretically and experimentally studied the problem of impact of an intruder into a shear-thickening dense suspension. In agreement with previous authors such as Mukhopadhyay et al. (2018), we find that the added-mass model (Waitukaitis & Jaeger 2012), which has been the dominant model used to explain the dynamics of impacts into shear-thickening dense suspensions, predicts $F_{\text{max}} \propto v_0^2$ and $t_{\text{max}} \propto v_0^{-1}$. In contrast, the experimental data show $F_{\text{max}} = A v_0^{1.5}$ and $t_{\text{max}} = B v_0^{-0.5}$. These exponents are consistent with a class of models where the dominant force is not added mass but viscous-like forces at the boundary of the jammed suspension.

We have also studied how the prefactors $A$ and $B$ depend on intruder mass, size and shape. These results are either consistent with both added-mass and viscous models (e.g. in the case of varying intruder mass) or more consistent with viscous models (e.g. in the case of cylindrical intruders with varying diameter). Our results suggest that the added-mass model should be revised to include large, viscous-like terms at the boundary, since these forces may play a dominant role. These results do not change certain aspects of the underlying physical picture for impact into dense suspensions: a solid-like region grows outward from the point of impact and dominates the intruder dynamics. If large, viscous-like forces (corresponding to the large, nearly constant viscosity observed in the shear-thickening regime of dense suspensions) are dominant, this also has the advantage of conceptually unifying impact with steady-state rheology descriptions such as DST.

We emphasize that the viscous models we analyse here are not applicable to simple, high-viscosity liquids. The models we discuss are predicated on the existence of a growing solid-like region and a large viscosity that is ‘turned on’ only in regions of very high strain rate. These features would not exist in the case of a simple liquid with very large viscosity, as the viscosity would be constant everywhere (independent of local shear rate) and there would be no solid-like region.

As mentioned in § 1, we note that figure 6 of Peters & Jaeger (2014) shows that added mass is sufficient to explain the forces measured by an external sensor for velocity controlled impact into a 2-D layer of dense suspension. However, in a 2-D experiment, the viscous-like forces that we propose would act over a 1-D boundary between the jammed region and the uncompressed suspension; in a 3-D experiment, the surface area of the 3-D jammed solid is much bigger, leading to significantly larger viscous forces. Thus, their
findings (that added mass was sufficient to explain the resisting force on the intruding object in a 2-D situation) are consistent with the results we have shown for 3-D impacts.

Our results are limited to the case where the jammed region does not span the system, which leads to a ‘bounce’, as shown in figure 2 of Waitukaitis & Jaeger (2012). The mechanical response of a system-spanning dynamically jammed region was analysed by Allen et al. (2018) and Maharjan et al. (2018) in the case of a much smaller container with an intruder driven at constant speed. Thus, we expect the impact process to be significantly different in the case of a shear-thickening suspension in a smaller container or confined to a thin layer.

Finally, we reiterate that we have made several approximations throughout our analysis, such as neglecting buoyant and gravitational forces. Thus, inclusion of these or other forces may help resolve the discrepancies between the models and experimental results. Future work is needed to better characterize the relative contribution of these forces; to better characterize the magnitude of and the length scale over which the viscous forces act; and to understand how the solidified mass relaxes back into a fluid-like state.

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REFERENCES

Allen, B., Sokol, B., Mukhopadhyay, S., Maharjan, R. & Brown, E. 2018 System-spanning dynamically jammed region in response to impact of cornstarch and water suspensions. Phys. Rev. E 97, 052603.

Barnes, H.A. 1989 Shear–thickening (‘dilatancy’) in suspensions of nonaggregating solid particles dispersed in Newtonian liquids. J. Rheol. 33 (2), 329–366.

Bi, D., Zhang, J., Chakraborty, B. & Behringer, R.P. 2011 Jamming by shear. Nature 480 (7377), 355–358.

Brown, E. & Jaeger, H.M. 2009 Dynamic jamming point for shear thickening suspensions. Phys. Rev. Lett. 103, 086001.

Brown, E. & Jaeger, H.M. 2014 Shear thickening in concentrated suspensions: phenomenology, mechanisms and relations to jamming. Rep. Prog. Phys. 77 (4), 046602.

Chen, D.Z., Zheng, H., Wang, D. & Behringer, R.P. 2019 Discontinuous rate-stiffening in a granular composite modeled after cornstarch and water. Nat. Commun. 10 (1), 1283.

Clark, A.H., Kondic, L. & Behringer, R.P. 2012 Particle scale dynamics in granular impact. Phys. Rev. Lett. 109, 238302.

Clark, A.H., Petersen, A.J. & Behringer, R.P. 2014 Collisions model for granular impact dynamics. Phys. Rev. E 89, 012201.

Fall, A., Bertrand, F., Ovarlez, G. & Bonn, D. 2012 Shear thickening of cornstarch suspensions. J. Rheol. 56 (3), 575–591.

Fall, A., Lemaître, A., Bertrand, F., Bonn, D. & Ovarlez, G. 2010 Shear thickening and migration in granular suspensions. Phys. Rev. Lett. 105, 268303.

Goldman, D.I. & Umbanhowar, P. 2008 Scaling and dynamics of sphere and disk impact into granular media. Phys. Rev. E 77, 021308.

Guazzelli, É. & Pouliquen, O. 2018 Rheology of dense granular suspensions. J. Fluid Mech. 852, 1.

Han, E., James, N.M. & Jaeger, H.M. 2019a Stress controlled rheology of dense suspensions using transient flows. Phys. Rev. Lett. 123, 248002.
Viscous-like forces control impact into dense suspensions

HAN, E., PETERS, I.R. & JAEGGER, H.M. 2016 High-speed ultrasound imaging in dense suspensions reveals impact-activated solidification due to dynamic shear jamming. Nat. Commun. 7 (1), 12243.

HAN, E., VAN HA, N. & JAEGGER, H.M. 2017 Measuring the porosity and compressibility of liquid-suspended porous particles using ultrasound. Soft Matter 13 (19), 3506–3513.

HAN, E., WYART, M., PETERS, I.R. & JAEGGER, H.M. 2018 Shear fronts in shear-thickening suspensions. Phys. Rev. Fluids 3, 073301.

HAN, E., ZHAO, L., VAN HA, N., HSIEH, S.T., SZYLD, D.B. & JAEGGER, H.M. 2019b Dynamic jamming of dense suspensions under tilted impact. Phys. Rev. Fluids 4, 063304.

VAN HECKE, M. 2009 Jamming of soft particles: geometry, mechanics, scaling and isostaticity. J. Phys.: Condens. Matter 22 (3), 033101.

HOFFMAN, R.L. 1972 Discontinuous and dilatant viscosity behavior in concentrated suspensions. I. Observation of a flow instability. Trans. Soc. Rheol. 16 (1), 155–173.

JEROME, J.J.S., VANDENBERGHE, N. & FORTERRE, Y. 2016 Unifying impacts in granular matter from quicksand to cornstarch. Phys. Rev. Lett. 117 (9), 098003.

KRIZOU, N. & CLARK, A.H. 2020 Power-law scaling of early-stage forces during granular impact. Phys. Rev. Lett. 124, 178002.

LEE, Y.S., WETZEL, E.D. & WAGNER, N.J. 2003 The ballistic impact characteristics of Kevlar–Woven fabrics impregnated with a colloidal shear thickening fluid. J. Mater. Sci. 38 (13), 2825–2833.

MAHARJAN, R., MUKHOPADHYAY, S., ALLEN, B., STORZ, T. & BROWN, E. 2018 Constitutive relation for the system-spanning dynamically jammed region in response to impact of cornstarch and water suspensions. Phys. Rev. E 97, 052602.

MUKHOPADHYAY, S., ALLEN, B. & BROWN, E. 2018 Testing constitutive relations by running and walking on cornstarch and water suspensions. Phys. Rev. E 97, 052604.

O’HERN, C.S., SILBERT, L.E., LIU, A.J. & NAGEL, S.R. 2003 Jamming at zero temperature and zero applied stress: the epitome of disorder. Phys. Rev. E 68 (1), 011306.

PETERS, I.R. & JAEGGER, H.M. 2014 Quasi-2d dynamic jamming in cornstarch suspensions: visualization and force measurements. Soft Matter 10 (34), 6564–6570.

PRADIPTO & HAYAKAWA, H. 2021 Impact-induced hardening in dense frictional suspensions. Phys. Rev. Fluids 6, 033301.

SETO, R., MARI, R., MORRIS, J.F & DENN, M.M. 2013 Discontinuous shear thickening of frictional hard-sphere suspensions. Phys. Rev. Lett. 111 (21), 218301.

TORQUATO, S. & STILLINGER, F.H. 2010 Jammed hard-particle packings: from kepler to bernal and beyond. Rev. Mod. Phys. 82, 2633–2672.

UEHARA, J.S., AMBROSO, M.A., OHBA, R.P. & DURIAN, D.J. 2003 Low-speed impact craters in loose granular media. Phys. Rev. Lett. 90, 194301.

WAITUKAITIS, S.R. 2014 Impact-Activated Solidification of Cornstarch and Water Suspensions. Springer.

WAITUKAITIS, S.R. & JAEGGER, H.M. 2012 Impact-activated solidification of dense suspensions via dynamic jamming fronts. Nature 487 (7406), 205–209.

WAITUKAITIS, S.R., ROTH, L.K., VITELLI, V. & JAEGGER, H.M. 2013 Dynamic jamming fronts. Europhys. Lett. 102 (4), 44001.

WALSH, A.M., HOLLOWAY, K.E., HABDAS, P. & DE BRUYN, J.R. 2003 Morphology and scaling of impact craters in granular media. Phys. Rev. Lett. 91, 104301.

WANG, D., REN, J., DUKSMAN, J.A., ZHENG, H. & BEHRINGER, R.P. 2018 Microscopic origins of shear jamming for 2d frictional grains. Phys. Rev. Lett. 120, 208004.

WYART, M. & CATES, M.E. 2014 Discontinuous shear thickening without inertia in dense non-Brownian suspensions. Phys. Rev. Lett. 112, 098302.

ZHAO, R., ZHANG, Q., TJUGITO, H. & CHENG, X. 2015 Granular impact cratering by liquid drops: understanding raindrop imprints through an analogy to asteroid strikes. Proc. Natl Acad. Sci. 112 (2), 342–347.