Approximation algorithms for the generalized incremental knapsack problem

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Abstract
We introduce and study a discrete multi-period extension of the classical knapsack problem, dubbed generalized incremental knapsack. In this setting, we are given a set of $n$ items, each associated with a non-negative weight, and $T$ time periods with non-decreasing capacities $W_1 \leq \cdots \leq W_T$. When item $i$ is inserted at time $t$, we gain a profit of $p_{it}$; however, this item remains in the knapsack for all subsequent periods. The goal is to decide if and when to insert each item, subject to the time-dependent capacity constraints, with the objective of maximizing our total profit. Interestingly, this setting subsumes as special cases a number of recently-studied incremental knapsack problems, all known to be strongly NP-hard. Our first contribution comes in the form of a polynomial-time $(1 - \frac{1}{2} - \epsilon)$-approximation for the generalized incremental knapsack problem. This result is based on a reformulation as a single-machine sequencing problem, which is addressed by blending dynamic programming techniques and the classical Shmoys–Tardos algorithm for the generalized assignment problem. Combined with further enumeration-based self-reinforcing ideas and new structural properties of nearly-optimal solutions, we turn our algorithm into a quasi-polynomial time approximation scheme (QPTAS). Hence, under widely believed complexity assumptions, this finding rules out the possibility that generalized incremental knapsack is APX-hard.

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1 Introduction

In many scenarios, classical optimization models are too simplistic to faithfully capture applications arising in real-life environments. Much research has therefore been devoted to extend fundamental well-studied models to more realistic, yet still algorithmically tractable settings. A very common extension along these lines introduces time-dependent components, adding a computationally-challenging layer on top of the inherent complexity of the underlying problem. For instance, maximum flow over time, originally introduced in the seminal work of Ford and Fulkerson [23], has recently received a great deal of attention [27,31,36]. Additional examples for such settings include time-expanded versions of various packing problems [1,12,18], network scheduling over time [2,9], adaptive routing over time [24,30], and facility location over time [20,32], just to mention a few.

Incremental knapsack problems. In this paper, we investigate a multi-period extension of the classical knapsack problem. To provide initial intuition for the inner-workings of our model, consider the problem faced by urban planners, who intend to build infrastructural facilities over the course of several years, under budget constraints. Once an infrastructure has been built, its construction cost cannot be recovered. Given each infrastructure’s annual contribution to welfare once it is in place, the goal is to maximize the total benefit over the course of the planning horizon (hence, the mayor’s chances of being re-elected). A host of additional applications, such as planning the incremental growth of highways and networks, community development, and memory allocation can be found within several of the papers mentioned in Sect. 1.2 and the references therein.

Computational questions of this nature can be modeled via multi-period knapsack extensions, collectively dubbed as incremental knapsack problems. In such settings, the input ingredients consist in a set of \( n \) items with strictly positive weights \( \{w_i\}_{i \in [n]} \), a collection of \( T \) time periods with non-decreasing capacities \( W_1 \leq \cdots \leq W_T \), and a set of item-period profits, on which we further elaborate below. We say that a sequence of item sets \( \mathcal{S} = (S_1, \ldots, S_T) \) is a chain when \( S_1 \subseteq \cdots \subseteq S_T \subseteq [n] \); here, \( S_t \) represents the subset of items inserted into the knapsack up to and including time period \( t \). As such, the chain \( \mathcal{S} \) is feasible when \( w(S_t) \leq W_t \) for every \( t \in [T] \).

Our fundamental assumption is that, for each item \( i \in [n] \) and time \( t \in [T] \), we are given a non-negative parameter \( p_{it} \), corresponding to the profit we obtain when item \( i \) is inserted at time \( t \) (i.e., when \( i \in S_t \setminus S_{t-1} \), with the convention that \( S_0 = \emptyset \)). Hence, the cumulative profit of any chain \( \mathcal{S} = (S_1, \ldots, S_T) \) over all time periods is captured by \( \Phi(\mathcal{S}) = \sum_{t \in [T]} \sum_{i \in S_t \setminus S_{t-1}} p_{it} \). We refer to the resulting formulation as the generalized incremental knapsack problem.

Due to its double-dependency on both the item and time period in question, the above-mentioned profit structure makes generalized incremental knapsack the most
inclusive incremental knapsack problem studied so far. In support of this statement, we provide an overview of recently-studied special cases and directly-related settings in Sect. 1.2. Moreover, we are not aware of any way to leverage existing techniques within this line of work for dealing with the broad generality of our profit structure. Along these lines, we further explain in Sect. 1.2 (also see “Appendix A”) how one can obtain a slightly sublogarithmic approximation via an appropriate reduction to unsplittable flow on a path with bag constraints. However, this performance guarantee is still far from our intended objective.

1.1 Contributions and techniques: high-level overview

*Constant-factor approximation.* Our first contribution comes in the form of a polynomial-time constant-factor approximation for the generalized incremental knapsack problem, whose specifics are provided in Sect. 2.

**Theorem 1** For any fixed $\epsilon \in (0, \frac{1}{2})$, the generalized incremental knapsack problem can be approximated in polynomial time within factor $\frac{1}{2} - \epsilon$.

From a technical perspective, a common approach to handling various packing problems consists of dividing items into *light* and *heavy*, depending on how their weight relates to the overall capacity. This idea leads to two restricted problems, one for heavy items and the other for light ones, that can be separately addressed, given subadditivity properties of the objective function. Typically, the heavy-items problem is approached via enumeration techniques, while its light-items counterpart may be approximated via appropriate LP-rounding methods.

However, in the generalized incremental knapsack problem, it is unclear whether a dichotomy of this nature makes sense, due to having multiple time-dependent capacities. To overcome this issue, we present an equivalent *single-machine sequencing* formulation, where feasible chains are mapped to permutations, with $\pi(i_1) < \pi(i_2)$ indicating that the insertion time of item $i_1$ precedes that of $i_2$. Based on this reformulation, we decompose the optimal permutation into heavy and light parts, depending on how item weights compare against the combined weight of all previously-inserted items. This sequencing reformulation is, to our knowledge, very different from how existing related problems were approached up until now. At a high level, to compete against the heavy part, we exploit dynamic programming ideas, whereas for the light part, we propose a further reformulation as a highly-structured generalized assignment instance, which is handled by leveraging the Shmoys–Tardos algorithm [35] and an additional greedy truncation phase. Consequently, we show that both subproblems can be approximated within factor $1 - \epsilon$ of optimal, thus leading to Theorem 1.

*Approximation schemes.* As explained in Sect. 1.2, even seemingly-simple special cases of the generalized incremental knapsack problem are known to be strongly NP-hard, admitting a PTAS under specific profit-structure assumptions. A natural question is whether one can design efficient algorithms with the same degree of accuracy for generalized incremental knapsack, without any mitigating assumptions. Towards this goal, our second main contribution establishes the existence of a quasi-PTAS across Sects. 3 and 4.
Theorem 2  The generalized incremental knapsack problem admits a quasi-polynomial time approximation scheme.

Hence, under widely believed complexity assumptions, this finding rules out the possibility that generalized incremental knapsack is APX-hard, thus making it substantially different from other knapsack extensions, such as the generalized assignment problem (see brief discussion in Sect. 1.2).

Our main technical idea in this context resides in showing how to boost the approximation guarantee of Theorem 1 via a suitable guessing procedure. More precisely, we argue that one can efficiently enumerate over the identities and insertion times of all heavy items, thereby creating a residual instance, which can then be handled by employing Theorem 1. Clearly, this approach is attractive when a substantial portion of the optimal profit is coming from heavy items; in the opposite case, our approximation scheme for light items (applied to the original instance) is necessarily extracting a substantial profit. Balancing between these two algorithms, we show that picking the best resulting solution guarantees roughly $2/3$ of the optimal profit. We then explain how repeated applications of this self-reinforcing procedure lead to a $(1-\epsilon)$-approximation.

That said, from a running time perspective, the above-mentioned approach is exponentially dependent on $\log(\frac{w_{\max}}{w_{\min}})$, forming a quasi-PTAS only when the ratio $\frac{w_{\max}}{w_{\min}}$ is polynomial in the input size. To establish a quasi-PTAS for any possible instance, further structural and algorithmic insights are required, and we postpone their formal statements to Sect. 4. Still, at a high level, our algorithmic approach and its analysis rely on proving a fundamental result regarding the existence of a highly structured near-optimal solution. Essentially, we show that a sufficiently-profitable subcollection of items can be partitioned into clusters $C_1, \ldots, C_M$ such that: (1) The weight ratio within each cluster is $O(n^{1/\epsilon})$; and (2) There is a near-optimal solution where, for every $m \in [M]$, only $O\left(\frac{1}{\epsilon} \log M\right)$ items in $C_{m+1}, \ldots, C_M$ could be crossing, i.e., appear before items belonging to $C_m$. From an algorithmic angle, the existence of such sparse crossing solutions allows us to account for the interaction between different clusters by means of dynamic programming. At the same time, since the weight ratio within any given cluster is only $O(n^{1/\epsilon})$, our previously-mentioned approximation scheme, which is exponentially dependent on $\log(\frac{w_{\max}}{w_{\min}})$, can be employed as a black box quasi-PTAS for each cluster by itself.

1.2 Related knapsack extensions

Probably the simplest incremental knapsack problem studied so far is time-invariant incremental knapsack, where each item $i$ is assumed to contribute a profit of $\phi_i$ to each period starting at its insertion time, corresponding to product-form profits, $p_{it} = (T+1-t) \cdot \phi_i$. Surprisingly, unlike the basic knapsack problem, Bienstock et al. [8] showed that this extension is strongly NP-hard. On the positive side, Faenza and Malinovic [19] proposed a polynomial-time approximation scheme (PTAS) based on rounding fractional solutions to an appropriate disjunctive relaxation. In the broader incremental knapsack problem, we have $p_{it} = \phi_i \cdot \sum_{\tau=t}^{T} \Delta_\tau$, where $\Delta_\tau \geq 0$ is a time-dependent scaling factor; in this context, Aouad and Segev [4] have very recently obtained a PTAS, leveraging approximate dynamic programming ideas. We refer the
reader to a number of additional resources related to incremental knapsack problems [16,17,28,34,37] for a deeper look into these settings.

In contrast, the flexibility of our item- and time-dependent profit structure allows us to capture a variety of situations. For instance, when an item \( i \) gains a profit of \( \phi_i \tau \) for each period \( \tau \), starting at its insertion time, we can set \( p_{it} = \sum_{\tau=t}^{T} \phi_i \tau \). If, moreover, the per-period profits \( \phi_i \tau \) are discounted by a factor of \( c_{\tau-t} \) after \( \tau - t \) time units have elapsed since the insertion of item \( i \), we set \( p_{it} = \sum_{\tau=t}^{T} c_{\tau-t} \phi_i \tau \). More broadly, the generalized incremental knapsack problem allows the profits \( p_{it} \) to be completely unrelated, and in particular, to possibly be non-monotone in \( t \).

As a preliminary background, it is worth highlighting two additional generalizations of the classical knapsack problem. In the maximum generalized assignment problem, we are given \( n \) items and \( m \) capacitated buckets. Assigning an item \( j \) to a bucket \( i \) takes \( w_{ij} \) capacity units while generating a profit of \( p_{ij} \). The goal is to compute a feasible item-to-bucket assignment whose overall profit is maximized. For the minimization variant of this problem, Shmoys and Tardos [35] proposed an LP-based 2-approximation, which was observed by Chekuri and Khanna [14] to be easily adaptable to obtain a 1/2-approximation for the maximization variant. Interestingly, these algorithmic ideas will be useful within one of the subroutines employed by our approach. Feige and Vondrák [21] attained a \((1 - 1/e + \delta)\)-approximation, for some absolute constant \( \delta > 0 \), which is currently the best known performance guarantee for maximum generalized assignment. Earlier constant-factor approximations were obtained in [15,22,33].

In the unsplittable flow on a path problem, we are given an edge-capacitated path as well as a collection of tasks. Each task is characterized by its own subpath, profit, and demand. The goal is to select a subset of tasks of maximum total profit, under the constraint that the overall demand of the selected tasks along each edge resides within its capacity. The currently best polynomial-time approximation for the unsplittable flow on a path problem is \( \frac{5}{3} + \epsilon \), for any fixed \( \epsilon > 0 \), due to Grandoni et al. [26], who improved on earlier constant-factor guarantees [3,10,11]. In parallel, unsplittable flow on a path admits a quasi-PTAS, as shown by Bansal et al. [6] and by Batra et al. [7]. In “Appendix A”, we describe an unfruitful attempt of reducing the generalized incremental knapsack problem to unsplittable flow on a path, explaining what the main technical issues are. That said, we further present a reduction to a generalization of the latter problem, with so-called “bag constraints”. In the latter setting, the best known polynomial-time algorithm attains an approximation factor of \( O \left( \frac{\log \log n}{\log n} \right) \).

2 A polynomial-time \((\frac{1}{2} - \epsilon)\)-approximation

In this section, we present our first approximability result for the generalized incremental knapsack problem, showing that the optimal profit can be efficiently approached within a factor arbitrarily close to \( \frac{1}{2} \). The specifics of this finding, along with its corresponding running time, are formally stated in the next theorem.
Theorem 3  For any error parameter $\epsilon \in (0, \frac{1}{2})$, the generalized incremental knapsack problem can be approximated within factor $\frac{1}{2} - \epsilon$. The running time of our algorithm is $O(n^{O(1/\epsilon^2)} \cdot |I|^{O(1)})$, where $|I|$ stands for the input size.

Outline. For simplicity of presentation, we start off Sect. 2.1 by proposing an equivalent formulation of the generalized incremental knapsack problem as a single-machine sequencing problem. Given this reformulation, we explain in Sect. 2.2 how the profit function can be decomposed into “heavy” and “light” item contributions. Somewhat informally, with respect to an unknown optimal sequencing solution, the marginal contribution of each item to the overall profit will be classified as being either heavy or light, depending on the item’s weight and position on the timeline. Guided by this decomposition, our approach consists of devising two approximation schemes, one competing against the best-possible profit due to heavy contributions (Sect. 2.3) and the other against the analogous quantity due to light contributions (Sect. 2.4). The best of these algorithms will be shown to provide an approximation guarantee of $\frac{1}{2} - \epsilon$, thereby deriving Theorem 3. It is worth pointing out that the techniques involved in competing against light contributions will be further utilized in Sects. 3 and 4 to obtain an approximation scheme for general instances, albeit in quasi-polynomial time.

2.1 An equivalent sequencing formulation

In what follows, we present an equivalent sequencing reformulation for the generalized incremental knapsack problem. As explained in subsequent sections, the interchangeability between these formulations allows us to describe our algorithmic ideas and to analyze their performance guarantees with greater ease. For this purpose, we proceed by arguing that the generalized incremental knapsack problem can be rephrased as a sequencing problem on a single machine as follows:

– Let $\pi : [n] \rightarrow [n]$ be a permutation of the underlying items, where $\pi(i)$ stands for the position of item $i$.

– By viewing the weight of each item as its processing time, we define the completion time of item $i$ with respect to $\pi$ as $C_\pi(i) = \sum_{j \in [n], \pi(j) \leq \pi(i)} w_j$. Accordingly, the profit $\varphi_\pi(i)$ of this item is given by the largest profit we can gain by inserting $i$ at a time period whose capacity occurs is at least $C_\pi(i)$, namely, $\varphi_\pi(i) = \max\{p_{i,t} : t \in [T+1] \text{ and } W_t \geq C_\pi(i)\}$, with the convention that $W_{T+1} = \infty$ and $p_{i,T+1} = 0$ for every item $i$.

– The overall profit of the permutation $\pi$ is specified by $\Psi(\pi) = \sum_{i \in [n]} \varphi_\pi(i)$. Our objective is to compute a permutation whose profit is maximized.

The next lemma captures the equivalence between the item-introducing perspective of the generalized incremental knapsack problem and the sequencing perspective described above.

Lemma 1  Any feasible chain $S$ can be mapped to a permutation $\pi_S$ with $\Psi(\pi_S) \geq \Phi(S)$. Conversely, any permutation $\pi$ of a subset of the items can be mapped to a feasible chain $S_\pi$ with $\Phi(S_\pi) = \Psi(\pi)$.

Proof  First, given a feasible chain $S$, we construct the permutation $\pi_S$ as follows:
For each $t \in [T]$, let $\pi^t$ be an arbitrary permutation of the items introduced in this period, $S_t \setminus S_{t-1}$. In addition, let $\pi^{T+1}$ be an arbitrary permutation of the remaining items, i.e., those in $[n]\setminus S_T$.

The permutation $\pi_S$ is defined as the concatenation of $\pi^1, \ldots, \pi^{T+1}$ in this order. Namely, for $i \in S_t \setminus S_{t-1}$ with $t \in [T]$, we have $\pi_S(i) = \pi^t(i) + |S_{t-1}|$, whereas for $i \in [n]\setminus S_T$, we have $\pi_S(i) = \pi^{T+1}(i) + |S_T|$.

To prove that $\Psi(\pi_S) \geq \Phi(S)$, it suffices to argue that $\varphi_{\pi_S}(i) \geq p_{i,t_i}$ for every item $i \in S_T$, where $t_i$ stands for the insertion time of item $i$ with respect to the chain $S$. To derive this relation, note that $C_{\pi_S}(i) \leq w(S_{t_i}) \leq W_{t_i}$ for any such item, where the last inequality follows from the feasibility of $S$. Therefore, $\varphi_{\pi_S}(i) = \max\{p_{i,t} : t \in [T+1] \text{ and } W_t \geq C_{\pi_S}(i)\} \geq p_{i,t_i}$.

Conversely, given a permutation $\pi$ of any subset of items, we construct a chain $S_\pi$ that includes all items whose completion time is at most $W_T$. Specifically, the insertion time $t_i$ of each such item $i$ will be the time period that maximizes $p_{i,t_i}$ over the set $\{t \in [T] : W_t \geq C_\pi(i)\}$. As such, the chain $S_\pi$ is indeed feasible, since $w(S_t) \leq \sum_{i \in [n] : C_\pi(i) \leq W_t} w_i \leq W_t$ for every $t \in [T]$. To show that $\Phi(S_\pi) = \Psi(\pi)$, it remains to explain why $p_{i,t_i} = \varphi_\pi(i)$ for inserted items and why $\varphi_\pi(i) = 0$ for non-inserted ones. To this end, note that our choice for the insertion time $t_i$ follows the definition of $\varphi_\pi(i)$ to the letter, meaning that $p_{i,t_i} = \varphi_\pi(i)$. On the other hand, for any item $i$ we do not insert to $S_\pi$, one has $\varphi_\pi(i) = 0$, since $C_\pi(i) > W_T$. 

\section{2.2 Profit decomposition and high-level overview}

In what follows, we focus our attention on the sequencing formulation and present a decomposition of the profit function $\Psi$ into “heavy” and “light” contributions, collected over geometrically-increasing intervals. With the necessary definitions in place, we outline how a decomposition of this nature guides us in proposing two approximation schemes, to separately compete against heavy and light contributions. The main result of this section, as stated in Theorem 3, will eventually be derived by taking the more profitable of these approaches.

For simplicity of presentation, we assume without loss of generality that $\epsilon \in (0, \frac{1}{2})$, and moreover, that $\frac{1}{\epsilon}$ is an integer. In addition, we assume that $w_{\min} = \min_{i \in [n]} w_i = 3$; the latter property can easily be enforced through scaling all item weights $w_i$ and time period capacities $W_t$ by a factor of $\frac{3}{w_{\min}}$.

Profit decomposition. We begin by geometrically partitioning the interval $[0, \sum_{i \in [n]} w_i]$ by powers of $1 + \epsilon$ into a collection of intervals $I_0, \ldots, I_K$, where $K = \lceil \log_{1+\epsilon}(\sum_{i \in [n]} w_i) \rceil$. Specifically, $I_0 = [0, 1]$ and $I_k = [(1 + \epsilon)^{k-1}, (1 + \epsilon)^k]$ for $k \in [K]$. With this definition, the profit $\Psi(\pi) = \sum_{i \in [n]} \varphi_\pi(i)$ of any permutation $\pi$ can be expressed by summing item contributions according to the interval in which their completion times fall, i.e.,

$$\Psi(\pi) = \sum_{k \in [K]_0} \sum_{i \in [n] : C_\pi(i) \in I_k} \varphi_\pi(i).$$
We say that item $i$ is $k$-heavy when $w_i \geq \epsilon^2 \cdot (1 + \epsilon)^k$; otherwise, this item is $k$-light. We denote the sets of $k$-heavy and $k$-light items by $H_k$ and $L_k$, respectively, noting that $H_0 \supseteq H_1 \supseteq \cdots \supseteq H_k$ and that $L_k = [n] \setminus H_k$ for every $k$. As a side note, one can easily verify that all items are $0$-heavy (i.e., $H_0 = [n]$), by recalling that $w_{\text{min}} = 3$ and $\epsilon < \frac{1}{2}$. Consequently, the profit $\Psi(\pi)$ can be refined by separating $k$-heavy and $k$-light items, namely,

$$\Psi(\pi) = \sum_{k \in [K]} \left( \sum_{i \in H_k : C_{\pi}(i) \in I_k} \varphi_{\pi}(i) \right) + \sum_{k \in [K]} \left( \sum_{i \in L_k : C_{\pi}(i) \in I_k} \varphi_{\pi}(i) \right). \quad (1)$$

As shown above, we designate the first and second terms in the above expression by $\Psi_{\text{heavy}}(\pi)$ and $\Psi_{\text{light}}(\pi)$, respectively.

Overview. Let $\pi^*$ be an optimal permutation, with $\Psi(\pi^*) = \Psi_{\text{heavy}}(\pi^*) + \Psi_{\text{light}}(\pi^*)$. The remainder of this section is dedicated to presenting two approximation schemes that would separately compete against $\Psi_{\text{heavy}}(\pi^*)$ and $\Psi_{\text{light}}(\pi^*)$:

- **Heavy contributions**: Section 2.3 explains how dynamic programming ideas allow us to efficiently compute a permutation $\pi_{\text{heavy}} : [n] \rightarrow [n]$ satisfying $\Psi(\pi_{\text{heavy}}) \geq (1 - \epsilon) \cdot \Psi_{\text{heavy}}(\pi^*)$. The resulting running time will be $O(n^{O(1/\epsilon^2)} \cdot |I|)$.

- **Light contributions**: Section 2.4 argues that the generalized assignment algorithm of Shmoys and Tardos [35] can be leveraged to compute a permutation $\pi_{\text{light}} : [n] \rightarrow [n]$ satisfying $\Psi(\pi_{\text{light}}) \geq (1 - \epsilon) \cdot \Psi_{\text{light}}(\pi^*)$. This algorithm can be implemented in $O\left(\frac{|I|}{\epsilon} \cdot O(1)\right)$ time.

Consequently, to establish the approximation guarantee stated in Theorem 3, we pick the more profitable permutation out of $\pi_{\text{heavy}}$ and $\pi_{\text{light}}$, to obtain a profit of

$$\max \left\{ \Psi(\pi_{\text{heavy}}), \Psi(\pi_{\text{light}}) \right\} \geq \frac{1}{2} \cdot \left( \Psi(\pi_{\text{heavy}}) + \Psi(\pi_{\text{light}}) \right) \geq \frac{1 - \epsilon}{2} \cdot \left( \Psi_{\text{heavy}}(\pi^*) + \Psi_{\text{light}}(\pi^*) \right) = \frac{1 - \epsilon}{2} \cdot \Psi(\pi^*).$$

### 2.3 Algorithm for heavy contributions

In what follows, we present a dynamic programming approach for computing a permutation that competes against $\Psi_{\text{heavy}}(\pi^*)$, as formally stated in the next theorem.

**Theorem 4** For any error parameter $\epsilon \in (0, 1)$, there is an $O(n^{O(1/\epsilon^2)} \cdot |I|)$-time algorithm for constructing a permutation $\pi_{\text{heavy}}$ with a profit of $\Psi(\pi_{\text{heavy}}) \geq (1 - \epsilon) \cdot \Psi_{\text{heavy}}(\pi^*)$. 

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2.3.1 Preliminaries

The intuition behind our algorithm begins with the observation that, in order to compete against $\Psi_{\text{heavy}}(\pi^*)$, we can safely eliminate items that are classified as light with respect to the interval in which their completion time falls. While the remaining items will be shifted back in the residual permutation, potentially being completed in a lower-index interval, each of them will still be heavy. To formalize these notions, for a subset of items $S \subseteq [n]$ and a permutation $\pi : S \rightarrow [|S|]$, we say that the pair $(S, \pi)$ is bulky if, for every $k \in [K]_0$, all items with a completion time in $I_k$ are $k$-heavy, i.e., $\{i \in S : C_{\pi}(i) \in I_k\} \subseteq H_k$. The next claim shows that bulky pairs can attain a total profit of at least $\Psi_{\text{heavy}}(\pi^*)$.

**Lemma 2** There exist a subset of items $S \subseteq [n]$ and a permutation $\pi : S \rightarrow [|S|]$ such that $(S, \pi)$ is bulky and $\sum_{i \in S} \varphi_{\pi}(i) \geq \Psi_{\text{heavy}}(\pi^*)$.

**Proof** With respect to the optimal permutation $\pi^*$, we define a new permutation $\pi$ by eliminating, for every $k \in [K]_0$, all items $i \in L_k$ with $C_{\pi^*}(i) \in I_k$. The subset $S$ will consist of the remaining items. To see why $(S, \pi)$ is bulky, note that $C_{\pi}(i) \leq C_{\pi^*}(i)$ for any $i \in S$, meaning that each such item is still heavy with respect to the interval that contains $C_{\pi}(i)$, since $H_0 \supseteq \cdots \supseteq H_K$. In terms of profit, the latter observation implies that, for every item $i \in S$,

$$\varphi_{\pi}(i) = \max \left\{ p_{i,t} : t \in [T + 1] \text{ and } W_t \geq C_{\pi}(i) \right\} \geq \max \left\{ p_{i,t} : t \in [T + 1] \text{ and } W_t \geq C_{\pi^*}(i) \right\} = \varphi_{\pi^*}(i).$$

Summing the above inequality over all items in $S$, we have $\sum_{i \in S} \varphi_{\pi}(i) \geq \sum_{i \in S} \varphi_{\pi^*}(i) = \Psi_{\text{heavy}}(\pi^*)$, where the latter equality holds since every eliminated item does not contribute toward $\Psi_{\text{heavy}}(\pi^*)$ but rather toward $\Psi_{\text{light}}(\pi^*)$. \qed

**Additional notation.** For a bulky pair $(S, \pi)$, we define its top index as $\text{top}(S, \pi) = \max\{k \in [K]_0 : \{C_{\pi}(i) : i \in S\} \cap I_k \neq \emptyset\}$, that is, the largest index of an interval that contains at least one completion time. In addition, we define $\text{core}(S)$ as the set of the $\min\{\frac{1}{e^2}, |S|\}$ heaviest items in $S$, breaking ties by adding to $\text{core}(S)$ small-index items before large-index ones. Finally, the makespan of $(S, \pi)$ corresponds to the maximum completion time of an item in $S$ with respect to the permutation $\pi$; in our case, this measure identifies with $w(S)$.

2.3.2 The continuous dynamic program

The technical crux in restricting attention to bulky pairs will be exhibited through our dynamic programming formulation. As formally explained below, by focusing on the dual objective of makespan minimization, we prove the existence of a well-hidden optimal substructure within the sequencing problem.

**States.** Each state $(k, \psi_k, Q_k)$ of our dynamic program consists of the following parameters, whose precise role will be clarified once their corresponding value function is presented:

\[ \psi_k = \max \left\{ \frac{1}{e^2}, |S| \right\} \text{ and } \psi_k \leq \cdots \leq \psi_0 \leq |S| \]

\[ Q_k = \left\{ (S, \pi) : \psi_k \geq \psi_{\pi^*}(i) \text{ for all } i \in S \right\} \]
- The index of the current interval \( k \), taking values in \([K]\).
- The total profit \( \psi_k \) collected thus far, due to items whose completion time falls in \( \mathcal{I}_0, \ldots, \mathcal{I}_k \). For the time being, \( \psi_k \) will be treated as a continuous parameter, taking values in \([0, \sum_{i \in [n]} \max_{t \in [T]} p_{i,t}]\).
- The core \( \mathcal{Q}_k \) of the set of items whose completion time falls in \( \mathcal{I}_0, \ldots, \mathcal{I}_k \). By definition of \( \text{core}(\cdot)\), this parameter is restricted to item sets of cardinality at most \( \frac{1}{\epsilon^2} \).

It is important to emphasize that, since \( \psi_k \) is a continuous parameter, the dynamic programming formulation below is still not algorithmic in nature, and should be viewed as a characterization of optimal solutions. We remark that when the profits \( p_{i,t} \) are all integers, we can restrict \( \psi_k \) to integer values in \([0, \sum_{i \in [n]} \max_{t \in [T]} p_{i,t}]\), and our dynamic program can be solved in pseudo-polynomial time. In either case, we explain in Sect. 2.3.3 how to discretize the parameter \( \psi_k \) to take polynomially-many values while incurring only an \( \epsilon \)-loss in profit.

**Value function.** The value function \( F(k, \psi_k, \mathcal{Q}_k) \) represents the minimum makespan \( w(S) \) that can be attained over all bulky pairs \((S, \pi)\) that satisfy the following conditions:

1. **Top index:** \( \text{top}(S, \pi) \leq k \).
2. **Total profit:** \( \Psi(\pi) \geq \psi_k \).
3. **Core:** \( \text{core}(S) = \mathcal{Q}_k \).

For ease of presentation, we denote the collection of bulky pairs that meet conditions 1–3 by Bulky\((k, \psi_k, \mathcal{Q}_k)\). When the latter set is empty, we define \( F(k, \psi_k, \mathcal{Q}_k) = \infty \).

With these definitions, Lemma 2 proves in retrospect the existence of a bulky pair \((S, \pi) \in \text{Bulky}(K, \psi_{\text{heavy}}(\pi^*), \text{core}(S))\) with \( F(K, \psi_{\text{heavy}}(\pi^*), \text{core}(S)) < \infty \). Therefore, had we been able to compute the maximal value \( \psi^* \) that satisfies \( F(K, \psi^*, \mathcal{Q}_K) < \infty \) over all possible cores \( \mathcal{Q}_K \), its corresponding bulky pair would have guaranteed a profit of at least \( \psi^* \geq \Psi_{\text{heavy}}(\pi^*) \).

**Optimal substructure.** To this end, we proceed by unveiling the optimal substructure that allows us to compute the value function \( F \) by means of dynamic programming. In order to gain intuition, suppose that \((S, \pi)\) is a bulky pair that attains \( F(k, \psi_k, \mathcal{Q}_k) \). Then, we argue that, by eliminating from \( S \) the set of items \( Q \) whose completion time falls within the interval \( \mathcal{I}_k \), one obtains a bulky pair that attains \( F(k - 1, \psi_{k-1}, \mathcal{Q}_{k-1}) \), where the residual profit \( \psi_{k-1} \) is obtained by removing from \( \psi_k \) the contribution of items in \( Q \) and \( \mathcal{Q}_{k-1} \) is an appropriately chosen core. In this regard, the obvious question is: To attain \( F(k - 1, \psi_{k-1}, \mathcal{Q}_{k-1}) \), why would our dynamic program not pick any of the items in \( Q \)? The crux of our argument would be that, since the intervals \( \mathcal{I}_0, \ldots, \mathcal{I}_k \) are geometrically increasing in length, the \( k \)-heaviness of all items in \( Q \) forces each such item to reside within the core \( \mathcal{Q}_k \), meaning that we will indeed prevent it from being picked when \( F(k - 1, \psi_{k-1}, \mathcal{Q}_{k-1}) \) is computed by a suitable choice of the core \( \mathcal{Q}_{k-1} \) that, in particular, will be disjoint from \( Q \).

Formally, suppose that \( \text{Bulky}(k, \psi_k, \mathcal{Q}_k) \neq \emptyset \), and let \((S, \pi)\) be a bulky pair that minimizes \( w(S) \) over this set. Let \( Q = \{i \in S : C_{\pi}(i) \in \mathcal{I}_k\} \) be the set of items in \( S \) whose completion time with respect to \( \pi \) falls in the interval \( \mathcal{I}_k \). Note that since \( \text{top}(S, \pi) \leq k \), completion times cannot fall in \( \mathcal{I}_{k+1}, \ldots, \mathcal{I}_K \). We first argue that
$|Q| \leq \frac{1}{\epsilon}$. To verify this claim, note that since $(S, \pi)$ is bulky, $Q \subseteq H_k$. As a result, every item in $Q$ has a weight of at least $\epsilon^2 \cdot (1 + \epsilon)^k$, while $\mathcal{I}_k = ((1 + \epsilon)^k - 1, (1 + \epsilon)^k]$, meaning that we necessarily have $|Q| \leq \frac{(1 + \epsilon)^k - 1}{\epsilon^2 \cdot (1 + \epsilon)^k} \leq \frac{1}{\epsilon}$.

Now, let us define the pair $(\hat{S}, \hat{\pi})$, where $\hat{S} = S \setminus Q$ and $\hat{\pi} : \hat{S} \to |\hat{S}|$ is the permutation where items in $\hat{S}$ follow their relative order in $\pi$, that is, for any pair of items $i_1$ and $i_2$, we have $\hat{\pi}(i_1) < \hat{\pi}(i_2)$ if and only if $\pi(i_1) < \pi(i_2)$. In addition, let $\psi_{k-1} = [\psi_k - \sum_{i \in Q} \varphi_\pi(i)] +$ and $Q_{k-1} = \text{core}(\hat{S})$, where $[x]^+ = \max(x, 0)$. These definitions directly ensure that $(\hat{S}, \hat{\pi}) \in \text{Bulky}(k - 1, \psi_{k-1}, Q_{k-1})$. Moreover, as we show in Lemma 3 below, $(\hat{S}, \hat{\pi})$ forms an optimal solution with respect to the latter state. Intuitively, the key idea for proving this claim shows that, had there been a bulky pair $(\tilde{S}, \tilde{\pi}) \in \text{Bulky}(k - 1, \psi_{k-1}, Q_{k-1})$ with $w(\tilde{S}) < w(S)$, it can be extended to a bulky pair $(\tilde{S}^+, \tilde{\pi}^+) \in \text{Bulky}(k, \psi_k, Q_k)$ by adding the items in $Q$ following their internal order in $\pi$, to obtain $w(\tilde{S}^+) < w(S)$, thereby contradicting the optimality of $(S, \pi)$.

**Lemma 3** \[w(\hat{S}) = F(k - 1, \psi_{k-1}, Q_{k-1}).\]

**Proof** Suppose there exists some bulky pair $(\tilde{S}, \tilde{\pi}) \in \text{Bulky}(k - 1, \psi_{k-1}, Q_{k-1})$ with $w(\tilde{S}) < w(S)$. We first claim that $\tilde{S} \cap Q = \emptyset$. To verify this property, had there been an item $i \in \tilde{S} \cap Q$, its weight would satisfy $w_i \geq \epsilon^2 \cdot (1 + \epsilon)^k$, since $Q \subseteq H_k$. On the other hand, since top$(\tilde{S}, \tilde{\pi}) \leq k - 1$, the completion times of all items in $\tilde{S}$ with respect to $\tilde{\pi}$ reside within the union of $\mathcal{I}_0, \ldots, \mathcal{I}_{k-1}$, which is the interval $[0, (1 + \epsilon)^k - 1]$, implying that $w(\tilde{S}) \leq (1 + \epsilon)^k - 1$. Therefore, since core$(\tilde{S})$ is the set of min$\{\frac{1}{\epsilon^2}, |\tilde{S}|\}$ heaviest items in $\tilde{S}$, regardless of how ties are broken we must have $i \in \text{core}(\tilde{S})$. We have just arrived at a contradiction: Since core$(\tilde{S}) = Q_{k-1} = \text{core}(S) = \text{core}(S \setminus Q)$, it follows that $i \notin Q$.

Knowing that $\tilde{S} \cap Q = \emptyset$, we can extend the permutation $\tilde{\pi} : \tilde{S} \to |\tilde{S}|$ to $\tilde{S}^+ = \tilde{S} \cup Q$ by appending the set of items $Q$ in exactly the same order as they appear in $\pi$. Letting $\tilde{\pi}^+ : \tilde{S}^+ \to |\tilde{S}^+|$ be the resulting permutation, we next argue that $(\tilde{S}^+, \tilde{\pi}^+)$ is in fact a feasible solution to precisely the same subproblem with respect to which $(S, \pi)$ is optimal. The proof of this structural result is provided in “Appendix B.1”.

**Claim 1** $(\tilde{S}^+, \tilde{\pi}^+) \in \text{Bulky}(k, \psi_k, Q_k)$.

We have just arrived at a contradiction to the fact that $(S, \pi)$ minimizes $w(S)$ over the set Bulky$(k, \psi_k, Q_k)$, by observing that $w(\tilde{S}^+) = w(\tilde{S}) + w(Q) < w(\tilde{S}) + w(Q) = w(S)$.

**Recursive equations.** In light of this structural characterization, to obtain a recursive equation for $F(k, \psi_k, Q_k)$, it suffices to “guess” the collection of items $Q$, their internal permutation $\pi_Q$, the residual profit requirement $\psi_{k-1}$, and the resulting core $Q_{k-1}$. Formally, $F(k, \psi_k, Q_k)$ is given by minimizing $F(k - 1, \psi_{k-1}, Q_{k-1}) + w(Q)$ over all choices of $Q, \pi_Q, \psi_{k-1}$, and $Q_{k-1}$ that simultaneously satisfy the following conditions:

1. Top index: $F(k - 1, \psi_{k-1}, Q_{k-1}) + w(Q) \leq (1 + \epsilon)^k$. This constraint ensures that, with the addition of $Q$, all items can still be packed within $\mathcal{I}_0, \ldots, \mathcal{I}_k$. 

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2. Total profit: $\psi_{k-1} \geq [\psi_k - \sum_{i \in Q} \varphi_{\pi Q}^\rightarrow(i)]^+$, where the term $\varphi_{\pi Q}^\rightarrow(i)$ denotes the profit of item $i$ with respect to the permutation $\pi_Q$, when its completion time is increased by $F(k-1, \psi_{k-1}, Q_{k-1})$. This constraint guarantees that, by appending $\pi_Q$, we obtain a total profit of at least $\psi_k$.

3. Core: $Q_{k-1} \cap Q = \emptyset$, $\text{core}(Q_{k-1} \cup Q) = Q_k$, $Q \subseteq H_k$, and $|Q| \leq \frac{1}{\epsilon}$. These constraints ensure a correct core update as a result of adding the item set $Q$, where the latter set consists of at most $\frac{1}{\epsilon}$ items, each restricted to being $k$-heavy. To better understand the requirement $\text{core}(Q_{k-1} \cup Q) = Q_k$, note that the core resulting from the addition of $Q$ can be computed without a complete knowledge of all previously packed items, as all those outside the current core $Q_{k-1}$ are irrelevant for this purpose (i.e., too light to be one of the $\frac{1}{\epsilon}$ heaviest).

2.3.3 Discretization and final algorithm

As previously mentioned, due to the continuity of the profit requirement $\psi_k$, it remains to propose an appropriate discretization of this parameter, so that we obtain a polynomially-sized state space with only negligible loss in profit.

The discrete program $\tilde{F}$. To this end, we alter the underlying state space of our dynamic program, by restricting the continuous parameter $\psi_k$ to a finite set of values, $D_\psi = \{d : \frac{\epsilon p_{\text{max}}}{n} : d \in \lceil \frac{\psi_{\text{max}}}{\epsilon} \rceil \}$. Here, $p_{\text{max}}$ is the maximum profit attainable by any single item, i.e., $p_{\text{max}} = \max\{p_{it} : i \in [n], t \in [T], \text{ and } w_i \leq W_t\}$. We make use of $\tilde{F}(k, \psi_k, Q_k)$ to designate the value function $F$ restricted to the resulting set of states, and similarly, Bulky($k, \psi_k, Q_k$) will stand for the collection of bulky pairs that meet conditions 1–3. As a side note, beyond the additional restriction on $\psi_k$, both $\tilde{F}$ and Bulky are defined identically to $F$ and Bulky.

Analysis. We remind the reader that, in Sect. 2.3.2, the quantity $\psi^*$ was defined as the maximal value satisfying $F(K, \psi^*, Q_K) < \infty$ over all possible cores $Q_K$, noting that its corresponding bulky pair guarantees a profit of at least $\psi^* \geq \psi_{\text{heavy}}(\pi^*)$. In order to establish a parallel claim with respect to the discretized program $\tilde{F}$, we prove in Lemma 4 a lower bound of $(1 - \epsilon) \cdot \psi^*$ on the analogous quantity $\tilde{\psi}$ that satisfies $\tilde{F}(K, \tilde{\psi}, Q_K) < \infty$; the proof is provided in “Appendix B.2”. It follows that our dynamic program computes a bulky pair $(S, \pi)$ in which the permutation $\pi$ has a profit of $\Psi(\pi) \geq \tilde{\psi} \geq (1 - \epsilon) \cdot \psi^* \geq (1 - \epsilon) \cdot \psi_{\text{heavy}}(\pi^*)$.

Lemma 4 There exists a value $\tilde{\psi} \in D_\psi$ such that $\tilde{\psi} \geq (1 - \epsilon) \cdot \psi^*$ and such that $\tilde{F}(K, \tilde{\psi}, Q_K) < \infty$ for some core $Q_K$.

Running time. We first observe that the function $\tilde{F}(k, \psi_k, Q_k)$ needs to be evaluated over $O(n^{O(1/\epsilon^3)} \cdot |I|)$ possible states. Indeed, there are $O(K)$ choices for the interval index $k$, where by definition, $K = \lceil \log_{1+\epsilon}(\sum_{i \in [n]} w_i) \rceil = O\left(\frac{|I|}{\epsilon}\right)$. As for the profit parameter $\psi_k$, following its restriction to the set $D_\psi$, we ensure that $\psi_k$ takes only $|D_\psi| = O\left(\frac{n^2}{\epsilon^2}\right)$ values. Finally, since the core $Q_k \subseteq [n]$ consists of at most $\frac{1}{\epsilon}$ items, there are only $O(n^{O(1/\epsilon^3)})$ subsets to consider.
Now, evaluating each state requires minimizing the restricted function $\tilde{F}(k - 1, \psi_{k-1}, Q_{k-1}) + w(Q)$ over all choices of $Q$, $\pi_Q$, $\psi_{k-1}$, and $Q_{k-1}$ that simultaneously satisfy conditions 1–3 of the recursive equations (see Sect. 2.3.2). In this context, the number of joint configurations for these parameters is $O(n^{O(1/\epsilon^2)})$. Specifically, the profit parameter $\psi_{k-1}$ and the core $Q_{k-1}$ respectively take $O(n^2 \epsilon)$ and $O(n^{O(1/\epsilon^2)})$ values as before. In addition, the number of choices for the augmenting set $Q$ is $O(n^{O(1/\epsilon^2)})$, due to being comprised of at most $\frac{1}{\epsilon}$ items, and there are only $O((\frac{1}{\epsilon})^{O(1/\epsilon^2)})$ permutations $\pi_Q$ of these items. To summarize, we incur an overall running time of $O(n^{O(1/\epsilon^2)} \cdot |\mathcal{I}|)$.

### 2.4 Algorithm for light contributions

In this section, we construct a suitably-defined instance of the maximum generalized assignment problem, intended to compete against $\Psi_{\text{light}}(\pi^*)$. We show that, when applied to this highly-structured instance, the LP-based algorithm of Shmoys and Tardos [35] can be leveraged for computing a permutation that competes against $\Psi_{\text{light}}(\pi^*)$ along the lines of the next theorem.

**Theorem 5** For any error parameter $\epsilon \in (0, 1)$, there is an $O((\frac{|\mathcal{I}|}{\epsilon})^{O(1)})$-time algorithm for constructing a permutation $\pi_{\text{light}}$ with a profit of $\Psi(\pi_{\text{light}}) \geq (1 - 13 \epsilon) \cdot \Psi_{\text{light}}(\pi^*)$.

#### 2.4.1 Instance construction

**Intuition.** The general intuition behind our construction resides in viewing the intervals $\mathcal{I}_1, \ldots, \mathcal{I}_{K-1}$ as distinct buckets, to which items should be assigned subject to capacity constraints. Clearly, this perspective lacks the extra flexibility of the sequencing formulation, where items may be crossing between multiple successive intervals. In addition, any item-to-bucket assignment has to be associated with a specific profit a-priori, whereas the sequencing-related profits depend on the exact completion time of each item. As explained in the sequel, our approach bypasses the first obstacle by focusing on light items, for which greedy repacking of rounded solutions will be argued to be near-optimal. In regard to the second obstacle, we will allow seemingly unattainable profits, showing that appropriately scaled fractional solutions can be rounded to attain these profits up to negligible loss in optimality.

**The construction.** Guided by this intuition, we define an instance of the maximum generalized assignment problem as follows:

- **Buckets:** For every $k \in [K - 1]$, we set up a bucket $\mathcal{B}_k$. The capacity of this bucket is $\text{capacity}(\mathcal{B}_k) = (1 + \epsilon)^k - (1 + \epsilon)^{k-1}$, i.e., precisely the length of the interval $\mathcal{I}_k$. It is worth mentioning that there are no buckets corresponding to the intervals $\mathcal{I}_0$ and $\mathcal{I}_K$.

- **Items:** The set of items is still $[n]$, where each item has a weight of $w_i$.

- **Allowed assignments and profits:** An item $i$ can be assigned to bucket $\mathcal{B}_k$ only when $i$ is $(k + 1)$-light. For such an assignment, our profit is $q_{ik} = \max\{\psi_{i,t} : t \in [T + 1] \text{ and } W_t \geq (1 + \epsilon)^k\}$. 


The goal is to compute a capacity-feasible assignment whose total profit is maximized.

**IP formulation.** Moving forward, it is instructive to represent this instance through its standard integer programming formulation:

\[
\max \sum_{i \in [n]} \sum_{k \in [K-1]} q_{ik} x_{ik} \\
\text{s.t.} \sum_{k \in [K-1]} x_{ik} \leq 1 \quad \forall i \in [n] \\
\sum_{i \in L_{k+1}} w_i x_{ik} \leq \text{capacity}(B_k) \quad \forall k \in [K-1] \\
x_{ik} \in \{0, 1\} \quad \forall k \in [K-1], i \in L_{k+1}
\]

In this formulation, each decision variable \( x_{ik} \) indicates whether item \( i \) is assigned to bucket \( B_k \). The first constraint guarantees that every item is assigned to at most one bucket, and the second constraint ensures that the total weight of the items assigned to each bucket fits within its capacity. The next lemma shows that any feasible assignment can be efficiently mapped to a permutation for our sequencing formulation that collects at least as much profit; the proof is provided in “Appendix B.3”.

**Lemma 5** Any feasible solution \( x \) to (IP) can be translated in \( O(nK) \) time to a permutation \( \pi_x : [n] \rightarrow [n] \) satisfying \( \Psi(\pi_x) \geq \sum_{i \in [n]} \sum_{k \in [K-1]} q_{ik} x_{ik} \).

**LP-relaxation and lower bound.** The linear relaxation of this integer program, (LP), is obtained by replacing the integrality constraints \( x_{ik} \in \{0, 1\} \) with non-negativity constraints, \( x_{ik} \geq 0 \). To have a better intuition for how the fractional optimum of (LP) is related to \( \Psi_{\text{light}}(\pi^*) \), let \( C^*_k = \{ i \in L_k : C_{\pi^*}(i) \in I_k \} \) be the subset of \( k \)-light items whose completion time with respect to the optimal permutation \( \pi^* \) falls in \( I_k \). Then, within the proof of Lemma 6 below, we argue that a \( 1 - O(\epsilon) \) fraction of each such item can be assigned to bucket \( B_{k-1} \). This claim would follow by observing that capacity \( (B_{k-1}) \) nearly matches the length of the interval \( I_k \), in which all items in \( C^*_k \) are known to fit, potentially except for one item that crosses into \( I_k \). However, the latter item is \( k \)-light, meaning that its weight is very small in comparison to capacity \( (B_{k-1}) \), and scaling down all items in \( C^*_k \) by a factor of \( 1 - O(\epsilon) \) clears sufficient capacity for this item as well. The next claim formally shows that this fractional solution is indeed feasible in (LP) and earns a profit of nearly \( \Psi_{\text{light}}(\pi^*) \).

**Lemma 6** \( \text{OPT}(\text{LP}) \geq (1 - 5\epsilon) \cdot \Psi_{\text{light}}(\pi^*) \).

**Proof** In order to derive the desired bound, we prove that (LP) has a feasible fractional solution \( x \) with an objective value of at least \( (1 - 5\epsilon) \cdot \Psi_{\text{light}}(\pi^*) \). To this end, recalling that \( C^*_k = \{ i \in L_k : C_{\pi^*}(i) \in I_k \} \), we have

\[
\Psi_{\text{light}}(\pi^*) = \sum_{k \in [K]} \sum_{i \in C^*_k} \varphi_{\pi^*}(i) = \sum_{k=2}^{K} \sum_{i \in C^*_k} \varphi_{\pi^*}(i),
\]
where the second equality follows by observing that completion times cannot fall in either of the intervals \(I_0\) and \(I_1\), since their union is \([0, 2 + \epsilon]\) whereas \(w_{\min} = 3\), by our initial assumption in Sect. 2.2.

We define a fractional solution \(x\) to (LP) by setting \(x_{i,k-1} = 1 - 5\epsilon\) for every \(2 \leq k \leq K\) and \(i \in \mathcal{C}_k\); all other variables take zero values. To verify the feasibility of this solution, note that we clearly have

\[
\sum_{k \in [K-1]: i \in L_{k+1}} w_{i} x_{i,k-1} = (1 - 5\epsilon) \cdot \sum_{i \in \mathcal{C}_k^*} q_{i,k-1} \leq (1 - 5\epsilon) \cdot \left((1 + \epsilon)^k - (1 + \epsilon)^{k-1} + \epsilon^2 \cdot (1 + \epsilon)^k\right) \leq (1 - 5\epsilon) \cdot (1 + 5\epsilon) \cdot \left((1 + \epsilon)^{k-1} - (1 + \epsilon)^{k-2}\right) \leq \text{capacity (}B_{k-1}\text{)}.
\]

Here, the first inequality holds since all items in \(\mathcal{C}_k^*\) have completion times in \(I_k\), implying that their total weight is upper bounded by the length \((1 + \epsilon)^k - (1 + \epsilon)^{k-1}\) of this interval plus the maximum weight of any item in \(\mathcal{C}_k^*\), which is at most \(\epsilon^2 \cdot (1 + \epsilon)^k\) due to being \(k\)-light. The second inequality can easily be verified to hold for every \(\epsilon \in (0, 1)\).

Consequently, the fractional optimum can be lower-bounded by the objective function of \(x\), to obtain

\[
\OPT (LP) \geq \sum_{i \in [n]} \sum_{k \in [K-1]: i \in L_{k+1}} q_{i,k} x_{i,k} = (1 - 5\epsilon) \cdot \sum_{k=2}^{K} \sum_{i \in \mathcal{C}_k^*} q_{i,k-1} \geq (1 - 5\epsilon) \cdot \sum_{k=2}^{K} \sum_{i \in \mathcal{C}_k^*} \varphi_{\pi^*}(i) = (1 - 5\epsilon) \cdot \Psi_{\text{light}}(\pi^*) ,
\]

where the last equality is precisely (2). To understand the second inequality, note that for every item \(i \in \mathcal{C}_k^*\),

\[
\varphi_{\pi^*}(i) = \max \left\{ p_{i,t} : t \in [T + 1] \text{ and } W_t \geq C_{\pi^*}(i) \right\} \leq \max \left\{ p_{i,t} : t \in [T + 1] \text{ and } W_t \geq (1 + \epsilon)^{k-1} \right\} = q_{i,k-1} ,
\]

where the above inequality holds since \(C_{\pi^*}(i) \geq (1 + \epsilon)^{k-1}\). \qed
2.4.2 Employing the Shmoys–Tardos algorithm

The rounding algorithm. We proceed by utilizing the LP-rounding approach of Shmoys and Tardos [35, Sect. 2], which was originally proposed for the minimum generalized assignment problem. Specifically, given an optimal fractional solution to the linear program (LP), their algorithm computes an integral vector \( \hat{x} \) that satisfies the following properties:

1. Objective value: \( \hat{x} \) has a super-optimal objective value, i.e.,

\[
\sum_{i \in [n]} \sum_{k \in [K-1]} q_{ik} \hat{x}_{ik} \geq \text{OPT}(LP) .
\]

2. Item assignment: \( \hat{x} \) assigns each item to at most one bucket, namely, \( \sum_{k \in [K-1]} \hat{x}_{ik} \leq 1 \) for every \( i \in [n] \).

3. Fixable capacity: For every bucket \( B_k \), if its capacity is violated (i.e., \( \sum_{i \in L_{k+1}} w_i \hat{x}_{ik} > \text{capacity}(B_k) \)), there exists a single infeasibility item \( i_{\inf(k)} \) with \( \hat{x}_{i_{\inf(k)},k} = 1 \) whose removal restores the feasibility of that bucket, i.e.,

\[
\sum_{i \in L_{k+1}} w_i \hat{x}_{ik} - w_{i_{\inf(k)}} \leq \text{capacity}(B_k) .
\]

Restoring feasibility with negligible profit loss. Given the above-mentioned properties, a feasible integral solution can obviously be obtained by eliminating the infeasibility item of each bucket with violated capacity. However, this straightforward approach may decrease the objective value by a non-\( \epsilon \)-bounded factor. Instead, the final step of our algorithm greedily defines an integral solution \( \hat{x}^- \) which is feasible for (IP) and has an objective value of at least \( (1 - 8\epsilon) \cdot \text{OPT}(LP) \). To this end, for every bucket \( B_k \) whose capacity is not violated by \( \hat{x} \), we simply have \( \hat{x}_{ik}^- = \hat{x}_{ik} \) for all \( i \in L_{k+1} \). In contrast, for every bucket \( B_k \) whose capacity is violated, we proceed as follows:

- Let \( i_1, \ldots, i_M \) be an indexing of the set \( \{ i \in L_{k+1} : \hat{x}_{ik} = 1 \} \) such that

\[
\frac{q_{i_1,k}}{w_{i_1}} \geq \cdots \geq \frac{q_{i_M,k}}{w_{i_M}} .
\]

- Let \( \mu \) be the maximal index for which \( \sum_{m \in [\mu]} w_{im} \leq \text{capacity}(B_k) \).

- Then, our solution sets \( \hat{x}_{i_1,k}^- = \cdots = \hat{x}_{i_{\mu},k}^- = 1 \) and \( \hat{x}_{ik}^- = 0 \) for any other item. Clearly, \( \hat{x}_{ik}^- \leq \hat{x}_{ik} \) for all \( i \in L_{k+1} \).

In Lemma 7, we show that the profit collected by \( \hat{x}^- \) nearly matches the fractional optimum. To gain some intuition for the proof of this claim, the main idea is that the capacity of each bucket \( B_k \) will be shown to be violated in \( \hat{x} \) only by an \( \epsilon \)-related fraction, if at all, due to being assigned only \((k+1)\)-light items. For the same reason, our greedy procedure will be shown to nearly exhaust the entire capacity of each violated bucket. In this case, packing items by their profit-to-weight ratio guarantees that we are very close to matching the original profit contribution of such buckets.

Lemma 7 \( \sum_{i \in [n]} \sum_{k \in [K-1]} q_{ik} \hat{x}_{ik}^- \geq (1 - 8\epsilon) \cdot \text{OPT}(LP) \).
Proof Recall that the super-optimality property of \( \hat{x} \), as stated in (3), corresponds to having \( \sum_{i \in [n]} \sum_{k \in [K - 1]} q_{ik} \hat{x}_{ik} \geq \text{OPT}(LP) \). Therefore, by changing the order of summation, we can establish the desired claim by proving that
\[
\sum_{i \in L_{k+1}} q_{ik} \hat{x}_{ik} \geq (1 - 8\epsilon) \cdot \sum_{i \in L_{k+1}} q_{ik} \hat{x}_{ik} \quad \text{for every } k \in [K - 1].
\]
Moreover, since one has \( \hat{x}_k = \hat{x}_k \) with respect to buckets whose capacity is not violated by \( \hat{x} \), it remains to focus on violated buckets.

For such buckets, we first observe that, by the maximality of \( \mu \),
\[
\sum_{m \in [\mu]} w_{im} > \text{capacity}(B_k) - w_{i_{\mu+1}} \geq (1 - 4\epsilon) \cdot \text{capacity}(B_k),
\]
where the second inequality holds since \( i_{\mu+1} \in L_{k+1} \), and therefore \( w_{i_{\mu+1}} \leq \epsilon^2 \cdot (1 + \epsilon)^{k+1} \leq 4\epsilon \cdot ((1 + \epsilon)^k - (1 + \epsilon)^{k-1}) = 4\epsilon \cdot \text{capacity}(B_k) \) for \( \epsilon \in (0, 1) \). On the other hand,
\[
\sum_{m \in [\mu]} w_{im} = \sum_{i \in L_{k+1}} w_{i \hat{x}_{ik}} \leq \text{capacity}(B_k) + w_{i_{\text{inf}(k)}} \leq (1 + 4\epsilon) \cdot \text{capacity}(B_k),
\]
where the equality above follows from how the indices \( i_1, \ldots, i_M \) were defined, the first inequality is precisely the fixable capacity property of \( \hat{x} \) (see (4)), and the second inequality holds since \( w_{i_{\text{inf}(k)}} \leq 4\epsilon \cdot \text{capacity}(B_k) \), as explained earlier for \( w_{i_{\mu+1}} \). Consequently,
\[
\sum_{i \in L_{k+1}} q_{ik} \hat{x}_{ik} = \sum_{m \in [\mu]} q_{im,k} \geq \frac{\sum_{m \in [\mu]} w_{im}}{\sum_{m \in [M]} w_{im}} \cdot \sum_{m \in [M]} q_{im,k} \geq \frac{1 - 4\epsilon}{1 + 4\epsilon} \cdot \sum_{i \in L_{k+1}} q_{ik} \hat{x}_{ik},
\]
where the first inequality holds since \( \frac{\sum_{i=1}^M q_{i1,k} w_{i1}}{w_{i1}} \geq \cdots \geq \frac{\sum_{i=1}^M q_{iM,k} w_{i1}}{w_{i1}} \), and the second inequality is obtained by plugging in (5) and (6).

Performance guarantee. We conclude by noting that, since \( \hat{x}_k \) is a feasible solution to (IP), Lemma 5 allows us to construct a permutation \( \pi_{\text{light}} \) with an overall profit of
\[
\Psi(\pi_{\text{light}}) \geq \sum_{i \in [n]} \sum_{k \in [K-1]} \sum_{i \in L_{k+1}} q_{ik} \hat{x}_{ik}.
\]

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\[ \geq (1 - 8\epsilon) \cdot \text{OPT}(LP) \]
\[ \geq (1 - 13\epsilon) \cdot \Psi_{\text{light}}(\pi^*) , \]

where the second and third inequalities follow from Lemmas 7 and 6, respectively.

From a running time perspective, the computational bottleneck of our approach is the Shmoys–Tardos algorithm [35]. As the latter is applied to a maximum generalized assignment instance consisting of \( n \) items and \( O(K) = O(\frac{|I|}{\epsilon}) \) buckets, it requires \( O\left(\frac{|I|}{\epsilon}\right)^{O(1)} \) time in total. Beyond that, restoring the feasibility of \( \hat{x} \) and translating the resulting solution \( \hat{x}^- \) back to a permutation can both be implemented in \( O((nK)^{O(1)}) \) time.

### 3 QPTAS for bounded weight ratio

In this section, we develop an approximation scheme for the generalized incremental knapsack problem by embedding our LP-based approach for competing against light contributions within a self-improving algorithm. As formally stated in Theorem 6 below, the running time of this algorithm will be exponentially-dependent on \( \log(n \cdot \frac{w_{\max}}{w_{\min}}) \), meaning that it provides a quasi-polynomial time approximation scheme (QPTAS) when the ratio between the extremal item weights is polynomial in the input size. In Sect. 4, these ideas will be exploited within an approximate dynamic programming framework to derive a true QPTAS, without making any assumptions on the ratio \( \frac{w_{\max}}{w_{\min}} \).

**Theorem 6** For any error parameter \( \epsilon \in (0, 1) \), the generalized incremental knapsack problem can be approximated within a factor of \( 1 - \epsilon \) in time \( O\left(\frac{1}{\epsilon^5 \log(n \cdot \frac{w_{\max}}{w_{\min}})}\right) \cdot |\mathcal{I}|^{O(1)} \).

**Outline.** As an instructive step, we dedicate Sect. 3.1 to explaining how, given any feasible chain, one can define a residual instance on the remaining (non-inserted) items. In this context, we establish a number of structural properties that relate between the solution spaces of the original and residual instances, which will be useful moving forward. As explained in Sect. 3.2, the basic idea behind our “self-improving” algorithm resides in arguing that, given a black-box \( \alpha \)-approximation for the generalized incremental knapsack problem, efficient guessing methods can be utilized to construct a solution that optimally competes against heavy contributions, and simultaneously, \( \alpha \)-competes against light contributions. In Sect. 3.3, we combine this result with our near-optimal algorithm for light contributions and attain a performance guarantee of \( \frac{1}{2 - \alpha} \), up to lower-order terms. Repeated applications of these \( \alpha \mapsto \frac{1}{2 - \alpha} \) improvements will be shown to obtain a \( (1 - \epsilon) \)-fraction of the optimal profit within \( O\left(\frac{1}{\epsilon} \right) \) rounds. It is important to mention that each such application by itself incurs an exponential dependency on \( \log(n \cdot \frac{w_{\max}}{w_{\min}}) \), meaning that the results of this section are incomparable to those stated in Theorem 3, where the running time involved is truly polynomial for any fixed \( \epsilon > 0 \).
3.1 Residual instances and their properties

**Instance representation.** Due to working with modified instances in subsequent sections, we will designate the underlying set of items in a given instance by \( \mathcal{N} \). As before, each item \( i \in \mathcal{N} \) is associated with a weight of \( w_i \), each time period \( t \in [T] \) has a capacity of \( W_t \), and we gain a profit of \( p_{it} \) for introducing item \( i \) in period \( t \). That said, what differentiates between one instance and the other are two ingredients: The item set \( \mathcal{N} \) and the time period capacities \( W = (W_1, \ldots, W_T) \) with respect to which these instances are defined. It is important to point out that, regardless of the instance being considered, the item weights \( w_i \), the number of time periods \( T \), and the item-to-period profits \( p_{it} \) will be kept unchanged. For these reasons, we denote a generalized incremental knapsack instance simply by \( \mathcal{I} = (\mathcal{N}, W) \).

**The \( |G\)-operator.** In the following, we introduce additional definitions, notation, and structural properties related to modified instances and their solution space. For a pair of chains, \( S = (S_1, \ldots, S_T) \) and \( G = (G_1, \ldots, G_T) \), we define the union of \( S \) and \( G \) as \( S \cup G = (S_1 \cup G_1, \ldots, S_T \cup G_T) \), which is clearly a chain itself. For a chain \( S \) and a subset of items \( G \subseteq \mathcal{N} \), we denote by \( S|_G \) the restriction of \( S \) to \( G \), namely, \( S|_G = (S_1 \cap G, \ldots, S_T \cap G) \); one can easily verify that \( S|_G \) is a chain as well. The next claim, whose straightforward proof is omitted, establishes the feasibility of \( S|_G \) whenever \( S \) is feasible.

**Observation 1** Let \( S \) be a feasible chain for \( \mathcal{I} \). Then, for any set of items \( G \subseteq \mathcal{N} \), the chain \( S|_G \) is feasible as well.

**The residual instance.** Given a feasible chain \( G = (G_1, \ldots, G_T) \) for an instance \( \mathcal{I} = (\mathcal{N}, W) \), we define the residual generalized incremental knapsack instance \( \mathcal{I}^{-G} = (\mathcal{N}^{-G}, W^{-G}) \) as follows:

- The new set of items is \( \mathcal{N}^{-G} = \mathcal{N} \setminus G_T \). Namely, we eliminate all items that were introduced at any point in time by \( G \).
- The residual capacity of every time \( t \in [T] \) is set to \( W^{-G}_t = \min_{t \leq \tau \leq T} (W_\tau - w(G_\tau)) \).
- As previously mentioned, all item weights and profits remain unchanged.

To verify that the residual instance \( \mathcal{I}^{-G} \) is well defined, it suffices to show that the residual capacities \( W^{-G}_t \) are non-negative and non-decreasing over time. The former property holds since \( w(G_\tau) \leq W_\tau \) for every \( t \in [T] \), by feasibility of \( G \). The latter property follows by observing that

\[
W^{-G}_t = \min_{t \leq \tau \leq T} (W_\tau - w(G_\tau)) \leq \min_{t+1 \leq \tau \leq T} (W_\tau - w(G_\tau)) = W^{-G}_{t+1}.
\]

The next two claims, whose respective proofs appear in Appendices C.1 and C.2, explain the relationship between the solution spaces of the original instance \( \mathcal{I} \) and its residual instance \( \mathcal{I}^{-G} \). For our purposes, the main implication of this relationship will be that, whenever we are able to “guess” a chain \( \mathcal{G} = \mathcal{S}^*|_G \), where \( \mathcal{S}^* \) is an optimal chain for \( \mathcal{I} \), it suffices to focus on solving the residual instance \( \mathcal{I}^{-G} \). With an
appropriate guess for the set of items $G$, this property will be a key idea within the approximation scheme we devise in the remainder of this section.

**Lemma 8** Let $\mathcal{G}$ be a feasible chain for $\mathcal{I}$ and let $\mathcal{R}$ be a feasible chain for $\mathcal{I} \setminus \mathcal{G}$. Then, $\mathcal{G} \cup \mathcal{R}$ is a feasible chain for $\mathcal{I}$ with profit $\Phi(\mathcal{G} \cup \mathcal{R}) = \Phi(\mathcal{G}) + \Phi(\mathcal{R})$.

**Lemma 9** Let $S$ be a feasible chain for $\mathcal{I}$ and let $G = S|_G$, for some set of items $G \subseteq \mathcal{N}$. Then, $S|_{\mathcal{N} \setminus G}$ is a feasible chain for $\mathcal{I} \setminus G$ with profit $\Phi(S|_{\mathcal{N} \setminus G}) = \Phi(S) - \Phi(G)$. Moreover, if $S$ is optimal for $\mathcal{I}$, then $S|_{\mathcal{N} \setminus G}$ is optimal for $\mathcal{I} \setminus G$.

### 3.2 The boosting algorithm

Given a generalized incremental knapsack instance $\mathcal{I} = (\mathcal{N}, W)$, let us focus our attention on a fixed optimal chain $S^*$. As argued in Lemma 1, this chain can be mapped to a permutation $\pi_{S^*} : \mathcal{N} \to [|\mathcal{N}|]$ whose objective value with respect to the corresponding sequencing formulation is $\Psi(\pi_{S^*}) \geq \Phi(S^*)$. By decomposing the overall profit $\Psi(\pi_{S^*})$ into heavy and light contributions, as prescribed by Equation (1), we have:

$$\Psi(\pi_{S^*}) = \sum_{k \in [K]} \sum_{\{i \in I_k : c_{\pi_{S^*}(i)} \in I_k\}} \varphi_{\pi_{S^*}}(i) + \sum_{k \in [K]} \sum_{\{i \in I_k : c_{\pi_{S^*}(i)} \not\in I_k\}} \varphi_{\pi_{S^*}}(i).$$

Given these quantities, for $\alpha_H, \alpha_L \in [0, 1]$, we say that an algorithm $\mathcal{A}$ guarantees an $(\alpha_H, \alpha_L)$-approximation with respect to $S^*$ when it computes a feasible chain $S$ with $\Phi(S) \geq \alpha_H \cdot \Psi_{\text{heavy}}(\pi_{S^*}) + \alpha_L \cdot \Psi_{\text{light}}(\pi_{S^*})$. We mention in passing that this definition depends on the specific permutation $\pi_{S^*}$, and is generally different from the standard notion of an $\alpha$-approximation, where the chain $S$ is required to satisfy $\Phi(S) \geq \alpha \cdot \Phi(S^*)$.

**From $\alpha$-approximation to $(1, \alpha)$-approximation.** In what follows, we show how to boost the profit performance of any approximation algorithm for the generalized incremental knapsack problem. For every $\alpha \in [0, 1]$, we explain how to combine a black-box $\alpha$-approximation with further guesses for the positioning of heavy items with respect to the permutation $\pi_{S^*}$ in order to derive a $(1, \alpha)$-approximation, incurring an extra multiplicative factor of $O((nT)^O(1/\epsilon \log(n\rho)))$ in running time, where $\rho = w_{\text{max}} / w_{\text{min}}$. This result can be formally stated as follows.

**Lemma 10** Suppose that the algorithm $\mathcal{A}$ constitutes an $\alpha$-approximation for generalized incremental knapsack, for some $\alpha \in [0, 1]$. Then, there exists a $(1, \alpha)$-approximation whose running time is $O((nT)^O(1/\epsilon \log(n\rho))) \cdot \text{Time}_\mathcal{A}(n, T))$. Here, $\text{Time}_\mathcal{A}(n, T)$ designates the worst-case running time of $\mathcal{A}$ for instances with $n$ items and $T$ time periods.

**Preliminaries.** We remind the reader that Sect. 2.2 has previously defined the intervals $I_0 = [0, 1)$ and $I_k = [(1 + \epsilon)^{k-1}, (1 + \epsilon)^k]$ for $k \in [K]$, where $K = \left\lceil \frac{\log(n\rho)}{\epsilon} \right\rceil$.

}\n
\[\log_{1+\epsilon}(\sum_{i\in[n]} w_i); \text{ similarly, we assume without loss of generality that } w_{\min} \geq 3.\]

In this regard, an item \( i \) is \( k \)-heavy when \( w_i \geq \epsilon^2 \cdot (1 + \epsilon)^k \), with the convention that \( H_k \) stands for the collection of \( k \)-heavy items. Let \( G^{\text{heavy}} \) be the set of items that are heavy for the interval that contains their completion time with respect to the permutation \( \pi S^* \), i.e., \( G^{\text{heavy}} = \bigcup_{k \in [K]} \{ i \in H_k : C_{\pi S^*}(i) \in I_k \} \). The following lemma, whose proof appears in “Appendix C.3”, provides an upper bound on the cardinality of this set.

**Lemma 11** \[|G^{\text{heavy}}| \leq \frac{3\log(n\rho)}{\epsilon^2}.\]

We proceed by considering the restriction of the optimal chain \( S^* \) to the set of items \( G^{\text{heavy}} \), which will be denoted by \( \mathcal{H}^* = S^*| G^{\text{heavy}} \). By Observation 1, we know that \( \mathcal{H}^* \) is a feasible chain for \( \mathcal{I} \). The next lemma, whose proof can be found in “Appendix C.4”, relates between the profit of this chain and heavy contributions with respect to the permutation \( \pi S^* \).

**Lemma 12** \[\Phi(\mathcal{H}^*) = \Psi_{\text{heavy}}(\pi S^*).\]

**The algorithm.** At a high level, our algorithm relies on “knowing” the restricted chain \( \mathcal{H}^* \) in advance, which will be justified by guessing all items in \( G^{\text{heavy}} \) and their insertion times with respect to the optimal chain \( S^* \). This procedure will be implemented by enumerating over all possible configurations of these parameters. For each such guess, we construct the residual generalized incremental knapsack instance, to which the \( \alpha \)-approximation algorithm \( A \) is applied. Formally, given an instance \( \mathcal{I} = (N, W) \) and an error parameter \( \epsilon > 0 \), we proceed as follows:

1. For every feasible chain \( G = (G_1, \ldots, G_T) \) with \( |G_T| \leq \frac{3\log(n\rho)}{\epsilon^2} \):
   (a) Construct the residual instance \( \mathcal{I}^{-G} \).
   (b) Apply the algorithm \( A \) to obtain an \( \alpha \)-approximate feasible chain \( S^{-G} = (S_1^{-G}, \ldots, S_T^{-G}) \) for \( \mathcal{I}^{-G} \).

2. Return the chain \( G^* \cup S^{-G^*} \) of maximum profit among those considered above.

**Analysis: Feasibility and running time.** We first observe that, for any feasible chain \( G \) constructed in step 1, since \( S^{-G} \) is a feasible chain for \( \mathcal{I}^{-G} \), the feasibility of \( G \cup S^{-G} \) for \( \mathcal{I} \) follows by Lemma 8. In terms of running time, we are considering only chains that introduce at most \( \frac{3\log(n\rho)}{\epsilon^2} \) items over all time periods. Thus, the number of chains being enumerated is \( O((nT)^{O(\frac{1}{\epsilon^2} \log(n\rho))}) \). For each residual instance, consisting of \( T \) time periods and at most \( n \) items, we apply the algorithm \( A \) once, implying that the overall running time is indeed \( O((nT)^{O(\frac{1}{\epsilon^2} \log(n\rho))} \cdot \text{Time}_A(n, T)) \).

**Analysis: \((1, \alpha)\)-approximation guarantee.** We conclude the proof of Lemma 10 by arguing that \( G^* \cup S^{-G^*} \) is a \((1, \alpha)\)-approximate chain with respect to \( S^* \) for the original instance \( \mathcal{I} \).

**Lemma 13** \[\Phi(G^* \cup S^{-G^*}) \geq \Psi_{\text{heavy}}(\pi S^*) + \alpha \cdot \Psi_{\text{light}}(\pi S^*).\]
Proof We begin by observing that the feasible chain $\mathcal{H}^* = S^*|_{G^{\text{heavy}}}$ is one of those considered in step 1. To verify this claim, note that $|G^{\text{heavy}}| \leq \frac{3 \log(n\rho)}{\delta^2}$ by Lemma 11, meaning that $\mathcal{H}^*$ introduces at most that many items across all time periods. As a result, since the chain $G^* \cup S^*$ attains a maximum profit among those considered, we have $\Phi(G^* \cup S^*) \geq \Phi(\mathcal{H}^* \cup S^*)$, and it remains to prove that $\Phi(\mathcal{H}^* \cup S^*) \geq \Psi_{\text{heavy}}(\pi_{S^*}) + \alpha \cdot \Psi_{\text{light}}(\pi_{S^*})$.

For this purpose, let $L^* = S^*|_{\mathcal{N}\setminus G^{\text{heavy}}}$ be the restriction of $S^*$ to the set $\mathcal{N}\setminus G^{\text{heavy}}$, which is a feasible chain for $\mathcal{I}$ by Observation 1. We next show that $\Phi(L^*) = \Psi_{\text{light}}(\pi_{S^*})$. In order to derive this claim, note that since $L_T^*$ and $H_T^*$ are disjoint and $S^* = \mathcal{H}^* \cup L^*$, it follows that

$$\Phi(L^*) = \Phi(S^*) - \Phi(\mathcal{H}^*)$$

$$= \Phi(S^*) - \Psi_{\text{heavy}}(\pi_{S^*})$$

$$= \Psi(\pi_{S^*}) - \Psi_{\text{heavy}}(\pi_{S^*})$$

$$= \Psi_{\text{light}}(\pi_{S^*}),$$

where the second equality holds due to Lemma 12, the third equality is obtained by recalling that $\Psi(\pi_{S^*}) = \Phi(S^*)$, as shown along the proof of Lemma 12, and the last equality follows from the profit decomposition (7).

However, the crucial observation is that $L^*$ is a feasible chain for the residual instance $\mathcal{I} - \mathcal{H}^*$, by Lemma 9. Consequently, since the algorithm $\mathcal{A}$ computes an $\alpha$-approximate feasible chain $S - \mathcal{H}^*$ for the latter instance, $\Phi(S - \mathcal{H}^*) \geq \alpha \cdot \Phi(L^*) = \alpha \cdot \Psi_{\text{light}}(\pi_{S^*})$, implying that $\mathcal{H}^* \cup S - \mathcal{H}^*$ indeed has a profit of $\Phi(\mathcal{H}^* \cup S - \mathcal{H}^*) = \Phi(\mathcal{H}^*) + \Phi(S - \mathcal{H}^*) \geq \Psi_{\text{heavy}}(\pi_{S^*}) + \alpha \cdot \Psi_{\text{light}}(\pi_{S^*})$. \hfill $\Box$

### 3.3 The ratio improvement and final algorithm

We proceed by revealing the self-improving feature of our approach, by showing that a $(1, \alpha)$-approximation for generalized incremental knapsack leads in turn to a $\frac{1-\delta}{2-\alpha}$-approximation, when combined with our algorithm for light items, presented in Sect. 2.4. We will then show how to recursively apply this self-improving idea to eventually derive an approximation scheme.

**Lemma 14** Suppose that, for some $\alpha \in [0, 1]$, the algorithm $\mathcal{A}$ constitutes an $\alpha$-approximation. Then, for any error parameter $\delta > 0$, the generalized incremental knapsack problem can be approximated within factor $\frac{1-\delta}{2-\alpha}$ in time $O((nT)^{O(\frac{1}{\delta} \log(n\rho))} \cdot \text{Time}_{\mathcal{A}}(n, T))$.

**Proof** As explained in Sect. 3.2, the optimal chain $S^*$ can be mapped to a permutation $\pi_{S^*}$ whose overall profit $\Psi(\pi_{S^*})$ decomposes into heavy and light contributions, $\Psi(\pi_{S^*}) = \Psi_{\text{heavy}}(\pi_{S^*}) + \Psi_{\text{light}}(\pi_{S^*})$. Now, on the one hand, Lemma 10 provides us with a $(1, \alpha)$-approximation in $O((nT)^{O(\frac{1}{\delta} \log(n\rho))} \cdot \text{Time}_{\mathcal{A}}(n, T))$ time. That is, we obtain a feasible chain $S_{(1, \alpha)}$ with $\Phi(S_{(1, \alpha)}) \geq \Psi_{\text{heavy}}(\pi_{S^*}) + \alpha \cdot \Psi_{\text{light}}(\pi_{S^*})$. On the other hand, the main result of Sect. 2.4 allows us to compute in $O((\frac{1}{\delta})^{O(1)})$ time a
permutation π_{light} with a profit of \( \Psi(\pi_{light}) \geq (1 - \delta) \cdot \Psi_{light}(\pi_{S^*}) \). By converting this permutation to a feasible chain \( S_{(0, 1-\delta)} \) along the lines of Lemma 1, we clearly obtain a \((0, 1-\delta)\)-approximation, meaning that \( \Phi(S_{(0, 1-\delta)}) \geq (1 - \delta) \cdot \Psi_{light}(\pi_{S^*}) \). Our combined approach independently employs both algorithms and returns the more profitable of the two feasible chains computed, \( S_{(1, \alpha)} \) and \( S_{(0, 1-\delta)} \), to obtain a profit of

\[
\max \{ \Phi(S_{(1, \alpha)}), \Phi(S_{(0, 1-\delta)}) \} \geq \max \{ \Psi_{heavy}(\pi_{S^*}) + \alpha \cdot \Psi_{light}(\pi_{S^*}), (1 - \delta) \cdot \Psi_{light}(\pi_{S^*}) \}
\]

\[
\geq \frac{1}{2 - \alpha} \cdot (\Psi_{heavy}(\pi_{S^*}) + \alpha \cdot \Psi_{light}(\pi_{S^*})) + \left( 1 - \frac{1}{2 - \alpha} \right) \cdot (1 - \delta) \cdot \Psi_{light}(\pi_{S^*})
\]

\[
\geq \frac{1 - \delta}{2 - \alpha} \cdot (\Psi_{heavy}(\pi_{S^*}) + \Psi_{light}(\pi_{S^*}))
\]

\[
= \frac{1 - \delta}{2 - \alpha} \cdot \Psi(\pi_{S^*})
\]

\[
\geq \frac{1 - \delta}{2 - \alpha} \cdot \Phi(S^*),
\]

where the last inequality follows from Lemma 1. \( \square \)

The final approximation scheme. We conclude by explaining how our \( \alpha \mapsto \frac{1 - \delta}{\frac{1}{\alpha} - \delta} \) improvement, outlined in Lemma 14, can be iteratively applied to derive an approximation scheme for the generalized incremental knapsack problem, thereby completing the proof of Theorem 6.

For the purpose of ensuring a \((1 - \epsilon)\)-fraction of the optimal profit, we will set the error parameter \( \delta \) in Lemma 14 as a function of \( \epsilon \), where the exact dependency will be determined later on. Given this self-improving result, we define a sequence of algorithms \( A_0, A_1, \ldots \), with the convention that the approximation ratio of each such algorithm \( A_r \) is denoted by \( \alpha_r \). Specifically, this sequence begins with the trivial algorithm \( A_0 \) that returns an empty solution (\( \emptyset, \ldots, \emptyset \)), meaning that \( \alpha_0 = 0 \). Then, by applying Lemma 14 with respect to \( A_0 \), we obtain the algorithm \( A_1 \), for which \( \alpha_1 = \frac{1 - \delta}{2} \). Subsequently, by a similar application with respect to \( A_1 \), we obtain \( A_2 \), with \( \alpha_2 = \frac{1 - \delta}{\frac{1}{\alpha_1} - \delta} \). In general, for every integer \( r \geq 1 \), the resulting algorithm \( A_r \) guarantees an approximation ratio of \( \alpha_r = \frac{1 - \delta}{\frac{1}{\alpha_{r-1}} - \delta} \). The next lemma, whose proof is presented in “Appendix C.5”, provides a closed-form lower bound on \( \alpha_r \).

**Lemma 15** \( \alpha_r \geq \frac{r}{r + 1} - r \delta, \) for every \( r \geq 0 \).

By choosing \( \delta = \frac{\epsilon^2}{2} \), the above lemma implies that \( \lceil \frac{2}{\epsilon} \rceil \) self-improving rounds produce an algorithm \( A_{\lceil \frac{2}{\epsilon} \rceil} \) for computing a feasible chain \( S \) with a profit of \( \Phi(S) \geq \left( \frac{\lceil \frac{2}{\epsilon} \rceil}{\lceil \frac{1}{\alpha} \rceil + 1} - \frac{\epsilon^2}{2} \right) \cdot \Phi(S^*) \geq (1 - \epsilon) \cdot \Phi(S^*) \), thereby deriving the approximation guarantee of Theorem 6. Furthermore, it is not difficult to verify that algorithm \( A_{\lceil \frac{2}{\epsilon} \rceil} \) runs in \( O((nT)^{O(\frac{1}{\epsilon^2 \log(n\rho)})) \cdot |I|O(1)) \) time, by induction on \( r \).
4 QPTAS for general instances

Thus far, we have developed an approximation scheme whose running time includes an exponential dependency on \( \log(n \cdot \frac{w_{\text{max}}}{w_{\text{min}}}) \), leading to a quasi-PTAS for problem instances where the ratio \( \frac{w_{\text{max}}}{w_{\text{min}}} \) is polynomial in the input size. In what follows, we show how to obtain a true quasi-PTAS, without any assumptions on \( \frac{w_{\text{max}}}{w_{\text{min}}} \).

**Theorem 7** For any error parameter \( \epsilon \in (0, 1) \), the generalized incremental knapsack problem can be approximated within a factor of \( 1 - \epsilon \) in time \( O(|I|^{O(1/\epsilon \log |I|^{O(1)})}) \).

Interestingly, our algorithmic approach shows that any \( (1 - \epsilon) \)-approximation running in \( T(n, T, \frac{w_{\text{max}}}{w_{\text{min}}}) \) time can be executed in black-box fashion on appropriately-constructed instances with \( \frac{w_{\text{max}}}{w_{\text{min}}} = O(n^{1/\epsilon}) \), leading to a \( (1 - \epsilon) \)-approximation for the general problem formulation in \( O(|I|^{O(1/\epsilon \log |I|^{O(1)})}) \cdot T(n, T, n^{1/\epsilon}) \) time.

4.1 Technical overview

**Step 1: Creating a well-spaced instance.** We begin by slightly altering a given instance \( I = (N, W) \), with the objective of creating nearly-ideal circumstances for the approximation scheme of Sect. 3 to operate, losing negligible profits along the way. For this purpose, given an error parameter \( \epsilon > 0 \), we say that the instance \( I \) is well-spaced when its set of items \( N \) can be partitioned into clusters \( C_1, \ldots, C_M \) satisfying the following properties:

1. **Weight ratio within clusters:** For every \( m \in [M] \), the weights of any two items in cluster \( C_m \) differ by a multiplicative factor of at most \( n^{1/\epsilon} \).
2. **Weight gap between clusters:** For every \( m_1, m_2 \in [M] \) with \( m_1 < m_2 \), the weight of any item in cluster \( C_{m_2} \) is greater than the weight of any item in cluster \( C_{m_1} \) by a multiplicative factor of at least \( n^{1+(m_2-m_1-1)/\epsilon} \).

In Sect. 4.2, we show that one can efficiently identify a subset of items over which the induced instance is well-spaced, while still admitting a near-optimal solution. We derive this result, as formally stated below, through an application of the shifting method (see, for instance, [5,29]).

**Lemma 16** There exists an item set \( N_{\text{spaced}} \subseteq N \) for which \( I_{\text{spaced}} = (N_{\text{spaced}}, W) \) is a well-spaced instance, whose optimal chain \( S_{\text{spaced}} \) guarantees a profit of \( \Phi(S_{\text{spaced}}) \geq (1 - \epsilon) \cdot \Phi(S^*) \). Such a set can be determined in \( O((n/\epsilon)^{O(1)}) \) time.

**Step 2: Proving the sparse-crossing property.** For simplicity of notation, we assume from this point on that the instance \( I = (N, W) \) is well-spaced, with clusters \( C_1, \ldots, C_M \). Now suppose that the optimal permutation \( \pi^* \) for the sequencing-based formulation of this instance was known to be “crossing-free”, namely, items belonging to cluster \( C_1 \) appear first in \( \pi^* \), followed by those belonging to cluster \( C_2 \), so on and so forth. In other words, a left-to-right scan of the permutation \( \pi^* \) reveals that it is weakly-increasing by cluster. In this ideal situation, the approximation scheme we propose in Sect. 3 can be sequentially employed to the clusters \( C_1, \ldots, C_M \) in increasing
order. This way, we would have obtained a \((1 - \epsilon)\)-approximation in truly quasi-

poly-time, since the extremal weight ratio within each cluster is \(n^{1/\epsilon}\)-bounded, by property \(1\).

Unfortunately, elementary examples show that an optimal permutation \(\pi^*\) may not be crossing-free, in the sense that items in any given cluster can be preceded by items belonging to higher-index clusters. That said, a suitable relaxation of these ideas can still be exploited. Formally, let us denote by \(\text{cross}_m(\pi)\) the number of items in clusters \(C_{m+1}, \ldots, C_M\) that appear in the permutation \(\pi\) before the last item belonging to cluster \(C_m\); note that crossing-free is equivalent to having \(\text{cross}_1(\pi) = \cdots = \text{cross}_M(\pi) = 0\).

Our next structural result, formally established in Sect. 4.3, proves the existence of a near-optimal permutation with very few items crossing each cluster.

**Lemma 17** There exist an item set \(N_{\text{sparse}} \subseteq N\) and a permutation \(\pi_{\text{sparse}} : N_{\text{sparse}} \to [\vert N_{\text{sparse}}\vert]\) satisfying:

1. **Sparse crossing:** \(
   \max_{m \in [M]} \text{cross}_m(\pi_{\text{sparse}}) \leq \frac{\lceil \log_2 M \rceil}{\epsilon}.
   \)
2. **Near-optimal profit:** \(\Psi(\pi_{\text{sparse}}) \geq (1 - \epsilon) \cdot \Psi(\pi^*)\).

Technically speaking, our proof is based on applying a sequence of recursive

transformations with respect to the unknown optimal permutation \(\pi^*\). To convey the

high-level idea, let \(i_{\text{mid}}\) be the last-appearing item in \(\pi^*\) out of clusters \(C_1, \ldots, C_{M/2}\). When fewer than \(1/\epsilon\) items in clusters \(C_{(M/2)+1}, \ldots, C_M\) appear before \(i_{\text{mid}}\), each of the clusters \(C_1, \ldots, C_{M/2}\) has at most \(1/\epsilon\) crossings due to items in \(C_{(M/2)+1}, \ldots, C_M\). We can therefore recursively proceed into the left part of \(\pi^*\), stretching up to the item \(i_{\text{mid}}\), and into its right part, consisting of the remaining items. In the opposite case, where at least \(1/\epsilon\) items in clusters \(C_{(M/2)+1}, \ldots, C_M\) appear before \(i_{\text{mid}}\), the important observation is that we can eliminate the cheapest out of the first \(1/\epsilon\) such items while losing only an \(O(\epsilon)\)-fraction of their combined profit. However, since this item is heavier than any item in lower-index clusters by a factor of at least \(n\) (see property 2), the gap we have just created is sufficiently large to pull back each and every item in clusters \(C_1, \ldots, C_{M/2}\), only increasing their profit contributions. We can now recursively proceed into the left and right parts.

**Step 3: The external dynamic program.** Given the sparse-crossing property, we dedicate Sect. 4.4 to proposing a dynamic programming approach for computing a near-optimal permutation. For this purpose, by recycling some of the notation introduced in Sect. 2.3, our state description \((m, \psi_m, Q_{>m})\) will consists of the following parameters:

- The index of the current cluster, \(m\).
- The profit requirement, \(\psi_m\).
- The set of items \(Q_{>m}\) belonging to clusters \(C_{m+1}, \ldots, C_M\) that will be crossing into lower-index clusters, noting that Lemma 17 allows us to consider only small sets, of size \(O(\frac{\log M}{\epsilon})\).

At a high level, the value function \(F(m, \psi_m, Q_{>m})\) will represent the minimum makespan \(w(S)\) that can be attained, over all subset of items \(S\) within the union of \(Q_{>m}\) and the clusters \(C_1, \ldots, C_M\) (namely, \(S \subseteq Q_{>m} \cup (\bigcup_{\mu \in [M]} C_\mu)\)) and over all permutations \(\pi : S \to [\vert S\vert]\) that generate a total profit of at least \(\psi_m\). Clearly,
the best-possible profit of a sparse-crossing permutation corresponds to the maximal value $\psi_M$ that satisfies $F(M, \psi_M, \emptyset) < \infty$, which is at least $(1 - \epsilon) \cdot \psi(\pi^*)$, by Lemma 17.

As formally explained in Sect. 4.4, within the recursive equations for computing $F(m, \psi_m, Q_{>m})$, evaluating the marginal makespan increase of each possible action involves solving a single-cluster subproblem. Specifically for the latter, the approximation scheme we have devised in Sect. 3 will be shown to incur a quasi-polynomial running time. In parallel, the dominant factor in determining the underlying number of states emerges from the set of items $Q_{>m}$, taking $O(n^{O(\frac{1}{\epsilon} \log M)})$ possible values, respectively, thus forming the second source of quasi-polynomiality in our approach and concluding the proof of Theorem 7.

### 4.2 Proof of Lemma 16: Creating a well-spaced instance

**Bucketing.** For the purpose of identifying the desired subset $N_{\text{spaced}}$, we initially partition the overall collection of items $N$ into buckets $B_1, \ldots, B_L$ according to their weights. This partition will be geometric, by powers of $n$, meaning that $L = \lceil \log n \cdot \frac{\text{wmax}}{\text{wmin}} \rceil + 1$. Specifically, the first bucket $B_1$ consists of items whose weight resides in $[\text{wmin}, n \cdot \text{wmin}]$, the second bucket $B_2$ consists of those with weight in $[n \cdot \text{wmin}, n^2 \cdot \text{wmin}]$, so on and so forth, where in general, bucket $B_\ell$ corresponds to the interval $[n^{\ell-1} \cdot \text{wmin}, n^\ell \cdot \text{wmin}]$. It is easy to verify that $B_1, \ldots, B_L$ is indeed a partition of $N$.

**Creating clusters.** Now let $r \in \{0, \ldots, \frac{1}{\epsilon} - 1\}$ be an integer parameter whose value will be determined later. Accordingly, we create a subset of items $N_r \subseteq N$, that will be clustered into $C^r_1, \ldots, C^r_M$ with $M = O(\epsilon L)$, as follows. Intuitively, we introduce “gaps” within the sequence of buckets $B_1, \ldots, B_L$, spaced apart by $\frac{1}{\epsilon}$ indices, through eliminating every bucket $B_\ell$ with $\ell \mod \frac{1}{\epsilon} = r$; then, between every pair of successive gaps, buckets will be unified to form a single cluster. That is, the first cluster is defined as $C^r_1 = \bigcup_{\ell=1}^{\ell=r-1} B_\ell$, the second cluster is $C^r_2 = \bigcup_{\ell=r+1}^{\ell=r+1/\epsilon} B_\ell$, the third is $C^r_3 = \bigcup_{\ell=r+1+1/\epsilon}^{\ell=r+2/\epsilon} B_\ell$, and so on. Finally, we define the subset of items $N_r$ as the union of all clusters, i.e., $N_r = \bigcup_{m \in [M]} C^r_m$, with a corresponding generalized incremental knapsack instance $I_r = (N_r, W)$.

**Analysis.** In what follows, we argue that for every $r \in \{0, \ldots, \frac{1}{\epsilon} - 1\}$, the instance $I_r$ we have just constructed is in fact well-spaced, via the partition of $N_r$ into clusters $C^r_1, \ldots, C^r_M$. For this purpose, we separately prove each of the required well-spaced properties.

1. **Weight ratio within clusters:** Consider two items $i_1$ and $i_2$ belonging to the same cluster $C^r_m$. Letting $B_{\ell_1}$ and $B_{\ell_2}$ be the buckets containing these items, respectively, their weight ratio can be upper bounded by observing that

$$\frac{w_{i_2}}{w_{i_1}} \leq \frac{\max_{i \in B_{\ell_2}} w_i}{\min_{i \in B_{\ell_1}} w_i} \leq n^{\ell_2 - (\ell_1 - 1)}$$
where the second inequality holds since each bucket $B_\ell$ contains items whose weight falls within $[n^{\ell-1}, w_{\text{min}}, n^\ell \cdot w_{\text{min}})$, and the third inequality follows by noting that each cluster represents the union of at most $1/\epsilon - 1$ successive buckets, implying that $\ell_2 - \ell_1 \leq 1/\epsilon - 2$.

2. **Weight gap between clusters:** Similarly, let $i_1$ and $i_2$ be a pair of items that belong to clusters $C_{m_1}^r$ and $C_{m_2}^r$, respectively, with $m_1 < m_2$. In this case, when we denote the corresponding buckets by $B_{\ell_1}$ and $B_{\ell_2}$, their weight ratio can be lower bounded by

$$\frac{w_{i_2}}{w_{i_1}} \geq \frac{\min_{i \in B_{\ell_2}} w_i}{\max_{i \in B_{\ell_1}} w_i} \geq n^{(\ell_2-1)-\ell_1} \geq n^{1+(m_2-m_1-1)/\epsilon},$$

where the last inequality holds since $\ell_1 \in \{r + 1 + \frac{m_1-2}{\epsilon}, \ldots, r + 1 + \frac{m_1-1}{\epsilon}\}$ and $\ell_2 \in \{r + 1 + \frac{m_2-2}{\epsilon}, \ldots, r + 1 + \frac{m_2-1}{\epsilon}\}$, by definition of $C_{m_1}^r$ and $C_{m_2}^r$.

We conclude the proof of Lemma 16 by showing that at least one of the well-spaced instances $\mathcal{I}_0, \ldots, \mathcal{I}_{(1/\epsilon)-1}$ is associated with an optimal profit of at least $(1 - \epsilon) \cdot \Phi(S^*)$. To this end, with respect to the optimal chain $S^*$ for the original instance $\mathcal{I}$, note that the restriction of this chain $S^*|_{\mathcal{N}_r}$ to the item set $\mathcal{N}_r$ is clearly feasible for $\mathcal{I}_r$, by Observation 1. Letting $S^{r*}$ be an optimal chain for $\mathcal{I}_r$, we consequently have

$$\max_{0 \leq r \leq (1/\epsilon)-1} \Phi(S^{r*}) \geq \max_{0 \leq r \leq (1/\epsilon)-1} \Phi(S^*|_{\mathcal{N}_r}) \geq \epsilon \cdot \sum_{r=0}^{(1/\epsilon)-1} \Phi(S^*|_{\mathcal{N}_r}) = \epsilon \cdot \sum_{r=0}^{(1/\epsilon)-1} \sum_{t \in [T]} \sum_{i \in (S_t^* \setminus S_{t-1}^*) \cap \mathcal{N}_r} p_{it}$$

$$= \epsilon \cdot \sum_{r=0}^{(1/\epsilon)-1} \sum_{t \in [T]} \sum_{i \in S_t^* \setminus S_{t-1}^*} \left| \left\{ r \in \left\{ 0, \ldots, \frac{1}{\epsilon} - 1 \right\} : \right. \right.$$  

$$i \in \left( S_t^* \setminus S_{t-1}^* \right) \cap \mathcal{N}_r \left. \right\} \cdot p_{it}$$

$$= (1 - \epsilon) \cdot \sum_{r=0}^{(1/\epsilon)-1} \sum_{t \in [T]} \sum_{i \in S_t^* \setminus S_{t-1}^*} p_{it} = (1 - \epsilon) \cdot \Phi(S^*),$$

where the next-to-last equality holds since every item introduced in the optimal chain $S^*$ appears in all but one of the sets $\mathcal{N}_0, \ldots, \mathcal{N}_{(1/\epsilon)-1}$. 

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4.3 Proof of Lemma 17: The sparse-crossing property

Preliminaries. We begin by introducing some additional definitions and notation that will be utilized throughout this proof. For a set of cluster indices $\mathcal{M} \subseteq [M]$, we use $\mathcal{C}_\mathcal{M}$ to designate the union of $\mathcal{M}$-indexed clusters, i.e., $\mathcal{C}_\mathcal{M} = \biguplus_{m \in \mathcal{M}} \mathcal{C}_m$. Expanding upon the definition of $\text{cross}_m(\pi)$, given disjoint sets, $\mathcal{M}_1 \subseteq [M]$ and $\mathcal{M}_2 \subseteq [M]$, let $\text{cross}_{\mathcal{M}_1,\mathcal{M}_2}(\pi)$ denote the number of items in $\mathcal{C}_{\mathcal{M}_2}$ that appear in the permutation $\pi$ before the last item in $\mathcal{C}_{\mathcal{M}_1}$, namely,

$$\text{cross}_{\mathcal{M}_1,\mathcal{M}_2}(\pi) = \left\{ i \in \mathcal{C}_{\mathcal{M}_2} : \pi(i) < \max_{j \in \mathcal{C}_{\mathcal{M}_1}} \pi(j) \right\}.$$

When $\text{cross}_{\mathcal{M}_1,\mathcal{M}_2}(\pi) \geq \frac{1}{\varepsilon}$, we use $\mathcal{X}_{\mathcal{M}_1,\mathcal{M}_2}(\pi)$ to designate the set comprised of the first $\frac{1}{\varepsilon}$ items in $\mathcal{M}_2$-indexed clusters in the permutation $\pi$. When $\text{cross}_{\mathcal{M}_1,\mathcal{M}_2}(\pi) < \frac{1}{\varepsilon}$, we simply set $\mathcal{X}_{\mathcal{M}_1,\mathcal{M}_2}(\pi) = \emptyset$.

Fixing permutations. In order to formalize the notion of “pulling back” items within a given permutation, as briefly sketched in Sect. 4.1, we define a fixing procedure, FixCrossing$(\pi, \mathcal{M}^-, \mathcal{M}^+)$, and we then present how to eliminate crossings within a permutation. Here, we receive as input a permutation $\pi : \mathcal{Q} \to [|\mathcal{Q}|]$ over an item set $\mathcal{Q} \subseteq \mathcal{N}$, along with two disjoint sets of cluster indices, $\mathcal{M}^-$ and $\mathcal{M}^+$, which are assumed to satisfy $\max \mathcal{M}^- < \min \mathcal{M}^+$, i.e., any index in $\mathcal{M}^-$ is strictly smaller than any index in $\mathcal{M}^+$. As explained below, this procedure constructs in polynomial time a modified permutation $\bar{\pi} : \bar{\mathcal{Q}} \to [|\bar{\mathcal{Q}}|]$ over a subset $\bar{\mathcal{Q}} \subseteq \mathcal{Q}$, that satisfies the following properties:

(P₁) Sparse $(\mathcal{M}^-, \mathcal{M}^+)$-crossing: $\text{cross}_{\mathcal{M}^-,\mathcal{M}^+}(\bar{\pi}) \leq \frac{1}{\varepsilon}$.

(P₂) Completion times: $C_{\bar{\pi}}(i) \leq C_\pi(i)$, for every $i \in \bar{\mathcal{Q}}$.

(P₃) Difference: $\mathcal{Q} \setminus \bar{\mathcal{Q}}$ consists of at most one item, which is a member of $\mathcal{X}_{\mathcal{M}^-,\mathcal{M}^+}(\pi)$.

For this purpose, when $\text{cross}_{\mathcal{M}^-,\mathcal{M}^+}(\pi) < \frac{1}{\varepsilon}$, the procedure FixCrossing$(\pi, \mathcal{M}^-, \mathcal{M}^+)$ returns exactly the same permutation (i.e., $\bar{\pi} = \pi$), without any alterations. In the opposite case, when $\text{cross}_{\mathcal{M}^-,\mathcal{M}^+}(\pi) \geq \frac{1}{\varepsilon}$, let $i_{\mathcal{M}^-,\mathcal{M}^+}$ be the least profitable item in $\mathcal{X}_{\mathcal{M}^-,\mathcal{M}^+}(\pi)$ with respect to the permutation $\pi$, namely, $i_{\mathcal{M}^-,\mathcal{M}^+} = \arg\min\{\varphi_\pi(i) : i \in \mathcal{X}_{\mathcal{M}^-,\mathcal{M}^+}(\pi)\}$. Our construction consists of eliminating $i_{\mathcal{M}^-,\mathcal{M}^+}$ and placing instead all items in $\mathcal{C}_{\mathcal{M}^-}$ appearing in $\pi$ after $i_{\mathcal{M}^-,\mathcal{M}^+}$; this alteration results in a permutation $\bar{\pi}$ over $\mathcal{Q} \setminus \{i_{\mathcal{M}^-,\mathcal{M}^+}\}$. Formally, let $\mathcal{A}^-$ and $\bar{\mathcal{A}}^-$ be the items appearing after $i_{\mathcal{M}^-,\mathcal{M}^+}$ out of $\mathcal{C}_{\mathcal{M}^-}$ and $\mathcal{N} \setminus \mathcal{C}_{\mathcal{M}^-}$, respectively, i.e.,

$$\mathcal{A}^- = \left\{ i \in \mathcal{C}_{\mathcal{M}^-} : \pi(i) > \pi(i_{\mathcal{M}^-,\mathcal{M}^+}) \right\}$$

and

$$\bar{\mathcal{A}}^- = \left\{ i \in \mathcal{N} \setminus \mathcal{C}_{\mathcal{M}^-} : \pi(i) > \pi(i_{\mathcal{M}^-,\mathcal{M}^+}) \right\};$$

For simplicity, we index the items in $\mathcal{A}^-$ according to their order within the permutation $\pi$, which results in having $\mathcal{A}^- = \{i_1, \ldots, i_{|\mathcal{A}^-|}\}$ with $\pi(i_1) < \cdots < \pi(i_{|\mathcal{A}^-|})$. Now, the modified permutation $\bar{\pi}$ is constructed as follows:
– **Before** \( i_{\mathcal{M}^-,\mathcal{M}^+} \): Items in positions 1, \ldots, \( \pi(i_{\mathcal{M}^-,\mathcal{M}^+}) - 1 \) of the permutation \( \pi \) remain within their original positions, meaning that \( \bar{\pi}(i) = \pi(i) \) for every item \( i \) with \( \pi(i) \leq \pi(i_{\mathcal{M}^-,\mathcal{M}^+}) - 1 \).

– **Instead of** \( i_{\mathcal{M}^-,\mathcal{M}^+} \): Items in \( \mathcal{A}^- \) will appear in place of \( i_{\mathcal{M}^-,\mathcal{M}^+} \) following their relative order in \( \pi \). That is, \( \bar{\pi}(i_k) = \pi(i_{\mathcal{M}^-,\mathcal{M}^+}) - 1 + k \) for every \( k \in [\mathcal{A}^-] \).

– **After** \( i_{\mathcal{M}^-,\mathcal{M}^+} \): Items in \( \mathcal{A}^- \) will appear after those in \( \mathcal{A}^- \), again following their relative order in \( \pi \). In other words, \( \pi(i) = \pi(i) - 1 + |\{k \in [\mathcal{A}^-] : \pi(i_k) > \pi(i)\}| \) for every item \( i \in \mathcal{A}^- \).

In “Appendix D.1”, we show that the resulting permutation satisfies its desired properties, as formally stated below.

**Lemma 18** The permutation \( \bar{\pi} \) satisfies properties (P₁)–(P₃).

**The recursive construction.** We are now ready to explain how recursive applications of the fixing procedure allow us to conclude the proof of Lemma 17. At a high level, we bisect the cluster indices \( [M] \), such that in each step the indices being considered are split into their lower half \( \mathcal{M}^- \) and upper half \( \mathcal{M}^+ \), with respect to which the fixing procedure \( \text{FixCrossing}(\cdot, \mathcal{M}^-, \mathcal{M}^+) \) will be applied. The resulting permutation will then be divided into left and right parts, which are recursively bisected along the same lines.

To present the specifics of this bisection as simply as possible, we assume without loss of generality that the number of clusters \( M \) is a power of 2; otherwise, empty clusters can be appended to the sequence \( \mathcal{C}_1, \ldots, \mathcal{C}_M \). At the upper level of the recursion, we bisect the entire collection of cluster indices \( [M] \) into \( \mathcal{M}_{\lfloor \frac{M}{2} \rfloor} = \{1, \ldots, \lfloor \frac{M}{2} \rfloor \} \) and \( \mathcal{M}_{\lceil \frac{M}{2} \rceil} = \{\lceil \frac{M}{2} \rceil + 1, \ldots, M \} \). Designating the optimal permutation by \( \pi_{\lfloor \frac{M}{2} \rfloor} = \pi^* \), we employ our fixing procedure with \( \text{FixCrossing}(\pi_{\lfloor \frac{M}{2} \rfloor}, \mathcal{M}_{\lfloor \frac{M}{2} \rfloor}, \mathcal{M}_{\lceil \frac{M}{2} \rceil}) \), to obtain the permutation \( \bar{\pi}_{\lfloor \frac{M}{2} \rfloor} \). Now, we break the latter into its left and right part, \( \pi_{\lfloor \frac{M}{2} \rfloor} \) and \( \pi_{\lfloor \frac{M}{2} \rfloor + 1, M} \), such that the left permutation \( \pi_{\lfloor \frac{M}{2} \rfloor} \) is the prefix of \( \bar{\pi}_{\lfloor \frac{M}{2} \rfloor} \) ending at the last item in \( \mathcal{C}_{\lfloor \frac{M}{2} \rfloor} \cup \mathcal{X}_{\mathcal{M}_{\lfloor \frac{M}{2} \rfloor}} \cup \mathcal{X}_{\mathcal{M}_{\lceil \frac{M}{2} \rceil}}(\pi_{\lfloor \frac{M}{2} \rfloor}) \), whereas the right permutation \( \pi_{\lfloor \frac{M}{2} \rfloor + 1, M} \) is comprised of the remaining suffix.

In the second level of the recursion, for the left permutation \( \pi_{\lfloor \frac{M}{2} \rfloor} \), we bisect \( \mathcal{M}_{\lfloor \frac{M}{2} \rfloor} \) into \( \mathcal{M}_{\lfloor \frac{M}{4} \rfloor} = \{1, \ldots, \lfloor \frac{M}{4} \rfloor \} \) and \( \mathcal{M}_{\lfloor \frac{M}{2} \rfloor + \lfloor \frac{M}{4} \rfloor} = \{\lfloor \frac{M}{4} \rfloor + 1, \ldots, \lfloor \frac{M}{2} \rfloor \} \), followed by applying \( \text{FixCrossing}(\pi_{\lfloor \frac{M}{4} \rfloor}, \mathcal{M}_{\lfloor \frac{M}{4} \rfloor}, \mathcal{M}_{\lfloor \frac{M}{2} \rfloor + \lfloor \frac{M}{4} \rfloor}) \). Similarly, for the right permutation \( \pi_{\lfloor \frac{M}{2} \rfloor + 1, M} \), its corresponding set of cluster indices \( \mathcal{M}_{\lfloor \frac{M}{2} \rfloor + 1, M} \) is bisected into \( \mathcal{M}_{\lfloor \frac{M}{4} \rfloor + 1, \lfloor \frac{M}{2} \rfloor} = \{\lfloor \frac{M}{4} \rfloor + 1, \ldots, \lfloor \frac{3M}{4} \rfloor \} \) and \( \mathcal{M}_{\lfloor 3M/4 \rfloor + 1, M} = \{\lfloor \frac{3M}{4} \rfloor + 1, \ldots, M \} \), in which case we apply \( \text{FixCrossing}(\pi_{\lfloor \frac{3M}{4} \rfloor + 1, M}, \mathcal{M}_{\lfloor \frac{3M}{4} \rfloor + 1, \lfloor \frac{3M}{4} \rfloor}, \mathcal{M}_{\lfloor 3M/4 \rfloor + 1, M}) \). This recursive procedure continues up until the resulting sets of cluster indices are singletons. At that point in time, our final permutation \( \pi_{\text{sparse}} \) is obtained by concatenating \( \pi_{\lfloor 1 \rfloor}, \pi_{\lfloor 2, 2 \rfloor}, \ldots, \pi_{\lfloor M, M \rfloor} \).

**Analysis.** For ease of presentation, we make use of \( \Omega \) to denote the set of pairs of cluster index sets with respect to which \( \text{FixCrossing}(\cdot, \cdot, \cdot) \) is employed throughout our recursive construction, meaning that

\[ \bar{\pi}(i) = \pi(i) \text{ for every item } i \text{ with } \pi(i) \leq \pi(i_{\mathcal{M}^-,\mathcal{M}^+}) - 1. \]
\[
\Omega = \left\{ \left( \mathcal{M}_{[1, \frac{M}{2^1}]}, \mathcal{M}_{[\frac{M}{2^1} + 1, M]} \right), \text{ [level 1]} \right. \\
\left( \mathcal{M}_{[1, \frac{M}{2^2}]}, \mathcal{M}_{[\frac{M}{2^2} + 1, \frac{M}{2^1}]}, \mathcal{M}_{[\frac{M}{2^1} + 1, \frac{M}{2^2}]}, \mathcal{M}_{[\frac{M}{2^2} + 1, M]} \right), \text{ [level 2]} \\
\ldots \\
\left( \mathcal{M}_{[1,1]}, \mathcal{M}_{[2,2]}, \ldots, \mathcal{M}_{[M-1,M-1]}, \mathcal{M}_{[M,M]} \right) \right\}. \text{ [level } \log_2 M \]
\]

With this notation, we show in the next two claims that the permutation \(\pi_{\text{sparse}}\) indeed satisfies the sparse crossing and near-optimal profit properties of Lemma 17.

**Lemma 19** \(\text{cross}_m(\pi_{\text{sparse}}) \leq \frac{\log_2 M}{\epsilon} \), for every \(m \in [M]\).

**Proof** By construction of \(\pi_{\text{sparse}}\), every item belonging to one of the clusters \(C_{m+1}, \ldots, C_M\) that appears in this permutation before the last item in cluster \(C_m\) necessarily resides in \(\mathcal{X}_{M^-, M^+}(\pi_{\text{[min } M^-, \text{max } M^+]})\), for some pair \((M^-, M^+) \in \Omega\) with \(m \in M^-\). To verify this claim, consider such a crossing item \(i\), say belonging to cluster \(C_{m^+}\). By the way our recursive construction of \(\Omega\) is defined, there exists a unique pair of cluster index sets \((M^-, M^+) \in \Omega\) for which \(m \in M^-\) and \(m^+ \in M^+\); we argue that \(i \in \mathcal{X}_{M^-, M^+}(\pi_{\text{[min } M^-, \text{max } M^+]})\). Indeed, in the next recursion level, the left permutation \(\pi_{\text{[min } M^-, \text{max } M^+]}\) is the prefix of \(\pi_{\text{[min } M^-, \text{max } M^+]}\) ending with the last item in \(C_{m^+} \cup \mathcal{X}_{M^-, M^+}(\pi_{\text{[min } M^-, \text{max } M^+]})\). Furthermore, by construction, all items in the right permutation \(\pi_{\text{[min } M^+, \text{max } M^+]}\) will appear in \(\pi_{\text{sparse}}\) after all items in the left permutation \(\pi_{\text{[min } M^-, \text{max } M^+]}\). Therefore, since \(i \in C_{m^+}\) with \(m^+ \in M^+\) and since this item appears in \(\pi_{\text{sparse}}\) before the last item in cluster \(C_m\), we know that \(i\) appears as part of the left permutation \(\pi_{\text{[min } M^-, \text{max } M^+]}\), implying that \(i \in \mathcal{X}_{M^-, M^+}(\pi_{\text{[min } M^-, \text{max } M^+]})\).

As any such item \(i \in \mathcal{X}_{M^-, M^+}(\pi_{\text{[min } M^-, \text{max } M^+]})\) contributes at most once toward \(\text{cross}_{M^-, M^+}(\pi_{\text{[min } M^-, \text{max } M^+]})\), we have

\[
\text{cross}_m(\pi_{\text{sparse}}) \leq \sum_{(M^-, M^+) \in \Omega: m \in M^-} \text{cross}_{M^-, M^+}(\pi_{\text{[min } M^-, \text{max } M^+]}) \\
\leq \frac{1}{\epsilon} \cdot \left| \left\{ (M^-, M^+) \in \Omega : m \in M^- \right\} \right| \\
\leq \frac{\log_2 M}{\epsilon}.
\]

Here, the second inequality holds since \(\text{cross}_{M^-, M^+}(\pi_{\text{[min } M^-, \text{max } M^+]}) \leq \frac{1}{\epsilon} \) by property (P1) of the fixing procedure. The third inequality is obtained by observing that, as the definition of \(\Omega\) shows, all sets appearing in a single level of the recursion form a partition of \([M]\), implying that \(m \in M^-\) for at most one pair \((M^-, M^+)\) in that level. As there are \(\log_2 M\) levels overall, it follows that \(\left| \{ (M^-, M^+) \in \Omega : m \in M^- \} \right| \leq \log_2 M\).

**Lemma 20** \(\Psi(\pi_{\text{sparse}}) \geq (1 - \epsilon) \cdot \Psi(\pi^*)\).
Proof To prove the desired claim, we begin by relating the profits $\Psi(\pi_{\text{sparse}})$ and $\Psi(\pi^*)$, with the corresponding proof in “Appendix D.2”. The main idea is that our fixing procedure eliminates the least profitable item out of $X_{\mathcal{M}^-, \mathcal{M}^+}(\pi_{\text{min}, \mathcal{M}^-, \text{max}, \mathcal{M}^+})$, without increasing the completion time of any other item. Hence, every execution of this procedure loses a profit of at most $\epsilon \cdot \varphi_{\pi^*}(\pi_{\text{min}, \mathcal{M}^-, \text{max}, \mathcal{M}^+}(\pi_{\text{min}, \mathcal{M}^-, \text{max}, \mathcal{M}^+})).$

Claim 2 $\Psi(\pi_{\text{sparse}}) \geq \Psi(\pi^*) - \epsilon \cdot \sum_{(\mathcal{M}^-, \mathcal{M}^+) \in \Omega} \varphi_{\pi^*}(\pi_{\text{min}, \mathcal{M}^-, \text{max}, \mathcal{M}^+}(\pi_{\text{min}, \mathcal{M}^-, \text{max}, \mathcal{M}^+})).$

The next claim establishes the disjointness of $X_{\mathcal{M}^-_1, \mathcal{M}^+_1}(\pi_{\text{min}, \mathcal{M}^-_1, \text{max}, \mathcal{M}^+_1})$ and $X_{\mathcal{M}^-_2, \mathcal{M}^+_2}(\pi_{\text{min}, \mathcal{M}^-_2, \text{max}, \mathcal{M}^+_2})$, for all distinct pairs $(\mathcal{M}^-_1, \mathcal{M}^+_1)$ and $(\mathcal{M}^-_2, \mathcal{M}^+_2)$ in $\Omega$. Informally, this property holds since, once an item appears in some $X_{\mathcal{M}^-_1, \mathcal{M}^+_1}(\pi_{\text{min}, \mathcal{M}^-_1, \text{max}, \mathcal{M}^+_1})$, it will not appear in any item set belonging to future levels of the recursion, due to the specific way we are pulling back items in the fixing procedure. For ease of presentation, the formal proof is deferred to “Appendix D.3”.

Claim 3 For any two distinct pairs $(\mathcal{M}^-_1, \mathcal{M}^+_1)$ and $(\mathcal{M}^-_2, \mathcal{M}^+_2)$ in $\Omega$, the item sets $X_{\mathcal{M}^-_1, \mathcal{M}^+_1}(\pi_{\text{min}, \mathcal{M}^-_1, \text{max}, \mathcal{M}^+_1})$ and $X_{\mathcal{M}^-_2, \mathcal{M}^+_2}(\pi_{\text{min}, \mathcal{M}^-_2, \text{max}, \mathcal{M}^+_2})$ are disjoint.

Consequently, the profit attained by the permutation $\pi_{\text{sparse}}$ can be bounded by noting that

$$\Psi(\pi_{\text{sparse}}) \geq \Psi(\pi^*) - \epsilon \cdot \sum_{(\mathcal{M}^-, \mathcal{M}^+) \in \Omega} \varphi_{\pi^*}(\pi_{\text{min}, \mathcal{M}^-, \text{max}, \mathcal{M}^+}(\pi_{\text{min}, \mathcal{M}^-, \text{max}, \mathcal{M}^+})))$$

$$\geq \Psi(\pi^*) - \epsilon \cdot \sum_{i \in N} \varphi_{\pi^*}(i)$$

$$= (1 - \epsilon) \cdot \Psi(\pi^*),$$

where the first inequality is precisely Claim 2, and the second inequality follows from Claim 3. 

4.4 The external dynamic program

Given the sparse-crossing property of the near-optimal permutation $\pi_{\text{sparse}}$, whose existence has been established in Lemma 17, we turn our attention to formally presenting a dynamic programming approach for computing a permutation with a profit of at least $(1 - 2\epsilon) \cdot \Psi(\pi_{\text{sparse}})$.

States. Building on the intuition provided in Sect. 4.1, we remind the reader that each state $(m, \psi_m, Q_{\geq m})$ of our dynamic program consists of the following parameters:

- The index of the current cluster $m$, taking values in $[M]_0$.
- The total profit $\psi_m$ collected thus far. Initially, $\psi_m$ will be treated as a continuous parameter, taking values in $[0, np_{\text{max}}]$, where $p_{\text{max}}$ is the maximum profit attainable by any single item, i.e., $p_{\text{max}} = \max\{p_{it} : i \in [n], t \in [T]\}$, and $w_i \leq W_t$.
- The set of items $Q_{\geq m}$ belonging to clusters $C_{m+1}, \ldots, C_M$ that will be crossing into lower-index clusters. Motivated by the sparse-crossing property established in Lemma 17, we only consider sets $Q_{\geq m}$ of cardinality at most $\frac{\log_2 M}{\epsilon}$. 

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Value function. For a subset of items $S \subseteq \mathcal{N}$ and a permutation $\pi : S \rightarrow [|S|]$, we say that the pair $(S, \pi)$ is thin when $\text{cross}_m(\pi) \leq \frac{\log_2 M}{\epsilon}$ for all $m \in [M]$. Given this definition, the value function $F(m, \psi_m, Q_{>m})$ represents the minimum makespan $w(S)$ that can be attained over all thin pairs $(S, \pi)$ that satisfy the following conditions:

1. **Allowed items:** The set $S$ consists of items that belong to one of the clusters $C_1, \ldots, C_m$ or to $Q_{>m}$. In other words, $S \subseteq C_{[1,m]} \cup Q_{>m}$, where $C_{[1,m]} = \biguplus_{\mu \in [1,m]} C_\mu$ by convention.

2. **Required crossing items:** The set $S$ contains all items in $Q_{>m}$, meaning that $Q_{>m} \subseteq S$.

3. **Total profit:** $\Psi(\pi) \geq \psi_m$.

Recycling some of the notation introduced in Sect. 2.3.2, we use Thin$(m, \psi_m, Q_{>m})$ to denote the collection of thin pairs that meet conditions 1–3 above. When the latter set is empty, we define $F(m, \psi_m, Q_{>m}) = \infty$. With these definitions, Lemma 17 proves the existence of a thin pair $(S, \pi) \in \text{Thin}(M, \Psi(\pi_{\text{sparse}}), \emptyset)$ with $F(M, \Psi(\pi_{\text{sparse}}), \emptyset) \leq W_T$. It is worth pointing out that, for the item set $Q_{>m}$ to be well-defined, we should ensure that $\pi_{\text{sparse}}$ is a continuous parameter, we will eventually explain how to discretize $\pi_{\text{sparse}}$ with respect to $\pi_{\text{sparse}}$ can be eliminated, leaving us with a permutation that still satisfies Lemma 17. Therefore, had we been able to compute the maximal value $\psi^*$ for which $F(M, \psi^*, \emptyset) \leq W_T$, its corresponding permutation would have guaranteed a profit of at least $\psi^* \geq \Psi(\pi_{\text{sparse}}) \geq (1-\epsilon) \cdot \Psi(\pi^*)$. Once again, since $\psi_m$ is a continuous parameter, we will eventually explain how to discretize $\psi_m$ to take polynomially-many values, incurring only an $\epsilon$-loss in profit.

Optimal substructure. In what follows, we identify the optimal substructure that allows us to compute the value function $F$ by means of dynamic programming. To this end, suppose that $(S, \pi)$ is a thin pair that minimizes $w(S)$ over Thin$(m, \psi_m, Q_{>m})$. We will argue that by eliminating from $(S, \pi)$ a carefully-selected suffix of the permutation $\pi$ consisting of items in clusters $C_m, \ldots, C_M$, one obtains a thin pair that attains $F(m-1, \psi_{m-1}, Q_{>m-1})$ for an appropriately defined state $(m-1, \psi_{m-1}, Q_{>m-1})$. We proceed by first defining the latter state, for which a suitable alteration of $(S, \pi)$ will be shown to be optimal:

- **Crossing set:** $Q_{>m-1}$ is defined as the set of items in $C_m \cup Q_{>m}$ that appear before the last item in $C_1, \ldots, C_{m-1}$ with respect to the permutation $\pi$. Namely,

$$Q_{>m-1} = \left\{ i \in S \cap (C_m \cup Q_{>m}) : \pi(i) < \max_{j \in S \cap C_{[1,m-1]}} \pi(j) \right\}. \quad (8)$$

- **Profit requirement:** $\psi_{m-1} = [\psi_m - \sum_{i \in S \cap (C_{[1,m-1]} \cup Q_{>m-1})} \psi(\pi(i))]^+$. It is worth pointing out that, for this state to be well-defined, we should ensure that $Q_{>m-1}$ indeed consists of at most $\frac{|\log_2 M|}{\epsilon}$ items. To understand why this property is satisfied, note that since every item in $Q_{>m-1}$ appears in the permutation $\pi$ before the last item in $S \cap C_{[1,m-1]}$, we have $|Q_{>m-1}| \leq \max_{\mu \in [m-1]} \text{cross}_\mu(\pi) \leq \frac{|\log_2 M|}{\epsilon}$, where the last inequality holds since $(S, \pi)$ is thin.

Now, let us define the pair $(\hat{S}, \hat{\pi})$, in which $\hat{S} = S \cap (C_{[1,m-1]} \cup Q_{>m-1})$, meaning that $\hat{S}$ is the restriction of $S$ to items belonging to either one of the clusters $C_1, \ldots, C_{m-1}$.
or to $Q_{>m-1}$. It is not difficult to verify that any item in $\hat{S}$ appears in $\pi$ before any item in $S \setminus \hat{S}$, as any item in $S \cap C_{[m,M]}$ appears before an item in $C_{[1,m-1]}$ is necessarily a member of $Q_{>m-1}$. Therefore, the items in $\hat{S}$ form a prefix of $\pi$, whereas those in $S \setminus \hat{S}$ form the remaining suffix. Given this observation, we define the permutation $\hat{\pi} : \hat{S} \to [\lvert \hat{S} \rvert]$ as the former prefix, or equivalently, as the restriction of $\pi$ to the items in $\hat{S}$.

In Lemma 21 below, we show that the pair $(\hat{S}, \hat{\pi})$ indeed resides within $\text{Thin}(m-1, \psi_{m-1}, Q_{>m-1})$. Subsequently, we prove in Lemma 22 that this pair is in fact makespan-optimal over the latter set. At a high level, this claim will be established by showing that, for any pair $(\tilde{S}, \tilde{\pi}) \in \text{Thin}(m-1, \psi_{m-1}, Q_{>m-1})$, the item sets $\hat{S}$ and $S \setminus \hat{S}$ are disjoint. Therefore, had there been such a pair with $w(\tilde{S}) < w(\hat{S})$, it could be extended to a pair in $\text{Thin}(m, \psi_{m}, Q_{>m})$ via an appropriate addition of $S \setminus \hat{S}$, contradicting the optimality of $(S, \pi)$. To avoid deviating from the overall flow of this section, the proofs of the next two lemmas are presented in Appendices D.4 and D.5, respectively.

**Lemma 21** $(\hat{S}, \hat{\pi}) \in \text{Thin}(m-1, \psi_{m-1}, Q_{>m-1})$.

**Lemma 22** $w(\hat{S}) = F(m-1, \psi_{m-1}, Q_{>m-1})$.

**Recursive equations.** Given the optimal substructure characterization discussed above, we proceed by explaining how to express $F(m, \psi_{m}, Q_{>m})$ in recursive form. In essence, had we known what the preceding state $(m-1, \psi_{m-1}, Q_{>m-1})$ is, the remaining question would have been that of identifying the lightest set of “extra” items $E$ to be appended, along with their internal permutation $\pi_{E} : E \to (\lvert E \rvert)$, under a marginal profit constraint. Formally, to capture the agreement between crossing items, we say that state $(m-1, \psi_{m-1}, Q_{>m-1})$ is conceivable for state $(m, \psi_{m}, Q_{>m})$ when $Q_{>m-1} \setminus C_{m} \subseteq Q_{>m}$. In the opposite direction, $(m, \psi_{m}, Q_{>m})$ is reachable from $(m-1, \psi_{m-1}, Q_{>m-1})$ when there exist an item set $E$ and permutation $\pi_{E} : E \to (\lvert E \rvert)$ that simultaneously satisfy the following constraints:

1. **Extra items:** The collection of extra items can be written as $E = E_{m} \cup (Q_{>m} \setminus Q_{>m-1})$. Here, items in $E_{m}$ are to be picked out of cluster $C_{m}$, with the exclusion of those appearing in $Q_{>m-1}$, meaning that we have the constraint $E_{m} \subseteq C_{m} \setminus Q_{>m-1}$. Concurrently, each and every item in $Q_{>m} \setminus Q_{>m-1}$ should be picked.

2. **Marginal profit:** $\sum_{i \in E} \varphi_{\pi_{E}}(i) \geq \psi_{m} - \psi_{m-1}$, where the term $\varphi_{\pi_{E}}(i)$ denotes the profit of item $i$ with respect to the permutation $\pi_{E}$, when its completion time is increased by $F(m-1, \psi_{m-1}, Q_{>m-1})$. This constraint guarantees that, by appending $\pi_{E}$ to the permutation that achieves $F(m-1, \psi_{m-1}, Q_{>m-1})$, we obtain a total profit of at least $\psi_{m}$.

Letting $\text{Extra}[F(m, \psi_{m}, Q_{>m})]$ denote the collection of item sets and permutations that satisfy these constraints, we mention in passing that this set may be empty. Moreover, it will be utilized only for purposes of analysis, and in particular, we will not assume that $\text{Extra}[F(m-1, \psi_{m-1}, Q_{>m-1})]$ can be efficiently constructed. Nevertheless, the function value $F(m, \psi_{m}, Q_{>m})$ can still be expressed by minimizing...
F(m − 1, ψ_{m−1}, Q_{>m−1}) + w(\mathcal{E}) over all conceivable states (m − 1, ψ_{m−1}, Q_{>m−1}) and over all item sets and permutations (\mathcal{E}, π_\mathcal{E}) ∈ \text{Extra}[(m, ψ_{m}, Q_{>m})] \setminus \text{Best}(m, ψ_{m}, Q_{>m}). For convenience, when F(m, ψ_{m}, Q_{>m}) ≤ W_T, we use \text{Best}(m, ψ_{m}, Q_{>m}) to denote an arbitrary state (m − 1, ψ_{m−1}, Q_{>m−1}) chosen out of those for which the minimum value F(m, ψ_{m}, Q_{>m}) is attained. As mentioned earlier, we wish to compute the maximal value \psi^* that satisfies F(M, ψ^*, \emptyset) ≤ W_T, as its corresponding permutation guarantees a profit of at least (1 − \epsilon) \cdot \Psi(\pi^*).

Approximate recursion. That said, due to having a lower bound on the marginal profit, even when \text{Best}(m, ψ_{m}, Q_{>m}) is known, the recursive formulation above is expected to identify an item set and permutation (\mathcal{E}, π_\mathcal{E}) ∈ \text{Extra}[(m, ψ_{m}, Q_{>m})] \setminus \text{Best}(m, ψ_{m}, Q_{>m}) for which w(\mathcal{E}) is minimized. This setting can be viewed as an “inverse” generalized incremental knapsack problem, where the objective is to minimize makespan rather than to maximize profit. To deal with this obstacle, we employ our QPTAS for bounded weight ratio instances (see Sect. 3) in order to approximately solve these recursive equations.

Specifically, for \Delta ≥ 0, we say that constraint 2 is (\epsilon, \Delta)-satisfied when

\sum_{i∈\mathcal{E}} \varphi_{\pi_\mathcal{E}}^\pm \Delta(i) ≥ (1 − \epsilon) \cdot (\psi_m − ψ_{m−1}),

where \varphi_{\pi_\mathcal{E}}^\pm \Delta(i) is the profit of item i with respect to the permutation π_\mathcal{E}, when its completion time is increased by \Delta. As such, the standard sense of satisfying this constraint can be recovered by picking \epsilon = 0 and \Delta = F(m − 1, ψ_{m−1}, Q_{>m−1}). With this definition, we say that state (m, ψ_{m}, Q_{>m}) is (\epsilon, \Delta)-reachable from state (m − 1, ψ_{m−1}, Q_{>m−1}) when there exist an item set \mathcal{E} and permutation π_\mathcal{E} : \mathcal{E} → [\mathcal{I}] that satisfy constraint 1 and (\epsilon, \Delta)-satisfy constraint 2; as before, \text{Extra}_\epsilon,\Delta[(m−1,ψ_{m−1},Q_{>m−1})] will stand for the collection of such item sets and permutations. In what follows, we devise an auxiliary procedure for approximately solving the recursive equations, as summarized in the next claim; for readability purposes, the proof is deferred to “Appendix D.6”.

Lemma 23 Suppose that (m, ψ_{m}, Q_{>m}) and (m − 1, ψ_{m−1}, Q_{>m−1}) are two given states, such that F(m, ψ_{m}, Q_{>m}) ≤ W_T and (m − 1, ψ_{m−1}, Q_{>m−1}) = \text{Best}(m, ψ_{m}, Q_{>m}). Given a parameter \Delta ≤ F(m − 1, ψ_{m−1}, Q_{>m−1}) = \text{Best}(m, ψ_{m}, Q_{>m}), we can identify an item set \hat{\mathcal{E}} and permutation \hat{π}_\mathcal{E} : \hat{\mathcal{E}} → [\hat{\mathcal{I}}] for which:

1. (\hat{\mathcal{E}}, \hat{π}_\mathcal{E}) ∈ \text{Extra}_\epsilon,\Delta[(m−1,ψ_{m−1},Q_{>m−1})].
2. w(\hat{\mathcal{E}}) ≤ F(m, ψ_{m}, Q_{>m}) − F(m − 1, ψ_{m−1}, Q_{>m−1}).

The running time of our algorithm is O((nT)^{O(1/\epsilon^2)} \cdot (\log n + \log M) \cdot |\mathcal{I}|^{O(1)}), regardless of whether the assumptions above hold or not.

With this procedure in-hand, we define an approximate value function \hat{F}, whose state space is identical to that of F. However, rather than attempting to solve an inverse generalized incremental knapsack problem, the recursive equations through which \hat{F} is defined will tackle the latter problem in an approximate way via our auxiliary procedure. To formalize this approach, the function value \hat{F}(m, ψ_{m}, Q_{>m}) is evaluated as follows:

- **Terminal states (m = 0):** Here, we simply define \hat{F}(0, ψ_{0}, Q_{>0}) = F(0, ψ_{0}, Q_{>0}). While \hat{F}-values are unknown in general, \hat{F}(0, ψ_{0}, Q_{>0}) evaluates to either
Lemma 25

As it is nearly identical to that of Lemma 4.

Lemma 24

Approximation algorithms for generalized incremental knapsack

(through this dynamic program, showing that it indeed matches that of the permutation achieved this objective, recall that Lemma 17 proves the existence of a thin pair. Nevertheless, as we show in the next lemma, whose proof is provided in “Appendix D.8”, any profit requirement which is attainable by the original dynamic program can be attained up to factor 1 − \(\epsilon\) by our approximate program \(\tilde{F}\). The precise relationship we establish between these functions can be formally stated as follows.

Lemma 24

Let \((m, \psi_m, Q_{\geq m})\) be a state for which \(F(m, \psi_m, Q_{\geq m}) \leq W_T\). Then, \(\tilde{F}(m, \psi_m, Q_{\geq m}) \leq F(m, \psi_m, Q_{\geq m})\), where the makespan \(\tilde{F}(m, \psi_m, Q_{\geq m})\) is attained by an item set \(\tilde{S}_m\) and a permutation \(\tilde{\pi}_{\tilde{S}_m}: \tilde{S}_m \to [|\tilde{S}_m|]\) for which:

- Allowed and required items: \(\tilde{S}_m \subseteq \mathcal{C}_{[1,m]} \cup Q_{\geq m}\) and \(Q_{\geq m} \subseteq \tilde{S}_m\).
- Profit: \(\Psi(\tilde{\pi}_{\tilde{S}_m}) \geq (1 - \epsilon) \cdot \psi_m\).

As previously mentioned, the primary intent of this section is to compute a permutation with a profit of at least \((1 - 2\epsilon) \cdot \Psi(\pi_{\text{sparse}})\). To argue that we have nearly achieved this objective, recall that Lemma 17 proves the existence of a thin pair \((S, \pi)\) \(\in\) \(\text{Thin}(M, \Psi(\pi_{\text{sparse}}), \emptyset)\) with \(F(M, \Psi(\pi_{\text{sparse}}), \emptyset) \leq W_T\). Therefore, as an immediate consequence of Lemma 24, we infer that \(\tilde{F}(M, \Psi(\pi_{\text{sparse}}), \emptyset) \leq W_T\), which is attained by a permutation \(\pi\) with a profit of \(\Psi(\pi) \geq (1 - \epsilon) \cdot \Psi(\pi_{\text{sparse}})\).

The discrete program \(\tilde{F}\). That said, the above-mentioned existence proof still does not correspond to a constructive algorithm, due to the continuity of the profit requirement parameter \(\psi_m\). To discretize this parameter, similarly to Sect. 2.3.3, we restrict \(\psi_m\) to a finite set of values, \(\mathcal{D}_\psi = \{d : \frac{\epsilon \cdot \rho_{\text{max}}}{2\pi} : d \in [\frac{2\pi^2}{\epsilon}]\}\). In turn, we use \(\tilde{F}(m, \psi_m, Q_{\geq m})\) to denote the resulting dynamic program over the discretized set of states, whose recursive equations are identical to those of \(\tilde{F}\), except for instantiating Lemma 23 with \(\Delta = \tilde{F}(m - 1, \psi_{m-1}, Q_{\geq m-1})\).

We conclude our analysis by lower-bounding the best-possible profit achievable through this dynamic program, showing that it indeed matches that of the permutation \(\pi_{\text{sparse}}\) up to \(\epsilon\)-related terms. To avoid redundancy, we omit the corresponding proof, as it is nearly identical to that of Lemma 4.

Lemma 25

There exists a value \(\tilde{\psi} \in \mathcal{D}_\psi\) such that \(\tilde{\psi} \geq (1 - \epsilon) \cdot \Psi(\pi_{\text{sparse}})\) and such that \(\tilde{F}(M, \tilde{\psi}, \emptyset) \leq W_T\). This makespan is attained by an item set \(\tilde{S}\) and a permutation \(\tilde{\pi}_{\tilde{S}}\) whose profit is \(\Psi(\tilde{\pi}_{\tilde{S}}) \geq (1 - \epsilon) \cdot \tilde{\psi} \geq (1 - 2\epsilon) \cdot \Psi(\pi_{\text{sparse}})\).
Running time. We first observe that the function $\tilde{F}(m, \psi_m, Q_{>m})$ is being evaluated over $O(n^{O(1/\epsilon \log M)} \cdot |I|^{O(1)})$ possible states. To verify this claim, note that there are $O(M) = O(|I|)$ choices for the cluster index $m$, and that the discretized profit parameter $\psi_m$ takes values in $D_\psi$, with $|D_\psi| = O(\epsilon^2 / \epsilon)$. In addition, the set of crossing items $Q_{>m}$ is of cardinality at most $\left\lceil \log_2 M \right\rceil \epsilon$, implying that there are only $O(n^{O(1/\epsilon \log M)})$ subsets to consider for this parameter. Now, evaluating $\tilde{F}(m, \psi_m, Q_{>m})$ for a given state depends on its type:

- **Terminal states** ($m = 0$): As previously explained, such states are handled by enumerating over all permutations of $Q_{>0}$ in time $O(|I|^{O((1/\epsilon \log |I|)^{O(1)})})$.
- **General states** ($m \in [M]$): Here, each state $(m-1, \psi_{m-1}, Q_{>m-1})$ would involve a single application of our auxiliary procedure, running in $O((nT)^{O(1/\epsilon \log n + \log M)})$. According to Lemma 23. As argued above, there are only $O(n^{O(1/\epsilon \log M)}) \cdot |I|^{O(1)}$ states of the form $(m-1, \psi_{m-1}, Q_{>m-1})$ to be considered.

Overall, we incur a running time of $O(|I|^{O((1/\epsilon \log |I|)^{O(1)})})$, as stated in Theorem 7.

5 Concluding remarks

In this paper, we introduced and studied what is, to the best of our knowledge, the most general incremental packing problem examined in the literature thus far. Our algorithmic results imply that the generalized incremental knapsack problem can be approximated within a constant factor in polynomial time. Concurrently, we have shown that this setting admits a quasi-PTAS, meaning that it is unlikely to be APX-hard, given known complexity results. We conclude with a number of open questions.

**Improved constant-factor approximation?** A natural direction for future research would be to investigate whether the $(\frac{1}{2} - \epsilon)$-approximation we obtained in Sect. 2 can be improved. One possible approach to achieve such improvements lies in proposing an efficient way to combine $k$-light and $k$-heavy items within a single solution, rather than constructing separate solutions that compete against each of these contributions by themselves. Our efforts along these lines have not been successful to date, perhaps since significantly different methods appear to be required in each case.

**Obstacles toward obtaining a PTAS?** A particularly challenging direction to pursue is whether the quasi-PTAS we devised in Sect. 4 can be enhanced to admit a truly polynomial running time. In fact, we are unaware of any inapproximability result that rules out the existence of a PTAS for generalized incremental knapsack. To this end, the first bottleneck resides in the guessing step for bounded weight ratio instances in Sect. 3.2. The second bottleneck emerges from our $O(\log M / \epsilon)$ bound on the number of items crossing each cluster in a near-optimal permutation, formally established in Lemma 17. Bypassing these two sources for quasi-polynomial running time would result in a polynomial-time implementation of our overall approach.

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the relation to unsplittable flow on a path with bag constraints (see “Appendix A”), through which an $O\left(\frac{\log\log n}{\log n}\right)$-approximation can be attained.

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**Declaration**

**Conflict of interest** The authors declare that they have no conflict of interest.

**A Reduction to unsplittable flow on a path with bag constraints**

As a first attempt of reducing the generalized incremental knapsack problem to unsplittable flow on a path (see Sect. 1.2), we could construct a path over the sequence of vertices $0, \ldots, T$, where each edge $(t, t+1)$ has a capacity of $W_t$. Then, for each item-time pair $(i, t)$, one could create a corresponding task to capture the decision of inserting item $i$ at time period $t$; this task would extend across the subpath $(t, \ldots, T)$, with a profit of $p_{it}$ and a demand of $w_{it}$. However, as each item $i$ may be inserted into the knapsack only once, we are lacking an additional constraint, stating that at most one of the tasks corresponding to $\{(i, t)\}_{t \in [T]}$ can be picked, which makes the resulting problem fundamentally different.

To capture this additional constraint, consider a generalization of the unsplittable flow on a path problem with “bag constraints”, first studied by Chakaravarthy et al. [13]. In this setting, tasks are further partitioned into a set of bags $B_1, \ldots, B_k$, and we are allowed to pick at most one task from each bag. This way, we can model the generalized incremental knapsack problem by having a separate bag $B_i$ for each item $i \in [n]$, which contains all tasks corresponding to $\{(i, t)\}_{t \in [T]}$, thereby capturing the extra feature that each item may be inserted only once. For this generalization, Grandoni et al. [25] proposed an $O\left(\frac{\log\log n}{\log n}\right)$-approximation through an LP-based rounding approach, which is currently the best known performance guarantee.

**B Additional Proofs from Sect. 2**

**B.1 Proof of Claim 1**

We first show that $(\tilde{S}^+, \tilde{\pi}^+)$ is indeed a bulky pair. For this purpose, since $(\tilde{S}, \tilde{\pi})$ is bulky, it suffices to explain why each item $i \in Q$ is necessarily $k_i$-heavy, where $k_i$ is the unique index for which $C_{\tilde{\pi}^+}(i) \in I_{k_i}$. This claim follows by noting that, for such items, the way we construct $(\tilde{S}^+, \tilde{\pi}^+)$ leads to a completion time of

$$C_{\tilde{\pi}^+}(i) = w(\tilde{S}) + \sum_{j \in Q: \pi(j) \leq \pi(i)} w_j$$
Recalling that \( Q = \{ i \in S : C_\pi(i) \in \mathcal{I}_k \} \), we have just shown that \( k_i \leq k \), and since item \( i \) is \( k \)-heavy due to the bulkiness of \((S, \pi)\), it is \( k_i \)-heavy as well.

We proceed by showing that \((\hat{S}^+, \hat{\pi}^+)\) satisfies conditions 1–3:

1. **Top index:** \( \text{top}(\hat{S}^+, \hat{\pi}^+) \leq k \). To verify this property, note that when \( Q = \emptyset \), we clearly have \( w(\hat{S}^+) = w(\hat{S}) < w(\hat{S}) = w(S) \), and therefore, \( \text{top}(\hat{S}^+, \hat{\pi}^+) \leq \text{top}(S, \pi) \leq k \). In the opposite case, where \( Q \neq \emptyset \), the makespans of both \( \hat{S}^+ \) and \( S \) are attained by the respective completion times of precisely the same item in \( Q \). However, by inequality (9), we have \( C_{\hat{\pi}^+}(i) \leq C_\pi(i) \) for every \( i \in Q \), and it follows that \( \text{top}(\hat{S}^+, \hat{\pi}^+) \leq \text{top}(S, \pi) \leq k \).

2. **Total profit:** \( \Psi(\hat{\pi}^+) \geq \psi_k \). Along the same lines, since \( C_{\hat{\pi}^+}(i) \leq C_\pi(i) \) for every \( i \in Q \), it follows that \( \varphi_{\hat{\pi}^+}(i) \geq \varphi_\pi(i) \) for such items. Thus,

\[
\Psi(\hat{\pi}^+) = \sum_{i \in \hat{S}} \varphi_{\hat{\pi}^+}(i) + \sum_{i \in Q} \varphi_{\hat{\pi}^+}(i)
\]

\[
= \sum_{i \in \hat{S}} \varphi_\pi(i) + \sum_{i \in Q} \varphi_{\hat{\pi}^+}(i)
\]

\[
\geq \psi_{k-1} + \sum_{i \in Q} \varphi_{\hat{\pi}^+}(i)
\]

\[
= \left[ \psi_k - \sum_{i \in Q} \varphi_\pi(i) \right]^+ + \sum_{i \in Q} \varphi_{\hat{\pi}^+}(i)
\]

\[
\geq \psi_k .
\]

Here, the second equality holds since the permutations \( \hat{\pi}^+ \) and \( \hat{\pi} \) are identical when restricted to items in \( \hat{S} \). The first inequality follows by recalling that \((\hat{S}, \hat{\pi}) \in \text{Bulky}(k-1, \psi_{k-1}, Q_{k-1})\), meaning in particular that \( \sum_{i \in \hat{S}} \varphi_\pi(i) = \Psi(\hat{\pi}) \geq \psi_{k-1} \).

3. **Core:** \( \text{core}(\hat{S}^+) = Q_k \). One can easily verify that, for any pair of disjoint sets of items, \( S_1 \) and \( S_2 \), we have \( \text{core}(S_1 \cup S_2) = \text{core}(\text{core}(S_1) \cup \text{core}(S_2)) \). Therefore,

\[
\text{core}(\hat{S}^+) = \text{core}(\hat{S} \cup Q)
\]

\[
= \text{core}(\text{core}(\hat{S}) \cup \text{core}(Q))
\]

\[
= \text{core}(\text{core}(S \setminus Q) \cup \text{core}(Q))
\]

\[
= \text{core}(S)
\]

\[
= Q_k ,
\]

where the second equality follows by noting that \( \hat{S} \) and \( Q \) are disjoint, and similarly, the fourth equality holds since \( S \setminus Q \) and \( Q \) are clearly disjoint.
B.2 Proof of Lemma 4

Let us consider the sequence of states traversed by the dynamic program \( F \), as it arrives to the optimal state \((K, \psi_K^*, Q_K^*)\); the latter is “optimal” in the sense that \( \psi^*_K = \psi^* \) and \( F(K, \psi_K^*, Q_K^*) < \infty \). This sequence, along with the specific parameters and the bulky pair corresponding to each state will be designated by:

\[
\begin{align*}
(0, \psi_0^*, Q_0^*) & \xrightarrow{Q_1^*, \pi_{Q_1^*}} (1, \psi_1^*, Q_1^*) & \xrightarrow{Q_2^*, \pi_{Q_2^*}} \ldots & \xrightarrow{Q_k^*, \pi_{Q_k^*}} (k, \psi_k^*, Q_k^*) \\
(S_0^*, \pi_{S_0^*}) & \xrightarrow{Q_1^*, \pi_{Q_1^*}} (S_1^*, \pi_{S_1^*}) & \xrightarrow{Q_2^*, \pi_{Q_2^*}} \ldots & \xrightarrow{Q_k^*, \pi_{Q_k^*}} (S_k^*, \pi_{S_k^*})
\end{align*}
\]

To better understand this illustration, we note that for every \( k \in [K] \), the collection of items \( Q_k^* \) and their internal permutation \( \pi_{Q_k^*} \) are precisely those by which the dynamic program \( F \) transitions from state \((k - 1, \psi_{k-1}^*, Q_{k-1}^*)\) to state \((k, \psi_k^*, Q_k^*)\). Consequently, the resulting item set is \( S_k^* = S_{k-1}^* \cup \psi_{Q_k^*} \), whereas the resulting permutation \( \pi_{S_k^*} \) is obtained by appending \( \pi_{Q_k^*} \) to \( \pi_{S_{k-1}^*} \). In addition, for the starting state, we have \( \psi_0^* = 0 \) and \( Q_0^* = \emptyset \).

To prove the desired claim, we argue that one feasible sequence of states that can be traversed by the approximate program \( \tilde{F} \) is obtained when each profit parameter \( \psi_k \) is substituted by \( \tilde{\psi}_k = \left[ \psi_k^* - \min\{k, |S_k^*|\} \cdot \frac{p_{\max}}{n} \right] \cdot D_\psi \). Here, the operator \([\cdot]\) \( D_\psi \) rounds its argument up to the nearest value in \( D_\psi \). In other words, as shown in Claim 4 below, we prove that

\[
\begin{align*}
(0, \tilde{\psi}_0^*, Q_0^*) & \xrightarrow{Q_1^*, \pi_{Q_1^*}} (1, \tilde{\psi}_1^*, Q_1^*) & \xrightarrow{Q_2^*, \pi_{Q_2^*}} \ldots & \xrightarrow{Q_k^*, \pi_{Q_k^*}} (k, \tilde{\psi}_k^*, Q_k^*) \\
(S_0^*, \pi_{S_0^*}) & \xrightarrow{Q_1^*, \pi_{Q_1^*}} (S_1^*, \pi_{S_1^*}) & \xrightarrow{Q_2^*, \pi_{Q_2^*}} \ldots & \xrightarrow{Q_k^*, \pi_{Q_k^*}} (S_k^*, \pi_{S_k^*})
\end{align*}
\]

forms a feasible sequence of states, action parameters, and bulky pairs for \( \tilde{F} \). That is, we have \((S_k^*, \pi_{S_k^*}) \in \text{Bulky}(k, \tilde{\psi}_k^*, Q_k^*)\), for every \( k \in [K]_0 \). In light of this result, we conclude in particular that \( \tilde{F}(K, \tilde{\psi}_K^*, Q_K^*) < \infty \) with

\[
\tilde{\psi}_K = \left[ \psi_K^* - \min\{K, |S_K^*|\} \cdot \frac{p_{\max}}{n} \right] \cdot D_\psi \\
\geq \psi^* - \epsilon p_{\max} \\
\geq (1 - \epsilon) \cdot \psi^*.
\]

Here, the first inequality holds since \( \psi_k^* = \psi^* \) and \( |S_k^*| \leq n \). To understand the second inequality, note that for every item \( i \in [n] \), the pair that consists of introducing this item and nothing more is necessarily bulky. Indeed, as a result, the completion time of item \( i \) would fall within the interval \( \mathcal{I}_{ki} \), where \( k_i \) is the unique integer for which \( (1 + \epsilon)^{k_i-1} < w_i \leq (1 + \epsilon)^{k_i} \). However, since \( w_i > (1 + \epsilon)^{k_i-1} \geq \epsilon^2 \cdot (1 + \epsilon)^{k_i} \) for \( \epsilon \leq \frac{1}{2} \), it follows that item \( i \) is \( k \)-heavy, implying in turn that the pair in question is bulky.
Now, noting that this pair guarantees a profit of \( \max\{p_{it} : t \in [T] \text{ and } w_i \leq W_t\} \), any such expression provides a lower bound on \( \psi^* \), meaning that \( \psi^* \geq \max\{p_{it} : i \in [n], t \in [T], \text{ and } w_i \leq W_t\} = p_{\max} \).

**Claim 4** \((S^*_k, \pi_{S^*_k}) \in \overline{\text{Bulky}}(k, \tilde{\psi}_k, Q^*_k)\), for every \( k \in [K]_0 \).

**Proof** We first note that the parameter \( \tilde{\psi}_k \) is indeed well-defined for all \( k \in [K]_0 \), since \( \tilde{\psi}_k \leq \lceil \psi^*_k \rceil_{D_{\psi}} \leq np_{\max} = \max D_{\psi} \). Given this observation, we proceed to prove the claim by induction on \( k \).

In the base case of \( k = 0 \), the claim trivially holds since \( \tilde{\psi}_0 = 0 \), \( Q^*_0 = \emptyset \), \( S^*_0 = \emptyset \), and \( \pi_{S^*_0} \) is the empty permutation. In the general case of \( k \geq 1 \), to argue that \((S^*_k, \pi_{S^*_k}) \in \overline{B}(k, \tilde{\psi}_k, Q^*_k)\), we consider two scenarios, depending on whether \( Q^*_k \) is empty or not:

- **Case 1**: \( Q^*_k = \emptyset \). We first observe that, since \( S^*_k = S^*_{k-1} \cup Q^*_k \), we have \( S^*_k = S^*_{k-1} \) by the case hypothesis, implying in turn that \( \psi^*_k = \psi^*_{k-1} \) and \( Q^*_k = Q^*_{k-1} \). Consequently,

\[
\tilde{\psi}_k = \psi^*_k - \min\{k, |S^*_k|\} \cdot \frac{p_{\max}}{n} D_{\psi} \\
\leq \left[ \psi^*_{k-1} - \min\{k-1, |S^*_{k-1}|\} \cdot \frac{p_{\max}}{n} \right] D_{\psi} \\
= \tilde{\psi}_{k-1}.
\]

and it follows that \( \overline{\text{Bulky}}(k, \tilde{\psi}_k, Q^*_k) \supseteq \overline{\text{Bulky}}(k, \tilde{\psi}_{k-1}, Q^*_{k-1}) \supseteq \overline{\text{Bulky}}(k-1, \tilde{\psi}_{k-1}, Q^*_{k-1}) \), where the first inclusion holds since \( \tilde{\psi}_k \leq \tilde{\psi}_{k-1} \) and \( Q^*_k = Q^*_{k-1} \). Thus, \((S^*_k, \pi_{S^*_k}) = (S^*_{k-1}, \pi_{S^*_{k-1}}) \in \overline{\text{Bulky}}(k-1, \tilde{\psi}_{k-1}, Q^*_{k-1}) \) \( \subseteq \) \( \text{Bulky}(k, \tilde{\psi}_k, Q^*_k) \), where the middle transition is precisely our induction hypothesis.

- **Case 2**: \( Q^*_k \neq \emptyset \). In this case, \( |S^*_k| = |S^*_{k-1}| + |Q^*_k| \geq |S^*_{k-1}| + 1 \), as \( S^*_k \) is the disjoint union of \( S^*_{k-1} \) and \( Q^*_k \). By the inductive hypothesis, \((S^*_k, \pi_{S^*_k}) \in \overline{\text{Bulky}}(k-1, \tilde{\psi}_{k-1}, Q^*_{k-1}) \), meaning that for the purpose of proving \((S^*_k, \pi_{S^*_k}) \in \overline{\text{Bulky}}(k, \tilde{\psi}_k, Q^*_k)\), it suffices to show that \( \tilde{\psi}_{k-1} + \sum_{i \in Q^*_k} \varphi_{\pi_{S^*_k}}(i) \geq \tilde{\psi}_k \). We establish the latter inequality by noting that

\[
\tilde{\psi}_{k-1} + \sum_{i \in Q^*_k} \varphi_{\pi_{S^*_k}}(i) = \tilde{\psi}_{k-1} + \psi^*_k - \psi^*_{k-1} \\
\geq \left( \psi^*_{k-1} - \min\{k-1, |S^*_{k-1}|\} \cdot \frac{p_{\max}}{n} \right) + \psi^*_k - \psi^*_{k-1} \\
\geq \left[ \psi^*_{k} - \min\{k, |S^*_k|\} \cdot \frac{p_{\max}}{n} \right] D_{\psi} \\
= \tilde{\psi}_k,
\]
where the first equality holds since $\psi_k^* = \psi_{k-1}^* + \sum_{i \in Q_k^*} \varphi_{\pi_k^*}(i)$, by the optimality of $\psi_k^*$. 

\[ \Box \]

### B.3 Proof of Lemma 5

In order to construct the required permutation, for every $k \in [K-1]$, let $\pi_k$ be an arbitrary permutation of the items that were assigned by $x$ to bucket $B_k$, i.e., $\{i \in [n]: x_{ik} = 1\}$. In addition, let $\pi_-$ be an arbitrary permutation of the remaining items, i.e., those that were not assigned to any bucket. The permutation $\pi_x$ is now defined by concatenating these permutations in order of increasing index, with $\pi_-$ appended at the end, namely, $\pi_x = (\pi_1, \ldots, \pi_{K-1}, \pi_-)$. It is easy to verify that this construction can be implemented in $O(nK)$ time.

To obtain a lower bound of $\sum_{i \in [n]} \sum_{k \in [K-1]} i_{ik} x_{ik}$ on the profit of this permutation, $\Psi(\pi_x) = \sum_{i \in [n]} \varphi_{\pi_x}(i)$, note that since each item is assigned to at most one bucket, it suffices to show that for every $i \in [n]$ and $k \in [K-1]$ with $x_{ik} = 1$, we necessarily have $\varphi_{\pi_x}(i) \geq q_{ik}$. For this purpose, we observe that

\[
\varphi_{\pi_x}(i) = \max \left\{ p_{t,i} : t \in [T+1] \text{ and } W_t \geq C_{\pi_x}(i) \right\} \\
= \max \left\{ p_{t,i} : t \in [T+1] \text{ and } W_t \geq (1+\epsilon)^k \right\} \\
= q_{ik},
\]

where the inequality above holds since $C_{\pi_x}(i) \leq (1+\epsilon)^k$. Indeed, this bound on the completion time of item $i$ can be derived by observing that every item $j$ that appears before $i$ in the permutation $\pi_x$ (i.e., $\pi_x(j) < \pi_x(i)$) was assigned by the solution $x$ to one of the buckets $B_1, \ldots, B_k$, and therefore,

\[
C_{\pi_x}(i) = \sum_{j \in [n]: \pi_x(j) \leq \pi_x(i)} w_j \\
\leq \sum_{\kappa \in [k]} \sum_{j \in L_{\kappa+1}} w_j x_{j\kappa} \\
\leq \sum_{\kappa \in [k]} \text{capacity}(B_{\kappa}) \\
= \sum_{\kappa \in [k]} \left( (1+\epsilon)^{\kappa} - (1+\epsilon)^{\kappa-1} \right) \\
\leq (1+\epsilon)^k,
\]

where the second inequality follows from the second constraint of (IP).
C Additional Proofs from Sect. 3

C.1 Proof of Lemma 8

Clearly, $\mathcal{R} \cup \mathcal{G}$ is a chain for $\mathcal{I}$, as each of $\mathcal{R}$ and $\mathcal{G}$ is such a chain by itself. To verify the feasibility of $\mathcal{R} \cup \mathcal{G}$, note that for any time period $t \in [T]$, since $\mathcal{R}$ is feasible for $\mathcal{I} - \mathcal{G}$ we have

$$w(R_t) \leq W_t - G_t$$

$$= \min_{t \leq \tau \leq T} (W_\tau - w(G_\tau))$$

$$\leq W_t - w(G_t).$$

By recalling that $G_1 \subseteq \cdots \subseteq G_T$ and $R_1 \subseteq \cdots \subseteq R_T \subseteq \mathcal{N} \setminus \mathcal{G} = \mathcal{N} \setminus G_T$, it follows in particular that $G_t$ and $R_t$ are disjoint, implying in turn that $w(R_t \cup G_t) = w(R_t) + w(G_t) \leq W_t$ as required.

Now, to account for the profit of $\mathcal{R} \cup \mathcal{G}$, we conclude that

$$\Phi(\mathcal{R} \cup \mathcal{G}) = \sum_{t \in [T]} \sum_{i \in (R_t \cup G_t) \setminus (R_{t-1} \cup G_{t-1})} p_{it}$$

$$= \sum_{t \in [T]} \left( \sum_{i \in R_t \setminus R_{t-1}} p_{it} + \sum_{i \in G_t \setminus G_{t-1}} p_{it} \right)$$

$$= \Phi(\mathcal{R}) + \Phi(\mathcal{G}).$$

Here, the second equality holds again due to the observation above, since having both $G_1 \subseteq \cdots \subseteq G_T$ and $R_1 \subseteq \cdots \subseteq R_T \subseteq \mathcal{N} \setminus \mathcal{G}$ means that $(R_t \cup G_t) \setminus (R_{t-1} \cup G_{t-1})$ can be written as the disjoint union of $R_t \setminus R_{t-1}$ and $G_t \setminus G_{t-1}$.

C.2 Proof of Lemma 9

For convenience, let us denote the chain in question by $\mathcal{R} = S|_{\mathcal{N} \setminus \mathcal{G}}$. By observing that $R_T = (S_T \cap (\mathcal{N} \setminus \mathcal{G})) = (S_T \cap G_T) \subseteq \mathcal{N} \setminus G_T$, it follows that $\mathcal{R}$ is also a chain for $\mathcal{I} - \mathcal{G}$. We proceed by arguing that $\mathcal{R}$ is in fact feasible for the latter instance. To this end, note that for every $t \leq \tau$,

$$w(R_t) \leq w(R_\tau)$$

$$= w(S_\tau) - w(G_\tau)$$

$$\leq W_\tau - w(G_\tau),$$

where the middle equality follows by recalling that $S_\tau$ is the disjoint union of $G_t$ and $R_t$, and the last inequality is implied by the feasibility of $S$ for $\mathcal{I}$. As a result, $w(R_t) \leq \min_{t \leq \tau \leq T} (W_\tau - w(G_\tau)) = W_t^{-G}$, which proves that $\mathcal{R}$ is a feasible chain for $\mathcal{I} - \mathcal{G}$.
We now turn our attention to showing that $\Phi(\mathcal{R}) = \Phi(S) - \Phi(\mathcal{G})$. Again, based on the observation that $S_t$ is the disjoint union of $G_t$ and $R_t$ for every $t \in [T]$, we conclude that

$$\Phi(\mathcal{R}) + \Phi(\mathcal{G}) = \sum_{t \in [T]} \left( \sum_{i \in R_t \setminus R_{t-1}} p_{it} + \sum_{i \in G_t \setminus G_{t-1}} p_{it} \right)$$

$$= \sum_{t \in [T]} \sum_{i \in (R_t \cup G_t) \setminus (R_{t-1} \cup G_{t-1})} p_{it}$$

$$= \sum_{t \in [T]} \sum_{i \in S_t \setminus S_{t-1}} p_{it}$$

$$= \Phi(S).$$

Finally, suppose that $S$ is optimal for $\mathcal{I}$, but on the other hand, $\mathcal{R}$ is not optimal for $\mathcal{I}^{-\mathcal{G}}$, meaning that there exists a feasible chain $\mathcal{R}'$ for $\mathcal{I}^{-\mathcal{G}}$ with profit $\Phi(\mathcal{R}') > \Phi(\mathcal{R})$. Then, by Lemma 8, we infer that $\mathcal{R}' \cup \mathcal{G}$ is a feasible chain for $\mathcal{I}$, with profit $\Phi(\mathcal{R}' \cup \mathcal{G}) = \Phi(\mathcal{G}) + \Phi(\mathcal{R}') > \Phi(\mathcal{G}) + \Phi(\mathcal{R}) = \Phi(S)$, contradicting the optimality of $S$.

### C.3 Proof of Lemma 11

We say that an interval $\mathcal{I}_k$ is non-empty with respect to the permutation $\pi_{S_*}$ if it contains the completion time of at least one item. Note that, since the latter completion time is within $[w_{\min}, n w_{\max}]$ and we assume that $w_{\min} = 3$ (see Sect. 2.2), the interval $\mathcal{I}_0 = [0, 1]$ is clearly empty. Furthermore, any non-empty interval $\mathcal{I}_k = ((1 + \epsilon)^k - 1, (1 + \epsilon)^k]$ necessarily has $\lceil \log_{1+\epsilon} (w_{\min}) \rceil \leq k \leq \lceil \log_{1+\epsilon} (n w_{\max}) \rceil$. Therefore, the number of non-empty intervals with respect to $\pi_{S_*}$ is at most $\lceil \log_{1+\epsilon} (n w_{\max}) \rceil - \lceil \log_{1+\epsilon} (w_{\min}) \rceil + 1 \leq 2 \cdot \lceil \log_{1+\epsilon} (n \rho) \rceil$. Now, any such interval $\mathcal{I}_k$ is of length $(1 + \epsilon)^k - (1 + \epsilon)^{k-1}$, meaning that the number of $k$-heavy items with a completion time in this interval is at most $\frac{(1 + \epsilon)^k - (1 + \epsilon)^{k-1}}{\epsilon^2} \leq \frac{1}{\epsilon}$, as every $k$-heavy item has a weight of at least $\epsilon^2 \cdot (1 + \epsilon)^k$. All in all, we have just shown that $|S_*^{\text{heavy}}| \leq 2 \cdot \frac{\lceil \log_{1+\epsilon} (n \rho) \rceil}{\epsilon^2} \leq \frac{3 \log(n \rho)}{\epsilon^2}$. 

### C.4 Proof of Lemma 12

For every item $i \in \mathcal{N}$, let $t_i$ be its insertion time with respect to the optimal chain $S_*$. By convention, for non-inserted items (i.e., those in $\mathcal{N} \setminus S_*^T$), we say that their “insertion time” is $T + 1$, with a profit of $p_{i, T+1} = 0$. As explained during the proof of Lemma 1, our construction of the permutation $\pi_{S_*}$ guarantees that $\varphi_{\pi_{S_*}}(i) \geq p_{i, t_i}$ for every item $i \in \mathcal{N}$. While this inequality was established for any chain-to-permutation mapping, one can easily notice that, due to the optimality of $S*$, we actually have $\varphi_{\pi_{S_*}}(i) = p_{i, t_i}$ for every $i \in \mathcal{N}$. Otherwise, there would have been at least one item with $\varphi_{\pi_{S_*}}(i) > p_{i, t_i}$, implying that $\Psi(\pi_{S_*}) > \Phi(S^*)$. By Lemma 1, the permutation
\( \pi_{S^*} \) can then be mapped to a feasible chain \( S \) with \( \Phi(S) = \Psi(\pi_{S^*}) > \Phi(S^*) \), contradicting the optimality of \( S^* \). Thus, \( \Phi(H^*) = \sum_{i \in G_{\text{heavy}}} p_i t_i = \sum_{i \in G_{\text{heavy}}} \Phi_{\pi_{S^*}}(i) = \Psi_{\text{heavy}}(\pi_{S^*}) \).

### C.5 Proof of Lemma 15

We prove the lower bound \( \alpha_r \geq \frac{r}{r+1} - r\delta \) by induction on \( r \). For \( r = 0 \), we have \( \alpha_0 = 0 \) and the claim clearly holds. Now, for \( r \geq 1 \),

\[
\alpha_r = \frac{1 - \delta}{2 - \alpha_{r-1}} \geq \frac{1 - \delta}{2 - \left(\frac{r-1}{r} - (r - 1)\delta\right)} = \frac{r(1 - \delta)}{r + 1 + r(r - 1)\delta} \geq \frac{r + 1 + (r - 1)\delta}{(r + 1)(1 + (r - 1)\delta)} = \frac{r}{r + 1} \cdot \left(1 - \frac{r\delta}{1 + (r - 1)\delta}\right) \geq \frac{r}{r + 1} - r\delta.
\]

### D Additional Proofs from Sect. 4

#### D.1 Proof of Lemma 18

**Sparse \( (\mathcal{M}^-, \mathcal{M}^+) \)-crossing.** On the one hand, our construction guarantees that the last item in \( C_{\mathcal{M}^-} \) appears in position \( \pi(\mathcal{M}^-, \mathcal{M}^+) - 1 + |\tilde{A}^-| \) of the permutation \( \tilde{\pi} \). On the other hand, every item in \( C_{\mathcal{M}^+} \) that appears before this position necessarily belongs to \( \mathcal{X}_{\mathcal{M}^-, \mathcal{M}^+}(\pi) \). It follows that there are at most \( |\mathcal{X}_{\mathcal{M}^-, \mathcal{M}^+}(\pi)| = \frac{1}{\epsilon} \) such items, and therefore, \( \text{cross}_{\mathcal{M}^-, \mathcal{M}^+}(\tilde{\pi}) \leq \frac{1}{\epsilon} \).

**Completion times.** We establish this property by considering three cases, depending on whether the item in question appears before \( i_{\mathcal{M}^-, \mathcal{M}^+} \), belongs to \( \tilde{A}^- \), or belongs to \( \tilde{A}^- \).

- **Before \( i_{\mathcal{M}^-, \mathcal{M}^+} \):** For every item \( i \in \mathcal{N} \) with \( \pi(i) \leq \pi(i_{\mathcal{M}^-, \mathcal{M}^+}) - 1 \) we clearly have \( C_{\tilde{\pi}}(i) = C_{\pi}(i) \), since the permutations \( \tilde{\pi} \) and \( \pi \) are identical up to position \( \pi(i_{\mathcal{M}^-, \mathcal{M}^+}) - 1 \).
- **Items in \( \tilde{A}^- \):** For every item \( i \in \tilde{A}^- \), we have \( C_{\tilde{\pi}}(i) \leq C_{\pi}(i) \), since the collection of items appearing before \( i \) in \( \tilde{\pi} \) is a subset of those appearing before \( i \) in \( \pi \).
- **Items in \( \tilde{A}^- \):** For every item \( i \in \tilde{A}^- \), the important observation is that the collection of items appearing before \( i \) in \( \tilde{\pi} \) consists of: (1) The same items appearing before \( i \) in \( \pi \), except for the eliminated item \( i_{\mathcal{M}^-, \mathcal{M}^+} \); as well as (2) All items in \( \mathcal{A}^- \).
appearing after $i$ in $\pi$. Therefore,

$$C_{\pi}(i) \leq C_{\pi}(i) - w_{i,M^-,M^+} + w(A^-) \leq C_{\pi}(i).$$

To understand the last inequality, recall that $i_{M^-,M^+} \in X_{M^-,M^+}(\pi)$, meaning in particular that this item resides within $C_{M^+}$. Since $I = (N, W)$ is well-spaced, property 2 of such instances implies that $w_{i_{M^-,M^+}}$ is greater than the weight of any item in $C_{M^-}$ by a multiplicative factor of at least $n^{1+\frac{1}{\epsilon}} \geq n$, as $\max M^- < \min M^+$. Consequently, since all items in $A^-$ reside within $C_{M^-}$, we indeed have $w_{i_{M^-,M^+}} \geq n \cdot \max_j w_j \geq w(A^-)$.

**Difference.** This property is straightforward, by construction of $\bar{\pi}$.

**D.2 Proof of Claim 2**

For simplicity of notation, let $D = \{i_{M^-,M^+} : (M^-, M^+) \in \Omega, X_{M^-,M^+} \neq \emptyset\}$ be the collection of items that were removed throughout all recursive calls to our fixing procedure. Then, the profit of the resulting permutation $\pi_{\text{sparse}}$ can be lower-bounded by observing that

$$\Psi(\pi_{\text{sparse}}) = \sum_{i \in N \setminus D} \varphi_{\pi_{\text{sparse}}}(i) \geq \sum_{i \in N \setminus D} \varphi_{\pi^*}(i) = \Psi(\pi^*) - \sum_{i \in D} \varphi_{\pi^*}(i) \geq \Psi(\pi^*) - \epsilon \cdot \sum_{(M^-,M^+) \in \Omega} \varphi_{\pi^*}(X_{M^-,M^+}(\pi_{\min M^-,\max M^+})).$$

Here, the first inequality holds since, for any remaining item $i \in N \setminus D$, it is not difficult to verify (by induction on the recursion level) that property (P2) of the fixing procedure implies $C_{\pi_{\text{sparse}}}(i) \leq C_{\pi^*}(i)$, and we therefore have $\varphi_{\pi_{\text{sparse}}}(i) \geq \varphi_{\pi^*}(i)$. The second inequality is obtained by recalling that any item $i_{M^-,M^+} \in D$ was chosen as the least profitable item in $X_{M^-,M^+}(\pi_{\min M^-,\max M^+})$ with respect to $\pi^*$, thus

$$\varphi_{\pi^*}(i_{M^-,M^+}) \leq \frac{\varphi_{\pi^*}(X_{M^-,M^+}(\pi_{\min M^-,\max M^+}))}{|X_{M^-,M^+}(\pi_{\min M^-,\max M^+})|} = \epsilon \cdot \varphi_{\pi^*}(X_{M^-,M^+}(\pi_{\min M^-,\max M^+})).$$

**D.3 Proof of Claim 3**

By definition, $X_{M_1^-,M_1^+}(\pi_{\min M_1^-,\max M_1^+})$ and $X_{M_2^-,M_2^+}(\pi_{\min M_2^-,\max M_2^+})$ contain only items in $M_1^+$-indexed clusters and $M_2^+$-indexed clusters, respectively.
Thus, when $\mathcal{M}_1^+$ and $\mathcal{M}_2^+$ are disjoint, $\chi_{\mathcal{M}_1^- \mathcal{M}_1^+}(\pi_{\lfloor \min \mathcal{M}_1^- \max \mathcal{M}_1^+ \rfloor})$ and $\chi_{\mathcal{M}_2^- \mathcal{M}_2^+}(\pi_{\lfloor \min \mathcal{M}_2^- \max \mathcal{M}_2^+ \rfloor})$ must be disjoint as well. Hence, it remains to consider the scenario where $\mathcal{M}_1^+$ and $\mathcal{M}_2^+$ are not disjoint. In this case, the permutations $\pi_{\lfloor \min \mathcal{M}_1^- \max \mathcal{M}_1^+ \rfloor}$ and $\pi_{\lfloor \min \mathcal{M}_2^- \max \mathcal{M}_2^+ \rfloor}$ must have been created at different levels of the recursive construction; we assume without loss of generality that $\pi_{\lfloor \min \mathcal{M}_1^- \max \mathcal{M}_1^+ \rfloor}$ was created at a lower-index level. Therefore, $\mathcal{M}_2^+ \subseteq \mathcal{M}_1^+$, and $\chi_{\mathcal{M}_2^- \mathcal{M}_2^+}(\pi_{\lfloor \min \mathcal{M}_2^- \max \mathcal{M}_2^+ \rfloor})$ consists of only items in the right permutation, $\pi_{\lfloor \min \mathcal{M}_1^- \max \mathcal{M}_1^+ \rfloor}$. On the other hand, by construction, any item in $\chi_{\mathcal{M}_2^- \mathcal{M}_2^+}(\pi_{\lfloor \min \mathcal{M}_2^- \max \mathcal{M}_2^+ \rfloor})$ ends up in the left permutation, $\pi_{\lfloor \min \mathcal{M}_1^- \max \mathcal{M}_1^+ \rfloor}$, implying the disjointness of $\chi_{\mathcal{M}_2^- \mathcal{M}_2^+}(\pi_{\lfloor \min \mathcal{M}_2^- \max \mathcal{M}_2^+ \rfloor})$ and $\chi_{\mathcal{M}_1^- \mathcal{M}_1^+}(\pi_{\lfloor \min \mathcal{M}_1^- \max \mathcal{M}_1^+ \rfloor})$.

D.4 Proof of Lemma 21

We first observe that the pair $(\hat{S}, \hat{\pi})$ is indeed thin. To this end, note that since the permutation $\hat{\pi}$ is a prefix of $\pi$, for every $m \in [M]$ we clearly have $\text{cross}_m(\hat{\pi}) \leq \text{cross}_m(\pi) \leq \frac{[\log_2 M]}{\epsilon}$, where the last inequality holds since $(\hat{S}, \hat{\pi})$ is thin. Next, we show that $(\hat{S}, \hat{\pi})$ satisfies conditions 1–3:

1. **Allowed items**: By construction, $\hat{S} = S \cap (C_{[1, m-1]} \cup Q_{>m-1})$, implying that $\hat{S}$ forms a subset of $C_{[1, m-1]} \cup Q_{>m-1}$.

2. **Required crossing items**: An additional implication of our definition of $\hat{S}$ is that $Q_{>m-1} \subseteq \hat{S}$, since $Q_{>m-1} \subseteq S$ by (8).

3. **Total profit**: To obtain a lower bound on the profit of $\hat{\pi}$, we observe that

$$
\psi(\hat{\pi}) = \sum_{i \in \hat{S}} \varphi_{\hat{\pi}}(i)
= \sum_{i \in \hat{S}} \varphi_{\pi}(i)
= \psi(\pi) - \sum_{i \in S \setminus (C_{[1, m-1]} \cup Q_{>m-1})} \varphi_{\pi}(i)
\geq \left[ \psi_m - \sum_{i \in S \setminus (C_{[1, m-1]} \cup Q_{>m-1})} \varphi_{\pi}(i) \right]^{+}
= \psi_{m-1}.
$$

Here, the second equality holds since $\hat{\pi}$ is a prefix of $\pi$, as previously mentioned. The third equality follows by noting that $S \setminus \hat{S} = S \setminus (C_{[1, m-1]} \cup Q_{>m-1})$. The inequality above is obtained by observing that its left-hand-side is non-negative, and by recalling that $(S, \pi) \in \text{Thin}(m, \psi_m, Q_{>m})$, implying that $\psi(\pi) \geq \psi_m$. The last equality is precisely the definition of $\psi_{m-1}$.
D.5 Proof of Lemma 22

By way of contradiction, suppose there exists a pair \((\tilde{S}, \tilde{\pi})\) in \(\text{Thin}(m-1, \psi_{m-1}, Q_{>m-1})\) whose makespan is smaller than that of \(\hat{S}\), namely, \(w(\hat{S}) < w(\tilde{S})\). We begin by noticing that the item sets \(S\setminus \hat{S}\) and \(\tilde{S}\) are disjoint, since \(S\setminus \hat{S} \subseteq (C_m \cup Q_{>m}) \setminus Q_{>m-1} \subseteq C_{[m,M]} \setminus Q_{>m-1}\) whereas \(\tilde{S} \subseteq C_{[1,m-1]} \cup Q_{>m-1}\), as \((\hat{S}, \hat{\pi}) \in \text{Thin}(m-1, \psi_{m-1}, Q_{>m-1})\). Taking advantage of this observation, we define a new pair \((\tilde{S}^+, \tilde{\pi}^+)\) as follows:

- The underlying set of items is given by \(\tilde{S}^+ = \tilde{S} \cup (S\setminus \hat{S})\).
- The permutation \(\tilde{\pi}^+ : \tilde{S}^+ \to [|\tilde{S}^+|]\) is constructed by appending the items in \(S\setminus \hat{S}\) to \(\tilde{\pi}\), following their internal order in \(\pi\).

The next claim shows that the resulting pair is a feasible solution to exactly the same subproblem for which \((S, \pi)\) is optimal.

Claim 5 \((\tilde{S}^+, \tilde{\pi}^+) \in \text{Thin}(m, \psi_m, Q_{>m})\).

**Proof** First, we show that \((\tilde{S}^+, \tilde{\pi}^+)\) is a thin pair. To this end, for every \(\mu \in [M]\) with \(C_\mu \cap \tilde{S}^+ \neq \emptyset\), let \(i_\mu \in C_\mu\) be the item that appears last in \(\tilde{\pi}^+\) out of this cluster, i.e., \(i_\mu = \text{argmax}_{i \in \tilde{S}^+ \cap C_\mu} \tilde{\pi}^+(i)\). We proceed by considering two cases:

- **Item \(i_\mu\) appears in \(\tilde{\pi}\):** By construction, \(\tilde{\pi}\) is a prefix of \(\tilde{\pi}^+\), and therefore \(\text{cross}_\mu(\tilde{\pi}^+) = \text{cross}_\mu(\tilde{\pi}) \leq \frac{\log_2 M}{\epsilon}\), where the last inequality holds since \((\hat{S}, \hat{\pi})\) is a thin pair.

- **Item \(i_\mu\) does not appear in \(\tilde{\pi}\):** In this case, \(i_\mu \in S\setminus \hat{S} \subseteq C_{[m,M]} \setminus Q_{>m-1}\), implying that \(\mu \geq m\). Thus, all items in clusters \(C_{\mu+1}, \ldots, C_M\) that appear before \(i_\mu\) in the permutation \(\tilde{\pi}^+\) necessarily belong to \(Q_{>m}\), and we conclude that \(\text{cross}_\mu(\tilde{\pi}^+) \leq |Q_{>m}| \leq \frac{\log_2 M}{\epsilon}\).

Next, we show that \((\tilde{S}^+, \tilde{\pi}^+)\) satisfies conditions 1–3:

1. **Allowed items:** First note that \(\tilde{S} \subseteq C_{[1,m-1]} \cup Q_{>m-1} \subseteq C_{[1,m]} \cup Q_{>m}\), where the first inclusion holds since \((\tilde{S}, \tilde{\pi}) \in \text{Thin}(m-1, \psi_{m-1}, Q_{>m-1})\) and the second follows by definition of \(Q_{>m-1}\) in (8). In addition, \(S \subseteq C_{[1,m]} \cup Q_{>m}\), since \((S, \pi) \in \text{Thin}(m, \psi_m, Q_{>m})\). Combining these two observations, we have \(\tilde{S}^+ = \tilde{S} \cup (S\setminus \hat{S}) \subseteq C_{[1,m]} \cup Q_{>m}\) as required.

2. **Required crossing items:** To prove \(Q_{>m} \subseteq \tilde{S}^+\), we observe that

\[
Q_{>m} \subseteq Q_{>m-1} \cup (Q_{>m} \setminus Q_{>m-1}) \\
\subseteq \tilde{S} \cup (S\setminus \hat{S}) \\
= \tilde{S}^+.
\]

To better understand the second inclusion, note that \(Q_{>m-1} \subseteq \tilde{S}\), since \((\hat{S}, \hat{\pi}) \in \text{Thin}(m-1, \psi_{m-1}, Q_{>m-1})\). In addition, \(Q_{>m} \setminus Q_{>m-1} \subseteq S\setminus \hat{S}\), since \(Q_{>m} \subseteq S\) due to having \((S, \pi) \in \text{Thin}(m, \psi_m, Q_{>m})\), and since \((Q_{>m} \setminus Q_{>m-1}) \cap \hat{S} = \emptyset\), due to having \(Q_{>m} \subseteq C_{[m+1,M]}\) and \(S \subseteq C_{[1,m-1]} \cup Q_{>m-1}\), where the latter inclusion holds since \(\hat{S} \in \text{Thin}(m-1, \psi_{m-1}, Q_{>m-1})\).

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3. **Total profit**: By construction, any item $i \in S \setminus \hat{S}$ appears in the permutation $\tilde{\pi}^+$ after all items in $\hat{S}$, and moreover, the internal order between the items in $S \setminus \hat{S}$ is determined according to $\pi$. Hence, we can bound the completion time of any item $i \in S \setminus \hat{S}$ by noting that

$$C_{\tilde{\pi}^+}(i) = w(\tilde{S}) + \sum_{j \in S \setminus \hat{S}: \pi(j) < \pi(i)} w_j$$

$$< w(\hat{S}) + \sum_{j \in S \setminus \hat{S}: \pi(j) < \pi(i)} w_j$$

$$= C_\pi(i),$$

where the inequality above follows from our initial assumption that $w(\tilde{S}) < w(\hat{S})$. Consequently, $\varphi_{\tilde{\pi}^+}(i) \geq \varphi_\pi(i)$ for such items, and we have

$$\Psi(\tilde{\pi}^+) = \Psi(\tilde{\pi}) + \sum_{i \in S \setminus \tilde{S}} \varphi_{\tilde{\pi}^+}(i)$$

$$\geq \psi_{m-1} + \sum_{i \in S \setminus \tilde{S}} \varphi_\pi(i)$$

$$= \left[ \psi_m - \sum_{i \in S \setminus (C_{[1,m-1]} \cup Q_{>m-1})} \varphi_\pi(i) \right] + \sum_{i \in S \setminus \tilde{S}} \varphi_\pi(i)$$

$$\geq \psi_m - \left( \sum_{i \in S \setminus (C_{[1,m-1]} \cup Q_{>m-1})} \varphi_\pi(i) - \sum_{i \in S \setminus \tilde{S}} \varphi_\pi(i) \right)$$

$$= \psi_m.$$  

Here, equality (10) holds since $\pi$ is a prefix of $\tilde{\pi}^+$, with the items in $S \setminus \hat{S}$ forming the remaining suffix. Inequality (11) holds since $(\tilde{S}, \tilde{\pi}) \in \text{Thin}(m-1, \psi_{m-1}, Q_{m-1})$, meaning that $\Psi(\tilde{\pi}) \geq \psi_{m-1}$, and since $\varphi_{\tilde{\pi}^+}(i) \geq \varphi_\pi(i)$ for all $i \in S \setminus \hat{S}$, as shown above. Equality (12) follows from the definition of $\psi_{m-1}$. Equality (13) is obtained by noting that $S \setminus (C_{[1,m-1]} \cup Q_{>m-1}) = S \setminus \hat{S}$.

Consequently, by combining our initial assumption that $w(\tilde{S}) < w(\hat{S})$ along with Claim 5, we have just identified a pair $(\tilde{S}^+, \tilde{\pi}^+) \in \text{Thin}(m, \psi_m, Q_{>m})$ with a makespan of

$$w(\tilde{S}^+) = w(\tilde{S}) + w(S \setminus \tilde{S})$$

$$< w(\hat{S}) + w(S \setminus \hat{S})$$

$$= w(S),$$

contradicting the fact that $(S, \pi)$ minimizes $w(S)$ over the set $\text{Thin}(m, \psi_m, Q_{>m})$. 

\[ \square \]

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D.6 Proof of Lemma 23

Overview. Prior to delving into the nuts-and-bolts of our approach, we provide a high-level overview of its main ideas. For this purpose, to make sure condition 2 of Lemma 23 is satisfied, meaning that the item set \( \hat{\mathcal{E}} \) we compute has a total weight of at most \( F(m, \psi_{m}, \mathcal{Q}_{\geq m}) - F(m - 1, \psi_{m-1}, \mathcal{Q}_{\geq m-1}) \), our algorithm relies on “knowing” the latter difference, which will be justified through binary search. With this limitation, restricting ourselves to the item set \((\mathcal{C}_{m} \cup \mathcal{Q}_{\geq m}) \setminus \mathcal{Q}_{\geq m-1}\), we aim to identify a feasible chain whose associated permutation \((\epsilon, \Delta)\)-satisfies constraint 2. To this end, our algorithm “guesses” the insertion time of every item in \( \mathcal{Q}_{\geq m} \setminus \mathcal{Q}_{\geq m-1} \) by enumerating over all feasible chains \( \mathcal{G} = (G_1, \ldots, G_T) \) whose set of introduced items is \( G_T = \mathcal{Q}_{\geq m} \setminus \mathcal{Q}_{\geq m-1} \). Since there are at most \( \left\lceil \frac{\log_2 M}{\epsilon} \right\rceil \) such items, the number of required guesses is only \( O(TO(\frac{\log M}{\epsilon})) \). For each guess, we construct the residual generalized incremental knapsack instance, as explained in Sect. 3.1, which will be solved to near-optimality via the approximation scheme proposed in Theorem 6.

Algorithm. For ease of presentation, on top of all input ingredients mentioned in Lemma 23, we feed into the upcoming algorithm an additional parameter \( \omega \geq 0 \), whose role will be explained later on. With this parameter, our algorithm operates as follows:

1. We define the generalized incremental knapsack instance \( \hat{\mathcal{G}}^{\omega} = (\hat{\mathcal{N}}, \hat{\mathcal{W}}^{\omega}) \), where:
   - The set of items \( \hat{\mathcal{N}} \) is comprised of those allowed by constraint 1, namely, \( \hat{\mathcal{N}} = (\mathcal{C}_{m} \cup \mathcal{Q}_{\geq m}) \setminus \mathcal{Q}_{\geq m-1} \).
   - Additionally, we reduce the capacity \( W_t \) of each period \( t \in [T] \) by \( \Delta \), while ensuring that the maximum resulting capacity does not exceed \( \omega \), meaning that \( \hat{W}_{t}^{\omega} = \min\{W_{t} - \Delta, \omega\} \).

2. For every feasible chain \( \mathcal{G} = (G_1, \ldots, G_T) \) for the instance \( \hat{\mathcal{G}}^{\omega} \) with \( G_T = \mathcal{Q}_{\geq m} \setminus \mathcal{Q}_{\geq m-1} \), we construct the residual instance \( \hat{\mathcal{G}}^{\omega} \setminus \mathcal{G} = (\hat{\mathcal{N}} \setminus \mathcal{G}, \hat{\mathcal{W}}^{\omega} \setminus \mathcal{G}) \). The approximation scheme we proposed in Sect. 3 is now applied to this instance, thereby obtaining a feasible chain \( \mathcal{R}^{\mathcal{G}} \) whose profit is within factor \( 1 - \epsilon \) of the residual optimum (see Theorem 6). When there are no feasible chains with \( G_T = \mathcal{Q}_{\geq m} \setminus \mathcal{Q}_{\geq m-1} \), we abort and report this finding.

3. Out of all chains \( \mathcal{G} \) considered in step 2, let \( \mathcal{G}^{\omega} \) be the one for which the sum of profits \( \Phi(\mathcal{G}^{\omega}) + \Phi(\mathcal{R}^{\mathcal{G}^{\omega}}) \) is maximized. The item set we return is \( \mathcal{E}_{\omega} = \mathcal{R}_{T}^{\mathcal{G}^{\omega}} \setminus (\mathcal{Q}_{\geq m} \setminus \mathcal{Q}_{\geq m-1}) \), i.e., all items inserted by the chain \( \mathcal{R}^{\mathcal{G}^{\omega}} \) along with those in \( \mathcal{Q}_{\geq m} \setminus \mathcal{Q}_{\geq m-1} \). We define the corresponding permutation \( \pi_{\mathcal{E}_{\omega}} : \mathcal{E}_{\omega} \to |\mathcal{E}_{\omega}| \) as the one constructed by Lemma 1 for the chain \( \mathcal{G}^{\omega} \cup \mathcal{G}^{\omega} \).

The binary search. We assume without loss of generality that all item weights take integer values. This property can easily be enforced by uniform scaling, which produces an equivalent instance whose input length is polynomial in that of the original instance. Now, knowing in advance that the total weight of any item set is an integer within \([0, w(\mathcal{N})]\), we employ our \( \omega \)-parameterized algorithm to conduct a binary search over this interval, with the objective of identifying the smallest integer \( \omega_{\min} \) such that:

\( \omega_{\min} \)
Thus, \( \hat{\pi}_{\omega_{\min}} \) that satisfies \( \sum_{i \in E_{\omega_{\min}}} \varphi_{\pi_{\omega_{\min}}}^+ (i) \geq (1 - \epsilon) \cdot (\psi_m - \psi_{m-1}) \).

In contrast, for \( \omega_{\min} - 1/2 \), the algorithm either aborts at step 2, or returns a permutation \( \pi_{E_{\omega_{\min} - 1/2}} \) satisfying \( \sum_{i \in E_{\omega_{\min} - 1/2}} \varphi_{\pi_{E_{\omega_{\min} - 1/2}}}^+ (i) < (1 - \epsilon) \cdot (\psi_m - \psi_{m-1}) \).

To verify that this search procedure is well-defined, let us examine the endpoints of \( [0, w(N)] \). For \( \omega = 0 \), if we obtain a permutation \( \pi_{E_0} \) that satisfies \( \sum_{i \in E_0} \varphi_{\pi_{E_0}}^+ (i) \geq (1 - \epsilon) \cdot (\psi_m - \psi_{m-1}) \), our immediate conclusion is that \( \omega_{\min} = 0 \). For \( \omega = w(N) \), as shown in Lemma 26 below, we are guaranteed to obtain a permutation \( \pi_{E_{w(N)}} \) that satisfies \( \sum_{i \in E_{w(N)}} \varphi_{\pi_{E_{w(N)}}}^+ (i) \geq (1 - \epsilon) \cdot (\psi_m - \psi_{m-1}) \).

**Running time.** Clearly, the number of binary search iterations we incur is linear in the input size. Now, within each iteration, since there are \( O(T) \) guesses for the insertion time of every item \( i \in Q_{>m} \setminus Q_{>m-1} \) and since \( |Q_{>m} \setminus Q_{>m-1}| \leq \frac{\log M}{\epsilon} \), there are only \( O(T \frac{(\log M)}{\epsilon}) \) chains \( G \) to consider in step 2. The crucial observation is that, for each such chain, the residual instance \( \hat{T}_{\omega - G} \) is defined over the set of items

\[
\hat{N} - G = N \setminus GT = ((C_m \cup Q_{>m}) \setminus Q_{>m-1}) \setminus (Q_m \setminus Q_{>m-1}) \\
\subseteq C_m \setminus Q_{>m-1} \\
\subseteq C_m. \quad (14)
\]

Thus, \( \hat{T}_{\omega - G} \) is in fact a single-cluster instance, where the weights of any two items differ by a multiplicative factor of at most \( n^{1/\epsilon} \), by property 1 of well-spaced instances (see Sect. 4.1). By Theorem 6, the running time of our approximation scheme for such instances is truly quasi-polynomial, being \( O((nT)^{O(\frac{1}{\epsilon} \log n)} \cdot |I|^{O(1)}) \). All in all, we incur a running time of \( O((nT)^{O(\frac{1}{\epsilon} \log n + \log M)} \cdot |I|^{O(1)}) \), with room to spare.

**Final solution and analysis.** In the remainder of this section, we argue that the item set \( E_{\omega_{\min}} \) and its permutation \( \pi_{E_{\omega_{\min}}} \) satisfy the properties required by Lemma 23. For this purpose, recalling that the latter lemma assumes \( F(m, \psi_m, Q_{>m}) \leq W_T \) and \( (m-1, \psi_{m-1}, Q_{>m-1}) = \text{Best}(m, \psi_m, Q_{>m}) \), let \( E^* \) and \( \pi_{E^*} \) be the item set and permutation attaining the minimum makespan \( w(E^*) \) over \( \text{Extra}_{(m-1, \psi_{m-1}, Q_{>m-1})} \), noting that by definition,

\[
F(m, \psi_m, Q_{>m}) = F(m-1, \psi_{m-1}, Q_{>m-1}) + w(E^*) \quad (15)
\]

At the heart of our analysis lies the following claim, showing that whenever the \( \omega \)-parameterized algorithm is employed with \( \omega \geq w(E^*) \), we obtain a permutation whose \( \Delta \)-shifted profit is at least \( (1 - \epsilon) \cdot (\psi_m - \psi_{m-1}) \). We provide the proof in “Appendix D.7”.
Lemma 26  For any \( \omega \geq w(\mathcal{E}^*) \), the \( \omega \)-parameterized algorithm computes an item set \( \mathcal{E}_\omega \) and a permutation \( \pi_{\mathcal{E}_\omega} : \mathcal{E}_\omega \to |\mathcal{E}_\omega| \) that satisfy \( \sum_{i \in \mathcal{E}_\omega} \varphi^\omega_{\mathcal{E}_\omega}(i) \geq (1 - \epsilon) \cdot (\psi_m - \psi_{m-1}) \).

With this result in place, the properties required by Lemma 23 can easily be established, as we show next.

Lemma 27  The item set \( \mathcal{E}_{\omega_{\text{min}}} \) and permutation \( \pi_{\mathcal{E}_{\omega_{\text{min}}}} \) satisfy properties 1 and 2.

Proof  We begin by explaining why \( (\mathcal{E}_{\omega_{\text{min}}}, \pi_{\mathcal{E}_{\omega_{\text{min}}}}) \in \text{Extra}_{\epsilon, \Delta}[(m, \psi_m, Q_{>m})] \), as stated in property 1:

- **Constraint 1 is satisfied:** We first show that \( \mathcal{E}_{\omega_{\text{min}}} \subseteq (C_m \cup Q_{>m}) \setminus Q_{>m-1} \) and \( Q_{>m} \setminus Q_{>m-1} \subseteq \mathcal{E}_{\omega_{\text{min}}} \). Since the item set in question is defined in step 3 as \( \mathcal{E}_{\omega_{\text{min}}} = R_T^{G_{\omega_{\text{min}}}} \cap (Q_{>m} \setminus Q_{>m-1}) \), it suffices to explain why \( R_T^{G_{\omega_{\text{min}}}} \subseteq C_m \setminus Q_{>m-1} \).

The latter inclusion follows by noting that \( R_T^{G_{\omega_{\text{min}}}} \) is a feasible chain for the instance \( \mathcal{T}_{\omega_{\text{min}}}, G_{\omega_{\text{min}}} \), where the set of items is \( \mathcal{N} \setminus G_{\omega_{\text{min}}} \subseteq C_m \setminus Q_{>m-1} \), as shown in the first inclusion of (14).

- **Constraint 2 is \( (\epsilon, \Delta) \)-satisfied:** To argue that \( \sum_{i \in \mathcal{E}_{\omega_{\text{min}}}^\omega} \varphi^\omega_{\mathcal{E}_{\omega_{\text{min}}}}(i) \geq (1 - \epsilon) \cdot (\psi_m - \psi_{m-1}) \), following Lemma 26, there exists a value \( \omega \leq w(\mathcal{N}) \) for which \( \sum_{i \in \mathcal{E}_{\omega_{\text{min}}}^\omega} \varphi^\omega_{\mathcal{E}_{\omega_{\text{min}}}}(i) \geq (1 - \epsilon) \cdot (\psi_m - \psi_{m-1}) \), and the desired claim is implied by the termination condition of our binary search.

We now turn our attention to proving that \( w(\mathcal{E}_{\omega_{\text{min}}}) \leq F(m, \psi_m, Q_{>m}) - F(m - 1, \psi_{m-1}, Q_{>m-1}) \), as stated in property 2. To this end, since \( F(m, \psi_m, Q_{>m}) = F(m - 1, \psi_{m-1}, Q_{>m-1}) + w(\mathcal{E}^*) \) by equation (15), it remains to argue that \( w(\mathcal{E}_{\omega_{\text{min}}}) \leq w(\mathcal{E}^*) \). To verify this relation, note that

\[
w(\mathcal{E}_{\omega_{\text{min}}}) = w\left( R_T^{G_{\omega_{\text{min}}}} \cup (Q_{>m} \setminus Q_{>m-1}) \right) = w\left( R_T^{G_{\omega_{\text{min}}}} \right) + w\left( G_T^{\omega_{\text{min}}} \right) \\
\leq \hat{W}_T = \min \{ [W_T - \Delta]^+, \omega_{\text{min}} \} \leq \omega_{\text{min}} \leq w(\mathcal{E}^*) .
\]

Here, the second equality holds since \( G_T^{\omega_{\text{min}}} = Q_{>m} \setminus Q_{>m-1} \), as stated in step 2. The first inequality follows by observing that the chain \( R_T^{G_{\omega_{\text{min}}}} \cup G_{\omega_{\text{min}}} \) is feasible for \( \mathcal{T}_{\omega_{\text{min}}} \), due to Lemma 8, meaning in particular that for period \( T \) we have \( w(R_T^{G_{\omega_{\text{min}}}}) + w(G_T^{\omega_{\text{min}}}) \leq \hat{W}_T \). The final inequality is derived by combining Lemma 26 and the termination condition of our binary search.

D.7 Proof of Lemma 26

Constructing a feasible chain for \( \mathcal{T}^\omega \). With respect to the item set \( \mathcal{E}^* \) and permutation \( \pi_{\mathcal{E}^*} \), let us define a chain \( \mathcal{S}^* \) for the instance \( \mathcal{T}^\omega \) as follows:

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– The collection of inserted items is $S_{st} = \mathcal{E}^\ast$.
– The insertion time $t_i$ of each item $i \in S_{st}$ is the one maximizing $p_{iti}$ over $\{ t \in [T] : W_t \geq F(m - 1, \psi_{m-1}, Q_{>m-1}) + C_{\pi_{E_{\mathcal{E}^\ast}}(i)} \}$. Note that the latter set is indeed non-empty, since

$$C_{\pi_{E_{\mathcal{E}^\ast}}(i)} \leq w(\mathcal{E}^\ast) = F(m, \psi_{m}, Q_{>m}) - F(m - 1, \psi_{m-1}, Q_{>m-1}) \leq W_T - F(m - 1, \psi_{m-1}, Q_{>m-1}),$$

where the equality above is exactly (15), and the last inequality holds since $F(m, \psi_{m}, Q_{>m}) \leq W_T$, as assumed in Lemma 23.

The next claim establishes the feasibility and profit guarantee of $S_{st}$ with respect to $\hat{T}^\omega$. Below, $\Phi_{\omega}(\cdot)$ stands for the profit function with respect to this instance.

**Claim 6** The chain $S_{st}$ is feasible for $\hat{T}^\omega$, with a profit of $\Phi_{\omega}(S_{st}) = \sum_{i \in \mathcal{E}^\ast} \varphi^{\pi_{E_{\mathcal{E}^\ast}}}(i)$.

**Proof** To prove the feasibility of $S_{st}$, we first observe that, for every time period $t \in [T],$

$$w(S_{st}) \leq w(\mathcal{E}^\ast) \leq \omega,$$  \hspace{1cm} (16)

where the first inequality holds since $S_{st} \subseteq S_{st} = \mathcal{E}^\ast$, and the second inequality is precisely what Lemma 26 assumes. In addition, by definition of $S_{st}$, every item $i \in S_{st}$ is associated with a completion time of $C_{\pi_{E_{\mathcal{E}^\ast}}(i)} \leq W_t - F(m - 1, \psi_{m-1}, Q_{>m-1})$. Thus, when the latter difference is negative, we have $S_{st} = \emptyset$ and therefore $w(S_{st}) = 0 \leq [W_t - \Delta]^+$. In the opposite case,

$$w(S_{st}) \leq W_t - F(m - 1, \psi_{m-1}, Q_{>m-1}) \leq W_t - \Delta \leq [W_t - \Delta]^+,$$  \hspace{1cm} (17)

where the second inequality holds since $\Delta \leq F(m - 1, \psi_{m-1}, Q_{>m-1})$, as assumed in Lemma 23. Putting together inequalities (16) and (17), we have $w(S_{st}) \leq \min([W_t - \Delta]^+, \omega) = \hat{W}_t^\omega$, meaning that the chain $S_{st}$ is indeed feasible for $\hat{T}^\omega$.

Now, to derive the profit guarantee $\Phi_{\omega}(S_{st}) = \sum_{i \in \mathcal{E}^\ast} \varphi^{\pi_{E_{\mathcal{E}^\ast}}}(i)$, we observe that since $\Phi_{\omega}(S_{st}) = \sum_{i \in \mathcal{E}^\ast} p_{iti}$, it suffices to show that $p_{iti} = \varphi^{\pi_{E_{\mathcal{E}^\ast}}}(i)$ for each item $i \in \mathcal{E}^\ast$. To this end, note that our choice for the insertion time $t_i$ of each item $i \in \mathcal{E}^\ast$ exactly follows the definition of $\varphi^{\pi_{E_{\mathcal{E}^\ast}}}(i)$, implying that $p_{iti} = \varphi^{\pi_{E_{\mathcal{E}^\ast}}}(i)$. \hfill \Box

**Concluding the proof.** Having established this claim, we are now ready to show that the item set $\mathcal{E}_{\omega}$ and permutation $\pi_{E_{\mathcal{E}_{\omega}}}$ satisfy $\sum_{i \in \mathcal{E}_{\omega}} \varphi_{\pi_{E_{\mathcal{E}_{\omega}}}}(i) \geq (1 - \epsilon) \cdot (\psi_m - \psi_{m-1})$. For this purpose, similarly to $\Phi_{\omega}(\cdot)$, let $\Psi_{\omega}(\cdot)$ be the profit function of a given permutation with respect to the instance $\mathcal{E}_{\omega}$ in its sequencing formulation. With this notation, we obtain the required lower bound by arguing that

$$\sum_{i \in \mathcal{E}_{\omega}} \varphi_{\pi_{E_{\mathcal{E}_{\omega}}}}(i) = \Psi_{\omega}(\pi_{E_{\mathcal{E}_{\omega}}})$$
We prove the first equality and second inequality in Claims 7 and 8, respectively. To understand the first inequality, recall that the permutation $\pi_{\mathcal{E}_\omega}$ is constructed in step 3 according to Lemma 1 for the chain $G^\omega \cup R^{G^\omega}$, which guarantees $\psi_{\omega}(\pi_{\mathcal{E}_\omega}) \geq \Phi_{\omega}(G^\omega \cup R^{G^\omega})$.

Claim 7 $\sum_{i \in \mathcal{E}_\omega} \Psi_{\pi_{\mathcal{E}_\omega}}^+(i) = \Psi_{\omega}(\pi_{\mathcal{E}_\omega})$.

Proof Let us use $\phi_{\pi_{\mathcal{E}_\omega}}^+(i)$ to denote the profit contribution of item $i$ with respect to the permutation $\pi_{\mathcal{E}_\omega}$ in the instance $\hat{T}^\omega$. In other words, $\phi_{\pi_{\mathcal{E}_\omega}}^+(i) = \max\{p_{it} : t \in \{T + 1\} \text{ and } W_t \geq C_{\pi_{\mathcal{E}_\omega}}(i)\}$. With this notation, we have $\Psi_{\omega}(\pi_{\mathcal{E}_\omega}) = \sum_{i \in \mathcal{E}_\omega} \phi_{\pi_{\mathcal{E}_\omega}}^+(i)$, meaning that to prove the desired equality, it remains to show that $\phi_{\pi_{\mathcal{E}_\omega}}^+(i) = \phi_{\pi_{\mathcal{E}_\omega}}^+(i)$ for every item $i \in \mathcal{E}_\omega$. To verify this claim, note that

$$\phi_{\pi_{\mathcal{E}_\omega}}^+(i) = \max \left\{ p_{it} : t \in \{T + 1\} \text{ and } W_t - \Delta \geq C_{\pi_{\mathcal{E}_\omega}}(i) \right\}$$

$$= \max \left\{ p_{it} : t \in \{T + 1\} \text{ and } [W_t - \Delta]^+ \geq C_{\pi_{\mathcal{E}_\omega}}(i) \right\}$$

$$= \max \left\{ p_{it} : t \in \{T + 1\} \text{ and } \min\{[W_t - \Delta]^+, \omega \} \geq C_{\pi_{\mathcal{E}_\omega}}(i) \right\}$$

$$= \phi_{\pi_{\mathcal{E}_\omega}}^+(i) .$$

Here, the second equality holds since $C_{\pi_{\mathcal{E}_\omega}}(i) \geq 0$. The third equality is obtained by noting that $C_{\pi_{\mathcal{E}_\omega}}(i) \leq w(\mathcal{E}_\omega) = w(G^\omega_T) + w(R^{G^\omega}_T) \leq \hat{W}_T \leq \omega$, where the equality follows by definition of $\mathcal{E}_\omega$ and the second inequality is implied by the feasibility of $G^\omega \cup R^{G^\omega}$ for the instance $\hat{T}^\omega$. The last two equalities follow from the definitions of $\hat{W}_T$ and $\phi_{\pi_{\mathcal{E}_\omega}}^+(i)$.

Claim 8 $\Phi_{\omega}(G^\omega \cup R^{G^\omega}) \geq (1 - \epsilon) \cdot (\psi_m - \psi_{m-1})$.

Proof We begin by noting that since $(\mathcal{E}^*, \pi_{\mathcal{E}^*}) \in \text{Extra}_{(m, \psi_m, Q_{m-1})}$, this item set and permutation necessarily satisfy constraint 1, which informs us that $\mathcal{E}^* \subseteq (\mathcal{C}_m \cup Q_{m-1}) \setminus Q_{m-1}$ and $Q_{m-1} \setminus Q_{m-1} \subseteq \mathcal{E}^*$. As a result, recalling that the collection of items introduced by the chain $S_{\pi}$ is precisely $\mathcal{E}^*$, it follows that the latter chain can be expressed as $S_{\pi} = S_{\pi}|_{Q_{m-1} \setminus Q_{m-1}} \cup S_{\pi}|_{\mathcal{C}_m \setminus Q_{m-1}}$. We remind the reader that, based on the terminology of Sect. 3, the first term $S_{\pi}|_{Q_{m-1} \setminus Q_{m-1}}$ is the restriction of $S_{\pi}$ to the items in $Q_{m-1} \setminus Q_{m-1}$, whereas the second term $S_{\pi}|_{\mathcal{C}_m \setminus Q_{m-1}}$ is its restriction to $\mathcal{C}_m \setminus Q_{m-1}$.

The crucial observation is that, since the chain $S_{\pi}$ introduces all items in $Q_{m-1} \setminus Q_{m-1}$, its restriction $G_{\pi} = S_{\pi}|_{Q_{m-1} \setminus Q_{m-1}}$ is necessarily considered in step 2 of our algorithm; moreover, $S_{\pi}|_{\mathcal{C}_m \setminus Q_{m-1}}$ constitutes a feasible chain for the residual instance $\hat{T}^\omega - G_{\pi}$, by Lemma 9. As such, the corresponding chain $R^{G_{\pi}}$ we compute for the latter instance is guaranteed to have a profit of $\Phi_{\omega}(R^{G_{\pi}}) \geq \Phi_{\omega}(G^\omega \cup R^{G^\omega})$.
In addition, letting \( \Phi_\omega \left( \mathcal{G}^\omega \cup \mathcal{R}^\omega \right) \) be the corresponding item set and permutation \( \pi \) of \( \mathcal{G}^\omega \) and \( \mathcal{R}^\omega \), we conclude that \( \mathcal{G}^\omega \cup \mathcal{R}^\omega \) is a feasible chain for \( \hat{\mathcal{G}}^\omega \) with a profit of

\[
\Phi_\omega \left( \mathcal{G}^\omega \cup \mathcal{R}^\omega \right) = \Phi_\omega \left( \mathcal{G}^\omega \right) + \Phi_\omega \left( \mathcal{R}^\omega \right) \\
\geq \Phi_\omega \left( \mathcal{G}^\omega \right) + \Phi_\omega \left( \mathcal{R}^\omega \right) \\
\geq \left( 1 - \epsilon \right) \cdot \Phi_\omega \left( \mathcal{S}_m \right) \\
= \left( 1 - \epsilon \right) \cdot \sum_{i \in \mathcal{E}^*} \varphi_{\pi_{\hat{\mathcal{E}}}^*}^\omega \left( i \right) \\
\geq \left( 1 - \epsilon \right) \cdot \left( \psi_m - \psi_{m-1} \right).
\]

Here, the first and second equalities follow from Lemma 8 and Claim 6, respectively. The last inequality holds since \( \left( \mathcal{E}^*, \pi_{\hat{\mathcal{E}}}^* \right) \in \text{Extra}_{(m, \psi_{m-1}, Q_{>m-1})} \) by definition, and hence, constraint 2 is necessarily satisfied. \( \square \)

D.8 Proof of Lemma 24

We prove the lemma by induction on \( m \).

**Base case: \( m = 0 \).** In this case, for any state with \( F(0, \psi_0, Q_{>0}) \leq W_T \), we actually have \( \hat{F}(0, \psi_0, Q_{>0}) = F(0, \psi_0, Q_{>0}) \), by the way terminal states of \( \hat{F} \) are handled. In addition, letting \( \hat{\pi}_{\hat{S}_0} \) be the permutation of \( \hat{S}_0 = Q_{>0} \) that attains \( \hat{F}(0, \psi_0, Q_{>0}) \), it follows that \( \hat{S}_0 \subseteq C_{[1,0]} \cup Q_{>0} \subseteq \hat{S}_0 \), and \( \Psi(\hat{\pi}_{\hat{S}_0}) \geq \psi_0 \), again by definition.

**General case: \( m \geq 1 \).** Let \( (m, \psi_m, Q_{>m}) \) be a state for which \( F(m, \psi_m, Q_{>m}) \leq W_T \). We first show that \( \hat{F}(m, \psi_m, Q_{>m}) \leq F(m, \psi_m, Q_{>m}) \). To this end, recall that the function value \( \hat{F}(m, \psi_m, Q_{>m}) \) is determined by minimizing \( \hat{F}(m - 1, \psi_{m-1}, Q_{>m-1}) + w(\mathcal{E}) \) over all conceivable states \( (m - 1, \psi_{m-1}, Q_{>m-1}) \), where the item set \( \mathcal{E} \) and its permutation \( \pi_{\hat{\mathcal{E}}} : \mathcal{E} \rightarrow [\mathcal{E}] \) are obtained by instantiating Lemma 23 with \( \Delta = \hat{F}(m - 1, \psi_{m-1}, Q_{>m-1}) \) and satisfying \( (\mathcal{E}, \pi_{\hat{\mathcal{E}}}) \in \text{Extra}_\epsilon,\Delta_{(m-1, \psi_{m-1}, Q_{>m-1})} \). Therefore, specifically for the state \( (m - 1, \psi_{m-1}, Q_{>m-1}) = \text{Best}(m, \psi_m, Q_{>m}) \), we have \( \Delta = \hat{F}(m - 1, \psi_{m-1}^*, Q_{>m-1}^*) \leq F(m - 1, \psi_{m-1}, Q_{>m-1}) \) by the induction hypothesis. In turn, our auxiliary procedure computes a corresponding item set and permutation \( (\mathcal{E}^*, \pi_{\hat{\mathcal{E}}}^*) \in \text{Extra}_\epsilon,\Delta_{(m-1, \psi_{m-1}, Q_{>m-1})} \) with total weight \( w(\mathcal{E}^*) \leq F(m, \psi_m, Q_{>m}) - F(m - 1, \psi_{m-1}^*, Q_{>m-1}^*) \), as guaranteed by Lemma 23. Consequently,

\[
\hat{F}(m, \psi_m, Q_{>m}) \leq \hat{F}(m - 1, \psi_{m-1}^*, Q_{>m-1}^*) + w(\mathcal{E}^*) \\
\leq F(m - 1, \psi_{m-1}, Q_{>m-1}) \\
+ \left( F(m, \psi_m, Q_{>m}) - F(m - 1, \psi_{m-1}, Q_{>m-1}) \right)
\]
\[
F(m, \psi_m, Q_{>m}) = F(m, \psi_m, Q_{>m}) \,,
\]
which is precisely the required upper bound on \( \hat{F}(m, \psi_m, Q_{>m}) \).

Next, we show that \( \hat{F}(m, \psi_m, Q_{>m}) \) is attained by an item set \( \hat{S}_m \) and a permutation \( \hat{\pi}_{\hat{S}_m} \) satisfying \( \hat{S}_m \subseteq C_{[1,m]} \cup Q_{>m}, Q_{>m} \subseteq \hat{S}_m, \) and \( \Psi(\hat{\pi}_{\hat{S}_m}) \geq (1 - \epsilon) \cdot \psi_m. \) For

this purpose, let \((m - 1, \psi_{m-1}, Q_{>m-1}), \hat{E}, \) and \(\hat{\pi}_{\hat{E}}\) be the conceivable state, item set, and permutation at which \( \hat{F}(m, \psi_m, Q_{>m}) = \hat{F}(m - 1, \psi_{m-1}, Q_{>m-1}) + w(\hat{E}) \) is attained, meaning in particular that \( Q_{>m-1} \setminus C_m \subseteq Q_{>m} \) by definition of conceivable states, and that \((\hat{E}, \hat{\pi}_{\hat{E}}) \in \text{Extra}_{\epsilon, \Delta}[1, \psi_{m-1}, Q_{>m-1}] \) by the way general states of \( \hat{F} \) are handled. We proceed by observing that, by the induction hypothesis, \( \hat{F}(m - 1, \psi_{m-1}, Q_{>m-1}) \) is attained by an item set \( \hat{S}_{m-1} \) and a permutation \( \hat{\pi}_{\hat{S}_{m-1}} \) satisfying \( \hat{S}_{m-1} \subseteq C_{[1,m-1]} \cup Q_{>m-1}, Q_{>m-1} \subseteq \hat{S}_{m-1}, \) and \( \Psi(\hat{\pi}_{\hat{S}_{m-1}}) \geq (1 - \epsilon) \cdot \psi_{m-1}. \) With these ingredients, let us define the item set \( \hat{S}_m \) and permutation \( \hat{\pi}_{\hat{S}_m} \) as follows:

- The item set \( \hat{S}_m \) is given by \( \hat{S}_m = \hat{S}_{m-1} \cup \hat{E}. \) To understand why \( \hat{S}_{m-1} \) and \( \hat{E} \) are disjoint, recall that \( (\hat{E}, \hat{\pi}_{\hat{E}}) \in \text{Extra}_{\epsilon, \Delta}[1, \psi_{m-1}, Q_{>m-1}] \), which implies by constraint 1 that \( \hat{E} \subseteq (C_m \cup Q_{>m}) \setminus Q_{>m-1} \subseteq C_{[1,m]} \setminus Q_{>m-1}; \) however, \( \hat{S}_{m-1} \subseteq C_{[1,m-1]} \cup Q_{>m-1} \) by the induction hypothesis. These observations allow us to concurrently argue that \( \hat{S}_m \subseteq C_{[1,m]} \cup Q_{>m} \) as required, since \( \hat{E} \subseteq (C_m \cup Q_{>m}) \setminus Q_{>m-1} \subseteq C_{[1,m]} \cup Q_{>m} \) and since

\[
\hat{S}_{m-1} \subseteq C_{[1,m-1]} \cup Q_{>m-1} \\
\subseteq C_{[1,m]} \setminus (Q_{>m-1} \setminus C_m) \\
\subseteq C_{[1,m]} \cup Q_{>m} ,
\]

where the last inclusion follows by noting that \( Q_{>m-1} \setminus C_m \subseteq Q_{>m} \) due to state \((m - 1, \psi_{m-1}, Q_{>m-1})\) being conceivable. In addition,

\[
Q_{>m} \subseteq Q_{>m-1} \cup (Q_{>m} \setminus Q_{>m-1}) \\
\subseteq \hat{S}_{m-1} \cup \hat{E} \\
= \hat{S}_m ,
\]

where the second inclusion holds since \( Q_{>m-1} \subseteq \hat{S}_{m-1} \) by the induction hypothesis and since \( Q_{>m} \setminus Q_{>m-1} \subseteq \hat{E}, \) again by constraint 1.

- To define the permutation \( \hat{\pi}_{\hat{S}_m} : \hat{S}_m \to [\hat{S}_m], \) we simply append \( \hat{\pi}_{\hat{E}} \) to \( \hat{\pi}_{\hat{S}_{m-1}}. \) As a result, we obtain a profit of

\[
\Psi(\hat{\pi}_{\hat{S}_m}) = \Psi(\hat{\pi}_{\hat{S}_{m-1}}) + \sum_{i \in \hat{E}} \varphi_{\pi_{\hat{E}}}^{\hat{\pi}_{\hat{E}}} (i) \\
= \Psi(\hat{\pi}_{\hat{S}_{m-1}}) + \sum_{i \in \hat{E}} \varphi_{\pi_{\hat{E}}}^{\hat{\pi}_{\hat{E}}} (i)
\]
\[ \geq (1 - \epsilon) \cdot \psi_{m-1} + (1 - \epsilon) \cdot (\psi_m - \psi_{m-1}) \]
\[ = (1 - \epsilon) \cdot \psi_m . \]

Here, the second equality holds since \( w(\hat{S}_{m-1}) = \hat{F}(m-1, \psi_{m-1}, Q_{>m-1}) = \Delta. \)

To understand the inequality above, note that \( \Psi(\hat{\pi}_{\hat{S}_{m-1}}) \geq (1 - \epsilon) \cdot \psi_{m-1} \) by the inductive hypothesis, and in addition, \( \sum_{i \in \hat{E}} \Phi_{\hat{\pi}_{\hat{E}}}(i) \geq (1 - \epsilon) \cdot (\psi_m - \psi_{m-1}), \)

since \( (\hat{E}, \hat{\pi}_{\hat{E}}) \in \text{Extra}_{\epsilon, \Delta}[(m-1, \psi_{m-1}, Q_{>m-1})] \) implies that constraint 2 is \((\epsilon, \Delta)\)-satisfied.

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