Two-dimensional superintegrable mappings and integrable hierarchies in the \((2|2)\) superspace

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Abstract

The formalism of integrable mappings is applied to the problem of constructing hierarchies of \((1+2)\) dimensional integrable systems in the \((2|2)\) superspace. We find new supersymmetric integrable mappings and corresponding to them new hierarchies of integrable systems which, at the reduction to the \((1|2)\) superspace, possess \(N = 2\) supersymmetry. The general formulae obtained for the hierarchies are used to explicitly derive their first nontrivial equations possessing a manifest \(N = 2\) supersymmetry. New bosonic substitutions and hierarchies are obtained from the supersymmetric counterparts in the bosonic limit.

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1 Introduction

The method of discrete substitutions is one of the shortest and most direct ways of describing and solving equations of integrable systems [1], [2].

The integrable mappings and their properties play the key role in this approach. All other ingredients of the theory, such as the explicit form of integrable hierarchies and the solution of the corresponding systems of equations, are a direct corollary of the representation theory of the discrete group of integrable mappings.

The discrete substitution is the same for each system of equations in the given integrable hierarchy and thus contains detailed information about all of the sets of integrable systems belonging to it.

Of course, it is necessary to keep in mind that up to now there is no rigorous, from a mathematical point of view, classification theory of discrete substitutions themselves nor a representation theory of the group of integrable mappings. Nevertheless, if under some other consideration it is possible to obtain an explicit form of a discrete substitution, all other corollaries may be obtained by straightforward calculations.

The algorithm of these calculations is very simple and resembles a computer program: it is necessary to perform many identical operations that can be interrupted at an arbitrary step and thus obtain relevant information about some system of equations belonging to the hierarchy.

The goal of the present letter is to apply this approach to the problem of constructing two-dimensional supersymmetric hierarchies of integrable systems. We demonstrate this approach by supersymmetric integrable mappings connected with the two-dimensional supersymmetric Toda lattice. It is also possible to consider this construction as one of the possible generalizations of the Darboux transformation to the supersymmetric case.

2 Supersymmetric substitutions in the (2|2) superspace

Below, we briefly discuss the main points of the discrete transformational approach, taking as the simplest examples, the integrable supersymmetric substitutions related to the supersymmetric Toda lattice in two dimensions.

It is well known that under appropriate boundary conditions, the supersymmetric Toda lattice

\[ D_- D_+ \ln b = \vec{b} - \vec{b} \]  

is an exactly integrable system (see e.g., [3] and references therein). Here \( b \) is the bosonic superfield \( b = b^0 + \theta_+ b^- + \theta_- b^+ + \theta_+ \theta_- b^2; \ b^0, b^2 (b^-, b^+) \) are bosonic (fermionic) fields, and \( \theta_+, \theta_- \) are elements of the Grassman algebra. The \( D_\pm \) are the \( N = 1 \) supersymmetric fermionic covariant derivatives

\[ D_+ = \frac{\partial}{\partial \theta_+} + \theta_+ \partial_x, \quad D_- = \frac{\partial}{\partial \theta_-} + \theta_- \partial_y, \quad D_+^2 = \partial_x, \quad D_-^2 = \partial_y, \quad \{D_+, D_-\} = 0, \]  

where \( x \) and \( y \) are two independent space coordinates; the notation \( \vec{b} (\vec{b}) \) means that the index of variable \( b \) is shifted by +1 (−1) (for instance, \( \vec{b}_n \equiv b_{n+1} \), and so on).

Now let us introduce the fermionic \( f \), or bosonic \( \bar{b} \), superfields by the following chain of relations:

\[ \vec{f} = f + D_+ \ln \vec{b} \]  

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and using (2.3) or (2.4), identically rewrite the set of Toda lattice equations (2.1) as
\[ \tilde{b} = D_- f - \tilde{b}, \quad \tilde{f} = f + D_+ \ln \tilde{b} \] (2.5)
or
\[ \tilde{b} = b + D_- D_+ \ln \bar{b}, \quad \tilde{b} = \bar{b}, \] (2.6)
respectively. The relation (2.5) ((2.6)) can be considered as a definition of some mapping: it is a rule used to associate two arbitrary initial functions \(f, b(\bar{b}, \tilde{b})\) with the final ones \(\tilde{f}, \tilde{b}(\bar{b}, \tilde{b})\). It is not difficult to check that the mappings\(^1\) (2.5) and (2.6) are invertible and the inverse transformations have the form
\[ \tilde{b} = D_- \tilde{f} - b, \quad \tilde{f} = f - D_+ \ln b \] (2.7)
or
\[ \bar{b} = \tilde{b} + D_- D_+ \ln \tilde{b}, \quad b = \bar{b}. \] (2.8)
Substitutions (2.5) and (2.6) possess the inner automorphisms \(\sigma\) with the properties
\[ \sigma = \sigma_2 \sigma_1, \quad \sigma_1 b \sigma_1^{-1} = -b, \quad \sigma_1 f \sigma_1^{-1} = D_+ D_- f, \quad \sigma_1 D_+ \sigma_1^{-1} = D_+, \quad \sigma_1 D_- \sigma_1^{-1} = D_-, \]
\[ \sigma_2 b \sigma_2^{-1} = b, \quad \sigma_2 f \sigma_2^{-1} = f, \quad \sigma_2 D_+ \sigma_2^{-1} = D_-, \quad \sigma_2 D_- \sigma_2^{-1} = D_+; \] (2.9)
\[ \sigma b \sigma^{-1} = -b, \quad \sigma b \sigma^{-1} = \bar{b}, \quad \sigma D_+ \sigma^{-1} = D_-, \quad \sigma D_- \sigma^{-1} = D_+, \] (2.10)
respectively, which will be useful in what follows. The action of \(\sigma\) on the covariant derivatives \(D_{\pm}\) can be induced by the following transformation of the \(2|2\) superspace coordinates
\[ \sigma x \sigma^{-1} = y, \quad \sigma y \sigma^{-1} = x, \quad \sigma \theta_+ \sigma^{-1} = \theta_-, \quad \sigma \theta_- \sigma^{-1} = \theta_+. \] (2.11)

The next substitution connected with the super Toda lattice (2.1) is the literal generalization of the Darboux transformation (direct and inverse) to the supersymmetric case
\[ \tilde{u} = \frac{1}{v}, \quad \tilde{v} = v(D_- D_+ \ln v - uv); \]
\[ \bar{v} = \frac{1}{u}, \quad \bar{u} = -u(D_- D_+ \ln u + uv), \] (2.12)
where \(u\) and \(v\) are bosonic superfields. Substitution (2.12) possess the global \(U(1)\)-invariance. With respect to the \(U(1)\) group, the superfields \(u\) and \(v\) have opposite \(U(1)\)-charges. From (2.12), it immediately follows that the function \(T_0 = uv\) satisfies the equation for the supersymmetric Toda lattice (2.1):
\[ D_- D_+ \ln T_0 = \tilde{T}_0 - \bar{T}_0. \] (2.13)

The most important object for investigations closely connected with the substitution, and directly following from it, is the symmetry equation. Following [1], [2], let us recall that the

\(^1\)We also call them substitutions.
symmetry equation for a given substitution can be obtained by differentiation of the substitution with respect to an arbitrary independent argument, or parameter. Denoting a derivative from all functions involved in the substitution by new letters and considering them as independent functionals whose arguments are the above-mentioned functions and their superspace derivatives, one can find a set of equations for these functionals which are called the symmetry equations. If a symmetry equation possesses nontrivial solutions and it is possible to construct them, then one can produce an evolution-type system of integrable equations by a simple algorithmic procedure using only these solutions. Substitutions of this kind have been called integrable (for details, see [1], [2]).

In the case under consideration, the independent arguments of a substitution are the coordinates of the (2|2) superspace. Thus, as different from bosonic coordinate space, in the case of superspace, it is possible to differentiate with respect to its even or odd coordinates (parameters). As a consequence, knowledge of the symmetry equation solutions allows us to produce the evolution-integrable equations with either even or odd evolution parameters which, nevertheless, belong to the same integrable hierarchy. Here we discuss only the former case.

To illustrate the above-general discussion, for definiteness, we restrict ourselves to a concrete example of the substitution (2.5); however, at the end of the next section, we present the results of calculations for all of the substitutions discussed in this section.

Using the general rules described above, we obtain the following symmetry equation corresponding to the substitution (2.5):

\[
\begin{align*}
\tilde{B} &= D_F - B, \\
\tilde{F} &= F + D_+ (\frac{\tilde{B}}{b}).
\end{align*}
\]  

(2.14)

Here, \(F\) and \(B\) are fermionic and bosonic functionals whose independent arguments are \(f, b, D_\pm f, D_\pm b, b_x, b_y, f_x, f_y, \ldots \). The functionals \(\tilde{B}, \tilde{B}, \tilde{F}, \tilde{F}\) are the same functionals whose arguments are shifted \(s\)-times in the direct or inverse direction and are connected with the initial arguments \(b, f\) by relations (2.3) and (2.7).

By construction, the pair \(F = f', B = b'\) (the sign ‘ means differentiation with respect to each of the independent bosonic arguments of the problem) is a solution of eq. (2.14). This solution was called a trivial one in [1], [2] and, in this sense, this term has been used above.

Each solution of the symmetry equation (2.14) is connected with the evolution-type integrable system

\[
\begin{align*}
b_t &= B, \\
f_t &= F.
\end{align*}
\]  

(2.15)

Moreover, the last one is invariant with respect to the discrete transformation of the substitution (2.3), and the symmetry equation (2.14) is exactly the condition of this invariance.

The hierarchies of integrable systems are encoded in integrable substitutions and can be explicitly obtained by solving their symmetry equations. In the next section, we give an infinite set of partial solutions of the symmetry equation for substitutions (2.3), (2.6), and (2.12).

We would like to close this section with a few remarks.

First, all substitutions in the (2|2) superspace can be reduced to the (1|2) superspace. In this case, all of the considered functions must have a dependence on only three arguments: \(x + y, \theta_+\), and \(\theta_-\). At this reduction, the substitutions (2.3) and (2.12) become \(N = 2\) supersymmetric substitutions. This statement becomes evident if one takes into account that the fermionic covariant derivatives \(D_{\pm}\) enter (2.6) and (2.12) only in the \(N = 2\) superinvariant combination \(D_- D_+\) (for details, see section 4). It is easily to understand that the same statement is correct
with respect to the substitution (2.5) after introducing the new bosonic superfield \( \hat{b} \) by the relation
\[
f = D_+ \hat{b}.
\]
Indeed, in terms of the superfields \( b, \hat{b} \), the substitution (2.5) has the following desirable form:
\[
\hat{b} = D_- D_+ b - b, \quad \hat{b} = \hat{b} + \ln \hat{b}
\]
(2.17)
up to the possible unessential constant on the right hand side of the second relation. It is interesting to note that up to this constant, the substitution (2.17) possesses the local inner automorphism \( \sigma \)
\[
\sigma b \sigma^{-1} = -b, \quad \hat{b} \sigma^{-1} = \hat{b}, \quad \sigma D_+ \sigma^{-1} = D_-, \quad \sigma D_- \sigma^{-1} = D_+,
\]
(2.18)
as different from substitution (2.5), possessing the nonlocal automorphism (2.9).

Second, the \( N = 2 \) superinvariance of the substitutions (2.6), (2.12) and (2.17) guarantees the same invariance for the all integrable hierarchies related to these substitutions.

### 3 Solution of the symmetry equation

In the case of usual space, the problem of the title of this section was solved in [4]. An algorithm for recurrent calculations has been proposed which allows one to pass step by step. This process may be interrupted at an arbitrary step for which at least one obvious simple solution always exists. Here, we generalize these calculations to the case of superspace. One essential difference, compared to the case of usual space, consists of the fact that in superspace, the recurrent procedure can be interrupted only at an even step but not at an odd one.

For definiteness, we restrict ourselves to the substitution (2.3).

From the symmetry equation (2.14), it immediately follows that an unknown bosonic function may be represented in the form
\[
\frac{B}{b} = \alpha_0 - \alpha_0^r,
\]
(3.1)
and a new unknown bosonic function \( \alpha_0 \) is the solution of the equation
\[
D_- D_+ \alpha_0 = \hat{b} (\hat{\alpha}_0 - \alpha_0) + b (\alpha_0 - \alpha_0^r).
\]
(3.2)

In terms of the solution to eq.(3.2), the evolution-type integrable system (2.15) can be rewritten as
\[
b_t = b (\alpha_0 - \alpha_0^r), \quad f_t = D^{-1}_- \hat{b} (\hat{\alpha}_0 - \alpha_0) + b (\alpha_0 - \alpha_0^r)].
\]
(3.3)

Now let us describe the recurrent steps for the solution of the symmetry equation (3.2), linear with respect to an unknown function \( \alpha_0 \).

The symmetry equation, together with the equation for the bosonic function \( b \) (which arises after excluding the fermionic function \( f \) from the system (2.5)), may be rewritten in an equivalent form suitable for further calculations:
\[
D_+ \alpha_0 = D^{-1}_- [\hat{b} (\hat{\alpha}_0 - \alpha_0) + b (\alpha_0 - \alpha_0^r)], \quad D_+ b = b D^{-1}_- (\hat{b} - \hat{b}).
\]
(3.4)
Simple inspection of the eqs. (3.4) shows that they possess the inner automorphism \( \sigma \) with the properties

\[
\sigma \alpha_0 \sigma^{-1} = \eta_0, \quad \sigma b \sigma^{-1} = -b, \quad \sigma D_+ \sigma^{-1} = D_-, \quad \sigma D_- \sigma^{-1} = D_+,
\]

where \( \eta_0 \) is another solution of the eq. (3.2). We use this automorphism to construct one-parametric family of solutions of the eq. (3.2) (see the end of this section).

We present below a series of straightforward, simple transformations with short comments. First, let us introduce the new fermionic function \( \tilde{\alpha}_0 \):

\[
\alpha_0 = D_-^{-1} \tilde{\alpha}_0.
\]

Keeping in mind that the operator \( D_- \) is odd and the corresponding rules of working with such objects, we come to the equation for \( \tilde{\alpha}_0 \):

\[
-D_+ \tilde{\alpha}_0 = \tilde{b}D_-^{-1}(\tilde{\alpha}_0 - \tilde{\alpha}_0) + bD_-^{-1}(\tilde{\alpha}_0 - \tilde{\alpha}_0).
\]

Second, we use the following substitution:

\[
\tilde{\alpha}_0 = \tilde{b} \alpha_1 + b \beta_1,
\]

and, as a result, we get the corresponding system of equations for the unknown functions \( \alpha_1, \beta_1, \)

\[
-D_+ \alpha_1 + \alpha_1 D_-^{-1}(\tilde{b} - b) = D_-^{-1}(\tilde{\alpha}_0 - \tilde{\alpha}_0),
\]

\[
-D_+ \beta_1 + \beta_1 D_-^{-1}(\tilde{b} - b) = D_-^{-1}(\tilde{\alpha}_0 - \tilde{\alpha}_0),
\]

after equating the coefficient functions to zero at the \( \tilde{b}, b \) terms. Let us stress that this is an additional independent assumption. Subtracting the second equation, shifted by one step to the left, from the first, we obtain the following equation:

\[
-D_+(\alpha_1 - \beta_1) + (\alpha_1 - \beta_1)D_-^{-1}(\tilde{b} - b) = 0.
\]

From the last equation, we see that the system (3.7) possesses a partial solution for which \( \alpha_1 = \beta_1 \). In what follows, we work exactly with a solution of this kind, and (3.7), in this case, is equivalent to a single equation for the unknown function \( \alpha_1 \):

\[
-D_+ \alpha_1 + \alpha_1 D_-^{-1}(\tilde{b} - b) = D_-^{-1}(\tilde{b} \tilde{\alpha}_1 - b \tilde{\alpha}_1).
\]

From this place, it is necessary to repeat the circle of calculations mentioned in the introduction. Thus, after introducing the new function \( \tilde{\alpha}_1 \) in the following way:

\[
\alpha_1 = D_-^{-1} \tilde{\alpha}_1
\]

the equation for \( \tilde{\alpha}_1 \) takes the form

\[
D_+ \tilde{\alpha}_1 + \tilde{\alpha}_1 D_-^{-1}(\tilde{b} - b) = \tilde{b} D_-^{-1}(\tilde{\alpha}_1 + \tilde{\alpha}_1) - b D_-^{-1}(\tilde{\alpha}_1 + \tilde{\alpha}_1).
\]
Let us notice that the function $\tilde{\alpha}_1$ is bosonic in contrast to its first-step counterpart $\tilde{\alpha}_0$. After the substitution

$$\tilde{\alpha}_1 = b\,\alpha_2 + b\beta_2,$$
(3.11)

we come to a system of two equations, in the same way as to (3.7), for two unknown functions $\alpha_2, \beta_2$, which now possesses the partial solution $\alpha_2 = -\beta_2$ to be utilized. The single equation for the unknown function $\alpha_2$ takes the form

$$D_+\alpha_2 + \alpha_2 D_-^{-1}(b - b + b - b) = D_-^{-1}(b\,\alpha_2 - b\alpha_2 + b\,\alpha_2 - b\alpha_2),$$
(3.12)

In contrast to the analogous equation (3.8), eq. (3.12) possesses the partial solution $\alpha_2 = 1$. By this solution, it is possible to interrupt the calculation procedure or continue it according to the proposed scheme. As a result, we obtain the following recurrent relations:

$$\alpha_{2n} = D_-^{-1}(b\,\alpha_{2n+1} + b\alpha_{2n+1}), \quad \alpha_{2n+1} = D_-^{-1}(b\,\alpha_{2n+2} - b\alpha_{2n+2}),$$
(3.13)

where $\alpha_{2n}$ ($\alpha_{2n+1}$) are bosonic (fermionic) functions and, for every even step, there is a partial solution $\alpha_{2n} = 1$.

Applying the automorphism transformations (3.5) to the recurrent relations (3.13), we get the new recurrent relations

$$\eta_{2n} = -D_-^{-1}(b\,\eta_{2n+1} + b\eta_{2n+1}), \quad \eta_{2n+1} = -D_-^{-1}(b\,\eta_{2n+2} - b\eta_{2n+2}),$$
(3.14)

generating another solution $\eta_0$ to the eq. (3.2).

In terms of these two solutions of the symmetry equation (3.2), for the substitution (2.5), the integrable system of evolution equations (3.3) can be represented in the following form:

$$b_t = b(\gamma_0 - \gamma_0), \quad f_t = D_-^{-1}[b(\gamma_0 - \gamma_0) + b(\gamma_0 - \gamma_0)],$$
(3.15)

where

$$\gamma_n = \alpha_n + h\eta_n$$
(3.16)

and $h$ is an arbitrary parameter.

Thus, the expressions (3.13)-(3.15) give us explicit formulae for the one-parametric hierarchy of integrable equations corresponding to the integrable substitution (2.3). In what follows, we choose the parameter $h$ equal to zero for all of the discussed substitutions to simplify the formulae, keeping in mind that it is always possible to restore it by the action of an inner automorphism transformation, corresponding to a given substitution (see e.g., eqs. (2.9) and (2.10) for the substitutions (2.5) and (2.6), respectively), on the solution with $h = 0$ and then adding to it the obtained result multiplied by $h$.

To close this section, let us present the results of calculations for the substitutions (2.6) and (2.12).

For the substitution (2.6), the hierarchy of integrable equations can be written in the following form:

$$b_t = b(\gamma_0 - \gamma_0), \quad \bar{b}_t = \bar{b}(\gamma_0 - \gamma_0),$$
(3.17)

The coefficient at $\alpha_0$ in (3.13), (3.16) is ineffective and it is always possible to put it equal to unity by the corresponding rescaling of the evolution variable $t$. 

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where $\gamma_n$ is defined by eq. (3.16) and $\alpha_n (\eta_n)$ satisfies the same recurrent relations (3.13) (3.14).

The integrable equations of the hierarchy corresponding to the substitution (2.12) look like

$$u_t = u D^{-1}_- (T_0 \gamma_1 + T_0 \gamma_1), \quad v_t = -v D^{-1}_- (T_0 \gamma_1 + T_0 \gamma_1)$$

(3.18)

where $\alpha_n$ and $\eta_n$ satisfy the same recurrent relations (3.13) and (3.14), respectively, in which the function $b$ is replaced by $T_0$ (2.13).

It is easy to observe that at the interruption of the recurrent relations at the $n$-th step (i.e., at $\alpha_{2n} = \eta_{2n} = 1$), the maximal order of the linearly appearing bosonic derivative on the right-hand side of the eqs. (3.15), (3.17) and (3.18) is equal to just $n$. To use the terminology of the inverse scattering theory suitable to the one-dimensional case, one can say that the interruption condition $\alpha_{2n} = \eta_{2n} = 1$ extracts the $n$-th flow.

4 Examples: the $n = 2$ cases and their bosonic limits

Using the general formulae of the last section, we present here the results of the calculations for the first nontrivial equations of the integrable hierarchies corresponding to the substitutions (2.5), (2.6) and (2.12), as well as for their bosonic limits. These cases correspond to the interruption of the recurrent procedures (3.13) and (3.14) at the second step (at $n = 2$), i.e., we put $\alpha_4 = \eta_4 = 1$ in eqs. (3.13) and (3.14). Let us give the results of the calculations with short comments.

4.1 Substitution (2.5)

$$f_t = f_{xx} + 2D^{-1}_- \{(D_- f) D_+ f\}_x - D_+ \{f_x D_- f - (fb)_x + b D^{-1}_- b_x\} - 2D_+ (f D^{-1}_- b_x),$$

$$b_t = -b_{xx} + 2(b D_+ f)_x - 2D_+ (b D^{-1}_- b_x).$$

(4.1)

Under the reduction of (4.1) from the superspace (2|2) to superspace (1|2) (see discussion at the end of section 2), it can be represented in the following form:

$$f_t = f_{xx} + D_+ ((D_+ f)^2 - 2bD_- f + 2D_- D_+ b + 2b^2),$$

$$b_t = -b_{xx} + 2(b D_+ f)_x - D_+ D_- b^2.$$  

(4.2)

As was mentioned at the end of section 2, in this case one can get the $N = 2$ supersymmetric system of equations if one introduces the superfield $b$ (2.16) instead of $f$. Thus, in the new terms, one can rewrite (4.2) as

$$\tilde{b}_t = \tilde{b}_{xx} + \tilde{b}_x^2 - ib[D, \tilde{D}]\tilde{b} + i[D, \tilde{D}]b + 2b^2,$$

$$b_t = -b_{xx} + 2(b \tilde{b}_x)_x + \frac{i}{2} [D, \tilde{D}]b^2$$

(4.3)

and the $N = 2$ supersymmetry of (1.3) becomes manifest. Here $i$ is the imaginary unity and $D$, $\tilde{D}$ are two $N = 2$ supersymmetric fermionic covariant derivatives with opposite U(1)-charges, related to their $N = 1$ counterparts $D_\pm$ by the following formulae:

$$D = \frac{1}{\sqrt{2}} (D_- + iD_+), \quad \tilde{D} = \frac{1}{\sqrt{2}} (D_- - iD_+), \quad \{D, \tilde{D}\} = 2\partial_x, \quad D_- D_+ = \frac{i}{2} [D, \tilde{D}].$$

(4.4)
Of course, the eqs. (4.2) also possess the $N = 2$ supersymmetry; however, in this form, it is hidden. It is interesting that both (4.2) and (4.3) are local despite the fact that the $U(1)$ transformation from the $N = 2$ supergroup is nonlocally realized for (4.2).

We would next like to describe the bosonic limit of the substitution (2.3) and the eqs. (4.1). We define the bosonic components of the superfields $b, f$ as

$$ b| = a, \quad D_D b| = w, \quad D_f| = c, \quad D_- f| = d, \quad (4.5) $$

where $|$ means the $(\theta_x, \theta_\pm) \to 0$ limit. In terms of these components, the desirable expressions have the following form:

$$ \tilde{a} = d - a, \quad \tilde{w} = -c_y - w, \quad \tilde{c} = c + (\ln \tilde{a})_x, \quad \tilde{d} = d + \tilde{w} \frac{1}{\tilde{a}} \quad (4.6) $$

for the substitution and

$$ d_t = (d - 2a)_{xx} + 2cd_x + 2ac_x + 2(a - d)\partial_y^{-1}w_x, \quad a_t = -a_{xx} + 2(ac)_x + 2a\partial_y^{-1}w_x, $$

$$ c_t = [c_x + c^2 + \partial_y^{-1}(2w + b^2 - 2ad)]_x, \quad w_t = [-(w + b^2)_x + 2wc + 2ad_x]_x \quad (4.7) $$

for the eqs. (4.1).

From the substitution (2.3), as well as directly from (4.1)-(4.3), one can see that the scaling dimensions of all superfields are completely fixed and are defined by the relations: $[b] = cm^{-1}$, $[\tilde{b}] = cm^0$, $[f] = cm^{-1/2}$. As a consequence, for the components (4.5), we get the following values for the scaling dimensions: $[a] = [c] = [d] = cm^{-1}, \quad [w] = cm^{-2}$.

### 4.2 Substitution (2.6)

$$ b_t = -b_{xx} - 2(bD_D^{-1}D_D b)_x - 2D_D((D_D b)D_D^{-1}D_D b), $$

$$ \tilde{b}_t = \tilde{b}_{xx} - 2(\tilde{b}D_D^{-1}D_D \tilde{b})_x - 2D_D((D_D \tilde{b})D_D^{-1}D_D \tilde{b}). \quad (4.8) $$

At the reduction to the (1|2) superspace, one can rewrite (4.8) in the following form:

$$ b_t = -b_{xx} - i(b\partial^{-1}[D, \tilde{D}][b]_x - ib_x\partial^{-1}[D, \tilde{D}]b + 2i(Db)\tilde{D}b, $$

$$ \tilde{b}_t = \tilde{b}_{xx} - i(\tilde{b}\partial^{-1}[D, \tilde{D}]b)_x - i\tilde{b}_x\partial^{-1}[D, \tilde{D}]\tilde{b} + 2i(D\tilde{b})\tilde{D}\tilde{b}. \quad (4.9) $$

Equations (4.3) possess manifest, local $N = 2$ supersymmetry; however, they are nonlocal. Nevertheless, it is possible to localize them. In order to do this, let us introduce new fermionic superfields $\psi, \tilde{\psi}$ in the following way: $b = D_D \psi \tilde{b} = D_D \tilde{\psi}$. In these new terms, the systems (4.8), (4.9) have the form

$$ \psi_t = -\psi_{xx} + 2D_D^{-1}((D_D \psi)D_D \tilde{\psi})_x + D_D(D_D \psi)_x^2, $$

$$ \tilde{\psi}_t = \tilde{\psi}_{xx} + 2D_D^{-1}((D_D \tilde{\psi})D_D \psi)_x + D_D(D_D \tilde{\psi})_x^2; \quad (4.10) $$

$$ \psi_t = -\psi_{xx} + 2D_D^{-1}((D_D \psi)D_D \tilde{\psi}) + D_D(D_D \psi)_x^2, $$

$$ \tilde{\psi}_t = \tilde{\psi}_{xx} + 2D_D^{-1}((D_D \tilde{\psi})D_D \psi) + D_D(D_D \tilde{\psi})_x^2, \quad (4.11) $$

respectively. Inspection of (1.11) shows that, like in the case of (4.2), it possesses the $N = 2$ supersymmetry, and the $U(1)$ transformation from the $N = 2$ supergroup is realized nonlocally.

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3Let us remember that in the bosonic limit, all fermionic components must be put equal to zero.
In terms of $\psi$ and $\bar{\psi}$, the substitution (2.6) can be rewritten in the local form

$$\bar{\psi} = \psi + D_+ \ln D_- \bar{\psi}, \quad \bar{\psi} = \bar{\psi}. \quad (4.12)$$

It is instructive to consider its bosonic limit as well as the bosonic limit of the corresponding eqs. (1.11). In terms of the bosonic components of the fermionic superfields $\psi$ and $\bar{\psi}$, defined as

$$D_+ \psi| = \bar{q}, \quad D_- \psi| = \bar{p}, \quad D_+ \bar{\psi}| = q, \quad D_- \bar{\psi}| = p, \quad (4.13)$$

we get the following expressions for the substitution:

$$\bar{\bar{q}} = \bar{q} + (\ln p)_x, \quad \bar{\bar{q}} = q, \quad \bar{\bar{p}} = \bar{p} - \frac{q_y}{p}, \quad \bar{\bar{p}} = p, \quad (4.14)$$

and for the (1+2)-dimensional bosonic equations:

$$\bar{q}_t = -\bar{q}_{xx} + (q^2)_x + 2\partial_y^{-1}(\bar{q}_yq - \bar{p}px)_x, \quad \bar{p}_t = -\bar{p}_{xx} + 2(\bar{p}q)_x + 2\bar{p}_x\bar{q},$$

$$q_t = q_{xx} + (q^2)_x + 2\partial_y^{-1}(q_y\bar{q} - p\bar{p}_x)_x, \quad p_t = p_{xx} + 2(p\bar{q})_x + 2p_xq. \quad (4.15)$$

One can see from (4.15) that at the reduction to the one dimensional case, the (1+1)-dimensional equations are local. Obviously, (4.14) and (4.15) are integrable because they are obtained from integrable supersymmetric counterparts in the bosonic limit.

Equations (4.12)-(4.11) and (4.15) have complex structure. It is easy to find the following complex conjugation properties and scaling dimensions of all superfields and their components: $b^* = -b$, $b^* = -b$, $[b] = [\bar{b}] = cm^{-1}$; $\psi^* = -\bar{\psi}$, $\bar{\psi}^* = -\psi$, $[\psi] = [\bar{\psi}] = cm^{-1/2}$; $q^* = -\bar{q}$, $\bar{q}^* = -q$, $p^* = -\bar{p}$, $\bar{p}^* = -p$, $[q] = [\bar{q}] = [p] = [\bar{p}] = cm^{-1}$. Evolution variable $t$ is purely imaginary and under complex conjugation $t^* = -t$.

### 4.3 Substitution (2.12)

$$v_t = -v_{xx} + 2(D_+ v)D_+^{-1}(uv)_x - 2vD_+^{-1}\{(vD_+ u)_x + 2uvD_+^{-1}(uv)_x\},$$

$$u_t = u_{xx} + 2(D_+ u)D_+^{-1}(uv)_x - 2uD_+^{-1}\{(uD_+ v)_x + 2uvD_+^{-1}(uv)_x\}. \quad (4.16)$$

At the reduction to the (1|2) superspace, expressions (4.16) are localized and look like

$$v_t = -v_{xx} - 2i\{ (Dv)\bar{D}u - (\bar{D}v)Du - \frac{1}{2}v[\bar{D}, D]u - i(uv)^2 \} - 2iu(Dv)\bar{D}v,$$

$$u_t = u_{xx} - 2i\{ (Du)\bar{D}v - (\bar{D}u)Dv - \frac{1}{2}u[\bar{D}, D]v + i(uv)^2 \} - 2iv(Du)\bar{D}u. \quad (4.17)$$

One can see that the eqs. (4.17) are local and they admit the local $N = 2$ supersymmetry.

As in the previous cases, we now present the bosonic limits of the substitution (2.12) and the corresponding eqs. (1.10).

Let us define the bosonic components of the bosonic superfields $u$ and $v$ as

$$v| = s, \quad D_- D_+ v| = \bar{r}, \quad u| = s, \quad D_- D_+ u| = r. \quad (4.18)$$

In terms of these components, (2.12) and (4.16) become

$$\bar{s} = \frac{1}{s}, \quad \bar{s} = \bar{r} - ss^2, \quad \bar{r} = -\frac{\bar{r}}{s^2}, \quad \bar{r} = -\bar{s}(\bar{s} \bar{r} + \bar{s}\bar{r} + s\bar{r} + (\ln \bar{s})_{xy}); \quad (4.19)$$
\[ \bar{r}_t = -\bar{r}_{xx} + 2\bar{s}_x(s\bar{s})_x + 2\bar{s}(s_x\bar{s})_x + 2\bar{r}\partial_y^{-1}(s\bar{r} - (s\bar{s})^2)_x - 4\bar{s}\bar{s}^2\partial_y^{-1}(\bar{s}r + s\bar{r})_x, \]
\[ r_t = r_{xx} + 2s_x(s\bar{s})_x + 2s(s\bar{s}_x)_x + 2r\partial_y^{-1}(\bar{s}r + (s\bar{s})^2)_x + 4\bar{s}\bar{s}^2\partial_y^{-1}(\bar{s}r + s\bar{r})_x, \]
\[ \bar{s}_t = -\bar{s}_{xx} - 2\bar{s}\partial_y^{-1}(\bar{s}r + (s\bar{s})^2)_x, \quad s_t = s_{xx} - 2s\partial_y^{-1}(s\bar{r} - (s\bar{s})^2)_x, \]
respectively.

Equations (4.16), (4.17) and (4.20) possess the global U(1)-invariance and admit complex structure. With respect to the U(1) group, the superfields \( u \) and \( v \) have opposite U(1)-charges. Due to the U(1)-invariance, only scaling dimensions of U(1)-invariant products are fixed: \([uv] = [s\bar{s}] = cm^{-1}, [\bar{r}] = [\bar{s}] = cm^{-2}\), as are the relations among the components with the same U(1) charge, \([r] = [s] \times cm^{-1}, [r] = [s] \times cm^{-1}\). The complex conjugation properties of the superfields and their bosonic components are the following: \( v^* = \pm iu, u^* = \pm iv; s^* = \pm is, \)
\( \bar{s}^* = \pm is, r^* = \pm i\bar{s}, \bar{r}^* = \pm ir; t^* = -t. \)

It appears that the \( N = 2 \) supersymmetric integrable systems constructed here in the (1|2) superspace do not belong to the wide class of \( N = 2 \) supersymmetric integrable hierarchies that have recently been explicitly constructed using the Lax pair approach in [5], [6]. It would be interesting to understand how our hierarchies would be described in that formalism.

## 5 Conclusion

In this paper, we have applied the formalism of integrable mappings to the problem of the construction of supersymmetric integrable hierarchies in the (2|2) and (1|2) superspaces with manifest \( N = 2 \) supersymmetry. We have proposed three supersymmetric integrable mappings and found three, to our knowledge, new hierarchies of integrable supersymmetric systems corresponding to them in the (2|2) superspace possessing manifest \( N = 2 \) supersymmetry at the reduction to the (1|2) superspace. New bosonic substitutions and hierarchies are obtained from the constructed supersymmetric counterparts in the bosonic limit.

We attempted to demonstrate that almost all information about the integrable hierarchy is encoded in its integrable substitution. In this context, it seems to be very important to have some reserve integrable supersymmetric substitutions and integrable hierarchies corresponding to them in order to better understand their rigorous mathematical structure, the relations among them and their origins. An analysis of this intriguing problem is under way.

## 6 Acknowledgments

This work was partially supported by the RFFR grant 96-02-17634, INTAS grants 93-633, 94-2317, and by a grant from the Dutch NWO organization.

## References

[1] D.B. Fairlie and A.N. Leznov, Phys.Lett. A 199 (1995) 360.

[2] A.N. Leznov, Physica D 87 (1995) 48;

D.V. Fairlie and A.N. Leznov, The Theory of Integrable Systems from the Point of View of Representation Theory of Discrete Group of Integrable Mapping, Preprint IHEP-95-30, Protvino (1995).
[3] J. Evans and T. Hollowood, *Nucl. Phys. B* 352 (1991) 723.

[4] V.B. Derjagin, A.N. Leznov and E.A. Yuzbashyan, *Two-dimensional integrable mappings and explicit form of equations of (1 + 2)-dimensional hierarchies of integrable systems*, Preprint MPI 96-39, Bonn (1996).

[5] Z. Popowicz, *The extended supersymmetrization of the multicomponent Kadomtsev-Petviashvili hierarchy*, Preprint [hep-th/9510185](https://arxiv.org/abs/hep-th/9510185).

[6] L. Bonora, S. Krivonos and A. Sorin, *Towards the construction of N = 2 supersymmetric integrable hierarchies*, Preprint SISSA 56/96/EP, JINR E2-96-138, [hep-th/9604165](https://arxiv.org/abs/hep-th/9604165). *Nucl. Phys. B* (in press).