A Proof of the AdS-CFT Correspondence

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Abstract

The AdS-CFT correspondence is established as a re-assignment of localization to the observables which is consistent with locality and covariance.

1 Introduction

The “holographic principle”, or “AdS-CFT correspondence” has been considered as an amazing prediction on the basis of string theory, D-branes, and supergravity, setting off an enormous research activity. It states that the “degrees of freedom” of a quantum field theory in anti-deSitter (AdS) space-time of dimension $d$ can be completely identified with the degrees of freedom of a conformal quantum field theory in Minkowski space-time of one dimension less. It was first formulated for supergravity in 5 (+ 5 compactified) dimensions and super Yang-Mills theory in 4 dimensions by J. Maldacena [6], and was then conjectured by E. Witten to hold as a rather general principle [9].

The holographic principle arises by a confluence of several ideas arising from string theory and from gravity. It was discussed long ago by ’t Hooft [8] that the degrees of freedom of quantum fields above a black hole horizon are counted by the Bekenstein entropy of the horizon. In string theory, the horizons of certain extremal black hole solutions to classical supergravity with AdS geometry are considered as D3-branes. These appear as the supporting space-time for a 1+3-dimensional superconformal Yang-Mills theory, whose state space should contain that of the supergravity theory.

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1 Talk given at “Quantum Theory and Symmetries”, Goslar (D), July 1999
The AdS-CFT correspondence, yet, is not a feature of string theory. Although string theory happened to play a prominent role in its discovery, the conjecture itself is a statement about ordinary QFT, and its validity should be discussed in the framework of ordinary QFT.

It is well known that the AdS-CFT correspondence admits no simple identification of the respective quantum fields on the two space-time manifolds involved. It has also by now become a familiar conception with many of us that the quantum fields proper are somewhat ambiguous entities in QFT while the invariant entities are the local algebras they generate \([5]\). If we agree to take this idea seriously, a precise formulation of the conjectured correspondence is possible without any recourse to string theory or to perturbative supergravity, and a rigorous proof is astonishingly simple \([6]\).

The formulation of a QFT should be such that it admits the unambiguous and relativistically invariant computation of all the physical quantities (such as masses and cross sections) which specify the theory. It has been demonstrated that this possibility is guaranteed if, in a representation of positive energy, only the localization of each observable is known (see \([1]\)). The covariant assignment of localizations to the observables (or, what amounts to the same, the specification of all observables which are localized in a given region) provides the physical interpretation and therefore determines the theory. This formulation of a QFT does not rely on any idea of quantization of some classical physics.

The idea for the AdS-CFT correspondence is that the same observables can be given different assignments of localizations (in different space-times) in a covariant way, and are thus interpreted differently as a QFT in \(d\) dimensions or a conformal QFT in \(d-1\) dimensions. We shall establish the conjecture by showing that any covariant assignment of a localization in one space-time to the observables gives rise to another such assignment in the other space-time, and vice versa. But we shall see that it is not always adequate to think of localization in terms of smearing of fields.

So what is the role of string theory? It might turn out on the long run, that string theory is just an ingenious device to produce nonperturbative QFT’s, and might not at all exceed the realm of ordinary QFT. This must not be a reason to be disappointed but may be celebrated as a scheme which circumvents or soothens the notorious difficulties attached to Lagrangean perturbation theory. If this is correct, then it is no surprise if “stringy” pictures lead to statements about QFT.

While locality has a central position in QFT, the intuition prevails that string theory is “less local” than ordinary QFT, and hence is a new type of theory. But quite to the contrary, string theory is even more local than is necessary in QFT: namely, if one accepts that (free) fields carrying different inner degrees of freedom commute irrespective of their localization, then, treating transversal string excitations as inner degrees of freedom, they may cause excited string fields to commute even at time-like separation. This crude idea has been elaborated with some rigor for free strings \([3]\). I personally take it as another hint that, as the dust settles, string theory might bring us back to QFT, hopefully at an advanced level of insight.
2 Algebraic holography

We formulate the AdS-CFT conjecture as the assertion that the local QFT’s on AdS space-time $\text{AdS}_{1,d-1}$ are in 1:1 correspondence with the conformally invariant local QFT’s on compactified Minkowski space-time $\text{CM}_{1,d-2}$. Corresponding theories have the same state space.

AdS space-time is usually described as the hypersurface \( \{ x : x_0^2 - \vec{x}^2 + x_2^2 = R^2 \} \) in an ambient Minkowski space with two time directions (or rather, its quotient by identification of antipodal points \( x, -x \)). The embedded space has only one intrinsic time direction and a global time orientation. A serious reason for unease is that it has closed time-like curves. Yet, the Klein-Gordon field constructed in [4] is, for quantized values of the mass, a perfectly sensible QFT on AdS space-time.

QFT on AdS space-time has also been studied in [2]. No a priori conflict with the general framework of QFT was found, except that causal influences (signalled by non-commutation of local observables) can propagate only along geodesics. This feature (well known also from chiral observables in 2D conformal QFT’s) must presumably be interpreted as the absence of proper interaction. Thus, interacting QFT’s will, like the Klein-Gordon field of generic mass [4], live on a covering space.

One may also regard AdS space-time $\text{AdS}_{1,d-1}$ as a “cosmological deformation” of Minkowski space-time with the maximal number of isometric symmetries. The symmetry group (AdS group) is the Lorentz group $\text{SO}(2,d-1)$ of the ambient space, the boosts in the second time direction acting as deformed translations.

The very same group, $\text{SO}(2,d-1)$, is also the symmetry group of conformal QFT in $d-1$ dimensions, that is on compactified Minkowski space-time $\text{CM}_{1,d-2}$. In fact, the manifold $\text{CM}_{1,d-2}$ coincides with the conformal boundary of AdS space-time (defined by the conformal structure of $\text{AdS}_{1,d-1}$ induced by the AdS metric), and its conformal structure coincides with the one inherited by restriction. The AdS group preserves the boundary and acts on it like the conformal group.

This observation is pivotal for the asserted 1:1 identification of QFT’s: the Hilbert space and the representation of the group $\text{SO}(2,d-1)$ are the same for both theories. What differs is the interpretation of the group: e.g., the Hamiltonian of the AdS group (the rotation between the two ambient time directions) corresponds to a periodic subgroup of the conformal group generated by time translations and time-like special conformal transformations. Space-like AdS translations correspond to dilatation subgroups of the conformal group.

The set of observables, considered as operators on the common Hilbert space, is also the same for corresponding theories. What differs is the assignment of a localization to a given observable. The same operator is said to be localized in a suitable region $X \subset \text{AdS}_{1,d-1}$ as an observable of the AdS theory, and it is said to be localized in another region $Y \subset \text{CM}_{1,d-2}$ as an observable of the conformal theory.

For this reinterpretation to work, it must be compatible with covariance and locality, and this is the nontrivial issue. The group $\text{SO}(2,d-1)$ must act geometrically
in both interpretations, and hence the sets of regions \( X \) and \( Y \) above must be related by a bijective correspondence which respects the action of the group. Furthermore, this bijection must map causal complements with respect to one geometry into causal complements with respect to the other geometry since the geometric notion of causal independence is coded algebraically by local commutativity.

These constraints already fix the bijection between regions \( X \) in AdS space-time and regions \( Y \) in conformal Minkowski space-time of one dimension less. Here it is:

The basic regions in \( \text{AdS}_{1,d-1} \) are wedge-like regions which arise as the connected components of the intersection of AdS space-time with the ambient space region \( \{ x : x^1 > |x^0| \} \), and all \( \text{SO}(2,d-1) \) transforms thereof. Every such intersection provides a pair of wedge regions which turn out to be causally complementary in the sense that they cannot be connected by a causal geodesic.

The basic regions in \( \text{CM}_{1,d-2} \) are nonempty intersections of a forward and a backward light-cone, called double-cones. Each wedge region in \( \text{AdS}_{1,d-1} \) intersects the boundary in a double-cone in \( \text{CM}_{1,d-2} \). This yields a bijection between wedge regions in \( \text{AdS}_{1,d-1} \) and double-cone regions in \( \text{CM}_{1,d-2} \) which respects the action of the group \( \text{SO}(2,d-1) \) by construction, and causal complements by inspection.

Sofar, the discussion is entirely geometric. The algebraic part is very simple: We declare an operator which, as an observable of the boundary theory, is localized in some double-cone, to be localized, as an observable of the AdS theory, in the associated wedge region, or vice versa. A minute’s thought brings about that this prescription yields a 1:1 correspondence between local QFT’s on AdS space-time \( \text{AdS}_{1,d-1} \) and local QFT’s on conformal Minkowski space-time \( \text{CM}_{1,d-2} \). The associated theories have the same Hilbert space and the same representation of the group \( \text{SO}(2,d-1) \), while their respective physical interpretations are different.

One issue remains to be discussed, the positivity of the energy spectrum. As we mentioned before, the two interpretations go along with different Hamiltonians. By a standard argument, it is well known that if one of these Hamiltonians has positive spectrum, then so does the other. Yet, the spectra are very different! The AdS Hamiltonian has discrete spectrum due to periodicity in time, while the boundary Hamiltonian has conformally invariant, hence continuous spectrum.

We shall discuss that the notion of sharp localization acquires a very different meaning in corresponding theories. While sharp boundary localization obviously corresponds to localization at space-like infinity of AdS space-time, sharp AdS localization turns out to have a very delicate meaning in terms of boundary localization.

3 Examples

The proof sketched above is a structural proof for a structural statement. It does not tell us which particular AdS theory is associated with which particular conformal QFT. This has to be discussed case by case, as will be exemplified below.
Our proof also doesn’t tell us which observables are localized in bounded regions of AdS space-time. The standard procedure is to say that an observable is localized in an arbitrary AdS region $X$, if it is localized in all wedge regions which contain $X$. The according algebraic determination of localization in AdS double-cone regions yields the following general structural results [7].

In $d \geq 1 + 2$, if there are double-cone localized AdS observables, the boundary theory must violate the additivity property that the observables localized in small double-cones covering the space-like basis of a large double-cone generate the observables localized in the large double-cone. While this additivity should always hold for theories generated by gauge-invariant Wightman fields, its violation seems characteristic for non-abelian gauge theories, as Wilson loops cannot be expressed in terms of point-like gauge invariant quantities. Thus, AdS theories which are described in terms of proper local fields [4] and which consequently possess observables localized in bounded regions, should correspond to gauge-type conformal boundary theories.

Conversely, a boundary theory which is additive in the above sense must correspond to a QFT on AdS space-time which does not possess local fields. To prevent confusion: the latter still has as many wedge-localized observables as there are observables in the boundary theory, which however cannot be “detached” from infinity. Impossibility of point-like localization does not mean that the theory is non-local, since observables localized in causally complementary wedges do commute as they should. Topological field theories (such as Chern-Simons with Wilson lines attached to the boundary as wedge-localized observables) come to one’s mind [9].

The situation is much more favorable in $d = 1 + 1$. Then $CM_{1,d-2} = S^1$ and the boundary theory is a chiral conformal QFT. The Penrose diagram of two-dimensional AdS space-time is the strip $\mathbb{R} \times (0, \pi)$ with points $(t, x) \sim (t + \pi, \pi - x)$ identified. This is in fact a Möbius strip. Light rays are $45^\circ$ lines. A wedge region is a space-like triangular region enclosed by a future and a past directed light ray emanating from a point in the interior of the strip, and the associated boundary “double-cone” is the interval cut out of the boundary $S^1$ by this wedge.

A double-cone in the Möbius strip is the intersection of two wedges which cut out of the boundary $S^1$ two intervals $I_1, I_2$ which overlap at both ends: $I_1 \cap I_2 = I_1 \cup I_2$.

An AdS observable in $d = 1 + 1$ is thus localized in a double-cone if as a chiral observable it is at the same time localized in both the large intervals $I_1, I_2$. This is of course true for the chiral observables localized in $J_1$ or $J_2$, but in general there will be more than these. The additional AdS double-cone observables remain broadly localized (if considered as boundary observables) even if the double-cone, and hence the intervals $J_1, J_2$ are small. We shall see examples of such observables below. This again does not mean any non-locality of AdS double-cone observables, since they do commute whenever two double-cones sit within causally complementary wedges. It only means that their description in terms of boundary fields may be non-local.

We present an example in $d = 1 + 1$: a massless conserved vector current $j^\mu$ on AdS space-time represented by the Möbius strip. Its equations of motion are solved in the
plane by \( j^0(t, x) = j_R(t-x) + j_L(t+x) \) and \( j^1(t, x) = j_R(t-x) - j_L(t+x) \). Restricting this solution to the strip \( \mathbb{R} \times (0, \pi) \) and requiring it to respect the identification of points \( (t, x) \sim (t + \pi, \pi - x) \) amounts to put \( j_L = j_R \equiv j \) with period \( 2\pi \). Canonical quantization is achieved by representing \( [j(u), j(u')] = i\delta'(u - u') \) on a Fock space.

This is indeed a U(1) current, the simplest conformal QFT on \( S^1 \). But its degrees of freedom are redistributed (through the combinations \( j^u \)) over the Möbius strip.

We now compute observables localized in an AdS double-cone (giving rise to intervals \( I_1 \cap I_2 = J_1 \cup J_2 \) as before) in the interior of the strip. Typical boundary observables localized in an interval \( I \) are of the form \( W(f) = \exp ij(f) \) where \( f \) is a periodic smearing function which is constant outside the interval \( I \). Adding a constant \( c \) to \( f \) is immaterial since the charge operator \( \int j(u)du \) is a multiple \( q \) of unity in every irreducible representation, so \( W(f + c) = e^{i\alpha}W(f) \). Now, consider a smearing function which has constant but different values on both gaps between \( J_1, J_2 \). Then \( W(f) \) is localized as a boundary observable in both intervals \( I_1, I_2 \), but it is neither localized in \( J_1 \) nor in \( J_2 \), nor is it generated by such observables. As an AdS observable, \( W(f) \) is localized in a double-cone, and operators of this form generate all observables in the double-cone. Suitably regularized limits of \( W(f) \) even yield point-like local fields on the Möbius strip \( \phi(t, x) = \exp i\alpha \int_{t-x}^{t+x} j(u)du \).

As the size of a double-cone well within the strip shrinks, the correlations of the boundary observables within the intervals \( J_1, J_2 \) disappear, giving rise to two decoupled (chiral) current algebras, one for \( J_1 \) and one for \( J_2 \), among the AdS observables of the double-cone. The additional observables of the form \( W(f) \) as discussed above behave in this limit as vertex operators \( E_\alpha(t + x) \otimes E_{-\alpha}(t - x) \) carrying opposite left and right chiral charge. Thus, locally the QFT on AdS space-time looks like the bosonic sector of the Thirring (“U(1) WZW”) model.

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