The self-gravitating Fermi gas

Pierre-Henri Chavanis

Laboratoire de Physique Quantique, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France

Abstract. We discuss the nature of phase transitions in the self-gravitating Fermi gas at non-zero temperature. This study can be relevant for massive neutrinos in Dark Matter models and for collisionless self-gravitating systems experiencing a “violent relaxation”. Below a critical energy (in the microcanonical ensemble) or below a critical temperature (in the canonical ensemble), the system undergoes a gravitational collapse leading to a compact object with a massive degenerate nucleus (fermion ball).

1 Introduction

The self-gravitating Fermi gas model was introduced by Fowler in 1926 in order to explain the puzzling nature of white dwarf stars. When a star has exhausted its nuclear fuel, it undergoes a gravitational collapse until gravity is balanced by the pressure of a degenerate gas of electrons (on account of Pauli’s exclusion principle). This model was later on improved by Chandrasekhar (1930) who took into account special relativity effects in the degenerate equation of state of electrons and discovered that white dwarf stars have a limiting mass of \( \sim 1.4 \, M_{\odot} \). Then, in 1939, Oppenheimer & Volkoff considered the extension of this model to the framework of general relativity in connexion with the structure of neutron stars. They found again a limiting mass of \( \sim 0.7 \, M_{\odot} \) above which the pressure of the neutrons cannot sustain gravity anymore. In that case, the star is expected to collapse into a black hole. In these studies, the Fermi gas is completely degenerate since the thermal energy \( kT \) is much smaller than the Fermi energy. The self-gravitating Fermi gas at finite temperature was investigated by Hertel & Thirring (1971). They showed that below a critical temperature a homogeneous gas of fermions undergoes a first order phase transition leading to a compact object with a degenerate core. Application of these results to Dark Matter was considered by Bilic & Viollier (1997). They proposed that Dark Matter could be made of massive neutrinos (with \( m \sim 15\text{keV} \)) in equilibrium with a radiation background imposing its temperature. In the langage of statistical mechanics, this corresponds to the canonical ensemble. By cooling below a critical temperature, a condensed phase emerges consisting of a quasidegenerate “fermion ball”. These fermion balls may provide an alternative to black holes that are reported to exist at the center of galaxies. At large distances, the \( r^{-2} \) law of density decrease of an isothermal gas is consistent with the flat rotation curves of spiral galaxies. This could provide an attractive model of Dark Matter but the dynamical mechanism leading to a “fermion ball” remains to be clearly identified.
The self-gravitating Fermi gas model also appeared in a completely different context, independent of quantum mechanics. In 1967, Lynden-Bell argued that a self-gravitating system far from mechanical equilibrium would rapidly relax towards a virialized state, due to the strong fluctuations of the gravitational potential. Since this process of violent relaxation is essentially collisionless, the coarse-grained distribution function \( f \) cannot increase and it must satisfy an effective exclusion principle \( \bar{f} \leq \eta_0 \), where \( \eta_0 \) is the maximum value of the initial distribution function. This upper bound is a consequence of the Liouville theorem. Assuming ergodicity, Lynden-Bell predicted that \( \bar{f} \) should converge towards a Fermi-Dirac distribution, or a superposition of Fermi-Dirac distributions. Relaxation equations towards these maximum entropy states (on a coarse-grained scale) were proposed by Chavanis, Sommeria and Robert (1996). In that context, the proper thermodynamical ensemble is the microcanonical ensemble since energy is conserved during the course of the evolution. Fermi-Dirac spheres were computed by Chavanis & Sommeria (1997). They found a wide variety of nuclear concentration depending on the degree of degeneracy and on the value of energy. For high energies, the system is almost uniform. For intermediate energies, the core is partially degenerate. For low energies, the maximum entropy (i.e. most probable) state consists of a massive degenerate nucleus surrounded by an isothermal halo. They argued that degeneracy (in Lynden-Bell’s sense) could stabilize the system and play a role in galactic nuclei and Dark Matter. It is however unclear whether violent relaxation can lead to massive “fermion balls”. Indeed, it is in general advocated that the fluctuations of the gravitational potential fade before the system has developed high density contrasts. To our point of view, this remains a matter for further investigation.

Recently, we have carried on a detailed study of phase transitions in the self-gravitating Fermi gas (Chavanis 2002). We showed that this system exhibits a rich structure with the occurence of three types of phase transitions of zeroth, first and second order. We worked at a general level without specifying the source of degeneracy (quantum mechanics or violent relaxation) and we described the complete structure of the equilibrium phase diagram, for arbitrary values of control parameters and arbitrary degree of degeneracy. We worked both in the microcanonical and canonical ensembles, emphasizing the inequivalence of these ensembles for long-range systems. The description of the microcanonical ensemble and the relation between the structure of Fermi-Dirac spheres and classical isothermal spheres investigated by Antonov (1962) and Lynden-Bell & Wood (1968) is a specificity of our approach. In the following, we give a short summary of this study.

2 Phase transitions in self-gravitating systems

We consider a system of \( N \) fermions of mass \( m \) interacting via Newtonian gravity. At statistical equilibrium, the system is described by the Fermi-Dirac distribution

\[
f = \frac{\eta_0}{1 + \lambda e^{\beta (\frac{\mu}{2} + \phi)}}, \quad (\beta = m/kT)
\]
The self-gravitating Fermi gas coupled to the Poisson equation

$$\Delta \Phi = 4\pi G \int f \, d^3 v.$$  \hspace{1cm} (2)

This determines a self-consistent meanfield equation for the gravitational potential $\Phi$. For spherically symmetrical systems, this is just an ordinary differential equation, which can be solved numerically by usual means. Then, the fugacity $\lambda^{-1} > 0$ and the inverse temperature $\beta$ can be related to the total mass $M$ and total energy $E$ of the system. In Eq. (1), $\eta_0$ represents the maximum allowable value of the distribution function, i.e. $f \leq \eta_0$. For a quantum gas, $\eta_0 = (2s + 1)m^4/h^3$ where $s$ is the spin of the particles. In the fully degenerate limit $f \simeq \eta_0$, the system is equivalent to a polytrope of index $n = 3/2$. In the non-degenerate limit $f \ll \eta_0$, Eqs. (1)-(2) describe a classical isothermal gas

$$f = \frac{\eta_0}{\lambda} e^{-\beta(\frac{v^2}{2} + \Phi)} \quad \text{and} \quad \Delta \Phi = 4\pi G A e^{-\beta \Phi}.$$ \hspace{1cm} (3)

The Boltzmann-Poisson system (3) has been studied in relation with the structure of isothermal stellar cores and globular clusters. It is well-known that the density of an isothermal gas decreases at large distances like $r^{-2}$. Therefore, the total mass of the configuration is infinite and we need to introduce truncated models (Chavanis 1998) or confine the system within a box of radius $R$.

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**Fig. 1.** Equilibrium phase diagram for classical isothermal spheres. For $\Lambda > \Lambda_c$ or $\eta > \eta_c$, there is no hydrostatic equilibrium and the system undergoes a gravitational collapse.
In the non degenerate limit, the equilibrium phase diagram \((E, T)\) is represented in Fig. 1. The curve has a striking spiral behaviour parametrized by the density contrast \(\mathcal{R} = \rho(0)/\rho(R)\) going from 1 (homogeneous system) to \(+\infty\) (singular sphere) as we proceed along the spiral. There is no equilibrium state below \(E_c = -0.335GM^2/R\) or \(T_c = \frac{GMm}{4\pi^2kR}\). In that case, the system is expected to collapse indefinitely. This is called *gravothermal catastrophe* in the microcanonical ensemble (fixed \(E\)) and *isothermal collapse* in the canonical ensemble (fixed \(T\)). Dynamical models show that the collapse is self-similar and develops a finite time singularity (see Chavanis et al. 2002 and references therein). However, although the central density goes to \(+\infty\), the shrinking of the core is so rapid that the core mass goes to zero. Therefore, the singularity contains no mass and this process alone cannot lead to a black hole.

It is also important to stress that the statistical ensembles are not interchangeable for systems with long-range interaction, like gravity. In the microcanonical ensemble, the series of equilibria becomes unstable after the first turning point of energy (\(MCE\)) corresponding to a density contrast of 709. At that point, the solutions pass from local entropy maxima to saddle points. In the canonical ensemble, the series of equilibria becomes unstable after the first turning point of temperature (\(CE\)) corresponding to a density contrast of 32.1. At that point, the solutions pass from minima of free energy \((F = E - TS)\) to saddle points. It can be noted that the region of negative specific heats between \((CE)\) and \((MCE)\) is stable in the microcanonical ensemble but unstable in the canonical ensemble as expected on general physical grounds.
Fig. 3. Equilibrium phase diagram for Fermi-Dirac spheres with a degeneracy parameter $\mu = 10^5$. Points A form the "gaseous" phase. They are global entropy maxima (GEM) for $\Lambda < \Lambda_c(\mu)$ and local entropy maxima (LEM), i.e. metastable states, for $\Lambda > \Lambda_c(\mu)$. Points C form the "condensed" phase. They are LEM for $\Lambda < \Lambda_c(\mu)$ and GEM for $\Lambda > \Lambda_c(\mu)$. Points B are unstable saddle points (SP) and contain a "germ".

Fig. 4. Entropy of each phase versus energy for $\mu = 10^5$. A first order phase transition is expected at $\Lambda_c(\mu)$ at which the two stable branches (solutions A and C) intersect. However, the entropic barrier B probably prevents this phase transition [8,14].
When degeneracy is taken into account, the structure of the equilibrium phase diagram depends on the value of the degeneracy parameter \( \mu = \sqrt[3]{\frac{12 \pi^4 G^3 M R^3}{\eta_0}} \). The classical limit is recovered for \( \mu \to +\infty \). We see that the inclusion of degeneracy has the effect of unwinding the spiral (Fig. 2). For large values of the degeneracy parameter, the equilibrium phase diagram is depicted in Fig. 3. The solutions on the upper branch (points A) are non degenerate and have a smooth density profile; they form the “gaseous phase”. The solutions on the lower branch (points C) have a “core-halo” structure with a massive degenerate nucleus and a dilute atmosphere; they form the “condensed phase”. The density profiles of these solutions are given in Ref. [8]. By comparing their entropy (Fig. 4), we would expect that a first order phase transition from the gaseous phase to the condensed phase occurs at the transition energy \( E_t(\mu) \). This is, however, unlikely because the probability of a fluctuation able to induce this phase transition is extremely weak [14]. Therefore, the metastable gaseous states with \( E < E_t \) are probably physical for the time scales involved in astrophysical situations. In any case, a phase transition must occur at the critical energy \( E_c \) at which the gaseous branch disappears. Below that energy, the system undergoes a gravothermal catastrophe but, for self-gravitating fermions, the core ceases to shrink when it becomes degenerate. Since this collapse is accompanied by a discontinuous jump of entropy, this is sometimes called a zeroth order phase transition. The resulting equilibrium state (point D) possesses a small degenerate nucleus which contains a moderate fraction of the total mass (\( \approx 20\% \) for \( \mu = 10^3 \)). The rest of the mass is diluted in a hot envelope held by the box. In an open system, it would be dispersed at infinity.

**Fig. 5.** Equilibrium phase diagram for Fermi-Dirac spheres with \( \mu = 10^3 \).
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For smaller values of the degeneracy parameter, the equilibrium phase diagram is represented in Fig. 5. The curve $\eta(\Lambda)$ is now univalued and the first order phase transition in the microcanonical ensemble is suppressed: all the equilibrium states are global entropy maxima. For large energies, they are almost homogeneous and for smaller energies they have a “core-halo” structure with a partially degenerate nucleus. As energy is progressively decreased, the nucleus becomes more and more degenerate and contains more and more mass until a minimum energy $E_{\text{min}} = -2.36 G^2 M^{7/3} \eta_0^{2/3}$ at which all the mass is in the completely degenerate nucleus of radius $R_\ast = 0.181 \eta_0^{-2/3} G^{-1} M^{-1/3}$. In that case, the system has the same structure as a cold white dwarf star. Therefore, depending on the degree of degeneracy and on the value of energy, a wide variety of nuclear concentrations can be obtained in the Fermi gas.

We can also consider the situation in which the system is in contact with a heat bath which imposes its temperature (canonical description). Considering again the case $\mu = 10^3$ in Fig. 5, we note that the curve $\Lambda(\eta)$ is multivalued. We expect therefore a first order phase transition to occur at a transition temperature $T_t$ determined by a Maxwell construction. This phase transition should be accompanied by a huge release of latent heat. This may not be physically realizable and the true collapse will rather occur at $T_c$. The outcome of this
collapse is the formation of a fermion ball containing almost all the mass. Considering Fig. 2 again, we see that the phase transitions are suppressed in the microcanonical ensemble for \( \mu < \mu_{MTP} \approx 2600 \) and in the canonical ensemble for \( \mu < \mu_{CTP} \approx 82.5 \). At these tricritical points, the two phases (gaseous and condensed) merge. This characterizes a second order phase transition (see Figs. 6-7). The analogies and the differences with the liquid/gas transition are described in Ref. [9].

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