Detection of incompatible properties

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Abstract

In a typical two-slits experiments we face the question whether it is possible or not to attain knowledge about properties incompatible with Which-Slit property together with the measurement of the final impact point. A wide family of solutions is concretely found and an ideal experiment realizing such a detection is designed, relatively to the detection of three such incompatible properties. In the case of four incompatible properties, general conditions for the existence of solutions are singled out and a particular family of solutions is provided.

1 Introduction

Standard Quantum Theory [1] forbids simultaneous measurement of non-commuting observables; therefore, in a double-slit experiment it is not generally possible to measure which-slit property and the final impact point, since these properties are represented by non-commuting operator. This notwithstanding, Which-Slit knowledge can be inferred by measuring a suitable property \( T \), compatible with the measurement on the final impact point. A lot of devices exploit this idea to obtain indirect knowledge about Which-Slit property (Einstein’s recoiling slit [2], the light-electron scattering scheme of Feynman [3], the micro-maser apparatus of Englert, Scully and Walther [4]).

For the same reason, the direct measurement of the three cartesian components of a spin-\( \frac{1}{2} \) particle is denied. However, in [6] (VAA) a procedure is described allowing to make inferences about three such cartesian components. The ideal experiment proposed by VAA works as follows: one of the three non-commuting observables, \( \sigma_x, \sigma_y \) or \( \sigma_z \), is measured, on a system suitably prepared, by means of an apparatus which leaves the entire system in an eigenstate of the measured observable; after such a spin measurement, a suitable observable \( A \) is measured, having the property that the outcome of the measured spin can be inferred from the outcome of \( A \), without knowing which spin had been previously measured; this method yields inferences such as: if \( \sigma_x \) has been measured the outcome is \( +\frac{1}{2} \), if \( \sigma_y \) has been measured the outcome is \( -\frac{1}{2} \) and if \( \sigma_z \) has been measured the outcome is \( \frac{1}{2} \).

In [7] the problem of detecting more than two incompatible properties is investigated on a theoretical ground, within the framework of the double-slit experiment. Nisticò shows that, in a double-slit experiment, properties, incompatible with Which-slit property, can be detected together with the knowledge of which slit each particle passes through and together with the measurement of the point of impact on the final screen.
This kind of detection is made possible by the fact that besides the position of the centre-of-mass-motion, the system possesses further degrees of freedom. As a consequence, the Hilbert space describing the entire system can be decomposed as $\mathcal{H}_I \otimes \mathcal{H}_{II}$, where $\mathcal{H}_I$ is the Hilbert space used to represent the observable position, and $\mathcal{H}_{II}$ is the Hilbert space used to represent the observable arising from the further degrees of freedom. The detection of Which Slit property $E$ is obtained by means of an observable represented by a particular projection operator $T$ acting on $\mathcal{H}_{II}$. The possibility of detecting an incompatible property $G$ is provided by the existence of an observable represented by another projection operator $Y$ acting on $\mathcal{H}_{II}$, but which can be measured together with $T$. A systematic investigation establishes \cite{7} that the existence of such an observable (projection operator) depends on the dimension of space $\mathcal{H}_I$.

A real experiment realizing such a detection is not yet performed, but finding further solutions increases the possibility of a concrete realization of such an experiment. In this perspective, in the present work we present the results of an investigation aimed to find more general possibilities of detecting more than two incompatible properties, by using the same method proposed in \cite{7}.

Section 2 is devoted to the detection of three incompatible properties: Which slit property, an incompatible one and the final impact point. We translate such a problem into mathematical terms. We treat in details the case $\dim(\mathcal{H}_I) = 6$, neglecting the case $\dim(\mathcal{H}_I) < 6$ which provides no solution or correlated ones \cite{7}. A wide family of solutions is provided. We conclude the section by proposing an ideal experiment which realizes the detection at issue.

In section 3 the mathematical details of the derivation of the solutions proposed in section 2 is presented.

In section 4 the question whether two mutually incompatible properties, $G$ and $L$, both incompatible with Which Slit property $E$, can be detected, together with the measurement of the final impact point (four incompatible properties), is treated. In particular, we show that such a question has an affirmative answer. As in the previous case, the existence of solutions depends on dimension of space $\mathcal{H}_I$: we find a particular solution for $\dim(\mathcal{H}_I) = 10$, nevertheless, in such a case the properties $L$ and $G$ turn out to be correlated.

The details of the derivation of a family of this kind of solutions is presented in an appropriate section, namely section 5.

2 Simultaneous detection of incompatible properties

We briefly introduce the mathematical formalism to describe a two-slit experiment, allowing the detection of Which Slit property $E$ and an incompatible property, together with the final impact point.

The physical system consists of a localizable particle whose position observable is represented, at time $t$ in Heisenberg picture, by an operator $Q(t)$ of a suitable Hilbert space $\mathcal{H}_I$. Let us suppose that, besides the position of the centre-of-mass motion, the system possesses further degrees of freedom, described in a different Hilbert space $\mathcal{H}_{II}$. As a consequence, the Hilbert space describing the entire system can be decomposed as $\mathcal{H}_I \otimes \mathcal{H}_{II}$. Let us suppose that the Hamiltonian operator $H$ of the entire system is essentially independent of the degrees of freedom described by $\mathcal{H}_{II}$, so that we may assume the ideal case $H = H_I \otimes 1_{II}$.
Which Slit property is a position observable, so that it is represented by a projection operator $E$ acting on $\mathcal{H}_I$. One can make inferences about Which Slit property by means of measurements of an observable represented by a projection operator $T$, acting on $\mathcal{H}_{II}$, whose outcome is correlated with the outcome of $E$.

Let $G$ be another property, represented by a projection operator acting on $\mathcal{H}_I$, incompatible with Which Slit property; if this new property can be detected by means of an operator $Y$ acting on $\mathcal{H}_{II}$ and if the detections of $T$ and $Y$ can be carried out together, then we can make simultaneous inferences about $E$ and $G$.

The localization property “the particle passes through slit 1” (which-slit property, WS) is represented in the complete Hilbert space by the operator $E = E_I \otimes 1_{II}$ where $E_I$ is a localization operator in $\mathcal{H}_I$.

If at time $t_1$ the particle passes through the screen supporting the slits, we denote by $t_2$ the time of the impact on the final screen. Given any interval $\Delta$ on the final screen we denote by $F(\Delta)$ the projection operator representing the event “the particle hits the final screen in a point within $\Delta$”, hence concerning time $t_2$ ($F(\Delta) = F_{t_2}(\Delta)$). $F_{t_1}(\Delta)$ must have the form $F_{t_1}(\Delta) = J_{t_1} \otimes 1_{II}$, because it is a localization operator at time $t_1$; equation $[F_{t_1}(\Delta), E] = 0$ holds but we cannot assume $[F(\Delta), E] = 0$. This notwithstanding, since the Hamiltonian $H$ has the form $H = H_I \otimes 1_{II}$ then $F(\Delta)$ is a projection operator of the form $F(\Delta) = J \otimes 1_{II}$.

Let $G = G_I \otimes 1_{II}$ be a projection operator representing a property incompatible with WS property $E$, i.e $[E, G] \neq 0$.

The possibility of detecting $G$ and $E$, together with $F(\Delta)$, is ensured by detectors $T$ and $Y$ of $E$ and $G$ respectively, of the form $T = 1_I \otimes T_{II}$ and $Y = 1_I \otimes Y_{II}$, satisfying $[T, Y] = 0$ in such a way that $T$ and $Y$ can be measured together giving simultaneous information about $E$ and $G$.

The problem at issue can be set out as follows:

**Problem** (P) Given the property $E = E_I \otimes 1_{II}$ we want to find a projection operator $G_I$ of $\mathcal{H}_I$, $T_{II}$ and $Y_{II}$ of $\mathcal{H}_{II}$, and a state vector $\Psi \in \mathcal{H}_I \otimes \mathcal{H}_{II}$ such that the following conditions are satisfied:

(C.1) $[E, G] \neq 0$ i.e $[E_I, G_I] \neq 0$
(C.2) $[T, E] = 0$ and $T\Psi = E\Psi$
(C.3) $[Y, G] = 0$ and $Y\Psi = G\Psi$
(C.4) $[T, Y] = 0$
(C.5) $\Psi \neq E\Psi \neq 0$ and $\Psi \neq G\Psi \neq 0$
(C.6) $[T, F(\Delta)] = [Y, F(\Delta)] = 0$

Condition (C.1) and equation $[T, E] = 0$ in (C.2) express the incompatibility between properties represented by $E$ and $G$, and compatibility between properties represented by $T$ and $E$, respectively; equation $T\Psi = E\Psi$ in (C.2) entails an entanglement between $T$ and $E$, indeed it is mathematically equivalent to state that conditional probabilities satisfy

$$p(T \mid E) = \frac{\langle \Psi \mid TE\Psi \rangle}{\langle \Psi \mid E\Psi \rangle} = 1 = \frac{\langle \Psi \mid TE\Psi \rangle}{\langle \Psi \mid T\Psi \rangle} = p(E \mid T),$$

in other words, outcome 1 (0) for $T$ reveals the passage of the particle through slit 1 (2). Condition (C.3) has a similar interpretation; (C.4) states the simultaneous measurability.
of the two supplementary detections, $T$ and $Y$; condition (C.5) is added to exclude solutions corresponding to the unitesting case that $\Psi$ is an eigenvector of $E$ or $G$; conditions (C.6) states that the measurement of $T$ (and $Y$) can be performed together with the final impact point and, taking into account the form of the operators involved, it is automatically satisfied.

### 2.1 Matrix representation

Conditions (C.1)-(C.6) can be expressed in a more useful form if the following matrix representation is adopted.

Let $(e_1, e_2, e_3, \ldots; r_1, r_2, r_3, \ldots)$ be an orthonormal basis of $\mathcal{H}_I$, formed by eigenvectors of $E_I$, such that $E_I e_k = e_k$ and $E_I r_j = 0$ for all $k$ and $j$. Therefore, every vector $\Psi \in \mathcal{H}_I \otimes \mathcal{H}_{II}$ can be uniquely decomposed as $\Psi = \sum_j x_j \otimes x_j + \sum_k r_k \otimes y_k$, where $x_j, y_k \in \mathcal{H}_{II}$. Now, condition (C.4) $[T, Y] = 0$ implies that four projection operators, $A_i$ ($i = 1, \ldots, 4$), of $\mathcal{H}_{II}$ exist such that $\sum^4_i A_i = \mathbb{1}$, $T_{II} = A_1 + A_2$ and $Y_{II} = A_1 + A_3$. Thereby, we choose to represent vectors $x_j, y_k \in \mathcal{H}_{II}$ as column vectors $x_j = (a_j, b_j, c_j, d_j)^t$ and $y_k = (\alpha_k, \beta_k, \gamma_k, \delta_k)^t$, where $a_j = A_1 x_j$, $b_j = A_2 x_j$, $c_j = A_3 x_j$, $d_j = A_4 x_j$ and $\alpha_j = A_1 y_j$, $\beta_j = A_2 y_j$, $\gamma_j = A_3 y_j$, $\delta_j = A_4 y_j$. Then $\Psi \in \mathcal{H}$ shall be represented as a column vector

$$\Psi = (x_1, x_2, \ldots, x_j, \ldots; y_1, y_2, \ldots, y_k, \ldots)^t$$

Once introduced suitable matrices $P = (p_{ij})$, $U = (u_{ij})$, $V = (v_{ij})$ and $Q = (q_{ij})$, we can write a linear operator $G_I$ of $\mathcal{H}_I$ in the form $G_I = \begin{pmatrix} P & U \\ V & Q \end{pmatrix}$; let $X$ be linear operators of $\mathcal{H}_{II}$, then, according to such a representation, any factorized linear operator $G_I \otimes X$ can be identified with the matrix

$$G_I \otimes X = \begin{pmatrix} p_{ij}X & u_{ij}X \\ v_{ij}X & q_{ij}X \end{pmatrix}$$  \hspace{1cm} (1)$$

Since $E = E_I \otimes \mathbb{1}$, $T = \mathbb{1} \otimes T_{II}$, $G = G_I \otimes \mathbb{1}$ and $Y = \mathbb{1} \otimes Y_{II}$, then we have

$$E = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 1 & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \end{pmatrix} \hspace{1cm} T = \begin{pmatrix} T_{II} & 0 & \ldots & 0 & 0 & \ldots \\ 0 & T_{II} & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & T_{II} & 0 & \ldots \\ 0 & 0 & \ldots & 0 & T_{II} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$G = \begin{pmatrix} p_{11}1 & p_{12}1 & \ldots & u_{11}1 & u_{12}1 & \ldots \\ p_{21}1 & p_{22}1 & \ldots & u_{21}1 & u_{22}1 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{11}1 & v_{12}1 & \ldots & q_{11}1 & q_{12}1 & \ldots \\ v_{21}1 & v_{22}1 & \ldots & q_{21}1 & q_{22}1 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \end{pmatrix} \hspace{1cm} Y = \begin{pmatrix} Y_{II} & 0 & \ldots & 0 & 0 & \ldots \\ 0 & Y_{II} & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & Y_{II} & 0 & \ldots \\ 0 & 0 & \ldots & 0 & Y_{II} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \end{pmatrix}$$
From condition (C.2) \( T\Psi = E\Psi \), we obtain \( x_i = (a_i, b_i, 0, 0) \) and \( y_j = (0, 0, \gamma_j, \delta_j) \). Hence, condition (C.3) \( Y\Psi = G\Psi \) is equivalent to

\[
\begin{align*}
(i - A) & \quad \sum_i p_{ji} a_i = a_j \\
(ii - B) & \quad \sum_i p_{ji} b_i = 0 \\
(iii - C) & \quad \sum_i v_{ki} a_i = 0 \\
(iv - D) & \quad \sum_i v_{ki} b_i = 0
\end{align*}
\]

(2)

2.2 Solutions of problem \( (P) \)

So far we have established general constraints to be satisfied by every solution of the problem, independently of ranks of \( E, G \) and \( A_i \), with \( i = 1, \ldots, 4 \), and therefore of dimensions of space \( \mathcal{H}_I \) and \( \mathcal{H}_{II} \).

We restrict the search to the case that the two slit are symmetrical: this leads to exclude odd dimension of \( \mathcal{H}_I \) and, moreover, to assume that \( \text{rank}(L) = \text{rank}(I - L) = \dim(\mathcal{H}_I)/2 \).

In the rest of this section, we present the results of an investigation aimed to find general families of solutions, whose existence, according to [7], depends on dimension of \( \mathcal{H}_I \). The detailed analysis of the derivation of such solutions is a rather technical matter; so, they are displaced in the next section; here we only sketch the idea.

From (2), by using basic notions of linear algebra, it can be shown that, in order to attain non-correlated solutions, the triples of component-vectors, \( \{\delta_1, \delta_2, \delta_3\} \), \( \{\gamma_1, \gamma_2, \gamma_3\} \), \( \{b_1, b_2, b_3\} \), and \( \{\delta_1, \delta_2, \delta_3\} \), are generated by just one vector. Briefly, (ii-B) and \( U \neq 0 \) imply one of the three vectors \( y_1, y_2, y_3 \), say \( y_1 \), is a linear combination of the remaining two

\[
\begin{align*}
\gamma_1 &= \lambda_2 \gamma_2 + \lambda_3 \gamma_3 \\
\delta_1 &= \lambda_2 \delta_2 + \lambda_3 \delta_3
\end{align*}
\]

(3)

In system (iv-D) we get

\[
\begin{align*}
(q_{j1} \lambda_2 + q_{j2}) \gamma_2 + (q_{j1} \lambda_3 + q_{j3}) \gamma_3 &= \gamma_j \\
(q_{j1} \lambda_2 + q_{j2}) \delta_2 + (q_{j1} \lambda_3 + q_{j3}) \delta_3 &= 0
\end{align*}
\]

(4)

If vectors \( \delta_2 \) and \( \delta_3 \) are linearly independent, then second equation in (4) implies \( (q_{j1} \lambda_2 + q_{j2}) = (q_{j1} \lambda_3 + q_{j3}) = 0 \), so that first equation in (4) yields \( \gamma_j = 0 \) for all \( j \), hence \( y_j = (0, 0, 0, \delta_j)^t \); in a similar way, \( b_1 \) and \( b_2 \) linearly independent imply \( x_j = (0, b_j, 0, 0)^t \). If a solution of \( (P) \) exists, then \( G\Psi = Y\Psi = 0 \) (meaningless solution). If a solution exists such that \( \delta_2, \delta_3 \) are linearly independent and \( b_2, b_3 \) linearly dependent, then \( x_j = (a_j, b_j, 0, 0)^t \) and \( y_j = (0, 0, 0, \delta_j)^t \); in this case equation \( YTY = Y\Psi \) holds, which is equivalent to say that conditional probabilities satisfy

\[
p(T | Y) = \frac{\langle \Psi | T\Psi \rangle}{\langle \Psi | Y\Psi \rangle} = 1
\]

and this means that each time a particle is sorted by \( T \), then it is certainly sorted by \( Y \); therefore for all eventual solution, property \( G \) must be correlated with Which-Slit property \( E \).
Using equations 3 in (iv-D) we have that non-correlated solutions exist if and only if \( \delta_2 \) and \( \delta_3 \) (\( b_2 \) and \( b_3 \)) are linearly dependent, say \( \delta_3 = \lambda \delta_2 \) (\( b_3 = \mu b_2 \)); thereby, we find \((u_{j1}\lambda_3 + u_{j2}) = -\lambda(u_{j1}\lambda_3 + u_{j3}) = 0\). Moreover, (iv-D) implies \( \gamma_2 \) and \( \gamma_3 \) can be neither linearly independent nor dependent as \( \gamma_3 = \lambda \gamma_2 \); a constant \( \lambda_4 \neq \lambda \) must exists such that \( \gamma_3 = \lambda \gamma_2 \) (and similarly \( a_3 = \lambda a_2 \)).

As a consequence we also attain the form \( Q \) and \( U \) have to do in order to satisfy (ii-B) and (iv-D); nevertheless, self-adjointness of \( Q \) yields to rather difficult calculation; hence we prefer make easier the search with the choice \( \gamma_2 = 0 \).

Among general solutions of (2), we select those which satisfy \( G_1^* = G_1 \) only; we get

\[
P = \begin{pmatrix}
  p & -\mu_2 \left(p - \frac{|\mu_3|^2}{1 + |\mu_3|^2}\right) & \mu_3(1 - p) \\
  -\bar{\mu}_2 \left(p - \frac{|\mu_3|^2}{1 + |\mu_3|^2}\right) & |\mu_2|^2 \left(p - \frac{|\mu_3|^2}{1 + |\mu_3|^2}\right) & |\mu_3|^2 \bar{\mu}_2 \left(p - \frac{|\mu_3|^2}{1 + |\mu_3|^2}\right) \\
  \bar{\mu}_3(1 - p) & \bar{\mu}_3 \mu_2 \left(p - \frac{|\mu_3|^2}{1 + |\mu_3|^2}\right) & 1 - |\mu_3|^2(1 - p)
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
  q & -\lambda_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) & \lambda_3(1 - q) \\
  -\bar{\lambda}_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) & |\lambda_2|^2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) & \lambda_3 \bar{\lambda}_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) \\
  \bar{\lambda}_3(1 - q) & \bar{\lambda}_3 \lambda_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) & 1 - |\lambda_3|^2(1 - q)
\end{pmatrix},
\]

\[
U = \begin{pmatrix}
  u & -\lambda_2 u & -\lambda_3 u \\
  -\bar{\mu}_2 u & \lambda_2 \bar{\mu}_2 u & \lambda_3 \bar{\mu}_2 u \\
  -\bar{\mu}_3 u & \lambda_3 \bar{\mu}_3 u & \lambda_3 \bar{\mu}_3 u
\end{pmatrix},
\]

\[
\Psi = (x_1, x_2, x_3; y_1, y_2, y_3)^t,
\]

where

\[
x_1 = (\mu_3 a_3, -\frac{\mu_2}{|\mu_3|^2} + 1 b_2, 0, 0)^t \quad y_1 = (0, 0, \lambda_3 \gamma_3, \frac{\lambda_2}{|\lambda_3|^2 + 1} \delta_2, 0)^t
\]

\[
x_2 = (0, b_2, 0, 0)^t \quad y_2 = (0, 0, 0, \lambda_2)^t
\]

\[
x_3 = (a_3, -\frac{\mu_2}{|\mu_3|^2} + 1 b_2, 0, 0)^t \quad y_3 = (0, 0, \gamma_3, -\frac{\lambda_3}{|\lambda_3|^2 + 1} \delta_2)^t
\]

Self-adjointness of \( G_1 \) implies that \( V = U^t \), moreover \( \lambda = -\frac{\lambda_3}{|\lambda_3|^2 + 1} \) and \( \mu = -\frac{\mu_3}{|\mu_3|^2 + 1} \).

Such a solution completely solves the problem if numbers \( p, u, q \neq 0 \) can be chosen in such a manner that \( G \) turns out to be idempotent. It is easily shown that idempotence implies

1. \( |\mu_3|^2 < p < \frac{|\mu_3|^2}{1 + |\mu_3|^2} + \frac{1}{1 + |\mu_3|^2 + |\mu_3|^2}, \)

2. \( u = e^{i\theta} \sqrt{\frac{|\mu_3|^2}{1 + |\mu_3|^2} - \frac{|\mu_2|^2 + |\mu_3|^2}{1 + |\lambda_2|^2 + |\lambda_3|^2}} \), \( q = \frac{1 - (1 + |\mu_2|^2 + |\mu_3|^2)}{1 + |\lambda_2|^2 + |\lambda_3|^2} \), \( \bar{\lambda}_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) \), \( \lambda_3 \bar{\lambda}_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) \), \( \bar{\lambda}_3(1 - q) \), \( \bar{\lambda}_3 \lambda_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) \), \( 1 - |\lambda_3|^2(1 - q) \)
Therefore for every real number \( \frac{|\mu_3|^2}{1 + |\mu_3|^2} < p < \frac{|\mu_3|^2}{1 + |\mu_3|^2} + \frac{1}{1 + |\mu_2|^2 + |\mu_3|^2} \), every \( \theta \in \mathbb{R} \) and every \( \mu_2, \mu_3, \lambda_2, \lambda_3 \in \mathbb{C} \) we have a solution of \((P)\).

No constraint is imposed to \( \text{rank}(G_I) \), i.e. to the trace of the projection operator \( G \), hence these parameters are not all independent.

For every choice we attain a solution, \( G_I^1, G_I^2, \ldots \); then we can state there are several properties \( G^1, G^2, \ldots \) incompatible with Which-Slit property but detectable together with it. However, we notice that, taking into account (C.3), every \( G^i \) transforms \( \Psi \) in \( Y\Psi \).

Our solutions form a rather wide family; however, it is not exhaustive, because of the choice \( \gamma_2 = 0 \); if the case \( \gamma_2 \neq 0 \) is taken into account, the problem would be completely solved. The family singled out by Nisticò in [7] is just a subfamily of the present one corresponding to the particular choice \( \mu_3 = \lambda_3 = 0 \) and \( \lambda_2 = \mu_2 = 1 \).

In next subsection an ideal experiment, not concretely performable, that realizes the detection at issue, is proposed.

### 2.3 Ideal experiment

Until now the treatment has been carried out on a theoretical ground only. Now we describe an ideal apparatus following the results of the previous sections.

The experimental set-up corresponds to the particular solution with parameters \( \mu_2 = \lambda_2 = \sqrt{3} \) and \( \mu_3 = \lambda_3 = 1 \).

The system consists of an electrically neutral particle of spin \( \frac{3}{2} \); the position of its centre-of-mass is described in space \( \mathcal{H}_I \). The further degrees of freedom, described in \( \mathcal{H}_{II} \), concerns the spin of the particle.

Let us suppose that, after crossing the screen with the slits, each particle passes through a non-uniform magnetic field, with gradient along the direction \( z \) (fig.1). The beam splits into four beams and the deflection of each particle depend of the component of the spin in the direction of the magnetic field gradient. Hence the measurement of the amount of deflection of the particle indicates the value of its spin component.

Let \( A \) be the projection operator representing the event “the spin component in the \( z \)-direction is \( \frac{3}{2} \)”. Similarly we define operators \( B, C \) and \( D \) associated to the spin components \( \frac{1}{2}, -\frac{1}{2} \) and \( -\frac{3}{2} \), respectively. We denote their respective eigenvectors relative to the eigenvalue 1 by \( | \frac{3}{2} \rangle, | \frac{1}{2} \rangle, | -\frac{1}{2} \rangle \) and \( | -\frac{3}{2} \rangle \). By \( \psi_i, i = 1, \ldots, 6 \) we denote orthonormal eigenfunctions of \( \mathcal{H}_I \) such that \( \psi_1, \psi_2, \psi_3 \) lie in \( \mathcal{E}_I \mathcal{H}_I \) and \( \psi_4, \psi_5, \psi_6 \) lie

![Figure 1: Experimental set-up](image-url)
in \((I - E_l)\mathcal{H}_1\). Let the state vector of the entire system be

\[
\psi = \frac{1}{3} \left\{ (\psi_1 + \psi_2) | \frac{3}{2} \rangle + \left( \frac{2}{\sqrt{3}} \psi_1 + \frac{2}{\sqrt{3}} \psi_2 - \frac{2}{\sqrt{15}} \psi_3 \right) | \frac{1}{2} \rangle \right\} + \\
+ \frac{1}{3} \left\{ (\psi_4 + \psi_6) | -\frac{1}{2} \rangle + \left( \frac{2}{\sqrt{3}} \psi_4 + \frac{2}{\sqrt{3}} \psi_5 - \frac{2}{\sqrt{15}} \psi_6 \right) | -\frac{3}{2} \rangle \right\},
\]

which, within our representation, coincides with

\[
\Psi = \frac{1}{3} \left( 1, \frac{\sqrt{3}}{2}, 0, 0, 1, 0, 0, -\frac{\sqrt{3}}{2}, 0, 0, 1, \frac{\sqrt{3}}{2}, 0, 0, 1, 0, 0, 1, -\frac{\sqrt{3}}{2} \right)^t.
\]

According with the results of previous section, with respect to this state vector there exists a Which-Slit detector \(T = I \otimes (A + B)\) and a detector \(Y = I \otimes (A + C)\) of a property \(G = G_I \otimes I\), incompatible with property \(E; G_I\), with the particular choice, is the projection operator

\[
G_I = \\
\begin{pmatrix}
\frac{2}{3} & -\frac{1}{2\sqrt{3}} & \frac{1}{6\sqrt{3}} & -\frac{1}{2\sqrt{15}} & -\frac{1}{6\sqrt{15}} \\
-\frac{1}{2\sqrt{3}} & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{15}} & \frac{1}{2\sqrt{15}} \\
\frac{1}{6\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{2} & -\frac{1}{6\sqrt{3}} & \frac{1}{2\sqrt{15}} \\
\frac{1}{6\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{15}} & \frac{1}{2\sqrt{15}} \\
\frac{1}{6\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{15}} & \frac{1}{10\sqrt{3}} & \frac{1}{10\sqrt{3}} \\
\end{pmatrix}
\]

Therefore, the ideal experiment just described allows to make inferences about three non-commuting observables: the position of the final impact point is inferred from a direct measurement of \(F(\Delta)\); Which-Slit property \(E\) is inferred by the outcome of detector \(T\) and property \(G\) is inferred by the outcome of detector \(Y\).

We have to notice the difference between the meaning of our results with respect to that obtained by VAA. According to this latter, from the outcome of \(A\), the outcome of the performed spin-component measurement can be retrodicted; while the remaining inferences cannot be considered as detections; furthermore inferences can be drawn only under the hypothesis that the spin measurement actually performed leaves the system in an eigenstate of the measured observable.

We stress the ideal character of the experiment just described. In order to make it meaningful, we would be able to identify the observable represented by \(G\) by a physical point of view. Nevertheless, practical difficulties of creating the initial entangled state \(\Psi\) are the real obstacles in realizing the designed experiment. Hence, even if a real experiment for simultaneous detection of Which-Slit property, an incompatible one and the final impact point is not yet performed, a wider family of solutions is a contribution to increase the possibility of a concrete realization.
3 Derivation of a family of solutions

This section is devoted to find a detailed derivation of the family of solutions presented in previous section.

We are seeking for solutions such that the rank of $L$ is 3, so that $i, j, k, l$ take values in $\{1, 2, 3\}$. No constraint is imposed to the ranks of $A_i$, with $i = 1, 2, 3, 4$, and hence to the dimension of $H_{ij}$. If $H_{ij} = 6$, then $\Psi = (x_1, x_2, x_3; y_1, y_2, y_3)^t$, so that $P, U, V$ and $Q$ are $3 \times 3$ matrices. Since $U \neq 0$, (ii-B) implies that one of the three vectors, $y_2, y_3$ is linear combination of the remaining two, say $y_1$, so that complex numbers $\lambda_2, \lambda_3$ must exist such that

$$
\begin{align*}
\gamma_1 &= \lambda_2 \gamma_2 + \lambda_3 \gamma_3 \\
\delta_1 &= \lambda_2 \delta_2 + \lambda_3 \gamma_3
\end{align*}
$$

(5)

In system (iv-D) we get

$$
\begin{align*}
(q_j \lambda_2 + q_j \delta_2) \gamma_2 + (q_j \lambda_3 + q_j \delta_3) \gamma_3 &= \gamma_j \\
(q_j \lambda_2 + q_j \delta_2) \delta_2 + (q_j \lambda_3 + q_j \delta_3) \delta_3 &= 0
\end{align*}
$$

(6)

If vectors $\delta_2$ and $\delta_3$ are linearly independent, then second equation in (6) implies 

$$(q_j \lambda_2 + q_j \delta_2) = (q_j \lambda_3 + q_j \delta_3) = 0$$

so that first equation in (6) yields $\gamma_j = 0$ for all $j$, hence $y_j = t(0, 0, 0)$. In a similar way, $b_1$ and $b_2$ linearly independent imply $x_j = t(0, b_j, 0).$ If a solution of ($P$) exists, then $G \Psi = Y \Psi = 0$. A detailed analysis in (7) shows that the only case which can lead meaningful solutions without correlation is $b_2, b_3$ linearly dependent and $\delta_2, \delta_3$ linearly dependent.

Let us suppose $\delta_3 = \lambda \delta_2$ with $\lambda \neq 0$ (and similarly $b_3 = \mu b_2$ with $\mu \neq 0$). Thereby, in (ii-B) we get

$$
\begin{align*}
(u_j \lambda_2 + u_j \delta_2) \gamma_2 + (u_j \lambda_3 + u_j \delta_3) \gamma_3 &= 0 \\
[(u_j \lambda_2 + u_j \delta_2) + \lambda (u_j \lambda_3 + u_j \delta_3)] \delta_2 &= 0
\end{align*}
$$

(7)

If $\delta_2 = 0$ then second equation in (7) is satisfied and $\delta_j = 0$ for all $j$, so that $y_j = t(0, 0, 0).$ In a similar way, $b_2 = 0$ implies $b_j = 0$ for all $j$, so that $x_j = t(a_j, 0, 0, 0)$. Given a state $\Psi$, we obtain the following implications:

1. $b_2 = 0$ and $\delta_2 = 0$ imply $x_j = t(a_j, 0, 0, 0)$ and $y_j = t(0, 0, \gamma_j, 0)$; in this case if a solution exists, then $G \Psi = \Psi$. Therefore meaningful solutions cannot exist;

2. $b_2 \neq 0$ and $\delta_2 = 0$ imply $x_j = t(a_j, b_j, 0, 0)$ and $y_j = t(0, 0, \gamma_j, 0)$; if a solution exists, then $T \Psi = T' \Psi$, that is to say property $G$ must be correlated with WS property $E$;

3. $b_2 = 0$ and $\delta_2 \neq 0$ imply $x_j = t(a_j, 0, 0, 0)$ and $y_j = t(0, 0, \gamma_j, \delta_j)$; if a solution exists, then $TY \Psi = T \Psi$. As in the previous case property $G$ must be correlated with WS property $E$;

4. $b_2 \neq 0$ and $\delta_2 \neq 0$ imply $x_j = t(a_j, b_j, 0, 0)$ and $y_j = t(0, 0, \gamma_j, \delta_j)$; this is the only case that can lead to solution without correlation.

Hence, we are interested only in case (4).

Since $\delta_2 \neq 0$, second equation in (7) is satisfied if and only if

$$
(u_j \lambda_2 + u_j \delta_2) = -\lambda (u_j \lambda_3 + u_j \delta_3)
$$

(8)
so that first equation in (7) becomes
\[ (u_{j1} \lambda_3 + u_{j3})(\gamma_3 - \lambda \gamma_2) = 0. \] (9)

If we suppose \( \gamma_3 = \lambda \gamma_2 \), then in (6) we get
\[
\begin{cases}
[(q_{j1} \lambda_2 + q_{j2}) + \lambda(q_{j1} \lambda_3 + q_{j3})] \gamma_2 = \gamma_j \\
[(q_{j1} \lambda_2 + q_{j2}) + \lambda(q_{j1} \lambda_3 + q_{j3})] \delta_2 = 0.
\end{cases}
\] (10)

Since \( \delta_2 \neq 0 \), second equation of (10) implies that \( (q_{j1} \lambda_2 + q_{j2}) = -\lambda(q_{j1} \lambda_3 + q_{j3}) \); hence \( \gamma_j = 0 \), for all \( j \), follows from the first equation in (10). Analogous reasoning for equations in (i-A) leads to \( a_j = 0 \), for all \( j \); hence, if a solution exists, it corresponds to the uninteresting case \( G \Psi = 0 \).

If in (9) we consider all possibilities for \( \gamma_1 \) and \( \gamma_2 \), we get:

a. \( \gamma_3 = \lambda \gamma_2 \),
b. \( \gamma_2 \) and \( \gamma_3 \) linearly independent,
c. \( \gamma_3 = \lambda \gamma_2 \),
d. \( \gamma_2 = 0 \) and \( \gamma_3 \neq 0 \).

Now we draw the consequences of (b), (c) and (d), since in case (a) eventual solutions are uninteresting.

**Case (b).**

If vectors \( \gamma_2 \) and \( \gamma_3 \) are linearly independent, then (6) becomes
\[
\begin{cases}
(q_{j1} \lambda_2 + q_{j2}) \gamma_2 + (q_{j1} \lambda_3 + q_{j3}) \gamma_3 = \gamma_j \\
[(q_{j1} \lambda_2 + q_{j2}) + \lambda(q_{j1} \lambda_3 + q_{j3})] \delta_2 = 0.
\end{cases}
\] (11)

Since \( \delta_2 \neq 0 \), then second equation in (11) implies \( (q_{j1} \lambda_2 + q_{j2}) = -\lambda(q_{j1} \lambda_3 + q_{j3}) \), so that first equation in (11) yields \( (q_{j1} \lambda_3 + q_{j3})(\gamma_3 - \lambda \gamma_2) = \gamma_j \). Using this relation we get
\[
\begin{cases}
(q_{j1} \lambda_3 + q_{j3})(\gamma_3 - \lambda \gamma_2) = \lambda_2 \gamma_2 + \lambda_3 \gamma_3 \\
(q_{j1} \lambda_3 + q_{j3})(\gamma_3 - \lambda \gamma_2) = \gamma_2 \\
(q_{j1} \lambda_3 + q_{j3})(\gamma_3 - \lambda \gamma_2) = \gamma_3.
\end{cases}
\] (12)

Second equation in (12) implies
\[
\begin{cases}
(q_{j1} \lambda_3 + q_{j3}) = 0 \\
-\lambda(q_{j1} \lambda_3 + q_{j3}) = 1.
\end{cases}
\] (13)

Then system (13) is impossible.

**Case (c).**

If vectors \( \gamma_2 \) and \( \gamma_3 \) are linearly independent no solution exists. Hence we may suppose that vectors \( \gamma_2 \) and \( \gamma_3 \) are linearly dependent. Nevertheless, if \( \gamma_3 = \lambda \gamma_2 \) we proved that eventual solutions lead to correlated properties, so we may suppose the existence of a complex number \( \lambda_3 \neq \lambda \), such that \( \gamma_3 = \lambda \gamma_2 \). As a consequence (iv-D) becomes
\[
\begin{cases}
[(q_{j1} \lambda_2 + q_{j2}) + \lambda_4(q_{j1} \lambda_3 + q_{j3})] \gamma_2 = \gamma_j \\
[(q_{j1} \lambda_2 + q_{j2}) + \lambda(q_{j1} \lambda_3 + q_{j3})] \delta_2 = 0.
\end{cases}
\] (14)

10
Since \( \delta_2 \neq 0 \), second equation in (14) implies \((q_{j1} \lambda_2 + q_{j2}) = -\lambda(q_{j1} \lambda_3 + q_{j3})\), so that first equation in (14) yields \((q_{j1} \lambda_3 + q_{j3})(\lambda_1 - \lambda)\gamma_2 = \gamma_j\).

Straightforward calculations lead to a matrix \(Q\) of the form

\[
Q = \begin{pmatrix}
q_{11} - \frac{\lambda(\lambda_2 + \lambda_3 \lambda_4)}{\lambda_1 - \lambda} & -q_{11} \lambda_2 & \frac{\lambda_1 + \lambda_3 \lambda_4}{\lambda_1 - \lambda} - q_{11} \lambda_3 \\
q_{21} - \frac{\lambda_1 + \lambda_3 \lambda_4}{\lambda_2 - \lambda} & q_{21} \lambda_2 & \frac{\lambda_2 + \lambda_3 \lambda_4}{\lambda_2 - \lambda} - q_{21} \lambda_3 \\
q_{31} - \frac{\lambda_1 + \lambda_3 \lambda_4}{\lambda_3 - \lambda} & q_{31} \lambda_2 & \frac{\lambda_3 + \lambda_3 \lambda_4}{\lambda_3 - \lambda} - q_{31} \lambda_3
\end{pmatrix}.
\]

Nevertheless, self-adjointness of \(Q\) yields to rather difficult calculation.

**Case (d).**

Now we suppose \(\gamma_2 = 0\) and \(\gamma_3 \neq 0\) in order to make easier the search of solutions. Equations in (iv-D) become

\[
\begin{align*}
(q_{j1} \lambda_3 + q_{j3})\gamma_3 &= \gamma_j \\
[(q_{j1} \lambda_2 + q_{j2}) + \lambda(q_{j1} \lambda_3 + q_{j3})] \delta_2 &= 0.
\end{align*}
\]

Thus, first equation of (15) gets

\[
\begin{align*}
(q_{j1} \lambda_3 + q_{j3})\gamma_3 &= \lambda_3 \gamma_3 \\
(q_{21} \lambda_3 + q_{23})\gamma_3 &= 0 \\
(q_{31} \lambda_3 + q_{33})\gamma_3 &= \gamma_3
\end{align*}
\]

and, since \(\gamma_3 \neq 0\), this is equivalent to say

\[
\begin{align*}
q_{j1} &= \lambda_3 (1 - q_{11}) \\
q_{23} &= -\lambda_3 q_{21} \\
q_{31} &= 1 - \lambda_3 q_{31}
\end{align*}
\]

Similarly, second equation in (15) implies

\[
(q_{j1} \lambda_2 + q_{j2}) = -\lambda(q_{j1} \lambda_3 + q_{j3}),
\]

then

\[
\begin{align*}
q_{j2} &= -\lambda \lambda_3 - \lambda_2 q_{j1} \\
q_{22} &= -\lambda_2 q_{21} \\
q_{32} &= -\lambda - \lambda_2 q_{31}
\end{align*}
\]

By imposing self-adjointness, we find that \(q_{11} = q\) is a real number, \(\lambda = -\frac{\lambda_3 q_3}{|\lambda_3|^2 + 1}\) and

\[
Q = \begin{pmatrix}
q & -\lambda_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) & \lambda_3 (1 - q) \\
-\lambda_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) & |\lambda_2|^2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) & \lambda_3 \lambda_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) \\
\lambda_3 (1 - q) & \lambda_3 \lambda_2 \left(q - \frac{|\lambda_3|^2}{1 + |\lambda_3|^2}\right) & 1 - |\lambda_3|^2 (1 - q)
\end{pmatrix}.
\]
Taking into account (8), matrix $U$ has the form

$$U = \begin{pmatrix} u_{11} & -u_{11} \lambda_2 & -u_{11} \lambda_3 \\ u_{21} & -u_{21} \lambda_2 & -u_{21} \lambda_3 \\ u_{11} & -u_{31} \lambda_2 & -u_{31} \lambda_3 \end{pmatrix}.$$ 

Analogous reasonings, for systems (i-A) and (iii-C), provides matrices $P$ and $V$ of a similar form; however, self-adjointness of $G$ implies that $V = U'$, in such a manner that we attain the solution presented in section 2.2.

4 Detection of four incompatible properties

We already noticed the difference between the meaning of our results with respect to that obtained by VAA; moreover, they affirm [6] that, according to their method, it is no possible to produce inferences (such that described in [4]) for more than three observable. Since our method runs in a quite different matter, maybe it shall admit solutions.

In this section we treat the question whether two incompatible properties, $G$ and $L$, can be detected together with the further incompatible property $E$ (Wich-Slit property) and together with measurement of the final impact point, in the same kind of ideal experiment.

More precisely, we seek for a concrete Hilbert space $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$ where $E$ is the projection operator acting on $\mathcal{H}_I$ representing which-slit property, such that a concrete state $\Psi$, and concrete projection operators $G$ and $L$ representing mutually incompatible properties can be found in such a manner that:

- Property $E$ can be detected by means of a detector $T$, acting on $\mathcal{H}_{II}$
- Property $G$ can be detected by means of a detector $Y$, acting on $\mathcal{H}_{II}$
- Property $L$ can be detected by means of a detector $W$, acting on $\mathcal{H}_{II}$
- The three detections can be carried out together, i.e. $[T,Y] = 0$, $[T,W] = 0$, $[Y,W] = 0$.

The method used for such a research is similar to that presented in previous section. In the rest of this section we formulate the question in formal way as problem $(P)$; as before, we adopt a matrix representation and in this framework we establish the constraints to be satisfied in order that solutions exist. Then we present concrete solutions.

The systematic research of solution is in suitable section 6. We set out such a research in order to answer the question whether non-correlated solutions do exist or not. Hence, the case $\text{dim}(\mathcal{H}_I) = 10$ is investigated; a detailed analysis show that solutions exist, however, the three detections turn out to be always correlated.

4.1 Mathematical Formalism

Let $G = G_I \otimes 1_{II}$ and $L = L_I \otimes 1_{II}$ be properties incompatible with WS property $E$. Detection of both $G$, $L$ and which-slit property $E$ is possible if, with respect to the same state vector $\Psi$, there exists a which-slit detector $T = 1_I \otimes T_{II}$ of $E$, a detector $Y = 1_I \otimes Y_{II}$ of $G$ and a detector $W = 1_I \otimes W_{II}$ of $L$. Let detectors satisfy the
condition \([T, Y] = [T, W] = [Y, W] = 0\) in such a manner that \(T, Y\) and \(W\) can be measured together; hence they provide simultaneous informations about \(E, G\) and \(L\). Formally, we are asking if the following problem has solution

\[(P)\] Given the property \(E = E_I \otimes 1_{II}\) we have to find

- projection operators \(G_I\) and \(L_I\) of \(\mathcal{H}_I\)
- projection operators \(T_{II}, Y_{II}\) and \(W_{II}\) of \(\mathcal{H}_{II}\)
- a state vector \(\Psi \in \mathcal{H}_I \otimes \mathcal{H}_{II}\)

such that the following conditions are satisfied:

\[\text{(C.1)} \quad [E, G] \neq 0 \quad \text{i.e.} \quad [E_I, G_I] \neq 0\]
\[\text{(C.2)} \quad [E, L] \neq 0 \quad \text{i.e.} \quad [E_I, L_I] \neq 0\]
\[\text{(C.3)} \quad [L, G] \neq 0 \quad \text{i.e.} \quad [L_I, G_I] \neq 0\]
\[\text{(C.4)} \quad [T, E] = 0 \quad \text{and} \quad T\Psi = E\Psi\]
\[\text{(C.5)} \quad [Y, G] = 0 \quad \text{and} \quad Y\Psi = G\Psi\]
\[\text{(C.6)} \quad [W, L] = 0 \quad \text{and} \quad W\Psi = L\Psi\]
\[\text{(C.7)} \quad [T, Y] = 0\]
\[\text{(C.8)} \quad [T, W] = 0\]
\[\text{(C.9)} \quad [Y, W] = 0\]
\[\text{(C.10)} \quad \Psi \neq E\Psi \neq 0, \quad \Psi \neq G\Psi \neq 0 \quad \text{and} \quad \Psi \neq L\Psi \neq 0\]

Conditions (C.1)-(C.3) are equivalent to state that properties represented by projection operators \(E, G\) and \(L\) are mutually non-compatible. In the remaining items, the fact that the commutators are zero is expression of the compatibility (hence simultaneous knowledge) between properties represented by the projection operators involved. Moreover, for a given state \(\Psi\) if equation \(T\Psi = E\Psi\) in (C.4) (resp. \(Y\Psi = G\Psi\) in (C.5) and \(W\Psi = L\Psi\) in (C.6)) holds, then it is also possible to detect which slit each particle passes through (resp. to detect \(G\) or \(I - G\), to detect \(L\) or \(I - L\)), by means of \(T\) (resp. \(Y\) and \(W\)) ; indeed, the formula

\[
p(T \mid E) = \frac{\langle \Psi \mid TE\Psi \rangle}{\langle \Psi \mid E\Psi \rangle} = 1 = \frac{\langle \Psi \mid TE\Psi \rangle}{\langle \Psi \mid T\Psi \rangle} = p(E \mid T)
\]

represents the conditional probabilities allowing us to infer the passage of particle 1 through slit 1 from the occurrence of outcome 1 for \(T\): in this sense \(E\) and \(T\) are correlated properties (similarly in (C.5) and (C.6)). Condition (C.10) is added to exclude solutions corresponding to uninteresting case that \(\Psi\) is eigenvector of \(E, G\) or \(L\).

We introduce matrix representation, to make easier our task.

### 4.2 Matrix Representation

\(\mathcal{H}_I\) representation decribed in section II is general so that it can be adopted here without modifications. Therefore, projection operators \(E_I, G_I\) and \(L_I\) in (C.1)-(C.10) must have the following representations:

\[
E_I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_I = \begin{pmatrix} P & U \\ V & Q \end{pmatrix}, \quad L_I = \begin{pmatrix} M & Z \\ W & N \end{pmatrix}
\] (16)
where $U \neq 0$ and $Z \neq 0$; $G_I = G_I^* = G_I^2$ and $L_I = L_I^* = L_I^2$. Constraints $U \neq 0$ and $Z \neq 0$ above are equivalent to $[E_I, G_I] \neq 0$ and $[E_I, L_I] \neq 0$ required by (C.1) and (C.2).

Following the same argument of previous section for the representation of $\mathcal{H}_{II}$, eight projection operators, $A_i$ ($i = 1, \ldots, 8$) of $\mathcal{H}_{II}$ must exist, such that \[ \sum_i A_i = 1, \quad T_{II} = A_1 + A_2 + A_3 + A_5, \quad Y_{II} = A_1 + A_2 + A_4 + A_6, \quad W_{II} = A_1 + A_3 + A_4 + A_7 \] (fig. 2). Then we choose to represent every vector $x \in \mathcal{H}_{II}$ as a column vector $x = (a, b, c, d, e, f, g, h)^t$ where $a = A_1 x, b = A_2 x, \ldots, h = A_8 x$. As a consequence, projection operators $T_{II}$, $Y_{II}, W_{II}$ in (C.1)-(C.10) must satisfy the following constraints

\[
T_{II} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\quad Y_{II} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

and

\[
W_{II} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\] (17)

The $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$ representation described in section II can be adopted here taking into account that every vector $\Psi$ in the product space $\mathcal{H}_I \otimes \mathcal{H}_{II}$ shall be represented as a column vector

\[
\Psi = (x_1, x_2, \ldots, x_i, \ldots, y_1, y_2, \ldots, y_j, \ldots)
\]

where $x_i = (a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i)$ and $y_j = (\alpha_j, \beta_j, \gamma_j, \delta_j, \epsilon_j, \zeta_j, \eta_j, \theta_j)$ and $a_i = A_1 x_i, b_i = A_2 x_i, \ldots, h_i = A_8 x_i, \alpha_j = A_1 y_j, \beta_j = A_2 y_j, \ldots, \theta_j = A_8 y_j$. 

\[ \]
4.3 Constraints for $\Psi, G_I, L_I$

According to (11), (16) and (17) $E = E_I \otimes 1$ and $T = 1 \otimes T_{II}$ are represented in matrix form as

$$E = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 1 & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$T = \begin{pmatrix} T_{II} & 0 & \ldots & 0 & 0 & \ldots \\ 0 & T_{II} & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \ldots & T_{II} & 0 & \ldots \\ 0 & 0 & \ldots & 0 & T_{II} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}$$

(18)

Condition (C.4) $T\Psi = E\Psi$ implies $d_i = f_i = g_i = h_i = 0$ and $\alpha_j = \beta_j = \gamma_j = \epsilon_j = 0$ so that $x_i$ and $y_j$ take the form

$$x_i = (a_i, b_i, c_i, 0, e_i, 0, 0, 0)$$

$$y_j = (0, 0, 0, \delta_j, 0, \zeta_j, \eta_j, \theta_j)$$

(19)

Further constraints are imposed by condition (C.5) $Y \Psi = G \Psi$. Since $G_I = \begin{pmatrix} P & U \\ V & Q \end{pmatrix}$ then projection operators $G = G_I \otimes 1$ and $Y = 1 \otimes Y_{II}$ are represented as

$$G = \begin{pmatrix} p_{11} & p_{12} & \ldots & u_{11} & u_{12} & \ldots \\ p_{21} & p_{22} & \ldots & u_{21} & u_{22} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ v_{11} & v_{12} & \ldots & q_{11} & q_{12} & \ldots \\ v_{21} & v_{22} & \ldots & q_{21} & q_{22} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$Y = \begin{pmatrix} Y_{II} & 0 & \ldots & 0 & 0 & \ldots \\ 0 & Y_{II} & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \ldots & Y_{II} & 0 & \ldots \\ 0 & 0 & \ldots & 0 & Y_{II} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}$$

(20)

and

$$Y \Psi = ^t(\cdots, a_i, b_i, 0, 0, 0, 0, 0, 0, 0, \delta_j, 0, \zeta_j, 0, 0, \cdots)$$

$$G \Psi = ^t(z_1, z_2, \cdots, z_i, \cdots; w_1, w_2, \cdots, w_j, \cdots)$$

(21)

where

$$z_i = \begin{pmatrix} \sum_k P_{ik} a_k \\ \sum_k P_{ik} b_k \\ \sum_k P_{ik} c_k \\ \sum_k u_{ik} \delta_k \\ \sum_k u_{ik} \epsilon_k \\ \sum_k u_{ik} \eta_k \\ \sum_k u_{ik} \theta_k \end{pmatrix}$$

and

$$w_j = \begin{pmatrix} \sum_k v_{jk} a_k \\ \sum_k v_{jk} b_k \\ \sum_k v_{jk} c_k \\ \sum_k q_{jk} \delta_k \\ \sum_k q_{jk} \epsilon_k \\ \sum_k q_{jk} \eta_k \\ \sum_k q_{jk} \theta_k \end{pmatrix}$$

(22)
so that, taking into account \(^{20}\) and \(^{22}\), condition (C.5) \(Y\Psi = G\Psi\) can be explicited as

\[
(i - A) \quad \begin{cases} 
\sum_k p_{ik} a_k = a_i \\
\sum_k p_{ik} b_k = b_i \\
\sum_k p_{ik} c_k = 0 \\
\sum_k p_{ik} e_k = 0 
\end{cases} \quad (i - B) \quad \begin{cases} 
\sum_k u_{ik} \delta_k = 0 \\
\sum_k u_{ik} \zeta_k = 0 \\
\sum_k u_{ik} \eta_k = 0 \\
\sum_k u_{ik} \theta_k = 0 
\end{cases}
\]

\[
(iii - C) \quad \begin{cases} 
\sum_k v_{ik} a_k = 0 \\
v_{ik} b_k = 0 \\
\sum_k v_{ik} c_k = 0 \\
\sum_k v_{ik} e_k = 0 
\end{cases} \quad (iv - D) \quad \begin{cases} 
\sum_k q_{ik} \delta_k = \delta_i \\
\sum_k q_{ik} \zeta_k = \zeta_i \\
\sum_k q_{ik} \eta_k = 0 \\
\sum_k q_{ik} \theta_k = 0 
\end{cases}
\]

Further constraints are imposed by condition (C.6) \(Y\Psi = G\Psi\). Since \(L_I = \begin{pmatrix} M & Z \\ W & N \end{pmatrix}\)
then, projection operators \(L = L_I \otimes 1\) and \(W = 1 \otimes W_{II}\) are represented as

\[
L = \begin{pmatrix} 
m_{11} & m_{12} & \ldots & z_{11} & z_{12} & \ldots \\
m_{21} & m_{22} & \ldots & z_{21} & z_{22} & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
w_{11} & w_{12} & \ldots & n_{11} & n_{12} & \ldots \\
w_{21} & w_{22} & \ldots & n_{21} & n_{22} & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{pmatrix} \quad \text{and} \quad 
W = \begin{pmatrix} 
W_{II} & 0 & \ldots & 0 & 0 & \ldots \\
0 & W_{II} & \ldots & 0 & 0 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
0 & 0 & \ldots & W_{II} & 0 & \ldots \\
0 & 0 & \ldots & 0 & W_{II} & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]

\[
W\Psi = ^t (\ldots, a_i, 0, c_i, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \delta_j, 0, 0, \eta_j, 0, \ldots) \\
L\Psi = ^t (s_1, s_2, \ldots, s_i, \ldots, t_1, t_2, \ldots, t_j, \ldots)
\]

where

\[
s_i = \begin{pmatrix} 
\sum_k m_{ik} a_k \\
\sum_k m_{ik} b_k \\
\sum_k m_{ik} c_k \\
\sum_k m_{ik} \delta_k \\
\sum_k m_{ik} \zeta_k \\
\sum_k m_{ik} \eta_k \\
\sum_k m_{ik} \theta_k 
\end{pmatrix} \quad \text{and} \quad 
t_j = \begin{pmatrix} 
\sum_k w_{jk} a_k \\
\sum_k w_{jk} b_k \\
\sum_k w_{jk} c_k \\
\sum_k w_{jk} \delta_k \\
\sum_k w_{jk} \zeta_k \\
\sum_k w_{jk} \eta_k \\
\sum_k w_{jk} \theta_k 
\end{pmatrix}.
\]

Then condition (C.6) \(W\Psi = L\Psi\), can be written as

\[
(i - A') \quad \begin{cases} 
\sum_k m_{ik} a_k = a_i \\
\sum_k m_{ik} b_k = 0 \\
\sum_k m_{ik} c_k = c_i \\
\sum_k m_{ik} e_k = 0 
\end{cases} \quad (i - B') \quad \begin{cases} 
\sum_k z_{ik} \delta_k = 0 \\
z_{ik} \zeta_k = 0 \\
z_{ik} \eta_k = 0 \\
z_{ik} \theta_k = 0 
\end{cases}
\]

\[
(iii - C') \quad \begin{cases} 
\sum_k w_{ik} a_k = 0 \\
w_{ik} b_k = 0 \\
w_{ik} c_k = 0 \\
w_{ik} e_k = 0 
\end{cases} \quad (iv - D') \quad \begin{cases} 
\sum_k n_{ik} \delta_k = \delta_i \\
n_{ik} \zeta_k = 0 \\
n_{ik} \eta_k = \eta_j \\
n_{ik} \theta_k = 0 
\end{cases}
\]
5 A family of solutions

Until now we have established general constraints to be satisfied by any solution of the problem, independently of the ranks of matrices, and then of dimensions of the spaces \( \mathcal{H}_I \) and \( \mathcal{H}_{II} \). Here we present a concrete solution of the problem, whose derivation is displaced in next section; our research is not at all exhaustive: we analyze a particular situation and, according to it, the three detections turn out to be always correlated.

We notice that, if in correspondence of a given state \( \Psi \) satisfying (19), matrices \( G \) and \( L \), solutions of (23) and (27) exist such that \( [E, G], [G, L] \) and \( [L, E] \) are non zero, \( G = G^* = G^2 \) and \( L = L^* = L^2 \), then, all (C-4)-(C-9) are automatically satisfied.

As in previous case, we restrict our search to the case that the two slit are symmetrical: this leads to exclude odd dimensions of \( \mathcal{H}_I \) and, moreover, to assume that \( \text{rank}(L) = \text{rank}(I - L) = \dim(\mathcal{H}_I)/2 \).

We shall proceed as follows: at the beginning, no constrain is imposed about the dimension of \( \mathcal{H}_I \). We restrict our search by working with subsystems (ii-B) and (iv-D) of (23) and (ii-B') and (iv-D') of (27), rather than with the entire systems (23) and (27); then we take analogous results for the remaining subsystems (which have analogous forms). By using the elementary notion of linear combination, we make some hypothesis of linear dependence or independence among the vector-components of \( x \) and \( y \); since no hypothesis is made about the dimension of \( \mathcal{H}_I \), we can suppose \( i \) running from 1 to \( n \).

Our task would be more meaningful if we attain non-correlated solutions of (P); for such a reason, at this level of the search, we neglect the correlated ones we gradually find. So, we reduce systems (23) and (27) in a more useful form, which makes evident that, with our assumptions, if solutions exist, they are always correlated. Thereby, we shall see that concrete solutions of (C-1)-(C-10) exist; at this point, our task is made easier by fixing the dimension of \( \mathcal{H}_I \), \( \dim(\mathcal{H}_I) = 10 \), and searching solutions corresponding to a particular state vector \( \Psi \) (see (44)). Taking into account self-adjointness of \( G_I = \begin{pmatrix} P & U \\ V & Q \end{pmatrix} \) and \( L_I = \begin{pmatrix} M & Z \\ W & N \end{pmatrix} \) we get

\[
Q = \begin{pmatrix}
q & -\alpha_2 (q - \frac{1}{n}) & -\alpha_3 q & -\beta_4 \frac{q}{\alpha_3} & -\beta_5 \frac{q}{\alpha_3} \\
-\overline{\alpha_2} (q - \frac{1}{n}) & \Lambda_3 + |\alpha_2|^2 (q - \frac{1}{n}) & \alpha_2 \overline{\alpha_2} (q - \frac{1}{n}) & -\beta_4 \Lambda_3 & -\beta_5 \Lambda_3 \\
-\overline{\alpha}_3 q & -\overline{\alpha}_3 \alpha_2 (q - \frac{1}{n}) & |\alpha_3|^2 q & \overline{\alpha}_3 \beta_4 \frac{q}{\alpha_3} & \overline{\alpha}_3 \beta_5 \frac{q}{\alpha_3} \\
-\overline{\beta}_4 \frac{q}{\alpha_3} & -\overline{\beta}_4 \Lambda_3 & \overline{\beta}_4 \alpha_2 \frac{q}{\alpha_3} & |\beta_4|^2 \Lambda & \overline{\beta}_4 \beta_5 \Lambda \\
-\overline{\beta}_5 \frac{q}{\alpha_3} & -\overline{\beta}_5 \Lambda_3 & \overline{\beta}_5 \alpha_2 \frac{q}{\alpha_3} & \overline{\beta}_5 \beta_4 \Lambda & |\beta_5|^2 \Lambda
\end{pmatrix},
\]

where

- \( \Gamma = \left( 1 + |\alpha_3|^2 \right) + \left( |\beta_4|^2 + |\beta_5|^2 \right) \left( 1 + |\alpha_2|^2 + |\alpha_3|^2 \right) \),
- \( \Lambda_2 = \frac{|\alpha_3|^2}{2}, \Lambda_3 = \frac{1 + |\alpha_3|^2}{2} \) and \( \Lambda = \Lambda_2 + \Lambda_3; \)
- \( N = \begin{pmatrix}
n & -\alpha_2 (n - \frac{1}{n}) & -\alpha_3 (n - \frac{1}{n}) & -\beta_4 \frac{\alpha_2}{\alpha_3} - \lambda_4 \frac{\alpha_2}{\alpha_3} & -\beta_5 \frac{\alpha_2}{\alpha_3} - \lambda_5 \frac{\alpha_2}{\alpha_3} \\
-\overline{\alpha_2} (n - \frac{1}{n}) & \Delta_3 + |\alpha_2|^2 (n - \frac{1}{n}) & \alpha_2 \overline{\alpha_2} (n - \frac{1}{n}) & -\beta_4 \Delta_3 + \lambda_4 \overline{\alpha_2} \frac{\alpha_2}{\alpha_3} & -\beta_5 \Delta_3 + \lambda_5 \overline{\alpha_2} \frac{\alpha_2}{\alpha_3} \\
-\overline{\alpha}_3 (n - \frac{1}{n}) & \alpha_3 \overline{\alpha}_3 (n - \frac{1}{n}) & |\alpha_3|^2 (n - \frac{1}{n}) & \overline{\alpha}_3 \beta_4 \frac{\alpha_2}{\alpha_3} - \lambda_4 \overline{\alpha}_3 \frac{\alpha_2}{\alpha_3} & \overline{\alpha}_3 \beta_5 \frac{\alpha_2}{\alpha_3} - \lambda_5 \overline{\alpha}_3 \frac{\alpha_2}{\alpha_3} \\
-\overline{\beta}_4 \frac{\alpha_2}{\alpha_3} & -\overline{\beta}_4 \frac{\alpha_2}{\alpha_3} & \overline{\beta}_4 \Delta_3 + \alpha_2 \overline{\beta}_4 \frac{\alpha_2}{\alpha_3} & |\beta_4|^2 \Delta + |\lambda_4|^2 \Sigma & \overline{\beta}_4 \beta_5 \Delta + \overline{\lambda}_4 \overline{\lambda}_5 \Sigma \\
-\overline{\beta}_5 \frac{\alpha_2}{\alpha_3} & -\overline{\beta}_5 \frac{\alpha_2}{\alpha_3} & \overline{\beta}_5 \Delta_3 + \alpha_2 \overline{\beta}_5 \frac{\alpha_2}{\alpha_3} & \overline{\beta}_5 \beta_4 \Delta + \overline{\lambda}_5 \overline{\lambda}_4 \Sigma & |\beta_5|^2 \Delta + |\lambda_5|^2 \Sigma
\end{pmatrix}.\)
where

\[ \lambda_4 = \frac{\alpha_4 \omega_4}{\beta_4 (1 + |\alpha_2|^2 + |\alpha_3|^2)} - \lambda_0 \frac{\omega_4}{\beta_4} \]

\[ \Delta = \left( 1 + |\alpha_2|^2 \right) + \left( |\lambda_4|^2 + |\lambda_5|^2 \right) \left( 1 + |\alpha_2|^2 + |\alpha_3|^2 \right) \]

\[ \Sigma_2 = \frac{1 + |\alpha_2|^2}{\Delta}, \Sigma_3 = \frac{|\alpha_3|^2}{\Delta} \text{ and } \Sigma = \Sigma_2 + \Sigma_3; \]

and

\[ U = \begin{pmatrix}
    u & -\alpha_2u & -\alpha_3u & 0 & 0 \\
    -\pi_2u & \pi_2\alpha_2 & \pi_3\alpha_3 & 0 & 0 \\
    -\pi_3u & \pi_3\alpha_2 & \pi_3\alpha_3 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix}.\]

Matrices \( P, M \) and \( Z \) have the same form of \( Q, N \) and \( U \) respectively, with \( p, m, z, \alpha_i \) and \( b_j \) in place of \( q, n, u, \alpha_i \) and \( \beta_j \), where \( i = 2, 3 \) and \( j = 4, 5 \); \( A_i, B_i, C, D \) take the place of \( \Lambda_i, \Sigma_i, \Gamma, \Delta \) and are defined in analogous manner; moreover, \( V = U^t \) and \( W = \overline{Z} \). We notice that \( a_i, b_j, l_j, \alpha_i, \beta_j, \lambda_j \), with \( i = 2, 3 \) and \( j = 4, 5 \), are constant complex numbers arising from the linear dependence among the vector-components of \( x_i \) and \( y_j \), where \( i, j = 1, \ldots, 5 \), as we shall see in the next section.

In order to solve the problem, such solution must be idempotent. By imposing idempotence we find that

1. \[ \frac{A_2}{1 + |\alpha_2|^2 + |\alpha_3|^2} < p < \frac{A_2 + 1}{1 + |\alpha_2|^2 + |\alpha_3|^2} \]
2. \[ \frac{A_2 + B_3}{1 + |\alpha_2|^2 + |\alpha_3|^2} < m < \frac{A_2 + B_3 + 1}{1 + |\alpha_2|^2 + |\alpha_3|^2} \]
3. \[ u = e^{i\theta_1} \sqrt{\frac{(p - \frac{1}{p})(1 - 2A_3) - (p - \frac{1}{p})^2 (1 + |\alpha_2|^2 + |\alpha_3|^2) + A_3 |\alpha_2|^2 + |\alpha_3|^2}{1 + |\alpha_2|^2 + |\alpha_3|^2}} \]
4. \[ z = e^{i\theta_2} \sqrt{\frac{(m - \frac{1}{m})(1 - 2(A_3 - B_3)) - (m - \frac{1}{m})^2 (1 + |\alpha_2|^2 + |\alpha_3|^2) - (A_3 - B_3)^2 (1 + |\alpha_2|^2 + |\alpha_3|^2)}{1 + |\alpha_2|^2 + |\alpha_3|^2}} \]
5. \[ q = \frac{1 + A_2 + A_3 - p (1 + |\alpha_2|^2 + |\alpha_3|^2)}{1 + |\alpha_2|^2 + |\alpha_3|^2} \]
6. \[ n = \frac{1 + A_2 + B_3 + A_3 - m (1 + |\alpha_2|^2 + |\alpha_3|^2)}{1 + |\alpha_2|^2 + |\alpha_3|^2} \]

where \( \theta_1 \) and \( \theta_2 \) are real numbers. Our family of solutions completely solves the problem if it satisfies \( |G, L| \neq 0 \).

Therefore, for every real number \( \frac{A_2}{1 + |\alpha_2|^2 + |\alpha_3|^2} < p < \frac{A_2 + 1}{1 + |\alpha_2|^2 + |\alpha_3|^2} \), every real number \( \frac{A_2 + B_3}{1 + |\alpha_2|^2 + |\alpha_3|^2} < m < \frac{A_2 + B_3 + 1}{1 + |\alpha_2|^2 + |\alpha_3|^2} \), every \( \alpha_2, \alpha_3, b_1, b_2, l_2, \alpha_2, \alpha_3, \beta_1, \beta_2, \lambda_5 \in \mathbb{C} \) we have a solution of \( (P) \). Since \( G_f \) and \( L_f \) are projection operators and no constraint is imposed to their traces, i.e. to \( \text{Rank}(G_f) \) and \( \text{Rank}(L_f) \), these parameters are not all independent.
For instance, the following solution of \((P)\)

\[
G = \begin{pmatrix}
\frac{11}{12} & -\frac{1}{20} & -\frac{11}{12} & -\frac{1}{8} & -\frac{1}{8} & \frac{\sqrt{3}}{9} & -\frac{\sqrt{3}}{9} & -\frac{\sqrt{3}}{9} & 0 & 0 \\
-\frac{1}{36} & \frac{5}{18} & \frac{1}{36} & \frac{1}{4} & \frac{1}{4} & -\frac{\sqrt{7}}{9} & \frac{\sqrt{7}}{9} & \frac{\sqrt{7}}{9} & 0 & 0 \\
-\frac{11}{72} & \frac{1}{36} & \frac{11}{72} & \frac{1}{8} & \frac{1}{8} & -\frac{\sqrt{7}}{9} & \frac{\sqrt{7}}{9} & \frac{\sqrt{7}}{9} & 0 & 0 \\
-\frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 \\
\frac{\sqrt{7}}{9} & -\frac{\sqrt{7}}{9} & -\frac{\sqrt{7}}{9} & 0 & 0 & \frac{19}{72} & -\frac{5}{36} & -\frac{10}{72} & -\frac{1}{8} & -\frac{1}{8} \\
-\frac{\sqrt{7}}{9} & \frac{\sqrt{7}}{9} & \frac{\sqrt{7}}{9} & 0 & 0 & -\frac{5}{36} & \frac{7}{18} & \frac{5}{36} & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{\sqrt{7}}{9} & \frac{\sqrt{7}}{9} & \frac{\sqrt{7}}{9} & 0 & 0 & -\frac{19}{72} & \frac{5}{36} & \frac{19}{72} & \frac{1}{8} & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8}
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
\frac{67}{456} & -\frac{5}{228} & \frac{5}{456} & \frac{1}{152} & -\frac{3}{152} & -\frac{43}{152} & -\frac{4}{19\sqrt{3}} & -\frac{4}{19\sqrt{3}} & -\frac{4}{19\sqrt{3}} & 0 & 0 \\
-\frac{5}{228} & \frac{31}{111} & -\frac{31}{228} & \frac{27}{76} & -\frac{7}{76} & -\frac{4}{19\sqrt{3}} & -\frac{4}{19\sqrt{3}} & \frac{4}{19\sqrt{3}} & \frac{4}{19\sqrt{3}} & 0 & 0 \\
\frac{5}{456} & -\frac{31}{228} & \frac{159}{456} & \frac{51}{152} & \frac{29}{152} & -\frac{4}{19\sqrt{3}} & \frac{4}{19\sqrt{3}} & \frac{4}{19\sqrt{3}} & 0 & 0 \\
-\frac{3}{152} & -\frac{27}{76} & \frac{51}{152} & \frac{89}{152} & \frac{9}{152} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{43}{152} & -\frac{7}{76} & -\frac{29}{152} & \frac{9}{152} & \frac{129}{152} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{19\sqrt{3}} & -\frac{4}{19\sqrt{3}} & -\frac{4}{19\sqrt{3}} & 0 & 0 & \frac{3}{8} & -\frac{1}{4} & -\frac{33}{152} & -\frac{3}{152} & -\frac{43}{152} \\
-\frac{4}{19\sqrt{3}} & \frac{4}{19\sqrt{3}} & \frac{1}{19\sqrt{3}} & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & \frac{7}{152} & -\frac{7}{152} & \frac{7}{152} \\
-\frac{4}{19\sqrt{3}} & \frac{4}{19\sqrt{3}} & \frac{1}{19\sqrt{3}} & 0 & 0 & -\frac{33}{152} & \frac{7}{76} & \frac{81}{152} & \frac{51}{152} & -\frac{29}{152} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{152} & -\frac{7}{76} & -\frac{29}{152} & \frac{9}{152} & \frac{9}{152} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{152} & -\frac{7}{76} & -\frac{29}{152} & \frac{9}{152} & \frac{9}{152}
\end{pmatrix}
\]

\[
\Psi = (x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5)^t
\]
where

\[
\begin{align*}
x_1 &= (-\frac{1}{4}a_5, 0, -\frac{1}{2}c_5, 0, \frac{1}{4}e_4 + 2e_5, 0, 0, 0, 0, 0)^t \\
x_2 &= (-\frac{2}{3}a_5, 0, \frac{1}{3}c_5, 0, e_4 + e_5, 0, 0, 0, 0, 0)^t \\
x_3 &= (\frac{1}{2}a_5, 0, -\frac{2}{3}c_5, 0, -\frac{2}{3}e_4 + e_5, 0, 0, 0, 0, 0)^t \\
x_4 &= (a_5, 0, -\frac{2}{3}c_5, 0, e_4, 0, 0, 0, 0, 0, 0)^t \\
x_5 &= (a_5, 0, c_5, 0, e_5, 0, 0, 0, 0, 0, 0)^t \\
y_1 &= (0, 0, 0, 0, -\frac{1}{4}\delta_5, 0, 0, -\frac{1}{4}\eta_5, \frac{1}{4}\theta_4 + 2\theta_5)^t \\
y_2 &= (0, 0, 0, 0, -\frac{2}{3}\delta_5, 0, 0, \frac{1}{3}\eta_5, \theta_4 + \theta_5)^t \\
y_3 &= (0, 0, 0, 0, \frac{1}{3}\delta_5, 0, 0, -\frac{2}{3}\eta_5, -\frac{2}{3}\theta_4 + \theta_5)^t \\
y_4 &= (0, 0, 0, \delta_5, 0, 0, -\frac{2}{3}\eta_5, \theta_4)^t \\
y_5 &= (0, 0, 0, \delta_5, 0, 0, \eta_5, \theta_5)^t
\end{align*}
\]

is obtained in correspondence with the particular choice \(a_2 = a_3 = \alpha_2 = \alpha_3 = 1, \quad b_1 = b_5 = \beta_4 = \beta_5 = 1, \quad \lambda_1 = l_5 = 1, \quad \theta_1 = \theta_2 = 0, \quad p = \frac{1}{2}, \quad m = \frac{67}{356}\).

We stress that our family of solutions is a particular one, attained by making particular assumptions and corresponding to correlated detections. The question whether non-correlated solutions do exist or not, if \(\text{dim}({\mathcal{H}}_1) = 10\), remains open.

6 Derivation of a family of solutions

In this section we carry out the detailed derivation of the family of solutions of problem \((P')\) presented in previous section. Our treatment is not at all exhaustive. Indeed, we shall consider solutions characterized by particular conditions of linear independence between some of their components. In our derivation we analyze the equations involving vectors \(y_i\), i.e. (ii-B), (iv-D) in (23) and (ii-B'), (iv-D') in (27); in order to solve them, we make some assumptions (e.g. vectors \(\begin{pmatrix} \eta_3 \\ \theta_3 \end{pmatrix}, \ldots, \begin{pmatrix} \eta_n \\ \theta_n \end{pmatrix}\) are supposed linearly independent, as well as \(\theta_1, \ldots, \theta_n\); furthermore, \(\zeta_j = 0\) for all \(j = 1, \ldots, n\)) which lead to particular forms for vectors \(y_i\) and for matrices \(Q, U, N, Z\). Analogous results are taken also for (i-A), (iii-C) in (23) and (i-A'), (iii-C') in (27), which have the same form of the previous ones; so we obtain a particular form for vectors \(x_i\) and for matrices \(P, V, M, W\). Then we fix the dimension of \(\mathcal{H}_1\), \(\text{dim}({\mathcal{H}}_1) = 10\); by fixing the value of parameters (see (44)) we obtain the particular family of solutions of problem \((P')\), singled out in previous section.

6.1 General constraints for \(\Psi\) and \(G\)

Equations in (ii-B) imply \(u_{j1}y_1 + \cdots + u_{jn}y_n = 0\). Therefore, since \(U \neq 0\), vectors \(y_1, \ldots, y_n\) must be linearly dependent. Let us suppose \(y_1 = \alpha_2y_2 + \cdots + \alpha_ny_n\). By
using this relation in (iv-D) we get
\[
\begin{aligned}
Q_j \delta_2 + \cdots + Q_{jn} \delta_n &= \delta_j \\
Q_j \zeta_2 + \cdots + Q_{jn} \zeta_n &= \zeta_j \\
Q_j \left( \frac{\eta_2}{\theta_2} \right) + \cdots + Q_{jn} \left( \frac{\eta_n}{\theta_n} \right) &= \left( \begin{array}{c} 0 \\ 0 \end{array} \right).
\end{aligned}
\]  
(28)
where \( Q_{jk} = q_{jk} + \alpha_k q_j \), for all \( k = 2, \ldots, n \).

In the third equation of (28), if we suppose \( \frac{\eta_2}{\theta_2}, \ldots, \frac{\eta_n}{\theta_n} \) linearly independent, then \( Q_j \delta_2 = \cdots = Q_{jn} = 0 \), which implies \( \delta_j = \zeta_j = 0 \), for all \( j = 1, \ldots, n \), in the first and second equation of (28). A similar reasoning for (i-A) leads to \( a_j = b_j = 0 \), for all \( j = 1, \ldots, n \); as a consequence \( x_j = (0, 0, c_j, 0, e_j, 0, 0) \) and \( y_j = (0, 0, 0, 0, 0, 0, \eta_j, \theta_j) \) so that, if a solution exists, then \( \Psi = 0 \), i.e. it would be correspond to the uninteresting case excluded by (C.10). If we consider all cases, we obtain the following implications:

a.) \( (c_2, e_2, \ldots, c_n, e_n) \) linearly independent and \( (\eta_2, \theta_2, \ldots, \eta_n, \theta_n) \) linearly independent imply \( x_j = (0, 0, c_j, 0, e_j, 0, 0, 0) \) and \( y_j = (0, 0, 0, 0, 0, 0, \eta_j, \theta_j) \).

b.) \( (c_2, e_2, \ldots, c_n, e_n) \) linearly independent and \( (\eta_2, \theta_2, \ldots, \eta_n, \theta_n) \) linearly independent imply \( x_j = (0, 0, c_j, 0, e_j, 0, 0, 0) \) and \( y_j = (0, 0, 0, 0, 0, 0, \eta_j, \theta_j) \);

c.) \( (c_2, e_2, \ldots, c_n, e_n) \) linearly independent and \( (\eta_2, \theta_2, \ldots, \eta_n, \theta_n) \) linearly independent imply \( x_j = (a_j, b_j, c_j, 0, e_j, 0, 0, 0) \) and \( y_j = (0, 0, 0, 0, 0, 0, \eta_j, \theta_j) \);

d.) \( (c_2, e_2, \ldots, c_n, e_n) \) linearly independent and \( (\eta_2, \theta_2, \ldots, \eta_n, \theta_n) \) linearly independent imply \( x_j = (a_j, b_j, c_j, 0, e_j, 0, 0, 0) \) and \( y_j = (0, 0, 0, 0, 0, 0, \eta_j, \theta_j) \).

We shall search solutions for cases (b), (c) and (d), since in case (a) meaningful solutions cannot exist.

Cases (b) and (c)

According to (b), conditions (21) and (23) imply equation \( TW \Psi = 0 \) holds, which is equivalent to say that each time a particle is sorted by \( W \) than it is certainly not sorted by \( T \); therefore, for all eventual solutions corresponding to this case, property \( L \) must be correlated with WS property \( E \).

In case (c) \( T'W'Y \Psi = 0 \) holds; this equation expresses the impossibility that two probabilities are zero and the remaining is 1, for the occurrence of \( T', W' \) and \( Y \); in any case, properties \( L \) and \( G \) are correlated with WS property \( E \).

Case (d)

No correlated or meaningless solution immediately follows from case (d). We restrict our search by working with the equations in (ii-B), (iv-D) of (23), rather than with all
of them. Since (i-A), (iii-C) in (28) are formally identical to (ii-B), (iv-D) of (28), we can extend to them the results found for (ii-B), (iv-D).

Our task is more simple if we search solutions corresponding to particular state vectors \( \Psi \) satisfying (d); for this reason, among vectors \( \begin{pmatrix} \eta_2 \\ \theta_2 \end{pmatrix}, \ldots, \begin{pmatrix} \eta_n \\ \theta_n \end{pmatrix} \), we suppose that only \( \begin{pmatrix} \eta_2 \\ \theta_2 \end{pmatrix} \) is a linear combination of the remaining ones; let \( \beta_3, \ldots, \beta_n \) be complex numbers such that

\[
\begin{pmatrix} \eta_2 \\ \theta_2 \end{pmatrix} = \beta_3 \begin{pmatrix} \eta_3 \\ \theta_3 \end{pmatrix} + \cdots + \beta_n \begin{pmatrix} \eta_n \\ \theta_n \end{pmatrix},
\]

(29)

where \( \begin{pmatrix} \eta_3 \\ \theta_3 \end{pmatrix}, \ldots, \begin{pmatrix} \eta_n \\ \theta_n \end{pmatrix} \) are supposed linearly independent. Hence, (ii-B) and (iv-D) yield

\[
\begin{align*}
U_{j2} \begin{pmatrix} \delta_2 \\ \zeta_2 \end{pmatrix} + \cdots + U_{jn} \begin{pmatrix} \delta_n \\ \zeta_n \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
(U_{j2} \beta_3 + U_{j3}) \begin{pmatrix} \eta_3 \\ \theta_3 \end{pmatrix} + \cdots + (U_{j2} \beta_n + U_{jn}) \begin{pmatrix} \eta_n \\ \theta_n \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}
\]

(30)

where \( U_{jk} = u_{jk} + \alpha_k u_{j1} \), for all \( k = 2, \ldots, n \), and

\[
\begin{align*}
Q_{j2} \begin{pmatrix} \delta_2 \\ \zeta_2 \end{pmatrix} + \cdots + Q_{jn} \begin{pmatrix} \delta_n \\ \zeta_n \end{pmatrix} &= \begin{pmatrix} \delta_j \\ \zeta_j \end{pmatrix} \\
(Q_{j2} \beta_3 + Q_{j3}) \begin{pmatrix} \eta_3 \\ \theta_3 \end{pmatrix} + \cdots + (Q_{j2} \beta_n + Q_{jn}) \begin{pmatrix} \eta_n \\ \theta_n \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}
\]

(31)

where \( Q_{jk} = q_{jk} + \alpha_k q_{j1} \), for all \( k = 2, \ldots, n \). The linear independence among vectors in the second equation of (30) and (31) implies that

\[
U_{j2} \beta_3 + U_{j3} = \cdots = U_{j2} \beta_n + U_{jn} = 0,
\]

(32)

\[
Q_{j2} \beta_3 + Q_{j3} = \cdots = Q_{j2} \beta_n + Q_{jn} = 0
\]

(33)

respectively. Then (30) and (31) become

\[
\begin{align*}
U_{j2} \begin{pmatrix} \delta_2 \\ \zeta_2 \end{pmatrix} - \beta_3 \begin{pmatrix} \delta_3 \\ \zeta_3 \end{pmatrix} - \cdots - \beta_n \begin{pmatrix} \delta_n \\ \zeta_n \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
U_{j2} \beta_3 + U_{j3} = \cdots = U_{j2} \beta_n + U_{jn} = 0;
\end{align*}
\]

(34)

\[
\begin{align*}
Q_{j2} \begin{pmatrix} \delta_2 \\ \zeta_2 \end{pmatrix} - \beta_3 \begin{pmatrix} \delta_3 \\ \zeta_3 \end{pmatrix} - \cdots - \beta_n \begin{pmatrix} \delta_n \\ \zeta_n \end{pmatrix} &= \begin{pmatrix} \delta_j \\ \zeta_j \end{pmatrix} \\
Q_{j2} \beta_3 + Q_{j3} = \cdots = Q_{j2} \beta_n + Q_{jn} = 0.
\end{align*}
\]

(35)

First equation in (31) is satisfied if one of the two factor is zero; however, if we suppose

\[
\begin{pmatrix} \delta_2 \\ \zeta_2 \end{pmatrix} = \beta_3 \begin{pmatrix} \delta_3 \\ \zeta_3 \end{pmatrix} + \cdots + \beta_n \begin{pmatrix} \delta_n \\ \zeta_n \end{pmatrix}.
\]
\(35\) implies \(\delta_j = \zeta_j = 0\), for all \(j = 1, \ldots, n\); since we are taking analogous results for (i-A) and (iii-C), then \(a_j = b_j = 0\) for all \(j = 1, \ldots, n\). As a consequence, \(x_j = (0, 0, c_j, 0, e_j, 0, 0, 0)\) and \(y_j = (0, 0, 0, 0, 0, 0, \eta_j, \theta_j)\), which lead to meaningless solutions (case (a)). Hence, \(U_{j2} = 0\) follows from \(34\).

In \(35\) we can suppose that vectors \((\delta_2, \zeta_2), \ldots, (\delta_n, \zeta_n)\) are either linearly independent or linearly dependent (with coefficients different from \(b_3, \ldots, b_n\)). In case they are independent, first equation in \(35\) yields

\[
\begin{cases}
Q_{n2} = 0 \\
\beta_3 Q_{n2} = 0 \\
\vdots \\
\beta_{n-1} Q_{n2} = 0 \\
\beta_n Q_{n2} = 1
\end{cases}
\quad (36)
\]

in correspondence with \(j = n\), which has no solution. So we can suppose the existence of complex numbers, \(c_3, \ldots, c_n\) such that

\[
\begin{pmatrix}
\delta_2 \\
\zeta_2 \\
\vdots \\
\delta_n \\
\zeta_n
\end{pmatrix} = c_3 \begin{pmatrix}
\delta_3 \\
\zeta_3 \\
\vdots \\
\delta_n \\
\zeta_n
\end{pmatrix} + \ldots + c_n \begin{pmatrix}
\delta_n \\
\zeta_n
\end{pmatrix}.
\]

Again, with a reasoning similar to the previous one, carried out for \(j = n\) and \(j = n-1\), no solution is found for \(35\). The same conclusion can be drawn if just two vectors among \((\delta_2, \zeta_2), \ldots, (\delta_n, \zeta_n)\) are linearly independent, so we may conclude that only one is independent, say the last. Hence complex numbers \(\gamma_2, \ldots, \gamma_{n-1}\) must exist such that

\[
\begin{pmatrix}
\delta_2 \\
\zeta_2 \\
\vdots \\
\delta_n \\
\zeta_n
\end{pmatrix} = \gamma_2 \begin{pmatrix}
\delta_n \\
\zeta_n
\end{pmatrix} + \ldots + \gamma_{n-1} \begin{pmatrix}
\delta_n \\
\zeta_n
\end{pmatrix}.
\quad (37)
\]

As a consequence \(35\) can be written as

\[
\begin{cases}
Q_j \left[ \gamma_2 - \beta_3 \gamma_3 - \ldots - \beta_{n-1} \gamma_{n-1} - \beta_n \right] \begin{pmatrix}
\delta_n \\
\zeta_n
\end{pmatrix} = \begin{pmatrix}
\delta_j \\
\zeta_j
\end{pmatrix} \\
Q_j \beta_3 + Q_j \gamma_3 = \ldots = Q_j \beta_n + Q_j \gamma_n = 0.
\end{cases}
\quad (38)
\]

### 6.2 General constraints for \(\Psi\) and \(L\)

Equations in (ii-B’) have the same form of those in (ii-B); hence, following \(29\) - \(54\), conclusions drawn for (ii-B) continue to hold for (ii-B’).

Again, in order to make easier our task, we restrict the search by working with the equations in (iv-D’); since (i-A’) is formally identical, we can extend to it analogous results.

In (iv-D’), as a consequence of \(29\) and \(37\), we get

\[
\begin{cases}
(N_j \gamma_2 + \ldots + N_{jn-1} \gamma_{n-1} + N_j n) \begin{pmatrix}
\delta_n \\
\zeta_n
\end{pmatrix} = \begin{pmatrix}
\delta_j \\
0
\end{pmatrix} \\
(N_j \beta_3 + N_j \gamma_3) \begin{pmatrix}
\eta_3 \\
\theta_3
\end{pmatrix} + \ldots + (N_j \beta_n + N_j n) \begin{pmatrix}
\eta_n \\
\theta_n
\end{pmatrix} = \begin{pmatrix}
\eta_j \\
0
\end{pmatrix}
\end{cases}
\quad (39)
\]

23
where \( N_{jk} = n_{jk} + \alpha_k n_{j1} \). In order to solve first equation, either \( \zeta_j = 0 \), for all \( j \), or \( N_{ji1} = n_{ji1} + \gamma_i n_{j1} + N_j = 0 \), which imply \( \delta_j = 0 \), for all \( j \); in any case, solutions, if they exist, are correlated. Indeed, since we are taking symmetrical results for (i-A'), we obtain the following implications:

e.) if \( \delta_j = a_j = 0 \) for all \( j = 1, \ldots, n \) then \( WY\Psi = 0 \), i.e. each time a particle is sorted by \( Y \) than it is certainly not sorted by \( W \); therefore, for all eventual solutions corresponding to this case, property \( L \) must be correlated with the incompatible property \( G \);

f.) if \( \zeta_j = b_j = 0 \) for all \( j = 1, \ldots, n \) then \( W'Y\Psi = 0 \), i.e. each time a particle is sorted by \( Y \) than it is certainly not sorted by \( W' \); therefore, for all eventual solutions corresponding to this case, property \( L \) must be correlated with the incompatible property \( G \).

Let us suppose \( \zeta_j = 0 \), for all \( j \).

Equations in (39) can also be written as

\[
\begin{align*}
(N_{j2} \gamma_{j2} + \ldots + N_{jn-1} \gamma_{jn-1} + N_j) \delta_n &= \delta_j \\
N_{j3} \eta_3 + \ldots + N_j \eta_n &= \eta_j \\
N_{j3} \theta_3 + \ldots + N_j \theta_n &= 0
\end{align*}
\]

(40)

where \( N_{jk} = N_{j2} \beta_k + N_{jk}, \) for all \( k = 3, \ldots, n \). In the last two equations of (40), a reasoning similar to that carried out for (iv-D) (see equations (28)-(37)) implies the existence of coefficients \( \lambda_i, \ldots, \lambda_n \) and \( \mu_3, \ldots, \mu_{n-1} \) such that

\[
\begin{align*}
\theta_3 &= \lambda_4 \theta_4 + \ldots + \lambda_n \theta_n \\
\eta_k &= \mu_k \eta_n & \forall k = 3, \ldots, n - 1
\end{align*}
\]

(41)

where we have supposed \( \theta_4, \ldots, \theta_n \) linearly independent. Such an independence implies

\[
N_{j3} \lambda_4 + N_{j4} = \ldots = N_{j3} \lambda_n + N_j = 0,
\]

(42)

in the last equation of (40). Hence, (iv-D') can be written as

\[
\begin{align*}
(N_{j2} \gamma_{j2} + \ldots + N_{jn-1} \gamma_{jn-1} + N_j) \delta_n &= \delta_j \\
N_{j3} [\mu_3 - \lambda_4 \mu_4 - \ldots - \lambda_{n-1} \mu_n - \lambda_n] \eta_n &= \eta_j \\
N_{j3} \lambda_4 + N_{j4} = \ldots = N_{j3} \lambda_n + N_j &= 0.
\end{align*}
\]

(43)

### 6.3 Concrete solutions

So far we have established some constraints in the hypothesis that vectors \( \begin{pmatrix} \eta_3 \\ \theta_3 \end{pmatrix} \),

\ldots, \begin{pmatrix} \eta_n \\ \theta_n \end{pmatrix} \) are linearly independent, as well as \( \theta_4, \ldots, \theta_n \); furthermore, \( \zeta_j = 0 \) for all \( j = 1, \ldots, n \), independently of the ranks of matrices \( U_i \), \( Q_i \), \( Z \), \( N \) and \( A_i \), with \( i = 1, \ldots, 8 \), and therefore of the dimensions of spaces \( \mathcal{H}_i \) and \( \mathcal{H}_{11} \).

Now we fix \( \dim(\mathcal{H}_i) = 10 \), hence \( n, j, k \in \{1, \ldots, 5\} \), and we shall see that concrete
solutions exist. Our task is made easier if we search for solutions corresponding to a particular state vector $\Psi$ such that
\[
\gamma_3 = \lambda_4 \gamma_4 + \lambda_5, \quad \alpha_4 = \alpha_5 = \beta_3 = 0
\]
\[
c_3 = l_4 c_4 + l_5, \quad a_4 = a_5 = b_3 = 0.
\]
where the coefficients that appear in the second line are those corresponding to the vectors $x_i$ ($i = 1, \ldots, 10$). Conditions (44) and (32), together with $U_{j2} = 0$ (arising from (34)), imply that $U$ has the following form
\[
U = \begin{pmatrix}
  u_{11} & -\alpha_3 u_{11} & -\alpha_3 u_{11} & 0 & 0 \\
  \vdots  & \vdots  & \vdots & \vdots & \vdots \\
  u_{j1} & -\alpha_3 u_{j1} & -\alpha_3 u_{j1} & 0 & 0 \\
  \vdots  & \vdots  & \vdots & \vdots & \vdots 
\end{pmatrix};
\]
the first equation in (43), together with (32) and the choice (44) imply that $Q = (q_{ij})_{5 \times 5}$ where $q_{33} = -\alpha_3 q_{13}$, $q_{44} = -\beta_3 (q_{14} + q_{24})$ and $q_{54} = -\beta_5 (q_{14} + q_{24})$. Similarly, independence of $\theta_4, \ldots, \theta_9$ in (40) and the second equation of (43), imply that $N = (n_{ij})_{5 \times 5}$ where $n_{33} = q_{12} + \alpha_2 (q_{13} + q_{23})$, $n_{44} = -\lambda_4 \alpha_3 n_{13} - \beta_4 (q_{14} + q_{24}) - \lambda_4 n_{33}$ and $n_{54} = -\lambda_5 \alpha_3 n_{13} - \beta_5 (q_{14} + q_{24}) - \lambda_5 n_{33}$.

Since (ii-B') has the same form of (ii-B), matrix $Z$ can be obtained from $U$ by means of the substitution $u_{ik} = z_{jk}$. Matrices $P$, $V$, $N$, $W$ have similar forms. By imposing that $G_I$ and $L_I$ are self-adjoint matrices we find exactly matrices $U$, $Q$ and $N$ in section ??, and moreover $V = U^t$.

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