Charge densities for conducting ellipsoids

T L Curtright, Z Cao, S Huang, J S Sarmiento, S Subedi, D A Tarrence and T R Thapaliya

Department of Physics, University of Miami, Coral Gables, FL 33124-8046, United States of America

E-mail: curtright@miami.edu

Received 24 December 2019, revised 2 March 2020
Accepted for publication 17 March 2020
Published 20 April 2020

Abstract

The volume charge density for a conducting ellipsoid is expressed in simple geometrical terms, and then used to obtain the known surface charge density as well as the uniform charge per length along any principal axis. Corresponding results are presented for conducting hyperellipsoids in any number of spatial dimensions. The presentation is at a level suitable for use in graduate courses on electrostatics, as a supplement to more traditional material.

Keywords: electrostatics, ellipsoid, charge density

1. Introduction

The electrostatics of charged conducting ellipsoids embedded in three dimensions were first understood in the early part of the nineteenth century [1, 2]. The surface charge densities as well as the potentials and electric fields surrounding such objects have elegant geometrical properties, as discussed extensively in the literature [3–8]. In particular, the equipotentials are confocal ellipsoids surrounding the charged surface, with the electric field everywhere normal to those ellipsoids. More specifically, the electric field at any observation point outside a charged conducting ellipsoid of revolution, either prolate or oblate, is always directed along the bisector of a pair of straight lines drawn from either of two focal points of the ellipsoid to the observation point. Moreover, for any conducting ellipsoid in three spatial dimensions, the charge per length is always constant when projected along any of the three principal axes, a feature that is perhaps the one elementary property that is easiest to keep in mind. However, in any other number of spatial dimensions, this last statement must be modified, as discussed recently in [9].

In any case, it may not be so well known that the surface and linear charge densities for conducting ellipsoids follow easily from elementary volume charge densities that are simply expressed in geometrical terms using Dirac deltas. The purpose of this article is to present the
volume charge densities and obtain from these the known surface and linear densities, in any number of spatial dimensions.

For completeness, we also discuss the potentials and electric fields surrounding conducting ellipsoids, thereby confirming the surface charge density through the use of Gauss’ law. Finally, in an appendix, we discuss the geometry of hyperellipsoids from both intrinsic and extrinsic points of view. Overall, we have tried to present our discussion at a level suitable for use in graduate courses on electrostatics, as a supplement to more traditional material, similar to the presentations in two previous articles in this journal [9, 10].

2. Charge densities for ellipsoids in three dimensions

Let us begin with the charge distribution on an ideal, static, conducting ellipsoidal surface embedded in three spatial dimensions. The charged surface is specified by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where we have expressed the constraint that defines the ellipsoid in the most convenient Cartesian frame. If this surface is an ideal conductor, i.e. an equipotential surface, carrying a total static charge $Q$, then that charge is distributed according to a volume charge density given simply by

$$\rho(\mathbf{r}) = -\frac{Q}{4\pi abc} \delta\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right).$$

The Dirac delta $\delta\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right)$ restricts all the charge to lie on the surface, albeit not uniformly if any two of $a$, $b$, $c$ are unequal, and is an obvious property of $\rho$ for this system. On the other hand, it is remarkable and perhaps surprising that the coefficient of this $\delta$ is a constant. Of course, when $a = b = c$ the expression (2) reduces to the well-known $\rho$ for a uniformly charged, hollow spherical shell of radius $a$. But, as briefly discussed in our conclusions, by choosing a constant coefficient for a Dirac delta that defines other closed surface shapes that are not ellipsoids, one does not obtain a constant potential within the surface.

Nevertheless, although this form for $\rho(\mathbf{r})$ is remarkably simple even when $a$, $b$, and $c$ are all different, we are not aware of any previous literature that gives the explicit result (2). For a complete justification of (2), we next compute from $\rho(\mathbf{r})$ the corresponding surface charge density $\sigma(\mathbf{r})$ on the general triaxial ellipsoid defined by (1).

Our point is just that, for a volume density on a surface defined by $F(\mathbf{r}) = 0$, of the form

$$\rho(\mathbf{r}) = f(\mathbf{r}) \delta(F(\mathbf{r})),$$

the restriction of the function $f(\mathbf{r})$ to the surface is uniquely determined by the surface charge density $\sigma(\mathbf{r})$, and vice versa. It is only necessary to integrate $\rho(\mathbf{r})$ along a line normal to the charged surface to obtain $\sigma(\mathbf{r})$, and thereby determine $f(\mathbf{r})$. It follows from a straightforward calculation that the relation between $f$ and $\sigma$ is given by

$$\int \rho(\mathbf{r}) \, dx \, dy \, dz = Q$$

just by rescaling $x, y, z \rightarrow ax, by, cz$. 

1 Note that integration over all space immediately gives $\int \rho(\mathbf{r}) \, dx \, dy \, dz = Q$ just by rescaling $x, y, z \rightarrow ax, by, cz$. 


for points $\mathbf{r}$ on the surface. To obtain the known surface charge density for a triaxial ellipsoid from (2), for arbitrary $a$, $b$, and $c$, the calculation goes as follows.

The normal unit vector at any point on the surface is given by
\[
\hat{n} = \frac{\nabla(x/a^2 + y/b^2 + z/c^2)}{|\nabla(x/a^2 + y/b^2 + z/c^2)|} = \frac{x \hat{x}/a^2 + y \hat{y}/b^2 + z \hat{z}/c^2}{\sqrt{x^2/a^2 + y^2/b^2 + z^2/c^2}} ,
\]
while the three-space volume element in an infinitesimal neighborhood straddling the surface is
\[
dV = du \ dA, \quad du = \hat{n} \cdot d\mathbf{r}.
\]
Rewriting the Dirac delta in terms of the normal coordinate $u$ then gives
\[
\delta\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right) = \frac{1}{\frac{du}{\sqrt{x^2/a^2 + y^2/b^2 + z^2/c^2}}} \delta(u - u_0) = \frac{1}{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}} \delta(u - u_0),
\]
where $u = u_0$ when $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Using this last expression and integrating over the infinitesimal neighborhood $u \in (u_0 - \varepsilon, u_0 + \varepsilon)$ gives the expected result (e.g. see [7], section 5.02, equation (5)) for the surface charge density
\[
\sigma(\mathbf{r}) = \lim_{\varepsilon \to 0} \int_{u_0 - \varepsilon}^{u_0 + \varepsilon} \rho(\mathbf{r}) du = \frac{Q}{4\pi abc} \frac{1}{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}} ,
\]
thereby confirming that (2) is correct. In this last expression, it is to be understood that all points $\mathbf{r}$ are on the surface (1).

The volume charge density (2) is also convenient to show that the projected charge/length along any principal axis is constant. For example, using the Dirac delta property $\delta(f(z)) = \sum_{\text{roots of } f(z)} \delta(z - z_0)/|f'(z_0)|$, we have for any $x$ between $\pm a$,
\[
\frac{dQ}{dx} \equiv \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \ \rho(\mathbf{r}) = \frac{Q}{4\pi abc} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \ \delta\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right) = \frac{Q}{4\pi abc} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \ \left[\frac{c^2}{z}\right] \delta(z - c\sqrt{1 - x^2/a^2 - y^2/b^2}) + \delta(z + c\sqrt{1 - x^2/a^2 - y^2/b^2})|_{z=x_0} = \frac{Q}{2\pi ab} \int_{-\infty}^{+\infty} dy \sqrt{1 - x^2/a^2 - y^2/b^2} = \frac{Q}{2\pi a} \int_{-1}^{+1} \frac{ds}{\sqrt{1 - s^2}}.
\]
But then \[ \int_{-1}^{+1} \frac{dx}{\sqrt{1-x^2}} = \pi, \] so for \(-a \leq x \leq a\)
\[ \frac{dQ}{dx} = \frac{Q}{2a}. \tag{10} \]

Similarly, the projected linear charge densities along the other principal axes are given by
\[ \frac{dQ}{dy} = \frac{Q}{2b} \] for \(-b \leq y \leq b\) and
\[ \frac{dQ}{dz} = \frac{Q}{2c} \] for \(-c \leq z \leq c\).

### 3. Charge densities for hyperellipsoids

The results and analysis given above are easily extended to describe equipotential static conducting ellipsoids embedded in any number of spatial dimensions, \(D\). These are sometimes called ‘hyperellipsoids’ for \(D > 3\). At the risk of being somewhat repetitive, let us consider this generalization in detail.

The corresponding volume charge distribution, \(\rho_D(r^2)\), that is appropriate for equipotential hyperellipsoids in \(D\) dimensions, is given by an immediate generalization of (2), namely
\[ \rho_D(r^2) = \frac{Q}{\Omega_D a_1 a_2 \cdots a_D} \delta \left( \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_D^2}{a_D^2} - 1 \right) = \sigma_D(r^2) \delta(u - u_0), \tag{11} \]

\[ \sigma_D(r^2) = \frac{Q}{\Omega_D a_1 a_2 \cdots a_D} \left( \frac{1}{\sqrt{x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_D^2/a_D^2}} \right) \bigg|_{u=u_0}, \tag{12} \]
where \(u = u_0\) when \(x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_D^2/a_D^2 = 1\) and where
\[ \Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{13} \]

This \(\Omega_D\) is the total ‘solid angle’ in \(D\) spatial dimensions. The unit normal vector and the volume measure, for an infinitesimal neighborhood straddling the \(D - 1\) dimensional ‘hypersurface’ containing the charge, are also given by the obvious generalizations of (5) and (6)
\[ \hat{n} = \frac{x_1 \hat{x}_1/a_1^2 + x_2 \hat{x}_2/a_2^2 + \cdots + x_D \hat{x}_D/a_D^2}{\sqrt{x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_D^2/a_D^2}}, \quad d^D V = du \ d^{D-1} V, \quad du = \hat{n} \cdot d\hat{r^2}. \tag{14} \]

The volume charge density (11) can be projected along any principal axis to obtain a non-uniform linear charge density for \(D = 3\). As done before for \(D = 3\), the projection is defined by integrating over all but one direction. For instance

3 Note that integration over all space immediately gives \( \int_{\rho(r^2)} dV_1 \cdots dV_D = Q\) just by rescaling \(x_1, x_2, \cdots, x_D \to a_1 x_1, a_2 x_2, \cdots, a_D x_D\).

4 It is interesting to compare the hypersurface charge density \(\sigma_D\) with the product of the principal curvatures, \(\kappa_m, m = 1, 2, \cdots, D - 1\), for the same ellipsoidal hypersurface. A straightforward calculation (see the appendix) gives
\[ (\Omega_D \sigma_D)^{D-1} = \left( \prod_{m=1}^{D-1} \frac{1}{a_m} \right) \left( \prod_{m=1}^{D-1} \kappa_m \right). \]
\[ \frac{dQ}{dx_1} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx_2 \cdots dx_D \quad \rho_D(r) = \frac{Q}{\Omega_D a_1 a_2 \cdots a_D} \int_{-\infty}^{+\infty} dx_2 \]

\[ \times \cdots \int_{-\infty}^{+\infty} dx_D \quad \delta \left( \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_D^2}{a_D^2} - 1 \right). \]  

(15)

Now perform the integrations sequentially, beginning with \( \int_{-\infty}^{+\infty} dx_D \) using the same property of the Dirac delta as was used for \( D = 3 \):  

\[ \delta \left( \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_D^2}{a_D^2} - 1 \right) \]

\[ = \frac{a_D^2}{\omega_D} \left[ \delta(x_D - a_D \sqrt{1 - x_1^2/a_1^2 - \cdots - x_{D-1}^2/a_{D-1}}^2) ight]. \]  

(16)

This leads to the next integration:

\[ \frac{2}{a_{D-1}} \int_{a_{D-1}}^{+a_{D-1} \sqrt{1 - x_1^2/a_1^2 - \cdots - x_{D-2}^2/a_{D-2}^2}} dx_{D-1} \quad \frac{1}{\sqrt{1 - x_1^2/a_1^2 - \cdots - x_{D-2}^2/a_{D-2}^2}} = \frac{2\pi}{a_{D-1}}. \]  

(17)

and therefore

\[ \frac{dQ}{dx_1} = \frac{Q}{\Omega_D a_1 a_2 \cdots a_{D-2}} V(a_1, a_2, \ldots, a_{D-2}). \]  

(18)

The remaining integrations, easily performed sequentially, are given for \(-a_1 \leq x_1 \leq a_1\) by

\[ V(a_1, \ldots, a_{D-2}) = 2\pi \int_{-a_1}^{a_1} d x_2 \int_{a_2}^{+a_2 \sqrt{1 - x_1^2/a_1^2 - x_2^2/a_2^2}} dx_3 \]

\[ \cdots \int_{a_{D-2}}^{+a_{D-2} \sqrt{1 - x_1^2/a_1^2 - \cdots - x_{D-2}^2/a_{D-2}^2}} dx_{D-2} \]

\[ = \Omega_{D-1} a_2 a_3 \cdots a_{D-2} \left( 1 - x_1^2/a_1^2 \right)^{\frac{D-3}{2}}. \]  

(19)

Note that \( \Omega_{D-1} = \frac{2\pi}{D-3} \Omega_{D-3} \). Thus the final linear charge density projected along the axis is

\[ \frac{dQ}{dx_1} = \frac{Q}{\omega_D} \frac{\Omega_{D-1}}{\omega_D} (1 - x_1^2/a_1^2)^{\frac{D-3}{2}}, \]  

(20)

for \(-a_1 \leq x_1 \leq a_1\), in agreement with the results in section 6 of [9]. Clearly, the same form applies for any other principal axis, i.e. \( \frac{dQ}{dx_k} = \frac{Q \Omega_{k-1}}{\omega_D} (1 - x_k^2/a_k^2)^{\frac{D-3-k}{2}}, \) for \(-a_k \leq x_k \leq a_k \) and \( k = 1, \ldots, D \).

For \( D > 3 \) the non-uniform linear charge density (20) is rather counter-intuitive as it has a maximum at \( x_1 = 0 \) and falls monotonically to zero on either side of the maximum, vanishing at the end points \( x_1 = \pm a_1 \). Only for \( D = 2 \) does the result conform to what one would naively expect for mutually repelling charges placed on a line, namely, a charge distribution peaked at the ends. This is discussed in [9].

As a generalization of (20), consider projecting the charge onto any subset of the principal axes, for example, onto \( x_1, x_2, \ldots, x_k \) for \( k < D \), as may be accomplished by integrating \( \rho_D \) over \( x_m \) for \( m = k + 1, \ldots, D \). Following the same steps as given above, the result is readily seen to be
\[
\frac{dQ}{dx_1 dx_2 \cdots dx_k} = \frac{Q}{a_1 a_2 \cdots a_k} \frac{\Omega_{D-k}}{\Omega_D} (1 - x_1^2/a_1^2 - x_2^2/a_2^2 - \cdots - x_k^2/a_k^2)^{\frac{D-k}{2}},
\]

for all \( x_1, x_2, \cdots, x_k \) such that \( x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_k^2/a_k^2 \leq 1 \). Since the final answer here is independent of \( a_{k+1}, \cdots, a_D \), this would in fact be the correct charge density on an equipotential \( k \)-dimensional manifold obtained from the original equipotential hyperellipsoid by letting \( a_m = 0 \) for \( m = k + 1, \cdots, D \), i.e. by ‘squashing’ these \( D - k \) dimensions. In particular, if the original \( x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_2^2/a_2^2 = 1 \) hyperellipsoid were completely ‘flattened’ to a two-dimensional ellipse embedded in \( D \) spatial dimensions, the surface charge density that results would be

\[
\frac{dQ}{dA} = \frac{Q}{2\pi a_1 a_2 \sqrt{1 - x_1^2/a_1^2 - x_2^2/a_2^2}} \text{ for } D = 3.
\]

For both (22) and (23), the boundary of the flattened disk is the ellipse \( x_1^2/a_1^2 + x_2^2/a_2^2 = 1 \).

As was the case for the linear charge density in (20), the result (21) is rather counter-intuitive for squashed manifolds with \( D \geq k + 2 \). For \( D < k + 2 \), the charge density \( dQ/dx_1 dx_2 \cdots dx_k \) is peaked at the boundary of the squashed manifold, for which \( x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_k^2/a_k^2 = 1 \), where the density actually diverges. In our opinion, this would conform with naive expectations for mutually repelling charges placed on the manifold. But for the case \( D = k + 2 \), the charge density on the squashed manifold is uniform, exactly like that of an ideal conducting line segment in three dimensions. Or, as another particular case, a flat, elliptical, equipotential disk in four spatial dimensions would have a constant surface charge density. So (21) for \( D = k + 2 \) is again non-intuitive, although perhaps it can be reconciled with intuition using arguments similar to those advanced for the equipotential line segment in three dimensions, as discussed in [11] and references therein. On the other hand, for \( D > k + 2 \), the charge distribution (21) is peaked at the center of the squashed manifold and falls monotonically to zero at the boundary where \( x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_k^2/a_k^2 = 1 \), exactly like ideal line segments for \( D > 3 \). In our opinion, this is counter-intuitive. Nevertheless, it is what it is.

### 4. Potentials and electrostatic fields

An exact, single-parameter integral expression for the potential surrounding a static, equipotential, conducting hyperellipsoid carrying a total charge \( Q \), in \( D \) spatial dimensions, is in general an elliptic integral, although it may reduce to an elementary function if some of the \( a_k \) are equal. This fact is well-known for the \( D = 3 \) ellipsoid case [7]. Explicitly, if the charged surface is defined by

\[
\sum_{k=1}^{D} \frac{x_k^2}{a_k^2} = 1,
\]

(24)
then the potential is given by
\[
\Phi(\vec{r}) = \frac{kQ}{2} \int_{\Theta(\vec{r})}^{\infty} \left( \prod_{k=1}^{D} \frac{1}{\sqrt{a_k^2 + \Theta}} \right) d\theta,
\]
where the \(\Theta\)-equipotentials are a set of confocal ellipsoids consisting of all points \(\vec{r}\) outside the charged hyperellipsoid that satisfy, for a given \(\Theta\),
\[
\sum_{k=1}^{D} \frac{x_k^2}{a_k^2 + \Theta} = 1, \quad \text{for } \Theta > 0.
\]
(26)

Note the charged hyperellipsoid itself is defined to be at \(\Theta = 0\). If given an arbitrary point \(\vec{r}\) outside the charged hyperellipsoid, to compute the potential at that point it would first be necessary to find the appropriate \(\Theta\) for the given \(\vec{r}\), i.e. \(\Theta(\vec{r})\). In general, if all the \(a_k\) are distinct, this would require solving for the appropriate root of the \(D\)th order polynomial in \(\Theta\) implicit in (26), something that can always be done in principle (although in practice, perhaps only numerically, especially if \(D > 4\) and all the \(a_k\) are distinct).

The static electric field is the gradient of the potential, as usual. From (26) and (25) it follows that
\[
\vec{E}(\vec{r}) = -\nabla \Phi = -\left( \nabla \Theta \right) \frac{d\Phi}{d\Theta} = kQ \left( \prod_{k=1}^{D} \frac{1}{\sqrt{a_k^2 + \Theta}} \right)
\times \left( \sum_{n=1}^{D} \frac{x_n \tilde{e}_n}{a_n^2 + \Theta} \right) \left/ \left( \sum_{m=1}^{D} \frac{x_m^2}{(a_m^2 + \Theta)^2} \right) \right.,
\]
(28)

Once again, if only \(\vec{r}\) is specified, it is necessary to find \(\Theta(\vec{r})\) from (26) to evaluate this expression. But note that as \(r \to \infty\), it follows from (26) that \(\Theta \to \infty\) with \(\Theta \sim r^2\). So asymptotically
\[
\Phi(\vec{r}) \sim kQ \frac{1}{D-2} \frac{1}{r^{D-2}}, \quad \vec{E}(\vec{r}) \sim \frac{kQ}{r^D} \vec{r}.
\]
(29)

These asymptotic expressions are just the potential and electric field for a point charge in \(D\) spatial dimensions (e.g. see [6]) as should have been expected.

The direction of the electric field (when multiplied by the sign of \(Q\)) at a point on a given \(\Theta\)-equipotential hyperellipsoid, is given by the unit vector
\[
\vec{E}(\vec{r}) = \left( \sum_{n=1}^{D} \frac{x_n \tilde{e}_n}{a_n^2 + \Theta} \right) \left/ \sqrt{ \left( \sum_{m=1}^{D} \frac{x_m^2}{(a_m^2 + \Theta)^2} \right) } \right.
\]
(30)

As a check, when all the \(a_k\) are equal, \(\vec{E} = \vec{r}\), as expected. The (signed) strength of the electric field, at a point on that same equipotential, is given by
\[
E(\vec{r}) = kQ \left( \prod_{k=1}^{D} \frac{1}{\sqrt{a_k^2 + \Theta}} \right) \left/ \sqrt{ \left( \sum_{m=1}^{D} \frac{x_m^2}{(a_m^2 + \Theta)^2} \right) } \right.
\]
(31)

so that \(\vec{E}(\vec{r}) = E(\vec{r}) \vec{E}(\vec{r})\) for either sign of \(Q\). This reduces to \(E = kQ/r^{D-1}\) when all the \(a_k\) are equal, as expected.
Since the charge is located on the hyperellipsoid with $\Theta = 0$, by construction, the potential on the charge carrying hypersurface itself is

$$\Phi(\vec{r}(\Theta = 0)) = \frac{kQ}{2} \int_0^\infty \left( \prod_{k=1}^D \frac{1}{\sqrt{\alpha_k + \Theta}} \right) d\Theta. \quad (32)$$

That is to say, the capacitance of the isolated hyperellipsoid, defined by $Q = C\Phi(\vec{r}(\Theta = 0))$, is also given by an elliptic integral:

$$C = \frac{2}{k} \int_0^\infty \left( \prod_{k=1}^D \frac{1}{\sqrt{\alpha_k + \Theta}} \right) d\Theta. \quad (33)$$

Furthermore, the charge density on the ellipsoidal hypersurface may be obtained directly from Gauss’ law, with the normalization for a point charge determined by

$$\nabla \cdot \frac{\vec{r}}{r^D} = \Omega_D \delta^D(\vec{r}), \quad (34)$$

where again $\Omega_D$ is the total solid angle in $D$ spatial dimensions. Thus the hypersurface charge density is given as usual by the value of the normal electric field as the hypersurface is approached from the outside, that is to say by the limit: $k \cdot \Omega_D \sigma_D(\vec{r}) = \lim_{\vec{r} \to \text{hypersurface}} \vec{n} \cdot \vec{E}$ where $\vec{n}$ is the outward normal unit vector on the hypersurface. For the problem at hand this limit is just $\lim_{\theta \to 0} \vec{E} \cdot \vec{E} = E|_{\theta=0}$. So (31) gives the explicit result:

$$\sigma_D(\vec{r}) = \frac{1}{k} \Omega_D \frac{Q}{\prod_{k=1}^D a_k} \left. 1 \right|_{\Theta=0}, \quad (35)$$

thereby confirming both (11) and (12).5

5 Using notation that is more consistent with the previous section, $d\Theta = (\nabla \Theta) \cdot d\vec{r} = |\nabla \Theta| \hat{n}(\Theta) \cdot d\vec{r} = |\nabla \Theta| \, d\omega(\Theta)$, where $\hat{n}(\Theta) = \vec{E}$ is the local normal on the $\Theta$-equipotential, and $|\nabla \Theta| = 2/\sqrt{\sum_{k=1}^D a_k^2/(a_k^2 + \Theta)^2}$. Therefore $\vec{E} = -\hat{n}(\Theta) \, d\omega(\Theta) = -\vec{E} \left|\nabla \Theta\right| \, d\Phi/d\Theta$. This is in agreement with (28), (30), and (31).
Acknowledgments

For helpful comments, we thank K McDonald (in particular for pointing out [2]) and A Zangwill (in particular for pointing out [13]). This work was supported in part by a University of Miami Cooper Fellowship.

Appendix. Geometry of hyperellipsoids

For a hyperellipsoid embedded in D dimensions, as given by \( \sum_{j=1}^{D} x_j^2 / a_j^2 = 1 \), resolving the constraint by solving for \( x_j = \frac{a_j}{\sqrt{D}} \) gives simple expressions for the metric, inverse metric, and 2nd fundamental form of the manifold. The results are

\[
g_{km} = \delta_{km} + \frac{a_k^2}{x_D} \frac{x_k x_m}{a_k^2 a_m^2}, \quad g^{mn} = \delta_{mn} - \frac{1}{S} \frac{x_m x_n}{a_m a_n} \quad \text{for } k, m, n = 1, 2, \ldots, D - 1, \quad (36)
\]

where \( x_D^2 = a_D^2 \left( 1 - \sum_{j=1}^{D-1} \frac{x_j^2}{a_j^2} \right) \), \( \det g_{km} = \frac{a_D^4}{x_D^2} S \), and \( S = \sum_{j=1}^{D} x_j^2 > 0 \), \( (37) \)

\[
K_{mn} = -\hat{n} \cdot \left( \partial_m \hat{n} \times \partial_n \hat{n} \right)
\]

\[
= \frac{1}{\sqrt{S}} \frac{1}{a_n^2} \left( \delta_{mn} + \frac{a_k^2}{x_D} \frac{x_m x_n}{a_k^2 a_n^2} \right) \quad \text{with } \hat{r} \text{ on the manifold } \& \hat{n} \cdot \partial_m \hat{r} = 0, \quad (38)
\]

\[
\mathbb{K}_{kn} \equiv g^{kn} K_{nn} = \frac{1}{(S)^{3/2}} \left( S \delta_{kn} + \frac{x_k x_n}{a_k a_n} \left( \frac{1}{a_k^2} - \frac{1}{a_n^2} \right) \right), \quad \text{for } k, m, n = 1, \ldots, D - 1. \quad (39)
\]

Note the \( \left( \frac{1}{a_k^2} - \frac{1}{a_n^2} \right) \) factor in the matrix \( \mathbb{K}_{kn} \) breaks the \( k \leftrightarrow n \) symmetry. Also note the coordinate singularity (as opposed to a physical singularity) in the metric \( g_{mn} \) and \( K_{mn} \) on the \( x_D = 0 \) ‘equatorial’ submanifold. However, there is no such singularity in \( \mathbb{K}_{kn} \), whose eigenvalues \( \kappa_m \) for \( m = 1, \ldots, D - 1 \), are all finite so long as \( a_k > 0 \) for \( k = 1, \ldots, D \). The intrinsic curvature scalar densities on the manifold are encoded in

\[
\det (1 + \lambda \mathbb{K}) = 1 + \lambda \sum_{m=1}^{D-1} \kappa_m + \lambda^2 \sum_{m>n=1}^{D-1} \kappa_m \kappa_n + \cdots + \lambda^{D-1} \left( \prod_{m=1}^{D-1} \kappa_m \right), \quad (40)
\]

where \( R = \sum_{m>n=1}^{D-1} \kappa_m \kappa_n \) etc. The last term in the expansion of \( \det (1 + \lambda \mathbb{K}) \) is \( \lambda^{D-1} \det \mathbb{K} \), of course. From the above expression (39) it follows that

\[
\det \mathbb{K} = \left( \frac{1}{(S)^{D+1}} \right) \left( \prod_{j=1}^{D} \frac{1}{a_j^2} \right). \quad (41)
\]

Therefore, on a charged, conducting hyperellipsoid embedded in \( D \) dimensions, for which the hypersurface charge density is \( \sigma_0 = \frac{1}{16 \sqrt{3}} \frac{1}{a_j} \), it follows that \( \sigma^{D+1} \propto \det \mathbb{K} \). This generalizes the long-known result for ellipsoids embedded in three dimensions (e.g. see the footnote, p 191, [5]). More precisely

\[
(\Omega_D \sigma_0)^{D+1} = \left( \prod_{j=1}^{D} \frac{1}{a_j^{D-1}} \right) \det \mathbb{K} = \left( \prod_{j=1}^{D} \frac{1}{a_j^{D-1}} \right) \left( \prod_{m=1}^{D-1} \kappa_m \right). \quad (42)
\]
Here $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit radius sphere embedded in $D$ dimensions (i.e. the total 'solid' angle around the center of the sphere).

**ORCID iDs**

T L Curtright  [https://orcid.org/0000-0001-7031-5604](https://orcid.org/0000-0001-7031-5604)

**References**

[1] Green G 1871 An essay on the application of mathematical analysis to the theories of electricity and magnetism *Mathematical Papers of the Late George Green* ed N M Ferrers (Nottingham: MacMillan and Company) (1828)

[2] Murphy R 1833 *Elementary Principles of the Theories of Electricity, Heat, & Molecular Actions* (Cambridge: Pitt Press)

[3] Thomson W and Tait P G 1883 *Treatise on Natural Philosophy* (Cambridge: Cambridge University Press) Part I & Part II. 1879

[4] Routh E 1891 A *Treatise on Analytical Statics with Numerous Examples* vol 1 1st edn (Cambridge: Cambridge University Press) vol 2. 1892

[5] Kellogg O D 1953 *Foundations of Potential Theory* (New York: Dover) pp 188–91

[6] Sommerfeld A 1952 *Electrodynamics* (New York: Academic) p 56 and Problem II.1.

[7] Smythe W R 1968 *Static and Dynamic Electricity* 3rd edn (New York: McGraw-Hill) section 5.02

[8] Durand E 1953 *Électrostatique et magnétostatique*, Masson et Cie, pp 50-51

[9] Curtright T L, Aden N M, Chen X, Haddad M J, Karayev S, Khadka D B and Li J 2016 Charged line segments and ellipsoidal equipotentials *Eur. J. Phys.* **37** 035201

[10] Curtright T L, Alshal H, Baral P, Huang S, Liu J, Tamang K, Zhang X and Zhang Y 2019 The conducting ring viewed as a wormhole *Eur. J. Phys.* **40** 015206

[11] Jackson J D 2002 Charge density on a thin straight wire: the first visit *Am. J. Phys.* **70** 409–10

[12] Curtright T L unpublished

[13] Alawneh A D and Kanwal R P 1977 Singularity methods in mathematical physics *SIAM Rev.* **19** 437–71