BREZIS - LIEB SPACES AND AN OPERATOR VERSION OF BREZIS - LIEB'S LEMMA

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Abstract. The Brezis - Lieb spaces, in which Brezis - Lieb’s lemma holds true for nets, are introduced and studied. An operator version of Brezis - Lieb’s lemma is also investigated.

1. Introduction

Throughout the paper, $(\Omega, \Sigma, \mu)$ stands for a measure space in which every set $A \in \Sigma$ of nonzero measure possesses a subset $A_0 \subseteq A$, $A_0 \in \Sigma$, such that $0 < \mu(A_0) < \infty$. The famous Brezis - Lieb lemma [3, Thm.2] is known as Theorem 1 [3, Thm.2], and as its corollary, Theorem 2 [3, Thm.1], and also as Theorem 3 (cf. [12, Cor.3]), which is a corollary of Theorem 2.

Theorem 1 (Brezis - Lieb’s lemma). Let $j : \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function with $j(0) = 0$ such that, for every $\varepsilon > 0$, there exist two non-negative continuous functions $\phi_\varepsilon, \psi_\varepsilon : \mathbb{C} \rightarrow \mathbb{R}_+$ with

$$(1.1) \quad |j(x + y) - j(x)| \leq \varepsilon \phi_\varepsilon(x) + \psi_\varepsilon(y) \quad (\forall x, y \in \mathbb{C}).$$

Let $g_n$ and $f$ be ($\mathbb{C}$-valued) functions in $L^0(\mu)$ such that $g_n \xrightarrow{\text{a.e.}} 0$; $j(f)$, $\phi_\varepsilon(g_n)$, $\psi_\varepsilon(f) \in L^1(\mu)$ for all $\varepsilon > 0$, $n \in \mathbb{N}$; and let

$$\sup_{\varepsilon > 0, n \in \mathbb{N}} \int_{\Omega} \phi_\varepsilon(g_n(\omega))d\mu(\omega) \leq C < \infty.$$

Then

$$(1.2) \quad \lim_{n \rightarrow \infty} \int_{\Omega} |j(f + g_n) - (j(f) + j(g_n))|d\mu(\omega) = 0.$$ 

For a proof of Theorem 1 see [3].

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Theorem 2 (Brezis - Lieb’s lemma for $L^p$ ($0 < p < \infty$)). Suppose $f_n \xrightarrow{a.e.} f$ and $\int_{\Omega} |f_n|^p d\mu \leq C < \infty$ for all $n$ and some $p \in (0, \infty)$. Then

$$\lim_{n \to \infty} \int_{\Omega} \left( |f_n|^p - |f_n - f|^p \right) d\mu = \int_{\Omega} |f|^p d\mu.$$  

We reproduce here the arguments from [3] since they are short and instructive. Take $j(z) = \phi_{\varepsilon}(z) := |z|^p$ and $\psi_{\varepsilon}(z) = C_{\varepsilon} |z|^p$ for a sufficiently large $C_{\varepsilon}$. Theorem 1 applied to $g_n = f_n - f$ ensures $f \in L^p(\mu)$, which, in view of (1.2), completes the proof of Theorem 2. Theorem 3 below is an immediate corollary of Theorem 2 (cf. also [12, Cor. 3]).

Theorem 3 (Brezis - Lieb’s lemma for $L^p$ ($1 \leq p < \infty$)). Let $f_n \xrightarrow{a.e.} f$ in $L^p(\mu)$ and $\|f_n\|_p \to \|f\|_p$, where $\|f_n\|_p := \left[ \int_{\Omega} |f_n|^p d\mu \right]^{1/p}$ with $f_n \in L^p(\mu)$ and $f_n \in f_n$. Then $\|f_n - f\|_p \to 0$.

Theorem 3 is a Banach lattice type result if $a.e.$--convergence is replaced by $uo$--convergence (cf. [9, Prop. 3.1]). It motivates us to investigate the general class of Banach lattices, in which the statement of Theorem 3 yields. Even more important reason for such investigation relies on the fact that all the above versions of Brezis - Lieb’s lemma in Theorems 1, 2, and 3 are sequential due to the sequential nature of $a.e.$--convergence. It is worth to mention that Corollary 1 may serves as an extension of the Brezis - Lieb lemma (in form of Theorem 3) for nets.

In Section 2, we introduce Brezis - Lieb’s spaces and their sequential version. Then we prove Theorem 4 which gives an internal geometric characterization of Brezis - Lieb’s spaces. We also discuss possible extensions of Theorem 4 to locally solid Riesz spaces.

In Section 3, we prove Theorem 5 which can be seen as an operator version of Theorem 1 in convergence spaces.

For the theory of vector lattices we refer to [1, 2] and for unbounded convergences to [11, 10, 9, 8].

2. Brezis - Lieb spaces

Definition 1. A normed lattice $(E, \| \cdot \|)$ is said to be a Brezis - Lieb space (shortly, a BL--space) (resp. $\sigma$--Brezis - Lieb space ($\sigma$-BL--space)) if, for any net $x_\alpha$ (resp, for any sequence $x_n$) in $X$ such that $\|x_\alpha\| \to \|x_0\|$ (resp.
\[\|x_n\| \to \|x_0\|\] and \(x_n^{uo} \to x_0\) (resp. \(x_n^{uo} \to x_0\)) we have that \(\|x_n - x_0\| \to 0\) (resp. \(\|x_n - x_0\| \to 0\)).

Trivially, any normed Brezis-Lieb space is a \(\sigma\)-BL-space, and any finite-dimensional normed lattice is a \(BL\)-space. Taking into account that a.e.−convergence for sequences in \(L^p\) is the same as \(uo\)−convergence [9], Theorem 3 says exactly that \(L^p\) is a \(\sigma\)-BL-space for \(1 \leq p < \infty\).

**Example 1.** The Banach lattice \(c_0\) is not a \(\sigma\)−Brezis-Lieb space. To see this, take \(x_n = e_{2n} + \sum_{k=1}^{n} \frac{1}{k}e_k\) and \(x = \sum_{k=1}^{\infty} \frac{1}{k}e_k\) in \(c_0\). Clearly, \(\|x\| = \|x_n\| = 1\) for all \(n\) and \(x_n^{uo} \to x\), however \(1 = \|x - x_0\|\) does not converge to 0.

A slight change of an infinite-dimensional BL-space may turn it into a normed lattice which is even not a \(\sigma\)−BL-space.

**Example 2.** Let \(E\) be a Brezis-Lieb space, \(\dim(E) = \infty\). Let \(E_1 = \mathbb{R} \oplus_\infty E\). Take any disjoint sequence \((y_n)_{n=1}^{\infty}\) in \(E\) such that \(\|y_n\|_E \equiv 1\). Then \(y_n^{uo} \to 0\) in \(E\) [9] Cor.3.6]. Let \(x_n = (1, y_n) \in E_1\). Then \(\|x_n\|_{E_1} = \sup(1, \|y_n\|_E) = 1\) and \(x_n = (1, y_n)^{uo} \to (1, 0) =: x\) in \(E_1\), however \(\|x_n - x\|_{E_1} = \|(0, y_n)\|_{E_1} = \|y_n\|_E = 1\) and so, \(x_n\) does not converge to \(x\) in \((E_1, \| \cdot \|_{E_1})\). Therefore \(E_1 = \mathbb{R} \oplus_\infty E\) is not a \(\sigma\)−Brezis-Lieb space.

In order to characterize BL-spaces, we introduce the following definition.

**Definition 2.** A normed lattice \((E, \| \cdot \|)\) is said to have the Brezis-Lieb property (shortly, BL-property), whenever \(\limsup_{n \to \infty} \|u_0 + u_n\| > \|u_0\|\) for any disjoint normalized sequence \((u_n)_{n=1}^{\infty}\) in \(E_+\) and for any \(u_0 \in E\), \(u_0 > 0\).

Clearly, every finite dimensional normed lattice \(E\) has the \(BL\)−property. The Banach lattice \(c_0\) obviously does not have the \(BL\)−property. The modification of the norm in an infinite-dimensional Banach lattice \(E\) with the \(BL\)−property, as in Example 2 turns it into a Banach lattice \(E_1 = \mathbb{R} \oplus_\infty E\) without the \(BL\)−property. Indeed, take a disjoint normalized sequence \((y_n)_{n=1}^{\infty}\) in \(E_+\). Let \(u_0 = (1, 0)\) and \(u_n = (0, y_n)\) for \(n \geq 1\). Then \((u_n)_{n=0}^{\infty}\) is a disjoint normalized sequence in \((E_1)_+\) with \(\limsup_{n \to \infty} \|u_0 + u_n\| = 1\). Remarkably, it is not a coincidence.

**Theorem 4.** For a \(\sigma\)−Dedekind complete Banach lattice \(E\), the following conditions are equivalent:
(1) $E$ is a Brezis-Lieb space;
(2) $E$ is a $\sigma$-Brezis-Lieb space;
(3) $E$ has the BL-property and the norm in $E$ is order continuous.

Proof. (1) $\Rightarrow$ (2) It is trivial.
(2) $\Rightarrow$ (3) We show first that $E$ has BL-property. Notice that in this part of the proof the $\sigma$-Dedekind completeness of $E$ will not be used. Suppose that there exists a disjoint normalized sequence $(u_n)_{n=1}^{\infty}$ in $E_+$ and a $u_0 > 0$ in $E$ with
\[ \limsup_{n \to \infty} \| u_0 + u_n \| = \| u_0 \|. \]
Since $\| u_0 \| \leq \| u_0 + u_n \|$, then $\lim_{n \to \infty} \| u_0 + u_n \| = \| u_0 \|$. Denote $v_n := u_0 + u_n$. By [9, Cor.3.6], $u_n \xrightarrow{uo} 0$ and hence $v_n \xrightarrow{uo} u_0$.

Since $E$ is a $\sigma$-BL-space and $\lim_{n \to \infty} \| v_n \| = \| u_0 \|$, then $\| v_n - u_0 \| \to 0$, which is impossible in view of $\| v_n - u_0 \| = \| u_0 + u_n - u_0 \| = \| u_n \| = 1$.

Assume that the norm in $E$ is not order continuous. Then, by the Fremlin-Meyer-Nieberg theorem (see for example [2, Thm.4.14]) there exist $y \in E_+$ and a disjoint sequence $e_k \in [0, y]$ such that $\| e_k \| \neq 0$. Without lost of generality, we may assume $\| e_k \| = 1$ for all $k \in \mathbb{N}$. By the $\sigma$-Dedekind completeness of $E$, for any sequence $\alpha_n \in \mathbb{R}_+$ there exist the following vectors
\begin{equation}
(2.1) \quad x_0 = \bigoplus_{k=1}^{\infty} e_k, \quad x_n = \alpha_{2n} e_{2n} + \bigoplus_{k=1, k \neq n, k \neq 2n}^{\infty} e_k \quad (\forall n \in \mathbb{N}).
\end{equation}

Now, we choose $\alpha_{2n} \geq 1$ in (2.1) such that $\| x_n \| = \| x_0 \|$ for all $n \in \mathbb{N}$. Clearly, $x_n \xrightarrow{uo} x_0$. Since $E$ is a $\sigma$-BL-space then $\| x_n - x_0 \| \to 0$, violating
\[ \| x_n - x_0 \| = \| (\alpha_{2n} - 1) e_{2n} - e_n \| = \| (\alpha_{2n} - 1) e_{2n} + e_n \| \geq \| e_n \| = 1. \]

Obtained contradiction shows that the norm in $E$ is order continuous.

(3) $\Rightarrow$ (1) If $E$ is not a Brezis-Lieb space, then there exists a net $(x_\alpha)_{\alpha \in A}$ in $E$ such that $x_\alpha \xrightarrow{uo} x$ and $\| x_\alpha \| \to \| x \|$ but $\| x_\alpha - x \| \not\to 0$. Then $x_\alpha \xrightarrow{uo} |x|$ and $\| x_\alpha \| \to \| |x| \|$.

Notice that $\| x_\alpha | - |x| \| \not\to 0$. Indeed, if $\| x_\alpha | - |x| \| \to 0$ then $(x_\alpha)_{\alpha \in A}$ is eventually in $[|x|, |x|]$ and then $(x_\alpha)_{\alpha \in A}$ is almost order bounded. Since $E$ is order continuous and $x_\alpha \xrightarrow{uo} x$, then by [10, Pop.3.7.] $\| x_\alpha - x \| \to 0$, which is impossible. Therefore, without lost of generality, we may assume that $x_\alpha \in E_+$ and, by normalizing, also $\| x_\alpha \| = \| x \| = 1$ for all $\alpha$.

Passing to a subnet, denoted again by $x_\alpha$, we may assume
\begin{equation}
(2.2) \quad \| x_\alpha - x \| > C > 0 \quad (\forall \alpha \in A). \]


Notice that $x \geq (x - x_\alpha)^+ = (x_\alpha - x)^- \to 0$, and hence $(x_\alpha - x)^- \to 0$. The order continuity of the norm ensures

$$\|(x_\alpha - x)^-\| \to 0. \quad (2.3)$$

Denoting $w_\alpha = (x_\alpha - x)^+$ and using (2.2) and (2.3), we may also assume

$$\|w_\alpha\| = \|(x_\alpha - x)^+\| > C \quad (\forall \alpha \in A). \quad (2.4)$$

In view of (2.4), we obtain

$$2 = \|x_\alpha\| + \|x\| \geq \|(x_\alpha - x)^+\| = \|w_\alpha\| > C \quad (\forall \alpha \in A). \quad (2.5)$$

Since $w_\alpha \to (x - x)^+ = 0$ then, for any fixed $\beta_1, \beta_2, ..., \beta_n$,

$$0 \leq w_\alpha \wedge (w_{\beta_1} + w_{\beta_2} + ... + w_{\beta_n}) \to 0 \quad (\alpha \to \infty). \quad (2.6)$$

Since $x_\alpha \to x$, then $x_\alpha \wedge x \to x \wedge x = x$ and so $x_\alpha \wedge x \to x$. By the order continuity of the norm, there is an increasing sequence of indices $\alpha_n$ in $A$ with

$$\|x - x_\alpha \wedge x\| \leq 2^{-n} \quad (\forall \alpha \geq \alpha_n). \quad (2.7)$$

Furthermore, by (2.6), we may also suppose that

$$\|w_\alpha \wedge (w_{\alpha_1} + w_{\alpha_2} + ... + w_{\alpha_n})\| \leq 2^{-n} \quad (\forall \alpha \geq \alpha_{n+1}). \quad (2.8)$$

Since

$$\sum_{k=1, k \neq n}^{\infty} \|w_{\alpha_n} \wedge w_{\alpha_k}\| \leq \sum_{k=1}^{n-1} \|w_{\alpha_n} \wedge (w_{\alpha_1} + ... + w_{\alpha_{n-1}})\| + \sum_{k=n+1}^{\infty} \|w_{\alpha_k} \wedge (w_{\alpha_1} + ... + w_{\alpha_{k-1}})\| \leq (n - 1) \cdot 2^{-n+1} + \sum_{k=n+1}^{\infty} 2^{-k+1} = n \cdot 2^{-n+1}, \quad (2.9)$$

the series $\sum_{k=1, k \neq n}^{\infty} w_{\alpha_n} \wedge w_{\alpha_k}$ converges absolutely and hence in norm for any $n \in \mathbb{N}$. Take

$$\omega_{\alpha_n} := \left( w_{\alpha_n} - \sum_{k=1, k \neq n}^{\infty} w_{\alpha_n} \wedge w_{\alpha_k} \right)^+ \quad (\forall n \in \mathbb{N}). \quad (2.10)$$

First, we show that the sequence $(\omega_{\alpha_n})_{n=1}^{\infty}$ is disjoint. Let $m \neq p$, then

$$\omega_{\alpha_m} \wedge \omega_{\alpha_p} = \left( w_{\alpha_m} - \sum_{k=1, k \neq m}^{\infty} w_{\alpha_m} \wedge w_{\alpha_k} \right)^+ \wedge \left( w_{\alpha_p} - \sum_{k=1, k \neq p}^{\infty} w_{\alpha_p} \wedge w_{\alpha_k} \right)^+ \leq \left( w_{\alpha_m} - w_{\alpha_m} \wedge w_{\alpha_p} \right)^+ \wedge \left( w_{\alpha_p} - w_{\alpha_p} \wedge w_{\alpha_m} \right)^+ = \left( w_{\alpha_m} \wedge w_{\alpha_p} \right)^+ \wedge \left( w_{\alpha_p} \wedge w_{\alpha_m} \right)^+ = \left( w_{\alpha_m} \wedge w_{\alpha_p} \right)^+ \wedge \left( w_{\alpha_m} \wedge w_{\alpha_p} \right)^+ = \left( w_{\alpha_m} \wedge w_{\alpha_p} \right)^+. \quad (2.11)$$
$$(w_{\alpha m} - w_{\alpha m} \wedge w_{\alpha p}) \wedge (w_{\alpha p} - w_{\alpha m} \wedge w_{\alpha p}) = 0.$$ 

By (2.9),

$$\|w_{\alpha n} - \omega_{\alpha n}\| = \|w_{\alpha n} - \left( w_{\alpha n} - \sum_{k=1, k \neq n}^{\infty} w_{\alpha n} \wedge w_{\alpha k} \right) \| =$$

$$\|w_{\alpha n} - \left( w_{\alpha n} - \sum_{k=1, k \neq n}^{\infty} w_{\alpha n} \wedge w_{\alpha k} \right) \| = \sum_{k=1, k \neq n}^{\infty} w_{\alpha n} \wedge w_{\alpha k} \| \leq n2^{-n+1}. \quad (\forall n \in \mathbb{N}). \quad (2.11)$$

Combining (2.11) with (2.5) gives

$$2 \geq \|w_{\alpha n}\| \geq \|\omega_{\alpha n}\| \geq C - n2^{-n+1} \quad (\forall n \in \mathbb{N}). \quad (2.12)$$

Passing to further increasing sequence of indices, we may assume that

$$\|w_{\alpha n}\| \to M \in [C, 2] \quad (n \to \infty).$$

Now

$$\lim_{n \to \infty} \|M^{-1}x + \|\omega_{\alpha n}\|^{-1}\omega_{\alpha n}\| = M^{-1} \lim_{n \to \infty} \|x + \omega_{\alpha n}\| = [\text{by (2.11)}] =$$

$$M^{-1} \lim_{n \to \infty} \|x + w_{\alpha n}\| = [\text{by (2.3)}] = M^{-1} \lim_{n \to \infty} \|x + (x_{\alpha n} - x)\| =$$

$$M^{-1} \lim_{n \to \infty} \|x_{\alpha n}\| = M^{-1} = \|M^{-1}x\|,$$

violating the Brezis-Lieb property for $u_0 = M^{-1}x$ and $u_n = \|\omega_{\alpha n}\|^{-1}\omega_{\alpha n}$, $n \geq 1$. The obtained contradiction completes the proof. \qed

The next fact is a corollary of Theorem 4 which states a Brezis-Lieb’s type lemma for nets in $L^p$.

**Corollary 1.** Let $f_{\alpha} \xrightarrow{u_0} f$ in $L^p(\mu)$, $(1 \leq p < \infty)$, and $\|f_{\alpha}\|_p \to \|f\|_p$. Then $\|f_{\alpha} - f\|_p \to 0$.

We do not know where or not implication (2) $\Rightarrow$ (3) of Theorem holds true without the assumption that the Banach lattice $E$ is $\sigma$-Dedekind complete.

**Question 1.** Does every $\sigma$-Brezis-Lieb Banach lattice have order continuous norm?
In the proof of (2) ⇒ (3) σ—Dedekind completeness of $E$ has been used only for showing that $E$ has order continuous norm. So, any σ—Brezis—Lieb Banach lattice has the Brezis—Lieb property. Therefore, for answering in positive the question of possibility to drop σ—Dedekind completeness assumption in Theorem $[4]$ it is sufficient to have the positive answer to the following question.

**Question 2.** Does the Brezis—Lieb property imply order continuity of the norm?

In the end of the section we discuss possible generalizations of Brezis—Lieb spaces and Brezis—Lieb property. To avoid overloading the text, we restrict ourselves with the case of multi-normed Brezis—Lieb lattices, postponing the discussion of locally solid Brezis—Lieb lattices to further papers.

A multi-normed vector lattice (shortly, MNVL) $E = (E, \mathcal{M})$ (see $[5]$):

(a) is said to be a Brezis—Lieb space if

$$[x_\alpha \overset{u_0}{\rightharpoonup} x_0 \& m(x_\alpha) \to m(x_0) \quad (\forall m \in \mathcal{M})] \Rightarrow [x_\alpha \overset{\mathcal{M}}{\to} x_0].$$

(b) has the Brezis—Lieb property, if for any disjoint sequence $(u_n)_{n=1}^\infty$ in $E_+$ such $u_n$ does not converge in $\mathcal{M}$ to 0 and for any $u_0 > 0$, there exists $m \in \mathcal{M}$ such that $\limsup_{n \to \infty} m(u_0 + u_n) > m(u_0)$.

A σ-Brezis—Lieb MNVL is defined by replacing of nets with sequences.

By using the above definitions one can derive from Theorem $[4]$ the following result, whose details are left to the reader.

**Corollary 2.** For an MNVL $E$ with a separating order continuous multi-norm $\mathcal{M}$, the following conditions are equivalent:

1. $E$ is a Brezis—Lieb space;
2. $E$ is a σ—Brezis—Lieb space;
3. $E$ has the Brezis—Lieb property.

3. **Operator version of Brezis—Lieb’s lemma in convergent vector spaces**

In this section, we consider both complex and real vector spaces and vector lattices. A convergence “$\overset{c}{\rightharpoonup}$” for nets in a set $X$ is defined by the following conditions:

(a) $x_\alpha \equiv x \Rightarrow x_\alpha \overset{c}{\rightharpoonup} x$, and
(b) $x_\alpha \overset{c}{\to} x \Rightarrow x_\beta \overset{c}{\to} x$ for every subnet $x_\beta$ of $x_\alpha$.

A mapping $f$ from a convergence set $(X, c_X)$ into a convergence set $(Y, c_Y)$ is said to be $c_X c_Y$-continuous (or just continuous), if $x_\alpha \overset{c_X}{\to} x$ implies $f(x_\alpha) \overset{c_Y}{\to} f(x)$ for every net $x_\alpha$ in $X$.

A subset $A$ of $(X, c_X)$ is called $c_X$-closed if $A \ni x_\alpha \overset{c_X}{\to} x \Rightarrow x \in A$. If the set $\{x\}$ is $c_X$-closed for every $x \in X$ then $c_X$ is called $T_1$-convergence. It is immediate to see that $c_X \in T_1$ iff every constant net $x_\alpha \equiv x$ does not $c_X$-converge to any $y \neq x$.

Under convergence vector space $(X, c_X)$ we understand a vector space $X$ with a convergence $c_X$ such that the linear operations in $X$ are $c_X$-continuous. $(E, c_E)$ is a convergence vector lattice if $(E, c_E)$ is a convergence vector space which is a vector lattice where the lattice operations are also $c_E$-continuous.

For further references see [1, 2, 4].

Motivated by the proof of the famous Brezis - Lieb’s lemma [3, Thm.2], we present its operator version in convergent spaces.

Given a convergence complex vector space $(X, c_X)$; two convergence complex vector lattices $(E, c_E)$ and $(F, c_F)$, where $F$ is Dedekind complete; an order ideal $E_0$ in $E_+ - E_+$; and a $c_{E_0} a_F$-continuous positive linear operator $T : E_0 \to F$, where $a_F$ stands for the order convergence in $F$. Furthermore, let $J : X \to E$ be $c_X c_E$-continuous, $J(0) = 0$, and, for every $\varepsilon > 0$, let there exist two $c_X c_E$-continuous mappings $\Phi_\varepsilon, \Psi_\varepsilon : X \to E_+$ with

$$|J(x + y) - Jx| \leq \varepsilon \Phi_\varepsilon x + \Psi_\varepsilon y \quad (\forall x, y \in X).$$

**Theorem 5** (An operator version of Brezis - Lieb’s lemma for nets). Let $X, E, E_0, F, T : E_0 \to F$, and $J : X \to E$ satisfy the above hypothesis. Let $(g_\alpha)_{\alpha \in A}$ be a net in $X$ satisfying $g_\alpha \overset{c_X}{\to} 0$, let $f \in X$ be such that $|Jf|, \Phi_\varepsilon g_\alpha, \Psi_\varepsilon f \in E_0$ for all $\varepsilon > 0, \alpha \in A$, and let some $u \in F_+$ exist with $T^\Phi_\varepsilon g_\alpha \leq u$ for all $\varepsilon > 0, \alpha \in A$. Then

$$T \left( |J(f + g_\alpha) - (Jf + Jg_\alpha)| \right) \overset{c_F}{\to} 0 \quad (\alpha \to \infty).$$

**Proof.** It follows from (3.1) that

$$|J(f + g_\alpha) - (Jf + Jg_\alpha)| \leq |J(f + g_\alpha) - Jg_\alpha| + |Jf| \leq \varepsilon \Phi_\varepsilon g_\alpha + \Psi_\varepsilon f + |Jf|,$$

and hence

$$|J(f + g_\alpha) - (Jf + Jg_\alpha)| - \varepsilon \Phi_\varepsilon g_\alpha \leq \Psi_\varepsilon f + |Jf| \quad (\varepsilon > 0, \alpha \in A).$$
Thus
\begin{equation}
0 \leq w_{\varepsilon, \alpha} := \left( |J(f + g_\alpha) - (Jf + Jg_\alpha)| - \varepsilon \Phi_\varepsilon g_\alpha \right) + \leq \Psi_\varepsilon f + |Jf|
\end{equation}
for all \( \varepsilon > 0 \) and \( \alpha \in A \). It follows from (3.2) and from \( C_X c_E \)-continuity of \( J \) and \( \Phi_\varepsilon \), that \( E_0 \ni w_{\varepsilon, \alpha} \overset{c_E}{\to} 0 \) as \( \alpha \to \infty \). Furthermore, (3.2) implies
\begin{equation}
|J(f + g_\alpha) - (Jf + Jg_\alpha)| \leq w_{\varepsilon, \alpha} + \varepsilon \Phi_\varepsilon g_\alpha \quad (\varepsilon > 0, \alpha \in A).
\end{equation}
Since \( T \geq 0 \) and \( T \Phi_\varepsilon g_\alpha \leq u \), we get from (3.3)
\begin{equation}
0 \leq T \left( |J(f + g_\alpha) - (Jf + Jg_\alpha)| \right) \leq T w_{\varepsilon, \alpha} + \varepsilon T \Phi_\varepsilon g_\alpha \leq T w_{\varepsilon, \alpha} + \varepsilon u
\end{equation}
for all \( \varepsilon > 0 \) and \( \alpha \in A \). Since \( F \) is Dedekind complete and \( T \) is \( c_{E_0} o_F \)-continuous, \( T w_{\varepsilon, \alpha} \overset{op}{\to} 0 \), and in view of (3.4)
\[ 0 \leq (o_F) - \limsup_{\alpha \to \infty} T \left( |J(f + g_\alpha) - (Jf + Jg_\alpha)| \right) \leq \varepsilon u \quad (\forall \varepsilon > 0). \]
Then \( T \left( |J(f + g_\alpha) - (Jf + Jg_\alpha)| \right) \overset{op}{\to} 0. \quad \square \)

(1) Replacing nets by sequences one can obtain a sequential version of Theorem 5 whose details are left to the reader.

(2) In the case of \( F = \mathbb{R} \) and \( X = E = L^0(\mu) \) with the almost everywhere convergence, \( E_0 = L^1(\mu) \), \( T f = \int f d\mu \), and \( J : X \to E \) given by \( Jf = j \circ f \), where \( j : \mathbb{C} \to \mathbb{C} \) is continuous with \( j(0) = 0 \) such that for every \( \varepsilon > 0 \) there exist two continuous functions \( \phi_\varepsilon, \psi_\varepsilon : \mathbb{C} \to \mathbb{R}_+ \) satisfying
\[ |j(x + y) - j(x)| \leq \varepsilon \phi_\varepsilon(x) + \psi_\varepsilon(y) \quad (\forall x, y \in \mathbb{C}), \]
we obtain Theorem 11 which is the classical Brezis - Lieb’s lemma [3, Thm.2], from Theorem 5 by letting \( \Phi_\varepsilon(f) := \phi_\varepsilon \circ f \) and \( \Psi_\varepsilon(f) := \psi_\varepsilon \circ f \).

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