LOGARITHMIC DIFFERENCE LEMMA IN SEVERAL COMPLEX VARIABLES AND PARTIAL DIFFERENCE EQUATIONS

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Abstract. In this paper, we mainly propose improvements of the logarithmic difference lemma for meromorphic functions in several complex variables, and then apply them to investigate meromorphic solutions of partial difference equations.

1. Introduction

It is well known that the celebrated binomial function $C^m_n = \frac{n!}{(n-m)! m!} (1 \leq m \leq n)$ having the relation

(1) $C^m_n = C^{m-1}_{n-1} + C^{m-1}_m$

in the early history of mathematics, which was known to Shijie Zhu in China in 1303. The functional relation (1) is an example of partial difference equations which while was developed only after the 18th century (refer to, for examples [8, 12, 13]). If their continuous counterparts are considered, then it is easy to check that the entire function $f(z_1, z_2) = e^{z_1+z_2}$ on $\mathbb{C}^2$ is a nontrivial solution of the partial difference equation

$f(z_1, z_2) = f(z + c_1, z_2 + c_2) + f(z_1, z_2 + c_2)$,

where $c_1, c_2$ are values in $\mathbb{C}^2$ such that $e^{c_1+c_2} + e^{c_2} = 1$. Hence, it is worth in considering entire or meromorphic solutions of partial difference equations.

As early as over 30 years ago, several initial results on the existence of meromorphic solutions of some complex difference equations have been obtained by Bank, Kaufman, Shimomura, Yanagihara and other researchers. Later on, the researches in this field were developed slowly, almost in a state of stagnation. Until recent ten years, Nannanlinna theory (especially the difference analogues such as logarithmic derivative lemma, Tumura-Clunie theorem etc.) has been used as a powerful tool to investigate complex difference equations, and thus it becomes an interesting and hot direction. For this background, we refer to see [15, 10, 5].

As far as we know, there are very little of results on solutions of complex partial difference equations. In 2012, Korhonen [24] firstly obtained the difference version of logarithmic derivative lemma (shortly, we may say logarithmic difference lemma) for meromorphic functions on $\mathbb{C}^m$ with hyperorder strictly less than $\frac{1}{3}$, and then used it to consider a class of partial difference equations in the same paper. In [2], the work is supported by the National Natural Science Foundation of China (#11461042) and the outstanding young talent assistance program of Jiangxi Province (#20171BCB23002) in China.

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Cao and Korhonen improved the logarithmic difference lemma to the case where the hyperorder is strictly less than one. Meanwhile, Wang [35] considered some kinds of partial $q$-difference equations.

The main purpose of this paper is to improve the logarithmic difference lemma in Nevanlinna theory and use it to study complex partial difference equations. We first introduce some basic notations and definitions as follows. Let $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ with $\|z\|^2 = \sum_{j=1}^m |z_j|^2$. Define the differential operators $d = \partial + \overline{\partial}$ and $d^c = \frac{\partial - i\pi}{2\pi i}$. For a meromorphic function $f$ on $\mathbb{C}^m$, let $\nu^0_{f-a}$ be the zero divisor of $f - a$. Set $n(t, \frac{r}{f-a}) = \int_{\supp \nu^0_{f-a} \cap B_m(t)} \nu^0_{f-a}(z)(dd^c\|z\|^2)^m - 1$ if $m \geq 2$; and $n(t, \frac{1}{f-a}) = \sum_{|z| \leq t} \nu^0_{f-a}(z)$ if $m = 1$, where $B_m(t) = \{z : \|z\| \leq t\}$. Denote by $N(r, \frac{1}{f-a}) = \int_1^r n(r, \frac{1}{t})dt$ the counting function of zeros of $f-a$ on complex vector space $\mathbb{C}^m$, by $m(r, f)$ the proximity function of $f$ defined as $m(r, f) = \int_{\partial B_m(r)} \log^+ |f(z)| \sigma_m(z)$, where $\sigma_m(z) = d^c \log \|z\|^2 \wedge (dd^c\|z\|^2)^m - 1$ and $\log^+ x = \max\{\log x, 0\}$. Then the Nevanlinna characteristic function of $f$ is defined as $T(r, f) = N(r, f) + m(r, f)$. Then the first main theorem is said that

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1)$$

for any value $a \in \mathbb{C} \cup \{\infty\}$. A meromorphic function $f$ can be also seen as a holomorphic curve from $\mathbb{C}^m$ into $\mathbb{P}^1(\mathbb{C})$ with a reduced representation $f = (f_0, f_1)$, where $f_0$ and $f_1$ are entire function on $\mathbb{C}^m$ without common zeros. The Cartan characteristic function is defined by $T_f(r) = \int_{\partial B_m(r)} \log \max\{|f_0(z)|, |f_1(z)|\} \sigma_m(z) - \int_{\partial B_m(r)} \log \max\{|f_0(z)|, |f_1(z)|\} \sigma_m(z)$. The two characteristic functions have the relation $T_f(r) = T(r, f) + O(1)$. The defect $\delta_f(a)$ of zeros of $f - a$ is defined as

$$\delta_f(a) = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{r-a})}{T(r, f)}.$$  

The order $\rho(f)$ and hyperorder $\rho_2(f)$ of $f$ are defined respectively by

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r},$$

and

$$\rho_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$  

We assume that the readers are familiar with the basic notations and results on Nevanlinna theory for meromorphic functions in several complex variables (refer to see, for examples [14] [33] [32]).

Let $c \in \mathbb{C}^m \setminus \{0\}$. Motivated by the ideas of [4] [39], we will prove an improvement of the logarithmic difference lemma in several complex variables [24] [2] that

$$m \left( r, \frac{f(z + c)}{f(z)} \right) = o(T(r, f))$$

holds for all $r$ possible outside of a set $E$ with zero upper density measure, provided that the growth of the meromorphic function $f$ on $C^m$ satisfies

$$\limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0$$  

(which implies that the hyperorder is rather than just strictly less than one). Then from it, we get the relation
\[ T(r, f(z + c)) = T(r, f) + o(T(r, f)), \quad (r \not\in E), \]
under the assumption of (2). We will also show the explicit expression of \( o(T(r, f)) \) in the logarithmic difference lemma for the special case whenever \( f \) is of finite order as
\[ m \left( r, \frac{f(z + c)}{f(z)} \right) = O \left( r^{\rho(f) - 1 + \varepsilon} \right), \]
and thus obtain the relation
\[ T(r, f(z + c)) = T(r, f) + o \left( r^{\rho(f) - 1 + \varepsilon} \right) \]
for any \( \varepsilon(> 0) \). This is an extension of Chiang and Feng \[10\] from one variable to several variables.

In terms of our results on the logarithmic difference lemma, we can consider meromorphic solutions of partial difference equations. For the discrete KdV equations
\[ X_{i+1}^{j+1} = X_i^j + \frac{1}{X_{i+1}^j} - \frac{1}{X_{i+1}^{j+1}}, \]
in \[34\], we can consider the KdV type partial difference equation
\[ f(z_1 + c_1, z_2 + c_2) = f(z_1, z_2) + \frac{1}{f(z_1, z_2 + c_2)} - \frac{1}{f(z_1 + c_1, z_2)} \]
where \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \). In fact, we will obtain that any transcendental meromorphic solution of the equation \[3\] with the assumption \[2\] must satisfy \( \delta_f(\infty) > 0 \). Furthermore, we will prove the difference versions of the well-known Tumura-Clunie theorem in several complex variables which is a powerful tool for studying complex (partial) differential equations (see for examples \[27, 22, 21, 30\]).

There are many models of partial linear difference equations (see \[8\]), such as the two-level discrete heat equation
\[ u_{i+1}^j = au_{i-1}^j + bu_i^j + cu_{i+1}^j, \]
the nonsymmetric partial difference functional equation
\[ \frac{u_{x+y} - 2u_{x} + u_{x-y}}{t^2} = \frac{u_{x+y} - 2u_{x} + u_{x-y}}{s^2}, \]
and the steady state discrete Laplace equation
\[ u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - 4u_{m,n} = 0. \]
These equations impel us to study general linear partial difference equations
\[ A_n(z)f(z + c_n) + \ldots + A_1(z)f(z + c_1) + A_0(z)f(z) = 0, \]
where \( A_0, \ldots, A_n \) are meromorphic functions on \( \mathbb{C}^m \) and \( c_1, \ldots, c_n \in \mathbb{C}^m \setminus \{0\} \). According to the logarithmic difference lemma for finite order, we will obtain that any nontrivial meromorphic solution \( f \) of \[4\] satisfies \( \rho(f) \geq \rho(A_k) + 1 \), whenever one transcendental meromorphic coefficient \( A_k \) \((k \in \{0, 1, \ldots, n\})\) dominates the growths of all the meromorphic coefficients. Motivated by the the model of the discrete or finite Poisson equation (see \[8\])
\[ u_{i,j+1}^j + u_{i+1,j}^j + u_{i,j-1}^j + u_{i-1,j}^j - 4u_{i,j}^j = g_{ij}, \]
we also consider the linear partial difference equations

\[ A_n(z)f(z + c_n) + \ldots + A_1(z)f(z + c_1) + A_0(z)f(z) = F(z), \]

where meromorphic coefficients \( A_0, \ldots, A_n, F \neq 0 \) on \( \mathbb{C}^m \) are small functions with respect to meromorphic solutions \( f \). We will prove that if \( \limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0 \), then \( \delta_f(0) = 0 \).

This paper is organized as follows. Three forms of the logarithmic difference lemma for meromorphic function in several complex variables are proved in Section 2. By them, the relation of \( N(r, f) \sim N(r, f(z + c)) \) and \( T(r, f) \sim T(r, f(z + c)) \) are given in the same section. In Section 3, we firstly obtain an improvement of Korhonen’s result for a class of complex partial difference equations by our logarithmic difference lemma, then consider the KdV type complex partial difference equation, difference analogues of Tumura-Clunie theorem concerning partial difference polynomials, and finally investigate general partial linear difference equations. Some examples are given to show that the results of partial difference equations are sharp.

2. LOGARITHMIC DIFFERENCE LEMMA IN SEVERAL COMPLEX VARIABLES

In 2006, Halburd-Korhonen [15, Theorem 2.1] and Chiang-Feng [10] obtained independently the difference version of logarithmic derivative lemma (shortly say, logarithmic difference lemma) for meromorphic functions with finite order on the complex plane. In 2014, Halburd, Korhonen and Tohge [16, Theorem 5.1] extended it to the case for hyperorder strictly less than one. In the high dimensional case, Korhonen [24, Theorem 3.1] gave a logarithmic difference lemma for meromorphic functions in several variables of hyperorder strictly less than \( 2/3 \). In 2016, Cao and Korhonen [2] improved it to the case for meromorphic functions with hyperorder \(< 1 \) in several variables. Very recent, Zheng and Korhonen [39] improve the condition to the case when the meromorphic function \( f \) on the plane satisfies \( \limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0 \) (rather than just hyperorder strictly less than one) which is usually called minimal type. In fact, they proved a version of the subharmonic functions for the logarithmic difference lemma. Here, we improve and extend the known results on logarithmic difference lemma directly for meromorphic functions of one and several complex variables by using a growth lemma for nondecreasing positive logarithmic convex function due to Zheng-Korhonen, but avoiding the subharmonic function theory. A tropical version of logarithmic derivative lemma due to Cao and Zheng [4] was obtained very recently.

**Theorem 2.1.** Let \( f \) be a nonconstant meromorphic function on \( \mathbb{C}^n \), and let \( c \in \mathbb{C}^n \setminus \{0\} \). If

\[ \limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0, \]

then

\[ m \left( r, \frac{f(z + c)}{f(z)} \right) + m \left( r, \frac{f(z)}{f(z + c)} \right) = o(T(r, f)) \]

for all \( r \notin E \), where \( E \) is a set with zero upper density measure \( E \), i.e.,

\[ \overline{\text{dens}}E = \limsup_{r \to \infty} \frac{1}{r} \int_{E \cap [1, r]} dt = 0. \]
Remark 2.1. (i). We note that the condition (5) implies that $\rho_2(f) \leq 1$ and the equality can possibly take happened. In fact, assume that (5) holds, then there exists $r_0 > 0$ such that for any $r > r_0$, we have $\log T(r, f) < r$ and thus $\rho_2(f) \leq 0$. Moreover, whenever $f$ is taken to satisfy, for example $T(r, f) = \exp\{\frac{1}{(\log r)^m}\}$ where $m \geq 1$, one can easily get both (5) and $\rho_2(f) = 1$. Hence, Theorem 2.1 is an improvement of all the difference version of the logarithmic derivative lemma in several variables obtained before.

(ii). By the new version of the logarithmic difference lemma, all the second main theorem and Picard type theorem for meromorphic mappings from $\mathbb{C}^n$ into complex projective spaces $\mathbb{P}^n(\mathbb{C})$ obtained in [24, 12, 2] (including also [16, 10, 5] and [25, 3]) can be improved under the assumption of (5).

Before giving the proof, we show the following lemma proved recently by Zheng and Korhonen, by which they obtained an improvement of difference version of logarithmic derivative lemma for meromorphic functions of one variable under the assumption [5]. This lemma is an improvement of a result on growth properties of nondecreasing continuous real functions ([15, Lemma 2.1] and [16]). Here the properties of real logarithmic convex functions are considered. Note that the characteristic function $T(r, f)$ and counting function $N(r, f)$ for a meromorphic function on $\mathbb{C}^n$ are satisfies the properties of nondecreasing positive, logarithmic convex, continuous function for $r$.

Lemma 2.1. [59, Lemma 2.1] Let $T(r)$ be a nondecreasing positive function in $[1, +\infty)$ and logarithmic convex with $T(r) \to +\infty (r \to +\infty)$. Assume that

\begin{equation}
\liminf_{r \to \infty} \frac{\log T(r)}{r} = 0.
\end{equation}

Set

$$
\phi(r) = \max_{1 \leq t \leq r} \{\frac{t}{\log T(t)}\}.
$$

Then given a constant $\delta \in (0, \frac{1}{2})$, we have

$$
T(r) \leq T(r + \delta^\phi(r)) \leq \left(1 + 4\delta^\frac{t}{\log T(t)}\right) T(r), \ r \not\in E_\delta,
$$

where $E_\delta$ is a subset of $[1, +\infty)$ with the zero lower density. And $E_\delta$ has the zero upper density if (1) holds for $\limsup$.

Remark 2.2. Note that $\phi\phi(r) \to \infty$ and $\phi^\delta - \frac{1}{2}(r) \to 0$ as $r \to +\infty$ in Lemma 2.1. Then for sufficiently large $r$, we have $\phi\phi(r) \geq h$ for any positive constant $h$. Hence,

$$
T(r) \leq T(r + h) \leq T(r, \phi\phi(r)) \leq (1 + \varepsilon)T(r), \ r \not\in E,
$$

where $E$ is a subset of $[1, +\infty)$ with the zero lower density.

The following lemma was obtained by Korhonen [24]. Since the assumption of $f(0) \neq 0, \infty$ for a meromorphic function $f$ of one variable in [24, Lemma 5.1] can be omitted when the Poisson-Jensen formula is used, it does not matter with [24, Lemma 5.2]. Thus we delete it in the statement.

Lemma 2.2. [24, Lemma 5.2] Let $f$ be a nonconstant meromorphic function in $\mathbb{C}^n$, let $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, let $\frac{1}{4} < \delta < 1$, and let denote $\tilde{c}_j = (0, \ldots, 0, c_j, 0, \ldots, 0)$.
Then there exists a nonnegative constant $C(\delta)$, depending only on $\delta$, such that

$$\int_{\partial B_n(r)} \log^+ \left| \frac{f(z + \tilde{c}_j)}{f(z)} \right| \sigma_n(z)$$

$$\leq \frac{8\pi |c_j| \delta C(\delta)}{\delta(1 - \delta)} \left( \frac{R}{r} \right)^{2n-2} \frac{n_f(R, \infty) + n_f(R, 0)}{r^\delta} + 4\pi |c_j| \left( \frac{R}{r} \right)^{2n-2} \left( \frac{R}{R - (r + |c_j|)} \right) \left( \frac{R}{R - r} \right)^{1 - \delta} \frac{m_f(r, \infty) + m_f(r, 0)}{\sqrt{R^2 - r^2}}$$

for all $R > r + |c_j| > |c_j|$.

Now we give the proof of our version of logarithmic difference lemma.

**Proof of Theorem 2.1** By the definition of counting function, we have

$$n_f(r, \infty) + n_f(r, 0) \leq \frac{R}{R - r} \left( N(R, f) + N(R, \frac{1}{f}) \right)$$

for all $R > r$. Then it follows by Lemma 2.2 and the first main theorem that there exists a positive constant $K_1$, depending only on $c_j = (0, \ldots, 0, c_j, 0, \ldots, 0)$ and $\delta' \in (\frac{1}{2}, 1)$, such that

$$m(r, \frac{f(z + \tilde{c}_j)}{f(z)}) = \int_{\partial B_n(r)} \log^+ \left| \frac{f(z + \tilde{c}_j)}{f(z)} \right| \sigma_n(z)$$

$$\leq K_1 K_2(r, R) \left( T(R, f) + \log \frac{1}{|f(0)|} \right)$$

for all $R > r + |c_j| > |c_j|$, where

$$K_2(r, R) = \left( \frac{R}{r} \right)^{2n-2} \left( \frac{1}{R - (r + |c_j|)} \right) \left( \frac{R}{\sqrt{R^2 - r^2}} \right) \left( \frac{R}{R - r} \right)^{1 - \delta'} \frac{1}{r^{\delta'}}.$$

Under the assumption of (5). Take $R = (r + |c_j|) + (r + |c_j|)^\delta$, $\delta \in (0, \frac{1}{2})$. Then for sufficiently large $r$,

$$\frac{1}{R - (r + |c_j|)} = \left( \frac{\log T(r + |c_j|, f)}{r + |c_j|} \right)^\delta = o(1),$$

$$\frac{R}{r} = 1 + \frac{|c_j|}{r} + \frac{(r + |c_j|)^\delta}{r} \frac{1}{(\log T(r + |c_j|, f))^\delta} = o(1)$$

and

$$\frac{R}{\sqrt{R^2 - r^2}} \left( \frac{R}{R - r} \right)^{1 - \delta'} = \frac{\frac{R}{r}}{\sqrt{\frac{R}{r} - 1}} \left( \frac{\frac{R}{r}}{\frac{R}{r} - 1} \right)^{1 - \delta'} = o(1).$$

Combining these with (7),

$$m(r, \frac{f(z + \tilde{c}_j)}{f(z)}) \leq o(1) \left( T(R, f) + \log \frac{1}{|f(0)|} \right).$$
for all sufficiently large $r$. Moreover, under the assumption (5), it follows from Lemma 2.1 that for any $\varepsilon > 0$ and $\phi(r) = \frac{r + |c_j|}{\log T(r + |c_j|, f)}$

\[ T(R, f) \leq (1 + \varepsilon'(r))T(r + |c_j|, f) \leq (1 + \varepsilon'(r))^2T(r, f) \]

holds for all $r \notin E_1$ where $\text{d}ensE_1 = 0$. Hence, (8) yields

\[ m(r, \frac{f(z + \tilde{c}_j)}{f(z)}) = \int_{\partial B_n(r)} \log^+ \left| \frac{f(z + \tilde{c}_j)}{f(z)} \right| \sigma_n(z) = o(T(r, f)) \]

for all $r$ possibly outside the set $E_1$ with $\text{d}ensE_1 = 0$.

Now for any $c \in \mathbb{C}^n$ which can be written as $c = \tilde{c}_1 + \cdots + \tilde{c}_n$. Take $\tilde{c}_0 = 0$. Since

\[ \frac{f(z + c)}{f(z)} = \frac{f(z + (c_1, \ldots, c_n))}{f(z + (c_1, \ldots, c_n-1, 0))} \cdot \frac{f(z + (c_1, \ldots, c_{n-1}, 0))}{f(z + (c_1, \ldots, c_{n-2}, 0, 0))} \cdots \frac{f(z + (c_1, 0, \ldots, 0))}{f(z + (0, \ldots, 0))} \]

\[ = \frac{\sum_{j=0}^n \tilde{c}_j}{f(z + \sum_{j=0}^n \tilde{c}_j)} \cdot \frac{f(z + \sum_{j=0}^{n-1} \tilde{c}_j)}{f(z + \sum_{j=0}^{n-2} \tilde{c}_j)} \cdots \frac{f(z + \sum_{j=0}^1 \tilde{c}_j)}{f(z + \tilde{c}_0)} \]

we get from (9) that

\[ m(r, \frac{f(z + c)}{f(z)}) = \sum_{j=1}^n o(T(r, f(z + \sum_{k=0}^{j-1} \tilde{c}_k))) \]

for all $r$ possibly outside the set $E_1$ with $\text{d}ensE_1 = 0$.

Next, we assert that

\[ T(r, f(z + c)) = T(r, f) + o(T(r, f)) \]

for any $c = (c_1, c_2, \ldots, c_n)$ and for all $r$ possibly outside a set $F$ with $\text{d}ens(F) = 0$. In fact, by the fist main theorem and (5), we have

\[ \limsup_{r \to \infty} \frac{\log N(r, f)}{r} \leq \limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0. \]

Then by Lemma 2.1 we get that

\[ N(r + h, f) = (1 + o(1))N(r, f) \]

holds for any constant $h(> 0)$ independently on $r$ and all $r \notin E_2$ with $\text{d}ensE_2 = 0$. Hence, it follows from (9) and (12) that

\[ T(r, f(z + \tilde{c}_j)) = m(r, f(z + \tilde{c}_j)) + N(r, f(z + \tilde{c}_j)) \]

\[ \leq m(r, \frac{f(z + \tilde{c}_j)}{f(z)}) + m(r, f) + N(r + |\tilde{c}_j|, f) \]

\[ = m(r, \frac{f(z + \tilde{c}_j)}{f(z)}) + m(r, f) + N(r, f) + o(N(r, f)) \]

\[ = T(r, f) + o(T(r, f)) \]
for all \( r \) possibly outside the set \( E_1 \cup E_2 \) with \( \text{dens}(E_1 \cup E_2) = 0 \). Thus, it deduces that

\[
T(r, f(z + c)) \leq T(r, f(z + (c_1, \ldots, c_{n-1}, 0))) + o(T(r, f(z + (c_1, \ldots, c_{n-1}, 0))))
\]

\[
\vdots
\]

\[
\leq T(r, f(z + (c_1, 0, \ldots, 0))) + o(T(r, f))
\]

\[
\leq T(r, f) + o(T(r, f))
\]

for all \( r \) possibly outside the set \( F = E_1 \cup E_2 \) with \( \text{dens}F = 0 \). Note that \( f(z) = f((z + c) - c) \). Then we get the assertion.

Therefore, the theorem is got immediately from (10) and (11). \( \square \)

From the proof of Theorem 2.1, we have the assertion (11). Since the relation between \( T(r, f(z)) \) and \( T(r, f(z + c)) \) is very useful to study solutions of complex difference equations, we here rewrite it as a theorem.

**Theorem 2.2.** Let \( f \) be a nonconstant meromorphic function on \( \mathbb{C}^n \) with

\[
\limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0,
\]

then

\[
T(r, f(z + c)) = T(r, f) + o(T(r, f))
\]

holds for any constant \( c \in \mathbb{C}^n \setminus \{0\} \) and all \( r \not\in E \) with \( \text{dens}E = 0 \).

If using the Hinkkanen’s Borel type Growth Lemma but not Lemma 2.1, we can obtain another form of the logarithmic difference lemma as follows. A tropical version is also given by Cao and Zheng [4] at the same time.

**Theorem 2.3.** Let \( f \) be a nonconstant meromorphic function on \( \mathbb{C}^n \), and let \( c \in \mathbb{C}^n \setminus \{0\} \). If

\[
\limsup_{r \to \infty} \frac{\log T(r, f)(\log r)^{\varepsilon}}{r} = 0,
\]

for any \( \varepsilon(>0) \), then

\[
m \left( r, \frac{f(z + c)}{f(z)} \right) + m \left( r, \frac{f(z)}{f(z + c)} \right) = o(T(r, f))
\]

for all \( r \not\in E \), where \( E \) is a set with \( \int_E \frac{dt}{\log t} < +\infty \) which implies \( E \) with zero upper logarithmic density measure i.e.,

\[
\text{dens}E = \limsup_{r \to \infty} \frac{1}{\log r} \int_{E \cap [1, r]} \frac{dt}{t} = 0.
\]

The next lemma is the Hinkkanen’s Borel type growth lemma (or see also a similar lemma [9, Lemma 3.3.1].

**Lemma 2.3.** [19 Lemma 4] Let \( p(r) \) and \( h(r) = \varphi(r)/r \) be positive nondecreasing functions defined for \( r \geq \varrho > 0 \) and \( r \geq \tau > 0 \), respectively, such that \( \int_{\varrho}^{\infty} \frac{dr}{p(r)} = \infty \)
and \( \int_\tau^\infty \frac{dr}{\varphi(r)} < \infty \). Let \( u(r) \) be a positive nondecreasing function defined for \( r \geq r_0 \geq \varrho \) such that \( u(r) \to \infty \) as \( r \to \infty \). Then if \( C \) is real with \( C > 1 \), we have

\[
u(r + \frac{p(r)}{h(u(r))}) < Cu(r)
\]

whenever \( r \geq r_0, u(r) > \tau \), and \( r \not\in E \) where

\[
\int_E \frac{dr}{p(r)} \leq \frac{1}{h(w)} + \frac{C}{C - 1} \int_w^\infty \frac{dr}{\varphi(r)} < \infty
\]

and \( w = \max\{\tau, u(r_0)\} \).

**Proof of Theorem 2.3.** In Lemma 2.3, we take

\[
u(r) = T(r, f), \quad p(r) = r \log r,
\]

and

\[
h(r) = \frac{\varphi(r)}{r}
\]

where \( \varphi(r) = r \log r (\log \log r)^{1+\varepsilon} \) with \( \varepsilon > 0 \). Then it is obvious that \( \int_\tau^\infty \frac{dr}{p(r)} = \infty \) and \( \int_r^\infty \frac{dr}{\varphi(r)} < \infty \) for \( r \geq \varrho > 0 \) and \( r \geq \tau > 0 \). Let

\[
R := (r + |c_j|) + \frac{p(r + |c_j|)}{(r + |c_j|) \log T_f(r + |c_j|)} = (r + |c_j|) + \frac{(r + |c_j|) \log(r + |c_j|)}{\log T_f(r + |c_j|)(\log \log T_f(r + |c_j|))^{1+\varepsilon}}.
\]

Note that

\[
T(R, f) = T\left((r + |c_j|) + \frac{(r + |c_j|) \log(r + |c_j|)}{\log T_f(r + |c_j|)(\log \log T_f(r + |c_j|))^{1+\varepsilon}}, f\right).
\]

Applying Lemma 2.3 we have

\[
T(R, f) \leq CT(r + |c_j|, f)
\]

for all \( r \) possibly outside a set \( E_1 \) satisfying

\[
E_1 := \{r \in [r_0, \infty) : T(R, f) \geq CT(r + |c_j|, f)\}
\]

where

\[
\int_{E_1} \frac{dt}{p(t)} = \int_{E_1} \frac{dt}{t \log t} \leq \frac{1}{\log w(\log \log w)^{1+\varepsilon}} + \frac{C}{C - 1} \int_w^\infty \frac{dt}{t \log t(\log \log t)^{1+\varepsilon}} < +\infty.
\]

This gives

\[
\logdens E_1 = \limsup_{r \to \infty} \frac{1}{\log r} \int_{E_1 \cap [1, r]} \frac{dt}{t} 
\]

\[
\leq \limsup_{r \to \infty} \int_{E_1 \cap [1, \log r]} \frac{dt}{\log r} + \limsup_{r \to \infty} \int_{E_1 \cap [\log r, r]} \frac{dt}{t \log t}
\]

\[
\leq \limsup_{r \to \infty} \frac{\log r}{\log r} + 0 = 0.
\]
Under the condition (13), we get that for any \( \varepsilon' > 0 \) and sufficiently large \( r \),
\[
\frac{\log T(r + |c_j|, f)(\log(r + |c_j|))^{\varepsilon}}{r + |c_j|} < \varepsilon'.
\]
(15)

Since (13) implies \( \rho_2(f) \leq 1 \) according to Remark 2.1(i), we have
\[
\frac{\log T(r + |c_j|, f)}{\log(r + |c_j|)} \leq 1 + \varepsilon''
\]
(16)

for any \( \varepsilon'' > 0 \) and sufficiently large \( r \). Then (15) and (16) give that for sufficiently large \( r \),
\[
\frac{1}{R - (r + |c_j|)} = \frac{\log T(r + |c_j|, f)(\log T(r + |c_j|, f))^{1+\varepsilon}}{(r + |c_j|) \log(r + |c_j|)}
\]
\[
= \frac{\log T(r + |c_j|, f)(\log(r + |c_j|))^{\varepsilon}}{r + |c_j|} \left( \frac{\log T(r + |c_j|, f)}{\log(r + |c_j|)} \right)^{1+\varepsilon}
\]
\[
\leq \varepsilon'(1 + \varepsilon'')^{1+\varepsilon},
\]

and
\[
\frac{R}{r} = 1 + \frac{|c_j|}{r} + (1 + \frac{|c_j|}{r}) \frac{\log(r + |c_j|)}{\log T(r + |c_j|, f)(\log T(r + |c_j|, f))^{1+\varepsilon}}
\]
\[
= 1 + \frac{|c_j|}{r} + (1 + \frac{|c_j|}{r}) \frac{1}{(\log(r + |c_j|))^{\varepsilon}} \log T(r + |c_j|, f) \left( \frac{\log T(r + |c_j|, f)}{\log(r + |c_j|)} \right)^{1+\varepsilon}
\]
\[
= o(1)
\]

and
\[
\frac{R}{\sqrt{R^2 - r^2}} \left( \frac{R}{R - r} \right)^{1-\delta'} = \frac{R}{\sqrt{R - 1}} \left( \frac{R}{R - 1} \right)^{1-\delta'} = o(1).
\]

Combining these with (7) and (13),
\[
m(r, \frac{f(z + \tilde{c}_j)}{f(z)}) \leq o(1) \left( T(r + |c_j|, f) + \log \frac{1}{|f(0)|} \right)
\]
(17)

for all \( r \) possibly outside a set \( E_1 \) with \( \int_{E_1} \frac{dt}{t \log t} < +\infty \).

By (13), we also have
\[
T \left( r + \frac{r \log r}{\log T(r, f)(\log T(r, f))^{1+\varepsilon}}, f \right) \leq CT(r, f)
\]
for all \( r \notin E_1 \). It follows from (15) and (16) that
\[
\frac{\log T(r, f)(\log r)^{\varepsilon}}{r} < \varepsilon'
\]

and
\[
\frac{\log T(r, f)}{\log r} \leq 1 + \varepsilon''.
\]

Thus it yields that
\[
\frac{r \log r}{\log T(r, f)(\log T(r, f))^{1+\varepsilon}} \to \infty
\]
as \( r \to \infty \). Then we have
\[
r + |c_j| \leq r + \frac{r \log r}{\log T(r, f)(\log \log T(r, f))^{1+\varepsilon}}
\]
for sufficiently large \( r \). Hence,
\[
(18) \quad T(r + |c_j|, f) \leq T(r + \frac{r \log r}{\log T(r, f)(\log \log T(r, f))^{1+\varepsilon}}, f) \leq CT(r, f)
\]
for all \( r \not\in E_1 \). Therefore, we get from \((14)\) and \((15)\) that the equation \((9)\) is still valid for \( r \) possibly outside the set \( E_1 \). Using as the same reason as in the proof of Theorem \(2.3\) to get \((10)\), we then get immediately the conclusion of the theorem from \((10)\) and Theorem \(2.2\).

\[\square\]

In the proof of Theorem \(2.3\) we do not know how to improve the condition \((13)\) by \((6)\) whenever using the Hinkkanen’s Borel type Growth Lemma (Lemma \(2.3\)). The difficulty we met is how to give well defined functions \( p \) by \((5)\) whenever using the Hinkkanen’s Borel type Growth Lemma (Lemma \(2.3\)).

For study on the solutions of complex partial difference equations, we next prove another form of the logarithmic difference lemma for meromorphic functions with finite order in several complex variables. This is an extension of \([10, \text{Corollary 2.5}]\) from one variable to several variables.

**Theorem 2.4.** Let \( f \) be a nonconstant meromorphic function on \( \mathbb{C}^n \) and let \( c \in \mathbb{C}^n \setminus \{0\} \). If \( f \) is of finite order, then
\[
m \left( r, \frac{f(z + c)}{f(z)} \right) + m \left( r, \frac{f(z)}{f(z + c)} \right) = O \left( r^{\rho(f) - 1 + \varepsilon} \right)
\]
holds for any \( \varepsilon(> 0) \).

**Proof.** Since \( f \) is of finite order, \( T(r, f) \leq r^{\rho(f) + \varepsilon} \) holds for any \( \varepsilon > 0 \). Take \( R = 2r \). Then it follows from \((7)\) that
\[
m(r, \frac{f(z + \hat{c_j})}{f(z)}) = O(r^{\rho(f) - 1 + \varepsilon}).
\]
For any \( c \in \mathbb{C}^n \) which can be written as \( c = \hat{c}_1 + \cdots + \hat{c}_n \). Take \( \hat{c}_0 = 0 \). Since
\[
\frac{f(z + c)}{f(z)} = \frac{f(z + (c_1, \ldots, c_n))}{f(z + (c_1, \ldots, c_{n-1}, 0))} \cdots \frac{f(z + (c_1, 0, \ldots, 0))}{f(z + (0, \ldots, 0))} = \frac{f(z + \sum_{j=0}^n \hat{c}_j)}{f(z + \sum_{j=0}^{n-1} \hat{c}_j)} \cdots \frac{f(z + \hat{c}_0)}{f(z + \hat{c}_0)},
\]
Theorem 2.5. Let the convergence exponent of poles of a meromorphic function $f$ on $\mathbb{C}^n$ is finite, i.e.,

$$\lambda(\frac{1}{f}) := \limsup_{r \to \infty} \frac{\log N(r, f)}{\log r} < \infty,$$

then for any $c \in \mathbb{C}^n \setminus \{0\}$,

$$N(r, f(z + c)) = N(r, f) + O(r^{\lambda(\frac{1}{f})-1+\varepsilon})$$

holds for any $\varepsilon > 0$. The $\lambda(\frac{1}{f})$ can be changed by $\rho(f)$ whenever $f$ is of finite order.

Proof. Set $\|c\| = \sqrt{|c_1|^2 + \ldots + |c_m|^2}$. Since $\lambda(\frac{1}{f}) < \infty$, it is enough to take the same method due to Zheng and Korhonen [39] Pages 15-16 to obtain that

$$N(r + \|c\|, f) = N(r, f) + O(r^{\lambda(\frac{1}{f})-1+\varepsilon})$$

holds for any $\varepsilon > 0$. The original proof is owing to Chiang and Feng [10] Theorem 2.2] by the definition of Riemann-Stieltjes integral for counting functions (in fact, they proved this lemma for meromorphic functions of one variables). On the other hand, it is obvious that

$$N(r, f(z + c)) \leq N(r + \|c\|, f)$$

by the definition of counting function. Hence, we get that

$$N(r, f + c) \leq N(r, f) + O(r^{\lambda(\frac{1}{f})-1+\varepsilon})$$

and thus

$$N(r, f) \leq N(r, f + c) + O(r^{\lambda(\frac{1}{f})-1+\varepsilon})$$

holds for any $\varepsilon > 0$. The assumption $\lambda(\frac{1}{f}) < \infty$ implies that we can get from (21) that $\lambda(\frac{1}{f}) = \lambda(\frac{1}{f(z + \varepsilon c)})$. Therefore,

$$N(r, f + c) = N(r, f + c) + O(r^{\lambda(\frac{1}{f})-1+\varepsilon})$$

holds for any $\varepsilon > 0$. Obviously, the $\lambda(\frac{1}{f})$ can be changed by the order of $f$ from the above discussion whenever $f$ is of finite order. □
Finally in this section, we give the explicit relation $T(r, f(z + c)) \sim T(r, f)$ for a meromorphic function with finite order. This is an extension of \cite[Theorem 2.1]{10}.

**Theorem 2.6.** If a meromorphic function $f$ on $\mathbb{C}^n$ is of finite order, then
\[
T(r, f(z + c)) = T(r, f) + O(r^{\rho(f) - 1 + \varepsilon})
\]
for any $c \in \mathbb{C}^n \setminus \{0\}$ and for any $\varepsilon > 0$.

**Proof.** By Theorem 2.4 and Theorem 2.5, we have
\[
T(r, f(z + c)) = m(r, f(z + c)) + N(r, f(z + c)) \leq m(r, \frac{f(z + c)}{f(z)}) + m(r, f) + N(r, f) + O(r^{\rho(f) - 1 + \varepsilon}) = T(r, f) + O(r^{\rho(f) - 1 + \varepsilon}).
\]
This implies
\[
T(r, f) \leq T(r, f(z + c)) + O(r^{\rho(f(z + c)) - 1 + \varepsilon}).
\]
Since $\rho(f) < \infty$, it follows from Theorem 2.2 that $\rho(f(z + c)) = \rho(f)$. Hence the theorem is proved. \qed

3. Partial difference equations

In this section, we will consider meromorphic solutions of partial difference equations by making use of our results on logarithmic difference lemma.

3.1. Improvement of Korhonen’s result.

A meromorphic solution $w$ on $\mathbb{C}^n$ of a partial difference equation is called admissible if all coefficients $\{a_j\}$ of the equations are small functions with respect to $w$, that is, $\max \{T(r, a_j)\} = o(T(r, w))$. By applying Theorem 2.1 Theorem 2.2 and Valiron-Mohon’ko theorem in several complex variables \cite[Theorem 3.4]{20} into the following equation \cite{22}, it is easy to follow that
\[
T(r, w) = \deg_{w}(R) T(r, w) + o(T(r, w))
\]
for all $r \notin E$ with $\text{dens} E = 0$. We restate \cite[Theorem 4.1]{24} as follows.

**Theorem 3.1.** Let $c \in \mathbb{C}^n \setminus \{0\}$. If the difference equation
\[
w(z + c) = R(z, w(z)),
\]
where $R(z, u)$ is rational in $u$ having meromorphic coefficients in $\mathbb{C}^n$, has an admissible meromorphic solution $w$ on $\mathbb{C}^n$ with
\[
\limsup_{r \to \infty} \frac{\log T(r, w)}{r} = 0,
\]
then the degree $\deg_{w}(R)$ of $R(z, w(z))$ is equal to one.
3.2. KdV type Partial difference equations.

Next, motivated by the discrete KdV equations \( X_{j+1}^t = X_j^t + \frac{1}{X_j^t} - \frac{1}{X_{j+1}^t} \), in [31], we consider the KdV type partial difference equation as follows.

**Theorem 3.2.** Let \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \). Let \( f \) be a transcendental meromorphic solution of the KdV type partial difference equation

\[
(23) \quad f(z_1 + c_1, z_2 + c_2) = f(z_1, z_2) + \frac{1}{f(z_1, z_2 + c_2)} - \frac{1}{f(z_1 + c_1, z_2)}.
\]

If \( \delta_f(\infty) > 0 \), then \( \limsup_{r \to \infty} \frac{\log T(r,f)}{r} > 0 \).

**Proof.** Assume that a transcendental solution \( f \) satisfies \( \limsup_{r \to \infty} \frac{\log T(r,f)}{r} = 0 \). It follows from the equation (24) that

\[
f^2(z_1, z_2) = \left( \frac{f(z_1, z_2)}{f(z_1, z_2 + c_2)} - \frac{f(z_1, z_2)}{f(z_1 + c_1, z_2)} \right) \frac{f(z_1, z_2)}{f(z_1 + c_1, z_2 + c_2) - f(z, z_2)},
\]

and thus,

\[
m(r, f^2(z_1, z_2)) \leq m \left( r, \frac{f(z_1, z_2)}{f(z_1, z_2 + c_2)} \right) + m \left( r, \frac{f(z_1, z_2)}{f(z_1 + c_1, z_2)} \right) + O(1).
\]

By Theorem 2.1 one can deduce that

\[
m \left( r, \frac{f(z_1, z_2)}{f(z_1, z_2 + c_2)} \right) = o(T(r, f))
\]

\[
m \left( r, \frac{f(z_1, z_2)}{f(z_1 + c_1, z_2)} \right) = o(T(r, f))
\]

\[
m \left( r, \frac{f(z_1, z_2)}{f(z_1 + c_1, z_2 + c_2) - f(z, z_2)} \right) = o(T(r, f))
\]

hold for \( r \not\in E \) where \( E \) is a set with \( \text{den} E = 0 \). Hence,

\[
T(r, f^2) = N(r, f^2) + m(r, f^2) \leq N(r, f^2) + o(T(r, f)) \leq 2N(r, f) + o(T(r, f))
\]

holds for all \( r \not\in E \) where \( E \) is a set with \( \text{den} E = 0 \). Since \( \delta_f(\infty) > 0 \), we have

\[
N(r, f) < \left( 1 - \frac{\delta_f(\infty)}{2} \right) T(r, f).
\]

This gives

\[
T(r, f^2) \leq 2 \left( 1 - \frac{\delta_f(\infty)}{2} \right) T(r, f) + o(T(r, f))
\]

holds for all \( r \not\in E \) where \( E \) is a set with \( \text{den} E = 0 \). By the Valion-Mohon’ko theorem in several complex variables [20] Theorem 3.4, we get

\[
T(r, f) = 2T(r, f) + O(1).
\]

Therefore, it follows that

\[
\delta_f(\infty) T(r, f) \leq o(T(r, f))
\]

for all \( r \not\in E \) where \( E \) is a set with \( \text{den} E = 0 \). This is a contradiction. \( \Box \)
Since it is obvious of $\delta_f(\infty) > 0$ for an entire function $f$, we get immediately the following corollary.

**Corollary 3.1.** Let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Suppose that any transcendental entire solution $f$ of the KdV type partial difference equation

$$f(z_1 + c_1, z_2 + c_2) = f(z_1, z_2) + \frac{1}{f(z_1, z_2 + c_2)} - \frac{1}{f(z_1 + c_1, z_2)}$$

must satisfy $\limsup_{r \to \infty} \frac{\log T(r,f)}{r} > 0$.

**Example 3.1.** Let $c_1 = 2\pi i$, $c_2 = 4\pi i$. Then the meromorphic function $f(z_1, z_2) = \frac{1}{e^{z_1 + z_2} - 1}$ with $\delta_f(\infty) = 0$ and $\limsup_{r \to \infty} \frac{\log T(r,f)}{r} = 0$ is an transcendental solution of the KdV type partial difference equation. This example implies that the assumption $\delta_f(\infty) > 0$ in Theorem 3.2 is necessary.

3.3. Difference analogues of Tumura-Clunie theorem in several complex variables.

Now, we will extend difference version of Tumura-Clunie theorem from one variable to several variables. The Clunie lemma [11] for meromorphic functions of one variable in Nevanlinna theory has been a powerful tool of studying complex differential equations and related fields, particularly the lemma has been used to investigate the value distribution of certain differential polynomials; see [11] for the original versions of these results, as well as [17, 22]. A slightly more general version of the Clunie lemma can be found in [18, pp. 218–220] and [27, Lemma 2.4.5]. In 2007, the additional assumptions in the He-Xiao version of the Clunie lemma have been removed by Yang and Ye in [37, Theorem 1]. A generalized Clunie lemma for meromorphic functions of several complex variables was proved in [30]; for some special cases refer to see [23, 21]. Recently, Hu and Yang [22] extended the classical Tumura-Clunie theorem ([17, Theorem 3.9] and [31]) for meromorphic functions of one variable to that of meromorphic functions of several complex variables.

We firstly improve and extend Laine-Yang’s difference analogue of Clunie theorem in one variable [23] to high dimension by using Theorem 2.1. Define complex partial difference polynomials as follows

$$P(z, w) = \sum_{\lambda \in I} a_{\lambda}(z)w(z)^{\lambda_0}w(z + q_{\lambda_1})^{\lambda_1} \cdots w(z + q_{\lambda_i})^{\lambda_i},$$

(25)

$$Q(z, w) = \sum_{\mu \in J} b_{\mu}(z)w(z)^{\mu_0}w(z + q_{\mu_1})^{\mu_1} \cdots w(z + q_{\mu_j})^{\mu_j},$$

(26)

$$U(z, w) = \sum_{\nu \in K} c_{\nu}(z)w(z)^{\nu_0}w(z + q_{\nu_1})^{\nu_1} \cdots w(z + q_{\nu_k})^{\nu_k},$$

(27)

where all coefficients $a_{\lambda}(z)$, $b_{\mu}(z)$ and $c_{\nu}(z)$ are small functions with respect to the function $w(z)$ meromorphic on $\mathbb{C}^m$, $I, J, K$ are three finite sets of multi-indices, and $q_s \in \mathbb{C}^m \setminus \{0\}$, $(s \in \{\lambda_1, \ldots, \lambda_i, \mu_1, \ldots, \mu_j, \nu_1, \ldots, \nu_k\})$. Since the proof is closely similar as in [28], we omit it here.

**Theorem 3.3.** Let $w$ be a nonconstant meromorphic function on $\mathbb{C}^m$ with

$$\limsup_{r \to \infty} \frac{\log T(r,w)}{r} = 0,$$
and let \( P(z, w), Q(z, w), \) and \( U(z, w) \) be complex partial difference polynomials as (25), (26) and (27) satisfying a complex partial difference equation of the form
\[
U(z, w)P(z, w) = Q(z, w).
\]
Assume that the total degree of \( U(z, w) \) is equal to \( n \), and the total degree of \( Q(z, w) \) is less than or equal to \( n \), and that \( U(z, w) \) contains just one term of maximal total degree in \( w(z) \) and its shifts. Then we have
\[
m(r, P(z, w)) = o(T(r, w))
\]
for all \( r \notin E \) where \( E \) is a set with \( \text{dens}E = 0 \).

Below, we prove a difference counterpart of the Hu-Yang’s version \([22]\) of Tumura-Clunie theorem in several complex variables as follows. There exists a corresponding result of one variable \([29, \text{Theorem 1}]\) which later was modified by Chen, Huang and Zheng \([6]\) (see also \([5, \text{Theorem 4.3.4}]\)). Set a difference polynomial of several complex variables
\[
G(z, f) = \sum_{\lambda \in J} b_{\lambda}(z) \prod_{j=1}^{\tau_{\lambda}} f(z + q_{\lambda,j})^{\mu_{\lambda,j}},
\]
where \( \max_{\lambda \in J} \sum_{j=1}^{\tau_{\lambda}} \mu_{\lambda,j} = n \), and \( q_{\lambda,j} \neq 0 \) for at least one of the constants \( q_{\lambda,j} \).

Moreover, we assume that the coefficients in (29) are meromorphic functions on \( C^m \) and small with respect to the function \( f \), which is meromorphic on \( C^m \).

**Theorem 3.4.** Let \( f \) be a meromorphic function on \( C^m \) with
\[
\limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0,
\]
such that
\[
N \left( r, \frac{1}{f} \right) + N(r, f) = o(T(r, f)).
\]
Suppose that the difference polynomial (29) of \( f(z) \) and its shifts is of maximal total degree \( n \). If \( G \) also satisfies
\[
\sum_{\lambda \in J_{n-1}} b_{\lambda}(z) \prod_{j=1}^{\tau_{\lambda}} f(z + q_{\lambda,j})^{\mu_{\lambda,j}} \neq 0,
\]
where \( J_{n-1} = \{ \lambda : \sum_{j=1}^{\tau_{\lambda}} \mu_{\lambda,j} = n - 1 \} \), then \( G \) must satisfy
\[
N \left( r, \frac{1}{G} \right) \neq o(T(r, f)).
\]

For the proof of Theorem 3.4 we first need the Tumura-Clunie theorem of several complex variables due to Hu and Yang.

**Lemma 3.1.** \([22, \text{Theorem 2.1}]\) Suppose that \( f \) is meromorphic and not constant in \( C^m \), that
\[
g = f^n + P_{n-1}(f),
\]
where \( P_{n-1}(f) \) is a differential polynomial of degree at most \( n - 1 \) in \( f \), and that
\[
N(r, f) + N \left( r, \frac{1}{g} \right) = o(T(r, f)).
\]
Then
\[ g = \left( f + \frac{\alpha}{n} \right)^n, \]
where \( \alpha \) is a meromorphic function in \( \mathbb{C}^m \), small with respect to \( f \), and determined by the terms of degree \( n - 1 \) in \( P_{n-1}(f) \) and by \( g \).

We also need the second main theorem for meromorphic functions with small function targets on \( \mathbb{C}^m \). It is mentioned in [7, Theorem 2.1] that the conclusion is easily extended from the second main theorem for small function targets due to Yamanoi [38] by the standard process of averaging over the complex lines in \( \mathbb{C}^m \).

**Lemma 3.2.** Let \( f \) be a meromorphic function on \( \mathbb{C}^m \), and \( a_1, \ldots, a_q \) be distinct meromorphic functions "small" with respect to \( f \). Then we have
\[
(q - 2)T(r, f) \leq \sum_{j=1}^{q} N\left(r, \frac{1}{f - a_j}\right) + o(T(r, f))
\]
for all \( r \notin F \), where \( F \) is a set of finite Lebesgue logarithmic measure.

**Proof of Theorem 3.4.** Suppose that the conclusion is not true, that is,
\[ N\left(r, \frac{1}{G}\right) = o(T(r, f)). \]
To prove this theorem, we propose to follow the idea in the proof of [29, Theorem 1]. Since the difference polynomial (29) of \( f(z) \) and its shifts is of maximal total degree \( n \), we get
\[
G(z, f) = \sum_{\lambda \in J} b_\lambda(z) \prod_{j=1}^{n} f(z + q_{\lambda,j})^{\mu_{\lambda,j}}
\]
\[
= \sum_{\lambda \in J} b_\lambda(z) \prod_{j=1}^{n} \left[ \left( \frac{f(z + q_{\lambda,j})}{f(z)} \right)^{\mu_{\lambda,j}} \cdot f(z)^{\mu_{\lambda,j}} \right]
\]
\[
:= \sum_{j=0}^{n} \tilde{b}_j(z)f(z)^j,
\]
where each of the coefficients \( \tilde{b}_j(z) \) \((j = 1, \ldots, n)\) is the sum of finitely many terms of type
\[
b_\lambda(z) \left( \frac{f(z + q_{\lambda,j})}{f(z)} \right)^{\mu_{\lambda,j}}.
\]
It yields
\[
\frac{G(z, f)}{b_n(z)} = f(z) + \sum_{j=0}^{n-1} \frac{\tilde{b}_j(z)}{b_n(z)} f(z)^j.
\]
In terms of the assumption (31), we have \( \sum_{j=0}^{n-1} \frac{\tilde{b}_j(z)}{b_n(z)} f(z)^j \neq 0 \).

Note that all the coefficient functions \( b_\lambda(z) (\lambda \in J) \) are small with respect to \( f \). Then by Theorem 2.1 we get that for all \( j = 1, \ldots, n \),
\[
m(r, \tilde{b}_j) = o(T(r, f))
\]
holds for all \( r \notin E \) with \( \text{dens}E = 0 \). Moreover, by the assumption (30) and Lemma 2.1 we have
\[
N(r, \tilde{b}_j) = o(T(r, f)),
\]
and thus
\[ T(r, b_j) = o(T(r, f)), \quad j \in \{0, 1, \ldots, n\} \]
and
\[ N\left( r, \frac{1}{G(z, f)} \right) = o(T(r, f)). \]
for all \( r \not\in E \). Hence by Lemma 3.1 we may write
\[ \frac{G(z, f)}{b_n(z)} = \left( f(z) + \frac{\alpha(z)}{n} \right)^n, \]
where \( \alpha \neq 0 \) and \( T(r, \alpha) = o(T(r, f)) \). This implies that
\[ (32) \quad N\left( r, \frac{1}{f(z) + \frac{\alpha(z)}{n}} \right) = o(T(r, f)). \]
Together with \( (30) \) and \( (32) \), it follows from Lemma 3.2 that
\[ T(r, f) \leq N\left( r, \frac{1}{f} \right) + N(r, f) + N\left( r, \frac{1}{f(z) + \frac{\alpha(z)}{n}} \right) + o(T(r, f)) = o(T(r, f)) \]
for all \( r \not\in (E \cup F) \), where \( F \) is a set of finite Lebesgue logarithmic measure. Hence we get a contradiction. \( \square \)

3.4. Linear partial difference equations.
In the last subsection, we consider general partial linear difference equations, and obtain the following results. The first one is an extension of [10, Theorem 9.2] and [11, Theorem 6.2.3] from one variable to several variables.

**Theorem 3.5.** Let \( A_0, \ldots, A_n \) be meromorphic functions on \( \mathbb{C}^m \) such that there exists an integer \( k \in \{0, \ldots, n\} \) satisfying
\[ \rho(A_k) > \max\{\rho(A_j) : 0 \leq j \leq n, j \neq k\}, \quad \text{and} \quad \delta_{A_k}(\infty) > 0. \]
If \( f \) is a nontrivial meromorphic solution of linear partial difference equation
\[ (33) \quad A_0(z)f(z + c_n) + \ldots + A_1(z)f(z + c_1) + A_0(z)f(z) = 0 \]
where \( c_1, \ldots, c_n \) are distinct values of \( \mathbb{C}^m \setminus \{0\} \), then we have \( \rho(f) \geq \rho(A_k) + 1 \).

**Proof.** If \( \rho(A_k) = \infty \), then we obviously get from [35] that \( f \) must be of infinite order. Without loss of generality, we assume \( +\infty > \rho(A_k) > 0 \). In this case, it gives that \( A_k \) must be transcendental. We find that there is nothing to do if \( f \) is of infinite order. So, we may assume that \( \rho(f) < +\infty \). From the the equation (33), we get that the solution \( f \) of (33) can not be any nonzero rational function. Now we only need assume that \( f \) is a transcendental meromorphic function with finite order. The equation (33) gives
\[ (34) \quad -A_k = A_n \frac{f(z + c_n)}{f(z + c_k)} + \ldots + A_{k+1} \frac{f(z + c_{k+1})}{f(z + c_k)} + A_{k-1} \frac{f(z + c_{k-1})}{f(z + c_k)} + \ldots + A_0 \frac{f(z)}{f(z + c_k)}. \]
Since \( \delta := \delta_{A_k}(\infty) > 0 \), by the definition we get that
\[ (35) \quad N(r, A_k) < (1 - \frac{\delta}{2})T_{A_k}(r). \]
It yields by Theorem 2.4 that
\[
(36) \quad m(r, \frac{f(z + c_j)}{f(z + c_k)}) = O(r^{\rho(f)-1+\varepsilon})
\]
for any \(\varepsilon(> 0)\), where \(j \in \{0, 1, \ldots, n\} \setminus \{k\}\) and \(c_0 = 0\). Then from (34), (35) and (36), we have
\[
(37) \quad \frac{\delta}{2} T(r, A_k) \leq T(r, A_k) - N(r, A_k)
\]
\[
\leq \sum_{0 \leq j \leq n; j \neq k} m(r, A_j) + \sum_{0 \leq j \leq n; j \neq k} m(r, \frac{f(z + c_j)}{f(z + k)}) + O(1)
\]
\[
\leq \sum_{0 \leq j \leq n; j \neq k} T(r, A_j) + O(r^{\rho(f)-1+\varepsilon}).
\]
Set
\[
\max\{\rho(A_j) : 0 \leq j \leq n, j \neq k\} := \sigma < \rho(A_k) := \rho
\]
such that \(\rho - \sigma > 3\varepsilon > 0\). Then for the above \(\varepsilon > 0\),
\[
T(r, A_j) < r^{\sigma+\varepsilon} < r^{\rho-2\varepsilon}
\]
holds for all \(0 \leq j \leq n, j \neq k\). From the definition of order of \(A_k\), there exists a sequence \(\{r_m\}_{m=1}^{+\infty}\) (with \(r_m \to \infty\) as \(m \to \infty\)) such that
\[
T(r_m, A_k) > r_m^{\rho-\varepsilon}
\]
for sufficiently large \(r_m\). Hence, it follows (37) that
\[
\frac{\delta}{2} T(r, A_k) \leq (n - 1)r_m^{\sigma+\varepsilon} + O(r_m^{\rho(f)-1+\varepsilon})
\]
\[
\leq (n - 1)r_m^{\rho-2\varepsilon} + O(r_m^{\rho(f)-1+\varepsilon}),
\]
and thus
\[
\frac{\delta}{2} r_m^{\rho-\varepsilon} \leq O(r_m^{\rho(f)-1+\varepsilon}).
\]
This implies \(\rho(f) \geq \rho + 1 = \rho(A_k) + 1\). \(\square\)

Obvious, if the dominant coefficient \(A_k\) is holomorphic, then \(\delta_{A_k}(\infty) > 0\). Hence we get immediately the following corollary.

**Corollary 3.2.** Let \(A_0, \ldots, A_n\) be entire functions on \(\mathbb{C}^m\) such that there exists an integer \(k \in \{0, \ldots, n\}\) satisfying
\[
\rho(A_k) > \max\{\rho(A_j) : 0 \leq j \leq n, j \neq k\}.
\]
If \(f\) is a nontrivial entire solution of linear partial difference equation
\[
(38) \quad A_n(z)f(z + c_n) + \ldots + A_1(z)f(z + c_1) + A_0(z)f(z) = 0
\]
where \(c_1, \ldots, c_n\) are distinct values of \(\mathbb{C}^m \setminus \{0\}\), then we have \(\rho(f) \geq \rho(A_k) + 1\).

**Example 3.2.** Let \(c = (c_1, \ldots, c_m) \in \mathbb{C}^m\). Then \(w(z) = e^{z^2+c}\) is an entire solution of the linear partial difference equation
\[
\frac{1}{e^{z^2+c}} w(z_1 + c_1, \ldots, z_m + c_m) - e^{2z+c^2+c} w(z) = 0.
\]
Here $\rho(w) = 2$ and $\rho\left(\frac{1}{w}e^{2z+i}\right) = 0$ and $\rho\left(-e^{2z+c}+e\right) = 1$. This means that the conclusion $\rho(f) \geq \rho(A_k) + 1$ in Theorem 3.2 is sharp.

**Example 3.3.** Let $c_1 = (1,0), c_2 = (0,i) \in \mathbb{C}^2$. Then $w(z) = e^{i+i}z^2$ is an entire solution of linear partial difference equation

$$A_2(z)w(z + c_2) + A_1(z)w(z + c_1) + A_0w(z) = 0,$$

that is

$$A_2(z)w(z_1, z_2 + i) + A_1(z)w(z_1 + 1, z_2) + A_0w(z) = 0,$$

where $A_1(z) \equiv 1$, $A_2(z) = z_1 + z_2$ and $A_0(z) = -(z_1 + z_2)e^{2i+1} + e^{2iz_2-1}$. Here $\rho(w) = 2$ and $\rho(A_1) = \rho(A_2) = 0$ and $\rho(A_0) = 1$. This also means that the conclusion $\rho(f) \geq \rho(A_k) + 1$ in Theorem 3.2 is sharp.

**Example 3.4.** Let $c_1 = (1,i), c_2 = (i, -1) \in \mathbb{C}^2$. Then $w(z) = \frac{e^{i+z^2}}{z_1+z_2}$ is a meromorphic solution of linear partial difference equation

$$A_2(z)w(z + c_2) + A_1(z)w(z + c_1) + A_0w(z) = 0,$$

that is

$$A_2(z)w(z_1 + 1, z_2 + i) + A_1(z)w(z_1 + i, z_2 - 1) + A_0w(z_1, z_2) = 0,$$

where $A_1(z) = \frac{1}{z_1+z_2}, A_2(z) = \frac{z_1+z_2+1}{z_1+z_2}$ and

$$A_0(z) = -\left(\frac{e^{2z_1-3z_2}+1}{z_1+z_2+1+i} + e^{2iz_1+3z_2-3}\right).$$

Here $\rho(w) = 2$ and $\rho(A_1) = \rho(A_2) = 0$ and $\rho(A_0) = 1$. This also means that the conclusion $\rho(f) \geq \rho(A_k) + 1$ in Theorem 3.2 is sharp.

**Example 3.5.** Let $c_1 = (1,i), c_2 = (i, 1) \in \mathbb{C}^2$. Then $w(z) = e^{iz_1+z_2} - 1$ is an entire solution of linear partial difference equation

$$A_2(z)w(z + c_2) + A_1(z)w(z + c_1) + A_0w(z) = 0,$$

that is

$$A_2(z)w(z_1 + 1, z_2 + i) + A_1(z)w(z_1 + i, z_2 + 1) + A_0w(z_1, z_2) = 0,$$

where $A_1(z) = A_2(z) \equiv 1$ and

$$A_0(z) = -1 - \left(\frac{e^{iz_1+z_2+2i} - 1}{e^{iz_1+z_2} - 1}\right).$$

Here $\rho(w) = 1$ and $\rho(A_1) = \rho(A_2) = 0$, $\rho(A_0) = 1$ and $\delta_{A_0}(\infty) = 0$. This implies that the assumption $\delta_{A_n}(\infty) > 0$ in Theorem 3.2 is necessary.

**Question 3.1.** It is obvious that $w(z) = \frac{e^{iz_1+z_2}}{z_1+z_2}$ is a meromorphic solution of the partial difference equation

$$A(z)w(z_1 + 1, z_2 - 1) + B(z)w(z_1, z_2) = 0,$$

where the coefficients $B(z) = -z_1+z_2$ and $A(z) = z_1+2z_2-1$ are polynomials in $\mathbb{C}^2$. Obviously $\rho(w) = 1 = \rho(A)+1 = \rho(B)+1$. Thus we ask what can be said for a general partial difference equation $[\mathfrak{S}]$ with all polynomial coefficients $A_0, \ldots, A_n$?

Since there is the model of the discrete or finite Poisson equation (see [\mathfrak{S}])

$$u_{i,j+1} + u_{i+1,j} + u_{i,j-1} + u_{i-1,j} - 4u_{i,j} = g_{ij},$$

it is interesting to consider the following result on linear partial difference equations.
Theorem 3.6. Let a meromorphic function \( f \) on \( \mathbb{C}^m \) be a solution of linear partial difference equation

\[
A_n(z)f(z + c_n) + \ldots + A_1(z)f(z + c_1) + A_0(z)f(z) = F(z),
\]

where meromorphic coefficients \( A_0, \ldots, A_n, F \neq 0 \) on \( \mathbb{C}^m \) are small functions with respect to \( f \), and \( c_1, \ldots, c_n \) are distinct values of \( \mathbb{C}^m \setminus \{0\} \). If \( \limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0 \), then \( \delta_f(0) = 0 \).

Proof. Assume that the defect of zeros of \( f \) satisfies \( \delta_f(0) > 0 \). Then we have

\[
N(r, \frac{1}{f}) < (1 - \frac{\delta_f(0)}{2})T(r, f).
\]

It follows from the equation (39) that

\[
\frac{1}{f} = \frac{1}{F} \left( A_n f(z + c_n) f^{-1} + A_{n-1} f(z + c_{n-1}) f^{-1} + \ldots + A_1 f(z + c_1) f^{-1} + A_0 \right).
\]

Under the assumption of \( \limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0 \), we get from the first main theorem and Theorem 2.1 that

\[
m(r, \frac{1}{f}) \leq m\left(r, \frac{1}{F}\right) + \sum_{j=1}^{n} m\left(r, \frac{f(z + c_j)}{f}\right) + \sum_{j=0}^{n} m\left(r, A_j\right) + O(1)
\]

\[
\leq T(r, F) + \sum_{j=0}^{n} T(r, A_j) + \sum_{j=1}^{n} m\left(r, \frac{f(z + c_j)}{f}\right) + O(1)
\]

\[
= o(T(r, f)).
\]

for \( r \notin E \) where \( E \) is a set with \( \text{dens} E = 0 \). This gives

\[
T(r, f) + O(1) = T\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right)
\]

\[
\leq N\left(r, \frac{1}{f}\right) + o(T(r, f))
\]

\[
\leq (1 - \frac{\delta_f(0)}{2})T(r, f) + o(T(r, f))
\]

for all \( r \notin E \). Therefore, we get

\[
\delta_f(0)T(r, f) \leq o(T(r, f))
\]

for all \( r \notin E \) where \( E \) is a set with \( \text{dens} E = 0 \). This is a contradiction. \( \square \)

Example 3.6. It is obvious that \( f(z_1, z_2) = z_1 e^{z_2} \) with \( \limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0 \) and \( \delta_f(0) > 0 \) is an entire solution of partial difference equation

\[
\frac{1}{e^{z_2}} f(z_1, z_2 - 1) + \frac{1}{z_1 + 1} f(z_1 + 1, z_2) = e^{z_2} + \frac{z_1}{e}.
\]

Here the coefficient \( \frac{1}{e^{z_2}} \) is a small function with respect to \( f \), however the other coefficients \( \frac{1}{z_1 + 1} \) and \( e^{z_2} + \frac{z_1}{e} \) are not. This means that it is necessary of the assumption that the coefficients are small with respect to the solution in Theorem 3.6.
Example 3.7. Let $c_2 = (1, 0)$ and $c_1 = (0, -2i)$. Then $f(z) = e^{z_1 + z_2 + i}$ with

$$\limsup_{r \to \infty} \frac{\log T(r,f)}{r} = 0$$

and $\delta_f(0) = 1$ is an entire solution of partial difference equation

$$\frac{1}{e^{1+2z_1}} f(z_1 + 1, z_2) - e^{2iz_2} f(z_1, z_2 - 2i) = 0.$$ 

Here the coefficient $\frac{1}{e^{1+2z_1}}$ and $-e^{2iz_2}$ are small functions with respect to the solution $f$. This implies that the coefficient $F$ in Theorem 3.6 cannot be identical to zero.

References

1. T. B. Cao, Difference analogues of the second main theorem for meromorphic functions in several complex variables, Math. Nachr. 287(2014), no. 5-6, 530-545.
2. T. B. Cao and R. J. Korhonen, A new version of the second main theorem for meromorphic mappings intersecting hyperplanes in several complex variables, J. Math. Anal. Appl. 444(2016), no. 2, 1114-1132.
3. T. B. Cao and J. Nie, The second main theorem for homomorphic curves intersecting hypersurfaces with Casorati determinant into complex projective spaces, Ann. Acad. Sci. Fenn. Math. 42(2017), 979-996.
4. T. B. Cao and J. H. Zheng, Second main theorem with tropical hypersurfaces and defect relation, arXiv:1806.04294v1 [math.CV] 12 Jun 2018.
5. Z. X. Chen, Complex Differences and Difference Equations, Mathematics Monograph Series 29, Science Press, Beijing, 2014.
6. Z. X. Chen, Z. B. Huang and X. M. Zheng, On properties of difference polynomials, Acta. Math. Sci. Ser. B. 31(2011), 1282-1294.
7. Z. H. Chen and Q. M. Yan, Uniqueness problem for meromorphic functions sharing small functions, Proc. Amer. Math. Soc. 134(2006), no. 10, 2895-2904.
8. S. S. Cheng, Partial Difference Equations, Taylor & Francis, London, 2003.
9. W. Cherry and Z. Ye, Nevanlinna’s Theory of Value Distribution Second Main Theorem and its error terms, Springer-Verlag, Berlin, 2001.
10. Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane, Ramanujan J. 16(2008), no. 1, 105-129.
11. J. Clunie, On integral and meromorphic functions, J. London Math. Soc. 37(1962), 17-27.
12. H. Davis, Poisson’s partial difference equation, Quat. J. Math. Oxford 6(1955), no. 2, 232-240.
13. R. J. Duffin and J. Rohrer, A convolution product for the solutions of partial difference equations, Duke Math. J. 35(1968), 683-698.
14. P. A. Griffiths, Entire Holomorphic Mappings in One and Several Complex Variables, Anal. of Mathematics Studies 85, Princeton University Press, 1976.
15. R. G. Halburd and R. J. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. Math. 31(2006), no.2, 463-478.
16. R. G. Halburd, R. J. Korhonen, and K. Tohge, Holomorphic curves with shift-invariant hyperplane preimages, Trans. Amer. Math. Soc. 366(2014), no. 8, 4267-4298.
17. W. K. Hayman, On the characteristic of functions meromorphic in the plane and of their integral, Proc. London Math. Soc. 14(1965), 93-128.
18. Y. Z. He and X. Z. Xiao, Algebroid Functions and Ordinary Differential Equations, Science press, Beijing, 1988. (Chinese)
19. A. Hinkkanen, A sharp form of Nevanlinna’s second fundamental theorem, Invent. Math. 108(1992),549-574.
20. P. C. Hu, Malquist type theorem and factorization of meromorphic solutions of partial differential equations, Complex Var. 27(1995), 269-285.
21. P. C. Hu and C. C. Yang, Malquist type theorem and factorization of meromorphic solutions of partial differential equations, Complex Var. 27(1995), 269-285.
22. P. C. Hu and C. C. Yang, The Tumula-Clunie theorem in several complex variables, Bull. Aust. Math. Soc. 90(2014), 444-456.
23. P. C. Hu, P. Li and C. C. Yang, Unicity of Meromorphic Mappings, Kluwer 2003.
24. R. J. Korhonen, A difference Picard theorem for meromorphic functions of several variables, Comput. Methods Funct. Theory 12(2012), no. 1, 343-361.
25. R. J. Korhonen, N. Li, and K. Tohge, *Difference analogue of Cartan’s second main theorem for slowly moving periodic targets*, Ann. Acad. Sci. Fenn. Math. 41(2016), 523-549.

26. R. J. Korhonen, K. Tohge, Y. Zhang and J. H. Zheng, *A lemma on the difference quotients*, arXiv:1806.00210v1 [math.CV] 1 Jun 2018.

27. I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1993.

28. I. Laine and C. C. Yang, *Clunie theorems for difference and q-difference polynomials*, J. London Math. Soc. 76(2007), 556-566.

29. I. Laine and C. C. Yang, *Value distribution of difference polynomials*, Proc. Japan Acad. Ser. A 83(2007), 148-151.

30. B. Q. Li, *On reduction of functional-differential equations*, Complex Var. 31(1996), 311-324.

31. E. Mues and N. Steinmetz, *The theorem of Tumura-Clunie for meromorphic functions*, J. London Math. Soc. 23(1981), no. 2, 113-122.

32. J. Noguchi and J. Winkelmann, *Nevanlinna Theory in Several Complex Variables and Diophantine Approximation*, Springer, Japan, 2014.

33. M. Ru, *Nevanlinna Theory and its Relation to Diophantine Approximation*, Singapore: World Scientific Publishing Co., 2001.

34. S. Tremblay, B. Grammaticos and A. Ramani, *Integrable lattice equations and their growth properties*, Physics Letters A. 278(2001), 319-324.

35. Y. Wang, *Some results for meromorphic functions of several variables*, J. Comp. Anal. Appl. 21(2016), no.5, 967-979.

36. P. M. Wong, H. F. Law, and P. P. W. Wong, *A second main theorem on \( \mathbb{P}^n \) for difference operator*, Sci. China Ser. A 52(2009), no. 12. 2751-2758.

37. C. C. Yang and Z. Ye, *Estimates of the proximate functions of differential polynomials*, Proc. Japan. Acad. Ser. A. Math. Sci. 83(2007), 50-55.

38. K. Yamanoi, *The second main theorem for small functions and related problems*, Acta Math. 192(2004), 225-294.

39. J. H. Zheng and R. Korhonen, *Studies of differences from the point of view of Nevanlinna theory*, arXiv:1806.00212v1 [math.CV] 1 Jun 2018.

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