Flow equations for QED in the light front dynamics

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Abstract

The method of flow equations is applied to QED on the light front. Requiring that the partical number conserving terms in the Hamiltonian are considered to be diagonal and the other terms off-diagonal an effective Hamiltonian is obtained which reduces the positronium problem to a two-particle problem, since the particle number violating contributions are eliminated. No infrared divergences appear. The ultraviolet renormalization can be performed simultaneously.

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1 Introduction

We use the method of flow equations to construct an effective Hamiltonian starting from the light-front formulation which can be used to solve the bound state problem.

Recently Glazek and Wilson [1] and one of the authors [2] have suggested methods to diagonalize Hamiltonians continuously which have been called similarity renormalization and flow equations by the authors, resp.

It is common to both methods that they eliminate by means of a unitary transformation initially the off-diagonal matrix elements between states with large energy differences and continue with states closer and closer in energy, so that off-diagonal matrix elements between states of energy difference larger than \( \lambda \) are eliminated or strongly suppressed. The final aim is to eliminate them completely (\( \lambda \to 0 \)) and to obtain a diagonalized Hamiltonian.

Application of this method to an \( n \)-orbital model has shown, that a literal use of this concept can lead to convergency problems [2]. Instead one may leave the idea of diagonalizing immediately in favor of block-diagonalizing. If matrix-elements between states of equal particle number are considered diagonal, then the procedure brings the Hamiltonian into a block-diagonal form, a procedure which worked much better in the case of the \( n \)-orbital model, where block-diagonalization with respect to the quasiparticle number (number of electrons above the Fermi edge plus number of holes below the Fermi edge) was performed.

It becomes apparent from the calculations by Jones, Perry and Glazek [3] on the basis of the similarity transformation, that this scheme works well down to energy differences of the order of Rydberg, but if one goes below, then contributions in higher orders in the coupling become important.

Indeed it seems to be rather difficult to obtain bound states from plane waves by continuous unitary transformations. In eliminating only the terms which do not conserve the number of particles one postpones the diagonalization, but reduces the problem to one in the space of fixed particle number [4]. Thus for the positronium problem it is sufficient to determine the one- and two-particle contribution of the Hamiltonian for electrons and positrons.

Basically the procedure is very similar to that of the elimination of the electron-phonon interaction [5] which yields an effective attractive interaction between electrons responsible for superconductivity. Both the method of flow equations and the similarity renormalization [6] yield results different from Fröhlich’s original ones [8] but in very good agreement with more sophisticated methods. In QED it is the interaction of the electrons with the photons instead of the phonons which has to be eliminated.

A basic advantage of the methods of similarity renormalization and flow equations in comparison to conventional perturbation theory is, that one obtains normally less singular effective interactions.

This procedure is similar to the Tamm-Duncoff Fock space truncation [9, 10, 11, 12] in the sense that also in this truncation particle number changing interactions are eliminated.

We define an effective, renormalized Hamiltonian

\[
H_{\text{eff}} = \lim_{\lambda \to 0, \Lambda \to \infty} H_B(\lambda, \Lambda)
\]  

(1)

where \( H_B(\lambda, \Lambda) \) is obtained by means of the unitary transformation \( U(\lambda, \Lambda) \) from the bare Hamiltonian (regularized at the cutoff \( \Lambda \))

\[
H_B(\lambda, \Lambda) = U(\lambda, \Lambda) H_B(\Lambda) U^\dagger(\lambda, \Lambda).
\]  

(2)

The unitary transformation \( U(\lambda, \Lambda) \) is determined by the flow equations below.
Flow equations eliminate the far-off diagonal matrix elements in the 'energy space' \( (\text{the matrix elements between the states with large energy jumps } |E_i - E_j| > \lambda, \text{ where } \lambda \text{ is the 'running' UV cutoff}) \) but only for those blocks that change the number of quasiparticles. The value of \( \lambda = \Lambda \to \infty \) corresponds to the initial bare Hamiltonian, a finite \( \lambda \) determines the effective Hamiltonian at an intermediate stage, for \( \lambda = 0 \) the elimination of the 'non-diagonal' part in the Fock representation is complete fig. (1).

The method of flow equations in the perturbative theory frame is discussed in section 2. In section 3 we apply this scheme to QED on the light front. The effective Hamiltonian serves for the calculation of bound states, which due to the triviality of the vacuum state is now much simpler.

## 2 Flow equations in the perturbative frame

The flow equation reads

\[
\frac{dH(l)}{dl} = [\eta(l), H(l)].
\]  

(3)

This is the differential form of a continuous unitary transformation acting on the Hamiltonian in general; \( \eta(l) \) is antihermitean generator of the transformation and \( l \) is the flow parameter. The aim of the flow equations is to bring the initial Hamiltonian matrix \( H(l = 0) \) to a diagonal (or block-diagonal) form. Finite values of the flow parameter \( l \) correspond to intermediate stages of diagonalization with the band-diagonal structure of Hamiltonian (similarity renormalization) \[13\]. The result of the procedure at \( l \to \infty \) is a block-diagonal Hamiltonian. Also the transformation is designed in a way to avoid small energy denominators usually present in perturbation theory.

What is the choice of the generator \( \eta(l) \) that performs such a transformation? Break the Hamiltonian into a 'diagonal' and a 'rest' part \( H = H_d + H_r \). Then the prescription in \[2\] is

\[
\eta(l) = [H_d(l), H_r(l)].
\]  

(4)

At this step the unitary transformation is defined. The only freedom left is the principle of separation into the 'diagonal' and the 'rest' part. It depends on the problem one wants to treat.

Our goal is to transform the Hamiltonian into blocks with the same number of (quasi-)particles. This means, that we define the 'diagonal' part \( H_d \) as the part of the interaction which conserves the number of particles (electrons, positrons, photons), and the 'rest' \( H_r \) as the particle number changing part. In the case of QED(QCD), where the electron-photon (quark-gluon) coupling is present, the number of photons (gluons) is conserved in each block of the final effective Hamiltonian.

As a result of the unitary transformation new interactions are induced (see below). They are absent at \( l = 0 \) and are generated as \( l \) increases. They also give rise to new terms in the generator of transformation \( \eta(l) \). This in its turn generates new interactions again.

To be able to perform the calculations analytically we proceed in a perturbative frame and truncate the series assuming the coupling constant is small.

As illustration of the method we consider QED on the light front. For any finite value of \( l \) one has

\[
H(l) = H_d^{(0)}(l) + H_r^{(1)} + H_d^{(2)} + H_r^{(2)} + ...
\]  

(5)

1Under the 'energy space' we understand the basis of the free Hamiltonian \( H_d^{(0)} \), corresponding to the sum of kinetic energies of single noninteracting particles.
where the superscript denotes the order in the bare coupling constant, $H^{(n)} \sim \epsilon^n$; the indices 'd' and 'r' indicate the diagonal and the rest parts correspondingly. The part $H_d^{(0)}$ is the free Hamiltonian, corresponding to the single particle energies with the structure in secondary quantization $a^\dagger a, b^\dagger b, d^\dagger d$, where $a, b, d$ are the annihilation operators of the photons, electrons and positrons correspondingly; $H_{r}^{(1)}$ denotes the electron-photon coupling (of the type $a^\dagger b^\dagger b$); $H_d^{(2)}$ is the second order diagonal part of the Hamiltonian, having the structure $b^\dagger d^\dagger bd, b^\dagger b^\dagger bb, d^\dagger d^\dagger dd$ (in the light front they correspond to the canonical instantaneous (seagull) and to newly generated interactions in the diagonal sector in second order [14],[15]). Note, that the diagonal part in the flow equations is not only the free Hamiltonian but the full particle number conserving part of the effective Hamiltonian. The choice of only $H_d^{(0)}$ as the diagonal part gives rise to the band-diagonal structure of the effective Hamiltonian in each 'particle number' sector in the similarity renormalization scheme [13]. However, this makes a difference for the diagonal part only if one goes beyond third order in $\epsilon$.

The generator of the transformation is

$$\eta(l) = [H_d, H_r] = [H_d^{(0)}, H_{r}^{(1)}] + [H_d^{(0)}, H_d^{(2)}] + ... = \eta^{(1)} + \eta^{(2)} + ...$$ (6)

Up to second order the flow equation reads

$$\frac{dH(l)}{dl} = [\eta, H] = [[H_d^{(0)}, H_{r}^{(1)}], H_d^{(0)}] + [[H_d^{(0)}, H_{r}^{(1)}], H_d^{(1)}] + [[H_d^{(0)}, H_d^{(2)}], H_d^{(0)}] + ...$$ (7)

Also terms of higher orders in $\epsilon$ are generated by the flow equations.

In the basis of the eigenfunctions of the free Hamiltonian $H_d^{(0)}$

$$H_d^{(0)}|i> = E_i|i>$$ (8)

one obtains for the matrix-elements between the many-particle states

$$\eta_{ij} = (E_i - E_j)H_{r}^{(1)} + (E_i - E_j)H_{r}^{(2)} + ...$$

$$\frac{dH_{ij}}{dl} = -(E_i - E_j)^2H_{r}^{(1)} + [\eta^{(1)}, H_{r}^{(1)}]_{ij} - (E_i - E_j)^2H_{r}^{(2)} + ...$$ (9)

The energy differences are given by

$$E_i - E_j = \sum_{k=1}^{n_2} E_{i,k} - \sum_{k=1}^{n_1} E_{j,k}$$ (10)

where $E_{i,k}$ and $E_{j,k}$ are the energies of the created and annihilated particles, respectively.

The energy $E_i$ depends on the flow parameter $l$ only in second order in the coupling. Therefore one has

$$\frac{dH_{r}^{(1)}}{dl} = -(E_i - E_j)^2H_{r}^{(1)}$$

$$H_{r}^{(1)}(l) = H_{r}^{(1)}(l = 0)e^{-(E_i - E_j)^2l} = H_{r}^{(1)}(\lambda = \Lambda \rightarrow \infty)e^{-\frac{(E_i - E_j)^2}{\lambda^2}}$$ (11)

Here we have used the physical meaning of the flow parameter $l$. Namely, in the similarity renormalization scheme it defines the width of the band $\lambda$, corresponding to the UV-cutoff, where the matrix elements of the effective Hamiltonian are not zero ($|E_i - E_j| < \lambda$) [13]. The connection between these two quantities is

$$l = \frac{1}{\lambda^2}$$ (12)
In the flow equations as used here the matrix elements of the interactions, which change the number of particles, are strongly suppressed, if the energy difference exceeds \( \lambda \), while for the particle number conserving part of the effective Hamiltonian the matrix elements with all energy differences are present.

In the flow equations \( \lambda \) characterizes the smooth UV-cutoff. This fact insures the analytical behavior of the effective Hamiltonian with \( \lambda \), that helps in numerical calculations.

In second order one has to distinguish between the behavior of the 'diagonal' and the 'rest' term. For the 'rest' part one has

\[
\frac{dH^{(2)}_{rij}}{dl} = [\eta^{(1)}, H^{(1)}]_{rij} - (E_i - E_j)^2 H^{(2)}_{rij},
\]

where index 'r' by \([\eta^{(1)}, H^{(1)}]_r\) defines the particle number changing part of the commutator. Introduce

\[
H^{(2)}_{rij}(l) = e^{-(E_i - E_j)^2l} \tilde{H}^{(2)}_{rij}(l).
\]

Then the solution reads

\[
\tilde{H}^{(2)}_{rij}(l) = \tilde{H}^{(2)}_{rij}(l = 0) + \int_0^l dl' e^{(E_i - E_j)^2l'} [\eta^{(1)}, H^{(1)}]_{rij}(l').
\]

For the 'diagonal' part one has

\[
\frac{dH^{(2)}_{dij}}{dl} = [\eta^{(1)}, H^{(1)}]_{dij}
\]

and the solution is

\[
H^{(2)}_{dij}(l) = H^{(2)}_{dij}(l = 0) + \int_0^l dl' [\eta^{(1)}, H^{(1)}]_{dij}(l').
\]

Note, that though in general the commutator \([[H^{(0)}_d, H^{(2)}_d], H^{(0)}_d]]\) is not zero, it is not present in the flow equation due to the definition of the diagonal part. The corresponding commutator \([[H^{(0)}_d, H^{(2)}_r], H^{(0)}_d]]\) in the 'non-diagonal' sector insures the band-diagonal form for the 'rest' interaction and also gives rise to the different structure of the generated interaction (the integral term) in the 'rest' and 'diagonal' sectors.

The commutator \([\eta^{(1)}, H^{(1)}]\) gives rise to new terms in second order in the bare coupling \(e\). In the case of QED it induces new types of interactions and generates the renormalization group corrections to the electron (photon) masses. The coupling constant starts to run in third order in \(e\).

Note, that the method of flow equations (and also the similarity renormalization) enables one to build an effective low energy Hamiltonian together with all, 'canonical' and 'new' \([14]\) counterterms, found (from the coupling coherence condition \([14],[17]\)) order by order in the coupling constant \(e\). This defines an effective renormalized Hamiltonian which may be used for the numerical solution of the bound state problem.

### 3 Renormalized effective electron-positron interaction

In this section we give the effective Hamiltonian in the light front dynamics for the positronium system, generated by the unitary transformation in section 2, compare \([14]\)
The light front Schrödinger equation for the positronium model reads

\[ H_{LC} |\psi_n> = M_n^2 |\psi_n> \]  \hspace{1cm} (18)

where \( H_{LC} = P^\mu P_\mu \) is the invariant mass (squared) operator, refered for convenience to as the light front Hamiltonian of positronium and \( |\psi_n> \) being the corresponding eigenfunction; \( n \) labels all the quantum numbers of the state.

The canonical Hamiltonian of the system \( H_{LC} \) contains infinitely many Fock sectors (i.e. one has for the positronium wave function \( |\psi_n> = c_{ee}|(e\bar{e})_n> + c_{e\bar{e}\gamma}|(e\bar{e}\gamma)_n> + c_{\bar{e}e\gamma}|(\bar{e}e\gamma)_n> + ... \) and each Fock sector contains states with arbitrarily large energies. We now

1. introduce the bare cutoff (regularization) with the result \( H_{LC}^B(\Lambda) \) - the bare Hamiltonian;
2. perform the unitary transformation by means of flow equations eqs. (1) and (2) with the result \( H_{LC}^{eff} \) - the effective renormalized Hamiltonian (table 1 for finite value of \( \lambda \));
3. truncate the Fock space to the lowest Fock sector (\( |e\bar{e}> \)) with the result \( \tilde{H}_{LC}^{eff} \) - the effective renormalized Hamiltonian acting in the electron-positron sector.

Then the eigenvalue equation reads

\[ \tilde{H}_{LC}^{eff} |(e\bar{e})_n> = M_n^2 |(e\bar{e})_n>. \]  \hspace{1cm} (19)

The effective light front Hamiltonian consists of the free (noninteracting) part and the effective electron-positron interaction

\[ \tilde{H}_{LC}^{eff} = H_{LC}^{(0)} + V_{LC}^{eff} \]  \hspace{1cm} (20)

The light front equation eq. (19) is then expressed by the integral equation (the coordinates are given in fig. (2))

\[
\left( \frac{m^2 + k^2}{x'(1-x')} - M_n^2 \right) \psi_n(x', \vec{k}_\perp; s_3, s_4) \\
+ \sum_{s_1, s_2} \int_D \frac{dx^2 d^2k_L}{(2\pi)^3} <x', \vec{k}_\perp; s_3, s_4|V_{LC}^{eff}|x, \vec{k}_\perp; s_1, s_2> \psi_n(x, \vec{k}_\perp; s_1, s_2) = 0
\]  \hspace{1cm} (21)

The integration domain \( D \) is restricted by the covariant cutoff condition of Brodsky and Lepage

\[
\frac{m^2 + k^2}{x(1-x)} \leq \Lambda^2 + 4m^2
\]  \hspace{1cm} (22)

which allows for states which have a kinetic energy below the cutoff \( \Lambda \).

For the effective electron-positron interaction one has in the exchange and annihilation channels

\[ V_{LC}^{eff} = V_{exch} + V_{ann} = \sum_{channel} \lim_{\lambda \to 0} (V_{\lambda}^{gen} + V_{\lambda}^{inst} + V_{\lambda}^{PT}) \]  \hspace{1cm} (23)

where the terms in eq. (23) correspond to generated, instantaneous and perturbative photon exchange interactions in the electron-positron sector. The interactions generated by the flow equations from the Hamiltonian in the light front frame are, fig. (2).
in the exchange channel

\[ V_{\lambda}^{\text{gen}} = -e^2 N_1 \left( \frac{\int_{\Delta_1}^{\infty} df_{\lambda}(\Delta_1) f_{\lambda}(\Delta_2) d\lambda'}{\Delta_1} + \frac{\int_{\Delta_2}^{\infty} df_{\lambda}(\Delta_2) f_{\lambda}(\Delta_1) d\lambda'}{\Delta_2} \right) \]

\[ V_{\lambda}^{\text{inst}} = -\frac{4e^2}{(x-x')^2} \delta_{s_1s_3} \delta_{s_2s_4} \]

\[ V_{\lambda}^{\text{PT}} = -e^2 N_1 \frac{1}{\Delta_3} f_{\lambda}(\Delta_1) f_{\lambda}(\Delta_2) \]

(24)

in the annihilation channel

\[ V_{\lambda}^{\text{gen}} = e^2 N_2 \left( \frac{\int_{\Delta_1}^{\infty} df_{\lambda}(M_0^2) f_{\lambda}(M_0^2) d\lambda'}{M_0^2} + \frac{\int_{\Delta_2}^{\infty} df_{\lambda}(M_0^2) f_{\lambda}(M_0^2) d\lambda'}{M_0^2} \right) \]

\[ V_{\lambda}^{\text{inst}} = 4e^2 \delta_{s_1s_3} \delta_{s_2s_4} \]

\[ V_{\lambda}^{\text{PT}} = e^2 N_2 \frac{1}{M_0^2} f_{\lambda}(M_0^2) f_{\lambda}(M_0^2). \]

(25)

where \( f_{\lambda}(\Delta) \) is the similarity function

\[ f_{\lambda}(\Delta) = e^{-\frac{\Delta^2}{x^2}} \]  

(26)

where we have used the rescaled value of the UV cutoff \( \lambda \to \frac{\lambda^2}{m^2} \).

As long as \( \lambda \) is finite we have taken the still existing terms in \( H_1^{(1)} \) into account by the perturbative result obtained by Lepage and Brodsky [12].

The functions \( N_1, N_2 \) (current-current terms) and the energy denominators \( \tilde{\Delta}_i, i = 1, 2, 3, M_0^2, M_0^2, M_n^2 \) are defined in the light front dynamics [13] as follows [14] fig. (2)

\[ N_1 = \delta_{s_1s_3} \delta_{s_2s_4} \left( T_1^\perp \cdot T_2^\perp - \delta_{s_1s_3} \delta_{s_2s_4} \delta_{s_2s_4} 2m^2 \frac{(x-x')^2}{xx'(1-x)(1-x')} \right. \\
+ \left. im \sqrt{2} (x'-x) \left[ \delta_{s_1s_3} \delta_{s_2s_4} \frac{s_1}{xx'} T_1^\perp \cdot \varepsilon_{s_1}^\perp + \delta_{s_1s_3} \delta_{s_2s_4} \frac{s_2}{(1-x)(1-x')} T_2^\perp \cdot \varepsilon_{s_2}^\perp \right] \right) \]

\[ N_2 = \delta_{s_1s_3} \delta_{s_2s_4} \left( T_3^\perp \cdot T_4^\perp + \delta_{s_1s_3} \delta_{s_2s_4} \delta_{s_2s_4} 2m^2 \frac{1}{xx'(1-x)(1-x')} \right. \\
+ \left. im \sqrt{2} \left[ \delta_{s_3s_4} \cdot \delta_{s_2s_4} \frac{s_1}{(1-x)} T_3^\perp \cdot \varepsilon_{s_1}^\perp - \delta_{s_3s_4} \delta_{s_1s_2} \frac{s_3}{xx'(1-x')} T_4^\perp \cdot \varepsilon_{s_2}^\perp \right] \right) \]

\[ \varepsilon_{s}^i = - \frac{1}{\sqrt{2}} (s, i) \]

(27)

and

\[ T_1^i = - \left[ \frac{2(k_1 - k_1')^i}{(x-x')} + \frac{k_1^i(s_2)}{(1-x)} + \frac{k_1^i(s_2)}{(1-x')} \right]; \quad T_2^i = 2 \left[ \frac{k_1 - k_1'}{x-x'} \right] - \frac{k_1^i(s_1)}{x} - \frac{k_1^i(s_1)}{x'} \]

\[ T_3^i = - \frac{k_1^i(s_3)}{x'} + \frac{k_1^i(s_3)}{(1-x')}; \quad T_4^i = \frac{k_1^i(s_1)}{(1-x)} - \frac{k_1^i(s_1)}{x} \]

\[ k_1^i(s) = k_1^i + is \varepsilon_{ij} k_1^j; \quad \varepsilon_{ij} = \varepsilon_{ij3}; \quad \bar{s} = -s \]
with the definitions

\[
\begin{align*}
\tilde{\Delta}_1 &= \frac{(xk'_- - x'k_\perp)^2 + m^2(x - x')^2}{xx'}; \\
\tilde{\Delta}_2 &= \tilde{\Delta}_1 |_{x \rightarrow (1 - x)}, x' \rightarrow (1 - x') \\
\tilde{\Delta}_3 &= \frac{\tilde{\Delta}_2}{x' - x}; \\
\Delta_1 &= \frac{\tilde{\Delta}_1}{x' - x}; \\
\Delta_2 &= \frac{\tilde{\Delta}_2}{x' - x}; \\
\Delta_3 &= \frac{\tilde{\Delta}_3}{x' - x}.
\end{align*}
\]

Here \(x\) is the light front fraction of the electron momentum, \(P\) is the total momentum of positronium and \(B_n\) the binding energy of the positronium.

Explicitly the integrals in eqs. (24) and (25) are given

\[
\int_\infty^{-\infty} \frac{df_\lambda(\Delta_1)}{dl'} f_\lambda(\Delta_2) dl' = \tilde{\Delta}_2 \tilde{\Delta}_1 \Delta_2 \Delta_3.
\]

We use instead of the light front parameterization fig. (2), the instant form. Namely, we express the variable \((x, \vec{k}_\perp)\) in terms of the equal-time variable \(\vec{k} = (k_z, \vec{k}_\perp)\) as

\[
x = \frac{1}{2} \left( 1 + \frac{k_z}{\sqrt{\vec{k}^2 + m^2}} \right) \tag{30}
\]

\[
\vec{k}^2 = k_z^2 + \vec{k}_\perp^2 \tag{31}
\]

and similarly for \(x'\) and \(\vec{k}'^2\) as function of \(k_z'\).

Then the electron-positron interaction reads in the exchange and annihilation channels

\[
V_\lambda = V_{\lambda,exch} + V_{\lambda,ann}
\]

\[
= -e^2 N_{\lambda,\lambda} \left[ \left( \frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{\Delta_1^2 + \Delta_2^2} \right) (1 - e^{-\frac{\Delta_1^2 + \Delta_2^2}{x^4}}) + \frac{1}{\Delta_3} e^{-\frac{\Delta_3^2}{x^4}} \right]
\]

\[
+ \left( \frac{4e^2}{(x - x')^2} \delta_{s_1 s_2} \delta_{s_3 s_4} \right)
\]

\[
+ e^2 N_{2,\lambda} \left[ \left( \frac{M_0^2 + M_0'^2}{M_0^2 + M_0'^2} \right) (1 - e^{-\frac{M_0^2 + M_0'^2}{x^4}}) + \frac{1}{M_n^2} e^{-\frac{M_n^2}{x^4}} \right]
\]

\[
+ \left( 4e^2 \delta_{s_1 s_2} \delta_{s_3 s_4} \right) \tag{32}
\]

where one has in the instant frame

\[
M_0^2 = 4(\vec{k}^2 + m^2) \tag{33}
\]

and similarly for \(M_0'^2\) as function of \(\vec{k}'^2\). For all the quantities, defined in eqs. (27) and (28), the substitution \(x(k_z), x'(k'_z)\) is to be done. To the leading order of the nonrelativistic approximation \(|\vec{k}|/m \ll 1\) one obtains (for the exchange channel)

\[
\tilde{\Delta}_1 \sim \tilde{\Delta}_2 \sim \tilde{\Delta}_3 = \tilde{\Delta} = (\vec{k} - \vec{k}')^2
\]
\[ V_{\text{gen}}^{\alpha} \approx -e^2 \frac{N_1}{(k - k')^2} (1 - f_\lambda^2(\Delta)) \]
\[ V_{\lambda}^{PT} \approx -e^2 \frac{N_1}{(k - k')^2} f_\lambda^2(\Delta) \]
\[ \Delta = \frac{(\vec{k} - \vec{k}')^2}{x' - x} \]

This gives for the electron-positron interaction in the whole nonrelativistic range of \( \lambda, \lambda \ll m \)
\[ V_{\text{e-e}} \approx -e^2 \frac{N_1}{(k - k')^2} - \frac{4e^2}{(x - x')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} \]

the \( \lambda \)-independent result. Making use of the following expressions
\[ N_1^{\text{diag}} \approx -4\frac{(k_{1 \perp} - k'_{1 \perp})^2}{(x - x')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} \]
\[ (k - k')^2 = (k_{1 \perp} - k'_{1 \perp})^2 + (k_z - k_{1 z})^2 \approx (k_{1 \perp} - k'_{1 \perp})^2 + 4m^2(x - x')^2 \]

one obtains in leading order of the nonrelativistic approximation the 3-dimensional Coulomb interaction \( (e^2 = 4\pi\alpha) \)
\[ V_{\text{e-e}} \approx 16m^2 \left( -\frac{e^2}{(k - k')^2} \right) \delta_{s_1 s_3} \delta_{s_2 s_4} \]

hence the rotational invariance is restored in this order. This result eq. (37) is valid for any nonrelativistic value of cutoff \( \lambda \) and does not depend on the details of the similarity function \( f_\lambda(\Delta) \).

We perform the limit of \( \lambda \to 0 \) in the effective interaction, eqs. (24) and (25), that corresponds to the complete elimination of the electron-photon vertex. Then the perturbative term for the dynamical photon exchange \( V^{PT} \) vanishes. Therefore the effective interaction eq. (23), generated by the flow equations, is defined in the whole parameter region, (except maybe for the Coulomb singularity point \( \vec{q} = \vec{k} - \vec{k}' = 0 \)) as follows
\[ V_{\text{eff}}^{\alpha} = V_{\text{exch}} + V_{\text{ann}} \]
\[ = -e^2 N_1 \left( \frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{\Delta_1 + \Delta_2} \right) + \left( -\frac{4e^2}{(x - x')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} \right) \]
\[ + e^2 N_2 \left( \frac{M_0^2 + M_0'^2}{M_0^2 + M_0'^2} \right) \]
\[ + \left( 4e^2 \delta_{s_1 s_3} \delta_{s_2 s_4} \right) \]

(38)

The effective interaction \( V_{\text{eff}}^{\alpha} \) obtained in eq. (38) stands for the kernel in the integral equation eq. (21) for the calculation of the bound state spectrum and the wave functions of positronium. The integral equation eq. (21) with the effective interaction given above is to be used for the numerical calculations of positronium spectrum.

This interaction was also used for an analytical analysis of the positronium ground states. The standard singlet-triplet mass splitting for positronium \( \frac{7}{2} \alpha^2 \text{Ryd} \) was obtained and the degeneracy of the triplet ground state \( n = 1 \) was recovered \[14], [3].

Instead of the choice \( \eta_{ij}^{(1)} = (E_i - E_j)H_{rij}^{(1)} \) one can choose \( \eta_{ij}^{(1)} = \text{sign}(E_i - E_j)H_{rij}^{(1)}. \) With this choice in the above equations the ratio \( \frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{\Delta_1 + \Delta_2} \) has to be replaced by \( \frac{2}{\Delta_1 + \Delta_2}. \) If one approximates
$M_3^2 = \frac{1}{2}(M_0^2 + M_0'2)$ in the expression for $\tilde{\Delta}_3$, then $\tilde{\Delta}_3$ equals $\frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{2}$, and one obtains the perturbation theoretic result $V_{\lambda=\Lambda\to\infty}^{PT} + V^{\text{inst}}$ for the effective interaction. These both interactions with different choices of $\eta$ do not coincide. Both have a leading Coulomb behavior, but they differ in the order $e^2q^0$, which determines the fine structure splitting. Quite generally the two particle interaction is of order $q^{-2}$ and subleading terms in $q$. In the order of fine structure splitting $\alpha^4$ also terms of order $e^4q^{-1}$ and $e^6q^{-2}$ will in general be important. Thus we expect that the difference in order $e^2q^0$ will be compensated by differences in order $e^4$ and $e^6$. They should yield in total the correct fine structure splitting.

Using the similarity transformation Brisudova and Perry [19] have obtained the correct spin-spin interaction for the positronium from the effective interaction in order $e^2$. This had to be expected, since each spin enters the interaction with a factor of order $q/m$ as compared to the leading Coulomb interaction. Thus the two-spin interaction enters only in order $q^0$ or higher. The only contribution to order $\alpha^4$ comes from order $e^2$. The same holds for the spin-triplet splitting (which is quadratic in the spin), and has been obtained correctly for the ground state by Jones, Perry and Glazek [3]. By the same reasoning contributions to the spin-orbit coupling are of order $q^{-1}$, so that contributions from order $e^2$ and $e^4$ have to be expected, but not from order $e^6$. Spin independent contributions can be of order $q^{-2}$. To obtain all contributions of order $\alpha^4$ one has to consider the interaction in order $e^2$, $e^4$ and $e^6$.

4 Conclusions and outlook

In this work we have outlined a strategy to derive an effective renormalized Hamiltonian by means of flow equations. Application of the flow equations with the condition, that particle number conserving terms are considered diagonal and those changing the particle number off-diagonal led as in other cases to a useful effective Hamiltonian.

The main advantage of this procedure as compared with the similarity renormalization of Glazek and Wilson [1] is, that finally states of different particle number are completely decoupled, since the particle number violating contributions are eliminated down to $\lambda = 0$. Thus one is able to truncate the Fock space and the positronium problem reduces to a two particle problem which can be analyzed further analytically (since in leading order one obtains the nonrelativistic Coulomb problem) or numerically [?] for positronium bound states.

The effective Hamiltonian, obtained by the similarity transformation, is band-diagonal in the energy space. The width of the band $\lambda$ introduces the artificial parameter in the procedure, which is defined from the physical reasoning ($\lambda$ is low enough to neglect the contribution of high Fock states, but is restricted from below to stay in perturbation theory region). Flow equations as used here with the particle number conserving part of Hamiltonian to be diagonal, have no additional parameter and converge well as $\lambda \to 0$ [2] to the effective Hamiltonian, which is block-diagonal in particle number and can be used directly for the numerical calculations of the spectrum (work in preparation).

The procedure of elimination of nondiagonal blocks, that change the number of quasiparticles, is performed not just in one step as in the method of Tamm-Dancoff truncation but rather continuously for the states with different energies in sequence. This is the main advantage of the proposed method as compared with Tamm-Dancoff truncation, the possibility to perform simultaneously the ultraviolet renormalization of the initial Hamiltonian. In general, in the definite order of perturbation theory all counterterms, associated with canonical operators of the theory and also with possible new operators induced by unitary transformation, can be obtained in the procedure [14]. Since different sectors of the effective Hamiltonian are decoupled,
one does not encounter the usual difficulties of Tamm-Dancoff truncation and the methods related to it. Namely, the counterterms to be introduced are 'sector-' and 'state-' independent (work in preparation and [4]).

If one goes beyond the tree approximation then one obtains terms with ultraviolet divergences which have to be renormalized. This has not been considered in this paper. Simultaneously also terms describing interactions between more than two particles are generated. In this approach we were not faced with infrared problems (except infrared collinear divergences along the light front, that are removed in the considered sector, if all possible diagrams in this order are taken into account [4]).

By means of the flow equation method one can simultaneously renormalize the initial field theoretical Hamiltonian and construct the effective Hamiltonian, for which the Fock space truncation is valid. In order to solve the flow equations analytically we were forced to apply in this work the perturbative theory expansion. One is able to improve this approach systematically by going to higher orders in the coupling.

We consider flow equations as a method which can also be used beyond perturbation theory in a self-consistent way. Examples in solid-state physics are the flow of the tunneling-frequency in the spin-boson model [7] and of the phonon energies in the electron-phonon coupling [5]. Due to the flow the couplings decay even at resonance.

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**References**

[1] S. D. Glazek and K. G. Wilson, Phys. Rev. D48, 5863 (1993); S. D. Glazek and K. G. Wilson, Phys.Rev. D49, 4214 (1994).

[2] F. Wegner, Ann. Physik 3, 77 (1994).

[3] B. D. Jones, R. G. Perry and S. D. Glazek, Phys.Rev. D55, 6561 (1997), hep-th/9605231.

[4] K. Harada and A. Okazaki, Phys.Rev. D55, 6198 (1997), KYUSHU-HET-35, hep-th/9610020.

[5] P. Lenz and F. Wegner, Nucl. Phys. B 482, 693 (1996), cond-mat/9604087.

[6] A. Mielke, cond-mat/9609065, Ann. Phys. 6, 215 (1997), cond-mat/9709175, to be publ. in Europhys. Lett. (accepted)

[7] S. Kehrein, A. Mielke and P. Neu, Z.Phys. B99 269 (1996).

[8] H. Fröhlich, Proc.Roy.Soc. A215, 291 (1952)

[9] I. J. Tamm, J.Phys. (USSR) 9, 449 (1945); S. M. Dancoff, Phys.Rev. 78, 382 (1950).

[10] H. C. Pauli, MPIH-V25-1996, hep-th/9608039, MPIH-V8-1996

[11] H. C. Pauli, MPIH-V26-1997, hep-ph/9707361
[12] U. Trittmann and H. C. Pauli, hep-th/9705021
[13] E. L. Gubankova, F. Wegner, HD-TVP-97-09, hep-th/9702162
[14] E. L. Gubankova, F. Wegner, HD-TVP-97-11, hep-th/9708054
[15] W. M. Zhang and A. Harindranath, Phys.Rev. D48 4868 (1993);
[16] R. G. Perry, hep-th/9407056, hep-ph/9604318
[17] R. G. Perry and K. G. Wilson, Nucl.Phys. B403, 587 (1993). ibid. 4881; ibid. 4903.
[18] G. P. Lepage, S. J. Brodsky, Phys.Rev. D22 2157 (1980);
S. J. Brodsky, G. P. Lepage, World Scientific, Singapur, 1989.
[19] M. Brisudova and R. Perry, hep-th/9605363
Figure 1: Flow equations perform the block-diagonalization of the bare Hamiltonian of the canonical theory $H_B(\Lambda)$ into a Hamiltonian consisting of blocks with equal number of particles. For a finite value of $\lambda$ the matrix elements of the 'particle number changing' sectors are squeezed into an energy band with roughly $|E_i - E_j| < \lambda$ (left hand side picture) and are eliminated completely as $\lambda \to 0$ (right hand side picture).

Figure 2: The effective electron-positron interaction in the exchange channel; the diagrams correspond to the generated and the instantaneous interactions. The perturbative photon exchange with two different time orderings is also depicted.

Table 1: The effective light front QED Hamiltonian matrix up to second order in $e$ in the Fock space representation. The matrix elements of the 'diagonal' (Fock state conserving) sectors are unrestricted in the energy differences; the 'rest' (Fock state changing) sectors are squeezed roughly in an energy band of width $\lambda$. Black dots correspond to zero matrix elements in order $O(e^2)$. Instantaneous and disconnected diagrams are not included.
Table 1
\[ \begin{align*}
 p_1(x, k^\perp) & \quad p_3(x', k'^\perp) \\
 p_2(1-x, -k^\perp) & \quad p_4(1-x', -k'^\perp)
\end{align*} \]

Figure 2