Nonequilibrium wetting of finite samples

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Abstract. We study, as a canonical model for wetting far from thermal equilibrium, a Kardar–Parisi–Zhang interface growing on top of a hard-core substrate. Depending on the average growth velocity, the model can exhibit a nonequilibrium wetting transition which is characterized by an additional surface critical exponent $\theta$. Simulating the single-step model in one spatial dimension we provide accurate numerical estimates for $\theta$ and investigate the distribution of contact points between the substrate and the interface as a function of time. Moreover, we study the influence of finite size effects, in particular the time needed for a finite substrate to become completely covered by the wetting layer for the first time.

Keywords: nonequilibrium wetting (theory)

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1. Introduction

When a chemically inert surface is exposed to a gas phase, particles may preferentially attach to the surface, forming a wetting layer of a different phase. By changing physical parameters such as pressure or temperature these systems may be caused to undergo a wetting transition at which the thickness of the layer diverges and becomes macroscopic. In many experimental situations it is reasonable to assume that the wetting layer is in thermal equilibrium with the substrate so that methods of equilibrium statistical physics can be applied [1, 2]. More recently, however, wetting phenomena far from equilibrium attracted considerable attention [3]–[9]. These studies are motivated by the question of whether wetting processes under nonequilibrium conditions, for example caused by a flux of particles or energy, may show physical properties that are qualitatively different from those of equilibrium wetting, especially at or in the vicinity of a wetting transition.

In order to describe a wetting process far from equilibrium one has to resort to a dynamical description of the microscopic processes. This can be done either by introducing explicit solid-on-solid growth models [4, 5] where the microscopical dynamics is described by appropriate rules for deposition and absorption of adatoms over the substrate, or by studying appropriate Langevin equations for nonequilibrium growth [8]. As regards the universal properties, all these studies were so far related to the Kardar–Parisi–Zhang equation [10] in one spatial dimension with an additional potential term [3]:

\[
\frac{\partial h(\vec{x}, t)}{\partial t} = v_0 + \sigma \nabla^2 h(\vec{x}, t) - \frac{\partial V(h(\vec{x}, t))}{\partial h(\vec{x}, t)} + \lambda (\nabla h(\vec{x}, t))^2 + \zeta(\vec{x}, t),
\]

(1)

where \( h(\vec{x}, t) \) is the height of the interface defining the thickness of the growing layer, \( v_0, \sigma \) and \( \lambda \) are real parameters and \( \zeta(\vec{x}, t) \) is a white Gaussian noise. Here \( V(h) \) is a potential that takes the influence of the substrate into account. The nonequilibrium properties of this equation are caused by the nonlinear term \( \lambda (\nabla h)^2 \) which can be shown to be a relevant perturbation in the renormalization group sense that violates detailed balance.
Table 1. Critical exponents of wetting transitions in one spatial dimension.

| $\lambda$  | $\alpha$ | $\beta$ | $z$   | $\theta$ |
|------------|---------|-------|------|--------|
| $\lambda < 0$ | $1/2$   | $1/3$ | $3/2$ | 1.184(10) |
| $\lambda = 0$ | $1/2$   | $1/4$ | $2$   | $3/4$  |
| $\lambda > 0$ | $1/2$   | $1/3$ | $3/2$ | 0.228(5)  |

Obviously this term breaks the reflection symmetry $h \rightarrow -h$ so that in combination with a symmetry-breaking substrate the sign of $\lambda$ is expected to play a significant role. In the special case $\lambda = 0$, however, it can be shown that the dynamics in a stationary state obeys detailed balance and is thus at thermal equilibrium. Hence by varying $\lambda$ we can study the crossover from equilibrium to nonequilibrium wetting.

Without a substrate, i.e., for $V = 0$, a one-dimensional interface evolving according to the KPZ equation is known to roughen as $w(t) \sim t^{\beta}$, where $w(t)$ is the width (defined as the standard deviation of the heights) and $\beta$ is the so-called growth exponent. In a finite system of size $L$ this roughening process continues until the width saturates at $w_s \sim L^\alpha$ when the intrinsic correlation length $\xi \sim t^{1/z}$ reaches the system size. Here $z$ and $\alpha = z\beta$ are the dynamical exponent and the roughness exponent, respectively. In the case of KPZ growth in one spatial dimension, where a fluctuation-dissipation theorem holds, the exponents are given by simple rational values [11]; see table 1.

In order to study nonequilibrium wetting, one has to introduce a substrate by imposing a boundary in the microscopic dynamics or by choosing an appropriate form of the potential appearing in equation (1). The simplest way to introduce such a boundary is to impose the restriction $h(\vec{x}, t) \geq 0$, corresponding to an infinite step potential of the form

$$V(h) = \begin{cases} 0 & \text{if } h \geq 0 \\ \infty & \text{if } h < 0. \end{cases}$$

In order to avoid singularities in the continuous formulation, this infinite step potential is usually replaced in equation (1) by an exponential function

$$V(h) = \exp(-h)$$

that was found to display essentially the same physical properties.

Models defined by the addition of this simple boundary to a KPZ roughening surface are known to exhibit a continuous wetting transition. In an infinite system the critical point of this transition is determined by the asymptotic average velocity $v_\infty = \lim_{t \rightarrow \infty} v(t)$ of a free interface. For $v_\infty > 0$ the interface moves away from the substrate so that it no longer influences the dynamics. For $v_\infty < 0$, however, the interface is driven towards the substrate, approaching a steady state of finite roughness that is characterized by a certain height profile. At the transition point, where $v_\infty = 0$, a scale-invariant behaviour is observed. Obviously, the order parameter that characterizes this transition is the density of sites where the interface touches the substrate:

$$\rho_s(t) = \frac{1}{L} \sum_{i=1}^{L} \delta_{h_i(t),0}$$

or—equivalently—the spatial average $\exp(-h(\vec{x}, t))$ in the continuous formulation.
From a theoretical point of view the influence of a hard-core substrate without additional short-range forces can be interpreted as a boundary condition in the space spanned by $\vec{x}$, $t$ and $h$. As for other critical phenomena, this boundary condition does not change the underlying universality class and the associated bulk exponents, meaning that the process still belongs to the KPZ universality class. However, the substrate as an additional boundary condition adds a new feature, characterized by an order parameter $\rho_s(t)$ which is associated with an additional surface exponent $\theta$. This order parameter was found to decrease with increasing time as a power law [3]:

$$\rho_s(t) \sim t^{-\theta}. \quad (5)$$

The exponent $\theta$ is expected to take three different values depending on the sign of the parameter $\lambda$ of the nonlinear term. In fact, a visual inspection of typical interfaces at criticality already reveals significant differences depending on the sign of $\lambda$. As can be seen in figure 1, for $\lambda < 0$ only few contact points remain, leading to a fast decay of $\rho_s(t)$, while for $\lambda > 0$ the density of contact points decays only slowly.

Quite interestingly, it has recently been discovered, through both numerical simulations and analytical arguments [12,13], that the complete synchronization phase transition (CST) in most spatially extended chaotic dynamical systems also belongs to the same universality class of a bounded KPZ equation with a negative nonlinear term $\lambda$. It can be shown that, when the local dynamics is smooth enough (more precisely: the linearization of the system around the synchronized state correctly describes the propagation of finite perturbations), the critical behaviour of the CST is described by the so-called multiplicative noise (MN) Langevin equation

$$\frac{\partial}{\partial t} n(\vec{x}, t) = a n(\vec{x}, t) - n^2(\vec{x}, t) + D \nabla^2 n(\vec{x}, t) + n(\vec{x}, t) \zeta(\vec{x}, t) \quad (6)$$

where the order parameter $n(\vec{x}, t) \geq 0$ is the coarse grained density of unsynchronized regions. Here $a$ and $D > 0$ are real parameters, and $\zeta(\vec{x}, t)$ is the same noise as in

Figure 1. Upper row: typical spatio-temporal pattern of contact points (pinned sites) between interface and substrate in the SSM (see the text) at criticality, starting with a flat interface at $t = 0$. Lower row: final configuration of the interface at $t = 5000$. Note that for $\lambda < 0$ there are only few contact points, while for $\lambda > 0$ the density of contact points is much higher.
By a Cole–Hopf transformation [3],
\[ h(\vec{x}, t) - \log n(\vec{x}, t). \]

Equation (6) turns exactly into the bounded KPZ equation (1) with \( \lambda = -\sigma = -D \) and an exponential potential as in equation (3). This indicates that the MN equation is just a realization of the KPZ universality class and its nontrivial surface critical behaviour. In particular the order parameter \( n(\vec{x}, t) \) in equation (6) scales in exactly the same way as the density of contact points \( \rho_c(t) \) and therefore its spatial average will decay as \( t^{-\theta} \).

Equation (6) has been studied numerically in a series of papers [14]–[16] and its transition is said to belong to the multiplicative noise I (MN1) class. Moreover, the related multiplicative Langevin equation which can be derived via an inverse Cole–Hopf transformation from the bounded KPZ equation (1) with \( \lambda > 0 \) has been numerically analysed in [17]. The associated transition is said to belong to the multiplicative noise II (MN2) class.

At present, no exact analytical derivation of the surface exponent \( \theta \) in the nonequilibrium case is known. While in [18] the surface exponent has been obtained for both the MN1 and MN2 classes with a mean-field-like approximation suited for one spatial dimension, the presence of a strong-coupling fixed point and of essential divergences makes the exact values of the surface exponent not accessible by known renormalization group techniques. On the other hand, as we shall see, the equilibrium case \( \lambda = 0 \), sometimes referred to as bounded Edward–Wilkinson, yields the simple rational result for the surface exponent \( \theta_{EW} = 3/4 \). In the light of this result, it would be desirable to know whether the nonequilibrium surface exponents also have simple rational values, an indication that the surface critical behaviour is a trivial consequence of the known KPZ bulk properties. In particular, in [19] the value \( \theta = 7/6 \) has been conjectured for \( \lambda < 0 \). Moreover, it is not yet fully clear how the contact points between interface and substrate are distributed and how finite size effects influence the dynamics at the substrate.

As a first step toward a better comprehension of the bounded KPZ class, we present here a detailed analysis of a simple solid on solid model generally belonging to the KPZ universality class, the so-called single-step model (SSM). Once provided with an additional boundary restriction without additional short-range forces, it shows a simple wetting transition belonging to the bounded KPZ universality class [20]. In particular, the simplicity of the model allows for analytical knowledge of its wetting transition critical point \( v_\infty = 0 \). In section 2 the efficiency of its algorithm allows us to improve current numerical estimates of the exponent \( \theta \).

In section 3 we investigate the distribution of intervals between the contact points, identifying significantly different scaling behaviours depending on the sign of \( \lambda \). Finally, in section 4, we study the scaling properties of finite systems, analysing the time that is needed before the interface detaches globally for the first time. This kind of finite size scaling analysis is a common tool in numerical studies of nonequilibrium systems with an absorbing state, such as ones showing directed percolation (DP) [21], where the average absorbing time is known to scale according to the dynamical exponent, \( \tau \sim L^z \). However, supported by numerical results and analytical arguments, our argument says that in the case \( \lambda \leq 0 \) this detachment time does not grow as a power law with the system size. Conclusions are drawn in section 5, while in the appendix we analytically compute the surface exponent when the detailed balance requirement is imposed in the SSM, thus proving the equilibrium result \( \theta_{EW} = 3/4 \).
Figure 2. Single-step model in $1 + 1$ dimensions. Left panel: the simulation starts with a flat interface in the form of a horizontal sawtooth pattern. On selecting a site at a local minimum a diamond (rhombus) is deposited with probability $p$, flipping up the interface by two units. Similarly, selecting a site at a local maximum, a diamond evaporates with probability $1 - p$, flipping the interface downward by two units. Right panel: roughening interface after several deposition and evaporation events.

2. Measurement of the surface exponent in the single-step model

In what follows let us consider the so-called single-step model (SSM) in one spatial dimension, which is probably the simplest and most compelling lattice model for KPZ-type interface growth. In this model the growing interface is represented by a set of integer heights $h_i$ residing at the sites $i = 1, \ldots, L$ of a one-dimensional lattice of length $L$ with periodic boundary conditions, obeying the restriction

$$h_{i+1} - h_i = \pm 1. \quad (8)$$

The model is controlled by a single parameter $p \in [0, 1]$ and evolves by random sequential updates as follows. Choosing a random site $i$ the height $h_i$ is increased by two units with probability $p$ provided that the restriction (8) is not violated. As shown in figure 2 this move can be thought of as depositing a diamond. Similarly, the height $h_i$ is decreased by two units with probability $1 - p$ provided that the restriction (8) is not violated. As usual each local update attempt corresponds to a time increment of $d t = 1/L$.

For $p = 1/2$ the propagation velocity of the interface is zero, detailed balance holds in the stationary state and its dynamics is effectively described by an Edwards–Wilkinson equation [22]. For $p \neq 1/2$, however, the SSM exhibits KPZ growth with $\lambda$ being proportional to $\frac{1}{2} - p$. In this case the propagation velocity is nonzero and depends on the roughness and the average slope of the interface. For example, the initial velocity of a flat interface (a sawtooth pattern as shown in the left panel of figure 2) is given by

$$v(0) = 2p - 1. \quad (9)$$

As time proceeds, the interface roughens and the propagation velocity decreases until it saturates at a certain value $v_L(\infty)$ which depends on the system size. For the KPZ class the so-called excess velocity $v_L(t) - v_L(\infty)$ is known to decay with time as $t^{-2/3}$ [23]. As a major breakthrough, Prähofer and Spohn recently succeeded in computing the rescaled height profile of a roughening KPZ analytically [24].

In contrast to many other KPZ growth models the SSM allows one to calculate the asymptotic growth velocity $v_L(\infty)$ exactly. To this end one identifies segments of positive (negative) slope with particles (vacancies), mapping the single-step model onto a partially asymmetric exclusion process (ASEP) [25] of diffusing particles with density $1/2$. 

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Since the ASEP is known to evolve towards an uncorrelated product state with a current $j = p/2 - 1/4$, one can easily show that the propagation velocity of the interface in a finite system of size $L \gg 1$ tends to

$$v_L(\infty) = \lim_{t \to \infty} v_L(t) = \left( p - \frac{1}{2} \right) \left( 1 + \frac{1}{L} \right).$$

In order to study nonequilibrium wetting we need to introduce a hard-core substrate. As there is no parameter in the SSM to control the asymptotic growth velocity of the interface independently from the KPZ nonlinear term $\lambda$, the substrate itself has to move so that we can approach the wetting transition by tuning its velocity $v_s$. Obviously, this transition takes place even in finite systems and the critical point is exactly given by $v_s = v_L(\infty)$.

To determine the surface exponent $\theta$ we simulated the SSM for $p = 1/2$ as well as for $p = 0, 1$, where the nonlinear effects of the KPZ term are most pronounced. In order to minimize finite size effects we use a very large lattice size $L = 10^6$. For $p = 1/2$ (corresponding to the equilibrium case $\lambda = 0$ in the KPZ equation) the interface velocity at the transition vanishes so that the substrate can be implemented by rejecting all updates that would lead to a negative height. The process is then simulated as usual and the density of sites at zero height is averaged over many independent runs. For $p = 1$ (corresponding to $\lambda < 0$) we advance the substrate in temporal intervals $\Delta t = 2(1 - 1/L)$ by one unit, flipping all sites below the new baseline upward. Finally, for $p = 0$ (corresponding to $\lambda > 0$) the substrate moves backward by one unit every interval $\Delta t$, rejecting all updates that would yield a height below the actual baseline. Since the moving baseline effectively introduces a discrete time step in the algorithm, the density of contact points is measured immediately after (before) moving the wall when $p = 1$ ($p = 0$).

The numerical results are shown in figure 3. Simulating up to $8 \times 10^5$ time steps and averaging over 160 independent runs one obtains convincingly straight lines in a
double-logarithmic plot. Extrapolating local slopes we arrive at the estimates

$$\theta = \begin{cases} 
0.228(5) & \text{for } p = 0 \\
0.750(5) & \text{for } p = 1/2 \\
1.184(10) & \text{for } p = 1.
\end{cases}$$  \hspace{1cm} (11)$$

In order to illustrate the influence of scaling corrections, we multiplied each data set with $t^\theta$ in the right panel of figure 3 so that the curves become horizontal for large $t$. Our results have to be compared to the early numerical estimates $\theta_1 = 1.1(1)$ and $\theta_2 = 0.215(15)$ for the MN1 and MN2 Langevin equations [17]. A more recent numerical study of the MN1 equation yielded $\theta_1 = 1.21(3)$ [16].

The numerical result for $p = 1$ is particularly interesting as it seems to rule out an earlier conjecture by Droz and Lipowski [19]. Analysing a synchronization transition between two coupled lattices of tent maps and making use of a hyperscaling relation between stationary and spreading exponents, they proposed that for $\lambda < 0$ the exponent $\theta$ should be given by the exact value $7/6 \approx 0.1666$. With the accuracy of the present simulations, however, this value clearly lies outside the error margin. To illustrate this difference we plotted $\rho_s(t) t^{7/6}$ in the right panel of figure 3. As can be seen, the curve has a negative slope over at least two decades before finite size effects set in at $t \approx 10^5$.

Our result suggests that in both cases of $\lambda \neq 0$ the exponent $\theta$ is probably not a simple rational number; rather it seems to be irrational. For $\lambda = 0$ ($p = 1/2$), however, our numerical estimate is in fair agreement with $\theta = 3/4$. In fact, as shown in the appendix this value can be proven analytically provided that the two-point $C(\ell)$ function of contact points decays algebraically as $\ell^{-\theta_z}$.

3. Gap distribution of an infinite system

Let us now investigate how the contact points of a critical interface with the substrate are distributed in an infinite system at a given time. Here an important quantity is the gap distribution $G(\ell, t)$ of distances $\ell$ between neighbouring contact points measured at time $t$. As figure 1 suggests, this distribution depends significantly on the sign of $\lambda$.

Our numerical results are shown in figure 4. For $p = 0$ ($\lambda > 0$) one observes an ordinary algebraic decay with a cut-off at the correlation length $\xi \sim t^{1/z}$. This type of decay is the same as that in other nonequilibrium critical phenomena, for example, in DP. For $p \geq 1/2$ ($\lambda \leq 0$), however, the curves first decay algebraically, then reach a plateau, and finally cross over to an exponential cut-off.

The qualitative difference between these two cases is due to the presence of two different length scales, namely, the bulk correlation length $\xi$ and the average gap size $\bar{\ell}$. These two length scales grow with time as

$$\xi(t) \sim t^{1/z} \hspace{1cm} (12)$$

$$\bar{\ell}(t) \sim t^\theta, \hspace{1cm} (13)$$

the latter equation being a consequence of the relation $\bar{\ell}(t) = 1/\rho_s(t)$ which holds for any point distribution on a one-dimensional line [26]. Depending on the values of the exponents $z$ and $\theta$, two different scenarios are encountered:
Figure 4. Panels (A)–(C) show the gap distribution function depending on the gap size measured in the SSM for $p = 0, 1/2$ and $1$ at fixed times $t = 16, 32, 64, \ldots, 8192$. Panel (D) shows a collapse of the data for $\lambda < 0$ according to the scaling form (14). In contrast, panels (E) and (F) demonstrate that this collapse does not work for $\lambda \leq 0$.

- For $\lambda > 0$ ($p < 1/2$) we have $\theta < 1/z$ and thus the average distance between contact points $\ell(t)$ is smaller than the correlation length $\xi(t)$. In this case the gap distribution is a simple power law with an exponential cut-off at $\xi(t)$. As in other critical phenomena this suggests the usual scaling form

$$G(\ell, t) \sim \ell^{-(2-\theta z)} f \left( \frac{\ell}{\ell^{1/z}} \right).$$

In fact, plotting $G(\ell, t) \ell^{2-\theta z}$ versus $\ell/\ell^{1/z}$ one obtains a convincing data collapse, as shown in figure 4(D).

- For $\lambda \leq 0$ ($p \geq 1/2$), where $\theta > 1/z$, the average distance between contact points is larger than the correlation length. In this case the gap distribution function is found to decay initially as $\ell^{-\theta z}$. Reaching the bulk correlation length $\ell \approx \xi(t)$ it crosses over to an exponential function. However, in contrast to the previous case the associated length scale of this exponential function is $\ell \sim t^\theta$; hence in a double-logarithmic graph the curves exhibit a plateau at a level $\propto t^{-\theta}$ extending from $\xi(t)$ to $\ell(t)$, followed by an exponential cut-off.

In the following section we will use these postulated scaling forms for the gap distribution to derive the first detachment time in a finite system, showing that the two cases lead to very different types of finite size scaling. We note that the power law scaling of the
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gap distribution is a postulate, supported by numerical results shown in figure 4 and the generally accepted power law scaling of $\rho_s(t) = 1/\ell(t) \sim t^{-\theta}$. However, to our knowledge a rigorous proof of power law scaling in the gap distribution is still lacking.

4. First detachment in a finite system

We finally turn our attention to finite size scaling analysis, which is a common tool for analysing properties of systems at criticality. We recall that the correct order parameter describing the wetting transition is the density $\rho_s(t)$ of sites pinned to the substrate, as defined in equation (4). Comparison with DP naively leads one to draw an analogy between the pinned sites and the infected sites of systems with an absorbing state (i.e. a state in which the system dynamics get trapped forever). However, in the present case the depinned phase (i.e. the phase in which the substrate is completely covered) is not absorbing in the DP sense. In fact, since the wetting transition takes place at zero relative velocity, fluctuations can easily bring the interface back to the substrate, thus leaving the depinned state. Hence, instead of studying the average survival time $\tau_{\text{abs}}$ (i.e., the average time at which the system becomes trapped in an absorbing state), here we are forced to consider merely the first depinning time $\tau_f$, i.e., the average time at which an initially exposed substrate gets completely covered for the first time.

While in systems with an absorbing phase one expects the average survival time in a system of size $L$ to scale as $\tau_{\text{abs}} \sim L^z$, where $z$ is the dynamical exponent, our numerical results will show an apparent power law behaviour $\tau_f \sim L^\gamma$, where the exponent $\gamma$ depends on the sign of $\lambda$ and is generally different from the KPZ dynamical exponent. However, a more detailed analysis of the first depinning time probability distribution $P_L(t)$ will show that at least for $\lambda \leq 0$ the scaling form

$$P_L(t) \sim L^{-\gamma} g(t/L^z)$$

is not satisfied, thus indicating that the observed power law behaviour does not hold asymptotically.

4.1. Numerical results

In order to measure the first depinning time $\tau_f$, we used the single-step model with a moving hard-core substrate as described above. To test finite size scaling we simulated various system sizes ranging from $L = 64$ to 16384.

For each realization we determined the time $t$ at which the interface was completely depinned for the first time. The resulting histograms of the probability distribution $P_L(t)$ of first depinning times are displayed in figure 5. Moreover, we calculated the average depinning time for different system sizes and the relative width of the characteristic distributions:

$$\sigma_{\text{rel}} = \frac{\Delta \tau_f}{\bar{\tau}_f}$$

where $\Delta \tau_f$ is the standard deviation of first depinning times. The dependences of both quantities on $L$ are shown in figure 6. Our numerical results can be summarized as follows:
Figure 5. Panels (A)–(C) show the distributions of the first depinning times for \( p = 0, 0.5 \) and 1. The system size increases from left to right starting with \( L = 64 \) and going up to \( L = 1024, 2048 \) and 16384 respectively. In panels (D)–(F) we test a collapse of the above distributions according to equation (15). Only for \( p = 0 \) does the collapse work convincingly.

- In the case of \( \lambda > 0 \) we simulated the SSM with \( p = 0 \) (i.e., no absorption just desorption), averaging over \( 1.56 \times 10^6 \) realizations for each system sizes up to \( L = 1024 \). We find that there is a strong indication of an ordinary power law behaviour. The average depinning time scales with \( L^\gamma \), with \( \gamma = 1.41(2) \), while the probability distributions \( P_L(t) \) have a constant relative width and, as is shown in figure 5 (panel (D)), they can be cleanly collapsed according to equation (15).

- To study the case \( \lambda = 0 \) and \( \lambda < 0 \) we simulated the SSM with \( p = 0.5 \) and 1 for system sizes ranging from \( L = 64 \) to 2048 (\( p = 0.5 \)) or to \( L = 16384 \) (\( p = 1 \)), averaging over \( 8 \times 10^5 \) realizations. In these cases too, the average depinning time \( \tau_f \) shows an apparent power law scaling. However, the relative width \( \sigma_{\text{rel}} \) of the probability distributions shrinks with increasing system size so that the probability distribution curves cannot be convincingly collapsed, making it obvious that we have no ordinary scaling behaviour (see figures 5 and 6). As will be shown in the next section, this rather unconventional behaviour can be traced back to the particularities of the gap distribution.

4.2. Analytical arguments for the case \( \lambda \leq 0 \)

In the following we show how the unconventional behaviour observed for \( \lambda \leq 0 \) can be explained analytically in terms of the gap distribution function. The calculation is based on the observation that for \( \lambda \leq 0 \) the moment of first detachment in a finite system typically occurs long before the correlation length \( \xi(t) \) reaches the system size. This
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Figure 6. Left: mean depinning time $\tau_f$ plotted against the system size $L$. Right: relative width $\sigma_{\text{rel}}$ of the distributions as a function of $L$. As can be seen, the relative width stays constant for $p = 0$, in contrast to the cases for $p = 0.5$ and 1. The solid lines represent the theoretical predictions for the cases $p = 0.5$ (blue) and $p = 1$ (green) according to equation (22).

means that the interface may be thought of as being divided into $L/\xi(t)$ uncorrelated segments, each of them touching the substrate with a small probability (cf figure 7). The moment of first detachment is then interpreted as that of the first fluctuation in which all these segments happen to detach independently. This assumption justifies the following approximations:

(a) The distribution of first detachment times in a finite sample of size $L$ with periodic boundary conditions approximately coincides with the corresponding distribution of first detachment for an arbitrarily chosen segment of size $L$ in an infinite system.

(b) The distributions of contact points at different times are assumed to be uncorrelated.

These two assumptions allow us to use the gap distribution investigated in section 3 and to compute the distribution of first detachment times analytically. While assumption (a) is reasonable for systems much larger than the correlation length, assumption (b) is crucial as it neglects all temporal correlations of contact points.

Obviously, the probability of a segment of size $L$ in an infinite system detaching at time $t$ is given by

$$Q^d_L(t) = \frac{\int_0^\infty d\ell \left( \ell - L \right) G(\ell, t)}{\int_0^\infty d\ell \ell G(\ell, t)},$$

where $G(\ell, t)$ is the gap distribution discussed in section 3. Consequently, the probability $Q^n_L(t)$ that the segment never detached up to time $t$ is given by the product of $1 - Q^d_L(t)\, dt$ over all time steps or—in a continuous formulation—by an exponentiated integral

$$Q^n_L(t) = \exp\left( -k \int_0^t \, dt' Q^d_L(t') \right)$$

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where $k$ is a constant to be fitted to the numerical data. The quantity of interest, namely, the probability of first detachment, is then given by

$$P_L(t) = -\frac{d}{dt} Q^d_L(t). \quad (19)$$

An examination of these expressions reveals that the probability of first detachment is determined by the occasional occurrence of very large gaps. This means that $P_L(t)$ is governed by the right edge of the plateau shown in figures 4(B) and (C) while the initial power law decay of the gap distribution has virtually no influence. Therefore, we may approximate the gap distribution function by

$$G(\ell, t) \simeq A t^{-\theta} \exp \left( -B \frac{\ell}{t^\theta} \right) \quad (20)$$

where $A$ and $B$ are nonuniversal constants. With this ‘exponential’ approximation the above integrals can be solved, giving

$$Q^d_L(t) = \exp \left( -B \frac{L}{t^\theta} \right) \quad (21)$$

and

$$P_L(t) = k \exp \left( -B \frac{L}{t^\theta} - \frac{k(BL)^{1/\theta}}{\theta} \Gamma(-1/\theta, BL/t^\theta) \right). \quad (22)$$

where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^t \, dt$ is the incomplete gamma function. Here the scaling-invariant combination $L/t^\theta$ appears in two places whereas the prefactor of the gamma function is proportional to $L^{1/\theta}$. Obviously it is this prefactor which breaks scaling invariance so that $P_L(t)$ does not satisfy the scaling form (15).

In figure 6 the average depinning time and the relative width computed numerically from equation (22) for $p = 1, 1/2$ are found to be in good agreement with numerical data, while figure 8 shows a direct comparison between the analytical and the numerically derived probability distribution functions for the first depinning times. The parameters $B$ and $k$ have been determined by fitting to the numerical data and to grant normalization. Discrepancies between theory and simulations mainly occur for small system sizes and can be traced back to the above approximation of the gap distribution by an exponential
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Figure 8. Overlay of analytical predictions for the first depinning time probability distribution for $\lambda \leq 0$ ($p = 0.5$ and $1$) and numerical results of our simulation for selected system sizes.

function in equation (18). As can be seen in figures 4(B) and (C) the initial power law part of the gap distribution function neglected in the theory is expected to play a more pronounced role for small system sizes. However, for large system sizes, where the ‘exponential’ approximation used to derive equation (21) performs better, the agreement steadily improves.

5. Conclusions

Nonequilibrium wetting can be modelled by a KPZ growth process combined with a hard-core substrate that gives rise to an additional potential term in the KPZ equation. Adding such a term breaks up–down symmetry and leads to nontrivial surface features, which depend on the sign of the parameter $\lambda$ of the KPZ nonlinear term. For numerical studies of nonequilibrium wetting transitions we advocate the use of the single-step model because of its simplicity and efficiency.

With large-scale simulations presented in this paper we have improved numerical estimates of the surface exponent $\theta$. These estimates strongly suggest that $\theta$ takes nontrivial irrational values for $\lambda \neq 0$. In particular we can rule out the simple rational value $\theta = 7/6$ for $\lambda < 0$ suggested by Droz and Lipowski. The failure of their argument can probably be traced back to the absence of an absorbing state in the bounded KPZ transition, questioning the feasibility of spreading analysis and hyperscaling relations used in absorbing phase transitions in the present case. However, in the equilibrium case ($\lambda = 0$) our results confirm the simple rational value $\theta = 3/4$, in agreement with analytical arguments.

In order to investigate finite size scaling of wetting processes described by the bounded KPZ equation we introduced the average first depinning time as opposed to the average survival time in absorbing phase transitions. In one spatial dimension this naturally leads one to consider the distribution of gaps between pinned sites. We find that two different length scales are shaping the surface dynamics, namely, the spatial correlation length $\xi(t) \sim t^{1/z}$ and the average gap length $\bar{\ell}(t) \sim t^{\theta}$. If this latter length is larger than $\xi(t)$, as
in the case \( \lambda \leq 0 \), gaps between contact points effectively behave as if uncorrelated, thus breaking scale invariance for the first depinning times. This result comes as a surprise if one naively considers the analogy with absorbing phase transitions, where the scaling between temporal and spatial quantities is simply dictated by the dynamical exponent \( z \).

The case of \( \lambda > 0 \) on the other hand, shows a clean finite size scaling behaviour of the first depinning times. Since in this latter case the spatial correlation length is larger than the average gap size, the analytical arguments presented in section 4.2 no longer apply so that we are not able to predict the shape of the first depinning time probability distribution \( P_L(t) \). However, we note that our numerical result \( \gamma = 1.41(2) \) is less than 10% off the known value for the KPZ dynamical exponent \( z = 1.5 \). Since it has been proven in [17] that spatial and temporal correlations are not asymptotically altered by the additional boundary condition, we expect \( \gamma \) to converge gradually versus \( z \) in the limit \( L \to \infty \). However, for large but finite systems within the numerically accessible range the estimates may deviate.

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Appendix. Derivation of the exponent \( \theta \) in the equilibrium case

In this appendix we derive the exponent \( \theta \) for the equilibrium case \( p = 1/2 \) in the SSM analytically. In this case all rates of transition between mutually reachable interface configurations are equal. The system thus approaches a stationary equilibrium state obeying detailed balance in which all interface configurations compatible with the restrictions \( |h_i - h_{i+1}| = 1 \) and \( h_i \geq 0 \) occur with the same probability. Although in an infinite system this state is never reached, the conditional correlation function

\[
c(\ell) = \frac{\langle \delta_{h_i,0} \delta_{h_{i+\ell},0} \rangle}{\langle \delta_{h_i,0} \rangle},
\]

where \( \langle \cdot \rangle \) denotes the spatial average over the index \( i \), is a well-defined quantity. This correlation function may be interpreted as the conditional probability of finding site \( i + \ell \) at zero height given that site \( i \) is a contact point, too. As the interface roughens, the numerator and denominator tend to zero but their quotient tends to a well-defined nonzero value.

The present proof relies on the assumption that this two-point function—like other two-point functions in scale-free situations—decays according to the power law

\[
c(\ell) \sim \ell^{-\theta z},
\]

where \( z = 2 \) is the dynamical exponent. As will be shown, the exponent \( \theta \) can be computed by deriving an exact expression for \( c(\ell) \).
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Adopting a transfer matrix approach used in [27, 28], the correlation function (A.1) may be expressed as

\[ C(\ell) = \frac{\langle 1 | T^\ell | 1 \rangle}{\Lambda^\ell}, \quad (A.3) \]

where \( T \) is an infinite dimensional transfer matrix with matrix elements

\[ T_{h,h'} = \begin{cases} 1 & \text{if } |h - h'| = 1 \\ 0 & \text{otherwise} \end{cases} \quad h, h' \in \{0, 1, 2, \ldots\} \quad (A.4) \]

while \( \Lambda = -2 \) is its dominating eigenvalue and \( \langle 1 | = | 1 \rangle^\dagger = (1, 0, 0, \ldots) \) denotes the vector representing height zero. Because of the special structure of \( T \), where only the secondary diagonals are occupied, the evaluation of equation (A.3) reduces to counting all possible paths of the interface from site \( i \) to site \( i + \ell \). The number of such paths is given by Catalan numbers, leading to the exact solution

\[ C(\ell) = \begin{cases} \frac{\ell!}{(\ell/2)! (\ell/2 + 1)! 2^\ell} & \text{if } \ell \text{ even} \\ 0 & \text{if } \ell \text{ odd} \end{cases} \quad (A.5) \]

Inserting Stirling’s formula \( n! \approx \sqrt{2\pi n} n^{n+1/2} e^{-n} \), it is straightforward to show that this correlation function decays asymptotically as

\[ c(\ell) \approx \sqrt{\frac{8}{\pi}} \ell^{-3/2}. \quad (A.6) \]

Taking this together with the known exponent \( z = 2 \) we thus arrive at \( \theta = 3/4 \).

References

[1] Dietrich S, 1988 Phase Transitions and Critical Phenomena vol 12, ed C Domb and J L Lebowitz (London: Academic) p 1
[2] Fisher D S and Huse D A, 1985 Phys. Rev. B 32 247
Kroll D M and Lipowsky R, 1982 Phys. Rev. B 26 5289
Brézin E, Halperin B I and Leibler A, 1983 Phys. Rev. Lett. 50 1387
[3] Tu Y, Grinstein G and Muñoz M A, 1997 Phys. Rev. Lett. 78 274
Muñoz M A and Hwa T, 1998 Europhys. Lett. 41 147
[4] Hinrichsen H, Livi R, Mukamel D and Politi A, 1997 Phys. Rev. Lett. 79 2710
[5] Hinrichsen H, Livi R, Mukamel D and Politi A, 2000 Phys. Rev. E 61 R1032
[6] Giada L and Marsili M, 2000 Phys. Rev. E 62 6015
[7] Candia J and Albano E V, 2000 Eur. J. Phys. B 16 531
[8] de los Santos F, da Gama M M T and Muñoz M A, 2002 Europhys. Lett. 57 803
de los Santos F, da Gama M M T and Muñoz M A, 2003 Phys. Rev. E 67 021607
[9] Hinrichsen H, Livi R, Mukamel D and Politi A, 2003 Phys. Rev. E 68 041106
[10] Kardar M, Parisi G and Zhang Y-C, 1986 Phys. Rev. Lett. 56 889
[11] Barabási A-L and Stanley H E, 1995 Fractal Concepts in Surface Growth (Cambridge: Cambridge University Press)
[12] Baroni L, Livi R and Torcini A, 2001 Phys. Rev. E 63 036226
[13] Ahlers V and Pikovsky A, 2002 Phys. Rev. Lett. 88 254101
[14] Tu Y, Grinstein G and Muñoz M A, 1997 Phys. Rev. Lett. 78 274
[15] Muñoz M A, de los Santos F and Achalbar A, 2003 Braz. J. Phys. 33 443
[16] Dornic I, Chate H and Muñoz M A, 2004 Preprint cond-mat/0404105
[17] Muñoz M A, Nonequilibrium phase transitions and multiplicative noise, 2004 Advances in Condensed Matter and Statistical Mechanics ed E Korutcheva and R Cuerno (New York: Nova Science Publishers)
Nonequilibrium wetting of finite samples

[18] Ginelli F and Hinrichsen H, 2004 J. Phys. A: Math. Gen. 37 11085
[19] Droz M and Lipowski A, 2003 Phys. Rev. E 67 056204
[20] Ginelli F, Ahlers V, Livi R, Mukamel D, Pikovsky A, Politi A and Torcini A, 2003 Phys. Rev. E 68 065102(R)
[21] Hinrichsen H, 2000 Adv. Phys. 49 815 [cond-mat/0001070]
[22] Edwards S F and Wilkinson D R, 1982 Proc. R. Soc. A 381 17
[23] Krug J and Meakin P, 1990 J. Phys. A: Math. Gen. 23 L987
[24] Prähofer M and Spohn H, 2000 Phys. Rev. Lett. 84 4882
Prähofer M and Spohn H, 2000 Physica A 279 342
Prähofer M and Spohn H, 2002 J. Stat. Phys. 108 1071
Prähofer M and Spohn H, 2004 J. Stat. Phys. 115 255
[25] Liggett T M, 1985 Interacting Particle Systems (New York: Springer)
[26] See for instance the brief discussion in: Ginelli F, Hinrichsen H, Livi R, Mukamel D and Politi A, 2005 Phys. Rev. E 71 026121
[27] van Leeuwen J M J and Hilhorst H J, 1981 Physica A 107 318
[28] Burkhardt T W, 1981 J. Phys. A: Math. Gen. 14 L63

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