Self-Excited Ising Game *

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Abstract

Effects of activity spillover from neighbouring agents for a noisy binary choice game (Ising game) on a complete graph are studied. Binary choice games are very important for both economics and statistical physics. Here we investigate self-excited Hawkes activity applied to a noisy binary choice game. Particular attention is drawn to the affecting the decision-taking rate.

Keywords: Ising game, activity spillover, Hawkes process

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1. Introduction

Studies of noisy binary choice games are of special interest because of the existence of close parallels to statistical physics of spin systems, in particular to static and dynamic properties of phase transitions in them [1, 2, 3]. These parallels are particularly intriguing because of the fundamentally different origins of equilibria in game theory and statistical physics: in game theory equilibration is a result of balancing individual interests while in statistical physics equilibration is a search of a global minimum of free energy. For the noisy binary choice problem on complete graphs it is long known, see [1] and references therein, that for a special choice of noise game-theoretic equilibria are characterised by the mean-field Curie-Weiss equation, see e.g. [3], describing phase transitions in magnetics. Recently in [4] it was established that static game-theoretic equilibria in noisy binary choice games on graphs correspond to the so-called quantal response equilibria [5].

The dynamics of games can, however, be fundamentally different from conventional spin dynamics due to a variety of possible mechanisms. One of these is a possibility of activity spillover (self-excitation) that was intensively studied for so-called Hawkes processes [6] with applications to finance [7, 8], earthquakes [9] and other subjects, see the recent review in [10]. Recently a master equation formalism for such processes was developed in [11, 12].

The main idea of the present paper is studying the effects of an activity spillover different from the Hawkes self-excitation mechanism having is origin in the activity spillover from neighbouring agents for a noisy binary choice game (Ising game) on a complete graph.

2. Self-excited dynamical Ising game

2.1. Game description

In what follows we consider a dynamical noisy binary choice game of $N$ agents on a graph $\mathcal{G}$. The strategy space of each agent $i$ includes two strategies $s_i = \pm 1$ so at given time $t$ the system is fully characterised by the set $s_t = (s_1, \ldots, s_n)_t$. The temporal evolution of the strategies configuration $s_t \rightarrow s_{t+\delta t}$ within a small time interval $dt$ is assumed to be driven by a strategy flip $s_i \rightarrow -s_i$ of some agent $i$:

$$(s_1, \ldots, s_i, \ldots, s_n)_t \rightarrow (s_1, \ldots, -s_i, \ldots, s_n)_t$$

(1)

The flip probability $\pi_{i,t}$ within a time interval $(t, t + \delta t)$ is assumed to have a form

$$\pi_{i,t} = \lambda_i(t)\delta t \gamma(s_i \rightarrow -s_i|s_{-i,t})$$

(2)

where $\lambda_i(t)\delta t$ is a time-dependent probability of having a possibility for an agent $i$ to reconsider a strategy within a time interval $(t, t + \delta t)$ and $\gamma(s_i \rightarrow -s_i|s_{-i,t})$ is a probability of changing a strategy dependent of the current configuration $s_{-i,t}$ of strategies in the neighbourhood $\mathcal{V}_i$ of the node $i$. Let us denote the sequence of flip times of an agent $i$ by $\{t^i_k\}_{k \in \mathbb{N}}$. The main effect studied in the present paper is the acceleration of the flip activity of the node $i$ by the
preceding flip activity of its neighbours so that

\[ \lambda_i(t) = \lambda_0 + \sum_{k \in V_i} \sum_{i' \leq t} h(t - t'_k), \]  

(3)

where \( h(t - t') \) is a memory kernel. Equation (3) describes flip activity spillover from the neighbours of a node and is therefore a generalisation of the Hawkes activity spillover which is self-induced. The case of constant intensity \( \lambda_i(t) = \lambda_0 \) corresponds to a standard poissonian dynamics underlying the Ising game, see e.g. [11].

In what follows we shall assume an Ising-Glauber flip rate

\[ \gamma_i(t) = \frac{1}{2} \left[ 1 - s_i \tanh \left( \beta J \sum_{k \in V_i} s_k(t) \right) \right], \]  

(4)

where \( \beta = 1/T \) is an inverse temperature and \( J = J/N \) is an Ising coupling constant and a markovian exponential memory kernel

\[ h(t) = \mu e^{-bt} \]  

(5)

In the present study we consider a complete graph topology in which at large \( N \) equation (4) takes the form [2]:

\[ \gamma(t) = \frac{1}{2} \left[ 1 - s \tanh (\beta Jm(t)) \right] \rightarrow \gamma_{\pm}(m) = \frac{1}{2} (1 \pm \tanh(\beta Jm)) \]  

(6)

where \( \gamma_{\pm} = \gamma(\mp s \rightarrow \pm s) \) and

\[ m(t) = \frac{1}{N} \sum_{i=1}^{N} s_i \]  

(7)

As for the memory kernel, it turns out convenient to change normalisation so that

\[ \lambda(t) = \lambda_0 + \frac{\mu}{N} \sum_{\tau_k} e^{-b(t - \tau_k)} \]  

(8)

As all vertices in the complete graph are equivalent in describing the system evolution it is natural to use a collective intensity of the flip process \( \Lambda(t) = N\lambda(t) \).

2.2. Dynamical evolution

A state of the system at any given time is fully described by the values \((\Lambda(t), m(t))\). Their evolution in the time interval \([t; t + \delta t]\) is driven by the two following mechanisms:

- The strategy flips \(-1 \rightarrow 1 \) or \( 1 \rightarrow -1 \) leading to a constant shift in the process intensity \( \Lambda \rightarrow \Lambda + \mu \) and a change in the average strategy \( m \rightarrow m \pm 1/N \) taking place with the following probabilities \( \pi_{\pm} \)

\[ (\Lambda, m) \rightarrow \left( \Lambda + \mu, m \pm \frac{2}{N} \right) \leftrightarrow \pi_{\pm} = \Lambda \delta t \cdot \gamma_{\pm}(m) \frac{1 \mp m}{2} \]  

(9)
• Decrease of the rate $\Lambda$ taking place in the absence of a strategy flip

$$(\Lambda, m) \rightarrow ((\Lambda - \Lambda_0)e^{b\delta t} + \Lambda_0, m)$$

occurring with the probability

$$\pi_0 = 1 - \Lambda\delta t \left( \gamma_-(m)\frac{1+m}{2} + \gamma_+(m)\frac{1-m}{2} \right)$$

From equations (9,10,11) we obtain the following master equation for the probability density function $P(\Lambda, m; t)$

$$\frac{\partial P(\Lambda, m; t)}{\partial t} = b \frac{\partial}{\partial \Lambda} ((\Lambda - \Lambda_0)P(\Lambda, m; t))$$

$$+ \left( (\Lambda - \mu) \left( m + \frac{1}{2} + \frac{1}{N} \right) \gamma_- \left( m + \frac{2}{N} \right) P(\Lambda - \mu, m + \frac{2}{N}; t) \right)$$

$$+ \left( (\Lambda - \mu) \left( 1 - m + \frac{1}{N} \right) \gamma_+ \left( m - \frac{2}{N} \right) P(\Lambda - \mu, m - \frac{2}{N}; t) \right)$$

$$- \Lambda \left( m + \frac{1}{2} \gamma_-(m) + \frac{1-m}{2} \gamma_+(m) \right) P(\Lambda, m; t)$$

Let us note that if one neglects evolution of $m$, equation (12) coincides with the master equation for Poissonian self-excited processes obtained in [11].

From (12) a standard computation in the limit $N \rightarrow \infty$ leads us to the following Fokker-Planck equation describing the Brownian motion in space $(m, \lambda)$,

$$\partial_t P = \partial_i(f_i P) + \frac{1}{N} \partial_{ij} (g_{ij} P)$$

$$f_i = \begin{pmatrix} \lambda \left[ m - \tanh (\beta Jm) \right] \\ -\lambda \left[ 1 - m \tanh (\beta Jm) \right] \end{pmatrix}$$

$$g_{ij} = \begin{pmatrix} \lambda \left[ 1 - m \tanh (\beta Jm) \right] & -\lambda \left[ m - \tanh (\beta Jm) \right] \\ -\lambda \left[ m - \tanh (\beta Jm) \right] & \lambda \left[ 1 - m \tanh (\beta Jm) \right] \end{pmatrix},$$

and due to smallness $1/2N$, the following evolution equations for $m$ and $\lambda$:

$$\dot{m}(t) = -\lambda(t) \left[ m(t) - \tanh (\beta Jm(t)) \right]$$

$$\dot{\lambda}(t) = \lambda(t) \left[ 1 - m(t) \tanh (\beta Jm(t)) \right] - b \left[ \lambda(t) - \lambda_0 \right],$$

where we have performed the following rescaling of the variables:

$$\lambda \rightarrow \frac{2\lambda}{\mu}, \lambda_0 \rightarrow \frac{2\lambda_0}{\mu}, b \rightarrow \frac{2b}{\mu}, t \rightarrow \frac{\mu t}{2}.$$
Ising game under consideration are the same as in the constant intensity case \( \lambda = \lambda_0 \) and are described by by the Curie-Weiss equation \[1, 2, 4\]

\[ m_{\text{eq}} = \tanh(\beta J m_{\text{eq}}), \]  

(16)

the difference with the standard case being due to the temporal evolution of relaxation intensity described by equation \[15\]. The equilibrium configurations \( m_{\text{eq}}(\beta) \) described by the corresponding solutions of \[16\] are therefore the conventional ones: \( m_{\text{eq}} = 0 \) at high temperatures \( \beta J < 1 \) and \( m_{\text{eq}} = \pm m_0(\beta) \) at low temperatures \( \beta J > 1 \).

As to the characteristic regimes of evolution of the process intensity, it is convenient to consider the vicinity of equilibrium \( m \sim m_{\text{eq}} \) in which from equations \[15, 16\] there follows that

\[ \dot{\lambda}(t) \simeq [1 - m_{\text{eq}}^2 - b] \lambda(t) + b \lambda_0 \]  

(17)

so that

\[ \lambda(t) = \frac{\lambda_0}{1 - m_{\text{eq}}^2 - b} \left[ (1 - m_{\text{eq}}^2)e^{(1-m_{\text{eq}}^2-b)t} - b \right] \]  

(18)

The two characteristic regimes corresponding to growing and relaxing intensity are thus characterised by the following asymptotic behavior:

\[ 1 - m_{\text{eq}}^2 - b > 0 \quad \rightarrow \quad \lambda(t)_{t \rightarrow \infty} \simeq \frac{e^{(1-m_{\text{eq}}^2-b)t}}{1 - m_{\text{eq}}^2 - b} \lambda_0 \]  

\[ 1 - m_{\text{eq}}^2 - b < 0 \quad \rightarrow \quad \lambda(t)_{t \rightarrow \infty} \simeq \frac{b}{m_{\text{eq}}^2 + b - 1} \lambda_0 \]  

(19)

From the derived system of evolution equations \[14, 15\] it is clear that the main effect of time-dependent intensity should be in speeding up relaxation to the appropriate temperature - dependent equilibrium. A nontrivial part of this effect is its dependence on the characteristic memory timescale \( \tau_\lambda = 1/b \). At parametrically small \( \tau_\lambda \) (large \( b \)) one expects a rapid recovery of the base intensity \( \lambda_0 \) while at large \( \tau_\lambda \) (small \( b \)) one, on the contrary, expects a prolonged period of high-intensity evolution. Therefore at large \( b \) the influence of self-excitement in the game development should be small while at large \( b \) it should, on the contrary, be large.

In the high-temperature phase \( \beta J < 1 \) the game has only one equilibrium configuration \( m_{\text{eq}} = 0 \). In Fig. 1(a - d) we show relaxation from the initial state \( m(0) = 1 \) for a set of values of \( b \) at various temperatures \( T < 1/J \) in comparison to the Poisson case. As expected, at all temperatures the self-excited game relaxes faster than in the Poissonian case with relaxation becoming slower with diminishing \( b \). In Fig. 1(c,d) we also clearly observe the critical slowing down of relaxation characteristic for dynamics in the vicinity of the phase transition point \( \beta J = 1 \).

In the low-temperature phase \( \beta J > 1 \) the Curie-Weiss equation has three solutions: the (dynamically) unstable one at \( m_{\text{eq}} = 0 \) and a pair of temperature dependent stable equilibria at \( m_{\text{eq}} = \pm m_0(T) \) such that at the critical point \( T_c = J \) one has \( m_0(T_c) = 0 \) and at \( T \rightarrow 0 \) one has \( m_0 \rightarrow 1 \).

The relaxation to the stable equilibria is illustrated in Fig. 2(a-d) where for definiteness we have chosen the equilibrium corresponding to \( m_{\text{eq}} = m_0 \). Similar to the results on above-described case of high temperature equilibrium shown in Fig. 1(a-d), the bigger is \( b \), the faster is the relaxation to equilibrium.
Figure 1. Relaxation from the initial state \( m(0) = 1 \) to the equilibrium \( m_{eq} = 0 \) at \( b = 0.1, 0.5, 1, 2, 10 \) and Poisson case at different temperatures: a) \( \beta_J = 0.1 \), b) \( \beta_J = 0.5 \), c) \( \beta_J = 0.9 \), d) \( \beta_J = 0.99 \).

Figure 2. Relaxation from the initial state \( m = 0 \) to the equilibrium \( m_{eq} = m_0 \) at \( b = 0.1, 0.5, 1, 2, 10 \) and Poisson case at different temperatures: a) \( \beta_J = 1.1 \), b) \( \beta_J = 1.5 \), c) \( \beta_J = 2 \), d) \( \beta_J = 10 \).
3. Conclusion

The self-excited version of the Ising game on a complete graph was considered. The considered mechanism of self-excitation was an activity spillover from the neighbouring agents affecting the decision-taking rate at a node under consideration. Fokker-Planck equation describing Brownian motion in space of the average opinion and the process intensity, as well as system of evolution equations for the average opinion and the process intensity generalising the Curie-Weiss equations, were derived using the master equation formalism. A detailed dependence of the relaxation acceleration on the parameters of the Poissonian memory kernel was studied. A further interesting aspect for theoretical work is to study escape from one stable equilibria to another.

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