ON THE SUPPORT OF THE ASHTEKAR-LEWANDOWSKI MEASURE

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Abstract

We show that the Ashtekar-Isham extension $\mathcal{A}/\mathcal{G}$ of the configuration space of Yang-Mills theories $\mathcal{A}/\mathcal{G}$ is (topologically and measure-theoretically) the projective limit of a family of finite dimensional spaces associated with arbitrary finite lattices.

These results are then used to prove that $\mathcal{A}/\mathcal{G}$ is contained in a zero measure subset of $\mathcal{A}/\mathcal{G}$ with respect to the diffeomorphism invariant Ashtekar-Lewandowski measure on $\mathcal{A}/\mathcal{G}$. Much as in scalar field theory, this implies that states in the quantum theory associated with this measure can be realized as functions on the “extended” configuration space $\mathcal{A}/\mathcal{G}$.

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1 Introduction

The usual canonical approach to quantization of a (finite dimensional) system defines states as functions on a configuration space and defines an inner product of two such functions $\psi$ and $\phi$ through

$$(\psi, \phi) = \int_Q d\mu \, \psi^* \phi$$

where $\mu$ is some measure on the configuration space $Q$. Naively applying this procedure to Yang-Mills theories produces a "connection representation" with states that are functions of the Yang-Mills connection. In particular, these states are functions on the quotient space $A/G$, where $A$ is the space of ($C^1$-)connections and $G$ is the group of ($C^2$-)gauge transformations. The same is true for gravity formulated in terms of Ashtekar variables before one imposes the diffeomorphism and hamiltonian constraints [1,2].

A more sophisticated analysis of examples, such as scalar field theory [3-5], shows that the domain space of the wave functions may not be exactly the classical configuration space. Instead, some extension of $Q$ is required.

In order to define an inner product for a connection representation, one expects to give $A/G$, or some suitable extension, the structure of a measurable space (by choosing the measurable sets) and to define appropriate measures. Ashtekar and Isham described an algebraic program to construct such measures in [2]. They proposed, for a compact gauge group $G$, a compact extension $\overline{A/G}$ of $A/G$ on which regular Borel measures are well defined and are in one-to-one correspondence with positive continuous linear functionals on a certain $C^*$-algebra of connection observables known as the holonomy algebra $\mathcal{H}_A$. In [6], Ashtekar and Lewandowski constructed such a Borel measure $\mu_{AL}$ on $\overline{A/G}$ that is both diffeomorphism invariant and strictly positive on continuous cylindrical functions. To do so they, and independently Baez in [7], introduced the concepts of "cylindrical sets" and "cylindrical functions" on $\overline{A/G}$. Baez then generalized the Ashtekar-Lewandowski measure by finding an infinite dimensional space of diffeomorphism invariant measures. In [6] it was also shown that the Ashtekar-Isham space $\overline{A/G}$ is in one-to-one correspondence with the set of homomorphisms from the group of piecewise analytic hoops (i.e. based loops modulo an equivalence relation defined by the holonomies) $\mathcal{H}_G_{x_0}$ to the gauge group $K$, modulo conjugation.

In what follows, we reinterpret some of the results of [2, 6] in terms of the theory of projective limits. In particular, we consider projective limits of infinite families of finite-dimensional topological and measurable spaces associated with arbitrary finite lattices. This theory provides an appropriate framework for studying different properties of $\overline{A/G}$, both from the topological and measure theoretical points of view. Our main result is the use of this formalism to prove that the space $A/G$ is contained in a zero measure subset of $\overline{A/G}$ (with respect to the Ashtekar-Lewandowski measure).

The present work is organized as follows. In Sect. 2 we recall (mainly from [8]) some aspects of the theory of projective limits of infinite families of measurable spaces. Sect. 3 is devoted to reinterpreting some results of [2, 6] in the language of projective limits. In particular we consider projective limits of infinite families of finite dimensional topological and measurable spaces associated with arbitrary finite lattices. This theory provides an appropriate framework for studying different properties of $\overline{A/G}$, both from the topological and measure theoretical points of view. Our main result is the use of this formalism to prove that the space $A/G$ is contained in a zero measure subset of $\overline{A/G}$ (with respect to the Ashtekar-Lewandowski measure).

In Sect. 4 we prove the main result of the paper stated above. Sect. 5 is devoted to the study of the additive, but not $\sigma$-additive, measure $\hat{\mu}_{AL}$ induced by $\mu_{AL}$ on the (finite) algebra $\mathcal{C}$ of cylindrical sets of $A/G$

$$\mathcal{C} = \{ \overline{C} \cap A/G ; \overline{C} \subset \overline{\mathcal{C}} \}$$

where $\overline{\mathcal{C}}$ denotes the algebra of cylindrical sets on $\overline{A/G}$. We show that $\hat{\mu}_{AL}$ cannot be extended to a $\sigma$-additive measure on $A/G$ and that the space of square integrable (cylindrical) functions on $A/G$ is not
complete. We also prove that the Cauchy completion of this space is $L^2 \left( \overline{A/G}, \mu_{AL}, B(G) \right)$, justifying the use of the “generalized connections” in $\overline{A/G}$.

2. Projective limit measurable spaces

In the present section we recall, mainly from [8], the relevant aspects of a class of measures on infinite dimensional spaces which are obtained as rigorously defined limits of measures on finite dimensional spaces. This class contains the direct product measures (on $\mathbb{R}^\infty$ for example) and the projective limit measures. First, however, we introduce some more terminology and notation that will prove useful.

The pair $(X, \mathcal{B})$ (or $(X, \mathcal{F})$), where $X$ is a set and $\mathcal{B}$ (or $\mathcal{F}$) is a $\sigma$-algebra (or algebra) of subsets of $X$, will be called a $\sigma$-measurable (measurable) space. In the mathematical literature, definitions of a measurable $\sigma$-algebra have been given both that require $\mathcal{B}$ to be a $\sigma$-algebra and that require only that $\mathcal{B}$ be closed under finite operations. As we will be interested in a comparison of these two cases it will be convenient to use the above terminology to distinguish between them.

We will be interested in $\sigma$-additive probability measures on $\mathcal{B}$, which are, by definition non-negative, normalized and $\sigma$-additive functions on the $\sigma$-algebra $\mathcal{B}$. That is, such a measure $\mu$ satisfies:

\begin{align*}
\mu(B) &\geq 0 \quad , \quad B \in \mathcal{B} \\
\mu(X) &\equiv 1 \\
\mu(\bigcup_{i=1}^\infty B_i) &\equiv \sum_{i=1}^\infty \mu(B_i) \quad , \quad B_i \in \mathcal{B}, \quad B_i \cap B_j = \emptyset, \quad i \neq j. \quad (2.1c)
\end{align*}

Additive measures on an algebra $\mathcal{F}$ satisfy (2.1) with $\mathcal{B}$ replaced by $\mathcal{F}$ and with only finite unions and sums in (2.1c). For a given measure $\mu$ on $\mathcal{F}$, an important question is whether or not it can be extended to a $\sigma$-additive measure on $\mathcal{B}(\mathcal{F})$, the minimal $\sigma$-algebra that contains $\mathcal{F}$. A necessary and sufficient condition for extendibility is given by the Hopf theorem [8]:

**Theorem 2.1** (Hopf Theorem)
A measure $\mu$ on $\mathcal{F}$ can be extended to a $\sigma$-additive measure on $\mathcal{B}(\mathcal{F})$ if and only if for every decreasing sequence $\{F_i\}$ such that $F_i \in \mathcal{F}$, $F_1 \supset \ldots \supset F_n \supset \ldots$ with $\bigcap_{i=1}^\infty F_i = \emptyset$, we have

\[ \lim_{i \to \infty} \mu(F_i) = 0. \quad (2.2) \]

Essentially, the condition (2.2) allows an extension $\tilde{\mu}$ to be consistently defined on elements of $\mathcal{B}(\mathcal{F})$ as limits of $\mu$-measures of sets in $\mathcal{F}$. The triplet $\{(X, \mathcal{B}, \mu) \mid \{(X, \mathcal{F}, \mu)\}\}$, where $\mathcal{B}$ (or $\mathcal{F}$) is a $\sigma$-algebra (or algebra) and $\mu$ is $\sigma$-additive (additive) is called a $\sigma$-measure (measure) space.

The possibility of extending a measure $\mu$ on $\mathcal{F}$ to a $\sigma$-additive measure $\tilde{\mu}$ on $\mathcal{B}(\mathcal{F})$ is in particular relevant to physical applications in quantum mechanics. Recall that quantum mechanical systems are often defined by first giving a linear pre-Hilbert space and then completing this space with respect to an inner product. In general, if $\mu$ is cylindrical but not $\sigma$-additive, the space $\mathcal{H}$ of $\mu$-square integrable cylindrical functions on $X$ (denoted through $CL^2(X, \mathcal{F}, \mu)$) is only a pre-Hilbert space. Such spaces will be discussed in section 5. However, if $\mu$ is extendible to a $\sigma$-additive measure $\tilde{\mu}$ on $(X, \mathcal{B}(\mathcal{F}))$ then the Cauchy completion of $\mathcal{H}$ leads to the space $\tilde{\mathcal{H}} = L^2(\mathcal{F}, \mathcal{B}(\mathcal{F}), \tilde{\mu})$ (see section 5). On the other hand if $\mu$ is not extendible then the Cauchy completion of $CL^2(X, \mathcal{F}, \mu)$ leads in general to a space with state-vectors which cannot
be expressed as functions on the initial space $\mathbb{X}$. This is the case in scalar field theory if one considers $\mathbb{X} = \mathcal{S}(\mathbb{R}^3)$ (the Schwarz space of rapidly decreasing smooth $C^\infty$ functions on $\mathbb{R}^3$) and $\mu$ is a cylindrical measure defined with the help of a positive definite function on $\mathcal{S}(\mathbb{R}^3)$, continuous in the nuclear space topology (see [3, 5, 8]). As we shall see in Sect. 5 this is also the case in Yang-Mills theory if we take $\mathcal{H} = CL^2(\mathcal{A}/\mathcal{G}, \mathcal{F} = C, \hat{\mu}_{AL})$, where $\hat{\mu}_{AL}$ is the Ashtekar-Lewandowski measure on $\mathcal{A}/\mathcal{G}$. In the scalar field case the Cauchy completion of $CL^2(\mathcal{S}(\mathbb{R}^3), \mathcal{F}, \mu)$ gives the space of square integrable functions on $\mathcal{S}'(\mathbb{R}^3)$ (the space of tempered distributions), while in the Yang-Mills case the completion of $CL^2(\mathcal{A}/\mathcal{G}, C, \hat{\mu}_{AL})$ gives the space $L^2(\mathcal{A}/\mathcal{G}, \mathcal{B}(\mathbb{C}), \mu_{AL})$ of square integrable functions on the Ashtekar-Isham space $\mathcal{A}/\mathcal{G}$ of generalized “distributional” connections modulo gauge transformations.

Let $\{(X, \mathcal{B}, \mu)\}$ be a $\sigma$-measurable space. The subset $Y \subset \mathbb{X}$ is said to be $\mu$-thick in $\mathbb{X}$ if for every $B \in \mathcal{B}$ such that $B \cap Y = \emptyset$, $\mu(B) = 0$. If $Y$ is $\mu$-thick in $\mathbb{X}$ then $\mu$ induces a $\sigma$-additive measure $\mu_Y$ on the $\sigma$-measurable space

$$(Y, \mathcal{B}_Y) ,$$

where $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$, through

$$\mu_Y(B \cap Y) = \mu(B) , \quad \forall B \in \mathcal{B} .$$

(2.3b)

The measure $\mu_Y$ is called the trace of the measure $\mu$ on $Y$ [8]. If $Y$ is not $\mu$-thick on $\mathbb{X}$ then (2.3b) is not well defined.

Note that if $Y$ is $\mu$-thick in $\mathbb{X}$ then

$$L^p(\mathbb{X}, \mu, \mathcal{B}) \cong L^p(Y, \mu_Y, \mathcal{B}_Y)$$

so that if we are concerned only with such spaces we can restrict ourselves to $Y$ and $\mu_Y$. This is particularly convenient when the set $Y$ has advantages (for instance from the “differentiable” point of view) over $\mathbb{X}$. When a set $Y$ is $\mu$-thick in $\mathbb{X}$ we say that the support of the measure $\mu$ is contained in $Y$. An illustrative example is the one given by the Wiener measure on $\mathbb{R}^{0,1}$ used in the (euclidean) path integral formulation of quantum mechanics. In this case the support of the measure is contained in the space $Y = C^0([0,1])$ of continuous functions on the interval [3].

The inclusion map from $Y$ to $\mathbb{X}$ above is referred to as measurable. In general, a map between $\sigma$-measurable (measurable) spaces

$$\phi : \mathbb{X}_1 \to \mathbb{X}_2$$

is called measurable if for every measurable set $B_2 \in \mathcal{B}_2$ the set $\phi^{-1}(B_2)$ is measurable i.e. $\phi^{-1}(B_2) \in \mathcal{B}_1$. $\phi$ in (2.4) is called an isomorphism of $\sigma$-measurable (measurable) spaces if it is bijective and if both $\phi$ and $\phi^{-1}$ are measurable.

Let us now briefly review (see [8]) the construction of infinite products of $\sigma$-measurable spaces and of (projective) limits of infinite projective families of $\sigma$-measurable spaces. Let

$$\{(X^{(\lambda)}, \mathcal{B}^{(\lambda)})\}_{\lambda \in \Lambda}$$

be an indexed family of $\sigma$-measurable spaces. The product $\sigma$-measurable space $(X^{(\Lambda)}, \mathcal{B}^{(\Lambda)})$ is, by definition, given by

$$X^{(\Lambda)} = \prod_{\lambda \in \Lambda} X^{(\lambda)}$$

(2.6)

with $\mathcal{B}^{(\Lambda)}$ being the minimal $\sigma$-algebra for which all the projections

$$p_{\lambda_0} : X^{(\Lambda)} \to X^{(\lambda_0)}$$

(2.7)

$$\left( x_{\lambda} \right)_{\lambda \in \Lambda} \mapsto x_{\lambda_0}$$
are measurable. That is, \( \mathcal{B}^{(\Lambda)} \) is the \( \sigma \)-algebra generated by the inverse images of measurable sets in \( X^{(\Lambda)} \) under the projections \( p_\lambda \). If all the \( X^{(\lambda)} \), \( \lambda \in \Lambda \) are different copies of the same set \( Y \) with the same \( \sigma \)-algebras \( \mathcal{B}^{(\lambda)} = \mathcal{B} \) then the points of \( X^{(\Lambda)} = Y^\Lambda \), \( x \in Y^\Lambda \) are (arbitrary) maps from \( \Lambda \) to \( Y \):

\[
x \in X^{(\Lambda)} = Y^\Lambda \iff x : \Lambda \to Y
\]

Examples are the set of all sequences of real numbers

\[
\mathbb{R}^\infty = \prod_{j \in \mathbb{N}} \mathbb{R}_{(j)} \quad , \quad (\mathbb{R}_{(j)} = \mathbb{R})
\]

and the set of all real valued functions on the interval \([0, 1]\)

\[
\mathbb{R}^{[0,1]} = \prod_{t \in [0,1]} \mathbb{R}_t \quad , \quad (\mathbb{R}_t = \mathbb{R})
\]

Suppose we have a \( \sigma \)-additive measure \( \mu \) on \( X^{(\Lambda)} \) in (2.6). Let \( \mathcal{L} \) be the family of all finite subsets of \( \Lambda \) and for \( L \in \mathcal{L} \) let \((X^{(L)}, \mathcal{B}^{(L)})\) be the partial products of \( \sigma \)-measurable spaces with

\[
X^{(L)} = \prod_{\lambda \in L} X^{(\lambda)}
\]

and the corresponding \( \mathcal{B}^{(L)} \). Then all the projections

\[
p_L : X^{(\Lambda)} \to X^{(L)}
p_L((x_\lambda)_{\lambda \in \Lambda}) = (x_\lambda)_{\lambda \in L}
\]

are measurable. Consider the family \( \{\mu_L\}_{L \in \mathcal{L}} \) of \( \sigma \)-additive measures on \( X^{(L)} \) defined by the pushforwards of the measure \( \mu \)

\[
\mu_L(B) = \mu(p_L^{-1}(B))
\]

for \( B \in \mathcal{B}_L \), which, in the notation of measure theory is written as

\[
\mu_L = (p_L)^* \mu.
\]

This family satisfies a self consistency condition:

\[
L \subset L' \Rightarrow \mu_L = (p_{LL'})^* \mu_{L'} ,
\]

where \( p_{LL'} \) denote the measurable projections from \( X^{(L')} \) to \( X^{(L)} \). In [8] (see corollary to Th. 10.1) are found conditions for which the converse is also true:

**Proposition 2.2**

For a family of \( \sigma \)-compact or complete and separable, metric spaces every family of Borel measures that is consistent in the sense of (2.14) can be extended to a \( \sigma \)-additive measure on the product \( \sigma \)-measurable space.

Such a measure is in fact defined by (2.13), i.e. for \( B_L \in \mathcal{B}_L \), \( \mu(p_L^{-1}(B_L)) \) is defined to be just \( \mu_L(B_L) \). Recall that a topological space \((X, \tau)\) is said to be \( \sigma \)-compact if \( X \) can be represented as a countable union of compact sets.
Notice ([8]) that a measure $\mu$ satisfying (2.13) and given on the algebra
\[ F(L) = \bigcup_{L \in \mathcal{L}} p_{L}^{-1}(B_{L}) \] (2.15)
always exists. The only question is whether $\mu$ can be extended to a $\sigma$-additive measure $\tilde{\mu}$ on $B(L)$, which is the minimal $\sigma$-algebra that contains (2.15)
\[ B(L) = B(\bigcup_{L \in \mathcal{L}} p_{L}^{-1}(B_{L})) . \] (2.16)
For instance, in the example of $X^{(\Lambda)} = \mathbb{R}^{[0,1]}$ the question is to know when a self-consistent family of measures $\{\mu_{1},...,\mu_{n}\}_{t_{1},...,t_{n} \in [0,1]}$ on the finite dimensional spaces
\[ \mathbb{R}^{n} = \prod_{t_{1}=1}^{n} \mathbb{R}^{t_{i}} \] (2.17)
defines a $\sigma$-additive measure on the infinite dimensional space
\[ \mathbb{R}^{[0,1]} . \] (2.18)
Quite remarkably, in this example and in many others relevant to quantum field theory the answer is affirmative, as indicated by Proposition 2.2.
An infinite product $\sigma$-measure space can also be realized as a “projective limit” (which we will define next). However, the product space $X^{(\Lambda)}$ is a projective limit not of the family of spaces $X^{(\lambda)}$ labelled by $\lambda \in \Lambda$ but rather of the family of spaces $X_{L} = X(L)$ labelled by $L \in \mathcal{L}$, the set of all finite subsets of $\Lambda$. In general, a projective limit space can be defined for any “projective family” of $\sigma$-measurable spaces; that is, for any family
\[ \{(X_{L},B_{L})_{L,L' \in \mathcal{L}}\} \] (2.19)
of the following form. The set $\mathcal{L}$ is taken to be directed, i.e. partially ordered and such that for any two elements $L_{1}, L_{2} \in \mathcal{L}$, there is some $L$ such that $L_{1} \leq L$ and $L_{2} \leq L$. We will also assume that $\mathcal{L}$ does not have a maximum. Here $p_{LL'}$ are measurable projections, i.e. surjective mappings
\[ p_{LL'} : X_{L'} \to X_{L} \quad L < L' \] (2.20a)
satisfying
\[ p_{LL'} \circ p_{LL''} = p_{LL''} \quad \text{for } L < L' < L''. \] (2.20b)
Now, let $(X(L),B(L))$ denote the direct product of the family $\{(X_{L},B_{L})\}_{L \in \mathcal{L}}$
\[ X(L) = \prod_{L \in \mathcal{L}} X_{L} . \] Then the projective limit of the family (2.19) is by definition the $\sigma$-measurable space $(X_{\mathcal{L}},B_{\mathcal{L}})$, where
\[ X_{\mathcal{L}} \subset X(L) ; \quad X_{\mathcal{L}} = \{(x_{L})_{L \in \mathcal{L}} \in X(L) : L < L' \Rightarrow x_{L} = p_{LL'}(x_{L'})\} \] (2.21a)
and
\[ B_{\mathcal{L}} = \{B \cap X_{\mathcal{L}} : B \in B(L)\} . \] (2.21b)
That is, $X_{\mathcal{L}}$ is the subset of $X(L)$ that is consistent with the projections $p_{LL'}$. Note that a direct product space can also be thought of as a projective limit of the spaces formed by taking arbitrary finite products of the factors. A family of measures $(\mu_{L})_{L \in \mathcal{L}}$ is said to be self-consistent if it satisfies (2.14) with $L \subset L'$.
replaced by \( L < L' \). A measure \( \mu \) on \( \mathcal{B}_L \) always defines a self consistent family of measures \( (\mu_L)_{L \in \mathcal{L}} \) through (2.13) and a consistent family \( (\mu_L)_{L \in \mathcal{L}} \) defines a finitely additive measure on \( X_L \) through (2.13) as well. A measure on \( X_L \) defined by such a family is called cylindrical. An important result is (see [8] Corollary to Th. 10.1):

**Proposition 2.3**

Under the same conditions as in Proposition 2.2, a self-consistent family of Borel measures on a projective family (2.19) defines a cylindrical measure that can be extended to a \( \sigma \)-additive measure in the projective limit \( \sigma \)-measurable space (2.21) if for every increasing sequence

\[
\mathcal{M} = \{ L_i \}_{i=1}^{\infty} \subset \mathcal{L} : L_1 < L_2 < \ldots < L_n < \ldots
\]

with projective limit \((X_M, \mathcal{B}_M)\) the projection

\[
p_M : X_L \to X_M
\]

\[
p_M((x_L)_{L \in \mathcal{L}}) = (x_{L_n})_{L_n \in \mathcal{M}}
\]

is surjective.

3. Ashtekar-Isham space \( \mathcal{A}/\mathcal{G} \) as a projective limit

Let \( \mathcal{A}/\mathcal{G} \) denote the space \( \mathcal{A} \) of smooth \( C^1 \) \( G \)-connections modulo the group \( \mathcal{G} \) of gauge transformations on a three dimensional analytic manifold \( \Sigma \) where, as in [6], the gauge group \( G \) is assumed to be \( U(N) \) or \( SU(N) \). Following [6], we consider the \( G \)-hoop group \( \mathcal{H}\mathcal{G}_{x_0} = \mathcal{L}\Sigma_{x_0}/\sim \) where \( \mathcal{L}\Sigma_{x_0} \) is the space of piecewise analytic loops based at \( x_0 \) (see [6]) and the equivalence relation \( \sim \) is

\[
\alpha, \beta \in \mathcal{L}\Sigma_{x_0} , \alpha \sim \beta \text{ if and only if } H(\alpha, A) = H(\beta, A) , \forall A \in \mathcal{A} . \tag{3.1}
\]

Here, \( H(\alpha, A) \) denotes the holonomy corresponding to the connection \( A \) and the loop \( \alpha \). The Ashtekar-Isham space \( \mathcal{A}/\mathcal{G} \) as a “compactification” of \( \mathcal{A}/\mathcal{G} \) obtained as follows (see [2]). Let \( T_\alpha \), \( \alpha \in \mathcal{L}\Sigma_{x_0} \) denote the Wilson loop function on \( \mathcal{A}/\mathcal{G} \) defined by

\[
T_\alpha(A) = T_{|\alpha|}([A]) = \frac{1}{N} Tr H(\alpha, A) . \tag{3.2}
\]

where \([\alpha]\) denotes the equivalence class of \( \alpha \) in \( \mathcal{H}\mathcal{G}_{x_0} \), \([A]\) denotes the equivalence class of \( A \) in \( \mathcal{A}/\mathcal{G} \) and the trace is taken in the fundamental representation of the gauge group. In the following, for simplicity, \( \alpha, \beta \) will denote hoops. The holonomy algebra \( \mathcal{H}\mathcal{A} \) is the commutative \( C^* \)-algebra generated by the Wilson loop functions. The Ashtekar-Isham space \( \mathcal{A}/\mathcal{G} \) is the compact Hausdorff space that is the spectrum [2] of \( \mathcal{H}\mathcal{A} \) in which \( \mathcal{A}/\mathcal{G} \) is densely embedded [2,6,8].

Ashtekar and Lewandowski [6] obtained a useful algebraic characterization of the space \( \mathcal{A}/\mathcal{G} \). They proved that there is a one-to-one correspondence between \( \mathcal{A}/\mathcal{G} \) and the space of all homomorphisms from the hoop group \( \mathcal{H}\mathcal{G}_{x_0} \) to the gauge group \( G \), modulo conjugation. We will therefore identify these two sets and write

\[
\tilde{h} = [h_0] \in \overline{\mathcal{A}/\mathcal{G}} \iff
\]

\[
[h_0] = \{ h \in Hom(\mathcal{H}\mathcal{G}_{x_0}, G) : (h(\alpha))_{\alpha \in \mathcal{H}\mathcal{G}_{x_0}} = (gh_0(\alpha)g^{-1})_{\alpha \in \mathcal{H}\mathcal{G}_{x_0}} , \forall g \in G \} \tag{3.3}
\]
where \( g \) above does not depend on the hoop \( \alpha \). Notice that no continuity condition has been imposed on the homomorphisms \( h \) in (3.3). This will allow us to interpret \( \mathcal{A}/\mathcal{G} \) (both topologically and measure theoretically) as a projective limit of finite dimensional spaces.

Let \( \mathcal{L} \) denote the set of all subgroups of \( \mathcal{H}G_{x_0} \) generated by a finite number of hoops \( \beta_1, \ldots, \beta_n \) that are strongly independent in the sense of [6], i.e. such that loop representatives of the hoop equivalence classes \( \beta_i \) can be chosen in such a way that each contains an open segment which is traced exactly once and which intersects any of the other representative loops at most at a finite number of points. Then

\[
S^* \in \mathcal{L} \iff S^* = \{ \text{group generated by } \beta_1, \ldots, \beta_n \} \subset \mathcal{H}G_{x_0}
\]

and we write \( S^* = S^*[\beta_1, \ldots, \beta_n] \). Now let \( H_{S^*} = \text{Hom}(S^*, G)/\text{Ad} \) be the set of equivalence classes of homomorphisms from \( S^* \) to \( G \) under conjugation. If \( S^* = S^*[\beta_1, \ldots, \beta_n] \) then, as shown in [6], a homomorphism from \( S^* \) to \( G \) is known if and only if we know it on the hoops \( \beta_1, \ldots, \beta_n \) so that we have the one-to-one correspondence

\[
H_{S^*} \rightarrow G^n/\text{Ad}
\]

\[
[h] \mapsto [h(\beta_1), \ldots, h(\beta_n)] .
\]

Consider now the following projective family of finite dimensional spaces

\[
\{(H_{S^*}, p_{S^*S''})_{S^*, S'' \in \mathcal{L}} \}
\]

where \( p_{S^*S''}, S^* \subset S'' \), denotes the mapping

\[
p_{S^*S''} : H_{S''} \rightarrow H_{S^*}
\]

\[
p_{S^*S''}([h_{S''}]) = [h_{S''} |_{S^*}]
\]

and \( h_{S''} |_{S^*} \) denotes the restriction of \( h_{S''} \) to the subgroup \( S^* \) of \( S'' \). From [6] we see that these projections are surjective. According to (2.21) the projective limit \( H_{\mathcal{L}} \) of the family (3.6) is given by

\[
H_{\mathcal{L}} = \text{lim}_{\mathcal{L}} H_{S^*} = \prod_{S^* \in \mathcal{L}} H_{S^*}
\]

\[
H_{\mathcal{L}} = \{ ([h_{S^*}]_{S^* \in \mathcal{L}} \in H^{(\mathcal{L})} : S^* \subset S'' \Rightarrow [h_{S^*}] = p_{S^*S''}([h_{S''}]) \} .
\]

We will now show that this is just the Ashtekar-Isham space \( \mathcal{A}/\mathcal{G} \).

**Proposition 3.1**

There is a bijective map \( \phi \)

\[
\begin{align*}
\mathcal{A}/\mathcal{G} &\rightarrow H_{\mathcal{L}} \\
[h] &\mapsto ([h_{S^*}]_{S^* \in \mathcal{L}}) \quad h_{S^*} = h |_{S^*} .
\end{align*}
\]

**Proof**

Consider the space \( \text{Hom}(\mathcal{H}G_{x_0}, G) \) of all homomorphisms from \( \mathcal{H}G_{x_0} \) to \( G \) and the projective family

\[
\{ \text{Hom}(S^*, G), \tilde{p}_{S^*S''} \}_{S^*, S'' \in \mathcal{L}}
\]

where \( \tilde{p}_{S^*S''} : \tilde{p}_{S^*S''}(h_{S''}) = h_{S''} |_{S^*} \), \( S^* \subset S'' \) are surjective maps from \( \text{Hom}(S^*, G) \) to \( \text{Hom}(S^*, G) \). Let \( K^{(\mathcal{L})} \) be the infinite product space and \( K_{\mathcal{L}} \) be the projective limit space of this family.
\[ K_{\mathcal{L}} = \{(h_{S^*})_{S^* \in \mathcal{L}} \in K^{(\mathcal{L})} : S^* \subset S^* \Rightarrow h_{S^*} = \tilde{\phi}_{S^*}(h_{S^*})\} \]  

We will need the following lemmas.

**Lemma 3.2**

The map

\[ \tilde{\phi} : \text{Hom}(\mathcal{H}_{\mathcal{L},G}) \to K_{\mathcal{L}} \]

\[ \tilde{\phi}(h) = (h |_{S^*})_{S^* \in \mathcal{L}} \]

is bijective and \( Ad \)-equivariant, i.e. \( Ad \circ \tilde{\phi} = \phi \circ Ad \) for every \( g \in G \).

**Proof of Lemma 3.2**

The injectivity of \( \tilde{\phi} \) is trivial. Let us prove that \( \tilde{\phi} \) is surjective. Fix an arbitrary element \( (h_{S^*})_{S^* \in \mathcal{L}} \in K_{\mathcal{L}} \).

Let us construct the homomorphism \( h^0 \) which is the pre-image of this element. Let \( \alpha \) be an arbitrary hoop and \( S^*_1 \in \mathcal{L} \) such that \( \alpha \in S^*_1 \) (\( S^*_1 \) always exists for a piecewise analytic hoop \( \alpha \) [2]). Then choose \( h^0(\alpha) = h^0_{S^*_1}(\alpha) \). To see that \( h^0(\alpha) \) does not depend on the choice of the finitely generated group \( S^*_1 \supseteq \alpha \) let \( \alpha \in \hat{S}^*_1 \) and \( S^*_2 \) be a subgroup which contains both \( S^*_1 \) and \( \hat{S}^*_1 \). Then, according to the definition of \( K_{\mathcal{L}} \), we have \( h^0_{S^*_1}(\alpha) = h^0_{S^*_2}(\alpha) \) and \( h^0_{\hat{S}^*_1}(\alpha) = h^0_{\hat{S}^*_2}(\alpha) \) which implies that \( h^0_{\hat{S}^*_1}(\alpha) = h^0_{\hat{S}^*_2}(\alpha) \). We can easily show that \( h^0 \) constructed in this way is an homomorphism and that the map \( \tilde{\phi} \) is equivariant.

\[ Q.E.D. \]

Lemma 3.2 implies that the map \( \tilde{\phi} \) induces a bijective map \( \phi_1 \)

\[ \phi_1 : \text{Hom}(\mathcal{H}_{\mathcal{L},G})/Ad \to K_{\mathcal{L}}/Ad \]

\[ \phi_1([h]) = [(h |_{S^*})_{S^* \in \mathcal{L}}] \]

**Lemma 3.3**

The map

\[ \phi_2 : K_{\mathcal{L}}/Ad \to H_{\mathcal{L}} \]

\[ \phi_2([(h_{S^*})_{S^* \in \mathcal{L}}]) = (\{h_{S^*}\}_{S^* \in \mathcal{L}}) \]

is bijective.

**Proof of Lemma 3.3**

We will first show that \( \phi_2 \) is surjective. To do so, recall that any element of \( H_{\mathcal{L}} \) is a family \( (h_{S^*})_{S^*} \) of consistent equivalence classes in the sense of (3.7b). Now, choose a representative \( h^0_{S^*} \) from each \( [h_{S^*}] \) and construct the subgroup \( C^0_{S^*} \) of \( G \) that commutes with \( h^0_{S^*} \); that is, let

\[ C^0_{S^*} = \{g \in G : \forall \alpha \in S^* , gh^0_{S^*}(\alpha)g^{-1} = h^0_{S^*}(\alpha)\} \]

(3.14)

Note that \( C^0_{S^*} \) is closed in \( G \). Any closed subgroup of a Lie group is a Lie group and any closed subset of a compact space is compact, so that \( C^0_{S^*} \) is again a compact Lie group. Thus, \( C^0_{S^*} \) has some dimension \( d_{S^*} \geq 0 \) and, by compactness, some finite number \( m_{S^*} \geq 1 \) of connected components. There is then
some least value \( d_0 \) of \( d_{S^*} \) \((d_0 = \min_{S^* \in L} d_{S^*})\) and some \( m_0 \) that is the least value of \( m_{S^*} \) for which the dimension of \( C_0^0 \) is \( d_0 \) \( \text{(i.e. } m_0 = \min_{d_{S^*}=d_0} m_{S^*})\). Choose some \( S_0^* \) with \( d_{S_0^*} = d_0 \) and \( m_{S_0^*} = m_0 \).

Now, for every \( S^* \supset S_0^* \), choose another representative \( h_{S^*}^1 \) of \([h_{S^*}]\) such that
\[
h_{S^*}^1 \mid_{S_0^*} = h_{S_0^*}^0
\] (3.15)
and construct the corresponding \( C_{S^*}^1 \):
\[
C_{S^*}^1 = \{g \in G : \forall \alpha \in S_*, \ g h_{S^*}^1(\alpha) g^{-1} = h_{S^*}^1(\alpha)\} \tag{3.16}
\]

Note that \( C_{S^*}^1 \subset C_{S_0^*}^0 \) and that \( C_{S^*}^1 \) differs from \( C_{S^*}^0 \) only by conjugation. Thus, \( C_{S^*}^1 \) has dimension \( d_{S^*} \geq d_{S_0^*} \) and \( m_{S^*} \) connected components. But, since \( C_{S^*}^1 \) is contained in \( C_{S_0^*}^0 \), \( d_{S^*} \geq d_{S^*} \) so that \( C_{S^*}^1 \) and \( C_{S_0^*}^0 \) are of the same dimension. It follows that they agree in some neighborhood of the identity and thus on the entire component connected to the identity. Since \( C_{S^*}^0 \subset C_{S^*}^1 \), is a disjoint union of \( m_{S^*} \) copies of this component, \( m_{S^*} \leq m_{S_0^*} \). But, since \( C_{S^*}^1 \) has dimension \( d_0 \), we have \( m_{S^*} \geq m_{S_0^*} \) and in fact \( m_{S^*} = m_{S_0^*} \). We thus conclude that \( C_{S^*}^1 = C_{S_0^*}^0 \).

This means that \( h_{S^*}^1 \) is unique, since any \( g \) that commutes with \( h_{S_0^*}^0(\alpha) = h_{S_0^*}^1(\alpha) \) for all \( \alpha \in S_0^* \) lies in \( C_{S_0^*}^0 = C_{S_0^*}^1 \) and commutes with \( h_{S_0^*}^1(\alpha) \) for all \( \alpha \in S^* \). Thus, no other representative of \([h_{S^*}]\) satisfies (3.15). It now follows that for any \( S^{*'} \supset S^* \supset S_0^* \),
\[
h_{S^{*'}} \mid_{S^*} = h_{S^*}^1,
\] (3.17)
since \( h_{S^{*'}} \mid_{S^*} \) is the unique representative of \([h_{S^*}^1]\) that satisfies
\[
(h_{S^{*'}} \mid_{S^*}) \mid_{S_0^*} = h_{S^*}^1 \mid_{S_0^*} = h_{S_0^*}^0.
\] (3.18)

Finally, for any \( S^* \) that does not contain \( S_0^* \), let \( S^{*'} \) be any subgroup of \( H_{S_0^*} \) generated by a finite number of independent hoops that contains \( S^* \) and \( S_0^* \) (we see from [6] that such a group exists) and let
\[
h_{S^*}^1 = h_{S^*}^0 \mid_{S^*}.
\] (3.19)
Then the representatives \((h_{S^*}^1)_{S^* \in L} \in ([h_{S^*}])_{S^* \in L}\) form a consistent family of homomorphisms in \( K(L) \) and the equivalence class of this family under the adjoint action is a member of \( K_L / Ad \) that maps to \(([h_{S^*}])_{S^* \in L}\) under the map \( \phi_2 \). We conclude that \( \phi_2 \) is surjective.

Now, injectivity of \( \phi_2 \) follows in a straightforward fashion. Consider any other equivalence class of families \([\{h_{S^*}^1\}_{S^* \in L}] \in K_L / Ad \) that maps to the family \([h_{S^*}])_{S^* \in L}\) chosen above under \( \phi_2 \). As with the family constructed above, \( h_{S_0^*}^0 \) must be a representative of \([h_{S_0^*}^1]\). Let \((h_{S^*}^2)_{S^* \in L}\) be any family in \([\{h_{S^*}^1\}_{S^* \in L}] \) such that \( h_{S_0^*}^2 = h_{S_0^*}^1 \). We have just seen that \((h_{S_0^*}^1)_{S^* \in L}\) is the unique self-consistent family of homomorphisms that includes \( h_{S_0^*}^1 \) and satisfies \([h_{S_0^*}]) = [h_{S_0^*}^1]\). Therefore, \( h_{S_0^*}^2 = h_{S_0^*}^1 \), and the families \([h_{S_0^*}^2]_{S^* \in L}] \) and \([h_{S_0^*}^1]_{S^* \in L}] \) coincide, showing that \( \phi_2 \) is also injective.

\[\text{Q.E.D.}\]

We complete the proof of the proposition by noticing that the bijective map \( \phi \) is given by
\[
\phi = \phi_2 \circ \phi_1 \tag{3.20}
\]
\[\text{Q.E.D.}\]
Endowed with the natural topology, the spaces $H_{S^*}$ are compact topological spaces (see (3.6)). The Tychonov topology $\tau_T$ on the product space $H^{(L)}$ is the minimal topology for which all the projections

$$
\pi_{S^*} : H_L = \overline{A/G} \to H_{S^*}
$$

$$
\pi_{S^*}([h]) = [h |_{S^*}]
$$

are continuous. It coincides with the topology of pointwise convergence in $H^{(L)}$, i.e. the net $[h]^{(\nu)} = ([h_{S^*}]^{(\nu)})_{S^* \in L}$ is $\tau_T$-convergent

$$
[h]^{(\nu)} \xrightarrow{\tau_T} [h]
$$

if and only if

$$
[h_{S^*}]^{(\nu)} \to [h_{S^*}], \forall S^* \in L,
$$

where the last convergence is with respect to the topology on $H_{S^*} = G^n/Ad$. In this topology, the space $H^{(L)}$ is compact (see [8, Tychonov theorem]). Let us also refer to the topology induced on the projective limit $H_L \subset H^{(L)}$ from $H^{(L)}$ as the Tychonov topology $\tau_T$. Then from the continuity of the projections $p_{S^*S'^*}, H_L$ is closed in $H^{(L)}$ and therefore

$$
(H_L, \tau_T)
$$

is also a compact topological space. Since $H_L$ is compact in the Tychonov topology and $\overline{A/G}$ is compact in the Gel’fand topology $\tau_{Gd}$ it is natural to expect that the bijective map $\phi$ in (3.9) is actually a homeomorphism. Indeed we have

**Proposition 3.4**

The bijective map in (3.9) is a homeomorphism

$$
\phi : (\overline{A/G}, \tau_{Gd}) \to (H_L, \tau_T),
$$

where $\tau_{Gd}$ and $\tau_T$ denote the Gel’fand and Tychonov topologies respectively.
Proof
First let us obtain a more convenient characterization of the topology on the spaces $H_{S^*}$. As mentioned above, $H_{S^*}$ endowed with the standard topology induced from $G^n$ is a compact Hausdorff space. Consider on $H_{S^*}$ the continuous functions
\[ T^S_{\alpha}([h_{S^*}]) = Tr(h_{S^*}(\alpha)) , \quad \alpha \in S^* . \quad (3.25) \]
They separate the points in $H_{S^*}$ for the same reason that the $T^G_\alpha, \alpha \in HG_{x_0}$, separate the points in $A/G$ [2, 6]. Therefore, according to the Stone-Weierstrass theorem [9] the algebra $\tilde{HA}_{S^*}$ obtained by taking finite linear combinations (with complex coefficients) and products of $T^S_{\alpha}$ is dense in the $C^*$-algebra $C(H_{S^*})$ of all continuous functions on $H_{S^*}$, i.e.
\[ \tilde{HA}_{S^*} = C(H_{S^*}) . \quad (3.26) \]
Using the first Gel’fand-Naimark theorem [2, 9, 10] we then conclude that the spectrum of $\tilde{HA}_{S^*}$, endowed with the Gel’fand topology (see below) is homeomorphic to $H_{S^*}$. An equivalent description of the initial topology in $H_{S^*}$ is therefore given by the Gel’fand topology, which is, by definition, the weakest for which all the functions $T^S_{\alpha}, \alpha \in S^*$ are continuous.

Returning to (3.24) we see that, in accordance with (3.21), the Tychonov topology on $H_L$ is the weakest for which all the functions $T^S_{\alpha} \circ \pi_{S^*} : H_L \to \Phi , \alpha \in S^*, S^* \in \mathcal{L}$ are continuous. On the other hand the Gel’fand topology on $\tilde{A/G}$ is the weakest for which all the functions $T^G_\alpha, \alpha \in HG_{x_0}$ are continuous.
Since for all $\alpha \in HG_{x_0}$
\[ T^G_\alpha \circ \phi^{-1} = T^S_{\alpha} \circ \pi_{S^*}, \forall S^* : \alpha \in S^* . \quad (3.27) \]
we conclude that $\phi$ in (3.9) is a homeomorphism.

Q.E.D.

We now proceed to derive a measure theoretic analog of proposition 3.4. Let $B_{S^*}$ denote the Borel $\sigma$-algebra on $H_{S^*}$ so that, since the projections $p_{S^*, S^{'}}$ are measurable,
\[ \{(H_{S^*}, B_{S^*}), p_{S^*, S^{'}}\}_{S^*, S^{'}} \in \mathcal{L} \]
is a projective family of $\sigma$-measurable spaces (see (2.19)). Let
\[ (H_L, B_L) \]
denote the projective limit $\sigma$-measurable space. In $\tilde{A/G}$ we take the measurable sets to be generated by the class $\mathcal{C}$ of “cylindrical sets” used in [6, 7], i.e. the inverse images $C_B$ of Borel sets $B$ in $G^n/Ad$ with respect to $\pi_{S^*} \circ \phi$
\[ C_B \in \mathcal{C} \iff C_B = (\pi_{S^*} \circ \phi)^{-1}(B) = \{[h] \in \tilde{A/G} : [h(\beta_1), \ldots, h(\beta_n)] \in B \subset G^n/Ad\} , \quad (3.30a) \]
where, as in (3.5), we have identified $H_{S^*}$ with $G^n/Ad$ with the help of the independent hoops
\[ \beta_1, \ldots, \beta_n \in S^* . \]
Note that the complement of a cylindrical set is cylindrical, as are finite unions and intersections of cylindrical sets so that $\mathcal{C}$ is in fact a (finite) algebra. Denoting the minimal $\sigma$-algebra algebra containing the cylindrical sets by $B(\mathcal{C})$, the space
\[ (\tilde{A/G}, B(\mathcal{C})) \]

becomes a σ-measurable space. From the definition of $B_{\mathcal{L}}$ and $B(\mathcal{C})$, we see that

**Proposition 3.5**

The map (3.9)

$$\overline{(\mathcal{A}/G, B(\mathcal{C}))} \rightarrow (H_{\mathcal{L}}, B_{\mathcal{L}})$$

(3.32)

is an isomorphism of σ-measurable spaces.

**Corollary 3.6**

(i) $B(\mathcal{C})$ and $B_{\mathcal{L}}$ are contained in the Borel algebras corresponding to the Gel’fand and Tychonov topologies respectively. This follows from the fact that the cylindrical sets in $B_{\mathcal{L}}$ with open “base” $B$ in $B_{S^\ast}$ form a base in the topology $\tau_T$.

(ii) We call a function $f$ on $\overline{A/G}$ cylindrical if there exists $S^\ast \in \mathcal{L}$ such that $f$ is a pull back of a function $\tilde{f}$ on $H_{S}$.

$$f = (\pi_{S^\ast} \circ \phi)^* \tilde{f}$$

(3.33a)

i.e.

$$f([h]) = \tilde{f}([h \mid S^\ast])$$

(3.33b)

where $\tilde{f}$ is a measurable function on $H_{S^\ast}$. The Wilson loop functions $T_{\mu}([h]) = \frac{1}{N} Tr h(\alpha)$ (for $G = SU(N)$ or $G = U(N)$) are continuous cylindrical functions (6).

(iii) The projective limit $H_{\mathcal{L}}$ provides a generalization of the Ashtekar-Isham space $\overline{A/G}$ to the case where the gauge group $G$ is not compact.

(iv) There is a one-to-one correspondence between cylindrical measures $\mu$ on $\mathcal{C}$ (i.e. additive on $\mathcal{C}$ but σ-additive on the σ-subalgebras $(\pi_{S^\ast} \circ \phi)^{-1}(B_{S^\ast})$) and families of measures $\{(\mu_{S^\ast})_{S^\ast \in \mathcal{L}}\}$ ($\mu_{S^\ast}$ are Borel measures on the *finite dimensional* spaces $H_{S^\ast}$) satisfying the self-consistency condition

$$S^\ast \subset S^\ast' \Rightarrow \mu_{S^\ast} = (p_{S^\ast, S^\ast'})^* \mu_{S^\ast'}$$

(3.34)

The correspondence is given by

$$\mu_{S^\ast} = (\pi_{S^\ast} \circ \phi)^* \mu$$

(3.35)

Recall [11] that a Borel measure $\mu$ is called regular if for every Borel set $E$

$$\mu(E) = \inf \{\mu(V) : E \subset V, V \text{ open}\}$$

$$\mu(E) = \sup \{\mu(K) : E \supset K, K \text{ compact}\}$$

Also from [11, Theorem 2.18] it follows that on the spaces $H_{S^\ast}$ every Borel measure is regular. The following result (similar to [6, Theorem 4.4] and [7, Proposition 2]) holds.

**Proposition 3.7**

There is a one-to-one correspondence between regular Borel measures $\mu$ on $\overline{A/G}$ and self-consistent families of measures $\{(\mu_{S^\ast})_{S^\ast \in \mathcal{L}}\}$.

**Proof**

From (i) and (iv) we see that a regular Borel measure $\mu$ on $\overline{A/G}$ defines, by restriction, a σ-additive measure on $B(\mathcal{C})$ and therefore a consistent family of measures $\{(\mu_{S^\ast})_{S^\ast \in \mathcal{L}}\}$. Conversely let $\{(\mu_{S^\ast})_{S^\ast \in \mathcal{L}}\}$ be a consistent family of Borel measures on $\{H_{S^\ast}\}$ and $\mu_0$ be the cylindrical measure on $\mathcal{C}$ defined by this family. The family $\{(\mu_{S^\ast})_{S^\ast \in \mathcal{L}}\}$ (or equivalently the measure $\mu_0$) defines a positive functional on the continuous cylindrical functions $f = (\pi_{S^\ast} \circ \phi)^* f$ on $\overline{A/G}$.
\[ \Gamma_{\mu_0}(f) = \int_{H_S^*} \tilde{f} d\mu_{S^*}. \]  

(3.36)

This functional is bounded with respect to the sup-norm

\[ \| \Gamma_{\mu_0}(f) \| \leq \| f \|_{\infty}, \]  

(3.37)

where \( \| f \|_{\infty} = \sup_{[h] \in \mathcal{A}/G} | f([h]) |. \) Since the space of continuous cylindrical functions is dense in the \( C^* \)-algebra \( C(\mathcal{A}/G) \) of all continuous functions on \( \mathcal{A}/G \) (see [6]) the functional \( \Gamma_{\mu_0} \) can be extended in a unique way to a continuous positive (norm 1) functional on \( C(\mathcal{A}/G) \) (see [9]). But in accordance with the Riesz representation theorem (see [11]) there is then a unique regular Borel measure \( \mu \) on \( \mathcal{A}/G \) such that

\[ \Gamma_{\mu_0}(f) = \int_{\mathcal{A}/G} d\mu f \]  

(3.38)

for every \( f \in C(\mathcal{A}/G) \), where we denoted the extension of \( \Gamma_{\mu_0} \) to \( C(\mathcal{A}/G) \) with the same letter. Regular Borel measures are completely determined if the integral of continuous functions is known (see [11], p.41), which implies that \( \mu \) and \( \mu_0 \) coincide on \( \mathcal{A}/G \). Therefore \( \mu \) is the unique (see [12]) extension of \( \mu_0 \) to \( B(\mathcal{C}) \) and (as we have showed) the unique regular extension to a Borel measure.

\[ Q.E.D. \]

4. \( \mathcal{A}/\mathcal{G} \) is contained in a zero measure subset of \( \mathcal{A}/\mathcal{G} \).

The present section contains the main result of this paper. For simplicity we will use (3.32) to identify the \( \sigma \)-measurable spaces \( (\mathcal{A}/\mathcal{G}, B(\mathcal{C})) \) and \( (H_{\mathcal{E}}, B_{\mathcal{E}}) \) so that we will consider \( \mathcal{A}/\mathcal{G} \) to be the projective limit of the projective family of finite dimensional spaces (3.6).

In [6] Ashtekar and Lewandowski introduced the following measure \( \mu_{AL} \) on \( (\mathcal{A}/\mathcal{G}, B(\mathcal{C})) \). Let \( \mu_H \) be the normalized Haar measure on \( G \) and \( \mu^H_0 \) and \( \mu^H_{S^*} \) the corresponding measures on \( G^n/Ad \) and \( H_{S^*} \) (\( \mu^H_{S^*} \) is obtained from \( \mu^H_0 \) using (3.5)). Then the (uncountable) family \( (\mu^H_{S^*})_{S^* \in \mathcal{L}} \) satisfies the self-consistency conditions (2.20). The Ashtekar-Lewandowski measure \( \mu_{AL} \) is the corresponding (unique) measure on \( (\mathcal{A}/\mathcal{G}, B(\mathcal{C})) \) satisfying

\[ \mu^H_{S^*} = (\pi_{S^*})_* \mu_{AL}. \]  

(4.1)

The measure \( \mu_{AL} \) is \( \sigma \)-additive, \( \text{Diff}(\Sigma) \)-invariant, and strictly positive as a functional on the space continuous cylindrical functions on \( \mathcal{A}/\mathcal{G} \) (see [6]).

The space \( \mathcal{A}/\mathcal{G} \) is canonically embedded in \( \mathcal{A}/\mathcal{G} \) [2] and is topologically dense there [6, 10]. It is interesting to find out whether \( \mathcal{A}/\mathcal{G} \) is also \( \mu_{AL} \)-thick in \( \mathcal{A}/\mathcal{G} \); that is, whether \( \mathcal{A}/\mathcal{G} \) supports the measure \( \mu_{AL} \). We will in fact prove that this is far from being the case:
Theorem 4.1
There exists a measurable set
\[ Z \in B(C) \] (4.2a)
such that
\[ \mu_{AL}(Z) = 0 \] (4.2b)
and
\[ \mathcal{A}/\mathcal{G} \subset Z. \] (4.2c)

Proof
We need the following lemma

Lemma 4.2
For every \( q \in (0, 1] \) there exists \( Q^{(q)} \subset \mathcal{A}/\mathcal{G} \) such that
\[ \mu_{AL}(Q^{(q)}) = q \] (4.3)
and
\[ \mathcal{A}/\mathcal{G} \subset Q^{(q)} \] (4.4)

Proof of Lemma
The complement \( Q^{(q)^c} \) of \( Q^{(q)} \) will be constructed essentially (i.e. modulo dividing by \( Ad \)) by taking an infinite product of sets consisting of copies of \( G \) with holes cut out around the identity such that the “diameter” of the holes decreases to zero. These copies of \( G \) are chosen to correspond to a certain “convergent” sequence of hoops. In order to do this explicitly, choose \( r_0 \) such that the exponential map \( \exp : \mathcal{U}_{r_0}(0) \rightarrow \mathcal{O}_{r_0}(e) \subset G \), where \( e \) is the identity of the group.

Let us define a function on \( \mathcal{O}_{r_0}(e) \) that measures the “distance” to the identity
\[ d_e : \mathcal{O}_{r_0}(e) \rightarrow \mathbb{R}^+ \cup \{0\} \]
and denote by the same letter \( d_e \) the following extension to the whole group \( G \):
\[ d_e : G \rightarrow \mathbb{R}^+ \cup \{0\} \]
(4.7a)

and
\[ d_e(g) = \| \ln(g) \| \]
(4.7b)

The \( Ad \)-invariance of \( \| \cdot \| \) on \( Lie(G) \) implies that \( d_e(\cdot) \) is \( Ad \)-invariant on \( G \). Consider now the basic sets
\[ \Delta^t \subset G \]
\[ \Delta^t = \{ g \in G : d_e(g) \geq \epsilon \} \quad 0 \leq \epsilon \leq r_0. \] (4.8)
The function given by
\[
s : [0, r_0) \to \mathbb{R}^+
\]
\[
s(\epsilon) = \mu_H(\Delta^\epsilon)
\] (4.9)
is continuous, monotonically decreasing and \(s(0) = 1\). Now let \(\Delta_n^{\{\epsilon_i\}_{i=1}^n}\) be the subset of \(G^n\) given by
\[
\Delta_n^{\{\epsilon_i\}_{i=1}^n} = \{(g_1, \ldots, g_n) : d_e(g_i) \geq \epsilon_i\} = \prod_{i=1}^n \Delta^\epsilon_i.
\] (4.10)
Clearly we have
\[
\mu_n^H(\Delta_n^{\{\epsilon_i\}_{i=1}^n}) = \prod_{i=1}^n s(\epsilon_i).
\] (4.11)
Notice that the set \(\Delta_n^{\{\epsilon_i\}}\) is an Ad-invariant subset of \(G^n\). It is the inverse image of the set
\[
\tilde{\Delta}_n^{\{\epsilon_i\}_{i=1}^n} \subset G^n/Ad
\] (4.12a)
\[
\tilde{\Delta}_n^{\{\epsilon_i\}_{i=1}^n} = \{[g_1, \ldots, g_n] : d_e(g_i) \geq \epsilon_i\}
\] (4.12b)
under the quotient map \(\pi : G^n \to G^n/Ad\). By the definition of the measure \(\mu_n^H\) on \(G^n/Ad\) we thus have
\[
\mu_n^H(\tilde{\Delta}_n^{\{\epsilon_i\}_{i=1}^n}) = \prod_{i=1}^n s(\epsilon_i).
\] (4.13)
Now, for each \(q \in (0, 1]\) choose a sequence
\[
\{\epsilon_i^{(q)}\}_{i=1}^\infty
\] (4.14a)
such that \(\epsilon_i^{(q)} \neq 0\) but
\[
\lim_{i \to \infty} \epsilon_i^{(q)} = 0
\] (4.14b)
and
\[
1 - q = \lim_{n \to \infty} \prod_{i=1}^n s(\epsilon_i^{(q)})
\] (4.14c)
Let \(\{\beta_i\}_{i=1}^\infty\) be an arbitrary sequence of independent hoops. Then the sets
\[
\hat{\Delta}_n^{\{\epsilon_i^{(q)}\}_{i=1}^n} \subset \overline{A/G}
\]
\[
\hat{\Delta}_n^{\{\epsilon_i^{(q)}\}_{i=1}^n} = (\pi_{S^*[\beta_1, \ldots, \beta_n]} \circ \phi)^{-1}\left(\tilde{\Delta}_n^{\{\epsilon_i^{(q)}\}_{i=1}^n}\right),
\] (4.15a)
were we used (3.5) to identify \(H_{S^*}\) and \(G^n/Ad\), form a decreasing sequence
\[
\hat{\Delta}_1^{\{\epsilon_i^{(q)}\}_{i=1}^n} \supset \cdots \supset \hat{\Delta}_n^{\{\epsilon_i^{(q)}\}_{i=1}^n} \supset \cdots
\] (4.15b)
such that
\[
\mu_{AL}(\hat{\Delta}_n^{\{\epsilon_i^{(q)}\}_{i=1}^n}) = \mu_n^H(\hat{\Delta}_n^{\{\epsilon_i^{(q)}\}_{i=1}^n}).
\] (4.15c)
Now, introducing $\mathcal{R}(q)\{\beta_i\}$ whose complement in $\overline{\mathcal{A}/G}$ is

$$\mathcal{R}(q)\{\beta_i\}^c = \cap_{n=1}^{\infty} \Delta_n^{(\epsilon_i^{(q)})}_{i=1},$$

we conclude from (4.13) and the $\sigma$-additivity of $\mu_{AL}$ that

$$\mu_{AL}(\mathcal{R}(q)\{\beta_i\}^c) = \lim_{n \to \infty} \mu_{AL}(\Delta_n^{(\epsilon_i^{(q)})}_{i=1}) = 1 - q$$

and

$$\mu_{AL}(\mathcal{R}(q)\{\beta_i\}) = q \in (0, 1].$$

Let us now turn to the second part of the lemma namely the choice of $Q(q)$ satisfying (4.3) and (4.4). Take for $\hat{\beta}_i$ the hoops corresponding to coordinate squares (all parallel to a fixed coordinate plane) with a corner at $x_0$ and fix a metric. Choose $\hat{\beta}_i$ to have areas such that

$$\text{Area}(\hat{\beta}_i) = \epsilon_i^{(q)}\delta_i,$$

where $\{\epsilon_i^{(q)}\}_{i=1}^{\infty}$ is the same as in (4.14) and $\{\delta_i\}_{i=1}^{\infty}$ is any sequence with $\delta_i \to 0$. Let

$$Q(q) = \mathcal{R}(q)\{\hat{\beta}_i\}.$$

Then, for every $A \in \mathcal{A}$ we have (from the smoothness of $A$)

$$H(\hat{\beta}_i, A) = 1 + F(A)\epsilon_i^{(q)}\delta_i + O(\epsilon_i^{(q)}\delta_i^2),$$

where $F(A)$ denotes the component of the curvature at $x_0$ in the plane of the squares $\hat{\beta}_i$. Then for every $[A] \in \mathcal{A}/G$ there exists a constant $c([A]) > 0$ such that

$$d_e(H(\hat{\beta}_i, [A])) < c([A])\epsilon_i^{(q)}\delta_i.$$ (4.20)

and, since $\delta_n \to 0$, for $n$ large enough we have

$$d_e(H(\hat{\beta}_n, [A])) < \epsilon_n^{(q)}.$$ (4.20)

Thus, for every $[A] \in \mathcal{A}/G$, $[A] \in Q(q)$. We have therefore proved that with our choice (4.18) of $\hat{\beta}_i$ we have

$$\mathcal{A}/G \subset Q(q)$$

Q.E.D.

Let us now prove the theorem. From (4.4) we conclude that for every $q > 0$

$$\mathcal{A}/G \subset Q(q) \subset \overline{\mathcal{A}/G}$$

and

$$\mu_{AL}(Q(q)) = q.$$ (4.21b)
Considering now the decreasing sequence \( Q^{(1/n)} \). We have
\[
\mathcal{A}/\mathcal{G} \subset Z \equiv \bigcap_{n=1}^{\infty} Q^{(1/n)}.
\]
(4.22)
while the \( \sigma \)-additivity of \( \mu_{AL} \) implies that
\[
\mu_{AL}(Z) = \lim_{N \to \infty} \mu_{AL}(Q^{(1/n)}) = 0
\]
(4.23)

5. Completion of the space of square integrable functions on \( \mathcal{A}/\mathcal{G} \)

Although \( \mathcal{A}/\mathcal{G} \) is not a projective limit of the family (3.6) a procedure similar to that of (2.14), (2.19)-(2.21) can be used to define a measure \( \hat{\mu}_{AL} \) on \( \mathcal{A}/\mathcal{G} \) as was noted in [6]. This is done by returning to the notion of a cylindrical set (3.17) but now in \( \mathcal{A}/\mathcal{G} \). That is, we introduce (surjective) projections
\[
\hat{\pi}_{S^*} : \mathcal{A}/\mathcal{G} \to H_{S^*}[\beta_1, \ldots, \beta_n]
\]
\[
\pi_{S^*}([A]) = [H(\beta_1, A), \ldots, H(\beta_n, A)],
\]
(5.1)
where again we are identifying \( G^n/Ad \) with \( H_{S^*}, \) and take as measurable sets
\[
C_B \subset \mathcal{A}/\mathcal{G}
\]
\[
C_B = \hat{\pi}_{S^*}^{-1}(B),
\]
(5.2)
for some \( B \in \mathcal{B}_{S^*} \). Let \( \mathcal{C} \) be the collection of such cylindrical sets in \( \mathcal{A}/\mathcal{G} \). Note that \( \mathcal{C} \) is closed under union, intersection and complementation (i.e. forms an algebra) so that the pair
\[
(\mathcal{A}/\mathcal{G}, \mathcal{C})
\]
(5.3)
is a measurable space. The measure \( \hat{\mu}_{AL} \) is then defined by
\[
\hat{\mu}_{AL}(\hat{\pi}_{S^*}^{-1}(B)) = \mu_{H_{S^*}}^H(B).
\]
(5.4)
The additivity of \( \mu_{H_{S^*}}^H \) for every \( S^* \) implies additivity of \( \hat{\mu}_{AL} \). However, the \( \sigma \)-additivity of the \( \mu_{H_{S^*}}^H \) does not imply \( \sigma \)-additivity of \( \hat{\mu}_{AL} \). Indeed, we have the following

**Proposition 5.1**
The measure (5.4) on \( \mathcal{A}/\mathcal{G} \) cannot be extended to a \( \sigma \)-additive measure on \( \mathcal{B}(\mathcal{C}) \).

**Proof**
This theorem follows easily from lemma 4.2. Indeed consider the same sets \( \tilde{\Delta}_n^{(\epsilon)} \subset G^n/Ad \) as in (4.12)-(4.14) and define analogously to (4.15) the decreasing sequence
\[
\hat{\Delta}_n^{(\epsilon)} \subset \mathcal{A}/\mathcal{G}
\]
\[
\hat{\Delta}_n^{(\epsilon)} \subset \hat{\pi}_{S^*}^{-1}([\beta_1, \ldots, \beta_n])
\]
\[
\Delta_n^{(\epsilon)} \supset \cdots \supset \Delta_n^{(\epsilon)} \supset \cdots,
\]
(5.5a)
(5.5b)
(5.5c)
where the sequence $\{\beta_i\}_{i=1}^\infty$ is defined as in (4.18). Then for the same reason as in (4.20) there is not a single \([A]\) belonging to the intersection of all $\Delta_n^{(q)}$ i.e. now we have
\[
\cap_{n=1}^\infty \Delta_n^{(q)} = \emptyset
\]
even though
\[
\lim_{n \to \infty} \hat{\mu}_{AL} \left( \Delta_n^{(q)} \right) = 1 - q.
\]
Therefore, choosing $q : 0 < q < 1$ we conclude from the Hopf theorem 2.1 that $\hat{\mu}_{AL}$ is not extendible to a $\sigma$-additive measure on $B(C)$.

Q.E.D.

Let us recall aspects of integration theory for the so called (non-$\sigma$) measurable spaces with limit structure (see [13] def. 1.5). The measurable space $(X, \mathcal{F}_X)$ is said to be a space with limit structure if
\[
\mathcal{F}_X = \bigcup_{L \in \mathcal{L}} \mathcal{B}_L,
\]
where for all $L \in \mathcal{L}$, $\mathcal{B}_L$ is a $\sigma$-algebra and for every $L_1, L_2 \in \mathcal{L}$, there exists a $L_3$ such that $\mathcal{B}_{L_1} \cup \mathcal{B}_{L_2} \subset \mathcal{B}_{L_3}$. If the family $\{\mathcal{B}_L\}_{L \in \mathcal{L}}$ does not have a maximal element then $\mathcal{F}_X$ is not a $\sigma$-algebra. Obviously every projective limit defined as in (2.19)-(2.21) is a measurable space with limit structure. The converse is also true as we can see by taking as projective family of $\sigma$-measurable spaces (see [8] p. 20)
\[
\{(X_L, B_L), p_{LL'}\}_{L, L' \in \mathcal{L}} = \{(X, B_L), id\}_{L \in \mathcal{L}}.
\]
Though this makes the class of projective limit spaces equivalent to that of measurable spaces with limit structure the latter is more “natural” for integration theory.

In a measurable space with limit structure $(X, \mathcal{F}_X)$ the sets $F \in \mathcal{F}_X$ are called cylindrical sets and the map $f$ to a $\sigma$-measurable space $(Y, \mathcal{B})$ is called cylindrical if there is a $L \in \mathcal{L}$ such that $f : (X, \mathcal{B}_L) \to (Y, \mathcal{B})$ is measurable. A measure $\mu$ on $\mathcal{F}_X$ is called a quasi-$\sigma$-measure (quasi-measure in [13]) if its restriction $\mu_L = \mu |_{\mathcal{B}_L}$ to every $\mathcal{B}_L \subset \mathcal{F}_X$ is $\sigma$-additive. The triple $\{(X, \mathcal{F}_X), \mu_X\}$, where $(X, \mathcal{F}_X)$ is a measurable space with limit structure and $\mu_X$ is a quasi-measure is called a quasi-measure space. Let $L_0 \in \mathcal{L}$ be such that the (complex-valued) cylindrical function
\[
f : (X, \mathcal{B}_{L_0}) \to (\mathbb{C}, \mathcal{B})
\]
is measurable. A measure $\mu$ on $\mathcal{F}_X$ is called a quasi-$\sigma$-measure (quasi-measure in [13]) if its restriction $\mu_L = \mu |_{\mathcal{B}_L}$ to every $\mathcal{B}_L \subset \mathcal{F}_X$ is $\sigma$-additive. The triple $\{(X, \mathcal{F}_X), \mu_X\}$, where $(X, \mathcal{F}_X)$ is a measurable space with limit structure and $\mu_X$ is a quasi-measure is called a quasi-measure space. Let $L_0 \in \mathcal{L}$ be such that the (complex-valued) cylindrical function
\[
f : (X, \mathcal{B}_{L_0}) \to (\mathbb{C}, \mathcal{B})
\]
where $\mathcal{B}$ denotes the $\sigma$-algebra of the complex plane, is measurable. Then a function $f$ on the quasi-measure space $\{(X, \mathcal{F}_X), \mu\}$ is said to be $\mu$-integrable if it is $\mu_{L_0}$ integrable in the usual sense
\[
\int_X f d\mu(x) = \int_{X_{L_0}} f d\mu_{L_0}(x).
\]

**Definition 5.2**
The set of square-integrable cylindrical functions on the quasi-measure space $\{(X, \mathcal{F}_X), \mu\}$ will be denoted through $CL^2(X, \mathcal{F}_X, \mu)$. 

19
It is easy to see that $CL^2(X,\mathcal{F}_X,\mu)$ is a pre-Hilbert space with inner product given by

$$
(f,g) = \int_X \overline{f(x)}g(x)d\mu(x) = \int_X f(x)g(x)d\mu_L(x),
$$

(5.11) where $L_0$ is such that both $f : (X,\mathcal{B}_{L_0}) \to (\Phi,\mathcal{B})$ and $g : (X,\mathcal{B}_{L_0}) \to (\Phi,\mathcal{B})$ are measurable.

**Proposition 5.3**

Suppose that we are given two quasi-measure spaces $\{(X,\mathcal{F}_X,\mu_X)\}$ and $\{(Y,\mathcal{F}_Y,\mu_Y)\}$, where

$$
\mathcal{F}_X = \cup_{L \in \mathcal{L}} \mathcal{B}_L(X) \quad \text{and} \quad \mathcal{F}_Y = \cup_{L \in \mathcal{L}} \mathcal{B}_L(Y)
$$

and that $Y \subset X$. Let $\chi : \mathcal{F}_X \to \mathcal{F}_Y$ be an isomorphism of set algebras given by $\chi(B) = B \cap Y$ for $B \in \mathcal{F}_X$ and such that the restriction to every $\mathcal{B}_L(X)$ is an isomorphism of $\sigma$-algebras $\mathcal{B}_L(X) \to \mathcal{B}_L(Y)$. Assume also that $\mu_Y \circ \chi = \mu_X$.

Then if $\mu_X$ is extendible to a $\sigma$-additive measure $\tilde{\mu}_X$ on $\mathcal{B}(\mathcal{F}_X)$, the completion of $CL^2(Y,\mathcal{F}_Y,\mu_Y)$ is $L^2(X,\mathcal{B}(\mathcal{F}_X),\tilde{\mu}_X)$.

**Proof**

Note that the map $\chi : \mathcal{F}_X \to \mathcal{F}_Y$ induces a one-to-one correspondence between the sets $\mathcal{X}_Y$ of characteristic functions of sets in $\mathcal{F}_Y$ and $\mathcal{X}_X$ of characteristic functions of sets in $\mathcal{F}_X$. Further, since $\chi$ is an isomorphism of finite set algebras, this correspondence extends to an isomorphism over the linear spans of $\mathcal{X}_Y$ and $\mathcal{X}_X$. Finally, since $\chi$ preserves the measure of sets, this correspondence preserves the inner product in these linear spaces. We need the following lemma.

**Lemma 5.4**

(i) The completion of $\mathcal{X}_Y$ ($\mathcal{X}_X$) is equal to the completion of $CL^2(Y,\mathcal{F}_Y,\mu_Y)$ ($CL^2(X,\mathcal{F}_X,\mu_X)$).

(ii) The space $\mathcal{X}_X$ is dense in $L^2(X,\mathcal{B}(\mathcal{F}_X),\tilde{\mu}_X)$.

**Proof of Lemma**

(i) Obviously $\mathcal{X}_Y$ is a subset of $CL^2(Y,\mathcal{F}_Y,\mu_Y)$. It is sufficient to show that any $f \in CL^2(Y,\mathcal{F}_Y,\mu_Y)$ can be represented as

$$
f = \lim_{n \to \infty} \phi_n,
$$

(5.12) where $\phi_n \in \mathcal{X}_Y$ and the sequence converges in the norm of $CL^2(Y,\mathcal{F}_Y,\mu_Y)$. But for $f \in CL^2(Y,\mathcal{F}_Y,\mu_Y)$ there exists a $L_0 \in \mathcal{L}$ such that $f$ belongs to the (complete) space $L^2\left(Y,\mathcal{B}_{L_0}(Y),\mu_Y\mid_{\mathcal{B}_{L_0}(Y)}\right)$. Since $\mathcal{X}_Y \mid_{\mathcal{B}_{L_0}(Y)} \subset \mathcal{X}_Y$ is dense in $L^2\left(Y,\mathcal{B}_{L_0}(Y),\mu_Y\mid_{\mathcal{B}_{L_0}(Y)}\right)$ (see [12]) $f$ can be represented in the form (5.10).

(ii) For a quasi-measure space $\{(X,\mathcal{F}_X,\mu_X)\}$ satisfying the conditions of proposition 5.2 we have

$$
\mathcal{X}_X \subset CL^2(X,\mathcal{F}_X,\mu_X) \subset L^2(X,\mathcal{B}(\mathcal{F}_X),\tilde{\mu}_X),
$$

(5.13) where clearly all the inclusions are isometric. It will be sufficient to prove that for every set $B \in \mathcal{B}(\mathcal{F}_X)$ its characteristic function $\chi_B$ is in the $L^2$-closure of $\mathcal{X}_X$. But this result follows easily from Theorem 3.3 in [8].

*Q.E.D.*
**Proof of Proposition**

We have an isometric isomorphism (i.e., one which preserves the inner product) between the spaces \( \mathcal{X}_Y \) and \( \mathcal{X}_X \), which are dense in \( \tilde{\mathcal{X}}_Y = C^{L^2}(Y, \mathcal{F}_Y, \mu_Y) \) and \( L^2(\mathcal{X}, \mathcal{B}(\mathcal{F}_X), \tilde{\mu}_X) \) respectively. The isomorphism therefore extends to a natural isometric isomorphism

\[
\eta : C^{L^2}(Y, \mathcal{F}_Y, \mu_Y) \to L^2(\mathcal{X}, \mathcal{B}(\mathcal{F}_X), \tilde{\mu}_X) \tag{5.14}
\]

Q.E.D.

In the case of \( \mathcal{A}/\mathcal{G} \) since the projections \( \tilde{\pi}_S \) are surjective we have

\[
\tilde{\pi}_S^{-1}(B_1) = \tilde{\pi}_S^{-1}(B_2) \tag{5.15}
\]

if and only if there is some \( S^* \subset S^*_1 \cap S^*_2 \) and some \( B \subset H_{S^*} \) such that

\[
B_1 = \pi_{S^*_1}^{-1}(B), \quad B_2 = \pi_{S^*_2}^{-1}(B) \tag{5.16}
\]

Since the same is true for the algebra \( \mathcal{T} \) of cylindrical sets in \( \mathcal{A}/\mathcal{G} \), there is a one-to-one correspondence between \( \mathcal{C} \) and \( \mathcal{T} \) given by

\[
\tilde{\pi}_S^{-1}(B) = \chi((\pi_S \circ \phi)^{-1}(B)) \tag{5.17}
\]

where \( \phi \) and \( \pi_{S^*} \) have been defined in (3.9) and (3.11) respectively. Note that the map \( \chi \) is an isomorphism of set algebras, and that it preserves measures in the sense that

\[
\chi(\tilde{B}) = \mathcal{A}/\mathcal{G} \cap \tilde{B} \tag{5.18a}
\]

and that

\[
\hat{\mu}_{AL} \circ \chi = \mu_{AL} | \mathcal{T} \tag{5.18b}
\]

so that the conditions of proposition 5.3 are satisfied for this case. In this way, the completion of \( C(L^2(\mathcal{A}/\mathcal{G}, \mathcal{C}, \hat{\mu}_{AL}) \) is \( L^2(\mathcal{A}/\mathcal{G}, \mathcal{B}(\mathcal{T}), \mu_{AL}) \) and we arrive at the space \( \mathcal{A}/\mathcal{G} \).

Let us also show that \( C(L^2(\mathcal{A}/\mathcal{G}, \hat{\mu}_{AL}, \mathcal{C}) \) (hereafter referred to as simply \( C(L^2(\mathcal{A}/\mathcal{G}) \) is not complete. To see this, consider the sets

\[
\Delta_n = \Delta_n^{(\epsilon)} \subset G^n/Ad \tag{5.19a}
\]

and

\[
\hat{\Delta}_n = \hat{\Delta}_n^{(\epsilon)} \subset \mathcal{A}/\mathcal{G} \tag{5.19b}
\]

introduced above, for some \( q < 1 \), as well as the corresponding characteristic functions \( \chi_n \).

Since

\[
\hat{\mu}_{AL}(\hat{\Delta}_n) \to 1 - q > 0, \tag{5.20}
\]

given any \( \epsilon > 0 \) there is some \( N \in \mathbb{N} \) such that \( \forall n \geq m > N \),

\[
\| \chi_n - \chi_m \| = \int_{\mathcal{A}/\mathcal{G}} (\chi_n - \chi_m)^2 d\hat{\mu}_{AL} = \hat{\mu}_{AL}(\hat{\Delta}_n) - \hat{\mu}_{AL}(\hat{\Delta}_m) < \epsilon \tag{5.21}
\]

and the sequence \( \{\chi_n\}_{n=1}^{\infty} \) is Cauchy. Suppose that it converges to some

\[
f \in C(L^2(\mathcal{A}/\mathcal{G})
\]

which implies that \( f \) is itself a cylindrical function, \( f = \tilde{f} \circ \hat{\pi}_{S^*} \) for some function \( \tilde{f} \) on some \( H_{S^*} \).
Consider now the finitely generated subgroups $S^n_*=S^*[\hat{\beta}_1,\ldots,\hat{\beta}_n]$ used to define $\Delta_n^{(r)}_{i=1}$ and $\chi_n$. For large enough $N$, no $\hat{\beta}_m$ for $m \geq N$ lies in $S^*_0$. Thus, if $S^{*'}_m$, $m \geq N$, is the subgroup generated by hoops in $S^*_m$ and hoops in $S^*_0$, $\chi_m(h) = 0$ for any homomorphism $h$, $[h] \in H_{S^*_m}$ such that $d(e(h(\hat{\beta}_N))) \leq \epsilon_N$. Let $R_m$ be the set of all such $[h] \in H_{S^*_m}$. Then

$$
\|\chi_m - f\|^2 = \int_{H_{S^*_m}} d\mu_{S^*_m} |\chi_m - f|^2 \circ \pi_{S^*_m}^{-1}
\geq \int_{R_m} d\mu_{S^*_m} |f|^2 \circ \pi_{S^*_m}^{-1}
= s(\epsilon_N) \int_{S^*_0} d\mu_{S^*_0} |\tilde{f}|^2
$$

so that $\|\chi_m - f\|^2$ is bounded away from zero unless $\tilde{f}$ is the zero function. However, if $f$ is the zero function then

$$
\|\chi_m - f\|^2 = \|\chi_m\|^2 \geq q
$$

so that the Cauchy sequence $\{\chi_n\}_{n=1}^\infty$ does not converge in $\mathcal{CL}^2 (A/G)$ and $\mathcal{CL}^2 (A/G)$ is incomplete.

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