Anomalous dimensions of leading twist conformal operators

A.V. Belitsky\textsuperscript{a}, J. Henn\textsuperscript{b}, C. Jarczak\textsuperscript{c,d}, D. Müller\textsuperscript{e}, E. Sokatchev\textsuperscript{b}

\textsuperscript{a}Department of Physics and Astronomy, Arizona State University
Tempe, AZ 85287-1504, USA

\textsuperscript{b}Laboratoire d’Annecy-le-Vieux de Physique Théorique LAPTH
B.P. 110, F-74941 Annecy-le-Vieux, France

\textsuperscript{c}Institut Fourier, Laboratoire de Mathématiques
B.P. 74, F-38402 St Martin d’Hères, France

\textsuperscript{d}Laboratoire de Physique, Groupe de Physique Théorique, ENS Lyon
46, Allée d’Italie, F-69364 Lyon, France

\textsuperscript{e}Institut für Theoretische Physik II, Ruhr-Universität Bochum
D-44780 Bochum, Germany

Abstract

We extend and develop a method for perturbative calculations of anomalous dimensions and mixing matrices of leading twist conformal primary operators in conformal field theories. Such operators lie on the unitarity bound and hence are conserved (irreducible) in the free theory. The technique relies on the known pattern of breaking of the irreducibility conditions in the interacting theory. We relate the divergence of the conformal operators via the field equations to their descendants involving an extra field and accompanied by an extra power of the coupling constant. The ratio of the two-point functions of descendants and of their primaries determines the anomalous dimension, allowing us to gain an order of perturbation theory. We demonstrate the efficiency of the formalism on the lowest-order analysis of anomalous dimensions and mixing matrices which is required for two-loop calculations of the former. We compare these results to another method based on anomalous conformal Ward identities and constraints from the conformal algebra. It also permits to gain a perturbative order in computations of mixing matrices. We show the complete equivalence of both approaches.
1 Introduction

The framework of the Wilson-Kadanoff operator product expansion [1] for correlation functions of field operators in quantum field theory, which determine physical observables, enormously facilitates the analysis of their short- (or light-cone-) distance structure. Since its discovery it has found a large range of applications stretching from phase transition phenomena in condensed matter physics to scattering amplitudes in four-dimensional gauge theories. The main advantage of the formalism is that it allows one to evaluate a product of field operators near the short or light-cone distance singularity in terms of composite operators. In an interacting field theory, the latter mix among each other under renormalization and acquire nontrivial anomalous dimensions. Their evaluation at higher orders in the coupling constant is one of the goals of perturbative field theory.

The recent surge of interest in anomalous dimensions of Wilson operators in the maximally supersymmetric Yang-Mills theory (\( \mathcal{N} = 4 \) SYM) was inspired by the gauge/string correspondence which identifies them with the energies of excitations in the dual description in terms of a string theory on an AdS_5 \times S^5 background [2]. The strong/weak nature of this duality makes its straightforward tests difficult, since the perturbative coupling expansion windows in both theories do not overlap. Due to complications in the quantization of string theories on warped backgrounds, such direct tests would require an exact evaluation of anomalous dimensions in gauge theory. In practical terms, what one can realistically do is to perform multiloop computations, then use results as initial data for the conjectured integrability [3] of \( \mathcal{N} = 4 \) SYM. In this way one may be able to determine the strong-coupling asymptotics of the anomalous dimensions and compare it to the string theory predictions.

The goal of the present study is to develop a formalism for efficient multiloop calculations of anomalous dimensions of a certain class of Wilson operators in \( \mathcal{N} = 4 \) SYM, namely operators of leading twist. An important feature of this model is that it stays anomaly-free even quantum mechanically and thus preserves all classical symmetries of the Lagrangian. The central object of our consideration is a correlation function of two conformal primary operators of leading twist. The conformal symmetry of the model completely determines (up to normalization) their functional dependence on the space-time interval between these points with the exponent given by the scaling dimension of the composite operators (given by the sum of their canonical and anomalous dimensions). A key observation for our formalism is that twist-two conformal primary operators lie near the conformal unitarity bound and hence are conserved in the non-interacting theory. However, they acquire a non-vanishing divergence in the interacting theory. It is obtained by applying the field equations of motion and thus is proportional to the coupling constant. In conformal terms, this divergence defines a particular conformal descendant of the primary operator. The idea of the method is to compute the ratio of the two-point correlation functions of descendants and of their primaries. This ratio is proportional to the anomalous dimension and involves an overall factor of two powers of the coupling constant. Thus, in order to evaluate the anomalous dimension at order \( n \) in perturbation theory, it is sufficient to compute the two correlators at order \( n - 1 \) and thus gain an order of perturbation theory.

The method we describe here has been first used in [3] to calculate the one-loop anomalous dimensions of some twist-two operators of low spin in \( \mathcal{N} = 4 \) SYM and then generalized to arbitrary spin in [4]. In this paper we present a simpler version of [4] which does not require

1The literature on the subject is vast, we therefore refer the reader to comprehensive proceedings of a recent workshop for details, \( \text{http://www-spht.cea.fr/Meetings/Rencitz2007/agenda.php} \).
supersymmetry. A similar method was applied to obtain the two- \[5\] and three-loop \[6\] anomalous dimension of the Konishi operator and of the twist-three operator of the BMN series.

It is a common knowledge that conformal symmetry is broken provided that the renormalization group function \(\beta\) of the coupling constant is non-vanishing. This is a direct consequence of dimensional transmutation which generates an intrinsic mass scale in the theory, modifying the scaling behavior of correlation functions. In perturbative calculations one has to use a regularization procedure for ultraviolet divergences to render correlation functions finite. The only consistent regularization method for non-Abelian gauge theories is dimensional regularization or its spin-off, dimensional reduction. However, neither of them preserves all space-time symmetries of the regularized theory, in particular, they violate the scaling and special conformal boosts symmetries. This has profound consequences for the form of the correlation functions even when the regulator is eliminated. Namely, the non-zero \(\beta\)-function in \(D = 4 - 2\varepsilon\) dimensions is \(\beta_\varepsilon(g) = -2\varepsilon g + \beta(g)\) and it induces an anomaly in the trace of the energy-momentum tensor. The renormalization of the product of the renormalized energy momentum tensor and conformal operators generates anomalous dimensions of the latter and leads to their mixing under conformal boosts, as we will demonstrate in this study. Even when the four-dimensional \(\beta\)-function is zero to all orders of perturbation theory, the conformal symmetry is violated for \(\varepsilon \neq 0\). Subtracting divergences and sending \(\varepsilon \to 0\) afterwards, one generates symmetry breaking contributions to the dilatation operator coming from terms \(\sim \beta_\varepsilon(g)/\varepsilon\). This source of the symmetry breaking is a peculiar feature of dimensional regularization rather than an intrinsic property of the dilatation operator. In other words, in gauge theories with vanishing \(\beta\) function the conformal symmetry breaking terms can be removed by performing a scheme transformation of the dilatation operator and by going over to the so-called conformal scheme. This transformation does not affect the eigenvalues of the dilatation operator but it does change the form of the corresponding eigenstates, and is required for evaluation of correlation functions to preserve their diagonality.

Our subsequent presentation is organized as follows. In Sect. 2 we give a general introduction to the method of computing anomalous dimensions of leading twist operators by differentiation, applicable to any conformal any field theory. In Sect. 3 we begin with a one-loop calculation of anomalous dimensions of conformal operators in six-dimensional scalar field theory with cubic interaction. Then we turn to two-loop order and demonstrate that conformal symmetry is broken and bare conformal operators start to mix in the minimal subtraction scheme. We introduce the so-called conformal scheme which preserves the autonomous renormalization group equation for conformal operators. We demonstrate the mixing for the scalar theory making use of an appropriate two-point correlation function of a conformal operator and its descendant. We explain how this method allows us to gain an order of perturbation theory. We then turn to \(\mathcal{N} = 4\) SYM and compute the mixing matrix. Section 4 is dedicated to an alternative computation of the mixing matrix within a different formalism to evaluate the same quantity making use of conformal Ward identities and commutator constraints stemming from conformal algebra. Finally, we conclude.

**2 Anomalous dimensions by differentiation**

We are interested in renormalization properties of operators of leading twist (i.e., dimension minus spin) in \(D = 2h\) dimensions and will restrict ourselves to those that are build from scalar
fields. We are going to outline our method using a simple six-dimensional model ($\phi^3$ theory),

$$\mathcal{L} = \sum_{i=1}^{N_f} (\partial_\mu \phi_i) (\partial^\mu \bar{\phi}_i) + \frac{1}{2} (\partial_\mu \chi) (\partial^\mu \chi) + g \sum_{i=1}^{N_f} \phi_i \bar{\phi}_i \chi,$$

(2.1)

which is conformally invariant at the classical level. Moreover, its $\beta$ function \cite{7}

$$\beta(g) = -\beta_0 \frac{g^3}{(2\pi)^3} + O(g^5) \quad \beta_0 = \frac{1}{6} - \frac{1}{24} N_f$$

(2.2)

can be made vanish up to order $g^5$ in coupling constant by choosing $N_f = 4$. This allows us to use conformal symmetry arguments up to this order of perturbation theory. For the specific operators that we are interested in, the treatment of $\mathcal{N} = 4$ SYM requires only minor changes.

The $\mathcal{N} = 4$ SYM Lagrangian in Minkowski space has the form\footnote{We use conventions of Ref. \cite{8}.}

$$\mathcal{L}_{\mathcal{N}=4} = \text{tr} \left\{ -\frac{1}{2} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} (D_\mu \phi^{AB}) (D^\mu \phi_{AB}) + \frac{1}{8} g_{YM}^2 [\phi^{AB}, \phi^{CD}][\phi_{AB}, \phi_{CD}] 
+ 2i \hat{\lambda}_{A A} \sigma_\mu ^{A} D_\mu \lambda_B \right. 
- \sqrt{2} g_{YM} \lambda_{\alpha}^{A} [\phi_{AB}, \lambda_{\alpha}^{B}] + \sqrt{2} g_{YM} \bar{\lambda}_{\dot{A} A} [\phi^{AB}, \bar{\lambda}_{\dot{B}}^{\dot{A}}] \left. \right\}. $$

(2.3)

In the following we perform perturbative expansions with respect to the coupling $g = \sqrt{N_c g_{YM}}$.

## 2.1 Preliminaries

The composite operators under consideration are built from two field operators and are generically written as

$$O^{\mu_1 \cdots \mu_j} = \sum_{k=0}^{j} a_{jk} \partial^{(\mu_1} \cdots \partial^{\mu_k} \varphi \partial^{\mu_{k+1}} \cdots \partial^{\mu_j)} \, \varphi,$$

(2.4)

where $\{ \cdots \}$ stands for traceless symmetrization, and $\varphi$ stands for an elementary scalar field of canonical dimension $h - 1$ in $D = 2h$ space-time dimensions. In $\mathcal{N} = 4$ SYM they have leading twist two, in the $\phi^3$ theory they have twist four. From Eq. (2.4) we can see that for a given spin $j$ there are in general $j + 1$ coefficients $a_{jk}$ to determine (one of which is just an overall normalization).

There exists a continuous series of unitary irreducible representations of the conformal group characterized by the conformal dimension $d$ and the Lorentz spin $j > 0$ (symmetric traceless tensor). Unitarity requires that

$$d \geq 2(h - 1) + j.$$

(2.5)

When the bound is saturated, these representations become reducible and one can impose irreducibility conditions. For the operators $O^{\mu_1 \cdots \mu_j}$ this happens in the free field theory, where they possess the canonical dimension

$$d_{0,j} = 2(h - 1) + j.$$

(2.6)

Then the irreducibility condition implies that the $O^{\mu_1 \cdots \mu_j}$ are conserved tensors:

$$\partial_\mu O^{\mu \mu_2 \cdots \mu_j} = 0.$$

(2.7)
Equations (2.7) allows one to fix the coefficients $a_{jk}$ in (2.4) up to an overall normalization.

Before writing down the solution, however, let us introduce an efficient tool for traceless symmetrization of the vector indices [9]. It consists in projecting all the vector indices of a given tensor with a null vector $z^\mu$, $z^2 = 0$, thus automatically symmetrizing Lorentz indices and suppressing all the traces. Thus we define the projected operator

$$\hat{O}_j \equiv O^{\mu_1 \cdots \mu_j} z_{\mu_1} \cdots z_{\mu_j} = \sum_{k=0}^{j} a_{jk} \hat{\partial}^k \varphi \hat{\partial}^{j-k} \bar{\varphi},$$

(2.8)

where we introduced the notation $\hat{\partial} = z^\mu \partial_\mu$. To free some indices from contractions with $z^\mu$-vectors (e.g., in order to take a divergence), one has to differentiate with respect to the auxiliary vector $z^\mu$, in the presence of the constraint $z^2 = 0$. The lowest order differential operator which does this is second-order and reads

$$\Delta^\mu = (h - 1 + z \cdot \partial_z) \partial^\mu - \frac{1}{2} z^\mu \partial_z \cdot \partial_z.$$  

(2.9)

So, for instance, we may recover the operator in (2.4) from the projected one in (2.8) by

$$O^{\mu_1 \cdots \mu_j} = N \Delta^{\mu_1} \cdots \Delta^{\mu_j} \hat{O}_j,$$

(2.10)

where $N$ is an inessential normalization constant. The symmetrization and tracelessness of $O^{\mu_1 \cdots \mu_j}$ in (2.10) are automatic due to the properties

$$[\Delta^\mu, \Delta^\nu] = 0, \quad \Delta^\mu \Delta_\mu = 0$$

(2.11)

of the differential operator $\Delta^\mu$. Let us now rewrite (2.8) in a bi-local form,

$$\hat{O}_j = P_j \left( \hat{\partial}_a, \hat{\partial}_b \right) \varphi(x_a) \bar{\varphi}(x_b),$$

(2.12)

where

$$P_j(x, y) = \sum_{k=0}^{j} a_{jk} x^k y^{j-k} \equiv (x + y)^j p_j \left( \frac{x - y}{x + y} \right)$$

(2.13)

is a homogeneous polynomial of degree $j$.

Using the projection variables $z^\mu$ and the differential operator $\Delta^\mu$ introduced in (2.9), we can rewrite (2.7) in the following way:

$$\partial_\mu \Delta^\mu \hat{O}_j = 0.$$  

(2.14)

The advantage of (2.14) over (2.7) now becomes evident: the condition (2.14) turns into a simple differential equation [11] for the polynomial $p_j(x)$ defined in (2.13),

$$\left( (1 - x^2) \frac{d^2}{dx^2} - (2\nu + 1)x \frac{d}{dx} + j(j + 2\nu) \right) p_j(x) = 0,$$

(2.15)

where [3] $\nu = h - 3/2$. Its regular solution is the Gegenbauer polynomial $C_j^\nu(x)$ and hence the operators (2.12) read [12, 13]

$$\hat{O}_j = (\hat{\partial}_a + \hat{\partial}_b)^j C_j^\nu \left( \frac{\hat{\partial}_a - \hat{\partial}_b}{\hat{\partial}_a + \hat{\partial}_b} \right) \varphi(x_a) \bar{\varphi}(x_b), \quad \nu = h - 3/2.$$  

(2.16)

---

3In Minkowski space this is a light-like vector, in Euclidean space it is a complex isotropic vector.

4Owing to $[\Delta^\mu, z^2] = 2z^2 \frac{\partial}{\partial z^\mu} = 0$, this is compatible with the constraint $z^2 = 0$.

5For fields with a light-cone spin $s$ the index is $\nu = h - 3/2 + s$. 
2.2 Anselmi’s trick

Here we illustrate the main idea of our method [3]. We exploit conformal properties of quantum field theories supposing that the renormalization has already been done. When the regulator, e.g., $\varepsilon$ in dimensional regularization, is sent to zero, one expects conformal properties to emerge. As it is well known, renormalization introduces scheme dependence. We define our renormalization scheme (within dimensional regularization) such that conformal primary operators have conformal correlation functions [14].

When interactions are turned on ($g \neq 0$), the scaling dimension of the tensors $\hat{O}_j$ becomes in general coupling-dependent. In the following we assume that the quantum numbers of the composite operators are chosen in such a manner that they form a set closed under renormalization. After the diagonalization of the mixing matrix, the operators have well-defined conformal properties. In particular, their scaling dimension can be simply written as a sum of the canonical $d_{0,j}$ and anomalous $\gamma_j(g^2)$ dimension,

$$d_j = d_{0,j} + \gamma_j(g^2). \quad (2.17)$$

Then the unitarity bound (2.15) is no longer saturated and hence the conservation equation (2.14) turns into the non-conservation equation

$$\partial^\mu \Delta_\mu \hat{O}_j = g\hat{K}_{j-1}. \quad (2.18)$$

The right-hand side defines a (classical) conformal descendant (a state in the infinite-dimensional space of the conformal UIR). In our case, where the primary (lowest weight) state is a bilinear $O$ of conformal dimension $j$ and anomalous $\gamma_j(g^2)$ dimension,

$$\langle \hat{O}_j(x_1) \hat{O}_k(x_2) \rangle = \delta_{jk} C_j(g) \hat{I}^j (x_{12}^2)^{-d_j} \quad (2.19)$$

where

$$\hat{I} \equiv I^\mu z_1^\mu z_2^\nu, \quad I^\mu = \eta^{\mu\nu} - 2x_{12}^{\mu}x_{12}^{\nu}, \quad x_{12}^{\mu} = x_1^{\mu} - x_2^{\mu}, \quad (2.20)$$

and $C_j(g)$ are normalization constants. In (2.19) we have used two independent projection variables, $z_1^\mu$ and $z_2^\nu$, for the operators at points $x_1^\mu$ and $x_2^\nu$, respectively.

The idea now is to take the divergence at both points of (2.19) and to replace the result by the descendant (2.18) \footnote{This equation can be found by requiring covariance under the action of the conformal group, see, e.g., [11].}

$$\partial_1^\mu \Delta_1^{\mu} \partial_2^\nu \Delta_2^{\nu} \langle \hat{O}_j(x_1) \hat{O}_j(x_2) \rangle = g^2 \delta_{\hat{K}_{j-1}(x_1) \hat{K}_{j-1}(x_2)} \quad (2.21)$$

Using the expressions (2.19), (2.20), it is straightforward to carry out the differentiation in (2.21). When it is done, we equate the auxiliary vectors $z_1^\mu = z_2^\mu = z^\mu$, for simplicity. Then we evaluate the ratio to be

$$g^2 \frac{\delta_{\hat{K}_{j-1}(x_1) \hat{K}_{j-1}(x_2)}}{\langle \hat{O}_j(x_1) \hat{O}_j(x_2) \rangle} = -\gamma_j(g^2) j(j + h - 2) \left[ (j + h - 1)(j + 2h - 3) + \gamma_j(g^2)(j^2 + hj - 2j + h - 1) \right]. \quad (2.22)$$

\textsuperscript{7}Equation (2.21) is to be understood for non-coincident points, $x_1 \neq x_2$, otherwise one would need to consider possible contact terms. These terms are irrelevant for our discussion, so we can simply ignore them.
In a conformal field theory, the ratio (2.22) is considered as an exact (non-perturbative) expression. In practice we can calculate its left-hand side only perturbatively. It is important to realize the appearance of the factor \( g^2 \) in the left-hand side of (2.22). If we want to determine \( \gamma_j(g^2) = g^2 (2\pi)^{\gamma_j(0)} + g^4 (2\pi)^{2\gamma_j(1)} + \ldots \),

up to, say, order \((g^2)^n\), we only need to evaluate the correlators on the left-hand side of (2.22) up to order \((g^2)^{n-1}\). In other words, this method allows us to gain one perturbative order in the calculation of \( \gamma_j(g^2) \). This simple observation is the main point in our approach.

3 Evaluation of anomalous dimensions

In this section, we use the general formula (2.22) to perform actual loop calculations. We start with a simple scalar field-theory example and then turn to maximally supersymmetric gauge theory, which is the focus of our study.

3.1 Anomalous dimensions in \( \phi^3 \) theory in six dimensions

Let us consider conformal operators of leading twist,

\[
\hat{O}_j = (\hat{\partial}_a + \hat{\partial}_b) J^{3/2} \left( \frac{\hat{\partial}_a - \hat{\partial}_b}{\hat{\partial}_a + \hat{\partial}_b} \right) \phi_1(x_a) \bar{\phi}^2(x_b),
\]

(3.1)

which are ‘flavor’ non-singlets (to avoid additional mixing) and are thus closed under renormalization\(^8\). The descendants (2.18) are straightforwardly evaluated and are given by

\[
\hat{K}_{j-1} = (\hat{\partial}_a + \hat{\partial}_b + \hat{\partial}_c)^{j-1} k_{j-1} \left( \frac{\hat{\partial}_a + \hat{\partial}_b - \hat{\partial}_c}{\hat{\partial}_a + \hat{\partial}_b + \hat{\partial}_c}, \frac{-\hat{\partial}_a + \hat{\partial}_b - \hat{\partial}_c}{\hat{\partial}_a + \hat{\partial}_b + \hat{\partial}_c} \right) \chi(x_a) \phi_1(x_b) \bar{\phi}^2(x_c),
\]

(3.2)

\(^8\)The case \( j = 1 \) corresponds to the one of the conserved \( U(N_f) \) currents, \( O^\mu = \partial^\mu \phi_1 \bar{\phi}^2 - \phi_1 \partial^\mu \bar{\phi}^2 \), for which \( \gamma_{j=1}(g^2) = 0 \).
where
\[ k_{j-1}(x, y) = 3 \left[ 2C_{j-1}^{5/2}(x) - 5(1-x)C_{j-2}^{7/2}(x) - 2C_{j-1}^{5/2}(y) - 5(1+y)C_{j-2}^{7/2}(y) \right] \quad (3.3) \]
is now a polynomial of two variables rather than one.

We now utilize the ratio \((2.22)\) of two-point functions of primaries and descendants to evaluate the anomalous dimension. Since we gained a factor of \(g^2\), it is clear that at lowest order we have
\[ (2\pi)^h x^2 \langle \hat{K}_{j-1}(x_1) \hat{K}_{j-1}(x_2) \rangle_{g^0} = -\gamma_j^{(0)} j(j + h - 2)(j + h - 1)(j + 2h - 3), \quad (3.4) \]
where the left-hand side can be evaluated by a tree-level (order \(g^0\)) calculation. Depending on whether we want to use this formula in the \(\phi^3\) model or in \(\mathcal{N} = 4\) SYM, we can specialize it to either six \((h = 3)\) or four \((h = 2)\) dimensions to determine the corresponding leading order anomalous dimensions.

A comment on the evaluation of the two point functions that arise in \((3.4)\) is in order here. Although it is a tree level calculation, the spin dependence presents a technical difficulty. Namely, the composite operators \(\hat{O}_j, \hat{K}_{j-1}\) contain polynomials in the projected derivatives (see, e.g., \((2.16), (3.3)\), and expanding them one gets a result for their two-point functions in terms of multiple sums. Therefore, it is more convenient to use the Schwinger representation for the scalar Euclidean propagators,
\[ \langle \phi_i(x_1) \bar{\phi}^j(x_2) \rangle = \frac{\Gamma(h - 1)}{4\pi^h} \frac{\delta^j_i}{(x_{12})^{h-1}} = \delta^j_i \int_0^\infty \frac{d\alpha}{4\pi^h} \alpha^{h-2} e^{-\alpha x_{12}^2}, \quad (3.5) \]
so that all projected derivatives acting on propagators are essentially replaced by the \(\alpha\)-parameters. As a consequence, instead of infinite sums one obtains integrals over the polynomials that appear in the definition of the conformal operators, see, e.g., \((2.16), (3.3)\), which are then evaluated using standard properties of the Gegenbauer polynomials, in particular their orthogonality relation
\[ \int_0^1 dx [x(1 - x)]^{\nu - 1/2} C_j^\nu(x) C_k^\nu(x) = \delta_{jk} \frac{2^{1-4\nu} \pi \Gamma(j + 2\nu)}{\Gamma^2(\nu) \Gamma(j + 1)(j + \nu)}. \quad (3.6) \]

Let us now use \((3.4)\) (with \(h = 3\)) in order to calculate the one-loop anomalous dimensions of the twist-four operators \((2.16)\). We need to evaluate the two correlators at order \(g^0\), see Fig. 1 (a). The first one fixes the normalization:
\[ \langle \hat{O}_j(x_1) \hat{O}_k(x_2) \rangle = \frac{(-2\hat{\chi}_{12})^{j+k}}{(4\pi^3)^2} \int_0^\infty d\alpha \int_0^\infty d\beta (\alpha + \beta)^{j+k} \times C_{j}^{3/2} \left( \frac{\alpha - \beta}{\alpha + \beta} \right) C_{k}^{3/2} \left( \frac{\alpha - \beta}{\alpha + \beta} \right) \exp \left[ -(\alpha + \beta)x_{12}^2 \right]. \quad (3.7) \]
The integrals with respect to Schwinger parameters are straightforwardly evaluated using \((3.5)\) and yield the following result:
\[ \langle \hat{O}_j(x_1) \hat{O}_k(x_2) \rangle = \delta_{jk}(j + 1)(j + 2)(2j + 2)! \frac{2^{2j-2} (\hat{\chi}_{12})^{2j}}{(4\pi^3)^2 (x_{12}^2)^{2j+4}}. \quad (3.8) \]
It indeed possesses the orthogonal form \((2.19)\) expected on the basis of conformal symmetry. The correlation function for descendants, i.e., \(\langle \hat{K}_{j-1}(x_1) \hat{K}_{k-1}(x_2) \rangle\), is obtained in a similar fashion
upon evaluation of the Feynman diagram in Fig. 1(b). Its expression differs from Eq. (3.8) only by a factor and thus the ratio (3.4) gives the well-known leading order approximation for the anomalous dimensions [15, 7]:

\[ \gamma^{(0)}_j = \frac{1}{4} \left[ \frac{1}{6} - \frac{1}{(j+1)(j+2)} \right]. \] (3.9)

### 3.2 Scheme ambiguities

Up to now we have been able to avoid addressing renormalization issues because our method of calculating \( \gamma^{(0)}_j \) involved only tree-level calculations. However, by going higher up in perturbation theory, we need to discuss how the composite operators (2.16) are renormalized. As pointed out above, it is misleading to assume from the start that conformal symmetry is preserved and an ad hoc renormalization scheme will yield diagonal correlation functions. The Poincaré invariance alone imposes rather weak constrains on the mixing of operators: an operator with a given spin \( j \) will mix under renormalization with total derivatives of lower spin operators, such as

\[ \hat{O}_{jl} = \hat{\partial}^l - j \hat{O}_j \quad \text{(no summation)}. \] (3.10)

The bare operators \( \hat{O}_{jk} \), defined in terms of bare field operators, will be our basis states to discuss operator mixing. The renormalized operator is obtained as a superposition given by the renormalization matrix \( Z \),

\[ \hat{O}_j = \sum_{k=0}^{j} Z_{jk} \hat{O}_{kj}. \] (3.11)

The mixing pattern implied by Poincaré symmetry results in a lower triangular matrix \( Z \), i.e., \( Z_{jk} = 0 \) for \( k > j \). Our goal is to define a conformal scheme, that is, to choose \( Z \) such that the renormalized operators \( \hat{O}_j \) have conformal two-point functions (cf. (2.19))

\[ \langle \hat{O}_j(x_1) \hat{O}_k(x_2) \rangle = \delta_{jk} C_j(g) \hat{\delta}^j (x_1^{2} - d_j(g)). \] (3.12)

In Eq. (3.12) it is understood that the regulator has been removed, \( \epsilon \to 0 \) in dimensional regularization.

We determine the \( Z \) matrix from a calculation in the \( \overline{\text{MS}} \) scheme and then perform an additional finite scheme transformation. The rotation matrix is governed by the form (3.12) of the two-point correlation function. In the \( \overline{\text{MS}} \) scheme we define the renormalized operator insertion

\[ [\hat{O}_j] = \sum_{k=0}^{j} Z_{jk} \hat{O}_{kj} \] (3.13)

in terms of a renormalization matrix \( Z \). Perturbatively, it is given by the Laurent series:

\[ Z_{jk} = \delta_{jk} + \sum_{n=1}^{\infty} \frac{g^n}{(2\pi)^m} \sum_{m=1}^{n} \frac{Z_{jk}^{[n]}(\epsilon)}{\epsilon^m}. \] (3.14)

The anomalous dimension matrix is obtained directly from a scale variation applied to Eq. (3.13). Since the bare operator does not depend on the renormalization scale \( \mu \) we get the standard relation

\[ \gamma_{jk}(g^2) = - \lim_{\epsilon \to 0} \frac{d}{d\mu} \left( \ln Z_{jk}(g^2) \right). \] (3.15)
for computation of anomalous dimension matrix from the renormalization $Z$-matrix. The perturbative expansion of the anomalous dimension matrix is analogous to that in Eq. (2.23). Let us also recall that in this scheme it is entirely determined by the residue of the $Z$-matrix,

$$\gamma_{jk}(g^2) = g \frac{\partial}{\partial g} Z_{jk}^{[1]}(g^2),$$

while all higher order poles are fixed from the renormalizability of the composite operators. For instance, up to order $g^4$ we have (for a vanishing $\beta$ function):

$$Z_{jk}^{[1](1)} = \delta_{jk} \frac{1}{2} \gamma_{j}^{(0)} , \quad Z_{jk}^{[2](2)} = \delta_{jk} \frac{1}{8} \left( \gamma_{j}^{(0)} \right)^2 , \quad Z_{jk}^{[1](2)} = \frac{1}{4} \gamma_{jk} . \quad (3.16)$$

Beyond leading order approximation, the anomalous dimension matrix in the scheme is non-diagonal and has a triangular form. Note that the eigenvalues of the anomalous dimension matrix are given by the diagonal entries and coincide with the scale dimensions of the conformal operators in the conformal scheme. The scheme transformation to the latter from $\overline{\text{MS}}$ is given by

$$\hat{O}_{jj} = \sum_{k=0}^{j} B_{jk}^{-1} \hat{O}_{jk} , \quad Z_{jk} = \sum_{m=k}^{j} B_{jm}^{-1} Z_{mk} . \quad (3.17)$$

The finite renormalization matrix $B$ admits the perturbative series representation

$$B_{jk} = \delta_{jk} + \sum_{n=1}^{\infty} \frac{g^{2n}}{(2\pi)^{n} h} B_{jk}^{(n)} , \quad (3.19)$$

with the expansion coefficients $B_{jk}^{(n)}$ being triangular matrices. In a conformal field theory where the $\beta$–function vanishes, the anomalous dimension matrices in the two schemes are simply related by

$$\gamma_{j}(g^2) \delta_{jk} = \left( B^{-1} \gamma B \right)_{jk} (g^2) . \quad (3.20)$$

Hence, the $B$–matrix diagonalizes the anomalous dimension matrix evaluated in the $\overline{\text{MS}}$ scheme. In particular, to the first nontrivial order at which the mixing phenomena occurs we have:

$$\gamma_{jk}^{(1)} = - \left( \gamma_{j}^{(0)} - \gamma_{k}^{(0)} \right) B_{jk}^{(1)} . \quad (3.21)$$

Note that the knowledge of $\gamma_{j}^{(0)}$, and $B_{jk}^{(1)}$ is sufficient to reconstruct the off-diagonal entries $\gamma_{jk}^{(1)}$ up to order $g^4$.

Let us comment on the orthogonality of conformal operators in non-integer space-time dimensions. According to Eq. (2.16), the index of the Gegenbauer polynomials is shifted by $-\varepsilon$. In practical calculations, we find it convenient to use conformal covariance of the bare operator in $2\hbar$ dimensions and therefore we define an “$\varepsilon$–deformed” basis

$$\hat{O}_{j}^{(\varepsilon)} = (\hat{\partial}_a + \hat{\partial}_b)^{\varepsilon/2} C_j^{3/2-\varepsilon} \left( \frac{\hat{\partial}_a - \hat{\partial}_b}{\hat{\partial}_a + \hat{\partial}_b} \right) \phi_1(x_a) \tilde{\phi}^2(x_b) . \quad (3.22)$$
Then instead of (3.10) we have
\[ \hat{O}_j = \sum_{k=0}^{j} Z'_{jk} \hat{O}^{(e)}_{jk}. \] (3.23)

The practical advantage of the basis states (3.22) is that the two-point functions at order \( g^0 \) are orthogonal even in the presence of the regulator \( \varepsilon \),
\[ \langle \hat{O}^{(e)}_j(x_1) \hat{O}^{(e)}_k(x_2) \rangle|_{g^0} = 0, \quad j \neq k. \] (3.24)

It is easy to recover the mixing matrix in the standard basis from the \( \varepsilon \)-deformed one by expanding the Gegenbauer polynomials with respect to their index. We define
\[ \frac{\partial}{\partial \rho} C_{j}^{\nu+\rho}(x)|_{\rho=0} = -2 \sum_{k=0}^{j} d_{jk}^{\nu} C_{k}^{\nu}(x), \] (3.25)
and have for \( j > k \) the entries
\[ d_{jk}^{\nu} = -\left(1 + (-1)^{j-k}\right) \frac{k + \nu}{(j + k + 2\nu)(j-k)}. \] (3.26)

Then the desired relation reads
\[ B_{jk}^{(1)} = B_{jk}^{(1)} - d_{jk}^{(h-1/2)} \left( \gamma_{j}^{(0)} - \gamma_{k}^{(0)} \right). \] (3.27)

### 3.3 Operator mixing: perturbative calculations

As we explained in the previous section, the conformal renormalization scheme is defined by the requirement that the correctly renormalized conformal primary operator \( \hat{O}_j \) should lead to the diagonal two-point correlation functions (3.12). This fixes the renormalization matrix \( Z \) or, equivalently, \( Z' \). In this section we show how to determine \( B^{(1)} \) (which, due to (3.27), is equivalent to finding \( B^{(1)} \)), by a calculation at order \( g^1 \).

For this purpose it is sufficient to employ Eq. (3.12) for \( j > k \),
\[ \langle \hat{O}_j(x_1) \hat{O}_k(x_2) \rangle|_{g^1} = 0, \quad j > k. \] (3.28)

Let us now expand (3.28) up to order \( g^2 \) in the coupling constant. With the help of the expansion
\[ \hat{O}_j = \hat{O}^{(e)}_j + \frac{g^2}{(2\pi)^h} \frac{1}{2\varepsilon} \hat{O}_j^{(0)} - \frac{\gamma_{j}^{(0)}}{(2\pi)^h} \sum_{k=0}^{j} B_{jk}^{(1)} \hat{O}^{(e)}_{jk} + O(g^4), \] (3.29)
we obtain
\[ \langle \hat{O}_j(x_1) \hat{O}_k(x_2) \rangle|_{g^2} = -\frac{g^2}{(2\pi)^h} B_{jk}^{(1)} \langle \hat{O}_k(x_1) \hat{O}_k(x_2) \rangle|_{g^0} = 0, \quad j > k. \] (3.30)

Here we have used the orthogonality relation (3.24) of the operators \( \hat{O}_j^{(e)} \) at tree level, which is why the \( 1/\varepsilon \) terms from (3.29) have also disappeared. The finite part of (3.30) determines in principle \( B'_{1} \). However, we can gain an order in \( g \) by taking the divergence \( \Delta_{\mu}\partial^\mu \) at point \( x_1 \):
\[ \frac{g^2}{(2\pi)^h} B_{jk}^{(1)} \langle \partial_{\mu}\Delta^{\mu}\partial_{j-k-1} \hat{O}_k(x_1) \hat{O}_k(x_2) \rangle|_{g^0} = g \langle \hat{K}_{j-1}(x_1) \hat{O}_k(x_2) \rangle|_{g^1}, \quad j > k. \] (3.31)
In Eq. (3.31) the order $O(g)$ correlator has still to be regularized but it turns out to be finite (up to a contact term) when the regulator is set to zero, see Eq. (3.34). Notice that we cannot gain a further order in $g$ by taking the divergence at the second point in (3.31) because then the equation would be trivially satisfied.

We are going to determine $B^{(1)}$ from Eq. (3.31). On the right-hand side of this equation we need to evaluate $\langle \hat{K}_{j-1}(x_1) \hat{O}_k(x_2) \rangle|_g$ for $j > k$, see Fig. (c). It turns out that the same calculation, but for $j < k$ and $j = k$ provides a useful consistency check. Firstly, the trianularity of the matrix $B'$ requires

$$\langle \hat{K}_{j-1}(x_1) \hat{O}_k(x_2) \rangle|_g = 0, \quad j < k.$$  \hfill (3.32)

Secondly, for $j = k$, it follows from (3.12) that

$$g \hat{x}^2 \frac{\langle \hat{K}_{j-1}(x_1) \hat{O}_j(x_2) \rangle|_g}{\langle \hat{O}_j(x_1) \hat{O}_j(x_2) \rangle|_{g^0}} = - \frac{g^2}{(2\pi)^h} \gamma^{(0)} j(j + h - 2).$$  \hfill (3.33)

Of course, we could also have taken another divergence at the second point of the correlator, as in (3.4), to gain a further order in $g$, but our goal here is to provide a check for the calculation of the matrix elements. We have indeed verified that (3.33) reproduces the previously determined values (3.9), (3.41) of $\gamma^{(0)}$ in the $\phi^3$ theory and in $\mathcal{N} = 4$ SYM, respectively, and that the trianularity condition (3.32) is satisfied.

The calculation of $\langle \hat{K}_{j-1}(x_1) \hat{O}_k(x_2) \rangle|_g$ involves two Feynman diagrams, see Fig. (c), with just one interaction vertex in $x$-space. The only Feynman integral that arises,

$$\int d^{6-2\epsilon} x_3 \frac{1}{(x_{13})^{4-2\epsilon} (x_{23})^{2-\epsilon}} = - \frac{\pi^3}{3(x_{12})^3} + O(\epsilon), \quad x_1 \neq x_2$$  \hfill (3.34)

is finite in dimensional regularization, up to a contact term. Recall that we always keep $x_{12} \neq 0$ and so we can drop contact terms. Taking the ratio of $\langle \hat{K}_{j-1}(x_1) \hat{O}_k(x_2) \rangle|_g$ and $\langle \hat{O}_j(x_1) \hat{O}_k(x_2) \rangle|_{g^0}$ according to Eq. (3.31), we find

$$B^{(1)}_{jk} = \gamma^{(0)}_j d^{3/2}_{jk}.$$  \hfill (3.35)

Switching back to the undeformed basis using (3.27), we have the result :

$$B^{(1)}_{jk} = d^{3/2}_{jk} \gamma^{(0)}_k,$$  \hfill (3.36)

which coincides with both the explicit evaluation [7] and the conformal symmetry predictions [16], see Eq. (3.21).

Historically, there were attempts to determine the form of the renormalized operator insertion by a shift of the canonical dimensions in the Gegenbauer polynomials [17] [7]. We find that the renormalized conformal operator insertion can be written to this order as

$$\hat{O}_j = \left(1 + \frac{g^2}{(2\pi)^3} \frac{\gamma^{(0)}_j}{2\epsilon} \right) \left(\hat{\partial}_a + \hat{\partial}_b\right)^j C^{3/2-\epsilon+\gamma_j(g)/2}_{j} \left(\frac{\hat{\partial}_a - \hat{\partial}_b}{\hat{\partial}_a + \hat{\partial}_b}\right) \phi_1(x_a) \phi(x_b) + O(g^4).$$  \hfill (3.37)

The result we obtain in $\phi^3$ theory, starting with a conformal invariant operator in $6 - 2\epsilon$ dimensions, indeed exhibits the expectation that the index $\nu = h - 3/2$ of the Gegenbauer polynomials will be shifted by the amount of the anomalous dimensions, however, it contains explicitly the regularization parameter contrary to previous proposals. In gauge field theories such a simple recipe does not work [18], as we will demonstrate once more in the next section.
3.4 Application of the method to $\mathcal{N} = 4$ SYM

In $\mathcal{N} = 4$ SYM one builds twist-two operators by taking bi-linear combinations of the elementary fields $\phi_{AB}, \lambda_A, \tilde{\lambda}^A$ and $F^{\mu\nu}$. There are many different possibilities for constructing such operators [8, 19], and in general one faces an additional mixing problem between scalars, fermions and gluons. This happens for instance for the superconformal primary operators constructed in [20] [4]. There the mixing problem at tree level was resolved by exploiting a super-conservation condition required by superconformal symmetry. Here we avoid this additional mixing problem by considering a different member of this superconformal multiplet, which is a conformal primary operators in the $20'$ of $SU(4)$ [8, 21] (in which we choose the highest-weight projection of the $SU(4)$ indices of the scalar fields $\phi^{AB}$):

$$
\hat{O}_j = (\hat{\partial}_a + \hat{\partial}_b)^j \frac{1}{2} \left( \hat{\partial}_a - \hat{\partial}_b \right) \text{tr} \phi^{12}(a) \phi^{12}(b). \tag{3.38}
$$

Note that $j$ has to be even in (3.38), and that $j = 0$ is a special case, for which $\hat{O}_{j=0}$ is itself the superconformal primary operator of the protected energy-momentum supermultiplet. For $j \neq 0$, the $\hat{O}_j$ can be obtained by acting with four supercharges [8] on the superconformal primary operators of [20] [4]. The fact that these operators belong to the same supermultiplet implies that their anomalous dimensions are given by the same universal formula.

The expression of the descendants $\hat{K}_j$ is given in the Appendix, see Eq. (3.1). Schematically, they read

$$
\hat{K}_{j-1} = A_j(\hat{\partial}_a, \hat{\partial}_b, \hat{\partial}_c) \text{tr} \left( \{\lambda_3^a(x_a), \lambda_{a4}(x_c)\} \phi^{12}(x_b) + \{\tilde{\lambda}_3^a(x_a), \tilde{\lambda}^{a2}(x_c)\} \phi^{12}(x_b) \right) \\
+ B_j(\hat{\partial}_a, \hat{\partial}_b, \hat{\partial}_c) \text{tr} \left[ \partial^{\mu} \phi^{12}(x_a), \phi^{12}(x_b) \right] F^{\mu\nu}(x_c) z_\nu + O(g), \tag{3.39}
$$

Let us first calculate $\gamma_j^{(0)}$ from (3.4). The calculations presented here were done in the light cone gauge, i.e., $z \cdot A = 0$. However, because the correlators we calculate are gauge invariant, our results do not depend on the gauge choice. We have explicitly checked that a calculation in the standard covariant gauge leads to an identical result. For the $\langle \hat{K}_{j-1}(x_1) \hat{K}_{j-1}(x_2) \rangle$ correlator,
there are two contributing Feynman diagrams, one involving fermions in Fig. 2(b) and the other one gluons in see Fig. 2(c). Their respective contributions to $\gamma^{(0)}_j$ are

$$\gamma^{(0)}_j|_{\text{Fig. 2(a)}} = 2, \quad \gamma^{(0)}_j|_{\text{Fig. 2(b)}} = 2(S_j - 1),$$

(3.40)

where $S_j = \sum_{k=1}^j 1/k$ is a harmonic sum. Adding the two, we obtain the well-known result

$$\gamma^{(0)}_j = 2S_j.$$  

(3.41)

Now turning to the mixing matrix, in the left-hand side of (3.31), the prefactor of $B^{(1)}$ can be easily evaluated by a tree-level calculation,

$$\partial^\mu_1 \Delta_{1\mu} \partial^j j^{k} \langle \hat{O}_k(x_1) \hat{O}_k(x_2) \rangle g^\rho \equiv -\frac{1}{d_{jk}^{1/2}} M_{jk}(x_{12}),$$

(3.42)

where

$$M_{jk}(x_{12}) = 2(j + k + 1)! \left( \frac{N_c^2 - 1}{(4\pi^2)^2} \right) \left( \frac{\hat{j}^2}{x^2} \right)^{j+k-1},$$

(3.43)

and $d_{jk}^{1/2}$ is defined in (3.20). In order to evaluate the right-hand side of (3.31), we need to compute the correlator $\langle \hat{K}_{j-1} \hat{O}_j \rangle |_g$ (see Fig. 3). We obtain

$$\langle \hat{K}_{j-1}(x_1) \hat{O}_j(x_2) \rangle = M_{jk}(x) (I_{jk} + J_{jk}),$$

(3.44)

where the integrals $I_{jk}, J_{jk}$ come from Fig. 3(a) and the sum of Figs. 3(b,c), respectively. They are defined by (for $j, k$ even)

$$I_{jk} = 2 \int_{-1}^1 dt (1 + t)a_j(t)C^{1/2}_k(t) = -2 \quad (j > k),$$

(3.45)

and

$$J_{jk} = \int_{-1}^1 ds \int_{-1}^s dt (1 + t) b_j(s, t) C^{1/2}_k(s) + \int_{-1}^1 dt \frac{1 + t}{1 - t} \int_{t}^1 ds (1 - s)b_j(s, t) C^{1/2}_k(t).$$

(3.46)

The expressions for $a_j(t)$ and $b_j(s, t)$ can be found in (A.3). With the method described in the appendix of [22] one can evaluate $J_{jk}$, the result being

$$J_{jk} = 2 \left( 2S_{j-k} + S_{j+k} - S_{j-k} - 2S_j + 1 \right) \quad (j > k).$$

(3.47)

Combining the two contributions (3.45), (3.47) and using (3.31), we obtain for the mixing matrix

$$B_{jk}^{(1)} = 2d_{jk}^{1/2} \left( 2S_j + S_{j+k} - 2S_{j-k} - S_{j+k} \right).$$

(3.48)

Taking into account Eq. (3.27) and $\gamma_j^{(0)} = 2S_j$ we finally arrive at

$$B_{jk}^{(1)} = 2d_{jk}^{1/2} \left( S_j + S_k + S_{j+k} - S_{j+k} - 2S_{j-k} \right).$$

(3.49)

This result constitutes one of the main points of our paper. Namely, we have been able to determine the first correction to the mixing matrix of a particular type of twist-two operators in the $\mathcal{N} = 4$ SYM by just performing an order $g^1$ perturbative calculation. The standard quantum field theory approach would require going to order $g^2$ to obtain the same result. This clearly demonstrates the power of conformal invariance applied to higher-loop calculations.
4 Origin of the mixing matrix

As we established in the preceding sections, even in a theory with a vanishing $\beta$–function, conformal symmetry is broken in the $\overline{\text{MS}}$ scheme and leads to a mixing of conformal composite operators under scale transformations. To analyze this mechanism in greater detail, we employ conformal Ward identities \[23, 22\]. They can be simply derived from the reparameterization invariance of the generating functional, given as a path integral. To derive the true Ward identities it is crucial that the action contains a regulator, which in our case is done by changing the integration volume, i.e., $d^4x \rightarrow d^Dx$ and replacing the coupling $g \rightarrow \mu^{(D-4)/2}g$. Performing an infinitesimal field transformation $\Phi(x) \rightarrow \Phi'(x) = \Phi(x) + \delta\Phi(x)$ yields for a generic Green function of a (composite) operator $O(\Phi)$ to the Ward identities,

$$\langle O(\Phi)\delta(\Phi(x_1) \cdots \Phi(x_n)) \rangle = -\langle (\delta O(\Phi)) \Phi(x_1) \cdots \Phi(x_n) \rangle - \langle O(\Phi) (\delta iS(\Phi)) \Phi(x_1) \cdots \Phi(x_n) \rangle,$$

(4.1)

in the regularized theory. In a second step, we will perform the renormalization procedure, which yields anomalous terms that emerge from contact terms in the product of the symmetry variation of the regularized action functional and the operator insertion, i.e., $O(\Phi) (\delta iS(\Phi))$. In particular when the conformal variation is involved, it generates the anomalous dimensions and the mixing matrix. However, the source of the latter symmetry breaking is a peculiar feature of dimensional regularization. In a third step, we employ conformal constraints, arising from the conformal algebra, to perform a scheme transformation that restores conformal symmetry such that conformal operators obey autonomous renormalization group equations.

Let us first with the conformal variations of the dimensional regularized $\mathcal{N} = 4$ SYM action \[23\]. Dilatation and special conformal boost variations lead to the following anomalous operator insertions:

$$\delta^{D} S = \varepsilon \int d^Dx \left\{ \sum_{k=1}^{3} O_{A_k}(x) + O_{B} - \Omega_{\phi}(x) - \Omega_{\lambda \lambda}(x) \right\},$$

(4.2)

$$\delta^{K}_{\mu} S = \varepsilon \int d^Dx 2x_{\mu} \left\{ \sum_{k=1}^{3} O_{A_k}(x) + O_{B} - \Omega_{\phi}(x) - \Omega_{\lambda \lambda}(x) \right\} - 2(D-2) \int d^Dx O_{\mu, B}(x).$$

(4.3)
Here the relevant gauge invariant operator insertions, of type-A in terminology of Ref. [24], are
\[ O_{A_1}(x) = \text{tr} F_{\mu\nu}(x) F^{\mu\nu}(x), \] (4.4)
\[ O_{A_2}(x) = \frac{g^2}{4} \text{tr}[\phi^{AB}(x), \phi^{CD}(x)][\bar{\phi}_{AB}(x), \bar{\phi}_{CD}(x)], \] (4.5)
\[ O_{A_3}(x) = \sqrt{2} g \text{tr}\{\bar{\lambda}^{\dot{\alpha}A}(x)[\phi^{AB}(x), \bar{\phi}^{\dot{\alpha}B}(x)] - \lambda^{\alpha A}(x)[\bar{\phi}_{AB}(x), \bar{\phi}^{\dot{\beta}B}(x)]\}. \] (4.6)

For the composite operator (3.38) considered here, only the first two operator insertions \( O_{A_i} \) can potentially contribute to our leading order analyses of conformal anomalies. The type-B operators are BRST exact variations, and are irrelevant for the present study. Finally \( \Omega \) are equations of motion operator, e.g.,
\[ \Omega_{\phi}(x) = \frac{\delta S}{\delta \phi^{AB}(x)}. \] (4.7)

In the \( \overline{\text{MS}} \) scheme, the scale and special conformal Ward-identities for the Green functions with the conformal operator \( \hat{O}_j = \hat{O}_j(x = 0) \) insertion, see Eq. (3.38), can be found by plugging in the variations (4.2) and (4.3) into the generic Ward identity (4.1),
\[ \langle [\hat{O}_j] \delta^D \mathcal{X} \rangle = -\langle (\delta^D [\hat{O}_j]) \mathcal{X} \rangle - \langle [\hat{O}_j] (\delta^D iS) \mathcal{X} \rangle \] (4.8)
\[ \langle [\hat{O}_j] \delta^K \mathcal{X} \rangle = -\langle (\delta^K [\hat{O}_j]) \mathcal{X} \rangle - \langle [\hat{O}_j] (\delta^K iS) \mathcal{X} \rangle . \] (4.9)

Without loss of generality, we specify the field monomial of elementary fields to the two scalar fields of the theory \( \mathcal{X} = \phi^{AB}(x_1)\phi^{CD}(x_2) \). By definition, the left-hand side of these equations are finite, however, the separate terms on the right-hand side contain anomalous contributions or even divergencies. To obtain the renormalized Ward identities we will now renormalize the operator product of composite insertion and conformal variation of the action and finally remove the regularization by sending \( \varepsilon \) to zero.

In the case of the dilatation Ward identity (4.8), the variation in the first term on the right-hand side leads to the canonical dimension of the operator, while the second term in the right-hand side of Eq. (4.8) is ill-defined since it involves an operator product at coincident space-time points,
\[ [\hat{O}_j](\delta^D S) = \varepsilon \int d^D x \, O_{A_1}(x)[\hat{O}_j] - \int d^D x \, \Omega_{\phi}(x)[\hat{O}_j] + \cdots . \] (4.10)

Hence a subtractive renormalization procedure is required, where the divergence is a \( 1/\varepsilon \)−pole and its residue is nothing but the anomalous dimension of the operator. The pole will be cancelled by the \( \varepsilon \)−term, appearing in the variation of the action (4.2). A rigorous treatment can be done by means of differential operator vertex insertions [25]. As expected, it can be demonstrated in a straightforward manner that finally the Ward-identity (4.8) turns into the renormalization group equation for Green function with renormalized composite operator insertions,
\[ \left[ \mu \frac{\partial}{\partial \mu} + 2\gamma_\phi(g^2) \right] \langle [\hat{O}_j] \mathcal{X} \rangle = -\sum_{k=0}^{j} \gamma_{jk}(g^2) \langle [\hat{O}_{jk}] \mathcal{X} \rangle . \] (4.11)

Note that a \( \beta \)−function proportional term is absent in our conformal theory, nevertheless, the trace anomaly in the regularized theory turns into the anomalous dimensions and, as we will now see, it is also responsible for the mixing of operators.
The renormalization procedure for the right-hand side of the special conformal Ward identity (4.9) has two peculiarities. First, the variation of the renormalized operator insertion, i.e., $\delta K - \hat{O}_j$, leads to an infinite expression which has to be cancelled against a singularity arising from the renormalization of the operator product

$$[\hat{O}_j](\delta^K_S) = \varepsilon \int d^Dx \, 2x_\perp O_{A_1}(x)[\hat{O}_j] - \int d^Dx \, 2x_\perp \Omega_\varphi(x)[\hat{O}_j] + \cdots. \quad (4.12)$$

Second, the subtractive renormalization of the latter is not automatically fixed by the one appearing in the dilatation variation. The renormalized Ward identity can be written as

$$\langle [\hat{O}_{jl}] \delta^K S \rangle = -i \sum_{k=0}^{j} \left[a(l) + \gamma^c(l; g^2)\right]_{jk} \langle [\hat{O}_{k,l}] \mathcal{X} \rangle + \cdots, \quad (4.13)$$

where the ellipsis contain BRST variations, the scaling dimensions in the conformal variation $\delta^K$ is now given by $1 + \gamma^c$, $a_{jk}(l) = 2(l - k)(l + k + 1)\delta_{jk}$. Again in a conformal theory a $\beta$ proportional term is absent. The so-called special conformal anomaly

$$\gamma^c(l; g^2) = \frac{g^2}{(2\pi)^2} \left(Z_{A_1}^{[1]} - 2\gamma^\phi b + 2[Z_{A_1}^{[1]}, b]\right) + O(g^4), \quad (4.14)$$

where $b_{jk} = 2(j - k)(j + k + 1)d_{jk}^{1/2}$, see Eq. (3.26), is expressed by two subtractive renormalization constants $Z_{A_1}^{[1]} = \delta_{jk}(\gamma_{(0)}^\phi - 2\gamma_{(0)}^\phi) / 2$ and $Z_{A_1}^{-}$. While the former is nothing else as the residue of the renormalization matrix and is expressed in terms of the anomalous dimensions, cf. Eq. (3.17), the latter one is defined as

$$\int d^Dx \, 2x_\perp O_{A_1}(x)[\hat{O}_{jl}] = \left[\int d^Dx \, 2x_\perp O_{A_1}(x)\hat{O}_{jl}\right] + \sum_{k=0}^{j} Z_{A_1,jk}^{-} \hat{O}_{k,l-1} + \cdots. \quad (4.15)$$

and is straightforwardly evaluated to lowest order from the Feynman diagrams shown in Fig. 4.

The result is

$$Z_{A_1}^{-} = -\frac{g^2}{(2\pi)^2} \frac{1}{\varepsilon} \left(2Z_{A_1}^{[1]} b - w\right) + O(g^4), \quad (4.16)$$
where the matrix elements of the \( w \)-matrix are given for \( j > k \) by

\[
w_{jk} = 2(2k + 1) \left[ 1 + (-1)^{j-k} \right] \left( S_j + S_{j-k} - S_{j+k} - 2S_{j-k} \right). \tag{4.17}
\]

Combining Eqs. (4.14), (4.16), and (4.17), we find that the special conformal anomaly,

\[
\gamma^{c}_{jk}(j; g^2) = -2(j - k)(j + k + 1) \frac{g^2}{(2\pi)^2} B^{(1)}_{jk} + O(g^4), \tag{4.18}
\]

is expressed by means of the mixing matrix \( B^{(1)}_{jk} \), found earlier in Eq. (3.49).

The anomalous dimensions and the so-called special conformal anomaly are not entirely independent quantities. In the case of a conformal field theory it is easy to derive a constraint between them by acting with the differential operator \( \mu d/d\mu \) on the conformal boost Ward identity (4.13) and with the generator of the conformal boost, — a differential operator acting on the arguments of elementary fields, — on the renormalization group equation (4.11). Subtracting both equations yields zero on the left-hand side while the right-hand side is the desired constraint between the conformal anomalies, which we write as

\[
2(j - k)(j + k + 1) \gamma_{jk}(g) = \sum_{m=k}^{j} \left[ \gamma_{jm}(g) \gamma_{mk}(j; g) - \gamma_{jm}(j; g) \gamma_{mk}(g) \right]. \tag{4.19}
\]

The result can be understood as a consequence of the conformal commutator \([D, K_-] = iK_-\) applied to the Green function with conformal operator insertion. We note that the additional \( l \)-dependence in \( \gamma^{c}_{jk}(l; g) \) is governed by another conformal constraint that arises from the commutator relation \([K_-, P_+] = -2i(D + M_-)\) and reads

\[
\gamma^{c}(l + 1; g) = \gamma^{c}(l; g) - 2\gamma(g). \tag{4.20}
\]

The above conformal constraints guarantee that there exist a scheme in which the covariant behavior of conformal operator under conformal transformations is ensured [14]. Such a scheme is obtained by a finite renormalization group transformation (3.20) that diagonalizes the anomalous dimension matrix. Utilizing the constraints (4.19) and (4.20) one finds the \( B \)-matrix in terms of the special conformal anomaly:

\[
B = \frac{1}{1 + J \gamma^{c}} = 1 - J \gamma^{c} + J(\gamma^{c} \gamma^{c}) - \cdots, \quad \text{where} \quad (J \gamma^{c})_{jk} = \frac{\gamma^{c}_{jk}}{2(j - k)(j + k + 1)}. \tag{4.21}
\]

Needless to say that the mixing matrix to order \( g^2 \) follows by inserting the expression (4.18) for the special conformal anomaly into this equation and coincide with the result (3.49), obtained in Sect. 3.4. Rotating now the conformal operator via Eq. (3.18), the conformal boost Ward identities (4.13) turns into [14]:

\[
\langle \hat{O}_{jl} \delta^{K}_{l} X' \rangle = -i 2(j - l) \left( j + l + 1 + \gamma_{j}(g^2) \right) \langle [\hat{O}_{j,l-1}] X' \rangle + \ldots. \tag{4.22}
\]

Here the ellipsis stands for Green functions with BRST-exact operator insertions \( O_{B} \), which do not contribute in gauge invariant quantities. As we see, the conformal operators \( \hat{O}_{jl} \) transform covariantly under conformal boost and in particular the lowest state in the module \( \hat{O}_{j} \) is invariant under conformal boost.
5 Conclusions

As we demonstrated in this work, in a conformal field theory there exists a special renormalization scheme in which renormalized conformal operators possess diagonal two-point correlation functions (2.19), whose fall-off with distance is determined merely by their scale dimensions. We proposed to use these correlators for evaluation of anomalous dimensions of conformal operators at higher orders making use of the so-called Anselmi’s trick. The main advantage of this formalism for multi-loop calculations of leading twist anomalous dimensions, compared to diagrammatic evaluation of operator matrix elements, arises from simpler topologies of contributing Feynman graphs taking the form of bubble diagrams. Along this line of reasoning, the knowledge of the four loop diagrams, appearing in the descendants, would permit a three-loop calculation of anomalous dimensions.

We finally remark that in a non-conformal theory, e.g., QCD, the covariant behavior of conformal operators is spoiled by a term proportional to the $\beta$-function. Hence, the form of the correlation functions will change too. However, setting by hand the $\beta$-function to zero at a given order of perturbation theory, the conformal prescription still applies and can be used for highly non-trivial predictions. We can even argue that in certain cases the trace anomaly can be entirely incorporated into the conformal predictions. For instance, we can introduce a scheme in which the conformal operators of leading twist are multiplicatively renormalizable. Then the form of the two-point correlation function in the full theory is entirely fixed by the renormalization group equation:

$$\langle \hat{O}_j(x_1) \hat{O}_k(x_2) \rangle_{(\mu^2)} = \delta_{jk} C_{j} \left( \bar{g}(1/\sqrt{x_{12}; g}) \right) \hat{j} ^{j} \left( x_{12} \right)^{-d_j} \exp \left\{ \int_{1/\sqrt{x_{12}}}^{\mu} \frac{d\mu'}{\mu'} \gamma_j (\bar{g}(\mu'; g)) \right\}, \quad (5.1)$$

where the running coupling satisfies the initial condition $\bar{g}(\mu; g) = g$.

Another potential application of the method is the evaluation of the mixing matrix in full QCD to two-loop order accuracy, including the $\beta$-function (see, e.g., Ref. [22]). This allows one to restore the full anomalous dimension matrix and even the non-forward evolution kernels in the $\overline{\text{MS}}$ scheme to three-loop level. This piece of information is required in the application of perturbative QCD to exclusive processes and would allow for a complete next-to-next-to-leading analysis in the $\overline{\text{MS}}$ scheme.

We benefited from enlightening discussions with B. Eden, J.M. Drummond, G.P. Korchemsky and I.T. Todorov. This work was supported by the U.S. National Science Foundation under grant no. PHY-0456520 (A.B.) and the French Agence Nationale de la Recherche under contract ANR-06-BLAN-0142 (E.S.). Three of us (A.B., J.H. and E.S.) would like to thank LPT (Orsay) and (A.B. and D.M.) are grateful to LAPTH (Annecy) for the warm hospitality extended to them at different stages of the work. J.H. acknowledges the warm hospitality extended to him by the Theory Group of the Dipartimento di Fisica, Universit`a di Roma “Tor Vergata”.

A Appendix: Calculation of descendants

We calculate the divergence (2.18) of the twist-two operator (3.38) on the classical level, using the equations of motion. The terms in the descendant $K$ have two origins: the commutation relation
of covariant derivatives and the use of the Klein-Gordon equation derived from the action (2.3). The final result can be written as

\[
\hat{K}_{j-1} = A_j(\hat{\partial}_a, \hat{\partial}_b, \hat{\partial}_c) \text{tr} \left\{ \{\lambda_3(x_a), \lambda_4(x_c)\} \phi^{12}(x_b) + \{\bar{\lambda}_3(x_a), \bar{\lambda}_2(x_c)\} \phi^{12}(x_b) \right\} \\
+ B_j(\hat{\partial}_a, \hat{\partial}_b, \hat{\partial}_c) \text{tr} [\partial\phi^{12}(x_a), \phi^{12}(x_b)] F^{\mu\nu}(x_c) z_{\nu} + O(g),
\]

(A.1)

where \(A_j\) and \(B_j\) are homogeneous polynomials of degree \(j - 1\) and \(j - 2\), respectively. Here we have dropped \(O(g)\) terms, which are quadratic in \(\phi\) or contain two \(F^{\mu\nu}\)’s, since they are irrelevant for the considered perturbative order. In terms of new variables \(u = a + b + c\), \(s = (a + c - b)/u\) and \(t = (a - c - b)/u\) they can be written as

\[
A(a, b, c) = 2\sqrt{2} u^{j-1} a_j(s), \quad B_j(a, b, c) = -i 8 u^{j-2} b_j(s, t)
\]

(A.2)

with

\[
b_j(s, t) = \frac{1}{s-t} \left[ P_j'(s) + (s-1)P_j''(s) \right] + \frac{1}{(s-t)^2} \left[ (1-s)P_j'(s) - (1-t)P_j'(t) \right]
\]

(A.3)

\[
a_j(s) = P_j'(s) + (s-1)P_j''(s),
\]

(A.4)

and \(P_j(x) = C_j^{1/2}(x)\). Note there is no singularity in \(b_j(s,t)\) for \(s = t\), which can be seen by Taylor expanding around \(s = t\).

References

[1] K.G. Wilson, Phys. Rev. 179 (1969) 1499; L.P. Kadanoff, Phys. Rev. Lett. 23 (1969) 1430.

[2] J.M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231; S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Phys. Lett. B 428 (1998) 105; E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253.

[3] D. Anselmi, Nucl. Phys. B 541 (1999) 369.

[4] J. Henn, C. Jarczak, E. Sokatchev, Nucl. Phys. B 730 (2005) 191.

[5] B. Eden, Nucl. Phys. B 681 (2004) 195.

[6] B. Eden, C. Jarczak, E. Sokatchev, Nucl. Phys. B 712 (2005) 157.

[7] S.V. Mikhailov, A.V. Radyushkin, Theor. Math. Phys. 65 (1986) 999.

[8] A.V. Belitsky, S.E. Derkachov, G.P. Korchemsky, A.N. Manashov, Phys. Rev. D 70 (2004) 045021.

[9] V.K. Dobrev, V.B. Petkova, S.G. Petrova, I.T. Todorov, Phys. Rev. D 13 (1976) 887.

[10] V. Bargmann, I.T. Todorov, J. Math. Phys. 18 (1977) 1141.

[11] I.T. Todorov, M.C. Mintchev, V.B. Petkova, Conformal Invariance In Quantum Field Theory, Sc. Norm. Sup. (Pisa, 1978).
[12] Yu.M. Makeenko, Sov. J. Nucl. Phys. 33 (1981) 440.

[13] T. Ohrndorf, Nucl. Phys. B 198 (1982) 26.

[14] D. Müller, Phys. Rev. D 58 (1998) 054005;
    A.V. Belitsky, D. Müller, Phys. Lett. B 417 (1998) 129.

[15] T. Kubota, Nucl. Phys. B 165 (1980) 277;
    L. Baulieu, E.G. Floratos, C. Kounnas, Nucl. Phys. B 166 (1980) 321;
    B. Humpert, W.L. van Neerven, Nucl. Phys. B 178 (1981) 498.

[16] D. Müller, Z. Phys. C 49 (1991) 293.

[17] N.S. Craigie, V.K. Dobrev, I.T. Todorov, Annals Phys. 159 (1985) 411.

[18] S.J. Brodsky, P. Damgaard, Y. Frishman, G.P. Lepage, Phys. Rev. D 33 (1986) 1881.

[19] N. Beisert, Phys. Rept. 405 (2005) 1.

[20] M. Bianchi, P.J. Heslop, F. Riccioni, J. High Ener. Phys. 0508 (2005) 088.

[21] M. Bianchi, B. Eden, G. Rossi, Y.S. Stanev, Nucl. Phys. B 646 (2002) 69;
    M. Bianchi, G. Rossi, Y.S. Stanev, Nucl. Phys. B 685 (2004) 65;
    B. Eden, C. Jarczak, E. Sokatchev, Y.S. Stanev, Nucl. Phys. B 722 (2005) 119.

[22] A.V. Belitsky, D. Müller, Nucl. Phys. B 537 (1999) 397.

[23] D. Müller, Phys. Rev. D 49 (1994) 2525.

[24] J.A. Dixon, J.C. Taylor, Nucl. Phys. B 78 (1974) 552;
    H. Kluberg-Stern, J.B. Zuber, Phys. Rev. D 12 (1975) 467, 482, 3159;
    S.D. Joglekar, B.W. Lee, Ann. Phys. 97 (1975) 160.

[25] J.C. Collins, *Renormalization*, Cambridge Univ. Press, (Cambridge, 1984).