Shadow of charged wormholes in Einstein-Maxwell-dilaton theory

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The study of shadow is quite prominent nowadays because of the ongoing Event Horizon Telescope observations. We construct the shadow images of charged wormholes in Einstein-Maxwell-dilaton (EMD) theory. The spacetime metric of the charged wormholes contains three charges: magnetic charge $P$, electric charge $Q$, and dilaton charge $\Sigma$. We evaluate the photon geodesics around the charged wormholes. Also we calculate the effective potential and discuss its behavior with angular momentum $L$ and different values of charges $P$, $Q$, and $\Sigma$. A study of the shadow of charged wormholes reveals that the shadow is a perfect circle, and it has an effect of the charges $P$ and $Q$. We find that the radius of the shadow monotonically increases with magnetic charge $P$ as well as electric charge $Q$.

I. INTRODUCTION

Wormholes are one of the most interesting predictions of general relativity, which act as tunnel-like spacetime structures connecting different points in spacetime or even different universes. To date wormholes are largely considered as in science fiction, though it has a long history after Einstein proposed the general theory of relativity. From a theoretical perspective, wormhole physics can originally be traced back to Flamm in 1916 [1], soon after the discovery of the Schwarzschild solution. Further, wormhole-type solutions were considered by Einstein and Rosen [2] in 1935, they wanted to rid physics of singularities, and they introduced a bridge-like structure connecting two identical sheets. This mathematical construction came to be called “Einstein-Rosen bridges”. Some time ago Wheeler [3, 4] revived the subject and discussed wormholes that connected different regions of spacetime in terms of topological entities called geons, which were transformed later into Euclidean wormholes by Hawking [5] and others.

Modern interest in wormholes is mainly based on the pioneering work of Morris and Thorne [6], where observers may freely traverse, and the way to convert them into time machines [7]. They proposed wormholes as a tool for teaching general relativity, and for attracting young students into the field. These developments were boosted by the publication of the book “Lorentzian Wormholes: From Einstein to Hawking” by Visser [8], where the author reviewed the subject up to 1995, as well as proposing several new ideas. To make a Lorentzian wormhole traversable, it has to necessarily violate the null energy condition (NEC), at least in a neighborhood of the wormhole throat [6, 9], according to the needs of the geometrical structure. Regarding the energy conditions, the NEC is the weakest of the energy conditions, which implies the violation of other energy conditions also. However, such a matter appears in quantum field theory, since in the Casimir effect the null, weak, strong, and dominant energy conditions are all violated.

Various efforts have been taken to avoid it by using exotic matter, but all were in vain within the context of general relativity. Nevertheless, a popular approach had been taken by Visser et al. [10], namely the “volume integral quantifier” which quantifies the total amount of energy condition violating matter. Nandi et al. [11] suggested an exact integral quantifier for matter violating averaged null energy condition (ANECD). During the last decade, the existence of sufficiently stable wormhole solutions have been found in support of phantom-like matter [12–15]. An alternative interpretation was given in [16] with a variable equation of state parameter $\omega(r)$. On the other hand, dynamic wormholes geometries were analysed in specific cases [17–19], whereas self-dual Lorentzian wormholes have been studied in [20, 21]. An intriguing aspect about deflection of light rays gave rise to traversable wormhole solutions, yielding the possibility of identifying wormholes as compact objects possibly inhabiting our universe. Gravitational lensing for wormholes in the strong/weak field approximation has been developed in [22–35].

Based on these analyses, the most challenging problem is to construct traversable and stable wormhole solutions within classical gravitational physics. In this regard, a large number of works have been devoted to model and study wormhole geometries within the context of modified gravity including $f(R)$ gravity [36], Born-Infeld theory [37], noncommutative geometry [38], bumblebee gravity [39], and others. Also another gravity theory, namely, Einstein-Maxwell-Dilaton (EMD) theory, which provides various distinctive physical properties depending on the parameters present, have been extensively explored in different literatures. The field content of EMD theories have a $U(1)$ gauge field $A_{\mu}$ and a dilaton $\phi$, in addition to a metric $g_{\mu\nu}$, and the dilaton couples expo-
nentially to the field strength. Our interest was triggered by the work of Goulart [40], who found zero mass point-like solutions and charged wormholes that arise from the dyonic black hole solution in the development of EMD theory. Furthermore wormholes geometries were analysed [41], and the deflection of light by charged wormholes [42] was studied within the same context.

It is widely believed that most galaxies contain supermassive black holes at their centers [43, 44], e.g. Milky Way and Messier 87 have, namely, Sgr A* and M87. This opens the door toward the observations because direct image of these supermassive black holes will give strong evidence regarding their existence. Therefore the study of black hole shadow has received much attention in recent years. Black hole casts a shadow on the bright background as an optical appearance due to the strong gravitational lensing effect. Actually, an observer images the photon orbits around the event horizon. The Event Horizon Telescope (EHT) is the available observational setup regarding the detection of black hole shadow. BlackHoleCam\footnote{https://blackholecam.org/} is collaboratively working with the EHT astronomers team with the aim to make the first ever direct image of the supermassive black hole Sgr A*. Note that an observation of Sgr A* requires a earth-sized telescope which is impossible to create. This issue can be resolved with the help of Very Long Baseline Interferometry (VLBI) network. VLBI is an astronomical interferometry in which earth-based multiple radio telescopes collect signals from astronomical radio sources. These radio telescopes are distributed across different parts of earth such that a virtual earth-sized telescope can be constructed, and it is able to produce highest resolution at the mm/sub-mm wavelength scale. Schwarzschild black hole shadow is discussed by Synge [45] and it is further explored by Luminet [46]. Bardeen [47, 48] studied the shadow cast by the Kerr black holes and further extension was provided by Falcke [49]. The available literature on this topic contains the study of shadow for various types of black holes [50–72]. Naked singularity shadows are discussed in [54, 58, 73, 74]. Moreover, this phenomenon has also been considered in higher-dimensional black holes [73–75], and the authors in [76–78] considered different approaches to explore it more precisely.

As motivated by the black hole shadow, one can consider imaging of the shadow of wormholes because it could be helpful to probe the physics of wormholes. Recent studies tell us that the wormholes cast shadow just like the black holes do, and this is taken into account by several authors [79–85]. In this work, we are going to construct the shadow of charged wormholes in EMD theory, and explicitly bring out the effect of charges on the shapes of the shadow.

The paper is organized as follows. In Sec. II, we briefly review the charged wormholes solution in EMD theory. Section III is devoted to the calculation of photon geodesics around the charged wormholes and the evaluation of impact parameters required for the study of shadow. The shadow study of charged wormholes is the subject of Sec. IV. Finally, we conclude our results in Sec. V. We have used geometrized units, $G = c = 1$ throughout this paper.

II. DYONIC WORMHOLES IN THE EINSTEIN-MAXWELL-DILATON THEORY

We start with a brief review about dyonic wormholes in the EMD theory [40, 41]. The authors considered the EMD theory without a dilaton potential and the corresponding action is written as

$$S = \int d^4x\sqrt{-g}\left(R - 2\partial_\mu\phi\partial^\mu\phi - W(\phi)F_{\mu\nu}F^{\mu\nu}\right),$$  \hspace{1cm} (1)

where $\phi$ denotes the dilaton scalar field, $R$ is the Ricci scalar, and $F_{\mu\nu}$ represents the electromagnetic field strength with the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$  \hspace{1cm} (2)

where $A_\mu$ is the gauge potential. We consider the field equations for the metric, dilaton, and gauge fields, and Bianchi identities arising from the action (1) in the form

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}W(\phi)F_{\rho\sigma}F^{\rho\sigma} + 2W(\phi)F_{\mu\rho}F_{\rho\nu},$$  \hspace{1cm} (3)

$$\nabla_\mu(\partial^\mu\phi) - \frac{1}{4}\frac{\partial W(\phi)}{\partial\phi}F_{\mu\nu}F^{\mu\nu} = 0,$$  \hspace{1cm} (4)

$$\nabla_\mu(W(\phi)F^{\mu\nu}) = 0,$$  \hspace{1cm} (5)

$$\nabla_{[\mu}F_{\rho\sigma]} = 0.$$  \hspace{1cm} (6)

Among all solutions of the theory, we only consider here the particular case $W(\phi) = e^{-2\phi}$ [40]. With this input, one can obtain the bosonic sector $SU(4)$ of $N = 4$ supergravity theory [86] for constant axion field. An argument by Kallosh et al. [87], showed that the extreme solutions saturate the supersymmetry bound of $N = 4$, $d = 4$ supergravity, or dimensionally reduced superstring theory. Moreover, the references [88–90] include other solutions related to black holes where it was shown that the dilaton changes the causal structure of the black hole which leads to the curvature singularities at finite radii. Our interest in the problem was greatly enhanced by the result that low energy limit of string theory includes a scalar dilaton field, which is massless in all finite orders of perturbation theory [91]. The underlying non-extremal dyonic black hole solution for Einstein-Maxwell-dilaton theory, in absence of a scalar potential,
with integration constants which is constrained by the equations of motion, can be expressed as [40]:

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + h(r)d\Omega^2.
\] (7)

In the above \(d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2\) denotes the line element of the unit 2-sphere with the metric functions

\[
f(r) = \frac{(r - r_1)(r - r_2)}{(r + d_0)(r + d_1)}, \quad h(r) = (r + d_0)(r + d_1),
\] (8)

\[
e^{2\phi} = e^{2\phi_0} \frac{r + d_1}{r + d_0},
\] (9)

\[
F_{rt} = \frac{e^{2\phi_0} Q}{(r + d_0)^2}, \quad F_{\theta\phi} = P \sin \theta,
\] (10)

where the parameters \(Q, P\) and \(\phi_0\) are electric charge, magnetic charge, and the value of the dilaton at infinity, with four integrating constant \(r_1, r_2, d_0\) and \(d_1\). Note that this solution is free of boundary conditions.

We are interested here in a massless point like dyonic solution. Thus, for such solutions, we consider the case when \(d_1 = -d_0 = -\Sigma\) and \(r_1 = -r_2 \equiv r_H\), to obtain the following relation [40]

\[
e^{2\phi_0} = \pm \frac{P}{Q}.
\] (11)

Following the argument in [40], we only consider the case of a negative sign, to achieve the massless solution obtained in EMD theory. For that purpose, the nonextremal solution corresponding to the negative sign of Eq. (11), leads to

\[
f(r) = \frac{(r - r_+)(r - r_-)}{(r^2 - \Sigma^2)}, \quad h(r) = (r^2 - \Sigma^2),
\] (12)

\[
e^{2\phi} = -\frac{P}{Q} \left( \frac{r - \Sigma}{r + \Sigma} \right),
\] (13)

\[
F_{rt} = -\frac{P}{(r + \Sigma)^2}, \quad F_{\theta\phi} = P \sin \theta.
\] (14)

The horizon and singularity are located at

\[
r_+ = +\sqrt{\Sigma^2 + 2QP}, \quad r_S = |\Sigma|.
\] (15)

Note that this solution excludes \(r_- = -\sqrt{\Sigma^2 + 2QP}\), as an inner horizon, and the area of the two-sphere shrinks to zero at \(r_S\). Moreover, it is interesting that when the dilaton charge is zero, one obtains a non-extremal black hole, with horizon at \(r_+ = +\sqrt{2QP}\). To see this one can argue that massless solution seems physically acceptable, even with a complex dilaton field at infinity.

Noting Eq. (12), to describe the full massless nonextremal solution, one must choose the negative sign in Eq. (11) with constants \(d_1 = -d_0 = -\Sigma\). By so doing, we find the expression (12) leads to [41]

\[
ds^2 = -\left( \frac{r^2}{r^2 + 2QP} \right) dt^2 + \left( \frac{r^2 + 2PQ}{r^2 + \Sigma^2 + 2QP} \right) dr^2 + (r^2 + 2QP)(d\theta^2 + \sin^2 \theta d\phi^2).
\] (16)

The above metric represents the charged wormhole in EMD theory, which can be obtained from the massless nonextremal dyonic solution. Also, once we pick a value for \(\Sigma = 0\), we recover the solution of the Einstein-Rosen bridge with throat radius \(R_{thro} = \sqrt{2PQ}\).

This line element indicates that there is no singularity in this spacetime. To understand the geometry of the spacetime intuitively, we draw an embedding diagram as follows. Introducing a new radial coordinate as

\[
r^* \equiv \sqrt{r^2 + 2PQ}.
\] (17)

The respective line element (16), for a fixed moment of \(t = \text{const}\), yields

\[
ds_{\text{CW}}^2 = \frac{dr^*}{1 - \frac{2PQ}{r^*}r^*} + r^* d\varphi^2.
\] (18)

To describe the whole scenario, one embeds this metric into three-dimensional Euclidean space described by cylindrical coordinates \((r, \varphi, z)\) as

\[
ds_{\text{EW}}^2 = dz^2 + dr^2 + r^2 d\varphi^2.
\] (19)

Assuming the line elements are equal \(ds_{\text{CW}} = ds_{\text{EW}}\), one can obtain the following relation

\[
Z = \pm \sqrt{2PQ} \arccosh \frac{r^*}{\sqrt{2PQ}}.
\] (20)

This can best be seen visually in Fig. 1 with the \(\phi\) direction for a specific value when \(\Sigma = 0\). As a result the charged wormhole solution, which comes from nonextremal massless dyonic solution, does not violate the null energy conditions. This formula allows to consider wormholes solutions without exotic matter.

In the next section, we study shadows cast by a charged wormholes geometry within the context of the EMD theory.
III. PHOTON MOTION AROUND CHARGED WORMHOLES IN EMD THEORY

In this section, our aim is to discuss the motion of photons around charged wormholes that can be described by evaluating the geodesic equations. To determine the photon geodesics, we consider the conserved quantities with the spacetime metric of charged wormholes. It is clear that the spacetime metric (16) is invariant under time translation and spatial rotation which leads to the conservation of energy and conservation of angular momentum, respectively. Consequently, it is characterized by two constants of motion, i.e., energy \( E = -p_t \) and angular momentum \( L = p_\varphi \). Now one can derive the geodesic equations by using these conserved quantities

\[
\frac{dt}{d\sigma} = \frac{E(r^2 + 2PQ)}{r^2},
\]
\[
\frac{d\varphi}{d\sigma} = \frac{L \csc^2 \theta}{r^2 + 2PQ},
\]

where \( \sigma \) indicates the affine parameter along the geodesic. Since test particle geodesics around the wormholes satisfy the Hamilton-Jacobi equation

\[
\frac{\partial S}{\partial \sigma} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu},
\]

and for photons \( (m_0 = 0) \), Eq. (22) has a solution of the following form

\[
S = -Et + L\varphi + S_r(r) + S_\theta(\theta),
\]

where \( S_r \) and \( S_\theta \) are the functions of \( r \) and \( \theta \), respectively. On substituting (23) into (22) as well as contravariant component of metric \( g^\mu\nu \), and separating the terms of variables \( r \) and \( \theta \) equal to the Carter constant \( (\pm \mathcal{K}) \) [92], we obtain

\[
\frac{r\sqrt{r^2 + 2PQ}}{\sqrt{r^2 + \Sigma^2 + 2PQ}} \frac{dr}{d\sigma} = \pm \sqrt{\mathcal{R}},
\]
\[
(r^2 + 2PQ) \frac{d\theta}{d\sigma} = \pm \sqrt{\Theta},
\]

where \( \mathcal{R} \) and \( \Theta \) have the following forms

\[
\mathcal{R} = E^2(r^2 + 2PQ) - \frac{r^2}{(r^2 + 2PQ)}(\mathcal{K} + L^2),
\]
\[
\Theta = \mathcal{K} - L^2 \cot^2 \theta.
\]

Having the geodesic equations, we turn our attention towards the radial motion of photons around the throat of charged wormholes, which can be demonstrated by the calculation of the effective potential

\[
\left( \frac{dr}{d\sigma} \right)^2 + V_{\text{eff}} = 0, \tag{26}
\]

with

\[
V_{\text{eff}} = -\frac{r^2 + \Sigma^2 + 2PQ}{r^2} \left[ E^2 - \frac{r^2(\mathcal{K} + L^2)}{(r^2 + 2PQ)^2} \right]. \tag{27}
\]

It is clear from the expression of the effective potential that it has dependency on the charges \( P, Q \), and \( \Sigma \) as well angular momentum. One can visualize the behavior of the effective potential from Fig. 2 with angular momentum and dilaton charge \( \Sigma \). When there is an increase in the value of angular momentum \( L \) and dilaton charge \( \Sigma \), it turns out there is an increase in the peak of effective potential where the unstable circular orbits forms. There occurs three kinds of photon trajectories around the throat of wormholes: (i) scattered, or (ii) form circular orbits, or (iii) traversed to the other region (cf. Fig. 3).

It is be convenient to reduce the number of parameters by defining the impact parameters such that \( \xi = L/E \) and \( \eta = \mathcal{K}/E^2 \). Photon geodesics can be expressed in terms of these impact parameters \( (\xi, \eta) \). We can rewrite \( \mathcal{R} \), in terms of impact parameters as

\[
\mathcal{R} = -E^2 \left[ r^2 + 2PQ - \frac{r^2(\eta + \xi^2)}{(r^2 + 2PQ)} \right]. \tag{28}
\]

The study of effective potential reveals the existence of unstable circular orbits of constant radius around the throat of wormholes. These orbits are very important from the point of view of optical observations.
IV. SHADOW OF CHARGED WORMHOLES

In the preceding section, we investigate the shadow cast by the charged wormholes in EMD theory. As discussed earlier the wormhole is a tunnel-like structure connecting different region of spacetime and it does not contain any horizon or singularity. Moreover, wormholes have gravitational tidal forces so weak that can be assumed to be bearable by a human. The throat of the wormhole plays a crucial role, as we showed earlier there exists unstable photon orbits revolving around it infinite times before reaching the observer.

We consider the situation where one of the spacetime regions that is connected by the wormholes illuminated by a distant light source. A distant observer can be able to observe the scattered photons from the second visible region of the spacetime. The photons which are plunging, consequently traversing through the wormholes cannot be observed. These photons form a dark spot on the luminous background is known as shadow of wormhole. The unstable circular photon orbits give the boundary of the shadow. Equations that determine the unstable circular orbits can be expressed as

\[ R = 0, \quad R' = 0, \]

where prime (') represents derivative with respect to \( r \). Substituting (28) into (29), one can easily derive

\[ \eta = 8PQ - \xi^2, \quad r = \sqrt{2PQ}. \]

Note that the impact parameter \( \eta \) has a dependency on the magnetic charge \( P \) as well of the electric charge \( Q \), but it is independent on dilaton charge \( \Sigma \).

Equation (30) determines the boundary of shadow and a distant observer can observe its projection in ‘his sky’. Therefore one needs to define the celestial coordinates \((\alpha, \beta)\) in the observer’s sky and relate them with impact parameters \((\xi, \eta)\). These celestial coordinates can be defined [47, 48] as

\[
\alpha = \lim_{r_0 \to \infty} \left( r_0^2 \sin \theta_0 \frac{d\varphi}{dr} \right),
\]

\[
\beta = \lim_{r_0 \to \infty} r_0^2 \frac{d\theta}{dr}, \quad (31)
\]

where \( r_0 \) is the distance from the wormholes to the observer and \( \theta_0 \) is the angular coordinate of the observer or one can say the inclination angle. On substituting the expressions of four-velocities into (31), and after some straightforward calculations we obtain the following form of celestial coordinates

\[
\alpha = -\xi \csc \theta_0, \quad \beta = \pm \sqrt{\eta - \xi^2 \cot^2 \theta_0}. \quad (32)
\]

Having the expressions of celestial coordinates and impact parameters, we are in a position to construct the shadow of charged wormholes. In order to obtain the shape of shadow, we plot \( \alpha \) vs \( \beta \) that gives the boundary of shadow in the observer’s sky. The shapes of the shadow for the charged wormholes can be visualized from Fig. 4. We have shown the behavior of shadow by varying the magnetic \( P \) and electric \( Q \) charge in the equatorial plane \( \theta_0 = \pi/2 \). We find that the shape of the shadow is a perfect circle and it is affected by the presence of charge (cf. Fig. 4). As a consequence we can say that the shadow of charged wormholes monotonically increases with both of the charges \( P \) and \( Q \).

V. CONCLUSION

Imaging the shadow of compact object has astrophysical significance because it provides strong meaningful evidence regarding these objects, and further one can study the nature of these objects. This phenomenon is not limited to the black holes, one can assume it for wormholes as well. As, wormholes are the most exotic objects predicted by general relativity, which acts as tunnel linking two spacetimes as a two-way interface, in the same (or not) Universe. In this paper, we have constructed the shadow of charged wormholes in EMD theory. The spacetime metric of charged wormhole has three different types of charges, namely, magnetic charge \( P \), electric charge \( Q \), and dilaton charge \( \Sigma \). We started by calculating the photon geodesics and explained the possible trajectories of it around the charged wormholes in EMD theory. Then, we have discussed the behavior of the effective potential with angular momentum \( L \) and dilaton charge \( \Sigma \). Next we evaluated the impact parameters and celestial coordinates to image the boundary of shadow for charged wormholes. We found that the shape of shadow is a perfect circle for charged wormholes and it is affected by the charges \( P \) and \( Q \). A graphical illustration of wormhole’s shadow was constructed. As a consequence we found that the radius of the shadow monotonically increases with both magnetic charge \( P \) and electric charge \( Q \).
FIG. 4: Plots showing the shapes of the shadow of charged wormhole in EMD theory by varying the charges with $\theta_0 = \pi/2$.

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