Note on the real rootedness of polynomials

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Abstract. In this paper, by the generalized Bell umbra and Rolle’s theorem, we give some results on the real rootedness of polynomials. Some applications on partition polynomials and the sigma polynomials of graphs are given.

Keywords. Polynomials with real zeros, Generalized Bell umbra, Partition polynomials, Exponential polynomials.

2010 MSC. 11B73, 30C15.

1 Introduction

The real rootedness of polynomials has attracted researchers great interest. One of the reasons is that any polynomial of real zeros implies the log-concavity and the unimodality of its coefficients, which appear in various fields of mathematics, see [6, 17]. In this paper, we investigate the properties of the generalized Bell umbra and Rolle’s theorem to give a result on the real rootedness of polynomials. Partial and auxiliary results on partition polynomials and exponential polynomials are also given. For the partition polynomials, we consider the σ-polynomials of graphs and a class of polynomials linked to the partial r-Bell polynomials. The mathematical tools used here are the generalized Bell umbra and Rolle’s theorem. To use them, recall that the n-th Bell polynomial $B_n (x)$ and the n-th r-Bell polynomial $B_{n,r} (x)$ can be defined by Dobinski’s formula:

$$B_n (x) = e^{-x} \sum_{j \geq 0} j^n x^j / j!,$$

and $B_0^n (x) = B_n (x)$ be the generalized Bell umbra introduced by Sun et al. [18].

For further information about umbral calculus on Bell polynomials, one can also see [4, 9, 10]. To use later, recall that for any polynomial $f$ and integer $n \geq 0$, it is known [18] that

$$(B_x)^n f (B_x) = x^n f (B_x + n).$$

In particular, we have

$$B_{n+1}^n = x (B_x + 1)^n$$

and

$$(B_x)^n = x^n.$$

where $(x)_n = x (x-1) \cdots (x-n+1)$ if $n \geq 1$ and $(x)_0 = 1.$

The paper is organized as follows. In the next section we give a result on polynomials with real zeros. In the third section we present an application on the σ-polynomials and another on a class of polynomials linked to the partial r-Bell polynomials [15]. In the last section we give an application of a class of exponential polynomials.

2 Polynomials of real zeros via generalized Bell umbra

Let $RZ$ be the set of real polynomials having only real zeros.

The principal main result of this paper is the following theorem.
Theorem 1 Let $r$ be a non-negative integer and let $f$ be a polynomial with real coefficients such that $f(\mathbf{B}_x) \in \mathbb{R}Z$. Then, for any non-negative integers $r_1, \ldots, r_q$, there holds

$$(\mathbf{B}_x)_{r_p} \cdots (\mathbf{B}_x)_{r_1} f(\mathbf{B}_x) \in \mathbb{R}Z.$$ 

Proof. From the identity $f(\mathbf{B}_x) = e^{-x} \sum_{k \geq 0} f(k) \frac{x^k}{k!}$ we get

$$\frac{d}{dx} (e^{x} f(\mathbf{B}_x + r - 1)) = \frac{d}{dx} \left( \sum_{k \geq 0} f(k + r - 1) \frac{x^k}{k!} \right) = \sum_{k \geq 0} f(k + r) \frac{x^k}{k!} = e^{x} f(\mathbf{B}_x + r) .$$

The proof can be obtained by induction on $r$ by application of Rolle’s theorem on the function $e^{x} f(\mathbf{B}_x + r - 1)$. More generally, since $g(x) := (\mathbf{B}_x)_{r_1} f(\mathbf{B}_x) \in \mathbb{R}Z$, it follows $(\mathbf{B}_x)_{r_2} g(\mathbf{B}_x) = (\mathbf{B}_x)_{r_2} (\mathbf{B}_x)_{r_1} f(\mathbf{B}_x) \in \mathbb{R}Z$ and so on.

Example 1 Let $f(x) = x^n$, then $f(\mathbf{B}_x) = \mathcal{B}_n(x)$ which has only real zeros. We deduce the known result \cite{7} (see also \cite{4, 14})

$$(\mathbf{B}_x)_{r_p} \cdots (\mathbf{B}_x)_{r_1} \mathcal{B}_n = x^{\max(r_1, \ldots, r_p)} \mathcal{B}_{n; r_1, \ldots, r_p}(x) \in \mathbb{R}Z.$$ 

Example 2 Let $f(x) = (x)_n$, then $f(\mathbf{B}_x) = x^n \in \mathbb{R}Z$. It follows that the polynomial

$$(\mathbf{B}_x)_{r_p} f(\mathbf{B}_x) = x^r (\mathbf{B}_x + r)_n = x^r \sum_{k=0}^{\min(n, r)} \binom{n}{k} \frac{r!}{(r-k)!} x^{n-k}$$

is in $\mathbb{R}Z$. Also, the polynomial

$$(\mathbf{B}_x)_{r_p} \cdots (\mathbf{B}_x)_{r_1} \mathcal{B}_n = x^{r} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathcal{B}_{k; r_1, \ldots, r_p}(x)$$

is in $\mathbb{R}Z$, where $\binom{n}{k}$ are the unsigned Stirling numbers of the first kind.

Example 3 Let $f(x) = (x + n - 1)_n$, then $f(\mathbf{B}_x) = (\mathbf{B}_x + n - 1)_n = \mathcal{L}_n(x) \ [8]$ is the $n$-th Lah polynomial which is in $\mathbb{R}Z$. It follows that the polynomial

$$(\mathbf{B}_x)_{r_p} \cdots (\mathbf{B}_x)_{r_1} (\mathbf{B}_x + n - 1)_n = x^r \sum_{k=0}^{n} (-1)^{n-k} \left[ \frac{2n-1}{k+n-1} \right]_{n-1} \mathcal{B}_{k; r_1, \ldots, r_p}(x)$$

is in $\mathbb{R}Z$, where $\binom{n}{k}_{r}$ are the unsigned $r$-Stirling numbers of the first kind.

3 Partition polynomials with real zeros

We present in this section two applications of Theorem 1, one on the $\sigma$-polynomial associated to another on a class of polynomials linked to the partial $r$-Bell polynomials.

For the first application, recall that a $\lambda$-coloring of $G$, $\lambda \in \mathbb{N}$, is a mapping $f : V \rightarrow \{1, 2, \ldots, \lambda\}$ where $f(u) \neq f(v)$ whenever the vertices $u$ and $v$ are adjacent in $G$. Two $\lambda$-colorings $f$ and $g$ of $G$ are distinct if $f(x) \neq g(x)$ for some vertex $x$ in $G$, and, the number of $\lambda$-colorings of $G$ is called the chromatic polynomial $P(G, \lambda)$. The chromatic polynomial can be defined as $f(\lambda) = \sum_{k=0}^{n} \alpha_k(G) (\lambda)_k$, where $\alpha_k(G)$ is the number of ways of partitioning $V$ into $i$ nonempty sets. The $\sigma$-polynomial associated to $G$ is $\sum_{k=0}^{n} \alpha_k(G) x^k = \sum_{k=0}^{n} \alpha_k(G) (\mathbf{B}_x)_k = f(\mathbf{B}_x)$.

For more information about chromatic polynomials, see \cite{8}. 
Corollary 2 If the σ-polynomial $f(B_x)$ of a graph $G$ is in $\mathbb{R}Z$, then the σ-polynomial $(B_x)_{r_1} \cdots (B_x)_{r_p} f(B_x)$ of the graph $G \cup K_{r_1} \cup \cdots \cup K_{r_p}$ is in $\mathbb{R}Z$, where $K_r$ is the complete graph of $r$ vertices.

Example 4 A tree $T_n$ of $n \geq 1$ vertices has $f(x) = x(x-1)^{n-1}$. By the identity $(B_x)_n f(B_x) = x^n f(B_x + n)$, the σ-polynomial is to be $f(B_x) = B_x (B_x - 1)^{n-1} = xB_x^{n-1} = xB_{n-1}(x)$ which is in $\mathbb{R}Z$. Then, the σ-polynomial of the graph $T_n \cup K_r$ is in $\mathbb{R}Z$, and is to be

$$x^r f(B_x + r) = x^r (B_x + r)(B_x + r - 1)^{n-1} = x^r [xB_{n-1,r}(x) + rB_{n-1,r-1}(x)].$$

For the second application, let the $(n,k)$-th partial $r$-Bell polynomial

$$B_{n+r,k+r}^{(r)}(a;b) := B_{n+r,k+r}^{(r)}(a_1,a_2,\ldots;b_1,b_2,\ldots)$$

introduced by Mihoubi et al. [15] (see also [16]) and defined by

$$\sum_{n \geq k} B_{n+r,k+r}^{(r)}(a;b) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{j \geq 1} a_j \frac{t^j}{j!} \right)^k \left( \sum_{j \geq 0} b_j \frac{t^j}{j!} \right)^r.$$

This polynomial presents an extension of the $(n,k)$-th partial Bell polynomial $B_{n,k}(a_1,a_2,\ldots) := B_{n,k}(a)$ introduced by Bell [2] and studied later by several authors, see for example [7] [12] [13]. Let $(a_n)$ and $(b_n)$ be two sequences of real numbers linked as follows

$$\varphi(t) = \sum_{n \geq 1} a_n \frac{t^n}{n!}, \quad 1 + \varphi(t) = \sum_{n \geq 0} b_{n+1} \frac{t^n}{n!}.$$

Here $b = e + La$, where $e = (1,0,0,\ldots)$, $a = (a_1,a_2,\ldots)$, and the sequence $(L^n a)$ is defined by $L^0 a = (a_1,a_2,\ldots)$, $La = (0,a_1,a_2,\ldots)$, $L^2 a = (0,0,a_1,a_2,\ldots)$ and so on.

Proposition 3 Let $V_{n,r}(x)$ and $V_n(x)$ be the polynomials defined by

$$V_{n,r}(x) = \sum_{k=0}^n B_{n+r,k+r}^{(r)}(a;e + La) x^k, \quad V_n(x) = V_{n,0}(x) = \sum_{k=0}^n B_{n,k}(a) x^k.$$

If $V_n(x) \in \mathbb{R}Z$, then $V_{n,r}(x) \in \mathbb{R}Z$.

Proof. Then, from [15] Th. 4] we have

$$\sum_{n \geq 0} V_{n,r}(x) \frac{t^n}{n!} = (1 + \varphi(t))^r \exp (x\varphi(t)).$$

For $f_n(x) = \sum_{k=0}^n B_{n,k}(a)(x)_k$ we get $f_n(B_x + r) = V_{n,r}(x)$. Indeed, we have

$$\sum_{n \geq 0} f_n(B_x + r) \frac{t^n}{n!} = \sum_{k \geq 0} (B_x + r)_h \sum_{n \geq k} B_{n,k}(a) \frac{t^n}{n!}$$

$$= \sum_{k \geq 0} \binom{B_x + r}{k} (\varphi(t))^k$$

$$= (1 + \varphi(t))^{B_x + r}$$

$$= (1 + \varphi(t))^r \sum_{n \geq 0} f_n(B_x) \frac{t^n}{n!}$$

$$= (1 + \varphi(t))^r \exp (x\varphi(t)).$$
which is the exponential generating function of the sequence \((V_{n,r}(x))\).
Hence, the application of Theorem 1 completes the proof.

**Corollary 4**  For \(a = (1,1,1,\ldots)\), the following polynomials are in \(\mathbb{RZ}\)
\[ V_{n,r}(x) = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)} (La:e + L^2a) x^k, \quad U_{n,r}(x) = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)} (L^2a:e + L^3a) x^k \]

**Proof.** The 2-associated and 3-associated Bell polynomials
\[ V_{n}(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}^{(2)} x^k = \sum_{k=0}^{n} B_{n,k}(La) x^k, \quad U_{n}(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}^{(3)} x^k = \sum_{k=0}^{n} B_{n,k}(L^2a) x^k \]
are in \(\mathbb{RZ}\), where
\[ \sum_{n \geq k} \left\{ \begin{array}{c} n \\ k \end{array} \right\}^{(2)} \frac{t^n}{n!} = \frac{1}{k!} \left( e^t - 1 - t \right)^k, \quad \sum_{n \geq k} \left\{ \begin{array}{c} n \\ k \end{array} \right\}^{(3)} \frac{t^n}{n!} = \frac{1}{k!} \left( e^t - 1 - t - \frac{t^2}{2} \right)^k, \]
see [5, 19]. So, the corollary follows from Proposition 3.

\qed

**4 Exponential polynomials with real zeros**

The following proposition presents an application of Theorem 1 given in [3].

**Theorem 5** Let \((A_n(x))\) be a sequence of polynomials defined by
\[ 1 + \sum_{n \geq 1} A_n(x) \frac{t^n}{n!} = \exp(\chi h(t)), \quad h(t) = \sum_{j \geq 1} a_j \frac{t^j}{j!}, \]
and let \((A_n^{(s)}(x))\) be a sequence of polynomials defined by
\[ A_n^{(0)}(x) = A_n(x), \quad A_n^{(s)}(x) = A_n^{(s-1)}(B_x), \quad s \geq 1. \]
Then
\[ 1 + \sum_{n \geq 1} A_n^{(s)}(x) \frac{t^n}{n!} = \exp \left( x \sum_{j \geq 1} A_j^{(s-1)}(1) \frac{t^j}{j!} \right), \quad s \geq 1. \]
Furthermore, if the sequence \(\left( A_n^{(s)}(1) \right) / (n-1)! \) is log-concave, then for \(x > 0\), the sequence \(\left( A_n^{(s)}(x) \right) / n! \) is log-convex and the sequence \(\left( A_n^{(s)}(x) \right) / n! \) is log-concave.

**Proof.** The desired identity is true for \(s = 1\) because we have
\[ \sum_{n \geq 0} A_n^{(1)}(x) \frac{t^n}{n!} = \sum_{n \geq 0} A_n^{(0)}(B_x) \frac{t^n}{n!} \]
\[ = \exp(B_x h(t)) \]
\[ = \sum_{n \geq 0} \frac{B_n(x)}{n!} (h(t))^n \]
\[ = \exp \left( x \sum_{n \geq 1} A_n^{(0)}(1) \frac{t^n}{n!} \right). \]
Assume it is true for \( s, s \geq 1 \). Then
\[
\sum_{n \geq 0} A^{(s+1)}_n(x) \frac{t^n}{n!} = \sum_{n \geq 0} A^{(s)}_n(B_x) \frac{t^n}{n!} \]
\[
= \exp \left( B_x \sum_{j \geq 1} A^{(s-1)}_j(1) \frac{t^j}{j!} \right) \]
\[
= \sum_{n \geq 0} \frac{B_n(x)}{n!} \left( \sum_{j \geq 1} A^{(s-1)}_j(1) \frac{t^j}{j!} \right)^n \]
\[
= \exp \left( x \left( \exp \left( \sum_{j \geq 1} A^{(s-1)}_j(1) \frac{t^j}{j!} \right) - 1 \right) \right) \]
\[
= \exp \left( x \sum_{n \geq 1} A^{(s)}_n(x) \frac{t^n}{n!} \right),
\]
which proves the induction step. To rest of the proof can be obtained by induction on \( s \) upon using Theorem 1 given by Bender-Canfield [3]. \( \square \)

**Example 5** For \( a_n = 1 \), we get Theorem 1 given in [4, Th. 1].

**Example 6** For \( a_n = (n-1)! \), \( n \geq 1 \), we get \( h(t) = -\ln(1 - t) \) and
\[
A^{(0)}_n(x) = x(x+1) \cdots (x+n-1) := \langle x \rangle_n \text{ with } \langle x \rangle_0 = 1,
\]
\[
A^{(1)}_n(x) = A_n(B_x) = (B_x + n - 1)_n = \mathcal{L}_n(x),
\]
\[
A^{(2)}_n(x) = A^{(2)}_n(B_x) = \mathcal{L}_n(B_x) = \sum_{k=0}^n L(n, k) B_k(x), \text{ etc},
\]
where \( L(n, k) \) are the Lah numbers. Theorem 3 shows that the sequences \( \langle (x) \rangle_n \), \( \langle \mathcal{L}_n(x) \rangle_n \) and \( \left( \sum_{k=0}^n L(n, k) B_k(x) \right) \) are log-convex and the sequences \( \left( \frac{\langle x \rangle_n}{n!} \right) \), \( \left( \frac{\mathcal{L}_n(x)}{n!} \right) \) and \( \left( \sum_{k=0}^n \frac{L(n,k)}{n!} B_k(x) \right) \)
are log-concave.

**Remark 6** Let \( a_s = \left( A^{(s)}_1(1), A^{(s)}_2(1), \ldots \right) \) and let the polynomials
\[
\mathcal{V}^{(s)}_{n,r}(x) = \sum_{k=0}^n B^{(r)}_{n+r,k+r}(a_s; e + La_s) x^k, \ s \geq 0.
\]
If \( \mathcal{V}^{(s)}_{n,0}(x) \), then \( \mathcal{V}^{(s)}_{n,0}(x) \in \mathbb{R}Z \).

**Example 7** For \( a_n = (n-1)! \), \( n \geq 1 \), we get \( h(t) = -\ln(1 - t) \) and
\[
A^{(0)}_n(x) = x(x+1) \cdots (x+n-1) := \langle x \rangle_n,
\]
\[
A^{(1)}_n(x) = A_n(B_x) = (B_x + n - 1)_n = \mathcal{L}_n(x),
\]
are in \( \mathbb{R}Z \), then the polynomials
\[
\mathcal{V}^{(0)}_{n,r}(x) = \sum_{k=0}^n B^{(r)}_{n+r,k+r}(a_0; e + La_0) x^k, \ \mathcal{V}^{(1)}_{n,r}(x) = \sum_{k=0}^n B^{(r)}_{n+r,k+r}(a_1; e + La_1) x^k,
\]
are also in \( \mathbb{R}Z \), where \( a_0 = (1!, 2!, \ldots) \) and \( a_1 = (\mathcal{L}_1(1), \mathcal{L}_2(1), \ldots) \).
Theorem 7 Let \( r \) be a non-negative integer and \( \left( f_n^{(r)}(x) \right) \) be the sequence defined by

\[
\sum_{n \geq 0} f_n^{(r)}(x) \frac{t^n}{n!} = F(t) \left( h(t) \right)^r \exp(xh(t)), \quad h(t) = \sum_{j \geq 1} a_j \frac{t^j}{j!},
\]

for some power series \( F \). Then, for \( r \leq n - 1 \), if the polynomial \( f_n^{(0)}(x) \) is of degree \( n \) and is in \( \text{RZ} \), then

\[
f_n^{(r)}(x) = r! \sum_{k=r}^{n} \binom{n}{k} B_{k,r} f_{n-k}^{(0)}(x) \in \text{RZ}.
\]

Proof. From the definition of the sequence \( \left( f_n^{(r)}(x) \right) \) there holds \( f_n^{(r)}(x) = \frac{d}{dt} f_n^{(r-1)}(x) \). It follows that the polynomial \( f_n^{(r)}(x) \) is of degree \( n - r \). The proof can be deduced by induction on \( r \) and by application of Rolle’s theorem. \( \square \)

Example 8 Let \( \left( f_n^{(r)}(x) \right) \) be the sequence defined by

\[
\sum_{n \geq 0} f_n^{(r)}(x) \frac{t^n}{n!} = (e^t - 1)^r \exp(x(e^t - 1)).
\]

Then, the polynomial \( f_n^{(r)}(x) = r! \sum_{k=r}^{n} \binom{n}{k} E_{n-k}(x) \) is in \( \text{RZ} \).

Example 9 Let \( \left( f_n^{(r)}(x) \right) \) be the sequence defined by

\[
\sum_{n \geq 0} f_n^{(r)}(x) \frac{t^n}{n!} = (\ln(1+t))^r \exp(x(\ln(1+t))).
\]

Then, the polynomial \( f_n^{(r)}(x) = r! \sum_{k=r}^{n} (-1)^{k-r} \binom{n}{k} \binom{k}{r} L_{n-k}(x) \) is in \( \text{RZ} \).

Example 10 Let \( \left( f_n^{(r)}(x) \right) \) be the sequence defined by

\[
\sum_{n \geq 0} f_n^{(r)}(x) \frac{t^n}{n!} = \frac{1}{r!} \left( \frac{t}{1-t} \right)^r \exp \left( \frac{xt}{1-t} \right).
\]

Then, the polynomial \( f_n^{(r)}(x) = r! \sum_{k=r}^{n} (-1)^{k-r} \binom{n}{k} L(k,r) L_{n-k}(x) \) is in \( \text{RZ} \).

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