Communications in Analysis and Geometry
Volume 26, Number 4, 857–886, 2018

Hitchin’s equations on a nonorientable manifold

Nan-Kuo Ho, Graeme Wilkin, and Siye Wu

We define Hitchin’s moduli space $M_{\text{Hitchin}}(P)$ for a principal bundle $P$, whose structure group is a compact semisimple Lie group $K$, over a compact non-orientable Riemannian manifold $M$. We use the Donaldson-Corlette correspondence, which identifies Hitchin’s moduli space with the moduli space of flat $K^C$-connections, which remains valid when $M$ is non-orientable. This enables us to study Hitchin’s moduli space both by gauge theoretical methods and algebraically by using representation varieties. If the orientable double cover $\tilde{M}$ of $M$ is a Kähler manifold with odd complex dimension and if the Kähler form is odd under the non-trivial deck transformation $\tau$ on $\tilde{M}$, Hitchin’s moduli space $M_{\text{Hitchin}}(\tilde{P})$ of the pullback bundle $\tilde{P} \to \tilde{M}$ has a hyper-Kähler structure and admits an involution induced by $\tau$. The fixed-point set $M_{\text{Hitchin}}(\tilde{P})^\tau$ is symplectic or Lagrangian with respect to various symplectic structures on $M_{\text{Hitchin}}(\tilde{P})$. We show that there is a local diffeomorphism from $M_{\text{Hitchin}}(P)$ to $M_{\text{Hitchin}}(\tilde{P})^\tau$. We compare the gauge theoretical constructions with the algebraic approach using representation varieties.

1 Introduction 858
2 The gauge-theoretic perspective 862
3 The representation variety perspective 872
4 Comparison of representation variety and gauge theoretical constructions 880
References 883

2010 Mathematics Subject Classification: 53D30, 58D27.
Key words and phrases: moduli spaces, non-orientable manifolds, symplectic and hyper-Kähler geometry, representation varieties.
1. Introduction

Let $M$ be a compact orientable Riemannian manifold and let $K$ be a connected compact Lie group. Given a principal $K$-bundle $P \to M$, let $\mathcal{A}(P)$ be the space of connections and let $\mathcal{G}(P)$ be the group of gauge transformations on $P$. Consider Hitchin’s equations

\begin{equation}
F_A - \frac{1}{2} [\psi, \psi] = 0, \quad d_A \psi = 0, \quad d_A^* \psi = 0
\end{equation}

on the pairs $(A, \psi) \in \mathcal{A}(P) \times \Omega^1(M, \text{ad} P)$. Hitchin’s moduli space $\mathcal{M}^{\text{Hitchin}}(P)$ is the set of space of solutions $(A, \psi)$ to (1.1) modulo $\mathcal{G}(P)$ \cite{15, 29}. On the other hand, let $G = K^\mathbb{C}$ be the complexification of $K$ and let $P^\mathbb{C} = P \times_K G$, which is a principal bundle with structure group $G$. The moduli space $\mathcal{M}^{\text{dR}}(P^\mathbb{C})$ of flat $G$-connections on $P^\mathbb{C}$, also known as the de Rham moduli space, is the space of flat reductive connections of $P^\mathbb{C}$ modulo $\mathcal{G}(P^\mathbb{C})$. A theorem of Donaldson \cite{8} and Corlette \cite{7} states that the moduli spaces $\mathcal{M}^{\text{Hitchin}}(P)$ and $\mathcal{M}^{\text{dR}}(P^\mathbb{C})$ are homeomorphic. The smooth part of $\mathcal{M}^{\text{Hitchin}}(P)$ is a Kähler manifold with a complex structure $\overline{J}$ induced by that on $G$.

Suppose in addition that $M$ is a Kähler manifold. Then there is another complex structure $\overline{I}$ on $\mathcal{M}^{\text{Hitchin}}(P)$ induced by that on $M$, and a third one given by $K = IJ$. The three complex structures $\overline{I}, \overline{J}, \overline{K}$ and their corresponding Kähler forms $\overline{\omega}_I, \overline{\omega}_J, \overline{\omega}_K$ form a hyper-Kähler structure on (the smooth part of) $\mathcal{M}^{\text{Hitchin}}(P)$ \cite{15, 29}. This hyper-Kähler structure comes from an infinite dimensional version of a hyper-Kähler quotient \cite{16} of the tangent bundle $TA(P)$, which is hyper-Kähler, by the action of $\mathcal{G}(P)$, which is Hamiltonian with respect to each of the Kähler forms $\omega_I, \omega_J, \omega_K$ on $TA(P)$. When $M$ is a compact orientable surface, Hitchin’s moduli space $\mathcal{M}^{\text{Hitchin}}(P)$ is equal to the hyper-Kähler quotient $\mathcal{M}^{\text{HK}}(P)$ := $TA(P) / \mathcal{G}(P)$ \cite{15}. It plays an important role in mirror symmetry and geometric Langlands program \cite{14, 21}. When $M$ is higher dimensional, $\mathcal{M}^{\text{Hitchin}}(P)$ is a hyper-Kähler subspace in $\mathcal{M}^{\text{HK}}(P)$ \cite{29}.

For a compact Lie group $K$, the moduli space of flat $K$-connections on a compact orientable surface was already studied in a celebrated work of Atiyah and Bott \cite{1}. When $M$ is a compact, nonorientable surface, the moduli space of flat $K$-connections was studied in \cite{17, 19} through an involution on the space of connections over its orientable double cover $\tilde{M}$, induced by lifting the deck transformation on $\tilde{M}$ to the pull-back $\tilde{P} \to \tilde{M}$ of the given $K$-bundle $P \to M$ so that the quotient of $\tilde{P}$ by the involution is the original bundle $P$ itself. This involution acts trivially on the structure group $K$. If
Hitchin’s equations on a nonorientable manifold

instead one considers an involution on the bundle over \( \tilde{M} \) that acts nontrivially on the fibers (such as the complex conjugation), then the fixed points give rise to the moduli space of real or quaternionic vector bundles over a real algebraic curve. This was studied thoroughly in [4, 27], for example when \( K = U(n) \).

In this paper, we study Hitchin’s equations on a non-orientable manifold. Let \( M \) be a compact connected non-orientable Riemannian manifold and let \( P \to M \) be a principal \( K \)-bundle over \( M \), where \( K \) is a compact connected Lie group. The de Rham moduli space \( \mathcal{M}^{\text{dR}}(P^C) \), i.e., the moduli space of flat connections on \( P^C \), does not depend on the orientability of \( M \). On the other hand, Hitchin’s equations (1.1) on the pairs \((A, \psi) \in A(P) \times \Omega^1(M, \text{ad} P)\) still make sense (see Subsection 2.2). We define Hitchin’s moduli space \( \mathcal{M}_{\text{Hitchin}}(P) \) as the quotient of the space of pairs \((A, \psi)\) satisfying (1.1) by the group \( G(P) \) of gauge transformations on \( P \). We explain that the homeomorphism \( \mathcal{M}_{\text{Hitchin}}(P) \cong \mathcal{M}^{\text{dR}}(P^C) \) of Donaldson-Corlette remains valid when \( M \) is non-orientable (Theorem 2.2).

If the oriented cover \( \tilde{M} \) of \( M \) is a Kähler manifold, then for the pull-back bundle \( \tilde{P} := \pi^*P \) over \( \tilde{M} \), Hitchin’s moduli space \( \mathcal{M}_{\text{Hitchin}}(\tilde{P}) \) is hyper-Kähler with complex structures \( \bar{I}, \bar{J}, \bar{K} \) and Kähler forms \( \bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K \). If the Kähler form \( \omega \) on \( \tilde{M} \) satisfies \( \tau^*\omega = -\omega \) (the complex dimension of \( \tilde{M} \) must be odd for \( \tau \) to be orientation reversing), then \( \tau \) induces an involution (still denoted by \( \tau \)) on \( \mathcal{M}_{\text{Hitchin}}(\tilde{P}) \) that satisfies \( \tau^*\bar{\omega}_I = -\bar{\omega}_I, \tau^*\bar{\omega}_J = \bar{\omega}_J \) and \( \tau^*\bar{\omega}_K = -\bar{\omega}_K \). Consequently, the fixed-point set \( (\mathcal{M}_{\text{Hitchin}}(\tilde{P}))^\tau \) is Lagrangian in \( \mathcal{M}_{\text{Hitchin}}(\tilde{P}) \) with respect to \( \bar{\omega}_I, \bar{\omega}_K \) and symplectic with respect to \( \bar{\omega}_J \). This is known as an (A,B,A)-brane in [21]. We discover that Hitchin’s moduli space \( \mathcal{M}_{\text{Hitchin}}(P) \) (where \( M \) is non-orientable) is related to \( (\mathcal{M}_{\text{Hitchin}}(\tilde{P}))^\tau \) by a local diffeomorphism. Our main results are summarized in the following main theorem. For simplicity, we restrict to certain smooth parts \( \mathcal{M}_{\text{Hitchin}}(P)^\circ, \mathcal{M}_{\text{Hitchin}}(\tilde{P})^\circ \) and \( \mathcal{A}^{\text{flat}}(P^C)^\circ \) of the respective spaces (see Subsection 2.3 for details).

**Theorem 1.1.** Let \( M \) be a compact non-orientable manifold and let \( \pi: \tilde{M} \to M \) be its oriented cover on which there is a non-trivial deck transformation \( \tau \). Let \( K \) be a compact connected Lie group. Given a principal \( K \)-bundle \( P \to M \), let \( \tilde{P} = \pi^*P \) be its pull-back to \( \tilde{M} \). Suppose that \( \tilde{M} \) is a Kähler manifold of odd complex dimension and the Kähler form \( \omega \) on \( \tilde{M} \) satisfies \( \tau^*\omega = -\omega \). Then

1. \( \mathcal{M}_{\text{Hitchin}}(P)^\circ = \mathcal{A}^{\text{flat}}(P^C)^\circ / \theta_0 \mathcal{G}(P) \), which is a symplectic quotient.
(2) \( (\mathcal{M}_{\text{Hitchin}}(\tilde{P})^\circ)^\tau \) is Kähler and totally geodesic in \( \mathcal{M}_{\text{Hitchin}}(\tilde{P})^\circ \) with respect to \( J, \bar{\omega}_J \) and totally real and Lagrangian with respect to \( \bar{I}, \bar{K} \) and \( \bar{\omega}_I, \bar{\omega}_K \).

(3) there is a local Kähler diffeomorphism from \( \mathcal{M}_{\text{Hitchin}}(P)^\circ \) to \( (\mathcal{M}_{\text{Hitchin}}(\tilde{P})^\circ)^\tau \).

The theorem of Donaldson and Corlette in the non-orientable setup (Theorem 2.2) enable us to identify Hitchin’s moduli space associated to an orientable or non-orientable manifold with the moduli space of flat connections and therefore the representation varieties. Let \( \Gamma \) be a finitely generated group and let \( G \) be a connected complex semi-simple Lie group. The representation variety, \( \text{Hom}(\Gamma, G)/G := \text{Hom}_{\text{red}}(\Gamma, G)/G \), is the quotient of the space of reductive homomorphisms from \( \Gamma \) to \( G \) by the conjugation action of \( G \). When \( \Gamma \) is the fundamental group of a compact manifold \( M \), the representation variety is also called the Betti moduli space of \( M \); it is homeomorphic to the union of the de Rham moduli spaces \( M_{\text{dR}}(P) \) associated to principal \( G \)-bundles \( P \to M \) of various topology. When \( M \) is non-orientable, let \( \tilde{\Gamma} \) be the fundamental group of the oriented cover \( \tilde{M} \). Then there is a short exact sequence \( 1 \to \tilde{\Gamma} \to \Gamma \to \mathbb{Z}_2 \to 1 \) and \( \tau \) acts as an involution on the representation variety \( \text{Hom}(\tilde{\Gamma}, G)/G \) (Lemma 3.3). We study the relation of representation varieties associated to \( \Gamma \) and \( \tilde{\Gamma} \) from an algebraic point of view. Let \( PG = G/Z(G) \), where \( Z(G) \) is the center of \( G \). Our main results are summarized in the following theorem.

**Theorem 1.2.** Let \( G \) be a connected complex semi-simple Lie group. Let \( M \) be a compact non-orientable manifold and let \( \tilde{M} \) be its oriented cover on which there is a non-trivial deck transformation \( \tau \). Denote \( \Gamma = \pi_1(M) \) and \( \tilde{\Gamma} = \pi_1(\tilde{M}) \) with some chosen base points. Then

1. there exists a continuous map \( L \) from \( (\text{Hom}_{\text{good}}(\tilde{\Gamma}, G)/G)^\tau \) to \( Z(G)/2Z(G) \). Consequently, \( (\text{Hom}_{\text{good}}(\tilde{\Gamma}, G)/G)^\tau = \bigcup_{r \in Z(G)/2Z(G)} N_r^{\text{good}} \), where \( N_r^{\text{good}} \) is the preimage of \( r \in Z(G)/2Z(G) \).

2. there exists a \( |Z(G)/2Z(G)| \)-sheeted Galois covering map from \( \text{Hom}_{\text{good}}(\Gamma, G)/G \) to \( N_0^{\text{good}} \). In particular, if \( |Z(G)| \) is odd, then there exists a bijection from \( \text{Hom}_{\text{good}}(\tilde{\Gamma}, G)/G \) to \( (\text{Hom}_{\text{good}}(\Gamma, G)/G)^\tau \). The above statements are true if \( \text{Hom}_{\text{good}} \) is replaced by \( \text{Hom}_{\text{irr}} \). If in addition \( M = \Sigma \) is a compact non-orientable surface and \( G \) is simple and simply connected, then
Hitchin’s equations on a nonorientable manifold

3) there exists a surjective map from \((\text{Hom}^{\text{tr}}(\tilde{\Gamma}, G)/G)^\tau\) to \(\text{Hom}^{\text{irr}}(\Gamma, PG)/PG\) that maps \(N_0^{\text{tr}}\) to flat \(PG\)-bundles on \(\Sigma\) whose topological type is given by \(r \in Z(G)/2Z(G) \cong H^2(\Sigma, Z(G))\). In particular, \(N_0^{\text{tr}}\) maps to the topologically trivial flat \(PG\)-bundles on \(\Sigma\).

Here \(\text{Hom}^{\text{good}}\), following the terminology of [20], denotes the “good” part of the space of homomorphisms that are reductive and whose stabilizer is \(Z(G)\), whereas \(\text{Hom}^{\text{tr}}\) is the space of homomorphisms whose composition with the adjoint representation of \(G\) is an irreducible representation (see Subsection [3.1] for details). \(\text{Hom}^{\text{good}}(\Gamma, G)\) is the set of homomorphisms from \(\Gamma\) to \(G\) whose restriction to \(\tilde{\Gamma}\) is “good”. \(\text{Hom}^{\text{good}}(\Gamma, G)/G\) is not smooth in general, but contains a smooth part \((\mathcal{M}^{\text{flat}}(P^C))^\circ\) (upon identification of moduli spaces). By parts (1) and (2) of the theorem, there is a local homeomorphism \(\text{Hom}^{\text{good}}(\Gamma, G)/G \rightarrow (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau\) (see also Corollary [3.7]), which in fact restricts to the local diffeomorphism \(\mathcal{M}^{\text{flat}}(P^C)^\circ \rightarrow (\mathcal{M}^{\text{flat}}(P^C)^\circ)^\tau\) in part (3) of Theorem 1.1 but is now more accurately described using representation varieties. Also, for \(\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)\) such that \([\phi] \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G\) is fixed by \(\tau\), \(L([\phi])\) is the obstruction of extending \(\phi\) to a representation of \(\Gamma\). In the gauge-theoretic language, \(\phi\) corresponds to a flat connection on \(\tilde{M}\) and represents a point fixed by \(\tau\) in the de Rham moduli space \(\mathcal{M}^{\text{flat}}(P^C)\), while extension of \(\phi\) to \(\Gamma\) means that the flat connection on \(\tilde{M}\) is the pull-back of a flat connection on \(M\). Flat connections on \(M\) that are not pull-backs from \(\tilde{M}\) correspond to flat \(PG\)-bundles over \(M\) (where \(PG = G/Z(G)\)). This is shown in part (3) of Theorem 1.2 and then discussed in greater generality in the last section.

For example, let \(G = SL(2, C)\), \(M\) a compact nonorientable surface and \(\tilde{M}\) its orientable double cover. Then \((\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau\) is labeled by \(Z(G)/2Z(G) = \mathbb{Z}_2\), i.e., \((\text{Hom}^{\text{good}}(\pi_1(M), G)/G)^\tau = \bigcup_{r \in \mathbb{Z}_2} N_r^{\text{good}}\). An element of \((\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau\) is mapped by map \(L\) in Theorem 1.1 (defined in Proposition 3.4) to the null element of \(\mathbb{Z}_2\) if and only if it represents a flat connection on \(\tilde{M}\) that is the pull-back of a flat connection on \(M\). The natural map from \((\text{Hom}^{\text{good}}(\pi_1(M), G)/G)\) to \((\text{Hom}^{\text{good}}(\pi_1(\tilde{M}), G)/G)^\tau\) is not surjective; it is a \(\mathbb{Z}_2\)-sheeted Galois covering map onto \(N_0^{\text{good}}\), and \(N_1^{\text{good}}\) is not in the image. \(N_0^{\text{tr}}\) corresponds to the space of topologically trivial flat \(PSL(2, C)\)-bundles over \(M\) while \(N_1^{\text{tr}}\) corresponds to that of topologically nontrivial flat \(PSL(2, C)\)-bundles over \(M\).

The rest of this paper is organized as follows. In Section 2, we review the basic setup in the orientable case and explain the Donaldson-Corlette theorem for bundles over non-orientable manifolds. We then study finite dimensional symplectic and hyper-Kähler manifolds with an involution and
apply the results to the gauge-theoretical setting to prove Theorem 1.1. In Section 3 we study flat $G$-connections by representation varieties. We show that a flat connection on $M$ is reductive if and only if its pull-back to $\tilde{M}$ is reductive. We then define the continuous map in part (1) of Theorem 1.2 and prove the rest of the theorem. In Section 4 we relate the components $N^\text{good}_r$ ($r \neq 0$) in Theorem 1.2 to $G$-bundles over $\tilde{M}$ admitting an involution up to $Z(G)$.

We note that in order to study the moduli space of $G$-bundles over the nonorientable manifold $M$ itself, our involution is fixed-point free on $\tilde{M}$ and is the identity map on $G$. During the revision of this paper, we came across a few related works. We thank O. García-Prada for pointing out to us the paper [3], where their anti-holomorphic involution acts both on the manifold $M$ and on the structure group $G$, thus resulting in a different fixed-point set of the moduli space. In a more recent paper [2], which overlaps with paper [3], where their anti-holomorphic involution acts both on the manifold $M$ and on the structure group $G$, thus resulting in a different fixed-point set of the moduli space. In a more recent paper [2], which overlaps with paper [3], where their anti-holomorphic involution acts both on the manifold $M$ and on the structure group $G$, thus resulting in a different fixed-point set of the moduli space. In a more recent paper [2], which overlaps with paper [3], where their anti-holomorphic involution acts both on the manifold $M$ and on the structure group $G$, thus resulting in a different fixed-point set of the moduli space.

2. The gauge-theoretic perspective

2.1. Basic setup in the orientable case

Let $K$ be a connected compact Lie group and let $G = K^C$ be its complexification. Given a principal $K$-bundle $P$ over a compact orientable manifold $M$, $P^C = P \times_K G$ is a principal bundle whose structure group is $G$. The set $A(P)$ of connections on $P$ is an affine space modeled on $Ω^1(M, \text{ad}P)$. At each $A ∈ A(P)$, the tangent space is $T_A A(P) \cong Ω^1(M, \text{ad}P)$. The total space of the tangent bundle over $A(P)$ is $TA(P) = A(P) \times Ω^1(M, \text{ad}P)$.

At $(A, ψ) ∈ TA(P)$, the tangent space is $T_{(A, ψ)} TA(P) \cong Ω^1(M, \text{ad}P)^{g\text{old}}$. There is a translation invariant complex structure $J$ on $TA(P)$ given by $J(α, ϕ) = (ϕ, -α)$. The space $TA(P)$ can be naturally identified with $A(P^C)$, the set of connections on $P^C → M$, via $(A, ψ) ↦ A - √{-1}ψ$, under which $J$ corresponds to the complex structure on $A(P^C)$ induced by $G = K^C$. The covariant derivative on $Ω^•(M, \text{ad}P^C)$ is $D := d_A - √{-1}ψ$, where $d_A$ denotes the covariant derivative of $A ∈ A(P)$ and $ψ$ acts by bracket.

The group of gauge transformations on $P$ is $G(P) \cong Γ(M, \text{Ad}P)$. It acts on $A(P)$ via $A ↦ g ∙ A$, where $d_{g ∙ A} = g \circ d_A \circ g^{-1}$ and on $TA(P)$ via $g : (A, ψ) ↦ (g ∙ A, \text{Ad}_g ψ)$. Since the action of $G(P)$ on $TA(P)$ preserves $J$, there is a holomorphic $G(P)^C$ action on $(TA(P), J)$. In fact, the complexification $G(P)^C$ can be naturally identified with $G(P^C) \cong Γ(M, \text{Ad}P^C)$, and
Hitchin’s equations on a nonorientable manifold

The action of $\mathfrak{g}(P^C)$ on $TA(P)$ corresponds to the complex gauge transformations on $A(P^C)$, i.e., $g \in \mathfrak{g}(P^C) : D \mapsto g \circ D \circ g^{-1}$. Let

$$A_{\text{flat}}(P^C) = \{A - \sqrt{-1} \psi \in A(P^C) : F_{A - \sqrt{-1} \psi} = 0\} = \{(A, \psi) \in TA : F_A - \frac{1}{2} [\psi, \psi] = 0, d_A \psi = 0\}$$

be the set of flat connections on $P^C$. Since the vanishing of $F_{A - \sqrt{-1} \psi}$ is a holomorphic condition, $A_{\text{flat}}(P^C)$ is a complex subset of $A(P^C)$; it is also invariant under $\mathfrak{g}(P^C)$. The holonomy group $\text{Hol}(A)$ of $A \in A_{\text{flat}}(P^C)$ can be identified as a subgroup of $G$, up to a conjugation in $G$. A flat connection $A$ on $P^C$ is reductive if the closure of $\text{Hol}(A)$ in $G$ is contained in the Levi subgroup of any parabolic subgroup containing $\text{Hol}(A)$; let $A_{\text{flat, red}}(P^C)$ be the set of such. It can be shown that a flat connection is reductive if and only if its orbit under $\mathfrak{g}(P^C)$ is closed [7]. The de Rham moduli space, or the moduli space of reductive flat connections on $P^C$, is

$$\mathcal{M}^{\text{dR}}(P^C) = A_{\text{flat}}(P^C)/\mathfrak{g}(P^C) = A_{\text{flat, red}}(P^C)/\mathfrak{g}(P^C).$$

It has an induced complex structure $J$ on its smooth part.

Assume that $M$ has a Riemannian structure and choose an invariant inner product $(\cdot, \cdot)$ on the Lie algebra $\mathfrak{k}$ of $K$. Then there is a symplectic structure on $TA(P)$, with which $J$ is compatible, given by

$$\omega_J((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_M (\varphi_2, \wedge \ast \alpha_1) - (\varphi_1, \wedge \ast \alpha_2),$$

where $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \Omega^{1,0}(M, \text{ad } P)$, such that $(TA(P), \omega_J)$ is Kähler. The subset $A_{\text{flat}}(P^C)$ is Kähler in $A(P^C) \cong TA(P)$. We identify the Lie algebra $\text{Lie}(\mathfrak{g}(P)) \cong \Omega^0(M, \text{ad } P)$ with its dual by the inner product on $\Omega^0(M, \text{ad } P)$. The action of $\mathfrak{g}(P)$ on $(TA(P), \omega_J)$ is Hamiltonian, with moment map

$$\mu_J(A, \psi) = d^*_A \psi \in \Omega^0(M, \text{ad } P).$$

Let

$$A_{\text{Hitchin}}(P) = A_{\text{flat}}(P^C) \cap \mu_J^{-1}(0) = \{(A, \psi) \in TA : F_A - \frac{1}{2} [\psi, \psi] = 0, d_A \psi = 0, d^*_A \psi = 0\},$$

the set of pairs $(A, \psi)$ satisfying Hitchin’s equations (1.1), and let the quotient space $\mathcal{M}_{\text{Hitchin}}(P) = A_{\text{Hitchin}}(P)/\mathfrak{g}(P)$ be Hitchin’s moduli space. A
theorem of Donaldson [8] and Corlette [7] states that if $M$ is compact and if the structure group $G$ is semisimple, then $\mathcal{M}^\text{Hitchin}(P) \cong \mathcal{M}^\text{dR}(P^C)$.

Suppose that $M$ is a compact Kähler manifold of complex dimension $n$ and let $\omega$ be the Kähler form on $M$. Then there is a complex structure on $TA(P)$ given by

$$I: (\alpha, \varphi) \mapsto \frac{1}{(n-1)!}(* \omega^{n-1} \wedge (\alpha, -\varphi)) = \frac{1}{(n-1)!} A^{n-1}(*\alpha, -*\varphi),$$

where $(\alpha, \varphi) \in \Omega^1(M, \text{ad } P)^{\oplus 2} \cong T_{(A,\psi)} TA(P)$ and the map

$$A: \Omega^\bullet(M, \text{ad } P) \to \Omega^{\bullet-2}(M, \text{ad } P)$$

is the contraction by $\omega$. With respect to $I$, we have

$$T_{(A,\psi)}^{1,0} TA(P) \cong \Omega^{0,1}(M, \text{ad } P^C) \oplus \Omega^{1,0}(M, \text{ad } P^C)$$

for any $(A, \psi) \in TA(P)$. This complex structure $I$ is compatible with a symplectic form $\omega_I$ on $TA(P)$ given by

$$\omega_I((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_M \frac{\omega^{n-1}}{(n-1)!} \wedge ((\alpha_1, \wedge \alpha_2) - (\varphi_1, \wedge \varphi_2)),$$

where $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \Omega^1(M, \text{ad } P)$. The action of $G(P)$ on $TA(P)$ is also Hamiltonian with respect to $\omega_I$ and the moment map is

$$\mu_I(A, \psi) = A(F_A - \frac{1}{2}[\psi, \psi]) \in \Omega^0(M, \text{ad } P),$$

where $F_A \in \Omega^2(M, \text{ad } P)$ is the curvature of $A$. Since the action of $G(P)$ on $TA(P)$ preserves $I$, there is a holomorphic $G(P^C)$ action on $(TA(P), I)$. For any $(A, \psi) \in TA(P)$, write $\psi = \sqrt{-1}(\phi - \phi^*)$, where $\phi \in \Omega^{1,0}(M, \text{ad } P^C)$, $\phi^* \in \Omega^{0,1}(M, \text{ad } P^C)$. Here $\phi \mapsto \phi^*$ is induced by the conjugation on $G = K^C$ preserving the compact form $K$. Then $D = d_A - \sqrt{-1}\psi = D' + D''$, where $D' = \overline{\partial}_A - \phi^*$, $D'' = \overline{\partial}_A + \phi$. The action of $G(P^C)$ on $TA(P)$ $\cong A(P^C)$ can be described by $g \in G(P^C) \circ D'' \circ g^{-1}$.

Let $\mathcal{A}^\text{Higgs}(P^C)$ be the set of Higgs pairs $(A, \phi)$, i.e., $A \in \mathcal{A}(P)$ and $\phi \in \Omega^{1,0}(M, \text{ad } P^C)$ satisfying $(D'')^2 = 0$, or

$$\overline{\partial}_A^2 = 0, \quad \overline{\partial}_A \phi = 0, \quad [\phi, \phi] = 0.$$

Then $\mathcal{A}^\text{Higgs}(P^C)$ is a Kähler subspace of $\mathcal{A}(P^C) \cong TA(P)$ respect to $I$. Let $\mathcal{A}^\text{sst}(P^C)$ be the set of semistable Higgs pairs and let $\mathcal{A}^\text{sst}(P^C)$ be the set
Hitchin’s equations on a nonorientable manifold

polystable Higgs pairs. (The notions of stable, semistable and polystable Higgs pairs were introduced in [15, 30, 31].) The moduli space of polystable Higgs pairs or the Dolbeault moduli space is

$$\mathcal{M}^{\text{Dol}}(P^C) = (\mathcal{A}^{\text{Higgs}}(P^C) \cap \mathcal{A}^{\text{pol}}(P^C)) / \mathcal{G}(P^C).$$

It has a complex structure induced by $I$. It can be shown [30, Lemma 1.1] that $\mathcal{A}^{\text{Hitchin}}(P) = \mathcal{A}^{\text{flat}}(P^C) \cap \mu^{-1}_J(0) = \mathcal{A}^{\text{Higgs}}(P^C) \cap \mu^{-1}_I(0)$. A theorem of Hitchin [15] and Simpson [29] states that if $M$ is compact and Kähler and the bundle $P$ has vanishing first and second Chern classes, then $\mathcal{M}^{\text{Hitchin}}(P) \cong \mathcal{M}^{\text{Dol}}(P^C)$.

There is a third complex structure on $TA(P)$ defined by

$$K =IJ = -JI: (\alpha, \varphi) \mapsto \frac{1}{(n-1)!} \ast (\omega^{n-1} \wedge (\varphi, \alpha))$$

$$= \frac{1}{(n-1)!} \Lambda^{n-1}(\ast \varphi, \ast \alpha),$$

which is compatible with the symplectic form

$$\omega_K((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_M \frac{\omega^{n-1}}{(n-1)!} \wedge ((\alpha_1, \wedge \varphi_2) - (\alpha_2, \wedge \varphi_1)).$$

The action of $\mathcal{G}(P)$ on $TA(P)$ is Hamiltonian with respect to $\omega_K$ and the moment map is

$$\mu_K(A, \psi) = \Lambda(d_A \psi) \in \Omega^0(M, \text{ad } P).$$

Moreover, the action preserves $K$ and therefore extends to another holomorphic action of $\mathcal{G}(P)^C$. The three complex structures $I, J, K$ define a hyper-Kähler structure on $TA(P)$. Since the action of $\mathcal{G}(P)$ on $TA(P)$ is Hamiltonian with respect to all three symplectic forms, we have a hyper-Kähler moment map $\mu = (\mu_I, \mu_J, \mu_K): TA(P) \to (\Omega^0(M, \text{ad } P))^\mathbb{C}$. The hyper-Kähler quotient [10] is $\mathcal{M}^{\text{HK}}(P) = \mu^{-1}(0) / \mathcal{G}(P)$, with complex structures $\bar{I}, \bar{J}, \bar{K}$ and symplectic forms $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$. By the theorems of Donaldson-Corlette and of Hitchin-Simpson, the Hitchin moduli space $\mathcal{M}^{\text{Hitchin}}(P)$ is a complex space with respect to both $\bar{I}$ and $\bar{J}$. Therefore $\mathcal{M}^{\text{Hitchin}}(P)$ is a hyper-Kähler subspace in $\mathcal{M}^{\text{HK}}(P)$ [10, Theorem 8.3.1].

When $M = \Sigma$ is an orientable surface, $A: \Omega^2(\Sigma, \text{ad } P) \to \Omega^0(\Sigma, \text{ad } P)$ is an isomorphism. So $\mathcal{A}^{\text{Hitchin}}(P) = \mathcal{A}^{\text{flat}}(P^C) \cap \mu^{-1}_J(0) = \mathcal{A}^{\text{Higgs}}(P^C) \cap \mu^{-1}_I(0)$.
Now suppose $M$ is a compact non-orientable manifold. Let $\pi: \tilde{M} \to M$ be its oriented cover and let $\tau: \tilde{M} \to \tilde{M}$ be the non-trivial deck transformation. Given a principal $K$-bundle $P \to M$, let $\tilde{P} = \pi^* P \to M$ be its pull-back to $\tilde{M}$. Since $\pi \circ \tau = \pi$, the $\tau$ action can be lifted to $\tilde{P} = \tilde{M} \times_M P$ as a $K$-bundle involution (i.e., the lifted involution commutes with the right $K$-action on $\tilde{P}$), and hence to the associated bundles $\text{Ad} \tilde{P}$ and $\text{Ad} \tilde{P}$. Consequently, $\tau$ acts on the space of connections $\mathcal{A}(\tilde{P})$ by pull-back $A \mapsto \tau^* A$ and on the group of gauge transformations $\mathcal{S}(\tilde{P})$ by $g \mapsto \tau^* g := \tau^{-1} \circ g \circ \tau$. The $\tau$-invariant subsets are $(\mathcal{A}(\tilde{P}))^\tau \cong \mathcal{A}(P)$ and $(\mathcal{S}(\tilde{P}))^\tau \cong \mathcal{S}(P)$. In fact, the inclusion map $\mathcal{A}(P) \hookrightarrow \mathcal{A}(\tilde{P})$ onto the $\tau$-invariant part is the pull-back via $\pi$ of connections on $P$ to those on $\tilde{P}$. Since $\mathcal{A}(\tilde{P})$ is an affine space modeled on $\Omega^1(\tilde{M}, \text{ad} \tilde{P})$, the differential $\tau_\ast$ of $\tau: \mathcal{A}(\tilde{P}) \to \mathcal{A}(\tilde{P})$ can be identified with a linear involution on $\Omega^1(\tilde{M}, \text{ad} \tilde{P})$ given by $\alpha \mapsto \tau^* \alpha$.

A Riemannian metric on a non-orientable manifold $M$ pulls back to a Riemannian metric on $\tilde{M}$. Assuming that $M$ is compact, we define an inner product on the space $\Omega^\ast(M)$ of differential forms on $M$ by

$$\langle \alpha, \beta \rangle = \frac{1}{2} \int_{\tilde{M}} \pi^* \alpha \wedge \tilde{\ast} \pi^* \beta$$

for $\alpha, \beta \in \Omega^\ast(M)$, where $\tilde{\ast}$ is the Hodge star operator on $\tilde{M}$. Alternatively, the Hodge star $\ast$ on $M$ maps a form on $M$ to one valued in the orientation line bundle over $M$, and if $\alpha, \beta$ are of the same degree, then $\alpha \wedge \ast \beta$ is a top-degree form on $M$ valued in the orientation line bundle, which can be integrated over $M$. We still have $\langle \alpha, \beta \rangle = \int_M \alpha \wedge \ast \beta$. More generally, there is an inner product on the space $\Omega^\ast(M, \text{ad} P)$ of forms valued in $\text{ad} P$. Therefore $\mathcal{A}(P)$ admits a Riemannian structure, which is half of the restriction of the Riemannian structure on $\mathcal{A}(\tilde{P})$ to the $\tau$-invariant subspace $(\mathcal{A}(\tilde{P}))^\tau \cong \mathcal{A}(P)$.

Consider the tangent bundle $TA(\tilde{P}) = \mathcal{A}(\tilde{P}) \times \Omega^1(\tilde{M}, \text{ad} \tilde{P})$ of $\mathcal{A}(\tilde{P})$. It has a $\tau$-action given by $\tau: (A, \psi) \mapsto (\tau^* A, \tau^* \psi)$, which is holomorphic with respect to the complex structure $J$. Therefore the fixed point set $(TA(\tilde{P}))^\tau \cong TA(P)$ is a complex subspace in $TA(\tilde{P}) \cong \mathcal{A}(P^C)$. With respect
Hitchin’s equations on a nonorientable manifold

It is clear that to the induced Riemannian structure on $TA(\tilde{P})$, $\tau: TA(\tilde{P}) \to TA(\tilde{P})$ is an isometry. Since $\tau$ also acts holomorphically on $A(\tilde{P}^C) \cong TA(\tilde{P})$, $(TA(\tilde{P}))^\tau$ is a Kähler and totally geodesic subspace in $TA(\tilde{P}) \cong A(\tilde{P}^C)$. Moreover, $A^{\text{flat}}(\tilde{P}^C) \cong (A^{\text{flat}}(\tilde{P}^C))^\tau$ is also Kähler and totally geodesic in $A^{\text{flat}}(\tilde{P}^C)$.

We summarize the above discussion in the following lemma.

**Lemma 2.1.** Given a compact non-orientable manifold $M$ with oriented double cover $\pi: \tilde{M} \to M$ and a principal $K$-bundle $P \to M$, the non-trivial deck transformation $\tau$ on $\tilde{M}$ lifts to an involution (also denoted by $\tau$) on $\tilde{P} = \pi^*P$ and acts as involutions on the space of connections $A(\tilde{P})$ and on $TA(\tilde{P}) \cong A(\tilde{P}^C)$. Moreover, the $\tau$-invariant subspaces $A(\tilde{P}^C)_{\tau} \cong A(P^C)$ and $A^{\text{flat}}(\tilde{P}^C)_{\tau} \cong A^{\text{flat}}(P^C)$ are Kähler and totally geodesic subspaces in $A(P^C) \cong TA(\tilde{P})$ and $A^{\text{flat}}(\tilde{P}^C)$, respectively.

On a non-orientable manifold $M$, we still have Hitchin’s equations (1.1).

Here $d_A^*$ is defined as the (formal) adjoint of $d_A$ with respect to the inner products on $\Omega^*(M, ad P)$. Alternatively, $d_A^*$ is the first order differential operator on $M$ such that on any orientable open set in $M$, $d_A^* = \ast^{-1}d_A \ast$; the latter is actually independent of the choice of local orientation. Yet another but related way to explain the operator $d_A^*$ is to consider the Hodge star operator $\ast$ on a non-orientable manifold $M$ as a map from differential forms to those valued in the orientation bundle over $M$. Since the latter is a flat real line bundle, $d_A^* = \ast^{-1}d_A \ast$ maps $\Omega^2(M, ad P)$ to $\Omega^0(M, ad P)$. Finally, $d_A^*$ can be defined as $(\pi^*)^{-1} \circ d_{\pi^*A} \circ \pi^*$. Here $d_{\pi^*A} = \ast^{-1}d_{\pi^*A} \ast$ holds globally on $\tilde{M}$ and $\pi^*: \Omega^*(M, ad P) \to \Omega^*(\tilde{M}, ad \tilde{P})$ is injective. Let

$$A^{\text{Hitchin}}(P) := \{(A, \psi) \in TA : F_A - \frac{1}{2}[\psi, \psi] = 0, d_A \psi = 0, d_A^* \psi = 0\}.$$ 

It is clear that $A^{\text{Hitchin}}(P) = (A^{\text{Hitchin}}(\tilde{P}))^\tau$.

The notion of reductive connections on $P$ does not depend on the orientability of $M$, and we still have the moduli space of flat connections $\mathcal{M}^{\text{flat}}(P^C) = \mathcal{A}^{\text{flat}, \text{red}}(P^C)/\mathcal{G}(P^C)$. Let $\mathcal{M}^{\text{Hitchin}}(P) = A^{\text{Hitchin}}(P)/\mathcal{G}(P)$ be Hitchin’s moduli space. The following is the Donaldson-Corlette theorem that also applies to the case when $M$ is non-orientable. Equivalently, there exists a unique reduction of structure group from $G$ to $K$ admitting a solution to Hitchin’s equations.

**Theorem 2.2.** Let $M$ be a compact non-orientable Riemannian manifold. Then for every reductive flat connection $D$ on $P^C$, there exists a gauge transformation $g \in \mathcal{G}(P^C)$ (unique up to $\mathcal{G}(P)$ and the stabilizer of $D$) such that
868 Nan-Kuo Ho, Graeme Wilkin, and Siye Wu

\[ g \cdot D = d_A - \sqrt{-1}\psi \]  
with \((A, \psi) \in \mathcal{A}^{\text{Hitchin}}(P)\). As a consequence, we have a homeomorphism \(\mathcal{M}^{\text{dR}}(P^C) \cong \mathcal{M}^{\text{Hitchin}}(P)\).

We now explain that Corlette’s proof in [7] applies to the case when \(M\) is non-orientable. There is a symplectic form \(\omega_J\) on \(T_A(P)\), still given by (2.1), which is half of the restriction of the symplectic form on \(T\bar{A}(\bar{P})\). The action of \(\mathcal{G}(P)\) on \(T_A(P)\) is Hamiltonian, and the moment map remains \(\mathbb{R}^2\). Recall Corlette’s flow equations on the space of flat connections. Let \(D = d_A - \sqrt{-1}\psi\) be a flat connection of the \(G = K^C\) bundle \(P^C \to M\). Then the flow equations are

\[
\frac{\partial D}{\partial t} = -D\mu_J(D).
\]

Equivalently, one can look for a flow of the form \(g(t) \cdot D_0\) and solve for \(g(t) \in \mathcal{G}(\bar{P}^C)\) using (cf. [7, p. 369])

\[
\frac{\partial g}{\partial t} g^{-1} = -\sqrt{-1}\mu_J(g \cdot D_0).
\]

Corlette shows in [7] that we have existence and uniqueness of solutions to (2.3) and (2.4) for all time. If the initial condition is a reductive flat connection, then there is a sequence converging to a solution to \(\mu_J(D) = 0\). Also, the limit is gauge equivalent to the initial flat reductive connection [7]. These arguments are valid when \(M\) is non-orientable.

We remark that Theorem 2.2 for non-orientable manifolds also follows from the result of the orientable double cover. A flat connection on \(P\) is reductive if and only if the pull-back \(\pi^* A\) is a flat reductive connection on \(\bar{P}\). (We defer the proof of this statement to Corollary 3.2.) For the bundle \(\bar{P} \to \bar{M}\), it is easy to check that the right-hand sides of (2.3) and (2.4) define \(\tau\)-invariant vector fields on \(\mathcal{A}(\bar{P}^C)\) and \(\mathcal{G}(\bar{P}^C)\), respectively. Since the space \(\mathcal{A}^{\text{flat}}(\bar{P}^C)\) of \(\tau\)-invariant connections is closed in \(\mathcal{A}(\bar{P}^C)\) and the space \(\mathcal{G}(\bar{P}^C)\) of \(\tau\)-invariant gauge transformations is closed in \(\mathcal{G}(\bar{P}^C)\), Corlette’s results on the limit of the flow restrict to the \(\tau\)-invariant subset as well. That is, the flow on the space of connections is contained in the \(\tau\)-invariant subset and the limit is a \(\tau\)-invariant solution to Hitchin’s equation. Similarly, the gauge transformation relating to the initial condition is contained in the \(\tau\)-invariant part of the group of gauge transformations, and the limit is \(\tau\)-invariant.
2.3. The Hitchin moduli space and the hyper-Kähler quotient

Now consider a compact non-orientable manifold \( M \). Suppose its oriented cover \( \tilde{M} \) is a Kähler manifold of complex dimension \( n \). Let \( \omega \) be the Kähler form on \( \tilde{M} \). Throughout this subsection, we assume that \( n \) is odd and the deck transformation \( \tau \) on \( \tilde{M} \) is an anti-holomorphic involution such that \( \tau^* \omega = -\omega \). Then \( \tau^* \omega^n = -\omega^n \), which is consistent with the requirement that \( \tau \) is orientation reversing. The \( \tau \)-action on \( TA(\tilde{P}) = \mathcal{A}(\tilde{P}) \times \Omega^1(M, \text{ad} P) \), \( \tau : (A, \psi) \mapsto (\tau^* A, \tau^* \psi) \), is an isometry and its differential \( \tau_\ast : \Omega^1(M, \text{ad} P)^{\oplus 2} \to \Omega^1(M, \text{ad} P)^{\oplus 2} \) is \( \tau_\ast : (\alpha, \varphi) \mapsto (\tau^* \alpha, \tau^* \varphi) \). It is easy to see that \( \tau_\ast \circ I = - I \circ \tau_\ast \) since \( \tau \) reverses the orientation of \( \tilde{M} \) and that \( \tau_\ast \circ K = - K \circ \tau_\ast \) since \( K = IJ \). So \( \tau \) acts as an anti-holomorphic involution with respect to both \( I \) and \( K \), and \( \tau^* \omega_I = - \omega_I \), \( \tau^* \omega_K = - \omega_K \). Moreover, since the moment maps \( \mu_I \) and \( \mu_K \) on \( TA(\tilde{P}) \) involve the contraction \( A \) by \( \omega \), they satisfy \( \tau^* (\mu_I (A, \psi)) = - \mu_I (\tau^* A, \tau^* \psi) \), \( \tau^* (\mu_K (A, \psi)) = - \mu_K (\tau^* A, \tau^* \psi) \) for all \( (A, \psi) \in TA(\tilde{P}) \). The fixed point set \( \mathcal{A}(\tilde{P})^\tau \) is totally real with respect to the complex structures \( I \) and \( K \), and Lagrangian with respect to the symplectic forms \( \omega_I \) and \( \omega_K \).

A flat connection \( D = d A - \sqrt{-1} \psi \) on \( \tilde{P}^C \) defines an elliptic complex with \( D_I : \Omega^i(M, \text{ad} \tilde{P}^C) \to \Omega^{i+1}(M, \text{ad} \tilde{P}^C) \). Let \( \mathcal{A}^{\text{flat}}(\tilde{P}^C) \) be the set of flat connections on \( \tilde{P}^C \) such that (i) the stabilizer under the \( G(\tilde{P}^C) \) action is \( Z(G) \), and (ii) the linearization \( D_I \) of the curvature map surjects onto \( \ker D_2 \cap \Omega^2(M, [\text{ad} \tilde{P}^C, \text{ad} \tilde{P}^C]) \). Notice that when \( M \) is a surface, condition (i) implies (ii). The method in [22] and Chapter VII shows that \( \mathcal{A}^{\text{flat}}(\tilde{P}^C) \) is a smooth submanifold in \( \mathcal{A}(\tilde{P}^C) \), and as the action of \( G(\tilde{P}^C)/Z(G) \) on it is free, the subset

\[
\mathcal{M}^{\text{dR}}(\tilde{P}^C)^\circ := (\mathcal{A}^{\text{flat}}(\tilde{P}^C)^\circ \cap \mathcal{A}^{\text{flat,red}}(\tilde{P}^C))/G(\tilde{P}^C)
\]

is in the smooth part of the moduli space \( \mathcal{M}^{\text{dR}}(\tilde{P}^C) \) (see also [11] from the point of view of representation varieties). The free action of \( G(\tilde{P}^C)/Z(G) \) or \( G(\tilde{P})/Z(K) \) from condition (i) implies that 0 is a regular value of \( \mu_J \) on \( \mathcal{A}^{\text{flat}}(\tilde{P}^C) \), and the subset \( \mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ := \mathcal{A}^{\text{flat}}(\tilde{P}^C)^\circ \cap \mu_J^{-1}(0)/G(\tilde{P}) \) is in the smooth part of Hitchin’s moduli space \( \mathcal{M}^{\text{Hitchin}}(\tilde{P}) \) [15]. By the Donaldson-Corlette theorem, we have the homeomorphism \( \mathcal{M}^{\text{Hitchin}}(\tilde{P})^\circ \cong \mathcal{M}^{\text{dR}}(\tilde{P}^C)^\circ \).

On the other hand, for the non-orientable manifold \( M \), let \( \mathcal{A}^{\text{flat}}(P)^\circ = \{ A \in \mathcal{A}(P) : \pi^* A \in \mathcal{A}^{\text{flat}}(\tilde{P}^C)^\circ \} \), \( \mathcal{A}^{\text{Hitchin}}(P)^\circ = \mathcal{A}^{\text{Hitchin}}(P) \cap \mathcal{A}^{\text{flat}}(P)^\circ \). Then \( \mathcal{M}^{\text{Hitchin}}(P)^\circ := \mathcal{A}^{\text{Hitchin}}(P)^\circ / G(P) \) is in the smooth part of \( \mathcal{M}^{\text{Hitchin}}(P) \),
but we will not consider here the smooth points of $M^{\text{Hit}}(P)$ that are outside $M^{\text{Hit}}(P)^{0}$. By Theorem 2.2 (the analog of the Donaldson-Corlette theorem for non-orientable manifolds), we have a homeomorphism between $M^{\text{Hit}}(P)^{0}$ and $M^{\text{IR}}(P^{c})^{0} := (A^{\text{flat}}(P^{c})^{0} \cap A^{\text{flat,red}}(P^{c}))/\mathbb{G}(P^{c})$.

We now study a general setting. Let $(X,\omega)$ be a finite dimensional symplectic manifold with a Hamiltonian action of a compact Lie group $K$ and let $\mu: X \to \mathfrak{k}^{*}$ be the moment map. Suppose as in [25], that there are involutions $\sigma$ on $X$ and $\tau$ on $K$ such that $\sigma(k \cdot x) = \tau(k) \cdot \sigma(x)$ for all $k \in K$ and $x \in X$. Assume that $\omega$ is not empty. Then $K^{\tau}$ acts on $X^{\sigma}$. We note that $\tau$ acts on $\mathfrak{k}$, $\mathfrak{k}^{*}$, and $K^{\tau}$ is a closed Lie subgroup of $K$ with Lie algebra $\mathfrak{k}^{\tau}$. Contrary to [25], we assume that the action of $(K,K^{\tau})$ on $(X,X^{\sigma})$ is symplectic, i.e., we have $\sigma^{*}\omega = \omega$ and $\sigma^{*}\mu = \tau\mu$. Then $X^{\sigma}$ is a symplectic submanifold in $X$. Assume that $0$ is a regular value of $\mu$ and that $K$ acts on $\mu^{-1}(0)$ freely. Since $\sigma$ preserves $\mu^{-1}(0)$, it descends to a symplectic involution $\bar{\sigma}$ on the (smooth) symplectic quotient $X/\!/_{0}K = \mu^{-1}(0)/K$ at level $0$, and $(X/\!/_{0}K)^{\bar{\sigma}}$ is a symplectic submanifold.

**Lemma 2.3.** In the above setting, the action of $K^{\tau}$ on $X^{\sigma}$ is Hamiltonian and the symplectic quotient is $X^{\sigma}/\!/_{0}K^{\tau} = (\mu^{-1}(0) \cap X^{\sigma})/K^{\tau}$. If $\mu^{-1}(0) \cap X^{\sigma} \neq \emptyset$, then there exists a symplectic local diffeomorphism from $X^{\sigma}/\!/_{0}K^{\tau}$ to $(X/\!/_{0}K)^{\bar{\sigma}}$.

**Proof.** Let $\mathfrak{k} = \mathfrak{k}^{\tau} \oplus \mathfrak{q}$ such that $\tau = \pm 1$ on $\mathfrak{k}^{\tau}$, $\mathfrak{q}$, respectively. It is clear that the action of $K^{\tau}$ on $X^{\sigma}$ is Hamiltonian and the moment map $\mu_{\tau}$ is the composition $X^{\sigma} \hookrightarrow X \to \mathfrak{k}^{\tau} \to (\mathfrak{k}^{\tau})^{\ast}$. Since for any $x \in X^{\sigma}$, $\langle \mu(x) , \mathfrak{q} \rangle = 0$, we get $\mu_{\tau}^{-1}(0) = \mu^{-1}(0) \cap X^{\sigma} = (\mu^{-1}(0))^{\tau}$. By the assumptions, $0$ is a regular value of $\mu_{\tau}$, the action of $K^{\tau}$ on $\mu_{\tau}^{-1}(0)$ is free, and the symplectic quotient is $X^{\sigma}/\!/_{0}K^{\tau} = (\mu_{\tau}^{-1}(0) \cap X^{\sigma})/K^{\tau}$.

For any $x \in X^{\sigma}$, the map $\mathfrak{k} \to T_{x}X$ intertwines $\tau$ on $\mathfrak{k}$ and $\sigma$ on $T_{x}X$, and $T_{x}(K^{\tau} \cdot x) = (T_{x}(K \cdot x))^{\sigma}$. The inclusion $\mu_{\tau}^{-1}(0) \hookrightarrow \mu^{-1}(0)$ induces a natural map $X^{\sigma}/\!/_{0}K^{\tau} \to (X/\!/_{0}K)^{\bar{\sigma}}$, whose differentiation at $[x]$ is, after natural symplectic isomorphisms, the linear map $(T_{x}\mu^{-1}(0))^{\sigma}/(T_{x}(K \cdot x))^{\sigma} \to (T_{x}\mu_{\tau}^{-1}(0)/T_{x}(K \cdot x))^{\bar{\sigma}}$. The latter is clearly injective; to show surjectivity, we note that for any $V \in T_{x}\mu^{-1}(0)$, if $V + T_{x}(K \cdot x) \in (T_{x}\mu_{\tau}^{-1}(0)/T_{x}(K \cdot x))^{\bar{\sigma}}$, then it is the image of $\frac{1}{2}(V + \sigma V) + (T_{x}(K \cdot x))^{\sigma}$. The map $X^{\sigma}/\!/_{0}K^{\tau} \to (X/\!/_{0}K)^{\bar{\sigma}}$ is a local diffeomorphism; it is symplectic because the above linear map is so for each $x \in \mu_{\tau}^{-1}(0)$.

Now let $X$ be a hyper-Kähler manifold with complex structures $J_{i}$ and symplectic structures $\omega_{i}$ ($i = 1, 2, 3$). Suppose $K$ acts on $X$ and the action is Hamiltonian with respect to all $\omega_{i}$. Let $\mu = (\mu_{1}, \mu_{2}, \mu_{3}) : X \to (\mathfrak{k}^{*})^{\oplus 3}$ be
Hitchin’s equations on a nonorientable manifold

the hyper-Kähler moment map. Assume that there are involutions \( \sigma \) on \( X \) and \( \tau \) on \( K \) such that \( \sigma(k \cdot x) = \tau(k) \cdot \sigma(x) \) for all \( k \in K \) and \( x \in X \) and \( \sigma^*J_i = (-1)^iJ_i \), \( \sigma^*\omega_i = (-1)^i\omega_i \), \( \sigma^*\mu_i = (-1)^i\tau\mu_i \) for \( i = 1, 2, 3 \). So the action of \( (K, K^\tau) \) on \( (X, X^\sigma) \) is symplectic with respect to \( \omega_2 \) (as above) and anti-symplectic with respect to \( \omega_1, \omega_3 \) (as in \([25]\)). Then \( X^\sigma \), if non-empty, is Kähler and totally geodesic in \( X \) with respect to \( J_2, \omega_2 \) and is totally real and Lagrangian with respect to \( J_1, \omega_1 \) and \( J_3, \omega_3 \). If \( 0 \) is a regular value of \( \mu \) (i.e., \( 0 \) is a regular value of each \( \mu_i \)) and that \( K \) acts on \( \mu^{-1}(0) \) freely, then \( X/\!/K = \mu^{-1}(0)/K \) is the (smooth) hyper-Kähler quotient at level 0, which has complex structures \( \bar{J} \) and symplectic structures \( \bar{\omega}_i \) (i.e., \( 0 \) is a regular value of each \( \bar{J}_i \)) \( i \in \{1, 2, 3\} \) \([15]\).

**Proposition 2.4.** In the above setting, let \( Y = \mu_1^{-1}(0) \cap \mu_3^{-1}(0) \). Then

1. \( Y \) is a \( \sigma \)-invariant Kähler submanifold in \( X \) with respect to \( J_2, \omega_2 \) and the symplectic quotient \( Y^\sigma/\!/K^\tau = (\mu^{-1}(0))^\sigma/K^\tau \) is Kähler;
2. \( (X/\!/K)^\sigma \) is Kähler and totally geodesic in \( X/\!/K \) with respect to \( \bar{J}_2, \bar{\omega}_2 \) and is totally real and Lagrangian with respect to \( \bar{J}_1, \bar{J}_3 \) and \( \bar{\omega}_1, \bar{\omega}_3 \);
3. if \( (\mu^{-1}(0))^\sigma \neq \emptyset \), there is a Kähler (with respect to \( \bar{J}_2, \bar{\omega}_2 \)) local diffeomorphism \( Y^\sigma/\!/K^\tau \rightarrow (X/\!/K)^\sigma \).

**Proof.** 1k3. Let \( \mu_c = \mu_3 + \sqrt{-1}\mu_1 : X \rightarrow \mathfrak{t}^*\mathbb{C} \). Then \( \mu_c \) is holomorphic with respect to \( \bar{J}_2 \) and is equivariant under the action of \( K \). Since \( 0 \) is a regular value of \( \mu_c \), \( Y = \mu_c^{-1}(0) \) is a smooth Kähler submanifold in \( X \) on which the action of \( K \) is Hamiltonian. Applying Lemma \([2.3]\) to \( Y \), we conclude that the action of \( K^\tau \) on \( Y^\sigma \) is Hamiltonian and that \( (\mu^{-1}(0))^\sigma/K^\tau = (\mu_c^{-1}(0) \cap Y^\sigma)/K^\tau = Y^\sigma/\!/K^\tau \). Moreover, there is a local diffeomorphism from \( Y^\sigma/\!/K^\tau \) to \( Y^\sigma/\!/K^\tau = (X/\!/K)^\sigma \) which is symplectic. Since \( K^\tau \) acts holomorphically on \( (Y^\sigma, J_2) \), the symplectic quotient \( Y^\sigma/\!/K^\tau \) is Kähler, and the above local diffeomorphism is also Kähler.

2. Since \( \sigma \) preserves \( \mu^{-1}(0) \), it descends to an involution \( \bar{\sigma} \) on \( X/\!/K \) such that \( \bar{\sigma}^*\bar{J}_i = (-1)^i\bar{J}_i, \bar{\sigma}^*\bar{\omega}_i = (-1)^i\bar{\omega}_i \) for \( i = 1, 2, 3 \). The result then follows.

We now prove Theorem \([1.1]\).

**Proof.** 1k3. Note that \( A^{\flat}(\tilde{P}^\mathbb{C})^\circ \) is a \( \tau \)-invariant Kähler submanifold in \( TA(\tilde{P}) \cong A(\tilde{P}^\mathbb{C}) \). Following \([11, 15]\), we can apply the method in Lemma \([2.3]\) to \( A^{\flat}(\tilde{P}^\mathbb{C})^\circ \) on which \( \tau \) acts preserving \( \omega_J \) and \( J \). Since \( \tau \) also acts on \( \mathcal{H}(\tilde{P}) \) and \( \mathcal{H}(P) \cong (\mathcal{H}(\tilde{P}))^\tau \), \( \mathcal{H}(P)/Z(K) \) acts Hamiltonianly and freely on
A^\text{flat}(P^C)\circ \cong (A^\text{flat}(\tilde{P}^C)\circ)^\tau$, which is Kähler with respect to $J, \omega_J$. Thus

\[
\mathcal{M}^{\text{Hitchin}}(P)^\circ = (A^\text{flat}(P^C)\circ \cap \mu_I^{-1}(0))/\mathcal{S}(P) = A^\text{flat}(P^C)\circ /\mathcal{S}(P)
\]

is a symplectic quotient. Since the latter is non-empty, there is a local Kähler diffeomorphism $M^{\text{Hitchin}}(P)^\circ \to (A^\text{flat}(\tilde{P}^C)\circ /\mathcal{S}(\tilde{P}))^\tau = (M^{\text{Hitchin}}(\tilde{P})^\circ)^\tau$.

2. The space $TA(\tilde{P}) \cong A(\tilde{P}^C)$ with $I, J, K$ is hyper-Kähler and the action of $\mathcal{S}(\tilde{P})$ is Hamiltonian with respect to $\omega_I, \omega_J, \omega_K$. Let $(\mu^{-1}(0))^\circ$ be the subset of $\mu^{-1}(0)$ on which $\mathcal{S}(\tilde{P})/\mathcal{Z}(K)$ acts freely. Then $M^{\text{HK}}(\tilde{P})^\circ := (\mu^{-1}(0))^\circ /\mathcal{S}(\tilde{P})$ is the smooth part of the hyper-Kähler quotient $M^{\text{HK}}(\tilde{P})$. The involutions $\tau$ on $A(\tilde{P})$ and $\mathcal{S}(\tilde{P})$ satisfy the conditions of Proposition 2.4. So $M^{\text{HK}}(\tilde{P})^\circ ^\tau$ is Kähler and totally geodesic with respect to $J$ and $\omega_J$, and totally real and Lagrangian with respect to $\tilde{I}, \tilde{K}$ and $\tilde{\omega}_I, \tilde{\omega}_K$ in $M^{\text{HK}}(\tilde{P})^\circ$. If $M$ is a non-orientable surface, then $\mu_I^{-1}(0) \cap \mu_K^{-1}(0) = A^\text{flat}(P^C)$ which implies that $M^{\text{Hitchin}}(P)^\circ = M^{\text{HK}}(\tilde{P})^\circ$. In general, $M^{\text{Hitchin}}(P)^\circ$ is a $\tau$-invariant hyper-Kähler submanifold in $M^{\text{HK}}(\tilde{P})^\circ$. The results follow from $(M^{\text{Hitchin}}(\tilde{P})^\circ)^\tau = M^{\text{Hitchin}}(\tilde{P}) \cap (M^{\text{HK}}(\tilde{P})^\circ)^\tau$. □

3. The representation variety perspective

3.1. Representation variety and Betti moduli space

Let $\Gamma$ be a finitely generated group and let $G$ be a connected complex Lie group. Then $G$ acts on $\text{Hom}(\Gamma, G)$ by the conjugate action on $G$. A representation $\phi \in \text{Hom}(\Gamma, G)$ is reductive if the closure of $\phi(\Gamma)$ in $G$ is contained in the Levi subgroup of any parabolic subgroup containing $\phi(\Gamma)$; let $\text{Hom}^{\text{red}}(\Gamma, G)$ be the set of such. The condition $\phi \in \text{Hom}^{\text{red}}(\Gamma, G)$ is equivalent to the statement that the $G$-orbit $G \cdot \phi$ is closed [13]. It is also equivalent to the condition that the composition of $\phi$ with the adjoint representation of $G$ is semi-simple (see [26, Section 3] and [28, Theorem 30]). The quotient

$$\text{Hom}(\Gamma, G)/G = \text{Hom}^{\text{red}}(\Gamma, G)/G$$

is known as the representation variety or character variety. A reductive representation $\phi \in \text{Hom}^{\text{red}}(\Gamma, G)$ is good [26] if its stabilizer $G_{\phi} = Z(G)$; let $\text{Hom}^{\text{good}}(\Gamma, G)$ be the set of such. On the other hand, $\phi \in \text{Hom}(\Gamma, G)$ is Ad-irreducible if its composition with the adjoint representation of $G$ is an irreducible representation of $\Gamma$. Let $\text{Hom}^{\text{rr}}(\Gamma, G)$ be the set of such. Notice that this set is empty unless $G$ is simple. Clearly, $\text{Hom}^{\text{rr}}(\Gamma, G) \subset \text{Hom}^{\text{good}}(\Gamma, G)$. In general, $\text{Hom}^{\text{good}}(\Gamma, G)/G$ may not be smooth, but it is so when $\Gamma$ is the fundamental group of a compact orientable surface [28, Corollary 50].
Hitchin’s equations on a nonorientable manifold

Suppose $M$ is a compact manifold and $P^C \to M$ is a principal $G$-bundle over $M$. Choose a base point $x_0 \in M$ and let $\Gamma = \pi_1(M, x_0)$ be the fundamental group. Then $\text{Hom}(\Gamma, G)/G$ is known as the Betti moduli space \cite{Betti}, denoted by $\mathcal{M}^{\text{Betti}}(P^C)$. The identification $\mathcal{M}^{\text{dR}}(P^C) \cong \mathcal{M}^{\text{Betti}}(P^C)$, which we recall briefly now, is well known. Given a flat connection, let $T_\alpha : P_{\alpha(0)} \to P_{\alpha(1)}$ be the parallel transport along a path $\alpha$ in $M$. Fix a point $p_0 \in P_{x_0}$ in the fibre over $x_0$. For $a \in \pi_1(M, x_0)$, choose a loop $\alpha$ based at $x_0$ representing $a$, then $\phi(a)$ is the unique element in $G$ defined by $T_\alpha(p_0) = p_0 \phi(a)^{-1}$. If we choose another point in the fibre over $x_0$, then $\phi$ differs by a conjugation.

Finally, the flat connection is reductive if and only if the corresponding element in $\text{Hom}(\Gamma, G)/G$ is reductive. Upon identification of the de Rham moduli space $\mathcal{M}^{\text{dR}}(P^C)$ and the Betti moduli spaces $\mathcal{M}^{\text{Betti}}(P^C) = \text{Hom}(\Gamma, G)/G$, the subset $\text{Hom}^{\text{good}}(\Gamma, G)/G$ contains the smooth part $\mathcal{M}^{\text{dR}}(P^C)^{\circ}$ introduced in Subsection 2.3; they are equal when $M$ is a compact orientable surface.

If $M$ is non-orientable and $\pi : \tilde{M} \to M$ is the oriented cover, we choose a base point $\tilde{x}_0 \in \pi^{-1}(x_0)$ and let $\tilde{\Gamma} = \pi_1(\tilde{M}, \tilde{x}_0)$. Then there is a short exact sequence

$$1 \to \tilde{\Gamma} \to \Gamma \to \mathbb{Z}_2 \to 1$$

and $\tilde{\Gamma}$ can be identified with an index 2 subgroup in $\Gamma$. In the rest of this section, we will study the relation of the representation varieties $\text{Hom}(\Gamma, G)/G$ and $\text{Hom}(\tilde{\Gamma}, G)/G$ or the Betti moduli spaces $\mathcal{M}^{\text{Betti}}(P^C)$ and $\mathcal{M}^{\text{Betti}}(\tilde{P}^C)$. Some of the results, when $M$ is a compact non-orientable surface, appeared in \cite{17}, which used different methods.

We first establish a useful fact that was used in Subsection 2.2.

**Lemma 3.1.** Suppose $\Gamma$ is a finitely generated group and $\tilde{\Gamma}$ is an index 2 subgroup in $\Gamma$. Let $G$ be a connected, complex reductive Lie group. Then $\phi \in \text{Hom}(\Gamma, G)$ is reductive if and only if the restriction $\phi|_{\tilde{\Gamma}} \in \text{Hom}(\tilde{\Gamma}, G)$ is reductive.

**Proof.** Recall that $\phi \in \text{Hom}(\Gamma, G)$ is reductive if and only if the composition $\text{Ad} \circ \phi$ is a semisimple representation on $\mathfrak{g}$. Similarly, $\phi|_{\tilde{\Gamma}}$ is reductive if and only if $\text{Ad} \circ \phi|_{\tilde{\Gamma}}$ is semisimple. By $\Gamma/\tilde{\Gamma} \cong \mathbb{Z}_2$ and \cite{14}, Chap. 3, §9.8, Lemme 2, $\text{Ad} \circ \phi$ is semisimple if only if $\text{Ad} \circ \phi|_{\tilde{\Gamma}}$ is so. The result then follows. \qed

**Corollary 3.2.** Let $G$ be a connected, complex reductive Lie group. Suppose $P$ is a principal $G$-bundle over a compact non-orientable manifold $M$ whose oriented cover is $\pi : \tilde{M} \to M$. Then a flat connection $A$ on $P$ is reductive if and only if the pull-back $\pi^*A$ is a flat reductive connection on $\tilde{P} := \pi^*P$. 
3.2. Representation varieties associated to an index 2 subgroup

Let $\Gamma$ be a finitely generated group and let $\tilde{\Gamma}$ be an index 2 subgroup in $\Gamma$. Let $G$ be a connected complex Lie group and let $Z(G)$ be its center. For any $c \in \Gamma \setminus \tilde{\Gamma}$, we have $\text{Ad}_c|_{\tilde{\Gamma}} \in \text{Aut}(\tilde{\Gamma})$, and the class $[\text{Ad}_c|_{\tilde{\Gamma}}] \in \text{Aut}(\tilde{\Gamma})/\text{Inn}(\tilde{\Gamma})$ is independent of the choice of $c$. So we have a homomorphism $Z_2 \cong \{1, \tau\} \to \text{Aut}(\tilde{\Gamma})/\text{Inn}(\tilde{\Gamma})$ given by $\tau \mapsto [\text{Ad}_c|_{\tilde{\Gamma}}]$.

**Lemma 3.3.** $Z_2 \cong \{1, \tau\}$ acts on $\text{Hom}(\tilde{\Gamma}, G)/G$ and on $\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G$.

**Proof.** We define $\tau[\phi] = [\phi \circ \text{Ad}_c]$ for any $\phi \in \text{Hom}(\tilde{\Gamma}, G)$. The action is well-defined since if $[\phi'] = [\phi]$, i.e., $\phi' = \text{Ad}_g \circ \phi$ for some $g \in G$, then $\phi' \circ \text{Ad}_c = \text{Ad}_g \circ \phi \circ \text{Ad}_c$. The $\tau$-action is independent of the choice of $c$ because if $c' \in \Gamma \setminus \tilde{\Gamma}$ is another element, then $c' c^{-1} \in \tilde{\Gamma}$ and $\phi \circ \text{Ad}_{c'} = \text{Ad}_{\phi(c'^{-1})} \circ (\phi \circ \text{Ad}_c) \sim \phi \circ \text{Ad}_c$. We do have a $Z_2$-action because $\tau^2[\phi] = [\phi \circ \text{Ad}_c] = [\text{Ad}_{\phi(c'^{-1})} \circ \phi] = [\phi]$. Finally, if $\phi$ is in $\text{Hom}^{\text{red}}(\tilde{\Gamma}, G)$ or $\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$, then so is $\phi \circ \text{Ad}_c$. Thus $\tau$ acts on $\text{Hom}(\tilde{\Gamma}, G)/G$ and $\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G$. $\square$

**Proposition 3.4.** There exists a continuous map

$$ L: (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau \to Z(G)/2Z(G). $$

So $(\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau = \bigcup_{r \in Z(G)/2Z(G)} N^\text{good}_r$, where $N^\text{good}_r := L^{-1}(r)$.

**Proof.** If $\tau[\phi] = [\phi]$, then there exists $g \in G$ such that $\phi \circ \text{Ad}_c = \text{Ad}_g \circ \phi$. Since $c^2 \in \Gamma$, we have $\text{Ad}_g \circ \phi = \phi \circ \text{Ad}_c = \text{Ad}_{\phi(c^2)} \circ \phi$. Thus $z := g^2 \phi(c^2)^{-1} \in G_\phi = Z(G)$.

If $[\phi'] = [\phi]$, i.e., $\phi' = \text{Ad}_h \circ \phi$ for some $h \in G$, then $\phi' \circ \text{Ad}_c = \text{Ad}_{g'} \circ \phi$ for $g' = \text{Ad}_h g$. Since $g'^2 = \text{Ad}_h g^2 = z \text{Ad}_h \phi(c^2) = z \phi'(c^2)$, we obtain $$(g')^2 \phi'(c^2)^{-1} = z.$$ If $\phi \circ \text{Ad}_{c'} = \text{Ad}_{g'} \circ \phi$ holds for different choices of $c' \in \Gamma \setminus \tilde{\Gamma}$ and $g' \in G$, then $z' = (g')^2 \phi'(c'^2)^{-1} \in Z(G)$ from the above discussion. On the other hand, we have $\text{Ad}_{g^{-1} g'} \circ \phi = \text{Ad}_{\phi(c'^{-1} c)} \circ \phi$ as $c^{-1} c' \in \tilde{\Gamma}$. This gives us $t :=$
The restriction
Lemma 3.5.
Proof. First, the image \( \text{im}(\phi) \subset G \) is well-defined.

Since \( \phi \in \text{Hom}^\text{good}(\Gamma, G) \), the element \( g \in G/2Z(G) \) is uniquely determined by and depends continuously on \( \phi \). Therefore \( \phi \in \text{Hom}^\text{good}(\Gamma, G) \) depends continuously on \( \phi \in \text{Hom}^\text{good}(\Gamma, G) \).

If \( \phi \in \text{Hom}(\Gamma, G) \) satisfies \( \phi|_\Gamma \in \text{Hom}^\text{good}(\Gamma, G) \), then \( \phi \in \text{Hom}^\text{good}(\Gamma, G) \). However, \( \phi \in \text{Hom}^\text{good}(\Gamma, G) \) does not imply \( \phi|_\Gamma \in \text{Hom}^\text{good}(\Gamma, G) \). Let

\[
\text{Hom}^\text{good}(\Gamma, G) = \{ \phi \in \text{Hom}(\Gamma, G) : \phi|_\Gamma \in \text{Hom}^\text{good}(\Gamma, G) \}.
\]

We show that if \( \phi \in \text{Hom}^\text{good}(\Gamma, G) \), then \( L([\phi]) \) is the obstruction of extending \( \phi \) to a representation of \( \Gamma \).

Lemma 3.5. The restriction \( R : [\phi] \mapsto [\phi|_\Gamma] \) maps \( \text{Hom}^\text{good}(\Gamma, G)/G \) surjectively to \( N_0^\text{good} \).

Proof. First, the image \( \text{im}(R) \subset N_0^\text{good} \) because for any \( \phi \in \text{Hom}^\text{good}(\Gamma, G) \), \( \phi|_\Gamma \in \text{Hom}^\text{good}(\Gamma, G) \) by definition, so \( (\phi|_\Gamma) \circ \text{Ad}_c = \text{Ad}_{\phi(c)} \circ \phi|_\Gamma \sim \phi|_\Gamma \) and \( L([\phi|_\Gamma]) = [\phi(c)^2 \phi(c)^{-1}] = 0 \). We will show that in fact \( \text{im}(R) = N_0^\text{good} \).

Let \( \phi_0 \in \text{Hom}^\text{good}(\Gamma, G) \) such that \( \tau(\phi_0) = [\phi_0] \) and \( L([\phi_0]) = 0 \). Then there exist \( g \in G \) and \( t \in Z(G) \) such that \( \phi_0 \circ \text{Ad}_c = \text{Ad}_g \circ \phi_0 \) and \( g^2 \phi(c)^{-1} = t \). We can extend \( \phi_0 \) to \( \phi \in \text{Hom}(\Gamma, G) \) which is uniquely determined by the requirements \( \phi|_\Gamma = \phi_0 \) and \( \phi(c) = gt^{-1} \). Since \( \phi_0 \in \text{Hom}^\text{good}(\Gamma, G) \), \( \phi \in \text{Hom}^\text{good}(\Gamma, G) \) and therefore \( [\phi_0] \in \text{im}(R) \).

Proposition 3.6. \( R : \text{Hom}^\text{good}(\Gamma, G)/G \to N_0^\text{good} \) is a Galois covering map whose structure group is \( \{ s \in Z(G) : s^2 = e \} \).

Proof. We define an action of \( \{ s \in Z(G) : s^2 = e \} \) on \( \text{Hom}^\text{good}(\Gamma, G) \). For any such \( s \) and \( \phi \in \text{Hom}^\text{good}(\Gamma, G) \), we define \( s \cdot \phi \) by \( (s \cdot \phi)|_\Gamma = \phi|_\Gamma \) and \( (s \cdot \phi)|_{\Gamma \setminus \Gamma} = s(\phi|_{\Gamma \setminus \Gamma}) \) the group multiplication. It is clear that \( s \cdot \phi \in \text{Hom}^\text{good}(\Gamma, G) \).
Hom(Γ, G). Moreover, since

$$(s \cdot \phi)|_{\hat{\mathcal{R}}} = \phi|_{\hat{\mathcal{R}}} \in \text{Hom}^{\text{good}}(\hat{\mathcal{R}}, G), \quad s \cdot \phi \in \text{Hom}^{\text{good}}(\Gamma, G).$$

Clearly, the action descends to a well-defined action on $\text{Hom}^{\text{good}}(\Gamma, G)/G$ by $s \cdot [\phi] = [s \cdot \phi]$ preserving the fibres of $R$.

We show that this action is free. Suppose $s \cdot [\phi] = [\phi]$, then $s \cdot \phi = \text{Ad}_h \circ \phi$ for some $h \in G$. Since $\phi|_{\hat{\mathcal{R}}} = (s \cdot \phi)|_{\hat{\mathcal{R}}} = \text{Ad}_h \circ (\phi|_{\hat{\mathcal{R}}}) \in \text{Hom}^{\text{good}}(\hat{\mathcal{R}}, G)$, we get $h \in Z(G)$ and hence $s \cdot \phi = \phi$. Then $s\phi(c) = \phi(c)$ implies $s = e$.

It remains to show that the action is transitive on each fibre of $R$. Let $[\phi], [\phi'] \in \text{Hom}^{\text{good}}(\Gamma, G)$ such that $R([\phi]) = R([\phi'])$. Then there exists an $h \in G$ such that $\phi'|_{\hat{\mathcal{R}}} = \text{Ad}_h \circ (\phi|_{\hat{\mathcal{R}}})$. Thus

$$\text{Ad}_{\phi(c)h} \circ (\phi|_{\hat{\mathcal{R}}}) = \text{Ad}_{\phi'(c)} \circ (\phi|_{\hat{\mathcal{R}}}) = (\phi'|_{\hat{\mathcal{R}}}) \circ \text{Ad}_e = \text{Ad}_h \circ (\phi|_{\hat{\mathcal{R}}}) \circ \text{Ad}_e = \text{Ad}_{h\phi(c)} \circ (\phi|_{\hat{\mathcal{R}}}).$$

Hence $s := \phi(c)^{-1}h^{-1}\phi'(c)h \in Z(G)$ since $\phi|_{\hat{\mathcal{R}}} \in \text{Hom}^{\text{good}}(\hat{\mathcal{R}}, G)$. Furthermore

$$s^2 = \phi(c)^{-1}sh^{-1}\phi'(c)h = \phi(c^{-2})h^{-1}\phi'(c^2)h = \phi(c^{-2})\phi(c^2) = e.$$

Since we have $(s \cdot \phi)|_{\hat{\mathcal{R}}} = \phi|_{\hat{\mathcal{R}}} = \text{Ad}_{h^{-1}} \circ (\phi|_{\hat{\mathcal{R}}})$ and $(s \cdot \phi)(c) = s\phi(c) = \phi(c)s = (\text{Ad}_{h^{-1}} \circ \phi')(c)$, we get $s \cdot \phi = \text{Ad}_{h^{-1}} \circ \phi'$, or $[\phi'] = [s \cdot \phi]$. □

**Corollary 3.7.** *Under the above assumptions, there is a local homeomorphism from $\text{Hom}^{\text{good}}(\Gamma, G)/G$ to $(\text{Hom}^{\text{good}}(\hat{\mathcal{R}}, G)/G)^\tau$, which restricts to a local diffeomorphism on the smooth part. If $|Z(G)|$ is odd, this local homeomorphism (diffeomorphism, respectively) is a homeomorphism (diffeomorphism, respectively).*

**Proof.** The first statement follows easily from Propositions 3.4 and 3.6. If $|Z(G)|$ is odd, we get $Z(G)/2Z(G) \cong \{0\}$ and $(\text{Hom}^{\text{good}}(\hat{\mathcal{R}}, G)/G)^\tau = \text{N}^{\text{good}}_0$ by Proposition 3.4. Furthermore, since $\{s \in Z(G) : s^2 = e\} = \{e\}$, the covering map in Proposition 3.6 is a bijection. □

The involution $\tau$ also acts on $\text{Hom}^{\text{irr}}(\hat{\mathcal{R}}, G)/G$. Let

$$\text{Hom}^{\text{irr}}(\Gamma, G) = \{\phi \in \text{Hom}(\Gamma, G) : \phi|_{\hat{\mathcal{R}}} \in \text{Hom}^{\text{irr}}(\hat{\mathcal{R}}, G)\}.$$ 

By the same idea used in the proof of Propositions 3.4 and 3.6, we get
Corollary 3.8. If $G$ is simple, there exists a decomposition
\[
(\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G)^r = \bigcup_{r \in \mathbb{Z}(G)/2\mathbb{Z}(G)} N^r_{\text{irr}},
\]
where $N^r_{\text{irr}} = N^r_{\text{good}} \cap (\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G)^r$. Furthermore, there exists a Galois covering map $R: \text{Hom}^{\text{irr}}(\Gamma, G)/G \to N^0_{\text{irr}}$ with structure group $\{s \in \mathbb{Z}(G) : s^2 = e\}$. If $|\mathbb{Z}(G)|$ is odd, then there is a bijection from $\text{Hom}^{\text{irr}}(\tilde{\Gamma}, G)/G$ to $(\text{Hom}^{\text{irr}}(\Gamma, G)/G)^r$.

The results in this subsection show parts (1) and (2) of Theorem 1.2.

3.3. The Betti moduli space associated to a non-orientable surface

By Subsection 3.2 or parts (1) and (2) of Theorem 1.2 we know that a representation $\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ such that $\tau[\phi] = [\phi]$ can be extended to one on $\Gamma$ if and only if $L([\phi]) = 0$. When applied to $\Gamma = \pi_1(M)$ and $\tilde{\Gamma} = \pi_1(\tilde{M})$, where $M$ is non-orientable and $\tilde{M}$ is its oriented cover, we conclude that a $\tau$-invariant flat bundle over the $\tilde{M}$ corresponding to $\phi \in \text{Hom}^{\text{good}}(\tilde{\Gamma}, G)$ is the pull-back of a flat bundle over $M$ if and only if $L([\phi]) = 0$. We now consider the example when $M = \Sigma$ is a compact non-orientable surface, in which case we can characterize all the components $N^r_{\text{good}}$ explicitly. The principal $G$-bundles on $\Sigma$ are topologically classified by $H^2(\Sigma, \pi_1(G)) \cong \pi_1(G)/2\pi_1(G)$ whereas those on the oriented cover $\tilde{\Sigma}$ are classified by $H^2(\tilde{\Sigma}, \pi_1(G)) \cong \pi_1(G)$. The classes in these groups are the obstructions of lifting the structure group $G$ of the bundles to its universal cover group.

A compact non-orientable surface $\Sigma$ is of the form $\Sigma^k_\ell$ ($\ell \geq 0$, $k = 1, 2$), the connected sum of $2\ell + k$ copies of $\mathbb{R}P^2$. Then $\tilde{\Sigma}$ is a compact surface of genus $2\ell + k - 1$. For $k = 1$, we have
\[
\pi_1(\Sigma) = \langle a_i, b_i : 1 \leq i \leq \ell, c : c^{-2} \prod_{i=1}^{\ell} [a_i, b_i] \rangle,
\]
\[
\pi_1(\tilde{\Sigma}) = \langle a_i, b_i, a'_i, b'_i : 1 \leq i \leq \ell : \prod_{i=1}^{\ell} [a_i, b_i] \prod_{i=1}^{\ell} [a'_i, b'_i] \rangle.
\]

The inclusion $\pi_1(\tilde{\Sigma}) \to \pi_1(\Sigma)$ is given by $a_i \mapsto a_i$, $b_i \mapsto b_i$, $a'_i \mapsto \text{Ad}_c b_i$, $b'_i \mapsto \text{Ad}_c a_i$, $1 \leq i \leq \ell$. For $k = 2$, we have
\[
\pi_1(\Sigma) = \langle a_i, b_i : 1 \leq i \leq \ell, c, d : d^{-1} c d^{-1} e^{-1} \prod_{i=1}^{\ell} [a_i, b_i] \rangle,
\]
\[
\pi_1(\tilde{\Sigma}) = \langle a_0, b_0, a_i, b_i, a'_i, b'_i : 1 \leq i \leq \ell : [a_0, b_0] \prod_{i=1}^{\ell} [a_i, b_i] \prod_{i=1}^{\ell} [a'_i, b'_i] \rangle.
\]
The inclusion \( \pi_1(\hat{\Sigma}) \to \pi_1(\Sigma) \) is given by \( a_0 \mapsto d^{-1}, \; b_0 \mapsto c^2, \; a_i \mapsto a_i, \; b_i \mapsto b_i, \; a'_i \mapsto \text{Ad}_{a^{-1}} b_i, \; b'_i \mapsto \text{Ad}_{a^{-1}} a_i \) (1 \( \leq i \leq \ell \)). In both cases, \( c \in \pi_1(\Sigma) \setminus \pi_1(\hat{\Sigma}) \).

While a flat \( G \)-bundle over \( \Sigma \) may be non-trivial, its pull-back to \( \hat{\Sigma} \) is always trivial topologically [19]. We assume that \( G \) is semi-simple, simply connected and denote \( PG = G/Z(G) \). Then \( \pi_1(PG) = Z(G) \) and we have \( H^2(\Sigma, \pi_1(PG)) \cong Z(G)/2Z(G) \). The map

\[
O : \text{Hom}(\pi_1(\Sigma), PG)/PG \to Z(G)/2Z(G)
\]

that gives the obstruction class can be explicitly described as follows [18]. Let \( \phi \in \text{Hom}(\pi_1(\Sigma), PG) \). For \( k = 1 \), let \( \hat{\phi}(a_i), \; \hat{\phi}(b_i), \; \phi(c) \in G \) be any lifts of \( \phi(a_i), \; \phi(b_i), \; \phi(c) \in PG \), respectively. Then \( O([\phi]) \) is the element in \( Z(G)/2Z(G) \) represented by \( \phi(c)^2(\prod_{i=1}^{\ell} [\hat{\phi}(a_i), \hat{\phi}(b_i)])^{-1} \in Z(G) \). (It is easy to check that the class in \( Z(G)/2Z(G) \) is independent of the lifts.) The description of the case \( k = 2 \) is similar. Consequently, there is a decomposition

\[
\text{Hom}(\pi_1(\Sigma), PG)/PG = \bigcup_{r \in Z(G)/2Z(G)} M_r,
\]

where \( M_r = O^{-1}(r) \).

Let \( G \to PG, \; g \mapsto \bar{g} \) be the quotient map. Denote the induced map by \( \text{Hom}(\pi_1(\Sigma), G) \to \text{Hom}(\pi_1(\Sigma), PG), \; \phi \mapsto \bar{\phi} \). In this section, we need to be restricted to \( \text{Ad} \)-irreducible representations. The reason is that \( \phi \) is \( \text{Ad} \)-irreducible if and only if \( \bar{\phi} \) is so, whereas if \( \phi \) is good, \( \bar{\phi} \) is not necessarily so and its stabilizer may be larger than \( Z(G) \). We have

\[
\text{Hom}^\text{irr}(\pi_1(\Sigma), PG)/PG = \bigcup_{r \in Z(G)/2Z(G)} M_r^\text{irr},
\]

where \( M_r^\text{irr} = M_r \cap (\text{Hom}^\text{irr}(\pi_1(\Sigma), PG)/PG) \).

**Lemma 3.9.** There is a natural map

\[
\Psi : (\text{Hom}^\text{irr}(\pi_1(\hat{\Sigma}), G)/G)^\tau \to \text{Hom}^\text{irr}(\pi_1(\Sigma), PG)/PG
\]

satisfying \( L = O \circ \Psi \). Consequently, \( \Psi \) maps \( N_r^\text{irr} \) to \( M_r^\text{irr} \) for each \( r \in Z(G)/2Z(G) \).

**Proof.** Given \([\phi] \in (\text{Hom}^\text{irr}(\pi_1(\hat{\Sigma}), G)/G)^\tau\), there exists \( g \in G \) (which is unique up to \( Z(G) \) since \( G_{\bar{\phi}} = Z(G) \)) such that \( \text{Ad}_g \phi = \phi \circ \text{Ad}_g \). We define \( \hat{\phi} \in \text{Hom}(\pi_1(\Sigma), PG) \) by \( \hat{\phi} |_{\pi_1(\Sigma)} = \phi \) and \( \phi(c) = \bar{g} \). The representation \( \hat{\phi} \) is
a homomorphism because \( \hat{\phi}(c)^2 = g^2 = \tilde{\phi}(c^2) \), which follows from the result

\[
z = g^2\phi(c^2)^{-1} \in Z(G) \text{ in Proposition 3.4.}
\]

Since \( \tilde{\phi} \in \text{Hom}^{\text{irr}}(\pi_1(\tilde{\Sigma}), PG) \), we have \( \tilde{\phi} \in \text{Hom}^{\text{irr}}(\pi_1(\tilde{\Sigma}), PG) \). We define \( \Psi \) by \( \Psi([\phi]) = [\tilde{\phi}] \). To show that

\[
O([\phi]) = L([\phi]) = [z],
\]

we work in the case \( k = 1 \). By using the respective lifts \( \phi(a_i), \phi(b_i), g \in G \) of \( \hat{\phi}(a_i), \hat{\phi}(b_i), \hat{\phi}(c) \in PG \), we get

\[
O([\phi]) = \left[ g^2 \left( \prod_{i=1}^{\ell} [\phi(a_i), \phi(b_i)] \right)^{-1} \right] = [g^2\phi(c^2)^{-1}] = [z],
\]

where we have used the relation \( \prod_{i=1}^{\ell} [\phi(a_i), \phi(b_i)] = c^2 \) in \( \pi_1(\tilde{\Sigma}) \). The case

\[
k = 2
\]

is similar. \( \square \)

**Proposition 3.10.** The map

\[
\Psi : (\text{Hom}^{\text{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \to \text{Hom}^{\text{irr}}(\pi_1(\Sigma), PG)/PG
\]

is surjective. Consequently, \( \Psi : N^{\text{irr}}_r \to M^{\text{irr}}_r \) is surjective for each \( r \in Z(G)/2Z(G) \).

**Proof.** Let \( [\phi] \in \text{Hom}^{\text{irr}}(\Sigma, PG)/PG \). Although \( \phi(c) \in PG \), \( \text{Ad}_{\phi(c)} \) acts on

\( G \). We show the case \( k = 1 \) only. Fix the lifts \( \hat{\phi}(a_i), \hat{\phi}(b_i) \in G \) of \( \phi(a_i), \phi(b_i) \in PG \). Define \( \tilde{\phi} \in \text{Hom}(\pi_1(\tilde{\Sigma}), G) \) by setting \( \hat{\phi}(a_i) = \hat{\phi}(a_i), \hat{\phi}(b_i) = \hat{\phi}(b_i), \hat{\phi}'(a_i) = \text{Ad}_{\phi(c)} \hat{\phi}(b_i), \hat{\phi}'(b_i) = \text{Ad}_{\phi(c)} \hat{\phi}(a_i) \), for \( i = 1, \ldots, \ell \). This indeed defines a representation because

\[
\prod_{i=1}^{\ell} [\hat{\phi}(a_i), \hat{\phi}(b_i)] \prod_{i=1}^{\ell} [\hat{\phi}'(a_i), \hat{\phi}'(b_i)] = \prod_{i=1}^{\ell} [\hat{\phi}(a_i), \hat{\phi}(b_i)] \text{Ad}_{\phi(c)} \prod_{i=1}^{\ell} [\hat{\phi}(b_i), \hat{\phi}(a_i)] = e.
\]

The last equality is because \( \prod_{i=1}^{\ell} [\hat{\phi}(a_i), \hat{\phi}(b_i)] \in G \) projects to \( \phi(c)^2 \in PG \). Since \( \phi \) is Ad-irreducible, so is \( \tilde{\phi} \). \( [\phi] \) is \( \tau \)-invariant because

\[
\tilde{\phi} \circ \text{Ad}_c = \text{Ad}_{\phi(c)} \circ \tilde{\phi},
\]

which can be checked on the generators: \( \hat{\phi}(\text{Ad}_c a_i) = \hat{\phi}(b_i') = \text{Ad}_{\phi(c)} \hat{\phi}(a_i), \hat{\phi}(\text{Ad}_c a_i') = \text{Ad}_{\phi(c)} \hat{\phi}(b_i) = \text{Ad}_{\phi(c)} \hat{\phi}(a_i') \), etc. It is then obvious that \( \Psi([\tilde{\phi}]) = [\phi] \). \( \square \)
For the group $PG$, since $Z(PG)$ is trivial, $(\text{Hom}^{irr}(\pi_1(\hat{\Sigma}), PG)/PG)^\tau$ does not decompose according to Proposition 3.4 and the map

$$R: \text{Hom}^{irr}(\pi_1(\Sigma), PG)/PG \to (\text{Hom}^{irr}(\pi_1(\hat{\Sigma}), PG)/PG)^\tau$$

in Proposition 3.6 is bijective. The map $\Psi$ is in fact the composition of $(\text{Hom}^{irr}(\pi_1(\hat{\Sigma}), G)/G)^\tau \to (\text{Hom}^{irr}(\pi_1(\Sigma), PG)/PG)^\tau$ (induced by $G \to PG$) followed by $R^{-1}$. So for each $r \in Z(G)/2Z(G)$, the component $N^r$ of the fixed point set $(\text{Hom}^{irr}(\pi_1(\Sigma), G)/G)^\tau$ corresponds precisely to the component $M^r$ of $\text{Hom}^{irr}(\pi_1(\Sigma), PG)/PG$ which consists of flat $PG$-bundles over $\Sigma$ of topological type $r \in Z(G)/2Z(G)$. In particular, $N^0$ corresponds to the component $M^0$ of topologically trivial flat $PG$-bundles over $\Sigma$.

The results in subsection shows part (3) of Theorem 1.2.

4. Comparison of representation variety and gauge theoretical constructions

Suppose $M$ is a compact non-orientable manifold, $\pi: \tilde{M} \to M$ is the oriented cover, and $\tau: \tilde{M} \to \tilde{M}$ is the non-trivial deck transformation. In Subsection 2.2, we considered the natural lift of $\tau$ on $\tilde{P}C = \pi^*P^C$, where $P^C$ is a principal $G$-bundle over $M$. Such a lift, still denoted by $\tau$, is a $G$-bundle map satisfying $\tau^2 = \text{id}_{P^C}$ and induces involutions on the space $A(\tilde{P}^C)$ of connections on $\tilde{P}C$ and various moduli spaces. Moduli spaces associated to $P^C \to M$ are then related to the $\tau$-invariant parts of those associated to $\tilde{P}C \to \tilde{M}$ (cf. Theorem 1.1, especially part 3). This can also be seen in the language of representation varieties (cf. Lemma 3.5, Proposition 3.6 on $N^0$ and Corollary 3.7). To provide a geometric interpretation of the rest of the results in Subsections 3.2 and 3.3 on $N^0$ or $N^r$ when $r \neq 0$, we will need to generalize the setting in gauge theory.

Suppose $Q \to \tilde{M}$ is a principal $G$-bundle and the non-trivial deck transformation $\tau$ on $\tilde{M}$ is lifted to a bundle map $\tau_Q$ on $Q$, which is not necessarily an involution. Let $A$ be an irreducible connection on $Q$ that is invariant under $\tau_Q$ up to a gauge transformation, i.e., $\tau_Q^*A = \varphi^*A$ for $\varphi \in S(Q)$. Since $(\tau_Q \circ \varphi^{-1})^2$ is a gauge transformation on $Q$ which fixes $A$, it is in the center $Z(G)$. So by modifying $\tau_Q$ with a gauge transformation $\varphi$, we can assume that $\tau_Q$ satisfies $\tau_Q^2 = z \in Z(G)$. In this way, although $\tau_Q$ is not strictly an involution, it is so up to a gauge transformation, the right action of $z$ on $Q$. Since $\varphi$ and hence $\tau_Q$ can be adjusted by an element in $Z(G)$, $z = \tau_Q^2$ is well defined modulo $2Z(G)$. If $z = t^2 \in 2Z(G)$ ($t \in Z(G)$), then $z$ can be absorbed in $\tau_Q$ by a redefinition such that $\tau_Q$ is an honest involution, and
we are back to the situation before. In the general case when \( \tau_Q^2 = z \in Z(G) \) is not the identity element, since \( Z(G) \) acts trivially on the connections as gauge transformations, the action \( \tau_Q^2 : \mathcal{A}(Q) \to \mathcal{A}(Q) \) of \( \tau_Q \) on connections is still an honest involution. So we can define the invariant subspace \( \mathcal{A}(Q)^{\tau_Q} \) and much of the analysis in Subsections 2.2 and 2.3 applies.

We now consider flat connections and relate this generalized setting to our results on representation varieties. Choose base points \( x_0 \in M \) and \( \tilde{x}_0 \in \pi^{-1}(x_0) \subset \hat{M} \), and let \( \Gamma = \pi_1(M, x_0) \), \( \hat{\Gamma} = \pi_1(\hat{M}, \tilde{x}_0) \). We fix an element \( c \in \Gamma \setminus \hat{\Gamma} \).

**Proposition 4.1.** For any \( z \in Z(G) \), there is a 1-1 correspondence between the following two sets:

1. Isomorphism classes of pairs \((Q, A)\), where \( Q \to \hat{M} \) is a principal \( G \)-bundle with a \( G \)-bundle map \( \tau_Q \) lifting the deck transformation \( \tau \) on \( \hat{M} \) satisfying \( \tau_Q^2 = z \), \( A \) is a \( \tau_Q \)-invariant flat connection on \( Q \).

2. Equivalence classes of pairs \((\phi, g)\) under the diagonal adjoint action of \( G \), where \( \phi \in \text{Hom}(\hat{\Gamma}, G) \) and \( g \in G \) satisfy \( \phi \circ \text{Ad}_c = \text{Ad}_g \circ \phi \) and \( g^2 \phi(c^2)^{-1} = z \).

**Proof.** Given a bundle \( Q \) and a \( \tau_Q \)-invariant flat connection \( A \), let \( T_\alpha : Q_{\alpha(0)} \to Q_{\alpha(1)} \) be the parallel transport along a path \( \alpha : [0, 1] \to \hat{M} \). \( \tau_Q \)-invariance of the connection implies \( \tau_Q \circ T_\alpha = T_{\tau_\alpha} \circ \tau_Q \) for any path \( \alpha \). Let \( \gamma \) be a path in \( M \) from \( \tilde{x}_0 \) to \( \tau(\tilde{x}_0) \) so that \( [\pi \circ \gamma] = c \). Choose \( q_0 \in Q_{\tilde{x}_0} \) and let \( g \in G \) be defined by \( T_\gamma q_0 = \tau_Q(q_0)g^{-1} \). On the other hand, define \( \phi \in \text{Hom}(\hat{\Gamma}, G) \) by \( T_\alpha q_0 = q_0 \phi(a)^{-1} \) for any \( a \in \hat{\Gamma} \), where \( a \) is a loop in \( M \) based at \( \tilde{x}_0 \) such that \( [a] = a \). To check the conditions on \((\phi, g)\), we note that \( \tau_Q(T_\alpha q_0) = \tau_Q(q_0) \phi(a)^{-1} \) and

\[
T_{\tau_\alpha} \tau_Q(q_0) = T_{\gamma} \circ T_{\gamma(\tau_\alpha)\gamma^{-1}}(q_0 g) = (T_\gamma q_0) \phi(\text{Ad}_c a) g = \tau_Q(q_0) \text{Ad}_g^{-1} \phi(\text{Ad}_c a).
\]

So \( \tau_Q \)-invariance implies \( \phi(\text{Ad}_c a) = \text{Ad}_g \phi(a) \) for all \( a \in \hat{\Gamma} \). Similar calculations give \( \tau_Q(T_\gamma q_0) = \tau_Q(\tau_Q(q_0)g^{-1}) = q_0 g^{-1} \) and \( T_{\tau_\gamma(\tau_Q g)} = T_{\gamma(\tau_Q g)}(q_0 g) = q_0 \phi(c^2)^{-1} g \) which imply \( g^2 \phi(c^2)^{-1} = z \). If another point \( q'_0 = q_0 h \in Q_{\tilde{x}_0} \) is chosen (where \( h \in G \)), then the resulting pair is \((\phi', g') = (\text{Ad}_{h^{-1}} \circ \phi, \text{Ad}_{h^{-1}} g)\).

Conversely, given a pair \((\phi, g)\) satisfying the conditions, we want to construct a bundle \( Q \) together with a lifting \( \tau_Q \) of \( \tau \) such that \( \tau_Q^2 = z \) and a \( \tau_Q \)-invariant flat connection on \( Q \). Let \( \hat{M} \) be the universal covering space of \( \hat{M} \) (and of \( M \)). Then \( \hat{\Gamma} \) and \( \Gamma \) act on \( \hat{M} \), and \( M = \hat{M}/\hat{\Gamma}, \hat{M} = M/\Gamma \).
Let $Q = \tilde{M} \times_F G$, that is, points in $Q$ are equivalence classes $\{(x, h)\}$, where $x \in \tilde{M}$ and $h \in G$, and $(xa, h) \sim (x, \phi(a)h)$ for any $a \in \tilde{\Gamma}$. Let $\tau_Q : Q \to Q$ be defined by $\tau_Q : \{(x, h)\} \mapsto \{(xc^{-1}, gh)\}$. To check that $\tau_Q$ is well-defined, we note that for any $a \in \tilde{\Gamma}$, $(xac^{-1}, gh) \sim (xc^{-1}, \phi(Ad_c a)gh) = (xc^{-1}, g\phi(a)h)$. Clearly, $\tau_Q$ commutes with the right $G$-action on $Q$. Furthermore, $\tau_Q^2 = z$ because $\tau_Q^2 : \{(x, h)\} \mapsto \{(xc^{-2}, g^2h)\} = \{(x, \phi(c^{-2})g^2h)\} = \{(x, h)\}z$. It is easy to see that the trivial connection on $\tilde{M} \times G$ is $\tilde{\Gamma}$-invariant and descends to a flat connection on $Q$. The latter is invariant under $\tau_Q$ since the trivial connection on $\tilde{M} \times G$ is invariant under $(x, h) \mapsto (xc^{-1}, gh)$. Moreover, this connection induces the pair $(\phi, g)$. $\square$

**Remark 4.2.** We explain the gauge theoretic perspective of the results in Subsections 3.2 and 3.3 using the correspondence in Proposition 4.1.

1. As we noted, the $\tau$ is lifted to a $G$-bundle map $\tau_Q$ on $Q \to \tilde{M}$ such that $\tau_Q^2 = z \in Z(G)$, then $z$ is determined up to $2Z(G)$. Likewise, $z = g^2\phi(c^2)^{-1}$ is determined also modulo $2Z(G)$ by $[\phi] \in (\text{Hom}^{\text{good}}(\tilde{\Gamma}, G)/G)^\tau$ (Proposition 3.4). If $\tau_Q^2 = t^2$ for some $t \in Z(G)$, then $\tau_Q$ can be redefined as $\tau_Q' = \tau_Qt^{-1}$ so that $(\tau_Q')^2 = \text{id}_Q$. We then have a $G$-bundle $Q/\tau_Q' \to \tilde{M}$ over the non-orientable manifold $M$ whose pull-back of to $\tilde{M}$ is $Q$. If a flat connection is invariant under $\tau_Q$, it is also invariant under $\tau_Q'$ and hence descends to a flat connection on $Q/\tau_Q$. This is the situation in Lemma 3.5 and Proposition 3.6 (where $Q/\tau_Q$ was $P\Sigma$). In fact, from these results, we see that $[z] \in Z(G)/2Z(G)$ is the obstruction to the existence of a flat $G$-bundle on $M$ whose pull-back to $\tilde{M}$ is $Q$.

2. In general, $\tau_Q^2 \neq \text{id}_Q$ and the quotient of $Q$ by the subgroup generated by $\tau_Q$ is a bundle over $M$ with a fibre smaller than $G$. However, the $PG$-bundle $\tilde{Q} := Q/Z(G)$ over $\tilde{M}$ does have an honest involution $\tilde{\tau}_Q$. So $\tilde{Q}$ descends to a $PG$-bundle $Q/\tau_Q \to M$. Moreover, a $\tau_Q$-invariant flat connection on $Q$ descends to a $\tau_Q$-invariant flat connection on $\tilde{Q}$ and hence to a flat $PG$-connection on $\tilde{Q}/\tau_Q$. The bundle $\tilde{Q}/\tau_Q \to M$ is usually non-trivial as its structure group can not be lifted to $G$. (Otherwise, $Q$ would be its pull-back to $\tilde{M}$ and would admit a lift $\tau_Q$ so that $\tau_Q^2 = \text{id}_Q$.) Proposition 3.10 shows that when $G$ is simply connected and when $M = \Sigma$ is a non-orientable surface, the topological type, i.e., the obstruction to lifting the $PG$-bundle $\tilde{Q}/\tau_Q$ to a $G$-bundle over $M$ is precisely $[z] \in Z(G)/2Z(G)$.

**Remark 4.3.** 1. We can use $\tilde{x}_1 = \tau(\tilde{x}_0)$ as an another base point of the fundamental group of $\tilde{M}$ so that $\tilde{x}_0$ and $\tilde{x}_1$ play symmetric roles. The image of $\pi_1(\Sigma, \tilde{x}_1)$ under $\pi_*$ can be identified with $\tilde{\Gamma} \subset \Gamma$. The isomorphism $\tau_* : \tilde{\Gamma} \to \pi_1(\Sigma, \tilde{x}_1) \cong \tilde{\Gamma}$ is then $a \mapsto \text{Ad}_c^{-1} a$. Having chosen $q_0 \in Q_{\tilde{x}_0}$,
let $q_1 = \tau Q(q_0) \in Q_{\tilde{x}_1}$ and define $\phi_1: \pi_1(\tilde{\Sigma}, \tilde{x}_1) \to G$ by $T_{\alpha}q_1 = q_1\phi_1([\alpha])^{-1}$, where $\alpha$ is a loop in $\tilde{\Sigma}$ based at $\tilde{x}_1$. Using the identity $\tau Q \circ T_{\tau \alpha} = T_{\alpha} \circ \tau Q$, we obtain $\phi_1([\alpha]) = \phi([\tau \circ \alpha])$. Since $\tau_2^Q = z$, we also have the identity $T_{\alpha}z = T_{\tau \gamma} \circ \tau Q$. So upon the identification of $Q_{\tilde{x}_0}$ and $Q_{\tilde{x}_1}$ by $\tau Q$, the parallel transports along $\gamma$ and $\tau \circ \gamma$ differ by $z$.

2. When $M = \Sigma$ is a non-orientable surface, the approach of double base points was taken in [17, 19]. Consider for example the case $M = \Sigma^\ell$. Let $\alpha_i, \beta_i$ ($1 \leq i \leq \ell$) be loops in the oriented cover $\tilde{\Sigma}$ based at $\tilde{x}_0$ and let $\gamma$ be a path in from $\tilde{x}_0$ to $\tilde{x}_1$ so that $[\pi \circ \alpha_i] = a_i$, $[\pi \circ \beta_i] = b_i$, $[\pi \circ \gamma] = c$. Then an element in $N_r$ ($r = [z] \in Z(G)/2Z(G)$) can be represented by $(A_i, B_i, C; A'_i, B'_i, C') \in G^{4\ell+2}$ satisfying $A'_i = A_i$, $B'_i = B_i$, $C' = Cz$, where $A_i, B_i, C, A'_i, B'_i, C'$ are the holonomies along the loops or paths $\alpha_i, \beta_i, \gamma, \tau \circ \alpha_i, \tau \circ \beta_i, \tau \circ \gamma$, ($1 \leq i \leq \ell$), respectively. By the above discussion, we have the pattern $A_i = \phi([\alpha_i]) = \phi_1([\tau \circ \alpha_i]) = A'_i$, $B_i = \phi([\beta_i]) = \phi_1([\tau \circ \beta_i]) = B'_i$, ($1 \leq i \leq \ell$), $C' = Cz$ as in [17, 19].

Acknowledgments. We would like to thank U. Bruzzo, W. Goldman and E. Xia for useful discussions. The research of N.H. was supported by grant number 99-2115-M-007-008-MY3 from the National Science Council of Taiwan. The research of G.W. was supported by grant number R-146-000-200-112 from the National University of Singapore. The research of S.W. was partially supported by RGC grant HKU705612P (Hong Kong) and by MOST grant 105-2115-M-007-001-MY2 (Taiwan).

References

[1] M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1983), 523–615.

[2] D. Baraglia and L.P. Schaposnik, Higgs bundles and (A,B,A)-branes, Commun. Math. Phys. 331 (2014), 1271–1300.

[3] I. Biswas, O. García-Prada, and J. Hurtubise, Pseudo-real principal Higgs bundles on compact Kähler manifolds, Ann. Inst. Fourier 64 (2014), 2527–2562.

[4] I. Biswas, J. Huisman, and J. Hurtubise, The moduli space of stable vector bundles over a real algebraic curve, Math. Ann. 347 (2010), 201–233.

[5] N. Bourbaki, Groupes et Algèbres de Lie, Chap. II, III, Hermann, Paris, 1972.
884  Nan-Kuo Ho, Graeme Wilkin, and Siye Wu

[6] A. Clifford, Representation induced in an invariant subgroup, Ann. Math. 38 (1937), 533–550.

[7] K. Corlette, Flat G-bundles with canonical metrics, J. Diff. Geom. 28 (1988), 361–382.

[8] S. K. Donaldson, Twisted harmonic maps and the self-duality equations, Proc. London Math. Soc. 55 (1987), 127–131.

[9] J. J. Duistermaat, Convexity and tightness for restrictions of Hamiltonian functions to fixed point sets of an antisymplectic involution, Trans. Amer. Math. Soc. 275 (1983), 417–429.

[10] A. Fujiki, Hyperkähler structure on the moduli space of flat bundles, in: Prospects in Complex Geometry (Katata and Kyoto, 1989), Lecture Notes in Math. 1468, 1–83, Springer, Berlin, 1991.

[11] W. M. Goldman, The symplectic nature of fundamental groups of surfaces, Adv. Math. 54 (1984), 200–225.

[12] W. M. Goldman, Representations of fundamental groups of surfaces, in: Geometry and topology (College Park, Md., 1983/84), Lecture Notes in Math. 1167, 95–117, Springer, Berlin, 1985.

[13] W. M. Goldman and J. J. Millon, The deformation theory of representations of fundamental groups of compact Kähler manifolds, Publ. Math. IHES 67 (1988), 43–96.

[14] T. Hausel and M. Thaddeus, Mirror symmetry, Langlands duality, and the Hitchin system, Invent. Math. 153 (2003), 197–229.

[15] N. J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. 55 (1987), 59–126.

[16] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, Hyperkähler metrics and supersymmetry, Commun. Math. Phys. 108 (1987), 535–589.

[17] N.-K. Ho, The real locus of an involution map on the moduli space of flat connections on a Riemann surface, Inter. Math. Res. Notices 61 (2004), 3263–3285.

[18] N.-K. Ho and C.-C. M. Liu, Connected components of the space of surface group representations, Inter. Math. Res. Notices 44 (2003), 2359–2371.

[19] N.-K. Ho and C.-C. M. Liu, Yang-Mills connections on nonorientable surfaces, Commun. Anal. Geom. 16 (2008), 617–679.
Hitchin’s equations on a nonorientable manifold

[20] D. Johnson and J. J. Millson, Deformation spaces associated to compact hyperbolic manifolds, in: Discrete groups in geometry and analysis (New Haven, CT, 1984), Progr. Math. 67, 48–106, Birkhäuser, Boston, MA, 1987.

[21] A. Kapustin and E. Witten, Electric-magnetic duality and the geometric Langlands program, Commun. Number Theory Phys. 1 (2007), 1–236.

[22] H.-J. Kim, Moduli of Hermite-Einstein vector bundles, Math. Z. 195 (1987), 143–150.

[23] S. Kobayashi, Differential geometry of complex vector bundles, Princeton Univ. Press, Princeton, NJ, 1987.

[24] K. Meyer, Hamiltonian systems with a discrete symmetry, J. Diff. Equations 41 (1981), 228–238.

[25] L. O’Shea and R. Sjamaar, Moment maps and Riemannian symmetric pairs, Math. Ann. 317 (2000), 415–457.

[26] R. W. Richardson, Conjugacy classes of n-tuples in Lie algebras and algebraic groups, Duke Math. J. 57 (1988), 1–35.

[27] F. Schaffhauser, Real points of coarse moduli schemes of vector bundles on a real algebraic curve, J. Symplectic Geom. 10 (2012), 503–534.

[28] A. S. Sikora, Character Varieties, Trans. Amer. Math. Soc. 364 (2012), 5173–5208.

[29] C. T. Simpson, Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization, J. Amer. Math. Soc. 1 (1988), 867–918.

[30] C. T. Simpson, Higgs bundles and local systems, Publ. Math. IHES 75 (1992), 5–95.

[31] C. T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety II, Publ. Math. IHES 80 (1995), 5–79.
Department of Mathematics, National Tsing Hua University
No. 101, Sec. 2, Guangfu Rd., Hsinchu 300, Taiwan
and National Center for Theoretical Sciences
Taipei 106, Taiwan
E-mail address: nankuo@math.nthu.edu.tw

Department of Mathematics
National University of Singapore
Block S17, 10 Lower Kent Ridge Rd., Singapore 119076
E-mail address: graeme@nus.edu.sg

Department of Mathematics, University of Hong Kong
Current address:
Department of Mathematics, National Tsing Hua University
No. 101, Sec. 2, Guangfu Rd., Hsinchu 300, Taiwan
E-mail address: swu@math.nthu.edu.tw

Received August 7, 2015