DIRAC OPERATORS WITH EXPONENTIALLY DECAYING ENTROPY

PAVEL GUBKIN

ABSTRACT. We prove that the Weyl function of the one-dimensional Dirac operator on the half-line \( \mathbb{R}_+ \) with exponentially decaying entropy extends meromorphically into the horizontal strip \( \{ 0 \geq \text{Im} \ z > -\delta \} \) for some \( \delta > 0 \) depending on the rate of decay. If the entropy decreases very rapidly then the corresponding Weyl function turns out to be meromorphic in the whole complex plane. In this situation we show that poles of the Weyl function (scattering resonances) uniquely determine the operator.

1. Introduction

The main object of the present paper is the one-dimensional Dirac operator \( D_Q \) on the positive half-line \( \mathbb{R}_+ = [0, \infty) \),

\[
D_Q = J \frac{d}{dx} + Q,
\]

where \( Q = \begin{pmatrix} -q & p \\ p & q \end{pmatrix} \) is a symmetric \( 2 \times 2 \) zero-trace real potential and \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) is the square root of the minus identity matrix. We will use the entropy approach developed by R. Bessonov and S. Denisov in [1], [2], [3] to prove a version of the theorem by P. Nevai and V. Totik from [21] for Dirac operator (1). We start with the formulation of the original Nevai-Totik theorem to explain the motivation of the work and then introduce all necessary objects from the spectral theory of Dirac operators and state the results of the present paper.

1.1. Orthogonal polynomials on the unit circle. Nevai-Totik theorem. The basics of the theory of orthogonal polynomials on the unit circle can be found in the books [31] by G. Szegő and [27] by B. Simon. In this section we give some definitions and formulate the result by P. Nevai and V. Totik from [21], where they proved that the rate of exponential decay of the recurrence coefficients is the same as the radius of analyticity of the inverse Szegő function.

Let \( \mu = w \, dm + \mu_s \) be a probability measure on the unit circle \( \mathbb{T} = \{ \zeta : |\zeta| = 1 \} \); here \( m \) is a normalized Lebesgue measure on the unit circle, \( w \) and \( \mu_s \) are the density and the singular part of \( \mu \) with respect to \( m \). Assume that \( \mu \) is nontrivial, i.e., that the support of \( \mu \) is not a finite set. Then the functions \( 1, z, z^2 \ldots \) are linearly independent in \( L^2(\mu) \) and therefore there exists a sequence \( \{ \Phi_n \}_{n \geq 0} \) of monic polynomials orthogonal in \( L^2(\mu) \). They satisfy

\[
\Phi_{n+1}(z) = z\Phi_n(z) - \alpha_n^* \Phi_n^*(z), \quad z \in \mathbb{C},
\]

where \( \Phi_n^*(z) = z^n \overline{\Phi_n(1/\overline{z})} \) and \( \alpha_n \in \mathbb{D} = \{ \omega : |\omega| < 1 \} \) - the recurrence coefficients. The Szegő theorem states that \( \mu \) belongs to the Szegő class, i.e., \( \log w \in L^1(\mathbb{T}) \) if and only if \( \sum_{n \geq 0} |\alpha_n|^2 < \infty \).

2020 Mathematics Subject Classification. 34L40.
Key words and phrases. Krein system, Dirac operator, Entropy function.

The work is supported by Ministry of Science and Higher Education of the Russian Federation, agreement 075–15–2022–287.
Moreover, in this case for every $z \in \mathbb{D}$ the sequence $\Phi_n^*(z)/\|\Phi_n^*\|_{L^2(\mu)}$ converges to $\Pi(z)$, where $\Pi$ is an outer function in the unit disc satisfying $|\Pi(\zeta)|^{-2} = w(\zeta)$ for almost every $\zeta \in \mathbb{T}$, see Theorem 2.3.5 in [27]. If $\mu$ is an a. c. measure from the Szegő class on the unit circle let
\[
r_\Pi = \inf \left\{ r : \Pi \text{ is analytic in } \{ |w| < r^{-1} \} \right\}
\]
be the inverse radius of analyticity of $\Pi$. Otherwise put $r_\Pi = 1$. Also let
\[
r_\alpha = \limsup_{n \to \infty} |\alpha_n|^{1/n}
\]
be the rate of exponential decay of the recurrence coefficients.

**Theorem A** (Nevai-Totik theorem; Theorem 1, [21]; Chapter 7, [27]). For any nontrivial probability measure $\mu$ on the unit circle we have $r_\alpha = r_\Pi$.

Methods of the theory of orthogonal polynomials on the unit circle can be applied to the spectral theory of Dirac operators by means of Krein systems, see Section 2.1. In the original proof of Theorem A the relation $\alpha_n = -\Phi_{n+1}(0)$ plays a significant role. It does not have an analogue for Krein systems. Let us give a proof of the inequality $r_\alpha \leq r_\Pi$ in a way that will later work in the setting of spectral theory. The idea of the proof is similar to the idea in Chapter 12.3 of the Szegő’s book [31] where the asymptotic behaviour of orthogonal polynomials is studied via the Christoffel minimizing functions
\[
\lambda_n(z) = \lambda_n(\mu, z) = \min \left\{ \|P\|_{L^2(\mu)}^2 : \deg P \leq n, \ P(z) = 1 \right\}, \quad z \in \mathbb{C}.
\]
If $r_\Pi = 1$ then the claim $r_\alpha \leq r_\Pi$ follows immediately. Assume that $r_\Pi < 1$, then $\mu$ belongs to the Szegő class, $\Pi$ is well-defined and $d\mu(\zeta) = |\Pi(\zeta)|^{-2} d\mu$. We have, see Chapter 2.2 in [27],
\[
\lambda_n(0) = \prod_{n=0}^{n-1} (1 - |\alpha_n|^2), \quad \lambda_\infty(0) = \inf_n \lambda_n(0) = |\Pi(0)|^{-2}.
\]
Fix an arbitrary number $r_0 \in [1, r_\Pi^{-1})$. The Taylor series of $\Pi$ converges absolutely in $\{|z| \leq r_0\}$ and $\Pi$ does not have zeroes on $\mathbb{T}$, because the converse contradicts the assumption $\mu(\mathbb{T}) < \infty$. Hence for $\zeta \in \mathbb{T}$ we the uniform bound
\[
|\Pi(\zeta) - h_n(\zeta)| = O\left( r_0^{-n} \right), \quad n \to \infty
\]
holds, where $h_n$ is the $n$-th Taylor polynomial of $\Pi$. Substituting $h_n/h_n(0)$ into (2), we get
\[
\lambda_n(0) \leq \left\| \frac{h_n}{h_n(0)} \right\|_{L^2(\mu)}^2 = \frac{1}{|h_n(0)|^2} \int_\mathbb{T} |h_n(\zeta)|^2 d\mu(\zeta)
\]
\[
= \frac{1}{|\Pi(0)|^2} \int_\mathbb{T} |\Pi^{-1}(\zeta)|^2 dm(\zeta) = \frac{1}{|\Pi(0)|^2} \int_\mathbb{T} \left| 1 + \frac{h_n(\zeta) - \Pi(\zeta)}{\Pi(\zeta)} \right|^2 dm(\zeta)
\]
\[
= \frac{1}{|\Pi(0)|^2} \int_\mathbb{T} \left| 1 + 2 \text{Re} \left( \frac{h_n(\zeta) - \Pi(\zeta)}{\Pi(\zeta)} \right) \right|^2 dm(\zeta).
\]
Since $\frac{h_n - \Pi}{\Pi}$ is analytic in $\mathbb{D}$, the second term vanishes after the integration. Therefore we have
\[
\lambda_n(0) \leq \frac{1}{|\Pi(0)|^2} \int_\mathbb{T} \left| \frac{h_n(\zeta) - \Pi(\zeta)}{\Pi(\zeta)} \right|^2 dm(\zeta) = \frac{1}{|\Pi(0)|^2} + O\left( r_0^{-n} \right), \quad n \to \infty.
\]
We see that $\lambda_n(0) = \prod_{n=0}^{n-1}(1 - |\alpha_n|^2)$ is exponentially close to $\lambda_\infty(0) = \prod_{n=0}^{\infty}(1 - |\alpha_n|^2)$. Hence $|\alpha_n| = O \left(r_n^{-n}\right)$ as $n \to \infty$. Consequently $r_\alpha \leq r_0^{-1}$ and the inequality $r_\alpha \leq r_\Pi^{1}$ follows because $r_0$ can be taken arbitrary close to $r_\Pi^{1}$.

The sequence $\lambda_n(0)$ is connected to the entropy function $K(n) = -\log \prod_{k>n}(1 - |\alpha_n|^2)$ introduced by R. Bessonov and S. Denisov in [3]. They also transferred $K(n)$ to the theory of Krein systems; this object will be the key tool of the present paper.

D. Damanik and B. Simon in the paper [6] proved a version of the Nevai-Totik theorem for orthogonal polynomials on the real line. They showed the equality between the degree of exponential decay of Jacobi parameters $a_n, b_n$, i.e., $\lim \sup |a_n - 1 + b_n|^{1/n}$, and the inverse radius of analyticity of the corresponding Jost solution $u$, for details see [6] or Section 13.7 in [28].

1.2. Main results. Recall that we are studying one-dimensional Dirac operator (1). Throughout this paper we assume that the entries $p, q$ of $Q$ are real-valued and $p, q \in L^1_{\text{loc}}(\mathbb{R}^+)$ (sometimes we will simply write $Q \in L^1_{\text{loc}}(\mathbb{R}^+)$). The latter means that $p, q \in L^1([0, r])$ for every $r > 0$. Let us introduce the key objects of the spectral theory of Dirac operators; for a reference we use the book [19] by B. Levitan and I. Sargsjan. Consider the corresponding boundary value problem:

\begin{equation}
JN'(t, \lambda) + Q(t)N(t, \lambda) = \lambda N(t, \lambda), \quad N(0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad t \geq 0.
\end{equation}

Let $\theta_\pm$ and $\varphi_\pm$ be the entries of $N$, $N(t, \lambda) = \begin{pmatrix} \theta_+(t, \lambda) & \varphi_+(t, \lambda) \\ \theta_-(t, \lambda) & \varphi_-(t, \lambda) \end{pmatrix}$. For any potential $Q$ there exists a unique Borel measure $\sigma_Q$ on the real line such that $(1 + x^2)^{-1} \in L^1(\mathbb{R}, d\sigma_Q)$ and the mapping

\begin{equation}
(f_1, f_2) \mapsto \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^+} f_1(x)\theta_+(x, \lambda) + f_2(x)\theta_-(x, \lambda) \, dx, \quad f_1, f_2 \in L^2(\mathbb{R}^+)
\end{equation}

is an isometric operator between $L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}, \sigma_Q)$. The measure $\sigma_Q$ is called the spectral measure of $D_Q$. The Weyl function of $D_Q$ is an analytic function in the upper half-plane $\mathbb{C}_+ = \{z: \text{Im} z > 0\}$ defined by the relation

\begin{equation}
m(z) = - \lim_{t \to \infty} \frac{\varphi_+(t, z)}{\theta_+(t, z)}, \quad z \in \mathbb{C}_+.
\end{equation}

The spectral measure and the Weyl function are connected by the relation

\begin{equation}
m(z) - m(z_0) = \frac{1}{\pi} \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{1}{\lambda - z_0} \right) \, d\sigma_Q(\lambda), \quad z, z_0 \in \mathbb{C}_+.
\end{equation}

A Borel measure $\sigma = w \, dx + \sigma_s$ on the real line $\mathbb{R}$ belongs to the Szegő class if

\begin{equation}
\int_{\mathbb{R}} \frac{d\sigma(x)}{1 + x^2} < \infty, \quad \int_{\mathbb{R}} \frac{\log w(x)}{1 + x^2} \, dx > -\infty.
\end{equation}

Because of the inequality $\log w < w$, the second integral can’t diverge to $+\infty$ and therefore for any measure in the Szegő class we have $\frac{\log w}{1 + x^2} \in L^1(\mathbb{R})$. Furthermore, in this case there exists an outer function $\Pi$ in $\mathbb{C}_+$ such that $\Pi(i) > 0$ and $|\Pi(x)|^{-2} = w(x)$ for Lebesgue almost every point $x$ on the real line (see Section 4 in [11]). We will call $\Pi$ the inverse Szegő function of $\sigma$. 
Denote the solution of (5) for $\lambda = 0$ by $N_Q(t) = N_Q(t, 0)$. Let also

\begin{equation}
H_Q(t) = N_Q^*(t)N_Q(t), \quad E_Q(r) = \det \int_r^{r+2} H_Q(t) \, dt - 4.
\end{equation}

As we will see in Section 2.2.2, the matrix $H_Q$ is the Hamiltonian of the canonical system corresponding to $D_Q$ and $E_Q$ is the leading term of the entropy function $K_H$ of this Hamiltonian. If $Q \in L^2(\mathbb{R}^+)$ then $E_Q$ can be bounded via the Sobolev norm of the function $p + iq$, for details see Theorem 4 below. For $Q \in L^1_{\text{loc}}(\mathbb{R}^+)$ we prove that the exponential decay of $E_Q$ implies the existence of analytic continuation of $m$ into a strip in the lower half-plane; we denote the half-plane \( \{ z : \text{Im} \, z > -\Delta \} \) by $\Omega_\Delta$.

**Theorem 1.** Let $p, q \in L^1_{\text{loc}}(\mathbb{R}^+)$ be real-valued functions and $Q = (\begin{smallmatrix} -q & \ell \\ p & -\ell \end{smallmatrix})$. Assume that there exists $\delta > 0$ such that $E_Q(r) = O \left( e^{-\delta r} \right)$ as $r \to \infty$. Then

1. the spectral measure of $D_Q$ is absolutely continuous and belongs to the Szegő class \( (9) \);
2. the inverse Szegő function of $D_Q$ continues analytically into $\Omega_{\delta/4}$;
3. the Weyl function of $D_Q$ continues meromorphically into $\Omega_{\delta/4}$.

In particular, if $E_Q(r) = O \left( e^{-\delta r} \right)$ for every $\delta > 0$, then the Szegő function and the Weyl function of $D_Q$ extend into the whole complex plane $\mathbb{C}$.

We prove Theorem 1 and give details on the continuations of $\Pi$ and $m$ in Section 3. The constant 1/4 in the theorem is sharp for some potentials, see Section 5.2.

The simplest class of potentials with exponentially decaying entropy is a class of compactly supported potentials. Furthermore, the entropy function of exponentially decreasing or super-exponentially decreasing potentials, i.e., potentials that satisfy

\[ |p(r)| + |q(r)| = O \left( e^{-ar} \right), \quad r \to \infty, \]

for some $a > 0$ or for all $a > 0$ is also exponentially decaying, see Theorem 5 below. Theorem 1 is widely known in these situations, see [14], [29], [26], [10]. When $m_Q$ extends meromorphically below the real line one can speak about the scattering resonances. Namely, $z$ is a resonance of $D_Q$ of multiplicity $n$ if $m$ has a pole of order $n$ in $z$, see the book [8] by S. Dyatlov and M. Zworski for the general background on resonances and the papers [12], [13], [16] by E. Korotyaev for the case of Dirac operators. If $E_Q$ decays super-exponentially then, by Theorem 1, $m_Q$ is meromorphic in the whole complex plane and the resonances are also well-defined. Let us introduce the class

\begin{equation}
\mathcal{E} = \bigcup_{\alpha > 1} \mathcal{E}_\alpha, \quad \mathcal{E}_\alpha = \{ Q \in L^1_{\text{loc}}(\mathbb{R}^+) : E_Q = O \left( \exp(-r^\alpha) \right), \quad r \to \infty \}.
\end{equation}

In Section 5.3 we show that the resonances are exactly the zeroes of the corresponding Szegő function $\Pi$ if $E_Q$ decreases super-exponentially and prove the following result.

**Theorem 2.** Resonances of the Dirac operator uniquely determine its potential in the class $\mathcal{E}$.

Description of the sets which can be resonances sets is an open problem. We plan to continue working in this direction.

1.3. **Square summable potentials.** Denote by $\Delta_E$ the rate of exponential decay of the entropy function, i.e., let

\[ \Delta_E = \sup \{ \delta : E_Q(r) = O \left( e^{-\delta r} \right), \quad r \to \infty \}. \]
Furthermore, recall that in our notation $\Omega_\delta = \{ z : \text{Im } z > -\delta \}$ and define

$$\Delta_\Pi = \sup \left\{ \delta : \Pi \text{ extends analytically into } \Omega_\delta \text{ such that } \frac{\Pi(x - i\delta)}{x + i} \in H^2(\mathbb{C}+) \right\}$$

if $\sigma$ is an absolutely continuous measure from the Szegő class (9) and let $\Delta_\Pi = 0$ otherwise. The numbers $\Delta_E$ and $\Delta_\Pi$ play the roles of $\Gamma$ and $\Delta_0$ from Nevanlinna theorem 3. Recall that $\Gamma = \pi^{-1} R$; in the following theorem we state that $\Delta_E$ and $\Delta_\Pi$ satisfy the two-sided inequality. In this sense Theorem 3 can be regarded as a version of Theorem A for Dirac operators.

**Theorem 3.** Let $p, q \in L^2(\mathbb{R}_+)$ be real-valued functions. Then $4\Delta_\Pi \leq \Delta_E \leq 8\Delta_\Pi$.

If $Q \in L^2(\mathbb{R}_+)$, then $\Delta_E$ can be computed. Let $F$ be the isometric Fourier transform on the real line and define Sobolev space of tempered distributions as

$$W_2^{-1}(\mathbb{R}) = \left\{ f : \|f\|_{W_2^{-1}}^2 = \int_\mathbb{R} \frac{(\mathcal{F}f)^2(\eta)}{1 + \eta^2} d\eta < \infty \right\}.$$  

Theorem 4.1 in [4] states that the Sobolev norm of the function $p + iq$ is comparable to $K_H(r) = \sum_{n \geq 0} E_Q(r + n)$. More precisely, denote by $1_{[r,\infty)}$ the indicator function of the set $[r,\infty)$ and let $f_r = (p + iq)1_{[r,\infty)}$. Then the inequality $C^{-1}\|f_r\|_{W_2^{-1}}^2 \leq K_H(r) \leq C\|f_r\|_{W_2^{-1}}^2$ holds with some constant $C$ depending only on $\|p + iq\|_{L^2(\mathbb{R}_+)}$. It follows that

$$\Delta_E = \sup \left\{ \delta : \|f_r\|_{W_2^{-1}}^2 = O\left(e^{-\delta r}\right), \quad r \to \infty \right\} = 2\liminf_{r \to \infty} \left(-\frac{1}{r} \ln \|f_r\|_{W_2^{-1}}\right).$$

Therefore Theorem 3 can be rewritten in the following form.

**Theorem 4.** Let $p, q \in L^2(\mathbb{R}_+)$ be real-valued functions and let $f_r = (p + iq)1_{[r,\infty)}$. Then

$$2\Delta_\Pi \leq \liminf_{r \to \infty} \left(-\frac{1}{r} \ln \|f_r\|_{W_2^{-1}}\right) \leq 4\Delta_\Pi.$$

Proof of Theorem 3 can be found in Section 4. In Section 5.2 we show that the first inequality in Theorem 3 is sharp for some potentials just as the constant 1/4 in Theorem 1.

1.4. Examples. For a given $Q \in L^1_{\text{loc}}(\mathbb{R}_+)$ (especially, for $Q \notin L^2(\mathbb{R}_+)$) it can be hard to answer whether or not $E_Q$ decreases fast enough. The following theorem provides some examples of potentials with a rapidly decaying entropy.

**Theorem 5.** In the two following situations, the assertion $E_Q(r) = O\left(e^{-\delta r}\right)$ as $r \to \infty$ holds.

- $Q = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$, where $p$ is real-valued and
  $$\sup_{t \geq r} \int_r^t |p(s)| ds = O\left(e^{-\delta r/2}\right), \quad r \to \infty;$$

- $Q = \begin{pmatrix} -q & p \\ p & q \end{pmatrix}$, where $p$ and $q$ are real-valued, $\sup_{r \geq 0} \int_r^{r+1} (|p| + |q|) ds < \infty$ and
  $$\sup_{t \geq r} \int_r^t |p(s)| ds = O\left(e^{-\delta r}\right), \quad \sup_{t \geq r} \int_r^t |q(s)| ds = O\left(e^{-\delta r}\right), \quad r \to \infty.$$
The fact that the oscillation may compensate the growth of the potential and lead to the properties typical to the properties of decreasing potentials is not new, see [20], [30], [25] and Appendix 2 to XI.8 in [22]. If we consider the function \( p(x) = e^x \sin(e^{2x}) \) then
\[
\sup_{t \geq r} \left| \int_r^t p(x) \, dx \right| = \sup_{t \geq r} \left| \int_r^t e^x \sin(e^{2x}) \, dx \right| = \sup_{t \geq r} \left| \int_{e^{2t}} \frac{\sin(s)}{2\sqrt{s}} \, ds \right| = O(e^{-r}), \quad r \to \infty.
\]
Hence, by Theorems 1 and 5 the Weyl function of \( D_Q \) with \( Q = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix} \) is meromorphic in the half-plane \( \{z : \text{Im } z > -1/2\} \). A similar argument for \( p(x) = xe^{x^2} \sin(e^{2x^2}) \) gives an example of a “large” potential corresponding to a meromorphic Weyl function in the whole complex plane.

1.5. **Structure of the paper.** In Section 2 we introduce the main tools of the present paper – Krein systems, canonical Hamiltonian systems and the regularized Krein systems, see Sections 2.1, 2.2 and 2.3 respectively. Proof of Theorem 1 can be found in Section 3.1; in Section 3.2 we give more details on the extensions from Theorem 1. We prove Theorem 3 in Section 4, the large part of the proof is devoted to a rescaling argument, see Section 4.2. At the end of the paper, in Section 5, we prove Theorems 2 and 5 and discuss the sharpness of the constants in Theorems 1 and 3.

1.6. **Acknowledgements.** I am grateful to Roman Bessonov for numerous discussions and constant attention to this work.

## 2. Preliminaries

### 2.1. Krein systems

#### 2.1.1. General definitions

Let \( a \in L^1_{\text{loc}}(\mathbb{R}_+) \) be a complex-valued function on the positive half-line \( \mathbb{R}_+ \). The Krein system with the coefficient \( a \) is the following system of differential equations:

\[
\begin{align*}
\frac{\partial}{\partial r} P(r, \lambda) &= i\lambda P(r, \lambda) - a(r)P_*(r, \lambda), \quad P(0, \lambda) = 1, \\
\frac{\partial}{\partial r} P_*(r, \lambda) &= -a(r)P(r, \lambda), \quad P_*(0, \lambda) = 1.
\end{align*}
\]

(12)

After seminal work [17] of M. Krein, the solutions of Krein system (12) are called the continuous analogs of polynomials orthogonal on the unit circle. Using Krein systems, one can transfer methods from the theory of orthogonal polynomials on the unit circle to the spectral problems for self-adjoint differential operators. Detailed account of this approach can be found in the paper [7] by S. Denisov.

For any Krein system (12) there exists a unique Borel measure \( \sigma_a \) on the real line such that \((1 + x^2)^{-1} \in L^1(\mathbb{R}, \sigma_a)\) and the mapping

\[
f \mapsto \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(r)P(r, \lambda) \, dr,
\]

(13)

initially defined on simple measurable functions with compact support, can be continuously extended to an isometry from \( L^2(\mathbb{R}_+) \) to \( L^2(\mathbb{R}, \sigma_a) \). This measure is called the spectral measure of the Krein system. The following theorem is called the Krein theorem and can be regarded as an analog of the Szegő theorem in the setting of the Krein systems.

**Theorem B** (Krein theorem; Section 8 in [7]; [32]). Let \( \sigma_a \) be the spectral measure of Krein system (12) and let \( P, P_* \) be its solutions. Then the following assertions are equivalent:

(a) \( \sigma_a \) belongs to Szegő class (9) on the real line;
(b) for some point $\lambda_0$ in $\mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im } z > 0 \}$ we have
\[ \int_0^\infty |P(r, \lambda_0)|^2 \, dr < \infty; \]

(c) there exist a sequence $r_n \to \infty$ and a number $\gamma \in [0, 2\pi)$ such that for every $\lambda \in \mathbb{C}_+$ the limit
\[ \Pi(\lambda) = e^{-i\gamma} \lim_{n \to \infty} P_*(r_n, \lambda) \]

exists and defines an analytic in $\mathbb{C}_+$ function with $\Pi(i) > 0$.

If the equivalent assertions of the Krein theorem hold then (see Lemma 8.6 in [7]) $\Pi$ is the inverse Szegő function of $\sigma_a$, i.e., $\Pi$ is an outer function in $\mathbb{C}_+$ satisfying $|\Pi(x)|^{-2} = \sigma'_a(x)$ almost everywhere on $\mathbb{R}$, and
\[ \Pi(\lambda) = \exp \left[ -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \left( \frac{1}{s - \lambda} - \frac{s}{s^2 + 1} \right) \log \sigma'_a(s) \, ds \right], \quad \lambda \in \mathbb{C}_+. \]  

We will use the following enhancement of assertion (c) of the Krein theorem.

**Lemma 1** (Section 8, [7]). Assume that $\sigma_a$ belongs to the Szegő class on the real line and the sequence $r_n \to \infty$ is such that $P(r_n, \lambda_0) \to 0$ as $n \to \infty$ for some point $\lambda_0 \in \mathbb{C}_+$. Then there exists a constant $\gamma \in [0, 2\pi)$ and a subsequence $n_k$ such that $P_*(r_{n_k}, \lambda) \to e^{i\gamma} \Pi(\lambda)$ as $k \to \infty$ for every $\lambda \in \mathbb{C}_+$.

**Proof.** The claim immediately follows from Lemma 8.5 and the proof of Lemma 8.6 from [7]. \qed

### 2.1.2. Reproducing kernels and the minimization problem.

Let $PW_{[0, r]}$ denote the Paley-Wiener space of entire functions $f$ that can be represented in the form
\[ f(z) = \int_0^r \varphi(s) e^{izs} \, ds, \quad z \in \mathbb{C}, \quad \varphi \in L^2[0, r]. \]

The function
\[ k_r(z', z) = \frac{1}{2\pi} \int_0^r P(s, z) \overline{P(s, z')} \, ds \]

is the reproducing kernel in $PW_{[0, r]}$ at the point $z'$, see Lemma 8.1 in [7]. In other words, for every $f \in PW_{[0, r]}$ we have
\[ f(z') = \langle f, k_r(z', \cdot) \rangle_{L^2(\sigma_a)} = \int_{\mathbb{R}} f(x) k_r(z', x) \, d\sigma_a(x). \]

For $r > 0$, define
\[ m_r(z) = m_r(\sigma_a, z) = \inf \left\{ \frac{1}{2\pi} \| f \|^2_{L^2(\sigma_a)} : f \in PW_{[0, r]}, f(z) = 1 \right\}, \quad z \in \mathbb{C}. \]

The function $m_r$ is the analog of the Christoffel function (2). Lemma 8.2 in [7] says that
\[ m_r(z) = (2\pi k_r(z, z))^{-1} = \left( \int_0^r |P(s, z)|^2 \, ds \right)^{-1}, \quad z \in \mathbb{C}. \]
For every \(\lambda, \mu \in \mathbb{C}\), the functions \(P, P_*\) satisfy the following Christoffel-Darboux formula:

\[
\begin{align*}
P(r, \lambda)\overline{P(r, \mu)} - P_*(r, \lambda)\overline{P_*(r, \mu)} &= i(\lambda - \mu) \int_0^r P(s, \lambda)\overline{P(s, \mu)} \, ds, \\
|P_*(r, \lambda)|^2 - |P(r, \lambda)|^2 &= 2\operatorname{Im} \lambda \int_0^r |P(s, \lambda)|^2 \, ds,
\end{align*}
\]

see Lemma 3.6 in [7]. Furthermore, a simple calculation shows that

\[
P(r, z) = e^{izr}P_*(r, \overline{z}), \quad P_*(r, z) = e^{izr}\overline{P(r, \overline{z})}, \quad r \geq 0, \quad z \in \mathbb{C}.
\]

Together with Theorem B, relation (19) gives

\[
|\Pi(\lambda)|^2 = 2\operatorname{Im} \lambda \int_0^\infty |P(s, \lambda)|^2 \, ds, \quad \lambda \in \mathbb{C}_+.
\]

Define \(m_\infty(z) = \inf_r m_r(z)\). It follows that \(m_\infty\) can be represented by

\[
m_\infty(\lambda) = \left( \int_0^\infty |P(r, \lambda)|^2 \, dr \right)^{-1} = \frac{2\operatorname{Im} \lambda}{|\Pi(\lambda)|^2}, \quad \lambda \in \mathbb{C}_+.
\]

2.1.3. **Connections with the Dirac operator.** Consider a dual Krein system, i.e., Krein system (12) with the coefficient \(-a\), and denote its solutions by \(\hat{P}, \hat{P}_*\). It can be verified (see Section 4 in [7]) that \(\hat{P}\) and \(-\hat{P}_*\) solve the same differential system (12) as \(P\) and \(P_*\) but with the initial value \((\frac{1}{2}, \frac{1}{2})\). This can be rewritten in the form

\[
X'(r, \lambda) = \begin{pmatrix} i\lambda & -a(r) \\ -a(r) & 0 \end{pmatrix} X(r, \lambda), \quad X(0, \lambda) = (\frac{1}{2}, \frac{1}{2}),
\]

where

\[
X(r, \lambda) = \begin{pmatrix} P(r, \lambda) & \hat{P}(r, \lambda) \\ P_*(r, \lambda) & -\hat{P}_*(r, \lambda) \end{pmatrix}.
\]

Define

\[
p_a(r) = -2\operatorname{Re} a(2r), \quad q_a(r) = 2\operatorname{Im} a(2r), \quad Q_a = \begin{pmatrix} -q_a(r) & p_a(r) \\ p_a(r) & q_a(r) \end{pmatrix}.
\]

A calculation shows (see Chapter 13 in [7]) that

\[
Y(r, \lambda) = \frac{e^{-i\lambda r}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X(2r, \lambda)
\]

solves the differential system

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y'(r, \lambda) + Q_aY(r, \lambda) = \lambda Y(t, \lambda), \quad Y(0, \lambda) = (\frac{1}{2}, \frac{1}{2}).
\]

This differential system differs from Dirac system (5) only in the definition of the square root of the minus identity matrix. It can be easily seen that \(f = (f_1, f_2)^T\) solves \((\frac{1}{2}, \frac{1}{2})f + Q_a f = \lambda f\) if and only if \(f^# = (f_1, -f_2)\) solves \(Jf^# - Q_a f^# = \lambda f^#\). Therefore, the fundamental solution of Dirac system (5) with the potential \(-Q_a\) can be expressed in terms of \(P, P_*, \hat{P}, \hat{P}_*\):

\[
\begin{pmatrix} \theta_+(t, \lambda) & \varphi_+(t, \lambda) \\ \theta_-(t, \lambda) & \varphi_-(t, \lambda) \end{pmatrix} = \frac{e^{-i\lambda r}}{2} \begin{pmatrix} P(2r, \lambda) + P_*(2r, \lambda) & i\hat{P}(2r, \lambda) - i\hat{P}_*(2r, \lambda) \\ iP(2r, \lambda) - iP_*(2r, \lambda) & -P(2r, \lambda) - \hat{P}_*(2r, \lambda) \end{pmatrix}.
\]

Theorem 13.1 in [7] provides a relation between the spectral measures of \(D_{-Q_a}\) and the Krein system with the coefficient \(a\). Our normalization in definitions (6) and (13) of spectral measures
differs from the one in [7]. Because of that Theorem 13.1 actually implies the coincidence of the measures \( \sigma_{-Q_a} \) and \( \sigma_a \).

Using relation (24) the Dirac system can be rewritten in the Krein system and the other way around. We will say that \( D_{-Q_a} \) and the Krein system with the coefficient \( a \) correspond to each other.

By (7) and (25), for every Dirac system we have

\[
m(z) = -\lim_{r \to \infty} \frac{\varphi_+(r, z)}{\theta_+(r, z)} = -\lim_{r \to \infty} \frac{i\hat{P}(2r, z) - i\hat{P}_s(2r, z)}{P(2r, z) + P_s(2r, z)} = \lim_{r \to \infty} \frac{i\hat{P}_s(r, z) - i\hat{P}(r, z)}{P_s(r, z) + P(r, z)}, \quad z \in \mathbb{C}_+.
\]

In the Szegő case, both \( P(\cdot, z) \) and \( \hat{P}(\cdot, z) \) are in \( L^2(\mathbb{R}_+) \), consequently there exists a sequence \( r_n \to \infty \) such that \( P(r_n, z) \to 0 \) and \( \hat{P}(r_n, z) \to 0 \) as \( n \to \infty \). Hence

\[
m(z) = i \lim_{n \to \infty} \frac{\hat{P}(r_n, z)}{P_s(r_n, z)}, \quad z \in \mathbb{C}_+.
\]

By Lemma 1, we can choose a subsequence \( n_k \) such that both numerator and denominator converge. Therefore there exists a constant \( \gamma \in [0, 2\pi) \) such that

\[
m(z) = e^{i\gamma} \frac{\hat{\Pi}(z)}{\Pi(z)}, \quad z \in \mathbb{C}_+,
\]

where \( \Pi \) and \( \hat{\Pi} \) are the inverse Szegő functions of \( \sigma_{-Q_a} \) and \( \sigma_{Q_a} \) respectively. The latter equation will be crucial in the proof of part (3) of Theorem 1.

2.2. Canonical systems. The entropy approach is based on the reduction of Dirac and Krein systems to a more general form of differential equations – the canonical Hamiltonian system. Below we introduce key objects and definitions of the theory, for details see [23] and [24].

A Hamiltonian is a matrix-valued mapping of the form

\[
H = \left( \begin{array}{cc} h_1 & h \\ h & h_2 \end{array} \right),
\]

where \( h, h_1 \) and \( h_2 \) are real-valued functions from \( L^1_{\text{loc}}(\mathbb{R}_+) \). Moreover, it is assumed that \( H \) satisfies \( \text{trace } H(t) > 0 \) and \( \text{det } H(t) \geq 0 \) for every \( t \geq 0 \). A Hamiltonian \( H \) is called singular if \( \int_{\mathbb{R}_+} \text{trace } H(s) \, ds = +\infty \). We will call a Hamiltonian \( H \) trivial if it coincides with \( (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \) or \( (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \) and nontrivial otherwise. The canonical system with Hamiltonian \( H \) is the differential equation

\[
J \frac{\partial}{\partial t} M(t, z) = zH(t)M(t, z), \quad M(0, z) = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), \quad z \in \mathbb{C}, \quad t \geq 0.
\]

The solution \( M \) of (28) is often represented as

\[
M(t, z) = (\Theta(t, z), \Phi(t, z)) = \left( \begin{array}{c} \Theta_+(t, z) \\ \Phi_+(t, z) \end{array} \right), \quad \left( \begin{array}{c} \Theta_-(t, z) \\ \Phi_-(t, z) \end{array} \right).
\]

The Weyl function of the canonical system is defined by

\[
m(z) = \lim_{t \to \infty} \frac{w\Phi_+(t, z) + \Phi_-(t, z)}{w\Theta_+(t, z) + \Theta_-(t, z)}, \quad z \in \mathbb{C}_+, \quad w \in \mathbb{C} \cup \{\infty\}.
\]
If the Hamiltonian is singular, this limit is correctly defined and does not depend on \( w \). Furthermore, \( m \) has a strictly positive imaginary part in \( \mathbb{C}_+ \) and therefore admits the following Herglotz representation:

\[
m(z) = \frac{1}{\pi} \int_{\mathbb{R}} \left( \frac{1}{x - z} - \frac{x}{x^2 + 1} \right) d\sigma(x) + az + b,
\]

where \( a \in \mathbb{R} \) and \( b \geq 0 \) are constants and \( \sigma \) is a Borel measure satisfying \( (1 + x^2)^{-1} \in L^1(\mathbb{R}, \sigma) \). The measure \( \sigma \) is called the spectral measure of \( H \) and of canonical system \( (28) \).

### 2.2.1. Reduction of the Dirac system to the canonical system.

As we mentioned it earlier, \( (28) \) is a more general form of a differential system than \( (5) \) or \( (12) \). In this subsection we outline the reduction of Dirac system \( (5) \) to the Hamiltonian canonical system. We omit the calculations, for a more detailed explanation see [24] or Section 2.4 in [1].

Consider Dirac system \( (5) \) with the potential \( Q \) and define \( N_Q(t) = N(t, 0) \). Then \( M(t, z) = N_Q^{-1}(t)N(t, z) \) is the fundamental solution of canonical system \( (28) \) with the Hamiltonian

\[
H_Q(t) = N_Q^*(t)N_Q(t).
\]

Moreover, the spectral measure of the canonical system with Hamiltonian \( H_Q \) coincides with the spectral measure of the Dirac operator \( D_Q \) (see Section 2.4 in [1]).

### 2.2.2. Entropy of a canonical system.

In the papers [1], [2] R. Bessonov and S. Denisov described the class of Hamiltonians for which the corresponding spectral measure belongs to the Szegő class on the real line. They introduced the criterion in terms of the entropy function of the Hamiltonian. Let us define it.

Consider an arbitrary singular nontrivial Hamiltonian \( H \) and let \( H_r \) be its \( r \)-shift, i.e., let \( H_r \) be such that \( H_r(x) = H(r + x), r, x \geq 0 \). Denote by \( m_r, \sigma_r \) and \( w_r \) the Weyl function, the spectral measure and the density of the spectral measure corresponding tho the canonical system with the Hamiltonian \( H_r \). Next, define

\[
\mathcal{I}_H(r) = \text{Im} m_r(i), \quad \mathcal{R}_H(r) = \text{Re} m_r(i), \quad \mathcal{J}_H(r) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log(w_r(x))}{1 + x^2} dx.
\]

The entropy function \( \mathcal{K}_H \) of \( \sigma \) is defined as

\[
\mathcal{K}_H(r) = \log \mathcal{I}_H(r) - \mathcal{J}_H(r), \quad r \geq 0.
\]

If \( \sqrt{\det H} \notin L^1(\mathbb{R}_+) \) then we can define

\[
\mathbf{K}_H(r) = \sum_{n \geq 0} \left( \det \int_{\eta_n(r)}^{\eta_{n+2}(r)} H(t) dt - 4 \right), \quad \eta_n(r) = \min \left\{ x : \int_r^x \sqrt{\det H(t)} dt = n \right\}.
\]

The function \( \mathbf{K}_H \) is called the entropy function of the Hamiltonian \( H \). The main result of [2] is the following theorem.

**Theorem C** (Theorem 1.2, [2]). The spectral measure of a singular nontrivial Hamiltonian \( H \) belongs to Szegő class \( (9) \) if and only if \( \sqrt{\det H} \notin L^1(\mathbb{R}_+) \) and \( K_H(0) < \infty \). Moreover, we have

\[
c_1 \mathcal{K}_H(r) \leq \mathbf{K}_H(r) \leq c_2 \mathcal{K}_H(r) \cdot e^{c_2 \mathcal{K}_H(r)},
\]

where \( c_1 \) and \( c_2 \) are some absolute constants.
In the present paper we are interested in the case when \( K_H(r) \) decreases exponentially fast in \( r \). It is equivalent to the exponential decay of \( K_H(r) \) and, formally, can be written as

\[
\sum_{n \geq 0} \left( \det \int_{n(r)}^{n+2(r)} H(t) \, dt - 4 \right) = O \left( e^{-\delta r} \right), \quad r \to \infty
\]

for some \( \delta > 0 \). We have \( \eta_{n+1}(r) = \eta_n(\eta_1(r)) \) hence the latter in its turn is equivalent to the exponential decay of the first term of the sum, i.e.,

\[
\det \int_{n_0(r)}^{n_2(r)} H(t) \, dt - 4 = K_H(r) - K_H(\eta_1(r)) = O \left( e^{-\delta r} \right), \quad r \to \infty.
\]

Notice that if \( H = H_Q \) is the Hamiltonian constructed for the Dirac operator \( D_Q \) in Section 2.2.1, then \( \det H(t) = 1 \) for every \( t \geq 0 \) and consequently \( \eta_n(r) = n + r \) for every \( n, r \geq 0 \). In this situation assertion (33) becomes

\[
\det \int_r^{r+2} H_Q(t) \, dt - 4 = O \left( e^{-\delta r} \right), \quad r \to \infty,
\]

which is exactly the assertion \( E_Q(r) = O \left( e^{-\delta r} \right) \) from Theorem 1.

2.3. Regularized Krein systems. Fix a singular nontrivial Hamiltonian \( H \). Assume that the spectral measure \( \sigma \) corresponding to \( H \) belongs to the Szeg\'o class and let \( K_H, \mathcal{I}_H \) and \( \mathcal{R}_H \) be as in the previous section. In order to simplify the exposition we will omit the index \( H \) later on. The regularized Krein system corresponding to \( H \) is the following system of differential equations:

\[
\frac{\partial}{\partial r} \tilde{P}_r^*(z) = (z - i) f_1(r) \tilde{P}_r(z) + iz f_2(r) \tilde{P}_r^*(z), \quad \tilde{P}_0^*(z) = I_H(0)^{-1},
\]

\[
\frac{\partial}{\partial r} \tilde{P}_r(z) = iz \tilde{P}_r(z) + (z + i) f_1(r) \tilde{P}_r^*(z) - iz f_2(r) \tilde{P}_r(z), \quad \tilde{P}_0(z) = 0,
\]

where \( f_1(r) = -\frac{1}{4} e^{2i u(r)} \left( \mathcal{K}_r(\tau) + i \mathcal{L}_r(\tau) \right) \) with \( u(r) = \int_0^r \frac{\mathcal{K}(t)}{2t} \, dt \) and \( f_2(r) = \frac{\mathcal{K}(r/2)}{4} \). Notice that by Lemma 3 in [1], \( K_H, \mathcal{I}_H \) and \( \mathcal{R}_H \) are locally absolutely continuous and the differential system is well-defined.

First of all, let us show that the regularized Krein system defined in [1] coincides with the one defined by (34) and (35). Denote by \( \tilde{P}_r \) and \( \tilde{P}_r^* \) the regularized Krein system from [1]. We claim that \( \tilde{P}_r = \tilde{P}_r \) and \( \tilde{P}_r^* = \tilde{P}_r^* \). Indeed, the initial values of \( \tilde{P}_r \) and \( \tilde{P}_r^* \) is chosen so that it coincides with the initial values of \( P_r \) and \( P_r^* \) hence it is suffices to show that \( \tilde{P}_r \) and \( \tilde{P}_r^* \) satisfy differential equations (34) and (35). Equation (34) follow from Lemma 8 in [1] by the change of variables. To establish (35), notice that (37) in [1] yields \( \tilde{P}_r(z) = e^{irz} \bar{P}_r^*(\overline{z}) \) and

\[
\frac{\partial}{\partial r} \tilde{P}_r(z) = \frac{\partial}{\partial r} \left( e^{irz} \bar{P}_r^*(\overline{z}) \right) = iz \cdot e^{irz} \bar{P}_r^*(\overline{z}) + e^{irz} \cdot \frac{\partial}{\partial r} \bar{P}_r^*(\overline{z})
\]

\[
= iz \tilde{P}_r(z) + e^{irz} \left( (z - i) f_1(r) \tilde{P}_r(z) + iz f_2(r) \tilde{P}_r(z) \right)
\]

\[
= iz \tilde{P}_r(z) + (z + i) f_1(r) \tilde{P}_r^*(z) - iz f_2(r) \tilde{P}_r(z).
\]
2.4. Properties of the regularized Krein systems. In this subsection we discuss properties of the regularized Krein system. Lemmas 3 and 4 are respectively Lemmas 9 and 8 from [1]; in Lemma 4 we will need more accurate bounds than provided in [1] so we state it with a proof (which is almost identical to the one in [1]). The following lemma is a simple corollary of the differential equations for $\tilde{P}_r$ and $\tilde{P}_r$.

**Lemma 2.** The functions $\tilde{P}_r^*$ and $\tilde{P}_r$ satisfy the reflection formula
\[
\tilde{P}_r^*(z) = e^{irz}\tilde{P}_r(\overline{z}), \quad \tilde{P}_r(z) = e^{irz}\tilde{P}_r^*(\overline{z}).
\]
Moreover, the following differential equations hold for the absolute values of $\tilde{P}_r$ and $\tilde{P}_r$:
\[
\begin{align*}
\frac{\partial}{\partial r}|\tilde{P}_r^*(z)|^2 &= 2\text{Re} \left( (z - i)f_1(r)\tilde{P}_r(z)\overline{\tilde{P}_r^*(z)} \right) - 2\text{Im}zf_2(r)|\tilde{P}_r^*(z)|^2, \\
\frac{\partial}{\partial r}|\tilde{P}_r(z)|^2 &= -2\text{Im}z|\tilde{P}_r(z)|^2 + 2\text{Re} \left( (z + i)f_1(r)|\tilde{P}_r^*(z)\overline{\tilde{P}_r(z)}| + 2\text{Im}zf_2(r)|\tilde{P}_r(z)|^2. \\
\end{align*}
\]

**Proof.** Equations in (37) follow from (36). The rest of the proof is a straightforward calculation. Indeed, we have
\[
\begin{align*}
\frac{\partial}{\partial r}|\tilde{P}_r^*(z)|^2 &= 2\text{Re} \left( \tilde{P}_r^*(z)\frac{\partial}{\partial r}\tilde{P}_r(z) \right) = 2\text{Re} \left( \tilde{P}_r^*(z) \left( (z - i)f_1(r)\tilde{P}_r(z) + izf_2(r)\tilde{P}_r^*(z) \right) \right) \\
&= 2\text{Re} \left( (z - i)f_1(r)\tilde{P}_r(z)\overline{\tilde{P}_r^*(z)} \right) - 2\text{Im}zf_2(r)|\tilde{P}_r^*(z)|^2; \\
\frac{\partial}{\partial r}|\tilde{P}_r(z)|^2 &= 2\text{Re} \left( \tilde{P}_r(z)\frac{\partial}{\partial r}\tilde{P}_r(z) \right) \\
&= 2\text{Re} \left( \tilde{P}_r(z) \left( iz\tilde{P}_r(z) + (z + i)f_1(r)\tilde{P}_r^*(z) - izf_2(r)\tilde{P}_r(z) \right) \right) \\
&= -2\text{Im}z|\tilde{P}_r(z)|^2 + 2\text{Re} \left( (z + i)f_1(r)|\tilde{P}_r^*(z)\overline{\tilde{P}_r(z)}| + 2\text{Im}zf_2(r)|\tilde{P}_r(z)|^2. \\
\end{align*}
\]

Further in the paper we will use the symbols $\lesssim$ and $\gtrsim$ meaning that the corresponding inequality $\leq$ or $\geq$ holds with some multiplicative constant depending only on fixed parameters. We will use the symbol $\approx$ when both $\lesssim$ and $\gtrsim$ hold.

The spectral measure $\sigma$ belongs to Szegő class (9) hence the inverse Szegő function $\Pi$ is well defined by (14). The solutions of the regularized Krein system satisfy the following limit relations.

**Lemma 3** (Lemma 9, [1]). For $z \in \mathbb{C}_+$, we have
\[
\lim_{r \to \infty} \tilde{P}_r^*(z) = \Pi(z), \quad \lim_{r \to \infty} \tilde{P}_r(z) = 0, \quad \int_0^\infty |\tilde{P}_r(z)|^2 dv < \infty.
\]

**Lemma 4** (Lemma 8, [1]). The coefficient $f_1$ of the regularized Krein system satisfies
\[
|f_1(r)| \lesssim \sqrt{|\mathcal{K}'(r)| + |\mathcal{K}(r)|}
\]
uniformly for every $r \geq 0$.

**Proof.** Due to the definition of $f_1$, the claim of the lemma is equivalent to
\[
\left| \frac{T(r)}{I(r)} \right| + \left| \frac{\mathcal{R}'(r)}{I(r)} \right| \lesssim \sqrt{|\mathcal{K}'(r)| + |\mathcal{K}(r)|}.
\]
Formulas (39) and (40) in [1] give
\begin{align}
-\mathcal{K}' &= \left(1 - \frac{1}{2} \right) + \frac{1}{4} \left(\frac{\mathcal{R}'}{\mathcal{I}}\right)^2 \frac{1}{I} h_1, \\
\frac{\mathcal{I}'}{\mathcal{I}} &= \frac{1}{I} h_1 - \frac{1}{4} \left(\frac{\mathcal{R}'}{\mathcal{I}}\right)^2 \frac{1}{I} h_1,
\end{align}
where $h_1$ is the upper-left entry of $H$, see (27). The function $\mathcal{K}$ is non-increasing hence $-\mathcal{K} = |\mathcal{K}'|$ for every $t \geq 0$. Two terms in the right hand side of the first equality are nonnegative and therefore we have
\begin{align}
\frac{I}{I} h_1 + \frac{1}{I} h_1 - 2 \leq |\mathcal{K}'|,
\end{align}
\begin{align}
\frac{1}{4} \left(\frac{\mathcal{R}'}{\mathcal{I}}\right)^2 \frac{1}{I} h_1 \leq |\mathcal{K}'|, \quad \left(\frac{\mathcal{R}'}{\mathcal{I}}\right)^2 \leq 4 I h_1 |\mathcal{K}'|.
\end{align}
From the definition of $\mathcal{I}(r)$ we know $\mathcal{I}(r) = \text{Im} m_r(i) \geq 0$ because $m_r$ is a Herglotz function. Also overall assumptions on $H$ imply $h_1 \geq 0$. If $\mathcal{I}(r) h_1(r) \in [\frac{1}{2}, 2]$ then by (43) we have $(\mathcal{R}'/\mathcal{I})^2 \lesssim |\mathcal{K}'|$ and
\begin{align}
\left|\frac{I}{I} h_1 + \frac{1}{I} h_1 - 2 \leq \sqrt{\mathcal{I} h_1 + \frac{1}{I} h_1} \leq \sqrt{|\mathcal{K}'|};
\end{align}
else we have
\begin{align}
 IH_1 \lesssim \left|\frac{1}{I} h_1 - \mathcal{I}_h\right| \approx \mathcal{I}_h + \frac{1}{I} h_1 - 2 \leq |\mathcal{K}'|
\end{align}
and $(\mathcal{R}'/\mathcal{I})^2 \leq 4 I h_1 |\mathcal{K}'| \lesssim |\mathcal{K}'|^2$. The required inequalities for $\mathcal{R}'/\mathcal{I}$ immediately follow in both situations. To get the bound for $\mathcal{I}'(r)/\mathcal{I}(r)$ substitute (44), (45) and (43) into (41).

Consider Krein system (12) and construct a canonical system via the reductions from Sections 2.1.3 and 2.2.1 so that the spectral measure of the Krein system coincides with the spectral measure of the canonical system. The next lemma connects the Krein system and its regularized version.

**Lemma 5.** For $z_0 \in \mathbb{C}_+$ and $r \geq 0$ we have
\begin{align}
2 \text{Im } z_0 \int_r^\infty |P(x, z_0)|^2 \, dx = |\Pi(z_0)|^2 - \left(|\hat{P}_r(z_0)|^2 - |\hat{P}_r(z_0)|^2\right).
\end{align}

**Proof.** Let $\sigma$ be the spectral measure of the Krein system. Recall that the reproducing kernel in the space $PW_{[0, r]}$ with norm inherited from $L^2(d\sigma)$ at the point $z_0 \in \mathbb{C}_+$ is given by (15),
\[
k_r(z_0, z) = \frac{1}{2\pi} \int_0^r P(x, z) P(x, z_0) \, dx.
\]
On the other hand, the reproducing kernel admits the following representation in terms of the regularized Krein system (see formula (48) in [1]):
\[
k_r(z_0, z) = \frac{1}{2\pi i} \frac{\hat{P}_r(z_0) \hat{P}_r(z_0) - \hat{P}_r(z_0) \hat{P}_r(z_0)}{z - \bar{z}_0}.
\]
Therefore we have
\[
    k_r(z_0, z_0) = \frac{|\tilde{P}_r^*(z_0)|^2 - |\tilde{P}_r(z_0)|^2}{4\pi \text{Im } z_0} = \frac{1}{2\pi} \int_0^r |P(x, z_0)|^2 dx.
\]

The claim of the lemma now follows from (21). \(\square\)

3. Analytic Extension of the Szegő Function. Proof of Theorem 1

Consider canonical system (28) with spectral measure \(\sigma\) in the Szegő class. Let \(K\) be its entropy function (31). \(\Pi\) be the inverse Szegő function of \(\sigma\) and \(\tilde{P}_r, \tilde{P}_r^*\) be the solutions of the corresponding regularized Krein system. Before the proof of Theorem 1 let us show that the singular part of \(\sigma\) is absent under the weaker assumptions on \(K\).

Lemma 6. If \(\sqrt{|K|} \in L^1(\mathbb{R}_+)\) then \(\sigma\) is a. c. with respect to the Lebesgue measure on the real line and \(\Pi\) is continuous in \(\overline{C}_+\). Furthermore, uniformly for \(z \in \overline{C}_+\) we have
\[
    (47) \quad \left| \log |\tilde{P}_r^*(z)| \right| \lesssim |z| + 1,
\]
\[
    (48) \quad \left| \log |\Pi(z)| \right| \lesssim |z| + 1.
\]

Proof. Recall differential equation (34) and divide it by \(\tilde{P}_r^*(z)\):
\[
    \frac{\partial}{\partial r} \log \tilde{P}_r^*(z) = (z - i)f_1(r) \tilde{P}_r(z) + izf_2(r).
\]

The inequality \(|\tilde{P}_r(z)| \leq |\tilde{P}_r^*(z)|\) holds for \(z \in \overline{C}_+\), therefore
\[
    \left| \frac{\partial}{\partial r} \log \tilde{P}_r^*(z) \right| \leq |(z - i)f_1(r) + izf_2(r)| \lesssim (|z| + 1) \left( \sqrt{|K(r/2)|} + |K'(r/2)| \right),
\]
where the last inequality is by Lemma 4. Inequality (47) then follows by integration. We see that \(\frac{\partial}{\partial r} \log \tilde{P}_r^*(z) \in L_1(\mathbb{R}_+)\) and \(\left\| \frac{\partial}{\partial r} \log \tilde{P}_r^*(z) \right\|\) is uniformly bounded on compact subsets of \(\overline{C}_+\). This means that \(\tilde{P}_r^*(z)\) converge as \(r \to \infty\) uniformly on compact subsets of \(\overline{C}_+\). By Lemma 3, the limit coincides with \(\Pi\) in \(C_+\) therefore \(\Pi\) is continuous in \(\overline{C}_+\) and (48) follows from (47). Finally, \(|\tilde{P}_r^*(x)|^{-2} dx\) is a spectral measure of the Hamiltonian \(\tilde{H}\) from [1] hence Corollary 5.8 and Theorem 7.3 in [23] give \(|\tilde{P}_r^*(x)|^{-2} dx \overset{w}{\to} d\sigma\). Together with (47) this means that \(\sigma\) is absolutely continuous. The proof is concluded. \(\square\)

3.1. Proof of Theorem 1.

Proof. The Hamiltonian \(H = H_Q\) defined by (10) coincides with the Hamiltonian constructed in Section 2.2.1. At the end of Section 2.2.2 we showed that the assertion
\[
    E_Q(r) = \det \int_r^{r+2} H_Q(t) \, dt - 4 = O(e^{-\delta r}), \quad r \to \infty,
\]
of Theorem 1 is equivalent to the assertion
\[
    K_H(r) = \sum_{n \geq 0} \left( \det \int_{r+n}^{r+n+2} H_Q(t) \, dt - 4 \right) = O(e^{-\delta r}), \quad r \to \infty.
\]
By Theorem C, it follows that $\sigma$ is in the Szegő class and we have
\begin{equation}
\mathcal{K}_H(r) = O \left( e^{-\delta r} \right), \quad r \to \infty.
\end{equation}
In particular, $\sqrt{|K'|} \in L_1$. Hence part (1) of Theorem 1 immediately follows from Lemma 6.

Let us show that $\tilde{P}^*_r(z)$ converges as $r \to \infty$ uniformly on compact subsets of $\Omega_{\delta/4}$. By Lemma 3, the limit function will be the required continuation of $\Pi$. Let $z$ be with $\text{Im } z \leq 0$. Recall differential equation (34) and apply the bounds from Lemma 4 to it:
\begin{equation}
\left| \frac{\partial}{\partial r} \tilde{P}^*_r(z) \right| \lesssim (|z| + 1) \left( \sqrt{|K'|(r/2)|} + |K'(r/2)| \right) \left| \tilde{P}_r(z) \right| + |z||K'(r/2)| \left| \tilde{P}^*_r(z) \right|.
\end{equation}
By (47) and reflection formula (37), we get
\begin{equation}
\left| \tilde{P}^*_r(z) \right| \lesssim e^{|\text{Im } z|} e^{c(|z|+1)}, \quad \left| \tilde{P}_r(z) \right| \lesssim e^{|\text{Im } z|} e^{c(|z|+1)}, \quad \text{Im } z \leq 0,
\end{equation}
where $c$ does not depend on $z$. Substituting these bounds into the previous inequality, we obtain
\begin{equation}
\left| \frac{\partial}{\partial r} \tilde{P}^*_r(z) \right| \lesssim (|z| + 1) \left( \sqrt{|K'|(r/2)|} + |K'(r/2)| \right) e^{|\text{Im } z|} e^{c(|z|+1)}, \quad \text{Im } z \leq 0.
\end{equation}
Because of (49), the integral
\begin{equation}
\int_0^\infty \left( \sqrt{|K'(r/2)|} + |K'(r/2)| \right) e^{|\text{Im } z|} dr
\end{equation}
corresponds when $|\text{Im } z| < \delta/4$. Therefore $\frac{\partial}{\partial r} \tilde{P}^*_r(z) \in L^1$ for $z$ with $0 \geq \text{Im } z > -\delta/4$ and Part (2) of Theorem 1 follows.

Simple calculations show (see Lemma 3 in [1]) that for every $t \geq 0$ we have
\begin{equation}
H_{-Q}(t) = J^* H_Q(t) J, \quad K_{H_{-Q}}(t) = K_{H_Q}(t), \quad E_{-Q}(t) = E_Q(t).
\end{equation}
Hence $-Q$ also satisfies the assertions of the theorem and Part (2) can be applied for $-Q$ as well as for $Q$. It follows that both $\Pi_Q$ and $\Pi_{-Q}$ extend analytically into $\Omega_{\delta/4}$ and therefore the Weyl function of $D_Q$ can be meromorphically extended into the same domain via relation (26). This concludes the proof of the whole theorem. \qed

**Corollary 1.** Assume that the assertions on Theorem 1 hold. Then for every $\delta_1 < \delta/4$ there exists a constant $C$ such that $|\Pi(z)| \leq e^{C(|z|+1)}$ in $\Omega_{\delta_1}$.

**Proof.** The required inequality for $z \in \mathbb{C}_+$ and $z \in \Omega_{\delta_1} \setminus \mathbb{C}_+$ follows from (48) and (50) respectively. \qed

Let $P$ be the solution of the Krein system corresponding to $D_Q$. The next corollary is a quantitative version of Lemmas 3 and 5.

**Corollary 2.** Fix $z_0$ with $\text{Im } z_0 > \delta/4$. Under the assumptions of Theorem 1, for $r \geq 0$ we have
\begin{align*}
|\tilde{P}_r(z_0)| \lesssim e^{-\delta r/4}, \quad |\tilde{P}^*_r(z_0) - \Pi(z_0)| \lesssim e^{-\delta r/2}, \\
\int_r^\infty |P(x, z_0)|^2 dx \lesssim e^{-\delta r/2}.
\end{align*}
Proof. Integrating (50) for $z = z_0$, we get

$$|\tilde{P}_r^*(z_0)| \lesssim 1 + \int_0^r \left( \sqrt{|K'(\rho/2)| + |K'(\rho/2)|} \right) e^{\rho \text{Im} z_0} d\rho.$$  

The entropy decay $K(r) = O \left( e^{-\delta r} \right)$ implies $|\tilde{P}_r^*(z_0)| \lesssim e^{(\text{Im} z_0 - \delta/4)r}$ and the bound for $\tilde{P}_r(z_0)$ follows from (37). Recall differential equation (34). By Lemma 3, $\tilde{P}_r^*(z_0)$ converges as $r \to \infty$ and hence $\tilde{P}_r^*(z_0)$ is bounded in $r$. Thus, (34) and Lemma 4 give

$$\left| \frac{\partial}{\partial r} \tilde{P}_r^*(z_0) \right| \lesssim \left( \sqrt{|K'(r/2)| + |K'(r/2)|} \right) \left| \tilde{P}_r(z_0) \right| + |K'(r/2)|.$$  

If we integrate the latter inequality on $[r, +\infty)$ and apply Lemma 3 together with the obtained bound for $\tilde{P}_r(z_0)$, we will get the required bound for $\tilde{P}_r^*(z_0) - \Pi(z_0)$. To conclude the proof of the corollary notice that Lemma 5 yields

$$\int_r^\infty |P(x, z_0)|^2 dx \lesssim \left( |\Pi(z_0)|^2 - |\tilde{P}_r^*(z_0)|^2 \right) + |\tilde{P}_r(z_0)|^2 \lesssim e^{-\delta r/2}.$$  

\[\square\]

3.2. Analytic continuation via the Christoffel-Darboux formula.

**Theorem 6.** Let $p, q \in L^1_{\text{loc}}(\mathbb{R}_+)$ be real-valued functions and $Q = (\frac{p}{r} \frac{q}{r})$. Assume that there exists $\delta > 0$ such that $E_Q(r) = O \left( e^{-\delta r} \right)$ as $r \to \infty$. Fix an arbitrary number $h > \delta/4$. Then the integral

$$\int_0^\infty P(x, z) \overline{P(x, ih)} \, dx$$  

converges uniformly on compact subsets of $\Omega_{\delta/4} = \{ z : \text{Im} z > -\delta/4 \}$ and the function

$$z \mapsto z + ih \int_0^\infty P(x, z) \overline{P(x, ih)} \, dx, \quad z \in \Omega_{\delta/4}$$

is analytic in $\Omega_{\delta/4}$ and coincides with $\Pi$ in $\mathbb{C}_+$.  

**Proof.** Substitute $ih$ into Christoffel-Darboux formula (18). We have

$$\int_0^r \frac{i \, P_r^*(r, z) \overline{P_r^*(r, ih)} - P(r, z) \overline{P(r, ih)}}{z + ih} \, dx, \quad z \in \mathbb{C}.$$  

Take an arbitrary increasing sequence $\rho_n \to \infty$ such that $P(\rho_n, ih) \to 0$ as $n \to \infty$. Then from Lemma 1 we know that there exist a subsequence $n_k$ and $\gamma \in [0, 2\pi)$ such that

$$P(\rho_{n_k}, z) \to 0, \quad P_*(\rho_{n_k}, z) \to e^{i\gamma} \Pi(z),$$

uniformly on compact subsets of $\mathbb{C}_+$. Substituting $\rho_{n_k}$ for $r$ into (54) and taking the limit as $k \to \infty$, we obtain

$$\int_0^\infty \frac{i \, \Pi(z) \overline{\Pi(ih)}}{z + ih} \lim_{k \to \infty} \int_0^{\rho_{n_k}} P(x, z) \overline{P(x, ih)} \, dx, \quad z \in \mathbb{C}_+,$$

or, equivalently,

$$\Pi(z) = \frac{z + ih}{i \Pi(ih)} \lim_{k \to \infty} \int_0^{\rho_{n_k}} P(x, z) \overline{P(x, ih)} \, dx, \quad z \in \mathbb{C}_+.$$
Therefore, the fact that (53) defines an analytic continuation of $\Pi$ immediately follows from the convergence of integral (52).

For two positive real numbers $A \leq B$, define

$$F_{A,B}(z) = \int_{A}^{B} P(r, z) \overline{P(r, i\hbar)} \, dr.$$  

From Corollary 2 we have

$$\int_{r}^{\infty} |P(x, i\hbar)|^2 \, dx \lesssim e^{-\delta r/2}, \quad r \geq 0. \tag{56}$$

Then, for $z \in \mathbb{C}_+$, the Cauchy-Schwartz inequality gives

$$|F_{A,B}(z)| \lesssim \sqrt{\int_{A}^{B} |P(r, z)|^2 \, dr \int_{A}^{B} |P(r, i\hbar)|^2 \, dr} \lesssim \sqrt{\int_{0}^{\infty} |P(r, z)|^2 \, dr \cdot e^{-A\delta/4} \lesssim e^{-A\delta/4} \frac{|\Pi(z)|}{\sqrt{\text{Im } z}}. \tag{57}$$

Therefore, we have

$$|F_{A,B}(z)| \lesssim e^{-A\delta/4} \frac{|\Pi(z)|}{\sqrt{\text{Im } z}}, \quad z \in \mathbb{C}_+, \tag{58}$$

uniformly in $\mathbb{C}_+$. Because of reflection formula (20), $F_{A,B}$ admits the following representation:

$$F_{A,B}(z) = \int_{A}^{B} e^{izr} \overline{P_*(r, z)} P(r, i\hbar) \, dr.$$  

By definition, for $\Delta > 0$ put $\Omega^-_{\Delta} = \{ z : 0 > \text{Im } z > -\Delta \}$. Using the same Cauchy-Schwartz argument for $z \in \Omega^-_{\delta/4}$ as for $z \in \mathbb{C}_+$, we get

$$|F_{A,B}(z)| \lesssim \sqrt{\int_{A}^{B} e^{2\text{Im } z\overline{r}} |P(r, i\hbar)|^2 \, dr \int_{A}^{B} |P_*(r, z)|^2 \, dr} \lesssim \frac{e^{-A(\delta/4 - \text{Im } z)}}{\delta/4 - \text{Im } z} \cdot \sqrt{\int_{A}^{B} |P_*(r, z)|^2 \, dr}. \tag{58}$$
The function \(|P_*(r, \overline{z})|^2\) is not summable on \(\mathbb{R}_+\), however, the integral on the finite segment can be estimated by (19) and (21). In other words, we have

\[
\int_A^B |P_*(r, \overline{z})|^2 \, dr = \int_A^B \left( |P(r, \overline{z})|^2 + 2 \text{Im}(\overline{z}) \int_0^r |P(s, \overline{z})|^2 \, ds \right) \, dr
\]

\[
= \int_A^B |P(r, \overline{z})|^2 \, dr + 2 \text{Im}(\overline{z}) \int_A^B \int_0^r |P(s, \overline{z})|^2 \, ds \, dr
\]

\[
\leq \int_0^\infty |P(r, \overline{z})|^2 \, dr + 2 \text{Im}(\overline{z}) \int_0^\infty |P(s, \overline{z})|^2 \, ds \, dr
\]

\[
= \left( \frac{1}{2 \text{Im}(\overline{z})} + B - A \right) 2 \text{Im}(\overline{z}) \int_0^\infty |P(s, \overline{z})|^2 \, ds
\]

\[
\leq \left( \frac{1}{2 \text{Im}(\overline{z})} + B - A \right) |\Pi(\overline{z})|^2.
\]

If, in addition, \(|B - A| \leq 1\) then, for \(z \in \Omega_{\delta/4}\), we have

\[
\frac{1}{2 \text{Im}(\overline{z})} + B - A \leq \frac{1 + 2 \text{Im}(\overline{z})}{2 \text{Im}(\overline{z})} \leq \frac{1}{\text{Im}(\overline{z})},
\]

\[
\int_A^B |P_*(r, \overline{z})|^2 \, dr \leq \frac{|\Pi(\overline{z})|^2}{\text{Im}(\overline{z})}.
\]

Substituting the latter bound into (58), we get

\[
|F_{A,B}(z)| \lesssim e^{-A(\delta/4 - \text{Im}\overline{z})} \cdot \frac{|\Pi(\overline{z})|}{\sqrt{|\text{Im}\overline{z}|}}, \quad z \in \Omega_{\delta/4}, \quad B - A \leq 1.
\]

Fix a connected compact set \(K \subset \Omega_{\delta/4}\). Let us show that there exists a positive constant \(\alpha\) depending on \(K\) such that

\[
|F_{A,B}(z)| \lesssim e^{-A\alpha},
\]

uniformly for \(A \leq B\) and \(z \in K\). Three different situations are possible:

\(K \subset \mathbb{C}_+, \quad K \subset \Omega_{\delta/4} \cap \mathbb{C}_-, \quad K \cap \mathbb{R} \neq \emptyset\).

In the first and in the second situations bound (60) for \(B - A \leq 1\) easily follows from (57) and (59) respectively. If \(K\) intersects the real line then take a rectangle \(R\) with sides parallel to the real and imaginary axis of the complex plane such that \(K \subset R \subset \Omega_{\delta/4}\) and \(\text{dist}(K, \partial R) > 0\), see Figure 1. Let \(L_1, L_2, L_3\) and \(L_4\) be the left, top, right and bottom sides of \(R\) respectively and let \(x_1, x_2, y_1, y_2\) be such that

\[
L_1 \subset \{z: \text{Re} z = x_1\}, \quad L_3 \subset \{z: \text{Re} z = x_2\},
\]

\[
L_2 \subset \{z: \text{Im} z = y_1\}, \quad L_4 \subset \{z: \text{Im} z = y_2\}.
\]

By Lemma 6, \(\Pi\) is continuous in \(\overline{\mathbb{C}_+}\) hence

\[
\sup_{z \in \partial R \cap \mathbb{C}_+} |\Pi(z)| < \infty, \quad \sup_{z \in \partial R \cap \mathbb{C}_-} |\Pi(\overline{z})| < \infty.
\]
Denote $\delta/4 - |y_2| > 0$ by $\alpha$. Then, combining (57), (59) and (61), we obtain

$$|F_{A,B}(z)| \lesssim \frac{e^{-A\alpha}}{\sqrt{|\Im z|}}, \quad z \in \partial R \setminus \mathbb{R}, \quad B - A \leq 1.$$  

(62)

For every $z_0 \in K$ we have

$$F_{A,B}(z_0) = \frac{1}{2\pi i} \int_{\partial R} \frac{F_{A,B}(z)}{z - z_0} \, dz = \frac{1}{2\pi i} \sum_{n=1}^{4} \int_{L_n} \frac{F_{A,B}(z)}{z - z_0} \, dz.$$  

The inequality $|z - z_0| \geq \text{dist}(K, \partial R) > 0$ holds therefore

$$|F_{A,B}(z_0)| \lesssim \int_{\partial R} |F_{A,B}(z)||dz| = \sum_{n=1}^{4} \int_{L_n} |F_{A,B}(z)||dz|.$$  

(63)

It remains to bound the integrals over the sides of $R$. We have

$$\int_{L_2} |F_{A,B}(z)||dz| = \int_{x_1}^{x_2} |F_{A,B}(x + iy_1)||dx| \lesssim e^{-A\alpha},$$  

$$\int_{L_1} |F_{A,B}(z)||dz| = \int_{y_2}^{y_1} |F_{A,B}(x_1 + iy)||dy| \lesssim e^{-A\alpha} \int_{y_2}^{y_1} \frac{1}{\sqrt{\Im y}} \, dy \lesssim e^{-A\alpha}.$$  

$$\int_{L_3} |F_{A,B}(z)||dz| = \int_{y_2}^{y_1} |F_{A,B}(x_1 + iy)||dy| \lesssim e^{-A\alpha} \int_{y_2}^{y_1} \frac{1}{\sqrt{\Im y}} \, dy \lesssim e^{-A\alpha}.$$  

(62)
The integrals over the segments $L_3$ and $L_4$ can be bounded similarly. Therefore (60) holds for $B - A \leq 1$. If $n \leq B - A < n + 1$ for some positive integer $n$, then we have

$$|F_{A,B}(z)| \leq \sum_{k=0}^{n} |F_{A+k,A+k+1}(z)| + |F_{A+n,B}(z)| \lesssim \sum_{k=0}^{n} e^{-(A+k)\alpha} + e^{-(A+n)\alpha} \approx e^{-A\alpha},$$

hence (60) holds for all $A \leq B$ with some other constant in $\lesssim$. The convergence of integral (52) follows and the proof of Theorem 6 is finished.

\[ \square \]

4. Square summable potentials. Proof of Theorem 3

In Theorem 3 the entries $p$ and $q$ of the potential $Q$ are in $L^2(\mathbb{R}_+)$. The coefficient $a$ of the corresponding Krein system (recall Section 2.1.3) is also in $L^2(\mathbb{R}_+)$. As usual $\sigma$ is the spectral measure of the Dirac operator and of the Krein system, $\Pi$ is the inverse Szegő function of $\sigma$ and $P, P_*$ are the solutions of the Krein system with coefficient $a$. Let us start with a general result concerning Krein systems with $a \in L^2(\mathbb{R}_+)$. 

**Theorem D** (S. Denisov). Assume that the coefficient $a$ of Krein system (12) belongs to $L^2(\mathbb{R}_+)$. Then $\sigma$ belongs to the Szegő class and there exists $\gamma \in [0,2\pi)$ such that

$$P_*(r, \lambda) \to \Pi_\gamma(\lambda) = e^{\gamma \Pi(\lambda)}, \ r \to \infty,$$

$$|P_*(r, \lambda)| \approx 1, \ |\Pi(\lambda)| \approx 1, \ |P(r, \lambda)| \lesssim 1$$

uniformly in $\{|\text{Im} \lambda | > \varepsilon\}$ for every $\varepsilon > 0$. Additionally, we have

$$\int_{\mathbb{R}} \left| \frac{1}{\Pi_\gamma(x)} - 1 \right|^2 \, dx < \infty.$$

**Proof.** The Szegő condition and the convergence in $\{|\text{Im} \lambda | > \varepsilon\}$ is stated in Theorem 11.1 from [7]; the assertion $\Pi_{\gamma}^{-1} - 1 \in L^2(\mathbb{R}_+)$ is Theorem 11.2 in [7]. The boundedness of $P_*$ and $P_*^{-1}$ follows from the proof of Theorem 11.1 and Lemma 4.6 in [7] respectively. Christoffel-Darboux formula (19) gives $|P(r, z)| < |P_*(r, z)|$ for $z \in \mathbb{C}_+$. Finally, the convergence of $P_*$ to $\Pi_\gamma$ implies the inequalities for $\Pi$ and $\Pi^{-1}$. \[ \square \]

4.1. Auxiliary results.

**Lemma 7.** Assume that $a \in L^2(\mathbb{R}_+)$ and $E_Q(r) = O\left(e^{-\delta r}\right)$ as $r \to \infty$ for some $\delta > 0$. Then $\Pi$ is analytic in $\Omega_{\delta/4}$ and for every $\alpha \in (0,1)$ the inequality $|\Pi(z)| \lesssim (1 + |z|)^{\alpha}$ holds uniformly in the closed half-plane $\overline{\Omega_{\delta/4}}$.

**Proof.** The analyticity of $\Pi$ immediately follows from Theorem 1, hence it remains to prove the bound $(1 + |z|)^{\alpha}$. To simplify the exposition introduce $\delta_1 = \delta/4$. Fix an arbitrary $h > \delta_1$. By Theorem D, we have $P_*(r, ih) \to \Pi_\gamma(ih)$ as $r \to \infty$. An application of the Cauchy–Schwarz inequality to (12) gives

$$|P_*(r, ih) - \Pi_\gamma(ih)| = \left| \int_{r}^{\infty} a(x)P(x, ih) \, dx \right| \leq \|a\|_{L^2(\mathbb{R}_+)} \sqrt{\int_{r}^{\infty} |P(x, ih)|^2 \, dx}.$$

Corollary 2 yields

$$|P_*(r, ih) - \Pi_\gamma(ih)| \lesssim e^{-r\delta_1}, \ r \geq 0.$$
Reordering the terms in (54), we get
\( P_s(r, z) = \frac{z + ih}{i\Pi_\gamma(ih)} \int_0^r P(x, z)P(x, ih) \, dx + \frac{P(r, z)P(r, ih)}{P_s(r, ih)}, \quad z \in \mathbb{C}. \)

On the other hand, in Theorem 6 we showed (53) that
\[ \Pi_\gamma(z) = \frac{z + ih}{i\Pi_\gamma(ih)} \int_0^\infty P(x, z)P(x, ih) \, dx, \quad z \in \Omega_{\delta_1}. \]

The two latter equalities together give
\[
\Pi_\gamma(z) - P_s(r, z) = \frac{z + ih}{i\Pi_\gamma(ih)} \int_0^\infty P(x, z)P(x, ih) \, dx - \frac{P(r, z)P(r, ih)}{P_s(r, ih)} \\
+ \frac{(z + ih)}{i} \int_0^r P(x, z)P(x, ih) \, dx \left( \frac{1}{\Pi_\gamma(ih)} - \frac{1}{P_s(r, ih)} \right), \quad z \in \Omega_{\delta_1}.
\]

Let \( z \) be a point with \( \text{Im} \, z = -\alpha \delta_1 \). From (20) we have
\[ |P(r, z)| = |e^{irz}P_s(r, z)| = e^{\alpha \delta_1 r}|P_s(r, z)|. \]

The functions \( P_s(r, z) \) and \( P_s(r, ih)^{-1} \) are uniformly bounded for \( r \geq 0 \) by Theorem D. Therefore
\[
|\Pi_\gamma(z) - P_s(r, z)| \lesssim (|z| + 1) \int_r^\infty e^{\alpha \delta_1 x}|P(x, ih)| \, dx + e^{\alpha \delta_1 r}|P(r, ih)| \\
+ (|z| + 1)|\Pi_\gamma(ih) - P_s(r, ih)| \int_0^r e^{\alpha \delta_1 x}|P(x, ih)| \, dx,
\]
uniformly for \( z \) with \( \text{Im} \, z = -\alpha \delta_1 \). Corollary 2 gives us that the integral in the first term is \( O\left(e^{-(1-\alpha)\delta_1 r}\right) \) as \( r \to \infty \). For the same reason the integral in the last term is bounded for \( r \geq 0 \). Because of (64), the last term is \( O\left(e^{-r\delta_1}\right) \) as \( r \to \infty \). Therefore,
\[
|\Pi_\gamma(z) - P_s(r, z)| \lesssim (|z| + 1)e^{-(1-\alpha)\delta_1 r} + e^{\alpha \delta_1 r}|P(r, ih)|, \quad r \geq 0,
\]
\[
\int_r^{r+1} |\Pi_\gamma(z) - P_s(\rho, z)| \, d\rho \lesssim (|z| + 1)e^{-(1-\alpha)\delta_1 r} + \int_r^{r+1} e^{\alpha \delta_1 \rho}|P(\rho, ih)| \, d\rho, \quad r \geq 0.
\]

Applying Corollary 2 for \( ih \) one more time, we get
\[
\int_r^{r+1} |\Pi_\gamma(z) - P_s(\rho, z)| \, d\rho \lesssim (|z| + 1)e^{-(1-\alpha)\delta_1 r}.
\]

Therefore there exists a constant \( C \) such that
\[
\int_r^{r+1} |P_s(\rho, z)| \, d\rho \geq |\Pi_\gamma(z)| - C(|z| + 1)e^{-(1-\alpha)\delta_1 r}, \quad r \geq 0.
\]
Define \( r_0 = r_0(z) \) as the solution of the equation
\[
e^{-\delta_1(1-\alpha)r} = \frac{|\Pi_\gamma(z)|}{2C(|z| + 1)}.
\]
Then for every \( r \geq r_1 = \min(r_0, 0) \) we have \( \int_r^{r+1} |P_\gamma(z)| \, d\rho \geq |\Pi_\gamma(z)|/2 \). Formulas (21) and (20) for the point \( \overline{z} \in \mathbb{C}_+ \) and the latter inequality give

\[
|\Pi_\gamma(\overline{z})|^2 = 2 \text{Im} \int_0^\infty |P(x, \overline{z})|^2 \, dx \gtrsim \int_{r_1}^{r_1+1} e^{-2\alpha_1 x} |P_\gamma(x, z)|^2 \, dx \gtrsim |\Pi_\gamma(z)|^2 e^{-2\alpha_1 r_1} .
\]

From Theorem D we know that \( \Pi_\gamma(\overline{z}) \) is uniformly bounded for \( \text{Im} \, z = -\alpha \delta_1 \) hence the latter implies \( |\Pi_\gamma(z)| e^{-\alpha_1 r_1} \lesssim 1 \). If \( r_1 = 0 \) then we obtain \( |\Pi_\gamma(z)| \lesssim 1 \). If \( r_1 = r_0 \) we get

\[
e^{-\alpha_1 r_1} = e^{-\alpha_1 r_0} = (e^{-\delta_1 (1-\alpha) r_1})^\beta \approx \left( \frac{|\Pi_\gamma(z)|}{|z| + 1} \right)^\beta ,
\]

where \( \beta = \frac{\alpha}{1-\alpha} \). Hence for \( z \) with \( \text{Im} \, z = -\alpha \delta_1 \) we have

\[
|\Pi_\gamma(z)| \left( \frac{|\Pi_\gamma(z)|}{|z| + 1} \right)^\beta \lesssim 1,
\]

(67) \[ |\Pi_\gamma(z)| \lesssim (|z| + 1)^{\frac{\beta}{\delta}} = (|z| + 1)^{\alpha} . \]

The function \( z \mapsto \Pi_\gamma(z)(z + i\delta)^{-\alpha} \) is analytic in the strip \( \{z : \text{Im} \, z \leq \alpha \delta_1\} \); by Theorem D and (67), it is bounded on the boundary of this strip. Furthermore, from Corollary 1 it follows that this function grows no faster than the exponential function. Hence we can apply the Phragmén–Lindelöf principle in the strip to deduce that \( f \) is bounded in the strip. Consequently (67) holds in the whole closed half-plane \( \Omega_{\alpha \delta_1} \). The proof is finished. \( \square \)

**Corollary 3.** Assume that \( a \in L^2(\mathbb{R}_+) \) and \( E_Q(r) = O \left( e^{-\delta r} \right) \) as \( r \to \infty \) for some \( \delta > 0 \). Then \( \frac{\Pi(x - i\Delta)}{x + i} \in H^2(\mathbb{C}_+) \) for every \( \Delta < \delta / 8 \).

**Proof.** Fix an arbitrary \( \Delta < \delta / 8 \) and let \( \alpha = 4\Delta / \delta < 1 / 2 \). By Lemma 7, the inequality \( |\Pi(z)| \lesssim (1 + |z|)^{\alpha} \) holds in \( \Omega_{\alpha \delta/4} = \Omega_\Delta \). We have \( \alpha - 1 < -1/2 \) hence \( \Pi(1 + |z|)^{-1} \) is square integrable over horizontal lines in \( \Omega_\Delta \). \( \square \)

**Lemma 8.** Assume that \( a \in L^2(\mathbb{R}_+) \), \( \sigma \) is absolutely continuous and \( \frac{\Pi(x - i\Delta)}{x + i} \in H^2(\mathbb{C}_+) \) for some \( \Delta > 0 \). Then for every \( z_0 \in \mathbb{C}_+ \) and \( \delta < \min(\Delta, \text{Im} \, z_0) \) we have

\[
\int_r^{\infty} |P(r, z_0)|^2 \, dr \lesssim e^{-2\delta r}, \quad r \geq 0 .
\]

**Proof.** Fix a point \( z_0 \) in \( \mathbb{C}_+ \) and \( \delta < \min(\Delta, \text{Im} \, z_0) \). Denote \( \Pi_\gamma(z - \overline{z}_0) \) by \( G \). We know that \( G(z - i\delta) \in H^2(\mathbb{C}_+) \) hence there exists a function \( \varphi \in L^2(\mathbb{R}_+) \), such that

\[
G(z - i\delta) = \int_0^\infty \varphi(t) e^{it z} \, dt, \quad z \in \mathbb{C}_+ ,
\]

\[
G(z) = \int_0^\infty \varphi(t) e^{-it \delta} e^{it z} \, dt, \quad z \in \Omega_\delta .
\]

Let \( G_r \) be the projection of \( G \) onto the space \( PW_{[0, r]} \) or, in other words, let

\[
G_r(z) = \int_0^r \varphi(t) e^{-it \delta} e^{it z} \, dt, \quad z \in \mathbb{C} .
\]
Recall definition (16) of the minimizing function \( m_r \). We will show that \( m_r(z_0) \) converges to \( m_\infty(z_0) \) exponentially fast in \( r \). We have \( G_r \in PW_{[0,r]} \) therefore

\[
(69) \quad m_r(\sigma, z_0) \leq \frac{1}{2\pi} \| G_r / G_r(0) \|_{L^2(\sigma)} = \frac{1}{2\pi |G_r(z_0)|^2} \int_{-\infty}^{\infty} |G_r(t)|^2 dt.
\]

Let us examine the right hand side of the latter inequality. For \( z \in \mathbb{C}_+ \), we have

\[
G(z) - G_r(z) = \int_r^\infty \varphi(t) e^{-it} e^{itz} dt.
\]

This difference is uniformly bounded in \( \mathbb{C}_+ \):

\[
|G(z) - G_r(z)| \leq \int_r^\infty |\varphi(t)| e^{-it} e^{itz} \, dt = \int_r^\infty |\varphi(t)| e^{-t(\Im z + \delta)} \, dt
\]

\[
(70) \quad \leq e^{-r(\Im z + \delta)} \int_r^\infty |\varphi(t)| e^{-(t-r)\delta} \, dt \lesssim e^{-(\Im z + \delta) r}, \quad r \geq 0,
\]

where the last inequality follows from the Cauchy - Schwarz inequality. Hence

\[
|G(z_0)|^2 - |G_r(z_0)|^2 \leq |G(z_0) - G_r(z_0)| (|G(z_0)| + |G_r(z_0)|) \lesssim e^{-(\delta + \Im z_0)r}.
\]

Furthermore, \( |G(z_0)| \neq 0 \) therefore

\[
(71) \quad \frac{1}{|G_r(z_0)|^2} \leq \frac{1}{|G(z_0)|^2} + O \left( e^{-(\delta + \Im z_0)r} \right) = \frac{4(\Im z_0)^2}{|\Pi(z_0)|^2} + O \left( e^{-(\delta + \Im z_0)r} \right), \quad r \to \infty.
\]

Let us estimate the integral in (69). We have

\[
\int_{-\infty}^{\infty} \frac{|G_r(t)|^2}{|\Pi_\gamma(t)|^2} dt = \int_{-\infty}^{\infty} \left| \frac{G(t)}{\Pi_\gamma(t)} + \frac{G_r(t) - G(t)}{\Pi_\gamma(t)} \right|^2 dt = \int_{-\infty}^{\infty} \left| \frac{1}{t - z_0} + \frac{G_r(t) - G(t)}{\Pi_\gamma(t)} \right|^2 dt
\]

\[
(72) \quad = \int_{-\infty}^{\infty} \frac{1}{|t - z_0|^2} + 2 \Re \left( \frac{G_r(t) - G(t)}{\Pi_\gamma(t)} \right) \left( \frac{1}{t - z_0} \cdot \frac{G_r(t) - G(t)}{\Pi_\gamma(t)} \right) dt.
\]

Define the set \( S = \left\{ t : \frac{1}{\Pi_\gamma(t)} - 1 \leq \frac{1}{2} \right\} \subseteq \mathbb{R} \). Then \( |\Pi_\gamma|^{-1} < 2 \) on \( S \) and

\[
\int_S \left| \frac{G_r(t) - G(t)}{\Pi_\gamma(t)} \right|^2 dt \leq 4 \int_\mathbb{R} |G_r(t) - G(t)|^2 \approx \| e^{-\delta t} \varphi 1_{[r,\infty]} \|^2_{L^2(\mathbb{R})} \lesssim e^{-2\delta t}.
\]

On the other hand, on \( \mathbb{R} \setminus S \) we have

\[
3 \left| \frac{1}{\Pi_\gamma} - 1 \right| \geq \left| \frac{1}{\Pi_\gamma} - 1 \right| + 1 \geq \frac{1}{|\Pi_\gamma|}.
\]

Combining this with Theorem D, we get

\[
\int_{\mathbb{R} \setminus S} \frac{1}{|\Pi_\gamma(t)|^2} dt \leq 9 \int_{\mathbb{R}} \left| \frac{1}{\Pi_\gamma(t)} - 1 \right|^2 dt < \infty,
\]

\[
(74) \quad \int_{\mathbb{R} \setminus S} \left| \frac{G_r(t) - G(t)}{\Pi_\gamma(t)} \right|^2 dt \leq \sup_{t \in \mathbb{R}} |G_r(t) - G(t)|^2 \int_{\mathbb{R} \setminus S} \frac{1}{|\Pi_\gamma(t)|^2} dt \overset{(70)}{\lesssim} e^{-2\delta r}.
\]
Next, consider the integral
\[ I = \int_{-\infty}^{\infty} \frac{1}{t - z_0} \cdot \frac{G_r(t) - G(t)}{\Pi_\gamma(t)} \, dt. \]

We know that both \( G_r - G \) and \( \frac{1}{(z - z_0)\Pi_\gamma} \) belong to \( H^2(\mathbb{C}_+) \). Therefore \( \frac{G_r - G}{(z - z_0)\Pi_\gamma} \in H^1(\mathbb{C}_+) \) and consequently \( I = 0 \). Rewrite the second term in (72) in the following way:
\[
\int_{-\infty}^{\infty} \frac{1}{t - z_0} \cdot \frac{G_r(t) - G(t)}{\Pi_\gamma(t)} \, dt = \int_{-\infty}^{\infty} \frac{1}{t - z_0} \cdot \frac{G_r(t) - G(t)}{\Pi_\gamma(t)} \, dt - I
\]
\[= -2i \operatorname{Im} z_0 \int_{-\infty}^{\infty} \frac{1}{(t - z_0)(t - \overline{z_0})} \cdot \frac{G_r(t) - G(t)}{\Pi_\gamma(t)} \, dt. \]

Define
\[ F(z) = \frac{G_r(z) - G(z)}{(z - z_0)(z - \overline{z_0})\Pi_\gamma(z)}, \quad z \in \mathbb{C}_+. \]

Let us calculate \( \int_{C_R} F(t) \, dt \) using the contour integration. Denote by \( x_0 \) and \( y_0 \) the real and imaginary parts of \( z_0 \) and put \( z_1 = \frac{1}{2}(z_0 + x_0) \). The contour \( C_R \) is shown at Figure 2; it consists of the horizontal segment \( J_R = [x_0 - R, x_0 + R] \), two vertical segments \( L_R^{(1)} = [x_0 + R, z_1 + R] \), \( L_R^{(2)} = [z_1 - R, x_0 - R] \) and the semicircle \( A_R \) with center \( z_1 \) of radius \( R \).

\[ \text{Figure 2. an integration contour} \]

The only singularity of \( F \) inside \( C_R \) is a simple pole at \( z_0 \) hence
\[ \int_{C_R} F(z) \, dz = 2\pi i \operatorname{Res}_{z_0} F(z) = \frac{2\pi (G_r(z_0) - G(z_0))}{2 \operatorname{Im} z_0 \Pi_\gamma(z_0)}. \]
By Theorem D and (70), the function \( \frac{G_r - G}{\Pi_z} \) is uniformly bounded in \( \text{Im} \, z > y_0/2 \). Therefore

\[
\left| \int_{A_R} F(z) \, dz \right| \lesssim \int_{A_R} \frac{1}{|z - z_0(z - z_0)|^2} |dz| = O \left( \frac{1}{R} \right), \quad R \to \infty.
\]  

Next, consider \( V(R) = \int_{L_R^{(1)}} F(z) \, dz + \int_{L_R^{(2)}} F(z) \, dz \). We have

\[
|V(R)|^2 \lesssim \int_0^{\frac{y_0}{2}} |F(x - R + iy)|^2 dt + \int_0^{\frac{y_0}{2}} |F(x + R + iy)|^2 dy.
\]

Therefore

\[
\int_{R^+} |V(R)|^2 dR \lesssim \int_M |F(x + iy)|^2 \, dx \, dy,
\]

where \( M = \{ z : 0 \leq \text{Im} \, z \leq \frac{y_0}{2} \} = \mathbb{R} \times [0, \frac{y_0}{2}] \). For \( z = x + iy \in M \) we have

\[
|F(z)| \leq \sup_{w \in M} \left| \frac{G_r(w) - G(w)}{w - z_0} \right| \cdot \frac{1}{|z - z_0|^{\Pi_\gamma(z)}} \lesssim \frac{1}{|z - z_0|^{\Pi_\gamma(z)}}.
\]

Combining the two latter inequalities, we get

\[
\|V\|_{L^2(R^+)}^2 \leq \int_R \int_0^{\frac{y_0}{2}} \left| \frac{1}{(z - z_0)^{\Pi_\gamma(z)}} \right|^2 \, dy \, dx \lesssim \left\| \frac{1}{|z - z_0|^{\Pi_\gamma}} \right\|_{H^2(C^+)}^2 < \infty.
\]

We have

\[
\frac{2\pi(G_r(z_0) - G(z_0))}{2 \text{Im} \, z_0 \Pi(z_0)} = \int_{C_R} F(z) \, dz = V(R) + \int_{A_R} F(z) \, dz + \int_{J_R} F(z) \, dz,
\]

\[
\left| \int_{J_R} F(z) \, dz - \frac{2\pi(G_r(z_0) - G(z_0))}{2 \text{Im} \, z_0 \Pi(z_0)} \right| \leq |V(R)| + \left| \int_{A_R} F(z) \, dz \right|.
\]

Because of (76) the second term tends to 0 as \( R \to \infty; \) (77) implies that \( \lim_{R \to \infty} |V(R)| = 0 \). Thus, we have

\[
\left| \int_R F(z) \, dz \right| = \left| \frac{2\pi(G_r(z_0) - G(z_0))}{2 \text{Im} \, z_0 \Pi(z_0)} \right| \lesssim e^{-(\delta + \text{Im} \, z_0)t}.
\]

Substitution of (73), (74), (75) and (78) into (72) gives

\[
\int_{-\infty}^{\infty} \frac{|G_r(t)|^2}{|\Pi(t)|^2} dt = \int_{-\infty}^{\infty} \frac{1}{|t - z_0|} dt + O \left( e^{-2\delta t} \right) = \frac{\pi}{\text{Im} \, z_0} + O \left( e^{-2\delta t} \right), \quad r \to \infty.
\]

Substituting (71) and (79) into (69) we get

\[
\mathbf{m}_r(z_0) = \left( \frac{4|\text{Im} \, z_0|^2}{|\Pi(z_0)|^2} + O \left( e^{-(\delta + \text{Im} \, z_0)t} \right) \right) \left( \frac{1}{2 \text{Im} \, z_0} + O \left( e^{-2\delta t} \right) \right)
\]

\[
= \frac{2 \text{Im} \, z_0}{|\Pi(z_0)|^2} + O \left( e^{-2\delta t} \right) \quad r \to \infty.
\]

To conclude the proof, rewrite \( \mathbf{m}_r \) and \( \mathbf{m}_\infty \) using relations (17) and (22):

\[
\mathbf{m}_r(z_0) - \mathbf{m}_\infty(z_0) = \left( \int_0^r |P(r, z_0)|^2 \, dr \right)^{-1} - \left( \int_0^\infty |P(r, z_0)|^2 \, dr \right)^{-1} \gtrsim \int_r^\infty |P(r, z_0)|^2 \, dr.
\]
Inequality (68) immediately follows. \hfill \Box

**Corollary 4.** Under the assumptions of Lemma 8 for every $\delta < \min(4, 4\Delta)$, we have $E_Q(r) = O\left(e^{-\delta r}\right)$ as $r \to \infty$.

**Proof.** Differential equation (38) for $z = i$ becomes

$$\frac{\partial}{\partial r} |\hat{P}_r^*(i)|^2 = -2 \text{Im} i \text{Re} f_2(r)|\hat{P}_r^*(i)|^2 = \frac{|K'(r/2)|}{2} |\hat{P}_r^*(i)|^2.$$  

We see that $|\hat{P}_r^*(i)|^2$ is increasing and Lemma 3 gives $|\hat{P}_r^*(i)|^2 \leq \lim_{r \to \infty} |\hat{P}_r^*(i)|^2 = |\Pi(i)|^2$. Rewrite the equality in Lemma 5 at the point $z = i$ in a form

$$2 \int_r^\infty |P(x, i)|^2 \, dx = \left(|\Pi(i)|^2 - |\hat{P}_r^*(i)|^2\right) + |\hat{P}_r(i)|^2.$$  

Both terms in the right hand side are nonnegative therefore, from Lemma 8, it follows that

$$0 \leq |\Pi(i)|^2 - |\hat{P}_r^*(i)|^2 \lesssim e^{-2\delta r}$$

holds for every $\delta < \min(1, \Delta)$. On the other hand, $|\hat{P}_r^*(i)| \gtrsim 1$ for $r \geq 0$ and (80) yields

$$|\Pi(i)|^2 - |\hat{P}_r^*(i)|^2 = \int_r^\infty \frac{\partial}{\partial r} |\hat{P}_r^*(i)|^2 \, ds = \int_r^\infty \frac{|K'(s/2)|}{2} |\hat{P}_r^*(i)|^2 \, ds \gtrsim K(r/2).$$

This together with (81) and Theorem C finishes the proof:

$$K(r/2) \lesssim e^{-2\delta r}, \quad K(r) \lesssim e^{-4\delta r}, \quad E_Q(r) \lesssim e^{-4\delta r}. \hfill \Box$$

**Remark.** The restrictive assertion $\delta < 4$ is imposed because the point $z = i$ is special for differential equation (38). At any other point the term $2 \text{Re} \left((z - i)f_1(r)|\hat{P}_r(z)\bar{P}_r^*(z)\right)$ does not vanish preventing further argumentation. To overcome this obstacle, below we introduce the scaled entropy and do the rescaling procedure.

### 4.2. Rescaling

#### 4.2.1. Auxiliary estimates

For a segment $I$ and a measurable function $f$ on $I$ we use the notation $\langle f \rangle_I$ to denote the average of $f$ over $I$, i.e.,

$$\langle f \rangle_I = \frac{1}{|I|} \int_I f(x) \, dx.$$  

We fix some segment $I = [a, b]$ and define

$I_1 = [a, (a + b)/2], \quad I_2 = [(3a + b)/4, (a + 3b)/4], \quad I_3 = [(a + b)/2, b]$  

– the left, the middle and the right half of $I$ respectively. This notation is used throughout the whole subsection.

**Lemma 9.** Let $g$ be a positive measurable function on $I$. Then

$$\langle g \rangle_I \langle g^{-1} \rangle_I - 1 \leq \max_{n \in \{1, 2, 3\}} \left[ \langle g \rangle_{I_n} \langle g^{-1} \rangle_{I_n} - 1 \right],$$

provided that the latter maximum does not exceed 1.
Proof. Denote the maximum from the statement of the lemma by \( \varepsilon \). We have

\[
\langle g \rangle_I \langle g^{-1} \rangle_I = \frac{1}{4} \left( \langle g \rangle_I + \langle g \rangle_{I_3} \right) \left( \langle g^{-1} \rangle_I + \langle g^{-1} \rangle_{I_3} \right)
\]

\[
= \frac{1}{4} \left( \langle g \rangle_I - \langle g \rangle_{I_3} \right) \left( \langle g^{-1} \rangle_I - \langle g^{-1} \rangle_{I_3} \right) + \frac{1}{2} \left( \langle g \rangle_I \langle g^{-1} \rangle_I + \langle g \rangle_{I_3} \langle g^{-1} \rangle_{I_3} \right).
\]

Therefore

\[
\langle g \rangle_I \langle g^{-1} \rangle_I - 1 \leq \frac{\langle g \rangle_I \langle g^{-1} \rangle_I - 1}{2} + \frac{\langle g \rangle_{I_3} \langle g^{-1} \rangle_{I_3} - 1}{2}
\]

\[
+ \frac{1}{4} \left| \langle g \rangle_I - \langle g \rangle_{I_3} \right| \left| \langle g^{-1} \rangle_I - \langle g^{-1} \rangle_{I_3} \right|
\]

\[
\leq \varepsilon + \frac{1}{4} \left( \langle g \rangle_I \langle g^{-1} \rangle_I \right) \left( \left| 1 - \frac{\langle g \rangle_{I_3}}{\langle g \rangle_I} \right| \left| 1 - \frac{\langle g^{-1} \rangle_{I_3}}{\langle g^{-1} \rangle_I} \right| \right)
\]

\[
\leq \varepsilon + \frac{1 + \varepsilon}{4} \left( \left| 1 - \frac{\langle g \rangle_{I_3}}{\langle g \rangle_I} \right| \left| 1 - \frac{\langle g^{-1} \rangle_{I_3}}{\langle g^{-1} \rangle_I} \right| \right).
\]

(82)

Let \( f = \log g \). By Jensen’s inequality, for \( n \in \{1, 2, 3\} \) we have

\[
1 \leq \langle e^f \rangle_{I_n} e^{-|f|_{I_n}} \leq \langle e^f \rangle_{I_n} \langle e^{-f} \rangle_{I_n} = \langle g \rangle_{I_n} \langle g^{-1} \rangle_{I_n} \leq 1 + \varepsilon.
\]

It follows that (see formula (4.12) in [5] or (3.7) in [15])

\[
\langle |f - \langle f \rangle_{I_n} \rangle \rangle_{I_n} \lesssim \sqrt{\varepsilon}.
\]

As a corollary, we get

\[
\langle |f \rangle_{I_1} - \langle f \rangle_{I_2} \rangle_{|I_1 \cap I_2|} \leq \langle |f \rangle_{I_1} - f \rangle_{I_1 \cap I_2} + \langle |f - \langle f \rangle_{I_2} \rangle \rangle_{|I_1 \cap I_2|}
\]

\[
\leq \frac{1}{2} \langle |f \rangle_{I_1} - f \rangle_{I_1} + \frac{1}{2} \langle |f - \langle f \rangle_{I_2} \rangle \rangle_{I_2} \leq \sqrt{\varepsilon}.
\]

(84)

Inequalities (83) and (84) yield

\[
\left| \frac{\langle g \rangle_{I_1}}{\langle g \rangle_{I_2}} - 1 \right| \leq \left| \frac{\langle e^f \rangle_{I_1}}{\langle e^f \rangle_{I_2}} - 1 \right| = \left| \frac{\langle e^f \rangle_{I_1} \cdot e^{(f)}_{I_1}}{e^{(f)}_{I_1} \cdot e^{(f)}_{I_2}} - 1 \right| \lesssim \sqrt{\varepsilon}.
\]

Similarly we can obtain

\[
\left| \frac{\langle g^{-1} \rangle_{I_1}}{\langle g^{-1} \rangle_{I_2}} - 1 \right| \lesssim \sqrt{\varepsilon}, \quad \left| \frac{\langle g \rangle_{I_2}}{\langle g \rangle_{I_3}} - 1 \right| \lesssim \sqrt{\varepsilon}, \quad \left| \frac{\langle g^{-1} \rangle_{I_2}}{\langle g^{-1} \rangle_{I_3}} - 1 \right| \lesssim \sqrt{\varepsilon}.
\]

(85)

Therefore, we have

\[
\left| \frac{\langle g \rangle_{I_1}}{\langle g \rangle_{I_3}} - 1 \right| \lesssim \sqrt{\varepsilon}, \quad \left| \frac{\langle g^{-1} \rangle_{I_1}}{\langle g^{-1} \rangle_{I_2}} - 1 \right| \lesssim \sqrt{\varepsilon}.
\]

The substitution of (85) into (82) concludes the proof.

\[\square\]

**Lemma 10.** Let \( G \) be a matrix-valued function on \( I \) such that \( G(t) \geq 0 \) and \( \det G(t) = 1 \) for almost every \( t \in I \). Additionally assume that for every unit vector \( v \in \mathbb{R}^2 \) and \( t \in I \) we have

\[
\langle G(t)v, v \rangle \approx 1.
\]
Then the inequality
\begin{equation}
\det \langle G \rangle_I - 1 \lesssim \max_{n \in \{1,2,3\}} [\det (G)_{I_n} - 1]
\end{equation}
holds, provided that the latter maximum does not exceed 1.

**Proof.** Denote the maximum from the statement of the lemma by \( \varepsilon \). For every \( 2 \times 2 \) matrix \( M \) we have
\[
\frac{1}{\sqrt{\det M}} = \frac{1}{\pi} \int_{\mathbb{R}^2} e^{-\langle Mx, x \rangle} dx.
\]
Rewriting this in polar coordinates, we get
\[
\frac{1}{\sqrt{\det M}} = \frac{1}{\pi} \int_{\mathbb{T}} \left( \int_0^\infty e^{-r^2 \langle M\zeta, \zeta \rangle} r \, dr \right) \frac{d\zeta}{\sqrt{\det (M\zeta, \zeta)}}.
\]
Let \( J \) be some subinterval of \( I \) and define \( A_J = \langle G \rangle_J \). By Jensen’s inequality
\[
\frac{1}{\sqrt{\det A_J}} = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\zeta}{\langle A_J \zeta, \zeta \rangle} \leq \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1}{\langle G\zeta, \zeta \rangle} \frac{d\zeta}{\langle A_J \zeta, \zeta \rangle} = \left( \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\zeta}{\langle G\zeta, \zeta \rangle} \right)_J = 1.
\]
Therefore
\[
0 \leq \int_{\mathbb{T}} \left( \left( \frac{1}{\langle G\zeta, \zeta \rangle} \right)_J - \frac{1}{\langle A_J \zeta, \zeta \rangle} \right) d\zeta = 2\pi \left( 1 - \frac{1}{\sqrt{\det A_J}} \right).
\]
Denote \( \langle G(t)\zeta, \zeta \rangle \) by \( g_\zeta(t) \) then the latter equality can be rewritten as
\[
\int_{\mathbb{T}} \langle g_\zeta^{-1} \rangle_J - \langle g_\zeta \rangle_J^{-1} d\zeta = 2\pi \left( 1 - \frac{1}{\sqrt{\det A_J}} \right).
\]
Next, we see that
\[
\langle g_\zeta^{-1} \rangle_J - \langle g_\zeta \rangle_J^{-1} = \frac{\langle g_\zeta^{-1} \rangle_J \langle g_\zeta \rangle_J - 1}{\langle g_\zeta \rangle_J} \approx \langle g_\zeta^{-1} \rangle_J \langle g_\zeta \rangle_J - 1
\]
because of assumption (86). Therefore
\[
1 - \frac{1}{\sqrt{\det A_J}} \approx \int_{\mathbb{T}} F_J(\zeta) \, d\zeta,
\]
where \( F_J(\zeta) = \langle g_\zeta^{-1} \rangle_J \langle g_\zeta \rangle_J - 1 \). Moreover, (87) is equivalent to
\begin{equation}
\int_{\mathbb{T}} F_J(\zeta) \, d\zeta \lesssim \max_{n \in \{1,2,3\}} \left[ \int_{\mathbb{T}} F_{I_n}(\zeta) \, d\zeta \right].
\end{equation}
Define the set \( S \subset \mathbb{T} \) by
\[
S = \{ \zeta : F_n(\zeta) < 1, \ n = 1,2,3 \}.
\]
By Lemma 9, for \( \zeta \in S \) we have
\begin{equation}
F(\zeta) \lesssim \max_{n \in \{1,2,3\}} F_n(\zeta).
\end{equation}
On the other hand, we have $\max_{n \in \{1, 2, 3\}} F_n(\zeta) \geq 1$ on $\mathbb{T} \setminus S$ and $F_1 \lesssim 1$ on $\mathbb{T} \setminus S$ because of assumption (86). Hence (89) holds on the whole circle with some other constant in $\lesssim$. Inequality (88) follows and the proof is concluded.

4.2.2. **Ordered exponent.** Let $A$ be a matrix-valued function with real entries in $L^1_{\text{loc}}(\mathbb{R}_+)$. The solution of the differential equation

$$\frac{\partial}{\partial t} X_A(r, t) = A(t) X_A(r, t), \quad X_A(r, r) = I, \quad t \geq r \quad (90)$$

is called the ordered exponent of $A$. It follows from (90) that the relation

$$X_A(r, t) X_A(t_0, r) = X_A(t_0, t) \quad (91)$$

holds for every $t_0 \leq r \leq t$. Moreover, $X_A$ admits the following explicit representation:

$$X_A(r, t) = \sum_{m \geq 0} \int_r^t A(t_1) \int_r^{t_1} A(t_2) \int_r^{t_2} \cdots \int_r^{t_{m-1}} A(t_m) \, dt_m \cdots dt_2 \, dt_1, \quad (92)$$

for details see the book [9]. We remark that the series converges in the operator norm because the norm of the $m$-th term does not exceed $\frac{1}{m!} \left( \int_r^t \| A(s) \| \, ds \right)^m$.

Fix an arbitrary unit vector $v$ and consider

$$\frac{\partial}{\partial t} \| X_A(r, t) v \|^2 = 2 \left( \frac{\partial}{\partial t} X_A(r, t) v, X_A(r, t) v \right) = 2 \langle A X_A(r, t) v, X_A(r, t) v \rangle. \quad (93)$$

Hence

$$2 \| A(t) \| \| X_A(r, t) v \|^2 \geq \frac{\partial}{\partial t} \| X_A(r, t) v \|^2 \geq -2 \| A(t) \| \| X_A(r, t) v \|^2, \quad (94)$$

Integrating on $[r, t]$ and taking the exponent, we get

$$\exp \left[ \int_r^t \| A(s) \| \, ds \right] \geq \| X_A(r, t) v \| \geq \exp \left[ - \int_r^t \| A(s) \| \, ds \right].$$

4.2.3. **Entropy and rescaling.** Recall definition (10) of the entropy function. In this subsection we show what happens if we replace the integration segment $[r, r + 2l]$ with $[r, r + 2l]$ in (10) for different parameters $l > 0$. Namely, define the scaled entropy function by

$$E_Q^{(l)}(r) = \frac{1}{l^2} \det \int_r^{r+2l} H_Q(t) \, dt - 4, \quad (95)$$

so that $E_Q(r) = E_Q^{(1)}(r)$. Equation (5) for $\lambda = 0$ can be rewritten as $N_Q'(t) = J_Q(t) N_Q(t)$. This is a differential equation for the ordered exponent of $J_Q$ hence $N_Q(t) = X_{J_Q}(0, t)$. Using (91) and (10) for all $r < t$ we get

$$N_Q(t) = X_{J_Q}(r, t) X_{J_Q}(0, r), \quad (96)$$

$$H_Q(t) = (X_{J_Q}(0, r))^* (X_{J_Q}(r, t))^* X_{J_Q}(r, t) X_{J_Q}(0, r).$$
For \( r \geq 0 \) we have \( \text{trace}(JQ(r)) = 0 \) hence \( \det X_{JQ}(0, r) = 1 \). Therefore (94) can be rewritten as

\[
E_{Q}^{(l)}(r) = \frac{1}{l^2} \det \int_{r}^{r+2l} (X_{JQ}(r, t))^*X_{JQ}(r, t) \, dt - 4.
\]

**Theorem 7.** Assume that \( p, q \in L^2(\mathbb{R}_+) \) and let the scaled entropy \( E_{Q}^{(l)} \) be defined by (95). Let \( f \) be an arbitrary nonincreasing function with \( \lim f(r) = 0 \) as \( r \to \infty \). If the assertion

\[
E_{Q}^{(l)}(r) \lesssim f(r), \quad r \geq 0
\]

holds for some \( l > 0 \) then it holds for every \( l > 0 \).

**Proof.** Assume that (96) holds for \( l = l_0 \). Let us show that for all large \( r \), the assertions of Lemma 10 hold for \( G_r(t) = (X_{JQ}(r, t))^*X_{JQ}(r, t) \) on the segment \( I = [r, r + 4l_0] \). Indeed, the properties of the ordered exponent instantly give that \( G_r(t) \geq 0 \) and \( \det G_r(t) = 1 \) for all \( t \in I \). Furthermore, if \( v \in \mathbb{R}^2 \) is a unit vector, then

\[
\langle G_r(t)v, v \rangle = \|X_{JQ}(r, t)v\|_2^2 \approx 1,
\]

because of inequality (93) and the assertion \( Q \in L^2 \). An application of Lemma 10 gives

\[
\frac{1}{4}E_{Q}^{(2l_0)}(r) = \det \langle G_r \rangle_I - 1 \leq \max_{n \in \{1, 2, 3\}} [\det \langle G_r \rangle_{I_n} - 1]
\]

\[
= \frac{1}{4} \max \left[ E_{Q}^{(l_0)}(r), E_{Q}^{(l_0)}(r + l_0), E_{Q}^{(l_0)}(r + 2l_0) \right] \lesssim f(r).
\]

Therefore (96) holds for \( l = 2l_0 \). Iterating this procedure we get that it also holds for \( l = 2^n l_0 \), for every integer \( n > 0 \). It remains for us to notice that (96) for \( l = l_0 \) implies (96) for all \( l < l_0 \) (see formula 5.1 in [5]).

### 4.3. Proof of Theorem 3.

**Proof of Theorem 3.** As we mentioned at the beginning of Section 4, the assertion \( p, q \in L^2(\mathbb{R}_+) \) is equivalent to \( a \in L^2(\mathbb{R}_+) \), where \( a \) is the coefficient of the Krein system corresponding to \( D_Q \). Therefore Corollaries 3 and 4 give us the inequalities

\[
\min(4, 4\Delta_{\Pi}) \leq \Delta_E \leq 8\Delta_{\Pi}.
\]

If \( \Delta_{\Pi} < 1 \) then this is exactly the claim of Theorem 3 otherwise let us use the following rescaling argument. Let \( v > 0 \) and consider Krein system (12) with the coefficient \( a(v) : t \mapsto va(vr) \). Let

\[
P_{*}^{(v)}, \quad \Pi^{(v)}, \quad H_{Q}^{(v)}, \quad E_{Q}^{(v)}, \quad \Delta_{\Pi}^{(v)} \quad \text{and} \quad \Delta_{E}^{(v)}
\]

be the solution, the inverse Szegő function, the Hamiltonian, the entropy function and the parameters from Theorem 3 corresponding to this scaled Krein system. Simple calculation show that \( P_{*}^{(v)}(r, \lambda) = P_{*}^{(v)}(vr, \lambda/r) \) and therefore

\[
\Pi^{(v)}(\lambda) = \Pi(\lambda/v), \quad \Delta_{\Pi}^{(v)} = v\Delta_{\Pi}.
\]

If \( v \) is such that \( v\Delta_{\Pi} < 1 \), then the conclusions of the theorem hold for the scaled Krein system. From the definition of the Hamiltonian we have \( H_{Q}^{(v)}(t) = H_{Q}(vt) \) and hence \( E_{Q}^{(v)}(r) = E_{Q}^{(v)}(vr) \),
where $E_Q^{(v)}$ is the scaled entropy, see (94). From Theorem 7 it follows that $\Delta_E^{(v)} = v \Delta_E$. Therefore we have

$$\frac{\Delta_H}{\Delta_E} = \frac{\Delta_H^{(v)}}{\Delta_E^{(v)}} \in [4, 8].$$

The proof of the theorem is concluded.

\[\square\]

5. Some applications

In this section we prove Theorems 2 and 5 and discuss the sharpness of the inequalities in Theorems 1 and 3.

5.1. Proof of Theorem 5.

5.1.1. Off-diagonal potentials. Recall that we have the potential $Q = (p^t p^0)$ where $p$ is a real-valued function such that $\sup_{t \geq r} |\int_r^t p(s) ds| = O(e^{-δr})$ as $r \to \infty$. In this case both $N_Q$ and $H_Q$ defined by (10) are diagonal and can be calculated explicitly. Namely,

$$N_Q(t) = \begin{pmatrix} \exp(-g_0(t)) & 0 \\ 0 & \exp(g_0(t)) \end{pmatrix}, \quad H_Q(t) = \begin{pmatrix} \exp(-2g_0(t)) & 0 \\ 0 & \exp(2g_0(t)) \end{pmatrix},$$

where $g_r(t) = \int_r^t p(s) ds$. Moreover,

$$E_Q(r) = \det \int_r^{r+2} H_Q(t) dt - 4 = \int_r^{r+2} e^{-2g_0(t)} dt \cdot \int_r^{r+2} e^{2g_0(t)} dt - 4$$

(97)

The assumption on $p$ of Theorem 5 can be rewritten in the form $\sup_{t \geq r} |g_r(t)| \lesssim e^{-δr/2}$. Hence the inequalities $\sup_{t \geq r} |e^{2g_r(t)} - 2g_r(t) - 1| \lesssim e^{-δr}$ and $\sup_{t \geq r} |e^{-2g_r(t)} + 2g_r(t) - 1| \lesssim e^{-δr}$ hold and (97) gives $E_Q(r) \lesssim e^{-δr}$. This completes the proof in the off-diagonal case.

5.1.2. General situation. Recall (95) that the the entropy function can be calculated as

$$E_Q(r) = \det \int_r^{r+2} (X_{JQ}(r,t))^*X_{JQ}(r,t) dt - 4,$$

(98)

where $X_{JQ}$ is the ordered exponent of $JQ$. We claim that under the assumptions of Theorem 5 $X_{JQ}$ is close to the identity operator. Indeed, the operator $\int_r^t Q(s) ds$ is symmetric and has zero trace, hence $\left\| \int_r^t Q(s) ds \right\| = \sqrt{(\int_r^t p(s) ds)^2 + (\int_r^t q(s) ds)^2} \lesssim e^{-δr}$. Apply this norm inequality
to explicit formula (92) for the ordered exponent. We have
\[ \|X_{Q(t)}(t) - I\| = \left\| \sum_{m \geq 1} \int_r^t JQ(t_1) \int_r^{t_1} JQ(t_2) \int_r^{t_2} \cdots \int_r^{t_{m-1}} JQ(t_m) \, dt_m \cdots dt_3 \, dt_2 \, dt_1 \right\| \]
\[ \leq \sum_{m \geq 1} \int_r^t \int_r^{t_1} \cdots \int_r^{t_{m-2}} \prod_{k=1}^{m-1} \|Q(t_k)\| \cdot \int_r^{t_{m-1}} \|Q(t_m)\| \, dt_m \cdots dt_2 \, dt_1 \]
\[ \leq \sup_{s \geq r} \left\| \int_r^s Q(s) \, ds \right\| \cdot \prod_{m \geq 1} \left( f_r^{t} \|Q(t_1)\| \, dt_1 \right)^{m-1} \frac{1}{(m-1)!} \]
\[ \leq e^{-\delta r} \cdot \exp \left( \int_r^t \|Q(t_1)\| \, dt_1 \right) \leq e^{-\delta r} \cdot \exp \left( \int_r^t |p(t_1)| + |q(t_1)| \, dt_1 \right) . \]
This for \( t \in [r, r+2] \) implies \( \|X_{Q(t)}(t) - I\| \leq e^{-\delta r} \) hence \((X_{Q(t)}(t))\) \(X_{Q(t)}(t) = I + M_t(t)\), where \( M_t \) is a matrix-valued function such that \( \|M(t)\| \leq e^{-\delta r} \). To conclude the proof, substitute the latter into (98):
\[ E_Q(r) = \det \int_r^{r+2} (I + M_t(t)) \, dt - 4 = \det \left( 2I + \int_r^{r+2} M_t(t) \right) \, dt - 4 \leq e^{-\delta r} . \]

5.2. On the sharpness of results. It Theorems 3 and 1 we established inequalities between the rate of exponential decay of the entropy function \( E_Q \) and the width of the strip, where the inverse Szegö function and the Weyl function have proper extensions. In this section we give an example of a potentials for which some of the estimates in these two theorems are precise. Consider Krein system (12) with the coefficient \( a(r) = e^{-r} \). Then the potential of the corresponding Dirac operator \( D_Q \) is given by \( Q = (2e^{-2r} \begin{pmatrix} 0 & 2e^{-2r} \\ e^{-2r} & 0 \end{pmatrix} \) \. Let us show that for this \( Q \) the corresponding inverse Szegö function does not extend into \( \Omega_\delta \) for any \( \delta > \Delta_E/4 \) and, moreover, the equality \( 4\Delta_{\Pi} = \Delta_E \) holds.

The potential \( Q \) is off-diagonal hence the entropy can be calculated by formula (97). In our situation we have \( g_r(t) = \int_r^t 2e^{-2t_1} \, dt_1 = e^{-2r}(1 - e^{2(t-r)}) \). Then (97) gives \( E_Q(r) \approx e^{-4r} \), \( \Delta_E = 4 \) and, by Theorem 1, \( \Pi \) extends analytically into \( \Omega_1 \). Let us show that \( \Pi \) does not extend into \( \Omega_\delta \) for any \( \delta > 1 \). The inequality \( |P(r, \lambda)| \leq |P_\ast(r, \lambda)| \) in \( \overline{C}_+ \) and the Gronwall-Bellman inequality for differential equation (12) for \( P_\ast \) give \( |P_\ast(r, \lambda)| \approx 1 \) uniformly in \( \lambda \in \overline{C}_+ \) and \( r \geq 0 \). Furthermore, we have
\[ \left| \frac{\partial}{\partial r} P_\ast(r, \lambda) \right| = |a(r)P(r, \lambda)| \leq |a(r)e^{i\lambda r}P_\ast(r, \lambda)| = e^{-\Im \lambda r} |P_\ast(r, \lambda)| . \]

Therefore \( \|\frac{\partial}{\partial r} P_\ast(r, \lambda)\|_{L^1(\mathbb{R}_+)} \) is uniformly bounded in \( \{\lambda : 1 - \varepsilon \geq \Im \lambda \geq 0\} \) for every \( \varepsilon > 0 \). Consequently \( \Pi \) is uniformly bounded in \( \Omega_\delta \) for every \( \delta < 1 \) and \( \Delta_{\Pi} \geq 1 = \Delta_E/4 \). Theorem 3 then implies \( 4\Delta_{\Pi} = \Delta_E \).

On the other hand, for \( \lambda = i\epsilon \), the Krein system has only real-valued coefficients hence \( P_\ast(r, i\epsilon) \) is real-valued; \( P_\ast(r, i\epsilon) \) is positive for every \( r \geq 0 \) because \( P_\ast(r, i\epsilon) \neq 0 \) for every \( r \geq 0 \). Therefore
the relation
\[ \frac{\partial}{\partial r} p_r(r, -ih) = e^{-(1-h)r} p_r(r, ih) \approx e^{-(1-h)r} \]
holds uniformly for \( h \geq 0 \). It follows that
\[ |\Pi(-ih)| = \lim_{r \to \infty} |p_r(r, -ih)| \approx 1 + \int_0^\infty e^{-(1-h)r} d\rho = \frac{h}{1 - h}. \]
Therefore \( |\Pi(-ih)| \to \infty \) as \( h \to 1^- \) and \( \Pi \) cannot be analytically extended to the point \( \lambda = -i \)
and especially into any horizontal half-plane containing \(-i\).

5.3. **Resonances. Proof of Theorem 2.** Recall definition (11) of the class \( \mathcal{E} \) of potentials with super-exponentially decaying entropy. By Theorem 1, for any \( Q \in \mathcal{E} \) the corresponding inverse
Szegö function \( \Pi = \Pi_Q \) and the Weyl function \( \rho = m_Q \) extend into the whole complex plane.
We denote these extensions with the same symbols \( \Pi \) and \( m \). By the definition, the scattering
resonances of \( D_Q \) are the poles of \( m \). Let us show that they are precisely the zeroes of \( \Pi \). Indeed,
\( Q \in \mathcal{E} \) iff \( -Q \in \mathcal{E} \) hence relation (26) implies that if \( z \) is a pole of \( m_Q \) of multiplicity \( n \), then \( z \) is a zero of \( \Pi \) of multiplicity not greater than \( n \). To establish that the multiplicities are the same we need to show that \( \Pi \) and \( \Pi \) have no common zeroes, which in its turn follows from
\[ (99) \quad \Pi(\bar{z})\Pi(z) + \Pi(z)\Pi(\bar{z}) = 2, \quad z \in \mathbb{C}. \]
When \( Q \) has compact support the latter relation simply follows from (75) in [7]; in our situation we need to consider the dual regularized Krein system more carefully. The coefficients of the dual regularized Krein system are not much different from the original system of (34) and (35). Namely (see the remark after Lemma 7.3 in [2]), we need to change the sign of \( f_1 \) and leave \( f_2 \) the same.
In a matrix form two regularized systems become
\[ \frac{\partial}{\partial r} \begin{pmatrix} \tilde{p}_r^* \\ \tilde{p}_{r,d} \end{pmatrix} = \begin{pmatrix} iz f_2 & (z - i) f_1 \\ (z + i) f_1 & iz(1 - f_2) \end{pmatrix} \begin{pmatrix} \tilde{p}_r^* \\ \tilde{p}_r \end{pmatrix}, \]
\[ \frac{\partial}{\partial r} \begin{pmatrix} \tilde{p}_{r,d}^* \\ \tilde{p}_{r,d} \end{pmatrix} = \begin{pmatrix} iz f_2 & -(z - i) f_1 \\ -(z + i) f_1 & iz(1 - f_2) \end{pmatrix} \begin{pmatrix} \tilde{p}_{r,d}^* \\ \tilde{p}_{r,d} \end{pmatrix}, \]
where the functions with the subindex \( d \) form the regularized dual system. Denote the matrix in the first equality by \( V \). A simple transformation gives
\[ \frac{\partial}{\partial r} \begin{pmatrix} \tilde{p}_r^* \\ \tilde{p}_r \end{pmatrix} = V \begin{pmatrix} \tilde{p}_r^* \\ \tilde{p}_r \end{pmatrix}, \quad \frac{\partial}{\partial r} \begin{pmatrix} \tilde{p}_{r,d}^* \\ \tilde{p}_{r,d} \end{pmatrix} = V \begin{pmatrix} \tilde{p}_{r,d}^* \\ \tilde{p}_{r,d} \end{pmatrix}. \]
Hence the solution \( X \) of the \( X' = VX \) with \( X(0, z) = I \) is given by
\[ X = \frac{1}{2} \begin{pmatrix} \tilde{p}_r^* & \tilde{p}_{r,d}^* \\ \tilde{p}_r & -\tilde{p}_{r,d} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \]
We have
\[ \frac{1}{2} \left( \tilde{p}_r^*(z)\tilde{p}_{r,d}(z) + \tilde{p}_{r,d}^*(z)\tilde{p}_r(z) \right) = \det X(r, z) = \exp \left( \int_0^r \text{trace} V(t, z) \, dt \right) = \exp \left( \int_0^r iz \, dt \right) = e^{izr}. \]
By (37) the latter is equivalent to
\[ \bar{P}_r^*(z)\bar{P}_r^*(\bar{z}) + \bar{P}_r^*(z)\bar{P}_r^*(\bar{z}) = 2, \quad z \in \mathbb{C}. \]

If \( Q \in \mathcal{E} \) then from the proof of Theorem 1 it follows that \( \Pi(z) = \lim_{r \to \infty} \bar{P}_r^*(z) \) and \( \hat{\Pi}(z) = \lim_{r \to \infty} \bar{P}_r^*(z) \) for every \( z \in \mathbb{C} \). Hence (99) follows and \( \Pi(z) = \hat{\Pi}(z) = 0 \) is not possible.

Thus the resonances are the zeroes of the entire function \( \Pi \). The counting function of the zeroes is tightly connected to the order of \( \Pi \).

**Lemma 11.** If \( Q \in \mathcal{E}_\alpha \) for some \( \alpha > 1 \) then \( \Pi \) is of order not greater than \( \frac{\alpha}{\alpha - 1} \).

**Proof.** We need to show that for \( z \in \mathbb{C} \), the inequality
\[ |\Pi(z)| \lesssim \exp \left( C|z|^\frac{\alpha}{\alpha - 1} \right) \]
holds for some constant \( C \). From Lemma 6 we know the stronger bound \( \Pi(z) \lesssim \exp (C|z|) \) in \( \mathbb{C}_+ \).

The limit relation \( \Pi(z) = \lim_{r \to \infty} \bar{P}_r^*(z) \) holds for every \( z \in \mathbb{C} \) hence (50) gives
\[ |\Pi(z)| \lesssim (|z| + 1)e^{c(|z|+1)} \int_0^\infty \left( \sqrt{|K'(\rho/2)|} + |K'(\rho/2)| \right) e^{\rho|\text{Im } z|} d\rho, \quad \text{Im } z \leq 0. \]

We have \( E_Q(r) \lesssim \exp (-r^\alpha) \) hence \( K(r) \lesssim \exp (-r^\alpha) \). It follows that
\[ |\Pi(z)| \lesssim (|z| + 1)e^{c(|z|+1)} \int_0^\infty \exp \left( \rho \text{ Im } z - \rho^\alpha/2^{\alpha+1} \right) d\rho \lesssim (|z| + 1)e^{c(|z|+1)} \exp \left( c_1 |\text{Im } z|^\frac{\alpha}{\alpha - 1} \right) \lesssim \exp \left( c_2 |z|^\frac{\alpha}{\alpha - 1} \right). \]

**Proof of Theorem 2.** Let \( Q_1 \) and \( Q_2 \) be two potentials from \( \mathcal{E} \) with the same set of resonances with multiplicities \( \{z_k\}_{k=0}^\infty \). Let \( \sigma_i, \Pi_i, m_i \) and \( H_i \), for \( i = 1, 2 \) be the spectral measures, the inverse Szegő functions, the Weyl functions and the Hamiltonians corresponding to \( Q_1 \) and \( Q_2 \). Denote by \( E_n \) the elementary Weierstrass factor,
\[ E_n(z) = \begin{cases} (1 - z), & n = 0, \\ (1 - z) \exp \left( \frac{\dot{z}}{1} + \frac{\dot{z}^2}{2} + \cdots + \frac{\dot{z}^n}{n} \right), & n > 0. \end{cases} \]

By Lemma 11, both \( \Pi_1 \) and \( \Pi_2 \) are of finite exponential order; let \( n \) be an integer such that the order of \( \Pi_1 \) and \( \Pi_2 \) does not exceed \( n \). Hadamard factorization theorem, see Section 4.2 in [18], gives that
\[ \Pi_1(z) = e^{g_1(z)} \prod_{k=0}^\infty E_n \left( \frac{z}{z_k} \right) \quad \text{and} \quad \Pi_2(z) = e^{g_2(z)} \prod_{k=0}^\infty E_n \left( \frac{z}{z_k} \right), \]
where \( g_1 \) and \( g_2 \) are polynomials of degree not greater than \( n \). Hence \( \Pi_1(z)\Pi_2^{-1}(z) = e^g(z) \) with \( g = g_1 - g_2 \). By Lemma 6 there exists \( c > 0 \) such that \( |\Pi_1(z)| \lesssim e^{c|z|} \) and \( e^{-c|z|} \lesssim |\Pi_2(z)| \) for \( z \in \mathbb{C}_+. \) Consequently, \( |e^g(z)| \lesssim e^{2c|z|} \) or \( \text{Re } g(z) \lesssim |z| + 1 \) for \( z \in \mathbb{C}_+ \), which is possible only if
deg g ≤ 1. Let g(z) = a₁z + a₂, where a₁ and a₂ are some complex constants. Because of the Szegő class assumption (9) we have

\[
\frac{\text{Re } g}{x² + 1} = \frac{\log |Π₁| - \log |Π₂|}{x² + 1} \in L^1(\mathbb{R}).
\]

It follows that Re a₁ = 0 and |Π₁(x)| = |Π₂(x)|e^{Re a₂}, x ∈ ℝ. Next, by Lemma 6, both σ₁ and σ₂ are a.c. hence dσ₁ = e^{-2Re a₂}dσ₂; relation (8) then gives m₁ = am₂ + b for some constants a > 0 and b. Proof of Lemma 2.4 in [2] implies H₂(t) = A*H₁(t)A with A = \begin{pmatrix} 1/\sqrt{a} & b/\sqrt{a} \\ b/a & a+b/a \end{pmatrix}. For t = 0 it becomes I = A*A = \begin{pmatrix} 1/a & b/a \\ b/a & a+b/a \end{pmatrix}. Hence a = 1, b = 0 and the Weyl functions m₁ and m₂ coincide. Therefore Q₁ = Q₂ and the proof is finished.

□

References

[1] R. Bessonov. Szegő condition and scattering for one-dimensional Dirac operators. Constr. Approx., 51(2):273–302, 2020.
[2] R. Bessonov and S. Denisov. De Branges canonical systems with finite logarithmic integral. Anal. PDE, 14(5):1509–1556, 2021.
[3] R. Bessonov and S. Denisov. Zero sets, entropy, and pointwise asymptotics of orthogonal polynomials. J. Funct. Anal., 280(12):Paper No. 109002, 38, 2021.
[4] R. Bessonov and S. Denisov. Sobolev norms of L²-solutions to NLS. arXiv:2211.07051, 2022.
[5] R. Bessonov and S. Denisov. Szegő condition, scattering, and vibration of Krein strings. arXiv:2203.07132, 2022.
[6] D. Damanik and B. Simon. Jost functions and Jost solutions for Jacobi matrices. II. Decay and analyticity. Int. Math. Res. Not., pages Art. ID 19396, 32, 2006.
[7] S. Denisov. Continuous analogs of polynomials orthogonal on the unit circle and Krein systems. IMRS Int. Math. Res. Surv., pages Art ID 54517, 148, 2006.
[8] S. Dyatlov and M. Zworski. Mathematical theory of scattering resonances, volume 200 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2019.
[9] H. Fried. Green’s functions and ordered exponentials. Cambridge University Press, Cambridge, 2002.
[10] R. Froese. Asymptotic distribution of resonances in one dimension. J. Differential Equations, 137(2):251–272, 1997.
[11] J. Garnett. Bounded analytic functions, volume 96 of Pure and Applied Mathematics. Academic Press, New York-London, 1981.
[12] A. Iantchenko and E. Korotyaev. Resonances for 1D massless Dirac operators. J. Differential Equations, 256(8):3038–3066, 2014.
[13] A. Iantchenko and E. Korotyaev. Resonances for Dirac operators on the half-line. J. Math. Anal. Appl., 420(1):279–313, 2014.
[14] M. Klein. On the absence of resonances for Schrödinger operators with nontrapping potentials in the classical limit. Comm. Math. Phys., 106(3):485–494, 1986.
[15] M. Korey. Ideal weights: asymptotically optimal versions of doubling, absolute continuity, and bounded mean oscillation. J. Fourier Anal. Appl., 4(4-5):491–519, 1998.
[16] E. Korotyaev and D. Mokeev. Inverse resonance scattering for Dirac operators on the half-line. Anal. Math. Phys., 11(1):Paper No. 32, 26, 2021.
[17] M. Krein. Continuous analogues of propositions on polynomials orthogonal on the unit circle. Dokl. Akad. Nauk SSSR (N.S.), 105:637–640, 1955.
[18] B. Levin. Lectures on entire functions, volume 150 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1996.
[19] B. Levitan and I. Sargsjan. Sturm-Liouville and Dirac operators, volume 59 of Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the Russian.
[20] V. Matveev and M. Skriganov. Wave operators for a Schrödinger equation with rapidly oscillating potential. Dokl. Akad. Nauk SSSR, 202:755–757, 1972.
[21] P. Nevai and V. Totik. Orthogonal polynomials and their zeros. *Acta Sci. Math. (Szeged)*, 53(1-2):99–104, 1989.

[22] M. Reed and B. Simon. *Methods of modern mathematical physics. III*. Academic Press, New York-London, 1979. Scattering theory.

[23] C. Remling. *Spectral Theory of Canonical Systems*. De Gruyter, Berlin, Boston, 2018.

[24] R. Romanov. Canonical systems and de branges spaces. *arXiv:1408.6022*, 2014.

[25] I. Sasaki. Schrödinger operators with rapidly oscillating potentials. *Integral Equations Operator Theory*, 58(4):563–571, 2007.

[26] B. Simon. Resonances in one dimension and Fredholm determinants. *J. Funct. Anal.*, 178(2):396–420, 2000.

[27] B. Simon. *Orthogonal polynomials on the unit circle. Part 1*, volume 54 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2005. Classical theory.

[28] B. Simon. *Orthogonal polynomials on the unit circle. Part 2*, volume 54 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2005. Spectral theory.

[29] J. Sjöstrand. Geometric bounds on the density of resonances for semiclassical problems. *Duke Math. J.*, 60(1):1–57, 1990.

[30] M. Skriganov. The spectrum of a Schrödinger operator with rapidly oscillating potential. *Trudy Mat. Inst. Steklov.*, 125:187–195, 235, 1973. Boundary value problems of mathematical physics, 8.

[31] G. Szegő. *Orthogonal polynomials*. American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[32] A. Teplyaev. A note on the theorems of M. G. Krein and L. A. Sakhnovich on continuous analogs of orthogonal polynomials on the circle. *J. Funct. Anal.*, 226(2):257–280, 2005.

Pavel Gubkin: gubkinpavel@pdmi.ras.ru
St. Petersburg State University
Universitetskaya nab. 7-9, St. Petersburg, 199034, Russia
St. Petersburg Department of Steklov Mathematical Institute
Russian Academy of Sciences
Fontanka 27, St. Petersburg, 191023, Russia