Schwarzian for colored Jackiw-Teitelboim gravity

Konstantin Alkalaev, a Euihun Joung, b Junggi Yoon c,d,e

a I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute, Leninsky ave. 53, 119991 Moscow, Russia
b Department of Physics, Kyung Hee University, 26 Kyungheedae-ro Dongdaemun-gu, Seoul 02447, Korea
c Asia Pacific Center for Theoretical Physics, 77 Cheongam-ro, Nam-gu, Pohang-si, Gyeongsangbuk-do, 37673, Korea
d Department of Physics, POSTECH 77 Cheongam-ro, Nam-gu, Pohang-si, Gyeongsangbuk-do, 37673, Korea
e School of Physics, Korea Institute for Advanced Study 85 Hoegiro Dongdaemun-gu, Seoul 02455, Korea

E-mail: alkalaev@lpi.ru, euihun.joung@khu.ac.kr, junggi.yoon@apctp.org

Abstract: We study the boundary effective action of the colored version of the Jackiw-Teitelboim (JT) gravity. We derive the boundary action, which is the color generalization of the Schwarzian action, from the $su(N,N)$ BF formulation of the colored JT gravity. Using different types of the $SU(N,N)$ group decompositions both the zero and finite temperature cases are elaborated. We provide the semi-classical perturbative analysis of the boundary action and discuss the instability of the spin-1 mode and its implication for the quantum chaos. A rainbow-AdS$_2$ geometry is introduced where the color gauge symmetry is spontaneously broken.

1 Introduction

In recent years, Jackiw-Teitelboim (JT) gravity [1, 2] has enabled us to deepen the understanding of quantum gravity and the black hole information problem. The action of the
nearly-AdS$_2$ JT gravity \cite{3}

$$S_{JT} = - \frac{1}{16\pi G_N} \left[ \int_{\mathcal{M}_2} \sqrt{|g|} \phi(R + 2) + 2 \int_{\partial \mathcal{M}_2} \sqrt{|h|} \phi(K - 1) \right] \tag{1.1}$$

reduces to the Schwarzian boundary action which is dual to the low energy effective action of the Sachdev-Ye-Kitaev (SYK) model \cite{6?–9}. This Schwarzian mode plays a central role in the maximal quantum chaos of the SYK-like models \cite{6, 9?–13} and its holographic dual 2D gravity \cite{3}. Furthermore, the Schwarzian action features the one-loop exactness of the partition function \cite{14} and its genus expansion demonstrates the random matrix behavior of 2d gravity \cite{15}. It is remarkable that JT gravity (1.1) in the frame-like formulation can be described as $sl(2, \mathbb{R})$ BF theory \cite{16–18} which is the reminiscence of the Chern-Simons formulation for the AdS$_3$ gravity \cite{19, 20}. The BF formulation for JT gravity has a great advantage in extending the space-time symmetry$^1$ simply by taking other gauge algebras instead of $sl(2, \mathbb{R})$: $sl(N, \mathbb{R})$ for the higher-spin extension \cite{23–28}, $OSp(2, \mathcal{N})$ for the supersymmetric extension \cite{29–31}. Another interesting extension is a color generalization of the spacetime which can be thought of as a multi-graviton theory. In \cite{32} we introduced a colored version of JT gravity with $su(N, N)$, which is analogous to the colored extension of 3D Einstein gravity via $su(N, N)$ Chern-Simons gravity \cite{33–35}. Despite no-go results and interpretation problems related to higher-dimensional multi-graviton constructions, in lower dimensions, BF and Chern-Simons theories of gravity can be safely color extended just because of their pure topological nature. Nonetheless, there could be edge states interpreted as boundary “multi-gravitons”.

In this paper, we begin with the $su(N, N)$ BF action for the colored JT gravity and impose a suitable asymptotic boundary condition leading to the boundary effective action. Since $SU(1, 1) \approx SL(2, \mathbb{R})$ this construction can be considered as the color $N > 1$ generalization of the Schwarzian action in the nearly-AdS$_2$ \cite{3}. In the $SL(2, \mathbb{R})$ case, one can obtain the Schwarzian action at finite temperature by using the reparametrization transformation $\tau \to \tan(\frac{\tau}{\beta})$ from the zero-temperature one. However, for the case of the extended space-time symmetry like higher-spin or colored one, such a transformation to the finite temperature is not known. Hence, we consider an alternative way to derive boundary actions at finite temperature. At $N = 1$ one may solve the asymptotic AdS$_2$ condition by the Gauss decomposition of the respective $SL(2, \mathbb{R})$ group elements, which gives the zero temperature Schwarzian action \cite{15, 36}. It was shown in \cite{37} that the Iwasawa decomposition of $SL(2, \mathbb{R})$ group elements, instead of the Gauss decomposition, leads to the Schwarzian action at finite temperature. We propose the boundary effective action of the colored JT gravity at finite temperature which is built by using the Iwasawa decomposition of $SU(N, N)$ group elements.

The colored JT gravity accommodates interesting background solutions which could be referred to as rainbow-AdS$_2$ in which the colored gravitons do not vanish at asymptotic infinity.$^2$ In the rainbow-AdS$_2$ background, the broken color gauge symmetry makes a part

$^1$Also See Ref. \cite{21} as well as Ref. \cite{22} for their non-relativistic limits.

$^2$This rainbow-AdS$_3$ vacua were investigated in the colored AdS$_3$ gravity \cite{33–35}. 

--- 2 ---
of colored gravitons unstable that resembles partially-massless fields emerging via the Higgs mechanism in the colored AdS$_3$ Chern-Simons gravity [33–35]. Furthermore, those modes also transform according to the modified asymptotic symmetry of the rainbow-AdS$_2$ background which we refer to as the asymptotic rainbow-AdS$_2$ symmetry.

The paper is organized as follows. In Section 2 we review the Schwarzian boundary action of the $sl(2,\mathbb{R})$ BF action. We discuss the isometry, the finite temperature extension and the respective smooth geometry. In Section 3 we build the boundary effective action of the $su(N,N)$ BF action for the color generalization of JT gravity. The zero and finite temperature cases are analyzed by means of the Gauss and Iwasawa group decompositions. In Section 4 we develop the semi-classical perturbative analysis of the boundary action and find the form of the quadratic action which explicitly manifests the issues of (in)stability of the colored gravity. Also, we analyze the Lyapunov exponents and the bound on chaos. In Section 5 we introduce the notion of the rainbow-AdS$_2$ geometry and analyze the spectrum of fluctuations around the colored background. In Section 6 we summarize our results and discuss future directions.

2 Boundary Effective Action of $sl(2,\mathbb{R})$ JT gravity

In this section we review the uncolored JT gravity with emphasis on the points which are important in the subsequent construction of the color extension.

2.1 Boundary reduction of $sl(2,\mathbb{R})$ JT gravity

Let us discuss the boundary reduction of $sl(2,\mathbb{R})$ BF formulation of JT gravity. The bulk part of the action is given by

$$S_{BF} = \kappa \int_{M_2} \text{tr} (\Phi F),$$

(2.1)

where $F = dA + A \wedge A$ is a curvature 2-form, a connection 1-form $A$ and a dilaton 0-form $\Phi$ take values in $sl(2,\mathbb{R})$; the trace $\text{tr}$ stands for the invariant quadratic form of the gauge algebra. The above BF action can be interpreted as the frame-like formulation of JT gravity upon identifying the components of $A$ with the zweibein and Lorentz spin connection and one component of $\Phi$ as the dilaton $\phi$ [16–18]. The other two components of $\Phi$ play the role of Lagrangian multiplier to impose the torsion-free condition while the spin connection determines them algebraically through other fields. In this way, the $sl(2,\mathbb{R})$ BF action is reduced to JT action (1.1), up to a boundary term, where the dimensionless coupling constant $\kappa$ is related to the Newton constant $G_N$ by $\kappa = (8\pi G_N)^{-1}$.

The $sl(2,\mathbb{R})$ algebra basis elements are realized as

$$L_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

(2.2)

with the commutation relations $[L_1,L_{-1}] = 2L_0$, $[L_0,L_{\pm 1}] = \mp L_{\pm 1}$. Then, $\text{tr}$ is a matrix trace.
An appropriate boundary term is needed to be added to (2.1) since the two-dimensional manifold $M_2$ has an asymptotic boundary. $M_2$ is assumed to have the topology of a half plane or an open disc depending on whether the temperature of the theory is zero or not. Both cases can be parameterized by the spatial coordinate $r$ and the Euclidean time coordinate $u$. When the temperature of the theory is finite, $u \in S^1 (u \equiv u + 2\pi)$, otherwise $u \in \mathbb{R}$. The asymptotic boundary $\partial M_2$ is identified with the $r = \infty$ curve\(^4\) parameterized by the coordinate $u$. Then, the coordinate parameterization of the fields reads $\Phi = \Phi(u, r)$ and $A = \{A_u(u, r), A_r(u, r)\}$.

The boundary term should be chosen together with an associated boundary condition so that the variational principle is well-defined in the presence of the boundary.\(^5\) In this paper, we focus on the boundary term which is often used in the frame-like formulation,

$$S_{bdy} = -\frac{\kappa}{2} \int_{\partial M_2} \text{tr} (\Phi A), \quad (2.3)$$

along with its compatible boundary condition,

$$\left(\Phi + \gamma A_u\right)|_{\partial M_2} = 0, \quad (2.4)$$

which altogether lead to the well-defined variational principle defined by the total action

$$S_{tot} = S_{BF} + S_{bdy}, \quad (2.5)$$

(see e.g. a discussion in [15, 38]). Here, $\gamma$ is a real constant related to the boundary value of $\Phi$ which is determined by the boundary value of the dilaton field $\phi$ in the nearly-AdS$^2$ space (1.1). From (2.3) and (2.4) we can rewrite the boundary term as

$$S_{bdy} = \frac{\kappa \gamma}{2} \int_{\partial M_2} du \, \text{tr} (A_u^2). \quad (2.6)$$

In addition, we demand the asymptotic AdS$^2$ condition in the expansion of the frame-like formulation [39–41]

$$(A - A_{AdS})|_{\partial M_2} = \mathcal{O}(1), \quad (2.7)$$

where the AdS$^2$ connection $A_{AdS}$ is given by

$$A_{AdS} = (e^r L_1 + \mathcal{L}_0 e^{-r} L_{-1}) \, du + \mathcal{L}_0 \, dr. \quad (2.8)$$

Here, $\mathcal{L}_0$ is a real constant which determines global properties of the space (see section 2.2).

The Schwarzian action is obtained by evaluating the total action $S_{tot}$ on a particular flat connection supplemented with the asymptotic AdS$^2$ condition (2.7). Such a connection has the gauge fixed form [42, 43],

$$A(u, r) = b^{-1}(r) a(u) \, du \, b(r) + b^{-1}(r) \, db(r), \quad b(r) = e^{r L_0}, \quad (2.9)$$

\(^4\)The other bound of $r$ is determined by the background solution. See the following discussions.

\(^5\)Let us remark that the choice of boundary terms and conditions is not unique. The full generality of all consistent choices as well as the precise matching of such choices between the metric-like and frame-like formulations is yet to be scrutinized.
where \( a(u) \) is an \( sl(2, \mathbb{R}) \)-valued function independent of the radial variable \( r \) due to the flatness constraint satisfied by \( A(u, r) \) in the bulk. The function \( a(u) \) is further restricted by the asymptotic AdS\(_2\) condition (2.7) and, after a further gauge fixing, takes the form,

\[
a(u) = L_1 + \mathcal{L}(u) L_{-1} = \begin{pmatrix} 0 -\mathcal{L}(u) \\ 1 & 0 \end{pmatrix}.
\]  

(2.10)

Here, as opposed to the AdS\(_2\) connection (2.8), the function \( \mathcal{L}(u) \) does depend on \( u \). It is analogous to the boundary energy-momentum tensor arising in the asymptotic analysis of the 3D Chern-Simons gravity. Since the BF action in the bulk vanishes after solving the flatness condition, the total action \( S_{\text{tot}} \) is reduced to the boundary action \( S_{\text{bdy}} \) (2.6) and we find

\[
S_{\text{tot}} = -\kappa \gamma \int_{\partial M_2} du \mathcal{L}(u). 
\]  

(2.11)

Therefore, the task boils down to determining the function \( \mathcal{L}(u) \).

### 2.2 Background solutions

Remark that Schwarzian action can be viewed as a partial on-shell action because the other equation of motion, the covariant constancy condition of the dilaton, \( d\Phi + [A, \Phi] = 0 \), is not yet imposed. Imposing the latter is equivalent to imposing the equation of motion of the Schwarzian theory [3]. The covariant constancy condition reduces to

\[
\partial_u \phi(u, r) + [a(u), \phi(u, r)] = 0, \quad \partial_r \phi(u, r) = 0, 
\]  

(2.12)

where \( \phi(u, r) = b(r) \Phi(u, r) b^{-1}(r) \). The second condition removes the \( r \)-dependence of \( \phi \) that is consistent with the boundary condition (2.4): \( \phi(u) + \gamma a(u) = 0 \). In the end, the first equation of (2.12) yields the condition \( \partial_u a(u) = 0 \) which has only constant solutions. It follows that \( \mathcal{L}(u) = \mathcal{L}_0 \), and hence we recover the AdS\(_2\) connection (2.8) with a constant \( \mathcal{L}_0 \).

We can rewrite the AdS\(_2\) connections parameterized by different values of \( \mathcal{L}_0 \) into the metric,

\[
ds^2 = dr^2 + (e^r - \mathcal{L}_0 e^{-r})^2 du^2,
\]  

(2.13)

that leads to solutions of different topologies depending on the value of \( \mathcal{L}_0 \).

1. When \( \mathcal{L}_0 > 0 \), we can parameterize \( \mathcal{L}_0 = e^{2r_0} \) by some \( r_0 \) and find that

\[
ds^2 = dr^2 + \sinh^2(r - r_0) (2 e^{r_0} du)^2.
\]  

(2.14)

The metric is defined in the domain \( r \in (r_0, +\infty) \) and \( r = r_0 \) is the origin which has a conical singularity unless \( u \approx u + 2\pi/(2 e^{r_0}) \). The space has the topology of an open disk (a cigar-type geometry). The Euclidean AdS\(_2\) solution corresponds to the case \( \mathcal{L}_0 = \frac{1}{4} \) where the periodicity of \( u \) is \( 2\pi \) and the origin is regular: the global hyperboloid metric in the polar coordinate system.
(2) When $L_0 = 0$, we find the Poincaré half-plane metric,
\[ ds^2 = dr^2 + e^{2r} du^2, \]  
which is valid for $r, u \in (-\infty, +\infty)$. The space has the topology of a half-plane.

(3) When $L_0 < 0$, we can parameterize it as $L_0 = -e^{2r_0}$ and find
\[ ds^2 = dr^2 + \cosh^2(r - r_0) (2 e^{r_0} du)^2, \]  
which is valid for $r \in (-\infty, +\infty)$. In this case, the time coordinate $u$ is not restricted: it may belong to $\mathbb{R}$ or $S^1$ with any periodicity. Note that the hyperplane $r = -\infty$ is the other asymptotic boundary disconnected from the one at $r = +\infty$. The space has the topology of a plane or a cylinder, depending on whether $u \in \mathbb{R}$ or $S^1$.

In this work, we will concern the color generalization of the first two cases (1) and (2), i.e. the two-dimensional hyperboloid with $L_0 = \frac{1}{4}$ and the Poincaré half-plane with $L_0 = 0$. With suitable regularity conditions at $r \to +\infty$ and $u \to \pm\infty$, the latter case also provides a good coordinate system on the two-dimensional hyperboloid, namely, the Poincaré coordinates. Therefore, if we formulate everything in a coordinate independent manner and the metric and dilaton fields are sufficiently regular, the two cases cannot give different results. In fact, the difference of the two cases arises from the asymptotic boundary condition for $\Phi$ (2.4), which is coordinate dependent. Due to this coordinate dependence, regular field configurations in one case map to irregular ones in the other case, and vice versa, upon coordinate transformations.

**Gauge function and holonomy.** When the background topology is that of a half-plane, see (2), the flatness condition assures that there exists a gauge function $G(r, u)$ such that $A(r, u) = G^{-1}(r, u) dG(r, u)$. With the gauge choice (2.9), the connection $a(u)$ can be written as $a(u) = g^{-1}(u) \dot{g}(u)$, where $G(r, u) = g(u) b(r)$ and $\dot{g}(u) = \frac{d}{du} g(u)$. We may also write the action (2.11) with $\Gamma \equiv \gamma L_1 \in sl(2, \mathbb{R})$ as
\[ S_{tot} = \kappa \int_{\partial M_2} \text{tr} (\Gamma g^{-1} dg), \]  
which has the form of a coadjoint orbit action. However, $g(u)$ here is not arbitrary but subject to the asymptotic AdS$_2$ condition (2.10).

When the background topology is that of an open disk with a possible conical singularity at the origin, see (1), the gauge function cannot be uniquely defined within a single coordinate patch. Focusing on the background solutions, we find the gauge function as
\[ g_{AdS}(u) = g_0 \exp(a_{AdS} u), \]  
where $g_0$ is a constant group element, and the multi-valuedness of $g_{AdS}(u)$ is given by the holonomy,
\[ \text{Hol}(A) = \mathcal{P} \exp \left[ \oint A \right] \sim \exp \left( 2\pi a_{AdS} \right). \]
If we restrict to the trivial holonomy, then the gauge function is single-valued and defines a map from $S^1$ to $PSL(2, \mathbb{R})$, provided the flat connections were not subject to the asymptotic AdS$_2$ condition. On the other hand, the asymptotic AdS$_2$ condition restricts $PSL(2, \mathbb{R})$ to its one-dimensional subgroup $SO(2)$ generated by $a_{\text{AdS}}$ with $\mathcal{L}_0 > 0$. The solutions of the trivial holonomy are parameterized as
\[
\mathcal{L}_0 = \frac{\nu^2}{4}, \quad \text{where } \nu \in \mathbb{Z}_+.
\] (2.20)
Note that we have used the projectivity of $PSL(2, \mathbb{R})$. The $\nu = 1$ case corresponds to the AdS$_2$ background with $\mathcal{L}_0 = \frac{1}{4}$. Other connections with $\nu > 1$ correspond to the geometry with conical surplus which we will not discuss in this paper. \(^6\)

### 2.3 Derivation of Schwarzian action

Both the zero- and finite-temperature Schwarzian actions can be obtained from (2.11) by imposing the asymptotic AdS$_2$ condition (2.10). In what follows we review the relevant details.

#### 2.3.1 Zero-temperature Schwarzian

Using the Gauss decomposition of the gauge group element $g(u)$,
\[
g(u) = \begin{pmatrix} 1 & 0 \\ f(u) & 1 \end{pmatrix} \begin{pmatrix} b^{-1}(u) & 0 \\ 0 & b(u) \end{pmatrix} \begin{pmatrix} 1 & -e(u) \\ 0 & 1 \end{pmatrix},
\] (2.21)
the boundary connection $a = a(u)$ can be represented as follows
\[
a = g^{-1} \dot{g} = \begin{pmatrix} -\frac{\dot{b}}{b} + \frac{e}{b^2} \\ \frac{\dot{f}}{b^2} \end{pmatrix}, \quad \begin{pmatrix} -\dot{e} + 2 \frac{b}{\mathcal{L}} \mathcal{L} - \frac{e^2}{b^2} \\ \frac{\dot{b}}{b} - \frac{e}{b^2} \end{pmatrix}.
\] (2.22)
By equating $a(u)$ given by (2.22) with the one in (2.10), we obtain the relations
\[
f = \mathcal{L}^2, \quad b^{-1} \dot{b} = e, \quad \dot{e} - e^2 = \mathcal{L}.
\] (2.23)
These allow us to express $\mathcal{L}(u)$ in terms of $f(u)$, which turns out to coincide with the Schwarzian derivative
\[
\mathcal{L} = b^{-1} \dot{b} - 2 (b^{-1} \dot{b})^2 = \frac{1}{2} \left[ \frac{\dot{f}}{f} - \frac{3}{2} \left( \frac{f'}{f} \right)^2 \right] \equiv \frac{1}{2} \text{Sch} (f, u).
\] (2.24)
Finally, the total action on the nearly-AdS$_2$ background is reduced to the Schwarzian action [3]
\[
S_{\text{tot}} = -\frac{\kappa \gamma}{2} \int_{\partial M_2} \text{du} \text{ Sch} (f, u).
\] (2.25)
This form is seemingly that of the Schwarzian action at zero temperature. However, we have not imposed any condition on the function $f(u)$ so far. Only when we restrict $f(u)$ to be a non-periodic function from $\mathbb{R}$ to $\mathbb{R}$, this will describe the Schwarzian theory at zero temperature. \(^6\)This is related to the different coadjoint orbits considered in [44].
2.3.2 Finite-temperature Schwarzian

From the Schwarzian derivative (2.24) by using the compactifying coordinate transformation

\[ f = \tan \frac{\theta}{2}, \quad \theta \in (-\pi, \pi), \]  

one can get the finite temperature Schwarzian,

\[ \mathcal{L}(u) = \frac{1}{2} \left( \text{Sch}(\theta, u) + \frac{1}{2} \dot{\theta}^2 \right). \]  

(2.27)

However, this method of coordinate transformation cannot be immediately applied to Schwarzian-like systems with extended symmetries, as it would require a transformation beyond the diffeomorphism.\(^7\) Thus, in order to obtain the color-decorated Schwarzian at finite temperature from colored BF theory, we need to understand first the reasoning behind the transformation (2.26). It is instructive to consider once again background solutions in each cases.

- In the zero-temperature case, \( f \) is a function from \( \mathbb{R} \) to \( \mathbb{R} \). The function \( f(u) = u \) is a background solution and the corresponding group element is given by

\[ g(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}. \]  

(2.28)

The two endpoints of \( g(u) \) do not meet: \( g(-\infty) \neq g(\infty) \) and \( g(-\infty) \neq -g(\infty) \). The curve goes along an \( \mathbb{R} \) subgroup of \( PSL(2, \mathbb{R}) \).

- In the finite-temperature case, \( \theta(u) \) is a function from \( S^1 \) to \( S^1 \): \( \theta(u + 2\pi) = \theta(u) + 2\pi \). The functions \( \theta(u) = u \) or \( f(u) = \tan \frac{u}{2} \) are a background solution of this kind and the corresponding group element is given by

\[ g(u) = \begin{pmatrix} \cos \frac{u}{2} & -\sin \frac{u}{2} \\ \sin \frac{u}{2} & \cos \frac{u}{2} \end{pmatrix}. \]  

(2.29)

The two endpoints of \( g(u) \) coincide as elements of \( PSL(2, \mathbb{R}) \). That is, up to the quotient by \( \mathbb{Z}_2 \) of \( SL(2, \mathbb{R}) \): \( g(u + 2\pi) = -g(u) \). The curve goes along a \( SO(2) \) subgroup of \( PSL(2, \mathbb{R}) \) and it is non-contractible.

From the above, we conclude that a finite temperature theory can be obtained from the gauge functions which define non-contractible cycles of the group manifold, \( PSL(2, \mathbb{R}) \) in the present uncolored case. For the non-contractibility, the cycles should non-trivially wind the maximal compact subgroup because the fundamental group of a non-compact group is that of its maximal compact subgroup. For this reason, it will be convenient to use a group decomposition involving a maximal compact subgroup and use its angle coordinate

---

\(^7\)For example, the form of the higher-spin Schwarzian derivative at finite temperature is still an open question. Its perturbative expression was obtained in [36].
to parameterize the cycle. The Iwasawa decomposition precisely does this job. For any $g \in SL(2, \mathbb{R})$ it reads

$$g(\theta, c, e) = \begin{pmatrix}
\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\
\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix}
\begin{pmatrix}
1^{-1} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 - e \\
0 & c
\end{pmatrix}, \quad (2.30)$$

where $c, e \in \mathbb{R}$ and $g(\theta + 2\pi, c, e) = -g(\theta, c, e) \equiv g(\theta, c, e)$. Here, we used the $\mathbb{Z}_2$ equivalence relation $\equiv$ in order to address an element of $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\mathbb{Z}_2$. In the same way as in Section 2.3.1 with the Gauss decomposition, we evaluate the connection $a$ from the Iwasawa decomposition of $g(u) (2.30)$ as

$$a(u) = \dot{g}^{-1} \dot{g} = \begin{pmatrix}
-\frac{\dot{e}}{c} + \frac{e \dot{\theta}}{2c^2} \\
\frac{\dot{\theta}}{2c^2} - \frac{e \dot{\theta}}{2c^2}
\end{pmatrix}. \quad (2.31)$$

Then, imposing the asymptotic AdS$_2$ condition (2.10) we find the relations

$$\dot{\theta} = 2c^2, \quad c^{-1} \dot{c} = e, \quad \dot{e} - e^2 + \frac{1}{4} \dot{\theta}^2 = \mathcal{L}, \quad (2.32)$$

and the action,

$$S_{tot} = -\kappa \gamma \int du \mathcal{L}(u) = -\kappa \gamma \int du (c^{-1} \dot{c} - 2(c^{-1} \dot{c})^2 + e^4)$$

$$= -\frac{\kappa \gamma}{2} \int du \left( \text{Sch}(\theta, u) + \frac{1}{2} \dot{\theta}^2 \right). \quad (2.33)$$

Indeed, the Iwasawa decomposition successfully reproduces the finite temperature Schwarzian action. This method of employing the Iwasawa decomposition can be applied straightforwardly to a more general case.

### 2.4 Redundancy and Isometry

In this section we discuss the issues of the redundancy in the Schwarzian action and the isometry of JT gravity within the BF formulation as a preparatory step for the analysis of the colored counterpart.

The redundancy arises from the fact that the map

$$s : g \mapsto a = g^{-1} \dot{g} \quad (2.34)$$

is not injective: gauge functions $\tilde{g} \equiv k_0 g$ with different constants $k_0 \in SL(2, \mathbb{R})$ are all mapped to the same field $a(u)$. This implies that the resulting boundary action will have the corresponding symmetry which should be considered as an equivalence. In this sense, this symmetry is “gauge” rather than “global”. In other words, if one considers the path-integral quantization of the current theory one should not take into account the modes corresponding to this symmetry as it arises in the redefinition of the integration variable $a$ to $g$. In its turn, the redundancy in the gauge function $g$ induces a redundancy of the field variable $f(u)$

---

9
or \( \theta(u) \) of the Schwarzian theory, depending on the temperature of the theory. In the zero-temperature case, using the Gauss decomposition (2.21) we can decompose \( \tilde{g} = k_0 g \) into \( \tilde{f}, \tilde{c} \) and \( \bar{c} \). For the constant \( k_0 \in SL(2, \mathbb{R}) \) given by

\[
k_0 = \begin{pmatrix} c_4 & c_3 \\ c_2 & c_1 \end{pmatrix}, \quad c_1 c_4 - c_2 c_3 = 1,
\]

we find that

\[
\tilde{f}(u) = c_1 f(u) + c_2 c_3 f(u) + c_4.
\]

(2.35)

This is the well-known \( SL(2, \mathbb{R}) \) invariance of the Schwarzian derivative which could be regarded as a redundancy (or a finite-dimensional “gauge” symmetry) rather than a physical (global) symmetry of the Schwarzian theory (2.25). In the finite-temperature case, using the Iwasawa decomposition (2.30) of \( \tilde{g} = k_0 g \) we find

\[
\tan \frac{\theta}{2} \implies \tan \frac{\vartheta}{2} = \frac{c_1 \tan \frac{\theta}{2} + c_2}{c_3 \tan \frac{\theta}{2} + c_4}.
\]

(2.36)

\[
\tan \frac{\vartheta}{2} \implies \tan \frac{\theta}{2} = \frac{c_1 \tan \frac{\vartheta}{2} + c_2}{c_3 \tan \frac{\vartheta}{2} + c_4}.
\]

(2.37)

In fact, the redundant description of the gauge connection \( A(u, r) \) expressed in terms of \( a(u) \) (2.9) is related to the isometry of the AdS\(_2\) background. The dynamical variable \( f(u) \) of the Schwarzian theory can be viewed as the Nambu-Goldstone boson resulting from the symmetry breaking of diffeomorphism by the AdS\(_2\) background, as explained in the metric-like formulation of JT gravity for the nearly-AdS\(_2\) [3]. Therefore, Schwarzian modes correspond to

\[
\frac{\text{Diffeomorphism}}{(\text{Isometry of AdS}) \times (\text{Boundary preserving diffeomorphism})},
\]

(2.38)

where we made the further quotient by (Boundary preserving diffeomorphism) which is the genuine gauge symmetry.

Figure 2.1. A diagram of transformations underlying the relation between the redundancy and the isometry. The maps (2.40) and (2.41) define \( h \) and \( k \). A commutativity in the middle rectangle yields \( m = h^{-1}kh \) defining a stability transformation, \( a = m^{-1}a k + k^{-1} \dot{k} \), cf. (2.41). The gauge group elements on the left and right parts of the diagram are transformed by left multiplications by \( k_0 \). The map \( s \) assigns connections to group elements according to (2.34).

More explicit relations can be shown by decomposing the gauge field \( A \) into

\[
A = H^{-1} \hat{A} H + H^{-1} dH,
\]

(2.39)
where \( H(r, u) = b^{-1}(r) h(u) b(r) \) is a function determined by the Goldstone boson \( h(u) \). Although the gauge symmetry of \( A \) is restricted to that of the boundary preserving diffeomorphism, that of \( \tilde{A} \) is not restricted because it can be compensated by the transformation of \( h(u) \). Using the unrestricted gauge symmetry, we can fix the flat connection \( \tilde{A} \) as \( A_{\text{AdS}} \) given in (2.8) and express \( a \) of (2.9) as

\[
a = h^{-1} a_{\text{AdS}} h + h^{-1} \dot{h}.
\]  

(2.40)

The above manifests that there is an ambiguity in \( h \), namely, \( h \cong k h \), corresponding to the isometry of the background connection \( a_{\text{AdS}} \):

\[
k^{-1} a_{\text{AdS}} k + k^{-1} \dot{k} = a_{\text{AdS}}.
\]  

(2.41)

By expressing \( a_{\text{AdS}} \) using (2.34) as

\[
a_{\text{AdS}} = g_{\text{AdS}}^{-1} \dot{g}_{\text{AdS}}, \quad g_{\text{AdS}} = e^{a_{\text{AdS}} u},
\]  

(2.42)

we can parameterize \( k(u) \) by a constant \( k_0 \in SL(2, \mathbb{R}) \) as

\[
k = g_{\text{AdS}}^{-1} k_0 g_{\text{AdS}}.
\]  

(2.43)

This form of the solution can be easily understood if one rewrites the equation (2.41) as the covariance constancy condition \( \dot{k} + [a_{\text{AdS}}, k] = 0 \) which, therefore, defines the Killing symmetries of the background geometry.\(^8\) Also, for the gauge function \( g_{\text{AdS}} \) the isometry is seen as the redundancy \( g_{\text{AdS}} \cong k_0 g_{\text{AdS}} = \tilde{g}_{\text{AdS}} \). When we express \( a = g^{-1} \dot{g} \) (2.34), the function \( g \) is related to the Goldstone boson \( h \) by \( g = g_{\text{AdS}} h \), and the ambiguity \( h \cong k h \) due to the isometry is mapped to the left invariance \( g \cong k_0 g = \tilde{g} \). This consideration can be summarized in terms of a commutative diagram of maps depicted on Fig. 2.1.

### 3 Boundary Effective Action of the Colored JT Gravity

Let us build the color generalization of the \( sl(2, \mathbb{R}) \) BF theory for the nearly-AdS\(_2\) which we have reviewed in the previous section. Noting the group isomorphism \( SL(2, \mathbb{R}) \cong SU(1, 1) \) we can consider \( su(N, N) \) BF theory for the colored JT gravity. In [32] we developed the \( su(N, N) \) BF formulation and its relation to the second-order formulation of the colored JT gravity. In what follows we study how to reduce the colored JT gravity to the color-decorated Schwarzian theory along the lines of the standard \( SL(2, \mathbb{R}) \) case.

To implement the reduction in the \( su(N, N) \) case it is convenient to work in a representation isomorphic to the fundamental one which at \( N = 1 \) reduces to the fundamental representation of \( SL(2, \mathbb{R}) \). Using the fundamental representation the isomorphism

\(^8\)On the other hand, recall that the dilaton \( \Phi \) satisfies the covariant constancy condition which can be viewed as the gauge transformation preserving the form of the connection \( A \). Then, using the boundary condition (2.4) and the form of the gauge fixed component \( A_u \) of the background connection (2.9) one readily obtains the above Killing equation.
$SL(2, \mathbb{R}) \cong SU(1, 1)$ can be achieved by the unitary similarity transformation,

$$A \in SL(2, \mathbb{R}) \leftrightarrow B = U^{-1}AU \in SU(1, 1), \quad U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \quad (3.1)$$

The similar trick can be used in the $SU(N, N)$ case. To this end, we recall the standard matrix parameterization

$$B \in SU(N, N): \quad B^\dagger \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \det B = 1, \quad (3.2)$$

where $I$ is $N \times N$ unit matrix. Then, the similarity transformation

$$A = UBU^{-1}, \quad U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} I & -iI \\ -iI & I \end{pmatrix}, \quad U^\dagger = U^{-1}, \quad (3.3)$$

yields the following matrix representation:

$$A \in SU(N, N): \quad A^\dagger \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \det A = 1, \quad (3.4)$$

which is isomorphic to the fundamental one by construction. Such a representation can be viewed as a color extension of $SL(2, \mathbb{R})$ understood as replacing entries of $SL(2, \mathbb{R}) 2 \times 2$ matrix with $N \times N$ blocks representing colors. An infinitesimal expansion $A = I_{2N} + \epsilon X + O(\epsilon^2)$, where $I_{2N}$ is $2N \times 2N$ unit matrix, provides the corresponding representation of the algebra $su(N, N)$ as

$$\text{tr} X = 0, \quad X^\dagger = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} X \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \quad (3.5)$$

Any $X$ satisfying the condition (3.5) can be expressed as

$$X = \begin{pmatrix} -\beta + i\delta & -\epsilon \\ \varphi & \beta + i\delta \end{pmatrix}, \quad (3.6)$$

where $\beta, \delta, \epsilon, \varphi$ are Hermitian matrices and $\text{tr} \delta = 0$. Exponentiations of the generators corresponding to $\beta, \delta, \epsilon, \varphi$ can be given as, respectively,

$$\begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix}, \quad \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} I & -\epsilon \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ f & I \end{pmatrix}, \quad (3.7)$$

where matrices $b, d, f$ are Hermitian, whereas $d \in SU(N)$.

Yet another useful parameterization of the algebra $su(N, N)$ is given by the following decomposition

$$su(N, N) \cong [su(1, 1) \otimes I] \oplus [I \otimes su(N)] \oplus [su(1, 1) \otimes su(N)], \quad (3.8)$$
where $I$ is $2 \times 2$ identity matrix. An arbitrary element $M \in su(N, N)$ can be represented as

$$M = L^m (L_m \otimes I) + i J^A (I \otimes T_A) + K^{m, A} (L_m \otimes T_A),$$  \hspace{1cm} (3.9)$$

where $L^m$, $J^A$, $K^{m, A}$, with indices $m = 0, \pm 1$ and $A = 1, \ldots, N^2 - 1$, are real parameters, while matrices $(I, L_m)$ are $u(1, 1)$ basis elements (2.2), and Hermitian $N \times N$ matrices $(I, T_A)$ are $u(N)$ basis elements (for more details see, e.g. [32, 35]). The latter satisfy the product relation

$$T_A T_B = \frac{1}{N} \delta_{AB} I + \left( g_{ABC}^C + i f_{ABC}^C \right) T_C,$$  \hspace{1cm} (3.10)$$

where $g_{ABC}$ and $f_{ABC}$ are totally (anti-)symmetric real constants. The basis elements of $su(N, N)$ read

$$G_m = L_m \otimes I, \quad G_A = I \otimes T_A, \quad G_{m,A} = L_m \otimes T_A,$$  \hspace{1cm} (3.11)$$

where $G_m$ and $G_A$ form the subalgebras $sl(2, \mathbb{R})$ and $su(N)$, respectively.

### 3.1 Colored Schwarzian Action at Zero Temperature

As a colored extension of the $su(1, 1)$ BF theory (2.1), let us consider the $su(N, N)$ BF theory defined by the action

$$S_{BF} = \kappa \frac{1}{N} \int_{M_2} \text{tr} (\Phi F),$$  \hspace{1cm} (3.12)$$

where $F = dA + A \wedge A$ is the curvature 2-form, the connection 1-form $A$ and the dilaton 0-form $\Phi$ now take values in $su(N, N)$, the trace tr stands for the invariant quadratic form of the gauge algebra.

Using (3.9) and (3.11) the BF $p$-forms $\Upsilon := (\Phi, A, F)$ can be represented as

$$\Upsilon = \Upsilon^m G_m + i \Upsilon^A G_A + \Upsilon^{m, A} G_{m,A}.$$  \hspace{1cm} (3.13)$$

The component expansion (3.13) defines the spectrum of the colored JT gravity [32]:

- $\Upsilon^m$ are in the adjoint of $sl(2, \mathbb{R}) \approx su(1, 1) \subset su(N, N)$ subalgebra. This color-singlet sector describes JT dilatonic gravity.

- $\Upsilon^A$ are in the $su(N) \subset su(N, N)$ adjoint representation. This is the $su(N)$ BF sector.

- $\Upsilon^{m, A}$ are in the tensor product of $sl(2, \mathbb{R})$ and $su(N)$ adjoint representations. They form $su(N)$ adjoint multiplet of colored gravitons and dilatons.

In total, there are JT graviton and dilaton, $(N^2 - 1)$ colored gravitons and $(N^2 - 1)$ colored dilatons, $su(N)$ gauge fields and matter fields forming the $su(N)$ adjoint multiplet. We refer to them as (colored) “spin-2” and “spin-1” modes, respectively.
Gauge and asymptotic conditions. As in Section 2, we can choose a gauge

$$A(r, u) = b^{-1}(r) a(u) du b(r) + b^{-1} db(r), \quad b(r) = e^{r L_0 \otimes I}. \quad (3.14)$$

The difference with the $sl(2, \mathbb{R})$ case (2.9) is in the form of the gauge function $b(r)$ which is now exponentiation of the $sl(2, \mathbb{R})$ generator $G_0 = L_0 \otimes I$ which is trivially color-extended $sl(2, \mathbb{R})$ generator $L_0 \cdot (3.11)$. Noting that the theory admits a consistent truncation to the pure gravitational sector $sl(2, \mathbb{R}) \subset su(N,N)$ the global AdS$_2$ solution is given by the connection

$$a_{\text{AdS}} = (L_1 + L_0 L_{-1}) \otimes I. \quad (3.15)$$

It can be interpreted as describing the background geometry with all other fields of the theory vanishing. Using the representation (3.9) the asymptotic AdS$_2$ condition (2.7) can be translated into the following form of the connection $a(u)$ after we fix a suitable residual gauge symmetry,

$$a(u) = (L_1 \otimes I) + \mathcal{L}(u) (L_{-1} \otimes I) + i \mathcal{J}_A(u) (I \otimes T^A) + \mathcal{K}_A(u) (L_{-1} \otimes T^A)$$

$$= L_1 \otimes I + i I \otimes \mathcal{J}(u) + L_{-1} \otimes \mathcal{L}(u), \quad (3.16)$$

where we defined quantities

$$\mathcal{J}(u) = \mathcal{J}_A(u) T^A, \quad \mathcal{L}(u) = \mathcal{L}(u) I + \mathcal{K}_A(u) T^A, \quad (3.17)$$

which are traceless and traceful $N \times N$ Hermitian matrices, respectively. It follows that in the representation (3.6) the connection $a(u)$ takes the form

$$a(u) = \begin{pmatrix} i \mathcal{J} - \mathcal{L} \\ \mathcal{L} \end{pmatrix}. \quad (3.18)$$

Now, we want to identify a group element $g(u) \in SU(N,N)$ such that $a(u) = g^{-1}(u) \dot{g}(u)$ (2.34). Using the representation (3.7) the element $g(u)$ can be represented through the Gauss-like decomposition as$^9$

$$g(u) = \begin{pmatrix} I & 0 \\ f & I \end{pmatrix} \begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} I & -e \\ 0 & I \end{pmatrix}, \quad (3.19)$$

where matrices $b, e, f$ are Hermitian and $d \in SU(N)$. Then, we find that

$$a(u) = g^{-1} \dot{g} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad (3.20)$$

$^9$Note that here we do not use the Gauss decomposition $LDU$, where $D$ is the diagonal matrix, and $L$ and $U$ are the nilpotent subgroups corresponding to the lower and upper triangular matrix, respectively.
with
\[ p = d^{-1} \dot{d} - d^{-1} \dot{b}b^{-1}d + e d^{-1} b^{-1} f b^{-1} d, \]
\[ q = -\dot{e} - [d^{-1} \dot{d}, e] + d^{-1} \dot{b}b^{-1}d e + e d^{-1} b^{-1} b d - e d^{-1} b^{-1} f b^{-1} d e, \]
\[ r = d^{-1} b^{-1} \dot{f} b^{-1} d, \]
\[ s = d^{-1} \dot{d} + d^{-1} b^{-1} \dot{b} d - d^{-1} b^{-1} \dot{f} b^{-1} d e. \]

By equating (3.20) with (3.18) after some algebra we obtain the relations
\[ \dot{f} = b^2, \tag{3.22} \]
\[ d^{-1} \dot{b}d + d^{-1} \dot{d} = i \mathcal{J} + e, \tag{3.23} \]
\[ -d^{-1} \dot{b}b^{-1}d + d^{-1} \dot{d} = i \mathcal{J} - e, \tag{3.24} \]
\[ \dot{e} - e^2 + i [\mathcal{J}, e] = \mathcal{L}. \tag{3.25} \]

Having in total four matrix differential equations for six independent matrix functions we find out that \( b, e, \mathcal{J}, \mathcal{L} \) can be algebraically expressed in terms of \( d, f \) and their derivatives. What matters for us are the following traced quantities,
\[ \text{tr} (\mathcal{L}) = \frac{d}{du} \text{tr} e - \text{tr} (e^2) = \text{tr} \left[ \frac{d}{du} (b^{-1}b) - \left( \frac{b^{-1}b + bb^{-1}}{2} \right)^2 \right], \tag{3.26} \]
and
\[ \text{tr} (\mathcal{J}^2) = -\text{tr} \left[ (d^{-1} \dot{d})^2 + \dot{d} d^{-1} \left( b^{-1}b - bb^{-1} \right) + \left( \frac{b^{-1}b - bb^{-1}}{2} \right)^2 \right], \tag{3.27} \]

which directly follow from equations (3.23)-(3.25).

**Identifications via cyclic groups.** So far we have considered \( SU(N,N) \) as the color generalization of \( SL(2,\mathbb{R}) \cong SU(1,1) \), but the theory depends, in fact, only on the Lie algebra structure \( su(N,N) \) and hence \( PSU(N,N) \) will be sufficient because there are no matter fields in addition to the connection which may transform non-trivially under the center group \( Z[SU(N,N)] \). If we consider \( SU(N,N) \) instead of \( PSU(N,N) \), the action will acquire an integer factor corresponding to the order of the center group, which can be simply absorbed into the coupling constant \( \kappa \). Remark that \( Z[SU(N,N)] = \{ e^{2\pi i \frac{n}{2N}} I_{2N} \mid n = 0, \ldots, 2N - 1 \} \cong \mathbb{Z}_{2N} \) and, hence, \( PSU(N,N) = SU(N,N)/\mathbb{Z}_{2N} \). A few comments are in order.

- Since \( \mathbb{Z}_2 \subset \mathbb{Z}_{2N} \), we can first factor out \( \mathbb{Z}_2 = \{ \pm I_{2N} \} \) by disregarding the overall sign in \( b \) because \( g \cong -g \) in this case and \( b \) enters the decomposition (3.19) only through the block-diagonal second factor. This resolves the sign ambiguity in the solutions of \( b^2 = \dot{f} \) (3.22). However, there still remain multiple solutions for \( b \) even after factoring out \( \mathbb{Z}_2 \) and, hence, there are multiple branches of solutions of this matrix equation.\(^{10}\)

\(^{10}\)Suppose that the matrix \( \dot{f} \) is diagonalizable with eigenvalues \( \lambda_1, \ldots, \lambda_N \). Then \( b \) is a diagonal matrix with entries \( \pm \sqrt{\lambda_i} \) up to an overall sign. Therefore, we find as many solutions as possible sign choices. The number depends on the degeneracy: for the maximally degenerated case there are \( [(N + 1)/2] \) possibilities.
• After the initial $\mathbb{Z}_2$ factorization, we still have $\mathbb{Z}_N = \{ e^{2\pi i k/N} I_{2N} \mid k = 0, \ldots, N-1 \}$ because $\mathbb{Z}_2$ naturally acts on $\mathbb{Z}_{2N}$ so that $\mathbb{Z}_N = \mathbb{Z}_{2N}/\mathbb{Z}_2$. Obviously, these $\mathbb{Z}_N$ phase factors are not associated with the different solution branches of the equation $b^2 = \dot{f}$.

• Instead, the $\mathbb{Z}_N$ can be quotiented rather from the $SU(N)$ element $d$ which enters the decomposition (3.19) only through the block-diagonal third factor. The $\mathbb{Z}_N$ quotient will lead to $PSU(N)$ instead of $SU(N)$ for the spin-1 part of the BF theory.

**Boundary effective action.** As in Section 2, the bulk BF action (3.12) vanishes with the above asymptotic AdS$_2$ solutions (3.14) and (3.15). Hence, the effective action for the asymptotic AdS$_2$ solutions comes from the boundary term analogous to (2.3),

$$S_{\text{bdy}} = -\frac{\kappa}{2N} \int_{\partial M_2} \text{tr} (\Phi A),$$

(3.28)

together with the boundary condition (2.4). Using (3.14) and (3.18) along with the relations (3.26) and (3.27) we obtain the boundary effective action for the colored JT gravity as

$$S_{\text{tot}} = \frac{\kappa \gamma}{2N} \int_{\partial M_2} du \text{tr} (A_2^2) = -\frac{\kappa \gamma}{N} \int_{\partial M_2} du \text{tr} (\mathcal{L} + \mathcal{J}^2),$$

(3.29)

where $\text{tr} (\mathcal{L} + \mathcal{J}^2)$ is given by

$$\text{tr} (\mathcal{L} + \mathcal{J}^2) = \text{tr} \left[ b^{-1} \dot{b} - 2 (b^{-1} \dot{b})^2 - (d^{-1} \dot{d})^2 - d^{-1} \dot{d} - \left( b^{-1} \dot{b} - b^{-1} \dot{b} b^{-1} \right) \right].$$

(3.30)

Note that the boundary action (3.29) is a functional of $b$ and $d$, while $b^2$ itself is given by $f$ in (3.22). A few comments are in order.

Comparing the above result with the usual Schwarzian action (2.24), we find that the variable $b$ in (2.24) is generalized to the $N \times N$ Hermitian matrix $b$ where the trace part can be viewed as the “JT graviton” whereas the traceless parts are the colored spin-2 modes.

On the other hand, the group element $d \in (P)SU(N)$ can be viewed as a variable of the boundary reduced $(P)SU(N)$ BF theory. Indeed, turning off the gravitational part by setting $b = I$ we end up with a particle Lagrangian

$$\text{tr} (\mathcal{J}^2) = - \left( d^{-1} \dot{d} \right)^2,$$

(3.31)

treated as that of 1d non-linear $\sigma$-model on the $(P)SU(N)$ group manifold.

The above natural identifications generated by the cyclic subgroups, we find a novel subtlety: as opposed to the equation $b^2 = \dot{f}$ (2.23), the matrix equation $b^2 = \dot{f}$ (3.22) is not direct to solve. Nonetheless, it turns out that the boundary action (3.29) which is now given by the only contribution $\text{tr} (\mathcal{L})$ can be cast into a simple form in terms of $f$,

$$\text{tr} (\mathcal{L}) = \frac{1}{2} \text{tr} \left[ \dot{f}^{-1} \ddot{f} - \frac{3}{2} (\dot{f}^{-1} \dot{f})^2 \right].$$

(3.32)
This is the color generalization of the Schwarzian derivative for JT graviton and colored spin-2 modes. A few remarks are in order.

First, the matrix Schwarzian derivative in Eq. (3.32) was studied for $Sp(2N, R)$ case in Refs. [45, 46] as the higher-dimensional analogue of the Schwarzian derivative\(^{11}\). In this work, the $SU(N, N)$ Schwarzian (3.32) has been obtained from the colored JT gravity. Accordingly, the invariance of the matrix Schwarzian action under a $SU(N, N)$ transformation, which was proven in Refs. [45, 46] for the $Sp(2N, R)$ case, naturally follows from the physical condition (the isometry of the $AdS_2$ background) of the colored JT gravity (e.g. see Section 3.2). Also note that the Schwarzian derivatives for higher-rank groups have been considered perviously in the literature, see e.g. [47, 48] for $SL(N, R)$ groups in the context of $W$-gravity.

Second, we may consider a restriction of the $su(N, N)$ gauge algebra of BF theory to a certain subalgebra by reducing the spin-1 gauge algebra, according to the construction\(^{49, 50}\). For instance, if we consider the $so(N)$ color instead of $su(N)$, the $su(N, N)$ gauge algebra will be replaced by $sp(2N, R)$ with symmetric matrix $f$ and $d \in SO(N)$. For even $N$, we can also consider the $usp(N/2)$ color which requires the $so(N, N)$ gauge algebra with antisymmetric matrix $f$ and $d \in USp(N/2)$.

Let us now consider small fluctuations described by the Schwarzian (3.32). Indeed, when the traceless part of $f$ is sufficient small, we can do a perturbative analysis.\(^{12}\) We separate the trace and traceless parts as

$$f = f I + k, \quad \text{tr} k = 0,$$

and solve the relation $b^2 = \dot{f}$ perturbatively in $k$. In the end, the action can be expanded in the powers of $k$ or equivalently in the powers of $\alpha = \dot{f}^{-1} \dot{k}$. Using

$$b^{-1} \dot{b} - \dot{b} b^{-1} = -\frac{1}{4} [\alpha, \dot{\alpha}] + O(\alpha^3),$$

$$b^{-1} \dot{b} + \dot{b} b^{-1} = \dot{f}^{-1} \dot{f} + \dot{\alpha} - \frac{1}{4} \{\alpha, \dot{\alpha}\} + O(\alpha^3),$$

we find

$$-\text{tr} (\mathcal{L} + \mathcal{J}^2) = -\frac{N}{2} \left( \dot{f}^{-1} \dot{f} - \frac{3}{2} \left( \dot{f}^{-1} \dot{f} \right)^2 \right) + \frac{1}{2} \text{tr} \left( \alpha \dot{\alpha} + \frac{3}{2} \alpha^2 \right) + \text{tr} \left( d^{-1} \dot{d} \right)^2$$

$$- \frac{1}{4} \dot{f}^{-1} \dot{f} \text{tr} (\alpha \dot{\alpha}) - \frac{1}{4} \text{tr} \left( \dot{d} d^{-1} [\alpha, \dot{\alpha}] \right) + O(\alpha^3).$$

The first line consists of the Schwarzian action of the singlet graviton (2.24)-(2.25), a quadratic action of $\alpha$ and the action for a particle on the group manifold $(P)SU(N)$ corresponding to the spin-1 mode (3.31). The second term contains the action of free colored gravitons. Indeed, one can check that the kinematic nature of the singlet graviton $f$ and the colored gravitons

\(^{11}\)We thank the referee for pointing this out.

\(^{12}\)Here, we focus the branch connected to the solution with $b = I$. Note that there exist multiple branches satisfying $f = b^2.$
\( k \) are the same: they have the same kinetic terms up to total derivatives,

\[
- \left( \dot{j}^{-1} \ddot{j} - \frac{3}{2} (j^{-1} \dot{j})^2 \right) = -\dddot{h} + \left( \dddot{h}^2 + \frac{3}{2} \dddot{h}^2 \right) + \mathcal{O}(h^3),
\]

and

\[
\text{tr} \left( \alpha \dddot{\alpha} + \frac{3}{2} \dddot{\alpha}^2 \right) = \text{tr} \left( \dddot{k}^2 + \frac{3}{2} \dddot{k}^2 \right) + \mathcal{O}(k^2 h). \tag{3.38}
\]

Therefore, we can view the first line in (3.36) as the part of action free of mutual interactions, up to the \( \dot{f} \) dressing in \( \alpha \). Note that the action for the spin-1 mode is negative-definite, and hence it has “wrong” overall sign. This leads to the instability of the colored JT gravity which was also observed in the 3D colored gravity [33–35]. We will discuss the consequence of this “wrong” sign in Section 4. The second line in (3.36) describes the mutual interactions among the singlet graviton, the colored gravitons, and the spin-1 mode.

### 3.2 Colored Isometry and Gauge Symmetry

In Section 2.4 we have discussed the \( SL(2, \mathbb{R}) \) redundancy of \( g \) in \( a = g^{-1} \dot{g} \) (2.34). In the color extended case, we find the same kind of redundancy: \( a = g^{-1} \dot{g} = \tilde{g}^{-1} \dot{\tilde{g}} \), where \( \tilde{g} = k_0 g \) with a constant \( k_0 \in SU(N, N) \). Note that \( k_0 \) in the basis (3.4) is given by

\[
k_0 = \begin{pmatrix} D & C \\ B & A \end{pmatrix}; \quad k_0^\dagger \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} k_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{3.39}
\]

that can be rewritten as

\[
AC^\dagger = A^\dagger C, \quad BD^\dagger = DB^\dagger, \quad AD^\dagger - B^\dagger C = I. \tag{3.40}
\]

Similar to the \( SL(2, \mathbb{R}) \) case we need to mod out such a \( SU(N, N) \) redundancy. To this end, using the decomposition (3.19) for \( g \) and \( \tilde{g} \) related as \( \tilde{g} = k_0 g \) we find\(^{13} \)

\[
\tilde{f} = (Af + B)(Cf + D)^{-1}, \tag{3.41}
\]

\[
\tilde{d}^{-1} \tilde{b} = d^{-1} b (Cf + D)^{-1}. \tag{3.42}
\]

Let us note that one can find only the implicit expression for the transformation of \( b \) and \( d \). It is noteworthy that \( \tilde{f} \) is solely given by \( f \) through a form which generalizes the \( SL(2, \mathbb{R}) \) transformation of the Schwarzian mode (2.36). A common constant factor in matrices \( A, B, C, D \) – which is the \( \mathbb{Z}_{2N} \) phase \( e^{i \frac{2\pi n}{2N}} \) – does not affect the transformation of \( f \) (3.41), but not those of \( b \) and \( d \). Thus, the \( \mathbb{Z}_{2N} \) phase factor can be quotiented from \( b \) and \( d \).

Recall that the classical solutions of the Schwarzian action are connected to one another by the isometry. Since one should mod out the isometry, it is enough to choose the simplest saddle classical solution \( f(u) = u \). In the colored case, we shall also consider a series of

\(^{13}\)These matrix transformations describe a color generalization of the modular transformations \( SL(2, \mathbb{Z}) \), for example, of the elliptic genus.
classical solutions of the boundary effective action (3.29). Using the isometry (3.41) and (3.42) we can find a simplest solution which is connected to others.

Let us introduce the following class of saddle classical solutions of the boundary action (3.29):

\[ f = \mathcal{M} u, \quad d = \text{constant}, \quad \text{where} \quad \mathcal{M} = \text{constant}. \tag{3.43} \]

Then, we consider transformations which map solutions (3.43) to each other. Using the \( SU(N, N) \) defining conditions (3.40) we find that these are given by

\[ A = (D^\dagger)^{-1}, \quad \forall D: \quad \det D \neq 0 \quad \text{and} \quad B = C = 0. \tag{3.44} \]

Then, consider the transformations (3.41) and (3.42) in this particular case. By virtue of (3.44) \( f \) transforms as

\[ \tilde{f} = A f A^\dagger, \tag{3.45} \]

that allows us to diagonalize Hermitian matrix \( f \) and, hence, \( \dot{f} \) by choosing \( A \) to be unitary. Indeed, due to the equation \( \dot{f} = b^2 \) (3.22), where \( b \) is Hermitian (3.19), the eigenvalues of Hermitian matrix \( \dot{f} \) should be positive and can be normalized again using (3.45). In this way, we can set \( f = I u \) so that its stabiliser group is \( A \in U(N) \).

As for the transformation of \( d \), by virtue of (3.42) it is given implicitly through

\[ \tilde{b} \tilde{d} = A b d, \quad \tilde{b}^{-1} \tilde{d} = (A^\dagger)^{-1} b^{-1} d. \tag{3.46} \]

Restricting to the solution \( f = I u \) (hence, \( b^2 = I \)) and its stabiliser group \( A \in U(N) \), we find

\[ \tilde{d} = \tilde{A} d, \tag{3.47} \]

where \( \tilde{A} = \tilde{b} A b \in U(N) \). Since \( d \in SU(N) \), we can set \( d = I \) by choosing an appropriate \( \tilde{A} \). With the remaining freedom \( A \) in \( \tilde{b} = A b = b A^\dagger \) (3.46), we can also fix \( b = I \).

Finally, we find that a class of solutions (3.43) defined by the gauge function

\[ g(u) = \begin{pmatrix} I & 0 \\ u & I \end{pmatrix} = \exp \left[ a_{\text{AdS}} u \right], \quad a_{\text{AdS}} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \tag{3.48} \]

where \( a_{\text{AdS}} \) corresponds to the AdS\(_2\) background (set \( \mathcal{L}_0 = 0 \) in (3.15)).

### 3.3 Colored Schwarzian Action at Finite Temperature

As we have discussed in Section 2.3.2, the finite temperature JT gravity could be obtained by employing the Iwasawa decomposition which leads to a coordinate chart, where non-contractible cycle can be parameterized by a simple periodic boundary condition. Note that there is no simple expression which relate the Iwasawa decomposition to the Gauss decomposition for a general Lie group.\(^{14}\) In Section 3.1, we have used a Gauss-like decomposition

\(^{14}G = KAN, \) where \( K \) is the maximal compact subgroup, \( A \) and \( N \) are the Abelian and the nilpotent subgroups, respectively.
(3.19) for the zero temperature case. Therefore, in order to implement a finite temperature extension of the colored JT gravity we use an Iwasawa-like decomposition of $SU(N,N)$ which minimally modifies the decomposition (3.19) to incorporate the maximal compact subgroup $K \subset SU(N,N)$.

It is known that $K \cong U(1) \otimes SU(N) \otimes SU(N)$ which can be parametrized as follows

$$K = \begin{pmatrix} e^{i \theta} d_1 & 0 \\ 0 & e^{-i \theta} d_2 \end{pmatrix} = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}. \quad (3.49)$$

Here, $d_1, d_2 \in SU(N)$ and $\theta \in \mathbb{R}$ parameterizes $U(1)$ factor, while $\xi \in U(N)$ and $d \in SU(N)$. The unitary transformation (3.3) maps the matrix (3.49) to

$$Y = U K U^{-1} = \begin{pmatrix} \frac{\xi + \xi^{-1}}{2} & \frac{\xi - \xi^{-1}}{2} \\ \frac{\xi - \xi^{-1}}{2} & \frac{\xi + \xi^{-1}}{2} \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}. \quad (3.50)$$

Hence, we introduce the following Iwasawa-like decomposition of the group element $g(u) \in SU(N,N)$:

$$g(u) = \begin{pmatrix} \frac{\xi + \xi^{-1}}{2} & \frac{\xi - \xi^{-1}}{2} \\ \frac{\xi - \xi^{-1}}{2} & \frac{\xi + \xi^{-1}}{2} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} I & -e \\ e & 0 \end{pmatrix}. \quad (3.51)$$

As in Section 3.1, going from $SU(N,N)$ to $PSU(N,N)$ we impose $b \equiv -b$ and $d \in PSU(N)$. This time, we find that

$$a(u) = g^{-1} \hat{g} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad (3.52)$$

with

$$p = d^{-1} d - d^{-1} b b^{-1} d + d^{-1} b \Gamma b^{-1} d + e d^{-1} b^{-1} \theta b^{-1} d,$$

$$q = -\dot{e} - [d^{-1} \dot{d}, e] + d^{-1} \dot{b} b^{-1} d e + e d^{-1} b^{-1} \dot{b} d + e d^{-1} b^{-1} \theta b^{-1} d e,$$

$$r = d^{-1} b^{-1} \theta b^{-1} d,$$

$$s = d^{-1} \dot{d} + d^{-1} b^{-1} b d + d^{-1} b^{-1} \Gamma b d - d^{-1} b^{-1} \theta b^{-1} d e,$$

cf. (3.19)–(3.21). Here, $\theta$ and $\Gamma$ are defined in terms of $\xi \in U(N)$ as

$$\theta = \frac{\xi^{-1} \dot{\xi} + \dot{\xi} \xi^{-1}}{2i}, \quad \Gamma = \frac{\xi^{-1} \dot{\xi} - \dot{\xi} \xi^{-1}}{2}. \quad (3.54)$$

The asymptotic AdS$_2$ condition (3.18) gives

$$\theta = b^2, \quad (3.55)$$

$$d^{-1} \dot{d} + d^{-1} b^{-1} \partial_L b d = i \mathcal{J} + e, \quad (3.56)$$

$$d^{-1} \dot{d} - d^{-1} \partial_R b b^{-1} d = i \mathcal{J} - e, \quad (3.57)$$

$$\dot{e} - e^2 + i [\mathcal{J}, e] + d^{-1} \theta^2 d = \mathcal{L}, \quad (3.58)$$
where \( \partial_L b \equiv \dot{b} + \Gamma b \) and \( \partial_R b \equiv \dot{b} - b \Gamma \). From (3.55)–(3.58) we find the trace of \( \mathcal{L} \) and \( \mathcal{J}^2 \) as

\[
\text{tr} (\mathcal{L}) = \text{tr} \left[ \frac{d}{du} (b^{-1} \dot{b}) - \left( \frac{b^{-1} \partial_L b + \partial_R b b^{-1}}{2} \right)^2 - b^4 \right],
\]

\[
\text{tr} (\mathcal{J}^2) = - \text{tr} \left[ (\dot{d}^{-1} \dot{d})^2 + \dot{d} \dot{d}^{-1} (b^{-1} \partial_L b - \partial_R b b^{-1}) + \left( \frac{b^{-1} \partial_L b - \partial_R b b^{-1}}{2} \right)^2 \right].
\]

Combining these two terms we finally obtain the boundary effective action for the colored JT gravity at the finite temperature

\[
S_{\text{tot}} = - \frac{\kappa \gamma}{N} \int du \text{tr} (\mathcal{L} + \mathcal{J}^2)
= - \frac{\kappa \gamma}{N} \int du \left[ b^{-1} \dot{b} - 2(b^{-1} \dot{b})^2 - b^4 - \Gamma^2 + \Gamma \left( b^{-1} \dot{b} - \dot{b} b^{-1} \right) \right. \]
\[
\left. - (\dot{d}^{-1} \dot{d})^2 - \dot{d} \dot{d}^{-1} (b^{-1} \partial_L b - \partial_R b b^{-1}) \right].
\]

Let us compare this result to that in the colored gravity at zero temperature (3.29) as well as in the uncolored gravity at finite temperature (2.33). First, note that the role of the variable \( f \) in the zero temperature case is now played by the variable \( \xi \). The former is \( N \times N \) Hermitian matrix whereas the latter belongs to \( U(N) \). In a sense, we see a compactification of \( N^2 \) dimensional space by introducing a finite temperature. The equation \( \dot{f} = b^2 (3.22) \) is replaced by \( \dot{\xi} = \frac{\xi^{-1} \dot{\xi} + \dot{\xi} \xi^{-1}}{2i} = b^2 (3.54)-(3.55) \), and in the \( N = 1 \) case the identification \( \xi = e^{i \theta / 2} \) reproduces the equation (2.32) (up to rescaling) which leads to the (uncolored) Schwarzian action at finite temperature (2.33). The effective action (3.61) contains also the typical additional term \( b^4 \) of the finite temperature Schwarzian which is analogous to the term \( \frac{1}{2} \dot{\theta}^2 \) in (2.33), but there are also a few other deviations from the zero temperature case, which are proportional to \( \Gamma \).

If we re-introduce a variable \( f \) by parameterizing \( \xi = \exp(i f) \), then \( \theta \) and \( \Gamma \) (3.54) can be represented as\(^\text{15}\)

\[
\theta = f + \mathcal{O}(f^2), \quad \Gamma = \frac{1}{2} [f, f] + \mathcal{O}(f^3).
\]

(3.62)

This form implies that the terms proportional to \( \Gamma \) in (3.61) can be viewed as self-couplings of the boundary colored spin-2 modes. Isolating the singlet graviton by virtue of the decomposition (3.33) we see that it falls out of the leading contribution in \( \Gamma \). Nonetheless, it still contributes to the \( b^4 \) term among additional terms in the finite temperature case.

The isometry which needs to be modded out in the boundary action at finite temperature can be obtained by repeating the same calculation as in Section 3.2 using the Iwasawa-like decomposition (3.51). We find that

\[
\mathbb{T}N (\tilde{\xi}) = (A \mathbb{T}N (\xi) + B) (C \mathbb{T}N (\xi) + D)^{-1},
\]

(3.63)

\(^\text{15}\)Note that the leading terms here are similar to the momentum and the \( SU(N) \) angular momentum of the free matrix quantum mechanics of \( f \).
\[
\bar{d}^{-1} b \, SC(\xi) = d^{-1} b \, SC(\xi) \left[ C \, TN(\xi) + D \right]^{-1},
\]
(3.64)

where the tangent-like \( TN(\xi) \) and secant-like function \( SC(\xi) \) of the matrix \( \xi \) are defined by
\[
TN(\xi) \equiv \frac{1}{2i} (\xi - \xi^{-1}) \left[ \frac{1}{2} (\xi + \xi^{-1}) \right]^{-1}, \quad SC(\xi) \equiv \left[ \frac{1}{2} (\xi' + \xi'^{-1}) \right]^{-1}.
\]
(3.65)

The isometry (3.63), (3.64) generalizes the respective transformations (2.37) in the \( SL(2, \mathbb{R}) \) finite temperature case as well as transformations (3.41) and (3.42) in the \( SU(N,N) \) zero temperature case.

### 3.4 Holonomy for \( SU(N,N) \) Connection

In Section 3.2 we have considered a class of classical solutions of the boundary effective action which is related by the isometry to the AdS\(_2\) background at zero temperature (3.48). At finite temperature, we can also consider the following simple classical solutions of the boundary effective action (3.61):
\[
b = b_0, \quad \theta = b_0^2, \quad d = I, \quad \Gamma = e = 0,
\]
(3.66)

where \( b_0 \) is a constant \( N \times N \) Hermitian matrix. Using the relations (3.53) one can see that this solution corresponds to the constant connection \( a_0 \) given by
\[
a_0 = g^{-1} \dot{g} = \begin{pmatrix} 0 & -L_0 \\ I & 0 \end{pmatrix}, \quad \text{where} \quad L_0 \equiv b_0^4.
\]
(3.67)

However, for a given temperature \( T = \beta^{-1} = (2\pi)^{-1} \), not all the constant \( L_0 \) can be obtained from a single-valued gauge function as in Section 2, and this issue is captured by the holonomy of the gauge field \( A \) along the thermal circle,
\[
\text{Hol}(A) = P \exp \left[ \oint A \right] \sim \exp (2\pi a_0),
\]
(3.68)

where \( A = b^{-1}(d+a)b \), and \( a \) is a fluctuation around the constant background \( a_0 \) (3.67). Like in the \( SL(2, \mathbb{R}) \) case in Section 3.2, one can assume that a “nice geometry” in the colored gravity have a single-valued gauge function, and hence a trivial holonomy. Viewed as an \( SU(N,N) \) element, the trivial holonomy belongs to the center subgroup \( \mathbb{Z}_{2N} \) of \( SU(N,N) \):
\[
\text{Hol}(A) \sim e^{2\pi a_0} = e^{\frac{2\pi n}{N}} I_{2N} \in Z[SU(N,N)] \quad (n = 0, 1, \cdots, 2N - 1).
\]
(3.69)

Using a gauge transformation by a constant gauge parameter
\[
\begin{pmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{pmatrix} a_0 \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} 0 & -U^{-1} L_0 U \\ I & 0 \end{pmatrix}, \quad \text{where} \quad U \in U(N) : \text{constant},
\]
(3.70)
one can diagonalize the matrix $L_0$ as

$$L_0 = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N),$$

with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. Note that the eigenvalues of $L_0$ are non-negative because $L_0 = b_0^\dagger$ with Hermitian matrix $b_0$ (3.67). Then, the $2N$ eigenvalues of the constant $a_0$ are found to be

$$-i \sqrt{\lambda_1}, -i \sqrt{\lambda_2}, \cdots, -i \sqrt{\lambda_N}, i \sqrt{\lambda_1}, i \sqrt{\lambda_2}, \cdots, i \sqrt{\lambda_N}.$$  

Therefore, from the condition (3.69), we conclude that the holonomy $\text{Hol}(A)$ can be $\pm I_{2N}$, and the eigenvalues of the corresponding $L_0$ are given by

$$\lambda_j = \frac{\nu_j^2}{4}, \quad \text{where} \quad \nu_j \in \mathbb{Z}_+ \quad \text{and} \quad \nu_1 \leq \cdots \leq \nu_N.$$  

(3.73)

Here, all $\nu_j$'s are either even for $\text{Hol}(A) = I_{2N}$ or odd for $\text{Hol}(A) = -I_{2N}$.

The Hamiltonian of the Schwarzian theory is defined by the Schwarzian derivative up to a factor [14] and one can rewrite the Hamiltonian density in terms of the variable $a(u)$ (2.31) as

$$H = -\frac{\kappa \gamma}{2} \left( \text{Sch} \left[ f(u), u \right] + \frac{1}{2} f'(u)^2 \right) = \frac{\kappa \gamma}{2} \text{tr} (a^2).$$

(3.74)

Using this expression we define the energy of the colored gravity by

$$E \equiv \frac{\kappa \gamma}{2N} \int_0^\beta du \, \text{tr} (a^2).$$

(3.75)

Especially, the energy of the constant solution $a_0$ is

$$E_0 = \frac{\beta \kappa \gamma}{2N} \text{tr} a_0^2 = -\frac{\pi^2 \kappa \gamma}{\beta N} \sum_{j=1}^N \nu_j^2,$$

(3.76)

where we retrieved the temperature $T = \beta^{-1}$. The highest-energy smooth constant solution has $\nu_j = 1$ for all $j = 1, 2, \cdots, N$, and this is the global AdS$_2$ background with $L_0 = \frac{\gamma^2}{\beta^2} I$ (see discussion in Section 2.2). The other constant solutions have lower energy. These are analogous to the conical surplus in the 3D Chern-Simons (higher-spin) gravity [51], which is often considered as “unphysical” one. We also observed the signal of its unphysical nature in Section 4.

### 3.5 Asymptotic AdS$_2$ Symmetry

In Section 3.1 we have imposed the asymptotic AdS$_2$ condition and the gauge condition for $a(u)$ as follows (see (3.16), (3.18))

$$a(u) = L_1 \otimes I + i I \otimes \mathcal{J}(u) + L_{-1} \otimes \mathcal{L}(u) = \begin{pmatrix} i \mathcal{J}(u) - \mathcal{L}(u) \\ I \end{pmatrix}.$$  

(3.77)
In this section, we will consider the residual gauge symmetry which keeps the form of $a(u)$ (3.77) intact. This will lead to the asymptotic AdS$_2$ symmetry for the colored JT gravity. Using (3.6) the gauge parameter $h(u) \in su(N, N)$ can be represented as

$$h(u) = \begin{pmatrix} l & n \\ m & -l^\dagger \end{pmatrix}, \quad \text{where } l \equiv s + it.$$  

(3.78)

where $m, n, s, t$ are Hermitian $N \times N$ matrices. Then, an infinitesimal gauge transformation of $a(u)$ can be written as

$$\delta a(u) = \dot{h} + [a(u), h] = \begin{pmatrix} \dot{l} & \dot{n} \\ \dot{m} & -\dot{l}^\dagger \end{pmatrix} + \begin{pmatrix} -n - \mathcal{L} m + i[\mathcal{J}, l] \\ 2s + i[\mathcal{J}, m] \end{pmatrix} \begin{pmatrix} \mathcal{L} l^\dagger + l \mathcal{L} + i[\mathcal{J}, n] \\ n + m \mathcal{L} - i[\mathcal{J}, l^\dagger] \end{pmatrix}.$$  

(3.79)

Imposing the condition $\delta a(u) = 0$ we find the following constraints on the gauge parameters

$$s = -\frac{1}{2} \mathcal{D}_a m, \quad n = \mathcal{D}_a s - \frac{1}{2} \{ \mathcal{L}, m \},$$  

(3.80)

where $\mathcal{D}_a \equiv \partial_u + i[a, \mathcal{J}]$. Hence, the residual transformations can be parameterized by $m$ and $t$, and the corresponding transformations of $\mathcal{J}$ and $\mathcal{L}$ are found to be

$$\delta \mathcal{J} = \mathcal{D}_a t + \frac{i}{2} [\mathcal{L}, m], \quad \delta \mathcal{L} = \frac{1}{2} \mathcal{D}_a^3 m + \frac{1}{2} \{ \mathcal{D}_a \mathcal{L}, m \} + \{ \mathcal{L}, \mathcal{D}_a m \} + i \{ \mathcal{L}, t \}.$$  

(3.81)

To identify transformations of each component, we decompose $\mathcal{L}$ and $\mathcal{J}$ as well as the gauge parameters $m$ and $t$ in the $u(N)$ basis as follows

$$\mathcal{L} = \mathcal{L} I + \mathcal{K} = \mathcal{L} I + \mathcal{K} T_A, \quad m = \xi I + \zeta = \xi I + \zeta T_A,$$  

(3.82)

$$\mathcal{J} = \mathcal{J} T_A, \quad t = \lambda = \lambda T_A,$$  

(3.83)

(see the beginning of Section 3). The transformation of $\mathcal{J} T_A$ is found to be

$$\delta \mathcal{J} T_A = \dot{\lambda} T_A - 2f_{BC} A \mathcal{J} B \mathcal{X} A + f_{BC} A \mathcal{K}^B \mathcal{X} C,$$  

(3.84)

where $f_{ABC}$ are the $su(N)$ structure constants. In addition, the $\mathcal{L}$ is transformed as

$$\delta \mathcal{L} = \frac{1}{N} \text{tr} [\delta \mathcal{L}] = -\frac{1}{2} \hat{\lambda} \xi + \hat{\mathcal{L}} \xi + 2\mathcal{L} \hat{\xi} + \frac{1}{N} (\mathcal{D}_a \mathcal{K}) A \zeta_A + \frac{2}{N} (\mathcal{D}_a \zeta) A \mathcal{K} A,$$  

(3.85)

$$= \frac{1}{2} \hat{\xi} + \hat{\mathcal{L}} \xi + 2\mathcal{L} \hat{\xi} + \frac{1}{N} \hat{\mathcal{K}}^A \zeta_A + \frac{2}{N} \mathcal{K}^A \hat{\zeta} A + \frac{2}{N} f_{ABC} \mathcal{J} A \mathcal{K}^B \zeta C,$$  

where we used the product relation (3.10). One can also obtain the transformation

$$\delta \mathcal{K}^A = \text{tr} [\delta \mathcal{L} T^A] = \frac{1}{2} \hat{\mu}^A \zeta_A + \hat{\mathcal{L}} \zeta_A + 2\mathcal{L} \hat{\zeta} A + \hat{\mathcal{K}} \zeta_A + 2\mathcal{K} \hat{\zeta} A + \cdots.$$  

(3.86)

The transformations (3.84)-(3.86) are analogous to the asymptotic colored AdS$_2$ symmetry of the 3D colored Chern-Simons gravity [35].

– 24 –
4 Quantum Fluctuations around Classical Solutions

In Section 3.3 we have obtained the boundary effective action at finite temperature. However, it was difficult to express the action in terms of $\xi$ which is analogous to $f$ in the zero-temperature case, and, therefore, a perturbative expansion of the boundary action around the finite temperature background would not be straightforward as in the $sl(2,\mathbb{R})$ Schwarzian theory. A similar difficulty also appears in the higher-spin gravity where the closed form of the boundary effective action at finite temperature is still an open question. Nevertheless, one can work out a perturbative analysis for the case of higher-spin gravity at finite temperature [36].

For example, in the 3D Chern-Simons (higher-spin) gravity, starting from some background connection, one can perturbatively build an infinitesimal gauge transformation which keeps the asymptotic AdS condition intact. It maps the background connection to some new gauge connection parameterized by the gauge parameters which therefore describe the physical boundary modes. Then, provided the appropriate boundary condition, one can evaluate the total action on the resulting gauge connection. This leads to the boundary action expressed perturbatively in terms the boundary modes propagating on the fixed background (e.g. BTZ black hole). In the same way, we will also consider an infinitesimal gauge transformation (3.79) around the background solution found in Section 3.4 to derive perturbatively the quadratic boundary action of the colored JT gravity.

4.1 Boundary Effective Action

Recall that the finite temperature boundary action (3.61) was obtained by using the representation $a(u) = g^{-1} \dot{g}$ (2.34). As explained in Section 2.4, the connection $a(u)$ can also be parameterized by a smooth gauge transformation of the constant connection $a_0$ (3.67)\(^{16}\)

$$a_0 = \begin{pmatrix} 0 & -\mathcal{L}_0 \\ I & 0 \end{pmatrix}. \quad (4.1)$$

Therefore, an infinitesimal gauge transformation which keeps the asymptotic AdS$_2$ condition (3.77) intact can produce the perturbation of the boundary action (3.61). To this end, let us consider a smooth gauge transformation of the constant connection $a_0$ by the gauge group element $U(u) \in SU(N,N)$ which is expanded with respect to a small parameter $\epsilon$ as follows

$$U(u) = I_{2N} + \epsilon h(u) + \frac{1}{2} \epsilon^2 (k(u)^2 + k(u)) + \mathcal{O}(\epsilon^3), \quad (4.2)$$

where $I_{2N}$ is $2N \times 2N$ unit matrix. From the condition (3.4) for a $SU(N,N)$ element, we have

$$h(u) = \begin{pmatrix} s(u) + i\tau(u) & \nu(u) \\ m(u) & -s + i\tau(u) \end{pmatrix}, \quad k(u) = \begin{pmatrix} \sigma(u) + i\tau(u) & \mu(u) \\ \nu(u) & -\sigma(u) + i\tau(u) \end{pmatrix}. \quad (4.3)$$

\(^{16}\)In general, one may consider a background with non-zero $\mathcal{J}_0$ in $a_0 = \begin{pmatrix} i\mathcal{J}_0 & -\mathcal{L}_0 \\ I & i\mathcal{J}_0 \end{pmatrix}$. For simplicity, we analyze the background with $\mathcal{J}_0 = 0$. 

– 25 –
Here, matrices $m, n, s, t$ and $\mu, \nu, \sigma, \tau$ are Hermitian. The perturbative expansion of the smooth gauge transformation is found to be
\[
a(u) = U^{-1}a_0U + U^{-1}\dot{U}
= a_0 + \epsilon \left[ h + [a_0, h] \right] + \frac{1}{2} \epsilon^2 \left( k + [a_0, k] + \dot{h}h - \dot{h}h + a_0h^2 - 2ha_0h + h^2a_0 \right) + \mathcal{O}(\epsilon^3).
\] (4.4)

At order $\mathcal{O}(\epsilon)$, the asymptotic AdS$_2$ condition and the gauge condition (3.18) allow to express $s$ and $n$ in terms of $m$ as in Section 3.5,
\[
s = -\frac{1}{2} \dot{m}, \quad n = -\frac{1}{2} \ddot{m} - \frac{1}{2} \{\mathcal{L}_0, m\}. \quad \text{(4.5)}
\]

In the same way, the condition (3.18) at order $\mathcal{O}(\epsilon^2)$ gives
\[
\begin{align*}
\sigma &= -\frac{1}{2} \mu + \frac{i}{2} \{\mu, \dot{t} \} + \frac{1}{4} \{\mathcal{L}_0, m^2 \} - \frac{1}{2} \mathcal{L}_0 m, \\
\nu &= -\frac{1}{2} \mu - \frac{1}{2} \{\mathcal{L}_0, \mu \} + \frac{1}{4} \{\mathcal{L}_0, m^2 \} - \frac{1}{2} \mathcal{L}_0 m \\
&\quad - \frac{1}{4} \{m, \dot{m} \} + \frac{3}{4} \{[\mathcal{L}_0, m], \dot{m} \} - \frac{1}{2} \{m, \{m, \mathcal{L}_0 \} \} \\
&\quad + i \{m, \dot{t} \} - \frac{i}{2} \{\dot{m}, \dot{t} \} + \frac{i}{2} \{m, \{t, \mathcal{L}_0 \} \}.
\end{align*}
\] (4.6)

Expanding the connection $a$ in (4.4) yields the perturbative expansion of $\mathcal{L}$ and $\mathcal{J}$ with respect to $\epsilon$
\[
\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}^{(1)} + \epsilon^2 \mathcal{L}^{(2)} + \cdots, \quad \mathcal{J} = \epsilon \mathcal{J}^{(1)} + \epsilon^2 \mathcal{J}^{(2)} + \cdots. \quad \text{(4.8)}
\]

Similarly, the action can be expanded as follows
\[
S_{\text{tot}} = -\frac{\kappa \gamma}{N} \int du \, \text{tr} \left( \mathcal{L} + \mathcal{J}^2 \right)
= -\frac{\kappa \gamma}{N} \int du \, \left[ \mathcal{L}_0 - \epsilon \mathcal{L}^{(1)} - \epsilon^2 \left( \mathcal{L}^{(2)} + (\mathcal{J}^{(1)})^2 \right) + \mathcal{O}(\epsilon^3) \right]. \quad \text{(4.9)}
\]

The leading order term is the energy (3.75) of the constant background connection $a_0$ given by (4.1). Using the solution (4.5)–(4.7) for the asymptotic AdS$_2$ condition, we can express $\mathcal{L}^{(n)}$ and $\mathcal{J}^{(n)}$ ($n = 1, 2$) in terms of $m$ and $t$. Especially, we are interested in $\text{tr} (\mathcal{L}^{(1)})$, $\text{tr} (\mathcal{L}^{(2)})$ and $\left[ (\mathcal{J}^{(1)})^2 \right]$ in (4.9). Up to total derivatives we find
\[
\int du \, \text{tr} (\mathcal{L}^{(1)}) = 0, \quad \text{(4.10)}
\]
\[
\int du \, \text{tr} (\mathcal{L}^{(2)}) = \int du \, \text{tr} \left( -\frac{1}{4} \dot{m}^2 + \mathcal{L}_0 m^2 + \frac{1}{2} (\mathcal{L}_0 m)^2 - \frac{1}{2} \mathcal{L}_0^2 m^2 \right), \quad \text{(4.11)}
\]
\[
\int du \, \text{tr} (\mathcal{J}^{(1)})^2 = \int du \, \text{tr} \left( i^2 + i \{\mathcal{L}_0, m \} - \frac{1}{2} (\mathcal{L}_0 m)^2 + \frac{1}{2} \mathcal{L}_0^2 m^2 \right), \quad \text{(4.12)}
\]
In total, the perturbative expansion of the boundary action (3.61) up to quadratic order is found to be

\[ S_{\text{tot}} = -\frac{\kappa \gamma}{N} \int du \ tr \left( \mathcal{L} + \mathcal{J}^2 \right) = E_0 + \frac{\kappa \gamma \epsilon^2}{N} \int du \ tr \left( \frac{1}{4} \ddot{m}^2 - \mathcal{L}_0 \dot{m}^2 - t^2 - i t [\mathcal{L}_0, m] \right) + O(\epsilon^3), \]  

(4.13)

where the energy \( E_0 \) is given by (3.76). Note that in the AdS_2 background, \( \mathcal{L}_0 = \frac{\pi^2}{\beta^2} I \), the singlet graviton, colored graviton and spin-1 modes are decoupled at quadratic level. Moreover, the spin-1 mode \( t \) has a wrong-sign kinetic term that leads to instability of the colored JT gravity. This instability persists even for the other constant backgrounds \( \mathcal{L}_0 \) (see the next section).

### 4.2 Mode Expansion

As is shown in (3.73), the smooth background \( \mathcal{L}_0 \) can be chosen to be

\[ \mathcal{L}_0 = \frac{\pi^2}{\beta^2} \text{diag}(\nu_1^2, \nu_2^2, \ldots, \nu_N^2), \quad \text{where} \quad \nu_j \in \mathbb{Z}_+ \quad \text{and} \quad \nu_1 \leq \nu_2 \leq \cdots \leq \nu_N, \]  

(4.14)

where we retrieved the temperature \( \beta^{-1} \). Then, in terms of the matrix entries

\[ f_{jk} \equiv (m)_{jk} \quad \text{and} \quad \phi_{jk} \equiv (t)_{jk}, \quad j, k = 1, 2, \ldots, N, \]  

(4.15)

the quadratic part \( S_{\text{tot}}^{(2)} \) of the boundary action (4.13) can be written as

\[ S_{\text{tot}}^{(2)} = \frac{\kappa \gamma}{N} \int du \sum_{j=1}^{N} \left[ \frac{1}{4} \dddot{f}_{jj} \dddot{f}_{jj} - \frac{\pi^2 \nu_j^2}{\beta^2} \dddot{f}_{jj} \dddot{f}_{jj} - \dddot{\phi}_{jj} \dddot{\phi}_{jj} \right] \]

\[ + \frac{2\kappa \gamma}{N} \int du \sum_{j>k} \left[ \frac{1}{4} \dddot{f}_{jk} \dddot{f}_{jk}^* - \frac{\pi^2 (\nu_j^2 + \nu_k^2)}{2\beta^2} \dddot{f}_{jk} \dddot{f}_{jk}^* - \dddot{\phi}_{jk} \dddot{\phi}_{jk}^* \right] \]

\[ + \frac{2\kappa \gamma}{N} \int du \sum_{j>k} \left[ \frac{i \pi^2}{\beta^2} (\nu_j^2 - \nu_k^2)(\dddot{\phi}_{jk} \dddot{f}_{jk}^* - \dddot{\phi}_{jk}^* \dddot{f}_{jk}) \right]. \]  

(4.16)

Since the temperature is finite (i.e. \( u \in S^1 \)), we can use the Fourier mode expansion of \( f_{jk}(u) \) and \( \phi_{jk}(u) \),

\[ f_{jk}(u) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} f_{jk,n} e^{\frac{2\pi inu}{\beta}}, \]  

(4.17)

\[ \phi_{jk}(u) = \frac{1}{\sqrt{2\pi}} \frac{2\pi}{\beta} \sum_{n \in \mathbb{Z}} \phi_{jk,n} e^{\frac{2\pi inu}{\beta}}, \]  

(4.18)
so that the quadratic boundary action can be expanded as

\[
S_{tot}^{(2)} = \frac{\kappa \gamma}{N} \left( \frac{2\pi}{\beta} \right)^4 \left[ \frac{1}{4} \sum_{j=1}^{N} \sum_{n} n^2 (n^2 - \nu_j^2) f_{jj,-n} f_{jj,n} - \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} n^2 \phi_{jj,-n} \phi_{jj,n} \right] \\
+ \frac{\kappa \gamma}{2N} \left( \frac{2\pi}{\beta} \right)^4 \sum_{j>k} \sum_{n \in \mathbb{Z}} (f_{jk,n}^* \phi_{jk,n}) K_n^{(j,k)} \left( f_{jk,n} \right),
\]

where the \(2 \times 2\) matrix \(K_n^{(j,k)}\) is given by

\[
K_n^{(j,k)} \equiv \begin{pmatrix} n^2 & (\nu_j^2 - \nu_k^2) n \\ (\nu_j^2 - \nu_k^2) n & -n^2 \end{pmatrix}.
\]

In order to diagonalize the quadratic action (4.19) we find the eigenvalues \(\lambda_{n,\pm}^{(j,k)}\) of the matrix \(K_n^{(j,k)}\):

\[
\lambda_{n,\pm}^{(j,k)} = \frac{n}{16} \left\{ 2n^3 - n(2 + \nu_j^2 + \nu_k^2) \pm \sqrt{n^2(\nu_j^2 + \nu_k^2 - 2n^2 - 2)^2 + 16(\nu_j^2 - \nu_k^2)^2} \right\}.
\]

Introducing suitable eigenfunctions \(\xi_{jk,n;\pm}\) the quadratic action \(S_{tot}^{(2)}\) can be cast into the form

\[
S_{tot}^{(2)} = \frac{\kappa \gamma}{N} \left( \frac{2\pi}{\beta} \right)^4 \left[ \frac{1}{4} \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} n^2 (n^2 - \nu_j^2) f_{jj,-n} f_{jj,n} - \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} n^2 \phi_{jj,-n} \phi_{jj,n} \right] \\
+ \frac{\kappa \gamma}{2N} \left( \frac{2\pi}{\beta} \right)^4 \sum_{j>k} \sum_{n \in \mathbb{Z}} \left[ \lambda_{n,+}^{(j,k)} \xi_{jk,n,+}^* \xi_{jk,n,+} + \lambda_{n,-}^{(j,k)} \xi_{jk,n,-}^* \xi_{jk,n,-} \right].
\]

Note that eigenvalues \(\lambda_{n,-}^{(j,k)}\) become negative for sufficiently large \(n\) that leads to the instability of the colored gravity. From the quadratic action (4.22) one can read off the 2-point functions:

\[
\langle f_{jj,-n} f_{jj,n} \rangle \sim \frac{1}{n^2(n^2 - \nu_j^2)}, \quad \langle \phi_{jj,-n} \phi_{jj,n} \rangle \sim \frac{1}{n^2}, \quad \langle \xi_{jk,n;\pm}^* \xi_{jk,n;\pm} \rangle \sim \frac{1}{\lambda_{n,\pm}^{(j,k)}}.
\]

Note that some modes, e.g. \(f_{jj,n}\) at \(n = \nu_j\), vanish in the quadratic action, which leads to the divergence of the propagator. However, those modes correspond to the isometry of the constant background \(a_0\) (4.1), which we regard as “gauge symmetries”. Therefore, we exclude their contributions in the path-integral. In details, it is easy to see that \(f_{jj,0}, f_{jj,\nu_j}\), and \(\phi_{jj,0}\) are zero modes of the quadratic action (4.1), and we will exclude them in the semi-classical analysis. On the other hand, one can first see that \(\xi_{jk,0,\pm}\) \((j > k)\) are also zero modes of the quadratic action (4.1). To find the other zero modes among \(\xi_{jk,n;\pm}\), we find the zeros of the \(\lambda_{n,\pm}^{(j,k)}\) in addition to the trivial zero \(n = 0\). For \((j, k)\) such that \(\nu_j \neq \nu_k\), the eigenvalue \(\lambda_{n,\pm}^{(j,k)}\) has two non-trivial zeros denoted as \(\omega_{n,\pm}^{(j,k)}\) in addition to the zero \(n = 0\),

\[
\lambda_{n,+}^{(j,k)} = 0 : \quad \omega_{+,\pm}^{(j,k)} = \frac{1}{2} \sqrt{\nu_j^2 + \nu_k^2 \pm \sqrt{(\nu_j^2 + \nu_k^2)^2 - 16(\nu_j^2 - \nu_k^2)^2}},
\]

\[
\lambda_{n,-}^{(j,k)} = 0 : \quad \omega_{-,\pm}^{(j,k)} = -\frac{1}{2} \sqrt{\nu_j^2 + \nu_k^2 \pm \sqrt{(\nu_j^2 + \nu_k^2)^2 - 16(\nu_j^2 - \nu_k^2)^2}}.
\]
For \((j, k)\) with \(\nu_j = \nu_k\), the eigenvalue \(\lambda_{n, \pm}^{(j,k)}\) becomes simple
\[
\lambda_{n, \pm}^{(j,k)} = n^2(n^2 - \nu_j^2), \quad \lambda_{n, -}^{(j,k)} = n^2.
\]

Whence, \(\lambda_{n, +}^{(j,k)}\) has zero at \(n = \nu_j\) and zeros with double root at \(n = 0\) while \(\lambda_{n, -}^{(j,k)}\) has one double root at \(n = 0\). Note that for the case of \(\nu_j \neq \nu_k\) the zeros \(\omega_{\pm}^{(j,k)}\) (4.24) and \(\omega_{\pm}^{(j,k)}\) (4.25) are not integer in general, and \(\xi_{jk,0;\pm}\) is the only zero mode among \(\xi_{jk,n;\pm}\)’s. This implies that the isometry \(SU(N, N)/\mathbb{Z}_2\) of the AdS\(_2\) background is broken in the non-trivial colored gravity backgrounds. For special values of \(\nu\)’s, the zeros \(\omega_{\pm}^{(j,k)}\) (4.24)–(4.25) happen to become integers so that the broken isometry can be enhanced.

### 4.3 Lyapunov Exponents of the Colored Gravity

The zeros of kernel in the quadratic action (4.22), or, equivalently, the poles in the propagator of the modes (4.23), could appear as the Lyapunov exponent in the out-of-time-ordered correlator (OTOC) of the matter coupled to the colored gravity [36, 52–54]. In the field theory with the Hamiltonian \(H\), the Lyapunov exponent \(\lambda_L\) can be defined by the exponential growth rate in real time \(t\) of the (regularized) OTOC \(F(t)\) of two operators \(V(t)\) and \(W(0)\) [55, 56]:
\[
F_{\text{OTOC}}(t) \equiv \text{Tr} \left[ e^{-\frac{\beta H}{4}} V(t) e^{-\frac{\beta H}{4}} W(0) e^{-\frac{\beta H}{4}} V(t) e^{-\frac{\beta H}{4}} W(0) \right] \sim 1 - \varepsilon e^{\lambda_L t},
\]
where \(\varepsilon\) is proportional to \(\frac{1}{N^\alpha}\) in large-\(N\) models for an appropriate \(\alpha\) which is proportional to the Newton constant \(G_N\) in the holographic dual gravity. The Lyapunov exponent \(\lambda_L\) is bounded in the unitary QFTs [57]:
\[
\lambda_L \leq \frac{2\pi}{\beta}.
\]

This bound is violated in the 3D higher spin gravities with finite number of higher spin fields [54, 58–60]. We will examine the bound violation issue in the 2D colored gravity.

The OTOC (4.27) can be evaluated by an appropriate analytic continuation of the Euclidean 4-point function \(F_{\text{Euc}}(u_1, u_2, u_3, u_4)\). Assuming that the 4-point function is dominated by the OPE channel of which the intermediate operator is holographically dual to the graviton, one can approximate the 4-point function by the 2-point functions of the intermediate graviton. In our case we assume that the OTOC of the boundary operators is dominated by the OPE channel of the (singlet and colored) gravitons and spin-1 mode. It

---

17In this paper we assume that a matter field can be coupled to the colored gravity.
18In chaotic system we expect the universal behaviour of the OTOC of typical operators.
19We will take the analytic continuation \(u \rightarrow -\frac{2\pi i}{\beta} t\) of \(F_{\text{Euc}}(u - \pi/2, u + \pi/2, 0, \pi)\).
can be approximated by their 2-point functions (4.23) as

\[
F_{\text{Eucl}}(u_1, u_2, u_3, u_4) = \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}/\{0, \nu_j\}} \langle f_{jj,-n} f_{jj,n} \rangle \left[ \delta f_{jj,-n} G(1, 2) \right] \left[ \delta f_{jj,n} G(3, 4) \right] \\
+ \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}/\{0\}} \langle \phi_{jj,-n} \phi_{jj,n} \rangle \left[ \delta \phi_{jj,-n} G(1, 2) \right] \left[ \delta \phi_{jj,n} G(3, 4) \right] \\
+ \sum_{j > k}^{N} \sum_{n \in \mathbb{Z}/\{0, \omega_{(j,k)}\}} \langle \xi^*_{jk,n+} \xi_{jk,n+} \rangle \left[ \delta \xi^*_{jk,n+} G(1, 2) \right] \left[ \delta \xi_{jk,n+} G(3, 4) \right] \\
+ \sum_{j > k}^{N} \sum_{n \in \mathbb{Z}/\{0, \omega_{(j,k)}\}} \langle \xi^*_{jk,n-} \xi_{jk,n-} \rangle \left[ \delta \xi^*_{jk,n-} G(1, 2) \right] \left[ \delta \xi_{jk,n-} G(3, 4) \right] + \cdots ,
\]

where the soft mode eigenfunction, \( \delta f_{jj,n} G(1, 2) \equiv \frac{\delta G(u_1, u_2)}{\delta f_{jj,n}} \), corresponds to the OPE of the boundary matter operators with the soft mode \( f_{jj,n} \), and it can be obtained by the infinitesimal transformation of the boundary-to-boundary 2-point function \( G(u_1, u_2) \) of the boundary matter operator with respect to \( f_{jj,n} \). Note that the ellipse denotes the contributions from conformal blocks of other intermediate operators, and we assume that they are subleading in the conformal block expansion. It is crucial to note that since the isometry of the background is modded out then the mode corresponding to the isometry should be excluded in the summation in (4.29). The summation over \( n \) can be expressed as a contour integral around integer simple poles except for those isometry modes.\(^{20}\) By changing the contour such a contour integral is reduced to the residue at the zeros of the kernel in the quadratic action which was excluded in the original summation.\(^{21}\) After the analytic continuation to real time, this gives the exponential growth of the OTOC:

\[
F_{\text{OTOC}}(t) = \sum_{j=1}^{N} e^{\frac{\nu_j}{\beta} t} + \sum_{j > k} \left[ e^{\frac{2\pi \omega_{(j,k)^+}}{\beta} t} + e^{\frac{2\pi \omega_{(j,k)^-}}{\beta} t} \right] + \cdots ,
\]

where we included the terms which grow exponentially in real time \( t \) up to prefactor.\(^{22}\)

Therefore, the Lyapunov exponent of the colored gravity is given by

\[
\lambda_L = \max \left\{ \left\{ 2\pi \nu_N \over \beta \right\} \cup \left\{ \text{Re} \left( 2\pi \omega_{(j,k)^+} \over \beta \right) j > k \right\} \cup \left\{ \text{Re} \left( 2\pi \omega_{(j,k)^-} \over \beta \right) j > k \right\} \right\}
\]

where we used \( \nu_1 \leq \nu_2 \leq \ldots \leq \nu_N \). When \( \nu_N > 1 \), the Lyapunov exponent \( \lambda_L \) is larger than \( 2\pi / \beta \), and it violates the bound on chaos (4.28). The non-unitarity of the colored JT gravity,

\(^{20}\)The form of the boundary-to-boundary propagator and its infinitesimal transformation is not known. Nevertheless, the result for the Lyapunov exponent is independent of their detailed form.

\(^{21}\)The pole at \( n = 0 \) is subtle because the soft mode eigenfunction could vanish at \( n = 0 \). However, the contribution from \( n = 0 \) does not grow exponentially in time.

\(^{22}\)For the detailed calculations of the OTOCs, see [36].
in particular, the instability of the spin-1 mode, is responsible for the violation of the bound on chaos. When \( \nu_j = 1 \) for all \( j = 1, 2, \ldots, N \) which correspond to the AdS_2 background, we can see that the Lyapunov exponent \( \lambda_L = \frac{2\pi}{\beta} \) saturates the bound although the instability of the spin-1 mode still persists even for the AdS_2 background. The exponential growth of the OTOC comes from (singlet and colored) gravitons, which are decoupled from the unstable spin-1 mode in the AdS_2 background. Therefore, in spite of the instability of the colored JT gravity, the bound on chaos (4.28) seems to hold at the quadratic level of the boundary action. However, unlike the sl(2, \( \mathbb{R} \)) Schwarzian theory, quantum corrections\(^{23}\) to the Lyapunov exponent, which can be evaluated from the loop corrections to the propagators (4.23) of gravitons and spin-1 mode induced by interaction terms in the boundary effective action (3.61), might violate the bound on chaos even for the AdS_2 background because of the interaction among the unstable spin-1 mode and the gravitons.

5 Rainbow AdS_2

So far, we have considered the asymptotic AdS_2 condition (3.77) together with the gauge condition (3.14),

\[
A(r, u) = b^{-1}(r)(a(u)du + d) b(r) = \begin{pmatrix} i \mathcal{J}(u) - \mathcal{L}(u) e^{-r} \\ I e^r \end{pmatrix},
\]

where quantities \( \mathcal{J}(u) \) and \( \mathcal{L}(u) \) are defined by (3.17).\(^{24}\) It allows the background only for the spin-2 singlet at asymptotic infinity \( r \to \infty \). As a result, the trace part of \( \mathcal{L} \) containing both singlet and colored spin-2 modes appears in the boundary effective action (3.29).

We may also consider an ansatz for the connection where the colored spin-2 modes also acquire backgrounds at asymptotic infinity,

\[
a(u) = (L_1 + \mathcal{L}(u) L_{-1} + \mathcal{M}(u) L_0) \otimes I + (X_A L_1 + \mathcal{K}_A(u) L_{-1} + \mathcal{N}_A(u) L_0) \otimes T^A + i \mathcal{J}_A(u) I \otimes T^A,
\]

where the backgrounds of the colored spin-2 modes are given by the constants \( X_A \), and we have also introduced \( \mathcal{M}(u) \) and \( \mathcal{N}_A(u) \) to make the ansatz sufficiently general. The above ansatz can be represented as a matrix form,

\[
a(u) = \begin{pmatrix} i \mathcal{W} & -\mathcal{L} \\ Z & i \mathcal{W}^\dagger \end{pmatrix},
\]

where \( \mathcal{L}(u) \) and \( \mathcal{W}(u) \) are matrix functions while \( Z \) is a constant matrix:

\[
\mathcal{L}(u) = \mathcal{L}(u) I + \mathcal{K}_A(u) T^A,
\]

\[
\mathcal{W}(u) = \mathcal{J}_A(u) T^A - \frac{i}{2} \left( \mathcal{M}(u) I + \mathcal{N}_A(u) T^A \right),
\]

\[
Z = I + X_A T_A.
\]

\(^{23}\)See Ref. [61] for the sl(2, \( \mathbb{R} \)) case.

\(^{24}\)In the 3D higher-spin Chern-Simons gravity, \( \mathcal{L} \) is analogous to “charge” while “source” is located at the position of \( I \) in Eq. (5.1) [62–65].
The choice of constant $Z$ (i.e. $X_A$) corresponds to a stationary background at asymptotic infinity. Since this mimics the “rainbow solutions” of the 3D colored gravity [33] we will also keep this term in the 2D colored gravity. However, the main difference in the present case is that the 2D colored gravity action has no potential term for the variable $X_A$ to be minimized. Furthermore, unlike the ansatz (5.1) with $Z = I$ in the present case we need to check whether one can impose the gauge $W = W^\dagger$ (i.e. $M = N_A = 0$) by using the residual gauge symmetry.

5.1 Rainbow Effective Action

Let us first consider a gauge transformation of the rainbow solution (5.3):

$$a' = U^{-1} a U = \begin{pmatrix} i \mathcal{W}' - \mathcal{L}' \\ Z' \\ i \mathcal{W}'^\dagger \end{pmatrix}, \quad U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $U$ is taken to be a constant $SU(N,N)$ matrix (3.4). The resulting transformed components $\mathcal{W}', \mathcal{L}', Z'$ are collected in Appendix A, see (A.2)–(A.4). In general, such a transformation modifies the asymptotic behaviour of the connection $A(r,u)$ (3.14). To keep $Z'$ constant that guarantees a stationary background at asymptotic infinity one chooses a constant gauge parameter $U$ with $B = C = 0$ and $AD = I$ that yields the following transformation of $Z$ inherited from the relation (A.4):

$$Z' = A^\dagger Z A.$$

Then, one may consider all $A \in GL(N,\mathbb{C})$ such that there are left $N + 1$ different $GL(N,\mathbb{C})$ orbits in the space of $Z$ with the representatives

$$Z_{(p,q)} = \text{diag}(\underbrace{1,1,\cdots,1}_{p},\underbrace{-1,-1,\cdots,-1}_{q}) \quad p + q = N.$$  

(5.7)

Taking into account the $Z_2$ quotient which modes out the overall sign, there will be $[(N+1)/2]$ different orbits left. This reproduces the equation for $Z_{(p,q)}$ found in the context of the AdS$_3$ colored gravity [33],

$$Z^2_{(p,q)} = I.$$  

(5.8)

Choosing $Z_{(p,q)}$ (5.7) we introduce the asymptotic rainbow-AdS$_2$ condition

$$(A - A_{\text{AdS}})|_{\partial M_2} = \mathcal{O}(1),$$  

(5.9)

where the rainbow-AdS$_2$ background is defined by the connection

$$A_{\text{AdS}} = L_1 \otimes Z_{(p,q)} + L_{-1} \otimes \mathcal{L}_0, \quad \mathcal{L}_0 = \text{const}.$$  

(5.10)

In general, the $SU(N)$ color gauge symmetry is broken by the asymptotic rainbow-AdS$_2$ condition. To see this, let us consider a color gauge transformation $a' = h^{-1}ah + h^{-1}\dot{h}$ with the gauge parameter

$$U = \begin{pmatrix} V(u) & 0 \\ 0 & V(u) \end{pmatrix}, \quad \text{where} \quad V(u) \in SU(N),$$  

(5.11)
under which $Z_{(p,q)}$ transforms as
\begin{equation}
Z'_{(p,q)} = V^{-1} Z_{(p,q)} V.
\end{equation}
Hence, unless $Z_{(N,0)} = I$, the $SU(N)$ color gauge symmetry is spontaneously broken to the subgroup $U(p) \otimes U(q)/U(1)$.

This also affects the gauge condition that can be chosen by using residual gauge symmetry keeping the asymptotic rainbow-AdS$_2$ condition (5.9). To this end, let us consider the residual gauge transformation of $a(u)$ (5.3) by the gauge parameter $U(u)$ of the form
\begin{equation}
U(u) = \begin{pmatrix} I & h(u) \\ 0 & I \end{pmatrix} \quad \text{for some } h^\dagger = h.
\end{equation}
Then, we find that $\mathcal{W}$ transforms as
\begin{equation}
\mathcal{W}' = \mathcal{W} + i h Z \equiv \mathcal{W} + i \begin{pmatrix} h_{(p,p)} - h_{(p,q)} \\ h_{(q,p)} - h_{(q,q)} \end{pmatrix},
\end{equation}
where we used the block-matrix notation (A.1) for the matrix $h$: the Hermiticity of $h$ implies $h^\dagger_{(p,p)} = h_{(p,p)}$ and $h^\dagger_{(q,q)} = h_{(q,q)}$ while $h^\dagger_{(p,q)} = h_{(q,p)}$. Notice that unlike the case $Z = I$ one cannot choose a gauge where $\mathcal{W} = \mathcal{W}^\dagger$ because $\mathcal{W}^\dagger_{(p,q)} - \mathcal{W}_{(q,p)}$ is invariant under the transformation. Instead, we can set $\mathcal{W}^\dagger_{(p,q)} + \mathcal{W}_{(q,p)}$ to zero, and obtain
\begin{equation}
\mathcal{W} = \begin{pmatrix} \mathcal{J}_{(p,p)} & -i \mathcal{J}_{(p,q)} \\ -i \mathcal{J}_{(q,p)} & \mathcal{J}_{(q,q)} \end{pmatrix},
\end{equation}
with $\mathcal{J}^\dagger_{(p,p)} = \mathcal{J}_{(p,p)}$, $\mathcal{J}^\dagger_{(q,q)} = \mathcal{J}_{(q,q)}$ and $\mathcal{J}^\dagger_{(p,q)} = \mathcal{J}_{(q,p)}$.

Using the parameterization (3.19) for the group element $g$ in $a = g^{-1} \dot{g}$ and taking $a$ in the asymptotic rainbow-AdS$_2$ condition (5.9) in the form (5.15) we find
\begin{equation}
\dot{f} = b d Z d^{-1} b,
\end{equation}
\begin{equation}
\mathcal{L} = \dot{e} - e Z e + i \left( \mathcal{W} e - e \mathcal{W}^\dagger \right),
\end{equation}
\begin{equation}
i \mathcal{W} = e Z + d^{-1} \dot{d} - d^{-1} \dot{b} b^{-1} d.
\end{equation}
Then, the boundary effective action (3.29) is given again by
\begin{equation}
S_{tot} = \frac{\kappa \gamma}{2N} \int du \, \text{tr} \left( a^2 \right) = -\frac{\kappa \gamma}{N} \int du \, \text{tr} \left[ \mathcal{L} Z + \frac{1}{2} \left( \mathcal{W}^2 + \mathcal{W}^\dagger_2 \right) \right]
\end{equation}
\begin{equation}
= -\frac{\kappa \gamma}{N} \int du \, \text{tr} \left[ b^{-1} \dot{b} - 2(b^{-1} \dot{b})^2 - (d^{-1} \dot{d})^2 - \dot{d} \ddagger(b^{-1} \dot{b} - \dot{b} b^{-1}) \right].
\end{equation}
Remarkably, the last expression is formally identical to that one in the case $Z = I$ (3.30). However, when $Z \neq I$, there exists a crucial difference in the physical property compared to the $Z = I$ case.
In order to see this point, we expand the action (5.17) perturbatively around the branch $b^2 = \dot{f} I$, i.e.

$$b = \dot{f}^{\frac{1}{2}}(I + \eta), \quad (5.18)$$

where $\eta$ stands for a fluctuation and $f \equiv \text{tr}(Zf)$ is the singlet graviton mode in the symmetry-broken phase. In this way one obtains the perturbative expansion of the action not only in $\eta$, but also in $k \equiv f - fZ$ (or, equivalently, in $\alpha \equiv \dot{f}^{-1} \dot{k}$) and $\phi = -i \log d$. The first-order solution is found to be

$$\eta_d + i \phi_o = \alpha + O((\eta, \alpha, \phi)^2), \quad (5.19)$$

where

$$\eta_d = \frac{1}{2}(\eta + Z \eta Z), \quad \phi_o = \frac{1}{2}(\phi - Z \phi Z), \quad (5.20)$$

are diagonal and off-diagonal block parts of $\eta$ and $\phi$, respectively. The remaining block parts

$$\eta_o = \frac{1}{2}(\eta - Z \eta Z), \quad \phi_d = \frac{1}{2}(\phi + Z \phi Z) \quad (5.21)$$

are left independent. Using the following perturbative relations

$$\frac{1}{2}(b^{-1} \dot{b} - \dot{b} b^{-1}) = O((\eta, \alpha, \phi)^2), \quad (5.22)$$

we find

$$- \text{tr} \left[ L Z + \frac{1}{2} (W^2 + W^f + W^f)^2 \right] = N \left( \frac{\dot{f}}{f} \right)^2 + \frac{1}{4} \text{tr} (Z \dot{\alpha})^2 - \text{tr} (\dot{\phi}_d)^2 + \text{tr} (\ddot{\eta}_o)^2 + O((\eta, \alpha, \phi)^3) + \text{(total derivative)}, \quad (5.23)$$

where we used $\text{tr}(kZ) = 0$.

Two remarks are in order. First, in the off-diagonal block modes, the derivatives of the spin-two modes $k$ are expressed by $\phi_o$ rather than $\eta_o$. At the same time, $\eta_o$ plays the role of spin-1 modes. This means that the spin-connection of the spin-two modes and the spin-one modes are interchanged in the symmetry broken part. This is in agreement with the Higgs-like mechanism of the colored JT gravity in the bulk [32]. Second, compared to the $Z = I$ case (3.36), the overall signs of the off-diagonal block modes, $\alpha_o$ and $\eta_o$, are flipped (up to total derivatives). Therefore, the off-diagonal block spin-2 modes $\alpha_o$ gets to be unstable, while the off-diagonal block spin-1 modes $\eta_o$ becomes stable because of the broken color symmetry. We will also reach the same conclusion in Section 5.3 by the perturbative analysis at finite temperature.
5.2 Asymptotic Rainbow Symmetry

Let us now consider the asymptotic symmetry for the rainbow AdS$_2$. As in Section 3.5, we consider the infinitesimal gauge transformation of the connection $a(u)$ (5.15):

$$\delta a(u) = \dot{h} + [a(u), h], \quad h(u) = \begin{pmatrix} l & n \\ m & -l^{\dagger} \end{pmatrix}, \quad \text{where} \quad l \equiv s + it,$$

see (A.5). Imposing the asymptotic rainbow-AdS$_2$ condition $\delta a(u) = 0$ we find the components of the connection and the parameter are related by (A.6)–(A.10) given in terms of the block-diagonal decomposition (A.1). Unlike the case $Z = I$, one can solve the asymptotic rainbow-AdS$_2$ condition for the Hermitian matrices $n, s_{(p,p)}, s_{(q,q)}$ and the complex matrix $t_{(p,q)}$. Hence, the asymptotic rainbow-AdS$_2$ symmetry can be parametrized by the Hermitian matrices $m, t_{(p,p)}, t_{(q,q)}$ and the complex $s_{(p,q)}$. We conclude that the rainbow-AdS$_2$ has $2N^2 - 1$ boundary degrees of freedom (taking into account the traceless condition $\text{tr}(t_{(p,p)} + t_{(q,q)}) = 0$), which is the same as in the case $Z = I$. However, note a different organization of the degrees of freedom: in the rainbow-AdS$_2$ case, there are off-diagonal $s_{(p,q)}$ boundary degrees of freedom instead of $t_{(p,q)}$ in the $Z = I$ case.\footnote{Note that for $Z = I$ the $(p, q)$-decomposition (A.1) is superfluous and can be done only to compare with the rainbow-AdS$_2$ case.}

The residual transformation of the connection $a(u)$ (A.5) defines the asymptotic rainbow-AdS$_2$ transformation in the block-diagonal notation:

$$\delta \mathcal{J} = \mathcal{D} \mathcal{J}_{(p,p)} t_{(p,p)} + \frac{i}{2} \left( [\mathcal{L}, m] \right)_{(p,p)} - i \left( \mathcal{J}_{(p,q)} s_{(q,p)} - s_{(p,q)} \mathcal{J}_{(q,p)} \right),$$

$$\delta \mathcal{J}_{(q,q)} = \mathcal{D} \mathcal{J}_{(q,q)} t_{(q,q)} + \frac{i}{2} \left( [\mathcal{L}, m] \right)_{(q,q)} - i \left( \mathcal{J}_{(q,p)} s_{(p,q)} - s_{(p,q)} \mathcal{J}_{(p,q)} \right),$$

$$\delta \mathcal{J}_{(p,q)} = \left( s - \frac{1}{2} \mathcal{L} m - \frac{1}{2} m \mathcal{L} \right)_{(p,q)} + i \left( \mathcal{J}_{(p,p)} s_{(q,q)} - s_{(p,q)} \mathcal{J}_{(q,q)} \right) + i \left( \mathcal{J}_{(p,q)} t_{(q,q)} - t_{(p,q)} \mathcal{J}_{(q,q)} \right),$$

$$\delta \mathcal{L} = -\dot{\mathcal{L}} - \mathcal{L} \left[ \mathcal{J}, s \right] + i \left[ \mathcal{L}, t \right].$$

Here, as discussed before, parameters $n, s_{(p,p)}, s_{(q,q)}, t_{(p,q)}$ should be expressed in terms of independent parameters $m, t_{(p,p)}, t_{(q,q)}$ via (A.6)–(A.10). This is the rainbow-extension of the asymptotic AdS$_2$ symmetry in Section 3.5 and in Ref. [35]. In the rainbow-AdS$_2$ case, the matrix $\mathcal{J} = \mathcal{J}_{(p,q)}$ is the generator of the broken symmetry.

Furthermore, compared to the asymptotic AdS$_2$ transformations (3.81) of $\mathcal{J}$ and $\mathcal{L}$ in the $Z = I$ case, there are simple at first sight extra contributions: the last two terms in (5.25)–(5.26) and the last term in (5.28). However, due to the asymptotic rainbow-AdS$_2$ constraints
(A.6)–(A.10), the resulting asymptotic transformation of \( J \) and \( L \) parameterized by \( m, t_{(p,p)}, t_{(q,q)}, s_{(p,q)} \) turn out to be much more complicated.

The asymptotic rainbow-AdS\(_2\) transformations (5.25)–(5.28) are not enough to understand the consequence of the broken color symmetry and the organization of the resulting boundary degrees of freedom. Therefore, we will evaluate the boundary effective action for the rainbow-AdS\(_2\) case up to quadratic order as in Section 4. To this end, we first consider the smooth rainbow-AdS\(_2\) background \( a_0 \) (5.10) given by

\[
a_0 = \begin{pmatrix} 0 & -\mathcal{L}_0 \\ \mathcal{Z} & 0 \end{pmatrix},
\]

where \( \mathcal{L}_0 \) is a constant matrix. By means of the \( u \)-independent gauge transformation

\[
\begin{pmatrix} U^{-1} \mathcal{Z} & 0 \\ 0 & \mathcal{Z} U^{-1} \end{pmatrix} a_0 \begin{pmatrix} \mathcal{Z} U & 0 \\ 0 & U \mathcal{Z} \end{pmatrix} = \begin{pmatrix} 0 & -U^{-1} \mathcal{Z} \mathcal{L}_0 U \mathcal{Z} \\ \mathcal{Z} & 0 \end{pmatrix},
\]

with constant \( U \in U(N) \) one can also diagonalize\(^{26}\) the constant matrix \( \mathcal{Z} \mathcal{L}_0 \):

\[
\mathcal{Z} \mathcal{L}_0 = \text{diag}(\lambda_1, \ldots, \lambda_N).
\]

From the trivial holonomy condition discussed in Section 3.4, the eigenvalues of \( \mathcal{Z} \mathcal{L}_0 \) are determined to be

\[
\lambda_j = \frac{\pi^2 \nu_j^2}{\beta^2}, \quad \text{where } \nu_j \in \mathbb{Z}_+ \quad \text{and } \nu_1 \leq \cdots \leq \nu_N.
\]

Here all \( \nu_j \)’s are either even for \( \text{Hol}(A) = I_{2N} \) or odd for \( \text{Hol}(A) = -I_{2N} \). Note that the energy of the constant solution \( a_0 \) in (3.76)

\[
E_0 = \frac{\beta \kappa \gamma}{2N} \text{tr} a_0^2 = -\frac{\beta \kappa \gamma}{N} \text{tr} (\mathcal{Z} \mathcal{L}_0) = -\frac{\pi^2 \kappa \gamma}{\beta N} \sum_{j=1}^{N} \nu_j^2.
\]

Note that the value of the energy is not affected by the color symmetry breaking, which one could expect from the fact that the colored JT gravity does not have a potential term for the colored spin-2 modes as opposed to its 3D analog [32]. In what follows, we will focus on the highest-energy constant solution given by

\[
a_0 = \begin{pmatrix} 0 & -\frac{\pi^2 \mathcal{Z}}{\beta^2} \\ \mathcal{Z} & 0 \end{pmatrix},
\]

for the sake of simplicity.

\(^{26}\)We demand that the matrix \( a_0^2 \) instead of \( a_0 \) is Hermitian. It follows that the boundary effective action in (5.17) given by \( \text{tr} a^2 \) is real. Therefore, the matrix \( \mathcal{Z} \mathcal{L}_0 \) is Hermitian.
5.3 Perturbative Expansion

As in Section 4 we consider a small fluctuation around the rainbow-AdS$_2$ background (5.34) and evaluate the boundary effective action in the quadratic approximation. Similarly, we expand the gauge parameter $U(u)$ (4.2) with respect to small fluctuations $h(u)$ and $k(u)$ given by (4.3). A fluctuation around the background (5.34) can be parameterized by the gauge transformation of $a_0$ by $U(u)$ (4.4). At order $O(\epsilon)$, the asymptotic AdS$_2$ condition and the gauge condition (5.15) allow to express block-diagonal components of $s, t, n$ in terms of the block-diagonal components of $m$ as

$$s_{(p,p)} = -\frac{1}{2} \tilde{m}_{(p,p)}, \quad s_{(q,q)} = \frac{1}{2} \tilde{m}_{(q,q)}, \quad t_{(p,q)} = \frac{i}{2} \tilde{m}_{(p,q)}, \quad t_{(q,p)} = -\frac{i}{2} \tilde{m}_{(q,p)},$$  \hspace{0.5cm} (5.35)

These relations are obtained from (A.6)–(A.10) by choosing $\mathcal{L}_0 = \frac{\pi^2}{\beta^2} Z$ and $\mathcal{W}_0 = 0$. By their means one can perturbatively express $\mathcal{L}, \mathcal{W}$ (5.15) in terms of $m, t_{(p,p)}, t_{(q,p)}, s_{(p,q)}$. Expanding $\mathcal{L}$ and $\mathcal{W}$ with respect to $\epsilon$,

$$\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}^{(1)} + \epsilon^2 \mathcal{L}^{(2)} + \ldots, \quad \mathcal{W} = \epsilon \mathcal{W}^{(1)} + \epsilon^2 \mathcal{W}^{(2)} + \ldots,$$  \hspace{0.5cm} (5.38)

we can similarly expand the effective action (5.17) up to a quadratic order,

$$S_{\text{tot}} = \frac{\kappa \gamma}{2N} \int du \text{tr} (a^2) = -\frac{\kappa \gamma}{N} \int du \text{tr} \left[ \mathcal{L} Z + \frac{1}{2} \left( \mathcal{W}^2 + \mathcal{W}^t \mathcal{W} \right) \right]$$

$$= -\frac{\kappa \gamma}{N} \int du \text{tr} (\mathcal{L}_0 Z) - \frac{\epsilon \kappa \gamma}{N} \int du \text{tr} (\mathcal{L}^{(1)} Z)$$

$$- \frac{\epsilon^2 \kappa \gamma}{N} \int du \text{tr} \left[ \mathcal{L}^{(2)} Z + \frac{1}{2} \left( \mathcal{W}^{(1)2} + \mathcal{W}^{(1)\dagger 2} \right) \right] + O(\epsilon^3).$$  \hspace{0.5cm} (5.39)

The leading term here is the energy of the background $E_0 = -\frac{\pi^2 \kappa \gamma}{\beta^2}$, see (5.33), the first-order term vanishes similarly to (4.10), the second-order terms can be explicitly calculated according to (A.11)–(A.14). Finally, using (5.35)–(5.37) one obtains the boundary effective action (5.39) in terms of $m, t_{(p,p)}, t_{(q,p)}, s_{(p,q)}$:

$$S_{\text{tot}} = E_0 + \frac{\kappa \gamma \epsilon^2}{N} \int du \text{tr} \left[ \frac{1}{4} \tilde{m}^2_{(p,p)} - \frac{\pi^2}{\beta^2} \tilde{m}^2_{(p,p)} + \frac{1}{4} \tilde{m}^2_{(q,q)} - \frac{\pi^2}{\beta^2} \tilde{m}^2_{(q,q)} \right]$$

$$- \frac{2 \kappa \gamma \epsilon^2}{N} \int du \text{tr} \left[ \frac{1}{4} \tilde{m}_{(p,q)} \tilde{m}_{(q,p)} - \frac{\pi^2}{\beta^2} \tilde{m}_{(p,q)} \tilde{m}_{(q,p)} \right]$$

$$- \frac{\kappa \gamma \epsilon^2}{N} \int du \text{tr} \left[ t^2_{(p,p)} + t^2_{(q,q)} \right] + \frac{2 \kappa \gamma \epsilon^2}{N} \int du \text{tr} \left[ s_{(p,q)} \dot{s}_{(q,p)} \right] + O(\epsilon^3).$$  \hspace{0.5cm} (5.40)
One can evaluate the mode expansion of the quadratic boundary action along the lines of Section 4.2, see (4.22). First, it is easy to see that the \((p^2 + q^2 - 1)\) modes \(t_{(p,p)}\) and \(t_{(q,q)}\) have the same quadratic action as that of the unstable spin-1 mode in the colored JT graviton (4.22). However, the \(2pq\) modes, \(s_{(p,q)} = (s_{(q,p)})^\dagger\), also have the same quadratic action of \(t_{(p,p)}\) and \(t_{(q,q)}\) but with the opposite overall sign. Therefore, \(s_{(p,q)}\) corresponds to the stable spin-1 mode. Furthermore, the \(p^2 + q^2\) modes \(m_{(p,p)}\) and \(m_{(q,q)}\) have the same quadratic action as that of the stable colored graviton (4.22) while \(2pq\) modes, \(m_{(p,q)} = (m_{(q,p)})^\dagger\), have the same one but with the opposite overall sign. Due to this opposite sign, \(m_{(p,q)}\) is turned out to be the unstable spin-2 mode. The broken color symmetry made the broken part of the spin-2 unstable while the broken part of the spin-1 becomes stable. These results are summarized in Table 1, which are consistent with the perturbative analysis in (5.23) and the Higgs-like mechanism of the colored JT gravity in the bulk [32]. A similar phenomenon has been observed in the Higgs-like mechanism of the rainbow-AdS\(_3\) background in 3D Chern-Simons colored gravity [33–35] where the modes analogous to \(m_{(p,q)}\) become partially-massless through the Higgs mechanism.

### Table 1. The form of quadratic actions of each boundary mode.

| Mode          | Form of Quadratic Action                      |
|---------------|-----------------------------------------------|
| \(m_{(p,p)}\), \(m_{(q,q)}\) : \((p^2 + q^2)\) modes | \(\sum_n n^2(n^2 - 1)f_n^*f_n\) : stable     |
| \(s_{(p,q)}\) : \((2pq)\) modes                  | \(\sum_n n^2\phi_n^*\phi_n\) : stable       |
| \(m_{(p,q)}\) : \((2pq)\) modes                  | \(-\sum_n n^2(n^2 - 1)f_n^*f_n\) : unstable |
| \(t_{(p,p)}, t_{(q,q)}\) : \([p^2 + q^2 - 1]\) modes | \(-\sum_n n^2\phi_n^*\phi_n\) : unstable     |

6 Discussion

In this work we have studied the boundary effective action of the colored JT gravity for the nearly-colored-AdS\(_2\). Starting from the \(su(N, N)\) BF theory we have derived the boundary effective action which is the color generalization of the Schwarzian theory. We have also obtained the isometry of the colored JT gravity and have shown that due to the redundant description of the boundary degrees of freedom this isometry in the boundary effective action should be modded out. Furthermore, we have proposed the boundary action of the colored JT gravity at finite temperature by using the Iwasawa-like decomposition.
Also, we have investigated the colored asymptotic AdS$_2$ symmetry. The perturbative analysis of the boundary action have revealed the instability of the spin-1 mode in the colored JT gravity. In particular, we have demonstrated the influence of the instability on the quantum chaos measured by the Lyapunov exponent of the OTOC. We have also proposed the rainbow-AdS$_2$ geometry where the SU($N$) color gauge symmetry is spontaneously broken and have obtained the boundary effective action along with the asymptotic rainbow-AdS$_2$ symmetry.

Since the instability is originated from the spin-1 sector of the theory one can truncate it from the boundary action (3.29) to construct the matrix generalization of the Schwarzian action without the instability:

\[ S_{mSch} \equiv -\frac{\kappa \gamma}{2N} \int d\tau \text{tr} \left[ f^{-1} \dot{f} - \frac{3}{2} (f^{-1} \ddot{f})^2 \right]. \] (6.1)

It is intriguing to ask whether the one-loop exactness of the $sl(2, \mathbb{R})$ Schwarzian action still holds in the matrix-Schwarzian action (6.1). This might also be related to the question of whether the matrix-Schwarzian action can be interpreted as a coadjoint orbit action of a certain group. Such questions require the matrix-Schwarzian at finite temperature (3.59)

\[ S_{mSch} \equiv -\frac{\kappa \gamma}{N} \int d\tau \text{tr} \left[ \frac{d}{d\tau} (b^{-1} \dot{b}) - \left( b^{-1} \partial_L b + \frac{\partial_R b b^{-1}}{2} \right)^2 - b^4 \right]. \] (6.2)

But, as explained in Section 3.3, we found it difficult to express the matrix-Schwarzian at finite temperature (6.2) in an explicit closed form in terms of a single variable $f(u)$ unlike the zero-temperature case (6.1). Nevertheless, the perturbative analysis can still be applied to the matrix-Schwarzian theory at finite temperature, and, hopefully, it can shed light on its structure.

The instability of a quantum mechanical model can often be cured by supersymmetry. Hence, it is natural to consider a supersymmetric generalization of the colored JT gravity. Although the supersymmetry would not be helpful in avoiding the instability from the point of view of gravity, it is worthwhile to check the instability of the supersymmetric one-dimensional boundary action if exists. In addition, it would be interesting to study the higher-spin extension of the colored JT gravity as in the three-dimensional colored higher-spin gravity [34]. In spite of the instability, those extensions will provide a fruitful testing ground for algebraic structures induced by the presence of extended space-time symmetries.

We have shown that the color symmetry broken by the rainbow-AdS$_2$ background leads to $2pq$ unstable spin-2 modes $m_{(p,q)}$ and $2pq$ stable spin-1 modes $s_{(p,q)}$. In AdS$_3$ such modes become partially-massless by virtue of the Higgs mechanism [33–35]. In our case, it is not clear how to define a notion of “partially-massless” fields because of the degenerate classification of higher-spin fields in AdS$_2$.\footnote{Moreover, all higher-spin fields in two dimensions can be interpreted in some sense as “partially-massless fields of the maximal depth” [23].} At least, we have confirmed that the kinematic nature of
the analogous modes is changed by the rainbow-AdS$_2$ background. It will be interesting to investigate further the Higgs mechanism and the resulting “partially-massless” modes for the colored JT gravity. Furthermore, unlike the $Z = I$ case, the non-singlet part of $\mathcal{L}$ can appear in the boundary action, which is reminiscence of the chemical potential in the higher-spin gravity [62–65]. Such a boundary effective action at finite temperature could be derived from an Iwasawa-like decomposition of the group elements which define the rainbow-AdS$_2$ background connection.

The boundary effective action of the colored JT gravity has multiple branches which correspond to different solutions of the matrix equation $\dot{f} = b^2$ (or the analogous equation in the finite-temperature case). In the uncolored case, there are also two branches corresponding to the different signs of $c$ but these are isomorphic to each other and the $Z_2$ quotient of $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/Z_2$ identifies them. In the colored case, different branches have different natures, in particular, different branches have different dimensions. The relevant issue has been addressed within the context of the colorful particles in 3D [21], where such branches arise from the matrix mass-shell equation $P^2 = m^2 I$. In the present case, the perturbative dynamics has been analyzed for the branch connected to the solution with $b = I$. For other branches one would get different dynamics, in particular, different numbers of physical degrees of freedom.

Finally, it would be interesting to identify a quantum mechanical model of which the low-energy sector is described by the boundary effective action of the colored JT gravity (3.29). However, such a holographic model, if exists, would also suffer from the instability. Instead, one can consider the matrix-Schwarzian action (6.1) as the low-energy effective action and look for the generalizations of the SYK-like models to incorporate the “colored reparameterization symmetry” at the infinite coupling constant limit. We leave those questions for future work.

Acknowledgements. We are grateful to Xavier Bekaert, Cheng Peng, Dongwook Ghim for discussions. E.J. thanks Joaquim Gomis for his encouragement of the current work. The work of K.A. was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS”. The work of E.J. was supported by National Research Foundation (Korea) through the grant NRF-2019R1F1A1044065. The work of J.Y. was supported by KIAS individual Grant PG070102 at Korea Institute for Advanced Study and the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2019R1F1A1045971, 2022R1A2C1003182). J.Y. is supported by an appointment to the JRG Program at the APCTP through the Science and Technology Promotion Fund and Lottery Fund of the Korean Government. J.Y. is also supported by the Korean Local Governments – Gyeongsangbuk-do Province and Pohang City.
A Matrix relations from Section 5.

1. Block-diagonal matrix notation: for a given \( N \times N \) matrix \( M \), let \( M_{(a,b)} \) \((a,b = p,q)\) denote a submatrix of \( M \) such that

\[
M = \begin{pmatrix} M_{(p,p)} & M_{(p,q)} \\ M_{(q,p)} & M_{(q,q)} \end{pmatrix}.
\]

(A.1)

2. Constant gauge transformation (5.5):

\[
\mathcal{L}' = B^\dagger Z B - i D^\dagger \mathcal{W} B + i B^\dagger \mathcal{W}^\dagger B + D^\dagger \mathcal{L} D,
\]

\[
\mathcal{W}' = - B^\dagger Z A + i D^\dagger \mathcal{W} A - i B^\dagger \mathcal{W}^\dagger C - D^\dagger \mathcal{L} C,
\]

\[
Z' = A^\dagger Z A - i C^\dagger \mathcal{W} A + i A^\dagger \mathcal{W}^\dagger C + C^\dagger \mathcal{L} C.
\]

(A.2) (A.3) (A.4)

3. The gauge transformation (5.24):

\[
\delta a(u) = \mathcal{H} + [a(u), \mathcal{H}]
\]

\[
= \begin{pmatrix} \dot{i} & \dot{n} \\ \dot{m} & - \dot{l} \end{pmatrix} + \begin{pmatrix} - n Z - \mathcal{L} m + i [\mathcal{W}, l] & \mathcal{L} l^\dagger + i (\mathcal{W} n - n \mathcal{W}^\dagger) \\ Z l + l^\dagger Z + i (\mathcal{W}^\dagger m - m \mathcal{W}) & Z n + m \mathcal{L} - i [\mathcal{W}^\dagger, t] \end{pmatrix}.
\]

(A.5)

4. Asymptotic rainbow symmetry constraints:

\[
Z l + l^\dagger Z = 2 \begin{pmatrix} s_{(p,p)} & it_{(p,p)} \\ -it_{(q,p)} & -s_{(q,q)} \end{pmatrix} = - \mathbf{m} + i (m \mathcal{W} - \mathcal{W}^\dagger m),
\]

(A.6)

\[
n_{(p,p)} = (s - \frac{1}{2} \{\mathcal{L}, m\})_{(p,p)} + i [\mathcal{J}_{(p,p)}, s_{(p,p)}] + i (\mathcal{J}_{(p,q)} t_{(p,q)} - t_{(p,q)} \mathcal{J}_{(p,q)}),
\]

(A.7)

\[
n_{(q,q)} = (- s + \frac{1}{2} \{\mathcal{L}, m\})_{(q,q)} - i [\mathcal{J}_{(q,q)}, s_{(q,q)}] - i (\mathcal{J}_{(q,p)} t_{(q,p)} - t_{(q,p)} \mathcal{J}_{(q,p)}),
\]

(A.8)

\[
n_{(p,q)} = (i t + \frac{1}{2} \{\mathcal{L}, m\})_{(p,q)} + (\mathcal{J}_{(p,p)} t_{(p,q)} - t_{(p,q)} \mathcal{J}_{(p,q)})
\]

\[- (\mathcal{J}_{(p,q)} s_{(p,q)} - s_{(p,p)} \mathcal{J}_{(p,q)}),
\]

(A.9)

\[
n_{(q,p)} = (i t - \frac{1}{2} \{\mathcal{L}, m\})_{(q,p)} - (\mathcal{J}_{(q,q)} t_{(q,p)} - t_{(q,p)} \mathcal{J}_{(q,p)})
\]

\[+(\mathcal{J}_{(q,p)} s_{(p,p)} - s_{(q,q)} \mathcal{J}_{(q,p)}).
\]

(A.10)

5. Let us now calculate each term in the quadratic action (5.39). At order \( \mathcal{O}(\epsilon^1) \), we find

\[
\mathcal{L}^{(1)} = - n - \frac{\pi^2}{\beta^2} (l Z + Z l^*) = \begin{pmatrix} \frac{1}{2} \tilde{m}_{(p,p)} + \frac{2\pi^2}{\beta^2} \tilde{m}_{(p,p)} \frac{1}{2} \tilde{m}_{(p,q)} - \frac{2\pi^2}{\beta^2} \tilde{m}_{(p,q)} \\ -\frac{1}{2} \tilde{m}_{(q,p)} - \frac{2\pi^2}{\beta^2} \tilde{m}_{(q,p)} \frac{1}{2} \tilde{m}_{(q,q)} + \frac{2\pi^2}{\beta^2} \tilde{m}_{(q,q)} \end{pmatrix},
\]

\[
\mathcal{W}^{(1)} = \begin{pmatrix} i t_{(p,p)} - i s_{(p,q)} \\ -i s_{(q,p)} i t_{(q,q)} \end{pmatrix}.
\]

(A.11)

At order \( \mathcal{O}(\epsilon^2) \), we will also demand the asymptotic rainbow-AdS condition (5.9) together with the gauge condition (5.15) to determine \( k \) in (4.3). Especially, by demanding \( Z \) part
of $a$ in (5.3), the higher-order gauge parameters $\sigma_{(p,p)}, \sigma_{(q,q)}, \tau_{(p,q)}$ and $\tau_{(q,p)}$ in (4.3) can be obtained as follows:

\[
\begin{pmatrix}
\sigma_{(p,p)} & i\tau_{(p,q)} \\
-i\tau_{(q,p)} & -\sigma_{(q,q)}
\end{pmatrix}
= -\frac{1}{2}
\begin{pmatrix}
\mu_{(p,p)} & \mu_{(p,q)} \\
\mu_{(q,p)} & \mu_{(q,q)}
\end{pmatrix}
- \frac{1}{2}
\begin{pmatrix}
ml - l^i\dot{m} \\
\dot{l} - l^i\dot{m}
\end{pmatrix}
+ \frac{1}{2}
\begin{pmatrix}
ml - l^i\dot{m} \\
\dot{l} - l^i\dot{m}
\end{pmatrix}
- \frac{1}{2}
\begin{pmatrix}
Z(l^2 + mn) + (l^i)^2 + mn \right) Z - \left( l^i Zl + \frac{\pi^2}{\beta^2} mZm \right),
\end{align}
\]

where $l \equiv s + it$. Similarly, $\mathcal{L}^{(2)}$ is found to be

\[
\mathcal{L}^{(2)} = -\frac{1}{2}\dot{\nu} - \frac{\pi^2}{2\beta^2} \left( \{\sigma, Z\} + i[\tau, Z] \right)
+ \frac{1}{2}\left( \{s, \dot{n}\} - \{\dot{s}, n\} \right) + i\left( [t, \dot{n}] - [\dot{t}, n] \right) + nZ\dot{n}
+ \frac{\pi^2}{2\beta^2} \left( Zmn + nmZ + l^2 Z + Zl^2 + 2lZl^\dagger \right).
\]

Note that one could also determine $\nu$ from $\mathcal{W}$ and $\mathcal{W}^\dagger$. However, the solution for $\nu$ is not necessary in evaluating the quadratic action because the term related to $\nu$ in $\mathcal{L}^{(2)}$ is total derivative. Using (A.12) as well as (A.6)–(A.10) we find the respective part of the effective action $\text{tr} \left[ Z \mathcal{L}^{(2)} \right]$ (5.17) up to total derivatives:

\[
\int du \text{tr} \left[ Z \mathcal{L}^{(2)} \right] = \int du \text{tr} \left[ \frac{\pi^2}{\beta^2} Z(ml + l^i\dot{m}) - Z(ln + n\dot{l}) \right]
+ \int du \text{tr} \left[ \frac{\pi^2}{\beta^2} (Zl^\dagger Zl + \frac{\pi^2}{\beta^2} (Zm)^2) + (Zn)^2 + \frac{\pi^2}{\beta^2} ZlZl^\dagger \right]
+ \frac{2\pi^2}{\beta^2} \int du \text{tr} \left[ s^2 - t^2 + mn \right].
\]

References

[1] C. Teitelboim, Gravitation and Hamiltonian Structure in Two Space-Time Dimensions, Phys. Lett. 126B (1983) 41–45.
[2] R. Jackiw, Lower Dimensional Gravity, Nucl. Phys. B252 (1985) 343–356.
[3] J. Maldacena, D. Stanford and Z. Yang, Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space, PTEP 2016 (2016) 12C104, [1606.01857].
[4] A. Kitaev, A simple model of quantum holography (part 1), talk at KITP, April 7, 2015, http://online.kitp.ucsb.edu/online/entangled15/kitaev/.
[5] A. Kitaev, A simple model of quantum holography (part 2), talk at KITP, May 27, 2015, http://online.kitp.ucsb.edu/online/entangled15/kitaev2/.
[6] A. Kitaev, Hidden correlations in the Hawking radiation and thermal noise, http://online.kitp.ucsb.edu/online/joint98/kitaev/, KITP seminar, Feb. 12, (2015).
[7] J. Polchinski and V. Rosenhaus, *The Spectrum in the Sachdev-Ye-Kitaev Model*, *JHEP* **04** (2016) 001, [1601.06768].

[8] A. Jevicki, K. Suzuki and J. Yoon, *Bi-Local Holography in the SYK Model*, *JHEP* **07** (2016) 007, [1603.06246].

[9] J. Maldacena and D. Stanford, *Remarks on the Sachdev-Ye-Kitaev model*, *Phys. Rev. D* **94** (2016) 106002, [1604.07818].

[10] W. Fu, D. Gaiotto, J. Maldacena and S. Sachdev, *Supersymmetric Sachdev-Ye-Kitaev models*, *Phys. Rev. D* **95** (2017) 026009, [1610.08917].

[11] R. A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen and S. Sachdev, *Thermoelectric transport in disordered metals without quasiparticles: The Sachdev-Ye-Kitaev models and holography*, *Phys. Rev. B* **95** (2017) 155131, [1612.00849].

[12] J. Yoon, *SYK Models and SYK-like Tensor Models with Global Symmetry*, *JHEP* **10** (2017) 183, [1707.01740].

[13] P. Narayan and J. Yoon, *Supersymmetric SYK Model with Global Symmetry*, *JHEP* **08** (2018) 159, [1712.02647].

[14] D. Stanford and E. Witten, *Fermionic Localization of the Schwarzian Theory*, *JHEP* **10** (2017) 008, [1703.04612].

[15] P. Saad, S. H. Shenker and D. Stanford, *JT gravity as a matrix integral*, [1903.11115].

[16] T. Fukuyama and K. Kamimura, *Gauge Theory of Two-dimensional Gravity*, *Phys. Lett. B* **160B** (1985) 259–262.

[17] A. H. Chamseddine and D. Wyler, *Topological Gravity in (1+1)-dimensions*, *Nucl. Phys. B* **340** (1990) 595–616.

[18] A. H. Chamseddine and D. Wyler, *Gauge Theory of Topological Gravity in (1+1)-Dimensions*, *Phys. Lett. B* **228** (1989) 75–78.

[19] A. Achucarro and P. K. Townsend, *A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories*, *Phys. Lett. B* **180** (1986) 89.

[20] E. Witten, *(2+1)-Dimensional Gravity as an Exactly Soluble System*, *Nucl. Phys. B* **311** (1988) 46.

[21] J. Gomis, E. Joung, A. Kleinschmidt and K. Mkrtchyan, *Colourful Poincaré symmetry, gravity and particle actions*, *JHEP* **08** (2021) 047, [2105.01686].

[22] E. Joung and W. Li, *Nonrelativistic limits of colored gravity in three dimensions*, *Phys. Rev. D* **97** (2018) 105020, [1801.10143].

[23] K. B. Alkalaev, *On higher spin extension of the Jackiw-Teitelboim gravity model*, *J. Phys. A* **47** (2014) 365401, [1311.5119].

[24] D. Grumiller, M. Leston and D. Vassilevich, *Anti-de Sitter holography for gravity and higher spin theories in two dimensions*, *Phys. Rev. D* **89** (2014) 044001, [1311.7413].

[25] K. B. Alkalaev, *Global and local properties of AdS_2 higher spin gravity*, *JHEP* **10** (2014) 122, [1404.5330].
[26] H. A. González, D. Grumiller and J. Salzer, *Towards a bulk description of higher spin SYK*, *JHEP* **05** (2018) 083, [1802.01562].

[27] K. Alkalaev and X. Bekaert, *Towards higher-spin AdS$_2$/CFT$_1$ holography*, *JHEP* **04** (2020) 206, [1911.13212].

[28] K. Alkalaev and X. Bekaert, *On BF-type higher-spin actions in two dimensions*, *JHEP* **05** (2020) 158, [2002.02387].

[29] M. Astorino, S. Cacciatori, D. Klemm and D. Zanon, *AdS(2) supergravity and superconformal quantum mechanics*, *Annals Phys.* **304** (2003) 128–144, [hep-th/0212096].

[30] M. Cárdenas, O. Fuentelba, H. A. González, D. Grumiller, C. Valcárcel and D. Vassilevich, *Boundary theories for dilaton supergravity in 2D*, *JHEP* **11** (2018) 077, [1809.07208].

[31] Y. Fan and T. G. Mertens, *Supergroup Structure of Jackiw-Teitelboim Supergravity*, 2106.09353.

[32] K. Alkalaev, E. Joung and J. Yoon, *Color decorations of Jackiw-Teitelboim gravity*, *JHEP* **08** (2022) 286, [2204.10214].

[33] S. Gwak, E. Joung, K. Mkrtchyan and S.-J. Rey, *Rainbow Valley of Colored (Anti) de Sitter Gravity in Three Dimensions*, *JHEP* **04** (2016) 055, [1511.05220].

[34] S. Gwak, E. Joung, K. Mkrtchyan and S.-J. Rey, *Rainbow vacua of colored higher-spin (A)dS$_3$ gravity*, *JHEP* **05** (2016) 150, [1511.05975].

[35] E. Joung, J. Kim, J. Kim and S.-J. Rey, *Asymptotic Symmetries of Colored Gravity in Three Dimensions*, *JHEP* **03** (2018) 104, [1712.07744].

[36] P. Narayan and J. Yoon, *Chaos in Three-dimensional Higher Spin Gravity*, *JHEP* **07** (2019) 046, [1903.08761].

[37] F. Valach and D. R. Youmans, *Schwarzian quantum mechanics as a Drinfeld-Sokolov reduction of BF theory*, *JHEP* **12** (2020) 189, [1912.12331].

[38] T. G. Mertens, *The Schwarzian theory — origins*, *JHEP* **05** (2018) 036, [1801.09605].

[39] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207–226.

[40] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007, [1008.4744].

[41] D. Grumiller, R. McNees, J. Salzer, C. Valcárcel and D. Vassilevich, *Menagerie of AdS$_2$ boundary conditions*, *JHEP* **10** (2017) 203, [1708.08471].

[42] M. Banados, *Global charges in Chern-Simons field theory and the (2+1) black hole*, *Phys. Rev.* **D52** (1996) 5816, [hep-th/9405171].

[43] M. Banados, *Three-dimensional quantum geometry and black holes*, hep-th/9901148.

[44] T. G. Mertens and G. J. Turiaci, *Defects in Jackiw-Teitelboim Quantum Gravity*, *JHEP* **08** (2019) 127, [1904.05228].

[45] V. Ovsienko, *Lagrange schwarzian derivative and symplectic Sturm theory*, Annales de la Faculté des sciences de Toulouse : Mathématiques Ser. **6**, **2** (1993) 73–96.
[46] V. Ovsienko and S. Tabachnikov, *Projective Differential Geometry Old and New*. Cambridge University Press, dec, 2004, 10.1017/cbo9780511543142.

[47] A. Marshakov and A. Morozov, *A note on W(3) algebra*, *Nucl. Phys.* B339 (1990) 79–94.

[48] W. Li and S. Theisen, *Some aspects of holographic W-gravity*, *JHEP* 08 (2015) 035, [1504.07799].

[49] M. A. Vasiliev, *Extended Higher Spin Superalgebras and Their Realizations in Terms of Quantum Operators*, *Fortsch. Phys.* 36 (1988) 33–62.

[50] S. E. Konstein and M. A. Vasiliev, *Extended higher spin superalgebras and their massless representations*, *Nucl. Phys.* B331 (1990) 475–499.

[51] A. Castro, R. Gopakumar, M. Gutperle and J. Raeymaekers, *Conical Defects in Higher Spin Theories*, *JHEP* 02 (2012) 096, [1111.3381].

[52] G. Sárosi, *AdS2 holography and the SYK model*, *PoS Modave2017* (2018) 001, [1711.08482].

[53] V. Jahnke, K.-Y. Kim and J. Yoon, *On the Chaos Bound in Rotating Black Holes*, *JHEP* 05 (2019) 037, [1903.09086].

[54] J. Yoon, *A bound on chaos from stability*, *JHEP* 11 (2021) 097, [1905.08815].

[55] S. H. Shenker and D. Stanford, *Black holes and the butterfly effect*, *JHEP* 03 (2014) 067, [1306.0622].

[56] S. H. Shenker and D. Stanford, *Stringy effects in scrambling*, *JHEP* 05 (2015) 132, [1412.6087].

[57] J. Maldacena, S. H. Shenker and D. Stanford, *A bound on chaos*, *JHEP* 08 (2016) 106, [1503.01409].

[58] E. P. Perlmutter, *Bounding the Space of Holographic CFTs with Chaos*, *JHEP* 10 (2016) 069, [1602.08272].

[59] J. R. David, S. Khetrapal and S. P. Kumar, *Local quenches and quantum chaos from higher spin perturbations*, *JHEP* 10 (2017) 156, [1707.07166].

[60] S. Datta, *The Schwarzian sector of higher spin CFTs*, *JHEP* 04 (2021) 171, [2101.04980].

[61] Y.-H. Qi, S.-J. Sin and J. Yoon, *Quantum Correction to Chaos in Schwarzian Theory*, *JHEP* 11 (2019) 035, [1906.00996].

[62] M. Gutperle and P. Kraus, *Higher Spin Black Holes*, *JHEP* 05 (2011) 022, [1103.4304].

[63] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, *Spacetime Geometry in Higher Spin Gravity*, *JHEP* 10 (2011) 053, [1106.4788].

[64] A. Castro, E. Hijano, A. Lepage-Jutier and A. Maloney, *Black Holes and Singularity Resolution in Higher Spin Gravity*, *JHEP* 01 (2012) 031, [1110.4117].

[65] M. Henneaux, A. Perez, D. Tempo and R. Troncoso, *Chemical potentials in three-dimensional higher spin anti-de Sitter gravity*, *JHEP* 12 (2013) 048, [1309.4362].