Dimensional Regularization and Renormalization of Non-Commutative Quantum Field Theory

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Abstract. Using the recently introduced parametric representation of non-commutative quantum field theory, we implement here the dimensional regularization and renormalization of the vulcanized $\Phi^4_4$ model on the Moyal space.

1. Introduction and motivation

Non-commutative geometry (see [6]) is one of the most appealing frameworks for the quantification of gravitation. Quantum field theory (QFT) on these type of spaces, called non-commutative quantum field theory (NCQFT) – for a general review see [12, 36] – is now one of the most appealing candidates for new physics beyond the Standard Model. Also, NCQFT arises as the effective limit of some string theoretical models [7, 34].

Moreover, NCQFT is well suited to the description of the physics in background fields and with non-local interactions, like for example the fractional quantum Hall effect [25, 30, 35].

However, naive NCQFT suffers from a new type of non renormalizable divergences, known as the ultraviolet (UV)/ infrared (IR) mixing. The simplest example of this kind of divergences is given by the nonplanar tadpole: it is UV convergent, but inserting it an arbitrary number of times in a loop gives rise to IR divergences.

Interest in NCQFT has been recently revived with the introduction of the Grosse–Wulkenhaar scalar $\Phi^4_4$ model, in which the UV/IR mixing is cured: the model is renormalizable at all orders in perturbation theory [17, 18]. The idea of Grosse–Wulkenhaar was to modify the kinetic part of the action in order to satisfy the Langmann–Szabo duality [28] (which relates the infrared and ultraviolet regions). We refer to this modified theory as the vulcanized $\Phi^4_4$ model.
A general proof, using position space and multiscale analysis, has then been given in [21], and the parametric representation of this model was computed in [23]. Furthermore, it was recently proved that the vulcanized $\Phi^4_4$ is better behaved than the commutative $\phi^4_4$ model: it does not have a Landau ghost [10, 11, 20].

In commutative QFT dimensional renormalization is the only scheme which respects the symmetries of gauge theories see [2, 3, 5]. It also is the appropriate setup for the Connes–Kreimer Hopf algebra approach to renormalization (see [8, 9, 27] for the case of commutative QFT).

A second class of renormalizable NCQFT exists. These models, called covariant, are characterized by a propagator which decays in position space as $x - y$ tends to infinity (like the Grosse–Wulkenhaar propagator of (2.8)) but it oscillates when $x + y$ goes to infinity, rather than decaying. In this class of NCQFT models enters the non-commutative Gross–Neveu model and the Langmann–Szabo–Zarembo model [29]. The non-commutative orientable Gross–Neveu model was proven to be renormalizable at all orders in perturbation theory [38]. The parametric representation was extended to this class of models [33]. For a general review of recent developments in the field of renormalizable NCQFT see [32].

The parametric representation introduced in [23] is the starting point for the dimensional regularization and renormalization performed in this paper. Our proof follows that of the commutative $\Phi^4_4$ model, as presented in [2, 3].

This paper is organized as follows. Section 2 is a summary of the parametric representation of the vulcanized $\Phi^4_4$ model. The non-commutative equivalent $HU_G$ and $HV_G$ of the Symanzik polynomials $U_G$ and $V_G$ are recalled. In Section 3 we prove the existence in the polynomial $HU_G$ of some further leading terms in the ultraviolet (UV) regime. This is an improvement of the results of [23], needed to correctly identify the meromorphic structure of the Feynman amplitudes. In Section 4 we prove the factorization properties of the Feynman amplitudes. These properties are needed in order to prove that the pole extraction is equivalent to adding counterterms of the form of the initial lagrangean. This factorization is essential for the definition of a coproduct $\Delta$ necessary for the implementation of a Hopf algebra structure in NCQFT [37]. Section 5 uses the results of the previous sections to perform the dimensional regularization, prove the counterterm structure for NCQFT and complete the dimensional renormalization program. Section 6 is devoted to some conclusion and perspectives.

2. The non-commutative model

In this section we give a brief overview of the Grosse–Wulkenhaar $\Phi^4$ model. Our notations and conventions as well as some notions of diagrammatics and the results of the parametric representation follow [23].

To define the Moyal space of dimension $D$, we introduce the deformed Moyal product $\star$ on $\mathbb{R}^D$ so that

$$[x^\mu, x^\nu] = i\Theta^\mu\nu,$$  

(2.1)
where the matrix $\Theta$ is
\[
\Theta = \begin{pmatrix}
0 & \theta & 0 & 0 \\
-\theta & 0 & 0 & 0 \\
0 & 0 & 0 & \theta \\
0 & 0 & -\theta & 0
\end{pmatrix}.
\] (2.2)

The associative Moyal product of two functions $f$ and $g$ on the Moyal space writes
\[
(f \star g)(x) = \int \frac{d^Dk}{(2\pi)^D} d^Dy f \left( x + \frac{1}{2} \Theta \cdot k \right) g(x + y)e^{ik \cdot y} = \frac{1}{\pi^D |\det \Theta|} \int d^Dy d^Dz f(x + y)g(x + z)e^{-2y \Theta^{-1} z}. \] (2.3)

The Euclidean action introduced in [17] is
\[
S = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \ast \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \ast (\tilde{x}^\mu \phi) + \frac{1}{2} m^2 \phi \ast \phi + \phi \ast \phi \ast \phi \ast \phi \right), \] (2.4)
where
\[
\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu. \] (2.5)

The propagator of this model is the inverse of the operator
\[-\Delta + \Omega^2 x^2. \] (2.6)

The results we establish here hold for orientable models (in the sense of Subsection 2.1). This corresponds to a Grosse–Wulkenhaar model of a complex scalar field
\[
S = \int d^4x \left( \frac{1}{2} \partial_\mu \bar{\phi} \ast \partial^\mu \phi + \frac{\bar{\Omega}^2}{2} (\tilde{x}_\mu \bar{\phi}) \ast (\tilde{x}^\mu \bar{\phi}) + \frac{1}{2} m^2 \bar{\phi} \ast \phi + \bar{\phi} \ast \phi \ast \phi \ast \phi \right). \] (2.7)

Introducing $\bar{\Omega} = 2\Omega/\theta$, the kernel of the propagator is (Lemma 3.1 of [24])
\[
C(x, y) = \int_0^\infty \frac{\Omega \dd \alpha}{[2\pi \sinh(\alpha)]^{D/2}} e^{-\frac{\Omega}{2} \coth(\alpha)(x-y)^2 - \frac{\bar{\Omega}}{2} \tanh(\alpha)(x+y)^2}. \] (2.8)

Using (2.3) the interaction term in (2.7) leads to the following vertex contribution in position space (see [21])
\[
\delta(x_1 - x_2 + x_3 - x_4)e^{2i \sum_{1 \leq i < j \leq 4} (-1)^{i+j+1} x_i \Theta^{-1} x_j} \] (2.9)
with $x_1, \ldots, x_4$ the 4-vectors of the positions of the 4 fields incident to the vertex.

To any such vertex $V$ one associates a hypermomentum $p_V$ using the relation
\[
\delta(x_1 - x_2 + x_3 - x_4) = \int \frac{dp_V}{(2\pi)^4} e^{ip_V \sigma(x_1 - x_2 + x_3 - x_4)}. \] (2.10)
2.1. Some diagrammatics for NCQFT; orientability

In this subsection we introduce some useful conventions and definitions, some of them used in [21] and [24] but also some new ones.

Let a graph $G$ with $n(G)$ vertices, $L(G)$ internal lines and $F(G)$ faces. The Euler characteristic of the graph is

$$2 - 2g(G) = n(G) - L(G) + F(G),$$

where $g(G) \in \mathbb{N}$ is the genus of the graph. Graphs divide in two categories, planar graph with $g(G) = 0$, and non-planar graphs with $g(G) > 0$. Let also $B(G)$ denote the number of faces broken by external lines and $N(G)$ be number of external points of the graph.

The “orientable” form (2.9) of the vertex contribution of our model allows us to associate “+” sign to a corner $\bar{\phi}$ and a “−” sign to a corner $\phi$ of the vertex. These signs alternate when turning around a vertex. As the propagator always relates a $\bar{\phi}$ to a $\phi$, the action in (2.7) has orientable lines, that is any internal line joins a “−” corner to a “+” corner.

Consider a spanning tree $T$ in $G$. In has $n - 1$ lines and the remaining $L - (n - 1)$ lines form the set $L$ of loop lines. Amongst the vertices $V$ one chooses a special one $V_R$, the root of the tree. One associates to any vertex $V$ the unique tree line which hooks to $V$ and goes towards the root.

We introduce now some topological operations on the graph which allow one to reexpress the oscillating factors coming from the vertices of the graph $G$.

Let a tree line in the graph $\ell = (i, j)$ and its endpoints $i$ and $j$. Suppose it connects to the root vertex $V_R$ at $i$ and to another vertex $V$ at $j$. In Figure 5, $\ell_2$ is the tree line, $y_4$ is $i$ and $x_1$ is $j$. The first Filk move, inspired by [13], consists in removing such a line from the graph and gluing the two vertices together respecting the ordering. Thus the point $i$ on the root vertex is replaced by the neighbors of $j$ on $V$. This is represented in Figure 1 where the new root vertex is $y_2, y_3, x_4, x_1, x_2, y_1$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{The first Filk move: the line 2 is reduced and the 2 vertices merge.}
\end{figure}

The orientability of our theory allows us to simplify the proofs. It should however be possible, although tedious, to follow the same procedure for the non orientable model.
Note that the number of faces or the genus of the graph do not change under this operation.

A technical point to be noted here is that one must chose the field $j$ on the vertex $V$ to be either the first (if the line $\ell$ enters $V$) or the last (if the line $\ell$ exits $V$) in the ordering of $V$. Of course this is allways possible by the use of the $\delta$ functions in the vertex contribution.

Iterating this operation for the $n - 1$ tree lines, one obtains a single final vertex with all the loop lines hooked to it – a rosette (see Figure 2).

The rosette contains all the topological information of the graph. If no two lines cross (on the left in Figure 2) the graph is planar. If on the contrary we have at least a crossing (on the right in Figure 2) the graph is non planar (for details see [23]).

For a nonplanar graph we define a nice crossing in a rosette as a pair of lines such that the end point of the first is the successor in the rosette of the starting point of the other. A genus line of a graph is a loop line which is part of a nice crossing on the rosette (lines 2 and 4 on the right of Figure 2).

In the sequel we are interested in performing this operation in a way adapted to the scales introduced by the Hepp sectors: we perform the first Filk move only for a subgraph $S$ (we iterate it only for a tree in $S$). Thus, the subgraph $S$ will be shrunk to its corresponding rosette inside the graph $G$. If $S$ is not primitively divergent we have a convergent sum over its associated Hepp parameter.

We will prove later that $S$ is primitively divergent if and only if $g(S) = 0$, $B(S) = 1$, $N(S) = 2, 4$. For primitively divergent subgraphs the first Filk move above shrinks $S$ to a Moyal vertex inside the graph $G$.

For example, consider the graph $G$ of Figure 3 and its divergent sunshine subgraph $S$ given by the set of lines $\ell_4, \ell_5$ and $\ell_6$. Under the first Filk move for the subgraph $S$, $G$ will have a rosette vertex insertion like in Figure 4. Denote $G - S$ graph $G$ with its subgraph $S$ erased (see Figure 5). It becomes the graph $G/S$ with a Moyal vertex like in Figure 6.
2.2. Parametric representation for NCQFT

In this subsection we recall the definitions and results obtained in [23] for the parametric representations of the model defined by (2.7). First let us recall that, when considering the parametric representation for commutative QFT, one has translation invariance in position space. As a consequence of this invariance, the
The graph $G/S$ obtained by shrinking to a Moyal vertex the sunshine primitive divergent subgraph $S$.

first polynomial vanishes when integrating over all internal positions. Therefore, one has to integrate over all internal positions (which correspond to vertices) save one, which is thus marked. However, the polynomial is a still a canonical object, i.e., it does not depend of the choice of this particular vertex.

As stated in [23], in the non-commutative case translation invariance is lost (because of non-locality). Therefore, one can integrate over all internal positions and hypermomenta. However, in order to be able to recover the commutative limit, we also mark a particular vertex $\bar{V}$; we do not integrate on its associate hypermomenta $p_{\bar{V}}$. This particular vertex is the root vertex. Because there is no translation invariance, the polynomial does depend on on the choice of the root; however the leading ultraviolet terms do not.

We define the $(L \times 4)$-dimensional incidence matrix $\varepsilon^V$ for each of the vertices $V$. Since the graph is orientable (in the sense defined in Subsection 2.1 above) we can choose

$$\varepsilon^V_{\ell i} = (-1)^{i+1}, \text{ if the line } \ell \text{ hooks to the vertex } V \text{ at corner } i. \quad (2.12)$$

Let also

$$\eta^V_{\ell i} = |\varepsilon^V_{\ell i}|, \quad V = 1, \ldots, n, \quad \ell = 1, \ldots, L \quad \text{and} \quad i = 1, \ldots, 4. \quad (2.13)$$

From (2.12) and (2.13) one has

$$\eta^V_{\ell i} = (-1)^{i+1} \varepsilon^V_{\ell i}. \quad (2.14)$$

We introduce with the “short” $u$ and “long” $v$ variables by

$$v_\ell = \frac{1}{\sqrt{2}} \sum_V \sum_i \eta^V_{\ell i} x^V_i, \quad u_\ell = \frac{1}{\sqrt{2}} \sum_V \sum_i \varepsilon^V_{\ell i} x^V_i. \quad (2.15)$$

Conversely, one has

$$x^V_i = \frac{1}{\sqrt{2}} (\eta^V_{\ell i} v_\ell + \varepsilon^V_{\ell i} u_\ell). \quad (2.16)$$
From the propagator 2.8 and vertices contributions 2.9 one is able to write the amplitude $A_{G,\bar{V}}$ of the graph $G$ (with the marked root $\bar{V}$) in terms of the non-commutative polynomials $H_{U_{G,\bar{V}}}$ and $H_{V_{G,\bar{V}}}$ as (see [23] for details)

$$A_{G,\bar{V}}(x_e, p_{\bar{V}}) = \left( \frac{\tilde{\Omega}}{2^\tilde{\nu} - 1} \right)^L \int_0^1 \prod_{\ell=1}^L \, dt_\ell (1 - t_\ell^2)^{\tilde{\nu} - 1} \exp \left[ \frac{H_{U_{G,\bar{V}}}(t_\ell, x_e, p_{\bar{V}})}{H_{V_{G,\bar{V}}}(t_\ell)} \right], \quad (2.17)$$

with $x_e$ the external positions of the graph and

$$t_\ell = \tanh \frac{\alpha_\ell}{2}, \quad \ell = 1, \ldots, L. \quad (2.18)$$

where $\alpha_\ell$ are the parameters associated by (2.8) to the propagators of the graph. In [23] it was proved that $H_U$ and $H_V$ are polynomials in the set of variables $t$. The first polynomial is given by (see again [23])

$$H_{U_{G,\bar{V}}} = (\det Q)^\tilde{\nu} \prod_{\ell=1}^L t_\ell, \quad (2.19)$$

where

$$Q = A \otimes 1_D - B \otimes \sigma, \quad (2.20)$$

with $A$ a diagonal matrix and $B$ an antisymmetric matrix. The matrix $A$ writes

$$A = \begin{pmatrix} S & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.21)$$

where $S$ and resp. $T$ are the two diagonal $L$ by $L$ matrices with diagonal elements $c_\ell = \coth(\frac{\alpha_\ell}{2}) = 1/t_\ell$, and resp. $t_\ell$. The last $(n-1)$ lines and columns are have 0 entries.

The antisymmetric part $B$ is

$$B = \begin{pmatrix} sE & C \\ -C^t & 0 \end{pmatrix}, \quad (2.22)$$

with

$$s = \frac{2}{\Theta} = \frac{1}{\tilde{\Omega}} \quad (2.23)$$

and

$$C_{\ell V} = \left( \frac{\sum_{i=1}^4 (-1)^{i+1} \epsilon_{\ell i}}{\sum_{i=1}^4 (-1)^{i+1} \eta_{\ell i}} \right), \quad (2.24)$$

$$E = \begin{pmatrix} E_{uu} & E_{uv} \\ E_{vu} & E_{vv} \end{pmatrix}, \quad (2.25)$$
The blocks of the matrix $E$ are

$$E_{v,v}^{\ell,\ell'} = \sum_V \sum_{i,j=1}^4 (-1)^{i+j+1} \omega(i,j) \eta_i^V \eta_{i'}^V,$$

$$E_{u,u}^{\ell,\ell'} = \sum_V \sum_{i,j=1}^4 (-1)^{i+j+1} \omega(i,j) \epsilon_i^V \epsilon_{i'}^V,$$

$$E_{u,v}^{\ell,\ell'} = \sum_V \sum_{i,j=1}^4 (-1)^{i+j+1} \omega(i,j) \epsilon_i^V \eta_{i'}^V.$$

(2.26)

The symbol $\omega(i,j)$ takes the values $\omega(i,j) = 1$ if $i < j$, $\omega(i,j) = -1$ is $j < i$ and $\omega(i,j) = 0$ if $i = j$. In (2.24) of the matrix $C$ we have rescaled by $s$ the hypermomenta $p_V$. For further reference we introduce the integer entries matrix:

$$B' = \begin{pmatrix} E & C \\ -C^t & 0 \end{pmatrix}.$$ (2.27)

In [23] it was proven that

$$\det Q = (\det M)^D,$$ (2.28)

where

$$M = A + B.$$ (2.29)

Thus (2.19) becomes:

$$H U_{G,V} = \det M \prod_{\ell=1}^L t_\ell.$$ (2.30)

Let $I$ and resp. $J$ be two subsets of $\{1, \ldots, L\}$, of cardinal $|I|$ and $|J|$. Let

$$k_{I,J} = |I| + |J| - L - F + 1,$$ (2.31)

and $n_{I,J} = \text{Pf}(B'_{I,J})$, the Pfaffian of the matrix $B'$ with deleted lines and columns $I$ among the first $L$ indices (corresponding to short variables $u$) and $J$ among the next $L$ indices (corresponding to long variables $v$).

The specific form 2.21 allows one to write the polynomial $H U$ as a sum of positive terms:

$$H U_{G,V}(t) = \sum_{I,J} s^{2^g - k_{I,J}} n_{I,J}^2 \prod_{\ell \in I} t_\ell \prod_{v \in J} t_v.$$ (2.32)

In [23], some non-zero terms were identified. They correspond to subsets $I = \{1, \ldots, L\}$ and $J$ admissible, that is

- $J$ contains a tree $\tilde{T}$ in the dual graph,
- the complement of $J$ contains a tree $T$ in the direct graph.

Amongst this terms, the leading UV terms, (i.e., terms with the smallest global degree in the $t$ variables) are given by choosing $J$ minimal, that is

- $J = \tilde{T}$, tree in the dual graph,
• the complement of $J$ is the union of a tree $T$ in the direct graph and $2g$ genus lines.

However, the list of these leading terms, as already remarked in [23], is not exhaustive. In the next section we complete this list with further terms, necessary for the sequel.

We end this section with the explicit example of the bubble and the sunshine graph (Figure 7 and Figure 8).

For the bubble graph one has
\[ H_{UG,\bar{V}} = (1 + 4s^2) (t_1 + t_2 + t_1^2t_2 + t_1t_2^2) . \] (2.33)

For the sunshine graph one has
\[ H_{UG,\bar{V}} = \left[ t_1t_2 + t_1t_3 + t_2t_3 + t_1^2t_2t_3 + t_1t_2^2t_3 + t_1t_2t_3^2 \right] (1 + 4s^2)^2 + 16s^2 \left( t_2^2 + t_2t_3^2 \right) . \] (2.34)

For further reference we also give the polynomial of the graph of Figure 6
\[ H_{UG,\bar{V}} = (1 + 4s^2)(t_1 + t_2 + t_3 + t_1t_2t_3)(1 + t_2t_3 + t_1(t_2 + t_3)) . \] (2.35)

The polynomial $HV$ is more involved. One has
\[ \frac{H_{VG,\bar{V}}}{H_{UG,\bar{V}}} = \frac{\hat{\Omega}}{2} (x_e \quad p_V) PQ^{-1} P^t \left( x_e \quad p_V \right) , \] (2.36)

\[ ^2 \text{For the purposes of [23], the existence of some non-zero leading terms was sufficient.} \]
where $P$ is some matrix coupling the external positions $x_e$ and the root hypermomenta $p_V$ with the short $u$ and long $v$ variables and the rest of the hypermomenta $p_V (V \neq \bar{V})$. Explicit expressions can be found in [23].

3. Further leading terms in the first polynomial $H_U$

To proceed with the dimensional regularization one needs first to correctly isolate the divergent subgraphs in different Hepp sectors. Let a subgraph $S$ in the graph $G$. If $S$ is nonplanar the leading terms in $H_U_G$ suffice to prove that $S$ is convergent in every Hepp sector as will be explained in Section 5. This is not the case if $S$ is planar with more that one broken face. Let $\tilde{\ell}$ be the line of $G$ which breaks an internal face of $S$.

If $\tilde{\ell}$ is a genus line in $G$ (see Subsection 2.1), one could still use only the leading terms in $H_U_G$ to prove that $S$ is convergent. But one still needs to prove that $S$ is convergent if the line $\tilde{\ell}$ is not a genus line in $G$. This is for instance the case of the sunshine graph: in the Hepp sector $t_1 < t_3 < t_2$ one must prove that the subgraph formed by the lines $l_1$ and $l_3$ is convergent. This is true due to the term $16s^2t_2^2$ in (2.34). Note that the variable $t_2$ is associated to the line which breaks the internal face of the subgraph $l_1, l_3$. One needs to prove that for arbitrary $G$ and $S$ we have such terms.

If we reduce $S$ to a rosette there exists a loop line $\ell_2 \in S$ which either crosses $\tilde{\ell}$ or encompasses it. This line separates the two broken faces of $S$.

**Definition 3.1.** Let $J_0$ a subset of the internal lines of the graph $G$. $J_0$ is called pseudo-admissible if:

- its complement is the union of tree $T$ in $G$ and $\ell_2$,
- neither $\tilde{\ell}$ nor $\ell_2$ belong to $T$.

Let $I_0 = \{\ell_1, \ldots, \ell_L\} - \tilde{\ell} \equiv I - \tilde{\ell}$. This implies $|I| = L(G) - 1$ and $|J| = F(G) - 2 + 2g(G)$. For the sunshine graph (see Figure 8) $I = \{\ell_1, \ell_3\}$ and $J = \{\ell_2\}$. One has the theorem

**Theorem 3.1.** In the sum 2.32 the term associated to $I_0$ and $J_0$ above is

\[
n_{I_0,J_0}^2 = \begin{cases} 4 & \text{if } \tilde{\ell} \text{ is a genus line in } G \\ 16 & \text{if } \tilde{\ell} \text{ is not a genus line in } G. \end{cases}
\]

**Proof.** The proof is similar to the one concerning the leading terms of $H_U$ given in Lemma III.1 of [23], being however more involved.

The matrix whose determinant we must compute is obtained from $B$ by deleting the lines and columns corresponding to the subsets $I_0$ and $J_0$ (as explained in the previous section).

The matrix $B'_{I_0,J_0}$ has

- a line and column corresponding to $u_{\tilde{\ell}}$, the short variable of $\tilde{\ell}$
\begin{itemize}
  \item $n$ lines and columns corresponding to the $v$ variables of the $n - 1$ tree lines of $T$ and the supplementary line $\ell_2$
  \item $n - 1$ lines and columns associated to the hypermomenta.
\end{itemize}

We represent the determinant of this matrix by the Grassmann integral:

$$\det B'_{I_0, J_0} = \int d\psi^\alpha_\ell d\bar{\psi}^\alpha_\ell d\bar{\psi}^\beta_\ell d\psi^\beta_\ell e^{-\bar{\psi} B' \psi}. \tag{3.1}$$

The quadratic form in the Grassman variables in the above integral is:

$$- \sum_{V} \sum_{\ell, i,j} \bar{\psi}^\ell_i \epsilon^{i+1} \omega(i,j) \bar{\psi}^V_{\ell i} \psi^V_{\ell j} \psi^\ell_j$$

$$- \sum_{V} \sum_{\ell, i,j} \bar{\psi}^\ell_i \omega(i,j) \epsilon^{i+1} \psi^V_{\ell i} \psi^\ell_j$$

$$+ \sum_{V} \bar{\psi}^\ell_i (-1)^{i+1} \epsilon^{V}_{i} \psi^\ell_i$$

$$- \sum_{V} \sum_{\ell, \ell',i,j} \bar{\psi}^\ell_i \omega(i,j) \epsilon^{i+1} \psi^{V}_{\ell i} \psi^{\ell'}_{\ell' j}$$

$$+ \sum_{V} \sum_{\ell, i} \bar{\psi}^\ell_i \epsilon^{i+1} \psi^{V}_{\ell i} \psi^\ell_i - \sum_{V} \sum_{\ell, i} \bar{\psi}^\ell_i \epsilon^{i+1} \psi^{V}_{\ell i} \psi^\ell_i. \tag{3.2}$$

We implement the first Filk move as a Grassmann change of variables. At each step we reduce a tree line $\ell_1 = (i, j)$ connecting the root vertex $V_G$ to a normal vertex $V$ and gluing the two vertices. This is achieved by performing a change of variables for the hypermomenta and reinterpreting the quadratic form in the new variables as corresponding to a new vertex $V'_G$: the quadratic form essentially reproduces itself under the change of variables!

Take $\ell_1 = (i, j)$ a line connecting the “root” vertex $V_G$ to a vertex $V$. We make the change of variables

$$\psi^{PV} = \chi^{PV} + \sum_{\ell' \neq \ell_1} \sum_k \left( -\omega(i, k) \epsilon^{V}_{\ell' k} + \omega(j, k) \epsilon^{V}_{\ell' k} \right) \bar{\psi}^{\ell'}_k$$

$$+ \sum_k \left( -\omega(i, k) \epsilon^{V}_{\ell' k} + \omega(j, k) \epsilon^{V}_{\ell' k} \right) (-1)^{k+1} \psi^V_k$$

$$\bar{\psi}^{PV} = \bar{\chi}^{PV} + \sum_{\ell' \neq \ell_1} \sum_k \left( -\omega(i, k) \epsilon^{V}_{\ell' k} + \omega(j, k) \epsilon^{V}_{\ell' k} \right) \bar{\psi}^{\ell'}_k$$

$$+ \sum_k \left( -\omega(i, k) \epsilon^{V}_{\ell' k} + \omega(j, k) \epsilon^{V}_{\ell' k} \right) (-1)^{k+1} \bar{\psi}^{V}_k, \tag{3.3}$$

where the first sum is performed on the internal lines of $G$ (note that because of the presence of the incidences matrices $\epsilon$ this sum reduces to a sum on the lines hooked to the two vertices $V_G$ and $V$). The corners $i$ and resp. $j$ are the corners where the tree line $\ell_1$ hooks to the vertex $V_G$ and resp. $V$.

At each step, let us consider the coupling between the variables associated to the line $\ell_1$ and to the hypermomentum $p_V$ and the rest of the variables. Using (3.2)
one has

\[
- \bar{\psi}^{\nu}_{\ell} \sum_{p} (-1)^{p+1} \left( \omega(p, i) e_{\ell p}^{V G} e_{\ell_1}^{V G} + \omega(p, j) e_{\ell p}^{V} e_{\ell_1}^{V} \right) \psi_{\ell_1}^{\nu} - \bar{\psi}^{\nu}_{\ell_1} \sum_{p} \left( \omega(i, p) e_{\ell p}^{V G} e_{\ell_1}^{V G} + \omega(j, p) e_{\ell p}^{V} e_{\ell_1}^{V} \right) \psi_{\ell_1}^{\nu} + \sum_{V \neq \ell} \bar{\psi}^{\nu}_{V} (-1)^{p+1} e_{\ell p}^{V} \psi_{\ell}^{\nu} - \sum_{V \neq \ell_1} \bar{\psi}^{\nu}_{V} \left( \omega(i, k) e_{\ell p}^{V G} e_{\ell_1}^{V G} + \omega(j, k) e_{\ell p}^{V} e_{\ell_1}^{V} \right) \psi_{\ell_1}^{\nu} - \sum_{V \neq \ell_1} \bar{\psi}^{\nu}_{V} \left( \omega(i, p) e_{\ell p}^{V G} e_{\ell_1}^{V G} + \omega(j, p) e_{\ell p}^{V} e_{\ell_1}^{V} \right) \psi_{\ell_1}^{\nu} + \bar{\psi}^{\nu}_{V} e_{\ell_1}^{V} \psi_{\ell}^{\nu} - \sum_{V \neq \ell_1} \bar{\psi}^{\nu}_{V} e_{\ell_1}^{V} \psi_{\ell}^{\nu} - \bar{\psi}^{\nu}_{\ell_1} e_{\ell_1}^{V} \psi_{\ell}^{\nu} - \sum_{V \neq \ell_1} \bar{\psi}^{\nu}_{V} e_{\ell_1}^{V} \psi_{\ell}^{\nu} \right)
\]

which rewrites as

\[
- \left[ \bar{\psi}^{\nu}_{\ell} \sum_{p} (-1)^{p+1} \left( \omega(p, i) e_{\ell p}^{V G} e_{\ell_1}^{V G} + \omega(p, j) e_{\ell p}^{V} e_{\ell_1}^{V} \right) \right] \psi_{\ell_1}^{\nu} + \sum_{V \neq \ell_1} \bar{\psi}^{\nu}_{V} \left( \omega(i, k) e_{\ell p}^{V G} e_{\ell_1}^{V G} + \omega(j, k) e_{\ell p}^{V} e_{\ell_1}^{V} \right) \psi_{\ell_1}^{\nu} - \sum_{V \neq \ell_1} \bar{\psi}^{\nu}_{V} \left( \omega(i, p) e_{\ell p}^{V G} e_{\ell_1}^{V G} + \omega(j, p) e_{\ell p}^{V} e_{\ell_1}^{V} \right) \psi_{\ell_1}^{\nu} + \bar{\psi}^{\nu}_{\ell_1} e_{\ell_1}^{V} \psi_{\ell}^{\nu} - \sum_{V \neq \ell_1} \bar{\psi}^{\nu}_{V} e_{\ell_1}^{V} \psi_{\ell}^{\nu} - \bar{\psi}^{\nu}_{\ell_1} e_{\ell_1}^{V} \psi_{\ell}^{\nu} - \sum_{V \neq \ell_1} \bar{\psi}^{\nu}_{V} e_{\ell_1}^{V} \psi_{\ell}^{\nu} \right)
\]

Performing now in (3.4) the change of variable (3.3) for the hypermomentum \( p_{V} \) of the vertex \( V \) associated to the tree line \( \ell_1 \) of \( G \) and taking into account that \( \varepsilon_{\ell_1} = -\varepsilon_{\ell_1} \) the first two lines of 3.5 are simply

\[
- \bar{\chi}_{V}^{\nu} e_{\ell_1}^{V} \psi_{\ell_1}^{\nu} + \bar{\psi}_{\ell_1}^{\nu} e_{\ell_1}^{V} \chi_{V}^{\nu}.
\]
As $\tilde{\psi}_i^p$ and $\tilde{\psi}_i^p$ do not appear anymore in the rest of the terms we are forced to pair them with $\chi^p v$ and $\chi^p v$. The rest of the terms in the quadratic form are:

$$
\sum_{V,p} \sum_{\ell \neq \ell_1} \sum_{k} \left( -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right) \psi^v_{\ell} \\
+ \sum_{k} \left( -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right) (-1)^{k+1}\psi^u_{\ell} \\
+ \sum_{\ell \neq \ell_1, p} \sum_{\tilde{\ell}_1} \left( -\omega(i,k)\varepsilon^V_{\tilde{\ell}_1 k} + \omega(j,k)\varepsilon^V_{\tilde{\ell}_1 k} \right) \psi^v_{\tilde{\ell}} \\
+ \sum_{k} \left( -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right) (-1)^{k+1}\psi^u_{\ell} \\
- \sum_{V,p} \sum_{\ell \neq \ell_1} \sum_{k} \left( -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right) \tilde{\psi}^v_{p,\ell} \\
+ \sum_{k} \left( -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right) (-1)^{k+1}\tilde{\psi}^u_{p,\ell} \\
- \sum_{\ell \neq \ell_1} \sum_{k} \left( -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right) \tilde{\psi}^v_{\tilde{\ell}_1, p} \\
+ \sum_{k} \left( -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right) (-1)^{k+1}\tilde{\psi}^u_{\ell_1, p} \sum_{\ell \neq \ell_1, p} \varepsilon^V_{\ell p,\ell'} \psi^{v\prime}_{p,\ell'} .
$$

(3.7)

We analyse the different terms in the above equation. The term $\tilde{\psi}^v_{\ell} \tilde{\psi}^v_{\ell}$ is:

$$
\sum_{p,k} \varepsilon^V_{\ell p} (-1)^{p+1} \left[ -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right] (-1)^{k+1} \\
- \sum_{p,k} \left[ -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right] (-1)^{k+1}\varepsilon^V_{\ell p} (-1)^{p+1} = 0 .
$$

(3.8)

The term in $\tilde{\psi}^u_{p,\ell} \psi^{v\prime}_{p,\ell}$ is given by:

$$
\sum_{p} \varepsilon^V_{\ell p} (-1)^{p+1} \sum_{\ell \neq \ell_1} \sum_{k} \left( -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right) \psi^{v\prime}_{\ell} \\
- \sum_{p} \left( -\omega(i,k)\varepsilon^V_{\ell k} + \omega(j,k)\varepsilon^V_{\ell k} \right) (-1)^{p+1}\tilde{\psi}^u_{p,\ell} \sum_{\ell \neq \ell_1, k} \varepsilon^V_{\ell k} \psi^{v\prime}_{\ell} .
$$

(3.9)

Setting $j$ to be either the first or the last halfline on the vertex $V$ we see that the last terms in the two lines above cancel eachother. The first two terms hold:

$$
\sum_{p} \sum_{\ell \neq \ell_1} \sum_{k} \tilde{\psi}^u_{\ell} \psi^{v\prime}_{p,\ell} (-1)^{p+1} \left[ -\omega(i,k)\varepsilon^V_{\ell k} + \omega(i,k)\varepsilon^V_{\ell k} \right] .
$$

(3.10)
This can be rewritten in the form:
\[ - \sum_{p} \sum_{\ell \neq \ell_1} \sum_{k} \psi^u_{\ell} \psi^v_{\ell'} (-1)^{p+1} \omega (p, k) \epsilon_{\ell p} \epsilon_{\ell' k} \mu^G, \] (3.11)
where \( \tilde{V}^G \) is a new root vertex in which the vertex \( V \) has been glued to the vertex \( \tilde{V}^G \) and the halflines on the vertex \( V \) have been inserted on the new vertex at the place of the halfline \( i \).

The coupling between \( \psi^u \)'s with \( \psi^v \) is
\[ + \sum_{\ell \neq \ell_1, k} \tilde{\psi}^u_{\ell} \epsilon_{\ell k} \sum_{\ell'} \left( - \sum_{p} \epsilon_{\ell p} \omega (i, p) + \sum_{k'} 4 \epsilon_{\ell' k} \omega (j, k') \right) \tilde{\psi}^v_{\ell'} \]
\[ - \sum_{\ell \neq \ell_1} \left( - \sum_{p} \epsilon_{\ell p} \omega (i, p) + \sum_{k} 4 \epsilon_{\ell k} \omega (j, k) \right) \tilde{\psi}^u_{\ell} \sum_{\ell' \neq \ell_1} \epsilon_{\ell' k} \tilde{\psi}^v_{\ell'} . \] (3.12)

Again, as \( j \) is either the first or the last halfline on the vertex \( V \), the last two terms in the two lines of (3.12) cancel each other. The rest of the terms give exactly the contacts between the \( \psi^u \) and \( \psi^v \) on a new vertex \( \tilde{V}^G \) obtained by gluing \( V \) on \( \tilde{V}^G \) This is the first Filk move on the line \( \ell_1 \) and its associated vertex \( V \). One iterates now this mechanism for the rest of the tree lines of \( G \). Hence we reduce the graph \( G \) to a rosette (see Subsection 2.1).

The quadratic form writes finally as
\[ - \sum_{\ell', p, k} \tilde{\psi}^u_{\ell'} \epsilon_{\ell' k} \sum_{\ell} \tilde{\psi}^v_{\ell} \epsilon_{\ell p} \omega (i, p) \tilde{\psi}^u_{\ell} \omega (k, p) \epsilon_{\ell p} \epsilon_{\ell' k} \mu^G, \]
\[ - \sum_{\ell, \ell', j, k} \tilde{\psi}^u_{\ell} \tilde{\psi}^v_{\ell'} \omega (j, k) \tilde{\psi}^u_{\ell'} . \] (3.13)

The sum concerns only the rosette vertex \( \tilde{V}^G \). Therefore the last line is 0, as by the first Filk move we have exhausted all the tree lines of \( T \).

As \( \ell \) breaks the face separated from the external face by \( \ell_2 \) we have \( k_1 < p < k_2 \). By a direct inspection of the terms above, one obtains the requested result. \( \square \)

4. Factorization of the Feynman amplitudes

Take now \( S \) to be a primitively divergent subgraph. We now prove that the Feynman amplitude \( A_G \) factorizes into two parts, one corresponding to the primitive divergent subgraph \( S \) and the other to the graph \( G/S \) (defined in Section 2.1). This is needed in order to prove that divergencies are cured by Moyal counterterms.\(^3\)

In Section 5 we will prove that only the planar \( (g = 0) \), one broken face \( B = 1, N = 2 \) or \( N = 4 \) external legs subgraphs are primitively divergent. We now deal only with such subgraphs.

\(^3\)It is also the property needed for the definition of a coproduct \( \Delta \) for a Connes–Kreimer Hopf algebra structure (see [8,9,27]). Details of this construction are given elsewhere [37].
4.1. Factorization of the polynomial $HU$

We denote all the leading terms in the first polynomial associated to a graph $S$ by $HU^I_S$. If we rescale all the $t_i$ parameters corresponding to a subgraph $S$ by $\rho^2$, $HU_G$ becomes a polynomial in $\rho$. We denote the terms of minimal degree in $\rho$ in this polynomial by $HU^I_G(\rho)$. It is easy to see that for the subgraph $S$ we have $HU^I_S(\rho) = \rho^2 [L(S) - n(S) + 1] HU^I_S|_{\rho=1}$. We have the following theorem.

**Theorem 4.1.** Under the rescaling

$$t_\alpha \mapsto \rho^2 t_\alpha$$

of the parameter corresponding to a divergent subgraph $S$ of any Feynman graph $G$, the following factorization property holds

$$HU^I_G(\rho) = HU^I_{S,V} HU_{G/S,V}.$$  \hspace{1cm} (4.2)

**Proof.** In the matrix $M$ defined in (2.29) (corresponding to the graph $G$) we can rearrange the lines and columns so that we place the matrix $M_G$ (corresponding to the subgraph $S$) into the upper left corner. We place the line (and resp. the column) associated to the hypermomentum of the root vertex of $S$ to be the last line (and resp. column) of $M$ (without loss of generality we consider that the root of the subgraph $S$ is not the root of $G$). $M$ takes the form

$$
\begin{pmatrix}
E^u s u^s & E^v s v^s & E^v s p^s & E^u s q^s & E^u s q^{G-S} & E^u s q^G & E^u s p^{G-S} & E^u s p^G \\
E^v s u^s & E^v s v^s & E^v s p^s & E^v s q^s & E^v s q^{G-S} & E^v s q^G & E^v s p^{G-S} & E^v s p^G \\
C^p s u^s & C^p s v^s & C^p s p^s & C^p s q^s & C^p s q^{G-S} & C^p s q^G & C^p s p^{G-S} & C^p s p^G \\
E^u s s s & E^u s s v^s & E^u s s p^s & E^u s s q^s & E^u s s q^{G-S} & E^u s s q^G & E^u s s p^{G-S} & E^u s s p^G \\
E^v s s s & E^v s s v^s & E^v s s p^s & E^v s s q^s & E^v s s q^{G-S} & E^v s s q^G & E^v s s p^{G-S} & E^v s s p^G \\
C^p s s s & C^p s s v^s & C^p s s p^s & C^p s s q^s & C^p s s q^{G-S} & C^p s s q^G & C^p s s p^{G-S} & C^p s s p^G \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

(4.3)

where we have denoted by $E^u s u^s$ a coupling between two short variables corresponding to internal lines of $S$ etc..

I. We first write the determinant of the matrix above under the form of a Grassmannian integral

$$\det M = \int d\bar{\psi}^a d\psi^a d\bar{\psi}^s d\psi^s d\bar{\psi}^p d\psi^p e^{-\bar{\psi}M\psi}.$$ \hspace{1cm} (4.4)

Denote a generic line of the subgraph $S$ by $\ell_S$ and a generic line of the subgraph $G - S$ by $\ell_{G-S}$.

We perform a Grassmann change of variables of Jacobian 1. The value of the integral (4.4) does not change under this change of variables. We will prove that the following properties hold for the different terms in the Grassmannian quadratic form

$$
\begin{align*}
E^u s s u^s &= \text{diag}(t_i), \\
E^u s s v^s &= E^{u s s u^s}, \\
E^u s s p^s &= E^{u s s p^s}, \\
E^u s s q^s &= 0, \\
E^{u s s q^{G-S}} &= 0, \\
E^{u s s q^G} &= 0. \\
\end{align*}
$$

(4.5)
The new matrix of the quadratic form $M'$ will now be

$$
\begin{pmatrix}
E_{u}^{V}S u^{S} & E_{u}^{V}S u^{S} & C\bar{\omega}^{p} & E_{u}^{V}S u^{G-S} & E_{u}^{V}S u^{G-S} & E_{u}^{V}S p^{G-S} \\
E_{u}^{V}S u^{S} & E_{u}^{V}S u^{S} & 0 & 0 & 0 \\
C\bar{\omega}^{p} & C\bar{\omega}^{p} & 0 & C\bar{\omega}^{p} & C\bar{\omega}^{p} & C\bar{\omega}^{p} \\
E_{u}^{G-S} u^{S} & 0 & C\bar{\omega}^{p} & E_{u}^{G-S} u^{G-S} & E_{u}^{G-S} u^{G-S} & E_{u}^{G-S} p^{G-S} \\
E_{u}^{G-S} u^{S} & 0 & C\bar{\omega}^{p} & E_{u}^{G-S} u^{G-S} & E_{u}^{G-S} u^{G-S} & E_{u}^{G-S} p^{G-S} \\
C\bar{\omega}^{p} & C\bar{\omega}^{p} & 0 & C\bar{\omega}^{p} & C\bar{\omega}^{p} & C\bar{\omega}^{p}
\end{pmatrix}.
$$

(4.6)

The first part of the proof follows that of Section 3, where we replace

$$
\psi^{u}_{\ell} \rightarrow \sum_{\ell' \in G-S} \psi^{u}_{\ell'}, \quad \bar{\psi}^{u}_{\ell} \rightarrow \sum_{\ell' \in G-S} \bar{\psi}^{u}_{\ell'}.
$$

(4.7)

Thus the appropriate change of variables is now

$$
\bar{\psi}^{pV} = \chi^{pV} + \sum_{\ell' \notin \ell, \bar{k}} \left(-\omega(i, k)\epsilon_{V_{\ell' k}}^{V_{\ell k}} + \omega(j, k)\epsilon_{V_{\ell' k}}^{V_{\ell k}}\right) \psi^{p}_{\ell'}
$$

+ \sum_{\ell' \in G-S \setminus \ell, \bar{k}} \left(-\omega(i, k)\epsilon_{V_{\ell' k}}^{V_{\ell k}} + \omega(j, k)\epsilon_{V_{\ell' k}}^{V_{\ell k}}\right) (-1)^{k+1} \psi^{p}_{\ell'}

$$
\bar{\psi}^{pV} = \chi^{pV} + \sum_{\ell' \notin \ell, \bar{k}} \left(-\omega(i, k)\epsilon_{V_{\ell' k}}^{V_{\ell k}} + \omega(j, k)\epsilon_{V_{\ell' k}}^{V_{\ell k}}\right) \bar{\psi}^{p}_{\ell'}
$$

+ \sum_{\ell' \in G-S \setminus \ell, \bar{k}} \left(-\omega(i, k)\epsilon_{V_{\ell' k}}^{V_{\ell k}} + \omega(j, k)\epsilon_{V_{\ell' k}}^{V_{\ell k}}\right) (-1)^{k+1} \bar{\psi}^{p}_{\ell'}.

(4.8)

We emphasize that this Grassmann change of variables can be viewed as forming appropriate linear combinations of lines and columns. As we only use lines and columns associated to hypermomenta $p_{S}$, this manipulations can not change the value of the determinant in the upper left corner: it will always correspond precisely to the first polynomial of the subgraph $S$.

The relevant terms in the quadratic form are those of (3.4) and (3.5), with the substitutions (4.7). Again, after the change of variables (4.8) the only surviving contacts of $\psi^{p}_{\ell'}$ and $\bar{\psi}^{u}_{\ell}$ are given by (3.6). Finally, the remaining terms are given by (3.7) with the substitutions (4.7).

One needs again to analyse the different terms in this equation. The quadratic term in $\bar{\psi}^{pV} \psi^{p}$ is:

$$
\sum_{\ell, \ell' \in G-S} \bar{\psi}^{p}_{\ell'} \psi^{p}_{\ell} (-1)^{k+p+1} \left(\omega(i, k)\epsilon_{V_{\ell' k}}^{V_{\ell k}} - \omega(j, k)\epsilon_{V_{\ell' k}}^{V_{\ell k}}\right) \epsilon_{V_{\ell' k}}^{V_{\ell k}}
$$

$$
- \omega(i, k)\epsilon_{V_{\ell k}}^{V_{\ell' k}} - \omega(j, k)\epsilon_{V_{\ell k}}^{V_{\ell' k}}.
$$

(4.9)

As $j$ is either the first or the last halfline on the vertex $V$, we see that the last terms in the two lines above cancel. The remaining two terms give the contacts...
amongst \( \tilde{\psi}^u \) and \( \psi^u \) on the rosette new vertex \( \tilde{V}_S' \), obtained by gluing \( \tilde{V}_S \) and \( V \) respecting the ordering.

The contacts between \( \tilde{\psi}^u \) and \( \psi^u \) become

\[
+ \sum_{\ell'} \sum_{p} \tilde{\psi}_{\ell'}^u (-1)^{p+1} \varepsilon_{\ell' p}^V \sum_{\ell'' \neq \ell_1} \sum_{k} \left( -\omega(i, k) \varepsilon_{\ell'' k}^{\tilde{V}_S} + \omega(j, k) \varepsilon_{\ell'' k}^{V} \right) \psi_{\ell'}^u,
\]

\[
- \sum_{\ell''} \sum_{p} \left( -\omega(i, p) \varepsilon_{\ell'' p}^{\tilde{V}_S} + \omega(j, p) \varepsilon_{\ell'' p}^{V} \right) (-1)^{p+1} \tilde{\psi}_{\ell''}^u \sum_{\ell'' \neq \ell_1, k} \varepsilon_{\ell'' k}^{\tilde{V}_S} \psi_{\ell''}^u. \tag{4.10}
\]

Again that the last terms in the two lines above cancel. Rearranging the rest as before we end recover again the terms corresponding to a rosette.

Finally for the \( \tilde{\psi}^v \) and \( \psi^v \) contacts (3.12) goes through.

We iterate the change of variables only for a tree in the subgraph \( S \), hence we reduce the subgraph \( S \) to a rosette (see Subsection 2.1). As the quadratic form reproduced itself, the root vertex \( \tilde{V}_S \) is now a Moyal vertex, with either 2 or 4 external legs.

Let \( r \) be an external half line of the subgraph \( S \). As \( S \) is planar one broken face any line \( l_S = (p,q) \) will either have \( r < p, q \) or \( p,q < r \). Thus

\[
E_{v^G, v^G}^{v^S} = \sum_{l,p,r} \omega(p, r) \varepsilon_{l p}^{\tilde{V}_S} \varepsilon_{l r}^{V} = 0. \tag{4.11}
\]

The reader can check by similar computations that (4.5) holds.

**II.** To obtain in the lower right corner the matrix corresponding to the graph \( G/S \), we just have to add the lines and columns of the hypermomenta corresponding to the vertices of \( S \) to the ones corresponding to the root of \( S \). Furthermore, by performing this operation, the block \( C^{u^S p^G, s^G} \) (which had only one non-trivial column, the column corresponding to the hypermomentum \( p_{\tilde{V}_S} \)) becomes identically 0. Forgetting the primes, the matrix in the quadratic becomes:

\[
\begin{pmatrix}
E^{u^S u^S} & E^{u^S v^S} & C^{u^S p^S} & E^{u^S w^G} & E^{u^S v^G} & C^{u^S p^G} \\
E^{v^S u^S} & l \delta_{u^S, v^S} & C^{u^S p^S} & 0 & 0 & 0 \\
C^{p^S u^S} & 0 & C^{p^S p^G} & C^{p^S v^G} & C^{p^S v^G} & 0 \\
E^{w^G u^S} & 0 & C^{w^G p^S} & E^{w^G u^G}_{/w^G} & E^{w^G v^G}_{/w^G} & C^{w^G v^G}_{/w^G} \\
E^{w^G v^S} & 0 & C^{w^G p^S} & E^{w^G v^G}_{/w^G} & E^{w^G v^G}_{/w^G} & C^{w^G v^G}_{/w^G} \\
C^{w^G w^G} & 0 & 0 & C^{w^G w^G}_{/w^G} & C^{w^G w^G}_{/w^G} & 0
\end{pmatrix}. \tag{4.12}
\]

This is equivalent to the Grassmannian change of variables:

\[
\chi^{p^S} = \chi^{p^S} + \tilde{\psi}_{p^S} v^S, \\
\tilde{\chi}^{p^S} = \tilde{\chi}^{p^S} + \tilde{\psi}_{p^S} v^S. \tag{4.13}
\]

**III.** We finally proceed with the rescaling with \( p^2 \) of all the parameter \( t_\alpha \) corresponding to the divergent subgraph \( S \) (see 4.1). Recall that these parameters are present as \( \frac{1}{t_\alpha} \) on the diagonal of the block \( E^{v^S u^S} \) and as \( t_\alpha \) on the diagonal of the block \( E^{v^S v^S} \).
We factorize \( \frac{1}{\rho} \) on the first \( L(S) \) lines of columns of \( M \) (corresponding to the \( u^S \) variables) and \( \rho \) on the next \( L(S) \) lines and columns (corresponding to the \( v^S \) variables). We also factorize \( \frac{1}{\rho} \) on the \( n(S) - 1 \) lines and columns corresponding to the hypermomenta \( p^S \). This is given by the Grassmann change of variables

\[
\psi'_{u} = \frac{1}{\rho} \psi_{u} \\
\psi'_{v} = \rho \psi_{v} \\
\chi'_{p} = \frac{1}{\rho} \chi_{p}.
\]

(4.14)

In the new variables the matrix of the quadratic form is

\[
\begin{pmatrix}
 c_4 \delta_{uu'} + \rho^2 E_{uu'} & E_{uv} & \rho^2 C_{uuu'p} & \rho E_{uuv} & \rho E_{vuv} & \rho C_{uuvp} \\
 E_{tvu} & t_4 \delta_{uu'} & C_{uu} & 0 & 0 & 0 \\
 \rho^2 C_{uuu'} & 0 & C_{uuu'} & 0 & 0 & 0 \\
 \rho E_{uuu'} & 0 & \rho C_{uuu'} & E_{nS} & E_{nS} & E_{nS} \\
 \rho E_{vuu'} & 0 & \rho C_{vuu'} & E_{nS} & E_{nS} & E_{nS} \\
 \rho C_{uuu'p} & 0 & 0 & C_{uuu'} & C_{uuu'} & 0
\end{pmatrix}.
\]

(4.15)

The determinant of the original matrix is obtained by multiplying the overall factor \( \rho^{-2n(S)+2} \) (coming from the Jacobian of the change of variables) with the determinant of the matrix (4.15). To obtain \( HU_G \), according to (2.30) we must multiply this determinant by a product over all lines of \( t_\ell \).

The determinant upper left corner, corresponding to the subgraph \( S \), multiplied by the appropriate product of \( t_\ell \) as in (2.30) and by the Jacobian factor holds the complete polynomial \( HU_S \). At leading order in \( \rho \) it is \( HU_S^{(\rho)} \).

The determinant of the lower right corner, multiplied by its corresponding product of \( t_\ell \) holds the complete polynomial \( HU_{G/S} \) and no factor \( \rho \).

At the leading order in \( \rho \) the off diagonal blocks become 0. Therefore we have

\[
HU_{G}^{(\rho)}(p) = HU_{S}^{(\rho)}(p) HU_{G/S}.
\]

(4.16)

Let us illustrate all this with the example of the graph of Figure 3, where the primitive divergent subgraph is taken to be the sunshine graph of lines \( \ell_4, \ell_5 \) and \( \ell_6 \). A direct computation showed that, under the rescaling

\[
t_4 \rightarrow \rho^2 t_4, \quad t_5 \rightarrow \rho^2 t_5, \quad t_6 \rightarrow \rho^2 t_6,
\]

(4.17)

the leading terms in \( \rho \) of the polynomial \( HU_G \) factorize as

\[
\rho^4 \left[ (1 + 4s^3) t_4 (t_5 + t_6) + t_5 (t_6 + 8s^2 (2t_5 + t_5 + 2s^2 t_6)) \right] \times (1 + 4s^2) (t_1 + t_2 + t_3 + t_1 t_2 t_3) (1 + t_2 t_3 + t_1 (t_2 + t_3)).
\]

(4.18)
The first line of this formula corresponds to the leading terms under the rescaling with $\rho$ of $HU_S$, while the second line is nothing but the polynomial $HU_{G/S}$ of (2.35).

4.2. The exponential part of the Feynman amplitude

In order to perform the appropriate subtractions we need to check the factorization also at the level of the second polynomial. Throughout this subsection we suppose that $S$ is a completely internal subgraph, that is none of its external points is an external point of $G$. The general case is treated by the same methods with only slight modifications. We have the following lemma.

**Proposition 4.2.** Under the rescaling $t_\alpha \mapsto \rho^2 t_\alpha$ of all the lines of the subgraph $S$ we have:

$$HV_G\left|_{\rho=0}\right. = HV_{G/S}/H_{G/S}. \quad (4.19)$$

**Proof.** The ratio $HV_G/H_{G/S}$ is given by the inverse matrix $Q^{-1}$ due to (2.36) which in turn is given by $M^{-1}$ (see [23] for the exact relation). Thus any property which holds for $M^{-1}$ will also hold for $Q^{-1}$.

We write the matrix elements of the inverse of $M$ with the help of Grassmann variables

$$(M^{-1})_{ij} = \frac{\int d\bar{\psi}\psi\psi\bar{\psi}e^{\bar{\psi}M\psi}}{\int d\bar{\psi}\psi e^{\bar{\psi}M\psi}}. \quad (4.20)$$

As $S$ is a completely internal subgraph we only must analyse the inverse matrix entries

$$(M^{-1})_{G-S,G-S} = \frac{\int d\bar{\psi}\psi\psi\bar{\psi}e^{\bar{\psi}M\psi}}{\int d\bar{\psi}\psi e^{\bar{\psi}M\psi}}. \quad (4.21)$$

the only ones which intervene in the quadratic form due to the matrix $P$ in (2.36). None of the changes of variables of the previous section involve any Grassmann variable associated with the $G-S$ sector. We conclude that

$$(M^{-1})_{G-S,G-S} = (M')^{-1}_{G-S,G-S}, \quad (4.22)$$

with $M'$ in (4.12). After the rescaling with $\rho$, the matrix $M'$ becomes (4.15). At leading order we set $\rho$ to zero so that $M'$ becomes block diagonal. Consequently

$$M^{-1}_{G-S,G-S} = M'^{-1}_{G/S,G/S}. \quad (4.23)$$

□

4.3. The two point function

The results proven above must be refined further for the two point function. The reason is that, as explained in Section 5, the two point functions have two singularities so that one needs also to analyse subleading behaviour.

In the sequel we replace $Q_G$ by $M_G$, $Q_{G/S}$ by $M_{G/S}$, etc., the difference between the $Q$'s and the $M$'s being inessential.
When integrating over the internal variables of $G$ we start by integrating over the variables associated to $S$ first. All the variables $u$, $v$ and $p$ appearing in the sequel belong then to $G/S$. The amplitude of the graph $G$ will then write, after rescaling of the parameters of the subgraph $S$ and having performed the first Filk move

$$A_G = \int [dt d\rho] \int [dudvdp]^{G/S} \rho^{2L(S)} e^{-\left(\frac{u}{p} + p^2 \delta M'\right)} \frac{e^{-\left(2 \delta M'\right)\left(\frac{u}{p} + p^2 \delta M'\right)}}{HU_S^2(\rho)}$$

where $[dt d\rho]$ is a shorthand notation for the measure of integration on the Schwinger parameters, to be developed further in the next section. Here $[dudvdp]^{G/S}$ is the measure of integration for the internal variables of $G/S$, and $M'$ is given in (4.15). We explicitate $\delta M'$ as

$$\delta M' = \begin{pmatrix}
E_{uG-S}^{uG-S} & 0 & C_{uG-S}^{uG-S} \\
E_{vG-S}^{uG-S} & 0 & C_{vG-S}^{uG-S} \\
0 & 0 & 0
\end{pmatrix}^{-1} \begin{pmatrix}
E_{uG-S}^{uG-S} & E_{vG-S}^{vG-S} & 0 \\
t_{uG-S}^{uG-S} & t_{vG-S}^{vG-S} & C_{vG-S}^{vG-S} \\
0 & 0 & 0
\end{pmatrix}.$$

(4.25)

The Taylor development in $\rho$ of the exponential gives

$$\int [dt] \int [dudvdp]^{G/S} e^{-\left(\frac{u}{p} + p^2 \delta M'\right)} \frac{1}{HU_S^2(\rho)}.$$

(4.26)

The first term in the integral over $\rho$ above corresponds to a (quadratic) mass divergence.

The second term (logarithmically divergent) corresponds to the insertion of some operator which we now compute.

The interaction is real. This means that we should symmetrize our amplitudes over complex conjugation of all vertices. For instance, at an even loop one should always symmetrize the left and right tadpoles [10, 11, 20].

Consequently, the inverse matrix in (4.25) is actually a sum over the two possible choices of orientation of vertices. We must also sum over all possible choices of signs for the entries in the contact matrices in (4.25) as a similar symmetrization must be performed for the hypermomenta.

Following [21] one can prove using this argument that if terms like $x\partial$ do not appear in the initial Lagrangean for the complex orientable model, they will not be generated by radiative corrections, which is not proven there.
As $\tilde{\psi} u^s$ couples only to the linear combination $\psi^G_{\ell} + \tilde{\psi} v^G_{\ell}$ in the initial matrix as well as in the change of variables, we have $E u^s = \epsilon^G_{\ell} E \tilde{\psi} v^G_{\ell}$.

Due to the sums over choices of signs, the only non-zero entries in $\delta M'$ are $\delta M'_{u^s u^s}$, $\delta M'_{v^s v^s}$, $\delta M'_{u^s v^s}$, and $\delta M'_{p^s p^s}$.

We denote the two external lines of $S$ by $\ell_1$ and $\ell_2$. A tedious but straightforward computation holds

$$(uvp)\left(\begin{array}{c} u \\ v \\ p \end{array}\right)_{\delta M'} = (A_1 \epsilon^{\psi_{\ell_1}} u_{\ell_1} + v_{\ell_1})^2 + A_2 \epsilon^{\psi_{\ell_2}} u_{\ell_2} + v_{\ell_2})^2 + B_1 p^2_{\ell_1}.$$

(4.27)

The first two terms are an insertion of the operator $\Omega x^2$ whereas the last is the insertion of an operator $-\Delta + x^2$, being of the form of the initial Lagrangean.

Take the graph $G/S$ with the insertion of this operator at $S$, and with the addition of eventual mass subdivergencies (due to the first term). We denote it by an operator $O_S$ acting on the graph $G/S$.

We can then sum up the results of this section in the formula

$$e^{-HV_{G/S}(\rho)} \approx 1 \left[ HU_{G/S}(\rho) \right]^{D/2} (1 + \rho^2 O_S) e^{-HV_{G/S}(\rho)} HU_{G/S}^{D/2}$$

(4.28)

where by $\approx$ we mean the divergent part.

5. Dimensional regularization and renormalization of NCQFT

In this section we proceed to the dimensional regularization and renormalization of NCQFT. We detail the meromorphic structure and give the form of the subtraction operator. Dimensional regularization and meromorphic structure of Feynman amplitudes for this model was also established in [22]. However, for consistency reasons we will give here an independent proof of this results.

However, as the proof of convergence of the renormalized integral for this $\Phi^4_1$ model is identical with that for the commutative $\Phi^4_4$ (up to substituting the commutative subtraction operator with our subtraction operator) we will not detail it here.

5.1. Meromorphic structure of NCQFT

In this subsection we prove the meromorphic structure of a Feynman amplitude $A$. We follow here the approach of [2]. We express the amplitude by (2.17)

$$A_{G, V}(x_c, p_V, D) = \left(\frac{\tilde{\Omega}}{2^{D-1}}\right) \int_0^1 \prod_{t=1}^L dt (1 - t^2)_{D-1} e^{-\frac{H\rho_G V(t)}{H_G, G(\rho)}}. $$

(5.1)

We restrict our analysis to connected non-vacuum graphs. As in the commutative case we extend this expression to the entire complex plane. Take a Hepp sector $\sigma$
defined as
\[ 0 \leq t_1 \leq \cdots \leq t_L, \]  
and perform the change of variables
\[ t_\ell = \prod_{j=1}^{L} x_j^2, \quad \ell = 1, \ldots, L. \]

We denote by \( G_i \) the subgraph composed by the lines \( t_1 \) to \( t_i \). As before, we denote \( L(G_i) = i \) the number of lines of \( G_i \), \( g(G_i) \) its genus, \( F(G_i) \) its number of faces, etc. The amplitude is
\[
A_{G_i} = (\frac{\tilde{\Omega}}{2^{(D-4)/2}})^L \int_0^1 \prod_{i=1}^{L} \left( 1 - \left( \prod_{j=1}^{L} x_j^2 \right) \right)^{2^n - 1} dx_i \\
\prod_{i=1}^{L} x_i^{2L(G_i) - 1} e^{-\frac{NV_G(x^2)}{H_U(x^2)}}.
\]  

(5.3)

In the above equation we factor out in \( H_{G_i} \) the monomial with the smallest degree in each variable \( x_i \)
\[
A_{G_i}(x_e, p_e) = (\frac{\tilde{\Omega}}{2^2})^L \int_0^1 \prod_{i=1}^{L} dx_i \left( 1 - \left( \prod_{j=1}^{L} x_j^2 \right) \right)^{2^n - 1} \\
x_i^{2L(G_i) - 1} e^{-\frac{NV_G(x^2)}{H_U(x^2)}}.
\]  

(5.4)

The last term in the above equation is always bounded by a constant. Divergences can arise only in the region \( x_i \) close to zero (it is known that this theory does not have an infrared problem, even at zero mass).

The integer \( b'(G_i) \) is given by the topology of \( G_i \). It is
\[
b'(G_i) = \begin{cases} 
L(G_i) - [n(G_i) - 1] - 2g(G_i) & \text{if } g(G_i) > 0 \\
L(G_i) - n(G_i) & \text{if } g(G_i) = 0 \text{ and } B(G_i) > 1 \\
L(G_i) - [n(G_i) - 1] & \text{if } g(G_i) = 0 \text{ and } B(G_i) = 1
\end{cases}
\]  

(5.5)

To prove the first and the third line, let \( I = \{1 \ldots L\} \). We will exhibit a \( J \) admissible in \( G \) with the right scaling in the \( x_i \).

Let a tree in \( G_i \), \( T(G_i) \) and complete it to a tree \( T(G) \) in the graph \( G \). Furthermore, let \( 2g(G_i) \) genus lines in \( G_i \). Let \( J \) the complement of this set. It obviously contains a tree in the dual graph of \( G \) being therefore admissible and has the right scaling in \( x_i \).
For the second line one must take \( I = \{1 \ldots L\} - \ell \) and \( J \) pseudo-admissible as in Section 3, choosing again the tree \( T(G) \) contained in the complement of \( J \) to be a subtree in the component \( G_i \).

We see that \( b'(G_i) \) is at most \( L(G_i) - n(G_i) + 1 \) and that the maximum is achieved if and only if \( g(G_i) = 0 \) and \( B(G_i) = 1 \).

The convergence in the UV regime \((x_i \to 0)\) is ensured if

\[
\Re[2L(G_i) - Db'(G_i)] > 0, \quad i = 1 \ldots L.
\]  

(5.7)

As

\[
\Re[2L(G_i) - Db'(G_i)] > \Re \left( 2L(G_i) - D \left( L(G_i) - n(G_i) + 1 \right) \right),
\]

we always have convergence provided

\[
\Re D < 2 \leq \frac{4n(G_i) - N(G_i)}{n(G_i) - N(G_i)/2 + 1} \leq \frac{2L(G_i)}{L(G_i) - n(G_i) + 1},
\]  

(5.9)

where \( N(G_i) \) is the number of external points of \( G_i \). Thus \( A_{G,V}(D) \) is analytic in the strip

\[
\mathcal{D}' = \{ D | 0 < \Re D < 2 \}.
\]  

(5.10)

We extend now \( A \) as a function of \( D \) for \( 2 \leq \Re D \leq 4 \). We claim that if

- \( g(G_i) > 0 \)
- \( g(G_i) = 0 \) and \( B(G_i) > 1 \)
- \( N(G_i) > 4 \),

the strip of analyticity can be immediately extended up to

\[
\mathcal{D}' = \{ D | 0 < \Re D < 4 + \varepsilon_G \}.
\]  

(5.11)

for some small positive number \( \varepsilon_G \) depending on the graph. Indeed, for the first two cases we have \( b'(G_i) \leq L(G_i) - n(G_i) \) so that the integral over \( x_i \) converges for

\[
\Re D < 4 \leq \frac{4n(G_i) - N(G_i)}{n(G_i) - N(G_i)/2} = \frac{2L(G_i)}{L(G_i) - n(G_i) + 1}.
\]  

(5.12)

whereas in the third case, as \( N(G_i) > 4 \) the integral over \( x_i \) converges for

\[
\Re D < 4 \leq \frac{4n(G_i) - N(G_i)}{n(G_i) - N(G_i)/2 + 1} = \frac{2L(G_i)}{L(G_i) - n(G_i) + 1}.
\]  

(5.13)

The only possible divergences in \( A_{G,V}(D) \) are generated by planar two or four external legs subgraphs with a single broken face. They are called primitively divergent subgraphs.

Let \( S \) be such a two-point primitively divergent subgraph and call \( \rho \) its associated Hepp parameter. Using (4.28) its contribution to the amplitude writes:

\[
A_{G,V_G}^p \approx \int_0 \frac{d\rho}{\rho^2} \frac{1}{L^{2L(S)-1-D[L(S)-n(S)+1]} H_{S,V_G}^{\rho} \rho^{1 - \rho^2} \Omega} H_{U_G}^{\rho} \left( \frac{m_G}{H_{U_G}} \right). \]

(5.14)

\( \rho \) is a parameter of the graph and \( \rho \) is the parameter of the subgraph.

5We have used here the topological relation \( 4n(G_i) - N(G_i) = 2L(G_i) \).
where again by \( \approx \) we mean the divergent part. The integral over \( \rho \) is a meromorphic operator in \( D \) with the divergent part given by

\[
h(D) = \frac{r_1}{2L(S) - D[L(S) - n(S)] + 1} \pm \frac{r_2}{2L(S) - D[L(S) - n(S)] + 1} + O_S.
\]

For \( S \) a four-point primitively divergent subgraph, the treatment goes along the same lines and one obtains only the first term in (5.14). We have a pole at \( D = 4 \) if \( S \) is a four point subgraph. If \( S \) is a two point subgraph we have poles at \( D = 4 - 2/n(S) \) and \( D = 4 \). As all the singularities are of this type we conclude that \( \mathcal{A}_G \) is a meromorphic function in the strip

\[
D^\sigma = \{ D | 0 < \Re D < 4 + \varepsilon_G \}.
\]

### 5.2. The subtraction operator

The subtraction operator is similar to the usual one (see [2,3]), with the notable difference that the set of primitively divergent subgraphs are different. We give here a brief overview of its construction. For all functions \( \nu^\rho g(\rho) \) with \( g(0) \neq 0 \), denote \( E(\nu) \) the smallest integer such that \( E(\nu) \geq \Re \nu \). Let

\[
T^q_{\nu} = \sum_{k=0}^{q} \frac{1}{k!} g^{(k)}(0), \quad q \geq 0; \quad T^q_{\nu} = 0, \quad q < 0,
\]

be the usual Taylor operator. We define a generalized Taylor operator of order \( n \) by

\[
\tau^{n}_{\nu} [\rho^\nu g(\rho)] = \rho^\nu T_{\nu}^{n-E(\nu)} [g(\rho)].
\]

To each primitively divergent subgraph we associate a subtraction operator \( \tau^{S-2L(S)}_S \) acting on an integrand like

\[
\tau^{-2L(S)}_S \left( e^{\frac{\nu G/S}{H U_G}} H U_G^{D/2} \right) = \left[ \tau^{-2L(S)}_\rho \left( e^{\frac{\nu G/S}{H U_G}} H U_G^{D/2} \right) \right]_{\rho=1}.
\]

Take the example of a bubble subgraph \( S \). It is primitively divergent, and taking into account the factorization properties we have

\[
\tau^{-4}_S \left( e^{\frac{\nu G/S}{H U_G}} H U_G^{D/2} \right) = \left[ \tau^{-4}_\rho \left( \frac{1}{\rho^{D/2}} \right) \right]_{\rho=1}.
\]

If \( D < 4 \), \( E(D) = -3 \) and if \( D \geq 4 \), \( E(D) = -4 \). Consequently

\[
\tau^{-4}_S = \begin{cases} 0 & \text{if } D < 4, \\ \frac{\nu G/S}{H U_G^{D/2}} \frac{1}{(H U_G^{D/2})^{D/2}} & \text{if } D \geq 4. \end{cases}
\]

As expected the operator subtracts only for \( D \geq 4 \), and it exactly compensates the divergence in the expression (5.14).
We then define the complete subtraction operator as

$$R = 1 + \sum_{F} \prod_{S \in F} \left( -\tau_{S}^{-2L(S)} \right),$$  \hspace{1cm} (5.21)

where the sum runs over all forests of primitively divergent subgraphs.

From this point onward the classical proofs of (see [2,3]) go through. Theorems 1, 2 and 3 of [3] so that we have the theorem

**Theorem 5.1.** The renormalized amplitude

$$A_{G}^{r} = RA_{G},$$  \hspace{1cm} (5.22)

is an analytic function of $D$ in the strip:

$$D^{\sigma} = \{ D \mid 0 < \Re D < 4 + \varepsilon_{G} \},$$  \hspace{1cm} (5.23)

for some small positive number $\varepsilon_{G}$.

### 6. Conclusion and perspectives – towards a non-commutative Standard Model?

We have presented in this paper the dimensional regularization and dimensional renormalization for the vulcanized $\Phi_{4}^{4}$ model. The factorization results we have proven are the starting point for the implementation of a Hopf algebra structure for NCQFT [37].

The implementation of the dimensional renormalization program for covariant NCQFT (e.g., non-commutative Gross–Neveu or the Langmann–Szabo–Zarembo model [29], see Section 1) should follow the layout presented here.

One should try to extend the techniques presented here to non-commutative L-S dual gauge theories. Such models have recently been proposed, [14,16] but the reader should be aware that no proof of renormalizability of this models yet exists.

All the results mentioned here are obtained on a particular choice of non-commutative geometry, the Moyal space. One should try to extend this results to the NCQFT’s on more involved geometries, like for example the non-commutative tori.

As mentioned in the introduction, NCQFT is a strong candidate for new physics beyond the Standard Model. One could already study possible phenomenological implications for Higgs physics of such non commutative renormalizable $\Phi_{4}^{4}$ models. The absence of Landau ghost makes this theory better behaved than its commutative counterpart. Also the Langmann–Szabo symmetry responsible for supressing the ghost could play a role similar to supersymmetry in taming UV divergencies.
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