A New Type of $\xi$-Open Sets Based on Operations

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The aim of this paper is to introduce a new type of $\xi$-open sets in topological spaces which is called $\xi_{\gamma}$-open sets and we study some of their basic properties and characteristics.

1. Introduction

Ogata [9], introduced the concept of an operation on a topology, then after authors defined some other types of sets such as $\gamma$-open [9], $\gamma$-semi-open [6], $\gamma$-pre semi-open [6] and $\gamma$-$\beta$-open [1] sets in a topological space by using operations. In [4] the concept of $\xi$-open set in a topological space is introduced and studied.

The purpose of this paper is to introduce a new class of $\xi$-open sets namely $\xi_{\gamma}$-open sets and establish basic properties and relationships with other types of sets, also we define the notions of $\xi_{\gamma}$-neighbourhood, $\xi_{\gamma}$-derived, $\xi_{\gamma}$-closure and $\xi_{\gamma}$-interior of a set and give some of their properties which are mostly analogous to those properties of open sets. Throughout this paper, $(X, \tau)$ or briefly $(X)$ mean a topological space on which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of a topological space $X$, $\text{Cl}(A)$ and $\text{Int}(A)$ are denoted respectively the closure and interior of $A$.

2. Preliminaries.

We start this section by introducing some definitions and results concerning sets and spaces which will be used later.

**Definition 2.1.** A subset $A$ of a space $(X, \tau)$ is called:
1) semi-open [7], if $A \subseteq \text{Cl}(\text{Int}(A))$.
2) regular open [2], if $A = \text{Int}(\text{Cl}(A))$.

The complement of semi-open (resp., regular open, preopen and $\alpha$-open) set is said to be semi-closed (resp., regular closed, preclosed and $\alpha$-closed).

**Definition 2.2.** [4] An open subset $U$ of a space $X$ is called $\xi$-open if for each $x \in U$, there exists a semi-closed set $F$ such that $x \in F \subseteq U$. The family of all $\xi$-open subsets of a topological space $(X, \tau)$ is denoted by $\xi O(X, \tau)$ or (briefly $\xi O(X)$). The complement of each $\xi$-open set is called $\xi$-closed set. The family of all $\xi$-closed subsets of a topological space $(X, \tau)$ is denoted by $\xi C(X, \tau)$ or (briefly $\xi C(X)$).
Definition 2.3. [5] Let \((X, \tau)\) be a topological space. An operation \(\gamma\) on the topology \(\tau\) is a mapping from \(\tau\) into power set \(P(X)\) such that \(V \subseteq \gamma(V)\) for each \(V \in \tau\), where \(\gamma(V)\) denotes the value of \(\gamma\) at \(V\).

Definition 2.4. [8]
1) A subset \(A\) of a topological space \((X, \tau)\) is called \(\gamma\)-open set if for each \(x \in A\) there exists an open set \(U\) such that \(x \in U\) and \(\gamma(U) \subseteq A\). Clearly \(\tau_\gamma \subseteq \tau\).
2) The point \(x \in X\) is in the \(\gamma\)-closure of a set \(A \subseteq X\), if \(\gamma(U) \cap A \neq \emptyset\), for each open set \(U\) containing \(x\). The \(\gamma\)-closure of a set \(A\) is denoted by \(Cl_\gamma(A)\).
3) Let \((X, \tau)\) be a topological space and \(A\) be subset of \(X\), then \(\tau_\gamma \text{-Cl}(A) = \bigcap \{F: A \subseteq F, X \setminus F \in \tau_\gamma\}\).

Definition 2.5. [11] Let \((X, \tau)\) be a topological space and \(A\) be subset of \(X\), then \(\tau_\gamma \text{-Int}(A) = \bigcup \{U: U\ is \ \gamma\text{-open set and } U \subseteq A\}\).

Definition 2.6. [1] Let \((X, \tau)\) be a topological space with an operation \(\gamma\) on \(\tau\):
1) The \(\gamma\)-derived set of \(A\) is defined by \(\{x: \ for \ every \ \gamma\text{-open set } U \ containing \ x, \ U \cap (A \setminus \{x\}) \neq \emptyset\}\).
2) The \(\gamma\)-boundary of \(A\) is defined as \(\tau_\gamma \text{-Cl}(A) \cap \tau_\gamma \text{-Cl}(X \setminus A)\).

Definition 2.7. [4] Let \((X, \tau)\) be a topological space and \(A \subseteq X\), then:
1) \(\xi\text{-interior of } A\) is the union of all \(\xi\text{-open sets contained in } A\).
2) \(\xi\text{-closure of } A\) is the intersection of all \(\xi\text{-closed sets containing } A\).

Lemma 2.8. [4]
1) Let \((Y, \tau_Y)\) be a subspace of \((X, \tau)\). If \(F \in SC(X, \tau)\) and \(F \subseteq Y\), then \(F \in SC(Y, \tau_Y)\).
2) Let \((Y, \tau_Y)\) be a subspace of \((X, \tau)\). If \(F \in SC(Y, \tau_Y)\) and \(Y \in SC(X, \tau)\), then \(F \in SC(X, \tau)\).

Lemma 2.9 [4]
1) Let \(Y\) be a regular open subspace of a space \(X\). If \(G \in \xi O(Y)\), then \(G \in \xi O(X)\).
2) Let \(Y\) be a subspace of a space \(X\) and \(Y \in SC(X)\). If \(G \in \xi O(X)\) and \(G \subseteq Y\), then \(G \in \xi O(Y)\).

3. \(\xi_\gamma\)-Open Sets

In this section, a new class of \(\xi\text{-open sets}\) called \(\xi_\gamma\text{-open sets}\) in topological spaces is introduced. We define \(\gamma\) to be a mapping on \(\xi O(X)\) into \(P(X)\) and we say that \(\gamma: \xi O(X) \rightarrow P(X)\) is an \(\xi\text{-operation}\) on \(\xi O(X)\) if \(\forall V \subseteq \gamma(V)\), for each \(V \in \xi O(X)\).

Definition 3.1 A subset \(A\) of a space \(X\) is called \(\xi_\gamma\text{-open}\) if for each point \(x \in A\), there exist an \(\xi\text{-open set } U\) such that \(x \in U \subseteq \gamma(U) \subseteq A\).

The family of all \(\xi_\gamma\text{-open subset of a topological space } (X, \tau)\) is denoted by \(\xi_\gamma O(X, \tau)\) or (briefly \(\xi_\gamma O(X)\)).

A subset \(B\) of a space \(X\) is called \(\xi_\gamma\text{-closed}\) if \(X \setminus B\) is \(\xi_\gamma\text{-open}.\) The family of all \(\xi_\gamma\text{-closed subsets of a topological space } (X, \tau)\) is denoted by \(\xi_\gamma C(X, \tau)\) or (briefly \(\xi_\gamma C(X)\)).

Remark 3.2 From the definition of the operation \(\gamma\), it is clear that \(\gamma(X) = X\) for any \(\xi\text{-operation } \gamma\). For competence, it is assumed that \(\gamma(\phi) = \phi\) for any \(\xi\text{-operation } \gamma\).
Remark 3.3 It is clear from the definition that every $\xi_\gamma$-open subset of a space $X$ is $\xi$-open, but the converse is not true in general as shown in the following example:

Example 3.5. Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$. Define an $\xi$-operation $\gamma$ by

$$\gamma(A) = \begin{cases} A & \text{if } a \in A \\ X & \text{if } a \notin A \end{cases}$$

Then $\{c\}$ is open and $\xi_\gamma$-open but $\{c\} \notin \xi_\gamma O(X)$.

Proposition 3.6. Every $\xi_\gamma$-open set of a space $X$ is $\gamma$-open.

Proof. Let $A$ be $\xi_\gamma$-open in a topological space $(X, \tau)$, then for each point $x \in A$, there exists an $\xi$-open set $U$ such that $x \in U \subseteq \gamma(U) \subseteq A$. Since every $\xi$-open set is open, this implies that $A$ is a $\gamma$-open set.

The following example shows that the converse of the above proposition is not true in general.

Example 3.7 Consider $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a\}\}$. Define an $\xi$-operation $\gamma$ by $\gamma(A) = A$, for any subset $A$ of $X$. Then, $\{a\}$ is $\gamma$-open set but not $\xi_\gamma$-open set. Hence, it is not $\xi_\gamma O(X)$.

The following result shows that any union of $\xi_\gamma$-open sets in a topological space $(X, \tau)$ is $\xi_\gamma$-open.

Proposition 3.8 Let $\{A_\lambda\}_{\lambda \in \Delta}$ be a collection of $\xi_\gamma$-open sets in a topological space $(X, \tau)$. Then, $\bigcup_{\lambda \in \Delta} A_\lambda$ is $\xi_\gamma$-open.

Proof. Let $x \in \bigcup_{\lambda \in \Delta} A_\lambda$, then $x \in A_\lambda$ for some $\lambda \in \Delta$. Since, $A_\lambda$ is an $\xi_\gamma$-open set, then there exists an $\xi_\gamma$-open set $U$ containing $x$ and $\gamma(U) \subseteq A_\lambda \subseteq \bigcup_{\lambda \in \Delta} A_\lambda$. Therefore, $\bigcup_{\lambda \in \Delta} A_\lambda$ is an $\xi_\gamma$-open set in a topological space $(X, \tau)$.

The following example shows that the intersection of two $\xi_\gamma$-open sets need not be an $\xi_\gamma$-open set.

Example 3.9 Consider $X = \{a, b, c\}$ with discrete topology on $X$. Define an $\xi$-operation $\gamma$ by

$$\gamma(A) = \begin{cases} \{a, b\} & \text{if } A = \{a\} \text{ or } \{b\} \\ A & \text{otherwise} \end{cases}$$

Let $A = \{a, b\}$ and $B = \{b, c\}$, it is clear that $A$ and $B$ are $\xi_\gamma$-open sets, but $A \cap B = \{b\}$ is not $\xi_\gamma$-open set.

From the above example, we notice that the family of all $\xi_\gamma$-open subsets of a space $X$ is a supratopology and need not be a topology in general.

Proposition 3.10 The set $A$ is $\xi_\gamma$-open in the space $(X, \tau)$ if and only if for each $x \in A$, there exists an $\xi$-open set $B$ such that $x \in B \subseteq A$. 

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**Proposition 3.14** Let \((X, \tau)\) be a topological space. A mapping \(\gamma : \xi O(X) \rightarrow P(X)\) is said to be:

1) \(\xi\)-identity on \(\xi O(X)\) if \(\gamma(A) = A\) for all \(A \in \xi O(X)\).
2) \(\xi\)-monotone on \(\xi O(X)\) if for all \(A, B \in \xi O(X)\), \(A \subseteq B \implies \gamma(A) \subseteq \gamma(B)\).
3) \(\xi\)-additive on \(\xi O(X)\) if \(\gamma(\bigcup_{i \in I} A_i) = \gamma(A_1) \cup \ldots \cup \gamma(A_i)\).
4) \(\xi\)-open on \(\xi O(X)\) if \(\gamma(A \cup B) = \gamma(A) \cup \gamma(B)\) for all \(A, B \in \xi O(X)\).

Conversely, suppose that for each \(x \in A\), there exists an \(\xi\)-open set \(B_x\) such that \(x \in B_x \subseteq A\), thus \(A = \bigcup_{x \in A} B_x \in \xi \gamma O(X)\) for each \(x \in A\). Therefore, \(A\) is an \(\xi\)-open set.

**Definition 3.11** Let \((X, \tau)\) be a topological space. A mapping \(\gamma : \xi O(X) \rightarrow P(X)\) is said to be:

1) \(\xi\)-identity on \(\xi O(X)\) if \(\gamma(A) = A\) for all \(A \in \xi O(X)\).
2) \(\xi\)-monotone on \(\xi O(X)\) if for all \(A, B \in \xi O(X)\), \(A \subseteq B \implies \gamma(A) \subseteq \gamma(B)\).
3) \(\xi\)-additive on \(\xi O(X)\) if \(\gamma(\bigcup_{i \in I} A_i) = \gamma(A_1) \cup \ldots \cup \gamma(A_i)\).
4) \(\xi\)-open on \(\xi O(X)\) if \(\gamma(A \cup B) = \gamma(A) \cup \gamma(B)\) for all \(A, B \in \xi O(X)\).

If \(\bigcup_{i \in I} \gamma(A_i) \subseteq \gamma(\bigcup_{i \in I} A_i)\) for any collection \(\{A_i\}_{i \in I} \subseteq \xi O(X)\), then \(\gamma\) is said to be \(\xi\)-subadditive on \(\xi O(X)\).

**Proposition 3.12.** Let \(\gamma\) be an \(\xi\)-operation. Then, \(\gamma\) is \(\xi\)-monotone on \(\xi O(X)\) if and only if \(\gamma\) is subadditive on \(\xi O(X)\).

**Proof.** Let \(\gamma\) be \(\xi\)-monotone on \(\xi O(X)\) and let \(\{A_i\}_{i \in I} \subseteq \xi O(X)\). Then, for each \(i \in I\), \(\gamma(A_i) \subseteq \gamma(\bigcup_{i \in I} A_i)\), and thus \(\bigcup_{i \in I} \gamma(A_i) \subseteq \gamma(\bigcup_{i \in I} A_i)\). Therefore, \(\gamma\) is \(\xi\)-subadditive on \(\xi O(X)\).

Conversely, if \(\gamma\) is subadditive on \(\xi O(X)\), and \(A, B \subseteq \xi O(X)\) with \(A \subseteq B\), then \(\gamma(\bigcup_{i \in I} A_i) \subseteq \gamma(A) \cup \gamma(B)\). Hence, \(\gamma\) is \(\xi\)-monotone on \(\xi O(X)\).

The following result shows that if \(\gamma\) is \(\xi\)-operation, then the family of \(\xi\)-open sets is a topology on \(X\).

**Proposition 3.13** If \(\gamma\) is \(\xi\)-monotone, then the family of \(\xi\)-open sets is a topology on \(X\).

**Proof.** Clearly \(\phi, X \in \xi \gamma O(X)\) and by Proposition 3.8, the union of any family \(\xi\)-open sets is \(\xi\)-open set. To complete the proof, it is enough to show that the finite intersection of \(\xi\)-open sets is an \(\xi\)-open set. Let \(A\) and \(B\) be two \(\xi\)-open sets and let \(x \in A \cap B\), then \(x \in A\) and \(x \in B\), so there exists \(\xi\)-open sets namely \(U\) and \(V\) such that \(x \in U \subseteq \gamma(U) \subseteq A\) and \(x \in V \subseteq \gamma(V) \subseteq B\). Since \(U\) and \(V\) are \(\xi\)-open sets then \(U \cap V\) is \(\xi\)-open set, but \(\gamma(U \cap V) \subseteq \gamma(U) \cap \gamma(V) \subseteq A \cap B\). Thus, \(A \cap B\) is an \(\xi\)-open set. This completes the proof.

**Proposition 3.14** Let \(Y\) be a semi-closed subspace of a space \(X\). If \(A \in \xi \gamma O(X, \tau)\) and \(A \subseteq Y\), then \(A \in \xi \gamma O(Y, \tau_Y)\), where \(\gamma\) is \(\xi\)-identity on \(\xi O(Y)\).

**Proof.** Let \(A \in \xi \gamma O(X, \tau)\), then \(A \in \xi \gamma O(X, \tau)\) and for each \(x \in A\) there exists an \(\xi\)-open set \(U\) in \(X\) such that \(x \in U \subseteq \gamma(U) \subseteq A\). Since, \(A \in \xi \gamma O(X, \tau)\) and \(A \subseteq Y\), where \(Y\) is semi-closed in \(X\), then by Proposition 2.14, \(U \in \xi \gamma O(Y, \tau_Y)\). Hence, \(A \in \xi \gamma O(Y, \tau_Y)\).

**Proposition 3.15** Let \(Y\) be a regular open subspace of a space \((X, \tau)\) and \(\gamma\) is an \(\xi\)-identity on \(\xi O(X)\). If \(A \in \xi \gamma O(Y, \tau_Y)\) and \(Y \in \xi O(X, \tau)\), then \(A \in \xi \gamma O(X, \tau)\).

**Proof.** Let \(A \in \xi \gamma O(Y, \tau_Y)\), then \(A \in \xi O(Y, \tau_Y)\) and for each \(x \in A\) there exists an \(\xi\)-open set \(U\) in \(Y\) such that \(x \in U \subseteq \gamma'(U) \subseteq A\). Since, \(Y \in \xi O(X, \tau)\) and \(A \in \xi O(Y, \tau_Y)\), then by Proposition 2.13, \(U \in \xi O(X, \tau)\). Hence, \(A \in \xi \gamma O(X, \tau)\).
4. Other Properties of $\xi$-$\gamma$-Open Sets

In this section, we define and study some properties of $\xi$-$\gamma$-neighbourhood of a point, $\xi$-$\gamma$-derived, $\xi$-$\gamma$-closure and $\xi$-$\gamma$-interior of sets via $\xi$-$\gamma$-open sets.

**Definition 4.1** Let $(X, \tau)$ be a topological space and $x \in X$, then a subset $N$ of $X$ is said to be $\xi$-$\gamma$-neighbourhood of $x$, if there exists an $\xi$-$\gamma$-open set $U$ in $X$ such that $x \in U \subseteq N$.

**Proposition 4.2** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is $\xi$-$\gamma$-open if and only if it is an $\xi$-$\gamma$-neighbourhood of each of its points.

**Proof.** Let $A \subseteq X$ be an $\xi$-$\gamma$-open set. Since, for every $x \in A$, $x \in A \subseteq A$ and $A$ is $\xi$-$\gamma$-open, then $A$ is an $\xi$-$\gamma$-neighbourhood of each of its points. Conversely, suppose that $A$ is an $\xi$-$\gamma$-neighbourhood of each of its points. Then, for each $x \in A$, there exists $B_x \in \xi$-$\gamma$-$O(X)$ such that $B_x \subseteq A$. Then, $A = \bigcup \{ B_x : x \in A \}$. Since, each $B_x$ is $\xi$-$\gamma$-open, it follows that $A$ is an $\xi$-$\gamma$-open set.

**Definition 4.3** Let $(X, \tau)$ be a topological space with an operation $\gamma$ on $\xi$-$O(X)$. A point $x \in X$ is said to be $\xi$-$\gamma$-limit point of a set $A$ if for each $\xi$-$\gamma$-open set $U$ containing $x$, then $U \cap (A \setminus \{x\}) \neq \emptyset$. The set of all $\xi$-$\gamma$-limit points of $A$ is called $\xi$-$\gamma$-derived set of $A$ and denoted by $\xi$-$\gamma$-$D(A)$.

**Proposition 4.5** Let $A$ and $B$ be subsets of a space $X$. If $A \subseteq B$, then $\xi$-$\gamma$-$D(A) \subseteq \xi$-$\gamma$-$D(B)$.

**Proof.** Obvious.

Some properties of $\xi$-$\gamma$-derived sets are stated in the following proposition.

**Proposition 4.6** Let $A$ and $B$ be any two subsets of a space $X$, and $\gamma$ be an operation on $\xi$-$O(X)$. Then, we have the following properties:

1) $\xi$-$\gamma$-$D(\emptyset) = \emptyset$.
2) If $x \in \xi$-$\gamma$-$D(A)$, then $x \in \xi$-$\gamma$-$D(A \setminus \{x\})$.
3) $\xi$-$\gamma$-$D(A) \cup \xi$-$\gamma$-$D(B) \subseteq \xi$-$\gamma$-$D(A \cup B)$.
4) $\xi$-$\gamma$-$D(A \cap B) \subseteq \xi$-$\gamma$-$D(A) \cap \xi$-$\gamma$-$D(B)$.
5) $\xi$-$\gamma$-$D(\xi$-$\gamma$-$D(A)) \setminus A \subseteq \xi$-$\gamma$-$D(A)$.
6) $\xi$-$\gamma$-$D(A \cup \xi$-$\gamma$-$D(A)) \subseteq A \cup \xi$-$\gamma$-$D(A)$.

**Proof.** Straightforward.

In general, the equalities of (3), (4) and (6) of the above proposition do not hold, as is shown in the following examples.

**Example 4.7** Consider $X = \{a, b, c\}$ with discrete topology on $X$. Define an operation $\gamma$ on $\xi$-$O(X)$ by

$$\gamma(A) = \begin{cases} A & \text{if } A = \{b\} \text{ or } \{a, b\} \text{ or } \{a, c\} \\ X & \text{otherwise} \end{cases}$$

Now, if $A = \{a, b\}$ and $B = \{a, c\}$, then $\xi$-$\gamma$-$D(A) = \{c\}$, $\xi$-$\gamma$-$D(B) = \{c\}$ and $\xi$-$\gamma$-$D(A \cup B) = \{a, c\}$, where $A \cup B = X$, this implies that $\xi$-$\gamma$-$D(A) \cup \xi$-$\gamma$-$D(B) \neq \xi$-$\gamma$-$D(A \cup B)$.

**Example 4.8** Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define an operation $\gamma$ on $\xi$-$O(X)$ by.
Now, if we let \( A = \{a, b\} \) and \( B = \{c, d\} \), then \( \xi D(A) = \{a, c, d\} \), \( \xi D(B) = \{d\} \), hence \( \xi D(A) \cap \xi D(B) = \{d\} \), but \( \xi D(A \cap B) = \emptyset \), where \( A \cap B = \emptyset \), this implies that \( \xi D(A \cap B) \neq \xi D(A) \cap \xi D(B) \). Also \( \xi D(A) = \{d\} \), therefore \( \xi D(A) \not\subseteq \xi D(A) \).

**Definition 4.9** Let \( A \) be a subset of a topological space \((X, \tau)\) and \( \gamma \) be an operation on \( \mathcal{O}(X) \). The intersection of all \( \xi \gamma \)-closed sets containing \( A \) is called the \( \xi \gamma \)-closure of \( A \) and denoted by \( \xi \gamma Cl(A) \).

Here, we introduce some properties of \( \xi \gamma \)-closure of the sets.

**Proposition 4.10** Let \((X, \tau)\) be a topological space and \( \gamma \) be an operation on \( \mathcal{O}(X) \). For any subsets \( A \) and \( B \) of \( X \), we have the following:

1) \( A \subseteq \xi \gamma Cl(A) \).
2) \( \xi \gamma Cl(A) \) is an \( \xi \gamma \)-closed set in \( X \).
3) \( A \) is an \( \xi \gamma \)-closed set if and only if \( A=\xi \gamma Cl(A) \).
4) \( \xi \gamma Cl(\emptyset) = \emptyset \) and \( \xi \gamma Cl(X) = X \).
5) \( \xi \gamma Cl(A) \cup \xi \gamma Cl(B) \subseteq \xi \gamma Cl(A \cup B) \).
6) \( \xi \gamma Cl(A \cap B) \subseteq \xi \gamma Cl(A) \cap \xi \gamma Cl(B) \).

**Proof.** They are obvious.

In general, the equalities of (5) and (6) of the above proposition does not hold, as is shown in the following examples:

**Example 4.11** Consider \( X = \{a, b, c\} \) with discrete topology on \( X \). Define an operation \( \gamma \) on \( \mathcal{O}(X) \) by

\[
\gamma (A) = \begin{cases} 
  A & \text{ if } A=\{a,b\} \text{ or } \{a,c\} \\
  X & \text{ otherwise}
\end{cases}
\]

Then, \( \xi \gamma O(X) = \{\emptyset, X, \{a, b\}, \{a, c\}\} \). Now, if we let \( A = \{b\} \) and \( B = \{c\} \), then \( \xi \gamma Cl(A) = A \), \( \xi \gamma D(B) = B \) and \( \xi \gamma Cl(A \cup B) = X \), where \( A \cup B = \{b, c\} \), this implies that \( \xi \gamma Cl(A \cup B) \neq \xi \gamma Cl(A \cup B) \).

**Example 4.12** Consider \( X = \{a, b, c, d\} \) with the topology \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \). Define an operation \( \gamma \) on \( \mathcal{O}(X) \) by

\[
\gamma (A) = \begin{cases} 
  A & \text{ if } b \in A \\
  X & \text{ otherwise}
\end{cases}
\]

It is clear that \( \xi \gamma O(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}\} \). Now, if we let \( A = \{c\} \) and \( B = \{d\} \), then \( \xi \gamma Cl(A) = \{c, d\} \) and \( \xi \gamma Cl(B) = \{d\} \), hence \( \xi \gamma Cl(A) \cap \xi \gamma Cl(B) = \{d\} \), but \( \xi \gamma Cl(A \cap B) = \emptyset \), where \( A \cap B = \emptyset \), this implies that \( \xi \gamma Cl(A \cap B) = \xi \gamma Cl(A \cap B) \).

Now, if we let \( A = \{b\} \), we see that \( \xi Cl(A) = \{b, d\} \), but \( \xi \gamma Cl(A) = X \). Hence, \( \xi \gamma Cl(A) \not\subseteq \xi Cl(A) \).
Proposition 4.13 A subset $A$ of a topological space $X$ is an $\xi_r$-closed set if and only if it contains the set of its $\xi_r$-limit points.

Proof. Assume that $A$ is an $\xi_r$-closed set and if possible that $x$ is an $\xi_r$-limit point of $A$ which belongs to $X \setminus A$, then $X \setminus A$ is an $\xi_r$-open set containing the $\xi_r$-limit point of $A$, therefore, $A \cap (X \setminus A) \neq \emptyset$, which is contradiction.

Conversely, assume that $A$ is containing the set of its $\xi_r$-limit points. For each $x \in X \setminus A$, there exists an $\xi_r$-open set $U$ containing $x$ such that $A \cap U = \emptyset$, implies that $x \in U \subseteq X \setminus A$, so by Proposition 3.10, $X \setminus A$ is an $\xi_r$-open set hence, $A$ is an $\xi_r$-closed set.

Proposition 4.14 Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma$ be an $\xi$-operation. Then, $x \in \xi_\gamma Cl(A)$ if and only if for every $\xi_\gamma$-open set $V$ of $X$ containing $x$, $A \cap V \neq \emptyset$.

Proof. Let $x \in \xi_\gamma Cl(A)$ and suppose that $A \cap V = \emptyset$, for some $\xi_\gamma$-open set $V$ of $X$ containing $x$. Then, $(X \setminus V)$ is $\xi_\gamma$-closed and $A \subseteq (X \setminus V)$, thus $\xi_\gamma Cl(A) \subseteq (X \setminus V)$. But, this implies that $x \in (X \setminus V)$ which is contradiction. Therefore, $A \cap V \neq \emptyset$.

Conversely, Let $A \subseteq X$ and $x \in X$ such that for each $\xi_\gamma$-open set $V$ of $X$ containing $x$, $A \cap V \neq \emptyset$. If $x \notin \xi_\gamma Cl(A)$, then there exists an $\xi_\gamma$-closed set $F$ such that $A \subseteq F$. Then, $(X \setminus F)$ is an $\xi_\gamma$-open set with $x \in (X \setminus F)$, and thus $(X \setminus F) \cap A \neq \emptyset$, which is a contradiction.

The proof of the following two results is obvious.

Proposition 4.15 Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma$ be an $\xi$-operation on $\xi O(X)$. Then, $\xi_\gamma Cl(A) = A \cup \xi_\gamma D(A)$.

Proposition 4.16 If $A$ and $B$ are subsets of a space $X$ with $A \subseteq B$. Then, $\xi_\gamma Cl(A) \subseteq \xi_\gamma Cl(B)$.

Definition 4.17 Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma$ be an operation on $\xi O(X)$. The union of all $\xi_\gamma$-open sets contained in $A$ is called the $\xi_\gamma$-Interior of $A$ and denoted by $\xi_\gamma Int(A)$.

Here, we introduce some properties of $\xi_\gamma$-Interior of the sets.

Proposition 4.18 Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\xi O(X)$. For any subsets $A$ and $B$ of $X$, we have the following:
1) $\xi_\gamma Int(A)$ is an $\xi_\gamma$-open set in $X$.
2) $A$ is $\xi_\gamma$-open if and only if $A = \xi_\gamma Int(A)$.
3) $\xi_\gamma Int(\xi_\gamma Int(A)) = \xi_\gamma Int(A)$.
4) $\xi_\gamma Int(\emptyset) = \emptyset$ and $\xi_\gamma Int(X) = X$.
5) $\xi_\gamma Int(A) \subseteq A$.
6) If $A \subseteq B$, then $\xi_\gamma Int(A) \subseteq \xi_\gamma Int(B)$.
7) $\xi_\gamma Int(A) \cup \xi_\gamma Int(B) \subseteq \xi_\gamma Int(A \cup B)$.
8) $\xi_\gamma Int(A \cap B) \subseteq \xi_\gamma Int(A \cap \xi_\gamma Int(B)$.

Proof. Straightforward.

In general, the equalities of (7) and (8) of the above proposition do not hold, as is shown in the following examples:

Example 4.19 Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define an $\xi$-operation $\gamma$ by.

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It is clear that $\xi_\gamma^\ast O(X) = \{\phi, X, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Now, if we let $A = \{a\}$ and $B = \{b\}$, then $\xi_\gamma^\ast \text{Int}(A) = \phi$ and $\xi_\gamma^\ast \text{Int}(B) = \{b\}$, hence $\xi_\gamma^\ast \text{Int}(A) \cup \xi_\gamma^\ast \text{Int}(B) = \{b\}$, but $\xi_\gamma^\ast \text{Int}(A \cup B) = \{a, b\}$, where $A \cup B = \{a, b\}$, this implies that $\xi_\gamma^\ast \text{Int}(A \cup B) \neq \xi_\gamma^\ast \text{Int}(A) \cup \xi_\gamma^\ast \text{Int}(B)$.

**Example 4.20** Consider $X = \{a, b, c\}$ with discrete topology on $X$. Define an $\xi$-operation $\gamma$ on $\xi O(X)$ by

$$
\gamma(A) = \begin{cases} 
A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\
X & \text{otherwise}
\end{cases}
$$

Then, $\xi\gamma^\ast O(X) = \{\phi, X, \{a, b\}, \{a, c\}\}$. Now, if we let $A = \{a, b\}$ and $B = \{a, c\}$, then $\xi_\gamma^\ast \text{Int}(A) = \{a, b\}$ and $\xi_\gamma^\ast \text{Int}(B) = \{a, c\}$, therefore $\xi_\gamma^\ast \text{Int}(A) \cap \xi_\gamma^\ast \text{Int}(B) = \{a\}$, but $\xi_\gamma^\ast \text{Int}(A \cap B) = \phi$, where $A \cap B = \{a\}$, this implies that $\xi_\gamma^\ast \text{Int}(A) \cap \xi_\gamma^\ast \text{Int}(B) \neq \xi_\gamma^\ast \text{Int}(A \cap B)$.

The following two results can be easily proved.

**Proposition 4.21** For any subset $A$ of a topological space $X$, $\xi_\gamma^\ast \text{Int}(A) \subseteq \xi \text{Int}(A) \subseteq \text{Int}(A)$.

**Proposition 4.22** Let $A$ be any subset of a topological space $X$, and $\gamma$ be an operation on $\xi O(X)$. Then, $\xi_\gamma^\ast \text{Int}(A) = A \setminus \xi_\gamma^\ast D(X \setminus A)$. 
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