The solar LMA neutrino oscillation solution in the Zee model

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Abstract

We examine the neutrino mass matrix in the version of Zee model where both Higgs doublets couple to the leptons. We show that in this case one can accommodate the large mixing angle (LMA) MSW solution of the solar neutrino problem, while avoiding maximal solar mixing and conflicts with constraints on lepton family number-violating interactions. In the simplified scenario we consider, we have the neutrino mass spectrum characterized by $m_1 \simeq m_2 \simeq \sqrt{\Delta m^2_{atm}}/\sin 2\theta$ and $m_3/m_1 \simeq \cos 2\theta$, where $\theta$ is the solar mixing angle.

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1 Introduction

The results of the atmospheric [1] and solar neutrino experiments (for a recent presentation of the solar results see, e.g., Ref. [2]) indicate that neutrinos have non-zero masses. A suitable environment for obtaining Majorana neutrino masses is to extend the Higgs sector of the Standard Model [3]. Extending the Standard Model with a singly charged gauge singlet scalar and adding a second Higgs doublet allows to write down an explicitly lepton number-violating interaction in the Higgs potential and leads to 1-loop neutrino masses [4]. In the following we will only consider the lepton sector of the Zee model.

The Zee model is traded in two versions in the literature: the original model [4], which we will call general Zee model (GZM) in this paper, and a simpler version of the Zee model where only one of the two Higgs doublets couples in the lepton sector [5]. The latter version, which we will call restricted Zee model (RZM), can naturally be achieved with a discrete symmetry [6]; it has the advantage that the family lepton number-violating interactions mediated by the couplings of the three physical neutral scalars are absent. The interesting point is that the RZM leads to a symmetric neutrino mass matrix with zeros in the diagonal\(^1\); this mass matrix is called Zee mass matrix in the literature. One can easily check that all phases in the Zee mass matrix can be absorbed into the left-handed neutrino fields and, therefore, in neutrino oscillations no CP violation is observable if the Zee mass matrix is the correct neutrino mass matrix.

The Zee mass matrix has a special feature: it has been shown [7, 8] that it allows to accommodate only bimaximal mixing, i.e., solar and atmospheric mixing angles are both very close to 45°. While this is in excellent agreement with the atmospheric neutrino data, results from the solar neutrino experiments are not that well compatible with a solar mixing angle \(\theta \simeq 45^\circ\), though this value is also not excluded [2, 9]. The purpose of the present paper is to show that in the GZM it is no problem to accommodate the large mixing angle (LMA) MSW solution of the solar neutrino deficit, while at the same time all constraints on the additional couplings in the lepton sector stemming from the second Higgs doublet are respected. In the GZM also the diagonal elements of the neutrino mass matrix are non-zero in general. Since the GZM is a quite rich and intricate model, we restrict ourselves rather to an “existence proof” of the LMA MSW solution within the GZM instead of discussing the GZM in full generality.

Let us outline our procedure:

Point 1: We assume that \(U_{e3} = 0\) (\(U\) is the neutrino mixing matrix) and the atmospheric mixing angle is exactly 45° [10]. Then in the mixing matrix the only free parameter is the solar mixing angle \(\theta\). This gives a certain form of the mass matrix with 4 complex parameters (Section [2]).

Point 2: The general problem is still quite intricate, so we set one of these parameters of the mass matrix equal to zero and assume that the remaining ones are real. In this scenario we can relate in a simple way the three real parameters with the physical quantities \(\theta\), the solar mixing angle, \(\Delta m^2_\odot = |m^2_1 - m^2_2|\), the solar mass-squared difference, \(\Delta m^2_\text{atm} = |m^2_3 - (m^2_1 + m^2_2)/2|\), the atmospheric mass-squared difference

\(^1\)We always work in a basis where the charged lepton mass matrix is diagonal.
and $m_3$ ($m_j$ with $j = 1, 2, 3$ are the neutrino masses) and it is possible to have the LMA angle solution of the solar neutrino problem\(^2\) (Section 3).

**Point 3:** Now the GZM is brought into play. After a discussion of the neutrino mass matrix in this model (Section 4) we assume that all quantities in the model are real and we set to zero all but two of the additional Yukawa couplings present in the GZM; this has the purpose of avoiding as many family lepton number-violating neutral scalar interactions as possible. We show that with these two additional coupling constants the restricted mass matrix mentioned in Point 2, which allows for the LMA MSW solution, can be accommodated (Section 4).

**Point 4:** We complete our procedure by a numerical discussion of the parameters of our scenario and by estimates of the rates of (family) lepton number-violating processes (Section 6).

We also review the features of the RZM as a limit of the GZM (Section 7) and summarize the results (Section 8). In the appendix we present the general formulas for the 1-loop Majorana neutrino mass matrix induced by charged scalar loops.

## 2 Neutrino mixing and the mass matrix

The Majorana neutrino mass matrix $M_\nu$ is diagonalized with a unitary matrix $V$ by

$$V^T M_\nu V = \hat{m} = \text{diag} \left( m_1, m_2, m_3 \right).$$

With the assumptions mentioned in the introduction in Point 1 we can write the matrix $V$ as

$$V = e^{i\hat{\alpha}} U e^{i\hat{\beta}} \quad \text{with} \quad U = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta/\sqrt{2} & \cos \theta/\sqrt{2} & 1/\sqrt{2} \\ \sin \theta/\sqrt{2} & -\cos \theta/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \quad (2)$$

The phases in $V$ suggest the definitions

$$M'_\nu = U \hat{\mu} U^T \quad \text{with} \quad M'_\nu \equiv e^{i\hat{\alpha}} M_\nu e^{i\hat{\alpha}} \quad \text{and} \quad \hat{\mu} = e^{-2i\hat{\beta}} \hat{m}. \quad (3)$$

Then $M'_\nu$ can be expressed by $\hat{\mu}$ and the parameters of $U$ in the following way:

$$M'_\nu = \begin{pmatrix} c^2 \mu_1 + s^2 \mu_2 & -cs(\mu_1 - \mu_2)/\sqrt{2} & cs(\mu_1 - \mu_2)/\sqrt{2} \\ -cs(\mu_1 - \mu_2)/\sqrt{2} & (s^2 \mu_1 + c^2 \mu_2 + \mu_3)/2 & (-s^2 \mu_1 - c^2 \mu_2 + \mu_3)/2 \\ cs(\mu_1 - \mu_2)/\sqrt{2} & (s^2 \mu_1 - c^2 \mu_2 + \mu_3)/2 & (s^2 \mu_1 + c^2 \mu_2 + \mu_3)/2 \end{pmatrix} \quad (4)$$

with $c \equiv \cos \theta$ and $s \equiv \sin \theta$ (see, e.g., Refs. [7, 11]). Thus, the mass matrix has the structure

$$M'_\nu = \begin{pmatrix} a & -b & b \\ -b & c & d \\ b & d & c \end{pmatrix}. \quad (5)$$

\(^2\)This means that $\theta$ is large but safely smaller than 45°.
Consequently, for having $U_{e3} = 0$ and an atmospheric mixing angle of exactly 45° the following conditions on $\mathcal{M}'_\nu$ are necessary:

1. Condition 1: $\mathcal{M}'_{\nu e\mu} + \mathcal{M}'_{\nu e\tau} = 0$, (6a)
2. Condition 2: $\mathcal{M}'_{\nu \mu\mu} = \mathcal{M}'_{\nu \tau\tau}$, (6b)
3. Condition 3: $\tan 2\theta = 2\sqrt{2} \frac{b}{a - c + d} \in \mathbb{R}$. (6c)

Finally, with the parameterization (5) the complex masses $\mu_j$ are found as

$$\mu_1 = \frac{1}{2} \left( a + c - d \pm \left[ (a - c + d)^2 + 8b^2 \right]^{1/2} \right), \quad (7a)$$
$$\mu_2 = \frac{1}{2} \left( a + c - d \mp \left[ (a - c + d)^2 + 8b^2 \right]^{1/2} \right), \quad (7b)$$
$$\mu_3 = c + d. \quad (7c)$$

### 3 A simplified mass matrix

Having discussed the general form of the mass matrix which leads to the mixing matrix (2), we now investigate the consequences of the following simplifying assumptions:

$$a, b, d \in \mathbb{R} \quad \text{and} \quad c = 0.$$ (8)

In the next section, this scenario will be reproduced in the framework of the Zee model. With the reality assumptions, the quantities $\mu_j$ (3) are identical with the neutrinos masses apart from possible signs. The experimentally accessible quantities are expressed as

$$\tan 2\theta = 2\sqrt{2} \frac{b}{a + d}, \quad (9a)$$
$$\Delta m^2_\odot = |a - d| \left[ (a + d)^2 + 8b^2 \right]^{1/2}, \quad (9b)$$
$$\Delta m^2_{\text{atm}} = \frac{1}{2} (a^2 - d^2) + 2b^2, \quad (9c)$$
$$m_3 = |d|. \quad (9d)$$

by the parameters $a, b, d$. We have chosen $m_3$ as representative of the absolute neutrino mass values, since it is simply given by Eq. (9d). Without loss of generality we will adopt henceforth the following conventions: $0^\circ \leq \theta \leq 90^\circ$, $m_1 < m_2$, $\mu_3 = m_3$. It follows from the last relation and from Eq. (7c) that $d$ is positive. Note that in Eq. (9c) no absolute value of the right-hand side of the equation is necessary, because it must be positive. The argument goes as follows. Suppose that $(a^2 - d^2)/2 + 2b^2 = -\Delta m^2_{\text{atm}}$. Then it follows that $d^2 - a^2 \geq 2\Delta m^2_{\text{atm}}$. Therefore, $d^2 - a^2$ is positive, which allows to derive the inequality $\Delta m^2_\odot \geq d^2 - a^2 \geq 2\Delta m^2_{\text{atm}}$ from Eq. (9d). This is a contradiction to the values of the mass-squared differences, fitted from the data (2).

In Eqs. (9a), (9b), (9c) and (9d), four physical quantities are expressed by three parameters. Therefore, a consistency condition exists, which is given by

$$\Delta m^2_{\text{atm}} = \frac{1}{2} \eta \Delta m^2_\odot |\cos 2\theta| + \frac{1}{4} \left( \sqrt{m^2_3 + \eta \Delta m^2_\odot |\cos 2\theta| + \eta' m_3} \right)^2 \tan^2 2\theta. \quad (10)$$
The signs $\eta$ and $\eta'$ occurring in this equation are $\eta = \text{sign} \left( a^2 - d^2 \right)$ and $\eta' = \text{sign} \ a$. In the context of the Zee model we will finally need the relations

$$a^2 = m_3^2 + \eta \Delta m_{\odot}^2 \left| \cos 2\theta \right|, \quad (11a)$$

$$b^2 = \frac{1}{2} \Delta m_{\text{atm}}^2 - \frac{1}{4} \eta \Delta m_{\odot}^2 \left| \cos 2\theta \right|. \quad (11b)$$

Looking at the consistency condition (10) and assuming that $m_3^2$ is of the order of $\Delta m_{\odot}^2 |\cos 2\theta|$ or smaller, we obtain $\sin^2 2\theta / |\cos 2\theta| \sim \Delta m_{\text{atm}}^2 / \Delta m_{\odot}^2$, which amounts to bimaximal mixing. Since we want to show that the Zee model allows to avoid bimaximal mixing we concentrate on

$$m_3^2 \gg \Delta m_{\odot}^2 |\cos 2\theta|. \quad (12)$$

With this assumption it is easy to obtain an approximate expression for $m_3^2$. One can check that for $\eta' = -1$ one arrives again at bimaximal mixing. Using $\eta' = 1$, we have $a > 0$ and we easily calculate

$$m_3^2 = \Delta m_{\text{atm}}^2 \left\{ \frac{1}{\tan^2 \theta} - \eta \frac{1}{2} \frac{\Delta m_{\odot}^2}{\Delta m_{\text{atm}}^2} \left| \cos 2\theta \right| + \frac{1}{16} \left( \frac{\Delta m_{\odot}^2}{\Delta m_{\text{atm}}^2} \right)^2 \sin^2 2\theta + \ldots \right\}. \quad (13)$$

Using this equation the condition (12) implies

$$|\cos 2\theta| \gg \frac{\Delta m_{\odot}^2}{\Delta m_{\text{atm}}^2}. \quad (14)$$

Equation (13) together with $a > 0$ and $d > 0$, allows to estimate from Eq. (11a) that

$$a - d \simeq \frac{1}{2} \eta \frac{\Delta m_{\odot}^2}{\sqrt{\Delta m_{\text{atm}}^2}} \sin 2\theta. \quad (15)$$

From Eqs. (9a) and (9b) and the convention $m_1 < m_2$, we can express the masses $m_1$ and $m_2$ as

$$m_1 = \frac{1}{2} \left( -|a - d| + \frac{\Delta m_{\odot}^2}{|a - d|} \right), \quad (16a)$$

$$m_2 = \frac{1}{2} \left( |a - d| + \frac{\Delta m_{\odot}^2}{|a - d|} \right). \quad (16b)$$

Inspection of Eqs. (16a) and (16b) reveals that our conventions fix the signs of $\mu_{1,2}$: $\text{sign} \mu_1 = -\text{sign} \mu_2 = -\eta$. Then, with Eq. (15), an estimate of the neutrino masses which neglects the solar mass-squared difference is given by

$$m_1 \simeq m_2 \simeq \sqrt{\frac{\Delta m_{\text{atm}}^2}{\sin 2\theta}} \quad \text{and} \quad m_3 \simeq \sqrt{\frac{\Delta m_{\text{atm}}^2}{|\tan 2\theta|}}. \quad (17)$$

This equation tells us that $m_3 < m_{1,2}$, at least in the regime of large solar mixing.
4 Neutrino masses in the general Zee model

In the previous section we have discussed the mass matrix determined by Eqs. (5) and (8) without reference to any specific model of neutrino masses. Now we introduce the Zee model \[4\] and discuss the neutrino mass matrix in the case that both scalar doublets of the Zee model couple in the lepton section. The Yukawa Lagrangian is given by

$$\mathcal{L}_Y = -\sum_{a=1}^{2} \bar{L}_a \phi_a \ell_R + L^T C^{-1} i \tau_2 f L h^+ + \text{h.c.},$$

where \( f \) is an antisymmetric \( 3 \times 3 \) matrix \[4\]. The mass matrix of the charged leptons arises at tree level through

$$\langle \phi_a \rangle_0 = \frac{v_a}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad M_\ell = \frac{1}{\sqrt{2}} \sum_{a=1}^{2} v_a \Gamma_a$$

with \( v \equiv \sqrt{|v_1|^2 + |v_2|^2} \simeq 246 \text{ GeV} \). The physical charged scalar fields \( H_1^+, H_2^+ \) with masses \( M_1, M_2 \), respectively, and the would-be Goldstone boson \( \phi_w^+ \) are obtained by the unitary transformation \[4, 6, 12\]

$$\begin{pmatrix} \phi_1^+ \\ \phi_2^+ \\ h^+ \end{pmatrix} = \begin{pmatrix} \frac{v_2}{v} & -\frac{v_2}{v} W_{11} & -\frac{v_2}{v} W_{12} \\ \frac{v_1}{v} W_{11} & \frac{v_1}{v} W_{12} & 0 \\ 0 & W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} \phi_w^+ \\ H_1^+ \\ H_2^+ \end{pmatrix}. \quad (20)$$

As anticipated in the introduction, we assume to be in a basis where the charged lepton mass matrix is diagonal, i.e., \( M_\ell = \hat{M}_\ell \), where the hat symbolizes that this mass matrix is diagonal. In this basis we have

$$\Gamma_1 = \frac{1}{v_1} (\sqrt{2} \hat{M}_\ell - v_2 \Gamma_2).$$

In our parameterization the flavour-changing Higgs couplings are given by the off-diagonal elements of \( \Gamma_2 \). Furthermore, off-diagonal elements of the \( 2 \times 2 \) unitary matrix \( W \) are present because of the vacuum expectation value of the lepton number-violating term

$$\lambda h^+ \phi_1^+ \phi_2 + \text{h.c.} \quad (22)$$

in the Higgs potential.

The physical charged Higgses couple to the leptons in the following way:

$$-\mathcal{L}_Y(H^+) = \bar{\nu}_L \left( -\sqrt{2} \frac{v_2}{v_1} \hat{M}_\ell + \frac{v}{v_1} \Gamma_2 \right) \ell_R (W_{11} H_1^+ + W_{12} H_2^+) + (\bar{\nu}_L \ell) (2f) \ell_R (W_{21} H_1^+ + W_{22} H_2^+) + \text{h.c.} \quad (23)$$

With the formulas in the appendix we obtain \[4\]

$$\mathcal{M}_\nu = \sum_{j=1}^{2} A^j_L \hat{M}_\ell I(M_j^2, \hat{M}_\ell^2) A^j_R \text{ transp.} \quad (24)$$
with
\[ A^J_R = \left( -\sqrt{2} \frac{v}{v_1 v} \hat{M}_\ell + \frac{v}{v_1} \Gamma_2 \right) W_{1j} \quad \text{and} \quad A^J_L = 2 f W_{2j}. \] (25)

The infinity in Eq. (24) cancels \cite{4} because \[ \sum_{j=1}^2 W_{2j} W_{1j}^* = 0. \] Defining
\[ J(M^2_1, M^2_2, m^2) = \frac{M^2_1}{M^2_1 - m^2} \ln \frac{M^2_1}{m^2} - \frac{M^2_2}{M^2_2 - m^2} \ln \frac{M^2_2}{m^2}, \] (26)
from Eq. (24) we obtain the final result \cite{4, 12}
\[ M_\nu = 2 W_{21} W_{11}^* \times \frac{1}{(4\pi)^2} \]
\[ \times \left\{ \sqrt{2} \frac{v}{v_1 v} \left( \hat{M}^2_\ell \hat{J} f - \hat{f} \hat{M}_\ell \right) - \frac{v}{v_1} \left( \gamma \hat{M}_\ell \hat{J} f - \hat{f} \hat{M}_\ell \gamma^T \right) \right\} \] (27)
with \[ \hat{J} \equiv \text{diag} \left( J(M^2_1, M^2_2, m^2_\alpha) \right) \] \( (\alpha = e, \mu, \tau) \) and \[ \gamma \equiv \Gamma^2_2. \] Note that for \( m^2 \ll M^2_{1,2} \) the function \( J \) simplifies to
\[ J \simeq 2 \ln(M_1/M_2). \] (28)

For the product \( W_{21} W_{11}^* \) of elements of the charged-scalar mixing matrix \( W \), we obtain the relation
\[ W_{21} W_{11}^* = \frac{\lambda^* v}{\sqrt{2}(M^2_1 - M^2_2)}. \] (29)

It shows explicitly that the Majorana neutrino masses are proportional to the coupling \( \lambda \) in the Higgs potential \cite{22}.

In the GZM considered here, there are family lepton number-violating processes induced by the charged and the neutral scalar interactions. Experimental bounds constrain the coupling matrices \( f \) and \( \Gamma_2 \).

5 The simplified mass matrix within the Zee model

In this section, our aim is to reproduce the neutrino mass matrix defined by Eqs. (5) and (8) within the GZM. In order to save the amount of writing we introduce the notation

\[ M_\nu = A \left( \hat{r}^2 f - \hat{f} \hat{r}^2 \right) - B \left( \gamma \hat{r} f - \hat{f} \gamma^T \right) \] with \[ \hat{r} = \text{diag} \left( m_e, m_\mu, m_\tau \right)/v. \] (30)

Both constants \( A \) and \( B \) are of the order of 1 GeV, resulting from dividing the electroweak scale by \( 16\pi^2 \):
\[ B = 2 W_{21} W_{11}^* \times \frac{1}{(4\pi)^2} \ln \frac{M^2_2}{M^2_1} \times \frac{v^2}{v_1^2} \] and \[ A = \sqrt{2} \frac{v_2}{v} B. \] (31)

Since the off-diagonal elements of \( \gamma \) introduce flavour-changing neutral interactions, we adopt the philosophy to set to zero as many of them as possible. As we will see, it turns out that
\[ \gamma_{e\tau} \neq 0 \quad \text{and} \quad \gamma_{\tau\tau} \neq 0, \] (32)
and all other elements of $\gamma$ being equal to zero, is sufficiently general to avoid bimaximal mixing, which necessarily happens for $\gamma = 0$ \cite{7}. It can easily be checked that with this assumption Condition 2 \cite{13b} is fulfilled by having $M_{\nu\mu\mu} = M_{\nu\tau\tau} = 0$. Furthermore, we assume that all quantities we deal with are real: $A, B$, the elements of the matrix $f$, and $\gamma_{\tau\tau}$ and $\gamma_{\tau\tau}$. Thus we identify $M_{\nu}$ with $M'_{\nu}$.

We have to reproduce with the mass matrix \cite{30} the parameters $a$ \cite{11a}, $b$ \cite{11b}, $d$ \cite{13d} and Condition 1 \cite{6a}. Thus we have five coupling constants, three in $f$ and $\gamma_{\tau\tau}$, $\gamma_{\tau\tau}$ and four relations. It is convenient to express the other four coupling constants as a function of $\gamma_{\tau\tau}$:

$$f_{e\mu} = \frac{b}{A(r^2_\mu - r^2_\tau)} \left\{ 1 + \frac{ad}{2b^2} A(r^2_\tau - r^2_\mu) - Br_\tau \gamma_{\tau\tau} \right\},$$

$$f_{e\tau} = \frac{-b}{A(r^2_\tau - r^2_\mu) - Br_\tau \gamma_{\tau\tau}},$$

$$f_{\mu\tau} = \frac{-d}{A(r^2_\tau - r^2_\mu) - Br_\tau \gamma_{\tau\tau}},$$

$$\gamma_{\tau\tau} = -\frac{a}{2Br_\tau b} \left\{ A(r^2_\tau - r^2_\mu) - Br_\tau \gamma_{\tau\tau} \right\}.$$  

6 Numerical estimates

Let us now estimate the values of the coupling constants. For definiteness we take $\Delta m^2_{\text{atm}} = 3 \times 10^{-3}$ eV$^2$ and $A = 2$ GeV. Furthermore, we need the values of $r^2_\tau \approx 5.22 \times 10^{-5}$ and $m^2_\mu/m^2_\tau \approx 3.535 \times 10^{-3}$. Defining $x = B\gamma_{\tau\tau}/Ar_\tau$ and assuming that\footnote{This avoids some finetuning for $\gamma_{\tau\tau}$. Note, however, that $x = 0$ is also possible.} $x \sim 1$ and $(1 - m^2_\tau/m^2_\mu - x)/(1 - m^2_\mu/m^2_\tau - x) \simeq 1$, we obtain

$$f_{e\mu} \simeq 1.0 \times 10^{-4} \times \frac{\text{sign } b}{\sin^2 2\theta},$$

$$f_{e\tau} \simeq -3.7 \times 10^{-7} \times \frac{\text{sign } b}{1 - x},$$

$$f_{\mu\tau} \simeq -5.2 \times 10^{-7} \times \frac{1}{\tan 2\theta (1 - x)},$$

$$\gamma_{\tau\tau} \simeq -7.2 \times 10^{-3} \times \frac{v_2}{v \tan 2\theta}.$$  

These equations serve to see the orders of magnitude and any effects of $\Delta m^2_{\text{atm}}$ are neglected. As can be seen from Eq. (15), a considerable amount of finetuning is involved in order to reproduce the solar mass-squared difference.

Now we concentrate on the LMA MSW solution of the solar neutrino problem, where $\theta$ is in the first octant. With Eq. (2a) it follows that $b > 0$. In this case a representative value of the mixing angle $\theta$ is given by the best fit value $\sin^2 2\theta \approx 0.75$ (tan $2\theta \approx 1.71$) of Ref. [2], with the corresponding mass-squared difference $\Delta m^2_{\text{atm}} \approx 3.2 \times 10^{-5}$ eV$^2$. We note that in this case we have $|f_{e\tau}/f_{e\mu}| \sim m^2_\mu/m^2_\tau$, which is similar to the case of the Zee mass matrix \cite{7}. On the other hand, in our scenario we have $f_{e\tau} \sim f_{\mu\tau}$, whereas for the
Zee mass matrix the relation $|f_{e\tau}| \gg |f_{\mu\tau}|$ holds [7]. As far as $\gamma$ is concerned, with $x \sim 1$ we can have $\gamma_{e\tau} \sim \gamma_{\tau\tau} \sim 10^{-3} \div 10^{-2}$.

Due to our assumption (32), flavour-changing neutral scalar interactions at the tree level are very constrained. Among the charged lepton decays we only have $\tau^- \to e^- \ell^+ \ell^-$ with $\ell = e,\mu$. A generous estimate of the branching ratio of this decay for $\ell = \mu$ is obtained by

$$
\text{Br}(\tau^- \to e^- \mu^+ \mu^-) \sim \frac{1}{G_F^2 M_0^4} \gamma_{e\tau}^2 \left( \frac{m_{\mu}}{v} \right)^2 < 10^{-8},
$$

(35)

where we have taken $G_F^2 M_0^4 \sim 10^{-2}$ ($M_0 \sim 100$ GeV) and $M_0$ is a generic neutral Higgs mass. At the 1-loop level, neutral Higgs exchange also induces the decay $\tau^- \to e^- \gamma$.

Making again an estimate, we obtain [13, 14]

$$
\text{Br}(\tau^- \to e^- \gamma) \sim \frac{\alpha}{48\pi} \frac{1}{G_F^2 M_0^4} (\gamma_{e\tau} \gamma_{\tau e})^2 < 10^{-10}.
$$

(36)

Both estimates are well compatible with the experimental upper bounds on these branching ratios of the order of $10^{-6}$ [15]. More detailed discussions of these decays are found in Refs. [14, 16]. We have used $|\gamma_{e\tau}| \lesssim 10^{-2}$ and $|\gamma_{\tau\tau}| \lesssim 10^{-2}$ in Eqs. (35) and (36).

Also the charged scalars participate in various charged lepton decays as intermediate particles. Numerous decays of the type $\ell_1^- \to \ell_2^- + 2$ neutrinos proceed at tree level via the Lagrangian (23). According to the couplings in this Lagrangian we can distinguish between $f$, $\gamma$ and $\hat{r}$ vertices, and we can have decay amplitudes with all possible combinations of these vertices, except $\gamma$-$\gamma$, which is forbidden due to the restricted form of $\gamma$, Eq. (32). E.g., to the Standard Model amplitude of ordinary muon decay there is an amplitude with couplings $r_{\mu}r_e$ and another one with $f_{e\tau}f_{\mu\tau}$; another example is $\tau^- \to \bar{\nu}_e \nu_\mu \mu^-$ with couplings $f_{e\tau}r_\mu$, which is lepton number violating. The branching ratios of all these decays are negligible because of the smallness of the coupling constants $f_{\alpha\beta}$ and $\gamma_{e\tau}$ and the ratios $r_\alpha$. Also negligible are radiative decays $\ell_1^- \to \ell_2^- + \gamma$ induced by charged Higgs loops [13]. In this case one always has two $f$-vertices in the loop graph, except in the case of the amplitude for $\tau^- \to e^- + \gamma$, where there are two contributions, proportional to $f_{\mu\tau}f_{\mu\tau}$ and $\gamma_{e\tau}r_e$. Recent reviews of the restrictions on the coupling constants $f_{\alpha\beta}$ are found in Refs. [17, 18].

Scalar contributions to the anomalous magnetic moments of the electron and muon involving $f$ [18] and $\hat{r}$-couplings are totally negligible because these constants are too small. A contribution from the $\gamma$-couplings to the electron magnetic moment coming from $\tau$ exchange is proportional to $\gamma_{e\tau} \gamma_{\tau e}$ [19] and is thus zero in view of $\gamma_{\tau e} = 0$.

In our scenario we have $M_{\nu ee} = a \neq 0$. The matrix element $M_{\nu ee}$ is identical with the effective neutrino mass $\langle m_\nu \rangle$ probed in neutrinoless double-beta decay. Therefore, this decay is allowed and the effective neutrino mass is given by

$$
|\langle m_\nu \rangle| \simeq m_3 \simeq \frac{0.05 \text{ eV}}{\tan 2\theta}.
$$

(37)

This represents an order of magnitude which is accessible in future experiments (for recent reviews see, e.g., Ref. [20]).
Having seen that the LMA MSW solution of the solar neutrino problem can be accommodated in our scenario, we now proceed to the small mixing angle (SMA) MSW solution. In this case we take as illustration the best fit value of Ref. [2], \( \sin^2 2 \theta \simeq 2.3 \times 10^{-3} \) \((\tan 2 \theta \simeq 4.8 \times 10^{-2})\), which has a corresponding \( \Delta m^2_{\odot} \simeq 0.5 \times 10^{-5} \) eV\(^2\). From Eq. (34a) we see that now \( f_{e\mu} \simeq 0.05 \) becomes relatively large and barely compatible with the requirement that the Zee boson does not have an effect on the muon decay rate so that it does not destroy the agreement in electroweak precision tests [17]. Thus in our simple scenario we cannot incorporate safely the SMA solution.

### 7 The limit \( \Gamma_2 \to 0 \)

To make contact with Refs. [7, 8], we explore the effect of \( \Gamma_2 = 0 \). This means that all diagonal elements of the neutrino mass matrix are zero [1] and, in our notation, we have \( a = 0 \), i.e., \( m_3^2 = \Delta m^2_{\odot} |\cos 2 \theta| \) and \( \eta = -1 \) (see Eq. (11a)). These relations and inspection of the consistency condition (10) leads to

\[
m_3 \simeq \frac{1}{2} \frac{\Delta m^2_{\odot}}{\sqrt{\Delta m^2_{\text{atm}}}}, \quad m_1 \simeq m_2 \simeq \sqrt{\Delta m^2_{\text{atm}} \text{ and } |\cos 2 \theta| \simeq \frac{1}{4} \frac{\Delta m^2_{\odot}}{\Delta m^2_{\text{atm}}}}.
\]

From the last relation we read off that \( \theta \) is 45° for all practical purposes. The consequences for the coupling matrix \( f \) are

\[
\left| \frac{f_{e\tau}}{f_{e\mu}} \right| \simeq \frac{m_\mu^2}{m_\tau^2} \text{ and } \left| \frac{f_{e\tau}}{f_{\mu\tau}} \right| \simeq \sqrt{2} \frac{\Delta m^2_{\text{atm}}}{\Delta m^2_{\odot}}.
\]

The first of these two relations is obtained by taking the ratio of Eq. (33b) and Eq. (33a), where \( a = 0 \) is taken into account. In the second relation, Eq. (11b) has been used. The results (38) and (39) agree with those of Refs. [7, 8].

### 8 Summary

In this paper we have discussed neutrino masses and mixing in the general Zee model, where both Higgs doublets couple in the lepton sector. In this endeavour, we were motivated by the result that for \( \Gamma_2 = 0 \), where \( \Gamma_2 \) denotes the Yukawa coupling matrix of the second Higgs doublet, the Zee mass matrix leads to bimaximal mixing. It is true that a mixing angle of 45° is perfect for the description of the atmospheric neutrino data, but it does not represent a very good fit for the solar neutrino data. The general Zee model is a rather rich and intricate model. Therefore, in order to simplify the analysis, we have assumed that all quantities appearing in the neutrino mass matrix are real and that in the second Yukawa coupling matrix \( \Gamma_2 \), which is set to zero usually, only two elements, the \( e\tau \) and \( \tau\tau \) elements, are non-zero. We have shown that this is sufficient to accommodate solutions of the solar neutrino problem with a large mixing angle instead of maximal mixing; on the other hand, in the atmospheric sector maximal mixing remains. At the same time, all dangerous processes induced by the new couplings are sufficiently suppressed. Actually, we could have even set the \( \tau\tau \) element of \( \Gamma_2 \) equal to zero and used
a single non-zero element in this coupling matrix. However, the $\tau \tau$ element represents a possibility to achieve a further suppression of the potentially more dangerous $\epsilon \tau$ element. As in the case $\Gamma_2 = 0$, the neutrino mass $m_3$ is the smallest neutrino mass, and the resulting mass spectrum is rather of the type which is called "inverted hierarchy". However, whereas for $\Gamma_2 = 0$ one has $m_3 \ll m_1 \simeq m_2$, now $m_3 \simeq \sqrt{\Delta m_{\text{atm}}^2 / \tan 2\theta}$ is of the same order of magnitude as the other two masses.

Numerous finetunings are involved in our scenario. Implementing the condition (6a) and setting all but two elements of $\Gamma_2$ equal to zero represent rather severe finetunings. These procedures were useful for exactly having $U_{e3} = 0$ and an atmospheric mixing angle of 45°, and for avoiding a dangerous class of lepton family number-violating interactions. In order to reproduce the neutrino masses within our scenario, we need $|f_{\epsilon \tau}| \sim |f_{\mu \tau}| \sim |f_{e\mu}| (m_\mu / m_\tau)^2$ and a quite drastic finetuning between $a$ and $d$ in the mass matrix (5), in order to accommodate the solar mass-squared difference. Of course, the mass-squared difference $\Delta m_2^\odot \sim 10^{-8}$ eV$^2$ of the quasi-vacuum oscillation solution [2] would require a stronger finetuning than $\Delta m_2^\odot \sim 10^{-5}$ eV$^2$ of the large mixing angle solution. Examination of the Zee model with a general $\Gamma_2$ and with small deviations of the neutrino mass matrix from the form (5) should reveal how much our finetunings could be relaxed. In the present work we have confined ourselves to prove that it is possible to reproduce the LMA MSW solution within the Zee model.

In conclusion, even in the case that bimaximal mixing is ruled out, the Zee model will remain a viable and interesting scenario in order to accommodate the neutrino oscillation solutions of the solar and atmospheric neutrino problems.

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A Charged scalar exchange and the 1-loop neutrino mass

We proceed from the general Lagrangian
\begin{equation}
- \mathcal{L}_Y (\nu_L, \ell, \phi^+) = (\bar{\nu}_L A_R \ell_R + (\bar{\nu}_L)^c A_L \ell_L) \phi^+ + \text{h.c., (A1)}
\end{equation}
where the $A_{L,R}$ are $3 \times 3$ coupling matrices in the case of 3 families and $\ell_{L,R}$ denote chiral charged fermion fields. The neutrino Majorana mass Lagrangian is defined by
\begin{equation}
\mathcal{L}_\nu^{\text{mass}} = \frac{1}{2} \nu_L^T C^{-1} \mathcal{M}_\nu \nu_L + \text{h.c. (A2)}
\end{equation}
With the scalar interactions (A1) we obtain
\begin{equation}
\mathcal{M}_\nu^{1\text{-loop}} = A_L \tilde{M}_I (M_\phi^2, \tilde{M}_I^2) A_R^\dagger + \text{transp.}, (A3)
\end{equation}
where we have defined

$$I(M^2, m^2) = \frac{1}{(4\pi)^2} \times \left\{ -\frac{1}{\epsilon} + \ln(4\pi) + \gamma_E - 1 + \frac{1}{M^2 - m^2} \left( M^2 \ln \frac{M^2}{\mu^2} - m^2 \ln \frac{m^2}{\mu^2} \right) \right\}. \tag{A4}$$

We have used dimensional regularization; thus, $\epsilon = (4-n)/2$ with $n$ being the number of space-time dimensions, $\gamma_E$ is Euler’s constant and $\mu$ is an arbitrary mass scale.

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