NOTE

The perturbative scalar massless propagator in Schwarzschild spacetime

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Abstract

A short derivation of the weak gravitational field approximation to the scalar massless propagator in Schwarzschild spacetime obtained by Paszko using the path-integral approach is given. The contribution from the direct coupling of the quantum field to the scalar curvature is explicitly included. The propagator complies with Hadamard’s pattern, and the vacuum state is consistent with the perturbative version of the Boulware vacuum. The momentum space propagator is computed for massless or massive particles to the same perturbative order. The renormalized value of $\langle \phi^2(x) \rangle$ for the massless case is reproduced.

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In [1], Paszko carried out a perturbative calculation of the Feynman propagator of a scalar particle in the presence of a Schwarzschild metric, to first order in the black hole mass, obtaining an explicit analytical expression for massless particles. A path-integral approach was adopted for that calculation. In this note, we show that this result can be obtained with considerably less effort by means of the operator formalism.

For simplicity, we will work out the Euclidean version and perform the Wick rotation at the end to obtain the Feynman propagator. We use units $\hbar = c = 1$, and $M$, with dimension of length, denotes the black hole mass times Newton’s constant.

Our starting equation is [2]

$$(-\Box + m^2 + \xi R)G(x, x') = \frac{1}{\sqrt{g(x)}}\delta(x-x').$$

(1)

Here, $\Box = g^{-1/2}\partial_\mu g^{1/2}g^{\mu\nu}\partial_\nu$ is the Laplacian operator, $R$ is the scalar curvature and $G(x, x')$ is the (Euclidean) propagator from $x'$ to $x$ of the scalar particle with mass $m$.

Following [1], for the (Euclidean) Schwarzschild metric with mass $M$, we use isotropic coordinates [3],

$$g_{\mu\nu} \, dx^\mu \, dx^\nu = \left(\frac{1 - \frac{M}{r}}{1 + \frac{M}{r}}\right)^2 \, d\tau^2 + \left(1 + \frac{M}{2r}\right)^4 \, dx^2.$$

(2)
Here, we use $\tau$ to denote the Euclidean time (not the proper time), in order to distinguish it from the Lorentzian time $t$, below.

To first order in an expansion in powers of $M$, the propagator equation reduces to
\[
\left[-\left(1 + \frac{2M}{r}\right)\partial_{\tau}^2 - \left(1 - \frac{2M}{r}\right)\nabla^2 + m^2 + \xi R\right] G(x, x') = \left(1 - \frac{2M}{r}\right) \delta(x - x') + O(M^2),
\tag{3}
\]
Let us expand also the propagator and the scalar curvature,
\[
G = G^{(0)} + G^{(1)} + O(M^2), \quad R = R^{(1)} + O(M^2),
\tag{4}
\]
where $G^{(n)}$ and $R^{(n)}$ contain precisely $n$ powers of $M$. Substituting in equation (3) and equating terms with equal powers of $M$ produces the following relations:
\[
\left(-\partial_{\tau}^2 - \nabla^2 + m^2\right) G^{(0)}(x, x') = \delta(x - x'),
\tag{5}
\]
\[
\left(-\partial_{\tau}^2 - \nabla^2 + m^2\right) G^{(1)}(x, x') = \left[\frac{4M}{r} \left(\frac{\partial_{\tau}^2}{2} - \frac{1}{2} m^2\right) - \xi R^{(1)}\right] G^{(0)}(x, x').
\tag{6}
\]
From these relations it immediately follows that
\[
G^{(1)}(x, x') = \int d^4x'' G^{(0)}(x, x'') \left[\frac{4M}{r''} \left(\frac{\partial_{\tau}^2}{2} - \frac{1}{2} m^2\right) - \xi R^{(1)}(x'')\right] G^{(0)}(x'', x').
\tag{7}
\]
Of course, the same result can be obtained by applying standard perturbation theory to first order. See figure 1. The Feynman rule for the interaction vertex of the particle with the static gravitational field can be read off from the Lagrangian
\[
\mathcal{L}(x) = \sqrt{g} \left(\frac{1}{2} \delta^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{1}{2} (m^2 + \xi R) \phi^2\right)
= \frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{4M}{r} \left(\frac{\partial_{\tau} \phi}{2} + \frac{1}{2} m^2 \phi^2\right) + \frac{1}{2} \xi R^{(1)} \phi^2 + O(M^2).
\tag{8}
\]
Note that in the Euclidean perturbative computation the metric is just $\delta_{\mu\nu}$.
In order to obtain a closed expression in $x$-space (as opposed to momentum space), from now on we consider only the massless case, $m = 0$. In this case
\[
G^{(0)}(x, x') = \frac{1}{(2\pi)^2} \frac{1}{(x - x')^2}.
\tag{9}
\]
\footnote{Once the Minkowski vacuum has been adopted at zeroth order, the perturbative expansion automatically selects a privileged choice of boundary conditions at any other perturbative order. We do not over-ride that choice here.}
As follows from equation (7), the effect of \( \xi R \) on the propagator is additive to first order, so we define
\[
G^{(1)}(x, x') = G_0^{(1)}(x, x') + G_1^{(1)}(x, x').
\]

Let us consider first the effect of the scalar curvature term. From Einstein’s equations, a Dirac delta mass distribution as a source of the Schwarzschild metric corresponds to a scalar curvature \( R(x) = 8\pi M \delta(x) + O(M^2) \). When this is inserted in equation (7), one readily obtains
\[
G_1^{(1)}(x, x') = -\xi \frac{2M}{(2\pi)^2} \int \frac{1}{rr'} \frac{r + r'}{\Delta r^2 + (r + r')^2},
\]
where \( \Delta r = r - r' \). For the \( \xi = 0 \) part, equation (7) reduces to
\[
G_0^{(1)}(x, x') = 4M \delta_x^2 F(x, x'),
\]
with
\[
(2\pi)^4 F(x, x') = \int d^4x'' \frac{1}{r''} \frac{1}{(x'' - x')(x'' - x')^2}.
\]

Here, it is appropriate to use a standard Feynman parameterization
\[
\frac{1}{(\xi a + (1 - \xi)b)^2},
\]
which yields
\[
(2\pi)^4 F(x, x') = \int d^4x'' \frac{1}{r''} \int_0^{r''} d\xi \frac{1}{((x'' - y)^2 + \xi(1 - \xi)l^2)^2},
\]
where \( y = \xi x + (1 - \xi)x' \) and \( l^2 = (x - x')^2 \). Now it is straightforward to carry out the \( x'' \) integration, starting with the angular part, then \( r'' \) and lastly \( r'' \). This gives
\[
(2\pi)^4 F(x, x') = \int_0^1 d\xi \pi^2 \log \left( \frac{\sqrt{x^2 + \xi(1 - \xi)l^2} + |y|}{\sqrt{x^2 + \xi(1 - \xi)l^2} - |y|} \right) .
\]

The remaining integral produces a rather complicated expression. However, the function \( F \) itself is not needed. As it turns out, the first time derivative of \( F \) takes a simple form:
\[
(4\pi)^2 \partial_\tau F(x, x') = -\frac{2\Delta \tau}{l^2} \int_0^1 d\xi \frac{1}{\sqrt{x^2 + \xi(1 - \xi)l^2}}
\]
\[
= -\frac{4}{l^2} \arctan \left( \frac{\Delta \tau}{r + r'} \right) .
\]

Taking another time derivative yields the final expression for \( G^{(1)} \) (adding the \( \xi \) dependent term):
\[
G^{(1)}(x, x') = \frac{1}{(2\pi)^2} \frac{4M}{l^2} \left( 2\Delta \tau \arctan \left( \frac{\Delta \tau}{r + r'} \right) - \frac{r + r'}{(\Delta \tau)^2 + (r + r')^2} \left( 1 + \frac{\xi l^2}{2r'r'} \right) \right) .
\]

To obtain the propagator with Lorentzian signature, \( \Delta_F \), it only remains to perform the Wick rotation. This entails taking an analytical continuation
\[
i\Delta_F(\Delta\tau; x, x') = G(\Delta\tau = i\Delta\tau; x, x').
\]

Specifically, for the Feynman propagator, for positive (negative) \( \Delta t = t - t' \), the continuation is to be taken from the positive (negative) \( \text{Re}(\Delta\tau) \) half-planes, respectively. Hence, \( \Delta\tau = i\Delta\tau + \text{sgn}(\Delta\tau)\eta, \eta = 0^+ \). For the zeroth order, this yields the free Feynman propagator for a massless scalar particle
\[
\Delta_F^{(0)}(x, x') = \frac{i}{(2\pi)^2} \frac{1}{x^2 - i\eta^+}.
\]
where \( s^2 = \Delta t^2 - \Delta x^2 \) with \( \Delta x = x - x' \). For the first-order term, the Wick rotation gives (leaving \( \eta = 0 \) implicit)

\[
\Delta_F^{(1)}(x, x') = \frac{i}{(2\pi)^2} \frac{4M}{s^2} \left( \frac{r + r'}{\Delta t^2 - (r + r')^2} \left( 1 - \frac{s^2}{2rr'} \right) + \frac{2\Delta t}{s^2} \arctan \left( \frac{\Delta t}{r + r'} \right) \right).
\]

(21)

This form applies to the case \( |\Delta t| > r + r' \) (taking for \( \arctan \) the real branch). When \( |\Delta t| > r + r' \), the expression is\(^2\)

\[
\Delta_F^{(1)}(x, x') = \frac{i}{(2\pi)^2} \frac{4M}{s^2} \left( \frac{r + r'}{\Delta t^2 - (r + r')^2} \left( 1 - \frac{s^2}{2rr'} \right) \right) + \frac{2\Delta t}{s^2} \arctan \left( \frac{r + r'}{\Delta t} \right) - i\pi \frac{|\Delta t|}{s^2}.
\]

(22)

When \( \xi = 0 \), this is the same result already obtained in [1].

We have explicitly verified that \( \Delta_F^{(0)}(x, x') + \Delta_F^{(1)}(x, x') \) solves the Klein–Gordon differential equation to \( O(M) \).

From the symmetries of the problem, the propagator will be a function of \( r \) and \( r' \), as well as \( s^2 \) and \( \Delta t \). To \( O(M) \), the \( \xi = 0 \) part of the propagator depends only on the combination \( r + r' \), but this property is not preserved by the contribution from \( \xi R \).

To the same order in \( M \), the explicit result in momentum space can also be given, even for massive particles, as it follows immediately from the Feynman vertex in equation (8) (in its Lorentzian signature version). Letting

\[
D(p) = \frac{1}{p^2 - m^2 + i\eta},
\]

(23)

the propagator is given by

\[
\Delta_F^{(0)}(p, p') = (2\pi)^4 \delta(p - p')D(p),
\]

\[
\Delta_F^{(1)}(p, p') = 2\pi \delta(p^0 - p'^0)8\pi MD(p) \left[ \frac{m^2 - 2p^0p'^0}{(p - p')^2} + \xi \right] D(p').
\]

(24)

Next, we investigate whether the propagator conforms to Hadamard’s pattern,

\[
-2i(2\pi)^2 \Delta_F(x, x') = \frac{U}{\sigma} + V \log(\sigma) + W,
\]

(25)

where \( U, V \) and \( W \) are regular functions of \( x \) and \( x' \) at \( x = x' \) (at the points where the metric itself is nonsingular), with \( U(x, x) = 1 \), and \( \sigma(x, x') \) is Syngue’s function, half the square of the length along the geodesic joining \( x \) and \( x' \) (and \( \sigma \to \sigma - i\eta \) is implicit). To be a Hadamard state is a well-known requirement for the vacuum being sufficiently regular in the ultraviolet region to ensure a non-singular stress–energy tensor [4, 5].

The momenta \( q = p - p' \) and \( Q = (p + p')/2 \) control, respectively, \( (x + x')/2 \) and \( x - x' \) in \( x \)-space. The large \( Q \) behavior \( \Delta_F^{(0)} \sim Q^0 \) competes with that of \( \Delta_F^{(1)} \sim 1/Q^2 \), and this could give rise to a singular coincidence limit in the propagator in \( x \)-space. The corresponding situation manifests in \( s^2 \Delta_F^{(1)}(x, x') \), which behaves as \( \Delta t^2/s^2 \) and so it is not a continuous function in the coincidence limit.

In order to clarify this issue, \( \sigma \) has to be computed up to order \( M \). Being an extremal curve, the length of the geodesic can be obtained to first order by using the zeroth-order path, i.e., the straight line. This gives

\[
\sigma(x, x') = \frac{1}{2} s^2 - M(\Delta t^2 + \Delta x^2)h + O(M^2),
\]

(26)

\(^2\) This is most easily obtained by using first the identity \( \arctan \left( \frac{1}{2} \right) = -\arctan(s) + \frac{\pi}{4} \sgn(s) \) in equation (18), and noting that under the Wick rotation \( \sgn(\Delta t) \) goes to \( \sgn(\Delta t) \).

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where $\hbar$ is the average value of $1/\rho^\nu$ along the path:

$$h(x, x') = \int_0^1 \frac{d\lambda}{|2x + (1 - \lambda)x'|} = \frac{2}{|\Delta x|} \arctanh \left( \frac{|\Delta x|}{r + r'} \right).$$

(27)

When combined with the propagator in equation (21), we obtain the following relation:

$$-i(2\pi)^2 2\sigma \Delta_F = 1 + 8M \frac{\Delta^2}{s^2} \left( \frac{\arctanh \left( \frac{\Delta t}{\Delta r} \right)}{\Delta t} - \frac{\arctanh \left( \frac{\Delta t}{\Delta s} \right)}{\Delta s} \right)$$

$$+ 4M \left( \frac{r + r'}{\Delta t^2 - (r + r')^2} \left( 1 - \frac{x^2}{2r^2} \right) + \frac{\arctanh \left( \frac{|\Delta x|}{r + r'} \right)}{|\Delta x|} \right) + O(M^2).$$

(28)

The term with $4M$ is manifestly regular as a function of $\Delta x^\mu$ at $x = x'$. In the term with $8M$, the structure $(f(\Delta^2) - f(\Delta^2))/((\Delta^2 - \Delta x^2))$, with $f$ being a regular function, is itself regular. It follows that the propagator is of Hadamard type with $V = O(M^2)$, and $U(x, x) = 1 + O(M^2)$ is also checked.

To further expose the vacuum structure, we adopt the point of view that the Minkowski vacuum, $|0\rangle$, and the corresponding Hamiltonian $H_0$ are subject to an $O(M)$ time-independent perturbation. Specifically, $H = H_0 + H_1 + O(M^2)$ with (the canonical momentum being $\pi(x) = (1 + 4M/r)\partial_\phi(x) + O(M^2))$

$$H_1 = \int d^3x \frac{1}{2} \left[ \frac{4M}{r} \left( \frac{1}{2} m^2 \phi^2(x) - \pi^2(x) \right) + \xi R^{(1)}(x) \phi^2(x) \right].$$

(29)

Straightforward application of time-independent perturbation theory to first order yields the new perturbed ground state:

$$|\Omega\rangle = |0\rangle - \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{2\omega} \sqrt{2\omega'}} \frac{8\pi M}{\omega + \omega'}$$

$$\times \left[ \frac{\omega \omega' + \frac{1}{2}m^2}{|k + k'|^2} + \frac{\xi}{2} \right] a^\dagger(k) a^\dagger(k') |0\rangle + O(M^2),$$

(30)

where $\omega = \sqrt{k^2 + m^2}$ and $\omega' = \sqrt{k'^2 + m^2}$, and $a^\dagger(k)$ is the creation operator of the unperturbed Fock vacuum, with normalization $[a(k), a^\dagger(k')] = (2\pi)^3 \delta(k - k')$.

The perturbed vacuum is annihilated by the perturbed annihilation operators

$$\tilde{a}(k) = a(k) + \int \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{2\omega} \sqrt{2\omega'}} \frac{8\pi M}{\omega + \omega'}$$

$$\times \left[ \frac{\omega \omega'}{|k + k'|^2} + \frac{\xi}{2} \right] a^\dagger(k')$$

$$+ \frac{8\pi M}{\omega + \omega'} \left[ \frac{\omega^2 + \omega'^2}{|k + k'|^2} + \frac{\xi}{2} \right] a^\dagger(k') + O(M^2).$$

(31)

These operators satisfy $[H, \tilde{a}(k)] = -\omega \tilde{a}(k)$ to $O(M)$. The single particle spectrum is unchanged since it is determined by the asymptotic scattering region, far from the black hole. The fact that $\tilde{a}(k) |\Omega\rangle = 0$ and the commutator of the perturbed creation and annihilation operators are still c-numbers allows us to use Wick’s theorem for the field correlation functions, and this implies that the state $|\Omega\rangle$ is Gaussian.

A direct calculation shows that the Fourier transform of $-i(\Omega | T (\phi(x)\phi(x')) | \Omega\rangle$ just reproduces the propagator in equation (24). (Note that $H$, and not $H_0$, has to be used here to evolve the fields in Heisenberg picture.) The previous arguments strongly suggest that $|\Omega\rangle$ is just the Boulware vacuum, the stationary non-thermal and non-radiating ground state of the Schwarzschild metric [6, 7], albeit to $O(M)$. Therefore, the propagator applies to a static
spherical star, in the regime \( r, r' \gg M \), rather than to a proper black hole, which would be in a thermal state with a temperature \( T = 1/(8\pi M) \) [8].

The renormalized value of \( \langle \phi^2(x) \rangle \) has been computed in [9] for a massless particle in the weak gravitational field of a static spherical star. Local approximations have been avoided in [9].

From the definition of the propagator, the bare value of \( \langle \phi^2(x) \rangle \) follows from taking the coincidence limit in \( G(x,x') \) or \( i\Delta_F(x,x') \). Here, we take the alternative approach of going back to equation (7) and setting \( x = x' \) there (for \( m = 0 \)). Straightforward integration over \( \tau'' \) gives

\[
G^{(1)}(x,x) = -\pi M (2\pi)^d \int d^d x'' \left[ \frac{1}{|x'| |x-x'|^5} + 4\pi \xi \delta(x'') \frac{1}{|x-x''|^3} \right].
\] (32)

The term with \( \xi \) is ultraviolet finite and it just reproduces the result of taking the coincidence limit directly in equation (11). The term without \( \xi \) is divergent. However, as a distribution \( |x-x'|^{-3} = (1/6)\nabla^2 |x-x'|^{-3} \). Integration by parts and use of \( \nabla^2 |x'|^{-1} = -4\pi \delta(x') \) immediately gives

\[
\langle \phi^2(x) \rangle = -\left(\xi - \frac{1}{6}\right) \frac{1}{(2\pi)^d} \frac{M}{r^3} + O(M^2).
\] (33)

This result is in agreement with that obtained in [9] by using dimensional regularization plus standard subtraction of the singular part [2].

The Feynman propagator at \( x = x' \) but arbitrary \( \Delta t \) for the same setting (weak static spherical gravitational field and massless quantum field) follows from the results in [10]. Use of \( R^{(1)}(x) = 8\pi M \delta(x) \) in equation (6.9) of [10] yields a result that almost agrees with our value for \( \Delta_F^{(1)}(t,x,t',x) \) from equation (21)\(^3\). The result in [10] is nonsingular in the limit \( \Delta t \to 0 \), and it is consistent with equation (33). Instead, we obtain the same non-singular part plus a divergent contribution:

\[
i\Delta_F^{(1)}(t,x,t',x)|_{\text{div}} = -\frac{1}{(2\pi)^d} \frac{2M}{r} \frac{1}{\Delta t^2}.
\] (34)

This expression is consistent with equation (A1) of [11].

As a final comment, we note that ultraviolet divergences are expected to arise at higher orders in a strict expansion of the propagator in powers of \( M \). These would come from small radii in the intermediate point integrations. Inspection of the integrals involved suggests that such an expansion breaks down already at \( O(M^3) \) due to the presence of terms \((M/r')^3\).

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References

[1] Paszko R 2012 Class. Quantum Grav. 29 045005
[2] Birrell N D and Davies P C W 1982 Quantum Fields In Curved Space (Cambridge: Cambridge University Press)

\(^3\) Both results would agree if a factor 1/2 were added to the middle term in the integrand of equation (6.9). It can be noted that this middle term would correspond to a Dirac delta function in equation (6.8) of [10], rather than to an ordinary function.
[3] Landau L D and Lifshitz E M 1971 The Classical Theory of Fields (A Course of Theoretical Physics vol 2) (Oxford: Pergamon)
[4] Wald R M 1994 Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics (Chicago, IL: Chicago University Press)
[5] Decanini Y and Folacci A 2008 Phys. Rev. D 78 044025
[6] Boulware D G 1975 Phys. Rev. D 11 1404
[7] Christensen S M and Fulling S A 1977 Phys. Rev. D 15 2088
[8] Hartle J B and Hawking S W 1976 Phys. Rev. D 13 2188
[9] Satz A, Mazzitelli F D and Alvarez E 2005 Phys. Rev. D 71 064001
[10] Louko J and Satz A 2008 Class. Quantum Grav. 25 055012
[11] Breen C and Ottewill A C 2012 Phys. Rev. D 85 084029