Regge calculus: a unique tool for numerical relativity

Adrian P. Gentle

Theoretical Division (T-6, MS B288), Los Alamos National Laboratory, Los Alamos, NM 87545, USA

The application of Regge calculus, a lattice formulation of general relativity, is reviewed in the context of numerical relativity. Particular emphasis is placed on problems of current computational interest, and the strengths and weaknesses of the lattice approach are highlighted. Several new and illustrative applications are presented, including initial data for the head on collision of two black holes, and the time evolution of vacuum axisymmetric Brill waves.

I. NUMERICAL RELATIVITY

The complexity of the Einstein equations, combined with the sparsity of relevant analytic solutions, necessitates a range of other tools with which to explore complex physical scenarios. These include series expansions and perturbation techniques, together with numerical solutions of the fully non-linear equations. Unfortunately, the numerical solution of Einstein’s equations has proved to be an exceedingly difficult problem. Over the last three decades numerous schemes have been developed, or adapted, to tackle the vast range of problems which fall within the purview of numerical relativity.

The classic approach, involving a three-plus-one dimensional split of space and time, was first expounded by Arnowitt, Deser and Misner (ADM). This approach is natural in the context of our Newtonian intuition, and is also directly applicable to computer simulations. Whilst the traditional ADM approach has dominated numerical relativity, questions regarding its long-term stability have lead to the development of many other formulations of the Einstein equations in three-plus-one dimensions.

Techniques of current interest include those developed by Sasaki and Nakamura, and later by Baumgarte and Shapiro, known generically as Conformal ADM (CADM) formulations. Incorporating insights from York’s analysis of the initial value problem, these algorithms have shown superior stability properties compared with ADM in some applications. Other recent formulations are based on symmetric-hyperbolic forms of the Einstein equations, where it is hoped that the mathematical proofs of stability and well-posedness confer numerical advantages compared with techniques whose mathematical structures are more uncertain. The recent review by Lehner discusses many of these issues.

Despite the enormous effort invested in these techniques (and many others, including characteristic formulations), most modern numerical relativity codes continue to suffer from problems with long-term stability and lack of accuracy. Lack of resolution is only a partial solution; new insights or techniques seem to be required to overcome many of these problems.

II. REGGE’S FORMULATION OF GENERAL RELATIVITY ON A LATTICE

Regge calculus is a formulation of general relativity on a piecewise flat simplicial complex, rather than a differentiable manifold. In general this complex is built from four-simplices, the four dimensional generalisation of triangles and tetrahedra. The interior of each three and four dimensional lattice element is intrinsically flat, with curvature concentrated on the two-dimensional faces (triangles). In n-dimensions, curvature resides on lattice elements of co-dimension two.

Regge calculus would appear to be ideally suited to numerical simulations; it is an inherently discrete formulation of general relativity, and complex topologies are easily incorporated. In addition to its facility in classical numerical relativity, Regge calculus has also seen application in the sum-over-histories formulation of quantum gravity. In this review we concentrate on the application of Regge calculus to classical numerical relativity; details on quantum applications can be found elsewhere. The remainder of this section is devoted to the description of classical gravity on a simplicial lattice.

Given a lattice spacetime, which consists of a list of vertices, the corresponding connectivity matrix and the lengths of all edges, the Gaussian curvature $\kappa$ can be locally defined by the parallel transportation of a test vector about a closed loop. Calculating the angle through which the vector rotates, we obtain

$$\kappa = \frac{\text{angle rotated}}{\text{area of loop}} = \frac{\epsilon}{A^*}. \quad (1)$$

The first equality is the definition of Gaussian curvature; the second provides the equivalent expression on the lattice. The natural loop to choose, as indicated in figure 1(a), is defined by the area dual to the triangle on which the curvature resides. This is a portion of the dual-lattice, and may in principle be calculated for a given triangulation. The test vector is rotated through an angle $\epsilon$ in the plane orthogonal to the triangle (that is, in the plane of the dual area $A^*$). We refer to $\epsilon$ as the “deficit angle”, the lattice representation of the curvature concentrated on the triangle. The deficit angle is computed, given the edge lengths of the lattice, using

$$\epsilon = 2\pi - \sum_k \theta_k \quad (2)$$

$\theta_k$ is the angle at vertex $k$. The remainder of this section is devoted to the description of classical gravity on a simplicial lattice.
Regge equations

variational principle; the action is varied with respect to the lattice once all edges are known. In direct analogy to the metric, and thus any geometric quantity may be calculated between the two tetrahedral faces which hinge on the signature \( \theta_{012} \), measured between the tetrahedra 0123 and 0124 which hinge on 012 within the four-simplex 01234.

where the summation is over all four-simplices \( k \) which contain the triangle, and \( \theta_k \) is the hyper-dihedral angle between the two tetrahedral faces which hinge on the triangle within \( k \). Figure 1(b) shows this structure for the simplex 01234 which hinges on the triangle 012. The form given above applies for spacetimes with Euclidean signature; similar expressions apply for a spacetime with signature \( - + + + \).

Regge derived the simplicial equivalent of the Hilbert action \( \mathcal{H} \) for a lattice spacetime,

\[
\int_{\mathcal{M}} \sqrt{-g} R \, d^4x \rightarrow 2 \sum_i \epsilon_i A_i
\]

where the summation is over all triangles \( i \), \( A_i \) is the area and \( \epsilon_i \) is the deficit angle of the \( i \)th triangle. This simplicial expression can be intuitively understood by noting that on a simplicial complex the Hilbert integrand \( \mathcal{H} \) has compact support on triangles, each triangle has the volume element \( dV = \sqrt{-g} \, d^4x \approx AA^* \) and the scalar curvature is \( R \approx \kappa = \epsilon / A^* \). A more thorough proof in this spirit has been provided by Miller \( \mathcal{B} \).

Lattice edges are the discrete representation of the metric, and thus any geometric quantity may be calculated once all edges are known. In direct analogy to the continuum, the equations of motion are obtained by noting that on a simplicial complex the Hilbert integrand has compact support on triangles, each triangle has the volume element \( dV = \sqrt{-g} \, d^4x \approx AA^* \) and the scalar curvature is \( R \approx \kappa = \epsilon / A^* \). A more thorough proof in this spirit has been provided by Miller \( \mathcal{B} \).

From this basic structure it is possible, in principle, to construct the simplicial counterparts of any geometric object of interest: \( K_{\mu\nu} \), \( R_{\alpha\beta\mu\nu} \) and so forth. However, careful averaging is required if these simplicial definitions are to converge smoothly (and pointwise) to their continuum counterparts.

Finally, we note that the theory of Regge outlined above is one among many conceivable approaches to “lattice gravity”. A related method has recently been developed by Brewin \( \mathcal{F} \), where the metric is locally constructed in Riemann normal coordinates from an underlying lattice structure. Brewin is then able to use any of the standard formulations (ADM, CDM, etc) to evolve the lattice. These approaches are related in general structure to the finite element and finite volume methods in wide use in computational fluid dynamics and engineering. It is not yet entirely clear how lattice approaches to general relativity relate to these well-developed numerical techniques, but such an understanding would provide a useful bridge between the lattice theory and continuum methods.

III. \((3+1)\)-DIMENSIONAL LATTICE GRAVITY

The \((3 + 1)\)-dimensional formulation of Regge calculus is reasonably well understood, although to date there have been few applications. Significant new studies will be required to obtain a full understanding of such issues as convergence and long-term stability.

The initial value problem is a vital precursor to the generic evolution problem. A general technique for solving the initial value problem in the thin-sandwich, \((3+1)\)-dimensional formulation of Regge calculus has been described by Gentle and Miller \( \mathcal{R} \). Their approach is based on an identification of the geometric degrees of freedom and conformal structures employed in the York initial value formalism \( \mathcal{R} \). By associating lattice elements with the continuum conformal structure, Gentle and Miller describe one possible (although far from unique) approach to the construction of two-slice simplicial initial data. They successfully benchmark their approach on the Kasner cosmology \( \mathcal{R} \).
Once initial data has been constructed a consistent evolution scheme is required. The Regge equations, a set of coupled non-linear algebraic equations, can in principle be evolved by iteratively solving the entire coupled system at each timestep. However, it was recently realised that the lattice can be structured in such a way as to allow the implicit, parallel, decoupled evolution of sets of vertices \[^{11}\]. This algorithm is known as the “Sorkin evolution scheme”.

A general (3 + 1)-dimensional lattice may be constructed from a given three-dimensional lattice by the “evolution” (not true geometrodynamics) of individual vertices. Each vertex is “dragged forward”, off the initial hypersurface, and in the process addition edges are created to join the “evolved” vertex to its counterpart on the initial hypersurface. This process is repeated until all vertices have been “evolved”. In this way the connectivity of the initial hypersurface is replicated, whilst the lattice structure between the two surfaces is constructed naturally as part of the algorithm. A lattice of this type, referred to as a “Sorkin triangulation” \[^{11}\], allows the parallel evolution of individual vertices described above.

General relativity is fundamentally coordinate invariant, although one must choose a convenient frame in which to perform numerical calculations. In the standard (3 + 1)-dimensional ADM formulation, this is encoded in the freedom to lay down coordinates on the initial Cauchy surface, together with the freedom to choose how these coordinates are propagated to each future time slice (lapse and shift freedom). In a lattice formulation these correspond to the freedom to distribute vertices on the initial hypersurface and the freedom to propagate those vertices, respectively. The latter freedom requires the provision of four conditions per vertex during each evolution step to determine the propagation of the vertices; the simplicial lapse and shift freedom.

Regge calculus has been successfully applied in (3 + 1)-dimensions in the context of the anisotropic, homogeneous, $T^3$ Kasner cosmology \[^{8,11}\]. These initial studies provide confidence in the technique, which is able to recover the homogeneity and anisotropy of the analytic solution, and also demonstrate both stability and second-order convergence to the continuum solution.

IV. TEST-BED APPLICATIONS

In the development of any numerical technique one considers a variety of test problems which provide insight into the strengths and weaknesses of the approach. These should be of increasing complexity, and preferably, cover a wide range of physical scenarios which model, in a simplified manner, the actual physical problems which motivate the code development. In this way one develops confidence in the solutions obtained when the code is applied to real physical problems. Such a programme has been under way in the case of Regge calculus since its inception. Successful early applications included highly symmetric test problems with only a few degrees of freedom, together with the first numerical construction of axisymmetric binary black hole initial data. The review by Williams and Tuckey contains a full discussion of these early studies \[^{8}\].

More recently Regge calculus has been applied to increasingly complex problems of both physical and computational import. It is these studies on which we concentrate in this review. Successful applications to the Brill wave \[^{12,14}\] and black hole plus Brill wave \[^{15,16}\] initial data sets have validated computational lattice gravity, whilst also providing independent confirmation of the physical predictions of previous finite-difference studies. In this section we discuss these and other test-bed applications of the lattice approach, allowing us to investigate its accuracy and flexibility.

A. Spherical symmetry

The Schwarzschild solution is a classic test-bed for numerical relativity. The combination of non-trivial topology with an event horizon covering the central singularity has proved a challenging problem even for modern codes. In this section we briefly describe an initial evolution of the static Schwarzschild solution using lattice gravity.

The imposition of symmetry conditions in a lattice simulation is challenging. While providing enormous flexibility to model complex topologies, a simplicial lattice (built, in three dimensions, from tetrahedra) requires careful construction if it is to respect the spherical symmetry of the single black hole spacetime. The approach usually taken involves the construction of a full four-dimensional geometry constructed from simplices, after which the lattice is collapsed along the symmetry axes to obtain spherical symmetry in the appropriate limit. This approach is described elsewhere in axisymmetry \[^{10}\], and will also be mentioned briefly below.

![FIG. 2: Evolution of the Schwarzschild solution in isotropic coordinates using a lattice of 200 vertices. (a) The radial edge length $l$ (circles; every fourth point shown) plotted against the exponential radius coordinate $\eta$ at $t = 10M$. The exact solution is also shown (solid line). (b) The time evolution of the fractional rate of change of the radial edge lengths. After an initial period of relaxation, the evolution settles down to a static solution.](image-url)

In figure 2 we display a sample evolution of Schwarzschild initial data. The initial data is expressed
in isotropic coordinates, and the evolution is performed using the (fixed) analytic lapse and an area locking condition ($\partial_t q_{\theta\theta} = 0$). The lattice implementation of area locking demands that the edges which locally span the $\theta$-axis remain constant, thereby fixing the area of the two-sphere at that vertex. After an initial relaxation phase, during which the maximum fractional changes in the edges (metric) are less than $10^{-6}$, the solution settles into an entirely static configuration. This application of lattice gravity demonstrates that the approach can yield accurate, stable evolutions.

B. Axisymmetric initial data

1. Brill waves

The axisymmetric vacuum initial value problem first posed by Brill [12] contains gravitational radiation in an otherwise flat initial three-geometry. Conformal decomposition using a flat, simply-connected base metric leads only to trivial solutions: Brill introduced a metric of the form

$$dl^2 = \psi^4 \left\{ e^{2q}(d\rho^2 + d\zeta^2) + \rho^2 d\phi^2 \right\},$$

where the arbitrary function $q(\rho, \zeta)$ can be considered the distribution of gravitational wave amplitude, and is subject to certain boundary conditions to ensure that the mass is asymptotically well defined [12]. With this choice of background metric, the Hamiltonian constraint takes the form

$$\nabla^2 \psi = -\frac{\psi}{4} \left( \frac{\partial^2 q}{\partial \rho^2} + \frac{\partial^2 q}{\partial \zeta^2} \right),$$

which is solved for $\psi(\rho, \zeta)$ once $q(\rho, \zeta)$ is given.

The construction of lattice-based Brill wave initial data was originally considered by Dubal [13] using a lattice built from prisms, and later by Gentle [14,16] using a tetrahedral lattice. The approach, modeled on the analysis of Brill, uses a conformal decomposition technique and solves for the single conformal factor per vertex. The lattice is constructed to mirror the cylindrical polar coordinate system in which the continuum Brill wave metric is written. The conformal decomposition is obtained by integrating spacelike geodesics between vertices of the lattice, and assigning these lengths directly to the “base” lattice edges.

The solutions obtained for the tetrahedral three-geometry were in excellent agreement with previous finite-difference studies [14,16], thus confirming both the applicability of simplicial gravity, and the earlier numerical results. The prism-based calculations were considerably less accurate than the corresponding tetrahedral solution. Apparent horizons and ADM masses were also calculated for the tetrahedral initial data, again finding agreement with previous studies. The critical wave amplitude at which an apparent horizon first forms on the initial surface was found to be in complete agreement with earlier numerical values [14].

2. Distorted black holes

In this section we consider another axisymmetric, non-rotating configuration which contains a moment of time symmetry: the “distorted black hole” spacetime first considered by Bernstein [15]. These solutions represent an initial slice containing a black hole together with Brill waves perturbations, a natural generalisation of the preceding section, where Brill waves were considered on a flat Euclidean space.

The distorted black hole spacetime is astrophysically interesting, as it captures the “ring-down” phase of the merger and coalescence of a binary black hole system. Following merger, a single distorted (non-spherical) black hole forms, then evolves towards the static Schwarzschild solution by the emission of gravitational radiation. This problem is also amenable to perturbation approaches applied to the standard black hole solutions, providing a regime in which numerical relativity, theoretical analysis and gravitational wave observations combine to shed light on the foundations of general relativity.

The Brill wave plus black hole spacetime [15] is obtained by mirroring the original work of Brill [12]. A perturbation is introduced onto a “background metric”, and the conformal factor is calculated from the single initial value equation. Bernstein wrote the physical metric on the initial surface in the form

$$dl^2 = \psi^4 \left\{ e^{2q} (d\eta^2 + d\theta^2) + \sin^2 \theta d\phi^2 \right\},$$

where the exponential radius coordinate $\eta$ is defined by $\rho = m \exp(\eta)/2$. The black hole topology is obtained by demanding that the two-sphere at $\eta = 0$ is a minimal surface (isometry surface), which connects two asymptotically flat sheets. The “mass” $m$ is that of the black hole alone, and corresponds to the ADM mass measured at spatial infinity when $q(\eta, \theta) = 0$.

Distorted black hole initial data has been successfully constructed using Regge calculus [10]. It is natural to represent the three-metric in spherical polar coordinates; the lattice used to model the distorted black hole spacetime is also matched to the symmetry of the problem. Once again, the lattice three-geometry was obtained by locally aligning the background edges of the lattice with a spherical polar coordinate system. This implies, for example, that the $\zeta$-axis is locally aligned with the azimuthal $\theta$-axis. This approach was used as a matter of convenience; there is no requirement that the lattice be constructed from an underlying coordinate system, but doing so simplifies both the application of boundary conditions and comparison with the corresponding solution of the Einstein equations.

The simplicial distorted black hole initial data was found to agree well with finite-difference calculations, and estimates of the ADM mass were in excellent agreement
with previous studies. Convergence estimates indicate that the lattice solution converges as the second power of the typical lattice discretisation scale towards the true solution of the Einstein equations.  

3. Binary black holes

Misner [17] obtained an axisymmetric solution to the initial value equations of general relativity suitable for the study of the head-on collision of equal mass, non-rotating black holes. Due to the complexity of the problem there is no known analytic solution for the time development of this initial data. The maximal extension of the standard Schwarzschild black hole solution involves two asymptotically flat sheets, joined through the black hole throat. The spacetime has an isometry through the throat. The binary black hole solution obtained by Misner [17] is the natural generalisation of this solution; it consists of two asymptotically flat sheets joined to one another by two throats. The solution represents two black holes with equal mass and zero angular momentum at a moment of time-symmetry ($K_{a b} = 0$). The holes are located on the $z$-axis at $z = \pm \coth \mu$ and their throats have radius $a = 1/\sinh \mu$, which also determines the ADM mass of each individual hole if measured in isolation. The three-metric takes the conformally-flat form

$$dl^2 = \psi^4 \left( d\rho^2 + dz^2 + \rho^2 d\phi^2 \right)$$

(10)

where the conformal factor is given by

$$\psi = 1 + \sum_{n=1}^{\infty} \frac{1}{\sinh (n\mu)} \left( \frac{1}{\sqrt{\rho^2 + z_+^2}} + \frac{1}{\sqrt{\rho^2 + z_-^2}} \right)$$

(11)

and $z_\pm = z \pm z_n$ with $z_n = \coth(n\mu)$. This is a solution of the vacuum Hamiltonian constraint at a moment of time symmetry, $R = 0$. The single free parameter $\mu$ controls the separation, mass and radius of the black holes, with larger $\mu$ corresponding to greater separation and smaller bare mass [17].

In this section we present lattice solutions which correspond to the Misner initial data. Unlike the axisymmetric initial data presented in the previous sections, the existence of an analytic solution allows us to directly evaluate the accuracy and convergence of the lattice solution. The lattice is built using Čadež coordinates [22], which are spherical about both black hole throats and reduce to standard polar coordinates far from the origin. This structure necessarily introduces a coordinate singularity; in Čadež coordinates the singularity occurs at the origin.

The singular nature of the “physical” coordinate system hindered finite-difference studies of the Misner spacetime — we argue that the lattice approach overcomes this problem. While we are free to use the Čadež coordinates to construct the initial lattice (“location of vertices”; see figure 3(a)), the resulting lattice consists entirely of scalar edge lengths. Vertex positions are assigned (for example) using a regularly spaced grid in the Čadež coordinate system, and then integration of geodesics is used to assign the background edge lengths of the lattice given the flat background metric. From this point on the simplicial simulation consists entirely of evaluating scalar functions of the edge lengths. There is no singularity in the lattice construction.

A typical Regge solution for the conformal factor $\psi$ is shown in figure 3(b), with $\mu = 2.2$. This parameter is used to specify the centre ($z = \pm \cosh \mu$) and radius ($a = \operatorname{csch} \mu$) of the black holes. This choice guarantees that the initial data under consideration is of the Misner type, although it is clearly possible to investigate other configurations once the code has been benchmarked on the analytic solution.

The lattice solution shown in figure 3(b) is found to be in good agreement with the analytic solution, and shows almost second order convergence towards the analytic solution. The inset to figure 3(b) shows that the convergence rate is approximately 1.8. We find that as the number of vertices $N$ along each axis is increased, lattice edges far from the holes scale as $1/N$. Near the saddle point, however, the edges scale as $1/\sqrt{N}$. It is precisely this behaviour which leads to the (slightly) anomalous convergence rate; the slowly converging terms near the origin dominate. This “problem”, inherited from the singularity in the Čadež coordinates, can be easily overcome by further refinement of the lattice near the saddle point. Despite these minor issues, the Regge approach recovers a second-order accurate approximation to the solution of Einstein’s equations for the Misner binary black hole initial data. Following further refinement of the lattice near the saddle point, the method should yield precisely second order convergence. The Čadež coordinates are used only as a convenience when constructing the initial
Evolution is shown for a wave of amplitude $a = 0.01$ on a $100 \times 100$ grid, timestep $\Delta t = \Delta \rho/4$, and with the outer boundary at $\rho = z = 10$. (b) Mean fractional difference between magnitude of the first “bump” in (a) and the corresponding “continuum value”, plotted as a function of the grid resolution $N \times N$. The continuum value is estimated using Richardson extrapolation of the two most accurate values. We observe close to second order convergence; the gradient of the line of best fit is approximately 1.9.

C. Brill wave evolutions

In this section we describe an application of lattice gravity to the evolution of the (low amplitude) Brill wave initial data discussed above. Although all calculations are performed using lattice gravity, in this section we describe the techniques and gauge choices in the standard language of numerical relativity. Although the Regge framework is complete and self-contained, we find it useful to highlight connections with the continuum and more “standard” approaches to numerical relativity.

The spatial metric is written in cylindrical polar coordinates, and assumed to take the form

$$ds^2 = \psi^4 \left\{ \xi^2 \left( d\rho^2 + dz^2 \right) + \rho^2 d\phi^2 \right\}$$

throughout the evolution, where $\xi = \xi(\rho, z, t)$ and $\psi = \psi(\rho, z, t)$. The lattice is adapted to these coordinates, as described elsewhere [14]. This form of the three-metric $\gamma_{ab}$ was also chosen by Garfinkle and Duncan [19], and implicitly involves the imposition of the two conditions

$$\gamma_{\rho\rho} = \gamma_{zz}, \quad \gamma_{\rho z} = 0,$$

in addition to the initial assumption of axisymmetry. These two conditions are effectively a choice of gauge, and through the Einstein equations, determine the shift vector $\beta^i$ [19]. Whilst the same holds true in the lattice approach, we follow a different strategy in this initial evolution of gravitational radiation on a lattice.

The conditions [13] are applied directly to the lattice edges corresponding to the spatial components of the three-metric. A zero shift condition, $\beta^i = 0$, is also implemented on the lattice to ensure that the timelike worldlines generated by the vertices are orthogonal to the lower hypersurface. This is an over-specification of the available degrees of freedom, but we argue that for weak Brill waves ($a << 1$) the spacetime is a perturbation of flat space, and thus all of the above conditions can be enforced to within a reasonable accuracy. The choice of zero shift, together with the extra gauge choice [13], will in general violate the Einstein equations and their simplicial counterparts. For small perturbations away from flat space we expect that these errors are controllable. The generic evolution of medium to large amplitude waves will require a consistent specification of the available gauge freedoms. The choices outlined above, however, are sufficient to perform the initial low-amplitude test evolutions presented in this paper.

The remaining gauge freedom is determined by setting $\alpha = 1$ (unit lapse). This condition is not ideal, as it is known that even weak waves will impart non-zero velocities to the vertices of the lattice (or grid). Once the wave has propagated away from the centre of the grid we expect to recover a non-trivial, non-static coordinatisation of flat space. For strong amplitude waves we will require more advanced slicing conditions, such as maximal ($\text{Tr}K = 0$) or harmonic slicing. Geodesic slicing is sufficient, however, for this initial low-amplitude test problem.

Boundary conditions are imposed on the lattice to enforce the reflection symmetry about the $\rho = 0$ axis, together with reflection symmetry about the $z = 0$ axis to simplify the computations. These conditions are implemented using ghost cells centred about the axis. This approach has been used successfully in previous studies [19], and is found to work well. The outer boundary conditions are implemented using a radiative “Sommerfeld” condition, implemented in differential form on $\psi$ and $\xi$ along the outer boundary [21].

Results are shown in figures 4 and 5 and demonstrate that Regge calculus can successfully and accurately propagate a gravitational wave through the lattice. An initial convergence estimate is performed by examining the convergence of the magnitude of the first “bump” in figure 4(a). This feature of the solution is found to converge at close to second order, demonstrating a degree of consistency in the solutions obtained. Figure 5 displays the deviation of the conformal factor ($\psi^2 - 1$) at various times; the propagation of the wave can be observed, with the inner region returning to a roughly constant value (“flatness”) once the wave has escaped towards the boundary. Reflections from the outer boundary currently prevent long term evolutions.
FIG. 5: The deviation of the conformal factor, $\psi^2 - 1$, plotted as a function of $\rho$ along the line $z = 0$ for various times (a) $t = 0$ to 0.5, (b) $t = 1$ to 2, (c) $t = 3$ to 4, (d) $t = 5$ to 6) for a Brill wave with amplitude $a = 0.01$. We use the "Eppley-type" Brill function $q(\rho, z)$, and show solutions for a lattice with $100 \times 100$ vertices. Note that the $x$-axis scale changes in parts (c) and (d), and the outer boundary is placed at $\rho = z = 10$.

V. THE FUTURE OF LATTICE APPROACHES TO NUMERICAL RELATIVITY

We have reviewed several recent applications of Regge calculus, and demonstrated that it provides a unique, and thus far successful, alternative to the more standard techniques employed in numerical relativity.

In particular, we have presented new solutions of lattice gravity corresponding to initial data for the head-on collision of black holes, together with the first successful time evolution of gravitational radiation on a lattice. Despite these and the other studies described above, significant questions remain regarding the fundamental structure of Regge calculus.

Much work remains to be done to address the dual and diffeomorphic structures of the lattice, the relation of this and other lattice approaches to standard finite element and finite volume discretisations of differential equations, and the inclusion of matter. On the numerical front, further applications in three-plus-one dimensions are vital, requiring the development of a modern parallel code.

Despite these issues, every indication is that the lattice approach to gravity developed by T. Regge will continue to provide an interesting and complementary approach to numerical relativity.

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