FIRST EXCITED STATE WITH MODERATE RANK DISTRIBUTION

DANIEL C. MAYER

Abstract. Evidence is provided for the existence of infinite periodic sequences of Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq C_3^e \times C_3$, $e \geq 7$, and logarithmic order $lo(G) = 10 + e$. With respect to their maximal subgroups $H_1, \ldots, H_4$, they have moderate rank distribution $\varrho(G) = (\text{rank}_2(H_i/H'_i))_{1 \leq i \leq 4} \sim (2, 2, 3, 3)$ and represent the first excited state of their punctured transfer kernel types $\hat{\varrho}(G)$, which is characterized by a polarized component of the abelian quotient invariants $\alpha_1(G) = (H_i/H'_i)_{1 \leq i \leq 4}$ with $lo = 6 + e$ in contrast to the ground state with $lo = 4 + e$.

1. Introduction

This is the third (and last) of a series of three articles devoted to periodic sequences of Schur $\sigma$-groups $G$ with bicyclic commutator quotients $G/G' \simeq C_3^e \times C_3$ having one non-elementary component with logarithmic exponent $e \geq 2$. The periodicity appears in the shape of an infinite chain of immediate $p$-descendants of finite 3-groups with variable $e \geq e_0$ bigger than a starting value. The Schur $\sigma$-groups arise as leaves of finite twigs with constant structure emanating from the vertices of the chain.

In the first article [22] of the trilogy, periodicity of pairs of metabelian Schur $\sigma$-groups sets in with $e_0 = 3$, and the ground state of non-metabelian Schur $\sigma$-groups $G$ with moderate rank distribution $\varrho(G) \in \{(2, 2, 2, 3), (2, 2, 3, 3)\}$ begins to become periodic for $e_0 = 5$. The typical bifurcation between $G$ and its metabelianization $M = G/G''$ degenerates to a simple $p$-descendant relation, already for $e \geq 4$, i.e., the siblings topology becomes a child topology [19].

The primary motivation for the investigations in the present article was the question what will happen with the more complicated fork topology between $G$ and $M = G/G''$ for the first excited state of non-metabelian Schur $\sigma$-groups $G$ with moderate rank distribution $\varrho(G) \in \{(2, 2, 2, 3), (2, 2, 3, 3)\}$, which was conjectured to become periodic for $e_0 = 7$ in the conclusion of [22 § 12]. We shall see that $e_0 = 7$ is confirmed, and the bifurcation degenerates to an iterated $p$-descendant relation, already for $e \geq 6$.

The most difficult situation of non-metabelian Schur $\sigma$-groups $G$ with elevated rank distribution $\varrho(G) = (3, 3, 3, 3)$ was completely clarified in [23]. Periodicity sets in with $e_0 = 9$ and the extremely complicated fork topology freezes to a common bifurcation of infinite order for all values $e \geq 4$. An additional complication with decisive negative impact on experimental arithmetical realizations by 3-class tower groups $G \simeq \text{Gal}(F_3^\infty(K)/K)$ of imaginary quadratic number fields $K$ is the requirement of logarithmic abelian quotient invariants $\alpha_2(G)$ of second order, whereas in the present article and in [22] we have the immense benefit that invariants $\alpha_1(G)$ of first order suffice.

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2. Main Theorems: periodic Schur $\sigma$-groups

In order to emphasize logical independence, we partition our main result into existence, uniqueness, explicit construction in virtue of periodicity, and structural invariants.

**Theorem 1. (Existence Theorem.)** For each logarithmic exponent $e \geq 2$, and for each of three punctured transfer kernel types (pTKT),

(1) \[ D.5, \varphi(G) \sim (112; 3), \quad C.A, \varphi(G) \sim (113; 3), \quad D.10, \varphi(G) \sim (114; 3), \]

there exists a unique pair of Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq C_{3^e} \times C_3$ and logarithmic order $\log(G) = 10 + e$.

Observe that existence is warranted also in the pre-periodic range $2 \leq e \leq 6$.

**Theorem 2. (Periodicity Theorem.)** For each logarithmic exponent $e \geq 7$, the unique pair of Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq C_{3^e} \times C_3$ and logarithmic order $\log(G) = 10 + e$ is given explicitly by the periodic sequence of descendants (notation according to $[\ref{1}, \ref{2}]$)

(2) \[ G \simeq W_{\ell}[-#1;1]^e - #1;i - #1;1 - #1;1, \quad i \in \{2,3\} \]

with periodic root $W_{\ell} = \text{SmallGroup}(6561,93) - #2;1 - #2;1 - #2;\ell$, where

- $\ell = 2$ for type $D.10, \varphi(G) \sim (114;3)$,
- $\ell = 4$ for type $C.4, \varphi(G) \sim (113;3)$,
- $\ell = 5$ for type $D.5, \varphi(G) \sim (112;3)$.

The metabelianization of $G$ is given by

(3) \[ M = G/G'' \simeq W_{\ell}[-#1;1]^e - #1;i, \quad i \in \{2,3\}, \]

of log order $\log(G) = 8 + e$, in fact, $G''$ is cyclic of order 9 and is contained in the center of $G$.

**Theorem 3. (Structure Theorem.)** For each logarithmic exponent $e \geq 4$, the unique pair of Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq C_{3^e} \times C_3$ and logarithmic order $\log(G) = 10 + e$ possesses logarithmic abelian invariant of first order

(4) \[ \alpha_1(G) = [(e + 1)1, (e + 1)1, e33; e11] \text{ for type } D.10, \text{ and } \]

\[ \alpha_1(G) = [(e + 1)1, (e + 1)1, (e + 1)32; e11] \text{ for type } C.4 \text{ and } D.5. \]

thus, in both cases, moderate rank distribution $\varrho(G) \sim (2,2,3;3)$, and soluble length $\text{sl}(G) = 3$.

Observe that the periodic invariants partially also occur in the pre-periodic range $4 \leq e \leq 6$, but in different form for $2 \leq e \leq 3$. The proof will be developed in $\S 5$, illustrated by Figure II.

3. Pre-periodic Schur $\sigma$-groups

Since periodicity in Theorem II sets in with $e_0 = 7$, we must provide supplementary results for the pre-periodic range $2 \leq e \leq 6$. They are also justified in $\S 5$, and illustrated by Figure 11.

**Theorem 4.** The unique pair of Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq C_9 \times C_3$ and logarithmic order $\log(G) = 12$ has first abelian quotient invariants $\alpha_1(G) \sim (31,31,43;211)$ and is given by $G \simeq \text{SmallGroup}(2187,168) - #2;7 - #1;4 - #2;1$ with metabelianization $M \simeq \text{SmallGroup}(2187,168) - #1;7 - #1;4 - #1;i$, where $i \in \{2,9\}$ for type $D.5$, $i \in \{3,8\}$ for type $C.4$, and $i \in \{5,6\}$ for type $D.10$. Thus, $F = \text{SmallGroup}(2187,168)$ is fork between $M$ and $G$.

**Theorem 5.** The unique pair of Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq C_{27} \times C_3$ and logarithmic order $\log(G) = 13$ has first abelian quotient invariants $\alpha_1(G) \sim (41,41,432;311)$ and is given by $G \simeq \text{SmallGroup}(6561,98) - #2;1 - #1;1 - #2;1$ with metabelianization $M \simeq \text{SmallGroup}(6561,98) - #1;3 - #1;1 - #1;i$, where $i \in \{2,3\}$ for type $D.10$, $i \in \{5,9\}$ for type $C.4$, and $i \in \{6,8\}$ for type $D.5$. Thus, $F = \text{SmallGroup}(6561,98)$ is the fork between $M$ and $G$. 
The unique pair of Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq C_{243} \times C_3$ and logarithmic order $\log(G) = 15$ has first abelian quotient invariants $\alpha_1(G) \simeq (61, 61, 632; 511)$ or $\alpha_1(G) \simeq (61, 61, 533; 511)$ and is given by $G \simeq F - \#1; i - \#1; i$, in terms of the fork $F = \text{SmallGroup}(6561, 93) - \#2; 6$, where $i \in \{2, 3\}$ for type D.10, $i \in \{5, 9\}$ for type C.4, and $i \in \{6, 8\}$ for type D.5.

Theorem 8. The unique pair of Schur $\sigma$-groups $G$ with commutator quotient $G/G' \simeq C_{729} \times C_3$ and logarithmic order $\log(G) = 16$ has first abelian quotient invariants $\alpha_1(G) \simeq (71, 71, 732; 611)$ or $\alpha_1(G) \simeq (71, 71, 633; 611)$ and is given by $G \simeq M - \#1; i - \#1; i$ as an iterated p-derivative of the metabelianization $M \simeq \text{SmallGroup}(6561, 93) - \#2; 1 - \#2; 2 - \#2; 6$, where $i \in \{7, 8\}$ for type D.10, $i \in \{10, 14\}$ for type C.4, and $i \in \{11, 13\}$ for type D.5.

4. Arithmetical realizations

The automorphism group $\text{Gal}(F^\infty(K)/K)$ of the maximal unramified pro-3 extension $F^\infty(K)$ of an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{d})$ with negative fundamental discriminant $d < 0$ must be a Schur $\sigma$-group with restricted arithmetic properties.

With the aid of the computational algebra system Magma [6, 7, 14], we conducted a search for fundamental discriminants $-10^8 < d < 0$ such that the 3-class group $C_3(K) \simeq C_3 \times C_3$ is non-elementary bicyclic with $2 \leq e \leq 8$, and the capitulation in the four unramified cyclic cubic extensions of $K$ is the first excited state of one of the three types C.4, D.5, D.10 under investigation or of the closely related type D.6. The class field routines by Fieker [10] were employed.

In order to enable comparison with analogous cases of imaginary quadratic number field $K$ with elementary bicyclic 3-class group $C_3(K) \simeq C_3 \times C_3$, we begin with a recall of arithmetical information in [17] Fig. 1–2, pp. 24–25 concerning the first excited state of capitulation types in section E which are characterized by the polarization 43 of $\alpha_1(K)$. The rank distribution is either $g(K) \sim (2, 3, 2, 2)$ for the former two types or $g(K) = (2, 2, 2, 2)$ for the latter two types.

Example 1. For $e = 1$, the hits with absolutely minimal discriminants are

$\begin{align*}
\text{d} &= -262744 \text{ for type E.14}, \ x(K) \sim (3122), \ \alpha_1(K) \sim (43, 111, 21, 21), \\
\text{d} &= -258040 \text{ for type E.6}, \ x(K) \sim (1122), \ \alpha_1(K) \sim (43, 111, 21, 21), \\
\text{d} &= -297079 \text{ for type E.9}, \ x(K) \sim (2334), \ \alpha_1(K) \sim (21, 43, 21, 21), \\
\text{d} &= -370740 \text{ for type E.8}, \ x(K) \sim (2234), \ \alpha_1(K) \sim (21, 43, 21, 21).
\end{align*}$

Now we come to examples for non-elementary bicyclic 3-class groups $C_3(K) \simeq C_3 \times C_3$, $e \geq 2$.

For $e = 2$, the polarization of $\alpha_1(K)$ is irregular but uniform for all types. It is given by $(e+2)3!$ instead of $(e+1)3!$.

Example 2. For $e = 2$, the hits with absolutely minimal discriminants are

$\begin{align*}
\text{d} &= -210164 \text{ for type D.6}, \ x(K) \sim (123; 1), \ \alpha_1(K) \sim (31, 31, 31; 431), \\
\text{d} &= -320968 \text{ for type C.4}, \ x(K) \sim (113; 3), \ \alpha_1(K) \sim (31, 31, 431; 211), \\
\text{d} &= -354232 \text{ for type D.10}, \ x(K) \sim (114; 3), \ \alpha_1(K) \sim (31, 31, 431; 211), \\
\text{d} &= -776747 \text{ for type D.5}, \ x(K) \sim (112; 3), \ \alpha_1(K) \sim (31, 31, 431; 211).
\end{align*}$

For $e = 3$, the polarization of $\alpha_1(K)$ is uniform for all types. It is given by $(e+1)3!$.

Example 3. For $e = 3$, the hits with absolutely minimal discriminants are

$\begin{align*}
\text{d} &= -642491 \text{ for type D.5}, \ x(K) \sim (112; 3), \ \alpha_1(K) \sim (41, 41, 432; 311), \\
\text{d} &= -1021523 \text{ for type D.6}, \ x(K) \sim (123; 1), \ \alpha_1(K) \sim (41, 41, 432; 432), \\
\text{d} &= -1052072 \text{ for type C.4}, \ x(K) \sim (113; 3), \ \alpha_1(K) \sim (41, 41, 432; 311), \\
\text{d} &= -1265747 \text{ for type D.10}, \ x(K) \sim (114; 3), \ \alpha_1(K) \sim (41, 41, 432; 311).
\end{align*}$
For all $d \geq 4$, the polarization of $\alpha_d(K)$ depends on the type. For $C.4$ and $D.5$ it is $(e + 1)32$, whereas for $D.10$ and $D.6$ we have the variant $e33$.

**Example 4.** For $e = 4$, the hits with absolutely minimal discriminants are
\begin{align*}
d = -2249263 & \text{ for type } D.6, \quad \varkappa(K) \sim (123; 1), \quad \alpha_d(K) \sim (51, 51, 433), \\
d = -2959235 & \text{ for type } C.4, \quad \varkappa(K) \sim (113; 3), \quad \alpha_d(K) \sim (51, 51, 532; 411), \\
d = -4076823 & \text{ for type } D.10, \quad \varkappa(K) \sim (114; 3), \quad \alpha_d(K) \sim (51, 51, 433; 411), \\
d = -5231284 & \text{ for type } D.5, \quad \varkappa(K) \sim (112; 3), \quad \alpha_d(K) \sim (51, 51, 532; 411). \end{align*}

**Example 5.** For $e = 5$, the hits with absolutely minimal discriminants are
\begin{align*}
d = -5593787 & \text{ for type } D.10, \quad \varkappa(K) \sim (114; 3), \quad \alpha_d(K) \sim (61, 61, 533; 511), \\
d = -1880751 & \text{ for type } C.4, \quad \varkappa(K) \sim (113; 3), \quad \alpha_d(K) \sim (61, 61, 632; 511), \\
d = -18597255 & \text{ for type } D.5, \quad \varkappa(K) \sim (112; 3), \quad \alpha_d(K) \sim (61, 61, 632; 511), \\
d = -18731096 & \text{ for type } D.6, \quad \varkappa(K) \sim (123; 1), \quad \alpha_d(K) \sim (61, 61, 533). \end{align*}

**Example 6.** For $e = 6$, the hits with absolutely minimal discriminants are
\begin{align*}
d = -11591183 & \text{ for type } D.5, \quad \varkappa(K) \sim (112; 3), \quad \alpha_d(K) \sim (71, 71, 732; 611), \\
d = -17740111 & \text{ for type } C.4, \quad \varkappa(K) \sim (113; 3), \quad \alpha_d(K) \sim (71, 71, 732; 611), \\
d = -33942367 & \text{ for type } D.6, \quad \varkappa(K) \sim (123; 1), \quad \alpha_d(K) \sim (71, 71, 713; 633). \end{align*}

**Example 7.** For $e = 7$, the hits with absolutely minimal discriminants are
\begin{align*}
d = -11733415 & \text{ for type } D.10, \quad \varkappa(K) \sim (114; 3), \quad \alpha_d(K) \sim (81, 81, 733; 711), \\
d = -116407871 & \text{ for type } D.5, \quad \varkappa(K) \sim (112; 3), \quad \alpha_d(K) \sim (81, 81, 832; 711). \end{align*}

**Example 8.** For $e = 8$, the hits with absolutely minimal discriminants are
\begin{align*}
d = -98311919 & \text{ for type } D.5, \quad \varkappa(K) \sim (112; 3), \quad \alpha_d(K) \sim (91, 91, 932; 811). \end{align*}

**Theorem 9. (Three Stage Tower Theorem.)** Any imaginary quadratic number field $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with non-elementary bicyclic 3-class group $\text{Cl}_3(K) \simeq C_{3^e} \times C_3$, $e \geq 2$, and Artin pattern $\text{AP}(K) = (\varkappa(K), \alpha_d(K))$ given by Formulas (1) for $\varkappa(K)$ and (5) for $\alpha_d(K)$ has a finite 3-class field tower $F_3^\infty(K)$ with precisely three stages, $\ell_3(K) = 3$. For $e \geq 7$, the 3-class tower group $G = \text{Gal}(F_3^\infty(K)/K)$ is given by Formula (2), and the second 3-class group $M = G/G'' \simeq \text{Gal}(F_3^2(K)/K)$ by Formula (4), both in dependence on Formula (3).

**Proof.** According to the Galois correspondence of field theory and the Artin reciprocity law of class field theory [2], the maximal self-conjugate subgroups $H_1, \ldots, H_4$ of the automorphism group $G = \text{Gal}(F_3^\infty(K)/K)$ of the maximal unramified pro-3 extension of an algebraic number field $K$ with bicyclic 3-class group $\text{Cl}_3(K) \simeq C_{3^e} \times C_3$ correspond to the unramified cyclic cubic extensions $L_1, \ldots, L_3; L_4$ of $K$, and the abelian quotient invariants $\alpha_1(G) = [G/G'; (H_i/H_i')_{1 \leq i \leq 4}]$ coincide with the abelian type invariants of 3-class groups $\alpha_1(K) = [\text{Cl}_3(K); (\text{Cl}_3(L_i))_{1 \leq i \leq 4}]$. According to Artin’s theory of the transfer [3], the Schur transfer homomorphisms $T_i : G/G' \twoheadrightarrow H_i/H_i'$ correspond to the extension homomorphisms $\tau_i : \text{Cl}_3(K) \rightarrow \text{Cl}_3(L_i)$ of 3-ideal classes, and the punctured transfer kernel type $\varkappa(G) = (\ker(T_i))_{1 \leq i \leq 4}$ coincides with the punctured capitulation type $\varkappa(K) = (\ker(\tau_i))_{1 \leq i \leq 4}$. This was discussed in more detail in [15] and is the foundation of the strategy of pattern recognition via Artin transfers [21], which is due to the coincidence of Artin patterns $\text{AP}(G) = (\varkappa(G), \alpha_1(G)) = (\varkappa(K), \alpha_1(K)) = \text{AP}(K)$. Finally, the soluble length $s(G)$ is equal to the length $\ell_3(K)$ of the 3-class field tower of $K$. For an imaginary quadratic field $K$, the 3-class tower group $G$ must be a Schur $\sigma$-group [20] [13].

5. **Proof and tree diagram**

The proof of the preperiodic Theorems [4 – 8] and finally of the periodic Main Theorems [11 – 3] can be developed in accordance with the tree diagram in Figure [4]. All directed edges of this tree lead from descendants $D$ to $p$-parents $\pi_p(D) = D/P_{p-1}(D)$, $c_p = \text{cl}_p(D)$, rather than to parents $\pi(D) = D/\gamma_c(D)$, $c = \text{cl}(D)$. Consequently, the figure admits actual descendant construction.
The $p$-group generation algorithm [12] by Newman [24] and O'Brien [25] is implemented in the ANUPQ package [11] of the computational algebra system Magma [14, 21, 6]. This algorithm is used to construct all immediate $p$-descendants of an assigned finite $p$-group. Repeated recursive applications of the algorithm, guided by the strategy of pattern recognition via Artin transfers [24], eventually produce Figure 1. In each step, only $\sigma$-descendants are allowed to pass the filter. The figure shows an infinite main trunk and finite twigs emanating from the vertices of the trunk. Propagation along the trunk is exo-genetic with increasing commutator quotient $(3^e, 3)$, whereas all vertices of a twig share a common abelianization and the propagation is endo-genetic. The leftmost twig is included in order to point out analogy to the elementary commutator quotient $(3, 3)$. It was computed in [20, Fig. 6, p. 110], where historical information is provided for type E.9, $\kappa \sim (2231) \sim (3231)$, in [20] § 4, pp. 107–111. 

**Figure 1.** Schur $\sigma$-groups $G$ with $\rho(G) \sim (2, 2, 3; 3)$, $G/G' \simeq (3^e, 3)$, $2 \leq e \leq 9$
All the other twigs concern non-elementary commutator quotients \((3^e, 3), 2 \leq e \leq 9\), restricted to the particular punctured transfer kernel type D.10, \(\varpi \sim (114; 3)\). Figure 11 remains unchanged for the other two types C.4 and D.5 when the parameter \(i\) is selected according to the preperiodic Theorems 1 – 5 for \(2 \leq e \leq 6\), and the periodic root \(W_\ell\) if \(e \geq 7\) is replaced according to Formula 9 in the periodic Main Theorem 2. More changes are required for type D.6, \(\varpi \sim (123; 1)\), which is only included in the number theoretic §1 but not in the group theoretic §§2 and 4. For instance, the coclass tree with root \((729, 13)\) and bifurcation at \((2187, 168)\) [22] Fig. 5], which is responsible for three types D.10, C.4 and D.5, must be replaced by the coclass tree with root \((729, 21)\) and bifurcation at \((2187, 191)\) [22 Fig. 6] or with root \((729, 18)\) and bifurcation at \((2187, 181)\). Both of the latter coclass trees give rise to the single relevant type D.6. The initialization of the construction process at \(e = 2\) is described in [23 § 5], but now the scaffold type b.31, \(\varpi \sim (044; 4)\), must be replaced by type d.10, \(\varpi \sim (110; 3)\), associated with types D.10, C.4 and D.5. The search immediately leads to the root \((729, 13)\) and the fork \((2187, 168)\). Capable descendants of both step sizes \(s \in \{1, 2\}\) have relative identifiers \(\{1, 4, 7\}\), but only 7 is relevant for the desired types.

In the leftmost three twigs with \(1 \leq e \leq 3\), the fork topologies, are isomorphic as directed graphs. However, in the next two twigs with \(4 \leq e \leq 5\), the fork topologies shrink gradually, and for all \(e \geq 6\), the bifurcation vanishes leaving a descendant scaffold type. The vertices on the bifurcations are BCF-groups [23].

The vertices of the main trunk are CF-groups [2] with type a.1, \(\varpi \sim (000; 0)\), and \(\varpi \sim (2, 2, 3, 3)\).

For \(2 \leq e \leq 5\), the first vertex of the twig has scaffold type d.10, \(\varpi \sim (110; 3)\), but for \(e \geq 6\), the twigs entirely consist of vertices sharing a common type, D.10, C.4 or D.5.

Up to \(e \leq 6\), the construction process is straightforward, but for \(e = 7\) considerable difficulties arise, because it is hard to determine the next vertex on the main trunk.

Since an attempt with hypothetical next main trunk root \((6561, 93) - \#2; 1 - \#2; 1 - \#2; 1\) and 1709 descendants \(D\) up to \(p\)-class \(\text{cl}_p(D) \leq 11\) only led to three pairs of Schur \(\sigma\)-groups with commutator quotient \((2187, 3)\) and \(\text{lo} = 20\) in the second excited state \(\alpha_1 \sim (81, 74; 711)\) for type D.10, respectively \(\alpha_1 \sim (81, 843; 711)\) for type C.4 and D.5, we returned to a tour de force computation starting at the previous vertex \((6561, 93) - #2; 1 - #2; 1\) on the main trunk.

**Lemma 1.** Among the 1708 descendants \(D\) up to \(p\)-class \(\text{cl}_p(D) \leq 10\) of the main trunk vertex \((6561, 93) - \#2; 1 - \#2; 1\), there occur the following Schur \(\sigma\)-groups \(S\):

1. the expected three pairs with \(S/S' \simeq (729, 3)\) and \(\text{lo}(S) = 16\) in the first excited state \(\alpha_1 \sim (71, 71, 633; 611)\) for type D.10, respectively \(\alpha_1 \sim (71, 71, 732; 611)\) for type C.4, D.5,
2. the desired three pairs with \(S/S' \simeq (2187, 3)\) and \(\text{lo}(S) = 17\) in the first excited state \(\alpha_1 \sim (81, 81, 733; 711)\) for type D.10, respectively \(\alpha_1 \sim (81, 81, 832; 711)\) for type C.4, D.5,
3. three unexpected pairs with \(S/S' \simeq (729, 3)\) and \(\text{lo}(S) = 19\) in the second excited state \(\alpha_1 \sim (71, 71, 644; 611)\) for type D.10, respectively \(\alpha_1 \sim (71, 71, 743; 611)\) for type C.4, D.5,
4. and (as a superfluous byproduct) a quartet of type B.2, \(\varpi(S) \sim (111; 2)\), with \(S/S' \simeq (729, 3)\) and \(\text{lo}(S) = 19\) in the first excited state \(\alpha_1 \sim (71, 71, 732; 611)\).

A backtrack search, starting from item (2) in Lemma 1 finally reveals three suitable periodic roots \(W_\ell\) given in Formula 9 of Theorem 2.

6. Conclusion

In the present article, our hypothesis in [22 § 12] that periodicity of Schur \(\sigma\)-groups \(G\) with \(G/G' \simeq C_3 \times C_3\) and one of the punctured transfer kernel types D.10, C.4 and D.5 in the first excited state will set in with \(e \geq e_0 = 7\) was verified.

The question concerning the evolution of the fork topology between \(G\) and its metabelianization \(M = G/G''\), for which we were really unable to make any prediction, was answered by a gradual shrinking of the twigs in Figure 11 for \(4 \leq e \leq 7\), coming along with a last non-metabelian (second) bifurcation for \(e = 4\) and a last semi-metabelian (first) bifurcation for \(e = 5\). The fork topology completely degenerates to a descendant topology for all \(e \geq 6\) and periodicity of the shortest possible twigs sets in with \(e \geq e_0 = 7\).
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NAGLERGASSE 53, 8010 Graz, Austria

Email address: algebraic.number.theory@algebra.at

URL: http://www.algebra.at