Order preserving property of moment estimators

Piotr Nowak
Mathematical Institute, University of Wrocław
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail: nowak@math.uni.wroc.pl

Abstract

Balakrishnan and Mi [2] considered order-preserving property of maximum likelihood estimators. In this paper there are given conditions under which the moment estimators have the property of preserving stochastic orders. The property of preserving for usual stochastic order as well as for likelihood ratio order is considered. Sufficient conditions are established for some parametric families of distributions.

Keywords: moment estimator, maximum likelihood estimator, exponential family, stochastic ordering, total positivity.

2010 MSC: 60E15, 62F10.

1. Introduction and preliminaries

Suppose that $X = (X_1, X_2, \ldots, X_n)$ is a sample from a population with density $f(x; \theta)$, where $\theta \in \Theta \subset \mathcal{R}$. We now consider the estimation of $\theta$ by the method of moments (see, for example, Borovkov [4]). Let $g : \mathcal{R} \to \mathcal{R}$ be a function such that the function $m : \Theta \to \mathcal{R}$

$$m(\theta) = \int_{-\infty}^{\infty} g(x)f(x; \theta)dx$$

is monotone and continuous on $\Theta$. Particularly, if $m$ is strictly monotone, then $m$ is one-to-one and $m^{-1}$ exits. Then for every $t \in m(\Theta)$ exists a unique solution of the equation $m(\theta) = t$, i.e. $\theta = m^{-1}(t)$. In the case when $m$ is nondecreasing or nonincreasing we can take $m^{-1}(t) = \inf\{\theta : m(\theta) \leq t\}$, for any $t \in m(\Theta)$. Let

$$\bar{g} = \frac{1}{n} \sum_{i=1}^{n} g(X_i)$$

be the generalized empirical moment based on $X$. 
If \( \bar{g} \in m(\Theta) \), then the estimator obtained by the method of moments (to be short \textit{moment estimator}) is of the form

\[
\hat{\theta} = m^{-1}(\bar{g}).
\]

We know that these estimators are strongly consistent. We often put \( g(x) = x^k, \ k \geq 1 \). Then \( \bar{g} \) reduces to \( m_k - k \)th empirical moment, i.e. \( m_k = \frac{1}{n} \sum_{i=1}^{n} x_i^k \). Particularly, \( m_1 = \bar{X} \).

In this paper our aim is to give conditions under which the moment estimators have the property of preserving stochastic orders. We shall deal with stochastic orders, so recall their definitions.

Let \( X \) and \( Y \) be two random variables, \( F \) and \( G \) their respective distribution functions and \( f \) and \( g \) their respective density functions, if they exist. We say that \( X \) is stochastically smaller than \( Y \) (denoted by \( X \leq_{st} Y \)) if \( F(x) \geq G(x) \) for all \( x \in \mathbb{R} \). The stronger order than usual stochastic order is likelihood ratio order. We say that \( X \) is less dispersed than \( Y \) (denoted \( X \leq_{disp} Y \)) if \( g(x)/f(x) \) is increasing in \( x \). Let \( F^{-1} \) and \( G^{-1} \) be quantile functions of \( F \) and \( G \) respectively. We say that \( X \) is smaller than \( Y \) in the likelihood ratio order (denoted \( X \leq_{lr} Y \)) if \( g(x)/f(x) \) is increasing in \( x \). Let \( F^{-1} \) and \( G^{-1} \) be quantile functions of \( F \) and \( G \) respectively. We say that \( X \) is less dispersed than \( Y \) (denoted \( X \leq_{disp} Y \)) if \( G^{-1}(\alpha) - F^{-1}(\alpha) \) is an increasing function in \( \alpha \in (0, 1) \). The family of distributions \( \{f(x; \theta), \theta \in \Theta \subset \mathbb{R}\} \) is stochastically increasing in \( \theta \) if \( X(\theta_1) \leq_{st} X(\theta_2) \) for all \( \theta_1 < \theta_2 \in \Theta \), where \( X(\theta) \) has a density \( f(x; \theta) \). For example, it is well known that location parameter family \( \{f(x - \theta), \theta \in \mathbb{R}\} \) and scale parameter family \( \{1/\theta f(x/\theta), \theta > 0\}, \ x > 0 \), are stochastically increasing in \( \theta \). For further details on stochastic orders we refer to Shaked and Shanthikumar [8] and Marshall and Olkin [6].

Furthermore, we say that the family of distributions \( \{f(x; \theta), \theta \in \Theta \subset \mathbb{R}\} \) has monotone likelihood ratio if \( f(x; \theta_2)/f(x; \theta_1) \) is increasing function in \( \theta \) for any \( \theta_1 < \theta_2 \in \Theta \), i.e. \( X(\theta_1) \leq_{lr} X(\theta_2) \). For example, the following families have the monotone likelihood ratio in \( \theta \):

(a) the family of distributions with density \( f(x; \theta) = c(\theta)h(x)I_{(-\infty, \theta]}(x) \);
(b) the family of distributions with density \( f(x; \theta) = c(\theta)h(x) \exp(\eta(\theta)t(x)) \);

provided that both \( \eta \) and \( t \) are increasing.

The likelihood ratio order is closely related with total positivity, (see Karlin [5]). Let \( k(x, y) \) be a measurable function defined on \( X \times Y \), where \( X \) and \( Y \) are subsets of \( \mathbb{R} \). We say that \( k(x, y) \) is totally positive of order \( r \) (to be short \( k(x, y) \) is \( TP_r \)) if for all \( x_1 < \cdots < x_m, \ x_i \in X \); and for all \( y_1 < \cdots < y_m, \ y_i \in Y \); and all
1 \leq m \leq r$, we have

\[
\begin{vmatrix}
  k(x_1, y_1) & \ldots & k(x_1, y_m) \\
  \vdots & \ddots & \vdots \\
  k(x_m, y_1) & \ldots & k(x_m, y_m)
\end{vmatrix} \geq 0.
\]

It is clear, that the ordering $X(\theta_1) \leq_{lr} X(\theta_2)$ whenever $\theta_1 < \theta_2 \in \Theta$ is equivalent that the density $f(x; \theta)$ is $TP_2$. We refer to Karlin \[5\] for proofs of basic facts in the theory of total positivity.

From the definition immediately follows that if $g$ and $h$ are nonnegative functions and $k(x, y)$ is $TP_r$ then $g(x)h(y)k(x, y)$ is also $TP_r$. Similarly, if $g$ and $h$ are increasing functions and $k(x, y)$ is $TP_r$, then $k(g(x), h(y))$ is again $TP_r$.

Twice differentiable positive function $f(x; \theta)$ is $TP_2$ if and only if

\[
\frac{\partial^2}{\partial x \partial \theta} \log f(x; \theta) \geq 0.
\]

Total positivity of many functions that arise in statistics follows from the basic composition formula. If $g(x, z)$ is $TP_m$, $h(z, y)$ is $TP_n$ and if the convolution

\[
k(x, y) = \int_{-\infty}^{\infty} g(x, z)h(z, y)dz
\]

is finite, then $k(x, y)$ is $TP_{\min(m,n)}$.

A particular and important case is when $k(x, y) = f(y - x)$. A nonnegative function $f$ is said to be $PF_k$ (Pólya frequency function of order $k$) if $f(y - x)$ is $TP_k$.

Recall that a real valued function $f$ is said to be logconcave on interval $A$ if $f(x) \geq 0$ and $\log f$ is an extended real valued concave function (we put $\log 0 = -\infty$). It is well known (see, for example, Schoenberg \[9\]) that the function $f$ is $PF_2$ if and only if $f$ is nonnegative and logconcave on $\mathcal{R}$. Recall also a very important property of $PF_2$ functions which we will use in the sequel that if $g$ and $h$ are logconcave functions on $\mathcal{R}$, such that the convolution

\[
h(x) = \int_{-\infty}^{\infty} g(x - z)h(z)dz
\]

is defined for all $x \in \mathcal{R}$, then the function $h$ is also logconcave on $\mathcal{R}$.

Another very important property of $TP$ functions is the variation diminishing property.

**Lemma 1.** Let $g$ be given by absolutely convergent integral

\[
g(x) = \int_{-\infty}^{\infty} k(x, y)f(y)dy
\]
where \( k(x, y) \) is \( TP_r \) and \( f \) change sign at most \( j \leq r - 1 \) times. Then \( g \) changes sign at most \( j \) times. Moreover, if \( g \) changes sign \( j \) times, then \( f \) and \( g \) have the same arrangement of signs.

The basic tool for proving stochastic ordering of estimators are order preserving properties of underlying stochastic orders, for the proof see Shaked and Shanthikumar [8].

**Lemma 2.** Assume that \( g \) is an increasing function.

(a) If \( X \leq_{st} Y \), then \( g(X) \leq_{st} g(Y) \).

(b) If \( X \leq_{lr} Y \), then \( g(X) \leq_{lr} g(Y) \).

**Lemma 3.** Let \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \) be independent random variables.

(a) If \( X_i \leq_{st} Y_i, i = 1, \ldots, n \), then \( X_1 + \cdots + X_n \leq_{st} Y_1 + \cdots + Y_n \).

(b) If \( X_i \leq_{lr} Y_i, i = 1, \ldots, n \), then \( X_1 + \cdots + X_n \leq_{lr} Y_1 + \cdots + Y_n \), provided these random variables have logconcave densities.

Let \( \hat{\theta} \) be a moment estimator of \( \theta \) based on a sample from population with density \( f(x; \theta) \), \( \theta \in \Theta \subset \mathbb{R} \). Let now \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) be the moment estimators obtained on the basis of the sample \( X \) from population with density \( f(x; \theta_1) \) and of the sample \( Y \) from population with density \( f(x; \theta_2) \) respectively, where \( \theta_1 < \theta_2 \in \Theta \). We say, that \( \hat{\theta} \) is stochastically increasing in \( \theta \) if the family of its distributions is stochastically increasing in \( \theta \), i.e. \( \hat{\theta}_1 \leq_{st} \hat{\theta}_2 \). We also shall be interested whether the stronger property holds \( \hat{\theta}_1 \leq_{lr} \hat{\theta}_2 \), i.e. \( \hat{\theta} \) is increasing in \( \theta \) with respect to likelihood ratio order.

Note, that in general the moment estimator may not be stochastically monotone as the following example indicates.

**Example 1.** Consider a sample \( X = (X_1, \ldots, X_n) \) from the uniform distribution on the interval \((-\theta, \theta)\), \( \theta > 0 \). Then \( E(X) = 0 \) and the first moment contains no information about \( \theta \). We use the second moment. Then we have \( E(X^2) = \theta^2/3 = m(\theta) \). Thus the moment estimator of \( \theta \) is of the form \( \hat{\theta} = \sqrt{3/n \sum_{i=1}^n X_i^2} \). Since the random variable \( X \) is not stochastically increasing in \( \theta \) and \( m(\theta) \) is increasing for \( \theta > 0 \), the estimator \( \hat{\theta} \) is not stochastically monotone.

**2. Results**

Assume for simplicity that for a given function \( g \) the generalized empirical moment belongs to \( m(\Theta) \), the domain of the values \( m(\theta) \), \( \theta \in \Theta \), for every resulting set of observations.
The following theorem gives a sufficient conditions for likelihood ordering of moment estimators for one-parameter family in the case when the estimator is based on the sample mean.

**Theorem 1.** Assume that the function \( m(\theta) = \int_{-\infty}^{\infty} x f(x; \theta) \, dx < \infty \) for all \( \theta \), \( f(x; \theta) \) is TP₂ and \( f(x; \theta) \) is logconcave in \( x \). Then the moment estimator \( \hat{\theta} \) is increasing in \( \theta \) with respect to likelihood ratio order.

**Proof.** Since \( f(x; \theta) \) is TP₂ we have ordering \( X_i(\theta_1) \leq_{lr} X_i(\theta_2) \) for all \( \theta_1 < \theta_2 \in \Theta \) and \( i = 1, \ldots, n \). From the variation diminishing property (see Lemma 1) we deduce that function

\[
m(\theta) - c = \int_{-\infty}^{\infty} (x - c) f(x; \theta) \, dx
\]

changes sign at most one for any \( c \), from + to − if occurs. This implies that \( m(\theta) \) is increasing. Hence the moment estimator is of the form \( \hat{\theta} = m^{-1}(\bar{X}) \). Since \( f \) is logconcave, we conclude from Lemma 3 (b) that \( \bar{X} \) is increasing in \( \theta \) with respect to likelihood ratio order. Theorem follows from the preserving property of monotone likelihood order under increasing operations.

If we weakness the assumptions of Theorem 1 we can obtain the following theorem.

**Theorem 2.** Assume that the function \( m(\theta) = \int_{-\infty}^{\infty} x f(x; \theta) \, dx < \infty \) for all \( \theta \) and \( f(x; \theta) \) is TP₂. Then the moment estimator of \( \theta \) is stochastically increasing in \( \theta \).

In the general case we can formulate the following corollary. We omit the proof because it follows by the same method as in Theorem 1.

**Corollary 1.** Assume that the function \( g \) is increasing and the integral \( m(\theta) = \int_{-\infty}^{\infty} g(x) f(x; \theta) \, dx < \infty \) for all \( \theta \). If \( f(x; \theta) \) is TP₂, then the moment estimator \( \hat{\theta} \) of the form (1.1) is stochastically increasing in \( \theta \). Moreover, if the random variable \( g(X) \) has logconcave density, then the estimator \( \hat{\theta} \) is increasing in \( \theta \) with respect to monotone likelihood ratio order.

The next Corollary follows immediately from Corollary 1 and form Proposition 2 due to An 1.

**Corollary 2.** Let \( X \) be a random variable with a density \( f(x; \theta) \) and \( g \) be a function such the integral \( m(\theta) = \int_{-\infty}^{\infty} g(x) f(x; \theta) \, dx < \infty \) for all \( \theta \). Assume that the following conditions are satisfied:
(a) \( g \) is strictly increasing, concave and differentiable;
(b) \( \left| \frac{\partial}{\partial x} v(x) \right| \) is logconcave, where \( v = g^{-1} \);
(c) \( f \) is logconcave, decreasing on the support of \( X \);
(d) \( f(x; \theta) \) is \( TP_2 \).

Then the moment estimator \( \hat{\theta} \) is increasing in \( \theta \) with respect to likelihood ratio order.

In many situations we deal with one-parameter exponential family with densities of the form

\begin{equation}
(2.1) \quad f(x; \theta) = h(x)c(\theta) \exp(\eta(\theta) T(x)), \quad \theta \in \Theta, \ x \in (a, b).
\end{equation}

Recall the well known formula for moments of \( T(X) \), see Berger and Casella [3]:

\begin{equation}
(2.2) \quad E_\theta(T(X)) = -\left[ \frac{\log c(\theta)}{[\eta(\theta)]'} \right]',
\end{equation}

\begin{equation}
(2.3) \quad Var_\theta(T(X)) = \frac{-[\log c(\theta)]'' + E_\theta[T(X)] \cdot [\eta(\theta)]'''}{([\eta(\theta)]')^2}.
\end{equation}

It is easy to prove that if both \( T \) and \( \eta \) are increasing (decreasing), then \( f(x; \theta) \) is \( TP_2 \) and from the variation diminishing property it follows that \( m(\theta) = E_\theta(T(X)) \) is increasing (decreasing). Combining those facts with order preserving properties of the usual stochastic order we can formulate the following theorem.

Theorem 3. For the one-parameter exponential family with densities of the form \( (2.1) \), where both \( \eta \) and \( T \) are increasing (decreasing), the moment estimator \( \hat{\theta} = m^{-1}(1/n \sum_{i=1}^n T(X_i)) \) is stochastically increasing in \( \theta \).

Let us make the following observations. Now we consider the maximum likelihood estimation of \( \theta \) for the one-parameter exponential family \( (2.1) \). Let \( x = (x_1, \ldots, x_n) \) be a value of a sample \( X \). The maximum likelihood function is of the form

\[ L(x; \theta) = \prod_{i=1}^n h(x_i)[c(\theta)]^n \exp \left( \eta(\theta) \sum_{i=1}^n T(x_i) \right). \]

So we have

\[
\frac{\partial \log L(x; \theta)}{\partial \theta} = n[\log c(\theta)]' + [\eta(\theta)]' \sum_{i=1}^n T(x_i).
\]

It is clear that \( \frac{\partial \log L(x; \theta)}{\partial \theta} = 0 \) if and only if

\begin{equation}
(2.4) \quad \frac{1}{n} \sum_{i=1}^n T(x_i) = -\frac{[\log c(\theta)]'}{[\eta(\theta)]'}.
\end{equation}
Let \( \hat{\theta} \) be a solution of the equation (2.4). Then \( \hat{\theta} \) is the maximum likelihood estimator (MLE) since using (2.3) we have

\[
\frac{\partial^2 \log L(x; \theta)}{\partial \theta^2} \bigg|_{\theta = \hat{\theta}} = -n([\log c(\theta)]')^2 |_{\theta = \hat{\theta}} \cdot \text{Var}_\theta(T(X)) < 0.
\]

On the other hand \( \hat{\theta} \) is the moment estimator, since \( E_\theta(T(X)) = -\frac{[\log c(\theta)]'}{[\eta(\theta)]'} \).

Thus from Theorem 3 we have the following result.

**Theorem 4.** For the one-parameter exponential family with densities of the form (2.1), where both \( \eta \) and \( T \) are increasing (decreasing), the maximum likelihood estimator \( \hat{\theta} \) is stochastically increasing in \( \theta \).

Particularly, Theorems 2 and 3 of Balakrishnan and Mi [2] give conditions under which maximum likelihood estimators for the one-parameter exponential family of the form (2.1) are stochastically increasing. In the Theorem 4 we have proved it but under weaker assumptions.

**Example 2.** Let \( X = (X_1, \ldots, X_n) \) be a sample from the distribution with density

\[
f(x; \theta) = \sqrt{\frac{\theta}{\pi}} x^3 \exp(-\theta/x), \quad x > 0, \ \theta > 0.
\]

This is one-parameter exponential family with \( T(x) = 1/x \) and \( \eta(\theta) = -\theta \). Using (2.2) we get moment estimator \( \hat{\theta} = (2/n \sum_{i=1}^{n} 1/X_i)^{-1} \). By Theorem 3 the estimator \( \hat{\theta} \) is stochastically increasing in \( \theta \). Of course, \( \hat{\theta} \) is also maximum likelihood estimator.

**Example 3.** Let \( X = (X_1, \ldots, X_n) \) be a sample from the gamma distribution with density

\[
f(x; \lambda) = \frac{1}{\Gamma(\alpha)\lambda^\alpha} x^{\alpha-1} \exp(-x/\lambda), \quad x > 0, \ \lambda > 0,
\]

where \( \alpha > 0 \) is known. This is clearly exponential family with \( T(x) = x \) and \( \eta(\lambda) = -1/\lambda \), so

\[
E_\lambda(X) = \frac{\partial}{\partial \lambda}(-\alpha \log \lambda - \Gamma(\alpha)) = \alpha \lambda.
\]

By Theorem 1 the estimator \( \hat{\lambda} = \bar{X}/\alpha \) is stochastically increasing in \( \lambda \). Moreover, if \( \alpha \geq 1 \), then the density \( f \) is logconcave and by Theorem 1 the estimator \( \hat{\lambda} \) for \( \alpha \geq 1 \) is also increasing with respect to likelihood ratio order.
On the other hand, assume now that $\lambda$ is fixed and $\alpha$ is unknown. This is also one-parameter exponential family with $T(x) = \log x$ and $\eta(\alpha) = \alpha$. Using (2.2) we get

$$E_\alpha(\log(X)) = \frac{\partial}{\partial \alpha}(\log \Gamma(\alpha) + \alpha \log \lambda) = \Psi(\alpha) + \log \lambda = m(\alpha),$$

where $\Psi(\alpha) = \frac{\partial}{\partial \alpha} \log \Gamma(\alpha)$ is the digamma function. By Theorem 3 the estimator $\hat{\alpha} = m^{-1}(\bar{T})$ stochastically increasing in $\alpha$. This estimator obtained by the method of maximum likelihood was considered in Example 2 of Balakrishnan and Mi [2].

Example 4. Let $X = (X_1, \ldots, X_n)$ be a sample from the logistic distribution with density

$$f(x; \theta) = \frac{\theta \exp(-x)}{(1 + \exp(-x))^{(\theta+1)}}, \quad x \in \mathcal{R}, \ \theta > 0.$$  

This is the one-parameter exponential family with $T(x) = \log(1 + \exp(-x))$ and $\eta(\theta) = \theta$. From (2.2) we get $E_\theta(T(X)) = 1/\theta$, hence from Theorem 1 the estimator $\hat{\theta} = 1/\bar{T}$ is stochastically increasing in $\theta$.

Example 5. Let $X = (X_1, \ldots, X_n)$ be a sample from the uniform distribution on the interval $(0, \theta)$, $\theta > 0$. Then moment estimator of $\theta$ based on the first empirical moment is of the form $\hat{\theta} = 2\bar{X}$ and by Theorem 1 this estimator is increasing in $\theta$ with respect to likelihood ratio order. Consider another moment estimator based on generalized moment $\bar{g} = 1/n \sum_{i=1}^n \log X_i$. Easy calculations show, that $E_\theta(g(X)) = \log \theta - 1$, where $g(x) = \log x$, thus from Corollary 2 the moment estimator $\hat{\theta} = \exp(1/n \sum_{i=1}^n \log X_i - 1)$ is also increasing in $\theta$ with respect to likelihood ratio order.

Now we consider the case when our family of distribution is the location family. Then we can formulate the following theorem.

Theorem 5. Let $\mathcal{F} = \{f(x - \theta), \theta \in \mathcal{R}\}$ be the location family. Suppose that $\mu_1 = E(X(0)) = \int_{-\infty}^{\infty} x f(x) dx < \infty$. Then the estimator $\hat{\theta} = \bar{X} - \mu_1$ is stochastically increasing in $\theta$. Moreover, if $f$ is logconcave then $\hat{\theta}$ is also increasing in $\theta$ with respect to likelihood ratio order.

Proof. The estimator $\theta$ is stochastically increasing in $\theta$ since the family $\mathcal{F}$ is stochastically increasing in $\theta$. If we assume that $f$ is logconcave, i.e. $f(x - \theta)$ is $TP_2$, then the family $\mathcal{F}$ has monotone likelihood ratio. Since the convolution of logconcave functions is the logconcave function we deduce that the estimator $\hat{\theta}$ is also increasing with respect to likelihood ratio order. \(\square\)
Similar results we may obtain for a scale parameter family. The ordering property of the moment estimators of \( \theta \) in this case is described by the following theorem.

**Theorem 6.** Let \( \mathcal{F} = \{1/\theta f(x/\theta), \theta \in R_+\} \), \( x > 0 \), be the scale parameter family. Suppose that exists \( k \)-th moment of the random variable \( X(1) \) with density \( f(x) \) and \( E(X^k(1)) = \mu_k \). Then the moment estimator \( \hat{\theta} = \sqrt{m_k/\mu_k} \) is stochastically increasing in \( \theta \). Moreover, if \( \text{Var}(X(1)) = \sigma^2 < \infty \), where \( S^2 = m_2 - m_1^2 \) is the sample variance, then the another moment estimator for \( \theta \) given by \( \tilde{\theta} = \sqrt{S^2/\sigma} \), is also stochastically increasing in \( \theta \).

**Proof.** The first part of theorem is obvious since the family \( \mathcal{F} \) is stochastically increasing in \( \theta \). It is also easy to prove that if \( \theta_1 < \theta_2 \in \Theta \), and then the vector of spacings \( U = (U_1, \ldots, U_n) \), where \( U_i = X_{i:n}^\theta - X_{i-1:n}^\theta \), \( i = 1, \ldots, n \) (we put \( X_{0:n} = 0 \)), is stochastically increasing in \( \theta \) (see Oja [7]). Since

\[
S^2 = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (X_{(j)} - X_{(i)})^2 = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (U_j + U_{j-1} + \cdots + U_{i+1})^2,
\]

then the sample variance is an increasing function of vector \( U \). Thus the theorem follows from the stochastic preserving property of multivariate stochastic ordered vectors under monotone operations. \( \square \)

**Example 6.** Let \( X = (X_1, \ldots, X_n) \) be a sample from the logistic distribution with density

\[
f(x; \theta) = \frac{\exp(-(x - \theta))}{1 + \exp(-(x - \theta))}, \quad x \in \mathbb{R}, \quad \theta \in \mathbb{R}.
\]

It is not difficult to see that density \( f \) is logconcave and \( E_{\theta}(X) = \theta \). By Theorem 5 the estimator \( \hat{\theta} = \bar{X} \) is increasing in \( \theta \) with respect to likelihood ratio order.

**Example 7.** Let \( X = (X_1, \ldots, X_n) \) be a sample from the Weibull distribution with density

\[
f(x; \theta) = (1/\theta)x^{1/\theta-1}\exp(-x^{1/\theta}), \quad x > 0, \quad \theta > 0.
\]

It is obvious that the family of these distributions is not stochastically ordered in the parameter \( \theta \). Let \( T_i = -\log X_i \), \( i = 1, \ldots, n \). After easy calculation we have that \( T_1 \approx_{st} \theta W_1 \), where \( W_1 \) is the Gumbel distribution with cumulative distribution function \( e^{-e^{-x}} \), \( x \in \mathbb{R} \). Thus the distribution of \( T_1 \) belongs to the scale parameter family. It is known that \( E(W_1) = \gamma \), where \( \gamma \) is Euler constant and \( \text{Var}(W_1) = \pi^2/6 \), hence \( \hat{\theta} = \sqrt{6S_T^2}/\pi \) and \( \tilde{\theta} = T/\gamma \) are moment estimators for \( \theta \).
where \( S^2_T = 1/n \sum_{i=1}^n (T_i - \bar{T})^2 \), but we can not apply here Theorem \( \text{[3]} \) since the support of \( W_1 \) is not \( \mathcal{R}_+ \). So, let us consider \( Z_i = |\log X_i|, i = 1, \ldots, n \). Then we have \( Z_1 =_{st} \theta |W_1| \) and \( \mu = E(Z_1) = \gamma - 2Ei(-1) = 1.01598 \) approximately, where \( Ei(x) = -\int_{-x}^{\infty} e^{-t}/tdt \). Also, after calculations we have \( \sigma^2 = Var(Z_1) = \pi^2/6 + 4(\gamma - Ei(-1))Ei(-1) = 0.945889 \) approximately. By Theorem \( \text{[3]} \) the estimator \( \hat{\theta} = \sqrt{S^2_T} / \sigma \) is stochastically increasing in \( \theta \). The same is true for the estimator \( \hat{\theta} = \bar{T} / \mu \).

References

[1] M. Y. An, Logconcavity versus Logconvexity: A complete Characterization, Journal of Economic Theory 80 (1998), 350–369.
[2] N. Balakrishnan, J. Mi, Order-preserving property of maximum likelihood estimator, Journal of Statistical Planning and Inference 98 (2001), 88–99.
[3] R. L. Berger, G. Casella, Statistical inference, 2nd ed., Boston: Duxbury Press 2002.
[4] A. A. Borovkov, Mathematical Statistics, Gordon and Breach Science Publishers, Amsterdam, 1998.
[5] S. Karlin, Total Positivity, Stanford University Press, California, 1968.
[6] A. W. Marshall, I. Olkin, Inequalities: Theory of Majorization and Its Applications, Academic Press, New York, 1979.
[7] H. Oja, On location, scale, skewness and kurtosis of univariate distributions, Scandinavian Journal of Statistics 8 (1981), 154–168.
[8] M. Shaked, J. G. Shanthikumar, Stochastic Orders, Springer Verlag, New York, 2007.
[9] I. J. Schoenberg, On Pólya frequency functions, I. The totally positive functions and their Laplace transforms, J. Analyse Math. 1 (1951), 331–374.