NEW SOLUTIONS OF HYPERBOLIC TELEGRAPH EQUATION

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Abstract. We present a new method based on unification of fictitious time integration (FTI) and group preserving (GP) methods. The GP method is applied in numerically discretized ordinary differential equations obtained from application of FTI method to a given partial differential equation (PDE). The algorithm is applied to hyperbolic telegraph equation and utilizes the Cayley transformation and the Pade approximations in the Minkowski space. It avoids unauthentic solutions and ghost fixed points which is one of the advantages of the present method over other related numerical methods in the literature. The technique is tested on three specific examples for various parameter values appearing in the telegraph equation and discretization steps. Such solutions of the telegraph equation are obtained first time in this paper. Illustrative figures are provided. Efficiency of the method is determined by an error analysis which is achieved by comparing numerical solutions with exact solutions.

1. Introduction. Partial differential equations (PDEs) are extremely useful tools in the modeling of a large number of scientific events and experiments. PDEs and their solution algorithms became a phenomenal subject for centuries. In the scientific literature of mathematics, there are various and highly efficient solution techniques for PDEs. In this work, we are concerned with numerical solutions of hyperbolic telegraph equation. In order to achieve this, we propose a new method which may be considered as a synthesis of group preserving and fictitious time integration schemes. Research works regarding application and fundamental concepts of these techniques may be seen in the studies, e.g. [1, 2, 3, 4, 5, 6, 7, 8]. In these references, an interested reader can see the applications of GP and FTI methods to some equations including diffusion wave, Schrödinger, Klein-Gordon and a gas model.

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The GPS is a geometric technique which is introduced first by Liu in [15]. This method also is applied for different equations such as [16, 17]. The hyperbolic telegraph equation, see e.g. [9, 10, 11, 12], expressed as a boundary value problem considered in this paper is:

\[ u_{tt}(x,t) - 2\alpha u_t(x,t) + \beta^2 u(x,t) - u_{xx}(x,t) = F(x,t), \quad (x,t) \in \Omega \subset \mathbb{R}^2, \quad (1) \]

with

\[
\begin{align*}
    u(x,0) &= g_1(x), & x & \in \Omega_x, \\
    u(x,T) &= g_2(x), & x & \in \Omega_x, \\
    u(a,t) &= h_1(t), & t & \in \Omega_t, \\
    u(b,t) &= h_2(t), & t & \in \Omega_t,
\end{align*}
\]

where \( \Omega_t \) and \( \Omega_x \) are the boundaries of \( \Omega := \{(x,t) : a \leq x \leq b, \ 0 \leq t \leq T\} \), \( F(x,t) \) is continuously differentiable \( C^2 \) function. The \( C^1 \) functions \( g_i(x) \) and \( h_i(t) \) are assumed to satisfy compatibility condition on the boundaries of the domain. Telegraph equation is a linear PDE used for modeling of electromagnetic waves, radio frequencies, voltage and current on transmission lines. Some numerical and analytical solution approaches proposed for this problem in the history of research may be seen in [18, 19, 20]. At the next section, we describe fictitious time integration method shortly.

2. The fictitious time integration method. We shortly overview the fictitious time integration method, see e.g. [13, 14]. Let the parameter \( \zeta \) be a fictitious damping coefficient and multiply each term in the Eq. (1) with \( \zeta \):

\[ \zeta u_{tt}(x,t) - 2\alpha \zeta u_t(x,t) + \zeta \beta^2 u(x,t) - \zeta u_{xx}(x,t) = \zeta F(x,t) \quad (3) \]

Applying the transformation

\[ \Xi(x,t,\xi) = (1 + \xi)^\kappa u(x,t), \quad 0 < \kappa \leq 1, \quad (4) \]

for Eq. 3, we get:

\[ \frac{\zeta}{(1 + \xi)^\kappa} \left[ \Xi_{tt}(x,t,\xi) - 2\alpha \Xi_t(x,t,\xi) + \beta^2 \Xi(x,t,\xi) - \Xi_{xx}(x,t,\xi) \right] - \zeta F(x,t) = 0. \quad (5) \]

for

\[ \frac{\partial \Xi}{\partial \xi} = \kappa(1 + \xi)^{\kappa-1} u(x,t) \quad (6) \]

By Eq. (5), we have:

\[ \frac{\partial \Xi}{\partial \xi} = \frac{\zeta}{(1 + \xi)^\kappa} \left[ \Xi_{tt}(x,t,\xi) - 2\alpha \Xi_t(x,t,\xi) + \beta^2 \Xi(x,t,\xi) - \Xi_{xx}(x,t,\xi) \right] - \zeta F(x,t) + \kappa(1 + \xi)^{\kappa-1} u. \quad (7) \]

By setting \( u = \Xi/(1 + \xi)^\kappa \), we have

\[ \frac{\partial \Xi}{\partial \xi} = \frac{\zeta}{(1 + \xi)^\kappa} \left[ \Xi_{tt}(x,t,\xi) - 2\alpha \Xi_t(x,t,\xi) + \beta^2 \Xi(x,t,\xi) - \Xi_{xx}(x,t,\xi) \right] - \zeta F(x,t) + \kappa \Xi/1 + \xi. \quad (8) \]
Using
\[ \frac{\partial}{\partial \xi} \left( \frac{\Xi}{(1 + \xi)^\kappa} \right) = \frac{\Xi_\xi}{(1 + \xi)^\kappa} - \frac{\kappa \Xi}{(1 + \xi)^{1+\kappa}} , \] (9)
and by applying \(1/(1 + \xi)^\kappa\) to Eq. (8), we obtain:
\[ \frac{\partial}{\partial \xi} \left( \frac{\Xi}{(1 + \xi)^\kappa} \right) = \frac{\zeta}{(1 + \xi)^\kappa} \left[ \Xi_{tt}(x, t, \xi) - 2\alpha \Xi_t(x, t, \xi) + \beta^2 \Xi(x, t, \xi) - \Xi_{xx}(x, t, \xi) \right] - \zeta F(x, t). \] (10)
by employing \(u = \frac{\Xi}{(1 + \xi)^\kappa}\), we get:
\[ u_\xi = \frac{\zeta}{(1 + \xi)^\kappa} \left[ u_{tt}(x, t, \xi) - 2\alpha u_t(x, t, \xi) + \beta^2 u(x, t, \xi) - u_{xx}(x, t, \xi) \right] - \zeta F(x, t). \] (11)
Suppose \(u_i^j(\xi) := u(x_i, t_j, \xi), \Delta x = \frac{b - a}{m}, \Delta t = \frac{T}{n}, x_i = a + i\Delta x, t_j = j\Delta t.\)
Then, we can write Eq. (11) as:
\[ \frac{du_i^j}{d\xi} = \frac{\zeta}{(1 + \xi)^\kappa} \left[ \frac{u_i^{j+1}(\xi) - 2u_i^j(\xi) + u_i^{j-1}(\xi)}{\Delta t^2} - 2\alpha \frac{u_i^{j+1}(\xi) - u_i^{j-1}(\xi)}{\Delta t} + \beta^2 u_i^j(\xi) - \frac{u_i^{j+1}(\xi) - 2u_i^j(\xi) + u_i^{j-1}(\xi)}{\Delta x^2} \right] - \zeta F(x_i, t_j). \] (12)
Next part overviews the group preserving method.

3. The group preserving scheme (GPS). Suppose that
\[ u = (u_1^1, u_1^2, ..., u_m^n)^T. \] (13)
Then, we can write the Eq. (12) as:
\[ \frac{du}{d\xi} = E(u, \xi), \quad u \in \mathbb{R}^N, \xi \in \mathbb{R}, \] (14)
where
\[ E(u, \xi) = \frac{\zeta}{(1 + \xi)^\kappa} \left[ \frac{u_i^{j+1}(\xi) - 2u_i^j(\xi) + u_i^{j-1}(\xi)}{\Delta t^2} - 2\alpha \frac{u_i^{j+1}(\xi) - u_i^{j-1}(\xi)}{\Delta t} + \beta^2 u_i^j(\xi) - \frac{u_i^{j+1}(\xi) - 2u_i^j(\xi) + u_i^{j-1}(\xi)}{\Delta x^2} \right] - \zeta F(x_i, t_j). \] (15)
in which \(E\) indicates a vector with \(ij\)-elements being the r.h.s. of the Eq. (12) and \(N = m \times n\) is the number of total grid points. The notion of a geometric structure, Lie groups is helpful in building some accurate and powerful numerical methods to integrate ODEs, with the preservation of invariant feature. In fact, preserving the Lie group structure under discretization performs an important role in the recovery of a suitable trait in the error minimization. So, by sharing the geometric
construction and invariance of the original ODEs, new methods can be devised, which are more powerful, accurate and stable than the conventional numerical approaches. From the class of numerical Lie group approaches, the GPS, invented by Liu [15], is a numerical approach which utilizes the Cayley transformation and the Pade approximations in the Minkowski space. Avoiding the spurious solutions and ghost fixed points are major benefits of this method. In [17], the authors combined a semi-discretization method and GPS to acquire the approximate solutions of a harshly ill-posed Laplace equation, while Abbasbandi et al. have applied the Lie group shooting method (LGSM) to solve the Bratu equation.

\[ u_{\kappa+1} = u_{\kappa} + \frac{(a_{\kappa} - 1)E_{\kappa} \cdot u_{\kappa} + b_{\kappa} \| u_{\kappa} \| E_{\kappa}}{\| E_{\kappa} \|^2} E_{\kappa} = u_{\kappa} + \Pi_{\kappa} E_{\kappa}. \]  

(16)

with

\[ a_{\kappa} = \cosh \left( \frac{\Delta \xi \| E_{\kappa} \|}{\| u_{\kappa} \|} \right), \]

(17)

\[ b_{\kappa} = \sinh \left( \frac{\Delta \xi \| E_{\kappa} \|}{\| u_{\kappa} \|} \right). \]

We can control the convergence of \( u_{\kappa} \) at the \( \kappa \) and \( \kappa + 1 \) steps by following the criterion:

\[ \sqrt{\sum_{i,j=1}^{m,n} \left[ u_{\kappa}^j(l+1) - u_{\kappa}^j(\kappa) \right]^2} \leq \varepsilon, \]

(18)

where \( \varepsilon \) is known as convergence criteria. Obviously, the final solution \( u \) can be obtained from

\[ u_{\kappa} = u_{\kappa}^j(\xi_0) \left( 1 + \xi_0 \right)^{\kappa}, \]

(19)

where \( \xi_0 (\leq \xi_f) \) satisfies at Eq. (19), where \( \xi_0 \) and \( \xi_f \) are the initial and final values of \( \xi \).

4. Applications. Example 1: Consider

\[ u_{tt}(x,t) - 2u_t(x,t) + u(x,t) - u_{xx}(x,t) = 0 \]

with the exact solution

\[ u(x,t) = exp(x - t), \]

and with boundary and initial conditions:

\[ u(x,0) = exp(x), \]

\[ u(x,T) = exp(x - T), \]

\[ u(-2,t) = exp(-2 - t), \]

\[ u(2,t) = exp(2 - t). \]

(20)

Fig. 1 is allocated to display the exact and numerical solutions and error obtained by our method by choosing \( T = 0.05, m = n = 25, \xi = 60000, \kappa = 0.1 \) and \( u_{\kappa}(0) = 0.1 \) and \( \Delta \xi = 3/100000000000 \). The graph of error displayed in Fig.1 proves the accuracy of current method.

Example 2:

Consider

\[ u_{tt}(x,t) + 12u_t(x,t) + 4u(x,t) - u_{xx}(x,t) = 0 \]
by the exact solution

\[ u(x,t) = \cos(t)\sin(x), \]
with boundary conditions and initial conditions given by

\[
\begin{align*}
    u(x, 0) &= \cos(0)\sin(x), \\
    u(x, T) &= \cos(T)\sin(x), \\
    u(0, t) &= \cos(t)\sin(0), \\
    u(10, t) &= \cos(t)\sin(10).
\end{align*}
\]

Fig. 2 is allocated to display the exact and numerical solutions and error obtained by our method by choosing \( m = n = 30, \zeta = 8000, T = 0.05, \kappa = 0.1 \) and \( u(0) = 0.001 \) and \( \Delta \xi = 3/10000000000 \). The graph of error displayed in Fig. 2 shows the accuracy of the current method. Moreover, results for \( m = n = 40, \zeta = 7995, \kappa = 0.1 \) and \( u(0) = 0.001 \) and \( \Delta \xi = 3/10000000000 \) can be found in Fig. 3.

**Example 3:**
Consider

\[
    u_{tt}(x, t) - u_{xx}(x, t) - u(x, t) + u^3(x, t) = 0
\]

with

\[
    u(x, t) = \exp(-t)\sin(x),
\]
Figure 3. Exact and numerical solutions and error for $T = 0.05$, $m = n = 40$, $\zeta = 7995$, $\kappa = 0.1$ and $u_1'(0) = 0.001$ and $\Delta \zeta = 3/10000000000$ for Ex.2.

Table 2. Numerical and exact solutions and error values for $m = n = 30$, and $u_1'(0) = 0.001$ for Ex.2.

| (x, t)   | Numerical | Exact   | Error       |
|----------|-----------|---------|-------------|
| (0,0)    | 0         | 0       | 0           |
| (2.0,01) | 0.9289    | 0.9289  | 1.9657e-13  |
| (4.02)   | -0.6876   | -0.6876 | 9.7778e-14  |
| (6.0,03) | -0.4197   | -0.4197 | 3.7708e-14  |
| (8.0,04) | 0.9628    | 0.9628  | 2.2468e-13  |
| (10,0.05)| 3.6693e-16| 3.6693e-16| 2.9738e-44 |

with

\[
\begin{align*}
    u(x, 0) &= 0, \\
    u(x, T) &= \exp(-T)\sin(x), \\
    u(0, t) &= \exp(-t)\sin(-2), \\
    u(10, t) &= \exp(-t)\sin(2).
\end{align*}
\]
Table 3. Solutions with error values for \( m = n = 40 \) and \( u_i^j(0) = 0.001 \) for Ex. 2.

| (x, t)       | Numerical | Exact     | Error      |
|--------------|-----------|-----------|------------|
| (-10,0)      | -3.6739e-16 | -3.6739e-16 | 2.9813e-44 |
| (-5,0.01)    | 0.9350    | 0.9350    | 4.9081e-12 |
| (0.02)       | -0.9924   | -0.9924   | 1.3054e-11 |
| (5,0.03)     | -0.9346   | -0.9346   | 9.7228e-12 |
| (10,0.04)    | 3.6708e-16 | 3.6708e-16 | 2.9763e-44 |

We reveal the exact and numerical solutions and error obtained by the current technique by selecting \( m = n = 25, \zeta = 5995, \kappa = 0.01 \) and \( u_i^j(0) = 0.01 \) and \( \Delta \xi = 3/10000000000 \) for Ex. 3.

We propose a new method which combines the fictitious time integration (FTI) and group preserving (GP) methods. The method...
Table 4. Solutions with error values for $m = n = 25$, $u_i^j(0) = 0.001$ for Ex.3

| $(x, t)$ | Numerical | Exact | Error       |
|---------|-----------|-------|-------------|
| (-2.0)  | -0.9093   | -0.9093 | 2.6858e-13 |
| (-1.0,01) | -0.8328   | -0.8328 | 6.0082e-12 |
| (0,0.02) | 0         | 0     | 3.1758e-18 |
| (1.0,03) | 0.8173    | 0.8173 | 1.3564e-11 |
| (2.0,04) | 0.8740    | 0.8740 | 2.4814e-13 |

utilizes the Cayley transformation and Pade approximations in the Minkowski space. It avoids unauthentic solutions and ghost fixed points which is one of the advantages of the present method over other related numerical methods in the literature. The technique is tested on three specific examples for various parameter values appearing in the telegraph equation and discretization steps. Illustrative figures were provided. An error analysis is achieved by comparing numerical solutions with exact solutions. In a future extension of the present work, we will study on convergence, stability and consistency of the present method. We will also investigate the applicability of the method to some fractional-stochastic PDEs.

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