TRACE FORMULA FOR COMPONENT GROUPS OF NÉRON MODELS

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Abstract. We study a trace formula for tamely ramified abelian varieties $A$ over a complete discretely valued field, which expresses the Euler characteristic of the special fiber of the Néron model of $A$ in terms of the Galois action on the $\ell$-adic cohomology of $A$. If $A$ has purely additive reduction, the trace formula yields a cohomological interpretation for the number of connected components of the special fiber of the Néron model.

1. Introduction

We denote by $R$ a complete discrete valuation ring, by $K$ its quotient field, and by $k$ its residue field. We assume that $k$ is algebraically closed, and we denote its characteristic exponent by $p$. We denote by $K_s$ a separable closure of $K$, and by $K^t$ the tame closure of $K$ in $K_s$. We fix a topological generator $\phi$ of the tame monodromy group $G(K^t/K)$ and a prime $\ell$ invertible in $k$. We denote by $\chi_{\ell}(X)$ the $\ell$-adic Euler characteristic of a $k$-variety $X$ (it is independent of $\ell$). For each $R$-scheme $Z$, we denote by $Z_s$ its special fiber over $k$.

For each $K$-variety $X$, we denote by $H(X \times_K K^t, \mathbb{Q}_\ell)$ the graded $\ell$-adic cohomology space $\bigoplus_{i \geq 0} H^i(X \times_K K^t, \mathbb{Q}_\ell)$. We say that $X$ is cohomologically tame if the wild inertia $P \subset G(K^t/K)$ acts trivially on $H(X \times_K K^t, \mathbb{Q}_\ell)$. In this case, we have

$$H(X \times_K K^t, \mathbb{Q}_\ell) \cong H(X \times_K K^t, \mathbb{Q}_\ell)^P = H(X \times_K K^t, \mathbb{Q}_\ell)$$

If $X$ is a smooth and proper $K$-variety, and $X'$ is a weak Néron model for $X$ over $R$ [4, 3.5.1], then the value $\chi_{\ell}(X')$ is independent of the choice of weak Néron model, since it is a specialization of the motivic Serre invariant $S(X)$ of $X$ (see Section 2.1). In [16, 6.3] we established a trace formula for smooth and proper $K$-varieties $X$ that satisfy a certain geometric tameness condition. The trace formula states, in particular, that

$$\chi_{\ell}(X') = \text{Trace}(\varphi | H(X \times_K K^t, \mathbb{Q}_\ell)) = \sum_{i \geq 0} (-1)^i \text{Trace}(\varphi | H^i(X \times_K K^t, \mathbb{Q}_\ell))$$

The value $\chi_{\ell}(X')$ can be seen as a measure for the set of rational points on $X$, and the trace formula gives a cohomological interpretation for this measure.

If $p = 1$ the geometric tameness condition is void, but unfortunately, it is quite strong if $p > 1$. In [16, §7] we showed that if $X$ is a geometrically connected smooth projective $K$-curve, we can weaken the condition to cohomological tameness, unless $X$ is a genus one curve with additive reduction and without $K^t$-point. It seems plausible that the trace formula holds for any cohomologically tame, smooth, proper and geometrically connected $K$-variety $X$ that has a rational point over $K^t$. 
The purpose of the present note is to prove this for abelian varieties. Some of the results are certainly known to experts (even though we could not find them in the literature), but we believe that the trace formula provides a valuable context which casts new light on the results presented here, inserting them in a general theory. Conversely, the case of abelian varieties constitutes a first step towards a “motivic” proof of the general case, avoiding the use of resolution of singularities.

Let $A$ be an abelian variety over $K$. We denote by $\mathcal{A}$ its Néron model over $R$, and by $\phi_A$ the cardinality of the group of connected components $\Phi_A = \mathcal{A}_s/A_0^s$. We denote by $\phi_A'$ the prime-to-$p$ part of $\phi_A$, and by $(\phi_A)_q$ the $q$-primary part, for each prime number $q$. We say that $A$ is tamely ramified if it is cohomologically tame; this is equivalent to the property that $A$ acquires semi-abelian reduction on a finite tame extension of $K$. The toric, unipotent and abelian rank of $A$ are by definition the toric, unipotent and abelian rank of $A_0^s$.

We consider the following property for the abelian $K$-variety $A$.

(Trace) $\chi_{\text{ét}}(\mathcal{A}_s) = \text{Trace}(\varphi|H(A \times_K \mathbb{Q}_\ell))$

In [16, §7.3] we formulated the following conjecture.

Conjecture 1.1. The trace formula (Trace) holds for any tamely ramified abelian $K$-variety $A$.

In the present article, we will prove this conjecture. In fact, we show that both sides of the expression (Trace) vanish unless $A$ has purely additive reduction (Proposition 2.5), and that in the latter case, both sides equal $\phi_A$ (Theorem 2.8).

2. The trace formula

2.1. Chevalley decomposition in the Grothendieck ring. We denote by $K_0(\text{Var}_k)$ the Grothendieck ring of $k$-varieties. See, for instance, [16, 2.1] for its definition. As usual, we denote by $L$ the class in $K_0(\text{Var}_k)$ of the affine line $\mathbb{A}_k^1$.

Proposition 2.1. Let $G$ be a smooth connected commutative algebraic $k$-group and consider its Chevalley decomposition

\[
\begin{array}{cccc}
0 & \longrightarrow & (L = U \times_k T) & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0
\end{array}
\]

with $U$ unipotent, $T$ a torus, and $B$ an abelian variety. If we denote by $u$ and $t$ the dimensions of $U$, resp. $T$, then

$$[G] = L^u([L] - 1)^t[B]$$

in $K_0(\text{Var}_k)$.

Proof. As a $k$-variety, $U$ is isomorphic to $\mathbb{A}_k^n$. [19 VII n° 6]. By the scissor relations in the Grothendieck ring, it suffices to show that $\pi$ is a $L$-torsor w.r.t. the Zariski-topology. But $\pi$ is a $L$-torsor w.r.t. the fppf topology, and hence also w.r.t. the Zariski topology because $L$ is a successive extension of $\mathbb{G}_m$ and $\mathbb{G}_a$ [15 III.3.7+4.9].

Corollary 2.2. If $G$ is a smooth connected commutative algebraic $k$-group, then $\chi_{\text{ét}}(G) = 1$ if $G$ is unipotent, and $\chi_{\text{ét}}(G) = 0$ else.

Corollary 2.3. For each abelian variety $A$ over $K$, we have $\chi_{\text{ét}}(\mathcal{A}_s) = \phi_A$ if $A$ has purely additive reduction, and $\chi_{\text{ét}}(\mathcal{A}_s) = 0$ else.
Recall that the motivic Serre invariant $S(X)$ of a smooth and proper $K$-variety $X$ is defined as

$$S(X) = [X] \in K_0(Var_k)/(L - 1)$$

where $X$ is any weak Néron model of $X$ over $R$. This definition is independent of the choice of weak Néron model $[12, 4.5.3][16, 5.1]$. The motivic Serre invariant may be seen as a measure for the set of rational points on $X$, and the trace formula in $[16, 6.3]$ gives a cohomological interpretation of this measure.

**Corollary 2.4.** Let $A$ be an abelian variety over $K$, denote by $u$ and $t$ its unipotent, resp. toric rank, and denote by $B$ the abelian quotient in the Chevalley decomposition of $A^q$. Then we have the following equalities in $K_0(Var_k)/(L - 1)$:

$$S(A) = \begin{cases} 0 & \text{iff } t > 0 \\ \phi_A \cdot [B] & \text{iff } t = 0 \end{cases}$$

**Proof.** We only have to show that $\phi_A \cdot [B] \neq 0$ in $K_0(Var_k)/(L - 1)$. Consider the Poincaré polynomial

$$P(\cdot; T) : K_0(Var_k) \rightarrow \mathbb{Z}[T] : [X] \mapsto P(X; T)$$

from $[16, 2.10]$. It is a morphism of rings, mapping the class $[Y]$ of a smooth and proper $k$-variety $Y$ to the polynomial

$$P(Y; T) = \sum_{i \geq 0} (-1)^i b_i(Y) T^i$$

with $b_i(Y)$ the $i$-th $\ell$-adic Betti number of $Y$ (it is independent of $\ell$). We have $P(L; T) = T^2 - 1$, so evaluating at $T = -1$ we obtain a ring morphism

$$P(\cdot; -1) : K_0(Var_k)/(L - 1) \rightarrow \mathbb{Z} : [X] \mapsto P(X; -1)$$

We have

$$P(\phi_A \cdot [B]; -1) = \phi_A \cdot \sum_{i \geq 0} b_i(B) = \phi_A \cdot 4^{\dim(B)} \neq 0$$

\[\square\]

### 2.2. Trace formula for component groups

Assume that $A$ is an abelian $K$-variety of dimension $g$. We denote by $P_\varphi(T)$ the characteristic polynomial

$$P_\varphi(T) = \det(T \cdot Id - \varphi | H^1(A \times_K K^t, \mathbb{Q}_\ell))$$

of $\varphi$ on $H^1(A \times_K K^t, \mathbb{Q}_\ell)$. The polynomial $P_\varphi(T)$ belongs to $\mathbb{Z}[T]$, and it is independent of $\ell$, by $[13, 2.10]$. By quasi-unipotency of the $G(K^t/K)$-action on $T_1A [11, IX.4.3]$ we know that the zeroes of $P_\varphi(T)$ are roots of unity, so that $P_\varphi(T)$ is a product of cyclotomic polynomials. Since the pro-$p$-part of $G(K^t/K)$ is trivial, the orders of the zeroes of $P_\varphi(T)$ as roots of unity are prime to $p$.

If $A$ is tamely ramified, it follows immediately from the canonical isomorphism

$$H(A \times_K K^t, \mathbb{Q}_\ell) \cong \bigwedge H^1(A \times_K K^t, \mathbb{Q}_\ell)$$

that we have

$$Trace(\varphi | H(A \times_K K^t, \mathbb{Q}_\ell)) = P_\varphi(1)$$

**Proposition 2.5.** If $A$ is a tamely ramified abelian $K$-variety that does not have purely additive reduction, then

$$\chi_{et}(A) = Trace(\varphi | H(A \times_K K^t, \mathbb{Q}_\ell)) = 0$$

and Conjecture $[1, 1]$ holds for $A$. 
Proof. It is well-known that \( P_\varphi(1) = 0 \) iff \( A \) does not have purely additive reduction (see for instance [3, 1.3]) so the right hand side of trace formula vanishes. The left hand side vanishes as well, by Corollary 2.3. □

In order to investigate the case where \( A \) has purely additive reduction, we'll need some elementary lemmas.

**Lemma 2.6.** Fix an integer \( d > 1 \) and let \( \Phi_d(T) \in \mathbb{Z}[T] \) be the cyclotomic polynomial whose roots are the primitive \( d \)-th roots of unity. Then \( \Phi_d(1) \in \mathbb{Z}_{>0} \) and \( \Phi_d(1)|d \).

**Proof.** We proceed by induction on \( \Phi \) and \( \otimes M \) endomorphism of \( M \) and evaluating at polynomial whose roots are the primitive \( \Phi \).

\( \square \)

**Lemma 2.7.** Let \( q \) be a prime, \( M \) a free \( \mathbb{Z}_q \)-module of finite type, and \( \alpha \) an endomorphism of \( M \). Then \( M/\alpha M \) is torsion iff \( \alpha \) is an automorphism on \( M \otimes_{\mathbb{Z}_q} \mathbb{Q}_q \). In this case, the cardinality \( \sharp(M/\alpha M) \) of \( M/\alpha M \) satisfies

\[
\sharp(M/\alpha M) = |\det(\alpha \mid M \otimes_{\mathbb{Z}_q} \mathbb{Q}_q)|q^{-1}
\]

where \(| \cdot |_q \) denotes the \( q \)-adic absolute value.

**Proof.** The module \( M/\alpha M \) is torsion if \( (M/\alpha M) \otimes_{\mathbb{Z}_q} \mathbb{Q}_q = 0 \), i.e. if \( \alpha \) is a surjective and, hence, bijective endomorphism on \( M \otimes_{\mathbb{Z}_q} \mathbb{Q}_q \). In this case, we have

\[
M/\alpha M \cong \mathbb{Z}_q/q^{c_1} \mathbb{Z}_q \oplus \ldots \oplus \mathbb{Z}_q/q^{c_r} \mathbb{Z}_q
\]

where \( q^{c_1}, \ldots, q^{c_r} \) are the invariant factors of \( \alpha \) on \( M \). Since \( \det(\alpha \mid M \otimes_{\mathbb{Z}_q} \mathbb{Q}_q) \) equals \( q^{c_1+\ldots+c_r} \) times a unit in \( \mathbb{Z}_q \), we find

\[
\sharp(M/\alpha M) = q^{c_1+\ldots+c_r} = |\det(\alpha \mid M \otimes_{\mathbb{Z}_q} \mathbb{Q}_q)|q^{-1}
\]

\( \square \)

**Theorem 2.8.** If \( A \) is an abelian \( K \)-variety with purely additive reduction, then

\[
P_\varphi(1) = \phi_A'
\]

If, moreover, \( A \) is tamely ramified, then

\[
\text{Trace}(\varphi \mid H(A \times_K K^t, \mathbb{Q}_l)) = \phi_A' = \phi_A
\]

**Proof.** By [2, 1.8], the component group \( \Phi_A \) is killed by \( [K' : K]^2 \), with \( K' \) the minimal extension of \( K \) where \( A \) acquires semi-abelian reduction. If \( A \) is tamely ramified, then \( [K' : K] \) is prime to \( p \), so that \( \phi_A = \phi_A' \).

Hence, it suffices to prove the first assertion. Let \( q \) be a prime different from \( p \), and denote by \( T_q^t A = (T_q A)^t \) the tame \( q \)-adic Tate module of \( A \). By [1] IX.11.3.8], the \( q \)-primary part \( (\phi_A)_q \) of \( \phi_A \) equals the cardinality of the torsion part of

\[
H^1(G(K^t/K), T_q A) \cong H^1(G(K^t/K), T_q A) = T_q A/(Id - \varphi)T_q A
\]
Since \( A \) has purely additive reduction, 1 is not an eigenvalue of \( \varphi \) on \( V^t_q A = T^t_q A \otimes \mathbb{Z}_q \mathbb{Q}_q \) by [8, 1.3], so by Lemma 2.7 the \( \mathbb{Z}_q \)-module \( T^t_q A/(Id - \varphi)T^t_q A \) is torsion, and its cardinality is given by

\[
|\det(1 - \varphi | V^t_q A)|^{-1}_q
\]

where \( | \cdot |_q \) denotes the \( q \)-adic absolute value. Since \( P_{\varphi}(T) \) is independent of \( \ell \), we find

\[
(\phi_A)_q = |P_{\varphi}(1)|^{-1}_q
\]

for each prime \( q \neq p \). Moreover, the characteristic of \( k \) does not divide the order (as a root of unity) of any zero of \( P_{\varphi}(T) \), so if \( p > 1 \), we have \( |P_{\varphi}(1)|_p = 1 \) by Lemma 2.6.

Hence, taking the product of (2.1) over all primes \( q \neq p \), we get

\[
\phi'_A = \prod_{q \neq p} |P_{\varphi}(1)|^{-1}_q = \prod_{r \text{ prime}} |P_{\varphi}(1)|^{-1}_r = |P_{\varphi}(1)| = P_{\varphi}(1)
\]

where the last equality follows from Lemma 2.6. This concludes the proof. \( \square \)

As a consequence, we obtain a proof of Conjecture 1.1.

Corollary 2.9. The trace formula (Trace) holds for every tamely ramified abelian \( K \)-variety \( A \).

Proof. Combine Corollary 2.3, Proposition 2.5, and Theorem 2.8. \( \square \)

Corollary 2.10. If \( A \) has purely additive reduction, then \( \phi'_A \) is invariant under isogeny.

Remark. In the proof of Theorem 2.8, we invoked [9, 1.8]. The proof of [9, 1.8] is based on [9, 5.6+9]. Unfortunately, it is known that [9, 4.2] is not correct (see [6, 4.8(b)]), and the proofs of [9, 5.6+9] rely on this result. However, the authors of [9] have informed me that they’ve written a proof of [9, 1.8] that avoids the use of the erroneous part of [9, 4.2]. The proof can be found in [10]. \( \square \)

3. The monodromy zeta function

To conclude this note, we give an alternative proof of Conjecture 1.1 and Theorem 2.8 if \( A \) is a Jacobian \( \text{Jac}(C) \). The proof is based on an explicit expression for \( P_{\varphi}(T) \) in terms of an \( \text{sncld} \)-model of the curve \( C \).

Let \( C \) be a smooth projective geometrically connected \( K \)-curve of genus \( g(C) \). An \( \text{sncld} \)-model of \( C \) is a regular flat proper \( R \)-model of \( C \) whose special fiber is a divisor with strict normal crossings. Let \( C \) be a relatively minimal \( \text{sncld} \)-model of \( C \), with \( C_s = \sum_{i \in I} N_i E_i \). We put \( \delta(C) = \gcd\{N_i \mid i \in I\} \). For each \( i \in I \) we put \( E^o_i = E_i \setminus \cup_{j \neq i} E_j \) and we denote by \( d_i \) the cardinality of \( E_i \setminus E^o_i \). Let \( A \) be the Jacobian \( \text{Jac}(C) \) of \( C \). Then \( A \) is tamely ramified iff \( C \) is cohomologically tame.

We denote by \( \zeta_C(T) \) the reciprocal of the monodromy zeta function of \( C \), i.e.,

\[
\zeta_C(T) = \prod_{i=0}^{2} \det(T \cdot Id - \varphi \mid H^i(X \times_K K^1, \mathbb{Q}_\ell))^{(-1)^{i+1}} \in \mathbb{Q}_\ell(T)
\]
Theorem 3.1. Put $A = \text{Jac}(C)$. For each $i \in I$, we denote by $N'_i$ the prime-to-$p$ part of $N_i$. We have
\begin{equation}
\zeta_C(T) = \prod_{i \in I} (T^{N'_i} - 1)^{-\chi_{et}(E'_i)}
\end{equation}
(3.1)
\begin{equation}
P_{\varphi}(T) = (T - 1)^2 \prod_{i \in I} (T^{N'_i} - 1)^{-\chi_{et}(E'_i)}
\end{equation}
(3.2)
If $C$ is cohomologically tame, and either $\delta(C)$ is prime to $p$, or $g(C) \neq 1$, then
\begin{equation}
\zeta_C(T) = \prod_{i \in I} (T^{N'_i} - 1)^{-\chi_{et}(E'_i)}
\end{equation}
(3.3)
\begin{equation}
P_{\varphi}(T) = (T - 1)^2 \prod_{i \in I} (T^{N'_i} - 1)^{-\chi_{et}(E'_i)}
\end{equation}
(3.4)
Proof. The expressions for $P_{\varphi}(T)$ follow immediately from the expressions for $\zeta_C(T)$, since $\varphi$ acts trivially on the degree 0 and degree 2 cohomology of $C$.

By [2, 3.3], the complex of $\ell$-adic tame nearby cycles $R\psi_{et}^i(Q_\ell)$ associated to $C$ can be endowed with a finite filtration such that the successive quotients are tamely constructible in the sense of [17], and such that $\varphi$ has finite order on these quotients. By [17, 6.1], this implies that
\begin{equation}
\zeta_C(T) = \int_{C_x} \bigoplus_{i=0}^{\infty} \zeta(\varphi | R\psi_{et}^i(Q_\ell)_x)
\end{equation}
(3.5)
where $\int^{\infty}$ is multiplicative integration w.r.t. the Euler characteristic, and
\begin{equation}
\zeta(\varphi | R\psi_{et}^i(Q_\ell)_x)
\end{equation}
is the constructible function on $C_x$ mapping a closed point $x$ of $C_x$ to
\begin{equation}
\prod_{i=0}^{1} \det(T \cdot Id - \varphi | R\psi_{et}^i(Q_\ell)_x)^{(-1)^{i+1}} \in \mathbb{Q}_\ell(T)
\end{equation}
(3.6)
Using the local computation of the tame nearby cycles in [1 I.3.3] we see that the expression (3.6) equals $(T^{N'_i} - 1)^{-1}$ if $x \in E'_i$ for some $i \in I$, and 1 else. Computing (3.5), we obtain the formula (3.1) for the zeta function (compare to A’Campo’s formula [3] in the complex geometric setting).

Now assume that $C$ is cohomologically tame. If $g(C) \neq 1$, then Saito’s geometric tameness criterion [18, 3.11] implies that $E'_i \cong \mathbb{G}_{m,k}$ if $N_i \neq N'_i$, and hence $\chi_{et}(E'_i) = 0$. Therefore, (3.3) follows from (3.1).

Finally, assume that $C$ is cohomologically tame, $g(C) = 1$, and $\delta(C)$ is prime to $p$. By [11 6.6], we know that $\delta(C)$ is the order of the class of the torsor $C$ in $H^1(K, A)$, and that the type of the model $C$ is $\delta(C)$ times the type of the minimal $\text{sned}$-model of $A$. Combining Saito’s criterion [18, 3.11] with the Kodaira-Néron reduction table for elliptic curves, we see that tameness of $C$ implies that $\chi_{et}(E'_i) = 0$ for each $i \in I$ such that $N_i \neq N'_i$.

Remark. Assume that $g(C) = 1$, and denote by $\delta'(C)$ the prime-to-$p$ part of $\delta(C)$. If $A = \text{Jac}(C)$, then
\begin{equation}
\zeta_C(T) = \zeta_A(T)
\end{equation}
On the other hand, [3.1] and [11 6.6] yield that
\begin{equation}
\zeta_C(T) = \zeta_A(T^{\delta'(C)})
\end{equation}
So, either $\delta'(C) = 1$, or $\zeta_A(T) = 1$, i.e. $A$ has semi-abelian reduction. We recover the well-known fact that $H^1(K, E)$ is a $p$-group if $E$ is an elliptic curve with additive reduction.

As an immediate corollary, we obtain an alternative proof of Theorem 2.1(i) in [13]. Denote by $g(E_i)$ the genus of $E_i$, for each $i \in I$. We put $a = \sum_{i \in I} g(E_i)$. We denote by $\Gamma$ the dual graph of $C$, and by $t$ its first Betti number.

**Corollary 3.2** (Lorenzini, [13]). We have

$$P_{\varphi}(T) = (T - 1)^{2a + 2t} \prod_{i \in I} \left( \frac{T^{N_i'} - 1}{T - 1} \right)^{2g(E_i) + d_i - 2}$$

**Proof.** This follows immediately from Theorem 3.1, the formula $\chi_{top}(E_i) = 2 - 2g(E_i) - d_i$, and the fact that $2 - 2t = \sum_{i \in I} (2 - d_i)$ (this is twice the Euler characteristic of $\Gamma$).

In [13], it was assumed that $\delta(C) = 1$, but our arguments show that this is not necessary. If $\delta(C) = 1$, then $a$ equals the abelian rank of $A = \text{Jac}(C)$ and $t$ its toric rank (see [14, p. 148]). We obtain an alternative proof of Conjecture 1.1 and Theorem 2.8.

**Corollary 3.3.** Assume that $\delta(C) = 1$, and put $A = \text{Jac}(C)$.

1. If $A$ does not have purely additive reduction, then $P_{\varphi}(1) = 0$. If, moreover, $A$ is tamely ramified, then (Trace) holds for $A$.
2. If $A$ has purely additive reduction, then $\phi_A = P_{\varphi}(1)$. If, moreover, $A$ is tamely ramified, then $\phi_A$ equals $\phi_A'$ and (Trace) holds for $A$.

**Proof.** (1) It follows from Corollary 3.2 that the order of 1 as a root of $P_{\varphi}(T)$ equals $2a + 2t$. Hence, if $A$ does not have purely additive reduction, then $P_{\varphi}(1) = 0$. Combining this with Corollary 2.8, we see that (Trace) holds for $A$ if $A$ is tamely ramified.

2. By [14] 1.5], we have

$$\phi_A = \prod_{i \in I} N_i^{d_i - 2}$$

because the toric rank of $A$ is zero. By Corollary 3.2, we know that

$$P_{\varphi}(T) = \prod_{i \in I} \left( \frac{T^{N_i'} - 1}{T - 1} \right)^{d_i - 2}$$

because $a = t = 0$. This yields

$$P_{\varphi}(1) = \prod_{i \in I} (N_i')^{d_i - 2} = \phi'_A$$

If $A$ is tamely ramified, then Saito’s criterion [18 3.11] implies that $d_i = 2$ if $N_i \neq N_i'$, so that

$$\phi'_A = \prod_{i \in I} (N_i')^{d_i - 2} = \prod_{i \in I} N_i^{d_i - 2} = \phi_A$$

Combining this with Corollary 2.8, we see that (Trace) holds for $A$.

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