Factorization theorems for strong maps between matroids of arbitrary cardinality

1 Introduction and preliminaries

Much attention has been given to strong maps in the category of finite matroids (cf. [1–5], [6, Ch.7], [17, Ch.8]). The most well known results for strong maps in the category of finite matroids are factorization theorems.

Matroid theory is a theory not only of finite matroids but also of infinite matroids. If the factorization theorems do not extend to the case of infinite matroids they would appear to be incomplete. Therefore we need to find the factorization theorems of matroid theory for the class of infinite matroids.

Oxley points out in [8] that there is no single class of structures given the name infinite matroids. A variety of classes of matroid-like structures on infinite sets have been studied for different reasons by various authors.

Many of the recent results in infinite matroid theory are for matroids of arbitrary cardinality (cf. [9–15]). Matroids of arbitrary cardinality seem therefore to be the most studied classes of infinite matroids yielding fruitful results. In this paper we will adopt the definition of infinite matroid given in [9], i.e. the definition of matroid of arbitrary cardinality, and study the factorization theorems for strong maps of infinitie matroids of this type.

We start by reviewing those aspects of matroid theory which we will need. Firstly, we assume that $E$ is some arbitrary, possibly infinite set; $\mathcal{P}(E)$ is the set of all subsets of $E$; $\mathbb{N}_0 = \{0, 1, 2, \ldots \}$.

Definition 1.1 ([9]).
1. Assume $m \in \mathbb{N}_0$ and $F \subseteq \mathcal{P}(E)$. Then the pair $M := (E, F)$ is called a matroid of rank $m$ with $F$ as its closed sets, if the following axioms hold:
   (F1) $E \in F$;
   (F2) If $F_1, F_2 \in F$, then $F_1 \cap F_2 \in F$;
   (F3) Assume $F_0 \in F$ and $x_1, x_2 \in E \setminus F_0$. Then one has either:
      (i) $\{F \in F | F_0 \cup \{x_1\} \subseteq F\} = \{F \in F | F_0 \cup \{x_2\} \subseteq F\}$ or
      (ii) $F_1 \cap F_2 = F_0$ for certain $F_1, F_2 \in F$ containing $F_0 \cup \{x_1\}$ or $F_0 \cup \{x_2\}$, respectively;
   (F4) $m = \max\{n \in \mathbb{N}_0 | \text{there exist } F_0, F_1, \ldots, F_n \in F \text{ with } F_0 \subset F_1 \subset \cdots \subset F_n = E\}$. 

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(2) Let \( M = (E, \mathcal{F}) \) be a matroid. The closure operator \( \sigma = \sigma_M : \mathcal{P}(E) \to \mathcal{F} \) of \( M \) is defined by \( \sigma(A) := \bigcap_{F \in \mathcal{F}} F \). \( M \) is called simple, if any subset \( A \subseteq E \) with \(|A| \leq 1 \) lies in \( \mathcal{F} \).

Remark. The definition of a finite matroid can be found in [5, Ch.1], [6, Ch.1] and [7, Ch.2] that of identity map is cf. [16, p.12] and rank function of a matroid of arbitrary cardinality is come from [9]. From these definitions and their some relative properties, it is easy to know that any finite matroid on \( E \) is cf. \([16, p.12]\) and rank function of a matroid of arbitrary cardinality is come from \([9]\). From these definitions and their some relative properties, it is easy to know that any finite matroid on \( E \) is a matroid of arbitrary cardinality on \( E \). Hence, in this paper, except explaining in a special way, a matroid always means a matroid of arbitrary cardinality defined in Definition 1.1.

Lemma 1.2 ([9]). Assume \( M = (E, \mathcal{F}) \) is a matroid with \( \sigma \) as its closure operator. Then for any family \((F_i)_{i \in I}\) of closed sets in \( M \), one has also \( F := \bigcap_{i \in I} F_i \in \mathcal{F} \). \( \sigma(A) \) is the smallest set in \( \mathcal{F} \) containing \( A \subseteq E \). In particular, one has \( \sigma(A) = A \Leftrightarrow A \in \mathcal{F} \). Moreover, \( A \subseteq \sigma(A) = \sigma(\sigma(A)) \) holds for \( A \subseteq E \); for \( A \subseteq B \subseteq E \), one has \( \sigma(A) \subseteq \sigma(B) \).

In the rest of this section we give a few results and properties of matroids which will be needed in the next section.

Lemma 1.3.

1. Let \( M_1 = (E_1, \mathcal{F}_1), M_2 = (E_2, \mathcal{F}_2) \) be matroids on disjoint sets \( E_1, E_2 \) with \( m_1, m_2 \) as their ranks, respectively. Let \( \mathcal{F} = \{F_1 \cup F_2 | F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\} \). Then \( M = (E = E_1 \cup E_2, \mathcal{F}) \) is a matroid defined on \( E \) with rank \( m_1 + m_2 \). We call \( M \) the direct sum of \( M_1 \) and \( M_2 \), written \( M_1 \oplus M_2 \).

2. Let \( M = (E, \mathcal{F}) \) be a matroid with \( \sigma \) as its closure operator, \( T \subseteq E \) and \( \mathcal{F}_T = \{A \cap T | A \subseteq T\} \). Then \( (T, \mathcal{F}_T) \) is a matroid on \( T \). We denote this matroid by \( M|T \) and call it the restriction of \( M \) to \( T \).

Proof. One only need check that (F1)-(F4) are satisfied by \( M \) and \( (T, \mathcal{F}_T) \) respectively. These checks are straightforward. \( \square \)

The following definitions are for matroids of arbitrary cardinality. They are simply generalizations of the corresponding definitions for finite matroids (cf. [6, 7]).

Definition 1.4.

1. Let \( M = (E, \mathcal{F}) \) be a matroid with \( \sigma \) as its closure operator. Then

A loop of \( M \) is an element \( x \) of \( E \) such that \( x \in \sigma(0) \).

If \( x, y \in E \) and \( x \neq y \), then \( x \) is parallel to \( y \) if and only if \( x \in \sigma(y) \) and \( y \in \sigma(x) \) and neither \( x \) nor \( y \) are loops.

Let \( T \subseteq E \) and \( N = (T, \mathcal{F}_T) \) be a matroid with \( \sigma' \) as its closure operator satisfying the following statement: if for any \( A \subseteq T \) and \( x \in T \setminus A \), it is always true that \( x \in \sigma'(A) \Leftrightarrow x \in \sigma(A \cup (E \setminus T)) \), then \( N \) will be called the contraction of \( M \) to \( T \) denoted by \( M \cdot T \).

2. Two matroids \( M_1 \) and \( M_2 \) on \( E_1 \) and \( E_2 \) are isomorphic, denoted by \( M_1 \cong M_2 \), if there is a bijection \( \varphi : E_1 \to E_2 \) which preserves the closure operation.

3. Let \( M \) be a matroid on \( E \) and \( 0 \) be an element not in \( E \). The matroid \( M_0 \) is the direct sum \( M \uplus \{0\} \) on the set \( E \uplus \{0\} \), where \( \{0\} \) is the matroid of rank zero on the single-element set \( \{0\} \). We use the same symbol \( 0 \) for the zero of any matroid \( M_0 \).

Let \( M = (E, \mathcal{F}_M) \) and \( N = (T, \mathcal{F}_N) \) be matroids. A strong map \( f \) from \( M \) to \( N \) is a function \( f : E \uplus \{0\} \to T \uplus \{0\} \), mapping \( 0 \) to \( 0 \), and satisfying the following condition: the inverse image of any closed set of \( N_0 \) is a closed set of \( M_0 \).

Note. We use \( f \) to denote both the function \( E \uplus \{0\} \to T \uplus \{0\} \) and its restriction to \( E \). Also by a map \( g : E \to T \) we will mean a map \( g : E \uplus \{0\} \to T \uplus \{0\} \) in which \( g(0) = 0 \).
By Lemma 1.3 and Definition 1.4, it is easy to obtain the following Lemma 1.5 which is a result about a type of strong map which arises from “submatroids”. We will see in the following section that these maps play an important role in factorization theorems for strong maps.

**Lemma 1.5.**

1. Let $M_i$ be a matroid on $E_i$ $(i = 1, 2)$ where $E_1 \cap E_2 = \emptyset$. Then $M_i = (M_1 + M_2)|_{E_i}$, $i = 1, 2$.

2. Let $M_i$ be a matroid on $E_i$ $(i = 1, 2, 3)$ and $g : M_1 \rightarrow M_2$ and $f : M_2 \rightarrow M_3$ be strong maps. Then $fg : M_1 \rightarrow M_3$ is a strong map.

3. Let $M = (E, \mathcal{F})$ be a matroid. Then for $T \subseteq E$ and $N = M \mid T$, the inclusion map $i : T \rightarrow E$ defines a strong map, the injection map. For $S \subseteq E$ and $N = M \cdot S$, define $c : E \rightarrow S$ by $c(x) = \begin{cases} x, & x \in S \\ 0, & x \notin S \end{cases}$. The contraction map $c$ is a strong map. Let $s(M)$ be the simplification of $M$, that is all mutually parallel elements of $M$ are identified in $s(M)$ and all loops of $M$ are sent to the zero element in $s(M)$. Any simplification is a strong map. Specifically, when $M \cdot T$ exists, we define the projection of $M$ to $T \subseteq E$ as the composite map $s : M \rightarrow s(M) \cdot T$. Then the projection map is a strong map.

**2 Factorization theorems**

In this section we discuss factorization theorems for strong maps between matroids and present the two main results of this paper, Theorem 2.1 and Theorem 2.3.

To begin we note that the result of [6, p. 315, Theorem 3] is that of Crapo [1] and Higgs [2]. In [4], Mao and Liu have pointed out that [2, Theorem III] is wrong and as noted in [4] the discussion in Crapo is correct but the definition of strong map in Crapo is only a special case of that in [6]. Therefore [1] and [2] are of no use to the present discussion.

Other references to factorization theorems for strong maps are [3, 8.2.7 and 8.2.8], [5, pp. 267-268, Proposition 7.3.1.1, Corollary 7.3.12 and Corollary 7.3.13], [6, Ch.17, §4]. But the content of [3, 5] is the same as that of [6, Ch.17, §1] which in turn is a special case of [6, pp. 315-316, Theorem 3 and Theorem 4]. In other words it is not useful to revise the contents of [6, pp. 315-316, Theorem 3 and Theorem 4], [3, 5] and [6, Ch.17, §4].

We will see that Theorem 2.1 is simply a generalization of the corresponding result for finite matroids (cf. [6, p.313,Theorem 1]). In order to draw a comparison between the proof of our Theorem 2.1 and that of [6, p.313,Theorem 1] we mimic the structure of the proof in [6, p.313,Theorem 1]. Of course the mathematical content differs.

**Theorem 2.1.** Let $M_1 = (E_1, \mathcal{F}_1)$ and $M_2 = (E_2, \mathcal{F}_2)$ be matroids on disjoint sets $E_1, E_2$ and suppose there exist strong maps $f_i : M_i \rightarrow M$ $(i = 1, 2)$, where $M = (E, \mathcal{F}_M)$ is a matroid on $E$. Then there is a unique map $f : (M_1 + M_2) \rightarrow M$ such that the figure below commutes where $i_1$ and $i_2$ are the inclusion maps.

**Proof.** Define $f : E_1 \cup E_2 \rightarrow E$ by $f(x) = \begin{cases} f_1(x), & x \in E_1 \\ f_2(x), & x \in E_2 \end{cases}$. For any $F \in \mathcal{F}_M$, by the strong property of $f_i$, one has $f_i^{-1}(F) \in \mathcal{F}_i$ $(i = 1, 2)$, and further, $f^{-1}(F) = f_1^{-1}(F) \cup f_2^{-1}(F) \in \mathcal{F}_{M_1} + M_2$ according to Lemma 1.3. Hence $f$ is strong. Evidently, $f_1 =fi_1$ and $f_2 = fi_2$. The uniqueness of $f$ is easy to see.

Before presenting the second main result of this section, Theorem 2.3, we consider the following preparatory results.

**Lemma 2.2.** Let $f : M = (E, \mathcal{F}_M) \rightarrow N = (T, \mathcal{F}_N)$ be a surjective and strong map and let $N$ be a simple map. Then $f$ is (up to isomorphism) a projection.
Proof. Let $\mathcal{F}_1 = \{ f^{-1}(B) | B \in \mathcal{F}_N \}$ and $M_1 = (E, \mathcal{F}_1)$. Then it is easy to check that $M_1$ satisfies (F1)-(F4). So, $M_1$ is a matroid.

Let $\sigma_{M_1}$ be the closure operator of $M_1$. For any $x \in E$, if $f(x) = a$, then one has $x \in \sigma_{M_1}(x) \subseteq f^{-1}(a)$ in view of Lemma 1.2. $\sigma_{M_1}(x) \in \mathcal{F}_1$ implies that there is $B \in \mathcal{F}_N$ satisfying $f^{-1}(B) = \sigma_{M_1}(x)$, and so $f(x) \in B \subseteq \{a\}$. Hence $B = \{a\}$. Furthermore, it follows that $\sigma_{M_1}(x) = f^{-1}(a)$.

Since $N$ is simple, one that if $x$ is a loop of $M_1$, then $x \in \sigma_{M_1}(\emptyset) = \sigma_{M_1}(0) = f^{-1}(0)$. Conversely, suppose $0 \neq y \in f^{-1}(0)$. One has $\sigma_{M_1}(y) \subseteq f^{-1}(0) = \sigma_{M_1}(0) = \sigma_{M_1}(\emptyset)$, and so $y$ is a loop of $M_1$. That is $f^{-1}(0) = \{x \in E | x \text{ is a loop of } M_1 \cup \{0\} \}$ is true. In addition, if $x$ is parallel to $y$ in $M_1$, i.e. $x \in \sigma_{M_1}(y), y \in \sigma_{M_1}(x)$ and $x, y \notin f^{-1}(0)$, we must have $f(x) = f(y)$. We notice that if $y \in f^{-1}(a)$ where $a = f(x)$, then $f(y) = a$, and furthermore $\sigma_{M_1}(y) = f^{-1}(a) = \sigma_{M_1}(x)$. This implies that $x$ is parallel to $y$.

Denote $f^{-1}(0)$ by $K$ and set $\mathcal{F}_2 = \{ X \subseteq E \setminus K | X \cup K \in \mathcal{F}_1 \}$. Then it is straightforward to show that (F1)-(F4) are verified by $(E \setminus K, \mathcal{F}_2)$.

Let $\sigma_2$ be the closure operator of $(E \setminus K, \mathcal{F}_2)$. One obtains $\sigma_2(A) \cup K \in \mathcal{F}_1$ holds for $\forall A \subseteq E \setminus K$, that is there exists $B \in \mathcal{F}_N$ such that $\sigma_2(A) \cup K = f^{-1}(B) = f^{-1}(B) \cup K = \sigma_{M_1}(A \cup K)$.

Let $x \in (E \setminus K) \cup A = E \setminus (K \cup A)$. Then $x \in \sigma_2(A)$ if and only if $x \in \sigma_{M_1}(A \cup K) = \sigma_{M_1}(A \cup (E \setminus (E \setminus K)))$. Consequently, $(E \setminus K, \mathcal{F}_2) = M_1 \cdot (E \setminus K) = c(M_1)$ holds true.

Let $s(c(M_1)) = (E_s, \mathcal{F}_s)$ be the simplification of $c(M_1)$. By the properties of $c(M_1)$ and $M_1$, one obtains that $E_s = \{a^{-1} | \exists a \in T, s(c(f^{-1}(a))) = a^{-1}\}$. Let $g : (E_s, \mathcal{F}_s) \to N$ be defined by $g : a^{-1} \mapsto a$. Clearly $g$ is a bijection. For $\emptyset \neq A = \{a_\alpha \in T | \alpha \in A \} \in \mathcal{F}_N$, one has $f^{-1}(A) = \{ f^{-1}(a_\alpha) | a_\alpha \in A \} \cup K$ by the property of $f$. Evidently $K \cap \{ f^{-1}(a_\alpha) | a_\alpha \in A \} = \emptyset$, and so $f^{-1}(a_\alpha) \in \mathcal{F}_2$. Further $s\{ f^{-1}(a_\alpha) | a_\alpha \in A \} = \{ a^{-1}_\alpha | a_\alpha \in A \} \in \mathcal{F}_s$, i.e. $g^{-1}(A) \in \mathcal{F}_s$. Similarly, by the properties and definitions of $M_1$, $c(M_1)$, and $s(c(M_1))$, one obtains that for $\{a^{-1}_\alpha | a_\alpha \in A \} \in \mathcal{F}_s$, we have $A = \{a_\alpha \in T | a_\alpha \in A \} \in \mathcal{F}_N$. Hence $g$ is an isomorphism.

Let $id$ be the identity map on $E \cup \{0\}$ from $M$ to $M_1$. It is easy to see that $id$ is a strong map. Therefore $(gsc(id))$ is a strong map from $M$ to $N$. Furthermore, for every $x \in E$, if $f(x) = a \in T$ then $(gsc(id))(x) = (gsc)(id(x)) = (gsc)(x)$. We see that when $a = 0$ one has $x \in K$ and so $c(x) = 0$, and furthermore $s(c(x)) = 0, g(s(c(x))) = 0 = f(x)$; when $a \neq 0$, one obtains $x \in E \setminus K$, i.e. $x \in f^{-1}(a)$, and so $c(x) = x$, and furthermore $s(c(x)) = a^{-1}, g(s(c(x))) = g(a^{-1}) = a = f(x)$.

Consequently, $f(x) = g(s(c(id(x)))) = g(s(c(x))) = g((s(c))(x))$ for $x \in E$. Hence $f$ is (up to isomorphism) a projection $sc$.}

Using Lemma 2.2, we obtain Theorem 2.3.
**Theorem 2.3.** Let $M = (E, \mathcal{F}_M)$ and $N = (T, \mathcal{F}_N)$ be matroids. Let $N$ furthermore be simple. If $f : M \rightarrow N$ is a strong map, then $f$ is (up to isomorphism) the composition of an injection and a projection.

**Proof.** Let $h : T \rightarrow T'$ be a bijection and $T' \cap E = \emptyset = T' \cap T$, $\mathcal{F}_{N'} = \{h(F) \subseteq T'|h(F) = \{h(x)|x \in F}\}$. Clearly, $N' = (T', \mathcal{F}_{N'})$ is a matroid on $T'$ and $N \simeq N'$. Here, up to isomorphism, $f$ is a strong map from $M$ to $N'$. From this we can assume that $M$ and $N$ are on disjoint sets $E$ and $T$.

Let $j, j'$ be the natural injections, i.e. $j : M \rightarrow M + N$ and $j' : N \rightarrow M + N$. Let $id$ be the identity function $id : N \rightarrow N$. By Theorem 2.1, there is a strong map $g : M + N \rightarrow N$ such that the diagram below commutes and $g(x) = \begin{cases} f(x), & x \in E \\ x, & x \in T \end{cases}$

Fig. 2

By Lemma 1.5, $N = (M + N) \cdot T$ holds. Clearly $T \subseteq E \cup T$ and $g : M + N \rightarrow N$ is surjective. Using Lemma 2.2 we deduce that up to isomorphism, $g$ is a projection, so that $f$ is (up to isomorphism) the composition of an injection and a projection.

A further result for this section is based on the following observation. By [4] it is evident that the proof of [6, p.315, Theorem 3] is incorrect. The proof of the important result [6, p.316, Theorem 4] relies on [6, p.315, Theorem 3]. This makes us question the validity of [6, p.316, Theorem 4]. By Theorem 2.3 and the property that any finite matroid is a matroid of arbitrary cardinality it is evident that [6, p.316, Theorem 4] should read as follows:

“Let $f : M \rightarrow N$ be a strong map where $M, N$ are finite matroids and $N$ is simple. Then $f$ is (up to isomorphism) the composition of an injection and a projection”.

So in the finite case the method presented here differs from that of Welsh in [6].

In this paper we used the definition of infinite matroid given in [9]. Comparing the definition of infinite matroid given in [9] with that in [17] (i.e. finitary matroids in [17]), we find that the definition in [17] is more general than that in [9]. In addition, the authors of [17] not only compared the properties of finite matroids with those of ‘B-matroids’ by Oxley [8], but also sought to find a formulation of the axioms of infinite matroids. Hence, in future work we may consider the factorization theorems for strong maps between infinite matroids which satisfy the axioms of a finite matroid given in [17].

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