Scheduling with Complete Multipartite Incompatibility Graph on Parallel Machines

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Abstract

In this paper we consider the problem of scheduling on parallel machines with a presence of incompatibilities between jobs. The incompatibility relation can be modeled as a complete multipartite graph in which each edge denotes a pair of jobs that cannot be scheduled on the same machine. Our research stems from the work of Bodlaender et al. Bodlaender et al. (1994); Bodlaender and Jansen (1993). In particular, we pursue the line investigated partially by Mallek et al. Mallek et al. (2019), where the graph is complete multipartite so each machine can do jobs only from one partition.

We provide several results concerning schedules, optimal or approximate with respect to the two most popular criteria of optimality: $C_{\text{max}}$ (makespan) and $\sum C_j$ (total completion time). We consider a variety of machine types in our paper: identical, uniform and unrelated. Our results consist of delimitation of the easy (polynomial) and NP-hard problems within these constraints. We also provide algorithms, either polynomial exact algorithms for easy problems, or algorithms with a guaranteed constant worst-case approximation ratio or even in some cases a PTAS for the harder ones.

In particular, we fill the gap on research for the problem of finding a schedule with the smallest $\sum C_j$ on uniform machines. We address this problem by developing a linear programming relaxation technique with an appropriate rounding, which to our knowledge is a novelty for this criterion in the considered setting.

Index Terms

job scheduling, uniform machines, makespan, total completion time, approximation schemes, NP-hardness, incompatibility graph

I. INTRODUCTION

A. An example application

Imagine that we are treating some people ill with contagious diseases. There are quarantine units containing people ill with a particular disease waiting to receive some medical services. We also have a set of nurses. We would like the nurses to perform the services in a way that no nurse will travel between different quarantine units, to avoid spreading of the diseases. Also, we would like to provide to each patient the required services, which correspond to the time to be spent by a nurse.

Consider two sample goals: The first might be to lift the quarantine in the general as fast as possible. The second might be to minimize the average time of a patient waiting and treatment.

The problem can be easily modeled as a scheduling problem in our model. The jobs are the medical services to be performed. The division of jobs into partitions of the incompatibility graph is the division of the tasks into the quarantine units. The machines are the nurses. The sample goals correspond to $C_{\text{max}}$ and $\sum C_j$ criteria, respectively.

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B. Notation and the problems description

We follow the notation and definitions from Brucker (2007), with necessary extensions. Let the set of jobs be $J = \{j_1, \ldots, j_n\}$ and the set of machines be $M = \{m_1, \ldots, m_m\}$. We denote the processing requirements of the jobs $j_1, \ldots, j_n$ as $p_1, \ldots, p_n$.

Now let us define a function $p : J \times M \rightarrow \mathbb{N}$, which assigns a time needed to process a given job for a given machine. We distinguish three main types of machines, in the ascending order of generality:

- identical – when $p(j_i, m) = p_j$ for all $j_i \in J, m \in M$,
- uniform – when there exists a function $s : M \rightarrow \mathbb{Q}_+$, in this case $p(j_i, m) = \frac{p_j}{s_j}$ for any $j_i \in J, m \in M$,
- unrelated – when there exists $s : J \times M \rightarrow \mathbb{Q}_+$, which assigns $p(j_i, m) = \frac{p_j}{s_{ji}}$ for any $j_i \in J, m \in M$.

The incompatibility between jobs form a relation that can be represented as a simple graph $G = (J, E)$, where $J$ is the set of jobs, and $(j_1, j_2)$ belongs to $E$, iff $j_1$ and $j_2$ are incompatible. In this paper we consider complete multipartite graphs, i.e. graphs whose sets of vertices may be split into disjoint independent sets $J_1, \ldots, J_k$ (called partitions of the graph), such that for every two vertices in different partitions there is an edge between them. Due to the fact that the structure is simple, we omit the edges and we identify the graph with the partition of the jobs.

We differentiate between the cases when the number of the partitions is fixed, and when it is not the case. In the first case we denote the graph as $G = \text{complete } k\text{-partite}$, and in the second as $G = \text{complete multipartite}$.

A schedule $S$ is an assignment from jobs in the space of machines and starting times. Hence if $S(j) = (m, t)$, then $j$ is executed on the machine $m$ in the time interval $[t, t + p(j, m)]$ and $t + p(j, m) = C_j$ is the completion time of $j$ in $S$. No two jobs may be executed at the same time on any machine. Moreover, no two jobs which are connected by an edge in the incompatibility graph may be scheduled on the same machine. By $C_{\text{max}}(S)$ we denote maximum $C_j$ in $S$ over all jobs. By $\sum C_j(S)$ we denote sum of completion times of jobs in $S$. These are two criteria of optimality of a schedule commonly considered in the literature. Note that in both cases we are interested in finding or approximating the minimum value of respective measure.

Interestingly, an assignment from jobs in machines it sufficient to determine values of these measures in any reasonable schedule consistent with this assignment. By reasonable we mean a schedule in which there are no unnecessary delays between processed jobs; and the jobs forming a load on any machine are in optimal order given Smith’s Rule Smith (1956), in the case of $\sum C_j$ criterion. Under such an assignment the ordering of the jobs either has no impact on the $C_{\text{max}}$ criterion; or is in a sense determined in the case of $\sum C_j$ criterion.

We use the well-known three-field notation of Lawler et al. (1982). We are interested in problems described by $\alpha|\beta|\gamma$, where

- $\alpha$ is $P$ (identical machines), $Q$ (uniform machines) or $R$ (unrelated machines),
- $\beta$ contains either $G = \text{complete multipartite}$ or $G = \text{complete } k\text{-partite}$, or some additional constraints, e.g. $p_j = 1$ (unit jobs only),
- $\gamma$ is either $C_{\text{max}}$ or $\sum C_j$.

C. An overview of previous work

We recall that the $P|C_{\text{max}}$ is NP-hard even for two machines Garey and Johnson (1979). However, $Q|C_{\text{max}}$ (and therefore $P|C_{\text{max}}$ as well) does admit a PTAS [Hochbaum and Shmoys (1988). Moreover, $R_m|C_{\text{max}}$ admits a FPTAS [Horowitz and Sahni (1976). There is $(2 - \frac{1}{m})$-approximation algorithm for $R|C_{\text{max}}$ Shchepin and Vakhania (2005); however there is no polynomial algorithm with approximation ratio better than $\frac{3}{2}$, unless $P = \text{NP}$ [Lenstra et al. (1990). On the other hand, $Q|p_j = 1|C_{\text{max}}$ and $Q|\sum C_j$ (with $P|p_j = 1|C_{\text{max}}$ and $P|\sum C_j$ as their special cases) can be solved in $O(\min\{n + m \log m, n \log n\})$ Dessouky et al. (1990) and $O(n \log n)$ (Brucker, 2007, p. 133–134) time, respectively. $R|\sum C_j$ can be regarded as a special case of an assignment problem Bruno et al. (1974), which can be solved in polynomial time.

The problem of scheduling with incompatible jobs for identical machines was introduced by Bodlaender et al. in Bodlaender et al. (1994). They provided a series of polynomial time approximation algorithms for $P|G = \text{k-colorable}|C_{\text{max}}$. For bipartite graphs they showed that $P|G = \text{bipartite}|C_{\text{max}}$ has a polynomial 2-approximation algorithm, and this ratio of approximation is the best possible if $P \neq \text{NP}$. They also proved that there exist FPTAS in the case when the number of machines is fixed and $G$ has constant treewidth.
The special case $P|G,p_j=1|C_{\text{max}}$ was treated extensively in the literature under the name \textbf{Bounded Independent Sets}: for given $m$ and $t$, determine whether $G$ can be partitioned into at most $t$ independent sets with at most $m$ vertices in each. More generally, $P|G|C_{\text{max}}$ is equivalent to a weighted version of \textbf{Bounded Independent Sets}. We note also that $P|G,p_j=1|C_{\text{max}}$ is closely tied to Mutual Exclusion Scheduling, where we are looking for a schedule in which no two jobs connected by an edge in $G$ are executed in the same time. For the unrestricted number of machines it is the case that $P|G,p_j=1|C_{\text{max}}$ has a polynomial algorithm for a certain class of graphs $G$ if and only if Mutual Exclusion Scheduling has a polynomial algorithm for the same class of graphs. When all this is taken into account, there are known polynomial algorithms for solving $P|G,p_j=1|C_{\text{max}}$ when $G$ is restricted to the following classes: forests [Baker and Coffman Jr (1990), split graphs [Lond (1991), complements of bipartite graphs and complements of interval graphs [Bodlaender and Jansen (1995)]. However, the problem remains NP-hard when $G$ is restricted to bipartite graphs (even for 3 machines), interval graphs and cographs [Bodlaender and Jansen (1995).

Recently another line of research was established for $G$ equal to a collection of cliques (bags) in [Das and Wiese (2017). The authors considered $C_{\text{max}}$ criterion and presented a PTAS for identical machines together with $(\log n)^{1/4-\epsilon}$-inapproximability result for unrelated machines. They also provided an 8-approximate algorithm for unrelated machines with additional constraints. This approach was further pursued in [Grace et al. (2019), where an EPTAS for identical machines case was presented. The last result is a construction of PTAS for uniform machines with some additional restrictions on machine speeds and bag sizes [Page and Solis-Oba (2020).

Unfortunately, the case of complete multipartite incompatibility graph was not studied so extensively. It may be inferred from [Bodlaender et al. (1994) that for $P|G=\text{complete multipartite}|C_{\text{max}}$ there exists a PTAS, which can be easily extended to EPTAS; and that there is a polynomial time algorithm for $P|G=\text{complete multipartite},p_j=1|C_{\text{max}}$.

In the case of uniform machines Mallek et al. [Mallek et al. (2019) proved that $Q|G=\text{complete 2-partite},p_j=1|C_{\text{max}}$ is NP-hard, but it may be solved in $O(n)$ time when the number of machines is fixed. Moreover, they showed an $O(mn+m^2\log m)$ algorithm for the particular case $Q|G=\text{star},p_j=1|C_{\text{max}}$. However, their result implicitly assumed that the number of jobs $n$ is encoded in binary on $\log n$ bits (thus making the size of schedules exponential in terms of the input size), not – as it is customary assumed – in unary.

In this paper we provide several results: first, we prove that $P|G=\text{complete multipartite}|\sum C_j$, unlike its $C_{\text{max}}$ counterpart, can be solved in polynomial time. Next, we show that $Q|G=\text{complete multipartite},p_j=1|\sum C_j$ is Strongly NP-hard and that the same holds for $C_{\text{max}}$ criterion. However, it turns out that both $Q|G=\text{complete k-partite},p_j=1|\sum C_j$ and $Q|G=\text{complete k-partite},p_j=1|C_{\text{max}}$ admit polynomial time algorithms. Also, we propose 2-approximation and 4-approximation algorithms for $Q|G=\text{complete multipartite},p_j=1|C_{\text{max}}$ and $Q|G=\text{complete multipartite},p_j=1|\sum C_j$, respectively.

The first of our two main results is a 4-approximation algorithm for $Q|G=\text{complete k-partite}|\sum C_j$, based on a linear programming technique. The second one is a PTAS for $Q|G=\text{complete multipartite},p_j=1|C_{\text{max}}$.

We conclude by showing that the solutions for $R|G|C_{\text{max}}$, or for $R|G|\sum C_j$, cannot be approximated within any fixed constant, even when $G=\text{complete 2-partite}$ and there are only two processing times.

\section{Identical Machines}

We recall that $P|G=\text{complete multipartite}|C_{\text{max}}$ is NP-hard as a generalization of $P||C_{\text{max}}$, but it admits an EPTAS [Bodlaender et al. (1994).

Focusing our attention on $\sum C_j$ let us define what we mean by a \textit{greedy assignment} of machines to partitions: 1) assign to each partition a single machine, 2) assign remaining machines one by one to the partitions in a way that it decreases $\sum C_j$ as much as possible. To see why this approach works we need the following lemma.

\textbf{Lemma 1.} For any set of jobs, let $S_i$ be an optimal schedule in an instance of $P|\sum C_j$ determined by $J$. Then $\sum C_j(S_1) - \sum C_j(S_2) \geq \sum C_j(S_2) - \sum C_j(S_3) \geq \ldots \geq \sum C_j(S_{m-1}) - \sum C_j(S_m)$.

\textbf{Proof.} Assume for simplicity that $n$ is divisible by $i(i+1)(i+2)$. If this is not the case, then we add dummy jobs with $p_j=0$; obviously, this does not increase $\sum C_j$. 

Fix the ordering of jobs with respect to nonincreasing processing times. Now we may associate with each job its multiplier corresponding to the position in the reversed order on its machine. If a job $j_i$ has a multiplier $l$, then it contributes $l p_i$ to $\sum C_j$, and it is scheduled as the $l$-th last job on a machine.

Now think of the multipliers in the terms of blocks of size $i+1$. For $S_i$ the multipliers with respect to job order are:

$$1, \ldots, 1, 1; 2, \ldots, 2, 3, 3; \ldots; i, \ldots, i+1, i+1, i+1; \ldots$$

For $S_{i+1}$ the multipliers are:

$$1, \ldots, 1, 1, 1; 2, \ldots, 2, 2, 2; \ldots; i, \ldots, i, i, i; \ldots$$

For $S_{i+2}$ the multipliers are:

$$1, 1, 1, \ldots, 1; 1; 2, 2, 2, \ldots; i-1, \ldots, i-1, i, i; \ldots$$

Also, let the sum of multipliers of the $k$-th block in $S_i$ be $s_i^k$.

By some algebraic manipulations we prove that

$$s_i^k = (i+1)k + k + \lfloor (k-1)/i \rfloor,$$

$$s_{i+1}^k = (i+1)k,$$

$$s_{i+2}^k = (i+1)k - k + \lfloor k/(i+2) \rfloor.$$

It follows directly that $s_{i-1}^k - s_{i+1}^k = s_{i+2}^k - s_{i+1}^2$, for $k \geq 2$.

The smallest processing time in the $k$-th block is at least $p_{(i+1)k} \geq p_{(i+1)k+1}$, therefore the contribution of the $k$-th block to $\sum C_j(S_i) - \sum C_j(S_{i+1})$ is at least $p_{(i+1)k} s_i^k - s_{i+1}^k$. Similarly, the largest processing time in the $(k+1)$-th block is at most $p_{(i+1)k+1}$ so the contribution of the $(k+1)$-th block to $\sum C_j(S_{i+1}) - \sum C_j(S_{i+2})$ is at most $p_{(i+1)k+1} s_{i+1}^k - s_{i+2}^k$. Thus the contribution of the $(k+1)$-th block to $\sum C_j(S_{j+1}) - \sum C_j(S_{j+1})$, for all $k \geq 1$. Also, the first block does not contribute to $\sum C_j(S_{i+1}) - \sum C_j(S_{i+2})$, which proves the lemma.

Corollary 1. For a given instance of the problem $P|G=\text{complete multipartite}|\sum C_j$ a schedule constructed by the greedy method has optimal $\sum C_j$.

Proof. Let $S_{alg}$ and $S_{opt}$ be the greedy and optimal schedules, respectively. If the numbers of machines assigned to each of the partitions are equal in $S_{alg}$ and $S_{opt}$, then the theorem obviously holds.

Assume that there is a partition $J_i$ that has more machines assigned in $S_{opt}$ than in $S_{alg}$. It means that there is also a partition $J_j$ that has less machines assigned in $S_{opt}$ than in $S_{alg}$. Let us construct a new schedule $S_{opt}$ by assigning one more machine to $J_i$ and one less to $J_j$. By Lemma I we decreased $\sum C_j$ on partition $J_i$ no less than we increased it on partition $J_j$. Hence, the claim follows.

III. Uniform machines

It turns out that for an arbitrary number of partitions the problem is hard, even when all jobs have equal length:

Theorem 1. $Q|G=\text{complete multipartite}, p_j=1|\sum C_j$ is Strongly NP-hard.

Proof. We proceed by reducing Strongly NP-complete 3-Partition [Garey and Johnson (1979)] to our problem.

Recall that an instance of 3-Partition is $(A, b, s)$, where $A$ is a set of $3m$ elements, $b$ is a bound value, and $s$ is a size function such that for each $a \in A$, $\frac{1}{2} < s(a) < \frac{3}{2}$ and $\sum_{a \in A} s(a) = mb$.

The question is whether $A$ can be partitioned into disjoint sets $A_1, \ldots, A_m$, such that $\forall 1 \leq i \leq m, \sum_{a \in A_i} s(a) = b$.

For any $(A, b, s)$ we let $G = (J_1 \cup \ldots \cup J_m, E) = \text{complete } m\text{-partite}$, where $|J_i| = b$ for all $i = 1, 2, \ldots, m$.

Moreover, let $M = \{m_1, \ldots, m_{3mb}\}$ with speeds $s(m_i) = s(a_i)$. Finally, let the limit value be $\sum C_j = \frac{m(b+1)}{2}$.

Suppose now that an instance $(A, b, s)$ admits a 3-partition and let the sets be $A_1, \ldots, A_m$. Then if $a_i \in A_j$, we assign exactly $s(a_i)$ jobs from $J_j$ to the machine $m_i$. Since for every $i$ it holds that $\sum_{a \in A_i} s(a) = b$, we know...
that all jobs are assigned. Moreover, we never violate the incompatibility graph conditions, as we assign to any machine only jobs from a single partition.

By assigning \( s(a_i) \) jobs to a machine \( m_i \) we ensure that

\[
\sum C_j = \sum_{i=1}^{m} \left( \frac{s(a_i)+1}{2} \right) = \sum_{i=1}^{m} \frac{s(a_i) + 1}{2} = \frac{m(b+1)}{2}.
\]

Conversely, suppose that we find a schedule \( S \) with \( \sum C_j \leq \frac{m(b+1)}{2} \). Now let \( l_i \) be the number of jobs assigned to \( m_i \). Let us consider the following quantity:

\[
X := \sum_{i=1}^{m} \left( \frac{l_i + 1}{2} \right) \frac{1}{s(m_i)} - \frac{m(b+1)}{2}
\]

\[
= \sum_{i=1}^{m} \frac{l_i + s(m_i) + 1}{s(m_i)} (l_i - s(m_i)).
\]

\( X \) is the difference \( \sum C_j(S) \) and \( \sum C_j \) of a schedule where each machine \( m_i \) is assigned \( s(m_i) \) jobs. Now, we note that \( \sum_{i=1}^{m} (l_i - s(m_i)) = 0 \) as every job is assigned somewhere. Moreover,

\[
l_i + s(m_i) + 1 > 2s(m_i) \quad \text{if } l_i \geq s(m_i),
\]

\[
l_i + s(m_i) + 1 \leq 2s(m_i) \quad \text{if } l_i < s(m_i).
\]

By combining the last two facts we note that: every element \( l_i - s(m_i) \geq 0 \) in \( X \) gets multiplied by some number greater than 2, and every \( l_i - s(m_i) < 0 \) gets multiplied by some number not greater than 2. Therefore \( \sum_{i=1}^{m} (l_i - s(m_i)) = 0 \) implies \( X \geq 0 \). Moreover, if there exists any element, such that \( l_i - s(m_i) > 0 \), then \( X > 0 \). However, a schedule with \( \sum C_j \leq \frac{m(b+1)}{2} \) satisfies \( X \leq 0 \), therefore it holds that \( X = 0 \) and \( l_i = s(m_i) \) for all machines.

Each machine has jobs from exactly one partition assigned. Let \( M_j \) be the set of machines on which the jobs from \( J_j \) are executed. By the previous argument a machine \( m_i \) has exactly \( s(m_i) \) jobs assigned in \( S \). By this and the bounds on \( a \in A \), we have \( |M_j| = 3 \). Since the correctness of the schedule guarantees that all jobs are covered by some machines, we know that the division into \( M_1, M_2, \ldots, M_m \) corresponds to a partition.

**Theorem 2.** \( Q|G = \text{complete multipartite}, p_j = 1|C_{\text{max}} \) is Strongly NP-hard.

**Proof.** The proof is almost identical to that of Theorem 1 only using \( C_{\text{max}} \) as the criterion and 1 as the bound.

When the number of partitions is fixed, we show that there are polynomial algorithms for solving the respective problems.

**Theorem 3.** There exists a \( O(mn^{k+1} \log(mn)) \) algorithm for \( Q|G = \text{complete } k\text{-partite}, p_j = 1|C_{\text{max}} \).

**Proof.** We adopt the Hochbaum-Shmoys framework \cite{Hochbaum and Shmoys 1988}, i.e. we guess \( C_{\text{max}} \) of a schedule and check whether this is a feasible value. There are only up to \( O(mn) \) possible values of \( C_{\text{max}} \) to consider, e.g. using binary search, as it has to be determined by the number of jobs loaded on a single machine.

Now assume that we check a single possible value of \( C_{\text{max}} \). Fix any ordering of the machines. We store an information, if there is a feasible assignment of the first \( 0 \leq l \leq m \) machines such that there are \( a_i \) unassigned jobs for partition \( J_l \). In each step there is a set of tuples \((a_1, \ldots, a_k)\) corresponding to the remaining jobs of \( O(n^k) \) size. We start with \((|J_1|, \ldots, |J_k|)\) if no machines are used, all jobs are unassigned.

For \( l \geq 1 \) we take any tuple \((a_1, \ldots, a_k)\) from the previous iteration. We try all possible assignments of \( m_l \) to the partitions, then we try all feasible assignments of the remaining jobs to the machine. This produces an updated tuple, determining some feasible assignment of the first \( l \) machines. For each \((a_1, \ldots, a_k)\) we construct at most \( kn \) updated tuples, each in time \( O(1) \), so the work is bounded by \( O(kn^{k+1}) \). Note that we do need to store only one copy of each distinct \((a_1, \ldots, a_k)\).

After considering \( m \) machines it is sufficient to check if the tuple \((0, 0, \ldots, 0)\) is feasible. Clearly the total running time for a single guess of \( C_{\text{max}} \) is \( O(mkn^{k+1}) \).

**Theorem 4.** There exists a \( O(mn^{k+1}) \) algorithm for \( Q|G = \text{complete } k\text{-partite}|\sum C_j \).
Proof. Let us process machines in any fixed ordering. Let the state of the partial assignment be identified by a tuple \((a_1, \ldots, a_k, c)\), where \(a_i\) denotes the number of vertices remaining to be covered and \(c\) denotes \(\sum C_j\) of the jobs scheduled so far.

Assume that two partial assignments \(P_1\) and \(P_2\) on \(m'\) first machines are described by the same state \((a_1, \ldots, a_k)\); and two values \(c_1\) and \(c_2\), respectively. If \(c_2 \geq c_1\), then any extension of \(P_2\) on \(m'' > m'\) first machines cannot be better than the exact same extension of \(P_1\) on \(m''\) machines. Therefore for any \((a_1, \ldots, a_k)\) it is sufficient to store only the tuple \((a_1, \ldots, a_k, c)\) with the smallest \(c\).

We may proceed with a dynamic programming similar to the one used for \(C_{max}\). That is, we start with a single state \(((J_1, \ldots, J_k), 1, 0)\). In the \(l\)-th step \((l = 1, \ldots, m)\) we take states \((a_1, \ldots, a_k, c)\), corresponding to feasible assignments for the first \(l - 1\) machines. We try all \(k\) possible assignments of \(m_l\) to partitions. If \(m_l\) is assigned to \(J_i\), for some \(i\), then we try all choices of the number \(n' \in \{0, \ldots, a_i\}\) of remaining jobs from \(J_i\). Such a choice together with the assignment of \(m_l\) determines an assignment of \(n'\) unassigned jobs to \(m_l\). If the tuple constructed \((a_1', \ldots, a_k', c')\) has \(\sum C_j\) inferior to the already produced we do not store it.

Finally, after considering all machines we obtain exactly one tuple of the form \((0, \ldots, 0, c)\), which determines the optimal schedule.

At each step there are at most \(n^k\) states for every \((a_1, \ldots, a_k)\) we store only the smallest \(c\). There are up to \(n\) possible new assignments generated from each state. Each requires \(O(k)\) time. Therefore, it is clear that for any fixed \(k\) the time complexity of the algorithm is \(O(mn^{k+1})\).

**Lemma 2.** Let \(J\) be any set of jobs. Let \(M\) be a set of uniform machines with respective speeds \(s(m_1) = s(m_2) = 2, s(m_i) = 2^{-i}\) for \(3 \leq i \leq m\). Then it holds that \(\sum C_j\) of an optimal schedule of \(J\) on \(M\) is at least as big as the optimal \(\sum C_j\) for \(J\) and a single machine with speed \(2^m\).

Proof. It is sufficient to observe that by replacing two machines \(m'\) and \(m''\) with \(s(m') = s(m'') = 2^i\) with one machine \(m\) with \(s(m) = 2^{i+1}\) we never increase the total completion time of the optimal schedule.

Without loss of generality assume that there are exactly \(k\) jobs assigned to both \(m'\) and \(m''\), as we may always add jobs with \(p_j = 0\) at the beginning. The contribution of \(m'\) and \(m''\) to the total completion time is equal to \(\sum C_j(m', m'') = \sum_j \frac{j p'_j}{2^2} + \sum_j \frac{j p''_j}{2^2}\) where \(p'_j\) and \(p''_j\) are the \(j\)-th last jobs scheduled on \(m'\) and \(m''\), respectively.

Now if we consider a set of machines where \(m\) replaced \(m'\) and \(m''\) and consider a schedule where all jobs from \(m'\) and \(m''\) are scheduled on \(m\) in an interleaving manner the first is the first one from \(m'\), the second is the first one from \(m''\) etc. Then the contribution of \(m\) to the total completion time is equal to \(\sum C_j(m) = \sum_j \frac{(2j - 1) p'_j}{2^2} + \sum_j \frac{2j p''_j}{2^2} < \sum C_j(m', m'')\). Since this schedule is clearly a feasible one for a new set of machines we conclude that this holds also for the optimal schedule for the same set of machines.

Using Lemma 2 exhaustive search, linear programming, and rounding we were able to prove Theorem 5. Roughly speaking, the main idea lies in the fact that we may guess the speeds of the machines for each of the partitions. Then construct a linear (possibly fractional) relaxation of the assignment of the machines to the partitions and round it to integer. The rounding consists of rounding up the number of the machines of each speeds assigned to the partition. By knowledge what is the speed of fastest machine assigned to the partition in \(S_{opt}\) and by the previous lemma, we may schedule the jobs assigned to fractions of the machines on a machine with the highest speed in the partition, increasing the total completion by at most 2. This together with rounding proves the following theorem.

**Theorem 5.** There exists a 4-approximation algorithm for \(Q|G = \text{complete } k\text{-partite}| \sum C_j\).

Proof. Consider Algorithm 1. First notice that the proposed program is a linear relaxation of the scheduling problem. Precisely, \(n_{pr, tp}\) means how many machines from a group \(tp\) are assigned to the partition \(pr\); \(x_{jb, lr, tp}\) means what part of a job \(jb\) is assigned as the \(lr\)-th last on a machine of type \(tp\). Notice that jobs assigned to machines of a given type form layers, i.e. jobs assigned as last contribute their processing times once, as the last by one contributes twice, etc.

About the conditions:
- Condition 1 guarantees that all the machines are assigned, fractionally at worst.
- Condition 2 provides that no machine with speed higher than maximum possible (i.e. guessed) is assigned to the partition.
Algorithm 1 4-approximate algorithm for the problem

\[ \text{Algorithm } 1 \]

\[ Q[G = \text{complete } k\text{-partite}] \sum C_j \]

\textbf{Require:} \( J = (J_1, \ldots, J_k) \), \( M = \{m_1, \ldots, m_m\} \).

1. Round the speeds of the machines up to the nearest multiple of 2.
2. Let the nonempty group of the machines, ordered by the speeds be \( M_1, \ldots, M_l \).
3. For each of the partitions, guess the speed of the machine with the highest speed and the number of the machines of this speed assigned. Let the indices of their speed groups be \( s'_1, \ldots, s'_k \) and numbers be \( n'_1, \ldots, n'_k \), respectively. Dismiss the guesses that are unfeasible.
4. Solve the following linear program.

5. Let variables be:
   - \( n_{pr,tp} \), where \( pr \in \{1, \ldots, k\} \), \( tp \in \{1, \ldots, l\} \).
   - \( x_{jb,lr,tp} \), where \( jb \in J_1 \cup \ldots \cup J_k \), \( lr \in \{1, \ldots, n\} \), \( tp \in \{1, \ldots, l\} \).
6. Let the conditions be:

   \[
   \sum_{pr} n_{pr,tp} = |M_{tp}| \quad \forall tp
   \]

   \[
   n_{pr,tp} = 0 \quad \forall pr \forall tp > s_{pr}
   \]

   \[
   0 \leq n_{pr,tp} \leq |M_{tp}| - \sum_{i \in \{i|s_i = tp\}} n'_i \quad \forall pr, tp < s_{pr}
   \]

   \[
   n'_{pr} = n_{pr,tp} \quad \forall pr, tp = s_{pr}
   \]

   \[
   \sum_{lr,tp} x_{jb,lr,tp} = 1 \quad \forall jb
   \]

   \[
   \sum_{jb \in J_{pr}} x_{jb,lr,tp} \leq n_{pr,tp} \quad \forall pr \forall tp \forall lr
   \]

   \[
   0 \leq x_{jb,lr,tp} \quad \forall jb \forall lr \forall tp
   \]

   \[
   0 \leq n_{pr,tp} \quad \forall pr, tp
   \]

7. Let the cost function be:

   \[
   \sum_{jb,lr,tp} x_{jb,lr,tp} \cdot lr \cdot p(jb) \cdot \frac{1}{s(tp)}, \quad \text{where } p(jb) \text{ is the processing requirement of job } jb, \ s(tp) \text{ is the speed factor of machine of type } tp.
   \]

8. Solve the jobs assignment for each partition separately using the optimal solution of LP.

   - Condition (6) guarantees that each of the partitions can be given any number of not preassigned (not assigned by guessing) machines of a given type.
   - Condition (4) guarantees that the given number of machines of guessed type are assigned to a given partition as the fastest ones.
   - Condition (5) ensures that any job is assigned completely, in a fractional way at worst.
   - Condition (6) guarantees that for a given layer, partition, and machine type there is no more jobs assigned than the machines of this type to the partition.

The cost function corresponds to an observation that a job \( jb \) assigned as the \( l \)-th last on the machine of type \( tp \) contributes exactly \( \frac{tp(jb)}{s(tp)} \) to \( \sum C_j \).

An optimal solution to LP \((x^*, n^*)\) corresponds to a fractional assignment of machines to the partitions. We now construct for each partition \( J_i \), separately a partition fractional scheduling problem in the following way:

- A new set of variables \( y_{jb,lr,m} \) indicating a fractional assignment of \( jb \in J_i \) as the \( lr \)-th last job on machine \( m \in M' \).
- A cost function \( \sum_{jb,lr,m} y_{jb,lr,m} \cdot \frac{lr \cdot p(jb)}{s(m)} \).
- The conditions:
  - \( \forall jb,lr,m \ y_{jb,lr,m} \geq 0 \),
  - \( \forall jb \sum_{lr,m} y_{jb,lr,m} = 1 \) – each job has to be assigned completely,
  - \( \forall lr,m \sum_{jb} y_{jb,lr,m} \leq 1 \) – each layer on each machine cannot contain more than a full job in total.
Here the set of machines $M'$ consists of exactly $[n^*_{tp}]$ machines for each $1 \leq tp \leq l$. Hence, for each type we add at most one „virtual” machine due to rounding, except the machine with the highest speed per partition, which were preassigned exactly.

Now we rearrange jobs within layers for machines of the same speed to construct some feasible solution to partition fractional scheduling. Hence, we can change them into a single virtual machine with a speed no greater than the speed of the fastest (by the arc $y$). Here the set of machines $M'$ consists of exactly $[n^*_{tp}]$ machines for each $1 \leq tp \leq l$. Hence, we can change them into a single virtual machine with a speed no greater than the speed of the fastest (by the arc $y$). Here the set of machines $M'$ consists of exactly $[n^*_{tp}]$ machines for each $1 \leq tp \leq l$. Hence, we can change them into a single virtual machine with a speed no greater than the speed of the fastest (by the arc $y$). Hence an optimal solution can have only at most this cost.

Let us model this linear program as a flow network. Precisely, we construct:

- a set of vertices $V = J_j \cup (M' \times |J_j|)$
- a set of arcs $J_j \times (M' \times |J_j|)$ with capacity 1 each, and with the cost of the flow by an arc $(jb, (m, lr))$ equal to $br \cdot p(jb) \cdot \frac{1}{s(m)}$.

Any fractional solution corresponds to a fractional flow by the network, i.e. a value of $y_{jb,lr,m}$ is exactly the flow by the arc $(jb, lr, m)$. We know that e.g. Successive Shortest Path Algorithm (Ahuja et al. 1993) finds an integral minimum cost flow in the network corresponding directly to partition fractional scheduling. We know that this solution is at least as good as the solution for the general relaxation of the scheduling problem. We can treat the flow as an assignment for all the jobs in $J_j$. Due to rounding of $n^*$ it may use some virtual machines, but by Lemma 2 we can change them into a single virtual machine with a speed no greater than the speed of the fastest machine. Finally, by moving all jobs from the virtual machine and any fastest machine to this fastest machine we increase $\sum C_j$ of these jobs by at most 2 times. This together with rounding of the machine speeds allows to bound the approximation ratio by 4.

\[ \Box \]

A. Simple Algorithms for Unit Time Jobs

In this subsection we sketch outlines of two methods: a 2-approximation algorithm for the problem $Q|G = complete multipartite, p_j = 1|C_{max}$ and a 4-approximation algorithm for $Q|G = complete multipartite, p_j = 1|\sum C_j$. It is more convenient to express the algorithms in terms of covering the parts of $G$ by capacities of the machines.

Let us first sketch briefly the necessary notation. Let $J = J_1, \ldots, J_k$ be the parts. Let $M = m_1, \ldots, m_m$ be the machines. Let $c : M \to \mathbb{N}_+$ be a function giving capacities of the machines.

By a cover we mean any subset of the machines. For cover $M' \subseteq M$, a part $J_j$, and $c$ we define the cover ratio $CR_j^c(M') = \frac{\min\{|J_j|, \sum_{m \in M'} c(m)\}}{|J_j|}$. We say that a cover $M' \subseteq M$ is an $\alpha$-cover for $J_j$ under $c$ if $CR_j^c(M') \geq \alpha$. A cover $M' \subseteq M$ is said to be an exact cover for $J_j$ under $c$ if $CR_j^c(M') = 1$.

By a covering of $J' \subseteq J$ we mean a partial function $F : M \to J'$, i.e. a function which not necessarily assigns all the machines. We say that a covering $F$ of $J'$ is an $\alpha$-covering for $J'$ under $c$ if $\forall_{J_i \in J'} F^{-1}(J_i)$ is an $\alpha$-cover of $J_i$. Similarly, a covering $F$ of $J'$ is an exact covering for $J'$ under $c$ if $\forall_{J_i \in J'} F^{-1}(J_i)$ is an exact cover of $J_i$.

When defining particular covers (coverings) we usually omit the function $c$, it is clearly stated in the context of a given cover (covering). Also, we sometimes do not explicitly specify $J'$ if it is clear from the context.

Now, let us proceed to the algorithms that we propose and the proofs of the quality of solutions produced by the algorithms.

**Lemma 3.** For a given $(J, M, c)$ Algorithm 2 either verifies that there is no exact covering or it constructs a $\frac{1}{2}$-covering. The algorithm works in $O(n + m)$ time.

**Proof.** First, let us assume that for the given $(J, M, c)$ there exists an exact covering $F_{opt}$. We establish the following invariant under this assumption: during the execution of Algorithm 2 after constructing a $\frac{1}{2}$-covering $F$ for $J_1, \ldots, J_j$ the sum of capacities of $M \setminus F^{-1}(\{J_1, \ldots, J_j\})$ at least $\sum_{j'=j+1}^k |J_{j'}|$. First, let us prove a particular case: $\sum_{i \in [m]} c(m_i) = n$, i.e. that the instance is tight. Before the assignment of the first machine the invariant obviously holds. Now suppose that the invariant holds for all values $1, \ldots, j - 1$ and let us use a few of the remaining unassigned machines to construct a cover for $J_j$. \[ \Box \]
Algorithm 2 Let $J$ be the parts ordered by their sizes. Let $M$ be the machines ordered by their capacities $c$. The following procedure verifies whether there is no exact covering for the instance, or else it constructs a $\frac{1}{2}$-covering.

1: procedure GREEDY-COVERING($J = \{J_1, \ldots, J_k\}$, $M = \{m_1, \ldots, m_m\}$, $c$)
2: Let $F \leftarrow \emptyset$
3: Let $j \leftarrow 1$
4: for $i = 1, \ldots, m$ do
5: $F \leftarrow F \cup (m_i, J_j)$
6: if $CR^*_j(F) \geq \frac{1}{2}$ then $j \leftarrow j + 1$
7: end for
8: if $F$ is $\frac{1}{2}$-cover for $J$ then return $F$; else return NO
9: end procedure

- Assume that for the first machine $m_i$ assigned to $J_j$ in $F$ the following inequality holds: $c(m_i) \leq |J_j|$. Then the invariant is preserved after the assignment, since by assigning the sequence of machines $m_i, \ldots, m_{i'}$ to $J_j$ the sum of the remaining capacities was decreased by $\sum_{i'=i}^{i'} c(m_{i'}) \leq |J_j|$ and $\sum_{j'=j}^{m} |J_{j'}|$ was decreased by $|J_j|$.
- Assume the opposite, that $c(m_i) > |J_j|$ holds for the first $m_i$ used for $J_j$. Note that $J_j$ is covered by exactly one machine in $F$. Observe, the assignment of any machine $m_{i''} \leq i$ to any part $J_{j''} \geq j$ in $F_{opt}$ would force that $F_{opt}^{-1}(J_{j''})$ would have total capacity greater than $|J_{j''}|$, which is impossible, because the instance is tight. This means that $F_{opt}^{-1}(\{J_j, \ldots, J_k\})$ consist only of machines of smaller capacity, which are in $M \setminus F^{-1}(\{J_j, \ldots, J_k\})$.

Hence, the invariant again holds.

Finally, consider the general case $\sum_{i \in [m]} c(m_i) \geq \sum_{j \in [k]} |J_j|$. In such case let us consider an arbitrary exact covering $F_{opt}$. If there are some machines unassigned, then let us assign them arbitrary. Now, $F_{opt}$ would be also an exact covering if for any $j$ the number of jobs in $J_j$ was exactly $\sum_{i:F(m_i)=J_j} c(m_i)$. Let us modify the instance, that is let us assume that it consists of parts $J' = \{J'_1, \ldots, J'_k\}$ where $J'_j$ is of cardinality $\sum_{i:F(m_i)=J_j} c(m_i)$. Let us mark the modified instance, perhaps after resorting the parts, as $(M, (J'_1, \ldots, J'_k), c)$. By the observation for the tight case, the algorithm would succeed in constructing a $\frac{1}{2}$-covering $F'$ for $J'$. Note that for any $j \in [k]$ we have $|J'_j| \geq |J_j|$, due to the fact that no part size is lower than previously. Observe, by the fact that algorithm has to succeed in constructing $F'$ it has to succeed in constructing a $\frac{1}{2}$-covering $F$ for $(J_1, \ldots, J_k)$. This is by an observation (a formal proof of which we omit) that for any $j \in [k]$ $M \setminus F^{-1}(\{J_1, \ldots, J_k\}) \subseteq M \setminus F^{-1}(\{J_1, \ldots, J_k\})$. And vice versa, if the algorithm returns NO, there is no exact covering.

Example III-A.1. In order to illustrate the reasoning for the general case confer:

- A set of machines $M = \{m_1, m_2, m_3, m_4, m_5, m_6\}$.
- The function $c$ with values equal to $7, 4, 4, 3, 3, 2$, for $m_1, m_2, m_3, m_4, m_5, m_6$, respectively.
- Two sets of parts $J' = (J'_1, J'_2, J'_3)$, where the parts have sizes $10, 8, 5$, respectively; and $J = (J_1, J_2, J_3)$, where the parts have sizes $7, 6, 4$, respectively.

The covering produced for $J'$ is $m_1; m_2; m_3; m_4$ and the covering produced for $J$ is $m_1; m_2; m_3$.

Theorem 6. There exists 2-approximation algorithm for $Q|G = complete multipartite, p_j = 1|C_{\max}$ running in $O(m \log m + n \log n)$ time.

Proof. Assume that we suspect that there exits a schedule for an instance $(J, M)$ of $Q|G = complete multipartite, p_j = 1|C_{\max}$ within time $T$. Let us calculate the capacities of the machines, i.e. $c(m_i) = \lfloor s(m_i) \cdot T \rfloor$. If the assumption is true, then there exists an exact covering. Now we apply Algorithm 2 to $(J, M, c)$ and we may get two results:

1) the algorithm returned $F$ - a $\frac{1}{2}$-covering of $J$,
2) or the algorithm returned NO, which guarantees that there is no exact covering of $F$ in the time $T$.

By Lemma 3 the second case cannot occur if there exists a schedule in the time $T$.

Now, let us take $\frac{1}{2}$-covering $F$ for this $T$. We translate it to a schedule as follows: for $i = 1, \ldots, m$ if $F(m_i) = J_j$, we schedule up to $2c(m_i)$ jobs from $J_j$ on $m_i$. In total, the space on the machines assigned to $J_j$ in the time $2T$ ensures that all jobs are scheduled. Moreover, each machine gets jobs only from a single part. Finally, it is clear that the makespan of this schedule is at most $2T$. 
Let \( C_{max}^* \) be \( C_{max} \) of an optimum schedule. Observe, that \( C_{max}^* \) is determined by a number of jobs \( n' \in [n] \) assigned to some machine \( m_i \). This means that we have only \( O(mn) \) candidates for \( C_{max}^* \). By checking the candidates we can find the smallest \( T \) for which exists a \( \frac{1}{2} \)-covering \( F \). Using \( F \) it is easy to construct a schedule with \( C_{max} \) equal to at most \( 2T \leq 2C_{max}^* \).

Initially we have to sort the machines and parts, which can be done in \( O(m \log m) \) and \( O(n \log n) \) time, respectively. From this point that we can assume that \( m \leq n \); in the other case we can always discard all but \( n \) fastest machines without affecting the optimal solution. By this we have \( O(\log n) \) iterations of binary search over candidates for \( T \). Each application of Algorithm 2 requires \( O(n) \) time, also by \( m \leq n \). Clearly, the sketched 2-approximation algorithm requires \( O(m \log m + n \log n) \) time.

**Theorem 7.** There exists a 4-approximation algorithm for \( Q(G = \text{complete multipartite}, p_j = 1) \sum C_j \) running in \( O(m^2n^3 \log m) \) time.

**Proof.** Assume that the parts and machines are sorted in order of their nonincreasing sizes and speeds, respectively. Suppose that we knew in advance the numbers of jobs \( c_1, \ldots, c_m \) assigned to a machines \( m_1, \ldots, m_m \) in some optimal schedule. Without loss of generality, we could assume that if \( s_1 \geq \ldots \geq s_m \), then so \( c_1 \geq \ldots \geq c_m \). Observe, that the values \( c_i \) can be also interpreted to form capacities of the machines. In this case we could apply Algorithm 2 to \((J, M, c)\) and obtain \( F - a \) \( \frac{1}{2} \)-covering of \( J \). We could translate \( F \) to a schedule as follows: if \( F(m_i) = J_j \), then assign up to \( 2c_i \) jobs from \( J_j \) to \( m_i \). In total, the capacity of the machines assigned to \( J_j \) is at least \( \left\lfloor \frac{1}{2} |J_j| \right\rfloor \) so every job would be scheduled. Now observe that \( m_i \) in the optimal schedule contributes exactly \( \left( \frac{c_i+1}{2} \right) \frac{1}{s(m_i)} \) to \( \sum C_j \), but in constructed schedule it would contribute at most \( \left( \frac{2c_i+1}{2} \right) \frac{1}{s(m_i)} \leq 4 \left( \frac{c_i+1}{2} \right) \frac{1}{s(m_i)} \). So this would be schedule with \( \sum C_j \) at most 4 times the optimum. Unfortunately, by Theorem 7 it is NP-hard to obtain such information.

However, observe that in \( F \) the assignment of the machines to the parts would be ordered. That is, \( J_1 \) gets \( n_1 \) machines of biggest capacity, \( J_2 \) gets next \( n_2 \) machines of biggest capacity, etc. This is by the observation that in Algorithm 2 the machines are considered in a fixed order, determined by the order of capacities, which w.l.o.g. is determined by the order of speeds. Let us call by ordered covering any such covering.

This observation allows us to construct a covering that corresponds to a schedule with \( \sum C_j \) at most 4 times the optimum without knowledge of \( c \). We proceed by using dynamic programming over all ordered coverings. Precisely, let the states of this program be defined by \((j, i, c, F)\). Where \( F \) is ordered covering in which the first \( i \) machines are assigned to the first \( j \) parts, and whose associated schedule has lowest \( \sum C_j \) among all schedules corresponding to an ordered covering of first \( i \) machines to the first \( j \) parts. Notice that \((1, 1, cost_1, F_1), \ldots, (1, m-(k-1), cost_{m-(k-1)}, F_{m-(k-1)})\) are well defined. Precisely, \( cost_i \) is the total completion time of the jobs from \( J_1 \) scheduled on \( i \) fastest machines, \( F_i = \bigcup_{i \in [i]} (m_i, J_i) \). For \( k' \geq 2, m-(k-k') \geq m' \geq k' \) and \( m'-1 \geq m'' \), construct \((k', m', cost, F')\) as the best with respect to \( \sum C_j \) ordered covering corresponding to \((k'-1, m'', cost'', F'')\) and the assignment of \( m_{m'+1}, \ldots, m_{m'} \) to \( J_k \). Every such an assignment is feasible.

Moreover, for any \( 1 \leq k' \leq k \) and \( k' \leq m' \leq m-(k'-k) \) holds that there is no ordered covering of \( \{J_1, \ldots, J_{k'}\} \) using \( \{m_1, \ldots, m_{m'}\} \) with smaller \( \sum C_j \) of associated schedule than \( cost \) following from \((k', m', cost, F)\). For \( k' = 1 \) and any \( m' \) it obviously holds. Consider a counter-example with the minimum number of the parts and the minimum number of the machines. In this case, let there be an ordered covering \( F_{opt} \) associated with some schedule of minimum \( \sum C_j \) defined by the numbers of the machines assigned to \( J_1, \ldots, J_k \), and let these numbers be \( n_1, \ldots, n_k \), respectively. Consider an ordered covering \( F_{alg} \) determined by \((k-1, n_1 + \ldots + n_{k-1}, cost, F')\) and by the assignment of \( M_k = \bigcup_{i=n_i+\ldots+n_{k-1}+1}^{n_i+\ldots+n_{k-1}+1} m_i \) to \( J_k \). Notice that the contributions to \( \sum C_j \) of the \( J_k \) scheduled on \( M_k \) are equal in the schedule associated with \( F_{opt} \) and in the schedule associated with \( F_{alg} \). This means that there is a minimum counter-example on \( k-1 \) parts and \( n_1 + \ldots + n_{k-1} \) machines.

The sorting of parts and machines can be done in \( O(n) \) and \( O(m \log m) \) time, respectively. After this operation we can assume that \( m \leq n \). If \(|M| < |J|\), then no schedule can exist. At each step of our dynamic program there are at most \( mn \) states since for every \((i, j)\) we store only the smallest \( c \). There are up to \( m \) possible new coverings generated from each state, each requiring \( O(n \log m) \) time to generate. Therefore each step requires \( O(m^2n^2 \log m) \) operations and the total running time of the algorithm is \( O(km^2n^2 \log m) = O(m^2n^3 \log n) \).
B. A PTAS for $Q|G = \text{complete multipartite}, p_j = 1|C_{\text{max}}$

Now let us return to $Q|G = \text{complete multipartite}, p_j = 1|C_{\text{max}}$ problem. We can significantly improve on Theorem 6 and construct a PTAS inspired by the ideas of the PTAS for Machine Covering (Azar and Epstein, 1998).

Algorithm 3 The main part of the PTAS for $Q|G = \text{complete multipartite}, p_j = 1|C_{\text{max}}$. The following algorithm either verifies that there is no schedule of length at most $T$ or it constructs a schedule with $C_{\text{max}}$ close to $T$.

1: procedure PTAS($J, M, T, \epsilon$)  
2: \quad $\epsilon \leftarrow \min\{\frac{4}{3}, \epsilon\}$  
3: \quad Calculate rounded capacities $e^*(m)$ for all $m \in M$  
4: \quad $l_{\text{min}} \leftarrow \lceil 3 \log_{1 + \epsilon} \frac{1}{\epsilon} \rceil + 1$  
5: \quad Split $J$ into ranges $\{P_i\}_{i=0}^{l_{\text{max}}}$  
6: \quad Find $SV_{l_{\text{min}}+1}$ for $(J, M, e^*)$ (see Algorithm 4 and Lemma 5)  
7: \quad for $l = l_{\text{min}} + 1, \ldots, l_{\text{max}}$ do  
8: \quad \quad for $sv \in SV^l$ do  
9: \quad \quad \quad Generate $CSV(sv)$ (see Algorithms 5 and 6, Lemmas 6 and 7)  
10: \quad \quad end for  
11: \quad \quad Find $SV^{l+1}$ as a subset of $\bigcup_{sv \in SV^l} CSV(sv)$ (see Lemma 8)  
12: \quad end for  
13: \quad if $SV^{l_{\text{max}}+1} = \emptyset$ then return NO  
14: \quad Pick any $sv \in SV^{l_{\text{max}}+1}$ with its respective $(1 - \epsilon)$-covering $F$  
15: \quad return a schedule $S$ constructed from $F$  
16: end procedure

A high-level overview of the algorithm is presented as Algorithm 3. As previously, we state the algorithm in terms of constructing an approximate covering. As in the case of the application of the 2-approximation algorithm for $Q|G = \text{complete multipartite}, p_j = 1|C_{\text{max}}$, we state the algorithm in the framework of Hochbaum and Shmoys (1988). That is, we assume that a guess of a value $T$ is given such that there exists a schedule with $C_{\text{max}} \leq T$. Naturally, such a schedule corresponds to an exact covering of $J$ by $M$ under capacities given by $c(m) = \lceil s(m) \cdot T \rceil$. The method that we propose either verifies that the guess is incorrect, i.e. that there is no exact covering in capacities determined by the time $T$, or it constructs a covering with capacities determined by the time $T$ that can be transformed into a schedule with $C_{\text{max}}$ near $T$. By applying the method over a set of candidate makespans we find the smallest $T$ such that there exists a schedule with makespan close to $T$.

For any fixed guess of $T$ we can distinguish the following basic steps of the algorithm:

1) Applying some preprocessing, in particular to divide parts into ranges and to calculate rounded capacities of the machines.

2) Finding a set of vectors $SV^{l_{\text{min}}+1}$ such that at least one vector in the set describes machines that are not assigned to small parts in some exact covering and each vector in the set describes an exact covering of small parts.

3) Applying iteratively a procedure consisting of two steps:
   a) The first step is to find for each $sv^l \in SV^l$ a set of candidate state vectors $CSV(sv^l)$ such that it contains at least one good state vector (a term defined later) if $sv^l$ is a good state vector.
   b) The second step is to calculate $SV^{l+1}$ as a subset of $\bigcup_{sv \in SV^l} CSV(sv)$ such that the constructed set contains at least one good state vector.

4) Constructing a schedule with $C_{\text{max}} \leq T(1 + 7\epsilon)$ using a nice $(1 - \epsilon)$-covering, corresponding to a vector in $SV^{l_{\text{max}}+1}$ – the set $SV^{l_{\text{max}}+1}$ is nonempty provided that there exists a schedule with $C_{\text{max}} \leq T$.

1) Basic definitions: In order to prove the result formally, we state a suitable notation and a few notions tailored to our problem. As previously, for any fixed $T$ let us define the capacity of $m \in M$ by $c(m) = \lceil s(m) \cdot T \rceil$. Let us also define rounded capacity of $m$ by $e^*(m)$, equal to $c(m)$ rounded up to the nearest value of the form $\lceil (1 + \epsilon)^i \rceil$. Clearly, $c(m) \leq e^*(m) \leq (1 + \epsilon)c(m)$ so for convenience from now on we will refer to rounded capacities exclusively, and the covers are constructed with respect to $e^*$. 
Now, we group parts into sets (also called ranges) \( P_l = \{ J_k : |J_k| \in [(1 + \epsilon)^l, (1 + \epsilon)^{l+1}] \} \) and we will consider these ranges in order of increasing \( l \). For convenience, let \( l_{\text{max}} \) be the largest value such that its range is nonempty.

Next, given \( c^* \) and \( l \) we divide the machines into several types:

1. tiny – with \( c^*(m_i) < \epsilon^{-2} \),
2. small – with \( \epsilon^{-2} \leq c^*(m_i) \leq (1 + \epsilon)^l \),
3. average – with \( \max\{ \epsilon(1 + \epsilon)^l, \epsilon^{-2} \} \leq c^*(m_i) < (1 + \epsilon)^{l+1} \),
4. large – with \( \max\{ (1 + \epsilon)^{l+1}, \epsilon^{-2} \} \leq c^*(m_i) \).

The division is unequivocal only with respect to the given \( l \). For clarity of the notation we sometimes write that \( m \) is \( l \)-small (\( l \)-average) /\( l \)-large/ to denote that \( m \) is small (average) /large/ with respect to \( l \) under \( c^* \). Sometimes we do not use the \( l \) explicitly when stating that some machine is small, average, etc., but it is always given implicitly.

We use the notation of covers and coverings used in the description of Algorithm 2. However, we have to add a few other types of covers and a few other types of coverings. For a part \( J_k \) a set \( M' \subseteq M \) is a tiny exact cover if it is an exact cover and \( M' \) consists of tiny machines alone. Also, we say that \( M' \subseteq M \) is slack exact cover of a part \( J_j \) in \( P_l \) when it is exact cover of \( J_j \), \( M' \) consists of \( l \)-small machines or tiny machines, and there is at least one \( l \)-small machine in \( M' \). Also, for a cover \( M' \subseteq M \) of \( J_j \in P_l \) where \( M' \) consists of at least one \( l \)-small or \( l \)-average machine by \text{slack capacity} we mean the total capacity of all \( l \)-small and tiny machines in \( M' \).

We define that two covers \( M' \) and \( M'' \) are equivalent under capacity function \( c^* \) if there is a bijection \( f : M' \rightarrow M'' \) such that for any \( \forall m \in M' \) \( c^*(m) = c^*(f(m)) \). Hence, for a set \( M' \) of machines of equal capacity under \( c^* \) there is \(|M'| + 1\) nonequivalent subsets of \( M' \).

Due to the rounding we have only at most \( d_{\text{average}} = \lceil \log_{1+\epsilon}(\frac{1}{\epsilon}) \rceil + 1 \) distinct capacities for average machines, regardless of \( l \). Their capacities are equal to \( \{(1 + \epsilon)^{l-d_{\text{average}}} \}, \ldots , \{(1 + \epsilon)^l \} \) – since is easy to check that \( \{(1 + \epsilon)^{l-d_{\text{average}}} \} < \epsilon(1 + \epsilon)^l \). Similarly we have only at most \( d_{\text{tiny}} = \lceil \log_{1+\epsilon}(\frac{1}{\epsilon}) \rceil \) distinct capacities of tiny machines (the maximum number is of the form \( (1 + \epsilon)^{2\log_{1+\epsilon}(\epsilon^{-1})-1} \), the numbers are counted from 0). We write “at most” due to the fact that when \( \epsilon \) is small, then a few values \( (1 + \epsilon)^i \) for small \( i \) may be rounded to the same integer, hence there is no reason to duplicate entries. To avoid unnecessary details we assume that there are always \( d_{\text{average}} \) distinct capacities of average machines and \( d_{\text{tiny}} \) distinct capacities of tiny machines.

2) State vectors: The crucial concept for our algorithm and its proof is the state vector for the \( l \)-th range with the fields:

\[
(M_{\text{exact}}; M_{\text{slack}}, n_{\text{small}}; M_{\text{average}}; M_{\text{large}}; F),
\]

The meanings of the fields are as follows:

- \( M_{\text{exact}} \) – a set of unassigned tiny machines designed to form exact covers for some parts;
- \( M_{\text{slack}} \) – a set, disjoint with \( M_{\text{exact}} \), of unassigned tiny and \( l \)-small machines;
- \( n_{\text{small}} \) – the number of \( l \)-small machines in \( M_{\text{slack}} \);
- \( M_{\text{average}} \) – a set of unassigned \( l \)-average machines;
- \( M_{\text{large}} \) – a set of unassigned \( l \)-large machines;
- \( F \) – a \((1 - \epsilon)\)-covering for \( P_0 \cup \ldots \cup P_{l-1} \).

First at all, the set \( M_{\text{exact}} \) is necessary in our construction to guarantee that even parts of high cardinality can be covered exactly by tiny machines without excessive spending of machine capacity. Otherwise the intuition is clear; we would like to track the unassigned machines with a suitable precision: high enough that for the next ranges we can find a nice \((1 - \epsilon)\)-covering (a term defined later), but spending a polynomial amount of time. Also, keep in mind that we would like to track the unassigned machines with respect to equivalence relation defined. In particular, there can be \( O(m^{d_{\text{tiny}}}) \) not equivalent sets of \( M_{\text{exact}} \), \( O(m^{d_{\text{average}}}) \) nonequivalent sets of \( M_{\text{average}} \) and it is enough to consider \( O(m) \) nonequivalent sets of \( M_{\text{large}} \). The last observation is justified by the observations in the next subsection.

For clarity, for a state vector \( sv^l \) we use expressions like \( sv^l.M_{\text{exact}} \) or \( sv^l.n_{\text{small}} \) to refer to the values of its fields. Also, we denote the set of vectors as \( SV^l \) to emphasize that it consists of state vectors for \( l \)-th range.

3) Good vectors and \( \epsilon \)-approximate coverings: Let us consider parts in a non-decreasing order of their sizes. Assume that there exists an exact covering \( F \) of \( J \). Then, if \( F^{-1}(J_j) \) for any \( J_j \in P_l \) contains a large machine, then we may assume that \( F^{-1}(J_j) \) consists of exactly one large machine \( m \) – simply we can drop other machines.
Also, we may assume that \( m \) is the smallest large machine assigned to \( J_j \) or a later part. If this is not the case, then we can do as follows.

- As long as there is a \( l \)-large machine \( m' \) of smaller capacity that is unassigned, then exchange \( m \) with \( m' \).
- As long as there is a \( l \)-large machine \( m' \) of smaller capacity used for a later part then exchange \( m \) with \( m' \).

Let us fix any such exact covering and to differentiate it from other coverings use \( F_{\text{opt}} \). Now, with respect to the optimal covering we can define two additional types of the machines. For convenience we define \( m \) to be tiny-exact if in the optimal covering \( F_{\text{opt}} \) \( m \) is assigned and the set \( F_{\text{opt}}^{-1}(F_{\text{opt}}(m)) \) is a tiny exact cover; otherwise we call \( m \) tiny-non-exact machine.

We use the optimal covering to form conditions for desirable state vectors at each step of the algorithm. We say that a state vector \( sv' \) is good if:

- \( sv',M_{\text{exact}} \) is equivalent to the tiny-exact machines in \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_{l-1}) \),
- \( sv',M_{\text{large}} \) is equivalent to the \( l \)-large machines in \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_{l-1}) \),
- \( sv',M_{\text{average}} \) is equivalent to the \( l \)-average machines in \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_{l-1}) \),
- \( sv',n_{\text{small}} \) is at least the number of parts in \( P_l \cup \ldots \cup P_{l_{\text{max}}} \) that are covered by slack exact cover in \( F_{\text{opt}} \) and where the fastest machine assigned in \( F_{\text{opt}} \) is \( l \)-small,
- The capacity of \( M_{\text{slack}} \) at least the capacity of \( l \)-small and tiny-non-exact machines in \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_{l-1}) \). Intuitively, we would like a good state vector \( sv' \) to describe the set of unassigned machines similar to \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_{l-1}) \). Most importantly, the condition on \( n_{\text{small}} \), as we will see, is designed to guarantee that the number of \( l \)-small machines unassigned is at least as big as the number of parts in \( P_l \cup \ldots \cup P_{l_{\text{max}}} \) covered by slack exact cover for which the fastest machine assigned in the optimal covering is \( l \)-small. We use the condition to guarantee that each such part can be covered by a cover similar to slack exact cover.

We search the space of feasible coverings for a \((1-\epsilon)\)-covering which is nice. Formally, \( M' \) is an nice \((1-\epsilon)\)-cover of a part \( J_i \in P_l \) if:

- \( M' \) is an exact cover of \( J_i \), i.e., \( \sum_{m \in M'} c^*(m) \geq |J_i| \);
- or \( M' \) is relatively almost exact cover of \( J_i \), i.e., \( \sum_{m \in M'} c^*(m) \geq (1-\epsilon)|J_i| \) and \( c^*(m) > \epsilon^{-1} \) for all \( m \in M' \);
- or \( M' \) is absolutely almost exact cover of \( J_i \), i.e., \( \sum_{m \in M'} c^*(m) \geq |J_i| - \epsilon^{-1} \) and \( c^*(m) > \epsilon^{-2} \) for some \( m \in M' \).

We call a covering \( F \) of \( P_l = \{ J_1, \ldots, J_{|P_l|} \} \) a nice \((1-\epsilon)\)-covering if for any \( i \in [|P_l|] \) the set \( F^{-1}(J_i) \) is nice \((1-\epsilon)\)-cover for \( J_i \). Using a nice \((1-\epsilon)\)-covering it is easy to construct a schedule, perhaps increasing \( T \) a bit. To present the desirable property more clearly let us consider the following lemma.

**Lemma 4.** Assume that for an instance \((J,M,c^*,\epsilon)\), where \( c^* \) is an integer-valued function and \( \epsilon \in (0,1) \), there is a nice \((1-\epsilon)\)-covering \( F \). Then, \( F \) is exact covering of \( J \) with \( M \) under \( c' = [c^* \left( \frac{1}{1-\epsilon} + \epsilon \right)] \).

**Proof.** For \( J_i \in P_l \) consider what type of cover \( F^{-1}(J_i) \) is:

- If \( F^{-1}(J_i) \) is an exact cover, then there is nothing to prove.
- Assume that all machines in \( F^{-1}(J_i) \) have capacity at least \( \epsilon^{-1} \). In this case the \((1-\epsilon)\)-cover has to be (at least) relatively almost exact cover. Therefore, we can multiply \( c^* \) by \( \frac{1}{1-\epsilon} \) to increase the total capacity of \( F^{-1}(J_i) \) by \( \frac{1}{1-\epsilon} \). The total capacity of \( F^{-1}(J_i) \) is at least \(|J_i|\) after the operation. However, this could lead to some fractional capacities, so additionally we would like the capacities to be rounded up to the nearest integer. Summing up, we would like to replace \( c^*(m) \) by \( \lceil c^*(m) / (1-\epsilon) \rceil \). This can be done e.g. by using \( [c^* \left( \frac{1}{1-\epsilon} + \epsilon \right)] \) instead of \( c^* \) due to the fact that \[
\left\lceil \frac{c^*(m)}{1-\epsilon} \right\rceil \leq \left\lceil \frac{c^*(m)}{1-\epsilon} \right\rceil + 1 < \left\lceil \frac{c^*(m)}{1-\epsilon} \right\rceil
\]
where we used the property that \( c^*(m) > \epsilon^{-1} \) for all machines used to cover \( J_i \).
- Assume that at least one machine with capacity less than \( \epsilon^{-1} \) is present in \( F^{-1}(J_i) \). However, this means that the total capacity of \( F^{-1}(J_i) \) is at least equal to \(|J_i| - \frac{1}{2}\), by the requirement that \((1-\epsilon)\)-cover has to be absolutely almost exact cover in such case. Moreover, due to the fact that \( c^* \) is integer-valued, the missing capacity has to be at most \( \frac{1}{2} \) in fact – both \(|J_i|\) and capacities are integers. Additionally, \( F^{-1}(J_i) \) contains at least one machine \( m_i \) with capacity \( c^*(m_i) \geq \epsilon^{-2} \). Thus, it is sufficient to use \([c^*(1+\epsilon)]\) instead of \( c^* \),
because we get \[ |c^*(m_i)(1 + \epsilon)| \geq |c^*(m_i) + \frac{1}{e}| = c^*(m_i) + \frac{1}{e^2} - \text{ the additional capacity of } m_i \text{ brings the total capacity of cover to at least } |J_i|.

Overall, it is sufficient to pick the scaling according to the worst possible case. Hence, every nice \((1 - \epsilon)\)-covering of \(J\) with respect to \(c^*\) is exact covering with respect to \[ c^*\left(1 + \frac{1}{e^2}\right) + \epsilon \].

4) Parts with small sizes and slow machines: Let us turn our attention to the first nontrivial part of the Algorithm 3: finding \(SV^{l_{min}+1}\) - a set of state vectors for \((l_{min} + 1)\)-th range guaranteed to contain a good state vector, under the condition that \(C_{max} \leq T\). This state vector is our starting point for further iterations.

At first, let us describe the idea of Algorithm 4: we split the machines into slow and fast ones and perform a dynamic programming in order to find all combinations of slow and fast machines which can be used to construct an exact covering of \(J' = \bigcup_{i=1}^{l_{min}} P_i\). By the discussion on the optimal covering for any part from \(J'\) we use only the fast machine with the smallest capacity unused yet.

**Algorithm 4** An algorithm calculating the set \(SV^{l_{min}+1}\).

```plaintext
1: procedure FIND-SV^{l_{min}+1}(J, M, c*)
2: Sort \(J\) according to their cardinalities.
3: \(l_{min} \leftarrow \lfloor 3\log_{1 + \epsilon} \frac{1}{\epsilon^2} \rfloor + 1\)
4: for \(i = 0, 1, \ldots, l_{min}\) do
5: \(M_i \leftarrow \{m; c^*(m) = \lfloor (1 + \epsilon)^i \rfloor\}\) \hspace{1cm} \(\triangleright\) assign each \(m \in M\) to minimum feasible \(M_i\)
6: \(J' \leftarrow \{J_k; |J_k| < \lfloor (1 + \epsilon)^{l_{min}+1} \rfloor\}\)
7: \(M_{fast} \leftarrow M \setminus \bigcup_{i=1}^{l_{min}} M_i\)
8: \(S_0 \leftarrow \{(M_0, \ldots, M_{l_{min}}, M_{fast}; \emptyset)\}\)
9: for \(i = 1, \ldots, |J'|\) do
10: \(S_i \leftarrow \emptyset\).
11: for each \(s = (M_0, \ldots, M_{l_{min}}, M_{fast}; F) \in S_{i-1}\) do
12: for each \(M'_0 \subseteq M_0, \ldots, M'_{l_{min}} \subseteq M_{l_{min}}\) do
13: if \(\sum_{j=0}^{l_{min}} |M'_j| \cdot \lfloor (1 + \epsilon)^j \rfloor \geq |J_i|\) then
14: Let \(F' := F \cup (M'_0, J_i) \cup \ldots \cup (M'_{l_{min}}, J_i)\)
15: Add \((M_0 \setminus M'_0, \ldots, M_{l_{min}} \setminus M'_{l_{min}}, M_{fast}; F')\) to \(S_i\)
16: end if
17: end for
18: if \(M_{fast} \neq \emptyset\) then
19: Let \(m\) be the slowest machine in \(M_{fast}\)
20: Add \((M_0, \ldots, M_{l_{min}}, M_{fast} \setminus \{m\}; F \cup (m, J_i))\) to \(S_i\)
21: end if
22: end for
23: Trim \(S_i\)
24: end for
25: \(SV^{l_{min}+1} \leftarrow \emptyset\)
26: for each \(s = (M_0, \ldots, M_{l_{min}}, M_{fast}; F) \in S_{|J'|}\) do
27: \(M_{exact} \leftarrow \emptyset, M_{unused} \leftarrow \emptyset\)
28: for \(M'_0 \subseteq M_0, \ldots, M'_{d_{tiny}} \subseteq M_{d_{tiny}}\) do
29: for \(i = 0, \ldots, d_{tiny}\) do \(M_{exact} \leftarrow M_{exact} \cup M'_i\), \(M_{unused} \leftarrow M_{unused} \cup M'_i\)
30: end for
31: Transform \(M_{unused}\) to \(M_{slack}, M_{average}\) and \(M_{large}\) for range \(l_{min} + 1\)
32: Calculate \(n_{small}\) and \(c\) from \(M_{slack}\)
33: Add \((M_{exact}; M_{slack}, n_{small}; M_{average}; M_{large}; F)\) to \(SV^{l_{min}+1}\)
34: end for
35: end for
36: return The set \(SV^{l_{min}+1}\) after trimming
37: end procedure
```
Formally, the properties of the algorithm are summed in the following lemma:

**Lemma 5.** Let \( l_{\min} = \lceil 3 \log_{1+\epsilon} \frac{1}{\epsilon} \rceil + 1 \). Algorithm 4 finds a set of state vectors \( SV^{l_{\min}+1} \) with the following properties:

(i) \( SV^{l_{\min}+1} \) contains at least one good state vector if there is an exact covering of \( J \).

(ii) Every \( sv^{l_{\min}+1} \in SV^{l_{\min}+1} \) contains an exact covering of \( P_0 \cup \ldots \cup P_{l_{\min}} \).

(iii) \( |SV^{l_{\min}+1}| = O(m^{d_{\text{tiny}}+d_{\text{average}}+2}) \) and it can be computed in polynomial time.

**Proof.** First at all, let us clarify the trimming operation of sets \( S_i \) and \( SV^{l_{\min}+1} \). In the case of former set it means: for each nonequivalent set \( M'_0, M'_1, \ldots, M'_{l_{\min}}; M_{\text{large}} \) preserve only one tuple in \( S_i \). In the case if the latter set it means: for each nonequivalent set \( M_{\text{exact}}; M_{\text{average}}; M_{\text{large}} \), and a number \( n_{\text{small}} \in \{0, \ldots, m\} \) preserve only the tuple \( (M_{\text{exact}}; M_{\text{slack}}; n_{\text{small}}; M_{\text{average}}; M_{\text{large}}) \) in \( SV^{l_{\min}+1} \) where the capacity of \( M_{\text{slack}} \) is maximum, if any such tuple exists at all.

Let the optimal covering be given as \( F_{\text{opt}} \). Let us prove that each \( S_i \) contains a tuple representing a set of machines equivalent to \( M \setminus F_{\text{opt}}^{-1}(J_1 \cup \ldots \cup J_i) \). This claim is obviously true for \( i = 0 \). Assume that the claim is true for \( S_{i-1} \), we prove that it holds for \( S_i \). Consider the tuple \( s \) representing a set equivalent to \( M \setminus F_{\text{opt}}^{-1}(J_1 \cup \ldots \cup J_{i-1}) \).

- Assume that \( F_{\text{opt}}^{-1}(J_i) \) consists of slow machines, that is, \( F_{\text{opt}}^{-1}(J_i) \subseteq M_0 \cup \ldots \cup M_{l_{\min}} \). In this case the second nested for loop generates all subsets of slow machines which are exact cover for \( J_i \), hence in particular a set equivalent to \( F_{\text{opt}}^{-1}(J_i) \).
- Assume that \( F_{\text{opt}}^{-1}(J_i) \) consists of a single fast machine. Observe that the if generates an exact cover by a fast machines.

Thus in either case there exists a tuple \( s \) in \( S_i \) representing the set equivalent to \( M \setminus F_{\text{opt}}^{-1}(J_i) \). As a consequence, there exists a tuple \( s \in SV^{l_{\min}+1} \) that represents the machines that are equivalent to \( M \setminus F_{\text{opt}}^{-1}(J') \) establishes \((i)\)

An observation that for any \( i \in [l_{\min}+1] \) each tuple \( s \in S_i \) represents an exact covering of \( P_0 \cup \ldots \cup P_{i-1} \) establishes \((ii)\). The observation can be directly inferred from Algorithm 4.

In order to prove \((iii)\) we start by noting that \( |S_i| \leq (m+1)^{l_{\min}+2} \) as any coordinate of any vector \( s \in S_i \) can be expressed as a value from \( \{0, \ldots, m\} \) (remember that we identify sets equivalent under \( c^* \)) and the number of coordinates of \( s \) is equal to \( l_{\min} + 2 \). Moreover, each \( s \in S_{i-1} \) generates at most \( (m+1)^{l_{\min}+1} + 1 \) potential elements in \( S_i \). Checking if the generated element corresponds to a feasible covering of \( J_i \) and copying the covering of \( J_1, \ldots, J_i \) can be done in \( O(m) \) time. Thus the total time complexity of constructing \( S_i \) from \( S_{i-1} \) is also \( O(m^{2l_{\min}+4}) \). The trimming operation can be done in time \( O(m^{2l_{\min}+4}) \), due to the fact that the target set has size \( O((m+1)^{l_{\min}+2}) \) and the number of entries that have to be visited is \( O(m^{2l_{\min}+3}) \) and the entries are of length \( O(m) \). These observations follow from the fact that \( \subseteq \) is taken with respect to the equivalence relation, hence only the number of the machines taken from each group matters. Since \( |J'| \leq n \), finding \( S_{|J'|} \) requires \( O(nm^{2l_{\min}+4}) \) time.

The transformation of every \( s \in S_{|J'|} \) to a set of its corresponding state vectors in \( SV^{l_{\min}+1} \) is simple. The first step is composed of two parts. First is to divide the set of unassigned machines into \( M_{\text{exact}} \), the tiny machines designed to form exact covers and others (there are \( O(m^{d_{\text{tiny}}} \)) nonequivalent partitions. After the division, the second step is to turn the divided sets to a state vector for \( l_{\min}+1 \) directly using the definition of the state vector. Again the division can be done in total time \( O(m^{d_{\text{tiny}}+1}) \). Hence for all the tuples this gives time \( O(m^{l_{\min}+d_{\text{tiny}}+2}) \). After the construction the set of tuples is trimmed so that for each nonequivalent \( (M_{\text{exact}}; n_{\text{small}}; M_{\text{average}}; M_{\text{large}}) \) (at most \( O(m^{d_{\text{tiny}}+1+d_{\text{average}}+1}) \) entries) only the entry with biggest capacity of \( M_{\text{slack}} \) is preserved. Together this gives time \( O(n \cdot m^{2l_{\min}+4} + m^{l_{\min}+d_{\text{tiny}}+3} + m^{d_{\text{tiny}}+d_{\text{average}}+2}) = O(nm^{2l_{\min}+4}) \). This requires a polynomial time in \( n \) and \( m \) for each element of \( S_{|J'|} \), thus establishing the result.

5) Finding a good state vector for range \( l+1 \) using a good state vector for range \( l \): Now we proceed to the essence of the algorithm: generating and merging sets of state vectors after constructing a covering of \( P_i \) based on state vectors for range \( l \). During generation of sets of candidate state vectors for every \( sv^l \in SV^l \) (denoted as \( CSV(sv^l) \)) three invariants are preserved:

\(^2\)Taking into account equivalence relation.
If \( sv^l \) contains a nice \((1 - \epsilon)\)-covering for \( P_0 \cup \ldots \cup P_{l-1} \), then its every candidate state vector contains a nice \((1 - \epsilon)\)-covering for \( P_0 \cup \ldots \cup P_l \).

If \( sv^l \) was good, then at least one state vector among its candidate state vectors is good.

For any \( sv^l \in SV^l \) the cardinality of \( CSV(sv^l) \) is polynomial with respect to \( n \) and \( m \).

By these invariants, it is sufficient to merge all \( CSV(sv^l) \) into \( SV^{l+1} \) in such a way that if at least one \( CSV(sv^l) \) contains a good vector, then \( SV^{l+1} \) also contains a good vector.

The key ideas for generation of all candidate state vectors from a given state vector \( sv^l \), presented in Algorithm 6, are as follows:

1) First is to consider all nonequivalent sets of tiny machines reserved for tiny exact covers, \( l \)-average machines, and \( l \)-large machines. By checking all possibilities one have to match sets equivalent to the present in \( F_{\text{opt}}^{-1}(P_l) \).

2) The second is to consider all possible values of two other numbers, checking of course whether they are compatible with \( sv^l \):
   - The number of machines which are \( l \)-small and used for slack exact covers of parts in \( P_l \) in \( F_{\text{opt}} \) (hence everywhere below denoted as \( m_{su} \)) as the fastest machines,
   - The number of machines which are \( l \)-small and transferred (hence everywhere below denoted as \( m_{st} \)) and used for slack exact covers of parts in \( P_{l+1} \cup \ldots \cup P_{l_{\text{max}}} \) in \( F_{\text{opt}} \) as the fastest machines. In the algorithm we reserve them to guarantee that some state vector constructed after constructing covers for \( P_l \) is good.
   A correct guess of last value guarantees that if in the optimal covering in further ranges there is some number of parts that are covered by slack exact cover where the fastest machine is \( l \)-small, then the set of unused machines allows to construct a slack \((1 - \epsilon)\)-cover for such number of parts.

3) The last idea is to verify (using Algorithm 5) whether the guess is feasible.

The intuition behind Algorithm 5 can be summed up as the following 2 sentences. Assume that for a given range \( P_l \) there is given amount of resources, and number of parts that have to be covered by slack exact cover. Then there exists an algorithm which:
   - either calculates a lower bound on minimum total capacity of slack machines required in any exact covering of \( P_l \) under specified resources and conditions, and constructs a nice \((1 - \epsilon)\)-covering of \( P_l \) using such an amount of slack capacity;
   - or it proves that no exact covering of \( P_l \) with the provided resources exists.

Let us start with the description of the key procedure.
Algorithm 5 The following algorithm either proves that there is no exact covering for a range $P_l$, or calculates a nice $(1 - \epsilon)$-covering for this range.

1: procedure CHECK-COVERING($M^*_{exact}, M^*_{slack}, n^*_{msu}, M^*_{average}, M^*_{large}, P_l = \{J_1, \ldots, J_{|P_l|}\}$)
2: Let $M^*_{msu}$ be the set of $n^*_{msu}$ fastest $l$-small machines in $M^*_{slack}$
3: $S_0 \leftarrow \{(M^*_{exact}, M^*_{slack} \setminus M^*_{msu}, M^*_{average}, M^*_{large}, \emptyset)\}$
4: for $i = 1, \ldots, |P_l|$ do
5: \hspace{1em} $S_i = \emptyset$
6: \hspace{1em} for $s = (M^*_{exact}, M^*_{slack}, M^*_{msu}, M^*_{average}, M^*_{large}, F) \in S_{i-1}$ do
7: \hspace{2em} \textbf{Case I: tiny exact cover:}
8: \hspace{3em} if $M^*_{exact}$ is an exact cover of $J_i$ then add $s \setminus M^*_{exact}$ to $S_i$
9: \hspace{1em} end for
10: \hspace{1em} \textbf{Case II: a single $l$-large machine as an exact cover:}
11: \hspace{2em} if $m \in M^*_{large}$ then add $s \setminus \{m\}$ to $S_i$
12: \hspace{1em} \textbf{Case III:} $(1 - \epsilon)$-cover consisting of by $l$-average and slack machines:
13: \hspace{2em} for each nonempty $M^*_{average} \subseteq M^*_{average}$ do
14: \hspace{3em} \textbf{Case IV: slack $(1 - \epsilon)$-cover:}
15: \hspace{4em} Let $m^*_{msu}$ be the fastest machine from $M^*_{msu}$
16: \hspace{4em} Let $M^*_{slack}$ be the maximal (inclusion-wise) set of fastest machines from $M^*_{slack}$ such that
17: \hspace{5em} $\sum_{m \in M^*_{slack} \setminus m^*_{msu}} c^*(m) \leq |J_i|$
18: \hspace{4em} if $M^*_{average} \cup M^*_{slack}$ is an $(1 - \epsilon)$-cover of $J_i$ then
19: \hspace{5em} \textbf{end for}
20: \hspace{5em} Add $s \setminus (M^*_{average} \cup M^*_{slack})$ to $S_i$
21: \hspace{2em} end for
22: \hspace{2em} \textbf{end for}
23: \hspace{2em} if there exists $s = (\emptyset, M^*_{slack}, \emptyset, \emptyset, \emptyset, F)$ in $S_{|P_l|}$ then
24: \hspace{3em} return $s$ where $\sum_{m \in s} c^*(m)$ is maximum else return NO
25: end procedure

Lemma 6. Let the $l \geq l_{min} + 1$-th range of parts $P_l = \{J_1, \ldots, J_{|P_l|}\}$ be given. Let a number $n^*_{msu}$ be given. Let also the following sets of distinct machines be given:
- $M^*_{exact}$ be a set of tiny machines that can be used for tiny exact cover exclusively.
- $M^*_{slack}$ be a set of tiny and $l$-small machines containing at least $n^*_{msu}$ $l$-small machines.
- $M^*_{average}$ be a set of $l$-average machines.
- $M^*_{large}$ be a set of $l$-large machines.

Then Algorithm 5:
1) Either determines that there is no exact covering $F_{opt}$ for $P_l$ with machines equivalent to $M^*_{exact}$ forming tiny exact covers, $l$-average machines equivalent to $M^*_{average}$ assigned, and $|M^*_{large}|$ $l$-large machines assigned, and assigning the amount of slack capacity bounded by the amount given in $M^*_{slack}$ in a way that $n_{msu}$ parts are covered by slack exact covers.
2) Or it calculates a nice $(1 - \epsilon)$-covering of $P_l$ with an assignment of $M^*_{exact}, M^*_{average}, M^*_{large}$, where $n_{msu}$ parts are covered by slack $(1 - \epsilon)$-covers, and the amount of slack capacity assigned is bounded by the capacity

\* As a shorthand, $s \setminus M$ denotes a tuple $s = (M_{exact}, M_{slack}, M_{msu}, M_{average}, M_{large}, F)$ with machines from $M$ removed and where $\cup_{m \in M}(m, J_i)$ is added to $F$. 
of \( M^*_\text{slack} \). Moreover the amount of slack capacity assigned is a lower bound on the slack capacity used in any exact covering of \( P_1 \) described in (I), provided that any such exact covering exists.

**Proof.** First let us clarify what means to trim \( S_i \). It means that the algorithm considers all nonequivalent sets \( M_{\text{exact}}, M_{\text{average}}, M_{\text{large}} \) and number of machines in \( M_{\text{msu}} \). For each unique quadruple considered it preserves only a tuple with the biggest capacity of \( M_{\text{slack}} \).

Assume that there exists an exact covering \( F_{\text{opt}} \) of \( P_1 \) with the advertised properties. Let the set of machines assigned to \( P_1 \) in \( F_{\text{opt}} \) be denoted as \( M \) (keep in mind that all the machines have to be assigned). By a slight abuse of the notation we use the same symbols as for the optimal covering and for the set of machines. Keep in mind that the sets \( M_{\text{exact}}, M^*_\text{average} \) and \( M^*_\text{large} \) have equivalent sets in \( M \). This might be not the case for \( M^*_\text{slack} \), in the case of this set we are only interested in the total capacity of the machines in the set. Moreover, the capacity of tiny-non-exact and \( l \)-small machines in \( M \) may be less than the capacity of \( M^*_\text{slack} \). In particular, let us assume that the capacity of tiny-non-exact and \( l \)-small in \( M \) is least possible under the specified conditions.

We analyze \( F_{\text{opt}} \) part by part considering how the set of unassigned yet machines looks like. To prove the theorem we prove the following invariant. For every \( i \in [|P_1|] \) in \( S_i \) there exists a tuple \( s_i = (M_{\text{exact}}; M_{\text{slack}}; M_{\text{msu}}; M_{\text{average}}; M_{\text{large}}; F) \) such that:

- The set \( M_{\text{exact}} \) is equivalent to the machines forming tiny exact covers in \( M \setminus F_{\text{opt}}^{-1}(J_1 \cup \ldots \cup J_i) \).
- The set \( M_{\text{average}} \) is equivalent to the average machines in \( M \setminus F_{\text{opt}}^{-1}(J_1 \cup \ldots \cup J_i) \).
- The number \( |M_{\text{average}}| \) is exactly equal to the number of large machines in \( M \setminus F_{\text{opt}}^{-1}(J_1 \cup \ldots \cup J_i) \).
- \( |M_{\text{msu}}| \) is exactly equal to the number of parts \( J_j \) in \( P_1 \) that are covered by slack exact cover in \( F_{\text{opt}} \).
- The capacity of \( M^*_\text{slack} \setminus (M_{\text{slack}} \cup M_{\text{msu}}) \) (that is, the slack capacity assigned by the algorithm) is at most the slack capacity in \( F_{\text{opt}}^{-1}(J_1 \cup \ldots \cup J_i) \).

Moreover each \( S_i \) consists of tuples containing a nice \((1-\varepsilon)\)-covering of \( J_1, \ldots, J_i \).

For \( i = 0 \) it is trivially true since \((1-\varepsilon)\)-covering constructed is empty and \( M \setminus F_{\text{opt}}^{-1}(\emptyset) = M \). Assume that the invariant holds for \( i-1 \) and let \( s_{i-1} \) be the corresponding tuple. Consider how \( F_{\text{opt}}^{-1}(J_i) \) looks like:

**Case I** It consists of tiny machines only. In this case the algorithm also constructs exact cover for \( J_i \) using a set of machines equivalent to \( F_{\text{opt}}^{-1}(J_i) \).

**Case II** It consists of one \( l \)-large machine. In this case the algorithm also constructs exact cover for \( J_i \) as one large machine.

**Case III** It consists of \( l \)-average, and perhaps some tiny and \( l \)-small machines. In this case the algorithm constructs a nice \((1-\varepsilon)\)-cover for \( J_i \) with an equivalent set of \( l \)-average machines and some set of \( l \)-small and tiny machines.

**Case IV** It consists of tiny and \( l \)-small machines (in particular, it contains at least one \( l \)-small machine). The algorithm also produces a nice \((1-\varepsilon)\)-cover for \( J_i \) consisting of a machine from \( M_{\text{msu}} \) and some \( l \)-small and tiny machines.

Observe that the algorithm produces exact cover in the first two cases. In Cases III and IV it produces relatively almost exact cover (perhaps even exact cover) if only machines with capacity greater than \( \varepsilon^{-1} \) are assigned, or it produces absolutely almost exact cover (perhaps even exact cover) when smaller machines are assigned. The fact that in Case III a nice \((1-\varepsilon)\)-covering is constructed follows from the fact that the algorithm is applied from range \( P_{l_{\text{min}}+1} \). Recall that \( l_{\text{min}} = \lceil 3 \log_{1+\varepsilon} \frac{d}{\varepsilon} \rceil + 1 \) and \( d_{\text{average}} = \lceil \log_{1+\varepsilon} \frac{d}{\varepsilon} \rceil + 1 \). Thus, for any \( l > l_{\text{min}} \) we know that \( \varepsilon^{-2} < (1+\varepsilon)^l_{\text{average}} \), and any \( l \)-average machine has capacity greater than \( \varepsilon^{-2} \). Finally, the fact that in Case IV a nice \((1-\varepsilon)\)-covering is constructed follows from the observation that all the machines in \( M_{\text{slack}} \) that are \( l \)-small, hence also those reserved in \( M_{\text{msu}} \), have capacity that is greater than \( \varepsilon^{-2} \).

In the last two cases there is no waste of capacity of tiny and \( l \)-small machines, due to the fact that the parts are large compared to the capacities of such machines of those types and due to the way of assigning the machines – the algorithm never overfills the required capacity. Moreover in both cases the cover can be constructed. That is, despite the fact that some slack is stored in \( M_{\text{msu}} \) and potentially unavailable, the amount of slack is still sufficient.

In Case III:

- Either there are no more parts covered by slack exact covers in \( F_{\text{opt}} \). In this case \( M_{\text{slack}} \) contains the necessary slack by the inductive assumption.
• Or there are more parts covered by slack \((1-\epsilon)\)-cover in \(F_{opt}\), in this case each machine in \(M_{msu}\) corresponds to part that is covered by slack exact cover later in \(F_{opt}\). For each part that is covered by slack exact cover in \(F_{opt}\), but it is uncovered yet “almost all” slack capacity used to cover it in \(F_{opt}\) is stored in \(M_{slack}\).

Case IV is similar to Case III:
• Either \(J_t\) is the last part covered by slack exact cover in \(F_{opt}\). Then, by induction the capacity in \(M_{slack} \cup M_{msu}\) is sufficient to form a cover for \(J_t\).
• Or there are more parts covered by slack exact cover in \(F_{opt}\), but in this case every machine left in \(M_{msu}\) certifies that there is more than enough slack.

In each case the desired tuple exists before the trimming and by an easy observation the trimming rule has to preserve tuple of not lower remaining slack capacity. Due to this, we are sure that \(s_i\) has at least as much capacity present in \(M_{slack} \cup M_{msu}\) as capacity of \(l\)-small and tiny-non-exact machines present in \(M \setminus F_{opt}^{-1}(\{J_1, \ldots, J_t\})\). Therefore, the invariant holds for \(i\) as well.

Hence, consider the tuple \((\emptyset, M_{slack}^{**}, \emptyset, \emptyset, F)\), where \(M_{slack}^{**}\) has maximum capacity, present after constructing covers for all parts. By the invariant it has to be the case that the capacity of \(M_{slack}^{**} \setminus M_{slack}^{**}\) is at most the slack capacity assigned to \(P_i\) in \(F_{opt}\).

**Corollary 2.** Algorithm 5 is polynomial time.

**Proof.** The first loops makes \(O(n)\) iterations. The second loops is over a set of \(O(m^{d_{tiny}+d_{average}+2})\) entries. Then there are the following cases in parallel. In Case I there is loop over \(O(m^{d_{tiny}})\) entries. The next case has complexity \(O(1)\). In Case III there is outer loop over \(O(m^{d_{average}})\) entries and inner loop over \(O(m)\) entries. In Case IV there is only loop over \(O(m)\) entries. The entries have size that is \(O(m)\), due to the fact that we have to store the covering constructed. The trimming rule simply proceeds over all produced entries (a set of \(O(m^{2d_{tiny}+2d_{average}+3})\) entries of size \(O(m)\)) and produces the set that again has \(O(m^{d_{tiny}+d_{average}+2})\) entries. Together this gives time complexity \(O(nm^{2d_{tiny}+2d_{average}+4})\). Hence, the algorithm is polynomial time.

**Algorithm 6** The following algorithm generates a set of candidate state vectors for a given state vector

1: **procedure** GENERATE-CANDIDATE-STATE-VECTORS\((sv^I \in SV^I)\)
2: 
3: for each \(M_{exact} \subseteq sv^I.M_{exact}, M_{average} \subseteq sv^I.M_{average}, M_{large}^* \subseteq sv^I.M_{large}^*\) do
4:   for each \(n_{msu}, n_{mst}\) do
5:     if \(n_{msu}, n_{mst}\) inconsistent with \(sv^I\) then continue
6:     Let \(M_{mst}\) be \(n_{mst}\) machines from \(sv.M_{slack}\) of biggest capacity
7:     Apply Algorithm 5 for \((M_{exact}, sv, M_{slack} \setminus M_{mst}, n_{msu}, M_{average}, M_{large}, P_i)\)
8:     if Algorithm 5 returned a valid \(M_{slack}^{**}\) and covering \(F\) then
9:       \(csv' \leftarrow sv^I \setminus (M_{exact} \cup (sv^I.M_{slack} \setminus (M_{slack}^{**} \cup M_{mst}))) \cup M_{average}^* \cup M_{large}^*\)
10:      Add \(csv'\) to \(CSV'\)
11:     end if
12:   end for
13: end for
14: Transform \(CSV'\) to \(CSV\), i.e. to state vectors for the \((l + 1)\)-th range
15: **return** \(CSV\)
16: **end procedure**

**Lemma 7.** If \(sv^I \in SV^I\) is a good state vector, then Algorithm 6 returns a set of vectors \(CSV(sv^I)\) such that at least one \(sv^I \in CSV(sv^I)\) is also a good state vector. Moreover, if each \(sv^I\) contains an \((1-\epsilon)\)-covering of \(P_0 \cup \ldots \cup P_l\), then each \(csv^I \in CSV^I(sv^I)\) contains an \((1-\epsilon)\)-covering of \(P_0 \cup \ldots \cup P_{l+1}\).

**Proof.** Let us consider the optimal covering of \(J\). Let the number of parts in \(P_l\) that are covered by slack exact cover in the optimal covering be exactly equal to \(n_{msu}\). There are also \(n_{mst}\) parts in \(P_{l+1} \cup \ldots \cup P_{l+1}\) covered by

2As a shorthand, \(s.F\) denotes a tuple \(s = (M_{exact}, M_{slack}, M_{msu}, M_{average}, M_{large}, F)\) with machines from \(M\) removed \(s\), and where \(F\) is added to \(s.F\).
slack exact cover consisting of machines that are tiny or l-small. In particular, this means that there are \( n_{\text{mst}} + n_{\text{msu}}^* \) machines in \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_{l-1}) \) that are l-small. This is because each of the mentioned \( n_{\text{mst}} \) parts can be only covered by tiny-non-exact and l-small machines \( M' \) present in \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_{l-1}) \). This also means that the capacity of tiny-non-exact and l-small machines that is assigned to parts in \( P_l \) in the optimal covering is upper bounded by the capacity of tiny-non-exact and l-small machines present in \( (M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_{l-1})) \) with the capacity of \( M' \) (lower bounded by \( n_{\text{mst}} \cdot \lfloor (1 + \epsilon)^l \rfloor \) removed.

**Algorithm 6** tries every possible combination of \( M_{\text{exact}}^*, M_{\text{average}}^*, M_{\text{large}}^* \). Hence, we are certain that at some point the algorithm proceeds with sets \( M_{\text{exact}}^*, M_{\text{average}}^*, M_{\text{large}}^* \) equivalent to the ones assigned to \( P_l \) in the optimal covering. Moreover, since it tries every value of \( n_{\text{mst}}^* \) (again from 0 to \( m \)) we are certain to go through the iteration in which \( n_{\text{msu}}^* \) would be equal to the number of parts in \( P_l \) covered by slack exact cover in the optimal covering. Finally, since it tries every value of \( n_{\text{mst}} \) (from 0 to \( m \)) we are sure that we go through an iteration such that \( n_{\text{mst}}^* \) is exactly equal to the value derived from optimal covering. By the fact that the vector \( sv' \) is good we have to have \( sv'.n_{\text{small}} \geq n_{\text{msu}}^* + n_{\text{mst}} \); hence the algorithm proceeds with the values for which the inequality holds. Hence, the amount of slack available in \( sv'.M_{\text{slack}} \setminus M_{\text{mst}} \) at least \( (M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_{l-1})) \) (because the vector is good) with the capacity of \( M_{\text{mst}} \) upper bounded by \( n_{\text{mst}}^* \cdot \epsilon \lfloor (1 + \epsilon)^{l+1} \rfloor \) removed. Moreover, the number of l-small machines in \( sv'.M_{\text{slack}} \setminus M_{\text{mst}} \) at least \( n_{\text{msu}}^* \), again by the fact that the vector is good.

By this we know that we can apply **Algorithm 5** to construct a nice \( (1 - \epsilon) \)-covering of \( P_l \) and calculate a lower bound on capacity of tiny-non-exact and l-small machines that has to be assigned in any exact covering (hence in particular in the optimal covering) on \( P_l \). This means that after the execution the algorithm constructs a tuple representing unassigned machines such that:

- It has equivalents set of tiny machines reserved for tiny exact covers to the set of tiny-exact machines in \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_l) \). And similarly, it has equivalents sets of l-average and l-large machines.
- It has at least \( n_{\text{mst}}^* \) unassigned machines that are l-small. Moreover, the number of unassigned machines that are \( (l + 1) \)-small and l-average is exactly (due to the guess) the number of machines that are \( (l + 1) \)-small and l-average in \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_l) \).
- It has the amount of capacity remaining (in \( M_{\text{slack}}^* \cup M_{\text{mst}} \)) that is at least equal to the total capacity of tiny-non-exact and l-small machines in \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_l) \).

After the transformation of vectors the unassigned machines that are \( (l + 1) \)-small but are not l-small contribute exactly the same capacity as machines from \( M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_l) \) that are \( (l + 1) \)-small but are not l-small. Clearly, such a tuple in \( \text{CSV}^l \subset \text{CSV} \) is transformed to a good tuple \( \text{CSV} \). Each tuple in \( \text{CSV} \) corresponds to an \((1 - \epsilon)\)-covering of \( P_0 \cup \ldots \cup P_l \), by the properties of Lemma 8 and the assumption that each tuple of \( SV^l \) contains an \((1 - \epsilon)\)-covering of \( P_0 \cup \ldots \cup P_{l-1} \).

As a sidenote observe, that we did not included any trimming rules. Hence, it follows that the sets \( \text{CSV} \) may contain duplicated entries. However, this does not matter in the light of the further considered Lemma 8.

**Corollary 3.** Algorithm 6 is polynomial time and produces a set of size \( O(m^{d_{\text{tiny}}+d_{\text{average}}+2}) \).

**Proof.** The first loop is over \( O(m^{d_{\text{tiny}}+d_{\text{average}}+2}) \) entries. The second loop is over \( O(n^2) \) entries. Checking whether the guess is feasible can be done in \( O(1) \) time. Taking subsets of \( n_{\text{mst}} \) machines can be done in \( O(m) \) time. **Algorithm 5** has \( O(m^{n^{2d_{\text{tiny}}+2d_{\text{average}}+4}}) \) time complexity. Construction of new tuples can be done in \( O(m) \) time. Transformation can be done in \( O(m) \) time. Together this gives \( O(m^3 m^{3d_{\text{tiny}}+3d_{\text{average}}+6}) \) time. The produced set has cardinality \( O(n^2 m^{d_{\text{tiny}}+d_{\text{average}}+2}) \) entries of size \( O(m) \).

Let now \( SV^l \) for every \( l = l_{\text{min}} + 1, \ldots, l_{\text{max}} \) be defined as a subset of \( \bigcup_{sv \in SV^{l-1}} \text{CSV}(sv) \) such that for every \( M_{\text{exact}}, M_{\text{average}}, M_{\text{large}} \) (there are \( O(m^{d_{\text{tiny}}}), O(m^{d_{\text{average}}}), O(m) \) nonequivalent sets), and \( n_{\text{small}} \in \{0, \ldots, m\} \), we keep only the state vector \( (M_{\text{exact}}, M_{\text{slack}}, n_{\text{small}}, M_{\text{average}}, M_{\text{large}}) \) with the biggest capacity of \( M_{\text{slack}} \) machines (with ties broken arbitrarily).

**Lemma 8.** For any \( l \in \{l_{\text{min}} + 1, \ldots, l_{\text{max}}\} \):

(i) If \( SV^{l-1} \) has at least one good state vector, then \( SV^l \) also has at least one good state vector.

(ii) If any vector in \( SV^{l-1} \) contains a nice \((1 - \epsilon)\)-covering of \( P_0 \cup \ldots \cup P_{l-1} \), then any vector in \( SV^l \) contains a nice \((1 - \epsilon)\)-covering of \( P_0 \cup \ldots \cup P_l \).

(iii) The set \( \bigcup_{sv \in SV^{l-1}} \text{CSV}(sv) \) has polynomial size and can be calculated in polynomial time.
(iv) $SV^l$ has $O(m^{d_{tiny}+d_{average}+2})$ state vectors for every $l = l_{\min} + 1, \ldots, l_{\max}$.

Proof. In order to prove the first property observe, if $sv \in SV^{l-1}$ was a good state vector, then $CSV(sv)$ contains at least one good state vector, by Lemma 7. Now it is sufficient to note that for every $sv \in SV^{l-1}$ and every good state vector $sv'$ not in $SV^l$ there has to be another state vector $sv'' \in SV^l$ with the same values of $M_{\text{exact}}$, $n_{\text{small}}$, $M_{\text{average}}$ and $M_{\text{large}}$ but at least as large capacity of $M_{\text{slack}}$ so $sv''$ has to be a good state vector as well.

The second property follows directly from Lemma 7.

By the construction, there is at most one state vector in $SV^l$ for each $M_{\text{exact}}, n_{\text{small}}, M_{\text{average}}$ and $M_{\text{large}}$. Thus the cardinality of $SV^l$ is $O(m^{d_{tiny}+d_{average}+2})$. Hence, number of produced candidate state vectors is $O(n^2 m^{2d_{tiny}+2d_{average}+4})$ for each vector. Therefore, before the trimming the produced set has cardinality $O(n^2 m^{2d_{tiny}+2d_{average}+4})$. Also, they are produced in total time $O(n^2 m^{3d_{tiny}+3d_{average}+6}) = O(n^3 m^{d_{tiny}+4d_{average}+8})$. Hence, similarly to the previous cases, the trimming can be done by passing the constructed set $\bigcup_{sv \in SV^{l-1}} CSV(sv)$ of $O(n^2 m^{2d_{tiny}+2d_{average}+4})$ entries of size $O(m)$ once. Together this gives the total time required to produce $SV^l$ from $SV^{l-1}$ equal to $O(n^3 m^{d_{tiny}+4d_{average}+8})$. This proves the last two points.

The following conclusion follows directly from Lemma 5 and Lemma 8.

Corollary 4. If there is an exact covering of $J$, then $SV^{l_{\max}+1}$ is nonempty. Any state vector from $SV^{l_{\max}+1}$ contains an $(1-\epsilon)$-covering of $J$.

By summing all the observations up we obtain the following theorem.

Theorem 8. There exists a PTAS for $Q|G = \text{complete multipartite}, p_j = 1|C_{\text{max}}$.

Proof. To avoid unnecessary details we always execute the presented algorithms with $\epsilon \leq \frac{1}{2}$. Assume that there exists an exact covering within time $T$. Under such assumptions $SV^{l_{\max}+1}$ has to contain at least one good state vector with some nice $(1-\epsilon)$-covering $F$, by Corollary 4.

Observe that if we multiply $T$ by $\left(\frac{1}{1-\epsilon} + \epsilon\right)$, then the new capacities of the form $[c^* \left(\frac{1}{1-\epsilon} + \epsilon\right)]$ guarantee that $F$ is an exact covering, by Lemma 4. Finally, since we used the rounded capacities $c^*(m)$, we need to get back to capacities $c(m)$. By the fact that $c(m) \leq c^*(m) \leq (1+\epsilon)c(m)$ for all $m \in M$, if there is an exact covering of $J$ for rounded capacities with respect to $T \left(\frac{1}{1-\epsilon} + \epsilon\right)$, then it is exact covering of $J$ under the true capacities for $T(1+\epsilon) \left(\frac{1}{1-\epsilon} + \epsilon\right)$.

To complete the proof of the approximation ratio, we note that for all $\epsilon \in (0, \frac{1}{2})$ we have

$$ (1+\epsilon) \left(\frac{1}{1-\epsilon} + \epsilon\right) \leq (1+\epsilon)(1+3\epsilon) < (1+7\epsilon), $$

so our algorithm is $(1 + 7\epsilon)$-approximation algorithm.

The complexity of the algorithm is polynomial in $n$ and $m$. Lemma 5 establishes that $SV^{l_{\min}+1}$ can be found in time $O(nm^{2l_{\min}+4})$. By Lemma 6 Lemma 7 and Lemma 8 each $SV^l$ for $l = l_{\min} + 1, \ldots, l_{\max}$ and can be found in $O(n^3 m^{4d_{tiny}+4d_{average}+8})$ time. Clearly, the number of nonempty ranges between $P_{l_{\min}+1}$ and $P_{l_{\max}+1}$ is at most $n$ and we can even optimize the algorithm to iterate over non empty ranges. Together this gives an algorithm of time complexity $O(nm^{2l_{\min}+4} + n^4 m^{4d_{tiny}+4d_{average}+8}) = O(n^4 m^{4d_{tiny}+4d_{average}+8}) = O(n^4 m^{12} \log_{1+\epsilon}\frac{1}{\epsilon} + 12)$ for a given $T$. By combining it with a binary search over possible values of $T$ (there are $O(nm)$ candidates in total) we complete the proof obtaining the overall time complexity $O(\log nm \cdot n^4 m^{12} [\log_{1+\epsilon}\frac{1}{\epsilon} + 12])$. 


TABLE I

| Group | $M_0$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ | $M_9$ | $M_{10}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|--------|
| Number | 0 | 0 | 0 | 0 | 39 | 0 | 2 | 4 | 2 | 1 | 0 | 1 |

By $M_i$ we denote the set of machines of rounded capacity $\lfloor (1 + \epsilon) i \rfloor$; here $\epsilon = 0.5$. The set of tiny machines is additionally separated by a vertical line.

| Range | $P_1$ | $P_2$ | $P_3$ | $P_4$ |
|-------|-------|-------|-------|-------|
| Sizes in Part | $[1, 2]$ | $[17, 25]$ | $[25, 38]$ | $[38, 57]$ |
| Size | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ | $J_7$ | $J_8$ |
| The optimal covering | | | | |
| | $3$ | $25$ | $3^i$ | $3^i$ | $3^i$ | $3^i$ | $17$ | $17$ |
| An $(1 - \epsilon)$-covering I | $3$ | $25$ | $3^i$ | $3^i$ | $3^i$ | $3^i$ | $17$ | $17$ |
| An $(1 - \epsilon)$-covering II | $3$ | $25$ | $3^i$ | $3^i$ | $3^i$ | $3^i$ | $17$ | $17$ |

TABLE II

Parts, their cardinalities, a sample exact covering referred to as the optimal covering, a $(1 - \epsilon)$-covering I corresponding to the optimal covering, and a $(1 - \epsilon)$-covering II unrelated to the optimal covering. The parts that are small, i.e., they are in the first $l_{\text{min}}$ ranges are separated by a double line.

TABLE III

Here the sets $M_{\text{exact}}, M_{\text{average}}$ and $M_{\text{large}}$ are represented by numbers of machines of given capacity - due to the observation of equivalence of machines under given capacity. The “vectors” $M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_2)$ and $M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_8)$ were constructed for convenience, they describe the optimal covering presented in Table II.

That is, they characterize exactly $M_{\text{exact}}, M_{\text{average}}$ and $M_{\text{large}}$ of a good vector and give lower bounds on $n_{\text{small}}$ and capacity of $M_{\text{slack}}, n_{\text{small}}$ for $F_{\text{opt}}$ represents the 2 parts in ranges $P_9$ or later for which a 8-small machine (i.e. small from perspective of the 8-th range) is the fastest machine. Notice that $F_{\text{opt}}^{-1}(J_9)$ is slack exact cover, but the fastest machine is 9-small but not 8-small. Finally, observe that both $s_{8}^{a}$ and $s_{8}^{b}$ are good, despite the fact that they have fewer 9-small machines available than there is in $M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_8)$. Moreover, during construction of $s_{8}^{a}$ both $n_{\text{mas}}$ and $n_{\text{maa}}$ were guessed incorrectly; despite this a good vector was constructed. However, only the existence of $s_{8}^{b}$ is guaranteed by Lemma 7.

Example III-B.1. As an example confer the data given by:

- A precision parameter $\epsilon = 0.5$.
- Guessed $T = 1$, determining the presented capacities.
- A set of machines given in Table I. The machines are grouped into sets of respective cardinalities. To avoid introducing excessive number of symbols we identify the machines with their capacities and we will refer to the machines by the numbers only.
- A set of parts grouped into ranges given in Table II.
- A sample exact covering chosen to be the optimal covering $F_{\text{opt}}$.
- Good vectors $s_{8}^{a}, s_{8}^{a}$ and $s_{8}^{b}$ given in Table III.

Observe that here $l_{\text{min}} = 7$, hence $P_0 \cup \ldots \cup P_7$ are covered by exact covers using Algorithm 4 and there is a, potentially good, vector $s_{8}^{a}$ constructed.

As presented in Table II notice that $F_{\text{opt}}^{-1}(J_3), F_{\text{opt}}^{-1}(J_4), F_{\text{opt}}^{-1}(J_5), F_{\text{opt}}^{-1}(J_6)$ are: a set of tiny machines, a 8-large machine, 8-average, tiny-non-exact, and l-small machines, and tiny and 8-small machines, respectively. Moreover $F_{\text{opt}}^{-1}(J_7)$ and $F_{\text{opt}}^{-1}(J_8)$ are slack exact covers. However, in $F_{\text{opt}}^{-1}(J_7)$ the fastest machine is 8-small. This means that at least one 8-small is in $M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_8)$. Moreover, it means that the capacity of tiny-non-exact and 8-small machines in $M \setminus F_{\text{opt}}^{-1}(P_0 \cup \ldots \cup P_8)$ is at least 38, by the definition of range.

Hence, in at least one iteration Algorithm 6 considers a good state vector; for example $s_{8}^{a}$, presented in Table III. Moreover, tiny exact, 8-average, and 8-large machines that are to be assigned to parts in $P_8$ are equivalent
to the assigned in the optimal covering; in the example 39, 17, 57, respectively. The algorithm guesses the value

\( n_{msu} \), the number of parts covered by slack exact cover in \( P_9 \) or later range (here only \( P_9 \)) where the fastest machines are 8-small machines; here it is equal to 1. Moreover, the number of parts in \( P_8 \) that are covered by slack exact cover are guessed; in the example \( n_{msu} = 1 \). Observe that using the guesses the algorithm constructs the set \( M_{msu} \) (in the example \( M_{msu} = \{ 11 \} \)). The amount of capacity reserved in \( M_{msu} \) is much less than the capacity of tiny-non-exact 8-small machines assigned in \( F_{opt}^{-1}(P_9) \).

Hence, Algorithm 5 in at least one iteration is applied with equivalent set of tiny machines reserved for tiny exact covers, 8-large, and 8-average machines, a sufficient number of 8-small machines, equal at least to \( n_{msu} \) and at least the same amount of capacity in \( M_{msu} \) as the capacity of tiny-non-exact and \( l \)-small in \( F_{opt}^{-1}(P_8) \). Therefore, Algorithm 6 has to return a good vector, perhaps \( s_{v,a}^{0} \). Observe that \( s_{v,a}^{0} \) may perhaps have fewer 9-small machines than present in \( M \setminus F_{opt}^{-1}(P_8) \). However, by reserving \( M_{msu} \) and guessing the 8-average and 9-small machines in \( M \setminus F_{opt}^{-1}(P_0 \cup \ldots \cup P_8) \), still the number of 9-small machines is enough to construct slack \((1 - \epsilon)\)-covers for every part covered by slack exact covers in optimal covering in \( P_9 \cup \ldots \cup P_{1_{\max}} \). Of course, during the execution it might be the case that a superior good state vector is constructed, for example \( s_{v,a}^{0} \), even from guesses not corresponding to the optimal covering. By Lemma 3 it might be used as well to construct an \((1 - \epsilon)\)-covering in further ranges.

Considering the constructed \((1 - \epsilon)\)-covering, observe that \( J_{5} \) is covered by slack \((1 - \epsilon)\)-cover and it is relatively almost cover. Notice that \( J_{7}, J_{8} \) are covered by slack \((1 - \epsilon)\)-covers and they are absolutely almost covers.

IV. UNRELATED MACHINES

We prove that there is no good approximation algorithm possible in the case of unrelated machines.

Theorem 9. There is no constant approximation ratio algorithm for \( R \mid G = \text{complete 2-partite} \mid \sum C_j \) \( (R \mid G = \text{complete 2-partite} \mid \sum C_j) \)

Proof. Assume that there is \( d \)-approximate algorithm for the \( R \mid G = \text{complete 2-partite} \mid \sum C_j \) problem \( (R \mid G = \text{complete 2-partite} \mid \sum C_j) \). Consider an instance of 3-SAT with the set of variables \( V \) and the set of clauses \( C \), moreover where for each \( v \in V \) there are at most 5 clauses containing \( v \). This version is still NP-complete. We construct the scheduling instance as follows: let \( M = \{ v^T, v^F : v \in V \} \). Let also \( G = \text{complete 2-partite} \) with partitions \( J_{1} = \{ j_{v,1} : v \in V \} \cup \{ j_{v,c} : c \in C \} \) and \( J_{2} = \{ j_{v,2} : v \in V \} \). Hence \( n = 2|V| + |C| \leq 7|V| \), by \( |C| \leq 5|V| \).

Let \( p_{j} = 1 \) for all jobs. Let \( s_{1} \geq 1 \) be a value determined by an instance of 3-SAT, but polynomially bound by the size of the instance. Let now \( s(j_{v,1}, v^T) = s(j_{v,2}, v^F) = s(j_{v,c}, v^F) = s(j_{v,c}, v^F) = s_{1} \), for any \( v \in V \). Let for any \( c \in C \) \( s(j_{v,c}, v^F) = s_{1} \) if \( v \) appears in \( c \), and \( s(j_{v,c}, v^F) = s_{1} \) if \( v \) appears in \( c \). Set all other \( s(j,m) \) to 1.

Consider any instance of the scheduling problem, corresponding to an instance of 3-SAT with answer YES. Then we can schedule \( J \) on the machines according to fulfilling valuation, in the following way: If \( v \) has value \( T \) then we assign \( v^T \) to \( J_{1} \) and \( v^F \) to \( J_{2} \), otherwise we assign \( v^F \) to \( J_{1} \) and \( v^T \) to \( J_{2} \). Hence any job in \( J_{2} \) can be processed with speed 1 similarly for any job in \( J_{1} \). Hence, \( \sum C_j \leq \frac{(n+1)}{2} \frac{|V|^2}{s_1} \leq \frac{49|V|^2}{s_1} \) \( (C_{max} \leq \frac{7}{8} \leq \frac{7}{7} \) \( s_1 \)), for an optimal schedule. Now we can state that it is sufficient to set \( s_{1} = 49d|V|^2 + 1 \) \( (s_{1} = 7d|V| + 1) \) to prove the theorem.

On the other hand, assume that the answer for an instance of 3-SAT is NO. Assume that there exists a schedule with \( \sum C_j < 1 \) \( (C_{max} < 1) \). Assume that there is a partition, such that both \( v^T \) and \( v^F \) have no jobs from it assigned in the schedule, then \( \sum C_j \geq 1 \) \( (C_{max} \geq 1) \), a contradiction. Thus assume then, that \( j_{c} \in J_{1} \) is a job assigned to a machine \( m \) with \( s(j,c,m) = 1 \); in this case we also clearly have a contradiction. Hence, each \( j_{c} \in J_{1} \) is assigned to a machine corresponding to a valuation of the variable fulfilling \( c \), hence there exists a fulfilling valuation, a contradiction. Hence for such an instance for any schedule \( \sum C_j \geq 1 \) \( (C_{max} \geq 1) \).

Clearly, by using this \( d \)-approximate algorithm on an instance of the scheduling problem corresponding to an YES instance of 3-SAT we would be able to obtain a schedule with \( \sum C_j < 1 \) \( (C_{max} < 1) \). On the other hand, for an instance corresponding to a NO instance there is no schedule with \( \sum C_j \leq 1 \) \( (C_{max} \leq 1) \).

\( \square \)

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