Perturbation of FRW Spacetime in NP Formalism

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Abstract

Perturbation of FRW spacetime is carried out in NP formalism. The equation governing the scalar, vector and tensor modes take on a very simple and transparent form. All of them can be combined in one master equation for all helicities. The solutions for the closed, flat and open FRW are analytic continuation of the same functions, so only the solutions in the closed model are described. The scalar equation is the same as that of the conformally coupled massless Klein-Gordon field, the vectorial ones are the same as Maxwell equations, and the tensorial ones are for spin-2 fields. The corresponding eigen-functions are all determined, and in particular, the Green’s function and the Lienard-Wiechert type potential also solved. These solutions reduce to the familiar form in flat space.

1 Introduction

Motivated by the success of the Newman-Penrose(NP) method in investigating the perturbation of various black hole geometries, we thought that the application of this method to study the perturbation of the Friedmann-Robertson-Walker spacetime may give some further insight into structure formation. One of the authors [1, 2] had used this formalism to study free Maxwell and Dirac fields in FRW space. After setting up the notations below, we write down the perturbation equations in the next section and solve them. Section 3 discusses the scalar perturbation, and the Green’s function
as well as the Lienard-Wiechert type potential in closed space are found in the final section we discuss some consequences.

Writing the line element as \( ds^2 = a^2 [d\eta^2 - dr^2 - S^2 (d\theta^2 + \sin^2 \theta d\phi^2)] \) with \( S = \frac{\sin(\sqrt{Kr})}{\sqrt{K}} \), we may devise the null tetrad to be \( l_\mu = [1, -1, 0, 0], n_\mu = a^2 \hat{e}_x, m_\mu = -\frac{aS}{\sqrt{2}}[0, 0, 1, i \sin \theta] \) and the complex conjugate \( \bar{m}_\mu \), to express the directional derivatives as \( D = \eta \mu \mu^\mu = \frac{1}{a} \Phi^-, \Delta = \mu \mu^\mu = -\frac{1}{2} D^+, \delta = m^\mu \mu^\mu = \frac{1}{\sqrt{2aS}} \mathcal{L}^- \), \( \delta^* = \bar{m}^\mu \mu^\mu = \frac{1}{\sqrt{2aS}} \mathcal{L}^+ \) with \( \mathcal{D}^\pm = \frac{\partial}{\partial r} \mp \frac{1}{\sin \eta} \frac{\partial}{\partial \eta} \) and \( \mathcal{L}^\pm = \frac{\partial}{\partial S} \mp \frac{1}{\sin \eta} \frac{\partial}{\partial \eta} \). Also, the non-vanishing spin coefficients are given by \( \beta = -\alpha = \frac{\cos \theta}{\sqrt{2S}}, \gamma = -\frac{a}{2a}, 2\mu = \frac{a}{a} - \frac{s'}{S}, -a^2 \rho = \frac{a}{a} + \frac{s'}{S}, \) where the overdot and prime denote derivatives with the conformal time \( \dot{\eta} \) and \( r \) respectively. We will also have to use the operators \( \mathcal{D}^\pm_n = \mathcal{D}^\pm + n \frac{s'}{S} \) and \( \mathcal{L}^\pm_n = \mathcal{L}^\pm + n \cot \theta \).

With the change of variables \( \sqrt{K} \eta = x \), and \( \sin (\sqrt{K}r) = \cosec \xi \), we get \( \dot{\Delta}^\pm_n = i \Delta^\pm_n = i \left( \frac{\partial}{\partial x} \pm \frac{\partial}{\sin \xi \partial x} - n \cot \xi \right) \), and \( \Delta \) looks the same as \( \mathcal{L} \).

Of the NP quantities representing the curvature of the spacetime, all the Weyl scalers \( \Psi \)'s vanish in the homogeneous and isotropic FRW background, and the non-vanishing background Ricci scalers are given through \( R = 6 \left( \frac{a+Ka}{a^2} \right), \Phi_{00} = \frac{1}{a^2} \Phi_{11} = \frac{1}{a^2} \Phi_{22} = \frac{1}{\sqrt{a^2} (Ka^2 + 2\dot{a}^2 - a\ddot{a})} = \frac{4\pi}{a^2} (\rho + p) \). In the work below, we just consider the closed Universe with \( K = 1 \), as the other two, \( K = (0, -1) \) turn out to be analytic continuation of the same solution.

## 2 Perturbation of the space-time

Now we look at the perturbation equations. In combination with the \( \Psi \)'s that vanish in the background, we should take the unperturbed directional derivatives and spin coefficients for first order perturbation equations. Then we can write the first order perturbed Bianchi identities as

\[
\begin{align*}
i \Delta^\pm_s \Phi^\pm_s &\pm \mathcal{L}^\pm_{1-s} \Phi^\pm_{s-1} = Q^\pm_s, \quad (1) \\
i \Delta^\mp_{1-s} \Phi^\mp_{s-1} &\pm \mathcal{L}^\mp_s \Phi^\mp_s = U^\pm_s \quad (2)
\end{align*}
\]

where either the upper or lower signs are taken once at a time, the \( Q \)'s and \( U \)'s are the source terms comprising the perturbed energy-momentum; the \( \Phi \)'s
are related to the Weyl scalars $\Psi$'s, with $\Phi^\pm_2 = \begin{cases} \frac{a^5}{2} S^3 \Psi_0, & + \\ 2aS^3 \Psi_4, & - \end{cases}$ representing the gravitational or tensorial perturbations, $\Phi^\pm_1 = \begin{cases} \frac{a^4}{\sqrt{2}} S^3 \Psi_1, & + \\ \sqrt{2a^2} S^3 \Psi_4, & - \end{cases}$ representing the vectorial or vortical perturbations, and $\Phi_0 = a^3 S^3 \Psi_2$ the scalar perturbations. The equations for the vectorial perturbations are the same as the Maxwell equations discussed in Ref. [1]. It is an interesting proposition that the vortical perturbations arise from the electromagnetic interaction between the Maxwell and Dirac fields. Similarly, the equation governing the scalar perturbation is the same as that of the conformally coupled massless Klein-Gordon field. There are 10 degrees of gauge freedom in the problem as we have formulated, four relating to the general co-variance of general relativity and six corresponding to rotations of the tetrad frame as discussed in Ref. [3]. These conditions can best be used to simplify the source term, but we will not be concerned with the sources in this work, and will only discuss the eigen-modes of these perturbations. Inspecting Eqs. (1 and 2), we see that all the perturbations are related to each other, in that one contributes to the other; existence of any one single type of perturbation gives rise to all.

Eqs. (1 and 2) can be decoupled by operating on the former with $i \Delta^\pm_1$ and the latter with $\mp \mathcal{L}^\pm_s$ and adding the two. The result is

$$\left[ -\Delta^\pm_1 \Delta^\pm_s + \mathcal{L}^\pm_1 \mathcal{L}^\pm_s \right] \Phi^\pm_s = J^\pm_s. \quad (3)$$

Thus the variables are found to be separable. It is easy to establish the operational identity $\mathcal{L}^\pm_1 \mathcal{L}^\pm_s = \mathcal{L}^\pm_1 \mathcal{L}^\pm_s - 2s$, and a similar one for $\Delta$. Letting $p = \pm s$ for the appropriate helicities, all the perturbation equations contained in Eq. (3) can be written in a master equation as

$$\left[ -\Delta^\pm_{1+p} \Delta^-_{-p} + \mathcal{L}^\pm_{1+p} \mathcal{L}^-_{-p} \right] \Phi_p = J_p. \quad (4)$$

Changing the sign of $p$ together with the signs on the operator gives the equation for fields with helicity reversed. The eigenfunctions are separable, and we may write $\Phi_p = p Z \times_p Y$ where $p Y$ are the angular part represented by the spin-weighted spherical harmonics satisfying the

$$\mathcal{L}^\pm_{1+p} \mathcal{L}^-_{-p} p Y^m_l = -(l - p)(l + p + 1) p Y^m_l;$$

the solution is $p Y^m_l(\theta, \phi) = Ne^{in\phi}(1 - \cos \theta)^{m+\frac{p+1}{2}}(1 + \cos \theta)^{m+\frac{p+1}{2}} P^{(m+p,m-p)}_{l-m}$, allowing us to identify the spin-weighted spherical harmonics as the spherical harmonics formed with the Jacobi polynomial $P_n^{(\alpha,\beta)}$. It can easily
be established that $pY_{l}^{m*} = (-1)^{m+p} pY_{l}^{-m}$, and $qY_{l}^{m} = Y_{l}^{m}$ are just the usual spherical harmonics. These are normalized to give $\int_{p} pY_{l}^{m*} pY_{l}^{m} d\Omega = \delta_{ll'} \delta_{mm'}$, and they form a complete set in that $\sum_{l,m} pY_{l}^{m}(\Omega')^{*} pY_{l}^{m}(\Omega) = \delta(\Omega-\Omega')$. Koornwinder[5] developed an addition formula for Jacobi polynomials, and there is a generalized addition theorem for $pY_{l}^{m}$ given in Ref.[6], but we can devise a degenerate form for our purpose by noting that $\delta(\Omega-\Omega')$ can depend only on the angle $\beta$ between the directions $\Omega$ and $\Omega'$ where $\cos \beta = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$. So, multiplying $\delta(\Omega-\Omega') = \sum_{l} b_{l} pY_{l}^{-p}(\beta, 0)$ on both sides by $pY_{-p}^{-p}(\beta, 0)$ and integrating over the solid angle to find $b_{l} = p Y_{l}^{-p}(0, 0)$, we may write

$$\sum_{m} pY_{l}^{m*}(\Omega') pY_{l}^{-m}(\Omega) = pY_{l}^{-p}(0, 0) pY_{l}^{-p}(\beta, 0).$$

For the eigen-functions of $\Delta$ we may substitute $pZ_{\omega}^{\omega}(\eta, r) = e^{-i\omega r} pR_{k}^{\omega}(r)$ to separate the temporal part and write the operator $\Delta_{\pm} = \left( \frac{\partial}{\partial \xi} \pm \frac{\omega}{\sin \xi} + n \cot \xi \right)$ giving the radial eigen-equation

$$\Delta_{l+p}^{\pm} \Delta_{-p}^{\omega} pR_{k}^{\omega} = (k - p) (k + p + 1) pR_{k}^{\omega}$$

that determines the eigen-functions to be

$$pR_{k}^{\omega} = Ai^{b-k-1} (1 - \cos \xi)^{-\frac{(\omega+p)}{2}} (1 + \cos \xi)^{-\frac{(\omega-p)}{2}} P_{\omega-k-1}^{(-\omega-p,-\omega+p)} (\cos \xi)$$

$$= Ai^{b-k-1+p} \sin \omega r e^{-i\omega r} P_{\omega-k-1}^{(-\omega-p,-\omega+p)} (i \cot \xi);$$

$$A$$ is the normalization constant, and the parameters of the Jacobi polynomials are chosen to make the function regular at $r = 0$ and $\pi$. Ref. [7] has proved the non-Hermitian orthogonality of Jacobi polynomials with general parameters. In our case, we find that $A^{2} \int_{i\infty}^{i\infty} (pR_{k}^{\omega} pR_{k}^{\omega}) id (\cos \xi) = \delta_{kk'}$ gives us $A = 2^{\omega-1} \left[ (2k+1)(\omega-k-1)!(k+p)!(k-p)! \right]^{\frac{1}{\pi(\omega+k)!}}$. So $pR_{k}^{\omega*} = -pR_{k}^{\omega}$. Both the angular and radial functions satisfy the spin raising and lowering operations

$$L_{\pm p}^{\pm} Y_{l}^{m} = \pm L_{\pm p}^{(p+1)} Y_{l}^{m} \quad \text{and}$$

$$\Delta_{\pm p}^{\pm} Z_{k}^{\omega} = \pm K_{\pm p}^{(p+1)} Z_{k}^{\omega},$$

where the eigenvalues are $L_{p} = \sqrt{(l-p)(l+p+1)}$ and $K_{p} = \sqrt{(k-p)(k+p+1)}$. In Eqs. [8], the upper signs lower the helicities while the lower sign raise it. Also the eigen-values satisfy $L_{-p} (or K_{p}) = L_{p-1} (or K_{p-1})$. Some of the
eigen-functions representing the various modes of perturbation are displayed in the accompanying figures. The scalar perturbations, (1), represent the density perturbations. These give rise to the structures. Fig. (2) show the vectorial perturbations. These are responsible for generating rotational or vortical effects on the perturbations. The next mode shown in Fig. (3) are the tensorial perturbation representing the gravitational radiation content. All these modes are inter-related; e.g., any scalar density enhancement due to gravitational collapse generates a rotational component and induces the release of binding energy through gravitational radiation.

3 Scalar Perturbation

Let us look at the scalar perturbation in some greater detail. To solve the inhomogeneous equation with source, we can work out the Green’s function by writing Eq. (4) for $p = 0$ with the delta function source:

$$G = \frac{1}{2\pi} \sum_{k,l,m} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 - (k + l + 1)^2} e^{-i\omega(\eta - \eta')} Y_{l}^{m*} (\Omega') Y_{l}^{m} (\Omega) \times N \sin r \sin r' \frac{(k-l)!}{(k+l+1)!} \frac{2^l (2l+1)!}{(k+l)!} (\cos r \cos r')^{l} C_{k-l}^{(l+1)} (\cos r) C_{k-l}^{(l+1)} (\cos r')$$

(9)

The temporal eigen-functions are $e^{-i\omega\eta}$, the angular are the spherical harmonics $0 Y_{m}^{l} = Y_{m}^{l}$ and the radial ones are the normalized and appropriately weighted Gegenbauer polynomial $\sqrt{N} \sin^{l+1} r C_{n}^{l+1}$. Now we can immediately write down the eigen-function expansion of the Green’s function as

$$G = -\frac{1}{2\pi} \sum_{k=0}^{\infty} \sin [(k+1)(\eta - \eta')] \times \sum_{l=0}^{k} \frac{2^l (2l+1)! l!^2 (k-l)!}{(k+l+1)!} (\sin r \sin r')^{l} C_{k-l}^{(l+1)} (\cos r) C_{k-l}^{(l+1)} (\cos r')$$

5
Here, we may use the addition theorem for Gegenbauer polynomials\[^9\] to replace the second summation with $C_n^{(1)}(\cos \rho) = \frac{\sin[(n+1)\rho]}{\sin \rho}$, where $\cos \rho = \cos r \cos r' + \sin r \sin r' \cos \beta$. Thus,

$$
G = -\frac{1}{4\pi^2 \sin \rho} \sum_{k=1}^{\infty} \{ \cos k[\rho - (\eta - \eta')] - \cos k[\rho - (\eta - \eta')] \}
$$

$$
= -\frac{1}{4\pi^2 \sin \rho} \{ \delta[\rho - (\eta - \eta')] - \delta[\rho + (\eta - \eta')] \}.
$$

(11)

The first term within the braces is the retarded, and the second is advanced, Green’s function. In the limit to the flat case $(r, r') \to 0$, we find $\sin \rho \to \rho$, and $\cos \rho \to 1 - \frac{r^2}{2} = \left(1 - \frac{r^2}{2}\right) \left(1 - \frac{r'^2}{2}\right) + rr' \cos \beta \Rightarrow \rho^2 \to |\vec{r} - \vec{r}'|^2$; hence we recover the familiar Green’s function from classical electrodynamics. With Eq. (11), we can consider the potential $s(\eta', r', \theta', \phi') = \delta[r' - \zeta(\eta')] \delta[\theta' - \theta(\eta')] \delta[\phi' - \phi(\eta')]$ of a point scalar perturbation moving along the trajectory $(\zeta(\eta), \theta(\eta), \phi(\eta))$ to find a retarded Lienard-Wiechert type potential. Thus, $\Phi_0(\rho, \eta) = \int G d\eta' d\theta' d\phi' d\eta' = -\frac{1}{4\pi} \int d\eta' \frac{1}{\sin(\rho(\eta'))} \delta[\rho(\eta') - (\eta - \eta')]$.

The integral over $\eta'$ can now be easily done to give

$$
\Phi_0(\rho, \eta) = -\frac{1}{4\pi \sin[\rho(\eta)']} \left[ 1 + \frac{d}{d\eta}\rho(\eta) \right]_{\eta' = \eta - \rho(\eta')},
$$

(12)

where the variables are to be evaluated at the retarded time $\eta' = \eta - \rho(\eta')$; here, $\cos \rho(\eta') = \cos r \cos r'(\eta') + \sin r \sin r'(\eta') \cos \beta(\eta')$ and $\cos \beta(\eta') = \cos \theta \cos \theta'(\eta') + \sin \theta \sin \theta'(\eta') \cos[\phi - \phi(\eta')]$, with the primed co-ordinates being the location of the source that produce the potential at the un-primed location of the observer; also the right hand side is to be evaluated at the retarded time. Without loss of generality, we may simplify the notation by locating the observer at the origin, in which case $\cos \rho(\eta') = \cos |r'(\eta')]$. Again it can be easily checked that Eq. (12) reduces to the familiar form in the limit $r \to$ small.

4 Conclusions

In this work we have written the equations of perturbation of the FRW spacetime in Newman-Penrose formalism. We find that the tensorial, vectorial and
scalar perturbations decompose into forms that reveal the spin content very transparently. The perturbation modes are also inter-related, in that the presence of any one mode gives rise to the rest. The eigenfunctions of all these modes of perturbation can be solved in terms of appropriate spin-weighted spherical harmonics and radial functions of similar spin content.

The scalar mode gives rise to the density perturbations that are responsible for the large scale structures. We are able to solve for the Green’s function of this mode, and the result reduces to the familiar form from classical electrodynamics in the limit to flat space. With the Green’s function, we should be able to solve for the potential driving the scalar perturbations. Also, the scalar perturbation $\Psi_0 = \frac{\Phi_0}{a^3 \sin^3 r}$ may be interpreted to represent the Newtonian potential, as in the Schwarzschild spacetime $\Psi_0 = -\frac{M}{r}$.

The vectorial perturbation equations are the same as Maxwell equations. So it is worthwhile to investigate the possibility that the vortical modes and the rotational effects on the large, and even small, scale structures could have their origin in electrodynamic interactions in very early Universe. Although the free Dirac and Maxwell equations have been solved in another work, it is also necessary to incorporate the quantum corrections into the model. All the known massive particles are fermionic, so study of the Dirac current and energy-momentum as well as their interaction with the Maxwell field will reveal many features of structures. These and similar problems will be taken up in future work.

Regarding the tensorial perturbations, all that can be said at this moment is that they must represent the gravitational radiation content of the Universe, and the gravitational waveforms radiated by the structures present in the Universe.

The perturbation equations, Eq. (12), show that source of any one mode of perturbation produces the other modes. So the whole picture will become consistent only in an integrated view. This work can be considered only as an initial phase.

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Figure 1: The scalar radial eigen-functions. The dashed curves are the normalized functions and the solid curves are the squares of them. The upper two are $n = \omega - k - 1 = 0$, the lowest possible state on the with $k = 0$ on the left, and $k = 6$ on the right. The lower ones are for $n = 6$, with $k = 3$ on the left and $k = 8$ on the right. It is seen that the higher $k$ states are confined towards the middle, nearer to $r = \pi/2$. 
Figure 2: The vectorial radial eigenfunctions. The dashed curves are the real part, the dot-dashed are imaginary and the solid are the square of the modulus. $n = 0$ for the upper two graphs, with the lowest state of $k = 1$ on the left, and $k = 4$ on the right. The lower graphs are for $n = 3$, with $k = 1$ on the left and $k = 6$ on the right. Again, the higher $k$’s are confined nearer to the middle.
Figure 3: Same as Fig. (2), with same values of $n$ for the wave forms of the tensorial gravitational radiation. The upper left is the lowest state with $k = 2$, and the right is $k = 6$. The lower are $k = 4$ and $10$ respectively. Higher $k$ values show similar tendency as the scalar and vectorial.