A spectral sequence for André–Quillen cohomology of algebraic operads

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Operadic cohomology generalizes the existing theories of Harrison cohomology, Chevalley–Eilenberg cohomology and Hochschild cohomology. These are usually non-trivial to compute. We complement the existing computational techniques by producing a spectral sequence that converges to the operadic cohomology of a fixed algebra. Our main technical tool is that of filtrations arising from towers of cofibrations of algebras, which play the same role cell attaching maps and skeletal filtrations do for topological spaces.

As an application, we consider the rational Adams–Hilton construction on topological spaces, where our spectral sequence gives rise to a seemingly new and completely algebraic description of the Serre spectral sequence, which we also show is multiplicative and converges to the Chas–Sullivan loop product. Finally, we consider relative Sullivan–de Rham models of a fibration $p$, where our spectral sequence converges to the rational homotopy groups of the identity component of the space of self-fiber-homotopy equivalences of $p$.

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1 Introduction

Algebraic operads are an effective gadget to study different types of algebras through a common language. In particular, they provide us with tools to study the deformation theory of such algebras. For example, the operads controlling Lie, associative and commutative algebras, —collectively known as the three graces of J.-L. Loday—, along with the operadic formalism,
A spectral sequence for André–Quillen cohomology of algebraic operads

recover for us swiftly the usual (co)homology theories of Chevalley–Eilenberg [14], Hochschild [30] and Harrison [27], and shed light on the relation between these three. We point the reader to [26] for an interesting example of this.

More generally, for an operad \( \mathcal{P} \) and a \( \mathcal{P} \)-algebra \( A \), there is a dg Lie algebra \( \text{Def}_{\mathcal{P}}(\text{id} : A \rightarrow A) \), the deformation complex of \( A \) (also known as the cotangent complex), that codifies all deformation problems over \( A \), in the spirit of [13, 17, 18, 25]. In particular, given a deformation problem, classes in the cohomology groups \( \mathcal{H}^*(A) \) of \( \text{Def}_{\mathcal{P}}(\text{id} : A \rightarrow A) \) — which is now known as the André–Quillen cohomology of \( A \) — allow us, among other things, to determine obstructions to the existence of solutions of such deformation problems. In this paper, we construct a spectral sequence whose input is, in a precise sense, a “cellular decomposition of \( A \)”, that converges to the cohomology groups \( \mathcal{H}^*(A) \).

There is a rich interplay [28, 49] between the homological algebra that arises when studying such deformation complexes and the homotopical algebra of D. Quillen [42] and, in particular, with the study of the homotopy category types of algebras. Some time after the work of Quillen, H.-J. Baues [4] developed the theory of cofibration categories, which plays a similar role in the development of homotopical algebra (and algebraic homotopy) as model categories do. The definition of Baues, which is not self dual (as opposed to that of Quillen) allows us to focus our attention on a class of cofibrations. This asymmetry, although perhaps slightly artificial in our case, appears naturally when doing, for example, ‘proper’ homotopy theory [6, 40, 41].

Our main interest lies in towers of cofibrations of \( \mathcal{P} \)-algebras for a fixed operad \( \mathcal{P} \). For simplicity, we assume \( \mathcal{P} \) is non-dg, although most of what we do can be extended without too much effort for dg operads. To take this point of view, we prove in Theorem 2.2 that the category of (dg) \( \mathcal{P} \)-algebras is a cofibration category in the sense of Baues. This gives a very broad generalization of the pioneering work done in [5], where the authors deal do this for Lie and associative algebras. With this theoretical framework at hand, we focus ourselves on the computation of the André–Quillen cohomology of \( \mathcal{P} \)-algebras.

The relation between André–Quillen cohomology and (co)derivations of types of algebras has appeared many times in the literature, starting with the pioneering work of J. Stasheff and M. Schlessinger in [43, 46]. A definition for algebras over operads then naturally followed [38].

**Definition.** Let \( f : B_0 \longrightarrow B \) be a morphism of \( \mathcal{P} \)-algebras, and \( M \) a \( B \)-module. The André–Quillen cohomology of \( B \) relative to \( B_0 \) with values in \( M \) is the cohomology of the complex \( \text{Def}_{\mathcal{P}}(f : B_0 \longrightarrow B) : = \text{Der}_{Q_0}(Q,A) \) where \( Q_0 \longrightarrow Q \) is a cofibrant replacement of \( f : B_0 \longrightarrow B \), and we write it \( \mathcal{H}^*_{B_0}(B) \).

As we will explain, there is a functor from triples of algebras to short exact sequences — given by the usual Jacobi–Zariski sequence in geometry [31, Section 2.4] — which yields a short
exact sequence as long as the second map is a cofibration. This interplay between derivations and cofibrations is at the very heart of the construction of the spectral sequence appearing in our main statement, Theorem 3.7. There, we show that if one can exhibit $Q_0 \rightarrow Q$ as a colimit of a tower of cofibrations

$$\mathcal{T}: Q_0 \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_s \rightarrow A_{s+1} \longrightarrow \cdots \longrightarrow \lim_{s} Q_s = Q,$$

then one obtains a corresponding spectral sequence in terms of the André–Quillen cohomology of the skeleta of $\mathcal{T}$, where we allow coefficients to take values in a $B$-module, and not necessarily $B$ itself.

**Theorem A.** Let $Q$ be the colimit of the tower $\mathcal{T}$ of cofibrations of $\mathcal{P}$-algebras. There is a functorial right half-plane spectral sequence with first page

$$E_1^{s,t} = H^{s+t}(\text{Der}_{Q_s}(Q_{s+1}, -)) \Rightarrow H^{s+t}(\text{Der}_{Q_0}(Q, -))$$

that is conditionally convergent in the sense of Boardman.

Since $Q_0 \rightarrow Q$ is a cofibration of $\mathcal{P}$-algebras that is a cofibrant replacement of the morphism of $\mathcal{P}$-algebras $B_0 \rightarrow B$, the target of this spectral sequence is $\mathcal{H}^e_{B_0}(B, -)$. We record this immediate reinterpretation of the theorem in Corollary 3.8; this is the situation we are interested in general.

Having done this, we study how our construction specializes when considering the algebraic models in rational homotopy of J. F. Adams and P.J. Hilton [1], and D. Sullivan [47]. One of the main features of these algebraic models of rational homotopy types is that they are built through towers of cofibrations in terms of a cellular decomposition of choice. Our first result in this direction is Theorem 4.1, which the reader is invited to compare with that of S. Shamir [44] and to the eponymous spectral sequence of J.-P. Serre. We expect this spectral sequence, in which all coefficients are rational, to be isomorphic to the one obtained in [16] by R. L. Cohen, J. D. S. Jones and J. Yan, in which case it gives us a completely algebraic description of a multiplicative spectral sequence converging to the Chas–Sullivan loop product in string topology for simply connected oriented closed manifolds.

**Theorem B.** Let $X$ be a CW complex with exactly one 0-cell, no 1-cells and all whose attaching maps are based with respect to the only 0-cell. There is a first quadrant spectral sequence with

$$E_2^{s,t} = \text{hom}(H_s(X), H_t(\Omega X)) \Rightarrow H_{s+t}(LX),$$
which is conditionally convergent in the sense of Boardman. Moreover, this is a spectral sequence of algebras whose product in the $E_2$-page is given by the convolution product of $	ext{hom}(H_*(X), H_*(\Omega X))$. Whenever $X$ is a simply connected oriented closed manifold, this product converges to the Chas–Sullivan loop product.

We now turn our attention to Sullivan models. Let $p : E \to B$ be a fibration between simply-connected CW-complexes, with $E$ having finitely many cells. The grouplike topological monoid $\text{Aut}(p)$ of self-fibre-homotopy equivalences of $p$ is an important object in algebraic topology; we point out as an example the classification theorem of J. Stasheff [45], [34, Chapter 9], and the more recent [9]). In [21], the authors show that the rational homotopy type of the connected component $\text{Aut}_1(p)$ of the identity of $\text{Aut}(p)$ is determined completely in terms of the Harrison cohomology of the Sullivan model of $p$. We also point the reader to [8], where a general relation between the Harrison cohomology of Sullivan–de Rham algebras and homotopy types of function spaces is given, and to [12], where the authors show the rational homotopy groups of function spaces are also determined completely in terms of the Harrison cohomology of a Sullivan model. Building on top of the main result of [21], we obtain the following theorem (Theorem 4.7).

**Theorem C.** Let $F \hookrightarrow E \twoheadrightarrow B$ be a fibration of 1-connected CW-complexes, with $E$ finite. There is a spectral sequence with

$$E_2^{s,t} = \text{hom}(\pi_s(F), H^t(E)) \Rightarrow \pi_{s-t}(\text{Aut}_1(p)).$$

Again, all coefficients above are rational. In this case, it should be possible to show that the spectral sequence carries a multiplicative structure inherited from the convolution product on the $E_2$-page making it a spectral sequences of Lie algebras by obtaining a result in the lines of Theorem 3 in [12], by replacing the space of maps $\mathcal{F}(X,Y)$ by $\text{Aut}_1(p)$.

We remark that A. Berglund and B. Saleh [7, Proposition 4.4] have independently considered a spectral sequence of the same shape as ours in order to determine a Quillen (that is, dg Lie) model of the classifying space of the grouplike monoid of homotopy automorphisms of a space that fix a given subspace. This gives a third direction in which our methods can be pushed towards, and the interested reader can consult that paper for further details.

The paper is organized as follows. In Section 2 we recall the elements of operad theory and show that the category of $\mathcal{P}$-algebras carries the structure of a cofibration category in the sense of Baues [4]. The main technical result we prove there is the fact the category of $\mathcal{P}$-algebras satisfies the pushout axiom of Baues. In Section 3 we develop the main technical tool that we introduce in this paper: the spectral sequence of Theorem 3.7. Having done this, we describe the different pieces of the spectral sequence in that section. This includes the description of
its differentials the maps involved in the exact couple that gives rise to it. We conclude this section by showing that our spectral sequence degenerates in some natural situations. Finally, in Section 4, we give the mentioned applications to rational homotopy theory.

Throughout, we write $\mathcal{P}$ for a symmetric operad, which we assume is reduced, and identify $\mathcal{P}$ with the corresponding Schur endofunctor $V \mapsto \mathcal{P}(V)$. All $\mathcal{P}$-algebras are connected and dg unless stated otherwise. We also make use of the following functors: $\text{Sing}_*$ is the singular chains functor, $\text{Sing}^*$ is the singular cochains functor, $A$ is the Adams–Hilton construction, $\text{Cell}$ is the cellular chain complex of a CW-complex, $\Omega$ is the based loop space, and $L$ is the free loop space. The dual of a graded vector space $V$ is $V^\vee$. For a topological space $X$, we write $\pi_*(X)^\vee = (\pi_*(X) \otimes \mathbb{Q})^\vee$, omitting the rational coefficients. All tensor products and hom-sets are understood to be taken over a fixed base ring $\mathbb{k}$ of characteristic zero, which in Section 3 will be $\mathbb{Q}$. Similarly, in that section, all (co)homology groups are assumed to have rational coefficients.

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2 $\mathcal{P}$-Alg as a cofibration category

This section starts with a recollection of the necessary background on operad theory, mainly to set up notation. An excellent reference is [33]. We then recall the basics of cofibration categories as introduced by Baues [4] and prove that the category of $\mathcal{P}$-algebras satisfies the pushout axiom. Therefore, the category of $\mathcal{P}$-algebras admits the structure of a cofibration category where the weak equivalences are the quasi-isomorphisms and the cofibrations are maps obtained by freely adjoining variables.
2.1 \(\Sigma\)-modules and operads

Write \(\Sigma\text{dgMod}\) for the category of dg \(\Sigma\)-modules. Recall there is a monoidal product, the composition product

\[-\circ- : \Sigma\text{dgMod} \times \Sigma\text{dgMod} \to \Sigma\text{dgMod}\]

with unit 1 the module concentrated in arity 1 where its value is \(k\). It is useful to think of an element in \(X \circ Y\) as a corolla whose only vertex is labelled by \(x \in X\) of some arity \(k \in \mathbb{N}\) and whose leaves are labelled in order by \(y_1, \ldots, y_k \in Y\). This product restricts to the subcategory of non-graded \(\Sigma\)-modules, and a symmetric operad \(P\) is a monoid in this category, whose product we usually denote by

\[\gamma : P \circ P \to P.\]

We can of course talk about graded or differential graded symmetric operads, but symmetric operads will suffice for our purposes. An operad is reduced if \(P(0) = 0\), and we will only consider these kind of operads in what follows.

We recall that there is a unital monad \(T\) in \(\Sigma\text{dgMod}\) and that symmetric operads are precisely the \(T\)-algebras. It follows in particular that if \(X\) is any dg \(\Sigma\)-module, \(T_X\) is the free operad on \(X\). Concretely, for \(n \in \mathbb{N}\), the space \(T_X(n)\) consists of rooted trees \(t\) so that for each vertex of \(t\) is decorated by an element in \(X(d)\), where \(d\) is the number of inputs of it. The product

\[\mu : T \circ T \to T\]

is obtained by substitution of trees into vertices, and the unit

\[\eta : 1 \to T\]

sends an element of \(X\) to the corresponding corolla. We refer the reader to [33] for details. From this we can present operads as quotients of free operads by ideals of relations. Classical examples include the operads As, Lie, Comm, PreLie, Poiss, Ger and Grav, whose presentations can be found in the literature.

2.2 Algebras over operads

Recall we have fixed a reduced symmetric operad \(P\) and write \(P\)-Alg for the category of dg \(P\)-algebras. In particular, every \(P\)-algebra \(A\) includes the data of a square zero derivation \(d\) of
A, that is, is an endomorphism $d : A \to A$ so that $d^2 = 0$ and such that for each operation $\mu$ of $O$ of arity $n \in \mathbb{N}$, $d\mu = \mu d^{[n]}$, where $d^{[n]}$ is the induced differential on $A^{\otimes n}$. For example, if $\mu$ is binary (so that $n = 2$), the requirement is that

$$d\mu = \mu(d \otimes 1 + 1 \otimes d).$$

Note that Koszul signs will appear in the previous formula when evaluating $1 \otimes d$ on elements of $A$. Alternatively, we require that the structure map

$$\gamma^A : \mathcal{P} \to \text{End}_A$$

is one of complexes, where we view $\mathcal{P}$ as a dg operad concentrated in degree zero. Adjoint to this is a map

$$\gamma_A : \mathcal{P}(A) \to A$$

which we call the structure map of $A$. We will drop the prefix “dg” and speak simply of $\mathcal{P}$-algebras. In all of what follows, with the exception of the “tilde” construction of Proposition 3.6, will only consider the category of $\mathcal{P}$-algebras $A$ that are connected and non-negatively homologically graded. Precisely, this means that $A_0 = k$ is the base ring and $A$ vanishes in negative homological degrees.

### 2.3 Modules over algebras

Fix a dg $\mathcal{P}$-algebra $A$ as before. An operadic $A$-module is a dg vector space $M$ along with a unital action $\gamma_M : \mathcal{P} \circ (A, M) \to M$ so that

$$\gamma_M(1 \circ (\gamma_A, \gamma_M)) = \gamma_M(\gamma \circ (1, 1)).$$

It is useful to note that if $\mathcal{P} = \mathcal{A}s$ and if $A$ is an $\mathcal{P}$-algebra or, what is the same, an associative algebra, then an operadic $A$-module is the same as an $A$-bimodule and not a left (or right) $A$-module. Similarly, the operadic modules for commutative algebras are the symmetric bimodules, and the operadic modules for Lie algebras coincide with the usual notion of Lie algebra representation.
2.4 Cofibration categories

In this section we show that one may endow the category of $\mathcal{P}$-algebras with the structure of a cofibration category in the sense of Baues, see Theorem 2.2. The explicit description of the cofibrations given is useful to streamline the presentation of our results. One reason to take this point of view is that the cofibrations are simpler than those of [28]; they correspond to his ‘standard cofibrations’.

Cofibration categories, introduced by Baues in [4], provide a framework for doing axiomatic homotopy theory in the spirit of Quillen [42] but under weaker assumptions; in particular, as its name indicates, the cofibration categories of Baues are defined by choosing a class of cofibrations and a class of weak-equivalences, but their definition does not require a class of fibrations.

**Definition 2.1** A cofibration category is a category endowed with two classes of morphisms, the cofibrations and weak equivalences, so that the following four axioms are satisfied:

(C1) **Composition axiom:** The isomorphisms are weak equivalences and cofibrations, and weak equivalences satisfy the two out of three property. Moreover, cofibrations are closed under composition.

(C2) **Pushout axiom:** If $i : B \rightarrowtail A$ is a cofibration and $f : B \twoheadrightarrow Y$ is any map, the pushout of $A \leftarrowtail B \twoheadrightarrow Y$ exists, and $\bar{i}$ is a cofibration. Moreover, (i) if $f$ is a weak equivalence so is $\bar{f}$, and (ii) if $i$ is a weak equivalence, so is $\bar{i}$.

\[ B \xrightarrow{i} A \]
\[ f \downarrow \quad \bar{f} \]
\[ Y \xrightarrow{\bar{i}} P \]

(C3) **Factorization axiom:** Every arrow can be factored into a cofibration followed by a weak equivalence.

(C4) **Axiom on fibrant models:** For every object $X$ there is an arrow $X \leftarrowtail RX$ where $RX$ is fibrant.

An object $R$ is fibrant if every trivial cofibration $R \leftarrowtail Q$ splits. All $\mathcal{P}$-algebras are fibrant (but we will not need this) and the cofibrant objects are the $\mathcal{P}$-algebras which are free as graded algebras. For $A, B \in \mathcal{P}\text{-Alg}$, we write $A \star B$ their coproduct as $\mathcal{P}$-algebras. The work done in [28] shows that all axioms except possibly the pushout axiom hold for this model structure. We check it in Section 2.5, and hence deduce the following result. We remark that this was done, without mention to cofibration categories, by Baues–Lemaire [5] for associative and Lie algebras.
Theorem 2.2 The category $\mathcal{P}$-Alg of $\mathcal{P}$-algebras carries a structure of cofibration category. The cofibrations are those maps $B \to A$ for which there exists a submodule $X$ of $A$ such that the induced map $f \ast 1 : B \ast TX \to A$ is an isomorphism, and the weak-equivalences are the quasi-isomorphisms.

Let us introduce some useful definitions:

- A cofibration $B \hookrightarrow A$ is *elementary of height* $d$ if $A$ is obtained by adjoining finitely many generators in degree $d + 1$.

- An morphism $f : B \to A$ is is *n-connected* if it induces isomorphisms on homology in degree $i < n$ and a surjection in degree $n$.

- An algebra $A$ is *n-connected* if the unique map $\ast \to A$ is $n$-connected, and it is *n-truncated* if $H_s(A) = 0$ for $s > n$.

The elementary cofibrations correspond to the geometric situation where we add cells to a space. Note that we are not imposing any condition on $B$, which may very well have generators in arbitrary degrees.

**Definition 2.3** If $A \to \ast$ is an augmented $\mathcal{P}$-algebra, we write $\overline{A}$ for the kernel of this augmentation, and $\text{Ind}_\mathcal{P} A$ for the cokernel of the structure map of the non-unital algebra $\overline{A}$, which we call the *space of indecomposables* of $A$. In this way we obtain a functor $\text{Ind}_\mathcal{P} : \mathcal{P}$-Alg$_\ast \to \text{Ch}_k$ from augmented $\mathcal{P}$-algebras to chain complexes, which we call the *functor of indecomposables*.

It is clear, for example, that if $A = (\mathcal{P}(V), d)$ is quasi-free then $\text{Ind}_\mathcal{P} A = (V, d_{(1)})$ where $d_{(1)}$ is the linear part of the differential $d$ of $A$, and we will use this later. In this way, the (derived) functor $\text{Ind}_\mathcal{P} A$ captures the “first order information” of a cofibrant resolution of the $\mathcal{P}$-algebra $A$.

**Definition 2.4** We define the *Quillen homology of a $\mathcal{P}$-algebra* $A$, which we write $H_\ast(\mathcal{P}, A)$, as the left derived functor of $\text{Ind}_\mathcal{P}$. That is, if $B \xrightarrow{\sim} A$ is a weak equivalence and $B$ is a cofibrant $\mathcal{P}$-algebra, the homology of $\text{Ind}_\mathcal{P} B$ is by definition $H_\ast(\mathcal{P}, A)$.

To illustrate, if $A$ is an augmented associative algebra, this is just $\text{Tor}_\ast^{A}(k, k)$. Indeed, we can take $\Omega BA$ as a model of $A$, and then $H_\ast(\mathcal{P}, A)$ is the homology of the shifted reduced bar complex $\overline{BA}$ of $A$, which is precisely $\text{Tor}_\ast^{A}(k, k)$.

### 2.5 The pushout axiom

This is a technical subsection which the uninterested reader may very well skip, since it is not going to play a role in obtaining our main result. We have decided to include it for completeness,
to obtain Theorem 2.2. For details on the bar construction on operads we refer the reader to the book [33].

Write $B(P)$ for the dg cooperad whose underlying cooperad is free on the suspension $sP$ of $P$. Its differential collapses an edge of a tree and composes the elements of $P$ accordingly. We call $B(P)$ the bar construction on $P$. There is a twisting cochain $\tau : B(P) \to P$ and the twisted complex $E(P) = P \circ \tau B(P)$ is acyclic. These objects define functors $E_P$ and $B_P$ on the category of $P$-algebras to complexes and $B(P)$-coalgebras, respectively. If, moreover, $X$ is a quasi-free $P$-algebra, the quotient map

$$B_P(X) \to \text{Ind}_P(X)$$

is a surjective quasi-isomorphism. We begin with a lemma. Compare with [5, Proposition 1.5].

**Lemma 2.5** Let $f : X \to Y$ be a map between quasi-free $P$-algebras generated in non-negative degrees. Then $f$ is a quasi-isomorphism if, and only if, the induced map on indecomposables $\text{Ind}_P f$ is a quasi-isomorphism.

One can think of this result as a “Whitehead Lemma” in the lines of [11, 4.2]. Indeed, in the case of Lie algebras (in the category the reader may prefer), we have that $\text{Ind}_P(L) = L/[L,L]$ is the Abelianization functor in that paper.

**Proof.** The complexes $\text{Ind}_P(X)$ and $\text{Ind}_P(Y)$ compute the homology of the (non-unital) bar construction $B_P(X)$ and $B_P(Y)$ and the maps $B_P(X) \to \text{Ind}_P(X)$ are surjective quasi-isomorphisms. In this way, we can consider a commutative diagram of ‘fibration sequences’:

$$
\begin{array}{cccccc}
0 & \to & X & \to & E_P(X) & \to & B_P(X) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & Y & \to & E_P(Y) & \to & B_P(Y) & \to & 0.
\end{array}
$$

The middle map is a quasi-isomorphism of contractible complexes, so the claim follows by Moore’s comparison theorem: the map $B_P(X) \to B_P(Y)$ is a quasi-isomorphism if and only if the map on indecomposables $\text{Ind}_P(X) \to \text{Ind}_P(Y)$ is one, if and only if $X \to Y$ is one. ▶

**Proof of the Pushout Axiom.** If $f : B \to Y$ is a quasi-isomorphism then, by the lemma, so is $\text{Ind}_P f \oplus 1$ and, since the functor $\text{Ind}_P$ of indecomposables commutes with coproducts, we have that

$$\text{Ind}_P \bar{f} = \text{Ind}_P f \oplus 1 : QB \oplus X \to \text{Ind}_P Y \oplus X,$$
and hence $\overline{f}$ is a quasi-isomorphism. Assume now that $i$ is a quasi-isomorphism. Since the homotopy cofibres of $i$ and $\overline{i}$ are both isomorphic to $P(X)$ with the induced differential, we see that $\overline{i}$ is a quasi-isomorphism if $i$ is one.

3 A spectral sequence of derivations

This section is the core of the paper. We pave the way for constructing the spectral sequence of Theorem 3.7, the main result. We start by recalling the relevant facts on derivation complexes in Subsection 3.1. We refresh André–Quillen cohomology in Subsection 3.2. The main result and some corollaries are proven in Subsection 3.3. We take a closer look to the items of the spectral sequence in Subsection 3.4. We finish in Subsection 3.5 by studying some natural situations in which the spectral sequence degenerates.

3.1 The complex of derivations

In this section, we collect some general facts on complexes of derivations, which we then use to construct the spectral sequence of Theorem 3.7. Some applications of the spectral sequence will require, for convergence reasons, complexes concentrated in non-negative degrees. However, the results of this section do not need this constraint.

**Definition 3.1** Let $B \rightarrow A$ be a map of $P$-algebras, and let $M$ be an operadic $A$-module. We write $\text{Der}_B(A,M)$ for the cohomologically graded complex of derivations $A \rightarrow M$ that vanish on $B$. We write elements in $\text{Der}_B(A,M)$ by $F : A | B \rightarrow M$. The differential of such an $F$ is $\partial F = d_MF - (-1)^{|F|}Fd_A$.

In case $u : A \rightarrow U$ is a map of $P$-algebras, which in particular makes $U$ into an $A$-bimodule, we will write $\text{Der}_B(A,U)$ without explicit mention to the map $u$, which will be understood from context. In case $U = A$ and $u$ is the identity, we write this complex simply by $\text{Der}_B(A)$. Observe that in this case, the differential $\partial F$ is given by the bracket $[d_A,F]$, and that $\text{Der}_B(A)$ is a dg Lie algebra under the Lie bracket of derivations. Moreover, for each operadic $A$-module $M$, the complex $\text{Der}_B(A,M)$ is a left Lie module over $\text{Der}_B(A)$.

The following lemma says that this functor is well behaved when its arguments are cofibrations of algebras. The resulting exact sequence is called the *Jacobi–Zariski sequence* of the triple $B \rightarrow A \rightarrow A'$. It is straightforward to see that it is functorial on maps of algebras $A' \rightarrow U$. 


Lemma 3.2 Let $B \rightarrow A \rightarrow A'$ be a sequence of morphisms of $\mathcal{P}$-algebras, and let $u : A' \rightarrow U$ be a map of $\mathcal{P}$-algebras. Then, there is a left exact sequence

$$0 \longrightarrow \text{Der}_A(A', U) \longrightarrow \text{Der}_B(A', U) \longrightarrow \text{Der}_B(A, U) \longrightarrow 0$$

and it is exact if $A \rightarrow A'$ is a cofibration.

Proof. The map $A \rightarrow A'$ induces a map $\text{Der}_B(A', U) \rightarrow \text{Der}_B(A, U)$ by precomposition $F \mapsto F \circ i$ whose kernel is tautologically isomorphic to $\text{Der}_A(A, U)$. Hence, it suffices to show the first map is surjective in case $A \rightarrow A'$ is a cofibration. To do this, we observe that we can always lift a derivation $A \rightarrow U$ that vanishes on $B$ to one in $A'$ by declaring it to vanish on the module of generators of $A \rightarrow A'$, and we will always assume, for consistency, that this is our choice of lift. This finishes the proof of the lemma. ◀

3.2 André–Quillen cohomology

Let us now introduce the cohomology theory that will concern us and serves as the unifying concept to present our results. We recommend the article [38] for a thorough account on André–Quillen cohomology for operads.

For a morphism of $\mathcal{P}$-algebras, $B \rightarrow A$ and an operadic $A$-module $M$, we define the André–Quillen cohomology of $A$ relative to $B$ with values in $M$ as follows. Let $P \rightarrow Q$ be a cofibrant replacement of the map $f : B \rightarrow A$: by this we mean that this map is a cofibration between cofibrant $\mathcal{P}$-algebras that fits into a commutative diagram

$$
\begin{array}{ccc}
P & \longrightarrow & Q \\
\downarrow & & \downarrow \\
B & \longrightarrow & A
\end{array}
$$

where the vertical maps are quasi-isomorphisms. The second vertical map makes $M$ into an operadic $Q$-module, and hence we can consider the complex $\text{Der}_P(Q, M)$ of derivations $Q \rightarrow M$ that vanish on $P$.

Definition 3.3 The cohomology of this complex is, by definition, the André–Quillen cohomology of $A$ relative to $B$ with values in $M$, and we write it $\mathcal{H}_B^*(A, M)$. In case we take $A = M$, we will speak about the André–Quillen cohomology of the map $f : B \rightarrow A$, and write it $\mathcal{H}^*(f)$. 
Note that the Jacobi–Zariski sequence shows that for a triple \( B \longrightarrow A \longrightarrow A' \) of algebras, we have a corresponding long exact sequence
\[
\mathcal{H}^*_B(A', -) \longrightarrow \mathcal{H}^*_A(A', -) \longrightarrow \mathcal{H}^*_A(B, -) \longrightarrow \mathcal{H}^*_B(A', -)[1].
\]

**Remark 3.4** Consider the situation when \( A \) is an associative algebra (without differential), which we think of as concentrated in degree zero, and take \( Q \overset{\sim}{\longrightarrow} A \) a cofibrant replacement of \( A \) in \( \text{Alg} \), the cofibration category of dga algebras. Then we have identifications for \( n \in \mathbb{Z} \),
\[
\mathcal{H}^n(A, A) = \begin{cases} 
\text{HH}^{n+1}(A) & \text{for } n \geq 1 \\
\text{Der}(A) & \text{for } n = 0.
\end{cases}
\]
where \( \text{HH}^*(A) \) are the Hochschild cohomology groups of \( A \).

The way to remedy this difference between André–Quillen cohomology of associative algebras and their classical Hochschild cohomology is as follows.

**Lemma 3.5** Let \( \text{Ad}_Q : Q \longrightarrow \text{Der}(Q) \) be the adjoint map of \( Q \). We have an isomorphism of (shifted) Lie algebras \( H^*(\text{cone}(\text{Ad}_Q)) \simeq \text{HH}^*(A) \).

Observe that \( \text{cone}(\text{Ad}_Q) \) is a (shifted) Lie algebra obtained as the semidirect product of \( Q \) and \( s^{-1} \text{Der}(Q) \), where the bracket is given, for \( x + \varphi \) and \( y + \psi \in \text{cone}(\text{Ad}_Q) \), by the formula \([x + \varphi, y + \psi] = [x, y] + \varphi(y) - \psi(x) - [\varphi, \psi]\).

**Proof.** Since \( H_*(Q) = A \), the long exact sequence of the cone complex gives us isomorphisms
\[
H^n(\text{cone } \text{Ad}_Q) = H^{n-1}(\text{Der } Q)
\]
for \( n \geq 2 \), giving the claim of the theorem in that range. The remainder of the long exact sequence is a four term exact sequence
\[
0 \longrightarrow H^0(\text{cone } \text{Ad}_Q) \longrightarrow H^0(Q) \longrightarrow H^0(\text{Der } (Q)) \longrightarrow H^1(\text{cone } \text{Ad}_Q) \longrightarrow 0.
\]
Now observe that \( H^0(\text{Ad}_Q) \) is just the adjoint map \( A \longrightarrow \text{Der}(A) \) of \( A \). It follows the above exact sequence identifies \( H^0(\text{cone}(\text{Ad}_Q)) \) with the kernel of the map \( A \longrightarrow \text{Der}(A) \) and \( H^1(\text{cone } \text{Ad}_Q) \) with its cokernel, which is what we wanted.

It will be useful for us to have an alternative description of this cone. It is shown in \([22]\) that to compute \( \text{HH}^*(A) \)—even in the case we allow \( A \) to be dg— one may instead consider a complex
of derivations obtained from the algebra \(Q\) as follows. If \(Q = (\mathcal{P}(V), d)\), consider the algebra

\[
\tilde{Q} = (\mathcal{P}(V \oplus \varepsilon), \tilde{d}) \quad \text{where} \quad |\varepsilon| = -1, \quad \tilde{d}\varepsilon = \varepsilon^2, \quad \tilde{d}v = dv + [\varepsilon, v]
\]

Note that \(\varepsilon\) is a Maurer–Cartan element and that \(F = (\mathcal{P}(\varepsilon), \tilde{d})\) is acyclic —that is, \(H_*(F) = \mathbb{k}\)— since for each \(n \in \mathbb{N}\) we have that \(d(\varepsilon^{2n-1}) = \varepsilon^{2n}\).

**Proposition 3.6** For any quasi-free \(\mathcal{P}\)-algebra \(Q = (\mathcal{P}(V), d)\) the Lie algebra \(\text{Der}(\tilde{Q})\) is quasi-isomorphic to Lie algebra given by the cone of the adjoint map \(\text{Ad}_Q\).

**Proof.** There is a quotient map \(\pi: \tilde{Q} \to Q\) with fibre \(F\) and a cofibration \(F \to \tilde{Q}\), which gives us a short exact sequence

\[
0 \to \text{Der}_F(\tilde{Q}) \to \text{Der}(\tilde{Q}) \to \text{Der}(F, \tilde{Q}) \to 0
\]

On the other hand, we have an identification and a quasi-isomorphism

\[
\text{Der}(F, \tilde{Q}) = \tilde{Q} \quad \text{and} \quad \text{Der}_F(\tilde{Q}) \to \text{Der}(Q).
\]

coming from the fact that \(F\) is quasi-free and contractible. The takeaway is a commutative diagram whose rows are exact and whose columns consist of quasi-isomorphisms:

\[
\begin{array}{cccccc}
0 & \to & \text{Der}_F(\tilde{Q}) & \to & \text{Der}(\tilde{Q}) & \to & \text{Der}(F, \tilde{Q}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow^{\text{ev}_\varepsilon} & & \downarrow & & \\
0 & \to & \text{Der}(Q) & \to & \text{Der}(\tilde{Q}) & \to & \tilde{Q} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow^{\pi} & & \downarrow & & \\
0 & \to & \text{Der}(Q) & \to & \text{cone}(\text{Ad}_Q) & \to & Q & \to & 0 \\
\end{array}
\]

This shows that the Jacobi–Zariski short exact sequence above is quasi-isomorphic to the short exact sequence for the cone of \(Q\), and hence that \(\text{Der}(\tilde{Q})\) is quasi-isomorphic to \(\text{cone}(\text{Ad}_Q)\). 

### 3.3 The spectral sequence

In this section, we present our main technical tool, a spectral sequence converging to the cohomology of the derivations of a \(\mathcal{P}\)-algebra that is the colimit of a tower of cofibrations. The broad generality in which this spectral sequence appears means we can only guarantee it to be conditionally convergent in the sense of Boardman [10]. We will study the differentials in this spectral sequence, and give natural conditions that ensure its strong convergence in Section 3.5.
Theorem 3.7 Let $A$ be the colimit of a tower of cofibrations of $\mathcal{P}$-algebras

$$A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_s \rightarrow A_{s+1} \rightarrow \cdots \rightarrow \varinjlim A_s = A.$$ 

There is a functorial right half-plane spectral sequence with first page

$$E_1^{s,t} = H^{s+t}(\text{Der}_{A_s}(A_{s+1},-)) \Rightarrow H^{s+t}(\text{Der}_{A_0}(A,-))$$

that is conditionally convergent in the sense of Boardman.

A consequence of this result is the following corollary which involves André–Quillen cohomology. In fact, we will be mostly concerned in the situation described in the statement of this corollary:

Corollary 3.8 Let $f : A_0 \rightarrow A$ be a cofibration of $\mathcal{P}$-algebras that is a cofibrant replacement of a map of $\mathcal{P}$-algebras $B_0 \rightarrow B$. If $f$ is the colimit of a tower of cofibrations as above, there is a functorial right half-plane spectral sequence with first page

$$E_1^{s,t} = H^{s+t}(\text{Der}_{A_s}(A_{s+1},-)) \Rightarrow H^{s+t}(\text{Der}_{B_0}(B,-))$$

that is conditionally convergent in the sense of Boardman. ▲

We now proceed to prove Theorem 3.7.

Proof. Let $M$ be an operadic $A$-module, and for each $s \geq 0$, consider the exact sequence of derivations induced by the triple $A_0 \rightarrow A_s \rightarrow A$, as in Lemma 3.2:

$$0 \rightarrow \text{Der}_{A_s}(A_{s+1},M) \rightarrow \text{Der}_{A_0}(A_{s+1},U) \rightarrow \text{Der}_{A_0}(A_s,M) \rightarrow 0.$$ 

Form the exact couple associated to the long exact sequence in cohomology, explicitly given for every $s,t \in \mathbb{N}$ by

$$D^{s,t} = H^{s+t}(\text{Der}_{A_0}(A_{s+1},M)), \quad \text{and} \quad E^{s,t} = H^{s+t}(\text{Der}_{A_s}(A_{s+1},M)).$$

The maps $i,j,k$ forming the exact couple $(D,E,i,j,k)$ have bidegrees $(-1,1), (1,0)$ and $(0,0)$, respectively and are described after the proof. In the bidegree pair $(s,t)$, we are denoting by $s$ the filtration degree and by $t$ the complementary degree. The differential $d_1 = jk$ produced on $E = E_1$ of bidegree $(1,0)$. The general procedure of forming iterated derived couples produces the spectral sequence, finding that the $r$th differential $d_r$ has bidegree $(r,1-r)$. Thus, differentials entering into a fixed module $E^{s,t}_r$ originate at points outside the right-half-plane for all $r > t$, so
eventually vanish. This guarantees there is conditional convergence in the sense of Boardman; see [10, Theorem 7.3].

The spectral sequence is functorial, for if $M \rightarrow N$ is map of $A$-modules, then the morphism induced between the corresponding short exact sequences of derivations induces a morphism of the corresponding exact couples, producing the desired morphism of spectral sequences.

Another corollary of our main result is a spectral sequence that in many situations degenerates and exhibits André–Quillen cohomology as a twisted algebra defined in terms of Quillen homology and ordinary homology of $P$-algebras, in the same spirit as the usual way of computing Hochschild cohomology of an associative algebra $A$ through a twisted complex $\hom_\tau (C,A)$ where $C$ is an $A_\infty$-coalgebra model of $BA$ with associative twisting cochain $\tau : C \rightarrow A$.

To state it, let us take $A$ to be a cofibrant $P$-algebra of the form $P(V)$ that is a cofibrant replacement of the $P$-algebra $B$, and write $A$ as the colimit of the tower of cofibrations of the form

$$k \rightarrow P(V_{\leq 0}) \rightarrow P(V_{\leq 1}) \rightarrow \cdots \rightarrow P(V_{\leq s}) \rightarrow \cdots,$$

where for each $s \in \mathbb{N}$, we write $A_{s+1} = P(V_{\leq s})$ for the subalgebra of $A$ generated by elements of degree at most $s$.

**Corollary 3.9** Assume that $B$ is $0$-connected. There is a functorial right half-plane spectral sequence with second page

$$E_2^{s,t} = \hom(H_s(P,B),H_t(\cdot)) \Rightarrow \mathcal{H}^{s+t}(B,\cdot)$$

that is conditionally convergent in the sense of Boardman.

We remark that if both $B$ and $M$ have zero differential, the spectral sequence collapses and yields an isomorphism

$$H^*(\hom(H_*(P,B),M)) \rightarrow \mathcal{H}^*(B,M)$$

that exhibits André–Quillen cohomology as the cohomology of a twisted complex. This follows immediately from Lemma 3.12 which shows that the $E_1$-page is concentrated in one row. In this sense, the tower of cofibrations above is rather crude, and it is perhaps not unreasonable to consider other more refined towers to expect a more meaningful spectral sequence.

**Proof of Corollary 3.9.** Let us consider the cofibration $k = A_0 \rightarrow P(V) = A$ and compute the homology of the complex of derivations of $A$ relative to $A_0$. For each $s \in \mathbb{N}_0$ the cofibration $A_s \rightarrow A_{s+1}$ is obtained by adding generators in degree $s$, and it is elementary.
We can easily identify the $E_1$-page: a closed derivation of degree $s + t$ in

$$\text{Der}_{A_s}(A_{s+1}, M)$$

is determined on the generators of $A_{s+1}$ living in degree $s$, whose image are in homological degree $t$. From this and the fact $f(dv) = 0$ for a generator $v$ of degree $s$, we see that $f$ must have image in the cycles of $M$. Therefore, we have an identification

$$E_1^{s,t} = \text{hom}(V_s, H_t(M)).$$

Since $B$ is 0-connected, we can assume that $V_0 = 0$. Using this, we now check that the differential on the first page is $\text{hom}(d(1), 1)$ where $d(1) : V \rightarrow V$ is the linear part of the differential of $\mathcal{P}(V)$. Indeed, let us take a cocycle representative $f : A_{s+1} \mid A_s \rightarrow M$, let $F$ be extension by zero to $A_{s+2}$, so that $d_1[f] = [[d, F]]$. We now observe that if $w$ is a generator of $A_{s+1} \rightarrow A_{s+2}$, then $F$ vanishes on $w$. The remaining term is $(-1)^{|F|+1}f(dw)$, and now we note that any term in $dw$ that is not linear must be a product of lower degree terms, which $f$ vanishes on. Since $d(1)$ computes the Quillen homology of $B$, the description of the $E_2$-page is what we have claimed. ▶

The spectral sequence of Theorem 3.7 can also be obtained from a filtration. Indeed, to filter $\text{Der}_{A_0}(A, M)$, consider the exact sequence of derivations induced by the triple $A_0 \rightarrow A_s \rightarrow A$. It identifies the kernel of the restriction

$$\text{Der}_{A_0}(A, M) \rightarrow \text{Der}_{A_0}(A_s, M)$$

with $\text{Der}_{A_s}(A, M)$. If we let $F_s = \text{Der}_{A_s}(A, M)$ we obtain a complete decreasing filtration

$$\cdots \subseteq F_s \subseteq F_s^{-1} \subseteq \cdots \subseteq F^0 = \text{Der}_{A_0}(A, M).$$

Then the exact sequence for the triple $A_s \rightarrow A_{s+1} \rightarrow A$ identifies the quotient $F_s/F_{s+1}$ with the space $E_{0}^{s,*} = \text{Der}_{A_s}(A_{s+1}, M)$, giving rise to a spectral sequence with the same first page.

We have chosen to construct the spectral sequence from an exact couple that does not arise from the filtration above. The relationship between these two approaches is that we instead consider the tower of surjections

$$\mathcal{T} : \cdots \rightarrow F^0/F^s \rightarrow \cdots \rightarrow F^0/F^2 \rightarrow F^0/F^1 \rightarrow 0$$
whose inverse limit is \( F^0 = \text{Der}_{A_0}(A, M) \). From the discussion above, see that \( F^0 / F^s \) is naturally isomorphic to \( \text{Der}_{A_0}(A_s, M) \) so that we have exact sequences

\[
W_s : 0 \rightarrow \text{Der}_{A_s}(A_{s+1}, M) \rightarrow \text{Der}_{A_0}(A_{s+1}, M) \rightarrow \text{Der}_{A_0}(A_s, M) \rightarrow 0.
\]

The following proposition follows from the standard way in which an exact couples is built from a family of exact sequences \((W_s)_{s \in \mathbb{N}}\) as above, and we omit its proof.

**Proposition 3.10** These short exact sequences give rise to the cohomological exact couple \( \langle D, E; i, j, k \rangle \) where, for each \( s, t \in \mathbb{N} \) we have

\[
D^{s, t} = H^{s+t} \text{Der}_{A_0}(A_{s+1}, M), \quad E^{s, t} = H^{s+t} \text{Der}_{A_s}(A_{s+1}, M),
\]

and the maps \( i, j \) and \( k \) are of the form

\[
i : D^{s, t} \rightarrow D^{s-1, t+1}, \quad j : D^{s, t} \rightarrow E^{s+1, t}, \quad k : E^{s, t} \rightarrow D^{s, t}.
\]

Thus, \(|i| = (-1, 1)\), \(|j| = (1, 0)\) and \(|k| = (0, 0)\).

### 3.4 A closer look

Let us now take a closer look at the exact couple that gives rise to our spectral sequence.

**The first differential.** The differential \( d = jk \) on \( E \) is of bidegree \((1, 0)\). Explicitly, if a class \([f] \in E^{s, t}\) is represented by a cocycle \( f : A_{s+1} \rightharpoonup A_s \rightarrow M \), then \([d[f]]\) is represented by \( \partial f^{(s+2)} : A_{s+2} \rightharpoonup A_{s+1} \rightarrow M \); that is, we extend \( f \) by zero to \( A_{s+2} \) and then take its usual differential. In other words, \( d[f] = [[d, f^{(s+2)}]] \).

**The derived couple.** Let us write \( E' \) for the homology of \( E, D' = \text{im}(i) \) and explain how to obtain the first derived couple \( \langle D', E', i', j', k' \rangle \):

1. First, \( i' \) is just induced by \( i \), since \( i(D') \subseteq D' \). Note that \( i' \) has the same bidegree \((-1, 1)\) as that of \( i \).
2. To define \( j' \), write an element of \( D' \) as \( ix \), and consider the class of \( jx \) in \( E' \); note that this is a cycle since \( k j = 0 \).
3. Finally, define \( k'[x] = kx \). Note that since \( jkx = 0 \), \( kx \) is indeed in \( D' \).

One can see that \( j' \) has bidegree \((2, -1)\) and \( k' \) has bidegree \((0, 0)\), so that \( d_2 = j'k' \) has bidegree \((2, -1)\).
More generally, when forming the \((r-1)\)th derived couple of \(E\), which gives rise to the \(r\)th page \(E_r\) of our spectral sequence, obtained by simply iterating the prescription we just gave, the bidegree of the differential \(d_r : E_r \rightarrow E_r\) is \((r, 1-r)\), as expected.

**Figure 1:** A half-plane spectral sequence.

The \(r\)th page of the spectral sequence. Recall, see for instance [35, Proposition 2.9], that we can describe the \(r\)th page of our spectral sequence as a subquotient of \(E\) by using the iterated image spaces

\[ D_r = \text{im}(i \circ \cdots \circ i) = \text{im}(i_{r-1}) \]

where \(i\) appears \(r-1\) times, the boundary map \(k\) and the map \(j\), as follows. If we put

\[ Z^{s,t}_r = k^{-1}\text{im}(i_{r-1} : D^{s+r,t-r} \rightarrow D^{s,t}), \quad B^{s,t}_r = j(\ker i_{r-1} : D^{s-1,t} \rightarrow D^{s-r-1,t+r}) \]

then we have the following identifications:

\[ E^{s,t}_r = Z^{s,t}_r / B^{s,t}_r, \quad \text{and} \quad E^{s,t}_\infty = \bigcap_r Z^{s,t}_r / \bigcup_r B^{s,t}_r. \]

Diagrammatically, the \(r\)th page is the homology of the diagonal of the commutative diagram with exact rows and exact column of Figure 2. We now observe that for each \(r \in \mathbb{N}\), the term \(D^{s,t}_r\) consists of classes \([f]\) represented by a cocycle \(f : A_{s+1} | A_0 \rightarrow M\) of degree \(s+t\) which is the restriction of a cocycle \(F : A_{s+r+1} | A_0 \rightarrow M\) of the same degree, and the map

\[ i_{r-1} : D^{s+r,t-r}_r \rightarrow D^{s,t}_r \]
is just the map induced by restriction on homology.

The higher differentials. From our construction we see that the kernel of the map $jk = d_1 : E_1 \rightarrow E_1$ consists of classes of those $(s + t)$-cocycles $f : A_{s+1} \mid A_s \rightarrow M$ that are restrictions of cocycles $A_{s+2} \mid A_s \rightarrow M$. Indeed, if $jk[f] = 0$ then, by exactness, we must have that $k[f] = i[G]$ for some cocycle

$$G : A_{s+2} \mid A_0 \rightarrow M,$$

and this with some rearranging of terms implies $f$ is the restriction of some cocycle $g : A_{s+2} \mid A_0 \rightarrow M$ in $D^{s+1,t-1}$. In fact, we can make it so $g = f^{(s+2)} + h$ for some derivation $h$ which is not necessarily a cocycle. The map $k' : (E')^{s,t} \rightarrow (D')^{s,t}$ then sends the class of $g$ to $g$ and hence the differential

$$d_2^{s,t} : E_2^{s,t} \rightarrow E_2^{s+2,t-1}$$

is obtained by applying $j'$ to the class of $g$ or, what is the same, the connecting morphism $j$ to $g$ itself. This can be iterated to describe the successive differentials.

Let us now consider the second derived couple and obtain a description of $d_3$ before stating a general result describing the successive differentials in the spectral sequence. To do this, let us continue with the notation at the beginning of the section, and assume that we are given a cocycle

$$f : A_{s+1} \mid A_s \rightarrow M$$
for which \( d_1[f] = 0 \) and \( d_2[f]_2 = 0 \). As explained there, the first conditions implies there exists an extension of \( f \) to a cocycle

\[
h : A_{s+2} | A_s \longrightarrow M,
\]

and the definition says now that \( d_2[f]_2 \) is the class in \( E' \) of the connecting morphism \( j \) applied to \([h] \). If this vanishes, then we have an equality

\[
j[h] = j'k'(b')
\]

in \( E \) for some \( b' = [f_1] \). Unwinding the definitions, it follows that there are

- a cocycle \( f_1 : A_{s+2} | A_{s+1} \longrightarrow M \),
- a derivation \( h_1 : A_{s+3} | A_{s+2} \longrightarrow M \),

so that

\[
f^{(s+3)} + h^{(s+3)} + f_1^{(s+3)} + h_1 : A_{s+3} | A_s \longrightarrow M
\]

is a cocycle. The end result is that we have extended \( f \) to a cocycle up to \( A_{s+3} \). Since \( d_r \) is induced by \( k, j, \) and \((r - 1)\)-preimages of \( i \), we obtain the following:

**Proposition 3.11** For each \( r \geq 1 \), the kernel of \( d_r \) is generated by classes \( \eta \) of cocycles

\[
f : A_{s+1} | A_s \longrightarrow M
\]

of degree \( s + t \) that admit a lift to a cocycle \( g : A_{s+r+1} | A_s \longrightarrow M \) of the same degree. In this situation, the differential \( d_{r+1}(\eta) \) is the class of the connecting morphism on \( g \). □

In particular, if the class of some \( f : A_{s+1} | A_s \longrightarrow M \) of degree \( s + t \) survives to \( Z^M_{s+t} \), what we obtain is an extension to a cocycle \( f' : A | A_s \longrightarrow M \) of degree \( s + t \) in \( F^s = \text{Der}_{A_s}(A, M) \).

### 3.5 Collapse results

In Theorem 3.7, we have given the most general form of the spectral sequence of derivations. When giving applications, it is convenient that it be concentrated in a single quadrant in order to have strong, rather than conditional, convergence. To this end, we collect in this section some natural conditions that guarantee certain vanishing patterns on the \( E_1 \)-page of the spectral sequence.
A spectral sequence for André–Quillen cohomology of algebraic operads

Recall that a $\mathcal{P}$-algebra $U$ is $b$-truncated if $H_p(U) = 0$ for all $p \geq b + 1$, and that a cofibration $B \to A$ is elementary of height $s$ if $A = B \star \mathcal{P}(V)$ with $V = V_{s+1}$. In this case, one has that $d(V) \subseteq B$.

**Lemma 3.12** Let $B \to A$ be an elementary cofibration of height $s$, and $u : A \to U$ a morphism of $\mathcal{P}$-algebras. The following holds:

1. If $U = U_{\geq k}$ for some $k$, then $\text{Der}^p_B(A,U) = 0$ for all $p \geq s + 2 - k$.
2. If $U$ is $b$-truncated for some $b$, then $H^p(\text{Der}_B(A,U)) = 0$ for all $p \leq s - b$.

**Proof.** The fact that $B \to A$ is an elementary cofibration implies that any $F \in \text{Der}_B(A,U)$ is completely determined by its image on $V$, where $A = B \star TV$ and $V = V_{s+1}$.

To prove the first statement, let us take $F \in \text{Der}^p_B(A,U)$. Since $F$ has degree $-p$ the image of $F$ lies in $U_{s+1-p}$. It follows that $F$ vanishes if $p \geq s + 2 - k$. For the second statement, let us take a cocycle $F \in \text{Der}^p_B(A,U)$ of degree $p \leq s - t$, and show that $F = \partial G$ for some $G \in \text{Der}^{p-1}_B(A,U)$.

Recall that $F$ is determined by its image on $V$, and that for any generator $v \in V$ we have $F(v) \in U_{s+1-p}$. Given that $F$ is a cocycle, for any such generator $v$ we have that

$$\partial F(v) = dF(v) - (-1)^pFd(v) = 0.$$ 

But $d(v) \in B$, because $B \to A$ is elementary, so $Fd(v) = 0$ and thus $F(v)$ is a cycle in $U_{s+1-p}$. Since $p \leq s - b$, it turns out that $F(v) \in U_{\geq b+1}$ is a boundary. Let $G(v) \in U$ be such that $dG(v) = F(v)$. This defines $G \in \text{Der}^{p-1}_B(A,U)$ such that $\partial G = F$, which is what we wanted. Indeed, $\partial G$ and $F$ are derivations that coincide on every $v \in V$, and hence coincide on all of $A$. □
Definition 3.13 A tower of cofibrations

\[ A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_s \rightarrow \cdots \rightarrow \lim_{s} A_s = A \]

is \textit{cellular} if each \( A_s \rightarrow A_{s+1} \) is elementary of height \( s - 1 \). That is, if for every \( s \in \mathbb{N} \) the algebra \( A_{s+1} \) is obtained from \( A_s \) by freely adjoining a space of generators in homological degree \( s \).

Algebras that are quasi-free and triangulated are good examples of colimits of cellular towers of cofibrations, and in particular the three algebraic models of Adams–Hilton, Quillen and Sullivan, respectively, are good examples of colimits of towers of cofibrations relevant to topology. In Section 4 we will exploit this observation to apply our methods to rational homotopy theory.

Corollary 3.14 Let \( A \) be the colimit of a cellular tower of cofibrations of \( \mathcal{P} \)-algebras

\[ A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_s \rightarrow \cdots \rightarrow \lim_{s} A_s = A, \]

and let \( u : A \rightarrow U \) be a morphism of \( \mathcal{P} \)-algebras. The following holds:

1. If \( U = U_{\geq k} \) for some \( k \), then \( E_{1}^{s,t} = 0 \) for all \( t \geq 1 - k \).
2. If \( U \) is \( b \)-truncated for some \( b \), then \( E_{1}^{s,t} = 0 \) for all \( t \leq -(b+1) \).

In particular, if \( U \) is \( 0 \)-truncated and \( U = U_{\geq 0} \), the spectral sequence degenerates at the second page.

Proof. The description of the \( E_1 \)-page of our spectral sequence and Lemma 3.12 imply the first two statements, if we recall that in a cellular tower each \( A_s \rightarrow A_{s+1} \) is elementary of height \( s - 1 \). In case \( b = k = 0 \), we obtain that \( E_{1}^{s,t} = 0 \) unless \( t = 0 \), so that the first page is concentrated in one row, and the claim follows.

4 Applications to rational homotopy theory

In this section, we apply the spectral sequence of Theorem 3.7 to classical rational homotopy theory using the models for spaces due to Adams–Hilton and Sullivan. We assume the reader has certain familiarity with rational homotopy theory and recommend the book [19] as a reference to this subject.

For many types of algebras, it makes sense to consider cofibration sequences and derivation complexes. This gives rise to very useful spectral sequences in several contexts. Instead of
constructing the spectral sequence for each type of algebra, we have given it in the very general language of operads. The advantage is that a single argument provides spectral sequences readily applicable to many categories of interest.

To illustrate this point, we specialize our spectral sequence in the context of dg associative algebras and dg commutative algebras. Nevertheless, such a spectral sequence is attained under more general hypotheses than the ones we consider here for simplicity—that is, over a field of characteristic zero. In particular, it works for associative algebras over any field. This provides further applications to $p$-local homotopy theory by exploiting, for instance, the Adams-Hilton model of a space, see for example [2, 3, 39].

**Conventions for the section.** In this section, we change the bidegree in the spectral sequence of Theorem 3.7 by $(s,t) \mapsto (s,-t)$. This gives more natural formulas, avoiding negative signs in the applications that follow. The spectral sequence is still right-half-plane, but the differential $d_r$ changes its bidegree to $(r,r-1)$, and under convergence assumptions, the $E_{\infty}$-page recovers the cohomology of the target by the formula

$$(\text{gr}H)^p = \bigoplus_{s=p+t} E^{s,t}_{\infty}.$$ 

We also revert to using the qualifier “dg” in front of algebras to avoid any confusion when dealing with (co)chain algebras, for example.

### 4.1 Three flavours of homology

Let us now draw up some simple connections between the existing geometrical models of spaces from the work of F. Adams and P. Hilton, D. Quillen, and D. Sullivan, and the various homology functors we have at hand, namely:

1. The functor $H_*(-) : \mathcal{P}\text{-Alg} \to \mathcal{P}\text{-Alg}$ that assigns to a (dg) $\mathcal{P}$-algebra $A$ its homology groups $H_*(A)$. As our notation suggests, these are also (non dg) $\mathcal{P}$-algebras.
2. The functor $H_*(\mathcal{P},-): \mathcal{P}\text{-Alg} \to \mathcal{P}_\infty\text{-Cog}$ that assigns to a $\mathcal{P}$-algebra $A$ its Quillen homology groups $H_*(\mathcal{P},A)$. As our notation suggests, these are homotopy $\mathcal{P}$-coalgebras. We will use this only for $\mathcal{P}$ the associative operad.
3. The assignment (but not functor) $H_\mathcal{A} : \mathcal{P}\text{-Alg} \to \text{Lie-Alg}$ that assigns to a $\mathcal{P}$-algebra $A$ its André–Quillen cohomology groups with values in itself $H_\mathcal{A}(A,A)$.

On the other hand, we have three classical constructions on topological spaces:
The Quillen functor $\lambda : \text{Top}_{*,1} \longrightarrow \text{Lie-Alg}$ that assigns to a pointed 1-connected $X$ a dg Lie algebra $\lambda(X)$ that models the Lie algebra of homotopy groups $\pi_*(\Omega X)$.

(2) The Sullivan functor $A_{\text{PL}} : \text{Top} \longrightarrow \text{Com-Alg}$ that assigns to a space $X$ a commutative dga algebra $A_{\text{PL}}^*(X)$ that is a model of the cochain algebra $\text{Sing}^*(X)$.

(3) The Adams–Hilton construction $A : \text{Top}_{*,1} \longrightarrow \text{Ass-Alg}$ that assigns to a pointed 1-connected $X$ a dga algebra $A^*(X)$ that is a model of the Pontryagin algebra of chains $\text{Sing}_*(\Omega X)$.

In the following table we record some of the relations between these algebraic models of spaces and the three flavours of homology above. We point the reader to [7, 8, 43] for useful references where these relations are studied in detail. We now proceed to recall the last three functors above and apply our operadic formalism to obtain spectral sequences to compute their André–Quillen cohomology groups, which we will identify with invariants of known geometrical objects.

| Models       | Homology theory                                                                 |
|--------------|---------------------------------------------------------------------------------|
|             | Ordinary $\pi_*(\Omega X)$ $H_{*+1}(X)$ $\pi_*(LX)$                           |
| Quillen      |                                                                              |
| Sullivan–de Rham | $H^*(X)$ $\pi_{*+1}(X)^\vee$ $\pi_*(\mathcal{F}(X,X))$                         |
| Adams–Hilton | $H_*(\Omega X)$ $H_{*+1}(X)$ $H_*(LX)$                                              |

### 4.2 The Adams-Hilton model

In [22] the authors exploit the identification of Hochschild cohomology and the homology of derivations of Remark 3.4 along with the identification of the loop homology of a closed orientable manifold with the Hochschild cohomology of $\text{Sing}_*(\Omega X)$ to compute the loop bracket of $X$. This is done under the hypothesis the identification of Jones and Cohen [15] between loop homology and Hochschild cohomology of $\text{Sing}^*(X)$ commutes with the bracket.

Explicitly, one can compute Hochschild cohomology of $\text{Sing}_*(\Omega X)$ through the Adams–Hilton model $A_*(X)$ of $X$, and then the loop space bracket as the Lie bracket of derivations of its associated Lie algebra of derivations $\text{Der}(A_*(X))$. Since the model $A_*(X)$ has generators that are in bijection with the cells of a CW decomposition of the manifold $X$, this method lends itself quite nicely to computations. We now recall the construction of Adams and Hilton.

Let $X$ be a CW complex with exactly one 0-cell, no 1-cells, and such that all the attaching maps of higher dimensional cells are based with respect to this only 0-cell. In [1], the authors construct a cofibrant model $A_*(X)$ of the dg algebra $\text{Sing}_*(\Omega X)$, where $\Omega X$ is the Moore loop-space of $X$. In the following, for each $n \in \mathbb{N}$ write $X_n$ for the $n$-skeleton of $X$. 
Theorem (Adams–Hilton [1]) There is a cofibrant model

\[ f_X : A_*(X) = (TV, d) \longrightarrow (\text{Sing}_*(\Omega X), d) \]

of the Pontryagin dga algebra of \( X \) such that for each \( n \in \mathbb{N}_0 \),

\begin{align*}
\text{(L1)} & \quad \text{the space } V_n \text{ has basis the } (n+1)\text{-cells of } X, \text{ so that } V = V_{\geq 1}, \\
\text{(L2)} & \quad \text{the map } f_X \text{ restricts to quasi-isomorphisms } A_*(X_n) \longrightarrow (\text{Sing}_*(\Omega X_n), d), \\
\text{(L3)} & \quad \text{if } g : S^n \longrightarrow X_n \text{ is the attaching map of a cell } e \text{ in } X, \text{ then}
\end{align*}

\[ (f_X)_* [dv_e] = K [g], \]

where \( K \) is the isomorphism \( \pi_n X_n \rightarrow \pi_{n-1} \Omega X_n \) followed by the Hurewicz map. ▶

These conditions determine the dg algebra \( A_*(X) \) uniquely up to isomorphism, and we call it the Adams–Hilton model of \( X \). Of course, this model depends on the CW structure of \( X \), which we take as part of the input data for its construction. As we noted, homology of the shift of indecomposables \( s \text{Ind}_p A_*(X) \) of the Adams–Hilton model is the (reduced) homology \( H_*(X) \). In other words, \( s \text{Ind}_p A_*(X) \) is the reduced cellular chain complex \( \text{Cell}_*(X) \) of \( X \).

We remark that in [1] the authors actually produce a model of the fibration sequence of a CW complex \( X \). That is, there is a commutative diagram of complexes of \( k \)-modules

\[
\begin{array}{cccc}
A_*(X) & \longrightarrow & \text{Cell}_*(X) \otimes A_*(X) & \longrightarrow & \text{Cell}_*(X) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Sing}_*(\Omega X) & \longrightarrow & \text{Sing}_*(LX) & \longrightarrow & \text{Sing}_*(X)
\end{array}
\]

where all the vertical maps are quasi-isomorphisms, the top row is the classical “algebraic fibration” coming from a cobar construction, and the bottom row is obtained by applying the singular chains functor to the based path-space fibration. Moreover, the first vertical map is a map of dga algebras and the second vertical map is a map of right modules with respect to the obvious action of \( A_*(X) \) on \( \text{Cell}_*(X) \otimes A_*(X) \) and the action of \( \Omega X \) on \( LX \). The Adams-Hilton model and, more generally, non-commutative dg algebra models have proven successful in the study of the \( p \)-local homotopy theory of spaces, for example; see the article [2] and the book [3].

We now show how to run our spectral sequence to compute loop space homology of a space \( X \) by using the filtration by degree in the Adams–Hilton model. In doing so, we obtain the following result, which the reader can compare with the spectral sequence of S. Shamir [44] and to the eponymous spectral sequence of J.-P. Serre. We remark that our spectral sequence has differentials that we can control provided we can control the Adams–Hilton model. This gives
us a spectral sequence of the same shape as that in [16], produced by R. L. Cohen, J. D. S. Jones and J. Yan.

**Theorem 4.1** Let $X$ be a CW complex with exactly one 0-cell, no 1-cells and all whose attaching maps are based with respect to the only 0-cell. There is a first quadrant spectral sequence with

$$E_2^{s,t} = \text{hom}(H_s(X), H_t(\Omega X)) \Rightarrow H_{s+t}(LX),$$

which is conditionally convergent in the sense of Boardman.

**Proof.** Since $X$ is simply connected, the Adams–Hilton model of $X$ is a 0-connected dga algebra, and then Corollary 3.9 applies. We already noted that the Quillen homology of $A_\ast(X)$ is $H_{s+1}(X)$, while its homology is $H_\ast(\Omega X)$.

To conclude, we need only address the identification of the target of our spectral sequence and the fact we have replaced the positive homology groups of $X$ with with the homology groups of $X$. To do the second, we recall from Proposition 3.6 that we may pass from André–Quillen cohomology to Hochschild cohomology of the dga algebra $A_\ast(X)$ by taking the cone of the adjoint map

$$\text{Ad} : A_\ast(X) \longrightarrow \text{Der}A_\ast(X).$$

We then filter the cone by only filtering the summand corresponding to derivations. This has little effect in our spectral sequence, except that it adds the summand corresponding to the homology of $A_\ast(X)$, namely

$$H_\ast(\Omega X) = \text{hom}(H_0(X), H_\ast(\Omega X)),$$

in the second page and throughout the computation, and produces the desired shift (there is a shift in the cone).

Having addressed this, we now recall from [32, Chapter 4] and [15] that the Hochschild cohomology groups of the Pontryagin algebra $\text{Sing}_\ast(\Omega X)$ of $X$ are functorially isomorphic to the homology groups of the free loop space $LX = \text{Map}(S^1, X)$ of $X$. ◀

**Remark 4.2** Observe that if $A_\ast(X)$ is minimal or, what is the same, if the linear part of its differential is zero, then the description of the $E_2$-page is greatly simplified. If there are only cells in even degree or only in odd degree, then $A_\ast(X)$ will be minimal, for example.
4.3 Multiplicative structure

For $X$ a simply connected closed oriented manifold of dimension $m$, write

$$\mathbb{H}_* (LX) = H_{*+m}(LX).$$

We recall from [24] that there is a loop product $-\ast - : \mathbb{H}_*(LX) \otimes \mathbb{H}_*(LX) \to \mathbb{H}_*(LX)$. One of the main results of that paper is as follows:

**Theorem** For every simply connected closed oriented manifold $X$, there is a natural isomorphism of graded commutative associative algebras

$$\left( \mathbb{H}_*(LX), -\ast - \right) \longrightarrow \left( \mathbb{HH}^*(\text{Sing}^* (X)), -\circ - \right).$$

Following the notation of Section 4.2, let $A_*(X) = (TV,d)$ be an Adams–Hilton model for $X$, so that the cone of the adjoint map of $A_*(X)$ computes $\mathbb{H}_*(LX)$. In our spectral sequence, the $E_2$-page has the form

$$E_2^{s,t} = \text{hom}(H_s(X), H_t(\Omega X))$$

and, since $H_*(X)$ is a associative coalgebra and $H_*(\Omega X)$ an associative algebra, this page inherits an associative *convolution product* given by the following composite:

$$H_*(X) \xrightarrow{\Delta} H_*(X) \otimes H_*(X) \xrightarrow{f \otimes g} H_*(\Omega X) \otimes H_*(\Omega X) \xrightarrow{\mu} H_*(\Omega X).$$

Observe that in case we consider elements with domain in $H_0(X) = \mathbb{Q}$, this identifies with the Pontryagin product of $H_*(\Omega X)$. We remark that this is in line with Theorem 1 in [16], and also point the reader to [36].

**Theorem 4.3** The spectral sequence of Theorem 4.1 is multiplicative. The convolution product on the $E_2$-page converges to the cup product on the Hochschild cohomology of $A_*(X)$, which equals the Chas–Sullivan product in the loop homology groups of $X$.

**Proof.** The cup product on $\mathbb{HH}^*(A_*(X))$ is induced from the differential $d$ of $\text{Der}(A_*(X))$ as follows. Let $f$ and $g$ be linear maps $V \to TV$ which correspond to derivations $F$ and $G$. Then $F \circ G$ is determined by the derivation induced from the brace operation $\{d; f, g\}$ [29, 37]. Explicitly, this is obtained by all possible ways of inserting $f$ and $g$ (in this order) into the operators $d = d_2 + d_3 + d_4 + \cdots$. The first few terms are as follows:

$$\{d; f, g\} = d_2(f, g) + d_3(f, g, 1) + d_3(f, 1, g) + d_3(1, f, g) + \cdots.$$
Using the filtration of that theorem, on the $E_2$-page we are left only with the term induced in homology by the derivation associated to $d_2(f, g)$, which on $V$ restricts precisely to $\mu \circ (f \otimes g) \circ \Delta$, which is what we wanted.

We point out a similar description for a Lie bracket in the cohomology groups of derivations of Sullivan models of spaces through brace operations is given in [12, Theorem 3].

### 4.4 The Sullivan model

The Sullivan model of a topological space has proved to be quite successful and versatile in studying rational homotopy theory and its connection to other fields. We refer the reader to [19, 23] for a survey on the various results obtained through this formalism. In this section, we study the spectral sequence of Theorem 3.7 in this context, that is, when considering cofibrant commutative dga algebras that model the rational homotopy type of topological spaces and fibrations between these.

**The cohomological conventions.** As a chain algebra, a Sullivan algebra is concentrated in non-positive degrees, and the complexes of derivations of [21], which we will be using, are the same as ours. Since we intend to apply Corollary 3.14 to Sullivan algebras, which are naturally cohomologically graded and concentrated in non-negative degrees, we substitute the “truncated” condition following Theorem 2.2 by the following.

**Definition 4.4** Let $b \in \mathbb{N}$. A dga algebra $A$ is $b$-truncated if $H^p(A)$ vanishes for all $p \geq b + 1$.

For example, an $n$-manifold has a $n$-truncated Sullivan model. Using these cohomological conventions, the items of Lemma 3.12 read as follows:

**Lemma 4.5** Let $U$ be a cohomologically graded $\mathbb{P}$-algebra.

1. If $U = U \geq k$ for some $k$, then $\text{Der}_B^p(A, U) = 0$ for all $p \leq k - s - 2$,
2. If $U$ is $b$-truncated for some $b$, then $H^p(\text{Der}_B(A, U)) = 0$ for all $p \geq b - s$.

Similarly, the items of Corollary 3.14 read as follows, where again $t$ is replaced by $-t$:

**Lemma 4.6** Let $U$ be a cohomologically graded $\mathbb{P}$-algebra.

1. If $U = U \geq k$ for some $k$, then $E_1^{s,t} = 0$ for all $2s \leq k + t - 1$.
2. If $U$ is $b$-truncated for some $b$, then $E_1^{s,t} = 0$ for all $2s \geq t + b + 1$.

Since Sullivan algebras are concentrated in non-negative degrees, we will have that $E_1^{s,t} = 0$ whenever $t \geq 2s + 1$. This implies strong convergence in the spectral sequence of Theorem 4.7.
If moreover $X$ is an $n$-manifold, or any space whose rational cohomology is concentrated in degrees $\leq n$, then we have the sharper vanishing of $E^{s,t}_{1}$ whenever $t + n + 1 \leq 2s$.

Let us fix a fibration $p : E \to B$, where $E$, $B$ are 1-connected CW-complexes and $E$ is finite, and let us take $(\Lambda W, d) \to (\Lambda W \otimes \Lambda V, D)$ a relative Sullivan model of this fibration. In other words, the following diagram of cdga algebras commutes and the vertical maps are quasi-isomorphisms of cdga algebras:

$$
\begin{align*}
A_{PL}(B) & \xrightarrow{A_{PL}(p)} A_{PL}(E) \\
\cong & \xleftarrow{\cong} (\Lambda W, d) \longleftarrow (\Lambda W \otimes \Lambda V, D).
\end{align*}
$$

The main result in [21] show that this cofibration codifies the homotopy type of $\text{Aut}_1(p)$, the component of the identity in the topological monoid $\text{Aut}(p)$ of fibre-homotopy self equivalences $f : E \to E$. We recall this means that $f$ is a homotopy equivalence such that $pf = p$.

**Theorem** (Theorem 1 in [21]) There is an isomorphism of graded Lie algebras

$$
H^*(\text{Der}_{\Lambda W}(\Lambda W \otimes \Lambda V)) \to \pi_*(\text{Aut}_1(p)).
$$

In the language of this paper, this result states the following:

**Theorem** (Operadic version) The André-Quillen homology of the map

$$
A_{PL}(p) : A_{PL}(B) \to A_{PL}(E)
$$

is isomorphic, as a graded Lie algebra, to the rational Samelson Lie algebra $\pi_*(\text{Aut}_1(p))$: there is an isomorphism of graded Lie algebras

$$
\mathcal{H}^*(A_{PL}(p)) \to \pi_*(\text{Aut}_1(p)).
$$

It is useful to remark that, since each connected component of $\text{Aut}(p)$ has the same rational homotopy type, the construction above determines the rational homotopy type of the space $\text{Aut}(p)$ in terms of the rational homotopy type of $\text{Aut}_1(p)$ and the group structure in $\pi_0(\text{Aut}(p))$. See [21] for complete details.

We now observe that we can exhibit the cofibration

$$
(\Lambda W, d) \to (\Lambda W \otimes \Lambda V, D)
$$
as the tower of cofibrations obtained by adding the cells of $\Lambda V$ “one by one”. With this in mind, let us put the technical tools developed in Section 3.1 into the appropriate context to apply them to Sullivan algebras.

The construction of the relative Sullivan model [19, Proposition 15.6], does not require any finite type hypotheses on the spaces involved in the fibration, see [20, Theorem 3.1]. On the other hand, for any simply connected space $X$ with Sullivan model $(\Lambda V, d)$, we do need $\pi_*(X)$ finite dimensional in each degree in order to have an identification $V^* = \pi_*(X)$. Otherwise, we can only conclude that there is an isomorphism

$$V^* \longrightarrow \text{hom}(\pi_*(X), \mathbb{Q}).$$

Since in most applications we will take $X$ to be a finite type CW-complex, a classical result of Serre, see for instance [48, Theorem 20.6.3], let us identify $V^* = \pi_*(X)$, and we will write it like this in the statements.

**Theorem 4.7** Let $F \hookrightarrow E \overset{p}{\rightarrow} B$ be a fibration of 1-connected CW-complexes, with $E$ finite. There is a spectral sequence with

$$E_2^{s,t} = \text{hom}(\pi_s(F), H^t(E)) \Rightarrow \pi_{t-s}(\text{Aut}_1(p)).$$

**Proof.** Let $(\Lambda W, d) \hookrightarrow (\Lambda W \otimes \Lambda V, D)$ be a relative Sullivan model of the fibration. The Sullivan condition provides a filtration $V(k)$ of $V$ such that

$$DV(k) \subseteq \Lambda W \otimes \Lambda V(k - 1) \quad \text{for all } k \geq 0.$$ 

Since $W = W \geq 2$, we can assume that its Sullivan filtration is given by degree, $W(k) = W \leq k$. Equally, we can assume the same degree filtration on $V$. It thus makes sense to consider the cellular tower of cofibrations of cdga’s given by $A_s = \Lambda W \otimes \Lambda V \leq s$, whose colimit is the relative Sullivan algebra

$$(\Lambda W, d) \hookrightarrow (\Lambda V \otimes \Lambda W, D).$$

We are in the cohomological situation of Corollary 3.9, and this gives us the second page if we note that the cohomology groups of $\Lambda W \otimes \Lambda V$ are those of $E$. On the other hand, the quotient algebra $(\Lambda V, d')$ is a model for the fibre, so its Quillen homology gives the homotopy groups of $F$. To finish, recall that the target of the spectral sequence is the homology of $\text{Der}_{\Lambda W}(\Lambda W \otimes \Lambda V)$, so an application of [21, Theorem 1] finishes the proof.

Applying this result to the trivial fibration $X \longrightarrow *$ yields the following result.
Corollary 4.8 Let $X$ be a finite, 1-connected CW-complex. There is a conditionally convergent first quadrant spectral sequence with

$$E_2^{s,t} = \text{hom}(\pi_s(X), H^t(X)) \Rightarrow \pi_{s-t}(\text{Aut}_1(X))$$

To obtain the corollary above, we just feed a specific fibration to the spectral sequence of Theorem 4.7. The following table collects some other fibrations and the target of the corresponding spectral sequence.

| Input Shape               | Target                                      |
|---------------------------|---------------------------------------------|
| loop-space fibration      | $\pi_*(\Omega X)$                          |
| trivial fibration         | $\pi_*(\mathcal{F}(X, \text{Aut}(F)))$     |
| principal $G$-bundle      | $\pi_*(G_o)$                               |

We remark that the spectral sequence of Theorem 4.7 is multiplicative for a graded Lie bracket that, on the second page, identifies with the convolution bracket obtained from the Lie coalgebra $\pi_*(F)$ with the Whitehead cobracket and the commutative algebra $H^*(E)$ with the cup product. A result analogous to Theorem 3 in [12] for $\mathcal{F}(X, Y)$ replaced with $\text{Aut}_1(p)$ should then give the conclusion analogous to that of Theorem 4.3 that this product converges to the Whitehead product in the target homotopy groups $\pi_*(\text{Aut}_1(p))$.

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