FRW Cosmology in $F(R,T)$ Gravity

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Abstract

In this paper, we consider a theory of gravity with a metric-dependent torsion namely the $F(R,T)$ gravity, where $R$ is the curvature scalar and $T$ is the torsion scalar. We study the geometric root of such theory. In particular we give the derivation of the model from the geometrical point of view. Then we present the more general form of $F(R,T)$ gravity with two arbitrary functions and give some of its particular cases. In particular, the usual $F(R)$ and $F(T)$ gravity theories are particular cases of the $F(R,T)$ gravity. In the cosmological context, we find that our new gravitational theory can describe the accelerated expansion of the Universe.

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1 Introduction

The discovery of the accelerated expansion of the Universe has revolutionized modern cosmology. It is generally assumed that this cosmic acceleration is due to some kind of energy-matter with negative pressure known as ‘dark energy’ (DE). The nature of DE as well as its cosmological origin remain elusive so far. In order to explain the nature of the DE and the accelerated expansion, a variety of theoretical models have been proposed in the literature, such as quintessence [1], phantom energy [2], k-essence [3], tachyon [4], f-essence [5], Chaplygin gas [6], g-essence, etc. Among these

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Section 5 is devoted to the conclusion. Analytic solutions of the cosmological equations describe the accelerated expansion of the Universe. In particular, the exact de Sitter solution is found. These exact solutions of the cosmological equations are divided into two classes. Each of them is related with gravity theories like $F(R)$ gravity, $F(G)$ gravity, $F(T)$ gravity and so on (see e.g. Refs. [7]–[9]). In our opinion, one of interesting and prospective versions of modified gravity theories is the $F(R, T)$ gravity, where $R$ is the curvature scalar and $T$ is the torsion scalar. It can reproduce the unification of $F(R)$ and $F(T)$ gravity theories. Recently one of the versions of $F(R, T)$ gravity was proposed in [10] and its some properties was studied in [11]. In this paper we continue our work on $F(R, T)$ gravity.

In the previous papers [10]–[11], we introduced our $F(R, T)$ gravity models in an "ad hoc" manner. Contrary to previous, here we demonstrate (in the example of the M$_{37}$ - model) that $F(R, T)$ gravity models can be derived from geometry. So we put $F(R, T)$ gravity models in the geometrical language which is traditionally done for the modern gravity theories [12]–[13]. Also we show that $F(R, T)$ gravity models can describe the accelerated expansion of the Universe including the phantom crossing case. Finally we find a cosmological solution of the $F(R, T)$ gravity model corresponding to the de Sitter Universe.

As is well-known, General Relativity (GR) is described by Riemannian geometry which is torsion-free. In literature, several gravitational theories with torsion were proposed (see e.g. Refs. [18]–[22]) which show that the torsion effects should be included in the extensions of GR. In these theories, usually torsion is not propagating, since it is given algebraically in terms of the spin matter fields. As a consequence, the torsion can only be detected in the presence of spin matter fields. In contrast, in $F(T)$ gravity (here $T$ is the torsion scalar) torsion has no source and it can propagate without any matter source. This property of $F(T)$ gravity is same as the corresponding property of $F(R)$ gravity where the curvature ($R$) can propagates without the source. In our $F(R, T)$ gravity, at least at cosmological level it means that in a flat FRW space-time, the torsion can propagate in the absence of spin matter fields as in $F(T)$ gravity. In other words, $F(R, T)$ gravity inherits the relevant properties of its constituent two theories - $F(R)$ and $F(T)$ gravity theories. In other words, in our $F(R, T)$ gravity torsion as well as curvature can propagate without any matter source. It is a main merit (advantage) of our model. In fact this is a crucial point, otherwise the additional scalar torsion degree of freedom are not different from the additional metric gravitational degree of freedom present in extended $F(R)$ models. In other words, torsion $T$ is a fundamental quantity like curvature $R$. The $F(R, T)$ theory is considered as a fundamental gravitational theory, which describes the evolution of our Universe. Another important advantage of our $F(R, T)$ gravity models is that they contain as particular cases - two well-known modified gravity theories namely $F(R)$ gravity and $F(T)$ gravity. It indicates that our $F(R, T)$ gravity models are direct generalizations of well-studied and well-known $F(R)$ and $F(T)$ gravity theories and is the unified description of them.

A few words about our notations. First we would like to draw the attention to the existence to two types of gravity theory with the same name; that is, there are two type of $F(R, T)$ gravity theories. One was proposed in [32] but in this case, $T$ is the trace of the energy-momentum tensor. Second type is proposed in [10] (see also [11]) and also called $F(R, T)$ gravity and which we will study in this work. Lastly, we note that in some of our papers including this one, we used some conventional notations to distinguish the different gravitational (cosmological) models that were proposed (see [10]). For example in this paper we use notations like the M$_{37}$-model, the M$_{37}$-model and so on. These notations do not bring any physical or mathematical meaning and used just to distinguish and fix different models for our convenience.

This paper is organized as follows: In section-2, the gravitational action of $F(R, T)$ gravity and its arguments are derived from the geometrical point of view. Using them to the spatially flat FRW metric, an action with the curvature and torsion scalars is obtained. The Lagrangian formulation of the generalized $F(R, T)$ gravity model is given in section-3. Section-4 is devoted to construct some cosmological solutions for a particular model: $F = \mu R + \nu T$. In this case, the solutions of the cosmological equations are divided into two classes. Each of them is related with some torsion scalar functions. In particular, the exact de Sitter solution is found. These exact analytic solutions of the cosmological equations describe the accelerated expansion of the Universe. Section 5 is devoted to the conclusion.
2 Geometrical roots of $F(R, T)$ gravity

We start from the $M_{43}$ model (about our notations, see e.g. Refs. [10]-[11]). This model is one of the representatives of $F(R, T)$ gravity. The action of the $M_{43}$ model reads as

$$S_{43} = \int d^4x \sqrt{-g} [F(R, T) + L_m],$$

where $L_m$ is the matter Lagrangian, $\epsilon_i = \pm 1$ (signature), $R$ is the curvature scalar, $T$ is the torsion scalar (about our notation see below). In this section we try to give one of the possible geometric formulations of $M_{43}$ model. Note that we have different cases related with the signature: (1) $\epsilon_1 = 1, \epsilon_2 = 1$; (2) $\epsilon_1 = 1, \epsilon_2 = -1$; (3) $\epsilon_1 = -1, \epsilon_2 = 1$; (4) $\epsilon_1 = -1, \epsilon_2 = -1$. Also note that $M_{43}$ model is a particular case of $M_{37}$ model having the form

$$S_{37} = \int d^4x \sqrt{-g} [F(R, T) + L_m],$$

where $R_s = \epsilon_1 g^{\mu\nu} R_{\mu\nu}, T_s = \epsilon_2 S_{\rho}^{\mu\nu} T^\rho_{\mu\nu}$ (2.3)

are the standard forms of the curvature and torsion scalars.

2.1 General case

To understand the geometry of the $M_{43}$ - model, we consider some spacetime with the curvature and torsion so that its connection $G^{\lambda}_{\mu\nu}$ is a sum of the curvature and torsion parts. In this paper, the Greek alphabets ($\mu, \nu, \rho, ... = 0, 1, 2, 3$) are related to spacetime, and the Latin alphabets ($i, j, k, ... = 0, 1, 2, 3$) denote indices, which are raised and lowered with the Minkowski metric $\eta_{ij} = \text{diag} (-1, 1, 1, 1)$. For our spacetime the connection $G^{\lambda}_{\mu\nu}$ has the form

$$G^{\lambda}_{\mu\nu} = e^i_{\lambda} \partial_{\mu} e^i_{\nu} + e_j^{\lambda} e^i_{\nu} \omega^{\lambda}_{i\mu} = \Gamma^{\lambda}_{\mu\nu} + K^{\lambda}_{\mu\nu}. (2.4)$$

Here $\Gamma^i_{\mu\nu}$ is the Levi-Civita connection and $K^j_{i\mu}$ is the contorsion. Let the metric has the form

$$ds^2 = g_{ij} dx^i dx^j. (2.5)$$

Then the orthonormal tetrad components $e_i(x^\mu)$ are related to the metric through

$$g_{\mu\nu} = \eta_{ij} e_i^\mu e_j^\nu, (2.6)$$

so that the orthonormal condition reads as

$$\eta_{ij} = g_{\mu\nu} e_i^\mu e_j^\nu. (2.7)$$

Here $\eta_{ij} = \text{diag}(-1, 1, 1, 1)$, where we used the relation

$$e_i^\mu e_i^\nu = \delta^j_i. (2.8)$$

The quantities $\Gamma^i_{j\mu}$ and $K^j_{i\mu}$ are defined as

$$\Gamma^i_{j\mu} = \frac{1}{2} g^{ir} \left( \partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{jk} \right)$$

and

$$K^\lambda_{\mu\nu} = -\frac{1}{2} \left( T^\lambda_{\mu\nu} + T_{\mu\nu}^\lambda + T_{\nu\mu}^\lambda \right), (2.9)$$

and

$$K^\lambda_{\mu\nu} = -\frac{1}{2} \left( T^\lambda_{\mu\nu} + T_{\mu\nu}^\lambda + T_{\nu\mu}^\lambda \right), (2.10)$$
respectively. Here the components of the torsion tensor are given by

\[ T^\lambda_{\mu\nu} = e_i^\lambda T^i_{\mu\nu} = G^\lambda_{\mu\nu} - G^\lambda_{\nu\mu}, \]  \( (2.11) \)

\[ T^i_{\mu\nu} = \partial_\mu e^i_\nu - \partial_\nu e^i_\mu + G^i_{\mu\nu} - G^i_{\nu\mu}. \]  \( (2.12) \)

The curvature \( R^\rho_{\sigma\mu\nu} \) is defined as

\[ R^\rho_{\sigma\mu\nu} = e^i_\rho e^j_\sigma R^i_{\mu\nu} = \partial_\mu G^\rho_{\sigma\nu} - \partial_\nu G^\rho_{\sigma\mu} + G^\rho_{\lambda\mu} G^\lambda_{\sigma\nu} - G^\rho_{\lambda\nu} G^\lambda_{\sigma\mu}, \]  \( (2.13) \)

\[ \bar{R}^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\lambda\mu} \Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\lambda\nu} \Gamma^\lambda_{\sigma\mu}. \]  \( (2.14) \)

Now we introduce two important quantities namely the curvature (\( R \)) and torsion (\( T \)) scalars as

\[ R = g^{ij} R_{ij}, \]  \( (2.15) \)

\[ T = S^\rho_{\mu\nu} T^\rho_{\mu\nu}, \]  \( (2.16) \)

where

\[ S^\rho_{\mu\nu} = \frac{1}{2} \left( K^\rho_{\nu\lambda} T^\lambda_{\mu\sigma} - K^\rho_{\sigma\mu} T^\lambda_{\nu\lambda} - K^\lambda_{\nu\lambda} T^\rho_{\sigma\mu} \right). \]  \( (2.17) \)

Then the M43 - model we write in the form \( (2.1) \). To conclude this subsection, we note that in GR, it is postulated that \( T^\lambda_{\mu\nu} = 0 \) and such 4-dimensional spacetime manifolds with metric and without torsion are labelled as \( V_4 \). At the same time, it is a general convention to call \( U_4 \), the manifolds endowed with metric and torsion.

### 2.2 FRW case

From here we work with the spatially flat FRW metric

\[ ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2), \]  \( (2.18) \)

where \( a(t) \) is the scale factor. In this case, the non-vanishing components of the Levi-Civita connection are

\[ \Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{ij} = \Gamma^i_{0j} = \Gamma^i_{jk} = 0, \]
\[ \Gamma^0_{ij} = a^2 H \delta_{ij}, \]
\[ \Gamma^i_{j0} = \Gamma^i_{ij} = H \delta^i_j, \]  \( (2.19) \)

where \( H = (\ln a)_t \) and \( i, j, k, \ldots = 1, 2, 3 \). Now let us calculate the components of torsion tensor. Its non-vanishing components are given by:

\[ T_{110} = T_{220} = T_{330} = a^2 h, \]
\[ T_{123} = T_{231} = T_{312} = 2a^2 f, \]  \( (2.20) \)

where \( h \) and \( f \) are some real functions (see e.g. Refs. [12]). Note that the indices of the torsion tensor are raised and lowered with respect to the metric, that is

\[ T_{ijk} = g_{kl} T^l_{ij}. \]  \( (2.21) \)

Now we can find the contortion components. We get

\[ K^1_{10} = K^2_{20} = K^3_{30} = 0, \]
\[ K^1_{01} = K^2_{02} = K^3_{03} = h, \]
\[ K^0_{11} = K^0_{22} = a^2 h, \]
\[ K^1_{13} = K^2_{12} = K^3_{13} = -af, \]
\[ K^1_{32} = K^2_{13} = K^3_{21} = af. \]  \( (2.22) \)
The non-vanishing components of the curvature $R^\rho_{\sigma\mu\nu}$ are given by

\begin{align*}
R^{0\,101} &= \hat{R}^{0\,202} = \hat{R}^{0\,303} = a^2(\dot{H} + H^2 + Hh + \dot{h}), \\
R^{0\,123} &= -R^{0\,213} = R^{0\,312} = 2a^3f(H + h), \\
R^{1\,203} &= -R^{1\,302} = R^{2\,301} = -a(Hf + \dot{f}), \\
R^{1\,212} &= R^{1\,313} = R^{2\,323} = a^2[(H + h)^2 - f^2].
\end{align*}

Similarly, we write the non-vanishing components of the Ricci curvature tensor $R_{\mu\nu}$ as

\begin{align*}
R^{00} &= -3\dot{H} - 3\dot{h} - 3H^2 - 3Hh, \\
R^{11} = R^{22} = R^{33} &= a^2(\dot{H} + \dot{h} + 3H^2 + 5Hh + 2h^2 - f^2).
\end{align*}

At the same time, the non-vanishing components of the tensor $S_{\mu\nu}^{\rho}$ are given by

\begin{align*}
S^{10}_{10} &= \frac{1}{2} (K^{10}_{10} + \delta^1_0 T^0_{\theta} - \delta^0_0 T^\theta_{\theta}) = \frac{1}{2} (h + 2h) = h, \\
S^{10}_{10} &= S^{20}_{20} = S^{30}_{30} = 2h, \\
S^{23}_{1} &= \frac{1}{2} (K^{23}_{1} + \delta^2_1 + \delta^3_1) = -\frac{f}{2a}, \\
S^{23}_{1} &= S^{31}_{2} = S^{21}_{3} = -\frac{f}{2a}
\end{align*}

and

\begin{align*}
T &= T^{1}_{10} S^{10}_{10} + T^{2}_{20} S^{20}_{20} + T^{3}_{30} S^{30}_{30} + T^{23}_{1} S^{1}_{23} + T^{31}_{2} S^{2}_{31} + T^{12}_{3} S^{3}_{12}.
\end{align*}

Now we are ready to write the explicit forms of the curvature and torsion scalars. We have

\begin{align*}
R &= 6(\dot{H} + 2H^2) + 6\dot{h} + 18Hh + 6h^2 - 3f^2 \\
T &= 6(h^2 - f^2).
\end{align*}

So finally for the FRW metric, the $M_{43}$ - model takes the form

\begin{align*}
S^{43} &= \int d^4x \sqrt{-g} [F(R, T) + L_m], \\
R &= 6(\dot{H} + 2H^2) + 6\dot{h} + 18Hh + 6h^2 - 3f^2, \\
T &= 6(h^2 - f^2).
\end{align*}

It (that is the $M_{43}$ - model) is one of geometrical realizations of $F(R, T)$ gravity in the sense that it was derived from the purely geometrical point of view.

### 2.3 Particular cases

The $M_{43}$ - model admits some important features from the physical and geometrical point of view. In this subsection we want to present some particular reductions of the $M_{43}$ - model \(2.32\) such as the FRW metric case.

#### 2.3.1 $F(R)$ gravity

Let $h = f = 0$. Then the $M_{43}$ - model \(2.32\) reduces to the case

\begin{align*}
S^{43} &= \int d^4x \sqrt{-g} [F(R_m) + L_m], \\
R &= R_m = 6(\dot{H} + 2H^2), \\
T &= 0,
\end{align*}

which is the usual $F(R)$ gravity.
2.3.2 The M\textsubscript{43A} - model
Let $f = 0$. Then the M\textsubscript{43} - model \ref{eq:2.32} reduces to the case
\begin{align}
S_{43A} &= \int d^4x \sqrt{-g} [F(R, T) + L_m], \\
R &= R_s = 6(\dot{H} + 2H^2) + 6\dot{h} + 18Hh + 6h^2, \\
T &= 6h^2.
\end{align}
(2.34)

We can note that for this particular case $T \geq 0$.

2.3.3 The M\textsubscript{43B} - model
Let $h = 0$. Then we get the M\textsubscript{43B} - model with the action
\begin{align}
S_{43B} &= \int d^4x \sqrt{-g} [F(R, T) + L_m], \\
R &= 6(\dot{H} + 2H^2) - 3f^2, \\
T &= -6f^2.
\end{align}
(2.35)

For this particular model $T \leq 0$ as follows from the last equation.

2.3.4 The M\textsubscript{43C} - model
Let $f = \pm h$. Then $T = 0$ and obtain the following M\textsubscript{43C} - model:
\begin{align}
S_{43C} &= \int d^4x \sqrt{-g} [F(R) + L_m], \\
R &= 6(\dot{H} + 2H^2) + 6\dot{h} + 18Hh + 3h^2.
\end{align}
(2.36)

Hence as $h = 0$ we get the usual $F(R)$ gravity with the scalar curvature $R = R_s = 6(\dot{H} + 2H^2)$.

2.3.5 The M\textsubscript{43D} - model
Let $f = \pm \sqrt{2(\dot{H} + 2H^2) + 2\dot{h} + 6Hh + 2h^2}$. Then $R = 0$ and we have the M\textsubscript{43D} - model:
\begin{align}
S_{43D} &= \int d^4x \sqrt{-g} [F(T) + L_m], \\
T &= -12(\dot{H} + 2H^2 + \dot{h} + 3Hh + 0.5h^2).
\end{align}
(2.38)

2.3.6 The M\textsubscript{43E} - model
Now we assume that $h = 0$ and $f = \pm \sqrt{2(\dot{H} + 2H^2)}$. Then $R = 0$ and we have the M\textsubscript{43D} - model:
\begin{align}
S_{43E} &= \int d^4x \sqrt{-g} [F(T) + L_m], \\
T &= -12(\dot{H} + 2H^2).
\end{align}
(2.40)

3 Lagrangian formulation of $F(R, T)$ gravity
Of course, we can work with the form \ref{eq:2.32} of $F(R, T)$ gravity. But a more interesting and general form of $F(R, T)$ gravity is the so-called M\textsubscript{37} - model. The action of the M\textsubscript{37} - gravity reads as \[10\]
\begin{align}
S_{37} &= \int d^4x \sqrt{-g} [F(R, T) + L_m], \\
R &= u + R_s = u + 6\epsilon_1(\dot{H} + 2H^2), \\
T &= v + T_s = v + 6\epsilon_2H^2,
\end{align}
(3.1)
Therefore, the action \( (3.5) \) can be rewritten as

\[
S_{37} = \int dt \, a^3 \left\{ F(R, T) - \lambda \left[ T - v - 6\epsilon_2 \frac{\dot{a}^2}{a^2} \right] - \gamma \left[ R - u - 6\epsilon_1 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] + L_m \right\},
\]

where \( \lambda \) and \( \gamma \) are Lagrange multipliers. If we take the variations with respect to \( T \) and \( R \) of this action, we get

\[
\lambda = F_T, \quad \gamma = F_R.
\]

Therefore, the action \( (3.5) \) can be rewritten as

\[
S_{37} = \int dt \, a^3 \left\{ F(R, T) - F_T \left[ T - v - 6\epsilon_2 \frac{\dot{a}^2}{a^2} \right] - F_R \left[ R - u - 6\epsilon_1 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] + L_m \right\}.
\]

Then the corresponding point-like Lagrangian reads

\[
L_{37} = a^3 \left[ F - (T - v) F_T - (R - u) F_R + L_m \right] - 6(\epsilon_1 F_R - \epsilon_2 F_T) a \dot{a}^2 - 6\epsilon_1 (F_{RR} \dot{R} + F_{RT} \dot{T}) a^2 \ddot{a}.
\]

As is well known, for our dynamical system, the Euler-Lagrange equations read as

\[
\frac{d}{dt} \left( \frac{\partial L_{37}}{\partial \dot{R}} \right) - \frac{\partial L_{37}}{\partial R} = 0,
\]

\[
\frac{d}{dt} \left( \frac{\partial L_{37}}{\partial \dot{T}} \right) - \frac{\partial L_{37}}{\partial T} = 0,
\]

\[
\frac{d}{dt} \left( \frac{\partial L_{37}}{\partial \dot{a}} \right) - \frac{\partial L_{37}}{\partial a} = 0.
\]

Hence, using the expressions

\[
\frac{\partial L_{37}}{\partial R} = -6\epsilon_1 F_{RR} a^2 \dot{a},
\]

\[
\frac{\partial L_{37}}{\partial T} = -6\epsilon_1 F_{RT} a^2 \dot{a},
\]

\[
\frac{\partial L_{37}}{\partial a} = -12(\epsilon_1 F_R - \epsilon_2 F_T) a \ddot{a} - 6\epsilon_1 (F_{RR} \dddot{R} + F_{RT} \dddot{T} + F_{R\psi} \dddot{\psi}) a^2 + a^3 F_T v_a + a^3 F_R u_a,
\]

we get

\[
a^3 F_{TT} \left(T - v - 6\epsilon_2 \frac{\dot{a}^2}{a^2}\right) = 0,
\]

\[
a^3 F_{RR} \left(R - u - 6\epsilon_1 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right) = 0,
\]

\[
U + B_2 F_{TT} + B_1 F_T + C_2 F_{RR} + C_1 F_{RT} + C_0 F_R + M \dot{F} + 6\epsilon_2 a^2 \dot{p} = 0,
\]

\[
\dot{\dot{U}} = 0.
\]
respectively. Here

\begin{align}
U &= A_3 F_{RRR} + A_2 F_{RR} + A_1 F_R, \\
A_3 &= -6\epsilon_1 R^2 a^2, \\
A_2 &= -12\epsilon_1 \dot{R}a - 6\epsilon_1 \dot{R}a + a^3 \dot{R}u_a, \\
A_1 &= 12\epsilon_1 a^2 + 6\epsilon_1 a\dot{a} + 3a^2 \dot{u}_a + a^3 \ddot{a} - a^3 u_a, \\
B_2 &= 12\epsilon_2 \dot{T}a + a^3 \dot{T}v_a, \\
B_1 &= 24\epsilon_2 a^2 + 12\epsilon_2 \dot{a}a + 3a^2 \dot{v}_a + a^3 \ddot{v}_a - a^3 v_a, \\
C_2 &= -12\epsilon_1 a^2 \dot{T}, \\
C_1 &= -6\epsilon_1 a^2 \dot{T}, \\
C_0 &= -12\epsilon_1 \dot{T}a - 12\epsilon_2 \dot{R}a - 6\epsilon_1 a^2 \dot{T} + a^3 \dot{R}v_a + a^3 \dot{T}u_a, \\
M &= -3a^2.
\end{align}

If \( F_{RR} \neq 0, \quad F_{TT} \neq 0, \) from Eqs. (3.17) and (3.17), it is easy to find that

\[ R = u + 6\epsilon_1 (H + 2H^2), \quad T = v + 6\epsilon_2 H^2, \]

so that the relations (3.17) are recovered. Generally, these equations are the Euler constraints of the dynamics. Here \( a, R, T \) are the generalized coordinates of the configuration space. On the other hand, it is also well known that the total energy (Hamiltonian) corresponding to Lagrangian \( L_{37} \) is given by

\[ H_{37} = \frac{\partial L_{37}}{\partial \dot{a}} \dot{a} + \frac{\partial L_{37}}{\partial \dot{R}} \dot{R} + \frac{\partial L_{37}}{\partial \dot{T}} \dot{T} - L_{37}. \]

Hence using (3.12) - (3.14) we obtain

\[ H_{37} = [-12(\epsilon_1 F_R - \epsilon_2 F_T)a\dot{a} - 6\epsilon_1 (F_{RR}\dot{R} + F_{RT}\dot{T} + F_{R\psi}\dot{\psi})a^2 + a^3 F_T v_a + a^3 F_{R} u_a] \dot{a} \\
-6\epsilon_1 F_{RR}a^2 \dot{a} \dot{R} - 6\epsilon_1 F_T a^2 \dot{a} \dot{T} - [a^3 (F - T F_T + R F_R + \psi F_T + u F_R + L_m) - 6(\epsilon_1 F_R - \epsilon_2 F_T)a\dot{a}^2 - 6\epsilon_1 (F_{RR} \dot{R} + F_{RT} \dot{T} + F_{R\psi} \dot{\psi})a^2 \dot{a}]]. \]

Let us rewrite this formula as

\[ H_{37} = D_2 F_{RR} + D_1 F_R + J F_{RT} + E_1 F_T + K F + 2a^3 \rho, \]

where

\begin{align}
D_2 &= -6\epsilon_1 \dot{R}a^2 \dot{a}, \\
D_1 &= 6\epsilon_1 a\ddot{a} + a^3 \ddot{u}_a, \\
J &= -6\epsilon_1 a^2 \dot{T}, \\
E_1 &= 12\epsilon_2 a\dot{a}^2 + a^3 \dot{v}_a, \\
K &= -a^3.
\end{align}

As usual we assume that the total energy \( H_{37} = 0 \) (Hamiltonian constraint). So finally we have the following equations of the \( M_{37} \) - model [10]-[11]:

\begin{align}
D_2 F_{RR} + D_1 F_R + J F_{RT} + E_1 F_T + K F &= -2a^3 \rho, \\
U + B_2 F_{TT} + B_1 F_T + C_2 F_{RRT} + C_1 F_{RTT} + C_0 F_{RRT} + M F &= 6a^2 \rho, \\
\dot{\rho} + 3H(\rho + p) &= 0.
\end{align}

It deserves to note that the \( M_{37} \) - model[3.11] admits some interesting particular and physically important cases. Some particular cases are now presented.

\[ i) \text{The M}_{44} \text{- model}. \] Let the function \( F(R, T) \) be independent from the torsion scalar \( T \):

\[ F = F(R, T) = F(R). \]

Then the action (3.1) acquires the form

\[ S_{44} = \int d^4 x \epsilon[F(R) + L_m], \]
where
\[ R = u + R_s = u + \epsilon_1 g^{\mu \nu} R_{\mu \nu}, \] (3.39)
is the curvature scalar. It is the M$_{44}$ - model. We work with the FRW metric. In this case $R$ takes the form
\[ R = u + 6\epsilon_1 (\dot{H} + 2H^2). \] (3.40)
The action can be rewritten as
\[ S_{44} = \int dt L_{44}, \] (3.41)
where the Lagrangian is given by
\[ L_{44} = \left[ F - (R - u) F_R + L_m \right] - 6\epsilon_1 F_R a \dot{u}^2 - 6\epsilon_1 F_{RR} \dot{R} a \dot{u}. \] (3.42)
The corresponding field equations of the M$_{44}$ - model read as
\[ D_2 F_{RR} + D_1 F_R + K F = -2a^3 \rho, \] (3.43)
\[ A_3 F_{RRR} + A_2 F_{RR} + A_1 F_R + M F = 6a^2 p, \]
\[ \dot{\rho} + 3H (\rho + p) = 0. \] (3.44)
Here
\[ D_2 = -6\epsilon_1 \dot{R} a^2 \dot{u}, \] (3.45)
\[ D_1 = 6\epsilon_1 a^2 \ddot{u} + a^3 u \dot{a}, \] (3.46)
\[ K = -a^3 \] (3.47)
\[ A_3 = -6\epsilon_1 \dot{R}^2 a^2, \] (3.48)
\[ A_2 = -12\epsilon_1 \dot{R} \dot{a} - 6\epsilon_1 \dot{R} a^2 + a^3 \ddot{R} \dot{u}, \] (3.49)
\[ A_1 = 12\epsilon_1 a^2 + 6\epsilon_1 \dot{a} \dot{a} + 3a^2 \dddot{u} + a^3 \dddot{u} - a^3 u \dot{u}, \] (3.50)
\[ M = -3a^2. \] (3.51)
If $u = 0$ then we get the following equations of the standard $F(R_s)$ gravity (after $R = R_s$):
\[ 6\dot{R} H F_{RR} - (R - 6H^2) F_R + F = \rho, \] (3.51)
\[ -2\dot{R}^2 F_{RRR} + [-4\dot{R} H - 2\dot{R}] F_{RR} + [-2H^2 - 4a^{-1} \dot{a} + R] F_R - F = p, \] (3.52)
\[ \dot{\rho} + 3H (\rho + p) = 0. \] (3.53)

**ii) The M$_{45}$ - model.** The action of the M$_{45}$ - model looks like
\[ S_{45} = \int d^4 x e [F(T) + L_m], \] (3.54)
where $e = \det (e^i_j) = \sqrt{-g}$ and the torsion scalar $T$ is defined as
\[ T = v + T_s = v + \epsilon_2 S^\rho_{\mu \nu} T^\rho_{\mu \nu}. \] (3.55)
Here
\[ T^\rho_{\mu \nu} \equiv -\epsilon_5^\rho (\partial_\mu e^i_j - \partial_\nu e^i_j), \] (3.56)
\[ K^\mu_{\rho \nu} \equiv -\frac{1}{2} (T^{\mu \nu}_{\rho} - T^{\nu \mu}_{\rho} - T^{\mu \nu}_{\rho}), \] (3.57)
\[ S^\rho_{\mu \nu} \equiv \frac{1}{2} \left( K^{\mu \nu}_{\rho} + \delta^\rho_{\mu} T^\theta_{\nu} - \delta^\nu_{\rho} T^\theta_{\mu} \right). \] (3.58)
For a spatially flat FRW metric (2.18), we have the torsion scalar in the form
\[ T = v + T_s = v + 6\epsilon_2 H^2. \] (3.59)
The action \( S_{45} \) can be written as
\[
S_{45} = \int dt L_{45},
\]
where the point-like Lagrangian reads
\[
L_{45} = a^3[F - (T - v) F_T + L_m] + 6\epsilon_2 F_T a\dot{a}^2.
\]
So finally we get the following equations of the \( M_{45} \)-model:
\[
\begin{align*}
E_1 F_T + K F & = -2a^3 \rho, \\
B_2 F_T T + B_1 F_T + MF & = 6a^2 p, \\
\dot{\rho} + 3H(\rho + p) & = 0.
\end{align*}
\]
Here
\[
\begin{align*}
E_1 & = 12\epsilon_2 a\dot{a}^2 + a^3 \dot{v}_a \dot{a}, \\
K & = -a^3
\end{align*}
\]
and
\[
\begin{align*}
B_2 & = 12\epsilon_2 \dot{T} a \dot{a} + a^3 \dot{v}_a \dot{a}, \\
B_1 & = 24\epsilon_2 \dot{a}^2 + 12\epsilon_2 a\dot{a} + 3a^2 \ddot{v}_a + a^3 \dot{v}_a - a^3 v_a, \\
M & = -3a^2.
\end{align*}
\]
If we put \( v = 0 \) then the \( M_{45} \)-model reduces to the usual \( F(T_s) \) gravity, where \( T_s = 6\epsilon_2 H^2 \). As is well-known the equations of \( F(T_s) \) gravity are given by
\[
\begin{align*}
12H^2 F_T + F & = \rho, \\
48H^2 F_T T - F_T \left(12H^2 + 4H\right) - F & = p, \\
\dot{\rho} + 3H(\rho + p) & = 0,
\end{align*}
\]
where we must put \( T = T_s \). Finally we note that it is well-known that the standard \( F(T_s) \) gravity is not local Lorentz invariant \[33\]. In this context, we have a very meager hope that the \( M_{45} \)-model \[3.54\] is free from such problems.

## 4 Cosmological solutions

In this section we investigate cosmological consequences of the \( F(R, T) \) gravity. As example, we want to find some exact cosmological solutions of the \( M_{37} \)-gravity model. Since its equations are very complicated we here consider the simplest case when
\[
F(R, T) = \mu R + \nu T, \tag{4.1}
\]
where \( \mu \) and \( \nu \) are some constants. Then equations \[3.37\] take the form
\[
\begin{align*}
\mu D_1 + \nu E_1 + K(\nu T + \mu R) & = -2a^3 \rho, \\
\mu A_1 + \nu B_1 + M(\nu T + \mu R) & = 6a^2 p, \\
\dot{\rho} + 3H(\rho + p) & = 0,
\end{align*}
\]
where
\[
\begin{align*}
D_1 & = 6\epsilon_1 a^2 \ddot{a} + a^3 \dot{u}_a \dot{a}, \\
E_1 & = 12\epsilon_2 a\dot{a}^2 + a^3 \dot{v}_a \dot{a}, \\
K & = -a^3, \\
A_1 & = 12\epsilon_1 \dot{a}^2 + 6\epsilon_1 a\dot{a} + 3a^2 \ddot{u}_a + a^3 \ddot{v}_a - a^3 u_a, \\
B_1 & = 24\epsilon_2 \dot{a}^2 + 12\epsilon_2 a\dot{a} + 3a^2 \ddot{v}_a + a^3 \dot{v}_a - a^3 v_a, \\
M & = -3a^2.
\end{align*}
\]

We can rewrite this system as
\[3\sigma H^2 - 0.5(\dot{a}z_a - z) = \rho,\]
\[-\sigma(2\dot{H} + 3H^2) + 0.5(\dot{a}z_a - z) + \frac{1}{6}a(\dot{z}_a - z_a) = p,\]
\[\dot{\rho} + 3H(\rho + p) = 0,\]  \hspace{1cm} (4.9)
where \(z = \mu u + \nu v, \quad \sigma = \mu \epsilon_1 - \nu \epsilon_2.\) This system contains two independent equations for five unknown functions \((a, \rho, p, u, v).\) But in fact it contains 4 unknown functions \((H, \rho, p, z).\) The corresponding EoS parameter reads as
\[\omega = \frac{p}{\rho} = -1 + \frac{2\sigma \dot{H} + \frac{1}{6}a(\dot{z}_a - z_a)}{3\sigma H^2 - 0.5(\dot{a}z_a - z)}.\]  \hspace{1cm} (4.10)

Let us find some simplest cosmological solutions of the system (4.9).

4.1 Example 1

We start from the case \(\sigma = 0.\) In this case the system (4.9) takes the form
\[-0.5(\dot{a}z_a - z) = \rho,\]
\[0.5(\dot{a}z_a - z) + \frac{1}{6}a(\dot{z}_a - z_a) = p,\]
\[\dot{\rho} + 3H(\rho + p) = 0.\]  \hspace{1cm} (4.11)
At the same time the EoS parameter becomes
\[\omega = \frac{p}{\rho} = -1 + \frac{a(\dot{z}_a - z_a)}{3(\dot{a}z_a - z)}.\]  \hspace{1cm} (4.12)

Now we assume that
\[z = \kappa a^l,\]  \hspace{1cm} (4.13)
where \(\kappa\) and \(l\) are some real constants. Then
\[\omega = -1 - \frac{l}{3}.\]  \hspace{1cm} (4.14)

This result tells us that in this case our model can describes the accelerated expansion of the Universe for some values of \(l.\)

4.2 Example 2

Now consider the de Sitter case that is \(H = H_0 = \text{const}\) so that \(a = a_0 e^{H_0 t}.\) Then the system (4.9) reads as
\[3\sigma H_0^2 - 0.5(\dot{a}z_a - z) = \rho,\]
\[-3\sigma H_0^2 + 0.5(\dot{a}z_a - z) + \frac{1}{6}a(\dot{z}_a - z_a) = p,\]
\[\dot{\rho} + 3H(\rho + p) = 0.\]  \hspace{1cm} (4.15)
The EoS parameter takes the form
\[\omega = \frac{p}{\rho} = -1 + \frac{a(\dot{z}_a - z_a)}{18\sigma H_0^2 - 3(\dot{a}z_a - z)}.\]  \hspace{1cm} (4.16)
If \(z\) has the form (4.13) that is \(z = \kappa e^{H_0 t}\) then we have
\[\omega = \frac{p}{\rho} = -1 + \frac{\kappa}{18\sigma H_0^2 e^{-3H_0 t} + 3\kappa}.\]  \hspace{1cm} (4.17)
If $H_0 l > 0$ then as $t \to \infty$ we get again
\[
\omega = -1 - \frac{l}{3},
\]  
which is same as equation (4.14). This last equation tells us that our model can describes the accelerated expansion of the Universe e.g. for $l \geq 0$. Also it corresponds to the phantom case if $l > 0$. Finally we present other forms of the generalized Friedmann equations in the system (4.9). Let us rewrite these equations in the standard form as
\[
3H^2 = \rho + \rho_z, \tag{4.19}
\]
\[
-(2\dot{H} + 3H^2) = p + p_z. \tag{4.20}
\]
Here
\[
\rho_z = 0.5\sigma^{-1}(a\dot{z}_a - z), \tag{4.21}
\]
\[
p_z = -0.5\sigma^{-1}(a\dot{z}_a - z) - \frac{1}{6\sigma}a(\dot{z}_a - z_a) \tag{4.22}
\]
are the $z$ or $u - v$ contributions to the energy density and pressure, respectively.

5 Conclusion

In [30], Buchdahl proposed to replace the Einstein-Hilbert scalar Lagrangian $R$ with a function of the scalar curvature. The resulting theory is nowadays known as $F(R)$ gravity. Almost 40 years later, Bengochea and Ferraro proposed to replace the TEGR that is the torsion scalar Lagrangian $T$ with a function $F(T)$ of the torsion scalar, and studied its cosmological consequences [31]. This type of modified gravity is nowadays called as $F(T)$ gravity theory. These two gravity theories [that is $F(R)$ and $F(T)$] are, in some sense, alternative ways to modify GR. From these results arises the natural question: how can we construct some modified gravity theory which unifies $F(R)$ and $F(T)$ theories? Examples of such unified curvature-torsion theories were proposed in [10]-[11]. Such type of modified gravity theory is called the $F(R,T)$ gravity. In this $F(R,T)$ gravity, the curvature scalar $R$ and the torsion scalar $T$ play the same role and are dynamical quantities. In this paper, we have shown that the $F(R,T)$ gravity can be derived from the geometrical point of view. In particular, we have proposed a new method to construct particular models of $F(R,T)$ gravity. As an example we have considered the $M_{43}$ - model, deriving its action in terms of the curvature and torsion scalars. Then in detail we have studied the $M_{37}$ - model and presented its action, Lagrangian and equations of motion for the FRW metric case. Finally we have shown that the last model can describes the accelerated expansion of the Universe.

Concluding, we would like to note that in the paper, we present a special class of extended gravity models depending on arbitrary function $F(R,T)$, where $R$ is the Ricci scalar and $T$ the scalar torsion. While in the traditional Einstein-Cartan theory, the role of the torsion depends on the non trivial source associated with spin matter density, in our $F(R,T)$ gravity models, the torsion can propagate without the presence of spin matter density. In fact this is a crucial point, otherwise the additional scalar torsion degree of freedom are not different from the additional metric gravitational degree of freedom present in extended $F(R)$ models. Finally we would like to note that all results of this paper are new and different than results of our previous papers [10]-[11] on the subject.

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