THE OPTICAL ANALOGUE OF CP–VIOLATION

D. Cocolicchio\textsuperscript{(1,2)}, L. Telesca\textsuperscript{(3)} and M. Viggiano\textsuperscript{(1)}

\textsuperscript{(1)} Dipartimento di Matematica, Univ. Basilicata, Potenza, Italy
Via N. Sauro 85, 85100 Potenza, Italy

\textsuperscript{(2)} Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy
Via G. Celoria 16, 20133 Milano, Italy

\textsuperscript{(3)} Consiglio Nazionale delle Ricerche, Istituto di Metodologie Avanzate
C/da S. Loya, 85050 Tito, Potenza, Italy

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Abstract
The peculiar features of the mixing in the neutral pseudoscalar mesons \( K^0 - \overline{K^0} \) can be introduced by the analogy to the optical polarization. The time-reversed not-invariant processes and the related phenomenon of CP-nonconservation can be then joined to the dissipative effects which yield a not vanishing imaginary part in the relevant propagation of electromagnetic radiation in a medium. Thus, the propagation of the two transverse polarization states can reproduce the peculiar asymmetries which are so common in the realm of high energy particle physics.
1 Introduction

Mechanical models that provide a viable description of the non-unitary time evolution can be related to the problem of two coupled degenerate oscillators with dissipation [1]. The physical intuition underlying any proposed model stems back from the fact that any time-reversed non invariant process can yield a non vanishing imaginary part for the relevant Hamiltonian. Although, the Hamiltonian of a complete sensible quantum system is expected to be a Hermitian operator, under suitable conditions we may recover the time evolution of a subsystem according to an effective non hermitian Hamiltonian like in the case of metastable states [2]. A celebrated example where this description has proved extremely useful is the two-states kaon complex [3]. In this case, the single pole approximation of the Weisskopf-Wigner method [2] can be applied to derive the eigenstates $K_S$ and $K_L$ of a $2 \times 2$ effective Hamiltonian $H$ [4]. Although such puzzle system requires the formalism of density matrix, and the notion of the rest frame for an oscillating unstable composite system appears difficult to be implemented, nevertheless there has been always a steady activity towards understanding these processes at the level of wave-function [4].

The time evolution of the flavour states $K^0$ and $\bar{K}^0$ can be written as

$$i\hbar \frac{d}{dt} \left( \begin{array}{c} |K^0(t)\rangle \\ |\bar{K}^0(t)\rangle \end{array} \right) = H \left( \begin{array}{c} |K^0(t)\rangle \\ |\bar{K}^0(t)\rangle \end{array} \right) = \left( \begin{array}{cc} H_{11} & H_{12} \\ H_{21} & H_{22} \end{array} \right) \left( \begin{array}{c} |K^0(t)\rangle \\ |\bar{K}^0(t)\rangle \end{array} \right)$$  \hspace{1cm} (1)

where the effective $2 \times 2$ matrix Hamiltonian $H$ is linear but not necessarily Hermitian and it can be uniquely rewritten in terms of two Hermitian matrices $M$ and $\Gamma$: $H = M - i\Gamma/2$. $M$ is called the mass matrix and $\Gamma$ the decay matrix. Their explicit expressions can be derived by the weak scattering theory responsible of the decay [4]. The dynamical behaviour of this system can be described in the Schrodinger picture by means of a time evolution operator $O$:

$$\left( \begin{array}{c} |K^0(t)\rangle \\ |\bar{K}^0(t)\rangle \end{array} \right) = O(t) \left( \begin{array}{c} |K^0\rangle \\ |\bar{K}^0\rangle \end{array} \right)$$  \hspace{1cm} (2)

which can be written by means of the well known exponential solution

$$O(t) = \left\{ \exp \left[ -\frac{i}{\hbar} H t \right] \right\},$$  \hspace{1cm} (3)

whose matrix elements are given by

$$U_{ij} = \langle K_i | \exp \left[ -\frac{i}{\hbar} H t \right] | K_j \rangle$$  \hspace{1cm} (4)

where as usual the latin indices denote $K_i$, $\bar{K}_i$. In order to evaluate this matrix exponential form, we can use the Sylvester’s formula of the matrix spectral decomposition for a function $f(A)$ of a generic $n \times n$ square matrix $A$. It is easy to express this Sylvester’s formula for the special case of $2 \times 2$ matrix with two distinct complex eigenvalues $\alpha_1$ and $\alpha_2$

$$f(A) = -\frac{\alpha_2 \alpha_1}{\alpha_2 - \alpha_1} \left[ \frac{f(\alpha_2)}{\alpha_2} - \frac{f(\alpha_1)}{\alpha_1} \right] I + \frac{1}{\alpha_2 - \alpha_1} \left[ f(\alpha_2) - f(\alpha_1) \right] A$$  \hspace{1cm} (5)

which specializes in the limit $\alpha_2 \to \alpha_1$ in

$$f(A) \to -\alpha_1^2 \frac{d}{d\alpha_1} \left[ \frac{f(\alpha_1)}{\alpha_1} \right] I + \frac{df(\alpha_1)}{d\alpha_1} A = f(\alpha_1) I + f'(\alpha_1)(A - \alpha_1 I)$$  \hspace{1cm} (6)
This formula becomes rewarding in connection with the expansion of the exponential representation of the evolution operator. In fact, in this case, we get

\[ O(t) = \exp \left( -\frac{i}{\hbar} \mathcal{H} t \right) = \frac{1}{\alpha_2 - \alpha_1} \left\{ \left[ e^{-\frac{i}{\hbar} \alpha_2 t} - e^{-\frac{i}{\hbar} \alpha_1 t} \right] \mathcal{H} + \left[ \alpha_2 e^{-\frac{i}{\hbar} \alpha_1 t} - \alpha_1 e^{-\frac{i}{\hbar} \alpha_2 t} \right] I \right\} . \] (7)

It is convenient to represent the matrix \( \mathcal{H} \) in terms of the Pauli spin matrices

\[ \mathcal{H} = h_\mu \sigma_\mu = h_0 \mathcal{I} + \mathbf{h} \cdot \mathbf{\sigma} \] (8)

where we let

\[
\begin{align*}
h_0 &= \frac{1}{2} (H_{11} + H_{22}) \\
h_1 &= \frac{1}{2} (H_{12} + H_{21}^*) \\
h_2 &= \frac{i}{2} (H_{12} - H_{21}^*) \\
h_3 &= \frac{1}{2} (H_{11} - H_{22})
\end{align*}
\]

being \( \text{tr} \mathcal{H} = \alpha_1 + \alpha_2 = 2h_0 \), \( \det \mathcal{H} = \alpha_1 \alpha_2 = (h_0)^2 - |\mathbf{h}|^2 \) and then \( \alpha_2 - \alpha_1 = 2|\mathbf{h}| \). It is easy then proven that \( \mathcal{H} \) is a normal matrix with a complete set of orthogonal eigenvectors if and only if \( \mathbf{h} \times \mathbf{h}^* = 0 \). Rewriting the solution in terms of the \( \mathbf{h} \) vector, we have

\[
O(t) = \exp \left[ -\frac{i}{\hbar} \mathcal{H} t \right] = \exp \left( -\frac{i}{\hbar} h_0 t \right) \exp \left( -\frac{i}{\hbar} \mathbf{h} \cdot \mathbf{\sigma} t \right) = \frac{1}{2} \left[ (1 + \mathbf{h} \cdot \mathbf{\sigma}) e^{-\frac{i}{\hbar} \alpha_2 t} + (1 - \mathbf{h} \cdot \mathbf{\sigma}) e^{-\frac{i}{\hbar} \alpha_1 t} \right] .
\] (10)

where \( \mathbf{h} = \mathbf{h}/|\mathbf{h}| \). Except for the overall phase \( \exp (-i h_0 t) \), this is just a rotation \( \mathcal{R}[\omega t] = \exp [-i(\omega \cdot \mathbf{\sigma} t/2)] \) of the initial state \( |\psi(0)\rangle \): \( |\psi(t)\rangle = e^{-i h_0 t} \mathcal{R}[\omega t] |\psi(0)\rangle \). (11)

In particle physics, \( K^0 \) and \( \bar{K}^0 \) are expected to be distinct particles from the point of view of the strong interactions, they could transform into each other through the action of the weak interactions. In fact, the system \( K^0 - \bar{K}^0 \) results degenerate due to a coupling to common final states (direct \( CP \)-violation) \( (K^0 \leftrightarrow \pi\pi \leftrightarrow \bar{K}^0) \) or by means of a mixing \( (K^0 \leftrightarrow \bar{K}^0) \) (indirect \( CP \) violation). Even if this is an effect of second order in the weak interactions, the transition from the \( K^0 \) to \( \bar{K}^0 \) becomes important just because this interference in evaluating decay widths. The mass eigenstates are linear combinations of the flavour states \( |K^0\rangle \) and \( |\bar{K}^0\rangle \) namely \( |K^0_S\rangle > \) and \( |K^0_L\rangle > \), which have definite masses and lifetimes. A convenient way to discuss the \( K^0 - \bar{K}^0 \) mixing problem is then to express \( K_S \) and \( K_L \) as eigenstates of the \( 2 \times 2 \) effective Hamiltonian. It can be shown that if \( CP \) holds then the restriction \( \mathcal{H}_{11} = \mathcal{H}_{22} \) and hence \( M_{11} = M_{22} \), \( \Gamma_{11} = \Gamma_{22} \) must be adopted. Furthermore if \( CP \) invariance holds too, then besides \( H_{11} = H_{22} \) we get also \( H_{12} = H_{21} \), and consequently \( \Gamma_{12} = \Gamma_{21} = \Gamma_{21}^* \), \( M_{12} = M_{21} = M_{21}^* \), so that \( \Gamma_{ij} \), \( M_{ij} \) will result all real numbers. The time evolution of the mass eigenstates

\[
\left( \begin{array}{c} |K_S(t)\rangle \\ |K_L(t)\rangle \end{array} \right) = V(t) \left( \begin{array}{c} |K_S\rangle \\ |K_L\rangle \end{array} \right) \] (12)
is governed by the matrix elements
\[ V_{\alpha\beta} = \langle K_\alpha | \exp \left[ -\frac{i}{\hbar} \mathcal{H} t \right] | K_\beta \rangle \]  
(13)

where greek letters denote \( K_S, K_L \). The evolution matrices \( U \) and \( V \) are then related by the following similarity transformation
\[ U = \mathcal{R} V \mathcal{R}^{-1} \]  
(14)

where the complex scaling \( \mathcal{R} \) in any CPT invariant theory can be found by means of the eigenvalues of the effective Hamiltonian \( \mathcal{H} \)
\[ \lambda_S = H_{11} - \sqrt{H_{12}H_{21}} = M_{11} - \frac{i}{2} \Gamma_{11} - Q = m_S - \frac{i}{2} \Gamma_S \]  
(15)
\[ \lambda_L = H_{11} + \sqrt{H_{12}H_{21}} = M_{11} - \frac{i}{2} \Gamma_{11} + Q = m_L - \frac{i}{2} \Gamma_L \]  
(16)

where
\[ Q = \sqrt{H_{12}H_{21}} = \sqrt{(M_{12} - \frac{i}{2} \Gamma_{12})(M_{12}^* - \frac{i}{2} \Gamma_{12}^*)}. \]  
(17)

These real \((m_{S,L})\) and imaginary \((\Gamma_{S,L})\) components will define the masses and the decay width of the \( \mathcal{H} \) eigenstates \( K_S e K_L \). These short- and long-lived particles result then a linear combination of \( K^0 \) and \( \bar{K}^0 \):
\[ \left( \begin{array}{c} |K_S\rangle \\ |K_L\rangle \end{array} \right) = \mathcal{R}^t \left( \begin{array}{c} |K^0\rangle \\ |\bar{K}^0\rangle \end{array} \right) \]  
(18)

where usually \( \mathcal{R}^t \) is parameterized according to the following relation
\[ \mathcal{R}^t = \begin{pmatrix} p & -q \\ p & q \end{pmatrix} = \frac{1}{\sqrt{1 + |\alpha|^2}} \begin{pmatrix} 1 & -\alpha \\ 1 & \alpha \end{pmatrix}, \]  
(19)

being \(|p|^2 + |q|^2 = 1\). With a proper phase choice, the mixing parameter
\[ \alpha = \frac{q}{p} = \sqrt{\frac{H_{21}}{H_{12}}} = \sqrt{\frac{M_{12}^* - \frac{i}{2} \Gamma_{12}^*}{M_{12} - \frac{i}{2} \Gamma_{12}}} \]  
(20)

manifests its presence in the non orthogonality between the oblique independent states \( K_L \) and \( K_S \):
\[ \langle K_S | K_L \rangle = \frac{1 - |\alpha|^2}{1 + |\alpha|^2}. \]  
(21)

The experimental evidence \( \# \) in 1964 that both the short-lived \( K_S \) and long-lived \( K_L \) states decayed to \( \pi \pi \) upset this tidy picture. It means that the states of definite mass and lifetime are never more states with a definite \( CP \) character. With the conventional choice of phase, the \( CP \) eigenstates \( K_1 \) and \( K_2 \) enter into play so that the mass eigenstates can be parameterized by means of an impurity complex parameter \( \epsilon \) which encodes the
indirect mixing effect of CP violation in the neutral kaon system. The general expression of this impurity parameter

\[ \epsilon = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{2}} \left( \frac{\text{Im}M_{12}}{\text{Re}M_{12}} - i \frac{\text{Im}\Gamma_{12}}{2 \text{Re}M_{12}} \right) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\Delta m}} \left( \text{Im}M_{12} + 2\xi_0\text{Re}M_{12} \right) \]  

(22)

where \( \xi_0 \) represents the additional manifestations of the CP violation due to the effective isospin decay amplitudes. Neglecting these decay effects, to derive the amount of this complex parameter, it appears more convenient to write

\[ \alpha = \frac{q}{p} = \frac{1 - \epsilon}{1 + \epsilon} \]  

(23)

and then

\[ \epsilon = \frac{p - q}{p + q} = \frac{1 - \alpha}{1 + \alpha} = \frac{\sqrt{H_{12} - \sqrt{H_{21}}}}{\sqrt{H_{12} + \sqrt{H_{21}}}} = \frac{2\text{Im}M_{12} + \text{Im}\Gamma_{12}}{(2\text{Re}M_{12} - i\text{Re}\Gamma_{12}) + (\Delta m - \frac{i}{2}\Delta\Gamma)} \approx \frac{i \text{Im}M_{12} + \text{Im}(\Gamma_{12}/2)}{(\Delta m - \frac{i}{2}\Delta\Gamma)} \]  

(24)

Thus it results evident that the CP-violation parameter \( \epsilon \) arises from a relative imaginary part between the off-diagonal elements \( M_{12} \) and \( \Gamma_{12} \) i.e. if \( \text{arg}(M_{12}^*\Gamma_{12}) \neq 0 \). Although the discovery of CP-violation [5] indicated that the kaon system is somewhat more complex than the typical oscillating two-state problem and involves considerably more subtle complications; nevertheless there are many attempts to illustrate this system in classical physics [7]. In this context, the origin of the space-time discrete symmetries and their violation represent one of the most controversial topic in the realm not only of physics.

For a long time, the fact that Maxwell equations were invariant under space-inversion or parity (\( P \)) and time-reversal (\( T \)) bolstered the idea that all the laws of physics are invariant under those discrete operations. It was easily seen that electromagnetic equations possess another discrete symmetry since they are unaffected by a charge conjugation (\( C \)) operation which reverses the sign of all the charges and converts a particle into its antiparticle. However, since 1964 [5], we know that CP is also violated (although to a much lesser extent than the amount of parity violation [6]) in weak interactions among fundamental particles. On the other hand, the origin of CP-violation is still not explained since CP-violating tiny effects are known to be smaller than the usual weak interaction strength by about three orders of magnitude and it is not excluded that CP-violation could be an indication of some effects of new physics at a much higher energy scale. The only almost overwhelming theoretical prejudice comes against CPT violation. There are very strong reasons [8] to believe that fundamental interactions can never violate CPT invariance. Thus, except some loosing theoretical models, the validity of CPT is assumed and consequently the \( T \) violation is supposed to be reflected immediately in a CP counterpart. However, it should be borne in mind that observation of a \( T \) odd asymmetry or correlation is not necessarily an indication of CP (or even \( T \)) violation. The reason for this is the anti-unitary nature of the time reversal operator in quantum mechanics. As a consequence of this, a \( T \) operation not only reverses spin and three-momenta of all particles, but also interchanges initial and final states. Put differently, this means that final-state interactions can mimic \( T \) violation, but not genuine CP violation. Presently, genuine experimental evidence of CP-violation comes from the mixing and decays of the neutral kaon system. In fact, the intrinsic dissipative nature of this unstable system and
its decay sector, faced with the problem of the complex eigenvalue of the Hamiltonian and therefore with the extension of the Hilbert space which is a common tool to deal with open systems far from equilibrium.

In general, we can resume that there are three complementary ways to describe the evolution of the complex neutral kaon system:

1) In terms of the mass eigenstates $K_{L,S}$, which do not possess definite strangeness

$$|K_S(t)\rangle = |K_S(0)\rangle e^{-i\lambda_S t} \quad i\lambda_S = im_S + \frac{\Gamma_S}{2} \quad (25)$$

$$|K_L(t)\rangle = |K_L(0)\rangle e^{-i\lambda_L t} \quad i\lambda_L = im_L + \frac{\Gamma_L}{2} . \quad (25)$$

2) In terms of the flavour eigenstates, whose time evolution is more complex

$$|K^0(t)\rangle = f_+(t)|K^0(0)\rangle + \frac{1}{\alpha} f_-(t)|\bar{K}^0(0)\rangle \quad (26)$$

$$|\bar{K}^0(t)\rangle = \alpha f_-(t)|K^0(0)\rangle + f_+(t)|\bar{K}^0(0)\rangle$$

with

$$f_{\pm}(t) = \frac{1}{2}(e^{-i\lambda_L t} \pm e^{-i\lambda_S t}) = \frac{1}{2}e^{-i\lambda t} \left[ 1 \pm e^{-i(\lambda_S - \lambda_L)t} \right] . \quad (27)$$

3) In terms of the $CP$–eigenstates $K_1$ and $K_2$

$$|K^0_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle) \quad \text{CP}|K^0_1\rangle = + |K^0_1\rangle \quad (28)$$

$$|K^0_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle) \quad \text{CP}|K^0_2\rangle = - |K^0_2\rangle$$

which we let us express the mass eigenstates as

$$|K^0_S\rangle = \frac{1}{\sqrt{2}} \left[ (p + q)|K^0_1\rangle + (p - q)|K^0_2\rangle \right]$$

$$|K^0_L\rangle = \frac{1}{\sqrt{2}} \left[ (p - q)|K^0_1\rangle + (p + q)|K^0_2\rangle \right] . \quad (29)$$

The three bases $\{K_S, K_L\}, \{K^0, \bar{K}^0\}$ and $\{K_1, K_2\}$, are completely equivalent.

### 2 The Dissipative Effects in Coupled Oscillations.

In classical physics, an analogue of the two-state mixing problem which leads to a non-zero value of $\epsilon$ seems to strain its purport more than it is necessary, since the equations of motion in classical mechanics are time-reversal invariant. The main features of irreversibility enter only considering the effects of dissipation. Anyway, these results seem to reflect the well known requirements of additional complementarity relations, which occur at classical
level, to make the equations of motion of dissipative systems derivable from a variational principle ⁹. We can recover the theory for the non conservative systems in classical physics in terms of the standard formalism of analytical mechanics. In the case of \( n \) independent generalized coordinates \( q_k \), the equations of motions are given by

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j
\]

(30)

where the generalized momenta are of the form

\[
Q_j = \sum_{i=1}^{n} A_{ij} q_i + B_{ij} \frac{dq_i}{dt}.
\]

(31)

In the special case of conservative system, a configuration of stable equilibrium is specified by the following Lagrangian equations

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} + \frac{\partial U}{\partial q_k} = 0 \quad \text{or} \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \quad L = T - U.
\]

(32)

As usual, the kinetic tensor

\[
T = \frac{1}{2} \sum_{j,k} T_{jk} \dot{q}_j \dot{q}_k
\]

(33)

and the potential function

\[
U = \frac{1}{2} \sum_{j,k} U_{jk} q_j q_k
\]

(34)

are supposed bilinear without any loss of generality. Of course, the exact form of \((T_{ij})\) and \((U_{ij})\) specifies the way in which the motions of the various coordinates are coupled. The problem of determining the normal solutions reduces to that of simultaneous diagonalization of these two real symmetric matrices associated with each of the kinetic and potential energies. The problem consists practically in finding a coordinate transformation which simultaneously diagonalizes both \((T_{ij})\) and \((U_{ij})\). This is equivalent to the following set of second order homogeneous differential equations

\[
\sum_{j} (U_{jk} q_j + T_{jk} \ddot{q}_j) = 0.
\]

(35)

This underlying mathematical problem can be formulated elegantly in matrix language since it can be compactly associated with the following Lagrangian

\[
L = \mathcal{T} - U = \frac{1}{2} \left[ q' \mathbf{T} \dot{q} - q' \mathbf{U} q \right].
\]

(36)

Then the problem to determine the normal modes reduces to an eigenvalue problem involving the simultaneous diagonalization of two real symmetric matrices, classically associated with the kinetic and potential energies. The recipe for carrying out this diagonalization is straightforward. The kinetic and potential energy matrices are simultaneously diagonalized by means of a sequence of three transformations from the original to the normal coordinates \( \xi \)

\[
q = R_1 q' = R_1 S q'' = R_1 S R_2 \xi.
\]

(37)
In this new normal coordinates $\xi$, the equations of motion are uncoupled. The net transformation matrix $G = R_1 S R_2$ is not orthogonal because $S$ is not orthogonal. This matrix $G$ represents the most general (homogeneous) linear transformation which includes two rotation matrices $R_1, R_2$ and a contraction $S$. We have to sacrifice the property of orthogonality in order to construct a matrix that can simultaneously diagonalize both $T$ and $U$. The effect of this transformation is to diagonalize the kinetic and potential energy matrices:

$$\text{diag } T = G^{t} T G$$

$$\text{diag } U = G^{t} U G .$$

To be more explicit, in the case of normal solution of the form $q_j = a_j e^{i(\omega t - \delta)}$ we obtain $n$ linear homogeneous algebraic equations:

$$\sum_j (U_{ij} q_j - \omega^2 T_{ij}) a_j = 0 ,$$

where the common phase factor $e^{i(\omega t - \delta)}$ has been cancelled. A non trivial solution corresponds to the secular equation

$$\text{det}(U_{ij} - \omega^2 T_{ij}) = 0 .$$

In general, this $n$-order spectral equation in the eigenfrequencies $\omega^2$ cannot be solved simply. Nevertheless, in the case of dissipative systems the coupled differential equations that describe the damped motion can be obtained only by means of the use of the generalized momenta in Eq. (31). In general we must consider three matrices $T_{ij}$, $A_{ij}$ and $B_{ij}$. A normal mode solution of the kind $q_j = Q_j e^{-i\omega t}$ can be derived from the secular equation

$$\text{det}(T_{ij} \omega^2 + A_{ij} + i\omega B_{ij}) = 0 .$$

Using the modern Dirac notation, although we are discussing classical quantities, its physical solution derives from the characteristic equation

$$\mathcal{H}|q> = \omega^2 |q> ,$$

being

$$\mathcal{H} = i\omega B - A .$$

Thus the equations of the motion can be rewritten as

$$T \frac{d^2|q>}{dt^2} = A|q> - iB \frac{d|q>}{dt} .$$

This set of equations resembles the case of a system of coupled damped harmonic oscillators [10]. The dissipative effects are parameterized by the presence of small off-diagonal terms in $\mathcal{H}$, as it was carefully illustrated in a recent publication [1] in the classical framework of mechanics and by means of electromagnetic coupled circuits. The time-reversed not-invariant processes can be induced by dissipative effects which yield a not vanishing imaginary part for the relevant Hamiltonian.
3 Electromagnetic Constitutive Relations.

The description of the fundamental nature of the electromagnetic interactions induced in a medium is obscured by several mathematical problems \[11\]. These difficulties are simply an expression of our ignorance about the dynamics of all processes going on in the medium or our inability to quantify them in detail. From a purely macroscopic point of view, the crucial question is to append certain constitutive relations in dependence with an effective parameterization of the induced charges and current densities and feeding them into the Maxwell’s equations. The presence of the induced sources is taken into account by introducing a few empirical constants in the equations. The number of such constants necessary depends on the nature of the medium. For the simpler linear isotropic materials, it is well known that one needs two scalar quantities: the dielectric constant \(\varepsilon\) and the magnetic permeability \(\mu\), to describe all phenomena of macroscopic electrodynamics, according to the relations

\[
D \equiv \varepsilon_0 E + P = \varepsilon E , \quad H = \frac{B}{\mu_0} - M = \mu^{-1}B .
\]  

Once the empirical constants \(\varepsilon\) and \(\mu\) are known, one can then derive the electric \(E\) and magnetic \(B\) vector fields. However, for complex systems one needs in general more than these two constants to describe electrodynamical phenomena in them. In many instances this inadequacy may be addressed by postulating a more general set of constitutive relations to replace those in Eq. (45):

\[
D = \varepsilon E + \beta B , \quad H = \mu^{-1}B + \gamma E .
\]

We should comment that the restriction of the number of electromagnetic constants depends on the nature of the medium in question, since in the most general case they are rank-2 tensors \[11\]. While the approach of postulating these general constitutive relations are well known \[11\], until now, it is not deeply discussed in connection with the symmetry requirements which could be of great help in analyzing the dynamics of propagation. The symmetry requirements under spatial translation or under rotations are well-documented experimentally, and no physicist has serious doubts about their validity as a consequence of the conservation of momentum and angular momentum. Invariance under spatial inversion and time reflection, on the other hand, seems to hold only approximately, for some classes of phenomena. The point is that spatial inversion symmetry, which is associated with parity (\(P\)) invariance, is preserved by strong and electromagnetic interactions but violated by the weak force \[6\]. Since the latter interaction is at least six orders of magnitude weaker than its strong and electromagnetic counterpart, parity violation can generally be neglected. Similar comments apply to the case of time reversal (\(T\)) symmetry, which is found \[5\] to be violated only by an effect even more feeble than the weak force. Maxwell’s equations have another discrete symmetry since they are unaffected by the operation under which \(\rho, j, E\) and \(B\) all change sign. This operation, called charge conjugation (\(C\)) because the sign of all charges are reversed, plays an important role in quantum field theory, where it converts a particle into its associated antiparticle. The transformation properties of the electromagnetic fields, potentials and currents, with respect to the discrete symmetries \(C, P\) and \(T\), and some important combinations like \(CP\) and \(CPT\) (Table \[1\]), can be better understood to consider the Fourier-space version of the Maxwell’s equations.

In fact in many field of physics, including electrodynamics, it is convenient to re-write the (Maxwell) differential equations of motion in terms of their Fourier components. In
Table I: The transformation properties of the electromagnetic currents, fields and potentials, under the discrete symmetries $P$, $T$ and $C$, and some important combinations like $CP$ and $CPT$. If for a generic $\tilde{\Phi}$ of these quantities, including $\omega$ and $k$ themselves, we write the above transformation rules in the form $\tilde{\Phi} \xrightarrow{X} \eta_X \tilde{\Phi}$, then the value of the multiplicative constants $\eta_X$ is given for the various transformations. The same procedure is followed for the pulsation and wave vector that transforms, for example, under the time reversal symmetry, according to $(\omega, k) \xrightarrow{T} (\omega, k)$.

| Transformation | Quantity | $\eta_P$ | $\eta_T$ | $\eta_C$ | $\eta_{CP}$ | $\eta_{CPT}$ |
|---------------|----------|-----------|-----------|-----------|-----------|-----------|
| $\omega$      | +        | -         | +         | +         | -         |           |
| $k$           | -        | +         | +         | -         | -         |           |
| $\rho$        | +        | +         | -         | -         | -         |           |
| $j$           | -        | -         | -         | +         | -         |           |
| $E$           | -        | +         | -         | +         | +         |           |
| $B$           | +        | -         | -         | -         | +         |           |
| $A$           | -        | -         | -         | +         | -         |           |

This linear approximation, from the usual results of Fourier analysis, we express a general solution as a superposition of plane waves through the following relation

$$\mathbf{E}(t, \mathbf{x}) = \frac{1}{(2\pi)^2} \int d\omega \int d^3k \tilde{\mathbf{E}}(\omega, \mathbf{k}) \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \tag{48}$$

and similarly for other quantities. Now, if the Fourier transform of a linear equation is taken, the space and time drop out. By these means, the set of coupled (Maxwell) differential equations are replaced by a set of coupled algebraic equations, which of course greatly reduces the difficulty of the problem. The solutions of the algebraic equations are the Fourier transforms of the required quantities which can then be obtained by Fourier inversion. The Fourier momentum space version of the Maxwell’s equations in a medium is then given by

$$\mathbf{k} \cdot \tilde{\mathbf{B}} = 0 \quad , \quad \mathbf{k} \times \tilde{\mathbf{E}} = \omega \tilde{\mathbf{B}} \tag{49}$$

$$i\mathbf{k} \cdot \tilde{\mathbf{D}} = \tilde{\rho}_{\text{ext}} \quad , \quad i\mathbf{k} \times \tilde{\mathbf{B}} = \mu_0 \tilde{\mathbf{J}} - i\frac{\omega}{c^2} \tilde{\mathbf{E}} \tag{50}$$

with source terms just the external charges and currents. In writing these relations it is assumed that the response of the system to the applied electromagnetic fields is linear. Even when non linear effects are considered, they could be usually treated as corrections to linear ones. In this approach, it is assumed that all quantities which vary with time and space differ only slightly from an average value. The source Maxwell’s equations reduce to the wave form

$$\frac{\omega^2}{c^2} \tilde{\mathbf{E}} + \mathbf{k} \times \left( \mathbf{k} \times \tilde{\mathbf{E}} \right) = -i\omega\mu_0 \tilde{\mathbf{J}} \tag{51}$$

Eliminating the induced part $\tilde{\mathbf{J}}_{\text{ind}}$ of the current $\tilde{\mathbf{J}} = \tilde{\mathbf{J}}_{\text{ind}} + \tilde{\mathbf{J}}_{\text{ext}}$ in terms of general conductivity tensor $\tilde{\sigma}_{ij}$
\[
\left( \mathbf{J}_{\text{ind}} \right)_i = \sigma_{ij} \tilde{\mathbf{E}}_j
\]
then, Eq.(51) reduces to the inhomogeneous wave equation

\[
\tilde{\Lambda}_{ij} \tilde{\mathbf{E}}_j = -\frac{\mu_0 c^2}{\omega} \left( \mathbf{J}_{\text{ext}} \right)_i,
\]
with

\[
\tilde{\Lambda}_{ij} (\omega, \mathbf{k}) = \frac{k^2 c^2}{\omega^2} (k_i k_j - \delta_{ij}) + \tilde{K}_{ij} (\omega, \mathbf{k})
\]
being the equivalent response tensor

\[
\tilde{K}_{ij} (\omega, \mathbf{k}) = \delta_{ij} + \frac{i}{\omega\epsilon_0} \sigma_{ij} (\omega, \mathbf{k})
\]
If the medium is dissipative, the tensor \(\tilde{K}_{ij}\) may be separated into a hermitian and an antihermitian part

\[
\tilde{K}_{ij} = \tilde{K}_{ij}^H + \tilde{K}_{ij}^A
\]
with

\[
\tilde{K}_{ij}^H = \frac{1}{2} \left( \tilde{K}_{ij} + \tilde{K}_{ij}^* \right)
\]
\[
\tilde{K}_{ij}^A = \frac{1}{2} \left( \tilde{K}_{ij} - \tilde{K}_{ij}^* \right)
\]
which corresponds to a separation into a dissipative and non-dissipative part. The Kramers-Kronig relations assure that the hermitian and antihermitian parts are not independent. In fact,

\[
\tilde{K}_{ij}^A = \frac{i}{\pi} \mathcal{P} \int_{-\infty}^{+\infty} \frac{\tilde{K}_{ij}^H - \delta_{ij}}{\omega - \omega'} d\omega'
\]
\[
\tilde{K}_{ij}^H - \delta_{ij} = \frac{i}{\pi} \mathcal{P} \int_{-\infty}^{+\infty} \frac{\tilde{K}_{ij}^A}{\omega - \omega'} d\omega'.
\]
For an isotropic medium, that is only spatially dispersive, we have three independent (transverse, longitudinal and rotatory) parts in \(\tilde{K}_{ij}^H\)

\[
\tilde{K}_{ij}^H = \varepsilon^T (\delta_{ij} - k_i k_j) + \varepsilon^L k_i k_j + \varepsilon^R \epsilon_{ijr} k_r = \begin{pmatrix}
\varepsilon^T & \varepsilon^R & 0 \\
-\varepsilon^R & \varepsilon^T & 0 \\
0 & 0 & \varepsilon^L
\end{pmatrix}
\]
where the last matrix form holds if we choose \(\mathbf{k}\) along the \(z\)-axis. The fields \(\tilde{\mathbf{E}}\) and \(\tilde{\mathbf{B}}\) depend on \((\omega, \mathbf{k})\), whereas the quantities \(\varepsilon^T\), \(\varepsilon^L\) and \(\varepsilon^R\), which represent the properties of the medium, are supposed to depend on \(\mathbf{k}\) only through its magnitude \(k \equiv |\mathbf{k}|\). In fact we are considering a medium in which only the amplitudes of the electric and magnetic polarizations depend on direction. The electromagnetic constants \(\varepsilon^T\), \(\varepsilon^L\) and \(\varepsilon^R\) have no anisotropy and are the same in all the directions. Before we proceed any further, we
want to note that there is one relation that \( \varepsilon^T, \varepsilon^L \) and \( \varepsilon^R \) must satisfy based on very general grounds. It is simply the requirement that the fields \( \mathbf{E}, \mathbf{B} \) and the current \( j \) are, in coordinate space, real quantities. This yields that, for real values of \( \omega \), the electromagnetic constants satisfy

\[
 f^*(-\omega, \mathbf{k}) = f(\omega, \mathbf{k}),
\]

where \( f \) stands for any of the electromagnetic constants \( \varepsilon^T, \varepsilon^L \) and \( \varepsilon^R \) and the dependence is supposed through its magnitude \( k \), as noted earlier. All that means that, the real parts of these electromagnetic constants are even in \( \omega \) whereas the imaginary parts are odd. We stress that this is a very general property, not tied up in any way with the discrete \( C, P \) and \( T \) symmetries of space-time. Of course, for the validity of causality and the consequent Kramers-Kronig dispersion relations, \( \omega \) can in general be allowed to be complex and this feature must be included in the analysis.

The implication of \( C, P \) and \( T \) symmetries on the electromagnetic constants \( f \) imposes that the reversed equations result invariant under the \( C, P \) and \( T \) constraints if

\[
 C \text{ invariance } \Rightarrow \text{ no constraints on } \varepsilon^T, \varepsilon^L, \text{ and } \varepsilon^R,
\]

\[
 P \text{ invariance } \Rightarrow \varepsilon^R = 0,
\]

\[
 T \text{ invariance } \Rightarrow \begin{cases} 
 \varepsilon^T(-\omega, \mathbf{k}) = \varepsilon^T(\omega, \mathbf{k}) \\
 \varepsilon^L(-\omega, \mathbf{k}) = \varepsilon^L(\omega, \mathbf{k}) \\
 \varepsilon^R(-\omega, \mathbf{k}) = -\varepsilon^R(\omega, \mathbf{k})
\end{cases},
\]

and being in general \( f^*(-\omega, \mathbf{k}) = f(\omega, \mathbf{k}) \), thus implies that \( \varepsilon^R \) must be purely imaginary and an odd function of \( \omega \), whereas \( \varepsilon^T \) and \( \varepsilon^L \) should be real and even function of \( \omega \). In the case of a dissipative media, the response tensor components will respect the Onsanger relations

\[
 K_{ij}^H(-\omega, \mathbf{k}) = K_{ij}^H(\omega, \mathbf{k}) \\
 K_{ij}^A(-\omega, \mathbf{k}) = -K_{ij}^A(\omega, \mathbf{k})
\]

whereas in general \( K_{ij}(\omega, -\mathbf{k}) = K_{ij}(\omega, \mathbf{k}) \).

\( CP \) invariance obviously gives the same constraint as parity alone does, since charge conjugation does not affect.

\[
 \text{CPT invariance } \Rightarrow \begin{cases} 
 \varepsilon^T(-\omega, \mathbf{k}) = \varepsilon^T(\omega, \mathbf{k}) \\
 \varepsilon^L(-\omega, \mathbf{k}) = \varepsilon^L(\omega, \mathbf{k}) \\
 \varepsilon^R(-\omega, \mathbf{k}) = -\varepsilon^R(\omega, \mathbf{k})
\end{cases}.
\]

From these relations, it is quite clear that the presence of the \( \varepsilon^R \)-term implies some properties that are asymmetric under \( P \) and \( CP \) transformations. Similarly, since \( T \) and \( CPT \) imply mutually inconsistent constraints on \( \varepsilon^R \), at least one of them must be violated.
in order for $\varepsilon^R$ to exist. $\varepsilon^R$ might be expected to vanish if all interactions conserve $C$, $P$ and $T$. However since 1957 \[8\] we know that parity is violated by weak interactions and since 1964 \[5\] we know that $CP$ is also violated although to a much lesser extent. On the other hand the induced currents arise because of complicated processes taking place within the medium, including (super) weak interactions, so that there is no reason why the $\varepsilon^R$-term should not be present. Of course since the $\varepsilon^R$-term (if present) violates $CP$ it would seem to be extremely tiny. The only constraint on the value of $\varepsilon^R$ seems $\varepsilon^R(\omega,k) = \varepsilon^R(-\omega,k)$ which follows from $CPT$ symmetry, since there are strong reasons to believe that fundamental interactions between particles can never violate $CPT$. But in the presence of matter, the $CPT$ breaking effects could be relevant due to the effects of the medium in the underlying effective theory. So that, $CPT$ asymmetries could be large and their invariant constraints become irrelevant. As a result, parity violation in the weak interactions of the medium can give rise to a non vanishing value of $\varepsilon^R$ and similarly a much smaller $T$-violating effects should be taken also seriously into account. As we found before in Eq. (65), if $T$-invariance holds, $\varepsilon^R$ must be purely imaginary and an odd function of $\omega$. Then, it is worth noting to the extent that $T$-violation can be neglected in the fundamental interactions, it is conceivable to think a medium where $\varepsilon^R$ and $\varepsilon^L$ are constants, (being both even functions of $\omega$), but $\varepsilon^R$ cannot have a non zero constant term. In this case a medium becomes dispersive according only to $\varepsilon^R$. When we consider more complicated media, it is possible to obtain $P$-asymmetries from the medium itself. Therefore it is important to consider the passage of electromagnetic radiation through a generic medium with a built-in asymmetry. This is the case, for example, of natural sugar solution, where one helicity of the radiation is in excess over the other, and physically, it manifests itself as the phenomenon of optical activity.

4 The Optical Analogue of the $K^0 - \bar{K^0}$ Complex System.

We seek in electromagnetism an analogue of the two-state mixing problem which leads to a non-zero value of an optical parameter which could be associated to the impurity parameter $\epsilon$. The problem is quite difficult since the Maxwell’s equations are time-reversal invariant. The main features of irreversibility enter only considering the effects of anisotropy and dissipation. In an isotropic medium, transverse electromagnetic waves, have two degenerate states of polarization, and in anisotropic media, the natural modes are non degenerate and not necessarily transverse. Radiation propagating through such a medium splits into components in the two natural modes and the difference between the complex refractive indices of the two modes produces radiation with interfering effects when the components are recombined. In the case of optically active media, the dynamic behavior is connected to dissipation more than anisotropy, as far as their electromagnetic properties are concerned. We review here the fundamental mathematical description of the propagation of optical polarization in a medium. In a complex dielectric material, the nature of the wave propagation is learned, in general, from the dispersion relations. These are the relations that $\omega$ and $k$ must satisfy in order to solve the homogeneous wave equation for the $E$ and $B$ fields:

$$\tilde{\Lambda}_{ij} \tilde{E}_j = 0 \quad (68)$$

In a general complex medium, the existence of a solution is connected to the dispersion relation.
When this dispersion relation is satisfied, there exists a solution of the homogeneous wave equation. In the vacuum, this vector solution $\vec{E}$ is perpendicular to the direction of propagation of the electromagnetic wave. Thus, one can discuss this solution (with complex coefficients)

$$\vec{E} = \vec{E}_1 \epsilon_1 + \vec{E}_2 \epsilon_2$$

in terms of the components

$$\vec{E}_1 = \vec{E}_x = \text{Re} \left( \vec{E} \cdot \epsilon_1 \right)$$

$$\vec{E}_2 = \vec{E}_y = \text{Re} \left( \vec{E} \cdot \epsilon_2 \right)$$

of the electric field transverse to the coherent beam propagating in the $\hat{k} = \hat{z}$ direction. In this description of polarization, we refer to two basic polarization orthonormal unit vectors $\epsilon_1$ and $\epsilon_2$, both being perpendicular to the wave unit vector $\hat{k}$. An equivalent set of basic vectors is given by the following right and the left circular basis

$$\epsilon_+ = \frac{1}{\sqrt{2}} (\epsilon_1 + i \epsilon_2)$$

$$\epsilon_- = \frac{1}{\sqrt{2}} (\epsilon_1 - i \epsilon_2)$$

Since $\epsilon_2 = \hat{k} \times \epsilon_1$, these vectors are of the form $\epsilon_1 \pm i \hat{k} \times \epsilon_1$. In this language, the effect of parity transformation is to reverse the sign of the transverse polarization vector $\epsilon_2$ in a suitable gauge. Whereas, under parity, the circular polarization vectors interchange: $\epsilon_+ \leftrightarrow \epsilon_-$. In non dispersive media ($\varepsilon^R = 0$), the two transverse modes have the same physical characteristics, with the same speed of propagation $c/\sqrt{\varepsilon \mu}$ and, if parity is a good symmetry, there would be also no difference in the physical properties of a right-circular and left-circular polarization state of the transverse electromagnetic wave. Wherever $\varepsilon^R = 0$, in an isotropic homogeneous medium, the response tensor is given by

$$\tilde{K}_{ij} = \varepsilon^L \hat{k}_i \hat{k}_j + \varepsilon^T \left( \delta_{ij} - \hat{k}_i \hat{k}_j \right)$$

In this case, the dispersion relation

$$\det \tilde{\Lambda}_{ij} = \det \left[ \varepsilon^L \hat{k}_i \hat{k}_j + \left( \varepsilon^T - n^2 \right) \left( \hat{k}_i \hat{k}_j - \delta_{ij} \right) \right] = \varepsilon^L \left( \varepsilon^T - n^2 \right)^2 = 0$$

yields a longitudinal dispersion relation

$$\varepsilon^L (\omega, k) = 0$$

and a transverse dispersive equation

$$n^2 = \varepsilon^T (\omega, k) = \frac{\varepsilon (\omega) \mu (\omega)}{\varepsilon_0 \mu_0} = c^2 \varepsilon (\omega) \mu (\omega)$$

It is worth noting that only in an isotropic medium, all solutions are polarized strictly transverse. However, in an isotropic optically active media ($\varepsilon^R \neq 0$), parity, and, in a less
extent time-reversal are broken. If also dissipative effects are considered, the two transverse, (and circular), birefringent (or dichroic) modes have different propagating properties. For example, they travel with different speeds through matter and have different indices of refraction and coefficients of absorption. It is evident that the two transverse modes have different properties when \( \varepsilon^{R} \) does not vanish. The effect of dissipation in an isotropic medium will cause a non trivial complex form of \( \varepsilon^{R}(\omega, k) \). In fact, if time-reversed invariance is satisfied, \( \varepsilon^{T} \) and \( \varepsilon^{L} \) are real, whereas \( \varepsilon^{R} \) is purely imaginary only neglecting dissipation. In this case, for a large range of \( \varepsilon^{R} \), the dispersion relation is solved for a real \( \omega \) and \( k \). Thus, the amplitude of the wave will remain unchanged during the propagation, which means that there is no absorption in the medium. In the more general case of dissipation, time-reversal symmetry can be violated and the space-time evolution of the transverse electric field becomes governed by a set of coupled second order differential equations

\[
\vec{E}_i = \Lambda_{ij} \vec{E}_j .
\]

The two transverse modes no longer have the same dispersion relation. The transverse components are governed by the following inverse matrix propagator

\[
\Lambda_{ij}^{\perp} = -\frac{k^2 c^2}{\omega^2} \delta_{ij} + (\delta_{ij} + \chi_{ij})
\]

which is defined for the transverse \( \vec{E} \) components Eq. (70) with respect to \( \hat{k} \). However, the degree and type of the polarization of this resulting transverse electric wave is associated with a unit vector and therefore with a point on the unit sphere \([12]\). The stereographic projection of this unit vector onto the equatorial plane can be consequently associated to the ratio of the complex components of a Poincaré spinor \([13]\). Nothing else that the usual isomorphism between the rotation group and its covering group SU(2). The two-components Poincaré spinor requires the standard matrix representation in terms of the Pauli matrices and it is associated to a vector representing the angular momentum for a spin-\( \frac{1}{2} \) state. In fact, this vector results just the generator of the internal rotations and it is called Jones vector \([14]\), when it is considered in the Cartesian basis. Its components give the Cartesian components of the complex polarized electric field. This \( 2 \times 2 \) unimodular matrix representation of the Pauli algebra to describe the polarization, can be replaced by the \( 4 \times 4 \) Mueller matrix method \([15]\). Another modern method using \( 2 \times 2 \) matrices to represent polarization is represented by Fano’s density operator approach \([16]\) analogously to the case of spin density theory for the spin-\( \frac{1}{2} \) particles. It is worth noting that while the fundamental mathematical description of spin \( \frac{1}{2} \) and optical polarization are practically the same, their physical interpretations are quite different. In the description of spin \( \frac{1}{2} \) we have a spin vector whose direction in a cartesian 3-dimensional space depends upon a state vector in a complex 2-dimensional (spin-up, spin-down) spinor space. Anyway, in the description of polarization the two ordinary components \( x \) and \( y \) play the same role which the spinor bases played while describing spin \( \frac{1}{2} \). Although the problem to describe the evolution of the two transverse polarization states of the electromagnetic radiation propagating into a complex medium can be described in these several manners, nevertheless, a dispersion relation governing the propagation is required.

In our case of a wave propagating inside a medium with complex \( \varepsilon^{R} \neq 0 \), the space propagation of the transverse optical polarization states is governed by the transverse response tensor
\[ K^\perp_{ij} = \delta_{ij} + \chi_{ij} = D^2 \]  
which is given in terms of a general non Hermitian susceptibility tensor

\[ \chi = \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{11} \end{pmatrix}. \]  

Consider a plane polarized wave incident on a medium at \( z = 0 \), moving in the positive \( z \)-direction, the transverse electric components can be assumed in a normal form of the type \( \vec{E}(z,t) = \vec{E}(z)e^{-i\omega t} \) where

\[
\begin{pmatrix} E_1(z) \\ E_2(z) \end{pmatrix} = \exp[-izD] \begin{pmatrix} E_1(0) \\ E_2(0) \end{pmatrix}
\]

where

\[
D = \frac{\omega \sqrt{\mu \epsilon}}{c} \begin{pmatrix} 1 + \chi_{11} & \chi_{12} \\ \chi_{21} & 1 + \chi_{11} \end{pmatrix}^{\frac{1}{2}} = \frac{\omega \sqrt{\mu \epsilon}}{c(\sqrt{\lambda_1} + \sqrt{\lambda_2})} \begin{pmatrix} 1 + \chi_{11} + \sqrt{\lambda_1 \lambda_2} & \chi_{12} \\ \chi_{21} & 1 + \chi_{11} + \sqrt{\lambda_1 \lambda_2} \end{pmatrix}
\]

In the matrix square-root we use the eigenvalues of \( D^2 \)

\[
\lambda_{1,2} = \frac{1}{2}(\Sigma \pm \Delta)
\]

with

\[
\Sigma = \text{tr}D^2 = 2 + \text{tr}\chi = 2(1 + \chi_{11})
\]

\[
\Delta^2 = \text{tr}^2D^2 - 4\det D^2 = \text{tr}^2\chi - 4\det \chi = 4\chi_{12}\chi_{21}
\]

Therefore, the electric field will propagate as

\[
\begin{align*}
E_+(z) &= E_+(0)e^{-i\lambda_+z} \\
E_-(z) &= E_-(0)e^{-i\lambda_-z}
\end{align*}
\]

where the \( D \) eigenvalues \( \lambda_+ \) and \( \lambda_- \) are given by

\[
\lambda_{\pm} = \lambda_{1,2} \pm \sqrt{\lambda_1 \lambda_2}
\]

Here, \( E_+ \) and \( E_- \) are given by

\[
\begin{align*}
E_+ &= \frac{1}{\sqrt{1 + |\alpha|^2}} (E_1 + \alpha E_2) \\
E_- &= \frac{1}{\sqrt{1 + |\alpha|^2}} (E_1 - \alpha E_2)
\end{align*}
\]

We recognize that the parameter \( \alpha \) could be put in relation with the off-diagonal terms of the propagation matrix \( D \) by the following equation

\[
\alpha = \sqrt{\frac{D_{21}}{D_{12}}} = \sqrt{\frac{K^\perp_{21}}{K^\perp_{12}}} = \sqrt{\frac{K^H_{12} - K^A_{12}}{K^H_{12} + K^A_{12}}} = \sqrt{\frac{\chi_{21}}{\chi_{12}}}
\]

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It can be viewed that it depends upon the off-diagonal terms of the susceptibility tensor $\chi$. In a non-dissipative medium $K_{ij} = 0$ and $|\alpha| = 1$. In particular, in an isotropic medium which is only dispersive, we have: $\alpha = i$. The analogy with the neutral kaon system is quite evident; the eigenstates the space propagation resemble the mass eigenstates of the effective Hamiltonian $\mathcal{H}$ governing the evolution of $K_L$ and $K_S$.

In summary, it was presented a successful attempt to realize the optical analogue which reproduces the peculiar asymmetric effects of the $CP$–violating propagation. The absorptive and dispersive contributions in isotropic media can then resemble the intriguing features of the kaon complex evolution in the realm of high energy particle physics. A last remark merits the eventuality to introduce an absorptive effect by means of the space anisotropy whose consequent longitudinal propagation, however, seems outside an abelian and strictly massless effective theory of the electromagnetic interactions.

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