NORM INFLATION FOR THE CUBIC NONLINEAR HEAT EQUATION
ABOVE THE SCALING CRITICAL REGULARITY

ILYA CHEVYREV, TADAHIRO OH, AND YUZHAO WANG

Abstract. We consider the ill-posedness issue for the cubic nonlinear heat equation and prove norm inflation with infinite loss of regularity in the Hölder-Besov space $C^s = B^s_{\infty, \infty}$ for $s \leq -\frac{2}{3}$. In particular, our result includes the subcritical range $-1 < s \leq -\frac{2}{3}$, which is above the scaling critical regularity $s = -1$ with respect to the Hölder-Besov scale. In view of the well-posedness result in $C^s$, $s > -\frac{2}{3}$, our ill-posedness result is sharp.

1. Introduction

We consider the cubic nonlinear heat equation (NLH):

\[
\begin{cases}
\partial_t u - \Delta u + u^3 = 0 \\
u|_{t=0} = u_0,
\end{cases}
\quad (x, t) \in \mathcal{M} \times \mathbb{R}_+,
\]

where $\mathcal{M} = \mathbb{R}^d$ or $\mathbb{T}^d$ with $\mathbb{T}^d = \mathbb{R}^d/(2\pi \mathbb{Z})^d$. The equation (1.1) and its variants (see, for example, (1.7) and (1.8) below) appear in various physical settings and have been studied extensively from both theoretical and applied points of view. See [37, 15, 5, 33] for the classical well-posedness results on (1.1). Over the last two decades, there has been an increasing interest in studying the equation (1.1) with (singular) stochastic forcing and/or random initial data; see for example [13, 19, 17, 18]. Such study often requires the understanding of solutions to (1.1) in the low regularity setting, where the equation (1.1) is believed to be ill-posed. However, the aforementioned ill-posedness seems to be part of the folklore, and our main goal in this paper is to provide a rigorous proof of this ill-posedness by exhibiting a strong instability known as a norm inflation, thus completing the well-posedness study of (1.1).

In the following, we study the equation (1.1) with respect to the Hölder-Besov space $C^s(\mathcal{M}) = B^s_{\infty, \infty}(\mathcal{M})$, where the latter space $B^s_{\infty, \infty}(\mathcal{M})$ denotes the usual Besov space defined
by the norm:

\[ \|u\|_{B^s_{\infty, \infty}} = \sup_{j \in \mathbb{N} \cup \{0\}} 2^{js} \|P_j u\|_{L^\infty}. \]  

(1.2)

Here, \( P_j \) denotes the non-homogeneous Littlewood-Paley projector onto the frequencies \( \{ \xi \in \hat{\mathcal{M}} : |\xi| \sim 2^j \} \) for \( j \geq 1 \) and \( \{ \xi \in \hat{\mathcal{M}} : |\xi| \lesssim 1 \} \) for \( j = 0 \), where \( \hat{\mathcal{M}} \) denotes the Pontryagin dual of \( \mathcal{M} \), i.e. \( \hat{\mathcal{M}} = \mathbb{R}^d \) if \( \mathcal{M} = \mathbb{R}^d \) and \( \hat{\mathcal{M}} = \mathbb{Z}^d \) if \( \mathcal{M} = \mathbb{T}^d \). Recall the following Schauder estimate:

\[ \|e^{t\Delta} f\|_{C^\sigma(\mathcal{M})} \lesssim t^{-\frac{d-2\sigma}{2}} \|e^{t\Delta} f\|_{C^\sigma(\mathcal{M})} \]  

(1.3)

for any \( \sigma \geq s \) and \( 0 < t \leq 1 \). When \( \mathcal{M} = \mathbb{R}^d \), (1.3) follows from Young’s inequality and estimating the \( W^{\sigma-s,1,\text{crit}} \) norm of the heat kernel for \( e^{t\Delta} \); see [1] Lemma 2.4] (for the case \( \sigma = s \)). When \( \mathcal{M} = \mathbb{T}^d \), (1.3) follows from Young’s inequality and the Poisson summation formula to pass an estimate on (fractional derivatives of) the heat kernel on \( \mathbb{T}^d \) to that in a weighted Lebesgue space on \( \mathbb{R}^d \); see the proof of Lemma 2.3 in [23] for such a computation (but done for \( e^{-\gamma t\Delta} \)). Then, using the Schauder estimate (1.3), one can easily show that (1.1) is locally well-posed in \( C^\sigma(\mathcal{M}) \) for \( s > -\frac{2}{3} \). The well-posedness issue of (1.1) for \( s \leq -\frac{2}{3} \), however, appears unresolved.

As is well known, symmetries provide important heuristics in studying well-posedness issues of nonlinear PDEs. We now recall the scaling symmetry for (1.1); if \( u(x, t) \) is a solution to (1.1) on \( \mathbb{R}^d \), then \( u^\lambda(x, t) := \lambda^{-1} u(\lambda^{-1} x, \lambda^{-2} t) \) is also a solution to (1.1) on \( \mathbb{R}^d \) with the scaled initial data \( u_0^\lambda := \lambda^{-1} u(\lambda^{-1} x) \). A direct computation shows that this scaling symmetry leaves the homogeneous Besov norm \( \|u_0^\lambda\|_{B^{-s}_{\infty, \infty}} \) invariant, thus inducing the scaling critical regularity \( s_{\text{crit}} := -1 \) with respect to the Hölder-Besov scale. In studying an evolution equation with initial data in a Banach space \( B^s \) of regularity \( s \) (think of \( B^s \) as \( C^s \), the Sobolev space \( H^s \), etc.), it is commonly conjectured that the equation is well-posed in \( B^s \) for \( s > s_{\text{crit}} \), while it is ill-posed for \( s < s_{\text{crit}} \). In fact, on function spaces of positive regularities, this conjecture has been confirmed affirmatively for many parabolic and dispersive PDEs. However, working on function spaces of negative regularities, this heuristics provided by the scaling argument is known to break down in some instances; see 20, 26, 32, 10, 21, 14 for ill-posedness results above critical regularities. See also 3, 8, 20, 29, 31, 30 for ill-posedness at critical regularities.

In this paper, we establish a norm inflation with infinite loss of regularity for (1.1) above the scaling critical regularity \( s_{\text{crit}} = -1 \).

**Theorem 1.1.** Given \( d \in \mathbb{N} \), let \( \mathcal{M} = \mathbb{R}^d \) or \( \mathbb{T}^d \). Let \( s \leq -\frac{2}{3} \). Then, given any \( \varepsilon > 0 \), there exist a solution \( u_\varepsilon \) to (1.1) on \( \mathcal{M} \) and \( t_\varepsilon \in (0, \varepsilon) \) such that

\[ \|u_\varepsilon(0)\|_{C^s(\mathcal{M})} < \varepsilon \quad \text{and} \quad \|P_0 u_\varepsilon(t_\varepsilon)\|_{L^\infty(\mathcal{M})} > \varepsilon^{-1}. \]  

(1.4)

Hölder-Besov spaces play a dominant role in parabolic singular SPDEs. The threshold \( s > -\frac{2}{3} \) appears in the study of the parabolic \( \Phi_3 \) model [7, 17] (see also [18]), and our results prove that this threshold is optimal.

We also remark that, since \( \|u_\varepsilon(t_\varepsilon)\|_{C^s} \geq \|P_0 u_\varepsilon(t_\varepsilon)\|_{L^\infty} \), the second estimate in (1.3) implies \( \|u_\varepsilon(t_\varepsilon)\|_{C^s} > \varepsilon^{-1} \) for any \( \sigma \in \mathbb{R} \). When \( \sigma = s \), the statement corresponds to the usual norm inflation introduced in [12] in the context of the nonlinear Schrödinger equations (NLS) and the nonlinear wave equations. In proving Theorem 1.1 we exploit a robust high-to-low

---

1The upper bound on \( t \) is needed on \( \mathbb{T}^d \).
energy transfer, which allows us to prove (1.4), thus leading to the so-called “infinite loss of
regularity”; see, for example, [6,14].

Note that Theorem 1.1 in particular implies discontinuity of the solution map \( \Phi : u_0 \in C^s(M) \mapsto u \in C([0,T];C^s(M)) \) at the trivial function \( u_0 \equiv 0 \) for \( s \leq -\frac{2}{3} \), thus establishing ill-posedness of the cubic NLH (1.1). Furthermore, in view of the aforementioned local well-posedness in \( C^s(M) \) for \( s > -\frac{2}{3} \), Theorem 1.1 is sharp. See also Remark 1.5 (iii).

Our proof of Theorem 1.1 is based on the Fourier analytic approach [20,21,29]. In [20], Iwabuchi and Ogawa introduced a new method for proving ill-posedness of evolution equations, exploiting high-to-low energy transfer in the second Picard iterate. In [21], Kishimoto systematically applied this approach to prove ill-posedness of NLS. See the work by Bejenaru and Tao [2] for a precursor of this approach. We also mention the work by Bourgain and Pavlović [3] on the Navier-Stokes equations, where an analogous high-to-low energy transfer was exploited. See also [38,35]. In [20,21], the (scaled) modulation space \( M_{2,1} \) and its algebra property played an important role. In [29], the second author implemented a slight simplification of this approach by working on the Wiener algebra (instead of the modulation spaces). We point out that the H"{o}lder-Besov space \( C^s(M) \) is not a Fourier lattice\(^3\) and thus is not quite suitable for this Fourier analytic approach [20,21,29]. For example, in [20,26], this method was applied to prove ill-posedness of the (fractional) heat equations but the results were restricted to the \( L^2 \)-based Sobolev spaces \( H^s \) and the \( L^2 \)-based Besov spaces \( B^s_{2,q} \), both of which are Fourier lattices. As we see below, our choice of initial data turns out to be rather simple (see (3.1) and (4.1) below), enabling us to adapt this Fourier analytic approach to the \( L^\infty \)-based function spaces similar in spirit to [3].

As in [20,21,29], we first express a solution \( u \) to (1.1) with \( u|_{t=0} = u_0 \) in the power series expansion:

\[
    u = \sum_{j=0}^{\infty} \Xi_j(u_0),
\]

where \( \Xi_j(u_0) \) denotes homogeneous multilinear terms (in \( u_0 \)) of degree \( 2j+1 \). In [20,21], \( \Xi_j \)'s were defined by a recursive relation. In proving Theorem 1.1 we instead follow the approach in [29] and define the \( j \)-th term \( \Xi_j \) directly via the power series expansion indexed by trees. See Section 2. There are several advantages to this latter approach. First, it allows us to establish nonlinear estimates without an induction. By a slight modification, this approach allows us to treat the Allen-Cahn equation (see (1.7) below) in a straightforward manner. Furthermore, arguing as in [29], we can easily extend Theorem 1.1 to a norm inflation based at a general initial condition \( u_0 \in C^s(M) \); given any \( u_0 \in C^s(M) \) and any \( \varepsilon > 0 \), there exist a solution \( u_\varepsilon \) to (1.1) and \( t_\varepsilon \in (0,\varepsilon) \) such that

\[
    \|u_\varepsilon(0) - u_0\|_{C^s(M)} < \varepsilon \quad \text{and} \quad \|P_0 u_\varepsilon(t_\varepsilon)\|_{L^\infty(M)} > \varepsilon^{-1}, \tag{1.5}
\]

thus establishing discontinuity of the solution map everywhere in \( C^s(M) \).

We conclude this introduction by several remarks.

**Remark 1.2.** (i) Unlike the dispersive case treated in [20,21,29,10], we work in the \( L^\infty \)-based function spaces and thus our argument is insensitive to dimensions \( d \geq 1 \).

\(^2\)The work [14] does not state their result as infinite loss of regularity but it follows from the proof.

\(^3\)A Fourier lattice is a space, where a norm depends only on the absolute value of the Fourier transform.
(ii) For the standard cubic nonlinearity, Choffrut and Pocovnicu \[10\] proved a norm inflation for the fractional nonlinear Schrödinger equation:

\[ i\partial_t u + (-\Delta)^\alpha u + |u|^2 u = 0 \]

above the scaling critical regularity, provided that \( \alpha > 2 \). Namely, a dispersion stronger than the standard Laplacian was needed. It is interesting to compare this with the parabolic case (Theorem 1.1), where a norm inflation above the scaling critical regularity hold true for the standard Laplacian. See also \[21\] for a norm inflation above the scaling critical regularity for the quadratic NLS with the less physical nonlinearity \( |u|^2 \).

Remark 1.3. (i) While we present the result for the defocusing case (i.e. with the + -sign in (1.1)), our argument is insensitive to the defocusing / focusing nature of the nonlinearity and thus it also applies to the focusing case:

\[ \partial_t u - \Delta u - u^3 = 0. \] (1.6)

Furthermore, our argument is robust so that it can be applied to treat a general power-type nonlinearity \( u^k \). See \[21\] \[14\] in the case of the nonlinear Schrödinger equations and the nonlinear wave equations with a general power-type nonlinearity.

(ii) As mentioned above, a slight modification of the proof of Theorem 1.1 yields a norm inflation with a general initial data as stated in (1.5). See \[29\] for such an argument.

Remark 1.4. Consider the Allen-Cahn equation:

\[ \partial_t u - \Delta u - u^3 = 0. \] (1.7)

By the Schauder estimate, one can easily prove local well-posedness of (1.7) in \( \mathcal{C}^s(\mathcal{M}) \) for \( s > -\frac{2}{3} \). On the other hand, a slight modification of the proof of Theorem 1.1 yields a norm inflation for (1.7) in \( \mathcal{C}^s(\mathcal{M}) \) for \( s \leq -\frac{2}{3} \). See Remark 3.1.

Next, consider the Cahn-Hilliard equation:

\[ \partial_t u + \Delta^2 u - \Delta u^3 = 0. \] (1.8)

and its variants:

\[ \partial_t u + \Delta^2 u - u^2 \Delta u = 0 \quad \text{or} \quad \partial_t u + \Delta^2 u - u(\partial u)(\partial u) = 0. \] (1.9)

These equations all scale the same way and the scaling critical regularity is \( s = -1 \) in the Hölder-Besov scale. By the Schauder estimate:

\[ \|e^{-t\Delta^2}u_0\|_{\mathcal{C}^s} \lesssim t^{-\frac{2s+2}{s}}\|u_0\|_{\mathcal{C}^s} \] (1.10)

for any \( \sigma \geq s \) and \( 0 < t \leq 1 \)\footnote{As in (1.3), the upper bound on \( t \) is needed on \( T_4 \).}, we can prove local well-posedness of the Cahn-Hilliard equation (1.8) in \( \mathcal{C}^s(\mathcal{M}) \) for the full subcritical range \( s > -1 \). On the other hand, as for the equations in (1.9), the Schauder estimate (1.10) yields local well-posedness in \( \mathcal{C}^s(\mathcal{M}) \) only for a partial range \( s > -\frac{2}{5} \). In fact, for these variant equations in (1.9), our argument works and yields a norm inflation as in Theorem 1.1 holds for \( s \leq -\frac{2}{3} \). See Remark 3.1.

Due to the presence of the Laplacian on the cubic nonlinearity in (1.8), our strategy for proving norm inflation via the high-to-low energy transfer (or using high-to-high energy transfer) does not work for the Cahn-Hilliard equation (1.8). It is of interest to note how the equations in (1.8) and (1.9) scale the same way but that their well / ill-posedness results are quite different. We point out that, as for proving a norm inflation, it seems easier to treat a nonlinearity that is not a total derivative, as in (1.9). See also \[9\] \[36\].
Remark 1.5. (i) It is worthwhile to note that the threshold regularity \( s = -\frac{\delta}{2} \) also appears in the study of long-time behavior of small data solutions. In [25], Miao, Yuan, and Zhang proved small data global well-posedness of (1.1) and (1.6) with small initial data in \( B^s_{p,q}({\mathbb R}^d) \), where \( s = \frac{d}{p} - 1, \ d < p < 3d, \) and \( 1 \leq q \leq \infty \). Here, the regularity \( s = \frac{d}{p} - 1 \) corresponds to the scaling critical regularity for the \( L^p \)-based Besov spaces. Note that the condition on \( p \) implies that \( -\frac{2}{3} < s < 0 \). In [3], Brandolese and Cortez proved existence of a (smooth) finite time blowup solution to (1.6) with small initial data in \( \dot{B}^{-\frac{4}{3}}_{3d,q}({\mathbb R}^d), \ 3 < q \leq \infty \), thus providing a negative answer to Meyer’s question [24].

(ii) By a slight modification of the proof, we can extend the norm inflation to the \( L^p \)-based Besov spaces \( B^s_{p,q}(T^d) \) for any \( 1 \leq p \leq \infty \) in the case of the torus \( T^d \) for (a) \( s < -\frac{2}{3} \) and \( 1 \leq q \leq \infty \) and (b) \( s = -\frac{2}{3} \) and \( 3 < q \leq \infty \). On \( {\mathbb R}^d \), we need to slightly modify our choice of initial data by localizing in space, but we expect the same result also holds on \( {\mathbb R}^d \).

(iii) Recall the following characterization of the Besov norm ([22, Theorem 5.3]):

\[
\|f\|_{B^s_{\infty,3}} \sim \|e^{\Delta} f\|_{L^\infty} + \left( \int_0^1 t^{-1} \|t^{\frac{1}{2}} e^{t \Delta} f\|_{L^3}^3 \, dt \right)^{\frac{1}{3}} \\
= \|e^{\Delta} f\|_{L^\infty} + \left( \int_0^1 \|e^{t \Delta} f\|_{L^3}^3 \, dt \right)^{\frac{1}{3}}.
\]  

Using this characterization, it is straightforward to prove local well-posedness of (1.1) in \( B^{-\frac{4}{3}}_{\infty,3}(T^d) \). For example, one can first run a contraction argument in \( L^3([0,T]; L^\infty(T^d)) \) to construct a solution \( u \in L^3([0,T]; L^\infty(T^d)) \) to (1.1), and then a posteriori show that \( u \) belongs to \( C([0,T]; B^{-\frac{4}{3}}_{\infty,3}(T^d)) \). Therefore, our results (Theorem 1.1 and Remark 1.5(ii) above) are sharp across the entire Besov scale \( B^{s}_{\infty,q}(T^d) \) for \( s \in {\mathbb R} \) and \( 1 \leq q \leq \infty \).

Remark 1.6. Using probabilistic methods, the first author [9] recently showed the same type of norm inflation as (1.14) in the scaling subcritical space \( C^{-\frac{1}{2}}(T^d) \) (but not in \( B^{-\frac{1}{2}}_{\infty,q}(T^d) \) for finite \( q < \infty \)) for a heat equation with dominant nonlinearity \( u \times D Du \) that is not a total derivative, where the scaling critical space is \( C^{-\frac{1}{2}}(T^d) \).

2. Preliminary analysis

In this section, we review a basic local well-posedness result for (1.1) and the power series expansion of a solution. While the following presentation closely follows that in [29], we include details for readers’ convenience.

Footnotes:
5 In the endpoint case \( s = -\frac{2}{3} \), we need to slightly modify the proof by replacing \( \log K \) in (1.1) by \( K^{\frac{1}{4}} \). The restriction on \( q \) is needed to have the \( B^s_{p,q} \)-norm of the initial data tend to 0 as in (1.3).
6 Here, as in [27, 28], we view the Besov space \( B^{s}_{\infty,3}(T^d) \) as the completion of \( C^\infty(T^d) \) under the norm (1.11). Alternatively, i.e. if we view the Besov space \( B^{s}_{\infty,3}(T^d) \) as the collection of elements in \( \mathcal{D}’(T^d) \) with finite \( B^{s}_{\infty,3} \)-norms, then a solution \( u \) to (1.1) does not belong to \( C([0,T]; B^{s}_{\infty,3}(T^d)) \) in general. With this latter definition of the Besov space, even the linear solution \( e^{t \Delta} u_0 \) does not belong to \( C([0,T]; B^{s}_{\infty,3}(T^d)) \) in general. In this case, we can actually show that, given \( u_0 \in B^{s}_{\infty,3}(T^d) \), the nonlinear part \( v(t) := u(t) - e^{t \Delta} u_0 \) belongs to \( C([0,T]; L^\infty(T^d)) \) and that the map \( u_0 \mapsto v = v(u_0) \) is locally Lipschitz with respect to \( u_0 \in B^{s}_{\infty,3}(T^d) \).
We say that $u$ is a solution to (1.1) with $u|_{t=0} = u_0$ if $u$ satisfies the following Duhamel formulation:

$$u(t) = e^{t \Delta} u_0 + \mathcal{I}[u](t), \quad (2.1)$$

where $\mathcal{I}$ denotes the Duhamel integral operator defined by

$$\mathcal{I}[u_1, u_2, u_3](t) := -\int_0^t e^{(t-t') \Delta} \left( \prod_{j=1}^3 u_j(t') \right) dt'\quad (2.2)$$

with a shorthand notation $\mathcal{I}[u] := \mathcal{I}[u, u, u]$ when all the three arguments $u_1, u_2,$ and $u_3$ are identical. Then, from the boundedness of the heat semigroup and the algebra property of $C^{s_0}(\mathcal{M}), s_0 > 0$, we have the following local well-posedness of (1.1) in $C^{s_0}(\mathcal{M}), s_0 > 0$.

**Lemma 2.1.** Let $s_0 > 0$. Then, the cubic NLH (1.1) is locally well-posed in $C^{s_0}(\mathcal{M})$. More precisely, given $u_0 \in C^{s_0}(\mathcal{M})$, there exist $T \sim \|u_0\|_{C^{s_0}}^{-2} > 0$ and a unique solution $u \in C([0,T];C^{s_0}(\mathcal{M}))$ satisfying (2.1).

Fix $u_0 \in C^{s_0}(\mathcal{M})$ for some $s_0 > 0$, and let $u$ be the solution to (1.1) with $u|_{t=0} = u_0$ constructed in Lemma 2.1. Following the approach in [29], we express the solution $u$ in the power series expansion; see [34, 16, 11], where the power series expansion was used to construct solutions to parabolic and dispersive equations. We first recall the following definition of (ternary) trees.

**Definition 2.2.** (i) Given a partially ordered set $\mathcal{T}$ with partial order $\leq$, we say that $b \in \mathcal{T}$ with $b \leq a$ and $b \neq a$ is a child of $a \in \mathcal{T}$, if $b \leq c \leq a$ implies either $c = a$ or $c = b$. If the latter condition holds, we also say that $a$ is the parent of $b$.

(ii) A tree $\mathcal{T}$ is a finite partially ordered set, satisfying the following properties:

(a) Let $a_1, a_2, a_3, a_4 \in \mathcal{T}$. If $a_4 \leq a_2 \leq a_1$ and $a_4 \leq a_3 \leq a_1$, then we have $a_2 \leq a_3$ or $a_3 \leq a_2$.

(b) A node $a \in \mathcal{T}$ is called terminal, if it has no child. A non-terminal node $a \in \mathcal{T}$ is a node with exactly three children.

(c) There exists a maximal element $r \in \mathcal{T}$ (called the root node) such that $a \leq r$ for all $a \in \mathcal{T}$.

(d) $\mathcal{T}$ consists of the disjoint union of $\mathcal{T}^0$ and $\mathcal{T}^\infty$, where $\mathcal{T}^0$ and $\mathcal{T}^\infty$ denote the collections of non-terminal nodes and terminal nodes, respectively.

Note that the number $|\mathcal{T}|$ of nodes in a tree $\mathcal{T}$ is $3j + 1$ for some $j \in \mathbb{N} \cup \{0\}$, where $|\mathcal{T}^0| = j$ and $|\mathcal{T}^\infty| = 2j + 1$. Let us denote the collection of trees in the $j$th generation (i.e. with $j$ parental nodes) by $\mathcal{T}(j)$, i.e.

$$\mathcal{T}(j) := \{ \mathcal{T} : \mathcal{T} \text{ is a tree with } |\mathcal{T}| = 3j + 1 \}.$$  

Then, there exists $C_0 > 0$ such that

$$\#\mathcal{T}(j) \leq C_0^j \quad (2.3)$$

for any $j \in \mathbb{N} \cup \{0\}$. See [29] for the proof of (2.3).

We now define a map $\Psi = \Psi_{u_0} : \bigcup_{j=0}^\infty \mathcal{T}(j) \to \mathcal{D}'(\mathcal{M} \times [0,T])$ as follows. Given a tree $\mathcal{T} \in \mathcal{T}(j), j \in \mathbb{N} \cup \{0\}$, we define $\Psi(\mathcal{T})$ by the following rules:

(i) Replace a non-terminal node “*” by the Duhamel integral operator $\mathcal{I}$ defined in (2.2) with its three children as arguments $u_1, u_2,$ and $u_3$.

(ii) Replace a terminal node “." by the linear solution $e^{t \Delta} u_0$. 


Note that, if $T \in T(j)$, then $\Psi(T)$ is $(2j+1)$-linear in $u_0$. For example, we have $\Psi(\cdot) = e^{t\Delta}u_0$, where “•” denotes the trivial tree, consisting only of the root node. Similarly, we have $\Psi(\mathcal{A}) = I[e^{t\Delta}u_0]$ and $\Psi(\mathcal{A}^2) = I[I[e^{t\Delta}u_0], e^{t\Delta}u_0, e^{t\Delta}u_0]$. Consequently, we have $\Psi(\mathcal{A}^2) = I[I[e^{t\Delta}u_0], e^{t\Delta}u_0, e^{t\Delta}u_0]$. Lastly, we define $\Xi_j$ by $\Xi_j(u_0) := \sum_{T \in T(j)} \Psi(T)$. (2.4)

Denoting the solution $u$ by a star-shaped terminal node “•” we can express the Duhamel formulation (2.1) as

$$u = \Psi(\cdot) + \Psi(\mathcal{A}) + \Psi(\mathcal{A}^2) + \Psi(\mathcal{A}^3) + \cdots.$$ (2.5)

By recursively applying (2.5) and eliminating the occurrence of “•” from younger trees, we have

$$u = \Psi(\cdot) + \Psi(\mathcal{A}) + \Psi(\mathcal{A}^2) + \Psi(\mathcal{A}^3) + \cdots$$ (2.6)

By extending the definition of $\Psi$ to formal sums of trees via linearity and applying $\Psi$ to (2.6), we obtain the following (formal) power series expansion of the solution $u$ to (1.1) with $u|_{t=0} = u_0$:

$$u = \sum_{j=0}^{\infty} \Xi_j(u_0) = \sum_{j=0}^{\infty} \sum_{T \in T(j)} \Psi(T)$$ (2.7)

Then, from (2.7) with (2.2), (2.3), (2.4), the boundedness of the heat semigroup, and the algebra property of $C^{s_0}(M), s_0 > 0$, we have the following convergence result.

**Lemma 2.3.** Let $s_0 > 0$. Then, given $u_0 \in C^{s_0}(M)$, the power series expansion in (2.7) converges absolutely in $C([0, T]; C^{s_0}(M))$, provided that $T \leq c_0 \|u_0\|^{-2}_{C^{s_0}}$ for some small $c_0 > 0$.

It is easy to check that $u$ defined by the power series in (2.7) indeed satisfies the Duhamel formulation (2.1). Note that the time $T \sim \|u_0\|^{-2}_{C^{s_0}} > 0$ for the local well-posedness in Lemma 2.1 and for the convergence of the power series in Lemma 2.3 can be chosen to be the same. Thanks to the unconditional uniqueness of the solution in the class $C([0, T]; C^{s_0}(M))$, we conclude that $u$ defined by the power series in (2.7) coincides with the solution constructed in Lemma 2.1.

3. Non-endpoint case: $s < -\frac{2}{3}$

In this section, we present the proof of Theorem 1.1 for $s < -\frac{2}{3}$. We will treat the endpoint case $s = -\frac{2}{3}$ in the next section. Given $s < -\frac{2}{3}$, the $C^s$-norm is controlled by the $C^{-\frac{2}{3}}$-norm and thus Theorem 1.1 for the non-endpoint case $s < -\frac{2}{3}$ follows from that for the endpoint case $s = -\frac{2}{3}$ whose proof is presented in the next section. For readers’ convenience, however, we decided to include the proof of Theorem 1.1 for the non-endpoint case, since the relevant argument is much simpler than that for the endpoint case and thus is easier to follow. As

---

7 More precisely, we set $\Psi(\cdot) = u$.
in the previous work \cite{20, 21, 29}, the main strategy is to choose a suitable initial condition $u_0 = u_0(N)$, depending on a large parameter $N \gg 1$ and vanishing as $N \to \infty$, and show that the Picard second iterate $\Xi_1(u_0)$ diverges as $N \to \infty$, while keeping the other terms under control. In order to achieve the growth of the Picard second iterate, we exploit the high-$\times$high-$\times$high-to-low energy transfer mechanism of the cubic nonlinearity.

Note that in proving Theorem 1.1, the $C^s$-norm appears only in the first estimate of (1.4), where we need to establish an upper bound. Thus, in view of $\|u\|_{C^s} \leq \|u\|_{C^s}$ for $\sigma \leq s$, it suffices to only consider the subcritical case $-1 < s < -\frac{2}{3}$. Fix a large integer $N \gg 1$. Then, we set the initial condition $u_0 = u_0(N)$ as

$$u_0(x) = N^{-s-\varepsilon} (\cos(n_1 \cdot x) + \cos(2n_1 \cdot x))$$

for some small $\varepsilon > 0$, where $n_1 = (N, 0, \ldots, 0) \in \mathbb{Z}^d$. Then, from (1.2), we have

$$\|u_0\|_{C^s} \sim N^{-\varepsilon} \to 0,$$

as $N \to \infty$. This shows the first claim in (1.4).

From (3.1), we can write the linear solution as

$$e^{t\Delta} u_0(x) = N^{-s-\varepsilon} \left( e^{-t|n_1|^2} \cos(n_1 \cdot x) + e^{-t4|n_1|^2} \cos(2n_1 \cdot x) \right)$$

$$= N^{-s-\varepsilon} \left( e^{-tN^2} \cos(n_1 \cdot x) + e^{-4tN^2} \cos(2n_1 \cdot x) \right),$$

and thus we have

$$P_0 e^{t\Delta} u_0 = 0. \quad (3.3)$$

By a direct computation, we have

$$(e^{t\Delta} u_0(x))^3 = N^{-3s-3\varepsilon} \left( e^{-3tN^2} \cos^3(n_1 \cdot x) + 3e^{-6tN^2} \cos^2(n_1 \cdot x) \cos(2n_1 \cdot x) \right.$$

$$+ 3e^{-9tN^2} \cos(n_1 \cdot x) \cos^2(2n_1 \cdot x) + e^{-12tN^2} \cos^3(2n_1 \cdot x) \big).$$

Using the trigonometric identities:

$$\cos^3 x = \frac{1}{4} (\cos 3x + 3 \cos x), \quad (3.4)$$

$$\cos(x_1) \cos(x_2) \cos(x_3) = \frac{1}{4} \sum_{\varepsilon_2, \varepsilon_3 \in \{1,-1\}} \cos(x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3),$$

we then have

$$(e^{t\Delta} u_0(x))^3 = \frac{1}{4} N^{-3s-3\varepsilon} \left[ e^{-3tN^2} (\cos(3n_1 \cdot x) + 3 \cos(n_1 \cdot x)) \right.$$

$$+ 3e^{-6tN^2} \left( 1 + \cos(4n_1 \cdot x) + 2 \cos(2n_1 \cdot x) \right) + 3e^{-9tN^2} \left( \cos(5n_1 \cdot x) + 2 \cos(n_1 \cdot x) + \cos(3n_1 \cdot x) \right)$$

$$+ e^{-12tN^2} \left( \cos(6n_1 \cdot x) + 3 \cos(2n_1 \cdot x) \right) \big]. \quad (3.5)$$
Hence, from (2.4) and (3.5), we obtain
\[
\Xi_1(u_0)(t) = \int_0^t e^{(t-t')\Delta}(e^{t'}\Delta u_0)^3 dt' \\
= \frac{1}{8}N^{-2-3\varepsilon}(1-e^{-6tN^2}) + \frac{1}{4}N^{-3s-3\varepsilon}\left[ e^{-9tN^2} \frac{e^{6tN^2} - 1}{6N^2} \cos(3n_1 \cdot x) \\
+ 3e^{-tN^2} \frac{1-e^{-2tN^2}}{2N^2} \cos(n_1 \cdot x) \right] + 3e^{-16tN^2} \frac{e^{10tN^2} - 1}{10N^2} \cos(4n_1 \cdot x) \\
+ 3e^{-12tN^2} \frac{1-e^{-2tN^2}}{N^2} \cos(2n_1 \cdot x) + 3e^{-25tN^2} \frac{e^{15tN^2} - 1}{16N^2} \cos(5n_1 \cdot x) \\
+ 3e^{-tN^2} \frac{1-e^{-8tN^2}}{4N^2} \cos(n_1 \cdot x) + 3te^{-9tN^2} \cos(3n_1 \cdot x) \\
+ e^{-36tN^2} \frac{e^{24tN^2} - 1}{24N^2} \cos(6n_1 \cdot x) + 3e^{-4tN^2} \frac{1-e^{-8tN^2}}{8N^2} \cos(2n_1 \cdot x) \right].
\]

In particular, for \( N \gg 1 \), we have
\[
P_0 \Xi_1(u_0)(t) = \frac{1}{8}N^{-2-3\varepsilon}(1-e^{-6tN^2}),
\]
where \( P_0 \) denotes the Littlewood-Paley frequency projector onto the frequencies \( \{ \xi \in \hat{\mathcal{M}} : |\xi| \lesssim 1 \} \). Therefore, by choosing \( t = N^{-2+2\varepsilon} \) such that \( e^{-6tN^2} \ll 1 \) for \( N \gg 1 \), we have
\[
\| P_0 \Xi_1(u_0)(t) \|_{L^\infty} \sim N^{-2-3\varepsilon} \rightarrow \infty,
\]
as \( N \rightarrow \infty \), provided that \( s < \frac{2}{3} - \varepsilon \). Given \( s < \frac{2}{3} \), this last condition can be guaranteed by choosing \( \varepsilon > 0 \) sufficiently small.

Next, we estimate the contribution from the higher order terms in (2.7). From (2.4), (2.3), the boundedness of the heat semigroup on \( L^\infty(\mathcal{M}) \), and the algebra property of \( L^\infty(\mathcal{M}) \) together with (3.1) and \( t = N^{-2+2\varepsilon} \), we have
\[
\left\| \sum_{j=2}^{\infty} P_0 \Xi_j(u_0)(t) \right\|_{L^\infty} \lesssim \sum_{j=2}^{\infty} \| \Xi_j(u_0)(t) \|_{L^\infty} \\
\leq \sum_{j=2}^{\infty} C_0^j t^j \| u_0 \|_{L^\infty}^{2j+1} \\
\lesssim \sum_{j=2}^{\infty} C_0^j N^{(-2-2s)j-s-\varepsilon} \\
\lesssim N^{-4-5s-\varepsilon}
\]
for \( N \gg 1 \), provided that \( -2 - 2s < 0 \), namely \( s > -1 \). Here, we used that, by Lemmas 2.1 and 2.3 with (3.1), it suffices to have \( t = N^{-2+2\varepsilon} \lesssim \| u_0 \|_{C_0^s}^{-2} \sim N^{2s+2\varepsilon-2s_0} \) for some \( s_0 > 0 \) to guarantee local well-posedness and convergence of the power series expansion (2.7)\(^8\). This condition reduces to \( s \geq -1 + s_0 \), which can be guaranteed for given \( s > -1 \) by choosing \( s_0 > 0 \) sufficiently small. We point out that the right-hand side of (3.7) is controlled by (3.6), provided that \( s > -1 + \varepsilon \). Given \( s > -1 \), this last condition can be guaranteed by choosing \( \varepsilon > 0 \) sufficiently small.

\(^8\)Here, we need \( s_0 > 0 \) such that \( C_0^s(\mathcal{M}) \) is an algebra, which is crucial for Lemmas 2.1 and 2.3.
Finally, putting \((2.7), (3.3), (3.6),\) and \((3.7)\) together, we have

\[
\|P_0 u(t)\|_{L^\infty} \geq \|P_0 \Xi_1(u_0)(t)\|_{L^\infty} - \|P_0 e^{t \Delta} u_0\|_{L^\infty} - \sum_{j=2}^\infty \|P_0 \Xi_j(t)\|_{L^\infty}
\]

\[
\gtrsim N^{-2-3s-3\epsilon} \to \infty,
\]

as \(N \to \infty\), provided that

\[
-1 + \epsilon < s < -\frac{2}{3} - \epsilon
\]

and \(t = N^{-2+2\epsilon}\). This proves Theorem 1.1 for \(s < -\frac{2}{3}\).

**Remark 3.1.** In this remark, we indicate how to modify the argument to treat the Allen-Cahn equation \((1.7)\) and the Cahn-Hilliard-type equation \((1.9)\). We only discuss the case \(-1 < s < -\frac{2}{3}\).

We first consider the Allen-Cahn equation \((1.7)\). Note that the difference between the heat semigroup and the linear semigroup \(e^{t(1+\Delta)}\) for the Allen-Cahn equation appears only in the low frequencies. Moreover, in proving a norm inflation, we study the dynamics only for very short times and thus their difference becomes negligible. Hence, the argument presented above can be applied directly to \((1.7)\). In the following, however, we present an alternative argument, where we view \(u - u^3\) as a combined power-type nonlinearity and modify the structure of trees to allow nodes with different numbers of children. We decided to include this argument since it can be adapted to study the nonlinear heat equation with a combined power-type nonlinearity (such as the cubic-quintic nonlinearity). See also [36], where ternary-quinary trees were used to prove a norm inflation for the derivative nonlinear Schrödinger equation.

By writing \((1.7)\) in the Duhamel formulation, we have

\[
u(t) = e^{t \Delta} + I^1[u] + I[u],
\]

where \(I\) is as in \((2.2)\) and

\[
I^1[u] := -I[u, 1, 1].
\]

Thus, we need to modify Definition 2.2 to take this extra term \(I^1[u]\) into account. More precisely, we need to replace the condition (b) by

(b') A node \(a \in \mathcal{T}\) is called terminal, if it has no child. A non-terminal node \(a \in \mathcal{T}\) is a node with exactly one child or three children.

Given a tree \(\mathcal{T}\) with this new definition, let \(|\mathcal{T}|_1\) (and \(|\mathcal{T}|_3\), respectively) denotes the numbers of nodes which have one child (and three children, respectively). Then, for \(j_1, j_3 \in \mathbb{N} \cup \{0\}\), we set

\[
\mathcal{T}^{1,3}(j_1, j_3) := \{\mathcal{T} : \mathcal{T} \text{ is a tree with } (|\mathcal{T}|_1, |\mathcal{T}|_3) = (j_1, j_3)\}.
\]

Note that for \(\mathcal{T} \in \mathcal{T}^{1,3}(j_1, j_3)\), we have \(|\mathcal{T}| = j_1 + 3j_3 + 1\). Furthermore, arguing as in the proof of Lemma 2.3 in [29], the following bound on the size of \(\mathcal{T}^{1,3}(j_1, j_3)\) holds; there exists \(C_0 > 0\) such that

\[
\sum_{j=j_1+j_3} \#\mathcal{T}^{1,3}(j_1, j_3) \leq C_0^j
\]

for any \(j \in \mathbb{N} \cup \{0\}\).
We define a map $\Psi = \Psi_{u_0}: \bigcup_{j_1,j_3=0}^{\infty} T^{1,3}(j_1,j_3) \to \mathcal{D}'(\mathcal{M} \times [0,T])$ by replacing the definition (i) by

(i') Replace a non-terminal node "*" with three children by the Duhamel integral operator $I$ defined in (2.2) with its three children as arguments $u_1, u_2,$ and $u_3,$ and replace a non-terminal node "*" with one child by $I_1$ defined in (3.10) with its child as an argument $u.$

We then set

$$\Xi_j(u_0) := \sum_{T \in T^{1,3}(j_1,j_3)} \sum_{j = j_1 + j_3}^{\infty} \Psi(T).$$

Then, the power series expansion for a (smooth) solution $u$ to (1.7) is given by

$$u = \sum_{j=0}^{\infty} \Xi_j(u_0) = \sum_{j=0}^{\infty} \sum_{T \in T^{1,3}(j_1,j_3)} \Psi(T).$$

We point out that Lemmas 2.1 and 2.3 also hold for (1.7). Then, with the bound (2.3), we can simply repeat the argument presented above to conclude a norm inflation for (1.7).

Next, we consider the Cahn-Hilliard-type equation (1.9). Let us consider the first equation. A similar argument applies to the second equation in (1.9). In this case, we need to use the following Duhamel integral operator:

$$\tilde{I}[u_1, u_2, u_3](t) := \int_0^t e^{-\Delta(t-t')}u_1(t')u_2(t')\Delta u_3(t')dt'.$$  \hspace{1cm} (3.11)

With the same choice (3.1) for the initial data, by repeating the computation above, we have

$$\Pi_0 \tilde{\Xi}_1(u_0)(t) = c_0 N^{-2-3s-3\varepsilon}(1 - e^{-6tN^4}),$$

where $\tilde{\Xi}_j$ is defined as in (2.4) but with the Duhamel integral operator $\tilde{I}$ in defined (3.11). Hence, by choosing $t = N^{-4+2\varepsilon},$ the growth (3.6) still holds for $s < -\frac{2}{3} - \varepsilon.$

On the other hand, we claim that

$$\|\tilde{\Xi}_j(u_0)(t)\|_{L^\infty} \leq C_0^j t^{\frac{j}{2}} \|u_0\|^{2j+1}_{L^\infty}. \hspace{1cm} (3.12)$$

For now, assume (3.12) and we bound the higher order terms. With the choice of $t$ as above, (3.12), and (3.1), we have

$$\left\|\sum_{j=2}^{\infty} \Pi_0 \tilde{\Xi}_j(u_0)(t)\right\|_{L^\infty} \leq \sum_{j=2}^{\infty} C_0^j t^{\frac{j}{2}} \|u_0\|^{2j+1}_{L^\infty}$$

$$\lesssim \sum_{j=2}^{\infty} C_0^j N^{(-4-2s)j-s-\varepsilon}$$

$$\lesssim N^{(-4-5s-\varepsilon)}$$

just as in (3.7), provided that $-4 - 2s < 0,$ namely $s > -2.$ Hence, a norm inflation holds for the first equation in (1.9).

It now remains to prove (3.12). With (2.3) (for $\tilde{\Xi}_j(u_0)$) and (2.3), it suffices to prove

$$\|\Psi(T)(t)\|_{L^\infty} \lesssim t^{\frac{j}{2}} \|u_0\|^{2j+1}_{L^\infty} \hspace{1cm} (3.13)$$
for each $T \in T(j)$. We prove (3.13) by induction. When $j = 1$, the bound (3.13) follows from the Schauder estimate (1.10) (but for $L^\infty(M)$):

$$\|\Delta e^{-t\Delta^2} f\|_{L^\infty} \lesssim t^{-\frac{1}{2}} \|f\|_{L^\infty}. \quad (3.14)$$

Now, fix $j_0 \geq 2$ and assume that (3.13) holds for any $j \leq j_0 - 1$. Given $T \in T(j_0)$, suppose that there exists a terminal node $a \in T^\infty$ whose parent $b$ is not the third child of its parent. Let $T_a$ denote the sub-tree in $T$ of one generation, containing $a \in T^\infty$ as one of its terminal nodes and $b$ as the parental node. Then, let $T'$ be the tree obtained from $T$ by replacing the sub-tree $T_a$ with a node $b$ (and thus $b$ becomes a terminal node). Note that $T$ can be constructed by replacing the terminal node $b \in (T')^\infty$ by the tree $T_a$. By applying the inductive hypothesis twice, we have

$$\|\Psi(T)(t)\|_{L^\infty} \lesssim t^{\frac{j_0 - 1}{2}} \|u_0\|_{L^\infty}^{2j_0 - 2} \|\Psi(T_a)\|_{L^\infty} \lesssim j_0 \|u_0\|_{L^\infty}^{2j_0 + 1},$$

yielding (3.13) for $j = j_0$ in this case. Hence, it remains to consider the case when there is no terminal node whose parent is not the third child of its parent. In other words, every parent in this (unbalanced) tree is the third child of its parent. In this case, by the Schauder estimate (3.14), we have

(i) for the last generation\(^\text{10}\)

$$\|\Delta \overline{\mathcal{I}}[e^{-t\Delta^2} u_0]\|_{L^\infty} \lesssim \|u_0\|_{L^\infty}^3,$$

(ii) for the intermediate generations:

$$\|\Delta \overline{\mathcal{I}}'[e^{-t\Delta^2} u_0, e^{-t\Delta^2} u_0, u_3]\|_{L^\infty} \lesssim t^{\frac{1}{2}} \|u_0\|_{L^\infty}^2 \|u_3\|_{L^\infty},$$

(iii) for the first generation:

$$\|\overline{\mathcal{I}}'[e^{-t\Delta^2} u_0, e^{-t\Delta^2} u_0, u_3]\|_{L^\infty} \lesssim t \|u_0\|_{L^\infty}^2 \|u_3\|_{L^\infty},$$

where $\overline{\mathcal{I}}'$ is defined by

$$\overline{\mathcal{I}}'[u_1, u_2, u_3](t) := \int_0^t e^{-(t-t')\Delta^2} \left( \prod_{j=1}^3 u_j(t') \right) dt'.$$

Putting (i), (ii), and (iii) together, we directly obtain (3.13) in this case (without using the inductive hypothesis). Therefore, by induction, (3.13) holds for any $j \in \mathbb{N}$.

4. Endpoint case: $s = -\frac{2}{3}$

In this section, we present the proof of Theorem 1.1 at the endpoint regularity $s = -\frac{2}{3}$. When $s = -\frac{2}{3}$, the construction in the previous section (with $\varepsilon = 0$) only gives the $O(1)$ growth of the Picard second iterate. In order to overcome this difficulty, we consider a lacunary sequence of initial data of type (3.1), where each component yields $O(1)$ contribution to the Picard second iterate at the frequency 0.

\(^9\)Here, the third child corresponds to $u_3$ in (3.11) at the level of the Duhamel integral operator.

\(^{10}\)Here, we used the fact that $\int_0^t (t-t')^{-\frac{1}{2}} (t')^{-\frac{1}{2}} dt' = B(\frac{1}{2}, \frac{1}{2}),$ where $B(a, b)$ is the beta function.
Fix large integers $N, K \gg 1$. Then, we set the initial condition $u_0 = u_0(N, K)$ as
\[ u_0(x) = \frac{1}{\log K} \sum_{k=1}^{K} (a_k N)^2 \left( \cos(a_k n_1 \cdot x) + \cos(2a_k n_1 \cdot x) \right), \quad (4.1) \]
where $n_1 = (N, 0, \ldots, 0) \in \mathbb{Z}^d$ as before and
\[ a_k = 2^{2^k}. \quad (4.2) \]

Then, from (1.2), we have
\[ \|u_0\|_{L^{\frac{2}{3}}} \sim \frac{1}{\log K} \to 0, \quad (4.3) \]
as $K \to \infty$. This shows the first claim in (1.4).

Proceeding as in (3.2) and (3.3) with (4.1), we have
\[ P_0 \Delta u_0(t) = 0. \quad (4.4) \]

By repeating the computation in the previous section and making use of the lacunarity of the summands in (4.1), we have
\[ P_0(e^{t \Delta} u_0)^3 = \frac{3}{4(\log K)^3} \sum_{k=1}^{K} (a_k N)^2 e^{-6t(a_k N)^2}. \]

Hence, by choosing $t = N^{-2+\delta}$ for some small $\delta > 0$, we have
\[ \|P_0 \Xi_1(u_0)(t)\|_{L^\infty} = \left\| \int_0^t e^{(t-t') \Delta} P_0(e^{t' \Delta} u_0)^3 dt' \right\|_{L^\infty} \]
\[ = \frac{1}{8(\log K)^3} \sum_{k=1}^{K} (1 - e^{-6t(a_k N)^2}) \quad (4.5) \]
\[ \gtrsim \frac{K}{(\log K)^3} \to \infty, \]
as $K \to \infty$.

It remains to estimate the contribution from the higher order terms in (2.7). Proceeding as in the previous section with (1.1), (1.2), and $t = N^{-2+\delta}$ and choosing $K = c_0 \log \log N$ for some small $c_0 > 0$, we have
\[ \left\| \sum_{j=2}^{\infty} P_0 \Xi_j(u_0)(t) \right\|_{L^\infty} \lesssim \sum_{j=2}^{\infty} \|\Xi_j(u_0)(t)\|_{L^\infty} \leq \sum_{j=2}^{\infty} C_0^j t^j \|u_0\|_{L^\infty}^{2j+1} \]
\[ \lesssim \sum_{j=2}^{\infty} C_0^j K^{2j+1} (\log K)^{2j+1} N^{(-\frac{2}{3}+\delta)j + \frac{2}{3}} \quad (4.6) \]
\[ \lesssim N^{-\frac{2}{3}} \to 0, \]
as $N \to \infty$.

Finally, putting (4.3), (4.4), (4.5), and (4.6) together with $t = N^{-2+\delta}$ and $K = c_0 \log \log N$ for some small $\delta, c_0 > 0$ and proceeding as in (3.8), we conclude the proof of Theorem 1.1 at the endpoint case $s = -\frac{2}{3}$.  

Acknowledgments. The authors would like to thank the anonymous referee for the helpful comments which improved the quality of the paper. I.C. was supported by the EPSRC New Investigator Award EP/X015688/1. T.O. was supported by the European Research Council (grant no. 864138 “SingStochDispDyn”). Y.W. was supported by the EPSRC New Investigator Award (grant no. EP/V003178/1). T.O. would like to thank the Centre de recherches mathématiques, Canada, for its hospitality, where this manuscript was prepared.

References

[1] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier analysis and nonlinear partial differential equations, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343. Springer, Heidelberg, 2011. xvi+523 pp.
[2] I. Bejenaru, T. Tao, Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation, J. Funct. Anal. 233 (2006), no. 1, 228–259.
[3] J. Bourgain, N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D, J. Funct. Anal. 255 (2008), no. 9, 2233–2247.
[4] L. Brandolese, F. Cortez, Blowup for the nonlinear heat equation with small initial data in scale-invariant Besov norms, J. Funct. Anal. 276 (2019), no. 8, 2589–2604.
[5] H. Brezis, T. Cazenave, A nonlinear heat equation with singular initial data, J. Anal. Math. 68 (1996), 277–304.
[6] R. Carles, T. Kappeler, Norm-inflation for periodic NLS equations in negative Sobolev spaces, Bull. Soc. Math. France 145 (2017), no. 4, 623–642.
[7] R. Catellier, K. Chouk, Paracontrolled Distributions and the 3-dimensional Stochastic Quantization Equations, Ann. Probab. 46 (2018), no. 5, 2621–2679.
[8] A. Cheskidov, M. Dai, Norm inflation for generalized Navier-Stokes equations, Indiana Univ. Math. J. 63 (2014), no. 3, 869–884.
[9] I. Chevyrev, Norm inflation for a non-linear heat equation with Gaussian initial conditions, Stoch. Partial Differ. Equ. Anal. Comput. 12 (2024), no. 3, 1745–1768.
[10] A. Choffrut, O. Pocovnicu, Ill-posedness of the cubic half-wave equation and other fractional NLS on the real line, Int. Math. Res. Not. 2018, no. 3, 699–738.
[11] M. Christ, Power series solution of a nonlinear Schrödinger equation, Mathematical aspects of nonlinear dispersive equations, 131–155, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.
[12] M. Christ, J. Colliander, T. Tao, Ill-posedness for nonlinear Schrödinger and wave equations, arXiv:math/0311048 [math.AP].
[13] D. Da Prato, A. Debussche, Strong solutions to the stochastic quantization equations, Ann. Probab. 31 (2003), no. 4, 1900–1916.
[14] J. Forlano, M. Okamoto, A remark on norm inflation for nonlinear wave equations, Dyn. Partial Differ. Equ. 17 (2020), no. 4, 361–381.
[15] Y. Giga, Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations 62 (1986), no. 2, 186–212.
[16] M. Gubinelli, Rooted trees for 3D Navier-Stokes equation, Dyn. Partial Differ. Equ. 3 (2006), no. 2, 161–172.
[17] M. Hairer, A theory of regularity structures, Invent. Math. 198 (2014), no. 2, 269–504.
[18] M. Hairer, K. Lê, T. Rosati, The Allen-Cahn equation with generic initial datum, Probab. Theory Related Fields 186 (2023), no. 3–4, 957–979.
[19] M. Hairer, M. Ryser, H. Weber, Triviality of the 2D stochastic Allen-Cahn equation, Electron. J. Probab. 17 (2012), no. 39, 14 pp.
[20] T. Iwabuchi, T. Ogawa, Ill-posedness for the nonlinear Schrödinger equation with quadratic non-linearity in low dimensions, Trans. Amer. Math. Soc. 367 (2015), no. 4, 2613–2630.
[21] N. Kishimoto, A remark on norm inflation for nonlinear Schrödinger equations, Commun. Pure Appl. Anal. 18 (2019), no. 3, 1375–1402.
[22] P.G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem, Chapman & Hall/CRC Research Notes in Mathematics, 431. Chapman & Hall/CRC, Boca Raton, FL, 2002. xiv+395 pp.
[23] R. Liu, T. Oh, On the two-dimensional singular stochastic viscous nonlinear wave equations, C. R. Math. Acad. Sci. Paris 360 (2022), 1227–1248.
[24] Y. Meyer, Oscillating patterns in some nonlinear evolution equations, Mathematical foundation of turbulent viscous flows, 101–187, Lecture Notes in Math., 1871, Springer, Berlin, 2006.

[25] C. Miao, B. Yuan, B. Zhang, Strong solutions to the nonlinear heat equation in homogeneous Besov spaces, Nonlinear Anal. 67 (2007), no. 5, 1329–1343.

[26] L. Molinet, S. Tayachi, Remarks on the Cauchy problem for the one-dimensional quadratic (fractional) heat equation, J. Funct. Anal. 269 (2015), no. 8, 2305–2327.

[27] J.-C. Mourrat, H. Weber, Global well-posedness of the dynamic $\Phi^4$ model in the plane, Ann. Probab. 45 (2017), no. 4, 2398–2476.

[28] J.-C. Mourrat, H. Weber, The dynamic $\Phi^3_3$ model comes down from infinity, Comm. Math. Phys. 356 (2017), no. 3, 673–753.

[29] T. Oh, A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces, Funkcial. Ekvac. 60 (2017) 259–277.

[30] T. Oh, M. Okamoto, N. Tzvetkov, Uniqueness and non-uniqueness of the Gaussian free field evolution under the two-dimensional Wick ordered cubic wave equation, to appear in Ann. Inst. Henri Poincaré Probab. Stat.

[31] T. Oh, Y. Wang, On the ill-posedness of the cubic nonlinear Schrödinger equation on the circle, An. Științ. Univ. Al. I. Cuza Iași. Mat. (N.S.). 64 (2018), no. 1, 53–84.

[32] M. Okamoto, Norm inflation for the generalized Boussinesq and Kawahara equations, Nonlinear Anal. 157 (2017), 44–61.

[33] F. Ribaud, Semilinear parabolic equations with distributions as initial data, Discrete Contin. Dynam. Systems 3 (1997), no. 3, 305–316.

[34] Y. Sinai, Power series for solutions of the 3D-Navier-Stokes system on $\mathbb{R}^3$, J. Stat. Phys. 121 (2005), no. 5-6, 779–803.

[35] B. Wang, Ill-posedness for the Navier-Stokes equations in critical Besov spaces $\dot{B}_{\infty, q}^{-1}$, Adv. Math. 268 (2015), 350–372.

[36] Y. Wang, Y. Zine, Norm inflation for the derivative nonlinear Schrödinger equation, to appear in C. R. Math. Acad. Sci. Paris.

[37] F. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, Israel J. Math. 38 (1981), no. 1-2, 29–40.

[38] T. Yoneda, Ill-posedness of the 3D-Navier-Stokes equations in a generalized Besov space near $\text{BMO}^{-1}$, J. Funct. Anal. 258 (2010), no. 10, 3376–3387.

Ilya Chevyrev, School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King’s Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

Email address: ichevyrev@gmail.com

Tadahiro Oh, School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King’s Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom, and School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China

Email address: hiro.oh@ed.ac.uk

Yuzhao Wang, School of Mathematics, Watson Building, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom

Email address: y.wang.14@bham.ac.uk