I. INTRODUCTION

Quantum field theory as we know it is expected to break down near the Planck scale, \( l_{\text{Pl}} \sim m_{\text{Pl}}^{-1} \), where dimensional analysis indicates that gravity becomes as strong as the standard model forces. At this scale, nature is believed by many to contain an intrinsic cutoff of the fundamental degrees of freedom that will ultimately help to cure the divergencies of quantum field theory and general relativity. The manifestation of this cutoff in quantum gravity is as yet unknown; it may, for instance, render the theory discrete or invalidate the notion of locality as suggested by matrix theory. It is also unclear at which energy or momentum scale the cutoff begins to noticeably affect the behavior of quantum fields. For simplicity, the terms “Planck scale” and “cutoff scale” will be considered equivalent in this paper, bearing in mind that they might turn out to be very different concepts.

Under almost all circumstances, long wavelength phenomena are sufficiently decoupled from small scales to hide the short distance behavior of the fundamental theory. There are two well known situations where low energy quantum states potentially retain a memory of their Planckian origin: Hawking radiation and cosmological expansion (see for a recent overview of the trans-Planckian puzzle in both contexts).

In the original derivation of Hawking radiation, the exponential redshift of outgoing radiation immediately outside of a black hole event horizon requires the existence of a trans-Planckian mode reservoir. If Planck scale physics does indeed provide a short distance or high frequency cutoff, where do these modes come from? Beginning with Unruh’s sonic black hole analogy, whose basic philosophy we will employ in Sec. III, several papers have addressed this question by studying linear scalar field theories with nonlinear dispersion relations, short distance uncertainty relations, and discretized spatial degrees of freedom. Most of these models produce Hawking radiation by mode conversion of ingoing modes into outgoing ones without affecting the thermal Hawking spectrum, provided that the cutoff scale is well separated from the black hole horizon scale. In the lattice model, mode conversion takes place by means of Bloch oscillations. However, in the present formulation the lattice is constantly expanding, invalidating the notion of a fixed cutoff length. The nonlinear dispersion models achieve mode conversion by a reversal of group velocity near the horizon, but they fail to fully account for the origin of Hawking radiation owing to their time independence and hence the conservation of Killing frequency. Thus it seems that in order to formulate a self-contained model of Hawking radiation that avoids any reference to trans-Planckian modes one needs to include an explicit mechanism for the creation and dissipation of modes. It will be argued below that this statement is even more relevant in the cosmological context where mode conversion cannot take place.

In this work, an attempt is made to use a similar approach to the other setting that involves strong redshifting, i.e. cosmological expansion. Apart from the obvious overlap with the black hole problem, there are at least three different motivations for this inquiry. The most striking implications of a Planck scale cutoff arise in scenarios where the inflationary phase lasts sufficiently long to redshift trans-Planckian wavelengths to cosmological scales as in the simplest models for chaotic inflation. This case is especially interesting since, at least in principle, allows for observable Planck scale relics reflected by non-Gaussianity and broken scale invariance of the distribution of cosmic microwave background perturbations (Sec. II). In Sec. III, a simple model for inflation near the cutoff scale is analyzed, featuring a modified nonlinear dispersion relation of the form employed in that imposes an upper bound on the proper frequency of a free scalar field on a de Sitter background.

A detailed analysis of the sensitivity of inflation to trans-Planckian behavior using a very similar approach is described in Refs. Choosing three different types of dispersion relations and two alternative vacuum states, the authors study the scale dependence of the predicted perturbation power spectrum, finding significant
deviations from scale invariance under certain conditions. While our work was done independently, the relation of our results to those of Refs. [18, 19] will be highlighted at the appropriate points.

But even in the more benign case of ordinary power law cosmological expansion the introduction of a proper frequency or distance cutoff presents a conceptual challenge. As shown in [24] and further discussed in [25], the formulation of a Hamiltonian lattice field theory in an expanding space-time is highly problematic. Furthermore, a dynamically expanding or contracting universe with a proper cutoff implies a growing or shrinking Hilbert space of the effective field theory living in it [3] and it is not obvious how this can be realized while preserving unitarity and local Lorentz invariance on large scales. As always, the final answer must be deferred to quantum gravity, but it may be useful to take a phenomenological stance and ask the following question: Can a field theory in an expanding universe be modified in an ad hoc, but simple and well controlled way such that its proper energy and momentum eigenvalues are bounded but it asymptotically reduces to a standard relativistic field theory for measurements that involve only large scales? Just like in similar investigations of the Hawking effect and its sonic analogies, we expect to learn something about the requirements for a full theory by trying (and often by failing) to answer this question. This program is, of course, beyond the scope of the present paper but we hope to raise some of the relevant questions in Sec. II.

Finally, issues regarding Planck scale effects on the dynamics of inflation may be even more relevant in the framework of brane cosmology [26] and large extra dimensions [27], where the fundamental energy scale might be substantially lower than the conventional value for \( m_{\text{pl}} \), and where the effective Planck scale felt on our brane is a time variable quantity [5]. In these models, it is conceivable that Planck scale physics of the form discussed in Sec. III has left an imprint on the cosmic microwave background.

II. SOME IMPLICATIONS OF A FIXED CUTOFF IN EXPANDING SPACE-TIMES

A. The problem and its possible circumvention

The trans-Planckian puzzle in the context of inflationary cosmology was pointed out by Brandenberger [24]. In many scenarios for inflation, most notably the chaotic variety, the inflationary phase commences when the Hubble length \( H^{-1} \) is of the order of \( l_{\text{pl}} \) and lasts for \( N \sim 10^8 \) e-foldings [13, 25] (only the second property is relevant at this point, the first will become important in Sec. III). If a cutoff were present in the theory, metric perturbations on cosmological scales today would have redshifted from modes that were not part of our universe when inflation began. To see this, assume that at some initial time \( t_i \) the mode expansion of linear inflaton perturbations in terms of the comoving wavenumber \( k \) is truncated at some high wavenumber \( k_0 = a(t_i) m_{\text{pl}} \), where \( a(t) \) is the scale factor of the Friedmann-Robertson-Walker (FRW) line element. Due to the absence of mode interactions in a linear theory the comoving cutoff wavenumber \( k_0 \) is conserved, implying that at the time \( t_i \) after inflation the cutoff corresponds to a physical scale \( \ell_c = a(t_i) k_0^{-1} = e^N l_{\text{pl}} \).

Of course, one cannot naively impose a cutoff on an initial spatial hypersurface at the onset of inflation and evolve the fields without modifying the theory, as this would lead not only to a suppression of inflaton fluctuations but of all quantum modes in the universe today [3]. Moreover, all of the mechanisms for mode conversion of ingoing into outgoing modes that salvage Hawking radiation in truncated theories [6, 7, 8, 14] hinge on the spatial inhomogeneity of the black hole space-time and therefore seem unlikely to work in a FRW universe where no meaningful separation of “ingoing” and “outgoing” modes exists. Hence, in order to keep the cutoff scale pegged to a fixed absolute value, new modes must be added dynamically as the universe expands or deleted as it collapses. We will refer to this process as “mode creation” or “mode dissipation”.

Since only \( \sim 60 \) e-foldings of inflation are needed to explain the observed flatness of the universe, the problem can be evaded for purely practical purposes by demanding that inflation terminated before the initial cutoff length had crossed the particle horizon [22]. While this solution effectively hides Planck scale physics from cosmological observations, it doesn’t alleviate the need for mode creation and must therefore be considered incomplete.

In order to clarify the role of ultra high frequencies in the Hawking effect, Jacobson [20] demonstrates an alternative derivation that makes no reference to frequencies above a certain cutoff. Instead, it imposes a boundary condition in a timelike region near the horizon that forces quantum fields to be in their vacuum state for frequencies larger than the inverse black hole mass but smaller than the cutoff frequency, where all frequencies are measured in the frame falling freely into the black hole. One can use a very similar construction to discuss the dynamics of linear inflaton-driven metric perturbations in the presence of a cutoff. In the gauge invariant formalism [27], the quantized degree of freedom \( \delta \) is a linear combination of inflaton and scalar metric perturbations obeying the equation of motion for a minimally coupled scalar field with a time dependent mass. At large wavenumbers, \( k \gg aH \), its plane wave mode functions \( v_k \) asymptotically behave like those of a massless scalar field on a de Sitter background, justifying the choice of the adiabatic massless de Sitter-invariant vacuum for these modes.

Instead of prescribing the vacuum condition as an initial condition for integrating the mode equations as in Ref. [27], it can be re-interpreted as a boundary condition in k-space that is imposed at all times during inflation. In situations where the horizon length is far removed from
the cutoff, \( H^{-1} \gg l_{pl} \), it is possible to separate scales between those close to the cutoff and those where the \( v_k \) are still de Sitter-like. In other words, if the adiabatic vacuum boundary condition holds for \( aH \ll k \ll am_{pl} \), the predictions for the amplitude of \( |v_k|^2 \) at horizon crossing, and hence for the power spectrum of cosmological perturbations, remain unaltered.

Lacking a theory of mode creation, the best we can do to protect low-energy effective field theories in an expanding universe from cutoff anomalies is to demand the validity of the above boundary condition in cases where \( H^{-1} \gg l_{pl} \) is true (which it certainly is today). We can now ask the following questions: first, is it possible to isolate some key properties of a mode creation process that satisfies the boundary condition in this asymptotic regime and second, once we have constructed such a theory, how do the predictions for cosmological perturbations change in cases where the scale separation fails, i.e. if \( H^{-1} \sim l_{pl} \) ?

Constraining the investigation to theories that obey the asymptotic boundary condition, it is clear that the effects of a cutoff on the spectrum of cosmological perturbations should be strongest on those scales that crossed the horizon when it was smallest and hence closest to the cutoff scale. As \( H^{-1} \) grows during standard inflation, those scales would now be the largest observational ones.

\[ \text{C. The fluid analogy: dispersion and dissipation} \]

Looking for guidance in the construction of a phenomenological model for mode creation we turn to the fluid analogy that proved so useful for analyzing the effects of a cutoff on Hawking radiation. Unruh [5] showed that the quantized sound field of a fluid flow containing a sonic horizon becomes thermally excited for the same reason that a black hole emits Hawking radiation. Providing a natural short distance cutoff due to the breakdown of the continuum assumption, fluid models represent a well-defined testbed for Hawking radiation.

Sound waves with ever decreasing wavelengths propagating in a fluid first perceive the approach of the molecular scale \( l_m \) by a change in the dispersion relation coupling the wavenumber \( k \) and the frequency \( \omega \). Typically, the dispersion relation is linear in the low wavenumber regime and dips over toward smaller frequencies as \( k \rightarrow k_0 \sim l_m^{-1} \), never exceeding a maximum frequency \( \omega_0 \). For studies of Hawking radiation, this behavior was mimicked by invoking an artificial nonlinear dispersion relation in a linear scalar field theory living on a two-dimensional black hole space-time \([6, 7, 8]\). It was found that wave packets propagated backwards in time are “reflected” off the horizon by virtue of the group velocity dropping below the speed of sound (or light) as the waves are blueshifted toward the cutoff. The origin of the outgoing Hawking modes was thus revealed to be exotic ingoing modes that were “converted” into outgoing ones by the spatially varying group velocity.

In contrast, all modes in an expanding or collapsing spatially homogeneous FRW universe are red- or blueshifted at the same rate, making mode conversion impossible. There is little hope to avoid an explicit prescription for mode creation and dissipation if both energy and momentum are to be bounded. Turning again to fluids for intuition, here viscosity is responsible for both dispersion and dissipation, i.e. the dispersion relation becomes both nonlinear and complex \([29, 38]\). In spite of being manifestly non-unitary, this is a possible model for a collapsing universe where we want to get rid of existing modes. Mode creation in an expanding space-time, on the other hand, would involve inverse dissipation, i.e. exponential growth of the wave amplitudes from initial data arbitrarily close, but not identical, to zero. There is no obvious choice for assigning this initial data in the very early universe \([39]\). Perhaps even more importantly, in order to satisfy the adiabatic vacuum boundary condition of Sec. II, the redshifted modes ultimately need to be stabilized at a given amplitude and phase. This implies either tremendous fine-tuning of the initial data or nonlinear mode interactions. The latter appears possible in principle, but finding a viable form for the interaction that fulfills all the above requirements may be a formidable challenge.

The situation is little better for lattice theories on expanding backgrounds \([20]\). It seems to be necessary to dynamically introduce new lattice points as the universe expands \([36]\). Doing so, one basically faces the same dilemma for assigning field values to these points as in the inverse dissipation model above.

The first and simplest choice for analyzing the effects
of a Planck scale cutoff on inflationary perturbations is thus to ignore the actual creation of the modes and focus only on their frequency evolution. Since no dissipation is involved, this case is entirely analogous to the sonic black hole analysis in [3]. Just like there, we want to construct a model whose proper frequency is bounded when the proper wavelength drops below a critical value. Evolving this mode backward in time, we would see it oscillate more and more slowly with respect to the cosmological expansion until it freezes in, while its wavelength continues to be blueshifted without bound.

The same general approach to the trans-Planckian problem in inflationary cosmology was chosen by the authors of Refs. [13, 14]. In addition to the asymptotically constant Unruh dispersion relation described below, two additional forms of dispersion relation (the sub/superluminal cases of Ref. [1]) were analyzed in two different vacuum states.

### III. A SIMPLE MODEL FOR INFLATION NEAR THE CUTOFF SCALE

The setting for the model discussed in this section is the slow roll phase of inflation, driven by a scalar field rolling down a very flat potential. At the level of simplicity we are seeking it is unnecessary to specify any details of the potential or the amplitude of the homogeneous inflaton mode. Keeping in mind that in a more realistic scenario, the Hubble parameter $H$ is a slowly varying function of time, it will be assumed constant for our purpose of finding the first order effects of a nonlinear inflaton dispersion relation. The cosmological background is thus given by exact de Sitter space. As a further simplification, the gauge invariant inflaton-metric perturbation variable $v$ [2] will be replaced by a free, massless scalar field mimicking linear inflaton perturbations. This is a very good approximation for a weakly self-coupled inflaton field during slow roll [23].

#### A. Equations of motion

In order to establish a framework for the model that includes the proper frequency cutoff, let us first recall the standard theory for a scalar field in de Sitter space where $\omega$ and $k$ are both unbounded. A minimally coupled massless scalar field $\phi$ obeys the equation of motion

$$\Box \phi = 0 \ ,$$

where $\Box$ stands for the covariant Laplace-Beltrami operator. In terms of the conformally flat metric for de Sitter space,

$$ds^2 = a(\eta)^2(d\eta^2 - dx^2) \ ,$$

with the conformal time $\eta$, the cosmological scale factor $a(\eta) = -(H\eta)^{-1}$, and the 3-dimensional Euclidian space element $dx$, Eq. (1) becomes:

$$\phi'' - 2\alpha \phi' - \nabla^2 \phi = 0 \ ,$$

where $\alpha \equiv a'(\eta)/a(\eta)$, and the prime denotes differentiation with respect to $\eta$. $\phi$ can be expanded into comoving plane waves [30],

$$\phi(\mathbf{x}, \eta) = (2\pi)^{-3/2} \int_k \left( a_k u_k(\mathbf{x}, \eta) + a_k^* u_k^*(\mathbf{x}, \eta) \right) dk$$

where

$$u_k(\mathbf{x}, \eta) = a(\eta)^{-1} \chi(\eta) e^{ik\mathbf{x}}$$

and the mode functions $\chi(\eta)$ satisfy $(k = |k|)$:

$$\chi'' + \left( k^2 - \frac{\alpha''(\eta)}{a(\eta)} \right) \chi = 0 \ .$$

On subhorizon scales, $k \gg a(\eta) H$, Eq. (3) reduces to a harmonic oscillator equation with the solution $\chi \sim e^{i\omega \eta}$ and the linear (comoving) dispersion relation $\omega = \pm k$. Equivalently, the dispersion relation expressed in terms of the proper frequency, $\nu = \omega/a(\eta)$, and wavenumber, $\kappa = k/a(\eta)$ is simply $\nu = \pm \kappa$.

The theory is quantized by treating the field as a self-adjoint operator $\phi$ and imposing the equal-time commutation relations on the field and its canonically conjugate momenta or, equivalently, on $\hat{a}_k$ and $\hat{a}^+_k$ [8]. In addition, a vacuum state for $\phi$ must be specified. Since no unique representation of the vacuum exists in non-stationary space-times, one usually picks the state that minimizes the measurement of inflaton “particles” at the beginning of inflation by imposing the adiabatic positive frequency condition at past conformal infinity ($\eta \rightarrow -\infty$) [27]. Together with the normalization provided by the commutation relations, this gives [8]

$$\chi(\eta) = \frac{1}{2} \sqrt{\pi \eta} \mathcal{H}^{(2)}_{3/2}(k\eta) \ ,$$

in terms of the Hankel function $\mathcal{H}^{(2)}$. We can obtain an estimate for the strength of metric perturbations by evaluating $|u_k|$ at horizon crossing, i.e. $k = a(\eta)H = -1/\eta$, yielding the well-known Hawking amplitude:

$$\left( \frac{k^3}{2\pi^2} |u_k|^2 \right)^{1/2} \approx \frac{H}{2\pi} \ .$$

Our goal is to construct the cosmological analogue of the Unruh model [3], i.e. a modification of the scalar wave equation that keeps $\omega$ from exceeding an upper bound. We can then check whether the prediction for the horizon crossing amplitude deviates from Eq. (8). To this end, we replace the spatial derivative operator $\nabla$ in Eq. (3) with $F(\nabla, a(\eta))$ where $F(k,a)$ is an odd, analytic function that has the following properties:

$$F(k,a) = \begin{cases} k & \text{for } k \ll a \kappa_0 \\ a \kappa_0 & \text{for } k \gg a \kappa_0 \end{cases}$$
thereby explicitly breaking local Lorentz invariance for high wavenumbers. The preferred frame is the one where \( \kappa_0 \) is specified, i.e. the cosmic rest frame. As discussed, one would typically expect \( \kappa_0 \approx m_{pl} \). After going through the mode expansion, the new equation for the mode functions is given by

\[
\chi'' + \left( F^2(k, a(\eta)) - \frac{a''(\eta)}{a(\eta)} \right) \chi = 0 ,
\]

yielding the modified dispersion relation

\[
w = \pm F(k, a(\eta))
\]
on subhorizon scales. As promised, the proper frequency is constant, \( \nu = \kappa_0 \), above the proper cutoff scale \( \kappa_0 \), while converging to the conventional result far below the cutoff. The intermediate region around the cutoff may be everything from a smooth transition over many decades of \( k \) to a sharp turnover.

B. Choice of the vacuum state

Like in the non-dispersive case, a unique state that minimizes the probability for particle detection and hence most closely resembles the standard notion of a vacuum state can be constructed with the help of an adiabatic expansion \[\text{[54]}\]. To lowest order, an equation of the form

\[
\chi'' + \omega^2(\eta)\chi = 0
\]

has the positive frequency WKB solution

\[
\chi = \frac{1}{\sqrt{2\omega_+}}\exp\left(-i\int \omega_+(\eta)d\eta\right)
\]

where \( \omega_+(\eta) \) is the positive root of \( \omega^2 \). The following order correction to \( \omega_+ \) in Eq. (13) is \( O(\omega'/\omega) \).

Eq. (13) shows that the cutoff is exceeded by any mode with fixed comoving wavenumber \( k \) at very early times when \( a(\eta) \to 0 \), or at any fixed time \( \eta \) for very large \( k \). For the purpose of constructing the vacuum state it therefore suffices to analyze Eq. (13) far above the cutoff where it becomes

\[
\chi'' + \left( a^2\kappa_0^2 - \frac{a''(\eta)}{a(\eta)} \right) \chi = 0 .
\]

Introducing the dimensionless parameter

\[
\sigma \equiv \frac{\kappa_0}{H} \approx \frac{m_{pl}}{H}
\]

characterizing the size of the de Sitter horizon compared to the cutoff scale, Eq. (14) can be written in the form of Eq. (12) with

\[
\omega^2(\eta) = \frac{\sigma^2 - 2}{\eta^2} ,
\]

possessing the exact solutions

\[
\chi_{\pm} = C_{\pm} \eta^\frac{\sigma}{2} \left( 1 \pm \sqrt{1 - 4\left(\sigma^2 - 2\right)} \right) .
\]

The small amplitude of cosmic microwave background fluctuations indicates that while our current Hubble scale crossed the horizon during inflation, \( \sigma \) was very large, of order \( \gtrsim 10^5 \). Assuming that this condition was true also at the onset of inflation where the vacuum condition is evaluated, one finds

\[
\omega_+ \approx -\frac{\sigma}{\eta} ,
\]

with corrections of order \( O(\eta^{-2}) \) which quickly vanish as \( \eta \to -\infty \). The adiabatic vacuum for \( \sigma \gg 1 \) is thus given by

\[
\chi(\eta) = \frac{1}{\sqrt{2\sigma\kappa_0}}\exp\left(-i\sigma\kappa_0 t\right) ,
\]

where we made the coordinate transformation to FRW time \( t \) by replacing \( a\,d\eta \) with \( dt \). This solution is also admitted by the large-\( \sigma \) limit of Eq. (17). As intuitively expected, modes far above the cutoff and far inside the horizon behave like free harmonic oscillators with the fixed, \( k \)-independent proper frequency \( \kappa_0 \).

However, some scenarios for inflation invoke an initial phase where \( \sigma \approx 1 \) (e.g., [17]). In this case, Eqs. (13) and (14) show that no oscillating solutions exist as cosmic expansion dominates the field dynamics on all scales. A particle interpretation is unavailable even locally, making the construction of an adiabatic vacuum with the usual meaning impossible.

One can instead resort to postulating a phase of slow expansion prior to the onset of inflation, taking place in an approximately flat region of space time. Evidently, as long as \( a''/a \) is small compared to \( a^2\kappa_0^2 \), Eq. (14) has the approximate positive frequency root

\[
\omega_+ \approx a\kappa_0 ,
\]

reproducing the adiabatic vacuum solution of the large-\( \sigma \) case, Eq. (19). The interpretation in terms of harmonic oscillators with \( k \)-independent frequencies is, of course, identical.

The authors of Ref. [18] obtain essentially the same vacuum state by minimizing the energy density of the scalar field. They also analyze the evolution of the dispersive theory in an alternative state, the “instantaneous Minkowski vacuum”.

C. Comparison of horizon crossing amplitudes

It remains to be shown that the dispersive theory conforms to the asymptotic boundary condition of Sec. (II) for \( \sigma \gg 1 \). While the equations of motion, Eqs. (13) and (14), converge if \( a \gg k/\kappa_0 \) by virtue of Eq. (17), this is
not obviously true for the field amplitudes evolving from different initial conditions.

Subhorizon modes, i.e. those with wavenumbers \( k \gg aH \), approximately obey a harmonic oscillator equation with fixed frequency \( \omega \sim k \) (linear dispersion, Eq. (1)) or time dependent frequency \( \omega \sim F(k, a) \) (nonlinear dispersion, Eq. (10)). In the latter case, we will focus our attention on slowly varying functions \( F \) with respect to \( a(\eta) \), such that we can take advantage of the adiabatic invariant \( I = A^2 \omega \) of a harmonic oscillator obeying \( \omega'/\omega \ll \omega \) whose amplitude is denoted by \( A \) (e.g., [22]). The value of \( I \) is set by the initial conditions and remains conserved as long as the evolution of the field is adiabatic, i.e. as long as the frequency change is slow compared with the frequency itself.

According to the initial conditions Eq. (\ref{eq:10}) and Eq. (\ref{eq:19}), \( I = 1/2 \) initially for both the standard and the dispersive case. If the physical wavelength for a mode \( k \) becomes larger than the cutoff length while \( k \) is still well inside the horizon, i.e. if

\[
\sigma \gg k \gg \frac{1}{\eta},
\tag{21}
\]

for some \( \eta \) the frequencies of both theories converge and the invariance of \( I \) implies the convergence of the amplitudes. Consequently, the boundary condition of Sec. II is satisfied for large \( \sigma \), and the predictions of both theories for the cosmological power spectrum are identical. This is in agreement with the findings of Refs. [18, 19].

However, if the transition between the asymptotically constant and the linear regime is very smooth, \( F(k = aH, a) \) may be significantly smaller than the horizon crossing frequency in the non-dispersive case, \( \omega = k = aH \), leading to deviations from the standard result even if \( \sigma \gg 10^5 \). Although strictly speaking, the conservation of \( I \) is violated as \( k/a \) becomes comparable to \( H \) due to cosmological expansion, the effect will be similar in both cases and we can assume the near equality of \( I \). Consequently, the ratio of the horizon crossing amplitudes of the dispersive and non-dispersive theories is approximately

\[
\frac{\langle |\chi|^2 \rangle_{\text{disp}}}{\langle |\chi|^2 \rangle_{\text{non-disp}}} \approx \frac{\omega_{\text{non-disp}}(k = aH)}{\omega_{\text{disp}}(k = aH)} = \frac{aH}{F(aH, a)} > 1,
\tag{22}
\]

giving rise to possible deviations from scale invariance depending on the actual shape of \( F(k, a) \) and the time dependence of \( aH \) in a more realistic model for inflation. This effect becomes important if a residual quantum gravitational deformation of the dispersion relation is noticeable at energies approximately 5 orders of magnitude below the Planck scale. Present experimental bounds do not strongly constrain this regime; a more quantitative analysis is under way.

In addition to the incomplete (but fully adiabatic) convergence in case of a very smoothly varying \( F(k, a) \), there exists a second potential source of deviations from the standard theory: if \( F(k, a) \) changes too quickly to allow adiabatic adjustment of the modes, the conservation of \( I \) can be violated even on subhorizon scales. The significance of this effect will again depend on the particular form chosen for the dispersion relation.

IV. DISCUSSION

In this work, the cosmological analogue of Unruh’s dumb hole model is analyzed in order to address two issues, a purely cosmological one and a more fundamental problem concerning the origin of Planckian modes in expanding space times.

The cosmological question, also investigated in Refs. [18, 19], probes the (in)sensitivity of the predictions of inflation – most notably the scale invariance of the spectrum of cosmological fluctuations – with regard to changes of physics near the Planck scale. Even within the highly idealized framework of the Unruh model the answer is twofold. Assuming that the cosmological horizon is much larger than the cutoff scale at all times including the onset of inflation, one finds an adiabatic vacuum state of the modified theory whose corresponding \( k \)-mode amplitude adiabatically evolves toward the standard one. If full convergence is achieved before the mode crosses the horizon, the standard predictions of inflation are unchanged (in agreement with Refs. [18, 19]). Otherwise, the perturbation amplitude will be slightly larger than usual, at a level inversely related to the proper frequency of the mode at horizon crossing (Eq. 22).

If, however, the horizon and cutoff scales are comparable at the beginning of inflation, the notion of an adiabatic de Sitter vacuum is invalid as the modes become overdamped at past conformal infinity and a local particle interpretation ceases to exist. In Sec. (III B), this problem is formally solved by demanding that the de Sitter stage be preceded by a nearly flat and slowly expanding phase in which the quantum modes attain their ground state. One can try to justify this choice on the basis of the initial conditions for inflation but it remains less satisfactory than in the standard picture. In this scenario, the adiabatic vacuum is identical to the one in the large-horizon case, and so are the consequences for the cosmic fluctuation power spectrum.

We also made an attempt to shed light on the following, more fundamental question, also raised in Ref. (2): assuming that the number of degrees of freedom of the universe is finite due to a Planck scale cutoff, cosmological expansion inevitably implies that this number grows in time. What is the mechanism that assigns a particular state, e.g. adiabatic vacuum, to newly born modes? To avoid conflict with standard model physics, the vacuum state need only be attained at low wavenumbers and on scales far inside the cosmological horizon. In certain scenarios for inflation, however, these two requirements may not be satisfied. The challenge is thus to find a simple, self-contained model that includes a process for the cre-
ation of modes that converge to the adiabatic vacuum on low wavenumber, subhorizon scales.

In the analogous black hole problem, an explicit mechanism of mode creation can be avoided by conversion of ingoing into outgoing modes (16), at least at first sight (see Ref. [2]). This is not possible in the cosmological case. Furthermore, initial data for the black hole problem can always be assigned in the asymptotically flat region far away from the hole, in contrast with cosmology where the vacuum is usually defined exactly where we need to modify the theory, namely at $k \to \infty$.

The Unruh dispersion relation employed in this work sidesteps the mode creation problem by allowing the proper wavelength to become infinitely small. It is therefore an inadequate model for the more interesting case in which both frequency and wavelength are bounded by some fundamental scale. There, the cosmological ultraviolet problem forces us to think explicitly about how the vacuum modes of ordinary quantum field theory emerge from the “space-time foam” or a similar short distance concept. There are a number of complementary ways that this question may be addressed. Phenomenological quantum gravity in the spirit of Refs. [33, 34] might yield some intuition about how to make contact with current candidate theories for quantum gravity. “Fuzzy” short distance structures, implemented as space-time uncertainty relations (33), can be used to define the cutoff in a purely quantum mechanical manner as was successfully done for Hawking radiation (34). Finally, it is conceivable that the fluid analogy of “inverse dissipation”, i.e. the generation of finite amplitude modes out of thermal (or space-time) noise, will allow us to gain insight into the kinds of weak nonlinear mode interactions that manage to stabilize the waves in their vacuum state. A possible, albeit highly speculative, side effect of such nonlinearities may be some degree of non-Gaussianity in the distribution of cosmic microwave background perturbations.

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[36] We shall use natural units, $c = \hbar = 1$, throughout this paper so that $m_{pl}$ represents the Planck mass, energy, frequency, wavenumber, and momentum.
[37] This is in contrast with the black hole case where a given mode detected by a distant observer was redshifted exponentially in time since the black hole collapsed.
In order to reduce the problem of the modified Hawking effect to solving single-mode ODEs numerically \[9\] or analytically \[10\], Corley & Jacobson use a nonlinear dispersion relation that becomes complex above the cutoff wavenumber. Its motivation is different from the (inverse) dissipation framework discussed here, and its validity in the complex regime is unclear.

The closest analogy to thermal noise in fluids might be to prescribe some sort of “space-time noise” near the Planck scale. Perhaps a phenomenological approach similar to the one sketched in Refs. \[33, 34\] may turn out useful.