Power Flow as Intersection of Circles: A new Fixed Point Method

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Abstract—The power flow problem is a fundamental problem in power system engineering with a wide range of applications. In this paper, we present a new class of algorithms obtained from writing the power flow problem as finding the solution of a fixed point equation. Fixed point equations can be solved iteratively by simple function evaluations and can potentially converge to the solution exponentially fast. In recent years, a number of fixed point methods for the power flow problem have been proposed, but each of the methods only applied to a restricted class of networks. Using new a geometric insight into the power flow equations under rectangular coordinates, the algorithms presented in this paper can be applied to fully AC networks with no restrictions on the network topology or bus types. Furthermore, each iteration of the algorithm consists of finding intersections of circles, which can be computed efficiently with high numerical accuracy. We compare the performance of our fixed point algorithm with existing state-of-the-art methods, showing that the proposed method can correctly find the solutions when other methods cannot. In addition, we empirically show that the fixed point algorithm is much more robust to bad initialization points than the existing methods.

Index Terms—Power flow, fixed-point equation, intersection of circles, ill-conditioned problems

I. INTRODUCTION

The power flow problem is one of the canonical problems in power engineering and is frequently solved in both system operation and planning [1], [2]. Existing power flow methods mostly rely on iterative algorithms such as Newton-Raphson (NR) [3] or fast decoupled load flow (FDLF) [4], [5]. Despite being the workhorses of power system analysis, these algorithms can suffer from convergence issues, especially when the system operates close to its loadability limits or the initial guess is poor [6], [7]. Both of these issues are exacerbated by the large-scale development of renewable resources and distributed generation and the need for fast and robust power flow solvers remains despite decades of studies [8], [9].

Both NR and FDLF methods can be thought as variants of descent algorithms (or approximate descent in the case of FDLF) that modifies the solution iteratively. A fundamental reason for why these algorithms can fail to converge to a power flow solution is simply because the geometry of the feasible voltage and power flow regions is nonconvex [9], [10]. Because of the nonconvexities, the initial starting point plays an important role in the performances of the algorithms and “bad” initial starting points can prevent them from converging [11]–[14]. Recognizing this, a series of studies have used techniques such as homotopy, current injection and factorized methods to overcome the sensitive dependence on the initial guess [15]–[18].

A second challenge in adopting NR and FDLF methods comes from the nature of the power flow Jacobian. As expected, the Jacobian plays the key role of providing a direction of update in the iterations. However, when the Jacobian becomes singular or ill-conditioned (e.g., when the loading is heavy) [19]–[21], the algorithms tend to diverge. To avoid this phenomenon, a class of non-divergent power flow algorithms was developed to accelerate or decelerate the updates based on the conditioning of the Jacobian [22]–[25]. However, these approaches can still be sensitive of the initial guess and sometimes exhibit oscillatory behavior, where the solutions neither converge nor diverge.

Recently, a new class of power flow formulations based on fixed point equations has been proposed to overcome the algorithmic challenges present in descent algorithms [26], [27]. The basic idea is to write the power flow equations in a form of \( \mathbf{v} = f(\mathbf{v}) \), where \( \mathbf{v} \) is the complex voltage and a fixed point of the function \( f \) [28]. If this relationship can be found, then a simple algorithm to find the fixed point is to repeatedly apply the function \( f \). Furthermore, if the iterates converge to the fixed point, then it will converge exponentially quickly. Currently, fixed point equations have only been developed for restricted class of systems. For example, the results in [27] applies to networks with only PQ buses and the results in [26] only applies to purely inductive (lossless) radial networks.

In this paper, we present a new fixed point formulation by viewing the solution of power flow problems as the intersection of circles. Using rectangular voltage coordinates, we show that the power balance equations at each bus can be thought as describing two circles that intersect with each other. The parameters (center and radius) of the circles depend linearly on the voltages of the neighboring buses. A fixed point is a vector of complex voltages where the intersections of the circles at each bus are consistent with the parameters of the neighbors. Using this geometric intuition, our formulation is valid for full AC networks with arbitrary topology and arbitrary mix of PV and PQ buses.

Based on the fixed point equation, a simple coordinate-wise update can be used to find the power flow solutions. During each iterate, only the intersection of two circles needs to be calculated, which involves a series of simple algebraic computations. Therefore, this step is much cheaper...
than algorithms (e.g., NR) that require matrix calculations. It is important to note that even though we suspect that our method is a contraction mapping, due to the somewhat complicated dependence between the buses, we cannot provide a theoretical guarantee of its convergence. This is in contrast to the theoretical results in [26], [27], which have less generality and stricter assumptions, but do provide some guarantees on convergence. Therefore, to verify the performance of our algorithm, we test it on the standard IEEE 14, 30 and 118 buses and compare with NR, FDLF and non-divergent power flow algorithms. We show that when the loading is heavy, our algorithm is able to converge to the right solution while the other algorithm either diverge or become unstable. In addition, we show that our method is much more robust to random initialization points than the other methods.

The paper is organized as follows: Section II introduces the rectangular power flow equations and shows how they can be thought as intersections of circles. Section III discusses the fixed point formulation of the power flow equations and walks through a three-bus example. Section IV presents the main algorithm. Section V introduces a 3-tuple vector form of circles and shows how closed-form formulas with good numerical properties can be found using the vector notation. Section VI shows numerical results of our proposed algorithm compared against existing state-of-the-art algorithms on different IEEE benchmark networks. Section VII concludes the paper.

II. POWER FLOW EQUATIONS AS INTERSECTING CIRCLES

A. Power Flow Equations in Rectangular Coordinates

Throughout this paper we use rectangular coordinates where a bus is indexed by \( d \); \( p_d \) and \( q_d \) are the active and reactive powers, respectively; \( v_{d,r} \) and \( v_{d,i} \) are the real and imaginary parts of the bus voltage, respectively; and \( \mathcal{N}(d) \) is the set of neighboring buses connected to bus \( d \). We adopt the standard II model of transmission lines [2] and write the admittance of a line between buses \( d \) and \( k \) as \( g_{dk} + j b_{dk} \). We assume that \( b_{dk} \leq 0 \) for all lines (lines are inductive). In these notations, the power flow equations become [29], [30]:

\[
p_d = t_{d,1} \cdot v_{d,r}^2 + t_{d,2} \cdot v_{d,r} + t_{d,1} \cdot v_{d,i}^2 + t_{d,3} \cdot v_{d,i},
\]

\[
q_d = t_{d,A} \cdot v_{d,r}^2 - t_{d,3} \cdot v_{d,r} + t_{d,A} \cdot v_{d,i}^2 + t_{d,2} \cdot v_{d,i}.
\]  

The parameters \( t_{d,1}, t_{d,2}, t_{d,3} \) are always negative, (1) and (2) describe two circles in the variables \( v_{d,r} \) and \( v_{d,i} \). We call the circle described by (1) the active power circle parametrized by its center \( C_p \) and radius \( r_p \); similarly, we say that (2) describes the reactive power circle parameterized by center \( C_q \) and radius \( r_q \). These parameters are given by:

\[
C_p = \left( \frac{-t_{d,2}}{2t_{d,1}}, \frac{-t_{d,3}}{2t_{d,1}} \right), \quad C_q = \left( \frac{t_{d,3}}{2t_{d,4}}, \frac{-t_{d,2}}{2t_{d,4}} \right),
\]

\[
r_p = \sqrt{\frac{p_d}{t_{d,1}} + \frac{(t_{d,2})^2 + (t_{d,3})^2}{4t_{d,1}}}, \quad r_q = \sqrt{\frac{q_d}{t_{d,4}} + \frac{(t_{d,3})^2 + (t_{d,2})^2}{4t_{d,4}}},
\]

Since the terms \( t_{d,1} \) and \( t_{d,4} \) are always negative, (1) and (2) are given by:

\[\begin{align}
\mathbf{v}_d & = (v_{d,r}, v_{d,i}^T) \mathbf{C}_p + \mathbf{r}_p,
\mathbf{v}_d & = (v_{d,r}, v_{d,i}^T) \mathbf{C}_q + \mathbf{r}_q.
\end{align}\]

Figure 1 shows a three bus network and the associated active and reactive power cycles are buses 2 and 3 (bus 1 is assumed to be the slack bus). The intersection points A and B in Fig. 1b and Fig. 1c represent the potential power flow solutions.

B. PV Buses

The discussions in the above section focused on PQ buses. In a power flow problem, PV buses are also frequently used to describe generators [31]. In this case, the reactive power balance equation in (2) is replaced by a condition on the voltage magnitude:

\[v_{d,r}^2 + v_{d,i}^2 = V_{ref}^2,\]  

where \( V_{ref} \) is the reference voltage. Again, we can think of PV buses in term of circles, since (6) is a circle centered at the origin with a fixed radius. Therefore, our framework does not require different treatment of PQ and PV buses.
III. FIXED POINT EQUATION FOR POWER FLOW

The geometric representation of the power flow equations as the intersection of circles leads to a simple fixed point view of power flow solutions. Suppose that a vector of complex voltages is given. Then, the voltage at a particular bus \( d \) is determined by its neighbors as the intersection of the active power circle with the reactive power circle (for a PQ bus) or with the voltage magnitude circle (for a PV bus). A natural question arises: if the circles intersect in two points as shown in Figs. 1b and 1c which of the point should we choose as the voltage at that bus? To make this choice, we follow two common assumptions made in power flow calculations.

The first assumption we make is that we are interested in solutions at higher voltage magnitudes \([32, 33]\). These solutions are seen as more practical and stable for an actual system. For example, in both Figs. 1b and 1c we would chose point \( B \) as the solution. For a PV bus, all points of intersection obviously has the same voltage magnitude. In this case, we would choose the solution with a smaller angle, following the intuition that these are the more stable solutions \([35, 36]\).

With these choices, the complex voltage at a bus is uniquely determined by the complex voltages of its neighbors, which leads to a natural consistency condition for a solution. Given \( \mathbf{v} \), let \( f \) be a function that takes \( \mathbf{v} \) and performs the circle intersection operation (choosing a unique solution as described in the last paragraph). Then a vector \( \mathbf{v} \) is a solution to the power flow problem if and only if \( \mathbf{v} = f(\mathbf{v}) \). That is, \( \mathbf{v} \) is a fixed point of \( f \). Note that if two circles do not intersect at a bus, then we can declare that \( \mathbf{v} \) is not a fixed point.

Here we use the three bus network in Fig. 1 to illustrate an algorithm to solve the power flow problem. The line admittance of all the branches are \( 1 - j1.5 \). Bus 1 is considered to be a slack bus with a voltage of 1 p.u., while buses 2 and 3 are considered to be PQ buses. Initially, the voltage \( v_2 = v_2, r + j v_2, i \) at bus 2 is fixed with an initial guess. Based on \( v_2 \), the real and reactive power circles at bus 3 can be calculated. If these circles intersect with each other, the one with the higher voltage magnitude would be assigned as the value for \( v_3 \). Then, the voltage at bus 3 is fixed and intersections of the two circles at bus 2 are used to update \( v_2 \). This is repeated until the convergence is achieved. Next, we describe the algorithm for a general network.

IV. MAIN ALGORITHM

For an \( n \)-bus system, to start the algorithm, the voltages at all the buses in the system are fixed with an initial guess. Then the voltage solution at a bus is updated using its neighbors. This is repeated for all buses, which we call a round of the algorithm. The algorithm terminates if none of the buses update their complex power in a round or when the complex power mismatch is less than the tolerance set by the user. Algorithm 1 presents the pseudo code for a system with only PQ buses.

In our implementation of Algorithm 1 we sweep through all of the buses in one round, as illustrated in Fig. 2. The exact order of updates is not constrained by the algorithm, although it is an interesting question to see if there exist an “optimal” update order in some sense.

Algorithm 1 Fixed point algorithm for system with only PQ buses.

\[
\text{Input : } P_i, Q_i \text{ for bus } i = 2, \ldots, n, \\
\text{Tolerance } \delta \text{ for the stopping criterion.} \\
\text{Output: } v_i \text{ for bus } i = 2, \ldots, n. \\
1: \text{Initialize voltages at all buses, } v_i \text{ for } i = 2, \ldots, n; \\
2: \text{Let the neighboring bus index be } k; \\
3: \text{where } k \in \mathbb{N}_m. \\
4: \text{Calculate the power mismatch } (\Delta \Sigma); \\
5: \text{while } (\Delta \Sigma) > \delta \text{ do} \\
6: \text{for } m = 2, \ldots, n \text{ do} \\
7: \text{Calculate } (\epsilon_p, \epsilon_p) \text{ at bus } m, \forall k \in \mathbb{N}_m; \\
8: \text{Calculate } (\epsilon_q, \epsilon_q) \text{ at bus } m, \forall k \in \mathbb{N}_m; \\
9: \text{Calculate the voltage } v_m \text{ for bus } m; \\
10: \text{Update current bus } (m) \text{ voltage.} \\
11: \text{Calculate } (\Delta \Sigma); \\
12: \text{end while} \\
13: \text{return } v_m \forall \text{ buses } m = 2, \ldots, n; \\
\]

It is possible that the circles do not intersect at a bus either at the start of the algorithm or during one of the iterations. In these cases, we simply restart the algorithm with a new initial guess. We note that for feasible problems with PQ buses, we have never observed the non-intersection of the
circles. For mixed PQ and PV systems, it is possible for bad initial guesses to lead to non-intersection behaviors, especially when the loading is very heavy. We go into more details in Section VI

V. FINDING INTERSECTION POINTS

In Algorithm 1, the main computation step is to find the intersection of two points. At a first glance, this operation is almost trivial and there exist multiple short programs for it (see, e.g. [37]). However, the numerical implementation of an intersection algorithm can experience subtle but critical issues. For example, if lines are close to being purely inductive (high \( X/R \) ratio), then the reactive circle becomes a circle with a very large radius and most existing intersection algorithms would run into numerical instabilities. In addition, for a large network, the numerical precision and error propagation of the calculations become important for the convergence speed of the algorithm. Finally, since finding the intersections takes most of the time in Algorithm 1, it would be desirable to get a closed form solution. Therefore, we use an unconventional representation of circles developed by [38] to provide a robust and efficient algorithm to find the intersection of circles later in this section. For ease of exposition, we focus on system with PQ bus. Analogous results can be derived for PV buses.

Fig. 3 outlines the steps we take to find the intersection of the active and reactive power circles. First, we find the line through the two circles (Fig. 3a). Then, we find the smallest circle (called the orthogonal circle) that passes through the intersecting points of the original circles (Fig. 3b). Next, we find the intersection of the line with the orthogonal circle (Fig. 3c). It turns out that if we view the circles as vectors in some vector space, the above computations can be thought as vector manipulations, which is simple to perform and numerically stable. In the rest of this section, we develop this theory based on the materials in [38].

Remark 1. Note that we do not necessarily require the step in Fig. 3b. The key reason to compute another circle is that the numerical accuracy of the algorithm improves if the smallest possible circle that contains the intersection points in our algorithm.

A. 3-Tuple Vector Representation of Circles

Instead of the traditional center/radius parameterization, we can describe all of the points \( x \in \mathbb{R}^2 \) on a circle by the following equation:

\[
a(x \cdot x) + b \cdot x + c = 0, \quad (7)
\]

where \( \cdot \) denotes the dot product between two vectors. The form in (7) allows us to describe a circle using a three tuple \((a,b,c)\). Note that this presentation is not unique, since scaling all of the parameters by a scalar does not change the points that satisfy (7). If \( a \) is not zero, we will scale parameters such that \( a = 1 \). In this notation, the circles described by the real and reactive power equations (1) and (2) becomes

\[
(a_p, b_p, c_p) = \left(1, \frac{t_{d,2}}{t_{d,1}}, \frac{t_{d,3}}{t_{d,1}} \right)^T, \quad -\frac{P_d}{t_{d,1}}, \quad (8)
\]

and

\[
(a_q, b_q, c_q) = \left(1, \frac{t_{d,3}}{t_{d,4}}, \frac{t_{d,2}}{t_{d,4}} \right)^T, \quad -\frac{Q_d}{t_{d,4}}. \quad (9)
\]

In these representations, the circles shift gracefully and the same calculations can be applied to a wide range of parameter values.

Next, we separate the fixed parameters in the system (e.g., admittance values) and the voltages. Given a bus \( d \), let \( d_1, d_2, \ldots, d_k \) its neighboring nodes. Let \( g_d = [g_{d_1,d} \ g_{d_2,d} \ \cdots \ g_{d_k,d}] \) denote the vector of conductances between bus \( d \) and its neighbors. Similarly, let \( b_d = [b_{d_1,d} \ b_{d_2,d} \ \cdots \ b_{d_k,d}] \) denote the vector of susceptances. Let \( \mathbf{1} \) denote the vector of all \( 1 \)'s of the appropriate length. To represent the voltages of the neighboring buses, we use a vector \( \mathbf{u} \) formed by concatenating the real and imaginary voltages:

\[
\mathbf{u} = \left[u_{d,s}, r \ u_{d,r} \ \cdots \ u_{d,s}, r \ u_{d_1,i} \ u_{d_2,i} \ \cdots \ u_{d_k,i} \right]^T.
\]

Then, we can rewrite (8) and (9) as

\[
(a_p, b_p, c_p) = \left(1, \begin{bmatrix} -\alpha & \delta \\ \delta & -\alpha \end{bmatrix} \mathbf{u}, -\frac{P_d}{1 \cdot g_d} \right), \quad (10)
\]
where
\[ \alpha = \frac{g_d}{1 \cdot g_d}, \quad \beta = \frac{b_d}{1 \cdot b_d}, \quad \gamma = \frac{g_d}{1 \cdot b_d}, \quad \delta = \frac{b_d}{1 \cdot g_d}. \]

If needed, the centers and radii of power flow circles can be computed easily from (10) and (11):
\[ \text{Center}_p = -\frac{b_p}{2}, \quad \text{Center}_q = \frac{-b_q}{2}, \]
\[ r_p^2 = \left( \frac{b_p \cdot b_p}{4} - c_p \right), \quad r_q^2 = \left( \frac{b_q \cdot b_q}{4} - c_q \right), \]
where Center\(_p\) and Center\(_q\) are the centers of the real and reactive power circles, \(r_p\) and \(r_q\) are the radii, respectively.

### B. Line Passing Through Intersection Points of the Power Flow Circles

Given two circles \(C_1 = (1, b_1, c_1)\) and \(C_2 = (1, b_2, c_2)\), the line passing through their points of intersection is described by \(C_1 - C_2\), provided the circles intersect. More formally, \(C_1 - C_2\) is
\[ C_1 - C_2 = (0, b_1 - b_2, c_1 - c_2) = (0, L_2, L_3) \]
and describes the points \(v_d\) that satisfies the equation
\[ L_2 \cdot v_d + L_3 = 0, \]
where
\[ v_d = \begin{bmatrix} v_{d,r} \\ v_{d,i} \end{bmatrix}. \]
Substituting (10) and (11) into (15), we have the line described by
\[ \begin{pmatrix} -\alpha + \beta & \gamma + \delta \\ -\gamma + \delta & -\alpha + \beta \end{pmatrix} u = \frac{p_d}{1 \cdot g_d} + \frac{q_d}{1 \cdot b_d}. \]

### C. Orthogonal Circle

In principle, we can use the line computed in (16) to find the intersection points by intersecting that line with one of the active or reactive circles. However, the numerical accuracy and stability can suffer because the line may intersect the circles at a very acute angle. Therefore, it is more desirable to use the orthogonal circle for calculations. Geometrically, the orthogonal circle is the smallest circle that passes through the two intersection points. Algebraically, we label it as \(C^\perp\).

Again, the parameters of this circle can be computed from (10) and (11) via simple algebra [38, 39, 39]
\[ C^\perp = (a^\perp, b^\perp, c^\perp) \]
\[ \begin{pmatrix} a^\perp \\ b^\perp \\ b^\perp \end{pmatrix} = \begin{pmatrix} 1, \frac{b_1 + b_2}{2} + \frac{(b_2 - b_1) (k_1^2 - k_2^2)}{2 \|b_1 - b_2\|^2}, \frac{c_1 + c_2}{2} + \frac{(c_2 - c_1) (k_1^2 - k_2^2)}{2 \|b_1 - b_2\|^2} \end{pmatrix}, \]
\[ \begin{pmatrix} c^\perp \\ r^2 \end{pmatrix} = \begin{pmatrix} \frac{k_1^2}{\|b_1\|^2 - 4a_1c_1}, \frac{k_2^2}{\|b_2\|^2 - 4a_2c_2} \end{pmatrix}. \]
Here, \(\|\|\) is the standard \(L-2\) norm. The center and the radius of the orthogonal circle is given by
\[ \text{Center}^\perp = \frac{-b^\perp}{2}, \]
and
\[ r_u = \sqrt{\frac{b^\perp \cdot b^\perp}{4} - c^\perp}. \]
Substituting (10) and (11) into (17), we get
\[ C^\perp = \begin{pmatrix} 1, \frac{1}{2} M_B u + \frac{\left( \|u\|^2 \cdot K_c - 2 \cdot l \right) M_A u}{\|M_A u\|^2}, \frac{1}{2} \left( \frac{p_d}{1 \cdot g_d} - \frac{c^\perp}{1 \cdot b_d} \right) + \frac{l \left( 2l - \|u\|^2 K_c \right)}{\|M_A u\|^2} \end{pmatrix}, \]
where
\[ M_A = \begin{pmatrix} \alpha - \beta & -\gamma - \delta \\ \gamma + \delta & \alpha - \beta \end{pmatrix}, \quad K_c = (\|\alpha\|^2 + \|\delta\|^2) - (\|\gamma\|^2 + \|\beta\|^2). \]

### D. Point of Intersection

Next, we find the point of intersection (Fig. 36). These points are found at a distance of \(r_u\) from the center of the orthogonal circle along the line computed in (16). Through simple algebra, we have that the point of intersection, that is, the updated voltage at bus \(d\) is given by
\[ \begin{bmatrix} v_{d,r} \\ v_{d,i} \end{bmatrix} = \text{Center}^\perp + \frac{R L_2}{\|L_2\|^2}, \]
\[ = \frac{b^\perp}{2} + \sqrt{\frac{b^\perp \cdot b^\perp}{4} - c^\perp} \frac{R L_2}{\|L_2\|^2}, \]
where
\[ R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]
and \(b^\perp\) is given by (17) and \(L_2\) is given by (14). To choose one solution or a sign in (21), we will pick the one that leads to the higher voltage magnitude. Note that in (21), most of the computation can be done offline since they only involve the admittance parameters. Applying (21) to every bus \(d\) also gives us an analytical form of the fixed point equation for the complex voltages.
VI. Numerical Results

In this section, simulation studies on standard IEEE test cases are compared with other power flow methods. The comparative studies include heavy loading, and effect of random initialization.

First of all, the convergence of the proposed fixed point (FP) solver on the standard 14, 30 and 118-IEEE bus systems are shown in Fig. 4 under normal loading conditions. More precisely, we take the data from MatPower [40] and run our fixed point algorithm to test its convergence. As shown in Fig. 4 for these standard cases, the fixed point algorithm converges in tens of iterations. We note that the time it takes to finish one iteration is much faster than say a conventional Newton type algorithm, where the inverse of the Jacobian need to be computed.

![Fig. 4: Semi-log plot of the convergence of the fixed point algorithm for IEEE standard systems at base case loading.](image)

**A. Heavily Loaded Networks**

The more challenging case is to compare our algorithm against existing ones under nonstandard loading conditions. We take the 14 bus network and increase all of the loads by a factor of 3.99. This loading is still feasible, but is very close to the loadability boundary of the system. The proposed method is compared with other methods such as Fast-Decoupled XB version (FDXB) [41], Newton Raphson (NR) and optimal multiplier [22]. Fig. 5 presents the convergence comparison of the methods. As expected, both FDXB and NR diverges [42], [43]. Interestingly, the optimal multiplier method also becomes unstable. In our simulation, we use (22) is used to compute the solution of an optimal multiplier based method. Even though the optimal multiplier \( \mu \) changes the step size of the Jacobian, the Jacobian becomes highly sensitive when nearing to singularity and an incorrect \( \Delta V \) is predicted. This is verified by calculating the condition number of the Jacobian matrix under the loading conditions.

\[
\Delta V = \mu \cdot (J^{-1}) \cdot (\Delta S) \quad (22)
\]

In contrast, our proposed method is able to converge even under these conditions since it does not use the Jacobian. Similar behavior is observed for other networks.

![Fig. 5: Power flow convergence comparison with others methods such as FDXB, NR and Optimal multiplier method at heavy loading.](image)

**B. Sensitivity to Initial Conditions**

In addition to convergence, it is important for an algorithm to be robust to the initial conditions, especially as the randomness in the system increases due to renewable integration [17]. To test the performance of various algorithms to initial conditions, we take the IEEE 30-bus system at its standard loading and randomly select the starting voltages. In our experiments, we set the initial guess to be random samples from the uniform distribution on the interval \([1 - \alpha, 1 + \alpha]\) for various values of \( \alpha \) (we always set the imaginary part to be 0), independently for each bus. Table I reports the number of successful convergences (defined as power mismatch convergence less than 0.001 p.u.) for the FDXB, NR, optimal multiplier and our proposed FP methods for 100 trials.

| Initialization spread \( \alpha \) | NR | FDXB | optimal multiplier | FP |
|-----------------|-----|------|--------------------|-----|
| 0.05            | 100 | 98   | 100                | 100 |
| 0.1             | 64  | 62   | 100                | 100 |
| 0.2             | 4   | 0    | 0                  | 100 |
| 0.3             | 0   | 0    | 0                  | 100 |
| 0.4             | 0   | 0    | 0                  | 100 |
| 0.6             | 0   | 0    | 0                  | 100 |
| 0.9             | 0   | 0    | 0                  | 100 |

**TABLE I: Convergence test of the power flow methods with random initialization for 100 trials. The initial voltages are generated identical and independently from uniform distribution of \([1 - \alpha, 1 + \alpha]\). Our proposed method converged for every trial, where as other methods quickly stopped working once the range become moderately large.**

As we see in Table I our proposed FP method is much more robust to the value of the initial guesses than the other methods: it always converged while the other methods quickly stopped working when \( \alpha \) becomes large. This hints that the fixed point method may avoid being trapped in local optima that can impact descent algorithms since local optima are not fixed points by definition.
VII. CONCLUSION

A new fixed-point formulation of the power flow equation is developed in this paper. In contrast to existing fixed point formulations, it includes all possible cases of PV/PQ buses, mesh networks, resistive and inductive lines. Geometrically, our formulation treats the active and reactive power flow equations in rectangular voltage coordinates as circles and the power flow solutions as the intersection of these circles. Using a 3-tuple vector representation of circles, we derive simple, efficient and numerically stable formulas to find their intersection points. Based on the fixed point equations, we develop a natural iterative fixed point algorithm to solve the power flow problem. Numerical studies on the standard IEEE benchmarks show that our algorithm is able to converge when other state-of-the-art algorithms diverges or becomes unstable.

In addition, we show that our algorithm is much more robust to the initial starting point, able to converge for a wide range of starting conditions while other algorithms diverged.

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A. Handling PQ and PV buses

Here we present the fixed-point algorithm for a system with both PQ and PV buses. In the case of PV buses, Section II-B discusses the replacement of the reactive power balance equation by a condition on the voltage magnitude \( q \). The voltage circle is centered at origin \( (c_p) \) with fixed radius of \( V_{ref} \). Thus the voltage solution for a PV bus can be calculated similarly to PQ bus by the intersection of two circles, real power \( (1) \) and specified voltage magnitude circles \( (6) \) at bus \( d \). When there are no common points between the real power and specified voltage magnitude circles, the reactive power at bus \( d \) is calculated to check for the violation of reactive power limits. In such a scenario, the PV bus is converted to a PQ bus by fixing its reactive power with the violated limit and it is then solved as a PQ bus.

Algorithm 2 RFPFF method for PQ and PV buses.

**Input**: \( P_i \) for bus \( i = 2, \ldots, n \), 
\( Q_i \forall \) PQ buses, 
\( VSpec \forall \) PV buses, 
\( Q_{max} \) and \( Q_{min} \) for PV buses, 
Bus type information \( B\{i\} \) for bus \( i = 2, \ldots, n \), 
Tolerance \( \delta \) for the stopping criterion.

**Output**: \( v_i \) for bus \( i = 2, \ldots, n \).

1: Initialize voltages at all buses, \( v_i \) for \( i = 2, \ldots, n \);
2: Let the neighboring bus index be \( k \);
   where \( k \in N_m \); \( \triangleright N_m \): Neighboring buses of bus \( m \).
3: Calculate the power mismatch \( (\Delta s) \);
4: while \( (\Delta s) > \delta \) do \( \triangleright \) Convergence criteria.
5: \( \text{for } m = 2, \ldots, n \) do
6: \( \text{if } B\{m\} == \text{PQ bus then} \)
7: \( \text{Calculate } (c_p, r_p) \text{ at bus } m, \forall k \in N_m; \)
8: \( \text{Calculate } (c_q, r_q) \text{ at bus } m, \forall k \in N_m; \)
9: \( \text{Calculate the voltage } v_m \text{ for bus } m; \)
10: \( v_m = v_m; \triangleright \text{Update bus } m \text{ voltage.} \)
11: else \( \triangleright \text{PV bus.} \)
12: \( \text{Calculate } (c_p, r_p) \text{ at bus } m, \forall k \in N_m; \)
13: \( \text{Calculate } (c_q, r_q) \text{ at bus } m, \forall k \in N_m; \)
14: \( \text{if Circles } (c_p, r_p) \text{ and } (c_q, r_q) \text{ intersect then} \)
15: \( \text{Calculate the voltage } v_m \text{ for bus } m; \)
16: \( v_m = v_m; \triangleright \text{Update bus } m \text{ voltage.} \)
17: else \( \triangleright \text{else} \)
18: \( \text{Convert bus } m \text{ to PQ bus & update } B\{m\}; \)
19: \( \text{Calculate } (c_p, r_p) \text{ at bus } m; \)
20: \( \text{Calculate } (c_q, r_q) \text{ at bus } m; \)
21: \( \text{Calculate the voltage } v_m \text{ for bus } m; \)
22: \( v_m = v_m; \triangleright \text{Update bus } m \text{ voltage.} \)
23: end if
24: end if
25: end for
26: \( \text{Calculate } (\Delta s); \)
27: end while
28: return \( v_m \forall \) buses \( m = 2, \ldots, n \);