Intrinsic Harmonic Spaces: A Solution to the Ancient Problem of Perfect Tuning

Diederik Aerts

Center Leo Apostel (CLEA)
Vrije Universiteit Brussel,
Krijgskundestraat 33, 1160 Brussels.
e-mail: diraerts@vub.ac.be

Abstract

In this article we solve this ancient problem of perfect tuning in all keys and present a system were all harmonies are conserved at once. It will become clear, when we expose our solution, why this solution could not be found in the way in which earlier on musicians and scientist have been approaching the problem. We follow indeed a different approach. We first construct a mathematical representation of the complete harmony by means of a vector space, where the different tones are represented in complete harmonic way for all keys at once. One of the essential differences with earlier systems is that tones will no longer be ordered within an octave, and we find the octave-like ordering back as a projection of our system. But it is exactly by this projection procedure that the possibility to create a harmonic system for all keys at once is lost. So we see why the old way of ordering tones within an octave could not lead to a solution of the problem. We indicate in which way a real musical instrument could be built that realizes our harmonic scheme. Because tones are no longer ordered within an octave such a musical instrument will be rather unconventional. It is however a physically realizable musical instrument, at least for the Pythagorean harmony. We also indicate how perfect harmonies of every dimension could be realized by computers.

1 Introduction

In earlier times instrument tuning was anything but a standard art [1, 2, 3]. Already in the days of Pythagoras it was recognized that there are problems with creating a perfectly tuned scale [4, 5]. Over the centuries there have been many attempts to create tuning schemes that preserve the harmony of perfectly tuned intervals. The main approach was to try to minimize the errors that naturally occur when trying to solve the problem of harmony. That is the reason that during the history of music many different tuning systems have been proposed. They differ mostly with respect to what type of harmonies they try to conserve at the cost of losing other ones. But none of them could produce a full harmony for all keys, and it has been generally accepted that it is impossible to do so [1, 4, 5].

The tuning scheme that is now generally adopted in our western society is called the ‘equal temperament’ system. But many tuning systems have been common place during the history of western music, and in non-western music even nowadays alternative tuning systems are used. In this article we will study the general problem of harmony and clarify how the struggle with this problem has lead to all the different tuning systems. Although the solution that we present in this article, the construction of intrinsic harmonic spaces, does not really fit into the historical scheme, i.e. it is not a new tuning scheme, to be able to explain our solution as compared to traditional musical theory, we will have to use the concepts that have been introduced over the years. Let us therefore start by introducing these concepts and also their standard notations.

Notes, tones and tuning systems are the subject of our reflection. The collection of notes consists of the traditional twelve half notes in the different octaves. A tone can be uniquely identified by giving its frequency. Frequency is measured in a unity that is called Hertz, expressing the number of cycles per second. Traditionally the tone a of the central octave of a keyboard is tuned to 440 Hertz. A tuning systems defines a way to make correspond to each note a specific tone.
1.1 The Notes

We will use the common naming for the twelve half notes that are used in an octave in western music with $c$ as base note: $d$, $d_b$, $e$, $f$, $f_b$, $g$, $a$, $a_b$, $b$, and $b_b$. This are the notes of one octave, that we will consider as the basic octave, and we will refer to the notes of this octave as the basic notes. The notes of higher octaves to this basic octave we denote by using a positive whole number as subscript, the number indicating the different higher octaves in increasing order. This means for example that $a_1$ is the note $a$ of the first octave higher than the basic octave and $b_3$ is the note $b$ of the third octave higher than the basic octave. The notes of the lower octaves to this basic octave we denote by using a negative whole number as subscript, the absolute value of the number indicating the different lower octaves in decreasing order. This means for example that $c_{-2}$ is the note $c$ of the second octave lower to the basic octave. Taking the logic of this notation to its limit, we should notate a note of the basic octave, for example the note $a$, as $a_0$, and sometimes, if this comes out better for the formulas we will do so. Sometimes, when we want to make a statement about a general note, we will indicate this note by a variable, and use the variables $q, r$ and $s$ to indicate notes. We have introduced all the rules of notation to define now the set of all notes, that we will denote by $\mathcal{N}$. The set of all notes that we consider is given by

$$\mathcal{N} = \{q_k \mid q \in \{c, d_b, d, e_b, e, f, f_b, g, a, a_b, a_b, b, b_b\}, k \in \mathbb{Z}\}$$  \hspace{1cm} (1)

where \{c, d_b, d, e_b, e, f, f_b, g, a, a_b, a_b, b, b_b\} is the set of notes in the basic octave, and $\mathbb{Z}$ is the set of integers, positive as well as negative.

1.2 The Tones and the Tuning System

Tones will be expressed basically in Hertz, and as we mentioned already, the tone $a$ corresponding to the note $a$ is usually tuned at 440 Hertz. We will indicate tones by means of the letters \{u, v, w, \ldots\}.

The harmonic content of two different tones is directly related to the ratio of their frequencies being a simple number. If we consider two tones that have the distance of an entire octave between them, for example the tones corresponding to the notes $c$ and $c_1$, then the frequency of the highest tone, the one corresponding to the note $c_1$, is exactly the double of the frequency of the lower tone, the one corresponding to the note $c$. This is the case for all tuning systems. This means that from now on, independent of the adopted tuning system, the relation between $q_k$ where $q$ is one of the half notes of octave $x$, and $q_{x+1}$, which is the same notes of octave $x + 1$, is such that that the frequency of the tone corresponding to the note $q_{x+1}$ is the double of the frequency of the tone corresponding to the note $q_x$. That is the reason that $c$ and $c_1$ sound very harmonically together; in this case we even hear it as the same tone, but lower and higher in pitch.

If we consider the notes $c$ and $g_1$, then, for a perfect tuning system, the frequency corresponding to $g_1$ is the triple of the frequency corresponding to $c$. This is not the case for the ‘equal temperament’ tuning system, the one that is now adopted in our western society. That is the reason that in most ancient tuning systems also these two notes, $c$ and $g_1$ sound very harmonically together. These notes sound less harmonically together when played on actual instruments, because of equal temperament that is now universally adopted.

Let us explain what ‘equal temperament’ is. For reasons that we will come back to in section 4 it was decided to have twelve half notes in one octave, and to tune these half notes in such a way that the ratio between the frequencies corresponding to two consecutives half notes is the same for all couples of consecutives half notes. Let us introduce first some more notations. When we talk of an arbitrary note, we will denote this note by using one of the letters $q, r$, or $s$. Octaves we will denote by $x$ running over the set of whole numbers \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, where 0 corresponds to what we decide to be our basic octave. When we want to specify for a note $q$ that it is a note of the octave $x$ we write $q_x$. We denote arbitrary tones by the letters $u, v, w, \ldots$. The different tuning systems that we will talk about, we will indicate by capital letters, for example the equal temperament tuning system is denoted by $E$, the perfect harmonic system by $H$, and the Pythagorean tuning system by $P$. If we want to denote the tone $u$ that within a tuning system $K$ corresponds to the note $q$ we write $u = u(K, q)$. Hence $u(E, q)$ is the tone that corresponds to the note $q$ in the equal temperament system, and $u(H, q)$ is the tone that corresponds to note $q$ in the perfect harmonic system, while $u(P, q)$ is the tone that corresponds to the note $q$ in the Pythagorean tuning system. When the frequency of the tone $v$ corresponding to note $r$ within the tuning system $K$ is $k$ times the frequency of the tone $u$ corresponding to the note $q$ within the tuning system $L$ we denote

$$v(K, r)/u(L, q) = k$$  \hspace{1cm} (2)

This means that we can write now for the notes $c$ and $c_1$ the following

$$u(E, c_1)/u(E, c) = 2$$  \hspace{1cm} (3)

2
and also

\[ u(H, c_1)/u(H, c) = u(P, c_1)/u(P, c) = 2 \]  

(4)

Let us calculate the frequency ratio between two consecutive half notes of the same octave in the case of equal temperament. Hence \( c, d, e, f, g, a, b, c_1 \) is the set of twelve half notes that we consider. Equal temperament means that

\[ k = u(E, d)/u(E, c) = u(E, d)/u(E, d) = u(E, c)/u(E, c) \]

(5)

\[ = u(E, f)/u(E, e) = u(E, g)/u(E, f) = u(E, g)/u(E, g) = u(E, a)/u(E, g) \]

(6)

\[ = u(E, a)/u(E, a) = u(E, a)/u(E, a) = u(E, a)/u(E, a) \]

(7)

We also have

\[ 2 = u(E, c_1)/u(E, c) = u(E, d)/u(E, c_1) \cdot u(E, d)/u(E, d) \cdot u(E, c_1)/u(E, c_1) \cdot u(E, d)/u(E, d) \cdot u(E, c_1)/u(E, c_1) \cdot u(E, d)/u(E, d) \cdot u(E, c_1)/u(E, c_1) \]

(8)

\[ \cdot u(E, a)/u(E, a) \cdot u(E, b)/u(E, b_0) \cdot u(E, c_1)/u(E, b) \]

(9)

\[ \cdot u(E, a)/u(E, a) \cdot u(E, b)/u(E, b_0) \cdot u(E, c_1)/u(E, b) \]

(10)

\[ = \cdot u(E, b_0)/u(E, a) \cdot u(E, b)/u(E, b_0) \cdot u(E, c_1)/u(E, b) \]

(11)

\[ = k^{12} \]

(12)

and as a consequence

\[ k^{12} = 2 \iff k = \sqrt[12]{2} \]  

(13)

If we calculate \( \sqrt[12]{2} \) we find \( k = 1.059463 \). What is more important to remark however is that \( \sqrt[12]{2} \) is an irrational number, which means that there does not exist a fraction that equals this number. It also means that the complete equal tempering deviates in principle completely from the base of harmony itself. It only allows a rational number (fraction of whole numbers) for the relation of the frequencies of basic tones of the octaves, but no rational numbers for all of the other relations of frequencies. What is it that made adopt standard western music this completely non harmonic tuning system? There is a long history of trying out different tuning systems involved and a long struggle with harmony. We will count partly this story in the following of this article while we introduce our solution to the problem, the intrinsic harmonic spaces.

1.3 Harmonic Possibilities

Before the completely equal tempered tuning system was adopted, many other much more harmonic systems have been in use. Let us analyse the possible harmonic solutions to tuning. The fifth represents, after the octave, the next building block of harmony. If we consider the note \( g_1 \), hence the sol of the octave which is one higher than the basic octave, in a perfect harmonic tuning system, the frequency corresponding to this note \( g_1 \) is three times the frequency of the basic note \( c \). Hence

\[ u(PYT, g_1)/u(PYT, c) = 3 \]  

(14)

This gives us the information to calculate the ratio of the fifth itself, namely the ratio \( u(PYT, g)/u(PYT, c) \). Indeed, we have

\[ u(H, g_1)/u(H, g) = 2 \]  

(15)

which shows that

\[ u(H, g)/u(H, c) = u(H, g)/u(H, g_1) \cdot u(H, g_1)/u(H, c) = 1/2 \cdot 3 = 3/2 \]

(16)

So the frequency relation between a tone and its fifth is 3/2 in a harmonically tuned system. We also know that \( c \) is the fifth for \( f_{-1} \), which means that

\[ u(H, f_{-1})/u(H, c) = 2/3 \]  

(17)

and hence

\[ u(H, f)/u(H, c) = 4/3 \]  

(18)

The relation between \( c \) and \( f \) is called a fourth in music theory. These results are presented in Table III.

In the fourth column of Table III we have represented the ratio’s in cent, which is the commonly used measure for pitch differences in traditional music theory. It is an algorithmic measure, where ratio 1 corresponds to 0.000
cent and ratio 2 corresponds to 1200.000 cent. The following formula can be used to calculate the difference in pitch given the ratio’s of the two tones. If u and v are the two tones, and hence their ratio is given by v/u, then

\[ \frac{1200}{\log 2} \log \frac{v}{u} \]  

(19)
gives the difference in pitch of tone u and v measured in cent. The fraction \(\frac{1200}{\log 2}\) is a renormalization factor, otherwise the octave would correspond to the interval \([0, \log 2]\) instead of the interval \([0, 1200]\). Remark that the cent measure is introduced with reference to the completely equal tempered tuned system. Indeed, suppose that in case we take \(c\) as basic tone, Table 1 gives us the perfect tuning ratio’s for the different tone ratio’s, and complicated cent measures (see Table 1 Harmonic), while the tempered system gives simple cent measures, and complicated ratio’s (see Table 1 Tempered).

Let us proceed analyzing harmony. Starting with c as basic tone, Table I gives us the perfect tuning ratio’s corresponding to f (fourth), g (fifth) and c₁ (octave). What about the third, or e in case we take c as basic tone? Here different possible perfect tuning solutions appear. The oldest proposal dates from the time of Pythagoras, and proposes the ratio 81/64 for the third. This is far from being a fraction of simple numbers, but it comes about in a natural way from the Pythagorean system of perfect tuning, as we will see in the next section. In the late 15th century the most common tuning system in Europe was called meantone tuning, and in this system the third was tuned according to the ratio 5/4. This gives rise to a much more harmonic sound as the Pythagorean proposal for the third, and that is why we will from now on call it the perfect third. The third tuned according to the ratio 81/64 we will call the Pythagorean third. Let us consider for a moment the choice of the perfect third. In the case of e as basic tone, Table 2 gives us the perfect tuning ratio’s corresponding to f (fourth), g (fifth) and c₁ (octave). In Table 2 we have added the perfect third.

\[u(H, e)/u(H, c) = 5/4\]  

(23)
In Table 2 we have added the perfect third.

We can easily see now the type of problems that perfect tuning is confronted with. Indeed, if perfect tuning would be possible using the twelve standard tones of the octave, the tone \(a_0\) should be a perfect third with respect to e, and then the tone \(c_1\) should be a perfect third with respect to \(a_0\). This would mean however that

\[u(H, c_1)/u(H, c) = u(H, c_1)/u(H, a_0) \cdot u(H, a_0)/u(H, e) \cdot u(H, e)/u(H, c) = 5/4 \cdot 5/4 \cdot 5/4 = 125/64\]  

(24)
But we know that

\[u(H, c_1)/u(H, c) = 2 = 128/64\]  

(25)
which means that if we would tune e, \(a_0\) and \(c_1\) following the perfect third ratio, the tone \(u(H, c_1)\) would fall 3/64 short for a perfect octave. Hence it is impossible to use perfect thirds all the way within the octave if we only have available the twelve traditional notes. But why has western music been constructed around these twelve traditional notes? Could this perhaps be the basis of the problem? Let us analyse in the next section the Pythagorean perfect tuning system, because the reason to put twelve half notes into an octave is to be found in this system.

---

Table 1: The ratio’s of basic, fourth, fifth and octave in the case of perfect and tempered tuning

| Tone Ratio | Example | Freq. Fraction | Cent | Freq. Fraction | Cent |
|------------|---------|----------------|------|----------------|------|
| Basic      | \(u(c)/u(e)\) | 1 | 0.000 | 1 | 0.000 |
| Fourth     | \(u(f)/u(c)\) | \(4/3 = 1.3333\) | 498.044 | \((\sqrt[2]{3})^4 = 1.3348\) | 500.000 |
| Fifth      | \(u(g)/u(c)\) | \(3/2 = 1.5\) | 701.955 | \((\sqrt[2]{3})^3 = 1.4983\) | 700.000 |
| Octave     | \(u(c_1)/u(e)\) | 2 | 1200.000 | 2 | 1200.000 |
We start from

2.1 The Basic Pythagorean System

The Pythagorean system is a musical system where the tones are created out of ratios that only contain the numbers 2 and 3. In the 13th century the French Academy at Notre Dame decreed that only the Pythagorean tuning was accepted as the correct musical tuning. As a consequence till the 15 century musical instruments have been tuned following the Pythagorean system.

Let us calculate the tones that flow naturally out of this Pythagorean approach. We will do this again by starting with c as the basic note, but obviously the whole process can be repeated with any other choice of basic note.

2 Twelve Notes in an Octave: The Pythagorean System

The Pythagorean system is a musical system where the tones are created out of ratios that only contain the numbers 2 and 3. In the 13th century the French Academy at Notre Dame decreed that only the Pythagorean tuning was accepted as the correct musical tuning. As a consequence till the 15th century musical instruments have been tuned following the Pythagorean system.

Let us calculate the tones that flow naturally out of this Pythagorean approach. We will do this again by starting with c as the basic note, but obviously the whole process can be repeated with any other choice of basic note.

2.1 The Basic Pythagorean System

We start from \( u(P, c) \) and have constructed already \( u(P, c_1) \) with ratio 2, \( u(P, g) \) with ratio 3/2 and \( u(P, f) \) with ratio 2/3. Since these ratios only contain numbers 2 and 3, they fit into the Pythagorean system. The perfect third that we constructed in section 1.3 corresponds to ratio 5/4, does not appear in the Pythagorean system. If we move an octave plus a fifth further up we have \( u(P, d_3)/u(P, g_1) = 3 \), hence \( u(P, d_3)/u(P, c) = 3^2 = 9. \) Since \( u(P, d_3)/u(P, d) = 3^3 = 8 \) we have

\[
u(P, d)/u(P, c) = 9/8
\]

(26)

This gives us the next tone of the Pythagorean system, \( u(P, d) \) with ratio 9/8. Sometimes, in analogy with the other namings, this is also called the Pythagorean second. Let us proceed in a analogous way and calculate \( u(P, a)/u(P, c) \). We have \( u(P, a_4)/u(P, c) = 3^5 = 27 \) and \( u(P, a_4)/u(P, a) = 2^4 = 16 \) which gives

\[
u(P, a)/u(P, c) = 27/16
\]

(27)

and this ratio is referred to as the Pythagorean sixth. Further we know that \( u(P, e_6)/u(P, c) = 3^4 = 81 \) and \( u(P, e_6)/u(P, e) = 2^6 = 64 \) which gives

\[
u(P, e)/u(P, c) = 81/64
\]

(28)

and we find back the Pythagorean fifth that we mentioned already in the foregoing section. We have \( s_i(7)/c = 3^5 = 243 \) and \( u(P, b_7)/u(P, b) = 2^7 = 128 \) which gives

\[
u(P, b)/u(P, c) = 243/128
\]

(29)

and this is called the Pythagorean seventh. We calculated now the complete set of whole tones of the octave and proceeding with the same method will now start to give us half tones. We have \( u(P, f^\#_8)/u(P, c) = 3^6 = 729 \) and \( u(P, f^\#_8)/u(P, f^\#) = 2^3 = 512 \) which gives

\[
u(P, f^\#)/u(P, c) = 729/512
\]

(30)

By moving an octave plus a fifth upwards in frequency – and hence multiplying the basic tone by powers of 3 – and then moving down again by octaves – and hence dividing the basic tone by powers of 2 – we have gathered 7 of the notes of the basic octave, namely \( c, d, e, f^\#, g, a, \) and \( b \). The note \( f \) we had obtained by moving in the inverse way, first an octave plus a fifth down, which means dividing the basic tone by powers of 3, and then moving up by octaves, which means multiplying the basic tone by powers of 2. Let us proceed in this way to get the other tones in the basic octave. We have \( u(P, b_6-4)/u(P, c) = 1/3^2 = 1/9 \) and \( u(P, b_6)/u(P, b_6-4) = 2^4 = 16 \) which gives

\[
u(P, b_6)/u(P, c) = 16/9
\]

(31)
Next we start from \( u(P, e_b - 5)/u(P, c) = 1/3^3 = 1/27 \) and \( u(P, e_b)/u(P, e_b - 5) = 2^5 = 32 \) which gives
\[
u(P, e_b)/u(P, c) = 32/27
\]
(32)
The following step leads us to \( a_b \) by noting that \( u(P, a_b - \gamma)/u(P, c) = 1/3^4 = 1/81 \) and \( u(P, a_b)/u(P, a_b - \gamma) = 2^7 = 128 \) which gives
\[
u(P, a_b)/u(P, c) = 128/81
\]
(33)
The last note that we have to calculate is \( d_b \) and this one we get by noting that \( u(P, d_b - 8)/u(P, c) = 1/3^5 = 1/243 \) and \( u(P, d_b)/u(P, d_b - 8) = 2^6 = 256 \) which gives
\[
u(P, d_b)/u(P, c) = 256/243
\]
(34)
Let us present the twelve tones that we have calculated in this way in Table 3:

| Tone Ratio | Example | Freq. Fraction | Cent | Freq. Fraction | Cent |
|------------|---------|----------------|------|----------------|------|
| 0          | Basic   | \( u(c)/u(c) \) | 1    | 0.00           | 0.00 |
| 1          | Second | \( u(d)/u(c) \) | 9/8  | 1.125          | 203.91 |
| 2          | Third  | \( u(e)/u(c) \) | 32/27 | 1.1851 | 294.13 |
| 3          | Fourth | \( u(f)/u(c) \) | 81/64 | 1.2656 | 407.82 |
| 4          | Fifth   | \( u(g)/u(c) \) | 2    | 1.500          | 701.95 |
| 5          | Sixth   | \( u(h)/u(c) \) | 81/64 | 1.5802 | 792.17 |
| 6          | Seventh | \( u(i)/u(c) \) | 27/16 | 1.6875 | 905.86 |
| 7          | Eighth  | \( u(j)/u(c) \) | 16/9 | 1.7777 | 996.08 |
| 8          | Ninth   | \( u(k)/u(c) \) | 243/128 | 1.8984 | 1109.77 |
| 9          | Octave  | \( u(l)/u(c) \) | 2 | 2.0000 | 1200.00 |

Table 3: The twelve notes of the basic octave tuned in the Pythagorean and the tempered system

We see that the twelve Pythagorean tones are close to the tempered tunes, but they are different. None of them is equal. The difference of the fifth is 1.95 cent – the tempered fifth is a little bit too low – and the difference of the fourth is 1.96 cent – the tempered fourth is a little bit too high. These are small differences, and that is the reason that in the completely equal tempered system, both fifth and fourth still sound harmonically well. Since the fifth and the fourth are the base of a lot of simple popular music, our completely tempered system does not perform very badly for this type of music. For the third the difference is 7.82 cent – the tempered third is too low – which is much more already. But for the third we have to remark that the perfect third (see Table 2) is even lower than the tempered third, with a difference of 13.69 cent. This means that the tempered third comes closer to the beautiful harmonic sound of the perfect third as it is the case for the Pythagorean third. So also here the tempered system does not perform very badly, although the difference with the perfect third is still substantial. The tempered second is too low with 3.91 cent while the tempered \( b_5 \) is too high with 3.92. This are significant differences. The tempered seventh is too low with 9.77 cent which is almost a disaster harmonically speaking (or listening).

But, after all, now that we have understood how much trickery is involved to making the harmonic system into a tempered system, we should be amazed that it works at all. All these twelve tempered tunes are reasonably close to the Pythagorean ones. How does this comes about? To shed more light on this we better develop the Pythagorean system to its completeness. This is what we do in next section.

### 2.2 The Extended Pythagorean System

We could have proceeded further along the same lines with the construction of the Pythagorean system. Let us do this, and for example construct \( g_5 \). We have \( u(P, g_5)/u(P, c) = 1/3^6 = 1/728 \) and \( u(P, g_5)/u(P, g_5 - 10) = 2^{10} = 1024 \) which gives
\[
u(P, g_5)/u(P, c) = 1024/728 = 1.4065
\]
(35)
We see that \( u(P, g_b)/u(P, c) \) does not coincide with \( u(P, f^\#)/u(P, c) \). Let us proceed. Next step we find \( b \) again, indeed \( b_{-12} \). And we have \( u(P, b_{-12})/u(P, c) = 1/3^7 = 1/2187 \) and \( u(P, b)/u(P, b_{-12}) = 2^{12} = 4096 \). This gives

\[
u(P, b)/u(P, c) = 4096/2187 = 1.8728
\]

(36)

So we have another outcome for \( b \) than the one we had found already in \([22]\). We call this new tone \( b^* \) just to distinguish it from \( b \). Next comes \( e^* \), and we have \( u(P, e_{-13})/u(P, c) = 1/3^9 = 1/6561 \) and \( u(P, e)/u(P, e_{-13}) = 2^{13} = 8192 \), hence

\[
u(P, e^*)/u(P, c) = 8192/6561 = 1.2485
\]

(37)

Next is \( a^* \) and we have \( u(P, a_{-15})/u(P, c) = 1/3^9 = 1/19683 \) and \( u(P, a)/u(P, a_{-15}) = 2^{15} = 32786 \), which gives

\[
u(P, a^*)/u(P, c) = 32786/19683 = 1.6657
\]

(38)

Next comes \( d^* \) and we have \( u(P, d^*)/u(P, c) = 1/3^{10} = 1/59049 \) and \( u(P, d)/u(P, d_{-16}) = 2^{16} = 65572 \), which give

\[
u(P, d^*)/u(P, c) = 65572/59049 = 1.1104
\]

(39)

Then comes \( g^* \) and we have \( u(P, g_{-18})/u(P, c) = 1/3^{11} = 1/177147 \) and \( u(P, g)/u(P, g_{-18}) = 2^{18} = 262288 \), which gives

\[
u(P, g^*)/u(P, c) = 262288/177147 = 1.4806
\]

(40)

And finally we arrive at \( c^* \). We have \( u(P, c_{-19})/u(P, c) = 1/3^{12} = 1/531441 \) and \( u(P, c)/u(P, c_{-19}) = 2^{19} = 524576 \), which gives

\[
u(P, c^*)/u(P, c) = 524576/531441 = 0.9870
\]

(41)

Now we start to see better what the problem is. The \( c^* \) that we have constructed has a frequency that is definitely different from \( c \) which we started with. This difference is called the Pythagorean ‘COMMA’ in music theory. The difference is due to the fact that \( 2^{19} = 524288 \) is a number that is ‘close’ to \( 3^{12} = 531441 \), ‘but’ different. And, it is indeed impossible that these two numbers would be equal, since one of the numbers is a power of 2, while the other is a power of 3.

We can also proceed with the symmetrical construction, let us do this, to see even more clear what the problem is. The next tone will be \( c^{\#} \). We have \( u(P, c^{\#}_{11})/u(P, c) = 3^7 = 2187 \) and \( u(P, c^{\#}_{11})/u(P, c^{\#}) = 2^{13} = 2048 \) which gives

\[
u(P, c^{\#})/u(P, c) = 2187/2048 = 1.0678
\]

(42)

Next comes \( g^{\#} \), and we have \( u(P, g^{\#}_{12})/u(P, c) = 3^8 = 6561 \) and \( u(P, g^{\#}_{12})/u(P, g^{\#}) = 2^{12} = 4096 \) which gives

\[
u(P, g^{\#})/u(P, c) = 6561/4096 = 1.6018
\]

(43)

Next comes \( d^{\#} \), and we have \( u(P, d^{\#}_{14})/u(P, c) = 3^9 = 19683 \) and \( u(P, d^{\#}_{14})/u(P, d^{\#}) = 2^{14} = 16384 \) which gives

\[
u(P, d^{\#})/u(P, c) = 19683/16384 = 1.2013
\]

(44)

Next comes \( a^{\#} \), and we have \( u(P, a^{\#}_{16})/u(P, c) = 3^{10} = 59049 \) and \( u(P, a^{\#}_{16})/u(P, a^{\#}) = 2^{15} = 32768 \) which gives

\[
u(P, a^{\#})/u(P, c) = 59049/32768 = 1.8020
\]

(45)

Next comes \( f^* \), and we have \( u(P, f_{17})/u(P, c) = 3^{11} = 177147 \) and \( u(P, f_{17})/u(P, f) = 2^{17} = 131072 \) which gives

\[
u(P, f^*)/u(P, c) = 177147/131072 = 1.3515
\]

(46)

And then we are again back at \( c \), notated this time \( c^{**} \). We have \( u(P, c_{19})/u(P, c) = 3^{12} = 531441 \) and \( u(P, c_{19})/u(P, c) = 2^{19} = 524288 \) which gives

\[
u(P, c^{**})/u(P, c) = 531441/524288 = 1.0136
\]

(47)

This means that we have gathered in this way 25 different notes that lay within the octave interval between \( c \) and \( c_1 \). We represent this situation in Table [4].

How to choose now the tones that we will finally use to tune our musical instrument? And here the problem arrives. A good choice would be to first choose the tones that have the most simple ratio towards the basic tone. In case of basic note \( c \), we would then choose in the following way. There is one note, namely \( c_1 \), that is our first
make this choice for the system. The problem is however that this choice 'depends' on the basic tone that we start with. So we could choose just to drop the others, and tune our instrument with these 12 most simple fractions. These give us already all the twelve common notes that also exist in a tempered tuning system.

Modern musical instruments like synthesizers, it is possible to tune the instrument electronically again just by possible to each time tune an instrument again when a music piece would be played in a different key. But with

### Table 4: The 25 tones that a further calculation of the Pythagorean system gives rise to

| Tone Ratio | Example | Pythagorean Fraction | Cent |
|------------|---------|----------------------|------|
| 1          | $u(c^2)/u(c)$ | $2^{19} \cdot 3^{-12} = 524576/531441 = 0.9870$ | -23.46 |
| 2          | $u(c)/u(c)$   | 1                    | 0.00  |
| 3          | $u(c^2)/u(c)$ | $2^{-19} \cdot 3^{12} = 531441/524288 = 1.0136$ | 23.46 |
| 4          | $u(c^3)/u(c)$ | $2^{3} \cdot 3^{-5} = 256/243 = 1.0534$ | 90.22 |
| 5          | $u(c^4)/u(c)$ | $2^{-11} \cdot 3^2 = 2187/2048 = 1.0678$ | 113.68 |
| 6          | $u(c^6)/u(c)$ | $2^{18} \cdot 3^{-11} = 65572/59049 = 1.1104$ | 180.44 |
| 7          | $u(d)/u(c)$   | $2^{-3} \cdot 3^2 = 9/8 = 1.125$ | 203.91 |
| 8          | $u(e)/u(c)$   | $2^5 \cdot 3^{-3} = 32/27 = 1.1851$ | 294.13 |
| 9          | $u(d^2)/u(c)$ | $2^{-14} \cdot 3^2 = 19683/16384 = 1.2013$ | 317.59 |
| 10         | $u(e^2)/u(c)$ | $2^{13} \cdot 3^{-8} = 8192/6561 = 1.2485$ | 384.35 |
| 11         | Pyth. Third   | $u(e)/u(c)$ | $2^{-9} \cdot 3^2 = 81/64 = 1.2656$ | 407.82 |
| 12         | Fourth        | $u(f)/u(c)$ | $2^2 \cdot 3^{-1} = 4/3 = 1.3333$ | 498.04 |
| 13         | $u(f^2)/u(c)$ | $2^{-17} \cdot 3^{11} = 177147/131072 = 1.3515$ | 521.50 |
| 14         | $u(g)/u(c)$   | $2^{10} \cdot 3^{-6} = 1024/728 = 1.4065$ | 588.26 |
| 15         | $u(f^3)/u(c)$ | $2^{-10} \cdot 3^3 = 729/512 = 1.4338$ | 611.73 |
| 16         | $u(g^2)/u(c)$ | $2^{18} \cdot 3^{-11} = 262288/177147 = 1.4806$ | 678.49 |
| 17         | Fifth         | $u(g)/u(c)$ | $2^1 \cdot 3^3 = 3/2 = 1.5$ | 701.95 |
| 18         | $u(a)/u(c)$   | $2^7 \cdot 3^{-4} = 128/81 = 1.5802$ | 792.17 |
| 19         | $u(g^2)/u(c)$ | $2^{-10} \cdot 3^3 = 6561/4096 = 1.6018$ | 815.64 |
| 20         | $u(a^2)/u(c)$ | $2^{15} \cdot 3^{-9} = 32786/19683 = 1.6657$ | 882.40 |
| 21         | Sixth         | $u(a)/u(c)$ | $2^{-4} \cdot 3^2 = 27/16 = 1.6875$ | 905.86 |
| 22         | $u(b)/u(c)$   | $2^4 \cdot 3^{-2} = 16/9 = 1.7777$ | 996.08 |
| 23         | $u(a^3)/u(c)$ | $2^{-10} \cdot 3^{10} = 59049/32768 = 1.8020$ | 1019.55 |
| 24         | $u(b^2)/u(c)$ | $2^{12} \cdot 3^{-7} = 4096/2187 = 1.8728$ | 1086.31 |
| 25         | Seventh       | $u(b)/u(c)$ | $2^{-7} \cdot 3^5 = 243/128 = 1.8984$ | 1109.77 |
| 26         | Octave        | $u(c_1)/u(c)$ | 2 | 1200.00 |

There are three notes $g, f$ and $d$ that only have integers smaller than 10 in their fraction, so they come next. There are 4 notes, $e_b, e, a$ and $b$, that have integers smaller than 100 in their fractions, so they come next. There are 4 notes, $d_b, f^\#$, $a_b$ and $b$, that have integers smaller than 1000 in their fractions, so next we choose them. And this gives us already all the twelve common notes that also exist in a tempered tuning system. So we could choose just to drop the others, and tune our instrument with these 12 most simple fractions. These are the ones that we have presented in Table 4. That is also how an instrument is tuned in the Pythagorean system. The problem is however that this choice 'depends' on the basic tone that we start with. So we could make this choice for the $c$ octave, but the same way of choosing would lead to a different set of tones for another octave. And that is the reason that by this method is it not possible to realize a Pythagorean tuning system that is valid independent of the basic note that is chosen. If we tune following our Pythagorean calculus with basic note $c$ our music instrument will be well tuned for music played in this basic tone. But if on this instrument we want to play in another basic tone (another key), we would have to tune all over again. This means that the Pythagorean tuning, as we have introduced it here, and as it has been introduced in all of history, is not resistant against transportation between different keys. How can we proceed?

### 2.3 Pythagorean Tuning and Different Keys

If the problem with the Pythagorean tuning would reveal itself if transportation of a music piece is made to another key, it would perhaps be still possible to cope with it. Not in the old times, were it was technically not possible to each time tune an instrument again when a music piece would be played in a different key. But with modern musical instruments like synthesizers, it is possible to tune the instrument electronically again just by pushing a button. Within the community of musicians that are interested in microtonal music, which is music using types of tuning systems different from the completely tempered system that a standard synthesizer is tuned in, these possibilities have been explored. So this would make it technically possible to adopt a new tuning each
time that a music piece is played in a different key. The real hard problem of perfect tuning is however not this one. In more complex musical works keys can change within the works itself. Now even here one might think that it would be possible to change the tuning at the moment that a change of key appears, and so a solution would exist when these works are played by such a sophisticated synthesizer. The problem is however that this only would be a solution for musical works where abruptly a change of key appears, so when really a new theme in a new key starts at a specific moment. Real complex works that explore harmony in depth will however contain gradual change, more a kind of slow deviation out of the old key towards a new key. And then it would become a matter of taste or purely subjective decision of the musician when to change tuning. There does not seem to be a systematic way to do this as long as we stick with the type of Pythagorean tuning as the one that we have introduced here, and hence the one that is historically known. It is this problem that gets solved in depth by the introduction of the intrinsic harmonic spaces that we present in this article. To be able to prove this, we first have to introduce these intrinsic harmonic spaces, and that is what we will do in the next section.

3 Intrinsic Harmonic Spaces

In this section we introduce an intrinsic harmonic space for the Pythagorean system. As we will show it is a two dimensional vector space. In section 4 we show how the procedure is general and can be elaborated for any kind of harmonic tuning system, also the more elaborated ones than the Pythagorean system. In general we will then need a vector space of higher dimensions, the number of dimensions being equal to the number of prime numbers different from 1 that are allowed for the ratio’s of the frequencies of different tones.

3.1 The Pythagorean Intrinsic Harmonic Space

For the Pythagorean intrinsic harmonic space we consider ratio’s between frequencies of tones that only contain prime numbers 1, 2 and 3. Let us look again at equation (2), which we have introduced for arbitrary tones \( u \) and \( v \). In principle, for arbitrary tones, \( k \) can be any real positive number. If we want to analyse Pythagorean harmonized tones, \( k \) must be the product of a power of 2 and a power of 3. This means that we can always write, for \( m, n \in \mathbb{Z} \)

\[
    k = 2^m \cdot 3^n
\]

These numbers \( m, n \) represent in a unique way the relation between \( u \) and \( v \), because there is only one way to write an arbitrary number that is a product of powers of 2 and powers of 3. We can thus take \( (m, n) \) as a representative of this relation.

If we want to remain general, we want to include all the tones that can be formed in this way, starting from a basic tone. We shall see that there is an intrinsic way to represent these tones in a two dimensional vector space. Indeed, suppose that we take the basic tone \( u \), and represent this basic tone by the \((0,0)\) vector of the two dimensional vector space (the origin). Then we can represent any other tone that is related to \( v \) by equation (2) by means of the point \((m,n)\) where \( m \) and \( n \) are the two whole numbers that appear in equation (15). Let us give some examples. If \( u \) is \( u(P, c) \), then in this two dimensional vector space \( u(P, c) \) will represented by the centre \((0,0)\). The tone \( u(P, c_1) \) will be represented by the point \((1,0)\). Indeed:

\[
    u(P, c_1)/u(P, c) = 2 = 2^1 \cdot 3^0
\]

The tone \( u(P, g_1) \) is represented by the point \((0,1)\) because

\[
    u(P, g_1)/u(P, c) = 3 = 2^0 \cdot 3^1
\]

Hence \( u(P, c_1) \) and \( u(P, g_1) \) form a canonical base of our vector space. The tone \( u(P, c_{-1}) \) is represented by the point \((-1,0)\) because

\[
    u(P, c_{-1})/u(P, c) = 1/2 = 2^{-1} \cdot 3^0
\]

and the tone \( u(P, f_{-2}) \) is represented by the point \((0,-1)\). Indeed it is the tone \( u(P, c) \) which is a fifth higher than the tone \( u(P, f_{-2}) \), hence

\[
    u(P, f_{-2})/u(P, c) = 1/3 = 2^0 \cdot 3^{-1}
\]

What is the place of the tone \( u(P, g) \) for example in our vector space? We have

\[
    u(P, g)/u(P, c) = 3/2 = 2^{-1} \cdot 3^1
\]

and therefore \( u(P, g) \) will be represented by the point \((-1,1)\). What will be the point representing \( u(P, f) \)?
We have
\[ \frac{u(P, f_{-2})}{u(P, c)} = \frac{1}{3} = 2^0 \cdot 3^{-1} \] (54)
which shows that the point \((0, -1)\) represents \(u(P, f_{-2})\). From this follows that
\[ \frac{u(P, f_{-1})}{u(P, c)} = \frac{2}{3} = 2^1 \cdot 3^{-1} \] (55)
which shows that the point \((1, -1)\) represents \(u(P, f_{-1})\), and as a consequence
\[ \frac{u(P, f)}{u(P, c)} = \frac{4}{3} = 2^2 \cdot 3^{-1} \] (56)
which shows that \(u(P, f)\) is represented by the point \((2, -1)\). In Figure 1 we represent this situation.

3.2 The Octave Projection

We want to investigate now how we get back our traditional Pythagorean system, the one we have presented in section 2. This means that we have to investigate which tones are in which octave. This we can see in the following way. A tone is in the octave \([c, c_1]\) if its relation to \(c\) is in the interval \([1, 2]\). So we can say in general that a tone is in the octave \([m, 0), (m+1, 0)\] if its relation to the basic tone \((0, 0)\) is in the interval \(2^{m-1}, 2^{m+1}\).

Let us consider then a general tone represented by \((x, y)\). The relation between this tone and the basic tone is given by \(2^x \cdot 3^y\). So to have this tone in the octave \([m, 0), (m+1, 0)\] we must have:
\[ 2^m \leq 2^x \cdot 3^y \leq 2^{m+1} \] (57)
This is equivalent to
\[ m \log 2 \leq x \log 2 + y \log 3 \leq (m+1) \log 2 \] (58)
Let us draw the lines given by the equations
\[ x \log 2 + y \log 3 = m \log 2 \] (59)
Suppose that \( y = 0 \) then \( x = m \), which means that the straight lines that are determined by equation (59) cut the \( x \)-axis in the point \( m \). On the other hand, when \( x = 0 \), then \( y = m \cdot \log 2/\log 3 \). This means that these straight lines cut the \( y \)-axis in the points \( m \cdot \log 2/\log 3 \). The direction coefficient of all the lines is given by \( -\log 2/\log 3 = -0.6309 \). In Figure 2 we have drawn the tones that are contained in the octave \([c, c_1]\), and are between the two lines, one with \( m = 0 \) and the other with \( m = 1 \).

In Figure 2 we represent tones as they have been projected along a line with direction coefficient \( \log 2/\log 3 = -0.6309 \) onto the \( x \)-axis. To be able to find back the cent value for each projected tone on the interval \([(0, 0), (0, 1)]\) we have to multiply the unit of the \( x \)-axis with 1200 such that \( c_1 \) is now in the point \((0, 1200)\). Let us calculate the values of the projections on the \( x \)-axis of the different tones of octave \([c, c_1]\).

For \( g \) we get

\[
\text{proj}(g) = (\log 3 - \log 2) \cdot 1200/\log 2 = 701.955
\] (60)

For \( d \) we have

\[
\text{proj}(d) = (2 \log 3 - 3 \log 2) \cdot 1200/\log 2 = 203.910
\] (61)

For \( a \) we have

\[
\text{proj}(a) = (3 \log 3 - 4 \log 2) \cdot 1200/\log 2 = 905.865
\] (62)

For \( e \) we have

\[
\text{proj}(e) = (4 \log 3 - 6 \log 2) \cdot 1200/\log 2 = 407.820
\] (63)

For \( b \) we have

\[
\text{proj}(b) = (5 \log 3 - 7 \log 2) \cdot 1200/\log 2 = 1109.775
\] (64)

For \( f\# \) we have

\[
\text{proj}(f\#) = (6 \log 3 - 9 \log 2) \cdot 1200/\log 2 = 611.730
\] (65)

![Figure 2: The Pythagorean tones in the vector space representation and the octave projection of these tones](image)

For \( f \) we have

\[
\text{proj}(f) = (2 \log 2 - \log 3) \cdot 1200/\log 2 = 498.045
\] (66)
For $b_b$ we have 

$$
\text{proj}(b_b) = (4 \log 2 - 2 \log 3) \cdot 1200 / \log 2 = 996.090
$$

(67)

For $e_b$ we have 

$$
\text{proj}(e_b) = (5 \log 2 - 3 \log 3) \cdot 1200 / \log 2 = 294.135
$$

(68)

For $a_b$ we have 

$$
\text{proj}(a_b) = (7 \log 2 - 4 \log 3) \cdot 1200 / \log 2 = 792.180
$$

(69)

For $d_b$ we have 

$$
\text{proj}(d_b) = (8 \log 2 - 5 \log 3) \cdot 1200 / \log 2 = 90.225
$$

(70)

And finally for $g_b$ we have 

$$
\text{proj}(g_b) = (10 \log 2 - 6 \log 3) \cdot 1200 / \log 2 = 588.270
$$

(71)

We see that the projections slowly deviate more and more from the tempered tones, that correspond in the representation of Figure 2 with sums of pieces of 1/12 of the interval $[c, c_1]$. For example the deviation of $g$ is only 1.956 cent, while the deviation of $f^#$ is already 11.730 cent. In Table 5 we represent the different tones and the measures of their intervals.

Table 5: A representation of the octave projection giving rise to the tones within the octave interval as they appear in the traditional Pythagorean system

| Vector | Tone Ratio | Example | Projection Value | Deviation |
|--------|------------|---------|------------------|-----------|
| (0, 0) | Basic      | c       | 0.000            | 0.000     |
| (8, -5)| Second     | $d_b$   | 90.225           | 9.775     |
| (-3, 2)| Third      | $d$     | 203.910          | 3.910     |
| (5, -3)| Minor Third| $e_b$   | 294.135          | 5.865     |
| (-6, 4)| Fourth     | $e$     | 407.820          | 7.820     |
| (2, -1)| Fourth     | $f$     | 498.045          | 1.955     |
| (10, -6)| Fifth     | $g_b$   | 588.270          | 11.730    |
| (-9, 6)| $f^#$      | $g_b$   | 611.730          | 11.730    |
| (-1, 1)| Fifth      | g       | 701.955          | 1.955     |
| (7, -4)| Sixth      | $a_b$   | 792.180          | 7.820     |
| (4, -2)| Sixth      | a       | 905.865          | 5.865     |
| (4, -2)| Sixth      | $b_b$   | 996.090          | 3.910     |
| (-7, 5)| Seventh    | b       | 1109.775         | 9.775     |
| (1, 0)| Octave     | $c_1$   | 1200.000         | 0.000     |

3.3 The Intrinsic Musical Structure

Let us show by means of a concrete example the difference between the intrinsic musical structure that we can make correspond to each musical piece in our intrinsic harmonic space and the traditional way that music pieces are represented.

We choose a part of the leading theme of Beethoven’s Fifth Symphony of which the consecutive notes are the following: $e, f, g, f, e, d, c, c, d, c, e, d, d$.

Let us write this in our vector representation: $(-6, 4), (2, -1), (-1, 1), (2, -1), (-6, 4), (-3, 2), (0, 0), (0, 0), (-3, 2), (-6, 4), (-6, 4), (-3, 2), (-3, 2)$. We also want to calculate the vectors that carry us in the vector space from one tone to another, because it is the collection of these vectors that determines completely the harmonic structure of the musical pattern formed by this leading theme of Beethoven’s Fifth Symphony. Consider for example the first tone and second tone of the theme, which are $e$ and $f$ represented by the vectors $(-6, 4)$ and $(2, -1)$. It is the vector $(8, -5) = (2, -1) - (-6, 4)$ that carries the $e$ to the $f$. In a similar way, it is the vector $(-3, 2) = (-1, 1) - (2, -1)$ that carries the second tone $f$ to the third tone $g$ of the theme. In this way we can represent the tone by these ‘difference’ vectors in a way that is independent of the starting tone. In Table 6 we have represented tones, vectors and difference vectors for the Beethoven theme.

In Figure 3 we have drawn the intrinsic musical pattern corresponding to this theme.
Table 6: The Beethoven theme with the first column the vectors, the second column the names of the tones, the third column the difference vectors and the fourth column the actions as named in the traditional way.

| Vector   | Tone | Difference Vector | Action       |
|----------|------|-------------------|--------------|
| (-6, 4)  | e    | (8, -5) = (2, -1) - (-6, 4) | half tone up |
| (2, -1)  | f    | (-3, 2) = (-1, 1) - (2, -1) | tone up      |
| (-1, 1)  | g    | (0, 0) = (-1, 1) - (-1, 1) | stay         |
| (-1, 1)  | g    | (3, -2) = (2, -1) - (-1, 1) | tone down    |
| (2, -1)  | f    | (-8, 5) = (-6, 4) - (2, -1) | half tone down |
| (-6, 4)  | e    | (3, -2) = (-3, 2) - (-6, 4) | tone down    |
| (3, -2)  | d    | (3, -2) = (0, 0) - (-3, 2) | tone down    |
| (0, 0)   | c    | (0, 0) = (0, 0) - (0, 0) | stay         |
| (-3, 2)  | d    | (-3, 2) = (-6, 4) - (-3, 2) | tone up      |
| (0, 0)   | c    | (-3, 2) = (-3, 2) - (0, 0) | tone down    |
| (-6, 4)  | e    | (3, -2) = (-6, 4) - (-6, 4) | tone down    |
| (-3, 2)  | d    | (0, 0) = (-3, 2) - (-3, 2) | stay         |

Figure 3: The intrinsic musical pattern corresponding to the Beethoven theme

We can transpose this pattern all over the two dimensional vector space, and each time it will represent the Beethoven theme played in a different key. This is exactly also what the tempered tuning system allows. But our scheme allows this transposition within a full harmonic system. In our scheme the transposition takes place while at the same time the full harmony of the Pythagorean system is conserved. Let us transpose the theme for example such that the starting note is $b_b$, hence vector $(4, -2)$. 

13
In Table 7 we present this situation and in Figure 4 this new situation is represented in the vectorspace.

Table 7: A transposition of the Beethoven theme

| Vector | Tone | Difference Vector | Action   |
|--------|------|-------------------|----------|
| (4, -2) b_b | (8, -5) | half tone up       |
| (12, -7) b | (-3, 2) | tone up            |
| (9, -5) d_b | (0, 0) | stay              |
| (9, -5) d_b | (3, -2) | tone down          |
| (12, -7) b | (-8, 5) | half tone down     |
| (4, -2) b_b | (3, -2) | tone down          |
| (7, -4) a_b | (3, -2) | tone down          |
| (10, -6) g_b | (0, 0) | stay              |
| (10, -6) g_b | (-3, 2) | tone up            |
| (7, -4) a_b | (-3, 2) | tone up            |
| (4, -2) b_b | (0, 0) | stay              |
| (4, -2) b_b | (3, -2) | tone down          |
| (7, -4) d | (0, 0) | stay              |
| (7, -4) d |                |                    |

Figure 5: A transposition of the Beethoven theme. The harmonic space shows that the note b changes in pitch through this transposition, we have called this new note b*
The transposition of the Beethoven theme such that it starts with $b_5$ uses the tones of the original Pythagoras system as we have represented in Table 1 and Table 3 except for the tone $b^\ast$. Indeed the note $b$ corresponds for this theme with the vector $(12, -7)$, which is not included in Table 1 or Table 3. The tone $b$ as included in Table 1 or Table 3 comes from the vector $(-7, 5)$, and not from the vector $(12, -7)$. Let us calculate the difference between the tone coming from the vector $(-7, 5)$, denoted $b$, and the tone coming from vector $(12, -7)$, denoted $b^\ast$. The difference is the length of the difference vector $(-19, 12) = (-7, 5) - (12, -7)$, and $\|(-19, 12)\| = (-19 \log 2 + 12 \log 3) \cdot 1200 / \log 2 = 23.460$ cent.

We can understand now very well what happens in general. Once a tone of a theme after transposition leaves the rectangle of 10 by 7 (the rectangle that is shown in Figure 3) in which the traditional Phytagorean system is constructed before being projected by the octave projection onto the octave, it will not coincide with the tones of Table 1 or Table 3. The difference will always be the same, namely 23.460 cent.

The musical pattern drawn in the vector space corresponding to the intrinsic harmonic spaces contains all the harmonic content of a musical theme. Transposing just means transposing geometrically this pattern, as we have illustrated for the Beethoven theme (see Figure 5). If this geometric pattern corresponding to a musical theme gets transposed over the vector space, the projection into the octave, as illustrated in Figure 2, gives rise to different possible tones corresponding to the same note, as it is the case for the note $b$ and the tones $b$ and $b^\ast$ in Figure 5. Two such different tones are separated by a multiple of the Pythagorean COMMA. If a musical theme is represented in the intrinsic harmonic space we can always find out easily, just watching and following the geometrical pattern, which notes will correspond to which different tones in which parts of the musical theme. The rule is that we just have to minimize the difference vectors in the vector space. This rule allows a uniquely determined and complete harmonic performance of every possible musical theme, but equal notes will correspond to different tones for different parts of the theme as prescribed by the geometry of the intrinsic harmonic space, and the way the musical theme is represented in this space.

4 General Intrinsic Harmonic Spaces

In the foregoing section we have analyzed in detail the intrinsic harmonic space corresponding to the Pythagorean system. We saw that this space is realized as a two dimensional vector space. Historically it is the Pythagorean system that has given rise to the twelve half notes that exist in an octave in the tempered system, because $2^{19} \approx 3^{12}$. Our musical system is so much biased by this pattern of twelve half notes inside an octave, that is is difficult for us to take fully into account the relative nature of this choice. That is also the reason that the tempered system has become the system of reference now for all western music.

The story goes that the tempered system was propagated strongly by Johann Sebastian Bach. Indeed, one of the oeuvres of Bach is called ‘Das Wohltemperierte Klavier’ (The Well-Tempered Clavier), where he writes 24 fuga’s and preludes for piano, each of them in a different minor and major key. A closer look at the history shows however that Bach was not really interested in equal tempering, as we know it now, but he was interested in a system that would allow to play in all keys, but at the same time would conserve as much as possible the important harmonies. At the time of Bach it was in effect meantone tuning that was used, and Bach wanted to show that if fuga’s are written well, it is possible to use this system for all keys. The complete equal temperament system was not used in these times, because the consensus was that it sounded awful, out of tune and characterless. Only during the 19th century, keyboard tuning drifted closer and closer to equal temperament over the protest of many of the more sensitive musicians. Only in 1917 was a method devised for tuning exact equal temperament.

4.1 Meantone Tuning

Meantone tuning can be considered to be Europe’s most successful tuning. It appeared sometime around the late 15th century, and was used widely through the early 18th century. It survived in the tuning of English organs, all the way through the 19th century.

The principle of meantone tuning is that preserving the consonance of the thirds ($c$ to $e$, $f$ to $a$, $g$ to $b$) is more important than preserving the purity of the fifths ($c$ to $g$, $f$ to $c$, $g$ to $d$). The acoustical reason for this preference is that the notes in a slightly out-of-tune third, being closer together than those in a fifth, create faster and more disturbing beats than those in a slightly out-of-tune fifth [7]. In a perfect harmonic third, the two strings vibrate at a frequency ratio of 5 to 4. In section 1.3 we analyzed already the possibility of the perfect harmonic third (see Table 2), and found that its value in cent equals 386.314. As we mentioned already in section 1.3 the fact that three perfect fifths do not give an octave, since $3 \times 386.314 = 1158.941 < 1200$, was one of the basic problems of perfect harmony that musicians have been confronted with. This means indeed that it is not
possible to adopt a tuning system where, for example, the three thirds c to e, e to aₜ, and aₜ to c₁, would all be tuned as perfect thirds. Meantone tuning tried to find a solution where as much as possible of the thirds are tuned as perfect thirds, by giving up somewhat on the perfect tuning of the fifths as compared to Pythagorean tuning.

There is not one unique meantone tuning system, several variations on the basic ideas have been tried out. Let us explain the one invented by Pietro Aaron in 1523, as counted in [11] (see Table 8 for a presentation of this meantone tuning system).

Fixing c₁, e and aₜ:

We start with the octave c₁/c = 2, hence c and c₁ are separated by 12,000 cent, as usual. The second step is to tune a perfect third, which means that e/c = 5/4, and hence c and e are separated by 386.314 cent. Also the third with e as basic note is tuned in perfect harmony, which means that aₜ/e = 5/4 and hence aₜ/c = aₜ/e·c/e = 25/16. This means that c and aₜ are separated by 2 × 386.314 = 772.627 cent. There cannot be a perfect third with aₜ as basic note, if we want to respect the octave, hence the aₜ key is sacrificed, it will not be used in the meantone tuning system. More specifically c₁/aₜ = c₁/c·c/aₜ = 2·16/25 = 32/25. This means that aₜ and c are separated by 427.372 cent, which is way too much for a third. Since this distance comes however from a fraction that is still simple, namely 32/25, it sounds all right, but not like a third. This is the simple part of the meantone tuning.

Fixing d, f# and b₉:

Remark that in the Pythagorean system we have d/c = 9/8, and we also have c/b₉ = 9/8. This means that d/b₉ = 81/64. This means that b₉ and d are separated in the Pythagorean system by 407.820 cent, and hence do not form a perfect third, because then they would have to be separated by 386.314 cent. To rescue the perfect third b₉ to d it is decided to sacrifice the Pythagorean second of 9/8. More specifically, it is decided to tune the seconds by just taking half of the perfect third. This does not correspond any longer to a fraction, which means that in meantone tuning for the first time there is this departure from fractions, as later happens completely for tempered tuning. Hence the note d is tuned with distance 386.314/2 = 193.157 cent from c and distance 386.314/2 = 193.157 cent from e. It is this decision that has given its name to the meantone tuning, taking the meantone for d of e and c. In a similar way f# is put in between e and aₜ. This gives a difference between c and f# of 386.314 + 193.157 = 579.471 cent. By doing so we have obviously also created a perfect third from d to f#. We have no intention to create a harmonious a₉ key, because the distance between a₉ and c₁ is already spoiled by our compromise between the perfect thirds and the octave. This means that we can as well choose b₉ in such a way that also the b₉ key contains a perfect third. This means, for example, that we have to choose the distance between b₉₋₁ and c equal to 193.157 cent, which means that the distance between c and b₉ must equal 1200.000 - 193.157 = 1006.843 cent.

The meantone tuning wants to pay attention also to the minor thirds, as well to the fifths of course. A perfect minor third corresponds to the ratio 6/5, which is a distance of 315.641 cent. A perfect fifth corresponds to the ratio 3/2, which is a distance of 701.955 cent. Some harmony theory comes in here. For the overall harmony within one specific key there are two minor thirds of other keys that are the most important ones. For example for the overall harmony in key c, the most important minor thirds are the one of key a, called the relative one, and the one of key e. A perfect minor third of key e would fix g at cent distance 386.314 + 315.641 = 701.955 cent, which would create perfect harmony within key c, because also the fifth is perfect in this case. This is however not possible for other keys, and a compromise has to be made if also in other keys one wants to keep close to the perfect minor thirds and the perfect fifths.

Fixing g:

The cent distance between e and g is chosen to be 310.500 cent, which fixes g at 386.314 + 310.500 = 696.814 cent. This creates two fifths, one in key c with value 696.814 cent, and another in key g with value 1200.000 + 193.157 - 696.814 = 696.343 cent. Both fifths are too small, but sufficiently close to the perfect fifth to be all right.

Fixing a:

The cent distance between a and c₁ is chosen to be 310.300 cent, which fixes a at a distance 1200.000 - 310.300 = 889.700 cent, and creates again two fifths. One in key d of value 889.700 - 193.157 = 696.543 cent and another in key a with value 1200.000 + 386.314 - 889.700 = 696.614 cent.Fixing a we have also fixed the minor third in key f#, with value 889.700 - 579.471 = 310.229 cent, which is an important minor key for the overall harmony in key d.

Fixing b:

We choose b by giving the value of 310.300 cent to the minor third in b, which is the relative minor third to the third in key d, and hence the most important minor third for the overall harmony in key d. This fixes b at
within the meantone tuning system. The fixing of \( e \) itself, with value 579.471 - 310.300 = 269.171 cent. This is also a bad value.

\[ e \]

of the third in key \( f \) to \( i(2, 5) \) and hence with ratio 5\( ^{th} \) of this construction is the perfect third of what the Pythagorean tuning is with respect to the space \( i(2, 3) \). We take \( = 696.543 \) cent. Both fifths are reasonable approximations of the perfect fifth. The fixing of \( e \) also introduces a new minor third in key \( a \) with value 1082.857 - 772.627 = 310.230 cent.

Fixing \( e \):

In a similar way the distance of \( e \) is chosen such that the minor third in key \( c \), which is the relative minor third of the third in key \( e \), fits the value 310.300. Hence this fixes \( e \) on 310.300 cent. This creates again two new fifths, the one in key \( e \), with value 1006.843 - 310.300 = 696.543 cent, and the one in key \( a \), with value 310.300 + 1200.000 - 772.627 = 737.673 cent. And here we have arrived at a point where the meantone tuning system fails completely. Key \( a \) has a third and a fifth that are both way to big. It will not be possible to use this key within the meantone tuning system. The fixing of \( e \) also creates a new minor third, namely the one of key \( e \) itself, with value 579.471 - 310.300 = 269.171 cent. This is also a bad value.

\[ 193.157 - 310.300 + 1200.000 = 1082.857 \text{ cent. The fixing of } b \text{ creates two new fifths, namely the fifth of key } b, \text{ with value } 579.471 - 1082.857 + 1200.000 = 696.614 \text{ cent, and the fifth of key } e, \text{ with value } 1082.857 - 386.314 = 696.543 \text{ cent. Both fifths are reasonable approximations of the perfect fifth. The fixing of } b \text{ also introduces a new minor third in key } a, \text{ with value } 1082.857 - 772.627 = 310.230 \text{ cent.} \]

4.2 The Intrinsic Harmonic Space \( I(2, 5) \)

The intrinsic harmonic space related to the Pythagorean tuning system only uses the prime numbers 2 and 3, and that is why we will denote it \( I(2, 3) \). Meantone tuning introduces the perfect third by allowing the prime number 5 to play a role in the harmony. Let us see how the perfect harmonic space \( I(2, 5) \) would look like if working now in \( I(2, 5) \), and this also means that we can indicate the perfect third by \( e(5) \). Let us see now we are confronted with a very similar type of trouble than the one we encountered already with the attempt of putting the Pythagorean intrinsic harmonic space \( I(2, 3) \) onto the one dimensional octave keyboard. Indeed, the third tone that gives rise to a projection inside the octave is given by the point \( -4, 2 \), hence the ratio 25/16, which is 772.627 cent. This is again in the region of the tempered \( a \) at 800.000 cent, which means that we have two candidates now for a \( a \) within the octave projection of \( I(2, 5) \), one at 813.686 cent, which we called \( a(5) \) and another one at 772.627, which we will call \( a^*_5(5) \). The difference between both, 813.686 - 772.627 = 41.059 cent, is the value of the 5 – COMMA, which is the comma of the 5-system in analogy with the Pythagorean comma.
The fourth tone corresponds to the point (5, -2), hence the ratio 32/25 which is 427.373 cent and is a second perfect version of the tempered e at 400.000 cent, which we will call e∗. Its difference with the perfect e equals 427.373 - 386.314 = 41.059 cent, again the value of the 5 – COMMA.

This means that only three tones can be constructed harmonically within the construction which is the equivalent one with respect to I(2, 3) as the Pythagorean is with respect to I(2, 3), which we could expect in fact, because the power of 5 that comes close to a power of 2 is 3. The tone system that results in this way is too small to be able to lead to a tone system that can be used to play melodies. However the meantone tuning system uses some of the tones that result in this way. However, meantone tuning also makes use of the ratio 6/5 for the perfect minor third. Probably nobody was aware of this at that time, but the use of a perfect minor third was a first step towards the use of a higher than two dimensional harmonic space. We will analyse this space in the next section.

### 4.3 The Intrinsic Harmonic Space I(2, 3, 5)

To introduce the harmony of the perfect third and the perfect minor third within the intrinsic harmonic spaces we have to consider the harmonic space built by using the prime numbers 2, 3 and 5. This is a three dimensional space, each vector of it is written as a triple \((x, y, z)\) of three numbers. Meanwhile it is clear that music written in this new intrinsic harmonic space should make use directly of the vectors and the difference vectors within this three dimensional vector space. However, because of the ingrained tradition of the one dimensional octave system corresponding to traditional keyboards we are in need to look at the octave projection of this new intrinsic harmonic space I(2, 3, 5) to be able to see and understand its reach. Hence, let us mention some of the octave projections of I(2, 3, 5) with c as basic key. Hence in this case c is represented by the vector \((0, 0, 0)\). The vector \((-1, 0, 0)\) represents \(c_{0}\), and the vector \((-1, -1, 0)\) represents \(g\). The vector \((-2, 0, 1)\) represents \(e\) when tuned as a perfect third, while the vector \((-6, 4, 0)\) represents \(e\) when tuned within the Pythagorean system. We mentioned already that the fact that we have twelve half notes in a octave is related to the fact that \(2^{19} \approx 3^{12}\). The vector \((-19, 12, 0)\) corresponds to the new \(c\), denoted \(c^*\), within the Pythagorean tuning. The difference between \((0, 0, 0)\) and \((-19, 12, 0)\) in the octave projection equals 23.460 cent and is called the Pythagorean COMMA. We also mentioned that the I(2, 5) intrinsic harmonic space attributes 2 extra tones to each note, one being the perfect third of each of the considered notes and the other being what is the \(a_5(5)\) with respect to \(c\) taken with respect to each of the considered notes. However, meantone tuning also had introduced the perfect minor third, at a fraction of 6/5 with respect to the \(c\). This tone corresponds to the vector \((1, 1, -1)\), which is a genuine vector of the intrinsic harmonic space I(2, 3, 5), not existing in neither I(2, 3) and nor I(2, 5).

In Table 9 we collect some of the tones that result if anyhow such a projection is attempted, and we take \(c\) as a basic tone. Hence we have added the perfect thirds and perfect minor thirds for some of the Pythagorean tones, more specifically for \(c, d, e, f, g, a\) and \(b\). It is interesting to note that some of these ‘new’ tones come closer to the tempered ones than the original Pythagorean tones and the ones of the meantone system.

Table 9: The intrinsic harmonic space I(2, 3, 5)

| Vector    | Tone Ratio    | Example       | Projection Value | Deviation |
|-----------|---------------|---------------|------------------|-----------|
| (0, 0, 0) | Basic         | c             | 0.000            | 0.000     |
| (-4, 4, -1) | Meantone Minor Third for \(a\) | \(d_{b}^{path}\) | 90.225           | 9.775     |
| (-7, 3, 1) | Meantone Third for \(a\) | \(d_{b}^{path}\) | 92.179           | 7.821     |
| (-7, 6, -1) | Pythagorean Second | \(d_{b}^{path}\) | 203.910          | 3.910     |
| (5, -3, 0) | Pythagorean Minor Third | \(e_{b}^{path}\) | 294.135          | 5.865     |
| (-10, 5, 1) | Meantone Third for \(\hat{b}\) | \(e_{b}^{mean}\) | 296.089          | 3.911     |
| (1, 1, -1) | Meantone Minor Third for \(c\) | \(e_{b}^{mean}\) | 315.641          | 15.641    |
4.4 The Beethoven Theme in I(2, 3, 5)

Let us write a version of the Beethoven theme in the intrinsic harmonic space I(2, 3, 5), making use of the presence of the perfect third which is absent in the Pythagorean intrinsic harmonic space I(2, 3).

Table 10: The Beethoven theme in I(2, 3, 5). The first column represents the vectors, the second column the naming of the tones, the third column the difference vectors, and the fourth column the actions as named in the traditional way.

| Vector  | Tone   | Difference Vector | Action  |
|---------|--------|-------------------|---------|
| (-2, 0, 1) | e      | (-4, -1, -1) = (2, -1, 0) - (-2, 0, 1) | half tone up |
| (2, -1, 0) | f      | (-3, 2, 0) = (-1, 1, 0) - (2, -1, 0) | tone up   |
| (1, 1, 0)  | g      | (0, 0, 0) = (-1, 1, 0) - (-1, 1, 0) | stay     |
| (-1, 1, 0) | g      | (3, -2, 0) = (2, -1, 0) - (-1, 1, 0) | tone down |
| (2, -1, 0) | f      | (-4, 1, 1) = (-2, 0, 1) - (2, -1, 0) | half tone down |
| (-2, 0, 1) | e      | (-1, 2, -1) = (-3, 2, 0) - (-2, 0, 1) | tone down |
| (3, 2, 0)  | d      | (3, -2, 0) = (0, 0, 0) - (-3, 2, 0) | tone down |
| (0, 0, 0)  | c      | (0, 0, 0) = (0, 0, 0) - (0, 0, 0) | stay     |
| (0, 0, 0)  | c      | (-3, 2, 0) = (-3, 2, 0) - (0, 0, 0) | tone up   |
| (-3, 2, 0) | d      | (1, -2, 1) = (-2, 0, 1) - (-3, 2, 0) | tone up   |
| (-2, 0, 1) | e      | (0, 0, 0) = (-2, 0, 1) - (-2, 0, 1) | stay     |
| (-2, 0, 1) | e      | (-1, 2, -1) = (-3, 2, 0) - (-2, 0, 1) | tone down |
| (-3, 2, 0) | d      | (0, 0, 0) = (-3, 2, 0) - (-3, 2, 0) | stay     |

The difference with the theme in the Pythagorean tone system can be seen very easily now and is represented in Table 10. We have used the perfect third for e instead of the Pythagorean third. The other tones have remained Pythagorean, because for them the Pythagorean tones are the best choice. Remark that the actions ‘half tone up’ and ‘half tone down’ are characterized by different vectors than this was the case for the Beethoven tune in the Pythagorean space I(2, 3). Also two different actions correspond to ‘tone up’ represented by the vectors (-3, 2, 0) and (1, -2, 1), and the opposite actions (3, -2, 0) and (-1, 2, -1) correspond to ‘tone down’.

Again, like in the Pythagorean case, this Beethoven theme can be transposed to any starting tone within the intrinsic harmonic space I(2, 3, 5), and keep its perfect harmonic content.
5  The Prime Number 7 and the Space I(2, 3, 5, 7)

Pythagorean tuning only considers the prime numbers 2 and 3. Meantone tuning had discovered the prime number 5 for some of its tones. What if we allow the prime number 7 to join the scene? In principle, this should give rise to harmonies that are of a completely new type, neither present within the Pythagorean system nor within the meantone system.

5.1  The Space I(2, 7)

Let us start with the most simple of all ways to incorporate the prime number 7, namely in the intrinsic harmonic space I(2, 7). The question we have to investigate first is whether there is a power of 2 that comes close to a power of 7. The space I(2, 3) gives rise to 12 tones because $2^{19} \approx 3^{12}$. The space I(2, 5) gives rise to 3 tones because $2^7 \approx 5^3$. For I(2, 7) we get $2^{23} \approx 7^{26}$. This means that if we construct a set of tones in a similar way than is done within the Pythagorean system, we will find 26 tones in an octave for I(2, 7). See Table 11 for the construction of the 26 tones.

| Vector | Tone Ratio | Example | Projection Value | Deviation |
|--------|------------|---------|------------------|-----------|
| (0, 0, 0, 0) | Basic | c | 0.000 | 0.000 |
| (-14, 0, 0, 5) | | | 44.130 | 44.130 |
| (-28, 0, 0, 10) | | | 88.259 | 11.741 |
| (31, 0, 0, -11) | | | 142.915 | 42.915 |
| (17, 0, 0, -6) | Second | d | 187.045 | 12.955 |
| (3, 0, 0, -1) | | | 231.174 | 31.174 |
| (-11, 0, 0, 4) | | | 275.304 | 24.696 |
| (-25, 0, 0, 9) | Minor Third | e | 319.433 | 19.433 |
| (34, 0, 0, -12) | | | 374.089 | 25.911 |
| (20, 0, 0, -7) | Third | e | 418.218 | 18.218 |
| (6, 0, 0, -2) | Fourth | f | 462.348 | 37.652 |
| (-8, 0, 0, 3) | | | 506.478 | 6.478 |
| (-22, 0, 0, 8) | | | 550.607 | 49.393 |
| (-36, 0, 0, 13) | | | 594.737 | 5.263 |
| (37, 0, 0, -13) | | | 605.263 | 5.263 |
| (23, 0, 0, -8) | | | 649.393 | 49.393 |
| (9, 0, 0, -3) | Fifth | g | 693.522 | 6.478 |
| (-5, 0, 0, 2) | | | 737.652 | 37.652 |
| (-19, 0, 0, 7) | | | 781.781 | 18.209 |
| (-33, 0, 0, 12) | | | 825.911 | 25.911 |
| (26, 0, 0, -9) | Sixth | a | 880.567 | 19.433 |
| (12, 0, 0, -4) | | | 924.696 | 24.696 |
| (-2, 0, 0, 1) | | | 968.826 | 31.174 |
| (-16, 0, 0, 6) | | | 1012.956 | 12.956 |
| (-30, 0, 0, 11) | | | 1057.085 | 42.915 |
| (29, 0, 0, -10) | Seventh | b | 1111.741 | 11.741 |
| (15, 0, 0, -5) | | | 1155.870 | 44.130 |
| (1, 0, 0, 0) | Octave | c | 1200 | 0 |

We can construct the intrinsic harmonic spaces I(2, 3, 7), I(2, 5, 7) which will be three dimensional vector spaces each of them, and also the intrinsic harmonic space I(2, 3, 5, 7) which is a four dimensional vector space, combining all the constructions we did before and using the same method. Already the space I(2, 3, 5) will allow to play a type of harmony that never has been able to be realized before, and was attempted to by the different versions of the meantone tuning. However, when we also introduce the harmonies offered by the intrinsic harmonic space I(2, 3, 5, 7) a totally new type of harmony becomes available able to produce chord and tones that never before were able to be produced in a systematic way.
5.2 Musical instruments for perfect harmonic spaces

The reason that the old problem of perfect tuning was never solved is strictly related to the limitation that sets in if one wants to achieve such tuning using a musical instrument that generates tones with a keyboard that is one dimensional. In principle, music played on violins produce such harmonies. However, violin music is also limited in performance and especially in notation by the 12 notes of the octave. Although a violin can play all tones in principle there is no systematic way to indicate these tones let alone to form chords and keys with them. All of this becomes possible with the help of the intrinsic harmonic spaces presented here. If, in a first step, we limit ourselves to the intrinsic harmonic space \( I(2, 3) \) built around the Pythagorean scale, we can better understand what the real limitation is. In Figure 2 it becomes clear that classical thinking takes place in a one dimensional projection of the true harmonic space, which is two dimensional. So it is the attempt to squeeze harmony into the linear one-dimensional space of frequencies, the space that goes from low frequencies linear on a keyboard to high frequencies, on the keyboard from left to right, that makes it impossible. The intrinsic harmonic space \( I(2, 3) \) needs two dimensions. If we build a musical instrument that is two dimensional instead of one dimensional we can play the total intrinsic harmonic space \( I(2, 3) \) on that musical instrument. What does such a musical instrument look like? Consider again Figure 2 where the plane is shown in which \( I(2, 3) \) lies. Imagine that each point on this plane with integer coordinates, i.e., where the lines intersect, is a key such that when this key is pushed the tone corresponding to the point is played. Playing these keys then allows us to directly play the intrinsic harmonic space \( I(2, 3) \). The Beethoven theme worked out in Figure 3, Figure 4 and Figure 5 gives a good example of how to play such an instrument. Note that the tones are not ordered from low to high frequency, but the order is according to the \( I(2, 3) \) harmony.

Note, however, that for \( I(2, 3, 5) \) we already need three dimensions. This could still be realized with a physical keyboard-like instrument since the 5 harmony only adds a few notes. For example, one could work with a pedal that switches for the vertical up and down of the dimension where the 5 harmony belongs. And the 2, 3 harmony is then realized in the plane with keys at the points of the plane, as in Figure 2. However, if we go to higher intrinsic harmonic spaces, such as \( I(2, 3, 5, 7) \), then four dimensions are required and a physical instrument becomes difficult to realize. However, now that we have computers available, even this limitation can be removed. Programming tones by using the vectors of the vector spaces as denoting a tone is no problem at all in a computer, even for \( I(2, 3, 4, 7) \). Even \( I(2, 3, 4, 7, 11) \) can be explored that way. In further work, we plan to program some well-known pieces of music in such a way, in order to get to performances of these pieces of music with perfect harmony, which was probably never heard before by human ears.

References

[1] Owen H. Jorgensen, 1991, *Tuning: Containing the Perfection of Eighteenth-Century Temperament, the Lost Art of Nineteenth-Century Temperament, and the Science of Equal Temperament*, Michigan State University Press, Michigan.

[2] Anita T. Sullivan, 1986, *The Seventh Dragon: The Riddle of Equal Temperament*, Metamorphous Press, Portland.

[3] Stuart M. Isacoff, 2002, *Temperament: The Idea That Solved Music’s Greatest Riddle*, Alfred Knopf, New York.

[4] Abraham, G., 1979, *The Concise Oxford History of Music*, Oxford University Press, Oxford.

[5] Grout, D. J. and Palisca, C. V., 1960, *A History of Western Music*, J. M. Dent and Sons, North Carolina.

[6] Eric Regener, *Pitch Notation and Equal Temperament: A Formal Study*, University of California Press

[7] Rayleigh, J. W. S., 1945, *The Theory of Sound*, Dover Publications, New York.