Superoscillating response of a nonlinear system on a harmonic signal

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Abstract. We demonstrate that a superoscillating in time signal may be obtained as a nonlinear response on a harmonic low-frequency input. Using the realization of a superoscillating function proposed by (Huang et al. 2007 J. Opt. A: Pure Appl. Opt. 9 S285-8) as an example, we synthesize the response function of such a nonlinear transformer and investigated its robustness with respect to the frequency and amplitude variations of the input signal.

Superoscillations is a counterintuitive mathematical effect, in which a band-limited function \( f(t) \) (the function whose Fourier transform satisfies the condition \( \hat{f}(\omega) = 0 \) for all frequencies \( |\omega| > \omega_{\text{max}} \) ) may oscillate with a frequency much greater than \( \omega_{\text{max}} \). After the discovery of such functions by Berry [1], mathematical properties of superoscillating functions have been studied in detail [2-7] and various mathematical approaches to their construction have been suggested [8, 9]. The concept of superoscillations has proven to be extremely fruitful in nanophotonics, where superoscillations enable deep subwavelength focusing of electromagnetic fields without use of evanescent waves [10-12] (for review see Ref. [13]). More recently, superoscillations were studied in the time domain. Particularly, it has been shown that a quantum two-level emitter can be excited by a superoscillating electric field whose spectral components lie below the transition frequency of the emitter [14]. In another study, it has been found that a superoscillating electromagnetic signal can propagate trough absorbing media over length scales far exceeding the absorption length [15].

Here we explore a method of nonlinear synthesis of a superoscillating signal from a low-frequency single-harmonic input. We employ the technique of the harmonic synthesis [16] and explicitly construct the transformation function \( f(z) \) which transforms a low-frequency harmonic \( z(t) \) into a superoscillating function \( y(t) = f(z(t)) \).

Let us formulate the problem more rigorously. At the input of an inertialess nonlinear system we have a harmonic oscillation \( z(t) = \cos \omega_0 t \). The output function \( y(t) \) is expected to show a superoscillating behaviour. The problem is to find a nonlinear transformation function \( f(z) \) which relates values of the input and output signals at the current time \( t \). We are interested in the case when the output superoscillating function is represented as a superposition of \( N \) harmonic oscillations which frequencies are multiple of \( \omega_0 \). An example of such a superoscillating function is presented in Ref. [10]:

\[
y(t) = \sum_{n=0}^{5} A_n \cos n\omega_0 t, \quad A_0 = 1, \quad A_1 = 13295000, \quad A_2 = -30802818, \quad A_3 = 26581909, \quad A_4 = -10836909, \quad A_5 = 1762818, \quad \omega_0 = 1.\]

This function is plotted in figure 1, where the fastest harmonic \( \cos(5t) \) is also shown for comparison. It is clearly seen that in time interval \(-0.1 < t < 0.1\) the superoscillating function...
$y(t)$ is well approximated by function $f_{app}(t) = (\cos 43t + 1)/2$, which oscillates nearly 9 times faster than the cut-off component.

![Figure 1](image)

**Figure 1** Superoscillating function $y(t)$ containing spectral components $\omega_n = n$ (solid line) and the fastest component with frequency $\omega_5 = 5$ (black dashed line). The red dashed line shows the approximation of $y(t)$ near $t = 0$.

We seek for the desired transformation function in the form of a polynomial:

$$f(z) = a_0 + a_1 z + \ldots + a_N z^N.$$  \hspace{1cm} (1)

Then, equality $y(t) = f\left(z(t)\right)$ can be recast in the form

$$\sum_{n=0}^{N} a_n \cos n \omega t = \sum_{n=0}^{N} A_n \cos n \omega_0 t.$$ \hspace{1cm} (2)

Having the set of values $A_n$ which yields the superoscillating feature of $y(t)$, we can establish general expression for $a_n$, which constitutes the transformation. Below we show explicit expressions for the case of $N = 6$ harmonic components of the output signal, which can be found in Ref. [16]. It is convenient to write expressions for the odd and even coefficients separately:

$$a_0 = 2^0 (A_0 - A_5 + A_4),$$
$$a_2 = 2^1 (A_2 - 4A_4),$$
$$a_4 = 2^2 (A_4),$$
$$a_6 = 2^4 (A_0),$$
$$a_1 = 2^0 (A_1 - 3A_4 + 5A_5),$$
$$a_3 = 2^2 (A_1 - 5A_4),$$
$$a_5 = 2^4 (A_1).$$ \hspace{1cm} (3a, 3b)
These formulas can be generalized for a greater number of components using rules developed in Refs. [16, 17].

Equations (3a) and (3b) together with equation (1) determine the desired transformation function \( f(z) \) of the non-delay system. The resulting transformation for the given superscissating function takes the form:

\[
f(z) = 10^7 (1.9965910 - 5.7636637z - 2.5089636z^2 \\
+ 7.1071276z^3 - 8.6695272z^4 - 2.8205088z^5)
\] (4)

Nonlinear characteristic of this transformation yielding superscissating output is shown in figure 2. For the range of the input signal values \(-1 \leq z \leq 1\) the output signal lies in the 7 orders of magnitude broader range. However, such extreme amplification may be eliminated by scaling down the transformation by a factor of \(10^7\) [see equation (4)]. This decreases the superscissations amplitude, but leaves the shape of the superscissating function unchanged. One of features of the nonlinear transformation visible from figure 2 is that \( f(0) \neq 0 \), i.e., the current form of the transformation requires non-zero response of the system on zero input. Obviously, this is an unphysical property of the system. In order to fix this, one can subtract the constant \(a_0\) from the nonlinear transformation \(f(z)\). Obviously, the shifted output \(y(t) - a_0\) is still superscissating, but the resulting system with the transformation \(F(z) = f(z) - a_0\) does not generate any signal at zero input.

![Figure 2](image)

**Figure 2** Input-output characteristic of nonlinear transformation (4).

Now let us briefly discuss how robust the obtained transformation is against a variation of the input signal. Firstly, we note that transformation (4) is frequency scalable, i.e., if the input signal is given by \(z_o(t) = \cos \alpha \omega_o t\), then one would obtain the output in the form \(y_o(t) = y(\alpha t)\), so that the superscissating behaviour of the output signal is preserved.

Figure 3 shows the output generated by the transformation \(f(z)\) for different amplitudes \(z_o\) of the input signal \(z(t) = z_o \cos \omega_o t\). Due to its nonlinear character, transformation (4) distorts the signal when its amplitude differs from the reference value \(z_o = 1\) for which the initial transformation is designed by equations (1) – (4). For 1% increase of the input amplitude, the output is still superscissating while
the distance between the neighboring minima increases (orange and red curves). The phenomenon is more sensitive, however, to a decrease of the input amplitude: its 1% variation drastically changes shape of the output function and kills superoscillations (the green curve). Overall, the current form of the nonlinear transformation is tolerant to \( \sim 0.5\% \) variation of the input signal amplitude.

![Figure 3](image)

**Figure 3** Effect of variations of the input signal amplitude on the shape of the output signal \( y(t) \). Black thick curve shows the output signal for the perfectly matched input amplitude \( z_0 = 1 \) also shown in figure 1. Numbers at the curves indicate the values of the input signal amplitude.

To conclude, we have demonstrated synthesis of a superoscillating function from a single-frequency input signal in a generic nonlinear inertialess system. We have derived an expression for the system transformation function which performs such synthesis and discussed robustness of superoscillation synthesis against variations of the input signal.

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