LINKING CURVES, SUTURED MANIFOLDS AND THE AMBROSE CONJECTURE FOR GENERIC 3-MANIFOLDS

PABLO ANGULO ARDOY

Abstract. We present a new strategy for proving the Ambrose conjecture, a global version of the Cartan local lemma. We introduce the concepts of linking curves, unequivocal sets and sutured manifolds, and show that any sutured manifold satisfies the Ambrose conjecture. We then prove that the set of sutured Riemannian manifolds contains a residual set of the metrics on a given smooth manifold of dimension 3.

1. Introduction

Let $(M_1,g_1)$ and $(M_2,g_2)$ be two complete Riemannian manifolds of the same dimension, with selected points $p_1 \in M_1$ and $p_2 \in M_2$. A linear isometry $L : T_{p_1}M_1 \rightarrow T_{p_2}M_2$ induces a map between the pointed manifolds $(M_1,p_1)$ and $(M_2,p_2)$: $\varphi = \exp_{p_2} \circ L \circ (\exp_{p_1} |_{O_1})^{-1}$ defined in $\varphi(O_1)$, for any domain $O_1 \subset T_{p_1}M_1$ such that $e_1 |_{O_1}$ is injective (for example, if $\exp_{p_1} O_1$ is a normal neighborhood of $O_1$).

A classical theorem of E. Cartan [C] identifies a situation where this map is an isometry.

For $x \in T_{p_1}M_1$, let $\gamma_1$ be the geodesic on $M_1$ defined in the interval $[0,1]$, starting at $p_1$ with initial speed vector $x$ and $\gamma_2$ be the geodesic on $M_2$ starting at $p_2$ with initial speed $L(x)$.

Let $P_\gamma : T_{p_1}M_1 \rightarrow T_{\gamma(1)}M_1$ denote parallel transport along a curve $\gamma$.

Definition 1.1. The curvature tensors of $(M_1,p_1)$ and $(M_2,p_2)$ are $L$-related if and only if for any $x \in T_{p_1}M_1$:

$$P_{\gamma_1}^* (R_{\gamma_1(1)}) = L^* P_{\gamma_2}^* (R_{\gamma_2(1)}) \quad (1.1)$$

In the definition, $P_\gamma^* (R_\gamma(1))$ is the pull back of the $(0,4)$ curvature tensor at $\gamma(1) \in M_1$ by the linear isometry $P_\gamma$, for $i = 1,2$

$$P_{\gamma_i}^* (R_{\gamma_i(1)})(v_1, v_2, v_3, v_4) = R_{\gamma_i(1)}(P_{\gamma_i}(v_1), P_{\gamma_i}(v_2), P_{\gamma_i}(v_3), P_{\gamma_i}(v_4))$$

for any four vectors $v_1, v_2, v_3, v_4$ in $T_{p_i}M_i$, and $L^*$ is used to carry the tensor $P_{\gamma_2}^* (R_{\gamma_2(1)})$ from $p_2 \in M_2$ to $p_1 \in M_1$.

The usual way to express that the curvature tensors of $(M_1,p_1)$ and $(M_2,p_2)$ are $L$-related is to say that the parallel translation of curvature along corresponding geodesics on $M_1$ and $M_2$ coincides. This certainly holds if $L$ is the differential of a global isometry between $M_1$ and $M_2$.

Theorem 1.2. If the curvature tensors of $(M_1,p_1)$ and $(M_2,p_2)$ are $L$-related, and $\exp_1 |_{O_1}$ is injective for some domain $O_1 \subset T_{p_1}M_1$, then $\varphi = \exp_2 \circ L \circ (\exp_1 |_{O_1})^{-1}$ is an isometric immersion.

Proof. The proof of lemma 1.35 in [CE] works for any domain $O$ such that $\exp_1 |_{O}$ is injective.

The author was partially supported by research grant ERC 301179, and by INEM.
In 1956 (see [A]), W. AmbrOSE proved a global version of the above theorem, but with stronger hypothesis.

A broken geodesic is the concatenation of a finite amount of geodesic segments. The Ambrose's theorem states that if the parallel translation of curvature along broken geodesics on $M_1$ and $M_2$ coincide, and both manifolds are simply connected, then the above construction gives a global isometry $\varphi : M_1 \to M_2$ whose differential at $p_1$ is $L$. It is enough if the hypothesis holds for broken geodesics with only one "elbow" (the reader can find more details in [CE]).

In [Hi], in 1959, the result of W. AmbrOSE was generalized to parallel transport for affine connections; in [BH], in 1987, to Cartan connections; and in [PT], in 2002, to manifolds of different dimensions.

The Ambrose's theorem has found applications to inverse problems (see [HHILU] and [KLU]).

Ambrose also posed the following conjecture:

**Conjecture 1.3 (Ambrose Conjecture).** Let $(M_1, p_1)$ and $(M_2, p_2)$ be two simply-connected Riemannian manifolds with $L$-related curvature. Then there is a global isometry whose tangent at $p_1$ is $L$.

Ambrose himself was able to prove the conjecture if all the data is analytic. In 1987, in the paper [H87], James Hebda proved that the conjecture was true for surfaces that satisfy a certain regularity hypothesis, that he was able to prove true in 1994 in [H94]. J.I. Itoh also proved the regularity hypothesis independently in [I].

The latest advance came in 2010, after we had started our research on the Ambrose conjecture, when James Hebda proved in [H10] that the conjecture holds if $M_1$ is a heterogeneous manifold. Such manifolds are generic.

The strategy of James Hebda in [H87] can be rephrased in the following terms: in any Riemannian surface, for any cleave point $q$, there is always a cut locus linking curve (see definition 3.4) that joins the two minimizing geodesics that reach $q$. We prove in Theorem 3.5 that this strategy does not carry over to higher dimension, and present a new strategy towards a proof for the Ambrose conjecture in dimension greater than 2.

We refer the reader to section 5.1 for the terms used in the following definition:

**Definition 1.4.** A pointed manifold $(M_0, p_0)$ is **sutured** (resp. strongly sutured) if and only if for any $x \in T_{p_0} M_0$, there is an unequivocal $y \in T_{p_0} M_0$ with $\|y\| \leq \|x\|$ that is linked to $x$ (resp. strongly linked).

**MAIN THEOREM A.** The Ambrose conjecture holds if $(M_1, p_1)$ is a sutured manifold.

Our conjecture is that all manifolds are strongly sutured, but in this paper we only prove it for manifolds whose exponential map has some generic transversality properties (see definition 6.6):  

**MAIN THEOREM B.** The set of strongly sutured Riemannian metrics on a 3 dimensional differentiable manifold $M$ contains a residual set of metrics.

The proof of Main Theorem B involves several technical difficulties but it is also quite constructive: we build linking curves using the linking curve algorithm (although there is a non-deterministic step at which we have to choose one curve that avoids some obstacles). In theorem 7.21, we prove that the algorithm always produces a special type of linking curve, starting on any conjugate point.

The proof of the Ambrose conjecture in [H10] certainly works for a generic class of Riemannian manifolds of any dimension, and is shorter than the proof presented here. However, that proof does not seem to be extendable to arbitrary metrics.
Indeed, the class of Riemannian manifolds for which we prove the Ambrose conjecture is not contained in the corresponding class in [H10], so this is truly a different approach. In the last section, we show how the ideas in this paper could be used to complete the proof of the conjecture for all Riemannian manifolds.

For the proof of these results we have introduced some new concepts that we believe are interesting in their own sake, such as linking curves and synthesis manifolds in section 5 or conjugate descending flow in section 7.2.

The outline of the paper is as follows: In section 3 we interpret the proof in [H82] in our own terms and show why it only works in dimension 2. In section 4 we study tree-formed curves and prove lemma 4.3 about the affine development of curves in manifolds with \( L \)-related curvature. In section 5 we define quasi-continuous linking curves, unequivocal sets and synthesis manifolds, and prove our Main Theorem A. In section 6 we collect useful results about the exponential map for a generic metric. In section 7 we define conjugate descending curves, prove that they are unbeatable, define finite conjugate linking curves (FCLCs), and prove that they can be built for a generic metric using the linking curve algorithm.

The results in this paper are mostly included in the author's thesis [An], but they have been reorganized to make it more clear and more general, and a few short but powerful results have been added. We warn the reader of that document that some definitions have changed.

1.1. Acknowledgments. We thank Yanyan Li and Biao Yin, who introduced the author to the Ambrose conjecture. We also thank Luis Guijarro and James Hebda for their support and suggestions.

We wish to thank the referee for a detailed inspection of the paper, which uncovered a mistake in lemma 7.11. The lemma was fixed according to suggestions by the same referee.

2. Notation

\( M \) is an arbitrary Riemannian manifold, \( p \) a point of \( M \), \((M_1,p_1)\) and \((M_2,p_2)\) are two Riemannian manifolds with \( L \)-related curvature.

Let \( e_1 \) stand for \( \exp_{p_1} \) and \( e_2 \) for \( \exp_{p_2} \circ L \).

We denote by \( \text{Cut}_p \), the cut locus of \( M \) with respect to \( p \) (see chapter 5 of [CE] for definitions and basic properties). Let us define also the injectivity set \( O_p \subset T_pM \), consisting of those vectors \( x \) in \( T_pM \) such that \( d(\exp_p(tx),p) = t \) for all \( 0 \leq t \leq 1 \), and let \( T\text{Cut}_p = \partial O_p \) be the tangent cut locus. The set \( T\text{Cut}_p \) maps onto \( \text{Cut}_p \) by \( \exp_p \).

In our proof, we will make heavy use of a subset of \( T_pM \) bigger than the injectivity set, defined as follows. We define the functions \( \lambda_k : S_{p_1}M_1 \to \mathbb{R} \), where \( \lambda_k(x) \) is the parameter \( t_\ast \) for which \( t_\ast \cdot x \) is the \( k \)-th conjugate point along \( t \to \exp_p(tx) \) (counting multiplicities). These functions were shown to be Lipschitz in [IT00]. In [CR], it was shown that \( \lambda_1 \) is semiconcave. Together with L. Guijarro, the author proved in [AGH] that these functions are also Lipschitz in Finsler manifolds. We define \( V_1 \) as the set of tangent vectors \( x \) such that \( |x| \leq \lambda_1(x/|x|) \), a set with Lipschitz boundary. It is well known that \( O_p \subset V_1 \).

3. James Hebda's tree formed curves

3.1. Tree formed curves. Let \( AC_p(X) \) be the space of absolutely continuous curves in the manifold \( M \) starting at \( p \), with the topology defined as in [H87].

Affine development \( \text{Dev}_p : AC_p(M) \to AC_0(T_pM) \) for absolutely continuous curves is also defined in that reference, extending the common definition in [KN].
Tree-formed curves are similar to the tree-like paths of the theory of rough paths (see [HL]), but we will stick to the original definition in [H87]. The model for a tree-formed curve $u : [0, 1] \to M$ is an absolutely continuous curve that factors through a finite topological tree $\Gamma$. In other words, $u = \bar{u} \circ T$ for some continuous map $\bar{u} : \Gamma \to M$ and a map $T : [0, 1] \to \Gamma$ with runs through each edge of the tree exactly twice, in opposite directions. The tree $\Gamma$ is the topological quotient of the unit interval by the map $T$.

The definition also allows for a “partial identification”.

**Definition 3.1.** Let $T : [0, 1] \to \Gamma$ be a quotient map, and $u : [0, 1] \to M$ an absolutely continuous curve that factors through $\Gamma$ (so that $u = \bar{u} \circ T$ for some $\bar{u} : \Gamma \to M$). Then $u$ is **tree-formed** with respect to $T$ if and only if

$$\int_{t_1}^{t_2} \varphi(T(s))(u'(s))ds = 0$$

for any continuous $1$-form $\varphi : \Gamma \to T^*M$ along $\bar{u}$ ($\varphi(x) \in T^*_{\bar{u}(x)}M$ for $x \in \Gamma$) and for any $t_1$, $t_2$ such that $T(t_1) = T(t_2)$.

If $T(0) = T(1)$, we say the curve is **fully tree-formed**.

If $\Gamma = [0, 1]$ and $T$ is the identity, the definition is empty, and we will rather use the definition saying that a certain curve $u$ is tree-formed with respect to an identification map with $T(t_1) = T(t_2)$ as another way to say that $u|_{[t_1,t_2]}$ is a fully tree-formed curve.

**Theorem 3.2 ([H87, Theorem 3.3]).** Tree formedness is preserved by affine development:

- If $u \in AC_p(M)$ is tree formed for an identification $T$, then $Dev_p(u) \in AC(T_pM)$ is also tree formed for the same $T$.
- If $v \in AC(T_pM)$ is tree formed for an identification $T$, then $Dev^{-1}_p(v) \in AC_p(M)$ is also tree formed for the same $T$.

### 3.2. The proof of the Ambrose conjecture for surfaces by James Hebda

In this section we give a sketch of the paper [H87], which is important for later sections. The reader can find more details in that paper.

Theorem 1.2 shows that $\varphi = \exp_2 \circ L \circ (\exp_1|_{U_{p_1}})^{-1}$ is an isometric immersion from $U_{p_1} = M_1 \setminus \text{Cut}_{p_1}$ into $M_2$. The starting idea is to prove that whenever a point in $\text{Cut}_{p_1}$ is reached by two geodesics $\gamma_1$ and $\gamma_2$, meaning that $e_1(\gamma_1'(0)) = e_1(\gamma_2'(0))$, then $e_2(\gamma_1'(0)) = e_2(\gamma_2'(0))$. Then the formula $\varphi(p) = e_2(x)$, for any $x \in (O_p \cup \text{TCut}_{p}) \cap c^{-1}(p)$ gives a well-defined map $\varphi : M_1 \to M_2$ that is an isometry at least on $U_{p_1}$.

It is a well-known fact that the cut locus looks specially simple at the cleave points, for which there are exactly two minimizing geodesics from $p$, and both are non-conjugate (see [Oz], for example). Near a cleave point, the cut locus is a smooth hypersurface. The rest of the cut locus is more complicated, but we know that $\mathcal{H}^{n-1}(\text{Cut} \setminus \text{Cleave}) = 0$ and, indeed, that $\text{Cut} \setminus \text{Cleave}$ has Hausdorff dimension at most $n - 2$, for a smooth Riemannian manifold.

An isometric immersion from $M_1 \setminus A$ into a complete manifold, for any set $A$ such that $\mathcal{H}^{n-1}(A) = 0$, can be extended to an isometric immersion from $M_1$. Thus, it only remains to show that, for a cleave point $q = e_1(x_1) = e_1(x_2)$, we have $e_2(x_1) = e_2(x_2)$.

The way to do this is to find for each cleave point $q$ as above, a curve $Y$ whose image is contained in $\text{TCut}_{p_1}$ (in the metric space $AC(T_pM)$ of absolutely continuous curves) such that $Y(0) = x_1$, $Y(1) = x_2$, and $e_1 \circ Y : [0, 1] \to M_1$ is fully tree-formed.
James Hebda proves in lemma 4.1 of [H87] that this implies that \( e_2(x_1) = e_2(x_2) \).

We extend that lemma in our lemma 4.4, so that it is simpler to use, and more general. This is an important concept for this paper:

**Definition 3.3.** A linking curve is an absolutely continuous curve \( Y : [0, l] \to T_pM \) such that \( e_1 \circ Y \) is a fully tree formed curve.

**Definition 3.4.** A cut locus linking curve (CutLC) is a linking curve \( Y \) whose image is contained in the tangent cut locus, so that \( e_1 \circ Y \) is a fully tree formed curve with image contained in the cut locus.

J. Hebda’s way to find the cut locus linking curves works only in dimension 2. Let \( S_pM_1 \) be the set of unit vectors in \( T_pM_1 \) parametrized with a coordinate \( \theta \), and define \( \rho : S_pM_1 \to \mathbb{R} \) as the first cut point along the ray \( t \to tv \) for \( t > 0 \), and \( \rho(\theta) = \infty \) if there is no cut point on the ray. The tangent cut locus is parametrized by \( \theta \to (\rho(\theta), \theta) \), defined on the subset of \( S_pM_1 \) where \( \rho \) is finite. Given a cleave point \( q = e_1(x_1) = e_1(x_2) \), with \( x_i = (\rho(\theta_i), \theta_i) \), then \( \rho \) is finite in at least one of the two arcs in \( S_pM_1 \) that join \( \theta_1 \) and \( \theta_2 \), which we write \( [\theta_1, \theta_2] \). Then the curve \( Y(\theta) = (\rho(\theta), \theta) \) defined in \( [\theta_1, \theta_2] \), satisfies the previous hypothesis.

It is important that \( Y \) be absolutely continuous, which follows once it is proved that \( \rho \) is. This was shown independently in [H94] and [I], and later generalized to arbitrary dimension in [IT00].

### 3.3. Difficulties to extend the proof to dimension higher than 2

In dimension higher than 2, there is no natural choice for a cut locus linking curve joining the speed vectors of the two minimizing geodesics that reach a cleave point. Indeed, we can show that for some manifolds it is impossible to do so:

**Theorem 3.5.** Let \( M \) be a smooth manifold of dimension 3, and \( p \) a point in \( M \). There is an open subset of the set of smooth Riemannian metrics such that for any cleave point \( q = \exp_p(x_1) = \exp_p(x_2) \) from \( p \), there is no CutLC whose extrema are \( x_1 \) and \( x_2 \).

**Proof.** Using the main theorem in [We2], there is a metric \( g_1 \) on \( M \) whose tangent cut locus from \( p \) does not contain conjugate points (in other words, all segments with \( P \) as one endpoint are non-conjugate). Any metric sufficiently close to \( g_1 \) will also have disjoint cut and conjugate locus.

Let \( q = \exp_p(x_1) = \exp_p(x_2) \) be a cleave point and \( Y : [0, L] \to T_pM \) be a CutLC joining \( x_1 \) and \( x_2 \), where \( T : [0, 1] \to \Gamma \) is the identification map of \( \exp_p \circ Y \).

We can change the parameter \( t : [0, l] \to [0, L] \) so that \( (u \circ t)(s) \) has unit speed (the identification is reparametrized accordingly \( (T \circ t)(s) \)). We simply assume that the speed vector of \( u \) has norm one wherever it is defined and keep the letter \( t \) for the parameter.

Let \( t_\ast = L/2 \) The following possibilities may occur:

1. There is some \( t_0 \neq t_\ast \) such that \( T(t_0) = T(t_\ast) \).
2. There is some \( \varepsilon > 0 \) such that any point in \( [t_\ast - \varepsilon, t_\ast + \varepsilon] \) is not identified to other points by \( T \).
3. There is a sequence \( t_n \to t_\ast \) and a sequence \( r_n \to t_\ast \) such that \( r_n \neq t_n \) and \( T(r_n) = T(t_n) \).
4. There is some \( \varepsilon > 0 \), a sequence \( t_n \to t_\ast \) and a sequence \( r_n \to t_\ast \) such that \( |r_n - t_n| > \varepsilon \) and \( T(r_n) = T(t_n) \).

The second option is in contradiction with the hypothesis. The reason is that for any continuous 1-form \( \varphi \) along \( u|_{[t_\ast - \varepsilon, t_\ast + \varepsilon]} \), we must have

\[
\int_0^1 \varphi(s)(u'(s))ds = \int_{t_\ast - \varepsilon}^{t_\ast + \varepsilon} \varphi(s)(u'(s))ds = 0
\]
but since $T$ does not identify points in $[t_\ast - \varepsilon, t_\ast + \varepsilon]$ to other points, we can choose the continuous 1-form $\varphi |_{[t_\ast - \varepsilon/2, t_\ast + \varepsilon/2]}$ freely, and this implies that $u'$ is null on that interval, which is in contradiction with having unit speed.

The fourth option implies the first, since a subsequence of the $r_n$ will converge to some $r_0$ which is not in $(t_\ast - \varepsilon, t_\ast + \varepsilon)$ and so is different from $t_\ast$. If the third option holds, since $T(r_n) = T(t_n)$ implies $u(r_n) = u(t_n)$, or $\exp_p(Y(t_n)) = \exp_p(Y(r_n))$, any neighborhood of $Y(t_\ast)$ contains a pair of different points with the same image, which implies that $\exp_p$ is not a local diffeomorphism at $Y(t_\ast)$, in contradiction with the fact that the image of $Y$ is contained in the tangent cut locus, which does not contain conjugate points.

Only the first option remains. In this case, it follows from definition 3.1 that the curve $Y|_{[t_0, t_\ast]}$ is tree formed, for the identification $T|_{[t_0, t_\ast]}$ (if $t_\ast < t_0$, we restrict to $[t_\ast, t_0]$). The length of $[t_0, t_\ast]$ is smaller than $L/2$ and $Y|_{[t_0, t_\ast]}$ is also a CutLC. We can iterate the argument to get a sequence of nested closed intervals whose length decreases to 0. The point in the intersection of that sequence is a conjugate point, by a similar argument as in the third option above, and this is again a contradiction. \hfill $\square$

4. Affine development and tree formed curves

In this section we extend the main results of sections 3 and 4 in [H87].

For this whole section, let $(M_1, p_1)$ and $(M_2, p_2)$ be two manifolds with $L$-related curvature.

**Definition 4.1.** The local linear isometry induced by $x \in T_{p_1} M_1$ is defined by

$$I_x = P_{p_2} \circ L \circ P_{-\gamma_1}$$

where $\gamma_1$ is the geodesic on $M_1$ with $\gamma_1(0) = x$, $\gamma_2$ is the geodesic on $M_2$ with $\gamma_2(0) = L(x)$ and $P_n$ is the parallel transport along the curve $\alpha$.

**Remark.** Since parallel transport along $\gamma \in AC_p(M)$ depends continuously on $\gamma$ (see [H87, 6.1.6,3]), the map $x \mapsto I_x$ is continuous.

**Lemma 4.2.** Let $x \in T_{p_1} M_1$ be a regular point of $e_1$, and $O$ any neighborhood of $x$ in $T_{p_1} M_1$ such that $e_1|_{O}$ is injective. Let $f_x$ be the local isometry $e_2 \circ L \circ (e_1|_{O})^{-1}$ from $e_1(O)$ in $M_1$ to $e_2(L(O))$ in $M_2$. Then

$$I_x = d_{e_1(x)} f_x$$

**Proof.** See lemma 1.35 in [CE] \hfill $\square$

We define $\Dev_i : AC(M_i) \to AC(T_{p_i} M_i)$ as the affine development of absolutely continuous curves in $M_i$ based at $p_i$, for $i = 1, 2$.

**Lemma 4.3.** Let $Y : [0, l] \to V_1$ be an absolutely continuous curve such that $Y(0) = 0$. Define $u = e_1 \circ Y$ and $v = e_2 \circ L \circ Y$. Then:

1. $I_Y(t) = P_p \circ L \circ P_{-u}$.
2. At any point $t$ where $u'(t)$ and $v'(t)$ are defined, $I_Y(t)(u'(t)) = v'(t)$.
3. $v = (Dev_2)^{-1} \circ L \circ Dev_1(u)$.

**Proof.** We first assume that the image of the curve $Y$ is contained in the interior of $V_1$. Notice that if $Y$ is a radial line, the first statement is just the definition of $I_Y(t)$. Define:

$$J = \left\{ t : L \circ P_{-v|_{[0, t]}} \circ P_{-u|_{[0, t]}} \circ I_Y(t) \right\}$$

We will prove that $J = [0, l]$ by proving it is open and closed.
If $[0, t) \subset J$, we take a sequence $t_j \nearrow t$ to find by continuity of parallel transport and $L \circ P_{-w[0,t]} = P_{-w[0,t]} \circ I_Y(t_j)$ that $L \circ P_{-w[0,t]} = P_{-w[0,t]} \circ I_Y(t)$, so closedness follows. Assume now $[0, l] \subset J$. $Y(t)$ is in the interior of $V_1$ by hypothesis, so there is $\varepsilon > 0$ and a neighborhood $O$ of $Y([0, \varepsilon, \varepsilon + \varepsilon])$, and an isometry $\varphi : e_1(O) \to e_2(O)$ with $\varphi \circ e_1 = e_2$. Then for any $t < t_1 < t + \varepsilon$

$$P_{-w[t,t_1]} = P_{-w[0,t]} \circ P_{-w[t,t_1]}$$

and similarly for $v$. By hypothesis,

$$L \circ P_{-w[0,t]} = P_{-v[0,t]} \circ I_Y(t).$$

We have $\varphi \circ e_1 = e_2 \circ \varphi$ so, as parallel transport commutes with isometries, we have

$$P_{-v[t,t_1]} \circ I_Y(t_1) = \frac{d_\varphi(t)}{d_\varphi} P_{-u[t,t_1]} \circ I_Y(t)$$

It follows that $[t, t + \varepsilon] \subset J$, so $J$ is open and the first item follows when the image of $Y$ is contained in the interior of $V_1$.

We next prove that $I_Y(O)(v(t')) = v(t)$ for any $t \in [0, l]$ such that $Y(t)$ is defined. This is clear when $Y(t)$ is in the interior of $V_1$, because, for an isometry $f$ defined in a neighborhood $O$ of $Y(t)$ where $e_1(O)$ is injective and $Y((t - \varepsilon, t + \varepsilon)) \subset O$, we have $v|_{[t - \varepsilon, t + \varepsilon]} = f \circ u|_{[t - \varepsilon, t + \varepsilon]}$ and $I_Y(t) = d_\varphi(t)f$.

We now deal with curves whose image intersects the boundary of $V_1$. Such a curve $Y$ can be approximated in $AC_0(T, M)$ by curves $\gamma_k(t) = (1 - \frac{1}{k})Y(t)$ that stay in $int(V_1)$, so that $Y_k - Y_{AC_0(T, M)} \to 0$. Taking limits as $k$ goes to infinite, the first item follows by continuity of parallel transport, the second because $x \to I_x$ is continuous and by a standard use of the chain rule.

The third claim is equivalent to Dev$_2(v) = L \circ$ Dev$_1(v)$, and this follows by integration if we prove

\[(4.1) \quad L(P_{-w[0,t]}(v(t))) = P_{-w[0,t]}(v(t)) \quad \text{for almost every } t \in [0, 1] \]

But this clearly follows from the two earlier items.

\[\square\]

**Remark.** We will only need the above lemma, but it is worth mentioning that the above also holds for a more general path $Y : [0, 1] \to T_p, M_1$. There are (at least) two ways to do it:

1. As the set of singular points of $e_1$ is a Lipschitz multi-graph (see Theorem A of [IT00]), we can approximate $Y$ by paths that are transverse to the set of conjugate points. The proof that $J$ is open at an intersection point $t_0$ consists of gluing two intervals $(t_0 - \varepsilon, t_0)$ and $(t_0, t_0 + \varepsilon)$, where $e_1$ is not singular, and continuity of $I_X$ makes the gluing possible.

2. The above approach is straightforward but poses some technical difficulties. An alternative approach is to approximate the metric by a generic one and $Y$ by a generic path in $T_p, M_1$. The manifold $M_1$ with the new metric will no longer have curvature $L$-related to that of $M_2$, but the local maps $X$ can still be defined as a continuous family of linear isomorphisms. The path $Y$ will cross the set of conjugate points transversally, and only at $A_2$ singularities, which will simplify the proof.

**Lemma 4.4.** Let $Y : [0, l] \to T_p, M_1$ be a linking curve whose image is contained in $V_1$. Then:

- $e_2(Y(0)) = e_2(Y(l))$. 

7
\[ I_Y(0) = I_Y(l) \]

**Proof.** Let \( r : [0, 1] \to V_1 \) be the radial path from 0 to \( Y(0) \). Define \( u = e_1(r * \alpha) \) and \( v = e_2(r * \alpha) \), which are absolutely continuous curves defined on the interval \([0, l + 1] \). Then \( v = \text{Dev}_2^{-1} \circ L \circ \text{Dev}_1(u) \) by the previous lemma.

By its definition, \( u \) is tree-formed for an identification map \( T \) with \( T(1) = (l + 1) \), so it follows that \( v \) also is, by theorem 3.2. It follows that \( e_2(Y(0)) = v(1) = v(l + 1) = e_2(Y(l)) \).

For the second claim, observe that:
\[
I_Y(l) = P_{e_2 \circ r * Y} \circ L \circ P_{-e_1 \circ (r * Y)} = P_{e_2 \circ r} \circ P_{e_2 \circ Y} \circ L \circ P_{-e_1 \circ Y} \circ P_{-e_1 \circ r}
\]

We can simplify this expression, since both \( e_1 \circ Y \) and \( e_2 \circ Y \) are fully tree formed:
\[
I_Y(l) = P_{e_2 \circ r} \circ L \circ P_{-e_1 \circ r}
\]
and that is the definition of \( I_Y(0) \).

\[ \Box \]

5. **Synthesis**

For any point \( x \in \text{int}(V_1) \), the Cartan lemma provides an isometry from a neighborhood of \( e_1(x) \) to one of \( e_2(x) \). Our plan is to collect those local mappings to build a covering space.

**Definition 5.1.** A Riemannian covering is a local isometry that is also a covering map (see [O] for a motivation).

**Definition 5.2.** Let \( A \) be a topological manifold, \( X_1, X_2 \) are Riemannian manifolds, and \( e_1 : A \to X_1, e_2 : A \to X_2 \) are continuous surjective maps.

A **synthesis** of \( X_1 \) and \( X_2 \) through \( e_1 \) and \( e_2 \) is a Riemannian manifold \( X \), together with a continuous map \( e : A \to X \), and Riemannian coverings \( \pi_i : X \to X_i \) for \( i = 1, 2 \) such that \( \pi_1 \circ e = e_1 \).

\[
\begin{array}{c}
\text{A} \\
\downarrow e_1 \\
\downarrow e \\
\downarrow \pi_1 \\
\downarrow \pi_2 \\
\downarrow e_2 \\
X_1 \quad X_2
\end{array}
\]

If \( \pi_i \) are only local isometries, then \( X \) is called a **weak synthesis**.

We will use this extension of the Ambrose conjecture in terms of synthesis manifolds (see section 3 in [O]).

**Conjecture 5.3 (Ambrose Conjecture following O’Neill).** Let \( (M_1, p_1) \) and \( (M_2, p_2) \) be two Riemannian manifolds with \( L \)-related curvature. Define \( e_1 = \exp_{p_1} \) and \( e_2 = L \circ \exp_{p_2} \).

Then there is a synthesis \( M \) of \( M_1 \) and \( M_2 \) through \( e_1 \) and \( e_2 \), and a point \( p \in M \) such that \( \pi_i(p) = e_i(0) \) for \( i = 1, 2 \).

In particular, if \( M_1 \) and \( M_2 \) are simply-connected, then \( \pi_2 \circ \pi_1^{-1} : (M_1, p_1) \to (M_2, p_2) \) is the unique isometry whose tangent at \( p_1 \) is \( L \).

If \( e_1 \) has no singularities, we can pull the metric from \( M_1 \) onto \( T_{p_1} M_1 \) and the desired Riemannian coverings are \( \pi_1 = e_1 \) and \( \pi_2 = e_2 \). In the presence of singularities, the idea is to build the synthesis as a quotient of \( A = V_1 \) that identifies pairs of points with the same image by both \( e_1 \) and \( e_2 \).
5.1. Unequivocal points and linked points.

**Definition 5.4.** We say that an open set \( O \subset V_1 \) is *unequivocal* if and only if \( e_1(O) \) is an open set, and there is an isometry \( \varphi_O : e_1(O) \to e_2(O) \) such that \( \varphi_O \circ e_1|_O = e_2|_O. \)

**Definition 5.5.** We say \( x \in V_1 \) is *unequivocal* if there is a sequence of unequivocal sets \( W_n \) such that \( e_1(W_n) \) is a neighborhood base of \( e_1(x) \).

**Remark.** The above definition allows for points \( x \in V_1 \) that are not isolated in \( e_1^{-1}(e_1(x)) \). This is important if we want a definition of sutured manifold that may hold for all Riemannian manifolds.

We plan to identify points in \( T_p M \) that are joined by a linking curve. However, in order to build a quotient space, we need some kind of openness as in Lemma 5.11.

In order to define a *relaxed version* of the above relation for which Lemma 5.11 holds, we need to allow curves with some sort of “controlled” discontinuities.

**Definition 5.6.** A quasi-continuous linking curve is a bounded curve \( Y : [0, l] \to V_1 \) such that:

1. The composition \( e_1 \circ Y \) is an absolutely continuous tree formed curve.
2. For every point \( t_0 \), there is an \( \varepsilon > 0 \) such that either \( Y|_{[0,l]\cap[|t_0-\varepsilon, t_0+\varepsilon]} \) is absolutely continuous, or its image is contained in an unequivocal set \( W. \)

**Definition 5.7.** Two points \( x, y \in V_1 \) are *strongly linked* (by the curve \( Y \)) iff there is a linking curve \( Y : [0, l] \to V_1 \) such that \( Y(0) = x \) and \( Y(l) = y. \)

Two points \( x, y \in V_1 \) are *linked* \((x \leftrightarrow y)\) if and only if there is a quasi-continuous linking curve \( Y : [0, l] \to V_1 \) such that \( x \) is the limit of \( Y(t_j) \) for some sequence \( t_j \nearrow 0 \) and \( y \) is the limit of \( Y(t_j) \) for some sequence \( t_j \nearrow l. \)

5.2. Main properties of unequivocal sets and linked points. In this section we extend some results from section 4.

**Lemma 5.8.** Let \( W \) be any unequivocal neighborhood of \( x \in T_p M_1 \). Let \( \varphi : e_1(W) \to e_2(W) \) be the local isometry such that \( \varphi \circ e_1 = e_2. \) Then

\[ I_x = d_{e_1(x)} \varphi \]

In particular, it depends only on \( e_1(x) \).

**Proof.** If \( x \) is a regular point of \( e_1 \), we know from lemma 4.2 that \( I_x = d_{e_1(x)} f_x. \)

Both \( f_x \) and \( \varphi \) are isometries that agree on the open set \( e_1(U) \), for an open set \( U \subset W \) such that \( e_1 \) is injective when restricted to \( U \). Thus \( f_x \) and \( \varphi \) agree on \( U \) and the result follows.

For a conjugate point \( x \), we take limits of a sequence of regular points, since \( I_x = d_{e_1(z)} \varphi \) for any regular point \( z \in W \), and \( z \to I_x \) is continuous. \( \square \)

**Lemma 5.9.** Let \( Y : [0, l] \to V_1 \) be a bounded curve such that:

- \( Y(0) = 0. \)
- \( u = e_1 \circ Y \) and \( v = e_2 \circ Y \) are absolutely continuous.
- For every point \( t_0 \), there is an \( \varepsilon > 0 \) such that either \( Y|_{[0-t_0-\varepsilon, t_0+\varepsilon]} \) is absolutely continuous, or its image is contained in an unequivocal set \( W. \)

Then:

1. \( I_Y(t) = P_x \circ L \circ P_u. \)
2. At any point \( t \) where \( u(t) \) and \( v(t) \) are defined, \( I_Y(t)(u(t)) = v(t). \)
3. \( v = (\text{Dev}_2)^{-1} \circ L \circ \text{Dev}_1(u). \)

9
Proof. Define $J$ as in lemma 4.3:

$$J = \left\{ t : L \circ P_{-u|[0,1]} = P_{-v|[0,1]} \circ I_{Y(t)} \right\}$$

We know that $I_{Y(t)}$ is continuous at every $t$ where $Y$ is continuous. By the last hypothesis and the previous lemma, $I_{Y(t)}$ is also continuous at points where $Y$ is discontinuous. It follows that $J$ is closed and it remains to prove that it is open. Let $t_0 \in J$.

If $Y|_{[t_0-\varepsilon,t_0+\varepsilon]}$ is absolutely continuous and its image is contained in $\text{int}(V)$, we prove that $[t_0, t_0+\varepsilon] \subset J$ as we did in Lemma 4.3. If $Y|_{[t_0-\varepsilon,t_0+\varepsilon]}$ is contained in an unequivocal set $W$, there is an isometry $\varphi : e_1(W) \to e_2(W)$ such that $\varphi \circ u|_{[t_0-\varepsilon,t_0+\varepsilon]} = v|_{[t_0-\varepsilon,t_0+\varepsilon]}$ so, as parallel transport commutes with isometries, and using Lemma 5.8, we have, for $t_0 < t_1 < t_0 + \varepsilon$

$$P_{-v|[0,t_1]} \circ I_{Y(t_1)} = P_{-v|[0,t_1]} \circ d_{u(t_1)} \varphi = d_{u(t_0)} \varphi \circ P_{-u|[0,t_1]} = I_{Y(t_0)} \circ P_{-u|[0,t_1]}$$

and we deduce that $[t_0, t_0+\varepsilon] \subset J$ as in Lemma 4.3.

Finally, if $Y|_{[t_0-\varepsilon,t_0+\varepsilon]}$ is absolutely continuous but its image is not contained in $\text{int}(V)$, we define for every $k$ a modified curve:

$$Y_{t_0,\varepsilon,k}(t) = \begin{cases} Y(t) & t \leq t_0 \\ (1 - \frac{t-t_0}{\varepsilon}) Y(t) & t_0 < t \leq t_0 + \varepsilon \\ (1 - \frac{t-t_0}{\varepsilon}) Y(t) & t_0 + \varepsilon < t \end{cases}$$

Since $Y_{t_0,\varepsilon,k}|_{[t_0-\varepsilon,t_0+\varepsilon]}$ is absolutely continuous and its image is contained in $\text{int}(V)$, we learn that for any $t < t_0 + \varepsilon$

$$L \circ P_{-u_k|[0,1]} = P_{-v_k|[0,1]} \circ I_{Y_{t_0,\varepsilon,k}(t)}$$

and since $Y_k$ converges to $Y$ in $AC_0(T_\partial M)$, we have proven that $[t_0, t_0 + \varepsilon] \subset J$.

We now turn to the proof that $I_{Y(t)}(u'(t)) = v'(t)$ for almost every $t \in [0,1]$. We have already shown this if $Y|_{[t-\varepsilon,t+\varepsilon]}$ is absolutely continuous and $Y(t)$ belongs to $\text{int}(V)$. If $Y|_{[t-\varepsilon,t+\varepsilon]}$ is absolutely continuous but $Y(t)$ does not belong to $\text{int}(V)$, we construct the same curves $Y_{t_0,\varepsilon,k}$: we know that for every $t \in [0,1]$ for which $u'(t)$ and $v'(t)$ are defined, we have $I_{Y_{t_0,\varepsilon,k}(t)}(u'(t)) = v'(t)$. Since $I_{Y_{t_0,\varepsilon,k}(t)}$ converges to $I_{Y(t)}$ as $k$ goes to infinity, it follows that $I_{Y(t)}(u'(t)) = v'(t)$.

Finally, if $Y|_{[t-\varepsilon,t+\varepsilon]}$ is contained in an unequivocal set $W$, let $\varphi : e_1(W) \to e_2(W)$ be the isometry in the definition of unequivocal set. By lemma 5.8

$$I_{Y(t)}(u'(t_0)) = (d_{u(t_0)} \varphi)(u'(t_0)) = (\varphi \circ u)'(t_0) = v'(t_0)$$

The third item follows from the first and the second as in Lemma 4.3.

\[\square\]

Lemma 5.10. Let $x, y \in V$ be linked points. Then

1. $e_1(x) = e_1(y)$
2. $e_2(x) = e_2(y)$
3. $I_x = I_y$

Proof. Let $Y$ be a quasi-continuous linking curve that links $x$ and $y$.

The first part is obvious from the definition because $e_1(x)$ and $e_1(y)$ are the extrema of the fully tree formed curve $e_1 \circ Y$.

The second and third parts follow as in 4.4, because the curve $r \ast Y$ satisfies the hypothesis of lemma 5.9.

\[\square\]
Lemma 5.11. Let \( x \in V_1 \) be linked to some \( z \in W \) for an unequivocal set \( W \). Then there is a neighborhood \( U \subset V_1 \) of \( x \) and such that every \( y \in U \) is linked to some \( w \in W \).

Proof. We define \( U \) as the connected component of \( e^{-1}_1(e_1(W)) \cap V_1 \) that contains \( x \). For \( y \in U \), we want to prove that \( y \) is linked to some \( w \in W \).

Let \( Z : [0, l] \) be a quasi-continuous linking curve that joins \( x \) and \( z \). We want to find curves \( A : [0, a] \to U \) and \( B : [0, a] \to W \) such that \( e_1 \circ (A * Z * B) \) is fully tree formed, \( A(0) = y \). This holds if we choose an arbitrary absolutely continuous path \( A \) with \( A(0) = y \) and \( A(a) = x \), and \( B(t) \) so that \( e_1(B(t)) = e_1(A(a - t)) \). We may not be able to choose an absolutely continuous path \( B \), but since its image is contained in \( W \), \( Y \) is a quasi-continuous linking curve.

 Remark. Such a choice of \( B \) is not very elegant, and requires using the axiom of choice. The interested reader can find a more constructive alternative in the proof of Proposition 5.15.

5.3. Construction of the synthesis.

Theorem 5.12. Let \( M_1, M_2 \) be Riemannian manifolds with \( L \)-related curvature, such that for every \( x \in V_1 \) is linked to some unequivocal point \( y \in V_1 \).

Then there is a weak synthesis of \( M_1 \) and \( M_2 \) through \( e_1 \) and \( e_2 \).

Proof. Define a set \( M \) as a quotient by the equivalence relation:

\[
M = (A/\sim)
\]

Let \( e : A \to M \) be the projection map. We define maps \( \pi_i : M \to M_i \) by \( \pi_i([x]) = e_i(x) \). Both maps are well defined by lemma 5.10.

We give \( M \) a topology where the basic open sets are \( [W] = \{[x], x \in W\} \), for unequivocal open set \( W \).

- By hypothesis, every point belongs to some open set, so this is a good basis for a topology.
- \( e \) is continuous at every point \( x \in A \): Let \( [W] \) be a basis open neighborhood of \([x]\), for \( W \) unequivocal. There is \( z \in W \) such that \( x \sim z \), by a quasi-continuous linking curve \( \rho \). By lemma 5.11, there is an open neighborhood \( U \) such that any point in \( U \) is linked to some point in \( W \). Thus \( U \) is contained in \( e^{-1}(W) \).
- \( \pi_1([W]) \) is injective for any basis open set \([W]\): Let \( [x_1], [x_2] \in [W] \) be such that \( \pi_1([x_1]) = \pi_1([x_2]) \). This means \( e_1(x_1) = e_1(x_2) \). We can assume \( x_1, x_2 \in W \), which implies \([x_1] = [x_2] \) (using a curve \( Y \) that only takes the values \( x_1 \) and \( x_2 \)).
- \( \pi_2([W]) \) is injective for any basis open set \([W]\): If \( \pi_2([x_1]) = \pi_2([x_2]) \) for \( x_1, x_2 \in W \), it follows that \( e_2(x_1) = e_2(x_2) \), which implies \( \varphi_W(e_1(x_1)) = \varphi_W(e_1(x_2)) \), for the isometry \( \varphi_W \) in the definition of unequivocal set, which implies \( e_1(x_1) = e_1(x_2) \) and \([x_1] = [x_2] \).
- \( \pi_1([W]) \) is continuous, for a basis set \([W]\): let \( U \) be an open subset of \( \pi_1([W]) = e_1(W) \). Then \( (\pi_1([W]))^{-1}(U) = \pi_1^{-1}(U) \) is an open set, because \( e_1(W \cap e_1^{-1}(U)) = e_1(W \cap U) \) is an open set and \( W \cap e_1^{-1}(U) \subset W \), so \( W \cap e_1^{-1}(U) \) is unequivocal.
- \( \pi_2([W]) \) is continuous, for a basis set \([W]\): let \( U \) be an open subset of \( \pi_2([W]) = e_2(W) \), and let \( \varphi_W : e_1(W) \to e_1(W) \) be the isometry associated with \( W \). Then \( (\pi_2([W]))^{-1}(U) = \pi_2^{-1}(U) = e_2^{-1}(U) \). This is an open set, because \( e_1(W \cap e_2^{-1}(U)) = \varphi_W^{-1}(e_2(W \cap e_2^{-1}(U))) = e_1(W \cap e_2^{-1}(U)) \)
\[ \varphi_W^{-1}(\varepsilon(W) \cap U) = \varphi_W^{-1}(\varepsilon_W(e_1(W)) \cap U) \] is an open set and \( W \cap e_2^{-1}(U) \subset W \), so \( W \cap e_2^{-1}(U) \) is unequivocal.

- For a basis open set \([W]\), \( \pi_1([W]) = e_1(W) \) is open by hypothesis, and \( \pi_2([W]) = e_2(W) = \varphi(e_1(W)) \) is also open. Hence, \( \pi_i \) is open for \( i = 1, 2 \). Thus, \( \pi_i|_W \) is an homeomorphism onto its image.

- Since \( \pi_1 \) and \( \pi_2 \) are local homeomorphisms, we can use \( \pi_1 \), for instance, to give \( M \) the structure of a smooth Riemannian manifold, which trivially makes \( \pi_1 \) a local isometry. For an unequivocal set \( W \), with \( e_2|_W = \varphi \circ e_1|_W \), then \( \pi_2 \circ (\pi_1|_W)^{-1} = \varphi \) is an isometry from \( \pi_1([W]) = e_1(W) \) onto \( \pi_2([W]) = e_2(W) \), so \( \pi_2 \) is also a local isometry.

\[ \square \]

5.4. Compactness. In order to prove Theorem A, we still have to prove that \( \pi_1 \) and \( \pi_2 \) given by theorem 5.12 are covering maps. This requires some sort of “compactness” result, and using the extra hypothesis in the definition of a sutured manifold. We start with a general lemma:

**Lemma 5.13.** Let \( \exp_p : T_pM \to M \) be the exponential map from a point \( p \) in a Riemannian manifold \( M \). Then for any absolutely continuous path \( x : [0, t_0] \to T_pM \), the total variation of \( t \to |x(t)| \) is not greater than the length of \( t \to \exp_p(x(t)) \). In particular:

\[ |x(t_0)| - |x(0)| \leq \text{length}(\exp_p \circ x) \]

**Proof.** For an absolutely continuous path \( x \):

\[ \text{length}(\exp_p \circ x) = \int_0^{t_0} |(\exp_p \circ x)'| = \int_0^{t_0} |d\exp_p(x')| \]

The speed vector \( x' = \dot{x} + v \) is a linear combination of a multiple of the radial vector and a vector \( v \) perpendicular to the radial direction. By the Gauss lemma, \( |d\exp_p(x')| = |a^2 + |d\exp_p(v)|^2 |a| \). On the other hand, \( v \) is tangent to the spheres of constant radius, so:

\[ TV^{t_0}_0(|x|) = \int_0^{t_0} \frac{d}{dt}|x| = \int_0^{t_0} |a| \leq \text{length}(\exp_p \circ x) \]

\[ \square \]

Let us now come back to our hypothesis.

**Definition 5.14.** Let \((M_1, p_1)\) be saturated, and \((M_2, p_2)\) a manifold with L-related curvature. Let \( M \) be the weak synthesis obtained by application of Theorem 5.12 and \( p = e(0) \in M \).

Then the synthesis-distance to \( p \) is the function \( d : M \to \mathbb{R} \) given by

\[ d(q) = \inf_{x \in e^{-1}(q)} |x| \]

If we could prove that \( e \) is the exponential map of the Riemannian manifold \( M \) at the point \( p = e(0) \), it would follow that \( d \) is the distance to \( p \), and the following proposition would be trivial.

**Proposition 5.15.** The synthesis-distance \( d \) is a 1-Lipschitz function on \( M \):

\[ |d(q_2) - d(q_1)| \leq d_M(q_1, q_2) \]

**Proof.** Given \( q_1, q_2 \in M \) and \( \varepsilon > 0 \), there is a family of absolutely continuous paths \( \beta_k : [0, t_k] \to V_1 \), for \( k \) integer, with the following properties:
the curves \( \beta_k \) are parametrized so that \( e \circ \beta_k \) has unit speed. This is equivalent to asking that \( e \circ \beta_k \) has unit speed, since \( \pi_1 \) is a local isometry.

In particular, \( l_k = \text{length}(e \circ \beta_k) = \text{length}(e \circ \beta_k) \).

- \( \beta_1(0) \in e^{-1}(q_1) \) and \( |\beta_1(0)| < d(q_1) + \varepsilon/2 \).
- \( e(\beta_k(l_k)) \to q_2 \).
- for each \( k \): \( |\beta_{k+1}(0)| \leq |\beta_k(l_k)| \).
- \( \sum_{k=1}^{\infty} l_k < d_M(q_1, q_2) + \varepsilon/2 \).

The family of curves may be finite or infinite. We will assume the latter, since the former is strictly simpler.

From this and Lemma 5.13 it follows that

\[
|\beta_N(\ell_N)| = |\beta_1(0)| + \sum_{k=1}^{N-1} (|\beta_{k+1}(0)| - |\beta_k(0)|) + (|\beta_N(\ell_N)| - |\beta_N(0)|)
< |\beta_1(0)| + \sum_{k=1}^{N-1} (|\beta_k(l_k)| - |\beta_k(0)|)
< |\beta_1(0)| + \sum_{k=1}^{N-1} l_k
< d(q_1) + d_M(q_1, q_2) + \varepsilon
\]

Thus the points \( \beta_N(\ell_N) \) are bounded and a subsequence of them converge to some \( x \in V_1 \) which belongs to \( e^{-1}(q_2) \) and satisfies the same bound. This proves the result, and it remains to prove the claim.

We define \( C \) as the set of conjugate points for \( e_1 \) (the points where the exponential map \( e_1 \) is singular), and \( A_2 \) as the set of points where \( e_1 \) is a singular map, but the singularity is “of \( A_2 \) type”. The reader is refereed to subsection 6.2 and to definition 6.4 in particular for these and related definitions.

By [H87, 1.1], \( e_1(C \setminus A_2) \) has null \( H^{n-1} \)-measure. We define \( \mathcal{N} = e(C \setminus A_2) \). It follows that \( \mathcal{N} \) has null \( H^{n-1} \)-measure because \( \mathcal{N} \subset \pi^{-1}_1(e(C \setminus A_2)) \), and the image of a \( H^{n-1} \)-null set by a local isometry is also \( H^{n-1} \)-null.

Let \( R = d(q_1) + d_M(q_1, q_2) + \varepsilon \), and let \( B_0(R) \subset T_0 \mathcal{M} \) be the open ball of radius \( R \). The set \( A_2 \cap B_0(R) \) is a smooth \( n-1 \)-manifold (non-compact without boundary). Any \( x \in A_2 \) has a neighborhood \( U \) such that \( e_1(U \cap A_2 \cap B_0(R)) \) and \( e(U \cap A_2 \cap B_0(R)) \) are (isometric) smooth \( n-1 \)-manifolds.

We start the construction of \( \beta \) choosing a starting point \( \beta_1(0) \in e^{-1}(q_1) \) such that \( |\beta_1(0)| < d(q_1) + \varepsilon/2 \). The point \( \beta_1(0) \) may be singular (that is, \( \beta_1(0) \in \partial V_1 \)), in which case we start \( \beta_1 \) with a short straight path that reaches a new \( \beta_1(\varepsilon/4) \in \text{int}(V_1) \cap B_0(R) \) \( e^{-1}_1(e_1(C \cap B_0(R))) \). We can choose \( \beta_1 \) so that its derivative makes a positive angle with the kernel of the exponential, and this is all we need to change the parameter so that \( e_1 \circ \beta_1 \) has unit speed. So we assume that the length of \( e_1 \circ \beta_1 \) is \( \varepsilon/4 \).

By [H82, 4.3], if \( q_2 \notin \mathcal{N} \), we can find a curve \( c \) disjoint from \( \mathcal{N} \) joining \( e(\beta_0(\varepsilon/4)) \) with \( q_2 \) whose length is not greater than \( d_M(e(\beta_1(\varepsilon/4)), q_2) + \varepsilon/4 < d_M(q_1, q_2) + \varepsilon/2 \). We remark that [H82, 4.3] requires that \( M \) is complete, something that we have not proved yet. However, the proof of [H82, 4.3] is valid also without this hypothesis with minor modifications:

- let \( v \) be a path in \( M \) joining \( e(\beta_1(\varepsilon/4)) \) and \( q_2 \), of length smaller than \( d_M(e(\beta_1(\varepsilon/4)), q_2) + \varepsilon/8 \).
- find a finite partition \( 0 = t_0 < t_1 < \cdots < t_N \) of the domain of the curve so that two consecutive points \( v(t_i), v(t_{i+1}) \) lie in a strongly convex open set.
- choose points \( c(t_0) = v(t_0), c(t_N) = v(N) \) and \( c(t_i) \) using [H82, 4.2] (for \( K = \mathcal{N} \)) so that the length of \( c|_{[t_i, t_{i+1}]} \) is smaller than the length of \( v|_{[t_i, t_{i+1}]} \).

The resulting curve \( c \) does not intersect \( \mathcal{N} \) and has length smaller than \( \text{length}(v) + \varepsilon/8 < d_M(e(\beta_1(\varepsilon/4)), q_2) + \varepsilon/4 \).
If $q_2 \in \mathcal{N}$, we can use a similar procedure: take a curve $v : \left[\varepsilon/8, 0\right] \to M$ from a nearby point $v(\varepsilon/8)$ into $q_2$ of length smaller than $\varepsilon/8$ and split it by intervals of length $\varepsilon/2^{k+1}$ (we start from $k = 3$ for convenience). We can then replace $v$ by a broken geodesic $c$ that avoids $\mathcal{N}$ such that the length of the segment $c|_{[\varepsilon/2^k, \varepsilon/2^{k+1}]}$ is no more than $\varepsilon/2^k$. In this way we find a continuous curve $c$ of length smaller than $\varepsilon/4$ that joins $q_2$ to a point not in $\mathcal{N}$.

We also want a curve that is transverse to $c(A_2 \cap B_0(R))$. This is equivalent to being transverse to each of the countably many smooth manifolds $c(U \cap A_2 \cap B_0(R))$ that we mentioned before. Since transversality to a smooth manifold is a residual property [H, 3.2.1], and a countable intersection of residual sets is residual, and in particular dense, we can find a curve $u : [0, l] \to M$ joining $\beta_1(\varepsilon/4)$ and $c(\varepsilon/8)$ that is close to $c$ in the $C([0, l], M)$ topology, so that, in particular, the length of $u$ is not greater than $d_M(q_1, q_2) + \varepsilon/2$, that is transverse to $c(A_2 \cap B_0(R))$ and does not intersect $\mathcal{N}$ except possibly at the final point.

Assume that the intersection points of $u$ and $c(A_2 \cap B_0(R))$ cluster at a point $u(t_*) \neq u(l)$. Then there is a sequence of points $x_j \in A_2 \cap B_0(R)$ and times $t_j \to t_*$ such that $c(x_j) = u(t_j)$ converge to $u(t_*)$. Since the $x_j$ are bounded, there is a subsequence converging to $x_* \in C \cap B_0(R)$. If $x_* \in A_2$, since $u(t_*) = c(x_*)$, and $u$ is transverse to $c(U \cap A_2)$ at $t_*$, there is $\delta > 0$ such that $u|_{[t_* - \delta, t_* + \delta]}$ does not intersect $c(U \cap A_2)$, which is a contradiction with the fact that a subsequence of the $x_j$ converge to $x_*$. If $x_* \in C \setminus A_2$, the contradiction is with the fact that the image of $u$ does not intersect $\mathcal{N}$. Note however that it is perfectly possible that the intersection points of $u$ and $c(A_2 \cap B_0(R))$ cluster at the final point $u(l)$.

We have shown that the set of intersection points of $u$ and $c(A_2 \cap B_0(R))$ is discrete $0 < t_1 < \cdots < t_j < \cdots$ except at the limit $j \to \infty$, and is bounded by $l$.

Since $\beta_1(\varepsilon/4)$ is in $int(V_1)$, we can start a lift of $u$ from that point. Thus, the lift will stay in $B_0(R)$. The lift will stay in $int(V_1)$ up to $t_1$ since $u|_{[0, t_1]}$ does not intersect $c(0) \cap B_0(R)$, so we get a curve $\beta_2 : [0, l_2] \to V_1$ that may end in a point in $A_2 \cup int(V_1)$.

If $\beta_2(l_2)$ is in $A_2$, we can find a new unequivocal point $\beta_2(0)$ that is linked to $\beta_2(l_2)$ and with $|\beta_2(0)| < |\beta_2(l_2)|$. Since the point $e(\beta_2(0)) = e(\beta_2(l_2))$ belongs to the image of $u$ and is unequivocal, it can only be a nonsingular point, so we can start a new lift of $u|_{[t_2, t_3]}$, and so on. The claim follows easily.

\[ \square \]

\textit{Proof of Main Theorem A.} We only need to prove that the weak synthesis $M$ built using Theorem 5.12 is complete when $(M_1, p_1)$ is sutured. It is well known that a local isometry is a covering map when the domain is complete (see for example corollary 2 in [G]).

As mentioned in conjecture 5.3, this implies the original Ambrose conjecture when both manifolds are simply connected.

Let $q_n$ be a Cauchy sequence in $M$. Then there is $R > 0$ such that $d_M(q_n, q_1) < R$. Thanks to proposition 5.15, we can find $x_n \in e^{-1}(q_n) \cap B_{|q_n|+R}$. As $x_n$ is bounded, we can assume by passing to a subsequence that $x_n$ converges to some $x_0$, and then $q_n \to e(x_0)$. \[ \square \]

6. Generic exponential maps

A generic perturbation of a Riemannian metric greatly simplifies the types of singularities that can be found on the exponential map ([We], [K]) or the cut locus with respect to any point ([B77]). In [We], A. Weinstein showed that for a generic metric, the set of conjugate points in the tangent space near a singularity of order

14
$k$ is given by the equations:

\[
\begin{vmatrix}
  x_1 & x_2 & \ldots & x_k \\
  x_2 & x_{k+1} & \ldots & x_{2k-1} \\
  \vdots & \vdots & & \vdots \\
  x_k & x_{2k-1} & \ldots & x_{k(k+1)}
\end{vmatrix} = 0
\]

where $x_1, \ldots, x_n$ are coordinates in $T_{p_1}M_1$, and $k(k+1)/2 \leq n$. This is called a conical singularity.

In [B77], M. Buchner studied the energy functional on curves starting at $p_1$ and the endpoint fixed at a different point of the manifold, as a family of functions parametrized by the endpoint. The singularities of the exponential map can be detected as degenerate critical points of the energy functional with both endpoints fixed, so his results also apply to our setting. He also proved a multitransversality statement about this family of functions that we will comment on later, and then used this information to provide a description of the cut locus of a generic metric.

It is well known that an exponential map only has Lagrangian singularities. In [K], Fopke Klok showed that the generic singularities of the exponential maps are the generic singularities of Lagrangian maps. These singularities are, in turn, described by means of the generalized phase functions of the singularities. This is the approach most useful to our purposes. We also wish to mention [JM] for a different approach and generalizations of some of these results.

### 6.1. Generalized phase functions

A generalised phase function is a map $F : U \times \mathbb{R}^k \to \mathbb{R}$ such that $D_qF = \left( \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k} \right) : U \times \mathbb{R}^k \to \mathbb{R}^k$ is transverse to $\{0\} \subset \mathbb{R}^k$. We will use a result that relates generalized phase functions defined at $U \times \mathbb{R}^k$ and Lagrangian subspaces of $T^*U$.

**Proposition 6.1.** If $L \subset T^*U$ is a Lagrangian submanifold and $p \in L$, it is locally given as the graph of $\phi : C \to T^*U$, where $C = (D_qF)^{-1}(0)$ and $\phi(x,q) = (x, D_qF(x,q))$, for some generalized phase function $F$.

Furthermore, we can assume:

- $k = \text{corank}(L,p)$
- $F(0,0) = 0$
- $0 \in \mathbb{R}^k$ is a critical point of $F(0,\cdot) : \mathbb{R}^k \to \mathbb{R}$
- $\frac{\partial^2 F}{\partial q_i \partial q_j} = 0$ for all $i$ and $j$ in $1, \ldots, k$

**Proof.** This is found in section 1 of [K], specifically in proposition 1.2.4 and the comments in page 320 after proposition 1.2.6.

Given a germ of generalized phase function $F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, the Lagrangian map is built in this way: $D_qF$ is transverse to $\{0\}$, and we can assume the last $k$ coordinates are such that the derivative of $D_qF$ in those coordinates is an invertible matrix. Let us split the $x$ coordinates in $(y,z) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. Our hypothesis is that $D_qF$ is invertible.

The implicit equations $D_qF = 0$ defines functions $f_j : \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}$ such that, locally near $0$, $D_qF(y,f(y,q),q) = 0$.

**Definition 6.2.** A Lagrangian map $\lambda : L \to M$ is the composition of a Lagrangian immersion $i : L \to T^*M$ with the projection $\pi : T^*M \to M$ (a Lagrangian immersion is an immersion such that the image of sufficiently small open sets are Lagrangian submanifolds).

**Definition 6.3.** Two Lagrangian maps $\lambda_j = : L_j \to M_j$, with corresponding immersions $i_j : L \to T^*M$, $j = 1, 2$, are Lagrangian equivalent if and only if there
are diffeomorphisms $\sigma : L_1 \rightarrow L_2$, $\nu : M_1 \rightarrow M_2$ and $\tau : T^*M_1 \rightarrow T^*M_2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
L_1 & \xrightarrow{\iota_1} & T^*M_1 & \xrightarrow{\pi_1} & M_1 \\
\downarrow{\sigma} & & \downarrow{\tau} & & \downarrow{\nu} \\
L_2 & \xrightarrow{\iota_2} & T^*M_2 & \xrightarrow{\pi_2} & M_2
\end{array}
$$

and $\tau$ preserves the symplectic structure.

Lagrangian equivalence corresponds to equivalence of generalized phase functions (this is proposition 1.2.6 in [K]). Two generalized phase functions are equivalent if and only if we can get one from the other composing three operations:

1. Add a function $g(x)$ to $F$. This has no effect on the functions $f_j$.
2. Pick up a diffeomorphism $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and replace $F(x, q)$ by $F(G(x), q)$. If the map $G$ has the special form $G(x) = G(y, z) = (g(y), h(z))$, the effect is to replace the map $(y, q) \rightarrow (y, f(y, q))$ by $(y, q) \rightarrow (y, h^{-1}(f(g(y), q)))$.
3. Pick up a map $H : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ such that $D_pH$ is invertible, and replace $F(x, q)$ by $F(x, H(x, q))$. If the map $H$ does not depend on the $z$ variables, the effect is to replace the map $(y, q) \rightarrow (y, f(y, q))$ by $(y, q) \rightarrow (y, f(y, H(y, q)))$.

6.2. The singularities of a generic exponential map. Using theorem 1.4.1 in [K], we get the following result: fix a smooth manifold $M$, a point $p \in M$. For a residual set of metrics in $M$ the exponential map $T_pM \rightarrow M$ is nonsingular except at a set $C$, which is a smooth stratified manifold with the following strata (we describe the different singularities in some detail below):

- A stratum of codimension 1 consisting of folds, or Lagrangian singularities of type $A_2$.
- A stratum of codimension 2 consisting of cusps, or Lagrangian singularities of type $A_3$.
- Strata of codimension 3 consisting of Lagrangian singularities of types $A_4$ (swallowtail), $D_4^*$ (elliptical umbilic) and $D_4^+$ (hyperbolic umbilic).
- We do not need to worry about the rest, which consists of strata of codimension at least 4.

**Definition 6.4.** We define the sets $A_2$, $A_3$, etc as the set of all points of $V_1$ that have a singularity of type $A_2$, $A_3$, etc. We also define $C$ as the set of conjugate (singular) points and $NC$ as the set of non-conjugate (non-singular) points.

Thus, $C$ is a smooth hypersurface of $T_pM$ near a conjugate point of order 1 (including $A_2$, $A_3$ and $A_4$ points), and is diffeomorphic to the product of a cone in $\mathbb{R}^3$ with a cube near a conjugate point of order 2 (including $D_4^+$). The $A_2$ points are characterized as those for which the kernel of the differential of the exponential map is a vector line transverse to the tangent plane to $C$.

Furthermore, the image by $\exp_p$ of each stratum of canonical singularities is also smooth. There might be strata of high codimension that are not uniform, in the sense that the exponential map at some points in those strata may not have the same type of singularity (in other words, the singularities are non-determinate). This only happens in some strata of codimension at least 5, and is not a problem for our arguments.

There is also another generic property that interests us: the images of the different strata intersect “transversally”:

Take two different points $x_1, x_2 \in T_pM$ mapping to the same point of $M$, and assume $x_1$ and $x_2$ lie in $A_2 \cup A_3 \cup A_4 \cup D_4$. Then the points $x_1$ and $x_2$ have
neighboring domains \( U_1, U_2 \) such that \( \exp_p(U_1 \cap C) \) and \( \exp_p(U_2 \cap C) \) are transverse (each pair of strata intersect transversally). This follows from Proposition 1 in page 215 of [B77], with \( p = 2 \), so that \( \tilde{J}^k_H(\alpha) \) is transverse to the orbit in \( \mathbb{R}^2 \times [J_k^0(n, 1)]^2 \) where the first jet is of type \( T_1 \) and the second one is of type \( T_2 \).

For any singularity in the above list, we can choose coordinates near \( x \) and \( \exp_p(x) \) so that \( \exp_p \) is expressed by standard formulas. For example, the formulas near an \( A_3 \) point are \((x_1, \ldots, x_{n-1}, x_n) \to (x_1^2 \pm x_1 x_2, x_2, \ldots, x_n)\).

The coordinates that we will use are derived using generalized phase functions. We list the generalized phase functions and the corresponding coordinates for the exponential function that derives from it for the singularities \( A_2, A_3, A_4 \) and \( D_4^+ \):

- **A_2**:
  \[
  F(x_1, x_2, x_3, \ldots, x_n) = \frac{1}{4} x_1^4 - x_1 x_2
  \]
  \[
  \exp_p : (x_1, x_2, x_3, \ldots, x_n) \to (x_1^2, x_2, x_3, \ldots, x_n)
  \]

- **A_3**:
  \[
  F(x_1, x_2, x_3, \ldots, x_n) = \frac{1}{4} x_1^4 + \frac{1}{2} x_2 x_1^2 - x_1 x_2
  \]
  \[
  \exp_p : (x_1, x_2, x_3, \ldots, x_n) \to (x_1^2, x_2, x_3, \ldots, x_n)
  \]

- **A_4**:
  \[
  F(x_1, x_2, x_3, \ldots, x_n) = x_1^4 + \frac{1}{4} x_2 x_1^3 + \frac{1}{3} x_3 x_1^2 - x_1 x_2
  \]
  \[
  \exp_p : (x_1, x_2, x_3, \ldots, x_n) \to (x_1^2 + x_2 x_3 + x_1 x_3, x_2, x_3, \ldots, x_n)
  \]

- **D_4^+**:
  \[
  F(x_1, x_2, x_3, \ldots, x_n) = \frac{1}{4} x_1^4 - \frac{1}{2} x_1 x_2^2 + x_3 \left( \frac{1}{4} x_1^2 + \frac{1}{2} x_2^2 \right) - x_1 x_2 - x_2 x_3
  \]
  \[
  \exp_p : (x_1, x_2, x_3, \ldots, x_n) \to (\frac{1}{4} x_1^2 - \frac{1}{2} x_2^2 + x_1 x_3, -x_1 x_2 + x_2 x_3, x_3, \ldots, x_n)
  \]

**Definition 6.5.** The above expression is the canonical form of the exponential map at the singularity. The canonical form is only defined for the singularities in the above list.

We call adapted coordinates any set of coordinates on \( U \supset T_p M \) and \( V \supset \exp_p(U) \) for which the expression of the exponential map is canonical.

Although the adapted coordinates make the exponential map simple, radial geodesics from \( p \) are no longer straight lines, and the spheres of constant radius in \( T_p M \) are also distorted. We do not know of any result that gives an explicit canonical formula for the exponential map and also keeps radial geodesics in \( T_p M \) simple. The results of section 7.8 suggest that this might be possible to some extent, but the classification that might derive from it must be finer than the one above.

We will find examples showing that the radial vector can be placed in different, non-equivalent positions.

For example, near an \( A_3 \) point, \( C \) is given by \( 3 x_1^2 = x_2 \). The radial vector \( r = (r_1, \ldots, r_n) \) at \((0, \ldots, 0)\) is transverse to \( C \), and thus must have \( r_2 \neq 0 \). There are two possibilities:

- A point is \( A_3(I) \) if and only if \( r_2 > 0 \).
- A point is \( A_3(II) \) if and only if \( r_2 < 0 \).

Even though the exponential map has the same expression in both cases (for adequate coordinates), they differ for example in the following:

Let \( x \in A_3 \cap V_1 \) (a first conjugate point), and let \( U \) be a neighborhood of \( x \) of adapted coordinates. Then \( \exp_p(V_1 \cap U) \) is a neighborhood of \( \exp_p(x) \) if and only if \( x \) is \( A_3(I) \). A proof for this fact will be trivial after section 7.1.

In fact, the above can be used as a characterization (for points in \( A_3 \cap V_1 \)) that shows that the definition is independent of the adapted coordinates chosen.
remark that in a sufficiently small neighborhood of an $A_3(I)$ point, there are no $A_3(II)$ points, and vice versa.

We will get back to this distinction later, and we will also make a similar distinction with $D_4^\pm$ points.

**Remark.** In the literature, it is common to see singularities of real functions of type $A_3$ further subdivided into $A_3^+$ and $A_3^-$ points. A canonical form for an $A_3^\pm$ singularity is

$$F^\pm(x_1, \tilde{x}_1, x_2, x_3, \ldots, x_n) = \frac{1}{4}x_1^4 \pm \frac{1}{2}x_2x_1^2 - \tilde{x}_1x_1$$

When $F^\pm$ are generalized phase functions, both subtypes give equivalent singularities (in the sense of definition 6.3). Thus we only need to consider the minus sign, and we only talk about $A_3$ singularities.

In the work of Buchner, however, the same singularities appear, now as the energy function in a finite dimensional approximation to the space of paths with fixed endpoints. In this second context, it is not equivalent if a geodesic is a local minimum, or a maximum, of the energy functional, and it would make sense to use the distinction between $A_3^+$ and $A_3^-$, rather than the similar-but-not-the-same distinction between $A_3(I)$ and $A_3(II)$.

This can also serve as an illustration that the classifications of singularities of the exponential map by F. Klok and M. Buchner are not equivalent, even though the final result is indeed quite similar. In the classification of F. Klok, the $A_3$ singularities are not divided into the two subclasses $A_3^+$ and $A_3^-$.  

**Definition 6.6.** We define $\mathcal{H}_M$ as the set of Riemannian metrics for the smooth manifold $M$ such that the singular set of $\exp_p$ is stratified by singularities of types $A_2$, $A_3$, $A_4$ and $D_4^\pm$ with the codimensions listed above, plus strata of different types with codimension at least 4, and such that the images of any two strata intersect transversely as described above.

**Theorem 6.7.** $\mathcal{H}_M$ is residual in the set of all Riemannian metrics on $M$.

**Proof.** This is the work of M. Buchner and F. Klok, as we have shown in this section. \hfill $\square$

7. **Proof of Theorem B**

In the previous section we have classified the points of $T_pM$ for a generic Riemannian manifold according to the singularity of the exponential map at that point. We use that classification to split $T_pM$ into two sets, according to the role that they play when proving that the manifold is sutured:

**Definition 7.1.**

$$\mathcal{I} = (\mathcal{NC} \cup A_3(I)) \cap V_1$$

$$\mathcal{J} = (A_2 \cup A_3(II) \cup A_4 \cup D_4^\pm) \cap V_1$$

**Theorem 7.2.** Points in $\mathcal{I}$ are unequivocal, and any point in $\mathcal{J}$ is strongly linked to a point in $\mathcal{I}$ of smaller radius.

**Proof of Main Theorem B.** By definition, $V_1 = \mathcal{I} \cup \mathcal{J}$ for a metric in $\mathcal{H}_M$. By Theorem 7.2, all metrics on $\mathcal{H}_M$ are sutured. Main Theorem B follows by application of Theorem 6.7. \hfill $\square$
7.1. $A_3(I)$ first conjugate points are unequivocal.

Lemma 7.3. Any $x \in V_1$ of type $A_3(I)$ is unequivocal.

Proof. Consider an $A_3(I)$ point $x$ in a manifold $(M_1, p_1)$ whose curvature is $L$-related to the curvature of $(M_2, p_2)$, and use adapted coordinates near $x = (0, 0, 0)$, in an arbitrarily small neighborhood $O$:

- Define $\gamma(x_1, x_3) = x_3^2$.
- Let $A$ be the subset of $O$ given by $x_2 < \gamma(x_1, x_3)$. $e_1$ maps diffeomorphically $A$ onto a big subset of $e_1(O)$. Only the points with $x_1 = 0, x_2 \geq 0$ are missing. $x$ is $A_3(I)$, so $A \subset V_1$, and $e_1(O \cap V_1)$ is open.
- For any $(x_1, x_3)$, the pair of points $(x_1, x_3)$ and $(-x_1, x_3^*)$ map to the same point by $e_1$, the curve $t \to (t, t^2, x_3)$, $t \in [-x_1, x_1]$ maps to a tree-formed curve. This shows that the two points map to the same point by $e_2$ as well.
- Define a map $\phi : e_1(O) \to e_2(O)$ by $\phi(p) = e_2(a)$, for any $a \in A$ such that $p = e_1(a)$. By the above, this is unambiguous.
- For a pair of linked points $x = (x_1, x_2^*, x_3, \ldots, x_n)$ and $\bar{x} = (-x_1, x_1^*, x_3, \ldots, x_n)$, we have two different local isometries from a neighborhood of $p = e_1(x) = e_1(\bar{x})$ into $M_2$, given by $e_2 \circ (e_1(O_1))^{-1}$, for neighborhoods $O_1$ of $x$ and $\bar{x}$ such that $e_1(O_1) = e_1(O_2)$ and we need to show that they agree. They both send $p$ to the same point, and we only need to check that their differential at $p$ is the same. These are the linear isometries $I_x$ and $I_{\bar{x}}$, and they agree by 4.4.
- We know that $\phi \circ e_1(x) = e_2(\bar{x})$, for $x \in A$. Let $y \in O \setminus A$. There is a unique point $x$ in the radial line through $y$ in $\partial A$. We know $\phi \circ e_1(x) = e_2(\bar{x})$, and the radial segment from $x$ to $y$ map by both $\phi \circ e_1$ and $e_2$ to a geodesic segment with the same length, starting point and initial vector. We conclude $\phi \circ e_1(y) = e_2(y)$.

\[\square\]

Remark. The only place where we used that the point is $A_3(I)$ is when we assumed that $A \subset V_1$.

7.2. Conjugate flow. We now introduce the main ingredient in the construction of the linking curves. The idea in the definition of conjugate flow was used in lemma 2.2 of [H82] for a different purpose.

Near a conjugate point of order 1, the set $C$ of conjugate points is a smooth surface. Furthermore, we know ker $dF$ does not contain $r$ by Gauss’ lemma. Thus we can define a one dimensional distribution $D$ within the set of points of order 1 by the rule:

\[(7.1)\quad D = (\text{ker } dF \oplus r) \cap TC\]

Definition 7.4. A conjugate descending curve (CDC) is a smooth curve, consisting only of $A_2$ points, except possibly at the endpoints, and such that the speed vector to the curve is in $D$ and has negative scalar product with the radial vector $r$. Therefore, the radius is decreasing along a CDC.

The canonical parametrization of a CDC $\gamma$ is the one that makes $d\exp_p(\gamma')$ a unit vector. By Gauss lemma, it is also the one that makes $dR(\gamma') = 1$.

Definition 7.5. Let $\alpha : [0, t_1] \to T_pM$ be a smooth curve, and $x \in T_pM$ be a point such that $\exp_p(x) = \exp_p(\alpha(t_1))$. A smooth curve $\beta : [0, t_1] \to T_pM$ is a retort of $\alpha$ starting at $x$ if and only if $\alpha(t) \neq \beta(t_1 - t)$ for any $t \in [0, t_1]$, but $\exp_p(\alpha(t)) = \exp_p(\beta(t_1 - t))$ for any $t \in [0, t_1]$, and $\beta(t)$ is NC for any $t \in (0, t_1)$.
Whenever $\beta$ is a retort of $\alpha$, we say that $\beta$ replies to $\alpha$. A partial retort of $\alpha$ is a retort of the restriction of $\alpha$ to a subinterval $[t_0, t_1]$, for $0 < t_0 < t_1$.

We have seen that near an $A_2$ point $x$, there are coordinates near $x$ and $\exp_p(x)$ such that $\exp_p$ reads $(x_1, x_2, \ldots, x_n) \rightarrow (x_1^2, x_2, \ldots, x_n)$. The $A_2$ points are given by $x_1 = 0$, and no other point $y \neq x$ maps to $\exp_p(x)$. Thus, there is a neighborhood $U$ of any CDC such that any CDC contained in $U$ has no non-trivial retorts contained in $U$.

**Lemma 7.6.** Let $x$ be an $A_2$ point. Then there is a $C^\infty$ CDC $\alpha : [0, t_0) \rightarrow T_pM$ with $\alpha(0) = x$. The CDC is unique, up to reparametrization. Furthermore:

- $|\alpha(0)| - |\alpha(t_0)| = \text{length}(\exp_p \circ \alpha)$
- If $\beta$ is a non-trivial retort of $\alpha$, then of course, $\text{length}(\exp_p \circ \alpha) = \text{length}(\exp_p \circ \beta)$, but $|\beta(t_0)| - |\beta(0)| < \text{length}(\exp_p \circ \beta)$.

We say that conjugate descending curves are unbeatable.

**Proof.** Both $A_2$ and the distribution $D$ are smooth near $x$, so the first part is standard.

We also compute:

$$\text{length}(\exp_p \circ \alpha) = \int |(\exp_p \circ \alpha)'| = \int |d\exp_p(\alpha')|$$

By definition of $D$, $\alpha' = ar + v$ is a linear combination of a multiple of the radial vector and a vector $v \in \ker(d\exp_p)$. By the Gauss lemma, $|d\exp_p(\alpha')| = a$. On the other hand, $v$ is tangent to the spheres of constant radius, so:

$$|\alpha(0)| - |\alpha(t_0)| = \int \frac{d}{dt}|\alpha| = \int a = \text{length}(\exp_p \circ \alpha)$$

For a retort $\beta : [0, t_1) \rightarrow T_pM$, we also have $\beta' = br + v$ for a function $b : [0, t_1) \rightarrow \mathbb{R}$ and a vector $v(t) \in T_{\beta(t)}(T_pM)$ that is always tangent to the spheres of constant radius, and $v(t)$ is not identically zero because $e_1 \circ \beta$ is not a geodesic. However, $\beta(s)$ is non-conjugate, so $|d\exp_p(\beta')| = \sqrt{b^2 + |d\exp_p(v)|^2} > b$. The result follows. □

**Remark.** We recall that the plan is to build linking curves, whose composition with the exponential is fully tree formed. If a linking curve contains a CDC, it must also contain a retort for that CDC. The “unbeatable” property of CDCs is interesting, because the radius decreases along a CDC and along the retort it never increases as much as it decreased in the first place.

### 7.3. CDCs in adapted coordinates near $A_3$ points

As we mentioned in section 6, the radial vector field, and the spheres of constant radius of $T_pM$, that have very simple expressions in standard linear coordinates in $T_pM$, are distorted in canonical coordinates. Thus, the distribution $D$ and the CDCs do not always have the same expression in adapted coordinates. In this section, we study CDCs near an $A_3$ point. We will use the name $R : T_pM \rightarrow \mathbb{R}$ for the radius function, and $r$ for the radial vector field, and we assume that our conjugate point is a first conjugate point (it lies in $\partial V_1$).

In a neighborhood $O$ of special coordinates of an $A_3$ point, $C$ is given by $3x_1^2 = x_2$. At each $A_3$ point, the kernel is spanned by $\frac{\partial}{\partial x_1}$. At points in $C$, we can define a 2D distribution $D_2$, spanned by $r$ and $\frac{\partial}{\partial x_1}$. We extend this distribution to all of $O$ in the following way:

**Definition 7.7.** For any point $x \in O$, there are $y \in C$ and $t_0$ such that $x = \phi_{t_0}(y)$, where $\phi_t$ is the radial flow, and $y$ and $t$ are unique. Define $D_2(x)$ as $(\phi_{t_0})_*(D_2(y))$. 20
Linking curves, saturated manifolds and the Ambrase conjecture

The reader may check that $D_2$ is integrable. Let $P$ be the integral manifold of $D_2$ through $x_0 = (0,0,0)$. We can assume $P$ is a graph over the $x_1, x_2$ plane: $x_3 = p(x_1, x_2)$. $A_3$ is transverse to $D_2$, so $\{x_0\} = A_3 \cap P$. The integral curve $C$ of $D$ through $x_0$ is contained in $P$, and $C \setminus \{x_0\}$ consists of two CDCs. We claim that if the point is $A_3(I)$, the two CDCs descend into $x_0$, but if the point is $A_3(II)$, they start at $x_0$ and flow out of $O$. $P$ is also obtained by flowing the CDC with the radial vector field.

We can assume that $r$ is close to $r(x_0)$ in $O$. The tangent $T_2$ to the sphere of constant radius $\{ y : R(y) = R(x) \}$ must contain $\frac{\partial}{\partial x_1}$ (the kernel of $d\exp_p$) if $x \in C$, by Gauss lemma, and we can assume that the angle between $T_2$ and $\frac{\partial}{\partial x_1}$ is small if $x \not\in C$.

The curves $\{ R(x) = R_1 \} \cap P$, for any $R_1$, are all smooth graphs over the $x_1$ axis. We claim that the curve $\{ R(x) = R(x_0) \} \cap P$ may not intersect $3x_1^2 < x_2$. Assume that $R(y) = R(x_0)$ for some $y = (y_1, y_2)$ with $y_1 < 0$ and $y_2 > 3y_1^2$. Then there is a curve $\{ R(x) = R(x_0) - \varepsilon \} \cap P$, for some $0 < \varepsilon < 1$, must intersect $C$ at a point $(x_1, 3x_1^2, p(x_1, 3x_1^2))$ with $x_1 < 0$, and the tangent to $\{ R(x) = R_1 \} \cap P$ must be $\frac{\partial}{\partial x_1}$.

Taking coordinates, $x_1, x_2$ in $P$, we see it is not possible that a graph $(x_1, t(x_1))$ over the $x_1$ axis has $t(0) < 0$, $t(y_1) > 3y_1^2$, and intersect the curve $C \cap P$ only with horizontal speed.

It follows that $x_0$ is a local maximum, or minimum, of $R$, within $C$. If $r_2 > 0$ ($A_3(I)$ points), then $R(x) \geq R(x_0)$ for $x \in C$, while $r_2 < 0$ ($A_3(II)$ points), implies $R(x) \leq R(x_0)$ for $x \in C$.

Thus, $A_3(I)$ points are terminal for the conjugate flow, but $A_3(II)$ points are not. We have proved the following:

**Lemma 7.8.** In a neighborhood $O$ of adapted coordinates near an $A_3$ point $x_0$:

- $C$ is foliated by integral curves of $D$.
- $A_2$ is foliated by CDCs.
- If $x_0$ is $A_3(I)$, exactly two CDCs in $O$ flow into each $A_3$ point. If $x_0$ is $A_3(II)$, exactly two CDCs in $O$ flow out of each $A_3$ point.
- If $x_0$ is $A_3(I)$, every CDC in $O$ flows into some $A_3$ point. If $x_0$ is $A_3(II)$, every CDC in $O$ flows out of some $A_3$ point.

7.4. $A_3$ joins. We can continue a CDC as long as it stays within a stratum of $A_2$ points. As we have seen, a CDC may enter a different singularity. The most important situation is when the CDC reaches an $A_3$ point, because then we can start a non-trivial retort right after the CDC.

The set of conjugate points is a graph over the $x_1, x_3$ plane: $x_2 = \alpha(x_1, x_3) = 3x_3^2$. A CDC is written $t \to (t, 3t^2, x_3(t))$, for $t \in [0, 0]$, finishing at an $A_3$ point $(0, 0, x_3(0))$. We can start a retort for this segment of CDC starting at the $A_3$ point. The retort for this CDC is given explicitly by $t \to (-2t, 3t^2, x_3(t))$.

These curves, composed of a segment of CDC plus the corresponding retort, map to a fully tree-formed map that shows that the point $(t, 3t^2, x_3)$ is linked to $(-2t, 3t^2, x_3)$. We say that the CDC and the retort given above are joined with an $A_3$ join.

7.5. Avoiding some obstacles. In order to build linking curves, it is simpler to replace CDCs with curves that are close to CDC curves, but avoid certain "obstacles". The following remark helps in that respect:

A curve that is sufficiently $C^1$-close to a CDC is also unbeatable. Actually, we can say more: the greater the angle between $r_x$ and $\ker d_x e_1$, the more we can depart from the CDC.
Definition 7.9. The slack $A_x$ at a first order conjugate point $x$ is the absolute value of the sine of the angle between $D_x$ and $\ker(d_xe_1)$.

Remark. The above angle is with respect to the euclidean metric in $T_x M$. Using this metric for this purpose does not have intrinsic geometric meaning, but since this is only required to find bounds as in lemma 7.11, any other metric could be used.

Remark. The slack is positive if and only if the point is $A_2$.

The following definition is required for lemma 7.11. The reason for using all $s$ in $(0, 2]$ will become clear in the proof of that lemma.

Definition 7.10. Let $\gamma_x(s) = \exp_p(sx)$, for $s \in \mathbb{R}$, be the radial geodesic from $p$ with initial speed $x \in T_p M$.

We say that $\gamma_x$ does not return to the base if $\gamma_x(s) \neq p$ for all $s \in (0, 2]$.

Let $\delta > 0$ be a positive number smaller than the injectivity radius from $p$, and $R$ a positive real number.

We say that $\gamma_x$ stays $(\delta, R)$-away from the base if $\exp_p(sx)$ is not in the ball of radius $\delta$ around $p$ for any $s \in \mathbb{R}$ such that $\delta < s|x| \leq 2R$.

Remark. If $\gamma_x$ does not return to the base, then there is some $\delta > 0$ such that $\gamma_x$ stays $(\delta, R)$-away from the base for $R = |x|$.

Furthermore, if $\gamma_{x_0}$ does not return to the base, then there is a neighborhood $O$ of $x_0$ and some $\delta > 0$ such that for any $x \in O$, $\gamma_x$ stays $(\delta, R)$-away from the base for $R = |x_0|$.

This follows easily from the continuity of $\exp_p$ and the distance function.

Lemma 7.11. For any positive numbers $R > 0$, $\delta > 0$ and $a > 0$ there are constants $c > 0$ and $\varepsilon > 0$ depending on $M$, $R$, $\delta$ and $a$ such that the following holds:

Let $\alpha : [0, T] \rightarrow T_p M$ be a smooth curve of $A_2$ points with the following properties:

1. $|\alpha(0)| \leq R$
2. $\langle \alpha'(t), r_{\alpha(t)} \rangle < 0$ for all $t \in [0, T]$
3. $A_{\alpha(t)} > a$ for all $t \in [0, T]$
4. $\alpha'(t)$ is within a cone around $D$ of amplitude $c$ for all $t \in [0, T]$
5. $\alpha_{\gamma(t)}$ stays $(\delta, R)$-away from the base for all $t \in [0, T]$.

Then it holds that $\alpha$ is $\varepsilon$-unbeatable:

- any retort $\beta$ contained in $B_0(R)$ satisfies

$$|\beta(T) - \beta(0)| \leq (|\alpha(0)| - |\alpha(T)|) (1 - \varepsilon)$$

Proof. Fix a neighborhood $U$ of adapted $A_2$ coordinates that contains the image of $\alpha$. We assume that such one $U$ contains all of the image of $\alpha$, otherwise we split $\alpha$ into parts.

Let $v(t)$ be the vector at $\alpha(t)$, orthogonal to $r(\alpha(t))$, and such that $d(t) = -r(\alpha(t)) + v(t)$ belongs to $D_x$. Then the slack $A_{\alpha(t)}$ is $|r(\alpha(t))| = |r(\alpha(t)) + v(t)| = \frac{1}{|d(t)|}$.

We assume that $\alpha$ has the canonical parametrization, so that $\alpha'(t) = d(t) + p(t)$, with $p(t)$ is orthogonal to $d$. Then $|p(t)| = c|d(t)| = \frac{T}{a} < \frac{T}{2}$.

We compute:

$$|\alpha(T)| - |\alpha(0)| = \int_0^T \frac{d}{dt} |\alpha| = \int_0^T \langle r, \alpha' \rangle = -T + \int_0^T \langle r, p \rangle,$$

where

$$\left| \int_0^T \langle r, p \rangle \right| \leq T \frac{c}{a}$$

Write $\beta'(t) = b(t)r(\beta(t)) + w(t)$, where $w$ is a vector orthogonal to $r(\beta(t))$. 

Pablo Angulo

22
Claim: There are $\varepsilon > 0$ and $c > 0$, that depend only on $M$, $R$, $\delta$ and $a$, such that $|b(t)| < 1 - 2\varepsilon$.

Assuming that the claim is true, we compute:

$$|eta(T)| - |\beta(0)| = \int_0^T \frac{d}{dt} |\beta| = \int_0^T b(t) \leq T(1 - 2\varepsilon) \leq T(1 - \varepsilon)^2$$

We can make $c$ smaller so that $c < \varepsilon a$. Then it follows from (7.2):

$$|\alpha(0)| - |\alpha(T)| \geq T(1 - \varepsilon)$$

The result follows.

Proof of claim:

An $A_2$ point only has one preimage in $U$, so any retort $\beta$ of $\alpha$ lies outside of $U$. Since $\exp_p(\alpha(t)) = \exp_p(\beta(T - t))$, if it also holds that $d\exp_p(r(\alpha(t))) = d\exp_p(r(\beta(T - t)))$, then the geodesics $\gamma_{\alpha(t)}$ and $\gamma_{\beta(T - t)}$ agree, up to reparametrization, near their terminal point. Since $\alpha(t) \neq \beta(T - t)$, we deduce that the same geodesic passes through $p$ twice, which leads to two possibilities:

1. $\gamma_{\alpha(t)}$ is a restriction and reparametrization of $\gamma_{\beta(T - t)}$
2. viceversa: $\gamma_{\beta(T - t)}$ is a restriction and reparametrization of $\gamma_{\alpha(t)}$

The first option is not possible: $\alpha(t)$ is a conjugate point, hence the geodesic $\gamma_{\beta(T - t)}$ would have at least one conjugate point from $p$, which is incompatible with the hypothesis that the image of $\beta$ is contained in $V_1$. The second option is excluded by the last hypothesis on $\alpha$. Hence we deduce that for all $t$:

$$d\exp_p(r(\alpha(t))) \neq d\exp_p(r(\beta(T - t))).$$

If for some $t \in [0, T]$, we have $d\exp_p(r(\alpha(t))) = -d\exp_p(r(\beta(T - t)))$, we use a similar argument: since $\exp_p(\alpha(t)) = \exp_p(\beta(T - t))$, then the concatenation of the geodesics $\gamma_{\alpha(t)}$ and $\gamma_{\beta(T - t)}$, but in reverse direction, is a closed geodesic loop of length at most $2R$, which contradicts the fact that $\gamma_{\alpha(t)}$ stays $(\delta, R)$-away from the base.

We define two functions

$$F^+(x, y) = \text{dist}((\exp_p(x), d\exp_p(r(x))), (\exp_p(y), d\exp_p(r(y)))),$$

$$F^-(x, y) = \text{dist}((\exp_p(x), d\exp_p(r(x))), (\exp_p(y), -d\exp_p(r(y))))$$

where dist is a distance in the tangent bundle $TM$ induced by some Riemannian metric on $TM$. For convenience, we choose a Riemannian metric that restricts to the Riemannian metric $g$ of $M$ on every fiber $T_qM$.

Both $F^+$ and $F^-$ are continuous in the set $V_1 \times V_1$. By the previous argument, they are strictly positive in the compact set $(A_{R, \delta} \times V_1 \cap B_0(R))$ where $A_{R, \delta}$ is the set of vectors $x \in V_1 \cap B_0(R)$ that stay $(\delta, R)$-away from the base:

$$A_{R, \delta} = \{ x \in V_1 \cap B_0(R) : \forall t \in [0, 2], \exp_p\left(\frac{tx}{|x|}\right) \notin B_p(\delta) \}.$$

We deduce that both $F^+$ and $F^-$ are greater that some $\varepsilon_1 = \varepsilon_1(R, \delta)$ on $(A_{R, \delta} \times V_1 \cap B_0(R))$ and hence:

$$\text{dist}((\exp_p(\alpha(t)), d\exp_p(r(\alpha(t)))), (\exp_p(\beta(T - t)), d\exp_p(r(\beta(T - t)))) > \varepsilon_1,$$

$$\text{dist}((\exp_p(\alpha(t)), d\exp_p(r(\alpha(t)))), (\exp_p(\beta(T - t)), -d\exp_p(r(\beta(T - t)))) > \varepsilon_1,$$

and, since $q = \exp_p(\alpha(t)) = \exp_p(\beta(T - t))$:

$$|d\exp_p(r(\alpha(t))) - d\exp_p(r(\beta(T - t)))/_{TM} > \varepsilon_1$$

$$|d\exp_p(r(\alpha(t))) + d\exp_p(r(\beta(T - t)))/_{TM} > \varepsilon_1$$

_linking curves, sutured manifolds, and the ambrose conjecture_
It follows from our definitions that:

\[ d \exp_p(\beta'(T - t)) = b(t)d \exp_p(r(\beta(T - t))) + d \exp_p(w(t)) = -d \exp_p(\alpha'(t))) = -(d \exp_p(r(\alpha(t))) + d \exp_p(p(t))) \]

and thus, by Gauss lemma:

\[ |d \exp_p(\beta'(T - t))|^2 = b(t)^2 + |d \exp_p(w(t))|^2 \]

and also:

\[ |d \exp_p(\beta'(T - t))|^2 = |d \exp_p(\alpha'(t)))|^2 = 1 + |d \exp_p(p(t))|^2 \]

Since \(|p(t)| \leq \frac{\varepsilon}{a}\), we get:

\[ 1 \leq |d \exp_p(\beta'(T - t))|^2 \leq 1 + \left( \frac{C}{a} \right)^2 \]

where \(C\) is a bound for the operator norm of \(d \exp_p\) at every point in \(B_0(R)\). Hence:

\[ 1 - |d \exp_p(w(t))|^2 \leq b(t)^2 \leq 1 + \left( \frac{C}{a} \right)^2 - |d \exp_p(w(t))|^2 \]

We set \(c = \frac{\varepsilon a}{4C}\). We split the argument depending on which of \(|d \exp_p(w(t))|\) and \(\frac{\varepsilon}{4}\) is bigger.

If \(|d \exp_p(w(t))| < \frac{\varepsilon}{4}\), then:

\[ |d \exp_p(r(\alpha(t))) - d \exp_p(r(\beta(T - t))))| \leq |d \exp_p(r(\alpha(t))) + d \exp_p(\alpha'(t))| + |d \exp_p(\beta'(T - t)) - d \exp_p(r(\beta(T - t)))| \leq |d \exp_p(p(t))| + |b(t) - 1| + |d \exp_p(w(t))| \leq C\frac{\varepsilon}{a} + |b(t) - 1| + \frac{\varepsilon}{4} \leq |b(t) - 1| + \frac{3\varepsilon}{8a} \]

and also:

\[ |d \exp_p(r(\alpha(t))) + d \exp_p(r(\beta(T - t))))| \leq |d \exp_p(r(\alpha(t))) + d \exp_p(\alpha'(t))| + |d \exp_p(\beta'(T - t)) + d \exp_p(r(\beta(T - t)))| \leq |d \exp_p(p(t))| + |b(t) + 1| + |d \exp_p(w(t))| \leq C\frac{\varepsilon}{a} + |b(t) + 1| + \frac{\varepsilon}{4} \leq |b(t) + 1| + \frac{3\varepsilon}{8a} \]

Combining (7.3), (7.4), (7.6) and (7.7), we get:

\[ |b(t)|^2 - 1 = |b(t) + 1||b(t) - 1| \geq \left( \frac{5\varepsilon_1}{8} \right)^2 \]

but \(c = \frac{\varepsilon a}{4C}\) and (7.5) imply:

\[ b(t)^2 - 1 \leq \frac{\varepsilon^2}{64} \]

which is a contradiction with (7.8) if \(|b(t)| \geq 1\).

If, on the other hand, \(|b(t)| < 1\), then (7.8) becomes:

\[ 1 - b(t)^2 \geq \left( \frac{5\varepsilon_1}{8} \right)^2 \]

which, using the inequality \(\sqrt{1 + t} \leq 1 + \frac{1}{2}t\), yields:

\[ |b(t)| \leq 1 - \frac{25\varepsilon_1^2}{128} \]

and the claim holds for \(\varepsilon = \frac{25\varepsilon_1^2}{256}\).
It remains the possibility that \( |d \exp_p(w(t))| \geq \frac{c}{t} \). Then if it follows from (7.5) and the inequality \( \sqrt{1 + t} \leq 1 + \frac{1}{2}t \) that:

\[
|b(t)| \leq 1 + \frac{c_1^2}{128} - \frac{c^2}{32}
\]

and the claim is proven for \( \varepsilon = \frac{3c^2}{256} \) and \( c = \frac{ac_1}{8c} \).

\[\square\]

The following lemma makes lemma 7.11 easier to use.

**Lemma 7.12.** Let \( R > 0, \delta > 0 \) and \( \alpha > 0 \) be positive numbers, and \( \varepsilon > 0 \) be the constants given by lemma 7.11.

Let \( \alpha : [0, T] \rightarrow T_pM \) be a smooth curve of \( A_2 \) points that satisfies the conditions in lemma 7.11.

Let \( \beta \) be any retort of \( \alpha \) such that \( |\beta(0)| \leq |\alpha(T)| \).

Then \( \beta \) is contained in \( B_0(R) \).

**Proof.** Assume that \( \alpha \) and \( \beta \) satisfy the hypothesis.

The conditions \( |\alpha(0)| \leq R \) and \( \langle \alpha'(t), r_{\alpha(t)} \rangle < 0 \) for all \( t \in [0, T] \) assure that \( |\alpha(t)| \leq R \) for any \( t \in (0, T) \). It follows from \( |\beta(0)| \leq |\alpha(T)| \) that \( |\beta(0)| < R \).

Suppose that \( \beta \) is not contained in \( B_0(R) \). Then we let \( T^* \) be the supremum of all \( t \in [0, T] \) such that \( |\beta(t)| \leq R \).

These assumptions imply that \( 0 < T^* < T \) and \( |\beta(T^*)| = R \).

We can apply lemma 7.11 to \( \alpha \) restricted to \( [T - T^*, T] \) and \( \beta \) restricted to \( [0, T^*] \) to obtain:

\[ |\beta(T^*)| - |\beta(0)| < |\alpha(T)| - |\alpha(T)| (1 - \varepsilon) < |\alpha(T - T^*)| - |\alpha(T)|, \]

but this leads to the contradiction that

\[ R = |\beta(T^*)| < |\alpha(T - T^*)| + |\beta(0)| - |\alpha(T)| \leq R + 0 = R. \]

This proves that \( \beta \) is indeed contained in \( B_0(R) \).

\[\square\]

With these lemmas, we can perturb a CDC slightly to avoid some points:

**Definition 7.13.** An elementary approximately conjugate descending curve is a \( C^1 \) curve \( \alpha \) of \( A_2 \) points, defined in a compact interval, such that for all \( t \) in its domain:

1. \( \alpha'(t) \) points to the interior of the surfaces of constant radius.
2. \( \gamma_{\alpha(t)} \) stays (\( \delta, R \))-away from the base for some \( \delta > 0 \) and \( R > 0 \).
3. \( \alpha'(t) \) is within a cone around \( D \) of amplitude \( c \), where \( c \) is the constant in lemma 7.11 for \( R = |\alpha(0)| \).

An approximately conjugate descending curve (ACDC) is the concatenation of elementary ACDCs and CDGs.

7.6. Conjugate Locus Linking Curves. Let us assume that we have an ACDC \( \alpha : [0, t_0] \rightarrow T_pM \) starting at a point \( x \in J \), whose interior consists only of \( A_2 \) points and ending up in an \( A_3 \) point. We know that we can start a retort \( \overline{\alpha} \) at the \( A_3 \) point.

We can continue the retort while it remains in the interior of \( V_i \), where \( e_1 \) is a local diffeomorphism and we can lift any curve. However, we might be unable to continue the retort up to \( x \) if the returning curve hits the set of conjugate points.

If we hit an \( A_2 \) point \( y = \overline{\alpha}(t_1) \), we can take an ACDC \( \beta : [0, t_2] \rightarrow V_i \) starting at this point and ending in an \( A_3 \) point. If \( \beta \) has a retort \( \tilde{\beta} : [0, t_2] \rightarrow T_pM \) that ends up in a non-conjugate point \( \tilde{\beta}(t_2) \), we can continue with the retort \( \overline{\alpha}_2 \) of \( \alpha_{[0,t_0-t_1]} \) starting at \( \tilde{\beta}(t_2) \). If \( \overline{\alpha}_2 \) can be continued up to \( x = \alpha(0) \), the concatenation of \( \alpha, \overline{\alpha}, \beta, \tilde{\beta} \) and \( \overline{\alpha}_2 \) is a linking curve (see figure 1).
There are a few things that may go wrong with the above procedure: the retort $\tilde{\alpha}_i$ may meet $J \setminus A_2$, or $\beta$ may not admit a full retort starting at $\tilde{\beta}(t_2)$, or $\alpha|_{[0,t_0-t_1]}$ may not admit a full retort starting at $\tilde{\beta}(t_2)$. The first problem can be avoided if the ACDCs are built to dodge some small sets, as we will see later. Then, if we assume that a retort never meets $J \setminus A_2$, we can iterate the above argument whenever a retort is interrupted upon reaching an $A_2$ point. We will prove later that the argument only needs to be applied a finite number of times.

This is the motivation for the following definitions:

**Definition 7.14.** A finite conjugate linking curve (or FCLC, for short) is a continuous linking curve $\alpha : [0, l] \to T_p M$ that is the concatenation $\alpha = \alpha_1 \ast \ldots \ast \alpha_n$ of ACDCs and non-trivial retorts of those ACDCs, all of them of finite length.

We will build the FCLCs in an iterative way, as hinted at the beginning of this section, by concatenation of ACDCs and retorts of those ACDCs.

**Definition 7.15.** An aspirant curve is an absolutely continuous curve $\alpha : [0, l] \to T_p M$ that is the concatenation $\alpha = \alpha_1 \ast \ldots \ast \alpha_n$ of ACDCs and non-trivial retorts of those ACDCs, such that:

- Starting with the tuple $(\alpha_1, \ldots, \alpha_n)$ consisting of the curves that $\alpha$ is made of in the same order, we can reach a tuple with no retorts, by iteration of the cancellation rule:
  - Cancel an ACDC together with a retort of that ACDC that follows right after it: $(\alpha_1, \ldots, \alpha_{j-1}, \alpha_j, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_n) \to (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+2}, \ldots, \alpha_n)$, if $\alpha_{j+1}$ is a retort of $\alpha_j$.
- The extremal points of the $\alpha_i$ are called the vertices of $\alpha$. The vertices of $\alpha$ fall into one of the following categories:
  - starting point (first point of $\alpha_1$): a point in $J$.
  - end point (last point of $\alpha_n$): a point in $\tilde{\beta}$.
  - $A_3$ join, as explained in section 7.4.
  - a splitter: a vertex that joins two ACDCs whose concatenation is also an ACDC.
  - a hit: a vertex that joins a retort that reaches $A_3(I)$ transversally, and an ACDC starting at the intersection point.
  - a reprise: a vertex that joins a retort that completes its task of replying to a ACDC $\alpha_j$, and the retort for a different ACDC $\alpha_i$ (it follows from the first condition that $i < j$).

The tip of alpha is its endpoint $\alpha(l)$.

The loose ACDCs in $\alpha = \alpha_1 \ast \ldots \ast \alpha_k$ are the ACDC curves $\alpha_j$ for which there is no retort in $\alpha$.

An aspirant curve is saturated if there are no loose ACDCs.

The three new types of vertices: splitters, hits and reprises, always come in packs. We have already shown one example of how they could appear, but we formalize that construction in the following definition.

**Definition 7.16.** A standard $T$ consists of three vertices: a splitter, a hit and a reprise, such that the six curves $\alpha_i$ contiguous to the three points map to a $T$-shaped curve, with two curves mapping into each segment of the $T$ (see figure 1).

**Proposition 7.17.** Let $\alpha = \alpha_1 \ast \ldots \ast \alpha_k$ be a saturated aspirant curve between $x, y \in T_p M_1$. Then:

- $|x| > |y|
- \alpha$ is an FCLC.
1.2 We deduce curves that can be cancelled, the same property. Furthermore, using (7.9), we know that for any two consecutive (7.9) \[ \alpha \] followed by 7.11, and learn that:

\[ (\alpha) \] and its retort \( \alpha \) is continuous.

Indeed, we can apply lemma 7.11 to each pair of an ACDC \( \alpha \) and its retort \( \alpha \), but we need to verify that the image of \( \alpha \) is contained in the ball of radius \( |\alpha(0)| \).

If \( j = i + 1 \), this holds by lemma 7.12 and \( |\alpha(i+1)(0)| = |\alpha(i)(t)| \), which holds because \( \alpha \) is continuous.

Whenever we cancel a pair of an ACDC \( \alpha \) and its retort \( \alpha \), we can apply lemma 7.12 followed by 7.11, and learn that:

\[ |\alpha_j(0)| \leq |\alpha_i(l_i)| \Rightarrow |\alpha_j(l_j)| < |\alpha_i(0)| + |\alpha_j(0)| - |\alpha_i(l_i)| \leq |\alpha_i(0)|. \tag{7.9} \]

We note that \( \alpha_i(0) = \alpha_{i-1}(l_{i-1}) \), and \( \alpha_j(l_j) = \alpha_{j+1}(0) \), if \( \alpha_i \) is not the first and \( \alpha_j \) is not the last curve. Thus, we start with a sequence of ACDCs and retorts \( (\alpha_1, \ldots, \alpha_n) \) with the property that, for any two consecutive curves \( \alpha_i, \alpha_j \), it holds that \( |\alpha_j(0)| \leq |\alpha_i(l_i)| \). After cancellation of one pair, we move from \( (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_n) \) to \( (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+2}, \ldots, \alpha_n) \), which has the same property. Furthermore, using (7.9), we know that for any two consecutive curves that can be cancelled, \(|\alpha_j(l_j)| \leq |\alpha_i(0)|\).

Since by definition of saturated aspirant curve, it is possible to cancel all possible pairs, we deduce \(|y| = |\alpha_n(l_n)| < |\alpha_1(0)| = |x|\).

For the second part, we write \( x \sim y \) whenever \( x = \alpha_i(s) \) lies on an ACDC \( \alpha_i \) and \( y = \alpha_j(l_j - s) \) lies on its retort \( \alpha_j \), so that \( \alpha_i(t) = \alpha_j(l_j - t) \). We also identify the triples of vertices that belong to each standard T. Let \( T : [0, l] \to \Gamma \) be the identification map associated to the relation \( \sim \).

We must show that \( u = \exp \phi \) is tree-formed with respect to \( T \): let \( t_1, t_2 \) such that \( T(t_1) = T(t_2) \), and \( \varphi \) a continuous 1-form along \( \varphi(s) \in T_{\varphi(s)}^* M \) that factors
through $\Gamma$. Then we claim that:

\begin{equation}
\int_{t_1}^{t_2} \varphi(s)(u'(s))ds
\end{equation}

splits as a sum of integrals over the image by $\exp$ of an ACDC and the image of its matching retort. The curves in each such pair have the same image, and the integrals cancel out, as the integral of a 1-form is independent of the parametrization, and only differs by sign.

Suppose first that $t_1$ is in the domain of an ACDC $\alpha_i$ and $t_2$ lies in the retort $\alpha_j$ of $\alpha_i$. We recall it is possible to reach an empty tuple by canceling adjacent pairs of an ACDC and its retort. Thus, in order to cancel $\alpha_i$ and $\alpha_j$, it must be possible to cancel all the curves $\alpha_k$ with $i < k < j$. These curves can be matched in pairs $\{(\alpha_n, \alpha_m)\}_{(n,m) \in P}$ of ACDC and retort, with $i < n < m < j$ for each pair $(n, m) \in P$. Then we have:

\[
\int_{t_1}^{t_2} \varphi(s)(u'(s))ds = \int_{t_1}^{t_2} \varphi(s)((\exp \circ \alpha_i)'(s))ds + \sum_{(n,m) \in P} \left( \int_{t_1}^{t_2} \varphi(s)((\exp \circ \alpha_n)'(s))ds + \int_{t_1}^{t_2} \varphi(s)((\exp \circ \alpha_m)'(s))ds \right) + \int_{t_1}^{t_2} \varphi(s)((\exp \circ \alpha_j)'(s))ds
\]

The remaining two integrals also cancel out, proving the claim.

If $t_1$ and $t_2$ are two of the three points of a standard $T$, we can take points $t_1^*$ and $t_2^*$ as close to $t_1$ and $t_2$ as we want, but in an ACDC and its retort, respectively, and such that $T(t_1^*) = T(t_2^*)$. The result follows because the integral 7.10 depends continuously on $t_1$ and $t_2$.

\[\square\]

### 7.7. Existence of FCLCs

The goal of this section is to prove the existence of an FCLC starting at an arbitrary point $x \in J$. The set \( \{y: |y| < |x|, \exp_p(y) = \exp_p(x)\} \) is finite. This follows because \( \{y: |y| \leq |x|\} \) can be covered with a finite amount of neighborhoods of adapted coordinates, and in any of them the preimage of any point is a finite set. At least one $y_j$ realizes the minimum distance from $p$ to $q = \exp_p(x)$, and must be either $A_3(I)$ or NC (in other words, $y \in I$). We will show that there is an FCLC joining $x$ and one $y_j \in I$ with smaller radius, though it may not be the one with minimal radius.

**Definition 7.18.** We define some important sets:

\[
S_R = B_R \cap \exp^{-1}_p(A_2 \cap B_R)
\]

\[
V^0_1 = \left\{ x \in V_1 : \exp^{-1}_p(\exp_p(x)) \cap B_{|x|} \subset NC \cup A_2 \right\}
\]

\[
SA_2 = \{ x \in A_2 : \exists y \in A_2, \exp(y) = \exp_p(x), |y| < |x| \}
\]

In other words, $V^0_1$ consists of those points $x \in V_1$ such that:

1. All preimages of $\exp_p(x)$ with radius smaller than $|x|$ are NC or $A_2$.
2. The geodesic $\gamma_x$ does not return to the base.

**Definition 7.19.** A GA CDC is an ACDC $\alpha$ such that

- $\text{Im}(\alpha)$ is contained in $C \cap V^0_1$.

28
• for any \( y \in B_{\alpha(t_0)} \cap A_2 \) such that \( \exp_p(\alpha(t_0)) = \exp_p(y) \), \( \exp_p \alpha \) is transverse to \( \exp_p(A_2 \cap B(y)) \) at \( t_0 \), for some \( \varepsilon > 0 \).

In words: all possible retorts of a GACDC avoid all singularities that are not \( A_2 \) and only meet \( A_2 \) transversally.

**Definition 7.20.** The linking curve algorithm is a procedure that attempts to build an FCLC starting at a given point \( x \in V_1 \) (see figure 2).

It starts with the trivial aspirant curve \( \alpha = \{x\} \) and updates it at each segment by addition of one or more segments, to get a new aspirant curve. It only stops if the aspirant curve is saturated, and its tip is in \( \mathcal{I} \).

The aspirant curve \( \alpha = \alpha_1 \ast \ldots \ast \alpha_k \) is updated following the only rule in the following list that can be applied:

**Descent:** If the tip of \( \alpha_k \) is a point in \( \mathcal{J} \), let \( \gamma \) be a GACDC that starts at \( x \) and ends up in an \( A_3 \) point. We know that \( \gamma \) intersects \( SA_2 \) in a finite set and, for convenience, we split \( \gamma \) into \( r \) GACDCs \( \alpha_{k+1}, \ldots, \alpha_{k+r} \) such that each of these curves intersects \( SA_2 \) only at its extrema. The new curve \( \alpha \ast \alpha_{k+1} \ast \ldots \ast \alpha_{k+r} \) ends up in an \( A_3 \) point. The next step is an \( A_3 \) join.

**\( A_3 \) join:** If \( \alpha_k : [0,T] \to V_1 \) is a ACDC ending up in an \( A_3 \) point, add the retort \( \alpha_{k+1} \) of \( \alpha_k \) that starts at the \( A_3 \) join \( \alpha_k(T) \). This is always possible, since \( \alpha_k \) does not intersect \( SA_2 \). The new tip of \( \alpha \ast \alpha_{k+1} \) will be \( NC \), \( A_2 \) or \( A_3 \), but the latter can only happen if \( \alpha \ast \alpha_{k+1} \) is a linking curve.

**Reprise:** If the tip of \( \alpha \) is \( NC \) and \( \alpha \) is not a linking curve, let \( \alpha_j \) be the latest loose curve in \( \alpha \). We add the retort \( \alpha_{k+1} \) of \( \alpha_j \) starting at the tip of \( \alpha \). This is possible for the same reason as before and, again, the new tip of \( \alpha \ast \alpha_{k+1} \) will be \( NC \), \( A_2 \) or \( A_3 \), and the latter can not happen unless \( \alpha \ast \alpha_{k+1} \) is a linking curve.

**Success!** If \( \alpha \) is saturated and its tip is in \( \mathcal{I} \), then \( \alpha \) is an FCLC, so we report success and stop the algorithm. For completeness, the algorithm also reports success if \( \alpha = \{x\} \), for \( x \in \mathcal{I} \).

**Figure 2.** Flow diagram for the linking curve algorithm

**Remark.** The algorithm can also be presented in a recursive fashion. We start with some definitions:

- \( \text{Tip(} \alpha \text{)} = \alpha(T) \), for any curve \( \alpha \) defined in an interval \([0,T]\).
Let $M$ be a manifold with a Riemannian metric in $\mathcal{H}_M$.

1. For any $R > 0$ there is $L > 0$ such that any GACDC starting at $x \in J \cap B_R$ has length at most $L$, and can be extended until it reaches an $A_3$ point.

2. The algorithm 7.20 always reports “success!” after a finite number of steps, for any starting point $x \in J$.

Definition 7.22. A pair $(S, O)$ of open subsets of $T_pM$ with $\tilde{S} \subset O$, is transient if and only if, for any point $x$ in $S \cap J$, a finite number of steps of the linking curve algorithm will extend any aspirant curve that starts at $x$ to either an aspirant curve with endpoint outside of $O$, or an FCLC contained in $O$. The gain of a transient pair $(S, O)$ is the infimum of all $|x| - |y|$, for all $x \in S$, $y \in V_1 \setminus O$ such that there is an aspirant curve starting at $x$ and ending at $y$.

A transient pair is positive if it has positive gain.

A transient pair is bounded if there is a uniform bound for the length of any aspirant curve contained in $O$.

Lemma 7.23. For any point $x$ of type NC, $A_2$, $A_3$, $A_4$, $D^+_4$ or $D^-_1$, there is a positive bounded transient pair $(S, O)$, with $x \in S$.

Lemma 7.23 is all we need to complete the proof of Main Theorem B.

Proof of theorem 7.21. We prove the second part first.

Define:

$$R_0 = \sup \left\{ R : \text{for all } x \in B_R, \text{ the algorithm starting at } x \text{ reports success! after a finite amount of steps} \right\}$$

We will assume that $R_0$ is finite and derive a contradiction, thus showing that the algorithm always reports success after a finite amount of steps. Using lemma 7.23, we cover $B_{R_0}$ by a finite number of neighborhoods $\{S_i\}_{i=1}^N$, where $(S_i, O_i)$ are bounded positive transient pairs. Then $B_{R_0 + \varepsilon}$ is also covered by $\cup S_i$ for some $\varepsilon > 0$. Let $\varepsilon_0$ be the minimum of $\varepsilon$, and all the gains of the $N$ pairs $(S_i, O_i)$.

Take a point $x \in B_{R_0 + \varepsilon_0}$ and assume, without loss of generality, that $x \in S_1$. If, after a finite number of steps, the algorithm finds an FCLC starting at $x$ and contained in $O_1$, it will report success. By hypothesis, the only other possibility is that after a finite number of steps, we find an aspirant curve $\alpha$ with endpoint $y$ outside of $O_1$.

Since $\varepsilon_0$ is smaller than the gain of the pair $(S_1, O_1)$, and there is an aspirant curve starting at $x \in S_1 \cap B_{R_0 + \varepsilon_0}$ and ending at $y \in V_1 \setminus O_1$, we deduce that $|y| < R_0$. By hypothesis, after a finite number of steps, the algorithm starting at the trivial aspirant curve $\{y\}$ finds a saturated aspirant curve $\beta$ that joins $y$ to some point $z$. The algorithm, starting with the aspirant curve $\alpha$, would take the same decisions, since they depend only on the tip, and produce $\alpha * \beta$, which is an aspirant curve starting at $x$ and ending at $z \in I$. We remark that this may not be necessary if $y$ already belongs to $I$, in which case we can take $z = y$ and $\beta$ to be trivial.
Linking curves, sutured manifolds and the Ambrose conjecture

If we want to complete this aspirant curve to get a saturated one, it remains to reply to all the loose ACDCs in $\alpha$. Let $\alpha_1, \ldots, \alpha_k$ be the loose ACDCs in $\alpha$.

Each $\alpha_i$, except possibly its endpoints, is contained in $V_1^0 \setminus S A_2$, so we can reply to $\alpha_i$ starting at any point in $N C$. Suppose that after replying to one of them, we hit an $A_2$ point $y_0$. Then, since $e_0$ is smaller than the gain of the pair $(S_1, O_1)$, and the aspirant curve $\alpha$ has been expanded into a new aspirant curve starting at $x$ and ending at $y_0$, we deduce that $y_0 \notin O_1$ and $|y_0| < R_0$. By hypothesis, we can append a saturated aspirant curve that joins $y_0$ to some $z_0 \in N C \cap B_{|y_0|}$, and continue to reply to the remaining loose ACDCs.

Thus, starting from $x$, the algorithm takes a finite number of steps to produce an aspirant curve $\alpha$ ending at $y$, which has only a finite number of loose ACDCs. Then, again in a finite number of steps, the algorithm produces $\alpha \ast \beta$, finishing at $z \in Z$.

For each loose ACDCs $\alpha_i$, it takes one step to find a retort $\beta_i$ to $\alpha_i$, plus a finite number of steps to link the tip of $\beta_i$ to a point in $N C$, if that is necessary.

Altogether, the process takes a finite number of steps. Since $x \in B_{R_0+\varepsilon_0}$ is arbitrary, this is the desired contradiction that shows that $R_0 = \infty$ and completes the proof of the second part.

The first part follows trivially because the covering is by bounded pairs.

Proof of theorem 7.2. The second part of theorem 7.21 shows that the algorithm always stops after a finite number of steps.

Thus, we can always produce an FCLC starting at any point in $J$. Theorem 7.17 shows that an FCLC is a linking curve.

This, together with lemma 7.3 completes the proof of theorem 7.2.

It only remains to prove lemma 7.23. Before we can prove it, we need to look at $A_4$ and $D_4$ points more closely.

7.8. CDCs in adapted coordinates for $A_4$ and $D_4$ points. As we mentioned in section 6, the radial vector field, and the spheres of constant radius of $T_p M$, which have very simple expressions in standard linear coordinates in $T_p M$, are distorted in adapted coordinates. Thus, the distribution $D$ and the CDCs do not always have the same expression in adapted coordinates. In this section, we study them qualitatively. We will use the name $R: T_p M \to \mathbb{R}$ for the radius function, and $r$ for the radial vector field, and we assume that our conjugate point is a first conjugate point (it lies in $\partial V_1$).

7.8.1. $A_4$ points. In a neighborhood $O$ of an $A_4$ point, $T_{p_1} M_1$ can be stratified as an isolated $A_4$ point, inside a stratum of dimension 1 of $A_3$ points, inside a smooth surface consisting otherwise on $A_2$ points. The conjugate points are given by $4x_1^2 + 2x_1x_2 + x_3 = 0$, and the $A_3$ points are given by the additional equation $12x_1^2 + 2x_2 = 0$. The kernel is generated by the vector $\frac{\partial}{\partial x_1}$ at any conjugate point and we can assume that $D$ is close to $\frac{\partial}{\partial x_1}$ in $O \cap C$.

The radial vector field does not have a fixed expression in adapted coordinates, but the distribution $D$ is a smooth line distribution and its integral curves are smooth. Thus, the $A_4$ point belongs to exactly one integral curve of $D$.

As we saw, $A_3(I)$ (resp $A_3(II)$) points have neighborhoods without $A_3(II)$ (resp $A_3(I)$) points. The $A_4$ point splits $A_3$ into two branches, and it can be shown easily that they must be of different types. Composing with the coordinate change $(x_1, x_2, x_3) \to (-x_1, x_2, x_3)$ if necessary, we can assume that the CDCs travel in the directions shown in figure 3.
7.8.2. $D_4^-$ points. In a neighborhood $O$ of adapted coordinates near a $D_4^-$ point, $C$ is a cone given by the equations $0 = -x_1^2 - x_2^2 + x_3^3$. The kernel of $de_1$ at the origin is the plane $x_3 = 0$, which intersects this cone only at $(0, \ldots, 0)$. Three generatrices of the cone consist of $A_3$ points (they are given by the equations \{$x_2 = 0, x_1 - x_3 = 0$\}, \{$2x_1 + x_3 = 0, x_2 = \sqrt{3}x_1$\} and \{$2x_1 + x_3 = 0, x_2 = -\sqrt{3}x_1$\}), and the rest of the points of the cone are $A_2$.

The radial vector field $(r_1, r_2, r_3)$ at the origin must lie within the solid cone $-r_1^2 - r_2^2 + r_3^3 > 0$, because the number of conjugate points (counting multiplicities) in a radial line through a point close to $(0, 0, 0)$, must be 2. In particular, $|r_3| > 0$. Composing with the coordinate change $(x_1, x_2, x_3) \rightarrow (-x_1, -x_2, -x_3)$ to the left and $(x_1, x_2, x_3) \rightarrow (x_1, x_2, -x_3)$ to the right, if necessary, we can assume that $r_3 > 0$.

The kernel at the origin is contained in the tangent to the surface $T_0 = \{R(y) = R(0)\}$, and the radius always decreases along a CDC. Thus a CDC starting at a first conjugate point moves away from the origin and may either hit an $A_3$ point, or leave the neighborhood. Thus these points are not sinks of CDCs starting at points in $V_1$.

We now claim that there are three CDCs that start at any $D_4^-$ point and flow out of $O$, and these three CDCs that flow into any $D_4^-$ point, but the latter ones are contained in the set of second conjugate points.

Recall that the $D_4$ point is the origin. We write the radial vector as its value at the origin plus a first order perturbation:

$$r = r^0 + P(x)$$

with $|P(x)| < C|x|$ for some constant $C$.

We will consider angles and norms in $O$ measured in the adapted coordinates in order to derive some qualitative behavior, even though these quantities do not have any intrinsic meaning.

We can measure the angle between a generatrix $G$ and $D$ by the determinant of a vector in the direction of $G$, the radial vector $r$ and the kernel $k$ of $e_1$: the determinant is zero if and only if the angle is zero. The angle between $k$ and $r$ in
this coordinate system is bounded from below, and the norm of \( r \) is bounded close to 1. Thus if we use unit vectors that span \( G \) and \( k \), we get a number \( d(x) \) that is comparable to the sine of the angle between \( G \) and the plane spanned by \( r \) and \( k \). Thus \( c(d(x)) \) is a bound from below to \( |\sin(\alpha)| \), where \( \alpha \) is the angle between \( G \) and \( D \), for some \( c > 0 \).

The kernel is spanned by \((-x_1 + x_3, x_2, 0)\) if \(-x_1 + x_3 \neq 0\). The generatrix of \( C \) at a point \((x_1, x_2, x_3) \in C\) is the line through \((x_1, x_2, x_3)\) and the origin. So \( d \) is computed as follows:

\[
d(x) = \frac{1}{x_1^2 + x_2^2 + x_3^2} \begin{vmatrix} x_1 & x_2 & x_3 \\ -x_1 + x_3 & x_2 & 0 \\ r_1 & r_2 & r_3 \end{vmatrix}
\]

Let us look for the roots of the lower order (0-th order) approximation:

\[
d_0(x) = \frac{1}{x_1^2 + x_2^2 + x_3^2} \begin{vmatrix} x_1 & x_2 & x_3 \\ -x_1 + x_3 & x_2 & 0 \\ r_1^0 & r_2^0 & r_3^0 \end{vmatrix}
\]

where \((r_1^0, r_2^0, r_3^0)\) are the coordinates of \( r^0 \).

The equation \( \begin{vmatrix} x_1 & x_2 & x_3 \\ -x_1 + x_3 & x_2 & 0 \\ r_1^0 & r_2^0 & r_3^0 \end{vmatrix} = 0 \) is homogeneous in the variables \( x_1 \), \( x_2 \), and \( x_3 \), so we can make the substitution \(-x_1 + x_3 = 1\) in order to study its solutions. We only miss the direction \( \lambda(1,0,1) \), where \( D \) is not aligned with \( G \) because it consists of \( A_3 \) points.

Points in \( C \) now satisfy \( 1 + 2x_1 - x_2^2 = 0 \), and \( d_0(x) = 0 \) becomes \( p(x_2) = -\frac{1}{2} (a - 1)x_3^2 + \frac{1}{2} bx_3^2 - \frac{1}{2} (a + 3)x_2^2 + \frac{1}{2} b = 0 \), for \( a = \frac{r_1^0}{r_3^0} \) and \( b = \frac{r_2^0}{r_3^0} \). The roots of \( A_3 \) points correspond to \( x_2 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}, \) and the third line lies at \( \infty \). We prove that \( p \) has three different roots, one in each interval: \((-\infty, -\frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \infty) \). This follows immediately if we prove \( \lim_{x_2 \to -\infty} p(x_2) = -\infty, p(\frac{1}{\sqrt{3}}) > 0, p(\frac{1}{\sqrt{3}}) < 0 \) and \( \lim_{x_2 \to \infty} p(x_2) = \infty \) for all \( a \) and \( b \) such that \( a^2 + b^2 < 1 \). The first and last ones are obvious, so let us look at the second one. The minimum of

\[
p(\frac{-1}{\sqrt{3}}) = \frac{2\sqrt{3}}{9} a + \frac{2}{3} b + \frac{4\sqrt{3}}{9}
\]

in the circle \( a^2 + b^2 \leq 1 \) can be found using Lagrange multipliers: it is exactly 0 and is attained only at the boundary \( a^2 + b^2 = 1 \). The third inequality is analogous.

Thus, there is exactly one direction where \( D \) is aligned with \( G \) in each sector between two lines of \( A_3 \) points. Take polar coordinates \((\phi, r)\) in \( C \cap V_1 \). The roots of \( d_0 \) are transverse, and thus if \( \phi_0 \) corresponds to a root of \( d_0 \), then at a line in direction \( \phi \) close to \( \phi_0 \), the angle between \( D \) and \( G \) is at least \( c(\phi - \phi_0) + \eta(\phi, r) \), for \( c > 0 \) and \( \eta(\phi, r) = o(r) \). If, at a point in the line with angle \( \phi \) and sufficiently small \( r > 0 \), we move upwards in the direction of \( D \) (in the direction of increasing radius), we hit the line of \( A_3 \) points, not the center. There are two CDCs starting at each side of every \( A_3 \) point. A continuity argument shows that there must be one CDC in each sector that starts at the origin (see figure 4).

Reversing the argument, we see that there are three CD's that descend into the elliptic umbilic point, one in each sector, all contained in the set of second conjugate points.

7.8.3. \( D^+ \) points. The conjugate points in a neighborhood of adapted coordinates lie in the cone \( C \) given by \( 0 = x_1 x_2 - x_3^2 = \frac{1}{4}(x_1 + x_2)^2 - \frac{1}{4}(x_1 - x_2)^2 - x_3^2 \). This time, the kernel of \( d \exp_{\mathfrak{p}} \) at the origin intersects this cone in two lines through
Figure 4. CDCs in the half-cone of first conjugate points near an elliptic umbilic point, using the chart $(x_1, x_2) \rightarrow (x_1, x_2, -\sqrt{x_1^2 + x_2^2})$, for $r_0 = (0, 0, 1)$. The distribution $D$ makes half turn as we make a full turn around $x_1^2 + x_2^2 = 1$, spinning in the opposite direction.

As before, the radius decreases along a CDC, but this time, a CDC starting at a first conjugate point might end up at the origin. Let $F$ be the half cone of first conjugate points (given by the equations $x_1 x_2 = x_3^2$ and $\frac{1}{2}(x_1 + x_2) < 0$). Let $F_+$ be the points of $F$ with radius greater than the origin. Its tangent cone at the origin is $F \cap \{ x_3 < 0 \}$ or $F \cap \{ x_3 > 0 \}$, depending on the sign of the third coordinate of $r_0$.

As in the previous case, we can measure the angle between a generatrix $G$ and $D$ by the determinant of a vector in the direction of $G$, the radial vector $r$ and the kernel $k$ of $e_1$. This time, the kernel is spanned by $(-x_3, x_1, 0)$ in the chart $x_1 \neq 0$.

$$d(x) = \frac{1}{x_1^2 + x_2^2 + x_3^2} \begin{vmatrix} x_1 & x_2 & x_3 \\ -x_3 & x_1 & 0 \\ r_1 & r_2 & r_3 \end{vmatrix}$$
Again, we look for the roots of the lower order (0-th order) approximation, which is equivalent to looking for the zeros of:

\[ \tilde{d}(x) = \begin{vmatrix} x_1 & x_2 & x_3 \\ -x_3 & x_1 & 0 \\ a & b & 1 \end{vmatrix} \]

in the cone \( C \), for \( a = \frac{x_0}{\chi} \) and \( b = \frac{x_1^2}{\chi} \). We can make the substitution \( x_1 = -1 \) in order to study the zeros of the polynomial (we choose \( x_1 < 0 \) because we are interested in the half cone of first conjugate points). This implies \( x_2 = -x_3^2 \) for a point in \( C \), and we are left with \( p(x_3) = -x_3^3 - bx_3^2 + ax_3 + 1 = 0 \). If \( b^2 + 3a > 0 \), \( p \) has two critical points \( \frac{-b \pm \sqrt{b^2 + 3a}}{3} \), otherwise it is monotone decreasing. But even when \( p \) has two critical points, the local maximum may be negative, or the local minimum positive, with one real root.

The vector \( r^0 \) must satisfy \( r^0_3 \neq 0 \) and \( r_1 r_2 - r_3^2 > 0 \), or \( ab > 1 \). There are two chambers for \( r^0 \): \( r^0_1 > 0 \) and \( r^0_3 < 0 \). We will say that a \( D^+_3 \) point such that \( r^0_3 > 0 \) (resp, \( r^0_3 < 0 \)) is of type I (resp, type II).

If \( r^0_3 > 0 \) (or \( a, b > 0 \)), then \( r^0 \) and \( L \cap F \) lie at opposite sides of the kernel of \( dc_1 \) at the origin. The cubic polynomial \( p \) has limit \( \mp \infty \) at \( \pm \infty \), and \( p(0) > 0 \). The line of \( A_3 \) points intersects \( x_1 = -1 \) at \( x_3 = -1 \). We check that \( p(x_3 = -1) = 2 - a - b \) is always negative in the region \( a > 0, b > 0, ab > 1 \). Thus there is exactly one positive root, and two negative ones, one at each side of the line of \( A_3 \) points. This corresponds to the top right picture in figure 5, where the \( x_3 \) axis is vertical, and the CDCs descend, because \( r^0_3 > 0 \).

The positive root gives a direction that is tangent to a CDC that enters into the \( D^+_3 \) point, but moving to a nearby point we find CDCs that miss the origin, and approach either of the two CD Cs that depart from the origin, corresponding to the negative roots of \( p \).

However, if \( r^0_3 < 0 \) (type II), \( p \) may have one or three roots. We note that \( p(0) = 1 > 0 \), and \( p'(x_3) < 0 \) for \( x_3 < 0 \), \( a < 0 \) and \( b < 0 \), so there cannot be any negative root. A CDC starting at a point in \( F \) flows away from the stratum of \( A_3 \) points and out of the neighborhood (see the bottom pictures at figure 5). It can be checked by example that both possibilities do occur.

We want to remark that if there are three roots, the \( D^+_3 \) point is the endpoint of the CD Cs starting at any point in a set of positive \( H^2 \) measure. Fortunately, all of these points are second conjugate points. This is the main reason why we build the synthesis as a quotient of \( V_1 \) rather than all of \( T^0_p M \) but more important: this is a hint of the kind of complications we might find in arbitrary dimension, or for an arbitrary metric, where we cannot list the normal forms and study each possible singularity separately.

**Remark.** In order to find out the number of real roots of \( p \), for any value of \( a \) and \( b \), we used Sturm’s method. However, once we found out the results, we found alternative proofs and did not need to mention Sturm’s method in the proof. The precise boundary between the sets of \( a, b \) such that \( p \) has one or three real roots is found by Sturm’s method. It is given by:

\[ p_3 = -9a^2b^2 - 36a^3 - 36b^3 - 162ab + 243 = 0 \]

7.9. **Proof of lemma 7.23.** Let \( x_0 \in V_1 \) be a point and \( O \) be a cubical neighborhood of adapted coordinates around it. Let \( R \) be a positive number such that \( O \subset B_R(0), \) and \( S \) will be a “small enough” subset of \( O \):

**NC1:** The algorithm reports success! in one step for any non-conjugate point, so any \( S \subset O \), such that \( O \) has no conjugate points, satisfies the claim. The gain is the infimum of the empty set, \(+\infty\), so the pair is positive.
Figure 5. A hyperbolic umbilic point.

Explanation of figure 5. In the TopLeft corner, the cone $C$ appears in blue, the line of $A_3$ points in green, the radial vector at the origin in red, and the CDCs in red.

The other pictures show the CDCs in the parametrization of the half cone of first conjugate points, obtained by projecting onto the plane spanned by $(1, -1, 0)$ and $(0, 0, 1)$. The red dots indicate the directions where $D$ is parallel to the generatrix of the cone. The $A_3$ points lie in the half vertical line with $x_3 < 0$.

TopRight: $a > 0, b > 0$.  
BottomLeft: $a < 0, b < 0$, $p$ has only one real root.  
BottomRight: $a < 0, b < 0$, $p$ has three distinct real roots.

$A_2$: The set of preimages of $p$ by $\exp_p$ is finite on any bounded set, since it can be covered with a finite amount of neighborhoods of adapted coordinates, and in any of them the preimage of any point is a finite set. Covering $B_0(2R)$ with a finite number of neighborhoods of adapted coordinates, it follows that there are only a finite number of unit vectors $v \in F_0 \subset T_pM$ such that for some $0 < s \leq 2R$ it holds that $\exp_p(sv) = p$. The radial lines $L_v = \{sv \ | \ s \in (0, 2R)\}$, for $v \in F_0$, intersect $O \cap C$ in a finite set of points $F$. The set of points $x \in O \cap C$ whose corresponding radial geodesic $\gamma_x$ returns to the base is contained in $F$.

Let $X$ be the union of all the points in $O \cap C$ that belong to a CDC that contains a point in $F$. If $x_0$ is not in $X$, the CDC $\alpha_0$ starting at $x_0$ reaches
We recall that the set of singular points $y \in \partial O$, has a length $\varepsilon > 0$, and none of its points return to the base. Furthermore, for some $\delta > 0$, $\gamma_{\alpha_0(t)}$ is $(\delta, R)$-away from the base for all $t$.

If $x_0$ is in $X$ but does not return to the base, we start with a short ACDC whose tip does not belong to $X$, and then concatenate with a CDC found as before.

We have found an ACDC $\alpha_0$ starting at $x_0$ and reaching $\partial O$ that avoids all points in $C$ that returns to the base. By Lemma 7.11, we can find a GACDC close to $\alpha_0$ that reaches $\partial O$ with length $\varepsilon > 0$. For $x$ in a sufficiently small neighborhood $V$ of $x_0$, there is a GACDC $\alpha$ that reaches $\partial O$ and has length at least $\varepsilon/2$.

The only missing possibility is that $x_0$ itself returns to the base. In that case, we can follow a CDC for a short time, before it reaches another point in $C \setminus V_1$, and then continue with a GACDC in the same way. This can also be done for any $x$ in a sufficiently small neighborhood of $x_0$.

If there is an aspirant curve that starts with $\alpha$, and later has a retort of $\alpha$, starting at a point $z$, then $|z| < |g|$ by Lemma 7.17, because the restriction of the curve from $y$ to $z$ is a linking curve.

Furthermore, $\alpha$ is unbeatable, so that any non-trivial retort of $\alpha$ will increase the radius at most $|x| - |g| - \delta$ for some $\delta > 0$.

So if we take $S$ as the intersection of $V$ and a ball of radius $\delta/2$, then $(S, O)$ is transient, and the gain is at least $\delta/2$.

Any ACDC contained in $O$ is the graph over any CDC of a Lipschitz function with derivative bounded by $\epsilon$, so it has finite length. It follows that the pair is bounded.

**A3:** We recall that the set of singular points $C$ near an $A_3$ point is a surface, and the stratum of $A_2$ points is a smooth curve. An ACDC starting at any $A_2$ point will flow either into the stratum of $A_3$ points transversally (within $C$), or into the boundary of $O$.

For points in a smaller neighborhood $V \subset O$, one of the following things happen:

- If an ACDC starting at $x \in V \cap A_2$ flows into an $A_3$ point, then it can be replied in one step, and the algorithm stops. The algorithm also stops if $x \in A_1$. 
- If the ACDC starting at $x \in V \cap A_2$ flows into $y \in \partial O$, the argument is the same as that for an $A_2$ point.

The length and gain of any GACDC in $O$ are bounded for the same reason as for $A_2$ points, and this is enough to get bounds for aspirant curves contained in $O$.

**A4:** Near an $A_4$ point, $C$ is a smooth surface and $A_3$ is a smooth curve sitting inside $C$. The $A_4$ point is isolated and splits the curve $A_3$ into two parts. One of them, which we call Branch I, consists of $A_3(I)$ points, and the other branch consists of $A_3(II)$ points. The conjugate distribution $D$ coincides with the kernel of $\exp_p$ at the $A_4$ point, and is contained in the tangent to the manifold of $A_3$ points.

As we saw in Subsection 7.8.1, in $O$ there is only one CDC $\alpha_0$ that flows into the $A_4$ point. Hence, if we start at an $A_2$ point that does not return to the base, it is possible to find a GACDC $\alpha_1$ whose tip is not in the image of $\alpha_0$, while still some distance away from the $A_4$ point, so that the slack is positive and we can use Lemma 7.11. Flowing along a CDC $\alpha_2$ that starts at a point which is not in the image of $\alpha_0$ will not meet the $A_4$ point. Thus, $\alpha_1$, followed by a GACDC $\tilde{\alpha}_2$ close to $\alpha_2$, is a GACDC that avoids the $A_4$ point, and it will either hit an $A_3$ point, or leave the neighborhood.
If $x_0$ is the $A_4$ point or returns to the base, we can follow the CDC that starts at that point for a short time, before it reaches another point in $C \setminus V^0_1$, and then continue with a GACDC as in the previous paragraph.

Let $H$ be the integral submanifold of the distribution $D$ in definition 7.1 that contains the $A_4$ point. $H$ contains the image of $\alpha_0$, plus a different CDC that starts at the $A_4$ point.

$H$ is a smooth curve, and splits $O$ into two parts (see figure 6). One of them, $O_1$, contains only $A_2$ points, while the other, $O_2$, contains all the $A_3$ points.

![Figure 6](image)

**Figure 6.** This picture shows a neighborhood of an $A_4$ point in $T_pM$, together with the linking curves that start at $x$ and $y$ (to the left) and the image of the whole sketch by $\exp_p$ (to the right).

As the reader may see in figure 6, a CDC starting at a point $y \in O_1$ flows into the boundary of $O$ without meeting any obstacle. A CDC $\alpha$ starting at a point $x \in O_2$, however, flows into the branch I of $A_3$. We can start a retort $\beta$ at that point, but it will get interrupted when $\exp \circ \beta$ reaches the stratum of the queue d’ordonne that is the image of two strata of $A_2$ points meeting transversally. The retort cannot go any further because only the points “above $\exp(C)$” (the side of $C$) have a preimage, and points in the main sheet of $\exp(C)$ have only one preimage, that is $A_2$. When we hit the stratum of $A_2$ points, we follow a CDC to get a curve that leaves the neighborhood in a similar way as the curve starting at $y$ did.

**$D_4$:** Any CDC starting at any point in a neighborhood of a $D^-_I$, or $D^+_I$ of type I point leaves the neighborhood without meeting other singularities. A nearby GACDC will also do. We only have to worry about the one CDC that flows into the $D^+_I$ of type II, but we always take a nearby GACDC that avoids the center, by a similar argument as the one we used for $A_4$ points.

### 8. Further questions

We have proposed a new strategy for proving the Ambrose conjecture. If our only goal had been to prove that the Ambrose conjecture holds for a generic family...
of metrics, we could have simplified the definitions of unequivocal point and linked points. We have chosen the definitions so that the sutured property does not exclude some common manifolds.

There is a weaker form of the sutured property that may be simpler to prove, allowing for a remainder set $K$ consisting of points that are neither unequivocal nor linked to an unequivocal point, but such that the Hausdorff dimension of $e_1(K)$ is smaller than $n - 2$. We have decided not to include it here, but the reader can find details in chapter 6.5 of [An].

Our approach also hides a fact that James Hebda remarked: the results in this paper show that for a simply connected 3-dimensional manifold with a metric in a generic set, whenever two vectors $x, y \in T_p M$ map through $\exp_p$ to the same point in $M$, they are joined by a linking curve. We can only conjecture that this holds for an arbitrary metric, though we know already that the linking curves will not come from the algorithm presented here.

8.1. Bounding the length of the linking curves. It doesn’t seem likely that a uniform bound can be found for the lengths of the FCLCs built with the linking curve algorithm. Let us show how a naive argument for bounding the length fails at giving a uniform bound.

Let $B_R$ be the maximum length of a linking curve starting at a point $x$ of radius $R$. The algorithm starting at $x$ first adds a GACDC $\alpha$ of length $l$ that leaves a transient neighborhood $U$ of $x$. The next iterations of the algorithm add a linking curve $\beta$ at the tip of $\alpha$, and it only remains to reply to $\alpha$. If this could be done in one step, we would have:

$$B_R < 2l + B_{R - \varepsilon}$$

but unfortunately, $\alpha = \alpha_1 \ast \ldots \ast \alpha_k$ might cross $SA_2$ $k$ times. After adding $\beta$ to the tip of $\alpha$, we can always reply to $\alpha_k$, but then we may have to iterate the algorithm until we add a linking curve starting at the tip of $\alpha_{k - 1}$ before we can reply to the $\alpha_{k - 1}$. This means we may have to add $k$ linking curves, and our bound is only:

$$B_R < 2l + k \cdot B_{R - \varepsilon}$$

This is of little use unless we can bound $k$.

However, it may be enough to find a uniform bound of the composition of the linking curve with $e_1$. Then a metric can be approximated by generic ones, obtaining sequences of linking curves for the approximate metrics, and then using [HL, Lemma 4.2], for instance.
References

[AA] W. Ambröse: Parallel translation of riemannian curvature. Ann. of Math. (2) 64 (1956), 337–363.

[AN] P. Angulo Ardoy: Cut and conjugate points of the exponential map, with applications. arxiv.org/abs/1411.3399, Ph.D. Dissertation at Universidad Autónoma de Madrid (2014)

[AG] P. Angulo Ardoy and L. Guijarro: Cut and singular loci up to codimension 3. Ann. Inst. Fourier (Grenoble) 61 (2011), no. 4, 1655–1681. arxiv.org/abs/0806.3229. (2008)

[AGII] P. Angulo Ardoy and L. Guijarro: Balanced split sets and Hamilton-Jacobi equations. Calc. Var. Partial Differential Equations 40 (2011), no. 1-2, 223–252. arxiv.org/abs/0807.2046. (2008-2009)

[BH] R. A. Blumen thal and J. J. Heb da: The gener alized Cartan-Ambröse-Hicks the or em. C. R. Acad. Sci. Paris Sér. I Math 305 (1987), no. 14, 647–651.

[B] M. A. Buchner: The structur e of the cut lo cus in dimension less than or equal to six. Compositio Math. 37 (1978), no. 1, 103–119.

[B77] M. A. Buchner: Stability of the cut locus in dimensions less than or equal to 6. Invent. Math. 43 (1977), no. 3, 199–231.

[C] É. Cartan: Leçons sur la géométrie des espaces de Riemann. Les Grands Classiques Gauthier-Villars. Éditions Jacques Gabay, Sceaux, 1988.

[CR] M. Castelpietra and L. Riord: Regularity properties of the distance functions to conjugate and cut loci for viscosity solutions of Hamilton-Jacobi equations and applications in Riemannian geometry ESAIM Control Optim. C. R. Math. 16 (2010), no. 3, 695–718. arXiv:0812.4107. (2008).

[CE] J. Cheeger and D. G. Ebin: Comparison theorems in Riemannian geometry. Revised reprint of the 1975 original. AMS Chelsea Publishing, Providence, RI, 2008.

[G] P. Griffiths and J. Wolf: Complete maps and differentiable coverings. Michigan Math. J. 10 (1963), 253–255.

[HL] B. Hambly and T. Lyons: Uniqueness for the signature of a path of bounded variation and the reduced path group. Ann. of Math. (2) 171 (2010), no. 1, 109–167.

[Hi] N. Hicks: A theorem on affine connections. Illinois J. Math. 3 (1959), 242–254.

[IT98] J. Itoh and M. Tanaka: The dimension of a cut locus on a smooth Riemannian manifold. Tôhoku Math. J. 50 (1998), no. 4, 571–575.

[IT00] J. Itoh and M. Tanaka: The Lipschitz continuity of the distance function to the cut locus. Trans. Amer. Math. Soc. 353 (2001), p. 21–40.

[JM] S. Janeczko and T. Mostowski: Relative generic singularities of the exponential map. Compositio Mathematica 96, no. 3 (1995), p. 345–370.

[K] F. Klok: Generic singularities of the exponential map on Riemannian manifolds. Geom. Dedicata 14 (1983), no. 4, 317–342.

[KN] Sh. Kobayashi and K. Nomizu: Foundations of differential geometry. I. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.

[KLU] S. Kurek, M. Lassas and U. Uhlmann: Rigidity of broken geodesic flow and inverse problems. American Journal of Mathematics 132 (2010), no. 2, 529–562.

[O] B. O’Neill: Construction of Riemannian coverings. Proc. Amer. Math. Soc. 19 (1968), 1278–1282.

[Oz] V. Ozolin: Cut loci in Riemannian manifolds. Tôhoku Math. J. (2) 26 (1974), 219–227.
[PT] K. Pawel and H. Reckziegel: Affine submanifolds and the theorem of Cartan-Ambrose-Hicks. Kodai Math. J. 25 (2002), no. 3, 341–356.

[We] A. Weinstein: The generic conjugate locus. In Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968), 299–301. Amer. Math. Soc., Providence, R.I., 1970.

[We2] A. Weinstein: The cut locus and conjugate locus of a riemannian manifold. Ann. of Math. (2) 87 (1968), 29–41.
Universidad Politécnica de Madrid
E-mail address: pablo.angulo@upm.es