Smooth manifolds vs differential triads

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Dedicated to Professor Serban Stratila, on the occasion of his seventieth birthday

Abstract

We consider differentiable maps in the setting of Abstract Differential Geometry and we study the conditions that ensure the uniqueness of differentials in this setting. In particular, we prove that smooth maps between smooth manifolds admit a unique differential, coinciding with the usual one. Thus smooth manifolds form a full subcategory of the category of differential triads, a result with physical implications.

1 Introduction

Circa 1990 A. Mallios used sheaf–theoretic methods to extend the mechanism of the classical differential geometry (CDG) of smooth manifolds to spaces, which do not admit the usual smooth structure (smooth atlas); see [5, 6]. In this new setting of abstract differential geometry (ADG) a large number of notions and results of CDG have already been extended ([7, 23]), becoming at the same time applicable to spaces with singularities and to quantum physics (see for instance, [10, 11, 13, 14]).

In ADG, the ordinary structure sheaf of smooth functions is replaced by a sheaf of abstract algebras $\mathcal{A}$, admitting a differential $\partial$ (in the algebraic sense), which takes values in an $\mathcal{A}$–module $\Omega$. A triplet $(\mathcal{A}, \partial, \Omega)$ like that is called a differential triad. Suitably defined morphisms organize the differential triads into a category denoted by $\mathcal{DT}$ ([17]). Every smooth manifold defines a differential triad and every smooth map between manifolds defines a morphism of the respective differential triads, so that the category $\text{Man}$ of smooth manifolds is embedded in $\mathcal{DT}$ (ibid.).

In the present paper we study the conditions assuring that a morphism in $\mathcal{DT}$ over a differentiable (in the abstract sense) map is uniquely determined, a situation analogous to the classical “uniqueness of differentials”. Especially we prove that a continuous map between manifolds, which is differentiable in $\mathcal{DT}$, is also smooth in the usual sense, and its abstract differential coincides with the ordinary one (Theorem 4.5). This result makes differentials of maps between manifolds unique in both the abstract and the classical setting, while $\text{Man}$ becomes a full subcategory of $\mathcal{DT}$ (Theorem 4.6).

As a consequence, we have that phenomena described via CDG, have exactly the same interpretation in the more general setting of ADG.

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2 Preliminaries

For the terminology applied the reader is mainly referred to [5]. However, for the reader’s convenience, we recall the basic notions we use throughout the paper as those of differential triads and of their morphisms; we also give a brief account of the way they form a category. Note that a smooth manifold will always be considered finite dimensional and 2nd countable. Moreover, all algebras considered are unital, commutative, associative and over the field \( \mathbb{C} \) of complex numbers; their units are denoted by 1.

2.1 Definition ([7]). Let \( X \) be a topological space. A differential triad over \( X \) is a triplet \( \delta = (A, \partial, \Omega) \), where \( A \) is a sheaf of algebras over \( X \), \( \Omega \) is an \( A \)-module and \( \partial : A \rightarrow \Omega \) is a Leibniz morphism, i.e., a \( \mathbb{C} \)-linear sheaf morphism, that satisfies the Leibniz condition:

\[
\partial(\alpha \beta) = \alpha \partial(\beta) + \beta \partial(\alpha), \quad (\alpha, \beta) \in A \times_X A,
\]

where “\( \times_X \)” means fiber product over \( X \).

2.2 Examples. (1) Smooth manifolds. Every smooth manifold \( X \) defines a differential triad

\[
(2.1) \quad \delta^\infty_X = (\mathcal{C}^\infty_X, d_X, \Omega^1_X).
\]

In this respect, \( \mathcal{C}^\infty_X \) is the structure sheaf of germs of smooth \( \mathbb{C} \)-valued functions on \( X \); \( \Omega^1_X \) is the sheaf of germs of its smooth \( \mathbb{C} \)-valued 1-forms, namely, it consists of the smooth sections of the complexification of the cotangent bundle; and, \( d_X \) is the sheafification of the usual differential. For details we refer to [7, Vol. II, p. 9]. We shall call (2.1) the smooth differential triad of \( X \).

Apart of the smooth differential triads over manifolds, the above abstraction includes also differential triads on arbitrary topological spaces. We give a brief account of some basic examples. Details are found in [20].

(2) Projective limits of manifolds. It is known that the projective limit of a projective system of manifolds is not necessarily a manifold. M.E. Verona [26] introduced a class of functions on such limits and defined a differential on these functions, in order to apply differential geometric considerations to the limits. Verona’s construction is a (real) differential triad.

(3) Sheafification of Kähler’s differential. Assume that \( A \) is a unital, commutative and associative \( \mathbb{C} \)-algebra. We denote by \( \mu \) the algebra multiplication, by \( \phi \) the canonical bilinear map \( \phi(x, y) = x \otimes y \) and by \( m \) the linear map that corresponds to \( \mu \), that is,

\[
(m \circ \phi)(x, y) = m(x \otimes y) = \mu(x, y) = xy,
\]

for every \( (x, y) \in A \times A \); we also set \( I := \ker m \). Then \( I \) is an ideal of \( A \otimes \mathbb{C} A \) and the map

\[
\delta_A : A \rightarrow I/I^2 : x \mapsto (x \otimes 1 - 1 \otimes x) + I^2.
\]

is a derivation (\( : \) Kähler’s differential) ([11, A.III. p.132]).
Let now \((A_U, r_U^V)\) be a presheaf of algebras of the above type over an arbitrary topological space \(X\), generating a sheaf \(\mathcal{A}\). Then the family \((A_U \otimes_{\mathbb{C}} A_U)\) generates \(\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}\). The respective families of maps \((\mu_U), (\phi_U)\), and \((\mu_U)\) are compatible with the presheaf restrictions, hence the family of kernels \((I_U := \ker m_U)\) is a presheaf of \(A(U)\)-submodules of \((A_U \otimes_{\mathbb{C}} A_U)\) generating a sheaf \(\mathcal{I}\). Besides, \(\delta_A = (\delta_{A_U})\) is proved to be a presheaf morphism whose factors are derivations. According to our definition, the triplet \((A, \delta_A, \mathcal{I}/\mathcal{I}^2)\) is a differential triad, called the sheafification of Kähler’s differential.

(4) Differential spaces. Differential spaces and the subsequent differential-geometric concepts on them have been introduced by R. Sikorski ([22, 23]). Their sheaf-theoretic generalization, due to M. A. Mostow ([15]), defines a differential triad.

(5) Differentiable spaces. Here by “differentiable spaces” we refer to Spallek’s \(\infty\)-standard differential spaces (see, for instance, [24]), described below: An \(\mathbb{R}\)-algebra \(A\) is a differentiable algebra, if there is some \(n \in \mathbb{N}\) and some closed ideal \(a\) of \(C^\infty(\mathbb{R}^n)\), so that \(A\) is (algebraically) isomorphic to the quotient

\[ A \cong C^\infty(\mathbb{R}^n)/a. \]

Let \(\mathcal{M}(A)\) be the spectrum of such an algebra. We denote by \(A_U\) the ring of (equivalence classes of) fractions \(\frac{a}{s}\), with \(a, s \in A\) and \(s(x) \neq 0\), for every \(x \in U\). Then \((A_U)_{U \in \tau_{\mathcal{M}(A)}}\) is a presheaf of algebras on \(\mathcal{M}(A)\) generating a sheaf \(\mathcal{A}\) called the structural sheaf on \(\mathcal{M}(A)\).

Now a pair \((X, \mathcal{O})\), where \(\mathcal{O}\) is a sheaf of algebras over a topological space \(X\), is called a differentiable space, if every point \(x \in X\) has an open neighbourhood \(U\) such that \((U, \mathcal{O}_U)\) is isomorphic to a pair \((\mathcal{M}(A), \mathcal{A})\), as above.

The sheafification of Kähler’s differential provides the spectrum \(\mathcal{M}(A)\) of any differentiable algebra \(A\) with a differential triad \((\mathcal{A}, \partial_A, \Omega_A)\), and the local coincidence of a differentiable space \((X, \mathcal{O})\) with some \((\mathcal{M}(A), \mathcal{A})\) provides an \(\mathcal{O}\)-module \(\Omega\) and a sheaf morphism \(\partial : \mathcal{O} \to \Omega\), so that \((\mathcal{O}, \partial, \Omega)\) is a differential triad over \(X\); for details, see [10].

(6) Differential algebras of generalized functions. A special case of an algebra which is a quotient of a functional algebra by a certain ideal has been defined by E. E. Rosinger [21]: Let \(\emptyset \neq X \subseteq \mathbb{R}^n\) open. The corresponding nowhere dense differential algebra of generalized functions on \(X\) is the quotient

\[ A_{nd}(X) := (C^\infty(X, \mathbb{R}))^N / \mathcal{I}_{nd}(X) \]

where \(\mathcal{I}_{nd}(X)\) is the nowhere dense ideal consisting of all the sequences \(w = (w_m)_{m \in \mathbb{N}}\) of smooth functions \(w_m \in C^\infty(X, \mathbb{R})\) which satisfy an “asymptotic vanishing property”. Then \(A_{nd} = (A_{nd}(U))_{U \in \tau_X}\) with the obvious restrictions is a presheaf of associative, commutative, unital algebras, inducing a (fine and flabby) sheaf \(\mathcal{A}\) on \(X\). Next, an \(\mathcal{A}\)-module \(\Omega\) is defined: for every \(U \in \tau_X\), \(\Omega(U)\) is the free \(A_{nd}(U)\)-module of rank \(n\), with the free generators \(d_i x_1, \ldots, d_i x_n\). Consequently, the elements of \(\Omega(U)\) take the form

\[ \sum_{i=1}^n V_i d_i x_i, \]
where $V_i \in A_{\operatorname{nd}}(U)$. Finally, the differential $\partial : \mathcal{A} \to \Omega$ is defined by the presheaf morphism $(\partial_U)_{U \in \tau_X}$, with
\[
\partial_U(V) = \sum_{i=1}^{n} \partial_i(V) d_i x_i
\]
where $\partial_i$ denotes the usual $i$-partial derivation. Hence a differential triad $(\mathcal{A}, \partial, \Omega)$ is obtained “including the largest class of singularities dealt with so far”. The algebras consisting the present structure sheaf “contain the Schwartz distributions”, while they also “provide global solutions for arbitrary analytic nonlinear PDEs. Moreover, unlike the distributions, and as a matter of physical interest, these algebras can deal with the vastly larger class of singularities which are concentrated on arbitrary closed, nowhere dense subsets and, hence, can have an arbitrary large positive Lebesgue measure” (\cite{12, Abstract}).

Due to specificities of sheaf theory and the adjunction of the functors $f_*$ and $f^*$ induced by a continuous map $f$, there are three equivalent ways to introduce the notion of a morphism of differential triads (cf. \cite{17, 18, 19}). The one in \cite{17} is a straightforward generalization of the situation we have in the theory of manifolds, and it is the most suitable for our purposes here. First we notice that if $\delta_X := (\mathcal{A}_X, \partial_X, \Omega_X)$ is a differential triad over $X$ and $f : X \to Y$ is a continuous map, then the push-out of $\delta_X$ by $f$
\[
f_* (\delta_X) \equiv (f_*(\mathcal{A}_X), f_* (\partial_X), f_* (\Omega_X))
\]
is a differential triad over $Y$.

2.3 Definition. Let $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X)$ and $\delta_Y = (\mathcal{A}_Y, \partial_Y, \Omega_Y)$ be differential triads over the topological spaces $X$ and $Y$, respectively. A morphism of differential triads $\hat{f} : \delta_X \to \delta_Y$ is a triplet $\hat{f} = (f, f_A, f_\Omega)$, where

(i) $f : X \to Y$ is continuous;

(ii) $f_A : \mathcal{A}_Y \to f_* (\mathcal{A}_X)$ is a unit preserving morphism of sheaves of algebras;

(iii) $f_\Omega : \Omega_Y \to f_* (\Omega_X)$ is an $f_A$-morphism, namely, it is a morphism of sheaves of additive groups, satisfying
\[
f_\Omega (aw) = f_A(a) f_\Omega(w), \quad \forall (a, w) \in \mathcal{A}_Y \times_Y \Omega_Y;
\]

(iv) The diagram

\[
\begin{array}{ccc}
\mathcal{A}_Y & \xrightarrow{f_A} & f_* (\mathcal{A}_X) \\
\downarrow \partial_Y & & \downarrow f_* (\partial_X) \\
\Omega_Y & \xrightarrow{f_\Omega} & f_* (\Omega_X)
\end{array}
\]

is commutative.

Extending the standard terminology, we shall say that a continuous map $f : X \to Y$ is differentiable, if it is completed into a morphism $\hat{f} = (f, f_A, f_\Omega)$. Besides, we say that $f_\Omega$ is a differential of $f$. 
If $\delta_X, \delta_Y, \delta_Z$ are differential triads over the topological spaces $X, Y, Z$, respectively, and $f = (f, f_A, f_\Omega) : \delta_X \rightarrow \delta_Y$, $g = (g, g_A, g_\Omega) : \delta_Y \rightarrow \delta_Z$ are morphisms, setting

\[(g \circ f)_A := g_*(f_A) \circ g_A \quad \text{and} \quad (g \circ f)_\Omega := g_*(f_\Omega) \circ g_\Omega \]

we obtain a morphism

(2.3) \quad \widehat{g \circ f} = (g \circ f, (g \circ f)_A, (g \circ f)_\Omega) : \delta_X \rightarrow \delta_Z.

The differential triads, their morphisms and the composition law defined by (2.2) and (2.3) form a category, which will be denoted by $\mathcal{DT}$. Note that the identity $\text{id}_\delta$ of a differential triad $\delta = (A, \partial, \Omega)$ over $X$ is the triplet $(\text{id}_X, \text{id}_A, \text{id}_\Omega)$.

2.4 Example. Consider the smooth manifolds $X$ and $Y$ provided with their smooth differential triads $\delta^\infty_X = (C^\infty_X, d_X, \Omega^1_X)$ and $\delta^\infty_Y = (C^\infty_Y, d_Y, \Omega^1_Y)$, respectively, and let $f : X \rightarrow Y$ be a smooth map. Then, for every $V \subseteq Y$ open, set

\[
\begin{align*}
C^\infty_Y(V) &\equiv C^\infty(V, \mathbb{C}) \quad \text{and} \\
f_*(C^\infty_X)(V) &:= C^\infty_X(f^{-1}(V)) \equiv C^\infty(f^{-1}(V), \mathbb{C}).
\end{align*}
\]

The map

(2.4) \quad f_A^* : C^\infty(Y) \rightarrow C^\infty(f^{-1}(Y)) : \alpha \mapsto \alpha \circ f

is a unit preserving algebra morphism, while the family $(f_A^*)_V$ is a presheaf morphism giving rise to a unit preserving algebra sheaf morphism $f_A : C^\infty_Y \rightarrow C^\infty_X$. On the other hand, the respective tangent map $Tf : TX \rightarrow TY$ defines the so-called pull–back of the smooth 1–forms by $f$

(2.5) \quad f^*_\Omega : \Omega^1_Y(V) \rightarrow \Omega^1_X(f^{-1}(V)) : \omega \mapsto \omega \circ Tf.

Note that $\omega \circ Tf$ is a standard notation in CDG, with

\[
(\omega \circ Tf)_x(u) = \omega_{f(x)}(T_xf(u)), \quad x \in X, \ u \in T^C_xX,
\]

where $T^C_xX$ is the complexification of the tangent space of $X$ at $x$ and $T_xf$ stands also for the extension of the tangent map $T_xf$ on $T^C_xX$. Then $f^*_\Omega$ is an $f_A^*$–morphism and the family $(f^*_\Omega)_V$ is a presheaf morphism yielding an $f_A$–morphism $f_\Omega : \Omega^1_Y \rightarrow f_*(\Omega^1_X)$. Note that, if $(V, \psi)$ is a chart of $Y$ with coordinates $(y_1, \ldots, y_n)$, and $\omega \in \Omega^1_Y(V)$, then there are $\alpha_i \in C^\infty(V, \mathbb{C})$, $i = 1, \ldots, n$, with $\omega = \sum_{i=1}^n \alpha_i \cdot dYy_i$. In this case the pull–back of $\omega$ by $f$ is given by

(2.6) \quad f^*_\Omega(\omega) = \sum_{i=1}^n (\alpha_i \circ f) \cdot (dYy_i \circ Tf).

The commutativity of Diagram 1 is equivalent to the chain rule, therefore

\[
(f, f_A, f_\Omega) : \delta^\infty_X \rightarrow \delta^\infty_Y
\]

is a morphism in $\mathcal{DT}$.

Clearly, if $\text{Man}$ is the category of smooth manifolds, the functor

(2.7) \quad F : \text{Man} \rightarrow \mathcal{DT},

where $F(X)$ is the smooth differential triad $\delta^\infty_X$ and $F(f) = (f, f_A, f_\Omega)$, described in Example 2.4, is an embedding.
3 Existence and uniqueness of morphisms

In the above abstract approach of differentiability two problems arise:

1. For arbitrary algebra sheaves, the existence of a morphism extending a map is not ensured, even for very simple mappings (for instance, the constant map);

2. likewise, the uniqueness of a morphism over a fixed \( f \) (the analogue of the uniqueness of differentials) is not ensured.

Regarding (1), let \( X, Y \) be topological spaces provided by the differential triads \( \delta_X \) and \( \delta_Y \) and let \( c \) denote some fixed element in \( Y \) and \( c : X \to Y \) the respective constant map. Then, for every \( V \subseteq Y \) open, with \( c \in V \),

\[
c_\ast(A_X)(V) = A_X(c^{-1}(V)) = A_X(X),
\]

while, if \( c \notin V \), then \( c_\ast(A_X)(V) = \emptyset \). Thus \( c_\ast(A_X) \) is a sheaf over the space \( \{c\} \) (at least for \( T_1 \)-spaces), whose (unique) stalk is the space of global sections \( A_X(X) \), and the question of differentiability of the constant map \( c \) reduces to the question of the existence of a unit preserving algebra morphism

\[
c_A : A_Y,c \to A_X(X),
\]

and, similarly, of a \( c_A \)-morphism

\[
c_\Omega : \Omega_Y,c \to \Omega_X(X),
\]

making Diagram 1 commutative. Note that \( A_{Y,c}, \Omega_{Y,c} \) stand for the corresponding stalks of \( A_Y, \Omega_Y \) at \( c \). The existence of a non–trivial (\( \) unit preserving) algebra morphism is not assured, of course, in the general case.

However, we obtain the differentiability of the constant map, if the sheaf \( A_Y \) is functional. In this respect, we recall that a functional algebra sheaf over \( Y \) (see [5, Vol. I, p. 49] and [9]) we mean a sheaf of algebras which is a subsheaf of the sheaf \( \mathcal{C}_Y \) of germs of continuous \( \mathbb{C} \)-valued functions on \( Y \). In this respect, we have

3.1 Proposition. If \( \delta_I = (A_I, \partial_I, \Omega_I) \) are differential triads over the spaces \( I = X, Y \) and \( A_Y \) is functional, then every constant map \( c : X \to Y \) is differentiable.

Proof. Since \( A_X \) has a unit, it contains the constant sheaf \( X \times \mathbb{C} \). Thus every \( k \in \mathbb{C} \) can be considered as a global section of \( A_X \). Besides, for every \( a \in A_{Y,c} \), there is an open neighborhood \( V \) of \( c \) in \( Y \) and \( \alpha \in A_Y(V) \subseteq \mathcal{C}(V, \mathbb{C}) \), with \( a = [\alpha]_c \). Setting

\[
c_A : A_{Y,c} \to A_X(X) : a \mapsto \alpha(c)
\]

we obtain a unit preserving algebra morphism.

On the other hand, \( c_A(a) \in \mathbb{C} \) implies \( \partial_X \circ c_A = 0 \). Thus the zero morphism \( 0 : \Omega_{Y,c} \to \Omega_X(X) \) completes the triplet \( (c, c_A, c_\Omega = 0) \), so that Diagram 1 is commutative.
Concerning (2), although there may be infinitely many pairs \((f_A, f_\Omega)\) making \(f\) differentiable, due to the commutativity of Diagram 1, in certain cases the pairs need to satisfy some restrictions: For instance, if the sheafification of Kähler’s differential is considered, then \(f_A\) determines \(f_\Omega\) (see [7, Vol. II, p. 327] and [20]). More generally, we have the following

3.2 Proposition. If \((f, f_A, f_\Omega)\) is a morphism in \(\mathcal{D}T\), then \(f_\Omega\) is uniquely determined by \(f_A\) on the image \(\text{Im} \partial_Y\) of \(\partial_Y\).

Conversely, if \(\partial_X\) vanishes only on the constant subsheaf \(X \times \mathbb{C} \subseteq A_X\), then \(f_A\) is uniquely determined by \(f_\Omega\).

Proof. The former assertion is obvious. Regarding the latter, if \((f, f_A, f_\Omega)\) and \((f, f'_A, f_\Omega)\) are morphisms in \(\mathcal{D}T\), the equality

\[
 f_\Omega(\partial_Y(a)) = f_*(\partial_X)(f_A(a)) = f_*(\partial_X)(f'_A(a))
\]

implies

\[
 f_*(\partial_X)(f_A(a) - f'_A(a)) = 0.
\]

Consequently, \(f_A(a) - f'_A(a) = c \in \mathbb{C}\), for every \(a \in A_Y\). Since \(f_A - f'_A\) is \(\mathbb{C}\)–linear, \(f_A - f'_A = 0\). \(\square\)

4 Morphisms over manifolds

In this section we need some concepts from the general theory of (non–normed) topological algebras (see [4]). We briefly introduce them. A topological algebra is a complex associative algebra \(A\), which is also a topological vector space such that the ring multiplication in \(A\) is separately continuous. A topological algebra \(A\) with a unit element is called a \(Q\)–algebra if the group \(G_A\) of its invertible elements is open (ibid., p. 139). The algebra \(C^\infty[0, 1]\) of all smooth functions on \([0, 1]\) is a \(Q\)–algebra [2, Example 6.23(3)]. A topological algebra \(A\) whose topology is defined by a directed family of submultiplicative seminorms is called locally \(m\)–convex. The preceding algebra is of this kind. If \(A\) is an algebra, a non–zero complex multiplicative linear functional of \(A\) is called character of \(A\). If \(A\) is a commutative locally \(m\)–convex \(Q\)–algebra with unit, we denote by \(\mathfrak{M}(A)\) the topological spectrum or simply spectrum of \(A\), consisting of all continuous characters of \(A\); the spectrum endowed with the relative weak* topology from the topological dual of \(A\), turns into a Hausdorff completely regular topological space. The spectrum of a commutative locally \(m\)–convex \(Q\)–algebra with unit is always a compact space (ibid., p. 87, Lemma 1.3). Moreover, each character of a \(Q\)–algebra is continuous [4, p. 72, Corol. 7.3].

As we have seen in Example 2.2, every smooth manifold \(X\) gives rise to a differential triad \(\delta_X^\infty = (C^\infty_X, d_X, \Omega_X^1)\), while a smooth map \(f : X \to Y\) between such manifolds is completed to the morphism \(F(f)\) given in Example 2.4. But there rises the following question: If \(f : X \to Y\) is a continuous map between smooth manifolds, can it be differentiable in the abstract setting, without being such in the classical sense?

The answer is no. This is due, on the one hand, to the fact that every character of a \(Q\)–algebra is automatically continuous and on the other hand, to the fact that
the continuous characters of our algebras of smooth functions are uniquely determined by point evaluations, corresponding to the elements of the domain of the respective smooth functions (see [2] p. 56, (4.43)], [4] p. 227, (2.6)]. This implies that all unit preserving morphisms, whose domain is a suitable function $Q$–algebra have the form (2.4), for an appropriate $f$ between the spectra of the topological algebras involved. In our case, for a smooth manifold $Y$, the algebra $C^\infty(Y)$ is a Fréchet (i.e., metrizable and complete) locally $m$–convex algebra [4] p. 131, discussion around (4.19)]. In particular, its spectrum is homeomorphic to $Y$ (see [2], Example 4.20(2)]. Thus, if $Y$ is compact, $C^\infty(Y)$ is a $Q$–algebra [4] p. 143, Prop. 1.1; p. 183, Corol. 1.1 and p. 187, Lemma 1.3] and every algebra morphism $h : C^\infty(Y) \to C^\infty(X)$ takes the form (2.4), for a suitable $f$ between the spectra of the above algebras, i.e., between $X$ and $Y$. In the case when $Y$ is not compact, the nice way that sheaf morphisms localize does the trick (see Theorem 4.4, below).

Let $S$ be a sheaf over $X$. For every open $U \subseteq X$, $S(U)$ is the set of all sections of $S$ over $U$. If $K \subseteq X$ is closed, we denote by $S(K)$ the inductive limit of all $S(U)$, with $U \subseteq X$ open and $K \subseteq U$, i.e.,

\[(4.1)\quad S(K) := \lim_{K \subseteq U} S(U).\]

Besides, if $f : S \to T$ is a sheaf morphism, we denote by

\[(4.2)\quad f_K := \lim_{K \subseteq U} f_U : S(K) \to T(K)\]

the inductive limit of all $f_U : S(U) \to T(U)$, with $U$ as above. If $A, B$ are open sets in $U$ with $B \subseteq A$, by definition $(r_A^K)$ and $(\rho_B^K)$ are the restrictions of the presheaves of sections of $S$ and $T$, respectively, and

\[r_V^K : S(V) \to S(K), \quad \rho_V^K : T(V) \to T(K)\]

are the canonical maps. Then the following diagram

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Diagram 2
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\[
\begin{array}{ccc}
S(V) & \xrightarrow{f_V} & T(V) \\
\downarrow r_V^K & & \downarrow \rho_V^K \\
S(K) & \xrightarrow{f_K} & T(K)
\end{array}
\]

commutes. That is,

\[(4.3)\quad f_K([\alpha]_K) = f_K \circ r_V^K(\alpha) = \rho_V^K \circ f_V(\alpha) = [f_V(\alpha)]_K,\]

for every $V \subseteq X$ open, with $K \subseteq V \subseteq U$, and every $\alpha \in S(V)$.

Suppose now that $K$ as before is compact. Consider the inductive system $\{C^\infty_X(V)\}_V$, $V \subseteq X$ open, with $K \subseteq V$ (where the connecting maps are the obvious ones) and put

\[C^\infty_X(K) := \lim_{K \subseteq V} C^\infty_X(V).\]

In this regard, we have
4.1 Proposition. Let $X$ be a smooth manifold and let $C^\infty_X$ denote the sheaf of germs of smooth $\mathbb{C}$–valued functions on $X$. Let $(U, \phi)$ be a chart on $X$ and $K \subseteq U$ compact. Then $C^\infty_X(K)$ is a locally $m$–convex $\mathbb{Q}$–algebra, whose spectrum coincides with $K$.

Proof. Since the families $\{V \subseteq X \text{ open : } K \subseteq V\}$ and $\{V \subseteq U \text{ open : } K \subseteq V\}$ are cofinal, we have

$$C^\infty_X(K) := \lim_{K \subseteq V} C^\infty_X(V) = \lim_{K \subseteq V \subseteq U} C^\infty_X(V).$$

Every open $V$ with $K \subseteq V \subseteq U$ is still the domain of a chart, namely of $(V, \phi|_V)$, thus on every $C^\infty_X(V)$ one can define the family of seminorms

$$(4.4) \quad N_{m,C}(f) := \sup_{|p| \leq m} (\sup_{x \in C} |D^p(f \circ \phi^{-1})(\phi(x))|),$$

for every $f \in C^\infty_X(V)$, where $C$ is a compact subset of $V$, $m \in \mathbb{N} \cup \{0\}$, $|p|$ stands for the length $|p| = p_1 + \cdots + p_n$ of the multi–index $p = (p_1, \ldots, p_n)$ and

$$D^p(f \circ \phi^{-1})(\phi(x)) = \left. \frac{\partial^{|p|}}{\partial u_1^{p_1} \cdots \partial u_n^{p_n}}(f \circ \phi^{-1}) \right|_{\phi(x)}$$

(see [4] p. 130, (4.14)). Topologized by the above family of seminorms, $C^\infty_X(V)$ becomes a Fréchet locally $m$–convex algebra (ibid., p. 130, (4.12)). We consider the inductive limit algebra $C^\infty_X(K)$ topologized by the locally $m$–convex inductive limit topology, that is, the finest locally $m$–convex topology on $C^\infty_X(K)$, making all the canonical maps

$$r^K_V : C^\infty_X(V) \to C^\infty_X(K)$$

continuous (ibid., p. 120, Def. 3.1); thus $C^\infty_X(K)$ becomes a locally $m$–convex algebra.

We set

$$N_K : C^\infty_X(K) \longrightarrow \mathbb{C} : a \longmapsto N_{0,K}(a),$$

where $a = [\alpha]_K$, with $\alpha \in C^\infty_X(V)$, for some open $V$ in $X$, with $K \subseteq V \subseteq U$ and $N_{0,K}$ is given by (4.4). Then $N_K$ is a submultiplicative seminorm on $C^\infty_X(K)$, with

$$N_K \circ r^K_V = N_{0,K} : C^\infty_X(V) \longrightarrow \mathbb{R},$$

for every $V$ as above, hence the topology induced by $N_K$ on $C^\infty_X(K)$ is a locally $m$–convex topology making all canonical maps $r^K_V$ continuous. Consequently, it is coarser than the locally $m$–convex inductive limit topology.

We prove that the topology defined by $N_K$ makes $C^\infty_X(K)$ a $\mathbb{Q}$–algebra: Let $a \in C^\infty_X(K)$ be invertible and let $b$ be its inverse. Then there are an open set $V$ in $X$, with $K \subseteq V \subseteq U$, and $\alpha, \beta \in C^\infty_X(V)$, such that $a = [\alpha]_K$, $b = [\beta]_K$ and $\alpha \cdot \beta = 1|_V$. Since $\alpha$ is invertible on $V \ni K$, $[\alpha]$ takes a least value

$$\varepsilon := \min\{|\alpha(y)| : y \in K\} > 0.$$

Then the open ball $B(a, \varepsilon/2)$ with respect to $N_K$ is an open neighborhood of $a$ contained in the group $G^\infty_{C^\infty_X(K)}$ of the invertible elements of $C^\infty_X(K)$. This makes $G^\infty_{C^\infty_X(K)}$ open.
with respect to the $N_K$–topology and, consequently, to the locally $m$–convex inductive limit topology.

Regarding the spectrum of $C^\infty_X(K)$, we have (see [4, p. 156, Th. 3.1] and [2, Example 4.20(2)], for the spectrum of $C^\infty_X(V)$)

$$
\mathfrak{M}(C^\infty_X(K)) = \lim_{K \subseteq V \subseteq U} \mathfrak{M}(C^\infty_X(V)) = \lim_{K \subseteq V \subseteq U} V = K,
$$

which completes the proof.

Taking into account that every character of a $Q$–algebra is continuous (see discussion at the beginning of this section), we obtain

**4.2 Corollary.** Let $X$ be a smooth manifold and let $C^\infty_X$ denote the sheaf of germs of smooth $\mathbb{C}$–valued functions on $X$. Then, for every $x \in X$, the stalk $C^\infty_{X,x}$ topologized with the locally $m$–convex inductive limit topology is a locally $m$–convex $Q$–algebra, with

$$
\mathfrak{M}(C^\infty_{X,x}) = \{ x \}.
$$

As a result, $C^\infty_{X,x}$ has a unique character, which is also continuous.

If $x \in V$, the usual evaluation map at $x$,

$$
ev_x^V : C^\infty_X(V) \to \mathbb{C} : \alpha \mapsto \alpha(x)
$$

is a continuous character of $C^\infty_X(V)$ [2, p. 58], and the family $(\ev_x^V)_V$ commutes with the presheaf restrictions, hence the inductive limit map

$$
ev_x := \lim_{x \in V} \ev_x^V : C^\infty_{X,x} \to \mathbb{C}
$$

exists and it is a (continuous) character of the stalk $C^\infty_{X,x}$. According to the preceding corollary, it is the unique (continuous) character in $\mathfrak{M}(C^\infty_{X,x})$. That is, we have the following

**4.3 Corollary.** The unique character of $C^\infty_{X,x}$ coincides with that given by (4.5).

**4.4 Theorem.** Let $X, Y$ be smooth manifolds and let $f : X \to Y$ be a continuous map. If there is a unit preserving morphism of algebra sheaves $f_* : C^\infty_Y \to f_*(C^\infty_X)$, then $f$ is smooth and

$$
f_*\alpha = \alpha \circ f,
$$

for every $V \subseteq Y$ open and every $\alpha \in C^\infty_Y(V)$.

**Proof.** Let $x \in X$ and $(U, \phi)$ a chart of $Y$ at $f(x)$. Consider the stalks

$$
\mathfrak{M}(C^\infty_Y(V)), \quad f_*(C^\infty_X)(f(x)) = \lim_{f(x) \in V \subseteq U} C^\infty_X(f^{-1}(V))
$$

and

$$
C^\infty_{Y,f(x)} = \lim_{f(x) \in V \subseteq U} C^\infty_Y(V),
$$

as follows.
and the algebra morphism
\[ f_{A,f(x)} := \lim_{f(x) \in V \subseteq U} f_{AV} : C_X^{\infty}(Y,f(x)) \to f_*(C_X^{\infty})f(x) \]
(cf. (4.1) and (4.2)), where \((f_{AV} : C_Y^{\infty}(V) \to f_*(C_X^{\infty})(V))_V\) is the presheaf morphism induced by \(f_A\) (see (2.4)). We now set
\[ \overline{ev}_x := \lim_{f(x) \in V \subseteq U} ev^{-1}_x : f_*(C_X^{\infty})f(x) \to \mathbb{C}, \]
where \(ev^{-1}_x : C_X^{\infty}(f^{-1}(V)) \to \mathbb{C}\) is the usual evaluation at \(x \in f^{-1}(V)\). Then \(\overline{ev}_x\) is an algebra morphism, therefore,
\[ \overline{ev}_x \circ f_{A,f(x)} : C_{Y,f(x)}^{\infty} \to \mathbb{C} \]
is a character of \(C_{Y,f(x)}^{\infty}\). By virtue of Corollary 4.3, this character coincides with the one given by
\[ ev_{f(x)} := \lim_{f(x) \in V \subseteq U} ev^V_{f(x)} : C_{Y,f(x)}^{\infty} \to \mathbb{C}. \]

Further, if \((r_V^A)\) and \((\rho_V^A)\), \(A,B\) open in \(Y\) with \(B \subseteq A\), are the restrictions of the presheaves of sections of \(C_Y^{\infty}\) and of \(f_*(C_X^{\infty})\), respectively, by the definition of \(ev_{f(x)}\), we have
\[ ev_{f(x)}([\alpha]_{f(x)}) = ev_{f(x)} \circ r^V_{f(x)}(\alpha) = ev^V_{f(x)}(\alpha) = \alpha(f(x)) \]
for every \(V \subseteq Y\) open with \(f(x) \in V\) and every \(\alpha \in C_Y^{\infty}(V)\), while
\[ (\overline{ev}_x \circ f_{A,f(x)})([\alpha]_{f(x)}) = (\overline{ev}_x \circ f_{A,f(x)})(r^V_{f(x)}(\alpha)) = ev^V_{f(x)}(f_{AV}(\alpha)) = ev^{-1}_x(f_{AV}(\alpha)) = (f_{AV}(\alpha))(x). \]

That is,
\[ (f_{AV}(\alpha))(x) = \alpha(f(x)), \quad \forall x \in X, \]
which proves (4.6).

Regarding the smoothness of \(f\), it suffices to notice that, for every \(V \subseteq Y\) open and every \(\alpha \in C_Y^{\infty}(V)\), we have that
\[ \alpha \circ f = f_{AV}(\alpha) \in C_X^{\infty}(f^{-1}(V)), \]
so \(\alpha \circ f\) is smooth, and this completes the proof. \(\square\)

4.5 Theorem. Let \(X, Y\) be smooth manifolds provided with their smooth differential triads \(\delta_X^{\infty}\) and \(\delta_Y^{\infty}\). Besides, let \(\hat{f} = (f,f_A,f_{f}) : \delta_X^{\infty} \to \delta_Y^{\infty}\) be a morphism in \(\mathcal{DT}\). Then \(f\) is smooth in the ordinary sense and \(\hat{f} = F(f)\), where \(F\) is the embedding (2.7).
Proof. By the preceding Theorem 4.4, $f$ is smooth and the presheaf morphisms $(f_{AV})_V$ are given by (2.4).

In order to prove that $f_{AV}$ satisfies (2.6), it suffices to prove it for the domains of the charts of the maximal atlas of $Y$, since they form a basis for its topology. Thus let $(V, \psi)$ be a chart with coordinates $(y_1, \ldots, y_n)$, and let $\omega = \sum_{i=1}^n \alpha_i \cdot dy_i \in \Omega^1_Y(V)$. Applying the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{C}^\infty(V, \mathbb{C}) & \xrightarrow{f_{AV}} & \mathcal{C}^\infty(f^{-1}(V), \mathbb{C}) \\
d_{YV} & & d_{Xf^{-1}(V)} \\
\Omega^1_Y(V) & \xrightarrow{f_{AV}} & \Omega^1_X(f^{-1}(V))
\end{array}
\]

Diagram 3

we obtain

\[
f_{AV}(\omega) = f_{AV}\left(\sum_{i=1}^n \alpha_i \cdot dy_i\right) = \sum_{i=1}^n f_{AV}(\alpha_i) \cdot f_{AV}(dy_i)
\]

\[
= \sum_{i=1}^n (\alpha_i \circ f) \cdot (f_{AV} \circ d_{YV})(y_i)
\]

\[
= \sum_{i=1}^n (\alpha_i \circ f) \cdot (d_{Xf^{-1}(V)} \circ f_{AV})(y_i)
\]

\[
= \sum_{i=1}^n (\alpha_i \circ f) \cdot d_{Xf^{-1}(V)}(y_i \circ f)
\]

\[
= \sum_{i=1}^n (\alpha_i \circ f) \cdot (d_{YV}y_i \circ Tf)
\]

which proves our assertion.

The last two results imply the following

4.6 Theorem. Man is a full subcategory of $\mathcal{D}T$. In other words, when smooth manifolds $X$ and $Y$ are considered, the sets of morphisms between them in the categories $\text{Man}$ and $\mathcal{D}T$ coincide; that is,

\[
\text{Hom}_{\text{Man}}(X,Y) \cong \text{Hom}_{\mathcal{D}T}(X,Y).
\]

(4.7)

For the term “full subcategory”, see [3, p. 15].

Final remark. As we noticed in the introduction, ADG applies in large scale phenomena (i.e. general relativity) when singularities appear [8, 10, 13] as well as in quantum mechanics [14]. It also embodies phenomena usually studied by the ordinary CDG. Theorem 4.5 implies that ADG applied on the latter gives the same results with CDG.

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