BOCHNER’S SUBORDINATION AND FRACTIONAL CALORIC SMOOTHING IN BESOV AND TRIEBEL–LIZORKIN SPACES

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Abstract. We use Bochner’s subordination technique to obtain caloric smoothing estimates in Besov- and Triebel–Lizorkin spaces. Our new estimates extend known smoothing results for the Gauß–Weierstraß, Cauchy–Poisson and higher-order generalized Gauß–Weierstraß semigroups. Extensions to other function spaces (homogeneous, hybrid) and more general semigroups are sketched.

1. Introduction

Let \((W^f_t)_{t \geq 0}\) be the \(f\)-subordinated Gauß–Weierstraß semigroup; by this we mean the family of operators which is defined through the Fourier transform

\[ F(W^f_t u)(\xi) = e^{-tf(|\xi|^2)} F u(\xi), \quad u \in S(\mathbb{R}^n), \]

where the function \(f: (0, \infty) \to (0, \infty)\) is a so-called Bernstein function, see Section 3. Typical examples are \(f(x) = x\) (which gives the classical Gauß–Weierstraß semigroup), \(f(x) = \sqrt{x}\) (which gives the Cauchy–Poisson semigroup) or \(f(x) = x^\alpha, 0 < \alpha < 1\) (which leads to the stable semigroups). In this note we prove the caloric smoothing of \((W^f_t)_{t \geq 0}\) in Besov and Triebel–Lizorkin spaces, see Section 2. “Caloric smoothing” refers to the smoothing effect of the semigroup which can be quantified through inequalities of the following form

\[ C_{f,d}(t) \| W^f_t u \|_{A^s_{p,q}} \leq \| u \|_{A^s_{p,q}} \]

for all \(0 < t \leq 1\) and \(u \in A^s_{p,q}\),

where \(d \geq 0\) is arbitrary, \(C_{f,d}(t)\) is a constant depending only on \(f\) and \(d\), and \(C_{f,d}(t) \to 0\) as \(t \to 0\); the symbol \(A^s_{p,q} = A^s_{p,q}(\mathbb{R}^n)\) stands for a Besov space \(B^s_{p,q}(\mathbb{R}^n)\) or a Triebel–Lizorkin space \(F^s_{p,q}(\mathbb{R}^n)\).

Results of this type are known for the Gauß–Weierstraß semigroup \(W_t\), i.e. for \(f(x) = x\) (see Triebel [15, Theorem 3.35]) and for the generalized Gauß–Weierstraß semigroup \(W_t^{(m)}\) where \(m \in \mathbb{N}\); these operators are also given through the relation (1.1) if we take \(f(x) = x^m\), cf. [15, Remark 3.37], but note that for \(m > 1\) this is not a Bernstein function, and recent results by Baaske & Schmeißer [1] Theorem 3.5].

We will use Bochner’s subordination technique to prove (1.2) for arbitrary Bernstein functions \(f(x)\) and arbitrary powers \(f(x) = x^\beta, \beta > 0\). The constant \(C_{f,d}(t)\) is comparable with \([f^{-1}(1/t)]^{-d/2}\). As an application we generalize the result of Baaske & Schmeißer [1, Theorem 3.5] on the existence and uniqueness of the mild and strong solutions of a nonlinear Cauchy problem with arbitrary (fractional) powers of the Laplacian \((-\Delta)^\beta, \beta \geq 1\).

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2. Function spaces

Let us briefly recall some notation. \( L_p(\mathbb{R}^n) \), resp., \( \ell_q(\mathbb{N}_0) \) denote the spaces of \( p \)-th order integrable functions on \( \mathbb{R}^n \), resp., \( q \)-th order summable sequences indexed by \( \mathbb{N}_0 \); we admit \( 0 < p, q \leq \infty \). Since we are always working in \( \mathbb{R}^n \), we will usually write \( L_p \) instead of \( L_p(\mathbb{R}^n) \). If \( p, q < 1 \) these spaces are quasi-Banach spaces, their (quasi-)norms are denoted by \( \| \cdot \|_{L_p} \) and \( \| \cdot \|_{\ell_q} \), respectively. We write

\[
\| u(k, x) \|_{L_p} = \ell_q(\mathbb{N}_0) \| \| u(k, x) \|_{\ell_q(\mathbb{N}_0)} \|_{L_p}
\]

to indicate that we take first the \( L_p \)-norm and then the \( \ell_q(\mathbb{N}_0) \)-norm [resp. first the \( \ell_q(\mathbb{N}_0) \)-norm and then the \( L_p \)-norm]. Throughout, we use \( j, k, m \) for discrete and \( x, y, z \) for continuous variables, so there should be no confusion as to which variable is used for the \( \ell_q(\mathbb{N}_0) \)-norm or \( L_p \)-norm.

We follow Triebel [15, Definition 1.1 and Remark 1.2 (1.14), (1.15)] for the definition of the scales of Besov- and Triebel–Lizorkin spaces. Let \( \mathcal{F} \) denote the Fourier transform of a function \( u \); the extension to the space of tempered distributions \( S'(\mathbb{R}^n) \) is again denoted by \( \mathcal{F} \). Fix some \( \phi_0 \in C_0^\infty \) such that \( 1_{B(0,1)} \leq \phi_0 \leq 1_{B(0,3/2)} \) and set \( \phi_k(x) := \phi_0(2^{-k}x) - \phi_0(2^{-(k+1)}x) \). Since \( \sum_{k=0}^\infty \phi_k(x) = 1 \), the sequence \( (\phi_k)_{k \geq 0} \) is a dyadic resolution of unity. By

\[
\phi_k(D)u(x) := \mathcal{F}^{-1}(\phi_k \mathcal{F} u)(x)
\]

we denote the pseudo-differential operator (Fourier multiplier operator) with symbol \( \phi_k \). We will also need the dyadic cubes \( Q_{J,M} = 2^{-J}M + 2^{-J}(0,1)^n \), where \( J \in \mathbb{Z} \), \( M \in \mathbb{Z}^n \) and \( (0,1)^n \) is the open unit cube in \( \mathbb{R}^n \).

**Definition 2.1.** Let \( (\phi_k)_{k \geq 0} \) be any dyadic resolution of unity.

a) Let \( p \in (0, \infty) \), \( q \in (0, \infty) \) and \( s \in \mathbb{R} \). The **Besov space** \( B^s_{p,q} \) is the family of all \( u \in S'(\mathbb{R}^n) \) such that the following (quasi-)norm is finite:

\[
\| u \|_{B^s_{p,q}} := \| 2^{ks} \phi_k(D)u(x) \|_{L_p} = \ell_q(\mathbb{N}_0) \| u \|_{\ell_q(\mathbb{N}_0)} \|_{L_p}
\]

b) Let \( p \in (0, \infty) \), \( q \in (0, \infty) \) and \( s \in \mathbb{R} \). The **Triebel–Lizorkin space** \( F^s_{p,q} \) is the family of all \( f \in S'(\mathbb{R}^n) \) such that the following (quasi-)norm is finite:

\[
\| u \|_{F^s_{p,q}} := \| 2^{ks} \phi_k(D)u(x) \|_{\ell_q(\mathbb{N}_0)} = \ell_q(\mathbb{N}_0) \| u \|_{\ell_q(\mathbb{N}_0)} \|_{L_p}
\]

c) Let \( p = \infty \), \( q \in (0, \infty) \) and \( s \in \mathbb{R} \). The **Triebel–Lizorkin space** \( F^s_{\infty,q} \) is the family of all \( f \in S'(\mathbb{R}^n) \) such that the following (quasi-)norm is finite

\[
\| u \|_{F^s_{\infty,q}} := \sup_{J \in \mathbb{N}_0} \sup_{M \in \mathbb{Z}^n} \left( \int_{Q_{J,M}} \sum_{k=J}^{\infty} 2^{ksq} |\phi_k(D)u(x)|^q dx \right)^{1/q}
\]

d) Let \( p = q = \infty \) and \( s \in \mathbb{R} \). The **Triebel–Lizorkin space** \( F^s_{\infty,\infty} \) is the family of all \( f \in S'(\mathbb{R}^n) \) such that the following norm is finite

\[
\| u \|_{F^s_{\infty,\infty}} := \sup_{J \in \mathbb{N}_0} \sup_{M \in \mathbb{Z}^n} \sup_{x \in Q_{J,M}} \sup_{k \geq J} \| \phi_k(D)u(x) \|.
\]

Note that \( F^s_{\infty,\infty} = B^s_{\infty,\infty} \) for all \( s \in \mathbb{R} \) and that the norms appearing in Definition [2.1](#) coincide if \( p = q = \infty \): \( \| u \|_{F^s_{\infty,\infty}} = \| u \|_{B^s_{\infty,\infty}} \). Definition [2.1](#) does not depend on the choice of \( (\phi_k)_{k \geq 0} \) since different resolutions of unity lead to equivalent (quasi-)norms. Various properties of these spaces as well as
their relation to other classical function spaces can be found in Triebel [13], see also [13] and [13].

Consider the heat kernel (Gaussian probability density) related to the Laplace operator on $\mathbb{R}^n$

\[(2.1) \quad g_t(x) := \frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/(4t)}, \quad t > 0, \quad x \in \mathbb{R}^n.\]

We can use $g_t(x)$ to define a convolution operator on the space $B_b$ of bounded Borel functions $u : \mathbb{R}^n \to \mathbb{R}$

\[(2.2) \quad W_t u(x) := g_t * u(x) = \int_{\mathbb{R}^n} g_t(y - x) u(y) \, dy.\]

For positive $u \geq 0$ the above integral always exists in $[0, \infty]$ and extends $W_t$ to all positive Borel functions. It is not difficult to see that $g_{t+s} = g_t * g_s$, i.e. $(W_t)_{t \geq 0}$ is a semigroup. The operators are positivity preserving $(W_t u \geq 0$ if $u \geq 0$) and conservative $(W_t 1 \equiv 1)$. If $u \in S$, then $\mathcal{F}(W_t u)(\xi) = e^{-t|\xi|^2} \mathcal{F}u(\xi)$. We will need the following simple lemma. We provide the short proof for the readers’ convenience.

**Lemma 2.2.** Let $(W_t)_{t \geq 0}$ be the Gauß–Weierstraß semigroup.

a) $W_t : L_p \to L_p$, $p \in [1, \infty]$ is a contraction, i.e.

\[\|W_t u \| \leq \|u \| \quad \|L_p\|\]

b) Let $\psi_k(\cdot)$ be a sequence of positive measurable functions on $\mathbb{R}^n$ such that $(\psi_k(x))_{k \geq 0} \in \ell_q(\mathbb{N}_0)$ for some $q \in [1, \infty]$ and all $x \in \mathbb{R}^n$. Then

\[\|W_t \psi_k(x) \| \leq \|W_t \psi_k(\cdot) \| \quad \|L_q(\mathbb{N}_0)\|\]

**Proof.** Part a) follows immediately from Jensen’s inequality, see [10] Theorem 13.13, for the probability measure $g_t(y) dy$:

\[\|W_t u \| \leq \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| u(x - y) g_t(y) \right| dy \right)^p dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x - y)|^p g_t(y) \, dy \, dx\]

\[= \|u \| \quad \|L_p\|\]

If $0 < p < 1$, the inequality is reversed.

In order to prove Part b) we fix $x$ and pick a sequence $(a_k)_{k \in \mathbb{N}_0}$ from $\ell_{q'}$ where $q' = \frac{q}{1-q}$. We have

\[|\langle W_t \psi_k(\cdot), a_k \rangle| \leq W_t (|\psi_k(\cdot)|, a_k \rangle \leq W_t \|\psi_k(\cdot) \| \cdot \|a_k \| \cdot \|\ell_q\|\]

In the estimate we use the fact that $W_t$ is linear and positivity preserving, implying that $u \mapsto W_t u$ is monotone. Taking the supremum over all sequences such that $\|a_k \| \cdot \|\ell_{q'}\| = 1$ gives

\[\|W_t \psi_k(x) \| \leq \sup_{\|a_k \| \cdot \|\ell_{q'}\| = 1} \|W_t \psi_k(x), a_k \rangle \leq W_t (|\psi_k(\cdot) \| \cdot \|\ell_q\|)(x). \quad \square\]

**Lemma 2.2** is the key ingredient for our proof that $W_t$ is a contraction in the scales of Besov- and Triebel–Lizorkin spaces.

The next theorem is well-known for indices $1 \leq p, q \leq \infty$. Our elementary proof also covers $0 < q < 1$ in the case of Besov spaces.

**Theorem 2.3.** Let $(W_t)_{t \geq 0}$ be the Gauß–Weierstraß semigroup and $s \in \mathbb{R}$. 

...
a) \( \|W_t u \| B^s_{p,q} \leq \| u \| B^s_{p,q} \) for all \( p \in [1, \infty) \) and \( q \in (0, \infty] \).

b) \( \|W_t u \| F^s_{p,q} \leq c_n \| u \| F^s_{p,q} \) for all \( p, q \in [1, \infty) \) with \( c_n = 2^n \) if \( q = \infty \) and \( c_n = 1 \) otherwise.

Proof. We use Definition \ref{def:2.1} to introduce the respective (quasi-)norms. Note that the operators \( W_t \) and \( \phi_k(D) \) commute since their symbols (Fourier multipliers) do not depend on \( x \).

[a) Fix \( p \in [1, \infty) \), \( q \in (0, \infty] \) and \( s \in \mathbb{R} \) and let \( u \in B^s_{p,q} \). Note that \( \phi_k(D)u \in L_p \). Since \( W_t \) is a contraction in \( L_p \)—see Lemma \ref{lem:2.2.a}—, we get
\[
\|W_t u \| B^s_{p,q} \leq \| 2^{ks} W_t \phi_k(D)u(x) \|_{\ell_q(\mathbb{N}_0)} \leq \| 2^{ks} \phi_k(D)u(x) \|_{\ell_q(\mathbb{N}_0)} L_p \| = \| u \| B^s_{p,q} \|
\]
The calculation above uses only \( p \geq 1 \) and does not impose any restriction on \( q > 0 \) and \( s \in \mathbb{R} \).

[b) Fix \( p \in [1, \infty) \), \( q \in [1, \infty] \), \( s \in \mathbb{R} \), and let \( u \in F^s_{p,q}(\mathbb{R}) \). Note that \( \phi_k(D)u \) is measurable. Using the contractivity properties of \( W_t \) from Lemma \ref{lem:2.2} we get
\[
\|W_t u \| F^s_{p,q} \leq \| 2^{ks} W_t \phi_k(D)u(x) \|_{\ell_q(\mathbb{N}_0)} \leq \| 2^{ks} \phi_k(D)u(x) \|_{\ell_q(\mathbb{N}_0)} L_p \| = \| u \| F^s_{p,q} \|
\]
We will now consider the case \( p = \infty \) and \( q \in [1, \infty) \). As before, we write \( Q_{J,M} \) for the open cube in \( \mathbb{R}^n \) with side-length \( 2^{-J} \) and “lower left corner” \( 2^{-J} M \in \mathbb{Z}^n \). Below we use the notation \( \int_Q u(x) dx \) to denote \( \text{Leb}(Q)^{-1} \int_Q u(x) dx \). We can rewrite the norm for \( F^s_{\infty,q} \) as
\[
\| u \| F^s_{\infty,q} = \sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} \left( \int_{Q_{J,M}} \sum_{k \geq J} 2^{ksq} \| \phi_k(D)u \|^q dx \right)^{1/q}.
\]
In order to estimate the norm \( \|W_t u \| F^s_{\infty,q} \), we begin with an auxiliary estimate. Fix \( J \in \mathbb{N}_0 \) and \( M \in \mathbb{Z}^n \). By Jensen’s inequality,
\[
\int_{Q_{J,M}} |W_t w(x)|^q dx \leq \int_{Q_{J,M}} \int_{\mathbb{R}^n} g_t(x-y) |w(y)|^q dy dx
\]
\[
= \int_{\mathbb{R}^n} \int_{Q_{J,M}} g_t(y) |w(x-y)|^q dx dy
\]
\[
\leq \sup_{y \in \mathbb{R}^n} \int_{Q_{J,M}} |w(x-y)|^q dx \cdot \int_{\mathbb{R}^n} g_t(y) dy
\]
\[
= \sup_{y \in \mathbb{R}^n} \int_{y+Q_{J,M}} |w(x)|^q dx.
\]
The shifted cube \( Q := y + Q_{J,M} \) does, in general, not coincide with any of the \( Q_{J,N}, N \in \mathbb{Z}^n \). Since \( Q \) has side-length \( 2^{-J} \) it intersects at most \( 2^n \) of the \( Q_{J,N}, N \in \mathbb{Z} \). Define
\[
\lambda_{Q,J,N} := \frac{\int_{Q \cap Q_{J,N}} |w(x)|^q dx}{\int_{Q_{J,N}} |w(x)|^q dx},
\]
and observe that $0 \leq \lambda_{Q_{J,N}} \leq 1$ and $\sum_{N \in \mathbb{Z}^n} \lambda_{Q_{J,N}} \leq \sum_{N \in \mathbb{Z}^n} \mathbb{1}_{Q_{J,N}} \leq 2^n$ since the sum contains at most $2^n$ non-zero elements. Since $\text{Leb}(Q) = \text{Leb}(Q_{J,N})$, we get

$$\int_Q |w(x)|^q \, dx = \sum_{N \in \mathbb{Z}^n} \frac{1}{\text{Leb}(Q)} \int_{Q \cap Q_{J,N}} |w(x)|^q \, dx = \sum_{N \in \mathbb{Z}^n} \lambda_{Q_{J,N}} \int_{Q_{J,N}} |w(x)|^q \, dx.$$ 

Moreover, observing that $Q = y + Q_{J,M}$ and $\lambda_{N,J,Q} \leq 1$ we have

$$\int_{Q_{J,M}} |W_t w(x)|^q \, dx \leq \sup_{y \in \mathbb{R}^n} \left( \sum_{N \in \mathbb{Z}^n, Q \cap Q_{J,N} \neq \emptyset} \int_{Q_{J,N}} |w(x)|^q \, dx \right).$$

Now repeat the above calculations with $|W_t w|^q = |W_t \phi_k(D) u|^q$ and $|w|^q = |\phi_k(D) u|^q$, multiplied by $2^{kq}$ and summed over $k \geq J$. Since we have only positive terms, the summation and integration signs can be freely interchanged. Thus,

$$\int_{Q_{J,M}} \sum_{k \geq J} 2^{kq} |W_t \phi_k(D) u(x)|^q \, dx \leq \sup_{y \in \mathbb{R}^n} \left( \sum_{N \in \mathbb{Z}^n, Q \cap Q_{J,N} \neq \emptyset} \int_{Q_{J,N}} \sum_{k \geq J} 2^{kq} |\phi_k(D) u(x)|^q \, dx \right) \leq 2^n \|u\|^{F_{\infty,q}} ||^{q}$$

Finally, for $p = q = \infty$, the estimate is immediate using Lemma 2.2 and Definition 2.1(d).

Remark 2.4. In the proof of Theorem 2.3 and Lemma 2.2 we only use the following properties of the semigroup $(W_t)_{t \geq 0}$:

$$0 \leq u \leq 1 \implies 0 \leq W_t u \leq 1 \quad \text{and} \quad \phi_k(D) W_t = W_t \phi_k(D).$$

This means that Theorem 2.3 holds for every positivity preserving, Markovian semigroup $T_t$ which is given by a convolution: $T_t u = u * \pi_t$ where $(\pi_t)_{t \geq 0}$ is a convolution semigroup of probability measures on $\mathbb{R}^n$. These semigroups can be completely characterized using the Fourier transform. One has, see [7, Section 3.6]

$$\mathcal{F}(T_t u)(\xi) = e^{-t \psi(\xi)} \mathcal{F} u(\xi), \quad t > 0, \ \xi \in \mathbb{R}^n$$

where $\psi : \mathbb{R}^n \to \mathbb{C}$ is a continuous, negative definite function (in the sense of Schoenberg) such that $\psi(0) = 0$. All such $\psi$ are uniquely characterized by their Lévy–Khintchine representation

$$\psi(\xi) = i\ell \cdot \xi + \frac{1}{2} \xi \cdot Q \xi + \int_{y \neq 0} \left( 1 - e^{iy \cdot \xi} + iy \cdot \xi 1_{(0,1)}(|y|) \right) \nu(dy)$$

such that $\ell \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n \times n}$ is positive semidefinite and $\nu$ is a Radon measure on $\mathbb{R}^n \setminus \{0\}$ such that $\int_{y \neq 0} \min\{|y|^2, 1\} \nu(dy) < \infty$. Typical examples are $\psi(\xi) = |\xi|^2$ (leading to the Gauß–Weierstraß semigroup), $\psi(\xi) = |\xi|$ (leading to the Cauchy–Poisson semigroup), $\psi(\xi) = |\xi|^\alpha$, $0 < \alpha < 2$ (leading to the symmetric stable semigroups), but also $\psi(\xi) = \log(1 + |\xi|)$ and many others. These semigroups appear in the study of Lévy processes, see e.g. [8, 7, 9].

It is worth noting that $\psi(\xi)$ can grow at most like $|\xi|^2$ as $|\xi| \to \infty$. Although the multipliers $e^{-t|\xi|^\beta}$, $\beta > 2$, will lead to semigroups, these semigroups are not any longer positivity preserving.
3. Bochner’s subordination

In the paper [4] S. Bochner started to study initial-value problems of the form

\begin{equation}
\begin{cases}
\frac{\partial}{\partial t} u(t, x) = -f(-\Delta)x u(t, x), & t > 0, \ x \in \mathbb{R}^n, \\
u(0, x) = u_0(x), & t = 0, \ x \in \mathbb{R}^n,
\end{cases}
\end{equation}

where \( \Delta_x \) denotes the Laplace operator on \( \mathbb{R}^n \) and \( f : [0, \infty) \to [0, \infty) \) is a Bernstein function (see Theorem 3.1 below). Typical examples are \( \alpha < 0 \)

\[ \frac{\partial}{\partial t} u(t, x) = -f(-\Delta_x)u(t, x), \quad t > 0, \ x \in \mathbb{R}^n, \]

\[ u(0, x) = u_0(x), \quad t = 0, \ x \in \mathbb{R}^n, \]

where \( \Delta_x \) denotes the Laplace operator on \( \mathbb{R}^n \) and \( f \) is a Bernstein function (see Theorem 3.1 below). Typical examples are \( f(\lambda) = \lambda^\alpha, \quad 0 < \alpha < 1 \) or \( f(\lambda) = \sqrt{\lambda + c} - \sqrt{c} \). We may study the problem (3.1) in any of the Banach spaces \( L_p, 1 \leq p < \infty \) or \( C_\infty = \{ u \in C : \lim_{|x| \to \infty} u(x) = 0 \} \); throughout this section we write just \( X \).

From Bochner’s representation theorem for positive definite functions we know that there is a family of probability measures \( (\mu_t^f)_{t \geq 0} \) on \( [0, \infty) \) such that \( \phi_t(\lambda) = \exp(-tf(\lambda)) \) is their Laplace transform:

\[ \int_0^\infty e^{-\lambda r} \mu_t^f(dr) = e^{-tf(\lambda)}, \quad t > 0, \ \lambda \geq 0. \]

Since \( t \mapsto e^{-tf} \) is continuous and satisfies \( e^{-(t+s)f} = e^{-tf} e^{-sf} \), it is clear that \( (\mu_t^f)_{t \geq 0} \) is a semigroup w.r.t. convolution of measures on \( [0, \infty) \) which is vaguely (i.e. in the weak-* sense) continuous in the parameter \( t > 0 \). Notice that all vaguely continuous convolution semigroups are uniquely determined by their exponent \( f \). We may even characterize all such exponents.

**Theorem 3.1** (Schoenberg). A function \( f : (0, \infty) \to (0, \infty) \) such that \( f(0^+) = 0 \) is the characteristic exponent of a vaguely continuous convolution semigroup if, and only if, one of the following equivalent conditions hold

a) \( f \) is a Bernstein function, i.e. \( f \in C_\infty(0, \infty), \ f \geq 0 \) and \( (-1)^{n-1} f^{(n)} \geq 0, \ n \in \mathbb{N} \);

b) \( e^{-tf} \) is for each \( t > 0 \) a positive definite function;

c) \( f \) has the following Lévy–Khintchine representation

\[ f(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda r}) \nu(dr), \quad \lambda > 0, \]

with \( b \geq 0 \) and a measure \( \nu \) on \( (0, \infty) \) such that \( \int_0^\infty \min\{r, 1\} \nu(dr) < \infty \).

This is a standard result, see e.g. [11] Chapter 3 or Jacob [7] Sections 3.9.2–3.9.7. Notice that Bernstein functions are automatically strictly increasing.

Bochner showed that the problem (3.1) is solved by the semigroup

\[ W_t^f u_0(x) := \int_0^\infty W_t u_0(x) \mu_t^f(dr) \]

where \( (W_t)_{t \geq 0}, \ W_t = e^{t\Delta}, \) is the Gauß–Weierstraß semigroup generated by the Laplacian \( \Delta \). The integral appearing in (3.3) is understood in a pointwise sense. Moreover, the family \( (W_t^f)_{t \geq 0} \) inherits many properties of the semigroup \( (W_t)_{t \geq 0} \): it is a semigroup on the same Banach space \( X \) as \( (W_t)_{t \geq 0} \), it is again strongly continuous, contractive, positivity preserving and conservative. The infinitesimal generator of \( (W_t^f)_{t \geq 0} \) is a function of the Laplacian \(-f(-\Delta)) \), e.g. in the sense of spectral calculus, see [11] Chapter 13.
Remark 3.2. The formula (3.3) still makes sense for general strongly continuous contraction semigroups \((T_t)_{t \geq 0}\) on abstract Banach spaces \((\mathcal{X}, \| \cdot \|)\). The resulting subordinate semigroup \((T_t')_{t \geq 0}\) inherits all essential properties of \((T_t)_{t \geq 0}\) such as strong continuity and contractivity and—if applicable—it preserves positivity and is conservative whenever \((T_t)_{t \geq 0}\) is. Using the Lévy–Khintchine representation of \(f\) it is possible to give an explicit formula of the infinitesimal generator of \((T_t')_{t \geq 0}\) as a function of the generator of \((T_t)_{t \geq 0}\), see [11, Theorem 13.6].

Let us return to the Gauß–Weierstraß semigroup. Recall from (2.1) and (2.2) that

\[
\mathcal{F}(W_t u)(\xi) = e^{-t|\xi|^2} \mathcal{F} u(\xi)
\]

and

\[
W_t u(x) = g_t * u(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-\langle x-y \rangle^2/(4t)} u(y) dy
\]

whenever these expressions make sense, e.g. if \(u \in S\) (for the first formula) and \(u \in L_p\) or \(u \geq 0\) and measurable (for the second).

Lemma 3.3. Let \(f : (0, \infty) \to (0, \infty)\) be a Bernstein function. The semigroup \((W_t^f)_{t \geq 0}\) subordinate to the heat semigroup \((W_t)_{t \geq 0}\) satisfies

\[
\mathcal{F}(W_t^f u)(\xi) = e^{-tf(|\xi|^2)} \mathcal{F} u(\xi), \quad t > 0, \ u \in S,
\]

and if \(g_t^f(x) := \int_0^\infty g_r(x) \mu_t^f(dr) = \int_0^\infty (4\pi r)^{-n/2} e^{-x^2/(4r)} \mu_t^f(dr)\) is the generalized heat kernel,

\[
W_t^f u(x) = g_t^f * u(x) = \int_{\mathbb{R}^n} \int_0^\infty (4\pi r)^{-n/2} e^{-\langle x-y \rangle^2/(4t)} u(y) \mu_t^f(dr) dy, \quad t > 0, \ u \in L_p.
\]

Proof. Taking Fourier transforms on both sides of (3.3) with \(u_0 = u \in S\) gives

\[
\mathcal{F}(W_t^f u)(\xi) = \int_0^\infty \mathcal{F}(W_r u)(\xi) \mu_t^f(dr) = \int_0^\infty e^{-r|\xi|^2} \mu_t^f(dr) \mathcal{F} u(\xi) = e^{-tf(|\xi|^2)} \mathcal{F} u(\xi)
\]

where we use Theorem 3.1. The second assertion follows from a similar Fubini-argument. \(\square\)

Example 3.4. Let \(f(\lambda) = f_\alpha(\lambda) = \lambda^\alpha\) for \(\lambda \geq 0\) and \(0 < \alpha < 1\). In this case we write \(W^{(\alpha)}\) and \(g^{(\alpha)}_t\) instead of \(W_t^f\) and \(g_t^f\).

The Lévy–Khintchine representation of \(f_\alpha\) is

\[
\lambda^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1-e^{-\lambda r}) r^{-\alpha-1} dr
\]

and the Fourier transform of the generalized heat kernel is

\[
(2\pi)^{n/2} \mathcal{F} g^{(\alpha)}_t(\xi) = e^{-t|\xi|^{2\alpha}}.
\]

It is obvious, that \(g^{(\alpha)}_t(y)\) is a function, but only for \(\alpha = \frac{1}{2}\) there seems to be a closed representation with elementary functions

\[
g^{(1/2)}_t(x) = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{(n+1)/2} (t^2 + |x|^2)^{(n+1)/2}}.
\]
Remark 3.5. It is possible to associate with every vaguely continuous convolution semigroup of measures \((\mu_t^f)_{t \geq 0}\) on \([0, \infty)\) a random process \((S_t^f)_{t \geq 0}\) such that
\[
P(S_t^f \in A) = \mu_t^f(A), \quad A \in \mathcal{B}[0, \infty).
\]
The processes \((S_t^f)_{t \geq 0}\) are called subordinators. One can show that a subordinator is a random process with stationary and independent increments and right-continuous trajectories \(t \mapsto S_t\) (Lévy process) such that \(S_0 = 0\) and \(t \mapsto S_t\) is increasing. This allows us to write for any bounded or positive Borel function \(g\)
\[
\int_0^\infty g(r) \mu_t^f(dr) \quad \text{as an expected value} \quad \mathbb{E}[g(S_t^f)];
\]
this will be useful later on, in order to calculate certain constants.

If \(f(\lambda) = \lambda^\alpha\), the corresponding process \((S_t^{(\alpha)})_{t \geq 0}\) is usually called an \(\alpha\)-stable subordinator.

4. Fractional caloric smoothing

Let \(s \in \mathbb{R}\) and \(0 < p, q \leq \infty\). Denote by \(A_{p,q}^s\) one of the spaces \(B_{p,q}^s\) or \(F_{p,q}^s\) and write \(\|u \mid A_{p,q}^s\|\) for its (quasi-)norm. As before, \((W_t)_{t \geq 0}\) is the heat semigroup. We have seen in Theorem 2.3 that \(W_t\) is a contraction in the \(B\)-scale if \(s \in \mathbb{R}\), \(1 \leq p \leq \infty, 0 < q \leq \infty\) and in the \(F\)-scale if \(s \in \mathbb{R}\), \(1 \leq p, q \leq \infty\). The following caloric smoothing estimate can be found in [15, Theorem 3.35]: For every \(d \geq 0\) there is a constant \(c > 0\) such that
\[
\|W_t u \mid A_{p,q}^{s+d}\| \leq ct^{-d/2}\|u \mid A_{p,q}^s\| \quad \text{for all } 0 < t \leq 1 \text{ and } u \in A_{p,q}^s.
\]

If we want to prove the analogous result for the semigroup \(W_t^{(\alpha)}\) generated by the fractional Laplacian \(-\Delta)\alpha\), \(0 < \alpha < 1\), it is not clear how to define \(W_t^{(\alpha)}u\) for \(u \in S'\) since \(\xi \mapsto e^{-t|\xi|^{2\alpha}}\) is not smooth at the origin, hence it is no multiplier on \(S'\).

If we can restrict ourselves, however, to \(u \in S\) or \(u \in L_p\), \(W_t^{(\alpha)}u\) is well defined, as it is a convolution semigroup on all spaces \(L_p\), \(1 \leq p \leq \infty\).

Theorem 4.1. Denote by \((W_t^{(\alpha)})_{t \geq 0}\), \(W_t^{(\alpha)} = e^{-t(-\Delta)^\alpha}\) the ‘fractional’ heat semigroup of order \(\alpha \in (0, 1)\) generated by the fractional Laplace operator \(-\Delta)^\alpha\). Let \(s \in \mathbb{R}\) and \(1 \leq p, q < \infty\).

With the constant \(c > 0\) from \([4.1]\) one has for all \(d \geq 0, t > 0\) and \(u \in A_{p,q}^s\)
\[
\|W_t^{(\alpha)}u \mid A_{p,q}^{s+d}\| \leq c \left( t^{-d/(2\alpha)} \frac{\Gamma(1 + d/(2\alpha))}{\Gamma(1 + d/2)} + 1 \right) \|u \mid A_{p,q}^s\|.
\]
In particular, the fractional counterpart of \([4.1]\) holds for some constant \(c' = c'_{p,q,s,\alpha}\)
\[
\|W_t^{(\alpha)}u \mid A_{p,q}^{s+d}\| \leq c't^{-d/(2\alpha)} \|u \mid A_{p,q}^s\|, \quad 0 < t \leq 1.
\]

Proof. Since \(p, q < \infty\), the Schwartz functions \(S\) are dense in \(A_{p,q}^s\). This means that we have to prove \([4.3]\) only for \(u \in S\). Using Bochner’s subordination we can write
\[
W_t^{(\alpha)}u(x) = \int_0^\infty W_r u(x) \mu_t^{(\alpha)}(dr), \quad t > 0.
\]
Since the measures $\mu_t^{(\alpha)}(dr)$ are probability measures, we can use the vector-valued triangle inequality for the norm $\|u | A_{p,q}^s\|$ to deduce

\[
\|W_t^{(\alpha)}u \|_{A_{p,q}^{s+d}} \leq \int_0^1 \|W_r u \|_{A_{p,q}^{s+d}} \mu_t^{(\alpha)}(dr) + \int_1^\infty \|W_r u \|_{A_{p,q}^{s+d}} \mu_t^{(\alpha)}(dr)
\]

\[= \int_0^1 \|W_r u \|_{A_{p,q}^{s+d}} \mu_t^{(\alpha)}(dr) + \int_1^\infty \|W_r^{-1} u \|_{A_{p,q}^{s+d}} \mu_t^{(\alpha)}(dr).\]

Using first (4.1) for both terms (with $t=1$ in the second term), and then Theorem 2.3 for the second term, yields

\[
\|W_t^{(\alpha)}u \|_{A_{p,q}^{s+d}} \leq c \int_0^1 r^{-d/2} \mu_t^{(\alpha)}(dr) : \|u \|_{A_{p,q}^{s}} + c \int_1^\infty \|W_r^{-1} u \|_{A_{p,q}^{s}} \mu_t^{(\alpha)}(dr)
\]

\[\leq c \int_0^1 r^{-d/2} \mu_t^{(\alpha)}(dr) : \|u \|_{A_{p,q}^{s}} + c \int_1^\infty \mu_t^{(\alpha)}(dr) \cdot \|u \|_{A_{p,q}^{s}}.\]

In order to estimate the integral expressions we recall that $\mu_t^{(\alpha)}(dr)$ is the transition semigroup of an $\alpha$-stable subordinator $(S_t^{(\alpha)})_{t \geq 0}$. Therefore,

\[
\int_0^1 r^{-d/2} \mu_t^{(\alpha)}(dr) \leq \int_0^\infty r^{-d/2} \mu_t^{(\alpha)}(dr) = \mathbb{E} \left[ (S_t^{(\alpha)})^{-d/2} \right] = t^{-d/(2\alpha)} \frac{\Gamma(1+d/(2\alpha))}{\Gamma(1+d/2)},
\]

see Lemma 4.3 in the appendix. Since

\[
\int_1^\infty \mu_t^{(\alpha)}(dr) = \mathbb{P}(S_t^{(\alpha)} > 1) \leq 1,
\]

we get (4.2); the estimate (4.3) is now obvious. \qed

Observe that $A_{p,q}^s \subset L_p$ if $s > 0$. If we use in the proof of Theorem 4.1 $u \in L_p$, $1 \leq p \leq \infty$, instead of $u \in S$, we immediately get the following result.

**Corollary 4.2.** Denote by $(W_t^{(\alpha)})_{t \geq 0}$, $W_t^{(\alpha)} = e^{-t(-\Delta)^{\alpha}}$ the ‘fractional’ heat semigroup of order $\alpha \in (0,1)$ generated by the fractional Laplace operator $-(\Delta)^{\alpha}$. The estimates (4.2) and (4.3) of Theorem 4.1 remain valid if $s > 0$ and $1 \leq p, q \leq \infty$.

In order to treat the remaining cases $A_{p,q}^s$ where $s \leq 0$ and $\max\{p, q\} = \infty$ we use a lifting trick; we are grateful to H. Triebel for pointing this out to us (private communication), see also the discussion in [13, p. 104]. Recall that the lifting operator $(1 - \Delta)^{-r/2}$ is a bijection between $A_{p,q}^s$ and $A_{p,q}^{s-r}$ for all $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. On the Schwartz space $S$ the lifting operator and $W_t^{(\alpha)}$ commute,

\[
W_t^{(\alpha)}u = (1 - \Delta)^{-r/2} W_t^{(\alpha)} (1 - \Delta)^{r/2} u \quad \text{for all } u \in S.
\]

Let $s \in \mathbb{R}$ and pick $r$ with $s > r$. The operator $\overline{W}_t^{(\alpha)} := (1 - \Delta)^{-r/2} W_t^{(\alpha)} (1 - \Delta)^{r/2}$ is well-defined on $A_{p,q}^s$, extends $W_t^{(\alpha)}$ and makes the following diagram commutative:

\[
\begin{array}{ccc}
A_{p,q}^s & \xrightarrow{(1-\Delta)^{-r/2}} & A_{p,q}^{s-r} \\
W_t^{(\alpha)} \downarrow & & \downarrow w_t^{(\alpha)} \\
A_{p,q}^{s+d} & \xleftarrow{(1-\Delta)^{-r/2}} & A_{p,q}^{s-r+d}
\end{array}
\]
It is not hard to see that, for any fixed \( s \in \mathbb{R} \), the extension \( \overline{W}_t^{(\alpha)} \) onto \( A_{p,q}^s \) does not depend on \( r < s \), i.e. we may understand \( \overline{W}_t^{(\alpha)} \) as an operator on \( A_{p,q}^{-\infty} := \bigcup_{s \in \mathbb{R}} A_{p,q}^s \). Together with the previous considerations we get

**Corollary 4.3.** Denote by \( (\overline{W}_t^{(\alpha)})_{t \geq 0} \) the ‘extension by lifting’ of the fractional heat semigroup \( W_t^{(\alpha)} = e^{-t(-\Delta)^{\alpha}} \) of order \( \alpha \in (0, 1) \). The estimates (4.2) and (4.3) of Theorem 4.1 remain valid for \( \overline{W}_t^{(\alpha)} \) for all \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \).

If \( s > 0 \) and \( 1 \leq p, q \leq \infty \) or \( s \in \mathbb{R} \) and \( 1 \leq p, q < \infty \), these estimates are true for the original semigroup operators \( W_t^{(\alpha)} \).

## 5. Two Extensions of the Subordination Technique

The subordination technique which we have developed in the previous Section 4 can be extended into two directions: (i) We may give up the concept of fractional powers in favour of general Bernstein functions, or (ii) we may look at higher-order ‘fractional’ semigroups \( W_t^{(\beta)} \) where \( \beta > 0 \).

The extension from fractional powers \( \lambda \mapsto \lambda^\alpha \) to arbitrary Bernstein functions \( \lambda \mapsto f(\lambda) \), see Section 3, is straightforward. Using general subordinate semigroups \( (W_t^f)_{t \geq 0} \) instead of the fractional heat semigroup \( (W_t^{(\alpha)})_{t \geq 0} \), the arguments of Section 4 go through almost literally. As before, \( \overline{W}_t^f \) denotes the ‘extension by lifting’ of \( W_t^f \). Note that \( \overline{W}_t^f = W_t^f \) if \( \xi \mapsto f(|\xi|^2) \) is smooth at the origin. Typical examples are the ‘relativistic’ semigroups of the form \( f(\lambda) = (\lambda + 1)^\alpha - 1 \) for \( 0 < \alpha < 1 \).

**Theorem 5.1.** Let \( (W_t)_{t \geq 0} \) be as in Lemma 4.1, let \( f \) be a Bernstein function, \( (S_t^f)_{t \geq 0} \) the corresponding subordinator, and denote by \( (W_t^f)_{t \geq 0} \) the subordinate semigroup extended by lifting. For the constant \( c = c_{p,q,s} \) appearing in (4.1) and \( s \in \mathbb{R} \), \( 1 \leq p, q \leq \infty \) and \( d \geq 0 \) we have

\[
(5.1) \quad \|\overline{W}_t^f u \|_{A_{p,q}^{s+d}} \leq c E \left( (S_t^f)^{-d/2} \right) + P(S_t^f > 1) \|u\|_{A_{p,q}^s}, \quad t > 0.
\]

In particular, there exists some constant \( c' = c'_{p,q,s} \) such that for \( d > 0 \)

\[
(5.2) \quad \|\overline{W}_t^f u \|_{A_{p,q}^{s+d}} \leq c' E \left( (S_t^f)^{-d/2} \right) \|u\|_{A_{p,q}^s}, \quad 0 < t \leq 1.
\]

If \( s \geq 0 \) and \( 1 \leq p, q \leq \infty \) or \( s \in \mathbb{R} \) and \( 1 \leq p, q < \infty \), these estimates remain valid for the non-extended semigroup \( (W_t^f)_{t \geq 0} \).

**Proof.** The estimate (5.1) follows just as in the proof of Lemma 4.1. In order to see (5.2) observe that by monotone convergence and the fact that \( S_0^f = 0 \)

\[
\lim_{t \to 0} E \left( (S_t^f)^{-d/2} \right) = \infty \quad \text{and, trivially,} \quad P(S_t^f > 1) \leq 1.
\]

Using Lemma 5.2 we can control the growth of the expectation appearing in (5.2).

**Corollary 5.2.** If, in the setting of Corollary 5.1, the Bernstein function \( f \) satisfies \( \lim_{\lambda \to 0} f(2\lambda)/f(\lambda) > 1 \), there is some constant \( C' = C'_{p,q,s} \) such that

\[
(5.3) \quad [f^{-1}(1/t)]^{-d/2} \cdot \|\overline{W}_t^f u \|_{A_{p,q}^{s+d}} \leq C' \|u\|_{A_{p,q}^s}, \quad 0 < t \leq 1.
\]
Remark 5.3. Bochner’s subordination is an abstract technique that works in all Banach spaces. The essential ingredient in the proof of Theorem 4.1 is the generalized triangle inequality which allows us to estimate the norm of an integral \(\| \int \ldots \|\) by the integral of the norm \(\int \| \ldots \|\). This shows that our results can be extended to (tempered) homogeneous spaces of the form \(A^{s}_{p,q}(\mathbb{R}^n)\) as well as hybrid spaces \(A^{\tau}_{p,q}(\mathbb{R}^n) = L^{r}A^{s}_{p,q}\), \(\tau = p^{-1} + rn^{-1}\). The admissible parameters should be \(p, q \in [1, \infty), s \in \mathbb{R}\) and \(-np^{-1} \leq r < \infty\). As standard reference of these spaces we refer to [15] Section 4.1, Section 1.1.2] and the literature given there.

Let us now discuss higher-order generalized heat equations. In a series of papers, Baaske & Schmeißer [1, 2, 3] studied semigroups \((W^{(m)}_{t})_{t \geq 0}, m \in \mathbb{N}\), which are defined via

\[
\mathcal{F}W^{(m)}_{t}(u)(\xi) := e^{-t|\xi|^{2m}}\mathcal{F}u(\xi), \quad u \in S, \xi \in \mathbb{R}^n, t > 0.
\]

It is clear that \((W^{(m)}_{t})_{t \geq 0}\) is a semigroup which is given by a convolution kernel, \(W^{(m)}_{t}u = K_{tm} * u\), but while \(K_{tm}(x) = (2\pi)^{-n/2}e^{-|x|^2}e^{-|\xi|^{2m}}\) is from \(S\), it may have arbitrary sign; in particular, \(W^{(m)}_{t}\) is a uniformly bounded semigroup on \(L^p\), \(1 \leq p < \infty\), but it is not positivity preserving. This means, in particular, that there is no Markov process which has \(W^{(m)}_{t}\) as a transition semigroup. Nevertheless, Bochner’s subordination formula (3.3) is still applicable; if we use \(f(\lambda) = \lambda^{\alpha}\) for some \(\alpha \in (0, 1)\), we get a (in general, not positivity preserving) subordinate semigroup \((W^{(\alpha)}_{t})_{t \geq 0}\). The calculation used in the proof of Lemma 5.3 shows that

\[
\mathcal{F}W^{(\alpha)}_{t}(u)(\xi) = e^{-t|\xi|^{2\alpha}}\mathcal{F}u(\xi) = \mathcal{F}W^{(\alpha m)}_{t}(u)(\xi) \quad \text{for all } 0 < \alpha < 1, m \in \mathbb{N}.
\]

A key result of Baaske & Schmeißer [1] Theorem 3.5] is the following caloric smoothing estimate for the operators \((W^{(m)}_{t})_{t \geq 0}\): Let \(1 \leq p, q \leq \infty (p < \infty\) for the F-scale), \(s \in \mathbb{R}, d \geq 0\) and \(m \in \mathbb{N}\). There is a constant \(c > 0\) such that

\[
||W^{(m)}_{t}u||_{A^{s+d}_{p,q}} \leq ct^{-d/(2m)}||u||_{A^{s}_{p,q}} \quad \text{for all } t \in (0, 1).
\]

If we use (5.1) instead of (4.1) and write \(\beta := \alpha m\), we get immediately the following corollary to Theorem 4.1.

Corollary 5.4. Denote by \((W^{(\beta)}_{t})_{t \geq 0}, W^{(\beta)}_{t} = e^{-t(-\Delta)^{\beta}}\) the generalized ‘fractional’ heat semigroup of order \(\beta > 0\) generated by the higher-order fractional Laplace operator \(-(-\Delta)^{\beta}\). Let \(s \in \mathbb{R}\) and \(1 \leq p, q < \infty\).

With the constant \(c > 0\) from (5.1) one has for every \(d \geq 0\), \(t > 0\) and \(u \in A^{s}_{p,q}\)

\[
||W^{(\beta)}_{t}u||_{A^{s+d}_{p,q}} \leq c\left(t^{-d/(2\beta)}\Gamma\left(1 + d/(2\beta)\right)/\Gamma\left(1 + d/2\right) + 1\right)||u||_{A^{s}_{p,q}}.
\]

In particular, the fractional counterpart of (4.1) holds for some constant \(c' = c'_{p,q,s,\alpha}\)

\[
||W^{(\beta)}_{t}u||_{A^{s+d}_{p,q}} \leq c't^{-d/(2\beta)}||u||_{A^{s}_{p,q}}, \quad 0 < t \leq 1.
\]

The cases \(p = \infty, 1 \leq q < \infty\) (for the F-scale) and \((p,q) = (\infty, \infty)\) (for the B-scale) are special and require the ‘extension by lifting’ \(\tilde{W}_{t}^{(\beta)}\) explained at the end of Section 5. The analogues of (5.5) and (5.6) should be clear. If \(\xi \mapsto |\xi|^{2\beta}\) is smooth, i.e. if \(\beta \in \mathbb{N}\), there is no need for an extension. At the moment, there is no subordination version for the spaces \(F_{p,\infty}\), since in these cases (5.4) is yet unknown.
6. An application of the caloric smoothing estimate

The result (5.3) was used in [1] to prove the existence and uniqueness of a mild solution to the non-linear equation

\[ \partial_t u(x, t) + (-\Delta_x)^m u(x, t) = \text{div}[u^2](x, t), \quad x \in \mathbb{R}^n, \quad t \in (0, T) \]

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n, \]

where \( \text{div}[u^2] = \sum_{i=1}^n \frac{\partial}{\partial x_i} u^2 \) is the divergence, \( \Delta_x \) the Laplacian and \( m \in \mathbb{N} \). A mild solution is an element \( u \in S'((\mathbb{R}^{n+1}) \), which is a fixed point for the operator

\[ Q^{(m)} u(x, t) = W_t^{(m)} u_0(x) + \int_0^t W_{t-\tau}^{(m)} (\text{div}[u^2]) (x, \tau) d\tau, \quad x \in \mathbb{R}^n, \quad t \in (0, T) \]

in the space

\[ L_a \{ (0, T), b, A_{p,q}^s \} := \left\{ u : (0, T) \to A_{p,q}^s; \int_0^T t^a \| u(\cdot, t) \|_{A_{p,q}^s}^s \ dt < \infty \right\} \]

with some \( a, b > 0 \) (and the usual modification of the norm if \( a = \infty \)). A solution is called strong, if it is mild and if for any initial value \( u_0 \in A_{p,q}^{s_0} \) it belongs to \( C \left( [0, T), A_{p,q}^{s_0} \right) \) for some \( s_0 \). For suitable parameters \( a, b, p, q, s \), a mild solution will be a strong solution, see [1] Theorem 3.8.(ii).

The caloric estimate (5.4) was used in the proof of the existence of the mild solution, in order to show the contractivity of \( Q^{(m)} \) and to apply a fixed point argument. Corollary 5.4 enables us to follow the same procedure for the fractional equation

\[ \partial_t u(x, t) + (-\Delta_x)^\beta u(x, t) = \text{div}[u^2](x, t), \quad x \in \mathbb{R}^n, \quad t \in (0, T) \]

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n, \]

where \( \beta = m\alpha \) where \( m = 1, 2, 3, \ldots \) and \( \alpha \in (0, 1) \); the solution is understood as an element of the space \( A_{p,q}^{s-\infty} = \bigcup_{s \in \mathbb{R}} A_{p,q}^s \). To do so, we extend the notion of a mild solution in the following way: \( u(x, t) \) is a mild solution if \( u(\cdot, t) \in A_{p,q}^{-\infty}, \quad u(x, \cdot) \in C(0, T), \) and \( u \) is a fixed point of \( Q^{(\beta)} \). Note that \( Q^{(\beta)} \) corresponds to the semigroup \( W_t^{(m, \alpha)} = W_t^{(m)\alpha}, \quad t \geq 0 \), obtained by subordination from \( (W_t^{(m)})_{t \geq 0} \).

Corollary 5.4 allows us to extend the result of Baaske & Schmeiser from \( m \in \mathbb{N} \) to all real \( \beta \geq 1 \). We state this result without proof; the proof of [1] Theorem 3.8 transfers literally to the new situation. The only change is at the very end of the proof in [1] Eq. (3.79). Here we establish the continuity first for \( u_0 \in S(\mathbb{R}^n) \) and argue then by density. Notice that \( \| W_t^{(\beta)} u \|_{A_{p,q}^{s-\beta+\delta}} \leq c \| u \|_{A_{p,q}^{s-\beta+\delta}} \) by (5.3) with \( d = 0 \) and \( s \to s - \beta + \delta \) for all \( u \in S(\mathbb{R}^n) \) with a uniform constant \( c \). This is necessary since \( e^{-|t|\beta} \) is, in general, not a multiplier on \( S(\mathbb{R}^n) \). The restriction \( \beta \geq 1 \) is needed in the proof of the contraction property [1] proof of Theorem 3.8, Step 1], while all other steps do work for \( \beta > 0 \).

**Theorem 6.1.** Let \( n \geq 2, \beta \in [1, \infty) \), \( 1 \leq p, q \leq \infty \) (for the \( F \)-scale) and \( s \in \mathbb{R} \) is such that \( A_{p,q}^s(\mathbb{R}^n) \) is a multiplication algebra. Let

\[ a = \beta - \frac{1}{v} - \beta \lambda, \quad \text{where} \quad \frac{2}{\beta} < v \leq \infty, \quad 0 < \lambda < \epsilon \leq 1, \]
and $u_0 \in A_{p,q}^{\alpha-\beta,\beta}(\mathbb{R}^n)$ be the initial data. There exists some $T > 0$ such that (6.2) has a unique mild solution

$$u \in L_{2\beta_v}((0,T), \frac{\partial}{\partial t}, A_{p,q}^{\alpha}(\mathbb{R}^n)) \cap C^\infty((0,T) \times \mathbb{R}^n).$$

The mild solution is a strong solution if, in addition, $p,q < \infty$ and $\frac{1}{2} \epsilon \leq \lambda < \epsilon \leq 1$ (if $v < \infty$), resp., $\frac{1}{2} \epsilon < \lambda < \epsilon \leq 1$ (if $v = \infty$).

**7. Appendix – Some moment estimates**

We need the following moment estimate for $\alpha$-stable subordinators. Although the result is well-known, see e.g. Sato [8, Eq. (25.5), p. 162], we include the proof for our readers’ convenience. The short argument given below seems to be new.

**Lemma 7.1.** Let $(S_t^{(\alpha)})_{t \geq 0}$ be a stable subordinator with Bernstein function $f(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$, and transition semigroup $(\mu_t^{(\alpha)})_{t \geq 0}$. The moments $E\left[(S_t^{(\alpha)})^\kappa\right]$ exist for any $\kappa \in (-\infty, \alpha)$ and $t > 0$. Moreover,

$$E\left[(S_t^{(\alpha)})^\kappa\right] = \frac{\Gamma(1 - \frac{\kappa}{\alpha})}{\Gamma(1 - \kappa)} t^{\kappa}, \quad t > 0.$$  

**Proof.** In this proof we write $S_t$ and $\mu_t$ instead of $S_t^{(\alpha)}$ and $\mu_t^{(\alpha)}$. Recall that the Laplace transform of $S_t$ is $E e^{-xS_t} = \int_0^\infty e^{-x \mu_t} ds = e^{-tx^\alpha}$, $x, t > 0$. Substituting $\lambda = S_t$ in the well-known formula [11] p. vii]

$$\lambda^{-r} = \frac{1}{\Gamma(r)} \int_0^\infty e^{-\lambda x} x^{r-1} dx, \quad \lambda > 0, \quad r > 0,$$

and taking expectations yields, because of Tonelli’s theorem,

$$E S_t^{-r} = \frac{1}{\Gamma(r)} \int_0^\infty E e^{-xS_t} x^{r-1} dx = \frac{1}{\Gamma(r)} \int_0^\infty e^{-tx^\alpha} x^{r-1} dx,$$

Now we change variables according to $y = tx^\alpha$, and get

$$E S_t^{-r} = \frac{1}{\Gamma(r)} \frac{1}{\alpha} t^{-\frac{r}{\alpha}} \int_0^\infty e^{-y} y^{\frac{r}{\alpha}} dy = t^{-\frac{r}{\alpha}} \frac{1}{\Gamma(r)} \Gamma\left(\frac{r}{\alpha}\right) = t^{-\frac{r}{\alpha}} \frac{\Gamma\left(1 + \frac{r}{\alpha}\right)}{\Gamma(1 + r)}.$$  

Setting $\kappa = -r$ proves the assertion for $\kappa \in (-\infty, 0)$. This formula extends (analytically) to $-r = \kappa < \alpha$. Alternatively, we use a similar calculation and the Lévy–Khintchine formula from Example [3.4]

$$\lambda^{\kappa} = \frac{r}{\Gamma(1 - r)} \int_0^\infty (1 - e^{-\lambda x}) x^{-r-1} dx, \quad \lambda > 0, \quad r \in (0,1),$$

to get the assertion for $\kappa \in (0, \alpha)$. \hfill $\square$

The upper bound of the following lemma appears in the proof of [6, Theorem 2.1].

**Lemma 7.2.** Let $(S_t^f)_{t \geq 0}$ be a subordinator with Bernstein function $f$ and transition semigroup $(\mu_t^f)_{t \geq 0}$. Assume that $\lim_{\lambda \to \infty} f(\lambda) = \infty$ and that the inverse of $f$ satisfies $\limsup_{t \to \infty} f^{-1}(2t)/f^{-1}(t) < \infty$, (one can show, cf. [6, Lemma 2.2]), that these two conditions are equivalent to $\lim inf_{\lambda \to 0} f(2\lambda)/f(\lambda) > 1$, respectively.

Under these assumptions, the moments $E\left[(S_t^f)^{-r}\right]$ exist for any $r > 0$ and $t > 0$. Moreover,

$$\frac{1}{3\Gamma(1+r)} [f^{-1}\left(\frac{1}{t}\right)]^r \leq E\left[(S_t^f)^{-r}\right] \leq C \frac{1}{\Gamma(1+r)} [f^{-1}\left(\frac{1}{t}\right)]^r, \quad 0 < t \leq 1.$$
Proof. We write $S_t$ and $\mu_t$ instead of $S^f_t$ and $\mu^f_t$. Using the argument from the proof of Lemma 7.1, we get the following analogue of (7.1)

$$E \left[ (S_t)^{-r} \right] = \frac{1}{\Gamma(r)} \int_0^\infty e^{-tf(x)} x^r \frac{dx}{x}.$$ 

Changing variables according to $y = f(x)$—observe that $f$ is strictly monotone, $f(0) = 0$ and $f(\infty) = \infty$—and using the fact that $f^{-1}(2y) \leq cf^{-1}(y)$ for, say, $y \geq 1$, we get for all $t \in (0, 1)$

$$E \left[ (S_t)^{-r} \right] = \frac{1}{r \Gamma(r)} \left( \int_0^{1/t} + \sum_{n=0}^{\infty} \int_{2^n/t}^{2^{n+1}/t} \right) e^{-ty} dy [f^{-1}(y)]^r.$$

Since $r \Gamma(r) = \Gamma(r + 1)$, this implies

$$e^{-1} [f^{-1}(1/t)]^r \leq \Gamma(r + 1) E \left[ (S_t)^{-r} \right] \leq [f^{-1}(1/t)]^r + \sum_{n=0}^{\infty} e^{-2^n} [f^{-1}(2^{n+1}/t)]^r$$

$$\leq \left( 1 + \sum_{n=0}^{\infty} e^{-2^n} e^{(n+1)r} \right) [f^{-1}(1/t)]^r. \quad \Box$$

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Note added in proof: Meanwhile, the question just before Section 6, on the validity of (5.4) for the spaces $F_{p, \infty}$ has been affirmatively answered, see F. Kühn and R.L. Schilling: Convolution inequalities for Besov and Triebel-Lizorkin spaces, and applications to convolution semigroups. Studia Math. 262 (2022) 93–119.

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