Transverse-Momentum Dependent Factorization for $\gamma^*\pi^0 \to \gamma$

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Abstract

With a consistent definition of transverse-momentum dependent (TMD) light-cone wave function, we show that the amplitude for the process $\gamma^*\pi^0 \to \gamma$ can be factorized when the virtuality of the initial photon is large. In contrast to the collinear factorization in which the amplitude is factorized as a convolution of the standard light-cone wave function and a hard part, the TMD factorization yields a convolution of a TMD light-cone wave function, a soft factor and a hard part. We explicitly show that the TMD factorization holds at one loop level. It is expected that the factorization holds beyond one-loop level because the cancelation of soft divergences is on a diagram-by-diagram basis. We also show that the TMD factorization helps to resum large logarithms of type $\ln^2 x$. 
1. Introduction

Factorization is an important tool to make predictions from QCD for an exclusive process if it contains both long- and short-distance effects, the latter involving one or more large momentum transfers, denoted generically as $Q$. It has been proposed long time ago that the amplitude of such a process can be factorized by an expansion of the amplitude in $1/Q$, corresponding to an expansion of QCD operators in twist[1, 2]. The leading term can be factorized as a convolution of a hard part and light-cone wave functions of hadrons. The light-cone wave functions are defined with QCD operators, and the hard part describes hard scattering of partons at short distances. In this factorization the transverse momenta of partons in parent hadrons are also expanded in the hard scattering part. At leading twist they are neglected. This results in that the light-cone wave functions depend only on the longitudinal momentum fractions carried by partons, not on transverse momenta of partons. This is the so-called collinear factorization.

The collinear factorization has been also used for inclusive processes. At leading twist the transverse momenta of partons are also neglected, and the resulting standard parton distributions only depend on longitudinal momentum fractions. It is interesting to note that a transverse-momentum dependent(TMD) factorization can be made by introducing TMD parton distributions[3, 4, 5]. With a consistent definition, TMD parton distributions have some interesting relations to the standard Feynman distributions. For example, non-light-like gauge links shall be used to avoid some inconsistencies as discussed in the detail in [6]. The TMD factorization has many advantages. Beside those for different aspects of relevant physics, a prominent advantage is that it leads to the so-called CSS resummation formalism for resumming large logarithms appearing in the collinear factorization[3, 4, 5]. This implies that the two factorization approaches are closely related to each other.

In the collinear factorization for an exclusive process one usually has large contributions from end-point regions where the longitudinal momentum fraction $x$ is small or close to 1, signified by large logarithms $\ln^2 x$ in the perturbative expansion of the hard part. As such the expansion converges slowly and predictions can not be reliably made by taking few leading terms in the expansion only. It has been shown that those large logarithms can be resummed by using TMD- or $k_T$- factorization[7, 8]. Similarly to inclusive cases, one introduces TMD light-cone wave functions (LCWF) by using non-light-like gauge links. Recently, such a factorization has attracted many attentions related to studies of exclusive $B$-meson decays. For these decays, one can use collinear factorization to factorize short- and long-distance effects [9]. Beside large logarithms, there are even end-point singularities in the factorization of certain decays. This makes the factorization at the end-point useless. In [10] it has been suggested to use $k_T$- or TMD factorization to resum large logarithms or to eliminate those end-point singularities.

Comparing to TMD factorization for inclusive processes which has been studied extensively (see [3, 4, 5] and references therein), the TMD or $k_T$-factorization for exclusive processes have been not studied with consistent definitions of TMD LCWF’s as much as that for inclusive processes. For a given process the amplitude in collinear factorization is factorized in terms of a hard part and LCWF’s, latter representing all nonperturbative effects in the process. In [8, 10] the TMD factorization can be taken as a "direct" generalization of the results of collinear factorization in the sense that the amplitude is factorized as a convolution only with a hard part and TMD LCWF’s. From the study of TMD factorization for inclusive processes[3, 4, 5] one knows that the differential cross-section for a given process can be factorized as a convolution of a hard part, TMD parton distributions and a soft factor. The nonperturbative effects in the inclusive process are not only represented by the TMD parton distributions, but also by the soft factor. This is in contrast
In the study of TMD factorization for radiative leptonic decay of B-meson [11], it has been shown that the decay amplitude takes the convolution form, with a hard part, the TMD LCWF of B-meson, and a soft factor. The TMD LCWF of B-meson has been consistently defined and its relation to the standard LCWF has been studied in detail in [12]. From the result in [11] the soft factor is really needed in order to make the factorization complete. A complete factorization means that the hard part does not contain any soft divergence like collinear- and infrared (I.R.) divergence. It has been shown that two factorizations for this simple case are equivalent and large logarithms can be conveniently resummed.

In this work we study TMD factorization for the process $\gamma^* \pi \rightarrow \gamma$, where only one light hadron is involved. With the present study and that in [11, 12], we have examined TMD factorization for an exclusive process involving one hadron. This can be thought as a first step to the goal to examine TMD factorization for processes involving more than one light hadrons and to study the factorization for B-decays into light hadrons. A consistent definition of the TMD LCWF of $\pi$ was first given in [7]. The TMD LCWF has been studied in [13], where some interesting relations to the standard LCWF have been obtained. We will consider the case that the virtual photon has the negatively large virtuality $-Q^2 (Q^2 > 0)$, which converts $\pi$ into a real photon. Because of the large $Q^2$, one can expand the amplitude in the inverse of $Q^2$. We will show that the amplitude of $\gamma^* \pi \rightarrow \gamma$ at the leading order of the inverse $Q^2$ can be factorized at one-loop level as the convolution

$$
\phi_+ \otimes \tilde{S} \otimes H \cdot \left\{1 + O(\Lambda^2/Q^2)\right\}.
$$

(1)

In the above $\phi_+$ is the TMD LCWF, $\tilde{S}$ is the soft factor and $H$ is a coefficient function which can be safely calculated in perturbative QCD. The soft factor is defined with gauge links and it is needed to ensure $H$ free from I.R. singularities. The correction to the factorized form is power-suppressed and is proportional to $\Lambda^2/Q^2$. $\Lambda$ is any possible soft scale characterizing nonperturbative scale, like $\Lambda_{QCD}$, the mass of $\pi$ etc.. This scale is about several hundreds MeV. Therefore, the correction to the factorized form is small with the large $Q^2$.

In the factorized form, the TMD light-cone wave function only depends on the specific hadron and is universal for all possible processes. The coefficient function $H$ depends on the process but does not depend on the specific hadron. It represents a hard transition of a $q\bar{q}$ with the virtual photon into the real photon, in which the pairs move along one light-cone direction and has a small relative transverse momentum. The coefficient function $H$ is obtained from the transition amplitude where the nonperturbative effects represented by soft divergences are subtracted. The soft-factor depends neither on the process nor on the specific hadron, it is completely determined by the infrared behavior of QCD. The TMD light-cone wave function and the soft factor are nonperturbative objects and can only be studied with nonperturbative methods of QCD, like Lattice QCD or QCD sum rule method. Phenomenologically, they can also be extracted from relevant processes. In the case of a pion, they can be extracted, e.g., from the process studied here, and those like $\pi$ form factor, exclusive decays of $B$-mesons, etc... However, beside the process studied here, one needs first to show that the TMD factorization for other processes can be consistently
performed. In this work we will show that the proposed factorization for $\gamma^*\pi \rightarrow \gamma$ holds at one-loop level and the cancelation of all soft divergences is on a diagram-by-diagram basis. The later is important to extend the factorization beyond one-loop level. We will also show that the factorization provides a simple way to resum large logarithms.

Our paper is organized as the following: In Sect.2 we give the definition of TMD LCWF and its one-loop results which will be used to show the factorization. In Sect.3 we introduce our notation and show the TMD factorization at tree level. In Sect.4 we show that the TMD factorization can be completed at one-loop level by introducing a soft factor. The soft factor is defined as products of gauge links. In Sect.5 we show that possible large logarithms around end-point regions can be resummed with our method. Sect.6 contains a summary.

2. The Definition of TMD LCWF and Its One-loop Results

We will use the light-cone coordinate system, in which a vector $a^\mu$ is expressed as $a^\mu = (a^+, a^-, \vec{a}_\perp) = ((a^0 + a^3)/\sqrt{2}, (a^0 - a^3)/\sqrt{2}, a^1, a^2)$ and $a_\perp^2 = (a^1)^2 + (a^2)^2$. We introduce a vector $u^\mu = (u^+, u^-, 0, 0)$ and the definition of TMD LCWF for $\pi^0$ can be given as in the limit $u^+ << u^-:
\phi_+(x, k_\perp, \zeta, \mu) = \int dz^- d^2 z_\perp e^{ik^- x^- - i\vec{k}_\perp \cdot \vec{x}_\perp} \langle 0|\bar{q}(0)L_u^+(\infty, 0)\gamma^+\gamma_5 L_u(\infty, z)q(z)|\pi^0(P)\rangle|_{z^+ = 0},$

\begin{equation}
k^+ = xP^+, \quad \zeta^2 = \frac{2u^-(P^+)^2}{u^+} \approx \frac{4(u \cdot P)^2}{u^2},
\end{equation}

where $q(x)$ is the light-quark field. $L_u$ is the gauge link in the direction $u$:

$L_u(\infty, z) = P \exp \left(-ig_s \int_0^\infty d\lambda u \cdot G(\lambda u + z)\right).$

The definition has been given in [7]. In the above, $\pi^0$ has the momentum $P^\mu = (P^+, P^-, 0, 0)$. The limit $u^+ << u^-$ should be understood that we do not take the contributions proportional to any positive power of $u^+/u^-$ into account. The definition is gauge invariant in any non-singular gauge in which the gauge field is zero at space-time infinity. It has no light-cone singularities as we will show through our one-loop result, but it has an extra variable $\zeta^2$. The evolution with the renormalization scale $\mu$ is simple:

$\mu \frac{\partial \phi_+(x, k_\perp, \zeta, \mu)}{\partial \mu} = 2\gamma_q \phi_+(x, k_\perp, \zeta, \mu),$

where $\gamma_q$ is the anomalous dimension of the light quark field $q$ in the axial gauge $u \cdot G = 0$. In the above we do not specify if the light quark $q$ is a $u$- or $d$-quark. The definition can be generalized to other light hadrons and there can be various TMD LCWF’s. These TMD LCWF’s of various hadrons have been classified in [14], and their asymptotic behavior with $k_\perp \rightarrow \infty$ has been derived[15].

To perform TMD factorization one needs to calculate the wave function with perturbative QCD, in which $\pi^0$ is replaced by a partonic state. We take the partonic state $|q(k_q), \bar{q}(k_{\bar{q}})\rangle$ to replace $\pi^0$ in the definition, the momenta are given as

$k^\mu_q = (k^+_q, k^-_q, \vec{k}_{\perp q}), \quad k^\mu_{\bar{q}} = (k^+_q, k^-_q, -\vec{k}_{\perp q}), \quad k^+_q = x_0 P^+, \quad k^-_q = (1 - x_0)P^- = \bar{x}_0 P^+.$

These partons are on-shell. In this work we will keep the quark mass $m$ for light quarks to regularize collinear singularities and give a small gluon mass $\lambda$ to regularize I.R. singularities. We will omit the $\mu$-dependence in various quantities in this work and this dependence is always implied.
At tree-level, the wave function reads:

$$\phi^{(0)}_+(x, k_\perp, \zeta) = \delta(x - x_0)\delta^2(\vec{k}_\perp - \vec{k}_\perp)\phi_0, \quad \phi_0 = \bar{v}(k_q)\gamma^+\gamma_5u(k_q)/P^+. \quad (6)$$

We will always write a quantity $A$ as $A = A^{(0)} + A^{(1)} + \cdots$, where $A^{(0)}$ and $A^{(1)}$ stand for tree-level and one-loop contribution respectively. At one-loop one can divide the corrections into a real part and a virtual part. The real part comes from contributions of Feynman diagrams given in Fig.1. The virtual part comes from contributions of Feynman diagrams given in Fig.2., those contributions are proportional to the tree-level result.

![Figure 1](image.png)

**Figure 1:** The real part of one-loop contribution to the TMD LCWF. The double lines represent the gauge links.

The contributions from Fig.1b, Fig.1c and Fig.1d read:

$$\phi_+(x, k_\perp, \zeta)_{1b} = -\frac{\alpha_s C_F}{2\pi^2} \left\{ \frac{1}{x - x_0} \theta(x_0 - x) + \frac{1}{x_0\Delta_q} + \frac{1}{2}\delta(x - x_0) - \frac{1}{2}\delta(x - x_0) \right\} \frac{1}{q_\perp^2 + \lambda^2} \ln \frac{q_\perp^2 + \lambda^2}{\zeta^2 x_0^2} \phi_0,$$

$$\phi_+(x, k_\perp, \zeta)_{1c} = -\frac{\alpha_s C_F}{2\pi^2} \left\{ \frac{1}{x - x_0} \theta(x - x_0) + \frac{1}{1 - x_0\Delta_q} - \frac{1}{2}\delta(x - x_0) \right\} \frac{1}{q_\perp^2 + \lambda^2} \ln \frac{q_\perp^2 + \lambda^2}{\zeta^2 x_0^2} \phi_0,$$

$$\phi_+(x, k_\perp, \zeta)_{1d} = -\frac{\alpha_s C_F}{2\pi^2} \frac{1}{\lambda^2 + q_\perp^2} \delta(x - x_0)\phi_0, \quad (7)$$

with

$$\Delta_q = (1 - y)^2 m^2 + y\lambda^2 + (k_\perp - yk_{q,\perp})^2, \quad y = \frac{x}{x_0}, \quad q_\perp = k_\perp - k_{q,\perp}.$$  

$$\Delta_{\bar{q}} = (1 - \bar{y})^2 m^2 + \bar{y}\lambda^2 + (k_\perp - \bar{y}k_{q,\perp})^2, \quad \bar{y} = \frac{x}{x_0}, \quad \bar{q}_\perp = k_\perp - k_{q,\perp}. \quad (8)$$

It should be noted that if we take $u^+ = 0$ in the definition at the beginning there are light-cone singularities, represented by terms like $1/x$ and $1/(1 - x)$. With a nonzero, but small $u^+$ these singularities are regularized. All these contributions behave like $k_{\perp}^{-2}$ when $k_\perp$ becomes large. The contribution from Fig.1a is too lengthy. But its calculation is standard because the gauge link is not involved. The contribution also behaves like $k_{\perp}^{-2}$ when $k_\perp$ becomes large. The leading contribution for large $k_\perp$ is:

$$\phi_+(x, k_\perp, \zeta)_{1a} = \frac{2\alpha_s}{3\pi^2} \frac{1}{k_\perp^2 + \Lambda^2} \left\{ \frac{x}{x_0}\theta(x_0 - x) + \frac{1}{1 - x_0}\theta(x - x_0) \right\} \phi_0$$

$$+ \text{"finite terms"}, \quad (9)$$

where we present only the U.V. divergent part explicitly and introduce a “cut-off” $\Lambda$, which also appear in the finite term so that the sum does not depend on it. All terms in the second line are
proportional $k_\perp^{-4}$ when $k_\perp \to \infty$. Therefore, the one-loop correction of $\phi_+(x, k_\perp, \zeta)$ is proportional to $k_\perp^{-2}$ when $k_\perp \to \infty$, as expected from the general power counting rule [15].

The virtual part of the one-loop correction is from the Feynman diagrams in Fig.2. Contributions from each diagrams are:

\begin{align*}
\phi_+(x, k_\perp, \zeta)|_{2a} &= \phi_+(x, k_\perp, \zeta)|_{2d} = \phi_+(0)(x, k_\perp, \zeta) \cdot \frac{\alpha_s}{6\pi} \left[ -\ln \frac{\mu^2}{m^2} + 2\ln \frac{m^2}{\lambda^2} - 4 \right], \\
\phi_+(x, k_\perp, \zeta)|_{2b} &= \phi_+(x, k_\perp, \zeta)|_{2e} = \phi_+(0)(x, k_\perp, \zeta) \cdot \frac{\alpha_s}{3\pi} \ln \frac{\mu^2}{\lambda^2}, \\
\phi_+(x, k_\perp, \zeta)|_{2c} &= \phi_+(0)(x, k_\perp, \zeta) \cdot \frac{\alpha_s}{6\pi} \left[ 2\ln \frac{\mu^2}{m^2} + 2\ln \frac{\zeta^2 x_0^2}{m^2} + \ln^2 \frac{m^2}{\lambda^2} - \ln^2 \frac{\zeta^2 x_0^2}{\lambda^2} - \frac{2\pi^2}{3} + 4 \right], \\
\phi_+(x, k_\perp, \zeta)|_{2f} &= \phi_+(0)(x, k_\perp, \zeta) \cdot \frac{\alpha_s}{6\pi} \left[ 2\ln \frac{\mu^2}{m^2} + 2\ln \frac{\zeta^2 x_0^2}{m^2} + \ln^2 \frac{m^2}{\lambda^2} - \ln^2 \frac{\zeta^2 x_0^2}{\lambda^2} - \frac{2\pi^2}{3} + 4 \right](10)
\end{align*}

The complete one-loop contribution $\phi_+^{(1)}$ is the sum of contributions from the 10 Feynman diagrams in Fig.1. and Fig.2.

From the above results one can derive some interesting relations between the TMD LCWF and standard LCWF and the evolution with $\zeta$. The evolution with $\zeta$ has been shown to be very useful for resumming large logarithms. Details about these can be found in [13].

3. TMD Factorization for $\gamma^*\pi^0 \to \gamma$ at Tree-Level

We consider the process:

$$\pi^0 + \gamma^* \to \gamma$$ \hspace{1cm} (11)

where $\pi^0$ carries the momentum $P$ and the real photon momentum $p$. We take a coordinate system in which these momenta are:

$$P^\mu = (P^+, P^-, 0, 0), \quad p^\mu = (0, p^-, 0, 0). \hspace{1cm} (12)$$

We will consider the case that the virtual photon has the large negative virtuality $q^2 = (P - p)^2 = -2P^+p^- = -Q^2$. The process can be described by matrix element $\langle \gamma(p, e^*)| q\gamma^\mu q|\pi^0(P) \rangle$, which

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Figure 2: The virtual part of one-loop contribution to the TMD LCWF. The double lines represent the gauge links.
can be parameterized with the form factor $F(Q^2)$:

$$\langle \gamma(p, \epsilon^*) | \bar{q} \gamma^\mu q | \pi^0(P) \rangle = i \epsilon^{\mu
u\rho\sigma} \epsilon_\nu^\rho P_\sigma F(Q^2).$$  \hspace{1cm} (13)$$

For simplicity, we only consider one flavor quark $q$ with electric charge 1.

The form factor has been studied in experiments[16]. Theoretically it has been studied in the frame work of collinear factorization extensively [17, 18, 19, 20, 21, 22, 23]. In [22] the coefficient function in the collinear factorization has been calculated at one-loop, its two-loop result can be found in [23]. In [19, 20, 21] an attempt has been made to take the transverse momentum into account in order to make predictions more reliable. In [21], following [7, 8] the resummation of large logarithms has been performed. However, in these studies, the issue whether the TMD factorization holds or not has not been investigated. The main goal of the paper is to address the issue, and we will not perform a phenomenological study with the factorization to make any numerical predictions.

To find a factorized form for the form factor, we replace the hadronic state with the partonic state as before in calculation of the matrix element. The momenta of the partonic state is specified in Eq.(5). In general one should neglect the transverse momentum $\mathbf{k}_{q\perp}$ in the numerators in calculating $F(Q^2)$, while one keeps the most important terms of $\mathbf{k}_{q\perp}$ in the denominators[8]. Therefore, what neglected here is proportional to $k^2_{q\perp}/Q^2$. However, at tree-level, the amplitude does not depend on $\mathbf{k}_{q\perp}$. At tree-level there are two Feynman diagrams shown in Fig.3.

![Feynman diagrams of tree-level contributions to the partonic scattering. The black dot denotes the insertion of the electric current operator corresponding to the virtual photon.](image)

It is straightforward to obtain the tree-level result from Fig.3:

$$F(Q^2)|_{3a} = \phi_0 \frac{1}{Q^2 x_0}, \quad F(Q^2)|_{3b} = \phi_0 \frac{1}{Q^2 (1 - x_0)} = \phi_0 \frac{1}{Q^2 \bar{x}_0}. \hspace{1cm} (14)$$

Combining it with the tree-level result of the TMD LCWF, we can obtain a factorized form for the form factor:

$$F(Q^2) = \frac{1}{Q^2} \int dx dy k_{\perp} \phi_+ (x, k_{\perp}, \zeta) \left[ \frac{1}{x} + \frac{1}{\bar{x}} \right]. \hspace{1cm} (15)$$

As discussed before, such a factorization may not hold beyond tree-level, and a possible soft factor should be implemented. We expect that the form factor in TMD factorization can be written as:

$$F(Q^2) = \frac{1}{Q^2} \int dx dy d^2 l_{\perp} \phi_+ (x, l_{\perp}) S(y, l_{\perp}, \zeta) \theta(x + y) H(x + y) + \mathcal{O}(Q^{-4})$$

$$= \frac{1}{Q^2} \phi_+ \otimes \hat{S} \otimes H + \mathcal{O}(Q^{-4}), \hspace{1cm} (16)$$
where we only give the dependence on convolution variables explicitly. The complete dependence of different objects on different variables will be given in the next section. With the above factorization one can easily find the tree-level results of the hard part $H$ and the soft factor:

$$H^{(0)}(x) = \frac{1}{x} + \frac{1}{1 - x} = \frac{1}{xx}, \quad \tilde{S}^{(0)}(y, l_\perp) = \delta(y)\delta^2(\vec{l}_\perp).$$

(17)

It should be noted that at tree-level the hard part does not depend on transverse momenta. This is because we do not have an actual hadron in the final state of this simple case. When both initial- and final states contain hadrons, the hard part will depend on transverse momenta of partons entering hard scattering.

4. TMD Factorization at One-Loop Level

In this section we examine the TMD factorization at one-loop level and show that the soft factor is indeed needed. We will give its definition in terms of gauge links. At one-loop level, the factorization formula in the previous section can be written as:

$$Q^2 F^{(1)}(Q^2) = \phi^{(1)}_+ \otimes \tilde{S}^{(0)} \otimes H^{(0)} + \phi^{(0)}_+ \otimes \tilde{S}^{(1)} \otimes H^{(0)} + \phi^{(0)}_+ \otimes \tilde{S}^{(0)} \otimes H^{(1)}.$$  

(18)

If the factorization is right, $H^{(1)}$ will not contain any soft divergence and can be then calculated in perturbative theory. If all soft divergences in $Q^2 F^{(1)}(Q^2)$ are already contained in the first term, i.e., in $\phi^{(1)}_+ \otimes \tilde{S}^{(0)} \otimes H^{(0)}$, then there is no need to introduce the soft factor. However, this is not the case. In this section we will study different factors introduced above.

In deriving the factorization formula at tree-level, one should neglect the transverse momentum $k_{q\perp}$ in the numerators in calculating $F(Q^2)$, while one keeps the most important terms of $k_{q\perp}$ in the denominators. As discussed in the last section, the hard part at tree level does not depend on the transverse momenta of partons. In calculating the one-loop contribution of $F(Q^2)$ with the partonic state, we will therefore take the transverse momenta as small quantities and expand it in $k_{q\perp}^2/Q^2$. We only take the leading order to extract the hard part. The higher orders are as power-suppressed contributions and they are neglected.

4.1. The One-Loop Correction of $F(Q^2)$

There are 12 Feynman diagrams for the one-loop correction of the form factor, 6 of them are given in Fig.4. The other 6 diagrams are obtained from those in Fig.4 by reversing the direction of the quark line. The diagrams in Fig.4 represent the correction to Fig.3a, and the other 6 diagrams for the correction to Fig.3b. Two corrections are related each other by charge conjugation. Hence we will need to study how the contributions from Fig.4 can be factorized.

![Figure 4: Feynman diagrams of the one-loop corrections to Fig.3a.](image-url)
The contributions except that of Fig.4a can be calculated in a straightforward way. They are:

\[ F(Q^2)|_{4c} = F(Q^2)|_{4f} = F(Q^2)|_{3a} \cdot \frac{\alpha_s}{6\pi} \left[ -\ln \frac{\mu^2}{m^2} - 2 \ln \frac{\lambda^2}{m^2} - 4 \right], \]

\[ F(Q^2)|_{4d} = F(Q^2)|_{3a} \cdot \frac{2\alpha_s}{3\pi} \left[ \ln \frac{\mu^2}{Q^2} - \ln x_0 + 1 \right], \]

\[ F(Q^2)|_{4b} = F(Q^2)|_{3a} \cdot \frac{2\alpha_s}{3\pi} \left[ \frac{1}{2} \ln \frac{\mu^2}{Q^2} + \ln \frac{Q^2}{m^2} + \frac{1}{2} \ln x_0 \right], \]

\[ F(Q^2)|_{4c} = F(Q^2)|_{3a} \cdot \frac{2\alpha_s}{3\pi} \left\{ \frac{1}{2} x_0 \left[ \frac{1}{2} \ln^2 x_0 + 2 \text{Li}_2(-\frac{x_0}{x_0}) + \ln \frac{Q^2}{m^2} \ln x_0 \right] + \frac{1}{2} \ln \frac{\mu^2}{Q^2} - \ln \frac{m^2}{Q^2} - \frac{2 + x_0}{2x_0} \ln x_0 \right\}. \] (19)

It is obvious that the contributions from Fig.4e and Fig.4f are already contained in the contribution of Fig.2a and Fig. 2d of the TMD LCWF, respectively.

The contribution from Fig.4a is difficult to calculate and has a complicated expression. However, for the purpose of the factorization, we only need the contribution at the leading power of \(Q^2\). We have the result for the combination:

\[ F(Q^2)|_{4a} - \frac{1}{Q^2} \phi_+^{(1)}|_{1a} \otimes \tilde{S}^{(0)} \otimes H^{(0)}|_{3a} = -F(Q^2)|_{3a} \frac{2\alpha_s}{3\pi} x_0 \left[ -\ln x_0 \int d^2 k_\perp \frac{1}{\lambda^2 + k_\perp^2} - \pi \ln x_0 + \pi \ln x_0 \left( \ln \frac{Q^2}{\lambda^2} - 1 + \frac{1}{2} \ln x_0 \right) \right] (1 + \mathcal{O}(Q^{-2})). \] (20)

In the above expression the I.R. divergent term \(\ln \lambda^2\) will be canceled from the integration of \(k_\perp\). The expression contain an U.V. divergence. This divergence comes from TMD LCWF of Fig.1a and needs to be subtracted.

Since the contributions from Fig.4e and Fig.4f are already contained in the corresponding contributions of TMD LCWF, i.e., they are already subtracted for the hard part, we have only collinear divergences in \(F(Q^2)|_{4b}\) and \(F(Q^2)|_{4c}\), signalled by \(\ln m\), as soft divergences which need to be subtracted.

4.2. The Convolution \(\phi_+^{(1)} \otimes \tilde{S}^{(0)} \otimes H^{(0)}\)

In this section we present the results for the convolution \(\phi_+^{(1)} \otimes \tilde{S}^{(0)} \otimes H^{(0)}\). We will see that the convolution contains the same collinear divergences as those in \(F^{(1)}(Q^2)\). Therefore, with our TMD LCWF the collinear singularities are correctly factorized. Moreover, the collinear divergence in a given diagram for \(F^{(1)}(Q^2)\) appears exactly in the same class of diagrams for TMD LCWF. This fact can make it possible to extend the factorization of collinear divergences beyond one-loop level. We will also see that there are still some I.R. divergences in the convolution. Therefore, a soft factor is needed to subtract them.

We find that it is convenient to present the results in the following combinations:

\[ \mathcal{W}_b = [\phi_+|_{1b} + \phi_+|_{2b} + \phi_+|_{2c}] \otimes \tilde{S}^{(0)} \otimes H^{(0)}|_{3a} = \frac{2\alpha_s}{3\pi} x_0 \left\{ \ln \frac{\mu^2}{m^2} + \frac{1}{2} \ln \frac{\zeta^2 x_0^2}{\lambda^2} - \frac{1}{4} \ln^2 \frac{\zeta^2 x_0^2}{\lambda^2} - \frac{\pi^2}{4} + 1 + \frac{1}{2} \int \frac{d^2 q_\perp}{\pi} \frac{1}{q_\perp^2 + \lambda^2} \ln \frac{\zeta^2 x_0^2}{q_\perp^2 + \lambda^2} \right\}, \]
Comparing the above with the one-loop correction to the form factor in Eq. (19), we see that the vectors represent the trajectories of $q$ and $v$ should be understood that we neglect all terms with a positive power of $\phi$. But, in the convolution the hard part are subtracted, one can expect that the U.V. divergences are subtracted too. Because of these U.V. divergences are closely combined with certain I.R. divergences. Once these I.R. divergences are subtracted, one can expect that the U.V. divergences are subtracted too. Because of these divergences, one needs to introduce the soft factor so that the factorization can be completed, i.e., the hard part $H^{(1)}$ subtracted from Eq. (18) is free from any soft divergence.

4.3. The Soft Factor

As discussed in the above, the I.R. divergences which need to be subtracted are from TMD LCWF. In general these divergences can be analyzed by applying an eikonal approximation. Using the approximation for those diagrams in Fig. 1. one can find that the I.R. divergences in these diagrams just represent the contributions to the product of the gauge links:

$$S_4(z^-, z_\perp, v_1, v_2) = \frac{1}{3} \text{Tr}(0|T[L^\dagger_{v_2}(0, -\infty)L^\dagger_u(\infty, 0)L_u(\infty, z)L_{v_1}(z, -\infty)]|0)_{z^+ = 0},$$

$$L_{v_1, 2}(z, -\infty) = P \exp(-ig_s \int_{-\infty}^{0} d\lambda v_{1, 2} \cdot G(\lambda v_{1, 2} + z)),$$

$$S_4(y, q_\perp, \zeta_1, \zeta_2) = P^+ \int dz^- d^2z_\perp e^{iy P^+ z^- - iq_\perp \cdot z} S_4(z^-, z_\perp),$$

(22)

the two vectors $v_{1, 2}$ and the parameters $\zeta_{1, 2}$ are defined as:

$$v_{1, 2}^\mu = (v_{1, 2}^+, v_{1, 2}^- - v_{1, 2}^-, 0, 0), \quad v_{1, 2}^+ << v_{1, 2}^-,$$

$$\zeta_{1, 2}^2 = \frac{2v_{1, 2}^-(P^+)^2}{v_{1, 2}^+}. \quad (23)$$

these vectors represent the trajectories of $q$ and $\bar{q}$ in Fig. 1, respectively. The limit $v_{1, 2}^+ >> v_{1, 2}^-$ should be understood that we neglect all terms with a positive power of $\zeta_{1, 2}$. The ordering of the limits is relevant for calculating $S_4$. We will take the ordering that the limit $v_1^+ >> v_1^-$ is taken before taking the limit $v_2^+ >> v_2^-$. We have calculated $S_4$ in perturbation theory. At tree-level we have

$$S_4^{(0)}(y, q_\perp, \zeta_1, \zeta_2) = \delta(y)\delta^2(\bar{q}_\perp), \quad S_4^{(0)}(z^-, z_\perp, v_1, v_2) = 1.$$

(24)

At one-loop level, the contributions to $S_4$ can be represented by those diagrams in Fig.1 and Fig.2, where one replaces the quark line with the gauge link $L_{v_1}$ and the antiquark line with the gauge
link $L_{\nu_2}$. The contributions from Fig.1 are:

\begin{align}
S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2)|_{1a} &= \frac{2\alpha_s}{3\pi^2} \left[ \frac{\theta(-y)}{y(q_1^2 + \lambda^2 + \zeta_1^2 y^2)} + \frac{\theta(y)}{y(q_1^2 + \lambda^2 + \zeta_1^2 y^2)} \right], \\
S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2)|_{1b} &= -\frac{2\alpha_s}{3\pi^2} \left[ \frac{\theta(-y)}{q_1^2 + \lambda^2 + \zeta_1^2 y^2} \left( \frac{1}{y} \right) + \frac{\theta(y)}{q_1^2 + \lambda^2} \ln \frac{\zeta_1^2 x_0^2}{q_1^2 + \lambda^2} \right], \\
S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2)|_{1c} &= \frac{2\alpha_s}{3\pi^2} \left[ \frac{1}{q_1^2 + \lambda^2 + \zeta_2^2 y^2} \theta(y) \left( \frac{1}{y} \right) + \frac{\theta(y)}{q_1^2 + \lambda^2} \ln \frac{\zeta_2^2 x_0^2}{q_1^2 + \lambda^2} \right], \\
S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2)|_{1d} &= -\frac{2\alpha_s}{3\pi^2} \frac{\delta(y)}{q_1^2 + \lambda^2}.
\end{align}

The ordering of the limits mentioned above only affects the contribution $S_4|_{1a}$. If we change the ordering, $S_4|_{1a}$ changes its sign. The above expressions, when they are used to calculate the convolution $\phi_+^{(0)} \otimes \tilde{S}^{(1)} \otimes H^{(0)}$, will be taken as distributions for $-x_0 \leq y \leq 1 - x_0$. This can be seen from the convolution

\begin{equation}
\phi_+^{(0)} \otimes \tilde{S}^{(1)} \otimes H^{(0)} = \phi_0 \int_0^1 dy dq_\perp^2 \tilde{S}^{(1)}(y, q_\perp)\theta(x_0 + y)H^{(0)}(x_0 + y).
\end{equation}

In Eq.\((25)\) we have already take the limit $\zeta \to \infty$, i.e., $u^+ << u^-$ by taking them as distributions for $-x_0 \leq y \leq 1 - x_0$.

The contributions from the diagrams corresponding to those in Fig.2 are:

\begin{align}
S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2)|_{2a} &= S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2)|_{2b} = S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2)|_{2d} \\
&= S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2)|_{2e} = S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2) \cdot \frac{\alpha_s}{3\pi} \ln \frac{\mu^2}{\lambda^2}, \\
S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2)|_{2c} &= -S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2) \cdot \frac{\alpha_s}{3\pi} \ln \frac{\zeta_1^2}{\zeta_2}, \\
S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2)|_{2f} &= -S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2) \cdot \frac{\alpha_s}{3\pi} \ln \frac{\zeta_1^2}{\zeta_2^2}.
\end{align}

$S_4$ has the charge conjugation symmetry $S_4(y, \vec{q}_\perp, \zeta_1, \zeta_2) = S_4(-y, -\vec{q}_\perp, \zeta_1, \zeta_2)$. We can use the symmetry to simplify our definition of the soft factor by taking the limit $\zeta_2 \to \zeta_1 = \zeta_\nu$.

If we identify the soft factor $\tilde{S}$ with $S_4$ as $\tilde{S}(z^-, z_\perp) = [S_4(z^-, z_\perp, v_1, v_2)]^{-1}$, the I.R. divergences in $\phi_+(x, k_\perp, \zeta)$ are indeed subtracted if we integrate over $x$ and $k_\perp$ and extend the integration region of $x$ from $-\infty$ to $\infty$. However, in these convolutions $\phi_+^{(0)} \otimes \tilde{S}^{(0)} \otimes H^{(0)}$ and $\phi_+^{(0)} \otimes \tilde{S}^{(1)} \otimes H^{(0)}$, the integrands are not $\tilde{S}^{(1)}$ or $\phi_+^{(1)}$ alone, but multiplied with $H^{(0)}$, and the integration region of $x$ is restricted from 0 to 1. This results in that the I.R. singularities in $\phi_+^{(1)} \otimes \tilde{S}^{(0)} \otimes H^{(0)}$ and $\phi_+^{(0)} \otimes \tilde{S}^{(1)} \otimes H^{(0)}$ are not exactly reproduced in $\phi_+^{(0)} \otimes \tilde{S}^{(1)} \otimes H^{(0)}$, and there are end-point singularities in $\phi_+^{(0)} \otimes \tilde{S}^{(1)} \otimes H^{(0)}$ when $y$ approaches to $-x_0$ or $1 - x_0$. The problem is related that the proposed soft factor does not vanish at the end points, while the TMD LCWF does. Therefore the convolution with $\tilde{S}(z^-, z_\perp) = [S_4(z^-, z_\perp, v_1, v_2)]^{-1}$ is ill defined and the proposed soft factor is not right for the case.

To overcome the problem, one can construct quantities from $S_4$ and their contributions have a one-to-one correspondence to the one-loop contributions to $\phi_+(x, k_\perp, \zeta)$ in Eq.(5) and Eq.(9). The
correspondence means that the I.R. divergences in Eq. (8) and the U.V. divergences in Eq. (9) are exactly reproduced. For this we first define:

\[
S_1(y, x, \bar{q}, \zeta_v) = \left[ 1 - \theta(-y) \frac{y}{y-x} - \theta(y) \frac{y}{y+1-x} \right]
\]

\[
\frac{1}{2} \lim_{\zeta_2 \to \zeta_1 = \zeta_v} [S_4(q^+, \bar{q}_- \perp, \zeta_1) + S_4(q^+, \bar{q}_- \perp, \zeta_2, \zeta_1)]. \tag{28}
\]

The contributions from Fig.1 to \( S_1 \) can be easily obtained from those of \( S_4 \). We have:

\[
S_1(y, x, \bar{q}_- \perp, \zeta_v)_{|_{1a}} = 0,
\]

\[
S_1(y, x, \bar{q}_- \perp, \zeta_v)_{|_{1b}} = -\frac{2\alpha_s}{3\pi^2 P^+} \left[ \frac{\theta(x_0 - x)}{q^2_1 + \lambda^2 + \zeta_0^2 (x - x_0)^2 x_0} \left( \frac{1}{x - x_0} \right) + \frac{1}{2} \delta(x - x_0) \frac{1}{q^2_1 + \lambda^2} \ln \frac{\zeta^2 x_0^2}{q^2_1 + \lambda^2} \right],
\]

\[
S_1(y, x, \bar{q}_- \perp, \zeta_v)_{|_{1c}} = \frac{2\alpha_s}{3\pi^2 P^+} \left[ \frac{\theta(x - x_0)}{q^2_1 + \lambda^2 + \zeta_0^2 (x - x_0)^2 x_0} \left( \frac{1}{x - x_0} \right) + \frac{1}{2} \delta(x - x_0) \frac{1}{q^2_1 + \lambda^2} \ln \frac{\zeta^2 x_0^2}{q^2_1 + \lambda^2} \right],
\]

\[
S_1(y, x, \bar{q}_- \perp, \zeta_v)_{|_{1d}} = -\frac{2\alpha_s}{3\pi^2 P^+} \delta(x - x_0) \frac{1}{q^2_1 + \lambda^2}, \tag{29}
\]

where we have taken \( y = x - x_0 \) because of Eq. (23). The contributions from Fig.2 to \( S_1 \) are the same as those of \( S_4 \) by taking \( \zeta_2 = \zeta_1 = \zeta_v \). Comparing Eq. (29) with Eq. (8), we can see that the I.R. singularities in \( \phi_+ |_{1b}, \phi_+ |_{1c} \) and \( \phi_+ |_{1d} \) are exactly reproduced \( S_1 |_{1b}, S_1 |_{1c} \) and \( S_1 |_{1d} \), respectively.

Next we define

\[
S_2(y, x, \bar{q}_- \perp, \zeta_v) = y \left[ 1 - \theta(-y) \frac{y}{y-x} - \theta(y) \frac{y}{y+1-x} \right]
\]

\[
\frac{1}{2} \lim_{\zeta_2 \to \zeta_1 = \zeta_v} [S_4(q^+, \bar{q}_- \perp, \zeta_1, \zeta_2) - S_4(q^+, \bar{q}_- \perp, \zeta_2, \zeta_1)]. \tag{30}
\]

One easily finds that \( S_2 \) only receives a nonzero contribution from Fig.1a:

\[
S_2 = S_2 |_{1a},
\]

\[
S_2(y, x, \bar{q}_- \perp, \zeta_v) |_{1a} = \frac{2\alpha_s}{3\pi^2 P^+} \left[ \frac{\theta(x_0 - x)}{q^2_1 + \lambda^2 + \zeta_0^2 (x - x_0)^2 x_0} \left( \frac{1}{x - x_0} \right) + \frac{\theta(x - x_0)}{q^2_1 + \lambda^2 + \zeta_0^2 (x - x_0)^2 x_0} \left( \frac{1}{x - x_0} \right) \right]. \tag{31}
\]

where we have taken \( y = x - x_0 \) again. It is noted that the U.V. divergence in \( \phi_+ |_{1a} \) in Eq. (9) is reproduced by \( S_2 |_{1a} \).

There are extra I.R. divergences introduced by the contributions from Fig.2a and Fig.2d for \( S_4 \). These can be subtracted by introducing

\[
S_2(\zeta_1, \zeta_2) = \frac{1}{3} \text{Tr}(0 | T \left[ L_{v_2}^{+} (0, -\infty) L_{v_{1}} (0, -\infty) \right] | 0). \tag{32}
\]
Figure 5: One-loop contribution to $S_2$. The double lines represent the two gauge links. One is for $L_{v_1}$, the other one is for $L_{v_2}$.

As for $S_4$, the same limits about the vectors $v_1$ and $v_2$ are taken. At leading order $S_2^{(0)} = 1$. At one-loop level, the contributions are from the diagrams given in Fig.5. They are:

$$S_{2|5a} = \frac{\alpha_s}{3\pi} \ln \frac{\mu^2}{\lambda^2} \ln \frac{\zeta_1^2}{\zeta_2^2},$$

$$S_{2|5b} = S_{2|5c} = \frac{\alpha_s}{3\pi} \ln \frac{\mu^2}{\lambda^2}. \quad (33)$$

In $S_1$ we have taken the limit $\zeta_2 \rightarrow \zeta_1 = \zeta_e$. Before the limit there is an I.R. singularity in $S_{1a}$. This singularity is exactly the same as $S_{2|5a}$. We take the limit $\zeta_2 \rightarrow \zeta_1 = \zeta_e$ for $S_2$ and define:

$$S_{2s}(\zeta_e) = \lim_{\zeta_2 \rightarrow \zeta_1 = \zeta_e} \frac{1}{2} \left( S_2(\zeta_1, \zeta_2) + S_2(\zeta_2, \zeta_1) \right). \quad (34)$$

The I.R. singularities in Fig.2a and Fig.2d for $S_1$, hence for $S_1$, are reproduced by $S_{2s}$. It should be noted that the limit $\zeta_2 \rightarrow \zeta_1$ for $S_{2s}$ is nontrivial. If we take $\zeta_1 = \zeta_2$ at the beginning, we will always have $S_2 = 1$.

Finally we can define the correct soft factor as:

$$\tilde{S}(z^-, z_\perp, x, \zeta_e^v) = \left( \frac{S_1(z^-, z_\perp, x, \zeta_e^v) + S_2(z^-, z_\perp, x, \zeta_e^v)}{S_{2s}} \right)^{-1},$$

$$\tilde{S}(y, x, \bar{q}_\perp, \zeta_v^e) = P^+ \int dz^- d^2 z_\perp e^{i y P^+ z^- - i \bar{q}_\perp \cdot \vec{z}_\perp} \tilde{S}(z^-, z_\perp, x, \zeta_e^v). \quad (35)$$

With this we will show that the extracted hard part is free from any soft divergence.

### 4.4. The Convolution $\phi_+^{(0)} \otimes \tilde{S}^{(1)} \otimes H^{(0)}$ and The Result for $H^{(1)}$

With the soft factor we define the following combinations of convolution,

$$U_b = \phi_+^{(0)} \otimes \left( \tilde{S}_{1b}^{(1)} + \tilde{S}_{2b}^{(1)} + \tilde{S}_{2c}^{(1)} \right) \otimes H_{3a}^{(0)}$$

$$= 2\alpha_s \left[ \frac{\pi^2}{12} + \frac{1}{4} \ln^2 \frac{\mu^2}{\zeta_v^e x_0^2} \right] - \frac{1}{2} \ln \frac{\lambda^2}{\mu^2} \ln \frac{\zeta_1^2}{\zeta_2^2} + \frac{1}{2} \ln \frac{\lambda^2}{\mu^2} + \frac{1}{2} \int \frac{d^2 q_\perp}{\pi} \frac{1}{q_\perp^2 + \lambda^2} \ln \frac{q_\perp^2 + \lambda^2}{\zeta_v^e x_0^2} \right], \quad (36)$$

$$U_c = \phi_+^{(0)} \otimes \left( \tilde{S}_{1c}^{(1)} + \tilde{S}_{2c}^{(1)} + \tilde{S}_{2f}^{(1)} \right) \otimes H_{3a}^{(0)}$$

$$= 2\alpha_s \left[ \frac{\pi^2}{12} + \frac{1}{4} \ln^2 \frac{\mu^2}{\zeta_v^e x_0^2} \right] - \frac{1}{2} \ln \frac{\lambda^2}{\mu^2} \ln \frac{\zeta_1^2}{\zeta_2^2} + \frac{1}{2} \ln \frac{\lambda^2}{\mu^2} - \frac{1}{x_0} \left( 2\text{Li}(\frac{-\bar{x}_0}{x_0}) + \ln x_0 \ln \frac{\lambda^2}{\zeta_v^e x_0^2} \right).$$

13
\[ \frac{\ln x_0}{x_0} \int \frac{d^2 q_1}{\pi} \frac{1}{q_1^2 + \lambda^2} + \frac{1}{2} \int \frac{d^2 q_1}{\pi} \frac{1}{q_1^2 + \lambda^2} \ln \left( \frac{q_1^2 + \lambda^2}{\xi^2 x_0^2} \right) \}, \]

\[ U_d = \frac{\phi_+^{(0)} \otimes (\bar{S}_1^{(1)} \otimes H_{3a}^{(0)})}{3\pi x_0} = \frac{2\alpha_s}{3\pi x_0} \int \frac{d^2 q_1}{\pi} \frac{1}{q_1^2 + \lambda^2}. \]

\[ U_a = \frac{\phi_+^{(0)} \otimes (\bar{S}_1^{(1)} \otimes H_{3a}^{(0)})}{3\pi x_0} = \frac{2\alpha_s}{3\pi} \left\{ -\ln \frac{x_0}{x_0} \int \frac{d^2 q_1}{\pi} \frac{1}{q_1^2 + \lambda^2} + 2\ln \frac{x}{x_0} - \frac{1}{x_0} \left[ 2Li_2(-\frac{x_0}{x}) - \ln x_0 \ln \left( \frac{\xi^2 x_0^2}{\lambda^2} \right) \right] \right\}. \]

The convolution \( \phi_+^{(0)} \otimes \bar{S}^{(1)} \otimes H^{(0)} \) is the sum \( U_a + U_b + U_c + U_d \). Comparing Eq. (21) and Eq. (36), one can see that the I.R. divergence and U.V. divergence in \( W_b, W_c, \) and \( W_d \) is canceled by \( U_b, U_c \) and \( U_d \), respectively. The sum \( \sum_{i=b,c,d} [U_i + W_i] \) contains only collinear singularities which are the same in \( F^{(1)}(Q^2) \). The U.V. divergence in Eq. (21) is also canceled by that in \( U_a \). Hence, with the soft factor the hard part \( H \) will be infra-red finite. It is clear that the soft divergences are canceled or subtracted in a diagram-by-diagram basis, e.g., the collinear singularity in Fig. 4b for the form factor is subtracted by that in Fig. 1b and Fig. 1c for the TMD LCWF, and the I.R. singularity in in Fig. 1b and Fig. 1c for the TMD LCWF is canceled by that in in Fig. 1b and Fig. 1c for the soft factor. The contribution to the TMD LCWF from Fig. 2b is completely subtracted by the contribution from Fig. 2b to the soft factor. Therefore, after the subtraction and cancelation the contribution from Fig. 4b to the hard part \( H^{(1)} \) is free from any soft divergence.

We now can subtract the contribution from Fig. 4 to get \( H^{(1)} \):

\[ H(x)|_{4e} = H(x)|_{4f} = 0, \]

\[ H(x)|_{4d} = -\frac{\alpha_s}{3\pi x} \ln \left( \frac{\mu^2}{Q^2} - \ln x + 1 \right), \]

\[ H(x)|_{4b} = \frac{\alpha_s}{3\pi x} \left\{ -2 + \frac{\pi^2}{3} + \ln \left( \frac{Q^2}{\zeta^2} \right) - \ln x + \frac{1}{2} \ln \left( \frac{\zeta^2}{\mu^2} \right) - 4 \ln x \right\}, \]

\[ H(x)|_{4c} = -\frac{\alpha_s}{3\pi x} \left\{ 2 - \frac{\pi^2}{3} + 2\ln(x\bar{x}) - \frac{1}{x} \ln^2 x - \frac{4}{x} Li_2(-\frac{x}{x}) + \ln \frac{\zeta^2}{Q^2} \right\}, \]

\[ H(x)|_{4a} = \frac{2\alpha_s}{3\pi \bar{x}} \left\{ \ln x \left[ \ln \left( \frac{\zeta^2}{Q^2} \right) + 2 - \frac{1}{2} \ln x \right] - 2Li_2(-\frac{x}{x}) + 2Li_2(\frac{x}{x}) \right\}. \]

Combining the results from diagrams obtained by reversing the direction of quark lines in Fig. 4 we obtain the complete contribution to \( H^{(1)} \):

\[ H^{(1)}(x, Q^2, \zeta, \zeta_0, \mu^2) = \frac{2\alpha_s}{3\pi x} \frac{1}{x\bar{x}} \left\{ \frac{1}{2} \ln \frac{Q^2}{\mu^2} + \ln \frac{Q^2}{\zeta} + \frac{1}{2} \ln \frac{\zeta^2}{\mu^2} + \frac{1}{2} \ln \frac{\zeta^2}{\zeta_0^2} \right\} + \ln \left( \frac{\zeta^2}{Q^2} \right) \ln(x\bar{x}) \]

\[ -2\ln \left( \frac{\zeta^2}{Q^2} \right) \ln(x\bar{x}) + \ln \left( \frac{\zeta^2}{Q^2} \right) (x\bar{x} \ln x + x x) \]

\[-x \ln^2 x + x x + x \ln x + \bar{x} \ln \bar{x} - 2\ln(x\bar{x}) + 4\bar{x} Li_2(x) + 4x Li_2(\bar{x}) \]
The factorization formula with complete parameter-dependence now reads:

\[
F(Q^2) = \frac{1}{Q^2} \int dx dy dz d^2k \phi_+(x, k, \zeta, \mu) S(y, x, l, \zeta, \zeta_v, \mu) \theta(x + y) = H(x + y, Q^2, \zeta, \zeta_v, \mu) + O(Q^{-4}),
\]

with

\[
H(x, Q^2, \zeta, \zeta_v, \mu) = H^{(0)}(x, Q^2, \zeta, \zeta_v, \mu) + H^{(1)}(x, Q^2, \zeta, \zeta_v, \mu) = \frac{1}{x \bar{x}} + H^{(1)}(x, Q^2, \zeta, \zeta_v, \mu),
\]

from Eq. (38) it is clear that the hard part \( H \) does not contain any soft divergence and we complete the TMD factorization at one-loop level. Since the soft divergences are canceled in a diagram-by-diagram basis, the factorization can be argued to hold beyond one-loop level. From our results in this section, it is clear that a soft factor is needed so that the TMD factorization with the TMD LCWF defined in Eq. (2) can be completed.

5. Resummation of Large Logarithms

We note that our factorization formula can be simplified by introducing \( \tilde{\phi}_+(x, k, \zeta, \zeta_v, \mu) \):

\[
\tilde{\phi}_+(x, k, \zeta, \zeta_v, \mu) = \int dz^2 d^2k \overleftrightarrow{e}^{i k + z^--iz^+} \overleftrightarrow{S}(z^-, z^+, \zeta, \zeta_v, \mu) \cdot (0) \bar{q}(0) L_u^\dagger(\infty, 0) e^{z^+ \mu L} L_u(\infty, z) q(z) |0(P)| \big|_{z^+ = 0}.
\]

With \( \tilde{\phi}_+ \) the factorization formula reads:

\[
F(Q^2) = \frac{1}{Q^2} \int dx dy dz d^2k \tilde{\phi}_+(x, k, \zeta, \zeta_v, \mu) H(x, Q^2, \zeta, \zeta_v, \mu) + O(Q^{-4}).
\]

In principle one can re-define the TMD LCWF as that given in Eq. (11). With the redefined TMD LCWF the factorization takes a simple form. However, we do not intend to do so because it is unclear if the same re-definition is useful for other processes whose TMD factorization have been not studied in detail yet. We will call \( \phi_+ \) as a modified TMD LCWF in this work. It should be kept in mind that the soft factor is there.

With the one-loop result for every factor, we can work out the evolution equations for the hard part,

\[
\mu \frac{\partial}{\partial \mu} \ln H(x, Q^2, \zeta^2, \zeta_v^2, \mu^2) = \frac{2\alpha_s}{3\pi} \left[ 1 + 2 \ln \frac{\zeta^2}{\zeta_v^2} \right] + O(\alpha_s^2)
\]

\[
\zeta \frac{\partial}{\partial \zeta} \ln H(x, Q^2, \zeta^2, \zeta_v^2, \mu^2) = \frac{2\alpha_s}{3\pi} \left[ -2 + 2 \ln \frac{\zeta^2}{\mu^2} + 4 \ln(x \bar{x}) \right] + O(\alpha_s^2)
\]

\[
\zeta_v \frac{\partial}{\partial \zeta_v} \ln H(x, Q^2, \zeta^2, \zeta_v^2, \mu^2) = \frac{2\alpha_s}{3\pi} \left[ -2 \ln \frac{\zeta^2}{\mu^2} - 4 \ln(x \bar{x}) + 2 \ln \bar{x} \ln x \right] + O(\alpha_s^2).
\]

The corresponding evolutions of \( \tilde{\phi}_+ \) reads:

\[
\mu \frac{\partial}{\partial \mu} \ln \tilde{\phi}_+(x, k, \zeta^2, \zeta_v^2, \mu^2) = \frac{2\alpha_s}{3\pi} \left[ 1 + 2 \ln \frac{\zeta^2}{\zeta_v^2} \right] + O(\alpha_s^2)
\]

\[
\zeta \frac{\partial}{\partial \zeta} \ln \tilde{\phi}_+(x, k, \zeta^2, \zeta_v^2, \mu^2) = \frac{2\alpha_s}{3\pi} \left[ -2 - 2 \ln \frac{\zeta^2}{\mu^2} - 2 \ln(x \bar{x}) \right] + O(\alpha_s^2),
\]

\[
\zeta_v \frac{\partial}{\partial \zeta_v} \ln \tilde{\phi}_+(x, k, \zeta^2, \zeta_v^2, \mu^2) = \frac{2\alpha_s}{3\pi} \left[ -2 \ln \frac{\zeta^2}{\mu^2} - 2 \ln(x \bar{x}) \right] + O(\alpha_s^2).
\]
the evolution with $\zeta_v$ does not take a simple form, it takes a convolution form:

$$\zeta_v \frac{\partial}{\partial \zeta_v} \tilde{\phi}_+(x, k_\perp, \zeta^2, \zeta_v^2, \mu^2) = \int_0^1 dy C_v(x, y, \zeta_v, \mu) \tilde{\phi}_+(y, k_\perp, \zeta^2, \zeta_v^2, \mu^2),$$  \hspace{1cm} (45)

with

$$C_v(x, y) = \frac{2\alpha_s}{3\pi} \left\{ \delta(x - y) \left( 2 \ln \frac{\zeta_v^2 y \bar{y}}{\mu^2} - 4 \right) - 2 \left[ \frac{1}{x - y} \left( \theta(y - x) \frac{x}{y} - \theta(x - y) \frac{\bar{x}}{y} \right) \right]_+ + 2 \left( \theta(y - x) \frac{x}{y} + \theta(x - y) \frac{\bar{x}}{y} \right) \right\} ,$$  \hspace{1cm} (46)

where the last line comes from the box diagram. The $+$ prescription is for $x$. It should be noted that:

$$\int_0^1 dx \frac{1}{x \bar{x}} \left[ \frac{1}{x - y} \left( \theta(y - x) \frac{x}{y} - \theta(x - y) \frac{\bar{x}}{y} \right) \right]_+ = -\frac{1}{yy} [2 + \ln(yy)].$$

$$\int_0^1 dx \frac{1}{x \bar{x}} \left( \theta(y - x) \frac{x}{y} + \theta(x - y) \frac{\bar{x}}{y} \right) = -\frac{1}{yy} (\bar{y} \ln \bar{y} + y \ln y).$$  \hspace{1cm} (47)

With the above equations one can easily check that the form factor does not depend on $\mu, \zeta$ and $\zeta_v$, as expected.

In Eq.(38) there are different possibly large logarithms. Beside those resulted from different scales like $\zeta^2, \zeta_v^2$... etc., there are large logarithms around the end points of $x$, like $\ln^2 x$ and $\ln x$ if $x$ goes to 0, and $\ln^2 (1 - x)$ and $\ln (1 - x)$ if $x$ goes to 1. These logarithms also appear in the collinear factorization. They make the perturbative expansion of $H$ unreliable and must be resummed. With the TMD factorization, the resummations can be accomplished in a simple way.

The result for $H$ can be simplified by noting that the modified TMD LCWF is symmetric under the exchange $x \leftrightarrow 1 - x$. It is convenient to introduce a parameter $\rho$ to relate $\zeta_v^2$ and $\zeta^2$:

$$\zeta_v^2 = \frac{u^2 v^2}{4(v \cdot u)^2} \zeta^2 = \frac{\zeta^2}{\rho^2}, \quad \rho = \sqrt{\frac{v^+ u^-}{v^- u^+}}.$$  \hspace{1cm} (48)

The factorization in Eq.(42) now reads:

$$F(Q^2) = \frac{1}{Q^2} \int dx d^2 k_\perp \tilde{\phi}_+(x, k_\perp, \zeta, \rho, \mu) H(x, Q, \zeta, \rho, \mu) + O(Q^{-4}).$$  \hspace{1cm} (49)

and

$$H^{(1)}(x, Q, \zeta, \rho, \mu) = \frac{2\alpha_s}{3\pi x \bar{x}} \left[ \frac{1}{2} \ln \frac{Q^2}{\zeta^2} + \ln \frac{Q^2}{\rho^2} \right] - 2 \ln \frac{\zeta^2}{Q^2} \frac{1}{2} \ln (x - x \ln x) + 2 \ln^2 (2 \ln x - x \ln x) \] + \frac{2\alpha_s}{3\pi x \bar{x}} \left[ -\frac{5}{2} \bar{x} \ln^2 x + x \ln x - 2 \ln x + 4 \bar{x} \ln \bar{x} \right],$$  \hspace{1cm} (50)
where we have used the property \( \tilde{\phi}_+(x, k, \zeta, \rho, \mu) = \tilde{\phi}_+(1 - x, k, \zeta, \rho, \mu) \) to simplify the hard part \( H \). The evolution of \( H \) with various scales reads:

\[
\begin{align*}
\mu & \frac{\partial}{\partial \mu} \ln H(x, Q, \zeta, \rho, \mu) = -\frac{2\alpha_s}{3\pi} \left[ 1 + 2 \ln \rho^2 \right] + \mathcal{O}(\alpha_s^2) \\
\zeta & \frac{\partial}{\partial \zeta} \ln H(x, Q, \zeta, \rho, \mu) = -\frac{4\alpha_s}{3\pi} \left[ 1 - \ln \rho^2 + 2 \ln x - 2x \ln x \right] + \mathcal{O}(\alpha_s^2) \\
\rho & \frac{\partial}{\partial \rho} \ln H(x, Q, \zeta, \rho, \mu) = \frac{4\alpha_s}{3\pi} \left[ -\ln \rho^2 + \ln \frac{\zeta^2}{\mu^2} + 4 \ln x - 2x \ln x \right] + \mathcal{O}(\alpha_s^2). \\
\end{align*}
\]

(51)

To avoid large logarithms in the wave function, we first choose appropriate scales to eliminate them

\[
F(Q^2) = \frac{1}{Q^2} \lim_{b \to 0} \int_0^1 dx \tilde{\phi}_+(x, \zeta_L, \rho_L, \mu_L) H(x, Q, \zeta_L, \rho_L, \mu_L) + \mathcal{O}(Q^{-4}).
\]

(52)

If we use the result in Eq. (50), there will be large logarithms in \( H(x, Q, \zeta_L, \rho_L, \mu_L) \), especially terms like \( \ln^2 x, \ln x \) when \( x \to 0 \). It is possible to resum those large logarithms terms by using the evolutions in Eq. (51) to express \( H(x, Q, \zeta_L, \rho_L, \mu_L) \) as

\[
H(x, Q, \zeta_L, \rho_L, \mu_L) = e^{-S} H(x, Q, \zeta_H, \rho_H, \mu_H).
\]

(53)

This represents the resummed form for the hard part. By taking appreciated scales \( \zeta_H, \rho_H \) and \( \mu_H \) so that \( H(x, Q, \zeta_H, \rho_H, \mu_H) \) does not contain any large logarithms. It should be noted that \( \ln \rho_L \) is not a large logarithms. We first use the \( \mu \)-evolution to evolute the hard part to a high scale \( \mu_H \) and take \( \mu_H = Q \). As the next we use the \( \zeta \)-evolution to evolute \( H \) to a high scale \( \zeta_H = Q \). After these the hard part reads:

\[
H(x, Q, \zeta_L, \rho_L, \mu_L) = \exp \left\{ - \int_{\mu L}^{\mu H} \frac{d\mu}{\mu} \cdot \frac{2\alpha_s(\mu)}{3\pi} \left[ 1 + 2 \ln \rho_L^2 \right] \right\} \cdot H(x, Q, \zeta_H, \rho_H, \mu_H).
\]

(54)

Now the hard part only contains possibly large logarithms \( \ln^2 x \) and \( \ln x \) when \( x \) goes to zero. To resum these logarithms we note that

\[
-\frac{1}{2} \ln^2 \rho^2 + 4 \ln \rho^2 \ln x - \ln^2 x - 2 \ln x = -\frac{1}{2} \left[ \ln^2 \frac{\rho^2}{x^4} - \ln^2 \frac{\rho_H^2}{x^4} \right] - \frac{1}{7},
\]

\[
\rho_H^2 = \rho_H^2(x) = x^{4+\sqrt{11}} e^{-\sqrt{2/7}}.
\]

(55)

Using the \( \rho \)-evolution we have:

\[
H(x, Q, \zeta_L, \rho_L, \mu_L) = \exp(-S) \cdot H(x, Q, \rho_H, Q),
\]

\[
S = \frac{2\alpha_s(Q)}{3\pi} \left\{ \left[ 1 - \ln \rho_L^2 + 2\bar{x} \ln x \right] \ln \frac{\zeta_H^2}{\zeta_L^2} + \ln^2 x + 2 \ln x - \frac{1}{7} - 2 \ln x \ln \rho_H^2 \right\}.
\]

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\[ H(x, Q, Q, \rho_H, Q) = \frac{1}{x^2} \left\{ 1 + \frac{2\alpha_s(Q)}{3\pi} \left[ -\frac{1}{x} - 2x \ln \rho_L^2 \ln x + x \ln^2 x \right. \right. \\
\left. \left. + x \ln x + 4\bar{x} \text{Li}_2(x) - \frac{5}{2} \right] \right\}, \] (56)

Now there is no large logarithms in \( H(x, Q, Q, \rho_H, Q) \). It is instructive to compare the result of \( H(x, Q, \zeta_L, \rho_L, \rho_L) \) with and without the resummation for \( x \to 0 \). Without the resummation the result for \( x \to 0 \) reads:

\[ xH(x, Q, \zeta_L, \rho_L, \rho_L) \sim 1 - \frac{2\alpha_s(\mu_L)}{3\pi} \ln^2 x, \] (57)

therefore the one-loop correction can be very large and is divergent. After the resummation the result for \( x \to 0 \) reads:

\[ xH(x, Q, \zeta_L, \rho_L, \mu_L) \sim x^{-\frac{2\alpha_s(Q)}{3\pi} \ln x}, \] (58)

which will approaches to 0 faster than any positive power of \( x \).

6. Summary

In this work we start with a consistent definition of TMD LCWF’s to explicitly show that the TMD factorization for the process \( \gamma^* \pi^0 \to \gamma \) can be made at one-loop level, where a soft factor must be implemented. The soft factor is defined as a product of gauge links. It makes the hard part free from I.R. singularities. In contrast to the corresponding collinear factorization, where the amplitude is factorized as a convolution with the standard light-cone wave function and a hard part, the amplitude in the TMD factorization is factorized as a convolution of a TMD light-cone wave function, a soft factor and a hard part. From our results, the cancelation of all soft divergences happens in a diagram-by-diagram way, hence it is possible to extend the factorization beyond one-loop level.

With the factorization, the hard part can be safely calculated in a perturbative expansion in \( \alpha_s \). But in general results from perturbative expansions have large logarithms, which can make the results unreliable. A resummation of these large logarithms is needed. With the TMD factorization we show that the resummation in our case can be done simply. It should be mentioned that it is possible to show that the two factorization are equivalent, as the study of TMD factorization for the radiative leptonic decay of B-meson shows[11].

After the detailed examination of TMD factorization for the simple cases involving one hadron only, we can proceed to study more complicated cases involving more hadrons, particularly \( B \)-decays into light hadrons.

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