Involutory Hopf algebras and 3-manifold invariants

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Abstract
We establish a 3-manifold invariant for each finite-dimensional, involutory Hopf algebra. If the Hopf algebra is the group algebra of a group $G$, the invariant counts homomorphisms from the fundamental group of the manifold to $G$. The invariant can be viewed as a state model on a Heegaard diagram or a triangulation of the manifold. The computation of the invariant involves tensor products and contractions of the structure tensors of the algebra. We show that every formal expression involving these tensors corresponds to a unique 3-manifold modulo a well-understood equivalence. This raises the possibility of an algorithm which can determine whether two given 3-manifolds are homeomorphic.

1 Introduction
This paper describes some new invariants for triangulations and Heegaard diagrams of 3-manifolds. Specifically, for every finite-dimensional, involutory Hopf algebra $H$, we define an invariant $\sharp(M, H)$ for closed, oriented 3-manifolds $M$. The invariants can be extended in various ways to invariants of oriented manifolds with boundary.

We will convert a Heegaard diagram to an expression in terms of the structure tensors of a Hopf algebra. The value of the expression is then the value of the invariant. Using index notation, we write $M_{ab}^c$ and $\Delta_{a}^{bc}$ for the multiplication and comultiplication tensors of some involutory Hopf algebra, and $S_{a}^{b}$ for the antipode map. We may consider the set of all expressions in terms of these tensors, modulo the axioms of a Hopf algebra, which consist of identities which these tensors satisfy. We will show that if two (prime, closed, oriented) 3-manifolds produce equivalent formal expressions, then they are homeomorphic. Conversely, we may consider the question of determining whether two such expressions are formally equivalent. For example, is the equation:

$$M_{ab}^c S_{c}^{d} \Delta_{d}^{ab} = M_{ab}^c S_{c}^{d} \Delta_{d}^{ba}$$

an identity that follows directly from the axioms of an involutory Hopf algebra? The answer is yes if and only if two certain oriented 3-manifolds are homeomorphic (in this case $L(3,1)$ and $L(3,2)$).

This work is part of a new area of mathematics which we will call state model topology, as described in [12, 10]. As the starting point of state model topology, we are given a topological object, for example a knot, with a combinatorial description, for example a knot projection. We wish to find a state model (or collection of state models) which we can assign to all such combinatorial descriptions. We then hope that the partition function of the state model is a topological invariant, i.e. is independent of the chosen combinatorial description. Typically, we demonstrate invariance under an appropriate set of local moves on the combinatorial description, for example the three Reidemeister moves. (Strictly speaking, the partition function may be a topological “covariant” because its value may change in a simple way under some of the moves.)
A state model $M$ consists of a (commutative) ring $R$ (usually the complex numbers), a bipartite graph $G$, the connectivity graph, whose vertices are labeled as atoms and interactions; a set $S_A$ for each atom $A$, called the state set of $A$; and a function $w_I : A_1 \times A_2 \times \ldots \times A_n \to R$ for each interaction $I$ (where $A_1, \ldots, A_n$ are the neighbors of $I$), called the weight function or the Boltzmann weights of $I$. A state of $M$ is a function $s$ on the atoms of $M$ such that $s(A) \in S_A$. The weight $w(s)$ of a state $s$ is defined as the product of the $w_I$’s evaluated at the state $s$ when this product converges, and in particular when $G$ is finite. Finally, the partition function $Z(M)$ is defined to be the sum of $w(s)$ over all states $s$ when this sum converges, and in particular all state sets are finite.

![Figure 1](image)

To obtain a topological state model, we choose a prescription for assigning a connectivity graph and weight functions to each combinatorial description of a topological object. For example, if we are given a projection $P$ of a link, we can declare the arcs between crossings to be atoms and the crossings themselves to be the interactions. We choose some $n$-element set $S$ to be the common state set for all atoms. If the states of the arcs incident to a given crossing are labeled $a, b, c,$ and $d$ as in Figure 1, we may define the weight function $w$ of a crossing by:

$$w(a, b, c, d) = t \delta(a, b) \delta(c, d) + t^{-1} \delta(a, c) \delta(b, d),$$

where $t$ is chosen so that $n = -(t^2 + t^{-2})$, and $\delta(a, b) = 1$ when $a = b$ and 0 otherwise. This state model is a “link covariant” called the Kauffman bracket, which is essentially the Jones polynomial up to normalization.

Surprisingly, there are many non-trivial topological state models. We may start with the general form of such a state model for some kind of combinatorial description, and interpret combinatorial moves as a list of constraints on the weights of the model. Usually there are many more equations than unknowns, and yet the equations have solutions. For example, the Jones and HOMFLY polynomials and most of their variants and generalizations can be defined as topological state models.

We can conclude that the constraints on the weights of a topological state model have a deep algebraic structure. Indeed, the constraints imposed by invariance under the third Reidemeister move are also known as the Yang-Baxter equations, considered independently in the related area of exactly solved statistical mechanical models. More recently, many solutions to the Yang-Baxter equations have been discovered via Hopf algebras, or quantum groups, as is summarized in an inspiring paper by Drinfel’d. Recently, Turaev and Reshetikhin have extended this analysis to produce a 3-manifold invariant which generalizes the Jones polynomial.

The new invariants are closely related to other topological invariants. If we restrict to the special case when $M$ is a link complement, we may convert the invariant $\sharp(M, H)$ to a special case of the machinery of Drinfel’d, Turaev, and Reshetikhin. This conversion involves a bigger Hopf algebra $D(H)$, the quantum double of $H$, which was defined by Drinfel’d.
The following is an instructive example of a topological state model. Let $T$ be a triangulation of a connected $n$-manifold $M$. We orient the edges of $T$. We choose a finite group $G$ to be the state set, and we declare the edges of $T$ to be the atoms. If we assign $g$ to an edge with one orientation, we declare this equivalent to assigning $g^{-1}$ to the same edge with the opposite orientation. We assign interactions to the faces (2-simplices) of $T$. Given a state of the model, the weight of a face is 1 if the cyclically-ordered product of the states of the edges (when the edges are oriented in the same direction around the face) of the face is the identity and 0 otherwise. We leave it as an exercise that the partition function is simply $|G|^{v-1}\left|\text{Hom}(\pi_1(M), G)\right|$, where $v$ is the number of vertices of $T$. Thus, the partition function is an invariant after dividing by $|G|^{v-1}$. We will see later that this invariant equals $\sharp(M, H)$ when $H$ is the group algebra of $G$.

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2 Tensor notation

Let $V$ be a finite-dimensional vector space over a field $k$. For most of this paper, a tensor will mean an element of some tensor product space $V_1 \otimes V_2 \otimes \ldots \otimes V_n$, where each $V_i$ is either $V$ or $V^*$. We will adopt index notation for tensors as it is defined in [17]. In index notation, a tensor $T \in V_1 \otimes \ldots \otimes V_n$ is written as $T_{ab\ldots}^{cd\ldots}$, with $n$ indices total, where the $k$th index is a superscript (or contravariant index) if $V_k = V$ and a subscript (or covariant index) if $V_k = V^*$. The indices should be distinct letters which have no meaning independent of their use as a description of the tensor $T$. For example, a vector is written $v^a$, a dual vector is $w_a$, a linear operator is $L_{ab}$, a bilinear form is $g_{ab}$, and so on. If a tensor $T$ has $k$ superscripts and $n-k$ subscript, we say that $T$ has type $(k, n-k)$, and total type $n$.

The indices of a tensor can be used for three different operations on tensors:

1. We denote the factor-switching map $a \otimes b \mapsto b \otimes a$ and its cousins for higher tensor products by permuting the corresponding indices. For example, if $g_{ab} = g_{ba}$, then $g$ is a symmetric bilinear form.

2. We write the tensor product of two tensors by juxtaposing the two tensors and giving them distinct indices. For example, one may write $L_{ab} = w_a v_b$ if $L$ is a linear operator of rank 1.

3. The canonical map from $V^* \otimes V$ to $k$ which is called the trace map, the contraction map, or the evaluation map is denoted by a repeated index. The index must appear exactly once as a superscript and once as a subscript. Used together, contractions and tensor products subsume most of the usual operations in linear algebra. For example, $w_a v^a$ is the value of $w$ at $v$, $L_{ab} v^a$ is the operator $L$ applied to $v$, $L_{a}^{a}$ is the trace of $L$, and so on.

Although index notation only allows basis-independent computations with tensors, the notation is motivated by computations in a specific basis. Suppose the vector space $V$ has dimension $n$ and has a given basis. We may interpret the notation $v^a$ to mean the coordinates of the vector $v$, with $a$ an integer from 1 to $n$, and in general we may interpret $T_{ab\ldots}^{cd\ldots}$ to be the matrix entries of the tensor $T$. In this case a tensor formula in index notation may be interpreted as a formula for the matrix
entries of the tensors, and the two formulas are equivalent given the convention that we sum over any repeated indices (the Einstein summation convention).

We may use the summation convention to arrive at a re-interpretation of certain state models. Suppose that $M$ is a state model such that every atom is adjacent to precisely two interactions, and suppose for simplicity that there is only one state set $S$. We may declare $S$ to be the index set of a basis (or perhaps $S$ is the basis) of a vector space $V$. We declare each atom $a$ to be an index letter. We may then re-interpret the values of each weight function $w_I$ as the matrix entries of a tensor $T_I$ of total type $n$. We arbitrarily declare each of the indices of each $T_I$ to be covariant or contravariant, provided that each atom is used once as a superscript and once as a subscript. In this case the partition function $Z(M)$ is the result of tensoring together all of the interaction tensors and then contracting all of the (atom) indices in the natural manner.

Thus, we define a tensorial state model to be a list of tensors over one vector space, together with a formula which is some product of these tensors with all indices contracted. The partition function is defined to be the value of this formula. More generally, we say that $M$ is a tensorial state model with boundary if $M$ is a formula which is a product of tensors with some subset of the indices contracted. We call the uncontracted indices the boundary of the model, and the contracted indices the interior. We define the partition tensor $Z_{a:b..}(M)$ to be the value of the formula. The partition tensor $M$ should not be confused with the partition function of $M$.

The state models defined in this paper will all be tensorial state models.

To conclude this section, we present a more visual version of index notation which we will actually use. To avoid using many different letters for contracted indices, we may draw each tensor with an inward arrow for every subscript and an outward arrow for every superscript. For example, we write the tensor $T_{a b}^{c d}$ as follows:

\[
\begin{array}{c}
a \\
b \\
c \\
d
\end{array}
\begin{array}{c}
\rightarrow T \\
\rightarrow d
\end{array}
\begin{array}{c}
a \\
c
\end{array}
\begin{array}{c}
b \\
d
\end{array}
\]

or

\[
\begin{array}{c}
a \\
b \\
c \\
d
\end{array}
\begin{array}{c}
\rightarrow T \\
\rightarrow d
\end{array}
\begin{array}{c}
a \\
c
\end{array}
\begin{array}{c}
b \\
d
\end{array}
\]

We denote the contraction of two indices by connecting the corresponding arrows. For example, $M_{a b}^{c} v^{a} w^{b}$ and $M_{a b} b$ are written as:

\[
\begin{array}{c}
v \\
w
\end{array}
\begin{array}{c}
\rightarrow M \\
\rightarrow M
\end{array}
\begin{array}{c}
v \\
w
\end{array}
\begin{array}{c}
\rightarrow M
\end{array}
\]

For consistency, we should interpret the strange-looking expression:

\[
\begin{array}{c}
\rightarrow
\end{array}
\]

as the scalar dim $V$.

Arrows can and will cross. Strictly speaking, the visual notation can be ambiguous. For example, it is not clear if the following denotes $T_{a b c}$ or $T_{c a b}$, or perhaps even $T_{a c b}$:

\[
\begin{array}{c}
a \\
b
\end{array}
\begin{array}{c}
\rightarrow T \\
\rightarrow c
\end{array}
\begin{array}{c}
a \\
b
\end{array}
\begin{array}{c}
\rightarrow T
\end{array}
\]

We declare that indices should be read off in counter-clockwise order. The meaning of a diagram is still ambiguous if the arrows incident to some tensor are symmetric under rotation. However, all such tensors in this paper will be symmetric under cyclic permutation of their indices, so this ambiguity will be irrelevant to us.
3 Hopf algebras

A finite-dimensional Hopf algebra is a finite-dimensional vector space $H$ (over some ground field $k$) such that $H$ and $H^*$ are both algebras with unit, and the two algebra structures satisfy certain compatibility axioms. We will describe the axioms with the fancy notation of the previous section. Firstly, we may describe the algebra structure on $H$ by a multiplication tensor $M$ defined so that

$$
\begin{array}{c}
\text{v} \\
\text{w} \\
\end{array}
\text{M} \\
\xrightarrow{\text{v}} \\
\xrightarrow{\text{w}} \\
\xrightarrow{\text{M}}
$$

is the product of $v$ and $w$. $M$ satisfies the associativity axiom:

$$
\begin{array}{c}
\text{M} \\
\xrightarrow{\text{M}} \\
\xrightarrow{\text{M}} \\
\end{array}
\xrightarrow{\text{M}} =
\begin{array}{c}
\text{M} \\
\xrightarrow{\text{M}} \\
\xrightarrow{\text{M}} \\
\end{array}
\xrightarrow{\text{M}}
$$

Similarly, the algebra structure on $H^*$ is described by the comultiplication tensor $\Delta$ which is coassociative:

$$
\begin{array}{c}
\text{\Delta} \\
\xleftarrow{\text{\Delta}} \\
\xleftarrow{\text{\Delta}}
\end{array}
\xleftarrow{\text{\Delta}} =
\begin{array}{c}
\text{\Delta} \\
\xleftarrow{\text{\Delta}} \\
\xleftarrow{\text{\Delta}}
\end{array}
\xleftarrow{\text{\Delta}}
$$

In addition, there are three other tensors $i$, $\epsilon$, and $S$ called the unit, the counit, and the antipode of $H$. The five tensors together satisfy these axioms:

$$
\begin{array}{c}
\text{i} \\
\xrightarrow{\text{i}} \\
\xrightarrow{\text{i}}
\end{array}
\xrightarrow{\text{M}} =
\begin{array}{c}
\text{i} \\
\xrightarrow{\text{i}} \\
\xrightarrow{\text{i}}
\end{array}
\xrightarrow{\text{M}} =
\begin{array}{c}
\text{\Delta} \\
\xleftarrow{\text{\Delta}} \\
\xleftarrow{\text{\Delta}}
\end{array}
\xleftarrow{\text{\Delta}} =
\begin{array}{c}
\text{\Delta} \\
\xleftarrow{\text{\Delta}} \\
\xleftarrow{\text{\Delta}}
\end{array}
\xleftarrow{\text{\Delta}} =
\begin{array}{c}
\epsilon \\
\xleftarrow{\epsilon} \\
\xleftarrow{\epsilon}
\end{array}
\xleftarrow{\epsilon}
$$

$$
\begin{array}{c}
\text{M} \\
\xrightarrow{\text{M}} \\
\xrightarrow{\text{M}}
\end{array}
\xrightarrow{\text{\Delta}} =
\begin{array}{c}
\text{M} \\
\xrightarrow{\text{M}} \\
\xrightarrow{\text{M}}
\end{array}
\xrightarrow{\text{\Delta}} =
\begin{array}{c}
\text{\Delta} \\
\xleftarrow{\text{\Delta}} \\
\xleftarrow{\text{\Delta}}
\end{array}
\xleftarrow{\text{\Delta}} =
\begin{array}{c}
\epsilon \\
\xleftarrow{\epsilon} \\
\xleftarrow{\epsilon}
\end{array}
\xleftarrow{\epsilon}
$$

$$
\begin{array}{c}
\text{\Delta} \\
\xleftarrow{\text{\Delta}} \\
\xleftarrow{\text{\Delta}}
\end{array}
\xleftarrow{\text{S}}
\begin{array}{c}
\text{M} \\
\xrightarrow{\text{M}} \\
\xrightarrow{\text{M}}
\end{array}
\xrightarrow{\text{\Delta}} =
\begin{array}{c}
\text{\Delta} \\
\xleftarrow{\text{\Delta}} \\
\xleftarrow{\text{\Delta}}
\end{array}
\xleftarrow{\text{S}}
\begin{array}{c}
\text{M} \\
\xrightarrow{\text{M}} \\
\xrightarrow{\text{M}}
\end{array}
\xrightarrow{\text{\Delta}} =
\begin{array}{c}
\epsilon \\
\xleftarrow{\epsilon} \\
\xleftarrow{\epsilon}
\end{array}
\xleftarrow{\epsilon}
$$

We also assume as an axiom that $S$ has an inverse, although this fact follows from the other axioms in the finite-dimensional case [16]. Given $M$ and $\Delta$, the other tensors are unique when they exist, so we may say that $M$ and $\Delta$ are the “meat” of a Hopf algebra. A morphism of Hopf algebras is defined to be a linear transformation of the underlying vector spaces which commutes with the five structure tensors.

Since we have not defined infinite-dimensional Hopf algebras and will not use them, we will assume that a Hopf algebra is finite-dimensional unless explicitly stated otherwise.
We adopt the following abbreviations in light of the associativity and coassociativity axioms:

\[
\begin{align*}
\mathcal{M} & \leftrightarrow \mathcal{M} \mathcal{M} \mathcal{M} \cdots \mathcal{M} \leftrightarrow \mathcal{M} \leftrightarrow \mathcal{M} = \iota \leftrightarrow \\
\Delta & \leftrightarrow \Delta \Delta \Delta \cdots \Delta \leftrightarrow \Delta \leftrightarrow \Delta = \epsilon \leftrightarrow \\
-\mathcal{T} & \leftrightarrow \mathcal{M} \leftrightarrow \mathcal{C} \leftrightarrow \Delta \leftrightarrow \mathcal{M} = \mathcal{M} \mathcal{T} \leftrightarrow \Delta \leftrightarrow \Delta \leftrightarrow \Delta = \mathcal{C} 
\end{align*}
\]

The dual vector \( \mathcal{T} \) is called the *trace*, and the vector \( \mathcal{C} \) is the *cotrace*. We will call the last two tensors the *tracial product* and *tracial coproduct*. Like the diagrams, the tensors themselves are cyclically symmetric.

One motivation for the definition of a Hopf algebra is the observation that every group algebra is naturally a Hopf algebra. Let \( G \) be a finite group. We may let the elements of \( G \) be both a basis and a dual basis for a vector space \( k[G] \), and define multiplication and comultiplication by:

\[
\begin{align*}
g \mapsto \mathcal{M} & \leftrightarrow \mathcal{M} \mathcal{T} = \mathcal{M} \mathcal{M} \mathcal{M} \cdots \mathcal{M} = \mathcal{M} \leftrightarrow \mathcal{M} = \iota \\
g \mapsto \Delta & \leftrightarrow \Delta \Delta \Delta \cdots \Delta = \Delta \leftrightarrow \Delta = \mathcal{C} \\
\end{align*}
\]

The antipode is induced by the group inverse. This construction has the nice property that the morphisms between the Hopf algebras of two groups correspond to the homomorphisms between the groups themselves.

In general, the antipode plays the role of the group inverse in a Hopf algebra. However, the antipode need not satisfy \( S^2 = I \). A Hopf algebra is called involutory if this equation is satisfied. Many important Hopf algebras are not involutory, and in particular those used in the analysis of the Jones polynomial are non-involutory \( \mathbb{1} \). However, we will only obtain results about involutory Hopf algebras in this paper.

Finite-dimensional Hopf algebras possess an eight-fold duality. If \( H \) is a Hopf algebra, we obtain a new Hopf algebra \( H^{\text{op}} \) on the same vector space by reversing multiplication:

\[
\begin{align*}
\mathcal{M} \leftrightarrow \mathcal{M}^{\text{op}} & \leftrightarrow \mathcal{M}^{\text{cop}} \\
\end{align*}
\]

We will show below that \( S^{-1} \) is the antipode of \( H^{\text{op}} \). Similarly, we define \( H^{\text{cop}} \) by reversing comultiplication. Finally, we define the *dual* of \( H \), a Hopf algebra structure on the vector space \( H^* \), by switching \( \mathcal{M} \) with \( \Delta \) and \( \epsilon \) with \( \iota \) and reversing all the arrows (Actually, what we are calling \( H^* \) is usually defined as \( H^{*, \text{op}, \text{cop}} \), and vice-versa):

\[
\begin{align*}
\mathcal{M} \leftrightarrow \rightarrow \Delta & \leftrightarrow \iota \leftrightarrow \epsilon \leftrightarrow \mathcal{S} \leftrightarrow \leftarrow \mathcal{S} \leftrightarrow \\
\end{align*}
\]

We establish some basic identities involving Hopf algebras. The proof of each identity will use the following observation:
Lemma 3.1. The tensors: 

\[
\begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} 
\]

when viewed as vector space endomorphisms of \( H \otimes H \) and \( H \otimes H^* \), are invertible.

Proof. We compute:

\[
\begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} \quad \Delta \quad \begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} \quad \varepsilon \quad \begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} 
\]

\[
\begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} \quad \begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} \quad \varepsilon \quad \begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} 
\]

\square

Lemma 3.2. The following identities hold in any Hopf algebra:

a. \( M \varepsilon = \varepsilon M \rightarrow \)

\[
\begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} 
\]

b. \( MS \rightarrow = S M^\text{op} \rightarrow \)

\[
\begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} 
\]

c. \( i S \rightarrow = i \rightarrow \)

\[
\begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} 
\]

d. \( \Delta M^\text{op} \rightarrow = \varepsilon i \rightarrow \)

\[
\begin{array}{c}
\rightarrow M \rightarrow \\
\uparrow & \uparrow \\
S & \Delta \\
\downarrow & \downarrow \\
\rightarrow \Delta \leftarrow \\
\end{array} 
\]
Proof. We compute:

a. \[ \mu_L : M \to \Delta M \to \Delta \to \Delta \to M \to \varepsilon = \varepsilon \Delta = \varepsilon \Delta = \varepsilon \Delta M \to \varepsilon \]

\[ \Rightarrow \quad \mu_L = \varepsilon \]

b. \[ \mu_R : M \to \Delta M \to \Delta M \to \Delta S \to M \to \varepsilon i \]

\[ \Rightarrow \quad \mu_R = \varepsilon i \]

c. \[ \iota : S \to M \to i \Delta S \to M \to i \Delta S \to M \to \varepsilon i \]

d. \[ \varepsilon : \Delta S M \to \Delta S M \to \varepsilon i \]

\[ \square \]

We will need a few other definitions involving Hopf algebras. If \( H \) is a Hopf algebra, a left integral of a Hopf algebra is a dual vector \( \mu_L \) that satisfies:

\[ \Delta \mu_L = \varepsilon \]

One can similarly define a right integral, a left cointegral, and right cointegral:

\[ \mu_R : M \to \Delta S M \to \varepsilon e_R \]

A Hopf algebra always has a non-zero integral and cointegral. By Lemma 3.2, \( \varepsilon \) is an algebra homomorphism. In general, if \( A \) is a finite-dimensional algebra over \( k \) and \( \varepsilon : A \to k \) is any homomorphism, then there exists a non-zero element \( e \) such that \( ae = \varepsilon(a)e \) for all \( a \in A \).
A Hopf algebra $H$ is called *semisimple* if it is semisimple as an algebra, and cosemisimple if $H^*$ is semisimple. Equivalently, $H$ is semisimple if the tensor:

$$\begin{bmatrix} M \\ S^2 \end{bmatrix}$$

is a non-degenerate bilinear form. We say that $H$ has *invertible dimension* if $\dim H \neq 0$ in the ground field of $H$.

In an involutory Hopf algebra, the trace $T$ is also a left and right integral \[14\]. To prove this, we use the fact that there exists some non-zero integral, and we show that $T$ is a multiple of it:

**Lemma 3.3 (Radford and Larson).** The tensor:

$$\begin{bmatrix} M \\ S^2 \end{bmatrix}$$

is always a right integral (which may be zero).

**Proof.** Let $\mu_R$ be a non-zero right integral and let $e_L$ be a non-zero left cointegral. We first observe that:

$$\begin{array}{c}
\begin{array}{c}
eq \\
\downarrow = \\
\downarrow = \\
\downarrow \\
\downarrow = \\
\downarrow = \\
\downarrow
\end{array}
\end{array}$$

Therefore:

$$\begin{array}{c}
\begin{array}{c}
eq \\
\downarrow = \\
\downarrow = \\
\downarrow = \\
\downarrow = \\
\downarrow = \\
\downarrow
\end{array}
\end{array}$$

by Lemma 3.1. Since $\mu_R$ and $e_L$ are non-zero, the last equation shows that $\mu_R(e_L)$ must be non-zero. Finally, we obtain:

$$\begin{array}{c}
\begin{array}{c}
eq \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\end{array}$$

as desired. \[\square\]

By virtue of the fact that:

$$C \rightarrow T = \bigcirc$$

we obtain:

**Corollary 3.4.** If $H$ is involutory, then:

$$\begin{array}{c}
\begin{array}{c}
eq \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
eq \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\end{array}$$
Thus, Lemma 3.3 demonstrates not only that $T$ is a right integral, but also that $H$ is semisimple and cosemisimple. Moreover, if $H$ has invertible dimension, we can express all of the structure tensors of a Hopf algebra in terms of tracial product and coproduct!

4 Heegaard diagrams and triangulations

In the section (and the rest of the paper) we will work in the smooth category of manifolds. We will use the term $n$-manifold to mean an oriented (not merely orientable) and compact $n$-manifold which may have boundary and may or may not be connected.

We recall some basic elements of 3-manifold topology as described in [7]. A Heegaard splitting is a decomposition of a closed 3-manifold into two handlebodies of the same genus glued along their boundary; the boundary is called a Heegaard surface. Given such a decomposition, we may arbitrarily label the two handlebodies upper and lower. If the handlebodies are labeled, then an orientation of the Heegaard surface induces an orientation of the manifold: At a point $p$, we complete a positively-oriented basis $(e_1, e_2)$ of the tangent space of $S$ at $p$ to a (positive) basis $(e_1, e_2, e_3)$ of $T M |_p$ by choosing $e_3$ to point from the upper handlebody to the lower one.

If $S$ is a surface, we define a handlebody diagram $d$ on $S$ to be a prescription for gluing a handlebody to $S$. Specifically, it is a family of disjoint circles $\{c_i\}$ which divides $S$ into planar regions. (Often it is assumed that $\{c_i\}$ does not separate $S$, but we do not want this hypothesis.) We obtain a handlebody $H_d$ from $d$ by the following procedure: We start with $S \times I$ and glue a 2-handle along a tubular neighborhood of each circle $c_i \times \{0\}$. The boundary of the result is $S \times \{1\}$ plus a number of spheres. We eliminate the spherical boundary components by gluing in balls to obtain $H_d$.

We say that $D$ is a Heegaard diagram of a closed 3-manifold on a surface $S$ is a prescription for a Heegaard splitting of a manifold $M_D$, i.e. it is a pair of handlebody diagrams $d_l$ and $d_u$ which describe the upper and the lower handlebodies of the splitting. We assume that $d_l$ and $d_u$ are transverse.

We will need to generalize the definition of a Heegaard diagram to account for manifolds with boundary, and to relax the constraint that the circles of a handlebody diagram divide the surface of the diagram into planar regions. To this end, we also label the boundary of a 3-manifold $M$ by arbitrarily dividing the boundary components of $M$ into two disjoint subsets: The upper boundary and the lower boundary. We also define a puncture move on a 3-manifold $M$ with labeled boundary to consist of removing a ball from $M$ and labeling the new boundary as upper or lower. We define a bordism to be an equivalence class of 3-manifolds with labeled boundary under the puncture move. By abuse of terminology, we will not distinguish between a bordism and a representative of a bordism.

A compression body is a manifold which is obtained from $S \times I$ by gluing 2-handles on one side, where $S$ is a surface (usually one requires that all spherical boundary components of a compression body are capped by balls, but in the context of bordisms this does not matter). A compression diagram is then a generalization of a handlebody diagram: It is a prescription for gluing a compression body to a surface $S$ consisting of a collection of disjoint circles. However, the circles are not required to divide the surface into planar regions. We define a generalized Heegaard diagram to be a pair of compression diagrams (an upper diagram and a lower diagram) on one surface.

If we are given a generalized Heegaard diagram $D$ on $S$, we may form a bordism $M$ by gluing together the corresponding compression bodies along $S$. The boundary of the lower compression body becomes the lower boundary of the bordism, and the same for the upper compression body.
We will need a set of moves to convert any Heegaard diagram of a bordism to any other diagram for the same bordism. Also, it may not be obvious that every bordism has a Heegaard diagram. We may solve both of these problems using the theory of Morse functions, as described in [3]. We summarize the relevant parts of the theory: If \( M \) is a bordism, we first choose a Riemannian metric for \( M \). We choose a Morse function \( f \) on \( M \) which attains its minimum on the lower boundary of \( M \) and its maximum on the upper boundary of \( M \). Using the Riemannian metric, we can consider the flow \( \phi \) on \( M \) given by the gradient of \( f \). For each fixed point \( p \) of \( \phi \), we consider the descending manifold of \( p \), the set of all points that flow into \( p \). The descending manifold is necessarily a disk of some dimension, and in this way we obtain a handle decomposition of \( M \). If \( f \) is in general position, each \( n \)-handles is attached only to \( m \)-handles with \( m < n \) (not merely \( m \leq n \)) and possibly to the lower boundary. Finally, it is easy to convert such a handle decomposition to a generalized Heegaard diagram: We let the lower compression body be the union of the 0-handles, the 1-handles, and a tubular neighborhood of the lower boundary, and we let the upper compression body be the complement of the lower compression body. We obtain an upper circle for each 2-handle by intersecting the core (descending manifold) of the 2-handle with the Heegaard surface, and similarly we intersect the cocore (ascending manifold) of each 1-handle with the Heegaard surface to obtain the lower circles.

Since every bordism has a Morse function, every bordism has a Heegaard diagram. On the other hand, given a Heegaard diagram, it is easy to construct a Morse function which reproduces it. We can therefore obtain a set of moves on Heegaard diagrams from a set of moves on Morse functions. The upshot is the following theorem:

\[
\begin{align*}
(a) & \quad \Rightarrow \\
& \quad \Rightarrow \\
& \quad \Rightarrow \\
(b) & \quad \Rightarrow \\
& \quad \Rightarrow \\
& \quad \Rightarrow
\end{align*}
\]

Figure 2

**Theorem 4.1.** Let \( D \) be a generalized Heegaard diagram on a surface \( S \) and consider the following moves on \( D \):

1. **Homeomorphism of the diagram.** Using a homeomorphism from a surface \( S \) to a surface \( T \), we carry \( D \) to a diagram on \( T \).
2. The two-point move. We isotop the lower circles of $D$ relative to the upper circles. If this isotopy is in general position, it reduces to a sequence of two-point moves, as shown in Figure 2(a).

3. Sliding one circle past another. Suppose $C_1$ and $C_2$ are two circles of $D$, both lower or both upper, and $A$ is an arc which connects $C_1$ to $C_2$ but does not cross any other circle. We add a detour to $C_1$ that follows $A$, goes around $C_2$, and then goes back along $A$, as shown in Figure 2(b).

4. Creating a trivial circle. We add a contractible circle to $D$ which is disjoint from all other circles of $D$.

5. Stabilization. We remove a disk from $S$ which is disjoint from all circles of $D$ and replace it by a punctured torus with one lower circle and one upper circle, as shown in Figure 2(c).

Proof. Given a bordism $M$, we can construct a Heegaard diagram from a function $f$ if $f$ has three properties: $f$ is a Morse function, the descending manifold of each critical point of $f$ avoids other critical points of the same degree, and the descending manifolds index 2 critical points are transverse to the ascending manifolds of the index 1 critical points. Following [3], the space $\mathcal{F}$ of all smooth functions on $M$ has a stratification $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ in which $\mathcal{F}_0$ is the set of functions $f$ with all three properties. $\mathcal{F}_1$ is a codimension 1 subspace of $\mathcal{F}$ which is divided into three pieces: In each piece, an element $f$ fails to have one of the three desired properties. Given two elements $f_0$ and $f_1$ of $\mathcal{F}_0$, we choose a path of functions $f_t$ in general position. The path avoids the codimension 2 subspace $\mathcal{F}_2$, and is transverse to $\mathcal{F}_1$. Each time it crosses $\mathcal{F}_1$, the effect on the corresponding Heegaard diagram is one of the above moves. For example, if $f_t$ fails to be a Morse function, then a pair of critical points is created or destroyed, the Heegaard diagrams corresponding to $f_{t+\epsilon}$ and $f_{t-\epsilon}$ differ by stabilization or creation of a trivial circle. If a descending manifold of a critical point lands on another critical point of the same degree, the corresponding move is sliding one circle past another. Finally, if a descending manifold and an ascending manifold fail to be transverse, the effect is a two-point move.

In order to say that our state models can be defined on a triangulation of a 3-manifold, we construct a canonical way to convert a triangulation to a Heegaard diagram. Since a triangulation is a kind of handle decomposition, we can let the lower handlebody be a tubular neighborhood of the 1-skeleton as usual. We add a lower circle for each edge and an upper circle for each face. An intersection between a lower circle and an upper circle corresponds to an ordered pair consisting of an edge and a face that contains the edge.

5 The invariant and the proof that it works

Let $D$ be a (generalized) Heegaard diagram on a surface $S$. We assume that $D$ is an oriented diagram, i.e. its circles are oriented (we include reversing the orientation of a circle as a move on oriented Heegaard diagrams).

Let $H$ be a finite-dimensional involutory Hopf algebra of invertible dimension. We define the quantity $\sharp(D, H)$ as follows: To each upper circle $l$, we assign the tensor:

\[
\begin{array}{c}
c_2 \\
c_1 \\
\cdots \\
c_n
\end{array}
\]

12
where the indices $c_1, \ldots, c_n$ correspond to the crossings on $l$ in the order that they are encountered if we travel along $l$ in the positively oriented direction. We assign the tensor:

$$
\begin{array}{c}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{array}
\rightarrow \Delta

$$

to each lower circle in the same way. If, at a given crossing $c$, the tangent vectors of the lower circle and the upper circle, in that order, form a positively oriented basis for $T S$ at $c$, we contract the tensors assigned to the circle at the index corresponding to $c$. If the vectors form a negatively oriented basis, we interpose the antipode map before contracting:

$$
M \rightarrow S \rightarrow \Delta
$$

Let $Z(H)$ be the result of this series of tensor products and contractions. (Note that if we choose a basis for $H$, the summations corresponding to the contractions can alternatively be viewed as a sum over states, with $Z(H)$ the resulting partition function.) Define

$$
\sharp(D, H) = Z(H)(\dim H)^{g(S) - n_u - n_l},
$$

where $n_u$ is the number of upper circles, $n_l$ is the number of lower circles, and $g(S)$ is the genus of $S$.

**Theorem 5.1.** $\sharp(D, H)$ is an invariant of bordisms. In particular, it is an invariant of closed 3-manifolds.

**Proof.** We show invariance under the different moves:

1) Adding a trivial circle. This has the effect of contributing the following factor to $Z(H)$:

$$
i \rightarrow T = \bigcirc \text{ or } C \rightarrow \varepsilon = \bigcirc
$$

The factor is canceled by the normalization of $\sharp(D, H)$.

2) Stabilization. This contributes the following factor to $Z(H)$:

$$
C \rightarrow T = \bigcirc
$$

which is also canceled by normalization.

3) Orientation reversal. An upper circle corresponds to a factor of $Z(H)$ of the form:

$$
\begin{array}{c}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{array}
\rightarrow \Delta

\begin{array}{c}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{array}
\rightarrow M
$$

13
The equality, which follows from Lemma 3.2, demonstrates invariance. The same argument works for lower circles.

4) The two-point move. If the two circles are oriented properly, part of the expression for $Z(H)$ is of the form:

$$\Delta S \quad = \quad \Delta M$$

The equality demonstrates invariance.

5) Sliding a circle. We can assume that we are sliding one lower circle past another lower circle (the corresponding move for upper circles is equivalent by duality). We assume as a representative case that both circles have three crossings. Figure 3 depicts the move after a possible change of orientation of the circles involved. The “before” picture corresponds to the following factor of $Z(H)$:

\[
\begin{align*}
&\text{b} \rightarrow \text{M} \\
&\text{c} \quad \text{d} \\
&\text{f} \rightarrow \text{M} \quad \text{a} \\
&\text{e} \\
\end{align*}
\]

and the “after” picture corresponds to the following replacement for this factor:

\[
\begin{align*}
&\text{d} \rightarrow \Delta \\
&\text{e} \quad \text{f} \\
&\text{a} \rightarrow \Delta \\
&\text{b} \\
\end{align*}
\]

The following algebra demonstrates the equality of these two expressions:

\[
\begin{align*}
&\Delta \quad = \quad \Delta \\
&\text{f} \\
&\text{e} \rightarrow \Delta \\
&\text{d} \rightarrow \Delta \\
&\text{c} \quad \text{b} \quad \text{a} \\
&\text{M} \rightarrow \text{T} \\
&\text{M} \rightarrow \text{T} \\
&\text{M} \rightarrow \text{T} \\
&\text{M} \rightarrow \text{T} \\
&\text{T} \\
\end{align*}
\]

This completes the proof of invariance.
The theorem warrants the notation $\sharp(M, H) = \sharp(D, H)$, where $M$ is a bordism and $D$ is a generalized Heegaard diagram for $M$.

6 Properties of the invariant and special cases

We first observe that for a closed manifold $M$, $\sharp(M, H) = \sharp(M, H^*)$, and $\sharp(-M, H) = \sharp(M, H^{\text{op}}) = \sharp(M, H^\text{cop})$, where $-M$ is $M$ with the opposite orientation. Also, for any two bordisms, $M_1$, and $M_2$, $\sharp(M_1 \sharp M_2, H) = \sharp(M_1, H) \sharp(M_2, H)$, because we can choose a diagram for $M_1 \sharp M_2$, the connected sum of $M_1$ and $M_2$, which is a connected sum of diagrams of $M_1$ and $M_2$.

We will investigate the invariant $\sharp(M, H)$ in the special case that $H$ is a group algebra, and in the special case that $M$ is a link complement.

Let $G$ be a finite group and let $k[G]$ be its group algebra. As before, view the elements of $G$ as a basis for $k[G]$. In this basis, the matrix entry $\Delta_{ab}^c$ is 1 when $a = b = c$ and 0 otherwise, while $M_{ab}^c$ is 1 when $ab = c$ (as group elements), and 0 otherwise. The trace $T_a$ equals $\dim H_G = |G|$ when $a = e$, the identity, and 0 otherwise. The cotrace $C_a$ is 1 for all $a$. Thus, the tracial product $M_{a_1 a_2 \ldots a_n}$ equals $|G|$ when $\prod a_i = e$ and 0 otherwise, while the tracial coproduct $\Delta_{a_1 a_2 \ldots a_n}$ is 1 when all indices are equal and zero otherwise.

Let $M$ be a closed manifold or a manifold with some upper boundary (but no lower boundary). Viewing the invariant $\sharp(M, k[G])$ as a state model on a Heegaard diagram, we observe that the weight of a state is zero unless for each lower circle the states assigned to all crossings on the circle are equal. Thus, we may say that the states are assigned to the lower circles themselves. If the Heegaard diagram comes from a triangulation of $M$, the lower circles correspond to edges, so we may say that states are assigned to the edges. Meanwhile, the interaction on the upper circles dictates that a state has zero weight unless the product of the group elements around a face is the identity. Thus, the state model reduces to the one described in the introduction.

![Figure 4](image)

Let $M_L$ be the complement of (an open neighborhood of) a link $L$, where we declare that the boundary of $M_L$ is upper boundary. We wish to reduce $\sharp(M_L, H)$ to a state model on a projection $P$ of $L$ (we assume $P$ is a projection of $L$ onto the sphere rather than the plane). To construct a convenient handle decomposition for $M_L$, we first consider the tiling of the sphere by squares which is dual to the link projection, as illustrated in Figure 4. There will be one 3-handle $u$, and one 0-handle $v$; imagine $u$ as the region above $P$ and $v$ as a big, fat vertex below $P$. Figure 5(a) shows a piece of $M_L$ corresponding to a given square in Figure 4. We add a 1-handle for every edge in the tiling by squares, all attached to the lone 0-handle $l$, as in Figure 5(b). Next, for each square we add a 2-handle which is attached to the 1-handles on the left and right sides in
Figure 5(c). Finally, we add a second 2-handle which is attached to all four neighboring 1-handles, as in Figure 5(d). As before, we convert the handle decomposition to a Heegaard diagram by taking the lower handlebody to be a tubular neighborhood of the 1-skeleton.

Our goal is a model in which states are assigned to the arcs between crossings of $P$ and interactions are assigned to the crossings themselves. Equivalently, we can orient the link $L$ and assign a tensor of type (2,2) to each crossing, and wherever two crossings are connected, we contract the corresponding tensors. We convert the assembly of handles in Figure 5 to tracial product and coproduct tensors and we modify the coproduct tensors to obtain a tensor of type (2,2), to obtain the following replacement rule for a left-handed crossing:

\[
\begin{array}{c}
\text{d} \\
\downarrow
\end{array}
\begin{array}{c}
\text{a} \\
\downarrow
\end{array}
\begin{array}{c}
\text{c} \\
\downarrow
\end{array}
\Rightarrow
\begin{array}{c}
\text{d} \rightarrow M \rightarrow S \leftarrow S \leftarrow \Delta \\
\uparrow
\end{array}
\begin{array}{c}
\text{a} \\
\uparrow
\end{array}
\begin{array}{c}
\text{c} \\
\circ^2
\end{array}
\end{array}
\]

Using Corollary 3.4, we may simplify this tensor:

\[
\begin{array}{c}
\text{d} \\
\downarrow
\end{array}
\begin{array}{c}
\text{a} \\
\downarrow
\end{array}
\begin{array}{c}
\text{c} \\
\downarrow
\end{array}
\Rightarrow
\begin{array}{c}
\text{d} \rightarrow \Delta \rightarrow M \rightarrow \text{c} \\
\downarrow
\end{array}
\begin{array}{c}
\text{a} \\
\downarrow
\end{array}
\begin{array}{c}
\text{S} \\
\text{S}
\end{array}
\]

In [13], Kirillov and Reshetikhin describe a link invariant for any representation of any Hopf algebra with a special property called quasi-triangularity. In particular, for any Hopf algebra $H$ (involutory or not), there is a bigger Hopf algebra $D(H)$, called the quantum double of $H$, which can be defined on the vector space $H \otimes H^*$ and which is always quasi-triangular. The algebras $H$ and $H^*$ are contained in $D(H)$ by the inclusions $a \rightarrow a \otimes \epsilon$ and $a \rightarrow a \otimes i$. Consider the representation of $D(H)$ on itself given by left multiplication, and let $\rho$ be the restriction of this representation to $H$ by the inclusion $a \rightarrow a \otimes T$, where the trace $T$ is interpreted as an element of $H^*$. In this context it is easy to show that Reshetikhin and Turaev’s invariant in the special case of the representation $\rho$ of $D(H)$ is the same as our invariant $\sharp(M, H)$. Presumably our invariant, in its full generality, is a special case of Reshetikhin and Turaev’s extension of this invariant to 3-manifolds.
7 Tensor notation revisited

To make the invariant $\sharp(M, H)$ look good, we define a formal calculus of expressions involving structure tensors of a Hopf algebra, or more generally arbitrary expressions involving tensor products and contractions of tensors. We start by defining a contraction category as a natural setting for such computations.

A \textit{contraction category} is a set $T$, whose elements may be called tensors, with additional structure consisting of two functions $i$ and $o$ from $T$ to $\mathbb{Z}^+$, where $i(t)$ is called the \textit{in-degree} of $t$ and $o(t)$ is called the \textit{outdegree} (we will also say that $t$ has \textit{type} $(o(t), i(t))$), and a function $C : \mathcal{G}(T) \to T$, the evaluation function of $T$, where $\mathcal{G}(T)$ is the set of contraction graphs of $T$.

A \textit{contraction graph} of $T$ is an oriented graph $G$ whose edges and vertices are labeled in a certain way. $G$ can be a graph in the broadest sense of the word: $G$ need not be connected (and may be empty), an edge may go from a vertex to of vertices, an edge may go from a vertex to itself, and there may be edges whose heads or tails (or both) are not attached to any vertex. $G$ may also have degenerate “edges” which are oriented circles without heads or tails at all:

The vertices of $G$ should be labeled by elements of $T$. At a vertex labeled by $t$, there should be $i(t)$ inward edges and $o(t)$ outward edges, the inward edges should be numbered from 1 to $i(t)$, and similarly for the outward edges. We also number the free heads of edges of $G$ from 1 to $o$ for some $o$, and the free tails of edge of $G$ from 1 to $i$ for some $i$. We call $i$ the in-degree of $G$ and $o$ the outdegree. In short, a contraction graph looks very much like a tensorial expression in arrow notation, where the “tensors” are elements of $T$:

\[
\begin{align*}
\text{A} & \quad \text{B} \\
\text{2} & \quad \text{3} \\
\text{1} & \quad \text{2} \\
\end{align*}
\]

The evaluation function $C$ should map a contraction graph $G$ to an element of $T$ of the same in-degree and outdegree as $G$. Moreover, it should satisfy an axiom of substitution: If $G$ and $H$ are contraction graphs and $v$ is a vertex with the same in-degree and outdegree as $H$, we may define the \textit{composition} $G \circ_v H$ by replacing the vertex $v$ by the graph $H$:

\[
\begin{align*}
\text{A} & \quad \text{B} \\
\text{C} & \quad \text{D} \\
\end{align*}
\]

\[
\begin{align*}
\text{X} & \quad \text{Y} \\
\text{C} & \quad \text{D} \\
\end{align*}
\]

The axiom of substitution says that if $C(H)$ is the label of $v$, then $C(G \circ_v H) = C(G)$.

We give names to two distinguished tensors which exist in any contraction category. We call the value of the empty graph the \textit{identity scalar} (or one), and the value of a circular edge the \textit{dimension scalar}.

If $T$ and $U$ are contraction categories, a morphism from $T$ to $U$ is a function from the tensors of $T$ to the tensors of $U$ which respects in-degree and outdegree and which commutes with the evaluation map.
The term “contraction category” may seem inappropriate since we have not defined it as a kind of category. However, we can view it a category whose objects are ordered lists of left-pointing arrows and right-pointing arrows (arrows in the sense of contraction graphs, that is; we call category-theoretic arrows “morphisms”), and whose morphisms are contraction graphs with an ordered list of arrows on the left side and another ordered list on the right side:

\[ \text{A} \xrightarrow{\text{C}} \text{D} \]

We could modify the definition of a contraction category in natural ways: We could replace occurrences of the word “set” by “class”, and we could label the arrows by elements of a new set (or class) \( A \). In its full generality, a contraction category appears to be the same as a strict, symmetric, compact, monoidal category \([11]\).

As an example of a contraction category, we may define \( T(V) \) to be the set of all tensors over a finite-dimensional vector space \( V \) with ground field \( k \). In other words, \( T = k \cup V \cup V^* \cup V \otimes V \cup \ldots \). The evaluation function for contraction graphs may be constructed from the usual contraction and tensor product operations. We see that the value of the dimension scalar is the dimension of \( V \).

If \( T \) is any contraction category, one can consider a vector space \( V \) together with a morphism \( T \to T(V) \) to be a representation of \( T \).

A Hopf contraction category is a contraction category with distinguished elements called \( M, \Delta, S, \epsilon, \) and \( i \) which satisfy the usual identities of a Hopf algebra. We may define the universal Hopf category \( U \) to be the contraction category generated by these five letters with the axioms they satisfy interpreted as relations. That is, the elements of \( U \) are equivalence classes of contraction graphs on these five letters, where two contraction graphs \( G \) and \( H \) are equivalent if one can turn \( G \) into \( H \) by using the axioms of a Hopf algebra as replacement rules.

We can equivalently say that a Hopf contraction category is a contraction category \( T \) together with a morphism from the universal Hopf category to \( T \). A Hopf contraction category which happens to be a vector space is a Hopf algebra. We can also define the universal, involutory Hopf category of invertible dimension in which trace is a left and right integral, cotrace is a cointegral, and the dimension scalar has an inverse. Anticipating the next section, we call the latter category the Heegaard category, written \( \mathcal{H} \). One can check that the Heegaard category satisfies all of the identities of involutory Hopf algebras of invertible dimension presented in section 3.

We can define the “invariant” \( \sharp(M, \mathcal{H}) \) as before, although its value is not a number, but rather a formal expression which might be as difficult to analyze as the original manifold. We will see that \( \sharp(M, \mathcal{H}) \) has the surprising property of being a complete invariant on closed, irreducible 3-manifolds.

8 \( \sharp(M, \mathcal{H}) \) as a complete formal invariant

The value of \( \sharp(M, \mathcal{H}) \) involves factors of the dimension scalar for normalization. For simplicity we pass to the augmented Heegaard model \( \bar{\mathcal{H}} \), in which we assume that the dimension scalar equals the identity, and we consider \( \sharp(M, \bar{\mathcal{H}}) \).

Section 3 gives a procedure for converting a generalized Heegaard diagram \( D \) to a contraction graph over the generating tensors of \( \bar{\mathcal{H}} \). We wish to construct an inverse to this procedure.

Proposition 8.1. Each element of \( \bar{\mathcal{H}} \) is realized as \( \sharp(M, \bar{\mathcal{H}}) \) for some bordism \( M \).
Proof. Let $C$ be a graph composed of the generators of $\tilde{\mathcal{H}}$. We may re-write $i^a$, $e_a$, $M_{ab}^c$ and $\Delta_{abc}$ using Corollary 3.4:

$$
i \rightarrow = M \leftarrow \Delta \rightarrow \varepsilon = \rightarrow M \leftarrow \Delta \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
Proposition 8.1 assigns a unique equivalence class (under the sphere and circle moves) of Heegaard diagrams to each contraction graph $C$.

By extension, we can say that two bordisms $M_1$ and $M_2$ differ by an elementary move if there exist diagrams for $M_1$ and $M_2$ which differ by a sphere or a circle move. We can then consider equivalence classes of bordisms. It is easy to check that if $M$ is a bordism, then the bordisms produced by Proposition 8.1 from $\natural(M, \bar{H})$ is equivalent to $M$. One can also check that if two contraction graphs $C_1$ and $C_2$ differ by one of the defining relations of $\bar{H}$, then the corresponding bordisms are equivalent (thereby establishing a converse to the proof of invariance in section 3). Thus, $\natural(M_1, \bar{H}) = \natural(M_2, \bar{H})$ if and only if $M_1$ and $M_2$ are equivalent under sphere and circle moves.

We examine the effect of sphere and circle move on the topology of a bordism $M$. The sphere move corresponds to discarding a component of $M$ which is a 3-sphere; we may call this move on manifolds a 3-sphere move. The circle move is more complicated. Let $D$ be a diagram for $M$ and let $S$ be the surface of $D$. Recall that $M$ is obtained by gluing disks and then balls to $S \times [0, 1]$. Let $C$ be a circle on the surface of $D$ suitable for a circle move. The circle move corresponds to cutting $M$ along the annulus $C \times [0, 1]$, and then gluing a 2-handle to each of the two annular scars that result (see Figure 7). The operation may produce spherical boundary components, which are irrelevant by the definition of a bordism. We call this operation an annulus move on $M$.

It may happen that one or both of the circles $C \times \{0\}$ and $C \times \{1\}$ is trivial in its respective boundary component. In this case the annulus move simplifies $M$ and we give it a second name as follows:

Suppose that only one of the circles $C \times \{1\}$ and $C \times \{0\}$ is non-trivial, and assume that it is $C \times \{1\}$. We choose an embedded disk $S$ in the lower handlebody which bounds $C \times \{0\}$. Then the circle move corresponds to compressing the boundary of $M$ along the disk $S \cup C \times [0, 1]$. We call this a disk move on $M$.

If both circles are trivial, then we can find a disk $S_0$ in the lower boundary and a disk $S_1$ in the upper boundary so that $S_0 \cup C \times [0, 1] \cup S_1$ is a 2-sphere, and the circle move on the diagram of $M$ corresponds to cutting along this sphere, i.e. the inverse connect-sum operation. We call this a 2-sphere move on $M$.

If a 2-sphere in $M$ bounds a ball, then the only effect of the corresponding 2-sphere move is to add a 3-sphere component to $M$, which is reversed by the 3-sphere move. We call such a 2-sphere move degenerate. We will also refer to disk moves and 2-sphere moves as degenerate annulus moves, and to 2-sphere moves as degenerate disk moves.

We would like to determine when two bordisms are rendered equivalent by a sequence of annulus moves, disk moves, and non-degenerate 2-sphere moves and the inverse of these moves (and we apply the 3-sphere move whenever possible). We first observe that these three moves are destructive in the sense that they simplify a bordism, and their inverses are constructive; only degenerate 2-sphere moves are neither destructive nor constructive. Specifically, we define the height of a bordism $M$ to be the ordinal $\omega + s$, where $c$, the circle number of $\partial M$, is the maximum number of disjoint, non-trivial, non-parallel circles in the boundary of $M$, and $s$, the Kneser number of $M$, is the maximum
number of disjoint, non-trivial, non-parallel 2-spheres in $M$. (Since $M$ is actually an equivalence class of 3-manifolds and some members of this class may have 2-sphere boundary, we must stipulate that a 2-sphere which is parallel to the boundary of a representative of $M$ is trivial.) We may take $c = \dim H_1(\partial M) - \dim H_0(\partial M)$; see [7] for a proof that $s$ is finite. Clearly, destructive moves decrease the height of a bordism. Thus, any sequence of destructive moves, or path of destruction, is finite. The main result of this section is a uniqueness theorem:

**Theorem 8.2.** If $M$ is a bordism, then any two maximal paths of destruction produce the same result.

Thus if no destructive moves can be performed on two different bordisms $M_1$ and $M_2$ (e.g. if $M_1$ and $M_2$ are closed and prime), then they are either equivalent or $\sharp(M_1, \mathcal{H}) \neq \sharp(M_2, \mathcal{H})$.

This result is a special case of the Jaco-Shalen-Johannson decomposition theorem, as presented in [8] and [9]. We give a self-contained proof here:

![Diagram](image)

**Figure 8**

**Proof.** We outline the plan for a complicated proof by induction. Let $M$ be a counterexample of minimum height, and consider two possible destructive moves $x$ and $y$ on $M$, and call the resulting manifolds $M_x$ and $M_y$. If we can find two more moves $w$ and $z$ (which may or may not be degenerate) such that $M_{xw} = M_{yz}$, then by induction any two maximal paths of destruction beginning with $x$ and $y$ produce the same result, as indicated in Figure 8(a).

The moves $x$ and $y$ involve cutting along surfaces $S_x$ and $S_y$. If these two surfaces are disjoint, we can directly induct on the height of $M$, because we can perform $x$ after $y$ and vice-versa, with $M_{xy} = M_{yx}$. If $S_x$ and $S_y$ intersect, we put them in general position and induct on the number of components of their intersection (each of which is an arc or a circle). It suffices to find another destructive move $z$ (possibly after switching $x$ and $y$) such that $S_y \cap S_z$ has fewer components than $S_x \cap S_y$, and there exist moves $v$ and $w$ such that $M_{zw} = M_{xw}$, as indicated in Figure 8(b).

Suppose that one of the components of $S_x \cap S_y$ is a circle which is contractible in one of $S_x$ and $S_y$, say $S_y$. We may choose an innermost circle $C \subset S_x \cap S_y$ inside this one. $C$ bounds a disk $D \subset S_y$ whose interior does not intersect $S_x$, and $C$ divides $S_x$ into two surfaces $S_1$ and $S_2$, as shown in Figure 8(a). The move $x$ consists of cutting along $S_x$ and gluing in either 2-handles or balls, and in both cases we can find a disk $D_x$ in the region attached to $S_x$ which makes a sphere with $D$, as shown in Figure 8(b). We choose two surfaces $S'_1$ and $S'_2$ which run parallel to $S_1 \cup D$ and $S_2 \cup D$ but which do not intersect them or each other, as shown in Figure 8(c). Cutting along $S'_1$ and cutting along $S'_2$ are valid moves, and cutting along both is equivalent to performing $x$ and
then cutting along the sphere $D \cup D_x$. Therefore at least one of $S'_1$ and $S'_2$ must be a destructive move, and we may assume it is $S'_1$. At the same time, $S'_1$ has less intersection with $S_y$ than $S_x$ does. We let $z$ be the move corresponding to $S'_1$.

From this point on, we suppose that there is no circular component of $S_x \cap S_y$ which is contractible in either $S_x$ or $S_y$; in particular, neither $x$ nor $y$ can be a 2-sphere move. Suppose instead that some component is an arc with both endpoints in upper or lower boundary, say upper. This arc, together with part of the boundary of $S_y$, bounds a disk in $S_y$. We let $A \subset S_x \cap S_y$ be the innermost arc in this disk. $A$, together with part of the boundary of $S_y$, bounds a disk $D \subset S_y$ whose interior does not intersect $S_x$, and it divides $S_x$ into two surfaces $S_1$ and $S_2$. We argue as in the previous case, the only difference being that $D \cup D_x$ is now a disk instead of a 2-sphere, as shown in Figure 10.

Thus, we may assume that both $x$ and $y$ are annulus moves. There are two possibilities: Either $S_x$ and $S_y$ intersect in concentric circles which lie between the upper and lower boundary of $S_x$ and $S_y$, as in Figure 11(a); or they intersect in radial arcs, each arc connecting the upper and lower boundary of $S_x$ and $S_y$, as in Figure 11(b). In the first case, let $C$ be the circle of $S_x \cap S_y$ which is uppermost in $S_y$. $C$ bounds an annulus $A$ in $S_y$ which does not intersect $S_x$ in the interior. It divides $S_x$ into an upper annulus $A_u$ and a lower annulus $A_l$, as in Figure 12. Cutting along $A \cup A_l$ is necessarily a destructive move, because its upper boundary is a non-trivial circle. We call this move $z$. The circle $C$ bounds a disk $D_z$ in the 2-handle which is attached to $A \cup A_l$ in the move $z$, and $C$ bounds a disk $D_z$ in the 2-handle attached in the move $x$. We see that the move $z$ followed
by compressing the disk $D_x \cup A$ yields the same result as the move $x$ followed by compressing the disk $D_x \cup A_t$. As before, if we move $A \cup A_t$ away from the annulus $A$, it intersects $S_y$ less than $S_x$ does.

Finally, suppose that $S_x$ and $S_y$ intersect in radial arcs. Let $\mathcal{A}$ be the boundary of a tubular neighborhood of $S_x \cup S_y$. $\mathcal{A}$ is the union of disjoint annuli which do not intersect either $S_x$ or $S_y$. If we perform all of the corresponding annulus moves, then $S_x$ and $S_y$ lie in a component of the resulting bordism which is equivalent to $S \times [0, 1]$ for some closed surface $S$. Any maximal path of destruction removes this component entirely. Thus, $S_x$ and $S_y$ are rendered equivalent by the annulus moves represented by $\mathcal{A}$.

9 Unresolved questions

Although $\natural(M, \mathcal{H})$ is a complete formal invariant for closed 3-manifolds, it is not necessarily a complete invariant in the computational sense, because it may be as difficult to distinguish elements of $\mathcal{H}$ as it is to distinguish 3-manifolds by the standard means.

**Conjecture 9.1.** The Heegaard category $\mathcal{H}$ is residually representable. That is, if $a$ and $b$ are distinct elements of $\mathcal{H}$, there exists a vector space $V$ and a morphism $m : \mathcal{H} \to T(V)$ such that $m(a) \neq m(b)$.

If this conjecture is true, $\natural(M, \mathcal{H})$ is a complete invariant in a more meaningful sense. A few special cases of this conjecture might be easy to settle. Is there an involutory Hopf algebra $H$ for which $\natural(L(3, 1), H) \neq \natural(L(3, 2), H)$, i.e. $\Delta^{abc}M_{abc} \neq \Delta^{cde}M_{abc}$? Even if $\natural(M, H)$ is insensitive to orientation, perhaps there is a Hopf algebra that distinguishes $L(7, 1)$ from $L(7, 2)$, or $L(8, 1)$.
from $L(8,3)$ (correspondingly, $\Delta^{abcdefg} M_{abcdefg}$ from $\Delta^{abcdefg} M_{aceghdf}$ and $\Delta^{abcdefg} M_{abcdefg}$ from $\Delta^{abcdefg} M_{adgbehcf}$).

The construction given in this paper is almost certainly not as general as it could be. For example, Dijkgraaf mentions the following state model on triangulations which was described to me by Vaughan Jones: As before, we choose a finite group $G$ to be the state set, we assign states to the oriented edges of a triangulation, and we assign an interaction to each face which is 0 unless the product of the states of the face’s edges is the identity. However, we also have an interaction for each tetrahedron, which we assume is non-zero whenever the value of the tetrahedron’s faces is non-zero. We may view this interaction as a complex-valued function $I(a, b, c, d, e, f)$, with $a, b, c, d, e, f \in G$, and therefore as a cocycle on the canonical triangulation of $K(G,1)$ with coefficients in the multiplicative group $\mathbb{C}^*$. The constraint that the state model is a topological invariant is equivalent to the condition that this cocycle is actually a cocycle, and two different cocycles yield the same invariant if they are cohomologous. Thus, we get an invariant for every pair consisting of a finite group and an element of $H^3(G, \mathbb{C}^*)$. If we choose the trivial cohomology class, we get $\sharp(M, \mathbb{C}[G])$. Is there a mutual generalization of this invariant and $\sharp(M, H)$?

Can the invariant be extended to non-involutory Hopf algebras? As was mentioned in section 6, $\sharp(M, H)$ corresponds to a certain link invariant involving the quantum double of $H$. Can it be generalized to quasi-triangular Hopf algebras other than the quantum double?

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