GLOBAL WEAK SOLUTIONS FOR THE THREE-DIMENSIONAL CHEMOTAXIS-NAVIER-STOKES SYSTEM WITH SLOW $p$-LAPLACIAN DIFFUSION

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ABSTRACT. This paper investigates an incompressible chemotaxis-Navier-Stokes system with slow $p$-Laplacian diffusion

\[
\begin{aligned}
  n_t + u \cdot \nabla n &= \nabla \cdot (|\nabla n|^{p-2} \nabla n) - \nabla \cdot (n \chi(c) \nabla c), & x \in \Omega, \ t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - n f(c), & x \in \Omega, \ t > 0, \\
  u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla P + n \nabla \Phi, & x \in \Omega, \ t > 0, \\
  \nabla \cdot u &= 0, & x \in \Omega, \ t > 0
\end{aligned}
\]

under homogeneous boundary conditions of Neumann type for $n$ and $c$, and of Dirichlet type for $u$ in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. Here, $\Phi \in W^{1,\infty}(\Omega)$, $0 < \chi \in C^2([0, \infty))$ and $0 \leq f \in C^1([0, \infty))$ with $f(0) = 0$. It is proved that if $p > \frac{32}{15}$ and under appropriate structural assumptions on $f$ and $\chi$, for all sufficiently smooth initial data $(n_0, c_0, u_0)$ the model possesses at least one global weak solution.

1. Introduction

In this paper, we consider the following chemotaxis-Navier-Stokes system with $p$-Laplacian diffusion

\[
\begin{aligned}
  n_t + u \cdot \nabla n &= \nabla \cdot (|\nabla n|^{p-2} \nabla n) - \nabla \cdot (n \chi(c) \nabla c), & x \in \Omega, \ t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - n f(c), & x \in \Omega, \ t > 0, \\
  u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla P + n \nabla \Phi, & x \in \Omega, \ t > 0, \\
  \nabla \cdot u &= 0, & x \in \Omega, \ t > 0
\end{aligned}
\]

in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, where the scalar functions $n = n(x, t)$ and $c = c(x, t)$ denote bacterial density and the concentration of oxygen, respectively. The vector $u = (u_1, u_2, u_3)$ is the fluid velocity field and the associated pressure is denoted by $P = P(x, t)$. The function $\chi$ represents the chemotactic sensitivity, $f$ is the oxygen consumption rate by the bacteria and $\kappa \in \mathbb{R}$ measures the strength of nonlinear fluid convection. The given function $\Phi$ stands for the gravitational potential produced by the action of physical forces on the cell. If $p > 2$, the nonlinear diffusion $|\nabla n|^{p-2} \nabla n$ is called the slow $p$-Laplacian diffusion, whereas $1 < p < 2$, it is called the fast $p$-Laplacian diffusion.

The Keller-Segel model was first presented in [18] to describe the chemotaxis of cellular slime molds. Let $n$ denote the cell density and $c$ describe the concentration of the chemical signal secreted...
by cells. The mathematical model reads as

\[
\begin{align*}
  n_t &= \Delta n - \nabla \cdot (n \nabla c), \quad x \in \Omega, \ t > 0, \\
  c_t &= \Delta c - c + n, \quad x \in \Omega, \ t > 0,
\end{align*}
\]  

which is called Keller-Segel system. It is known that whether the classical solutions of the system exist globally depends on the size of initial data (cf. \[4, 15, 28, 38\]). A large number of variants of the classical form have been investigated, including the system with the logistic terms (see \[20, 31, 46\], for instance), two-species chemotaxis system (see \[19, 21, 27, 45\], for instance), attraction-repulsion chemotaxis system (see \[22, 30\], for instance) and so on. We refer to \[2, 12–14\] for the further reading.

The chemotaxis-Navier-Stokes system was first introduced in \[35\]. Aerobic bacteria such as Bacillus subtilis often live in thin fluid layers near solid-air-water contact line, in which the biology of chemotaxis, metabolism, and cell-cell signaling is intimately connected to the physics of buoyancy, diffusion, and mixing \[35\]. Both bacteria and oxygen diffuse through the fluid, and they are also transported by the fluid (cf. \[9\] and \[26\]). Taking into account all these processes, in \[35\] the authors proposed the model

\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \chi(c) \nabla c), \quad x \in \Omega, \ t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - nf(c), \quad x \in \Omega, \ t > 0, \\
  u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla P + n \nabla \Phi, \quad x \in \Omega, \ t > 0, \\
  \nabla \cdot u &= 0, \quad x \in \Omega, \ t > 0
\end{align*}
\]  

in a domain \(\Omega \subset \mathbb{R}^d\), where the vector \(u = (u_1(x, t), u_2(x, t), \ldots, u_d(x, t))\) is the fluid velocity field and the associated pressure is represented by \(P = P(x, t)\).

The chemotaxis fluid system has been studied in the last few years. In \[26\], local-in-time weak solutions were constructed for a boundary-value problem of (1.3) in the three-dimensional setting. In \[10\], global classical solutions near constant states were constructed with \(\Omega = \mathbb{R}^3\). For the chemotaxis-Navier-Stokes system in two space dimensions, the authors in \[25\] obtained global existence for large data. For the case of bounded domain \(\Omega \subset \mathbb{R}^d\), Winkler \[39\] proved that the initial-boundary value problem of (1.3) possesses a unique global classical solution for \(d = 2\) and possesses at least one global weak solution for \(d = 3\) under the assumption that \(\kappa = 0\). In \[40\] the same author showed that in bounded convex domains \(\Omega \subset \mathbb{R}^2\), the global classical solutions obtained in \[39\] stabilize to the spatially uniform equilibrium \((\bar{n}_0, 0, 0)\) with \(\bar{n}_0 := \frac{1}{|\Omega|} \int_{\Omega} n_0(x) dx\) as \(t \to \infty\). Recently, Zhang and Li \[47\] proved that such solution converges to the equilibrium \((\bar{n}_0, 0, 0)\) exponentially in time. By deriving a new type of entropy-energy estimate, Jiang et al. \[17\] generalized the result of \[40\] to general bounded domains. (If both \(\chi\) and \(f\) are supposed to be nonnegative and nondecreasing, it was shown by Chae, Kang and Lee \[5\] that the Cauchy problem of (1.3) admits a global classical solution under the assumption that \(d = 2\) and \(\sup_{c} |\chi'(c) - \mu f'(c)|\) be sufficiently small for some \(\mu > 0\). It was showed in \[23\] that the 2-dimensional Cauchy problem of (1.3) admits global classical bounded solutions for regular initial data. For more results of the well-posedness of the Cauchy problem to (1.3) in the whole space we refer the reader to \[6, 10, 25, 44, 50\].)

The diffusion of bacteria sometimes depend nonlinearly on their densities \[12, 33, 34, 36\]. Introducing this into the model (1.3) leads to the chemotaxis-Navier-Stokes system with nonlinear
diffusion \cite{8}

\[
\begin{aligned}
&n_t + u \cdot \nabla n = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (n\chi(c)\nabla c), \quad x \in \Omega, \ t > 0, \\
&c_t + u \cdot \nabla c = \Delta c - nf(c), \quad x \in \Omega, \ t > 0, \\
&u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla \Phi, \quad x \in \Omega, \ t > 0, \\
&\nabla \cdot u = 0, \quad x \in \Omega, \ t > 0.
\end{aligned}
\] (1.4)

Under the assumption \(D(n) = n^{m-1}\), Di Francesco et al. \cite{8} proved that when \(m \in (\frac{3}{7}, 2]\) the chemotaxis-Stokes system admits a global-in-time solution for general initial data in the bounded domain \(\Omega \subset \mathbb{R}^2\), while the same result holds in three-dimensional setting under the constraint \(m \in (\frac{7+\sqrt{217}}{12}, 2]\). Intuitively, the nonlinear diffusion \(\Delta u^m\) for \(m > 1\) can prevent the occurrence of blow up. Based on this intuition, Winkler \cite{41} revealed the condition \(m > \frac{7}{2}\) to be sufficient to guarantee the boundedness of global weak solutions to the chemotaxis-stokes system for all reasonably regular initial data in three-dimensional bounded convex domains (see also \cite{25}). This partially extended a precedent result which asserted global solvability within the larger range \(m > \frac{8}{7}\), but only in a class of weak solutions locally bounded in \(\Omega \times [0, \infty)\) (cf. \cite{34}). In \cite{49}, Zhang and Li studied the system (1.4) under the assumption \(D(n) = mn^{m-1}\) and they proved that the model possesses at least one global weak solution under the condition that \(m \geq \frac{7}{3}\). Recently, Winkler \cite{43} considered (1.4) under the assumption that \(\chi(s) \equiv 1, f(s) \equiv s\) for \(s \geq 0\) and that \(\kappa = 0\). It was shown that the chemotaxis-Stokes system admits global bounded weak solutions to an associated initial-boundary value problem under the assumption that \(m > \frac{9}{7}\). Moreover, the obtained solutions are shown to approach the spatially homogeneous steady state \((\frac{1}{|\Omega|} \int_{\Omega} n_0, 0, 0)\) in the large time limit. For smaller values of \(m > 1\), up to now existence results are limited to classes of possibly unbounded solutions (cf. \cite{11}).

Other forms of diffusion operator are also considered in Keller-Segel model, and \(p\)-Laplacian diffusion is one of them. Cong et al. \cite{7} studied the following \(p\)-Laplacian Keller-Segel model in \(d \geq 3\):

\[
\begin{aligned}
&n_t = \nabla \cdot (|\nabla n|^{p-2}\nabla n) - \nabla \cdot (n\chi(c)\nabla c), \quad x \in \mathbb{R}^d, \ t > 0, \\
&\Delta c = n, \quad x \in \mathbb{R}^d, \ t > 0, \\
&n(x, 0) = n_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\] (1.5)

and they proved the existence of a uniform in time \(L^\infty\) bounded weak solution for system (1.5) with the supercritical diffusion exponent \(1 < p < \frac{3d}{d+1}\) in the multi-dimensional space \(\mathbb{R}^d\) under the condition that the \(L^{\frac{d(3-p)}{p}}\) norm of initial data is smaller than a universal constant. They also proved the local existence of weak solutions and a blow-up criterion for general \(L^1 \cap L^\infty\) initial data. By the way, Jian-Guo et al. \cite{16} investigated the system (1.5) with the nonlocal diffusion term instead by \(-(-\Delta)^\frac{\alpha}{2} n\) (\(1 < \alpha < 2\)) in dimension \(d > 2\).

In contrast to the chemotaxis-(Navier-)Stokes system with porous-medium-type diffusion or the classical Keller-Segel model with \(p\)-Laplacian diffusion, very few results of global solvability are available for the full chemotaxis-Navier-Stokes system with \(p\)-Laplacian diffusion. This inspires us to study (1.1).

**Main results.** In order to formulate our result, we specify the precise mathematical setting: we shall subsequently consider (1.1) along with initial conditions

\[
n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x) \quad \text{and} \quad u(x, 0) = u_0(x), \quad x \in \Omega
\] (1.6)
and under the boundary conditions
\[ |\nabla n|^{p-2} \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial \Omega \] (1.7)
in a bounded convex domain \( \Omega \subset \mathbb{R}^3 \) with smooth boundary, where \( \nu \) is the exterior unit normal vector on \( \partial \Omega \) and we assume that
\[ \begin{cases} n_0 \in L^2(\Omega) \quad \text{is nonnegative with} \quad n_0 \not\equiv 0, \quad \text{that} \\ c_0 \in L^\infty(\Omega) \quad \text{is nonnegative and such that} \quad \sqrt{c_0} \in W^{1,2}(\Omega), \quad \text{and that} \\ u_0 \in L^2(\Omega) \end{cases} \] (1.8)
with \( L^p_0(\Omega) := \{ \varphi \in (L^2(\Omega))^3 | \nabla \cdot \varphi = 0 \} \) denotes the Hilbert space of all solenoidal vector in \( L^2(\Omega) \).

With respect to the parameter function in (1.2), we shall suppose throughout the paper that
\[ \begin{cases} \chi \in C^2([0, \infty)), \quad \chi > 0 \quad \text{in} \quad [0, \infty), \quad \text{that} \\ f \in C^1([0, \infty)), \quad f(0) = 0, \quad f > 0 \quad \text{in} \quad [0, \infty), \quad \text{and that} \\ \Phi \in W^{1,\infty}(\Omega) \end{cases} \] (1.9)
as well as
\[ (\frac{f}{\chi})' > 0, \quad (\frac{f}{\chi})'' \leq 0, \quad \text{and} \quad (\chi \cdot f)' \geq 0 \quad \text{on} \quad [0, \infty). \] (1.10)

Our main result reads as follows.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded convex domain with smooth boundary, \( \kappa \in \mathbb{R} \) and \( p > \frac{32}{15} \). Suppose that the assumptions (1.8)-(1.10) hold. Then there exists at least one global weak solution (in the sense of Definition 6.1 below) of (1.1), (1.6) and (1.7) such that
\[ n \in L^p_{loc}([0, \infty); W^{1,p}(\Omega)) \quad \text{and} \quad c^\frac{2}{4} \in L^4_{loc}([0, \infty); W^{1,4}(\Omega)). \]

**Remark 1.1.** Compared with the result of [49], one may wonder why the condition of \( p \) in this paper is not less than 2, but larger than \( \frac{32}{15} \). There are two reasons. One is a technical reason, and the other is more essential: the main difficulties appear when we want to show that the solutions \((n_\varepsilon, c_\varepsilon, u_\varepsilon)\) of regularized system satisfies
\[ |\nabla n_\varepsilon|^{p-2} \nabla n_\varepsilon \rightharpoonup |\nabla n|^{p-2} \nabla n \quad \text{in} \quad L^q_{loc}(\bar{\Omega} \times [0, \infty)) \] (1.11)
for some \( q > 1 \). This requires a higher regularity of the solutions \((n_\varepsilon, c_\varepsilon, u_\varepsilon)\) than that of [49]. Monotonic method (cf. [24, Section 2.1], see also [3]) is used to handle this problem, but it’s crucial to derive the a priori estimates. In fact, we need to show that
\[ \int_0^t \int_\Omega |\nabla n_\varepsilon|^p \leq C(t + 1) \quad \text{for all} \quad t > 0, \] (1.12)
but using the same method used in [42] and [49], we can not get the desired result, which can only obtain
\[ \int_0^t \int_\Omega |\nabla n_\varepsilon|^{p-\frac{4}{2}} \leq C(t + 1) \quad \text{for all} \quad t > 0. \] (1.13)
We overcome this difficulty by means of a bootstrap argument, but we need a stronger hypotheses: we need \( n_0 \in L^2(\Omega) \) rather than \( n_0 \in L \log L(\Omega) \). And also the condition \( p > \frac{32}{15} \) is required for the sake of deriving a crucial priori estimates. Actually, we shall prove that (see Section 6)
\[ |\nabla n_\varepsilon|^{p-2} \nabla n_\varepsilon \rightharpoonup |\nabla n|^{p-2} \nabla n \quad \text{in} \quad L^p_{loc}(\bar{\Omega} \times [0, \infty)). \] (1.14)
The rest of this paper is organized as follows. In Section 2, we introduce a family of regularized problems and give some preliminary properties. Based on an energy-type inequality, a priori estimates are given in Section 3. Section 4 is devoted to showing the global existence of the regularized problems. In Section 5, we further establish useful uniform estimates. Finally, we give the proof of the main result in Section 6.

Notations. Throughout the paper, for any vectors $v \in \mathbb{R}^3$ and $w \in \mathbb{R}^3$, we denote by $v \otimes w$ the matrix $A_{3 \times 3}$ with $a_{ij} = v_i w_j$ for $i, j \in \{1, 2, 3\}$. We set $L \log L(\Omega)$ is the standard Orlicz space and $L^2_{\sigma}(\Omega) := \{ \varphi \in (L^2(\Omega))^3 | \nabla \cdot \varphi = 0 \}$ denotes the Hilbert space of all solenoidal vector in $L^2(\Omega)$. As usual $\mathcal{P}$ denotes the Helmholtz projection in $L^2(\Omega)$. We write $W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}(\Omega) \cap L^2_{\sigma}(\Omega)$ and $C^\infty_{0,\sigma}(\Omega) := C^\infty(\Omega) \cap L^2_{\sigma}(\Omega)$. We represent $A$ as the realization of Stokes operator $-\mathcal{P}\Delta$ in $L^2(\Omega)$ with domain $\mathcal{D}(A) := W^{2,2}(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega)$. Also $n(\cdot, t)$, $c(\cdot, t)$ and $u(\cdot, t)$ will be denoted sometimes by $n(t)$, $c(t)$ and $u(t)$.

2. Regularized problem

Our intention is to construct a global weak solution as the limit of smooth solutions of appropriately regularized problem. According to the idea from [42] (see also [34, 49]), let us first consider the approximate problem

$$
\begin{aligned}
\begin{cases}
\partial_t n_\varepsilon + u_\varepsilon \cdot \nabla n_\varepsilon = \nabla \cdot \left( (|\nabla n_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla n_\varepsilon \right) - \nabla \cdot (n_\varepsilon F'_\varepsilon(n_\varepsilon) \chi(c_\varepsilon) \nabla c_\varepsilon), & x \in \Omega, \ t > 0, \\
\partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon - \Delta c_\varepsilon = F_\varepsilon(n_\varepsilon)f(c_\varepsilon), & x \in \Omega, \ t > 0, \\
\partial_t u_\varepsilon + (Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \Phi, & x \in \Omega, \ t > 0, \\
\nabla \cdot u_\varepsilon = 0, & x \in \Omega, \ t > 0, \\
\frac{\partial n_\varepsilon}{\partial t} = \frac{\partial u_\varepsilon}{\partial t} = 0, \ u_\varepsilon = 0, & x \in \partial \Omega, \ t > 0, \\
n(\varepsilon, x, 0) = n_0, \ c_\varepsilon(x, 0) = c_0, \ u_\varepsilon(x, 0) = u_0, & x \in \Omega
\end{cases}
\end{aligned}
$$

(2.1)

for $\varepsilon \in (0, 1)$, where the families of approximate initial data $n_0 \geq 0$, $c_0 \geq 0$ and $u_0$ have the properties that

$$
\begin{aligned}
\begin{cases}
n_0 \in C^\infty_0(\Omega), \int_\Omega n_0 = \int_\Omega n_0 \text{ for all } \varepsilon \in (0, 1) \text{ and } \\
n_0 \rightarrow n_0 \text{ in } L^2(\Omega) \text{ as } \varepsilon \downarrow 0,
\end{cases}
\end{aligned}
$$

(2.2)

that

$$
\begin{aligned}
\begin{cases}
\sqrt{c_0} \in C^\infty_0(\Omega), \ |c_0|_{L^\infty(\Omega)} \leq |c_0|_{L^\infty(\Omega)} \text{ for all } \varepsilon \in (0, 1) \text{ and } \\
\sqrt{c_0} \rightarrow \sqrt{c_0} \text{ a.e. in } \Omega \text{ and } W^{1,2}(\Omega) \text{ as } \varepsilon \downarrow 0,
\end{cases}
\end{aligned}
$$

(2.3)

and that

$$
\begin{aligned}
\begin{cases}
u_\varepsilon \in C^\infty_0(\Omega), \text{ with } |u_\varepsilon|_{L^2(\Omega)} = |u_0|_{L^2(\Omega)} \text{ for all } \varepsilon \in (0, 1) \text{ and } \\
u_\varepsilon \rightarrow u_0 \text{ in } L^2(\Omega) \text{ as } \varepsilon \downarrow 0.
\end{cases}
\end{aligned}
$$

(2.4)

The approximate function $F_\varepsilon$ in (2.1) [42] was chosen as

$$
F_\varepsilon(s) := \frac{1}{\varepsilon} \ln(1 + \varepsilon s), \text{ for all } s \geq 0,
$$

and the standard Yosida approximate $Y_\varepsilon$ [29] was defined by

$$
Y_\varepsilon v := (1 + \varepsilon A)^{-1}v, \text{ for all } v \in L^2(\Omega).
$$
Let us furthermore note that the choice of $F_\varepsilon$ ensures that

$$0 \leq F_\varepsilon'(s) = \frac{1}{1 + \varepsilon s} \leq 1, \quad \text{and} \quad 0 \leq F_\varepsilon(s) \leq s \quad \text{for all } s \geq 0, \quad (2.5)$$

$$s F_\varepsilon'(s) = \frac{s}{1 + \varepsilon s} \leq \frac{1}{\varepsilon}, \quad \text{for all } s \geq 0, \quad (2.6)$$

and that

$$F_\varepsilon'(s) \not> 1 \quad \text{and} \quad F_\varepsilon(s) \not> s \quad \text{as} \quad \varepsilon \searrow 0 \quad \text{for all } s \geq 0. \quad (2.7)$$

All the above approximate problems admit for local-in-time smooth solutions:

**Lemma 2.1.** Let $p \geq 2$, then for each $\varepsilon \in (0, 1)$, there exist $T_{\text{max}, \varepsilon} \in (0, \infty]$ and functions $n_\varepsilon > 0$, $c_\varepsilon > 0$ in $\bar{\Omega} \times (0, T_{\text{max}, \varepsilon})$, and $u_\varepsilon$ fulfilling

$$n_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\text{max}, \varepsilon}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}, \varepsilon})), \quad (2.8)$$

$$c_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\text{max}, \varepsilon}]) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}, \varepsilon})), \quad \text{and}$$

$$u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\text{max}, \varepsilon}] ; \mathbb{R}^3) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}, \varepsilon}) ; \mathbb{R}^3)$$

such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ is a classical solution of (2.1) in $\Omega \times (0, T_{\text{max}, \varepsilon})$ with some $P_\varepsilon \in C^{1,0}(\Omega \times (0, T_{\text{max}, \varepsilon}))$. Moreover, if $T_{\text{max}, \varepsilon} < \infty$, then

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} + \|A^\alpha u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \to \infty, \quad t \not> T_{\text{max}, \varepsilon}$$

for all $q > 3$ and $\alpha > \frac{3}{4}$.

The proof of this lemma is based on well-established methods involving the Schauder fixed point theorem, the standard regularity theories of parabolic equations and the Stokes system (see [34, 39, 42] for instance).

The following estimates of $n_\varepsilon$ and $c_\varepsilon$ are basic but important in the proof of our result.

**Lemma 2.2.** For each $\varepsilon \in (0, 1)$, the solution of (2.1) satisfies

$$\int_{\Omega} n_\varepsilon(\cdot, t) = \int_{\Omega} n_0 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \quad (2.9)$$

and

$$\|c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} := s_0 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}). \quad (2.10)$$

**Proof.** Integrating the first equation in (2.1) and using (2.2), we obtain (2.9). And an application of the maximum principle to the second equation in (2.1) gives (2.10). \qed

### 3. An energy-type inequality

This section is devoted to establish an energy-type inequality which will play a key role in the derivation of further estimates in Section 4 where we will show that the solution of the approximate problem (2.1) is actually global in time.

The following inequality is decisive in our proof, which is derived from the first two equations in (2.1). The main idea of the proof is similar to the strategy introduced in [42, Section 3](see also [39, 49]).
Lemma 3.1. Assume that $p \geq 2$. Let (1.9) and (1.10) hold. There exists $K \geq 1$ such that for any $\varepsilon \in (0, 1)$, the solution of (2.1) satisfies
\[
\frac{d}{dt} \left( \int_{\Omega} n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_{\Omega} |\nabla \Psi(c_\varepsilon)|^2 \right) + \frac{1}{K} \left\{ \int_{\Omega} (|\nabla n_\varepsilon|^2 + \varepsilon) \frac{p-2}{2} |\nabla n_\varepsilon|^2 + \int_{\Omega} \left| \frac{D^2 c_\varepsilon}{c_\varepsilon} \right|^2 + \int_{\Omega} \left| \frac{\nabla c_\varepsilon}{c_\varepsilon^3} \right|^2 \right\} \leq K \int_{\Omega} |\nabla u_\varepsilon|^2,
\] (3.1)
where $\Psi(s) := \int_1^s \frac{ds}{\sqrt{g(s)}}$ with $g(s) := f(s)/\chi(s)$.

Proof. The proof is based on the first two equations in (2.1) and integration by parts, we refer readers to [42, Lemmas 3.1-3.4] for details. \qed

Using the third equation of (2.1), we can absorb the term on the right hand of (3.1) appropriately and yield the following energy-type inequality which contains all the components $n_\varepsilon$, $c_\varepsilon$ and $u_\varepsilon$.

Lemma 3.2. Assume that $p \geq 2$. Let (1.9), (1.10) hold, and $\Psi$ and $K$ be given in Lemma 3.1. Then for any $\varepsilon \in (0, 1)$ there exist $K_0 > 0$ large enough such that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_\varepsilon|^2 + \int_{\Omega} |\nabla u_\varepsilon|^2 = - \int_{\Omega} (y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon \cdot u_\varepsilon + \int_{\Omega} n_\varepsilon \nabla \Phi \cdot u_\varepsilon \quad \text{for all } t \in (0, T_{\max, \varepsilon}).
\]
Since $\nabla \cdot u_\varepsilon = 0$ implies $\nabla \cdot Y_{\varepsilon} u_\varepsilon = 0$, we thereby obtain (see also [42, Lemma 3.5])
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_\varepsilon|^2 + \int_{\Omega} |\nabla u_\varepsilon|^2 = \int_{\Omega} n_\varepsilon \nabla \Phi \cdot u_\varepsilon \quad \text{for all } t \in (0, T_{\max, \varepsilon}).
\]
Substituting this into (3.1), we have
\[
\frac{d}{dt} \left( \int_{\Omega} n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_{\Omega} |\nabla \Psi(c_\varepsilon)|^2 + K \int_{\Omega} |u_\varepsilon|^2 \right) + \frac{1}{K} \left\{ \int_{\Omega} (|\nabla n_\varepsilon|^2 + \varepsilon) \frac{p-2}{2} |\nabla n_\varepsilon|^2 + \int_{\Omega} \left| \frac{D^2 c_\varepsilon}{c_\varepsilon} \right|^2 + \int_{\Omega} \left| \frac{\nabla c_\varepsilon}{c_\varepsilon^3} \right|^2 \right\} \leq 2K \int_{\Omega} n_\varepsilon \nabla \Phi \cdot u_\varepsilon \quad \text{for all } t \in (0, T_{\max, \varepsilon}).
\] (3.3)
Since $p \geq 2$, we have for each $\varepsilon \in (0, 1)$
\[
\int_{\Omega} (|\nabla n_\varepsilon|^2 + \varepsilon) \frac{p-2}{2} |\nabla n_\varepsilon|^2 \geq \int_{\Omega} \frac{|\nabla n_\varepsilon|^p}{n_\varepsilon} = \left( \frac{p}{p-1} \right)^p \int_{\Omega} |\nabla n_\varepsilon|^{p-1} |n_\varepsilon|^p \quad \text{for all } t \in (0, T_{\max, \varepsilon}).
\] (3.4)
Using Hölder’s inequality, (1.9), the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and the standard Poincaré inequality, one can find a constant $C_1 > 0$ such that
\[
2K \int_{\Omega} n_\varepsilon \nabla \Phi \cdot u_\varepsilon \leq 2K \|\nabla \Phi\|_\infty \|n_\varepsilon\|_2 \|u_\varepsilon\|_6 \leq \frac{K}{2} \left\{ \|\nabla u_\varepsilon\|^2_2 + C_1 \|n_\varepsilon\|^2_2 \varepsilon \right\} \quad \text{for all } t \in (0, T_{\max, \varepsilon}).
\] (3.5)
Let $\theta := \frac{p-1}{4(p-1)} \in (0, 1)$, then $\theta$ satisfies $\frac{5(p-1)}{2p} = \theta(\frac{1}{p} - \frac{1}{4}) + (1 - \theta)\frac{p-1}{p}$. An application of the Gagliardo-Nirenberg inequality shows that

$$
\|n_\varepsilon\|^2_p \leq \|n_\varepsilon\|^{\frac{p-1}{p}} \|\nabla n_\varepsilon\|^{\frac{2p}{p-1}} + C_2\|n_\varepsilon\|^{\frac{p-1}{p}} \|\nabla n_\varepsilon\|^{\frac{2p}{p-1}} + C_2\|n_\varepsilon\|^{\frac{p-1}{p}} \|\nabla n_\varepsilon\|^{\frac{2p}{p-1}}
$$

for all $t \in (0, T_{\max, \varepsilon})$ (3.6) with some $C_2 > 0$. Since $\frac{2p}{p-1} < p$ due to our assumption $p \geq 2 > \frac{7}{4}$, we use Young’s inequality together with (3.6) to estimate

$$
\|n_\varepsilon\|^2_p \leq \delta\|n_\varepsilon\|^{\frac{p-1}{p}} + C_3C_\delta \quad \text{for all } t \in (0, T_{\max, \varepsilon})
$$

with some $C_3 > 0$ and some $\delta > 0$ to be fixed later. Combining (3.3)-(3.7) with $\delta = \frac{1}{2KC_\varepsilon}(\frac{p}{p-1})^p$, we arrive at (3.2).

We can thereby establish the following consequences of Lemma 3.2.

**Lemma 3.3.** Assume that $p \geq 2$. Let (1.9), (1.10) hold, and $\Psi$ and $K$ be given in Lemma 3.1. Then there exists $C \geq 0$ such that for any $\varepsilon \in (0, 1)$ we have

$$
\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_\Omega |\nabla \psi(c_\varepsilon)|^2 + K \int_\Omega |u_\varepsilon|^2 \leq C \quad \text{for all } t \in (0, T_{\max, \varepsilon})
$$

and

$$
\int_0^T \int_\Omega \frac{n_\varepsilon}{2} |\nabla n_\varepsilon|^{\frac{p-1}{p}}p + \int_0^T \int_\Omega \frac{D^2 c_\varepsilon}{c_\varepsilon} \leq C(T + 1)
$$

for all $t \in (0, T_{\max, \varepsilon})$.

**Proof.** Set

$$
y_\varepsilon(t) := \int_\Omega \left\{ n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_\Omega |\nabla \psi(c_\varepsilon)|^2 + K \int_\Omega |u_\varepsilon|^2 \right\} (\cdot, t) \quad \text{for all } t \in (0, T_{\max, \varepsilon})
$$

and

$$
h_\varepsilon(t) := \int_\Omega \left\{ |\nabla n_\varepsilon|^{\frac{p-1}{p}} |p + \int_\Omega \frac{D^2 c_\varepsilon}{c_\varepsilon} + \int_\Omega |\nabla c_\varepsilon|^4 + \int_\Omega |\nabla u_\varepsilon|^2 \right\} (\cdot, t)
$$

for all $t \in (0, T_{\max, \varepsilon})$. Then (3.2) implies that

$$
y_\varepsilon(t) + \frac{1}{K_0}h_\varepsilon(t) \leq K_0 \quad \text{for all } t \in (0, T_{\max, \varepsilon}).
$$

In order to introduce dissipative term in (3.10), we show that $y_\varepsilon(t)$ is dominated by $h_\varepsilon(t)$. We first use the elementary inequality $z \ln z \leq 2z^\frac{3}{2}$ for all $z \geq 0$. And invoking the Young’s inequality together with (3.7) (pick $\delta = 1$), we infer that

$$
\int_\Omega n_\varepsilon \ln n_\varepsilon \leq 2 \int n_\varepsilon \frac{6}{5} = 2\|n_\varepsilon\|^{\frac{5}{6}} \leq \|n_\varepsilon\|^2_\varepsilon + 2\frac{5}{6} \leq \|\nabla n_\varepsilon\|^{\frac{p-1}{p}}(\frac{p}{p-1})^p + C_7
$$

with some $C_7 > 0$ for all $t \in (0, T_{\max, \varepsilon})$. From the standard Poincaré inequality, there exists $C_8 > 0$ such that

$$
K \int_\Omega |u_\varepsilon|^2 \leq C_8 \int_\Omega |\nabla u_\varepsilon|^2 \quad \text{for all } t \in (0, T_{\max, \varepsilon})
$$

(3.12)
Recalling the definitions of $\Psi$ and $g$ in Lemma 3.1, using (1.9), we have
\[
g(s) := \frac{f(s)}{\chi(s)} \in C^1([0, \infty)) \quad \text{and} \quad g(0) = 0.
\]
Hence there exist two constants $C^{-}_g > 0$ and $C^{+}_g > 0$ such that $C^{-}_g s \leq g(s) \leq C^{+}_g s$, which yields
\[
\frac{1}{2} \int_{\Omega} |\nabla \Psi(c_\varepsilon)|^2 = \frac{1}{2} \int_{\Omega} \frac{|\nabla c_\varepsilon|^2}{g(c_\varepsilon)} \leq \int_{\Omega} \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3 g(c_\varepsilon)} + \frac{1}{16} \int_{\Omega} \frac{c_\varepsilon^3}{g(c_\varepsilon)} \leq \int_{\Omega} \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3 g(c_\varepsilon)} + \frac{1}{16(C^{-}_g)^2} \int_{\Omega} c_\varepsilon \leq \int_{\Omega} \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{s_0|\Omega|}{16(C^{-}_g)^2} \quad \text{for all } t \in (0, T_{max, \varepsilon}). \tag{3.13}
\]
In conjunction with (3.11) and (3.12), we obtain
\[
y_\varepsilon(t) \leq C_9 h_\varepsilon(t) + C_9 \quad \text{for all } t \in (0, T_{max, \varepsilon})
\]
with $C_9 := \max\{1, C_8, C_7 + \frac{s_0|\Omega|}{16(C^{-}_g)^2}\}$, which together with (3.10) imply that $y_\varepsilon$ satisfies the ODI
\[
y_\varepsilon'(t) + \frac{1}{2K_0} h_\varepsilon(t) + \frac{1}{2C_9K_0} y_\varepsilon(t) \leq K_0 + \frac{1}{2K_0} := C_{10} \quad \text{for all } t \in (0, T_{max, \varepsilon}). \tag{3.14}
\]
This firstly shows that
\[
y_\varepsilon(t) \leq \max\{ \sup_{\varepsilon \in (0,1)} y_\varepsilon(0), 2C_9K_0C_{10}\} \quad \text{for all } t \in (0, T_{max, \varepsilon}).
\]
In view of (2.2)-(2.4) and [42, Lemma 3.7], we obtain (3.8). Secondly, another integration of (3.14) proves (3.9). \hfill \square

4. Global existence for the regularized problem (2.1)

Now we are in the position to show that the solution of the approximate problem (2.1) is actually global in time. The idea of the proof is based on the argument in [42, Lemma 3.9] (see also [49, Section 4]) for the linear case $p = 2$. Throughout this section, all the constants below possibly depending on $\varepsilon$.

**Lemma 4.1.** Assume that $p \geq 2$. For each $\varepsilon \in (0,1)$, the solutions of (2.1) are global in time.

**Proof.** Assume that $T_{max, \varepsilon} < \infty$ for some $\varepsilon \in (0,1)$. As an particular consequence of Lemma 3.3 and (2.10) we can find $C_1 > 0$ and $C_2 > 0$ such that
\[
\int_{0}^{T_{max, \varepsilon}} \int_{\Omega} |\nabla c_\varepsilon(\cdot, t)|^4 \leq C_1 \quad \text{and} \quad \int_{\Omega} |u_\varepsilon(\cdot, t)|^2 \leq C_2 \quad \text{for all } t \in (0, T_{max, \varepsilon}). \tag{4.1}
\]

We first multiply the first equation in (2.1) by $\beta n_\varepsilon^{\beta-1}$ with $\beta \in (1, 8(1 - \frac{1}{p}))$ and using integration by parts, we have
\[
\frac{d}{dt} \int_{\Omega} n_\varepsilon^{\beta} + \beta(\beta - 1) \int_{\Omega} (|\nabla n_\varepsilon|^2 + \varepsilon)^{\frac{\beta-2}{2}} n_\varepsilon^{\beta-2} |\nabla n_\varepsilon|^2
\]
\[
= \beta(\beta - 1) \int_{\Omega} n_\varepsilon F'_\varepsilon(n_\varepsilon) \chi(c_\varepsilon) \nabla c_\varepsilon n_\varepsilon^{\beta-2} \nabla n_\varepsilon
\]
for all \( t \in (0, T_{\text{max}, \varepsilon}) \). A combination of (2.6) and Young’s inequality shows that

\[
\frac{d}{dt} \int_{\Omega} n_{\varepsilon}^\beta + \beta(\beta - 1) \int_{\Omega} (|\nabla n_{\varepsilon}|^2 + \varepsilon) \frac{\beta^2}{2} n_{\varepsilon}^\beta - 2|\nabla n_{\varepsilon}|^2 \\
\leq \frac{\beta(\beta - 1)}{\varepsilon} \max_{s \in [0, s_0]} \chi(s) \int_{\Omega} n_{\varepsilon}^{\beta - 2} |\nabla c_{\varepsilon}| |\nabla n_{\varepsilon}| \\
\leq \int_{\Omega} n_{\varepsilon}^{\beta - 2} \left( \beta(\beta - 1)|\nabla n_{\varepsilon}|^p + C_3 |\nabla c_{\varepsilon}|^{\frac{p}{p - 1}} \right)
\]

for all \( t \in (0, T_{\text{max}, \varepsilon}) \) with some \( C_3 > 0 \). Obviously, \( \int_{\Omega} n_{\varepsilon}^{\beta - 2}|\nabla n_{\varepsilon}|^p \leq \int_{\Omega} (|\nabla n_{\varepsilon}|^2 + \varepsilon) \frac{\beta^2}{2} n_{\varepsilon}^\beta - 2|\nabla n_{\varepsilon}|^2 \), which implies that

\[
\frac{d}{dt} \int_{\Omega} n_{\varepsilon}^\beta + \beta(\beta - 1) \int_{\Omega} n_{\varepsilon}^{\beta - 2}|\nabla n_{\varepsilon}|^p \leq \int_{\Omega} n_{\varepsilon}^{\beta - 2} \left( \beta(\beta - 1)|\nabla n_{\varepsilon}|^p + C_3 |\nabla c_{\varepsilon}|^{\frac{p}{p - 1}} \right)
\]

for all \( t \in (0, T_{\text{max}, \varepsilon}) \). Observing that \( p > 2 \) implies \( \frac{p}{p - 1} < 4 \), we obtain there exist \( C_4, C_5 > 0 \) such that

\[
\frac{d}{dt} \int_{\Omega} n_{\varepsilon}^\beta \leq C_3 \int_{\Omega} n_{\varepsilon}^{\beta - 2}|\nabla c_{\varepsilon}|^{\frac{p}{p - 1}} \\
\leq C_3 \int_{\Omega} \left(|\nabla c_{\varepsilon}|^4 + C_4 n_{\varepsilon}^{(\beta - 2)(\frac{4(p - 1)}{3p - 4})} \right) \\
\leq C_3 \int_{\Omega} |\nabla c_{\varepsilon}|^4 + C_5 \int_{\Omega} (n_{\varepsilon}^\beta + |\Omega|) \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\]

(4.2)

since \( \beta \leq 8(1 - \frac{1}{p}) \) ensures that \( (\beta - 2)(\frac{4(p - 1)}{3p - 4}) \leq \beta \). Integrating (4.2) and using (4.1) yields \( C_6 > 0 \) such that

\[
\int_{\Omega} n_{\varepsilon}^\beta \leq C_6 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\]

(4.3)

where \( \beta \in (1, 8(1 - \frac{1}{p})) \).

We next use (4.3) to estimate \( \|c_{\varepsilon}\|_{W^{1, \infty}(\Omega)} \). Obviously, \( p > 2 \) implies \( 8(1 - \frac{1}{p}) > 4 \), thus as a particular consequence of (4.3), we have

\[
\int_{\Omega} n_{\varepsilon}^4 \leq C_6 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\]

(4.4)

which is enough to obtain

\[
\|u_{\varepsilon}\|_{L^\infty(\Omega)} \leq C_7 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon})
\]

(4.5)

and

\[
\|\nabla c_{\varepsilon}\|_{L^4(\Omega)} \leq C_8 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon})
\]

(4.6)

with some \( C_7 > 0 \) and \( C_8 > 0 \). Since the proof of (4.5) and (4.6) is exactly the same as in [42, Lemma 3.9], we refer the reader to it for more details. Notice that the second equation of (2.1) is equivalent to

\[
\partial_t c_{\varepsilon} + u_{\varepsilon} \cdot \nabla c_{\varepsilon} = (\Delta - 1)c_{\varepsilon} + c_{\varepsilon} - F_{\varepsilon}(n_{\varepsilon}) f(c_{\varepsilon})
\]

and then we rewrite \( c_{\varepsilon} \) by the variation-of-constant formula

\[
c_{\varepsilon}(t) = e^{t(\Delta - 1)} c_{0, \varepsilon} + \int_0^t e^{(t-s)(\Delta - 1)} (c_{\varepsilon} - F_{\varepsilon}(n_{\varepsilon}) f(c_{\varepsilon}) - u_{\varepsilon} \cdot \nabla c_{\varepsilon})(s) ds
\]
for all $t \in (0, T_{\text{max}, \varepsilon})$. For $p \in (1, \infty)$, let $A := A_p$ denote the sectorial operator defined by

$$A_p u := -\Delta u \quad \text{for} \quad u \in D(A_p) := \left\{ \varphi \in W^{2,p}(\Omega); \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial \Omega \right\}.$$  

Then the operator $(A + 1)$ possesses fractional powers $(A + 1)^{\beta}$, $\beta \geq 0$, the domains of which have the embedding property $D((A_1 + 1)^{\theta}) \hookrightarrow W^{1,\infty}(\Omega)$ for $\theta \in \left(\frac{2}{p^*}, 1\right)$ (cf. [37, 48]). Hence, by virtue of $L^p - L^q$ estimates associated heat semigroup (cf. [37]), (2.10), (1.9), (2.5), (4.4), (4.5) and (4.6), there exist positive constants $C_9, C_{10}, C_{11}$ and $C_{12}$ such that

$$\|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq \|(-\Delta + 1)^{\theta}c_\varepsilon(\cdot, t)\|_{L^4(\Omega)} \leq C_9 t^{-\theta} e^{-\varepsilon t} \|c_\varepsilon\|_{L^4(\Omega)} + C_{10} \int_0^t (t-s)^{-\theta} e^{-\varepsilon(t-s)} ds + C_{10} \int_0^t (t-s)^{\theta} e^{-\varepsilon(t-s)} ds + C_9 t^{-\theta} e^{-\varepsilon t} \|c_\varepsilon\|_{L^4(\Omega)} + C_{11} \Gamma(1 - \theta) \leq C_{12} \quad \text{for all} \quad t \in (\tau, T_{\text{max}, \varepsilon}) \quad (4.7)$$

with some $\nu > 0$, where $\Gamma(\cdot)$ is the Gamma function.

We finally use (4.7) to estimate $\|n_\varepsilon\|_{L^\infty(\Omega)}$. For any $\beta > 1$, from (4.7) we know that there exist positive constants $C_{13}, C_{14}$ and $C_{15}$ such that

$$\frac{d}{dt} \int_\Omega n_\varepsilon^\beta + \beta(\beta - 1) \int_\Omega n_\varepsilon^{\beta-2} |\nabla n_\varepsilon|^p$$

$$\leq \frac{d}{dt} \int_\Omega n_\varepsilon^{\beta} + \beta(\beta - 1) \int_\Omega (|\nabla n_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} n_\varepsilon^{\beta-2} |\nabla n_\varepsilon|^2$$

$$= \beta(\beta - 1) \int_\Omega n_\varepsilon F'_e(n_\varepsilon) \chi(c_\varepsilon) \nabla c_\varepsilon n_\varepsilon^{\beta-2} \nabla n_\varepsilon$$

$$\leq C_{13} \int_\Omega n_\varepsilon^{\beta-2} |\nabla n_\varepsilon|$$

$$\leq \int_\Omega n_\varepsilon^{\beta-2}(\beta(\beta - 1)|\nabla n_\varepsilon|^p + C_{14})$$

$$\leq \beta(\beta - 1) \int_\Omega n_\varepsilon^{\beta-2} |\nabla n_\varepsilon|^p + \int_\Omega n_\varepsilon^{\beta} + C_{15} \quad \text{for all} \quad t \in (\tau, T_{\text{max}, \varepsilon}). \quad (4.8)$$

Therefore, integrating (4.8) with respect to $t$ yields $C_{16} > 0$ such that

$$\int_\Omega n_\varepsilon^\beta \leq C_{16} \quad \text{for all} \quad t \in (\tau, T_{\text{max}, \varepsilon})$$

with any $\beta \geq 1$. Upon an application of the well-known Moser-Alikakos iteration procedure [1, 32], we see that

$$\|n_\varepsilon\|_{L^\infty(\Omega)} \leq C_{17} \quad \text{for all} \quad t \in (\tau, T_{\text{max}, \varepsilon}) \quad (4.9)$$

with some $C_{17} > 0$. In view of (4.5), (4.7) and (4.9), we apply Lemma 2.1 to reach a contradiction. □
5. Further Uniform Estimates for (2.1)

As mentioned in Remark 1.1, it’s crucial to prove (1.12). The following lemma plays a key role in the proof.

**Lemma 5.1.** Let \(m_0 \geq 1\), \(p > \frac{25}{12}\), \(m \geq 1\) be such that
\[
\frac{m_0(3p - 4)}{4(p - 1)} + \frac{p - 2}{p - 1} < m \leq m_0\left(p - \frac{4}{3}\right) + 3(p - 2),
\]
and \(\int_\Omega m_0 \leq C\). Then for all \(K > 0\) there exist \(C = C(m, p, K) > 0\) such that if for all \(\varepsilon \in (0, 1)\), there hold
\[
\int_\Omega \varepsilon^{m_0}(\cdot, t) \leq K \quad \text{for all } t \geq 0
\]
and
\[
\int_0^t \int_\Omega |\nabla \varepsilon(\cdot, s)|^4 ds \leq K(t + 1) \quad \text{for all } t \geq 0,
\]
then we have
\[
\int_\Omega \varepsilon^m(\cdot, t) \leq C(t + 1) \quad \text{for all } t \geq 0
\]
and
\[
\int_0^t \int_\Omega |\nabla \varepsilon^{m_+}|^p \leq C(t + 1) \quad \text{for all } t \geq 0,
\]
where \(m_+ = \frac{m - 2}{p} + 1\).

**Proof.** Test the first equation in (2.1) by \(n_\varepsilon^{m-1}\) and use Young’s inequality along with (1.9), (2.5) and (2.10) to see that for all \(t > 0\),
\[
\frac{1}{m} \frac{d}{dt} \int_\Omega \varepsilon^m + (m - 1) \int_\Omega |\nabla \varepsilon|^p \varepsilon^{m-2} \leq C_1(m - 1) \int_\Omega \varepsilon^{m-1}|\nabla \varepsilon| \cdot |\nabla \varepsilon|
\leq (m - 1) \int_\Omega \left(\frac{1}{2}|\nabla \varepsilon|^p \varepsilon^{m-2} + C_2(\varepsilon^{\frac{2-m}{p}} + m \varepsilon^{m-1}|\nabla \varepsilon|)^{p'}\right)
\]
so that
\[
\frac{1}{m} \frac{d}{dt} \int_\Omega \varepsilon^m + \frac{m - 1}{2} \int_\Omega |\nabla \varepsilon|^p \varepsilon^{m-2} \leq C_2(m - 1) \int_\Omega (\varepsilon^{\frac{2-m}{p}} + m \varepsilon^{m-1}|\nabla \varepsilon|)^{p'} \quad \text{for all } t > 0,
\]
where \(p' = \frac{p}{p - 1}\). Let
\[
\beta = \left(\frac{2 - m}{p} + m - 1\right) \cdot p' = m + \frac{1}{p - 1} - 1
\]
and
\[
m_* = \frac{m - 2}{p} + 1.
\]
Noticing that
\[
\int_\Omega |\nabla \varepsilon|^p \varepsilon^{m-2} = \int_\Omega |\varepsilon^{\frac{m-2}{p}} \nabla \varepsilon|^p = \frac{1}{m_+} \int_\Omega |\nabla \varepsilon^{m_+}|^p \quad \text{for all } t > 0,
\]
(5.6) turns into
\[
\frac{1}{m} \frac{d}{dt} \int_\Omega \varepsilon^m + \frac{m - 1}{2m_*} \int_\Omega |\nabla \varepsilon^{m_*}|^p \leq C_2(m - 1) \int_\Omega \varepsilon^\beta |\nabla \varepsilon|^{p'} \quad \text{for all } t > 0.
\]
Now in order to further estimate the right-hand side herein, we invoke the Hölder’s inequality to obtain
\[
\int_\Omega \varepsilon^\beta |\nabla \varepsilon|^{p'} \leq \left(\int_\Omega \varepsilon^{\beta n}|\nabla \varepsilon|^4\right)^{\frac{1}{4}} \left(\int_\Omega |\nabla \varepsilon|^4\right)^{\frac{1}{4}} \quad \text{for all } t > 0,
\]
with \( \alpha = \frac{4(p-1)}{3p-1} \), and \( \alpha' = \frac{\alpha}{\alpha - 1} = \frac{4(p-1)}{p} \).

Due to the left part of our assumption (5.1) we have

\[
\frac{m_0}{m^*} - \frac{\beta \alpha}{m^*} = -\frac{p(4m(p-1) + m_0(4 - 3p) - 4p + 8)}{(3p - 4)(m + p - 2)} < 0.
\]

And from

\[
\frac{m_0}{\beta \alpha} - \left( \frac{1}{p} - \frac{1}{3} \right) = \frac{m(4p - 7) + 5(p - 2)}{12(m(p - 1) - p + 2)} \geq 0
\]

we know that \( W^{1,p}(\Omega) \hookrightarrow L^\frac{\alpha 0}{m^*} (\Omega) \hookrightarrow L^\frac{\alpha}{m^*} (\Omega) \), whence in particular the number

\[
\theta := \frac{3(m + p - 2)(4m(p-1) + m_0(4 - 3p) - 4p + 8)}{4(m(p-1) - p + 2)(3m + (m_0 + 3)p - 3(m_0 + 2))}
\]

satisfies \( \theta > 0 \), since the left part of our assumption (5.1) ensures

\[
m - \left( m_0(1 - \frac{p}{3}) - p + 2 \right) > \frac{m_0(3p - 4)}{4p - 4} + \frac{p - 2}{p - 1} - \left( m_0(1 - \frac{p}{3}) - p + 2 \right) = \frac{p(m_0(4p - 7) + 12(p - 2))}{12(p - 1)} > 0,
\]

which is enough to warrant that

\[3m + (m_0 + 3)p - 3(m_0 + 2) > 0.\]

And also \( \theta < 1 \) since

\[
\theta - 1 = -\frac{m_0 p (5p - 2) + (4p - 7)m}{4(m(p-1) - p + 2)(3m + (m_0 + 3)p - 3(m_0 + 2))} < 0,
\]

and accordingly the Gagliardo-Nirenber inequality provides \( C_3 > 0 \) such that

\[
\left( \int_{\Omega} n_\epsilon^{\beta \alpha} \right)^{\frac{1}{\beta \alpha}} \leq \|n_\epsilon^{m^*}\|^{\frac{\beta}{m^*}}_{\Omega} \frac{\beta \alpha}{m^*} \\
\leq C_3\|\nabla n_\epsilon^{m^*}\|_{p^*}^{\frac{\beta}{m^*}} \|n_\epsilon^{m^*}\|^{\frac{\beta}{m^*}}_{m^*} (1 - \theta) + C_3\|n_\epsilon^{m^*}\|_{m^*}^{\frac{\beta}{m^*}} \\text{ for all } t > 0,
\]

where \( \theta \) given by (5.9) satisfies

\[
\frac{m^*}{\beta \alpha} = \theta \left( \frac{1}{p} - \frac{1}{3} \right) + (1 - \theta) \frac{m^*}{m_0}.
\]

As

\[
\|n_\epsilon^{m^*}\|_{m^*}^{m_0} = \int_{\Omega} n_\epsilon^{m_0} \leq K \quad \text{for all } t > 0
\]

by (5.2), together with (5.8), \( C_p \) inequality, and Young’s inequality this shows that we can find \( C_4 > 0 \) fulfilling

\[
\int_{\Omega} n_\epsilon^\beta |\nabla c_\epsilon|^{p'} \leq C_4(\|\nabla n_\epsilon^{m^*}\|_{p^*}^{\frac{\beta}{p^*}} + 1)(\int_{\Omega} |\nabla c_\epsilon|^{4})^{\frac{1}{4}} \\
\leq \delta(\|\nabla n_\epsilon^{m^*}\|_{p^*}^{\frac{\beta}{p^*}} + 1) + C_5 \int_{\Omega} |\nabla c_\epsilon|^{4} \quad \text{for all } t > 0,
\]

where \( \delta > 0 \) and \( C_5 \) are two constant to be chosen later. From (5.1) we have

\[
\frac{\beta}{m^*} \theta \alpha - p = \frac{p^2(3m - m_0(3p - 4) - 9(p - 2))}{(3p - 4)(3m - m_0 + (m_0 + 3)(p - 2))} \leq 0,
\]
whence another application of Young’s inequality and an appropriate choice of \( \delta \) yield \( C_5 > 0 \) satisfying
\[
C_2(m - 1) \int_\Omega n_\varepsilon^\beta |\nabla c_\varepsilon|^p' \leq \frac{(m - 1)}{4m_*^p} \int_\Omega |\nabla n_\varepsilon^m|^p + C_5 \int_\Omega |\nabla c_\varepsilon|^4 + C_5 \quad \text{for all } t > 0.
\]
This shows that (5.7) implies that
\[
\frac{1}{m} \frac{d}{dt} \int_\Omega n_\varepsilon^m + \frac{(m - 1)}{4m_*^p} \int_\Omega |\nabla n_\varepsilon^m|^p \leq C_5 \int_\Omega |\nabla c_\varepsilon|^4 + C_5 \quad \text{for all } t > 0. \tag{5.11}
\]
Let \( y(t) := \int_\Omega n_\varepsilon^m(\cdot, t) > 0 \), and \( h(t) := C_5 \int_\Omega |\nabla c_\varepsilon|^4 + C_5 \), then in view of the nonnegativity of \( h(t) \) and our assumption (5.3), (5.11) thus firstly implies that
\[
y(t) \leq y(0) + m \int_0^t h(s) ds \leq y(0) + C_5 K m(t + 1) + C_5 t \quad \text{for all } t > 0, \tag{5.12}
\]
whereupon (5.3), (5.11) and (5.12) secondly entail that
\[
\int_0^t \int_\Omega |\nabla n_\varepsilon^m|^p \leq \frac{4m_*^p}{m - 1} \int_0^t h(s) ds + \frac{4m_*^p}{(m - 1)m} y(0)
\leq \frac{4m_*^p}{m - 1} (C_5 K (t + 1) + C_5 t) + \frac{4m_*^p}{(m - 1)m} y(0) \quad \text{for all } t > 0,
\]
so that indeed both (5.4) and (5.5) hold with some conveniently large \( C = C(m, p, K) > 0 \). \( \square \)

**Remark 5.1.** We shall point out that there exist \( m \geq 1 \) satisfies (5.1). Indeed, a formal computation shows that
\[
m_0 \left( p - \frac{4}{3} \right) + 3(p - 2) - \left( \frac{m_0(3p - 4)}{4p - 4} + \frac{p - 2}{p - 1} \right) = \frac{(3p - 4)(m_0(4p - 7) + 12(p - 2))}{12(p - 1)} > 0.
\]
and
\[
m_0 \left( p - \frac{4}{3} \right) + 3(p - 2) - 1 = m_0 \left( p - \frac{4}{3} \right) + 3p - 7 \geq 4p - \frac{25}{3} > 0,
\]
since \( p > \frac{25}{12} \).

We shall see that this indeed leads to improved information whenever \( p > \frac{32}{15} > \frac{25}{12} \). Repeatedly applying Lemma 5.1, we can gradually raise the parameter \( m \) up to 2, which is equivalent to \( m_\ast = 1 \), so we can get a higher order gradient estimate of \( n_\varepsilon \).

**Corollary 5.2.** Let \( p = \frac{32}{15} + \delta \) for some \( \delta > 0 \) small such that \( \delta_1 = \frac{\log(\frac{25\delta}{12})}{\log(\delta + \frac{4}{3})} \) is a positive integer and \( \int_\Omega n_\varepsilon^2 \leq C \). Then for all \( K > 0 \) there exist \( C = C(m, p, K) > 0 \) such that if for some \( \varepsilon \in (0, 1) \) we have
\[
\int_\Omega n_\varepsilon(\cdot, t) \leq K \quad \text{for all } t \geq 0
\]
and
\[
\int_0^t \int_\Omega |\nabla c_\varepsilon(\cdot, s)|^4 ds \leq K(t + 1) \quad \text{for all } t \geq 0,
\]
then
\[
\int_0^T \int_\Omega |\nabla n_\varepsilon|^p \leq C(T + 1) \quad \text{for all } t \geq 0. \tag{5.13}
\]

**Proof.** We define \( (m_k)_{k \in \mathbb{N}_0} \subset \mathbb{R} \) by letting \( m_0 = 1 \) and
\[
m_{k+1} = m_k \left( \frac{32}{15} + \delta - \frac{4}{3} \right) + 3 \left( \frac{32}{15} + \delta - 2 \right) = m_k \left( \delta + \frac{4}{5} \right) + 3(\delta + \frac{2}{15}) \quad \text{for } k \in \mathbb{N}_0.
\]
A simple computation shows that
\[ m_k = (\delta + \frac{4}{5})^k (1 + \frac{15\delta + 2}{5\delta - 1}) - \frac{15\delta + 2}{5\delta - 1} \quad \text{for } k \in \mathbb{N}. \] (5.14)

Solving the equation
\[ (\delta + \frac{4}{5})^x (1 + \frac{15\delta + 2}{5\delta - 1}) - \frac{15\delta + 2}{5\delta - 1} = 2 \quad \text{for } x \geq 0, \] (5.15)
we have
\[ x = \frac{\log \left( \frac{25\delta^2 + 1}{20\delta + 1} \right)}{\log (\delta + \frac{4}{5})} > 0. \] (5.16)

Since \( \delta_1 = \delta_1(\delta) \) is a continuous and monotonically decreasing real function on the interval \((0, \frac{1}{10})\), and \( \lim_{\delta \to 0} \delta_1 = +\infty \) as well as \( \delta_1|_{\delta=\frac{1}{10}} = \frac{\log(\frac{16}{5})}{\log(\frac{16}{5})} \approx 7.44 \). Hence we can take \( \delta > 0 \) small such that \( \delta_1 \) is a positive integer, and apply Lemma 5.1 \( \delta_1 \) times, then we obtain \( m_{\delta_1} = 2 \) hence \( m_\ast = 1 \) and get the desired result. \( \square \)

A direct application of Corollary 5.2 enable us to get a higher order regularity of \( n_\varepsilon \).

**Lemma 5.3.** There exists \( C > 0 \) such that for any \( \varepsilon \in (0, 1) \), we have
\[ \int_0^T \int_\Omega |u_\varepsilon|^\frac{r}{2} \leq C(T + 1) \quad \text{for all } T > 0, \] (5.17)
and
\[ \int_0^T \int_\Omega n_\varepsilon^r \leq C(T + 1), \quad \text{for all } T > 0, \] (5.18)
where
\[ \begin{cases} \ r \geq 1, \quad p \geq 3, \\ r \in [1, \frac{3p}{3p-3}), \quad p \in (\frac{32}{15}, 3). \end{cases} \] (5.19)

**Proof.** (i) Let \( \theta = \frac{3p(\varepsilon-1)}{(4p-3)p} \), then \( \theta \in [0, 1) \) satisfies
\[ \frac{1}{r} = \theta \left( \frac{1}{p} - \frac{1}{3} \right) + (1 - \theta). \] (5.20)
Thus, invoking the Gagliardo-Nirenberg inequality along with (2.2), Corollary 5.2 and Young’s inequality we obtain \( C_1 > 0 \) and \( C_2 > 0 \) such that
\[ \|n_\varepsilon\|_{L^r((0,T)\times \Omega)} \leq C_1 (\|\nabla n_\varepsilon\|_{L^p((0,T)\times \Omega)}^{\theta} \|n_\varepsilon\|^{(1-\theta)}_{L^1((0,T)\times \Omega)} + \|n_\varepsilon\|_{L^1((0,T)\times \Omega)}) \] (5.21)
\[ \leq C(T + 1) \quad \text{for all } T > 0. \] (5.22)

(ii) As a particular consequence of Lemma 3.3, we can find \( C_3 > 0 \) and \( C_4 > 0 \) such that
\[ \int_\Omega |u_\varepsilon(\cdot, t)|^2 \leq C_3 \quad \text{for all } t > 0 \quad \text{and} \quad \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \leq C_4(T + 1) \quad \text{for all } T > 0. \]
Upon an application of the Gagliardo-Nirenberg inequality together with Poincaré inequality we see that with some $C_5 > 0$ and $C_6 > 0$ we have
\[
\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} = \int_0^T \|u_\varepsilon\|_{\frac{10}{3}}^{\frac{10}{3}} \\
\leq C_5 \int_0^T \left(\|u_\varepsilon\|_{W^{1,2}(\Omega)}\|u_\varepsilon\|_2^2\right)^{\frac{10}{7}} \\
\leq C_6 \int_0^T \|\nabla u_\varepsilon\|_2^2 \\
\leq C_6 C_4 (T + 1) \quad \text{for all } T > 0,
\]
whereby the proof is completed. \hfill \square

Since $3 > p \geq 2$ implies $\frac{3p}{3-p} \geq 6$, we immediately obtain:

**Corollary 5.4.** There exists $C > 0$ such that for $\varepsilon \in (0, 1)$ we have
\[
\int_0^T \int_\Omega n_\varepsilon^6 \leq C(T + 1), \quad \text{for all } T > 0.
\] (5.23)

The following estimates concerning the time derivative can allow us to apply Aubin-Lions lemma later to derive strong compactness properties.

**Lemma 5.5.** There exists $C > 0$ such that for all $\varepsilon \in (0, 1)$ we have
\[
\int_0^T \|\partial_t n_\varepsilon(\cdot, t)\|_{W^{1, p}(\Omega)^*}^{p'} \leq C(T + 1) \quad \text{for all } T > 0,
\] (5.24)
\[
\int_0^T \|\partial_t c_\varepsilon(\cdot, t)\|_{W^{1, 3}(\Omega)}^{\frac{10}{3}} \leq C(T + 1) \quad \text{for all } T > 0,
\] (5.25)
\[
\int_0^T \|\partial_t u_\varepsilon(\cdot, t)\|_{W^{1, 5}(\Omega)^*}^{\frac{5}{2}} \leq C(T + 1) \quad \text{for all } T > 0,
\] (5.26)
where $p' = \frac{p}{p-1}$.

**Proof.** (i) For arbitrary $t > 0$ and $\varphi \in C^\infty(\bar{\Omega})$, multiplying the first equation in (2.1) by $\varphi$, integrating by parts and using the Hölder’s inequality we obtain
\[
|\int_\Omega \partial_t n_\varepsilon(\cdot, t)\varphi| \\
= | -\int_\Omega (|E_n\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla n_\varepsilon \cdot \nabla \varphi + \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon) \chi(c_\varepsilon) \nabla c_\varepsilon \cdot \nabla \varphi + \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \varphi| \\
\leq \int_\Omega (|E_n\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla n_\varepsilon| |\nabla \varphi| + C |n_\varepsilon \nabla c_\varepsilon|_{p'}|\nabla \varphi|_p + |n_\varepsilon u_\varepsilon|_{p'}|\nabla \varphi|_p \\
\leq C_1 (|E_n\varepsilon|^2 + \varepsilon)^{\frac{p-1}{2}} + |n_\varepsilon \nabla c_\varepsilon|_{p'} + |n_\varepsilon u_\varepsilon|_{p'})|\nabla \varphi|_p
\]
with some $C_1 > 0$. By lemma 3.3 and (2.10) we have
\[
\int_0^T \int_\Omega |\nabla c_\varepsilon|^4 = \int_0^T \int_\Omega \left|\frac{\nabla c_\varepsilon}{c_\varepsilon^3}\right|^{4} c_\varepsilon^3 \leq C_2 (T + 1) \quad \text{for all } T > 0
\] (5.27)
with some $C_2 > 0$. Notice that $\frac{1}{6} + \frac{1}{4} < \frac{1}{6} + \frac{3}{10} < \frac{1}{p'}$. Thus, invoking (5.27) along with (5.13), (5.23), Lemma 3.3, Hölder’s inequality and Young’s inequality we obtain positive constants $C_3, C_4,$
and \( C_5 \) such that
\[
\int_0^T \| \partial_t \phi(\cdot, t) \|_{(W^{1,p}(\Omega))^*}^{p'} dt 
\]
\[
\leq C_3 \left( \int_0^T \int_\Omega (|\nabla n| + 1) + \int_0^T \int_\Omega |n| \nabla c \right) + \int_0^T \int_\Omega |n u| \nabla c \right) 
\]
\[
\leq C_4 \left( \int_0^T \int_\Omega (|\nabla c| + 1) + \int_0^T \int_\Omega |n| \nabla c \right) + \int_0^T \int_\Omega |n u| \nabla c \right) 
\]
\[
\leq C_5(T + 1) \quad \text{for all } T > 0. \quad (5.28)
\]

(ii) Likewise, given any \( \varphi \in C^\infty(\Omega) \), multiplying the second equation in (2.1) by \( \varphi \) to see that
\[
\left| \int_\Omega \partial_t c \varphi \right| = - \int_\Omega \nabla c \cdot \nabla \varphi - \int_\Omega F(c) \varphi + \int_\Omega u c \varphi \nabla \varphi \right| 
\]
\[
\leq C_6 \left( \| \nabla c \|_{W^{1,4}(\Omega)} + \| n c \|_{W^{1,4}(\Omega)} + \| u c \|_{W^{1,4}(\Omega)} \right) \| \varphi \|_{W^{1,4}(\Omega)} 
\]
\[
\leq C_7 \left( \| \nabla c \|_{W^{1,4}(\Omega)} + \| n c \|_{W^{1,4}(\Omega)} + \| u c \|_{W^{1,4}(\Omega)} \right) \| \varphi \|_{W^{1,4}(\Omega)},
\]
with some \( C_6 > 0 \) and \( C_7 > 0 \). A combination of (5.17), (5.23) and (5.27) shows that there exist positive constants \( C_8 \) and \( C_9 \) such that
\[
\int_0^T \| \partial_t c \|_{W^{1,4}(\Omega)}^{10} dt 
\]
\[
\leq C_8 \left( \int_0^T \int_\Omega |\nabla c|^{4} + \int_0^T \int_\Omega |n|^{6} + \int_0^T \int_\Omega |u|^{10/3} + |\Omega| T \right) 
\]
\[
\leq C_9(T + 1) \quad \text{for all } T > 0. \quad (5.29)
\]

(iii) (see also [34, Lemma 3.11]) Given \( \varphi \in C^\infty(\Omega; \mathbb{R}^3) \), we infer form the third equation in (2.1) that there exist positive constants \( C_{10} \) and \( C_{11} \) such that
\[
\int_\Omega \partial_t u \varphi \leq C_{10} \left( \int_\Omega |\nabla u|^2 + C_{11} \int_\Omega |Y u| \varphi \right) 
\]
\[
+ C_{11} \int_\Omega n^6 + C_{11} |\Omega| T \quad \text{for all } T > 0, \quad (5.32)
\]
where we use Young’s inequality and \( \nabla \Phi \in L^\infty(\Omega) \). Since \( \| Y v \|_{L^2(\Omega)} \leq \| v \|_{L^2(\Omega)} \) for all \( v \in L^2(\Omega) \) and hence \( \int_0^T \int_\Omega |Y u|^2 \leq \int_0^T \int_\Omega |u|^2 + |\Omega| T \) for all \( T > 0 \), (5.26) results from this upon another application of Lemma 3.3, Lemma 5.3 and Corollary 5.4. \( \square \)

6. Passing to the limit. Proof of Theorem 1.1

In this section we construct global weak solutions for (1.1), (1.6) and (1.7). Before going into detail, let us first give the definition of weak solution.

**Definition 6.1.** We call \((n, c, u)\) a **global weak solution** of (1.1), (1.6) and (1.7) if
\[
n \in L^1_{loc}(\Omega \times [0, \infty)), \quad c \in L^1_{loc}([0, \infty); W^{1,1}(\Omega)), \quad u \in (L^1_{loc}([0, \infty); W^{1,1}(\Omega)))^3
\]
such that \( n \geq 0 \) and \( c \geq 0 \) a.e. in \( \Omega \times (0, \infty) \), and that
\[
|\nabla n|^{p-2}\nabla n, \ n\chi(\nabla c, \ nu \text{ and } cu \text{ belong to } (L_{\text{loc}}^1([0, \infty); L^1(\Omega)))^3,
\]
\[
\n(u \otimes u) \in (L_{\text{loc}}^1([0, \infty); L^1(\Omega)))^{3 \times 3}
\]
and that
\[
\int_0^\infty \int_{\Omega} n_\epsilon \phi_1 - \int_0^\infty \int_{\Omega} nu \cdot \nabla \phi_1 = - \int_0^\infty \int_{\Omega} |\nabla n|^{p-2}\nabla n \cdot \nabla \phi_1 + \int_0^\infty \int_{\Omega} n\chi(\nabla c, \nabla \phi_1,
\]
\[
\int_0^\infty \int_{\Omega} c \chi \phi_2 - \int_0^\infty \int_{\Omega} cu \cdot \nabla \phi_2 = - \int_0^\infty \int_{\Omega} \nabla c \cdot \nabla \phi_2 - \int_0^\infty \int_{\Omega} nf(c) \phi_2,
\]
\[
\int_0^\infty \int_{\Omega} u \otimes \phi_3 - \int_0^\infty \int_{\Omega} u \otimes u \cdot \nabla \phi_3 = - \int_0^\infty \int_{\Omega} \nabla u \cdot \nabla \phi_3 + \int_0^\infty \int_{\Omega} n \nabla \phi_3,
\]
hold for all \( \phi_1, \phi_2 \in C_0^\infty(\Omega \times [0, \infty)) \) and \( \phi_3 \in (C_0^\infty(\Omega \times [0, \infty)))^3 \) satisfying \( \nabla \phi_3 = 0 \).

We need the following auxiliary lemma before proving Theorem 1.1.

**Lemma 6.1.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^d \) with \( d \geq 1 \) and \( f_k \rightharpoonup f \) in \( L^p(\Omega) \) with \( p \in (1, \infty) \), if there hold
\[
f_k \rightarrow f \quad \text{a.e. in } \Omega, \quad \text{as } k \rightarrow \infty, \quad \text{(6.1)}
\]
then there exists a subsequence \( k = \{k_j\}(j = 1, 2, 3, \cdots) \) such that
\[
f_{k_j} \rightarrow f \quad \text{in } L^q(\Omega), \quad \text{as } k_j \rightarrow \infty \quad \text{(6.2)}
\]
for any \( 1 \leq q < p \).

**Proof.** For any \( 1 \leq q < p \), we have \( |f_k|^q \) uniformly bounded in \( L^\infty(\Omega) \), hence there exists a subsequence \( (k_j)_j \in \mathbb{N} \) such that \( |f_{k_j}|^q \rightharpoonup |f|^q \) in \( L^\infty(\Omega) \). This proves that
\[
\lim_{k_j \rightarrow \infty} \|f_{k_j}\|_q^q = \lim_{k_j \rightarrow \infty} \int_\Omega |f_{k_j}|^q = \lim_{k_j \rightarrow \infty} \int_\Omega 1_{\Omega}|f_{k_j}|^q = \int_\Omega |f|^q = \|f\|_q^q,
\]
which together with (6.1) is enough to complete the proof. Indeed, applying the inequality \(|a - b|^q \leq 2^{q-1}(|a|^q + |b|^q)\) we obtain
\[
2^{q-1}(|f_{k_j}(x)|^q + |f(x)|^q) - |f_{k_j}(x) - f(x)|^q \geq 0, \quad \forall x \in \Omega.
\]

In view of Fatou’s lemma, we obtain
\[
2^q \int_\Omega |f|^q \leq \liminf_{k_j \rightarrow \infty} \int_\Omega [2^{q-1}(|f_{k_j}|^q + |f|^q) - |f_{k_j} - f|^q]
\]
\[
\leq 2^{q-1} \liminf_{k_j \rightarrow \infty} \int_\Omega |f_{k_j}|^q + 2^{q-1} \int_\Omega |f|^q + \liminf_{k_j \rightarrow \infty} \int_\Omega (-|f_{k_j} - f|^q)
\]
\[
= 2^q \int_\Omega |f|^q - \limsup_{k_j \rightarrow \infty} \int_\Omega |f_{k_j} - f|^q,
\]
which means
\[
\limsup_{k_j \rightarrow \infty} \int_\Omega |f_{k_j} - f|^q \leq 0,
\]
and this get the desired result. \( \square \)
We can now prove our main result.

**Proof of Theorem 1.1.** By Lemma 3.3, Corollary 5.4, Lemma 5.3 and Lemma 5.5, for some $C > 0$ independent of $\varepsilon$, there hold

$$
\|n_\varepsilon\|_{L^p_{loc}((0, \infty); W^{1,p}(\Omega))} \leq C(T + 1),
$$
(6.3)

$$
\|(n_\varepsilon)_t\|_{L^p_{loc}((0, \infty); (W^{1,p}(\Omega))^*)} \leq C(T + 1),
$$
(6.4)

$$
\|c_\varepsilon\|_{L^2_{loc}((0, \infty); W^{2,2}(\Omega))} \leq C(T + 1),
$$
(6.5)

$$
\|(c_\varepsilon)_t\|_{L^p_{loc}((0, \infty); (W^{1,p}(\Omega))^*)} \leq C(T + 1),
$$
(6.6)

$$
\|u_\varepsilon\|_{L^2_{loc}((0, \infty); W^{1,2}(\Omega))} \leq C(T + 1),
$$
(6.7)

$$
\|(u_\varepsilon)_t\|_{L^p_{loc}((0, \infty); (W^{1,5}_0(\Omega))^*)} \leq C(T + 1)
$$
(6.8)

for all $T > 0$. Therefore, the Aubin-Lions lemma [24] asserts that

$$
(n_\varepsilon)_{\varepsilon \in (0,1)} \text{ is strongly precompact in } L^p_{loc}(\bar{\Omega} \times [0, \infty)),
$$

$$
(c_\varepsilon)_{\varepsilon \in (0,1)} \text{ is strongly precompact in } L^2_{loc}((0, \infty); W^{1,2}(\Omega)) \text{ and}
$$

$$
(u_\varepsilon)_{\varepsilon \in (0,1)} \text{ is strongly precompact in } L^2_{loc}((\bar{\Omega} \times [0, \infty))).
$$

This together with (2.10), Lemma 3.3, Corollary 5.2 and Corollary 5.4, yields a subsequence $\varepsilon := \varepsilon_j \in (0, 1)$ ($j = 1, 2, 3, \cdots$) and functions $n$, $c$ and $u$ such that

$$
n_\varepsilon \rightarrow n \quad \text{in } L^p_{loc}(\bar{\Omega} \times [0, \infty)), \text{ and a.e. in } \Omega \times (0, \infty),
$$
(6.9)

$$
n_\varepsilon \rightarrow n \quad \text{in } L^r_{loc}(\bar{\Omega} \times [0, \infty)),
$$
(6.10)

$$
|\nabla n_\varepsilon|^{p-2} \nabla n_\varepsilon \rightarrow \nabla n \quad \text{in } L^r_{loc}(\bar{\Omega} \times [0, \infty))
$$
(6.11)

$$
|\nabla n_\varepsilon|^{p-2} \nabla n_\varepsilon \rightarrow \Gamma_1 \quad \text{in } L^r_{loc}(\bar{\Omega} \times [0, \infty))
$$
(6.12)

with $\Gamma_1 = |\nabla n|^{p-2} \nabla n$ which will be showed in Lemma 6.2 and $r$ is given by Lemma 5.3, and

$$
c_\varepsilon \rightarrow c \quad \text{in } L^2_{loc}([0, \infty); W^{1,2}(\Omega)) \text{ and a.e. in } \Omega \times (0, \infty),
$$
(6.13)

$$
c_\varepsilon^* \rightarrow c^* \quad \text{in } L^\infty(\Omega \times (0, \infty)),
$$
(6.14)

$$
\nabla c_\varepsilon \rightarrow \nabla c \quad \text{in } L^4_{loc}(\bar{\Omega} \times [0, \infty))
$$
(6.15)

as well as

$$
u_\varepsilon \rightarrow u \quad \text{in } L^2_{loc}(\bar{\Omega} \times [0, \infty)) \text{ and a.e. in } \Omega \times (0, \infty),
$$
(6.16)

$$
u_\varepsilon^* \rightarrow u^* \quad \text{in } L^\infty([0, \infty); L^2_{\sigma}(\Omega)),
$$
(6.17)

$$
u_\varepsilon \rightarrow u \quad \text{in } L^{\frac{p}{r}}_{loc}(\bar{\Omega} \times [0, \infty))
$$
(6.18)

$$
\nabla \nu_\varepsilon \rightarrow \nabla u \quad \text{in } L^2_{loc}(\Omega \times [0, \infty))
$$
(6.19)

as $\varepsilon \searrow 0$.

According to Lemma 5.3 and (6.9), an application of lemma 6.1 provides a sequence $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ and the limit function $n$ such that $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ and such that

$$
n_\varepsilon \rightarrow n \quad \text{in } L^5_{loc}(\bar{\Omega} \times [0, \infty)), \text{ and a.e. in } \Omega \times (0, \infty).
$$
(6.20)
Similarly, according to (1.9), (2.5)-(2.7), (2.10), (6.3)-(6.19) and Lemma 6.1 we can obtain

\[ F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon \to \chi(c)\nabla c \quad \text{in } L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)), \]  
\[ f(c_\varepsilon) \to f(c) \quad \text{in } L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)), \]  
\[ c_\varepsilon \to c \quad \text{in } L^2_{\text{loc}}(\bar{\Omega} \times [0, \infty)), \]  
\[ F_\varepsilon(n_\varepsilon) \to n \quad \text{in } L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)), \]  
\[ Y_\varepsilon u_\varepsilon \to u \quad \text{in } L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \]  

as \( \varepsilon \searrow 0 \). And moreover we have

\[ n_\varepsilon u_\varepsilon \to nu \quad \text{in } L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)), \]  
\[ n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon \to n\chi(c)\nabla c \quad \text{in } L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)), \]  
\[ F_\varepsilon(n_\varepsilon)f(c_\varepsilon) \to nf(c) \quad \text{in } L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)), \]  
\[ c_\varepsilon u_\varepsilon \to cu \quad \text{in } L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)), \]  
\[ Y_\varepsilon u_\varepsilon \otimes u_\varepsilon \to u \otimes u \quad \text{in } L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \]  

as \( \varepsilon \searrow 0 \). Based on the above convergence properties, we can pass to the limit in each term of weak formulation for (2.1) to construct a global weak solution of (1.1)-(1.7) and thereby completes the proof.

Now it remains to show that \( (6.12) \) holds with \( \Gamma_1 = |\nabla n|^{p-2}\nabla n \). Monotonic method (cf. [24, Section 2.1], see also [3]) is used to prove the convergence. Actually, we have

**Lemma 6.2.** Under the assumptions of Theorem 1.1, we have

\[ |\nabla n_\varepsilon|^{p-2}\nabla n_\varepsilon \to |\nabla n|^{p-2}\nabla n \quad \text{in } L^p_{\text{loc}}(\bar{\Omega} \times [0, \infty)). \]

**Proof.** We define \( \Sigma_T := \{(t, s, x) : (x, s) \in \Omega \times (0, t), t \in [0, T]\} \). It’s equivalent to show that

\[ \int_{\Sigma_T} (\Gamma_1 - |\nabla \sigma|^{p-2}\nabla \sigma) \cdot (\nabla n - \nabla \sigma) dxdsdt \geq 0, \quad \forall \sigma \in L^p(0, T; W^{1, p}(\Omega)). \]  

(6.31)

For all fixed \( \varepsilon > 0 \), we have the decomposition

\[ \int_{\Sigma_T} (|\nabla n_\varepsilon|^{p-2}\nabla n_\varepsilon - |\nabla \sigma|^{p-2}\nabla \sigma) \cdot (\nabla n - \nabla \sigma) dxdsdt = I_1 + I_2 + I_3, \]

with

\[ I_1 = \int_{\Sigma_T} |\nabla n_\varepsilon|^{p-2}\nabla n_\varepsilon \cdot (\nabla n - \nabla n_\varepsilon) dxdsdt, \]
\[ I_2 = \int_{\Sigma_T} (|\nabla n_\varepsilon|^{p-2}\nabla n_\varepsilon - |\nabla \sigma|^{p-2}\nabla \sigma) \cdot (\nabla n_\varepsilon - \nabla \sigma) dxdsdt, \]
\[ I_3 = \int_{\Sigma_T} |\nabla \sigma|^{p-2}\nabla \sigma \cdot (\nabla n_\varepsilon - \nabla n) dxdsdt. \]

Clearly, \( I_2 \geq C|\nabla n_\varepsilon - \nabla \sigma|^p \geq 0 \), where \( C \) is a positive constant only depending on \( p \), and from (6.11) we deduce that \( I_3 \to 0 \) as \( \varepsilon \searrow 0 \).
For $I_1$, if we multiply the first equation of (2.1) by $(n - n_\varepsilon)$ and integrate over $\Sigma_T$, we obtain
\[
\int_{\Sigma_T} \left( (|n_i|^2 + \varepsilon) \frac{p-2}{2} \nabla n_\varepsilon \cdot (\nabla n - \nabla n_\varepsilon) \right) dx ds dt
= - \int_0^T \int_0^t \left< \partial_t n_\varepsilon, n > ds dt + \int_0^T \int_{\Sigma_T} < \partial_t n_\varepsilon, n_\varepsilon > ds dt
- \int_{\Sigma_T} u_\varepsilon \cdot \nabla (n - n_\varepsilon) dx ds dt + \int_{\Sigma_T} n_\varepsilon F_\varepsilon'(n_\varepsilon) \chi(c_\varepsilon) \nabla c_\varepsilon \cdot (\nabla n - \nabla n_\varepsilon) dx ds dt
= - \int_0^T \int_0^t < \partial_t n_\varepsilon, n > ds dt + \frac{1}{2} \int_0^T \int_{\Sigma_T} n_\varepsilon^2 dx ds dt - \frac{T}{2} \int_{\Omega} n_0^2 dx
- \int_{\Sigma_T} u_\varepsilon \cdot \nabla (n - n_\varepsilon) dx ds dt + \int_{\Sigma_T} n_\varepsilon F_\varepsilon'(n_\varepsilon) \chi(c_\varepsilon) \nabla c_\varepsilon \cdot (\nabla n - \nabla n_\varepsilon) dx ds dt
=: J_1 + J_2 + J_3 + J_4 + J_5.
\]
From (5.13), (5.17), (6.9) we know that $J_4 \to 0$ as $\varepsilon \searrow 0$ and from (6.11), (6.27) we obtain $J_5 \to 0$ as $\varepsilon \searrow 0$ since $\frac{1}{2} + \frac{3}{4} + \frac{2}{5} \leq 1$. Therefore, using (2.2), (5.18), (6.3), (6.4), (6.9), (6.11) and the Lebesgue dominated convergence theorem, we obtain
\[
\lim_{\varepsilon \searrow 0} \int_{\Sigma_T} \left( (|n_i|^2 + \varepsilon) \frac{p-2}{2} \nabla n_\varepsilon \cdot (\nabla n - \nabla n_\varepsilon) dx ds dt
= \lim_{\varepsilon \searrow 0} (J_1 + J_2 + J_3)
= - \int_0^T \int_0^t < \partial_t n, n > ds dt + \frac{1}{2} \int_0^T \int_{\Omega} n_\varepsilon^2 dx ds dt - \frac{T}{2} \int_{\Omega} n_0^2 dx = 0.
\]
Hence we have
\[
\lim_{\varepsilon \searrow 0} \int_{\Sigma_T} \left( (|n_i|^2 + \varepsilon) \frac{p-2}{2} \nabla n_\varepsilon \cdot (\nabla n - \nabla n_\varepsilon) dx ds dt = 0,
\]
which is equivalent to $\lim_{\varepsilon \searrow 0} I_1 = 0$. Consequently, we have shown that
\[
\lim_{\varepsilon \searrow 0} \int_{\Sigma_T} \left( (|n_i|^2 + \varepsilon) \frac{p-2}{2} \nabla n_\varepsilon \cdot (\nabla n - \nabla n_\varepsilon) dx ds dt \geq 0,
\]
which proves (6.31). Choosing $\sigma = n - \lambda \xi$ with $\lambda \in \mathbb{R}$ and $\xi \in L^p(0, T; W^{1,p}(\Omega))$ and combining the two inequalities arising from $\lambda > 0$ and $\lambda < 0$, we obtain the assertion of the lemma. \hfill \Box

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