Bose-Einstein condensation in rainbow gravity

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In this paper, we study the effects of rainbow gravity on relativistic Bose-Einstein condensation and thermodynamics parameters. We initially discussed some formal aspects of the model to only then compute the corrections to the Bose-Einstein condensation. The calculations were carried out by computing the generating functional, from which we extract the thermodynamics parameters. The corrected critical temperature \( T_c \) that sets the Bose-Einstein Condensation was also computed for the three mostly adopted cases for the rainbow functions. We have also obtained a phenomenological upper bound for a combination of the quantities involved in the model, besides showing the possibility of occurrence of the Bose-Einstein condensation in two spatial dimensions under appropriate conditions on those functions. Finally, we have discussed how harder is for the particles at an arbitrary temperature \( T < T_c \) to enter the condensed state when compared with the usual scenario.

I. INTRODUCTION

The rainbow models of quantum gravity presuppose that there exists an invariant energy scale associated with Planck’s one \([1–3]\). According to these theoretical proposals, the nonlinear representation of the Lorentz transformations in momentum space yields an energy-dependent spacetime geometry, leading to a modification of the relativistic energy-momentum relation \([4]\), in a type of spacetime back-reaction. This means that the trajectory of a particle will be affected by that spacetime according to its energy, and thus we can talk about a ‘running geometry’.

Such a semi-classical approach allows explaining, for instance, the currently observed ultra-high-energy cosmic rays, whose origin is still unknown, suggesting that the dispersion relation is in fact modified, which opens up new possibilities for theoretical developments in astrophysics and cosmology \([5–13]\). In this sense the rainbow gravity can also play a decisive role in the very early Universe, in which the involved energies are next to the Planck scale, contributing thus to avoid the initial singularity \([9, 11, 12]\).

The Bose-Einstein condensed (BEC) can have been present in the primordial Universe \([14]\) at some critical redshift parameter \( z \). It can even be identified to the early dark matter, if this was constituted by bosons, with its perturbations possibly leaving an imprint in CMB \([15]\), and references therein). Therefore, in such cosmological scenarios, it is quite reasonable to admit that the early BEC was under the influence of the rainbow gravity, as well as in actual high energy events.

In the context of high-energy physics, the BEC was recently studied in connection with Lorentz symmetry violation \([16–18]\). It was shown in \([16, 17]\) how the CPT-even and CPT-odd Lorentz violating terms modify the BEC critical temperature and the thermodynamic parameters. In \([19]\) the influence of the anisotropic scaling in the BEC was discussed. The possibility of probing low-energy Lorentz violating effects in dipolar BEC was addressed in \([18]\). Also, the BEC in Rindler space and its occurrence due to Unruh effect was studied in \([20]\).

In this paper, we study the effects of rainbow gravity on Bose-Einstein condensation and on the thermodynamics parameters. We initially discussed some formal aspects of the model to only then compute the corrections to the Bose-Einstein condensation. The calculations were carried out by computing the generating functional, from which we extract the thermodynamics parameters. The corrected critical temperature \( T_c \) that sets the Bose-Einstein Condensation was also computed for the three mostly adopted cases for the rainbow functions. Moreover, we have obtained an upper bound for a combination of the quantities involved in the model besides showing the possibility of occurrence of the Bose-Einstein condensation in two dimensions under appropriate conditions on the rainbow functions. Finally, we have discussed how harder is for the particles at an arbitrary temperature \( T < T_c \) to enter the condensed state when compared with the usual scenario.

The model we are considering consists of rainbow gravity extension of the complex scalar sector. Hence the lagrangian describing the system is

\[
\mathcal{L} = f^2(\epsilon)(\partial_0 \phi)^4(\partial_0 \phi) - g^2(\epsilon)(\partial_0 \phi)^4(\partial_0 \phi) - m^2 \phi^4 \phi, \tag{1}
\]
where \( f(\epsilon) \) and \( g(\epsilon) \) are the so-called called rainbow functions, and \( \epsilon = E/E_p \) being \( E \) the energy of the probe particle and \( E_p \) the Planck energy.

The lagrangian \([1]\) possesses a clear \( U(1) \) symmetry, so that
\[
\phi \rightarrow \phi' = e^{-i \alpha} \phi,
\]
with \( \alpha \in \mathbb{R} \). From Noether's theorem it is known that for any given continuous symmetry there is a conserved quantity in connection. To find out such conserved quantity let us consider \( \alpha = a(x) \), i.e., a spacetime position-dependent function. The Euler-Lagrange equation gives us the equation of motion for the “field” \( a(x) \). Since the contribution \( \partial \mathcal{L}/\partial \alpha = 0 \), we can find the following charge density
\[
Q = \int d^3 x f^0 \phi \partial \phi \phi_0 \phi_0 \phi_0 - \frac{g^2(\epsilon)}{2} \partial_\phi \phi_0 \phi_0 - \frac{m^2}{2} \phi_0 \phi_0 .
\]

The equations of motion for \( \phi \) and \( \phi^\dagger \), given by
\[
-\frac{f^2(\epsilon)}{2} \partial_\phi \phi_0 \phi_0 + \frac{g^2(\epsilon)}{2} \partial_\phi \phi_0 \phi_0 - m^2 \phi = 0
\]
\[
-\frac{f^2(\epsilon)}{2} \partial_\phi \phi_0 \phi_0 + \frac{g^2(\epsilon)}{2} \partial_\phi \phi_0 \phi_0 - m^2 \phi = 0,
\]
shows us, by a straightforward calculation, the conservation of the four current, i.e., \( \partial_\mu f^\mu = 0 \). Splitting the fields \( \phi \) and \( \phi^\dagger \) into two real components \( \phi_1 \) and \( \phi_2 \) as
\[
\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2),
\]
\[
\phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2),
\]
we can rewrite the Lagrangian, more conveniently, in terms of \( \phi_\alpha \) with \( \alpha = 1, 2 \) as follows
\[
\mathcal{L} = \frac{f^2(\epsilon)}{2} \partial_\phi \phi_0 \phi_0 - \frac{g^2(\epsilon)}{2} \partial_\phi \phi_0 \phi_0 - \frac{m^2}{2} \phi_0 \phi_0 .
\]
The canonically conjugated momenta are:
\[
\pi_\alpha = \frac{\partial \mathcal{L}}{\partial \phi_\alpha}
\]
then the Hamiltonian becomes
\[
\mathcal{H} = \frac{1}{2} \left[ \pi_\alpha \pi_\alpha + f^2(\epsilon) \phi_0 \phi_0 \phi_0 \phi_0 + m^2 \phi_0 \phi_0 \right].
\]
We can also express the charge density in terms of \( \phi_\alpha \),
\[
Q = \int d^3 x \epsilon_{\mu\nu\rho} \pi_\mu \pi_\nu \phi_\rho.
\]
Letting \( \mathcal{H}(\phi, \pi) \rightarrow \mathcal{H}(\phi, \pi) - \mu N(\phi, \pi) \), where \( N(\phi, \pi) \) is the conserved charge density, identified as \( Q \), and \( \mu \) is the chemical potential, the partition function becomes,
\[
Z = \int D\phi \int \text{periodic} \ D\pi \exp \left\{ \int_0^\beta d\tau \int d^3 x \left[ i \pi_\alpha \partial_\tau \phi_\alpha - \mathcal{H}(\phi_\alpha, \pi_\alpha) + \mu \epsilon_{\mu\nu\rho} \pi_\mu \phi_\rho \right] \right\}.
\]
The term “periodic” stands for the fact that the field is constrained in such way that \( \phi(x, 0) = \phi(x, \beta) \) with \( \beta = 1/T \). The partition function can be written as
\[
Z = \int D\pi \int \text{periodic} \ D\phi \exp \left\{ \int_0^\beta d\tau \int d^3 x \left[ -\frac{1}{2 f^2(\epsilon)} \pi_\alpha^2 + \left( \frac{\partial \phi_\alpha}{\partial \tau} - \mu \epsilon_{\mu\nu\rho} \pi_\mu \right) \pi_\alpha - \frac{g^2(\epsilon)}{2} \left( \nabla \phi_\alpha \right)^2 - \frac{m^2}{2} \phi_\alpha^2 \right] \right\}
\]
\[
(13)
\]
The integration over the momenta can be directly done. Then we obtain,
\[
Z = (N')^2 \int D\phi \exp \left\{ \int_0^\beta d\tau \int d^3 x \left[ -\frac{f^2(\epsilon)}{2} \left( \frac{\partial \phi_\alpha}{\partial \tau} - \mu \epsilon_{\mu\nu\rho} \pi_\mu \right)^2 - \frac{g^2(\epsilon)}{2} \left( \nabla \phi_\alpha \right)^2 - \frac{m^2}{2} \phi_\alpha^2 \right] \right\}.
\]
\[
(14)
\]
The factor \( N' \) is a normalization constant, but since multiplication of \( Z \) by any constant does not change the thermodynamics the factor \( N' \) is irrelevant. The components of \( \phi \) can be Fourier-expanded as,
\[
\phi_1 = \sqrt{2} \zeta \cos \theta + \sqrt{2} \sum_n \sum_\rho e^{i \beta n\omega a \tau} \phi_{1,n}(\vec{p}),
\]
\[
\phi_2 = \sqrt{2} \zeta \sin \theta + \sum_n \sum_\rho e^{i \beta n\omega a \tau} \phi_{2,n}(\vec{p}),
\]
where \( \omega_n = 2\pi n T \), owing to the constraint of periodicity that \( \phi(x, \beta) = \phi(x, 0) \) for all \( x \). Here \( \zeta \) and \( \theta \) are spacetime position independent parameters and determine the full infrared behaviour of the field; that is, \( \phi_{1,0}(\vec{p} = 0) = \phi_{2,0}(\vec{p} = 0) = 0 \). This allows for the possibility of condensation of the bosons into the zero-momentum state. Substituting \([15]\) into \([13]\) the partition function becomes
\[
Z = (N')^2 \prod_n \prod_\rho \int D\phi_{1,n}(\vec{p}) D\phi_{2,n}(\vec{p}) e^S.
\]
\[
(17)
\]
being \( S \) is given by
\[
S = \beta V \left( f^2(\epsilon) \mu^2 - m^2 \right) \zeta^2 - \frac{1}{2} \sum_n \sum_\rho \left( \phi_{1,-n}(\vec{p}) \phi_{2,-n}(\vec{p}) \right) D \left( \phi_{1,n}(\vec{p}) \phi_{1,n}(\vec{p}) \right),
\]
\[
(18)
\]
where \( D \) is
\[
D = \beta^2 \left( f^2(\epsilon) \omega_n^2 + \omega_n^2 - f^2(\epsilon) \mu^2 \right) \frac{2 f^2(\epsilon) \mu \omega_n}{2 f^2(\epsilon) \omega_n} - 2 f^2(\epsilon) \mu \omega_n \frac{2 f^2(\epsilon) \omega_n}{2 f^2(\epsilon) \omega_n}
\]
\[
(19)
\]
with \( \omega_n = \sqrt{f^2(\epsilon) \beta^2 + m^2} \). Performing the integrations over \( \phi_{1,n} \) and \( \phi_{2,n} \), we have,
\[
\ln Z = \beta V (f^2(\epsilon) \mu^2 - m^2) \zeta^2 + \ln(\det D)^{-1/2},
\]
\[
(20)
\]
so that we can rewrite the following form

$$\ln Z = \beta V(f^2(\mu)^2 - m^2)\xi^2 - V \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{g}{f} \right\} \omega $$

$$+ \ln \left[ 1 - e^{-\beta(\mu_+ - \mu)} \right] + \ln \left[ 1 - e^{-\beta(\mu_+ + \mu)} \right] \right\} \right\}$$

(21)

Let us highlight here that the above expression for \(\ln Z\) was obtained under the consideration of a convergence criteria which states that

$$|f(\mu)| \leq m.$$  

(22)

This is modification of the usual convergence condition \(|\mu| \leq m\) very well known in the literature, which was first stated in the work of Haber and Weldon [21]. The usual relation

$$\frac{PV}{T} = \ln Z,$$

(23)

gives us the equation of state for the system. Hence the pressure is given by

$$P = \langle f^2(\mu)^2 - m^2 \rangle \xi^2 - \frac{1}{\beta} \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{g}{f} \right\} \omega $$

$$+ \ln \left[ 1 - e^{-\beta(\mu_+ - \mu)} \right] + \ln \left[ 1 - e^{-\beta(\mu_+ + \mu)} \right] \right\} \right\}$$

(24)

The internal energy can be written as

$$E = -\frac{\partial}{\partial \beta} \ln Z$$

$$= -V(f^2(\mu)^2 - m^4)\xi^2$$

$$-V \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\omega}{f} e^{\beta(\mu_+ - \mu)} - 1 \right\}$$

(25)

The specific heat at constant volume is expressed by

$$C_v = \frac{\partial E}{\partial T}$$

$$= -V \beta^2 \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\mu - \frac{\omega}{f} - \mu}{1 - e^{-\beta(\mu_+ - \mu)}} \right\} $$

$$+ \frac{\mu - \frac{\omega}{f} - \mu}{e^{-\beta(\mu_+ + \mu)} - 1} - \frac{\mu - \frac{\omega}{f} - \mu}{e^{-\beta(\mu_+ + \mu)} - 1}$$

(26)

The charge density is written as

$$\rho = \frac{1}{\beta V} \frac{\partial \ln Z}{\partial \mu}$$

$$= 2f^2(\mu)^2 + \int \frac{d^3p}{(2\pi)^3} \left\{ e^{-\beta(\mu_+ - \mu)} - e^{-\beta(\mu_+ + \mu)} \right\}$$

(27)

At this point it is important to discuss the role played by the rainbow functions in the critical temperature that sets the BEC. Hence, in what follows we will consider the three mostly adopted cases for the rainbow functions.

A. case I

Let us consider the case when

$$f(\epsilon) = E/\epsilon, \quad g(\epsilon) = \sqrt{1 - \xi(E/E_\rho)},$$

(28)

where \(E_\rho\) is the Planck energy, that works as a natural cutoff of the system. Thus, in order to solve the integral in Eq. (27), we will use spherical coordinates and change its integration measure for perform it with respect to \(E = \omega\), such that \(d^3p = 4\pi p^2 dp = 4\pi p^2(E)(dp/dE)dE\). Following, we expand the integrand around \(\xi = 0\) up to first order (making \(s = 1\)) and then around \(m/T = 0\) up to same order. After these steps, we operate the integration from \(E = 0\) to \(E = \infty\) and expand the result for high temperatures. The obtained leading terms are

$$\rho \approx \frac{1}{3} \frac{mT_c^3}{\pi^2} \left[ 1 + \frac{36\xi(3)}{\pi^2 T_c} \right].$$

(29)

To obtain this result, we have taken into account that the chemical potential must be equal to the upper limit of the convergence criteria, i.e., \(\mu = \pm m\). Moreover, in this case we can see that \((\xi = 0)\) recovers the usual BEC critical temperature [21].

We can infer an upperbound in the factor \(\xi T_c/E_\rho\) given in Eq. (29) by considering experiments in high energy collisions involving pions, for which the corresponding BEC was reached at a range of critical temperatures of the order of MeV [22]. Thus, based on the uncertainty in the current measurements of the charged pion mass given in [23], \(\Delta m_\pi/m_\pi \approx 10^{-6}\), we obtain

$$\xi T_c/E_\rho \leq 10^{-7}.$$  

(30)
In the 2+1 dimensional Minkowski spacetime, the relativistic BEC cannot exist, according to [21]. However, when we consider UV improvement through introduction of the rainbow functions, the condensed formation is possible, at least for $s > 1$, in the case I. In FIG. 1 we depict the charge density as a function of the critical temperature.

![Charge density as a function of the critical temperature](image)

**FIG. 1:** Charge density as a function of the critical temperature, for some values of $s$, of a rainbow relativistic BEC in 2 + 1 spacetime dimensions, in the case I. We have considered $m = 0.1, \xi = 0.01$, in Planckian units ($E_P = 1$).

B. case II

Considering the case when

$$ f(\epsilon = E/E_P) = g(\epsilon = E/E_P) = \frac{1}{1 - \xi(E/E_P)} \quad (31) $$

For this case, the same procedure employed in the previous case yields the following correction to the critical temperature that sets the BEC,

$$ \rho \approx \frac{1}{3} m T_c^2 \left(1 + \xi \frac{54 \xi(3)}{\pi^2 E_P T_c}\right). \quad (32) $$

The upper bound in the factor $\xi T_c/E_P$ for the present case keeps unchanged since the slight modification of the equation (32) in comparison with (29) does not affect the magnitude order of (30).

C. case III

Finally let us consider the case when

$$ f(\epsilon = E/E_P) = g(\epsilon = E/E_P) = \frac{e^{\xi(E/E_P)} - 1}{\xi(E/E_P)}, \quad g(\epsilon = E/E_P) = 1 \quad (33) $$

Similarly, for the BEC to occur we must have the chemical potential equals to the upper limit of the convergence criteria, i.e., $\mu = \pm \frac{m}{E_P}$. Setting such condition into (27) we obtain the corrected critical temperature for the present case as

$$ \rho \approx \frac{1}{3} m T_c^2 \left(1 + \xi \frac{63 \xi(3)}{\pi^2 E_P T_c}\right). \quad (34) $$

Also there is no significant modification in the bound presented in (30).

Lastly, the fraction of the particles which are not in the 4-D relativistic BEC at an arbitrary temperature, $T < T_c$, is given by

$$ \frac{\Delta N}{N} = 1 - \frac{T^2}{T_c^2} \left[1 - k(T_c - T)\right], \quad (35) $$

in first order of $\xi$, with $k \geq 0$ corresponding to the factors which multiply the temperature in the parenthesis given in Eqs. (29), (32), and (34). We can notice that such a fraction is greater than the one registered in the usual case (i.e., without the rainbow functions, $k = 0$).

### III. FINAL REMARKS

In this work, we study the the Bose-Einstein condensation in the context of rainbow gravity. By computing the generating functional we extract the thermodynamics parameters, such as pressure, energy, specific heat at constant volume and charge density. From the charge density we computed the corrected critical temperature $T_c$ that sets the Bose-Einstein Condensation for the three mostly adopted cases for the rainbow functions. For each rainbow function case we obtained a cubic correction on the critical temperature.

The corrected critical temperature allowed us to obtain an upper bound for a combination of the quantities involved in the model from the experiments in high energy collisions involving pions. The upper bound obtained is the same for the three rainbow function cases since the slight modification in the critical temperature for the three cases considered does not affect the order of magnitude of the parameter $\xi$.

Another important feature discussed graphically is the possibility of occurrence of the Bose-Einstein condensation in two spatial dimensions under appropriate conditions on the rainbow function. Thus, the case I was investigated, allowing to compare different ways of the energy cutoff to influence the relativistic 2+1 BEC.

Finally, we have analyzed the fraction of the particles out of the BEC at a temperature lower than the critical one, and verified that such a fraction is greater than the one observed in the usual case, meaning that the barrier imposed by the $E_P$ cutoff turns harder the entry of these particles in the condensed state.

A future perspective of this work consists in the consideration of self-interacting scalar fields.
