Majorana dc Josephson current mediated by a quantum dot

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Abstract

The Josephson supercurrent through a hybrid Majorana-quantum dot-Majorana junction is investigated. We particularly analyze the effect of spin-selective coupling between the Majorana and quantum dot states, which only emerges in the topological phase and will influence the current through bent junctions and/or in the presence of magnetic fields in the quantum dot. We find that the characteristic behavior of the supercurrent through this system is quite counterintuitive, differing remarkably from the resonant tunneling, e.g. through the similar (normal phase) superconductor-quantum dot-superconductor junction. Our analysis is carried out under the influence of the full set-up parameters and for both the $\frac{\pi}{2}$ and $\frac{\pi}{4}$ periodic currents. The present study is expected to be relevant to the future exploration of applications of Majorana-nanowire circuits.

Keywords: Majorana state, Josephson current, nanowire

(Some figures may appear in colour only in the online journal)

1. Introduction

Majorana fermions (MFs) are exotic self-Hermitian particles with non-Abelian statistics, which have the property of being their own antiparticles [1–3] and hold promise as robust building blocks for topological quantum computation [4, 5]. A remarkable insight predicts that MFs can emerge as novel excitations of the Majorana zero modes or Majorana bound states in condensed matter systems, e.g. from the non-Abelian excitations of the 5/2 fractional quantum Hall effect in semiconductor heterostructures [6], and based on exotic superconductors, where MFs correspond to the zero-energy states of the effective Bogoliubov–de Gennes Hamiltonian [7, 8].

More recent proposals employ the proximity effect from a conventional superconductor, either in nanowires in the presence of strong spin–orbit interaction and Zeeman splitting [9–17], or in topological insulators [18–22]. These efforts have brought MFs closer to experimental realization and predict more reliable experimental signatures of their presence. These signatures include half-integer conductance quantization [23], the zero-bias peak in the tunneling conductance [24–28], and the $4\pi$ Josephson effect in superconductor–superconductor junctions [14, 29–31]. In particular, in order to distinguish MFs from other quasi-particle states, in recent years, some interest has turned to the spin-selective Andreev reflection [32–35].

In this work, we consider the hybrid system of a Majorana-quantum dot-Majorana junction, which can be realized from semiconductor nanowires in proximity-contact with $s$-wave superconductors, as schematically shown in figure 1. Similar systems involving Majorana nanowires coupled to quantum dots (QDs) have been investigated for phenomena such as teleportation [36], the anomaly of the conductance peak [37], the characteristic signatures in the current noise spectrum [38], and the featured Josephson current [15]. Our present interest is the dc Josephson current under the influence of the spin-selective coupling between the Majorana and QD states,
which is most relevant to bent junctions and the presence of a magnetic field in the QD area.

Due to the helical property of MFs, an MF at the end of a nanowire only couples to a unique spin state in the normal region, e.g. the spin-up QD state, as shown in figure 1. Actually, this spin-selective coupling is the origin of the spin-dependent Andreev reflection [32, 39]. In previous studies, the set-up configuration is usually assumed to be either a single Majorana nanowire coupled to normal leads or QDs, or a straight Majorana-normal region-Majorana junction with a bent angle \( \theta = 0 \). In both cases, only the spin-up states (in the normal parts) couple to the MF and contribute to the current. However, if the Majorana-QD-Majorana junction is not straight (with a bent angle \( \theta \neq 0 \) between the nanowires), the two MFs at the ends of the nanowires (see figure 1) will couple to the spin states in the QD with different orientations, leading thus to both spin-up and spin-down states participating in the transport. This is anticipated to result in the different current behavior of the straight and bent junctions. On the application aspect (e.g. in topological quantum computations), the Majorana nanowires will possibly have an orientational angle, or one would want to employ this orientational angle to modulate the charge transfer properties. This thus makes the bending structure studied here relevant to possible real circuits.

We will also analyze the effect of magnetic fields in the QD area. This is motivated by the fact that in order to induce the emergence of an MF, a magnetic field needs to be applied to the nanowires, and this must spill over into the QD area owing to their close separation. Similar to the consequence of junction bending, we expect that the non-z-axial direction magnetic field will involve the spin-down state in the transport as well. We will show that, remarkably, owing to the spin-selective coupling, both the magnetic field in the QD area and the junction bending will result in the counterintuitive behavior of the Josephson current. For instance, in the case of a QD level aligned with the Fermi energy (under resonant tunneling), the Josephson current is to be strongly suppressed by the magnetic field and junction bending. However, as the QD level deviates from the Fermi level (violating the resonant-tunneling condition), the oscillation amplitude of the Josephson current always shows an enhanced value, together with robust jumps in the current-phase curves.

![Figure 1. A set-up sketch of the Majorana-QD-Majorana junction, realized by semiconductor nanowires in contact with s-wave superconductors. The two nanowires may have a mutual orientation angle, which is expected to affect the supercurrent via the unique spin-selective coupling between the Majorana and QD states.](image)

2. Model and methods

2.1. Set-up model

In this work we consider a simple setup, as shown in figure 1, where two semiconductor nanowires are connected through a QD. The nanowires are contacted in proximity with an s-wave superconductor. Then, the superconductivity caused by the proximity effect, together with the strong Rashba spin-orbit coupling (SOC) and Zeeman splitting inside the nanowires, can possibly induce the emergence of a pair of MFs at the end of each nanowire [13, 14], as denoted in figure 1 by \( \gamma_{L1,2} \) for the left wire and \( \gamma_{R1,2} \) for the right one. In order to generate these Majoranas, taking the coordinate system labeled in figure 1, if the direction of the Rashba SOC is in the plane (e.g. in the y-direction), the Zeeman field can be along either the x- or the z-direction. In our work, we assume the Zeeman field to be along the x-direction, which thus leads to the spin of the Majorana fermion along the axial direction of the nanowire, as proved in appendix by a simple calculation for the Bloch vector of the Majorana fermion using the solution of [13].

In this work, we employ an effective low-energy description for the nanowires which accommodate the induced Majoranas at the ends. Then, the entire tunnel-coupled system is described by the Hamiltonian

\[
H = H_M + H_{\text{dot}} + H_{AL} + H_{DR},
\]

more explicitly with

\[
H_M = i\varepsilon_L \gamma_{L1} \gamma_{L2} + i\varepsilon_R \gamma_{R1} \gamma_{R2},
\]

\[
H_{\text{dot}} = \sum_\sigma \varepsilon_d d_\sigma^\dagger d_\sigma + (d_\sigma^\dagger)^\dagger \mathbf{\hat{B}} \cdot \mathbf{\hat{d}}_\sigma
\]

\[
H_{AL} = (\lambda_\sigma d_\sigma - \chi_\sigma^d d_\sigma^\dagger) \gamma_{L1},
\]

\[
H_{DR} = i(\chi_{\sigma} d_\sigma^\dagger + \chi_{\sigma}^* d_\sigma) \gamma_{R2}.
\]

Here, \( H_M \) is the effective low-energy Hamiltonian for the two pairs of Majorana states that have emerged at the ends of the nanowires. Each Majorana pair may have nonzero coupling energy owing to the overlap of their spatial wavefunctions, i.e. \( \varepsilon_L \sim e^{-i/h}L \) and \( \varepsilon_R \sim e^{-i/h}R \), where \( L_{LR} \) and \( \zeta_{LR} \) are, respectively, the length of the nanowire and the superconductor coherence length. \( H_{\text{dot}} \) denotes the QD Hamiltonian which contains a single quantized level \( \varepsilon_d \) (tunable by gate voltage), and the electron spin in the dot is possibly affected (rotated) by the magnetic field in the dot area \( \mathbf{B} = (B_x, B_y, B_z) \). For generic purposes, here do not we assume the magnetic field in the quantum dot to be precisely along the x-direction.

In the dot Hamiltonian, \( d_\sigma^\dagger \) and \( d_\sigma \) are the creation and annihilation operators of the electron with spin \( \sigma \).

\( H_{AL} \) and \( H_{DR} \) describe the tunnel coupling between the dot and the nearby Majorana states. In general, the coupling amplitudes can be expressed as \( \lambda_{LR} = |\lambda_{LR}|e^{i\phi_{LR}}/2 \), where the phase factors are determined by the phase of the substrate superconductors and their difference will result in the famous Josephson current. Another important issue to be
noted is that due to the helical property of an MF, it only couples to the spin-up state in the QD (defined in the same \( z \)-representation of the associated nanowire), owing to the unique direction of the MF [32, 39]. As mentioned in the introduction, our special interest in this work is to consider the two nanowires which are not aligned in the same orientation, but with an angle \( \theta \), as shown in figure 1. Thus, the left and right MFs will only couple to the spin-polarized QD state, respectively, along the \( z \)- and \( z' \)-axes, with the associated electron operators connected by the following unitary transformation:

\[
\begin{bmatrix}
  d_1 \\
  d_2
\end{bmatrix} = \begin{bmatrix}
  \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\
  -\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{bmatrix}
\begin{bmatrix}
  d_1 \\
  d_2
\end{bmatrix}
\] (2)

This implies that if the bent angle \( \theta \neq 0 \), the spin-down state also couples to the MFs.

In practice, it would be convenient to convert the MFs to a regular fermion representation, via the simple transformation: \( c_L = (\gamma_{L1} + i \gamma_{L2}) \sqrt{2} \), and \( c_R = (\gamma_{R1} + i \gamma_{R2}) \sqrt{2} \). The creation operators are their Hermitian conjugate and satisfy \( \{c_{LR}, c_{LR}^\dagger\} = 1 \). Now let us apply the generalized Nambu representation, by introducing the field creation and annihilation operators as \( \psi^\dagger = (d_1, d_1^\dagger, d_2, d_2^\dagger, c_L, c_L^\dagger, c_R, c_R^\dagger)^T \), and \( \psi = (d_1, d_2^\dagger, d_1^\dagger, c_L, c_L^\dagger, c_R, c_R^\dagger)^T \). Then, the Hamiltonian of the whole system can be rewritten as

\[
\mathcal{H} = \frac{1}{2} \psi^\dagger \mathcal{H} \psi,
\] (3)

where the Hamiltonian matrix has a block form given by

\[
\mathcal{H} = \begin{pmatrix}
\mathcal{H}_{dd} & \mathcal{H}_{dl} & \mathcal{H}_{dr} \\
\mathcal{H}_{dl} & \mathcal{H}_{LL} & 0 \\
\mathcal{H}_{dr} & 0 & \mathcal{H}_{RR}
\end{pmatrix}
\] (4)

and each sub-matrix reads, respectively,

\[
\mathcal{H}_{dd} = \begin{pmatrix}
\varepsilon_d + B_x - iB_y & 0 & 0 \\
B_x + iB_y & -\varepsilon_d - B_x & 0 \\
0 & 0 & -\varepsilon_d - B_x - B_y \\
0 & 0 & -B_x - iB_y - \varepsilon_d - B_x
\end{pmatrix}
\] (5a)

\[
\mathcal{H}_{LL} = \begin{pmatrix}
\varepsilon_L & 0 \\
0 & -\varepsilon_L
\end{pmatrix}
\] (5b)

\[
\mathcal{H}_{RR} = \begin{pmatrix}
\varepsilon_R & 0 \\
0 & 0 & -\varepsilon_R
\end{pmatrix}
\] (5c)

\[
\mathcal{H}_{dl} = \frac{1}{\sqrt{2}} \begin{pmatrix}
-\lambda_L & -\lambda_L \\
0 & 0 \\
\lambda_L & \lambda_L
\end{pmatrix}
\] (5d)

The other two off-diagonal sub-matrices are given by \( \mathcal{H}_{ld} = \mathcal{H}_{dl}^\dagger \) and \( \mathcal{H}_{dr} = \mathcal{H}_{dr}^\dagger \).

2.2. Josephson current

From the description of the above low-energy effective Hamiltonian, it seems that this system, which accommodates discrete energy levels, can only support unitary evolution (quantum oscillations). However, this is not true. Note that the MFs at the ends of the nanowire are induced via contact with the superconductor which has the Cooper pair reservoir. Even under zero bias voltage, both the superconductors (see figure 1) can support the stationary dc Josephson current, if a phase difference between the superconductors is maintained. This understanding allows us to apply the quantum transport formalism of the nonequilibrium Green’s function (nGF) to the present set-up with only discrete energy levels.

Following the nGF technique outlined in [40, 41], the Josephson current reads

\[
I = \frac{e}{h} \int \text{d} \epsilon \text{Re} \text{Tr}[\sigma_3 \Sigma(e) G_d(e)]^\dagger \].
\] (6)

This result is expressed in the Nambu representation. Accordingly, \( \tilde{\sigma} = \text{diag}(1, 1, -1, -1) \), which makes the Keldysh equation \( [\Sigma G_d]^\dagger = \Sigma^* G_d^\dagger + \Sigma^\dagger G_d^\prime \) applicable in the compact form. \( G_d \) is the ‘reduced’ effective Green’s function of the quantum dot, by accounting for the effect of the ‘leads’ (the Majoranas connected at both sides) as self-energies (as usual in the nGF formalism for transport). Here, we use \( \Sigma \) to denote the difference of the self-energies from the left and right wires (leads), \( \Sigma = \Sigma_L - \Sigma_R \). The associated retarded and advanced self-energies are given, respectively, by

\[
\Sigma^{(a)}_{LR}(\epsilon) = \mathcal{H}_{LL} \Sigma^{(a)}_{LR}(\epsilon) \mathcal{H}_{LR,d}^\dagger + \mathcal{H}_{dd} \Sigma^{(a)}_{LR}(\epsilon) \mathcal{H}_{dl,d}^\dagger
\] (7)

where \( \Sigma^{(a)}_{LR}(\epsilon) = [\epsilon - \mathcal{H}_{LL} \pm i0^+]^{-1} \) are the retarded (advanced) Green’s functions of the isolated left/right wires. For the lesser self-energies, one can apply: \( \Sigma^{(c)}_{LR}(\epsilon) = f(\epsilon) \Sigma^{(a)}_{LR}(\epsilon) - \Sigma^{(r)}_{LR}(\epsilon) \), where \( f(\epsilon) \) is the Fermi-Dirac distribution function. The retarded and advanced Green’s functions read \( G^{(a)}_{LR}(\epsilon) = [\epsilon - \mathcal{H}_{dd} - (\Sigma^{(a)}_{LR} + \Sigma^{(r)}_{LR})]^{-1} \), and the lesser Green’s function can be similarly obtained by using \( G^{(c)}_{LR}(\epsilon) = f(\epsilon) [G^{(a)}_{LR}(\epsilon) - G^{(c)}_{LR}(\epsilon)] \). Using these relations together with some algebras, we obtain

\[
I = \frac{e}{h} \int \text{d} \epsilon \text{Re} \text{Tr}[\tilde{\sigma}_3 \Sigma^{(a)}_{LR}(\epsilon)] f(\epsilon).
\] (8)
3. Results

In the numerical investigations, the results will be calculated under the influence of a couple of set-up parameters: the gate voltage, which would affect the QD level \( \varepsilon_d \), the bent angle \( \theta \) of the junction, the magnetic field in the QD area, and the overlap strength of the Majorana wavefunctions \( \varepsilon_{LR} \). We would like to set \( |\lambda_L| = |\lambda_R| = \lambda = 1 \) and scale all the energies by \( \lambda \). Also, we denote the Fermi level of the entire set-up as the reference (zero) energy, the phase difference between the superconductors as \( \Delta \phi = \phi_R - \phi_L \), and assume a zero temperature and identical nanowires \( \varepsilon_L = \varepsilon_R \).

3.1. The effect of dot level modulation

We first display the result for the straight junction with \( \theta = 0 \). In this case, the unique Majorana feature is revealed, as shown in figure 2(a), by the robust ‘jump’ of the Josephson current at \( \Delta \phi = \pi \) and the nonvanishing current when modulating the dot level \( \varepsilon_d \) (via modulation of the gate voltage applied). We see that with the increase of \( \varepsilon_d \) (even far away from the Fermi energy, i.e. \( \varepsilon_d = 0 \), the Josephson current only reduces by a small amount, and the jump at \( \Delta \phi = \pi \) always survives there, being robust against the variation of \( \varepsilon_d \). For instance, for \( \varepsilon_d = 2.0 \), the amplitude of the Josephson current is only reduced to about 70% of the value at \( \varepsilon_d = 0 \).

Both features revealed here are entirely different from the normal superconductor-QD-superconductor junction. In the normal (trivial) case, the jump only appears at \( \varepsilon_d = 0 \) and the current is to be strongly suppressed when \( \varepsilon_d \) deviates a long way from the Fermi energy [41]. We thus conclude that the features revealed in figure 2(a) are closely associated with the superconductor-proximity-induced nontrivial (topological) phase of the nanowire where the MF emerges. In this case, the structure of the energy diagram is as shown in figure 2(b), where we find the remarkable zero-energy crossing points at \( \Delta \phi = \pm \pi \), which are responsible for the current ‘jumps’ as observed in figure 2(a).

Qualitatively speaking, the current is the sum of all contributions from the occupied energy levels, with each individual proportional to the derivative of the associated energy curve (with respect to \( \Delta \phi \)) [29]. In this work, we calculate the current using equation (8). We integrate the energy from \( -\infty \) to the Fermi level, implying that the system is always in thermal equilibrium, especially when \( \Delta \phi \) passes through \( \pi \). This treatment corresponds to the certain relaxation mechanism involved, as a consequence of particle addition/loss from/to the surrounding environment. As a consequence, we obtain the \( 2\pi \) periodic current (as a function of \( \Delta \phi \)), figure 2(a). Only under the fermion number parity conservation, can the so-called \( 4\pi \) periodic current be expected. We will address this issue later in more detail.

3.2. The effect of magnetic fields

We now consider the possible effects of the magnetic field in the dot area. For the Zeeman splitting of the dot level caused by the magnetic field along the \( z \)-axis, we can understand that the effect is the same as the electric gate-modulation of the dot level, as shown in figure 2(a).

The effect of \( B_z \) (magnetic field along the \( x \)-axis) is shown in figure 3. For the ideal configuration with \( \varepsilon_d = 0 \), the jump at \( \Delta \phi = \pi \) disappears and evolves to a rounded transition behavior. Furthermore, with the increase of \( B_z \), the Josephson current quickly decreases and finally vanishes. For \( \varepsilon_d \neq 0 \), the similar modulation effect of \( B_z \) on the amplitude of the Josephson current is caused by the spin-selective coupling between the Majorana and QD states, by noting that the magnetic field \( B_z \) has the role of rotating the electron spin, thus leading to a change of its \( z \)-component. However, in this more ‘relaxed’ case \( \varepsilon_d \neq 0 \), the jump behavior at \( \Delta \phi = \pi \) survives. This indicates that the Majorana characteristic jump behavior is robust against the deviation of the dot level from the Fermi energy, even under the influence of the magnetic field in the QD area.
Then, the zero-energy crossing structure of the dot level decreases with the increase of the bent angle \( \theta \), of the Josephson current will increase with the deviation of the dot level from the Fermi energy. This means that for a bent Majorana–Josephson junction, the more strongly the resonance condition is violated, the larger the supercurrent that will flow through the junction. We finally mention that for all cases, the parameter \( \theta = \pi \) simply means rotating the right wire to the same side of the quantum dot and in parallel to the left wire, which will result in a current that vanishes completely.

3.4. An energy-diagram-based interpretation

We found through figures 2–4 that regardless of the magnetic field \( B \) and the mutual angle \( \theta \), the current jump can safely survive and the amplitude of the current maintains a large value, if the dot level \( \varepsilon_d \) violates the resonance condition. We may further understand the behavior as follows, with the help of the energy diagrams of figures 2(b) and 5.

First, in figure 2(b) for \( \varepsilon_d = 0 \) and \( \theta = 0 \), the flat zero-energy levels are of four-fold degeneracy, i.e. for the states \( \gamma_{L1}, \gamma_{R1}, d \) and \( d^\dagger \). The other four phase-difference (\( \Delta \phi \))-dependent eigenenergies are from the coupling of the states \( \gamma_{L2}, \gamma_{R2}, d \) and \( d^\dagger \). To be more specific, the two Majoranas \( \gamma_{L1} \) and \( \gamma_{R2} \) couple commonly to the spin-up dot state, resulting in the ‘crossing’ structure at zero-energy at \( \Delta \phi = \pm \pi \), which is similar to the result of the direct coupling of two Majoranas [44]. It is this zero-energy crossing (at the Fermi energy) which results in the large Josephson current with an abrupt jump in the presence of ‘relaxation’ or thermal equilibrium.

Second, for the dot level \( \varepsilon_d = 0 \) but \( \theta = 0 \) (see figure 5(d)), the Majorana states \( \gamma_{L1} \) and \( \gamma_{R1} \) also have flat zero-energy (independent of \( \Delta \phi \)); however, the energies of \( d \) and \( d^\dagger \) move to \( \varepsilon_d \) and \( -\varepsilon_d \), respectively. This will lead to an opening of the high-energy crossing at \( \Delta \phi = 0 \) (or \( \pm 2\pi \)), but still keep the zero-energy crossings at \( \Delta \phi = \pm \pi \) caused by \( \gamma_{L1} \) and \( \gamma_{R2} \). As a result, the Josephson current keeps a large value and the jump survives, because the current is dominantly contributed to the zero-energy crossing states.

Third, if \( \theta = 0 \) and \( \varepsilon_d = 0 \) (see figure 5(a)), we see that the zero-energy degeneracy of \( \gamma_{L1} \) and \( \gamma_{R2} \) at \( \Delta \phi = \pm \pi \) is removed. The disappearance of the zero-energy crossings leads to a strong reduction of the Josephson current and the rounding of the jump to a smooth transition, as shown in figure 4(a).

Finally, for the case of both \( \theta = 0 \) and \( \varepsilon_d = 0 \), as shown in figures 5(b) and (c), the absence of efficient energy level interaction between the Majorana and QD states (owing to \( \varepsilon_d = 0 \)) does not remove the zero-energy degeneracy of \( \gamma_{L1} \) and \( \gamma_{R2} \) at \( \Delta \phi = \pm \pi \). Then, the zero-energy crossing structure of the energy spectrum at \( \Delta \phi = \pm \pi \) results in a large Josephson current with abrupt jumps, despite the dot level \( \varepsilon_d \) deviating far from the Fermi energy.

3.5. The effect of Majorana interaction

Below we show that in order to observe the featured behavior discussed above, the overlap of the Majorana wavefunctions at the ends of the same nanowire should be negligibly small.
The results for the Josephson current can be expected, which we obtained above, a remark-

We replace the occupation of the lower level under the influence of Majoranas in the same nanowire. Indeed, we find that the Majorana interaction should be negligible small. In figure 6, we display the result for \( \varepsilon_d = 0 \), which corresponds to the nonzero coupling between the two Majoranas in the same nanowire. Indeed, we find that the amplitude of the Josephson current is strongly reduced and the jump disappears, with the increase of \( \varepsilon_{L,R} \) (see figures 6(a) and (c)). The basic reason for this is that the Majorana interaction in the same nanowire destroys the zero-energy Majorana state.

We have also checked that the large Josephson current with jump behavior cannot be restored by altering the dot level, even when \( \varepsilon_d = 0 \) is in resonance with \( \varepsilon_L \) and \( \varepsilon_R \) (see figures 6(c) and (d)). The reason for this is that if \( \varepsilon_L = \varepsilon_R = 0 \), the zero-energy Majorana states \( \gamma_L \) and \( \gamma_R \) are destroyed and the zero-energy crossings at \( \Delta \phi = \pm \pi \) disappear. As a consequence, the jumps at \( \Delta \phi = \pm \pi \) are replaced by rounded transitions, and the current is strongly reduced.

In figures 6(b) and (d), taking the current at \( \Delta \phi = \pi/2 \), we again show the \( \theta \) dependence of current, in the presence of the Majorana coupling (\( \varepsilon_{L,R} \neq 0 \)). From this result we see clearly that in order to obtain a larger super-current, we should make the nanowire longer than the superconductor coherence length to ensure the emergence of Majorana zero modes at the ends of the nanowire.

3.6. 4\( \pi \) periodic current

So far we have assumed that the system always relaxes to a thermal equilibrium when we vary the phase difference \( \Delta \phi \). In particular, at the zero-energy crossings \( \Delta \phi = \pm \pi \), this relaxation is either accompanied by the addition or the loss of a single particle, which therefore changes the parity of the particle numbers (i.e. the fermion parity). If such a relaxation channel is blocked, or the fermion parity is conserved, rather than the \( 2\pi \) periodic current we obtained above, a remarkable 4\( \pi \) periodic Josephson current can be expected, which is usually regarded as one of the most prominent Majorana signatures.

The 4\( \pi \) periodic current can be calculated as well by using equation (8), based on the following technique. When we increase \( \Delta \phi \) after passing through \( \pm \pi \), for the state occupation of the levels crossing at zero-energy (the Fermi level) at \( \Delta \phi = \pm \pi \), we replace the occupation of the lower level under the Fermi energy by its counterpart above it (owing to the fermion parity conservation), while satisfying the condition of thermal equilibrium after this replacement.

The results of the 4\( \pi \) periodic current are shown in figure 7, for one period. Compared with the 2\( \pi \) periodic current, we find that the jumps at \( \Delta \phi = \pm \pi \) disappear for all the 4\( \pi \) periodic currents. However, for the case of \( \theta = 0 \) and \( \varepsilon_d = 0 \) similar jumps may appear near (but not at) \( \Delta \phi = \pm \pi \) as observed in figures 7(b) and (c), owing to the accidental energy crossings at the specific phases. Again, along the increase of the angle \( \theta \), the amplitude of the current decreases. However, the current is more strongly suppressed in the case of \( \varepsilon_d = 0 \), while for larger deviation of \( \varepsilon_d \) (from the Fermi level) the current is less reduced.

4. Summary and discussion

To summarize, we have investigated the dc Josephson super-current through the Majorana-quantum dot-Majorana junction. Our particular interest is the consequence of the unique spin-selective coupling between the Majorana and dot states, which emerges only in the topological phase and will drastically influence the current through bent junctions and/or in the presence of magnetic fields in the dot area. Differing from the typical resonant tunneling behavior of the supercurrent through a similar system in a normal phase such as the superconductor-quantum dot-superconductor junction, we uncovered some counterintuitive results associated with the exotic nature of the Majorana fermion.

For instance, even for a straight junction and without a magnetic field in the dot area, when the dot level deviates considerably from the Fermi energy, the Josephson supercurrent keeps a large amplitude of oscillation with the superconductor...
phase difference $\Delta \phi$, and reveals abrupt jumps of current at $\Delta \phi = \pm \pi$. This result differs drastically from the usual resonant tunneling behavior through a similar system in the normal phase. For a bent junction and/or in the presence of a magnetic field in the dot, richer unexpected behavior is found. In resonance (the dot level aligned with the Fermi energy), we find that the supercurrent is to be strongly reduced either in resonance (the dot level aligned with the Fermi energy), the supercurrent can, in contrast, maintain a large amplitude of current, with the current-jump robustly surviving at $\Delta \phi = \pm \pi$—even under the influence of junction bending and magnetic fields in the dot. We expect these findings to be useful in the future design of novel circuit devices, based on quantum dots and Majorana nanowires.

As a final remark, we point out that in similar time-reversal-symmetry breaking systems, it is possible to observe nonzero Josephson current at zero phase difference, i.e. to realize the Josephson $\phi_0$-junction [45–48]. However, in order to realize the Josephson $\phi_0$-junction, in addition to the breaking of time reversal symmetry, the so-called chiral symmetry must be broken at the same time as well—as clearly explained in [45]—based on a similar system, which also contains a central quantum dot in the Josephson junction. There, to violate the chiral symmetry, Rashba SOC and multiple orbitals were introduced in the central quantum dot to be involved in the transport process [45]. Since our system does not satisfy this second condition of chiral symmetry breaking, the supercurrent at zero phase difference is always zero in all the results of our present work.

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Appendix. Spin direction of the Majorana fermion

In this appendix we illustrate how to determine the Majorana spin direction shown in figure 1 by taking, as an example, the solution obtained in [13]. In general, in the $s$-wave superconductor-based hybrid system of semiconductor nanowire realization, the directions of the required Zeeman splitting field and the Rashba SOC should be perpendicular to each other. In [13], the quantum wire is along the $y$-axis, the Rashba SOC is along the $z$-axis, and the Zeeman splitting magnetic field $B$ is along the $x$-axis. Note that in our notation for these directions in figure 1 we simply exchange the $y$- and $z$-axes. Under this choice of direction, the key result of the MF solution is given by equation (6) in [13], i.e. the Bogoliubov quasi-particle operator

$$\gamma^\dagger = \gamma = \frac{1}{2}(\psi_\uparrow - i\psi_\downarrow + i\psi_\uparrow^\dagger + \psi_\downarrow^\dagger).$$  \hspace{1cm} (A.1)

More explicitly, the Majorana state (wavefunction) reads

$$|\gamma \rangle \equiv |\gamma^\dagger \rangle |\Omega \rangle = \frac{1}{2}|(e, \uparrow) + i |e, \downarrow \rangle \rangle + \frac{1}{2}|(h, \uparrow) - i |h, \downarrow \rangle \rangle,$$

(A.2)

where $|\Omega \rangle$ is the ground state of the superconductor and the four components in this quantum superposition are, respectively, $|e, \uparrow \rangle = |\psi_\uparrow \rangle |\Omega \rangle$, $|e, \downarrow \rangle = |\psi_\downarrow \rangle |\Omega \rangle$, $|h, \uparrow \rangle = |\psi_\uparrow^\dagger \rangle |\Omega \rangle$ and $|h, \downarrow \rangle = |\psi_\downarrow^\dagger \rangle |\Omega \rangle$. Here, we explicitly use ‘$e$’ and ‘$h$’ to denote the electron and hole components. Straightforwardly, the spin wavefunction of the electron sector reads

$$|\gamma_\alpha \rangle = \langle e |\gamma \rangle = (|\uparrow \rangle + i |\downarrow \rangle)/\sqrt{2}.$$  \hspace{1cm} (A.3)

Here, we have normalized this sector state. Using this projected wavefunction, we can easily check: $\langle \sigma_\alpha \rangle = \langle \gamma_\alpha |\sigma_\alpha \gamma_\alpha \rangle = 1$, and $\langle \sigma_\alpha \rangle = \langle \sigma_\alpha \rangle = 0$. This result clearly shows that the spin direction of the MF is along the axial direction of the quantum wire, which is the $y$-direction in [13] and the $z$-direction denoted in our figure 1.

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