DEDUCING THE POSITIVE ODD DENSITY OF $p(n)$ FROM THAT OF A MULTIPARTITION FUNCTION: AN UNCONDITIONAL PROOF

FABRIZIO ZANELLO

Abstract. A famous conjecture of Parkin-Shanks predicts that $p(n)$ is odd with density $1/2$. Despite the remarkable amount of work of the last several decades, however, even showing this density is positive seems out of reach. In a 2018 paper with Judge, we introduced a different approach and conjectured the “striking” fact that, if for any $A \equiv \pm 1 \pmod{6}$ the multipartition function $p_A(n)$ has positive odd density, then so does $p(n)$. Similarly, the positive odd density of any $p_A(n)$ with $A \equiv 3 \pmod{6}$ would imply that of $p_3(n)$.

Our conjecture was shown to be a corollary of an earlier conjecture of the same paper. In this brief note, we provide an unconditional proof of it. An important tool will be Chen’s recent breakthrough on a special case of our earlier conjecture.

One of the most fascinating and intractable problems in partition theory is study of the parity of the partition function $p(n)$. In particular, a classical conjecture of Parkin-Shanks \cite{6} predicts that $p(n)$ has odd density $1/2$ (see also \cite{2}). Despite a large amount of literature devoted to this problem, however, even showing that the odd density of $p(n)$ exists and is positive still appears out of reach. The best result currently available only guarantees that the number of odd values of $p(n)$ for $n \leq x$ has order at least $\frac{\sqrt{x}}{\log \log x}$, for $x$ large \cite{1}. This bound can be extended, for any $t \geq 1$, to the $t$-multipartition function $p_t(n)$, defined by

$$\sum_{n=0}^{\infty} p_t(n)q^n = \frac{1}{\prod_{i=1}^{\infty} (1-q^i)^t}.$$  

Note that $p_1(n) = p(n)$.

Recall that the odd density (or density of the odd values) of $p_t(n)$ is

$$\delta_t = \lim_{x \to \infty} \frac{\# \{n \leq x : p_t(n) \text{ is odd} \}}{x},$$

if this limit exists. In particular, $\delta_1$ is the odd density of $p(n)$, while $\delta_3$ is the odd density of the cubic partition function. Equivalently, assuming they both exist, $\delta_3$ is precisely 8 times

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\end{itemize}
the odd density of the Fourier coefficients of the Klein \(j\)-function \[7\]. For obvious parity reasons, since \(\delta_{2c-m} = \delta_m/2^c\), it suffices to restrict our attention to \(\delta_t\) for \(t\) odd.

In \[4\], joint with Judge and Keith, we conjectured that \(\delta_t = 1/2\) for all odd values of \(t\). Unfortunately, similarly to the case \(t = 1\), proving that \(\delta_t\) exists and is positive appears extremely difficult, for any \(t\).

In \[5\], joint with Judge, we introduced a new approach to the study of the parity of \(p(n)\). In Conjecture 2.4, we predicted the existence of a doubly-indexed, infinite family of identities modulo 2, which suitably related \(p(n)\) to the other multipartition functions. The main conjecture of \[5\], itself a corollary to 2.4, is the “striking” fact that, under reasonable existence assumptions, the positive odd density of \(p(n)\) follows from that of \(p_A(n)\), for any \(A \equiv \pm 1\) (mod 6); similarly, if \(\delta_A > 0\) for any \(A \equiv 3\) (mod 6), then \(\delta_3 > 0\).

Even though \[5\], Conjecture 2.4 is in general still open, the goal of this note is to establish its corollary unconditionally. A key ingredient will be a recent breakthrough by Chen \[3\], who proved an important infinite case of Conjecture 2.4.

We begin with the statement of 2.4. (\[5\], Conjecture 2.3 corresponds to the special case \(t = 1\) and will not be restated here.) Unless otherwise noted, all congruences are modulo 2.

**Conjecture 1** (\[5\], Conjecture 2.4). Fix odd positive integers \(a\) and \(t\), where \(t\equiv 3\) (mod 6) if \(a\equiv 3\) (mod 6). Let \(k = \left\lceil \frac{t(a^2-1)}{24a} \right\rceil\). Then

\[
q^k \sum_{n=0}^{\infty} p_t(an+b)q^n \equiv \sum_{d|a} \sum_{j=0}^{\lfloor k/d \rfloor} \epsilon_{a,d,j}^t q^{dj} \prod_{i \geq 1} (1-q^{di}) \frac{q^{i-24j}}{q^{24j}},
\]

where

\[
b = \begin{cases} 
0, & \text{if } a = 1; \\
\frac{t}{3} \cdot 8^{-1} \text{ (mod } a), & \text{if } t \equiv 3 \text{ (mod 6)}; \\
t \cdot 24^{-1} \text{ (mod } a), & \text{otherwise},
\end{cases}
\]

for a suitable choice of the \(\epsilon_{a,d,j}^t \in \{0,1\}\), with \(\epsilon_{a,1,0}^t = 1\) and \(\epsilon_{a,d,0}^t = 0\) if \(at/d - 24j < 0\).

The following is Chen’s result from \[3\].

**Lemma 2** (\[3\], Theorem 1.6). **Conjecture 7 holds for:**

i) \(a = p^\alpha\), with \(p \geq 5\) prime and \(\alpha \geq 1\), and any \(t \geq 1\) odd;
ii) \(a = 3\) and any \(t \geq 3\), \(t \equiv 3\) (mod 6).

The main conjecture of \[5\] can be stated as follows.

**Conjecture 3** (\[5\], Corollaries to Conjectures 2.3 and 2.4).
i) Suppose there exists an integer \( A \equiv \pm 1 \pmod{6} \) such that \( \delta_A > 0 \), and assume \( \delta_i \) exists for all \( i \leq A, i \equiv \pm 1 \pmod{6} \). Then \( \delta_1 > 0 \).

ii) Suppose there exists an integer \( A \equiv 3 \pmod{6} \) such that \( \delta_A > 0 \), and assume \( \delta_i \) exists for all \( i \leq A, i \equiv 3 \pmod{6} \). Then \( \delta_3 > 0 \).

We only remark here that, while Conjecture 1 is stronger than Conjecture 3, showing Conjecture 1 for specific values of \( a \) does not in general imply Conjecture 3 for the same values of \( A \). However, using Lemma 2 for \( a \) prime along with a careful inductive argument, we can now provide an unconditional proof of Conjecture 3 for all \( A \).

**Theorem 4.** Conjecture 3 is true.

**Proof.** We start with i), and proceed by induction on \( A \geq 5 \). The case \( A = 5 \) is already known (see [4], Theorem 2). Thus, assume the result holds up to \( A - 2 \) (or \( A - 4 \), depending on the value of \( A \) modulo 3), and let \( \delta_A > 0 \).

Let \( p \) be any prime dividing \( A \). Since \( p \geq 5 \), Lemma 2 i) guarantees that Conjecture 1 has a solution for \( a = p \) and \( t = A/p \). That is, we have an identity modulo 2 of the form

\[
q^k \sum_{n=0}^{\infty} p_{A/p}(pn + b)q^n \equiv \sum_{d=1,p} \sum_{j=0}^{[k/d]} \epsilon_{d,j} q^{dj} \prod_{i \geq 1} (1 - q^{di})^{\frac{d}{2} - 2A_j},
\]

for \( b \) and \( k \) determined by \( p \) and \( A \) as from the statement of Conjecture 1, all \( \epsilon_{d,j} = 0 \) or 1, and \( \epsilon_{1,0} = 1 \).

Notice that the condition \( \epsilon_{1,0} = 1 \) implies the existence of the summand

\[
\frac{1}{\prod_{i \geq 1} (1 - q^i)^A}
\]
on the right side of (2). Further, all other nonzero summands on the right side are easily seen to be of the form

\[
\frac{q^{dj}}{\prod_{i \geq 1} (1 - q^{di})^B},
\]

where \( 0 < B < A, B \equiv \pm 1 \pmod{6} \).

Hence, if any corresponding \( \delta_B > 0 \), by induction we are done. Otherwise, assume all \( \delta_B = 0 \), and as a consequence, note that the number of odd coefficients up to degree \( x \) on the right side of (2) is given by

\[
\delta_A \cdot x + o(x),
\]

for \( x \) large. Therefore, the odd coefficients on the right side of (2) have density \( \delta_A \), and the same obviously holds true for the left side.
Since \( p_{A/p}(pn + b) \) denotes the \((A/p)\)-multipartition function along the arithmetic progression \( pn + b \), it is clear that if this latter has odd density \( \delta_A \), then the multipartition function \( p_{A/p}(n) \) itself has odd density
\[
\delta_{A/p} \geq \frac{1}{p} \cdot \delta_A > 0.
\]
Since \( A/p < A, A/p \equiv \pm 1 \pmod{6} \), the inductive hypothesis again gives \( \delta_1 > 0 \), as desired.

The proof of part ii) is similar, so we will only sketch it. The base case to run the induction on \( A \equiv 3 \pmod{6} \) is that \( \delta_9 > 0 \) implies \( \delta_3 > 0 \), under the usual existence assumptions on the \( \delta_i \). This was again shown in [4], Theorem 2.

Next suppose that \( \delta_A > 0 \), and let \( p \) be the largest prime that divides \( A \). Set \( a = p \) and \( t = A/p \). Note that, here, \( p = 3 \) precisely when \( A \geq 9 \) is a power of 3, so we always have \( t \equiv 3 \pmod{6} \). Therefore, using both parts of Lemma 2, we are again guaranteed the existence of an identity of the form (2).

Now notice that, on the right side of (2), the exponent
\[
\frac{A}{d} - 24j
\]
is always positive, congruent to 3 \((\pmod{6})\), and except when \( d = 1 \) and \( j = 0 \), it is smaller than \( A \). The rest of the argument to show that \( \delta_3 > 0 \) is identical to part i). This completes the proof of the theorem.

**Remark 5.** Recall that, even though Conjecture 3 has been proven, Conjecture 1 remains open for most values of \( a \). We believe these identities modulo 2 to be of significant independent interest, so we hope Chen’s result from [3] (most likely combined with a new idea) can be extended to show Conjecture 1 in full.

Finally, despite multiple attempts, we have not been able to relate the conjectural positive density of \( p(n) \) to that of \( p_3(n) \). One issue in this sense is that no identity modulo 2, similar to those of Conjecture 1 but simultaneously involving \( p(n) \) and \( p_3(n) \), appears to exist. Thanks to Theorem 4, establishing that \( \delta_3 > 0 \) implies \( \delta_1 > 0 \) would immediately give us, under standard existence assumptions on the \( \delta_i \), that \( p(n) \) has positive odd density whenever any multipartition function \( p_A(n) \) does.

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DEPARTMENT OF MATHEMATICAL SCIENCES, MICHIGAN TECH, HOUGHTON, MI 49931-1295

Email address: zanello@mtu.edu