We develop the Ostrogradsky-Hamilton formalism for geodetic brane gravity, described by the Regge-Teitelboim geometric model in higher codimension. We treat this gravity theory as a second-order derivative theory, based on the extrinsic geometric structure of the model. As opposed to previous treatments of geodetic brane gravity, our Lagrangian is linearly dependent on second-order time derivatives of the field variables, the embedding functions. The difference resides in a boundary term in the action, usually discarded. Certainly, this suggests applying an appropriate Ostrogradsky-Hamiltonian approach to this type of theories. The price to pay for this choice is the appearance of second class constraints. We determine the full set of phase space constraints, as well as the gauge transformations they generate in the reduced phase space. Additionally, we compute the algebra of constraints and explain its physical content. In the same spirit, we deduce the counting of the physical degrees of freedom. We comment briefly on the naive formal canonical quantization emerging from our development.

Keywords: Ostrogradsky-Hamilton framework; Regge-Teitelboim model; Variational techniques.

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1. Introduction

In 1975, T. Regge and C. Teitelboim [RT] pictured our four-dimensional spacetime as the trajectory of a three-dimensional extended object evolving geodesically in a fixed higher-dimensional background Minkowski spacetime, close in spirit to the
worldline/worldsheet behavior of classical relativistic objects, like strings. They were motivated initially by a search for an alternative theory to pursue quantum gravity, rather than continue to quantize pure gravity described by the Einstein-Hilbert [EH] action in terms of a metric field variable, considering the unsurmountable difficulties encountered in a standard perturbative approach. In the following we will refer to standard General Relativity based on the EH action as metric-GR, see. The RT theoretical viewpoint replaces the components of the spacetime metric by the embedding functions of the worldvolume spanned by the three-dimensional object as field variables, so that the worldvolume metric becomes a composed field variable. The pioneering work of RT has received renewed interest in the context of braneworld scenarios, and in particular through the studies of Davidson and collaborators, that introduced the term ‘Geodetic Brane Gravity’ [GBG].

On geometrical grounds, in order to ensure the local existence of an embedding framework, at most \( N = n(n+1)/2 \) dimensions are needed for the ambient spacetime background. In addition, it is known that if the worldvolume metric admits Killing vectors, that number can be reduced. The RT model consists of the integral over the worldvolume of the Ricci scalar constructed from the induced worldvolume metric, so that one may have a misconception of continuing to work with metric-GR because the action is the same. However, one must keep in mind that in this framework the field variables are given by the embedding functions. Either way, the equations of motion remain second-order in derivatives of the fields, as in GR. In both cases, GBG and metric-GR, the action presents a problem due to the appearance of a term linear in the second derivative of the field variables.

After the RT proposal came up, the idea was criticized due to gauge dependence and, as Regge and Teitelboim themselves stated in their work, for the lack of an appropriate Hamiltonian formulation. Regarding the latter, many authors have made important advances in this direction, using several Hamiltonian strategies. Since the RT model is an embedded geometric model in terms of the geometry of sub-manifolds, where a divergence term can also be identified, perhaps avoiding that term as it is generally done may be detrimental to obtaining a clear canonical formulation.

In this paper we reconsider the canonical formalism of the RT model by making use of the appropriate Ostrogradsky-Hamilton framework developed for singular systems. Our approach has the advantage of bringing to the forefront the full geometrical content of the model in any codimension as well to consider the effect of all geometric terms in the RT model. In fact, we borrow the existent Hamiltonian formulation for relativistic extended objects that, in turn, was inspired by the Arnowitt-Deser-Misner Hamiltonian formulation of GR [ADM]. The canonical analysis does not involve reduction of the RT model by eliminating a total divergence. The resulting Lagrangian is simply linear in the accelerations of the extended object, that make this, in principle, a second-order derivative theory; fortunately, the accelerations enter into the game in such a way that they do not raise the order of the
equations of motion. Under this consideration and, according to the Ostrogradsky line of reasoning, the canonical approach needs to double the number of phase space variables. The advantage of this treatment is that we manage to keep the original geometric nature of the model intact. Additionally, this fact makes more evident the role that both the momenta and the Hamiltonian constraints play within the canonical structure. To the best of our knowledge, this is the first attempt in this direction, at least for the full theory.

After obtaining the full set of constraints in phase space, we separate them into first and second-class constraints. In particular, we compute the Dirac algebra of the constraints and analyze its content in addition to study the gauge transformations that the first-class constraints generate. Nevertheless, it should be mentioned that this point of view has been adopted earlier in an Hamiltonian formulation of the RT model minisuperspace by Cordero et al. and then by Banerjee et al. where the field theory is reduced to a finite number of degrees of freedom, allowing to substantial progress in the quantization of the reduced model. In passing, we would like to mention that, along this line of reasoning, Dutt and Dresden were interested in try to apply the Ostrogradsky-Hamilton formulation to metric-GR.

The paper is organized as follows. In Sec. 2 we introduce our notation and provide an overview of pure GBG, without additional brane matter fields, via the RT geometric model. In Sec. 3 we consider a $p+1$ ADM decomposition of the worldvolume geometry and obtain a suitable ADM Lagrangian for GBG in terms of ADM field variables, linear in the acceleration. The Ostrogradsky-Hamilton formulation is the subject of Sec 4, where we construct the phase space appropriate for a higher derivative theory, the Hamiltonian and we identify the primary and secondary phase space constraints for the theory. We also calculate Hamilton’s equation as a check for consistency. In Sec. 5, we separate the primary and secondary phase space constraints in first and second class constraints, and compute their algebra, using the standard Dirac-Bergmann approach. We also consider infinitesimal canonical transformations, as an help towards understanding the meaning of the constraints themselves. We end in Sec. 6 with a brief discussion. In three Appendices we have collected useful results used throughout the main text.

2. Geodetic brane gravity

We consider a $(p + 1)$-dimensional worldvolume $m$ spanned by the evolution of a $p$-dimensional spacelike extended object, or brane, $\Sigma$ in a flat $N$-dimensional Minkowski background spacetime, $\{M, \eta_{\mu\nu}\}$, with metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, \ldots, 1)$ ($\mu, \nu = 0, 1, 2, \ldots, N - 1$). $m$ is described by the embedding $y^\mu = X^\mu(x^a)$, where $y^\mu$ are local coordinates for $M$, $x^a$ are local coordinates for $m$, and $X^\mu$ are the embedding functions ($a, b = 0, 1, 2, \ldots, p$). The vectors $e^\mu_a := \partial_a X^\mu$ form a basis of tangent vectors to $m$. The inner product among them produce the components of the induced metric $g_{ab} = \eta_{\mu\nu} e^\mu_a e^\nu_b = e_a \cdot e_b$, that in this sense is a composed field variable. Here and henceforth a dot denotes inner product using the background
Minkowski metric. By $g^{ab}$ we denote the inverse of $g_{ab}$, and by $g$ its determinant. The worldvolume $m$ is assumed to be timelike so $g < 0$. In this framework, we introduce $N - p - 1$ normal vectors to the worldvolume $m$, denoted by $n^{\mu 1}$, and defined implicitly by $n^i \cdot e_a = 0$ and $n^i \cdot n^j = \delta^{ij} \ (i, j = 1, 2, \ldots, N - p - 1)$, up to a sign and a rotation. This gauge freedom does require the introduction of a gauge field, the twist potential, as shown below. We also introduce the extrinsic curvature for $m$ with $K^{ii} = g^{ii}K_{ab}^{ii}$ where $\nabla_a$ denotes the torsion-less metric compatible worldvolume covariant derivative, $\nabla_a n_b = 0$. In this spirit, the worldvolume scalar curvature depends explicitly on the extrinsic curvature by using the contracted Gauss-Codazzi integrability condition for surfaces $R = K^{ii}K_{ii} - K^{i\mu}K_{i\mu}^{ab}[25,26]$.

Geodetic brane gravity is described by the RT model defined by

$$ S_{\text{RT}}[X^\mu] = \frac{\alpha}{2} \int_m d^{p+1}x \sqrt{-g} R, \quad (1) $$

where $\alpha$ is a constant with dimensions $[L]^{1-p}$ in natural units. We could add some matter contribution through a Lagrangian $L_{\text{mat}}$ of fields living on the brane. In any case, such a term would not affect the geometric arguments in this note, that focus on the curvature contribution. As we are not coupling to any brane matter fields, we set $\alpha = 1$. Carrying out the first variation of the action we are able to obtain the equations of motion [eom] [27] The classical brane trajectories are obtained from the $N - p - 1$ compact relations

$$ G^{ab}K_{ab}^{ii} = 0, \quad (2) $$

where $G_{ab} = R_{ab} - (1/2)R g_{ab}$ is the worldvolume Einstein tensor. These eom are of second order in derivatives of the field variables $X^\mu$ because of the presence of the extrinsic curvature. Additionally, there are $p + 1$ tangential vanishing expressions related to the eom, reflecting the reparametrization invariance of the action [1]. Indeed, these are given by the divergence-free condition $\nabla_a G^{ab} = 0$. Another way of expressing the eom is by using the definition of the extrinsic curvature. In addition, the eom [2] can also be written as a set of projected conservation laws

$$ (\nabla_a P^a) \cdot n^i = 0, \quad (3) $$

where the conserved stress tensor, $P_\mu^a$, is given by [27]

$$ P_\mu^a := -\sqrt{-g} G^{ab} e_{\mu b}. \quad (4) $$

Notice that $P_\mu^a$ is purely tangential. In fact, this feature characterize theories leading to second-order eom. Indeed, [1] belongs to a family of conserved stress tensors associated to second-order derivative geometrical models leading to second-order eom, called Lovelock branes [28].

With an eye towards the Hamiltonian framework, when $\Sigma$ is viewed as a spacelike manifold immersed into $m$ (see Appendix A), the associated timelike unit normal, $\eta^a$, helps to construct the linear momentum density on $\Sigma$ with

$$ \pi_\mu := N^{-1} \eta_a P_\mu^a, \quad (5) $$
where \( N \) represents the lapse function that appears in the ADM decomposition for geometric extended object models depending on the extrinsic curvature, as we will see shortly.

3. The ADM Lagrangian for geodetic brane gravity

An adaptation of the ADM framework for metric-GR to branes, needed for a canonical formulation of GBG, is given in detail in.\textsuperscript{21,22} If we assume that \( m \) is globally hyperbolic, then it is possible to foliate it into a set of spacelike hyper-surfaces \( \Sigma_t \). This suggests to split in space and time derivatives the relevant worldvolume geometric quantities, close in spirit to the ADM formulation of metric-GR.

We describe \( \Sigma_t \) using an embedding formulation. First, using the embedding \( y^\mu = X^\mu(t = \text{const}, u^A) \), we split the \( p + 1 \) worldvolume coordinates \( x^a \) into an arbitrary time parameter \( t \) and \( p \) coordinates \( u^A \) with \( (A, B = 1, 2 \ldots, p) \), for \( \Sigma_t \). In this sense, \( \Sigma_t \) is viewed as the spacelike extended object \( \Sigma \) at fixed \( t \). Secondly, it can be described also by its embedding in itself, \( x^a = X^a(u^A) \). Both descriptions are related by composition. Indeed, in one picture, the tangent vectors to \( \Sigma_t \) are \( \epsilon^a_A = X^\mu_A = \partial X^\mu / \partial u^A \), and then the induced metric on \( \Sigma_t \) is \( h_{AB} = X_A \cdot X_B \). On the other hand, the tangent vectors to \( \Sigma_t \) are \( \epsilon^a_A = X^a_A = \partial X^a / \partial u^A \) and the induced metric is \( h_{AB} = g_{ab}X^a_AX^b_B \). Notice that \( h_{AB} = X_A \cdot X_B = (X_a \cdot X_b)X^a_AX^b_B \), and we see that \( \epsilon^a_A = \epsilon^\mu_a e^\mu_A \), from composition. Accordingly, the choice of the hypersurface vector basis depends on the particular description we are interested in. For the first description we have \( \{ \epsilon^a_A, n^a, \eta^a \} \), whereas for the second one we have \( \{ \epsilon^a_A, \eta^a \} \), where the appearance of the unit timelike vector accounts for the causal structure on \( \Sigma_t \). Note that \( \eta^a \) is defined implicitly by \( \epsilon_A \cdot \eta = 0, n_i \cdot \eta = 0 \) and \( \eta \cdot \eta = -1 \), and in the second description we have a single unit timelike normal vector, \( \eta^a \), defined implicitly by \( g_{ab} \epsilon^a_A \eta^b = 0 \) and \( g_{ab} \eta^a \eta^b = -1 \), up to a sign. Furthermore, note that \( g_{ab} \epsilon^a_A \eta^b = (e_a \cdot e_b) \epsilon^a_A \eta^b = \epsilon_A \cdot (\eta^a e_b) = 0 \) so that \( \eta^a = \eta^2 e^a_A \). In both descriptions, \( h^{AB} \) and \( h \) denotes the inverse metric and the determinant of \( h_{AB} \), respectively. We also define \( D_A \) as the torsion-less covariant derivative compatible with \( h_{AB} \), see Appendix A.

For our purposes, it is convenient to introduce the following projections of the extrinsic curvature of \( m \),

\[
L^i_{AB} = \epsilon^a_A \epsilon^b_B K^i_{ab} = -n^i \cdot D_A \epsilon_B, \quad (6)
\]

\[
L^i_A = \epsilon^a_A \eta^b K^i_{ab} = -n^i \cdot D_A \eta, \quad (7)
\]

in addition to

\[
k_{AB} = -g_{ab} \eta^a D_A \epsilon^b_B = k_{BA}, \quad (8)
\]

that is the \( \Sigma_t \) extrinsic curvature associated with the embedding of \( \Sigma_t \) in \( m \) given by \( x^a = \chi^a(u^A) \).

In a similar manner, in this geometrical framework the velocity vector, \( \hat{X}^a = \partial_t X^\mu \), is tangent to the worldvolume \( m \). In terms of the basis \( \{ \epsilon^a_A, \eta^a \} \) the velocity can
be written as
\[ \dot{X}^a = N \eta^a + N^A e^a_A, \] (9)
where, using familiar ADM terminology, \( N \) and \( N^A \) are the \textit{lapse} and the \textit{shift vector}, respectively. Since the lapse and the shift vector are expressed in terms of the derivatives of \( X^\mu \), \( i.e. \ N = -g_{ab} \dot{X}^a \eta^b \) and \( N^A = g_{ab} h^A B \dot{X}^a e^b_B \), neither \( N \) nor \( N^A \) is a canonical field variable. Indeed, contrary to what happens in the ADM treatment for the metric-GR, in the treatment adopted for extended objects both the lapse function and the shift vector are functions of the phase space, and not Lagrange multipliers.

When considering the evolution of \( \Sigma_t \) it is convenient to choose first the coordinate basis \{ \( e^a_A, \dot{X}^a \) \}. In this sense, the projections of the worldvolume metric \( g_{ab} \) with respect to this basis provide immediately its ADM form. We have
\[ g_{00} = g_{ab} \dot{X}^a \dot{X}^b = -N^2 + N^A N^B h_{AB}, \]
\[ g_{0A} = g_{ab} \dot{X}^a e^b_A = N^B h_{AB}, \]
\[ g_{AB} = g_{ab} e^a_A e^b_B = h_{AB}. \] (10)
In matrix form, the induced metric and its inverse are given by
\[ (g_{ab}) = \begin{pmatrix} -N^2 + N^A N^B h_{AB} & N^A h_{AB} \\ N^B h_{AB} & h_{AB} \end{pmatrix}, \] (11)
and
\[ (g^{ab}) = \frac{1}{N^2} \begin{pmatrix} -1 & N^A \\ N^A & N^2 h_{AB} - N^A N^B \end{pmatrix}, \] (12)
respectively. Its determinant is given by \( g = -N^2 h. \)

We can make a similar ADM decomposition of the extrinsic curvature
\[ K_{00}^i = K_{ab}^i \dot{X}^a \dot{X}^b = -n^i \cdot \ddot{X}, \]
\[ K_{0A}^i = K_{ab}^i \dot{X}^a e^b_A = -n^i \cdot D_A \dot{X}, \]
\[ K_{AB}^i = K_{ab}^i e^a_A e^b_B = -n^i \cdot D_A D_B X = L_{AB}^i. \] (13)
In matrix form we have
\[ K_{ab}^i = -\begin{pmatrix} n^i \cdot \ddot{X} \\ n^i \cdot D_A \dot{X} \\ -L_{AB}^i \end{pmatrix}. \] (14)
The mean extrinsic curvature, \( K^i = g^{ab} K_{ab}^i \), using (12) and (14), becomes
\[ K^i = \frac{1}{N^2} \left[ (n^i \cdot \ddot{X}) - 2N^A (n^i \cdot D_A \dot{X}) + (N^2 h_{AB} - N^A N^B) L_{AB}^i \right]. \] (15)
We observe the linear dependence of \( K^i \) on the accelerations of the extended object in the first term. In passing, we note that for pure normal evolution, \( N^A = 0 \), the previous expression specializes to
\[ N^2 K^i = n^i \cdot \ddot{X} + N^2 L^i, \] (16)
where \( L := h^{AB} L_{AB} \), that emphasizes the linear dependence on the acceleration.

By considering the contracted integrability conditions associated to the Gauss-Weingarten equations \([A.1]\), the worldvolume Ricci scalar can be expressed as a sum of a first-order function and a divergence term

\[
R = R + k_{AB} k^{AB} - k^2 + 2 \nabla_a (k \eta^a - \eta^b \nabla_b \eta^a),
\]

where \( k = h^{AB} k_{AB} \) and \( R \) is the Ricci scalar defined on \( \Sigma \). The presence of the last term, a total divergence, should not come as a surprise. In metric-GT it is the well known Gibbons-Hawking-York boundary term,\(^{29–31}\) that can either be subtracted from the outset, or kept as in the proof of the positivity of energy theorems by Schoen and Yau.\(^{32, 33}\)

Alternatively, using the integrability conditions associated to the Gauss-Weingarten equations \([A.3]\), the induced scale curvature can be expressed as a single second-order function, when the boundary term is kept,

\[
R = 2 L K^i - G^{ABCD} \Pi_{\mu \nu} \partial_A X^\mu \partial_B X^\nu - 2 h^{AB} \delta_{ij} \tilde{D}^A n^i \cdot \tilde{D}^B n^j.
\]

The linear dependence on the acceleration is hidden through \( K^i \) in the first term, see \([15]\). Here, \( \tilde{D}^A \) denotes the covariant derivative associated with the connection \( \sigma_{ij} := \epsilon^a_{ij} \omega_{aj}^a \), that takes into account the rotation freedom of the normal vector fields, see Appendix A. Furthermore, \( \Pi_{\mu \nu} := n^i \eta^i \) is a symmetric normal projector satisfying \( \Pi_{\mu \alpha} \Pi_{\alpha \nu} = \Pi_{\mu \nu} \), and

\[
G^{ABCD} := h^{AB} h^{CD} - \frac{1}{2} (h^{AC} h^{BD} + h^{AD} h^{BC}),
\]

is a Wheeler-DeWitt like metric associated to \( h_{AB} \). In passing, we must bear in mind the function dependence of the normals, \( n^\mu \eta^i = n^\mu \eta^i (X^\alpha, \dot{X}^\alpha) \). Clearly, the canonical formulation based on the expression \([17]\) that neglects the divergence term, assuming a brane without boundaries, as is commonly done, is a different starting point than the expression \([18]\) that includes it. This second avenue is the one taken in this paper.

Since our interest lies in performing a canonical description of the RT model respecting its second-order nature, we perform the ADM decomposition of all the terms of the action \([11]\) as follows

\[
S_{RT}[X^\mu] = \int dt L_{RT}(X^\mu_A, \dot{X}^\mu, \dot{X}^\mu_A, \ddot{X}^\mu),
\]

where we recall that \( \ddot{X}^\mu \) belongs to the configuration space from the Ostrogradsky-Hamilton viewpoint, and

\[
L_{RT} = \int d^p u \mathcal{L}_{RT} = \int \mathcal{L}_{RT}.
\]

For convenience in the notation, hereafter, the differential \( d^p u \) whenever a \( \Sigma \) integration is performed will be absorbed.
The Lagrangian density is

\[
\mathcal{L}_{\text{RT}} = \frac{1}{2} N \sqrt{h} \left[ 2 L_i K^i - G^{ABCD} \Pi_{\mu\nu} \mathcal{D}_A D_B X^\mu \mathcal{D}_C D_D X^\nu - 2 h^{AB} \delta_{ij} \mathcal{D}_A n^i \cdot \mathcal{D}_B n^j \right].
\]  

(22)

The structure of this Lagrangian density deserves a few comments. In the first term, through the mean extrinsic curvature, the linear acceleration dependence is hidden, see the first term in (15). The second term involves both the Wheeler-DeWitt like superspace metric and the normal projector \( \Pi_{\mu\nu} = n^{i\nu} \), defined earlier. Finally the last term is precisely the expression of a nonlinear sigma model built from \( n^{i\nu} \), with \( O(N-p-1) \) symmetry that reflects the invariance under rotations of the normal vectors \( n^{i\mu} = n^{i\nu}(X^{\alpha}, \dot{X}^{\alpha}) \), constrained to satisfy \( n^{i\mu} \cdot n^{j\nu} = \delta^{ij} \).

The Lagrangian density (22) is our starting point for obtaining the Hamiltonian formulation of GBG.

Regarding the ADM decomposition of the linear momentum density (5), using the tangential projector from \( m \) onto the hypersurface \( \Sigma \) defined as

\[
H_{ab} = h_{AB} \epsilon^a_{A} \epsilon^b_{B} = g_{ab} + \eta^{a} \eta^{b},
\]

we have

\[
\pi_{\mu} = -\sqrt{h} \eta^{a} G_{ab} \eta^{b} \epsilon_{\mu c} = \sqrt{h} \left[ (\eta^{a} G_{ab} \eta^{b}) \eta_{\mu} - \eta^{a} G_{ab} \epsilon^{b} \eta_{\mu} \right],
\]

(23)

where we have considered the fact that \( \sqrt{-g} = N \sqrt{h} \). Moreover, taking into account the integrability conditions associated to (A.3), the projections of the worldvolume Einstein tensor are given by

\[
\eta^{a} G_{ab} \eta^{b} = \frac{1}{2} \left( R - k_{AB} k^{AB} + k^2 \right),
\]

(24)

\[
\eta^{a} G_{ab} \epsilon^{b} = \mathcal{D}_A (k^{AB} - h^{AB} k) = -(L_i^{AB} - h^{AB} L_i) L_A^i,
\]

(25)

\[
\epsilon^{a} G_{ab} \epsilon^{b} = K_i L_A^{AB} - L^{AC} L^{C}_{B i} + L^{A}_{B i} - \frac{1}{2} R h_{AB},
\]

(26)

where we recall that \( R \) denotes the Ricci scalar of the hypersurface \( \Sigma_t \).

4. Ostrogradsky-Hamilton approach

According to the Ostrogradsky-Hamilton formulation, we have a \( 4N \)-dimensional phase space spanned by two conjugate pairs \{\( X^\mu, p_\mu \); \( \dot{X}^\mu, P_\mu \)\} where the momenta \( p_\mu \) and \( P_\mu \), conjugate to \( X^\mu \) and \( \dot{X}^\mu \) respectively, are defined in terms of the \( \Sigma \) basis as

\[
P_\mu = \frac{\delta L_{\text{RT}}}{\delta \dot{X}^\mu} = \frac{\sqrt{h}}{N} L_i^i n_{\mu i},
\]

(27)

\[
p_\mu = \frac{\delta L_{\text{RT}}}{\delta X^\mu} - \partial_i P_\mu = \pi_\mu + \partial_A \left( N^A P_\mu + \sqrt{h} h^{AB} L_B^i n_{\mu i} \right),
\]

(28)

where \( \pi_\mu \) is given by (3). Note that the momenta \( P_\mu \) and \( p_\mu \) are spatial densities of weight one because the presence of the factor \( \sqrt{h} \). Also, the integral over a spatial closed geometry of the momenta \( p_\mu \) and \( \pi_\mu \) differs by a boundary term. In this sense, whereas the momenta \( P_\mu \) are explicitly normal to the worldvolume, the momenta
\( p_\mu \) are tangential to the worldvolume, up to a spatial divergence. In our analysis, we will keep this setting as general as possible, so that we are not restricting our attention to a closed geometry, allowing for arbitrary boundary conditions.

In this extended phase space, the appropriate Legendre transformation is given by

\[
H_0 := p \cdot \dot{X} + P \cdot \dot{X} - \mathcal{L}_{\text{RT}},
\]

and it provides the canonical Hamiltonian density of weight one

\[
H_0 = p \cdot \dot{X} + 2N^A (P \cdot D_A X) + (N^2 h^{AB} - N^A N^B) (P \cdot D_A D_B X) + \frac{1}{2} N \sqrt{h} G^{ABCD} \Pi_{\mu\nu} D_A D_B X^\mu D_C D_D X^\nu + N \sqrt{h} A^{AB} \delta i^j \tilde{D}_{A \mu i} \cdot \tilde{D}_{B \nu j},
\]

so that the canonical Hamiltonian reads

\[
H_0[X^\mu, p_\mu; \dot{X}^\mu, P_\mu] = \int_\Sigma H_0(X^\mu, p_\mu, \dot{X}^\mu, P_\mu).
\] (30)

Note the linear dependence of the canonical Hamiltonian on the momenta \( p_\mu \) and \( P_\mu \). Classically, the physical momenta \( p_\mu \) can take both negative and positive values in phase space making the canonical Hamiltonian unbounded from below. In other words, the well known Ostrogradsky instabilities may be present in the dynamics of the theory (see e.g.\(^ {34} \)). Also observe the absence of a quadratic term, \( P^2 \), that would be a signature of an authentic second-order derivative brane model\(^ {21, 35, 38} \).

Moreover, \( H_0 \) involves a highly nonlinear dependence on the configuration variables \( X^\mu \) and \( \dot{X}^\mu \) through the lapse and shift functions as well as the last two terms in (29).

The presence of local symmetries manifests through the presence of constraints on the phase space variables. In principle, we can determine them by computing first the null eigenvectors of the Hessian matrix. In this case, this vanishes identically,

\[
H_{\mu\nu} = \frac{\delta^2 \mathcal{L}_{\text{RT}}}{\delta X^\mu \delta X^\nu} = 0.
\] (31)

This feature characterize theories affine in the acceleration\(^ {37, 38} \). The rank of the Hessian matrix is zero which is a signal that the phase space is fully constrained, i.e. we have the presence of \( N \) primary constraints. Clearly, we cannot invert any of the accelerations \( \ddot{X}^\mu \) in favour of the phase space variables so that the definition itself of the momenta \( P_\mu \)\(^ {27} \) provides the set of \( N \) primary constraints densities

\[
C_\mu := P_\mu - \frac{\sqrt{h}}{N} L^i n_{\mu i} = 0.
\] (32)

A more manageable approach to the computations with these constraints, without affecting their content, is to exploit the intrinsic geometric nature of the system. Indeed, using the tangential projector from \( \mathcal{M} \) onto the hypersurface \( \Sigma \), \( \mathcal{H}^{\mu\nu} = h^{AB} \epsilon^\mu_A \epsilon^\nu_B = \eta^{\mu\nu} + \eta^{\mu} \eta^{\nu} - n^{\mu i} n^{\nu i} \), written in terms of the hypersurface \( \Sigma \) basis \( \{ \dot{X}^\mu, \epsilon^\mu_A, n^{\mu i} \} \) we can rewrite them as \( \mathcal{C}_\mu = \eta_{\mu\nu} C^\nu = C_1 \dot{X}_\mu + C_A \epsilon^\mu_A + C_i n_{\mu i} = 0 \), where we have \( \eta^{\mu} = \epsilon^\mu_A \eta^A = (\dot{X}^\mu - N^A \epsilon^\mu_A)/N \). This linear combination helps to
identify a set of equivalent primary constraints densities

\[ C_1 := P \cdot \dot{X} = 0, \]  
\[ C_A := P \cdot \partial_A X = 0, \]  
\[ C_i := P \cdot n_i - \sqrt{\frac{h}{N}} L_i = 0. \]  

We will see below that these constraints do generate the expected local gauge transformations.

It is convenient to turn these constraints densities into functions in the phase space \( \Gamma \). To do this, we smear out the constraints (33), (34) and (35) by test fields \( \lambda, \lambda^A \) and \( \phi^i \) defined on \( \Sigma \), and then we integrate them over the entire spatial hypersurface \( \Sigma \) with

\[ S_\lambda := \int_\Sigma \lambda P \cdot \dot{X}, \]  
\[ V_\lambda := \int_\Sigma \lambda^A P \cdot \partial_A X, \]  
\[ W_{\phi} := \int_\Sigma \phi^i \left[ P \cdot n_i + \sqrt{\frac{h}{N}} L_i \right]. \]

According to the Dirac-Bergmann procedure for constrained systems, the time evolution in the phase space \( \Gamma \) is generated by the total Hamiltonian

\[ H[X^\mu, p_\mu; \dot{X}^\mu, P_\mu] = H_0 + S_\lambda + V_\lambda + W_{\phi}. \]  

The time evolution of any phase space function \( F \) is given by

\[ \partial_t F = \dot{F} \approx \{ F, H \}, \]

where we have used the Ostrogradsky-Poisson bracket [PB] appropriate for second-order derivative theories

\[ \{ F, G \} = \int_\Sigma \left[ \frac{\delta F}{\delta X} \cdot \frac{\delta G}{\delta p} + \frac{\delta F}{\delta \dot{X}} \cdot \frac{\delta G}{\delta P} - (F \leftrightarrow G) \right], \]

with \( F, G \in \Gamma \) as well as using the weak equality symbol \( \approx \) that means that after evaluating the PB, then one imposes the vanishing of the constraints, \( C_1, C_A, C_i = 0 \).

Hence, we can use (40) to efficiently compute the time evolution of any constraint function. Before we do this, we note that under the PB structure the primary constraints functions (36-38), are in involution with each other. We have

\[ \{ S_\lambda, S_{\lambda'} \} = 0, \]  
\[ \{ S_\lambda, V_\lambda \} = V_{\lambda'}, \]  
\[ \{ S_\lambda, W_{\phi} \} = W_{\phi'}, \]  
\[ \{ V_\lambda, V_{\lambda'} \} = 0, \]  
\[ \{ V_\lambda, W_{\phi} \} = 0, \]  
\[ \{ W_{\phi}, W_{\phi'} \} = 0, \]
where the functional derivatives of the primary constraints are

\[
\frac{\delta S_A}{\delta X^\mu} = \lambda P_\mu, \quad \frac{\delta S_A}{\delta \dot{X}^\mu} = -\partial_A(\Lambda^A P_\mu) = -\mathcal{L}_A P_\mu, \quad \frac{\delta S_A}{\delta \dot{X}^\mu} = \lambda \dot{X}^\mu, \quad \frac{\delta S_A}{\delta P_\mu} = \lambda^A \partial_A X^\mu = \mathcal{L}_A X^\mu, \quad \frac{\delta S_A}{\delta P_\mu} = \phi^i n^{i\mu}, \quad \frac{\delta W}{\delta P_\mu} = \phi^i n^{i\mu},
\]

In order for the primary constraints to be consistent, following the Dirac-Bergmann recipe, extended to higher-order Lagrangians, we require that their time evolution be vanishing. This procedure gives rise to the secondary constraints densities

\[
C_1 := \mathcal{H}_0 = 0, \quad \tag{44}
\]

\[
C_A := p \cdot \partial_A \dot{X} + P \cdot \partial_A X = 0, \quad \tag{45}
\]

\[
C_i := p \cdot n_i - n_i \cdot \partial_A \left( N^A P + \sqrt{\lambda} h^{AB} L_B j n_j \right) = 0. \quad \tag{46}
\]

It is worth mentioning that these constraints can also be obtained by projecting the momenta \( p_i \), given by (28), along the \( \Sigma_i \) basis \( \{ \dot{X}^\mu, \epsilon^{\mu}_{A,i}, n^{i\mu} \} \).

As above, we turn the local secondary constraints into secondary constraint functions in the phase space \( \Gamma \) by smearing them by the test fields \( \Lambda, \Lambda^A \) and \( \Phi^i \), defined on \( \Sigma_i \), and integrating over \( \Sigma \),

\[
S_\Lambda := \int_{\Sigma} \Lambda \mathcal{H}_0, \quad \tag{47}
\]

\[
V_{\dot{X}} := \int_{\Sigma} \Lambda^A \left( p \cdot \partial_A \dot{X} + P \cdot \partial_A X \right), \quad \tag{48}
\]

\[
W_{\Phi} := \int_{\Sigma} \Phi^i \left[ p \cdot n_i - n_i \cdot \partial_A \left( N^A P + \sqrt{\lambda} h^{AB} L_B j n_j \right) \right]. \quad \tag{49}
\]

Some of the functional derivatives of the secondary constraints are given by

\[
\frac{\delta V_{\dot{X}}}{\delta \dot{X}^i} = -\partial_A(\Lambda^A P_\mu) = -\mathcal{L}_A P_\mu, \quad \frac{\delta V_{\dot{X}}}{\delta \dot{X}^\mu} = \Lambda^A \partial_A \dot{X}^\mu = \mathcal{L}_A \dot{X}^\mu, \quad \tag{43}
\]

and

\[
\frac{\delta W_{\Phi}}{\delta p_\mu} = \Phi^i n^{i\mu}, \quad \frac{\delta W_{\Phi}}{\delta P_\mu} = N_A \mathcal{D}_A(\Phi^i n^{i\mu}) = \mathcal{S}_\Phi(\Phi^i n^{i\mu}), \quad \tag{43}
\]

\[
\frac{\delta W_{\Phi}}{\delta X^\mu} = (p \cdot n_i) \mathcal{D}_A \Phi^i h^{AB} \nu_{\mu B} + \frac{1}{N} \mathcal{D}_A \left( \mathcal{D}_B \Phi^i \sqrt{\lambda} h^{AB} \right) n_{\mu i} + \sqrt{\lambda} \Phi^i L_A L^A j n_j + \frac{1}{N} \Phi_i L_A L^A j n_j \right) \Phi_i n_{\mu i}, \quad \tag{43}
\]

\[
\frac{\delta W_{\Phi}}{\delta X^\mu} = \mathcal{D}_A \mathcal{W}^A, \quad \tag{43}
\]
where
\[
\bar{W}^A = h^{AB}(p \cdot \epsilon_B)\Phi_i n^i \mu_i + \frac{N^A}{N} (p \cdot \eta)\Phi_i n^i \mu_i + \bar{D}_B \Phi_i (P \cdot n^i) h^{AB} \dot{X}_\mu \\
- \left[ h^{AB} N^C L^j_{BC}(P \cdot n_j) + \sqrt{h} h^{AB} L^C_{BC} L^j_{BC} \right] \Phi_i n^i \mu_i + 2N(A h^B)C(P \cdot n^i) \bar{D}_C \Phi_i \epsilon_{\mu B} \\
- \left[ \frac{N^A N^B}{N} L_B^j (P \cdot n_j) + \sqrt{h} \frac{N^A}{N} L_B^j L^j_{\mu B} \right] \Phi_i n^i \mu_i + 2\sqrt{h} h^B(C)A \bar{D}_B \Phi^i \epsilon_{\mu C} \\
- \sqrt{h} h^{AB} L^C_{BC} \bar{D}_C \Phi^i \epsilon_{\mu B} + \sqrt{h} h^{AB} h^{CD} L^j_{BC} L^j_{BC} \bar{D}_D \Phi_i \eta_{\mu i} - \sqrt{h} h^{AB} \bar{D}_B \Phi_i n^i \mu_i \\
+ \sqrt{h} \frac{N^A}{N} h^{BC} \bar{D}_B \bar{D}_C \Phi_i n^i \mu_i + \sqrt{h} h^{AB} L^C_{BC} L^j_{BC} \Phi_i n^i \mu_j + \sqrt{h} \frac{N^A}{N} L^j_{j B} L^j_{\mu B} \Phi_i n^i \mu_j.
\]  
(50)
is a vector field with density weight +1. The functional derivatives of \( S_\Lambda \) have been listed in Appendix B. The following step is to check that the evolution in time of the secondary constraints does not generate further, tertiary constraints. By using the PB algebra listed in (42) we do not find tertiary constraints so that the Dirac-Bergmann algorithm terminates.

We make the following remarks. The constraints (33) and (34) are characteristic of second-order derivative brane models and just involve the momenta \( P_\mu \). Geometrically, they may be interpreted as a consequence of the orthonormality of the worldvolume basis. On the contrary, the constraints (44) and (45) involve all the phase space variables. Regarding these, (44) reflects the vanishing of the canonical Hamiltonian, that is expected because of the invariance under worldvolume reparametrization of the theory. Indeed, it generates diffeomorphisms out of \( \Sigma \) onto the worldvolume. On the other hand, (45) generates diffeomorphisms tangential to \( \Sigma \). This fact can be verified by considering the PB with the phase space variables as we will see shortly. These two constraints should be recognizable to the reader familiar with the ADM formulation of General Relativity. Regarding the remaining constraints (45) and (46), the first one represents a way of expressing the trace of the spatial-spatial projection of the extrinsic curvature, \( L^j \), in terms of the phase space variables and the second one reflects the orthogonality between the physical momenta \( \pi_\mu \) and the normal vectors to the worldvolume. In other words, the constraints (45) and (46) are characteristic of brane models linear in accelerations.

The remaining PB between the primary and secondary constraints, as well as between the secondary constraints themselves, are listed in Appendix C. These relationships are helpful in identifying the first and second class constraints of the model.

### 4.1. Hamilton’s equations

Here we obtain the field equations in the Hamiltonian formulation. This computation is helpful in order to fix some Lagrange multipliers that appear as test functions in the definition of the constraints as functions in phase space in terms of the phase space variables. In addition it provides a check, as it reproduces the form of the momenta \( P_\mu, p_\mu \) given by (27) and (28), respectively.
By considering the functional derivatives listed before and in Appendix B as well as the Hamiltonian \[(51)\] we have first that
\[
\partial_t X^\mu = \{X^\mu, H\} = \frac{\delta H_0}{\delta p^\mu} = \dot{X}^\mu.
\]
This result is obvious since the only dependence on \(p^\mu\) is through the term \(p \cdot \dot{X}\) appearing in \(H_0\). Secondly, we compute
\[
\partial_t \dot{X}^\mu = \{\dot{X}^\mu, H\} = \frac{\delta H_0}{\delta \dot{p}^\mu} + \frac{\delta S_\lambda}{\delta \dot{p}^\mu} + \frac{\delta V_\lambda}{\delta \dot{p}^\mu},
\]
\[
= 2N^A \mathcal{D}_A \dot{X}^\mu + (N^2 h^{AB} - N^A N^B) \mathcal{D}_A \mathcal{D}_B X^\mu + \lambda \dot{X}^\mu + \lambda^A \epsilon^A + \phi^i \eta^i.
\]
By contracting \[(52)\] with the momenta \(P^\mu\) and considering the identity \[(53)\] and the primary constraint densities \[(55)\], \[(56)\] and \[(57)\], we identify
\[
\phi^i = N^2 K^i.
\]
In order to fix the remaining Lagrange multipliers, it is useful to recall an important identity relating the acceleration in terms of the \(\Sigma_i\) basis \[(21)\]
\[
\dot{X}^\mu = (\dot{N}_A + N \mathcal{D}_A N - N^B \mathcal{D}_A N_B) \epsilon^A + (\dot{N} + N^A \mathcal{D}_A N + N^A N^B k_{AB}) \eta^A + (n^i \cdot \dot{X}) n^i.
\]
As before, by considering \[(54)\] and the primary constraints, when contracting \[(52)\] with \(\eta\) and \(\epsilon_A\) yields
\[
\lambda = \mathcal{D}_A N^A - \frac{N^2}{\sqrt{h}} \eta^a \nabla_a \left( \sqrt{h} \right) = \frac{1}{N} \left( \dot{N} - N^A \mathcal{D}_A N - N^2 k \right),
\]
\[
\lambda^A = N \mathcal{D}_A N - N^A \mathcal{D}_B N^B + \frac{N^2}{\sqrt{h}} \eta^a \nabla_a \left( \sqrt{h} N^A \right),
\]
where we have used the time derivative of the spatial metric, \(\dot{h}_{AB} = 2Nk_{AB} + 2\mathcal{D}_A N_B\) and its determinant, \(\partial_t (\sqrt{h}) = \sqrt{h}(NK + \mathcal{D}_A N^A)\). It is worthwhile to mention that the Lagrange multipliers \[(55)\] and \[(56)\] are inherent to second-order derivative brane models \[(21)\].

We turn now to compute the time evolution of the momenta \(P^\mu\). We obtain the lengthy expression
\[
\partial_t P^\mu = \{P^\mu, H\} = -\frac{\delta H_0}{\delta X^\mu} - \frac{\delta S_\lambda}{\delta X^\mu} - \frac{\delta V_\lambda}{\delta X^\mu},
\]
\[
= -p^\mu - 2(P \cdot \mathcal{D}_A \dot{X}) h^{AB} \epsilon^B + 2N \mathcal{D}_A (2N^A P^\mu) + 2Nh^{AB} (P \cdot \mathcal{D}_A \mathcal{D}_B X) \eta^\mu,
\]
\[
+ 2N^B (P \cdot \mathcal{D}_A \mathcal{D}_B X) h^{AC} \epsilon^C + \frac{1}{2} \sqrt{h} G^{ABCD} \Pi_{\alpha \beta} \mathcal{D}_A \mathcal{D}_B X^\alpha \mathcal{D}_C \mathcal{D}_D X^\beta \eta^\mu,
\]
\[
- \sqrt{h} G^{ABCD} l_{AB} k_{BCD} n^\mu n^i + \sqrt{h} h^{AB} \delta_{ij} (\tilde{D}_A n^i + \mathcal{D}_B n^i) \eta^\mu,
\]
\[
+ \left[ \mathcal{D}_A \left( 2N \sqrt{h} h^{AB} \delta_{ij} \mathcal{D}_B n^i \right) \cdot \eta \right] \frac{1}{N} n^\mu n^i - \lambda P^\mu + \phi^i \sqrt{h} L_i \eta^\mu + \phi^i \frac{\sqrt{h}}{N^2} k n^\mu n^i.
\]
By inserting (53) and (55) into the previous expression we get

\[ p_\mu = \left\{ \alpha \sqrt{h} \left[ L_i K^i - 2L_i L^i + \frac{1}{2} G^{ABCD} \Pi_{\alpha \beta} D_A D_B X^\alpha D_C D_D X^\beta \right. \right. \]
\[ \left. \left. + h^{AB} \delta_{ij} \left( \bar{D}_A n^i \cdot \bar{D}_B n^j \right) \right] \eta_{\mu} \right. \]
\[ + \sqrt{h} \left[ k K^i + \bar{D}_A \left( 2Nh^{AB} \delta_{ij} \bar{D}_B n^j \right) \cdot \left( \frac{1}{N} \eta \right) - G^{ABCD} L^i_{AB} k_{CD} \right] \eta_{\mu i} \]
\[ + 2\sqrt{h} h^{AB} L_i L^i_{A \mu} - \frac{1}{N} (\dot{N} - N^A \dot{D}_A N - N^2 k) \eta_{\mu} + D_A (2N^A P_\mu \right) \}
\[ - \partial_t P_\mu. \]

This expression matches the definition of \( p_\mu \) (28) for a higher derivative theory once we identify the term inside the curly brackets on the r.h.s. as \( \delta L_{RT} / \delta \dot{X}_\mu \). Also, this expression exhibits the linear dependence of the momenta \( p_\mu \) on the accelerations of the extended object. So far, the Hamilton’s equations do their job in that they correctly reproduce the expressions for the momenta as well the expressions for the velocity and the accelerations of the extended object. Finally, the time evolution of momenta \( p_\mu \)

\[ \partial_t p_\mu = \{ p_\mu, H \} = -\delta H_0 / \delta X^\mu - \delta S_\Lambda / \delta X^\mu - \delta V^\Lambda / \delta X^\mu - \delta W^\Lambda / \delta X^\mu, \]

are nothing but the field equations of the model (2) in its canonical form as it may shown after a very long, but straightforward, calculation after introducing the explicit form of the Lagrange multipliers (53), (55) and (56). We refrain from writing it down explicitly because it is too cumbersome.

5. First- and second-class constraints

In order to characterize the constraint surface we need to separate both the primary and secondary constraints into first and second class constraints. To begin with, we relabel the constraints functions as follows

\[ \varphi_I := \{ W^\Lambda_\Lambda, S_\Lambda, V^\Lambda_\Lambda, S_\Lambda, V^\Lambda_\Lambda, W^\Lambda_\Lambda \}, \quad I = 1, 2, \ldots, 6. \]

where we have chosen a convenient order for them. Then, we turn to construct the antisymmetric matrix composed of the PB of all the constraint functions, \( \Omega_{IJ} := \{ \varphi_I, \varphi_J \} \). Explicitly, the matrix \( \Omega_{IJ} \) reads, weakly on the constraint surface,

\[ (\Omega_{IJ}) \approx \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & C \\
0 & 0 & 0 & 0 & A & 0 \\
0 & 0 & 0 & 0 & B & 0 \\
0 & 0 & 0 & 0 & D & 0 \\
0 & 0 & 0 & 0 & E & 0 \\
-C -A -B -D -E & F
\end{pmatrix}, \]
where the nonvanishing entries $A, B, C, D, E$ and $F$ are defined in Appendix C. The rank of this matrix is 2, thus pointing out the existence of two second-class constraint functions. To select these it is necessary to determine first the 4 zero modes $\omega_{(u)}^I$ with $z = 1, 2, 3, 4$, so that $\Omega_{IJ}\omega_{(u)}^J = 0$. These can be taken as follows

$$
\omega_{(1)}^I = \begin{pmatrix} -A/C \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \omega_{(2)}^I = \begin{pmatrix} -B/C \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \omega_{(3)}^I = \begin{pmatrix} -D/C \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \omega_{(4)}^I = \begin{pmatrix} -E/C \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
$$

(61)

With these the functions $F_u := \omega_{(u)}^I \varphi_I$ are first-class constraints,

$$
F_1 = S_\lambda - \frac{4}{C} W_\phi, \quad F_2 = V_\lambda - \frac{8}{C} W_\phi, \\
F_3 = S_\lambda - \frac{4}{C} W_\phi, \quad F_4 = V_\lambda - \frac{8}{C} W_\phi.
$$

(62)

To formally obtain the second-class constraints we proceed as follows. If we choose a set of linearly independent vectors, $\omega_{(u')}^I$ with $u' = 5, 6$, such that they do not depend on the vectors $\omega_{(u)}^I$ and satisfy the condition $\text{det}(\omega_{(I)}) \neq 0$ with $I' = (u, u')$, then the functions $S_{u'} := \omega_{(u')}^I \varphi_I$ are second-class constraints. Indeed, by choosing

$$
\omega_{(5)}^I = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \omega_{(6)}^I = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
$$

(63)

we observe that the previously mentioned conditions are satisfied. Then,

$$
S_5 = W_{\bar{g}}, \quad S_6 = W_{\tilde{g}}.
$$

(64, 65)

are second-class constraints.

The constraints $F_u$ and $S_{u'}$ define an equivalent representation of the constrained phase space. In this new framework for the constraint surface we can introduce the matrix elements $S_{u'v'} := \{S_{u'}, S_{v'}\}$ with $u', v' = 5, 6$, and its inverse matrix components $(S^{-1})_{u'v'}$, given by

$$
(S_{u'v'}) = \begin{pmatrix} 0 & C \\ -C & F \end{pmatrix}, \quad \text{and} \quad (S^{-1})_{u'v'} = \frac{1}{C^2} \begin{pmatrix} F & -C \\ C & 0 \end{pmatrix},
$$

(66)

respectively. According to the theory for constrained systems, the matrix $(S^{-1})_{u'v'}$ allows us to introduce the Dirac bracket in the usual way

$$
\{F, G\}_D := \{F, G\} - \{F, S_{u'}\}(S^{-1})_{u'v'}\{S_{v'}, G\}.
$$

(67)
Once we have formally determined the second-class constrictions, we can set them strongly equal to zero, so that these merely becomes identities serving to express some phase space variables in terms of others. Accordingly, the first-class constraints (62) reduce to

\[
\begin{align*}
F_1 &= S_\lambda, \\
F_2 &= V'_\lambda, \\
F_3 &= S_\Lambda, \\
F_4 &= V'_\Lambda,
\end{align*}
\] (68)

as expected. It is worth observing that each of the first-class constraints functions, \(F_2\) and \(F_4\), includes \(p\) primary constraints and \(p\) secondary constraints, respectively. Similarly, each of the second-class constraints functions, \(S_5\) and \(S_6\), includes \((N - p - 1)\) primary constraints and \((N - p - 1)\) secondary constraints, respectively. In this sense, the counting of the physical degrees of freedom \([dof]\) is as follows: \(dof = (\text{total number of canonical variables}) - 2 (\text{number of first-class constraints}) - (\text{number of second-class constraints})\). That is, \(dof = N - p - 1 = i\). Hence, there are \(i\) degrees of freedom, one for each normal vector of the worldvolume. This number agrees with the number of physical transverse motions \(\sigma^i := n^i \cdot \delta X\) characterizing first-order derivative brane models, as expected.

With support with the gauge transformations that generate the first-class constraints, it is convenient to name \(S_\lambda\) the \textit{shift constraint} while \(V'_\lambda\) will be referred to as the \textit{primary vector constraint}. In the same spirit, \(S_\Lambda\) and \(V'_\Lambda\) may be thought of as being the \textit{scalar} and \textit{secondary vector} constraint, respectively, in comparison to the ones appearing in a canonical analysis of the Dirac-Nambu-Goto model.\[21\]

### 5.1. Algebra of constraints

Under the Dirac bracket, the algebra spanned by the first-class constraints is

\[
\begin{align*}
\{S_\lambda, S_{\lambda'}\}_D &= 0, \\
\{S_\lambda, V'_{\lambda}\}_D &= V'_{\lambda'}, \\
\{S_\lambda, S_\Lambda\}_D &= -S_\lambda - S_\Lambda, \\
\{S_\lambda, V'_{\Lambda}\}_D &= -S_{\ell\lambda\Lambda}, \\
\{V'_{\lambda}, V'_{\lambda'}\}_D &= 0, \\
\{V'_{\lambda}, S_\Lambda\}_D &= S_{\ell\lambda\Lambda} - V'_{\lambda_2} - V'_{\lambda_1}, \\
\{V'_{\lambda}, V'_{\Lambda}\}_D &= V'_{[\lambda,\Lambda]}, \\
\{S_\Lambda, S_{\lambda'}\}_D &= S_{\lambda_2}, \\
\{S_\Lambda, V'_{\lambda}\}_D &= -S_{\ell\lambda\Lambda} + V'_{\lambda_3}, \\
\{V'_{\Lambda}, V'_{\lambda'}\}_D &= V'_{[\Lambda,\lambda']},
\end{align*}
\] (69)
where we have introduced
\[
\begin{align*}
\lambda_1^A &= \lambda \lambda^A, \\
\lambda_2^A &= 2\Lambda N^B D_B \lambda^A, \\
\Lambda_1^A &= \Lambda \lambda^A, \\
\Lambda_1 &= \lambda \Lambda.
\end{align*}
\]

This algebra is equivalent to the algebra under under the PB, once we apply the property \(\{F, F_u\} \approx \{F, F_u\}_D\), for any phase space function \(F\).

The geometrical interpretation of this algebra can be illustrated as follows. Let us begin with (69h). We observe that two different orderings of the scalar constraints may only differ by a shift transformation which means that the time evolution with the scalar constraint is unique up to a rescaling. From (69i) we note that the PB of a vector with a scalar constraint is a scalar constraint with a test field given by the Lie derivative of the parameter \(\Lambda\) along the vector field \(\vec{\lambda}\); this is accompanied by tangential deformations provide by the primary vector constraint. Relationship (69j) shows that secondary vector constraints generate a proper subalgebra of their own, i.e., it exhibits the invariance under reparametrizations of the theory. Regarding (69a) and (69e), they show that shift and primary vector transformations each form a proper sub-algebra on their own, and their algebras are Abelian. Expression (69b) shows how the primary vector constraint changes under the shift transformation; indeed, it is observed that there is no substantial change since the vector constriction is still preserved but with a different test field. Relationships (69c) and (69d) reveal how the shift transformations change under the scalar and vector constraints. At this point, the role played by scalar and vector constraint as generators of diffeomorphisms, out and tangential, to \(\Sigma_t\), is evident. Likewise, (69f) and (69g), determine how the primary vector constraint changes under the scalar and vector constraints. To end this description, we mention that, despite the complete algebra is closed under the DB, this is an open algebra since several of the test fields, (70), depend on some of the phase space variables. Furthermore, this constraint algebra is not encountered in the usual gauge theories. This fact represents a difficulty towards a standard canonical quantization of GBG in the framework considered.

5.2. Infinitesimal canonical transformations

In order to further illustrate the role of the constraints in the theory, in this subsection we consider infinitesimal canonical transformations.

It is worth remembering that, for any classical observable \(F \in \Gamma\), the Hamiltonian vector field
\[
X_F := \int_\Sigma \left( \frac{\delta F}{\delta p} \cdot \frac{\delta}{\delta X} + \frac{\delta F}{\delta X} \cdot \frac{\delta}{\delta P} - \frac{\delta F}{\delta X} \cdot \frac{\delta}{\delta p} - \frac{\delta F}{\delta p} \cdot \frac{\delta}{\delta X} \right),
\]

generates a one-parameter family of canonical transformations \(G \rightarrow G + \delta_F G\), where \(\delta_F G := \epsilon\{G, F\}\), with \(\epsilon\) being an infinitesimal dimensionless quantity. The
Hamiltonian vector fields associated with the first-class constraints (68) induce the infinitesimal canonical transformations

$$
\begin{align*}
X_{F_1} & \rightarrow \left\{ \begin{array}{l}
\delta_{S_X} X^\mu = 0, \\
\delta_{S_X} \dot{X}^\mu = \epsilon_1 \lambda \dot{X}^\mu, \\
\delta_{S_X} P_\mu = 0, \\
\delta_{V_{\lambda}} X^\mu = 0, \\
\delta_{V_{\lambda}} \dot{X}^\mu = \epsilon_2 \dot{L}_X X^\mu, \\
\delta_{V_{\lambda}} P_\mu = \epsilon_2 \dot{L}_X P_\mu,
\end{array} \right. \\
X_{F_2} & \rightarrow \left\{ \begin{array}{l}
\delta_{S_X} X^\mu = 0, \\
\delta_{S_X} \dot{X}^\mu = \epsilon_1 \lambda \dot{X}^\mu, \\
\delta_{S_X} P_\mu = 0, \\
\delta_{V_{\lambda}} X^\mu = 0, \\
\delta_{V_{\lambda}} \dot{X}^\mu = \epsilon_2 \dot{L}_X X^\mu, \\
\delta_{V_{\lambda}} P_\mu = \epsilon_2 \dot{L}_X P_\mu,
\end{array} \right. \\
X_{F_3} & \rightarrow \left\{ \begin{array}{l}
\delta_{S_X} X^\mu = \epsilon_3 \Lambda \dot{X}^\mu, \\
\delta_{S_X} \dot{X}^\mu = \epsilon_3 \dot{L}_X X^\mu, \\
\delta_{S_X} P_\mu = -\epsilon_3 \dot{L}_X P_\mu, \\
\delta_{V_{\lambda}} X^\mu = 0, \\
\delta_{V_{\lambda}} \dot{X}^\mu = \epsilon_4 \dot{L}_X X^\mu, \\
\delta_{V_{\lambda}} P_\mu = \epsilon_4 \dot{L}_X P_\mu,
\end{array} \right. \\
X_{F_4} & \rightarrow \left\{ \begin{array}{l}
\delta_{S_X} X^\mu = \epsilon_3 \Lambda \dot{X}^\mu, \\
\delta_{S_X} \dot{X}^\mu = \epsilon_3 \dot{L}_X X^\mu, \\
\delta_{S_X} P_\mu = -\epsilon_3 \dot{L}_X P_\mu, \\
\delta_{V_{\lambda}} X^\mu = 0, \\
\delta_{V_{\lambda}} \dot{X}^\mu = \epsilon_4 \dot{L}_X X^\mu, \\
\delta_{V_{\lambda}} P_\mu = \epsilon_4 \dot{L}_X P_\mu,
\end{array} \right. 
\end{align*}
$$

(72)

where $\epsilon_u$, with $u = 1, \ldots, 4$, denotes arbitrary gauge parameters corresponding to each of the first-class constraints $F_u$, respectively. For instance,

$$
\dot{X}^\mu \mapsto \dot{X}^\mu + \epsilon_1 \lambda \dot{X}^\mu, \quad \text{and} \quad P_\mu \mapsto P_\mu - \epsilon_1 \lambda P_\mu,
$$

are the gauge transformations induced by the gauge function $\lambda$. From (72) we infer that the constraint $V_\lambda$ generates diffeomorphisms tangential to $\Sigma_t$, while $S_\lambda$ is the generator of diffeomorphisms out of $\Sigma_t$ onto the worldvolume $\Sigma$. On the other hand, $S_\lambda$ is a generator of a momentum reflection in the sub-sector of $\Gamma$ given by \{\dot{X}^\mu; P_\mu\} that is, the sector associated to the second-order derivative dependence; from another viewpoint, this constraint generates \textit{shift transformations} only in the velocity sector of the phase space. Finally, the constraint $V_\lambda$ only acts on the sub-sector \{\dot{X}^\mu; P_\mu\} by generating displacements in the orthogonal complement of this sub-sector, that is, in the sub-sector \{X^\mu; P_\mu\}.

6. Discussion

We have carried out a complete Ostrogradsky-Hamilton canonical study of geodetic brane gravity described by the RT model in which the embedding functions of the brane are the field variables instead of the components of the metric, as in metric-GR. An essential ingredient in our analysis is the construction of an ADM Lagrangian density for the model linear in the embedding functions acceleration. Usually this term is discarded as a boundary term contribution. By keeping it, we treat the RT model as a higher derivative theory, even though it has eom of second order. According to the Ostrogradsky-Hamilton canonical formulation, we have an extended phase space that has positions and velocities as configuration canonical variables, together with their conjugate momenta. We have derived the canonical Hamiltonian density for the model that contains terms linear in the conjugate momenta. This is a signal of the well known Ostrogradsky instability for higher derivative theories, \textit{i.e.} the Hamiltonian is unbounded from below. However, one can hope that a suitable canonical transformation can be found to deal with this issue and obtain an Hamiltonian bounded from below. A possible strategy has
been suggested by Paul, and consists in solving the second class constraints, but it appears to be non trivial in the present case. Another alternative is to implement a path integral quantization program adapted to second-order singular systems where second-class constraints are present in the theory, but, this deserves further investigation. As expected from the symmetry under reparametrization invariance of the theory, the Hamiltonian is a linear combination of constraints. We have determined the complete set of constraints, and separated them into first- and second-class constraints. The appearance of second class constraints is the price paid for keeping a term linear in the acceleration in the Lagrangian, however they take a form that is quite manageable. In addition, we show explicitly how the constraints generate the expected gauge transformations, and a correct counting of the physical degrees of freedom has been obtained. We also checked that Hamilton’s equations reproduce the Euler-Lagrange equations of the theory. It should be mentioned that, based on the expressions for the Lagrangian and Hamiltonian densities as well from from the constraint densities obtained, the codimension is left arbitrary. Many of the features of the RT model generalize to the larger class of theories linear affine in accelerations.

In principle, starting from our classical formulation, a formal canonical quantization program can be implemented. This would satisfy Regge and Teitelboim original motivation. With respect to quantum gravity one important technical advantage is the presence of a fixed background, that should come of help in a formal quantization. The phase space variables would be promoted to operators in a suitable Hilbert space. As appropriate for a theory with second class constraints, the Dirac brackets would turn into commutators for such operators. A point to be confronted would be to find suitable gauge fixing conditions, to arrive at a space of physical states. The issue of obtaining an Hamiltonian bounded from below, avoiding the presence of ghosts and lack of unitarity, would need to be resolved. In fact, ideally one would like to derive an Hamiltonian constraint quadratic in the momenta $p_\mu$. Another difficulty arises from the fact that the constraint algebra obtained is not a genuine Lie algebra, in addition to not being encountered in usual gauge theories. All of these issues would also appear in a path integral or BRST quantization of the model. Although aware of the difficulties ahead, we believe that our Ostrogradsky-Hamilton treatment of geodetic brane gravity provides a reliable stepping stone.

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Appendix A. Integrability conditions

Depending on the viewpoint, we have integrability conditions to describe the geometry of an extended object at a fixed time, $\Sigma_t$, once this undergo an ADM split.

Appendix A.1. $\Sigma_t$ embedded in $m$

If $\Sigma_t$ is embedded into $m$, $x^a = \chi^a(u^A)$, with $u^A$ being the local coordinates in $\Sigma_t$ and $A = 1, 2, \ldots, p$, the orthonormal basis is provided by $\{e^a_A = \partial_A \chi^a, \eta^a\}$. This satisfies $g_{ab} e^a_A \eta^b = 0$, $g_{ab} \eta^a \eta^b = -1$ and $g_{ab} e^a_A e^b_B = h_{AB}$ where $h_{AB}$ is the spacelike metric associated to $\Sigma$. The corresponding Gauss-Weingarten (GW) equations are

$$\nabla_A e^a_B = \Gamma^c_{AB} e^a_C + k_{AB} \eta^a, \quad \nabla_A \eta^a = k_{AB} h^{BC} e^a_C,$$

(A.1)

where $\nabla_A = e^a_A \nabla_a$, $k_{AB} = k_{BA}$ is the extrinsic curvature of $\Sigma_t$ associated to the normal $\eta^a$ and $\Gamma^c_{AB}$ stands for the connection compatible with $h_{AB}$.

The intrinsic and extrinsic geometries for the embedding under consideration must satisfy the integrability conditions

$$R_{abcd} e^a_A e^b_B e^c_C e^d_D = R_{ABCD} - k_{AD} k_{BC} + k_{AC} k_{BD}, \quad (A.2a)$$

$$R_{abcd} e^a_A e^b_B e^c_C \eta^d = D_A k_{BC} - D_B k_{AC}, \quad (A.2b)$$

where $R_{ABCD}$ is the Riemann tensor associated to the spacelike manifold $\Sigma_t$ and $D_A$ is the covariant derivative compatible with $h_{AB}$.

Appendix A.2. $\Sigma_t$ embedded in $\mathcal{M}$

If $\Sigma_t$ is embedded into $\mathcal{M}$, $x^\mu = X^\mu(u^A)$, the orthonormal basis is provided by $\{\epsilon^\mu_A = \partial_A X^\mu, \eta^\mu, n^\mu_i\}$. This satisfies $\epsilon_A \cdot \eta = \epsilon_A \cdot n_i = \eta \cdot n_i = 0$, $\eta \cdot \eta = -1$, $n_i \cdot n_j = \delta_{ij}$ and $\epsilon_A \cdot \epsilon_B = h_{AB}$. The corresponding GW equations are

$$D_A \epsilon^\mu_B = \Gamma^c_{AB} \epsilon^\mu_C + k_{AB} \eta^\mu - L_A^i n^\mu_i, \quad (A.3)$$

where $D_A = \epsilon^\mu_A D_\mu$ and $D_\mu$ being the background covariant derivative, $L_{AB}^i = L_{BA}^i$ is the extrinsic curvature of $\Sigma_t$ associated to the normal $n^\mu_i$. Additionally, we have introduced $L_A^i := \epsilon^a_A \eta^b K^i_{ab}$ and $\sigma_A^{ij} := \epsilon^a_A \omega_{a}^{ij}$. Observe that the tangent-normal projection of the worldvolume extrinsic curvature is in fact a piece of a non-trivial twist potential given by $L_A^i = n^i \cdot D_A \eta$.

The intrinsic and extrinsic geometries for the embedding under consideration
must satisfy the integrability conditions

\begin{equation}
0 = - R_{ABCD} - k_{AC}k_{BD} + k_{BC}k_{AD} + L_{AC}^j L_{BD}^i - L_{BC}^i L_{AD}^j, \tag{A.4a}
\end{equation}

\begin{equation}
0 = \mathcal{D}_A k_{BC} - \mathcal{D}_B k_{AC} + L_A^i L_{BC}^i - L_B^i L_{AC}^i, \tag{A.4b}
\end{equation}

\begin{equation}
0 = \mathcal{D}_A L_{BC}^i - \mathcal{D}_B L_{AC}^i + L_A^i k_{BC} - L_B^i k_{AC}, \tag{A.4c}
\end{equation}

\begin{equation}
0 = - \Omega_{AB}^j + L_A^C i L_{BC}^i - L_B^C i L_{AC}^j - L_A^i L_B^j + L_B^i L_A^j, \tag{A.4d}
\end{equation}

where $\Omega_{AB}^j := \mathcal{D}_B \sigma_A^{ij} - \mathcal{D}_A \sigma_B^{ij}$ is the curvature tensor associated with the gauge field $\sigma_A^{ij}$ and $\mathcal{D}_A$ is the $O(N - p - 2)$ covariant derivative acting on the normal indices associated with the connection $\sigma_A^{ij}$.

### Appendix B. Functional derivatives of $S_A$

Here we present the functional derivatives of the Hamiltonian constraint $S_A$

\begin{equation}
\frac{\delta S_A}{\delta \rho^\mu} = \Lambda \dot{X}^\mu, \quad \frac{\delta S_A}{\delta P^\mu} = 2\Lambda^A \mathcal{D}_A \dot{X}^\mu + \Lambda(N^2 h^{AB} - N^A N^B) \mathcal{D}_A \mathcal{D}_B X^\mu, \tag{B.1}
\end{equation}

where

\begin{align*}
\tilde{T}^A := -2\Lambda \eta^A B(P \cdot \mathcal{D}_A \dot{X}) \eta_\mu + 2\Lambda \eta^A h^{BC}(P \cdot \mathcal{D}_A \dot{X}) \eta_\mu + 2\Lambda \eta^A N^B(P \cdot \mathcal{D}_A \mathcal{D}_C X) \eta_\mu \\
+ 2\Lambda N^A N^B(P \cdot \mathcal{D}_A \mathcal{D}_C X) \eta_\mu + 2\Lambda N^A h^{BC} N^D(P \cdot \mathcal{D}_A \mathcal{D}_B \mathcal{D}_C X) \eta_\mu \\
+ 2\Lambda N^A h^{BC} h^{BD}(P \cdot \mathcal{D}_A \mathcal{D}_B \mathcal{D}_C X) \eta_\mu + \mathcal{D}_B \left[ \Lambda(N^2 h^{AB} - N^A N^B) \rho \right] \\
- \frac{1}{2} \Lambda \eta^A \eta^B \mathcal{D}_A \mathcal{D}_B X^\mu \eta^\alpha \mathcal{D}_\alpha \eta_\mu + 2\Lambda \eta^A \eta^B \eta^C \mathcal{D}_A \mathcal{D}_B \mathcal{D}_C X^\mu \eta_\mu \\
+ \Lambda \sqrt{\eta} h^{AB} h^{CD} \mathcal{D}_A \mathcal{D}_B \mathcal{D}_C \mathcal{D}_D X^\mu \eta_\mu \\
+ 2\Lambda \sqrt{\eta} h^{AB} h^{CD} \mathcal{D}_A \mathcal{D}_B \mathcal{D}_C \mathcal{D}_D \mathcal{D}_E \mathcal{D}_F X^\mu \eta_\mu \\
- \sqrt{\eta} h^{ABC} \delta_{ij} \left( \mathcal{D}_B n^i \cdot \mathcal{D}_C n^j \right) N^A \eta_\mu - \Lambda \sqrt{h} h^{ABC} \delta_{ij} \left( \mathcal{D}_C n^i \cdot \mathcal{D}_D n^j \right) \eta^\mu, \tag{B.2}
\end{align*}

is a vector field with density weight +1.
Appendix C. Constraint algebra

Primary-primary constraints

\[
\begin{align*}
\{S_\lambda, S_\lambda\} &= 0, & \{V_\lambda, V_\lambda\} &= 0, \\
\{S_\lambda, V_\chi\} &= V_\chi, & \chi^A &= \lambda^A, \quad \{V_\chi, W_{\phi^1}\} &= 0, \quad \phi'^i &= \lambda^i, \\
\{S_\lambda, W_{\phi^1}\} &= W_{\phi^1}, & \{W_{\phi^1}, W_{\phi^1}\} &= 0,
\end{align*}
\]

(C.1)

Primary-secondary constraints

\[
\begin{align*}
\{S_\lambda, S_\lambda\} &= -S_{\lambda_1} - S_{\lambda_2}, & \{W_{\phi^1}, S_\lambda\} &= S_{\lambda_2} + V_{\lambda_2} - W_{\phi^1}, \\
\{S_\lambda, V_\chi\} &= -S_{\chi_1}, & \{W_{\phi^1}, V_\lambda\} &= S_{\lambda_3} - V_{\lambda_3} - W_{\phi_2}, \\
\{S_\lambda, W_{\phi^1}\} &= A, & \{W_{\phi^1}, W_{\phi^1}\} &= W_{\phi^3} + C, \\
\{V_\chi, S_\lambda\} &= S_{\chi_1} - V_{\chi_1} - V_{\lambda_2}, & \{V_\chi, V_\lambda\} &= V_{[\chi, \lambda]}, \\
\{V_\chi, W_{\phi^1}\} &= B, & \{V_\chi, W_{\phi^1}\} &= 0,
\end{align*}
\]

(C.2)

where

\[
\begin{align*}
\lambda_1 &= 2 \lambda N^A \lambda, & \phi'^1 &= \lambda \phi^1 N k - \phi^1 N A D A + \lambda N A \bar{D}_A \phi^1, \\
\lambda_1 &= \lambda A, & \Phi^1 &= \lambda \phi^1, \\
\lambda_1^A &= 2 \lambda N B D_B \lambda^A, & \lambda_3 &= \lambda A \phi_i L_A, \\
\lambda_1^A &= \lambda \lambda^A, & \lambda_3 &= \lambda A \phi_i L_B, \\
\lambda_2^A &= \lambda \phi_i \left( N^A L^i - N B L_B A^i - \frac{N A N B}{N} L_B \right), & \phi^2 &= \lambda A \bar{D}_A \phi^1, \\
\lambda_2^A &= \lambda \phi_i \left( N^A L^i - N B L_B A^i - \frac{N A N B}{N} L_B \right), & \phi^2 &= \frac{N A}{N} \phi^1 \Phi_j L_A^i, \\
\lambda_3 &= \lambda 1, & \phi_4 &= \lambda A \Phi_i L_A^i.
\end{align*}
\]

and

\[
\begin{align*}
A &= \int_\Sigma \lambda \bar{D}_A \phi^i \sqrt{h} h^{AB} (n_i D_B \bar{X}), & (C.3) \\
B &= -\int_\Sigma \sqrt{h} h^{AB} \lambda^C \bar{D}_A \Phi_i L_{BC}, & (C.4) \\
C &= \int_\Sigma \left[ \sqrt{h} \phi^i \phi_i L_A^i (N^A L_j + N L_j) - \frac{1}{N} \phi^i \Phi_i (\eta \cdot \eta) + \sqrt{h} \phi^i \Phi_j (L_A^B L_{AB} - L_i L^j) \right]. & (C.5)
\end{align*}
\]

Secondary-secondary constraints

\[
\begin{align*}
\{S_\lambda, S_\lambda\} &= S_{\lambda_1} + W_{\phi^4} \\
\{S_\lambda, V_\chi\} &= V_{\chi_1} - S_{\chi_2}, & \{V_\chi, V_\chi\} &= V_{[\chi, \chi]}, \\
\{S_\lambda, W_{\phi^1}\} &= S_{\lambda_2} + V_{\lambda_3} + D, & \{V_\chi, W_{\phi^1}\} &= S_{\lambda_3} + V_{\lambda_3} + W_{\phi_4} + W_{\phi_4} + \mathcal{E}, \\
\{S_\lambda, W_{\phi^2}\} &= S_{\lambda_3} + V_{\lambda_3} + D, & \{W_{\phi^1}, W_{\phi^1}\} &= -W_{\phi_5} + \mathcal{E}, \\
\{V_\chi, W_{\phi^1}\} &= V_{[\chi, \lambda]}, & (C.6)
\end{align*}
\]
where
\begin{align}
\lambda_4 &= (N^2 h^{AB} - N^A N^B) (\mathcal{D}_A \mathcal{D}_B A' - \Lambda' \mathcal{D}_A \mathcal{D}_B A), \\
\phi_4^i &= 2 N^3 h^{AB} L_A^i (\mathcal{D}_B A' - \Lambda' \mathcal{D}_A A), \\
\lambda_4^i &= \Lambda (N^2 h^{BC} - N^B N^C) (\mathcal{D}_B \mathcal{D}_C A^A + R_{BDC} A^D), \\
\lambda_5 &= \frac{1}{N} [\Lambda^A N^B k_A^C L_{BC} \Phi_i - \mathcal{L}_X (N^A L_A^i) \Phi_i - \Lambda^A L_A^i L^i \Phi_i], \\
\lambda_6^C &= \Lambda^A \bar{\mathcal{D}}_A (N^B L_B^C \phi_i) + \Lambda^A \bar{\mathcal{D}}_A (N^B L_B^C) \frac{N C}{N} \Phi_i \\
&+ \Lambda^A N_B^C \left[ N \bar{\mathcal{D}}_B \Phi_i L_A^C i - k_A^D L_{BD} N^C \Phi_i + \frac{N_B}{N} (L_A^i \bar{\mathcal{D}}_B \Phi_i N^C - N k_A^C L_B^i \Phi_i) \right], \\
\phi_5^i &= 2 \Lambda^{[A} N^{B]} (L_A^C j L_{BC} - L_A^i L_B^i) \Phi_j, \\
\Phi_4^i &= \Lambda^A \bar{\mathcal{D}}_A \Phi_i, \\
\phi_6^i &= N h^{AB} k_B^j (\Phi_j \bar{\mathcal{D}}_A \Phi_i - \Phi_i \bar{\mathcal{D}}_A \Phi_j), \\
\lambda_6 &= \Lambda \bar{\mathcal{D}}_A \Phi_i \left[ (N^2 h^{AB} + N^A N^B) \left( \bar{\mathcal{D}}_A L^C_B - L_B^{iA} k_A^C \right) + N N^C \bar{\mathcal{D}}_A L^A_i \right] \\
&+ 2 N^4 \bar{\mathcal{D}}_A N^B + 2 N^A N^C \frac{N}{N} L_B^A D_A N^B + \frac{N A N B N C}{N} - \bar{\mathcal{D}}_A L^i B \right] \\
&+ \frac{\Lambda \bar{\mathcal{D}}_A \Phi_i \left[ 2 (h^{AB} N^2 + N^A N^B) \left( \frac{L^i C_B + L^i B N C}{N} \right) \right],} \\
\lambda_6 &= -\Lambda \bar{\mathcal{D}}_A \Phi_i \left[ (h^{AB} N^2 + N^A N^B) \bar{\mathcal{D}}_A L^i B \right] + 2 N^A L_B^i D_A N^B \\
&+ \frac{\Lambda \bar{\mathcal{D}}_A \Phi_i \left[ -2 (h^{AB} N^2 + N^A N^B) \frac{L^i B}{N} \right],} \\
\end{align}

and
\begin{align}
\mathcal{D} &= \int_\Sigma \left\{ \Lambda \Phi_i \left[ 2 N L_A^i (P \cdot \mathcal{D}^A X) - 2 N (N^B L^A i + N L A^B i) (P \cdot \mathcal{D}_A \mathcal{D}_B X) \\
&- [N^2 (L_A^i L^A L^j AB - L_B L^j K A) - N N^A (L_A^i L^j A) - L_B L_B A)] (P \cdot n_j) \\
&+ \sqrt{h} \left[ \frac{N}{2} L^i L^j L^j + \frac{N}{2} L^i L^i C D L_{j C D} - N L^j L^j A L^j A - N L^i A L^i A L^i D \right. \\
&- N L^i C L^j B L^j A + 4 N L_j A L^j B L^j A + 2 N^C L^j A L^i A C - 2 N L^j L^j A L^j A \left] \right] \\
&+ \Lambda \bar{\mathcal{D}}_A \Phi_i \left[ N^A (P \cdot n_i) - (P \cdot n^i) (N^A D_{C N} + N D^A N) - \bar{\mathcal{D}}_B (P \cdot n^i) (h^{AB} N^2 + N^A N^B) \\
&+ \sqrt{h} N \left( (L^i AB - h^{AB} k) L^B + \bar{\mathcal{D}}_B L^i AB \right) + 2 \sqrt{h} N^i B \bar{\mathcal{D}}_B L^i A - \sqrt{h} N C \bar{\mathcal{D}}_A L^i C \right. \\
&+ \frac{\Lambda \bar{\mathcal{D}}_A \bar{\mathcal{D}}_B \Phi_i \left[ 2 \sqrt{h} L^i (B N A) \right],} \\
\end{align}
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\begin{equation}
\mathcal{E} = \int_{\Sigma} \sqrt{h} \left\{ \frac{2}{N} A^{[AB]} \left( L_A^{[i} L_B^{j]} - L_A L_B \right) L^i_j \Phi_i - 2 \Lambda C L_C^{[i} L_A^{j]} L_A^{i} \Phi_i \\
+ 2h^{[AB]} h^{[CD]} h^{BC} L^D \tilde{D}_A \Phi_i + \Lambda D \left( L_C^{[i} L^{BC]} - k_{DC} L^{CA} \right) \tilde{D}_A \Phi_i \\
+ 2 \Lambda D L_A^{[i} h^{BC} L_B^{j]} L^i_j \Phi_i + 2h^{[AB]} h^{[CD]} \Lambda^B L_D^{[i} \tilde{D}_A \Phi_i \right\}, \tag{C.19}
\end{equation}

\begin{equation}
\mathcal{F} = \int_{\Sigma} \left\{ \left[ -h^{AB} \left( p \cdot \partial_A X \right) + N^A L_A^{i} h^{CB} \left( p \cdot n_i \right) + \sqrt{h} L_A^{[i} L^{AB]} \right] \times \right.
\left. \delta_{ij} \left( \Phi^i \tilde{D}_B \Phi^j - \Phi^{i} \tilde{D}_B \Phi^j \right) \right\} - 2\sqrt{h} L_A^{[i} \left( h^{AB} L_{j]} - L_{j]}^{AB} \right) \left( \Phi^i \tilde{D}_B \Phi^j - \Phi^{i} \tilde{D}_B \Phi^j \right) \right\}. \tag{C.20}
\end{equation}

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