SOLVABILITY IN CLASSICAL MECHANICS AND ALGEBRAIC HEUN OBSERVABLES

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Abstract. We construct a classical analog of the algebraic Heun operator $W$ associated to any bispectral pair of the operators $X$ and $Y$. We show that the dynamics of the classical variables $X$ or $Y$ is governed by elliptic functions if the classical $W$ is taken as the Hamiltonian. We demonstrate this general property by concrete examples of classical dynamical systems.

1. Introduction

Let $X, Y$ be a pair of bispectral operators. The algebraic Heun operator $W$ is defined as the most general bilinear combination of the operators $X$ and $Y$ [6]. This can be explicitly presented as

$$W = \tau_1 (XY + YX) + \tau_2 [X, Y] + \tau_3 X + \tau_4 Y + \tau_0 I,$$  \hspace{1cm} (1.1)

where $[X, Y] = XY - YX$ is the commutator and where $I$ is the identity operator.

The main motivation for introducing such an operator stems from the observation that in the special case when $X$ is the multiplication operator and $Y$ is the hypergeometric operator, the operator $W$ coincides with the generic Heun operator [5]. In [6] it was showed also that the algebraic Heun operator provides an illuminating explicit construction of the commuting operator in the practically important band-time limiting problem. From a quantum mechanical perspective, in this special case, the operator $W$ can always be presented (after an appropriate change of variable $x$) as the Schrödinger Hamiltonian

$$W = -\partial_x^2 + u(x) \hspace{1cm} (1.2)$$

with a potential $u(x)$ such that eigenfunctions of (1.2) can be expressed in terms of Heun functions [9], [10].

Let us here recall the remarkable observations made by Manning in a 1935 paper [7] which is not so widely known. These bear on the relation between exact solvability in classical and quantum mechanics. A first point made by Manning is that the situations for which the Schrödinger equation is solvable in terms of hypergeometric functions correspond to potentials such that in classical dynamics the Hamilton-Jacobi equation is solved in terms of elementary functions.

An interpretation of this connection was provided in [4] by showing that these dynamical systems are described by the same "hidden" quadratic Jacobi algebra both classically and quantum mechanically.
The second observation made by Manning is that for those potentials where the Hamilton-Jacobi equation is solved in terms of elliptic functions, the Schrödinger equation can be converted into the Heun equation. This last relation remained unexplained. We here offer a rationale on the basis of the underlying bispectrality and prove the converse of Manning’s statement namely that elliptic functions will describe the classical dynamics of quantum systems whose Schrödinger equation is essentially the Heun equation. Our construction is however, much wider and involves not only operators of Schrödinger type but the more general operators of Askey-Wilson type and their classical analogs. In particular, we demonstrate efficiency of our approach for the two models: the Zhukovsky-Volterra gyrostat and the relativistic $A_1$ model.

2. Classical analog of the algebraic Heun operator

We propose the following classical analog of the algebraic Heun operator. In the canonical classical-quantum correspondence, it is assumed that the operator $X$ becomes an ordinary dynamical variable. The anticommutator $XY + YX$ becomes $2XY$ while the commutator becomes the Poisson bracket $[X, Y] \rightarrow \{X, Y\}$. In the one-dimensional case the variables $X(q, p), Y(q, p)$ depend on the canonical variables $q, p$ and the Poisson bracket is expressed in terms of the Jacobian derivative

$$\{X, Y\} = \frac{D(X, Y)}{D(q, p)} = X_q Y_p - X_p Y_q,$$

where $F_s = \partial_s F$. Using this rule we can construct the classical ”Heun variable”

$$W = \tau_1 XY + \tau_2 Z + \tau_3 X + \tau_4 Y + \tau_0,$$

where

$$Z = \{X, Y\}$$

is the Poisson bracket of the variables $X$ and $Y$ and where $\tau_i$ are constants (not depending on time $t$ or the canonical variables $q, p$). Sometimes we will use term the ”classical Heun pencil” to indicate that $W$ is a linear combination of several terms with arbitrary coefficients $\tau_i$.

The variables $X$ and $Y$ are assumed to form a ”classical Leonard pair”. This notion was introduced in [11]. From an algebraic point of view the classical Leonard pair (CLP) can formally be defined by the condition that $X$ and $Y$ satisfy the following relations [11]

$$\{X, Z\} = \frac{1}{2} \Phi_Y(X, Y), \quad \{Z, Y\} = \frac{1}{2} \Phi_X(X, Y),$$

where $\Phi(X, Y)$ is an arbitrary polynomial of degree at most two in each of the variables $X$ and $Y$, i.e.

$$\Phi(X, Y) = \sum_{i,j=0}^2 \alpha_{ij} X^i Y^j.$$

with arbitrary complex coefficients $\alpha_{ij}$. Notations $\Phi_X(X, Y), \Phi_Y(X, Y)$ stand for the partial derivatives with respect to the subcripts. In the sequel, we will use also
the following polynomials:

\[ U_i(x) = \alpha_2^i x^2 + \alpha_1^i x + \alpha_0^i, \quad V_i(x) = \alpha_2^i x^2 + \alpha_1^i x + \alpha_0^i, \quad i = 0, 1, 2 \quad (2.6) \]

These polynomials can also be defined from the relations

\[ \Phi(X, Y) = U_2(X)Y^2 + U_1(X)Y + U_0(X), \]
\[ \Phi(X, Y) = V_2(Y)X^2 + V_1(Y)X + V_0(Y) \quad (2.7) \]

An important property of CLP’s is that the dynamical variable

\[ Q(q, p) = Z^2 - \Phi(X, Y) \quad (2.8) \]

has zero Poisson brackets with both \( X \) and \( Y \) \[11\]

\[ \{Q, X\} = \{Q, Y\} = 0 \quad (2.9) \]

and can be taken as a constant \( Q \) not depending on \( q, p \).

Equivalently, this means that the square of dynamical variable \( Z \) can be expressed as bi-quadratic polynomial in \( X, Y \):

\[ Z^2 = \Phi(X, Y). \quad (2.10) \]

Indeed, the constant \( Q \) can always be incorporated into the free term \( \alpha_{00} \) of the polynomial \( \Phi(X, Y) \). From this equation it follows that if the dynamical variable \( X \) is chosen as the Hamiltonian \( X = H \), the time dynamics of both variables \( Y(t) \) and \( Z(t) \) is then described by elementary functions (i.e. trigonometric, exponential or their degenerations). Indeed, the Hamilton equation of motion for the variable \( Y(t) \) is

\[ \dot{Y}(t) = \{Y, H\} = \{Y, X\} = -Z(t) \quad (2.11) \]

From (2.10) and (2.11) we obtain

\[ \dot{Y}(t)^2 = \mu_2(Y(t)) \quad (2.12) \]

with some polynomial \( \mu_2(Y(t)) \) of second degree with respect to the variable \( Y(t) \) with coefficients which may depend on \( X \) but not on time \( t \). In general, the solution of (2.12) is

\[ Y(t) = \xi_1 \exp(\omega t) + \xi_2 \exp(-\omega t) + \xi_0, \quad (2.13) \]

where the parameters \( \xi_i, \omega \) may depend on \( X \). In the degenerate case we obtain solution of the type

\[ Y(t) = \xi_1 t^2 + \xi_1 t + \xi_0. \quad (2.14) \]

The time dynamics of the variable \( Z(t) \) is obtained from the relation

\[ Z(t) = \{X, Y(t)\} = -\dot{Y}(t) \quad (2.15) \]

from which it follows that \( Z(t) \) is an elementary function in time \( t \) as well.

So, the choice of the variable \( X \) as the Hamiltonian leads to the elementary dynamics of the variables \( Y(t) \) and \( Z(t) \).

Due to the symmetry between \( X \) and \( Y \) in the CLP, the same statement is valid if one chooses \( Y \) as the Hamiltonian. In this case we have elementary dynamics of the variable \( X(t) \):

\[ \dot{X}(t)^2 = \nu_2(X(t)) \quad (2.16) \]
with a second-degree polynomial $\nu_2(X)$ whose coefficients do not depend on time $t$.

We thus have the "mutual integrability" property associated with the CLP: each of the variables $X$ or $Y$ displays an elementary dynamics with respect to time $t$ if the other variable is chosen as the Hamiltonian (see [4], [11] for details).

Consider now the dynamical system described by the Heun-type Hamiltonian $W$. What is time dependence of dynamical variables $X$ and $Y$?

Again, we have the Hamilton equations

$$\dot{X} = \{X, W\}, \quad \dot{Y} = \{Y, W\}. \quad (2.17)$$

The Poisson brackets (2.17) can be explicitly calculated using the CLP relations (2.4):

$$\{X, W\} = \tau_1 XZ + \frac{\tau_2}{2} \Phi_Y(X, Y) + \tau_4 Z \quad (2.18)$$

and

$$\{Y, W\} = -\tau_1 YZ - \frac{\tau_2}{2} \Phi_X(X, Y) - \tau_3 Z. \quad (2.19)$$

We first derive a simple relation between the Poisson bracket $\{X, W\}$ and variables $X$ and $W$.

Indeed, there are three polynomial equations: (2.2), (2.10) and (2.18) which relate 5 dynamical variables: $X, Y, Z, W, \{X, W\}$. Eliminating the two variables $Y$ and $Z$ from these equations we arrive at the relation

$$\{X, W\}^2 = \pi_2(X)W^2 + \pi_3(X)W + \pi_4(X), \quad (2.20)$$

where $\pi_i(X)$ are polynomials in $X$ of degree at most $i$. Their explicit expressions are

$$\pi_2(x) = U_2(x), \quad \pi_3(x) = (\tau_1 x + \tau_4)U_1(x) - 2(\tau_3 x + \tau_0)U_2(x) \quad (2.21)$$

and

$$\pi_4(x) = U_0(x)(\tau_1 x + \tau_4)^2 - U_1(x)(\tau_1 x + \tau_4)(\tau_3 x + \tau_0) + \frac{\tau_2}{4} \left[U_1^2(x) - 4U_0(x)U_2(x)\right], \quad (2.22)$$

where the polynomials $U_i(x)$ are given by (2.6).

Quite similarly, one can obtain that

$$\{Y, W\}^2 = \tilde{\pi}_2(Y)W^2 + \tilde{\pi}_3(Y)W + \tilde{\pi}_4(Y), \quad (2.23)$$

where the polynomials $\tilde{\pi}_i(x)$ are obtained from the polynomials $\pi_i(x)$ by the replacement $U_i(x) \rightarrow V_i(x)$ and $\tau_3 \rightarrow \tau_4, \tau_4 \rightarrow \tau_3$. From (2.20) it follows that if $W$ is the Hamiltonian, then the dynamical variable $X(t)$ satisfies the equation

$$\dot{X}^2 = \mathcal{P}_4(X), \quad (2.24)$$

with $\mathcal{P}_4(X)$ a polynomial of (in general) fourth degree. The coefficients of this polynomial do not depend on the time $t$ because they are expressed in terms of the time-independent parameters $\alpha_{iik}, \tau_i$ and $W$. 
The generic solution of (2.24) is hence expressed in terms of the elliptic function of second order [1]:

$$X(t) = \kappa \sigma(\mu(t - r_1)) \sigma(\mu(t - r_2)) \sigma(\mu(t - r_3)) \sigma(\mu(t - r_4)),$$

(2.25)

where \(\sigma(x)\) is the Weierstrass elliptic functions and \(\kappa, \mu, r_1, r_2, r_3, r_4\) are parameters (not depending on \(t\)) determined by the coefficients \(V_i\) of the equation (2.24) and by the initial condition \(X(0) = X_0\). The parameters \(r_i, i = 1, 2, 3, 4\) satisfy the standard balance condition of elliptic functions of second order [1]

$$r_1 + r_2 = r_3 + r_4.$$

(2.26)

Note that the Weierstrass elliptic functions \(\sigma(x)\) depend also of the so-called elliptic invariants \(g_2, g_3\) (which contain information about the periods of the elliptic functions). These elliptic parameters are also determined from the constants \(a_{ij}\) and \(\tau_i\).

The dynamical variable \(Y(t)\) satisfies the equation of the same type

$$Y^2 = \tilde{P}_4(Y),$$

(2.27)

with solutions similar to (2.25), i.e. \(Y(t)\) is an elliptic function of second order with different parameters \(\kappa, \mu, r_i\) but with the same elliptic invariants \(g_2, g_3\). The latter statement can be checked by routine calculations because the invariants \(g_2, g_3\) are expressed through the coefficients of the polynomial \(P_4(x)\) (see, e.g. [1]).

3. Simple applications

In this section we present three simple applications to physical systems.

3.1. Extension of the Poeschl-Teller Hamiltonian. The first example is connected with the original Manning paper [7]. We demonstrate how “elliptic” potentials can be constructed out of “elementary” ones.

We recall the main result of [4] on the ”elementary” potentials. Assume that \(X = \varphi(q)\) with some function \(\phi(q)\) and that \(Y = p^2 + u(q)\) is a classical one-particle Hamiltonian with potential \(u(q)\). The canonical variables \(q\) and \(p\) satisfy the standard relation

$$\{q, p\} = 1.$$

(3.1)

The relations of the classical Jacobi algebra which is behind elementary dynamics both classically and quantum mechanically (see [4]), are equivalent to the condition \(U_2(X) = 0\) in (2.7), i.e. to the

$$\{X, Y\}^2 = \Phi(X, Y) = U_1(X)Y + U_0(X),$$

(3.2)

where \(U_1(x), U_0(x)\) are arbitrary quadratic polynomials in \(x\).

Relation (3.2) is equivalent to two equations

$$4\varphi'(q)^2 = U_1(\varphi(q))$$

(3.3)

and

$$u(q) = -\frac{U_0(\varphi(q))}{U_1(\varphi(q))}.$$

(3.4)
Equation (3.3) determines the function \( \varphi(q) \). In the nondegenerate case (i.e. when \( U_1(x) \) is a quadratic polynomial with distinct roots) this function is either trigonometric or hyperbolic. We can put for simplicity
\[
\varphi(q) = \sinh^2 q \tag{3.5}
\]
Equation (3.4) yields then the generic Poeschl-Teller potential \([7],[4]\)
\[
u(q) = \beta_1 \sinh^{-2} q + \beta_2 \cosh^{-2} q + \beta_0 \tag{3.6}
\]
with three parameters \( \beta_0, \beta_1, \beta_2 \). From the general properties of the Jacobi algebra (see above), it follows that the dynamical variable \( X(t) = \sinh^2 q(t) \) is an elementary function of time \( t \).

Consider now the classical Heun pencil (2.2) with \( \tau_4 = 1, \tau_1 = 0 \). After appropriate canonical transformation \( q \to q, p \to p - \tau_2 \varphi'(q) \), we obtain the following Hamiltonian with 5 free parameters \( \beta_0, \ldots, \beta_4 \):
\[
W = p^2 + \beta_1 \sinh^{-2} q + \beta_2 \cosh^{-2} q + \beta_3 \sinh^2 q + \beta_4 \sinh^2 q \cosh^2 q + \beta_0 \tag{3.7}
\]
which is an extension of the Poeschl-Teller potential with two additional terms. As follows from the considerations of the previous section, the time dependence of the variable \( X(t) = \varphi(q(t)) \) is an elliptic function of the second order.

This example provides a partial explanation of Manning’s observation: the Heun pencil (2.2) in quantum case is equivalent to some confluent Heun operator, while in the classical case it leads to elliptic dynamics of the variable \( X \).

Remark. The case when the Hamilton-Jacobi equation can be reduced to “elliptic” form is wider than the case when time dynamics is elliptic. The Heun pencil approach can explain only the elliptic time dynamics.

3.2. The Zhukovsky- Volterra gyrostat. The second example is connected with the Heun pencil on the Poisson algebra \( su(2) \). This algebra is defined by the standard Poisson brackets relations
\[
\{s_i, s_k\} = \varepsilon_{ikl}s_l, \quad i,k,l = 1,2,3, \tag{3.8}
\]
where \( \varepsilon_{ikl} \) is the completely antisymmetric tensor. The Casimir element
\[
S^2 = s_1^2 + s_2^2 + s_3^2 \tag{3.9}
\]
has zero Poisson bracket with all \( su(2) \) generators
\[
\{S^2, s_i\} = 0, \quad i = 1,2,3 \tag{3.10}
\]
and hence can be chosen as a positive real parameter \( S > 0 \).

The classical Leonard pair on \( su(2) \) can be constructed in different ways. We choose the following pair
\[
X = s_1 + \beta s_2, \; Y = s_1 - \beta s_2 \tag{3.11}
\]
with an arbitrary real parameter \( \beta \).

We have
\[
Z = \{X, Y\} = -2\beta s_3 \tag{3.12}
\]
and
\[ Z^2 = 4\beta^2 s_2^2 = \Phi(X,Y) = 4S^2\beta^2 - (\beta^2 + 1)(X^2 + Y^2) + 2(1 - \beta^2)XY. \] (3.13)
Thus \( \Phi(X,Y) \) is a polynomial of total degree 2 and hence \( X \) and \( Y \) form a CLP.

The classical Heun pencil (2.2) is now (we put \( \tau_0 = 0 \))
\[ W = \tau_1 \left( s_1^2 - \beta^2 s_2^2 \right) - 2\beta\tau_2 s_3 + (\tau_3 + \tau_4) s_1 + \beta(\tau_3 - \tau_4)s_2. \] (3.14)
The first term \( s_1^2 - \beta^2 s_2^2 \) describes the Hamiltonian of the classical Euler top. The last three terms can be considered as ”linear perturbation” of the Euler top. The Hamiltonian (3.14) describes a completely integrable model which is known as the Zhukovsky-Volterra gyrostat [2].

From the above analysis it follows that both \( X(t) \) and \( Y(t) \) are elliptic functions of second order. Thus the classical Heun pencil approach allows to explain the integrability of the Zhukovsky-Volterra gyrostat in terms of elliptic functions. Without the approach based on the classical Heun pencil, it is not so easy to recognize which dynamical variable (constructed from \( s_1, s_2, s_3 \)) evolves as elliptic function of second order. See, e.g. analysis of this problem in [2].

Note that in [3] a quantum analog of the Heun pencil (3.14) was introduced. As expected, this leads to a quantum analog of the Zhukovsky-Volterra gyrostat.

### 3.3. Extension of the ”classical” version of the \( A_1 \) Ruijsenaars model.

The third example of the Heun pencil method is related to the relativistic \( A_1 \) model and can be considered as a ”dequantization” of the Askey-Wilson polynomials [8].

The Hamiltonian \( Y \) is chosen in the form [8]
\[ Y = u(q) \cosh p, \] (3.15)
where the term \( \cosh p \) corresponds to the relativistic kinetic energy and the term \( u(q) \) - to the relativistic potential.

In order to derive ”sufficiently good” potentials, we introduce the variable
\[ X = \varphi(q) \] (3.16)
with some unknown function \( \varphi(q) \). As in the first example we can try to determine when the variables \( X \) and \( Y \) realize the classical Askey-Wilson algebra (and then offer an example of classical Leonard pair [11]).

The necessary and sufficient condition (2.10) now reads
\[ Z^2 = \{X,Y\}^2 = U_2(X)Y^2 + U_0(X) \] (3.17)
with arbitrary polynomials \( U_2(X), U_0(X) \) of second degree. Note that the term \( U_1(X)Y \) disappears due to the concrete choice (3.15) of the variable \( Y \).

We have
\[ Z = u(q)\varphi'(q) \sinh p \] (3.18)
and hence condition (3.17) is equivalent to the two equations
\[ \varphi'(q)^2 = U_2(\phi(q)) \] (3.19)
and
\[ u^2(q) = \frac{U_0(\phi(q))}{U_2(\phi(q))} \] (3.20)
Equation (3.18) determines the function $\varphi(q)$ and has the same type as (3.3). Hence the function $\varphi(q)$ is the same as for the non-relativistic case. In general (nondegenerate) case we can choose

$$\phi(q) = \sinh^2 q$$  \hspace{1cm} (3.21)

Equation (3.20) determines the potential $u(q)$. It also has striking similarity with the non-relativistic equation (3.4). Hence we have the most general expression for the relativistic Poeschl-Teller potential

$$u^2(q) = \beta_1 \sinh^{-2} q + \beta_2 \cosh^{-2} + \beta_0,$$  \hspace{1cm} (3.22)

where the parameters $\beta_i$ should satisfy the positivity condition $u^2(q) > 0$. If these parameters are chosen in such a manner, then we have that the variable $X = \sinh^2 q$ evolves as an elementary function of the time $t$. This gives an elementary and purely algebraic explanation of the "nice" classical dynamics of the $A_1$ Hamiltonian [8] (the latter corresponds to the special choice $\beta_2 = 0$ in (3.22)).

We now can construct the classical Heun pencil corresponding to the model $A_1$. Using the Ansatz (2.2) we have the Hamiltonian

$$W = (\tau_1 \sinh^2 q + \tau_4) u(q) \cosh p + \tau_2 u(q) \varphi'(q) \sinh p + \tau_3 \sinh^2 q$$  \hspace{1cm} (3.23)

with potential $u(q)$ given by (3.22). From the general properties of the classical Heun pencil, it follows that the variable $X = \sinh^2 q$ evolves as an elliptic function of second order for the Hamiltonian (3.23) with arbitrary $\tau_i$, $i = 1, 2, 3, 4$.

The Hamiltonian (3.23) can be further simplified if one performs the canonical transformation $q \rightarrow q$, $p \rightarrow p + \chi(q)$ with some function $\chi(q)$. By an appropriate choice of this function, it is possible to kill the term with $\sinh p$ reducing the Hamiltonian to the form

$$W = \Phi_1(q) \cosh p + \Phi_0(q),$$  \hspace{1cm} (3.24)

where

$$\Phi_0(q) = \tau_3 \sinh^2 q, $$  \hspace{1cm} (3.25)

and

$$\Phi_1(q) = u(q) \sqrt{(\tau_1 \sinh^2 q + \tau_4)^2 - \tau_2^2 \sinh^2 q}.$$  \hspace{1cm} (3.26)

4. CONCLUSIONS

We have showed that time dynamics of two classical variables $X$ and $Y$ is described by elliptic functions of second order if the classical Heun pencil $W$ is chosen as the Hamiltonian. This result is universal: it holds for any type of classical Askey-Wilson algebra and for any choice of the parameters $\tau_i$ in the pencil (2.2). We also illustrated this result on three examples: the Heun extension of the Poeschl-Teller potentials (which explains Manning's observation [7]), the Zhukovsky-Volterra gyrostat which can be considered as a generic Heun pencil on the Lie-Poisson algebra $su(2)$ and the Heun extension of the classical relativistic $A_1$ model. There are many other non-trivial examples of elliptic classical dynamics related with the Heun pencil whose study will be considered in the future.
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