Scattering of tidally interacting bodies in post-Minkowskian gravity

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The post-Minkowskian approach to gravitationally interacting binary systems (i.e., perturbation theory in $G$, without assuming small velocities) is extended to the computation of the dynamical effects induced by the tidal deformations of two extended bodies, such as neutron stars. Our derivation applies general properties of perturbed actions to the effective field theory description of tidally interacting bodies. We compute several tidal invariants (notably the integrated quadrupolar and octupolar actions) at the first post-Minkowskian order. The corresponding contributions to the scattering angle are derived.

\section{I. INTRODUCTION}

The post-Minkowskian (PM) approach to gravitational interaction, which was pioneered some time ago \cite{1, 2}, has been recently revived \cite{3, 4} and has undergone many developments both in classical gravity \cite{5, 6}, and in the connection between classical gravity and quantum magnetic types. 

Tidal interactions are expected to play an important role in driving the dynamics of the last orbits of coalescing binary systems comprising at least one neutron star. Up to now, tidal effects in binary systems have been studied within either: i) the post-Newtonian (PN) approach \cite{7, 8}; ii) numerical relativity (see e.g., \cite{9, 10}); and iii) the gravitational self-force (SF) approach \cite{11, 12}. In addition, it was found useful \cite{13} to transcribe the results of the latter approaches within the effective-one-body (EOB) formalism \cite{14, 15}.

The starting point of our present computation is the effective field theory description of the dynamics of gravitationally interacting extended bodies \cite{16, 17, 18, 19}. This approach will be briefly recalled in Sec. III. It describes finite size effects by adding to the point-mass action of a two-body system certain non-minimal worldline couplings, defined as integrals of tidal invariants along the worldlines of the bodies. The coefficients appearing in front of these non-minimal worldline couplings are certain tidal polarizability parameters (linked to “Love numbers”), which can be computed given some equation of state for the nuclear matter \cite{20, 21}. Adopting such an effective action description of tidal effects, we will compute here several integrated tidal invariants associated with the worldlines of the two members of a binary system undergoing hyperbolic motion. Our calculations will be performed within PM theory, at the first PM approximation level (1PM), i.e., at first order in the gravitational constant $G$, but at all orders in velocities. We will focus on quadratic and cubic invariants of both electric and magnetic types.

We will generally use units where $c = 1$. The masses of the two gravitationally interacting bodies are denoted by $m_1$ and $m_2$. We then define the total rest mass of the system ($M$), the reduced mass ($\mu$) and the symmetric mass-ratio ($\nu$) as

$$ M \equiv m_1 + m_2, \quad \mu \equiv \frac{m_1 m_2}{M}, \quad \nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2} \leq \frac{1}{4}. \quad (1.1) $$

We will sometimes use the dimensionless mass ratios

$$ X_1 \equiv \frac{m_1}{M}, \quad X_2 \equiv \frac{m_2}{M} = 1 - X_1, \quad (1.2) $$

with the link $\nu = X_1 X_2$.

\section{II. PERTURBED ON-SHELL ACTION AND SCATTERING}

Here we shall follow Sec. II E of Ref. \cite{51} and show how some general properties of reduced actions allow one to simplify the discussion of additional effects perturbing a basic dynamics. We consider a two-body system whose interaction can be decomposed into some zeroth-order (unperturbed) dynamics modified by an additional interaction, of strength measured by a parameter $\epsilon$, say

$$ S = S_0 + \epsilon S_1. \quad (2.1) $$

When using an Hamiltonian formulation, such a perturbed dynamics is described by a Hamiltonian of the general form,

$$ H_0(p, q) = H_0(p, q) + \epsilon H_1(p, q). \quad (2.2) $$

Here $H_0(p, q)$ describes the unperturbed dynamics while $\epsilon H_1(p, q)$ describes a specific perturbation. Examples of
this very general setting are: i) free motion perturbed by the interaction mediated by some field; ii) geodesic motion in a black hole background perturbed by SF effects; iii) EOB description of 1PM gravity perturbed by higher PM interactions; iv) nonspinning dynamics perturbed by spin effects, etc. In the following we consider the case where the tidal deformation of two interacting bodies perturbs the dynamics of two pointlike objects. 

When studying, as we shall do, the relative dynamics of a two-body system considered in the center-of-mass (c.m.) frame, the phase-space variables reduce to that of one particle\(^1\) of position \(R = R_1 - R_2\) and momentum \(P = P_1 = -P_2\). Then, the energy conservation law yields

\[
E = H_0(R, P_R, P_\phi), \tag{2.3}
\]

where \(R = |R|\). Let us define \(P_R^{(0)}(R; E, P_\phi)\) as the unperturbed solution of the energy conservation law, i.e., of the equation

\[
E = H_0(R, P_R^{(0)}, P_\phi). \tag{2.4}
\]

Writing the solution of Eq. (2.3) as

\[
P_R(R; E, P_\phi) = P_R^{(0)}(R; E, P_\phi) + \epsilon_P^{(1)}(R; E, P_\phi) + O(\epsilon^2), \tag{2.5}
\]

leads to the following first-order equation for \(P_R^{(1)}\)

\[
E = H_0(R, P_R^{(0)}, P_\phi) + \frac{\partial H_0}{\partial P_R} P_R^{(1)} + \epsilon H_1 + O(\epsilon^2), \tag{2.6}
\]

so that

\[
\epsilon P_R^{(1)}(R; E, P_\phi) = -\epsilon \left[ \frac{H_1}{\epsilon} \right]_{P_R = P_R^{(0)}(R; E, P_\phi)} + O(\epsilon^2). \tag{2.7}
\]

When considering bound motions, a crucial invariant quantity is the radial action, integrated over one radial period,

\[
S_R(E, P_\phi) = \int P_R(R; E, P_\phi) dR. \tag{2.8}
\]

As pointed out in Ref. [9], the analog of this invariant for scattering motion is the (subtracted) radial action,

\[
S_R^{\text{subt}}(E, P_\phi) = \int_{-\infty}^{\infty} dR [P_R(R; E, P_\phi) - P_R^{\text{free}}(R; E, P_\phi)], \tag{2.9}
\]

where \(P_R^{\text{free}}(R; E, P_\phi)\) denotes the value of \(P_R\) in absence of any interaction \(^2\), and where the integral is taken over the full scattering motion [symbolically indicated by the time interval \(t \in (-\infty, +\infty)\)]. For instance, when using the (real) phase-space coordinates \(R = R_1 - R_2\) and \(P = P_1 = -P_2\), one defines \(P_R^{\nu}(R; E, P_\phi)\) as the solution of

\[
E = \sqrt{m_1^2 + P^2} + \sqrt{m_2^2 + P^2}, \tag{2.10}
\]

with \(P^2 = P_R^2 + P_\phi^2/R^2\). The corresponding equation within the EOB formalism would be simply

\[
\mu^2 + P_\phi^2 = E_{\text{eff}}^2, \tag{2.11}
\]

with \(P_\phi = (P_R^{\text{EOB}})^2 + (P_\phi^{\text{EOB}})^2/R^2\), \(P_\phi^{\text{EOB}} = P_\phi\), and with the EOB effective energy defined as \(\frac{E}{2(m_1 + m_2)}\).

The subtracted term \(P_R^{\text{free}}(R; E, P_\phi)\) has the effect both to render convergent \(^3\) the radial action (which would otherwise diverge linearly at large \(R\)) and to subtract the free-motion contribution, \(\pi\), from the scattering angle. Indeed,

\[
- \int_{-\infty}^{\infty} \frac{\partial}{\partial P_\phi} P_R^{\text{free}} dR = \pi. \tag{2.13}
\]

The total angular change during scattering is

\[
\Phi_{\text{scatt}} = - \int_{-\infty}^{\infty} dR \frac{\partial}{\partial P_\phi} P_R(R; E, P_\phi), \tag{2.14}
\]

where the integral is convergent because of the \(P_\phi\) differentiation in the integrand. The corresponding scattering angle \(\chi \equiv \Phi_{\text{scatt}} - \pi\) is then given by

\[
\chi = - \int_{-\infty}^{\infty} dR \frac{\partial}{\partial P_\phi} P_R(R; E, P_\phi) + \int_{-\infty}^{\infty} dR \frac{\partial}{\partial P_\phi} P_R^{\text{free}}(R; E, P_\phi) \tag{2.15}
\]

\[
= - \int_{-\infty}^{\infty} dR \frac{\partial}{\partial P_\phi} [P_R(R; E, P_\phi) - P_R^{\text{free}}(R; E, P_\phi)].
\]

Therefore

\[
\chi(E, P_\phi; \epsilon) = - \frac{\partial}{\partial P_\phi} S_R^{\text{subt}}(E, P_\phi), \tag{2.16}
\]

where the \(P_\phi\)-derivative could be taken out because the subtracted radial action is convergent for large \(R\).

The scattering angle \(\chi(E, P_\phi; \epsilon)\) is the full \(\epsilon\)-perturbed angle. When expanding \(\chi\) in series of \(\epsilon\),

\[
\chi(E, P_\phi; \epsilon) = \chi^{(0)}(E, P_\phi) + \epsilon \chi^{(1)}(E, P_\phi) + O(\epsilon^2), \tag{2.17}
\]

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\(^1\) Note, in passing, that this is the first element of the EOB approach to two-body dynamics.

\(^2\) Free motion here means motion in absence of any interaction, and not only of the additional interaction contained in \(H_1\).

\(^3\) Modulo the mild Coulomb logarithmic divergence when working in 3 space dimensions.
we find that
\[ \chi^{(0)}(E, P_\phi) = -\frac{\partial}{\partial P_\phi} \int_{-\infty}^{\infty} P_R^{(1)}(R; E, P_\phi) dR, \quad (2.18) \]
and
\[ \epsilon \chi^{(1)}(E, P_\phi) = -\epsilon \frac{\partial}{\partial P_\phi} \int_{-\infty}^{\infty} P_R^{(1)}(R; E, P_\phi) dR. \quad (2.19) \]

Inserting the expression of \( P_R^{(1)} \), Eq. (2.7), in Eq. (2.19) yields
\[ \epsilon \chi^{(1)}(E, P_\phi) = +\epsilon \frac{\partial}{\partial P_\phi} \int_{-\infty}^{\infty} H_1 \frac{\partial H}{\partial P_\phi} dR. \quad (2.20) \]

According to Hamilton’s equations \( \frac{\partial H}{\partial P_\phi} \) can be replaced by the time derivative \( \frac{dR}{dt} \) taken along the unperturbed, \( H_0 \)-driven, motion so that
\[ \chi^{(1)}(E, P_\phi) = -\frac{\partial}{\partial P_\phi} \delta \chi^{(1)}(E, P_\phi). \quad (2.21) \]

Here we have introduced the notation \( S_R^{(1)}(E, P_\phi) \) for the \( \epsilon \) piece of the subtracted radial action
\[ S_R^{\text{subtract}}(E, P_\phi; \epsilon) = S_R^{(0)}(E, P_\phi; 0) + \epsilon S_R^{(1)}(E, P_\phi) + O(\epsilon^2). \quad (2.22) \]

From the above results, we can write the following (equivalent) expressions for \( S_R^{(1)}(E, P_\phi) \)
\[ S_R^{(1)}(E, P_\phi) = \int_{-\infty}^{+\infty} P_R^{(1)}(R) dR 
- \int_{-\infty}^{+\infty} H_1 \frac{\partial H}{\partial P_\phi} dR 
= - \int_{-\infty}^{+\infty} dt^{(H_0)} H_1. \quad (2.23) \]

In addition, using the general property, \( \delta L(q, \dot{q}, \ldots) = -\delta \mathcal{L}(q, p) \), relating a first-order change in a Lagrangian to a change in the corresponding Hamiltonian, we can directly relate \( S_R^{(1)}(E, P_\phi) \) to the \( \epsilon \) piece of the original (Lagrangian-type) action, Eq. (2.21), namely
\[ S_R^{(1)}(E, P_\phi) = [S_1|_{E, P_\phi} - S_0 - \text{shell}], \quad (2.24) \]

where the notation on the right-hand side indicates that one must on-shell evaluate \( S_1 \) along a full \( S_0 \)-driven (or \( H_0 \)-driven) motion, with given total energy and angular momentum. In the case where \( \epsilon \) denotes the perturbation linked to the nonlocal tail effects in the orbital dynamics, this result was obtained in Ref. [61], where the gauge-invariant “potential” for the perturbed scattering angle was denoted as \( W^{(1)}(E, P_\phi) = -S_R^{(1)}(E, P_\phi) = + \int_{-\infty}^{+\infty} dt^{(H_0)} H_1. \)

The gauge-invariant nature of \( S_R^{(1)}(E, P_\phi) \) allows one to easily transcribe its value within the EOB framework.

We recall that EOB theory formulates the center-of-mass two-body dynamics in terms of a mass-shell constraint of the general form
\[ g_{\mu\nu} P_\mu^{\text{eob}} P_\nu^{\text{eob}} + \mu^2 + Q = 0, \quad (2.25) \]
where \( Q \) is a function in EOB phase-space (which is not simply quadratic in \( P_\text{eob}^{(0)} \)). When considering the \( \epsilon \)-perturbed version of the EOB mass-shell condition \( (2.25) \) one has the choice to parametrize perturbations either by modifying the effective metric \( g^{\text{eff}}_{\mu\nu} \) (when this is possible) or by changing the \( Q \) term in the mass-shell condition \( (2.25) \), or by doing both. In several previous papers dealing with tidal effects [47, 51] it was found convenient, when focusing on circular motions, to describe tidal effects by an additional term in \( g^{\text{eff}}_{\mu\nu} \) and more precisely in its main radial potential \( A(R) = -g^{\text{eff}}_0 \). By contrast, here, as we are considering hyperbolic motions, it will be more convenient to describe tidal effects by an additional (non-quadratic-in-momenta) term in \( Q \). Let us then consider a perturbed mass-shell of the form \( (2.25) \), with a perturbed \( Q \) of the general type
\[ Q = Q^{(0)} + \epsilon Q^{(1)}. \quad (2.26) \]

For simplicity, we shall assume here that the unperturbed effective metric is spherically symmetric (as is the case for non-spinning bodies):
\[ g_{\mu\nu} dx^\mu dx^\nu = -A(R) dt^{\text{eob}} + B(R) d\theta^{\text{eob}} + R_c^{(0)}(d\phi^{\text{eob}} + sin^2 \theta^{\text{eob}} d\phi^{\text{eob}}). \quad (2.27) \]

The effective Hamiltonian, \( H^{\text{eff}} \), is obtained by solving the EOB mass-shell condition, Eq. (2.25), with respect to \( P_0^{\text{eob}} = -H^{\text{eff}}, i.e., \)
\[ H^{\text{eff}} = A(R) \left( \frac{P_R^{\text{eob}}}{B(R^{\text{eob}})} + \frac{P_{\phi}^{\text{eob}}}{R_c^{\text{eob}}} \right)^2 + \mu^2 + Q^{(0)} + \epsilon Q^{(1)}. \quad (2.28) \]

If we assume that \( Q^{(0)} \) does not depend on \( P_0^{\text{eob}} \) (as, for example, was done in Ref. [10]), we then find that
\[ H^{\text{eff}}(R, P_0^{\text{eob}}, P_{\phi}^{\text{eob}}) = H_0^{\text{eff}} + \epsilon H_1^{\text{eff}} + O(\epsilon^2), \quad (2.29) \]
with
\[ (H_0^{\text{eff}})^2 = A(R^{\text{eob}}) \left( \frac{P_R^{\text{eob}}}{B(R^{\text{eob}})} + \frac{P_{\phi}^{\text{eob}}}{R_c^{\text{eob}}} \right)^2 + \mu^2 + Q^{(0)}, \quad (2.30) \]
and
\[ H_1^{\text{eff}} = \frac{A}{2H_0^{\text{eff}}} Q^{(1)}. \quad (2.31) \]

Let us recall the crucial facts that –because of their gauge-invariant properties– both the EOB effective (subtracted) radial action, the total EOB angular momentum, and the EOB scattering angle coincide with the
corresponding “real” physical quantities,

\[ P_{\phi}^{\text{cob}} = P_{\phi}^{\text{real}}, \]

\[ S_{R, \text{subt}}^{\text{eff}}(E_{\text{eff}}, P_{\phi}; \epsilon) = S_{R, \text{subt}}^{\text{real}}(E_{\text{real}}, P_{\phi}; \epsilon), \]

\[ \chi^{\text{cob}}(E_{\text{eff}}, P_{\phi}; \epsilon) = \chi^{\text{real}}(E_{\text{real}}, P_{\phi}; \epsilon). \]

(2.32)

We also recall that the effective energy, \( E_{\text{eff}} = -P_{\phi}^{\text{cob}} = H_{\text{eff}} \), is related to the real energy \( E_{\text{real}} = H_{\text{real}} \) by the energy map \([9, 48]\)

\[ H_{\text{real}} = M \sqrt{1 + 2 \nu \left( \frac{H_{\text{eff}}}{\mu} - 1 \right)}. \]

(2.33)

One then easily finds that the \( \epsilon \) piece of the total effective (subtracted) radial action (which is equal to the real one)

\[ S_{R, \text{subt}}^{\text{eff}}(E_{\text{eff}}, P_{\phi}; \epsilon) = S_{R, \text{subt}}^{\text{eff}}(E_{\text{eff}}, P_{\phi}; 0) + \epsilon S_{R}^{(1)\text{eff}}(E_{\text{eff}}, P_{\phi}) + O(\epsilon^2), \]

(2.34)

is equal to its real counterpart

\[ S_{R}^{(1)\text{eff}}(E_{\text{eff}}, P_{\phi}) = S_{R}^{(1)\text{real}}(E_{\text{real}}, P_{\phi}), \]

(2.35)

and is given by the following expressions

\[ S_{R}^{(1)\text{eff}}(E_{\text{eff}}, P_{\phi}) = \int_{-\infty}^{\infty} dP_{R}^{\text{cob}} \frac{dR_{\text{eff}}^{\text{cob}}}{dP_{R}^{\text{cob}}} R_{\text{eff}}^{\text{cob}}(1) \]

\[ = - \int \frac{dR_{\text{eff}}^{\text{cob}}}{dP_{R}^{\text{cob}}} H_{1}^{\text{eff}} \]

\[ = - \int dt H_{\text{eff}}^{\text{cob}}. \]

(2.36)

Here \( t_{\text{eff}}^{\text{cob}} \) denoted the effective time evolution parameter (defined by Hamilton’s equations), evaluated along the unperturbed effective motion, so that

\[ \frac{dR_{\text{eff}}^{\text{cob}}}{dt} = \frac{\partial H_{\text{eff}}^{\text{cob}}}{\partial P_{R}^{\text{cob}}}. \]

(2.37)

Substituting in Eq. (2.36) the value of the perturbed effective Hamiltonian, Eq. (2.31), also yields the expression

\[ S_{R}^{(1)\text{eff}} = \frac{1}{2} \int d\sigma_{(0)} Q^{(1)}, \]

(2.38)

where

\[ d\sigma_{(0)} = \frac{AdP_{\text{cob}}}{H_{0}^{\text{cob}}} \frac{dH_{R}^{\text{cob}}}{dP_{R}^{\text{cob}}}. \]

(2.39)

In the cases where the \( P_{R} \)-dependence of the unperturbed effective Hamiltonian is accurately described by

\[ (H_{0}^{\text{eff}})^{2} = A \left( \frac{(P_{R}^{\text{cob}})^{2}}{B} + P_{R}^{\text{cob} \text{-independent terms}} \right), \]

(2.40)

the parameter \( \sigma_{0} \) simplifies to

\[ d\sigma_{(0)} = B \frac{dR_{\text{cob}}^{\text{cob}}}{dP_{R}^{(0)\text{cob}}} = B \frac{dR_{\text{cob}}^{\text{cob}}}{dP_{R}^{(0)\text{cob}}}, \]

(2.41)

where \( P_{R} \equiv g_{\text{eff}}^{RR} P_{R} = \frac{d\sigma_{(0)}}{dP_{\phi}} \). The corresponding formula for the perturbation of the scattering angle,

\[ \chi^{(1)} = \frac{1}{2} \frac{\partial}{\partial P_{\phi}} \int d\sigma_{(0)} Q^{(1)}, \]

(2.42)

agrees with the result obtained in Eq. (4.22) of Ref. [10], where the unperturbed squared effective Hamiltonian was a Schwarzschild mass-shell condition \( g_{\text{eff}}^{\mu \nu} = g_{\text{Schw}}^{\mu \nu} \) and \( Q^{(0)} = 0 \) and where the \( \epsilon \) parameter was \( G^2 \), with \( Q^{(1)} = \sum_{k \geq 2} u_{k}(H_{\text{eff}}^{\text{cob}}) \) (PM energy gauge). In that case \( \sigma_{(0)} \) was the unperturbed \( (\mu \text{-normalized}) \) effective proper time along the geodesic Schwarzschild motion.

Summarizing: in a general \( \epsilon \)-perturbed situation, the crucial potential from which one can deduce scattering information is the on-shell perturbed radial action \( S_{R}^{(1)}(E, P_{\phi}) = -\int_{-\infty}^{\infty} dt (\hbar H_{1}) H_{1} \), integrated along the full hyperbolic motion. From the function \( S_{R}^{(1)}(E, P_{\phi}) \) one then deduces, by \( P_{R} \)-differentiation, the scattering angle. Let us note in passing that the \( E \)-differentiation of \( S_{R, \text{subt}}^{\text{cob}}(E, P_{\phi}; \epsilon) \) yields the (full) Wigner time delay \([62, 63]\), so that \( \partial S_{R}^{(1)}(E, P_{\phi})/\partial E \) is the \( O(\epsilon) \)-correction to this time delay. The time delay is a gauge-invariant observable quantity associated with a general scattering situation which has not received yet much attention in the gravitational physics literature. Let us also note, as recently pointed out in Ref. \([63, 64]\), that \( S_{R, \text{subt}}^{\text{cob}}(E, P_{\phi}) \) is equal to the classical limit of the quantum phase shift (entering the partial wave decomposition of the quantum scattering amplitude)

\[ S_{R, \text{subt}}^{\text{cob}}(E, P_{\phi}) = \frac{2 \delta(E)}{\hbar}, \]

(2.43)

where \( l = P_{\phi}/\hbar, \) see e.g., Eq. (4.5) of Ref. \([63, 64]\).

III. WORLDLINE TIDAL ACTIONS

In the following we will apply the general results of the previous section to the case where the unperturbed dynamics is that of two pointlike objects, and where the perturbation is due to the tidal deformations of two extended bodies, e.g., neutron stars.

In an effective field theory description of \( N \) extended (compact) objects, finite-size effects are treated by increasing the (leading-order) point-mass action \([57, 58]\)

\[ S_{0} = \int \frac{dx}{16\pi G} \sqrt{-gR} - \sum_{A=1}^{N} \int m_{A} d\tau_{A}, \]

(3.1)

by additional, nonminimal, couplings involving higher-order derivatives of the field evaluated along the worldlines of the bodies \([54, 55, 56, 71]\). Here \( d\tau_{A} = \)
$\sqrt{\eta^A_{\mu\nu}dz^\mu_A dz^\nu_A}$ is the (regularized) proper time along the worldline $z^\mu_A(t_A)$ of body $A$, with 4-velocity $u^\mu_A = dz^\mu_A/dt_A$. In the matched-asymptotic-expansion approach to $N$-body dynamics, body $A$ feels the gravitational field of the whole interacting $N$-body system via a smooth “external metric” $G_{\alpha\beta}^\text{ext}(X')$ defined in the local coordinate system $X'_A$ attached to body $A$. Non-minimal couplings are expressed in terms of two higher-order proper-time derivatives of $G_{\alpha\beta}$ and $H_{\alpha\beta}$, together with their proper-time derivatives (we follow here the star denoting the dual: $A \rightarrow \bar{A}$). We will also consider the invariants

$$S_{\text{nonmin}}^A = \sum_A S_{\text{nonmin}}^A = \sum_A \sum_{l \geq 2} S_{\text{nonmin}}^{A,l},$$

with $S_{\text{nonmin}}^{A,l}$ given by

$$S_{\text{nonmin}}^{A,l} = \frac{1}{2l!} \int d\tau_A (G_{\alpha\beta}^A(\tau_A))^2 + \frac{l}{l+1} \int d\tau_A (H_{\alpha\beta}^A(\tau_A))^2 + \mu_A^l \int d\tau_A (\dot{G}_{\alpha\beta}^A(\tau_A))^2 + \frac{l}{l+1} \int d\tau_A (\dot{H}_{\alpha\beta}^A(\tau_A))^2 + \ldots.$$  

In the normalization used to define the nonminimal action \cite{43,44,45,46}, the corresponding electric-type and magnetic-type tidal quadrupolar and octupolar tensors, $G_{\alpha\beta}$, $H_{\alpha\beta}$, $G_{\alpha\beta\gamma}$, $H_{\alpha\beta\gamma}$ read

$$G_{\alpha\beta}^A \equiv -E_{\alpha\beta}(u_A), \quad H_{\alpha\beta}^A \equiv 2B_{\alpha\beta}(u_A), \quad G_{\alpha\beta\gamma}^A \equiv -\tilde{E}_{\alpha\beta\gamma}(u_A), \quad H_{\alpha\beta\gamma}^A \equiv 2\tilde{B}_{\alpha\beta\gamma}(u_A).$$

These expressions use the fact that, at the approximation order where we work here, the Ricci tensor vanishes so that the various defined tensors are symmetric and tracefree.

The first few terms of the above expansion of the nonminimal worldline action are

$$S_{G_{\alpha\beta}} = \frac{1}{4} \mu_A^{(2)} \int d\tau_A G_{\alpha\beta} G_{\alpha\beta}^A,$$

$$S_{H_{\alpha\beta}} = \frac{1}{6} \sigma_A^{(2)} \int d\tau_A H_{\alpha\beta} H_{\alpha\beta}^A,$$

$$S_{G_{\alpha\beta\gamma}} = \frac{1}{12} \mu_A^{(3)} \int d\tau_A G_{\alpha\beta\gamma} G_{\alpha\beta\gamma}^A,$$

$$S_{G_{\alpha\beta\gamma}} = \frac{1}{4} \mu_A^{(2)} \int d\tau_A \tilde{G}_{\alpha\beta\gamma} \tilde{G}_{\alpha\beta\gamma}^A.$$
e.g., cubic in $G_{ab}^A$
\[ \int d\tau_A G_{ab}^A G^{ABc} G_{c\, a} \, . \] (3.9)

Below, we will explicitly consider only the tidal invariants associated with the body labeled 1 (with mass $m_1$) in a binary system (i.e., $N = 2$). The other tidal invariants are then simply obtained by a 1 ↔ 2 exchange. We shall sometimes suppress the body label $A = 1$.

The quadrupolar electric-type tidal tensor (3.7), in non-spinning comparable mass binary systems, has been computed to 1PN fractional accuracy in Refs. 34, 75, 76 (see also Refs. 39, 75 for more details). Ref. 76 has also computed to 1PN accuracy the octupolar electric-type tidal tensor, $G_{abc}$, and the quadrupolar magnetic-type tidal tensor $H_{ab}$. The significantly more involved calculation of tidal effects (along general orbits, but still in the case of non-spinning binary systems) at the 2PN fractional accuracy has been done in Ref. 51.

In the present work we will evaluate the tidal invariants

\[ E(u_1)^2 \equiv E(u_1)_{\alpha\beta} E(u_1)^{\alpha\beta} , \]
\[ B(u_1)^2 \equiv B(u_1)_{\alpha\beta} B(u_1)^{\alpha\beta} , \]
\[ E(u_1)^3 \equiv E(u_1)_{\alpha\beta\gamma} E(u_1)^{\alpha\beta\gamma} , \]
\[ B(u_1)^3 \equiv B(u_1)_{\alpha\beta\gamma} B(u_1)^{\alpha\beta\gamma} , \]
\[ \tilde{E}(u_1)^2 \equiv \tilde{E}(u_1)_{\alpha\beta} \tilde{E}(u_1)^{\alpha\beta} , \]
\[ \tilde{B}(u_1)^2 \equiv \tilde{B}(u_1)_{\alpha\beta} \tilde{B}(u_1)^{\alpha\beta} , \]
\[ \tilde{E}(u_1)^3 \equiv \tilde{E}(u_1)_{\alpha\beta\gamma} \tilde{E}(u_1)^{\alpha\beta\gamma} , \]
\[ \tilde{B}(u_1)^3 \equiv \tilde{B}(u_1)_{\alpha\beta\gamma} \tilde{B}(u_1)^{\alpha\beta\gamma} , \] (3.10)

along the worldline $\mathcal{L}_1$ of the mass $m_1$ using PM theory, at the first PM approximation level, i.e., limiting ourselves to the first-order in $G$ but including all orders in $v/c$. [The tidal coefficients $\mu_l^{(l)}$, etc., contain a factor $1/G$, so that the O($G$) tidal action is obtained by inserting the linearized gravity tidal tensors in the nonminimal worldline action (3.3).]

As explained in Sec. II, the integrated values of all the above quantities along the worldline $\mathcal{L}_1$ define gauge-invariant quantities of direct dynamical significance interesting for describing tidal effects in scattering situations.

IV. THE FIRST POST-MINKOWSKIAN APPROXIMATION

As proven in Sec. II above, see Eq. (2.21), the perturbed radial action $S^{(1)}_R(E, P_\phi)$ is simply equal to the on-shell value of the additional worldline action associated with tidal effects, taken along an unperturbed motion with given values of energy and angular momentum. In other words, we wish to integrate the tidal action $S_{\text{nom}}$, Eq. (3.3), along an unperturbed hyperbolic two-body motion with given energy and angular momentum. This computation can in principle be done to all orders of post-Minkowskian gravity, which would mean taking into account all powers of $G$ in the gravitational interaction of the two bodies. In the present paper we will solve this problem at the lowest order, where the interbody gravitational field entering the tidal action will be obtained by solving the linearized Einstein’s equations. At this leading PM order, when evaluating $S_{\text{nom}}$, the worldlines of the two bodies can be treated as free motion worldlines (i.e., straight lines in Minkowski spacetime).

At the PM order, i.e. when solving the linearized Einstein equations in harmonic coordinates, the metric generated by our binary system is of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(G^2)$, with

\[ h_{\mu\nu} = h_{1\mu\nu} + h_{2\mu\nu} , \] (4.1)

where $h_{1\mu\nu}$ is generated by $\mathcal{L}_1$ and $h_{2\mu\nu}$ by $\mathcal{L}_2$. When computing the tidal effects along $\mathcal{L}_1$, the external metric is simply the contribution $h_{2\mu\nu}$ from $\mathcal{L}_2$, containing a factor $Gm_2$. Finally, we deal along $\mathcal{L}_1$ with the (regular) metric

\[ g_{1\mu\nu} = \eta_{\mu\nu} + h_{2\mu\nu} , \] (4.2)

with $h_{2\mu\nu} \propto Gm_2$. It is straightforward to compute the PM-accurate tidal tensors from the PM metric (4.2) generated by $\mathcal{L}_2$. Using, for instance, the results given in Appendices A and B of Ref. 4, and using the simplifying fact that, at this order, we can consider that $\mathcal{L}_2$ is a straight worldline (with tangent $u_2$), we have (at an arbitrary field point $x'$)

\[ h_{2\mu\nu}(x) = 2 \frac{Gm_2}{R_2} (2 u_{2\mu} u_{2\nu} + \eta_{\mu\nu}) , \] (4.3)

which is conveniently rewritten in the form

\[ h_{2\mu\nu}(x) = \Phi(x) H_{2\mu\nu} , \]
\[ \Phi(x) \equiv \frac{2Gm_2}{R_2(x)} , \]
\[ H_{2\mu\nu} \equiv \eta_{\mu\nu} + 2 u_{2\mu} u_{2\nu} . \] (4.4)

Here $R_2 = R_2(x)$ denotes the Poincaré-invariant orthogonal distance between the field point $x$ and the straight worldline $\mathcal{L}_2$. Explicitly, $R_2(x) = |x - z_{2\perp}(x)|$, is the modulus of the four-vector

\[ R_2^a(x) = x^a - z_{2\perp}^a(x) , \] (4.5)

where $z_{2\perp}^a(x)$ denotes the foot of the perpendicular of the field point $x$ on the line $\mathcal{L}_2$.

The expressions of the two (straight) worldlines are the following

\[ z_1(\tau_1) = z_1(0) + u_1 \tau_1 + O(G) , \]
\[ z_2(\tau_2) = z_2(0) + u_2 \tau_2 + O(G) , \] (4.6)

with $u_1$ and $u_2$ constant vectors. [In the case where one must take into account the $O(G)$ curvature of $\mathcal{L}_2$ the
expression of $h_{2\mu\nu}(x)$ should involve the half-sum of retarded and advanced tensor potentials generated by $L_2$. It is convenient to choose the origins of the proper-time parameters along $L_1$ and $L_2$ such that the corresponding connecting four-vector

\[ b^\mu = z_0^\mu(0) - z_2^\mu(0), \quad (4.7) \]

is perpendicular both to $u_1$ and to $u_2$. The vector $b^{\mu}$ can be thought of as being the Poincaré-invariant four-vectorial impact parameter of $L_1$, with respect to $L_2$, corresponding to a moment of closest approach of the two bodies. [The vectorial impact parameter of $L_2$ with respect to $L_1$ is simply $z_2^\mu(0) - z_1^\mu(0) = - b^{\mu}$.]

As we are computing spacetime scalars, we can choose a coordinate system which simplifies our computations. We find convenient to use coordinates adapted to vectorial impact parameter of $L_1$ bodies. [The vectorial impact parameter of $L_2$ with respect to $L_1$ is simply $z_2^\mu(0) - z_1^\mu(0) = - b^{\mu}$]

The Lorentz gamma factor between the two worldlines, and hence
\[ \gamma \equiv - u_1 \cdot u_2, \quad (4.10) \]
will play an important role in all the formulas below. [Here, and below, the scalar product $u \cdot v$ is the Poincaré-Minkowski one.]

We have assumed that $b^{\mu}$ is along the $x$-axis, and that
\[ z_1(0) = b \partial_x, \quad z_2(0) = 0. \quad (4.11) \]
In this way we have, for example, that
\[ z_2^\perp(x) = \tau_2^\perp(x) u_2, \quad (4.12) \]
where $\tau_2^\perp(x)$ is identified by the condition
\[ (x - z_2^\perp(x)) \cdot u_2 = 0, \quad (4.13) \]
\[ u_2 \cdot (x - \tau_2^\perp u_2) = 0, \quad (4.14) \]
\[ \tau_2^\perp(x) = - u_2 \cdot x. \quad (4.15) \]
Consequently
\[ z_2^\perp(x) = - u_2(u_2 \cdot x) = T(u_2)x, \quad (x - z_2^\perp(x) = P(u_2)x, \quad (4.16) \]
where $T(u_2) = - u_2 \otimes u_2$ projects along $u_2$ while $P(u_2) = I - T(u_2) = I + u_2 \otimes u_2$ projects orthogonally to $u_2$. Explicitly, with respect to the chosen coordinate system,
\[ z_2^\perp(x) = (\gamma t + \sqrt{\gamma^2 - 1} y)u_2, \quad x - z_2^\perp(x) = x^\alpha \partial_\alpha - (\gamma t + \sqrt{\gamma^2 - 1} y)u_2. \quad (4.17) \]
Moreover,
\[ R_2(x)^2 = |x - z_2^\perp(x)|^2 = x^2 + (x \cdot u_2)^2 \]
\[ = (\gamma^2 - 1)t^2 + x^2 + \gamma^2 y^2 + z^2 + 2\gamma \sqrt{\gamma^2 - 1} y, \quad (4.18) \]
so that
\[ R_2(x) = |P(u_2)x| \]
\[ = \sqrt{(\gamma^2 - 1)t^2 + x^2 + \gamma^2 y^2 + z^2 + 2\gamma \sqrt{\gamma^2 - 1} y}. \quad (4.19) \]
The corresponding perpendicular distance between the field point $x$ and $L_1$ reads
\[ R_1(x) = |P(u_1)x| = \sqrt{x^2 + y^2 + z^2}. \quad (4.20) \]
One can also evaluate the four-vector $R_{12}^\perp(z_1(\tau_1))$ connecting a point of $L_1$ to its orthogonal foot on $L_2$, i.e.
\[ z_1(\tau_1) - z_2^\perp(z_1(\tau_1)) = P(u_2)z_1(\tau_1) \]
\[ = P(u_2)(b \partial_x + \tau_1 u_1) \]
\[ = b \partial_x + \tau_1 P(u_2)u_1. \quad (4.22) \]
The length of the latter (spacelike) vector reaches its minimum value $b$ when $\tau_1 = 0$.

One can also easily evaluate the retarded and advanced points on $L_1$ and $L_2$ associated with the spacetime point $x$, say $z_{1R,A} = z_1(\tau_{1R,A})$ and $z_{2R,A} = z_2(\tau_{2R,A})$. They correspond to the two roots of the null conditions
\[ (x - z_{1R,A})^2 = 0, \quad (x - z_{2R,A})^2 = 0. \quad (4.23) \]
These roots are respectively given by
\[ \tau_{1R,A} = -(x \cdot u_1) \pm \sqrt{(x \cdot u_1)^2 + x^2} \]
\[ = -(x \cdot u_1) \pm R_1(x) \]
\[ = t \pm R_1(x), \quad (4.24) \]
and
\[ \tau_{2R,A} = -(x \cdot u_2) \pm \sqrt{(x \cdot u_2)^2 + x^2} \]
\[ = -(x \cdot u_2) \pm R_2(x). \quad (4.25) \]
Well known dynamical quantities at the first post-Minkowskian approximation level, determined in previous work, are the scattering angle $\chi$ and the spin holonomy $\theta$. The first quantity, $\chi$, defines the direction of the final momentum in the center-of-mass frame after the full scattering process, while the second one, $\theta$, defines
the precession angle of a spin vector (for spinning bodies) again after the full scattering process (it is known today up to the 2PM level included). Their 1PM expressions are the following \[12\]

\[
\chi = \frac{2GM(2\gamma^2 - 1)}{b(\gamma^2 - 1)} + O(G^2),
\]

\[
\theta = \frac{2GM}{b(\gamma^2 - 1)} \left\{ \left( \frac{m_1}{M}(2\gamma^2 - 1)(h - 1) \right) + O(G^2) \right\}, \tag{4.26}
\]

where

\[
h(\gamma, \nu) \equiv \sqrt{1 + 2\nu(\gamma - 1)} = \frac{E}{M}, \tag{4.27}
\]

where \(E\) is the total c.m. energy (asymptotically given by \(E = \sqrt{m_1^2 + P^2} + \sqrt{m_2^2 + P^2}\)). Let us now extend the list of gauge-invariant scattering observables by computing integrated tidal scalars.

We must evaluate along \(L_1\) the combinations of partial derivatives of \(h_{2\mu\nu}(x)\) entering the tidal tensor expressions, and then integrate them over \(L_1\), using e.g., \(dx^\lambda = u_3 \, d\tau_1\).

The partial derivatives with respect to \(x^\alpha\) of \(h_{2\mu\nu}(x)\) (using again \(u_2 = O(G)\)) are given by

\[
\partial_\alpha h_{2\mu\nu}(x) = \Phi_{,\alpha} H_{2\mu\nu} + O(G^2),
\]

\[
\partial_{\alpha\beta} h_{2\mu\nu}(x) = \Phi_{,\alpha\beta} H_{2\mu\nu} + O(G^2), \tag{4.28}
\]

where (with \(R_{2\alpha} = \eta_{\alpha\beta} R_2^\beta\))

\[
\Phi_{,\alpha} = -2 \frac{Gm_2}{R_2^\alpha} R_{2\alpha},
\]

\[
\Phi_{,\alpha\beta} = \frac{6Gm_2}{R_2^\alpha} \left( R_{2\alpha} R_{2\beta} - \frac{1}{3} R_2^2 P(u_2)_{\alpha\beta} \right). \tag{4.29}
\]

After differentiation one must replace \(x \to x_1(\tau_1)\).

For example, the Riemann tensor components are given by

\[
R_{\alpha\beta\gamma\delta} = H_{2\alpha[\delta} \Phi_{,\gamma]\beta} - H_{2\beta[\delta} \Phi_{,\gamma]\alpha}. \tag{4.30}
\]

When writing down the explicit expressions of the electric and magnetic components of the Riemann tensor along \(u_1, E(u_1)_{\alpha\beta}\), and \(B(u_1)_{\alpha\beta}\), it is useful to define the following past-directed timelike vector

\[
V_2^\alpha \equiv 2(u_1 \cdot u_2) u_2^\alpha + u_1^\alpha, \quad V_1 \cdot V_2 = -1. \tag{4.31}
\]

Note that \(V_2\) is asymmetric under the \(1 \leftrightarrow 2\) exchange. We find

\[
E(u_1)_{\alpha\gamma} = \frac{1}{2} \left( V_{2\alpha} u_2^\beta \Phi_{,\gamma\beta} - (u_1 \cdot V_2) \Phi_{,\alpha\gamma} - H_{2\alpha\gamma} u_1^\beta \Phi_{,\beta\delta} + V_2 u_1^\beta \Phi_{,\beta\delta} \right),
\]

\[
B(u_1)^{\alpha\gamma} = \frac{1}{2} u_1^\rho \eta^{\mu\nu\rho\sigma} \left[ V_{2\rho} \Phi_{,\sigma\gamma} - H_{2\sigma\rho} \Phi_{,\beta\delta} u_2^\delta \right]. \tag{4.32}
\]

where

\[
H_{2\alpha\delta} u_1^\delta = V_{2\alpha},
\]

\[
H_{2\alpha\delta} u_1^\delta u_1^\rho = u_1 \cdot V_2 = -1 + 2(u_1 \cdot u_2)^2. \tag{4.33}
\]

With our choice of coordinates, we have \(u_1 \cdot u_2 = -\gamma, u_1 \cdot V_2 = 2\gamma^2 - 1, \) and

\[
E(u_1)_{\alpha\gamma} = \frac{1}{2} \left[ V_{2\alpha} \Phi_{,\gamma\beta} - (2\gamma^2 - 1) \Phi_{,\alpha\gamma} - H_{2\alpha\gamma} \Phi_{,00} + V_2 \Phi_{,00} \right],
\]

\[
B(u_1)^{\alpha\gamma} = \frac{1}{2} \epsilon^{\alpha\rho\sigma} \left[ V_{2\rho} \Phi_{,\sigma\gamma} - H_{2\sigma\rho} \Phi_{,00} \right]. \tag{4.34}
\]

where \(\epsilon^{\alpha\rho\sigma} \equiv u_1^\mu \eta^{\mu\nu\rho\sigma}\).

We first evaluate the values of several scalar tidal functions of the proper time \((\tau_1 = t)\), along the worldline \(L_1\): \(f(t) \equiv f(x(t))\). For example, the instantaneous value of the invariant \(E(u_1)^2\) at proper time \(\tau_1 = t\) is given by

\[
E(u_1)^2 = \frac{18G^2 m_2^2}{(\gamma^2 - 1) t^2 + b^4} \left( \frac{1}{3} \gamma^2 - 1 \right),
\]

\[
+ \left( \frac{1}{3} \gamma^2 - 1 \right)^2 \frac{b^4}{t^4}
\]

\[
+ \left( \frac{1}{3} \gamma^2 - 1 \right) \frac{b^4 t^2}{3}. \tag{4.35}
\]

It is then convenient to introduce the following shorthand notation for the corresponding full proper-time integral

\[
\langle f \rangle \equiv \int_{-\infty}^{\infty} f(x(\tau_1)) \, d\tau_1. \tag{4.36}
\]

Our final results have the form

\[
\langle E(u_1)^2 \rangle = \frac{G^2 m_2^2}{b^4} F_{E^2}(\gamma),
\]

\[
\langle B(u_1)^2 \rangle = \frac{G^2 m_2^2}{b^4} F_{B^2}(\gamma),
\]

\[
\langle E(u_1)^2 \rangle = \frac{G^2 m_2^2}{b^4} F_{E^2}(\gamma),
\]

\[
\langle B(u_1)^2 \rangle = \frac{G^2 m_2^2}{b^4} F_{B^2}(\gamma),
\]

\[
\langle \dot{E}(u_1)^2 \rangle = \frac{G^2 m_2^2}{b^4} F_{\dot{E}^2}(\gamma),
\]

\[
\langle \dot{B}(u_1)^2 \rangle = \frac{G^2 m_2^2}{b^4} F_{\dot{B}^2}(\gamma),
\]

\[
\langle \ddot{E}(u_1)^2 \rangle = \frac{G^2 m_2^2}{b^4} F_{\ddot{E}^2}(\gamma),
\]

\[
\langle \ddot{B}(u_1)^2 \rangle = \frac{G^2 m_2^2}{b^4} F_{\ddot{B}^2}(\gamma), \tag{4.37}
\]

where we have separated scaling prefactors from functions giving the dependence of the various quantities on the Lorentz factor \(\gamma\). Defining

\[
p_\infty \equiv \sqrt{\gamma^2 - 1}, \tag{4.38}
\]
the various functions \( F_\chi (\gamma) \) are given by

\[
F_{E^2}(\gamma) = \frac{9\pi(35\gamma^4 - 30\gamma^2 + 11)}{64\gamma^2 - 1},
\]
\[
= \frac{9\pi(16 + 40p_\infty^2 + 35p_\infty^4)}{64p_\infty},
\]
\[
F_{B^2}(\gamma) = \frac{45\pi\gamma^2 - 1(1 + 7\gamma^2)^2}{64},
\]
\[
= \frac{45\pi p_\infty(8 + 7p_\infty^2)}{64},
\]
\[
F_\xi(\gamma) = -\frac{192(40\gamma^4 - 36\gamma^2 + 7)}{385\sqrt{\gamma^2 - 1}},
\]
\[
= -\frac{192(11 + 44p_\infty^2 + 40p_\infty^4)}{385p_\infty},
\]
\[
F_{B^2}(\gamma) = 0,
\]
\[
F_{E^2}(\gamma) = \frac{75\pi(21\gamma^6 + 385\gamma^4 - 305\gamma^2 + 91)}{512\gamma^2 - 1},
\]
\[
= \frac{75\pi(192 + 528p_\infty^2 + 448p_\infty^4 + 21p_\infty^6)}{512p_\infty},
\]
\[
F_{B^2}(\gamma) = \frac{525\pi\sqrt{\gamma^2 - 1}(3\gamma^2 + 58\gamma^2 + 3)}{512},
\]
\[
= \frac{525\pi p_\infty(64 + 64p_\infty^2 + 3p_\infty^4)}{512},
\]
\[
F_{E^2}(\gamma) = \frac{225\pi\sqrt{\gamma^2 - 1}(21\gamma^4 - 14\gamma^2 + 9)}{512},
\]
\[
= \frac{225\pi p_\infty(16 + 28p_\infty^2 + 21p_\infty^4)}{512},
\]
\[
F_\xi(\gamma) = \frac{1575\pi(\gamma^2 - 1)^{3/2}(3\gamma^2 + 1)}{512},
\]
\[
= \frac{1575\pi p_\infty^3(3p_\infty^2 + 4)}{512}. \tag{4.39}
\]

The high energy (HE, i.e., \( \gamma \to \infty \)) behaviors of those functions are:

\[
F_{E^2}(\gamma) = \frac{315}{64}\gamma^3,
\]
\[
F_{B^2}(\gamma) = \frac{1575}{512}\gamma^5,
\]
\[
F_{E^2}(\gamma) = \frac{4725}{512}\gamma^5. \tag{4.40}
\]

while

\[
F_\xi(\gamma) = -\frac{1536}{77}\gamma^4. \tag{4.41}
\]

On the other hand, \( F_{B^2}(\gamma) = 0 \) at all energies because of its time-reversal antisymmetry.

Note the fact, visible on Eq. (4.39), that the high-energy behavior of the electric tidal tensors \( E_{\alpha\beta} \), \( E_{\alpha\beta\gamma} \), \( E_{\alpha\beta\gamma\delta} \) is, respectively, the same as the one of their magnetic counterparts, \( B_{\alpha\beta} \), \( B_{\alpha\beta\gamma} \), \( B_{\alpha\beta\gamma\delta} \). This can be understood from the variance property of those tensors under boosts of \( u_1 \), when keeping \( u_2 \) fixed. For instance, \( K \equiv E_{\alpha\beta}(u_1)E^{\alpha\beta}(u_1) - B_{\alpha\beta}(u_1)B^{\alpha\beta}(u_1) \) is a scalar invariant under boosts of \( u_1 \), i.e., independent of \( u_1 \). This implies that, contrary to the separate components of \( E_{\alpha\beta}(u_1) \) and \( B_{\alpha\beta}(u_1) \), \( K \) is not amplified by a factor \( \gamma^2 \), when boosting \( u_1 \) by a factor \( \gamma \) with respect to \( u_2 \). It is instead independent of \( \gamma \) and proportional to \( 1/[R_2(z_1(t_1))]^6 \). As a consequence, the integral \( \langle K \rangle \sim \int R_2(z_1(t_1)) \, dt_{\infty} \). This is also linked to the fact that, as \( \gamma \to \infty \), the Riemann curvature generated by \( u_2 \) and observed by \( u_1 \) is of the null (wave-like) type [77].

V. TRANSCRIPTION OF THE TIDAL ACTIONS WITHIN THE EOB FORMALISM

In Refs. [47, 51] all consideration were limited to circular motions, and tidal effects were translated into a modification of the main radial potential \( A = -g_{\mu\nu}^{\text{eff}} \) entering the general EOB mass-shell constraint (2.25). In the present, hyperbolic-motion, setting, it is more appropriate to translate tidal effects into an additional momentum-dependent contribution to the \( Q \) term in the general EOB mass-shell condition (2.25). More precisely, we consider here a mass-shell condition of the form

\[
g^{\mu\nu}_{\text{Schw}}P_\mu P_\nu + \mu^2 + Q_\nu + Q_{\text{tidal}} = 0. \tag{5.1}
\]

In Eq. (5.1) we have used as EOB effective metric the Schwarzschild metric of mass \( M \) (so that it incorporates the full 1PM gravitational interaction [9]); \( P_\mu \) denotes \( P_\mu^{\text{eob}} \) and \( Q = Q_\nu + Q_{\text{tidal}} \) is the sum of two types of contributions. The first one, \( Q_\nu \), represents the post-1PM, and actually, post-Schwarzschild, effects due to PM gravity [10],

\[
Q_\nu(u, H_S) = u^2 Q_{\nu2}(H_S) + u^3 Q_{\nu3}(H_S) + O(G^4), \tag{5.2}
\]

where \( u \equiv GM/R_{\text{eob}}, \) and where \( H_S \) denotes the Schwarzschild Hamiltonian (say in Schwarzschild coordinates), i.e.

\[
H_S(u, P_R, P_\nu) = \sqrt{A(R) \left( \frac{P_R^2}{B(R)} + \frac{P_\nu^2}{R^2} + \mu^2 \right)}, \tag{5.3}
\]

with \( A(R) = 1/B(R) = 1 - 2u \). The value of the 2PM, \( O(G^2) \), term \( u^2 Q_{\nu2}(H_S) \) has been computed in Ref. [10], while the value of the 3PM, \( O(G^3) \), term \( u^3 Q_{\nu3}(H_S) \) is still a matter of debate [15, 21, 22, 63]. The additional, tidal-related contribution \( Q_{\text{tidal}} \) in Eq. (5.1) is given by a sum of contributions corresponding to all the different terms in the non-minimal worldline action (3.2). When using the energy gauge of Ref. [10], each tidal term scaling like the time integral of some power of the real interbody distance, say \( \propto \int d\tau_\infty R_{12}^\alpha \) can be made to correspond to an energy-dependent contribution \( \propto u^\alpha \) in

\footnote{Actually it is proportional to the Kretschmann scalar.}
\[ Q_{\text{tidal}}: \text{It is convenient to work with the following dimensionless rescaled version of } Q_{\text{tidal}}, \]
\[ \hat{Q}_{\text{tidal}} = \frac{Q_{\text{tidal}}}{\mu^2}. \] (5.4)

We will therefore be considering an energy dependent \( \hat{Q}_{\text{tidal}} \) of the form
\[ \hat{Q}_{\text{tidal}}(u, \hat{H}_{\text{eff}}) = -u^6 \left[ q_1^{\text{eff}}(\hat{H}_{\text{eff}}) + q_2^{\text{eff}}(\hat{H}_{\text{eff}}) \right] - u^8 \left[ q_1^{\text{eff}}(\hat{H}_{\text{eff}}) + q_2^{\text{eff}}(\hat{H}_{\text{eff}}) \right] - u^9 \left[ q_1^{\text{eff}}(\hat{H}_{\text{eff}}) + q_2^{\text{eff}}(\hat{H}_{\text{eff}}) \right] \ldots + 1 \leftrightarrow 2. \] (5.5)

The precise choice of the energy argument \( \hat{H}_{\text{eff}} \) entering \( \hat{Q}_{\text{tidal}}(u, \hat{H}_{\text{eff}}) \) is a matter of choice. One could take for \( \hat{H}_{\text{eff}} \) the conserved Hamiltonian associated with the \( \nu \)-deformed mass-shell condition \( g^{\mu \nu}_{\text{Schw}} P_{\mu} P_{\nu} + \mu^2 + Q_{\text{eff}} = 0 \), or simply the Schwarzschild Hamiltonian \( \hat{H}_S \), Eq. (5.3), associated with the \( \nu \)-undeformed mass-shell condition \( g^{\mu \nu}_{\text{Schw}} P_{\mu} P_{\nu} + \mu^2 = 0 \). In the present paper, as we will work to leading-order PM accuracy, this choice will not matter and it will turn out that we can simply use for \( \hat{H}_{\text{eff}} \) the free-motion effective Hamiltonian \( \hat{H}_{\text{eff}} = \hat{H}_{\text{free}} + O(G) \), with \( \hat{H}_{\text{free}} = \mu^{-1} \sqrt{\mu^2 + P_I^2 + P_\Sigma^2} \).

The tool for converting real action terms into effective additional \( Q \) terms is given by Eq. (2.33) above that we rewrite here as
\[ S_{\text{tidal}}(E, P_0) = -\frac{1}{2} \int d\sigma(0) \hat{Q}_{\text{tidal}}, \] (5.6)

where, using the free-motion effective Hamiltonian \( \hat{H}_{\text{eff}} = \sqrt{\mu^2 + P_I^2 + P_\Sigma^2} + O(G) \), and \( A = 1 - 2u = 1 + O(G) \), one finds [with a plus (minus) sign for the incoming (outgoing) radial integral]
\[ d\sigma(0) = \frac{AdR_{\text{eoh}}}{\frac{\partial H_{\text{eff}}}{\partial P_0}} = \pm \frac{GM}{\mu} \frac{du}{u^2(\gamma^2 - 1) - \frac{G}{\mu} u^2} + O(G). \] (5.7)

Here we introduced the dimensionless angular momentum
\[ j = \frac{P_0}{GM\mu^2}, \] (5.8)

and denoted the constant value of the effective energy \( E_{\text{eff}} = \hat{H}_{\text{eff}} \) simply as \( \mu \gamma \), i.e., \( \gamma = E_{\text{eff}} \). [Indeed, asymptotically the conserved effective EOB energy is equal to the Lorentz gamma factor, i.e. to \(-\langle P_1 \cdot P_2 \rangle / (m_1 m_2)\].]

Therefore, denoting \( u_{\text{max}} = \sqrt{\frac{\gamma^2 - 1}{\gamma^2}} \), we find for the dimensionless integrated radial action
\[ \tilde{S}_{\text{tidal}}(\mu, \gamma) = -\int_{0}^{u_{\text{max}}} \frac{\hat{Q}_{\text{tidal}}(u)du}{u^2(\gamma^2 - 1) - \frac{G}{\mu} u^2}. \] (5.9)

When \( \hat{Q}_{\text{tidal}}(u, \gamma) = -q_{\text{tidal}}(\gamma) u^n \) (where the argument \( \hat{H}_S \) entering Eq. (5.5) has been replaced by \( \gamma \)) we find (as long as \( n > 1 \))
\[ \tilde{S}_{\text{tidal}}(\mu, \gamma) = +I_n q_{\text{tidal}}(\gamma) \frac{u^{2-n} - \frac{1}{2}}{n-1}, \] (5.10)
where
\[ I_n = \int_{0}^{1} \frac{x^{n-2} dx}{\sqrt{1-x^2}} = \frac{\sqrt{\pi} \Gamma \left( \frac{n-1}{2} \right)}{2 \Gamma \left( \frac{n}{2} \right)} \] (5.11)

Let us focus on the dominant tidal contribution associated with the electric quadrupole tensor. In that case, \( n = 6 \) and the left-hand-side of Eq. (5.10) is
\[ \frac{1}{4} \frac{m_2}{m_1} \hat{\mu}^{(2)} \left( \frac{GM}{b} \right)^5 F_{E_2}(\gamma), \] (5.12)
where
\[ F_{E_2}(\gamma) = \frac{9\pi(3\gamma^4 - 4\gamma^2 + 11)}{64\sqrt{\gamma^2 - 1}}, \] (5.13)
and where we introduced the dimensionless version of \( \hat{\mu}^{(2)} \), etc. defined as follows
\[ \hat{\mu}^{(2)} = \frac{G\mu^{(2)}}{(GM/c^2)^5}, \]
\[ \hat{\mu}^{(2)}' = \frac{G\mu^{(2)}}{(GM/c^2)^7}, \]
\[ \hat{\mu}^{(3)} = \frac{G\mu^{(3)}}{(GM/c^2)^7}, \] (5.14)

etc.

On the other hand, the right-hand-side is, using \( I_6 = \frac{3\pi}{16} \),
\[ 3\pi q_1^{\text{eff}} \frac{(\gamma^2 - 1)^2}{\gamma^3}. \] (5.15)

Using the link between \( b \) and \( j \) [10]
\[ \frac{GM}{b} = \sqrt{\frac{\gamma^2 - 1}{\gamma^2}}, \] (5.16)

where (using \( \hat{H}_{\text{eff}} = \gamma \))
\[ h(\gamma, \nu) = \frac{E_{\text{real}}}{M} = \sqrt{1 + 2\nu(\gamma - 1)}, \] (5.17)
one obtains
\[ q_1^{E^2} = \frac{m_2}{m_1} \mu_1 (2) \frac{35 \gamma^4 - 30 \gamma^2 + 11}{16h^6(\gamma, \nu)}. \]  
(5.18)

This is the contribution to the tidal influence of body 2 on body 1. Therefore, the coefficient of \( u^6 \) in \( \mathcal{Q}_{\text{tidal}} \), due to the quadrupolar-electric interaction between the two bodies will be the 1 \( \rightarrow \) 2 completion of this result, namely
\[
q_1^{E^2} = q_1^{E^2} + q_2^{E^2} = 3 \left( \frac{m_2}{m_1} \mu_1 (2) + \frac{m_1}{m_2} \mu_2 (2) \right) \frac{35 \gamma^4 - 30 \gamma^2 + 11}{16h^6(\gamma, \nu)} \\
= 3 \tilde{\mu}^{(2)}_\ast \frac{35 \gamma^4 - 30 \gamma^2 + 11}{16h^6(\gamma, \nu)}, \tag{5.19}
\]
(5.19)

where we defined
\[
\tilde{\mu}^{(2)}_\ast = \frac{m_2}{m_1} \mu_1 (2) + \frac{m_1}{m_2} \mu_2 (2). \tag{5.20}
\]

To complete our EOB formulation of tidal effects within the PM framework let us connect it to the previous PN-based EOB formulation. The latter \cite{47} was focusing on circular motions and was describing tidal effects by means of an additional radial function \( A_{\text{tidal}}(u) \) in the main EOB radial potential. This is equivalent to describing tidal effects by the following squared effective Hamiltonian
\[
H_{\text{eff}}^2 = (A(R) + A_{\text{tidal}}(R)) \left[ \mu^2 + \frac{P_R^2}{B} + \frac{P_\phi^2}{R^2} + Q_\nu \right]. \tag{5.21}
\]

By contrast our present approach consists of describing tidal effects by
\[
H_{\text{eff}}^2 = A(R) \left[ \mu^2 + \frac{P_R^2}{B} + \frac{P_\phi^2}{R^2} + Q_\nu + Q_{\text{tidal}} \right]. \tag{5.22}
\]

Comparing the two approaches we see that our \( Q_{\text{tidal}} \) contribution can be translated (along circular orbits) in the following equivalent tidal potential
\[
A_{\text{tidal}}(u) = \frac{A(u)}{1 + \frac{P_R^2}{B} + \frac{P_\phi^2}{R^2} + Q_\nu} \hat{Q}_{\text{tidal}} \tag{5.23}
\]
and in the phase space variables have been rescaled, notably, \( p_\nu = P_R / (\mu) \), \( p_\phi = P_\phi / (GM\mu) \) = \( j \).

Our PM approach to tidal effects, when restricting to electric quadrupolar effects, describes them by the \( H_{\text{eff}} \)-dependent contribution
\[
\hat{Q}_{\text{tidal}}(\tilde{H}_{\text{eff}}) = -q_1^{E^2} (\tilde{H}_{\text{eff}}) u^6, \tag{5.24}
\]
where \( q_1^{E^2} \) is given by Eq. \( \ref{5.19} \). Interpreting \( \tilde{H}_{\text{eff}}^2 \) as denoting \( H_{\text{eff}}^2 = A(1 + \frac{P_R^2}{B} + \frac{P_\phi^2}{R^2} + Q_\nu) \), this result is equivalent to an effective energy-dependent \( A_{\text{tidal}} \) potential equal to
\[
A_{\text{tidal}}(u, \tilde{H}_{\text{eff}}) = -u^6 \frac{A^2(u) q_1^{E^2} (\tilde{H}_{\text{eff}})}{\tilde{H}_{\text{eff}}^2}. \tag{5.25}
\]

One cannot directly compare the full energy-dependence predicted by this lowest-PM accuracy result to previous PN-based results which were PN-corrected, i.e., which included combinations of both \( p_{\Sigma}^2 = \gamma^2 - 1 \) and \( u = GM/R \) as corrections up to the 2PN level \cite{51}. One would need to compute higher PM gravity corrections (i.e. fractional corrections to Eq. \( \ref{5.19} \)) involving powers of \( u = GM/R \) to our PM-based result for doing a meaningful comparison. However, we will compare our new formulation to the previous one, at the Newtonian level, in the next section.

VI. TIDAL CONTRIBUTION TO THE SCATTERING ANGLE AND TO PERIASTRON PRECESSION

As we have seen above, the leading PM-order contribution to the tidal action is given by
\[
\frac{SE^2 (E, P_\phi)}{GM\mu} = \frac{1}{4} p^{(2)}_\ast F_{\Sigma^2} \left( \gamma \left( \frac{GM}{b} \right)^5 \right) \tag{6.1}
\]
\[
= \frac{1}{4} p^{(2)}_\ast F_{\Sigma^2} \left( \gamma \frac{p^5_{\Sigma}}{h^5 j^5} \right) \tag{6.1a}
\]
\[
= \frac{9\pi}{16} p^{(2)}_\ast \left( 1 + \frac{5}{2} p^2_{\Sigma} + \frac{35}{16} p^4_{\Sigma} \right) \frac{p^5_{\Sigma}}{h^5 j^5}. \tag{6.1b}
\]

Using Eq. \( \ref{2.21} \) gives the corresponding leading PM-order tidal contribution to the scattering angle \( \chi \) as a function of \( \gamma \) and \( j \), where we have used \( \gamma = \tilde{H}_{\text{eff}} \) and \( j = P_\phi / (GM\mu) \)
\[
\chi_{E^2}(p_{\Sigma}, j) = \frac{5}{4} p^{(2)}_\ast F_{\Sigma^2} \left( \gamma \frac{p^5_{\Sigma}}{h^5 j^5} \right) \tag{6.2a}
\]
\[
= \frac{45\pi}{256} p^{(2)}_\ast (16 + 40 p^2_{\Sigma} + 35 p^4_{\Sigma}) \frac{p^5_{\Sigma}}{h^5 j^5}. \tag{6.2b}
\]

Let us recall the beginning of the PM expansion of the scattering angle due to Einstein gravity
\[
\frac{1}{2} \chi(\gamma, j; \nu) = \frac{\chi_1(\gamma, \nu)}{j} + \frac{\chi_2(\gamma, \nu)}{j^2} + \frac{\chi_3(\gamma, \nu)}{j^3} + \cdots, \tag{6.3}
\]

where
\[
\chi_1(\gamma, \nu) = \frac{2\gamma^2 - 1}{\sqrt{\nu+1}}, \tag{6.4}
\]
does not depend on \( \nu \) and

\[
\chi_2(\gamma, \nu) = \frac{3\pi (5\gamma^2 - 1)}{8} \frac{\gamma}{h(\gamma, \nu)}. \tag{6.5}
\]

In the HE limit \( \gamma \to \infty \) this 2PM-accurate scattering angle has a finite limit (which is independent of the symmetric mass ratio \( \nu \)) if the ratio

\[
\alpha = \frac{\gamma}{j}, \tag{6.6}
\]

is kept fixed (and small). Namely,

\[
\frac{1}{2} \chi(\gamma, j; \nu) \overset{\text{HE}}{=} 2\alpha + \frac{15\pi}{8} \frac{\gamma^2}{h} + O(G^3), \tag{6.7}
\]

where the term of order \( \alpha^2 \) is negligible because \( h \approx \sqrt{2\nu\gamma} \to \infty \). There is a current debate about the recently computed 3PM \( O(G^3) \) contributions \[21, 22\] which yields a divergent \( G^3 \) contribution, \( O(\alpha^2 \ln(\gamma)) \to \infty \) as \( \gamma \to \infty \), while the computation of Ref. \[66\] (see also \[51\]) got a finite \( O(\alpha^2) \) correction in the massless limit. [The massless limit \( m_2 \to 0 \) corresponds to \( \gamma = -(P_1 \cdot P_2)/(m_1 m_2) \to \infty \).] See also the conjectured 3PM dynamics of Ref. \[63\] leading to a finite \( O(\alpha^4) \). In addition, both Refs. \[66\] and \[63\] argued that the HE limit of \( \chi/2 \) should be of the form \( 2\alpha + c_3\alpha^3 + c_5\alpha^5 + \ldots \), i.e., with odd powers only.

Let us then consider the high-energy limit of \( \chi^{E^2} \) when \( \alpha \) is kept fixed. It reads

\[
\chi^{E^2} \overset{\text{HE}}{=} \frac{1575}{256} \pi \mu^{(2)} \frac{\gamma^2}{h^5} \alpha^6. \tag{6.8}
\]

It contains a factor \( \alpha^6 \) as expected from a \( \sim j^{-6} \) effect. We note, however, the following limiting property of the energy-dependent ratio multiplying \( \alpha^6 \)

\[
\frac{\gamma^2}{h^5} = \frac{\gamma^2}{\left[1 + 2\nu(\gamma - 1)\right]^{3/2}} \overset{\text{HE}}{=} \frac{1}{(2\nu)^{3/2}} \frac{1}{\gamma^{3/2}} \to 0. \tag{6.9}
\]

From this point of view, we conclude that, in the HE limit, the tidal corrections to scattering behave in the way expected, i.e., similarly to the non-tidal \( \alpha^6 \)-correction coming from Einstein gravity. Indeed, one expects the latter to be suppressed compared to the terms involving odd powers of \( \alpha \) (in a way similar to the \( O(\alpha^2) \) contribution discussed above), so as to leave a final result independent of the masses (and of the internal structures) of the scattered HE bodies.

Let us note in passing that the HE behavior of tidal effects is also potentially important when considering circular motions. Indeed, Ref. \[51\] has pointed out the possible existence of a power-law divergence as the circular motion is formally allowed to approach the lightring, i.e., a circular orbit where \( H^2_{\text{eff}} = \gamma^2 \) goes to infinity. Even if such an orbit is never physically reached during the inspiral of a real binary system, such a power-law blow-up would imply an increase of the strength of tidal effects during the last inspiraling orbits before merger which seems to be needed to get a good agreement with numerical simulations. [For more discussions of this issue see Refs. \[43, 64\].] However, as already mentioned, our current 1PM-accurate result would need to be improved by \( \sim GM/R + (GM/R)^2 + \ldots \) PM corrections to meaningfully discuss the HE behavior happening near the lightring.

Recently, Ref. \[27\] has pointed out a simple link between scattering angle and periastron precession, namely

\[
\Delta \Phi(E, P_\nu) = \left[ \chi(E, P_\nu) + \chi(E, -P_\nu) \right] \overset{\text{analytically continued}}{\to} \Delta \Phi_{P_{\infty}, E}(p_{\infty}, j) \overset{\text{formally}}{=} \frac{45\pi}{128} \mu^{(2)} (16 + 40p_{\infty}^2 + 35p_{\infty}^4) \left( \frac{p_{\infty}^4}{h^5} \right)^{1/5}, \tag{6.10}
\]

under the assumption that one can define a suitable analytic continuation of the energy \( E \) from the scattering domain, \( E \geq M \), to the bound-state one, \( E \leq M \). In terms of \( \gamma = E_{\text{eff}}/\mu \) this corresponds to a continuation from \( \gamma \geq 1 \) to \( \gamma \leq 1 \), while in terms of the more directly relevant variable \( p_{\infty} = \sqrt{\gamma^2 - 1} \), this corresponds to a Wick rotation of \( p_{\infty} \) from the real axis to the imaginary one. If one applies this prescription to the above lowest-order PM estimate of the tidal scattering angle \( \chi^{E^2}(p_{\infty}, j) \) one formally gets a corresponding periastron precession equal to

\[
\Delta \Phi_{P_{\infty}, E} = \left[ \chi(E, P_{\infty}) + \chi(E, -P_{\infty}) \right] \overset{\text{analytically continued}}{\to} \Delta \Phi_{P_{\infty}, E}(p_{\infty}, j) \overset{\text{formally}}{=} \frac{45\pi}{128} \mu^{(2)} (16 + 40p_{\infty}^2 + 35p_{\infty}^4) \left( \frac{p_{\infty}^4}{h^5} \right)^{1/5}, \tag{6.11}
\]

without encountering ambiguities in the analytic continuation in \( p_{\infty} \) because the odd power of \( p_{\infty} \) contained in \( F_{E\nu} \) has been cancelled by the \( p_{\infty}^5 \) factor. However, one cannot directly compare the full structure of the formal 1PM expression \[6.11\] to any known, well-defined periastron advance result. Indeed, Eq. \[6.11\] is the first term in an expansion in powers of \( 1/j \propto G \), while keeping fixed \( p_{\infty} \). The missing fractional corrections to \[6.11\] include, in particular, powers of

\[
\epsilon \equiv \frac{1}{p_{\infty} j}. \tag{6.12}
\]

[The notation \( \epsilon \) introduced here should not be confused with the use of \( \epsilon \) as a generic small parameter in Sec. II above.] In other words, the 1PM expression \[6.11\] makes sense only if \( \epsilon \ll 1 \). On the other hand, the periastron advance \( \Delta \Phi_{E\nu} \) is an observable which makes sense only for ellipticlike, bound orbits, i.e., in the case where the Newtonian eccentricity, \( e \), whose square can be defined as (see below)

\[
e^2 \equiv 1 + p_{\infty}^2 j^2 = 1 + \frac{1}{\epsilon^2}, \tag{6.13}
\]

is smaller than 1. This conflicts with the domain of validity \( \epsilon \ll 1 \), of the PM expansion, which implies \( \epsilon \gg 1 \).

However, if we restrict ourselves to the PN regime, \( \epsilon \to \infty \), in which \( p_{\infty} \propto j^{-1} \ll 1 \), while \( j \propto c \gg 1 \), keeping fixed the product \( p_{\infty} j \) (and therefore the eccentricity), we can interpolate between the PM domain of validity,
\( \epsilon \gg 1 \), and the elliptic-motion domain, \( \epsilon < 1 \), where \( \Delta \Phi_{E^2} \) is defined. Let us then consider the non-relativistic (Newtonian-level) limits (obtained using \( p_\infty \ll 1 \) and \( h \approx 1 \)) of our above 1PM results for \( \chi \), and its formal counterpart \( \Delta \Phi \) (Eq. (6.11)), namely

\[
\chi_{E^2}^{\text{PM-Newton}}(p_\infty, j) = \frac{45 \pi}{16} \mu_s^{(2)}(p_\infty) \frac{P^4}{J^6},
\]

and

\[
\Delta \Phi_{E^2}^{\text{PM-Newton}}(p_\infty, j) = \frac{45 \pi}{8} \mu_s^{(2)}(p_\infty) \frac{P^4}{J^6}.
\]

We are going to compare these expressions to the corresponding complete Newtonian-level predictions for the (quadrupolar) tidal contributions, \( \chi_{E^2}^{\text{Newton}} \) and \( \Delta \Phi_{E^2}^{\text{Newton}} \), to the scattering angle and to periastron precession.

As far as we know, while \( \Delta \Phi_{E^2}^{\text{Newton}}(p_\infty, j) \) is known from classic works in Newtonian gravity \( \text{[13]} \), the corresponding scattering angle \( \chi_{E^2}^{\text{Newton}}(p_\infty, j) \) has never been obtained in the literature. We are going to derive \( \chi_{E^2}^{\text{Newton}}(p_\infty, j) \), and the corresponding \( \Delta \Phi_{E^2}^{\text{Newton}}(p_\infty, j) \), and then compare these Newtonian-level results to the above 1PM \( \cap \) Newton expressions (6.14), (6.15).

To derive \( \chi_{E^2}^{\text{Newton}}(p_\infty, j) \) (and the corresponding \( \Delta \Phi_{E^2}^{\text{Newton}}(p_\infty, j) \)) we must consider the effect of adding a \( O(1/R^6) \) perturbation to the Newtonian potential. We can easily do that within the EOB framework by considering a squared effective EOB Hamiltonian of the form

\[
(H_{\text{eff}})^{\text{Newton}} = 1 + p^2 + j^2 u^2 - 2u - C u^6. \tag{6.16}
\]

Such an effective EOB Hamiltonian is obtained from Eq. (6.1) by treating the Schwarzschild piece of the mass-shell contraint to leading Newtonian order (i.e. using \( A = 1 - 2u \) and \( B = 1 \)), by neglecting the 2PM correction \( Q_2 \), and by adding only the leading-order \( O(u^6) \) tidal term from \( Q_{\text{tidal}} \). From Eqs. (6.2) and (6.19), the coefficient \( C \) in front of \( -u^6 \) is the Newtonian limit of \( q_{E^2}^2 (\hat{H}_{\text{eff}}) \), namely

\[
C = 3 \mu_s^{(2)}. \tag{6.17}
\]

The conserved energy of \( (H_{\text{eff}})^{\text{Newton}} - 1 \) is \( \gamma^2 - 1 = p_\infty^2 \).

We can then apply the same approach as above to the effective Hamiltonian (6.14). Instead of Eq. (6.5), we now get the Newtonian-level tidal action

\[
\hat{S}_{\text{tidal}}^{\text{Newton}} = -\int_0^{u+} \frac{\hat{Q}_{\text{tidal}}(u) du}{u^2 \sqrt{p_\infty^2 - j^2 u^2 + 2u}}, \tag{6.18}
\]

where \( \hat{Q}_{\text{tidal}}(u) = -C u^6 \), and where \( u_+ \) is the positive root of the squared denominator.

The computation of the integral (6.18) is elementary. In terms of the above-defined \( \epsilon \), Eq. (6.12), it can be rewritten as

\[
\hat{S}_{\text{tidal}}^{\text{Newton}} = C \frac{P^4}{J^6} I_4(\epsilon), \tag{6.19}
\]

where

\[
I_4(\epsilon) = \int_0^{x_+} \frac{x^4 dx}{\sqrt{1 - x^2 + 2\epsilon x}}, \tag{6.20}
\]

with \( x_+ = \epsilon + \sqrt{1 + \epsilon^2} \). The explicit value of \( I_4(\epsilon) \) is

\[
I_4(\epsilon) = \left( \frac{3}{8} + \frac{15}{4} \epsilon^2 + \frac{35}{8} \epsilon^4 \right) B(\epsilon) - \frac{55}{24} \epsilon + \frac{35}{8} \epsilon^3, \tag{6.21}
\]

with \( B(\epsilon) \equiv \arctan(\epsilon) + \frac{\pi}{2} \).

From the action (6.19), one computes \( \chi \) by differentiating with respect to \( j \):

\[
\chi_{E^2}^{\text{Newton}}(p_\infty, j) = -\frac{\partial}{\partial j} \hat{S}_{\text{tidal}}^{\text{Newton}}(p_\infty, j). \tag{6.22}
\]

This leads to the following result for \( \chi \),

\[
\chi_{E^2}^{\text{Newton}}(p_\infty, j) = C \frac{P^4}{J^6} \left[ 5 I_4(\epsilon) + \epsilon \frac{\partial I_4(\epsilon)}{\partial \epsilon} \right], \tag{6.23}
\]

whose explicit form reads

\[
\chi_{E^2}^{\text{Newton}}(p_\infty, j) = C \frac{P^4}{J^6} \left[ \left( \frac{15}{8} + \frac{105}{4} \epsilon^2 + \frac{315}{8} \epsilon^4 \right) B(\epsilon) + \epsilon \left( \frac{113}{420} + \frac{247}{90} \epsilon^2 + \frac{315}{16} \epsilon^4 \right) \right]. \tag{6.24}
\]

This expression is a complicated function of \( \epsilon \). At this stage, one must remark that \( \epsilon = G m_1 m_2 / (p_\infty P_\odot) \) is of order \( O(G) \). Our previous 1PM result was obtained at leading order in the expansion in powers of \( G \). This corresponds to taking the leading order in the expansion of \( \chi_{E^2}^{\text{Newton}} \) in powers of \( \epsilon \). Alternatively (and more physically), we can remark that the quantity \( \epsilon \), defined above by Eq. (6.13), is the Newtonian eccentricity of the orbit, corresponding to the existence of two roots of \( p_\infty^2 \) (corresponding to the squared denominator of Eq. (6.18)).

More precisely, \( \epsilon^2 \) is larger than 1 when \( p_\infty \) is real, so that \( p_\infty^2 > 0 \) (hyperbolic motion), and smaller than 1 when \( p_\infty \) is purely imaginary, so that \( p_\infty^2 < 0 \) (elliptic motion). The leading order PM approximation (small scattering angle) corresponds to \( \epsilon \to \infty \), which indeed corresponds to \( \epsilon \to 0 \).

In the \( \epsilon \to 0 \) limit, it is enough to use the first term in the expansion of \( I_4(\epsilon) \):

\[
I_4(\epsilon) = \frac{3}{16} \pi + \frac{8}{3} \epsilon + \frac{15}{8} \epsilon^2 + 8 \epsilon^3 + \frac{35}{16} \epsilon^4 + O(\epsilon^5), \tag{6.25}
\]

namely \( I_4(0) = \frac{3}{16} \pi \). Inserting this value in Eq. (6.23) (or taking the \( \epsilon \to 0 \) limit of (6.24)) yields

\[
\chi_{E^2}^{\text{Newton large } \epsilon} = \frac{15}{16} C \frac{P^4}{J^6} = \frac{45}{16} \mu_s^{(2)}(p_\infty) \frac{P^4}{J^6}, \tag{6.26}
\]

in agreement with Eq. (6.14). Let us now consider the periastron advance \( \Delta \Phi \), as obtained from Eq. (6.10). The analytic continuation from
This replacement drastically simplifies $I_4(e)$ with respect to imaginary axis. In addition, Eq. (6.10) involves taking (twice) the even part with respect to $\epsilon$, and therefore

$$J_4(\epsilon) = I_4(\epsilon) + I_4(-\epsilon).$$

This replacement drastically simplifies $I_4(e)$ into

$$J_4(\epsilon) = \pi \left( 3 \frac{15}{8} + 35 \frac{4}{8} \right).$$

This then yields

$$\Delta \Phi_{\text{Newton}}^{E^2}(p_{\infty}, j) = \frac{45 \pi \hat{\mu}^{(2)}_{\mu}}{j^{10}} \left[ 1 + 3 e^2 + \frac{1}{8} e^4 \right].$$

The analytic continuation to imaginary values of $p_{\infty}$ is then unambiguous because the above expression is a function of $p_{\infty}^2$.

Finally, using $p_{\infty}^2 j^2 = e^2 - 1$, and $C = 3 \hat{\mu}^{(2)}_{\mu}$, one gets

$$\Delta \Phi_{\text{Newton}}^{E^2}(p_{\infty}, j) = \frac{45 \pi \hat{\mu}^{(2)}_{\mu}}{j^{10}} \left[ 1 + 3 e^2 + \frac{1}{8} e^4 \right].$$

On the one hand, this expression agrees with the classic Newtonian result of Ref. [78], when using the link [50]

$$G \mu_A^{(2)} = \frac{2}{3} \epsilon_A^{(2)} R_A.$$

On the other hand, if we formally consider the large eccentricity limit, $e \gg 1$, of the expression (6.20) (though it is physically defined only when $e < 1$), one gets

$$\Delta \Phi_{\text{Newton}}^{E^2}(p_{\infty}, j) \quad e \gg 1 \quad \frac{45 \pi}{8} \hat{\mu}_A^{(2)} \frac{e^4 j^{10}}{j^6}.$$

On the second line, we have used the definition (6.13) of $e^2$, leading to $e^4 \approx p_{\infty}^2 j^4$ in the large-eccentricity limit. We see that the latter final expression agrees with the formal 1PM \cap Newton expression (6.13).

This exercise has highlighted the fact that, in spite of the simple formal link (6.19), there is a long theoretical distance separating the PN-expansion of the scattering angle (valid in the large eccentricity limit, $e \gg 1$), and the PN-expansion of the periastron advance (meaningful only for $e < 1$).

VII. CONCLUDING REMARKS

We have extended the post-Minkowskian approach to the computation of tidally interacting binary systems. Our computation used the effective field theory description of tidally interacting bodies, and was simplified by using general properties of perturbed actions. We computed several tidal invariants (notably the integrated quadrupolar and octupolar actions) at the first post-Minkowskian order, and derived the corresponding contributions to the scattering angle, and to the periastron advance. We showed also how to transcribe our post-Minkowskian tidal results in the effective one body formalism, using the same type of energy gauge that was recently used in the post-Minkowskian approach to the dynamics of point masses. It would be interesting to extend our computation to higher post-Minkowskian levels so as, notably, to clarify the high-energy behavior of the tidal interaction of two bodies.

Appendix A: Four velocities, momenta and center-of-mass at the Minkowskian level

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FIG. 1: The figure shows the choice of the axes (adapted to $u^\mu_{\infty}$, and to the vectorial impact parameter $b^\mu$), and the origin (located on $L_2$), of the coordinates we use in the text. The $z$ coordinate of any point is assumed to be 0 and is omitted.

In the Minkowskian discussion of the two-body prob-
lem the following four-vectors are relevant

\[ u_1 = \partial_t, \]
\[ u_2 = \gamma \partial_t - \sqrt{\gamma^2 - 1} \partial_y, \]
\[ U = \frac{m_1 u_1 + m_2 u_2}{E_{\text{real}}} = \frac{X_1}{h} u_1 + \frac{X_2}{h} u_2, \]  
(A1)

where \( u_1 \) is the four-velocity of body 1, \( u_2 \) that of body 2, \( U \) that of the c.m. frame, and

\[ E_{\text{real}} = \sqrt{m_1^2 + m_2^2 + 2m_1m_2 \gamma} = M h(\gamma, \nu). \]  
(A2)

Here we used coordinates adapted to \( \mathcal{L}_1 \).

One can complete the unit timelike vectors \( u_1, u_2 \) and \( U \) by corresponding spatial, orthonormal vectorial frames (respectively orthogonal to \( u_1, u_2 \) and \( U \)) as follows

1. Spatial, orthonormal frame completing \( u_1 \)

\[ e(u_1)_1 = \partial_x, \quad e(u_1)_2 = \partial_y, \quad e(u_1)_3 = \partial_z. \]  
(A3)

2. Spatial, orthonormal frame completing \( u_2 \)

\[ e(u_2)_1 = \partial_x, \quad e(u_2)_2 = -\sqrt{\gamma^2 - 1} \partial_y + \gamma \partial_y, \quad e(u_2)_3 = \partial_z. \]  
(A4)

3. Spatial, orthonormal frame completing \( U \)

\[ e(U)_1 = \partial_x, \quad e(U)_2 = -\sinh \alpha \partial_t + \cosh \alpha \partial_y, \quad e(U)_3 = \partial_z, \]  
(A5)

where

\[ \sinh \alpha = \frac{m_2 \sqrt{\gamma^2 - 1}}{E_{\text{real}}}, \quad \cosh \alpha = \frac{m_1 + m_2 \gamma}{E_{\text{real}}}. \]  
(A6)

Note the expressions of \( \sinh \alpha \) and \( \cosh \alpha \) in terms of \( j \) and of the impact parameter \( b \):

\[ \sinh \alpha = \frac{G m_2 j}{b}, \quad \cosh \alpha = \frac{G j m_1 + m_2 \gamma}{\sqrt{\gamma^2 - 1}}, \]  
(A7)

implying

\[ \frac{G m_2 j}{b} = \sqrt{\gamma^2 - 1} \cosh \alpha - \gamma \sinh \alpha. \]  
(A8)

With this notation the center-of-mass 4-velocity \( U \) reads

\[ U = \cosh \alpha \partial_t - \sinh \alpha \partial_y. \]  
(A9)

These frames are obtained by boosting the spatial frame of \( u_1 \) into the local rest spaces of \( u_2 \) and \( U \). The spatial frame associated with \( U \) has the peculiarity that one leg of the triad \([e(U)_2]\) is aligned with the direction of the spatial momentum of each of the particles.

The spacetime vectorial frame \((U, e(U)_1, e(U)_2, e(U)_3)\) is the c.m. frame. Decomposing \( P_1 = m_1 u_1 \) and \( P_2 = m_2 u_2 \) along this frame gives

\[ P_1 = m_1 u_1 = E_1 U + P e(U)_2, \]
\[ P_2 = m_2 u_2 = E_2 U - P e(U)_2, \]  
(A10)

with \( E_1 + E_2 = E_{\text{real}} \), and

\[ E_1 = \sqrt{m_1^2 + P^2} = m_1 \cosh \alpha = \frac{m_1 m_2 \gamma + m_1}{E_{\text{real}}}, \]
\[ E_2 = \sqrt{m_2^2 + P^2} = m_2 \cosh \alpha' = \frac{m_2 m_1 \gamma + m_2}{E_{\text{real}}}, \]
\[ P = \frac{m_1 m_2 \sqrt{\gamma^2 - 1}}{E_{\text{real}}} = m_1 \sinh \alpha = m_2 \sinh \alpha'. \]  
(A11)

The Mandelstam variable \( s \) associated with \( P_1 \) and \( P_2 \) reads

\[ s = -(P_1 + P_2)^2 = E_{\text{real}}^2. \]  
(A12)

We also recall the following definitions for the spatial four-velocity\(^6\) of each body seen in the rest frame of the other one,

\[ u_{21} = P(u_1) u_2 = -\sqrt{\gamma^2 - 1} \partial_y, \]
\[ u_{12} = P(u_2) u_1 = \sqrt{\gamma^2 - 1} e(u_2)_2, \]  
(A13)

where we recall that \( P(u) = I + u \otimes u \) denotes the projector orthogonal to the unit timelike vector \( u \). Here \( e(u_2)_2 \), defined in Eq. (A4), is the boosted \( y \)-axis in the local rest space of \( u_2 \) with \( u_2 \cdot e(u_2)_2 = 0 \). Therefore,

\[ u_{12} = -(\gamma^2 - 1) \partial_t + \gamma \sqrt{\gamma^2 - 1} \partial_y, \]  
(A14)

and

\[ u_1 \cdot u_{12} = u_2 \cdot u_{21} = \gamma^2 - 1. \]  
(A15)

Note that \( |u_{12}| = |u_{21}| = \sqrt{\gamma^2 - 1} \) is equal to the EOB asymptotic momentum \( p_{\infty} \).

For completeness, let us write the parametric equations of the two worldlines in the coordinate system associated with \( \mathcal{L}_1 \)

\[ z_1(\tau_1) = \tau_1 \partial_t + b \partial_x, \]
\[ z_2(\tau_2) = \gamma \tau_2 \partial_t - \sqrt{\gamma^2 - 1} \tau_2 \partial_y. \]  
(A16)

We can also define coordinates, \((t_{\text{cm}}, x_{\text{cm}}, y_{\text{cm}}, z_{\text{cm}})\), adapted to the c.m. frame. They are related to the coordinates \((t, x, y, z)\) adapted to \( \mathcal{L}_1 \) via

\[ t_{\text{cm}} = \cosh \alpha t + \sinh \alpha y, \]
\[ x_{\text{cm}} = x - \frac{E_1}{E_{\text{real}}} b, \]
\[ y_{\text{cm}} = \sinh \alpha t + \cosh \alpha y, \]
\[ z_{\text{cm}} = z, \]  
(A17)

\(^6\) We denote some spacelike four-vectors by a boldface.
with inverse
\[
\begin{align*}
t &= \cosh \alpha t_{\text{cm}} - \sinh \alpha y_{\text{cm}}, \\
x &= x_{\text{cm}} + \frac{E_1}{E_{\text{real}}} b, \\
y &= -\sinh \alpha t_{\text{cm}} + \cosh \alpha y_{\text{cm}}, \\
z &= z_{\text{cm}} ,
\end{align*}
\] (A18)

As is standard, the origin of these coordinates has been taken as the center of energy of \(z_1\) and \(z_2\), when viewed in the c.m. frame, and at the same c.m. time \(t_{\text{cm}}\).

Finally, the Pauli-Lubanski pseudo-vector
\[
L_{\dot{a}} \equiv \eta_{\alpha \beta \mu \nu} U^\alpha(\tau_1) P_{\mu}^\nu(\tau_1) + z_{\text{cm}}^\alpha(\tau_2) P_{\alpha}^\nu(\tau_2)
\] (A19)

is independent of \(\tau_1\) and \(\tau_2\), and has, as only nonzero component,
\[
L_3 = b P = P_3 = G_{m_1 m_2} j \text{ both in the } \mathcal{L}_1 \text{ coordinate system, and in the c.m. one.}
\]
A. Cristofoli, “Post-Minkowskian Hamiltonians in
B. Maybee, D. O’Connell and J. Vines, “Observables
N. E. J. Bjerrum-Bohr, A. Cristofoli, P. H. Damgaard
J. Blümlein, A. Maier, P. Marquard, G. Schäfer and
G. Kalin and R. A. Porto, “From Boundary Data to
P. Di Vecchia, S. G. Naculich, R. Russo, G. Veneziano and C. D. White, “A tale of two exponents in \(N = 8\) supergravity at subleading level,” arXiv:1911.11710 [hep-th].
G. Kälin and R. A. Porto, “From Boundary Data to
T. Damour, M. Soffel and C. m. Xu, “General relativistic
T. Damour and C. D. White, “A tale of two exponentiations in


[28] N. E. J. Bjerrum-Bohr, A. Cristofoli and P. H. Damgaard, “Post-Minkowskian Scattering Angle in Einstein Gravity,” arXiv:1909.05600 [hep-th].
[29] P. Di Vecchia, A. Luna, S. G. Naculich, R. Russo, G. Veneziano and C. D. White, “A tale of two exponents in \(N = 8\) supergravity,” Phys. Lett. B 798, 134927 (2019) [arXiv:1908.05003 [hep-th]].
[30] P. Di Vecchia, S. G. Naculich, R. Russo, G. Veneziano and C. D. White, “A tale of two exponents in \(N = 8\) supergravity at subleading level,” arXiv:1911.11710 [hep-th].
[31] G. Kälin and R. A. Porto, “From Boundary Data to
[32] J. Blümlein, A. Maier, P. Marquard, G. Schäfer and C. Schneider, “From Momentum Expansions to Post-Minkowskian Hamiltonians by Computer Algebra Algorithms,” arXiv:1911.04411 [gr-qc].
[33] N. E. J. Bjerrum-Bohr, A. Cristofoli and P. H. Damgaard, “Generalized Hamiltonians in Modified Theories of Gravity,” Phys. Lett. B 800, 135095 (2020) doi:10.1016/j.physletb.2019.135095.
[34] P. Di Vecchia, S. G. Naculich, R. Russo, G. Veneziano and C. D. White, “Scalar-Graviton Amplitudes,” arXiv:1908.09755 [hep-th].
[35] B. Maybee, D. O’Connell and J. Vines, “Observables and amplitudes for spinning particles and black holes,” arXiv:1906.09200 [hep-th].
[36] A. Cristofoli, “Post-Minkowskian Hamiltonians in Modified Theories of Gravity,” Phys. Lett. B 800, 135095 (2020) doi:10.1016/j.physletb.2019.135095 [arXiv:1906.05209 [hep-th]].
[37] M. Ciafaloni, D. Colferai and G. Veneziano, “Infrared features of gravitational scattering andradiation in the eikonal approach,” Phys. Rev. D 99, no. 6, 066008 (2019) doi:10.1103/PhysRevD.99.066008 [arXiv:1812.08137 [hep-th]].
[38] T. Damour, M. Soffel and C. m. Xu, “General relativistic celestial mechanics. 3. Rotational equations of motion,” Phys. Rev. D 47, 3124 (1993). doi:10.1103/PhysRevD.47.3124.
[39] T. Damour, M. Soffel and C. m. Xu, “General relativistic
celestial mechanics. 4: Theory of satellite motion,” Phys.
Rev. D 49, 618 (1994). doi:10.1103/PhysRevD.49.618.
[40] J. E. Vines and E. E. Flanagan, “Post-1-
Newtonian tidal interactions in binary systems,” Phys. Rev. D 88, 024046 (2013) doi:10.1103/PhysRevD.88.024046 [arXiv:1009.4919 [gr-qc]].
[41] M. Shibata, “Fully general relativistic simulation of coalescing binary neutron stars: Preparatory tests,” Phys. Rev. D 60, 104052 (1999) doi:10.1103/PhysRevD.60.104052 [gr-qc/9908027].
[42] L. Baiotti, B. Giacomazzo and L. Rezzolla, “Accurate evolutions of inspiralling neutron-star binaries: prompt and delayed collapse to black hole,” Phys. Rev. D 78, 084033 (2008) doi:10.1103/PhysRevD.78.084033 [arXiv:0804.1054 [gr-qc]].
[43] S. Bernuzzi, A. Nagar, M. Thierfelder and B. Brugmann, “Tidal effects in binary neutron star coalescence,” Phys. Rev. D 86, 044030 (2012) doi:10.1103/PhysRevD.86.044030 [arXiv:1205.3408 [gr-qc]].
[44] D. Kavanagh, A. C. Ottewill and B. Wardell, “Analytical higher-order post-Newtonian expansions for extreme mass ratio binaries,” Phys. Rev. D 92, no. 8, 084025 (2015) doi:10.1103/PhysRevD.92.084025 [arXiv:1505.02344 [gr-qc]].
[45] P. Nolan, C. Kavanagh, S. R. Dolan, A. C. Ot-
tewill, N. Warburton and B. Wardell, “Octupolar invariants for compact binaries on quasicircular orbits,” Phys. Rev. D 92, no. 12, 123008 (2015) doi:10.1103/PhysRevD.92.123008 [arXiv:1505.04447 [gr-qc]].
[46] T. Damour and A. Nagar, “Effective One Body
description of tidal effects in inspiralling compact
binary systems,” Phys. Rev. D 81, 084016 (2010) doi:10.1103/PhysRevD.81.084016 [arXiv:0911.5041 [gr-qc]].
[47] A. Buonanno and T. Damour, “Effective one-body approach to general relativistic two-body dynamics,” Phys. Rev. D 59, 84006 (1999) [arXiv:gr-qc/9811091].
[48] A. Buonanno and T. Damour, “Transition from inspiral to plunge in binary black hole coalescences,” Phys. Rev. D 62, 064015 (2000) [arXiv:gr-qc/0001013].
[49] T. Damour, P. Jaranowski, and G. Schäfer, “On the determination of the last stable orbit for circular general relativistic binaries at the third post-Newtonian approximation,” Phys. Rev. D 62, 084011 (2000) arXiv:gr-qc/0005054.
[50] D. Bini, T. Damour, and G. Faye, “Effective action approach to higher-order relativistic tidal interactions in binary systems and their effective one body description,” Phys. Rev. D 85, 124034 (2012). [arXiv:1202.3565 [gr-qc]].
[51] D. Bini and T. Damour, “Dettweiler’s gauge-invariant redshift variable: analytic determination of the ninene-and-a-half post-Newtonian self-force contributions,” arXiv:1502.02450 [gr-qc].
[52] D. Bini and T. Damour, “Analytic determination of high-
order post-Newtonian self-force contributions to gravitational spin precession,” arXiv:1503.01272 [gr-qc].
[53] T. Damour and G. Esposito-Farèse, “Testing gravity to second post-Newtonian order: A Field theory approach,” Phys. Rev. D 53, 5541 (1996). [gr-qc/9506003].
[54] T. Damour and G. Esposito-Farèse, “Gravitational wave
versus binary-pulsar tests of strong field gravity,” Phys. Rev. D 58, 042001 (1998). [gr-qc/9803031].
[55] W. D. Goldberger and I. Z. Rothstein, “An Effective field theory of gravity for extended objects,” Phys. Rev. D 73, 104029 (2006). [hep-th/0409156].
[56] T. Damour, Gravitational Radiation And The
Modeling Of Compact Bodies, in Gravitational Radiation, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), pp. 59-144.
[57] T. Hinderer, “Tidal Love numbers of neutron stars” Astrophys. J. 677, 1216 (2008); erratum Astrophys. J. 697, 964 (2009).
[58] T. Damour and A. Nagar, “Relativistic tidal properties of neutron stars,” Phys. Rev. D 80, 044035 (2009) doi:10.1103/PhysRevD.80.044035 [arXiv:0906.0096].
[60] T. Binnington and E. Poisson, “Relativistic theory of tidal Love numbers” Phys. Rev. D **80**, 084018 (2009).
[61] D. Bini and T. Damour, “Gravitational scattering of two black holes at the fourth post-Newtonian approximation,” Phys. Rev. D **96**, no. 6, 064021 (2017) doi:10.1103/PhysRevD.96.064021 [arXiv:1706.06877 [gr-qc]].
[62] E. P. Wigner, “Lower Limit for the Energy Derivative of the Scattering Phase Shift,” Phys. Rev. **98**, 145 (1955). doi:10.1103/PhysRev.98.145
[63] T. Damour, “Classical and quantum scattering in post-Minkowskian gravity,” [arXiv:1912.02139 [gr-qc]].
[64] J. Steinhoff, T. Hinderer, A. Buonanno and A. Taracchini, “Dynamical Tides in General Relativity: Effective Action and Effective-One-Body Hamiltonian,” Phys. Rev. D **94**, no. 10, 104028 (2016) doi:10.1103/PhysRevD.94.104028 [arXiv:1608.01907 [gr-qc]].
[65] Q. Henry, G. Faye and L. Blanchet, “Tidal effects in the equations of motion of compact binary systems to next-to-next-to-leading post-Newtonian order,” [arXiv:1912.01920 [gr-qc]].
[66] D. Amati, M. Ciafaloni and G. Veneziano, “Higher Order Gravitational Deflection and Soft Bremsstrahlung in Planckian Energy Superstring Collisions,” Nucl. Phys. B **347**, 550 (1990). doi:10.1016/0550-3213(90)90375-N
[67] R.A. Porto, Phys. Rev. D **73**, 104031 (2006).
[68] M. Levi and J. Steinhoff, “An effective field theory for gravitating spinning objects in the post-Newtonian scheme,” [arXiv:1501.04956 [gr-qc]].
[69] R.A. Porto and I.Z. Rothstein, Phys. Rev. Lett. **97**, 021101 (2006).
[70] R.A. Porto and I.Z. Rothstein, Phys. Rev. D **78**, 044013 (2008).
[71] M. Levi and J. Steinhoff, “Leading order finite size effects with spins for inspiralling compact binaries,” [arXiv:1410.2601 [gr-qc]].
[72] Xiao-He Zhang, “Multipole expansions of the general-relativistic gravitational field of the external universe,” Phys. Rev. D **34**, 991 (1986).
[73] T. Damour, M. Soffel and C.-m. Xu, “General relativistic celestial mechanics. 1. Method and definition of reference systems,” Phys. Rev. D **43**, 3272 (1991).
[74] T. Damour, M. Soffel and C.-m. Xu, “General relativistic celestial mechanics. 2. Translational equations of motion,” Phys. Rev. D **45**, 1017 (1992).
[75] S. Taylor and E. Poisson, “Nonrotating black hole in a post-Newtonian tidal environment,” Phys. Rev. D **78**, 084016 (2008). [arXiv:0806.3052 [gr-qc]].
[76] N. K. Johnson-McDaniel, N. Yunes, W. Tichy, and B. J. Owen, “Conformally curved binary black hole initial data including tidal deformations and outgoing radiation,” Phys. Rev. D **80**, 124039 (2009). [arXiv:0907.0891 [gr-qc]].
[77] P. C. Aichelburg and R. U. Sexl, “On the Gravitational field of a massless particle,” Gen. Rel. Grav. **2**, 303 (1971). doi:10.1007/BF00758149
[78] T. E. Sterne, “Apsidal Motion in Binary Stars,” MNRAS, **99**, 451 (1939)