DETERMINATION OF GL(3) CUSP FORMS BY CENTRAL VALUES OF QUADRATIC TWISTED $L$-FUNCTIONS

SHENGHAO HUA AND BINGRONG HUANG

Abstract. Let $\phi$ and $\phi'$ be two GL(3) Hecke–Maass cusp forms. In this paper, we prove that $\phi = \phi'$ or $\tilde{\phi}'$ if there exists a nonzero constant $\kappa$ such that $L(\frac{1}{2}, \phi \otimes \chi_d) = \kappa L(\frac{1}{2}, \phi' \otimes \chi_d)$ for all positive odd square-free positive $d$. Here $\tilde{\phi}'$ is dual form of $\phi'$ and $\chi_d$ is the quadratic character $(8d)$. To prove this, we obtain asymptotic formulas for twisted first moment of central values of quadratic twisted $L$-functions on GL(3), which will have many other applications.

1. Introduction

Determining automorphic forms from central values of the twisted $L$-functions is a topic of much interest (see e.g. [14, 13, 2, 18, 12]). It was first considered by Luo and Ramakrishnan [14] for modular forms. They showed that if two cuspidal normalized newforms $f$ and $g$ of weight $2k$ (resp. $2k'$) and level $N$ (resp. $N'$) have the property that

$$L\left(\frac{1}{2}, f \otimes \chi_d\right) = L\left(\frac{1}{2}, g \otimes \chi_d\right)$$

for all quadratic characters $\chi_d$, then $k = k'$, $N = N'$ and $f = g$. Chinta and Diaconu [2] proved that self-dual GL(3) Hecke–Maass forms are determined by their quadratic twisted central $L$-values. They used the method of double Dirichlet series for the averaging process. Recently, Kuan and Lesesvre [12] generalized the analogous result to automorphic representations of GL(3, $F$) over number field which are self-contragredient. In this paper, we use a new method to give a general result on GL(3), without the assumptions of [12]. Instead of using double Dirichlet series as in [2] and [12], we introduce a twisted average of the central $L$-values and obtain its asymptotics for which we use a method based on Soundararajan’s work [20].

Let $\phi$ be a Hecke–Maass cusp form of type $\nu = (\nu_1, \nu_2)$ for SL(3, $\mathbb{Z}$) with the normalized Fourier coefficients $A(m, n)$. We have the conjugation relation $A(m, n) = \overline{A(n, m)}$, see [7, Theorem 9.3.11]. Any real primitive character to the modulus $q$ must be of the form $\chi(n) = (\frac{d}{n})$ where $d$ is a fundamental discriminant [17, Theorem 9.13], i.e., a product of pairwise coprime integers of the form $-4$, $\pm 8$, $(-1)^{\frac{p-1}{2}}p$ where $p$ is an odd prime. There are two primitive characters to the modulus $q$ if $8 \parallel q$ and only one otherwise. From [7, Theorem

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*They assume “Hypothesis 1” which is satisfied by self-contragredient forms.
7.1.3] we know
\[
\prod_{i=1}^{3} \Gamma\left(\frac{\frac{1}{2} - \gamma_i}{2}\right)L\left(\frac{1}{2}, \phi \otimes \chi_{sd}\right) = \prod_{i=1}^{3} \Gamma\left(\frac{\frac{1}{2} + \gamma_i}{2}\right)L\left(\frac{1}{2}, \tilde{\phi} \otimes \chi_{sd}\right)
\]

when \(d\) is positive where \(\tilde{\phi}\) is the dual form of \(\phi\), \(\gamma_1 = 1 - 2\nu_1 - \nu_2\), \(\gamma_2 = \nu_1 - \nu_2\), and \(\gamma_3 = -1 + \nu_1 + 2\nu_2\) are the Langlands parameters of \(\phi\). By unitarity and the standard Jacquet–Shalika bounds, the Langlands parameters of an arbitrary irreducible representation \(\pi \subseteq L^2(SL(3, \mathbb{Z})/\mathbb{H}^3)\) must satisfy \(\sum_{i=1}^{3} \gamma_i = 0\) and \(\{-\gamma_i\}_{i=1}^{3} = \{\overline{\gamma_i}\}_{i=1}^{3}\). Let
\[
\kappa_{\phi, \phi'} = \begin{cases} 
1, & \text{if } \phi = \phi', \\
\prod_{i=1}^{3} \frac{\Gamma\left(\frac{\frac{1}{2} + \gamma_i}{2}\right)}{\Gamma\left(\frac{\frac{1}{2} - \gamma_i}{2}\right)}, & \text{if } \phi = \overline{\phi}'.
\end{cases}
\]

Then we have
\[
L\left(\frac{1}{2}, \phi \otimes \chi_{sd}\right) = \kappa_{\phi, \phi'} L\left(\frac{1}{2}, \phi' \otimes \chi_{sd}\right),
\]

for all positive fundamental discriminants \(8d\). But we don’t know if the converse conclusion is true, i.e. if for normalized \(\phi\) and \(\phi'\), there exists a nonzero constant \(\kappa\) such that \(L\left(\frac{1}{2}, \phi \otimes \chi_{sd}\right) = \kappa L\left(\frac{1}{2}, \phi' \otimes \chi_{sd}\right)\) for all positive \(d\), is it then true that we have \(\phi = \phi'\) or \(\overline{\phi}'\)? Our main result in this paper is as follows.

**Theorem 1.1.** Let \(\phi\) and \(\phi'\) be two normalized Hecke–Maass cusp forms of \(SL(3, \mathbb{Z})\). Fix an integer \(M\) coprime to \(3, 5, 7, 11\). If there exists a nonzero constant \(\kappa\) such that
\[
L\left(\frac{1}{2}, \phi \otimes \chi_{sd}\right) = \kappa L\left(\frac{1}{2}, \phi' \otimes \chi_{sd}\right)
\]

hold for all positive odd square-free integers \(d\) coprime to \(M\), then we have \(\phi = \phi'\) or \(\overline{\phi}'\). If we further assume \(\prod_{i=1}^{3} \Gamma\left(\frac{\frac{1}{2} - \gamma_i}{2}\right) \notin \mathbb{R}\), then we have \(\phi = \phi'\) if and only if \(\kappa = 1\), and \(\phi = \overline{\phi}'\) if and only if \(\kappa = \prod_{i=1}^{3} \frac{\Gamma\left(\frac{\frac{1}{2} + \gamma_i}{2}\right)}{\Gamma\left(\frac{\frac{1}{2} - \gamma_i}{2}\right)}\).

**Remark 1.2.** Let \(\{\gamma_i\}\) be the Langlands parameters of \(\phi\). If \(\prod_{i=1}^{3} \Gamma\left(\frac{\frac{1}{2} - \gamma_i}{2}\right) \notin \mathbb{R}\), then \(\phi\) is not self-dual. Our method also works for \(GL(3)\) forms of any fixed level.

We prove Theorem 1.1 by using the following Theorem 1.3 on twisted first moment of central values of quadratic twisted \(GL(3)\) \(L\)-functions. Our arguments combine ideas from Soundararajan [20] and Chinta–Diaconu [2]. To extract the relevant information from the main term in Theorem 1.3, we will use Lemma 4.1 below. This argument is different from Chinta–Diaconu’s, where they used the fact that certain rational function is monotone for real variable (their Lemma 5.1), which may be special to the self-contragredient case since \(A(p, 1)\) are complex in the non self-contragredient case.

For an automorphic representation \(\pi\) of \(GL(3, \mathbb{A}_\mathbb{Q})\), \(L(s, \text{sym}^2 \pi)\) has a simple pole at \(s = 1\) if and only if \(\pi\) is the Gelbart–Jacquet lift [5] of an automorphic representation on \(GL(2, \mathbb{A}_\mathbb{Q})\) with trivial central character [6], i.e., it is a self-contragredient cuspidal automorphic representation [19]. So \(\phi\) is self-dual if and only if \(L(s, \text{sym}^2 \phi)\) has a simple pole at \(s = 1\), which is equivalent to \(L(s, \text{sym}^2 \phi)\) has a simple pole at \(s = 1\).
We denote \( \theta_3 \) be the least common upper bound of power of \( p \) for \(|A(p,1)|\), i.e., \(|A(p,1)| \leq 3^{p^{\theta_3}} \) for all prime \( p \). The Generalized Ramanujan Conjecture implies that \( \theta_3 = 0 \), and from Kim–Sarnak [11, Appendix 2] we know \( \theta_3 \leq \frac{5}{14} \).

Let \( \Phi \) be any smooth nonnegative Schwarz class function supported in the interval \((1, 2)\). For any integer \( \nu \geq 0 \) we define

\[
\Phi(\nu) = \max_{0 \leq j \leq \nu} \int_1^2 |\Phi^{(j)}(t)| \, dt.
\]

For any complex number \( w \), we define

\[
\bar{\Phi}(w) = \int_0^\infty \Phi(y) y^w \, dy,
\]

so \( \bar{\Phi}(w) \) is holomorphic. Integrating by parts \( \nu \) times, we have

\[
\bar{\Phi}(w) = \frac{1}{(w+1) \ldots (w+\nu)} \int_0^\infty \Phi^{(\nu)}(y) y^{w+\nu} \, dy,
\]

thus for \( \text{Re}(w) > -1 \) we have

\[
|\bar{\Phi}(w)| \ll \nu 2^{\text{Re}(w)} |w+1|^\nu \Phi(\nu).
\]

**Theorem 1.3.** Let \( \phi \) be a Hecke–Maass cusp form for \( \text{SL}(3, \mathbb{Z}) \) with normalized Fourier coefficients \( A(m, n) \). For sufficiently large \( X > 0 \), arbitrarily small \( \varepsilon > 0 \), and any odd integer \( l \ll X^{1/3-\varepsilon} \), if \( \phi \) is not self-dual then we have

\[
\sum_{2d \mid l} \chi_{8d}(l) L\left(\frac{1}{2}, \phi \otimes \chi_{8d} \right) \Phi\left(\frac{d}{X}\right) = \frac{2\Phi(0)X}{3\zeta(2) \sqrt{l_1}} \prod_{p \mid l} \frac{p}{p+1} \left( G_{\phi}(l) L(2, 1, \text{sym}^2 \phi) \right.
\]
\[
+ \left. \prod_{i=1}^{3} \frac{\Gamma\left(\frac{1}{2} + \gamma_i\right)}{\Gamma\left(\frac{1}{2} - \gamma_i\right)} G_{\phi}(l) L(2, 1, \text{sym}^2 \tilde{\phi}) \right) + O_X(\frac{4^{1+\varepsilon} X^{4\varepsilon}}{23}) ;
\]

and if \( \phi \) is self-dual then we have

\[
\sum_{2d \mid l} \chi_{8d}(l) L\left(\frac{1}{2}, \phi \otimes \chi_{8d} \right) \Phi\left(\frac{d}{X}\right) = \lim_{s \to 1} (s-1) L(2, s, \text{sym}^2 \phi) \Phi(0) \prod_{p \mid l} \frac{p}{p+1} X
\]
\[
	imes \left( G_{\phi}(l) \log \frac{X}{l^{3_1}} + C_{\phi}(l) \right) + O_X(\frac{4^{1+\varepsilon} X^{4\varepsilon}}{23}) ,
\]

where \( l = l_1 l_2^2 \) with \( l_1 \) is square-free, \( \sum_{2d \mid l} \) means summing over positive odd square-free \( d \), \( L(2, s, \text{sym}^2 \phi) = L(s, \text{sym}^2 \phi)/L_2(s, \text{sym}^2 \phi) \) and \( L_2(s, \text{sym}^2 \phi) \) is the local \( L \)-function in
2-place, $C_{\phi}(l)$ is defined as in [3, 18], $G_{\phi}(l) = \prod_{\text{odd prime } p} G_{\phi,p}(l)$ with

$$G_{\phi,p}(l) = \begin{cases} 
(A(p, 1) + \frac{1}{p})(1 - \frac{A(1, p)}{p} + \frac{A(p, 1)}{p^2} - \frac{1}{p^3}), & \text{if } p \mid l_1, \\
(1 + \frac{A(1, p)}{p})(1 - \frac{A(1, p)}{p} + \frac{A(p, 1)}{p^2} - \frac{1}{p^3}), & \text{if } p \mid l_2, p \nmid l_1, \\
(1 - \frac{p^2A(1, p)^2 - p^2A(1, p) - pA(1, p)^2 + 2pA(1, p) + 1}{(p + 1)p^2(p + A(1, p))}) \\
\times (1 + \frac{A(1, p)}{p})(1 - \frac{A(1, p)}{p} + \frac{A(p, 1)}{p^2} - \frac{1}{p^3}), & \text{if } p \nmid l.
\end{cases}$$

Moreover, there exists $c_\phi = 3^a \times 5^b \times 7^2 \times 11^2$ with some $a, b \in \{1, 2\}$ such that: For $l = c_\phi l'$ with $(2c_\phi, l') = 1$, $G_{\phi}(l) = 0$ if and only if there exists a prime $13 \leq p \mid l_1$ such that $A(p, 1) = -\frac{1}{p}$.

**Remark 1.4.** We did not try to optimize the exponent $19/20$ as it is enough for our main theorem. One may compare our result with the cubic moment of quadratic Dirichlet $L$-functions (see e.g. [20, 21, 3, 4]). Our case is more complicated as $\phi$ is undecomposable. We will use Soundararajan’s approach to prove Theorem [1, 3] which is based on the approximate functional equation and Poisson summation formula (Lemma [2, 4]). At present, one still can not unconditionally prove an asymptotic formula (with power saving) for the fourth moment of quadratic Dirichlet $L$-functions. To extend our result to $\GL_n$ for $n > 3$ seems hard.

**Remark 1.5.** In fact, our method of the proof also works for the functions $n \mapsto d_3(n) = (1 \ast 1 \ast 1)(n)$ and $n \mapsto (\lambda_f \ast 1)(n)$, which are the Hecke eigenvalues of certain non-cuspidal automorphic representations of $\GL(3, \mathbb{A}_\mathbb{Q})$, namely the isobaric representations $1 \boxplus 1 \boxplus 1$ and $\pi_f \boxplus 1$. Here $\pi_f$ is a cuspidal automorphic representation of $\GL(2, \mathbb{A}_\mathbb{Q})$. The method of the present paper can be generalized straightforwardly to show above results for an arbitrary irreducible automorphic representation $\pi \subseteq L^2(\SL(3, \mathbb{Z}) \backslash \mathbb{H}^3)$.

**Remark 1.6.** By using of the large sieve estimates for quadratic characters in [8], one may prove a nonvanishing result for central values of quadratic twisted $\GL(3)$ $L$-functions, i.e. there exist at least $O(X^{1/2-\varepsilon})$ fundamental discriminants $X \leq d \leq 2X$ for arbitrarily small $\varepsilon > 0$ such that $L\left(\frac{1}{2}, \phi \otimes \chi_{sd}\right) \neq 0$.

**Remark 1.7.** In [9], we give another application of Theorem [1, 3] where we prove the conjectured order lower bounds for the $k$-th moments of central values of quadratic twisted self-dual $\GL(3)$ $L$-functions for all $k \geq 1$.

To prove Theorem [1, 3], we need the following Generalized Ramanujan Conjecture on average for a special sequence of Fourier coefficients of $\phi$, which is closely related to the symmetric square lift of $\phi$. This may have its own interest.

**Theorem 1.8.** Let $\phi$ be a fixed normalized Hecke–Maass cusp form for $\SL(3, \mathbb{Z})$, and $A(m, n)$ be its Fourier coefficients. For any $\varepsilon > 0$ we have

$$\sum_{n \leq X} |A(n^2, 1)| \ll_{\phi, \varepsilon} X^{1+\varepsilon}.$$
to prove Theorem 1.3. In §4 we prove Theorem 1.1 by using Theorem 1.3. Finally, in §5, we prove Theorem 1.8 by using the Rankin–Selberg bounds on the averages of Fourier coefficients.

2. Notation and preliminary results

For any complex numbers sequence \( \{f_n\}_{n=1}^{\infty} \) and smooth nonnegative Schwarz class function \( \Phi \) supported in the interval \((1, 2)\). We define

\[
S(f_d; \Phi) = S_X(f_d; \Phi) = \frac{1}{X} \sum_{d \text{ odd}} \mu^2(d) f_d \Phi \left( \frac{d}{X} \right).
\]

For real parameter \( Y > 1 \) and we have \( \mu^2(d) = M_Y(d) + R_Y(d) \) where

\[
M_Y(d) = \sum_{l^2 | d, l \leq Y} \mu(l), \quad R_Y(d) = \sum_{l^2 | d, l > Y} \mu(l).
\]

Define

\[
S_M(f_d; \Phi) = S_{M,X,Y}(f_d; \Phi) = \frac{1}{X} \sum_{d \text{ odd}} M_Y(d) f_d \Phi \left( \frac{d}{X} \right),
\]

and

\[
S_R(f_d; \Phi) = S_{R,X,Y}(f_d; \Phi) = \frac{1}{X} \sum_{d \text{ odd}} R_Y(d) f_d \Phi \left( \frac{d}{X} \right),
\]

so \( S(f_d; \Phi) = S_M(f_d; \Phi) + S_R(f_d; \Phi) \).

Let \( H(s) \) be any function which is holomorphic and bounded in the strip \(-4 < \Re(u) < 4\), even, and normalized by \( H(0) = 1 \). From Kim–Sarnak [11, Appendix 2] we know

\[
\max \{ \Re(\gamma_i) \} \leq \frac{5}{14},
\]

and from [10] we know \( L(s, \phi \otimes \chi_{8d}) \) is entire and we have the following lemma.

Lemma 2.1 (Approximate functional equation). Let \( d \) be a positive odd square-free integer. Then we have

\[
L\left( \frac{1}{2}, \phi \otimes \chi_{8d} \right) = \sum_{n=1}^{\infty} \left( A(n, 1)V(n(\frac{\pi}{8d})^2) + \prod_{i=1}^{3} \frac{\Gamma\left( \frac{4+\gamma_i}{2} \right)}{\Gamma\left( \frac{4-\gamma_i}{2} \right)} A(n, 1)V(n(\frac{\pi}{8d})^2) \right) \chi_{8d}(n) \frac{\sqrt{n}}{n},
\]

where \( V(y) \) is defined by

\[
V(y) = \frac{1}{2\pi i} \int_{(u)} \prod_{i=1}^{3} \frac{\Gamma\left( \frac{4+\gamma_i}{2} \right)}{\Gamma\left( \frac{4-\gamma_i}{2} \right)} y^{-s} H(s) \frac{ds}{s}, \tag{2.1}
\]

with \( u > 0 \). Here, and in sequel, \( \int_{(u)} \) stands for \( \int_{u-i\infty}^{u+i\infty} \). We can choose \( H(s) = e^{s^2} \).

Proof. See [10, Theorem 5.3].

Lemma 2.2. The function \( V \) is smooth on \([0, \infty)\). Let \( h \in \mathbb{Z}_{\geq 0} \). For \( y \) near 0 and \( v < \frac{1}{2} - \theta_3 \) we have

\[
y^h V^{(h)}(y) = \delta_h + O_h(y^v),
\]

and for large \( y \), any \( A > 0 \), and any integer \( h \),

\[
y^h V^{(h)}(y) \ll_{h,A} y^{-A}.
\]
Here \( \delta_0 = 1 \), and \( \delta_h = 0 \) if \( h \geq 1 \).

**Proof.** See [10, Proposition 5.4]. \( \square \)

Let \( n \) be an odd integer. We define for all integers \( k \)

\[
G_k(n) = \left( \frac{1-i}{2} + \frac{1+i}{2} \right) \sum \left( \frac{a}{n} \right) e\left( \frac{ak}{n} \right),
\]

and put

\[
\tau_k(n) = \sum \left( \frac{a}{n} \right) e\left( \frac{ak}{n} \right) = \left( \frac{1+i}{2} + \frac{1-i}{2} \right) G_k(n).
\]

Here \( e(x) = \exp(2\pi i x) \). If \( n \) is square-free then \( \left( \frac{\cdot}{n} \right) \) is a primitive character with conductor \( n \). Here it is easy to see that \( G_k(n) = \left( \frac{\xi}{n} \right) \sqrt{n} \). For our later work, we require knowledge of \( G_k(n) \) for all odd \( n \).

For fundamental discriminant \( d \) we know Gauss sum of \( \chi_d \) is \( \tau(\chi_d) = \sqrt{d} \) where the square root is taken as its principal branch.

**Lemma 2.3.** (i) Suppose \( m \) and \( n \) are coprime odd integers, then

\[
G_k(mn) = G_k(m)G_k(n).
\]

(ii) Suppose \( p^\alpha \) is the largest power of \( p \) dividing \( k \). (If \( k = 0 \) then set \( \alpha = \infty \).) Then for \( \beta \geq 1 \)

\[
G_k(p^\beta) = \begin{cases} 
0, & \beta \leq \alpha \text{ is odd,} \\
\phi(p^\beta), & \beta \leq \alpha \text{ is even,} \\
\left( \frac{kp^\alpha - \alpha}{p} \right) p^\alpha \sqrt{p}, & \beta = \alpha + 1 \text{ is odd,} \\
-p^\alpha, & \beta = \alpha + 1 \text{ is even,} \\
0, & \beta \geq \alpha + 2.
\end{cases}
\]

**Proof.** See [20, Lemma 2.3]. \( \square \)

For a Schwarz class function \( F \) we define

\[
\tilde{F}(\xi) = \int_{-\infty}^{\infty} (\cos(2\pi \xi x) + \sin(2\pi \xi x)) F(x) dx.
\]

**Lemma 2.4** (Poisson summation formula). Let \( F \) be a nonnegative, smooth function supported in \((1,2)\). For any odd integer \( n \),

\[
S_M\left( \left( \frac{d}{n} \right); F \right) = \frac{1}{2n} \sum_{\alpha \leq Y} \sum_{k=-\infty}^{\infty} \frac{\mu(\alpha)}{\alpha^2} (-1)^k G_k(n) \tilde{F}\left( \frac{kX}{2\alpha^2 n} \right).
\]

**Proof.** See [20, Lemma 2.6]. \( \square \)
3. Proof of Theorem 1.3

By Lemma 2.1, we have
\[
\frac{1}{X} \sum_{d \leq X} L\left(\frac{1}{2}, \chi_{8d} \otimes \chi_{8d}\right) \Phi\left(\frac{d}{X}\right) = S(\chi_{8d}(l)B(d); \Phi) + \prod_{i=1}^{3} \frac{\Gamma\left(\frac{1+\gamma_i}{2}\right)}{\Gamma\left(\frac{1-\gamma_i}{2}\right)} S(\chi_{8d}(l)B(d); \Phi),
\]
where
\[
B(d) = \sum_{n=1}^{\infty} \chi_{8d}(n) \frac{A(n,1)}{\sqrt{n}} V\left(n\left(\frac{\pi}{8d}\right)^{\frac{3}{2}}\right).
\]
We first consider the main contribution in $S(\chi_{8d}(l)B(d); \Phi)$, that is,
\[
S_M(\chi_{8d}(l)B(d); \Phi) = \sum_{n=1}^{\infty} \frac{A(n,1)}{\sqrt{n}} S_M(\chi_{8d}(ln); \Phi_n),
\]
where $\Phi_y(t) = \Phi(t)V\left(y\left(\frac{\pi}{8d}\right)^{\frac{3}{2}}\right)$.

By Lemma 2.4, we obtain
\[
S_M(\chi_{8d}(ln); \Phi_n) = \frac{1}{2ln\left(\frac{16}{ln}\right)} \sum_{\alpha \leq Y} \frac{\mu(\alpha)}{\alpha^2} \sum_{\alpha \leq Y} (-1)^{k} G_k(ln) \Phi_n\left(kX\frac{2\alpha^2ln}{2}\right).
\]
Hence we deduce that
\[
S_M(\chi_{8d}(l)B(d); \Phi) = P(l) + R(l),
\]
where $P(l)$ are terms from $k = 0$ and $R(l)$ are terms include all the nonzero terms $k$. Thus
\[
P(l) = \frac{1}{2l} \sum_{n=1}^{\infty} \frac{A(n,1)}{n^{\frac{3}{2}}} \left(\frac{16}{ln}\right) \sum_{\alpha \leq Y} \frac{\mu(\alpha)}{\alpha^2} G_0(ln) \Phi_n(0),
\]
and
\[
R(l) = \frac{1}{2l} \sum_{n=1}^{\infty} \frac{A(n,1)}{n^{\frac{3}{2}}} \left(\frac{16}{ln}\right) \sum_{\alpha \leq Y} \frac{\mu(\alpha)}{\alpha^2} \sum_{k=-\infty}^{k=\infty} (-1)^{k} G_k(ln) \Phi_n\left(kX\frac{2\alpha^2ln}{2}\right).
\]

3.1. The principal $P(l)$ contribution. Note that $\Phi_n(0) = \Phi_n(0)$ and that $G_0(ln) = \phi(ln)$ if $ln = \square$ and $G_0(ln) = 0$ otherwise. Recall that $l = l_1l_2$ where $l_1$ and $l_2$ are odd, and $l_1$ is square-free. The condition $ln = \square$ is thus equivalent to $n = l_1m^2$ for some integer $m$. Hence by (3.3) we have
\[
P(l) = \frac{1}{\zeta(2)\sqrt{l_1}} \sum_{m=1}^{\infty} \frac{A(l_1m^2, \alpha)}{m} \left(\prod_{\nu|2m} \frac{p}{p+1}\right) \Phi_{l_1m^2}(0) + O\left(\frac{1}{Y\sqrt{l_1}} \sum_{m=1}^{\infty} \frac{|A(l_1m^2, \alpha)|}{m} |\Phi_{l_1m^2}(0)|\right).
\]
By Lemma 2.2 and Theorem 1.8 together with some arguments as in [1, §2], we have

\[
\sum_{m=1}^{\infty} \frac{|A(l_1m^2, 1)|}{m} |\tilde{\Phi}_{l_1m^2}(0)| \ll \sum_{m^2 \ll X^{\frac{3}{2} + \epsilon}/l_1} |A(l_1m^2, 1)|
\]

\[
\ll \sum_{d \leq l_1^\infty} \sum_{m^2 \ll X^{\frac{3}{2} + \epsilon}/l_1d^2} \frac{|A(l_1d^2, 1)|}{d} \sum_{m^2 \ll X^{\frac{3}{2} + \epsilon}/l_1d^2} \frac{|A(m^2, 1)|}{m} \ll l_1^{\theta_3 + \epsilon} \sum_{m \ll X^{\frac{3}{2} + \epsilon}} \frac{|A(m^2, 1)|}{m} \ll l_1^{\theta_3 + \epsilon} X^\epsilon.
\]

By (3.5) and (3.6), we get

\[
P(l) = \frac{1}{\zeta(2)\sqrt{l_1}} \sum_{m=1}^{\infty} \frac{A(l_1m^2, 1)}{m} \left(\prod_{p|2m} \frac{p}{p+1}\right) \tilde{\Phi}_{l_1m^2}(0) + O(l_1^{\theta_3 - \frac{1}{2} + \epsilon} Y^{-1}).
\]

For any \( u > 0 \) we have

\[
\tilde{\Phi}_{l_1m^2}(0) = \int_{0}^{\infty} \Phi(t)V(l_1m^2(\frac{\pi}{8Xt})^{\frac{3}{2}})dt
\]

\[
= \frac{1}{2\pi i} \int_{(u)} \prod_{i=1}^{3} \frac{\Gamma(\frac{s+\frac{3}{2} - \gamma_i}{2})}{\Gamma(\frac{s+\frac{1}{2} - \gamma_i}{2})} (\frac{1}{l_1m^2(\frac{8X}{\pi})^{\frac{3}{2}}}^s) \left(\int_{0}^{\infty} \Phi(t)t^{\frac{3s}{2}}dt\right) e^{\frac{s}{2}} ds.
\]

Thus

\[
P(l) = \frac{2}{3\zeta(2)\sqrt{l_1} 2\pi i} \int_{(u)} \prod_{i=1}^{3} \frac{\Gamma(\frac{s+\frac{3}{2} - \gamma_i}{2})}{\Gamma(\frac{s+\frac{1}{2} - \gamma_i}{2})} \left(\frac{1}{l_1} \frac{8X}{\pi}\right)^{\frac{3s}{2}} \Phi(\frac{3s}{2})
\]

\[
\times \sum_{m=1}^{\infty} \frac{A(l_1m^2, 1)}{m^{1+2s}} \left(\prod_{p|2m} \frac{p}{p+1}\right) e^{\frac{s}{2}} ds + O(l_1^{\theta_3 - \frac{1}{2} + \epsilon} Y^{-1}).
\]

Let \( \alpha(p), \beta(p), \gamma(p) \) be the local parameters of \( \phi \) at \( p \), and so \( \alpha(p)\beta(p), \alpha(p)\gamma(p), \beta(p)\gamma(p) \) are the local parameters of \( \tilde{\phi} \) at \( p \). Then we have

\[
\sum_{h=0}^{\infty} A(p^{h}, 1) p^{sh} = (1 - \alpha(p)p^{-s})^{-1}(1 - \beta(p)p^{-s})^{-1}(1 - \gamma(p)p^{-s})^{-1},
\]

The local Euler factor of the symmetric square lift of \( \phi \) is defined as

\[
L_p(s, \text{sym}^2 \phi) = (1 - \alpha(p)^2 p^{-s})^{-1}(1 - \beta(p)^2 p^{-s})^{-1}(1 - \gamma(p)^2 p^{-s})^{-1}
\]

\[
\times (1 - \alpha(p)\beta(p)p^{-s})^{-1}(1 - \alpha(p)\gamma(p)p^{-s})^{-1}(1 - \beta(p)\gamma(p)p^{-s})^{-1}.
\]
Let $S$ be a finite set of places of $\mathbb{Q}$. Define $L^S(s, \text{sym}^2 \phi) = \prod_{p \notin S} L_p(s, \text{sym}^2 \phi)$. We have the following lemma.

**Lemma 3.1.** Suppose that $l = l_1 l_2^3$ is as above. Then for Re$(s)$ sufficiently large

\[
\sum_{m=1 \atop m \text{ odd}}^{\infty} \frac{A(l_m^2, 1)}{m^s} \prod_{p \mid l_m} \left( \frac{p}{p+1} \right) = \prod_{p \mid l_1} \left( \sum_{h=0}^{\infty} \frac{A(p^{2h+1}, 1)}{(p+1)p^{2h-1}} \right) \prod_{p \mid l_1} \left( \sum_{h=0}^{\infty} \frac{A(p^{2h}, 1)}{(p+1)p^{2h-1}} \right) \prod_{p \mid l_2} \left( 1 + \sum_{h=1}^{\infty} \frac{A(p^{2h}, 1)}{(p+1)p^{2h-1}} \right).
\]  

(3.10)

The right hand side of (3.10) has analytic continuation to Re$(s) > \frac{1}{2}$. We have $|G_\phi(s;l)| \ll_{\phi} l_1^{2\varepsilon} l_2$ if Re$(s) = \sigma > 1/2 + \varepsilon$. Moreover, there exists $c_\phi = 3^a \times 5^b \times 7^2 \times 11^2$ with some $a, b \in \{1, 2\}$ such that: For any $l = c_\phi l'$ with $(2c_\phi, l') = 1$, $G_\phi(1;l) = 0$ if and only if there exists a prime $13 \leq p | l_1$ with $A(p, 1) = -\frac{1}{p}$.

**Proof.**

Expanding the Euler factors on the left, we get

\[
\sum_{m=1 \atop m \text{ odd}}^{\infty} \frac{A(l_m^2, 1)}{m^s} \prod_{p \mid l_m} \left( \frac{p}{p+1} \right) = \prod_{p \mid l_1} \left( \sum_{h=0}^{\infty} \frac{A(p^{2h+1}, 1)}{(p+1)p^{2h-1}} \right) \prod_{p \mid l_1} \left( \sum_{h=0}^{\infty} \frac{A(p^{2h}, 1)}{(p+1)p^{2h-1}} \right) \prod_{p \mid l_2} \left( 1 + \sum_{h=1}^{\infty} \frac{A(p^{2h}, 1)}{(p+1)p^{2h-1}} \right).
\]  

(3.10)

Recall from [7] the relationship between the coefficients of $\phi$:

\[
A(m_1, 1)A(1, m_2) = \sum_{d \mid (m_1, m_2)} A\left(\frac{m_1}{d}, \frac{m_2}{d}\right),
\]

\[
A(1, n)A(m_1, m_2) = \sum_{d_0 d_1 d_2 = n \atop d_1 \mid m_1 \atop d_2 \mid m_2} A\left(\frac{m_1 d_2}{d_1}, \frac{m_2 d_0}{d_2}\right),
\]

so we have

\[
A(p^{k+1}, 1) = A(p, 1)A(p^k, 1) - A(p^{k-1}, p),
\]

\[
A(p^{k-1}, p) = A(p^{k-1}, 1)A(1, p) - A(p^{k-2}, 1).
\]

(3.11)
thus
\[
\sum_{h=0}^{\infty} \frac{A(p^{2h+1}, 1)}{(p+1)p^{2h-1}} = \frac{p - 1 + p^s A(p, 1)}{p + 1} \sum_{h=0}^{\infty} \frac{A(p^{2h}, 1)}{p^{2h}},
\]
(3.12)
and
\[
1 + \sum_{h=1}^{\infty} \frac{A(p^{2h}, 1)}{(p+1)p^{2h-1}} = \sum_{h=0}^{\infty} \frac{A(p^{2h}, 1)}{p^{2h}} - \frac{1}{p + 1} \sum_{h=0}^{\infty} \frac{A(p^{2h}, 1)}{p^{2h}}
\]
\[
= \sum_{h=0}^{\infty} \frac{A(p^{2h}, 1)}{p^{2h}} - \frac{1}{(p+1)p^s} \sum_{h=0}^{\infty} \frac{A(p^{2h+2}, 1)}{p^{2h}}
\]
\[
= (1 + \frac{A(1,p)}{(p+1)p^s}) \sum_{h=0}^{\infty} \frac{A(p^{2h}, 1)}{p^{2h}} - \frac{1 + p^s A(p, 1)}{(p+1)p^{2s}} \sum_{h=0}^{\infty} \frac{A(p^{2h+1}, 1)}{p^{2h}}
\]
\[
= (1 - \frac{p^{2s} A(p, 1)^2 - p^{2s} A(1, p) - p^s A(1, p)^2 + 2p^s A(p, 1) + 1}{(p+1)p^{2s}(p^s + A(1, p))}
\]
\times \sum_{h=0}^{\infty} \frac{A(p^{2h}, 1)}{p^{2h}}.
\]
(3.13)

Recall the Euler factors of \( L(s, \phi) \) we know
\[
A(p^h, 1) = \sum_{a+b+c=h} \alpha(p)^a \beta(p)^b \gamma(p)^c.
\]

Note that if the sum of three integers is an even integer, then there will be zero or two odd integers. So we have
\[
A(p^{2h}, 1) = \sum_{a+b+c=2h} \alpha(p)^a \beta(p)^b \gamma(p)^c
\]
\[
= \sum_{a+b+c=h} \alpha(p)^{2a} \beta(p)^{2b} \gamma(p)^{2c}
\]
\[
+ (\alpha(p)\beta(p) + \alpha(p)\gamma(p) + \beta(p)\gamma(p)) \sum_{a+b+c=h-1} \alpha(p)^{2a} \beta(p)^{2b} \gamma(p)^{2c},
\]
thus
\[
\sum_{h=0}^{\infty} \frac{A(p^{2h}, 1)}{p^{2h}} = (1 + \frac{\alpha(p)\beta(p) + \alpha(p)\gamma(p) + \beta(p)\gamma(p)}{p^s}) \sum_{h=0}^{\infty} \frac{\alpha(p)^{2a} \beta(p)^{2b} \gamma(p)^{2c}}{p^{2h}}
\]
\[
= \frac{p^s + \alpha(p)\beta(p) + \alpha(p)\gamma(p) + \beta(p)\gamma(p)}{p^s}
\]
\[
\cdot (1 - \alpha(p)^2 p^{-s})^{-1} (1 - \beta(p)^2 p^{-s})^{-1} (1 - \gamma(p)^2 p^{-s})^{-1}.
\]

From
\[
(1 - \alpha(p)^2 p^{-s})^{-1} (1 - \beta(p)^2 p^{-s})^{-1} (1 - \gamma(p)^2 p^{-s})^{-1} = L_p(s, \phi \otimes \phi)/L_p(s, \tilde{\phi})^2
\]
\[
= L_p(s, \text{sym}^2 \phi)/L_p(s, \tilde{\phi}),
\]
and
\[
\alpha(p)\beta(p) + \alpha(p)\gamma(p) + \beta(p)\gamma(p) = A(1, p),
\]
we obtain
\[
\sum_{h=0}^{\infty} \frac{A(p^{2h}, 1)}{p^{2h}} = (1 + \frac{A(1, p)}{p^s})L_p(s, \phi)^{-1}L_p(s, \text{sym}^2 \phi)
\]
\[
= (1 + \frac{A(1, p)}{p^s})(1 - \frac{A(1, p)}{p^s} + \frac{A(p, 1)}{p^{2s}} - \frac{1}{p^{2s}})L_p(s, \text{sym}^2 \phi).
\]
\[
(3.14)
\]

By (3.10)-(3.14), we prove (3.8).

Recall that we have \(\theta_3 \leq \frac{\sigma}{11}\), thus when \(\text{Re}(s) = \sigma > \frac{1}{2}\) we have
\[
\log \prod_{2 < p \leq Z \atop p \nmid l} G_{\phi, p}(s; l) \ll \sum_{p \leq Z} |A(1, p)|^2 p^{-2\sigma} + |A(p, 1)|p^{-2\sigma} + |A(p, 1)|^2 p^{-1 - \sigma} + |A(p, 1)|^{-1 - \sigma}
\]
\[
\ll \sum_{n \leq Z} |A(1, n)|^2 n^{-2\sigma} + |A(1, n)|n^{-2\sigma} + |A(1, n)|^2 n^{-1 - \sigma} + |A(1, n)|^{-1 - \sigma},
\]
so \(\prod_{2 < p \leq Z \atop p \nmid l} |G_{\phi, p}(s; l)| \ll_{\sigma} 1\) for \(\text{Re}(s) > \frac{1}{2}\). For fixed \(l\) there only finite \(p \mid l_1\), thus \(G_{\phi}(s; l)\) converges for \(\text{Re}(s) > \frac{1}{2}\). Moreover, we have \(|G_{\phi}(s; l)| \ll_{\phi, \sigma} l_{13}^{\theta_3 + \varepsilon}\) if \(\text{Re}(s) = \sigma > 1/2 + \varepsilon\) by using the known Ramanujan bounds to the local factors at primes dividing \(l_1\).

Finally, we will prove the last claim. It is known that \(L_p(1, \phi)^{-1} \neq 0\), since we have \(L_p(1, \phi)^{-1} \sum_{h \geq 0} A(1, p)^h = 1\). From \(|A(p, 1)| \leq 3p^{\frac{1}{44}}\) we know that for \(p \geq 13\),
\[
|A(1, p)|/p < 1
\]
and
\[
\frac{p^2 A(p, 1)^2 - p^2 A(1, p) - A(p, 1)^2}{(p + 1)p^2(p + A(1, p))} + 2pA(p, 1) + 1 < 1.
\]
So \(G_{\phi, p}(1; l) \neq 0\) for \(13 \leq p \mid l\). Note that we have
\[
\log \prod_{13 \leq p \leq Z \atop p \nmid l} |G_{\phi, p}(1; l)| \gg - \sum_{13 \leq p \leq Z} \left(|A(1, p)|^2 p^{-2} + |A(p, 1)|p^{-2}\right)
\]
\[
\gg - \sum_{n \leq Z} |A(1, n)|^2 n^{-2} - \sum_{n \leq Z} |A(1, n)|n^{-2},
\]
which proves \(\prod_{13 \leq p \mid l} G_{\phi, p}(1; l) \neq 0\).

For \(p \mid l_1\), \(G_{\phi, p}(1; l) = 0\) if and only if \(A(p, 1) = -\frac{1}{p}\); and for \(p \mid l_2, p \nmid l_1\), we have \(G_{\phi, p}(1, l) = 0\) if and only if \(A(p, 1) = -p\) which may happen only if \(p = 3, 5\). Thus we know there exists \(c_{\phi} = 3^a \times 5^b \times 7^c \times 11^d\) with some \(a, b, c, d \in \{1, 2\}\) such that for any \(l = c_{\phi}l'\) with \((2c_{\phi}, l') = 1\), \(\prod_{3 \leq p \leq 11} G_{\phi, p}(1; l) \neq 0\). For such \(l\), we have \(G_{\phi}(1; l) = 0\) if and only if there is one prime \(13 \leq p \mid l_1\) such that \(A(p, 1) = -\frac{1}{p}\). This completes the proof of the lemma. □

We denote \(G_{\phi}(l) = G_{\phi}(1; l)\). By (3.7) and Lemma 3.1, we have
\[
P(l) = \frac{2}{3} \zeta(2) \sqrt{l_1} \prod_{p \mid l} \frac{p}{p + 1} I(l) + O(l_1^{\phi_3 - \frac{1}{2} + \varepsilon} Y^{-1}),
\]
\[
(3.15)
\]
where

\[
I(l) = \frac{1}{2\pi i} \int_{(u)} \prod_{i=1}^{3} \frac{\Gamma(\frac{s+1}{2} - \gamma_i)}{\Gamma(\frac{s+\theta_3}{2})} \left( \frac{1}{l_1} \frac{8X}{\pi} \right)^{\frac{3}{2}} e^{l_1 \frac{3s}{2} e^{s^2}} 
\times G_\phi(1 + 2s; l) L^{(2)}(1 + 2s, \text{sym}^2 \phi) \frac{ds}{s}.
\]

We move the line of integration to the \(\text{Re}(s) = -u\) line with \(u = \min\left\{ \frac{1}{3}, \frac{1}{2} - \theta_3 \right\} - \varepsilon\). From \cite{7} we know there is a pole of the integrand at \(s = 0\) and we shall evaluate the residue of this pole shortly. We now bound the integral on the \(-u\) line. From \cite{10} we know that on this line we have

\[
|L(1 + 2s, \text{sym}^2 \phi)| \ll \prod_{i=1}^{3} (|s| + |\gamma_i| + 3)^6, \quad -\frac{1}{4} < \text{Re} s < 0,
\]

\[
|L_2(1 + 2s, \text{sym}^2 \phi)| \gg (1 - 2^{-1 - 2 \text{Re}(s) + 2\theta_3})^6,
\]

and on the \(-u\) line we have \(|G_\phi(1 + 2s, l)| \ll \theta_3^{\theta_3 + \varepsilon}\). Hence the integral on the line is

\[
\leq \frac{l_1^{u + \theta_3 + \varepsilon}}{X^{\frac{3u}{2} - \varepsilon}} \int_{(-u)} \prod_{i=1}^{3} (|s| + |\gamma_i| + 3)^6 (s - 1 + 2^{-1 - 2 \text{Re}(s) + 2\theta_3})^6 |\Phi(3s/2)| e^{l_1 \frac{3s}{2} e^{s^2}} \left| \prod_{i=1}^{3} \Gamma\left( \frac{s + \frac{3}{2} - \gamma_i}{2} \right) \right| \frac{|ds|}{|s|}.
\]

When \(\phi\) is not self-dual, \(L(s, \text{sym}^2 \phi)\) has no pole or zero point at \(s = 1\), we evaluate residues of pole at \(s = 0\) are \(\tilde{\Phi}(0) G_\phi(l) L^{(2)}(1, \text{sym}^2 \phi)\). When \(\phi\) is self-dual, we know \(L(s, \text{sym}^2 \phi)\) has a simple pole at \(s = 1\), so we have the Laurent series expansions

\[
\prod_{i=1}^{3} \frac{\Gamma(\frac{s+\frac{3}{2} - \gamma_i}{2})}{\Gamma(\frac{s+\theta_3}{2})} = 1 + \alpha s + \ldots;
\]

\[
\left( \frac{1}{l_1} \frac{8X}{\pi} \right)^{\frac{3}{2}} e^{l_1 \frac{3s}{2} e^{s^2}} = 1 + \frac{3}{2} \log(\frac{8X}{l_1^{\frac{3}{2}} \pi}) s + \ldots;
\]

\[
G_\phi(1 + 2s; l) = G_\phi(l) + 2G'_\phi(1; l)s + \ldots;
\]

\[
L^{(2)}(1 + 2s, \text{sym}^2 \phi) = \lim_{s_1 \to 0} s_1 L^{(2)}(1 + 2s_1, \text{sym}^2 \phi) \frac{1}{s} + c_1 + c_2 s + \ldots;
\]

and \(\tilde{\Phi}(\frac{3a}{2}) e^{s^2} = \tilde{\Phi}(0) + \frac{3}{2} \tilde{\Phi}'(0) s + \ldots\). It follows that the residues may be written as

\[
\frac{3}{2} \lim_{s_1 \to 0} s_1 L^{(2)}(1 + 2s_1, \text{sym}^2 \phi) \Phi(0) \left( G_\phi(l) \log \frac{X}{l_1^{\frac{3}{2}}} + C_\phi(l) \right)
\]

where

\[
C_\phi(l) = G_\phi(l) \left( \frac{2}{3} a - \log \pi + \frac{\tilde{\Phi}'(0)}{\tilde{\Phi}(0)} \right) + \frac{4}{3} G'_\phi(1; l).
\]
By (3.15), (3.16), and (3.17), we conclude that if \( \phi \) is not self-dual, then

\[
P(l) = \frac{2\Phi(0)}{3\zeta(2)\sqrt{l_1}} \prod_{p|l} \frac{p}{p+1} G_{\phi}(l) L^{(2)}(1, \text{sym}^2 \phi) + O(\min\{l_1^{1/4+\delta+\varepsilon} X^{-3/8+\varepsilon}, l_1^{1/4+\varepsilon} X^{\beta_3-1/4+\varepsilon}\}) + O(l_1^{\beta_3-1/4+\varepsilon} Y^{-1});
\]

and if \( \phi \) is self-dual, then

\[
P(l) = \lim_{s_1 \to 0} s_1 L^{(2)}(1 + 2s_1, \text{sym}^2 \phi) \frac{\Phi(0)}{\zeta(2)\sqrt{l_1}} \prod_{p|l} \frac{p}{p+1} \left( G_{\phi}(l) \log \frac{X}{l_1^3} + C_{\phi}(l) \right) + O(\min\{l_1^{1/4+\delta+\varepsilon} X^{-3/8+\varepsilon}, l_1^{1/4+\varepsilon} X^{\beta_3-1/4+\varepsilon}\}) + O(l_1^{\beta_3-1/4+\varepsilon} Y^{-1}).
\]

3.2. The contribution of the remainder terms \( R(l) \). By using inverse Mellin transform, we have

\[
\sum_{n=1}^{\infty} a_n g(n) = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} \sum_{n=1}^{\infty} \frac{a_n}{n^w} (\int_0^\infty g(t) t^{w-1} dt) dw.
\]

By (3.4), we may recast the expression for \( R(l) \) as

\[
R(l) = \frac{1}{2l} \sum_{\alpha \leq Y} \frac{\mu(\alpha)}{\alpha^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{2\pi i} \int_{(c)} \sum_{n=1}^{\infty} \frac{A(n,1)}{n^{k+w}} G_{4k}(ln) \phi\left( \frac{kX}{2\alpha^2 l} \right) w dw
\]

for any \( c > 0 \), where

\[
\phi(\xi, w) = \int_0^\infty \tilde{\Phi}_t\left( \frac{\xi}{t} \right) t^{w-1} dt
\]

with

\[
\tilde{\Phi}_t(t) = \Phi(t) V\left( y\left( \frac{\pi}{8Xt} \right)^{\frac{3}{2}} \right).
\]

To estimate \( R(l) \), we will need the following results for the sum over \( n \) (Lemma 3.2) and for \( \phi(\xi, w) \) (Lemma 3.3).

**Lemma 3.2.** Write \( 4k = k_1 k_2^2 \) where \( k_1 \) is a fundamental discriminant (possibly \( k_1 = 1 \) is the trivial character), and \( k_2 \) is positive. In the region \( \text{Re}(s) > 1 \) we have

\[
\sum_{n=1}^{\infty} \frac{A(n,1) G_{4k}(ln)}{n^s} = L(s, \phi \otimes \chi_{k_1}) H(s; \phi, k, l, \alpha),
\]

where \( H(s; \phi, k, l, \alpha) \) has an analytic continuation to \( \text{Re}(s) > 1/2 \). For \( \text{Re}(s) > \frac{1}{2} + \varepsilon \) we have

\[
|H(s; \phi, k, l, \alpha)| \ll |k|^{\frac{1}{2}} \alpha^{\frac{1}{2}+\varepsilon} (l, k_2^2)^{\frac{1}{2}} \sum_{h_{\alpha}|h|} \sum_{h|k} |A(h_0 h, 1)|,
\]

where \( \text{rad } n = \prod_{p|n} p \) is the radical of \( n \).
Proof. By the multiplicativity of $G_{4k}(n)$, we have the following Euler product expansion

$$
\sum_{n=1}^{\infty} \frac{A(n, 1) G_{4k}(ln)}{n^s} = L(s, \phi \otimes \chi_{k_1}) \prod_{p} H_p(s; \phi, k, l, \alpha)
$$

$$
= L(s, \phi \otimes \chi_{k_1}) H(s; \phi, k, l, \alpha),
$$

where $H_p$ is defined as follows:

$$
H_p(s; \phi, k, l, \alpha) = \begin{cases} 
(1 - A(p, 1) \frac{k_1}{p} p^{-s} + A(1, p)(\frac{k_1}{p})^2 p^{-2s} - (\frac{k_1}{p}) p^{-3s}), & p \mid 2\alpha, \\
(1 - A(p, 1) \frac{k_1}{p} p^{-s} + A(1, p)(\frac{k_1}{p})^2 p^{-2s} - (\frac{k_1}{p}) p^{-3s}) \times \sum_{r=0}^{\infty} A(p^r, 1) G_k(p^{r + \text{ord}_p(l)}) \frac{1}{p^s}, & p \nmid 2\alpha. 
\end{cases}
$$

We see that for a generic $p \nmid 2\alpha kl$, from Lemma 2.3 we have $G_k(p^{r + \text{ord}_p(l)}) = 0$ for $r \geq 2$, so

$$
H_p(s; \phi, k, l, \alpha) = (1 - A(p, 1) \frac{k_1}{p} p^{-s} + A(1, p)(\frac{k_1}{p})^2 p^{-2s} - (\frac{k_1}{p}) p^{-3s})(1 + \frac{A(p, 1)}{p^s})
$$

$$
= 1 + (A(1, p) - A(p, 1)^2) p^{-2s} + (A(p, 1) A(1, p) - 1)(\frac{k_1}{p}) p^{-3s} - A(p, 1) p^{-4s}
$$

$$
= 1 - A(p^2, 1) p^{-2s} + A(p, p)(\frac{k_1}{p}) p^{-3s} - A(p, 1) p^{-4s}.
$$

Note that for $\text{Re}(s) = \sigma > \frac{1}{2} + \varepsilon$

$$
\log \prod_{p \leq Z \atop p \nmid 2\alpha kl} H_p(s; \phi, k, l, \alpha) \ll_{\varepsilon} \sum_{p \leq Z} |A(p^2, 1)| p^{-2\sigma} + |A(p, p)| p^{-3\sigma} + |A(p, 1)| p^{-4\sigma}
$$

$$
\ll_{\varepsilon} \sum_{p \leq Z} (|A(p, 1)^2| + |A(p, 1)|) p^{-2\sigma}
$$

$$
\ll_{\varepsilon} \sum_{n \leq Z} (|A(n, 1)^2| + |A(n, 1)|) n^{-2\sigma}
$$

$$
\ll_{\varepsilon} 1.
$$

This shows that $H(s; \phi, k, l, \alpha)$ is holomorphic in $\text{Re}(s) = \sigma > \frac{1}{2} + \varepsilon$.

It remains to prove the bound (3.21). From our evaluation of $H_p(s; \phi, k, l, \alpha)$ for $p \nmid 2\alpha kl$ we see that for $\text{Re}(s) > \frac{1}{2} + \varepsilon$,

$$
|H(s; \phi, k, l, \alpha)| \ll (|k| \alpha)^{\varepsilon} \prod_{p \mid kl \atop p^j \mid 2\alpha} |H_p(s; \phi, k, l, \alpha)|.
$$

Suppose now that $p^a \parallel k$ and $p^b \parallel l$ with $a + b \geq 1$ and $p \nmid 2\alpha$. By Lemma 2.3, we may suppose that $b \leq a + 1$, since otherwise we have $H_p(s; \phi, k, l, \alpha) = 0$. Note that $|G_k(p^j)| \leq p^j$ for $0 \leq j \leq a$, and consider the cases of $a$ even and $b = a + 1$; and $a$ odd and $b = a$, or
From the definitions of $\Phi$, we use this above, and interchange the integrals over $b$ conjecture when $b = 0$, we have

$$|H_p(s; \phi, k, l, \alpha)| \leq (1 + 3p^{-\frac{1}{4}} + 3p^{-\frac{1}{12}} + p^{-\frac{3}{4}})p^{\min\{b, \frac{1}{4}\}} \sum_{r=0}^{a+b} \frac{|A(p^r, 1)|}{p^{r(\sigma - \frac{1}{4})}}.$$  

By the Hecke relation (3.11), $|G_k(p^{n+1})| \leq p^{a+\frac{1}{4}}$, and the bounds toward to Ramanujan conjecture when $b = 0$, we have

$$|H_p(s; \phi, k, l, \alpha)| \leq (2 + 3p^{-\frac{1}{4}} + 3p^{-\frac{1}{12}} + p^{-\frac{3}{4}})p^{\min\{b, \frac{1}{4}\}} \sum_{r=0}^{a} |A(p^r, 1)|.$$  

For $a = 1$ from $|G_k(p^2)| = p$ we also have $|H_p(s; \phi, k, l, \alpha)| \leq 2p^{\frac{1}{2}}$ to handle with $p \mid (k_1, l), p \nmid k_2$. Hence we get

$$|H(s; \phi, k, l, \alpha)| \ll |k|^\varepsilon l^{\frac{1}{2} + \varepsilon} \sum_{h_0 | p | \left( \frac{b}{p} \right)} \sum_{h | l} |A(h_0 h, 1)|,$$

from which we finish the proof. \(\square\)

**Lemma 3.3.** We have

$$\phi(\xi, w) = \frac{1}{2\pi i} \int_{(u)} \left( \cos \left( \frac{\pi}{2} (s - w) \right) + \text{sgn}(\xi) \sin \left( \frac{\pi}{2} (s - w) \right) \right) \left( \frac{8X}{\pi} \right)^{\frac{3}{2}} (2\pi |\xi|)^{w-\gamma} \Gamma(s-w) \times \prod_{i=1}^{3} \frac{\Gamma \left( \frac{s+\frac{1}{2}-\gamma}{2} \right)}{\Gamma \left( \frac{1}{2}-\gamma \right)} \phi \left( \frac{s}{2} + w \right) e^{s^2} ds.$$  

**Proof.** By (2.22) and (3.22), we have

$$\phi(\xi, w) = \int_0^\infty \left( \int_0^\infty \Phi(y) V(t \left( \frac{\pi}{8Xy} \right)^{\frac{3}{2}}) (\cos(2\pi y \xi) + \text{sgn}(\xi) \sin(2\pi y \xi)) dy \right) t^{w-1} dt.$$  

In the inner integral over $y$, we make the substitution $z = |\xi| y / t$, so that this integral becomes

$$\frac{t}{|\xi|} \int_0^\infty \Phi(t \left( \frac{z}{|\xi|} \right)) (\cos(2\pi z) + \text{sgn}(\xi) \sin(2\pi z)) dz.$$  

We use this above, and interchange the integrals over $z$ and $t$. Thus

$$\phi(\xi, w) = \frac{1}{|\xi|} \int_0^\infty \left( \int_0^\infty \Phi(t \left( \frac{z}{|\xi|} \right)) t^w dt \right) (\cos(2\pi z) + \text{sgn}(\xi) \sin(2\pi z)) dz.$$  

From the definitions of $\Phi_t$ (3.23) and $V(2.1)$, the inner integral is

$$\int_0^\infty V(t \left( \frac{\pi |\xi|}{8Xtz} \right)^{\frac{3}{2}}) \Phi(t \left( \frac{z}{|\xi|} \right)) t^w dt = \frac{1}{2\pi i} \int_0^\infty \left( \prod_{i=1}^{3} \frac{\Gamma \left( \frac{s+\frac{1}{2}-\gamma}{2} \right)}{\Gamma \left( \frac{1}{2}-\gamma \right)} \left( \frac{8Xz}{\pi} \right)^{\frac{3}{2}} t^{\frac{s}{2} + w} \Phi(t \left( \frac{z}{|\xi|} \right)) e^{s^2} ds \right) dt$$

$$= \frac{1}{2\pi i} \int_0^\infty \left( \prod_{i=1}^{3} \frac{\Gamma \left( \frac{s+\frac{1}{2}-\gamma}{2} \right)}{\Gamma \left( \frac{1}{2}-\gamma \right)} \left( \frac{8Xz}{\pi} \right)^{\frac{3}{2}} \left( \int_0^\infty t^{\frac{s}{2} + w} \Phi(t \left( \frac{z}{|\xi|} \right)) dt \right) e^{s^2} ds \right)$$

$$= \frac{1}{2\pi i} \int_0^\infty \left( \prod_{i=1}^{3} \frac{\Gamma \left( \frac{s+\frac{1}{2}-\gamma}{2} \right)}{\Gamma \left( \frac{1}{2}-\gamma \right)} \Phi \left( \frac{s}{2} + w \right) \left( \frac{8Xz}{\pi} \right)^{\frac{3}{2}} \left( \frac{|\xi|}{z} \right)^{\frac{3}{2} + w + 1} e^{s^2} ds \right).$$

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Here in the last equality, we have made a change of variable from \( t z/|\xi| \) to \( t \) and used the definition of \( \Phi \). Thus

\[
\phi(\xi, w) = \frac{1}{2\pi i} \int_0^\infty \int_{(u)} \left( \cos(2\pi z) + \text{sgn}(\xi) \sin(2\pi z) \right) \left( \frac{8X}{\pi} \right)^{\frac{w}{2}} |\xi|^{w-s} z^{s-w-1} \times e^{sz} \prod_{i=1}^3 \frac{\Gamma\left(\frac{s+i-\eta_i}{2}\right)}{\Gamma\left(\frac{s-\eta_i}{2}\right)} \Phi\left(\frac{s}{2} + w\right) d\lambda dz.
\]

Interchange the integrals over \( s \) and \( z \), employing the expressions for the Fourier sine and cosine transforms of \( z^{s-w-1} \), we obtain the lemma.

From \([7]\) we know \( L \)-functions for Hecke–Maass forms are all entire. By Lemmas \([3, 2]\) and \([3, 3]\) and moving the lines of \( w \) and \( s \) such that \( \text{Re}(w-s) = -\frac{3}{4} - 2\varepsilon \) and \( \text{Re}(w) = -\frac{1}{2} + 2\varepsilon \) (so \( \text{Re}(s) = \frac{3}{4} + 4\varepsilon \)) in \([3, 21]\), we obtain

\[
R(l) = \frac{1}{4l\pi i} \sum_{\alpha \leq Y} \sum_{\alpha^2 \ell = 1} \sum_{(\alpha, 2l) = 1} \prod_{k=-\infty \atop k \neq 0}^{\infty} \frac{(-1)^k}{2\pi i} \int_{(-\frac{1}{2}+2\varepsilon)} \int_{(\frac{1}{4}+4\varepsilon)} L(1+w, \phi \otimes \chi_{k_1}) H(1+w; \phi, k, l, \alpha)
\times \left( \frac{k}{\alpha^2 \ell} \right)^{w-s} X^{\frac{3}{2}+w} \pi^{-\frac{3}{4}} \Phi\left(\frac{s}{2} - w\right) \gamma(|w|) e^{sz} \prod_{i=1}^3 \frac{\Gamma\left(\frac{s+i-\eta_i}{2}\right)}{\Gamma\left(\frac{s-\eta_i}{2}\right)} \Phi\left(\frac{s}{2} + w\right) d\lambda dw
\ll \frac{l^{3}+3\varepsilon}{X^{\frac{3}{4}-4\varepsilon}} \sum_{\alpha \leq Y} \alpha^{1/2+\varepsilon} \int_{(\frac{1}{4}+4\varepsilon)} \int_{(-\frac{1}{2}+2\varepsilon)} \sum_{k_2=1}^{\infty} \sum_{Z \geq 1} \sum_{Z \leq k_2 \leq 2Z} |L(1+w, \phi \otimes \chi_{k_1})|(l, k_2 \gamma(k_2))^{-\frac{3}{4}+\varepsilon} \times \sum_{h|k_1 k_2} |A(h, 1)| |\Phi\left(\frac{s}{2} + w\right)|(1 + |w-s|)^{\frac{3}{2}+2\varepsilon} \prod_{i=1}^3 |\Gamma\left(\frac{s+i-\eta_i}{2}\right)| e^{sz} |d\omega| |d\lambda|.
\]

Here we have used the Stirling’s formula to estimate \( \Gamma(s-w) \). We have

\[
\sum_{h_0|k_1 \ell} \sum_{h_0 \equiv p} |A(h_0 h_1, 1)| \leq \sum_{h_1|k_1 \ell} \sum_{h_1 \equiv p} \sum_{h_2|k_2 \ell} |A(h_1 h_2 h_3, 1)|.
\]

By using the approximate functional equations (cf. Lemma \([2,1]\) and the large sieve estimate for quadratic characters in \([8]\), for \( \text{Re}(w) = -1/2 + 2\varepsilon \), and in a similar way as \([3, 6]\), we get

\[
\sum_{Z \leq k_1 \leq 2Z} |L(1+w, \phi \otimes \chi_{k_1})| \sum_{h_1|k_1} |A(h_1 h_2 h_3, 1)| \ll \left( \sum_{Z \leq k_1 \leq 2Z} \sum_{h_1|k_1} \sum_{h_2|k_2 \ell \equiv p} |A(h_1 h_2 h_3, 1)|^2 \right)^{\frac{1}{2}} \ll Z^{\frac{3}{4}+\varepsilon} (3 + |w| + \sum_{i=1}^3 |\gamma_i|^{\frac{3}{4}+\varepsilon} (h_2 h_3)^{\theta_1+\varepsilon}.
\]
Hence we have

\[
R(l) \ll \frac{l^{4+3\varepsilon}Y^{\frac{3}{2}+2\varepsilon}}{X^{\frac{1}{6}-4\varepsilon}} \int_{(\frac{3}{2}+2\varepsilon)} \sum_{k_2=1}^{\infty} \sum_{h_2 | (l, k_2^2)} \sum_{(h_3, t) = 1} \frac{(l, k_2^2)^2 (h_2 h_3)^{6s+\varepsilon}}{k_2^{\frac{3}{2}-\varepsilon}} (1 + |\frac{3s}{2}| + |w + \frac{s}{2}|)^{\frac{3}{2}+2\varepsilon}
\]

\[
\times (3 + |w + \frac{s}{2}| + |\frac{s}{2}| + \sum_{i=1}^{3} |\gamma_i|)^{3+\varepsilon} |\Phi|^{\frac{s}{2} + w} \prod_{i=1}^{3} \Gamma\left(\frac{s + \frac{1}{2} + \gamma_i}{2}\right) |e^{s^2}/s||dwds|
\]

\[
\ll \frac{l^{\frac{3}{2}+3\varepsilon}Y^{\frac{3}{2}+2\varepsilon}}{X^{\frac{1}{6}-4\varepsilon}},
\]

(3.25)

because \((l, k_2^2)^{\frac{3}{2}+\varepsilon} (h_2 h_3)^{6s+\varepsilon} \leq (k_2, l^{\infty})^{1+\frac{3}{2}+\varepsilon} \).

### 3.3. The contribution of the remainder terms \(S_R(\chi_{sd}(l)B(d); \Phi)\).

Observe that \(R_Y(d)\) equals 0 unless \(d = d_1d_2\) where \(d_1\) is square-free and \(d_2 > Y\).

Hence

\[
S_R(\chi_{sd}(l)B(d); \Phi) = X^{-1} \sum_{Y < d_2 \leq \sqrt{2X}} \mu(d_2) \sum_{(d_1, t) = 1} \chi_{sd_1}(l) B(d_1 d_2^2) \Phi\left(\frac{d_1}{X/d_2}\right),
\]

and

\[
B(d_1 d_2^2) = \frac{1}{2\pi i} \left(\begin{array}{c} Y_{d_2} \\ d_2 \end{array}\right) \prod_{j=1}^{3} \frac{\Gamma\left(\frac{s + \frac{1}{2} - \gamma_j}{2}\right)}{\left(\frac{1}{2} - \gamma_j\right)} \left(\begin{array}{c} 8d_1 d_2^2 \pi \\ s \end{array}\right)^\frac{3}{2} s \sum_{n=1}^{\infty} \chi_{sd_1 d_2^2}(n) A(n, 1) n^{s+\frac{1}{2}} e^{s^2}/s.
\]

Plainly

\[
\sum_{n=1}^{\infty} \chi_{sd_1 d_2^2}(n) A(n, 1) n^{s+\frac{1}{2}} = L\left(\frac{1}{2} + s, \phi \otimes \chi_{sd_1}\right) I_{d_1}(s, d_2),
\]

where

\[
I_{d_1}(s, d_2) = \prod_{p | d_2} (1 - \frac{A(p, 1) \chi_{sd_1}(p)}{p^{s+\frac{1}{2}}} + \frac{A(1, p) \chi_{sd_1}(p)^2}{p^{2s+1}} - \frac{\chi_{sd_1}(p)}{p^{3s+\frac{1}{2}}}).
\]

Hence we may move the line of integration to the line \(\text{Re}(s) = \frac{1}{\log X}\). This gives

\[
B(d_1 d_2^2) = \frac{1}{2\pi i} \left(\begin{array}{c} Y_{d_2} \\ d_2 \end{array}\right) \prod_{j=1}^{3} \frac{\Gamma\left(\frac{s + \frac{1}{2} - \gamma_j}{2}\right)}{\left(\frac{1}{2} - \gamma_j\right)} \left(\begin{array}{c} 8d_1 d_2^2 \pi \\ s \end{array}\right)^\frac{3}{2} s L\left(\frac{1}{2} + s, \phi \otimes \chi_{sd_1}\right) I_{d_1}(s, d_2) e^{s^2}/s.
\]

Plainly

\[
|I_{d_1}(s, d_2)| \ll t^\varepsilon \prod_{p | d_2} (1 + p^{\frac{3}{2} - \frac{1}{2} - \text{Re}(s)}).
\]

From \(e^{s^2}\) is a Schwartz function, we truncate the integral from \(\frac{1}{\log X} - X^{\varepsilon}i\) to \(\frac{1}{\log X} + X^{\varepsilon}i\),

Thus

\[
B(d_1 d_2^2) = \frac{1}{2\pi i} \int_{\frac{1}{\log X} - X^{\varepsilon}i}^{\frac{1}{\log X} + X^{\varepsilon}i} \prod_{j=1}^{3} \frac{\Gamma\left(\frac{s + \frac{1}{2} - \gamma_j}{2}\right)}{\left(\frac{1}{2} - \gamma_j\right)} \left(\begin{array}{c} 8d_1 d_2^2 \pi \\ s \end{array}\right)^\frac{3}{2} s L\left(\frac{1}{2} + s, \phi \otimes \chi_{sd_1}\right) I_{d_1}(s, d_2) e^{s^2}/s + O\left(\frac{1}{X^{2022}}\right),
\]
Assume should use a recursive argument as in Young [21] to prove the following lemma.

By using the approximate functional equations of $L(\frac{1}{2} + s, \phi \otimes \chi_{8d})$ and the large sieve estimate for quadratic characters in [8], we have

$$\sum_{d \leq X} \mu(d) A(d, 1) \sum_{d \leq Y} A(1, d) \sum_{d \leq X} \frac{1}{d}$$

for some constant $B > 0$. Then we truncate the sum of $d_2$ in (3.26) from $Y$ to $X^{\frac{1}{4}}$, getting

$$S_R(\chi_{8d}(l)B(d); \Phi) = \frac{1}{2\pi i X} \prod_{j=1}^{3} \Gamma \left( \frac{s+j-\gamma_j}{2} \right) \prod_{j=1}^{3} \Gamma \left( \frac{s+j-\gamma_j}{2} \right)$$

But we cannot get strong enough estimate of $S_R$ by simply applying (3.27) to all $d_2$. We should use a recursive argument as in Young [21] to prove the following lemma.

**Lemma 3.4.** Assume $1 \leq Y \leq X^{1/4}$. Then we have

$$S_R(\chi_{8d}(l)B(d); \Phi) \ll l^2 \epsilon X^\epsilon.$$  

**Proof.** We would show that if for all $Z \leq 2X$ and $\frac{1}{log X} \leq \text{Re}(s_0) \leq \frac{1000}{log X}$, $-X^\epsilon \leq \text{Im}(s_0) \leq X^\epsilon$, we have

$$\sum_{d} \chi_{8d}(l_0) \left( \frac{8d}{\pi} \right)^s L \left( \frac{1}{2} + s_0, \phi \otimes \chi_{8d} \right) \Phi \left( \frac{d}{Z} \right) \ll l_0^2 Z^{1+\delta} |1 + s_0|^B + l_0^{\frac{3}{2} + \epsilon} Z^{\frac{3}{2} + \epsilon} |1 + s_0|^B$$

with some constants $\delta \geq 0$ and $B > 0$, then we have

$$\sum_{d} \chi_{8d}(l_0) \left( \frac{8d}{\pi} \right)^s L \left( \frac{1}{2} + s_0, \phi \otimes \chi_{8d} \right) \Phi \left( \frac{d}{Z} \right) \ll (l_0^2 Z^{1+\delta} + l_0^3 Z^{1+\epsilon} + l_0^4 Z^{3+\epsilon}) |1 + s_0|^B (1 + |v|)^B$$

with some constants $B' \geq B$ and $B'' > 0$. 

Now we could iterate from $\delta = \frac{1}{4} + \varepsilon$ as in (3.27), set $\text{Re}(s_0) = \frac{6}{\log X}$, and iterate five times, then we have

$$
\sum_{d_1}^{b} \chi_{sd_1}(ld_3d_5d_4^2)(\frac{8d_1}{\pi})^{\frac{3}{2}}L(\frac{1}{2} + s, \phi \otimes \chi_{sd_1})\Phi(\frac{d_1}{X/d_2})
$$

$$
\ll \left((ld_3d_5d_4^2)^{\frac{3}{2}+\varepsilon}X^{\frac{19}{30}+\varepsilon}d_2^{\frac{19}{10}+\varepsilon} + l^\varepsilon X^{1+\varepsilon}d_2^{-2+\varepsilon}\right)|1 + s|^A
$$

with some constant $A$. Now by (3.28) we prove (3.29).

Now we still need to prove (3.31) by using (3.30). As in (3.1), we need to calculate

$$
S_1 = \sum_{n=1}^{\infty} \frac{A(n, 1)}{n^{\frac{3}{2}+s_0}} \sum_{d}^{b} \chi_{sd}(l_0n)(\frac{8d}{\pi})^{\frac{3}{2}s_0} \frac{1}{2\pi i} \int_{(u)} \prod_{j=1}^{3} \frac{\Gamma(\frac{s_0+\varepsilon+\gamma_j}{2})}{\Gamma(\frac{s_0+\gamma_j}{2})} (n(\frac{\pi}{8d}))^{\frac{3}{2}s_0} e^{s_0^2 \frac{d}{s}} \Phi(\frac{d}{Z})
$$

and

$$
S_2 = \sum_{n=1}^{\infty} \frac{A(1, n)}{n^{\frac{3}{2}+s_0}} \sum_{d}^{b} \chi_{sd}(l_0n)(\frac{8d}{\pi})^{-\frac{3}{2}s_0} \frac{1}{2\pi i} \int_{(u)} \prod_{j=1}^{3} \frac{\Gamma(\frac{s+\varepsilon+\gamma_j}{2})}{\Gamma(\frac{s+\gamma_j}{2})} (n(\frac{\pi}{8d}))^{\frac{3}{2}s_0} e^{s_0^{2} \frac{d}{s}} \Phi(\frac{d}{Z}).
$$

We also use $\mu^2(d) = M_{YZ}(d) + R_{YZ}(d)$, and we have

$$
S_1 = ZS_{1,M} + ZS_{1,R}
$$

where

$$
S_{1,M} = \sum_{n=1}^{\infty} \frac{A(n, 1)}{n^{\frac{3}{2}+s_0}} S_{M,Z,YZ}(\chi_{sd}(l_0n); \Phi_{1,n,s}),
$$

$$
S_{1,R} = \sum_{n=1}^{\infty} \frac{A(1, n)}{n^{\frac{3}{2}+s_0}} S_{R,Z,YZ}(\chi_{sd}(l_0n); \Phi_{1,n,s}),
$$

with $S_{M,Z,YZ}$ and $S_{R,Z,YZ}$ in §2, and

$$
\Phi_{1,g,s}(t) = \frac{\Phi(t)}{2\pi i} (\frac{8Zt}{\pi})^{\frac{3}{2}s_0} \frac{1}{2\pi i} \int_{(u)} \prod_{j=1}^{3} \frac{\Gamma(\frac{s_0+\varepsilon+\gamma_j}{2})}{\Gamma(\frac{s_0+\gamma_j}{2})} (n(\frac{\pi}{8Zt}))^{\frac{3}{2}s_0} e^{s_0^{2} \frac{d}{s}}.
$$

Then by Poisson summation formula we have $S_{1,M} = P_1 + R_1$ according to $k = 0$ and $k \neq 0$, then using the same method as §3.1 from

$$
\frac{1}{\Gamma(\frac{s_0+\gamma_j}{2})} \ll \text{Im}(s)^{\frac{3}{2}},
$$

and

$$
G_0(1 + s_0; l_0) \ll l_0^{\frac{3}{2}+\varepsilon},
$$

we have $P_1 \ll l_0^{\varepsilon}Z^{\varepsilon} + l_0^{\frac{1}{2}}Z^{-\frac{3}{4}+\varepsilon}Y_Z^{\frac{3}{4}+\varepsilon}$. Also by using the same method as §3.2 we have

$$
R_1 \ll l_0^{\frac{3}{2}+\varepsilon}Y_Z^{\frac{1}{2}+\varepsilon}Z^{-\frac{1}{4}+\varepsilon}. \text{ By estimating terms in } S_{1,R} \text{ with } Y_Z < d_2 \leq Z^{\frac{1}{4}} \text{ like our previous discussion about (3.28) with (3.30), estimating terms in } S_{1,R} \text{ with } d_2 > Z^{\frac{1}{4}} \text{ with (3.27), and}
$$
taking $Y_Z = Z^{1/20}$, we have

$$S_{1,R}(\chi_{8d}(l)B(d); \Phi) \ll \frac{1}{Z} \left( \int_{\log X}^{X} \sum_{j=1}^{3} \prod_{i=1}^{3} \left| \frac{\Gamma(s+\delta+i\gamma_i)}{\Gamma(s+\delta/2)} \right| \right) \sum_{d_3|d_2} |\mu(d_2)| \sum_{d_3|d_2} d_3^{-\frac{1}{2}+\delta_3}$$

$$\times \sum_{d_4|d_2} d_4^{-1+\theta_3} \sum_{d_5|d_3d_4} d_5^{\frac{3}{2}} \left( (l_0d_3d_5d_4)^{\varepsilon} \frac{Z}{d_2^{1+\delta}} + (l_0d_3d_5d_4)^{\frac{3}{2}+\varepsilon} \frac{Z}{d_2^{1+\delta}} \right)$$

$$\times |1 + s_0 + s|^{B}e^{s|d_s/|s|} + O(Z^{-\frac{3}{2}+\varepsilon})$$

$$\ll l_0^{\frac{3}{2}+\varepsilon} Z^{-\frac{1}{20}+\varepsilon} + l_0^\varepsilon Z^{1+\varepsilon} + l_0^\varepsilon Z^{1+\delta-\frac{3}{2}}.$$ 

Hence we get

$$S_1 \ll l_0^{\varepsilon} Z^{1+\varepsilon} + l_0^\varepsilon Z^{\frac{19}{20}+\varepsilon} + l_0^\varepsilon Z^{1+\delta-\frac{3}{2}}.$$ 

Similarly for $S_2$. This proves (3.31), and completes the proof of the lemma. □

Now by (3.1), (3.19), (3.20), (3.25), and Lemma 3.4, if we take $Y = X^{1/20}$, we have when $\phi$ is not self-dual

$$\sum_{2|d} L \frac{1}{2} \phi \otimes \chi_{8d} \chi_{8d}(l) \Phi \left( \frac{d}{X} \right) = \frac{2\Phi(0)}{3\zeta(2)\sqrt{l_1}} \prod_{p|l} \frac{p}{p + 1} \left( G_{\phi}(l) L^{(2)}(1, \text{sym}^2 \phi) \right)$$

$$+ \prod_{i=1}^{3} \frac{\Gamma \left( \frac{1}{2} + \gamma_i \right)}{\Gamma \left( \frac{1}{2} - \gamma_i \right)} G_{\phi}(l) L^{(2)}(1, \text{sym}^2 \tilde{\phi}) + O_{\Phi}(l^{\frac{3}{2}+\varepsilon} X^{\frac{19}{20}+\varepsilon}), \quad (3.32)$$

and when $\phi$ is self-dual

$$\sum_{2|d} L \frac{1}{2} \phi \otimes \chi_{8d} \chi_{8d}(l) \Phi \left( \frac{d}{X} \right) = \lim_{s \to 1} \left( s - 1 \right) L^{(2)}(s, \text{sym}^2 \phi) \Phi(0) \prod_{p|l} \frac{p}{p + 1} \frac{X}{\zeta(2)\sqrt{l_1}}$$

$$\times G_{\phi}(l) \log \frac{X}{l_1^{\frac{3}{2}}} + C_{\phi}(l) + O_{\Phi}(l^{\frac{3}{2}+\varepsilon} X^{\frac{19}{20}+\varepsilon}). \quad (3.33)$$

This completes the proof of Theorem 1.3.

4. PROOF OF THEOREM 1.1

Lemma 4.1. Let $M$ be a fixed integer coprime with 3, 5, 7, 11, $G_{\phi}(l)$ be defined as in Theorem 1.3. Let $Q_{\phi}$ be a positive integer such that $(Q_{\phi}, 2M) = 1$ and $G_{\phi}(Q_{\phi}) \neq 0$. For normalized $\phi$ and $\phi'$, if we have $G_{\phi}(Q_{\phi}N) = a \cdot G_{\phi'}(Q_{\phi}N)$ for all $N$ coprime to $2Q_{\phi}M$, with some nonzero constant $a$, then $\phi = \phi'$. 

Proof. By Theorem 1.3 we know $G_{\phi}(c_{\phi}H^2) \neq 0$ for all $(H, 2c_{\phi}) = 1$. So there exists a positive integer $Q_{\phi}$ such that $(Q_{\phi}, 2M) = 1$ and $G_{\phi}(Q_{\phi}) \neq 0$. Let $A(m, n)$ be the coefficients of $\phi$ and $A'(m, n)$ be the coefficients of $\phi'$. From $G_{\phi}(Q_{\phi}) = a \cdot G_{\phi'}(Q_{\phi})$ with $a \neq 0$, we have $G_{\phi'}(Q_{\phi}) \neq 0$. For any odd prime $p \geq 13$ satisfying $(p, 2Q_{\phi}M) = 1$, by comparing both sides of $G_{\phi}(Q_{\phi}N) = a \cdot G_{\phi'}(Q_{\phi}N)$ with $N = p$ and $p^2$, and the fact $G_{\phi}(Q_{\phi}p^2), G_{\phi'}(Q_{\phi}p^2) \neq 0$, we have

$$G_{\phi}(Q_{\phi}p)/G_{\phi}(Q_{\phi}p^2) = G_{\phi'}(Q_{\phi}p)/G_{\phi'}(Q_{\phi}p^2).$$
Hence we have
\[ G_{\phi,p}(Q_{\phi}p)/G_{\phi,p}(Q_{\phi}p^2) = G_{\phi',p}(Q_{\phi}p)/G_{\phi',p}(Q_{\phi}p^2). \]
From the definition (3.9), we have
\[ \frac{1 + pA(p, 1)}{p + A(1, p)} = \frac{1 + pA'(p, 1)}{p + A'(1, p)}, \]
and hence
\[ p^2A(p, 1) + pA(p, 1)A'(p, 1) + A(p, 1) = p(1 + pA'(p, 1)A(p, 1) + A(p, 1). \]
(4.1)
By comparing the real parts of both sides, we get
\[ p^2 \text{Re}(A(p, 1)) + p \text{Re}(A(p, 1)) \text{Re}(A'(p, 1)) + p \text{Im}(A(p, 1)) \text{Im}(A'(p, 1)) + \text{Re}(A'(p, 1)) \]
\[ = p^2 \text{Re}(A'(p, 1)) + p \text{Re}(A(p, 1)) \text{Re}(A'(p, 1)) + p \text{Im}(A(p, 1)) \text{Im}(A'(p, 1)) + \text{Re}(A(p, 1)). \]
So we get \((p^2 - 1) \text{Re}(A(p, 1)) = (p^2 - 1) \text{Re}(A'(p, 1)),\) from which we obtain
\[ \text{Re}(A(p, 1)) = \text{Re}(A'(p, 1)). \]
(4.2)
Then by comparing the imaginary parts of both sides in (4.1), we get
\[ p^2 \text{Im}(A(p, 1)) - p \text{Re}(A(p, 1)) \text{Im}(A'(p, 1)) + p \text{Re}(A(p, 1)) \text{Im}(A(p, 1)) - \text{Im}(A'(p, 1)) \]
\[ = p^2 \text{Im}(A'(p, 1)) + p \text{Re}(A(p, 1)) \text{Im}(A'(p, 1)) - p \text{Re}(A'(p, 1)) \text{Im}(A(p, 1)) - \text{Im}(A(p, 1)). \]
Together with (4.2), we have
\[ (p^2 + 2p \text{Re}(A(p, 1)) + 1)(\text{Im}(A(p, 1)) - \text{Im}(A'(p, 1))) = 0. \]
(4.3)
By the bounds toward to the Ramanujan conjecture, we get \(|\text{Re}(A(p, 1))| \leq |A(p, 1)| \leq 3p^{\frac{3}{2}} < \frac{p^2 + 1}{2p},\) and hence \(p^2 + 2p \text{Re}(A(p, 1)) + 1 > 0,\) provided \(p \geq 17.\) Thus for \(p, 2Q_{\phi} M = 1\) and \(p \geq 17,\) we have
\[ \text{Im}(A(p, 1)) = \text{Im}(A'(p, 1)). \]
(4.4)
Thus we know that \(A(p, 1) = A'(p, 1)\) for all odd primes \(p \geq 17\) which are coprime to \(2Q_{\phi} M.\) By the strong multiplicity one theorem (see e.g. [7, §12.6]), we prove \(\phi = \phi'.\)

**Proof of Theorem 1.1.** We first claim that the equality of twisted \(L\)-values determines whether the forms are both self-dual or not. By taking \(l = c_{\phi} \) in Theorem 1.3, we get \(G_{\phi}(l) \neq 0.\) So we know \(S(\phi) : = \sum_{2|d} \chi_{sd}(l) L\left(\frac{1}{2}, \phi \otimes \chi_{sd}\right) \Phi\left(\frac{d}{X}\right) = \asymp X \log X\) if \(\phi\) is self-dual and \(\ll X\) if not. Since \(L\left(\frac{1}{2}, \phi \otimes \chi_{sd}\right) = \kappa L\left(\frac{1}{2}, \phi' \otimes \chi_{sd}\right)\) for all odd square-free \(d\) and \(\kappa \neq 0,\) we get \(S(\phi') = \kappa S(\phi)\) is \(\asymp X \log X\) if \(\phi\) is self-dual. By Theorem 1.3 again, we know \(S(\phi') \asymp X \log X\) holds only if \(\phi'\) is self-dual. Hence we show that if \(\phi\) is self-dual then \(\phi'\) is also. Similarly, we know if \(\phi'\) is self-dual then \(\phi\) is also. This proves our claim.

Case i). If \(\phi\) and \(\phi'\) are both self-dual, then by comparing the main terms in Theorem 1.3 we have
\[ G_{\phi}(l) = G_{\phi'}(l) = aG_{\phi'}(l) = aG_{\phi}(l) \]
with some nonzero constant \(a\) depending on \(\phi\) and \(\phi'\) but not \(l.\) Then in Lemma 4.1 let \(Q_{\phi} = c_{\phi}\) we have \(\phi = \phi'.\)

Case ii). If \(\phi\) and \(\phi'\) are both not self-dual, from Theorem 1.3 we have
\[ a_1 G_{\phi}(l) + a_2 G_{\phi}(l) = b_1 G_{\phi'}(l) + b_2 G_{\phi'}(l) \]
(4.5)
where $a_1 = L^{(2)}(1, \text{sym}^2 \phi)$, $a_2 = \prod_{i=1}^{3} \frac{\Gamma\left(\frac{i+\gamma_{\phi}}{2}\right)}{\Gamma\left(\frac{i}{2}\right)}L^{(2)}(1, \text{sym}^2 \tilde{\phi})$, $b_1 = L^{(2)}(1, \text{sym}^2 \phi')$, $b_2 = \prod_{i=1}^{3} \frac{\Gamma\left(\frac{i+\gamma_{\phi'}}{2}\right)}{\Gamma\left(\frac{i}{2}\right)}L^{(2)}(1, \text{sym}^2 \phi')$ are all non-zero.

By strong multiplicity one theorem and $\phi \neq \tilde{\phi}$, we know there are infinity primes $p_0 \geq 13$ such that $A(p_0, 1) \neq A(1, p_0)$. For such $p_0$, we have $\frac{1+p_0A(p_0)}{p_0+A(1, p_0)} \notin \mathbb{R}$, and $G_{\phi}(c_0 p_0) \neq 0$. Fix such a prime $p_0$ with $(p_0, M) = 1$. Then there exists a $d_0 \in \{c_0 p_0, c_0 p_0^2\}$ such that $a_1 G_{\phi}(d_0) + a_2 G_{\phi}(d_0) \neq 0$. In particular, we have $G_{\phi}(d_0) \neq 0$. Indeed, if not, then we have $a_1 G_{\phi}(c_0 p_0) + a_2 G_{\phi}(c_0 p_0) = a_3 G_{\phi}(c_0 p_0^2) + a_2 G_{\phi}(c_0 p_0^2) = 0$. Together with $G_{\phi}(c_0 p_0) = \frac{1+p_0A(p_0)}{p_0+A(1, p_0)} G_{\phi}(c_0 p_0^2)$ and $\frac{1+p_0A(p_0)}{p_0+A(1, p_0)} \notin \mathbb{R}$, we get $G_{\phi}(c_0 p_0^2) = 0$, which contradicts with Theorem 1.3. Note that by (4.5) we have $b_1 G_{\phi'}(d_0) + b_2 G_{\phi'}(d_0) \neq 0$. So $G_{\phi'}(d_0) \neq 0$.

If $\phi = \phi'$, then we finish the proof.

If $\phi \neq \phi'$, then by the strong multiplicity one theorem (see e.g. [7, §12.6]), there are infinity primes $p \geq 17$ satisfying $A(p, 1) \neq A'(p, 1)$. By the same argument as in Lemma 4.1 we have $\frac{1+pA(p, 1)}{p+A(1, p)} \neq 1\frac{1+pA'(p, 1)}{p+A'(1, p)}$ for such $p$'s. Hence for those $p$, we have

$$G_{\phi, p}(pd_0)/G_{\phi, p}(p^2d_0) \neq G_{\phi', p}(pd_0)/G_{\phi', p}(p^2d_0).$$

We fix one such $p \geq 17$ with $(p, d_0 M) = 1$. By (4.5) with $l = pd_0 N$ and $l = p^2d_0 N$, we obtain

$$a_1 G_{\phi}(pd_0 N) + a_2 G_{\phi}(pd_0 N) = b_1 G_{\phi'}(pd_0 N) + b_2 G_{\phi'}(pd_0 N),$$

and

$$a_1 G_{\phi}(p^2d_0 N) + a_2 G_{\phi}(p^2d_0 N) = b_1 G_{\phi'}(p^2d_0 N) + b_2 G_{\phi'}(p^2d_0 N)$$

for any integer $N$ satisfying $(N, 2d_0 p M) = 1$. Thus by a linear combination of the above two identities to eliminate $a_1$, and $G_{\phi}(pd_0 N) = \frac{G_{\phi, p}(pd_0)}{G_{\phi', p}(pd_0)} G_{\phi'}(p^2d_0 N)$, we get

$$
\left(\frac{G_{\phi, p}(pd_0)}{G_{\phi, p}(p^2d_0)} - \frac{G_{\phi, p}(pd_0)}{G_{\phi', p}(p^2d_0)}\right) a_2 G_{\phi}(p^2d_0 N) = \left(\frac{G_{\phi, p}(pd_0)}{G_{\phi', p}(pd_0)} - \frac{G_{\phi, p}(pd_0)}{G_{\phi', p}(pd_0)}\right) b_1 G_{\phi'}(p^2d_0 N) + \left(\frac{G_{\phi, p}(pd_0)}{G_{\phi', p}(pd_0)} - \frac{G_{\phi, p}(pd_0)}{G_{\phi', p}(pd_0)}\right) b_2 G_{\phi'}(p^2d_0 N).
$$

(4.7)

By (4.6), we get

$$\frac{G_{\phi, p}(pd_0)}{G_{\phi', p}(p^2d_0)} - \frac{G_{\phi, p}(pd_0)}{G_{\phi', p}(p^2d_0)} \neq 0.$$ 

(4.8)

Indeed, if (4.8) is not true, then by (4.7) we have

$$G_{\phi'}(p^2d_0 N) = a \cdot G_{\phi'}(p^2d_0 N)$$

for all $(N, 2d_0 p M) = 1$ with some constant $a$. Let $N$ be a square-full number then we must have $a \neq 0$. Then by Lemma 4.1 with $Q_{\phi'} = p^2d_0$ we have $\phi' = \tilde{\phi}'$, which contradicts to that $\phi'$ is not self-dual.

By (4.7) and (4.8), we can rewrite as

$$G_{\phi}(p^2d_0 N) = c_1 G_{\phi'}(p^2d_0 N) + c_2 G_{\phi'}(p^2d_0 N)$$

(4.9)
for all \((N, 2d \phi p M) = 1\) with some constants \(c_1, c_2\) independent of \(N\), and we know \(c_1 \neq 0\).

Note that as in (4.6), there are infinity many primes \(q\) with \((q, p M) = 1\) such that

\[ G_{\phi, q}(q^2 p^2 d_\phi)/G_{\phi, q}(q^2 p^2 d_\phi) \neq G_{\phi, q}(q^2 p^2 d_\phi)/G_{\phi, q}(q^2 p^2 d_\phi). \]

We fix one such \(q\). By similar arguments as above we can eliminate \(c_2\) in (1.9), getting

\[
\left( \frac{G_{\phi, q}(q^2 p^2 d_\phi)}{G_{\phi, q}(q^2 p^2 d_\phi)} - \frac{G_{\phi, q}(q^2 p^2 d_\phi)}{G_{\phi, q}(q^2 p^2 d_\phi)} \right) G_{\phi}(q^2 p^2 d_\phi, N) \]

for all \((N, 2 p q d_\phi M) = 1\). Let \(N\) be a square-full number then we know \(G_{\phi, q}(q^2 p^2 d_\phi, N) / G_{\phi, q}(q^2 p^2 d_\phi, N) \neq 0\). Thus we have \(G_{\phi}(q^2 p^2 d_\phi N) = a \cdot G_{\phi'}(q^2 p^2 d_\phi N)\) for all \((N, 2 p q d_\phi M) = 1\) with some nonzero constant \(a\) depends on \(\phi, \phi', p, q, M, d_\phi\). Then by Lemma 4.1 with \(Q_\phi = q^2 p^2 d_\phi\), we have \(\phi = \bar{\phi}'\).

In conclusion, if \(\phi\) and \(\phi'\) are both not self-dual, then we have \(\phi = \phi'\) or \(\bar{\phi}'\). This completes the proof of Theorem 1.8.

\[\Box\]

5. Proof of Theorem 1.8

Proof of Theorem 1.8. By the Rankin–Selberg theory, we have (see 16)

\[
\sum_{m^2 n \leq X} |A(m, n)|^2 \ll X^{1+\varepsilon},
\]

and hence

\[
\sum_{n \leq X} |A(n, 1)| \ll X^{1+\varepsilon}, \quad (5.1)
\]

Recall that

\[ L(s, \phi \otimes \phi) = \sum_{k, m, n} \sum_{1} A(m, n)^2 \left( \frac{k m^2 n}{s} \right)^s, \quad \text{Re}(s) > 1. \]

Denote

\[ \lambda_{\phi \otimes \phi}(q) = \sum_{k^2 m^2 n = q} A(m, n)^2. \]

Then we have

\[
\sum_{q \leq X} \lambda_{\phi \otimes \phi}(q) \leq \sum_{q \leq X} \lambda_{\phi \otimes \phi}(q) \ll X^{1+\varepsilon}. \quad (5.2)
\]

Denote \(L_p(s, \text{sym}^2 \phi) = \sum_{h=0}^\infty B(p^h, h)\) and \(L_p(s, \text{sym}^2 \phi) / L_p(s, \bar{\phi}) = \sum_{h=0}^\infty C(p^h, 1) = 0\) if \(h < 0\). From

\[
L_p(s, \text{sym}^2 \phi) = \frac{L_p(s, \phi \otimes \phi)}{L_p(s, \bar{\phi})} = \sum_{h \geq 0} \frac{\lambda_{\phi \otimes \phi}(p^h)}{p^{hs}} \left( 1 - A(1, p) \frac{1}{p^s} + A(p, 1) \frac{1}{p^{2s}} - \frac{1}{p^{3s}} \right)
\]

we have

\[
B(p^h, 1) = \lambda_{\phi \otimes \phi}(p^h) - A(1, p) \lambda_{\phi \otimes \phi}(p^{h-1}) + A(p, 1) \lambda_{\phi \otimes \phi}(p^{h-2}) - \lambda_{\phi \otimes \phi}(p^{h-3}). \quad (5.3)
\]
Similarly we have
\[
C(p^h, 1) = B(p^h, 1) - A(1, p)B(p^{h-1}, 1) + A(p, 1)B(p^{h-2}, 1) - B(p^{h-3}, 1),
\]
and by (3.14) we have
\[
A(p^{2h}, 1) = C(p^h, 1) + A(1, p)C(p^{h-1}, 1).
\]
Define \( B(n, 1) \) by multiplicativity. We first prove
\[
\sum_{n \leq X} |B(n, 1)| \ll X^{1+\varepsilon}
\]
(5.6) based on (5.2) and (5.3). By (5.3) we have
\[
|B(n, 1)| \leq \prod_{p \nmid n} (|\lambda_{\phi \circ \phi}(p^a)| + |A(p, 1)||\lambda_{\phi \circ \phi}(p^{a-1})| + |A(p, 1)||\lambda_{\phi \circ \phi}(p^{a-2})| + |\lambda_{\phi \circ \phi}(p^{a-3})|).
\]
Since \(|A(n, 1)| \) and \(|\lambda_{\phi \circ \phi}(n)| \) are multiplicative, we have
\[
\prod_{p^a \mid n} (|\lambda_{\phi \circ \phi}(p^a)| + |A(p, 1)||\lambda_{\phi \circ \phi}(p^{a-1})| + |A(p, 1)||\lambda_{\phi \circ \phi}(p^{a-2})| + |\lambda_{\phi \circ \phi}(p^{a-3})|) = \sum_{n=n_0n_1n_2n_3 \atop (n_i,n_j)=1, i \neq j} |\lambda_{\phi \circ \phi}(n_0)||A(rad n_1, 1)||\lambda_{\phi \circ \phi}(n_1/rad n_1)\|
\]
\[
\times |A(rad n_2, 1)||\lambda_{\phi \circ \phi}(n_2/\langle rad n_2 \rangle^2)||\lambda_{\phi \circ \phi}(n_3/\langle rad n_3 \rangle^3)|.
\]
Here \( \text{rad} n = \prod_{p \mid n} p \) is the radical of \( n \). So we have
\[
\sum_{n \leq X} |B(n, 1)| \leq \sum_{n_0 \leq X} |\lambda_{\phi \circ \phi}(n_0)| \sum_{n_1 \leq X/n_0 \atop (n_1,n_0)=1} |A(1, rad n_1)||\lambda_{\phi \circ \phi}(n_1/rad n_1)\|
\]
\[
\times \sum_{n_2 \leq X/n_0n_1 \atop (n_2,n_0n_1)=1, p \nmid n_2 \text{ if } p \mid n_3} |A(rad n_2, 1)||\lambda_{\phi \circ \phi}(n_2/\langle rad n_2 \rangle^2)||\lambda_{\phi \circ \phi}(n_3/\langle rad n_3 \rangle^3)\|
\]
\[
\leq \sum_{m_1 \leq X} |\lambda_{\phi \circ \phi}(m_1)| \sum_{m_2 \leq X/m_1} |A(1, m_2)| \sum_{m_3 \leq X/m_1m_2} |\lambda_{\phi \circ \phi}(m_3)|
\]
\[
\times \sum_{m_4 \leq X/m_1m_2m_3} |A(m_4, 1)| \sum_{m_5 \leq X/m_1m_2m_3m_4^2} |\lambda_{\phi \circ \phi}(m_5)|
\]
\[
\times \sum_{m_6 \leq X/m_1m_2m_3m_4^2m_5 \leq X/m_1m_2m_3m_4^2m_5m_6^3} |\lambda_{\phi \circ \phi}(m_7)|
\]
\[
\ll X^{1+\varepsilon}.
\]
Here we have used (5.1) and (5.2). This completes the proof of (5.6).

Define \( C(n, 1) \) by multiplicativity. By (5.4), (5.6) and the same process as the proof of (5.6), we prove
\[
\sum_{n \leq X} |C(n, 1)| \ll X^{1+\varepsilon}.
\]
(5.7)
From (5.5) we have
\[
\sum_{n \leq X} |A(n^2, 1)| \leq \sum_{n \leq X} \prod_{p \mid n} (|C(p^\alpha, 1)| + |A(1, p)||C(p^{\alpha-1}, 1)|).
\]
(5.8)

By (5.7) and a similar but simpler process as the proof of (5.6), we obtain \( \sum_{n \leq X} |A(n^2, 1)| \ll X^{1+\varepsilon} \), which completes the proof of Theorem 1.8.

\[\Box\]

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References

[1] Blomer, V. “Subconvexity for twisted \( L \)-functions on \( GL(3) \).” Amer. J. Math. 134 (2012), no. 5, 1385–1421.

[2] Chinta, G. and Diaconu, A. “Determination of a \( GL_3 \) cuspform by twists of central \( L \)-values.” Int. Math. Res. Not. IMRN 2005, no. 48, 2941–2967.

[3] Diaconu, A., Goldfeld, D. and Hoffstein, J. “Multiple Dirichlet series and moments of zeta and \( L \)-functions.” Compos. Math. 139 (2003), no. 3, 297–360.

[4] Diaconu, A. and Whitehead, I. “On the third moment of \( L(\frac{1}{2}, \chi_d) \) II: the number field case.” J. Eur. Math. Soc. (JEMS) 23 (2021), no. 6, 2051–2070.

[5] Gelbart, S. and Kowalski, E. “Analytic number theory.” American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004.

[6] Goldfeld, D. “Automorphic forms and \( L \)-functions for the group \( GL(n, \mathbb{R}) \).” Cambridge Studies in Advanced Mathematics, 99. Cambridge University Press, Cambridge, 2006.

[7] Luo, W. “Special \( L \)-values of Rankin-Selberg convolutions.” Math. Ann. 314 (1999), no. 3, 591–600.

[8] Luo, W. and Ramakrishnan, D. “Determination of modular forms by twists of critical \( L \)-values.” Invent. Math. 130 (1997), no. 2, 371–398.

[9] Luo, W., Rudnick, Z. and Sarnak, P. “On Selberg’s eigenvalue conjecture.” Geom. Funct. Anal. 5 (1995), no. 2, 387–401.

[10] Montgomery, H. and Vaughan, R. “Multiplicative number theory. I. Classical theory.” Cambridge Studies in Advanced Mathematics, 97. Cambridge University Press, Cambridge, 2007.

[11] Munshi, R. and Sengupta, J. “Determination of \( GL(3) \) Hecke-Maass forms from twisted central values.” J. Number Theory 148 (2015), 272–287.
[19] Ramakrishnan, D. “An exercise concerning the selfdual cusp forms on $GL(3)$.” *Indian J. Pure Appl. Math.* 45 (2014), no. 5, 777–785.

[20] Soundararajan, K. “Nonvanishing of quadratic Dirichlet $L$-functions at $s = 1/2$.” *Ann. of Math. (2)* 152 (2000), no. 2, 447–488.

[21] Young, M. “The third moment of quadratic Dirichlet $L$-functions.” *Selecta Math. (N.S.)* 19 (2013), no. 2, 509–543.

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