AN EXTENSION OF BROWN FUNCTOR TO COSPAN DIAGRAMS OF SPACES

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Abstract. Let $\mathcal{A}$ be a small abelian category. The purpose of this paper is to introduce and study a category $\overline{\mathcal{A}}$ which implicitly appears in construction of some TQFT’s where $\overline{\mathcal{A}}$ is determined by $\mathcal{A}$. If $\mathcal{A}$ is the category of abelian groups, then the TQFT’s obtained by Dijkgraaf-Witten theory of abelian groups or Turaev-Viro theory of bicommutative Hopf algebras factor through $\overline{\mathcal{A}}$ up to a scaling. In this paper, we go further by giving a sufficient condition for an $\mathcal{A}$-valued Brown functor to extend to a homotopy-theoretic analogue of $\overline{\mathcal{A}}$-valued TQFT for arbitrary $\mathcal{A}$. The results of this paper and our subsequent paper reproduce TQFT’s obtained by DW theory and TV theory.

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1. Introduction

Let $\mathcal{A}$ be a small abelian category. The purpose of this paper is to introduce and study a category $\overline{\mathcal{A}}$ which implicitly appears in construction of some TQFT’s where $\overline{\mathcal{A}}$ is determined by $\mathcal{A}$. If $\mathcal{A} = \text{Ab}$, i.e. the category of abelian groups, then the TQFT’s obtained by Dijkgraaf-Witten theory [2] [3] of abelian groups or Turaev-Viro theory [8] [11] of bicommutative Hopf algebras factor through $\overline{\mathcal{A}}$ up to a scaling. In this paper, we go further by giving a sufficient condition for an $\mathcal{A}$-valued Brown functor to extend to a homotopy-theoretic analogue of $\overline{\mathcal{A}}$-valued TQFT for arbitrary $\mathcal{A}$. The results of this paper and our subsequent paper reproduce TQFT’s obtained by DW theory and TV theory.
The category $\overline{\mathcal{A}}$ is a dagger symmetric monoidal category. The category $\mathcal{A}$ is bijectively and faithfully embedded into $\overline{\mathcal{A}}$. We denote by $\iota_\mathcal{A} : \mathcal{A} \to \overline{\mathcal{A}}$ the embedding functor. See Remark 3.15 for details.

A $d$-dimensional $\mathcal{A}$-valued Brown functor $E$ is a functorial assignment of an object $E(K)$ of $\mathcal{A}$ to a pointed finite CW-space $K$ with dim $K \leq d$. It assigns the direct sum in $\mathcal{A}$ to the wedge sum of spaces and satisfies the Mayer-Vietoris axiom (see Definition 6.4).

For $d \in \mathbb{N} \cup \{\infty\}$, a cospan diagram of pointed finite CW-spaces can be a $d$-dimensional cospan diagram of pointed finite CW-spaces. We denote the category by $\text{Cosp}_{\leq d}(\text{CW}_{\ast}^\text{fin})$ (see Definition 3.10).

Note that the homotopy category $\text{Ho}(\text{CW}_{\ast}^\text{fin})$ of pointed finite CW-spaces $K$ with dim $K \leq (d-1)$ is naturally a subcategory of $\text{Cosp}_{\leq d}(\text{CW}_{\ast}^\text{fin})$. Then both of the suspension and the proper restriction of a $d$-dimensional $\mathcal{A}$-valued Brown functor extend to $\text{Cosp}_{\leq d}(\text{CW}_{\ast}^\text{fin})$:

**Theorem 1.1.** For $d \in \mathbb{N} \cup \{\infty\}$, let $E : \text{Ho}(\text{CW}_{\ast, \leq d}^\text{fin}) \to \mathcal{A}$ be a $d$-dimensional $\mathcal{A}$-valued Brown functor.

1. There exists a unique dagger-preserving symmetric monoidal extension of $\iota_\mathcal{A} \circ E \circ \Sigma$ to $\text{Cosp}_{\leq d}(\text{CW}_{\ast}^\text{fin})$. Here, $\Sigma : \text{Ho}(\text{CW}_{\ast, \leq (d-1)}^\text{fin}) \to \text{Ho}(\text{CW}_{\ast, \leq d}^\text{fin})$ denotes the suspension functor.

2. There exists a unique dagger-preserving symmetric monoidal extension $\iota_\mathcal{A} \circ E \circ i$ to $\text{Cosp}_{\leq d}(\text{CW}_{\ast}^\text{fin})$. Here, $i : \text{Ho}(\text{CW}_{\ast, \leq (d-1)}^\text{fin}) \to \text{Ho}(\text{CW}_{\ast, \leq d}^\text{fin})$ denotes the inclusion functor.

The proof appears in section 6.

We give some examples of $\infty$-dimensional $\mathcal{A}$-valued Brown functors in Example 6.6, 6.7. The categories $\text{Cosp}_{\leq d}(\text{CW}_{\ast}^\text{fin})$ and $\overline{\mathcal{A}}$ are constructed from a general formulation on (co)span categories (see section 2).

In a forthcoming paper, we give a construction of a $\text{Vec}_{k}^\text{ssd}$-valued projective TQFT from a $\text{Hopf}_{k}^\text{bc,vol}$-valued Brown functor based on the results in this paper. For a field $k$, we denote by $\text{Vec}_{k}^\text{ssd}$ the category of vector spaces over $k$ equipped with a symmetric self-duality. $\text{Hopf}_{k}^\text{bc,vol}$ denotes the category of bicommutative Hopf algebras with a finite volume $[4]$. In particular, we show that a $\text{Hopf}_{k}^\text{bc}$-valued homology theory induces a (possibly, empty) family of projective TQFT’s. We also give some computations of the obstruction classes induced by the scalars appearing from compositions. It gives a generalization of Dijkgraaf-Witten theory of abelian groups and Turaev-Viro theory of bicommutative Hopf algebras.

This paper is organized as follows. In section 2, we give a way to construct a (dagger symmetric monoidal) category whose morphisms consist of some equivalence classes of a cospan diagram. We apply the results to obtain the category $\text{Cosp}_{\leq d}(\text{CW}_{\ast}^\text{fin})$ in section 3. In section 4, we introduce a preorder of (co)span diagrams in an abelian category. In section 5, we define $\text{Cosp}^\infty(\mathcal{A}), \text{Sp}^\infty(\mathcal{A})$ via the general formulation in section 2. Furthermore, in subsection 5.2, we give an isomorphism between $\text{Cosp}^\infty(\mathcal{A}), \text{Sp}^\infty(\mathcal{A})$. In section 6, we prove Theorem 1.1
2. A general construction of (co)span categories

2.1. Definitions.

Definition 2.1. Let $C$ be a small category. A diagram $\Lambda = \left( x_0 \xrightarrow{f_0} y \xleftarrow{f_1} x_1 \right)$ is called a cospan diagram in $C$. Let $\text{Cosp}(C)$ be the set consisting of cospan diagrams in the category $C$. We define the source and target of $\Lambda$ by $s(\Lambda) \overset{\text{def}}{=} x_0$ and $t(\Lambda) \overset{\text{def}}{=} x_1$.

Definition 2.2. Let $C$ be a small category. Denote by $\iota : \text{Mor}(C) \to \text{Cosp}(C)$ the induced cospan defined by $\iota(f) = \left( x \xrightarrow{f} y \xleftarrow{\text{Id}_y} y \right)$. For an object $x$ in $C$, we write $\iota(x) \overset{\text{def}}{=} \iota(\text{Id}_x)$ where $\text{Id}_x : x \to x$ is the identity on $x$.

Definition 2.3. Consider an equivalence relation $\sim$ on the set $\text{Cosp}(C)$. The equivalence relation $\sim$ is compatible with the source and target maps if $\Lambda \sim \Lambda'$ for $\Lambda, \Lambda' \in \text{Cosp}(C)$ implies $s(\Lambda) = s(\Lambda')$ and $t(\Lambda) = t(\Lambda')$ where $s, t$ appear in Definition 2.1.

Definition 2.4. A subset $U \subset \text{Cosp}(C)$ is admissible if the following conditions hold.

1. We have $s(U) = t(U)$.
2. For $x, y \in \partial U$ and a morphism $f : x \to y$, we have $\iota(f) \in U$.

Definition 2.5. Let $\sim$ be an equivalence relation on $\text{Cosp}(C)$ compatible with the source and target maps. Let $U$ be an admissible subset of $\text{Cosp}(C)$. Denote by $U_i \times_{s,U} U = \{ (\Lambda, \Lambda') : t(\Lambda) = s(\Lambda') \}$. A map $\circ : U_i \times_{s,U} U \to U$ is a weak composition with respect to the pair $(\sim, U)$ if the following conditions hold.

1. For $(\Lambda_0, \Lambda_0'), (\Lambda_1, \Lambda_1') \in U_i \times_{s,U}$, if $\Lambda_0 \sim \Lambda_1$ and $\Lambda_0' \sim \Lambda_1'$, then we have $\Lambda_0' \circ \Lambda_0 \sim \Lambda_1' \circ \Lambda_1$.
2. For $(\Lambda, \Lambda'), (\Lambda', \Lambda'') \in U_i \times_{s,U}$, we have $(\Lambda'' \circ \Lambda') \circ \Lambda \sim \Lambda'' \circ (\Lambda' \circ \Lambda)$.
3. For $\Lambda \in U$, we have $\Lambda \circ \iota(s(\Lambda)) \sim \Lambda$.
4. For $\Lambda \in U$, we have $\iota(t(\Lambda)) \circ \Lambda \sim \Lambda$.
5. For $x, y, z \in \partial U$ and morphisms $f : x \to y$, $g : y \to z$ in $C$, we have $\iota(g) \circ \iota(f) \sim \iota(g \circ f)$.

Definition 2.6. Define a small category $\text{Cosp}_{U_0}(C)$. Its object set is given by $\partial U$ and morphism set is given by the quotient set $U/\sim$. The source map $s : (U/\sim) \to \partial U$ is induced by the source map $s : \text{Cosp}(C) \to \text{Obj}(C)$. The target map $t : (U/\sim) \to \partial U$ is defined analogously. The source map and target map are well-defined since $U$ is admissible. The composition of $\text{Cosp}_{U_0}(C)$ is induced by the map $\circ : U_i \times_{s,U} U \to U$ as $[\Lambda] \circ [\Lambda'] \overset{\text{def}}{=} [\Lambda' \circ \Lambda]$. The category $\text{Cosp}_{U_0}(C)$ is well-defined since $\circ$ is a weak composition with respect to the pair $(\sim, U)$.

Definition 2.7. Denote by $C|_{\partial U}$ the full subcategory of $C$ whose objects are $\partial U$. We define a functor $\iota : C|_{\partial U} \to \text{Cosp}_{U_0}(C)$ which is the identity on objects and assigns $[\iota(f)]$ to a morphism $f$ of $C|_{\partial U}$.

Remark 2.8. Note that Definition 2.7 is well-defined functor due to the second part of Definition 2.4 and the latter three conditions in Definition 2.5.
2.2. Dagger structure. Recall that for a category $\mathcal{D}$, a dagger operation on $\mathcal{D}$ is given by an involutive functor $\dagger : \mathcal{D}^{op} \to \mathcal{D}$ which is an identity on objects. A category equipped with a dagger operation is a dagger category.

**Definition 2.9.** Let $\Lambda = \left( x_0 \xrightarrow{f_0} y \xleftarrow{f_1} x_1 \right)$ be a cospan in $C$. We define a dagger cospan $\Lambda^\dagger$ by

$$\Lambda^\dagger = \left( x_0 \xrightarrow{f_0} y \xleftarrow{f_1} x_1 \right)^\dagger \overset{\text{def}}{=} \left( x_1 \xrightarrow{f_1} y \xleftarrow{f_0} x_0 \right).$$

The assignment of dagger cospan to cospans gives an involution on the set $\text{Cosp}(C)$. The dagger operation on $\text{Cosp}(C)$ is normal in the sense that $\iota(x)^\dagger = \iota(x)$.

**Definition 2.10.** We say that $(~, U, \circ)$ is a triple if $\sim$ is an equivalence relation on $\text{Cosp}(C)$ compatible with the source and target maps, $U$ is an admissible subset of $\text{Cosp}(C)$ and $\circ$ is a weak composition with respect to the pair $(~, U)$.

**Definition 2.11.** A triple $(~, U, \circ)$ is compatible with the dagger operation on $\text{Cosp}(C)$ if the following conditions hold.

1. For $\Lambda_0, \Lambda_1 \in \text{Cosp}(C)$, an equivalence relation $\Lambda_0 \sim \Lambda_1$ implies $\Lambda_0^\dagger \sim \Lambda_1^\dagger$.
2. If $\Lambda \in U$, then $\Lambda^\dagger \in U$.
3. For $\Lambda_0, \Lambda_1 \in U$ with $t(\Lambda_0) = s(\Lambda_1)$, we have $(\Lambda_1 \circ \Lambda_0)^\dagger \sim \Lambda_0^\dagger \circ \Lambda_1^\dagger$.
4. For $x, y \in \partial U$ and an isomorphism $f : x \to y$ in $C$, we have $\iota(f^{-1}) \sim \iota(f)^\dagger$.

**Proposition 2.12.** If a triple $(~, U, \circ)$ is compatible with the dagger operation on the set $\text{Cosp}(C)$, then the dagger operation on $\text{Cosp}(C)$ induces a dagger operation on the category $\text{Cosp}^\sim_{U, \circ}(C)$. The functor $\iota$ in Definition 2.7 assigns a unitary isomorphism to every isomorphism in $\text{Cosp}^\sim_{U, \circ}(C)$.

**Proof.** For a morphism $[\Lambda]$ of $\text{Cosp}^\sim_{U, \circ}(C)$, i.e. an equivalence class of $\Lambda \in U$, let $[\Lambda]^\dagger \overset{\text{def}}{=} [\Lambda^\dagger]$. It induces an involutive functor $\dagger$ on the category $\text{Cosp}^\sim_{U, \circ}(C)$. It is immediate that the functor $\dagger$ is a dagger operation on the category $\text{Cosp}^\sim_{U, \circ}(C)$. \qed

2.3. Symmetric monoidal category structure.

**Definition 2.13.** Let $\Lambda = \left( x_0 \xrightarrow{f_0} y \xleftarrow{f_1} x_1 \right)$, $\Lambda' = \left( x_0' \xrightarrow{f_0'} y' \xleftarrow{f_1'} x_1' \right)$ be cospans in $C$. We define a tensor product of cospans $\Lambda \otimes \Lambda'$ by

$$\Lambda \otimes \Lambda' \overset{\text{def}}{=} \left( x_0 \otimes x_0' \xrightarrow{f_0 \otimes f_0'} y \otimes y' \xleftarrow{f_1 \otimes f_1'} x_1 \otimes x_1' \right).$$

**Definition 2.14.** A triple $(~, U, \circ)$ (see Definition 2.10) is compatible with the symmetric monoidal category structure on $C$ if the following conditions hold.

1. For $\Lambda_0, \Lambda_1, \Lambda_0', \Lambda_1' \in \text{Cosp}(C)$, equivalence relations $\Lambda_0 \sim \Lambda_1$ and $\Lambda_0' \sim \Lambda_1'$ imply $\Lambda_0 \otimes \Lambda_0' \sim \Lambda_1 \otimes \Lambda_1'$.
2. If $\Lambda, \Lambda' \in U$, then $\Lambda \otimes \Lambda' \in U$.
3. For $\Lambda_0, \Lambda_1, \Lambda_0', \Lambda_1' \in \text{Cosp}(C)$ with $t(\Lambda_0) = s(\Lambda_1)$ and $t(\Lambda_0') = s(\Lambda_1')$, we have $(\Lambda_1 \circ \Lambda_0) \otimes (\Lambda_1' \circ \Lambda_0') \sim (\Lambda_1 \otimes \Lambda_1') \circ (\Lambda_0 \otimes \Lambda_0')$.

**Proposition 2.15.** Let $C$ be a symmetric monoidal category. If a triple $(~, U, \circ)$ is compatible with the symmetric monoidal structure, then the symmetric monoidal category structure of $C$ induces a symmetric monoidal category structure on $\text{Cosp}^\sim_{U, \circ}(C)$. The functor $\iota : C|_{\partial U} \to \text{Cosp}^\sim_{U, \circ}(C)$ in Definition 2.7 is enhanced to a symmetric monoidal functor.
Proof. We define a tensor product on $\mathrm{Cosp}_{\Lambda,0}^\Lambda(C)$ by $[\Lambda] \otimes [\Lambda'] = [\Lambda \otimes \Lambda']$. It gives a well-defined functor $\otimes : \mathrm{Cosp}_{\Lambda,0}^\Lambda(C) \times \mathrm{Cosp}_{\Lambda,0}^\Lambda(C) \to \mathrm{Cosp}_{\Lambda,0}^\Lambda(C)$ due to Definition 2.14. We set the unit object $I$ of the SMC $C$ as a unit object of $\mathrm{Cosp}_{\Lambda,0}^\Lambda(C)$. The associator, unitors, and symmetry on $C$ induce them of $\mathrm{Cosp}_{\Lambda,0}^\Lambda(C)$ by the last part of Definition 2.5. For example, denote by $a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ the associator of the SMC $C$. Then the induced morphism $[i(a_{x,y,z})] : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ for $x,y,z \in \partial U$ is an associator. In fact, the pentagon diagram with respect to $[i(a_{x,y,z})]$ is immediate from that of the associator $a_{x,y,z}$ due to the last part of Definition 2.5. By the construction of symmetric monoidal category structure, the functor $\iota : C|_{\partial U} \to \mathrm{Cosp}_{\Lambda,0}^\Lambda(C)$ in Definition 2.17 is enhanced to a symmetric monoidal functor in the obvious way.

Proposition 2.16. Let $C$ be a symmetric monoidal category. If a triple $(\sim, U, \circ)$ is compatible with the dagger operation on $\mathrm{Cosp}(C)$ and the symmetric monoidal category structure of $C$, then the symmetric monoidal category $\mathrm{Cosp}_{\Lambda,0}^\Lambda(C)$ with the dagger operation is a dagger symmetric monoidal category.

Proof. It suffices to prove that
\begin{enumerate}
\item $([\Lambda] \otimes [\Lambda'])^\dagger = [\Lambda]^\dagger \otimes [\Lambda']^\dagger$.
\item $[i(a_{x,y,z})]^\dagger = [i(a_{x,y,z})]^{-1}$ where $a_{x,y,z}$ is the associator of $C$.
\item $[i(I_1)]^\dagger = [i(I_1)]^{-1}$ where $I_1$ is the left unitor of $C$.
\item $[i(r_x)]^\dagger = [i(r_x)]^{-1}$ where $r_x$ is the right unitor of $C$.
\item $[i(s_{x,y})]^\dagger = [i(s_{x,y})]^{-1}$ where $s_{x,y}$ is the symmetry of $C$.
\end{enumerate}
These are immediate from their definitions. In particular, the claims from (2) to (5) follows from the fourth condition in Definition 2.11.

3. A cospan category of pointed finite CW-spaces

In this section, we define a cospan category of pointed finite CW-spaces $\mathrm{Cosp}_{\Lambda,0}^\Lambda(\mathrm{CW}^\text{fin})$ by using the preliminaries in section 2. It is a homotopy theoretical analogue of cobordism categories. In the set-theoretical sense, the category of all of topological spaces (or finite CW-spaces) is not small. In this paper, we fix a small category of topological spaces (or finite CW-spaces) which is categorically equivalent with the whole.

Definition 3.1. Denote the category of pointed topological spaces as $\mathrm{Top}_\ast$. Let $\Lambda = \left( K_0 \xrightarrow{f_0} L \xleftarrow{f_1} K_1 \right)$ be cospans in $\mathrm{Top}_\ast$ with the same sources and targets. A map $g : L \to L'$ is a homotopy equivalence from $\Lambda$ to $\Lambda'$ if
\begin{enumerate}
\item $g : L \to L'$ are pointed homotopy equivalences.
\item There exists a pointed homotopy $g \circ f_0 \simeq f_0' \circ g$.
\item There exists a pointed homotopy $g \circ f_1 \simeq f_1' \circ g$.
\end{enumerate}
If there exists a homotopy equivalence from $\Lambda$ to $\Lambda'$, then we write $\Lambda \simeq \Lambda'$.

The homotopy equivalence relation $\simeq$ of cospans is an equivalent relation on $\mathrm{Cosp}(\mathrm{Top}_\ast)$. The equivalence relation $\simeq$ is compatible with the source and target maps.

Definition 3.2. For a pointed map $f : K \to L$, we define a mapping cylinder $\mathrm{Cyl}(f)$ by a pointed space,
\begin{equation}
\mathrm{Cyl}(f) \overset{\text{def}}{=} \mathrm{Cyl}(K) \bigvee_f L.
\end{equation}
In other words, it is the quotient pointed space by identifying $[k,1] \in \mathrm{Cyl}(K)$ with $f(k) \in L$. By the canonical inclusion $L \to \mathrm{Cyl}(f)$, we consider $L$ as a subspace of $\mathrm{Cyl}(f)$. We denote by $i_f : K \to \mathrm{Cyl}(f)$ the inclusion where we identify $K$ with $K \cap \{0\}^+ \subset K \cap [0,1]^+ = \mathrm{Cyl}(K)$. 

Definition 3.3. Let $\Lambda = \left( K_0 \xrightarrow{f_0} L \xleftarrow{f_1} K_1 \right)$, $\Lambda’ = \left( K_1 \xrightarrow{g_1} L’ \xleftarrow{g_2} K_2 \right)$ be cospars such that $t(\Lambda) = s(\Lambda’$. We define a cospan $\Lambda’ \circ \Lambda$ in $\text{Top}_*$ by $\left( K_0 \xrightarrow{k_0} L'' \xleftarrow{k_1} K_2 \right)$. Here, $L''$ is the quotient space of $\text{Cyl}(f_1) \cup \text{Cyl}(g_1)$ by identifying $[k,0] \in \text{Cyl}(K_1) \subset \text{Cyl}(f_1)$ with $[k,0] \in \text{Cyl}(K_1) \subset \text{Cyl}(g_1)$. The quotient space $L''$ is equipped with the obvious base-point. The pointed maps $k_0, k_1$ are given by compositions $k_0 = \left( K_0 \xrightarrow{f_0} L \xrightarrow{f_1} L’ \right)$ and $k_2 = \left( K_2 \xrightarrow{g_2} L’ \xleftarrow{f_1} L'' \right)$ where $i, j$ are the canonical inclusions. The assignment $(\Lambda, \Lambda’) \mapsto \Lambda’ \circ \Lambda$ determines a map $\circ : U \times_* U \to U$ where $U = \text{Cosp}(\text{Top}_*)$.

Definition 3.4. The symmetric monoidal category structure on the category $\text{Top}_*$ by the wedge sum induces a wedge sum of cospars in $\text{Top}_*$. In other words, for cospars $\Lambda = \left( K_0 \xrightarrow{f_0} L \xleftarrow{f_1} K_1 \right)$, $\Lambda’ = \left( K’_0 \xrightarrow{f’_0} L’ \xleftarrow{f’_1} K’_1 \right)$, we define the wedge sum of cospars by

\[
\Lambda \vee \Lambda’ \overset{\text{def}}{=} \left( K_0 \vee K_0 \xrightarrow{f_0 \vee f’_0} L \vee L’ \xleftarrow{f_1 \vee f’_1} K_1 \vee K’_1 \right).
\]

Proposition 3.5. Let $U = \text{Cosp}(\text{Top}_*)$. Recall the definition of triples in Definition 2.10

1. The map $\circ : U \times_* U \to U$ in Definition 3.3 is a weak composition with respect to the pair $(\sim, U)$.
2. The triple $(\sim, U, \circ)$ is compatible with the dagger operation on $\text{Cosp}(\text{Top}_*)$.
3. The triple $(\sim, U, \circ)$ is compatible with the symmetric monoidal structure on $\text{Top}_*$.

Proof. The proof is elementary so that we leave the proof to the readers. In particular, the first part is related with the homotopy invariance of homotopy colimits.

Definition 3.6. Let $X = \{X_K\}$ be a family of a pointed finite CW-complex structure $X_K$ for each pointed finite CW-space $K$. Let $d \in \mathbb{N} \cup \{\infty\}$. Denote by $U_{d,X} \subset \text{Cosp}(\text{CW}^{\text{fin}})$ a subset consisting of cospars $\Lambda = \left( K_0 \xrightarrow{f_0} L \xleftarrow{f_1} K_1 \right)$ in $\text{CW}^{\text{fin}}$ satisfying following conditions.

1. $\dim K_0 \leq (d-1), \dim K_1 \leq (d-1)$ and $\dim L \leq d$.
2. There exists a pointed finite CW-complex structure $X'_L$ on $L$ such that $f_0$ and $f_1$ are cellular with respect to the complex structures $X_{K_0}, X_{K_1}, X'_L$.\n
Proposition 3.7. The subset $U_{d,X} \subset \text{Cosp}(\text{CW}^{\text{fin}})$ is admissible in the sense of Definition 2.4. Moreover, $\partial U_{d,X}$ is the set of pointed finite CW-spaces with $\leq (d-1)$.

Proof. It is immediate from definitions.

Proposition 3.8. The map $\circ$ in Definition 3.3 induces a map $\circ : U \times_* U \to U$ where $U = U_{d,X}$. The induced map gives a weak composition with respect to the pair $(\sim, U_{d,X})$.

Proof. Consider $(\Lambda, \Lambda’) \in U \times_* U$. If $\Lambda = \left( K_0 \xrightarrow{f_0} L \xleftarrow{f_1} K_1 \right)$ and $\Lambda’ = \left( K_1 \xrightarrow{f_1} L’ \xleftarrow{f_2} K_2 \right)$, then there exist pointed finite CW-complex structures $X'_L, X'_L’$ on $L, L’$ such that $f_0, f_1, f_’_1, f_’_2$ are cellular maps with respect to the complex structure $X_{K_0}, X_{K_1}, X_{K_1}’, X_{K_1}’’$. Hence, the mapping cylinders $\text{Cyl}(f_1), \text{Cyl}(f’_1)$ have canonical pointed finite CW-complex structures, so the glued space $L'' = \text{Cyl}(f_1) \cup_{K_1} \text{Cyl}(f’_1)$ does. Denote by $X’_{L’}$ the complex structure of $L''$. Recall Definition 3.3. Then $k_0 : K_0 \to L’’$ and $k_2 : K_2 \to L’’$ are cellular maps with respect to $X_{K_0}, X_{K_1}, X’_{L’}$, hence $\Lambda’ \circ \Lambda$ belongs to $U_{d,X}$.

In the following proposition, recall the notations in Definition 2.6.
Proposition 3.9. For another family of pointed finite CW-complex structures \( Y = \{Y_K\} \), we have a canonical isomorphism of categories,
\[
\text{Cosp}_{U_{d,Y}}^\sim (\text{CW}_*^\text{fin}) \cong \text{Cosp}_{U_{d,X}}^\sim (\text{CW}_*^\text{fin}).
\]

Proof. Note that \( \partial U_{d,X} = \partial U_{d,Y} \) since \( \partial U_{d,X} \) is the set of pointed finite CW-spaces with \( \dim \leq (d - 1) \) by Proposition 3.7. Hence, the object set of \( \text{Cosp}_{U_{d,X}}^\sim (\text{CW}_*^\text{fin}) \) and \( \text{Cosp}_{U_{d,Y}}^\sim (\text{CW}_*^\text{fin}) \) coincide with each other.

On the other hand, the set \( U_{d,X} \) and \( U_{d,Y} \) essentially coincide with each other in the following sense: For any cospan \( \Lambda \) lying in \( U_{d,X} \), there exists a cospan \( \Lambda' \in U_{d,Y} \) such that \( \Lambda \simeq \Lambda' \). It follows from the definitions of \( U_{d,X} \) and \( U_{d,Y} \).

We construct a functor \( F_{X,Y} : \text{Cosp}_{U_{d,X}}^\sim (\text{CW}_*^\text{fin}) \to \text{Cosp}_{U_{d,Y}}^\sim (\text{CW}_*^\text{fin}) \) as follows. The functor \( F_{X,Y} \) assigns \( K \) itself to an object \( K \) of \( \text{Cosp}_{U_{d,Y}}^\sim (\text{CW}_*^\text{fin}) \). The functor \( F_{X,Y} \) assigns a morphism \([\Lambda']\) in \( \text{Cosp}_{U_{d,Y}}^\sim (\text{CW}_*^\text{fin}) \) to a morphism \([\Lambda]\) in \( \text{Cosp}_{U_{d,X}}^\sim (\text{CW}_*^\text{fin}) \) where \( \Lambda \simeq \Lambda' \). It is a well-defined functor due to the first part of Proposition 3.5.

It follows from definitions that the functor \( F_{X,Y} \) is an inverse functor of \( F_{Y,X} \). It completes the proof. \( \square \)

Definition 3.10. Let \( X = \{X_K\} \) be a family of a pointed finite CW-complex structure \( X_K \) for each pointed finite CW-space \( K \). We define a dagger symmetric monoidal category,
\[
\text{Cosp}_{\leq d}^\sim (\text{CW}_*^\text{fin}) \stackrel{\text{def.}}{=} \text{Cosp}_{U_{d,X}}^\sim (\text{CW}_*^\text{fin}).
\]
By Proposition 3.9, the definition is independent of the choice of \( X \) up to a canonical isomorphism.

4. A preorder of (co)span diagrams in an abelian category

4.1. Basic properties.

Definition 4.1. For cospans \( \Lambda = \left( A_0 \xrightarrow{f_0} B \xleftarrow{f_1} A_1 \right) \) and \( \Lambda' = \left( A_0' \xrightarrow{f_0'} B' \xleftarrow{f_1'} A_1' \right) \), we denote by \( \Lambda \preceq \Lambda' \) if \( A_0 = A_0' \), \( A_1 = A_1' \) and there exists a monomorphism \( g : B \to B' \) in \( \mathcal{A} \) such that \( g \circ f_0 = f_0' \) and \( g \circ f_1 = f_1' \). For such a monomorphism \( g \), we say that the monomorphism \( g \) gives \( \Lambda \preceq \Lambda' \).

Proposition 4.2. The relation \( \preceq \) gives a preorder of cospans in \( \mathcal{A} \), i.e. we have
(1) \( \Lambda \preceq \Lambda \).
(2) \( \Lambda \preceq \Lambda' \) and \( \Lambda' \preceq \Lambda'' \) implies \( \Lambda \preceq \Lambda'' \).

Proof. The first part is proved by the fact that the identity morphism is a monomorphism.

The second part is proved by the fact that the composition of monomorphisms is a monomorphism. In fact, if a monomorphism \( g \) gives \( \Lambda \preceq \Lambda' \) and a monomorphism \( g' \) gives \( \Lambda \preceq \Lambda'' \), then the monomorphism \( g' \circ g \) gives \( \Lambda \preceq \Lambda'' \).

Lemma 4.3. Let \( \Lambda, \Lambda' \) be cospans in \( \mathcal{A} \). Then the following two conditions are equivalent.
(1) There exists a lower bound of \( \{\Lambda, \Lambda'\} \).
(2) There exists an upper bound of \( \{\Lambda, \Lambda'\} \).

Proof. Let \( \Lambda = \left( A_0 \xrightarrow{f_0} B \xleftarrow{f_1} A_1 \right) \), \( \Lambda' = \left( A_0' \xrightarrow{f_0'} B' \xleftarrow{f_1'} A_1' \right) \). With out loss of generality, we assume that \( A_0 = A_0' \) and \( A_1 = A_1' \).

We prove the second part starting from the first part. Suppose that a cospan \( \Lambda_0 \) is a lower bound of \( \{\Lambda, \Lambda'\} \) where \( \Lambda_0 = \left( A_0 \xrightarrow{g_0} B_0 \xleftarrow{g_1} A_1 \right) \). Let \( m : B_0 \to B \) be a monomorphism giving \( A_0 \preceq \Lambda \) and \( m' : B_1 \to B \) be a monomorphism giving \( A_1 \preceq \Lambda \). In other words,
we have $m \circ g_0 = f_0$, $m \circ g_1 = f_1$, $m' \circ g_0 = f'_0$, $m' \circ g_1 = f'_1$. Define $B_1$ to be the cokernel of $u = (m + (-m')) \circ \Delta g_0 : B_0 \to B \oplus B'$. Put $g : B \to B_1$ as the composition of $B \xrightarrow{i_0} B \oplus B' \xrightarrow{\text{cok}(u)} B_1$, and $g' : B' \to B_1$ as the composition of $B' \xrightarrow{i_1} B \oplus B' \xrightarrow{\text{cok}(a)} B_1$. By definition, we have $g \circ m = g' \circ m'$.

\[
\begin{array}{c}
\text{m} \\
\text{C} \\
\text{B} \\
\text{0}
\end{array}
\quad
\begin{array}{c}
\text{B'} \\
\text{B} \\
\text{B_0} \\
\text{0}
\end{array}
\xrightarrow{\text{p_1}}
\begin{array}{c}
\text{B_1} \\
\text{B} \\
\text{B_0} \\
\text{0}
\end{array}
\]

(7)

Note that $g$ is a monomorphism since $m'$ is a monomorphism: If a morphism $e : C \to B$ satisfies $g \circ e = 0$, then there exists a unique morphism $e' : C \to B_0$ such that $u \circ e' = i_0 \circ e$ where $i_0 : B \to B \oplus B'$ is the inclusion. Since we have $0 = p_1 \circ i_0 \circ e = p_1 \circ u \circ e' = (-m') \circ e'$ and $m'$ is a monomorphism, we obtain $e' = 0$. Hence, $g$ is a monomorphism.

Similarly, the morphism $g'$ is a monomorphism since $m$ is a monomorphism.

Define $h_0 : A_0 \to B_1$ and $h_1 : A_1 \to B_1$ by $h_0 = g \circ f_0$ and $h_1 = g \circ f_1$. Then we have $g' \circ f'_0 = h_0$. In fact, $g' \circ f'_0 = g' \circ m' \circ g_0 = g \circ m \circ g_0 = g \circ f_0 = h_0$. Likewise $g' \circ f'_1 = h_1$ holds. Above all, the monomorphism $g$ gives $\Lambda \leq \Lambda_1$ and the monomorphism $g'$ gives $\Lambda' \leq \Lambda_1$.

The cospan $\Lambda_1$ is an upper bound of $\{\Lambda, \Lambda'\}$.

We prove the first part starting from the second part. Suppose that a cospan $\Lambda_1$ is an upper bound of $\{\Lambda, \Lambda'\}$. Let $\Lambda_1 = \left(A_0 \xrightarrow{h_0} B_1 \xleftarrow{h_1} A_1\right)$. Let $g : B \to B_1$ be a monomorphism giving $\Lambda \leq \Lambda_1$ and $g' : B' \to B_1$ be a monomorphism giving $\Lambda' \leq \Lambda_1$. In other words, we have $g \circ f_0 = h_0$, $g \circ f_1 = h_1$, $g' \circ f'_0 = h_0$, and $g' \circ f'_1 = h_1$. Put $B_0$ to be the kernel of $v = \nabla B_1 \circ (g \oplus (-g')) : B \oplus B' \to B_1$. Put $m : B_0 \to B$ as the composition $B_0 \xrightarrow{\text{ker}(v)} B \oplus B' \xrightarrow{\text{p_0}} B$, and $m' : B_0 \to B'$ as the composition $B_0 \xrightarrow{\text{ker}(v)} B \oplus B' \xrightarrow{\text{p_1}} B'$. By definitions, we have $g \circ m = g' \circ m'$.

\[
\begin{array}{c}
\text{B} \\
\text{B_0} \\
\text{C} \\
\text{0}
\end{array}
\xrightarrow{\text{p_0}}
\begin{array}{c}
\text{B_1} \\
\text{B} \\
\text{B_0} \\
\text{0}
\end{array}
\]

(8)

The morphisms $m$ is a monomorphism: Suppose that $m \circ l = 0$ for a morphism $l : C \to B_0$. Since $m \circ l = p_0 \circ (\text{ker}(v) \circ l)$, there exists a unique morphism $k : C \to B'$ such that $i_1 \circ k = \text{ker}(v) \circ l$. Since $g' \circ k = v \circ i_0 \circ k = v \circ \text{ker}(v) \circ l = 0$ and $g'$ is a monomorphism, we obtain $k = 0$, hence $\text{ker}(v) \circ l = 0$. Since $\text{ker}(v)$ is a monomorphism, we obtain $l = 0$.

Similarly, the morphism $m'$ is a monomorphism.
Define \( g_0 : A_0 \to B_0 \) by \( \ker(v) \circ g_0 = (f_0 \oplus f'_0) \circ \Delta_A \). Then by definition we have \( m \circ g_0 = f_0 \).
In fact, we have \( m \circ g_0 = p_0 \circ \ker(v) \circ g_0 = p_0 \circ (f_0 \oplus f'_0) \circ \Delta_A = f_0 \). Likewise, we obtain \( m' \circ g_0 = f'_0 \). Define \( g_1 : A_1 \to B_0 \) by \( \ker(v) \circ g_1 = (f_1 \oplus f'_1) \circ \Delta_A \). Then we also obtain \( m \circ g_1 = f_1 \) and \( m' \circ g_1 = f'_1 \).

Let \( \Lambda_0 = \left( A_0 \xrightarrow{g_0} B_0 \xleftarrow{f_0} A_1 \right) \). Then the monomorphism \( m \) gives \( \Lambda_0 \leq \Lambda \) and the monomorphism \( m' \) gives \( \Lambda_0 \leq \Lambda' \) by the previous discussion. The cospan \( \Lambda_0 \) is a lower bound of \( \{ \Lambda, \Lambda' \} \).

**Proposition 4.4.** The preorder \( \leq \) is compatible with the biproduct of cospans in \( \mathcal{A} \). In other words, for cospans \( \Lambda_0, \Lambda_1 \) and cospans \( \Lambda_2, \Lambda_3 \), if \( \Lambda_0 \leq \Lambda_1 \) and \( \Lambda_2 \leq \Lambda_3 \), then we have \( \Lambda_0 \oplus \Lambda_2 \leq \Lambda_1 \oplus \Lambda_3 \).

**Proof.** Let \( g, h \) be a monomorphism which gives \( \Lambda_0 \leq \Lambda_1 \) and \( \Lambda_2 \leq \Lambda_3 \) respectively. Note that the biproduct of monomorphisms is a monomorphism. Then the biproduct \( g \oplus h \) gives \( \Lambda_0 \oplus \Lambda_2 \leq \Lambda_1 \oplus \Lambda_3 \).

**Definition 4.5.** Let \( \Lambda, \Lambda' \) be cospans in \( \mathcal{A} \) with \( \Lambda = \left( A_0 \xrightarrow{g_0} B \xleftarrow{f_0} A_1 \right) \) and \( \Lambda' = \left( A_1 \xrightarrow{f_1} B' \xleftarrow{g_1} A_2 \right) \).

We define a **composition cospan** \( \Lambda' \circ \Lambda = \left( A_0 \xrightarrow{g_0} C \xleftarrow{f_1} A_2 \right) \) where \( C \) is the cokernel of the composition \( (f_1 \oplus (-f_1')) \circ \Delta_A : A_1 \to B \oplus B' \) and the morphisms \( g_0, g_2 \) are given by the following compositions,

\[
\begin{align*}
g_0 &= A_0 \xrightarrow{f_0} B \xrightarrow{h_0} B \oplus B', \\
g_2 &= A_2 \xrightarrow{f_2} B' \xrightarrow{h_2} B \oplus B'.
\end{align*}
\]

**Proposition 4.6.** The preorder \( \leq \) is compatible with the composition of cospans in \( \mathcal{A} \). In other words, for cospans \( \Lambda_0, \Lambda_1 \) from \( A_0 \) to \( A_1 \) and cospans \( \Lambda_2, \Lambda_3 \) from \( A_1 \) to \( A_2 \), if \( \Lambda_0 \leq \Lambda_1 \) and \( \Lambda_2 \leq \Lambda_3 \), then we have \( \Lambda_0 \circ \Lambda_2 \leq \Lambda_1 \circ \Lambda_3 \).

**Proof.** Let \( \Lambda_0 = \left( A_0 \xrightarrow{f_0} B_0 \xleftarrow{f_0} A_1 \right), \Lambda_1 = \left( A_0 \xrightarrow{f_0} B_1 \xleftarrow{f_1} A_1 \right), \Lambda_2 = \left( A_1 \xrightarrow{f_1} B_2 \xleftarrow{f_2} A_2 \right), \Lambda_3 = \left( A_1 \xrightarrow{f_1} B_3 \xleftarrow{f_3} A_2 \right) \). Suppose that monomorphisms \( g : B_0 \to B_1 \) and \( g' : B_2 \to B_3 \) give \( \Lambda_0 \leq \Lambda_2 \) and \( \Lambda_3 \leq \Lambda_3 \) respectively.

Denote by \( C_0, C_1 \) the cokernels of \( u = (f_{01} \oplus (-f_{21})) \circ \Delta_{A_1} : A_1 \to B_0 \oplus B_2 \) and \( v = (f_{11} \oplus (-f_{31})) \circ \Delta_{A_1} : A_1 \to B_1 \oplus B_3 \) respectively. The biproduct \( g \oplus g' : B_0 \oplus B_2 \to B_1 \oplus B_3 \) induces a morphism \( h : C_0 \to C_1 \) such that \( h \circ \text{cok}(u) = \text{cok}(v) \circ (g \oplus g') \). The morphism \( h \) is a monomorphism. In fact, consider concentrated chain complexes \( D_* = \left( \cdots \to 0 \to A_1 \xrightarrow{u} B_0 \oplus B_2 \to 0 \to \cdots \right), D'_* = \left( \cdots \to 0 \to A_1 \xrightarrow{v} B_1 \oplus B_3 \to 0 \to \cdots \right) \) where the 0-th components are \( B_0 \oplus B_2 \) and \( B_1 \oplus B_3 \) respectively. The identity on \( A_1 \) and the biproduct \( g \oplus g' \) gives a chain homomorphism \( j_* : D_* \to D'_* \) which is a monomorphism.

Then by the long exact sequence, we obtain an exact sequence \( H_1(D'_*/D_*) \xrightarrow{h} H_0(D_*) \xrightarrow{H_0(j_*)} H_0(D'_*) \). We have \( H_0(D_*) = \text{Cok}(u) = C_0 \) and \( H_0(D'_*) = \text{Cok}(v) = C_1 \). Under the identifications, we have \( H_0(j_*) = h \). Note that \( H_1(D'_*/D_*) = 0 \). The morphism \( h \) is a monomorphism.

The monomorphism \( h \) gives \( \Lambda_0 \circ \Lambda_2 \leq \Lambda_1 \circ \Lambda_3 \). Define \( m_0 : B_0 \to C_0 \) by the composition \( B_0 \leftarrow B_0 \oplus B_2 \xrightarrow{\text{cok}(u)} C_0 \) and \( m_1 : B_1 \to C_1 \) by the composition \( B_1 \leftarrow B_1 \oplus B_3 \xrightarrow{\text{cok}(v)} C_1 \). Then we obtain \( h \circ m_0 = m_1 \circ g \). In particular, we obtain \( h \circ (m_0 \circ f_{00}) = (m_1 \circ f_{10}) \). Similarly, we define \( m_2 : B_2 \to C_0 \) by the composition \( B_2 \leftarrow B_0 \oplus B_2 \xrightarrow{\text{cok}(u)} C_0 \) and \( m_3 : B_3 \to C_1 \) by the composition \( B_3 \leftarrow B_1 \oplus B_3 \xrightarrow{\text{cok}(v)} C_1 \). Then we also have \( h \circ (m_2 \circ f_{22}) = m_3 \circ f_{32} \). Above all, the monomorphism \( h \) gives \( \Lambda_0 \circ \Lambda_2 \leq \Lambda_1 \circ \Lambda_3 \).
Proposition 4.7. Consider a commutative diagram in $\mathcal{A}$.

\[
\begin{array}{ccc}
A_0 & \xrightarrow{f_0} & B \\
\downarrow{f} & & \downarrow{g} \\
A_1 & \xrightarrow{f_1} & B' \\
\downarrow{f_2} & & \downarrow{h} \\
A_2 & & C
\end{array}
\]

Let $\Lambda_0 = \left( A_0 \xrightarrow{f_0} B \xleftarrow{f_1} A_1 \right)$, $\Lambda_1 = \left( A_1 \xrightarrow{f_1} B' \xleftarrow{f_2} A_2 \right)$ and $\Lambda_2 = \left( A_0 \xrightarrow{g \circ f_0} C \xleftarrow{h \circ f_2} A_2 \right)$ be cospans in $\mathcal{A}$. If the square diagram in (11) is exact, then we have $\Lambda_1 \circ \Lambda_0 \leq \Lambda_2$.

4.2. Proof of Proposition 4.7

Definition 4.8. Let $\mathcal{A}$ be an abelian category. A square diagram is a quadruple $(g, f, g', f')$ of morphisms in $\mathcal{A}$ such that $g, f$ and $g', f'$ are composable respectively. Consider a following square diagram $\Box$ in $\mathcal{A}$.

\[
\begin{array}{ccc}
B & \xrightarrow{g} & D \\
\downarrow{f} & & \downarrow{g'} \\
A & \xrightarrow{f'} & C
\end{array}
\]

The morphism $f$ induces a morphism $k_\Box : \text{Ker}(f') \to \text{Ker}(g)$. The morphism $g'$ induces a morphism $c_\Box : \text{Cok}(f') \to \text{Cok}(g)$. The square diagram is exact if the morphism $k_\Box$ is an epimorphism and the morphism $c_\Box$ is a monomorphism.

Definition 4.9. Let $\Box$ be a square diagram in $\mathcal{A}$ as (12). We define a chain complex $C(\Box)$ by

\[
\begin{array}{ccc}
A & \xrightarrow{u_\Box} & B \oplus C \\
\downarrow{v_\Box} & & \downarrow{w_\Box} \\
A' & \xrightarrow{u'} & B \oplus C
\end{array}
\]

where $u_\Box \overset{\text{def}}{=} (f \oplus (-f')) \circ \Delta_A$ and $v_\Box \overset{\text{def}}{=} \nabla_D \circ (g \oplus g')$.

Proposition 4.10. Consider a square diagram $\Box$ in $\mathcal{A}$.

\[
\begin{array}{ccc}
B & \xrightarrow{g} & D \\
\downarrow{f} & & \downarrow{g'} \\
A & \xrightarrow{f'} & C
\end{array}
\]

(1) The square diagram $\Box$ is exact.
(2) The induced chain complex $C(\Box)$ is exact.

Proof. Denote by $C_\bullet$ the chain complex induced the morphism $f : A \to B$, i.e. $C_1 = A$, $C_0 = B$, $C_q = 0$ otherwise and the 1st differential is $f$. Similarly, we denote by $D_\bullet$ the chain complex induced by the morphism $g'$. The morphisms $g, f'$ induce a chain homomorphism $E_\bullet : C_\bullet \to D_\bullet$. Then the mapping cone complex $E_\bullet$ of the chain homomorphism $E_\bullet$ consists of

(1) $E_2 = A$, $E_1 = B \oplus C$, and $E_0 = D$,
(2) the second differential is $u_\Box : A \to B \oplus C$,
(3) the first differential is $v_\Box : B \oplus C \to D$. 

By the mapping cone exact sequence, we obtain an exact sequence:

\[ H_1(C_*) \xrightarrow{H(E_*)} H_1(D_*) \rightarrow H_1(\tilde{E}_*) \rightarrow H_0(C_*) \xrightarrow{H_0(E_*)} H_0(D_*) \]

(15)

\[ \text{Ker}(f) \xrightarrow{k} \text{Ker}(g') \rightarrow H_1(\tilde{E}_*) \rightarrow \text{Cok}(f) \xrightarrow{e} \text{Cok}(g') \]

By the exactness, the induced morphisms \( k \) and \( e \) are isomorphisms and monomorphisms respectively if and only if \( H_1(\tilde{E}_*) \) is a zero object. By the definition of the chain complex \( \tilde{E}_* \), \( H_1(\tilde{E}_*) \) is a zero object if and only if the chain complex \( C(\square) \) induced by the square diagram \( \square \) is exact. \( \square \)

Proof of Proposition 4.7. Denote by \( \square \) the square diagram in (11), i.e. \( \square \) is a commutative subdiagram consisting of \( g, h, f_1, f_1' \). Suppose that the square diagram \( \square \) is exact. By Proposition 4.10, the induced chain complex \( C(\square) \) (see Definition 4.9) is an exact sequence. Equivalently, the induced morphism \( k_{\square} : \text{Cok}(u_{\square}) \rightarrow C \) is a monomorphism. Note that the composition of cospans \( \Lambda_1 \circ \Lambda_0 \) is given by \( \Lambda_1 \circ \Lambda_0 = (A_0 \rightarrow \text{Cok}(u_{\square}) \leftarrow A_2) \) by definitions. Then the monomorphism \( k_{\square} \) gives \( \Lambda_1 \circ \Lambda_0 \leq \Lambda_2 \) in the sense of Definition 4.1. It completes the proof.

5. A (co)span category of an abelian category

5.1. Definitions.

Definition 5.1. We define a relation \( \approx \) of cospan diagrams in \( \mathcal{A} \). We define \( \Lambda \approx \Lambda' \) if there exists an upper bound of \( \{ \Lambda, \Lambda' \} \) with respect to the preorder \( \leq \) in Definition 4.1. By Lemma 4.3, \( \Lambda \approx \Lambda' \) is equivalent with the condition that there exists a lower bound of \( \{ \Lambda, \Lambda' \} \).

Proposition 5.2. The relation defined in Definition 5.1 is an equivalence relation.

Proof. Since \( \Lambda \leq \Lambda \) by Proposition 4.2, we have \( \Lambda \approx \Lambda \).

Suppose that \( \Lambda \approx \Lambda' \). Then there exists an upper bound \( \Lambda'' \) of a set \( \{ \Lambda, \Lambda' \} = \{ \Lambda', \Lambda \} \). Hence, we have \( \Lambda \approx \Lambda' \).

Suppose that \( \Lambda \approx \Lambda', \Lambda' \approx \Lambda'' \). Then we have upper bounds \( \Lambda_0, \Lambda_1 \) of \( \{ \Lambda \approx \Lambda' \} \), \( \{ \Lambda' \approx \Lambda'' \} \) respectively. Then the cospan \( \Lambda' \) is a lower bound of \( \{ \Lambda_0, \Lambda_1 \} \). By Lemma 4.3, the set \( \{ \Lambda_0, \Lambda_1 \} \) has an upper bound \( \Lambda_2 \). Since \( \Lambda \leq \Lambda_0 \leq \Lambda_2 \) and \( \Lambda'' \leq \Lambda_1 \leq \Lambda_2 \), we have \( \Lambda \leq \Lambda_2 \) and \( \Lambda'' \leq \Lambda_2 \). Hence, \( \Lambda_2 \) is an upper bound of \( \{ \Lambda, \Lambda'' \} \). We obtain \( \Lambda \approx \Lambda'' \). \( \square \)

Proposition 5.3. The equivalence relation \( \approx \) is compatible with the source and target maps in the sense of Definition 2.3.

Proof. It is immediate from definitions. \( \square \)

Proposition 5.4. The composition in Definition 4.5 is a weak composition with respect to the pair \( (\approx, \text{Cosp}(\mathcal{A})) \) in the sense of Definition 2.5.

Proof. We prove the first part of Definition 2.5. In fact, the equivalence relation \( \approx \) is preserved under the composition of cospans by Proposition 4.6.

The second and fifth parts of Definition 2.5 is obviously satisfied from Definition 4.5.

We prove the third and fourth parts of Definition 2.5. Let \( \Lambda \) be a cospan from \( A_0 \) to \( A_1 \). We have

\[ \Lambda \circ \iota(A_0) \approx \Lambda, \]

(16)

\[ \iota(A_1) \circ \Lambda \approx \Lambda. \]

(17)
We prove the first claim and leave the second claim to the readers. Note that the cokernel of the composition \( A_0 \xrightarrow{\Lambda_0} A_0 \oplus A_0 \xrightarrow{Id_0 \oplus (-f_0)} A_0 \oplus B \), which is the bulk part of the cospan \( \Lambda \circ \iota(A_0) \), is naturally isomorphic to \( B \). The natural isomorphism gives \( \Lambda \circ \iota(A_0) \cong \Lambda \). □

Recall the definition of triples in Definition 2.10.

**Proposition 5.5.** The triple \( (\approx, \text{Cosp}(A), \circ) \) is compatible with the dagger operation on \( \text{Cosp}(A) \) in the sense of Definition 2.11.

**Proof.** It is immediate from definitions. □

**Proposition 5.6.** The triple \( (\approx, \text{Cosp}(A), \circ) \) is compatible with the symmetric monoidal category structure on \( A \) in the sense of Definition 2.14.

**Proof.** The equivalence relation \( \approx \) is preserved under the biproduct of cospans as a corollary of Proposition 4.4. □

In the following definition, recall the notations in Definition 2.6.

**Definition 5.7.** Consider a triple \( (\approx, U, \circ) \) where the equivalence relation \( \approx \) is defined in Definition 5.1. \( U = \text{Cosp}(A) \) and the weak composition \( \circ \) is defined in Definition 4.5. We define a dagger symmetric monoidal category \( \text{Cosp}\nabla^\approx(A) \) by

\[
\text{Cosp}\nabla^\approx(A) \overset{\text{def}}{=} \text{Cosp}\nabla_{U,\circ}(A)
\]

We denote by \( \iota_{\text{cosp}} : A \to \text{Cosp}\nabla^\approx(A) \) the functor induced by \( \iota \) in Definition 2.2.

**Definition 5.8.** By repeating a dual construction in this subsection, one can define a category \( \text{Sp}\nabla^\approx(A) \). In particular, we have \( \text{Sp}\nabla^\approx(A) \cong \text{Cosp}\nabla^\approx(A^{op}) \).

The functor \( \iota_{\text{cosp}} : A^{op} \to \text{Cosp}\nabla^\approx(A^{op}) \) is regarded as a functor from \( A \) to \( \text{Cosp}\nabla^\approx(A^{op})^{op} = \text{Sp}\nabla^\approx(A) \). The composition of the functor with the dagger \( \dagger : \text{Sp}\nabla^\approx(A)^{op} \to \text{Sp}\nabla^\approx(A) \) is denoted by \( \iota_{\text{sp}} : A \to \text{Sp}\nabla^\approx(A) \).

5.2. **An isomorphism between cospan and span categories.** Recall Definition 5.7 and 5.8. In this subsection, we define the transposition of cospans to spans. It defines an isomorphism between the cospan category \( \text{Cosp}\nabla^\approx(A) \) and span category \( \text{Sp}\nabla^\approx(A) \).

**Definition 5.9.** Let \( \Lambda = \left( A_0 \xrightarrow{f_0} B \xleftarrow{f_1} A_1 \right) \) be a cospan in \( A \). Let \( C \) be the kernel of the composition \( v = V_B \circ (f_0 \oplus f_1) : A_0 \oplus A_1 \to B \) and \( g_0, g_1 \) be the components of the morphism \( v \). We define a span \( T(\Lambda) \) by

\[
T(\Lambda) \overset{\text{def}}{=} \left( A_0 \overset{g_0}{\leftarrow} C \overset{g_1}{\to} A_1 \right).
\]

We dually define a induced cospan \( T(V) \) for a span \( V \) in \( A \).

**Lemma 5.10.** Consider a square diagram \( \square \) in the sense of Definition 4.8:

\[
\begin{array}{ccc}
A_0 & \longrightarrow & B \\
\uparrow & & \uparrow \\
C & \longrightarrow & A_1
\end{array}
\]

Let \( \Lambda = (A_0 \to B \leftarrow A_1) \) and \( V = (A_0 \leftarrow C \to A_1) \) be the cospan and span diagrams contained in the square diagram. If the square diagram \( \square \) is exact, then we have \( T(\Lambda) \leq V \) and \( T(V) \leq \Lambda \).

**Proof.** We prove that \( T(\Lambda) \leq V \). By Proposition 4.10, the induced morphism \( u' : C \to \text{Ker}(v) \) is an epimorphism where \( v = V_B \circ (f_0 \oplus f_1) : A_0 \oplus A_1 \to B \). It is easy to check that the epimorphism \( u' \) gives \( T(\Lambda) \leq V \). The other claim \( T(V) \leq \Lambda \) is proved dually. □
Lemma 5.11. Consider following three square diagrams \( \square_0, \square_1, \square_2 \) in \( \mathcal{A} \):

\[
\begin{align*}
\square_0 &= \begin{array}{ccc}
B & \xrightarrow{g} & D \\
f' & & g' \\
A & \xrightarrow{f'} & C
\end{array}, & \square_1 &= \begin{array}{ccc}
D & \xrightarrow{h} & F \\
g' & & h' \\
C & \xrightarrow{g''} & E
\end{array}, & \square_2 &= \begin{array}{ccc}
B & \xrightarrow{h \circ g} & F \\
\Lambda & \xrightarrow{\Lambda} & \Lambda
\end{array}
\end{align*}
\]  
(21)

Especially the morphism in the left side of \( \square_0 \) and that in the right side of \( \square_1 \) coincide with each other, and the diagram \( \square_2 \) is induced by gluing \( \square_0, \square_1 \). If the square diagrams \( \square_0, \square_1 \) are exact, then the square diagram \( \square_2 \) is exact.

Proof. Suppose that the square diagrams \( \square_0, \square_1 \) are exact. We first prove that the induced morphism \( k_{\square_2} \) is an epimorphism. Consider the following commutative diagram where the sequence in each row is the exact sequence induced by the composable morphisms \( g, h \) and \( f', g'' \) respectively:

\[
\begin{array}{c}
\text{Ker}(g) \xrightarrow{k_{\square_2}} \text{Ker}(h \circ g) \xrightarrow{k_{\square_1}} \text{Ker}(h) \xrightarrow{k_{\square_0}} \text{Cok}(g)
\end{array}
\]
(22)

Since \( k_{\square_0}, k_{\square_1} \) are epimorphisms and \( c_{\square_2} \) is a monomorphism, the morphism \( k_{\square_2} \) is an epimorphism by the 4-lemma. We can dually prove that the morphism \( c_{\square_2} \) is an monomorphism by the 4-lemma. It completes the proof. \( \square \)

Lemma 5.12. Let \( \Lambda, \Lambda' \) be composable cospans in \( \mathcal{A} \). We have \( T(\Lambda' \circ \Lambda) \leq T(\Lambda') \circ T(\Lambda) \). In particular, we have \( T(\Lambda' \circ \Lambda) \simeq T(\Lambda') \circ T(\Lambda) \).

Proof. Consider cospans \( \Lambda = (A_0 \rightarrow B \leftarrow A_1) \) and \( \Lambda' = (A_1 \rightarrow B' \leftarrow A_2) \). Let \( (A_0 \rightarrow B'' \leftarrow A_2) \) be the composition \( \Lambda' \circ \Lambda \). Let \( T(\Lambda) = (A_0 \rightarrow C \leftarrow A_1) \) and \( T(\Lambda') = (A_1 \rightarrow C' \leftarrow A_2) \) be the induced spans. Let \( (A_0 \leftarrow C'' \rightarrow A_2) \) be the composition \( T(\Lambda') \circ T(\Lambda) \). Then we obtain the following commutative diagram.

\[
\begin{array}{c}
A_0 \xrightarrow{\Lambda} B \xrightarrow{\Lambda'} B'' \\
\uparrow & & \uparrow \\
\downarrow & & \downarrow \\
C \xrightarrow{\Lambda'} A_1 \xrightarrow{\Lambda'} B' \\
\uparrow & & \uparrow \\
\downarrow & & \downarrow \\
C'' \xrightarrow{\Lambda'} C' \xrightarrow{\Lambda'} A_2
\end{array}
\]  
(23)

The four square diagrams in the above commutative diagram are exact in the sense of Definition 4.8 by definitions of \( B, B'', B', C, C', C'' \) and Proposition 4.10. By Lemma 5.11 and its variants with respect to gluing sides, the induced square diagram below is exact.

\[
\begin{array}{c}
A_0 \xrightarrow{\Lambda_2} B'' \\
\uparrow & \\
C'' \xrightarrow{\Lambda_2} A_2
\end{array}
\]  
(24)

By Lemma 5.10 we obtain \( T(\Lambda' \circ \Lambda) \leq T(\Lambda') \circ T(\Lambda) \). \( \square \)

Definition 5.13. We define a functor \( T: \text{Cosp}^\kappa(\mathcal{A}) \rightarrow \text{Sp}^\kappa(\mathcal{A}) \) as follows. It assigns the object \( A \) itself to an object \( A \) of \( \text{Cosp}^\kappa(\mathcal{A}) \). It assigns a morphism \( [T(\Lambda)] \) in \( \text{Sp}^\kappa(\mathcal{A}) \) to a morphism \( [\Lambda] \) in \( \text{Cosp}^\kappa(\mathcal{A}) \). It is a well-defined functor due to \( [T(\text{cosp}(A))] = [\text{csp}(A)] \) for any object \( A \) and Lemma 5.12.
Theorem 5.14. The functor $T : \mathbf{Cosp}^\otimes(A) \to \mathbf{Sp}^\otimes(A)$ in Definition 5.13 gives an isomorphism of dagger symmetric monoidal categories.

Proof. For any cospan $\Lambda$, we have $T(\Lambda)^\dagger = T(\Lambda^\dagger)$ by definitions. Hence, the functor $T$ preserves the dagger structure.

The symmetric monoidal functor structure on the functor $T$ is induced by the natural isomorphism $T(\Lambda \otimes \Lambda') \cong T(\Lambda) \otimes T(\Lambda')$ for cospans $\Lambda, \Lambda'$ in $A$.

Note that the assignment $[V] \mapsto [T(V)]$ for a span $V$ in $\mathcal{A}$ induces a dagger symmetric monoidal functor $T : \mathbf{Sp}^\otimes(A) \to \mathbf{Cosp}^\otimes(A)$. This functor is an inverse functor of the functor in Definition 5.13. For a cospan $\Lambda$ in $A$, we have $T(T(\Lambda)) \cong \Lambda$ due to Lemma 5.10. In particular, we have $T(T(\Lambda)) \cong \Lambda$. Dually, we have $T(T(\Lambda)) \cong \Lambda$ for a span $V$ in $\mathcal{A}$. □

Remark 5.15. We give a remark about $\mathcal{A}$ in the introduction. The two categories $\mathbf{Cosp}^\otimes(A)$ and $\mathbf{Sp}^\otimes(A)$ are naturally isomorphic to each other by Theorem 5.14. The former one (latter one, resp.) consists of equivalence classes of cospan (span, resp.) diagrams in $A$. Then the category $\mathcal{A}$ is $\mathbf{Cosp}^\otimes(A)$ or equivalently $\mathbf{Sp}^\otimes(A)$; and $\iota_A$ is given by an embedding functor $\iota_{sp} : A \to \mathbf{Sp}^\otimes(A)$ or equivalently $\iota_{cosp} : A \to \mathbf{Cosp}^\otimes(A)$. See Definition 5.7, 5.8 for $\iota_{sp}, \iota_{cosp}$.

6. Spanical and cospanical extensions

6.1. Brown functor.

Definition 6.1. Consider a diagram in $\mathbf{CW}_{*, \leq r}^{\text{fin}}$ which commutes up to a homotopy:

\[
\begin{array}{ccc}
K_0 & \longrightarrow & L \\
\uparrow & & \uparrow \\
T & \longrightarrow & K_1
\end{array}
\]

(25)

The diagram (25) is approximated by a triad of spaces if there exists a triad of pointed finite CW-complexes $(L', K_0', K_1')$ such that the following induced diagram (26) is homotopy equivalent with the diagram (25); there exist pointed homotopy equivalences $K_0 \simeq K_0'$, $K_1 \simeq K_1'$, $T \simeq K' \cap K_1'$ and $L \simeq K_0' \cup K_1'$ which make the diagram (25) and (26) coincide up to homotopies.

\[
\begin{array}{ccc}
K_0' & \longrightarrow & K_0' \cup K_1' \\
\uparrow & & \uparrow \\
K_0' \cap K_1' & \longrightarrow & K_1'
\end{array}
\]

(26)

Definition 6.2. Let $\mathcal{A}$ be an abelian category. For $r \in \mathbb{N} \cup \{\infty\}$, a functor $E : Ho(\mathbf{CW}_{*, \leq r}^{\text{fin}}) \to \mathcal{A}$ satisfies the Mayer-Vietoris axiom if for an arbitrary diagram (26) in $\mathbf{CW}_{*, \leq r}^{\text{fin}}$, approximated by a triad of spaces, the induced chain complex in $\mathcal{A}$ is exact.

\[
E(T) \to E(K_0) \oplus E(K_1) \to E(L).
\]

(27)

Remark 6.3. The exactness in the previous definition could be equivalently rephrased as follows: The induced square diagram (28) in $\mathcal{A}$ is exact in the sense of Definition 4.8. The equivalence follows from Proposition 4.10.

\[
\begin{array}{ccc}
E(K_0) & \longrightarrow & E(L) \\
\uparrow & & \uparrow \\
E(T) & \longrightarrow & E(K_1)
\end{array}
\]

(28)
Definition 6.4. An \( \mathcal{A} \)-valued Brown functor is a symmetric monoidal functor \( E : \text{Ho}(\text{CW}_{*, \leq r}^{\text{fin}}) \to \mathcal{A} \) satisfying the Mayer-Vietoris axiom.

Remark 6.5. In Definition 6.1, we do not restrict the dimensions of \( L', K'_0, K'_1 \). It is sufficient to prove Lemma 6.14.

Example 6.6. Consider the category of abelian groups \( \mathcal{A} = \text{Ab} \). A generalized homology theory (for finite CW-spaces) induces a sequence of \( \infty \)-dimensional \( \text{Ab} \)-valued Brown functors. In fact, if \( E_\ast \) is a generalized homology theory, then the \( q \)-th homology theory functor \( E_q \) is an \( \text{Ab} \)-valued Brown functor.

Example 6.7. Let \( k \) be a field. Consider the category of bicommutative Hopf algebras over \( k \), \( \mathcal{A} = \text{Hopf}^{\text{bc}}_k \). In fact, it is known that the category \( \text{Hopf}^{\text{bc}}_k \) is an abelian category \([6][7]\). Analogously to Example 6.6, a \( \text{Hopf}^{\text{bc}}_k \)-valued homology theory induces a sequence of \( \infty \)-dimensional \( \text{Hopf}^{\text{bc}}_k \)-valued Brown functors. There are various examples of \( \text{Hopf}^{\text{bc}}_k \)-valued homology theory \([5]\).

6.2. Proof of the second part of Theorem 1.1. For convenience, we introduce following terminologies.

Definition 6.8. Let \( F : \text{Ho}(\text{CW}_{*, \leq r}^{\text{fin}}) \to \mathcal{A} \) be a symmetric monoidal functor.

(1) A spanical extension of the symmetric monoidal functor \( F \) is a dagger-preserving symmetric monoidal functor \( F' : \text{Cosp}_{\leq (r+1)}^{\leq (r+1)}(\text{CW}_{*, \leq r}^{\text{fin}}) \to \text{Sp}^\infty(\mathcal{A}) \) with the following commutative diagram.

\[
\begin{array}{ccc}
\text{Ho}(\text{CW}_{*, \leq r}^{\text{fin}}) & \xrightarrow{F} & \mathcal{A} \\
\downarrow & & \downarrow_{\imath_{\mathcal{P}}} \\
\text{Cosp}_{\leq (r+1)}^{\leq (r+1)}(\text{CW}_{*, \leq r}^{\text{fin}}) & \xrightarrow{F'} & \text{Sp}^\infty(\mathcal{A})
\end{array}
\]

(29)

(2) A cospanical extension of the symmetric monoidal functor \( F \) is a dagger-preserving symmetric monoidal functor \( F'' : \text{Cosp}_{\leq (r+1)}^{\leq (r+1)}(\text{CW}_{*, \leq r}^{\text{fin}}) \to \text{Cosp}^\infty(\mathcal{A}) \) with the following commutative diagram.

\[
\begin{array}{ccc}
\text{Ho}(\text{CW}_{*, \leq r}^{\text{fin}}) & \xrightarrow{F} & \mathcal{A} \\
\downarrow_{\imath} & & \downarrow_{\imath_{\text{cosp}}} \\
\text{Cosp}_{\leq (r+1)}^{\leq (r+1)}(\text{CW}_{*, \leq r}^{\text{fin}}) & \xrightarrow{F''} & \text{Cosp}^\infty(\mathcal{A})
\end{array}
\]

(30)

By using the above terminologies, Theorem 1.1 is reformulated as follows.

Theorem 1.1 (reformulation) For \( d \in \mathbb{N} \cup \{ \infty \} \), let \( E : \text{Ho}(\text{CW}_{*, \leq d}^{\text{fin}}) \to \mathcal{A} \) be a \( d \)-dimensional \( \mathcal{A} \)-valued Brown functor.

(1) There exists a unique spanical extension of \( E \circ \Sigma \).

(2) There exists a unique cospanical extension \( E \circ \imath \).

In the rest of this subsection, we prove the second part of theorem. The first part is proved in the next subsection.

Lemma 6.9. For a morphism \( [\Lambda] \) in \( \text{Cosp}_{\leq d}^{\leq d}(\text{CW}_{*, \leq d}^{\text{fin}}) \), there exist cospans \( \Lambda_0, \Lambda_1, \Lambda_2 \) of pointed finite CW-complexes subject to following conditions.

(1) The components of \( \Lambda_1 \) have dimension lower than or equal to \( (d - 1) \).

(2) We have \( [\Lambda] = [\Lambda_2] \circ [\Lambda_1] \circ [\Lambda_0] \) in the category \( \text{Cosp}_{\leq d}^{\leq d}(\text{CW}_{*, \leq d}^{\text{fin}}) \).
Proof. (1) Let $\Lambda \in U_{d, X}$ where $U_{d, X}$ is defined in Definition 3.6. In particular, $\Lambda = \left( K \xrightarrow{f} L \leftarrow \text{pt} \right)$ be a cospan of pointed finite CW-complexes with $\dim K \leq (d - 1)$ and $\dim L \leq d$. There exists a pointed finite CW-complex structure $X'_L$ on $L$ such that $f$ is a cellular map with respect to $X_K, X'_L$. If we denote by $L^{(d-1)}$ the $(d-1)$-skeleton of $L$ with respect to $X'_L$, we have $f(K) \subseteq L^{(d-1)}$. Denote by $f' : K \to L^{(d-1)}$ the induced map. Let $\varphi : D^d \to L, r = 1, 2, \cdots, k$ be characteristic maps of $d$-cells of $X_L$. Let $\psi : \bigvee_j (S^{d-1})^+ \to L^{(d-1)}$ be the pointed map induced by the wedge sum $\bigvee_j (\varphi_j)|_{S^{d-1}}$. Let $c$ be the pointed map induced by the collapsing maps $S' \to \text{pt}$. Then $L = L^{(d)}$ is a homotopy pushout of $L^{(d-1)} \xleftarrow{\psi} \bigvee_j (S^{d-1})^+ \xrightarrow{c} \bigvee_j S^0$, hence, we have $[\Lambda_1] \circ [\Lambda_0] = [\Lambda]$ where $\Lambda_0 = \left( K \xrightarrow{f'} L^{(d-1)} \xleftarrow{\psi} \bigvee_j (S^{d-1})^+ \right)$ and $\Lambda_1 = \left( \bigvee_j (S^{d-1})^+ \xleftarrow{c} \bigvee_j S^0 \xrightarrow{\text{pt}} \right)$. (2) We have $[\Lambda] = (ev_{K_1} \circ [\tau(K_1)]) \circ ([\Lambda] \circ [\tau(K_1)]) \circ ([\tau(K_0)] \circ ev_{K_0})$. See Figure 1. We have $(ev_{K_1} \circ [\tau(K_1)]) \circ ([\Lambda] \circ [\tau(K_1)]) \circ ([\tau(K_0)] \circ ev_{K_0}) = ([\tau(K_0)] \circ ev_{K_0} \circ [\Lambda] \circ [\tau(K_1)]) \circ ([\tau(K_0)] \circ ev_{K_0} \circ [\tau(K_1)])$. By the previous discussion, there exist cospans $\Lambda_2, \Lambda_1$ whose components are $\dim \leq (d - 1)$ and $(ev_{K_1} \circ [\Lambda] \circ [\tau(K_1)]) = [\Lambda_2] \circ [\Lambda_1]$. Put $\Lambda_0 = (ev_{K_1} \circ [\Lambda] \circ [\tau(K_1)])$ whose components also satisfy $\dim \leq (d - 1)$. Since $[\Lambda] = [\Lambda_2] \circ [\Lambda_1] \circ [\Lambda_0]$, it completes the proof.

\[ \Lambda = \Lambda = \Lambda \]

Figure 1.

Lemma 6.10. Let $F : \text{Ho}(\text{CW}^{\text{fin}}_{(r, \leq r)}) \to \mathcal{A}$ be a symmetric monoidal functor. If a cospanical extension of the symmetric monoidal functor $F$ exists, then it is unique.

Proof. Let $F''$ be a cospanical extension of $F$. Let $\Lambda = \left( K_0 \xrightarrow{f_0} L \xleftarrow{f_1} K_1 \right)$ be a cospan of pointed CW-spaces with $\dim \leq r$. Note that the homotopy equivalence classes of cospans $[\tau(f_0)]$ and $[\tau(f_1)]$ are morphisms of $\text{Cosp}^{\leq (r+1)}_{(r, \leq r)}(\text{CW}^{\text{fin}}_r)$ since we assume $\dim L \leq r$. Since we have $[\Lambda] = [\tau(f_1)] \circ [\tau(f_0)]$ by definitions, we obtain $F''([\Lambda]) = F''([\tau(f_1)] \circ [\tau(f_0)]) = F''([\tau(f_0)] \circ [\tau(f_1)]) = \psi(F([\tau(f_1)] \circ [\tau(f_0)]))$. Hence, $F''([\Lambda])$ is determined by the given symmetric monoidal functor $F$ if $\Lambda = \left( K_0 \xrightarrow{f_0} L \xleftarrow{f_1} K_1 \right)$ is a cospan whose components satisfy $\dim \leq r$.

By Lemma 6.9, all the morphisms in $\text{Cosp}^{\leq (r+1)}_{(r, \leq r)}(\text{CW}^{\text{fin}}_r)$ is decomposed into some morphisms of the above type. Thus, a cospanical extension $F''$ is determined uniquely by the symmetric monoidal functor $F$.

Lemma 6.11. Let $E : \text{Ho}(\text{CW}^{\text{fin}}_{(r, \leq r)}) \to \mathcal{A}$ be an $\mathcal{A}$-valued Brown functor. Let $\Lambda, \Lambda' \in U_{d, X}$ and $\Lambda' \circ \Lambda$ be the composition. Then we have

$$E(\Lambda') \circ E(\Lambda) \leq E(\Lambda' \circ \Lambda).$$

Proof. Let $\Lambda = (K_0 \to L \leftarrow K_1)$ and $\Lambda' = (K_1 \to L' \leftarrow K_2)$ be cospans lying in $U_{d, X}$. Then the composition $\Lambda' \circ \Lambda = (K_0 \to L'' \leftarrow K_2)$ is given by $L'' = \text{Cyl}(f_1) \bigvee_{K_1} \text{Cyl}(f'_1)$. See
Denote by \( E \) in \( \text{Def} 3.3 \) for details. It induces a commutative square diagram as follows.

\[
\begin{array}{ccc}
E(L) & \longrightarrow & E(L') \\
\uparrow & & \uparrow \\
E(K_1) & \longrightarrow & E(L')
\end{array}
\]

(32)

Note that the square diagram (33) in \( \text{CW}_{\leq r}^{\text{fin}} \) is approximated by a triad of spaces with \( \dim \leq r \) in the sense of Definition 6.1. Hence, the square diagram is exact since \( E \) is a Brown functor.

\[
\begin{array}{ccc}
L & \longrightarrow & L'' \\
\uparrow & & \uparrow \\
K_1 & \longrightarrow & L'
\end{array}
\]

(33)

Denote by \( u : E(K_1) \to E(L) \oplus E(L') \) the composition \((E(f_1) \oplus (-E(f_1'))) \circ \Delta_{E(K_1)} \). The commutativity induces a morphism \( \bar{u} : \text{Cok}(u) \to E(L'') \). The morphism \( \bar{u} \) is a monomorphism by Proposition 4.10. Since \( E(\Lambda') \circ E(\Lambda) = (E(K_0) \to \text{Cok}(u) \leftarrow E(K_2)) \) by definition, we have \( E(\Lambda') \circ E(\bar{\Lambda}) \leq E(\Lambda' \circ \Lambda) \). □

**Proof of the second part of Theorem 1.1** We define a cospanical extension \( \hat{E} : \text{Cosp}^\infty_{\leq d}(\text{CW}_{\leq r}^{\text{fin}}) \to \text{Cosp}^\infty(\mathcal{A}) \) of \( E \circ i \) as follows:

1. The functor \( \hat{E} \) assigns \( E(\Delta) \) to an object \( \Delta \) of \( \text{Cosp}^\infty_{\leq d}(\text{CW}_{\leq r}^{\text{fin}}) \).
2. The functor \( \hat{E} \) assigns the homotopy equivalence class \([E(\Lambda)]\) of the induced cospan \( E(\Lambda) \), which is a morphism in \( \text{Cosp}^\infty(\mathcal{A}) \), to a morphism \([\Lambda] \) of \( \text{Cosp}^\infty_{\leq d}(\text{CW}_{\leq r}^{\text{fin}}) \).

We prove that the above assignment gives a well-defined functor. The assignment \( \hat{E} \) assigns an identity in the target category to each identity in the source category. Let \((\Lambda, \Lambda') \in U \times U\) where \( U = U_{dX} \). We have \( E(\Lambda') \circ E(\Lambda) \leq E(\Lambda' \circ \Lambda) \) by Lemma 6.11. It implies \( \hat{E}([\Lambda']) \circ \hat{E}([\Lambda]) = \hat{E}([\Lambda'] \circ [\Lambda]) \).

The functor \( \hat{E} \) is enhanced to a dagger-preserving symmetric monoidal functor : The functor preserves dagger structures by definitions. The symmetric monoidal functor structure of \( E \) naturally induces a symmetric monoidal functor structure on \( \hat{E} \).

The dagger-preserving symmetric monoidal functor \( \hat{E} \) is a cospanical extension by definition. The uniqueness of a cospanical extension follows from Lemma 6.10. It completes the proof.

6.3. **Proof of the first part of Theorem 1.1**

**Definition 6.12.** Let \( \tau_K : \Sigma K \to \Sigma K; [t, k] \mapsto [\bar{t}, k] \) be the conjugate for a pointed finite CW-space \( K \) where we identify \( \Sigma K = S^1 \wedge K \). Let \( \Lambda = \left( K_0 \xrightarrow{f_0} L \xleftarrow{f_1} K_1 \right) \) be a cospan of pointed spaces. Denote by \( p_0, p_1 \) the collapsing maps from the mapping cone \( C(f_0 \vee f_1) \) to the suspensions \( \Sigma K_0 \) and \( \Sigma K_1 \) respectively. We define a span of pointed spaces \( T\Sigma(\Lambda) \) by

\[
T\Sigma(\Lambda) \overset{\text{def.}}{=} \left( \Sigma K_0 \xrightarrow{\tau_K \circ p_0} C(f_0 \vee f_1) \xrightarrow{p_1} \Sigma K_1 \right).
\]

**Remark 6.13.** Recall that there is an assignment of spans \( T(\Lambda) \) to cospans \( \Lambda \) in an abelian category by Definition 5.9. It is not obvious that there is an analogous assignment in the category of (pointed finite CW-)spaces but it motivates the notation in Definition 6.12. In fact, if the symmetric monoidal functor \( E \) satisfies the Mayer-Vietoris axiom, then we have

\[
[E(T\Sigma(\Lambda))] = [T(E(\Sigma(\Lambda)))].
\]
for a cospan of pointed finite CW-spaces $\Lambda$. Here $T$ in the right hand side is the transposition in Definition 5.9 and $\Sigma(\Lambda)$ denotes the suspension of cospan. The bracket $[-]$ denotes the equivalence class of spans defined analogously to Definition 5.7.

Lemma 6.14. Let $\Lambda, \Lambda' \in U_{d,X}$ and $\Lambda' \circ \Lambda$ be the composition cospan. Then we have

\[ E(T\Sigma(\Lambda')) \circ E(T\Sigma(\Lambda)) \leq E(T\Sigma(\Lambda' \circ \Lambda)). \]

Proof. Let $\Lambda = (K_0 \xrightarrow{f_0} L \xleftarrow{\tau} K_1)$ and $\Lambda' = (K_1 \xrightarrow{f_1} L' \xleftarrow{\tau} K_2)$ be cospan lying in $U_{d,X}$. Then the composition $\Lambda' \circ \Lambda = (K_0 \xrightarrow{f_0} L' \xleftarrow{\tau} K_2)$ is given by $L'' = Cyl(f_1) \vee K_1 Cyl(f_1')$. See Definition 5.3 for details. Denote by $p_0, p_1$ the collapsing maps from the mapping cone $C(f_0 \vee f_1)$ to the suspensions $\Sigma K_0, \Sigma K_1$, and by $p_1', p_2'$ the collapsing maps from the mapping cone $C(f_1' \vee f_2')$ to the suspensions $\Sigma K_1, \Sigma K_2$. Denote by $q_0, q_2$ the collapsing maps from the mapping cone $C(f_0'' \vee f_2'')$ to the mapping cones $C(f_0 \vee f_1)$ and $C(f_1' \vee f_2')$ respectively.

It is easy to verify that the diagram (37) commutes up to a homotopy. In fact, we define $T(\Lambda)$ by using the conjugate $\tau_{K_0}$ in Definition 6.12 for this diagram to commute. Note that we apply Lemma 6.14 instead of Lemma 6.11.

\[ C(f_0 \vee f_1) \xrightarrow{q_0} \Sigma K_0 \xrightarrow{\tau_{K_1} \circ p_1} C(f_1' \vee f_2') \]

It induces the following square diagram where the morphisms are induced by the canonical collapsing maps.

\[ \begin{array}{ccc}
E(C(f_0 \vee f_1)) & \xrightarrow{E(p_1)} & E(\Sigma K_1) \\
E(q_0) \uparrow & & \uparrow \tau_{K_1} \circ p_1 \\
E(C(f_0'' \vee f_2'')) & \xrightarrow{E(q_2)} & E(C(f_1' \vee f_2'))
\end{array} \]

The square diagram (38) is exact since the symmetric monoidal functor $E$ satisfies the Mayer-Vietoris axiom. The remaining proof is similar to that of Lemma 6.11. □

Proof of the first part of Theorem 1.1. We define a spanical extension $\tilde{E} : \text{Cosp}_{\leq d}(\text{CW}_{\ast}) \to \text{Sp}_{\leq d}(\mathcal{A})$ of $E \circ \Sigma$ as follows.

(1) The functor $\tilde{E}$ assigns $E(\Sigma K)$ to an object $K$ of $\text{Cosp}_{\leq d}(\text{CW}_{\ast})$. It is well-defined since the domain of $E$ consists of complexes $T$ with $\text{dim } T \leq d$.

(2) The functor $\tilde{E}$ assigns the induced morphism $[E(T\Sigma(\Lambda))]$ in $\text{Sp}_{\leq d}(\mathcal{A})$ to a morphism $[\Lambda]$ of $\text{Cosp}_{\leq d}(\text{CW}_{\ast})$.

The proof that the assignment $\tilde{E}$ is a well-defined dagger-preserving symmetric monoidal functor is parallel with that of the second part of Theorem 1.1. Note that we apply Lemma 6.14 instead of Lemma 6.11.

We show that the dagger-preserving symmetric monoidal functor $\tilde{E}$ is a spanical extension of the symmetric monoidal functor $E$ in the sense of Definition 6.8. It suffices to prove that $\tilde{E}(\iota(f)) = \iota(E(f))$ for a morphism $f : K \to L$ in $\text{CW}_{\ast, \leq d}$ where $\iota(f) = (K \xrightarrow{f} L \xleftarrow{id} L)$. Denote by $p_0, p_1$ the collapsing maps from the mapping cone $C(f \vee Id_L)$ to $\Sigma K$ and $\Sigma L$ respectively. Note that the map $p_0$ is a pointed homotopy equivalence, especially so the
composition $\tau_K \circ p_0$ is. Since the following diagram commutes up to a homotopy, the span $T\Sigma(\iota(f))$ is homotopy equivalent with the span $\left(\Sigma K \xrightarrow{Id} \Sigma K \xrightarrow{\Sigma f} \Sigma L\right)$.

![Diagram](39)

As a result, we obtain $\hat{E}(\iota(f)) = [E(T\Sigma(\iota(f)))] = \iota(E(f))$. It completes the proof.

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