REGULARITY OF POWERS OF EDGE IDEALS OF VERTEX-WEIGHTED ORIENTED UNICYCLIC GRAPHS

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Abstract. In this paper we provide some exact formulas for the regularity of powers of edge ideals of vertex-weighted oriented cycles and vertex-weighted unicyclic graphs. These formulas are functions of the weight of vertices and the number of edges. We also give some examples to show that these formulas are related to direction selection and the weight of vertices.

1. Introduction

A directed graph or digraph $D$ consists of a finite set $V(D)$ of vertices, together with a collection $E(D)$ of ordered pairs of distinct points called edges or arrows. If $\{u, v\} \in E(D)$ is an edge, we write $uv$ for $\{u, v\}$, which is denoted to be the directed edge where the direction is from $u$ to $v$ and $u$ (resp. $v$) is called the starting point (resp. the ending point). Given any digraph $D$, we can associate a graph $G$ on the same vertex set simply by replacing each arrow by an edge with the same ends. This graph is called the underlying graph of $D$, denoted by $G(D)$. Conversely, any graph $G$ can be regarded as a digraph, by replacing each of its edges by just one of the two oppositely oriented arrows with the same ends. Such a digraph is called an orientation of $G$. An orientation of a simple graph is referred to as a simple oriented graph.

Edge ideals of edge-weighted graphs were introduced and studied by Paulsen and Sather-Wagstaff [23]. In this work we consider edge ideals of graphs which are oriented and have weights on the vertices. In what follows by a weighted oriented graph we shall always mean a vertex-weighted oriented graph.

A vertex-weighted oriented graph is a triplet $D = (V(D), E(D), w)$, where $V(D)$ is the vertex set, $E(D)$ is the edge set and $w$ is a weight function $w : V(D) \rightarrow \mathbb{N}^+$, where $\mathbb{N}^+ = \{1, 2, \ldots\}$. Sometimes for short we denote the vertex set $V(D)$ and edge set $E(D)$ by $V$ and $E$ respectively. The weight of $x_i \in V$ is $w(x_i)$, denoted by $w_i$ or $w_{x_i}$. The edge ideal of a vertex-weighted digraph was first introduced by Gimenez et al. [13]. Let $D = (V, E, w)$ be a vertex-weighted digraph with the vertex set $V = \{x_1, \ldots, x_n\}$. We consider the polynomial ring $S = k[x_1, \ldots, x_n]$ in $n$ variables over a field $k$. The edge ideal of $D$, denoted by $I(D)$, is the ideal of $S$ given by

$$I(D) = (x_i x_j^{w_j} | x_i x_j \in E).$$

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Edge ideals of weighted digraphs arose in the theory of Reed-Muller codes as initial ideals of vanishing ideals of projective spaces over finite fields \[21, 23\]. If a vertex \(x_i\) of \(D\) is a source (i.e., has only arrows leaving \(x_i\)) we shall always assume \(w_i = 1\) because in this case the definition of \(I(D)\) does not depend on the weight of \(x_i\). If \(w_j = 1\) for all \(j\), then \(I(D)\) is the edge ideal of its underlying graph.

Our motivation to study the regularity of powers of edge ideals springs from a famous result: for any homogeneous ideal \(I\) in a polynomial ring, it is well known that the regularity of \(I^t\) is asymptotically a linear function in \(t\), that is, there exist constants \(a\) and \(b\) such that for all \(t \gg 0\), \(\text{reg}(I^t) = at + b\) (see [11]). Generally, the problem of finding the exact linear form \(at + b\) and the smallest value \(t_0\) such that \(\text{reg}(I^t) = at + b\) for all \(t \geq t_0\) has proved to be very difficult. There are few classes of graphs for which \(a, b\) and \(t_0\) are explicitly computed (see [1, 2, 3, 4, 5, 22]).

Our objective in this paper is to find \(a, b\) and \(t_0\) in terms of combinatorial invariants of the vertex-weighted digraph \(D\) when \(D\) is a vertex-weighted oriented unicyclic graph. The digraph \(D = (V(D), E(D), w)\) is called an oriented unicyclic graph, denoted by \(D = C_m \cup (\bigcup_{j=1}^s T_j)\), if its underlying graph is \(G = G_0 \cup (\bigcup_{j=1}^s G_j)\), and \(C_m\) is an oriented cycle with underlying graph \(G_0\) and \(T_j\) is an oriented tree with underlying graph \(G_j\), its orientation is as follows: if \(V(G_0) \cap V(G_j) = \{x_{ij}\}\), then \(x_{ij}\) is the root of \(T_j\), and all edges in \(T_j\) are oriented away from \(x_{ij}\) for \(1 \leq j \leq s\). In [28], the first three authors derive some exact formulas for the regularity of edge ideals of vertex-weighted rooted forests and oriented cycles. In [29], we provide some exact formulas for the regularity of powers of edge ideals of vertex-weighted rooted forests. To the best of our knowledge, few papers consider the regularity of \(I(D)^t\) for a vertex-weighted digraph.

In this article, we are interested in algebraic properties corresponding to the regularity of \(I(D)^t\) for some vertex-weighted oriented graphs. By using the approaches of Betti splitting and polarization, we derive some exact formulas for the regularity of powers of edge ideals of some directed graphs. The results are as follows:

**Theorem 1.1.** Let \(C_n = (V(C_n), E(C_n), w)\) be a vertex-weighted oriented cycle with \(w(x) \geq 2\) for any \(x \in V(C_n)\), then for any \(t \geq 1\)

\[
\text{reg}(I(C_n)^t) = \sum_{x \in V(C_n)} w(x) - |E(C_n)| + 1 + (t - 1)(w + 1)
\]

where \(w = \max \{w(x) \mid x \in V(C_n)\}\).

**Theorem 1.2.** Let \(D = (V(D), E(D), w)\) be a vertex-weighted oriented unicyclic graph with \(w(x) \geq 2\) for any \(d(x) \neq 1\), then for any \(t \geq 1\)

\[
\text{reg}(I(D)^t) = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1)
\]

where \(w = \max \{w(x) \mid x \in V(D)\}\).

Our paper is organized as follows. In section 2, we recall some definitions and basic facts used in the paper. In section 3, we provide a special order on the set of minimal
monomial generators of powers of edge ideals of vertex-weighted oriented cycles. Using this order, we give exact formulas for the regularity of powers of edge ideals of vertex-weighted oriented cycles in section 4. Moreover, we give some examples to show regularity of powers of edge ideals of vertex-weighted oriented cycles is related to direction selection and the assumption that \( w(x) \geq 2 \) for any vertex \( x \) cannot be dropped. In section 5, we give some exact formulas for regularity of powers of edge ideals of vertex-weighted oriented unicyclic graphs. Moreover, we also give some examples to show regularity of powers of edge ideals of vertex-weighted oriented unicyclic graphs are related to direction selection and the assumption that \( w(x) \geq 2 \) if \( d(x) \neq 1 \) cannot be dropped.

For all unexplained terminology and additional information, we refer to [18] (for the theory of digraphs), [6] (for graph theory), and [9, 16] (for the theory of edge ideals of graphs and monomial ideals). We gratefully acknowledge the use of computer algebra system CoCoA ([10]) for our experiments.

Throughout this paper, if \( C_n = (V(C_n), E(C_n), w) \) be an \( n \)-cycle such that \( w(x) \geq 2 \) for any \( x \in V(C_n) \), we set \( x_j = x_i \) if \( j \equiv i \mod n \) \((1 \leq i \leq n)\). The oriented unicyclic graph \( D = C_m \cup (\bigcup_{j=1}^{s} T_j) \) satisfying if its underlying graph is \( G = G_0 \cup (\bigcup_{j=1}^{s} G_j) \), and \( C_m \) is an oriented cycle with underlying graph \( G_0 \) and \( T_j \) is an oriented tree with underlying graph \( G_j \), its orientation is as follows: if \( V(G_0) \cap V(G_j) = \{x_i\} \), then \( x_{ij} \) is the root of \( T_j \), and all edges in \( T_j \) are oriented away from \( x_{ij} \) for \( 1 \leq j \leq s. \)

2. Preliminaries

In this section, we gather together needed definitions and basic facts, which will be used throughout this paper. However, for more details, we refer the reader to [1, 6] [12] [16] [18] [21] [24] [26] [28].

Every concept that is valid for graphs automatically applies to digraphs too. For example, let \( D = (V(D), E(D)) \) be a digraph, the degree of a vertex \( x \) in the digraph \( D \), denoted \( d(x) \), is simply the degree of \( x \) in \( G(D) \). Likewise, a digraph is said to be connected if its underlying graph is connected. An oriented path or oriented cycle is an orientation of a path or cycle in which each vertex dominates its successor in the sequence. An oriented acyclic graph is a simple digraph without oriented cycles. An oriented tree or polytree is a oriented acyclic graph formed by orienting the edges of undirected acyclic graphs. A rooted tree is an oriented tree in which all edges are oriented either away from or towards the root. Unless specifically stated, a rooted tree in this article is an oriented tree in which all edges are oriented away from the root. An oriented forest is a disjoint union of oriented trees. A rooted forest is a disjoint union of rooted trees.

For any homogeneous ideal \( I \) of the polynomial ring \( S = k[x_1, \ldots, x_n] \), there exists a graded minimal finite free resolution
\[
0 \to \bigoplus_j S(-j)^{\beta_{p,j}(I)} \to \bigoplus_j S(-j)^{\beta_{p-1,j}(I)} \to \cdots \to \bigoplus_j S(-j)^{\beta_0,j(I)} \to I \to 0,
\]
where the maps are exact, \( p \leq n \), and \( S(-j) \) is an \( S \)-module obtained by shifting the degrees of \( S \) by \( j \). The number \( \beta_{i,j}(I) \), the \((i,j)\)-th graded Betti number of \( I \), is an invariant of \( I \) that equals the number of minimal generators of degree \( j \) in the \( i \)th syzygy module of \( I \). Of particular interest is the following invariant which measures the size of the minimal graded free resolution of \( I \). The regularity of \( I \), denoted \( \text{reg}(I) \), is defined by
\[
\text{reg}(I) := \max \{ j - i \mid \beta_{i,j}(I) \neq 0 \}.
\]

Let \( I \) be a monomial ideal, \( \mathcal{G}(I) \) denote the unique minimal set of monomial generators of \( I \). We now derive some formulas for \( \text{reg}(I) \) in some special cases by using some tools developed in [12].

**Definition 2.1.** Let \( I \) be a monomial ideal, and suppose that there exist monomial ideals \( J \) and \( K \) such that \( \mathcal{G}(I) \) is the disjoint union of \( \mathcal{G}(J) \) and \( \mathcal{G}(K) \). Then \( I = J + K \) is Betti splitting if
\[
\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \quad \text{for all } i, j \geq 0,
\]
where \( \beta_{i-1,j}(J \cap K) = 0 \) if \( i = 0 \).

In [12], the authors describe some sufficient conditions for an ideal \( I \) to have a Betti splitting. We need the following lemma.

**Lemma 2.2.** ([12, Corollary 2.7]). Suppose that \( I = J + K \) where \( \mathcal{G}(J) \) contains all the generators of \( I \) divisible by some variable \( x_i \) and \( \mathcal{G}(K) \) is a nonempty set containing the remaining generators of \( I \). If \( J \) has a linear resolution, then \( I = J + K \) is Betti splitting.

When \( I \) is a Betti splitting ideal, Definition 2.1 implies the following results:

**Corollary 2.3.** If \( I = J + K \) is a Betti splitting ideal, then
\[
\text{reg}(I) = \max \{ \text{reg}(J), \text{reg}(K), \text{reg}(J \cap K) - 1 \}.
\]

The following lemmas is often used in this article.

**Lemma 2.4.** ([15, Lemma 1.3]) Let \( S \) be a polynomial ring over a field and let \( I \) be a proper non-zero homogeneous ideal in \( S \). Then
\[
\text{reg}(I) = \text{reg}(S/I) + 1.
\]

Let \( u \in S \) be a monomial, we set \( \text{supp}(u) = \{ x_i : x_i | u \} \). If \( \mathcal{G}(I) = \{ u_1, \ldots, u_m \} \), we set \( \text{supp}(I) = \bigcup_{i=1}^m \text{supp}(u_i) \). The following lemma is well known.
Lemma 2.5. ([15] Lemma 2.5) Let $S_1 = k[x_1, \ldots, x_m]$ and $S_2 = k[x_{m+1}, \ldots, x_n]$ be two polynomial rings, $I \subseteq S_1$ and $J \subseteq S_2$ be two non-zero homogeneous ideals. Then
\[
\text{reg}(I + J) = \text{reg}(I) + \text{reg}(J) - 1.
\]

Lemma 2.6. Let $I, J = (u)$ be two monomial ideals such that $\text{supp}(u) \cap \text{supp}(I) = \emptyset$. If the degree of monomial $u$ is $d$. Then
\begin{enumerate}
  \item $\text{reg}(J) = d$,
  \item $\text{reg}(JI) = \text{reg}(I) + d$.
\end{enumerate}

Definition 2.7. Suppose that $u = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial in $S$. We define the polarization of $u$ to be the squarefree monomial
\[ P(u) = x_{11}x_{12} \cdots x_{1a_1}x_{21} \cdots x_{2a_2} \cdots x_{n1} \cdots x_{na_n} \]
in the polynomial ring $S^P = k[x_{ij} | 1 \leq i \leq n, 1 \leq j \leq a_i]$. If $I \subset S$ is a monomial ideal with $G(I) = \{u_1, \ldots, u_m\}$, the polarization of $I$, denoted by $I^P$, is defined as:
\[ I^P = (P(u_1), \ldots, P(u_m)), \]
which is a squarefree monomial ideal in the polynomial ring $S^P$.

A monomial ideal $I$ and its polarization $I^P$ share many homological and algebraic properties. The following is a very useful property of polarization.

Lemma 2.8. ([16] Corollary 1.6.3) Let $I \subset S$ be a monomial ideal and $I^P \subset S^P$ its polarization. Then
\begin{enumerate}
  \item $\beta_{ij}(I) = \beta_{ij}(I^P)$ for all $i$ and $j$,
  \item $\text{reg}(I) = \text{reg}(I^P)$.
\end{enumerate}

The following lemma can be used for computing the regularity of an ideal.

Lemma 2.9. ([15] Lemma 1.1 and Lemma 1.2) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated graded $S$-modules. Then
\begin{enumerate}
  \item $\text{reg}(C) \leq \max\{\text{reg}(A) - 1, \text{reg}(B)\}$,
  \item $\text{reg}(B) \leq \max\{\text{reg}(A), \text{reg}(C)\}$,
  \item $\text{reg}(B) = \text{reg}(A)$ if $\text{reg}(A) > \text{reg}(C) + 1$,
  \item $\text{reg}(B) = \text{reg}(C)$ if $\text{reg}(C) \geq \text{reg}(A)$,
  \item $\text{reg}(C) = \text{reg}(A) - 1$ if $\text{reg}(A) > \text{reg}(B)$,
\end{enumerate}

3. ORDERING THE MINIMAL GENERATORS OF POWERS OF EDGE IDEALS OF VERTEX-WEIGHTED ORIENTED CYCLES

In this section, we provide a special order on the unique minimal set of monomial generators of powers of edge ideals of vertex-weighted oriented cycles. Using this order, we will give some exact formulas for the regularity of powers of edge ideals of vertex-weighted oriented cycles in next section.
Throughout this section, let $C_n = (V(C_n), E(C_n), w)$ be an $n$-cycle such that $w(x) \geq 2$ for any $x \in V(C_n)$ and $V(C_n) = \{x_1, \ldots, x_n\}$. We define an order $L_1 > \cdots > L_n$ on the set $\mathcal{G}(I(C_n))$ where $L_i = x_{i-1}x_i^{w_i}$ for $1 \leq i \leq n$ and $x_j = x_i$ if $j \equiv i \mod n$ (1 $\leq i \leq n$). For any integer $t \geq 1$, we define an order on the set $\mathcal{G}(I(C_n)^t)$ as follows: We say $M > N$ for $M, N \in \mathcal{G}(I(C_n)^t)$ if $M = L_1^{a_1} \cdots L_n^{a_n}$, $N = L_1^{b_1} \cdots L_n^{b_n}$ such that $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = t$, we have $(a_1, \ldots, a_n) >_{\text{lex}} (b_1, \ldots, b_n)$.

We denote by $L^{(t)}$ the totally ordered set of $\mathcal{G}(I(C_n)^t)$ ordered in the way above and by $L_k^{(t)}$ the $k$-th element of the set $L^{(t)}$.

According to the order defined above, we can sort the set of generators of the following ideal.

**Example 3.1.** Let $I(C_3) = (x_3x_1^2, x_1x_2^2, x_2x_3^2)$ be the edge ideal of 3-cycle $C_3$. Then $L^{(2)} = \{(x_3x_1^2)^2, (x_3x_1^2)(x_1x_2^2), (x_3x_1^2)(x_2x_3^2), (x_1x_2^2)(x_2x_3^2), (x_2x_3^2)^2\}$.

We have the following fundamental fact.

**Theorem 3.2.** Let $t$ be a positive integer and $M \in \mathcal{G}(I(C_n)^t)$, then $M$ can be shown as $M = L_{i_1}^{a_1} \cdots L_{i_\ell}^{a_\ell}$ with $\sum_{p=1}^{\ell} a_{i_p} = t$, $a_{i_p} > 0$ for $1 \leq p \leq \ell$ and $1 \leq i_1 < \cdots < i_\ell \leq n$. Moreover, the expression of this form is unique.

**Proof.** It is clear that $M$ can be shown as $M = L_{i_1}^{a_1} \cdots L_{i_\ell}^{a_\ell}$ with $\sum_{p=1}^{\ell} a_{i_p} = t$, $a_{i_p} > 0$ for $1 \leq p \leq \ell$ and $1 \leq i_1 < \cdots < i_\ell \leq n$. Assume $L_{i_1}^{a_1} \cdots L_{i_\ell}^{a_\ell}$ and $L_{j_1}^{b_1} \cdots L_{j_m}^{b_m}$ are two expressions of $M$ with $\sum_{p=1}^{\ell} a_{i_p} = \sum_{q=1}^{m} b_{i_q} = t$, where $a_{i_p}, b_{j_q} > 0$ for any $1 \leq p \leq \ell$, $1 \leq q \leq m$ and $1 \leq i_1 < \cdots < i_\ell \leq n$, $1 \leq j_1 < \cdots < j_m \leq n$. We will show that $\ell = m$, $i_p = j_p$ and $a_{i_p} = b_{j_p}$ for $1 \leq p \leq \ell$. We use induction on $t$. Case $t = 1$ is clear. Now we assume $t \geq 2$.

Claim: $\{i_1, \ldots, i_\ell\} \cap \{j_1, \ldots, j_m\} = \emptyset$, thus we assume that $i_1 = j_1$. It follows that

$$L_{i_1}^{a_1-1} \cdots L_{i_\ell}^{a_\ell} = L_{j_1}^{b_1-1} \cdots L_{j_m}^{b_m}$$

Therefore, by induction hypothesis, we obtain that $\ell = m$, $i_p = j_p$ and $a_{i_p} = b_{j_p}$ for $1 \leq p \leq \ell$, as desired.

In fact, if $\{i_1, \ldots, i_\ell\} \cap \{j_1, \ldots, j_m\} = \emptyset$. For any $1 \leq p \leq \ell$, $L_{i_p}$ is a factor of monomial $L_{i_1}^{a_1} \cdots L_{i_\ell}^{a_\ell}$, thus it is also a factor of monomial $L_{j_1}^{b_1} \cdots L_{j_m}^{b_m}$. Hence there exists $1 \leq s \leq m$ such that $L_{j_s} = x_{i_p}^{w_{i_p+1}}$. By the expression of $L_{i_1}^{a_1} \cdots L_{i_\ell}^{a_\ell}$, we obtain

$$L_{j_1}^{b_1} \cdots L_{j_m}^{b_m} = (x_{i_1}^{w_{i_1+1}} x_{i_2}^{w_{i_2+1}} \cdots x_{i_\ell}^{w_{i_\ell+1}})^{a_1} a_{i_1}^{w_{i_1}} \cdots a_{i_\ell}^{w_{i_\ell}} M'$$


where $M'$ is a monomial. By comparing the degree of monomials $L_{i_1}^{a_1} \cdots L_{i_\ell}^{a_\ell}$ and $L_{j_1}^{b_1} \cdots L_{j_m}^{b_m}$, we get
\[
\sum_{p=1}^\ell a_{i_p}(1 + w_{i_p}) \geq \sum_{p=1}^\ell a_{i_p}w_{i_p}(1 + w_{i_p+1}).
\]
This implies $\sum_{p=1}^\ell a_{i_p}(1 - w_{i_p}w_{i_p+1}) \geq 0$, a contradiction. \hfill \Box

**Definition 3.3.** Let $1 \leq k < t$ be two integers, and $M_1 \in \mathcal{G}(I(C_n)^k)$, $M_2 \in \mathcal{G}(I(C_n)^t)$. We denoted by $M_1 \mid_{\text{edge}} M_2$ if there exists $M_3 \in \mathcal{G}(I(C_n)^{t-k})$ such that $M_2 = M_1 M_3$. Otherwise, we denoted by $M_1 \nmid_{\text{edge}} M_2$.

The following three results are needed.

**Lemma 3.4.** Let $L_i^{(2)}$ and $L_j^{(2)}$ be two edges such that $L_i^{(2)} > L_j^{(2)}$, then there exists $L_k^{(2)} \in L^{(2)}$ such that $L_i^{(2)} > L_k^{(2)}$ and $(L_k^{(2)} : L_i^{(2)})$ has one of the following two forms:
\begin{enumerate}
  \item $(L_k^{(2)} : L_i^{(2)}) = (L_{\ell_2} : L_{\ell_1})$, where $L_{\ell_1} > L_{\ell_2} \mid_{\text{edge}} L_k^{(2)}$ and $L_{\ell_1} \mid_{\text{edge}} L_i^{(2)}$;
  \item $(L_k^{(2)} : L_i^{(2)}) = (L_{n-2}L_n : L_1L_{n-1})$, where $L_n \mid_{\text{edge}} L_k^{(2)}$, $L_{n-2} \mid_{\text{edge}} L_i^{(2)}$.
\end{enumerate}
Furthermore, $(L_j^{(2)} : L_i^{(2)}) \subseteq (L_k^{(2)} : L_i^{(2)})$.

**Proof.** Set $L_j^{(2)} = L_{j_1}L_{j_2}$ with $1 \leq j_1 \leq j_2 \leq n$. If there exists some $1 \leq a \leq 2$ such that $L_{j_a} \mid_{\text{edge}} L_i^{(2)}$. For convenience, we assume $a = 1$, thus we get $(L_j^{(2)} : L_i^{(2)}) = (L_{j_2}^{(1)} : L_{i_2}^{(1)})$ and $L_{i_2}^{(1)} > L_{j_1}^{(1)}$, where $L_{i_2}^{(1)} = \frac{L_i^{(2)}}{L_{j_1}}$. Choose $k = j$, $\ell_2 = j_2$ and $\ell_1 = i'$, the result holds.

Otherwise, if $L_{j_a} \nmid_{\text{edge}} L_i^{(2)}$ for any $1 \leq a \leq 2$, then $j_1 \geq 2$. In this case, we consider the following two cases:
\begin{enumerate}
  \item If there exists some $1 \leq r \leq 2$ such that $(L_{j_r}^{(2)} : L_i^{(2)}) \subseteq (x_{j_r}^{w_{j_r}})$. Let $L_i^{(2)} = L_{i_1}L_{i_2}$ with $1 \leq i_1 \leq i_2 \leq n$. Choose $b = 2$ if $L_{i_2} > L_{j_r}$, otherwise $b = 1$, and $L_k^{(2)} = \frac{L_i^{(2)}}{L_{i_b}}L_{j_r}$, thus we get $L_{i_{b+1}} \geq L_{j_r}$, $L_i^{(2)} > L_k^{(2)}$ and $(L_k^{(2)} : L_i^{(2)}) = (L_{j_r} : L_{i_b})$. Notice that
    \[(L_j^{(2)} : L_i^{(2)}) = \left(\frac{L_j^{(2)}}{\text{gcd}(L_j^{(2)}, L_i^{(2)})}\right) \quad \text{and} \quad (L_{j_r} : L_{i_b}) = \left\{\begin{array}{ll}
    (x_{j_r}^{w_{j_r}}) & \text{if } L_{j_r} = L_{i_{b+1}}, \\
    (L_{j_r}) & \text{if } L_{i_{b+1}} > L_{j_r}.
\end{array}\right.
\]
If $L_{j_r} = L_{i_{b+1}}$, then the result is true. Otherwise, it is enough to show that $x_{j_r-1}$ is not a factor of $\text{gcd}(L_j^{(2)}, L_i^{(2)})$. Thus we obtain $L_{j_r}$ is a factor of generator $\frac{L_i^{(2)}}{\text{gcd}(L_j^{(2)}, L_i^{(2)})}$ of $(L_j^{(2)} : L_i^{(2)})$, the assertion follows from the formula above. In fact, if $x_{j_r-1}$ is a factor of $L_i^{(2)}$. By the expression of $L_j^{(2)}$ and the hypothesis that $L_{j_a} \nmid_{\text{edge}} L_i^{(2)}$ for any $a = 1, 2$, we obtain $L_{j_r-1} \nmid_{\text{edge}} L_i^{(2)}$. It follows $L_{j_r-1} = L_{i_b}$ by the definition of $b$, contradicting with the hypothesis $L_{i_{b+1}} > L_{j_r}$.\]
(ii) If \((L_j^{(2)} : L_i^{(2)}) \not\subseteq (x_{rj}^{w^j})\) for any \(1 \leq r \leq 2\), then \(x_{ij}\) is a factor of \(\gcd(L_j^{(2)}, L_i^{(2)})\) from the expression of \(L_j^{(2)}\) and the formula of \((L_j^{(2)} : L_i^{(2)})\). This implies \(L_{j+1} \mid_{\text{edge}} L_i^{(2)}\) by the hypotheses that \(L_{ja} \mid_{\text{edge}} L_i^{(2)}\) for any \(a = 1, 2\). Thus \(L_i^{(2)}\) has the form

\[
L_i^{(2)} = L_{j+1}L_{j+2}.
\]

It follows \(j_2 = n\) by the expression of \(L_j^{(2)}\) and \(L_i^{(2)} > L_j^{(2)}\). Claim: \(j_1 \neq n-1, n\).

In fact, if \(j_1 = n-1\), then \(L_n \mid_{\text{edge}} L_i^{(2)}\) and \(L_n \mid_{\text{edge}} L_j^{(2)}\), contradicting with the hypothesis \(L_{ja} \mid_{\text{edge}} L_i^{(2)}\) for any \(a = 1, 2\). If \(j_1 = n\), then \((L_j^{(2)} : L_i^{(2)}) \subseteq (x_{rj}^{w^j})\), contradicting with \((L_j^{(2)} : L_i^{(2)}) \not\subseteq (x_{rj}^{w^j})\) for any \(1 \leq r \leq 2\). Hence \(j_1 \leq n-2\), which implies \(n \geq 4\) because of \(j_1 \geq 2\). We consider the following two cases:

(i) If \(2 \leq j_1 < n-2\), then \(n \geq 5\) and \(j_1 + 1 < n-1\). Choose \(L_k^{(2)} = L_{j+1}L_n\), we obtain \((L_k^{(2)} : L_i^{(2)}) = (L_n : L_1) = (x_{n-1}x_{rj}^{w^j}-1), L_i^{(2)} > L_k^{(2)}\) and \((L_j^{(2)} : L_i^{(2)}) \subseteq (x_{n-1}x_{rj}^{w^j}-1), which implies \((L_j^{(2)} : L_i^{(2)}) \subseteq (L_k^{(2)} : L_i^{(2)}).

(ii) If \(j_1 = n-2\), then we choose \(k = j\). Thus \(L_i^{(2)} > L_k^{(2)}\) and \((L_j^{(2)} : L_i^{(2)}) = (L_k^{(2)} : L_i^{(2)}) = (L_n-2L_n : L_{n-1}L_1)\).

The next two theorems are the most important technical results of this section. They play vital roles in calculating the regularity of powers of edge ideals of vertex-weighted oriented cycles in the next section.

**Theorem 3.5.** Let \(t\) be a positive integer, \(\ell = \min \{t, \left\lceil \frac{n}{2} \right\rceil \} - 1\), where \(\left\lceil \frac{n}{2} \right\rceil\) denotes the largest integer \(\leq \frac{n}{2}\), and let \(L_i^{(t)} \subseteq L_j^{(t)} \subseteq L^{(t)}\) with \(L_i^{(t)} > L_j^{(t)}\), then there exists some \(k_i^{(t)} \subseteq L^{(t)}\) such that \(L_i^{(t)} > L_k^{(t)}\) and \((L_k^{(t)} : L_i^{(t)})\) has one of the following two forms:

1. \((L_k^{(t)} : L_i^{(t)}) = (L_\ell^t : L_\ell^t), where \(L_\ell^t > L_\ell^t, L_\ell^t \mid_{\text{edge}} L_k^{(t)}\) and \(L_\ell^t \mid_{\text{edge}} L_i^{(t)}\);

2. \((L_k^{(t)} : L_i^{(t)}) = (\prod_{s=0}^{q} L_{n-2s} : \prod_{s=0}^{q} L_{n+1-2s} for some \(q \leq \ell\), where \(L_{n-2s} \mid_{\text{edge}} L_k^{(t)}\),

\[
L_{n+1-2s} \mid_{\text{edge}} L_i^{(t)}\]

for any \(0 \leq s \leq q\), and \(n + 1 - 2s \equiv j \mod n\) for some \(0 < j \leq n\).

Furthermore, \((L_j^{(t)} : L_i^{(t)}) \subseteq (L_k^{(t)} : L_i^{(t)}).

Proof. We proceed by induction on \(t\). Case \(t = 1\) holds if we choose \(k = \ell_2 = j, \ell_1 = i\). Case \(t = 2\) holds from Lemma 3.4. Now suppose that \(t \geq 3\). Set \(L_j^{(t)} = L_{j_1} \cdots L_{j_t}\), with \(1 \leq j_1 \leq \cdots \leq j_t \leq n\). Similar to Lemma 3.4, we consider the following two cases:

(I) If there exists some \(1 \leq a \leq t\) such that \(L_{ja} \mid_{\text{edge}} L_i^{(t)}\), then

\[
(L_j^{(t)} : L_i^{(t)}) = (L_j^{(t-1)} : L_i^{(t-1)})
\]

where \(L_j^{(t-1)} = L_j^{(t)} \mid_{\text{edge}} L_i^{(t)}\) and \(L_j^{(t-1)} = L_j^{(t)} \mid_{\text{edge}} L_i^{(t)}\). By induction hypothesis, there exists some \(k'\) such that \(L_i^{(t-1)} > L_k^{(t-1)}\) and \((L_k^{(t-1)} : L_i^{(t-1)})\) is one of the following two forms:

(i) \((L_k^{(t-1)} : L_i^{(t-1)}) = (L_\ell^t : L_\ell^t), with L_\ell > L_\ell^t, L_\ell^t \mid_{\text{edge}} L_k^{(t-1)}\) and \(L_\ell^t \mid_{\text{edge}} L_i^{(t-1)}\).

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(ii) \((L^{(t-1)}_{k'} : L^{(t-1)}_{i'}) = (\prod_{s=0}^{q'} L_{n-2s} : \prod_{s=0}^{q'} L_{n+1-2s})\) for some \(q' \leq \ell'\), where 
\(\ell' = \min\{t - 1, \lceil \frac{n}{2} \rceil\} - 1, L_{n-2s} \mid_{\text{edge}} L^{(t-1)}_{k'}, L_{n+1-2s} \mid_{\text{edge}} L^{(t-1)}_{i'}\) for any \(0 \leq s \leq q'\) and 
\(n + 1 - 2s \equiv j \mod n\) for some \(0 < j \leq n\).

We choose \(L^{(t)}_{k} = L_{j_a} \cdot L^{(t-1)}_{k'}\), then \(L^{(t)}_{i} > L^{(t)}_{j}\) and \((L^{(t)}_{k} : L^{(t)}_{i}) = (L^{(t-1)}_{k'} : L^{(t-1)}_{i'})\), as desired.

(II) If \(L_{j_a} \mid_{\text{edge}} L^{(t)}_{i}\) for any \(1 \leq a \leq t\), then \(j_1 \geq 2\) because of \(L^{(t)}_{i} > L^{(t)}_{j}\). We consider the following two cases:

(i) If there exists some \(1 \leq r \leq t\) such that \((L^{(t)}_{j} : L^{(t)}_{i}) \subseteq (x_{j_r}^{w_{jr}})\). Set \(L^{(t)}_{i} = L_{i_1} \cdots L_{i_t}\) with \(1 \leq i_1 \leq \cdots \leq i_t \leq n\). We choose \(L_{i_b} = \min\{L_{i_b} \mid L_{i_b} > L_{j_r}\}\) and \(L_{i_b} \mid_{\text{edge}} L^{(t)}_{i}\) and \(L^{(t)}_{k} = \frac{L^{(t)}_{j}}{L_{i_b}} L_{j_r}\), thus \(L^{(t)}_{i} > L^{(t)}_{k}\) and \((L^{(t)}_{k} : L^{(t)}_{i}) = (L_{j_r} : L_{i_b})\). Similar arguments as Lemma 3.3, we have \((L^{(t)}_{j} : L^{(t)}_{i}) \subseteq (L_{j_r} : L_{i_b})\).

Hence the conclusion holds.

(ii) If \((L^{(t)}_{j} : L^{(t)}_{i}) \not\subseteq (x_{j_r}^{w_{jr}})\) for any \(1 \leq r \leq t\), then \(n \geq 4\) because of \(L_{j_a} \mid_{\text{edge}} L^{(t)}_{i}\) for any \(1 \leq a \leq t\), which implies \(\ell \geq 1\). Since \((L^{(t)}_{j} : L^{(t)}_{i}) = \left(\frac{L^{(t)}_{j}}{\gcd(L^{(t)}_{j}, L^{(t)}_{i})}\right)\), we have \(x_{j_r}\) is a factor of \(L^{(t)}_{i}\). This implies \(L_{j_{r+1}} \mid_{\text{edge}} L^{(t)}_{i}\) for \(1 \leq r \leq t\) by the hypothesis that \(L_{j_a} \mid_{\text{edge}} L^{(t)}_{i}\) for any \(1 \leq a \leq t\). Thus \(L^{(t)}_{i}\) has the form

\[
L^{(t)}_{i} = L_{j_{1+1}} \cdots L_{j_{t+1}}.
\]

This implies \(L_{j_{t+1}} = L_{1}\) by the expression of \(L^{(t)}_{j}\) and \(L^{(t)}_{i} > L^{(t)}_{j}\). It follows that \(L_{j_{t+1}} = L_{n}\), i.e., \(j = n\).

Choose \(q = \max\{q \mid L_{j_{t-q}} = L_{n-2q}, L_{n-2q} \mid_{\text{edge}} L^{(t)}_{j}\} \text{ for any } 0 \leq q \leq \ell\}\), thus \(L_{n-2s} \mid_{\text{edge}} L^{(t)}_{j}\) and \(L_{n+1-2s} \mid_{\text{edge}} L^{(t)}_{i}\) for any \(0 \leq s \leq q\). Next, we consider the following two cases:

(i) If \(q = t - 1\), then \(\ell = t - 1\). In this case, \(L^{(t)}_{j} = \prod_{s=0}^{t-1} L_{n-2s}\) and \(L^{(t)}_{i} = \prod_{s=0}^{t-1} L_{n-2s+1}\). Choose \(k = j\), as desired.

(ii) If \(q \leq t - 2\), then \(n \geq 2\ell + 2 \geq 2q + 2\) by the definition of \(q\) and \(\ell\), \(L^{(t)}_{j} = Q_1 \prod_{s=0}^{q} L_{n-2s}\) and \(Q_2 \prod_{s=0}^{q} L_{n-2s+1}\), where \(Q_1 = \frac{L^{(t)}_{j}}{\prod_{s=0}^{n-2s}} = L_{j_{1+1}} \cdots L_{j_{t-q-1+1}}\) and

\[
Q_2 = \frac{L^{(t)}_{i}}{\prod_{s=0}^{n+1-2s}} = L_{j_{1+1}} \cdots L_{j_{t-q+1+1}}. \text{ Choose } L^{(t)}_{k} = Q_2 \prod_{s=0}^{q} L_{n-2s}, \text{ then } L^{(t)}_{i} > L^{(t)}_{k},
\]

\((L^{(t)}_{k} : L^{(t)}_{i}) = (\prod_{s=0}^{q} L_{n-2s} : \prod_{s=0}^{q} L_{n+1-2s})\) and \(L_{n-2s} \mid_{\text{edge}} L^{(t)}_{k}, L_{n+1-2s} \mid_{\text{edge}} L^{(t)}_{i}\) for any \(0 \leq s \leq q\).

Next we prove \((L^{(t)}_{j} : L^{(t)}_{i}) \subseteq (L^{(t)}_{k} : L^{(t)}_{i})\). This is equivalent to prove \(uL^{(t)}_{i} \subseteq (L^{(t)}_{k} : L^{(t)}_{i})\) for any \(u \in (L^{(t)}_{j} : L^{(t)}_{i})\). It is enough to prove \(\prod_{s=0}^{q} L_{n-2s} \mid u \prod_{s=0}^{q} L_{n-2s+1}\) by the expression of \(L^{(t)}_{i}\) and \(L^{(t)}_{k}\). In fact, let \(u \in (L^{(t)}_{j} : L^{(t)}_{i})\), then \(uL^{(t)}_{i} \subseteq (L^{(t)}_{k} : L^{(t)}_{i})\).
Theorem 3.6. Let $t$ be a positive integer, $\ell = \min \{t, \lfloor \frac{n}{2} \rfloor \} - 1$, $L_i(t) = \{L_1(t), \ldots, L_r(t)\}$ a totally ordered set of all elements of $G(I(C_n)^t)$ such that $L_1(t) > \cdots > L_r(t)$. For any $1 \leq i \leq r$, we write $L_i(t)$ as $L_i(t) = L_{i_1}^{a_{i_1}} \cdots L_{i_k}^{a_{i_k}}$ with $1 \leq i_1 < \cdots < i_k \leq n$, $\sum_{j=1}^{k_i} a_{ij} = t$ and $a_{ij} > 0$ for $j = 1, \ldots, k_i$. For $1 \leq i \leq r - 1$, let $J_i(t) = (L_{i+1}(t), \ldots, L_r(t))$, $K_i = ((L_{i+1}, \ldots, L_n) : L_{i_1}) + \sum_{j=1}^{p_i} (L_{i+j} : L_{i_j})$, where if $i_1 = n$, then $p_i = k_i - 1$, otherwise, $p_i = k_i$.

1. If $i_1 = 1$, then $(J_i : L_i(t)) = K_i + Q_i$ and $Q_i = \sum_{j=0}^{q_j} (\prod_{s=0}^{q_j} L_{n-2s} : \prod_{s=0}^{q_j} L_{n+1-2s})$, where $q_i = \max \{q : L_{n+1-2q}^{\text{edge}} L_i(t) \text{ for any } 0 \leq q \leq \ell\}$;

2. If $i_1 \geq 2$, then $(J_i : L_i(t)) = K_i$.

Proof. It is obvious for $t = 1$. Now assume that $t \geq 2$. Set $M_j = \frac{L_j(t)}{L_{i_1}} L_j$ for any $1 \leq j \leq n$, $N_j = \frac{L_j(t)}{L_{i_1}} L_{i+j+1}$ for any $1 \leq j \leq p_i$, then $(M_{i_1+1}, \ldots, M_n, N_1, \ldots, N_{p_i}) \subseteq J_i$. Hence

$$K_i = ((L_{i_1+1}, \ldots, L_n) : L_{i_1}) + \sum_{j=1}^{p_i} (L_{i+j+1} : L_{i_j})$$

$$= ((M_{i_1+1}, \ldots, M_n) : L_i(t)) + ((N_1, \ldots, N_{p_i}) : L_i(t))$$

$$= ((M_{i_1+1}, \ldots, M_n, N_1, \ldots, N_{p_i}) : L_i(t)) \subseteq (J_i : L_i(t)).$$

We distinguish into the following two cases:

(i) If $i_1 \geq 2$, then $L_1 \mid_{\text{edge}} L_i(t)$. For any monomial $u \in G(J_i : L_i(t))$, then by Theorem 3.5, there exists $L_{i_1}, L_{i_2}, L_a(t) \in J_i$ for some $i + 1 \leq a \leq r$ such that $u \in (L_{i_2} : L_{i_1}), L_{i_1} > L_{i_2}, L_{i_1} \mid_{\text{edge}} L_a(t)$ and $L_{i_1} \mid_{\text{edge}} L_i(t)$, which implies $\ell_2 > \ell_1 > i_1$. Hence $(L_{i_2} : L_{i_1}) \subseteq K_i$, as desired.
(ii) If $i_1 = 1$, then $L_1 |_{edge} L_i^{(t)}$. By the definition of $q_i$, we get
\[ \prod_{s=0}^{j} L_{n+1-2s} \mid L_i^{(t)} \]
for any $0 \leq j \leq q_i$. Set $T_j = \frac{L_i^{(t)}}{\prod_{s=0}^{j} L_{n+1-2s}} \prod_{s=0}^{j} L_{n-2s}$, we obtain $L_i^{(t)} > T_j$. It follows that $T_j \in J_i$. Hence
\[ Q_i = \sum_{j=0}^{q_i} \left( \prod_{s=0}^{j} L_{n-2s} : \prod_{s=0}^{j} L_{n+1-2s} \right) = \sum_{j=0}^{q_i} (T_j : L_i^{(t)}) = ((T_0, \ldots, T_{q_i}) : L_i^{(t)}) \subseteq (J_i : L_i^{(t)}). \]
On the other hand, $(J_i : L_i^{(t)}) = ((L_{i+1}^{(t)}, \ldots, L_r^{(t)}) : L_i^{(t)}) = \sum_{j=i+1}^r (L_j^{(t)} : L_i^{(t)})$. If there exists some $q_i \in \{1, \ldots, \ell\}$, then $(J_i : L_i^{(t)}) \subseteq K_i + Q_i$. Otherwise, $(J_i : L_i^{(t)}) \subseteq K_i$. We complete the proof. \qed

4. Regularity of powers of edge ideals of vertex-weighted oriented cycles

In this section, we give exact formulas for the regularity of powers of edge ideals of vertex-weighted oriented cycles. Meanwhile, we also give some examples to show the regularity of powers of edge ideals of vertex-weighted oriented cycles is related to direction selection and the assumption that $w(x) \geq 2$ for any vertex $x$ cannot be dropped.

A hypergraph $H = (X, \mathcal{E})$ over the vertex set $X = \{x_1, \ldots, x_n\}$ consists of $X$ and a collection $\mathcal{E}$ of nonempty subsets of $X$, these subsets are called the edges of $H$. Let $Y \subseteq X$, the induced subhypergraph of $H$ on $Y$, denoted by $H[Y]$, is the hypergraph with the vertex set $Y$ and the edge set $\{E \in \mathcal{E} \mid E \subseteq Y\}$. A hypergraph $H$ is simple if there is no containment between any pair of its edges.

We need the following two lemmas.

**Lemma 4.1.** ([14, Lemma 3.1]) Let $H$ be a simple hypergraph. Then $\text{reg}(H') \leq \text{reg}(H)$ for any induced subhypergraph $H'$ of $H$.

**Lemma 4.2.** ([20, Proposition 4.1]) Let $I \subseteq S = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal satisfying every minimal generator of $I$ contains at least one variable not dividing any other generator of $I$. Then
\[ \text{reg}(I) = |X| - |\mathcal{G}(I)| + 1 \]
where $X = \text{supp}(I)$.

For convenience, all of notations used in the following two propositions and Theorem 4.5 are as those of Theorem 3.6.

**Proposition 4.3.** Let $L_i^{(t)}, L_i^{(t)}, J_i, K_i$ and $Q_i$ be as Theorem 3.6. For any $1 \leq i \leq r - 1$,

1. If $i_1 = 1$ and $q_i = 0$, then $\text{reg}((J_i : L_i^{(t)})) = \sum_{j=2}^{n} w_j - n + 1$;
(2) If $i_1 \geq 2$, then $\text{reg} \left( \left( J_i : L_i^{(t)} \right) \right) = \sum_{j=i_1+1}^{n} w_j - (n - i_1) + 1$.

Proof. (1) If $q_i = 0$, then $L_{n-1} \nmid L_i^{(t)}$. If $i_1 = 1$, then $i_{p_i} < n - 1$ by the definition of $p_i$. Thus

$$K_i = ((L_2, \ldots, L_n) : L_1) + \sum_{j=1}^{p_i} (L_{i_{j+1}} : L_{i_j})$$

$$= \left( x_2^{w_2} x_3^{w_3}, \ldots, x_{n-1}^{w_{n-1}} x_n \right) + \sum_{j=1}^{p_i} (x_{i_{j+1}}^{w_{i_{j+1}}}),$$

$$\left( J_i : L_i^{(t)} \right) = K_i + Q_i = K_i.$$

Let $K_i^P$ be the polarization of the ideal $K_i$, then $|\text{supp}(K_i^P)| = \sum_{j=2}^{n} w_j - 1$ and $|\mathcal{G}(K_i^P)| = n - 1$. Notice a fact that $x_j^{w_j}$ is only a factor of the unique monomial $x_{j-1,1} \prod_{k=1}^{w_j} x_{j,k}$ or $\prod_{k=1}^{w_j} x_{j,k}$ of the set $\mathcal{G}(K_i^P)$ for any $2 \leq j \leq n - 1$ and $x_{n,w_{n-1}}$ is also only a factor of the unique monomial $x_{n,j} \prod_{j=1}^{w_{n-1}} x_{n,j}$ of the set $\mathcal{G}(K_i^P)$. Hence by Lemma 2.8 (2) and Lemma 4.2, we obtain

$$\text{reg} \left( \left( J_i : L_i^{(t)} \right) \right) = \text{reg}(K_i) = \text{reg}(K_i^P) = |\text{supp}(K_i^P)| - |\mathcal{G}(K_i^P)| + 1$$

$$= \left( \sum_{j=2}^{n} w_j - 1 \right) - (n - 1) + 1 = \sum_{j=2}^{n} w_j - n + 1.$$

(2) If $i_1 \geq 2$, then by Theorem 3.6 (2), we obtain

$$\left( J_i : L_i^{(t)} \right) = K_i = ((L_{i_{1}+1}, \ldots, L_n) : L_{i_1}) + \sum_{j=1}^{p_i} (L_{i_{j+1}} : L_{i_j})$$

$$= (x_{i_{1}+1}^{w_{i_{1}+1}}, x_{i_{1}+1}^{w_{i_{1}+2}}, \ldots, x_{n-1}^{w_{n-1}} x_n) + \sum_{j=1}^{p_i} (x_{i_{j+1}}^{w_{i_{j+1}}}),$$

Let $K_i^P$ be the polarization of the ideal $K_i^P$, then $|\text{supp}(K_i^P)| = \sum_{j=i_{1}+1}^{n} w_j$ and $|\mathcal{G}(K_i^P)| = n - i_1$. Similar arguments as the proof of (1), we get

$$\text{reg} \left( \left( J_i : L_i^{(t)} \right) \right) = \sum_{j=i_{1}+1}^{n} w_j - (n - i_1) + 1.$$
Proposition 4.4. Let $L^{(t)}$, $L_i^{(t)}$, $J_i$, $K_i$ and $Q_i$ be as Theorem 3.6. For any $1 \leq i \leq r - 1$. If $i_1 = 1$ and $q_i \geq 1$, then

$$\text{reg}((J_i : L_i^{(t)})) \leq \sum_{j=2}^{n} w_j - n + 1.$$  

Proof. Since $i_1 = 1$ and $q_i \geq 1$, we have $L_j \mid_{edge} L_i^{(t)}$ for $j = 1, n - 1$. It follows that $i_p = n - 1$. Thus

$$K_i = (x_2^{w_2}, x_2x_3^{w_3}, \ldots, x_{n-1}x_n^{w_{n-1}}) + \sum_{j=1}^{p_i} (x_{i_j+1}^{w_{i_j+1}}), \quad (1)$$

$$Q_i = \sum_{j=0}^{q_i} (\prod_{s=0}^{j} L_{n-2s} : \prod_{s=0}^{j} L_{n+1-2s}) = (u_0, u_1, \ldots, u_{q_i}), \quad (2)$$

where monomial $u_j = \frac{\prod_{s=0}^{j} L_{n-2s}}{\gcd(\prod_{s=0}^{j} L_{n-2s}, \prod_{s=0}^{j} L_{n+1-2s})}$ for $0 \leq j \leq q_i$.

Let

$$T_j = K_i + (u_0, u_1, \ldots, u_j) \text{ for any } 0 \leq j \leq q_i,$$

then $(J_i : L_i^{(t)}) = K_i + Q_i = T_{q_i}$. For $0 \leq j \leq q_i$, we will prove

$$\text{reg} (T_j) \leq \sum_{j=2}^{n} w_j - n + 1, \quad (4)$$

thus the result follows.

Now we prove formulas (4) by induction on $j$.

If $j = 0$, then

$$T_0 = K_i + (u_0) = K_i' + (x_n^{w_n}),$$

where $K_i' = (x_2^{w_2}, x_2x_3^{w_3}, \ldots, x_{n-1}x_n^{w_{n-1}}) + \sum_{j=1}^{p_i-1} (x_{i_j+1}^{w_{i_j+1}})$.

Let $K_i'^P$ be the polarization of the ideal $K_i'$, then $|\text{supp}(K_i'^P)| = \sum_{j=2}^{n} w_j - 1$ and $|\mathcal{G}(K_i'^P)| = n - 1$. Since $x_j, w_j$ is only a factor of the unique monomial $x_{j-1,1} \prod_{k=1}^{w_j} x_{j,k}$ or $\prod_{k=1}^{w_j} x_{j,k}$ of the set $\mathcal{G}(K_i'^P)$ for any $2 \leq j \leq n - 1$ and $x_{n, w_{n-1}}$ is also only a factor of the unique monomial $x_{n-1,1} \prod_{j=1}^{w_{n-1}-1} x_{n,j}$ of the set $\mathcal{G}(K_i'^P)$, we obtain by Lemma 2.8 (2) and Lemma 4.2

$$\text{reg} (K_i') = \text{reg} (K_i'^P) = |\text{supp}(K_i'^P)| - |\mathcal{G}(K_i'^P)| + 1 = \sum_{j=2}^{n} w_j - n + 1. \quad (5)$$
Notice that \((K'_i : x_n^{w_n}) = P_i + (x_n-1)\), where \(P_i = (x_2^{w_2}, x_2x_3^{w_3}, \ldots, x_{n-3}x_{n-2}^{w_{n-2}}) + \sum_{j=1}^{p'_i}(x_{ij+1}^{w_{ij+1}})\), where if \(i_{p-1} = n - 2\), then \(p'_i = p_i - 2\) otherwise, \(p'_i = p_i - 1\).

Let \(P_i^P\) be the polarization of the ideal \(P_i\), then \(|\text{supp}(P_i^P)| = \sum_{j=2}^{n-2} w_j\) and \(|\mathcal{G}(P_i^P)| = n - 3\). Similar arguments as above, we have

\[
\text{reg}((K'_i : x_n^{w_n})(-w_n)) = \text{reg}((P_i + (x_n-1)) + w_n = \text{reg}(P_i) + w_n = \text{reg}(P_i^P) + w_n
\]

\[
= \sum_{j=2}^{n-2} w_j - (n - 3) + 1 + w_n \leq \sum_{j=2}^{n} w_j - n + 2
\]

(6)

where the inequality holds because of \(w_{n-1} \geq 2\).

Using formulas (5) and (6), Lemma 2.4 and Lemma 2.9 (1) on the short exact sequence

\[
0 \to \frac{S}{(K'_i : x_n^{w_n})}(-w_n) \to \frac{S}{K'_i} \to \frac{S}{T_0} \to 0,
\]

we have

\[
\text{reg}(T_0) \leq \sum_{j=2}^{n} w_j - n + 1.
\]

Suppose the formulas (4) is true for any \(1 \leq j \leq q_i - 1\). Now assume \(j = q_i\). We first compute \((T_{q_i-1} : u_{q_i})\). Since \(K_i = ((L_{i+1}, \ldots, L_n) : L_{i+1}) + \sum_{j=1}^{p_i}(L_{i+1} : L_{i+1})\) and \(T_{q_i-1} = K_i + (u_0, u_1, \ldots, u_{q_i-1})\), we obtain by simple calculation

\[
((\sum_{j=0}^{q_i-1} u_j) : u_{q_i}) = \sum_{j=0}^{q_i-1} (u_j : u_{q_i}) = \sum_{j=0}^{q_i-1} (\prod_{s=0}^{j} L_n-2s : \prod_{s=0}^{j} L_{n+1-2s} : u_{q_i})
\]

\[
= \sum_{j=0}^{q_i-1} (x_{n-2q_i+1}, x_{n-2q_i+3}, \ldots, x_{n-1}),
\]

(7)

\[
((\sum_{j=1}^{p_i}(L_{i+1} : L_{i+1})) : u_{q_i}) = \sum_{j=1}^{p_i}((L_{i+1} : L_{i+1}) : u_{q_i}) = \sum_{j=1}^{p_i} (x_{ij+1}^{w_{ij+1}} : u_{q_i})
\]

\[
= \begin{cases} 
(x_2) + \sum_{j=0}^{q_i-1} (x_{n-2s}), & \text{if } n \text{ is even and } q_i = \left\lceil \frac{n}{2} \right\rceil - 1, \\
\sum_{j=1}^{p_i} (x_{ij+1}^{w_{ij+1}}) + \sum_{j=0}^{q_i-1} (x_{n-2s}), & \text{otherwise,}
\end{cases}
\]

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\[(T_{q_i-1} : u_{q_i}) = ((K_i + (u_0, u_1 \ldots, u_{q_i-1})) : u_{q_i})\]

\[= ((\sum_{j=i_1+1}^{n} L_j : L_{i_1}) : u_{q_i}) + \sum_{j=1}^{p_i}(L_{ij+1} : L_{i_1} : u_{q_i}) + (\sum_{j=0}^{q_i-1} u_j : u_{q_i})\]

\[= \sum_{j=i_1+1}^{n} ((L_j : L_{i_1}) : u_{q_i}) + \sum_{j=1}^{p_i}(L_{ij+1} : L_{i_1} : u_{q_i}) + (\sum_{j=0}^{q_i-1} u_j : u_{q_i})\]

\[
\begin{align*}
(A + (x_2)) + \sum_{j=3}^{n} (x_j), & \quad \text{if } n \text{ is even, } q_i = \lfloor \frac{n}{2} \rfloor - 1 \\
(A + (x_2^{w_2-1}, x_3)) + \sum_{j=4}^{n} (x_j), & \quad \text{if } n \text{ is odd, } q_i = \lfloor \frac{n}{2} \rfloor - 1 \\
(A + B + (x_{n-2q_1})) + \sum_{j=1}^{p_i}(x_{ij+1}^{w_{ij+1}}) + \sum_{j=n-2q_1+1}^{n} (x_j), & \quad \text{otherwise,}
\end{align*}
\]

where \(A = \sum_{j=2}^{q_i}(x_{n-2j+1}^{w_{n-2j+1}}, x_{n-2j+1}x_{n-2j+2}) + (x_{n-1}), B = (x_2^{w_2}, x_{n-2q_1-2}x_{n-2q_1-1}^{-1}) + \sum_{j=2}^{n-2q_1+1} (x_jx_{j+1}^{w_{j+1}}) \) and \(p_i'' = \max\{p_i' : 1 \leq p_i' \leq p_i \text{ and } i_{p_i'} \leq n - 2q_i - 2\}.\)

Next we compute \(\operatorname{reg}((T_{q_i-1} : u_{q_i})).\) Let \(d\) be the degree of monomial \(u_{q_i}.\) If \(q_i = \lfloor \frac{n}{2} \rfloor - 1\) and \(n\) is even, then \(d = \sum_{j=1}^{n-2} w_{2j} - \frac{n}{2},\) otherwise, \(d = \sum_{j=0}^{q_i} w_{n-2j} - q_i.\) We distinguish into the following three cases:

(i) If \(n = 2m\) and \(q_i = m - 1,\) then by Lemma 2.5

\[
\operatorname{reg}((T_{q_i-1} : u_{q_i})(-d)) = \operatorname{reg}((T_{q_i-1} : u_{q_i})) + d = \operatorname{reg}(\sum_{j=2}^{n} (x_j)) + d
\]

\[
= 1 + (\sum_{j=1}^{m} w_{2j} - m) = (\sum_{j=2}^{n} w_j - n + 1) + (m - \sum_{j=2}^{m} w_{2j-1})
\]

\[
\leq \sum_{j=2}^{n} w_j - n + 1.
\]
(ii) If \( n = 2m + 1 \) and \( q_i = m - 1 \), then by Lemma 2.5 (1),

\[
\text{reg} \left( \left( T_{q_i-1} : u_{q_i} \right)(-d) \right) = \text{reg} \left( \left( T_{q_i-1} : u_{q_i} \right) \right) + d = \text{reg} \left( \left( x_2^{w_2} - 1 \right) + \sum_{j=3}^{n} (x_j) \right) + d
\]

\[
= \sum_{j=0}^{m-1} w_{n-2j} - (m - 1) + w_2 - 1
\]

\[
= \sum_{j=2}^{n} w_j - n + 1 + (m - \sum_{j=1}^{m-1} w_{n-2j+1})
\]

\[
\leq \sum_{j=2}^{n} w_j - n + 1.
\]

(iii) In other cases, by Lemma 2.5, we have

\[
\text{reg} \left( \left( T_{q_i-1} : u_{q_i} \right)(-d) \right) = \text{reg} \left( \left( T_{q_i-1} : u_{q_i} \right) \right) + d
\]

\[
= \text{reg} \left( B + \sum_{j=1}^{p_i''} (x_{i+j+1}^{w_i+1}) + \sum_{j=n-2q_i}^{n} (x_j) \right) + d = \text{reg} \left( B + \sum_{j=1}^{p_i''} (x_{i+j+1}^{w_i+1}) \right) + d
\]

\[
\leq \sum_{j=2}^{n-2q_i-1} w_j - (n - 2q_i - 1) + 1 + \sum_{j=0}^{q_i} w_{n-2j} - q_i
\]

\[
= \sum_{j=2}^{n} w_j - n + 1 + (q_i + 1 - \sum_{j=1}^{q_i} w_{n-2j+1}) \leq \sum_{j=2}^{n} w_j - n + 1,
\]

where the first inequality holds because of \( \text{reg} \left( B + \sum_{j=1}^{p_i''} (x_{i+j+1}^{w_i+1}) \right) \leq \sum_{j=2}^{n-2q_i-1} w_j - (n - 2q_i - 1) + 1 \) by similar arguments as the calculation of \( \text{reg} \left( T_0 \right) \).

Using the above formulas of \( \text{reg} \left( \left( T_{q_i-1} : u_{q_i} \right)(-d) \right) \), Lemma 2.4, Lemma 2.9 (1) and the induction hypothesis on the short exact sequence

\[
0 \rightarrow S \rightarrow (T_{q_i-1} : u_{q_i})(-d) \rightarrow S \rightarrow \frac{S}{T_{q_i-1}} \rightarrow 0,
\]

we have

\[
\text{reg} \left( \left( J_i : L_i^{(t)} \right) \right) = \text{reg} \left( T_{q_i} \right) \leq \sum_{j=2}^{n} w_j - n + 1.
\]

The proof is complete. \( \Box \)

The following Theorem is main result in this section.

**Theorem 4.5.** Let \( C_n = (V(C_n), E(C_n), w) \) be a vertex-weighted oriented cycle, \( I(C_n) = (L_1, \ldots, L_n) \) an edge ideal of \( C_n \), where \( L_i = x_{i-1} x_i^{w_i} \) and \( w_i \geq 2 \) for
1 ≤ i ≤ n. Then
\[
\text{reg}(I(C_n)^t) = \sum_{x \in V(C_n)} w(x) - |E(C_n)| + 1 + (t - 1)(w + 1) \quad \text{for any } t \geq 1,
\]
where \( w = \max \{ w_i \mid 1 \leq i \leq n \} \).

Proof. Case \( t = 1 \) follows from [28, Theorem 4.1]. Now assume \( t \geq 2 \). Let \( w = w_1 \) without loss of generality and \( L^{(t)} = \{ L_1^{(t)}, \ldots, L_r^{(t)} \} \) a totally ordered set of all elements of \( G(I(C_n)^t) \) such that \( L_1^{(t)} > \cdots > L_r^{(t)} \). For \( 1 \leq i \leq r \), we write \( L_i^{(t)} \) as \( L_i^{(t)} = L_{i_1}^{a_{i_1}} \cdots L_{i_{k_i}}^{a_{k_i}} \) with \( 1 \leq i_1 < \cdots < i_{k_i} \leq n \), \( \sum_{j=1}^{k_i} a_{i_j} = t \) and \( a_{i_j} > 0 \) for \( j = 1, \ldots, k_i \). Let \( d_i \) be the degree of monomial \( L_i^{(t)} \), then we get \( d_i \leq (w_i + 1) + (t - 1)(w + 1) \) for \( 1 \leq i \leq r - 1 \) by the definition of \( w \). We prove this argument in the following two steps.

Step 1: We first show \( \text{reg}(I(C_n)^t) \leq \sum_{j=1}^{n} w_j - n + 1 + (t - 1)(w + 1) \). Let \( J_i = (L_{i+1}^{(t)}, \ldots, L_r^{(t)}) \) for \( 1 \leq i \leq r - 1 \). Since \( J_{r-1} = (L_r^{(t)}) = (x_{n-1}^t x_n^{tw_n}) \), we get
\[
\text{reg}(J_{r-1}) = t(w_n + 1) = (\sum_{j=1}^{n} w_j - n + 1 + (t - 1)(w_n + 1)) + (n - \sum_{j=1}^{n-1} w_j)
\]
\[
\leq \sum_{j=1}^{n} w_j - n + 1 + (t - 1)(w + 1),
\]
where the inequality above holds because of \( w_n \leq w \) and \( w_j \geq 2 \) for \( 1 \leq j \leq n - 1 \).

By Proposition [4.3] and Proposition [4.4], we have, for any \( 1 \leq i \leq r - 1 \),
\[
\text{reg}((J_i : L_i^{(t)})) \leq \begin{cases} 
\sum_{j=2}^{n} w_j - n + 1 & \text{if } i_1 = 1, \\
\sum_{j=i_1+1}^{n} w_j - (n - i_1) + 1 & \text{if } i_1 \geq 2,
\end{cases}
\]
\[
\leq \begin{cases} 
\sum_{j=1}^{n} w_j - n + 1 - w_1 & \text{if } i_1 = 1, \\
\sum_{j=1}^{n} w_j - n + 1 - w_{i_1} & \text{if } i_1 \geq 2,
\end{cases}
\]
where the last inequality holds because of \( w_j \geq 2 \) for \( 1 \leq j \leq n \). It follows that
\[
\text{reg}((J_i : L_i^{(t)})(-d_i)) = \text{reg}((J_i : L_i^{(t)})) + d_i
\]
\[
\leq \left( \sum_{j=1}^{n} w_j - n + 1 - w_{i_1} \right) + \left( (w_{i_1} + 1) + (t - 1)(w + 1) \right)
\]
\[
= \sum_{j=1}^{n} w_j - n + 1 + (t - 1)(w + 1) + 1.
\]
Using the formulas (1) and (2), Lemma 2.4 and Lemma 2.9 (1) on the following short exact sequences

\[
0 \rightarrow \frac{S}{(J_1:L_1^{(t)})}(-d_1) \xrightarrow{\cdot L_1^{(t)}} \frac{S}{J_1} \rightarrow \frac{S}{I(C_n)^P} \rightarrow 0
\]

\[
0 \rightarrow \frac{S}{(J_2:L_2^{(t)})}(-d_2) \xrightarrow{\cdot L_2^{(t)}} \frac{S}{J_2} \rightarrow \frac{S}{J_1} \rightarrow 0
\]

\[\vdots\]

\[
0 \rightarrow \frac{S}{(J_{r-1}:L_{r-1}^{(t)})}(-d_r) \xrightarrow{\cdot L_{r-1}^{(t)}} \frac{S}{J_{r-1}} \rightarrow \frac{S}{J_{r-2}} \rightarrow 0,
\]

we obtain

\[
\operatorname{reg}(I(C_n)^t) = \sum_{j=1}^n w_j - n + 1 + (t - 1)(w + 1). \tag{3}
\]

Step 2: We show \(\operatorname{reg}(I(C_n)^t) = \sum_{j=1}^n w_j - n + 1 + (t - 1)(w + 1).\)

We write \(I(C_n)^t\) as \(I(C_n)^t = J + K\) with \(\mathcal{G}(I(C_n)^t) = \mathcal{G}(J) \sqcup \mathcal{G}(K)\) and \(K = (L_1^{(t)})\). Let \(J^P\), \(K^P\) and \((I(C_n)^t)^P\) be the polarization of \(J\), \(K\) and \((I(C_n)^t)\) respectively, then \(K = (x_n^t x_1^{w_1})\) and

\[
(I(C_n)^t)^P = J^P + K^P \quad \text{and} \quad J^P \cap K^P = K^P L,
\]

where \(L = (\prod_{j=1}^{w_2} x_{2j}, \prod_{j=1}^{w_3} x_{3j}, \ldots, x_{n-1,1}, \prod_{j=t+1}^{t-1+w_n} x_{n,j})\). Then \(|\text{supp}(L)| = \sum_{j=2}^{n} w_j - 1\) and \(|\mathcal{G}(L)| = n - 1\). We distinguish into the following two steps:

Step (i): We first compute \(\operatorname{reg}(J^P \cap K^P)\).

Since \(x_{2,j_2}^t\) (resp. \(x_{n,t-1+w_n}\)) is only a factor of the unique monomial \(\prod_{j=1}^{w_2} x_{2j}\) (resp. \(x_{n-1,1}^{t-1+w_n} \prod_{j=t+1}^{t-1+w_n} x_{n,j}\)) of the set \(\mathcal{G}(L)\) and \(x_{j,w_j}\) is also only a factor of the unique monomial \(x_{j-1,1} \prod_{k=1}^{w_j} x_{j,k}\) of the set \(\mathcal{G}(L)\) for any \(3 \leq j \leq n - 1\) and the variables that appear in \(K^P\) and \(L\) are different, thus by Lemma 2.6 (2) and Lemma 11.2, we obtain

\[
\operatorname{reg}(J^P \cap K^P) = \operatorname{reg}(K^P L) = \operatorname{reg}(K^P) + \operatorname{reg}(L) = t(w + 1) + (|\text{supp}(L)| - |\mathcal{G}(L)| + 1)
\]

\[
= t(w + 1) + \left(\sum_{j=2}^{n} w_j - 1 - (n - 1) + 1\right)
\]

\[
= \sum_{j=1}^{n} w_j - n + 1 + (t - 1)(w + 1) + 1. \tag{4}
\]

Step (ii): We compute \(\operatorname{reg}(J^P)\).
Let \( H = (V(H), \mathcal{E}(H)) \) and \( H' = (V(H'), \mathcal{E}(H')) \) are hypergraphs associated to \( \mathcal{G}((I(C_n)^t)^P) \) and \( \mathcal{G}(J^P) \) respectively, then \( H' \) is an induced subhypergraph of \( H \). In fact, \( H' \) is a subhypergraph of \( H \) by the choice of \( \mathcal{G}((I(C_n)^t)^P) \) and \( \mathcal{G}(J^P) \). On the other hand, if \( E \in \mathcal{E}(H) \) with \( E \subseteq V(H') \), then monomial \( \prod_{x_{ij} \in E} x_{ij} \) associated to \( E \) belong to \( \mathcal{G}((I(C_n)^t)^P) \). Since \( \mathcal{G}((I(C_n)^t)^P) = \mathcal{G}(K^P) \cup \mathcal{G}(J^P) \), if \( \prod_{x_{ij} \in E} x_{ij} \in \mathcal{G}(K^P) \), then \( x_{1, tw_1} \in E \) by definition of \( \mathcal{G}(K^P) \), contradicting with \( x_{1, tw_1} \notin V(H') \). Thus \( \prod_{x_{ij} \in E} x_{ij} \in \mathcal{G}(J^P) \). Hence \( H' \) is an induced subhypergraph of \( H \). By Lemma 2.8 (2), Lemma 4.1 and the formula (3), we get

\[
\text{reg} (J^P) \leq \text{reg} ((I(C_n)^t)^P) = \text{reg} ((I(C_n)^t)) \leq \sum_{j=1}^{n} w_j - n + 1 + (t-1)(w+1). \tag{5}
\]

Let \( \alpha = \text{reg} (J^P \cap K^P) - 1 \) and \( \beta = \text{reg} (K^P) = t(w+1) \), then

\[
\alpha - \beta = \left( \sum_{j=1}^{n} w_j - n + 1 + (t-1)(w+1) \right) - t(w+1) = \sum_{j=2}^{n} w_j - n \geq 0, \tag{6}
\]

where the inequality holds because of \( w_j \geq 2 \) for \( 2 \leq j \leq n \).

Since the variable \( x_{1, tw_1} \) in \( \text{supp}(K^P) \) can not divided generators of \( J^P \) and \( K^P \) has a linear resolution. By Lemma 2.2 it follows that \( (I(C_n)^t)^P = J^P + K^P \) is Betti splitting. By Corollary 2.3 formulas (4), (5) and (6), we obtain

\[
\text{reg} ((I(C_n)^t)) = \text{reg} ((I(C_n)^t)^P) = \max \{ \text{reg} (J^P), \text{reg} (K^P), \text{reg} (J^P \cap K^P) - 1 \}
\]

\[
= \max \{ \text{reg} (J^P), t(w+1), \left( \sum_{j=1}^{n} w_j - n + 1 + (t-1)(w+1) + 1 \right) - 1 \}
\]

\[
= \sum_{j=1}^{n} w_j - n + 1 + (t-1)(w+1).
\]

This proof is completed. \( \square \)

As a consequence of Theorem 4.5 we have

**Corollary 4.6.** Let \( C_n = (V(C_n), E(C_n), w) \) be a vertex-weighted oriented cycle as in Theorem 4.5. Then

\[
\text{reg} (I(C_n)^t) = \text{reg} (I(C_n)) + (t-1)(w+1) \quad \text{for any } t \geq 1,
\]

where \( w = \max \{ w(x) \mid x \in V(C_n) \} \).

The following example shows the assumption that \( w(x) \geq 2 \) for any \( x \in V(C_n) \) in Theorem 4.5 cannot be dropped.

**Example 4.7.** Let \( I(C_5) = (x_5x_1, x_1x_2^3, x_2x_3^3, x_3x_4, x_4x_5^3) \) be an edge ideal of the vertex-weighted oriented cycle \( C_5 = (V, E, w) \), its weight function is \( w_2 = w_3 = \)
$w_5 = 3$, $w_1 = w_4 = 1$. Thus $w = 3$. By using CoCoA, we obtain $\text{reg}(I(C_5)^2) = 10$.

But we have $\text{reg}(I(C_5)^2) = \sum_{i=1}^{5} w_i - |E(C_5)| + 1 + w + 1 = 11$ by Theorem 4.5.

The following example shows that the regularity of powers of edge ideals of vertex-weighted oriented cycles as Theorem 4.5 is related to direction selection.

**Example 4.8.** Let $I(C_5) = (x_1x_2^3, x_1x_2^3, x_2x_3^3, x_3x_4^3, x_4x_5^3)$ be an edge ideal of the vertex-weighted oriented cycle $C_5 = (V, E, w)$ with $w_2 = w_3 = w_4 = w_5 = 3$, $w_1 = 1$. Thus $w = 3$. By using CoCoA, we obtain $\text{reg}(I(C_5)^2) = 14$. But we have $\text{reg}(I(C_5)^2) = \sum_{i=1}^{5} w_i - |E(C_5)| + 1 + w + 1 = 13$ by Theorem 4.5.

5. **Regularity of powers of edge ideals of vertex-weighted unicyclic graphs**

In this section, we consider a vertex-weighted oriented unicyclic graph $D = (V(D), E(D), w)$ satisfying its underlying graph $G$ is the union of a circle and some forests. We will provide the exact formulas for the regularity of powers of its edge ideal. We also give some examples to show the regularity of powers of edge ideals of vertex-weighted oriented unicyclic graphs is related to direction selection and the assumption that $w(x) \geq 2$ if $d(x) \neq 1$ cannot be dropped.

**Definition 5.1.** Let $G_i = (V_i, E_i)$ be some simple graphs for $1 \leq i \leq s$, their union is a graph $G = (V, E)$, denoted by $\bigcup_{i=1}^{s} G_i$, satisfying its vertex set is $V = \bigcup_{i=1}^{s} V_i$ and its edge set is $E = \bigcup_{i=1}^{s} E_i$.

**Definition 5.2.** Let $G = (V(G), E(G))$ be a unicyclic graph with $n$ vertices. We write $G$ as $G = G_0 \cup \bigcup_{j=1}^{s} G_j$, where $G_0$ is an $m$-cycle and $G_j$ is a tree for $1 \leq j \leq s$.

The digraph $D = (V(D), E(D), w)$ is called an oriented unicyclic graph, denoted by $D = C_m \cup \bigcup_{j=1}^{s} T_j$, if its underlying graph is $G$, and $C_m$ is an oriented cycle with underlying graph $G_0$ and $T_j$ is an oriented tree with underlying graph $G_j$, its orientation is as follows: if $V(G_0) \cap V(G_j) = \{x_i\}$, then $x_i$ is the root of $T_j$, and all edges in $T_j$ are oriented away from $x_i$ for $1 \leq j \leq s$.

Throughout this section, let $D = (V(D), E(D), w)$ be a vertex-weighted oriented unicyclic graph with vertex set $V(D) = \{x_1, \ldots, x_n\}$, where $C_m$ is the unique oriented cycle in $D$, its vertex set $V(C_m) = \{x_1, \ldots, x_m\}$, its edge set $E(C_m) = \{x_1x_2^{w_1}, \ldots, x_{m-1}x_1^{w_{m-1}}\}$. The orientation of $D$ defined as above and the weight $w(x_i) \geq 2$ of $x_i$ if $d(x_i) \neq 1$ for $1 \leq i \leq n$.

Let $D = (V(D), E(D), w)$ be a vertex-weighted oriented graph. For $T \subset V(D)$, we define the induced vertex-weighted subgraph $H = (V(H), E(H), w)$ of $D$ to be a
vertex-weighted oriented graph with \( V(H) = T \), for any \( u, v \in V(H) \), \( uv \in E(H) \) if and only if \( uv \in E(D) \) and its orientation in \( H \) is the same as in \( D \). For any \( u \in V(H) \) and \( u \) is not a source in \( H \), its weight in \( H \) equals to the weight of \( u \) in \( D \), otherwise, its weight in \( H \) equals to 1. For \( P \subseteq V(D) \), we denote \( D \setminus P \) the induced subgraph of \( D \) obtained by removing the vertices in \( P \) and the edges incident to these vertices. If \( P = \{ x \} \) consists of a element, then we write \( D \setminus \{ x \} \) for \( D \setminus P \). If \( x \in V(D) \), then we denote by \( N_D^+(x) = \{ y : (x, y) \in E(D) \} \), \( N_D^-(x) = \{ y : (y, x) \in E(D) \} \) and \( N_D(x) = N_D^+(x) \cup N_D^-(x) \).

We need the following two lemmas, see for instance [29, Lemma 3.4, Lemma 3.6, Lemma 3.6 and Theorem 4.2].

**Lemma 5.3.** Let \( t \geq 2 \) be a positive integer and \( D = (V(D), E(D), w) \) a vertex-weighted oriented graph, let \( z \) be a leaf with \( N_D^-(z) = \{ y \} \). Then,

1. \( (I(D)^t)^t, z^{uw} = (I(D \setminus z)^t, z^{uw}) \),
2. \( (I(D)^t)^t : y^{uw} = (I(D)^t)^t, y^{uw} \),
3. \( (I(D)^t : z^{uw}), y = (I(D \setminus y)^t : z^{uw}), y = (I(D \setminus y)^t, y) \).

**Lemma 5.4.** Let \( D = (V(D), E(D), w) \) be a vertex-weighted rooted forest such that \( w(x) \geq 2 \) if \( d(x) \neq 1 \). Let \( w = \max \{ w(x) \mid x \in V(D) \} \), then

\[
\text{reg}(I(D)^t) = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1).
\]

We need the following propositions to prove the main results.

**Proposition 5.5.** Let \( t \) be a positive integer and \( D = (V(D), E(D), w) \) a vertex-weighted oriented unicyclic graph, where \( D = C_m \cup T_1 \) and \( T_1 \) is an oriented line graph, its orientation is as follows: \( x_i \) is the root of \( T_1 \) if \( V(T_1) \cap V(C_m) = \{ x_i \} \) for some \( 1 \leq i \leq m \), otherwise, \( x_{m+1} \) is the root of \( T_1 \). Then

\[
\text{reg}(I(D)^t) \leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) \quad \text{for any} \quad t \geq 1
\]

where \( w = \max \{ w(x) \mid x \in V(D) \} \).

**Proof.** Let \( V(D) = \{ x_1, \ldots, x_m, x_{m+1}, \ldots, x_n \} \), \( V(C_m) = \{ x_1, \ldots, x_m \} \) and \( w_i = w(x_i) \) for \( 1 \leq i \leq n \).

If \( V(T_1) \cap V(C_m) = \emptyset \), then the result can be shown by similar arguments as case \( V(T_1) \cap V(C_m) = \{ x_i \} \) for some \( 1 \leq i \leq m \), so we only prove that the conclusion holds under the condition that \( V(T_1) \cap V(C_m) = \{ x_i \} \) for some \( 1 \leq i \leq m \). In this case, we set \( i = m \) for convenience. Thus \( E(D) = \{ x_1 x_2, x_2 x_3, \ldots, x_{m-1} x_m, x_m x_1, x_m x_{m+1}, x_{m+1} x_{m+2}, \ldots, x_{n-1} x_n \} \) and \( x_n \) is the unique leaf of \( D \). It follows that

\[
I(D) = (x_1 x_2^{w_2}, \ldots, x_{m-1} x_m^{w_m}, x_m x_1^{w_1}, x_m x_{m+1}^{w_{m+1}}, x_{m+1} x_{m+2}^{w_{m+2}}, \ldots, x_{n-1} x_n^{w_n}).
\]

We apply induction on \( t \) and \( |E(T_1)| \). Case \( t = 1 \) follows from [30, Theorem 3.4]. Now assume that \( t \geq 2 \). If \( |E(T_1)| = 1 \), then \( n = m + 1 \). Consider the following two
short exact sequences

\[
0 \longrightarrow \frac{S}{(I(D)^t : x_n^{w_n})}(-w_n) \xrightarrow{x_n^{w_n}} \frac{S}{I(D)^t} \longrightarrow \frac{S}{(I(D)^t, x_n^{w_n})} \longrightarrow 0 \quad (1)
\]

\[
0 \longrightarrow \frac{S}{(I(D)^t : x_n^{w_n})}(-1) \xrightarrow{x_m^{w_n}} \frac{S}{(I(D)^t : x_n^{w_n})} \longrightarrow \frac{S}{((I(D)^t : x_n^{w_n}), x_m)} \longrightarrow 0. \quad (2)
\]

Notice that \(D \setminus x_m\) is a vertex-weighted rooted forest. By Lemma 5.3, we have \((I(D)^t, x_n^{w_n}) = (I(C_m)^t, x_n^{w_n}), ((I(D)^t : x_n^{w_n}), x_m) = (I(D \setminus x_m)^t, x_m)\) and \((I(D)^t : x_m^{w_n}) = I(D)^{t-1}\). Thus by Lemma 2.5, Lemma 5.4, Theorem 4.5 and induction hypothesis on \(t\), we obtain

\[
\text{reg} ((I(D)^t, x_n^{w_n})) = \text{reg} ((I(C_m)^t, x_n^{w_n})) = \text{reg} (I(C_m)^t) + \text{reg} ((x_n^{w_n}) - 1
\]

\[
= \sum_{i=1}^m w_i - m + 1 + (t-1)(w' + 1)) + w_n - 1
\]

\[
= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t-1)(w' + 1)
\]

\[
\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t-1)(w + 1), \quad (3)
\]

where the fourth equality holds because of \(n = m + 1\) and the last inequality holds because of \(w' \leq w\), where \(w' = \max \{w_i \mid 1 \leq i \leq m\}\),

\[
\text{reg} ((I(D)^t : x_m^{w_n})(-w_n-1)) = \text{reg} (I(D)^{t-1}) + w_n + 1
\]

\[
= \left( \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t-2)(w + 1) \right) + w_n + 1
\]

\[
\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t-1)(w + 1), \quad (4)
\]

where the last inequality holds because of \(w_n \leq w\), and

\[
\text{reg} (((I(D)^t : x_n^{w_n}), x_m)(-w_n)) = \text{reg} ((I(D \setminus x_m)^t, x_m)) + w_n
\]

\[
= \sum_{x \in V(D \setminus x_m)} w(x) - |E(D \setminus x_m)| + 1 + (t-1)(w'' + 1) + w_n
\]

\[
= \sum_{x \in V(D \setminus x_m)} w(x) - |E(D)| + 1 + (t-1)(w'' + 1) + 4 - (w_1 + w_m)
\]

\[
\leq \sum_{x \in V(D \setminus x_m)} w(x) - |E(D)| + 1 + (t-1)(w + 1), \quad (5)
\]

where the forth equality holds because we have weighted one in vertex \(x_1\) in the expression \(\sum_{x \in V(D \setminus x_m)} w(x)\) and \(|E(D)| = |E(D \setminus x_m)| + 3\), and the last inequality holds because of \(w_1, w_m \geq 2\) and \(w'' \leq w\), here \(w'' = \max \{w_i \mid 2 \leq i \leq m-1\}\).
Using Lemma 2.34 and Lemma 2.9 (2) on the short exact sequences (1), (2) and formulas (3)~(5), we have

$$\text{reg} (I(D)^t) \leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t-1)(w+1).$$

Assume $|E(T)| \geq 2$, consider the short exact sequences

$$0 \to \frac{S}{(I(D)^t : x_n^w)}(-w_n) \to \frac{S}{I(D)^t} \to \frac{S}{(I(D)^t, x_n^w)} \to 0 \quad (6)$$

and

$$0 \to \frac{S}{((I(D)^t : x_{n-1}^w))}(-1)^{-x_n^{w-1}} \to \frac{S}{(I(D)^t : x_n^w)} \to \frac{S}{((I(D)^t : x_n^w), x_{n-1})} \to 0. \quad (7)$$

Notice that $(I(D)^t, x_n^w) = (I(D \setminus x_n)^t, x_n^w)$, $(I(D)^t : x_n^w)$, $x_{n-1} = (I(D \setminus x_{n-1})^t, x_{n-1})$ and $(I(D)^t : x_{n-1}^w) = I(D)^{t-1}$ by Lemma 5.3 both $D \setminus x_n$ and $D \setminus x_{n-1}$ are vertex-weighted oriented unicyclic graphs. Thus, by Lemma 2.34 Theorem 4.5 and induction hypotheses on $t$ and $|E(T)|$, we obtain

$$\text{reg} ((I(D)^t, x_n^w)) = \text{reg} ((I(D \setminus x_n)^t, x_n^w)) = \text{reg} (I(D \setminus x_n)^t) + \text{reg} ((x_n^w)) - 1$$

$$\leq \left( \sum_{x \in V(D \setminus x_n)} w(x) - |E(D \setminus x_n)| + 1 + (t-1)(w+1) + w_n - 1 \right.$$  

$$= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t-1)(w_a + 1)$$

$$\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t-1)(w+1), \quad (8)$$

where the last inequality holds because of $w_a \leq w$, where $w_a = \max \{w(x) \mid x \in V(D \setminus x_n)\}$,

$$\text{reg} ((I(D)^t : x_{n-1}^w)(-w_{n-1})) = \text{reg} (I(D)^{t-1}) + w_{n-1} + 1$$

$$\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t-2)(w+1) + w_{n-1} + 1$$

$$= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t-1)(w+1) + w_n - w$$

$$\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t-1)(w+1), \quad (9)$$
where the last inequality holds because of \( w_n \leq w \), and
\[
\text{reg} (\langle (D)^t, x_n^w \rangle, x_{n-1}) (-w_n) = \text{reg} (\langle (D \setminus x_{n-1})^t, x_{n-1} \rangle) + w_n \\
= \text{reg} (\langle (D \setminus x_{n-1})^t \rangle + w_n \leq \sum_{x \in V(D) \setminus x_{n-1}} w(x) - \left| E(D \setminus x_{n-1}) \right| + 1 \right) + (t-1) \langle w_b + 1 \rangle \right) + w_n \\
= \sum_{x \in V(D)} w(x) - |E(D)| + 1 \right) + (t-1) \langle w_b + 1 \rangle \right) + 2 - w_{n-1} \\
\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 \right) + (t-1) \langle w + 1 \rangle \\
(10)
\]
where the last inequality holds because of \( w_{n-1} \geq 2 \), \( w_b \leq w \), here \( w_b = \max \{w(x), x \in V(D \setminus x_{n-1})\} \). Using Lemma 2.4 and Lemma 2.9 (2) on the short exact sequences (6) (7) and inequalities (8) ~ (10), we have
\[
\text{reg} (I(D)^t) \leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 \right) + (w + 1).
\]

Now we are ready to present the main result of this section.

**Theorem 5.6.** Let \( D = (V(D), E(D), w) \) be a vertex-weighted oriented unicyclic graph as Proposition 5.5. Then
\[
\text{reg} (I(D)^t) = \sum_{x \in V(D)} w(x) - |E(D)| + 1 \right) + (t-1) \langle w + 1 \rangle \right) \text{ for any } t \geq 1
\]
where \( w = \max \{w(x) | x \in V(D)\} \).

**Proof.** Case \( t = 1 \) follows from [30, Theorem 3.5]. Now we assume \( t \geq 2 \). If \( V(T_1) \cap V(C_m) = \emptyset \), or \( V(T_1) \cap V(C_m) = \{x_i\} \) for some \( 1 \leq i \leq m \) and \( |E(T_1)| \leq 3 \), then the conclusion can be shown by similar arguments as case \( V(T_1) \cap V(C_m) = \{x_i\} \) for some \( 1 \leq i \leq m \) and \( |E(T_1)| \geq 4 \), so we only prove the conclusion holds under the condition that \( V(T_1) \cap V(C_m) = \{x_i\} \) for some \( 1 \leq i \leq m \) and \( |E(T_1)| \geq 4 \). In this case, we set \( i = m \) for convenience. Thus \( E(D) = \{x_1x_2, x_2x_3, \ldots, x_{m-1}x_m, x_mx_1, x_mx_{m+1}, x_{m+1}x_{m+2}, \ldots, x_{n-1}x_n\} \). It follows that
\[
I(D) = (x_1x_2^{w_2}, \ldots, x_{m-1}x_m^{w_m}, x_mx_1^{w_1}, x_mx_{m+1}^{w_m+1}, x_{m+1}x_{m+2}^{w_{m+2}}, \ldots, x_{n-1}x_n^{w_n}).
\]

Let \( L \) be an ideal satisfying
\[
\mathcal{G}(L) = \mathcal{G}(I(D)^t) \setminus \mathcal{G}(M),
\]
where \( M = ((x_1x_2^{w_2})^t, \ldots, (x_{m-1}x_m^{w_m})^t, (x_mx_1^{w_1})^t, (x_mx_{m+1}^{w_m+1})^t, \ldots, (x_{n-1}x_n^{w_n})^t) \). Let \( J_0 \) be the polarization of \( I(D)^t \), then
\[
J_0 = M^P + L^P
\]
with \( \mathcal{G}(J_0) = \mathcal{G}(M^P) \cup \mathcal{G}(L^P) \) and \( \mathcal{G}(M^P) \cap \mathcal{G}(L^P) = \emptyset \).
For $1 \leq i \leq n - 2$, we set $K_i = ((\prod_{j=1}^{t} x_{i,j})(\prod_{j=1}^{t} x_{i+1,j}))$,

\[ J_i = ((\prod_{j=1}^{t} x_{1,j})(\prod_{j=1}^{t} x_{2,j}), \ldots, (\prod_{j=1}^{t} x_{i,j})(\prod_{j=1}^{t} x_{i+1,j}), (\prod_{j=1}^{t} x_{i+1,j})(\prod_{j=1}^{t} x_{i+2,j}), \ldots, (\prod_{j=1}^{t} x_{n-1,j})(\prod_{j=1}^{t} x_{n,j}), (\prod_{j=1}^{t} x_{n,j})(\prod_{j=1}^{t} x_{1,j})) + L^P, \]

where $(\prod_{j=1}^{t} x_{i,j})(\prod_{j=1}^{t} x_{i+1,j})$ denotes the element $(\prod_{j=1}^{t} x_{i,j})(\prod_{j=1}^{t} x_{i+1,j})$ being omitted from $J_i$.

Remind: when $i = m$, we set $K_m = ((\prod_{j=1}^{t} x_{m,j})(\prod_{j=1}^{t} x_{m+1,j}))$,

\[ J_m = ((\prod_{j=1}^{t} x_{m+1,j})(\prod_{j=1}^{t} x_{m+2,j}), \ldots, (\prod_{j=1}^{t} x_{n-1,j})(\prod_{j=1}^{t} x_{m,j}), (\prod_{j=1}^{t} x_{m,j})(\prod_{j=1}^{t} x_{1,j})) + L^P. \]

Let $K_{n-1} = ((\prod_{j=1}^{t} x_{n-1,j})(\prod_{j=1}^{t} x_{n,j}), J_{n-1} = ((\prod_{j=1}^{t} x_{n,j})(\prod_{j=1}^{t} x_{1,j})) + L^P,$

$K_n = ((\prod_{j=1}^{t} x_{m,j})(\prod_{j=1}^{t} x_{1,j})), J_n = L^P.$ Then for $1 \leq i \leq n$, we have

\[ J_{i-1} = J_i + K_i \text{ and } J_i \cap K_i = K_i L_i, \]

\[ L_1 = (\prod_{j=1}^{w_3} x_{3,j}, x_{31}, \prod_{j=1}^{w_3} x_{4,j}, \ldots, x_{n-1,1}, \prod_{j=1}^{w_n} x_{n,j}, x_{m,1}, \prod_{j=1}^{t-1+w_1} x_{1,j}), \]

\[ L_2 = (x_{11}, \prod_{j=1}^{t-1+w_2} x_{2,j}, x_{21}, \prod_{j=1}^{w_3} x_{4,j}, \prod_{j=1}^{w_4} x_{41}, \prod_{j=1}^{w_5} x_{5,j}, \ldots, x_{n-1,1}, \prod_{j=1}^{w_n} x_{n,j}, x_{m,1}, \prod_{j=1}^{t-1+w_1} x_{1,j}), \]

\[ L_i = (x_{11}, \prod_{j=1}^{w_2} x_{2,j}, \ldots, x_{i-1,1}, \prod_{j=1}^{w_i} x_{i,j}, \prod_{j=1}^{w_{i+2}} x_{i+2,j}, \prod_{j=1}^{w_{i+3}} x_{i+3,j}, \ldots, x_{n-1,1}, \prod_{j=1}^{w_n} x_{n,j}, x_{m,1}, \prod_{j=1}^{w_1} x_{1,j}) \text{ for all } 3 \leq i \leq m - 2 \text{ or } m + 1 \leq i \leq n - 3, \]

\[ L_{m-1} = (x_{11}, \prod_{j=1}^{w_2} x_{2,j}, \ldots, x_{m-3,1}, \prod_{j=1}^{w_{m-2}} x_{m-2,j}, x_{m-2,1}, \prod_{j=1}^{t-1+w_{m-1}} x_{m-1,j}, x_{m,1}, \prod_{j=1}^{w_{m+1}} x_{1,j}, x_{m+1,1}, \prod_{j=1}^{w_{m+2}} x_{m+2,j}, \ldots, x_{n-1,1}, \prod_{j=1}^{w_n} x_{n,j}, \prod_{j=1}^{w_1} x_{1,j}), \]

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Lemma 2.6 (2), we have

\[ L_m = \prod_{j=1}^{w_{m+3}} x_{2j}, \ldots, x_{m-1, j}, x_{m-1, j}, \prod_{j=t+1}^{t-1+w_m} x_{m, j}, \prod_{j=1}^{w_{m+2}} x_{m+2, j}, \]

\[ x_{m+2, 1} \prod_{j=1}^{w_n} x_{m+3, j}, \ldots, x_{n-1, 1} \prod_{j=1}^{w_1} x_{1, j}, \]

\[ L_{n-2} = \prod_{j=1}^{w_{n-3}} x_{2j}, \ldots, x_{n-4, 1} \prod_{j=1}^{w_n} x_{n-3, j}, x_{n-3, 1} \prod_{j=t+1}^{t-1+w_{n-2}} x_{n-2, j}, \prod_{j=1}^{w_1} x_{n-1, j}, x_{n-1, 1} \prod_{j=1}^{w_1} x_{1, j}, \]

\[ L_{n-1} = \prod_{j=1}^{w_{n-2}} x_{2j}, x_{21} \prod_{j=1}^{w_3} x_{3j}, \ldots, x_{m-2, j}, x_{m-2, 1} \prod_{j=t+1}^{t-1+w_m} x_{m, j}, x_{m-1, 1} \prod_{j=1}^{w_1} x_{1, j}, \]

\[ x_{m+1, 1} \prod_{j=1}^{w_n} x_{m+2, j}, \ldots, x_{n-1, 1} \prod_{j=1}^{w_1} x_{1, j}. \]

Thus for \(1 \leq i \leq n\), \( |\text{supp}(L_i)| = \sum_{i=1}^{n} w(x) - w_{i+1} - 1 \) and \( |\mathcal{G}(L_i)| = n - 1 \). By similar arguments as Proposition 4.3, we obtain

\[ \text{reg} (L_i) = \left( \sum_{i=1}^{n} w(x) - w_{i+1} - 1 \right) - (n - 1) + 1 = \sum_{i=1}^{n} w(x) - n + 1 - w_{i+1}, \]

where \( w_{n+1} = w_1 \). Notice that the variables appear in \( K_i \) and \( L_i \) are different, by Lemma 2.6 (2), we have

\[ \text{reg} (J_i \cap K_i) = \text{reg} (K_i L_i) = \text{reg} (K_i) + \text{reg} (L_i) \]

\[ = t(w_{i+1} + 1) + \left( \sum_{i=1}^{n} w(x) - n + 1 - w_{i+1} \right) \]

\[ = \sum_{i=1}^{n} w(x) - n + 2 + (t - 1)(w_{i+1} + 1). \]  \hspace{1cm} (1)

Let \( H = (V(H), \mathcal{E}(H)) \) and \( H' = (V(H'), \mathcal{E}(H')) \) are hypergraphs associated to \( \mathcal{G}(J) \) and \( \mathcal{G}(L^P) \) respectively, then \( H' \) is an induced subhypergraph of \( H \) by similar arguments as Theorem 4.5. Thus by Lemma 2.8 (2), Lemma 4.1 and Proposition 5.5 we get

\[ \text{reg} (J_n) = \text{reg} (L^P) \leq \text{reg} ((I(D)^t)^P) = \text{reg} (I(D)^t) \]

\[ \leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1). \]  \hspace{1cm} (2)

For any \(1 \leq i \leq n\), the variable \( x_{i+1, tw_{i+1}} \) in \( K_i \) is not a factor of any minimal generator of \( J_i \) and \( K_i \) has a linear resolution. We have \( J_i = J_{i+1} + K_{i+1} \) is Betti...
splitting by Lemma 2.2. Hence by Corollary 2.3 we obtain
\[ \text{reg}(J_{i-1}) = \max\{\text{reg}(K_i), \text{reg}(J_i), \text{reg}(K_i \cap J_i) - 1\}. \]  
(3)

Let \( \alpha = \text{reg}(K_i), \beta = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) \), then
\[ \beta - \alpha = (\sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1)) - t(w_{i+1} + 1) \]
\[ \geq \sum_{x \in V(D), x \neq x_{i+1}} w(x) - |E(D)| \geq 0. \]  
(4)

By Lemma 2.8(2), repeated use of the above the equality (3) and comparing formulas (1), (2), (4), we obtain
\[ \text{reg}(I(D)^t) = \text{reg}(J_0) = \max\{\text{reg}(J_n), \text{reg}(K_i), \text{reg}(K_i \cap J_i) - 1, \text{ for } 1 \leq i \leq n\} \]
\[ = \max\{\text{reg}(J_n), t(w_{i+1} + 1), \sum_{x \in V(D)} w(x) - |E(D)| + 2 + (t - 1)(w_i + 1) - 1, \text{ for } 1 \leq i \leq n\} \]
\[ = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1). \]

The result follows. \( \square \)

**Theorem 5.7.** Let \( D = (V(D), E(D), w) \) be a vertex-weighted oriented unicyclic graph, where \( D = C_m \cup T \) and \( T \) is an oriented forest. Let \( w(x) \geq 2 \) for any \( d(x) \neq 1 \). Then
\[ \text{reg}(I(D)^t) = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) \]  
for any \( t \geq 1 \), where \( w = \max\{w(x) \mid x \in V(D)\} \).

**Proof.** We apply induction on \( t \) and \( |E(T)| \). The case \( t = 1 \) follows from [30, Theorem 3.5]. Now assume that \( t \geq 2 \). By Theorem 5.6, we just need to prove the results hold under the condition that there are at least two leaves in \( D \). Let \( x, z \) be leaves of \( D \) with \( w_z \leq w_x \) and \( N_D(z) = \{y\} \). We distinguish into two cases:

(1) If there exists a connected component \( T_1 \) of \( T \) such that \( E(T_1) = \{yz\} \) and \( y \notin V(C_m) \). Then
\[ I(D)^t = I(D \setminus z)^t + (yz^{w_x})I(D)^{t-1}. \]
Thus there exists a surjection \( \phi : I(D \setminus z)^t \oplus I(D)^{t-1}(-w_z - 1) \xrightarrow{(1, yz^{w_x})} I(D)^t \) and the kernel of \( \phi \) is \((yz^{w_x})I(D \setminus z)^t \) since \( yz^{w_x} \) is a non-zero divisor of \( S/I(D \setminus z) \). Therefore, we have the following short exact sequence
\[ 0 \rightarrow I(D \setminus z)^t(-w_z - 1) \rightarrow I(D \setminus z)^t \oplus I(D)^{t-1}(-w_z - 1) \xrightarrow{(1, yz^{w_x})} I(D)^t \rightarrow 0. \]
By induction hypotheses on \( t \) and \( |E(T)| \), we obtain
\[
\text{reg } (I(D)^{t-1}(-w_z - 1)) = \text{reg } (I(D)^{t-1}) + w_z + 1
\]
\[
= ( \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 2)(w + 1)) + w_z + 1
\]
\[
= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) + w_z - w,
\]
\[
\text{reg } (I(D) \setminus z)^t(-w_z - 1) = \text{reg } (I(D) \setminus z)^t + w_z + 1
\]
\[
= ( \sum_{x \in V(D)} w(x) - |E(D) \setminus z| + 1 + (t - 1)(w + 1)) + w_z + 1
\]
\[
= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) + 1.
\]
Since \( w_z \leq w_x \), we get
\[
\text{reg } (I(D) \setminus z)^t(-w_z - 1) > \max \{ \text{reg } (I(D)^{t-1}(-w_z - 1), \text{reg } (I(D) \setminus z)^t) \}.
\]
Thus the result follows from Lemma 2.39 (5).

(2) If there is no connected component containing \( z \) in \( T \) such as (1), then \( d(y) \geq 2 \). Consider the following short exact sequences
\[
0 \longrightarrow \frac{S}{(I(D)^t : z^{w_z})}(-w_z) \longrightarrow \frac{S}{I(D)^t} \longrightarrow \frac{S}{(I(D)^t, z^{w_z})} \longrightarrow 0 \tag{1}
\]
\[
0 \longrightarrow \frac{S}{((I(D)^t : z^{w_z}) : y)}(-1) \longrightarrow \frac{S}{(I(D)^t : z^{w_z})} \longrightarrow \frac{S}{((I(D)^t : z^{w_z}), y)} \longrightarrow 0 \tag{2}
\]
Note that \( D \setminus z \) is a vertex-weighted oriented unicyclic graph, \( D \setminus y \) is an oriented unicyclic graph or a rooted forest, and \( w_z \leq w_x \), thus, Lemma 2.35, Lemma 5.4 or Theorem 1.2 and induction hypotheses on \( t \) and \( |E(T)| \), we obtain
\[
\text{reg } ((I(D)^t, z^{w_z})) = \text{reg } ((I(D) \setminus z)^t, z^{w_z})) = \text{reg } ((I(D) \setminus z)^t) + \text{reg } ((z^{w_z})) - 1
\]
\[
= [ \sum_{x \in V(D \setminus z)} w(x) - |E(D \setminus z)| + 1 + (t - 1)(w + 1)] + w_z - 1
\]
\[
= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1), \tag{3}
\]
\[
\text{reg } ((I(D)^t : y z^{w_z})(-w_z - 1)) = \text{reg } ((I(D)^t)^{t-1}) + w_z + 1
\]
\[
= ( \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 2)(w + 1)) + w_z + 1
\]
\[
= \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1) + w_z - w
\]
\[
\leq \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1). \tag{4}
\]
where $w'' = \max \{w(x) \mid x \in V(D \setminus y)\}$.

Notice that $N_D^-(y) = \emptyset$ or $N_D^-(y) = \{y_1\}$. For case $N_D^-(y) = \emptyset$, it can be shown by similar arguments as case $N_D^-(y) = \{y_1\}$. So we only prove the conclusion holds under the condition that $N_D^-(y) = \{y_1\}$. In this case, $w_y \geq 2$ and we set $|E(D)| = |E(D \setminus y)| + \ell$, then $|N_D^+(y) \setminus \{z\}| = \ell - 2$. Let $\alpha = \text{reg}((I(D)^t, z^{w_z}))$, $\beta = \text{reg}(((I(D)^t : z^{w_z}), y)(-z_w))$, then

$$\alpha - \beta = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1)$$

$$- \left[ \sum_{x \in V(D \setminus y)} w(x) - |E(D \setminus y)| + 1 + (t - 1)(w'' + 1) + w_z \right]$$

$$\geq \left( \sum_{x \in V(D)} w(x) - \sum_{x \in V(D \setminus y)} w(x) - w_z \right) - (|E(D)| - |E(D \setminus y)|)$$

$$\geq (\ell - 2 + w_y) - \ell \geq 0,$$

where the first inequality holds because of $w'' \leq w$ and the second inequality holds because of $|N_D^+(y) \setminus \{z\}| = \ell - 2$. By formulas (3), (4) and (5), we get

$$\text{reg}((I(D)^t, z^{w_z})) \geq \max \{\text{reg}((I(D)^t : yz^{w_z})(-w_z - 1)), \text{reg}(((I(D)^t : z^{w_z}), y)(-w_z))\}.$$ 

Using Lemma 2.9 (2), (4) on the short exact sequences (1), (2) and the equality (3), we obtain

$$\text{reg}(I(D)^t) = \text{reg}((I(D)^t, z^{w_z})) = \sum_{x \in V(D)} w(x) - |E(D)| + 1 + (t - 1)(w + 1).$$

The proof is completed. \qed

As a consequence of Theorem 5.7, we have

**Corollary 5.8.** Let $D = (V(D), E(D), w)$ be a vertex-weighted oriented unicyclic graph as Theorem 5.7. Then

$$\text{reg}(I(D)^t) = \text{reg}(I(D)) + (t - 1)(w + 1) \quad \text{for any} \quad t \geq 1,$$

where $w = \max \{w(x) \mid x \in V(D)\}$.

The following example shows the assumption in Theorem 5.7 that $D$ is a vertex-weighted oriented unicyclic graph such that $w(x) \geq 2$ for any $d(x) \neq 1$ cannot be dropped.

**Example 5.9.** Let $I(D) = (x_1x_2^2, x_2x_3^2, x_3x_2^2, x_4x_2^2, x_4x_5, x_5x_6, x_6x_2^2)$ be the edge ideal of an oriented unicyclic graph, its weight function is $w_1 = w_2 = w_3 = w_4 = w_7 = 2$ and $w_5 = w_6 = 1$. Thus $w = 2$. By using CoCoA, we obtain $\text{reg}(I(D)^2) = 10$. But we have $\text{reg}(I(D)^2) = (\sum_{i=1}^{7} w_i - |E(D)| + 1) + (w + 1) = 9$ by Theorem 5.7.
The following example shows the regularity of powers of edge ideals of vertex-weighted oriented unicyclic graphs is related to direction selection in Theorem 5.7.

**Example 5.10.** Let $I(D) = (x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_4x_5, x_6x_2, x_6x_5, x_6x_7)$ be the edge ideal of an oriented unicyclic graph, its weight function is $w_1 = w_2 = w_3 = w_4 = w_5 = w_7 = 2$ and $w_6 = 1$. Thus $w = 2$. By using CoCoA, we obtain $\text{reg}(I(D)^2) = 11$. But we have $\text{reg}(I(D)^2) = \left(\sum_{i=1}^{7} w_i - |E(D)| + 1\right) + (w + 1) = 10$ by Theorem 5.7.

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