ON m-DIMENSIONAL TORIC CODES

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Abstract. Toric codes are a class of m-dimensional cyclic codes introduced recently by J. Hansen in [7, 8], and studied in [9, 5, 10]. They may be defined as evaluation codes obtained from monomials corresponding to integer lattice points in an integral convex polytope \( P \subseteq \mathbb{R}^m \). As such, they are in a sense a natural extension of Reed-Solomon codes. Several articles cited above use intersection theory on toric surfaces to derive bounds on the minimum distance of some toric codes with \( m = 2 \). In this paper, we will provide a more elementary approach that applies equally well to many toric codes for all \( m \geq 2 \). Our methods are based on a sort of multivariate generalization of Vandermonde determinants that has also been used in the study of multivariate polynomial interpolation. We use these Vandermonde determinants to determine the minimum distance of toric codes from rectangular polytopes and simplices. We also prove a general result showing that if there is a unimodular integer affine transformation taking one polytope \( P_1 \) to a second polytope \( P_2 \), then the corresponding toric codes are monomially equivalent (hence have the same parameters). We use this to begin a classification of two-dimensional cyclic toric codes with small dimension.

1. Introduction

In [7], J. Hansen introduced the notion of a toric code. Let \( P \subseteq \mathbb{R}^m \) be an integral convex polytope (the convex hull of some set of integer lattice points). We suggest [11] as a good general reference for the geometry of polytopes. Suppose that \( P \cap \mathbb{Z}^m \) is properly contained in the rectangular box \([0, q-2]^m\) (which we denote \( \Box_q \)), for some prime power \( q \). Then a toric code is obtained by evaluating linear combinations of the monomials with exponent vector in \( P \cap \mathbb{Z}^m \) at some subset (usually all) of the points of \((\mathbb{F}_q^*)^m\). We formalize this in the following definition.

Definition 1.1. Let \( \mathbb{F}_q \) be a finite field with primitive element \( \alpha \). For \( f \in \mathbb{Z}^m \) with \( 0 \leq f_i \leq q-2 \) for all \( i \), let \( p_f = (\alpha^{f_1}, \ldots, \alpha^{f_m}) \) in \((\mathbb{F}_q^*)^m\). For each \( e = (e_1, \ldots, e_m) \in P \cap \mathbb{Z}^m \), let \( x^e \) be the corresponding monomial and write

\[
(p_f)^e = (\alpha^{f_1})^{e_1} \cdots (\alpha^{f_m})^{e_m}.
\]

The toric code \( C_P(\mathbb{F}_q) \) over the field \( \mathbb{F}_q \) associated to \( P \) is the linear code of block length \( n = (q-1)^m \) with generator matrix

\[
G = ([p_f]^e),
\]

where the rows are indexed by the \( e \in P \cap \mathbb{Z}^m \), and the columns are indexed by the \( p_f \in (\mathbb{F}_q^*)^m \). In other words, letting \( L = \text{Span}\{x^e : e \in P \cap \mathbb{Z}^m\} \), we define the
evaluation mapping

\[ \text{ev} : L \rightarrow \mathbb{F}_q^{(q-1)m} \]

\[ g \mapsto (g(p_f) : f \in (\mathbb{F}_q^*)^m) \]

Then \( C_P = \text{ev}(L) \). If the field is clear from the context, we will often omit it in the notation and simply write \( C_P \). The matrix \( G \) will be called the standard generator matrix for the toric code.

Because of the close connection between integral polytopes and the theory of toric varieties, Hansen and others have proposed techniques from algebraic geometry such as intersection theory on algebraic surfaces and higher dimensional varieties to study toric codes and their parameters. The articles [7], [8], [9], and [10] have used this approach.

In this article, for the most part, we use a more elementary viewpoint, based on the observation that the square submatrices of the standard generator matrix of a toric code are examples of a multivariate generalization of the familiar univariate Vandermonde matrices. These multivariate Vandermonde matrices have been studied by a number of different techniques in the context of multivariate polynomial interpolation. The literature there is truly vast because of the many applications of interpolation in numerical analysis and other parts of applied mathematics. We direct the reader to the bibliography of [6]. To our knowledge, these techniques have not been used before in coding theory in this form.

Our contributions are as follows. In §2, we begin with a lower bound on the minimum distance of toric codes based on Vandermonde determinants (Proposition 2.1). We use this to study the minimum distances of toric codes from rectangular polytopes and simplices (see Theorems 2.4, 2.5, 2.9). We do this by identifying special configurations of points in \((\mathbb{F}_q^*)^m\) for which the Vandermonde determinant is nonzero. In the context of polynomial interpolation, such sets are called poised sets for the interpolation problem using linear combinations of monomials corresponding to the lattice points in some polytope. (They are the sets for which the interpolation problem has a unique solution for all function values assigned at those points.) Our methods here are suggested by the interpolation-theoretic computations in [4].

All of the 2-dimensional examples from [7] and [8] can be handled with our methods. But in fact our theorems are more general since they apply for all \( m \geq 2 \), not only the case \( m = 2 \).

In §3 we prove the general statement that lattice equivalent polytopes \( P_1, P_2 \) yield monomially equivalent toric codes \( C_{P_1}, C_{P_2} \). We apply this result to consider a classification of \( m = 2 \) toric codes up to monomial equivalence for small \( k \).

Finally, some comments about the utility of toric codes are probably in order. In the case \( m = 1 \), a toric code is just a Reed-Solomon code since \( P \) is a line segment in \([0, q - 2] \subset \mathbb{R} \) with integer endpoints. Higher dimensional toric codes are in a sense a natural extension of Reed-Solomon codes and have many similar properties. For instance, it is easy to see that they are all \( m \)-dimensional cyclic codes (see [5]). So one might hope that toric codes exist having similarly good parameters. And indeed, [9] contains a number of examples showing that toric codes can have very good parameters, equaling or bettering the best known minimum distance for a given \( n, k \) in [3]. Not all toric codes perform this well, however. Our main results on toric codes from rectangular polytopes and simplices show, in fact, that their minimum distances are often quite small for their dimensions. It is an interesting
problem, we believe, to determine criteria concerning which polytopes yield good toric codes.

We will use this notation in the following sections. Suppose that \( P \subset \mathbb{R}^{m} \) is an integral convex polytope. We will write \( \#(P) \) for the number of integer lattice points in \( P \) (that is, \( \#(P) = |P \cap \mathbb{Z}^m| \)). We will write

\[
P \cap \mathbb{Z}^m = \{ e(i) : i = 1, \ldots, \#(P) \}
\]

for the set of those integer lattice points. For any set \( S \subset \mathbb{R}^n \), \( \text{conv}(S) \) denotes the convex hull of \( S \).

2. Minimum distances via Vandermonde matrices

We begin by describing the Vandermonde matrices involved here. Using the notation introduced in \( \S \)1, let \( P \) be an integral convex polytope, and suppose \( P \cap \mathbb{Z}^m = \{ e(i) : i = 1, \ldots, \#(P) \} \), listed in some particular order. Let \( S = \{ p_j : j = 1, \ldots, \#(P) \} \) be any set of \( \#(P) \) points in \((\mathbb{F}_q^*)^m \), also ordered. Then \( V(P; S) \), the Vandermonde matrix associated to \( P \) and \( S \), is the \( \#(P) \times \#(P) \) matrix

\[
V(P; S) = \begin{pmatrix} p_j^{(i)} \end{pmatrix},
\]

where we use the standard multi-index notation \( p_j^{(i)} \) to indicate the value of the monomial \( x^{e(i)} \) at the point \( p_j \). For example, if \( P = \text{conv}\{(0,0),(2,0),(0,2)\} \) in \( \mathbb{R}^2 \), and \( S = \{(x_j,y_j)\} \) is any set of 6 points in \((\mathbb{F}_q^*)^2 \), for one particular choice of ordering of the lattice points in \( P \), we have

\[
V(P; S) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\
x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 \\
x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 \\
y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 \end{pmatrix}
\]

Since we assume \( P \subset \mathbb{P}^{m-1} \), the monomials \( x^{e(i)} \) define linearly independent functions on \((\mathbb{F}_q^*)^m \) and we can also view the \( V(P; S) \) as square submatrices of the standard generator matrix for the toric code \( C_P = C_P(\mathbb{F}_q) \). Our first observation is that Vandermonde determinants may be used to bound the minimum distance of a toric code.

**Proposition 2.1.** Let \( P \subset \mathbb{R}^m \) be an integral convex polytope. Let \( d \) be a positive integer and assume that in every set \( T \subset (\mathbb{F}_q^*)^m \) with \( |T| = (q-1)^m - d + 1 \) there exists some \( S \subset T \) with \( |S| = \#(P) \) such that \( \det V(P; S) \neq 0 \). Then the minimum distance satisfies \( d(C_P) \geq d \).

**Proof.** All codewords of \( C_P \) are linear combinations of the rows of the \( \#(P) \times (q-1)^m \) standard generator matrix. If a codeword has zeroes in the locations corresponding to the subset \( T \subset (\mathbb{F}_q^*)^m \), then by looking at the entries corresponding to \( S \subset T \) we get a system of \( \#(P) \) homogeneous linear equations in the coefficients of the linear combination, whose matrix is \( V(P; S) \). By hypothesis, this matrix is nonsingular, so all the coefficients in the linear combination must be zero. Since this is true for all \( T \), all nonzero codewords of \( C_P \) have at most \( (q-1)^m - d \) zero entries, which implies \( d(C_P) \geq d \). \( \square \)
We will apply this proposition by identifying particular configurations of points $S$ for which $\det V(P; S) \neq 0$, based on the particular monomials appearing in $P$. From examples such as (1) above, it should be relatively clear that identifying characterizations of $S$ such that $\det V(P; S) = 0$ or $\det V(P; S) \neq 0$ is difficult in general. Indeed, even in the context of multivariate polynomial interpolation, only partial results in special cases are well understood.

We will begin with the case where $P$ is a rectangular polytope in $\mathbb{R}^m$ (special cases are rectangles in the plane and rectangular solids in $\mathbb{R}^3$). For these polytopes, the extension from the case $m = 2$ to general $m \geq 2$ is almost immediate, so for notational simplicity, we will treat only the case $m = 2$ in detail.

Let $P_{k, \ell}$ be the rectangle $P_{k, \ell} = \text{conv}\{(0,0), (k,0), (0,\ell), (k,\ell)\}$. Note that $\#(P_{k, \ell}) = (k+1)(\ell+1)$. We will call any set $S$ of $(k+1)(\ell+1)$ points in $(\mathbb{R}^*)^2$ consisting of $(\ell+1)$ distinct points on each of $(k+1)$ distinct vertical lines $x = a_i$ a $(k+1) \times (\ell+1)$ configuration.

When we construct $V(P_{k, \ell}; S)$ for a $(k+1) \times (\ell+1)$ configuration we get a matrix as in the following proposition.

**Proposition 2.2.** Suppose $A = (c_{ij})$ is an $a \times a$ matrix and $B_1, B_2, \ldots, B_a$ are $b \times b$ matrices. Let $M$ be the $ab \times ab$ block matrix:

$$M = \begin{pmatrix}
c_{11}B_1 & c_{12}B_2 & \cdots & c_{1a}B_a \\
\vdots & \vdots & & \vdots \\
c_{a1}B_1 & c_{a2}B_2 & \cdots & c_{aa}B_a
\end{pmatrix}$$

Then $\det(M) = \pm \det(A)^b \det(B_1) \det(B_2) \cdots \det(B_a)$.

The matrix $M$ is similar to a tensor product matrix, but the $B_i$ may be different matrices, so this construction is somewhat more general.

**Proof.** If $\det(A) = 0$ or $\det(B_i) = 0$ for some $i$, then $\det(M) = 0$ as well. So, we assume that $\det(A) \neq 0$ and $\det(B_i) \neq 0$ for all $i$. In order to find the determinant of $M$, we may transform $M$ into a block upper triangular matrix using blockwise row operations, obtaining a matrix $M'$ in the following form:

$$M' = 
\begin{pmatrix}
c'_{11}B_1 & 0 & \cdots & * \\
0 & c'_{22}B_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & c'_{aa}B_a
\end{pmatrix},$$

in which the $c'_{ii}$'s are the same as the entries obtained by the corresponding row operations applied to the matrix $A$. Now we have a block-upper triangular matrix. This implies that the determinant of the matrix is the product of the determinants of the diagonal block entries. Thus,

$$\det(M') = (c'_{11})^b \det(B_1) \cdot (c'_{22})^b \det(B_2) \cdots (c'_{aa})^b \det(B_a)$$

$$= (c'_{11} \cdots c'_{aa})^b \det(B_1) \det(B_2) \cdots \det(B_a)$$

$$= \det(A)^b \det(B_1) \det(B_2) \cdots \det(B_a)$$
We know that $\det(A') = \pm \det(A)$ (some row interchanges might have been necessary in the reduction to upper triangular form). So, we can substitute to yield:

$$\det(M) = \pm \det(A)^b \det(B_1) \det(B_2) \cdots \det(B_a),$$

which is what we wanted to show. \hfill \Box

When $S$ is a $(k+1) \times (\ell + 1)$ configuration, consisting of $\ell + 1$ distinct points on each of $k+1$ distinct lines $x = a_u$, $u = 1, \ldots , k+1$, then $V(P_{k,\ell}; S)$ has the form given in the proposition, where $A$ is the ordinary $(k+1) \times (k+1)$ univariate Vandermonde matrix of the $x$-coordinates of the points on $x = a_u$. We know that $\det(V_{\text{Vandermonde}})$ of the Vandermonde matrix of the $y$-coordinates of the points on $x = a_u$. Hence we obtain the following consequence.

**Corollary 2.3.** Let $S$ be a $(k+1) \times (\ell + 1)$ configuration in $(\mathbb{F}_q^*)^2$. Then

$$\det(V(P_{k,\ell}; S)) \neq 0.$$

**Proof.** This follows from the factorization of the univariate Vandermonde determinant as products of differences of the $x$- and $y$-values at the points in $S$. \hfill \Box

We are now ready to prove the first major result of this section.

**Theorem 2.4.** Let $k, \ell < q - 1$ so that $P_{k,\ell} \subset \Box_{q - 1} \subset \mathbb{R}^2$. Then the minimum distance of the two-dimensional toric code $C_{P_{k,\ell}}$ is

$$d(C_{P_{k,\ell}}) = (q - 1)^2 - (k + \ell)(q - 1) + k\ell = ((q - 1) - k)((q - 1) - \ell).$$

**Proof.** We write $d = d(C_{P_{k,\ell}})$. In order to show equality, we will show that both $d \leq (q - 1)^2 - (k + \ell)(q - 1) + k\ell$ and $d \geq (q - 1)^2 - (k + \ell)(q - 1) + k\ell$. We start with the former. In $L = \text{Span}\{1, x, x^2, \ldots, x^k, y, xy, x^2y, \ldots, x^ky, \ldots, x^ky^\ell\}$ consider a polynomial $p(x, y) = q_1(x)q_2(y)$ where $q_1(x)$ and $q_2(y)$ factor completely:

$$p(x, y) = (x - a_1)(x - a_2) \cdots (x - a_k)(y - b_1)(y - b_2) \cdots (y - b_{\ell})$$

for some distinct $a_u$ and $b_v$ in $\mathbb{F}_q^*$. This means that the codeword $\text{ev}(p)$ has zeros in the positions corresponding to the points along $k$ distinct vertical lines and $\ell$ distinct horizontal lines in $(\mathbb{F}_q^*)^2$. We see that $\text{ev}(p)$ has $(k + \ell)(q - 1) - k\ell$ zeroes. Accordingly, we have $d \leq (q - 1)^2 - (k + \ell)(q - 1) + k\ell$.

Now we show $d \geq (q - 1)^2 - (k + \ell)(q - 1) + k\ell$. By Corollary 2.3 and Proposition 2.1, it suffices to show that every set $T$ of size $(k + \ell)(q - 1) - k\ell + 1$ in $(\mathbb{F}_q^*)^2$ contains a $(k+1) \times (\ell+1)$ configuration $S$. Since $q - 1 > k$, we see that $(k + \ell)(q - 1) - k\ell + 1 > (q - 1)$. So, by the pigeonhole principle, some vertical line $x = a_1$ contains $\ell + 1$ distinct points from $T$. There are at most $(q - 1) - (\ell + 1) = q - \ell - 2$ other points on that line. Therefore, there are at least $(k + \ell - 1)(q - 1) - k\ell + 1$ other points in $T$ not on $x = a_1$. To repeat this argument to find the rest of the configuration $S$, we must show that for all $1 \leq j \leq k$, $(k + \ell - j)(q - 1) - k\ell + 1 > (q - 1) - j$. To do this, we subtract $\ell(q - 1 - j)$ and perform some arithmetic:

$$(k + \ell - j)(q - 1) - k\ell + 1 - \ell(q - 1 - j) = (k - j)((q - 1) - \ell) + 1 > 0.$$ 

So, after we have found the first $j$ sets of $\ell + 1$ points on vertical lines in $T$, there are still enough additional points left in $T$ so that we can find the next set of $\ell + 1$ points. After $k + 1$ steps, we have a complete $(k + 1) \times (\ell + 1)$ configuration $S \subset T$. \hfill \Box
The method used in the proof of Theorem 2.4 extends without difficulty to toric codes constructed from \( k_1 \times k_2 \times \cdots \times k_m \) rectangular polytopes \( P_{k_1, \ldots, k_m} \subset \mathbb{R}^m \) for all \( m \geq 2 \). The result is as follows.

**Theorem 2.5.** Let \( k_1, \ldots, k_m \) be small enough so that \( P_{k_1, \ldots, k_m} \subset \mathbb{Q}^{q-1} \subset \mathbb{R}^m \). Then the minimum distance of the \( m \)-dimensional toric code \( C_{P_{k_1, \ldots, k_m}} \) is

\[
d(C_{P_{k_1, \ldots, k_m}}) = \prod_{i=1}^m ((q-1) - k_i).
\]

We next turn to the toric codes \( C_{P_{\ell}(m)} \) for \( P_{\ell}(m) \) an \( m \)-dimensional simplex of the form

\[ P_{\ell}(m) = \text{conv}\{0, \ell e_1, \ldots, \ell e_m\}, \]

where the \( e_i \) are the standard basis vectors in \( \mathbb{R}^m \). The monomials corresponding to the \( (m+\ell) \) integer lattice points in \( P_{\ell}(m) \) are all of the monomials in \( m \) variables of total degree \( \leq \ell \). Naturally enough, the corresponding Vandermonde matrices arise in the study of multivariate polynomial interpolation using polynomials of bounded total degree. The next recursive definition gives the special configurations of points where we will be able to compute Vandermonde determinants for the \( P_{\ell}(m) \).

**Definition 2.6.** If \( m = 1 \), an \( \ell \)th order simplicial configuration is any collection of \( \binom{1+\ell}{\ell} \) distinct points in \( \mathbb{F}_q^{*} \). For \( m \geq 2 \), we will say that a collection \( S \) of \( \binom{m+\ell}{\ell} \) points in \( \mathbb{F}_q^{*m} \) is an \( m \)-dimensional \( \ell \)th order simplicial configuration if the following conditions hold:

1. For some \( i, 1 \leq i \leq m \), there are hyperplanes \( x_i = a_1, x_i = a_2, \ldots, x_i = a_{\ell+1} \) such that for each \( 1 \leq j \leq \ell + 1 \), \( S \) contains exactly \( \binom{m-1+j-1}{j-1} \) points with \( x_i = a_j \). (Note that

\[
\binom{m+\ell}{\ell} = \sum_{j=1}^{\ell+1} \binom{m-1+j-1}{j-1}
\]

by a standard binomial coefficient identity.)

2. For each \( j, 1 \leq j \leq \ell + 1 \), the points in \( x_i = a_j \) form an \( (m-1) \)-dimensional simplicial configuration of order \( j - 1 \).

We call these special configurations of points simplicial configurations because they mimic, to an extent, the arrangement of the integer lattice points in the corresponding simplex \( P_{\ell}(m) \). For instance, Figure 1 shows a 2-dimensional simplicial configuration of order 2 in \( \mathbb{F}_8^{*2} \). It consists of six points. (We write \( \alpha \) for a primitive element in the field \( \mathbb{F}_8 \).) Note that there is 1 = \( \binom{1+0}{0} \) point in \( S \) on the line \( x_1 = a_1 = \alpha^4 \), 2 = \( \binom{1+1}{1} \) points on the line \( x_1 = a_2 = \alpha^3 \), and 3 = \( \binom{1+2}{2} \) on the line \( x_1 = a_3 = \alpha \).
In order to state our next result, a sort of recurrence relation for the Vandermonde determinants \( \det V(P_\ell(m); S) \) where \( S \) is a \( m \)-dimensional simplicial configuration, we introduce some notation.

Let \( S \) be an \( m \)-dimensional \( \ell \)th order simplicial configuration consisting of \( \binom{m+\ell}{\ell} \) points, in hyperplanes \( x_m = a_1, \ldots, x_m = a_{\ell+1} \). Write \( S = S' \cup S'' \) where \( S' \) is the union of the points in \( x_i = a_1, \ldots, a_\ell \), and \( S'' \) is the set of points in \( x_i = a_{\ell+1} \). Also, let \( \pi : \mathbb{F}_q^m \to \mathbb{F}_q^{m-1} \) be the projection on the first \( m-1 \) coordinates. By the definition, it follows that both \( S' \) and \( \pi(S'') \) are themselves simplicial configurations, with \( S' \) of dimension \( m \) and order \( \ell - 1 \), and \( \pi(S'') \) of dimension \( m-1 \) and order \( \ell \).

**Theorem 2.7.** Let \( P_\ell(m) \) be as above and let \( S \) be an \( \ell \)th order simplicial configuration of \( \binom{m+\ell}{\ell} \) points as in the paragraph above. Then writing \( p = (p_1, \ldots, p_m) \) for points \( p \in (\mathbb{F}_q^*)^m \),

\[
\det V(P_\ell(m); S) = \pm \prod_{p \in S'} (p_m - a_{\ell+1}) \det V(P_{\ell-1}(m); S') \det V(P_\ell(m-1); \pi(S'')).
\]

Before we give the proof of this theorem, we will do two things. First, we give an example to illustrate what the theorem is saying. The idea for this computation comes from [4], where corresponding sets of points in \( \mathbb{R}^m \) are identified as poised sets for interpolation by polynomials of degree bounded bounded by \( \ell \). Consider all polynomials of degree \( \leq 2 \) in three variables and the Vandermonde matrix \( V(P_2(3); S) \). For notational simplicity, write points in a 3-dimensional simplicial configuration \( S \subset (\mathbb{F}_q^*)^3 \) of order 2 as \( (x_i, y_i, z_i) \), for \( i = 1, \ldots, 10 = \binom{3+2}{2} \). Here \( S' \) consists of the first four points in \( S \), and \( S'' \) consists of the other six points. Under the hypothesis that \( S \) is a simplicial configuration, we have \( z_5 = z_6 = \cdots = z_{10} = c \) for some \( c = a_3 \). Noting this, but ignoring other equalities between the coordinates,
we see $V(P_2(3); S) =$

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} \\
z_1 & z_2 & z_3 & z_4 & c & c & c & c & c & c \\
x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 & x_8^2 & x_9^2 & x_{10}^2 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 & x_4 y_4 & x_5 y_5 & x_6 y_6 & x_7 y_7 & x_8 y_8 & x_9 y_9 & x_{10} y_{10} \\
y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 & y_7^2 & y_8^2 & y_9^2 & y_{10}^2 \\
x_1 z_1 & x_2 z_2 & x_3 z_3 & x_4 z_4 & x_5 c & x_6 c & x_7 c & x_8 c & x_9 c & x_{10} c \\
y_1 z_1 & y_2 z_2 & y_3 z_3 & y_4 z_4 & y_5 c & y_6 c & y_7 c & y_8 c & y_9 c & y_{10} c \\
z_1^2 & z_2^2 & z_3^2 & z_4^2 & c^2 & c^2 & c^2 & c^2 & c^2 & c^2
\end{pmatrix}.
$$

To evaluate the determinant of this matrix, we perform row operations to introduce zeroes. First subtract $c$ times row 4 from row 10, then $c$ times row 3 from row 9, $c$ times row 2 from row 8, and finally $c$ times row 1 from row 4. After rearranging rows, we find a matrix with a block of zeroes:

$$
\begin{pmatrix}
z_1 - c & \cdots & z_4 - c & 0 & 0 & 0 & 0 & 0 \\
x_1(z_1 - c) & \cdots & x_4(z_4 - c) & 0 & 0 & 0 & 0 & 0 \\
y_1(z_1 - c) & \cdots & y_4(z_4 - c) & 0 & 0 & 0 & 0 & 0 \\
z_1(z_1 - c) & \cdots & z_4(z_4 - c) & 0 & 0 & 0 & 0 & 0 \\
1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 \\
x_1 & \cdots & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\
y_1 & \cdots & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} \\
x_1^2 & \cdots & x_4^2 & x_5^2 & x_6^2 & x_7^2 & x_8^2 & x_9^2 & x_{10}^2 \\
x_1 y_1 & \cdots & x_4 y_4 & x_5 y_5 & x_6 y_6 & x_7 y_7 & x_8 y_8 & x_9 y_9 & x_{10} y_{10} \\
y_1^2 & \cdots & y_4^2 & y_5^2 & y_6^2 & y_7^2 & y_8^2 & y_9^2 & y_{10}^2
\end{pmatrix}.
$$

Up to a sign, the determinant of $V(P_2(3); S)$ is therefore equal to the product of

$$\det \begin{pmatrix} z_1 - c & z_2 - c & z_3 - c & z_4 - c \\ x_1(z_1 - c) & x_2(z_2 - c) & x_3(z_3 - c) & x_4(z_4 - c) \\ y_1(z_1 - c) & y_2(z_2 - c) & y_3(z_3 - c) & y_4(z_4 - c) \\ z_1(z_1 - c) & z_2(z_2 - c) & z_3(z_3 - c) & z_4(z_4 - c) \end{pmatrix},$$

which equals

$$(z_1 - c)(z_2 - c)(z_3 - c)(z_4 - c) \det V(P_1(3), S'),$$

and the determinant of the lower right $6 \times 6$ block in (3), which is $V(P_2(2), \pi(S''))$. Hence

$$\det V(P_2(3); S) = \pm \prod_{j=1}^{4} (z_j - c) \det V(P_1(3), S') \det V(P_2(2), \pi(S'')),$$

as in the statement of the theorem.

Second, we note the following immediate consequence of the theorem.

**Corollary 2.8.** Let $P_t(m)$ be as above and let $S$ be an $l$th order simplicial configuration of $\binom{m+t}{t}$ points. Then $\det V(P_t(m); S) \neq 0$. 
Proof. This is seen easily by a double induction on \( m \) and \( \ell \). In the base cases \( m = 1 \), \( \ell \) arbitrary, we have an ordinary univariate Vandermonde determinant, which is nonzero by the definition of a simplicial configuration. For the induction step, the recurrence given in the theorem then establishes the corollary since all the factors are nonzero under the hypothesis that \( S \) is a simplicial configuration. \( \Box \)

We will now give a proof of Theorem 2.7, following the methods from [4] used in the example above.

Proof. Since \( S \) is simplicial, the \( \binom{m+\ell-1}{\ell-1} \) points in \( S'' \subset S \) all have the same \( x_m \)-coordinate, say \( x_m = a_{\ell+1} = c \). For each pair of monomials \( x^e \) and \( x^e x_m \) corresponding to points in \( P_\ell(m) \), we perform a row operation on the Vandermonde matrix, subtracting \( c \) times the row of \( V(P_\ell(m), S) \) for \( x^e \) from the row for \( x^e x_m \) in decreasing order by the degree in \( x_m \). After rearranging rows to put all the zeroes created by these operations in the upper right block, the lower right block in the columns corresponding to \( S'' \) is the matrix \( V(P_\ell(m - 1), \pi(S'')) \). In the upper left block (in the columns corresponding to \( S' \)), all the entries in a column are divisible by one of the \( (p_m - c) \) for \( p \in S' \). Factoring out those factors from each column, the matrix that is left is \( V(P_{\ell-1}(m), S') \), and the statement of the theorem follows. \( \Box \)

We note that if we apply the recurrence relation repeatedly, another corollary of Theorem 2.7 is a closed formula for \( \det V(P_\ell(m); S) \) for \( S \) a simplicial configuration in terms of univariate Vandermonde determinants. We will not need this formula, so we leave the derivation of its exact form as an exercise.

We will now use Corollary 2.8 to establish the minimum distances of the toric codes \( C_{P_\ell(m)} \).

**Theorem 2.9.** Let \( \ell < q - 1 \), and let \( P_\ell(m) \) be the simplex in \( \mathbb{R}^m \) defined above. Then the minimum distance of the toric code \( C_{P_\ell(m)} \) is given by

\[
d(C_{P_\ell(m)}) = (q - 1)^m - \ell(q - 1)^{m-1}.
\]

Proof. Let \( d = d(C_{P_\ell(m)}) \). As in the proof of Theorem 2.4 above, we show both inequalities \( d \leq (q - 1)^m - \ell(q - 1)^{m-1} \) and \( d \geq (q - 1)^m - \ell(q - 1)^{m-1} \) hold. The first of these follows as in Theorem 2.4, since the completely reducible polynomials

\[
p(x_m) = (x_m - a_1) \cdots (x_m - a_\ell)
\]

for \( a_i \) distinct in \( \mathbb{F}_q^* \) are contained in \( L = \text{Span}\{x^e : e \in P_\ell(m)\} \). Such a polynomial has zeroes at all the \( \mathbb{F}_q \)-rational points with nonzero coordinates on the union of the hyperplanes \( x_m = a_i \). There are \( \ell(q - 1)^{m-1} \) such points, so \( d \leq (q - 1)^m - \ell(q - 1)^{m-1} \) as claimed.

To establish the reverse inequality, by Corollary 2.8 and Proposition 2.1, it suffices to show that every set \( T \) of \( \ell(q - 1)^{m-1} + 1 \) points contains an \( m \)-dimensional simplicial configuration \( S \) of order \( \ell \). By the pigeonhole principle, \( T \) contains some set of \( \ell(q - 1)^{m-2} + 1 \) points with the same \( x_m \)-coordinate \( x_m = a_{\ell+1} \). By an easy induction on \( m \), it follows that

\[
\ell(q - 1)^{m-2} + 1 > \binom{m - 1 + \ell}{\ell}
\]

(the first term on the right of (2)) for all \( m \geq 2 \) (and \( q - 1 > \ell \), of course) There are at least \((\ell-1)(q-1)^{m-1}+1\) points in \( T \) that do not lie on \( x_m = a_{\ell+1} \). Hence we can
apply the same pigeonhole principle argument repeatedly (or argue by induction on $\ell$) to see that $T$ contains
\[\sum_{j=1}^{\ell+1} \binom{m-1+j-1}{j-1} = \binom{m+\ell}{\ell}\]
points making up an $m$-dimensional $\ell$th order simplicial configuration. \qed

We also have the following consequence for more general simplices. Let $\ell_i \geq 1$ for all $i$ and define
\[P_{\ell_1, \ldots, \ell_m} = \text{conv}\{0, \ell_1 e_1, \ldots, \ell_m e_m\},\]
where again the $e_i$ are the standard basis vectors in $\mathbb{R}^m$.

**Corollary 2.10.** If $P_{\ell_1, \ldots, \ell_m} \subset \mathbb{R}^m$, and $\ell = \max_i \ell_i$, then
\[d(C_{P_{\ell_1, \ldots, \ell_m}}) = (q-1)^{\ell} - \ell(q-1)^{\ell-1}.\]

**Proof.** By definition, $P_{\ell_1, \ldots, \ell_m} \subseteq P_{\ell}(m)$. Hence $C_{P_{\ell_1, \ldots, \ell_m}}$ is a subcode of $C_{P_{\ell}(m)}$ and $d(C_{P_{\ell_1, \ldots, \ell_m}}) \geq d(C_{P_{\ell}(m)}) = (q-1)^{\ell} - \ell(q-1)^{\ell-1}$. But if $\ell = \ell_m$, then $C_{P_{\ell_1, \ldots, \ell_m}}$ also contains codewords of weight exactly $(q-1)^{\ell} - \ell(q-1)^{\ell-1}$ obtained from evaluation of completely reducible polynomials
\[p(x_i) = (x_{i_0} - a_1) \cdots (x_{i_0} - a_{\ell})\]
for some $i$ and distinct $a_j \in \mathbb{F}_q^*$. This establishes the corollary. \qed

The same sort of reasoning applies to any toric code from a polytope $P \subset P_{\ell}(m)$ that contains one complete edge conv$\{0, \ell e_i\}$ or conv$\{\ell e_i, \ell e_j\}$ of the simplex $P \subset P_{\ell}(m)$ and gives the same minimum distance for $C_P$.

3. **Classification of toric codes**

In this section we will begin by stating and proving a theorem guaranteeing that two toric codes have the same parameters. We begin by introducing some terminology.

**Definition 3.1.** Let $C_1$ and $C_2$ be two codes of block length $n$ and dimension $k$ over $\mathbb{F}_q$. Let $G_1$ be a generator matrix for $C_1$. Then $C_1$ and $C_2$ are said to be **monomially equivalent** if there is an invertible $n \times n$ diagonal matrix $\Delta$ and an $n \times n$ permutation matrix $\Pi$ such that
\[G_2 = G_1 \Delta \Pi\]
is a generator matrix for $C_2$.

It is easy to see that monomial equivalence is actually an equivalence relation on codes since a product $\Pi \Delta$ equals $\Delta' \Pi$ for another invertible diagonal matrix $\Delta'$. It is also a direct consequence of the definition that monomially equivalent codes $C_1$ and $C_2$ have the same dimension and the same minimum distance (indeed, the same full weight enumerator).

Next we turn to a natural notion of equivalence for polytopes. Recall that an **affine transformation** of $\mathbb{R}^m$ is a mapping of the form $T(x) = Mx + \lambda$, where $\lambda$ is a fixed vector and $M$ is an $m \times m$ matrix. The affine mappings $T$ where $M, \lambda$ have integer entries and $M \in \text{GL}(m, \mathbb{Z})$ (so det$(M) = \pm 1$) are precisely the bijective affine mappings from the integer lattice $\mathbb{Z}^m$ to itself.
Definition 3.2. We will say that two integral convex polytopes $P_1$ and $P_2$ in $\mathbb{R}^m$ are are lattice equivalent if there exists an invertible integer affine transformation $T$ as above such that $T(P_1) = P_2$.

This brings us to our next theorem which relates the two concepts we have just defined.

Theorem 3.3. If two polytopes $P_1$ and $P_2$ are lattice equivalent, then the toric codes $C_{P_1}$ and $C_{P_2}$ are monomially equivalent.

Proof. Suppose we have two lattice equivalent polytopes $P_1$ and $P_2$. Both $P_1$ and $P_2$ contain integer lattice points corresponding to monomials of the form $x^e$ where $e \in \mathbb{Z}^m$. By our hypothesis on $P_1$ and $P_2$, there exists an invertible integer transformation

$$T(x) = M(x) + \lambda$$

such that $T(P_1) = P_2$ and $M$ is an element of $GL(m, \mathbb{Z})$ so $\det(M) = \pm 1$. Hence $\#(P_1) = \#(P_2)$. Let $P_1 \cap \mathbb{Z}^m = \{e(i) : i = 1, \ldots, \#(P_1)\}$. So, $C_{P_1}$ is spanned by $\text{ev}(x^{e(i)})$ for $1 \leq i \leq n$, and similarly $C_{P_2}$ is spanned by $\text{ev}(x^{T(e(i))})$.

Write $\alpha$ for a primitive element in $\mathbb{F}_q^*$. Let $e(i) \in P_1 \cap \mathbb{Z}^m$ and define $\alpha^f = (\alpha^{f_1}, \ldots, \alpha^{f_m}) \in (\mathbb{F}_q^*)^m$. The component of $\text{ev}(x^{e(i)}) \in C_{P_1}$ corresponding to $\alpha^f$ is $\alpha^{(e(i), f)}$, where $(e(i), f)$ is the usual dot product. The corresponding entry in the codeword $\text{ev}(x^{T(e(i))})$ in $C_{P_2}$ is $\alpha^{(T(e(i)), f)}$. This can be rewritten as

$$\alpha^{(Me(i) + \lambda, f)} = \alpha^{(Me(i), f)} \cdot \alpha^{(\lambda, f)}$$

The second term of the product is not dependent on $e(i)$. These nonzero scalars are the diagonal entries in the matrix $\Delta$ as in the definition of monomially equivalent codes. By a standard property of dot products,

$$\alpha^{(Me(i), f)} = \alpha^{(e(i), M^t f)}.$$

The transposed matrix $M^t$ also defines a bijective mapping from $\mathbb{Z}^m$ to $\mathbb{Z}^m$ since $\det(M^t) = \det(M) = \pm 1$. Now we must show that $M^t$ induces a permutation of $(\mathbb{F}_q^*)^m$. Suppose $M^t f \equiv M^t g \pmod{q - 1}$. Since $\det(M^t) = \pm 1 \neq 0$, we know that $M^t$ is invertible and $(M^t)^{-1}$ is also an integer matrix. So, we can multiply by $(M^t)^{-1}$ on the left. Hence, $f \equiv g \pmod{q - 1}$ and $M^t$ defines a permutation of the points $\alpha^f$, as desired. Note that $M^t$ permutes all of the codewords in the same way. This gives the permutation matrix $\Pi$. Hence $C_{P_1}$ is monomially equivalent to $C_{P_2}$.

In the remainder of this section, we will show how this result leads to a complete classification for toric codes with $m = 2$ and $k \leq 5$. The classification could also be continued, of course, using a census of lattice equivalence classes of lattice polytopes with given $\#(P)$.

Proposition 3.4. Every toric surface code $C_P$ with $k = 2$ is monomially equivalent to the toric code $C_{P_2}$ for $P_2 = \text{conv}\{(0, 0), (1, 0)\}$.

Proof. Let $e(1), e(2) \in \mathbb{Z}^2$ be the integer lattice points in $P$. We can use a translation to map $e(1)$ to $(0, 0)$. Then let $e(2) = (a, b) \in \mathbb{Z}^2$. By convexity we have that $\gcd(a, b) = 1$, since otherwise there would be additional integer lattice points on the line from $e(1)$ to $e(2)$ and $k = \#(P)$ would be greater than 2. Since $\gcd(a, b) = 1$, there exist integers $r, s$ such that $ra + sb = 1$, and this implies that there exists an
invertible integer matrix \( M = \begin{pmatrix} r & s \\ -b & a \end{pmatrix} \) such that \( M \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Hence there is an affine equivalence between \( P \) and \( P_2 \). By Theorem 3.1, this completes the proof.

Next, we wish to find a “nice” lattice polygon in each possible lattice equivalence class with \( \#(P) = 3, 4, 5 \). One way is to add additional points to \( P_2 \). Using Pick’s Theorem: \( A(P) = \#(P) + \frac{1}{2} \partial(P) - 1 \), (where \( \partial(P) \) is the number of lattice points in the boundary of \( P \)) then eliminating lattice equivalent polygons, we obtain the following.

**Theorem 3.5.** Every toric surface code with \( 3 \leq k \leq 5 \) is monomially equivalent to one constructed from one of the following polygons.

![Figure 2. Polygons yielding toric codes with \( k = 3 \).](image)

![Figure 3. Polygons yielding toric codes with \( k = 4 \).](image)
Theorem 3.6. Let $q > 5$. No two of the toric codes $C_P(\mathbb{F}_q)$ constructed from the polygons in Theorem 3.5 are monomially equivalent.

Proof. If the dimensions are different, the toric codes are certainly not monomially equivalent. Hence we only need to consider each $k$ separately.

For the two codes with $k = 3$, Theorem 2.4 (which applies when $\ell = 0$ also) shows $d(C_{P_3^{(1)}}) = (q - 1)^2 - 2(q - 1)$. On the other hand, Theorem 2.9 gives $d(C_{P_3^{(3)}}) = (q - 1)^2 - (q - 1)$. Hence these two codes are not equivalent.

For the codes with $k = 4$, Theorem 2.4 shows $d(C_{P_4^{(1)}}) = (q - 1)^2 - 3(q - 1)$. Corollary 2.10 shows $d(C_{P_4^{(2)}}) = (q - 1)^2 - 2(q - 1)$. Theorem 2.4 applies to $C_{P_4^{(3)}}$ also, and shows $d(C_{P_4^{(5)}}) = (q - 1)^2 - (2q - 3)$. Finally, we must analyze $d(C_{P_4^{(4)}})$. Write $C_{P_4^{(4)}}(\mathbb{F}_q) = C(\mathbb{F}_q)$. In this case, some more advanced tools are needed. If we translate this polygon by $(1,1)$ to place it in $\square_{q-1}$, then we are evaluating polynomials in $\text{Span}\{1, xy, x^2y, xy^2\}$ to get the codewords of the corresponding (monomially equivalent) code. Any linear combination of these monomials in which the coefficient of $x^2y$ or $xy^2$ is nonzero defines an absolutely irreducible curve of degree 3, whose closure in $\mathbb{P}^2$ has arithmetic genus 1 by Theorem 4.2 of [2]. Hence by the general version of the Hasse-Weil bound from [1], there can be at most $1 + q + 2\sqrt{q}$ $\mathbb{F}_q$-rational points on the corresponding affine curve. This means that the minimum distance of $C$ is at least $(q - 1)^2 - (1 + q + 2\sqrt{q})$. On the other hand, the other $k = 4$ examples have minimum distance no larger than $d(C_{P_4^{(5)}}) = (q - 1)^2 - (2q - 3)$. It is easy to see from the quadratic formula that $(1 + q + 2\sqrt{q}) < 2q - 3$ for all $q > 11$. Hence $d(C)$ is strictly larger than any of the
others for \( q > 11 \). For the remaining small values of \( q \) we check directly that \( d(C) \) is different from the others using the Magma code described in [9]. The results are:

| \( q \) | \( d(C(\mathbb{F}_q)) \) | \((q - 1)^2 - (2q - 3)\) |
|---|---|---|
| 7 | 27 | 25 |
| 8 | 40 | 36 |
| 9 | 52 | 49 |
| 11 | 85 | 81 |

Table 1. \( d(C_{p(4)}(\mathbb{F}_q)) \)

These are also different from any of the other \( k = 4 \) codes over those fields.

For the \( k = 5 \) codes, Theorem 2.4 shows \( d(C_{P(5)}(c)) = (q - 1)^2 - 4(q - 1) \). Corollary 2.10 shows \( d(C_{P(5)}(c^1)) = (q - 1)^2 - 3(q - 1) \). \( C_{P(5)}(c^3) \) is a subcode of the code \( C_{P(5)}(c) \) from Theorem 2.9, and contains codewords of the same minimum weight as the supercode. So \( d(C_{P(5)}(c^3)) = (q - 1)^2 - 2(q - 1) \). The other three \( k = 5 \) codes also have \( d(C_{P(5)}(c_i)) = (q - 1)^2 - 2(q - 1) \), which can be seen, for example, using Minkowski-decomposable subpolytopes (the sets of three collinear points) as in [10]. In \( C_{p(5)} \), for example, we have codewords \( ev(b(xy - a_1)(xy - a_2)) \), where \( a_1, a_2, b \in \mathbb{F}_q^* \) and \( a_1 \neq a_2 \), which have \( 2(q - 1) \) zeroes in \((\mathbb{F}_q^*)^2\).

To show that the four codes with \( d = (q - 1)^2 - 2(q - 1) \) are not equivalent, we need to look at finer invariants. For instance, \( C_{P(5)}(c) \) can be distinguished from the other three by the number of words of minimum weight. In \( P(5) \), there are two different sets of three collinear lattice points while in the others, there is only one. This means that there will be more words of the minimum weight in \( C_{P(5)}(c) \) than in \( C_{P(5)}(c_i) \) for \( i = 3, 4, 6 \). \( C_{P(5)}(c) \) has at least \( 2^{(q - 1)/2}(q - 1) \) such words because there are two distinct families of reducible polynomials: \( b(x - a_1)(x - a_2) \) with \( b, a_i \in \mathbb{F}_q^* \) and \( a_1 \neq a_2 \) and \( b(y - a_1)(y^{-1} - a_2) \) with \( b, a_i \in \mathbb{F}_q^* \) and \( a_1 \neq a_2^{-1} \). On the other hand, \( C_{P(5)}(c_i) \) for \( i = 3, 4, 6 \) have (at least) \( (q - 1)/2 \) such words. There are more for some small \( q \), but never as many as \( 2^{(q - 1)/2}(q - 1) \). See the weight enumerators for \( C_{P(5)}(c) \) over \( \mathbb{F}_{11} \) and \( \mathbb{F}_{16} \) in Table 2 below. For sufficiently large \( q \), we claim in fact that there are exactly \( (q - 1)/2 \) \((q - 1)\) such words. This follows from the general Hasse-Weil bound from [11]. For instance, for \( C_{P(5)}(c) \), if \( q \) is sufficiently large, then we claim all words in \( C_{P(5)}(c) \) of weight \((q - 1)^2 - 2(q - 1)\) come from evaluations \( ev(b(y - a_1)(y - a_2)) \). Any other such word could come only from evaluating a linear combination of \( \{1, x, y, y^2, x^{-1}y^{-1}\} \) in which \( y^2, x, x^{-1}y^{-1} \) appear with nonzero coefficients (since otherwise we are in a case previously covered). Any such curve is absolutely irreducible, of arithmetic genus 2 (because of the 2 interior lattice points in this case, see Theorem 4.2 of [2]). A simple argument shows that \( 1 + q + 4\sqrt{q} < 2q - 2 \) for all \( q \geq 23 \). For smaller values of \( q \), we verify directly that the weight enumerators of \( C_{P(5)}(c, \mathbb{F}_q) \) do not match the weight enumerators of the other codes using the Magma code from [9]. See Table 2.

To distinguish \( C_{P(5)}(c^3) \) and \( C_{P(5)}(c^4) \), we use the codewords of weight \((q - 1)^2 - (2q - 3)\) (one more than the minimum weight). Both of these codes contain such words coming from evaluation of the polynomials corresponding to the \( 1 \times 1 \) squares
contained in these polygons (copies of $P_{3}^{(3)}$). Any such square yields $(q-1)^3$ words of this weight since the polynomials in question have the form $c(x-a)(y-b)$ and $a, b, c \in \mathbb{F}_q^*$ are arbitrary. However $C_{P_{3}^{(4)}}$ has precisely $(q-1)^3$ words of weight $(q-1)^2-(2q-3)$, while $C_{P_{3}^{(3)}}$ has more of them, $3(q-1)^3$ to be specific. This can be seen by considering the reducible polynomials $d(x-a)(y-bx-c)$ that evaluate to give codewords in $C_{P_{3}^{(3)}}$. We get codewords of weight $(q-1)^2-(2q-3)$ if $b = 0$, or if $c = 0$, or if $b, c \neq 0$ and $a = -c/b$.

Finally, to distinguish $C_{P_{5}^{(6)}}$ from the other three codes with $d = (q-1)^2 - 2(q-1)$, we must argue as in the last case of the $k = 4$ codes. If $q$ is sufficiently large, then we claim $C_{P_{5}^{(6)}}$ contains no words at all of weight $(q-1)^2-(2q-3)$. By Corollary 2.10 and the previous cases, we see that any such word could only come from a linear combination of $\{1, x, y, y^2, x^{-1}y^{-1}\}$ in which $y^2, x, x^{-1}y^{-1}$ all appear with nonzero coefficients. As before, any such curve is absolutely irreducible, of arithmetic genus 2. A simple argument shows that $1 + q + 4\sqrt{q} < 2q-3$ for all $q > 23$. Hence the Hasse-Weil bound from \cite{1} shows that there are no words of this weight for large $q$. For smaller values of $q$, we again verify directly that the weight enumerators of $C_{P_{5}^{(6)}}(\mathbb{F}_q)$ do not match the weight enumerators of the other codes. See Table 2 below.

The following table gives the first three nonzero terms in the weight enumerators:

\[
W_C(x) = \sum_{i=0}^{(q-1)^2} A_i x^i,
\]

where $A_i = |\{w \in C : \text{wt}(w) = i\}$, for the $k = 5$ toric codes with $d = (q-1)^2 - 2(q-1)$. These were all computed using the Magma code from \cite{9}.

**Over $\mathbb{F}_7$:**

- $P_{5}^{(3)}$  $1 + 90x^{24} + 648x^{25} + \cdots$
- $P_{5}^{(4)}$  $1 + 90x^{24} + 216x^{25} + \cdots$
- $P_{5}^{(5)}$  $1 + 180x^{24} + 324x^{26} + \cdots$
- $P_{5}^{(6)}$  $1 + 90x^{24} + 432x^{26} + \cdots$

**Over $\mathbb{F}_8$:**

- $P_{5}^{(3)}$  $1 + 147x^{35} + 1029x^{36} + \cdots$
- $P_{5}^{(4)}$  $1 + 147x^{35} + 343x^{36} + \cdots$
- $P_{5}^{(5)}$  $1 + 294x^{35} + 343x^{37} + \cdots$
- $P_{5}^{(6)}$  $1 + 147x^{35} + 1029x^{37} + \cdots$

**Over $\mathbb{F}_9$:**

- $P_{5}^{(3)}$  $1 + 224x^{48} + 1536x^{49} + \cdots$
- $P_{5}^{(4)}$  $1 + 224x^{48} + 512x^{49} + \cdots$
- $P_{5}^{(5)}$  $1 + 448x^{48} + 512x^{51} + \cdots$
- $P_{5}^{(6)}$  $1 + 224x^{48} + 512x^{50} + \cdots$
Over $\mathbb{F}_{11}$:

\[
P_5^{(3)} \ 1 + 450x^{80} + 3000x^{81} + \cdots \\
P_5^{(4)} \ 1 + 450x^{80} + 1000x^{81} + \cdots \\
P_5^{(5)} \ 1 + 900x^{80} + 1500x^{84} + \cdots \\
P_5^{(6)} \ 1 + 650x^{80} + 1000x^{82} + \cdots \\
\]

Over $\mathbb{F}_{13}$:

\[
P_5^{(3)} \ 1 + 792x^{120} + 5184x^{121} + \cdots \\
P_5^{(4)} \ 1 + 792x^{120} + 1728x^{121} + \cdots \\
P_5^{(5)} \ 1 + 1584x^{120} + 7776x^{126} + \cdots \\
P_5^{(6)} \ 1 + 792x^{120} + 1728x^{125} + \cdots \\
\]

Over $\mathbb{F}_{16}$:

\[
P_5^{(3)} \ 1 + 1575x^{195} + 10125x^{196} + \cdots \\
P_5^{(4)} \ 1 + 1575x^{195} + 3375x^{196} + \cdots \\
P_5^{(5)} \ 1 + 3150x^{195} + 13500x^{203} + \cdots \\
P_5^{(6)} \ 1 + 2250x^{195} + 13500x^{203} + \cdots \\
\]

Over $\mathbb{F}_{17}$:

\[
P_5^{(3)} \ 1 + 1920x^{224} + 12288x^{225} + \cdots \\
P_5^{(4)} \ 1 + 1920x^{224} + 4096x^{225} + \cdots \\
P_5^{(5)} \ 1 + 3840x^{224} + 5120x^{232} + \cdots \\
P_5^{(6)} \ 1 + 1920x^{224} + 4096x^{230} + \cdots \\
\]

Over $\mathbb{F}_{19}$:

\[
P_5^{(3)} \ 1 + 2754x^{288} + 17496x^{289} + \cdots \\
P_5^{(4)} \ 1 + 2754x^{288} + 5832x^{289} + \cdots \\
P_5^{(5)} \ 1 + 5508x^{288} + 32076x^{298} + \cdots \\
P_5^{(6)} \ 1 + 2754x^{288} + 5832x^{294} + \cdots \\
\]

Over $\mathbb{F}_{23}$:

\[
P_5^{(3)} \ 1 + 5082x^{440} + 31944x^{441} + \cdots \\
P_5^{(4)} \ 1 + 5082x^{440} + 10648x^{441} + \cdots \\
P_5^{(5)} \ 1 + 10164x^{440} + 154396x^{454} + \cdots \\
P_5^{(6)} \ 1 + 5082x^{440} + 21296x^{450} + \cdots \\
\]

Table 2.

Hence the enumerators never coincide for these four codes, even in exceptional cases for small $q$. □

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