Inferential models and the decision-theoretic implications of the validity property

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Abstract

Inferential models (IMs) are data-dependent, probability-like structures designed to quantify uncertainty about unknowns. As the name suggests, the focus has been on uncertainty quantification for inference, and on establishing a validity property that ensures the IM is reliable in a specific sense. The present paper develops an IM framework for decision problems and, in particular, investigates the decision-theoretic implications of the aforementioned validity property. I show that a valid IM’s assessment of an action’s quality, defined by a Choquet integral, will not be too optimistic compared to that of an oracle. This ensures that a valid IM tends not to favor actions that the oracle doesn’t also favor, hence a valid IM is reliable for decision-making too. In a certain special class of structured statistical models, further connections can be made between the valid IM’s favored actions and those favored by other more familiar frameworks, from which certain optimality conclusions can be drawn. An important step in these decision-theoretic developments is a characterization of the valid IM’s credal set in terms of confidence distributions, which may be of independent interest.

Keywords and phrases: Choquet integral; confidence distribution; imprecise probability; fiducial inference; possibility measure.

1 Introduction

In scientific applications, often the goal is to learn some feature of a system under investigation based on observed data. This was the setting Fisher (1973) had in mind when developing his widely-used framework for scientific inference. In other applications, however, the primary focus might be on decision-making, which can often be carried out successfully without fully understanding the system under investigation. At least intuitively, if one were able to reliably learn about the relevant features of the underlying system, in the sense of Fisher, then good—perhaps even optimal—decisions ought to be within reach. The goal of the present paper is to investigate this connection between reliable or valid probabilistic inference and decision-making in cases where inferences are not based on a precise probability distribution, Bayesian or otherwise.

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Inferential models (IMs) are data-dependent, probability-like structures designed to provide uncertainty quantification about unknowns. Familiar constructions of data-dependent probabilities, such as Bayesian (with genuine or default prior distributions) and fiducial/confidence distributions (Hannig et al. 2016; Xie and Singh 2013), are examples of IMs, but so are other less familiar constructions, such as Dempster–Shafer belief functions (Dempster 2008; Denœux and Li 2013; Shafer 1976; Wasserman 1990), possibility measures (Balch 2012; Hose and Hanss 2021; Liu and Martin 2021), or more general kinds of imprecise probabilities (Augustin et al. 2014; Walley 1991). In the most general case, the IM’s uncertainty quantification takes the form of a data-dependent lower and upper probability defined on a class of assertions/hypotheses about the unknown quantity. Then inference is made based on the magnitudes of the lower/upper probabilities assigned to various hypotheses, i.e., if the lower probability is large, then data supports the hypothesis and, if upper probability is small, then data doesn’t support the hypothesis. The quality of an IM should be judged based on how reliable its uncertainty quantification is. A precise notion of reliability, called validity, was first presented in Martin and Liu (2013, 2015b), and later elaborated on in Martin (2019, 2021a); see Section 2.1 below. Roughly, validity requires that the IM’s data-dependent lower and upper probabilities be suitably calibrated as functions of the data. More precisely, the lower probabilities assigned to false hypotheses don’t tend to be too large and, similarly, the upper probabilities assigned to true hypotheses don’t tend to be too small. Validity also implies that hypothesis tests and confidence regions derived from the IM exactly control frequentist error rates, not just approximately or asymptotically.

The developments described above are all focused on the inference problem; to my knowledge, the decision problem hasn’t yet been considered from the perspective of valid IMs. To set the scene, a statistical decision problem comes equipped with a set \( \mathcal{A} \) of possible actions along with a loss function that maps \( a \in \mathcal{A} \) and a particular value \( \vartheta \) of the unknown to the penalty/reward that results by taking action \( a \) when the state of nature is \( \vartheta \). If, in addition, a data-dependent probability distribution is available to quantify uncertainty about the unknown, then it’s common to choose an action that minimizes the corresponding expected loss. Generalizations of this basic framework to cases where uncertainty is quantified via an imprecise lower/upper probability have been developed, and I’ll adopt the same approach here; that is, I’ll rank actions according to a suitable upper expected loss, a Choquet integral (see Section 2.2). My main focus, however, is on the decision-theoretic implications of the IM’s validity property. As stated in the first paragraph above, if the IM reliably solves the inference problem, then, in some sense, it ought to do the same for the decision problem. The paper’s first key contribution is in Section 4 where I show that the valid IM’s upper expected loss tends to be not too small compared to that of a quasi-oracle who has partial information available about the state of nature. This is relevant because, if the decision-maker is overly optimistic about the quality of an action, i.e., his assessment is more favorable than the quasi-oracle’s, then there’s a chance that he’ll make a poor decision and incur a non-trivial loss. Theorems 2–3 show, however, that validity implies the decision-maker’s and quasi-oracle’s assessments of different actions tend not to be inconsistent and, consequently, the valid IM is reliable for the decision problem too. That is, existence of an action that the decision-maker deems to be relatively good and that the quasi-oracle deems to be relatively poor is a rare event with respect to the sampling distribution of the data.
An important step in these developments, and perhaps of independent interest, is a characterization of the set of precise probabilities compatible with the valid IM’s upper probability. Specifically, using existing results on connections between the level sets of a possibility contour and calibrated prediction sets (e.g., [Destercke and Dubois 2014, Proposition 4.1]), Theorem 1 establishes that precise probabilities compatible with a valid IM are confidence distributions. This characterization provides the following helpful intuition about what a valid IM does: the valid IM’s upper probability assigned to a hypothesis \(A\) is the maximum probability assigned to \(A\) by a confidence distribution. The “maximum” operation implies that the valid IM is more conservative than any individual confidence distribution, and it’s this conservatism that makes the IM valid while the individual confidence distributions are not. For statistical models having a special kind of additional structure, like the functional models studied in [Dawid and Stone 1982], the characterization result described above can be strengthened. What’s relevant about this class of statistical models is that they’re perfectly suited for Fisher’s fiducial argument, hence they admit a well-defined fiducial distribution. Then Theorem 4 in Section 5 shows that this fiducial distribution is where, in a certain sense, the aforementioned “maximum” is attained. That is, in this special class of models, the fiducial distribution is “maximal” among all the confidence distributions compatible with the valid IM. An important decision-theoretic consequence of this connection is Theorem 5, the paper’s second main contribution, which says that, under certain conditions on the loss function, the valid IM’s upper expected loss is minimized at the same action where the fiducial distributions expected loss is minimized. Combined with the main result in [Taraldsen and Lindqvist 2013] about decision rules derived from the fiducial distribution, Theorem 5 implies that actions chosen by applying the minimize (upper) expected loss principle to valid IMs leads to generally high-quality and, in some cases, optimal decision rules. Therefore, while validity requires some degree of conservatism, a valid IM is not so conservative that its recommended actions are much, if at all, different from those actions recommended by more familiar statistical frameworks which lack the validity property.

2 Background

2.1 Inferential models

Let \(Y\) be the observable data, taking values in \(\mathbb{Y}\), and let

\[ \mathcal{P} = \{P_{Y|\theta} : \theta \in \Theta\} \]

be the posited statistical model, which is just a family of probability distributions on \(\mathbb{Y}\) indexed by a parameter space \(\Theta\). As is customary, I’ll assume there is a true parameter value, \(\theta \in \Theta\), which is completely unknown. An observation \(Y = y\) is made, which provides some (limited) information about the unknowns, and the goal is to make inference and/or decisions in the presence of uncertainty about \(\theta\).

Inferential models (IMs) are data-dependent, probability-like structures designed to provide uncertainty quantification about unknowns. To cover the most general case where the IM output is something different from an ordinary/precise/additive probability, suppose the IM returns a pair \((\Pi_L, \Pi_U)\), which I’ll refer to as lower and upper probabilities,
respectively. Mathematically, both $\Pi_y$ and $\bar{\Pi}_y$ are *capacities* in the sense of Choquet (1954). But since one is “lower” and the other is “upper,” the two are necessarily linked together, so it suffices to describe only the upper probability, $\bar{\Pi}_y$.

Towards this, let $\mathcal{A} \subseteq 2^\Theta$ be a collection of subsets of $\Theta$ that represents all the assertions or hypotheses about $\theta$ can be considered; throughout, both $\mathcal{A}$ and “$\theta \in A$” will be called *assertions*. Specifically, I require $\mathcal{A}$ to contain $\emptyset$ and $\Theta$, and be closed under complementation. Since the goal is “probabilistic inference” on $\theta$, I’ll additionally require $\mathcal{A}$ to be at least as rich as the Borel $\sigma$-algebra on $\Theta$, so that it contains all the singletons. That way, the IM output is like a “posterior distribution” in the sense that it assigns (lower and upper) probabilities to a large collection of interesting hypotheses. In general, a set function $\gamma : \mathcal{A} \rightarrow [0, 1]$ is a capacity if $\gamma(\emptyset) = 0$, $\gamma(\Theta) = 1$, and $\gamma(A) \leq \gamma(A')$ if $A \subseteq A'$. Here I’ll also require the capacity to be sub-additive in the sense that

$$\gamma(A \cup B) \geq \gamma(A) + \gamma(B), \text{ for all } A, B \in \mathcal{A}, \text{ with } A \cap B = \emptyset.$$ 

This is a very mild requirement—all the standard upper probabilities models, e.g., Shafer’s plausibility functions, are sub-additive—so has virtually no effect the data analyst’s flexibility. For a given sub-additive $\bar{\Pi}_y$, if the corresponding $\Pi_y$ is defined as

$$\Pi_y(A) = 1 - \bar{\Pi}_y(A^c),$$

(1)

then it immediately follows that

$$\Pi_y(A) \leq \bar{\Pi}_y(A), \text{ for all } A \in \mathcal{A},$$

hence it makes sense to call these lower and upper probabilities. If the above inequality is an equality for all $A$, then the IM’s output is an ordinary probability. In such cases, property (1) reduces to the familiar complementation rule for ordinary probabilities.

The formal interpretation of lower and upper probability is as follows. Imagine a situation where assertions about $\theta$ correspond to gambles, i.e., where agents have the opportunity to buy or sell gambles $\$1(\theta \in A)$ that pay out $\$1 if “$\theta \in A$” turns out to be true, and pay out $\$0$ otherwise. Following Walley (1991) and others, $\Pi_y(A)$ can be interpreted as agent’s maximum buying price for the gamble $\$1(\theta \in A)$ and, similarly, $\bar{\Pi}_y(A)$ is his corresponding minimum selling price. So, if the “real price” is smaller than $\Pi_y(A)$ or larger than $\bar{\Pi}_y(A)$, then the agent would buy or sell the gamble, respectively; otherwise, to avoid sure loss, he neither buys nor sells. When $\Pi_y \equiv \bar{\Pi}_y$, the buying and selling prices coincide and the above description reduces to de Finetti’s classical subjective interpretation of probability.

Nothing prevents this behavioral/gambling from carrying over into the scientific inference or decision-making setting, and some would even argue that it should carry over (e.g., Crane 2018; Shafer 2021). But since gambling terminology is rarely used in scientific contexts, I should offer the following alternative interpretation:

- for a hypothesis $A \subseteq \Theta$ about the unknown $\theta$, the lower probability $\Pi_y(A)$ represents the data analyst’s degree of belief in the truthfulness of the assertion “$\theta \in A$,” given data $y$ and the posited model;
- similarly, the upper probability $\bar{\Pi}_y(A)$ the data analyst’s assessment of how plausible the truthfulness of “$\theta \in A$” is based on the data $y$ and posited model.
Armed with this, the data analyst would behave very much like the agent in the gambling situation above, just without a notion of “real price.” That is, if $\Pi_y(A)$ is relatively large, then the data strongly supports $A$, so infer $A$; similarly, if $\Pi_y(A)$ is small, then the data is inconsistent with $A$, so infer $A^c$. For the many real-life situations in which $\Pi_y(A)$ and $\Pi_y(A)$ are relatively small and large, respectively, the data just isn’t sufficiently informative to support either $A$ or $A^c$. In such situations, like the cautious gambler who prefers not participating in a gamble to losing, the data analyst ought to consider a “don’t know” conclusion (e.g., Dempster 2008), switch focus to a more basic assertion, $A'$, and/or collect more informative data.

Beyond simply constructing a data-dependent (precise or imprecise) probability, Martin (2019, 2021a) argued that in order to be a meaningful quantification of uncertainty, the IM must be valid in the sense that its inference is reliable. For a given statistical model, $\mathcal{P}$, and collection of assertions $A \subseteq 2^{\Theta}$, the IM is said to be valid if

$$\sup_{\theta \in A} P_{Y|\theta}\left\{ \Pi_Y(A) \leq \alpha \right\} \leq \alpha, \quad \text{for all } \alpha \in [0, 1] \text{ and all } A \in \mathcal{A}. \tag{2}$$

Under the present assumptions on $(\Pi_y, \Pi_y)$ and the support $\mathcal{A}$, there’s an equivalent definition of validity in terms of the lower probability,

$$\sup_{\theta \notin A} P_{Y|\theta}\left\{ \Pi_Y(A) > 1 - \alpha \right\} \leq \alpha, \quad \text{for all } \alpha \in [0, 1] \text{ and all } A \in \mathcal{A},$$

but here I’ll focus on the version in (2). The intuition behind these conditions is as follows. Based on the interpretation of the lower and upper probabilities offered above, the data analyst would be drawing an incorrect inference if the the upper probability of a true assertion was small—or if the lower probability of a false assertion was large. The validity property ensures that these undesirable events are rare in the sense of having small $P_{Y|\theta}$-probability. Moreover, there is an explicit connection between the meaning of “small/large” and “rare” since it’s the same $\alpha$ inside and outside of the curly braces in (2). Therefore, statistical methods that satisfy the validity property are reliable in the sense that error rates can be easily controlled. From the perspective of prices for gambles, the validity property (2) ensures that the agent’s minimum selling price, $\Pi_Y(A)$, for the gamble $\$1(\theta \in A)$ won’t tend to be too small when “$\theta \in A$” is true. A similar statement can be made about the maximum buying price.

The “for all $A \in \mathcal{A}$” part of the validity condition is important for several reasons. Here I’ll mention only that it allows for marginal inference on any relevant quantity of interest. Suppose that $\phi(\theta)$ is some feature of the unknown parameter that’s of interest, e.g., the norm of a mean vector. Of course, assertions about $\phi(\theta)$ can be expressed as assertions about $\theta$, i.e., $A = \{ \theta : \phi(\theta) \notin B \}$, for $B \subseteq \phi(\Theta)$, and this could be covered by the property (2) for all practically relevant choices of $\phi$ and $B$. Since it’s known (e.g., Fraser 2011, 2013) that Bayesian and other additive IMs are mis-calibrated for some (non-linear) functions $\phi$, there is reason to doubt that these are valid. Indeed, it was shown in Balch et al. (2019) that additive IMs suffer from what’s called false confidence, which implies they’re not valid in the sense of (2). For further details about the false confidence phenomenon, the reader is referred to Martin (2019, 2021a). The point is: in order to achieve validity, it’s necessary that the IM’s output be non-additive.

How to construct a valid IM and what kind of (non-additive) structure does it have? The original IM constructions presented in Martin and Liu (2013, 2015b) were based on
user-defined random sets on an auxiliary variable space. In particular, in the spirit of Fisher’s fiducial inference, Martin and Liu proposed to express the statistical model, $P_{Y|\theta}$, in terms of an association, or data-generating equation,

$$Y = a(\theta, U), \quad U \sim P_U,$$

where $a : \Theta \times U \to Y$ is a known mapping and $U$ is an auxiliary variable with known distribution $P_U$ supported $U$. The idea is that there are two aspects that influence the observable data $Y$, namely, the structural part—how the model depends on $\theta$—and the purely random part. In the formulation (3), these two parts are controlled by $a$ and by $U$, respectively. A similar setup can be found in Hannig et al. (2016) and elsewhere. What distinguishes Martin and Liu’s approach from these others is the handling of uncertainty about the auxiliary variable $U$ after the data $Y = y$ is observed. Fiducial-like approaches would “continue to regard” (Dempster 1963) the unobservable $U$ as a random variable with distribution $P_U$ after $Y = y$ is observed. However, there’s no logical basis for such an assumption because, in fact, it’s known that $U$ solves the deterministic equation $y = a(\theta, U)$, which would typically be a $P_U$-probability 0 event. In contrast, Martin and Liu recognize the change in the status of $U$—from a random variable with distribution $P_U$ before $Y = y$ is observed, to a fixed-but-unobserved realization after—and, correspondingly change how the uncertainty about it is quantified. It’s difficult to say what’s the “correct” way to quantify this uncertainty, but, fortunately, the solution is known even if the logical justification is not. Indeed, Martin and Liu recognized that valid inference on $\theta$ can be achieved by reliably predicting/guessing the unobserved value of $U$ associated with the observed $Y = y$, and they propose to carry out this “reliable” prediction using a random set $U$ with distribution $P_U$ supported on subsets of $U$. These random set-based predictions of the unobserved value of $U$ get mapped to a new and $y$-dependent random set, $\Theta_y(U)$, on the $\theta$-space through (3):

$$\Theta_y(U) = \bigcup_{u \in U} \{ \vartheta \in \Theta : y = a(\vartheta, u) \}.$$

That is, if $U$ represents a set of plausible values for the unobserved $U$, then any $\vartheta$ values consistent with the observed $Y = y$ and some $u \in U$ are equally plausible. Under certain mild conditions on the association (3) and on the random set $U$, this leads to

$$\Pi_y(A) = P_U\{\Theta_y(U) \cap A \neq \emptyset\}, \quad A \in \mathcal{A}.$$

Under some very mild and verifiable conditions on the user-specified random set $U$, it can be shown that the $\Pi_y$ as defined above achieves the validity condition (2) with $\mathcal{A} = 2^\Theta$; see, e.g., Martin and Liu (2013, 2015b) and Martin (2019, 2021a).

Being based on random sets, it turns out that the IM output $(\Pi_y, \Pi_y)$ described above is a special belief and plausibility function pair (e.g., Cuzzolin 2021; Molchanov 2005; Nguyen 2006), possessing a certain continuity property. It was later observed that, for efficiency reasons, the random set $U$ should be nested, which makes the IM output $(\Pi_y, \Pi_y)$ even more special: it has the mathematical form of a necessity and possibility measure pair (e.g., Dubois 2006; Dubois and Prade 1988; Liu and Martin 2021). This

\[1\] I do have some new ideas pertaining to this point, but I’ll report on these elsewhere.
connection between valid IMs and possibility measures is important because it may allow for simpler and more direct constructions. Indeed, a possibility measure $\Pi_y$ is determined by its so-called possibility contour function $\pi_y : \Theta \to [0, 1]$, given by

$$
\pi_y(\vartheta) = P_U\{\Theta_y(U) \ni \vartheta\}, \quad \vartheta \in \Theta,
$$

and the property

$$
\Pi_y(A) = \sup_{\vartheta \in A} \pi_y(\vartheta), \quad A \in \mathcal{A}. \quad (4)
$$

Consequently, the IM’s 100(1 $- \alpha$)% plausibility regions can be conveniently expressed as

$$
C_\alpha(y) = \{\vartheta : \pi_y(\vartheta) > \alpha\}, \quad \alpha \in [0, 1]. \quad (5)
$$

Validity implies that $C_\alpha(Y)$ is a nominal 100(1 $- \alpha$)% confidence region in the sense that

$$
\inf_{\theta \in \Theta} P_{Y|\theta}\{C_\alpha(Y) \ni \theta\} \geq 1 - \alpha. \quad (6)
$$

In what follows, I’ll focus exclusively on IMs of this latter type, which are both valid and whose upper probability $\Pi_y$ has the mathematical structure of a possibility measure. All of the examples of valid IMs presented in, e.g., Martin and Liu (2015b) are of this type, and the recent results in Martin (2021a) strongly suggest that valid IMs can only take this form, so there are no practical consequences of this restriction.

### 2.2 Decision theory

Decision theory, especially the normative form, is a mathematical framework analyzing the choices agents make in the presence of uncertainty. In particular, decision theory aims describe the “optimal” behavior for an agent faced with decision problem. This requires a way to compare the agent’s strategies and to (at least partially) rank them by preference. Early efforts, e.g., Bernoulli (1954), went directly to considerations of utility and, in particular, ranking strategies according to their expected utility and, hence, defining the optimal strategy as one that maximizes expected utility. Other authors took a different route, by considering general preference orders on strategies, but reached effectively the same destination. For example, the celebrated utility theorem of von Neumann and Morgenstern (1947) says that if the preference order satisfies certain rationality axioms, then there exists a utility function such that the ranking of strategies according to preferences is equivalent to the ranking by expected utility. Similar conclusions, under different formulations, have been reached by others, e.g., Savage (1972). Hence, the maximize expected utility principle. In a statistical decision problem, it is common to work with loss instead of utility, but the difference is really only superficial. Basically, if $\ell$ is a loss function, then $u = -\ell$ is an equivalent utility function. The negation, however, changes the directions of preference so, for the statistical decision problem, the maximize-expected-utility principle becomes the minimize-expected-loss principle.

The remainder of this subsection is devoted to a description of the statistical decision problem, including introducing the notation that will be needed in what follows. I don’t have anything to add to the discussions concerning the appropriateness or suitability of the minimize-expected-loss principle. My focus, again, is on how the IM’s validity
property manifests in the context of decision-making so, in what follows, I’ll simply adopt the standard minimize-expected-loss principle—and the appropriate and necessary generalizations described below—without question.

Let \( (\vartheta, a) \mapsto \ell_a(\vartheta) \) denote a loss function that measures the loss incurred by taking an action \( a \in A \) when the “state of nature” is \( \vartheta \). Common examples of loss functions include squared-error loss, with \( \ell_a(\vartheta) = \| a - \vartheta \|^2 \), and 0–1 loss, with \( \ell_a(\vartheta) = 1(\vartheta \in H, a = 1) + 1(\vartheta \notin H, a = 0) \), where \( H \subset \Theta \) is a (null) hypothesis, and \( a = 1 \) and \( a = 0 \) denote “reject” and “do not reject” \( H \), respectively. I’ll assume throughout that the loss is non-negative, which is not unreasonable. While negative losses might make sense in some contexts, these gains would be bounded and so the loss could be made non-negative by adding a constant. This additive constant won’t affect judgments of the relative quality of actions.

Of course, an oracle who knows the value of \( \theta \) could easily compare the different possible actions according to their corresponding loss values, \( \ell_a(\theta) \). I’ll refer to \( a \mapsto \ell_a(\theta) \) as the oracle’s assessment of action \( a \). Naturally, the oracle would take the action \( a^* = \arg \min_a \ell_a(\theta) \) and incur minimal loss. None of us have oracle powers, however, so the value of \( \theta \) is unknown and a different approach is required. When uncertainty about \( \theta \), given \( Y = y \), is quantified via an ordinary probability \( Q_y \), like a Bayesian posterior distribution (e.g., Berger [1985]; Ghosh et al. [2006]) or a fiducial/confidence distribution (Schweder and Hjort [2002]; 2016; Xie and Singh [2013]), a typical strategy is to rank the candidate actions according to their expected loss, \( a \mapsto Q_y \ell_a \). Then the obvious analogue to the known-\( \theta \) case mentioned above is to select the action that minimizes the posterior expected loss, \( \hat{a}(y) = \arg \min_a Q_y \ell_a \).

Other authors have argued that requiring a single precise probability to quantify uncertainty puts too much of a burden on the data analyst, and that a decision-theoretic framework based on more general imprecise probabilities is better suited for practical applications. Excellent reviews of decision theory from an imprecise probability perspective can be found in Huntley et al. [2014] and Denœux [2019]. At the risk of oversimplification, here I’ll explain only the key idea that will be relevant in what follows. Given a loss function like above, if uncertainty is quantified via an imprecise probability, then it’s only natural to extend the minimize-expected-loss principle by replacing the expected loss, \( Q_y \ell_a \), with an appropriate generalization, and then optimizing to find the best action (e.g., Gilboa [1987]; Gilboa and Schmeidler [1989]). It was shown in Gilboa and Schmeidler [1994] that an appropriate generalization of the expected loss is obtained via the Choquet integral; see Appendix C of Troffaes and de Cooman [2014] for full details, Section 4.1 of Denœux [2019] for a summary, and Section 4 below for the special case where the imprecise probability model is a possibility measure. Under an imprecise probability framework, there are actually two versions of “expected loss,” an upper and a lower. Given that the goal is to choose actions for which the loss is small, and that imprecise probability frameworks are largely motivated by conservatism, it’s only natural to define the optimal action in such a framework to be that which minimizes the upper expected loss. For reasons that will be made clear—see Equations (8) and (11) below—this strategy is commonly referred to as minimax, and there is extensive work about this in the literature on imprecise probability (e.g., Huntley et al. [2014] and the references therein) and in robust Bayesian inference (e.g., Berger [1984, 1985]; Vidakovic [2000]). Finally, spe-
cial decision-theoretic attention has been paid to “partially consonant” belief models, which include the (fully consonant) possibility measures considered here; for details, see Walley (1987) and Giang and Shenoy (2011).

3 Credal set corresponding to a valid IM

Before jumping in to the decision-theoretic formulation, I’ll first present a new perspective on how valid IMs and confidence distributions are related. I believe that this characterization is of independent interest, but it will also be helpful in what follows. Recall that, as mentioned at the end of Section 2.1 my focus in the remainder of this paper is on cases where the valid IM’s output has the form of a necessity and possibility measure pair, so that $\Pi_y$ below is determined by a possibility contour, $\pi_y$, via (4).

Since the IM with upper probability $\Pi_y$ is genuinely non-additive, there is a collection of probability distributions on $\Theta$ that are compatible with it. This set of probability measures compatible with $\Pi_y$ is called the credal set and is defined as

$$
\mathcal{C}(\Pi_y) = \{Q_y : Q_y \in Q \text{ and } Q_y(A) \leq \Pi_y(A) \text{ for all } A \in B_\Theta\},
$$

where $Q = Q(\Theta, B_\Theta)$ is the set of all probability distributions on the measurable space $(\Theta, B_\Theta)$, with $B_\Theta$ the Borel $\sigma$-algebra on $\Theta$. Though the distributions $Q_y$ in $\mathcal{C}(\Pi_y)$ can depend on $y$, they may not be Bayesian posterior distributions under any prior. It is possible, however, to give a practically meaningful characterization of the members of $\mathcal{C}(\Pi_y)$, which is the main goal of this section.

Recall the IM’s 100(1 $- \alpha$)% plausibility regions for $\theta$ defined in (5), i.e., $C_\alpha(y) = \{\theta : \pi_y(\theta) > \alpha\}$. Since the IM is valid, $\{C_\alpha(y) : \alpha \in [0, 1]\}$ is a family of nominal confidence regions for $\theta$ in the sense that (6) holds. If an ordinary/precise and necessarily data-dependent probability distribution $Q_y$ satisfies

$$
Q_y(C_\alpha(y)) \geq 1 - \alpha, \quad \text{for all } \alpha \in [0, 1],
$$

then I’ll call it a confidence distribution. That is, I’m defining a confidence distribution to be any probability measure that’s calibrated in the sense that the probability it assigns to the confidence set $C_\alpha(y)$ is at least the nominal coverage probability. This definition of confidence distributions is a little different/stronger than that found in, e.g., Schweder and Hjort (2002, 2016) and Xie and Singh (2013). In these references, the focus is primarily on the scalar-$\theta$ case, and a confidence distribution is a data-dependent distribution function whose upper-$\alpha$ quantile determines a 100(1 $- \alpha$)% confidence upper limit. The difference is that the above references define a data-dependent distribution to be a confidence distribution if its quantiles form nominal confidence intervals, whereas I’m starting with a given confidence region $C_\alpha(\alpha)$—which could be in more than one dimension—and defining a distribution to be a confidence distribution if it assigns probability at least $1 - \alpha$ to $C_\alpha(y)$ for each $\alpha \in [0, 1]$. In the scalar-$\theta$ case, if (7) holds, then $Q_y$ would be a confidence distribution in the sense of, e.g., Xie and Singh, so this is a generalization. A similar definition of confidence distributions is given in Taraldsen (2021, Sec. 2).

The advantage of this alternative perspective is that it provides some new insights into what confidence distributions are. In particular, Theorem [1] below establishes that
confidence distributions are simply those probability measures compatible with the possibility measure $\Pi_y$ from a valid IM. Therefore, properties developed for the valid IM should have implications for all compatible confidence distributions, and vice versa.

**Theorem 1.** For a valid IM whose upper probability $\Pi_y$ is a possibility measure, the credal set $\mathcal{C}(\Pi_y)$ is precisely the set of all confidence distributions satisfying (7).

**Proof.** An interesting and useful characterization of the set of probabilities compatible with a possibility measure, summarized in Proposition 4.1 of Destercke and Dubois (2014) and the subsequent discussion, is

$$Q_y \in \mathcal{C}(\Pi_y) \iff Q_y\{C_\alpha(y)\} \geq 1 - \alpha \text{ for all } \alpha \in [0, 1].$$

The claim now follows from my definition (7) of a confidence distribution. \hfill \Box

This general interpretation of the credal set $\mathcal{C}(\Pi_y)$ as the set of confidence distributions will be important in Section 4. I’ll revisit this connection in Section 5, where, with some additional structure on the statistical model, a much more precise characterization of the credal set can be given. This, in turn, will allow for a better understanding of the decision-theoretic properties developed next.

## 4 Inferential models and decisions

### 4.1 Setup

I’ll start here by developing the decision-theoretic framework for IMs. Since imprecise or non-additive probabilities are often used, even an appropriate extension of basic notions like “expected loss” requires some care. Recall that the decision problem comes equipped with a loss function $\ell_a(\vartheta)$ that maps $(a, \vartheta) \in A \times \Theta$ to non-negative real numbers. In the familiar Bayesian case, one considers a posterior expected loss, $Q_y\ell_a$, of an action $a$, where $Q_y$ is the $y$-dependent posterior distribution on $\Theta$. A similar thing can be done if the Bayesian posterior is replaced by a fiducial or confidence distribution. For more general kinds of IMs, those that quantify uncertainty via an imprecise probability $(\Pi_y, \Pi_y)$, a different formulation is required.

If one thinks of an upper probability $\Pi_y$ as an upper bound on a collection of ordinary probabilities, then its extension to an upper prevision/expectation is via

$$\Pi_y f = \sup\{Q_y f : Q_y \in \mathcal{C}(\Pi_y)\},$$

where $f$ is an appropriate real-valued function defined on $\Theta$. In light of Theorem 1, the upper envelope has some further intuition, that is, $\Pi_y f$ is largest of the ordinary expected values, $Q_y f$, corresponding to confidence distributions $Q_y$ compatible with $\Pi_y$. There is an associated lower prevision, $\Pi_y f$, which is given by $\Pi_y f = -\Pi_y(-f)$, but this won’t be considered in what follows.

According to, e.g., Chapter 7 in Troffaes and de Cooman (2014), extensions of an upper probability/plausibility to an upper prevision—which is akin to the jump from probability to expected value via Lebesgue integral in the classical theory—is carried out via the *Choquet integral*. As stated in Section 2.1, the focus here is primarily on
IMs whose upper probability $\Pi_y$ is a possibility measure. For these, Propositions 7.14 and 15.42, in increasing generality, establish that the Choquet integral of a non-negative function $f : \Theta \to \mathbb{R}$ with respect to the possibility measure $\Pi_y$, if it exists, is given by

$$\Pi_y f = \int_0^1 \sup \{ f(\vartheta) : \pi_y(\vartheta) > \alpha \} \, d\alpha,$$

where $\pi_y$ is the possibility contour corresponding to $\Pi_y$. The integral in the above display is a Riemann integral, which exists because the integrand is monotone in $\alpha$. Existence of the Choquet integral requires that the function $f$ be “previsible” in the sense of Definition 15.6 in Troffaes and de Cooman (2014), which is a notion of integrability with respect to a lower/upper prevision.

In what follows, I’ll silently assume that all loss functions are previsible in this sense. Finally, note that the Choquet integral definition of the upper prevision is equivalent to the previous one involving a supremum over $\mathcal{C}(\Pi_y)$, so that the right-hand sides of (8) and (9) are equal.

Since the loss function is assumed to be non-negative, either of the above two equivalent formulas can be applied directly to define an upper risk, or expected loss,

$$a \mapsto \Pi_y \ell_a. \tag{10}$$

Assuming $\Pi_y \ell_a$ is finite at least for some actions $a$, this can be used to assess the quality of different actions, relative to the given loss function and the IM’s data-dependent possibility measure. In particular, one can select a $y$-dependent “optimal” action as

$$\hat{a}(y) = \arg\min_a \Pi_y \ell_a, \tag{11}$$

which has a minimax connotation, being the action that minimizes the upper expected loss. The rationale behind the use of upper instead of lower expected loss is conservatism, i.e., if an action has small upper expected loss, then, by (8), the corresponding expected loss with respect to any compatible $Q_y \in \mathcal{C}(\Pi_y)$ is also small. Finally, as mentioned briefly in Section 2.2 and in more detail in Denœux (2019), the ranking of actions based on magnitudes of the upper expected loss is consistent with certain rationality axioms.

### 4.2 Implications of validity

Connections between validity and the no-sure-loss property in the sense of Walley (1991, Sec. 6.5.2) were recently made in Cella and Martin (2021a,b). But this doesn’t speak directly to the possible connections between validity and the IM’s assessment of the quality of various actions. Recall that validity ensures the user won’t make systematically misleading inferences or, in other words, that incorrect inferences are controllably rare. In the present context, a poor choice of action may result if the IM’s assessment of $a$, based on $\Pi_y \ell_a$ in (10), were very different from the oracle’s assessment $\ell_a(\theta)$. That is, I hope to avoid cases where, for some action $a$, my assessment $\Pi_y \ell_a$ in (10) of the loss associated with action $a$ is much more optimistic than the oracle’s, i.e., $\Pi_y \ell_a \ll \ell_a(\theta)$.

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2 Roughly, previsibility is a growth condition on the function $f$ relative to the upper probability $\Pi$. For example, according to Condition 15.31 in Troffaes and de Cooman (2014), $x \Pi(\{|f| \geq x\}) \to 0$ as $x \to \infty$ and one other similar property implies previsibility of $f$ with respect to $\Pi$. 

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Again, such cases are undesirable because a situation where my assessment of an action is much more optimistic than the oracle’s creates an unfortunate opportunity for me to choose a poor action and potentially suffer a non-trivial loss.

To be more precise about what it means for the IM’s assessment, \( \Pi_y \ell_a \), of action \( a \) to be “much more optimistic” than the oracle’s assessment, \( \ell_a(\theta) \), define the data-dependent local maximum loss,

\[
L_a(y, \theta) = \sup \{ \ell_a(\vartheta) : \vartheta \in C_{\pi_y(\theta)}(y) \} = \sup \{ \ell_a(\vartheta) : \pi_y(\vartheta) > \pi_y(\theta) \},
\]  

(12)

where the second equality follows by the definition (5) of the plausibility region \( C_\alpha(y) \). I’ll refer to \( L_a(y, \theta) \) as the quasi-oracle’s assessment of \( a \), based on data \( Y = y \). This corresponds to an oracle who doesn’t know the exact value of \( \theta \) but knows which values \( \vartheta \) are at least as \( \pi_y \)-plausible as \( \theta \) relative to data \( Y = y \). Then by “much more optimistic” I mean \( \Pi_y \ell_a \) being less than a small multiple of \( L_a(y, \theta) \), i.e.,

\[
\Pi_y \ell_a \leq \alpha L_a(y, \theta), \quad \alpha \in (0, 1).
\]

The proper oracle’s assessment, \( \ell_a(\theta) \), is out of reach: its magnitude is always much smaller than that of any reasonable data-driven assessment. So it’s not meaningful to directly compare \( \ell_a(\theta) \) and, say, \( \Pi_y \ell_a \). However, it’s easy to see from (12) that \( L_a(y, \theta) \geq \ell_a(\theta) \), so the quasi-oracle’s assessment is more conservative than the oracle’s and possibly within reach. Therefore, if my IM’s assessment, \( \Pi_y \ell_a \), of the loss associated with action \( a \) tends not to be much more optimistic than the quasi-oracle’s assessment, \( L_a(y, \theta) \), then it also won’t tend to be much more optimistic than the oracle’s assessment, \( \ell_a(\theta) \). And not tending to be much more optimistic than the oracle provides some assurance that poor actions won’t be favored by the IM.

Below is (to my knowledge) the first result showing the implications of validity in the context of decision theory. It says that the valid IM’s assessment \( \Pi_Y \ell_a \) tends to be not considerably smaller than \( L_a(Y, \theta) \) for all actions \( a \in \mathcal{A} \) and for any true \( \theta \in \Theta \).

**Theorem 2.** Let \( \ell_a : \Theta \to [0, \infty) \) be a non-negative loss function for each \( a \) in the action space \( \mathcal{A} \). For an upper probability \( \Pi_y \), define the minimum ratio

\[
R(y, \theta) = \inf_{a \in \mathcal{A}} \frac{\Pi_y \ell_a}{L_a(y, \theta)},
\]

where \( L_a(y, \theta) \) is as in (12). If the IM with upper probability \( \Pi_Y \) is valid, then

\[
\sup_\theta P_{Y|\theta} \{ R(Y, \theta) \leq \alpha \} \leq \alpha, \quad \text{for all } \alpha \in [0, 1].
\]

**Proof.** Define the function

\[
h_{y,a}(\alpha) = \sup\{ \ell_a(\vartheta) : \pi_y(\vartheta) > \alpha \},
\]

(13)

so that the Choquet integral \( \Pi_y \ell_a \) is just a Riemann integral of \( h_{y,a} \). It’s clear that
$\alpha \mapsto h_{y,a}(\alpha)$ is decreasing, which implies
\[
\Pi_y \ell_a = \int_0^1 h_{y,a}(\alpha) d\alpha \\
= \left( \int_0^{\pi_y(\theta)} h_{y,a}(\alpha) d\alpha + \int_{\pi_y(\theta)}^1 h_{y,a}(\alpha) d\alpha \right) \\
\geq \pi(\theta) h_{y,a}(\pi_y(\theta)).
\]

This is effectively a Markov inequality for the Choquet integral, a special case of that presented in [Wang (2011)]. Note that $L_a(y, \theta) = h_{y,a}(\pi_y(\theta))$. Therefore,

$$R(Y, \theta) \leq \alpha \iff \Pi_Y \ell_a \leq \alpha L_a(Y, \theta) \text{ for some } a$$

$$\implies \pi_Y(\theta) L_a(Y, \theta) \leq \alpha L_a(Y, \theta) \text{ for some } a$$

$$\iff \pi_Y(\theta) \leq \alpha.$$

Validity implies the latter event has $P_{Y|\theta}$-probability $\leq \alpha$, uniformly in $\theta$. $\square$

A simple consequence of Theorem 2, in particular, of the uniformity in actions $a \in \mathcal{A}$, is that the event $\{\Pi_Y \ell_a \leq \alpha L_a(Y, \theta)\}$ has $P_{Y|\theta}$-probability no more than $\alpha$ for every action $a \in \mathcal{A}$. This includes data-dependent actions, in particular, the “best” action $\hat{a}(y) = \arg \min_a \Pi_y \ell_a$, i.e.,

$$\sup_{\theta} P_{Y|\theta}(\Pi_Y \ell_{\hat{a}(Y)} \leq \alpha L_{\hat{a}(Y)}(Y, \theta)) \leq \alpha, \quad \text{for all } \alpha \in [0, 1].$$

For a quick recap, recall that it would be undesirable if the IM’s assessment of the quality of $a$ were much more optimistic than the oracle’s or the quasi-oracle’s—it would put the decision-maker at risk of suffering non-trivial loss. This is especially true for the “best” action, $a = \hat{a}(y)$, the one that the decision-maker is likely to take. Theorem 2 shows that validity of the IM implies that such undesirable cases are rare events with respect to the true distribution of $Y$. Therefore, validity provides some assurance that the IM’s data-driven assessment of action $a$ is not inconsistent with that of the oracle or quasi-oracle, hence the IM helps the decision-maker mitigate risk. This can be compared to the notion of “safety” in [Grünwald (2018)], e.g., his Theorem 4.

It turns out that the bound in Theorem 2 is rather conservative, at least in the special cases considered in Section 4.3. One explanation for this conservatism is that the theorem assumes almost nothing. The formulation described below assumes a bit more about both the decision problem and the IM, which leads to a stronger conclusion about the magnitude of $\Pi_Y \ell_a$ relative to a version of the oracle’s assessment. Towards this, some more notation and terminology is needed, which I provide next. I’ll also focus here on the case of a scalar $\theta$ since I’ve not yet been able to work out the technical details in the more general case; see the discussion below, and Appendix A.

First, I’ll need some special—but apparently fairly general—structure in the possibility contour $\pi_y$, namely, that

- $\pi_y$ is convex increasing to its maximum value 1;
- constant equal to 1 on an interval (perhaps just a singleton);
and convex decreasing thereafter.

I’ve not found an existing name for such a property so, for lack of any better options, I’ll refer to this as directional convexity, i.e., convex in both directions away from the mode(s)\(^3\). Both the possibility contours plotted in Section 4.3 below are directionally convex, as are most\(^4\) of the others I’ve seen in the literature.

Second, a more conservative version of the quasi-oracle’s assessment of the quality of different actions is required. Intuitively, while the quasi-oracle has much less information than the oracle who knows \(\theta\), being able to restrict attention to those \(\vartheta\) values that are at least as \(\pi_y\)-plausible as \(\theta\) is also rather unrealistic. Therefore, even comparing \(\Pi_Y \ell_a\) to \(L_a(Y, \theta)\), with \(L_a(Y, \theta) \gg \ell_a(\theta)\), is unfair. A more conservative comparison could be made if the quasi-oracle’s assessment was inflated by optimizing over a larger set that includes some \(\vartheta\) values that are only “almost” as \(\pi_y\)-plausible as \(\theta\). In particular, define

\[
\tilde{L}_a(y, \theta) = \sup \{ \ell_a(\vartheta) : \pi_y(\vartheta) > \frac{1}{2} \pi_y(\theta) \}.
\]

Clearly, \(\tilde{L}_a(y, \theta) \geq L_a(y, \theta)\), so this modified quasi-oracle assessment is a more conservative threshold. Then the IM’s assessment, \(\Pi_y \ell_a\), not being “much more optimistic” than the modified quasi-oracle’s assessment, \(\tilde{L}_a(y, \theta)\), is expressed as

\[
\Pi_y \ell_a \leq \alpha \tilde{L}_a(y, \theta), \quad \alpha \in [0, 1].
\]

Analogous to Theorem 2, the following result says that the IM’s assessment being much more optimistic than the modified quasi-oracle’s, in the sense described above, is a rare event relative to \(P_{Y|\theta}\).

**Theorem 3.** Suppose that \(\vartheta \mapsto \ell_a(\vartheta)\) is convex for each \(a\) and that the IM’s possibility contour \(\pi_y\) is directionally convex in the sense described above. Define the minimum ratio

\[
\tilde{R}(y, \theta) = \inf_{a \in \mathcal{A}} \frac{\Pi_y \ell_a}{L_a(y, \theta)}.
\]

(14)

If the IM with upper probability \(\Pi_Y\) is valid, then

\[
\sup_{\theta} P_{Y|\theta} \{ \tilde{R}(Y, \theta) \leq \alpha \} \leq \alpha, \quad \text{for all } \alpha \in [0, 1].
\]

**Proof.** Reinterpreting the Choquet integral \(\Pi_y \ell_a\) as an ordinary expectation of \(h_{y,a}(\alpha)\), defined in (13) with respect to \(\alpha \sim \text{Unif}(0, 1)\) leads to

\[
\Pi_y \ell_a \geq \int_0^{\pi_y(\theta)} h_{y,a}(\alpha) d\alpha = \pi_y(\theta) E\{h_{y,a}(\alpha) \mid \alpha \leq \pi_y(\theta)\}.
\]

\(^3\) An appropriate generalization beyond the scalar case being considered here is to assume convexity away from the mode(s) in all directions.

\(^4\) The only exceptions I’m aware of are the discrete data problems (e.g., Martin et al. 2012) where a “most efficient” possibility contour is used, which typically has some tiny local modes and jump discontinuities. The recent work in Balch (2020) was able to remove these tiny modes, so only the presence of jump discontinuities would need to be addressed, but I won’t attempt this here.
Lemma \[\text{1} \] in Appendix \[\text{A} \] shows that, under the stated assumptions, \( \alpha \rightarrow h_{y,a}(\alpha) \) is convex for each \( y \) and \( a \). Then it follows from Jensen’s inequality that

\[
\Pi_y^a \geq \pi_y(\theta) h_y(\pi_y(\theta) \bar{L}_a(y, \theta)).
\]

The remainder of the proof proceeds exactly like that of Theorem \[\text{2} \]. That is, \( \bar{R}(Y, \theta) \leq \alpha \) implies \( \pi_Y(\theta) \leq \alpha \), and the latter event has probability not exceeding \( \alpha \).

As with Theorem \[\text{2} \], the take-away message is that validity helps to prevent the IM’s assessment of an action from being overly optimistic compared to the (modified) quasi-oracle’s assessment, and this helps to protect the decision-maker from loss. The focus here was on scalar-\( \theta \) problems but nothing about the above proof depends on this. The challenge to extending beyond the scalar case is proving the claim that \( \alpha \rightarrow h_{y,a}(\alpha) \) is convex. I expect that this property holds more generally, at least under some reasonable conditions, but, as of now, I’ve only been able to establish convexity for the scalar case; see Appendix \[\text{A} \] for further details and discussion on this.

Finally, it’s possible to say more about the decision-theoretic implications of validity, e.g., how the IM’s suggested decision \( \hat{a}(y) \) relates to the “optimal” decision, but this requires some additional structure in the statistical model; see Section \[\text{5} \] below.

### 4.3 Examples

#### 4.3.1 Location model

Consider the case where \( P_{Y|\theta} \) is a location model, that is,

\[
Y = \theta + U, \quad U \sim P_U,
\]

where \( P_U \) a known distribution on \( \mathbb{R} \), with distribution and density functions \( F \) and \( f \), respectively. For technical reasons, I’ll assume that \( f \) is bounded and unimodal, strictly decreasing moving away from the mode in either direction. Also, I’ll focus here on the scalar case, but the case where \( Y = (Y_1, \ldots, Y_n) \) is a vector of iid samples, each with the same scalar location parameter \( \theta \), can be handled similarly, following a preliminary conditioning argument to reduce the problem to the case analyzed here; see Section \[\text{5} \].

The IM construction is relatively straightforward for this problem; see Martin and Liu (2013). Here I’ll proceed using the more “modern” formulation in Liu and Martin (2021) that focuses directly on the possibility contours, rather than the random set formulation in the earlier literature. Define the possibility contour on the auxiliary space as follows:

\[
\pi(u) = P_U \{ f(U) \leq f(u) \}, \quad u \in \mathbb{R}.
\]

This defines a possibility measure, \( \Pi \), on \( \mathbb{R} \), through \[\text{4} \]. One important feature of this possibility measure is that it forms a credal set \( \mathcal{C}(\Pi) \) that contains \( P_U \), i.e., \( P_U(B) \leq \Pi(B) \) for all measurable sets \( B \). Another important feature, which I’ll not discuss in detail, is that this particular construction of \( \Pi \), using the density \( f \) of \( P_U \) in \[\text{15} \], is the \textit{maximally specific} possibility measure corresponding to \( P_U \) (e.g., Dubois et al. 2004, Theorem 3.2), hence is optimal in a certain sense.
Next, by the extension principle of Zadeh (1978), for the observed data $Y = y$, the corresponding possibility contour on the parameter space is given by

$$\pi_y(\vartheta) = \pi(y - \vartheta), \quad \vartheta \in \mathbb{R}.$$ 

As this is the first example, to keep things sufficiently simple that the calculations can be done analytically, I’ll focus on the case where the auxiliary variable distribution $P_U$ is symmetric around 0 in the sense that $f(-u) = f(u)$. Then the possibility contours can be simplified and, in particular,

$$\pi_y(\vartheta) = 2\{1 - F(|y - \vartheta|)\}, \quad \vartheta \in \mathbb{R}.$$ 

This defines an IM whose upper probability $\Pi_y$ is given by the supremum in (4). This is equivalent to the “optimal” IM construction discussed in, e.g., Martin and Liu (2013), based on the use of a nested and symmetric random set $U$. Moreover, it’s easy to check that the IM’s $100(1 - \alpha)$% plausibility intervals are

$$C_\alpha(y) = \left[ y - F^{-1}(1 - \frac{\alpha}{2}), y + F^{-1}(1 - \frac{\alpha}{2}) \right], \quad \alpha \in [0,1].$$

To define a decision problem, consider the case of “point estimation” where the action space $\mathcal{A}$ is the parameter space, and the goal is to select an action $\hat{a}(y)$, depending on data $y$, that’s best in the sense of minimizing $a \mapsto \Pi_y \ell_a$ for a suitable loss function $\ell_a$. As is customary, I’ll use the squared error loss, i.e., $\ell_a(\vartheta) = (a - \vartheta)^2$. It’ll be clear in what follows that the upper risk $\Pi_y \ell_a$ is well-defined only if the density $f$ has sufficiently thin tails that its variance $V(U)$ is finite, so I’ll assume this throughout what follows.

Figure 1 shows both the contour function $\pi_y(\vartheta)$ and the loss function $\ell_a(\vartheta)$ for specific values of $y$ and $a$, with $F$ a Student-t distribution function. Note, in particular, that $\pi_y$ is directionally convex and, hence, on any level set of $\pi_y$, the loss function attains its
maximum at the boundary. Using this observation, and applying the Choquet integral formula in [9], I get
\[
\Pi_y \ell_a = \int_0^1 \sup\left\{ (\vartheta - a)^2 : 2\{1 - F(|\vartheta - y|)\} > \alpha \right\} d\alpha \\
= \int_0^1 \max\left\{ y - a - F^{-1}(1 - \frac{\alpha}{2}), y - a + F^{-1}(1 - \frac{\alpha}{2}) \right\}^2 d\alpha \\
= \int_0^1 \left\{ |y - a| + F^{-1}(1 - \frac{\alpha}{2}) \right\}^2 d\alpha \\
= \int_{-\infty}^{\infty} (|y - a| + |u|)^2 f(u) du \\
= (y - a)^2 + 2|y - a|\mathbb{E}|U| + \mathcal{V}(U).
\]

For comparison, there is a standard fiducial/confidence distribution for the location problem, which is also the Bayesian posterior distribution under the default, flat prior for \(\theta\). That posterior, denoted by \(Q^*_y\), has density function
\[
q^*_y(\vartheta) = f(\vartheta - y),
\]
and it’s easy to check that the corresponding fiducial expected loss is
\[
Q^*_y \ell_a = (y - a)^2 + \mathcal{V}(U).
\]

Note that \(Q^*_y \ell_a \leq \Pi_y \ell_a\) for all \(a\), with equality if and only if \(a\) equals \(y\). Indeed, the two expected loss functions are equal at their common minimizer \(\hat{a}(y) = y\). This minimizer is Pitman’s optimal equivariant estimator, the solution suggested by any reasonable approach to estimation of a location parameter. As I show in Section 5, it’s a coincidence that the valid IM’s suggested action based on the minimum upper risk principle aligns with that of other methods in this location model.

It’s a good sign that the IM’s choice of a best action, the one that minimizes upper expected loss, aligns with the standard solution in this problem. Beyond that, it’s of interest to compare the magnitude of IM’s assessment, \(\Pi_Y \ell_a\), to that of the (modified) quasi-oracle’s assessment, \(\tilde{L}_a(Y, \theta)\), to that of the (modified) quasi-oracle’s assessment, \(\tilde{L}_a(Y, \theta)\). Theorem 3 says that the former won’t be too small compared to the latter, in the sense that \(\tilde{R}(Y, \theta)\), defined in (14), is stochastically no smaller than \(\text{Unif}(0, 1)\) as a function of \(Y \sim P_{Y|\theta}\). Figure 2 shows the distribution function of \(\tilde{R}(Y, \theta)\), i.e.,
\[
\alpha \mapsto P_{Y|\theta}\{\tilde{R}(Y, \theta) \leq \alpha\}, \quad \alpha \in [0, 1],
\]
and the fact that this curve is below the diagonal confirms the result in Theorem 3. For comparison, Figure 2 also shows the distribution function of the random variable
\[
\inf_{a \in A} \frac{Q^*_Y \ell_a}{\tilde{L}_a(Y, \theta)},
\]
where \(Q^*_Y\) is the fiducial/confidence/Bayes posterior distribution described above. As is clear from the plot, the claim in Theorem 3 does not hold for the assessment based on the
additive $Q_\star^Y$. That is, the fiducial/Bayesian IM’s assessment $Q_\star^Y \ell_a$ tends to be a bit too optimistic compared to the quasi-oracle’s. Whether violation of the particular property in Theorem 3 has any practical consequences deserves further investigation. The take-away message is that the IM solution is sufficiently conservative in the sense that it provides provably valid inference in the sense of (2), which implies that its assessment of actions is not too optimistic in the sense of Theorem 3 yet efficient enough to clearly identify the optimal action for the decision problem.

There is nothing really special about the symmetric location problem, except that symmetry makes the calculations more manageable. Here I’ll give a short description of what happens in the case where the auxiliary variable distribution is not symmetric. All of the calculations that follow are based on Monte Carlo approximations.

Suppose that $U$ has a skew-normal distribution (e.g., Azzalini 1985) with slant parameter $\kappa$, so that the density is

$$f(u) = \phi(u) \Phi(\kappa u), \quad u \in \mathbb{R},$$

where $\phi$ and $\Phi$ denote the standard normal density and distribution functions, respectively. Here I take $\kappa = 3$, so that the distribution is skewed to the right. I can follow the same approach to define a $u$-space possibility contour $\pi$ and then setting $\pi_y(\vartheta) = \pi(y - \vartheta)$ as before. A plot of the latter possibility contour is shown in Figure 3(a), based on data $y = 0$, and the effect of the asymmetric density is apparent. For the decision problem, again I’ll consider the squared error loss, $\ell_a(\vartheta) = (a - \vartheta)^2$. Figure 3(b) shows the IM’s upper expected loss, $a \mapsto \Pi_y \ell_a$, along with the analogous fiducial expected loss $a \mapsto Q_\star^Y \ell_a$. Note that the shape of these curves are very similar, in particular, the minimizers are very close, but the IM’s curve is strictly above the fiducial’s, as expected based on the construction and the interpretation as an “upper” expected loss.
Figure 3: Skew-normal example with $y = 0$. Panel (a) shows the possibility contour $\pi_y(\vartheta)$ for the skew-normal illustration; overlaid is the squared error loss function $\ell_a(\cdot)$ with $a = -2$. Panel (b) shows the IM (solid) and fiducial (dashed) expected losses.

4.3.2 Binomial model

Consider a binomial model, $P_{Y|\theta} = \text{Bin}(n, \theta)$, where the number of trials, $n$, is known but the success probability, $\theta \in [0, 1]$, is unknown. There are a number of different strategies available to constructing a binomial IM, the most recent being in [Balch (2020)]. Here I’ll follow a simpler strategy, which has some close connections to the classical Clopper–Pearson confidence interval construction, that I believe was first presented in [Martin and Liu (2014), Sec. 4.3]. This binomial IM construction was reviewed and recast in more modern terms recently in [Cahoon and Martin (2021)], so I’ll not reproduce the details here. Roughly speaking, the starting point is the association

$$F_\theta(Y - 1) \leq U < F_\theta(U), \quad U \sim \text{Unif}(0, 1),$$

where $F_\theta$ is the binomial distribution function. A suitable (nested) random set $U$ is introduced to predict the unobserved value of $U$, which is pushed through the association, with $Y = y$, leading to a new $y$-dependent random set in the $\theta$-space. The end result is a possibility contour

$$\pi_y(\vartheta) = 1 - \max\{0, 2B_{y+1, n-y}(\vartheta) - 1\} - \max\{0, 1 - 2B_{n-y+1, y}(\vartheta)\},$$

where $B_{a,b}$ is the Beta$(a,b)$ distribution function. Figure 4(a) shows a plot of this possibility contour for the case $n = 18$ and $y = 7$. While it appears from the plot that the edges of the plateau are rounded, this is not the case; there are corners, so this function is directionally convex in the sense described above. The aforementioned plateau is around the maximum likelihood estimator, $\hat{\vartheta} = y/n \approx 0.39$.

For a decision problem, we can consider estimation of $\theta$ with respect to a weighted squared error loss function

$$\ell_a(\vartheta) = \frac{(a - \vartheta)^2}{\vartheta(1 - \vartheta)}, \quad \vartheta \in [0, 1].$$
Figure 4: Binomial example with \( n = 18 \) and \( y = 7 \). Panel (a) shows the possibility contour \( \pi_y(\vartheta) \); dashed line shows the weighted squared error loss function \( \ell_a(\vartheta) \), when \( a = 0.2 \). Panel (b) shows the (upper) expected loss functions, \( a \mapsto \Pi_y\ell_a \) (solid) and \( a \mapsto Q_y^*\ell_a \) (dashed), for the weighted squared error loss function.

This loss function is overlaid in Figure 4(a), for the case \( a = 0.2 \). For the IM’s assessment, \( \Pi_y\ell_a \), of an action \( a \) relative to the weighted squared error loss, no closed-form expression is available. This can, however, be readily evaluated numerically. A plot of that function \( a \mapsto \Pi_y\ell_a \), based on data \( n = 18 \) and \( y = 7 \), is shown in Figure 4(b). For a comparison, I’ll also consider a fiducial solution to this binomial problem. There are several different fiducial solutions available (e.g., [Murph et al. 2021]), all of which give similar numerical answers, at least for moderate to large \( n \). Here I’ll take the version where the fiducial distribution has a beta density

\[
q_y^*(\vartheta) = \text{Beta}(\vartheta \mid y + \frac{1}{2}, n - y + \frac{1}{2}).
\]

It’s also straightforward to numerically evaluate the fiducial expected loss \( a \mapsto Q_y^*\ell_a \) and a plot of that function is also displayed in Figure 4(b). It is apparent that these two curves are roughly the same shape, with very similar minimizers, but the IM’s expected loss curve is uniformly larger than the fiducial’s.

Is the difference in magnitudes of any consequence? Of course, there’s no closed-form expression for the modified quasi-oracle’s assessment \( \tilde{L}_a(y, \theta) \) either, so only numerical calculations are possible. The goal is to approximate the distribution function of \( \tilde{R}(Y, \theta) \), defined in (14), as a function of \( Y \sim \text{Bin}(n, \theta) \). For this experiment, I consider \( n = 50 \) and \( \theta = 0.3 \). Figure 5 shows the distribution function in (16) for the IM with possibility contour as described above. According to Theorem 3, this distribution function should be below that of \( \text{Unif}(0, 1) \), which is confirmed by comparing the black line to the dashed diagonal line. Also shown on the plot is the distribution function for a version of \( \tilde{R}(Y, \theta) \) that replaces the IM’s assessment, \( \Pi_y\ell_a \), in the numerator with the expected loss, \( Q_y^*\ell_a \), for the fiducial solution described above. Note that the curve corresponding the fiducial distribution passes above the diagonal line, so it doesn’t satisfy the property in Theorem 3.
The fiducial solution’s violation of the property in Theorem 3 is more severe in this case than in the location model example above but, again, it remains to understand what, if any, practical consequences there might be as a result of this violation. In any case, the IM solution in this binomial problem is provably valid in the sense of (2), satisfies the property in Theorem 3, and its “best” action is effectively the same as the fiducial’s. So there’s no reason not to prefer the valid IM solution in this example.

5 Special case: functional models

5.1 Setup

In this section, I follow up on some of the insights presented in Section 3 above. In particular, with some additional structure, a more precise characterization of the contents of $\mathcal{C}(\widehat{\Pi}_y)$ can be given. This is perhaps of independent interest, but it will also shed light on the new decision-theoretic developments in Section 4. The key observation is that the possibility measure constructed from a valid IM has some special features, at least when the statistical model is suitably structured. Below I will introduce this simplifying structure, which covers many practically useful models, and then proceed to give a more detailed characterization of the credal set’s contents.

As discussed in Section 2.1 above, suppose the statistical model is re-expressed as

$$Y = a(\theta, U), \quad U \sim P_U,$$

where $P_U$ is a known probability distribution on $U$; see Equation (3). Let $y$ denote the observed value of $Y$. In what follows, I’ll assume that the equation “$y = a(\vartheta, u)$” defines
continuously differentiable bijection between $\vartheta$ and $u$ for a given $y$. That is, let $\theta_y(u)$ and $u_y(\vartheta)$ be such that

$$a(\theta_y(u), u) = a(\vartheta, u_y(\vartheta)), \quad \text{for all } (u, \vartheta),$$

for each fixed $y$. In other words, $u_y$ and $\theta_y$ are inverse maps, i.e., $u_y = \theta_y^{-1}$. This is just the functional model setup in [Dawid and Stone (1982)](https://doi.org/10.1093/biomet/69.3.381), which is similar to the fiducial perspective taken in [Taraldsen and Lindqvist (2013)](https://doi.org/10.1093/biomet/90.3.537). Both of these contain the structural models of [Fraser (1968)](https://doi.org/10.1093/biomet/55.1.1), where each of $y$, $\vartheta$, and $u$ can be identified with elements in a group of transformations. Then the association (3) is often rewritten more succinctly as $y = \theta u$, where the concatenation $\theta u$ on the right corresponds to composition of the two transformations, $\vartheta$ and $u$. For technical reasons, it is necessary to assume that the distribution $P_U$ of $U$ has a density, $f$, with respect to Lebesgue measure, and that $f$ is not constant on sets with positive $P_U$-probability.

The above structure is rather restrictive, but often the original statistical model can be transformed into one that meets these conditions. The main obstacle is that a one-to-one connection between $\vartheta$ and $u$ for a given $y$ requires the dimensions of the two quantities to be the same. However, in the location model with $n$ iid samples, the baseline association (3) would have a scalar $\vartheta$ and a $n$-vector $u$. To side-step this problem, it is often possible to apply an initial dimension-reduction step to $u$ via a conditioning. This is described in detail in Chapter 2 of [Fraser (1968)](https://doi.org/10.1093/biomet/55.1.1) for the structural/group model case, as well as in [Dawid and Stone (1982)](https://doi.org/10.1093/biomet/69.3.381) and [Taraldsen and Lindqvist (2013)](https://doi.org/10.1093/biomet/90.3.537); a slightly different approach is taken in [Martin and Liu (2015a)](https://doi.org/10.1080/01621459.2014.972352). Suffice it to say, for the structural/functional models in consideration here, it is possible to convert the problem into one in which the simple one-to-one relationship holds. For the iid location model, if $P_U$ is normal, then a satisfactory dimension reduction can be achieved through sufficiency. More generally, in the location model, the residuals are ancillary statistics and conditioning on the vector of residuals leads to an effective reduction in dimension. Even more generally, for models with a group structure, there is a maximal invariant statistic which can be conditioned on. The point is just that, while the simple one-to-one association is rather restrictive, many problems can be transformed into one that satisfies this.

### 5.2 Inferential model constructions

Here I present the construction of two different IMs: a fiducial IM in the form of a precise probability and an IM based on the construction in [Liu and Martin (2021)](https://doi.org/10.1080/01621459.2019.1649952) of a data-dependent possibility measure; the latter is valid in the sense of Section 2.1 while the former is not. The structure of the functional model allows me to draw a clear connection between these two different solutions to the inference/decision problem.

I’ll start with the fiducial IM. In the present case, there is a well-defined fiducial distribution construction, a generalization of the Fisher’s ideas, as presented in [Fraser (1968)](https://doi.org/10.1093/biomet/55.1.1) and subsequently in, e.g., [Dawid and Stone (1982)](https://doi.org/10.1093/biomet/69.3.381) and [Taraldsen and Lindqvist (2013)](https://doi.org/10.1093/biomet/90.3.537). Indeed, modern presentations of fiducial inference suggest constructing a $y$-dependent probability $Q^*_y$ on $\Theta$ based on the formula

$$Q^*_y(A) = P_U\{\theta_y(U) \in A\}, \quad A \subseteq \Theta.$$  

(17)
Since $P_U$ has a density function $f$ with respect to Lebesgue measure, so does $Q^*_y$, and it’s given by the transformation formula
\[q^*_y(\vartheta) = f(u_y(\vartheta)) J(\vartheta; u_y),\]
where $J(\vartheta; u_y)$ is the transformation’s Jacobian, i.e.,
\[J(\vartheta; u_y) = \det \left| \frac{\partial u_y(\vartheta)}{\partial \vartheta} \right|.
\]
Alternatively, also from (3), the recently proposed construction in Liu and Martin (2021) would produce output in the form of a lower and upper probability pair, $(\Pi_y, \bar{\Pi}_y)$, where $\bar{\Pi}_y$ is determined by
\[\bar{\Pi}_y(A) = \sup_{\vartheta \in A} \pi_y(\vartheta), \quad A \subseteq \Theta,
\]
with the possibility contour function
\[\pi_y(\vartheta) = P_U \{ f(U) \leq f(u_y(\vartheta)) \}, \quad \vartheta \in \Theta.
\]
This IM, is valid according to the general theory in, e.g., Section 2.1. For convenience later on, define the function
\[\eta(z) = P_U \{ f(U) < z \}, \quad z \in [0, \sup f]. \tag{18}\]
Since the density $f$ is assumed to be non-constant on sets of positive $P_U$-probability, $\eta$ is strictly increasing and, hence, has an inverse $\eta^{-1}$. With this notation, the IM’s plausibility regions, which are needed in what follows, can be expressed as
\[C_\alpha(y) = \{ \vartheta : \pi_y(\vartheta) > \alpha \} = \{ \vartheta : f(u_y(\vartheta)) > \eta^{-1}(\alpha) \}. \tag{19}\]
This latter form for $C_\alpha(y)$ will be important to verifying the key identity (20) below.

## 5.3 Connections

Despite having the same starting point, namely (3), these two approaches seem entirely different. It turns out, however, that there is a close connection through the valid IM’s family of plausibility regions, $\{C_\alpha(y) : \alpha \in [0, 1]\}$. First note that, by definition,
\[\Pi_y \{ C_\alpha(y)^c \} = \sup_{\vartheta \notin C_\alpha(y)} \pi_y(\vartheta) = \sup_{\vartheta : \pi_y(\vartheta) \leq \alpha} \pi_y(\vartheta) = \alpha.
\]
Second, using the special structure of the model formulation above and, in particular, a change-of-variables from $\vartheta \to u$, I get
\[Q^*_y \{ C_\alpha(y)^c \} = \int_{\pi_y(\vartheta) \leq \alpha} q^*_y(\vartheta) d\vartheta \]
\[= \int_{f(u_y(\vartheta)) \leq \eta^{-1}(\alpha)} f(u_y(\vartheta)) J(\vartheta; u_y) d\vartheta \]
\[= \int_{f(u) \leq \eta^{-1}(\alpha)} f(u) du \]
\[= P_U \{ f(U) \leq \eta^{-1}(\alpha) \} \]
\[= \eta(\eta^{-1}(\alpha)) \]
\[= \alpha,
\]
where the range-of-integration change in the second inequality follows from (19), using the invertible function $\eta$ defined in (18). Consequently,

$$Q^*_y\{C_\alpha(y)^c\} = \overline{\Pi}_y\{C_\alpha(y)^c\}, \quad \text{for all } \alpha \in [0, 1].$$

This is important for several reasons. First, it implies that the fiducial distribution, $Q^*_y$, is a confidence distribution in the sense described in Section 3 which, by Theorem 1, implies that $Q^*_y \in \mathcal{C}(\overline{\Pi}_y)$. Second, the fact that both assign the same numerical probability/plausibility values to a wide class of assertions, namely, \{\,$C_\alpha(y)^c:\,\alpha \in [0, 1]$\}, implies that the fiducial distribution, $Q^*_y$, must be the “most extreme” of those members of $\mathcal{C}(\overline{\Pi}_y)$. Moreover, at least intuitively, the other members in the credal set should be those distributions that are dominated by $Q^*_y$ in some specific sense. I’ll refer to this special $Q^*_y$ as the maximally diffuse confidence distribution in $\mathcal{C}(\overline{\Pi}_y)$.

To make this intuitive notion of dominance more precise, some additional notation and terminology is needed. Following Bergmann [1991], suppose that $\Theta$ is equipped with a partial order, denoted by $\preceq$. Then a function $g : \Theta \to \mathbb{R}$ is said to be “increasing” (with respect to $\preceq$) if $\vartheta \preceq \vartheta'$ implies $g(\vartheta) \leq g(\vartheta')$. Similarly, a set $A \subseteq \Theta$ is said to be “increasing” (with respect to $\preceq$) if the indicator function $\vartheta \mapsto 1\{\vartheta \in A\}$ is increasing in the sense described above.

Returning to the statistical problem at hand, recall that the family of confidence regions is nested. From this nestedness property, a (data-dependent) partial order on $\Theta$ can be defined. That is, for two points $\vartheta_1$ and $\vartheta_2$ in $\Theta$, I’ll say that “$\vartheta_1 \preceq \vartheta_2$” if

$$\text{there exists an } \alpha \text{ such that } \vartheta_1 \in C_\alpha(y) \text{ but } \vartheta_2 \not\in C_\alpha(y).$$

(21)

In words, the points in $\Theta$ are ordered according to how small the confidence level, $\alpha$, needs to be for the $100(1-\alpha)\%$ confidence region, $C_\alpha(y)$, to contain them. Next, for two distributions on $\Theta$, say $Q$ and $Q'$, define the stochastic order relation $Q \preceq_{st} Q'$, relative to the aforementioned partial order “$\preceq$” on $\Theta$, as

$$Q(A) \leq Q'(A), \quad \text{for all } A \text{ increasing with respect to } \preceq.$$

This proves the following characterization of the valid IM’s credal set, $\mathcal{C}(\overline{\Pi}_y)$.

**Theorem 4.** Let $(\Pi_y, \overline{\Pi}_y)$ be the realization of a valid IM based on data $Y = y$, with maximally diffuse confidence distribution $Q^*_y$. Let $\preceq$ denote the (y-dependent) partial order determined by (21) and $\preceq_{st}$ the corresponding stochastic order. Then

$$Q_y \in \mathcal{C}(\overline{\Pi}_y) \iff Q_y \preceq_{st} Q^*_y.$$

For a quick illustration, reconsider the (symmetric) location model, $Y = \theta + U$, from Section 4.3.1. Since $\pi_y(\vartheta) = 2\{1 - F(|y - \vartheta|)\}$, where $F$ is the distribution function for the unobservable $U$, it’s easy to check that

$$C_\alpha(y) = [y - F^{-1}(1 - \frac{\alpha}{2}), y + F^{-1}(1 - \frac{\alpha}{2})], \quad \alpha \in [0, 1].$$

Under standard conditions on $F$—e.g., a special case of Example 2.1 in Dawid and Stone [1982]—the fiducial distribution for $\theta$, given $Y = y$, has the density function

$$q'_\vartheta(\vartheta) = -\frac{\partial}{\partial \vartheta} F(y - \vartheta) = f(\vartheta - y), \quad \vartheta \in \Theta.$$
This is also the Bayesian solution under the default, flat prior for $\theta$. It’s well known that the Bayes/fiducial solution in this case is a confidence distribution, and corresponds to the maximally diffuse element in the credal set $\mathcal{C}(\Pi_y)$. Furthermore, any distribution that’s symmetric and “appropriately more concentrated” than $Q^*_y$ would be contained in $\mathcal{C}(\Pi_y)$. For example, scaled versions of $Q^*_y$, having density

$$g_y(\vartheta) = s^{-1}f(s^{-1}(\vartheta - y)), \quad \text{for } s \in [0, 1],$$

belong to the credal set, even the special case “$s = 0$,” corresponding to a point mass distribution concentrated at $y$. It may also contain other symmetric distributions.

Beyond providing some understanding of the credal set’s contents, Theorem 4 also has some decision-theoretic implications. The following theorem and the subsequent discussion make this claim precise.

**Theorem 5.** Let $(\Pi_y, \Pi_{\hat{a}})$ be a valid IM based on data $Y = y$, with maximally diffuse confidence distribution $Q^*_y$, and let $\preceq$ denote the $y$-dependent partial order determined by (21). Let $(a, \vartheta) \mapsto \ell_a(\vartheta)$ be a convex, non-negative loss function such that there exists an action $\hat{a} = \hat{a}(y)$ for which $\vartheta \mapsto \ell_{\hat{a}}(\vartheta)$ is increasing with respect to $\preceq$. Then

$$\inf_{a \in A} \Pi y \ell_a = \Pi y \ell_{\hat{a}} = Q^*_y \ell_{\hat{a}} = \inf_{a \in A} Q^*_y \ell_a,$$

and, therefore, $\hat{a} = \hat{a}(y)$ is the minimum (upper) expected loss action for both the valid IM and the confidence distribution.

**Proof.** The proof proceeds in three steps, one for each of the three equalities in (22), starting with the second equality. By definition, $\Pi y \ell_{\hat{a}} = \sup_{a \in A} Q^*_y \ell_a$, where the supremum is over all $Q_y$ in the credal set $\mathcal{C}(\Pi_y)$. But since the credal set is determined by the stochastic partial order $Q_y \preceq_{st} Q^*_y$, and $\vartheta \mapsto \ell_{\hat{a}}(\vartheta)$ is increasing with respect to the same partial order, it follows from Theorem 3(b) in Bergmann (1991) that the supremum is attained at the stochastic upper bound, $Q^*_y \ell_{\hat{a}}$.

Next, for the third equality. The expected loss, $a \mapsto Q^*_y \ell_a$, is minimized at an action $a$ such that the level sets of $\ell_a$ most closely align with those of the confidence/fiducial density. But the latter level sets, recall, are the confidence regions that determine the orders $\preceq$ and $\preceq_{st}$. Since $\vartheta \mapsto \ell_{\hat{a}}(\vartheta)$ is increasing with respect to $\preceq$, its level sets exactly align with the confidence regions and, hence, with the level sets of the confidence/fiducial density. Therefore, the minimum is attained at $\hat{a}$.

Finally, for the first equality, note that, again by definition, $\Pi y \ell_{\hat{a}}$ can’t be smaller than $Q^*_y \ell_a$ for any $a$. So, since equality is attained at $\hat{a}$, and since this is the minimizer of $a \mapsto Q^*_y \ell_a$, it must also be the minimizer of $a \mapsto \Pi y \ell_a$. \hfill $\square$

I don’t expect the reader to be familiar with a condition like that on the loss function in Theorem 5, so some further remarks are in order.

- First, the simplest example of this is the symmetric location problem, with squared error loss, presented in Section 4.3.1 above. There it was shown directly, by evaluating the risks $\Pi y \ell_{\hat{a}}$ and $Q^*_y \ell_a$ and showing that they’re both minimized at the same point, $\hat{a}(y) = y$. For the case when the error distribution was not symmetric, the valid IM and fiducial suggest different actions under squared error loss. However,
with a different loss function that incorporates the shape of the error distribution, a connection between the valid IM and fiducial actions could be made. While the location model is indeed a very special case, note that virtually every fixed-dimensional inference problem takes this form when the data are sufficiently informative. That is, Le Cam’s deep insights about limiting experiments (e.g., van der Vaart 2002) suggest that all “regular” problems boil down to normal locations problem in the limit. Therefore, a characterization result for location models goes a long way toward understanding more general problems.

• Second, where might a loss function that satisfies the conditions of Theorem 5 come from? In the symmetric location problem, squared error loss would be appropriate because it, too, is symmetric. More generally, in a multivariate location model where the error distribution is elliptically symmetric, a generalized squared error loss, one that incorporates the error distributions dispersion matrix, would achieve the theorem’s “increasing” condition. But what about beyond the symmetric location model? For a decision problem whose ingredients can be identified with elements of a group, a reasonable strategy is to define

\[ \ell_a(\vartheta) = k \{1 - \pi(\vartheta^{-1} a)\}, \]

for \( k : [0, 1] \rightarrow [0, \infty) \) strictly increasing, such as \( k(x) = -c \log(1 - x) \), for a constant \( c > 0 \). For an illustration, reconsider the skew-normal location model from Section 4.3.1. Following the above strategy, I’ll define a loss function

\[ \ell_a(\vartheta) = -\frac{1}{5} \log \pi(a - \vartheta). \]  

(23)

Figure 6 shows this loss function, for \( a = -1 \) and \( a = 0 \), overlaid on the same possibility contour displayed in Figure 3(a) with \( y = 0 \). This plot demonstrates that the level sets of the loss function, with \( a = 0 \), are the same as those of the possibility contour. Therefore, the loss \( \ell_a \) defined above, with \( a = 0 \), is increasing in the sense considered in Theorem 5. Consequently, for a general data point \( y \), the action corresponding to the minimum (upper) expected loss is \( \hat{a}(y) = y \).

• Third, the “increasing with respect to \( \succeq \)” is just a sufficient condition for connecting the suggested actions from the valid IM and fiducial distribution based on the minimum (upper) expected loss principle. More generally, one can apply that principle with different loss functions, including those which are not “increasing” in this sense, and the resulting action would still satisfy this principle; it’s just that it may be different from—perhaps even better than—the rule one would get from applying the same principle to the fiducial distribution.

The message in Taraldsen and Lindqvist (2013) is that fiducial methods, which inherently have some desirable confidence properties, will also provide good frequentist decision procedures with some degree of generality. Here the message is similar but arguably goes further. First, IMs are specifically designed to achieve valid inference about unknowns, which is stronger than the confidence property achieved by fiducial methods (e.g., Martin 2021a; Martin et al. 2021). Second, the conservativeness that’s built in to the IM to achieve this validity property appears in the decision decision-making context.
through its assessment of the quality of different actions, and Theorem 2 ensures that the IM’s assessment will not be too optimistic compared to a suitable oracle’s assessment. This gives the decision-maker some comfort that the IM’s minimum (upper) expected loss rule is driven by signal rather than noise. Third, despite its conservativism, the conditions of Theorem 5 define a range of problems, wide enough to cover almost all problems in an asymptotic limit, in which the IM’s suggested action based on the minimum (upper) expected loss principle is the same as the one considered to be “optimal” based on other criteria. And beyond those problems covered by the previous theorem, like the skewed-normal or binomial examples above, the solution based on the valid IM’s upper probability may have some advantages which deserve further investigation.

6 Conclusion

To date, and to my knowledge, inferential models (IMs) have been used exclusively for inference—no formal theory on decision-making had been investigated in the literature. This paper aims to fill this gap by, first, developing a framework for decision-making under uncertainty based on an IM’s upper probabilities and, second, investigating the decision-theoretic implications of the IM’s validity property. For the first, I follow previous suggestions made in the imprecise probability literature to define an upper expected loss via the Choquet integral of the loss with respect to the IM’s upper probability, so that different actions can be compared using a natural variation of the minimize-expected-loss principle. For the second, it was shown in Theorems 2–3 that validity of the IM implies that its assessment, \( \Pi_Y \ell_a \), of an action \( a \) would, in a specific sense, not tend to be inconsistent with the quasi-oracle’s assessment, thus giving the IM a sense of reliability in the decision-making context which, in turn, provides some protection to the decision-
maker. While the validity property requires some degree of conservatism, the numerical examples and the developments in Section 5 strongly suggest that the IM’s suggested actions are high-quality, in some cases, optimal from a purely frequentist point of view. Specifically, Theorem 5 shows that, for a certain class of statistical decision problem, the valid IM’s suggested action agrees with that of the fiducial distribution, where the latter is known to be optimal in many cases where an “optimal” decision rule is known to exist.

In my numerical illustrations, computation of the Choquet integral was straightforward, whether using Monte Carlo methods or quadrature, as was the optimization to obtain the minimum (upper) expected loss rule, \( \hat{a} \). However, in higher dimensions, the same computations become more difficult. For a recent survey on some of the general computational tools available in the imprecise probability literature for evaluating Choquet integrals, see Troffaes and Hable (2014). That reference also describes a simplification in the special case where the upper probability is a possibility measure. A thorough investigation into how well these general algorithms work for the specific case of integration with respect to the valid IM’s lower/upper probability output is still needed.

The numerical examples in Section 4.3 demonstrated that the fiducial or confidence distribution’s assessment of the action quality does not satisfy the property in the conclusion of Theorem 3, while the valid IM’s assessment does. A relevant question is if this might serve as the basis for an analogue of the false confidence theorem (Balch et al. 2019) in the context of decision-making. Two things are needed to achieve this: (a) better understanding of the quasi-oracle’s assessment, i.e., why a failure to satisfy the conclusion of Theorem 3 may lead to poor practical decision-making performance, and (b) a proof that non-valid IM’s can’t satisfy that property.

An important notion in frequentist statistical decision theory is admissibility. For a decision rule \( \hat{a} : \mathcal{Y} \to \mathcal{A} \), define the (frequentist) expected loss

\[
r(\theta; \hat{a}) = \int \ell_{\hat{a}(y)}(\theta) \, P_{Y|\theta}(dy).
\]

Then \( \hat{a} \) is inadmissible if there exists another decision rule \( \tilde{a} \) such that \( r(\theta; \tilde{a}) \leq r(\theta; \hat{a}) \) for all \( \theta \), with strict inequality for some \( \theta \); otherwise, \( \hat{a} \) is admissible. Unfortunately, the valid IM’s recommended action, defined in (11), is not admissible in general. To see this, consider the \( d \)-dimensional normal mean problem. Under loss \( \ell_{a}(\vartheta) = \|a - \vartheta\|^2 \), the IM’s recommended action is \( \hat{a}(y) = y \), which corresponds to the least squares, maximum likelihood, fiducial, and flat-prior Bayes estimator, was shown by Stein (1956) to be inadmissible when \( d \geq 3 \); see, also, Brown (1971). This is not especially surprising since validity requires that the IM be “unbiased” in a certain sense which, in the present context implies the IM’s recommended action be an unbiased estimator in the usual sense. But the familiar bias-variance tradeoff suggests that \( r(\cdot; \hat{a}) \) can be reduced by choosing a biased \( \hat{a} \), e.g., like a proper-prior Bayes rule that’s biased and admissible. To introduce the appropriate “bias” into the IM construction, it’s natural to consider incorporation of suitable (partial) prior information, e.g., sparsity. A notion of validity under partial prior information is currently being developed (see Martin 2021b), and an extension of the decision-theoretic results presented here to that case, along with admissibility considerations, are part of my ongoing investigations.
A  Additional technical details

In the proof of Theorem 3 presented in Section 4.2 an important step was postponed. The relevant result is stated and proved below.

**Lemma 1.** As in the statement of Theorem 3, suppose that the loss function \( \vartheta \mapsto \ell_a(\vartheta) \) is convex and non-negative for each action \( a \), and the possibility contour \( \vartheta \mapsto \pi_y(\vartheta) \) is directionally convex. Then the function

\[
\alpha \mapsto h_{y,a}(\alpha) := \sup \{ \ell_a(\vartheta) : \pi_y(\vartheta) > \alpha \}, \quad \alpha \in [0,1],
\]

is convex.

**Proof.** For simplicity, drop the subscripts \((y,a)\) on the \( h \) function. Two key observations are as follows:

- the level sets \( \{ \vartheta : \pi_y(\vartheta) > \alpha \} \) of \( \pi_y \) are intervals;
- and the supremum in the definition of \( h(\alpha) \) is attained at the one or both of the endpoints of the aforementioned interval.

Suppose first that the supremum on each of those \( \alpha \)-indexed intervals is unique, i.e., that the loss evaluated at one endpoint is strictly larger than the loss at the other. Define

\[
\theta_\alpha = \arg\max \{ \ell_a(\vartheta) : \pi_y(\vartheta) > \alpha \}.
\]

Since both \( \ell_a \) and \( \pi_y \) are continuous, the function \( \alpha \mapsto \theta_\alpha \) is also continuous, with \( \theta_\alpha \) converging monotonically from either the left or right to left- or right-most mode, respectively, as \( \alpha \to 1 \). Also, again by continuity, \( \pi_y(\theta_\alpha) = \alpha \). If it happens that the loss function takes the same value on both endpoints of one of the level sets of the possibility contour, then the choice of \( \theta_\alpha \) could be made so that \( \alpha \mapsto \theta_\alpha \) is continuous.

Take arbitrary \( \alpha_1, \alpha_1 \in [0,1] \) and \( w_1 \in [0,1] \), and then set \( w_2 = 1 - w_1 \). The goal is to show that

\[
h(w_1\alpha_1 + w_2\alpha_2) \leq w_1 h(\alpha_1) + w_2 h(\alpha_2),
\]

where, with the notation defined above, \( h(\alpha) = \ell_a(\theta_\alpha) \), for \( \alpha \in [0,1] \). Let \( \bar{\alpha} = w_1\alpha_1 + w_2\alpha_2 \) and \( \bar{\theta}_\alpha = w_1\theta_{\alpha_1} + w_2\theta_{\alpha_2} \). Since \( \pi_y \) is assumed to be directionally convex, it follows that

\[
\pi_y(\bar{\theta}_\alpha) \leq w_1\pi_y(\theta_{\alpha_1}) + w_2\pi_y(\theta_{\alpha_2}) = \bar{\alpha}.
\]

This implies \( \bar{\theta}_\alpha \) is not in the interior of \( \{ \vartheta : \pi_y(\vartheta) > \bar{\alpha} \} \) and, in turn, that \( \ell_a(\bar{\theta}_\alpha) \) is no less than \( \ell_a(\theta_\alpha) \). From this, and convexity of the loss, it follows that

\[
h(\bar{\alpha}) = \ell_a(\bar{\theta}_\alpha)
\leq \ell_a(\theta_\alpha)
\leq w_1\ell_a(\theta_{\alpha_1}) + w_2\ell_a(\theta_{\alpha_2})
= w_1 h(\alpha_1) + w_2 h(\alpha_2),
\]

which proves the convexity claim. \( \square \)
Figure 7: Contour plot showing level sets of $\pi_y$ (gray) and of $\ell_a$ (dotted). The solid black line is the path $\alpha \mapsto \theta_\alpha$, which is a curve in this example.

The challenge to extending the above convexity argument to more than one dimension is that the path $\alpha \mapsto \theta_\alpha$ is generally not a line. Figure 7 shows an example of this. Then $\bar{\theta}_\alpha$ is not necessarily on this path, which makes it difficult to establish that $\ell_a(\bar{\theta}_\alpha)$ is no less than $\ell_a(\theta_\alpha)$. This was a crucial step in the proof of Lemma 1 above, so if it could still be made when the aforementioned path is not a line, then the convexity result and, hence, the claim in Theorem 3 would hold for the general vector-$\theta$ case.

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