Embeddings of $k$-complexes into $2k$-manifolds.*

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January 19, 2022

Abstract

We improve the bound on Kühnel’s problem to determine the smallest $n$ such that the $k$-skeleton of an $n$-simplex $\Delta^n(k)$ does not embed into a compact PL $2k$-manifold $M$ by showing that if $\Delta^n(k)$ embeds into $M$, then $n \leq (2k + 1) + (k + 1)\beta_k(M; \mathbb{Z}_2)$. As a consequence we obtain improved Radon and Helly type results for set systems in such manifolds.

Our main tool is a new description of an obstruction for embeddability of a $k$-complex $K$ into a compact PL $2k$-manifold $M$ via the intersection form on $M$. In our approach we need that for every map $f: K \to M$ the restriction to the $(k-1)$-skeleton of $K$ is nullhomotopic. In particular, this condition is satisfied in interesting cases if $K$ is $(k-1)$-connected, for example a $k$-skeleton of $n$-simplex, or if $M$ is $(k-1)$-connected. In addition, if $M$ is $(k-1)$-connected and $k \geq 3$, the obstruction is complete, meaning that a $k$-complex $K$ embeds into $M$ if and only if the obstruction vanishes. For trivial intersection forms, our obstruction coincides with the standard van Kampen obstruction. However, if the form is non-trivial, the obstruction is not linear but rather ‘quadratic’ in a sense that it vanishes if and only if certain system of quadratic diophantine equations is solvable. This may potentially be useful in attacking algorithmic decidability of embeddability of $k$-complexes into PL $2k$-manifolds.

1 Introduction

Motivation. This paper has three main goals:

1) Describe an obstruction for (almost)-embeddability of $k$-dimensional simplicial complexes into compact $2k$-dimensional PL manifolds (Theorems 4 and 6). This extends the standard van Kampen obstruction for embeddability into $\mathbb{R}^{2k}$.

2) Improve the bounds for so-called Kühnel problem: Provide an upper bound on $n$ such that the $k$-skeleton of the $n$-simplex embeds into compact $2k$-dimensional PL-manifold $M$ (Theorem 1).

3) Use the bounds from the previous item to obtain versions of Radon’s and Helly’s theorem on manifolds (Theorem 2 and Corollary 3).

Motivation for such research emerges from various directions:

The classical setting, related to the first goal, considers the case of embeddings of $k$-complexes into $\mathbb{R}^{2k}$ (a $k$-complex is always embeddable into $\mathbb{R}^{2k+1}$, thus the target dimension $2k$ is the first nontrivial dimension). This line of research was initiated by results of van Kampen and Flores [vK32, Flo34] on nonembeddability of the $k$-skeleton of the $(2k+2)$-simplex, $\Delta^{(k)}_{2k+2}$, and the $(k+1)$-fold join of three isolated points into $\mathbb{R}^{2k}$. This case is in general well understood: If $k \neq 2$, embeddability of $K$ in $\mathbb{R}^{2k}$ is characterized via vanishing of so-called van Kampen obstruction [vK32, Sha57, Wu65, Mel09], which is even efficiently computable (details on computability are given in [MTW11]). If $k = 2$, the obstruction

*This research is supported by the GAČR grant 19-04113Y.
is incomplete \cite{FKT94}, and it seems to be a challenging problem to determine whether embeddability of 2-complexes into $\mathbb{R}^k$ is decidable. (Only NP-hardness is known \cite{MTW11}.) However, there are many interesting target spaces that are not $\mathbb{R}^{2k}$. In geometry one often works with projective spaces, incidence problems lead to embeddings into Grassmannians or flag manifolds, etc. A possible concrete example where the ideas of this paper can be useful are considerations of Helly type results as in \cite{GPP17}. Here considerations of a general manifold $M$ become apparent, for example, when considering Helly-type theorems for line transversals as in \cite{CGHP08}.

In relation to our second main goal, embeddability of the $k$-skeleton of $n$-simplex $\Delta_n^k$ to a $2k$-manifold was considered in \cite{Kueh94, Vol96, GMP17}. Volovikov \cite{Vol96} shows, for quite general $M$, that there is no embedding $f : |\Delta_{2k+2}^k| \to M$ provided that $f$ induces a trivial map on (co)homology, which generalizes nonembeddability of $\Delta_{2k+2}^k$ in $\mathbb{R}^{2k+2}$. Given a $(k-1)$-connected $2k$-manifold $M$, Kühnel conjectured an upper bound on $n$ depending only on $k$ and the Euler characteristic of $M$; see equation (1) below. A weaker bound was proved in \cite{GMP17}. As an application of our tools, we will show how this bound can be significantly improved (for compact PL manifolds). Such improvement also yields improved Radon and Helly type theorems on manifolds, which is our third goal.

Finally, in a special case when $k = 1$ our research coincides with a classical topic of embeddings of graphs in surfaces \cite{MT01}; and, in particular, our work is related to Hanani–Tutte type results for graphs on surfaces \cite{PSS09, FK19, FK18}. In the language of these references, our algebraic description in this case provides a characterization of graphs admitting an independently even drawing into a given surface.

1.1 The Kühnel problem and Helly-type results

Before we explain the details of our description of embeddability of $k$-complexes into $2k$-manifolds, let us survey a few results that we can reach with our tools.

**Kühnel’s conjecture.** Kühnel conjectured \cite{Kueh94} that if the $k$-dimensional skeleton $K := \Delta_n^k$ can be embedded into a $(k-1)$-connected $2k$-manifold $M$, then

\[
\left( \frac{n - k - 1}{k + 1} \right) \leq (-1)^k \binom{2k + 1}{k} \chi(M) - 2.
\]  

Because of $(k-1)$-connectivity, this inequality is equivalent to

\[
\left( \frac{n - k - 1}{k + 1} \right) \leq \binom{2k + 1}{k} \beta_k(M; \mathbb{Z}_2),
\]

which seems to hold even without the connectivity assumption. (Here $\beta_k$ denotes the Betti number.)

The special case $k = 1$, of Kühnel’s conjecture is known as Heawood inequality and it is fully confirmed in this case (see \cite{Rin74} for discussion).\footnote{However, our work should be understood only as a first step towards an improvement of \cite{GPP17}. In particular, we did not attempt to upgrade our results to homological almost embeddings which are really used in \cite{GPP17}.} For $k \geq 2$, a recent far reaching work Adiprasito \cite{Adi18} proves the Kühnel bound under an additional assumption that the embedding is sufficiently tame. Without the tameness assumption, together with Goaoc, Mabillard, Patáková and Wagner \cite{GMP17}, we have obtained a bound $n \leq 2\beta_k(M; \mathbb{Z}_2)(2k+2) + 2k + 4$. Here we demonstrate how the ‘obstruction machinery’ may improve this bound (under an extra assumption that $M$ is PL). Once the machinery is set up, the main idea of the proof is relatively simple; see the sketch at the beginning of Section \ref{ss:obstruction}.

Given a simplicial complex $K$ and a manifold $M$ (or arbitrary topological space in general) an almost embedding of $K$ into $M$ is a map $f : |K| \to M$ such that $f(\sigma) \cap f(\tau) = \emptyset$ whenever $\sigma$ and $\tau$ are disjoint simplices of $K$. Every embedding is an almost embedding. By $\Omega_{\mathbb{Z}_2} : H_k(M; \mathbb{Z}_2) \times H_k(M; \mathbb{Z}_2) \to \mathbb{Z}_2$ we will denote the $\mathbb{Z}_2$-intersection form on $M$. The intersection form is discussed in more detail in Subsection \ref{ss:obstruction}.

\footnote{Volovikov’s result is in fact even more general in different directions.}

\footnote{In fact, we are not sure from the statement of the conjecture in \cite{Kueh94} Conjecture B) whether it regards arbitrary (possibly non-compact) manifolds or whether it regards polyhedral manifolds discussed in the paper which are PL and closed (a fortiori compact).}

\footnote{Note that $\Delta_n^k$ has $n + 1$ vertices, thus $n$ is shifted by one when compared with the standard statement of the Heawood inequality.}
The main properties are that $\Omega_{2z}$ is a symmetric bilinear form and if $z$ and $z'$ are two general position $k$-cycles in $M$, then $\Omega_{2z}(z, z')$ counts the number of crossings between $z$ and $z'$ modulo 2; here [.] stands for the corresponding homology class.

**Theorem 1.** If the $k$-skeleton $\Delta_n^{(k)}$ of an $n$-simplex can be almost embedded into a compact (possibly with boundary) PL $2k$-manifold $M$, then

(i) $n \leq (2k + 1) + (k + 1)\beta_k(M; \mathbb{Z}_2)$ and

(ii) $n \leq (2k + 1) + \frac{1}{2}(k + 2)\beta_k(M; \mathbb{Z}_2)$ if the intersection form on $M$ is alternating, that is $\Omega_{2z}(h, h) = 0$ for all $h \in H_k(M; \mathbb{Z}_2)$.

If $\beta_k(M; \mathbb{Z}_2) = 1$, our bounds agree with the value proposed by Kühnel and if the form is alternating the same is true for $\beta_k(M; \mathbb{Z}_2) = 2$. The condition that the form is alternating is a natural condition that occurs, for example, if $M$ is a connected sum of $S^k \times S^k$. One of the advantages of Theorem 1 is that it also applies to manifolds which are not $(k - 1)$-connected. This distinguishes it from Kühnel’s conjecture. Using Theorem 1 we can, for example, see that there is no (almost) embedding of $\Delta_{12}^{(3)}$ into $\mathbb{R}P^6$.

For $k = 1$, Theorem 1 does not recover the Heawood inequality. However, considering that Theorem 1 is stated for almost embeddings, it seems to say something new even for $k = 1$ as it is an open question whether embeddability and almost embeddability coincide for graphs on surfaces [FK19, Problem 5.2]. This makes the result relevant for example in context of Helly-type theorems; see Theorem 2 and Corollary 3 below.

There are several cases where the inequalities from Theorem 1 are tight: there is a 6-point triangulation of the real projective plane ($k = 1$, $\beta_1(M; \mathbb{Z}_2) = 1, n = 5$), a 9-point triangulation of the complex projective plane ($k = 2$, $\beta_2(M; \mathbb{Z}_2) = 1, n = 8$) [HK83] and 15-point triangulation of the quaternionic projective plane ($k = 4, \beta_4(M; \mathbb{Z}_2) = 1, n = 14$) [Cor19], and the torus can by triangulated using 7 vertices only ($k = 1, \beta_1(M; \mathbb{Z}_2) = 2, n = 6$). Quick computation of the number of faces reveals that each of these triangulations necessarily contains the complete $k$-skeleton of $\Delta_n$. (It is $(k + 1)$-neighbourly [Kuh94].)

In addition, there is a hope that bounds of Theorem 1 can be still improved significantly by using our tools, possibly giving a solution of the Kühnel conjecture. In Section 3 we pose a specific conjecture (purely in combinatorics and linear algebra), Conjecture 18 that implies Kühnel’s conjecture (in case that $M$ is a compact PL-manifold). A computer assisted search for small values of $k$ and $\beta_k(M; \mathbb{Z}_2)$ suggests that Conjecture 18 may hold.

**Radon and Helly type theorems.** Improved bounds on the Kühnel problem as in Theorem 1 immediately imply improved bounds on the Radon number (value $r$ in the statement below) in the theorem below. Consequently one obtains better bounds on Helly’s number [Lev51], Tverberg’s numbers [JW81], fractional Helly number [HL21], existence of weak $\varepsilon$-nets and $(p, q)$-theorems [HL21], AKKM09.

**Theorem 2.** Let $M$ be a compact PL $2k$-manifold. Let $cl: 2^M \to 2^M$ be a closure operator.

(i) If $r \geq 2k + 3 + (k + 1)\beta_k(M; \mathbb{Z}_2)$, or

(ii) if the intersection form of $M$ is alternating (over $\mathbb{Z}_2$) and $r \geq 2k + 3 + \frac{1}{2}(k + 2)\beta_k(M; \mathbb{Z}_2)$,

then there are two disjoint subsets $P_1, P_2 \subseteq P$ such that $cl(P_1) \cap cl(P_2) \neq \emptyset$.

**Corollary 3** (Helly-type theorem). Let $M$ be a compact PL $2k$-manifold. Let $\mathcal{F}$ be a finite collection of subsets of $M$ such that $\bigcap \mathcal{G}$ is $k$-connected or empty for every subfamily $\mathcal{G} \subseteq \mathcal{F}$. If $\bigcap \mathcal{G}$ is nonempty for every $\mathcal{G} \subseteq \mathcal{F}$ of cardinality less than $r$, where $r$ is as in the previous theorem, then $\bigcap \mathcal{F} \neq \emptyset$. 
We postpone the precise definition of a general position map, intersection number and intersection form. We start by letting the points \( P \) whose coordinates are indexed by the set \( \cal F \). Theorem 1(i) in case (i) and Theorem 1(ii) in case (ii). We define \( f \) to be the 0-skeleton of \( \Delta \) and \( \tau \) is an almost embedding of \( \Delta \) modulo 2. It turns out that the vectors \( f(\delta \sigma) \) belongs to \( cl I \), where \( I \) is the set of vertices of \( \sigma \). As \( cl I \) is k-connected, we can extend \( f \) to \( \sigma \) inside \( cl I \), thus we maintain the required property. It remains to show that the resulting \( f \) is an almost embedding of \( \Delta_{k-1} \) into \( M \). Given disjoint \( k \)-simplices \( \sigma \) and \( \tau \) of \( \Delta_{r-1} \), let \( I \) be set of vertices of \( \sigma \) and \( J \) be the set of vertices of \( \tau \). In particular \( I \) and \( J \) are disjoint. But then \( f(\tau) \) lies in \( cl I \) and \( f(\sigma) \) lies in \( cl J \) and these two sets are disjoint by our assumption.

### 1.2 Obstruction for embeddability

Now we describe an obstruction for embeddability of a \( k \)-complex into a compact PL \( 2k \)-manifold, which is our main technical tool. In general, we follow [Sha57, FKT94, Joh02, Sko08, Mel09] and [MTW11, App. D]; however the concrete interpretations of the van Kampen obstruction in these references somewhat vary. We choose to specify the details in a way convenient for working with intersection form later on. We postpone the precise definition of a general position map, intersection number and intersection form to Section 2 as they are not so essential for understanding this text in the introduction.

**The standard van Kampen obstruction.** Let \( k \geq 1 \) and \( K \) be a simplicial \( k \)-complex. Let \( f : [K] \to \mathbb{R}^{2k} \) be a general position map. Given two disjoint \( k \)-simplices \( \sigma \) and \( \tau \) of \( K \), the number of intersections \( f(\sigma) \) and \( f(\tau) \) is finite and each such intersection is transversal. One way how to express the idea of the van Kampen obstruction [vK32] is the following: Let \( \mathbb{Z}_2^P \) be the vector space over \( \mathbb{Z}_2 \) whose coordinates are indexed by the set \( P \) of all (ordered) pairs \( (\sigma, \tau) \) of disjoint \( k \)-simplices of \( K \). The general position map \( f \) induces a vector \( v_\ell \in \mathbb{Z}_2^P \) such that its coordinate corresponding to the pair \( (\sigma, \tau) \) is the number of intersections between \( f(\sigma) \) and \( f(\tau) \) modulo 2. It turns out that the vectors \( v_\ell \) when considering over all possible general maps \( f \) form an affine subspace of \( \mathbb{Z}_2^P \). In particular, if there is an embedding \( g \) of \( K \) into \( \mathbb{R}^{2k} \), this affine subspace \( A \) has to contain the zero vector \( v_o \). For concrete \( K \), it is possible to determine whether \( A \) contains the zero vector, and \( A \) is essentially the object that we will call the van Kampen obstruction (see below).

For practical purposes (computations), it is convenient to consider \( A \) as a certain cohomology class which we will overview below. In particular, \( P \) will be replaced with deleted product of \( K \); \( v_\ell \) with corresponding intersection cochain and \( A \) with certain cohomology class denoted \( o(K) \). In addition, the similar ideas as above may be performed over the integers \( \mathbb{Z} \) instead of \( \mathbb{Z}_2 \). The cost is that one has to...
consider intersections of $f(\sigma)$ and $f(\tau)$ carefully with signs but the benefit is that the integer valued obstruction is complete for $k \neq 2$.

From now on we perform all our considerations in a ring $R = \mathbb{Z}_2$ or $R = \mathbb{Z}$. All the orientation considerations can be skipped if $R = \mathbb{Z}_2$. (This specifically applies in the proof of Theorem 1 as the $\mathbb{Z}_2$-version of the obstruction is fully sufficient there.) Let $K := \{\sigma \times \tau : \sigma, \tau \in K, \sigma \cap \tau = \emptyset\}$ denote the deleted product of $K$. We fix an orientation of every simplex of $K$. This induces an orientation of the cells $\sigma \times \tau \in K$ by the product orientation.\footnote{If $(u_1, \ldots, u_p)$ is a positive basis of $\sigma$ and $(v_1, \ldots, v_q)$ is a positive basis of $\tau$, then $((u_1, 0), \ldots, (u_p, 0), (0, v_1), \ldots, (0, v_q))$ is a positive basis of $\sigma \times \tau$.} By $C_m(K; R)$ we denote the group of $m$-chains in $K$ (for some integer $m$).\footnote{We work with cellular homology, thus the group $C_m(K; R)$ should be understood as $H_m(\tilde{K}^{(m)}, \tilde{K}^{(m-1)}, R)$ and $\sigma \times \tau$ should be understood as an oriented generator corresponding to the cell $\sigma \times \tau$ with $\dim \sigma + \dim \tau = m$. The symbol $\tilde{K}^{(s)}$ stands for $s$-skeleton of $\tilde{K}$.} This essentially means that $C_m(K; R)$ is the group of formal $R$-combinations of products $\sigma \times \tau$ with the fixed orientation as above. The boundary operator on $C_m(K; R)$ is given by

$$\partial(\sigma \times \tau) = (\partial \sigma) \times \tau + (-1)^{\dim \sigma} \sigma \times (\partial \tau).$$

By $C^m(K; R)$ we denote the group of $m$-cochains in $\tilde{K}$. These are homomorphisms from $C_m(K; R)$ to $R$. The coboundary operator, dual to the boundary operator, is given by

$$\delta_2(\sigma \times \tau) = \xi((\partial \sigma) \times \tau) + (-1)^{\dim \sigma} \xi(\sigma \times (\partial \tau))$$

for $\xi \in C^m(K; R)$. We will also need (only in dimension $2k$) a subgroup $C^{2k}_{\text{alt-sym}}(\tilde{K}; R)$ of $C^{2k}(\tilde{K}; R)$ consisting of cochains $\xi \in C^{2k}(\tilde{K}; R)$ satisfying

$$\xi(\sigma \times \tau) = (-1)^k \xi(\tau \times \sigma).$$

We will also need (only in dimension $2k - 1$) a subgroup $C^{2k-1}_{\text{sym}}(\tilde{K}; R)$ of $C^{2k-1}(\tilde{K}; R)$ consisting of cochains $\xi \in C^{2k-1}(\tilde{K}; R)$ satisfying

$$\xi(\sigma \times \tau) = \xi(\tau \times \sigma).$$

We call $\xi \in C^{2k}_{\text{alt-sym}}(\tilde{K}; R)$ alternately-symmetric and $\xi \in C^{2k-1}_{\text{sym}}(\tilde{K}; R)$ symmetric. A simple computation reveals that $\delta_2(\xi) \in C^{2k-1}_{\text{sym}}(\tilde{K}; R)$ for $\xi \in C^{2k}_{\text{alt-sym}}(\tilde{K}; R)$ and vice versa $\delta_2(\xi) \in C^{2k}_{\text{alt-sym}}(\tilde{K}; R)$ for $\xi \in C^{2k-1}_{\text{sym}}(\tilde{K}; R)$. Then the cohomology group $H^{2k}_{\text{alt-sym}}(\tilde{K}; R)$ is defined in the standard way as $H^{2k}_{\text{alt-sym}}(\tilde{K}; R) = \ker \delta_{2k} / \text{im} \delta_{2k-1}$ with respect to the coboundary operator

$$C^{2k-1}_{\text{sym}}(\tilde{K}; R) \xrightarrow{\delta_{2k-1}} C^{2k}_{\text{alt-sym}}(\tilde{K}; R) \xrightarrow{\delta_{2k}} 0.$$ \hfill (5)

The operator $\delta_{2k}$ is in particular trivial, thus $\ker \delta_{2k} = C^{2k}_{\text{alt-sym}}(\tilde{K}; R)$.

Given a general position map $f: |K| \to \mathbb{R}^k$, we have the intersection cochain $\vartheta_f \in C^{2k}_{\text{alt-sym}}(\tilde{K}; R)$ given so that $\vartheta_f(\sigma \times \tau)$ is the intersection number of $f(\sigma)$ and $f(\tau)$. (The details are postponed to Section 4.) Intuitively, the intersection number is the number of intersections between $f(\sigma)$ and $f(\tau)$; however, if $R = \mathbb{Z}$, then the intersections have to be counted carefully with signs.) This cochain satisfies $\vartheta_f(\sigma \times \tau) = (-1)^k \vartheta_f(\tau \times \sigma)$; therefore it belongs to $C^{2k}_{\text{alt-sym}}(\tilde{K}; R)$. It turns out that the cohomology class $[\vartheta_f] \in H^{2k}_{\text{alt-sym}}(\tilde{K}; R)$ is independent of the choice of $f$. This class (for arbitrary $f$) is called the van Kampen obstruction for embeddability of $K$ into $\mathbb{R}^k$ and we will denote it $\mathbf{o}(K)$. If $f$ is an embedding, then $\vartheta_f = 0$ which also implies that $\mathbf{o}(K) = [\vartheta_f] = 0$. Thus $\mathbf{o}(K)$ is indeed an obstruction for embeddability of $K$ into $\mathbb{R}^k$.

**The obstruction in a manifold.** Now let us in addition assume that $M$ is a compact PL $2k$-manifold. Let us also assume that $M$ is $R$-orientable—this condition is vacuous if $R = \mathbb{Z}_2$ while this is the standard orientability if $R = \mathbb{Z}$. By $\Omega: H_k(M; R) \times H_k(M; R) \to R$ we denote the $R$-intersection form on $M$.

If $R = \mathbb{Z}_2$ the properties of the form were sketched in the previous subsection and they are analogous for $R = \mathbb{Z}$. In general, $\Omega$ is again alternately-symmetric, that is, $\Omega(h, h') = (-1)^k \Omega(h', h)$; this is of course the same as symmetric if $R = \mathbb{Z}_2$. Given a homomorphism $\psi: C_k(K; R) \to H_k(M; R)$, we define $\omega_\psi \in C^{2k}_{\text{alt-sym}}(\tilde{K})$ by $\omega_\psi(\sigma \times \tau) := \Omega(\psi(\sigma), \psi(\tau))$. By alternating symmetry of $\Omega$ we get that $\omega_\psi$ is indeed an alternately symmetric cochain.
Then there is a PL embedding $f$.

Assume also that the restriction of $f$ to the $(k-1)$-skeleton $K^{(k-1)}$ is nullhomotopic. Then there is a homomorphism $\psi: C_k(K; R) \to H_k(M; R)$ such that

$$[\omega_\psi] - o(K) = 0.$$ 

First, let us remark that the extra assumption that the restriction of $f$ to the $(k-1)$-skeleton $K^{(k-1)}$ is nullhomotopic is always satisfied in two important cases: if either $M$ or $K$ is $(k-1)$-connected. In particular, this occurs if $K := \Delta^{(k)}$ is the $k$-skeleton of an $n$-simplex. The latter one we use in the proof of Theorem 1 and a reader interested only in the proof of Theorem 1 and willing to accept Theorem 4 as a blackbox may immediately jump to Section 4.

With slight abuse of terminology, we can consider non-existence of a homomorphism $\psi$ from the theorem as an obstruction for (almost) embeddability of $K$ to $M$, and we say that this obstruction vanishes if such homomorphism exists.

**Remarks 5.** (a) If $\Omega$ is trivial, then $\psi$ must be a trivial homomorphism, thus our obstruction coincides with the standard van Kampen obstruction.

(b) The minus sign at $o(K)$ in the statement is not important as the van Kampen obstruction is an element of order 2, $o(K) = -o(K)$.

(c) We will show that our obstruction is ‘quadratic’ in a sense that it vanishes if and only if certain system of quadratic equations has a solution; see Theorem 1.5

(d) Given a map $f: |K| \to M$, there are several ways how to describe an obstruction, depending on $f$, for existence of a homotopy from $f$ to an embedding:

1. A necessary condition for existence of such homotopy is existence of an equivariant homotopy from $f^2: |K|^2 \to M^2$ to so called isovariant map; see Harris [Har69] for details. If $K$ is a $k$-complex and $M$ is an $m$-manifold and $3k \leq 2m - 3$ (in particular if $m = 2k$ and $k \geq 3$), then this is even ‘if and only if’ condition; see [Har69] Theorem 1. This gives rise to obstruction theories in this setting; see Corollary 6 and Corollary 8 in [Har69]. From this point of view, some description of an obstruction for embedding $k$-complexes into $2k$-manifolds is not new. However, the added value of Theorem 4 is that it provides quite concrete description for all maps $f: |K| \to M$ suitable for applications.

2. A more explicit description of such obstruction appears in a work of Johnson [Joh02] (in the setting when $K$ is a $k$-complex and $M$ is a $2k$-manifold). There are some mild differences in the assumptions on $M$. In particular, Johnson works in the smooth case. However, Johnson’s setting is overall closer to our setting than Harris’ setting because he essentially works with the van Kampen obstruction. When adapted to our notation, Johnson’s obstruction is a class in $H_{alt-sym}^k(K; \mathbb{Z})$. However, it does not seem that Johnson’s approach answers which class is it. We in principle provide this answer (see the proof of Proposition 1.4) as an intermediate step in a proof of Theorem 4 though we need to assume the nullhomotopy condition as in Theorem 1.

As a counterpart to Theorem 4 using the standard tools, we will show that our obstruction is complete, if $k \geq 3$ and $M$ is $(k-1)$-connected. (We will mainly follow [FKT94] but similar ideas go back at least to Whitney [Whi44], Shapiro [Sha57] and Wu [Wu65].)

**Theorem 6 (Completeness of the obstruction).** Let $k \geq 3$, $K$ be a $k$-complex, $M$ be a compact $(k-1)$-connected (in particular orientable) PL $2k$-manifold. Assume that there is a homomorphism $\psi: C_k(K; \mathbb{Z}) \to H_k(M; \mathbb{Z})$ such that $[\omega_\psi] - o(K) = 0$ (over the integers), that is, the obstruction vanishes. Then there is a PL embedding $f: |K| \to M$.

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As our obstruction is parametrized by homomorphisms $\psi: C_k(K; R) \to H_k(M; R)$ we have been asked whether Theorem 4 can be equivalently stated in (co)homological invariants instead of (co)chains. This is indeed possible: A homomorphism $\psi: C_k(K; R) \to H_k(M; R)$ is an element of the cochain group $C^k(K; H_k(M; R))$. Each such element is a cocycle because $K$ is $k$-dimensional. It can be computed that $[\omega_\psi]$ is independent of the choice of representative $\psi$ of a cohomology class in $H^k(K; H_k(M; R))$; thus $[\omega_\psi]$ could be defined only with respect to such a cohomology class.
The Kühnel problem revisited. Theorem 6 can be used to transfer the solution to Kühnel’s problem from one manifold to another, as we discuss now. Another case, where Theorem 6 could be useful are the computational aspects that we will discuss in next subsection. In particular, in both these cases, the consideration of the obstruction over the integers is unavoidable.

In the proof of Theorem 6, it was crucial that Theorem 1 holds for almost embeddings (and not only for embeddings). However, our approach allows, under mild conditions on the manifold, to extend an upper bound on the Kühnel problem from embeddings to almost embeddings. This would be in particular interesting, if it were possible to remove the additional assumption on the embeddings in Adiprasito’s proof of the Kühnel bound (mentioned early in the introduction).

Proposition 7. Assume that \( k \geq 3 \), \( M \) is a compact orientable PL 2\( k \)-manifold, and \( M' \) is a compact \((k-1)\)-connected orientable PL 2\( k \)-manifold such that \( M \) and \( M' \) have isomorphic intersection forms over the integers. If \( \Delta^{(k)}_n \) (topologically) almost embeds into \( M \), then \( \Delta^{(k)}_n \) PL embeds into \( M' \).

Proof. Given an embedding of \( \Delta^{(k)}_n \) to \( M \), Theorem 6 implies that there is a homomorphism \( \psi : C_k(\Delta^{(k)}_n; \mathbb{Z}) \to H_k(M; \mathbb{Z}) \) such that \( [\omega_\psi] - \alpha(\Delta^{(k)}_n) = 0 \). As the intersection forms of \( M \) and \( M' \) are isomorphic, there is also a homomorphism \( \psi' : C_k(\Delta^{(k)}_n; \mathbb{Z}) \to H_k(M'; \mathbb{Z}) \) such that \( [\omega_\psi'] - \alpha(\Delta^{(k)}_n) = 0 \). Therefore, we get the required PL embedding into \( M' \) from Theorem 6.

The intersection form on \( M \) is in particular very simple if \( k \) is odd and \( M \) is closed. The former property implies that the form is antisymmetric; the latter property implies that it is unimodular (after factoring out the torsion) [Pra07, Subsection 2.7]. Therefore for a suitable choice of the basis of \( H_k(M; \mathbb{Z}) \) (after factoring out the torsion) it can be represented by a block-diagonal matrix where each block is of the form \(
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\); this is a simple exercise (and probably a well known fact). The explicit reference containing the proof we were able to find are the online lecture notes [Mor18, Claim 2.1, Lecture 7]. On the other hand, the block diagonal matrix with \( b \) blocks \(
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\) is the matrix of the intersection form of the connected sum of \( b \) copies of \( S^k \times S^k \), which is \((k-1)\)-connected. If we take this connected sum as \( M' \), Proposition 7 gives the following corollary.

Corollary 8. Assume that \( k \geq 3 \) is odd and assume that \( \Delta^{(k)}_n \) (topologically) almost embeds into a closed orientable PL 2\( k \)-manifold \( M \), then \( \Delta^{(k)}_n \) PL embeds into the connected sum \((S^k \times S^k)^{\#} \cdots \# (S^k \times S^k)^{\#}\) of \( \beta_k(M; \mathbb{Z}) \) copies of \( S^k \times S^k \).

In other words, Corollary 8 says that it we want to solve the orientable variant of the Kühnel problem for \( k \) odd, it is sufficient to solve it in the very special cases for \( M = (S^k \times S^k)^{\#} \cdots \# (S^k \times S^k)^{\#}\).

1.3 Computational aspects

Part of our motivation for introducing the obstruction for embeddability of \( K \) into \( M \) was to understand an analogue of algorithmic embeddability question from [MTW11], when the target space is \( M \) (instead of Euclidean space as in [MTW11]). For this, let \( \text{Embed}(k, M) \), for fixed \( k \) and \( M \) denote the computational problem which asks whether a \( k \)-complex \( K \) on input is embeddable into \( M \).

Question 9. For which 2\( k \)-manifolds is \( \text{Embed}(k, M) \) decidable?

This problem of course makes sense even without the assumption that \( \dim M = 2k \) but we will stay in the world of 2\( k \)-manifolds as this is the first nontrivial case. As mentioned early in this section, \( \text{Embed}(k, \mathbb{R}^{2k}) \) is decidable, even polynomial time solvable, for \( k \neq 2 \). Also, if \( k = 1 \) and \( M \) is an arbitrary (closed) surface, then \( \text{Embed}(1, M) \) is decidable, even linear time solvable [Mol99, KMR08]. If \( k = 2 \), decidability of \( \text{Embed}(k, \mathbb{R}^{2k}) \) is unknown.

For \( k \geq 3 \), our approach allows us to reformulate each instance of this problem as a certain, very special, system of quadratic Diophantine equations. This follows from Theorems 1, 6 and 15 (stated later on). Unfortunately, it is in general undecidable to determine whether a system of quadratic equations over integers has a solution [Mat70]. However, the system of equations coming from Theorem 15 is somewhat special and we suspect that it can be solved algorithmically for sufficiently nice \( M \). In any case, the reformulation of Question 9 via Theorem 15 allows to try new tools when answering Question 9.
On the other hand, if we consider the same system of quadratic equations over \( \mathbb{Z}_2 \), then solvability of such a system is decidable (in worst case by trying all options). This reflects in decidability stated in the following theorem. The properties of maps stated in the theorem are generalizations of even drawings and independently even drawings of graphs [PT04, FKMP15, FK19].

**Theorem 10.** Let us assume that \( k \geq 3 \) and \( M \) is a compact \((k-1)\)-connected PL manifold. Then, it is algorithmically decidable to determine whether a given \( k \)-complex \( K \) admits

(i) a general position map \( f: |K| \to M \) such that whenever \( \sigma \) and \( \tau \) are disjoint \( k \)-simplices of \( K \), then \( f(\sigma) \) and \( f(\tau) \) intersect an even number of times;

(ii) a general position map \( f: |K| \to M \) such that whenever \( \sigma \) and \( \tau \) are \( k \)-simplices of \( K \), then \( f(\sigma) \) and \( f(\tau) \) intersect an even number of times.

Finally, if \( M \) is compact PL and simply connected then it can be efficiently decided whether a given map \( |K| \to M \) is homotopic to an embedding. For details we refer to Remark 22. However, this remark is not very new; this is essentially just Johnson’s [Joh02] description of the obstruction, though not stated this way explicitly. We add this remark for completeness.

**Organization.** In Section 2 we properly introduce the intersection number and the intersection form. Then, Theorem 1 is proved in Section 3, Theorem 1 is proved in Section 4, and Theorems 6 and 10 are proved in Section 5. In Section 6 we mention a few open problems.

## 2 Preliminaries

Throughout the paper, we work in the PL-category. In particular, all maps and manifolds are PL, unless stated otherwise. Simplicial complexes are geometric simplicial complexes, that is, triangulations of polyhedra as in [RS72]. We assume that \( k \geq 1 \) is an integer, and \( R \) is either the ring \( \mathbb{Z} \) of integers or \( \mathbb{Z}_2 \). We assume that \( M \) is \( R \)-orientable compact (possibly with boundary) \( 2k \)-manifold, unless explicitly stated otherwise. (\( Z \)-orientability is the standard orientability, \( \mathbb{Z}_2 \)-orientability is vacuous.) In sequel ‘oriented’ stays for \( R \)-oriented and all orientation considerations should be skipped if \( R = \mathbb{Z}_2 \). We also assume that \( K \) is \( k \)-complex in which each simplex has a fixed orientation but we do not require any compatibility conditions for orientations of different simplices. By \( K^{(k-1)} \) we denote the \((k-1)\)-skeleton of \( K \). The closed interval \([0,1]\) is denoted \( I \).

### 2.1 Intersection number

In the definitions of general position and intersection number below apart from our conventions on \( M \), we also allow \( M = \mathbb{R}^{2k} \). This fills the postponed details from Subsection 1.2.

**General position.** Let \( f: |K| \to M \) be a map. We say that \( f \) is a general position map if \( f|_{K^{(k-1)}} \) is injective; there are only finitely many \( x \) with more than one preimage; each such \( x \) has exactly two preimages, which both lie in \(|K| \setminus |K^{(k-1)}|\), and the crossing of \( f \) at \( x \) is transversal. In addition, if \( M \) has nonempty boundary, we assume that \( f((|K| \setminus |K^{(k-1)}|) \subseteq M \setminus \partial M \). We will sometimes need to perturb a map \( f \) to a general position map \( f' \) by a homotopy with a support in an arbitrarily close neighborhood of \( f(|K|) \). In such case we mean to use Lemma 4.8 of [Hud69].

Sometimes, we will need a mutually general position of two maps \( f: |K| \to M \) and \( f': |K'| \to M \) where \( K' \) is another \( k \)-complex. This will be equivalent with requiring that \( f \sqcup f': |K| \sqcup |K'| \to M \) is a general position map, where ‘\( \sqcup \)’ stands for disjoint union.

**Intersection number.** Let \( f: |K| \to M \) and \( f': |K'| \to M \) be maps. Let \( \sigma \in K \), \( \tau \in K' \) be two \( k \)-simplices such that \( f|_{\sigma} \sqcup f'|_{\tau} \) is in general position. Let \( x \in M \) be an intersection point of \( f(\sigma) \) and \( f'(\tau) \), that is, \( x = f(y) = f'(y') \) for some \( y \in \sigma \) and \( y' \in \tau \). By general position, the intersection is 11It would be perhaps more natural to assume \( f(|K|) \subseteq M \setminus \partial M \) for a general position map. However, allowing the nonempty intersection of \( \partial M \) and the image of \( K^{(k-1)} \) will be useful in one of the proofs.
transversal and \( y \) is in the interior of \( \sigma \) and \( y' \) is in the interior of \( \tau \). By \( \text{sgn}_{f,f'}(x) \) we denote the sign of this intersection:

- If \( R = \mathbb{Z}_2 \), then \( \text{sgn}_{f,f'}(x) = 1 \).
- If \( R = \mathbb{Z} \), because the intersection of \( f(\sigma) \) and \( f'(\tau) \) is transversal at \( x \), there is a neighborhood \( N(x) \) of \( x \) in \( M \) and an orientation preserving PL-embedding \( g: N(x) \to \mathbb{R}^{2k} \) such that both \( g(f(\sigma) \cap N(x)) \) and \( g(f'(\tau) \cap N(x)) \) are flat. Considering the orientations of \( \sigma \) and \( \tau \) as a choice of positively oriented bases, this gives positively oriented bases of (affine spans of) \( g(f(\sigma) \cap N(x)) \) and \( g(f'(\tau) \cap N(x)) \). Then by concatenation, taking a positively oriented basis of \( g(f(\sigma) \cap N(x)) \) first, we get an orientation of \( g(N(x)) \). We set \( \text{sgn}_{f,f'}(x) = 1 \) if this orientation agrees with the orientation of \( M \) (after applying \( g \)) and \(-1\) otherwise; see Figure 1 for an example. It turns out that \( \text{sgn}_{f,f'}(x) = (-1)^k \text{sgn}_{f,f'}(x) \).

Next, the intersection number of \( f(\sigma) \) and \( f'(\tau) \) is defined as

\[
f(\sigma) \cdot f'(\tau) := \sum_x \text{sgn}_{f,f'}(x)
\]

where the sum is over all \( x \) obtained as intersection points of \( f(\sigma) \) and \( f'(\tau) \). Consequently,

\[
f(\sigma) \cdot f'(\tau) = (-1)^k f'(\tau) \cdot f(\sigma).
\]

### 2.2 Intersection form.

By \( \Omega: H_k(M; R) \times H_k(M; R) \to R \) we denote the intersection form. Intuitively, given two cycles \( z_1, z_2 \in Z_k(M; R) \) in general position, the value \( \Omega([z_1], [z_2]) \) counts the intersection number of these two cycles, which could be defined similarly as for general position maps.

**Intersection form for closed manifolds.** Now, we temporarily assume that \( M \) is closed. In this case we refer to [Pra07, Chapter 2, §2.7] for precise definition; we use the dual form \( f^* \) in [Pra07]. However, if \( R = \mathbb{Z} \), we assume that \( \Omega \) is also defined on the torsion part of \( H_k(M; R) \) and it evaluates to 0 there. (Prasolov [Pra07] points out that the form vanishes on the torsion part and he factors out the torsion—then the form is nondegenerate.) We will use the following properties of the intersection form:

(i) \( \Omega \) is a bilinear form.

(ii) \( \Omega \) is alternately-symmetric, that is, \( \Omega(a, b) = (-1)^k \Omega(b, a) \).

(iii) \( \Omega \) evaluates to 0 on the torsion part of \( H_k(M; R) \) if \( R = \mathbb{Z} \).
(iv) Let \( f: |K| \to M \) and \( f': |K'| \to M \) be maps such that \( f \sqcup f' \) is in general position. Let \( z \in Z_k(|K|; R), z' \in Z_k(|K'|; R), z = \sum n_i \sigma_i, z' = \sum n'_i \sigma'_i \) be two \( k \)-cycles, where \( n_i, n'_i \in R \) and \( \sigma_i, \sigma'_i \) are all \( k \)-simplices of \( K \) and \( K' \) respectively. Then

\[
\Omega(f_*(z)), f'_*(z')) = \sum_{i,j} n_i n'_j f(\sigma_i) \cdot f'(\sigma'_j).
\]

Property (i) follows immediately from the definition and (ii) is the contents of Theorem 2.17(b) in [Pra07]; (iii) is due to our convention. Finally, (iv) comes from the definition of the intersection number in [Pra07] Chapter 1, §5.3. For getting formula (8) we need that \( z \) and \( z' \) are cycles in mutually dual cell decompositions of \( M \) but this can be achieved by considering sufficiently fine subdivision of \( M \) and a perturbation of \( z' \). For other properties of the intersection form, we also refer to [MAP].

**Intersection form for manifolds with nonempty boundary.** As Prasolov points out [Pra07], the intersection form can be also defined for manifolds with nonempty boundary. However, one has to be a bit careful because there are two natural ways how to do it, and in the case of manifolds with boundaries these two definitions are non-equivalent. Here we provide a definition for which formula (8) remains true.

Now we assume that \( M \) is compact with nonempty boundary. Let \( M' \) be the double of \( M \). (We take two copies of \( M \) and we glue them together along their common boundary.) Let \( \Omega_{M'} \) be the intersection form on \( M' \) and we aim to define the intersection form \( \Omega = \Omega_M \) on \( M \). Let \( h_1, h_2 \) be two homology classes in \( H_k(M; R) \). Let \( z_1, z_2 \) be \( k \)-cycles with \( [z_1]_M = h_1 \) and \( [z_2]_M = h_2 \) where the subscript \( M \) indicates that their homology class is taken in \( M \). (Analogously, we use subscript \( M' \) if the homology class is taken in \( M' \).) We define

\[
\Omega_M(h_1, h_2) := \Omega_{M'}([z_1]_{M'}, [z_2]_{M'}).
\]

We note that \( \Omega_{M'}(h_1, h_2) \) is well defined because if \( z \) and \( z' \) are homologous in \( M \), then they are homologous in \( M' \) as well. We also remark that all the properties (i), (ii), (iii), (iv) remain true for \( M \) with nonempty boundary—this can be easily checked because \( M' \) satisfies them.

## 3 Van Kampen obstruction in a manifold

Throughout this section we have the same assumptions as in the beginning of Section 2 regarding the notation \( K, M \) and \( R \).

Recall from the introduction that \( \tilde{K} \) denotes the deleted product of \( K; \ C_{alt-sym}^{2k}(\tilde{K}; R) \) is the group of alternately-symmetric cochains; \( H_{alt-sym}^{2k}(\tilde{K}; R) \) is the corresponding cohomology group; and \( \partial(K) \in H_{alt-sym}^{2k}(\tilde{K}; R) \) is the van Kampen obstruction. Let us also recall that \( \partial(K) = [\vartheta_f] \) where \( \vartheta_f \) is the intersection cocycle of an arbitrary general position map \( f: |K| \to \mathbb{R}^{2k} \).

We generalize the intersection cocycle to maps with codomain \( M \): Given a general position map \( f: |K| \to M \) we define the intersection cocycle for \( f \) as \( \vartheta_f \in C_{alt-sym}^{2k}(\tilde{K}; R) \) via

\[
\vartheta_f(\sigma \times \tau) = f(\sigma) \cdot f(\tau).
\]

It follows from (7) that \( \vartheta_f \) is alternately-symmetric as required. We also define the van Kampen obstruction of the homotopy class of \( f \) as the cohomology class \( \partial_f(K) := [\vartheta_f] \in H_{alt-sym}^{2k}(\tilde{K}; R) \).

**Lemma 11.** Let \( f, f': |K| \to M \) be homotopic general position maps. Then \( [\vartheta_f] = [\vartheta_{f'}] \). Equivalently, \( \partial_f(K) = \partial_{f'}(K) \).

The proof of Lemma 11 is given in [Sha57] Lemma 3.5] and reproduced, e.g., in [FKT94] Lemma 1] in the case that \( M = \mathbb{R}^{2k} \). The proof is based on an existence of a homotopy between \( f \) and \( f' \) and can be used essentially in verbatim in our setting.

**Lemma 12.** Let \( f: |K| \to M \) be a general position map such that the restriction of \( f \) to \( |K^{(k-1)}| \) is nullhomotopic. Then there is a PL \( 2k \)-ball \( B \) in \( M \) and a general position map \( f': \ |K| \to M \setminus B \) homotopic to \( f \) (in \( M \)) such that \( f'(|K^{(k-1)}|) \subseteq \partial B \).

10
Proof. Because the restriction of \( f \) to \( |K^{(k-1)}| \) is nullhomotopic, by the homotopy extension property \[ Hat01 \] Proposition 0.16 there is \( f'' : |K| \to M \) homotopic to \( f \) such that the restriction of \( f'' \) to \( |K^{(k-1)}| \) is constant. Let \( x = f''(|K^{(k-1)}|) \) and let \( B \) be 2k-ball such that \( x \in \partial B \). By further homotopy, we can get \( f''' : |K| \to M \) such that \( f''' \) restricted to \( |K^{(k-1)}| \) is in general position, and \( f'''(|K^{(k-1)}|) \subseteq \partial B \). (We first perform the homotopy on \( |K^{(k-1)}| \) and then we use the homotopy extension property again.) Finally, by next homotopy fixed on \( |K^{(k-1)}| \) we push the image of \( |K| \setminus |K^{(k-1)}| \) outside \( B \) so that the resulting map is in general position, obtaining the required \( f' \).

Now the shift of \( f \) to \( f' \) in the previous lemma allows us to easily compare the intersection cochain \( \vartheta' \) with the intersection cochain \( \vartheta_g \) of another map \( g \) which is fully inside \( B \). The advantage of using \( g \) is that this is essentially the case in \( \mathbb{R}^{2k} \). Therefore let \( B \subseteq M \) be a 2k-ball in \( M \). Let \( f' : |K| \to \overline{M \setminus B} \) and \( g : |K| \to B \) be two general position maps such that \( f'|_{|K^{(k-1)}|} = g|_{|K^{(k-1)}|} \). (In particular \( f'(K^{(k-1)}) = g(K^{(k-1)}) \subseteq \partial B \).

Now for a \( k \)-simplex \( \sigma \in K \) let \( z_\sigma \) be the (singular) \( k \)-cycle \( f'(\sigma) - g(\sigma) \). We also define \( \omega_{f',g} \in C_{alt-sym}^2(K;R) \) via

\[
\omega_{f',g}(\sigma \times \tau) := \Omega([z_\sigma],[z_\tau]).
\]

\textbf{Lemma 13.} \( \vartheta_{f'} = \omega_{f',g} - \vartheta_g \).

\textit{Proof.} For \( \sigma \times \tau \in K \) we have

\[
\omega_{f',g}(\sigma \times \tau) = \Omega([z_\sigma],[z_\tau]) = f'(\sigma) \cdot f'(\tau) + g(\sigma) \cdot g(\tau) = \vartheta_f(\sigma \times \tau) + \vartheta_g(\sigma \times \tau).
\]

The second equality follows from the fact that \( f'(\sigma) \cap g(\tau) = g(\sigma) \cap f'(\tau) = \emptyset \) and from [8].

\textbf{Proposition 14.} Let \( f : |K| \to M \) be a general position map with \( \vartheta_f = 0 \). Assume that the restriction of \( f \) to \( |K^{(k-1)}| \) is nullhomotopic. Then there is a homomorphism \( \psi : C_k(K;R) \to H_k(M;R) \) such that \( [\omega_f] = o(K) \) is trivial.

\textit{Proof.} Let \( f' : |K| \to \overline{M \setminus B} \) be the map obtained from Lemma 12. Take an arbitrary general position map \( g : |K| \to B \) which coincides with \( f' \) on \( \partial B \) and define \( z_\sigma \) and \( \omega_{f',g} \) as above. By Lemma 13 and Lemma 14 we get \([\omega_{f',g} - \vartheta_g] = [\vartheta_f] = [\vartheta_f] = [0] = 0 \). Let us define \( \psi(\sigma) \) to be the homology class of \( z_\sigma \) in \( H_k(M;R) \). Then, according to the earlier definition of \( \omega_f \), we get \( \omega_f = \omega_{f',g} \) and \( o(K) = [\vartheta_g] \). Therefore \([\omega_f] - o(K) = [\omega_{f',g} - \vartheta_g] = 0 \).
System of quadratic equations. Our next aim is to describe an existence of almost embedding via solvability of a certain system of quadratic equations.

Let $\eta, \mu \in K$ be a $(k-1)$-simplex and $k$-simplex respectively and assume that $\eta$ and $\mu$ are disjoint. For every such pair we define a variable $x_{\eta,\mu}$.

Next we need to distinguish whether $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$. If $R = \mathbb{Z}$, assume that $H_k(M; \mathbb{Z}) \cong \mathbb{Z}^b \oplus T_k(M; \mathbb{Z})$ where $T_k(M; \mathbb{Z})$ is the torsion. Let $\pi: H_k(M; \mathbb{Z}) \to \mathbb{Z}^b$ be the homomorphism obtained from the isomorphism above after factoring out the torsion. If $R = \mathbb{Z}_2$, then $H_2(M; \mathbb{Z}_2) \cong \mathbb{Z}_2^b$ for some $b$ and we take an arbitrary isomorphism $\pi: H_k(M; \mathbb{Z}_2) \to \mathbb{Z}_2^b$.

Let $A_\Omega \in R^{b \times b}$ be the matrix of $\Omega$, that is, for every $h, h' \in H_k(M; R)$ we have $\Omega(h, h') = \pi(h)^T A_\Omega \pi(h')$. For every $k$-simplex $\sigma$ and every $i \in \{1, \ldots, b\}$ we define an integer variable $y^i_\sigma$ and we set $y^i_\sigma := (y^1_\sigma, \ldots, y^b_\sigma)$. Let $\vartheta^i_\sigma$ be any fixed intersection cochain in the class $o_K$; an explicit choice is described in [MTW11, App. D].

Now consider a system of quadratic equations with variables $x_{\eta,\mu}$ and $y^i_\sigma$ over $R$ given by the following equation for each pair $(\sigma, \tau)$ of disjoint $k$-simplices.

$$\sum_{\eta,\mu} x_{\eta,\mu} \delta \varepsilon_{\eta,\mu}(\sigma \times \tau) + y^i_\sigma A_\Omega y^i_\tau = \vartheta^i_\sigma(\sigma \times \tau).$$

(11)

We remark that swapping $\sigma$ and $\tau$ gives the same equation as both sides are alternately-symmetric.

**Theorem 15.** Let $M$ be a compact $R$-orientable PL $2k$-manifold. Then there is a homomorphism $\psi: C_k(K; R) \to H_k(M; R)$ such that $[\omega_\psi] - o(K)$ is trivial (considered over $R$) if and only if the system of equations (11) has a solution in $R$.

**Proof.** First assume that $[\omega_\psi] - o(K)$ is trivial, hence $\omega_\psi$ and $\vartheta^i_\sigma$ differ by a linear combination of cochains $\delta \varepsilon_{\eta,\mu}$ by (10). Thus, there are $x_{\eta,\mu} \in R$, one for each $\delta \varepsilon_{\eta,\mu}$, such that for every $\sigma \times \tau \in \tilde{K}$ we get $\omega_\psi(\sigma \times \tau) - \vartheta^i_\sigma(\sigma \times \tau) = \sum_{\eta,\mu} x_{\eta,\mu} \delta \varepsilon_{\eta,\mu}(\sigma \times \tau)$. We also set $y^{i}_\sigma := (y^1_\sigma, \ldots, y^b_\sigma)$ and we get a solution of (11).

Now assume that we have a solution of (11). For a $k$-simplex $\sigma \in K$, we define $\psi(\sigma)$ as an arbitrary element in $\pi^{-1}(y^{i}_\sigma)$ and we extend $\psi$ to a homomorphism from $C_k(K; R)$ to $H_k(M; \mathbb{Z})$. We get $\omega_\psi(\sigma \times \tau) = y^i_\tau A_\Omega y^i_\tau$. Therefore, from (11), we get that $\omega_\psi - \vartheta^i_\sigma$ is a linear combination of cochains $\delta \varepsilon_{\eta,\mu}$ by (10). This gives that $\omega_\psi$ and $\vartheta^i_\sigma$ belong to the same cohomology class of $H^{2k}_{alt-sym}(\tilde{K}; R)$; that is, $[\omega_\psi] - o(K)$ is trivial.

\[ \square \]

4. Kühlne question

In this section we prove Theorem 4. However; first we sketch a proof of Theorem 4 with slightly weaker bound $n \leq 2k + 1 + (k + 2)\beta_k(M; \mathbb{Z}_2)$. For a full proof of Theorem 4, it is not necessary to follow this sketch. However, it may help to understand why do we prepare some auxiliary claims.

**Sketch of Theorem 4 with a weaker bound $n \leq 2k + 1 + (k + 2)\beta(M; \mathbb{Z}_2)$.** Let us assume that $K = \Delta_2^k$ almost embeds into $M$. By Theorem 4 there is $\psi: C_k(K; \mathbb{Z}_2) \to H_k(M; \mathbb{Z}_2)$ such that $[\omega_\psi] + o(K) = 0$. Consider an induced subcomplex $J$ of $K$ on $2k + 3$ vertices (if $n \geq 2k + 2$, otherwise we are done). It is well known that the $\mathbb{Z}_2$ van Kampen obstruction of this complex is nonzero. The formula $[\omega_\psi] + o(K) = 0$
and 

\[ (C_2) \]

vertices of \((C_1)\). Although we do not need it, we note that the proof of Theorem 3 in \([Kyn20]\) shows that the other

then by Theorem 4 there is \(\psi\) over \(Z\) and by \((\Psi)\) shows that the number of vertices \(K\) is at most \(k\) of the form by 2. These improvements require a more subtle analysis.

\[ k \]

k shows that the number of vertices \(K\) is at most \(k\) of the form by 2. These improvements require a more subtle analysis.

Throughout this section, let \(k,n \geq 1\). We set \(K := \Delta_{(k)}\) to be the \(k\)-skeleton of an \(n\)-simplex while \(M\) is a compact PL \(2k\)-manifold. We work only with \(Z_2\) coefficients, that is, we set \(R = Z_2\). Note the notions ‘alternately symmetric’ and ‘symmetric’ coincide over \(Z_2\). In particular, \(\Omega\) is a symmetric bilinear form over \(Z_2\) in this case. We also systematically replace ‘alt-sym’ with ‘sym’ in expressions such as \(C_{alt-sym}^k(K;Z_2)\).

Now we state the two main ingredients for the proof of Theorem 1. Given a vertex \(v\) of \(\Delta_k\), and a simplex \(\sigma \in \Delta_k\) not containing \(v\), by \(\sigma \ast v\) we denote the join of \(\sigma\) and \(v\), that is, the simplex formed by vertices of \(\sigma\) and \(v\). Note that if \(\kappa\) is a \((k+1)\)-simplex in \(\Delta_n\), then \(\partial \kappa\) belongs to \(K\).

**Proposition 16.** Assume that \(K\) almost embeds into \(M\), then there is a symmetric bilinear form \(\Lambda\) on \(Z_\kappa(K;Z_2) = H_\kappa(K;Z_2)\) of rank at most \(\beta_{(k)}(M;Z_2)\) satisfying the following conditions.

\(\text{(C1)}\) Let \(\kappa, \kappa'\) be disjoint \((k+1)\)-simplices in \(\Delta_n\). Then \(\Lambda(\partial \kappa, \partial \kappa') = 0\).

\(\text{(C2)}\) Let \(J\) be an induced subcomplex of \(K\) on \(2k+3\) vertices (that is, \(J\) is isomorphic to \(\Delta_{2k+2}\)). Then for every vertex \(v\) of \(J\) we get

\[ \sum_{(\sigma,\tau) \in P_{J,v}} \Lambda(\partial(\sigma \ast v), \partial(\tau \ast v)) = 1 \]

where \(P_{J,v}\) is the set of all unordered pairs \(\{\sigma, \tau\}\) of disjoint \(k\)-simplices in \(J\) avoiding \(v\).

In addition, if \(\Omega(h,h) = 0\) for every \(h \in H_\kappa(M;Z_2)\), then \(\Lambda(z,z) = 0\) for every \(z \in Z_\kappa(K;Z_2)\).

Relation between Proposition 15 and the preceding sketch is the following: If \(K\) embeds into \(M\), then by Theorem 3 there is \(\psi : C_\kappa(K;R) \to H_\kappa(M;R)\) such that \([\omega_{\psi}] = \Omega(K) = 0\). We will take \(\Lambda\) so that \(\Lambda(z,z') = \Omega(\psi(z), \psi(z'))\). Then the conditions (C1) and (C2) verify the conditions required in the sketch. Although we do not need it, we note that the proof of Theorem 3 in \([Kyn20]\) shows that the other implication can be partially reverted for \(k = 1\): If (C1) and (C2) are satisfied for \(\Lambda\) obtained from \(\Omega\) and \(\psi\) as above, then \([\omega_{\psi}] + \Omega(K)\) is trivial.

**Proposition 17.** Assume that \(\Lambda\) is a symmetric bilinear form on \(Z_\kappa(K;Z_2)\) satisfying conditions (C1) and (C2) in Proposition 16.

Then

\[ n \leq (2k + 1) + (k + 1) \text{ rank } \Lambda \quad \text{and} \quad \begin{equation} n \leq (2k + 1) + \frac{(k + 2) \text{ rank } \Lambda}{2} \quad \text{if } \Lambda(z,z) = 0 \text{ for all } z \in Z_\kappa(K;Z_2). \tag{12} \end{equation} \]

Assuming the two propositions above, Theorem 1 follows immediately:
Proof of Theorem 4 Assume that $K$ almost embeds into $M$. Let $\Lambda$ be the symmetric bilinear form on $Z_k(K;\mathbb{Z}_2)$ obtained from Proposition 16. Because $\text{rank } \Lambda \leq \beta_k(M;\mathbb{Z}_2)$, we immediately deduce Theorem 1 from (12). If, in addition, $\Omega(h,h) = 0$ for every $h \in H_k(M;\mathbb{Z}_2)$, then $\Lambda(z,z) = 0$ for every $z \in Z_k(M;\mathbb{Z}_2)$ and we deduce Theorem 1(iii) from (13).

Propositions 16 and 17 are proved in forthcoming subsections.

In fact we conjecture that the bounds given by Proposition 17 can be improved to the Kühnel bounds:

Conjecture 18. Assume that $\Lambda$ is a symmetric bilinear form on $Z_k(K;\mathbb{Z}_2)$ satisfying conditions (C1) and (C2) (in Proposition 16).

Then
\[
\begin{pmatrix}
(n-k-1) \\
(k+1)
\end{pmatrix} \leq \begin{pmatrix}
2k+1 \\
2k+1
\end{pmatrix} \text{rank } \Lambda.
\]

Proposition 16 and Conjecture 18 together imply Kühnel’s conjecture (for PL manifolds) in the same way as Theorem 1 is proved. In fact they imply even something stronger. (It is not necessary to assume $(k-1)$-connectedness and the conclusion holds for almost-embeddings.)

Computer-assisted bounds. In our proof of Theorem 1, we do not use Proposition 16 in full strength—at least for small values the bounds can be improved: If $b$ is odd, all non-degenerate symmetric bilinear forms on $Z_2^b$ are equivalent to the form with the identity matrix $I_b$. If $b = 2c$, we furthermore have symplectic forms—forms equivalent to
\[
Q_b = \begin{pmatrix}
0 & I_c \\
I_c & 0
\end{pmatrix}.
\]

These are the forms satisfying $\Lambda(z,z) = 0$ for every $z$. The matrix of $\Lambda$ can hence be written as $A_\Lambda = B^T XB$, where $X = I_{2b}$ or $X = Q_b$ for $b' = \text{rank } \Lambda \leq \beta_k(M;\mathbb{Z}_2)$, and $B$ is a $b' \times \dim Z_k(K;\mathbb{Z}_2)$ matrix over $\mathbb{Z}_2$. Proposition 16 then translates into equations over $\mathbb{Z}_2$ which $B$ needs to satisfy. For small values these equations can be turned into a CNF formula and checked by modern SAT solvers, preferably ones that support xor clauses, e.g. CryptoMiniSat [SNC09]. Using this technique we obtain computer assisted bounds in Table 1.

4.1 Proof of Proposition 16

Given $\psi$ as in Theorem 1, as announced earlier, we define a symmetric bilinear form $\Lambda$ on $Z_k(K;\mathbb{Z}_2)$ via $\Lambda(z,z') := \Omega(\psi(z),\psi(z'))$. We observe that the rank of $\Lambda$ is at most the rank of $\Omega$ which is at most $\beta_k(M;\mathbb{Z}_2)$. We also observe that if $\Omega(h,h) = 0$ for every $h \in H_k(M;\mathbb{Z}_2)$, then $\Lambda(z,z) = 0$ for every $z \in Z_k(K;\mathbb{Z}_2)$. These are some of the conditions required on $\Lambda$ in Proposition 16.

For the proof of Proposition 16 we also need the following lemma.

Lemma 19. Let $\Xi \in H^2_{\text{sym}}(\tilde{K};\mathbb{Z}_2)$ be a cohomology class. Let $c$ be a $2k$-chain in the ordinary chain group $C_{2k}(\tilde{K};\mathbb{Z}_2)$ such that $\partial c$ is symmetric. (That is, $\sigma \times \eta$ and $\eta \times \sigma$ appear with the same coefficient in $\partial c$ for any $k$-simplex $\sigma$ and $(k-1)$-simplex $\eta$.) Then the value $\xi(c)$ is independent of the choice of the representative $\xi \in C^k_{\text{sym}}(\tilde{K};\mathbb{Z}_2)$ with $[\xi] = \Xi$.

Proof. Let $\xi,\xi' \in C^k_{\text{sym}}(\tilde{K};\mathbb{Z}_2)$ be such that $[\xi] = [\xi']$. Let $\zeta = \xi - \xi'$. By the previous condition, $\zeta$ is cohomologically trivial, thus $\zeta = \partial \rho$ for some $\rho \in C^{k+1}_{\text{sym}}(\tilde{K};\mathbb{Z}_2)$. Then we get
\[
\xi(c) = \xi'(c) = \zeta(c) = \delta \rho(c) = \rho(\partial c) = 0.
\]

The last equality above follows from the facts that $\rho(\sigma \times \eta) = \rho(\eta \times \sigma)$ for any $k$-simplex $\sigma$ and $(k-1)$-simplex $\eta$ and that $\partial c$ is symmetric. Thus we get $\xi(c) = \xi'(c)$ as required.

Now, let $\psi$ be a homomorphism as in Theorem 1. The conclusion of Theorem 1 is that $[\omega_\psi]$ and $o(K)$ are the same homology class. On the one hand $\omega_\psi$ is a representative of this cohomology class; on the other hand an arbitrary intersection cochain $\theta_g$ of a general position map $g : |K| \to \mathbb{R}^{2k}$ is a representative as well. Consequently, Lemma 19 gives
\[
\omega_\psi(c) = \theta_g(c)
\]
for $c \in C_{2k}(\tilde{K};\mathbb{Z}_2)$ such that $\partial c$ is symmetric. (14)

Our strategy is that we will deduce Proposition 16 from (14) by suitable choices of $c$ and $\theta_g$. 

14
Kühnel’s conjecture

max $n$, $\Lambda \sim I$
max $n$, $\Lambda$ symplectic
Kühnel’s conjecture

| $k$ | $\beta$ | $n$ | $\Lambda \sim I$ | $n$ | $\Lambda$ symplectic | $\Lambda \sim I$ |
|-----|-----|-----|----------------|-----|---------------------|-----------------|
| 1   | 5   | 6   | (Prop 17)      | 5   | 6                   | 5               |
| 2   | 1   | 8   | (Prop 17)      | -   | -                   | -               |
| 3   | 2   | 1   | 8 (Prop 17)    | -   | -                   | -               |
| 4   | 1   | 9 ≤ $n$ ≤ 10 | -   | -               | 11             |
| 4   | 1   | 14  | (Prop 17)      | -   | 14                 |                 |

Table 1: The table gives maximal $n$ for which there is a symmetric bilinear form $\Lambda$ on $Z_k(\Delta_n^{(k)}, \mathbb{Z}_2)$ of rank $\beta$ satisfying conditions (C1) and (C2) of Proposition 16 distinguishing whether $\Lambda \sim I$ or whether $\Lambda$ is symplectic. Via Proposition 16, if $\Delta_n^{(k)}$ almost embeds into $M$ with $\beta_k(M, \mathbb{Z}_2) = \beta$ and the intersection form $\Omega \sim \Lambda$, then $n'$ ≤ $n$ where $n$ is the value in the table. (Strictly speaking, this requires checking that $\Omega \sim \Lambda$ in the proof of Proposition 16.)

For the values in bold, there is a matching lower bound in a strong sense: There exists a $(k-1)$-connected closed manifold $M$ with $\beta_k(M, \mathbb{Z}_2) = \beta$ and the intersection form $\Omega \sim \Lambda$ such that $\Delta_n^{(k)}$ embeds into $M$. Indeed for $k=1$ and $\beta \leq 4$, the lower bounds is given by the Ringel-Youngs theorem [RY68, Rin74]. For $k=2$ and $\beta_2 = 1$, there is Kühnel’s 9-point triangulation of $CP^2$ [KBS3]. Taking its two skeleton shows that $\Delta_n^{(2)}$ embeds into $CP^2$. For this reason $\Delta_n^{(2)}$ also embeds into $CP^2 \times CP^2$ (for which $\beta_2 = 2$). The case $k=4, \beta_k(M, \mathbb{Z}_2)$ corresponds to the 4-skeleton of the 15-point triangulation of the quaternionic projective plane [Gar19].

Last but not least, in the 5th column, we provide a bound that would follow from Kühnel’s conjecture (which does not distinguish type of $\Lambda$). It is interesting to note that in the cases $(k, \beta) = (2, 2), (3, 1)$ our computer assisted bounds even beat Kühnel’s conjecture.

Proof of Proposition 16(i). Because $K$ almost embeds into $M$, there is $\psi: C_k(K; \mathbb{Z}_2) \to H_k(M; \mathbb{Z}_2)$ such that $[\omega_\psi] = [K]$ by Theorem 4. Take $\Lambda$ via $\Lambda(z, z') := \Omega(\psi(z), \psi(z'))$ as described above.

Take the 2k-chain $c \in C_{2k}(K; \mathbb{Z}_2)$ given by $c = \partial k \times \partial k'$. Note that $c$ is actually a cycle. Thus $\partial c = 0$ and $c$ satisfies the assumptions of Lemma 19. Let $g: |K| \to \mathbb{R}^{2k}$ be an arbitrary general position map such that $g(\partial k)$ and $g(\partial k')$ are disjoint. (This is possible because $\kappa$ and $\kappa'$ are disjoint.) Then we get $\Lambda(\partial k, \partial k') = \Omega(\psi(\partial k), \psi(\partial k')) = \omega_\psi(c) = \partial g(c) = 0$ as required. The first equality is the definition of $\Lambda$; the second equality comes from the definition of $\omega_\psi$; the third equality is the contents of [4]; and the last equality follows from our choice of $g$ which implies that the intersection cochain $\psi_\partial$ is identically zero on $c$.

For a proof of Proposition 16(ii), we will need another auxiliary lemma which will be also reused in a proof of Proposition 17. Given an induced subcomplex $J$ of $K$ on $2k+3$ vertices let $P_J$ be the set of all unordered pairs $\{\sigma', \tau'\}$ of disjoint k-simplices in $J$. We also consider a chain $c_J \in C_{2k}(J; \mathbb{Z}_2) \subseteq C_{2k}(K; \mathbb{Z}_2)$ given by

$$c_J := \sum_{\{\sigma', \tau'\} \in P_J, \sigma' \prec \tau'} \sigma' \times \tau',$$

where $\prec$ is an arbitrary fixed order on k-simplices of $K$. We check that $c_J$ satisfies the assumptions of Lemma 19 that is, $\partial c_J$ is symmetric: Let $t$ be the isomorphism of $C_{2k}(K; \mathbb{Z}_2)$ swapping the coordinates. Then $c_J + t(c_J)$ is a cycle formed by all products $\sigma' \times \tau'$ over all ordered pairs $(\sigma', \tau')$ of disjoint k-simplices of $J$. Therefore $\partial c_J + \partial t(c_J) = 0$ which implies $\partial c_J = \partial t(c_J)$ (as we work over $\mathbb{Z}_2$). Consequently $\partial c_J$ is symmetric as required.

Lemma 20. Let $\Lambda$ be a symmetric bilinear form on $Z_k(K; \mathbb{Z}_2)$. Let $J$ be an induced subcomplex of $K$ on $2k+3$ vertices. Then for arbitrary vertex $v$ of $J$ the value:

$$\sum_{(\sigma, \tau) \in P_J, v} \Lambda(\partial (\sigma + v), \partial (\tau + v))$$

(15)
is independent of the choice of $v$. In addition, if $\psi: C_k(K; \mathbb{Z}_2) \to H_0(M, \mathbb{Z}_2)$ is a homomorphism and
\[\Lambda(z, z') = \Omega(\psi(z), \psi(z'))\]
for every $z, z' \in \mathbb{Z}_k(K; \mathbb{Z}_2)$, then the value of the sum (15) equals $\omega_\psi(c_J)$.

**Proof.** First, we extend $\Lambda$ to a symmetric bilinear form $\Lambda'$ on $C_k(K; \mathbb{Z}_2)$. If we are in the ‘in addition’
case that $\Lambda(z, z') = \Omega(\psi(z), \psi(z'))$, then we simply set $\Lambda'(\sigma, \tau) = \Omega(\psi(\sigma), \psi(\tau))$ for $k$-simplices $\sigma$ and $\tau$ of $J$. In any other case, $\Lambda'$ can be obtained in the following way: We pick a vertex $v_0 \in J$. Then for $k$-simplices $\sigma$ and $\tau$ in $J$ we set
\[\Lambda'(\sigma, \tau) = \begin{cases} 0 & \text{if } \sigma \text{ or } \tau \text{ contains } v_0; \\ \Lambda(\partial(\sigma \ast v_0), \partial(\tau \ast v_0)) & \text{otherwise}. \end{cases}\]

It is easy to check that $\Lambda(\partial(\sigma \ast v_0), \partial(\tau \ast v_0)) = \Lambda'(\partial(\sigma \ast v_0), \partial(\tau \ast v_0))$ if both $\sigma$ and $\tau$ avoid $v_0$. In addition, because the cycles $\partial(\sigma \ast v_0)$ for $\sigma$ avoiding $v_0$ generate $\mathbb{Z}_k(K; \mathbb{Z}_2)$, we get $\Lambda(z, z') = \Lambda'(z, z')$ for any $z, z' \in \mathbb{Z}_k(K; \mathbb{Z}_2)$.

Our aim is to show that
\[\sum_{(\sigma, \tau) \in P_{J, v}} \Lambda(\partial(\sigma \ast v), \partial(\tau \ast v)) = \sum_{(\sigma', \tau') \in P_J} \Lambda'(\sigma', \tau').\] (16)

Because the right-hand side of (16) is independent of the choice of $v$, this will prove the first part of the lemma. If we are in the ‘in addition’ case, then we further get
\[\sum_{(\sigma', \tau') \in P_J} \Lambda'(\sigma', \tau') = \sum_{(\sigma', \tau') \in P_J} \Omega(\psi(\sigma'), \psi(\tau')) = \omega_\psi(c_J)\]
where the last equality follows from the definitions of $\omega_\psi$ and $c_J$ (from $c_J$ we only need that for every $\{\sigma', \tau'\} \in P_J$ exactly one of the products $\sigma' \ast \tau'$ and $\tau' \ast \sigma'$ appears with coefficient 1 in $c_J$). This proves the lemma as soon as we show (16).

By bilinearity of the intersection form,
\[\sum_{(\sigma, \tau) \in P_{J, v}} \Lambda(\partial(\sigma \ast v), \partial(\tau \ast v)) = \sum_{(\sigma, \tau) \in P_{J, v}} \Lambda'(\partial(\sigma \ast v), \partial(\tau \ast v)) = \sum_{\{\sigma', \tau'\} \in Q_J} a_{\sigma', \tau'} \Lambda'(\sigma', \tau'),\]
where $Q_J$ is the set of all (unordered) pairs of distinct $k$-simplices in $J$ and $a_{\sigma', \tau'}$ is the number of appearances of $\sigma' \subseteq \sigma \ast v$, $\tau' \subseteq \tau \ast v$ or $\sigma' \subseteq \tau \ast v$, $\tau' \subseteq \sigma \ast v$ over all unordered pairs $\{\sigma, \tau\} \in P_{J, v}$, modulo 2. Therefore, for checking (16), it remains to show that $a_{\sigma', \tau'} = 1$ if $\{\sigma', \tau'\} \in P_J$ (that is, $\sigma'$ and $\tau'$ are disjoint) and $a_{\sigma', \tau'} = 0$ if $\{\sigma', \tau'\} \in Q_J \setminus P_J$ ($\sigma'$ and $\tau'$ are not disjoint). We also remark that for any $\{\sigma, \tau\} \in P_{J, v}$, only one of the two options above for appearance is possible, thus we can safely assume $\sigma' \subseteq \sigma \ast v$ and $\tau' \subseteq \tau \ast v$ when counting.

If $\sigma'$ and $\tau'$ share a vertex different from $v$, then there is no appearance as $\sigma$ and $\tau$ are required to be in disjunct and consequently $\sigma \ast v$ and $\tau \ast v$ share only $v$.

If $\sigma'$ and $\tau'$ share $v$ but no other vertex, then there are exactly two vertices $w_1, w_2$ of $J$ outside $\sigma' \cup \tau'$. Consequently, there are two appearances $\sigma = (\sigma' \ast v \ast w_1, \tau = (\tau' \ast v \ast w_2, \sigma = (\sigma' \ast w_1) \ast (\tau' \ast w_2)$, $\tau = (\tau' \ast w_1)$,

If neither $\sigma'$ nor $\tau'$ contains $v$, then there is the exactly one appearance $\sigma = \sigma', \tau = \tau'$.

If exactly one of the simplices $\sigma'$, $\tau'$ contains $v$, say $\sigma'$ contains $v$, then there is exactly one appearance $\sigma = (\sigma' \ast v \ast w, \tau = \tau'$ where $w$ is the vertex of $J$ not in $\sigma' \cup \tau$. \hfill $\Box$

**Proof of Proposition 16(ii).** We will take $\psi$ and $\Lambda$ in the same way as in the proof of (i). It is well known that there is a general position map $g: |K| \to \mathbb{R}^{2k}$ such that $\partial_{g}(c_J) = 1$ with $c_J$ defined above Lemma 20 see, e.g., [Mel09, Example 3,5]. Then by Lemma 20 and by (14) we get
\[\sum_{(\sigma, \tau) \in P_{J, v}} \Lambda(\partial(\sigma \ast v), \partial(\tau \ast v)) = \omega_\psi(c_J) = \partial_{g}(c_J) = 1\]
as required. \hfill $\Box$
4.2 Proof of Proposition 17

Proof. We will prove both items of Proposition 17 simultaneously by induction in the rank of \( \Lambda \). If \( \text{rank} \Lambda = 0 \), then \( \Lambda(z, z') = 0 \) for any two cycles \( z, z' \in Z_k(K; \mathbb{Z}) \). In particular, (12) can only be satisfied in this case if the there is no induced subcomplex \( J \) of \( K \) on \( 2k + 3 \) vertices, i.e., if \( n \leq 2k + 1 \). (Recall that \( \Delta_n \) has \( n + 1 \) vertices.) This yields the base for the induction in both cases.

Assume now that we are in the case (12) and not in (13), which has a better bound. Since \( \partial \kappa \), where \( \kappa \) runs through all \( (k + 1) \)-simplices in \( K \), generate \( Z_k(K; \mathbb{Z}) \), there is a \( (k + 1) \)-simplex \( \kappa \) for which \( \Lambda(\partial \kappa, \partial \kappa) = 1 \). Up to reordering the vertices, we may assume that \( \kappa \) is the simplex on the last \( (k + 2) \) vertices of \( K \). Define\(^{12}\)

\[
\Lambda'(x, y) := \Lambda(x, y) - \Lambda(x, \partial \kappa) \Lambda(\partial \kappa, y).
\]  

(17)

Then \( \Lambda' \) is a symmetric bilinear form. If for some \( y_0 \in Z_k(K; \mathbb{Z}) \), \( \Lambda(x, y_0) = 0 \) for all \( x \in Z_k(K; \mathbb{Z}) \), the same is true for \( \Lambda' \). Moreover \( \Lambda'(x, \partial \kappa) = 0 \) for every \( x \), yet \( \Lambda(\partial \kappa, \partial \kappa) \neq 0 \). In other words, \( \text{Ker} \Lambda \subseteq \text{Ker} \Lambda' \), and so \( \text{rank} \Lambda' \leq \text{rank} \Lambda - 1 \). Let \( K' \) be the subcomplex of \( K' \) formed by the first \( n - k - 1 \) vertices of \( K \).

We are going to show that \( \Lambda' \) restricted to \( Z_k(K'; \mathbb{Z}) \) satisfies both (11) and (12) (see Proposition 16). Due to (11) for \( \Lambda \) and (17), \( \Lambda(x, y) = \Lambda'(x, y) \) as long as at least one of the chains \( x, y \) is disjoint with \( \kappa \). This is clearly true when verifying (11) for \( \Lambda' \) on \( Z_k(K'; \mathbb{Z}) \). It also holds when verifying (12), if \( v \neq v_{n-k-1} \). However, Lemma 20 then tells us that (12) holds for \( v = v_{n-k-1} \) as well.

By induction, \( n - k - 1 \leq (2k + 1) + (k + 1) \text{rank} \Lambda' \), leading to

\[
n \leq (2k + 1) + (k + 1) (\text{rank} \Lambda' + 1) \leq (2k + 1) + (k + 1) \text{rank} \Lambda.
\]

Let us now prove the case (13). If \( \Lambda(z, z) = 0 \) for every \( z \in Z_k(K; \mathbb{Z}) \) and the rank of \( \Lambda \) is non-zero, there are two simplices \( \kappa \) and \( \kappa' \) for which \( \Lambda(\partial \kappa, \partial \kappa') = 1 \). By reordering the vertices, if necessary, we may assume that \( \kappa \) is the subcomplex on the last \( (k + 2) \) vertices of \( K \). Define\(^{13}\)

\[
\Lambda'(x, y) := \Lambda(x, y) - \Lambda(\partial \kappa, y) \Lambda(x, \partial \kappa') - \Lambda(\partial \kappa', y) \Lambda(x, \partial \kappa).
\]

(18)

This is a symmetric bilinear form. If for some \( y_0 \in Z_k(K; \mathbb{Z}) \), \( \Lambda(x, y_0) = 0 \) for all \( x \in Z_k(K; \mathbb{Z}) \), the same is true for \( \Lambda' \). Moreover \( \partial \kappa, \partial \kappa' \in \text{Ker} \Lambda' \text{Ker} \Lambda \). If \( \partial \kappa' \) could be written as a linear combination \( a \partial \kappa + b \cdot z \), where \( a, b \in Z_2 \) and \( z \in \text{Ker} \Lambda \), then \( 1 = \Lambda(\partial \kappa, a \partial \kappa + b \cdot z) = a \Lambda(\partial \kappa, \partial \kappa) + b \Lambda(\partial \kappa, z) = a \cdot 1 + b \cdot 0 = 0 \), a contradiction. It follows that the vectors \( \partial \kappa \) and \( \partial \kappa' \) are linearly independent modulo \( \text{Ker} \Lambda \). Hence \( \dim \text{Ker} \Lambda' \geq \dim \text{Ker} \Lambda + 2 \) and \( \text{rank} \Lambda' \leq \text{rank} \Lambda - 2 \).

Let now \( K' \) be the complex formed from \( K \) by deleting the vertices of \( \kappa' \). We are going to show that \( \Lambda' \) restricted to \( Z_k(K'; \mathbb{Z}) \) satisfies both (11) and (12). However, if \( z, z' \in Z_k(K'; \mathbb{Z}) \), then both cycles \( z \) and \( z' \) are disjoint with \( \kappa' \). Then (11) for \( \Lambda \) and (18) imply that \( \Lambda'(z, z') = \Lambda(z, z') \). In particular, \( \Lambda' \) satisfies both (11) and (12) on \( K' \). By induction, \( n - k - 2 \leq (2k + 1) + \frac{k + 2}{2} \text{rank} \Lambda', \) leading to

\[
n \leq (2k + 1) + \frac{k + 2}{2} (\text{rank} \Lambda' + 2) \leq (2k + 1) + \frac{k + 2}{2} \text{rank} \Lambda.
\]

\[\square\]

5 Completeness

The aim of this section is to prove Theorem 6 and then Theorem 10. Therefore, for this section, in addition to our standard conventions from Section 2, we assume that \( k \geq 3 \) and \( M \) is \( (k - 1) \)-connected; unless explicitly stated otherwise, which occurs only in Remark 22.

Proof of Theorem 6. All considerations in this proof are over \( \mathbb{Z} \). According to the statement, we also assume that we are given \( \psi : C_k(K; \mathbb{Z}) \to H_k(M; \mathbb{Z}) \) such that \( \omega_\psi = \omega(0) \) is trivial.

Let \( B \subseteq M \setminus \partial M \) be a (closed) \( 2k \)-ball. Assume that \( g : |K| \to B \) is a general position map with \( g(|K|^{(k-1)}) \subseteq \partial B \). Our first step will be to find a general position map \( f' : |K| \to (M \setminus \overline{B}) \setminus \partial M \), agreeing with \( g \) on \( |K|^{(k-1)} \) such that \( \omega_{f', g} = \omega_\psi \) where \( \omega_{f', g} \) is as in Section 3 see (9). The second step will be to

\[\text{The definition of } \Lambda' \text{ was obtained as follows. We considered the projection } \pi : x \to x - \Lambda(x, \partial \kappa) \partial \kappa \text{ to the "orthogonal" complement of } \partial \kappa \text{, and we took } \Lambda' \text{ as the pullback of } \Lambda \text{ under } \pi.\]

\[\text{Here we consider the pullback of } \Lambda \text{ by the "orthogonal" projection to } \{\partial \kappa, \partial \kappa'\}^{(1)}.\]
find a homotopy of \( f' \) to a general position map \( f'' \) such that \( \partial f'' = 0 \). The third step will be to remove the remaining self-intersections via standard tricks.

**Step 1.** We define \( f' \) on each \( k \)-simplex \( \sigma \in K \) separately. We only need that \( \psi(\sigma) = [f'(\sigma) - g(\sigma)] \). Then \( \omega_{f',g} = \omega_\psi \) via (10).

By Hurewicz theorem \( H_k(M;\Z) \cong \pi_k(M;\Z) \), let \( h: \pi_k(M) \to H_k(M;\Z) \) be the Hurewicz isomorphism. We also recall the definition of \( h \) (see [Prat07, Chap.3, §1.1]). Given a map \( \gamma: (S^k, s_0) \to (M, x_0) \) where \( s_0 \in S^k \), \( x_0 \in M \), we set \( h(\gamma) := \gamma_*(S^k) \) where \( \gamma_*: H_k(S^k) \to H_k(M) \) is the induced map on homology and \([S^k]\) is the fundamental class. The map \( \gamma \) can be also regarded as a map from \( B^k \) to \( M \), constant on \( \partial B^k \).

Consider temporarily \( \sigma \) as a simplex in \( \R^d \) containing the origin and let \( \sigma_* = 1/2 \cdot \sigma \) be a homothetic smaller copy of \( \sigma \). Let \( f_*: \sigma_* \to M \) be a map, constant on \( \partial \sigma_* \), representing the class \( h^{-1}(\psi(\sigma)) \) in \( \pi_k(M) \). Now we want to extend \( f_* \) to \( \sigma \). We have \( \sigma \setminus \sigma_* \cong \partial \sigma \times I \), thus we can describe the extension of \( f_* \) on \( \partial \sigma \times I \) identifying \( \partial \sigma \) with \( \partial \sigma \times \{0\} \) and \( \partial \sigma \times \{1\} \). Let \( f_* \) coincide with \( g \) on \( \partial \sigma \times \{0\} \), then we first extend \( f_* \) to \( \partial \sigma \times [0, 1/2] \) as a homotopy in \( B \) from \( g \) to a constant map. Now let \( p: [1/2, 1] \to M \) be an arbitrary path from \( f_*(\partial \sigma \times \{1/2\}) \) to \( f_*(\partial \sigma \times \{1\}) \) (recall that \( f_* \) is constant on both \( \partial \sigma \times \{1/2\} \) and \( \partial \sigma \times \{1\} \)). For \( s \in [1/2, 1] \) we define \( f_*(s, \partial \sigma \times 1) := p(s) \).

It follows from the construction that the homology class of \( f_*(\sigma) - g(\sigma) \) is \( \psi(\sigma) \). Now it is sufficient to consider a homotopy of \( f_* \), constant on \( \partial \sigma \), such that the resulting map maps the interior of \( \sigma \) to \( M \setminus (B \cup \partial M) \), and then perform a perturbation to a required general position map \( f'' \).

**Step 2.** From the assumption that \([\omega_{f',g} - \omega_\psi] = o(K) \) is trivial and by Lemma (13) we get that \([\partial f'] \) is trivial. By (10) this means that

\[
\partial f' = \sum_{\eta, \mu} n_{\eta, \mu} \varepsilon_{\eta, \mu}
\]

where the sum is over all pairs \((\eta, \mu)\), where \( \eta \) is a \((k-1)\)-simplex, \( \mu \) is a \(k\)-simplex and \( \eta \cap \mu = \emptyset \); \( \varepsilon_{\eta, \mu} \) are the elementary cochains defined above (10); and \( n_{\eta, \mu} \in \Z \). If \( M \) were \( \R^k \), then for any \((\mu, \eta)\) we could apply ‘van Kampen finger moves’ as described in [FKT94 §2.4] and which provide a homotopy from \( f' \) to another map \( f \) such that \( \partial f = \partial f' \pm \varepsilon_{\eta, \mu} \). (Both choices \( \pm \varepsilon_{\eta, \mu} \) are possible.) In order to adapt to our situation of general \( M \), we consider a general position PL-path \( p \) in the interior of \( M \) connecting a point in the interior of \( \eta \) with a point in the interior of \( \mu \). Then we consider a regular neighborhood \( N_p \) of \( p \), which is a ball by [RS72 Corollary 3.27]. We perform the finger-move as in [FKT94 §2.4] inside \( N_p \) which has exactly same effect on \( \partial f' \) as in \( \R^k \). Therefore we can get a homotopy from \( f' \) to \( f'' \) with the required property \( \partial f'' = 0 \) by successively applying finger moves. In addition, \( f''(\partial K) \) avoids \( \partial M \).

**Step 3.** Finally, we want to build the required embedding \( f'' \) of \( f' \). This can be done by standard tricks such as the Whitney trick. They are described in [FKT94 §2.4] for \( M = \R^k \). The key observation is that all tricks are based on finding a copy of \( S^k \) in \( f(\partial K) \) in general position, filling this \( S^k \) with a general position disk \( T \), taking a regular neighborhood \( N_T \) of \( D \), which is a ball, and removing the singularities inside \( N_D \). In a simply connected manifold, these steps work in verbatim. This finishes the proof of Theorem 8.

Now, we provide (somewhat weaker) analogy of Theorem 4 for the \( \Z_2 \) case used in the proof of Theorem 10.

**Proposition 21.** Let us assume that \( k \geq 3 \) and \( M \) is compact \((k-1)\)-connected. Then, the following conditions are equivalent.

(i) There is a homomorphism \( \psi: C_k(K;\Z_2) \to H_k(M;\Z_2) \) such that \([\omega_\psi] = o(K) \) (over \( \Z_2 \)).

(ii) There is a general position map \( f''': K \to M \) such that for every pair \((\sigma, \tau)\) of disjoint \( k \)-simplices, \( f'''(\sigma) \) and \( f'''(\tau) \) have an even number of intersections.

(iii) There is a general position map \( f'': K \to M \) such that for every pair \((\sigma, \tau)\) of \( k \)-simplices, \( f''(\sigma) \) and \( f''(\tau) \) have an even number of intersections. (We can even assume that \( f''(\sigma) \) is an embedding on every \( k \)-simplex \( \sigma \) and that \( f''(\sigma) \) and \( f''(\tau) \) share only \( f''(\sigma \cap \tau) \), if \( \sigma \) and \( \tau \) are \( k \)-simplices which are not disjoint.)

\[14\text{This step of obtaining } f'' \text{ out of } f' \text{ seems to be the bulk of the work in Johnson's work. However, the standard approach via finger moves presented here seems to be simpler. (We could not directly refer to Johnson's work in this paragraph, as Johnson works in smooth category.)}\]
Proof of Proposition 21. The implication (ii) ⇒ (i) follows from Proposition 14. (Any map from a (k − 1)-complex into a (k − 1)-connected manifold is nullhomotopic.) The implication (iii) ⇒ (ii) is obvious.

Thus it remains to prove (i) ⇒ (ii), and (ii) ⇒ (iii). Note that the condition on \( f'' \) from (iii) is equivalent with \( \vartheta f'' = 0 \).

The proof of (i) ⇒ (ii) is analogous to steps 1. and 2. in the proof of Theorem 6, thus we only point out the single difference: In step 1 for \( \mathbb{Z} \) we use the Hurewicz isomorphism \( h \); however, we only use that \( h \) is an epimorphism. If we consider \( h \) as a homomorphism \( h : \pi_k(M) \to H_k(M, \mathbb{Z}_2) \) then the proof that \( h \) is an epimorphism from \( [\text{Pra07}, \text{Theorem 3.2}] \) works in verbatim.

The proof of (ii) ⇒ (iii) follows the step 3 of the proof of Theorem 6. However, we only perform the tricks that remove self-intersections of simplices that share at least one vertex. (For comparison, the reason why we cannot get rid of all singularities is that we cannot perform the Whitney trick. Given two disjoint \( k \)-simplices \( \sigma \) and \( \tau \) in \( K \) the Whitney trick may remove a pair of intersection points \( \{x, x'\} \subseteq f''(\sigma) \cap f''(\tau) \) provided that the signs at \( x \) and \( x' \) are opposite. But we do not know whether we get opposite signs if we perform computations only over \( \mathbb{Z}_2 \).)

Now, Theorem 10 follows quickly.

Proof. By Theorem 15 and Proposition 21, it is sufficient to find out whether the system of equations (11) has a solution in \( \mathbb{Z}_2 \). This is decidable as \( \mathbb{Z}_2 \) is finite. \( \square \)

Remark 22. Let us consider another algorithmic question: Given a \( k \)-complex \( K \), a compact PL \( 2k \)-manifold \( M \), not necessarily \( (k − 1) \)-connected, and a general position map \( f : [K] \to M \). We would like to know whether \( f \) is homotopic to an embedding. Let us also assume that \( f \) is presented on the input via its intersection cochain \( \vartheta_f \in C_{alt-sym}(K; \mathbb{Z}) \) over the integers. Then deducing whether \( \vartheta_f = [\vartheta_f] = 0 \) is of course efficiently computable even over the integers. (In formula (11), the term \( y_j^2 A \Omega y_\tau \) on the left side disappears while we have \( \vartheta \vartheta_f \) on the right side, thus the equations become linear.)

Therefore, the question whether \( f \) is homotopic to an embedding can be solved efficiently whenever vanishing \( \vartheta_f \) is a complete obstruction for \( f \) being homotopic to an embedding. According to Johnson [Joh02, Theorem 4] this occurs when \( M \) is closed, smooth and simply connected. However, the assumption that \( M \) is compact, PL and simply connected is also sufficient by checking Step 2 of the proof of Theorem 6. we leave the details for the interested reader.

6 Conclusions and open problems

Here we mention few conclusions and open problems, sometimes touched in the introduction.

Existence of the obstruction and completeness. Given an almost embedding \( f : [K] \to M \), the obstruction class \( o_f \) is well defined even if we do not assume that \( f \) restricted to the \( (k − 1) \)-skeleton of \( K \) is nullhomotopic. However, we need to give nullhomotopy for describing the obstruction as in Theorem 4. In particular, our approach gives \( \Theta_{K,M} \subseteq \Theta_{K,M} \) where \( \Theta_{K,M} := \{[\omega_\varphi] : o(K); \psi \in \text{hom}(C_\Delta(K; R), H_k(M; R))\} \) and \( \Gamma_{K,M} := \{o_f; f : [K] \to M\} \) (considering only general position PL maps). In particular, if there is an almost embedding \( f : [K] \to M \), then the trivial class belongs to \( \Gamma_{K,M} \) and thereby to \( \Theta_{K,M} \) as well, which is in principle our obstruction.

Problem 23 (Existence). Is there an easy to describe superset \( \Theta_{K,M} \) of \( \Gamma_{K,M} \) even if we do not assume the nullhomotopy condition, perhaps via (co)homology of \( M \) or \( K \).

Problem 24 (Completeness). When \( 0 \in \Theta_{K,M} \) implies \( 0 \in \Gamma_{K,M} \)? When \( 0 \in \Gamma_{K,M} \) implies that there is an embedding \( f : K \to M \)?

If we do not assume that the restriction of every map \( f : K \to M \) to the \( (k − 1) \)-skeleton is nullhomotopic, the answer to the first question of Problem 24 may of course depend on the answer to Problem 24. In our proof of Theorem 6, the implication \( 0 \in \Theta_{K,M} \Rightarrow 0 \in \Gamma_{K,M} \) was the contents of steps 1 and 2 in the proof and there we really used \( (k − 1) \)-connectedness of the manifold. The implication \( 0 \in \Gamma_{K,M} \) implies that there is an embedding \( f : K \to M \) is the contents of step 3 and it seems to be generally well understood. There we used \( k \geq 3 \) and the fact that \( M \) is simply-connected. This implication does
not hold if $k = 2$ even if $M = \mathbb{R}^{2k}$; [FKT94]. We also do not expect that the requirement that $M$ is simply-connected can be removed in general.

Somewhat specific case occurs when $k = 1$, that is, $K$ is a graph and $M$ is a surface, for simplicity connected, otherwise we can treat every component separately (let us remark that in this case the nullhomotopy condition is satisfied). If $M = \mathbb{R}^2$ then even vanishing the $\mathbb{Z}_2$-version of the van Kampen obstruction implies that $K$ is a planar graph [CH34, Tut70]. When $M$ is a general surface, Fulek and Kyncl [FK19] in their noticeable work provide an example of $K$, $M$ such that $\vartheta_f = 0$ over $\mathbb{Z}_2$ whereas $K$ does not embed in $M$. This shows that the $\mathbb{Z}_2$-version of our obstruction is not a complete obstruction for embeddability of graphs into surfaces. The $\mathbb{Z}$-case is not answered yet and it is essentially equivalent to Problem 5.3 in [FK19] (restated in our language):

**Problem 25.** Assume that $K$ is a graph and $M$ a connected orientable surface. Assume that there is a general position map $f : K \to M$ with $\vartheta_f = 0$ (over $\mathbb{Z}$). Does it follow that $K$ embeds in $M$?

**Computational aspects.** We have already mentioned Question 3 in the introduction. Here we only specify a few concrete cases when $M$ is $(k - 1)$-connected and this question seems to be easiest to approach.

**Problem 26.** (i) Is $\text{Embed}(k, S^k \times S^k)$ decidable for $k \geq 3$?

(ii) Is $\text{Embed}(4, \mathbb{H}P^2)$ decidable, where $\mathbb{H}P^2$ is the quaternionic projective plane? (We remark that $\mathbb{H}P^2$ is an 8-dimensional manifold.)

In the first case the intersection form has matrix $A = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$. For (ii), $A = (1)$.

**Homological almost embeddings.** Motivated by approach in [GPP+17] we pose:

**Problem 27.** Can Theorem 4 be upgraded to homological almost embeddings? (We refer to [GPP+17] for a definition of homological almost embeddings.)

**Acknowledgments**

We would like to thank Xavier Goaoc, Zuzana Patákrová and Uli Wagner for discussions in early stages of this project. We also thank Karim Adiprasito for explaining us the consequences of his work in [Adi18].

**References**

[Adi18] K. Adiprasito. Combinatorial Lefschetz theorems beyond positivity. Preprint; https://arxiv.org/abs/1812.10454, 2018.

[AKMM02] N. Alon, G. Kalai, J. Matoušek, and R. Meshulam. Transversal numbers for hypergraphs arising in geometry. Adv. in Appl. Math., 29(1):79–101, 2002.

[CGHP08] O. Cheong, X. Goaoc, A. Holmsen, and S. Petitjean. Helly-type theorems for line transversals to disjoint unit balls. Discrete Comput. Geom., 39(1-3):194–212, 2008.

[CH34] Ch. Chojnacki (H. Hanani). Über wesentlich unplättbare Kurven im dreidimensionalen Raume. Fund. Math., 23(1):135–142, 1934.

[FK18] R. Fulek and J. Kynčl. The $\mathbb{Z}_2$-genus of Kuratowski minors. In 34th International Symposium on Computational Geometry (SoCG 2018). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.

[FK19] R. Fulek and J. Kynčl. Counterexample to an extension of the Hanani-Tutte theorem on the surface of genus 4. Combinatorica, 39(6):1267–1279, 2019.

[FKMP15] R. Fulek, J. Kynčl, I. Malinović, and D. Pálvölgyi. Clustered planarity testing revisited. Electron. J. Combin., 22(4):Paper 4.24, 29, 2015.
M. H. Freedman, V. S. Krushkal, and P. Teichner. Van Kampen’s embedding obstruction is incomplete for 2-complexes in $\mathbb{R}^4$. *Math. Res. Lett.*, 1(2):167–176, 1994.

A. Flores. Über $n$-dimensionale Komplexe die im $R_{2n+1}$ absolut selbstverschlungen sind. *Ergeb. Math. Kolloq.*, 4:6–7, 1932/1934.

X. Goaoc, I. Mabillard, P. Paták, Z. Patákova, M. Tancer, and U. Wagner. On generalized Heawood inequalities for manifolds: a van Kampen–Flores-type nonembeddability result. *Israel J. Math.*, 222(2):841–866, 2017.

D. Gorodkov. A 15-vertex triangulation of the quaternionic projective plane. *Discrete Comput. Geom.*, 62(2):348–373, 2019.

X. Goaoc, P. Paták, Z. Patáková, M. Tancer, and U. Wagner. Bounding Helly numbers via Betti numbers. In *A journey through discrete mathematics*, pages 407–447. Springer, Cham, 2017.

L. S. Harris. Intersections and embeddings of polyhedra. *Topology*, 8:1–26, 1969.

A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2001.

A. F. Holmsen and D. Lee. Radon numbers and the fractional Helly theorem. *Israel J. Math.*, 241(1):433–447, 2021.

J. F. P. Hudson. *Piecewise linear topology*. University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

C. M. Johnson. An obstruction to embedding a simplicial $n$-complex into a $2n$-manifold. *Topology Appl.*, 122(3):581–591, 2002.

R. E. Jamison-Waldner. Partition numbers for trees and ordered sets. *Pacific J. Math.*, 96(1):115–140, 1981.

W. Kühnel and T. F. Banchoff. The 9-vertex complex projective plane. *The Mathematical Intelligencer*, 5:11–22, 1983.

K. Kawarabayashi, B. Mohar, and B. Reed. A Simpler Linear Time Algorithm for Embedding Graphs into an Arbitrary Surface and the Genus of Graphs of Bounded Tree-Width. In *49th Annual IEEE Symposium on Foundations of Computer Science, 2008.*, pages 771–780, Oct 2008.

W. Kühnel. Manifolds in the skeletons of convex polytopes, tightness, and generalized Heawood inequalities. In *Polytopes: abstract, convex and computational (Scarborough, ON, 1993)*, volume 440 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 241–247. Kluwer Acad. Publ., Dordrecht, 1994.

J. Kynčl. Simple realizability of complete abstract topological graphs simplified. *Discrete Comput. Geom.*, 64(1):1–27, 2020.

F. W. Levi. On Helly’s theorem and the axioms of convexity. *The Journal of the Indian Mathematical Society*, 15(0):65–76, 1951.

The manifold atlas project: Intersection form. [http://www.map.mpim-bonn.mpg.de/Intersection_form](http://www.map.mpim-bonn.mpg.de/Intersection_form)

J. Matoušek. A Helly-type theorem for unions of convex sets. *Discrete Comput. Geom.*, 18(1):1–12, 1997.
[Mel09] S. A. Melikhov. The van Kampen obstruction and its relatives. Tr. Mat. Inst. Steklova, 260(Geometriya, Topologiya i Matematicheskaya Fizika. II):149–183, 2009.

[Moh99] B. Mohar. A Linear Time Algorithm for Embedding Graphs in an Arbitrary Surface. SIAM Journal on Discrete Mathematics, 12(1):6–26, 1999.

[Mor18] J. Morgan. Homotopy theory lecture notes. http://scgp.stonybrook.edu/archives/27538, 2018.

[MT01] B. Mohar and C. Thomassen. Graphs on surfaces. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001.

[MTW11] J. Matoušek, M. Tancer, and U. Wagner. Hardness of embedding simplicial complexes in $\mathbb{R}^d$. J. Eur. Math. Soc. (JEMS), 13(2):259–295, 2011.

[Pat19a] P. Paták. Properties of closure operators in the plane. Preprint; http://arxiv.org/abs/1909.08489, 2019.

[Pat19b] Z. Patáková. Bounding Radon’s number via Betti numbers. Preprint; http://arxiv.org/abs/1908.01677, 2019.

[Pra07] V. V. Prasolov. Elements of homology theory, volume 81 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2007. Translated from the 2005 Russian original by Olga Sipacheva.

[PSS09] M. J. Pelsmajer, M. Schaefer, and D. Stasi. Strong Hanani–Tutte on the Projective Plane. SIAM Journal on Discrete Mathematics, 23(3):1317–1323, 2009.

[PT04] J. Pach and G. Tóth. Monotone drawings of planar graphs. J. Graph Theory, 46(1):39–47, 2004.

[Rad21] J. Radon. Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. Mathematische Annalen, 83(1):113–115, Mar 1921.

[Rin74] G. Ringel. Map color theorem. Die Grundlehren der mathematischen Wissenschaften, Band 209. Springer-Verlag, New York-Heidelberg, 1974.

[RS72] C. P. Rourke and B. J. Sanderson. Introduction to piecewise-linear topology. Springer-Verlag, New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69.

[RY68] G. Ringel and J. W. T. Youngs. Solution of the Heawood map-coloring problem. Proceedings of the National Academy of Sciences, 60(2):438–445, 1968.

[Sha57] A. Shapiro. Obstructions to the imbedding of a complex in a euclidean space. I: The first obstruction. Ann. of Math., II. Ser., 66:256–269, 1957.

[Sko08] A. B. Skopenkov. Embedding and knotting of manifolds in Euclidean spaces. In Surveys in contemporary mathematics, volume 347 of London Math. Soc. Lecture Note Ser., pages 248–342. Cambridge Univ. Press, Cambridge, 2008.

[SNC09] M. Soos, K. Nohl, and C. Castelluccia. Extending SAT solvers to cryptographic problems. In O. Kullmann, editor, Theory and Applications of Satisfiability Testing - SAT 2009, pages 244–257, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg.

[Tut70] W. T. Tutte. Toward a theory of crossing numbers. J. Combin. Theory, 1(8):45–53, 1970.

[vK32] R. E. van Kampen. Komplexe in euklidischen Räumen. Abh. Math. Sem. Hamburg, 9:72–78, 1932. Berichtigung dazu, ibid. (1932) 152–153.

[Vol96] A. Yu. Volovikov. On the van Kampen-Flores theorem. Mat. Zametki, 59(5):663–670, 797, 1996.
[Whi44] H. Whitney. The self-intersections of a smooth \( n \)-manifold in \( 2n \)-space. *Ann. of Math. (2)*, 45:220–246, 1944.

[Wu65] W.-T. Wu. *A Theory of Imbedding, Immersion, and Isotopy of Polytopes in a Euclidean Space*. Science Press, Peking, 1965.