Bayesian Low Rank Tensor Ring Model for Image Completion

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Abstract

Low rank tensor ring model is powerful for image completion which recovers missing entries in data acquisition and transformation. The recently proposed tensor ring (TR) based completion algorithms generally solve the low rank optimization problem by alternating least squares method with predefined ranks, which may easily lead to overfitting when the unknown ranks are set too large and only a few measurements are available. In this paper, we present a Bayesian low rank tensor ring model for image completion by automatically learning the low-rank structure of data. A multiplicative interaction model is developed for the low-rank tensor ring decomposition, where core factors are enforced to be sparse by assuming their entries obey Student-T distribution. Compared with most of the existing methods, the proposed one is free of parameter-tuning, and the TR ranks can be obtained by Bayesian inference. Numerical Experiments, including synthetic data, color images with different sizes and YaleFace dataset B with respect to one pose, show that the proposed approach outperforms state-of-the-art ones, especially in terms of recovery accuracy.

Index Terms

image completion, tensor ring decomposition, low rank Bayesian learning, Student-T distribution, Bayesian variational inference

I. INTRODUCTION

Tensors, which are multi-dimensional generalizations of matrices, provide a natural representation for multidimensional data. Exploring the internal structure of tensors could help us obtain more latent information for high-dimensional data processing. For example, a color video is a forth-order tensor, which allows its temporal and spatial correlation could be simultaneously investigated. Recently, tensor-based methods have attracted interests in image processing problems [1]–[9]. Image completion is one of them, which recovers the missing entries during acquisition and transformation.

In recent works, the tensor-based methods of image completion are mostly addressed by assuming the data is low rank and mainly divided into two groups. One is based on the rank minimization model and can be presented as:

$$\min_{\mathcal{X}} \text{rank}(\mathcal{X}) \quad \text{s. t.} \quad \mathcal{X}_O = \mathcal{T}_O,$$

where $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is the estimated tensor, $\mathcal{T}$ is an $N$-th order observed tensor with the same size as $\mathcal{X}$, and $O$ is an index set with the entries observed. Traditional decompositions based on rank minimization model such as Tucker decomposition [10], [11] and tensor-Singular Value Decomposition (t-SVD) [12], [13] have been well studied. Rank minimization models are developed by minimizing the summation of nuclear norm regularization terms which correspond to the unfolding matrices of $\mathcal{X}$. For example, Tucker ranks minimization is firstly proposed in [14], followed by tubal rank minimization [15], [16]. Besides, some improvements for these rank minimization methods are proposed in [17]–[22]. Recently, an advanced tensor network named tensor train (TT) has been proposed in [23] and showed its advantage to capture the latent information for image completion [24]. Moreover, as one of the generation formats of TT decomposition, tensor tree ranks minimization for image completion is proposed in [25].

The other important class is factorization based methods which alternatively update the factor with the predefined rank. Mathematically, the low-rank tensor completion problem can be formulated as:

$$\min_{\mathcal{X}} \frac{1}{2} \| \mathcal{P}_O (\mathcal{X} - \mathcal{T}) \|^2_2 \quad \text{s. t.} \quad \text{rank} (\mathcal{X}) = R.$$

where $\mathcal{P}_O$ is the random sampling operator. In [26], the model with CANDECOMP/PARAFAC (CP) rank known in advance is proposed and solved by the alternating least squares (ALS) algorithm. Besides, considering the factorization based model and ALS approach, some works such as HOOI [11] with Tucker ranks, Tubal-ALS [27] with tubal rank, TT-ALS [28] with TT ranks and TR-ALS [29] with tensor ring (TR) ranks are investigated to fill
these fields. Aparting from the ALS framework, Riemannian optimization scheme with nonlinear conjugate gradient approach has been explored to tackle the factorization based tensor completion model, resulting in CP-WOPT [30], RMTC [31], RTTC [32], TR-WOPT [33] and HTTC [34].

In these methods, the advanced tensor networks such as TT and TR perform better than the CP/Tucker based methods in image completion because they can capture more correlations than the traditional tensor decompositions. Compared with TT, TR has more balanced and smaller ranks due to its ring structure, which may be beneficial to explore more latent structure. However, it is intractable to directly minimize tensor ring rank since its corresponding fold matrix is hard to find due to its circular dimensional permutation invariance. The existing TR based methods are based on the ranks pre-defined in advance, which may easily lead to overfitting when rank is set to be large and a few observations are available.

Motivated by these, we present a Bayesian inference (BI) model for inferring TR ranks to solve image completion problem. BI, which models the low-rank problem, shows a success in low-rank matrix factorization [35]–[38] by automatically adjusting the tradeoff between rank and fitting error. In addition, some tensor based works [39]–[41] reveal the superiority of BI framework on low-rank tensor completion. To the best of our knowledge, this is the first work to investigate low rank TR model for image completion on BI framework. Our objective is to infer the missing entries from observations by low rank tensor ring completion, while TR ranks can be determined automatically.

![A natural image](image)

![Derivation at horizontal](image)

![Derivation at vertical](image)

Fig. 1: The property of heavy-tailed for a natural image

The natural image has the characteristic of heavy-tailed. The phenomenon could be explained intuitively in Fig. 1. This property inspires us to assume the core factor, which is the potential part in TR formats, following Student-T distribution. Specifically, each slice of core factor is assumed to independently follow a Gaussian distribution with zero mean and a variance matrix. The variance matrix is treated as a random hyperparameter prior over core factor. Specifically, each slice of core factor is assumed to independently follow a Gaussian distribution. To model this problem, we propose TR decomposition framework with a sparsity-inducing hierarchical structure from images. In addition, the experiments also indicate that TR ranks inferred by our algorithm can be used in approach outperforms existing state-of-the-art works, which may imply our algorithm can explore more correlations from images. In addition, the experiments also indicate that TR ranks inferred by our algorithm can be used in TR-ALS, which avoids tuning parameters.

The rest of this paper is organized as follows. Sec. II introduces some notations and preliminaries for TR decomposition. In sec. III, the details of Bayesian TR decomposition are introduced, including the model description, solution and discussion. Sec. IV provides some experiments on multi-way data completion. The conclusion is concluded in sec. V.

II. Notations and Preliminaries

A. Notations

Firstly we give the notations to be used. A scalar, a vector, a matrix, and a tensor are written as $x$, $\mathbf{x}$, $\mathbf{X}$, and $\mathbf{X}$, respectively. The product of two scalars denotes $z = x \ast y$, $\mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ denotes an $N$-order tensor where $I_n$ is the dimensional size. The trace operation is denoted as $\text{Trace}(\mathbf{X}) = \sum_{i=1}^{I} x_{ii}$ where $\mathbf{X} \in \mathbb{R}^{I \times J}$ is a square matrix. $\text{Vec}(\mathbf{X})$ denotes the vectorization of tensor $\mathbf{X}$ and $\text{Ten}(\mathbf{X}) \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ denotes a tensor transforming from a vector $\mathbf{x} \in \mathbb{R}^{(I_1 \cdots J_N)}$ or a matrix $\mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$. The Kronecker product of two tensors can be denoted as $\mathbf{Z} = \mathbf{X} \otimes \mathbf{Y} \in \mathbb{R}^{I_1J_1 \times \cdots \times I_NJ_N}$, where $\mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, $\mathbf{Y} \in \mathbb{R}^{J_1 \times \cdots \times J_N}$. The Hadamard product of two tensors is defined as $(\mathbf{X} \circ \mathbf{Y})_{i_1, \ldots, i_N} = X_{i_1, \ldots, i_N} Y_{i_1, \ldots, i_N}$, where $X_{i_1, \ldots, i_N}$ and $Y_{i_1, \ldots, i_N}$ are the entries of $\mathbf{X}$ and $\mathbf{Y}$.
B. Tensor Ring Model

Definition 1. (TR decomposition) [42] For an N-order tensor $X \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, the TR decomposition is defined as
\[
X(i_1, i_2, \cdots, i_N) = \text{Trace}(G_1(:, i_1, :)G_2(:, i_2, :) \cdots G_N(:, i_N, :)),
\]
(3)
where $G_n \in \mathbb{R}^{I_n-1 \times I_n \times I_n}$, $n = 1, \cdots, N$ are the core factors, and the TR ranks are defined as $\{R_n, 0 \leq n \leq N\}$ with $R_0 = R_N$. For simplicity, we denote TR decomposition by $X = f(G_1, \cdots, G_N)$. The graphical illustration of TR decomposition is shown in Fig. 2.

![Fig. 2: TR decomposition](image)

Definition 2. (Tensor permutation) For an N-order tensor $X \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, the tensor permutation is defined as $X^{P_n} \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \cdots \times I_{n-1}}$:
\[
X^{P_n}(i_n, \cdots, i_N, i_1, \cdots, i_{n-1}) = X(i_1, \cdots, i_N).
\]

Theorem 1. (Cyclic permutation property) [42] Based on the definitions of tensor permutation and TR decomposition, the tensor permutation of $X$ is equivalent to its factors circular shifting, as follows:
\[
X^{P_n} = f(G_n, \cdots, G_N, G_1, \cdots, G_{n-1}),
\]
with entries
\[
X^{P_n}(i_n, \cdots, i_N, i_1, \cdots, i_{n-1}) = \text{Trace}(G_n(:, i_n, :) \cdots G_N(:, i_N, :)G_1(:, i_1, :) \cdots G_{n-1}(:, i_{n-1}, :)).
\]

Definition 3. (Tensor connection product (TCP)) [43] The tensor connection product for 3-order tensors $G_n \in \mathbb{R}^{I_n-1 \times I_n \times I_n}$ is defined as
\[
G = \text{TCP}(G_1, G_2, \cdots, G_N) \in \mathbb{R}^{R_0 \times (I_1 \cdots I_N) \times R_N},
\]
and $G^{\#n}$, which is the TCP of a set of core factors excepting $G_n$, is defined as:
\[
G^{\#n} = \text{TCP}(G_{n+1}, \cdots, G_N, G_1, \cdots, G_{n-1}) \in \mathbb{R}^{R_n \times (I_1 \cdots I_N \cdots I_{n-1}) \times R_{n-1}}.
\]
Theorem 2. (Expectation of Inner product) Letting a random tensor $X = f(G_1G_2\cdots G_N)$, we can calculate the expectation of inner product by:

$$
\mathbb{E}[(\text{Vec}(X), \text{Vec}(X))] = \sum_{i_1=1}^{N} \prod_{n=1}^{N} (E_{R_n} \otimes K_{R_n, i_{n-1}} \otimes E_{R_n, i_n}) (\mathbb{E}[(g_{n}(i_n)(g_{n}(i_n))^T)]
$$

with

$$
\mathbb{E}[(\text{Vec}(g_{n}(i_n)))^T(\text{Vec}(g_{n}(i_n)))] = \mathbb{E}[(\text{Vec}(g_{n}(i_n)))^T] + \text{Var}(\text{Vec}(g_{n}(i_n))).
$$

Proof.

$$
\mathbb{E}[(\text{Vec}(X), \text{Vec}(X))] = \mathbb{E}[(\sum_{i_1=1}^{N} \text{X}(i_1, \cdots, i_N))\text{X}(i_1, \cdots, i_N)]]
$$

$$
= \mathbb{E}[(\sum_{i_1=1}^{N} \text{Trace}(\prod_{n=1}^{N} G_n(i_n)))\text{Trace}(\prod_{n=1}^{N} G_n(i_n)))]
$$

$$
= \mathbb{E}[(\sum_{i_1=1}^{N} \text{Trace}(\prod_{n=1}^{N} G_n(i_n)) \otimes (\prod_{n=1}^{N} G_n(i_n)))]
$$

$$
= \mathbb{E}[(\sum_{i_1=1}^{N} \text{Vec}(G_n(i_n) \otimes G_n(i_n)))]
$$

$$
= \mathbb{E}[(\sum_{i_1=1}^{N} \text{Vec}(G_n(i_n) \otimes G_n(i_n))^T \text{Vec}(\prod_{i \neq n}^{N} (G_i(i_i) \otimes G_i(i_i)))]
$$

$$
= \sum_{i_1=1}^{N} \prod_{n=1}^{N} (E_{R_n} \otimes K_{R_n, i_{n-1}} \otimes E_{R_n, i_n}) (\mathbb{E}[(g_{n}(i_n)(g_{n}(i_n))^T)])
$$

(6)

where $g_{n}(i_n) = \text{Vec}(G_n(i_n))$, $E_n \in \mathbb{R}^{n \times n}$ is an unit matrix, $K_{mn}$ is the permutation matrix.

A. Model Description

In this section, we present the Bayesian low TR rank decomposition based on Student-T process. Given an incomplete tensor $\mathcal{T}_0 \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, its entry is denoted by $\mathcal{T}_{i_1, i_2, \cdots, i_N}[(i_1, i_2, \cdots, i_N) \in \mathcal{O}$], where $\mathcal{O}$ is the set of indices of available data in $\mathcal{T}$. Our goal is to find a Bayesian low-TR-rank approximation for the observed tensor $\mathcal{T}$ under probabilistic framework, which is formulated as

$$
p(\mathcal{T}_0|G_n) = \prod_{i_1=1}^{I_1} \cdots \prod_{i_N=1}^{I_N} N(\mathcal{T}_{i_1, i_2, \cdots, i_N}|f(G_1(i_1), \cdots, G_N(i_N)), \tau^{-1})^\mathcal{O}_{i_1, \cdots, i_N}
$$

(7)

where $N(x | \mu, \sigma^2)$ denotes a Gaussian density with mean $\mu$ and variance $\sigma^2$, $\tau$ denotes the noise precision, $G_n(i_n) \in \mathbb{R}_{X \times X}$ is the $i_n$-th slice of $G_n$, and $\mathcal{O}$ is the indicator tensor in which 1 represents observed entry and 0 represents missing entry. The likelihood model in (7) means the observed tensor is generated by two parts where one is the core factors in TR format and the other is noise.

Firstly, we assume entries of noise are independent and identically distributed random variables obeying Gaussian distribution with zero mean and noise precision $\tau$. To learn $\tau$, we place a hyperprior over the noise precision $\tau$, as follows,

$$
p(\tau) = \text{Ga}(\tau|a, b)
$$

(8)
where \( \text{Ga}(x|a, b) = \frac{b^a x^{a-1} e^{-bx}}{\Gamma(a)} \) denotes a Gamma distribution. The expectation of \( \tau \) is defined by \( \mathbb{E}[\tau] = \frac{a}{b} \). The parameters \( a \) and \( b \) are set to small values, e.g., \( 10^{-7} \), which makes the Gamma distribution a non-informative prior.

Secondly, we assume the recovered tensor has a low rank structure. To learn the low rank structure, we assume the core factor following Student-T distribution and propose a two-layer multiplicative interaction model over core factor. In the first layer, the entries in core factor \( G_n \) obey Gaussian distribution with zero mean and a precision matrix:

\[
p(G_n|\lambda^{(n-1)}, \lambda^{(n)}) = \prod_{i_n=1}^{I_n} \prod_{r_{n-1}}^{R_{n-1}} \prod_{r_n}^{R_{n}} N(G_n(r_{n-1}, i_n, r_n)|0, (\lambda^{(n-1)}_{r_{n-1}} + \lambda^{(n)}_{r_n})^{-1})
\]

where hyperparameters \( \lambda^{(n)} = [\lambda_1^{(n)}, \cdots, \lambda_R^{(n)}] \) and \( \lambda_{r_n}^{(n)} \) simultaneously control the components in \( G_n \). Specifically, we directly employ a sparsity-inducing prior over each slice of core factor, leading to the following probabilistic framework:

\[
p(G_n|\lambda^{(n-1)}, \lambda^{(n)})
= \prod_{i_n=1}^{I_n} N(\text{Vec}(G_n(i_n)))|0, (\lambda^{(n-1)} \otimes \Lambda^{(n)})^{-1})
\]

where \( \Lambda^{(n)} = \text{diag}(\lambda^{(n)}) \) denotes the precision matrix. The second layer specifies a Gamma distribution as a hyperprior over \( \lambda \), as follows:

\[
p(\lambda^{(n)}|c_n, d_n) = \prod_{r_n=1}^{R_n} \text{Ga}(\lambda^{(n)}_{r_n}|c_{r_n}, d_n)
\]

where the parameters \( c_n \) and \( d_n \) are set to small values for making Gamma distribution a non-informative prior. Besides, the expectation of \( \lambda^{(n)} \) is defined as:

\[
\mathbb{E}[\lambda^{(n)}] = \frac{c_n}{d_n}
\]

For simplification, we set \( \mathbb{G} = \{G_1, \cdots, G_N\} \) and \( \lambda = (\lambda^{(1)}, \cdots, \lambda^{(N)}) \), and all unknown parameters in Bayesian TR model are collected and denoted by \( Z = \{\mathbb{G}, \lambda, \tau\} \). By combining the stages of the hierarchical Bayesian model, the joint distribution \( p(\mathcal{T}_\circ, Z) \) can be written as:

\[
p(\mathcal{T}_\circ, Z) = p(\mathcal{T}_\circ|\mathbb{G}_n^{(N)}) p(\tau|a, b)
\times \prod_{n=1}^{N} p(G_n|\lambda^{(n-1)}, \lambda^{(n)}) p(\lambda^{(n-1)}|c_{n-1}, d_{n-1}) p(\lambda^{(n)}|c_n, d_n)
\]

Our objective is to compute the conditional density of the latent variables given the observation, as follows:

\[
p(Z|\mathcal{T}_\circ) = \frac{p(Z, \mathcal{T}_\circ)}{\int p(Z, \mathcal{T}_\circ) dZ}
\]

Therefore, the missing entries can be inferred by the following equation:

\[
p(\mathcal{T}_\circ) = \int p(\mathcal{T}_\circ|Z)p(Z)dZ
\]

However, the integral over variables \( Z \) is unavailable in closed form, which leads to posterior is intractable to address.
B. Variable Bayesian Inference

In this section, we apply variable Bayesian inference (VBI) to tackle this problem. In variational inference, we specify a family $\mathcal{Z}$ of densities over the latent variables. Each $q(\mathcal{Z}) \in \mathcal{Z}$ is a candidate approximation to the exact posteriors. Our goal is to find the best candidate, the one closed to $p(\mathcal{Z}|\mathcal{T}_O)$ in Kullback-Leibler (KL) divergence, that is:

$$ q^*(\mathcal{Z}) = \arg\min_{q(\mathcal{Z}) \in \mathcal{Z}} KL(q(\mathcal{Z})||p(\mathcal{Z}|\mathcal{T}_O)) $$

According to KL definition, the problem (15) can be rewritten as:

$$ q^*(\mathcal{Z}) = \arg\min_{q(\mathcal{Z}) \in \mathcal{Z}} \mathbb{E}[\ln q(\mathcal{Z})] - \mathbb{E}[\ln p(\mathcal{Z}, \mathcal{T}_O)] + \ln p(\mathcal{T}_O) \quad (16) $$

Since $\ln p(\mathcal{T}_O)$ is a constant with respect to $q(\mathcal{Z})$, the problem (16) can be formulated as an optimization model:

$$ \max_{q(\mathcal{G}_1), \ldots, q(\mathcal{G}_N), q(\mathcal{\Lambda}^{(n)}), \ldots, q(\mathcal{\Lambda}^{(N)}), q(\mathcal{\tau})} \mathbb{E}[\ln p(\mathcal{Z}, \mathcal{T}_O)] - \mathbb{E}[\ln q(\mathcal{Z})] \quad (17) $$

where $q(\mathcal{Z}) = q(\mathcal{\tau})\prod_{n=1}^N q(\mathcal{G}_n)\prod_{n=1}^N q(\mathcal{\Lambda}^{(n)})$ based on the mean-filed approximation. It means each parameter in $\mathcal{Z}$ is independent and these parameters could be developed by iteratively optimizing each one while keeping the others fixed.

1) Update $q(\mathcal{G}_n)$: By substituting the equation (12) into the optimization problem (17), we can obtain the following subproblem with respect to $\mathcal{G}_n$:

$$ \max_{q(\mathcal{G}_n)} \mathbb{E}[\ln p(\mathcal{T}_O|\mathcal{G}_n, \mathcal{G}^{\neq n}, \mathcal{\tau})] + \mathbb{E}[\ln p(\mathcal{G}_n|\mathcal{\Lambda}^{(n-1)}, \mathcal{\Lambda}^{(n)})] - \mathbb{E}[\ln q(\mathcal{G}_n)] \quad (18) $$

where

$$ q(\mathcal{G}_n) = \prod_{i_n=1}^{I_n} \mathcal{N}(\text{Vec}(\mathcal{G}_n(i_n))|\mathbf{\tilde{g}}_n(i_n), \mathbf{V}^n_{i_n}) \quad (19) $$

with mean $\mathbf{\tilde{g}}_n(i_n)$ and variance $\mathbf{V}^n_{i_n}$.

From (18), we could infer core factor $\mathcal{G}_n$ by receiving the messages from the observed data $\mathcal{T}_O$, the rest core factors $\mathcal{G}^{\neq n}$ and the noise precision $\mathcal{\tau}$, and incorporating the message from its hyperparameters $\mathcal{\Lambda}^{(n-1)}$ and $\mathcal{\Lambda}^{(n)}$. By utilizing these messages, the optimization model with respect to each $\mathcal{G}_n(i_n), i_n \in \{1, \ldots, I_n\}$ could be rewritten as (details of the derivation can be found in sec. 2 of supplemental materials):

$$ \max_{\mathbf{\tilde{g}}_n(i_n), \mathbf{V}^n_{i_n}} \frac{1}{2}[(\text{Vec}(\mathcal{G}_n(i_n)))^T(\mathbb{E}[\mathcal{\Lambda}^{(n-1)}] \otimes \mathbb{E}[\mathcal{\Lambda}^{(n)}])^{-1}\text{Vec}(\mathcal{G}_n(i_n))] $$

$$ + \mathbb{E}[((\mathcal{\Lambda}^{(n-1)} \otimes \mathcal{\Lambda}^{(n)}))\text{Vec}(\mathcal{G}_n(i_n))] $$

$$ - \mathbb{E}[\mathcal{T}_O^T\mathcal{G}^{\neq n}_n\text{Vec}(\mathcal{G}_n(i_n))] $$

$$ - \mathbb{E}[\mathcal{T}_O^T\mathcal{G}^{\neq n}_n\text{Vec}(\mathcal{G}_n(i_n))] $$

$$ + \frac{1}{2}[\text{Vec}(\mathcal{G}_n(i_n))^T(\mathbf{V}^n_{i_n})^{-1}\text{Vec}(\mathcal{G}_n(i_n))] $$

$$ - \mathbf{g}_n(i_n)^T(\mathbf{V}^n_{i_n})^{-1}\text{Vec}(\mathcal{G}_n(i_n)) - \text{Vec}(\mathcal{G}_n(i_n))^T(\mathbf{V}^n_{i_n})^{-1}\mathbf{g}_n(i_n)] \quad (20) $$

The maximum value reaches with:

$$ \mathbf{V}^n_{i_n} = (\mathbb{E}[\mathcal{T}_O^T\mathcal{G}^{\neq n}_n\text{Vec}(\mathcal{G}^{\neq n}_n)^T])^{-1} $$

$$ \mathbf{g}_n(i_n) = \mathbb{E}[\mathcal{T}_O^T\mathcal{G}^{\neq n}_n\text{Vec}(\mathcal{G}^{\neq n}_n)^T] \mathcal{T}_O. \quad (21) $$

where $\mathcal{T}_O_{i_n} \in \mathbb{R}^{(I_{n+1}-I_n) \times I_{n+1}}$ and $\mathcal{G}^{\neq n}_n \in \mathbb{R}^{(I_{n+1}-I_n) \times I_{n+1}} \times \mathcal{O}_{i_n}$, $\mathcal{O}_{i_n}$ is the observed entries in $\mathcal{T}_O_{i_n}$ and $|\mathcal{O}_{i_n}|$ is the number of observations $\mathcal{O}_{i_n}$. 
The main computational complexity of updating core factor comes from the operation of \( \mathbb{E}[(G_{\Omega_{\infty,\infty}}^{\infty})^T G_{\Omega_{\infty,\infty}}^{\infty}] \). Based on Theorem 2, we can calculate \( \mathbb{E}[(G_{\Omega_{\infty,\infty}}^{\infty})^T G_{\Omega_{\infty,\infty}}^{\infty}] \) by the following equation:

\[
\mathbb{E}[(G_{\Omega_{\infty,\infty}}^{\infty})^T G_{\Omega_{\infty,\infty}}^{\infty}] = \sum_{\Omega_{\infty}} \prod_{l \in n} (E_{R_l} \otimes K_{R_{l-1}} \otimes E_{R_{l-1}})(\mathbb{E}[(g(i))^{(i)}])
\]

Assuming \( I_n = I \) and \( R_n = R \), the computational complexity for the calculation of \( \mathbb{E}[(g_n(i))^{(i)}] = \tilde{g}_n(i) \tilde{g}_n(i)^T + V_n \) is \( O(R^n) \). The update of core factor \( G_n \) has a complexity of \( O((N - 1)O_1 R^6) \).

2) Update \( q(\lambda^{(n)}) \): Combining equation (12) with problem (17), we can obtain the subproblem with respect to \( \lambda^{(n)} \), as follows (details of the derivation can be found in sec. 3 of supplemental materials):

\[
\max_{q(\lambda^{(n)})} \frac{1}{2} \left[ \mathbb{E}[\ln p(G_n|\lambda^{(n-1)}, \lambda^{(n)}) + \ln p(G_{n+1}|\lambda^{(n)}, \lambda^{(n+1)}) + 2 \ln p(\lambda^{(n)}|c_n, d_n)] - \mathbb{E}[\ln q(\lambda^{(n)})] \right] = (23)
\]

where

\[
q(\lambda^{(n)}) = \prod_{r_n = 1}^{R_n} \text{Ga}(\lambda^{(n)}|\tilde{c}_n^{r_n}, \tilde{d}_n^{r_n})
\]

with parameters \( \tilde{c}_n^{r_n} \) and \( \tilde{d}_n^{r_n} \).

As shown in (23), the inference of \( \lambda^{(n)} \) can be obtained by receiving the messages from its corresponding core factors, which are \( G_{n-1} \) and \( G_n \), and a pair of its partners, including \( \lambda^{(n-1)} \) and \( \lambda^{(n+1)} \), meanwhile combining the information with its hyperparameters, which are \( c_n \) and \( d_n \). Therefore, for each posteriors of \( \lambda^{(n)} \), \( r_n \in \{1, \cdots, R_n\} \), the optimization model is

\[
\max_{c_n^{r_n}, d_n^{r_n}} \left( c_n^{r_n} + \frac{1}{2}(I_n R_{n-1} + I_{n+1} R_{n+1}) - 1 \right) \ln \lambda^{(n)} + \left[ (d_n^{r_n} + \frac{1}{4}(\mathbb{E}[\lambda^{(n-1)}]\mathbb{E}[(G_n(r_n)G_n(r_n)^T)])] + \mathbb{E}[\lambda^{(n)}]\mathbb{E}[(G_{n+1}(r_n)G_{n+1}(r_n)^T)])] \right] = (25)
\]

The optimization solutions are obtained by

\[
\tilde{c}_n^{r_n} = c_n^{r_n} + \frac{1}{2}(I_n R_{n-1} + I_{n+1} R_{n+1})
\]
\[
\tilde{d}_n^{r_n} = d_n^{r_n} + \frac{1}{4}(\mathbb{E}[\lambda^{(n-1)}]\mathbb{E}[(G_n(r_n)G_n(r_n)^T)])] + \mathbb{E}[\lambda^{(n)}]\mathbb{E}[(G_{n+1}(r_n)G_{n+1}(r_n)^T)])]
\]
\[
\mathbb{E}[(\lambda^{(n-1)})\mathbb{E}[(G_n(r_n)G_n(r_n)^T)])] = \sum_{i_n = 1}^{I_n} \mathbb{E}[(\lambda^{(n-1)})\mathbb{E}[(\tilde{g}_n(i_n))\tilde{g}_n(i_n)^T + V_n(r_n)])] = (27)
\]

where \( \tilde{g}_n(i_n) = G_n(\cdot, i_n, r_n) \in \mathbb{R}^{R_n} \). Therefore, for each \( \lambda^{(n)} \), the complexity is \( O(2IR^2) \) under the assumption that all \( I_n = I \) and \( R_n = R \).
3) Update $q(\tau)$: Similarly, the subproblem corresponding to $\tau$ can be converted into:

$$
\max_{q(\tau)} \mathbb{E}[\ln p(T_{\mathcal{O}}|G_{n=1}^N, \tau) + \ln p(\tau|a, b)] - \mathbb{E}[\ln q(\tau)].
$$

(28)

where

$$
q(\tau) = \text{Ga}(\tau|\bar{a}, \bar{b})
$$

(29)

with parameters $\bar{a}$ and $\bar{b}$.

Form (28), the inference of $\tau$ can be obtained via receiving messages from observed tensor and core factors, meanwhile, incorporating with the message from the hyperparameters $a$ and $b$. Applying these messages, the (28) could be reformulated as:

$$
\max_{\bar{a}, \bar{b}} (a + \frac{O}{2} - 1) \ln \tau - (b + \frac{1}{2} \mathbb{E}[\|O \odot (T - \hat{X})\|^2])\tau - \bar{a} \ln \tau + \bar{b} \tau
$$

(30)

The maximization value could be obtained when

$$
\bar{a} = a + \frac{O}{2}
$$

$$
\bar{b} = b + \frac{1}{2} \mathbb{E}[\|O \odot (T - \hat{X})\|^2]
$$

(31)

where $|O| = \sum_{(i_1, \ldots, j_N) \in O} O_{i_1, \ldots, j_N}$ is the number of total observations, $\hat{X} = f(G_1, G_2, \ldots, G_N)$.

It can be seen from (31), calculating $\bar{b}$ costs the most time, as follows:

$$
\mathbb{E}[\|O \odot (T - \hat{X})\|^2] = T_{\mathcal{O}}^2 - 2 * T_{\mathcal{O}} * \hat{X}_{\mathcal{O}} + \mathbb{E}[\hat{X}_{\mathcal{O}} \hat{X}_{\mathcal{O}}^T]
$$

(32)

with

$$
\mathbb{E}[\hat{X}_{\mathcal{O}} \hat{X}_{\mathcal{O}}^T] = \sum_{n=1}^N (E_{R_n} \otimes K_{R_n} \otimes E_{R_n-1}) (\mathbb{E}[(g_{n}(i_n)(g_{n}(i_n))^T)])
$$

(33)

we could see the main computational complexity of updating $\bar{b}$ is $O(|O|NR^6)$ with all $I_n = I$ and $R_n = R$.

For clarity, we call this algorithm low TR rank based on VBI framework (TR-VBI) for image completion and summarize it in Algorithm 1.

C. Discussion

We have developed an efficient algorithm to automatically determine the TR ranks for tensor completion. From the solution of (21), the update of $V_n$ is related with the noise precision $\mathbb{E}[\tau]$, the information from other core factors $\mathbb{E}[(G_{\mathcal{O}_{w_{\tau}},\tau})^T G_{\mathcal{O}_{w_{\tau}},\tau}]$ and the prior $\mathbb{E}[(\Lambda (n-1) \otimes \Lambda (n))]$. It can be easily inferred the lower the value of noise precision is, the more the information from the $\mathbb{E}[(G_{\mathcal{O}_{w_{\tau}},\tau})^T G_{\mathcal{O}_{w_{\tau}},\tau}]$ is. Meanwhile, we could observe the update of noise precision is impacted by fitting error from (31). Therefore, if the model fits well, there will be more information from other factors than the prior. In addition, from (26), the update of $\Lambda (n)$ is associated with its interrelated core factors, which are $G_n$ and $G_{n+1}$, and its partners $\Lambda (n-1)$ and $\Lambda (n+1)$. The values of $\Lambda (n-1)$ and $\Lambda (n)$ affect their corresponding core factors. Moreover, the smaller values of $\mathbb{E}[\Lambda (n-1)](\mathbb{E}[(G_{n}(r_n)G_{n}(r_n))^T])$ and $\mathbb{E}[\Lambda (n+1)](\mathbb{E}[(G_{n+1}(r_n))G_{n+1}(r_n))]$ lead to larger $\Lambda (n)$. The larger values of $\Lambda (n-1)$ and $\Lambda (n)$ will enforce the values in core factor $G_n$ smaller, which will influence the update of $\Lambda (n-1)$ and $\Lambda (n)$ in turn. Therefore, this model have a robust capability of automatically adjusting tradeoff between fitting error and TR ranks.
Algorithm 1 TR-VBI algorithm

Input: The observed tensor $T \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, index set $\emptyset$.
Initialization: $G_n, V_n, R_n, c_n^{(r)}, d_n^{(r)}, \lambda_n^{(r)} = c_n^{(r)}/d_n^{(r)}, 1 \leq r_n \leq R_n, 1 \leq n \leq N, a, b, \tau = a/b$, stopping criterion $\varepsilon$, the maximum iteration $K$.

while $k \leq K$ do
  $k = k + 1$
  for $n = 1 : N$ do
    update the posterior $q(G_n)$ via (21)
  end for
  for $n = 1 : N$ do
    update the posterior $q(\lambda_n^{(r)})$ via (26)
    reduce rank $R_n$ by eliminating zero-components of $G_n$ and $G_n + 1$
  end for
  update the posterior $q(\tau)$ via (31)
  if $\mathbb{E}[\tau] \leq \varepsilon$ then
    break
  end if
end while
Output: recovered tensor $\hat{X}$.

D. Complexity Analysis

Storage Complexity For an $N$-order tensor $X \in \mathbb{R}^{I_1 \times \cdots \times I_N}$, the storage complexity is $\prod_{n=1}^{N} I_n$, which increases exponentially with its order. Assuming all $I_n = I$ and $R_n = R$ in the TR model, we only need to store the core factors and hyperparameters, which are $G_1, \cdots, G_N$ and $\lambda_1, \cdots, \lambda_N$ respectively, leading to $O(NIR^2 + NR)$ storage complexity.

Computational Complexity The computation cost of our proposed algorithm divides into three parts which are the update of core factors $G_1, \cdots, G_N$, the update of noise precision $\tau$ operation and the update of hyperparameters $\lambda_1, \cdots, \lambda_N$. Combining these computation complexities together, the computational complexity of our algorithm is $O(O_I(N - 1)NR^6 + 2NIR^2 + ONR^6)$ for one iteration with $I_n = I$ and $R_n = R$.

IV. Experiments

To evaluate our algorithm TR-VBI, we conduct experiments on synthetic data and real data, and compare it with TR-ALS [29], FBCP [44], HaLRTC [14] and SiLRTCTT [24]. TR-ALS and SiLRTCTT are advanced tensor networks based methods, where the former one utilizes the model based on factorization with the TR ranks known in advance and SiLRTCTT is based on TT rank minimization model, addressed by the block coordinate descent method. FBCP and HaLRTC are based on traditional tensor decompositions, where FBCP uses the CP decomposition in BI framework while HaLRTC explores low Tucker rank structure using ADMM.

All experiments are tested with respect to different missing ratios (MR), which is:

$$\text{MR} = \frac{M}{\prod_{n=1}^{N} I_n}$$

where $M$ is the number of total missing entries which are chosen randomly in a uniform distribution.

For the experiment on synthetic data, we consider the relative standard error (RSE) as a performance metric. The RSE is defined as

$$\text{RSE} = \frac{||\hat{X} - X||_F}{||X||_F}$$

where $\hat{X}$ is the recovered tensor and $X$ is the original one. In addition, peak signal-to-noise ratio (PSNR) are used to evaluate the performance for image recovery experiments too, which is

$$\text{PSNR} = 10 \log_{10}(\frac{\text{MAX}_I^2}{\text{MSE}})$$
where $\text{MAX}_I^2$ is the possible maximum pixel value of the image, and MSE is mean squared error between the original image and reconstructed image, which is defined as $\|(\hat{X} - X)\|/N$.

All tests repeatedly are ran 10 times and accomplished using MatLab 2018a on a desktop computer with 3.30GHz Intel(R) Xeon(R)(TM) CPU and 256GB RAM. Besides, it is noticed that we assume all initial TR ranks $R_n = R, 0 \leq n \leq N$ for simplification in the following experiments.

A. Synthetic Data

In this section, we conduct experiments on 4-order synthetic data $X \in \mathbb{R}^{I \times I \times I \times I}$ which is generated by the equation (3) with a set of core factors $\{G_1, \cdots, G_N\}$ where $G_n \in \mathbb{R}^{R \times I \times R}$, $R$ is the TR rank, $I$ is the size of dimension and the entries of $G_n$ obey Gaussian distribution. Besides, a tensor with noise can be constructed by adding the noise entries with the clean one, e.g. $Y = X + \hat{E}$ where $\hat{E}$ is a noise tensor with the entry following random Gaussian distribution.

To evaluate the performances of our model, including the rank estimation accuracy and the recovery quality, we consider three groups of experiments on synthetic data in this section. For verifying the rank estimation accuracy, we design two groups of experiments under different conditions. The mean and variance of predictive ranks are utilized to measure the accuracy, which is defined by

$$\text{AIR} = \frac{1}{10} \sum_{i=1}^{10} \text{mean}(\hat{R}_i),$$

$$\text{Var} = \frac{1}{10} \sum_{i=1}^{10} \text{std}(\hat{R}_i),$$

where AIR and Var represent the mean and the variance respectively, and each $\hat{R}_i, i \in \{1, \cdots, 10\}$ are the inferred TR ranks and $i$ is the number of $i$-th tests. The mean and std functions calculate the mean and variance of $\hat{R}_i$ respectively. It is noticed that the rank determination is a success if the $R - 0.25 \leq \text{AIR} \leq R + 0.25$, where $R$ is the real data rank in this experiment.

The first group is tested on a 4-order tensor $X \in \mathbb{R}^{10 \times 10 \times 10 \times 10}$ with $R=3$ under different signal noise ratio (SNR) conditions when $\text{MR}=0.1$ and $\text{MR}=0$. The change of AIR and Var along with SNR could be seen in Fig. 3(a).

We could observe the inferred rank is reaching the real rank when $\text{SNR} \geq 10\text{dB}$ for complete tensor. However, for incomplete tensor with noise, our model can successfully determine the real rank when $\text{SNR} = 20\text{dB}$.

The second group considered 4-order tensors with $I = 10$ and $I = 15$, respectively. In this case, $\text{SNR}=20$ and $R = 3$, and the change of AIR with MR can be illustrated in Fig. 3(b). We could see AIR also tends to the true one with different sizes when $\text{MR} \geq 0.7$.

![AIR vs SNR](image1)

(a) AIR vs SNR

![AIR vs MR](image2)

(b) AIR vs MR

Fig. 3: AIR on different conditions.

The last one is verified on a 4-D tensor with $I = 10$, $R = 3$ and $\text{SNR}=30$ using our proposed approach and exiting state-of-the-art methods including TR-ALS, SiLRTCTT, HaLRTC and FBCP. Specifically, TR-ALS with known TR ranks can be seen as a benchmark in this experiment. The result of RSE changing with MR is shown on Fig. ??

We could see RSE increases with MR growing for all methods. Among these, TR-ALS and TR-VBI approaches outperform others in terms of RSE. Furthermore, the result of TR-VBI is reaching that of TR-ALS with all MRs, which means our proposed algorithm can successfully recover the missing data. On the other hand, the inferred ranks could give a guideline for TR-ALS approach on condition that the real TR ranks are unknown in advance.
B. Color Images

In this experiment, we consider the image completion of different sizes of RGB images, including “lena” with the size 256 × 256 × 3, “bird” and “dragonfly” with the size of 320 × 480 × 3 chosen from Berkeley Segmentation database [45], and “Einstein”\(^1\) with the size of 600 × 600 × 3. The testing image can be observed in Fig. 5.

\(^1\)https://imgur.com/gallery/5ttQu
Fig. 6: Performance comparison on image completion using different methods with different MR under different conditions. The first row shows PSNR vs MR; the second row shows RSE vs MR.

Fig. 7: Examples on image completion using different methods with different MR.

Different conditions where “N” represents the recovery result with noisy environment and “NF” indicates the noise-free one. It can be observed the recovery performance of our proposed one outperforms that of others for all tested images under noisy-free condition. Interestingly, the second-best result on image completion is produced by TR-ALS approach. This may imply the TR decomposition can explore more latent information from image. Compared with TR-ALS, the reason that the proposed one achieves better may be our framework has a good ability to balance fitting error and TR ranks. However, the recovery result based on advanced tensor networks under noisy environment, including TR-VBI, TR-ALS and SiLRTCTT, performs worse than that with noise-free condition.

The recovered results using different algorithms with MR from 70% to 95% have been shown in Fig. 7. We could observe recovered images by TR-based methods have a better resolution when MR=90% and MR=95%. In addition, TR-VBI and TR-ALS recover more details with MR=70% and MR=80%. For example, the light on the hat for “lena” image and the wrinkles on the forehead for “Einstein” image are more clear.
C. YaleFace Dataset

In this experiment, extended YaleFace Dataset B [47], [48], which contains the images of 38 people under 9 poses and 64 illumination conditions where the size of each image is 192 × 168, is chosen as a 4D data with respect to one pose (192 × 168 × 64 × 38) in this experiment. We downsample the image size to 48 × 42 as a result of computational limitation and reformat the 4D tensor in $\mathbb{R}^{48 \times 42 \times 64 \times 38}$. This can be illustrated in Fig. 8. The parameters of the HaLRTCTT are set as $w = b/\|b\|_1$, $b=[1, 1, 1, 1]$. And the parameters of other methods are also chosen as same as the ones in the color image experiments.

Parameters of the HaLRTCTT are set as $w = b/\|b\|_1$, $b=[1, 1, 1, 1]$. And the parameters of other methods are also chosen as same as the ones in the color image experiments.

Fig. 8: Extended YaleFace Dataset B with respect to one pose.

![Extended YaleFace Dataset B with respect to one pose.](image)

Fig. 9: The comparison of YaleFace Dataset completion using different approaches when MR=90%. From Fig. 9, we could see the methods based on traditional tensor decompositions, including FBCP and HaLRTC, perform worse in terms of recovery quality, which may imply that advanced tensor network based methods explore more information in the high-dimensional data than traditional tensor decompositions based one. Meanwhile, the recovery results of TR-based methods are superior to the one recovered by SiLRTCTT, which may show the advantage of TR decomposition framework on high dimensional data. Furthermore, compared with TR-ALS, TR-VBI could recover the image with a better resolution such as clear eyes and smooth face. Therefore, TR-VBI outperforms all state-of-the-art ones in terms of image recovery performance.
We have developed a Bayesian low rank TR framework for image completion, which offers a better tradeoff between fitting error and TR ranks. To the best of my knowledge, it is the first time applying TR decomposition in full BI framework on image completion. We utilize mean-field variational inference to approximate the full Bayesian inference and we drive a detailed solution to solve this optimization problem. Several experiments on synthetic data and real-world data demonstrate the superiority of our method over state-of-the-art algorithms.

V. CONCLUSION

We have developed a Bayesian low rank TR framework for image completion, which offers a better tradeoff between fitting error and TR ranks. To the best of my knowledge, it is the first time applying TR decomposition in full BI framework on image completion. We utilize mean-field variational inference to approximate the full Bayesian inference and we drive a detailed solution to solve this optimization problem. Several experiments on synthetic data and real-world data demonstrate the superiority of our method over state-of-the-art algorithms.

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Table I: The PSNR/RSE comparision of 90% missing data via different methods.

|         | TR-VBI | TR-ALS | SiLRTCTT | FBCP | HaLRTC |
|---------|--------|--------|----------|------|--------|
| PSNR    | 25.59± 0.08 | 24.18± 0.08 | 21.90± 0.009 | 19.26± 0.33 | 16.39± 0.013 |
| RSE     | 0.1477±0.0014 | 0.1792±0.0002 | 0.2205±2.2×10⁻⁴ | 0.30±0.0113 | 0.4141±6.29×10⁻⁴ |

Table I illustrates the average results for 10 experiments using different method on YaleFace dataset completion when MR=90%. We could observe that TR-VBI performs best compared with other approaches in terms of PSNR and RSE. Interestingly, the values of variance in SiLRTCTT and HaLRTC are smaller than that in TR-VBI, TR-ALS and FBCP. This may be caused by the initialization of the latter algorithms which solve problems by iteratively updating core factors.
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