Coevolution of a network and perception

Hang-Hyun Jo*  Eunyoung Moon†

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Abstract

How does individuals’ cognition change a system which is a collective behavior of individuals? Or how does a system affect individuals’ cognition? To examine the interplay between a system and individuals, we study a cognition-based strategic link formation. When a network is not fully observable, individuals’ perception of a network plays an important role in decision making. Assuming that a communication link is costly and more accurate perception yields higher network utility, one decides whether to form a link in order to get better knowledge. A newly added link is a change in a network, which affects individuals’ perception accuracy back. We characterize the early stage of dynamics that a ring network is a global structure and there exists an agent who keeps the full information once a network is connected. Moreover, we discuss a local linking process which causes clusters and the influence of a positive cost of linking in the coevolution between a network and perception.

Keywords: perceptual attachment; coevolution; perception updating; local connections.

JEL Classification Numbers: A12, D83, D85.

*BECS, Aalto University School of Science, P.O. Box 12200, Finland and BK21plus Physics Division and Department of Physics, Pohang University of Science and Technology, Pohang 790-784, Republic of Korea, jo-hanghyun@postech.ac.kr

†Corresponding author. Address: University of Liverpool Management School, Chatham Street, Liverpool L69 7ZH, UK. Phone: +44 0151 795 3530. Email: E.Moon@liverpool.ac.uk
1 Introduction

In the 1960’s, Stanley Milgram conducted a notable experiment to examine the average path length in a social network. In his experiment, a randomly selected person was asked to pass a packet to either a target or someone in his/her acquaintances who might be most likely to know the target. The result of this experiment, widely known as *six-degrees of separation*, shows how close people are in a social network. After his experiment, studies of distance in a network have an important and long-standing research tradition in the network theory. From empirical studies (Sampson, 1998; Hedström, Sandell, and Stern, 2000; Newman, 2001; Kretschmer, 2004; Kossinets and Watts, 2006) to a theoretic approach (Watts and Strogatz, 1998; Watts, 1999), a wide range of research has delved into distance in a network. However, research has missed another aspect of this experiment - cognition of a social network: If a participant A forwarded the packet to his/her friend B, why he/she chose B, why not another friend C? Following the experiment instruction, it is simply because A thought that B is closer to the target than C. In other words, in A’s perceived network, B is the node which has the shortest path length to the target. In general, perception of a social network is more influential in individuals’ decision making than the real network when people do not have the full information on the network structure.

In this research, we focus on an ecology of how a social network as a system of collective behaviors evolves with individuals’ cognition, questioning why people make a connection, how individuals’ perception converges to the reality, and what the stable network structure will be. Precisely, we consider a cognition-based strategic network formation model: In a reasonably large size of group\(^1\), people may not fully observe the entire relationships in a social network, instead, they have perception of the network. For describing each individuals’ perceived network, we follow the concept of “the cognitive social structure (CSS)” defined by Krackhardt (1987\(^2\)) as the relation among a sender, a receiver, and a perceiver. We also introduce the notion of perception accuracy, which captures the number of correctly perceived links among all possible relationships. Because

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\(^1\)As Hill and Dunbar (2003) empirically showed that human brain can cope with a limited number of social relations (approximately maximum 150 on average), unlike other growing network models (Barabasi and Albert, 1999; Vázquez, 2003), we fix the number of nodes and examine how these nodes create links to each other. In the simulation, we set 200 nodes which is natural to assume that everyone knows the existence of others in the group regardless of communication links.

\(^2\)He attempted to aggregate individuals’ cognition in order to derive a representative CSS. In our model, we keep individual agents’ cognition in \(n \times n \times n\) matrices for \(n\) nodes.
one’s perception does not necessarily coincide with the objective reality, how much one can utilize a network depends on the accuracy of one’s perception of a network.

Given a network utility as an increasing function of the perception accuracy, people infer others’ accuracy by observing their utility, thus we can consider the network utility as a proxy measure of the accuracy of perception. People improve their perception by communication with a more accurately perceiving agent. With a positive cost of linking, a link can be created only if the advantage of having more accurate perception outnumbers the cost of linking. Whenever a link is added to a current network, the newly added link affects individuals’ accuracy, hence either the advantage of high accuracy or the disadvantage of low accuracy is temporary as long as the network continues evolving. Starting with an empty network in reality and an Erdős-Rényi random network with probability $p$ in cognition, the network dynamics ends when no one would like to add a link.

Our model provides a theoretic framework for a cognition-based evolving network. Admittedly, the analytic results have a limit for describing the ultimate picture of the coevolution between cognition and a network due to the randomness of neighbor choice in part. Nonetheless, we precisely derive the early stage of an evolving network in the analysis and show a broad picture in the numerical simulation. We briefly introduce key findings as follows.

Firstly, a network forms a ring as a global structure in the early stage. Given the initial perception, the least accurately perceiving agent, labeled as 2, creates a link to the most accurate agent, labeled as 1, for perception updating. The original perception of 2 has been replaced with 1’s perception which is initially most accurate, and correct information of 2’s own link status is added, thus 2 becomes the most accurate agent after the update. Then, another least accurately perceiving agent, labeled as 3, creates a link to 2 and replaces own perception with the mixture of 2’s most accurate perception and correct information of 3’s own, and so on. Hence the early stage dynamics forms a line structure and ends up a ring by a link suggestion of agent 1 who initially

Network utility in this model is distinguishable from other strategic network formation models in which utility comes from a link itself. The utility here is oriented from the information about links, capturing that one uses a social network better if one knows it better.

Costly linking is commonly accepted in strategic network formation models (Jackson and Wolinsky, 1996; Bala and Goyal, 2000; Watts, 2001; Bloch and Jackson, 2007; Galeotti et al. 2010). In psychology, a cost in a network is often interpreted as a capacity of the social brain related to the size of a social network, i.e. “the social brain hypothesis” (Rose and Serafica, 1986; Stiller and Dunbar, 2007; Roberts and Dunbar, 2011; Sutcliffe et al. 2012), however, in this model, the cost, specifically, the cost of linking is confined to an effort to initiate social relations. Although the cost in this research does not include a cost of maintaining, it is a possible extension of the model that the number of connections has an upper bound.
had the highest accuracy to agent $n$ who is most recently connected so that $n$ holds the up-to-date information. Note that we use the word “global structure” because the ring structure is maintained even if a network continues evolving.

Secondly, when a chance for perception updates has given to everyone in the early stage, there exists an agent who possesses the full information. Since everyone knows his/her own link status correctly, the correct information is added on the updated perception. Thus the partly correct information everyone has is filed up and the full information is completed when a network is connected. This information is spreading with a small fluctuation of the accuracy by newly added links. That is, there are always at least two agents who hold the full information and the gap between the most and the least accurate perception is decreasing because the transmission of almost perfect information makes the least accurate perception almost perfect.

Thirdly, once the global ring structure has been completed, additional connections occur locally. In a ring network, the most and the least accurate agents are always neighboring, which implies that perception update can occur by already existing links. Thus if a new link is added on the ring structure, it must be the case that there are multiple neighboring agents with the lowest accuracy so that at least one of the least accurate agents does not have a link to the most accurate agent who is located closely. Especially the distance between one of the most accurate agent and one of the least accurate agent in a ring network is 2. Thus newly added links on a ring structure are a local connection rather than a shortcut.

This result suggests a plausible explanation of clusters (i.e. triadic relation) in a social network. So far, local linking and clusters have been explained that people may choose new acquaintances who are friends of friends (Granovetter, 1973; Vázquez, 2003; Jackson and Rogers, 2007). This model provides a different aspect of clustering in which clusters can appear without a mediation of already existing relations in cognition-based linking.

Lastly, we derive a condition for evolving in terms of a cost of linking. The cost of linking plays a significant role in evolving process because individual’s incentive to create a new link depends on whether the benefit of better information outnumbers the cost. If the cost is too high for the least

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5In a ring network, Watts and Strogatz (1998) introduces a shortcut link by a random rewiring process. A randomly reallocated link is called a shortcut because only a few randomly rewired links in a ring network significantly reduce the average path length. However, when the link formation is not random, the dramatic change in the average path length is not always true, rather, gradual decreases of the average path length is observed in the strategic linking process.
accurate agent to initiate a link, no changes in the network structure occur. Although the network reaches in a steady state, information transmission can be continued through existing links, thus the lowest accuracy converges to the highest accuracy which implies that it becomes even harder to create additional links. Oppositely, if the cost is sufficiently low for the least accurate agent to create a new link, other agents’ accuracies decrease because others do not know the information about the newly added link. It may trigger another new links and the new links affect accuracies again, and so forth. Indeed, we find many discontinuous jumps in terms of stable link density according to the cost of linking, which reveals nontrivial coevolutionary dynamics between perception and a network structure.

Our research contributes to a cognition-based link formation, which has not been explored yet. Though, there are plenty of studies on network formation models based on either randomness or a strategic choice. Especially, network dynamics has provided understanding of important aspects of evolving structure, starting from the seminal papers about mechanisms behind the small-world phenomena (Watts and Strogatz, 1998; Barabasi and Albert, 1999). On the other hand, a game theoretic approach emphasizes the importance of decision making in network formation (Jackson and Wolinsky, 1996; Bala and Goyal, 2000; Watts, 2001; Galeotti, Goyal, and Kamphorst, 2006).

Unlike network formation models, cognitive aspects in a network has not been discovered deeply. The relationship between size/layers of networks and human brain capacity has been relatively active (Dunbar, 1998; 2009; Stiller and Dunbar, 2007; Sutcliffe, Dunbar, Binder, and Arrow, 2012). There are a few works more directly related to this research: perception of a network is formalized (Krackhardt, 1987), Krackhardt and Kilduff (1999) discuss how people perceive social networks with regard to the perception of balance models, and Dessi, Gallo, and Goyal (2012) experimentally show a cognition bias in which perceived network structures are flatter than the reality.

This paper is organized as follows. Section 2 presents the model and discusses the analytic results and Section 3 contains simulation results. Concluding remarks are offered in Section 4.

2 The model

We consider a two-way flow communication network and individuals’ perception of it. Information transmission occurs between connected two people and each individual has own perception
on who connects with whom. One’s perception does not necessarily coincide with the actual network and it is natural to assume that the more accurately perceived utilize a network better for a certain purpose. Although individuals’ perception details are not observed unless directly shared by a communication link, how well an agent utilizes a network reveals his/her overall accuracy of the perception. Thus, we assume that perception itself is private information but the accuracy of perception is indirectly observable.

Once individuals observe others’ accuracies, less accurate agents may want to update their perception with the most accurate one in order to improve the network utility. For those who have a link to the most accurate agent, there is no additional cost for the perception update. Whereas, those who are not connected to the most accurate agent need to create a new link which is costly. Thus the least accurate agent is willing to link to the most accurate agent only if the additional utility by updating perception is greater than the cost of linking.

We are interested in how the network structure and individuals’ perception evolve together. We firstly analyze individuals’ perception update process and an evolving network structure. We shall then see a numerical simulation.

2.1 Settings

Networks. Let $N = \{1, \cdots, n\}$ be the set of individuals. In order to exclude perception updating without communication\footnote{For instance, when $N = \{1, 2, 3\}$ and $e_{12} = e_{23} = 1$ and $e_{13} = 0$, it is possible to reach the full information without communication in a few time steps.} we assume that $n$ is a sufficiently large number. At the same time, when it comes to the cognitive problem, $n$ cannot be a huge number in order for everyone in a network to recognize the existence of $n - 1$ others regardless of the link status. We open $n$ as an arbitrary number, however it is fixed at 200 for a simulation in the next section.

For any pair of $j, k \in N$, $e_{jk,t}$ represents the relation between $j$ and $k$ in time period $t$. When $e_{jk,t} = 1$, it means that $j$ and $k$ are connected, while $e_{jk,t} = 0$ refers to the case of no connection. Since a relation is two-way communication, $e_{ij} = e_{ji}$. By convention, $e_{ii} = 0$ for all $i \in N$. A network $G_t$ is a collection of link status at $t$, i.e., $G_t = \{e_{jk,t}\}_{j,k \in N}, t = 0, 1, \cdots, T$. For analytic simplicity, we confine the link formation to a sequentially link added time-evolving network $G_t$,
however, we relax it in the numerical simulation\footnote{Results in the next section are not meaningfully different with/without the assumption of a single link creation in one time period $t \leq n$, however, it makes the analysis clear to see the intuition. Once a network is connected (i.e. there exists a path between any two nodes in a network), we relax this assumption for the further analysis about links added on the global structure. Precisely, we allow maximum accuracy ties in time period $t > n$ in the analysis and do not restrict any minimum/maximum ties in the simulation.} Denoting $G + ij$ as a network $G$ with a new link between $i$ and $j$,

\[
G_{t+1} = \begin{cases} 
G_t + ij & \text{if } e_{ij,t+1} = 1 \text{ for } i,j \text{ such that } e_{ij,t} = 0 \\
G_t & \text{otherwise.}
\end{cases}
\]

Let $d^t_i$ be the degree of $i$ in $G_t$, i.e. $d^t_i = \sum_{j \in N \setminus \{i\}} e_{ij,t}$.

**Perception and accuracy.** In the same way of defining a network $G_t$, we define one’s perception on the network such that $G^t_i = \{e^t_{jk}\}$ for $i \in N$. The network $G^t_i$ refers how $i$ perceives the actual network $G_t$: $e^t_{jk} = 1$ implies that there is a link between $j$ and $k$ in $i$’s perceived network, while $e^t_{jk} = 0$ implies that $i$ thinks no link between $j$ and $k$.

Now we measure how accurate one’s perception is.

**Definition 1** Perception accuracy is defined as

\[
\rho^t_i = \frac{1}{L} \sum_{j \in N} \sum_{k \in N, k > j} I(e_{jk,t}, e^t_{jk}), \quad \frac{2}{n} \leq \rho^t_i \leq 1,
\]

where $L = \frac{n(n-1)}{2}$ denotes the maximal number of links and $I(x, y)$ is an index function, having a value of 1 if $x = y$, otherwise 0.

The accuracy $\rho^t_i$ captures $i$’s correct information out of all possibilities of pairs among $n - 1$ others. Note that $i$ has correct information on at least $n - 1$ link status related to $i$ itself (i.e. $e^t_{ij} = e_{ij}$ for all $j \neq i$) so that the lower bound of the accuracy becomes $\frac{2}{n}$.

**Payoffs and perception updates.** We define a utility of $G_t$ as a function of individuals’ accuracy, $u(\rho^t_i)$. To reflect the intuition that the more accurate information the better use of a network, assume that $\frac{du}{d\rho} > 0$. Since one can improve the perception accuracy by communication with the most accurate agent, the network utility can also be improved if one replaces own perception with the most accurate perception. The least accurate agent at each time period is assumed to
update because of the strongest need for improving perception accuracy. For the further analysis, let \((\rho_t^{(1)}, \ldots, \rho_t^{(n)})\) be the accuracy profile at \(t\) in descending order. If \(\rho_t^i = \rho_t^{(n)}\), \(i\) has a chance to replace its perception with \(l\)'s perception such that \(\rho_l^i = \rho_t^{(1)}\). If \(i\) already has a link to \(l\) \((e_{il,t} = 1)\), communication for the perception update is not costly. However, if \(e_{il,t} = 0\), \(i\) needs to create a link to \(l\), which is costly. We set an arbitrary non-negative cost of linking \(c \geq 0\).

In general, \(i\)'s accuracy and the expected payoff at \(t + 1\) are

\[
\rho_{t+1}^i = \begin{cases} 
\frac{1}{t} \sum_{j \in N} \sum_{k \in N, k > j} I(e_{jk,t+1}, e_{jk,t}^i) & \text{if } \rho_t^i \neq \rho_t^{(n)} \\
\frac{1}{t} \left[ n - 1 + \sum_{j \in N \setminus \{i\}} \sum_{k \in N \setminus \{i\}, k > j} I(e_{jk,t+1}, e_{jk,t}^i) \right] & \text{if } \rho_t^i = \rho_t^{(n)}
\end{cases}
\]

(1)

\[
\Pi_{t+1}^i = \begin{cases} 
E[u(\rho_{t+1}^i)] - c & \text{if } e_{il,t} = 0 \text{ and } e_{il,t+1} = 1 \\
E[u(\rho_{t+1}^i)] & \text{if } e_{il,t} = e_{il,t+1} = 1 \\
u(\rho_{t+1}^i) & \text{if } e_{il,t} = e_{il,t+1} = 0.
\end{cases}
\]

(2)

implies that a new link will be added if and only if

\[
E[u(\rho_{t+1}^i)] - c > u(\rho_t^i). \quad (3)
\]

For a linear utility \(u(\rho) = \rho\) as a simple example, \(E[u(\rho_{t+1}^i)] = \frac{2}{n} + (1 - \frac{2}{n})\rho_t^{(1)}\) if \(i\) creates a link to the most accurate agent, thus, (3) can be expressed as following:

\[
\frac{2}{n} + \left(1 - \frac{2}{n}\right)\rho_t^{(1)} > \rho_t^i + c \quad \Leftrightarrow \quad \Delta \rho_t + \frac{2}{n} \left(1 - \rho_t^{(1)}\right) > c, \quad (4)
\]

where \(\Delta \rho_t \equiv \rho_t^{(1)} - \rho_t^{(n)}\).

Note that the accuracy can be either improved or declined because of newly added links. Consider three agents \(i, j, k\) such that \(\rho_t^i = \rho_t^{(n)} < \rho_t^j < \rho_t^k = \rho_t^{(1)}\) and \(e_{jk,t} = 0\). If \(i\) originally has a correct perception on \(j\) and \(k\) that there is no link between them \((e_{jk,t}^i = e_{jk,t} = 0)\) but a link between them is newly formed in \(t\), \textit{ceteris paribus}, \(i\)'s accuracy at the beginning of \(t + 1\) is

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8It is obvious that \(i\) chooses the most accurate agent for a new link due to the best use of costly linking, thus we ignore the case that \(l\) is not the most accurate agent.

9Note that the cost of linking is an effort to initiate a relation. It implies two: the cost is imposed only on the one who suggests a link, not on the one who gets a link offer, and extinction of existing links does not occur because the cost of maintenance is out of the scope in this model.
\[
\rho_{t+1}^i = \rho_t^i - \frac{1}{L}. \quad \text{Oppositely, if the original perception was wrong} \quad (e_{j,k,t}^i = 1), \quad \text{the accuracy is improved} \quad (\rho_{t+1}^i = \rho_t^i + \frac{1}{L}) \quad \text{by an accidental correction.}
\]

**Decision flow.** In the beginning of each time period \( t \), the network \( G_t \) and perception \( G_t^i, \forall i \) are given. Individuals firstly observe the accuracy profile \((\rho_t^{(1)}, \cdots, \rho_t^{(n)})\). The second step is the perception update decision of the least accurate agent, if necessary, the least accurate agent decides whether to create a link. Once the link decision making of the least accurate agent has been done, the third step is the perception update of all individuals according to [1]. In the beginning of \( t+1 \), the network \( G_{t+1} \) which includes a newly formed link in \( t \) and perception \( G_{t+1}^i \) which reflects the change in \( t \) are given. The whole three steps are repeated until no one would like to update perception. Note that structural stability does not include cognitive stability. That is, if there is a link between the least and the most accurate agents, perception can be updated in a steady state.

### 2.2 Analysis

In this section, we show how individuals make a decision to connect, how perception affects a link formation, and how the entire network structure evolves. From now on, we consider the linear utility \( u(\rho) = \rho \). Additionally since the cost \( c \) plays only a secondary role for the network structure by setting a cutoff point of evolving, we ignore the cost \( (c = 0) \) for a while to get the ultimate structure of a stable network.\(^{10}\)

We start with an empty network and perception which is a random graph with average link density \( p \in [0, 1] \) (i.e. \( e_{j,k,0} = 0 \) for all \( j, k \in N \) and \( \Pr(e_{j,k,0} = 1) = p \) for all \( i \neq j, k \)).

**Network formation in the early stage** \((0 \leq t \leq n)\). Given a set of conditions at \( t = 0 \), the first link is formed between \( i \) and \( j \) such that \( \rho_0^i = \rho_0^{(1)}, \rho_0^j = \rho_0^{(n)} \). For parsimonious notations, we label \( i \) as agent 1 and \( j \) as agent 2. Additionally, we label the least accurate agent in each time period \( t = 0, \cdots, n - 2 \) as \( t + 2 \). Then in each time period \( t < n \), agent \( t + 2 \) suggests a link to **\(^{11}\)**

\[10\] In fact, the curvature of a utility function makes a different cutoff point of connections under a positive cost only. When the gap between \( \rho^{(1)} \) and \( \rho^{(n)} \) is sufficiently large and \( c > 0 \), the lower (higher) level of \( \rho^{(n)} \) is more (less) benefitted by replacing with \( \rho^{(1)} \) if a utility function is concave, thus the incentive to add a link becomes stronger (weaker) under the lower (higher) \( \rho^{(n)} \) and concave utility. If \( c = 0 \), the curvature of a utility function has no influence in the incentive for linking due to the free of charge.

\[11\] Once the least accurate agent updates its perception in each time period, the one who updates in that time period is not the least accurate agent anymore in the next period. Thus this labeling rule implies \((\rho_0^{(1)}, \rho_0^{(2)}, \rho_0^{(3)}, \cdots, \rho_0^{(n)}) = (\rho_0^1, \rho_0^2, \rho_0^{-1}, \cdots, \rho_0^n)\).
agent $t + 1$.

Initially the actual network is empty, while individuals perceive that there is a link between any two randomly chosen agents with probability $p$, thus we can induce that the initial accuracies are normally distributed with mean $1 - p + \frac{2}{n}p$. Once the accuracies are revealed at $t = 0$, agent 2 suggests a link to agent 1 to improve its network utility by updating its perception. By linking to 1, 2 becomes the most accurate agent in the next time period: agent 2 replaces its own perception with 1’s perception which is most accurate and adds up own correct information to the replaced perception, thus 2 has at least equal or higher accuracy than 1 in $t = 1$. In $t = 1$, the least accurate agent, labeled 3, adds a link to agent 2 because $\rho_1^2 = \rho_1^{(1)}$. Then $\rho_2^3 = \rho_2^{(1)}$ in $t = 2$ and agent 3 gets a link offer from 4, and so on. Hence, in the beginning of time period $t < n$, there will be $t$ links, connecting from agent 1 to $t + 1$ in a line network. In time period $t = n$, agent 1 whose accuracy was the highest initially has the least accurate perception because of no update until all others have done. Then in $t = n$, 1 links to agent $n$ who has updated most recently so that a ring network is formed.

We can summarize the early stage dynamics as following:

**Proposition 1** In the early stage ($t < n$), a network evolves in a line structure. If a network is connected, it must be either a line structure at $t = n - 1$ or a ring structure at $t = n$.

This result shows a ring network as the global structure of an evolving network. We use the word “global” because the ring structure is sustained as local connections are added if a network continues evolving in the further time period $t > n$. Details shall be explained later.

In fact, a ring shape as a global structure is not strange in any communication involved social networks. For example, as seen in Figure 1 the global structure of romantic relations (Bearman, Moody, and Stovel, 2004) and company directorship is a ring shape.

Unlike most evolving network models which show a few distinguishable nodes in terms of link density, a network develops a regular link distribution in this model. Although it is possible for an agent to have one or two links more than others accidentally, majority have two links in either a line structure or a ring structure in the early stage. The non-existence of link concentration can explain local interactions and relatively long average distance.\(^{12}\) The simulation in the next

\(^{12}\)Comparing two extreme network structures with the same density, the average distance of a ring network is $\frac{n}{2}$ or $\frac{n}{2} + \frac{1}{2}$, whereas the average distance in a star network (i.e. a single hub node connects all n-1 nodes) is less than 2.
section supports local connections by showing that a newly added link in this model less reduces the average distance than a randomly added link.

**Perception spreading via links.** Now focusing on the perception update, we review the early stage dynamics. When agent 2 connects with agent 1, 2 replaces own perception with 1’s perception as described in (1). Then correct information related to 2 itself ($e_{2k,1}$ for $k = 1, \cdots, n$) and 1’s original perception ($e_{jk,0}$ for all $j, k$) are mixed in 2’s perception $G^2_1$ as following:

$$G^1_1 = \begin{pmatrix} 0 & e_{12,1} & 0 & \cdots & 0 \\ e_{13,1} & e^1_{23,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{1n,1} & e^1_{2n,0} & e^1_{3n,0} & \cdots & 0 \end{pmatrix}, \quad G^2_1 = \begin{pmatrix} 0 & e_{12,1} & 0 & \cdots & 0 \\ e_{13,1} & e_{23,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{1n,1} & e_{2n,1} & e^1_{3n,0} & \cdots & 0 \end{pmatrix}. \quad (5)$$

Since agent 2 replaces own perception with $G^1_0$, but not the other way round, 1 keeps original perception which consists of correct information in the first column and the rest $n-1$ columns of original perception. Because $e^1_{1j,t} = e_{1j,t}$ for all $j \neq 1$ and $t = 0, 1, \cdots, T$, 1’s accuracy is unchanged in time period $t = 1$ (i.e. $\rho^1_t = \rho^1_1$) because the change in $G_1$ is $e_{12,1}$. Whereas, when agent 2 updates own perception, all in $G^2_0$ are replaced with $G^1_0$ except the second column which is 2’s actual link.
status with others, thus the first two columns in $G_1^2$ are correct and the rest $n-2$ columns are 1’s original perception. Thus 2 becomes the most accurate agent in $t=1$ unless 1’s original perception on connections of 2 with others is correct ($e_{2k,0}^1 = 0, k = 3, \cdots, n$). That is, 2 reaches the highest accuracy with probability $1 - (1 - p)^{n-2}$ by updating perception. Similarly, once 3 forms a link to 2 in $t=2$, the first three columns in $G_2^3$ are correct information and the rest $n-3$ columns are 1’s original perception. Then $\rho_3^2 = \rho_2^{(1)}$ with probability $1 - (1 - p)^{n-3}$.

In general, agent 1, 2, $\cdots$, $t+1$ form a line by sequential linking from 1 to $t+1$ in any arbitrary time period $t < n$. Since agent $t+1$ replaces own perception with $t$’s perception which contains first $t$ columns of correct information and the rest $n-t$ columns of 1’s original perception, agent $t+1$ perceives the network $G_t$ as following:

$$G_{t}^{t+1} = \begin{pmatrix}
0 \\
e_{12,1} & 0 \\
e_{13,1} & e_{23,2} \\
\vdots & \vdots & \ddots \\
e_{1t,1} & e_{2t,2} & \cdots & 0 \\
e_{1t+1,1} & e_{2t+1,2} & \cdots & e_{tt+1,t} & 0 \\
e_{1t+2,1} & e_{2t+2,2} & \cdots & e_{tt+2,t} & e_{tt+1,t+2,t+1} & 0 \\
e_{1t+3,1} & e_{2t+3,2} & \cdots & e_{tt+3,t} & e_{tt+1,t+3,t+1} & e_{tt+1,t+2,t+3,0} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
e_{1n,1} & e_{2n,2} & \cdots & e_{tn,t} & e_{tt+1,n,t+1} & e_{tt+1,n+2,0} & e_{tt+1,n+3,0} & \cdots & 0
\end{pmatrix}.$$  \hspace{1cm} (6)

In (6), the first $t$ columns, in which agent 1’s original perception has been replaced with correct link status in each time period, are passed to agent $t+1$, and ($t+1$)th column is agent $t+1$’s own link status, thus 1’s original perception is remained only in the rest $n-(t+1)$ columns. Proposition 2 states the accuracy of agent $t+1$ in the early stage.

**Proposition 2** The expected highest accuracy in time period $t < n-1$ is

$$\rho_0^{(1)} + \frac{1 - \rho_0^{(1)}}{L} (t+1) \left(n - \frac{t+2}{2}\right).$$
In time period \( t \geq n - 1 \), \( \rho_t^{(1)} = 1 \).

As seen in Proposition 2, the highest accuracy in each time period \( t < n \) is a function of the initially highest accuracy because the perception which is initially most accurate spreads all over connecting individuals in the early stage of dynamics. However, once a network is connected in period \( t = n - 1 \), none of the initial perception of agent 1 remains. Since agent 1 to \( n - 1 \) have sequentially filed up the correct information of one’s own link status, two agents \( n - 1 \) and \( n \) reach the full information in time period \( t = n - 1 \), i.e. \( \rho_{n-1}^{n-1} = \rho_{n-1}^n = 1 \). This result implies that there must be an agent who has the full information in a connected network and the full information spreads one-by-one with a small adjustment of newly added links until no new link is created. Note that the information transmission may occur if the most and the least accurate agents are already connected, thus, how far the full information will spread depends on the network stability.

**Local connections in period \( t > n \).** Now we examine further evolving process after a ring structure has been arisen. An important property of \( G_t \) in time period \( t \geq n \) is that the most accurate agent is always neighboring with the least accurate agent. In \( t = n \), agent 2 has the least accurate perception which updated in \( t = 0 \) so that it contains \( n - 2 \) columns of 1’s initial perception. Since the least and the most accurate agents have a link \((e_{12,n} = 1)\), the least accurate agent can update its perception without creating a new link. This implies that no change occurs in the network unless there are ties in the minimum accuracy. Thus the existence of multiple minimum accuracies are critical to form a new link in a ring structure.

We start with the probability that two agents \( k \) and \( k + 1 \) have the same accuracy. Comparing any neighboring two agent \( k \) and \( k + 1 \), \( k \) has less information than \( k + 1 \) as much as \( k + 1 \’s \) own link status in the early stage, i.e. \((e_{k+1,k+2,0}^1, \cdots, e_{k+1,n,0}^1)\). If 1’s initial perception on \( k + 1 \’s \) links is correct in period \( t > n \) \((e_{k+1,k+2,0}^1 = e_{k+1,k+2,t} = 1, e_{k+1,j,0}^1 = e_{k+1,j,t} = 0 \) for \( j = k + 3, \cdots, n \)), two agents have the same accuracy as the following:

**Definition 2** For \( k, k + 1 \in N \) in time period \( t \geq n + k - 1 \),

\[
Pr(\rho_t^k = \rho_t^{k+1}) = p(1-p)^{n-k-2} \equiv q_{k,k+1}.
\]

Note that \( q_{k,k+1} \) is higher for those who have formed a link later in the early stage \( t \leq n \) because
the correct information has been filed up in each time period.

Consider link formation in $t > n$. In time period $t = n + 1$, agent 2 updates its perception using the existing link to agent 1. Since 2’s perception update in $t = n + 1$ does not change the network ($G_{n+1} = G_{n+2}$), three agents $n$, 1, and 2 hold the full information in $t = n + 2$ (i.e. $\rho^n_t = \rho^1_t = \rho^2_t = 1$). In time period $t = n + 2$, agent 3’s accuracy becomes lowest.

Suppose that a neighboring agent 4 is in the lowest accuracy tie with agent 3. We allow 3 and 4 to update their perception simultaneously because both are equally desperate for the improvement of accuracy. Agent 3 is a neighbor of 2 whose accuracy is highest. However agent 4 needs to create a new link to one of those three fully informed agents 1, 2 and $n$ for the update. It is natural to allocate an equal probability to be selected to each of the maximum accuracy ties. Then in period $t = n + 2$, agent 4 becomes most accurate and agent 1, 2 and $n$ keep the highest accuracy with the probability 1/3. If 1 is chosen, the newly added link between 1 and 4 ($e_{14,n+2} = 1$) lowers the accuracy of the unchosen agents $n$, 2, and others because the information $e_{14,t} = 1$ is known to 1 and 4 only and the information all others have ($e_{i4,t} = e_{14,t-1} = 0 \forall i \neq 1, 4$) becomes incorrect.

In general, there is no new link created and the full information spreads from agent $n$ to $k − 1$ ($\rho^n_t = \cdots = \rho^{k-1}_t = 1$) until two agents $k$ and $k + 1$ are tie in the minimum accuracy in period $t = n + (k − 1)$. In period $t = n + (k − 1)$, agent $n, 1, \cdots, k − 1$ have an equal probability to be selected by agent $k + 1$, thus we can derive the probability for a new link as following:

$$\Pr(e_{k+1,j,t} = 1) = \prod_{i=1}^{k-1}(1 - q_{ii+1})q_{kk+1} \text{ for } j = n, 1, \cdots, k - 1 \text{ in } t = n + k - 1. \quad (7)$$

When the first new link after the ring structure is created in period $t = n + k − 1$, it is between agent $k + 1$ and $i \in \{n, 1, \cdots, k − 1\}$. Since the distance between agent $k + 1$ and $i$ is confined to $k$, the newly added link between them does not dramatically reduce the average distance of a network, unlike the well-known random rewiring model by Watts and Strogatz (1998). We shall show not-so-small-world in perceptual attachment process$^{13}$ by a simulation in the next section.

Using (7), the following result generalizes the probability of local linking:

**Proposition 3** When no new link is created consecutively for $\bar{k}$ periods from $t = n + k$, the

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$^{13}$We name this link formation process as “perceptual attachment”. The name “perceptual attachment” emphasizes that new links are added based on subjective cognition rather than correct information about the network structure, as Barabasi-Albert model (i.e. preferential attachment) assumes.
probability that the next new link makes a cluster (i.e. triadic relationship) is
\[ \frac{1}{k} \prod_{i=k+1}^{k+\bar{k}} (1 - q_{ii+1}) q_{k+k+\bar{k}+1}. \] (8)

More importantly, Proposition 3 provides a plausible explanation about clusters. A few studies (Vázquez, 2003; Jackson and Rogers, 2007) explain triadic relations that one connects with a new node through existing connections, thus any two nodes sharing a common neighbor are more likely to be connected. Our result adds another clustering mechanism which does not need a mediation node: if information transmission continues through a link, a cluster is formed because the distance between the most/least well informed agents is close. By the definition of $q_{k+k+1}$, the probability of minimum accuracy ties gets higher for high-labeled agents, thus more frequently minimum accuracy ties appear, more local connections are added so that clusters are formed often among the high-labeled agents as shown in (8).

**Evolving or steady state** When we start this analysis, we set zero cost to examine an evolving network structure. Now we recall the cost of linking $c > 0$ to consider the balance between the benefit of better information and the cost to obtain it. In any arbitrary time period $t \leq n$, there exist $t$ links, connecting from agent 1 to $t + 1$, and the agent who has initially $(n - t)^{th}$ highest accuracy will be the least accurate agent because $t$ agents whose initial accuracy is lower than this agent have already created a link and updated their perception. Since the average link density in the initial perception is $p \left(1 - \frac{2}{n}\right)$, we can derive agent $t + 2$’s accuracy in time period $t < n$ as following:

\[ E[r_{t+2}^t] = \frac{1}{L} \left\{ p_0^{t-n} (L - t) + pt - (1 - p)t \right\} = p_0^{t-n} \left(1 - \frac{t}{L}\right) + \frac{t(2p - 1)}{L} \] (9)

Note that (9) is the lowest accuracy in time period $t + 1 < n$.

To derive the condition for a steady state, we define a network in the steady state as following:

**Definition 3** A network reaches a steady state if no agent would like to create a new link.\[14\]

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\[14\]We emphasize that the concept of a steady state in this model does not imply cognitive stability. It only states that the network structure stops evolving, thus perception update continues in the steady state as long as there exists a link between the most and the least accurate agents.
mally, a network \( G \) is in a steady state if \( G_{t+n} = G_t \).

From (4), Proposition 2 and (9), we can derive a specific condition for a steady state as following:

**Corollary 4** Given a linear utility, a network reaches a steady state in the early stage at \( t < n \) if the initial accuracies and the cost of linking satisfies the following inequality:

\[
c > \left( \rho_0^{(1)} - \rho_0^{(n-t)} \right) + \frac{1}{L} \left( (t+1) \left( n - 1 - \frac{t}{2} \right) \left( 1 - \rho_0^{(1)} \right) - t \left( 2p - 1 - \rho_0^{(n-t)} \right) \right).
\]

When \( t \geq n \), the condition for a steady state is \( c > \Delta \rho_t \).

For example, consider the case of 100 agents and their initial perception with link density \( p = 0.1 \). Individual’s perception on 4851 links status are binomial trials with probability 0.1, then the distribution of \( Z \equiv \sum_{j \in N} \sum_{k \in N, k > j} I(e_{jk}, 0, e_{jk}, 0) \) can be approximated to \( N(485.1, 20.9^2) \), so the ex ante distribution of the initial accuracy is \( \rho_0 \sim N(0.098, 0.0004^2) \) for \( i \in N = \{1, \ldots, 100\} \). Once the initial accuracies have been announced that \( \rho_0^{(1)} = 0.0994 \) and \( \rho_0^{(n)} = 0.0968 \), we can expect an empty network if the cost of linking is higher than 0.0206.

As mentioned in Corollary 4, in time period \( t > n \), a network continues evolving if the gap between maximum and minimum accuracies is greater than the cost of linking. We shall show the threshold cost for an evolving network by a simulation in the next section.

### 3 Simulation

We perform numerical simulations of the model by setting \( n = 200 \) and \( p = 0.1 \), while the cost of linking \( c \) is the only relevant parameter. Initially, the network is empty, the perception accuracy of the network is normally distributed around \( E[\rho_0] = \frac{2}{n} + (1 - \frac{2}{n})(1 - p) = 0.901 \), and the average degree of perception \( G_0^i \) is also distributed around \( E[d_0^i] = (n - 1)(1 - E[\rho_0]) = 19.701 \).

**Threshold cost and network density.** Since the cost \( c \) plays a role of setting a cutoff point of evolving, we first describe a threshold cost for a stable network and the effect of positive \( c \). Recall the condition for a new link in (4). We measure the left hand side in (4) as a function of time \( t \),
Figure 2: Numerical results of the model with $n = 200$ and $p = 0.1$. The payoff range in Eq. \((10)\) as a function of time is shown with transition points or threshold values of $c$ (a,b). We plot the stable link density according to $c$ (c).

\[ \frac{2}{n} + \left(1 - \frac{2}{n}\right) \rho_t^{(1)} - \rho_t^{(n)} \equiv c_t. \] \hspace{1cm} (10)

By definition, $c_t$ in \((10)\) is the level of the linking cost which makes link and no link indifferent, thus if the given cost of linking is higher than $c_t$, we expect no further evolution. As the accuracy varies with time, $c_t$ is also time-dependent, shown in Figure 2(a). In the early stage, $\rho_t^{(1)}$ increases because the correct information sequentially replaces the initial perception, whereas $\rho_t^{(n)}$ increases only slightly as long as an agent who holds the initial perception remains. Thus, $c_t$ increases until the network is connected so that everyone has replaced own perception with the mixture of the initially most accurate perception and cumulated correct information. Once the network is connected, $\rho_t^{(1)}$ is 1 or close to 1$^{15}$ whereas $\rho_t^{(n)}$ is now increasing because the proportion of remaining initial perception in $G_t^i$ gets smaller as seen in the rest $n - (t + 1)$ columns in \((6)\), thus $c_t$ decreases. Note that for any cost $c \geq c_0$, a steady state is an empty network because the evolving process does not even get started. Precisely, starting from $c_0 = 0.0114$, although the value of $c_t$ is increasing until a network is connected, no links can be created if the cost of linking is given $c \geq c_0$, thus we can expect that given the cost of linking higher than $c_0$ a steady state is an empty network and perception does not spread. As we are interested in a non-empty network, we do not consider any cost higher than $c_0$.

A new link can be formed as long as $c_t > c_0$, which holds at least by $t = 340$. It indicates

\textsuperscript{15}Because we relax the restriction of single link creation in the simulation, $\rho_t^{(1)}$ may be less than 1 if multiple links are created in one time period.
a sharp transition from zero link density \( (c \geq c_0) \) to a finite link density \( (c < c_0) \) in the steady state, illustrated in Figure 2(c). Figure 2 also shows other sharp transitions for the link density in a steady state, which come from fluctuations in \( c_t \) in \( t > 340 \). As shown in Figure 2(b), the value of \( c_t \) drops to 0.00258 by \( t = 400 \), which we denote by \( c^{(1)} \). At this moment, the network can stop evolving if \( c \geq c^{(1)} \) or evolve further at least until \( t = 1050 \) if \( c < c^{(1)} \). This leads to another sharp transition for the link density in a steady state by the same reason of the discontinuous link density between empty and non-empty networks. For example, if \( c = 0.0026 \) \( (c \geq c^{(1)}) \), no links are added after time period \( t = 400 \), while perception can still spread along the existing links. Otherwise, e.g., if \( c = 0.0025 \), the network evolves until time period \( t = 1050 \) when \( c_t \) reaches \( c^{(1)} \) again. In general, the \( m + 1 \)-th sharp transition occurs whenever a local minimum of \( c_t \) becomes smaller than the previous local minimum \( c^{(m)} \) for \( m \geq 1 \). Some of them are depicted by vertical lines in Figure 2(c).

Eventually at \( t = 6500 \), \( c_t \) becomes 0 without further evolution of the network. That is, because the network is sufficiently dense with link density around 0.38, all agents have the full information of the network.

In the rest of this section, we illustrate relatively sparse/dense networks in the different range of the cost of linking.

**High cost of linking** \( (c^{(1)} \leq c < c_0) \). For deeper understanding, we consider the case of \( c = 0.01 \) in the range of \( c^{(1)} \leq c < c_0 \). The left panel of Figure 3 depicts a network in a steady state under the high cost of linking. Overall, the network in a steady state is globally ring structured with local triangular relations.

The perceptual evolution can be partly characterized by the highest and lowest accuracies as a function of time, as shown in Figure 3(a). The numerically obtained highest accuracy (denoted by a red solid line in the graph) supports the analytic result in Proposition 2 (denoted by a black solid line in the graph) with the numerical value of \( \rho^{(1)}_0 \) up to \( t = 120 \) in which the ring structure occurs. Once the network is connected, the least accurately perceiving agent improves own perception using the cumulated correct information, boosting the lowest accuracy (denoted by a blue dotted line in the graph). Accordingly, perception approaches the network, evidenced by the fact that the average degree of perception approaches that of the network, see Figure 3(b).

\[^{16}\text{The completion of the ring structure occurs earlier than } t = n \text{ because multiple links in one time period are allowed in the simulation.}\]
Figure 3(c) shows the change in the average path length as links are added. Newly added links on the global structure in time period $t < 320$ mostly lead to the local triangular structure, hence the average path length slowly decreases. On the other hand, a few shortcuts are added in $t \in [320, 340]$ which reduce the average path length by 10. Note that these shortcuts connect nodes with the only limited distance such that the global ring structure remains. In order to compare the strategic link formation to the random link formation, we create a null network which is exactly the same as the network at the moment when the ring structure is completed. Then, whenever a new link is formed in the model, we also add a new random link to the null network for comparing the average path lengths between our model and the null model under the same link density. Here the number of added links is denoted by $t'$. As shown in the inset of Figure 3(c), the null network shows much smaller average path length than the cognition-based network, which supports local connections due to neighboring between the most and the least accurately perceiving agents.

Later than $t = 340$, the condition for the steady state specified in Corollary 4 is satisfied as the lowest accuracy approaches the highest accuracy. The network reaches a steady state, however, perception continues being updated until $t = 440$, see Figure 3(a). Although the perception spreading is still occurring, the finite $c$ inhibits further link formation for spreading correct information. Thus, the final perception does not fully reflect the network. As a result, the average degree of the network in a steady state is 3.04, while that of the perception is 2.83. The corresponding degree distributions are shown in Figure 3(d). Note that in the steady state, one agent has 11 links, which thus can be called a local hub. Such local hubs may emerge because agents with more neighbors can be updated better, hence has a higher chance to get additional links.

**Low cost of linking ($c^{(2)} < c < c^{(1)}$).** Next, we consider the case of $c = 0.0025$ in the range of $c^{(2)} < c < c^{(1)}$. The results are summarized in Figure 4. The relatively low cost of linking enables a continuous change in the network structure by adding links, thus the full information under the low cost is just temporary. If a network continues evolving due to the low cost of linking, individuals are more willing to update their perception because their information becomes incorrect quickly. This leads to a higher benefit of link creation, thus the least accurate agent easily adds a link to the most accurate agent, which makes others’ perception more inaccurate.

The early stage dynamics for $t < 340$ is the same as when $c = 0.01$. Later than $t = 340$,
Figure 3: Numerical results of the model with $n = 200$, $p = 0.1$, and $c = 0.01$. The left panel shows the visualized network in the stable state. The right panel shows the highest and lowest accuracies in comparison with analysis (a), average degrees of the network and perception (b), average path length of the network (c) as functions of time $t$, and the degree distributions of the network and perception in the stable state (d). The average path length has been calculated only for pairs of nodes that are connected, i.e., when there exists a path connecting nodes. In the inset of (c), the average path length of the model (red solid curve) is compared to that of the null model with random link addition (blue dotted curve) as a function of the number of added links $t'$ for $t' \geq n$.

The lower cost of linking enables more links to be added at the higher rate, as inferred by the comparison between Figure 3(b) and Figure 4(b). The difference between perception (blue dotted line in both graphs) and the network (red solid line in both graphs) is larger under the low cost, which implies more active link creation. The more actively added links affect evolving process as well. In particular, due to actively added links, the accuracies are also strongly perturbed as shown in Figure 4(a), and thus the least accurate agent becomes quickly far-off from the most accurate agent. The larger distance between the most/least accurate agents necessitates new links between them, which explains higher probability of a shortcut. On the other hand, once a shortcut is added, it expands the range of local interactions which leads to another shortcuts, thus the average distance becomes shorter under the low cost of linking. Since these shortcuts connect nodes with the limited distance, they do not destroy the global ring structure, as shown in the left panel of Figure 4. We consistently observe that the average path length considerably drops from 32 to 6 in
time period $t \in [320,440]$, as seen in Figure 4(c). In addition, the higher rate of link creation slows down the spreading of accurate information, inhibiting the lowest accuracy from approaching the highest accuracy.

4 Concluding remarks

This paper has proposed a simple model to study the coevolution between a network and perception. Focusing on how individuals and a system affect each other, we have examined a cognition-based strategic link formation. Assuming that a link as a conduit of communication is costly and more accurate perception yields higher network utility, one decides whether to form a link in order to get better knowledge. A newly added link causes a change in a network, which affects individuals’ perception accuracy back.

We found that once a network is connected the global structure will be a ring and there must be at least two agents who possess the full information. We also showed locally added links which provides a plausible reasoning for clusters in a social network. Additionally the relationship between an evolving process and the cost of linking has been discussed and a simulation illustrated how a
network and perception coevolve.

This research is meaningful by revealing importance of cognition in the coevolution between individuals and a system. As we have suggested a simplified framework of the interplay between individuals’ perspective and a systemic change, there are potential ways to develop further models in psychology, economics, and sociology. We remain extensions of this model to diverse directions for the future research.

Appendix

Proof of Proposition 2

Proof. In time period $t$, $p_t^j < p_t^1 \leq p_t^2 \leq \cdots \leq p_t^{t+1}$, $j = t+2, \ldots, n$ and $\Pr(p_t^{t+1} > p_t^1) = 1 - (1-p)^{n-t}$.

Since the first $t + 1$ columns in $G_t^{t+1}$ are correct and the rest $n - (t + 1)$ columns are 1’s original perception, the expected accuracy of $p_t^{t+1}$ can be expressed with 1’s original accuracy as following:

$$E[p_t^{t+1}] = \frac{1}{L} \left[ \sum_{k=n-(t+1)}^{n-1} k + \rho_0^{(1)} \left( L - \sum_{k=n-(t+1)}^{n-1} k \right) \right] = \rho_0^{(1)} + \frac{1 - \rho_0^{(1)}}{L} (t + 1) \left( n - \frac{t + 2}{2} \right) \quad (11)$$

In $t = n - 1$, (11) becomes 1 so the fully correct information is delivered after $t = n - 1$, thus $\rho_t^{(1)} = 1$ for $t \geq n - 1$. ■

References

Bala and Goyal (2000), A non-cooperative model of network formation, Econometrica 68, 1181–1230.

Barabási and Albert (1999), Emergence of scaling in random networks, Science 286, 509–512.

Bearman, Moody, and Stovel (2004), Chains of affection: The structure of adolescent romantic and sexual networks, American Journal of Sociology 100, 49–91.

Bloch and Jackson (2007), The formation of networks with transfers among players, Journal of Economic Theory 133, 83–110.

Dessi, Gallo, and Goyal (2012), Network cognition, Toulouse IDEI Working Papers with number 691.

Dunbar (1998), The social brain hypothesis, Brain: A Journal Of Neurology 9, 1780-190.

Dunbar (2009), The social brain hypothesis and its implications for social evolution, Annals Of Human Biology 36, 562–572.

Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010), Network games, Review of Economic Studies 77, 218–244.
Galeotti, Goyal, and Kamphorst (2006), Network formation with heterogeneous players, Games and Economic Behavior 54, 353–372.

Granovetter (1973), The strength of weak ties, American Journal of Sociology 78, 1360–1380.

Jackson and Rogers (2007), Meeting strangers and friends of friends: How random are social networks? American Economic Review 97, 890–915.

Jackson and Wolinsky (1996), A strategic model of social and economic networks, Journal of Economic Theory 71, 44–74.

Hedström, Sandell, and Stern (2000), Mesolevel networks and the diffusion of social movements: the case of the Swedish social democratic party, American Journal of Sociology 106, 145–172.

Hill and Dunbar (2003), Social network size in humans, Human Nature 14, 53–72.

Kossinets and Watts (2006), Empirical analysis of an evolving social network, Science 6, 88–90.

Krackhardt (1987), Cognitive social structures, Social Networks 9, 109–134.

Kretschmer (2004), Author productivity and geodesic distance in bibliographic co-authorship networks, and visibility on the Web, Scientometrics 60, 409–420.

Newman (2001), Scientific collaboration networks: I. Network construction and fundamental results, Physical Review E 64, 016131.

Newman (2001), Scientific collaboration networks: II. Shortest paths, weighted networks, and centrality, Physical Review E 64, 016132.

Roberts and Dunbar (2010), Communication in social networks: Effects of kinship, network size and emotional closeness, Personal Relationships 18, 439–452.

Rose and Serafica (1986), Keeping and ending casual, close and best friendships, Journal of Social and Personal Relationships 3, 275–288.

Sampson (1988), Local friendship ties and community attachment in mass society: a multilevel systemic model, American Sociological Review 53, 766–779.

Stiller and Dunbar (2007), Perspective-taking and social network size in humans, Social Networks 29, 93–104.

Sutcliffe, Dunbar, Binder, and Arrow (2012), Relationships and the social brain: integrating psychological and evolutionary perspectives, British Journal of Psychology 103, 149–168.

Travers and Milgram (1969), An experimental study of the small world problem, Sociometry 32, 425–443.

Vázquez (2003), Growing network with local rules: preferential attachment, clustering hierarchy, and degree correlations, Physical Review E 67, 056104.

Watts (2001), A dynamic model of network formation, Games and Economic Behavior 24, 331–341.

Watts (1999), Networks, dynamics, and the small-world phenomenon, American Journal of Sociology 105, 493–527.
Watts and Strogatz (1998), Collective dynamics of ‘small-world’ networks, Nature 393, 440–442.