COVERING OF A REDUCED SPHERICAL BODY BY A DISK

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We prove the following theorems: (1) every spherical convex body $W$ of constant width $\Delta(W) \geq \frac{\pi}{2}$ can be covered by a disk of radius $\Delta(W) + \arcsin\left(\frac{2\sqrt{3}}{3} \cos \frac{\Delta(W)}{2}\right) - \frac{\pi}{2}$. (2) every reduced spherical convex body $R$ of thickness $\Delta(R) < \frac{\pi}{2}$ can be covered by a disk of radius $\arctan\left(\sqrt{2} \tan \frac{\Delta(R)}{2}\right)$.

1. Introduction

The subject of reduced bodies in the Euclidean space $E^d$ has been well researched; see, e.g., [6]. However, it is a natural idea to investigate bodies of this kind in non-Euclidean geometries. There are numerous articles aimed at describing reduced bodies on the sphere; see [3, 5, 7, 8]. A question arises whether the results established in the Euclidean space can be transferred to the sphere. In particular, the main theorem of the present paper gives a spherical version of the modification of Jung theorem for reduced bodies in $E^2$ proposed in [4].

Let $S^2$ be the unit sphere of the three-dimensional Euclidean space $E^3$. By a great circle of $S^2$ we mean the intersection of $S^2$ with any two-dimensional subspace of $E^3$. A set of points of the great circle of $S^2$ lying at distances of at most $\frac{\pi}{2}$ from a point $c$ of this great circle is called a semicircle with center $c$. Any pair of points obtained as the intersection of $S^2$ with a one-dimensional subspace of $E^3$ is called a pair of antipodes. Note that if two different points $a$ and $b$ are not antipodes, then there is exactly one great circle containing them. The shorter part of this great circle is called the spherical arc connecting $a$ and $b$, or simply arc. It is denoted by $ab$. By the spherical distance $|ab|$ (or simply distance) between these points, we mean the length of the arc $ab$. If $a$ and $b$ are antipodes, then we set $|ab| = \pi$. If $p$ is a point of $S^2$ and $F$ is a closed set containing at least two points, then we define $\text{dist}(p, F)$ as $\min_{q \in F} |pq|$.

A subset of $S^2$ is called convex if it does not contain any pair of antipodes of $S^2$ and if, together with every two points, it contains the arc connecting these points. A spherical convex body is defined as a closed convex set with nonempty interior. If there is no arc in the boundary of a spherical convex body, then we say that the body is strictly convex.

Let $\rho \in \left(0, \frac{\pi}{2}\right]$. A disk of radius $\rho$ with center $c$ is defined as the set of points of $S^2$ lying at distances of at most $\rho$ from the point $c \in S^2$. The boundary of a disk is called a spherical circle. A hemisphere is defined as any disk of radius $\frac{\pi}{2}$. A hemisphere with center $p$ is denoted by $H(p)$. If $p$ and $q$ are antipodes, then $H(p)$ and $H(q)$ are called the opposite hemispheres.

Let $t$ be a boundary point of a convex body $C \subset S^2$. We say that a hemisphere $H$ supports $C$ at $t$ if $C \subset H$ and $t$ belongs to the great circle bounding $H$. If the body $C$ is supported at $p$ by exactly one hemisphere, then we say that $p$ is a smooth point of the boundary of $C$. If all boundary points of $C$ are smooth, then we say that $C$ is smooth. A point $e$ is called an extreme point of $C$ if $C \setminus \{e\}$ is a convex set. If the set $A$ is contained in

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an open hemisphere, then we define \( \text{conv}(A) \) as the smallest convex set containing \( A \) (for details, see the definition presented before Lemma 1 in [5]).

If hemispheres \( G \) and \( H \) are different but not opposite, then \( L = G \cap H \) is called a lune. The two semicircles bounding \( L \) and contained in \( G \) and \( H \) are denoted by \( G/H \) and \( H/G \), respectively. The thickness \( \Delta(L) \) of \( L \subset S^2 \) is defined as the distance of the centers of \( G/H \) and \( H/G \). The two points of the set \( (G/H) \cap (H/G) \) are understood as the corners of \( L \).

For every hemisphere \( K \) supporting a convex body \( C \subset S^2 \), we find hemispheres \( K^* \) supporting \( C \) such that the lunes \( K \cap K^* \) are of the minimum thickness (by the compactness arguments at least one hemisphere of this kind \( K^* \) exists). The thickness of the lune \( K \cap K^* \) is called the width of \( C \) determined by \( K \). It is denoted by \( \text{width}_K(C) \) (see [5]). If, for all hemispheres \( K \) supporting \( C \), the numbers \( \text{width}_K(C) \) are equal, then we say that \( C \) is of constant width (see [5]). The thickness \( \Delta(C) \) of a convex body \( C \subset S^2 \) is defined as the minimum width of \( C \) determined by \( K \) over all supporting hemispheres \( K \) of \( C \) (see [5]).

After [5], we say that a spherical convex body \( R \subset S^2 \) is reduced if \( \Delta(Z) < \Delta(R) \) for every convex body \( Z \subset R \) different from \( R \). Simple examples of reduced spherical convex bodies on \( S^2 \) are spherical bodies of constant width and, in particular, the disks on \( S^2 \). Moreover, each of the four parts of a spherical disk on \( S^2 \) dissected by two orthogonal great circles through the center of the disk is a reduced spherical body. It is called a quarter of a spherical disk.

**Remark.** Most of the notions presented above can be also introduced in higher dimensions; for details see, e.g., [5].

For the sake of convenience of the reader, we now recall several formulas from spherical geometry frequently used in the present paper. Consider a right spherical triangle with hypotenuse \( C \) and legs \( A \) and \( B \). By \( \alpha \) and \( \beta \), we denote the angles opposite to \( A \) and opposite to \( B \), respectively. By [11], the following relations are true:

\[
\tan A = \cos \beta \tan C, \tag{1}
\]

\[
\sin A = \sin \alpha \sin C, \tag{2}
\]

\[
\cos C = \cos A \cos B, \tag{3}
\]

\[
\cos C = \cot \alpha \cot \beta. \tag{4}
\]

After [2], we define the circumradius of a convex body \( C \) as the smallest \( \rho \) for which \( C \) can be covered by a disk of radius \( \rho \). This disk is unique and is called the disk circumscribed about \( C \). The boundary of this disk is called the circle circumscribed about \( C \). In particular, the circle circumscribed about a spherical triangle contains all vertices of this triangle, which is a consequence of the spherical Ceva’s theorem (see Theorem 2 in [9]).

The results of evaluation of the circumradius in three special cases considered in Lemmas 1–3 presented in what follows are useful for our subsequent presentation.

**Lemma 1.** Let \( Q \subset S^2 \) be a quarter of a disk. The circumradius of \( Q \) is given by the formula

\[
\rho = \arctan \left( \sqrt{2} \tan \frac{\Delta(Q)}{2} \right).
\]

**Proof.** By \( c \) we denote the center of a disk whose quarter is analyzed. Moreover, by \( a \) and \( b \) we denote two different extreme points of \( Q \) such that \( ca \) and \( cb \) are subsets of the boundary of \( Q \). It is easy to see that if
a disk contains points $a$, $b$, and $c$, then it also contains $Q$. Therefore, our aim is to find the radius $\rho$ of the disk circumscribed about the triangle $abc$. Let $o$ denote the center of this disk and let $p$ be a point of $ac$ for which the angle $\angle{cpo}$ is right. Since the distances between $o$ and the points $a$ and $c$ are identical, the point $p$ is in the middle of $ac$ and, thus,

$$|cp| = \frac{\Delta(Q)}{2}.$$  

Clearly, the angle $\angle{pco}$ is equal to $\frac{\pi}{4}$ and $|oc|$ is equal to the radius of our disk. Hence, by analyzing the triangle $cop$, by (1) we get

$$\tan \frac{\Delta(Q)}{2} = \cos \frac{\pi}{4} \tan \rho.$$  

Evaluating $\rho$, we arrive at the assertion of the lemma.

Recall [5] that the Reuleaux triangle is the intersection of three disks of radius $\sigma$ such that the centers of these disks are pairwise located at distances equal to $\sigma$.

**Lemma 2.** The circumradius of a spherical Reuleaux triangle $R$ is equal to

$$\rho = \arcsin \left( \frac{2\sqrt{3}}{3} \sin \frac{\Delta(R)}{2} \right).$$

**Proof.** Denote by $a$, $b$, and $c$ three points of $bd(R)$ that are not smooth. It is easy to check that any disk containing these three points contains $R$. Thus, $\rho$ is equal to the radius of the disk circumscribed about the triangle $abc$. Let $o$ be the center of this disk and let $p$ be the middle of the arc $ab$. Clearly, the angle $\angle{opa}$ is right, the angle $\angle{aop}$ is equal to $\frac{\pi}{3}$, and $|ap| = \frac{\Delta(R)}{2}$. Therefore, for the triangle $aop$, by virtue of (2), we obtain

$$\sin \frac{\Delta(R)}{2} = \sin \frac{\pi}{3} \sin \rho.$$  

We can now easily find $\rho$, which establishes the required formula.

**Lemma 3.** The circumradius $\rho$ of a spherical equilateral triangle $T$ of thickness less than $\frac{\pi}{2}$ is equal to

$$\arctan \frac{\sqrt{9 + 8 \tan^2 \Delta(T)} - 3}{2 \tan \Delta(T)}.$$  

**Proof.** We denote by $o$ the center of the disk circumscribed about our triangle, by $d$ the center of a side of this triangle, and by $a$ the endpoint of this side. Clearly,

$$\angle{doa} = \frac{\pi}{3}, \quad |oa| = \rho, \quad \text{and} \quad |od| = \Delta(T) - \rho.$$  

Therefore, by virtue of (1), we obtain

$$\tan(\Delta(T) - \rho) = \frac{1}{2} \tan \rho.$$
By using the subtraction formula for the tangent function, we can rewrite this equation as follows:

\[
\frac{\tan \Delta(T) - \tan \rho}{1 + \tan \Delta(T) \tan \rho} = \frac{\tan \rho}{2}.
\]

This is equivalent to

\[
\tan \Delta(T)(\tan \rho)^2 + 3 \tan \rho - 2 \tan \Delta(T) = 0.
\]

Hence, by \( \tan \rho > 0 \), we get

\[
\tan \rho = \frac{\sqrt{9 + 8 \tan^2 \Delta(T)} - 3}{2 \tan \Delta(T)},
\]

which completes the proof.

2. Covering of a Body of Constant Width over \( \frac{\pi}{2} \) by a Disk

For any set \( F \) on the sphere \( S^2 \), we define the set \( F^\oplus \) as follows: \( \{ p : F \subset H(p) \} \).

In [3] and [12] there is used the notion of the polar set of the set \( F \) on the sphere, which is defined as

\[
F^\circ = \bigcap_{p \in F} H(p).
\]

**Proposition 1.** For every set \( F \) on the sphere, \( F^\circ = F^\oplus \).

**Proof.** For any point \( q \), we find

\[
q \in F^\circ \iff \forall p \in F \ q \in H(p) \iff \forall p \in F \ |pq| \leq \frac{\pi}{2} \iff \forall p \in F \ p \in H(q) \iff F \subset H(q) \iff q \in F^\oplus.
\]

However, the application of \( F^\oplus \) is more convenient in the present paper.

Here, we omit a simple proof of the following lemma:

**Lemma 4.** If \( C \) is a spherical convex body, then \( C^\oplus \) is also a spherical convex body.

By the way, we note that \( (C^\oplus)^\oplus = C \) for a spherical convex body \( C \).

**Proposition 2.** If \( W \) is a spherical convex body of constant width, then \( W^\oplus \) is a spherical convex body of constant width \( \pi - \Delta(W) \).

**Proof.** Consider a hemisphere \( H(a) \) supporting \( W^\oplus \). Our aim is to show that

\[
\text{width}_{H(a)}(W^\oplus) = \pi - \Delta(W).
\]

It is clear that \( a \) is a boundary point of \( W \). Thus, by Theorem 7 in [7] there exist hemispheres \( K \) and \( M \) supporting \( W \) such that the lune \( K \cap M \) is of thickness \( \Delta(W) \) and \( a \) is the center of \( K/M \). We denote the center of the semicircle \( M/K \) by \( b \) and the centers of the hemispheres \( K \) and \( M \) by \( a' \) and \( b' \), respectively. Since \( a, b, a', \) and \( b' \) are located on the same distance from both corners of \( K \cap M \), all these points belong to the same
great circle. Thus, we can easily conclude that

$$|ab| + |a'b'| = |aa'| + |bb'| = \pi$$

and, hence,

$$|a'b'| = \pi - \Delta(W).$$

Since the distance between every point of $W^\oplus$ and $b$ does not exceed $\frac{\pi}{2}$, the body $W^\oplus$ is contained in $H(b)$. Therefore, $W^\oplus$ is contained in the lune $H(a) \cap H(b)$ of thickness $|a'b'|$. This means that $\text{width}_{H(a)}(W^\oplus)$ is not greater than $\pi - \Delta(W)$.

Assume that

$$\text{width}_{H(a)}(W^\oplus) < \pi - \Delta(W).$$

Thus, there exists a point $\bar{b}$ such that the lune $H(a) \cap H(\bar{b})$ is of thickness less that $\pi - \Delta(W)$ and contains $W^\oplus$. Since $|a\bar{b}| + \Delta(H(a) \cap H(\bar{b})) = \pi$, we conclude that

$$|a\bar{b}| > \Delta(W).$$

However, this contradicts the fact that each body $W$ of constant width has the diameter $\Delta(W)$. Hence,

$$\text{width}_{H(a)}(W^\oplus) = \pi - \Delta(W),$$

which completes the proof.

In [2], the diameter of a spherical compact set was estimated by the function of its circumradius (see the second part of Theorem 2 in [2]). We now recall this result for $S^2$ in a different form: If $d$ is the diameter of a compact set and $\sigma$ is its circumradius, then

$$\sin \sigma \leq \frac{2\sqrt{3}}{3} \sin \frac{d}{2}.$$ 

In particular, this is true for every spherical convex body $W$ of constant width and, in this case, $d = \Delta(W)$. If $\Delta(W)$ does not exceed $\frac{2\pi}{3}$, then $\frac{2\sqrt{3}}{3} \sin \frac{d}{2}$ is not greater than 1. In this case, our inequality is equivalent to the statement that every spherical convex body of constant width not greater than $\frac{2\pi}{3}$ can be covered with a disk of radius

$$\arcsin \left( \frac{2\sqrt{3}}{3} \sin \frac{\Delta(W)}{2} \right).$$

If $\Delta(W)$ is greater than $\frac{2\pi}{3}$, then $\frac{2\sqrt{3}}{3} \sin \frac{d}{2}$ is greater than 1 and, in this case, our inequality does not estimate $\sigma$ in a nontrivial way.

According to Lemma 2, the example of spherical Reuleaux triangle shows that the estimate cannot be improved for bodies of constant width that does not exceed $\frac{\pi}{2}$. The following theorem describes the case of bodies of constant width not smaller than $\frac{\pi}{2}$ and, in particular, improves the estimate recalled from [2] for convex bodies of constant width greater than $\frac{\pi}{2}$. 
Theorem 1. Every spherical body $W$ of constant width not smaller than $\frac{\pi}{2}$ is contained in a disk of radius

$$\Delta(W) + \arcsin\left(\frac{2\sqrt{3}}{3} \cos \frac{\Delta(W)}{2}\right) - \frac{\pi}{2}.$$

Proof. Note that, for $\Delta(W) = \frac{\pi}{2}$, we have

$$\arcsin\left(\frac{2\sqrt{3}}{3} \cos \frac{\Delta(W)}{2}\right) + \Delta(W) - \frac{\pi}{2} = \arcsin\left(\frac{2\sqrt{3}}{3} \sin \frac{\Delta(W)}{2}\right).$$

Therefore, by virtue of the above-mentioned Dekster’s result, the assertion of the theorem is true for bodies of constant width $\frac{\pi}{2}$.

We now assume that $\Delta(W) > \frac{\pi}{2}$. By Proposition 2, the body $W^{\oplus}$ is of thickness $\pi - \Delta(W)$. Hence, by the above-mentioned Dekster’s result, $W^{\ominus}$ is contained in a disk of radius

$$\arcsin\left(\frac{2\sqrt{3}}{3} \sin \frac{\pi - \Delta(W)}{2}\right) = \arcsin\left(\frac{2\sqrt{3}}{3} \cos \frac{\Delta(W)}{2}\right).$$

We denote the center of this disk by $o$. Let $p$ be a boundary point of $W$. By Proposition 3 in [7], $W$ is smooth and, therefore, there exists exactly one hemisphere supporting $W$ at $p$. We denote its center by $a$ and notice that $a$ is a boundary point of $W^{\ominus}$. Moreover, $H(p)$ is a supporting hemisphere of $W^{\ominus}$, which supports $W^{\ominus}$ at $a$. By the proof of the first part of Theorem 1 in [5], there exists a unique point of $W^{\ominus}$ closest to $p$. We denote it by $b$. Again by Theorem 1 in [5], $b$ is the center of one of two semicircles bounding the lune of thickness

$$\text{width}_{H(p)}(W^{\ominus}) = \Delta(W^{\ominus}) = \pi - \Delta(W).$$

Thus, $|ab| = \Delta(W^{\ominus})$ and, clearly, $b \in ap$. Hence,

$$|op| \leq |ob| + |bp| = |ob| + |ap| = |ob| + \frac{\pi}{2} - (\pi - \Delta(W)) = |ob| + \Delta(W) - \frac{\pi}{2}.$$

Since $|ob|$ does not exceed $\arcsin\left(\frac{2\sqrt{3}}{3} \cos \frac{\Delta(W)}{2}\right)$, the distance between $o$ and $p$ is not greater than

$$\arcsin\left(\frac{2\sqrt{3}}{3} \cos \frac{\Delta(W)}{2}\right) + \Delta(W) - \frac{\pi}{2},$$

which completes the proof.

Note that, in general, we cannot improve the estimate from Theorem 1. By the proof of this theorem, we see that the value

$$\Delta(W) + \arcsin\left(\frac{2\sqrt{3}}{3} \cos \frac{\Delta(W)}{2}\right) - \frac{\pi}{2}$$

is attained for every $W$ such that $W^{\ominus}$ is a Reuleaux triangle.
3. Covering of Reduced Bodies with Thicknesses of at Most $\frac{\pi}{2}$ by a Disk

The main theorem of the present paper is analogous to Theorem in [4]. However, we are unable to present a proof similar to that used in [4] due to the absence of the notion of parallelism on the sphere. For this reason, the proof of our main theorem is based on a different idea.

Lemma 5. Let $c$ be a positive number smaller than $\frac{1}{4}$ and let $a$ be a number from the interval

$$\left(\frac{1}{2}, \frac{1}{2} + \sqrt{\frac{1}{4} - c}\right).$$

The function $f(x) = \sqrt{1 - x} - \sqrt{1 - x - \frac{c}{x}}$ satisfies the inequality

$$f(x) \leq \max\left(f\left(\frac{1}{2}\right), f(a)\right) \quad \text{for every} \quad x \in \left[\frac{1}{2}, a\right].$$

Proof. Note that if $x > 0$, then $1 - x - \frac{c}{x} \geq 0$ is equivalent to $x^2 - x + c \leq 0$. This inequality is satisfied for

$$x \in \left[\frac{1}{2} - \sqrt{\frac{1}{4} - c}, \frac{1}{2} + \sqrt{\frac{1}{4} - c}\right].$$

In particular, we conclude that

$$\sqrt{1 - x - \frac{c}{x}} = 0 \quad \text{for} \quad x = \frac{1}{2} + \sqrt{\frac{1}{4} - c},$$

and $f(x)$ is well defined in the interval $\left(\frac{1}{2}, \frac{1}{2} + \sqrt{\frac{1}{4} - c}\right)$.

In order to prove the required assertion, we check the sign of the first derivative of $f(x)$. Thus, we get

$$f'(x) = \frac{1}{2\sqrt{1-x}} - \frac{1}{2\sqrt{1-x-c/x}}(-1 + \frac{c}{x^2})$$

$$= \frac{1}{2\sqrt{1-x-c/x}}\left(1 - \frac{c}{x^2} - \sqrt{1 - \frac{c}{x(1-x)}}\right).$$

We set

$$g(x) = 1 - \frac{c}{x^2} - \sqrt{1 - \frac{c}{x(1-x)}}.$$

Clearly, for any $x$, the derivative $f'(x)$ has the same sign as $g(x)$. We obtain

$$g(x) = 0 \iff 1 - \frac{c}{x^2} = \sqrt{1 - \frac{c}{x(1-x)}} \iff 1 - \frac{2c}{x^2} + \frac{c^2}{x^4} = 1 - \frac{c}{x(1-x)}$$

$$\iff \frac{1}{1-x} - \frac{2}{x} + \frac{c}{x^3} = 0 \iff 3x^3 - 2x^2 + c(1-x) = 0.$$
For \( V(x) = 3x^3 - 2x^2 + c(1 - x) \), we have

\[
V(0) = c > 0 \quad \text{and} \quad V\left(\frac{1}{2}\right) = \frac{1}{2}\left(c - \frac{1}{4}\right) < 0.
\]

Thus, \( V(x) \) has three zeros: one zero is smaller than 0, the second zero lies in the interval \( \left(0, \frac{1}{2}\right) \), and the third zero is greater than \( \frac{1}{2} \). Hence, \( g(x) \) has exactly one zero in the interval \( \left(\frac{1}{2}, \frac{1}{2} + \sqrt{\frac{1}{4} - c}\right) \). We denote this zero by \( x_0 \). Due to the continuity of the function \( g(x) \) in the interval \( \left(\frac{1}{2}, \frac{1}{2} + \sqrt{\frac{1}{4} - c}\right) \), it has a constant sign in the interval \( \left[\frac{1}{2}, x_0\right) \) and a constant sign in the interval \( (x_0, \frac{1}{2} + \sqrt{\frac{1}{4} - c}) \). Note that

\[
g\left(\frac{1}{2}\right) = 1 - 4c - \sqrt{1 - 4c} = (1 - 4c)(\sqrt{1 - 4c} - 1) < 0
\]

and

\[
g\left(\frac{1}{2} + \sqrt{\frac{1}{4} - c}\right) = \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - c}} > 0.
\]

Hence,

\[
g(x) < 0 \quad \text{for} \quad x \in \left(\frac{1}{2}, x_0\right) \quad \text{and} \quad g(x) > 0 \quad \text{for} \quad x \in (x_0, a).
\]

Therefore, \( f(x) \) is decreasing in \( \left(\frac{1}{2}, x_0\right) \) and increasing in \( (x_0, a) \). The assertion of the lemma immediately follows from this statement.

**Lemma 6.** Let \( a \) and \( b \) be points on a spherical circle. Consider any point \( t \) such that \( a, t, \) and \( b \) lie in this circle in the indicated order according to the positive orientation. The measure of the angle \( \angle atb \) is maximum if the point \( t \) is equidistant from \( a \) and \( b \).

**Proof.** We denote the center of the circle by \( o \) and its radius by \( \rho \). Let \( t' \) be the point of this circle lying at identical distances from \( a \) and \( b \) and on the same side of the great circle containing \( ab \) as the point \( t \). We set

\[
\alpha = \frac{1}{2}|\angle aot| \quad \text{and} \quad \beta = \frac{1}{2}|\angle bot|.
\]

Let \( k \) be the midpoint of the arc \( at \). Note that \(|\angle aot'| = |\angle bot'| = \alpha + \beta\). Since \(|\angle tok| = \alpha\), in view of (4), we can easily conclude that

\[
|\angle ota| = \arccot(\cos \rho \tan \alpha).
\]

Analogously,

\[
|\angle otb| = \arccot(\cos \rho \tan \beta) \quad \text{and} \quad |\angle ot'a| = |\angle ot'b| = \arccot\left(\cos \rho \tan \frac{\alpha + \beta}{2}\right).
\]

Our aim is to show that \(|\angle at'b| \geq |\angle atb|\).
This inequality is equivalent to
\[
\arccot\left(\cos \rho \tan \frac{\alpha + \beta}{2}\right) \geq \frac{\arccot(\cos \rho \tan \alpha) + \arccot(\cos \rho \tan \beta)}{2}.
\]

In order to prove this fact, it is sufficient to show that the function \( f(x) = \arccot(\cos \rho \tan x) \) is concave in the interval \((0, \frac{\pi}{2})\). The reader may check that
\[
f''(x) = \frac{2 \cos \rho \sin x \cos x(\cos^2 \rho - 1)}{(\cos^2 x + \cos^2 \rho \sin^2 x)^2}.
\]

It is easy to see that \( f''(x) < 0 \) for \( x \in \left(0, \frac{\pi}{2}\right) \) and, therefore, the function \( f(x) \) is concave in the interval \((0, \frac{\pi}{2})\), which completes the proof.

**Theorem 2.** Every reduced spherical body \( R \) whose thickness is not greater than \( \frac{\pi}{2} \) is contained in a disk of radius \( \rho = \arctan\left(\sqrt{2} \tan \frac{\Delta(R)}{2}\right) \).

**Proof.** Note that every boundary point of \( R \) belongs to an arc whose ends are extreme points of \( R \). Therefore, if a disk contains all extreme points of \( R \), then it contains all boundary points of \( R \) and, hence, the entire body \( R \). Thus, it is sufficient to show that all extreme points of \( R \) are in a disk of radius \( \rho \). Moreover, according to the spherical Helly theorem (see [1, 10]), it is even sufficient to show that every three extreme points of \( R \) are contained in a disk of radius \( \rho \).

Let \( e_1, e_2, \) and \( e_3 \) be any three different extreme points of \( R \). By Theorem 4 in [5], there exist lunes \( L_1, L_2, \) and \( L_3 \) such that \( e_i \) is the center of one of the semicircles bounding \( L_i \) for \( i = 1, 2, 3 \). By \( f_i \) we denote the center of the other semicircle bounding \( L_i \) for \( i = 1, 2, 3 \).

First, we consider the case where two points from the triple of points \( e_1, e_2, \) and \( e_3 \), say, \( e_1 \) and \( e_2 \), lie in \( L_3 \) on the same side of the great circle containing \( e_3 \) and \( f_3 \). Since \( e_1 f_1 \) and \( e_3 f_3 \) intersect (see the proof of Lemma 2 in [7]), the distance from \( e_1 \) to \( e_3 f_3 \) is not greater than the distance from \( e_1 \) to the point of intersection of \( e_1 f_1 \) with \( e_3 f_3 \) and, hence, not greater than \( \Delta(R) \). For the same reason, the distance from \( e_2 \) to \( e_3 f_3 \) does not exceed \( \Delta(R) \). We denote by \( c \) the corner of \( L_3 \) lying on the same side of the great circle containing \( e_3 \) and \( f_3 \) as \( e_1 \) and \( e_2 \). Further, by \( k \) we denote the point of \( e_3 c \) such that \( |e_3 k| = \Delta(R) \) and by \( l \) we denote the point of \( f_3 c \) such that \( |f_3 l| = \Delta(R) \). Since \( e_1, e_2, e_3 \in \text{conv} \{e_3, f_3, k, l\} \), it is sufficient to show that \( \text{conv} \{e_3, f_3, k, l\} \) can be covered by a disk of radius \( \rho \). The triangle \( e_3 f_3 k \) is contained in a quarter of a disk of thickness \( \Delta(R) \) and, therefore, by Lemma 1 it can be covered with a disk of radius \( \rho \). Furthermore, this disk is unique by virtue of Lemma 1. For the same reason, the triangle \( e_3 f_3 l \) can be covered with a disk of radius \( \rho \) and this disk is unique. It is easy to see that the indicated disk is the same for both triangles. This disk covers \( \text{conv} \{e_3, f_3, k, l\} \), which completes the proof in this case.

Further, assume that, for any \( i, j, k \) such that \( \{i, j, k\} = \{1, 2, 3\} \), the points \( e_i \) and \( e_j \) lie in \( L_k \) on different sides of the great circle containing \( e_1 \) and \( f_k \). We use this assumption up to the end of the proof.

Consider the case where the triangle \( e_1 e_2 e_3 \) is obtuse or right. Without loss of generality, we can assume that the angle \( \angle e_1 e_3 e_2 \) is either obtuse or right. Let \( k \) be a point located on the same side of the great circle containing \( e_1 \) and \( e_2 \) as \( e_3 \) and such that \( |e_1 k| = |e_2 k| \) and the angle \( \angle e_1 k e_2 \) is right. By Theorem 8 in [7], \( |e_1 e_2| \) is not smaller than \( \arccos(\cos^2 \Delta(R)) \). Therefore, \( |e_1 k| \) and \( |e_2 k| \) do not exceed \( \Delta(R) \) because, otherwise, by (3)
the length $|e_1e_2|$ is greater than $\arccos(\cos^2 \Delta(R))$. Hence, by Lemma 1 the circumradius of the triangle $e_1ke_2$ is not greater than $\arctan\left(\sqrt{2} \tan \frac{\Delta(R)}{2}\right)$.

By virtue of Lemma 6, we can easily show that, for any point $t$ of the circle circumscribed about $e_1ke_2$ that lies on the same side of the great circle containing $e_1$ and $e_2$ as $k$, the angle $\angle e_1te_2$ is at least $\frac{\pi}{2}$. Thus, in view of the fact that $\angle e_1e_3e_2$ is obtuse or right, $e_3$ must lie inside this circumscribed circle. This completes the proof in the analyzed case.

Finally, we consider the case where the triangle $e_1e_2e_3$ is acute. Clearly, all heights of this triangle have lengths smaller that $\Delta(R)$. Let $g$ be the point closest to $e_2$ and such that $e_2$ lies on the arc $e_1g$. Moreover, let $g$ satisfy at least one of the following conditions: $\angle e_1eg$ is right, or $\text{dist}(g,e_1e_3) = \Delta(R)$, or $\text{dist}(e_1,ge_3) = \Delta(R)$. If $\angle e_1eg$ is right, then $|e_1e_3|$ and $|e_3g|$ are not greater than $\Delta(R)$. Therefore, the triangle $e_1ge_3$ is contained in a right triangle with legs of length $\Delta(R)$ and, hence, in the quarter of a disk of thickness $\Delta(R)$. Hence, by Lemma 1, $e_1$, $e_2$, and $e_3$ can be covered by a disk of radius $\rho$.

Otherwise, the angle $\angle e_1eg$ is acute. Since $|\angle e_1ge_3| < |\angle e_1e_2e_3|$, the triangle $e_1ge_3$ is acute. One height of this triangle, say, the height from the vertex $g$, is of length $\Delta(R)$. Moreover, the length of the height of this triangle from the vertex $e_1$ is not greater than $\Delta(R)$. Let $j$ be the point closest to $e_3$ and such that $e_3$ lies on the arc $e_1j$. Also let $j$ satisfy at least one of following conditions: $\angle e_1gj$ is right, or $\text{dist}(j,e_1g) = \Delta(R)$, or $\text{dist}(e_1,jg) = \Delta(R)$. If $\angle e_1gj$ is right (note that this is possible only for $\Delta(R) = \frac{\pi}{2}$), then we use the same arguments as in the previous case. Otherwise, the triangle $e_1gj$ is acute and has two heights of length $\Delta(R)$, say, from the vertices $g$ and $j$. Note that, from the construction of this triangle, we conclude that the length of the third height does not exceed $\Delta(R)$. Since the points $e_1$, $e_2$, and $e_3$ are contained in the triangle $e_1gj$, it remains to prove that this triangle can be covered by a disk of radius $\rho$.

Note that the triangle $e_1gj$ has two equal heights and, hence, it is an isosceles triangle. We denote the angle at $e_1$ by $2\alpha$, the center of the circle circumscribed about $e_1gj$ by $o$, and the radius of this circle by $\sigma$. We also denote the point in the middle of $je_1$ by $k$ and the point of $e_1g$ closest to $j$ by $h$. Clearly, the triangles $hje_1$ and $oke_1$ are right. We set $B = |je_1|$ and note that $|ke_1| = \frac{B}{2}$. Since $e_1gj$ is an isosceles triangle, we conclude that $|\angle oke_1| = \alpha$. The length of the arc $hj$ is $\Delta(R)$.

From (1), for the triangle $oke_1$, we obtain

$$\tan \frac{B}{2} = \cos \alpha \tan \sigma.$$  

Relation (2) for the triangle $hje_1$ gives

$$\sin \Delta(R) = \sin 2\alpha \sin B$$

or, in a different form,

$$\sin B \cos \alpha = \frac{\sin \Delta(R)}{2 \sin \alpha}.$$  

By using the last two formulas and the trigonometric formula $\tan \frac{B}{2} = \frac{1 - \cos B}{\sin B}$, we obtain

$$\tan \sigma = \frac{\tan \frac{B}{2}}{\cos \alpha} = \frac{1 - \cos B}{\sin B \cos \alpha} = \frac{1 - \cos B}{\sin \Delta(R)} = \frac{2 \sin \alpha \left(1 - \sqrt{1 - \sin^2 B}\right)}{\sin \Delta(R)}.$$
\[2 \sin \alpha \left(1 - \sqrt{1 - \frac{\sin^2 \Delta(R)}{\sin^2 2\alpha}}\right) = \frac{2 \left(\sin \alpha - \sqrt{\sin^2 \alpha - \frac{\sin^2 \alpha \sin^2 \Delta(R)}{\sin^2 2\alpha}}\right)}{\sin \Delta(R)}\]

\[= \frac{2 \left(\sqrt{1 - \cos^2 \alpha} - \sqrt{1 - \cos^2 \alpha - \frac{\sin^2 \Delta(R)}{4 \cos^2 \alpha}}\right)}{\sin \Delta(R)}\]

The greatest possible value of \( \alpha \) is \( \frac{\pi}{4} \), while the smallest value is attained for the case where the third height is of length \( \Delta(R) \). In the first case, the triangle \( e_1jg \) is right and, by Lemma 1, we conclude that

\[\tan \sigma_1 = \sqrt{2} \tan \left(\frac{\Delta(R)}{2}\right)\]

By using the formula

\[\tan \Delta(R) = \frac{2 \tan \left(\frac{\Delta(R)}{2}\right)}{1 - \tan^2 \left(\frac{\Delta(R)}{2}\right)}\]

we can easily show that, in this case,

\[\tan \sigma_1 = \sqrt{2} \frac{\sqrt{4 + 4 \tan^2 \Delta(R)} - 2}{2 \tan \Delta(R)} = \frac{\sqrt{2} + 2 \tan^2 \Delta(R) - \sqrt{2}}{\tan \Delta(R)}\]

In the second case, the triangle \( e_1jg \) is equilateral and, by Lemma 3, we get

\[\tan \sigma_2 = \frac{\sqrt{9 + 8 \tan^2 \Delta(R)} - 3}{2 \tan \Delta(R)}\]

For the sake of brevity, we denote \( \tan \Delta(R) = t \). Thus, we get

\[\tan \sigma_2 = \frac{\sqrt{9 + 8t^2} - 3}{2t} = \frac{1}{2t} \left(\sqrt{9 + 8t^2} - \sqrt{8 + 8t^2} - 3 + \sqrt{8 + 8t^2}\right)\]

\[= \frac{1}{2t} \left(\frac{1}{\sqrt{9 + 8t^2} + \sqrt{8 + 8t^2}} - 3 + \sqrt{8 + 8t^2}\right)\]

\[< \frac{1}{2t} \left(\frac{1}{\sqrt{9} + \sqrt{8}} - 3 + \sqrt{8t^2}\right)\]

\[= \frac{1}{2t} \left(3 - 2\sqrt{2} - 3 + \sqrt{8 + 8t^2}\right)\]

\[= \frac{\sqrt{2} + 2t^2 - \sqrt{2}}{t} = \tan \sigma_1\]
If we set
\[ \cos^2 \alpha = x \quad \text{and} \quad \frac{\sin^2 \Delta(R)}{4} = c, \]
then, by Lemma 5, it is possible to conclude that \( \tan \sigma \) takes the greatest value
\[ \max (\tan \sigma_1, \tan \sigma_2) = \sqrt{2} \tan \frac{\Delta(R)}{2}, \]
which yields the assertion of the theorem.

Note that, in general, by virtue of Lemma 1, we cannot improve the estimate from Theorem 2.

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