Boundedness of the Segal-Bargmann Transform on Fractional Hermite-Sobolev Spaces

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Received 21 November 2016; Accepted 4 January 2017; Published 19 January 2017

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Let \( s \in \mathbb{R} \) and \( 2 \leq p \leq \infty \). We prove that the Segal-Bargmann transform \( \mathcal{B} \) is a bounded operator from fractional Hermite-Sobolev spaces \( \mathcal{W}_s^p(\mathbb{R}^n) \) to fractional Fock-Sobolev spaces \( \mathcal{F}_s^p(\mathbb{C}^n) \).

1. Introduction

In quantum mechanics, the Schrödinger equation is a partial differential equation that describes how the quantum state of some physical system changes with time. The most famous example is the nonrelativistic Schrödinger equation for a single particle moving in a potential:

\[
\sqrt{-1} \hbar \frac{\partial}{\partial t} \Psi(x, t) = \left[ -\frac{\hbar^2}{2m} \Delta + V(x, t) \right] \Psi(x, t),
\]

where \( m \) is the particle’s mass, \( \hbar \) is the Planck constant, \( V \) is its potential energy, and \( \Psi \) is the wave function.

Let \( H \) be the most basic Schrödinger operator in \( \mathbb{R}^n \), \( n \geq 1 \), the Hermite operator (or the harmonic oscillator):

\[
H = -\Delta + |x|^2.
\]

Then the Schrödinger equation can be written by

\[
\sqrt{-1} \frac{\partial \Psi}{\partial t} = H \Psi.
\]

This is an important model in quantum mechanics (see, e.g., [1]).

For \( s \in \mathbb{R} \), we define the fractional Hermite operator \( H^s = (-\Delta + |x|^2)^s \) of order \( s \). Let \( 0 < p \leq \infty \). The Hermite-Sobolev space \( \mathcal{W}_H^s(\mathbb{R}^n) \) of fractional order \( s \) is the space of all tempered distributions for which the distribution \( H^{s/2} f \) is given by an \( L^p \) function on \( \mathbb{R}^n \).

Let \( \mathbb{C}^n \) be the complex \( n \)-space and let \( dV \) be the ordinary volume measure on \( \mathbb{C}^n \). If \( z = (z_1, \ldots, z_n) \) and \( w = (w_1, \ldots, w_n) \) are points in \( \mathbb{C}^n \), we write

\[
z \cdot w = \sum_{j=1}^{n} z_j w_j,
\]

\[
|z| = (z \cdot \overline{z})^{1/2}.
\]

For any \( 0 < p \leq \infty \) the Fock space \( \mathcal{F}_p \) denotes the space of entire functions \( f \) on \( \mathbb{C}^n \) such that the function \( f(z)e^{-\frac{1}{4}|z|^2} \) is in \( L^p(\mathbb{C}^n, dV) \). We define

\[
\|f\|_{\mathcal{F}_p} = \left( \frac{p}{4\pi} \right)^{n/2} \left( \int_{\mathbb{C}^n} |f(z)e^{-\frac{1}{4}|z|^2}|^p dV(z) \right)^{1/p}.
\]

For \( p = \infty \) the norm in \( \mathcal{F}_\infty \) is defined by

\[
\|f\|_{\mathcal{F}_\infty} = \sup \left\{ |f(z)e^{-\frac{1}{4}|z|^2}| : z \in \mathbb{C}^n \right\}.
\]

Let

\[
A_j f(z) = 2 \frac{\partial}{\partial z_j} f(z),
\]

\[
A_j^* f(z) = z_j f(z),
\]

\( 1 \leq j \leq n, \ f \in \mathcal{F}_p \).
Both \( A_j \) and \( A_j^* \), as defined above, are densely defined linear operators on \( F^p \) (unbounded though). We consider the radial derivative \( \mathcal{R} \) defined by
\[
\mathcal{R} = \frac{1}{2} \sum_{j=1}^{n} (A_j A_j^* + A_j^* A_j).
\]  
Let \( s \) be a real number and \( 0 < p \leq \infty \). The fractional Fock-Sobolev space \( F_{s,p}^\alpha \) of order \( s \) is the space of all entire functions for which \( \mathcal{R}^{s/2} f \) is given by an \( F^p \) function.

The Segal-Bargmann transform \( \mathcal{B} \) is defined by
\[
\mathcal{B} f (z) = \frac{1}{\pi^{n/4}} \int_{\mathbb{R}^n} f(x) e^{x z - (1/2)|x|^2} dV(x),
\]  
where \( dV(x) \) is the volume measure on \( \mathbb{R}^n \). It is well-known that the Segal-Bargmann transform is a unitary isomorphism between \( L^2(\mathbb{R}^n) \) and \( F^2 \) [2, 3].

We prove that the radial derivative \( \mathcal{R} \) has a parallel behavior to the Hermite operator \( H \). In particular, \( \mathcal{R} \) is densely defined, positive, self-adjoint and has the discrete spectrum; it generates a diffusion semigroup. Moreover, we show that the Segal-Bargmann transform intertwines fractional Hermite-Sobolev spaces with fractional Fock-Sobolev spaces as follows.

**Theorem 1.** Let \( s \in \mathbb{R} \) and \( 2 \leq p \leq \infty \). Then the Segal-Bargmann transform \( \mathcal{B} : W_{s,p}^{\alpha}(\mathbb{R}^n) \to F_{s,p}^\alpha \) is bounded.

**2. Fractional Hermite-Sobolev Spaces**

In one dimension, the Hermite polynomials \( H_k \) are defined by
\[
H_k(x) = e^{-x^2} \frac{d^k}{dx^k} e^{x^2}, \quad x \in \mathbb{R},
\]  
and by normalization we obtain the Hermite functions
\[
h_k(x) = (\sqrt{\pi} 2^k k!)^{-1/2} e^{-x^2/4} (-1)^k H_k(x), \quad x \in \mathbb{R}.
\]  
Note that
\[
\left( \frac{d^2}{dx^2} + x^2 \right) e^{-(1/2)x^2} H_k(x) = (2k + 1) e^{-(1/2)x^2} H_k(x), \quad x \in \mathbb{R}.
\]  
In higher dimensions, for each multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \), the Hermite functions \( h_\alpha \) are defined by
\[
h_\alpha(x) = \prod_{j=1}^{n} h_{\alpha_j}(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.
\]  
Here, \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) is the set of nonnegative integer. By (12), we know that these are the eigenfunctions of the Hermite operator defined in (2). In fact,
\[
H h_\alpha = (2|\alpha| + n) h_\alpha.
\]  
Moreover, \( \{ h_\alpha : \alpha \in \mathbb{N}_0^n \} \) is an orthonormal basis for \( L^2(\mathbb{R}^n) \).

Let \( \mathcal{H} \) be the space of finite linear combinations of Hermite functions
\[
f = \sum_{|\alpha| \leq N} \langle f, h_\alpha \rangle h_\alpha,
\]  
where
\[
\langle f, h_\alpha \rangle = \int_{\mathbb{R}^n} f(x) h_\alpha(x) dV(x).
\]  
The space \( \mathcal{H} \) is dense in \( L^2(\mathbb{R}^n) \), and so, by the orthonormality of the Hermite functions,
\[
\| f \|_{L^2(\mathbb{R}^n)} = \left( \sum_{\alpha \in \mathbb{N}_0^n} |\langle f, h_\alpha \rangle|^2 \right)^{1/2}.
\]  
For \( s \in \mathbb{R} \), we define the fractional Hermite operator \( H^s = (-\Delta + |x|^2)^s \) of order \( s \). For \( f \in \mathcal{S}(\mathbb{R}^n) \), the Hermite series expansion
\[
\sum_{\alpha \in \mathbb{N}_0^n} \langle f, h_\alpha \rangle h_\alpha
\]  
converges to \( f \) uniformly in \( \mathbb{R}^n \) (and also in \( L^2(\mathbb{R}^n) \)), since \( \| h_\alpha \|_{L^2(\mathbb{R}^n)} \leq C \), for all \( \alpha \in \mathbb{N}_0^n \), and each \( m \in \mathbb{N} \), and we have (see [4])
\[
|\langle f, h_\alpha \rangle| \leq \| H^m f \|_{L^2(\mathbb{R}^n)} (2|\alpha| + n)^m.
\]  
**Definition 2.** Let \( s \in \mathbb{R} \) and \( f \in \mathcal{S}(\mathbb{R}^n) \). One defines the fractional Hermite operator \( H^s \) by
\[
H^s f = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s \langle f, h_\alpha \rangle h_\alpha.
\]  
The fractional Hermite operators \( H^s \) were introduced in [5].

**Definition 3.** Let \( s \in \mathbb{R} \) and \( 0 < p \leq \infty \). The fractional Hermite-Sobolev space \( W_{s,p}^{\alpha}(\mathbb{R}^n) \) of order \( s \) is the space of all tempered distributions for which the distribution \( H^{s/2} f \) is given by an \( L^p \) function on \( \mathbb{R}^n \). The fractional Hermite-Sobolev norm of order \( s \) is defined accordingly,
\[
\| f \|_{W_{s,p}^{\alpha}(\mathbb{R}^n)} = \| H^{s/2} f \|_{L^p(\mathbb{R}^n)}.
\]  
The fractional Hermite-Sobolev spaces \( W_{s,p}^{\alpha}(\mathbb{R}^n) \) of order \( s \) were introduced in [6].

**3. Radial Derivative**

We consider the radial derivative \( \mathcal{R} \) defined on
\[
\text{Dom}(\mathcal{R}) = \{ f \in F^2 : \mathcal{R} f \in F^2 \}
\]  
by
\[
\mathcal{R} = \frac{1}{2} \sum_{j=1}^{n} (A_j A_j^* + A_j^* A_j),
\]  
where \( \mathcal{R} = \frac{1}{2} \sum_{j=1}^{n} (A_j A_j^* + A_j^* A_j) \).
where
\[ A_j f(z) = \frac{\partial}{\partial z_j} f(z), \]
\[ A_j^* f(z) = z_j f(z), \quad 1 \leq j \leq n, \ f \in F^2. \]

We have
\[ \mathcal{R} = 2 \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + n. \tag{25} \]

The following example tells us that \( \mathcal{D} \circ m(\mathcal{R}) \not\subset F^2 \). Thus \( \mathcal{R} \) is an unbounded operator on \( F^2 \).

**Example 4.** Let
\[ f(z) = \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}}. \tag{26} \]

Then \( f \in F^2 \), but \( \mathcal{R} f \notin F^2 \).

**Proof.** Note that
\[ \|f\|^2_{L^2} = \frac{1}{2\pi} \sum_{k=0}^{\infty} \int \frac{|z^k|^2}{\sqrt{k!}} e^{-|z|} dV(z) \]
\[ = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \zeta(2) < \infty, \]
where \( \zeta(2) \) is the Riemann zeta function. However, we have
\[ \|\mathcal{R} f\|^2_{L^2} = \sum_{k=0}^{\infty} \frac{(2k+n)^2}{(k+1)^2} \leq \sum_{k=0}^{\infty} (2k+n)^2 = \infty. \tag{28} \]

**Lemma 5.** \( \mathcal{R} \) is a positive, self-adjoint operator on \( \mathcal{D} \circ m(\mathcal{R}) \).

**Proof.** Let \( \mathcal{P}(C^\circ) \) be the set of all holomorphic polynomials on \( C^\circ \). We know that \( \mathcal{P}(C^\circ) \) is dense in \( F^2 \) and \( \mathcal{R} \) is self-adjoint on \( \mathcal{P}(C^\circ) \). Hence \( \mathcal{D} \circ m(\mathcal{R}) \) is the domain of its unique self-adjoint extension.

Note that
\[ \langle f, \mathcal{R} f \rangle_{L^2} = 2 \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial z_j} \right\|^2_{L^2} + n \left\| f \right\|^2_{L^2} \geq n \left\| f \right\|^2_{L^2}, \tag{29} \]
\[ \forall f \in \mathcal{D} \circ m(\mathcal{R}). \]

Thus \( \mathcal{R} \) is positive. \( \square \)

**Lemma 6.** \( \mathcal{R} \) has the discrete spectrum \( \sigma(\mathcal{R}) = \{2|\alpha| + n : \alpha \in \mathbb{N}_0^n\} \).

**Proof.** By (29), we have \( \sigma(\mathcal{R}) \subset [n, \infty) \).

We define
\[ \epsilon_\alpha(z) = \frac{z^\alpha}{\|z^\alpha\|^2_{L^2}} = \frac{z^\alpha}{\sqrt{2^{|\alpha|} |\alpha|!}}. \tag{30} \]

Then \( \{\epsilon_\alpha : \alpha \in \mathbb{N}_0^2\} \) is an orthonormal basis for \( F^2 \). It is easy to see that \( \{2|\alpha| + n : \alpha \in \mathbb{N}_0^2\} \) is the set of all eigenvalues.

Let \( \lambda \in [n, \infty) \setminus \{2|\alpha| + n : \alpha \in \mathbb{N}_0^2\} \). First, we show that \( \lambda I - \mathcal{R} : \mathcal{D} \circ m(\mathcal{R}) \to F^2 \) is injective and surjective.

Suppose that \( (\lambda I - \mathcal{R}) f = (\lambda I - \mathcal{R}) \tilde{f} \). Then
\[ 0 = (\lambda I - \mathcal{R}) f - (\lambda I - \mathcal{R}) \tilde{f} = \sum_{\alpha \in \mathbb{N}_0^2} \lambda - (2|\alpha| + n) \langle f - \tilde{f}, \epsilon_\alpha \rangle \epsilon_\alpha. \tag{31} \]

This implies \( f = \tilde{f} \). Thus \( \lambda I - \mathcal{R} : \mathcal{D} \circ m(\mathcal{R}) \to F^2 \) is injective.

For \( f \in F^2 \) let
\[ f(z) = \sum_{\alpha \in \mathbb{N}_0^2} \epsilon_\alpha \epsilon_{\alpha}(z) \tag{32} \]
be the orthonormal decomposition of \( f \). We define
\[ g = \frac{1}{\lambda} f + \frac{1}{\lambda} \sum_{\alpha \in \mathbb{N}_0^2} \frac{2|\alpha| + n}{\lambda - (2|\alpha| + n)} \epsilon_\alpha \epsilon_{\alpha}(z). \tag{33} \]

Since
\[ \varphi_N = \sum_{|\alpha| = 0}^{N} \frac{2|\alpha| + n}{\lambda - (2|\alpha| + n)} \epsilon_\alpha \epsilon_{\alpha}(z) \tag{34} \]
is a Cauchy sequence in \( F^2 \), the series in (33) converges in \( F^2 \). Hence
\[ g = \frac{1}{\lambda} f + \sum_{|\alpha| = 0}^{\infty} \frac{2|\alpha| + n}{\lambda - (2|\alpha| + n)} \epsilon_\alpha \epsilon_{\alpha}(z) \tag{35} \]
is a well-defined element of \( F^2 \) and it satisfies \( (\lambda I - \mathcal{R}) g = f \). This means that \( \lambda I - \mathcal{R} : \mathcal{D} \circ m(\mathcal{R}) \to F^2 \) is surjective.

Moreover,
\[ \left\| (\lambda I - \mathcal{R})^{-1} f \right\|_{L^2} \leq \frac{1}{\lambda} \left\| f \right\|_{L^2} + \frac{1}{\lambda} \beta \left\| f \right\|_{L^2} \tag{36} \]
\[ = \frac{1}{\lambda} (1 + \beta) \left\| f \right\|_{L^2}, \]
where \( \beta = \sup_{\alpha \in \mathbb{N}_0^2} |2|\alpha| + n|/(\lambda - (2|\alpha| + n)) \). Hence \( (\lambda I - \mathcal{R})^{-1} \) is bounded and so \( \sigma(\mathcal{R}) = \{2|\alpha| + n : \alpha \in \mathbb{N}_0^2\} \). \( \square \)

For \( f \in F^2 \) let
\[ f(z) = \sum_{\alpha \in \mathbb{N}_0^2} \epsilon_\alpha \epsilon_{\alpha}(z) \tag{37} \]
be the orthonormal decomposition of \( f \). Associated with the operator \( \mathcal{R} \) is a semigroup \( \{B_t\}_{t \geq 0} \) defined by the expansion
\[ B_t f(z) = \sum_{\alpha \in \mathbb{N}_0^2} e^{-(2|\alpha| + n)t} \epsilon_\alpha \epsilon_{\alpha}(z). \tag{38} \]
We can check that \( u(z, t) = B_t f(z) \) is the solution of the heat-type equation:

\[
(\partial_t + \mathcal{R}) u = 0 \quad \text{on } \mathbb{C}^n \times (0, \infty),
\]

\[
u(t, 0) = f \quad \text{on } \mathbb{C}^n.
\]

(39)

It is easy to see that

\[
\|B_t f - f\|_{F^2}^2 \leq e^{-2\nu t} \|f\|_{F^2}^2.
\]

(40)

Thus \( B_t \) is contractive.

**Proposition 7.** \( \{B_t\}_{t \geq 0} \) is a strongly continuous semigroup.

**Proof.** We note that

\[
\|B_t f - f\|_{F^2}^2 = \sum_{\alpha \in \mathbb{N}_0^s} |e^{-(2|\alpha| + n)t} - 1|^2 |c_{\alpha}|^2
\]

\[
= \sum_{k=0}^{\infty} |e^{-(2k+n)t} - 1|^2 \sum_{|\alpha| = k} |c_{\alpha}|^2.
\]

Thus we get the result.

By Proposition 8, we have

\[
B_t = e^{-t\mathcal{R}}.
\]

(49)

**4. Fractional Fock-Sobolev Spaces**

Since \( \mathcal{R} \) has discrete spectrum \([2|\alpha| + n : \alpha \in \mathbb{N}_0^s]\), by using the spectral theorem, we define the fractional radial derivative \( \mathcal{R}^s \) for \( s \in \mathbb{R} \) as follows.

**Definition 9.** Let \( s \in \mathbb{R} \). For \( f \in L^2 \) let

\[
f(z) = \sum_{\alpha \in \mathbb{N}_0^s} c_{\alpha} e_\alpha(z)
\]

be the orthonormal decomposition of \( f \). By the spectral theorem, \( \mathcal{R}^s \) is given by

\[
\mathcal{R}^s f(z) = \sum_{\alpha \in \mathbb{N}_0^s} (2|\alpha| + n)^s c_{\alpha} e_\alpha(z).
\]

(50)

Then

\[
\lim_{t \to 0^+} \|B_t f - f\|_{F^2}^2 = \lim_{t \to 0^+} \int_0^\infty |e^{-(2|\alpha| + n)t} - 1|^2 d\nu(\lambda),
\]

(51)

where \( \nu \) is a discrete measure defined by

\[
\nu = \sum_{k=0}^\infty \left( \sum_{|\alpha| = k} |c_{\alpha}|^2 \right) \delta_k.
\]

(44)

By Lebesgue dominate convergence theorem, we have

\[
\lim_{t \to 0^+} \|B_t f - f\|_{F^2}^2 = \int_0^\infty |e^{-(2|\alpha| + n)t} - 1|^2 d\nu(\lambda)
\]

(45)

\[
= 0.
\]

Hence \( \{B_t\}_{t \geq 0} \) is a strongly continuous semigroup.

**Proposition 8.** \( -\mathcal{R} \) is the infinitesimal generator of \( \{B_t\}_{t \geq 0} \).

That is,

\[
\lim_{t \to 0^+} \frac{B_t f - f}{t} = -\mathcal{R} f.
\]

(46)

**Proof.** By using the previous discrete measure \( \nu \), it follows that

\[
\frac{B_t f - f}{t} - (-\mathcal{R} f)
\]

\[
eq \int_0^\infty \left| e^{-(2|\alpha| + n)t} - 1 \right|^2 d\nu(\lambda).
\]

(47)

Taking limit on both sides and by Lebesgue dominate convergence theorem,

\[
\lim_{t \to 0^+} \left\| \frac{B_t f - f}{t} - (-\mathcal{R} f) \right\|_{F^2}^2 = \lim_{t \to 0^+} \int_0^\infty \left| e^{-(2|\alpha| + n)t} - 1 \right|^2 d\nu(\lambda)
\]

(48)

\[
= \int_0^\infty \left| e^{-(2|\alpha| + n)t} - 1 \right|^2 d\nu(\lambda) = 0.
\]

Thus we get the result.

We refer the reader to [7–10] for other Fock-Sobolev spaces.

**5. \( L^p \)-Boundedness of the Segal-Bargmann Transform**

The Hermite operator \( H \) is self-adjoint on the set of infinitely differentiable functions with compact support \( C_0^{\infty}(\mathbb{R}^n) \), and it can be factorized as

\[
H = \frac{1}{2} \sum_{j=1}^n (a_j a_j^* + a_j^* a_j),
\]

(53)
where
\[ a_j = \frac{\partial}{\partial x_j} + x_j, \]
\[ a_j^* = -\frac{\partial}{\partial x_j} + x_j, \]
1 ≤ j ≤ n.

Lemma 11. For each j = 1, ..., n, one has
\[ \mathcal{B}(a_j f) = A_j \mathcal{B}(f), \]
\[ \mathcal{B}(a_j^* f) = A_j^* \mathcal{B}(f). \]  (55)

Proof. Let \( f \in C^\infty_c(\mathbb{R}^n) \). By the integration by parts, we have
\[ \mathcal{B} \left( \frac{\partial}{\partial x_j} f \right)(z) = \frac{1}{\pi^{n/4}} \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) e^{x \cdot z - (1/2)|x|^2 - (1/4)z \cdot z} dV(x) \]
\[ = -z_j \mathcal{B}(f) + \mathcal{B}(x_j f). \]
This gives
\[ \mathcal{B}(a_j^* f) = A_j^* \mathcal{B}(f). \]  (57)

We differentiate
\[ \mathcal{B}(f(z)) = \frac{1}{\pi^{n/4}} \int_{\mathbb{R}^n} f(x) e^{x \cdot z - (1/2)|x|^2 - (1/4)z \cdot z} dV(x) \]  (58)
under the integral sign to obtain
\[ A_j \mathcal{B}(f(z)) = \frac{1}{\pi^{n/4}} \]
\[ \cdot \left\{ \int_{\mathbb{R}^n} (2x_j - z_j) f(x) e^{x \cdot z - (1/2)|x|^2 - (1/4)z \cdot z} dV(x) \right\}. \]  (59)
This gives
\[ A_j \mathcal{B}(f) = 2 \mathcal{B}(x_j f) - A_j^* \mathcal{B}(f). \]  (60)
By (57) and (60), it follows that
\[ A_j \mathcal{B}(f) = \mathcal{B}(a_j f). \]  (61)

Corollary 12. Consider
\[ \mathcal{B}H = \mathcal{B} \mathcal{R}. \]  (62)

Proof. By Lemma 11, we have
\[ \mathcal{B}(Hf) = \sum_{j=1}^n \left( A_j A_j^* + A_j^* A_j \right) \mathcal{B}(f) = \mathcal{B} \mathcal{R}. \]  (63)

Proposition 13. Let \( s \in \mathbb{R} \). Then
\[ \mathcal{B}H^s = \mathcal{R}^s \mathcal{B}. \]  (64)

Proof. We define
\[ e_\alpha(z) = \frac{z^\alpha}{\|z\|^\alpha}. \]  (65)
Then \{\( e_\alpha : \alpha \in \mathbb{N}_0^n \)\} is an orthonormal basis for \( F^2 \) and \( \mathcal{B}(h_\alpha) = e_\alpha \). For \( f \in \mathcal{B}(\mathbb{R}^n) \) we have
\[ H^s f = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s \langle f, h_\alpha \rangle h_\alpha \]  (66)
and so
\[ \mathcal{B}(H^s f) = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s \langle \mathcal{B}(f), e_\alpha \rangle e_\alpha. \]  (67)
Since \( \mathcal{B} \) is a unitary isomorphism, we have \( \langle f, h_\alpha \rangle = \langle \mathcal{B}(f), e_\alpha \rangle \). Hence
\[ \mathcal{B}(H^s f) = \sum_{\alpha \in \mathbb{N}_0^n} (2|\alpha| + n)^s \langle \mathcal{B}(f), e_\alpha \rangle e_\alpha \]
\[ = \mathcal{R}^s \mathcal{B}(f). \]
Thus we get the result.

We consider the mapping property of the Segal-Bargmann transform \( \mathcal{B} \) as a map from \( L^p(\mathbb{R}^n) \) to \( F^p \) for \( p \in [2, \infty] \). Note that one-dimensional case is in [11].

Theorem 14. Consider
\[ \|\mathcal{B} f\|_{\mathcal{B} F^p} \leq (4\pi)^{n/4} \|f\|_{L^\infty(\mathbb{R}^n)}. \]  (69)

Proof. We have
\[ |\mathcal{B} f(x)| \leq \frac{1}{\pi^{n/4}} e^{1/4} \sup_{x \in \mathbb{R}^n} |f(x)| \]
\[ \cdot \int_{\mathbb{R}^n} e^{Re(x \cdot z) - (1/2)|x|^2 + (1/4) Re(z \cdot z) - (1/4)|z|^2} dV(x). \]  (70)
Note that
\[ |Re(z)|^2 = \frac{1}{2} \left( |z|^2 + Re(z \cdot z) \right). \]  (71)
Hence
\[ Re(z \cdot x) - \frac{1}{2} |x|^2 - \frac{1}{4} Re(z \cdot z) - \frac{|z|^2}{4} \]
\[ = Re(z \cdot x) - \frac{1}{2} |x|^2 - \frac{1}{2} |Re(z)|^2 \]  (72)
and so
\[ |B f(z)| \leq \frac{1}{\pi^{n/4}} e^{\frac{|z|^2}{4}} \sup_{x \in \mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} e^{-\frac{1}{2} |\text{Re}(z) - x|^2} dV(x) \]
\[ = (4\pi)^{n/4} e^{\frac{|z|^2}{4}} \sup_{x \in \mathbb{R}^n} |f(x)|. \]

Thus we get the result. \( \Box \)

The following Stein-Weiss interpolation theorem is well-known. See, for example, [3, 12].

**Lemma 15.** Let \( w, w_0, \) and \( w_1 \) be positive weight functions on a measure space \((X, d\lambda)\). If \( 1 \leq p_0 \leq p_1 \leq \infty \) and \( 0 \leq \theta \leq 1 \), then
\[ [L^{p_0}(X, w_0 d\lambda), L^{p_1}(X, w_1 d\lambda)]_{\theta} = L^p(X, w d\lambda) \]
with equal norms, where
\[ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \]
\[ w^{1/p} = w_0^{(1-\theta)/p_0} w_1^{\theta/p_1}. \]

**Theorem 16.** Let \( 2 \leq p \leq \infty \). There exists \( C > 0 \) such that
\[ \|B f\|_{L^p} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \]

**Proof.** The \( L^2 \)-boundedness is followed by the unitary isomorphism of the Segal-Bargmann transform. In Theorem 14, we proved the \( L^\infty \)-boundedness of the Segal-Bargmann transform. By Lemma 15, we have the required result. \( \Box \)

By Proposition 13 and Theorem 16, we have the following result.

**Theorem 17.** Let \( s \in \mathbb{R} \) and \( 2 \leq p \leq \infty \). Then the Segal-Bargmann transform \( \mathcal{B} : W^s_{\text{F}}(\mathbb{R}^n) \to F^s_p \) is bounded.

**Disclosure**

An earlier version of this work was presented as an abstract at the International Conference on the 70th Anniversary of the Korean Mathematical Society, 2016.

**Competing Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgments**

The authors would like to thank the referee for his/her valuable remarks and suggestions. The first author was supported by NRF of Korea (NRF-2016R1D1A1B03933740).

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