A teleparallel model for the neutrino

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(Dated: October 17, 2018)

The main result of the paper is a new representation for the Weyl Lagrangian (massless Dirac Lagrangian). As the dynamical variable we use the coframe, i.e. an orthonormal tetrad of covector fields. We write down a simple Lagrangian – wedge product of axial torsion with a lightlike element of the coframe – and show that variation of the resulting action with respect to the coframe produces the Weyl equation. The advantage of our approach is that it does not require the use of spinors, Pauli matrices or covariant differentiation. The only geometric concepts we use are those of a metric, differential form, wedge product and exterior derivative. Our result assigns a variational meaning to the tetrad representation of the Weyl equation suggested by J.B. Griffiths and R.A. Newing.

Throughout this paper we work on a 4-manifold $M$ equipped with prescribed Lorentzian metric $g$. In the following two subsections we describe two different models for the neutrino.

Traditional model

The accepted mathematical model for a neutrino field is the following linear partial differential equation on $M$

$$i\sigma^a_{\alpha b}\{\nabla\}_{\alpha}^{\xi^a}=0.$$  (1)

The corresponding Lagrangian is

$$L_{\text{Weyl}}(\xi) := \frac{i}{2}(\xi^b \sigma^a_{\alpha b}\{\nabla\}^{\xi^a} - \xi^a \sigma^a_{\alpha b}\{\nabla\}^{\xi^b})*1.$$  (2)

Here $\sigma^a_{\alpha}$, $\alpha = 0, 1, 2, 3$, are Pauli matrices, $\xi$ is the unknown spinor field, and $\{\nabla\}$ is the covariant derivative with respect to the Levi-Civita connection: $\{\nabla\}^{\xi^a} := \partial_a \xi^\alpha + \frac{1}{2} \epsilon^{\alpha\beta\gamma}\partial_a \sigma^{\beta}_{\beta\gamma} + \{\Gamma^a_{\alpha\beta}\gamma\xi^\beta\} = 0$ where $\{\Gamma^a_{\alpha\beta}\}$ are Christoffel symbols uniquely determined by the metric.

Teleparallel model

The purpose of our paper is to give an alternative representation for the Weyl equation (1) and the Weyl Lagrangian (2). To this end, we follow [1] in introducing a teleparallel model. This is a field of orthonormal bases with orthonormality understood in the Lorentzian sense of the coframe $\xi$. We define an affine connection and corresponding covariant derivative $\nabla$ from the conditions

$$\nabla \vartheta^j = 0.$$  (4)

Let us emphasize that we follow [2, 3, 4, 5] in employing holonomic coordinates so in explicit form conditions (4) read $\partial_a \vartheta^j - \{\Gamma^a_{\alpha\beta}\vartheta^\beta\} = 0$ giving a system of linear algebraic equations for the unknown connection coefficients $\{\Gamma^a_{\alpha\beta}\}$. The connection defined by the system of equations (4) is called the teleparallel or Weitzenb"ock connection.

Let $l$ be a nonvanishing real lightlike teleparallel covector field ($l \cdot l = 0, \nabla l = 0$). Such a covector field can be written down explicitly as $l = l_j \vartheta^j$ where $l_j$ are real constants (components of the covector $l$ in the basis $\vartheta^j$), not all zero, satisfying

$$o^{jk}l_j l_k = 0.$$  (5)

We define our Lagrangian as

$$L(\vartheta^j, l_j) = l_j o_{jk} \vartheta^j \wedge \vartheta^k \wedge d \vartheta^k,$$  (6)

where $d$ stands for the exterior derivative. Note that $\frac{1}{d} o_{jk} \vartheta^j \wedge d \vartheta^k$ is the axial (totally antisymmetric) piece of torsion of the teleparallel connection. (The irreducible decomposition of torsion is described in Appendix B.2 of [1].) Let us emphasize that formula (6) does not explicitly involve connections or covariant derivatives.

The Lagrangian (6) is a rank 4 covariant antisymmetric tensor so it can be viewed as a 4-form and integrated over the manifold $M$ to give an invariantly defined action $S(\vartheta^j, l_j) := \int L(\vartheta^j, l_j)$. Independent variation with respect to the coframe $\vartheta^j$ and parameters $l_j$ subject to the constraints (3) and (5) produces a pair of Euler–Lagrange

\[ \frac{\partial L}{\partial \vartheta^j} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\vartheta}^j}, \quad \frac{\partial L}{\partial l_j} = 0. \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\vartheta}^j} + \frac{\partial L}{\partial l_j} = 0. \]
equations which we write symbolically as
\[ \partial S(\vartheta^j, l_j)/\partial \vartheta^j = 0, \] (7)
\[ \partial S(\vartheta^j, l_j)/\partial l_j = 0. \] (8)

Note that the Lagrangian (6) and constraints (3) are invariant under rigid (i.e. with constant coefficients) Lorentz transformations
\[ (\vartheta^j, l_j) \rightarrow (\Lambda^j_k \vartheta^k, (\Lambda^{-1})^j_k l_j) \] (9)
where \( o_{jk} \Lambda^j_p \Lambda^k_q = o_{pq} \) and \( (\Lambda^{-1})^j_k \Lambda^i_j = \delta^i_k \). This means that any variation of the parameters \( l_j \) can be compensated by a rigid variation of the coframe \( \vartheta^j \). Hence, the field equation (3) is a consequence of the field equation (7). So further on we assume the parameters \( l_j \) to be fixed and study the field equation (7) only.

**Equivalence of the two models**

Let us define the spinor field \( \xi \) as the solution of the system of equations
\[ |\nabla| \xi = 0, \] (10)
\[ \sigma_{abc} \xi^a \bar{\xi}^b = \pm l_{\alpha} = \pm l_j \vartheta^j_{\alpha} \] (11)
where \( |\nabla| \xi^a \ := \partial_\alpha \xi^a + \frac{1}{2} \sigma_{bc} \sigma^a \partial_\alpha \sigma^{bc} + [\Gamma]^{\alpha}_{\beta \gamma} \sigma^{\beta \gamma} \xi^b \) and the sign is chosen so that the RHS lies on the forward light cone. The system (10), (11) determines the spinor field \( \xi \) uniquely up to a complex constant factor of modulus 1. This non-uniqueness is acceptable because we will be substituting \( \xi \) into the Weyl equation (10) and Weyl Lagrangian (12) which are both U(1)-invariant. We will call \( \xi \) the spinor field associated with the coframe \( \vartheta^j \).

The main result of our paper is the following

**Theorem 1** For any coframe \( \vartheta^j \) we have
\[ L(\vartheta^j, l_j) = \pm 4 L_{\text{Weyl}}(\xi) \] (12)
where \( \xi \) is the associated spinor field. The coframe satisfies the field equation (7) if and only if the associated spinor field satisfies the Weyl equation (10).

The sign in Eq. (12) depends on the sign of the parameter \( l_0 \), on whether the covector \( l = l_j \vartheta^j \) lies on the forward or backward light cone, and on the orientation of the coframe (eight different combinations).

The proof of Theorem 1 is given below. The crucial point is explained in the section on \( B^2 \)-invariance, whereas technicalities are handled in a separate section. In the final section we discuss Theorem 1 within the context of known results from the theory of teleparallelism.

**NOTATION**

Our notation follows \[ \text{[2, 3, 4, 5]. In particular, in line with the traditions of particle physics, we use Greek letters to denote tensor (holonomic) indices. Details of our spinor notation are given in Appendix A of [5].}

All our constructions are local. In particular, we restrict changes of local coordinates on \( M \) to those preserving orientation. This restriction enables us to define the Hodge star \( * \) in the usual way.

We define the forward light cone as the span of \( \sigma_{abc} \xi^a \bar{\xi}^b, \xi \neq 0 \). We also define
\[ \sigma_{\alpha \beta \gamma} := (1/2)(\sigma_{\alpha \beta b} \delta_{\gamma c} - \sigma_{\beta \alpha b} \delta_{\gamma c} \sigma_{acd}). \]
These “second order” Pauli matrices are polarized, i.e. \( *\sigma = \pm i \sigma \) depending on the choice of “basic” Pauli matrices \( \sigma_{abc} \). We assume that \( *\sigma = + i \sigma \).

**EXCLUDING PARAMETER-DEPENDENCE**

We can always perform a restricted rigid Lorentz transformation (9) which turns an arbitrary set of parameters \( l_j \rightarrow (1, 0, 0, 1) \). Our model is invariant under such transformations so it is sufficient to prove Theorem 1 for this particular choice of parameters. Moreover, by changing, if necessary, the sign of \( L(\vartheta^j, l_j) \) we can always achieve
\[ l_j = (1, 0, 0, 1). \] (13)

Further on we assume the special choice of parameters (13) in which case our Lagrangian (12) takes the form
\[ L(\vartheta^j, l_j) = (\vartheta^0 + \vartheta^3) \wedge (\vartheta^0 \wedge d\vartheta^0 - \vartheta^1 \wedge d\vartheta^1 - \vartheta^2 \wedge d\vartheta^2 - \vartheta^3 \wedge d\vartheta^3). \] (14)

**\( B^2 \)-INVARIANCE**

The crucial step in the proof of Theorem 1 is the observation that our model is invariant under a certain class of local (i.e. with variable coefficients) Lorentz transformations of the coframe. In order to describe these transformations it is convenient to switch from the real coframe \((\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3)\) to the complex coframe \((l, m, \bar{m}, n)\) where
\[ l := \vartheta^0 + \vartheta^3, \quad m := \vartheta^1 + i \vartheta^2, \quad n := \vartheta^0 - \vartheta^3 \]
(here the definition of \( l \) is in agreement with Eq. (13)). In this new notation the Lagrangian (12) and constraint (3) take the form
\[ L(\vartheta^j, l_j) = (1/2) l \wedge (n \wedge dl - \bar{m} \wedge dm - m \wedge d\bar{m}), \] (15)
$g = (1/2)(l \otimes n + n \otimes l - m \otimes \bar{m} - \bar{m} \otimes m)$.  \hfill (16)

Let us perform the linear transformation of the coframe

$$
\begin{pmatrix}
  l \\
  m \\
  \bar{m} \\
  n
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
  l \\
  m + fl \\
  \bar{m} + f l \\
  n + f \bar{m} + fm + |f|^2 l
\end{pmatrix}
$$  \hfill (17)

where $f : M \rightarrow \mathbb{C}$ is an arbitrary scalar function. It is easy to see that both the Lagrangian \(15\) and the constraint \(16\) are taken from subsection 10.122 of \(7\). It is known that of coframes can be identified with cosets of $B^2$, hence the field equation \(17\) is also invariant.

Invariance of the field equation \(17\) means that solutions come in equivalence classes: two coframes are said to be equivalent if there exists a scalar function $f : M \rightarrow \mathbb{C}$ such that the transformation \(17\) maps one coframe into the other. In order to understand the group-theoretic nature of these equivalence classes we note that at every point of the manifold $M$ transformations \(17\) form a subgroup of the restricted Lorentz group. Moreover, this is a very special subgroup: it is the unique nontrivial abelian subgroup of the restricted Lorentz group. Here “unique” is understood as “unique up to conjugation”, whereas the meaning of “nontrivial abelian subgroup” is explained in Appendix C of \(7\). In SL$(2, \mathbb{C})$ notation the subgroup \(17\) is written, up to conjugation, as $B^2 := \left\{ \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \bigg| f \in \mathbb{C} \right\}$ where the notation $B^2$ is taken from subsection 10.122 of \(7\). It is known that $B^2$ is the subgroup of the restricted Lorentz group preserving a given nonzero spinor. Our equivalence classes of coframes can be identified with cosets of $B^2$, hence they are equivalent to spinors.

**Remark 1** The rigorous statement is that a coset of the subgroup $B^2$ is equivalent to a spinor up to choice of sign, i.e. spinors $\zeta$ and $-\zeta$ correspond to the same coset. This non-uniqueness is acceptable because it is known (see, for example, section 19 in \(3\) or section 3-5 in \(2\)) that the sign of a spinor does not have a physical meaning.

**Remark 2** Our construction does not allow us to deal with the zero spinor.

**TECHNICALITIES**

Arguments presented in the previous section show that even though our field equation \(17\) has no spinors appearing in it explicitly, it is, in fact, a first order differential equation for an unknown spinor field. From this point it is practically inevitable that equation \(17\) is, up to a change of variable, Weyl’s equation \(11\).

The actual proof of Theorem \(1\) is carried out by means of a straightforward (but lengthy) calculation. The calculation goes as follows.

The set of coframes has four connected components corresponding to two different orientations, $* (l \wedge m) = \pm i (l \wedge m)$, and to $l$ lying on the forward or backward light cone. We assume for definiteness that we are working with coframes satisfying $* (l \wedge m) = +i(l \wedge m)$ and with $l$ lying on the forward light cone.

It is easy to see that our transformation \(17\) preserves the tensor $l \wedge m$. Moreover, each equivalence class of coframes is completely determined by this tensor. Therefore, it is convenient to identify each equivalence class with a spinor field $\zeta$ in accordance with the formula

$$
(l \wedge m)_{\alpha\beta} = \sigma_{\alpha\beta ab} \zeta^a \zeta^b.
$$  \hfill (18)

The fact that a decomposable polarized antisymmetric tensor is equivalent to the square of a spinor is a standard one and was extensively used in \(2\) \(5\) \(6\) \(7\).

Resolving Eq. \(13\) for the coframe $\{l, m, \bar{m}, n\}$, we get the following formulas: $l$ is given by

$$
l_\alpha = \sigma_{\alpha\beta ab} \zeta^a \zeta^b,
$$  \hfill (19)

$n$ is an arbitrary real (co)vector field satisfying

$$
n \cdot n = 0, \quad l \cdot n = 2,
$$  \hfill (20)

and $m$ is given by

$$
m_\beta = (1/2) \sigma_{\alpha\beta ab} \zeta^a \zeta^b.
$$  \hfill (21)

Formula \(15\) implies

$$
* L(\vartheta^j, l_j) = (1/2) \sqrt{|\det g|} \varepsilon_{\alpha\beta\delta}\ n^\alpha \{\nabla\} \gamma^{\beta} - \bar{m}^\beta \{\nabla\} \gamma^m \delta - m^\beta \{\nabla\} \gamma^m \delta
$$  \hfill (22)

where $\varepsilon$ is the Levi-Civita symbol, $\varepsilon_{0123} := +1$. Here $\{\nabla\}$ stands for the Levi-Civita covariant derivative which should not be confused with the teleparallel covariant derivative $\nabla$. Substituting formulas \(19\) and \(21\) into formula \(22\) and using algebraic properties of Pauli matrices as well as conditions \(20\), we arrive at

$$
* L(\vartheta^j, l_j) = -2i(\bar{\zeta} \sigma_{\alpha\beta} \{\nabla\} \alpha \zeta^a - \zeta^a \sigma_{\alpha\beta} \{\nabla\} \alpha \zeta^b).
$$  \hfill (23)

Formulas \(23\), \(18\) show that our Lagrangian \(15\) is a function of $l \wedge m$ rather than of $l$ and $m$ separately. This is, of course, a consequence of the $B^2$-invariance described in the previous section.

Applying the Hodge star to Eq. \(23\) and comparing with Eq. \(2\), we get

$$
L(\vartheta^j, l_j) = 4L_{\text{Weyl}}(\zeta).
$$  \hfill (24)

We have $|\nabla|(l \wedge m) = 0$, so formula \(18\) implies

$$
|\nabla| \zeta = 0.
$$  \hfill (25)
Comparing Eqs. (10), (11) with Eqs. (20), (19) we conclude that the spinor fields $\xi$ and $\zeta$ coincide up to a complex constant factor of modulus 1. The Weyl Lagrangian is U(1)-invariant, so in Eq. (24) we can replace $\zeta$ by $\xi$, arriving at Eq. (12).

As we have established the identity (24) and as each equivalence class of coframes is equivalent to a spinor field $\zeta$, our field equation (7) is equivalent to

$$i\sigma^\alpha_{ab} (\nabla)_{\alpha} \xi^a = 0. \tag{26}$$

The Weyl equation is U(1)-invariant, so in Eq. (26) we can replace $\zeta$ by $\xi$, arriving at Eq. (1). This completes the proof of Theorem I.

The detailed calculation leading to Eq. (26) will be presented in a separate paper.

**DISCUSSION**

The subject of teleparallelism has a long history dating back to the 1920s. Its origins lie in the pioneering works of É. Cartan, A. Einstein and R. Weitzenböck. Modern reviews of the physics of teleparallelism are given in [11, 12, 13]. Note that Einstein’s original papers on the subject are now available in English translation [14].

However, the construction presented in our paper differs from the traditional one. The crucial difference is our choice of Lagrangian (11) which is parameter-dependent and linear in torsion. The vast majority of publications on the subject deal with parameter-independent Lagrangians quadratic in torsion. One particular parameter-independent quadratic Lagrangian has received special attention as it leads to a teleparallel theory of gravity equivalent (in terms of the resulting metric) to general relativity; the explicit formula for this Lagrangian can be found, for example, in [6, 12, 13, 15, 16].

Another difference is that in teleparallelism it is traditional to vary the coframe without any constraints. This is because teleparallelism is usually viewed as a framework for alternative theories of gravity and in this setting the metric (4) has to be treated as an unknown. We, on the other hand, vary the coframe subject to the metric constraint (3). This is because we view teleparallelism as a framework for the reinterpretation of quantum electrodynamics and in this setting the metric plays the role of a given background.

The question of whether spin can be incorporated into the teleparallel theory of gravity has long been the subject of debate among specialists. The latest contributions can be found in [15, 16] with further references therein. As we do not vary the metric our result is not directly related to this debate but it might motivate a fresh re-examination of the question.

It is interesting that our model exhibits similarities with Caroll–Field–Jackiw electrodynamics [17, 18]. Both involve a covariantly constant covector field: in our model it is the lightlike covector field $l$ which is covariantly constant with respect to the teleparallel connection, whereas in Caroll–Field–Jackiw electrodynamics it is a timelike covector field which is covariantly constant with respect to the Levi-Civita connection.

Our model also exhibits strong similarities with the “bumblebee model” discussed by V.A. Kostelecký [19]: our teleparallel lightlike covector field $l$ plays a role similar to that of the “bumblebee field”. Of course, in our case this covector field has a simple physical interpretation: according to Eq. (11) it is the neutrino current.

The fact that the Weyl equation can be rewritten in tetrad form is not in itself new: this was done by J.B. Griffiths and R.A. Newing [1]. Our new result is the tetrad representation (6) for the Weyl Lagrangian.

The author is grateful to F.W. Hehl, Y. Itin and J.B. Griffiths for stimulating discussions.

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[1] J. B. Griffiths and R. A. Newing, J. Phys. A 3, 269 (1970), ISSN 0305-4470.
[2] A. D. King and D. Vassiliev, Classical Quantum Gravity 18, 2317 (2001), ISSN 0264-9381.
[3] D. Vassiliev, Gen. Relativity Gravitation 34, 1239 (2002), ISSN 0001-7701.
[4] D. Vassiliev, Ann. Phys. (8) 14, 231 (2005), ISSN 0003-8040.
[5] V. Pasic and D. Vassiliev, Classical Quantum Gravity 22, 3961 (2005), ISSN 0264-9381.
[6] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne’eman, Phys. Rep. 258, 1 (1995), ISSN 0370-1573.
[7] A. L. Besse, Einstein manifolds, vol. 10 of Ergebnisse der Mathematik und ihrer Grenzgebiete (Springer-Verlag, Berlin, 1987), ISBN 3-540-15279-2.
[8] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Quantum Electrodynamics, vol. 4 of Course of theoretical physics (Pergamon Press, Oxford, 1982), 2nd ed., ISBN 0-80-26503-0.
[9] R. F. Streeter and A. S. Wightman, *PCT, spin and statistics, and all that*, Princeton Landmarks in Physics (Princeton University Press, Princeton, NJ, 2000), ISBN 0-691-07062-8, corrected third printing of the 1978 edition.
[10] F. W. Hehl, J. Nitsch, and P. von der Heyde, in General relativity and gravitation, Vol. 1 (Plenum, New York, 1980), pp. 329–355.
[11] T. Sauer, Field equations in teleparallel spacetime: Einstein’s fernparallelismus approach towards unified field theory (2004), URL http://arxiv.org/abs/physics/0405112
[12] F. Gronwald and F. W. Hehl, in Quantum gravity (Erice, 1995) (World Sci. Publishing, River Edge, NJ, 1996), vol. 10 of Sci. Cult. Ser. Phys., pp. 148–198.
[13] U. Muench, F. Gronwald, and F. W. Hehl, Gen. Relativity Gravitation 30, 933 (1998), ISSN 0001-7701.
[14] A. Unzicker and T. Case, *Translation of Einstein’s at-
A temt of a unified field theory with teleparallelism (2005), URL http://arxiv.org/abs/physics/0503046.

[15] E. W. Mielke, Phys. Rev. D (3) 69, 128501, 3 (2004), ISSN 0556-2821.

[16] Y. N. Obukhov and J. G. Pereira, Phys. Rev. D (3) 69, 128502, 3 (2004), ISSN 0556-2821.

[17] S. M. Carroll, G. B. Field, and R. Jackiw, Phys. Rev. D (3) 41, 1231 (1990), ISSN 0556-2821.

[18] Y. Itin, Phys. Rev. D (3) 70, 025012, 6 (2004), ISSN 0556-2821.

[19] V. A. Kostelecký, Phys. Rev. D (3) 69, 105009, 20 (2004), ISSN 0556-2821.