Superdiffusive transport of energy in generic Luttinger liquids

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Metals in one spatial dimension are described at the lowest energy scales by the Luttinger liquid theory. It is well understood that this free theory, and even interacting integrable models, can support ballistic transport of conserved quantities including energy. In contrast, realistic Luttinger-liquid metals, even without disorder, contain integrability-breaking interactions that are expected to lead to thermalization and conventional diffusive linear response. We show that the expansion of energy when such a non-integrable Luttinger liquid is locally heated above its ground state shows superdiffusive behavior (i.e., spreading of energy that is intermediate between diffusion and ballistic propagation), by combining an analytical anomalous diffusion model with numerical matrix product state calculations.

Quantum many-body systems are now, thanks to recent developments, understood to support multiple universal classes of dynamical behavior at long length and time scales. Systems may fail to thermalize to the conventional Gibbs ensemble because there exist an infinite number of (sufficiently local) conservation laws: two well-studied examples in one spatial dimension include many-body localized systems\(^ \text{1,2} \) and quantum integrable models\(^ \text{3,4} \). However, most realistic condensed matter systems do not have more than a few conservation laws, and the Gibbs ensemble or thermal state based on these is still believed to be the asymptotic state of the system. The approach to the thermal state in such a system is usually assumed to be described either by conventional hydrodynamics, if momentum is conserved, or by diffusion.

The point of this work is to argue that a simple problem of energy transport in realistic Luttinger liquids generates a type of anomalous or nonlinear diffusion, even though the system is non-integrable, thermalizing, and described in other aspects by conventional linear response. The Luttinger liquid is the generic metallic state of interacting one-dimensional fermions, analogous to the Fermi liquid in higher dimensions but with several fundamental differences\(^ \text{7} \). The low-energy limit of the Luttinger liquid is a free bosonic theory, but real Luttinger liquids contain integrability-breaking perturbations that are responsible for thermalization. The irrelevance of these perturbations leads to superdiffusive behavior when energy expands from an initial finite heated region into the ground state. We study this type of rapid energy spread in part because of experiments using laser irradiation of a small region to generate an outward flux of heat in a solid\(^ \text{5,6} \). These could be performed on spin chain materials or others where thermal transport has been argued to show signs of near-integrability, although disentangling disorder and open-system effects can be complex\(^ \text{1,3} \).

The problem of expansion of excitations into a region previously in the ground state has been studied in many models and received new impetus with the advent of dynamical measurements on ultracold atomic gases\(^ \text{12} \). Two illustrative classes of possible behaviors come from considering classical physics: first, the case of free particles whose different velocities lead to dispersion, and second, the classical fluid limit in which interactions lead to nonlinear behavior and propagating wavefronts. Both these cases lead to ballistic behavior, and some kinds of interactions in one dimension lead to integrable models that also have this ballistic property. A third class covers diffusive behavior, for example of Brownian particles. Diffusion implies a parametrically slower rate of spreading of either particles or energy, with finite linear-response transport coefficients. The results presented here show that even simple, well-studied problems in quantum condensed matter physics lead to long-time scaling that is distinct from these three standard possibilities. Note that the superdiffusion described in the present work is distinct from that known to exist in momentum-conserving many-body systems\(^ \text{13,14} \), in which linear-response coefficients are not finite in the thermodynamic limit but rather diverge as a power-law in system size.

Our approach is to start with an explicit example of a local lattice Hamiltonian that shows superdiffusive behavior and can be studied quantitatively using time-dependent density-matrix renormalization group methods. We then present a simplified model of this behavior that is equally applicable to a broad class of Luttinger liquids, because the superdiffusive behavior originates in the continuous variation of the scaling dimensions of irrelevant integrability-breaking operators in the Hamiltonian. (Recall that continuous variation of the electron operator’s scaling dimension leads to the well-known power-laws in electron tunneling into a Luttinger liquid\(^ \text{15} \).) The special aspect of energy expansion into the ground state of a realistic Luttinger liquid is that the system is never fully in the linear-response regime because of
the singular zero-temperature thermal conductivity. The result is an anomalous diffusion equation with solutions of Barenblatt-Pattle type, which exhibit superdiffusive space-time scaling.

**Explicit model and initial condition:** For a microscopic realization of universal Luttinger liquid physics that is amenable to numerical simulation, we consider a spin-1/2 XXZ chain in the presence of a staggered magnetic field, with Hamiltonian

\[
H = \sum_{i=1}^{N} S_i^{\sigma} S_{i+1}^{\sigma} + \sum_{i} S_i^{\nu} S_{i+1}^{\nu} + \Delta S_i^{z} S_{i+1}^{z} + (-1)^i \hbar \sigma_i^z. \tag{1}
\]

This model was studied in previous work, and the staggered field can be verified to break integrability of the spin-1/2 XXZ chain by a level-statistics analysis. Meanwhile, the effect of the staggered field perturbation on the low-energy physics of the system can be determined via bosonization. For infinitesimal \(h\), the bosonized Hamiltonian can be written as

\[
H = \frac{u}{2} \int_0^L dx \left( \Pi^2 + (\partial_x \phi)^2 \right) + c h \int_0^L dx \cos \left( 2\sqrt{\pi K} \phi \right) + H_{\text{umklapp}} + H_{\text{band curvature}} + H_{\text{higher terms in } h} \tag{2}
\]

where the momentum and phase degrees of freedom satisfy canonical commutation relations \([\Pi(x), \phi(y)] = i\hbar (x - y)\). Here, the Luttinger parameter \(K\) is given by the Bethe ansatz result \(2K \cos^{-1}(-\Delta) = \pi\), and various other coupling constants can be determined exactly. From the scaling dimension \(\hbar = 2 - K\), it follows that the staggered field is relevant and opens a gap for \(K < 2\) or \(-\sqrt{2}/2 < \Delta \leq 1\). However, for \(K > 2\), or \(-1 < \Delta < -\sqrt{2}/2\), this perturbation is irrelevant and the model remains in a gapless Luttinger liquid phase.

**Low-temperature hydrodynamics.** In earlier work, it was demonstrated using bosonization that in the gapless regime \(K > 2\) of the model, the DC conductivity scales with temperature as

\[
\sigma_c(T) \sim T^{\nu(K)}, \quad T \to 0, \tag{3}
\]

where \(\nu(K) = 3 - 2K\). This is an instance of a very general scenario whereby perturbing a Luttinger liquid with an irrelevant vertex operator leads to a non-trivial power-law dependence on temperature in the low-\(T\) charge conductivity, which scales continuously with the Luttinger parameter \(K\). It is natural to expect the same phenomenon for the thermal conductivity \(\kappa(T)\), with

\[
\kappa(T) \sim T^{\lambda(K)}, \quad T \to 0, \tag{4}
\]

for some exponent \(\lambda(K) < 0\) that depends on \(K\) and the scaling dimension of the irrelevant perturbation. For example, the assumption that \(\sigma_c(T)\) and \(\kappa(T)\) are related by Wiedemann-Franz scaling \(\kappa(T) \sim T \sigma_c(T)\) would imply that \(\lambda(K) = 1 + \nu(K)\). Indeed this holds for the tunneling electrical and thermal conductances through a single impurity in a Luttinger liquid, although the Lorenz number (the coefficient of the Wiedemann-Franz ratio) is modified from its Fermi liquid value.

In general, one should not assume that \(\lambda(K)\) and \(\nu(K)\) are always so simply related (at least it is not clear to us that this must be the case for all integrability-breaking perturbations), but even without a specific value for \(\lambda(K)\), the ansatz Eq. (1) has striking consequences for linear-response thermal transport. To see this, note that in the linear-response regime and to leading order in temperature, Eq. (4) implies that temperature gradients give rise to heat currents according to \(j_0(x) \sim -CT^\lambda \partial_x T(x)\). Let us now consider states of the model that are in local thermodynamic equilibrium, in the sense that upon coarse-graining over a suitably small microscopic length scale \(l \ll L\), they are well described by a smoothly-varying average temperature distribution \(T(x,t)\). For flows in such states that are driven purely by temperature gradients, the heat current coincides with the energy current, and we can write down a hydrodynamic equation

\[
\partial_t \rho_E = \partial_x \left( CT^\lambda \partial_x T \right) \tag{5}
\]

for the flow of energy density \(\rho_E(x,t)\), which is expected to hold to leading order in \(T\) and its gradients. At low temperatures, it is also true (by Fermi-Dirac statistics) that the temperature dependence of \(\rho_E(x,t)\) is fixed by a local equation of state, of the form \(\rho_E(x,t) \sim BT(x,t)^2\). This gives rise to the non-linear diffusion equation

\[
\partial_t \rho_E = D \partial_x^2 (\rho_E^m) \tag{6}
\]

for \(\rho_E\), where the exponent \(m\) is given in terms of \(\lambda\) by \(m = (\lambda + 1)/2\). For a Fermi liquid, \(\lambda = 1\) and we recover ordinary diffusion of heat. However, in the context of weakly perturbed Luttinger liquids, for which we expect that \(\lambda \neq 1\) in general, more exotic scenarios can arise. If \(\lambda > 1\), Eq. (6) is the porous medium equation, whose solutions are characterized by subdiffusive space-time scaling, while if \(\lambda < 1\), this equation becomes the fast diffusion equation, whose solutions show superdiffusive space-time scaling. A transparent way to see this is from the fundamental solution of Eq. (6), which for \(\lambda > -1\) is the so-called “Barenblatt-Pattle” solution to the non-linear diffusion equation. Such solutions are characterized by a space-time scaling that varies continuously with \(\lambda\),

\[
x \sim t^\alpha, \quad \alpha = \frac{2}{\lambda + 3}. \tag{7}
\]

Thus “weakly perturbed” Luttinger liquids, whose low temperature thermal conductivity exhibits the power law dependence of Eq. (4), may exhibit a continuous range of space-time scaling exponents in their thermal transport. We now present numerical evidence for superdiffusive transport of heat, in the regime of weak integrability breaking for the Hamiltonian Eq. (1). We find that...
within this model, the spreading of thermal wavepackets is characterized by a single superdiffusive exponent $2/3 < \alpha < 1$, which can be tuned by varying the strength of the integrability-breaking staggered field $h$.

Note that collapse with a single exponent is not consistent with spreading (characterized by moments of the distribution, for example) determined by a ballistically propagating front with a weight that decays as a power-law in time, plus a central thermalized region. Rather, there is a single limit shape that expands with a single scaling behavior. We illustrate this behavior and then discuss its detailed relation to non-linear diffusion.

**Numerical calculations.** In order to demonstrate anomalous low-temperature thermal transport in Luttinger liquids, we perform DMRG simulations of the microscopic model (2) at finite temperature. The model parameters are first set to $\Delta = -0.85$ and $h = 0.2$, which were found in previous work to generate a Luttinger liquid with effective Luttinger parameter $K \approx 2.4$. The initial data for our numerical simulation consists of a localized heated region, with temperature distribution

$$\beta(x) = \beta - (\beta - \beta_M) e^{-(x/L)^2}. \quad (8)$$

In Fig. 1, we find clear evidence for superdiffusive, rather than diffusive, transport, both at the level of a naive scaling of the thermal wavepacket, and in the scaled logarithmic time derivatives of its absolute moments, which for non-linear diffusion with a single exponent $\alpha$ should collapse to a single value at long times,

$$\frac{1}{n} \frac{d \log \langle |x|^n \rangle (t)}{d \log t} \to \alpha, \quad t \to \infty. \quad (9)$$

Both analyses are consistent with the superdiffusive exponent $\alpha \approx 0.9$.

We next consider the effect of varying the integrability-breaking staggered field $h$. The natural expectation is that increasing the strength of the integrability-breaking perturbation leads to a decrease in the exponent $\alpha$, bringing transport closer to normal diffusion. This is consistent with the numerical results depicted in Fig. 2.

We now discuss more carefully the relation between these numerical results and the model proposed in the previous section. Strictly speaking, our model predicts that superdiffusive spreading of a localized heated region will persist indefinitely if the bulk temperature $T = 0$. In the more realistic scenario of a small, non-zero bulk temperature $T > 0$, we expect that wavepacket spreading will transition from superdiffusive to diffusive behaviour after some characteristic timescale $t_D(T) \sim T^{-1}$, that diverges faster than $T^{-1}$ as $T \to 0$. The temperature dependence of this timescale follows by linearizing the non-linear diffusion model Eq. (5) about a constant bulk temperature. To corroborate this picture, we have checked numerically that by increasing the bulk temperature $T$, the timescale $t_D(T)$ can be brought down until the effective exponent begins to decrease towards $\alpha = 0.5$ on the numerically accessible timescale. An example for this is shown in the inset of Fig. 2. (One expects that the same holds true for sufficiently shallow wavepackets, but verifying this is beyond the reach of our numerics.) For the low bulk temperature $T = 1/12$ considered in Fig. 1 and the main plot of Fig. 2, our results indicate that the numerically accessible timescale ($t \sim 50$) is in a regime $t \ll t_D(T)$ during which the dynamics is superdiffusive. That this dynamics represents genuine anomalous diffusion, rather than a generic transient en route to diffusion, is demonstrated by the numerical observation that effective exponents obtained from different moments of the wavepacket converge to the same, superdiffusive value, as in Eq. (9). We have additionally checked that in the limit of bulk temperature $T = 0$, for which we expect $t_D \to \infty$, superdiffusion is observed on accessible timescales (results not shown). This was simulated by initializing the system in the ground state of the Hamiltonian $H' = H + \delta H$, with $H$ given by Eq. (1) and $\delta H$ a localized inhomogeneity near $x = 0$, before time-evolving numerically under $H$ using pure state tDMRG.
FIG. 2. Main figure: Decrease of the effective superdiffusion exponent as the strength of the integrability-breaking staggered field \( h \) is increased. The initial wavepacket is given by Eq. (8) with \( \beta = 12, \beta_M = 8, L = 2, \) the model anisotropy is set to \( \Delta = -0.99 \) and only \( h \) is varied. Effective exponents are computed from logarithmic time derivatives of absolute moments \( n = 2, 3, 4. \) Inset: Time evolution of a higher temperature wavepacket with \( \beta = 1, \beta_M = 0.2, L = 2, \) at anisotropy \( \Delta = -0.99 \) and staggered field \( h = 0.49. \) Absolute moments \( n = 2, 3, 4, 5 \) demonstrate near-diffusive exponent \( \alpha \approx 0.58 \) (dashed line).

Thus the numerical collapse to a single exponent depicted in Figs. 1 and 2 indicates that our simple hydrodynamic model for propagation of heat in weakly perturbed Luttinger liquids, Eq. (4), is at least qualitatively correct, since it predicts that spreading of localized initial wavepackets at low temperature should be controlled by a single superdiffusive exponent, \( \alpha. \) On the other hand, the splitting of the wavepacket into a doubly-peaked structure, as depicted in Fig. 1, is markedly different from the shape of the Barenblatt-Pattle fundamental solution to the fast diffusion equation\(^{21}\), which exhibits a single maximum for all time. Moreover, the doubly-peaked structure appears to be somewhat robust to the details of the localized initial wavepacket, as shown in Fig. 3. This suggests that a more refined model than Eq. (6) is required to capture the precise shape of the superdiffusing wavepacket.

Discussion. We have shown that superdiffusive spreading of heat can occur in a generic class of non-integrable, thermalizing, one-dimensional physical systems. This can be understood from a simple theoretical model, which assumes only that the temperature-dependence of the heat conductivity in a weakly perturbed Luttinger liquid is given by a power law, \( \kappa(T) \sim T^\lambda, \) that diverges at low temperature.

One desirable goal for future work is a direct calculation of the low-temperature behaviour of the thermal conductivity, \( \kappa(T) \). Analytical methods that capture the charge conductivity in a weakly perturbed Luttinger liquid\(^{16,19}\) rely on the Dyson series for computing correlation functions, and do not readily generalize to the four-point functions that are required for thermal conductivities. Similarly, obtaining the power law accurately appears to be beyond the present state of the art for tDMRG methods.

An interesting question concerns the importance of proximate integrability in the systems under consideration. The simple anomalous diffusion model that we propose above captures the key qualitative feature of thermal wavepacket spreading in these systems, namely superdiffusion characterized by a single scaling exponent. However, the shape of the spreading wavepacket at low temperature differs from the simplest Barenblatt-Pattle form. One possible explanation for the discrepancy is that the spreading of the energy distribution and the consequent decrease of energy density violate the local thermalization assumption in the anomalous diffusion model: energy moves through a region more rapidly than the region can fully thermalize. The recently developed hydrodynamics of quantum integrable systems\(^{27,28}\) might provide a starting point for analyzing such effects, although...
it is unclear how to account for integrability-breaking physics within this formalism.  

Another direction for future work is to extend the current treatment to coupled charge and energy transport in systems away from half-filling, when thermopower effects become important. A subtlety is that the scaling of thermopower in a generic Luttinger liquid will be controlled by the leading integrability-breaking perturbation that also breaks particle-hole symmetry, which for the model (2) is the band curvature correction, distinct from the perturbation that controls thermal conductivity.

Such refinements of the theory notwithstanding, our numerical results are consistent with an emerging understanding that for low-dimensional physical systems, the usual dichotomy between ballistic and diffusive transport can break down, in contexts ranging from classical integrable models, and disordered quantum one-dimensional systems near the many-body localization transition. The fact that anomalous heat transport can arise from generic perturbations to the well-studied Luttinger liquid indicates that the full richness of transport in low-dimensional quantum systems remains to be explored.

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