On the Dependence of Linear Coding Rates on the Characteristic of the Finite Field

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Abstract

It is known that for any finite/co-finite set of primes there exists a network which has a rate $1$ solution if and only if the characteristic of the finite field belongs to the given set. We generalize this result to show that for any positive rational number $k/n$, and for any given finite/co-finite set of primes, there exists a network which has a rate $k/n$ fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set. For this purpose we construct two networks: $\mathcal{N}_1$ and $\mathcal{N}_2$; the network $\mathcal{N}_1$ has a $k/n$ fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given finite set of primes, and the network $\mathcal{N}_2$ has a $k/n$ fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given co-finite set of primes.

Recently, a method has been introduced where characteristic-dependent linear rank inequalities are produced from networks whose linear coding capacity depends on the characteristic of the finite field. By employing this method on the networks $\mathcal{N}_1$ and $\mathcal{N}_2$, we construct two classes of characteristic-dependent linear rank inequalities. For any given set of primes, the first class contains an inequality which holds if the characteristic of the finite field does not belong to the given set of primes but may not hold otherwise; the second class contains an inequality which holds if the characteristic of the finite field belongs to the given set of primes but may not hold otherwise. We then use these inequalities to obtain an upper-bound on the linear coding capacity of $\mathcal{N}_1$ and $\mathcal{N}_2$.

I. Introduction

In the year 2000, Ahlswede et al. [1] showed that the min-cut bound on the capacity of multicast networks can be achieved by allowing the nodes of the network to compute functions of the incoming symbols. It has been later shown that a restricted version of network coding, called the linear network coding is sufficient to achieve the capacity of multicast networks [2].

In linear network coding, the source alphabet is a ring or a finite field, and all symbols outgoing from a node is a linear function of the symbols the node receive. Li et al. [2] showed that such functions always exist (for multicast networks) if the underlying finite field is sufficiently large. Moreover, there are efficient algorithms to design these linear functions [3], [4].

Though, recently it has been shown that a multicast network being linearly solvable over a sufficiently large finite field does not necessarily guarantee solvability over every larger field [5].

For non-multicast networks though, linear network coding may not always achieve the capacity of the network [7]. A network was presented in [7] where linear coding capacity is strictly less than the coding capacity. It has been shown that the linear coding capacity of a network cannot be improved even if the source alphabet is a ring instead of a field [21]. Reference [21] also shows that, over finite fields, linear coding capacity of a network depends only on the characteristic of the finite field.

In a network code a block of symbols (say $k$ symbols) is considered at every source, and each edge forwards a block of symbols (say $n$ symbols) to its outgoing edges where these symbols are functions of incoming symbols to the nodes. The ratio $k/n$ is called the rate of the network code. In this paper we consider two specific issues related to linear coding rates. The first problem is related to the dependency of the linear coding rate on the characteristic of the finite field in a linear network coding problem. The second problem deals with producing linear rank inequalities that bound the linear coding rates of a network. In the rest of this section we discuss prior works related to these issues and present our contributions. We end this section detailing the organization of the rest of the paper.

• Dependency on the characteristic of the finite field

In case of multicast networks, the characteristic of the finite field does not play an important role in the sense that there does not exist a multicast network which has a scalar/vector linear solution if and only if the characteristic of the finite field belongs to a certain set of values. However, this is not true for non-multicast networks. In [7], Dougherty et al. showed a network known as the Fano network which has a rate 1 linear solution over any finite field of even characteristic, but over finite fields of odd characteristics, no rate more than $4/5$ is achievable using linear network coding. References [8] and [7] show another network known as the non-Fano network which has a rate 1 linear solution over finite fields of odd characteristics, but over even characteristics no rate more than $5/6$ is achievable using linear network coding (5/6 upper-bound has been shown in [9]). Furthermore, it has been shown in [10] that given any system of polynomial equations over integers, there exists a network which has a scalar linear network coding solution over a finite field if and only if the system of polynomial equations has a root in the same finite field. This showed that for any finite/co-finite set of primes, there exists a network which has a scalar linear solution if and only if the characteristic of the finite field belongs to the given set of primes. Afterwards, Rai et al. showed in [11] that given finite/co-finite set of primes there exists a network which has a vector linear solution if and only if the characteristic of the finite field belong to the given set of primes.
• **Characteristic-dependent linear rank inequalities**

Determining the coding capacity (or even linear coding capacity) of a general network is considered to be a very difficult problem. Although the capacity/linear capacity computation of various small networks have been presented in the literature. However, in such computations ad-hoc methods have been used. Harvey et al. [14] presented a method to obtain an upper-bound on the coding capacity by combining Shannon inequalities with topological properties of the network (informational dominance and independence of source symbols). In some cases, the bound obtained from this method may be improved by additionally incorporating non-Shannon information inequalities. A network named as the Vámos network is a good example to see how these inequalities come together. Vámos network was first considered in [8], where by applying non-Shannon inequalities it has been shown that its coding capacity is upper-bounded by 10/11. This bound has been further improved to 19/21 by applying other non-Shannon inequalities in [15].

To determine an upper-bound on the linear coding capacity, in addition to the Shannon and non-Shannon information inequalities, linear rank inequalities may also be applied. Linear rank inequalities are inequalities that are obeyed by ranks (dimensions) of any collection of vector subspaces of a finite dimensional vector space. For example, if $A$ and $B$ are vector subspaces of $V$, then, $\dim(A) + \dim(B) \geq \dim(A + B) + \dim(A \cap B)$ is a linear rank inequality. On the contrary, information inequalities are the Shannon and the non-Shannon inequalities which are obeyed by random variables. When applying an information inequality to a network, the messages are taken as random variables distributed over the source alphabet. When applying a linear rank inequality, the messages are taken as vector subspaces of a finite dimensional vector space over a finite field. For any collection of vector subspaces of a finite dimensional vector space, in p. 452 of [16] a way is shown to construct a corresponding set of random variables such that the dimension of any collection of the vector subspaces is equal to the joint entropy of the corresponding random variables (upto a scale factor). As a result, all subspaces of a vector space also obey the information inequalities (assuming the underlying conversion from vector subspaces to random variables). This implies that all information inequalities are also linear rank inequalities. However, the opposite is not true, i.e. not all linear rank inequalities are information inequalities (Theorem 4 of [16]). This implies that the best upper-bound obtained using information inequalities may not be a tight upper-bound on the linear coding capacity.

Hammer et al. showed that for upto three variables, there exists no linear rank inequality which is not an information inequality (Theorem 3 of [16]). They also showed that for four variables, the only linear rank inequality that is not an information inequality is the Ingleton inequality upto permutations of the variables (Theorem 5 of [16]). A list of twenty four new linear rank inequalities on five variables which are not information inequalities has been shown in [17]. Reference [18] shows that even an incomplete list of six variable linear rank inequalities crosses one billion. For seven or more variables, it has been shown in [9], [19] and [20] that there exist linear rank inequalities that hold if the characteristic of the field is among a certain set of values, but may not hold otherwise (this is expected as linear coding capacity has been shown to be dependent on the characteristic of the finite field). Such an inequality is called as a characteristic-dependent linear rank inequality.

First, Blasiak et al. showed two such seven variable inequalities: one holds over finite fields of even characteristic, and the other holds over finite fields of odd characteristic. Thereafter, Dougherty et al. showed two more seven variable characteristic-dependent linear rank inequalities in [9]. Subsequently, two new eight variable inequalities has been presented in [20]. Application (finding upper-bounds on the linear coding capacity of networks) of the inequalities shown in [9] and [20] has been also shown in the respective papers.

For producing these inequalities, the authors of [9] and [20] developed a novel method where these inequalities were yielded from the very networks they intended to find the linear coding capacity of. Hereafter, we will refer this method as the DFZ method. In reference [9] two linear rank inequalities have been obtained: one holds over all finite fields of odd characteristic but may not hold otherwise (produced from the Fano network); and another holds over all finite fields of even characteristic but may not hold otherwise (produced from the non-Fano network). In reference [20], first an inequality that holds over all finite fields of characteristic not equal to 3 but may not hold otherwise was produced from the T8 network; and then another inequality that holds over all finite fields of characteristic equal to 3 but may not hold otherwise was produced from the non-T8 network.

### A. Contributions of this paper

- **First contribution of the paper**
  
  In the works of [10] and [11], the dependency on the characteristic of the field is shown only for either scalar linear network coding or for vector linear network coding. In this paper we show that for any positive rational number $\frac{k}{n}$ and for any given finite/co-finite set of prime numbers, there exists a network which has a rate $\frac{k}{n}$ fractional linear network code solution if and only of the characteristic of the finite belongs to the given finite/co-finite set of primes.

- **Second contribution of the paper**
  
  In the second result of this paper, we construct two classes of characteristic-dependent linear rank inequalities. Given a set of primes, the first class contains an inequality that holds if the characteristic of the finite field does not belong to
the given set; and the second class contains an inequality that holds if the characteristic belongs to the given set. We also show that the inequalities in the first class may not hold if the characteristic belongs to the given set of primes; and the inequalities in the second class may not hold if the characteristic does not belong to the given set of primes. This contribution can be seen as a generalization of the works in [9] and [20].

B. Organization of the paper

In Section II we reproduce the standard definitions of fractional linear network coding, vector linear network and scalar linear network coding. In Section III-A for any positive rational number \( \frac{k}{n} \), and for any finite set of primes, we present a network \( N_1 \) which has a rate \( \frac{k}{n} \) fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set of primes. For the ease of readability, a part of this proof is shifted to Appendix A. In Fig. 1 we show a network \( N_1' \) which we use to construct \( N_1' \). In Section III-B for any positive rational number \( \frac{k}{n} \), and for any finite set of primes, we present a network \( N_2 \) which has a rate \( \frac{k}{n} \) fractional linear network coding solution if and only if the characteristic of the finite field does not belong to the given set of primes. As earlier, a part of this proof is deferred until Appendix B. We construct the network \( N_2' \) by using another network \( N_2' \) shown in Fig. 2.

In Theorem 7 Section IV using the network \( N_1' \), we also construct a characteristic-dependent linear rank inequality that holds if the characteristic of the finite field does not belong to the given set of primes but may not hold otherwise. The proof of this theorem is presented in Appendix A-B. Then, using \( N_2' \), we construct a characteristic-dependent linear rank inequality that holds if the characteristic of the finite field belongs to the given set of primes but may not hold otherwise. This inequality is presented in Theorem 8 of Section V and proved in Appendix B-B. Usage of these inequalities in computing upper-bounds on the linear coding capacity of \( N_1' \) and \( N_2' \) are also shown in Section V.

II. PRELIMINARIES

A network is represented by a graph \( G(V, E) \). The set \( V \) is partitioned into three disjoint sets: the set of sources \( S \), the set of terminals \( T \), and the set of intermediate nodes \( V' \). Without loss of generality, the sources are assumed to have no incoming edge and the terminals are assumed to have no outgoing edge. Each source generates an i.i.d random process uniformly distributed over an alphabet \( A \). The source process at any source is independent of all source processes generated at other sources. Each terminal demands the information generated by a subset of the sources. An edge \( e \) originating from node \( u \) and ending at node \( v \) is denoted by \((u, v)\); where \( u \) is denoted by \( tail(e) \), and \( v \) is denoted by \( head(e) \). For a node \( v \in V \), the set of edges \( e \) for which \( head(e) = v \) is denoted by \( In(v) \). The information carried by an edge \( e \) is denoted by \( Y_e \). Without loss of generality it is assumed that all the edges in the network are unit capacity edges (meaning, in one usage of an edge it carries one symbol \( q \times n \)).

A network having a rate \( \frac{k}{n} \) fractional linear network coding solution then the network is said to be scalar linearly solvable. A network having a rate \( \frac{k}{n} \) fractional linear network coding solution if and only if the characteristic of the finite field belongs to a given finite/co-finite set of primes.

A. Network having a rate \( \frac{k}{n} \) fractional linear network coding solution iff the characteristic of the finite field belongs to a given finite/co-finite set of primes.

First we show that for any positive non-zero rational number \( \frac{k}{n} \), and for any given finite set of primes, there exists a network which has a rate \( \frac{k}{n} \) fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set. Our proof is constructive. Consider the network \( N_1' \) presented in Fig. 1. The network has \( (q + 1) \) sets of sources: \( S_a = \{a_1, a_2, \ldots, a_r\} \), \( S_{bi} = \{b_{i1}, b_{i2}, \ldots, b_{in}\} \) for \( 1 \leq i \leq (q - 1) \), and \( S_c = \{c_1, c_2, \ldots, c_r\} \). The source \( s_i \in S_a \) generates the message \( a_i \). For \( 1 \leq i \leq (q - 1) \) and \( 1 \leq j \leq n \) the source \( s_j \in S_{bi} \) generates the message \( b_{ij} \). And the source \( s_j \in S_c \) generates the message \( c_i \). In the figure, the source nodes are indicated by the message it generates. There are \( 2q \) sets of terminals: \( T_e, T_{a_i}, T_{b_{ij}} \) for \( 1 \leq i \leq (q - 1) \), and \( T_{c_i} \) for \( 1 \leq i \leq (q - 1) \). Each individual terminal is indicated by the source message it demands.

List of edges emanating from a source node:
Fig. 1. Network $\mathcal{N}_1$ which has a rate $\frac{1}{m}$ fractional linear network coding solution if and only if the characteristic of the finite field divides $q$.

1) $(s, u_1)$ for $\forall s \in \{S_0, S_h, S_{b_1}, \ldots, S_{b_{n-1}}\}$.
2) $(s, u_2)$ for $\forall s \in \{S_{b_1}, S_{b_2}, \ldots, S_{b_{n-1}}, S_r\}$.
3) $(u_i, u_{i1})$ for $1 \leq i \leq n$.
4) $(e_i, u_6)$ for $1 \leq i \leq n$.
5) $(b_{ij}, tail(e_k))$ for $1 \leq i, k \leq (q - 1), i \neq k, 1 \leq j \leq n$.
6) $(b_{ij}, v_k)$ for $1 \leq i, k \leq (q - 1), i \neq k, 1 \leq j \leq n$.
7) $(b_{ij}, w_i)$ for $1 \leq i \leq (q - 1), 1 \leq j \leq n$.

List the edges which originates at an intermediate node and ends at a intermediate node:

1) $(u_i, u_{i+2})$ for $1 \leq i \leq 7, i \neq 4$.
2) $(u_i, u_{i+1})$ for $i = 4, 8, 9, 11, 13$.
3) $(u_3, u_6), (u_7, u_{11})$, and $(u_8, u_{13})$.
4) $e_i$ for $1 \leq i \leq (q - 1)$.
5) $(u_4, tail(e_i))$ for $1 \leq i \leq (q - 1)$.
6) $(head(e_i), u_{13})$ and $(head(e_i), w_i)$ for $1 \leq i \leq (q - 1)$.
7) \((u_{10}, v_i)\) and \((v_i, v'_i)\) for \(1 \leq i \leq (q - 1)\)
8) \((w_i, w'_i)\) for \(1 \leq i \leq (q - 1)\)

For any terminal \(t_i \in T_c\) there exists an edge \((u_{12}, t_i)\) and \(t_i\) demands the message \(c_i\). For any terminal \(t_j \in T_b\) for \(1 \leq i \leq (q - 1), 1 \leq j \leq n\), there exists an edge \((v'_i, t_j)\) where the terminal \(t_j\) demands the message \(b_{ij}\). For any terminal \(t_i \in T_a\) there exists an edge \((u_{14}, t_i)\) and \(t_i\) demands the message \(a_i\). For \(1 \leq i \leq (q - 1)\), a terminal \(t_j \in T_{a_i}\) for \(1 \leq j \leq n\) is connected from the node \(w'_i\) by the edge \((w'_i, t_j)\) and \(t_j\) demands the message \(c_j\). The local coding matrices are shown alongside the edges.

**Lemma 1.** The network in Fig. 1 has a rate \(\frac{1}{n}\) fractional linear network coding solution if and only if the characteristic of the finite field divides \(q\).

The proof of this lemma is shown in Appendix A.

**Theorem 2.** For any non-zero positive rational number \(\frac{k}{n}\) and for any finite set of prime numbers \(\{p_1, p_2, \ldots, p_l\}\), there exists a network which has a rate \(\frac{k}{n}\) fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set of primes.

**Proof:** Let us consider the union of \(k\) copies of the network \(N_1^q\) shown in Fig. 1 each for \(q = p_1 \times p_2 \times \cdots \times p_l\). Denote the \(i\)th copy as \(N_i^q\). Note that each source and each terminal has \(k\) copies in the union. Join all copies of any source or terminal into a single source or terminal respectively. Name this new network as \(N_1\). We show below that \(N_1\) has a rate \(\frac{k}{n}\) fractional linear network coding solution if and only if the characteristic of the finite field belong to the set \(\{p_1, p_2, \ldots, p_l\}\). Before we proceed further, consider the following property of \(N_1^q\) and \(N_1^q\).

**Lemma 3.** If \(N_1^q\) has a \((dk, dn)\) fractional linear network coding solution for any non-zero positive integer \(d\), then \(N_1^q\) has a \((dk, dk)\) fractional linear network coding solution.

**Proof:** This is true since the information that can be sent using the network \(N_1^q\) in \(x\) times, can be sent using the network \(N_1^q\) in \(k\) times. This is because \(N_1^q\) has \(k\) copies of \(N_1\).

First consider the only if part. Say \(N_1^q\) has a rate \(\frac{k}{n}\) fractional linear network coding solution even if the characteristic does not belong to the set \(\{p_1, p_2, \ldots, p_l\}\). Then from Lemma 3 the network \(N_1^q\) has a rate \(\frac{1}{n}\) fractional linear network coding solution even if the characteristic does not belong to the given set of primes. However, as shown in Lemma 1 \(N_1^q\) has a rate \(\frac{1}{n}\) fractional linear network coding solution if and only if the characteristic of the finite field divides \(q\). But, as \(q = p_1 \times p_2 \times \cdots \times p_l\), the characteristic divides \(q\) if and only if the characteristic is one of the primes in the set. Hence this is a contradiction.

Now consider the if part. Since \(N_1^q\) for \(1 \leq i \leq k\) has a \((1, n)\) fractional linear network coding solution, a \((k, n)\) fractional linear network coding solution for \(N_1^q\) can be constructed by keeping the same local coding matrices in all of the copies and sending the \(i\)th component of each source through \(N_1^q\).

**B. Network having \(\frac{k}{n}\) solution iff the characteristic of the finite field belongs to a given co-finite set of primes.**

The outline of the contents in this sub-section is similar to that of the last sub-section. Consider the network \(N_1^q\) shown in Fig. 2. The sources are partitioned into \((q + 1)\) sets: \(S_a = \{a_1, a_2, \ldots, a_n\}\) and \(S_b = \{b_1, b_2, \ldots, b_n\}\) for \(1 \leq i \leq q\). A source node and the message generated by the node is indicated by the same notation. The set of terminals are partitioned into \((q + 2)\) disjoint sets: \(T_{a_1}, T_{a_2}\) and \(T_b\) for \(1 \leq i \leq q\) where each set has \(n\) terminals. Each individual terminal is indicated by the source message it demands. We have the following edges in the network.

1) \(e_a, e_b, e'_a\) and \(e'_b\)
2) \(e_i, e'_i\) for \(1 \leq i \leq q\)
3) \((s, \text{tail}(e_a))\) for \(\forall s \in S_a \cup \{\cup_{j=1}^{q} S_{b_i}\}\)
4) \((s, \text{tail}(e_b))\) for \(1 \leq i \leq q\) and \(\forall s \in S_a \cup \{\cup_{j=1, j \neq i}^{q} S_{b_i}\}\)
5) \((s, \text{tail}(e'_a))\) for \(\forall s \in \cup_{j=1}^{q} S_{b_i}\)
6) \((\text{head}(e_a), \text{tail}(e'_a))\) and \((\text{head}(e_b), \text{tail}(e'_a))\)
7) \((\text{head}(e_a), \text{tail}(e'_b))\)
8) \((\text{head}(e_b), \text{tail}(e'_b))\) for \(1 \leq i \leq q\)
9) \((\text{head}(e_a), \text{tail}(e'_b))\) for \(1 \leq i \leq q\)
10) \((\text{head}(e_b), \text{tail}(e'_b))\) for \(1 \leq i \leq q\)

From each of the nodes \(\text{head}(e'_a), \text{head}(e'_b)\) for \(1 \leq i \leq q\), and \(\text{head}(e'_b)\), \(n\) outgoing edges emanate, and the \(\text{head}\) node of all such edges is a terminal. The set of \(n\) terminals which have a path from node \(\text{head}(e'_a)\) are denoted by \(T_{a_1}\). Similarly, the set of \(n\) terminals which have a path from node \(\text{head}(e'_b)\) are denoted by \(T_{a_2}\). And the \(n\) terminals in the set \(T_b\) for \(1 \leq i \leq q\) are connected from the node \(\text{head}(e'_b)\) by an edge.

**Lemma 4.** The network shown in Fig. 2 has a rate \(\frac{1}{n}\) fractional linear network coding solution if and only if the characteristic of the finite field does not divide \(q\).
Fig. 2. A network \( \mathcal{N}_2 \) which has a rate \( 1/n \) fractional linear network coding solution if and only if the characteristic of the finite field does not divide \( q \).

The proof of this lemma is shown in Appendix \[B\].

**Theorem 5.** For any non-zero positive rational number \( \frac{k}{n} \) and for any finite set of prime numbers \( \{p_1, p_2, \ldots, p_l\} \), there exists a network which has a rate \( \frac{k}{n} \) fractional linear network coding solution if and only if the characteristic of the finite field does not belong to the given set of primes.

**Proof:** Let \( q \) be equal to \( p_1, p_2, \ldots, p_l \) in \( \mathcal{N}_2 \). Let us construct \( \mathcal{N}_2 \) by joining \( k \) copies of \( \mathcal{N}_2' \) at the corresponding sources and the terminals, in a similar way \( \mathcal{N}_1 \) was constructed from \( \mathcal{N}_1' \). It can be also seen that Lemma [3] holds true when \( \mathcal{N}_1 \) and \( \mathcal{N}_1' \) are replaced by \( \mathcal{N}_2 \) and \( \mathcal{N}_2' \) respectively. So if \( \mathcal{N}_2 \) has a rate \( \frac{k}{n} \) fractional linear network coding solution then \( \mathcal{N}_2' \) has a rate \( \frac{1}{n} \) fractional linear network coding solution.

Now say \( \mathcal{N}_2 \) has a rate \( \frac{k}{n} \) fractional linear network coding solution even if the characteristic of the finite belongs to the set \( \{p_1, p_2, \ldots, p_l\} \). Then, as \( q = p_1, p_2, \ldots, p_l \), the characteristic of the finite field divides \( q \). Then, \( \mathcal{N}_2' \) has a rate \( \frac{k}{ln} = \frac{1}{n} \) fractional linear network coding solution over a finite field even if the characteristic divides \( q \). However, this is in contradiction.
Fig. 3. Gadget which attaches to the two terminals (denoted by \( n_1 \) and \( n_2 \)) demanding the same message (denoted by \( b \)) of any arbitrary network (indicated by the dotted lines). Nodes \( x_1, s_1, \ldots, s_{n-1} \) are source nodes and source \( s_1 \) generates the messages \( y_i \) for \( 1 \leq i \leq (n-1) \); \( x_1 \) generates the message \( z \). The nodes \( t_1, \ldots, t_{n-1}, x_4, x_5 \) are terminals and \( t_i \) demands \( y_i \) for \( 1 \leq i \leq (n-1) \); \( x_4 \) demands \( b \), and \( x_5 \) demands \( z \). Nodes \( x_2 \) and \( x_3 \) are the intermediate nodes.

If however, the characteristic does not belong to the given set of primes, then, since there are \( k \) copies of \( \mathcal{N}'_2 \) in \( \mathcal{N}_2 \), and each copy has a \((1,n)\) fractional linear network coding solution, a \((k,n)\) fractional linear network coding solution can easily be constructed for \( \mathcal{N}_2 \).

IV. A MULTIPLE-UNICAST NETWORK HAVING A RATE \( \frac{k}{n} \) FRACTIONAL LINEAR NETWORK CODING SOLUTION IFF THE CHARACTERISTIC BELONGS TO A GIVEN FINITE/CO-FINITE SET OF PRIMES

In this section we show that for any non-zero positive rational number \( \frac{k}{n} \) and for any finite/co-finite set of primes, there exists a multiple-unicast network which has a rate \( \frac{k}{n} \) fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set. To prove this result, we first show that for each of the networks \( \mathcal{N}'_1 \) and \( \mathcal{N}'_2 \) presented in Section III, there exists a multiple-unicast network which has a \((1,n)\) fractional linear network coding solution if and only if the corresponding network \( \mathcal{N}_1 \) or \( \mathcal{N}_2 \) has a \((1,n)\) fractional linear network coding solution.

In a multiple-unicast network, by definition, each source process is generated at only one source node and is demanded by only one terminal. Additionally, each source node generates only one source process, and each terminal demands only one source process. In both the networks \( \mathcal{N}'_1 \) and \( \mathcal{N}'_2 \) there exists no source processes which is generated by more than one source node, and no source node generates more than one source process. Moreover, there does not exist any terminal which demands more than one source process. However, there exists more than one terminal which demands the same source process. This is fixed in the following way.

In [12] it has been shown that for any network there exists a solvably equivalent multiple-unicast network. To resolve the case of more than one terminals demanding the same source message, the authors considered two such terminals at a time and added a gadget to the two terminals. The same procedure is followed here, only the gadget has been modified. This modified gadget is shown in Fig. 3. It is assumed that the nodes \( n_1 \) and \( n_2 \) both demanded the same message \( b \) in the original network (network before attaching the gadget). After adding the gadget, the modified network has \( n \) more source nodes \( x_1, s_1, \ldots, s_{n-1} \), and \( n + 1 \) new terminal nodes \( x_4, x_5, t_1, \ldots, t_{n-1} \). Nodes \( n_1 \) and \( n_2 \) are intermediate nodes in the modified construction. This process has to be repeated iteratively for every two terminals in the original network that demand the same source process. In the same way as shown in Theorem II.1 of [12], it can be shown that after the completion of this process, the resulting network has a \((1,n)\) fractional linear network coding solution if and only if the original network has a \((1,n)\) fractional linear network coding solution.

Hence, as shown above, corresponding to each of the networks \( \mathcal{N}'_1 \) and \( \mathcal{N}'_2 \), there exist multiple-unicast networks \( \mathcal{N}_1^m \) and \( \mathcal{N}_2^m \) which have a \((1,n)\) fractional linear network coding solution if and only if \( \mathcal{N}_1^m \) and \( \mathcal{N}_2^m \) have a \((1,n)\) fractional linear network coding solution respectively. Now by connecting \( k \) copies of \( \mathcal{N}_1^m \) and \( \mathcal{N}_2^m \) in the same way as \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) was constructed from \( \mathcal{N}_1' \) and \( \mathcal{N}_2' \) respectively, the following theorem can be proved in a similar way to Theorem 2 and Theorem 5.

**Theorem 6.** For any non-zero positive rational number \( \frac{k}{n} \) and for any finite/co-finite set of prime numbers there exists a multiple-unicast network which has a rate \( \frac{k}{n} \) fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set of primes.
V. Characteristic-dependent linear rank inequality

In this section, for any finite or co-finite set of primes, we present a characteristic-dependent linear rank inequality that holds if the characteristic of the finite field belongs to the given set, but may not hold otherwise. First we introduce some notations. To denote the dimension of a finite dimensional vector space \( V \) the notations \( \dim(V) \) and \( H(V) \) are used interchangeably. \( H(U, V) \) denotes \( \dim(U + V) \). \( H(U | V) \) denotes \( \dim(U + V) - \dim(V) \).

Theorem 7. For any given set of primes \( \{p_1, p_2, \ldots, p_l\} \), let \( A, B_1, B_2, \ldots, B_{q-1}, C, U, W, X, Y, Z, V_1, V_2, \ldots, V_{q-1} \) for \( q = p_1 \times p_2 \times \cdots \times p_l \) be vector subspaces of a finite dimensional vector space \( V \). Then the following linear rank inequality holds if \( V \) is a vector space over a finite field whose characteristic does not belong to \( \{p_1, p_2, \ldots, p_l\} \), but may not hold otherwise:

\[
(2q - 1)H(A) + (2q - 2)H(C) + \sum_{i=1}^{q-1} 2H(B_i) \leq (q - 1)(H(U) + H(Y) + H(W)) + 2H(X) + \sum_{i=1}^{q-1} H(V_i)
\]

\[
+ (7q - 6)H(U | A, B_1, \ldots, B_{q-1}) + (6q - 5)H(Y | B_1, \ldots, B_{q-1}, C) + \sum_{i=1}^{q-1} (2q)H(V_i | Y, B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_{q-1})
\]

\[
+ (3q - 3)H(W | U, Y) + (4q - 3)H(X | U, C) + (2q - 2)H(Z | W, X) + (2q - 1)H(A | X, V_1, \ldots, V_{q-1})
\]

\[
+ (q - 1)H(C | A, W) + \sum_{i=1}^{q-1} 2H(B_i | Z, B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_{q-1}) + \sum_{i=1}^{q-1} H(C | V_i, B_i)
\]

\[
+ (5q - 4)(H(A) - H(A, B_1, B_2, \ldots, B_{q-1}, C)) + (6q - 5)(\sum_{i=1}^{q-1} H(B_i) + H(C)) - (q - 1)H(B_1, \ldots, B_{q-1}, C)
\]

(1)

The proof of this inequality can be found in Appendix [A-B]. Here we show that this inequality may not hold if \( q = 0 \) over the finite field (note \( q = 0 \) when the characteristic belong to the given set of primes). Let \( V \) be the vector space \( V(q + 1, \mathbb{F}_{p^n}) \) where \( p \in \{p_1, p_2, \ldots, p_l\} \) and \( \alpha \) is some positive integer. Let \( u_i \) be the 1 dimensional vector space spanned by the \( i \)th element, \( i \) and all other elements are zero. Now, consider the following vector subspaces of \( V \). (We construct these subspaces from the fact that the network \( N_1^1 \) has a rate 1 linear solution when \( q = 0 \) over the finite field and \( n = 1 \).)

\[
A = u_1 \quad \text{for } 1 \leq i \leq q - 1 : \quad B_i = u_{i+1} \quad C = u_{q+1} \quad U = \sum_{i=1}^{q} u_i \quad Y = \sum_{i=2}^{q+1} u_i
\]

\[
W = u_1 - u_{q+1} \quad X = \sum_{i=1}^{q} u_i - u_{q+1} \quad \text{for } 1 \leq i \leq q - 1 : \quad V_i = u_{i+1} + u_{q+1} \quad Z = \sum_{i=2}^{q} u_i
\]

Now note \( V_i = Y - \sum_{j=1, j \neq i}^{q} u_j \), \( W = U - Y \), \( X = U - u_{q+1} \), \( Z = X - W \), \( A = X - \sum_{i=1}^{q-1} V_i \), \( C = A - W \), \( B_i = Z - \sum_{i=1, j \neq i}^{q-1} B_j \), \( C = Y_i - B_i \). Hence all the conditional terms in equation (1) becomes zero; and the inequality returns \( (6q - 5) \leq (6q - 6) \), or, \( 6, \leq 5 \). Hence the inequality in equation (1) is not valid over such a finite field.

It can be easily seen that, when inequality (1) is applied to \( N_1^1 \), it results an upper-bound equal to \( \frac{(6q - 6)}{(6q - 6)n} \).

Theorem 8. For any given set of primes \( \{p_1, p_2, \ldots, p_l\} \), let \( A, B_1, B_2, \ldots, B_q, X, Z, Y_1, \ldots, Y_q \) for \( q = p_1 \times p_2 \times \cdots \times p_l \) be vector subspaces of a finite dimensional vector space \( V \). Then the following linear rank inequality holds if \( V \) is a vector space over a finite field whose characteristic belongs to \( \{p_1, p_2, \ldots, p_l\} \), but may not hold otherwise:

\[
2H(A) + (q + 1)H(B_1) + \sum_{i=2}^{q} 2H(B_i) \leq (2q - 1)H(X) + \sum_{i=1}^{q} H(Y_i) + H(Z) + (3q)H(X | A, B_1, \ldots, B_q)
\]

\[
+ (q + 2)H(Y_1 | A, \cup_{j=2}^{q} B_j) + \sum_{i=2}^{q} 3H(Y_1 | A, \cup_{j=1, j \neq i}^{q} B_j) + 2H(Z | B_1, \ldots, B_q) + H(A | Y_1, \ldots, Y_q, Z) + H(A | X, Z)
\]

\[
+ (q + 1)H(B_1 | X, Y_1) + \sum_{i=2}^{q} 2H(B_i | X, Y_1) + (3q + 1)(H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q))
\]

(2)

The proof of this inequality can be found in Appendix [B-B]. Here we show that this inequality may not hold if \( q \) has an inverse over the finite field (thereby meaning the characteristic of the finite field does not belong to \( \{p_1, p_2, \ldots, p_l\} \)). Let \( V \) be the vector space \( V(q + 1, \mathbb{F}_{p^n}) \) where \( p \notin \{p_1, p_2, \ldots, p_l\} \) and \( \alpha \) is some positive integer. Let \( u_i \) be the 1 dimensional vector space spanned by the \( q + 1 \)-length vector whose \( i \)th element is 1 and all other elements are zero. (We construct these subspaces by using the fact that the network \( N_2^1 \) has a rate 1 linear solution when \( q \neq 0 \) over the finite field and \( n = 1 \).)

\[
A = u_1 \quad \text{for } 1 \leq i \leq q : \quad B_i = u_{i+1} \quad X = \sum_{i=1}^{q+1} u_i \quad \text{for } 1 \leq i \leq q : \quad Y_i = \sum_{j=1, j \neq i+1}^{q+1} u_i \quad Z = \sum_{i=2}^{q+1} u_i
\]
Now note \( X = A + \sum_{i=1}^{q} B_i, \) \( Y_i = A + \sum_{j=1, j \neq i}^{q} B_i, \) \( Z = \sum_{i=1}^{q} B_i, \) \( A = q^{-1}(\sum_{i=1}^{q} Y_i - (q-1)Z), \) \( A = X - Z, \) and \( B_i = X - Y_i. \) Hence all the conditional terms in equation (2) becomes zero; and the inequality returns \((3q + 1) \leq (3q), \) or, \( 1 \leq 0. \)

Now, note inequality \( (2) \) when applied to \( N_2 \) results an upper-bound equal to \( (3q)^k \).\

VI. CONCLUSION

We have showed that for any given finite/co-finite set of primes, and for any given positive rational number \( k/n, \) there exists a network which has a \( k/n \) fractional linear network coding solution if and only if the characteristic of the finite field belongs to the given set.

Next, for any given set of primes we have presented two characteristic-dependent linear rank inequalities: one holds if the characteristic of the finite field does not belong to the given set but may not hold otherwise; and the other holds if the characteristic of the finite field belongs to the given set but may not hold otherwise.

APPENDIX A

A. Proof of lemma \( [7] \)

We prove this lemma by first forming a set of equations that the local coding matrices must satisfy for the network to be linearly solvable, and then we find an expression to show that these equations hold only if \( q = 0 \) over the finite field. The ‘if’ part is shown by forming a rate 1 linear solution when \( q = 0. \)

Consider a \((d,dn)\) fractional linear network coding solution of the network \( N'_1 \) where \( d \) is any positive integer. The sizes of the local coding matrices are as follows. For \( 1 \leq i \leq n, \) the matrices \( D_{4i} \) and \( D_{1i} \) are of size \( dn \times d, \) and it left multiplies the information \( a_i \) which is a \( d \) length vector. Matrices \( P_{ij}, Q_{ij}, U_{ijk}, J_{ijk} \) and \( E_{ij} \) for \( 1 \leq i, k \leq (q-1), i \neq k \) and \( 1 \leq j \leq n \) are of size \( dn \times d \) and it left multiplies the \( d \) length vector \( b_{ij}. \) For \( 1 \leq i \leq n, \) the matrices \( D_{2i} \) and \( D_{3i} \) are of size \( dn \times d \) and it left multiplies the information \( c_i. \) The following matrices are of size \( dn \times dn: D_5, D_6, M_i, K_i, R_i, V_i \) and \( W_i \) for \( 1 \leq i \leq (q-1). \) And, the following are the matrices of size \( d \times dn: G_j, X_{ij}, L_j \) and \( Z_{ij} \) for \( 1 \leq i \leq (q-1) \) and \( 1 \leq j \leq n. \)
Also let $I_d$ be a $d \times d$ identity matrix. Then, from the definition of network coding we have:

$$Y_{(u_1, u_3)} = \sum_{i=1}^{n} D_{1i}a_i + \sum_{i=1}^{q-1} \sum_{j=1}^{n} P_{ij}b_{ij}$$  \hspace{1cm} (3)

$$Y_{(u_2, u_4)} = \sum_{i=1}^{q-1} \sum_{j=1}^{n} Q_{ij}b_{ij} + \sum_{i=1}^{n} D_{2i}c_i$$  \hspace{1cm} (4)

$$Y_{(u_5, u_7)} = M_1Y_{(u_1, u_3)} + M_2Y_{(u_2, u_4)} = \sum_{i=1}^{n} M_1D_{1i}a_i + \sum_{i=1}^{q-1} \sum_{j=1}^{n} (M_1P_{ij} + M_2Q_{ij})b_{ij} + \sum_{i=1}^{n} M_2D_{2i}c_i$$  \hspace{1cm} (5)

$$Y_{(u_6, u_8)} = M_3Y_{(u_1, u_3)} + \sum_{i=1}^{n} D_{3i}c_i = \sum_{i=1}^{n} M_3D_{1i}a_i + \sum_{i=1}^{q-1} \sum_{j=1}^{n} M_3P_{ij}b_{ij} + \sum_{i=1}^{n} D_{3i}c_i$$  \hspace{1cm} (6)

$$Y_{(u_9, u_{10})} = M_4Y_{(u_5, u_7)} + M_5Y_{(u_6, u_8)} = \sum_{i=1}^{q-1} \sum_{j=1}^{n} (M_4M_1D_{1i} + M_5M_3D_{1i})a_i + \sum_{i=1}^{q-1} \sum_{j=1}^{n} (M_4(M_1P_{ij} + M_2Q_{ij}) + M_5M_3P_{ij})b_{ij} + \sum_{i=1}^{n} (M_4M_2D_{2i} + M_5D_{3i})c_i$$  \hspace{1cm} (7)

$$Y_{(u_{11}, u_{12})} = \sum_{i=1}^{n} D_{4i}a_i + D_5Y_{(u_5, u_7)} = \sum_{i=1}^{n} (D_{4i} + D_5M_1D_{1i})a_i + \sum_{i=1}^{q-1} \sum_{j=1}^{n} D_5(M_1P_{ij} + M_2Q_{ij})b_{ij} + \sum_{i=1}^{n} D_5M_2D_{2i}c_i$$  \hspace{1cm} (8)

for $1 \leq i \leq (q - 1)$:

$$Y_{e_i} = W_iY_{(u_2, u_4)} + \sum_{j=1,j \neq i}^{q-1} \sum_{k=1}^{n} U_{ji}b_{kj} = \sum_{k=1}^{n} W_iQ_{ik}b_{ik} + \sum_{j=1,j \neq i}^{q-1} \sum_{k=1}^{n} (W_iQ_{jk} + U_{ji})b_{jk} + \sum_{k=1}^{n} W_iD_{2k}c_k$$  \hspace{1cm} (9)

for $1 \leq i \leq (q - 1)$: $Y_{(v_i, v_i')} = K_iY_{(u_9, u_{10})} + \sum_{j=1,j \neq i}^{q-1} \sum_{k=1}^{n} J_{ji}b_{jk} = \sum_{k=1}^{n} K_i(M_4M_1D_{1k} + M_5M_3D_{1k})a_k + \sum_{j=1,j \neq i}^{q-1} \sum_{k=1}^{n} (K_i(M_4M_1P_{kj} + M_2Q_{kj}) + M_5M_3P_{kj})b_{jk} + \sum_{j=1,k \neq i}^{q-1} \sum_{k=1}^{n} (K_i(M_4M_2D_{2j} + M_5D_{3j})c_j$  \hspace{1cm} (10)

for $1 \leq i \leq (q - 1)$:

$$Y_{(w_i, w_i')} = V_iY_{e_i} + \sum_{j=1}^{q-1} E_{ji}b_{ij} = \sum_{k=1}^{n} (V_iW_iQ_{ik} + E_{ik})b_{ik} + \sum_{j=1,j \neq i}^{q-1} \sum_{k=1}^{n} (V_i(W_iQ_{jk} + U_{ji})b_{jk} + \sum_{k=1}^{n} V_iW_iD_{2k}c_k$$  \hspace{1cm} (11)

$$Y_{(u_{13}, u_{14})} = D_6Y_{(u_6, u_8)} + \sum_{i=1}^{q-1} R_iY_{e_i} = \sum_{i=1}^{n} D_6M_3D_{1i}a_i + \sum_{i=1}^{q-1} \sum_{j=1}^{n} (D_6M_3P_{ij} + R_iW_iQ_{ij} + \sum_{k=1,k \neq i}^{q-1} \sum_{k=1}^{n} (D_6D_{3i} + (R_kW_kQ_{ij} + U_{ijk})b_{ij} + \sum_{i=1}^{n} (D_6D_{3i} + (R_kW_kD_{2i})c_i$  \hspace{1cm} (12)

It can be seen that some of the above considered local coding matrices are rectangular matrices. As a rectangular matrix do not have a unique inverse (which would be all-important as we progress), we use the following lemma, which shows how in some cases many different rectangular matrices can be combined to form a square matrix having a unique inverse.

Let $A = [A_1 \ A_2 \ \cdots \ A_n]$ and $B = [B_1 \ B_2 \ \cdots \ B_n]$ where $A_i$ and $B_i$ for $1 \leq i \leq n$ are matrices of size $d \times dn$ and $dn \times d$ respectively (So $A$ and $B$ are both of size $dn \times dn$).

**Lemma 9.** For $1 \leq i, j \leq n, i \neq j$, if $A_iB_i = I_d$ and $A_iB_j = 0$, then $AB = I_{dn}$.

**Proof:**

$$AB = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix} = \begin{bmatrix} A_1B_1 & A_1B_2 & \cdots & A_1B_n \\ A_2B_1 & A_2B_2 & \cdots & A_2B_n \\ \vdots & \vdots & \ddots & \vdots \\ A_nB_1 & A_nB_2 & \cdots & A_nB_n \end{bmatrix} = \begin{bmatrix} I_d & 0 & \cdots & 0 \\ 0 & I_d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_d \end{bmatrix} = I_{dn}$$
Corollary 10. For \(1 \leq i, j \leq n\), if \(A_i B_j = 0\), then \(A B = 0\).

As the components of \(b_{ij}\) is also zero at all \(t_k \in T_c\), for \(1 \leq i \leq (q - 1)\) and \(1 \leq j, k \leq n\), from equation (8) and Fig. 1 we have:

\[
G_k \{D_5(M_1 P_{ij} + M_2 Q_{ij})\} = 0
\]  

(13)

Let,  
\[
G = \begin{bmatrix} G_1 & G_2 & \cdots & G_n \end{bmatrix}^T, \tag{14}
\]

\[
P = \begin{bmatrix} P_{11} & P_{21} & \cdots & P_{n1} \end{bmatrix}, \tag{15}
\]

and  
\[
Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} & \cdots & Q_{in} \end{bmatrix}. \tag{16}
\]

Hence,  
\[
D_5(M_1 P_i + M_2 Q_i) = [D_5(M_1 P_{11} + M_2 Q_{11}) \ D_5(M_1 P_{12} + M_2 Q_{12}) \ \cdots \ D_5(M_1 P_{in} + M_2 Q_{in})]
\]  

(17)

Then, applying corollary 10 on equations (13), (14) and (17) we have:

\[
G_i(D_5(M_1 P_i + M_2 Q_i)) = 0
\]

(18)

Now, since the terminal \(t_i \in T_c\) retrieves \(c_i\) for \(1 \leq i, j \leq n\), \(j \neq i\) from equation (8) and Fig. 1 we have:

\[
G_i(D_5 M_2 D_{2i}) = I_d
\]

(19)

\[
G_i(D_5 M_2 D_{2j}) = 0
\]

(20)

Let,  
\[
D_2 = \begin{bmatrix} D_{21} & D_{22} & \cdots & D_{2n} \end{bmatrix}. \tag{21}
\]

Then,  
\[
D_5 M_2 D_2 = \begin{bmatrix} D_5 M_2 D_{21} & D_5 M_2 D_{22} & \cdots & D_5 M_2 D_{2n} \end{bmatrix} \tag{22}
\]

Then, applying lemma 9 on equations (19) and (20) we have:

\[
G(D_5 M_2 D_2) = I_{dn}
\]

(23)

Now as equation (23) implies both \(G\) and \(D_5\) are invertible, from equation (18) we have:

\[
for \ 1 \leq i \leq (q - 1) : \ M_1 P_i + M_2 Q_i = 0
\]

(24)

Now consider the \(n\) terminals in the set \(T_b\), for \(1 \leq i \leq (q - 1)\). Since the component of \(a_k\) for \(1 \leq k \leq n\) at \(t_j \in T_b\), for \(1 \leq j \leq n\) is zero, for \(1 \leq i \leq (q - 1)\) and \(1 \leq j, k \leq n\), using equation (10) and Fig. 1 we have:

\[
X_{ij} K_i (M_4 M_1 D_{1k} + M_5 M_3 D_{1k}) = 0
\]

(25)

Let,  
\[
X_i = \begin{bmatrix} X_{i1} & X_{i2} & \cdots & X_{in} \end{bmatrix}^T \tag{26}
\]

and,  
\[
D_1 = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1n} \end{bmatrix}^T \tag{27}
\]

Then,  
\[
X_i K_i = \begin{bmatrix} X_{i1} K_i & X_{i2} K_i & \cdots & X_{in} K_i \end{bmatrix}^T \tag{28}
\]

and  
\[
M_4 M_1 D_1 + M_5 M_3 D_1 = [M_4 M_1 D_{11} + M_3 M_D_{11} \ M_4 M_1 D_{12} + M_3 M_D_{12} \ \cdots \ M_4 M_1 D_{1n} + M_3 M_D_{1n}] \tag{29}
\]

Now, applying corollary 10 on equations (25), (28), and (29) we have:

\[
for \ 1 \leq i \leq (q - 1) : \ X_i K_i (M_4 M_1 D_{1} + M_5 M_3 D_{1}) = 0
\]

(30)

Since the terminal \(t_j \in T_b\) computes the information \(b_{ij}\), we have for \(1 \leq i \leq (q - 1)\), \(1 \leq j, m \leq n\) and \(m \neq j\):

\[
X_{ij} K_i \{M_4 (M_1 P_j + M_2 Q_{ij}) + M_5 M_3 P_j \} = I_d
\]

(31)

\[
X_{ij} K_i \{M_4 (M_1 P_{1m} + M_2 Q_{1m}) + M_5 M_3 P_{1m} \} = 0
\]

(32)

From equations (15) and (16) we have:

\[
[M_4 (M_1 P_{1} + M_2 Q_{1}) + M_5 M_3 P_{1} \ 
\cdots \ M_4 (M_1 P_{m} + M_2 Q_{m}) + M_5 M_3 P_{m} \ 
\cdots \ M_4 (M_1 P_{n} + M_2 Q_{n}) + M_5 M_3 P_{n}]
\]

(33)

Using lemma 9 and equations (31), (32), (28) and (33) we have:

\[
for \ 1 \leq i \leq (q - 1) : \ X_i K_i \{M_4 (M_1 P_{1} + M_2 Q_{1}) + M_5 M_3 P_{1} \} = I_{dn}
\]

(34)

Substituting equation (24) in equation (34) we have:

\[
X_i K_i M_5 M_3 P_{1} = I_{dn}
\]

(35)
Since from equation (34) both $X_i$ and $K_i$ are invertible, we have from equation (30):
\[ M_4M_1D_1 + M_5M_2D_1 = 0 \]  
(36)

Since the component of $c_k$ for $1 \leq k \leq n$ is zero at $t_j \in T_b$, using equation (10) we have for $1 \leq i \leq (q-1)$ and $1 \leq j, k \leq n$:
\[ X_{ij}K_i(M_4M_2D_{2k} + M_5D_{3k}) = 0 \]  
(37)

Let, $D_2 = [D_{21} D_{22} \cdots D_{2n}]$, and $D_3 = [D_{31} D_{32} \cdots D_{3n}]$.

Then, $M_4M_2D_2 + M_5D_3 = [M_4M_2D_{21} + M_5D_{31} \quad M_4M_2D_{22} + M_5D_{32} \cdots \quad M_4M_2D_{2n} + M_5D_{3n}]$  
(40)

Using Corollary 10 on equations (37), (28) and (40) we have:

\[ \text{for } 1 \leq i \leq (q-1) : \quad X_iK_i(M_4M_2D_2 + M_5D_3) = 0 \]  
(41)

Since both $X_i$ and $K_i$ are invertible (equation (34)), from equation (41) we have:
\[ M_4M_2D_2 + M_5D_3 = 0 \]  
(42)

Let us consider the terminals in the set $T_a$. Since $t_i \in T_a$ computes the message $a_i$, for $1 \leq i, j \leq n$ and $j \neq i$, using equation (12) and Fig. 1 we have:
\[ L_iD_6M_3D_{1i} = I_d \]  
(43)

\[ L_iD_6M_3D_{1j} = 0 \]  
(44)

Let, $L = [L_1 \quad L_2 \cdots \quad L_n]^T$.

From equation (27) we have: $D_6M_3D_{1i} = [D_6M_3D_{11} \quad D_6M_3D_{12} \cdots \quad D_6M_3D_{1n}]$  
(46)

Then applying lemma 9 on equations (43), (44), (45) and (46) we have:
\[ LD_6M_3D_1 = I_{dn} \]  
(47)

Since from equation (47) $D_1$ is invertible, from equation (36):
\[ M_4M_1 + M_5M_3 = 0 \]  
(48)

Substituting (48) in (34) we have:
\[ \text{for } 1 \leq i \leq (q-1) : \quad X_iK_iM_4M_2Q_i = I_{dn} \]  
(49)

At any $t_i \in T_a$ for $1 \leq i \leq (q-1)$ and $1 \leq l, j \leq n$ the component of $b_{ij}$ is zero. So we have:
\[ L_i\{D_6M_3P_{ij} + R_iW_1Q_{ij} + \sum_{k=1, k\neq i}^{q-1} R_k(W_kQ_{ij} + U_{ijk})\} = 0 \]  
(50)

Let $U_{ik} = [U_{i1k} \quad U_{i2k} \cdots \quad U_{ink}]$.

Then from equations (15), (16), (51) we have: $D_6M_3P_i + R_iW_1Q_i + \sum_{k=1, k\neq i}^{q-1} R_k(W_kQ_i + U_{ik})$
\[ = [D_6M_3P_{11} + R_iW_1Q_{11} + \sum_{k=1, k\neq i}^{q-1} R_k(W_kQ_{11} + U_{1ik}) \quad \cdots \quad D_6M_3P_{in} + R_iW_1Q_{in} + \sum_{k=1, k\neq i}^{q-1} R_k(W_kQ_{in} + U_{ink})] \]

So using Corollary 10 on equation (50) for $1 \leq i \leq (q-1)$ we get:
\[ \text{for } 1 \leq i \leq (q-1) : \quad L(D_6M_3P_i + R_iW_1Q_i + \sum_{k=1, k\neq i}^{q-1} R_k(W_kQ_i + U_{ik})) = 0 \]  
(52)

Since from equation (47) $L$ is invertible, we have:
\[ \text{for } 1 \leq i \leq (q-1) : \quad D_6M_3P_i + R_iW_1Q_i + \sum_{k=1, k\neq i}^{q-1} R_k(W_kQ_i + U_{ik}) = 0 \]  
(53)
At a terminal $t_j \in T_o$, since the component of $c_i$ is zero, for $1 \leq i, j \leq n$ we have:

$$L_j \{D_0 D_3i + \sum_{k=1}^{q-1} R_k W_k D_{2k} \} = 0 \quad (54)$$

Using equations (38) and (39): $D_0 D_3 + \sum_{k=1}^{q-1} R_k W_k D_2$

$$= [D_0 D_{31} + \sum_{k=1}^{q-1} R_k W_k D_{21}] \quad D_0 D_{32} + \sum_{k=1}^{q-1} R_k W_k D_{22} \ldots \quad D_0 D_{3n} + \sum_{k=1}^{q-1} R_k W_k D_{2n}] \quad (55)$$

Using corollary 10 and equations (54), (45) and (55) we have:

$$L(D_0 D_3 + \sum_{k=1}^{q-1} R_k W_k D_2) = 0 \quad (56)$$

Since $L$ is invertible from equation (47) we have:

$$D_0 D_3 + \sum_{k=1}^{q-1} R_k W_k D_2 = 0 \quad (57)$$

Now consider the terminals in the set $T_c$, for $1 \leq i \leq q - 1$. Since at $t_l \in T_c$, for $1 \leq k \leq (q - 1), k \neq i$ the component of $b_{kj}$ for $1 \leq l, j \leq n$ is zero, we have:

$$Z_d \{V_i(W_i Q_k + U_{kji}) \} = 0 \quad (58)$$

Let, $Z_i = [Z_{i1} \quad Z_{i2} \ldots \quad Z_{in}]^T$

Then, $Z_i V_i = [Z_{i1}V_i \quad Z_{i2}V_i \ldots \quad Z_{in}V_i]^T$

From equations (16) and (51): $W_i Q_k + U_{ki} = [V_i(W_i B_{k1} + U_{k1i}) \quad V_i(W_i B_{k2} + U_{k2i}) \ldots \quad V_i(W_i B_{kn} + U_{kni})]$

Using corollary 10 on equations (58), (60) and (61) we get:

$$Z_i V_i (W_i Q_k + U_{ki}) = 0 \quad (62)$$

Since $t_l \in T_c$, computes $c_l$, for $1 \leq l, m \leq n, l \neq m$, we have:

$$Z_d V_i W_i D_{2l} = I_d \quad (63)$$

$$Z_d V_i W_i D_{2m} = 0 \quad (64)$$

From equation (38): $W_i D_2 = [W_i D_{21} \quad W_i D_{22} \ldots \quad W_i D_{2n}]$

Using Lemma 9 on equations (63), (64), (60) and (65) we have:

$$Z_d V_i W_i D_{2} = I_{dn} \quad (66)$$

Since from equation (66), $Z_i V_i$ is invertible, from equation (62) we have:

$$Z_i V_i W_i Q_k + U_{ki} = 0 \quad (67)$$

Substituting equation (67) in equation (53) we have:

$$D_0 M_3 P_i + R_i W_i Q_i = 0 \quad (68)$$

Hence, for $1 \leq i \leq (q - 1)$:

$$D_0 M_3 P_i Q_i^{-1} + R_i W_i = 0 \quad [Q_i \text{ is invertible from (49)}]$$

$$D_0 M_3 P_i X_i K_i M_4 M_2 + R_i W_i = 0 \quad \text{[from equation (49)]}$$

$$D_0 M_3 P_i X_i K_i M_4 M_2 D_2 + R_i W_i D_2 = 0 \quad \text{[multiplying both sides by $D_2$]}$$

$$-D_0 M_3 P_i X_i K_i M_5 D_3 + R_i W_i D_2 = 0 \quad \text{[from equation (42)]}$$

$$-D_0 M_3 P_i P_i^{-1} M_5^{-1} D_3 + R_i W_i D_2 = 0 \quad \text{[from equation (35)]}$$

$$-D_0 D_3 + R_i W_i D_2 = 0 \quad (69)$$

Substituting equation (69) in equation (57) we have:

$$q D_0 D_3 = 0 \quad (70)$$
From equation \( (47) \), \( D_3 \) is invertible. As \( M_1M_2 \) is invertible from equation \( (49) \), and as \( D_2 \) is invertible from equation \( (42) \), \( M_3D_3 \) is invertible from equation \( (42) \). This implies \( D_3 \) is also an invertible matrix. So for \( qD_3D_3 = 0 \) to hold \( q \) must be equal to zero. Now, in a finite field, an element is equal to zero if and only if the characteristic divides the element. This proves that the network in Fig. [1] has a rate \( \frac{1}{n} \) fractional linear network coding solution only if the characteristic of the finite field divides \( q \). Next, we show that the network \( \mathcal{N}_1 \) has a \((1,n)\) fractional linear network coding solution if \( q = 0 \).

For this section, let \( \bar{a}_i \) denote an \( n \)-length column vector whose \( i^{th} \) component is \( a_i \) and all other components are zero (since \( k = 1 \), \( a_i \) is an unit-length vector). Let \( \bar{c}_i \) denote an \( n \)-length column vector whose \( i^{th} \) component is \( c_i \) and all other components are zero. Also let \( \bar{b}_{ij} \) denote an \( n \)-length column vector whose \( j^{th} \) component is \( b_{ij} \) and all other components are zero. Now, by choosing the appropriate local coding matrices, the messages shown below can be transmitted by the corresponding edges.

Let \( \bar{u}(i) \) be a unit row vector of length \( n \) which has \( i^{th} \) component equal to one and all other components are zero. Then from the vector \( \sum_{i=1}^{n} \bar{a}_i \), \( a_i \) for any \( 1 \leq i \leq n \) can be determined by the dot product \( \bar{u}(i) \cdot (\sum_{i=1}^{n} \bar{a}_i) \). Similarly for any \( 1 \leq i \leq (q - 1) \), \( \bar{b}_{ij} = \bar{u}(j) \cdot (\sum_{j=1}^{n} \bar{b}_{ij}) \). For \( 1 \leq i \leq n \), \( c_i \) can be determined similarly from \( \sum_{i=1}^{n} \bar{c}_i \).

**B. Proof of theorem [9]**

To produce the desired characteristic-dependent linear rank inequality, we apply DFZ method to the network shown in Fig. [1] for \( n = 1 \) and \( p = p_1 \times p_2 \times \ldots \times p_t \). Let the message carried by an edge \((u_i, u_j)\) be denoted by \( Y_{i,j} \). Also let the massage carried by the edge \( e_i \) for \( 1 \leq i \leq q - 1 \) be denoted by \( Y_{e,i} \). Corresponding to each of the source messages and the massages carried by the edges, consider the vector subspaces \( A, B_1, \ldots, B_{q-1}, C, Y_{1,3}, Y_{2,4}, Y_{5,7}, Y_{6,8}, Y_{9,10}, Y_{e,1}, \ldots, Y_{e,q-2} \) and \( Y_{e,q-1} \) of a finite dimensional vector space \( V \). Corresponding to the matrices in Fig. [1] consider the following linear functions:

\[
\begin{align*}
    &f_{D_1} : Y_{1,3} \rightarrow A & f_{D_2} : Y_{2,4} \rightarrow C & f_{D_3} : Y_{6,8} \rightarrow C & f_{D_4} : C \rightarrow A & f_{D_5} : C \rightarrow Y_{5,7} & f_{D_6} : A \rightarrow Y_{6,8} \\
    &f_{M_1} : Y_{5,7} \rightarrow Y_{1,3} & f_{M_2} : Y_{5,7} \rightarrow Y_{2,4} & f_{M_3} : Y_{6,8} \rightarrow Y_{1,3} & f_{M_4} : Y_{9,10} \rightarrow Y_{5,7} & f_{M_5} : Y_{9,10} \rightarrow Y_{6,8} \\
\end{align*}
\]

for \( 1 \leq i \leq q - 1 \), \( f_{P_i} : Y_{i,3} \rightarrow B_i \)

\[
\begin{align*}
    &f_{Q_i} : Y_{2,4} \rightarrow B_i & f_{K_i} : B_i \rightarrow Y_{9,10} & f_{E_i} : C \rightarrow B_i \\
\end{align*}
\]

for \( 1 \leq i \leq q - 1 \), \( f_{W_i} : Y_{i,3} \rightarrow Y_{2,4} \)

\[
\begin{align*}
    &f_{R_i} : A \rightarrow Y_{e,i} & f_{V_i} : C \rightarrow Y_{e,i} \\
\end{align*}
\]

for \( 1 \leq i,j \leq q - 1, i \neq j \), \( f_{U_{ij}} : Y_{e,i} \rightarrow B_j \) \( f_{J_{ij}} : B_i \rightarrow B_j \)
The idea behind the DFZ method is as follows. First note that the linear functions shown above is in accordance with the topology of the network. Now, we have seen in the last subsection that over a finite field where \( q \neq 0 \) the network \( \mathcal{N}' \) does not have a rate 1 linear solution (note \( n = 1 \) in Fig. [1] for this current proof). This means that if the dimension of all the above considered vector subspaces are equal, then such a functional assignment won’t exists when \( q \neq 0 \) over the finite field (because if it had existed then the realization of these vector subspaces would have formed a rate 1 linear solution). The DFZ method starts with these linear functions and tries to find an equation (relating the dimension of the corresponding vector subspaces) that must hold true for such a functional assignment to exist over a finite filed where \( q \neq 0 \). This equation is the desired inequality.

Now to obtain this equation, the DFZ method requires to find a subspace (say \( S \)) that becomes a zero subspaces when \( q \neq 0 \). This subspace must also be expressible as an intersection of other subspaces. Then, applying Lemma 11 (shown below) on \( S \) results the desired inequality. At present, all the steps of the DFZ method sans finding the set \( S \) is algorithmic. Intuitively, when \( S \) becomes the zero subspace (which happens when \( q \neq 0 \) in our case) the dimension of the union of the subspaces whose intersection is equal to \( S \) increases; thereby meaning that more information has to be sent (more is reflected in the increment of the dimension) when \( q \neq 0 \). This ‘more’ information results the rate to be less than 1.

For this proof, to find \( S \), we use the proof of lemma 11 shown in the above subsection. Let us define some notations and introduce some lemmas which will be required for the rest of the proof.

If \( A \) is a subspace of \( V \) then co-dimension of \( A \) in \( V \) is \( \text{codim}_V(A) = \dim(V) - \dim(A) \). The following lemmas are reproduced from [9]. The proofs of these lemmas are omitted from here and can be found in [9]. In all of these lemmas, \( V \) is a finite dimensional vector space, and \( A, B, A_1, A_2, \ldots, A_m \) are subspaces of \( V \). Let \( f : A \to B \) be a linear function. If \( B' \) is a subspace of \( B \), then \( f^{-1}(B') \) denotes a vector subspace \( A' \) of \( A \) such that \( f(A') = B' \).

**Lemma 11.** [9] Lemma 2, p. 2501:
\[
\text{codim}_V(\cap_{i=1}^{m} A_i) \leq \sum_{i=1}^{m} \text{codim}_V(A_i)
\]

**Lemma 12.** [9] Lemma 3, p. 2501: If \( B' \) is a subspace of \( B \), then
\[
\text{codim}_A(f^{-1}(B')) \leq \text{codim}_B(B')
\]

**Lemma 13.** [9] Lemma 4, p. 2501: There exist linear functions \( f_i : A \to A_i \) for \( 1 \leq i \leq m \) such that \( f_1 + \cdots + f_m = I \) on a subspace \( A' \) of \( A \) with
\[
\text{codim}_A(A') \leq H(A|A_1, A_2, \ldots, A_m)
\]

**Lemma 14.** [9] Lemma 6, p. 2502: For \( 1 \leq i \leq m \), let \( f_i : A \to A_i \) be linear functions such that \( f_1 + f_2 + \cdots + f_m = 0 \) on \( A \). Then \( f_1 = \cdots = f_m = 0 \) on a subspace \( A' \) of \( A \) with
\[
\text{codim}_A(A') \leq H(A_1) + \cdots + H(A_m) - H(A_1, \ldots, A_m)
\]

According to Lemma 13 the following holds:

\[
f_{D_1} + \sum_{i=1}^{q-1} f_{P_i} = I \quad \text{over a subspace } Y_{1,3}' \text{ of } Y_{1,3} \text{ where } \text{codim}_{Y_{1,3}}(Y_{1,3}') \leq H(Y_{1,3}|A, B_1, \ldots, B_{q-1}) \tag{71}
\]
\[
\sum_{i=1}^{q-1} f_{Q_i} + f_{D_1} = I \quad \text{over a subspace } Y_{2,4}' \text{ of } Y_{2,4} \text{ where } \text{codim}_{Y_{2,4}}(Y_{2,4}') \leq H(Y_{2,4}|B_1, \ldots, B_{q-1}, C) \tag{72}
\]
\[
f_{M_1} + f_{M_2} = I \quad \text{over a subspace } Y_{5,7}' \text{ of } Y_{5,7} \text{ where } \text{codim}_{Y_{5,7}}(Y_{5,7}') \leq H(Y_{5,7}|Y_{1,3}, Y_{2,4}) \tag{73}
\]
\[
f_{M_3} + f_{D_3} = I \quad \text{over a subspace } Y_{6,8}' \text{ of } Y_{6,8} \text{ where } \text{codim}_{Y_{6,8}}(Y_{6,8}') \leq H(Y_{6,8}|Y_{1,3}, C) \tag{74}
\]

for \( 1 \leq i, j \leq q - 1, i \neq j \):
\[
f_{W_i} + \sum_{j=1, j \neq i}^{q-1} f_{U_{ij}} = I \quad \text{over a subspace } Y_{\epsilon_i}' \text{ of } Y_{\epsilon_i} \text{ where } \text{codim}_{Y_{\epsilon_i}}(Y_{\epsilon_i}') \leq H(Y_{\epsilon_i}|Y_{2,4}, B_1, \ldots, B_{q-1}) \tag{75}
\]
\[
f_{M_4} + f_{M_5} = I \quad \text{over a subspace } Y_{9,10}' \text{ of } Y_{9,10} \text{ where } \text{codim}_{Y_{9,10}}(Y_{9,10}') \leq H(Y_{9,10}|Y_{5,7}, Y_{6,8}) \tag{76}
\]
\[
f_{D_4} + f_{D_5} = I \quad \text{over a subspace } C' \text{ of } C \text{ where } \text{codim}_{C}(C') \leq H(C|A, Y_{5,7}) \tag{77}
\]

for \( 1 \leq i, j \leq q - 1, i \neq j \):
\[
f_{K_i} + \sum_{j=1, j \neq i}^{q-1} f_{J_{ij}} = I \quad \text{over a subspace } B_{i}' \text{ of } B_i \text{ where } \text{codim}_{B_i}(B_{i}') \leq H(B_i|Y_{9,10}, B_1, \ldots, B_{q-1}) \tag{78}
\]
\[
f_{D_6} + \sum_{i=1}^{q-1} f_{R_i} = I \quad \text{over a subspace } A' \text{ of } A \text{ where } \text{codim}_{A}(A') \leq H(A|Y_{6,8}, Y_{\epsilon_1}, \ldots, Y_{\epsilon_{q-1}}) \tag{79}
\]
\[
f_{I_{\lambda}} + f_{E_i} = I \quad \text{over a subspace } C' \text{ of } C \text{ where } \text{codim}_{C}(C') \leq H(C|Y_{\epsilon_i}, B_i) \tag{80}
\]
Now, let’s consider the following composite functions:
\[ f_{D_i} + f_{D_1}f_{M_i}f_{D_b} : C \to A \]
for \( 1 \leq i \leq (q-1) \): \((f_{p_1}f_{M_1} + f_{Q_0}f_{M_2})f_{D_b} : C \to B_i \)
\[ f_{D_2}f_{M_2}f_{D_b} : C \to C \]

Using (71), (72), and (75) we have:
\[
\begin{align*}
    f_{D_1}f_{M_1}f_{D_b} + \sum_{i=1}^{q-1} f_{p_i}f_{M_1}f_{D_b} &= f_{M_1}f_{D_b} \text{ over a subspace } f_{D_b}^{-1}f_{M_1}^{-1}(Y_{1,3}^{'}) \text{ of } C \\
    f_{D_2}f_{M_2}f_{D_b} + \sum_{i=1}^{q-1} f_{Q_i}f_{M_2}f_{D_b} &= f_{M_2}f_{D_b} \text{ over a subspace } f_{D_b}^{-1}f_{M_2}^{-1}(Y_{2,4}^{'}) \text{ of } C \\
    f_{M_1}f_{D_b} + f_{M_2}f_{D_b} &= f_{D_b} \text{ over a subspace } f_{D_b}^{-1}(Y_{5,7}^{'}) \text{ of } C
\end{align*}
\]

Also note, from equation (77), \( f_{D_4} + f_{D_b} = I \) over \( C' \). Then,
\[
\begin{align*}
    f_{D_1}f_{M_1}f_{D_b} + \sum_{i=1}^{q-1} f_{p_i}f_{M_1}f_{D_b} + f_{D_2}f_{M_2}f_{D_b} + \sum_{i=1}^{q-1} f_{Q_i}f_{M_2}f_{D_b} + f_{D_4} &= I \text{ over a subspace } C'' = f_{D_b}^{-1}f_{M_1}^{-1}(Y_{1,3}^{'}) \cap f_{D_b}^{-1}f_{M_2}^{-1}(Y_{2,4}^{'}) \cap f_{D_b}^{-1}(Y_{5,7}^{'}) \cap C' \text{ of } C
\end{align*}
\]

Then using lemma [11]
\[
\begin{align*}
    \text{codim}_C(C'') &\leq \text{codim}_C(f_{D_b}^{-1}f_{M_1}^{-1}(Y_{1,3}^{'}) + \text{codim}_C(f_{D_b}^{-1}f_{M_2}^{-1}(Y_{2,4}^{'}) + \text{codim}_C(f_{D_b}^{-1}(Y_{5,7}^{'}) + \text{codim}_C(C') \\
    \text{or, codim}_C(C'') &\leq \text{codim}_{Y_{1,3}}(Y_{1,3}^{'}) + \text{codim}_{Y_{2,4}}(Y_{2,4}^{'}) + \text{codim}_{Y_{5,7}}(Y_{5,7}^{'}) + \text{codim}_C(C') \quad \text{[using lemma [12]} \quad (81)
\end{align*}
\]

Now, according to Lemma [14] there exists a subspace \( \bar{C} \) of \( C'' \) over which:
\[
\begin{align*}
    f_{D_1} + f_{D_1}f_{M_1}f_{D_b} &= 0 \\
    \text{for } 1 \leq i \leq (q-1) : (f_{p_i}f_{M_1} + f_{Q_i}f_{M_2})f_{D_b} &= 0 \\
    f_{D_2}f_{M_2}f_{D_b} - I &= 0
\end{align*}
\]

such that
\[
\begin{align*}
    \text{codim}_{C''}(\bar{C}) &\leq H(A) + \sum_{i=1}^{q-1} H(B_i) + H(C) - H(A, B_1, B_2, \ldots, B_{q-1}, C) \quad (82)
\end{align*}
\]

Now, \( \text{codim}_C(\bar{C}) = \text{codim}_C(C'') + \text{codim}_C(\bar{C}) \). So, from equations (81) and (85) we get:
\[
\begin{align*}
    \text{codim}_C(\bar{C}) &\leq \text{codim}_{Y_{1,3}}(Y_{1,3}^{'}) + \text{codim}_{Y_{2,4}}(Y_{2,4}^{'}) + \text{codim}_{Y_{5,7}}(Y_{5,7}^{'}) + \text{codim}_C(C') + H(A) \\
    + \sum_{i=1}^{q-1} H(B_i) + H(C) &- H(A, B_1, B_2, \ldots, B_{q-1}, C) \quad (86)
\end{align*}
\]

Notice the similarity between equations (82) and (85); and between equations (84) and (23). We now want to find a subspace \( \bar{B}_i \) of \( B_i \) for \( 1 \leq i \leq q-1 \), over which the following identities hold:
\[
\begin{align*}
    f_{D_1}(f_{M_1}f_{M_2} + f_{M_3}f_{M_4})f_{K_i} &= 0 \\
    \{(f_{p_1}f_{M_1} + f_{Q_1}f_{M_2})f_{M_4} + f_p,f_{M_3}f_{M_4} \}f_{K_i} &= I \\
    \text{for } 1 \leq j \leq q-1, j \neq i : (f_{p_j}f_{M_1} + f_{Q_j}f_{M_2})f_{M_4}f_{K_i} &= 0 \\
    (f_{D_2}f_{M_3}f_{M_4} + f_{D_3}f_{M_5})f_{K_i} &= 0
\end{align*}
\]
Here also notice the similarity between equations (87) and (40); between equations (88) and (34); and between equations (90) and (41). From equations (71), (72), (73) and (74) we have:

for $1 \leq i \leq (q - 1)$: \[ f_{D_i}f_{M_i}f_{M_i}f_{K_i} + \sum_{i=1}^{q-1} f_{P_i}f_{M_i}f_{M_i}f_{K_i} = f_{M_i}f_{M_i}f_{K_i} \text{ over a subspace } f_{K_i}^{-1}f_{M_i}^{-1}(Y_{1,3}) \text{ of } B_i \]

\[ f_{D_i}f_{M_i}f_{M_i}f_{K_i} + \sum_{i=1}^{q-1} f_{P_i}f_{M_i}f_{M_i}f_{K_i} = f_{M_i}f_{M_i}f_{K_i} \text{ over a subspace } f_{K_i}^{-1}f_{M_i}^{-1}(Y'_{1,3}) \text{ of } B_i \]

\[ f_{D_i}f_{M_i}f_{M_i}f_{K_i} + \sum_{i=1}^{q-1} f_{Q_i}f_{M_i}f_{M_i}f_{K_i} = f_{M_i}f_{M_i}f_{K_i} \text{ over a subspace } f_{K_i}^{-1}f_{M_i}^{-1}(Y'_{2,4}) \text{ of } B_i \]

\[ f_{M_i}f_{M_i}f_{K_i} + f_{M_i}f_{M_i}f_{K_i} = f_{K_i} \text{ over a subspace } f_{K_i}^{-1}(Y'_{9,10}) \text{ of } B_i \]

Also note that from eqn. (78) we have: \[ f_{K_i} + \sum_{j=1,j \neq i}^{q-1} f_{J_j} = I \text{ over } B'_i \].

Hence, \[ f_{D_i}f_{M_i}f_{M_i}f_{K_i} + \sum_{i=1}^{q-1} f_{P_i}f_{M_i}f_{M_i}f_{K_i} + f_{D_i}f_{M_i}f_{M_i}f_{K_i} + \sum_{i=1}^{q-1} f_{Q_i}f_{M_i}f_{M_i}f_{K_i} = f_{K_i} \text{ over subspace } B''_i = f_{K_i}^{-1}f_{M_i}^{-1}(Y'_{1,3}) \cap f_{K_i}^{-1}f_{M_i}^{-1}(Y'_{2,4}) \cap f_{K_i}^{-1}f_{M_i}^{-1}(Y'_{5,7}) \cap f_{K_i}^{-1}f_{M_i}^{-1}(Y'_{6,8}) \cap f_{K_i}^{-1}(Y'_{9,10}) \cap B'_i \text{ of } B_i \]

So, applying Lemma (11):

\[ \text{codim}_{B_i}(B''_i) \leq \text{codim}_{B_i}(f_{K_i}^{-1}f_{M_i}^{-1}(Y'_{1,3})) + \text{codim}_{B_i}(f_{K_i}^{-1}f_{M_i}^{-1}(Y'_{2,4})) + \text{codim}_{B_i}(f_{K_i}^{-1}f_{M_i}^{-1}(Y'_{5,7})) + \text{codim}_{B_i}(f_{K_i}^{-1}f_{M_i}^{-1}(Y'_{6,8})) + \text{codim}_{B_i}(f_{K_i}^{-1}(Y'_{9,10})) + \text{codim}_{B_i}(B'_i) \]

Using Lemma (12) we get:

\[ \text{codim}_{B_i}(B''_i) \leq 2\text{codim}_{Y_{1,3}}(Y'_{1,3}) + \text{codim}_{Y_{2,4}}(Y'_{2,4}) + \text{codim}_{Y_{5,7}}(Y'_{5,7}) + \text{codim}_{Y_{6,8}}(Y'_{6,8}) + \text{codim}_{Y_{9,10}}(Y'_{9,10}) + \text{codim}_{B_i}(B'_i) \] (91)

So from Lemma (14) over a subspace \( B'_i \) of \( B''_i \) equations (87), (88), (89) and (90) holds where

\[ \text{codim}_{B''_i}(B'_i) \leq H(A) + \sum_{i=1}^{q-1} H(B_i) + H(C) - H(A, B_1, B_2, \ldots, B_{q-1}, C) \] (92)

Now, \[ \text{codim}_{B_i}(B'_i) = \text{codim}_{B_i}(B''_i) \]

from equations (91) and (92) we have:

\[ \text{codim}_{B_i}(B'_i) \leq 2\text{codim}_{Y_{1,3}}(Y'_{1,3}) + \text{codim}_{Y_{2,4}}(Y'_{2,4}) + \text{codim}_{Y_{5,7}}(Y'_{5,7}) + \text{codim}_{Y_{6,8}}(Y'_{6,8}) + \text{codim}_{Y_{9,10}}(Y'_{9,10}) + \text{codim}_{B_i}(B'_i) \]

(93)

Next, we want to find an upper-bound on the co-dimension of a subspace \( \tilde{A} \) of \( A'' \) over which the following relations hold:

\[ f_{D_1}f_{M_3}f_{D_6} = I \]

(94)

for \( 1 \leq i \leq q - 1 \):

\[ f_{P_i}f_{M_3}f_{D_6} + f_{Q_i}f_{W_i}f_{R_i} + \sum_{j=1,j \neq i}^{q-1} (f_{Q_i}f_{W_j} + f_{U_i})f_{R_j} = 0 \]

(95)

\[ f_{D_3}f_{D_6} + \sum_{i=1}^{q-1} f_{D_2}f_{W_i}f_{R_i} = 0 \]

(96)
Here also notice the similarity between equations \(^{94}\) and \(^{47}\); between equations \(^{95}\) and \(^{52}\); and between equations \(^{96}\) and \(^{56}\). Using equations \(^{71}\), \(^{72}\), \(^{74}\) and \(^{75}\) we have:

\[
f_{D,1}f_{M,6}f_{D,6} + \sum_{i=1}^{q-1} f_{R_i}f_{M,6}f_{D,6} = f_{M,6}f_{D,6} \text{ over a subspace } f^{-1}_{D,1}f^{-1}_{M,6}(Y'_{1,3}) \text{ of } A
\]

\[
\sum_{i=1}^{q-1} f_{D,2}f_{W,1}f_{R_i} = \sum_{i=1}^{q-1} (f_{Q_j}(\sum_{j=1}^{q-1} f_{W,1}f_{R_j})) = f_{D,2} \sum_{i=1}^{q-1} f_{W,1}f_{R_i} + (\sum_{j=1}^{q-1} f_{Q_j}(\sum_{j=1}^{q-1} f_{W,1}f_{R_j})) = \sum_{j=1}^{q-1} f_{W,1}f_{R_j} \text{ over a subspace } \sum_{j=1}^{q-1} f_{W,1}f_{R_j}^{-1}(Y'_{2,4}) \text{ of } A
\]

Now, \(\sum_{i=1}^{q-1} (\sum_{j=1,j\neq i}^{q-1} f_{U,ij}f_{R_j}) = \sum_{i=1}^{q-1} (\sum_{j=1,i\neq j}^{q-1} f_{U,ij}f_{R_j}) = \sum_{i=1}^{q-1} (\sum_{j=1,j\neq i}^{q-1} f_{U,ij}f_{R_j})\)

So,

\[
\sum_{i=1}^{q-1} (f_{W,1}f_{R_i} + \sum_{j=1,j\neq i}^{q-1} f_{U,ij}) = \sum_{i=1}^{q-1} f_{W,1}f_{R_i} + \sum_{j=1,j\neq i}^{q-1} f_{U,ij}f_{R_j} = \sum_{i=1}^{q-1} (f_{W,1}f_{R_i}) \text{ over a subspace } f^{-1}_{R_1}(Y'_{1,1}) \cap f^{-1}_{R_2}(Y'_{2,2}) \cap \cdots \cap f^{-1}_{R_{q-1}}(Y'_{q-1,1}) \cap f^{-1}_{D,6}(Y'_{6,8}) \cap A'
\]

Also, \(f_{M,6}f_{D,6} + f_{D,1}f_{D,6} = f_{D,6} \text{ over a subspace } f^{-1}_{D,1}(Y'_{6,8}) \text{ of } A
\]

Note that from equation \(^{79}\) we have: \(f_{D,6} + \sum_{i=1}^{q-1} f_{R_i} = I \text{ over a subspace } A' \text{ of } A
\]

So, \(f_{D,1}f_{M,6}f_{D,6} + \sum_{i=1}^{q-1} f_{P_i}f_{M,6}f_{D,6} + \sum_{i=1}^{q-1} f_{D,2}f_{W,1}f_{R_i} + \sum_{i=1}^{q-1} (f_{Q_j}(\sum_{j=1}^{q-1} f_{W,1}f_{R_j})) + \sum_{i=1}^{q-1} (\sum_{j=1,j\neq i}^{q-1} f_{U,ij})f_{R_j} + f_{D,1}f_{D,6} = I
\]

over a subspace \(A'' \text{ of } A
\)

where, \(A'' = f^{-1}_{D,6}f^{-1}_{M,6}(Y'_{1,3}) \cap f^{-1}_{W,1}(Y'_{2,4}) \cap f^{-1}_{R_1}(Y'_{1,1}) \cap \cdots \cap f^{-1}_{R_{q-1}}(Y'_{q-1,1}) \cap f^{-1}_{D,6}(Y'_{6,8}) \cap A'
\]

So, \(\text{codim}_A A'' \leq \text{codim}_A (f^{-1}_{D,6}f^{-1}_{M,6}(Y'_{1,3})) + \text{codim}_A f^{-1}_{W,1}(Y'_{2,4}) + \text{codim}_A f^{-1}_{R_1}(Y'_{1,1}) + \cdots + \text{codim}_A f^{-1}_{R_{q-1}}(Y'_{q-1,1}) \)

Using Lemma \(^{11}\) and Lemma \(^{12}\) we have:

\[
\text{codim}_A (A') \leq \text{codim}_Y Y_{1,3}(Y'_{1,3}) + \text{codim}_Y Y_{2,4}(Y'_{2,4}) + \text{codim}_Y Y_{1,1}(Y'_{1,1}) + \text{codim}_Y Y_{2,2}(Y'_{2,2}) + \cdots + \text{codim}_Y Y_{q-1,1}(Y'_{q-1,1})
\]

Then from Lemma \(^{14}\) over a subspace \(A \text{ equations } \(^{94}\), \(^{95}\) and \(^{96}\) hold, such that

\[
\text{codim}_A (A') \leq H(A) + \sum_{i=1}^{q-1} H(B_i) + H(C) - H(A, B_1, B_2, \ldots, B_{q-1}, C)
\]

Now since \(\text{codim}_A (A') = \text{codim}_A (A'') + \text{codim}_A (A'' - A)\), from equations \(^{97}\) and \(^{98}\) we have:

\[
\text{codim}_A (A) \leq \text{codim}_Y Y_{1,3}(Y'_{1,3}) + \text{codim}_Y Y_{2,4}(Y'_{2,4}) + \text{codim}_Y Y_{1,1}(Y'_{1,1}) + \cdots + \text{codim}_Y Y_{q-1,1}(Y'_{q-1,1})
\]

\[
+ \text{codim}_Y Y_{6,8}(Y'_{6,8}) + \text{codim}_A (A') + H(A) + \sum_{i=1}^{q-1} H(B_i) + H(C) - H(A, B_1, B_2, \ldots, B_{q-1}, C)
\]

For \(1 \leq i \leq q - 1\) we now find an upper-bound on the co-dimension of a subspace \(C_i \text{ of } C \text{ over which the following identities hold:}

\[
f_{Q_i}f_{W,1}f_{V_i} + f_{E_i} = 0 \quad \text{for } 1 \leq j \leq (q - 1), j \neq i : (f_{Q_j}f_{W,1} + f_{U,ij})f_{V_i} = 0 \quad \text{f}_{D,2}f_{W,1}f_{V_i} - I = 0
\]
Here also notice the similarity between equations (101) and (102); and between equations (102) and (103).

Using equations (72) and (75) we have:

\[ f_{D_2} f_{W_i} f_{V_i} + \sum_{j=1}^{q-1} f_{Q_j} f_{W_i} f_{V_i} = f_{W_i} f_{V_i} \text{ over a subspace } f_{V_i}^{-1} f_{W_i}^{-1} (Y_{2,A}) \text{ of } C \]

\[ f_{W_i} f_{V_i} + \sum_{j=1, j \neq i}^{q-1} f_{U_{ij}} f_{V_i} = f_{V_i} \text{ over a subspace } f_{V_i}^{-1} (Y_{e_i'}) \text{ of } C \]

And from equation (80) we know: \( f_{V_i} + f_{E_i} = I \) over a subspace \( C_i' \) of \( C \)

So over a subspace \( C_i'' \) of \( C \) we have

\[ f_{D_2} f_{W_i} f_{V_i} + \sum_{j=1}^{q-1} f_{Q_j} f_{W_i} f_{V_i} + \sum_{j=1, j \neq i}^{q-1} f_{U_{ij}} f_{V_i} + f_{E_i} = I \text{ where } C_i'' = f_{V_i}^{-1} f_{W_i}^{-1} (Y_{2,A}) \cap f_{V_i}^{-1} (Y_{e_i'}) \cap C_i' \]

So, applying Lemma 11 and Lemma 12 we have:

\[ \text{codim}_{C} (C_i'') \leq \text{codim}_{Y_{2,A}} (Y_{2,A}) + \text{codim}_{V_{e_i'}} (Y_{e_i'}) + \text{codim}_{C} (C_i') \]

Now according to Lemma 14 over a subspace \( \tilde{C}_i \) equations (100), (101), and (102) holds, where

\[ \text{codim}_{C_i''} (\tilde{C}_i) \leq \sum_{j=1}^{q-1} H(B_j) + H(C) - H(B_1, B_2, \ldots, B_{q-1}, C) \]

Now, \( \text{codim}_{C} (\tilde{C}_i) = \text{codim}_{C} (C_i'') + \text{codim}_{C} (\tilde{C}_i) \). So Using equations (103) we have:

\[ \text{codim}_{C} (\tilde{C}_i) \leq \text{codim}_{Y_{2,A}} (Y_{2,A}) + \text{codim}_{V_{e_i'}} (Y_{e_i'}) + \text{codim}_{C} (C_i') + \sum_{j=1}^{q-1} H(B_j) + H(C) - H(B_1, \ldots, B_{q-1}, C) \]

We now form some equations analogous to the equations that were pivotal for the proof lemma 1 Consider the following vector subspaces.

for \( 1 \leq i \leq q - 1 \): \( S_{B_i} = \{ u \in B_i | f_{M_i} f_{K_i} (u) \in f_{D_{mi}} (C) \} \)

(105)

Hence, equation (83) holds over \( S_{B_i} \) when \( f_{D_{mi}} \) is replaced by \( f_{M_i} f_{K_i} \). So over \( S_{B_i} \), we have:

for \( 1 \leq i \leq (q - 1) \): \( (f_{P_{mi}} f_{M_i} + f_{Q_{mi}} f_{M_i}) f_{M_i} f_{K_i} = 0 \)

(106)

Since equation (88) holds over \( B_i \), from equations (88) and (106), over a subspace \( B_i \cap S_{B_i} \) we have:

\[ f_{P_{mi}} f_{M_i} f_{M_i} f_{K_i} = I \]

(107)

Notice the similarity between equation (55) and equation (107).

Now consider the following subspaces.

for \( 1 \leq i \leq q - 1 \): \( R_{B_i} = \{ u \in B_i | f_{M_i} f_{K_i} (u) \in f_{D_{mi}} (\tilde{A}) \} \)

for \( 1 \leq i \leq q - 1 \): \( L_{B_i} = \{ u \in B_i | f_{M_i} f_{M_i} f_{K_i} (u) \in f_{M_i} f_{D_{mi}} (\tilde{A}) \} \)

So \( f_{M_i} f_{K_i} (R_{B_i}) \) is a subspace of \( f_{D_{mi}} (\tilde{A}) \). Then, since from equation (94) \( f_{D_1} \) is invertible over \( f_{M_i} f_{D_{mi}} (\tilde{A}) \); \( f_{D_1} \) is also invertible over \( f_{M_i} f_{M_i} f_{K_i} (R_{B_i}) \). Similarly, \( f_{M_i} f_{M_i} f_{K_i} (L_{B_i}) \) is a subspace of \( f_{M_i} f_{D_{mi}} (\tilde{A}) \). Hence \( f_{D_1} \) is also invertible over \( f_{M_i} f_{M_i} f_{K_i} (L_{B_i}) \). Hence over a subspace \( R_{B_i} \cap L_{B_i} \) from equation (87) we have:

\[ (f_{M_i} f_{M_i} + f_{M_i} f_{M_i}) f_{K_i} = 0 \]

(108)

Applying this equation in equation (88), over a subspace \( B_i \cap R_{B_i} \cap L_{B_i} \) we have:

\[ f_{Q_{mi}} f_{M_i} f_{M_i} f_{K_i} = I \]

(109)

Notice the similarity between equation (49) and equation (109). Now consider the following subspace:

for \( 1 \leq i \leq q - 1 \): \( S_{A_i} = \{ u \in A | f_{R_{mi}} (u) \in f_{V_i} (\tilde{C}_i) \} \)

Hence for \( 1 \leq i, j \leq (q - 1), i \neq j \), \( (f_{Q_{mi}} f_{W_i} + f_{U_{ij}}) f_{R_{mi}} (S_{A_i}) \) is a subspace of \( (f_{Q_{mi}} f_{W_i} + f_{U_{ij}}) f_{V_i} (\tilde{C}_i) \). Hence from equation (101), over \( S_{A_i} \) we have:

for \( 1 \leq j \leq (q - 1), j \neq i \): \( (f_{Q_{mi}} f_{W_i} + f_{U_{ij}}) f_{R_{mi}} = 0 \)

(110)
Applying equation (110) on equation (95), over a subspace \( \cap_{i=1}^{q-1} S_{A_i} \cap \bar{A} \) we have:

\[
\text{for } 1 \leq j \leq q-1 : \quad f_{P_j} f_{M_3} f_{D_6} + f_{Q_j} f_{W_j} f_{R_i} = 0
\]  

(111)

Note the similarity between equations (68) and (111). Let us now consider the following subspaces:

\[
\text{for } 1 \leq i \leq q-1 : \quad L_{A_i} = \{ u \in A | f_{D_6}(u) \in f_{M_3} f_{K_i}(\bar{B}_i \cap S_{B_i}) \}
\]  

(112)

\[
\text{for } 1 \leq i \leq q-1 : \quad R_{A_i} = \{ u \in A | f_{W_i} f_{R_i}(u) \in f_{M_2} f_{M_3} f_{K_i}(\bar{B}_i \cap R_{B_i} \cap L_{B_i}) \}
\]  

(113)

\[
S = \bar{A} \cap (\cap_{i=1}^{q-1} L_{A_i}) \cap (\cap_{i=1}^{q-1} R_{A_i}) \cap (\cap_{i=1}^{q-1} S_{A_i})
\]  

(114)

For any \( a \in S \), from equation (90) we have:

\[
f_{D_3} f_{D_6}(a) + \sum_{i=1}^{q-1} f_{D_2} f_{W_i} f_{R_i}(a) = 0
\]

From (113) we know there exists a \( b_i \in (\bar{B}_i \cap R_{B_i} \cap L_{B_i}) \) such that \( f_{W_i} f_{R_i}(a) = f_{M_2} f_{M_3} f_{K_i}(b_i) \). So,

\[
f_{D_3} f_{D_6}(a) + \sum_{i=1}^{q-1} f_{D_2} f_{M_2} f_{M_3} f_{K_i}(b_i) = 0
\]

From equation (109) we know that \( b_i = f_{Q_i} f_{M_2} f_{M_3} f_{K_i}(b_i) \) for any \( b_i \in (\bar{B}_i \cap R_{B_i} \cap L_{B_i}) \). So,

\[
f_{D_3} f_{D_6}(a) + \sum_{i=1}^{q-1} f_{D_2} f_{M_2} f_{M_3} f_{K_i} f_{Q_i} f_{M_2} f_{M_3} f_{K_i}(b_i) = 0
\]

or. \( f_{D_3} f_{D_6}(a) + \sum_{i=1}^{q-1} f_{D_2} f_{M_2} f_{M_3} f_{K_i} f_{Q_i} f_{W_i} f_{R_i}(a) = 0 \)

Using equation (111) we have:

\[
f_{D_3} f_{D_6}(a) - \sum_{i=1}^{q-1} f_{D_2} f_{M_2} f_{M_3} f_{K_i} f_{P_i} f_{M_3} f_{D_6}(a) = 0
\]

From (112) we know there exists a \( b_i' \in (\bar{B}_i \cap S_{B_i}) \) such that \( f_{D_6}(a) = f_{M_3} f_{K_i}(b_i') \). So,

\[
f_{D_3} f_{D_6}(a) - \sum_{i=1}^{q-1} f_{D_2} f_{M_2} f_{M_3} f_{K_i} f_{P_i} f_{M_3} f_{K_i}(b_i') = 0
\]

From equation (107) we know that \( b_i' = f_{P_i} f_{M_3} f_{M_3} f_{K_i}(b_i') \) for any \( b_i' \in (\bar{B}_i \cap S_{B_i}) \). So,

\[
f_{D_3} f_{D_6}(a) - \sum_{i=1}^{q-1} f_{D_2} f_{M_2} f_{M_3} f_{K_i}(b_i') = 0
\]

Since \( b_i' \in \bar{B}_i \), using equation (90) we have:

\[
f_{D_3} f_{D_6}(a) + \sum_{i=1}^{q-1} f_{D_3} f_{M_3} f_{K_i}(b_i') = 0
\]

or. \( f_{D_3} f_{D_6}(a) + \sum_{i=1}^{q-1} f_{D_3} f_{D_6}(a) = 0 \)

\[
q f_{D_3} f_{D_6}(a) = 0 \tag{115}
\]

We now argue that for equation (115) to hold for any \( a \in S \), \( S \) must be a zero subspace. From equation (94) we know that \( f_{D_6} \) is one-to-one over \( \bar{A} \). From equation (105) we know that \( f_{M_3} f_{K_i}(S_{B_i}) \) for \( 1 \leq i \leq (q-2) \) is a subspace of \( f_{D_6}(C) \). Because of equation (84), \( f_{D_2} f_{M_2} \) is one-to-one over \( f_{D_6}(C) \). So \( f_{D_2} f_{M_2} \) is also one-to-one over \( f_{M_3} f_{K_i}(S_{B_i}) \). Then, from equation (90) it can be concluded that \( f_{D_2} f_{M_2} f_{K_i} \) is one-to-one over \( S_{B_i} \). Now, from (112) we know \( f_{D_6}(S) \) is a subspace of \( f_{M_3} f_{K_i}(\bar{B}_i \cap S_{B_i}) \) for any \( 1 \leq i \leq q - 1 \). So \( f_{D_3} \) is one-to-one over \( f_{D_6}(S) \). Moreover, as a pre-condition, since the characteristic of the finite field does not belong to \( \{p_1, p_2, \ldots, p_t\} \), \( q \neq 0 \) over the finite field. Hence for equation (115) to hold, \( S \) must be a zero subspace. Now,

\[
dim(A) = \dim(A) - \dim(S) = \codim_A(S) = \codim_A(\bar{A} \cap (\cap_{i=1}^{q-1} L_{A_i}) \cap (\cap_{i=1}^{q-1} R_{A_i}) \cap (\cap_{i=1}^{q-1} S_{A_i}))
\]

Applying lemma (11) we have:

\[
dim(A) \leq \codim_A(\bar{A}) + \sum_{i=1}^{q-1} \codim_A(L_{A_i}) + \sum_{i=1}^{q-1} \codim_A(R_{A_i}) + \sum_{i=1}^{q-1} \codim_A(S_{A_i}) \tag{116}
\]
We now calculate some values that would help us in computing a bound over $\text{dim}(A)$.

$$
codim_{B_r}(S_{B_r}) = \text{codim}_B(f^{-1}_{K_r}f^{-1}_{M_r}(f_{B_r}(\tilde{C})))
$$

Applying lemma 12 and noting that from equation 84 $f_{D_n}$ is one-to-one over $\tilde{C}$ we have:

$$
codim_{B_r}(S_{B_r}) \leq \text{codim}_{Y_{5,7}}(f_{D_n}(\tilde{C})) = \text{dim}(Y_{5,7}) - \text{dim}(f_{D_n}(\tilde{C})) = \text{dim}(Y_{5,7}) - \text{dim}(\tilde{C})
$$
or, $\text{codim}_{B_r}(S_{B_r}) \leq H(Y_{5,7}) + \text{codim}_C(\tilde{C}) - H(C)$

(117)

$$
codim_{B_r}(R_{B_r}) = \text{codim}_B(f^{-1}_{K_r}f^{-1}_{M_r}(f_{D_n}(\tilde{A})))
$$

Applying lemma 12 and noting that from equation 94 $f_{D_n}$ is one-to-one over $\tilde{A}$ we have:

$$
codim_{B_r}(R_{B_r}) \leq \text{codim}_{Y_{6,8}}(f_{D_n}(\tilde{A})) = \text{dim}(Y_{6,8}) - \text{dim}(f_{D_n}(\tilde{A})) = \text{dim}(Y_{6,8}) - \text{dim}(\tilde{A})
$$
or, $\text{codim}_{B_r}(R_{B_r}) \leq H(Y_{6,8}) + \text{codim}_A(\tilde{A}) - H(A)$

(118)

$$
codim_{B_r}(L_{B_r}) = \text{codim}_B(f^{-1}_{K_r}f^{-1}_{M_r}f_{D_n}(f_{M_r}f_{D_n}(\tilde{A})))
$$

Applying lemma 12 and noting that from equation 94 $f_{M_r}f_{D_n}$ is one-to-one over $\tilde{A}$ we have:

$$
codim_{B_r}(L_{B_r}) \leq \text{codim}_{Y_{1,3}}(f_{M_r}f_{D_n}(\tilde{A})) = \text{dim}(Y_{1,3}) - \text{dim}(f_{M_r}f_{D_n}(\tilde{A})) = \text{dim}(Y_{1,3}) - \text{dim}(\tilde{A})
$$
or, $\text{codim}_{B_r}(L_{B_r}) \leq H(Y_{1,3}) + \text{codim}_A(\tilde{A}) - H(A)$

(119)

$$
codim_A(S_{A_{i}}) = \text{codim}_A(f^{-1}_{V_i}(f_{V_i}(C_{i})))
$$

Applying lemma 12 and noting from equation 102 that $f_{V_i}$ is one-to-one over $\tilde{C}$ we have:

$$
codim_A(S_{A_{i}}) \leq \text{codim}_{Y_{i}}(f_{V_i}(C_{i})) = \text{dim}(Y_{i}) - \text{dim}(f_{V_i}(C_{i})) = \text{dim}(Y_{i}) - \text{dim}(C_{i})
$$
or, $\text{codim}_A(S_{A_{i}}) \leq H(Y_{i}) + \text{codim}_C(C_{i}) - H(C)$

(120)

$$
codim_A(R_{A_{i}}) = \text{codim}_A(f^{-1}_{V_i}(f_{M_r}f_{K_{i}}(\tilde{B}_{i} \cap R_{B_r} \cap L_{B_r})))
$$

Applying lemma 12 we have:

$$
codim_A(R_{A_{i}}) \leq \text{codim}_{Y_{2,4}}(f_{M_r}f_{K_{i}}(\tilde{B}_{i} \cap R_{B_r} \cap L_{B_r})) = \text{dim}(Y_{2,4}) - \text{dim}(f_{M_r}f_{K_{i}}(\tilde{B}_{i} \cap R_{B_r} \cap L_{B_r}))
$$

From equation 109 we know that $f_{M_r}f_{K_{i}}$, is one-to-one over $\tilde{B}_{i} \cap R_{B_r} \cap L_{B_r}$. So,

$$
codim_A(R_{A_{i}}) \leq H(Y_{2,4}) - \text{dim}(\tilde{B}_{i} \cap R_{B_r} \cap L_{B_r}) = H(Y_{2,4}) + \text{codim}_B(\tilde{B}_{i} \cap R_{B_r} \cap L_{B_r}) - H(B_{i})
$$

Applying lemma 11 and then substituting $\text{codim}_B(R_{B_r})$ and $\text{codim}_B(L_{B_r})$ from equations [118] and [119] we have:

$$
codim_A(R_{A_{i}}) \leq H(Y_{2,4}) + \text{codim}_B(\tilde{B}_{i}) + \text{codim}_B(R_{B_r}) + \text{codim}_B(L_{B_r}) - H(B_{i})
$$
or, $\text{codim}_A(R_{A_{i}}) \leq H(Y_{2,4}) + \text{codim}_B(\tilde{B}_{i}) + H(Y_{6,8}) + \text{codim}_A(\tilde{A}) - H(A) + H(Y_{1,3}) + \text{codim}_A(\tilde{A}) - H(A) - H(B_{i})$
or, $\text{codim}_A(R_{A_{i}}) \leq H(Y_{1,3}) + H(Y_{2,4}) + H(Y_{6,8}) + \text{codim}_B(\tilde{B}_{i}) + 2\text{codim}_A(\tilde{A}) - 2H(A) - H(B_{i})$

(121)

$$
codim_A(L_{A_{i}}) = \text{codim}_A(f^{-1}_{D_i}(f_{M_r}f_{K_{i}}(\tilde{B}_{i} \cap S_{B_r})))
$$

Applying lemma 12 we have:

$$
codim_A(L_{A_{i}}) \leq \text{codim}_{Y_{6,8}}(f_{M_r}f_{K_{i}}(\tilde{B}_{i} \cap S_{B_r})) = \text{dim}(Y_{6,8}) - \text{dim}(f_{M_r}f_{K_{i}}(\tilde{B}_{i} \cap S_{B_r}))
$$

From equation 107 we know that $f_{M_r}f_{K_{i}}$, is one-to-one over $\tilde{B}_{i} \cap S_{B_r}$. So,

$$
codim_A(L_{A_{i}}) \leq H(Y_{6,8}) - \text{dim}(\tilde{B}_{i} \cap S_{B_r}) = H(Y_{6,8}) + \text{codim}_B(\tilde{B}_{i} \cap S_{B_r}) - H(B_{i})
$$

Applying lemma 11 and then substituting $\text{codim}_B(S_{B_r})$ from equation [117] we have:

$$
codim_A(L_{A_{i}}) \leq H(Y_{6,8}) + \text{codim}_B(\tilde{B}_{i}) + \text{codim}_B(S_{B_r}) - H(B_{i})
$$
or, $\text{codim}_A(L_{A_{i}}) \leq H(Y_{6,8}) + \text{codim}_B(\tilde{B}_{i}) + H(Y_{5,7}) + \text{codim}_C(\tilde{C}) - H(C) - H(B_{i})$

(122)

Substituting equation [122], [121], and [120] in equation [116] we have:

$$H(A) \leq (q - 1)(H(Y_{1,3}) + H(Y_{2,4}) + H(Y_{5,7}) + 2H(Y_{6,8}) + \sum_{i=1}^{q-1} H(Y_{e_i}) - 2(q - 1)H(A) - 2(q - 1)H(C) - \sum_{i=1}^{q-1} 2H(B_{i})
$$

+ $(2q - 1)\text{codim}_A(\tilde{A}) + (q - 1)\text{codim}_C(\tilde{C}) + \sum_{i=1}^{q-1} 2\text{codim}_B(\tilde{B}_{i}) + \sum_{i=1}^{q-1} \text{codim}_C(\tilde{C}_{i})$

(123)
Now substituting equations (86), (93), (99) and (104) in equation (123) we have:

\[
H(A) \leq (q - 1)(H(Y_{1,3}) + H(Y_{2,4}) + H(Y_{5,7}) + 2H(Y_{6,8})) + \sum_{i=1}^{q-1} H(Y_{e_i}) - 2(q - 1)H(A) - 2(q - 1)H(C) - \sum_{i=1}^{q-1} 2H(B_i)
\]

+ (7q - 6)codim_{Y_{1,3}}(Y_{1,3}^\prime) + (6q - 5)codim_{Y_{2,4}}(Y_{2,4}^\prime) + \sum_{i=1}^{q-1} (2q)codim_{Y_{e_i}^\prime}(Y_{e_i}^\prime) + (3q - 3)codim_{Y_{5,7}}(Y_{5,7}^\prime)

+ (4q - 3)codim_{Y_{6,8}}(Y_{6,8}^\prime) + (2q - 2)codim_{Y_{9,10}}(Y_{9,10}^\prime) + (2q - 1)codim_{A}(A^\prime) + (q - 1)codim_{C}(C^\prime) + \sum_{i=1}^{q-1} 2\text{codim}_{B_i}(Y_{e_i})

+ \sum_{i=1}^{q-1} \text{codim}_{C}(C_i^\prime) + (5q - 4)(H(A) - H(A, B_1, \ldots, B_{q-1}, C))

+ (6q - 5)(\sum_{i=1}^{q-1} H(B_i) + H(C)) - (q - 1)H(B_1, \ldots, B_{q-1}, C)

Substituting values from equations (71), (72), (73), (74), (79), (78), (77), and (80) we get:

\[
H(A) \leq (q - 1)(H(Y_{1,3}) + H(Y_{2,4}) + H(Y_{5,7}) + 2H(Y_{6,8})) + \sum_{i=1}^{q-1} H(Y_{e_i}) - 2(q - 1)H(A) - 2(q - 1)H(C) - \sum_{i=1}^{q-1} 2H(B_i)
\]

+ (7q - 6)H(Y_{1,3}\mid A, B_1, \ldots, B_{q-1}) + (6q - 5)H(Y_{2,4}\mid B_1, \ldots, B_{q-1}, C) + \sum_{i=1}^{q-1} (2q)H(Y_{e_i}\mid Y_{2,4}, \cup_{j=1,j\neq i}^{q-1} B_j)

+ (3q - 3)H(Y_{5,7}\mid Y_{1,3}, Y_{2,4}) + (4q - 3)H(Y_{6,8}\mid Y_{1,3}, C) + (2q - 2)H(Y_{9,10}\mid Y_{5,7}, Y_{6,8}) + (2q - 1)H(A\mid Y_{6,8}, Y_{e_1}, \ldots, Y_{e_{q-1}})

+ (q - 1)H(C\mid A, Y_{5,7}) + \sum_{i=1}^{q-1} 2H(B_i\mid Y_{9,10}, B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_{q-1}) + \sum_{i=1}^{q-1} H(C\mid Y_{e_i}, B_i)

+ (5q - 4)(H(A) - H(A, B_1, \ldots, B_{q-1}, C)) + (6q - 5)(\sum_{i=1}^{q-1} H(B_i) + H(C)) - (q - 1)H(B_1, \ldots, B_{q-1}, C)

Replacing Y_{1,3} by U, Y_{2,4} by Y, Y_{5,7} by W, Y_{6,8} by X, Y_{e_1} by V, and Y_{9,10} by Z we get the desired inequality (1) of theorem[7]

\[
H(A) \leq (q - 1)(H(U) + H(Y) + H(W) + 2H(X)) + \sum_{i=1}^{q-1} H(V_i) - 2(q - 1)H(A) - 2(q - 1)H(C) - \sum_{i=1}^{q-1} 2H(B_i)
\]

+ (7q - 6)H(U\mid A, B_1, \ldots, B_{q-1}) + (6q - 5)H(Y\mid B_1, \ldots, B_{q-1}, C) + \sum_{i=1}^{q-1} (2q)H(V_i\mid Y, B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_{q-1})

+ (3q - 3)H(W\mid U, Y) + (4q - 3)H(X\mid U, C) + (2q - 2)H(Z\mid W, X) + (2q - 1)H(A\mid X, V_1, \ldots, V_{q-1})

+ (q - 1)H(C\mid A, W) + \sum_{i=1}^{q-1} 2H(B_i\mid Z, B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_{q-1}) + \sum_{i=1}^{q-1} H(C\mid V, B_i)

+ (5q - 4)(H(A) - H(A, B_1, \ldots, B_{q-1}, C)) + (6q - 5)(\sum_{i=1}^{q-1} H(B_i) + H(C)) - (q - 1)H(B_1, \ldots, B_{q-1}, C)

**APPENDIX B**

**A. Proof of lemma[2]**

Consider a \((d, dn)\) fractional linear network coding solution of the network in Fig. 2. The local coding matrices are shown along the edges. The matrices \(Q_i\) for \(1 \leq i \leq n\) and \(A_{ij}\) for \(1 \leq i \leq n, 1 \leq j \leq q\) are of size \(dn \times d\), and left multiplies the massage vector \(a_i\). The matrices \(C_{ij}\) for \(1 \leq i \leq q, 1 \leq j \leq n\), \(B_{ijk}\) for \(1 \leq i, k \leq q, i \neq k, 1 \leq j \leq n\), and \(D_{ij}\) for \(1 \leq i \leq q, 1 \leq j \leq n\) left multiplies \(b_{ij}\) and are of size \(dn \times d\). The matrices \(M_1, M_2, M_3\) and \(K_i, R_i\) and \(U_i\) for \(1 \leq i \leq q\)
are of sizes $dn \times dn$. And the matrices $E_j, G_{ij}$ and $V_j$ for $1 \leq i \leq q$ and $1 \leq j \leq n$ are of sizes $d \times d$. Also let $I_d$ be a $d \times d$ identity matrix. The following comes from the definition of network coding.

$$Y_{ea} = \sum_{i=1}^{n} Q_i a_i + \sum_{i=1}^{q} \sum_{j=1}^{n} C_{ij} b_{ij}$$

(124)

for $1 \leq i \leq q$: $Y_{ei} = \sum_{j=1}^{n} A_{ji} a_j + \sum_{j=1, j \neq i}^{q} \sum_{k=1}^{n} B_{jk} b_{jk}$

(125)

$$Y_{eb} = \sum_{i=1}^{q} \sum_{j=1}^{n} D_{ij} b_{ij}$$

(126)

$$Y_{ea} = M_1 Y_{ea} + M_2 Y_{eb} = \sum_{i=1}^{n} M_1 Q_i a_i + \sum_{i=1}^{q} \sum_{j=1}^{n} (M_1 C_{ij} + M_2 D_{ij}) b_{ij}$$

(127)

for $1 \leq i \leq q$: $Y_{e'i} = K_i Y_{ea} + R_i Y_{eb} = \sum_{j=1}^{n} (K_i Q_j + R_i A_{ji}) a_j + \sum_{k=1}^{n} K_i C_{ik} b_{ik} + \sum_{j=1, j \neq i}^{q} \sum_{k=1}^{n} (K_i C_{jk} + R_i B_{jki}) b_{jki}$

(128)

$$Y_{eb'} = \sum_{i=1}^{q} U_i Y_{ei} + M_3 Y_{eb} = \sum_{i=1}^{n} \sum_{j=1}^{n} U_i A_{ji} a_j + \sum_{j=1, i \neq j}^{q} \sum_{k=1}^{n} (U_i B_{jki} + M_3 D_{jki}) b_{jki}$$

(129)

Because of the demands of the terminals the following inequalities must be satisfied. Since any terminal $t_i \in T_{a_1}$ computes $a_i$, using equation (127) we have, for $1 \leq i, j \leq n, j \neq i$:

$$E_i M_1 Q_i = I$$

(130)

$$E_i M_1 Q_j = 0$$

(131)

Let, $E = [E_1 \ E_2 \ \cdots \ E_n]^T$

and, $Q = [Q_1 \ Q_2 \ \cdots \ Q_n]$

Then, $M_1 Q = [M_1 Q_1 \ M_1 Q_2 \ \cdots \ M_1 Q_n]$

(132)

(133)

(134)

Applying Lemma 9 on equations (130) and (131) and using the matrices in equation (132) and (134) we get:

$$EM_1 Q = I$$

(135)

At $t_k \in T_a$ the component of $b_{ij}$ is zero. So for $1 \leq i \leq q, 1 \leq j, k \leq n$, using equation (127) we have:

$$E_k (M_1 C_{ij} + M_2 D_{ij}) = 0$$

(136)

Let $C_i = [C_{i1} \ C_{i2} \ \cdots \ C_{in}]$

and $D_i = [D_{i1} \ D_{i2} \ \cdots \ D_{in}]$

Then $M_1 C_i + M_2 D_i = [M_1 C_{i1} + M_2 D_{i1} \ M_1 C_{i2} + M_2 D_{i2} \ \cdots \ M_1 C_{in} + M_2 D_{in}]$

(137)

(138)

(139)

From Corollary 10 and equations (136), (132) and (139) we get:

for $1 \leq i \leq q$: $E(M_1 C_i + M_2 D_i) = 0$  

(140)

Now consider the terminals in the set $T_{b_1}$ for $1 \leq i \leq q$. Since at any terminal $t_j \in T_{b_i}$ for $1 \leq j \leq n$ the component of $a_k$ in equation (128) for $1 \leq k \leq n$ is zero, we have:

$$G_{ij}(K_i Q_k + R_j A_{ki}) = 0$$

(141)

Let $G_i = [G_{i1} \ G_{i2} \ \cdots \ G_{in}]^T$

and $A_i = [A_{i1} \ A_{i2} \ \cdots \ A_{in}]$

So, $K_i Q + R_i A_i = [K_i Q_1 + R_i A_{i1} \ K_i Q_2 + R_i A_{i2} \ \cdots \ K_i Q_n + R_i A_{in}]$

(142)

(143)

(144)

Using Corollary 10 and equations (141), (142) and (144) we get:

for $1 \leq i \leq q$: $G_i(K_i Q + R_i A_i) = 0$

(145)
Because $t_j \in T_{b_i}$ computes $b_{ij}$ for $1 \leq i \leq q, 1 \leq j, k \leq n, k \neq j$, from equation (128) we have:

$$G_{ij}(K_iC_{ij}) = I \quad (146)$$
$$G_{ij}(K_iC_{ik}) = 0 \quad (147)$$

From the matrix in (137) we already have: $K_iC_i = [K_iC_{i1} \ K_iC_{i2} \cdots \ K_iC_{in}] \quad (148)$

Using Lemma 9 and equations (146), (147), (142) and (148) we get:

$$G_{ij}(K_iC_{kr} + R_iB_{ki}) = 0 \quad (150)$$

As the component of any $b_{kr}$ at $t_j \in T_{b_i}$ is zero if $k \neq i$, for $1 \leq i, k \leq q, i \neq k, 1 \leq j, r \leq n$ we have:

$$G_{ij}(K_iC_{kr} + R_iB_{ki}) = 0 \quad (151)$$

Let $B_{ki} = [B_{k1i} \ B_{k2i} \cdots B_{kni}]$.

Then $K_iC_k + R_iB_{ki} = [K_iC_{k1} + R_iB_{k1i} \ K_iC_{k2} + R_iB_{k2i} \cdots \ K_iC_{kn} + R_iB_{kni}] \quad (152)$

From Corollary 10 and equations (150), (142) and (152) we have:

$$V_i(\sum_{k=1}^{q} U_kA_{ik}) = 0 \quad (153)$$

We now consider the set $T_{a_2}$. Since the terminal $t_i \in T_{a_2}$ computes $a_i$ we have, for $1 \leq i, j \leq n, j \neq i$, using equation (129) we have:

$$V_i(\sum_{k=1}^{q} U_kA_{jk}) = 0 \quad (154)$$

Let, $V = [V_1 \ V_2 \cdots \ V_n]^T \quad (155)$

Using the matrix in (143): $\sum_{k=1}^{q} U_kA_k = \left[ \sum_{k=1}^{q} U_kA_{1k} \sum_{k=1}^{q} U_kA_{2k} \cdots \sum_{k=1}^{q} U_kA_{nk} \right] \quad (156)$

Applying lemma 9 and equations (144), (155), (156) and (157) we have:

$$V(\sum_{k=1}^{q} U_kA_k) = I \quad (158)$$

The component of $b_{jk}$ is zero at $t_i \in T_{a_2}$ for $1 \leq j \leq q, 1 \leq i, k \leq n$, and hence using equation (129) we have:

$$V_i(\sum_{r=1,r\neq j}^{q} U_rB_{jkr} + M_3D_{jk}) = 0 \quad (159)$$

Using (138) and (151): $\sum_{r=1,r\neq j}^{q} U_rB_{jr} + M_3D_j = \left[ \sum_{r=1,r\neq j}^{q} U_rB_{j1r} + M_3D_{j1} \cdots \sum_{r=1,r\neq j}^{q} U_rB_{jnr} + M_3D_{jn} \right] \quad (160)$

From Corollary 10 and equations (159), (156) and (160) we have, for $1 \leq j \leq q$:

$$V(\sum_{r=1,r\neq j}^{q} U_rB_{jr} + M_3D_j) = 0 \quad (161)$$

The matrices $E, M_1$ and $Q$ are invertible from equation (135). Matrices $G_i, K_i$ and $C_i$ for $1 \leq i \leq q$ are invertible from equation (149). Matrix $V$ is invertible from equation (158). Since $E$ is invertible we have from equation (140):

$$M_1C_i + M_2D_i = 0 \quad (162)$$

As both $M_1$ and $C_i$ are invertible matrices, from equation (162) $M_2$ is an invertible matrix. Since $G_i$ is invertible for $1 \leq i \leq q$, we have from equation (145):

$$K_iQ + R_iA_i = 0 \quad (163)$$
Since both $K_i$ and $Q$ are invertible matrices, their product is a full rank matrix, and hence $R_i$ is an invertible matrix for $1 \leq i \leq q$. Also, from equation (153) we have, for $1 \leq i, k \leq q, i \neq k$:

$$K_iC_k + R_iB_{ki} = 0$$  \hspace{1cm} (164)

And since $V$ is invertible, we have from equation (161), for $1 \leq i \leq q$:

$$\left( \sum_{r=1,r \neq i}^{p} U_r B_{ir} \right) + M_3 D_i = 0$$  \hspace{1cm} (165)

Substituting $D_i$ from equation (162) in equation (165) we get, for $1 \leq i \leq q$:

$$\left( \sum_{r=1,r \neq i}^{q} U_r B_{ir} \right) - M_3 M_2^{-1} M_1 C_i = 0$$

Substituting $B_{ir}$ from equation (164) we get :

$$-( \sum_{r=1,r \neq i}^{q} U_r R_r^{-1} K_r C_i ) - M_3 M_2^{-1} M_1 C_i = 0$$

Substituting $R_r^{-1} K_i$ from equation (163) we get :

$$\left( \sum_{r=1,r \neq i}^{q} U_r A_r Q^{-1} C_i \right) - M_3 M_2^{-1} M_1 C_i = 0$$

Substituting $Q^{-1}$ from equation (135) we get:

$$\left( \sum_{r=1,r \neq i}^{q} U_r A_r E M_1 C_i \right) - M_3 M_2^{-1} M_1 C_i = 0$$

or, \( \left( \sum_{r=1,r \neq i}^{q} U_r A_r E - M_3 M_2^{-1} \right) M_1 C_i = 0 \)

Since $M_1$ and $C_i$ both are invertible, we have :

$$\sum_{r=1,r \neq i}^{q} U_r A_r E - M_3 M_2^{-1} = 0$$  \hspace{1cm} (166)

From equation (158) we have \( \sum_{r=1,r \neq i}^{q} U_r A_r + U_i A_i = V^{-1} \). Substituting this value in equation (166) we get:

$$\left( V^{-1} - U_i A_i \right) E - M_3 M_2^{-1} = 0$$

or, \( V^{-1} - U_i A_i = M_3 M_2^{-1} E^{-1} \)

or, \( U_i A_i = V^{-1} - M_3 M_2^{-1} E^{-1} \)  \hspace{1cm} (167)

Now, substituting equation (167) in equation (158) we get:

$$V \left( \sum_{i=1}^{q} V^{-1} - M_4 M_3^{-1} E^{-1} \right) = I$$

$$\sum_{i=1}^{q} V \left( V^{-1} - M_4 M_3^{-1} E^{-1} \right) = I$$

$$\sum_{i=1}^{q} \left( I - V M_4 M_3^{-1} E^{-1} \right) = I$$

$$q - 1 = q IV M_4 M_3^{-1} E^{-1}$$  \hspace{1cm} (168)

In equation (168), if $q = 0$, then the equation becomes $-I = 0$. Hence $q \neq 0$ is a necessary condition for the network $\mathcal{N}_2^\prime$ to have a rate $\frac{n}{n}$ fractional linear network coding solution. Then, from the fact that an element in a finite field is equal to zero if and only if the characteristic of the finite field divides that element (so $q \neq 0$ if and only if the characteristic of the finite field does not divide $q$), the “only if” part of the proposition is proved.

We now show that $\mathcal{N}_2^\prime$ has a $(1, n)$ fractional linear network coding solution if the $q$ has an inverse in the finite field. Let $\bar{a}_i$ be an $n$-length vector whose $i^{th}$ component is $a_i$ and all other components are zero. Also let $\bar{b}_{ij}$ be an $n$-length vector whose
Due to Lemma 13, the following holds:

\[ Y_{e_a} = \sum_{j=1}^{n} \bar{a}_j + \sum_{i=1}^{q} \sum_{j=1}^{n} \bar{b}_{ij} \]

for \( 1 \leq i \leq q \):

\[ Y_{e_i} = \sum_{j=1}^{n} \bar{a}_j + \sum_{k=1, k \neq i}^{q} \sum_{j=1}^{n} \bar{b}_{kj} \]

\[ Y_{e_b} = \sum_{i=1}^{q} \sum_{j=1}^{n} \bar{b}_{ij} \]

\[ Y_{e_a} - Y_{e_b} = \sum_{j=1}^{n} \bar{a}_j \]

for \( 1 \leq i \leq q \):

\[ Y_{e_i} = Y_{e_a} - Y_{e_i} = \sum_{j=1}^{n} \bar{b}_{ij} \]

\[ Y_{e_b} = q^{-1}\left\{ \sum_{i=1}^{q} Y_{e_i} - (q-1)Y_{e_b} \right\} = q^{-1}\left\{ \sum_{j=1}^{n} \bar{a}_j + (q-1)\left( \sum_{i=1}^{q} \sum_{j=1}^{n} \bar{b}_{ij} \right) \right\} = \sum_{j=1}^{n} \bar{a}_j \]

Let \( \bar{u}(j) \) be a unit row vector of length \( n \) which has \( j^{th} \) component equal to one and all other components are equal to zero. Then from the dot product of \( \bar{u}(j) \) and \( \sum_{j=1}^{n} \bar{a}_j \), message \( a_j \) can be retrieved. Similarly from the dot product of \( \bar{u}(j) \) and \( \sum_{j=1}^{n} \bar{b}_{ij} \), \( b_{ij} \) can be determined.

**B. Proof of Theorem 8**

To obtain this inequality, we apply DFZ method to the network shown in Fig. 2 for \( n = 1 \) and \( q = p_1 \times \cdots \times p_l \). Corresponding to each of the messages in the network, define vector subspaces \( A, B_1, B_2, \ldots, B_q, Y_{e_a}, Y_{e_1}, \ldots, Y_{e_q}, Y_{e_b} \) of a finite dimensional vector space \( V \). Now consider the following linear functions.

\[ f_{Q}, f_{M_i}, f_{M_2}, f_{M_3} : A \rightarrow Y_{e_a} \]

\[ f_{C_1}, f_{C_2}, f_{A_i}, f_{D_1}, f_{B_i} : Y_{e_a} \rightarrow B_i \]

\[ f_{B_{ji}}, f_{U_i} : Y_{e_i} \rightarrow B_j \]

Due to Lemma 13, the following holds:

\[ f_{Q} + \sum_{i=1}^{q} f_{C_i} = I \text{ over a subspace } Y'_{e_a} \text{ of } Y_{e_a} \text{ where } \text{codim}_{Y_{e_a}}(Y'_{e_a}) \leq H(Y_{e_a} | A, B_1, \ldots, B_q) \quad (169) \]

\[ f_{A_i} + \sum_{j=1, j \neq i}^{q} B_{ji} = I \text{ over a subspace } Y'_{e_i} \text{ of } Y_{e_i} \text{ where } \text{codim}_{Y_{e_i}}(Y'_{e_i}) \leq H(Y_{e_i} | A, \cup_{i=1, j \neq i}^{q} B_j) \quad (170) \]

\[ \sum_{i=1}^{q} f_{D_i} = I \text{ over a subspace } Y'_{e_b} \text{ of } Y_{e_b} \text{ where } \text{codim}_{Y_{e_b}}(Y'_{e_b}) \leq H(Y_{e_b} | B_1, \ldots, B_q) \quad (171) \]

\[ f_{M_5} + f_{M_2} = I \text{ over a subspace } A' \text{ of } A \text{ where } \text{codim}_A(A') \leq H(A | Y_{e_a}, Y_{e_b}) \quad (172) \]

\[ f_{K_i} + f_{R_i} = I \text{ over a subspace } B'_i \text{ of } B_i \text{ where } \text{codim}_{B_i}(B'_i) \leq H(B_i | Y_{e_a}, Y_{e_i}) \quad (173) \]

\[ \sum_{i=1}^{q} f_{U_i} + f_{M_3} = I \text{ over a subspace } A'' \text{ of } A \text{ where } \text{codim}_A(A'') \leq H(A | Y_{e_1}, \ldots, Y_{e_q}, Y_{e_b}) \quad (174) \]
Now we have:

\[ f_{Q_i} f_{M_1} + \sum_{i=1}^{q} f_{C_i} f_{M_1} = f_{M_1} \quad \text{over a subspace } f_{M_1}^{-1}(Y_{e_i}') \quad \text{of } A \]

\[ \sum_{i=1}^{q} f_{D_i} f_{M_2} = f_{M_2} \quad \text{over a subspace } f_{M_2}^{-1}(Y_{e_i}') \quad \text{of } A \]

So, \( f_{Q_i} f_{M_1} + \sum_{i=1}^{q} f_{C_i} f_{M_1} + \sum_{i=1}^{q} f_{D_i} f_{M_2} = f_{M_1} + f_{M_2} = I \quad \text{over a subspace } A'' = f_{M_1}^{-1}(Y_{e_i}') \cap f_{M_2}^{-1}(Y_{e_i}') \cap A'. \)

Hence from lemma 11, \( \text{codim}_A(A'') \leq \text{codim}_A(f_{M_1}^{-1}(Y_{e_i}')) + \text{codim}_A(f_{M_2}^{-1}(Y_{e_i}')) + \text{codim}_A(A') \)

or, using lemma 12, \( \text{codim}_A(A'') \leq \text{codim}_{Y_{e_i}}(Y_{e_i}') + \text{codim}_{Y_{e_i}}(Y_{e_i}') + \text{codim}_A(A') \) (175)

So, due to Lemma 14 there exists a subspace \( \bar{A} \) of \( A'' \) over which:

\[ f_{Q_i} f_{M_1} - I = 0 \quad \text{for } 1 \leq i \leq q : f_{C_i} f_{M_1} + f_{D_i} f_{M_2} = 0 \] (176) (177)

where,

\[ \text{codim}_{A''}(\bar{A}) \leq H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q) \]

So, \( \text{codim}_A(\bar{A}) = \text{codim}_A(A'') + \text{codim}_{A''}(\bar{A}) \leq \text{codim}_{Y_{e_i}}(Y_{e_i}') + \text{codim}_{Y_{e_i}}(Y_{e_i}') + \text{codim}_A(A') \)

\[ + H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q) \] (178)

Similarly, note the following.

\[ f_{Q_i} f_{K_i} + \sum_{j=1}^{q} f_{C_j} f_{K_i} = f_{K_i} \quad \text{over a subspace } f_{K_i}^{-1}(Y_{e_i}') \quad \text{of } B_i \]

\[ f_{A_i} f_{R_i} + \sum_{j=1, j \neq i}^{q} f_{B_j} f_{R_i} = f_{R_i} \quad \text{over a subspace } f_{R_i}^{-1}(Y_{e_i}') \quad \text{of } B_i \]

So, \( f_{Q_i} f_{K_i} + f_{A_i} f_{R_i} + f_{C_i} f_{K_i} + \sum_{j=1, j \neq i}^{q} (f_{C_j} f_{K_i} + f_{B_j} f_{R_i}) = I \quad \text{over a subspace } B''_i = f_{K_i}^{-1}(Y_{e_i}') \cap f_{R_i}^{-1}(Y_{e_i}') \cap B_i \) of \( B_i \).

So, from Lemma 11 and Lemma 12 we have: \( \text{codim}_{B_i}(B''_i) \leq \text{codim}_{Y_{e_i}}(Y_{e_i}') + \text{codim}_{Y_{e_i}}(Y_{e_i}') + \text{codim}_{B_i}(B'_i) \)

So according to Lemma 14 there exists a subspace \( \bar{B}_i \) over which the following identities hold:

\[ f_{Q_i} f_{K_i} + f_{A_i} f_{R_i} = 0 \] (179)

\[ f_{C_i} f_{K_i} = I \] (180)

\[ f_{A_i} f_{R_i} + \sum_{j=1, j \neq i}^{q} f_{B_j} f_{R_i} = 0 \] (181)

where,

\[ \text{codim}_{B''_i}(\bar{B}_i) \leq H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q) \]

So, \( \text{codim}_{B_i}(\bar{B}_i) = \text{codim}_{B_i}(B''_i) + \text{codim}_{B''_i}(\bar{B}_i) \)

or, \( \text{codim}_{B_i}(\bar{B}_i) \leq \text{codim}_{Y_{e_i}}(Y_{e_i}') + \text{codim}_{Y_{e_i}}(Y_{e_i}') + \text{codim}_{B_i}(B'_i) + H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q) \) (182)
Similarly we have,

for \(1 \leq i \leq q\): \(f_A, f_{U_i} + \sum_{j=1, j \neq i}^{q} f_{B_j, f_{U_i}} = f_{U_i}\) over a subspace \(f_{U_i}^{-1}(Y_{e_i})\) of \(A\)

\[
\sum_{i=1}^{q} f_{D_i, f_{M_3}} = f_{M_3} \text{ over a subspace } f_{M_3}^{-1}(Y_{e_3}) \text{ of } A
\]

Hence, \(\sum_{i=1}^{q} (f_{A, f_{U_i}} + \sum_{j=1, j \neq i}^{q} f_{B_j, f_{U_i}}) + \sum_{i=1}^{q} f_{D_i, f_{M_3}} = 1\) over a subspace \(A'''' = \cap_{i=1}^{q} f_{U_i}^{-1}(Y_{e_i}) \cap f_{M_3}^{-1}(Y_{e_3}) \cap A''\) of \(A\)

Using Lemma [11] and Lemma [12] we have:

\[
codim_A(A''') \leq \sum_{i=1}^{q} \text{codim}_{Y_{e_i}}(Y_{e_i}) + \text{codim}_{Y_{e_3}}(Y_{e_3}) + \text{codim}_A(A''')
\]

From Lemma [14] over a subspace \(\hat{A}\) of \(A\) we have:

\[
\sum_{i=1}^{q} f_{A, f_{U_i}} = I
\] for \(1 \leq i \leq q\):

\[
\sum_{j=1, j \neq i}^{q} f_{B_j, f_{U_i}} + f_{D_i, f_{M_3}} = 0
\]

where,

\[
codim_{A''''}(\hat{A}) \leq H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q)
\]

So, \(\text{codim}_A(\hat{A}) = \text{codim}_A(A''') + \text{codim}_{A''''}(\hat{A})\)

or, \(\text{codim}_A(\hat{A}) \leq \sum_{i=1}^{q} \text{codim}_{Y_{e_i}}(Y_{e_i}) + \text{codim}_{Y_{e_3}}(Y_{e_3}) + \text{codim}_A(A'') + H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q)\)

One way to find the respective set \(S\) for this proof is to use the proof of lemma [4] in the above subsection — like lemma [1] was used to find the set \(S\) for the proof of theorem [7]. In Section B-C we show the the inequality that would result if we indeed follow this method. However, we have found that using a different technique, which is a generalization of the proof of Theorem 15 in p. 2502 of [9], a tighter upper-bound on the linear coding capacity of \(N_2\) can be obtained. Towards this end, let us define the following subspaces:

\[
A^* = f_{M_1}(\hat{A}) \quad \text{for } 1 \leq i \leq q: \quad B_i^* = f_{K_i}(B_i)
\]

\[
A^{**} = A^* \cap B_1^* \quad B_i^{**} = B_i^* \cap B_2^* \cap \cdots \cap B_3^* \quad \text{for } 2 \leq i \leq q: \quad B_i^{**} = B_i^* \cap B_3^*
\]

\[
A^{***} = f_Q(A^{**}) \quad \text{for } 1 \leq i \leq q: \quad B_i^{***} = f_{K_i}(B_i^{**})
\]

From equation [176] we know that \(f_Q\) is one-to-one over \(A^*\). Then, as \(f_Q f_{M_1}(A^{***}) = A^{***} = f_Q(A^{**})\), we must have \(A^{**} = f_{M_1}(A^{***})\). With similar reasoning we have: \(B_i^{***} = f_{K_i}(B_i^{***})\) for \(1 \leq i \leq q\).

Let us define the following subspaces:

\[
S_a = \{a \in A | f_{M_3}(a) \in f_{M_2}(A^{***})\}
\]

for \(1 \leq i \leq q\): \(S_i = \{a \in A | f_{U_i}(a) \in f_{R_i}(B_i^{***})\}\)

\[
S = \hat{A} \cap S_a \cap S_1 \cap S_2 \cap \cdots \cap S_q
\]

Let \(\hat{a} \in S\). Then \(f_{U_i}((\hat{a})) = f_{R_i}(b_i)\) for some \(b_i \in B_i^{***}\) where \(1 \leq i \leq q\). Also \(f_{M_3}(\hat{a}) = f_{M_2}(a)\) for some \(a \in A^{***}\). So from equations [187] and [188] respectively we have:

\[
\sum_{i=1}^{q} f_{A, f_{R_i}(b_i)} = \hat{a}
\]

and for \(1 \leq i \leq q\):

\[
\sum_{j=1, j \neq i}^{q} f_{B_j, f_{R_i}(b_j)} + f_{D_i, f_{M_2}(a)} = 0
\]
Summing equation (179) for \( 1 \leq i \leq q \) we have:
\[
\sum_{i=1}^{q} (f_Q f_{K_i} + f_A f_{R_i})(b_i) = 0
\]  
(191)

Substituting \( \sum_{i=1}^{q} f_A f_{R_i}(b_i) \) from equation (189) in equation (191) we have:
\[
\sum_{i=1}^{q} f_Q f_{K_i}(b_i) = -\hat{a}
\]  
(192)

interchanging \( i \) and \( j \) in equation (181), and then summing for \( 1 \leq j \leq q, j \neq i \) we have:
\[
\text{for } 1 \leq i \leq q : \quad \sum_{j=1, j\neq i}^{q} (f_C f_{K_j} + f_{B_j} f_{R_j})(b_j) = 0
\]  
(193)

Substituting \( \sum_{j=1, j\neq i}^{q} f_{B_j} f_{R_j}(b_j) \) from equation (190) in equation (193) we have:
\[
\text{for } 1 \leq i \leq q : \quad \sum_{j=1, j\neq i}^{q} f_C f_{K_j}(b_j) - f_{D_i} f_{M_2}(a) = 0
\]  
(194)

Substituting \( f_{D_i} f_{M_2}(a) \) from equation (177) we have:
\[
\text{for } 1 \leq i \leq q : \quad \sum_{j=1, j\neq i}^{q} f_C f_{K_j}(b_j) + f_C f_{M_1}(a) = 0
\]  
(195)

For \( i = 1 \), from equation (195) we get:
\[
\sum_{j=2}^{q} f_C f_{K_j}(b_j) + f_C f_{M_1}(a) = 0
\]
\[
\text{or, } f_C (\sum_{j=2}^{q} f_{K_j}(b_j) + f_{M_1}(a)) = 0
\]
\[
\text{Since } f_{K_1}(b_j) \in (B_i^* \cap B_j^*) \text{ for } 2 \leq j \leq q; \text{ and } f_{M_1}(a) \in (A^* \cap B_1^*); \text{ and as } f_C \text{ is invertible over } B_1^*, \text{ we have:}
\]
\[
\sum_{j=2}^{q} f_{K_j}(b_j) + f_{M_1}(a) = 0
\]  
(196)

For \( 2 \leq i \leq q \), from equation (195) we get:
\[
f_C (\sum_{j=1, j\neq i}^{q} f_{K_j}(b_j) + f_{M_1}(a)) = 0
\]
\[
\text{or, } f_C (f_{K_i}(b_1) - f_{K_i}(b_i)) + \sum_{j=2}^{q} f_{K_j}(b_j) + f_{M_1}(a) = 0
\]
\[
\text{or, } f_C (f_{K_i}(b_1) - f_{K_i}(b_i)) = 0 \quad \text{[using equation (196)]}
\]
\[
\text{or, } f_{K_i}(b_1) - f_{K_i}(b_i) = 0 \quad \text{[Since } f_{K_i}(b_1) \in B_i^* \text{ and } f_C \text{ is invertible over } B_i^*]\]
\[
\text{or, } f_{K_i}(b_1) = f_{K_i}(b_i)
\]  
(197)

Substituting equation (197) in equation (192) we get:
\[
\sum_{i=1}^{q} f_Q f_{K_i}(b_1) = -\hat{a}
\]
\[
\text{or, } q f_Q f_{K_i}(b_1) = -\hat{a}
\]

Since the characteristic of the finite field divides \( q \), we must have \( q = 0 \). So, \( -\hat{a} = 0 \)

Since this is true for any arbitrary \( \hat{a} \in S \), we must have \( S = \{0\} \), which implies \( \dim(S) = 0 \).

Now, \( \dim(A) = \dim(A) - \dim(S) = \codim_A(S) = \codim_A(\hat{A} \cap S_a \cap S_1 \cap S_2 \cap \cdots \cap S_q) \)
\[
\leq \codim_A(\hat{A}) + \codim_A(S_a) + \sum_{i=1}^{q} \codim_A(S_i) \quad \text{[applying lemma 11]}
\]  
(198)
From (186) we have:

\[
\text{codim}_A(S_a) = \text{codim}_A(f_{M_1}^{-1}(f_{M_2}(A^**))) \leq \text{codim}_{Y_a}(f_{M_2}(A^**)) \quad \text{[from lemma 12]}
\]

or, \( \text{codim}_A(S_a) \leq \dim(Y_{a_1}) - \dim(f_{M_2}(A^**)) \quad (199) \)

Since \( f_{M_1}(A^{**}) \) is a subspace of \( B_1^* \), \( f_{C_1}f_{M_1} \) is invertible over \( A^** \); from equation (177) \( f_{D_1}f_{M_2} \), and hence \( f_{M_2} \) is invertible over \( A^{**} \). Then, \( \dim(f_{M_2}(A^{**})) = \dim(A^{**}) \). From eqn. (199) we have:

\[
\text{codim}_A(S_a) \leq \dim(Y_{a_1}) - \dim(A^{**}) = \dim(Y_{a_1}) - \dim(f_Q(A^*)) \quad (199)
\]

Now, \( A^* \) is a subspace of \( f_{M_1}(A) \), and over \( f_{M_1}(A) f_Q \) is invertible because of eqn. (176). So,

\[
\text{codim}_A(S_a) \leq \dim(Y_{a_1}) - \dim(A^{**}) = \dim(Y_{a_1}) - \dim(A \cap B_1^*) = \dim(Y_{a_1}) + \text{codim}_{Y_a}(A^* \cap B_1^*) - \dim(Y_{a_1}) \quad \text{[from lemma 11]}
\]

or, \( \text{codim}_A(S_a) \leq \dim(Y_{a_1}) + \text{codim}_{Y_a}(A^*) + \text{codim}_{Y_a}(B_1^*) - \dim(Y_{a_1}) \)

or, \( \text{codim}_A(S_a) \leq \dim(Y_{a_1}) + \dim(Y_{a_2}) - \dim(A^*) + \dim(Y_{a_2}) - \dim(B_1^*) - \dim(Y_{a_1}) \)

or, \( \text{codim}_A(S_a) \leq \dim(Y_{a_1}) + \dim(Y_{a_2}) - \dim(M_1(A)) - \dim(f_{K_1}(B_1)) \)

or, \( \text{codim}_A(S_a) \leq \dim(Y_{a_1}) + \dim(Y_{a_2}) - \dim(A) - \dim(B_1) \quad (200) \)

for \( 1 \leq i \leq q \), from (187) we have:

\[
\text{codim}_A(S_i) = \text{codim}_A(f_{U_i}^{-1}(f_{R_i}(B_i^{**}))) \leq \text{codim}_{Y_a}(f_{R_i}(B_i^{**})) \quad \text{[from Lemma 12]}
\]

or, \( \text{codim}_A(S_i) \leq \dim(Y_{a_1}) - \dim(f_{R_i}(B_i^{**})) \)

Since \( f_{K_i}(B_i^{**}) \) is a subspace of \( B_i^* \), \( f_{C_i}f_{K_i} \) is invertible over \( B_i^{**} \); from equation (181) \( f_{B_i}f_{R_i} \), and hence \( f_{R_i} \) must be invertible over \( B_i^{**} \). So,

\[
\text{codim}_A(S_i) \leq \dim(Y_{a_1}) - \dim(B_i^{**}) = \dim(Y_{a_1}) - \dim(f_{C_i}(B_i^{**})) \quad (201)
\]

Now, \( B_i^* \) is a subspace of \( f_{K_i}(B_i) \), and over \( f_{K_i}(B_i) f_{C_i} \) is invertible from equation (180). So, for \( 2 \leq i \leq q \) we have:

\[
\text{codim}_A(S_i) \leq \dim(Y_{a_1}) - \dim(B_i^{**}) = \dim(Y_{a_1}) - \dim(B_i^* \cap B_1^*) = \dim(Y_{a_1}) + \text{codim}_{Y_a}(B_i^* \cap B_1^*) - \dim(Y_{a_1}) \quad \text{[using lemma 11]}
\]

\[
= \dim(Y_{a_1}) + \dim(Y_{a_2}) - \dim(B_i^*) + \dim(Y_{a_2}) - \dim(B_1^*) - \dim(Y_{a_1})
\]

\[
= \dim(Y_{a_1}) + \dim(Y_{a_2}) - \dim(f_{K_i}(B_i)) - \dim(f_{K_1}(B_1))
\]

\[
= \dim(Y_{a_1}) + \dim(Y_{a_2}) - \dim(B_i) - \dim(B_1) \quad \text{[using equation (180)]}
\]

for \( i = 1 \) form equation (201) we have:

\[
\text{codim}_A(S_1) \leq \dim(Y_{a_1}) - \dim(B_1^{**}) = \dim(Y_{a_1}) - \dim(B_1^* \cap B_2^* \cap \cdots \cap B_q^*)
\]

\[
= \dim(Y_{a_1}) + \text{codim}_{Y_a}(B_1^* \cap B_2^* \cap \cdots \cap B_q^*) - \dim(Y_{a_1})
\]

\[
\leq \dim(Y_{a_1}) + \sum_{i=1}^{q} \text{codim}_{Y_a}(B_i^*) - \dim(Y_{a_1}) \quad \text{[from lemma 11]}
\]

\[
= \dim(Y_{a_1}) + (q)\dim(Y_{a_1}) - \sum_{i=1}^{q} \dim(B_i^*) - \dim(Y_{a_1})
\]

\[
= \dim(Y_{a_1}) + (q - 1)\dim(Y_{a_1}) - \sum_{i=1}^{q} \dim(f_{K_i}(B_i))
\]

\[
= \dim(Y_{a_1}) + (q - 1)\dim(Y_{a_1}) - \sum_{i=1}^{q} \dim(B_i) \quad \text{[using equation (180)]}
\]

\[
= \dim(Y_{a_1}) + (q - 1)\dim(Y_{a_1}) + \sum_{i=1}^{q} \text{codim}_{B_i}(B_i) - \sum_{i=1}^{q} \dim(B_i) \quad (203)
\]
So, substituting equations (200), (202), and (203) in equation (198) we have:

\[ H(A) \leq \text{codim}_A(\bar{A}) + \text{codim}_A(S_a) + \sum_{i=1}^{q} \text{codim}_A(S_i) \]

or, \( H(A) \leq \text{codim}_A(\bar{A}) + \dim(Y_{e_a}) + \dim(Y_{e_b}) + \text{codim}_A(\bar{A}) + \text{codim}_{B_i}(\bar{B}_1) - \dim(A) - \dim(B_1) + \dim(Y_{e_b}) \)

+ \((q-1)\dim(Y_{e_a}) + \sum_{i=1}^{q} \text{codim}_{B_i}(B_i) - \sum_{i=1}^{q} \dim(B_i) + \sum_{i=2}^{q} \dim(Y_{e_a}) + (q-1)\dim(Y_{e_a}) + \sum_{i=2}^{q} \text{codim}_{B_i}(B_i) \)

+ \((q-1)\text{codim}_{B_i}(\bar{B}_1) - \sum_{i=2}^{q} \dim(B_i) - (q-1)\dim(B_1) \)

or, \( H(A) \leq \text{codim}_A(\bar{A}) + H(Y_{e_a}) + (2q-1)H(Y_{e_a}) + \text{codim}_A(\bar{A}) - H(A) - (q+1)H(B_1) + \sum_{i=1}^{q} H(Y_{e_a}) \)

+ \sum_{i=2}^{q} 2\text{codim}_{B_i}(\bar{B}_1) + (q+1)\text{codim}_{B_i}(\bar{B}_1) - \sum_{i=2}^{q} 2H(B_i) \)

Substituting values from equations (178), (182), and (185) we get:

\[ H(A) \leq \sum_{i=1}^{q} \text{codim}_{Y_{e_i}}(Y_{e_i}) + \text{codim}_{Y_{e_b}}(Y_{e_b}) + \text{codim}_A(A'') + H(Y_{e_b}) + (2q-1)H(Y_{e_a}) + \text{codim}_{Y_{e_a}}(Y_{e_a}') \]

+ \text{codim}_{Y_{e_b}}(Y_{e_b}') + \text{codim}_A(A') - H(A) - (q+1)H(B_1) + \sum_{i=1}^{q} H(Y_{e_i}) + \sum_{i=2}^{q} 2\text{codim}_{Y_{e_a}}(Y_{e_a}') + \text{codim}_{Y_{e_a}}(Y_{e_a}') \]

+ \text{codim}_{B_i}(B_i') + (q+1)(\text{codim}_{Y_{e_a}}(Y_{e_a}') + \text{codim}_{Y_{e_1}}(Y_{e_1}') + \text{codim}_{B_i}(B_i')) - \sum_{i=2}^{q} 2H(B_i) \]

+ \((3q+1)(H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q)) \)

or, \( H(A) \leq (q+2)\text{codim}_{Y_{e_1}}(Y_{e_1}') + \sum_{i=2}^{q} 3\text{codim}_{Y_{e_b}}(Y_{e_b}') + 2\text{codim}_{Y_{e_b}}(Y_{e_b}') + \text{codim}_A(A'') + H(Y_{e_b}) + (2q-1)H(Y_{e_a}) \)

+ \( \text{codim}_{Y_{e_b}}(Y_{e_b}') + \text{codim}_A(A') - H(A) - (q+1)H(B_1) + \sum_{i=1}^{q} H(Y_{e_i}) + \sum_{i=2}^{q} 2\text{codim}_{B_i}(B_i') \)

+ \((q+1)\text{codim}_{B_i}(B_i') - \sum_{i=2}^{q} 2H(B_i) + (3q+1)(H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q)) \)

Substituting values from equations (169), (171), (170), (172), (173) and (174) we get:

or, \( H(A) \leq (q+2)H(Y_{e_1} | A, \cup_{j=2}^{q} B_j) + \sum_{i=2}^{q} 3H(Y_{e_i} | A, \cup_{j=1,j \neq i}^{q} B_j) + 2H(Y_{e_b} | B_1, \ldots, B_q) + H(A | Y_{e_1}, \ldots, Y_{e_i}, Y_{e_b}) \)

+ \( H(Y_{e_b}) + (2q-1)H(Y_{e_a}) + (3q)H(Y_{e_a} | A, B_1, \ldots, B_q) + H(A | Y_{e_b}, Y_{e_a}) - H(A) - (q+1)H(B_1) + \sum_{i=1}^{q} H(Y_{e_i}) \)

+ \sum_{i=2}^{q} 2H(B_i | Y_{e_a}, Y_{e_i}) + (q+1)H(B_1 | Y_{e_a}, Y_{e_i}) - \sum_{i=2}^{q} 2H(B_i) + (3q+1)(H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q)) \)

Replacing \( Y_{e_a} \) by \( X, Y_{e_b} \) by \( Y_i \) and \( Y_{e_b} \) by \( Z \) we get the desired inequality.

\[ H(A) \leq (q+2)H(Y_1 | A, \cup_{j=2}^{q} B_j) + \sum_{i=2}^{q} 3H(Y_i | A, \cup_{j=1,j \neq i}^{q} B_j) + 2H(Z | B_1, \ldots, B_q) + H(A | Y_1, \ldots, Y_q, Z) + H(Z) \]

+ \((2q-1)H(X) + (3q)H(X | A, B_1, \ldots, B_q) + H(A | X, Z) - H(A) - (q+1)H(B_1) + \sum_{i=1}^{q} H(Y_i) + \sum_{i=2}^{q} 2H(B_i | X, Y_i) \]

+ \((q+1)H(B_1 | X, Y_1) - \sum_{i=2}^{q} 2H(B_i) + (3q+1)(H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q)) \)


Rearranging terms we get:

\[ 2H(A) + (q + 1)H(B_1) + \sum_{i=2}^{q} 2H(B_i) \leq (2q - 1)H(X) + \sum_{i=1}^{q} H(Y_i) + H(Z) + (3q)H(X|A, B_1, \ldots, B_q) \]

\[ + (q + 2)H(Y_1|A, \cup_{j=2}^{q} B_j) + \sum_{i=2}^{q} 3H(Y_i|A, \cup_{j=1, j \neq i}^{q} B_j) + 2H(Z|B_1, \ldots, B_q) + H(A|Y_1, \ldots, Y_q, Z) + H(A|X, Z) \]

\[ + (q + 1)H(B_1|X, Y_1) + \sum_{i=2}^{q} 2H(B_i|X, Y_1) + (3q + 1)(H(A) + \sum_{i=1}^{q} H(B_i) - H(A, B_1, \ldots, B_q)) \]

C. Using the proof of lemma [4] to find the set S

We now show that if we had used the proof of lemma [4] for finding the set S, in order to compute a characteristic-dependent rank inequality from the network in Fig. 2 for C. Using the proof of lemma 4 to find the set S.

Multiplying both sides of equation (208) by \( \bar{\mathbf{f}} \) then over a subspace \( \bar{\mathbf{f}} \), this end, we first define some subspaces which will be required to obtain equations analogous to equations (167) and (168).

\[ f \bar{\mathbf{f}} = I \]

Multiplying both sides of equation (209) by \( f_{R_i} \) we have:

\[ f_{M_i} f_{Q} f_{K_i} + f_{M_1} f_{A_i} f_{R_i} = 0 \]

Consider the below subspaces:

\[ \text{for } 1 \leq i \leq q: \quad S_{A_i} = \{ u \in A | f_{M_i}(u) \in f_{K_i}(B_i) \} \]

Now note that over \( f_{K_i}(B_i) \), \( f_{C_i} \) is one-to-one from equation (180), and \( f_{M_i} \) is one-to-one over \( \bar{\mathbf{A}} \) from equation (176). Then, over a subspace \( \bar{\mathbf{A}} \cap S_{A_i} \), \( f_{C_i} f_{M_i} \) is one-to-one. Hence from equation (177), both \( f_{D_i} \) and \( f_{M_i} \) are one-to-one over \( \bar{\mathbf{A}} \cap S_{A_i} \).

Now note that from equation (176), we have, over \( f_{M_i}(\bar{\mathbf{A}}) \):

\[ f_{M_i} f_{Q} = I \]

Then over \( \bar{B}_i \cap S_{B_i} \), from equation (206) we have:

\[ f_{K_i} + f_{M_i} f_{A_i} f_{R_i} = 0 \]

Multiplying both sides of equation (208) by \( f_{C_i} \), we get:

\[ f_{C_i} f_{K_i} + f_{C_i} f_{M_i} f_{A_i} f_{R_i} = 0 \]

or, from equation (180): \( f_{C_i} f_{M_i} f_{A_i} f_{R_i} = -I \)

We define the set \( S \) as following:

\[ S_{\bar{\mathbf{A}}} = \{ u \in \bar{\mathbf{A}} | f_{M_2}(\bar{\mathbf{A}} \cap \bigcap_{i=1}^{q} S_{A_i}) \} \]

\[ R_{\bar{\mathbf{A}}} = \{ u \in \bar{\mathbf{A}} | f_{R_i}(B_i \cap S_{B_i}) \} \]

\[ S = \bar{\mathbf{A}} \cap \bar{\mathbf{A}} \cap \bigcap_{i=1}^{q} S_{A_i} \cap S_{\bar{\mathbf{A}}} \cap \bigcap_{i=1}^{q} R_{\bar{\mathbf{A}}} \]
Let \( \hat{a} \in S \). Then from equation (184), for \( 1 \leq i \leq q \) we have:

\[
\sum_{j=1, j \neq i}^{q} f_{B_i} f_{U_j}(\hat{a}) + f_{D_i} f_{M_3}(\hat{a}) = 0
\]

From (210) we know there exists a \( a \in \tilde{A} \cap \bigcap_{i=1}^{q} S_{A_i} \) such that \( f_{M_3}(\hat{a}) = f_{M_3}(a) \). So,

\[
\sum_{j=1, j \neq i}^{q} f_{B_i} f_{U_j}(\hat{a}) + f_{D_i} f_{M_2}(a) = 0
\]

Substituting \( f_{D_i}, f_{M_2}(a) \) from equation (177) we have:

\[
\sum_{j=1, j \neq i}^{q} f_{B_i} f_{U_j}(\hat{a}) - f_{C_i} f_{M_1}(a) = 0
\]

Since \( f_{M_2} \) is invertible over \( \tilde{A} \cap \bigcup_{i=1}^{q} S_{A_i} \), we can write:

\[
\sum_{j=1, j \neq i}^{q} f_{B_i} f_{U_j}(\hat{a}) - f_{C_i} f_{M_1} f_{M_2}^{-1} f_{M_3}(a) = 0
\]

or,

\[
\sum_{j=1, j \neq i}^{q} f_{B_i} f_{U_j}(\hat{a}) - f_{C_i} f_{M_1} f_{M_2}^{-1} f_{M_3}(\hat{a}) = 0
\]

From (210) we know there exists a \( b_j \in \tilde{B}_j \cap S_{B_j} \) such that \( f_{U_j}(\hat{a}) = f_{R_j}(b_j) \). So,

\[
\sum_{j=1, j \neq i}^{q} f_{B_i} f_{R_j}(b_j) - f_{C_i} f_{M_1} f_{M_2}^{-1} f_{M_3}(\hat{a}) = 0
\]

Substituting \( f_{B_i}, f_{R_j}(b_j) \) from equation (181) we have:

\[
\sum_{j=1, j \neq i}^{q} -f_{C_i} f_{K_j}(b_j) - f_{C_i} f_{M_1} f_{M_2}^{-1} f_{M_3}(\hat{a}) = 0
\]

Substituting \( f_{K_j}(b_j) \) from equation (208) we have:

\[
\sum_{j=1, j \neq i}^{q} f_{C_i} f_{M_1} f_{A_j} f_{R_j}(b_j) - f_{C_i} f_{M_1} f_{M_2}^{-1} f_{M_3}(\hat{a}) = 0
\]

or,

\[
\sum_{j=1, j \neq i}^{q} f_{C_i} f_{M_1} f_{A_j} f_{U_j}(\hat{a}) - f_{C_i} f_{M_1} f_{M_2}^{-1} f_{M_3}(\hat{a}) = 0
\]

or,

\[
f_{C_i} f_{M_1} \left( \sum_{j=1, j \neq i}^{q} f_{A_j} f_{U_j} - f_{M_2}^{-1} f_{M_3} \right)(\hat{a}) = 0
\]

Now from equation (183) we have \( (f_{A_i} f_{U_i} + \sum_{j=1, j \neq i}^{q} f_{A_j} f_{U_j})(\hat{a}) = \hat{a} \). So,

\[
f_{C_i} f_{M_1}(\hat{a}) - f_{A_i} f_{U_i}(\hat{a}) - f_{M_2}^{-1} f_{M_3}(\hat{a}) = 0
\]

Using (210), (209) and (210) we have:

\[
\hat{a} - f_{A_i} f_{U_i}(\hat{a}) - f_{M_2}^{-1} f_{M_3}(\hat{a}) = 0
\]

or,

\[
f_{A_i} f_{U_i}(\hat{a}) = \hat{a} - f_{M_2}^{-1} f_{M_3}(\hat{a})
\]

As equation (210) holds for \( 1 \leq i \leq q \) we have:

\[
\sum_{i=1}^{q} f_{A_i} f_{U_i}(\hat{a}) = q\hat{a} - q f_{M_2}^{-1} f_{M_3}(\hat{a})
\]

Using equation (183): \( \hat{a} = q\hat{a} - q f_{M_2}^{-1} f_{M_3}(\hat{a}) \)

As \( q = 0 \) over the finite field: \( \hat{a} = 0 \)

As this holds for any \( \hat{a} \in S \), we must have \( S = \{0\} \). Now we calculate some values that help us compute an upper-bound over \( \text{dim}(A) \).

\[
\text{codim}_A(S_{A_i}) = \text{codim}_A(f_{M_2}^{-1}(f_{K_i}(B_i))) \leq \text{codim}_{Y_{\hat{a}}}(f_{K_i}(B)) = \text{dim}(Y_{\hat{a}}) + \text{dim}(f_{K_i}(B))
\]

or, \( \text{codim}_A(S_{A_i}) \leq H(Y_{\hat{a}}) + \text{dim}(B) = H(Y_{\hat{a}}) + \text{codim}_{B_i}(B) - H(B_i) \)

(211)
\[ \text{codim}_B(S_{B_i}) = \text{codim}_B(f^{-1}_R(f_{M_i}(A))) \leq \text{codim}_{Y_{e_i}}(f_{M_i}(A)) = H(Y_{e_i}) - \dim(f_{M_i}(A)) \]
\[ \text{codim}_B(S_{B_i}) \leq H(Y_{e_i}) - \dim(A) = H(Y_{e_i}) + \text{codim}_A(A) - H(A) \] (212)

\[ H(A) = \dim(A) - \dim(S) = \text{codim}_A(S) \]
or, \[ H(A) \leq \text{codim}_A(\bar{A}) + \text{codim}_A(A) + \sum_{i=1}^q \text{codim}_A(S_{A_i}) + \text{codim}_A(f^{-1}_M(f_{M_2}(\bar{A} \cap \cap_{i=1}^q S_{A_i}))) \]
\[ + \sum_{i=1}^q \text{codim}_A(f^{-1}_R(\bar{B}_i \cap S_{B_i}))) \]
or, \[ H(A) \leq \text{codim}_A(\bar{A}) + \text{codim}_A(A) + \sum_{i=1}^q \text{codim}_A(S_{A_i}) + \text{codim}_{Y_{e_i}}(\bar{A} \cap \cap_{i=1}^q S_{A_i}) + \sum_{i=1}^q \text{codim}_{Y_{e_i}}(f_{R_i}(\bar{B}_i \cap S_{B_i})) \]

As over \( \bar{A} \cap \cap_{i=1}^q S_{A_i} \), \( f_{M_2} \) is one-to-one, and as from equation (208) over \( \bar{B}_i \cap S_{B_i} \), \( f_{R_i} \) is one-to-one:

or, \[ H(A) \leq \text{codim}_A(\bar{A}) + \text{codim}_A(A) + \sum_{i=1}^q \text{codim}_A(S_{A_i}) + \text{codim}_{Y_{e_i}}(\bar{A} \cap \cap_{i=1}^q S_{A_i}) + \sum_{i=1}^q \text{codim}_{Y_{e_i}}(\bar{B}_i \cap S_{B_i}) \]
or, \[ H(A) \leq \text{codim}_A(\bar{A}) + \text{codim}_A(A) + \sum_{i=1}^q \text{codim}_A(S_{A_i}) + H(Y_{e_i}) + \text{codim}_A(\bar{A} \cap \cap_{i=1}^q S_{A_i}) - H(A) + \sum_{i=1}^q H(Y_{e_i}) \]
\[ + \sum_{i=1}^q \text{codim}_B(\bar{B}_i \cap S_{B_i}) - \sum_{i=1}^q H(B_i) \]
or, \[ H(A) \leq \text{codim}_A(\bar{A}) + \text{codim}_A(A) + \sum_{i=1}^q \text{codim}_A(S_{A_i}) + H(Y_{e_i}) + \text{codim}_A(\bar{A}) + \sum_{i=1}^q \text{codim}_A(S_{A_i}) - H(A) \]
\[ + \sum_{i=1}^q H(Y_{e_i}) + \sum_{i=1}^q \text{codim}_B(\bar{B}_i) + \sum_{i=1}^q \text{codim}_B(S_{B_i}) - \sum_{i=1}^q H(B_i) \]
or, \[ H(A) \leq \text{codim}_A(\bar{A}) + 2\text{codim}_A(\bar{A}) + \sum_{i=1}^q 2\text{codim}_A(S_{A_i}) + H(Y_{e_i}) - H(A) + \sum_{i=1}^q H(Y_{e_i}) + \sum_{i=1}^q \text{codim}_B(\bar{B}_i) \]
\[ + \sum_{i=1}^q \text{codim}_B(S_{B_i}) - \sum_{i=1}^q H(B_i) \]

Substituting values from equations (212) and (211) we have:

or, \[ H(A) \leq \text{codim}_A(\bar{A}) + 2\text{codim}_A(\bar{A}) + \sum_{i=1}^q 2(H(Y_{e_i}) + \text{codim}_B(\bar{B}_i) - H(B_i)) + H(Y_{e_i}) - H(A) + \sum_{i=1}^q H(Y_{e_i}) \]
\[ + \sum_{i=1}^q \text{codim}_B(\bar{B}_i) + \sum_{i=1}^q (H(Y_{e_i}) + \text{codim}_A(\bar{A}) - H(A)) - \sum_{i=1}^q H(B_i) \]
or, \[ H(A) \leq \text{codim}_A(\bar{A}) + (q + 2)\text{codim}_A(\bar{A}) + 3qH(Y_{e_i}) + \sum_{i=1}^q 3\text{codim}_B(\bar{B}_i) - \sum_{i=1}^q 3H(B_i) + H(Y_{e_i}) - (q + 1)H(A) \]
\[ + \sum_{i=1}^q H(Y_{e_i})(213) \]

Now substituting values from equations (178), (182), and (185) it can be seen that when equation (213) is applied to the network \( \mathcal{N}_2 \), it results an upper-bound equal to \( \frac{(4q+1)k}{(4q+2)n} \).

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