THE HYPER ZAGREB INDEX OF SOME PRODUCT GRAPHS

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Abstract. In this paper some basic mathematical expressions for the Hyper Zagreb index of Product Graphs containing the symmetric difference, disjunction and tensor product have been computed.

Keywords: hyper-zagreb index; symmetric difference; disjunction; tensor product.

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1. INTRODUCTION

All graphs considered here are simple, connected and finite. Let \( V(G), E(G), d_G(v) \) and \( d_G(u, v) \) denote the vertex set, the edge set, the degree of a vertex and the distance between the vertices \( u \) and \( v \) of a graph \( G \) respectively. A graph with \( n \) vertices and \( m \) edges is called a \((n, m)\) graph. First we present the definitions and notations which are required throughout this paper. The complement \( \overline{G} \) of a graph \( G \) is the graph with vertex set \( V(G) \) in which 2 vertices are adjacent if they are not adjacent in \( G \).

Clearly \( d_{\overline{G}}(u) = n - 1 - d_G(u) \) where \( n \) is the number of vertices in \( G \).
A topological index is a numerical parameter mathematically derived from the graph structure. Topological indices and graph invariants based on the distances between vertices of a graph or vertex degrees are widely used for characterizing molecular graphs. The Wiener index is the first and most studied topological indices both from theoretical point of view and applications.

The Wiener index \[ W(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) \].

Gutman and Trinajstic in 1972 [5] introduced the first and second Zagreb indices to study the structure-dependency of the total \( \pi \)-electron energy and are defined as

\[ M_1(G) = \sum_{v \in V(G)} d_G(v)^2 = \sum_{uv \in E(G)} (d_G(u) + d_G(v)) \]
\[ M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v) \]

Ashrafi A.R in 2010 [1] defined the first and second Zagreb coindices as

\[ \overline{M}_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] \]
\[ \overline{M}_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v) \]

The forgotten index \( F \) [4] is defined as

\[ F(G) = \sum_{uv \in E(G)} [d_G^2(u) + d_G^2(v)] = \sum_{v \in V(G)} (d_G(v))^3 \]

G.H. Shirdel [10] defined the Hyper Zagreb index as

\[ HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2 \]

In this paper, the Hyper zagreb index of join, Cartesian product, composition and corona product of graphs are computed. The chemical properties of Hyper Zagreb indices were discussed in [2],[3]. Nilanjan De et al.[8] found the Hyper Zagreb index of bridge and chain graphs. In [9] K.Pattabiraman computed the Hyper Zagreb indices and its coindices of graphs. Here we continue the line of research by exploring the behaviour of the Hyper Zagreb index under several
important graph operations. In [6] Liu et al. found the Hyper-Zagreb index of Cacti with perfect matchings.

In [11] the hyper Zagreb co index is defined as

$$HM(G) = \sum_{uv \notin E(G)} (d_G(u) + d_G(v))^2$$

The tensor product $G \times H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and $E(G \times H) = \{(x, a)(y, b) : xy \in E(G) \text{ and } ab \in E(H)\}$. The symmetric difference $G \oplus H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and $E(G \oplus H) = \{(x, a)(y, b) : xy \in E(G) \text{ or } ab \in E(H) \text{ but not both.}\}$

The disjunction $G \lor H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and $E(G \lor H) = \{(x, a)(y, b) : xy \in E(G) \text{ or } ab \in E(H)\}$

In this paper, Hyper Zagreb index of symmetric difference, disjunction and tensor product of two graphs are computed.

We begin with the following lemma which can be used for computing the above graph operations.

**Lemma 1.1.** [3, 5]

(a) $d_{G \times H}(x, y) = d_G(x)d_G(y)$, where $(x, y) \in E(G \times H)$

(b) $d_{G \oplus H}(x, y) = n_2d_G(x) + n_1d_H(y) - 2d_G(x)d_H(y)$, where $(x, y) \in E(G \oplus H)$.

(c) $d_{G \lor H}(x, y) = n_2d_G(x) + n_1d_H(y) - d_G(x)d_H(y)$, where $(x, y) \in E(G \lor H)$.

**Remark 1.2.** [7] For a graph $G$, let $A(G) = \{(x, y) \in V(G) \times V(G) | x \text{ and } y \text{ are adjacent in } G\}$ and let $B(G) = \{(x, y) \in V(G) \times V(G) | x \text{ and } y \text{ are not adjacent in } G\}$. For each $x \in V(G)$, $(x, x) \in B(G)$. Clearly, $A(G) \cup B(G) = V(G) \times V(G)$. The summation $\sum_{(x, y) \in A(G)}$ runs over the ordered pairs of $A(G)$. For simplicity, we write the summation $\sum_{(x, y) \in A(G)}$ as $\sum_{xy \in G}$. Similarly, we write the summation $\sum_{(x, y) \in B(G)}$ as $\sum_{xy \notin G}$. Also the summation $\sum_{xy \notin E(G)}$ runs over the edges of $G$.

Let $G$ be a simple graph with $n$ vertices. Then

**Lemma 1.3.** [5]
(a) \( \sum_{xy \in G} 1 = 2e(G) \)

(b) \( \sum_{xy \in G} d_G(x) = M_1(G) \).

(c) \( \sum_{xy \in G} d_G(x)d_G(y) = 2M_2(G) \).

(d) \( \sum_{xy \in G} d_G^2(x) = F(G) \)

(e) \( \sum_{xy \in G} (d_G(x) + d_G(y))^2 = 2HM(G) \)

Here we find the following lemma to compute the graph operations.

**Lemma 1.4.** \( \sum_{xy \notin G} d_G(x) = 2(n - 1)e(\overline{G}) - M_1(\overline{G}) \)

**Proof:**

\[
\sum_{xy \notin G} d_G(x) = \sum_{xy \notin G} (n - 1 - d_{\overline{G}}(x)) = (n - 1)(2e(\overline{G})) - M_1(\overline{G})
\]

\[\square\]

**Lemma 1.5.** \( \sum_{xy \notin G} d_G^2(x) = 2(n - 1)^2e(\overline{G}) + F(\overline{G}) - 2(n - 1)M_1(\overline{G}) \)

**Proof:**

\[
\sum_{xy \notin G} d_G^2(x) = \sum_{xy \notin G} (n - 1 - d_{\overline{G}}(x))^2 = \sum_{xy \notin G} ((n - 1)^2 + d_{\overline{G}}^2(x) - 2(n - 1)d_{\overline{G}}(x)) = (n - 1)^2(2e(\overline{G})) + F(\overline{G}) - 2(n - 1)M_1(\overline{G})
\]

\[\square\]

**Lemma 1.6.** \( \sum_{xy \notin G} d_G(x)d_G(y) = 2[(n - 1)^2e(\overline{G}) - (n - 1)M_1(\overline{G}) + M_2(\overline{G})] \)

**Proof:**

\[
\sum_{xy \notin G} d_G(x)d_G(y) = \sum_{xy \notin G} (n - 1 - d_{\overline{G}}(x))(n - 1 - d_{\overline{G}}(y))
\]
\[
\sum_{xy \in G} [(n-1)^2 - (n-1)(d_G(x) + d_G(y)) + d_G(x)d_G(y)] \\
= 2[(n-1)^2e(G) - (n-1)M_1(G) + M_2(G)]
\]

\[\square\]

2. The Hyper Zagreb Index of Tensor Product of Graphs

**Theorem 2.1.** Let $G$ and $H$ be two connected graphs with $n_1$ and $n_2$ vertices and $m_1$ and $m_2$ edges respectively. Then

\[
HM(G \times H) = 2F(G)F(H) + 8M_2(G)M_2(H)
\]

**Proof:**

\[
2HM(G \times H) = \sum_{(x,a)(y,b) \in G \times H} (d_{G \times H}(x, a) + d_{G \times H}(y, b))^2
\]

\[= \sum_{xy \in G} \sum_{ab \in H} \left[ (d_{G \times H}(x, a) + d_{G \times H}(y, b))^2 + (d_{G \times H}(x, b) + d_{G \times H}(y, a))^2 \right]
\]

\[= \sum_{xy \in G} \sum_{ab \in H} (d_G(x)d_H(a) + d_G(y)d_H(b))^2
\]

\[+ \sum_{xy \in G} \sum_{ab \in H} (d_G(x)d_H(b) + d_G(y)d_H(a))^2
\]

\[= T_1 + T_2
\]

\[T_1 = \sum_{xy \in G} \sum_{ab \in H} (d_G(x)d_H(a) + d_G(y)d_H(b))^2
\]

\[= \sum_{xy \in G} \sum_{ab \in H} \left( d_G^2(x)d_H^2(a) + d_G^2(y)d_H^2(b) + 2d_G(x)d_H(a)d_G(y)d_H(b) \right)
\]

\[= 2F(G)F(H) + 8M_2(G)M_2(H)
\]

\[T_2 = \sum_{xy \in G} \sum_{ab \in H} (d_G(x)d_H(b) + d_G(y)d_H(a))^2
\]

\[= \sum_{xy \in G} \sum_{ab \in H} \left( d_G^2(x)d_H^2(b) + d_G^2(y)d_H^2(a) + 2d_G(x)d_H(b)d_G(y)d_H(a) \right)
\]

\[= 2F(G)F(H) + 8M_2(G)M_2(H)
\]
Adding $T_1$ and $T_2$ we have

$$HM(G \times H) = 2F(G)F(H) + 8M_2(G)M_2(H)$$

\[ \square \]

3. **The Hyper Zagreb Index of Symmetric Difference of Graphs**

Here we compute the hyper Zagreb index of symmetric difference of two graphs.

**Theorem 3.1.** Let $G$ and $H$ be two connected graphs with $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges respectively. Then

$$HM(G \oplus H) = 2n_2^2HM(G) + 8n_2^2m_1M_1(H) + 8M_1(H)HM(G) + 16n_1n_2m_2M_1(G)$$

$$- 16n_2m_2HM(G) - 16n_1M_1(G)M_1(H) + 2n_1^2HM(H) + 8n_2^2m_2M_1(G)$$

$$+ 8M_1(G)HM(H) + 16n_1n_2m_1M_1(H) - 16n_2M_1(G)M_1(H)$$

$$- 16m_1n_1HM(H) + 8n_2^2M_2HM(G) + 8m_1n_1^2HM(H)$$

$$+ 16F(G)(2(n_2 - 1)^2\overline{m_2} + F(\overline{H}) - 2(n_2 - 1)M_1(\overline{H}))$$

$$+ 8n_1n_2M_1(G)(4\overline{m_2}(n_2 - 1) - 2M_1(\overline{H}))$$

$$- 16n_2(F(G) + 2M_2(G))(2\overline{m_2}(n_2 - 1) - M_1(\overline{H}))$$

$$- 16n_1M_1(G)(4(n_2 - 1)^2\overline{m_2} - 4(n_2 - 1)M_1(\overline{H}) + 2M_2(\overline{H}) + F(\overline{H}))$$

$$+ 64M_2(G)((n_2 - 1)^2\overline{m_2} - (n_2 - 1)M_1(\overline{H}) + M_2(\overline{H}))$$

$$+ 8m_2n_2^2HM(G) + 8m_1n_1^2HM(H) + 16F(H)[2(n_1 - 1)^2\overline{m_1} + F(\overline{G})$$

$$- 2(n_1 - 1)M_1(\overline{G}] + 16n_1n_2(2(n_1 - 1)\overline{m_1} - M_1(\overline{G}))M_1(H)$$

$$- 16n_2M_1(H)(4(n_1 - 1)^2\overline{m_1} - 4(n_1 - 1)M_1(\overline{G}) + 2M_2(\overline{G}) + F(\overline{G}))$$

$$- 16n_1(F(H) + 2M_2(H))(2(n_1 - 1)\overline{m_1} - M_1(\overline{G}))$$

$$+ 64M_2(H)((n_1 - 1)^2\overline{m_1} - (n_1 - 1)M_1(\overline{G}) + M_2(\overline{G}))$$
Proof:

\[ 2HM(G \oplus H) = \sum_{xy \in G} \sum_{a \in V(H)} (d_{G \oplus H}(x, a) + d_{G \oplus H}(y, a))^2 \]
\[ + \sum_{ab \in H} \sum_{x \in V(G)} (d_{G \oplus H}(x, a) + d_{G \oplus H}(x, b))^2 \]
\[ + \sum_{xy \in G} \sum_{ab \in H} \{(d_{G \oplus H}(x, a) + d_{G \oplus H}(y, b))^2 + (d_{G \oplus H}(x, b) + d_{G \oplus H}(y, a))^2 \} \]
\[ + \sum_{xy \notin G} \sum_{ab \in H} \{(d_{G \oplus H}(x, a) + d_{G \oplus H}(y, b))^2 + (d_{G \oplus H}(x, b) + d_{G \oplus H}(y, a))^2 \} \]
\[ = S_1 + S_2 + S_3 + S_4 \]

where \( S_1, S_2, S_3 \) and \( S_4 \) denote the sums of the above terms in order.

\[ S_1 = \sum_{xy \in G} \sum_{a \in V(H)} (d_{G \oplus H}(x, a) + d_{G \oplus H}(y, a))^2 \]
\[ = \sum_{xy \in G} \sum_{a \in V(H)} [(n_2d_G(x) + n_1d_H(a) - 2d_G(x)d_H(a)) \]
\[ + (n_2d_G(y) + n_1d_H(a) - 2d_G(y)d_H(a))^2 \]
\[ = \sum_{xy \in G} \sum_{a \in V(H)} [n_2(d_G(x) + d_G(y)) + 2n_1d_H(a) - 2d_G(a)(d_G(x) + d_G(y))]^2 \]
\[ S_1 = 2n_2^3HM(G) + 8n_1^2m_1M_1(H) + 8M_1(H)HM(G) + 16n_1n_2m_2M_1(G) \]
\[ - 16n_2m_2HM(G) - 16n_1M_1(G)M_1(H) \]

Now,

\[ S_2 = \sum_{x \in V(G)} \sum_{ab \in H} (d_{G \oplus H}(x, a) + d_{G \oplus H}(x, b))^2 \]
\[ = \sum_{x \in V(G)} \sum_{ab \in H} (n_2d_G(x) + n_1d_H(a) - 2d_G(x)d_H(a) + n_2d_G(x) + n_1d_H(b) - 2d_G(x)d_H(b))^2 \]
\[ = \sum_{x \in V(G)} \sum_{ab \in H} (2n_2d_G(x) + n_1(d_H(a) + d_H(b)) - 2d_G(x)(d_H(a) + d_H(b))^2 \]
\[ S_2 = 8n_2^3M_1(G)m_2 + 2n_1^3HM(H) + 8M_1(G)HM(H) + 16n_1n_2m_1M_1(H) \]
\[ - 16n_2M_1(G)M_1(H) - 16n_1M_1HM(H) \]
and

\[ S_3 = \sum_{xy \in G} \sum_{ab \not\in H} (d_{G\oplus H}(x, a) + d_{G\oplus H}(y, b))^2 + (d_{G\oplus H}(x, b) + d_{G\oplus H}(y, a))^2 \]

\[ = \sum_{xy \in G} \sum_{ab \not\in H} (d_{G\oplus H}(x, a) + d_{G\oplus H}(y, b))^2 + \sum_{xy \in G} \sum_{ab \not\in H} (d_{G\oplus H}(x, b) + d_{G\oplus H}(y, a))^2 \]

\[ = S_{3,1} + S_{3,2} \]

\[ S_{3,1} = \sum_{xy \in G} \sum_{ab \not\in H} (d_{G\oplus H}(x, a) + d_{G\oplus H}(y, b))^2 \]

\[ = \sum_{xy \in G} \sum_{ab \not\in H} (n_2 d_G(x) + n_1 d_H(a) - 2d_G(x)d_H(a)) + (n_2 d_G(y) + n_1 d_H(b) - 2d_G(y)d_H(b))^2 \]

\[ = \sum_{xy \in G} \sum_{ab \not\in H} (n_2(d_G(x) + d_G(y)) + n_1(d_H(a) + d_H(b)) - 2d_G(x)d_H(a) - 2d_G(y)d_H(b))^2 \]

\[ S_{3,1} = 4n_2^2m_2HM(G) + 4m_1n_1^2HM(H) + 8F(G)[2(n_2 - 1)^2m_2 + F(H)] - 2(n_2 - 1)M_1(H) + 4n_1n_2M_1(G)(4m_2(n_2 - 1) - 2M_1(H)) - 8n_2(F(G) + 2M_2(G))(2m_2(n_2 - 1) - M_1(H)) - 8n_1M_1(G)(4(n_2 - 1)^2m_2 - 4(n_2 - 1)M_1(H) + 2M_2(H) + F(H)) + 32M_2(G)((n_2 - 1)^2m_2 - (n_2 - 1)M_1(H) + M_2(H)) \]

\[ S_{3,2} = \sum_{xy \in G} \sum_{ab \not\in H} (d_{G\oplus H}(x, b) + d_{G\oplus H}(y, a))^2 \]

\[ = \sum_{xy \in G} \sum_{ab \not\in H} (n_2 d_G(x) + n_1 d_H(b) - 2d_G(x)d_H(b)) + (n_2 d_G(y) + n_1 d_H(a) - 2d_G(y)d_H(a))^2 \]

\[ = \sum_{xy \in G} \sum_{ab \not\in H} (n_2(d_G(x) + d_G(y)) + n_1(d_H(a) + d_H(b)) - 2d_G(x)d_H(b) - 2d_G(y)d_H(a))^2 \]

\[ S_{3,2} = 4n_2^2m_2HM(G) + 4m_1n_1^2HM(H) + 8F(G)(2(n_2 - 1)^2m_2 + F(H)) - 2(n_2 - 1)M_1(H) + 4n_1n_2M_1(G)(4m_2(n_2 - 1) - 2M_1(H)) \]
\[-8n_2(F(G) + 2M_2(G))(2m_2(n_2 - 1) - M_1(\overline{H})) - 8n_1M_1(G)(4(n_2 - 1)^2m_2\]
\[-4(n_2 - 1)M_1(\overline{H}) + 2M_2(\overline{H}) + F(\overline{H})\]
\[+ 32M_2(G)((n_2 - 1)^2m_2 - (n_2 - 1)M_1(\overline{H}) + M_2(\overline{H}))\]

\[S_3 = S_{3,1} + S_{3,2}\]
\[= 8n_22m_2HM(G) + 8m_1n_1^2HM(H) + 16F(G)(2(n_2 - 1)^2m_2 + F(\overline{H})\]
\[-2(n_2 - 1)M_1(\overline{H}) + 8n_1n_2M_1(G)(4m_2(n_2 - 1) - 2M_1(\overline{H}))\]
\[-16n_2(F(G) + 2M_2(G))(2m_2(n_2 - 1) - M_1(\overline{H})) - 16n_1M_1(G)(4(n_2 - 1)^2m_2\]
\[-4(n_2 - 1)M_1(\overline{H}) + 2M_2(\overline{H}) + F(\overline{H})\]
\[+ 64M_2(G)((n_2 - 1)^2m_2 - (n_2 - 1)M_1(\overline{H}) + M_2(\overline{H}))\]

Similarly,

\[S_4 = \sum_{xy \not\in G} \sum_{ab \in H} (d_{G \oplus H}(x, a) + d_{G \oplus H}(y, b))^2 + (d_{G \oplus H}(x, b) + d_{G \oplus H}(y, a))^2\]
\[= S_{4,1} + S_{4,2}\]

\[S_{4,1} = \sum_{xy \not\in G} \sum_{ab \in H} (d_{G \oplus H}(x, a) + d_{G \oplus H}(y, b))^2\]
\[= \sum_{xy \not\in G} \sum_{ab \in H} (n_2d_G(x) + n_1d_H(a) - 2d_G(x)d_H(a)) + (n_2d_G(y) + n_1d_H(b)\]
\[-2d_G(y)d_H(b))^2\]
\[= \sum_{xy \not\in G} \sum_{ab \in H} (n_2(d_G(x) + d_G(y)) + n_1(d_H(a) + d_H(b)) - 2d_G(x)d_H(a)\]
\[-2d_G(y)d_H(b))^2\]

\[S_{4,1} = 4m_2n_2^2\overline{H}M(G) + 4m_1n_1^2HM(H) + 8F(H)(2(n_2 - 1)^2\overline{m}_1 + F(\overline{G})\]
\[-2(n_1 - 1)M_1(\overline{G})) + 8n_1n_2(2(n_1 - 1)\overline{m}_1 - M_1(\overline{G}))M_1(H)\]
\[-8n_2M_1(H)(4(n_1 - 1)^2\overline{m}_1 - 4(n_1 - 1)M_1(\overline{G}) + 2M_2(\overline{G}) + F(\overline{G}))\]
\[-8n_1(F(H) + 2M_2(H))(2(n_1 - 1)\overline{m}_1 - M_1(\overline{G}))\]
\[+ 32M_2(H)((n_1 - 1)^2\overline{m}_1 - (n_1 - 1)M_1(\overline{G}) + M_2(\overline{G}))\]
Adding $S_1, S_2, S_3$ and $S_4$ we get the desired result.

4. **The Hyper Zagreb Index of Disjunction of Graphs**

Here we compute the hyper Zagreb Index of disjunction of Graphs.

**Theorem 4.1.** Let $G$ and $H$ be connected graphs with $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges respectively. Then

$$2HM(G \oplus H) = 8n_2^2m_2M_1(G) + 2n_1^3HM(H) + 2M_1(G)HM(H)$$
\[ \begin{align*}
&+ 16m_1n_2M_1(H) - 8n_2M_1(G)M_1(H) - 8m_1HM(H) + 8m_2n_2HM(G) \\
&+ 8m_1n_2HM(H) + 4F(H)(2(n_1 - 1)^2m_1 + F(G) - 2(n_1 - 1)M_1(G)) \\
&+ 16n_1n_2(2(n_1 - 1)m_1 - M_1(G))M_1(H) - 8n_2M_1(H)(4(n_1 - 1)^2m_1 \\
&- 4(n_1 - 1)M_1(G) + F(G) + 2M_2(G)) - 8n_1(2M_2(H) + F(H)) \\
&(2(n_1 - 1)m_1 - M_1(G)) + 16M_2(H)((n_1 - 1)^2m_1 - (n_1 - 1)M_1(G) + M_2(G)) \\
&+ 2n^2HM(G) + 2n^2m_1(8m_2^2 + 2n_2)M_1(H) + 2n_2F(G)M_1(H) + 16m_2n_1n_2M_1(G) \\
&- 8m_2n_2^2(F(G) + 2M_2(G)) - 4n_1M_1(G)(n_2M_1(H) + 4m_2^2) + 16m_2^2M_2(G)
\end{align*} \]

**Proof:**

\[ 
\begin{align*}
2HM(G \vee H) &= \sum_{xy \in G} \sum_{ab \in V(H)} \sum_{b \in V(H)} (d_{G \vee H}(x, a) + d_{G \vee H}(y, b))^2 \\
&+ \sum_{ab \in H} \sum_{x \in V(G)} (d_{G \vee H}(x, a) + d_{G \vee H}(x, b))^2 \\
&+ \sum_{xy \in G} \sum_{ab \in H} ((d_{G \vee H}(x, a) + d_{G \vee H}(y, b))^2 + (d_{G \vee H}(x, b) + d_{G \vee H}(y, a))^2 \\
&= S_3 + S_1 + S_2
\end{align*} \]

where \( S_1, S_2 \) and \( S_3 \) denote the sums of the above terms in order. Next we calculate \( S_1 \)

\[ 
\begin{align*}
S_1 &= \sum_{x \in V(G)} \sum_{ab \in H} (d_{G \vee H}(x, a) + d_{G \vee H}(x, b))^2 \\
&= \sum_{x \in V(G)} \sum_{ab \in H} (n_2d_G(x) + n_1d_H(a) - d_G(x)d_H(a) + n_2d_G(x) + n_1d_H(b) - d_G(x)d_H(b))^2 \\
&= \sum_{x \in V(G)} \sum_{ab \in H} (2n_2d_G(x) + n_1(d_H(a) + d_H(b)) - d_G(x)(d_H(a) + d_H(b)))^2 \\
&= 8n_2^2m_2M_1(G) + 2n_1^3HM(H) + 2M_1(G)HM(H) + 16m_1n_2M_1(H) \\
&- 8n_2M_1(G)M_1(H) - 8m_1n_1HM(H)
\end{align*} \]

Now,

\[ 
S_2 = \sum_{ab \in H} \sum_{xy \in G} ((d_{G \vee H}(x, a) + d_{G \vee H}(y, b))^2 + (d_{G \vee H}(x, b) + d_{G \vee H}(y, a))^2
\]
Similarly,

\[ S_{2,1} = \sum_{xy \in G} \sum_{abc \in H} (d_{G\vee H}(x, a) + d_{G\vee H}(y, b))^2 \]
\[ = \sum_{xy \in G} \sum_{abc \in H} (n_2d_G(x) + n_1d_H(a) - d_G(x)d_H(a) + n_2d_G(y) + n_1d_H(b) - d_G(y)d_H(b))^2 \]
\[ = \sum_{xy \in G} \sum_{abc \in H} (n_2(d_G(x) + d_G(y) + n_1(d_H(a) + d_H(b))) - d_G(x)d_H(a) - d_G(y)d_H(b))^2 \]
\[ = 4m_2n_2^2HM(G) + 4m_1n_1^2HM(H) + 2F(H)(2(n_1 - 1)^2m_1 + F(G) - 2(n_1 - 1)M_1(G)) \]
\[ + 8n_1n_2(2(n_1 - 1)m_1 - M_1(G))M_1(H) - 4n_2M_1(H)[4(n_1 - 1)^2m_1 \]
\[ - 4(n_1 - 1)M_1(G) + F(G) + 2M_1(G)] - 4n_1(F(H) + 2M_1(H))(2(n_1 - 1)m_1 - M_1(G)) \]
\[ + 8M_2(H)[(n_1 - 1)^2m_1 - (n_1 - 1)M_1(G) + M_2(G)] \]

Similarly,

\[ S_{2,2} = \sum_{xy \in G} \sum_{abc \in H} (d_{G\vee H}(x, b) + d_{G\vee H}(y, a))^2 \]
\[ = \sum_{xy \in G} \sum_{abc \in H} (n_2d_G(x) + n_1d_H(b) - d_G(x)d_H(b) + n_2d_G(y) + n_1d_H(a) - d_G(y)d_H(a))^2 \]
\[ = \sum_{xy \in G} \sum_{abc \in H} (n_2(d_G(x) + d_G(y) + n_1(d_H(a) + d_H(b))) - d_G(x)d_H(a) - d_G(y)d_H(b))^2 \]
\[ = 4m_2n_2^2HM(G) + 4m_1n_1^2HM(H) + 2F(H)(2(n_1 - 1)^2m_1 + F(G) - 2(n_1 - 1)M_1(G)) \]
\[ + 8n_1n_2(2(n_1 - 1)m_1 - M_1(G))M_1(H) - 4n_2M_1(H)[4(n_1 - 1)^2m_1 \]
\[ - 4(n_1 - 1)M_1(G) + F(G) + 2M_1(G)] - 4n_1(F(H) + 2M_1(H))(2(n_1 - 1)m_1 - M_1(G)) \]
\[ + 8M_2(H)[(n_1 - 1)^2m_1 - (n_1 - 1)M_1(G) + M_2(G)] \]

\[ S_2 = S_{2,1} + S_{2,2} \]
\[ = 8m_2n_2^2HM(G) + 8m_1n_1^2HM(H) + 4F(H)(2(n_1 - 1)^2m_1 + F(G) - 2(n_1 - 1)M_1(G)) \]
\[ + 16n_1n_2(2(n_1 - 1)m_1 - M_1(G))M_1(H) - 8n_2M_1(H)[4(n_1 - 1)^2m_1 \]
\[ - 4(n_1 - 1)M_1(G) + F(G) + 2M_1(G)] - 8n_1(F(H) + 2M_1(H))(2(n_1 - 1)m_1 - M_1(G)) \]
+ 16M_2(H)[(n_1 - 1)^2m_1 - (n_1 - 1)M_1(\overline{G}) + M_2(\overline{G})]

Now,

\[ S_3 = \sum_{xy \in G} \sum_{a \in V(H)} \sum_{b \in V(H)} (d_{G\vee H}(x,a) + d_{G\vee H}(y,b))^2 \]

\[ = \sum_{xy \in G} \sum_{a \in V(H)} \sum_{b \in V(H)} (n_2d_G(x) + n_1d_H(a) - d_G(x)d_H(a) + n_2d_G(y) + n_1d_H(b) - d_G(y)d_H(b))^2 \]

\[ = \sum_{xy \in G} \sum_{a \in V(H)} \sum_{b \in V(H)} (n_2(d_G(x) + d_G(y)) + n_1(d_H(a) + d_H(b)) - d_G(x)d_H(a) - d_G(y)d_H(b))^2 \]

\[ = 2n_2^4HM(G) + 2n_1^2m_1(8m_2^2 + 2n_2)M_1(H) + 2n_2F(G)M_1(H) \]

\[ + 16m_2n_1n_2^2M_1(G) - 8m_2n_2^2(F(G) + 2M_2(G)) - 4n_1M_1(G)(n_2M_1(H) + 4m_2^2) \]

\[ + 16m_2^2M_2(G) \]

By adding \( S_1, S_2 \) and \( S_3 \) the desired result follows. \( \square \)

5. **Conclusion**

In this paper, some basic mathematical expressions for the Hyper Zagreb index of Product Graphs containing the symmetric difference, disjunction and tensor product are derived.

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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