Lagrangian Approach of the First Class Constrained Systems

Yong-Wan Kim†, Seung-Kook Kim* and Young-Jai Park†

† Department of Physics and Basic Science Research Institute
Sogang University, C.P.O. Box 1142, Seoul 100-611, Korea

and

* Department of Physics, Seonam University, Namwon, Chonbuk 590-170, Korea

ABSTRACT

We show how to systematically derive the exact form of local symmetries for the abelian Proca and CS models, which are converted into first class constrained systems by the BFT formalism, in the Lagrangian formalism. As results, without resorting to a Hamiltonian formulation we obtain the well-known U(1) symmetry for the gauge invariant Proca model, while showing that for the CS model there exist novel symmetries as well as the usual symmetry transformations.

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1 Introduction

Field-Antifield formalism [1] is based on an analysis of local symmetries of a Lagrangian, and has a great advantage of representing manifestly covariant formulation of a theory. However, in general, local gauge symmetries are not systematically obtained in an action while constructing a Lagrangian. Even though they are related with the generalized Bianchi-like identities [2], it may be difficult to see the full local symmetries for some complicated Lagrangians.

On the other hand, the Batalin, Fradkin, and Tyutin (BFT) Hamiltonian method [3] has been applied to several second class constrained systems [4, 5, 6], which yield the strongly involutive first class constraint algebra in an extended phase space by introducing new fields. Recently, we have quantized other interesting models including the Proca models by using our improved BFT formalism [7, 8]. However, the Hamiltonian approach [9, 10, 11, 3] to the quantization of constrained systems has the drawback of not necessarily leading to a manifestly Lorentz covariant partition function. This problem is also avoided in the Lagrangian field-antifield approach. In this respect, the systematic and exhaustive determinations of local symmetries constitute an integral part of the field-antifield quantization program.

In this paper, we will consider Lagrangian approach of the first class constrained systems. In section 2, after embedding the abelian Proca model on the extended phase space by the BFT method, we explicitly show how to derive the exact form of the well-known local symmetry of the first class Proca model as a simple example from the view of Lagrangian without resorting to a Hamiltonian formulation. In section 3, we apply the Lagrangian approach to abelian pure Chern-Simons (CS) model which is invariant under the U(1) gauge transformation but has still second class constraint due to the symplectic structure. As results, we show that for the embedded symplectic structure of the CS model there exist additional novel local symmetries as well as the usual U(1) gauge symmetry. Our conclusions are given in section 4.

2 Proca Model

We first consider the abelian Proca model [3] whose dynamics are given by

\[ S = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \right], \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and \( g_{\mu\nu} = \text{diag}(+,-,-,-) \). In the Hamiltonian formulation the canonical momenta of gauge fields are given by \( \pi_0 = 0 \), and \( \pi_i = F_{i0} \). Then, \( \Omega_1 = \pi_0 \approx 0 \) is a primary constraint, and the canonical Hamiltonian

\[ H_c = \int d^3x \left[ \frac{1}{2} \pi_i^2 + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} m^2 \{(A^0)^2 + (A_i)^2\} - A_0 \Omega_2 \right]. \]
Here $\Omega_2$ is the Gauss’ law constraint which comes from the time evolution of $\Omega_1$ with
the primary Hamiltonian $H_p = H_c + \int d^3x \, v^1 \Omega_1$ as

$$\Omega_2 = \partial^i \pi_i + m^2 A^0 \approx 0. \tag{3}$$

We now convert these second class constraints into the corresponding first class ones via
*a la* BFT Hamiltonian embedding. This BFT method is by now well-known, and thus we would like to avoid the explicit calculation here and quote the results of
Ref. [8]. The effective first class constraints $\tilde{\Omega}_i$ are given by

$$\tilde{\Omega}_i = \Omega_i + \Omega_i^{(1)} = \Omega_i + m\Phi^i. \tag{4}$$

with the introduction of auxiliary fields $\Phi^i$ satisfying $\{\Phi^i(x), \Phi^j(y)\} = \omega^{ij}(x, y) = \epsilon^{ij}\delta^3(x - y)$, and the first class Hamiltonian $\tilde{H}$ corresponding to the canonical Hamiltonian $H_c$ by

$$\tilde{H}(A^\mu, \pi_\nu; \Phi^i) = H_c(A^\mu, \pi_\nu) + \int d^3x \left[ \frac{1}{2}(\Phi^2)^2 + \frac{1}{2}(\partial_i \Phi^1)^2 + m\Phi^1 \partial_i A^i - \frac{1}{m} \Phi^2 \tilde{\Omega}_2 \right], \tag{5}$$

which is strongly involutive, i.e., $\{\tilde{\Omega}_i, \tilde{H}\} = 0$.

It seems appropriate to comment on generators of local symmetry transformation
in the Hamiltonian formulation, which are given by the first class constraints. Defining
the generators by

$$G = \sum_{\alpha=1}^2 \int d^2x \, (-1)^{\alpha+1} \epsilon^\alpha(x) \Omega_\alpha(x), \tag{6}$$

we have

$$\delta A^0 = \epsilon^1, \quad \delta \pi_0 = m^2 \epsilon^2,$$
$$\delta A^i = \partial^i \epsilon^2, \quad \delta \pi_i = 0,$$
$$\delta \rho = -\epsilon^2, \quad \delta \pi_\rho = -m^2 \epsilon^1. \tag{7}$$

Here we inserted $(-1)^{\alpha+1}$ factor in Eq. (6) in order to make the gauge transformation
usual, and also identified the new variables $\Phi^i$ as a canonically conjugate pair $(\rho, \pi_\rho)$
in the Hamiltonian formalism through $(\Phi^i) \rightarrow (m\rho, \frac{1}{m} \pi_\rho)$. Now, it can be easily seen
that the extended action

$$S_E = \int d^4x (\pi_\mu \dot{A}^\mu + \pi_\rho \dot{\rho} - \tilde{H}) \tag{8}$$

is invariant under the gauge transformations (7) with $\epsilon^1 = \partial^0 \epsilon^2$.

Next, from the partition function given by the Faddeev-Popov formula [13] as

$$Z = \int D A^\mu D \pi_\mu D \rho D \pi_\rho \prod_{i,j=1}^2 \delta(\tilde{\Omega}_i) \delta(\Gamma_j) \text{det} \left| \{\tilde{\Omega}_i, \Gamma_j\} \right| e^{iS}, \tag{9}$$

where

$$S = \int d^4x \left( \pi_\mu \dot{A}^\mu + \pi_\rho \dot{\rho} - \tilde{H} \right), \tag{10}$$

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and $\Gamma_j$ are the gauge fixing functions. One could integrate all the momenta out with the delta functional in Eq. (9). As results, we have the well-known action

$$S = \int d^4x \mathcal{L} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 (A_\mu + \partial_\mu \rho)^2 \right],$$

which is invariant under $\delta A^\mu = \partial^\mu \epsilon^2$ and $\delta \rho = -\epsilon^2$.

Now, we are ready to use a recently proposed Lagrangian approach [12] of constrained systems in the configuration space. Starting from the gauge invariant action (11), our purpose is to find the gauge transformation rules systematically without resorting to the language of the Hamiltonian formulation.

The equations of motion following from (11) are of the form

$$L_i^{(0)}(x) = \int d^3y \left[ W_{ij}^{(0)}(x, y) \ddot{\varphi}^j(y) + \alpha_i^{(0)}(y) \delta^3(x - y) \right] = 0, \quad i = 1, 2, \ldots, 5, \quad (12)$$

where $W_{ij}^{(0)}(x, y)$ is the Hessian matrix

$$W_{ij}^{(0)}(x, y) := \frac{\delta^2 \mathcal{L}}{\delta \dot{\varphi}^i(x) \delta \dot{\varphi}^j(y)}$$

which satisfies

$$\delta^3(x - y) = \widetilde{W}_{ij}^{(0)} \delta^3(x - y), \quad (13)$$

and

$$(\varphi^i)^T(x) = (A^0, A^1, A^2, A^3, \rho)(x), \quad (14)$$

$$(\alpha_i^{(0)})^T(x) := \int d^3y \left[ \frac{\partial^2 \mathcal{L}}{\partial \dot{\varphi}^i(y) \partial \dot{\varphi}^j(x)} \partial_\varphi^j(y) \right] - \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^i(x)}$$

$$(\alpha_i^{(0)})^T(x) = (\alpha_{A^0}, \alpha_{A^1}, \alpha_{A^2}, \alpha_{A^3}, \alpha_\rho)(x), \quad (15)$$

with

$$\alpha_{A^0} = \partial_i (\dot{A}^i + \partial_i A^0) - m^2 (\dot{A}^0 + \dot{\rho}),$$

$$\alpha_{A^i} = \partial_i \dot{A}^0 - \partial_0 F^{0i} + m^2 (\dot{A}^i - \partial_i \rho),$$

$$\alpha_\rho = m^2 \dot{A}^0 + m^2 \partial_i (A^i - \partial_i \rho). \quad (16)$$

Since the Hessian matrix (13) is of rank four, there exists a “zeroth generation” null eigenvector $\lambda^{(0)}(x, y)$ satisfying

$$\int d^3y \lambda_i^{(0)}(x, y) W_{ij}^{(0)}(y, z) = 0. \quad (17)$$

For simplicity, let us normalize it to have components

$$\lambda^{(0)}(x, y) = (1, 0, 0, 0, 0) \delta^3(x - y). \quad (18)$$
Correspondingly we have a “zeroth generation” constraint in the Lagrangian sense as
\[ \Omega_1^{(0)}(x) = \int d^3 y \lambda_i^{(0)}(x, y) L_i^{(0)}(y) = L_1^{(0)}(x) = \alpha_{A^0} = 0, \quad (19) \]
when multiplied with the equations of motion (12).

We now require the “zeroth generation” Lagrange constraint (19) to vanish in the evolution of time, and add this to the equations of motion (12) through the equation of \( \dot{\Omega}_1^{(0)} = 0. \) The resulting set of six equations may be summarized in the form of the set of “first generation” equations,
\[ L_i^{(1)}(x) = 0, \quad i_1 = 1, \ldots, 6, \]
with
\[ L_i^{(1)}(x) = \begin{cases} L_i^{(0)}, & i = 1, \ldots, 5, \\ \frac{d}{dt}(\lambda_i^{(0)} L_i^{(0)}). & \end{cases} \quad (20) \]

\( L_i^{(1)}(x) \) can be written in the general form
\[ L_i^{(1)}(x) = \int d^3 y \left[ W_{i_1j}(x, y) \dot{\varphi}^j(y) + \alpha_{i_1}^{(1)}(y) \delta^3(x - y) \right] = 0, \quad i_1 = 1, \ldots, 6, \quad (21) \]
where
\[ W_{i_1j}(x, y) = \begin{pmatrix} \hat{W}_{ij}^{(0)} \\ 0 - \partial^1_x - \partial^2_x - \partial^3_x - m^2 \end{pmatrix} \delta^3(x - y), \quad (22) \]
and
\[ (\alpha_{i_1}^{(1)})^T(x) = ((\alpha_{i_1}^{(0)})^T, \alpha_{6}^{(1)}(x)) \quad (23) \]
with
\[ \alpha_{6}^{(1)} = \partial_1 \partial_2 \hat{A}^0 - m^2 \hat{A}^0. \quad (24) \]

We now repeat the previous analysis taking Eq. (21) as a starting point, and looking for solutions of a first generation null eigenvector as
\[ \int d^3 y \lambda_{i_1}^{(1)}(x, y) W_{i_1j}^{(1)}(y, z) = 0. \quad (25) \]
Since \( W_{i_1j}(x, y) \) is still of rank four, there exit two null eigenvectors, \( \lambda^{(1)} \) and \( \Sigma^{(1)} \). The \( \lambda^{(1)} \) is the previous null eigenvector (15) with an extension such as \( \lambda_i^{(1)}(x, y) = (\lambda_i^{(0)}, 0) \), and the other null eigenvector \( \Sigma^{(1)} \) of \( W_{ij}^{(1)}(x, y) \) is of the form \( (0, \partial^1_x, \partial^2_x, \partial^3_x, 1, 1)\nu(x)\delta^3(x - y). \) We could thus choose it as
\[ \Sigma_{i_1}^{(1)}(x, y) = (0, \partial^1_x, \partial^2_x, \partial^3_x, 1, 1)\delta^3(x - y). \quad (26) \]
The associated constraint is found to vanish “identically”
\[ \Omega_2^{(1)}(x) = \int d^3 y \Sigma_{i_1}^{(1)}(x, y) L_i^{(1)}(y) = \partial^i \alpha_{A^i}^{(1)} + \alpha_{6}^{(1)} = 0. \quad (27) \]
The algorithm ends at this point.

The local symmetries of the action (11) are encoded in the identity (27). Recalling (12) and (21) we see that the identity (27) can be rewritten as follows

$$\Omega^{(1)}_2(x) = \partial^i L^{(0)}_i + L^{(0)}_5 + \frac{d}{dt} L^{(0)}_1 = 0.$$  \hspace{0.5cm} (28)

This result is a special case of a general theorem stating [12] that the identities $\Omega^{(l)}_k \equiv 0$ can always be written in the form

$$\Omega^{(l)}_k = \sum_{s=0} \int d^3 y \left( (-1)^{s+1} \frac{d}{dt} \phi^{i(s)}_k (x, y) L^{(0)}_{i} (y) \right).$$  \hspace{0.5cm} (29)

For the Proca case we have the following relations

$$\phi^{(0)}_2 (x, y) = -\partial^1 \delta^3 (x - y),$$
$$\phi^{(0)}_3 (x, y) = -\partial^2 \delta^3 (x - y),$$
$$\phi^{(0)}_4 (x, y) = -\partial^3 \delta^3 (x - y),$$
$$\phi^{(0)}_5 (x, y) = -\delta^3 (x - y),$$
$$\phi^{(1)}_2 (x, y) = \delta^3 (x - y).$$  \hspace{0.5cm} (30)

It again follows from general considerations [12] that the action (11) is invariant under the transformation

$$\delta \phi^i (y) = \sum_k \int d^3 x \left( \Lambda_k (x) \phi^{i(0)}_k (x, y) + \dot{\Lambda}_k (x) \phi^{i(1)}_k (x, y) \right).$$  \hspace{0.5cm} (31)

For the case in question this corresponds to the transformations

$$\delta A^\mu (x) = \partial^\mu \Lambda_2,$$
$$\delta \rho (x) = -\Lambda_2.$$  \hspace{0.5cm} (32)

These are the set of symmetry transformations which is identical with the previous result (7) of the extended Hamiltonian formalism, when we set $\varepsilon^1 = \partial^0 \varepsilon^2$ and $\varepsilon^2 = \Lambda_2$, similar to the Maxwell case [14]. As results, we have systematically derived the set of symmetry transformations starting from the Lagrangian of the first class Proca model.

### 3 Chern-Simons Model

Similar to the Proca case, we have recently applied the BFT method to the pure CS theory whose dynamics are given by

$$S = \int d^3 x \frac{K}{2} \epsilon_{\mu \nu \rho} A^\mu \partial^\nu A^\rho.$$  \hspace{0.5cm} (33)
This is invariant under the U(1) gauge transformation, \( \delta A^\mu = \partial^\nu \Lambda \), but has still second class constraints due to the symplectic structure of the CS theory. As a result, we have obtained the following fully first class CS action \[34\] as 

\[
S = \int d^3x \left( \kappa \epsilon_{\mu \nu \rho} A^\mu \partial^\nu A^\rho - \frac{1}{2} \epsilon_{ij} \Phi^i \dot{\Phi}^j + \sqrt{\kappa} \Phi^i F_{i0j} \right),
\]

where \( \Phi^i \) satisfy the relation \( \{ \Phi^i(x), \Phi^j(y) \} = \epsilon^{ij} \delta(x - y) \). Then, we may raise a question what is the symmetry transformation corresponding to the additional first class constraints originated from the symplectic structure of the CS theory. We would like to find them through the similar analysis of the previous Lagrangian approach.

The equations of motion following from \[34\] are of the form 

\[
\begin{align*}
(L^{(0)}_i)^T(x) &= (L_{A^0}, L_{A^1}, L_{A^2}, L_{\Phi^1}, L_{\Phi^2}), \\
(\alpha^{(0)}_i)^T(x) &= (\alpha_{A^0}, \alpha_{A^1}, \alpha_{A^2}, \alpha_{\Phi^1}, \alpha_{\Phi^2}).
\end{align*}
\]

The starting Hessian matrix is trivial for this pure CS case due to the first order Lagrangian as follows

\[
W^{(0)}_{ij}(x, y) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \delta^2(x - y) \equiv \tilde{W}^{(0)}_{ij} \delta^2(x - y),
\]

which shows that there are no true dynamical degrees of freedom. Explicitly, the equations of motion are given by

\[
\begin{align*}
L_{A^0} &= -\kappa \epsilon_{ij} \partial^i A^j - \sqrt{\kappa} \partial_1 \Phi^1 - \sqrt{\kappa} \partial_2 \Phi^2 = \alpha_{A^0}, \\
L_{A^1} &= \kappa \dot{A}^1 - \sqrt{\kappa} \dot{\Phi}^1 + \kappa \partial_1 A^0 = \alpha_{A^1}, \\
L_{A^2} &= -\kappa \dot{A}^2 - \sqrt{\kappa} \dot{\Phi}^2 - \kappa \partial_2 A^0 = \alpha_{A^2}, \\
L_{\Phi^1} &= \dot{\Phi}^2 + \sqrt{\kappa} (\dot{A}^1 + \partial_1 A^0) = \alpha_{\Phi^1}, \\
L_{\Phi^2} &= -\dot{\Phi}^1 + \sqrt{\kappa} (\dot{A}^2 + \partial_2 A^0) = \alpha_{\Phi^2}.
\end{align*}
\]

Since the Hessian matrix \[37\] is of rank zero, there exist five “zeroth generation” null eigenvectors as

\[
\lambda^{(0)\alpha}_i (x, y) = \delta^\alpha_i \delta^2(x - y), \quad i, \alpha = 1, \ldots, 5.
\]

Correspondingly we have “zeroth generation” Lagrange constraints

\[
\Omega^{(0)}_i = L^{(0)}_i = \alpha^{(0)}_i.
\]

Moreover, the equations of motion \[38\] are not independent. We can thus obtain the following identical relations as

\[
\begin{align*}
\alpha^{(0)}_2 - \sqrt{\kappa} \alpha^{(0)}_5 &= 0, \\
\alpha^{(0)}_3 + \sqrt{\kappa} \alpha^{(0)}_4 &= 0.
\end{align*}
\]
As results, we can rewrite the “zeroth generation” Lagrange constraints as follows

\[
Ω^{(0)}_i = \begin{cases} 
Ω^{(0)}_{i\bar{i}} = Ω^{(0)}_i, & i, \bar{i} = 1, 2, 3, \\
Ω^{(0)}_{\bar{i}\bar{i}} = Ω^{(0)}_{\bar{i}}, & \bar{i} = 1, 2,
\end{cases}
\]  

(42)

where the bar in the subscript denotes the independent constraints while the carrot identities as

\[
Ω^{(0)}_{1} = α^{(0)}_2 - \sqrt{κ}α^{(0)}_5 = L^{(0)}_{2} - \sqrt{κ}L^{(0)}_{5} = 0,
Ω^{(0)}_{2} = α^{(0)}_3 + \sqrt{κ}α^{(0)}_4 = L^{(0)}_{3} + \sqrt{κ}L^{(0)}_{4} = 0.
\]  

(43)

We now require the independent “zeroth generation” Lagrange constraints to vanish in time evolution. Then, the resulting set of eight equations may be summarized in the form of the set of “first generation” equations, \( L^{(1)}_{i_1} = 0 \), \( i_1 = 1, \cdots, 8 \), with

\[
L^{(1)}_{i_1}(x) = \begin{cases} 
L^{(0)}_{i_1}, & i_1 = 1, 5, \\
\frac{d}{dt}(Ω^{(0)}_{i_1}), & i_1 = 6, \cdots, 8
\end{cases}
\]  

(44)

\( L^{(1)}_{i_1}(x) \) can be written in the general form as

\[
L^{(1)}_{i_1}(x) = \int d^2 y \left[ W^{(1)}_{i_1j}(x, y) \ddot{\varphi}^j(y) + α^{(1)}_{i_1}(y) \delta^2(x - y) \right] = 0,
\]  

(45)

where the Hessian matrix is given by

\[
W^{(1)}_{i_1j}(x, y) = \begin{pmatrix}
\ddot{W}^{(0)}_{ij} \\
0 0 0 0 0 \\
0 0 \kappa - \sqrt{κ} 0 \\
0 - \sqrt{κ} 0 0 - \sqrt{κ}
\end{pmatrix} \delta^2(x - y),
\]  

(46)

and

\[
(α^{(1)}_{i_1})^T(x) = ( α^{(0)}_i^T, α^{(1)}_6, α^{(1)}_7, α^{(1)}_8 ),
\]  

(47)

with

\[
α^{(1)}_{6} = -κε_{ij} \partial^i \ddot{A}^j - κ\partial_1 \ddot{Φ}^1 - \sqrt{κ}\partial_2 \ddot{Φ}^2,
α^{(1)}_{7} = κ\partial_2 \ddot{A}^0,
α^{(1)}_{8} = -κ\partial_1 \ddot{A}^0.
\]  

(48)

We now repeat the previous analysis taking Eq. (45) as a starting point, and looking for solutions of a first generation null eigenvector as

\[
\int d^2 y \lambda^{(1)}_{i_1}(x, y) W^{(1)}_{i_1j}(y, z) = 0.
\]  

(49)
Since $W_{i_1j}(x, y)$ is of rank two, there exist six null eigenvectors, $\lambda_{(1)i}^{(a)}$ and $\Sigma^{(1)}$. These $\lambda_{(1)i}^{(a)}$ are the previous null eigenvector (39) with an extension such as $\lambda_{(1)i}^{(a)}(x, y) = (\lambda_{(0)i}^{(a)}, 0)$, and a new null eigenvector $\Sigma^{(1)}$ as

$$\Sigma^{(1)}(x, y) = (0, 0, 0, 0, 0, 1, 0, 0)\delta^2(x - y).$$  (50)

The associated constraint of $\Sigma^{(1)}(x, y)$ generates one more Lagrange constraint as

$$\Omega_{1}^{(1)}(x, y) = \int d^2 y \Sigma^{(1)}(x, y) L_{i_1}^{(1)}(y) = -\kappa \epsilon_{ij} \partial^j \dot{A}^i - \kappa \partial_1 \Phi^1 - \sqrt{\kappa} \partial_2 \Phi^2 = 0.$$  (51)

Now, the resulting set of nine equations may be summarized in the form of “second generation”, $L_{i_2}^{(2)} = 0, i_1 = 1, \cdots, 9$, with

$$L_{i_2}^{(2)}(x) = \begin{cases} L_{i_1}^{(1)}, & i_1 = 1, \cdots, 8, \\ \frac{d}{dx}(\Omega_{1}^{(1)}), & \end{cases}$$  (52)

which can be written in the general form

$$L_{i_2}^{(2)}(x) = \int d^2 y \left[ W_{i_2j}^{(2)}(x, y) \ddot{\phi}^j(y) + \alpha_{i_2}^{(2)}(y) \delta^2(x - y) \right] = 0,$$  (53)

where

$$W_{i_2j}^{(2)}(x, y) = \begin{pmatrix} \ddot{W}_{ij}^{(0)} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa & -\sqrt{\kappa} & 0 \\ 0 & -\kappa & 0 & 0 & -\sqrt{\kappa} \\ 0 & \kappa \partial^2_x & 0 & \sqrt{\kappa} \partial^1_x & \sqrt{\kappa} \partial^2_x \end{pmatrix} \delta^3(x - y),$$  (54)

and

$$\left(\alpha_{i_2}^{(2)}\right)^T(x) = \left((\alpha_{i_1}^{(1)})^T, \alpha_{9}^{(2)}\right).$$  (55)

with identically vanishing component of $\alpha_{9}^{(2)} = 0$.

We now repeat the analysis starting from Eq. (53). Since $W_{i_2j}^{(2)}(x, y)$ is still of rank two, there exists one more new null eigenvector $\Sigma^{(2)}$ as

$$\Sigma^{(2)}(x, y) = (0, 0, 0, 0, 0, \partial_1^1, \partial_2^2, 1)\delta^2(x - y).$$  (56)

with the properly extended previous null eigenvectors $\lambda_{(1)i}^{(a)}$ and $\Sigma^{(1)}$. The associated constraint is now found to vanish “identically”

$$\Omega_{1}^{(2)}(x) = \int d^3 y \lambda_{i_2}^{(2)}(x, y) L_{i_2}(y) = \partial^1 L_7(x) + \partial^2 L_8(x) + L_9(x) = 0,$$  (57)
and the algorithm stops at this stage. As results, we gather all the identities, which will provide the explicit form of symmetry transformations, as follows

\[
\begin{align*}
\Omega_1^{(0)} &= \Omega_1^{(0)} = L_2 - \sqrt{\kappa} L_5 = 0, \\
\Omega_2^{(0)} &= \Omega_2^{(0)} = L_3 + \sqrt{\kappa} L_4 = 0, \\
\Omega_1^{(2)} &= \Omega_3^{(2)} = \partial^1 L_7 + \partial^2 L_8 + L_9 = \frac{d}{dt}(\partial^1 L_2 + \partial^2 L_3) + \frac{d^2}{dt^2} L_1 = 0.
\end{align*}
\]

Comparing these with Eq. (29), we have the following relations

\[
\begin{align*}
\phi_1^{2(0)}(x, y) &= -\delta^2(x - y), \\
\phi_1^{5(0)}(x, y) &= \sqrt{\kappa}\delta^2(x - y), \\
\phi_2^{3(0)}(x, y) &= -\delta^2(x - y), \\
\phi_2^{4(0)}(x, y) &= -\sqrt{\kappa}\delta^2(x - y), \\
\phi_3^{3(1)}(x, y) &= -\partial_1^1 \delta^2(x - y), \\
\phi_3^{3(1)}(x, y) &= -\partial_2^1 \delta^2(x - y), \\
\phi_3^{1(2)}(x, y) &= -\delta^2(x - y),
\end{align*}
\]

and making use of the following general expression of

\[
\delta \varphi^i(y) = \sum_k \int d^2 x \left( \Lambda_k(x) \phi_k^{i(0)}(x, y) + \dot{\Lambda}_k(x) \phi_k^{i(1)}(x, y) + \ddot{\Lambda}_k(x) \phi_k^{i(2)}(x, y) \right),
\]

we finally obtain the extended symmetry transformations of the first class pure CS theory as

\[
\begin{align*}
\delta A_0(x) &= \partial^0 \Lambda_3, \\
\delta A_i(x) &= \partial^i \Lambda_3 + \Lambda_i, \\
\delta \Phi^i(x) &= \sqrt{\kappa} \epsilon^{ij} \Lambda_j.
\end{align*}
\]

Therefore, we see that the gauge parameter \( \Lambda_3 \) is related with the usual U(1) gauge transformation, while \( \Lambda_i \) (\( i = 1, 2 \)) generate novel symmetries originated from the symplectic structure of the CS theory. Note that if we further restrict the transformation as \( \dot{\Lambda}_3 = \Lambda \), and \( \Lambda_i = 0 \), then we easily recognize this novel extended symmetries reduce to the original well-known U(1) symmetry.

## 4 Conclusion

In this paper we have considered the Lagrangian approach of the first class abelian Proca and CS models. First, we have turned the second class Lagrangians into first class
ones following the BFT method. Although the gauge invariant Lagrangian for the simple Proca model corresponding to the first class Hamiltonian exhibits the well-known local symmetry, we have explicitly shown, following the version of the Lagrangian approach, how this symmetry could be systematically derived on a purely Lagrangian level, without resorting to a Hamiltonian formulation. On the other hand, we have also studied the fully first class CS model by embedding the so called symplectic structure on the extended space. As results, we have found that there exist novel symmetries as well as the usual U(1) gauge symmetry by the Lagrangian method. We hope that the Lagrangian approach employed in these derivation will be of much interest in the context of the field-antifield formalism.

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