Some insights into Vaidya’s problem

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Abstract In this paper we highlight peculiar features of the radiating Vaidya’s
metric being the result of algebraic deduction from the Einstein equations
rather than of their proper integration. Thus, the standard “first integral”
\( f(m) \) proves to be not arbitrary but a constrained function, and the choice
\( f(m) = \pm \dot{m} \) sometimes used in literature does not satisfy the entire system of
the Einstein equations. But still, the general Vaidya algorithm as a shortcut
to catch underlying symmetries may be successfully applied, in particular, to
the problems of imbedding (e.g. Vaidya-de Sitter spacetime) or to the case of
non-null (time-like or space-like) radial dust emission.

Keywords Vaidya metric · Integrability conditions · Kottler solution ·
Isotropic coordinates · Non-null radiation

1 Introduction

In spite of all the beauty and conciseness of Vaidya’s approach to the problem
of radiating masses as a whole, it cannot be applied to physically important
cases where, e.g., mass depends on time without space-coordinate dependence.
We will show that this issue is a non-trivial consequence of the fact that
the Vaidya metric does not follow from proper integration of the Einstein
equations but represents, in effect, an algebraic consequence of those. From
this ensue some peculiar features related to integrability conditions, preferred
coordinates, imbedding problems, etc., which we highlight in the present paper.

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2 Formulation of the Vaidya problem

The general relativistic radiative mass problem had been presented in Vaidya’s seminal work [1]. The null-dust Einstein equations in case of radial emission,

\[ G_{\mu}^{\nu} = R_{\mu}^{\nu} - \frac{1}{2} R \delta_{\mu}^{\nu} = \kappa T_{\mu}^{\nu} = \kappa \varepsilon l_{\mu} l^{\nu}, \quad l_{\alpha} l^{\alpha} = 0, \quad l^2 = l^1 = 0, \quad (1) \]

imply the following conditions due to symmetries of the considered problem:

\[ T_{\mu}^{\alpha} l^{\alpha} = 0, \quad \mu = 0 \quad \Rightarrow \quad G_{0}^{0} + G_{1}^{0} e^{-\lambda / 2} = 0, \quad (2) \]

\[ T_{\alpha}^{\alpha} = 0 \quad \Rightarrow \quad G_{0}^{0} + G_{1}^{1} = 0, \quad (3) \]

If one uses curvature coordinates,

\[ ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - r^2 d\Omega^2, \quad (4) \]

the field equations corresponding to relations (2, 3) become:

\[ e^{-\lambda} \left( \lambda' - \frac{1}{r} \right) + \frac{1}{r} \lambda + \dot{\lambda} e^{-(\lambda + \nu)/2} = 0, \quad (5) \]

\[ e^{-\lambda} \left( \lambda' - \nu' - \frac{2}{r} \right) + \frac{2}{r} = 0, \quad (6) \]

where overdot denotes differentiation with respect to \( t \), and prime – with respect to \( r \). Vaidya had considered also the third symmetry condition,

\[ T_{2}^{2} = T_{3}^{3} = 0 \quad \Rightarrow \quad G_{2}^{2} = G_{3}^{3} = 0, \quad (7) \]

that is

\[ e^{-\lambda} \left( \lambda' - \nu' - \frac{\nu'^2}{2} + \frac{\lambda \nu'}{2} \right) + e^{-\nu} \left( \frac{\lambda'^2}{2} - \frac{\dot{\lambda} \nu'}{2} \right) = 0, \quad (8) \]

but this one is, in fact, not independent and admits essential simplification taking into account the zero trace of the null-dust energy-momentum tensor,

\[ G_{2}^{2} = G_{3}^{3} = 0 \quad \Rightarrow \quad R_{2}^{2} = R_{3}^{3} = 0, \quad (9) \]

so that the relation (8) is reduced to (9) which in its turn proves to be equivalent (up to a constant factor) to equation (6).

The crucial point in Vaidya’s approach is the ansatz stating that \( g_{11} \)-component of the metric (4) should be represented in a Schwarzschild-like form, in which constant mass is replaced with certain “mass-function” \( m(r,t) \):

\[ -g_{11} = e^{\lambda(r,t)} = \left( 1 - \frac{2 m(r,t)}{r} \right)^{-1} = D^{-1}, \quad (10) \]

because Vaidya believed that this corresponds to the desired asymptotics at large scales.

The next step is in finding the remaining metric components from the field equations given above.
3 Algebraic deduction of Vaidya’s metric

For that we return to the null-dust symmetry condition (2) in the form (5) which might be explicitly rewritten for the $g_{00}$ component:

$$g_{00} = e^{\nu(r,t)} = \frac{\dot{\lambda}^2 e^{\nu^2}}{\left(e^\lambda + \nu r - 1\right)^2}. \quad (11)$$

Direct substitution of the ansatz (10) into (11) leads, after some cumbersome algebra, to the final result:

$$e^{\nu} = \frac{\dot{m}^2}{m^2} \left(1 - \frac{2m(r,t)}{r}\right)^{-1} = \frac{\dot{m}^2}{m^2} D^{-1}, \quad (12)$$

So, the resulting full Vaidya metric is obtained without integration of the field equations:

$$ds^2 = \frac{\dot{m}^2}{m^2} D^{-1} dt^2 - D^{-1} dr^2 - r^2 d\Omega^2. \quad (13)$$

Thus, by employing the null-dust symmetries (5, 6, 8), the two arbitrary functions $\lambda(r,t)$ and $\nu(r,t)$ in the given metric of type (4) have been mapped into a single mass-function $m = m(r,t)$ in accord with the relations (10) and (12).

And conversely, the Vaidya metric (13) as a result of this mapping automatically satisfies the Einstein equations, including relations (5, 6, 8), not being nevertheless the result of the “solution” of those in customary mathematical sense.

The described Vaidya’s approach can also be generalized to include the $\Lambda$-term as cosmological background [2]. In that case, the ansatz (10) acquires the extended Kottler (also known as Schwarzschild-de Sitter) form:

$$-g_{11} = e^{\lambda(r,t)} = \left(1 - \frac{2m(r,t)}{r} - \frac{\Lambda r^2}{3}\right)^{-1} = \tilde{D}^{-1}, \quad (14)$$

which is to be substituted into the extended relation of the type (2),

$$\kappa T_{\alpha}^{\mu} T^{\alpha} = \Lambda \delta_0^{\mu} \implies \ G^0_0 + G^1_0 e^{\frac{\nu r}{r}} - \Lambda = 0, \quad (15)$$

yielding eventually the Vaidya-type metric imbedded into the de Sitter background, similar to (11), (12) and (13):

$$ds^2 = \frac{\dot{m}^2}{m^2} \tilde{D}^{-1} dt^2 - \tilde{D}^{-1} dr^2 - r^2 d\Omega^2. \quad (16)$$

This might be considered as a superposition of corresponding (Vaidya’s and de Sitter’s) spacetime intervals, obtained without integration of the Einstein equations as well.
4 The gist of the “first integral”

Contrary to the expectation of Vaidya that the sought-after metric should be asymptotically related to the Schwarzschild spacetime, this, strictly speaking, does not follow with necessity from the expression (13). The “desired form” would be

\[ ds^2 = \left( 1 - \frac{2m(r,t)}{r} \right) dt^2 - \left( 1 - \frac{2m(r,t)}{r} \right)^{-1} dr^2 - r^2 d\Omega^2, \]  

(17)

and, indeed, it is customary to present the metric (13) in a similar form

\[ ds^2 = \frac{\dot{m}^2}{f^2(m)} \left( 1 - \frac{2m}{r} \right) dt^2 - \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 - r^2 d\Omega^2 \]  

(18)

containing \( f(m) = m'(1 - 2m/r) \) as the first integral of (6) - see [1] and Appendix A where and in what follows we write \( m \) instead of \( m(r,t) \) where this does not lead to ambiguities.

The metric (18) appears to be reduced to the desired Schwarzschild-like form (17) if we impose “by hands” the physically relevant condition \( f(m) = \pm \dot{m} \), as proposed, e.g., by Mallet [2]. However, substitution of the resulting metric (17) into the field equations reveals that while (6) is satisfied, the relation (5) is incompatible with \( f(m) = \pm \dot{m} \) (unless \( \dot{m} = 0 \)). Indeed, the last term in (5) disappears if and only if \( m \) does not depend on \( t \), which is beyond the scope of Vaidya’s problem.

In fact, this negative result is expected because from (10) and (12) the general functional relation follows:

\[ m'e^{\nu/2} = -\dot{m}e^{\lambda/2}, \]  

(19)

which demonstrates that the variable \( m(r,t) \) in the Vaidya metric (13) or (18) cannot be a function of \( r \) or \( t \) alone. Such peculiar feature implies, in particular, that the Eddington-Jeans power law \( \dot{m} = -km^n \), important in the study of stellar evolution, is out of scope of Vaidya’s approach. The same goes for any dependence of the concrete type \( m = m(t) \).

All this represents the consequence of a more general assertion that the “first integral” \( f(m) \) is not arbitrary but is a special function restricted to the functional relation given above: \( f(m) = m'(1 - 2m/r) \). But in such a case the customary solution (18) returns to the original form (13) with only one indeterminate mass-function \( m = m(r,t) \).

The admissible constraints on \( f(m) \) following from integrability conditions are considered in Appendix B. We have also obtained that, in general, the function \( m(r,t) \) is not arbitrary as well but is subject to the constraint

\[ \left( \frac{m''}{m'} - \frac{\dot{m}'}{m} \right) D + D' + \frac{2m'}{r} = \left( \frac{m''}{m'} - \frac{\dot{m}'}{m} \right) D - \frac{2m}{r^2} = 0, \]  

(20)

as follows from (56) below. For example, from (20) one can also find that \( m = m(t) \) is incompatible with Vaidya’s metric. Analogous conclusion is true for functional relations of the form \( m(r,t) = f(r)\phi(t) \) or \( m(r,t) = \alpha f(r) + \beta \phi(t) \).
The detailed information on the mass-function might be obtained only outside Vaidya’s approach by imposing subsidiary initial and boundary conditions and subsequent proper integration of the field equations. As a tentative example, one might consider the deduction of matching conditions for transition from Vaidya’s metric to the Friedmanian one [3].

5 Vaidya’s metric as a coordinate effect

Algebraic deduction of the Vaidya metric is definitely coordinate dependent, in the sense that it is specific to the curvature coordinates and cannot be obtained in general, for example, in isotropic coordinates (see Appendix C). Moreover, the transformation from (13) to isotropic coordinates

\[ ds^2 = e^{\nu(R,t)}dt^2 - e^{\mu(R,t)}(dR^2 + R^2d\Omega^2) \] (21)

does not exist at all. Indeed, the standard algorithm [4] based on general condition that

\[ \int \frac{dR}{R} = \int \frac{e^{\lambda(r,t)}}{r} dr \] (22)

fails because \( \lambda(r,t) \) is expressed via the unknown function \( m(r,t) \), so that integration in (22) cannot be carried out in principle.

For the same reason, the Vaidya metric on the de Sitter background also cannot be transformed to isotropic coordinates, unless \( m \) is constant. It is interesting to note that even for that simplest case, \( m = \text{const} \), the transformation of Kottler’s (Schwarzschild-de Sitter) metric from curvature coordinates to isotropic coordinates is non-trivial (see [5] and Appendix D).

There exists yet another widely used representation of Vaidya’s spacetime in Eddington-Finkelstein coordinates:

\[ ds^2 = \left(1 - \frac{2m(u)}{r}\right)du^2 \pm 2dudr - r^2d\Omega^2. \] (23)

This might be obtained from (13) with the differential transformation

\[ dt = du \pm \frac{dr}{1 - \frac{4m}{r}} = du \pm dr^*, \] (24)

where \( r^* \) is supposed to be the so-called ingoing (+) or outgoing (−) tortoise coordinate (see below). For the Schwarzschild problem in curvature coordinates this transformation is the known way to get (23), with \( m = \text{const} \). To implement this transformation for arbitrary \( m \neq \text{const} \), we first note that the factor \( \frac{m^2}{m'^2} \) in (13) can be represented as

\[ \frac{m^2}{m'^2} = \nu^2 = \left(1 - \frac{2m(r,t)}{r}\right)^2, \] (25)

which corresponds to (24) at lines of constant \( u \). After that, (13) acquires the “desired form” (17) and so the metric (23) follows in a standard way by
applying (24). But transfer to \( m(u) \) is somewhat doubtful because for the undefined variable \( m = m(r, t) \) the expression (24) (unlike the Schwarzschild case \( m = \text{const} \)) cannot be integrated, and so the transfer to integral representation of the type \( u = t \pm r^* \) with the tortoise coordinate

\[
r^* = \int \frac{dr}{1 - \frac{2m}{r}}
\]

does not exist in explicit form (3). Thus, the Eddington-Finkelstein form of the Vaidya metric bares symbolic rather than operational character. Generally speaking, Vaidya’s approach leads to a class of symbolic metrics appropriate for detailed consideration in concrete situations.

### 6 Non-null radiation

We will now consider the most general type of radially propagating emission (or absorption) fields, including, apart from the null dust, also time-like or space-like emissions (with zero velocity dispersion of emitted particles). In such case the Einstein equations (11) might be transformed as follows:

\[
G_{\mu}^{\nu} \equiv R_{\mu}^{\nu} - \frac{1}{2} R \delta_{\mu}^{\nu} = \kappa T_{\mu}^{\nu} = \kappa \varepsilon_k \varepsilon^k_{\nu} = 0
\]

(26)

\[
k_\nu k^\nu = \alpha^2 - \beta^2 = \begin{cases} 
1 & k^2 = k^3 = 0, \\
0 & k^2 = k^3 = 0,
\end{cases}
\]

(27)

where now apart from \( k_\nu k^\nu = 0 \) we also consider \( k_\nu k^\nu = 1 \) or \( k_\nu k^\nu = -1 \), correspondingly. Each of these three conditions requires separate parametrization. Thus, for the traditional null-dust case the required parametrization is \( \alpha = \beta \), with equal contributions from timelike and spacelike parts.

Now we begin with the time-like (“bradyonic emission”) case \( k_\nu k^\nu = 1 \) and adopt the following gauge conditions for the radial vector \( k^\mu \) aligned in 3-space along the initial radial particle velocity \( v \):

\[
k^\mu = \alpha u^\mu + \beta n^\mu = \frac{u^\mu}{\sqrt{1 - v^2}} + \frac{v n^\mu}{\sqrt{1 - v^2}}
\]

(28)

\[
u_\nu u^\nu = 1, \quad n_\nu n^\nu = -1, \quad u_\nu n^\nu = 0,
\]

with values \( 0 \leq v < 1 \) (in units of the speed of light \( c \)) and unit vectors

\[
u^\mu = \frac{\delta^\mu_0}{\sqrt{g_{00}}}, \quad n^\mu = \frac{\delta^\mu_1}{\sqrt{-g_{11}}},
\]

(29)

from which it follows

\[
k^0 = \frac{\alpha}{\sqrt{g_{00}}}, \quad k^1 = \frac{\beta}{\sqrt{-g_{11}}}, \quad k_0 = \alpha \sqrt{g_{00}}, \quad k_1 = -\beta \sqrt{-g_{11}}.
\]

(30)
The null-dust symmetry conditions (2) and (3) are transformed into
\[ T_\alpha^\mu k^\nu = \varepsilon k^\mu, \quad \mu = 0 \quad \implies \quad G_0^0 k^0 + G_1^0 k^1 = \varepsilon \varepsilon k^0 \] (31)

and
\[ T_\alpha^\alpha = \varepsilon \quad \implies \quad G_0^0 + G_1^1 = \varepsilon \varepsilon. \] (32)

The third Vaidya’s symmetry condition (7) will be represented as
\[ T_2^2 = T_3^3 = 0 \quad \implies \quad G_2^0 = G_3^0 = 0 \quad \implies \quad R_2^2 = R_3^3 = -\varepsilon \varepsilon / 2, \] (33)

and all relations (31)-(33) represent the field equations to be satisfied with the sought-after metric. Unlike the null-dust case, in Eqs. (31)-(33) one needs to exclude the proper energy density \( \varepsilon \). This might be achieved by employing a relevant component of the Einstein equations, e.g.:
\[ G_1^0 k^0 = \kappa \varepsilon k^0, \] (34)

So, substituting (34) into (31) and taking into account (30) and (4 ) we ultimately obtain in curvature coordinates:
\[ G_0^0 - k_0 G_1^0 = G_0^0 + \frac{\alpha \sqrt{\beta \varepsilon}}{\beta \sqrt{\gamma_1}} G_1^0 = G_0^0 + \frac{\alpha}{\beta} e^{\frac{\alpha}{\beta}} G_1^0 = 0, \] (35)

where
\[ G_0^0 = e^{-\lambda} \left( r \lambda' + e^\lambda - 1 \right), \quad G_1^0 = \frac{e^{-\nu} \dot{\lambda}}{r}. \]

Applying the same Vaidya’s ansatz (10) to (35) we get, after some algebra, the generalisation of (19):
\[ \beta m' e^\nu/2 = -\alpha \dot{m} e^{\lambda/2}, \] (36)

or, finally,
\[ g_{00} = e^{\nu} = \frac{\alpha^2}{\beta^2} \frac{\dot{m}^2}{m'^2} \left( 1 - \frac{2 m}{r} \right)^{-1} = \frac{\alpha^2}{\beta^2} \frac{\dot{m}^2}{m'^2} D^{-1} = \frac{1}{v^2} \frac{\dot{m}^2}{m'^2} D^{-1}, \] (37)

where for short we write again \( m \) instead of \( m(r, t) \). There is no divergence at \( v \to 0 \) resulting from the factor \( 1/v \) in (37) because relation (36) implies that if \( \beta/\alpha = v \to 0 \) then \( \dot{m} \to 0 \), as it should be also from physical considerations. Thus, the resulting Vaidya metric for the generalized radial emission is obtained without integration of the field equations as well:
\[ ds^2 = \frac{\alpha^2}{\beta^2} \frac{\dot{m}^2}{m'^2} D^{-1} dt^2 - D^{-1} dr^2 - r^2 d\Omega^2. \] (38)

Substitution of this metric into the second symmetry condition (32), taking into account the expression following from (34),
\[ \varepsilon = \frac{2}{\alpha} \frac{m'}{\varepsilon \beta^2 r^2}, \] (39)
yields the subsidiary constraint on the mass-function \( m(r, t) \):

\[
\left( \frac{m''}{m'} - \frac{\dot{m}'}{\dot{m}} \right) D + D' + \frac{2m'}{r} = \frac{1}{\alpha^2} \frac{m'}{r}.
\]

(40)

As it should, (40) is consistent with the third symmetry condition (33) as well. In the null-dust case we have \( v \to 1 \) and \( \alpha \to \infty \), and so (40) reduces to the well-known relation (56).

For completeness, we note that superposition (imbedding) of the obtained interval (38) with the de Sitter background reduces to the replacement of the factor \( D \) in (38) with the factor \( \tilde{D} \) in (16), as it follows from the deduction of metric (16) and a remark after (56).

As for the space-like (“tachyonic radiation”) case, i.e. \( k_\nu k^\nu = -1 \), analytically it differs from the above consideration only by changing the initial parametrization in the gauge condition (28):

\[
k^\mu = \alpha u^\mu + \beta n^\mu = \frac{u^\mu}{\sqrt{v^2 - 1}} + \frac{m^\mu}{\sqrt{v^2 - 1}}, \quad 1 < v < \infty.
\]

(41)

It should be noted that for general energy-momentum tensor (26) under the gauge (28) or (41) and taking into account (29) and (30) the energy density \( \rho \) observed within the congruence \( \{u^\mu\} \) differs from the proper value of \( \varepsilon \) (39) by the constant factor \( \alpha^2 \):

\[
\rho = T_{\mu\nu} u^\mu u^\nu = \varepsilon (k_\nu u^\nu)^2 = \alpha^2 \varepsilon = \frac{2 m'}{x^2 r^2}.
\]

(42)

Tachyonic radiation phenomenon gauged by (41) may be of interest because at present we cannot yet exclude the principal possibility of the existence of background neutrinos in tachyonic state (see, e.g., [7,8,9,10]). This phenomenon cannot be directly proved or disproved today with Earth-based experiments due to very little absolute values of the observable masses of neutrinos. E.g., for neutrino energies of order 10 GeV and masses approximately \( 10^{-2} \) eV the expected relative deviation of neutrino velocities from the speed of light proves to be about \((\pm)10^{-24}\) which is extremely difficult to reveal.

Non-null radiation described above might be considered also as a source with nonzero radial pressure for the Vaidya-type metrics [6].

7 Conclusion

We have considered peculiar properties of the mass-function \( m(r, t) \) as a consequence of the fact that derivation of the Vaidya metric does not follow from proper integration of the Einstein equations. Besides, we have considered some issues related to various generalizations of the Vaidya case.

Essentially, because the general algebraic Vaidya algorithm is intrinsically locked to curvature coordinates, this leads to its inapplicability in isotropic frames convenient for physical measurements, unlike, e.g., the related problem of transformation of the Kottler metric from curvature to isotropic coordinates.
Nevertheless, in curvature coordinates all Vaidya-type metrics considered here appear as very flexible and superposable with the cosmological de Sitter background. Sufficiently easy (with no integration of field equation), these metrics can be generalized by taking into account the radial emission of particles of different (time-like or space-like) nature.

A Deduction of the first integral

From the Einstein equations (41) written with the metric (39) it follows

$$\left( \frac{\dot{m}'}{m} - \frac{f'}{f} \right) D = \frac{2m'}{r}, \quad D = 1 - \frac{2m(r,t)}{r}. \quad (43)$$

Now, adopting that \( f = f(m) \) [1], one can write \( \dot{f}(m) = f_{mm} \dot{m} \) and \( f'(m) = f_mm' \), so we have

$$\dot{f} = \frac{f'}{f} \dot{m} \quad (44)$$

and multiplying (39) by \( \dot{m}/m' \), we get

$$\dot{f} - \frac{m'}{m} \dot{m} + \frac{2m\dot{r}}{D} = 0 \quad (45)$$

But \( 2\dot{m}/r = -\dot{D} \), and so from (45) the general form of the first integral will be

$$\left( \ln \frac{f}{m'D} \right)' = 0 \quad \Rightarrow \quad f = \frac{m'D}{C(r)} \quad (46)$$

up to some function \( C(r) \) which now allows freedom in the definition of general function \( f \).

Some arguments for the choice \( C(r) = 1 \) are given in [11]. We consider this issue in more detail below.

B Integrability conditions

Solving the system of (43) and (44) with respect to \( \dot{f} \) and \( f' \) we get

$$\dot{f} = pf, \quad f' = qf \quad (47)$$

where

$$q = \frac{m'}{m} - \frac{2m'}{rD}, \quad p = q \frac{\dot{m}}{m}. \quad (48)$$

The first equation in (47) is satisfied identically with the function (46). Substituting now (46) into the second equation in (47), we obtain

$$C' - uC = 0 \quad \Rightarrow \quad C(r) = C_0 \exp \left( \int_{r_0}^{r} u \, dr \right), \quad (49)$$

where

$$u = \frac{m''}{m'} - \frac{\dot{m}'}{m} + v = \left( \ln \frac{m'D}{m} \right)' + \frac{2m'}{rD}, \quad (50)$$

$$v = (\ln D)' + \frac{2m'}{rD} \quad (51)$$

How one proves that solution in (46) with constraint (50) is indeed independent of \( t \)? The most general constraints on the function \( C(r) \) arise from the requirement of complete
integrability of the system (47). For that, differentiating the first equation in (47) with respect to $r$, and the second one with respect to $t$, we obtain

$$p' = \dot{q}. \quad (52)$$

Taking into account (48) and (51) after some calculations this will be

$$\dot{m}'' \left( \frac{m'}{m^2} - \frac{m''}{m^2} \right) + \frac{2m'}{rD} \left( \frac{1}{r} + v \right) = 0. \quad (53)$$

But from (51) we have the identity

$$2 \dot{m} \frac{m'}{rD} \left( \frac{1}{r} + v \right) = \dot{v}, \quad (54)$$

and so, in accord with (50), the relations (52)-(53) eventually reduces to

$$\dot{u} = 0, \quad (55)$$

which proves that solution (49) indeed cannot depend on $t$. The simplest choice $C(r) = C_0 = 1$ according to (49) requires $u = 0$.

In explicit form this condition is

$$u = \left( \ln \frac{m'D}{m} \right)' + \frac{2m'}{rD} = \left( \frac{m''}{m} - \frac{m'}{m} \right) + \frac{D'}{D} + \frac{2m'}{rD} = 0, \quad (56)$$

and reduces to the well-known relation (11) which we already used above - see (20) and (40). For the case of Vaidya’s metric imbedded into the de Sitter background (2) nothing changes except the replacement of the variable factor $D = 1 - \frac{2m}{r}$ with the generalized one $\tilde{D} = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}$ (where $\Lambda$ is the cosmological constant and $m = m(r, t)$), exactly as in equations (16) and (38).

C Vaidya’s ansatz and isotropic coordinates

While the transformation from curvature coordinates to isotropic ones does not exist for Vaidya’s spacetime (see Sec. 5), one might attempt to apply Vaidya’s algorithm directly in isotropic coordinates

$$ds^2 = e^{\nu(R,t)} dt^2 - e^{\mu(R,t)} \left( dR^2 - R^2 dΩ^2 \right). \quad (57)$$

Because null-dust symmetries (2), (3), (7) preserve their form, we have from (57) the field equations corresponding to (19), (20), (23):

$$\frac{e^{\nu/2}}{R} \left[ \mu' \left( 2\mu' + 8 \right) + 4 \mu'' \right] + 2 e^{-\nu/2} \left( 2\nu' - 2\mu' \right) - 3 \mu^2 e^{-\nu/2} = 0, \quad (58)$$

$$\frac{e^{-\mu}}{R} \left[ \mu' \left( 2\mu' + 8 \right) + 2 \mu'' + 2 \nu' \right] + e^{-\nu} \left( \nu' \left( 2\nu' + 2\mu' \right) + e^{-\nu} \left( 2\mu' - 3 \mu^2 - 4 \mu \right) \right] = 0, \quad (59)$$

where now prime denotes derivative with respect to $R$. Following the logic of Vaidya, we start with the Schwarzschild metric in isotropic coordinates as asymptotic requirement,

$$ds^2 = \left( \frac{1 - m/2R}{1 + m/2R} \right)^2 dt^2 - \left( 1 + \frac{m}{2R} \right)^4 \left( dR^2 + R^2 dΩ^2 \right). \quad (61)$$
and adopt the ansatz a-la Vaidya

\[ e^\mu(R, t) = \left(1 + \frac{m(R, t)}{2R}\right)^4 \equiv B(R, t)^4, \]  

so that the constant mass is replaced with mass-function \( m = m(R, t) \).

However, unlike the curvature coordinates, the sought-after isotropic component \( g_{00} \) cannot now be deduced algebraically from the equation (58). Instead, substituting (62) into (58), we obtain the following differential equation with respect to \( g_{00} = e^\nu \):

\[ y' + 3y RB - \frac{9m^2 R}{2R} B^2 = 0, \]  

where \( y = e^{3\nu/2} \). The solution of this equation (and of the entire system (58) - (60)) cannot be obtained if the dependence \( m = m(R, t) \) is unknown. So, in isotropic coordinates Vaidya’s algorithm does not work.

D Kottler’s metric in isotropic coordinates

To our knowledge, the Schwarzschild-de Sitter (Kottler) metric in isotropic coordinates is absent in the literature except for [5]. For this reason, we show its deduction in some detail in this appendix.

By applying the algorithm (22) of transition from curvature coordinates (4), where now

\[ e^\lambda = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1}, \quad e^\nu = 1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2, \]  

to isotropic coordinates (21) with the simplest assumption \( m = \text{const} \), one obtains the elliptic integral of the first kind

\[ \int \frac{dR}{R} = \int \frac{dr}{\sqrt{(-\Lambda/3)r^4 + r^2 - 2mr}}. \]  

This expression by means of substitution

\[ r = \frac{6m}{1 - 12y} = -\frac{m}{2 \left(y - \frac{1}{12}\right)} \]  

might be reduced to the canonical Weierstrass form [13]

\[ \int x_0 \, dx = \int_{\ln R_0}^{\ln R} d\ln R = \int_{\Phi_0}^{\Phi} \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}}, \]  

the inverse of which is the Weierstrass elliptic function

\[ y = \wp(x, g_2, g_3), \quad x = \ln R/\ln R_0, \quad g_2 = 1/12, \quad g_3 = \Lambda m^2/12 - 1/6^3. \]  

Thus, in accordance with (60) and (68), the transformation of the Kottler (Schwarzschild-de Sitter) solution to isotropic coordinates becomes

\[ r(R) = -\frac{m}{2(\Phi - 1/12)}. \]  

Finally, we get for the metric coefficients:

\[ e^\nu = \frac{r^2(R)}{R^2} = \frac{m^2}{4R^2(\Phi - 1/12)^2}. \]
\( e^\nu = 1 - \frac{2m}{r(R)} - \frac{A}{3} \langle R \rangle = 1 + 4 \left( \frac{\mathcal{Q}}{\mathcal{Q} - 1/12} \right) - \frac{Am^2}{12} \left( \frac{\mathcal{Q}}{\mathcal{Q} - 1/12} \right)^{-2} = \left( \frac{\mathcal{Q}}{\mathcal{Q} - 1/12} \right)^2, \quad (71) \)

where we have used the identity for the derivative of the Weierstrass function:

\[
\mathcal{Q}'(x) = 4\mathcal{Q}^3(x) - g_2\mathcal{Q}(x) - g_3. \quad (72)
\]

So, the spacetime interval in isotropic coordinates is:

\[
ds^2 = \frac{\mathcal{Q}'^2}{(\mathcal{Q} - 1/12)^2} dt^2 - \frac{m^2}{4R^2(\mathcal{Q} - 1/12)^2} (dR^2 + R^2 d\Omega^2), \quad (73)
\]

\[
\mathcal{Q} = \mathcal{Q}(\ln R/R_0, g_2, g_3).
\]

The dependence on cosmological constant is now contained in the Weierstrass invariant \(g_3 = \Lambda m^2/12 - 1/6\), see (68). In the limit \(R \to R_0\), using the expansion of the Weierstrass function \(\mathcal{Q}(x) = 1/x^2 + \ldots\) near the pole \(x = 0\) [13],

\[
\mathcal{Q}(\ln R/R_0) \approx -2\ln R/R_0 \approx \left( \frac{R}{R_0} - 1 \right)^{-3}, \quad (74)
\]

and having in the same limit \((\mathcal{Q} - 1/12)^2 \approx (R/R_0 - 1)^{-4}\), this metric can be reduced to the form not containing the cosmological constant (in approximation \(R \to R_0\)):

\[
ds^2 = \frac{4}{(R/R_0 - 1)^2} dt^2 - \frac{m^2}{4R^2} \left( \frac{R}{R_0} - 1 \right)^4 (dR^2 + R^2 d\Omega^2). \quad (75)
\]

The isotropic Kottler-type metric (73) can also be obtained independently by solving the corresponding Einstein equations directly in isotropic coordinates [5].

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References

1. P.C. Vaidya, Proceedings of the Indian Academy of Sciences - Section A 33(5), 264 (1951). DOI 10.1007/BF03173260
2. R.L. Mallett, Physical Review D 31(2), 416 (1985). DOI 10.1103/PhysRevD.31.416
3. F. Fayos, X. Jaen, E. Llanta, J.M.M. Senovilla, Classical and Quantum Gravity 8(11), 2057 (1991). DOI 10.1088/0264-9381/8/11/015
4. R.C. Tolman, Relativity, Thermodynamics and Cosmology (Oxford Clarendon Press, London, 1949)
5. T. Kozhanov, E. Mychelkin, in Proceedings of the Astrophysical Institute, vol. 45 (Nauka, Alma-Ata, 1986), vol. 45, p. 85
6. H. Culetu, arXiv:1612.06009 [gr-qc] (2016)
7. P. Cahan, J. Rembielinski, K.a. Smoliński, Z. Walczak, Foundations of Physics Letters 19, 619 (2006). DOI 10.1007/s10702-006-1015-4
8. R. Ehrlich, Astroparticle Physics 66, 11 (2015). DOI 10.1016/j.astropartphys.2014.12.011
9. E.G. Mychelkin, M.A. Makukov, International Journal of Modern Physics D 24, 1544025 (2015). DOI 10.1142/S0218271815440253
10. M.A. Makukov, E.G. Mychelkin, V.L. Saveliev, International Journal of Modern Physics: Conference Series 41, 1660133 (2016). DOI 10.1142/S2010194516601332
11. Y.K. Gupta, S. Gupta, General Relativity and Gravitation 20(12), 1293 (1988). DOI 10.1007/BF00766054
12. V. Smirnov, Course of higher mathematics, vol. 4 (GITTL, Moscow, 1953)
13. A. Zhuravsky, Handbook on elliptic functions (Phys.-Mat. Lit., Moscow, 1941)