Quantum Brownian Motion on a Triangular Lattice and c=2 Boundary Conformal Field Theory

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We study a single particle diffusing on a triangular lattice and interacting with a heat bath, using boundary conformal field theory (CFT) and exact integrability techniques. We derive a correspondence between the phase diagram of this problem and that recently obtained for the 2 dimensional 3-state Potts model with a boundary. Exact results are obtained on phases with intermediate mobilities. These correspond to non-trivial boundary states in a conformal field theory with 2 free bosons which we explicitly construct for the first time. These conformally invariant boundary conditions are not simply products of Dirichlet and Neumann ones and unlike those trivial boundary conditions, are not invariant under a Heisenberg algebra.

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I. INTRODUCTION

Conformal field theory (CFT) with boundaries finds applications both to open string theory and to various quantum impurity problems in condensed matter physics. These generally describe gapless bulk excitations interacting with some localized degrees of freedom. In these problems the gapless bulk excitations can be represented by a conformal field theory in (1+1) space-time dimensions, often simply free bosons. It is generally found that the boundary dynamics renormalize, at low energies, to a conformally invariant boundary condition (b.c.).

Quantum Brownian motion (QBM) provides an intriguing example of such a problem. Here one considers a heavy particle moving in a d-dimensional periodic potential and interacting with a heat bath. A simplified model for the heat bath is an infinite set of harmonic oscillators coupled linearly to the particle co-ordinate. (This is the obvious generalization of the Caldeira-Leggett model for a particle in a double well potential.)

When the oscillator spectral weight vanishes linearly at low frequencies, corresponding to ohmic dissipation, the set of oscillators may be represented by a (1+1) dimensional quantum conformal field theory of free massless bosons living on a fictitious half-line with the heavy particle at the origin. The number of bosonic fields required is given by the dimensionality of the space in which the heavy particle is diffusing (normally three). The boson fields at the origin, \( \vec{\phi}(0) \), correspond to the momentum of the particle and the dual fields, \( \vec{\tilde{\phi}}(0) \) to the particle’s position. The boson CFT may be regarded as compactified on the lattice on which the heavy particle is diffusing. The dimensionless compactification radius (scaled by the coupling to the bath) is a crucial parameter for QBM. The fictitious extra dimension is analogous to the coordinate along the string, \( \sigma \), in open string theory, while the field \( \vec{\phi} \) plays the role of the string coordinates in \( D \) dimensional space time, \( X^h \). Conformal invariant boundary conditions obtained in the dissipative quantum mechanics framework have therefore immediate applications to open string theory, and potential interpretations in terms of D-branes.

Until recently it was generally believed that, depending on the strength of the dissipation relative to the period of the potential, the particle could only be in either localized or freely diffusing phases corresponding to Dirichlet (D):\( \vec{\phi}(0) = \text{constant} \), or Neumann (N): \( \vec{\tilde{\phi}}(0) = \text{constant} \), boundary conditions.

in the CFT, respectively. However, it was recently shown\textsuperscript{8} that, for certain lattices, other phases are possible with intermediate mobility, in which the particle is neither perfectly localized nor freely diffusing. These correspond to non-trivial conformally invariant boundary conditions in the free boson CFT which are neither D nor N. In fact, the existence of such phases was shown earlier in another quantum impurity problem: tunneling through a single impurity in a quantum wire.\textsuperscript{9}

While these interesting phases seem to cry out for a general boundary CFT solution this has, so far, eluded us. The problem of classifying all (or all physically relevant) conformally invariant boundary conditions in $c = 2$ conformal field theory (2 free bosons) remains very much open. In the much simpler case of 1 free boson, $c = 1$, it is widely believed that only D and N phases generally occur. On the other hand, for conformal field theories with a finite number of conformal towers, an elegant way of generating non-trivial conformal boundary conditions is provided by fusion. This method can be used to generate an apparently complete set of conformally invariant b.c.’s in, for example, the Ising model\textsuperscript{7} or 3-state Potts model\textsuperscript{7}. In the case of Wess-Zumino-Witten (WZW) models, one can obtain by fusion the boundary conditions that describe the low temperature phases of various generalized Kondo problems.\textsuperscript{10} However, the infinite number of conformal towers present in $c = 2$ CFT makes the generalization of the fusion technique rather subtle. Indeed, it is not even clear in general how to understand the simple D and N b.c.’s this way. Whether or not some generalized notion of fusion will ultimately provide a complete solution to this problem for $c = 2$ is unclear at present. Needless to say, the generalization to larger values of $c$, in particular $c=3$ corresponding to 3-dimensional QBM, is also an open problem.

The original argument\textsuperscript{11} for the existence of non-trivial phases came from a study of the phase diagram in the quantum wire problem. This was studied\textsuperscript{12} by a type of “ɛ-expansion”. By fine-tuning the compactification radius the non-trivial fixed points can be moved arbitrarily close to either N or D fixed points, allowing a perturbative expansion. The other solved case for $c=2$ was obtained in the context of QBM on a triangular lattice. For a special value of dissipation strength (compactification radius) a beautiful exact mapping of the problem onto the critical point of the 3-channel Kondo problem\textsuperscript{13} was obtained by Yi and Kane,\textsuperscript{14} allowing use of results from that problem\textsuperscript{14} based on boundary CFT in the $SU(2)_3$ WZW model. Although it can be seen that non-trivial phases exist for a range of dissipation strength on the triangular lattice, no general solution has been found.

Several years after the Caldeira-Leggett type model was proposed for QBM,\textsuperscript{15} it was argued\textsuperscript{16} that this model does not provide a valid description of a heavy particle interacting with phonons or electron-hole pairs. In particular, the localized phase does not occur in physical models of quantum Brownian motion.\textsuperscript{16} Thus physical applications of the model may only be to the quantum wire and related problems. Nevertheless, the Caldeira-Leggett type QBM model is perfectly consistent as an ultraviolet (UV) regularization of a boundary conformal field theory.

In this paper we re-examine the soluble triangular lattice QBM Yi-Kane model. Our purpose is both to understand this fascinating system better and also to shed light on the general boundary CFT problem. Rather than relating this problem to the $SU(2)_3$ WZW model we instead relate it to the 3-state Potts model. This is done via a conformal embedding whereby the bulk degrees of freedom are represented by a direct sum of the 3-state Potts model, an Ising model and a tri-critical Ising model, satisfying:

$$c = c_{\text{Potts}} + c_{\text{Ising}} + c_{\text{tri-critical}} = 4/5 + 1/2 + 7/10 = 2.$$  \hspace{1cm} (1.1)

The $Z_3$ symmetry of the Potts model corresponds to the point group symmetry of the appropriate triangular lattice model. This seems like a more natural formulation of the QBM problem since it does not contain an $SU(2)$ symmetry. The 4 conformally invariant boundary conditions in the Potts model: free, fixed, mixed and “new” are shown to correspond to the 4 phases that can occur in the QBM problem. In this way it is possible to obtain the various fixed points by fusion in the Potts sector of the theory. These four fixed points correspond to localized and freely diffusing fixed points, the non-trivial fixed point of Yi and Kane and one new fixed point not previously known. The mobilities of these phases and also the groundstate entropies can be calculated from the fusion approach. We also show that it is possible to use an alternative conformal embedding to Eq. (1.1) where the Ising and tricritical Ising sectors are replaced by a $Z_3^{(5)}$\textsuperscript{17} conformal field theory, which is a sort of multicritical Potts model.\textsuperscript{17} We demonstrate that some of the crossovers between these fixed points induced by relevant boundary operators can be studied exactly using integrability techniques. This would allow, for example, exact calculations of universal quantities at all temperatures for a system which is crossing over between freely diffusing behavior at

\hspace{1cm}
higher temperatures and localized or “non-trivial” behavior at \( T = 0 \). The integrability also helps to understand the phase diagram of the model by determining all RG flows from a given fixed point.

In the next section we review the fusion approach to boundary CFT and the standard D and N b.c.’s for free bosons. We also show there that it is possible to obtain the D b.c. from the N b.c. by fusion with a twist field for the \( c=1 \) case of a single periodic boson, for certain rational radii. In Sec. III we introduce the QB\( \text{M} \) problem and its connection with boundary CFT, with careful attention to the boundary conditions and the groundstate degeneracy. We show that this degeneracy scales as the size of the space on which the particle moves. We derive, apparently for the first time, the connection between the particle co-ordinate and the value of the dual field at the origin. In Section IV we introduce a model of QB\( \text{M} \) on a triangular lattice which generalizes the model of Yi and Kane, and conjecture its phase diagram, suggesting the analogy with the Potts model. In Section V we discuss our conformal embedding and show how fusion in the Potts sector gives the various fixed points conjectured in Sec. IV. We also calculate the groundstate entropies and mobilities of these fixed points and discuss the corresponding boundary states. We also discuss some additional fixed points, that occur for more general Hamiltonians, and are obtained by fusion in the Ising, tricritical Ising or \( Z_3 \) sectors. In Section VI we discuss the integrable flows between the various fixed points. In Section VII we discuss generalizations of this model to \( (n-1) \) bosons, corresponding to QB\( \text{M} \) in \( (n-1) \) dimensions and also discuss the relationship with the \( n \)-channel Kondo problem.

II. BOUNDARY CONFORMAL FIELD THEORY

As will be shown in the next section, QB\( \text{M} \) can be formulated in terms of \((1+1)\) dimensional massless bosons on the half-line, \( x > 0 \), with interactions only at the boundary, \( x = 0 \). As such it is related to a number of other problems in particle physics and condensed matter physics that have been successfully studied using boundary conformal field theory (BCFT). The basic assumption in this approach is that the boundary dynamics renormalize, at low energies, to an effective conformally invariant b.c. A constructive approach to classifying possible fixed points is then to attempt to enumerate all possible conformally invariant b.c.’s corresponding to a given critical bulk theory. This has met with great success in situations where the bulk theory can be formulated in terms of a finite number of conformal towers, with the set of conformally invariant b.c.’s being generally in one to one correspondence with the bulk conformal towers. In this case, the “fusion” technique is very useful in constructing the conformally invariant b.c.’s. Unfortunately, many interesting quantum impurity problems apparently cannot be formulated in terms of a finite number of bulk conformal towers, generally corresponding to integer values of \( c \). Consequently the problem of studying the fixed points in these cases remains very much open. As we will show in Sec. V, for a special choice of parameters, QB\( \text{M} \) on a triangular lattice can be reduced, using a conformal embedding, to a finite number of conformal towers.

In this section we review two quite different approaches to boundary conformal field theory. The fusion approach of Cardy which has been very successful for CFT’s with a finite number of conformal towers and the straightforward D and N b.c.’s for free bosons. It is not known, in general, how to apply the fusion approach for free bosons but we demonstrate that this is possible for a single free boson at a special (discrete, infinite) set of compactification radii.

A. Fusion Approach to Boundary CFT

We review briefly some important results due to Cardy on the case of a finite number of conformal towers.

Precisely what is meant by a conformally invariant boundary condition? Without boundaries, conformal transformations are analytic mappings of the complex plane:

\[
z \equiv \tau + ix,
\]

into itself:
We may Taylor expand an arbitrary conformal transformation around the origin:

\[ w(z) = \sum_n a_n z^n, \tag{2.3} \]

where the \( a_n \)'s are arbitrary complex coefficients.

They label the various generators of the conformal group. It is the fact that there is an infinite number of generators (i.e. coefficients) which makes conformal invariance so powerful in \((1+1)\) dimensions. Now suppose that we have a boundary at \( x = 0 \), the real axis. At best, we might hope to have invariance under all transformations which leave the boundary fixed. This implies the condition:

\[ w(\tau)^* = w(\tau) \tag{2.4} \]

where \( \tau \) denotes imaginary time.

We see that there is still an infinite number of generators, corresponding to the \( a_n \)'s of Eq. (2.3) except that now we must impose the conditions:

\[ a_n^* = a_n. \tag{2.5} \]

We have reduced the (still \( \infty \)) number of generators by a factor of 1/2. The fact that there is still an \( \infty \) number of generators, even in the presence of a boundary, means that this boundary conformal symmetry remains extremely powerful.

Boundary conformal invariance implies that the momentum density operator, \( T - \bar{T} \) vanishes at the boundary. This amounts to a type of unitarity condition. Since \( T(t, x) = T(t + x) \) and \( \bar{T}(t, x) = \bar{T}(t - x) \), it follows that

\[ \bar{T}(t, x) = T(t, -x). \tag{2.6} \]

i.e. we may regard \( \bar{T} \) as the analytic continuation of \( T \) to the negative axis. Thus instead of working with left and right movers on the half-line we may work with left-movers only on the entire line.

When the bulk theory contains a conserved current, \( J \), it is possible to impose additional symmetry requirements on the boundary conditions of the form:

\[ J(0) = \pm \bar{J}(0). \tag{2.7} \]

In the case of free bosons the generic symmetry consists of a produce of U(1) current algebras (one for each boson) known as a Heisenberg algebra. It is then possible, more generally, to impose an invariance of the form:

\[ \vec{J}(0) = \mathcal{R} \vec{J}(0), \tag{2.8} \]

where the vector \( \vec{J} \) contains the U(1) currents and \( \mathcal{R} \) is a rotation matrix. Since the chiral energy density operators, \( T \) and \( \bar{T} \) can be written in Sugawara form:

\[ T = \frac{1}{4\pi} \vec{J}^2, \quad \bar{T} = \frac{1}{4\pi} \vec{J}^2, \tag{2.9} \]

it follows that Eq. (2.6) is still obeyed. One of the lessons of this paper is that, while it is fairly straightforward to construct boundary conditions in free boson theories that are invariant with respect to the full Heisenberg algebra, it is much more difficult, apparently, to find the other boundary conditions which obey only the Virasoro condition of Eq. (2.6) and not the additional constraints of Eq. (2.8).

To exploit this symmetry, following Cardy, it is very convenient to consider a conformally invariant system defined on a cylinder of circumference \( \beta \) in the \( \tau \)-direction (imaginary time) and length \( l \) in the \( x \) direction, with conformally invariant boundary conditions \( A \) and \( B \) at the two ends. [See Figure (1).] From the quantum mechanical point of view, this corresponds to a finite temperature, \( T = 1/\beta \). The partition function for this system is:
\[ Z_{AB} = \text{tr} e^{-\beta H_{AB}^l}, \quad (2.10) \]

where we are careful to label the Hamiltonian by the boundary conditions as well as the length of the spatial interval, both of which help to determine the spectrum. (We sometimes refer to this as the open string channel). Alternatively, we may make a modular transformation, \( \tau \leftrightarrow x \). Now the spatial interval, of length, \( \beta \), is periodic (the closed string channel). We write the corresponding Hamiltonian as \( H_{\beta}^l \). The system propagates for a time interval \( l \) between initial and final states \( A \) and \( B \). Thus we may equally well write:

\[ Z_{AB} = \langle A | e^{-lH_{\beta}^l} | B \rangle. \quad (2.11) \]

Equating these two expressions, Eq. (2.10) and (2.11) gives powerful constraints which allow us to determine the conformally invariant boundary conditions.

Due to the condition of Eq. (2.6), in the purely left-moving formulation, the energy momentum density, \( T \) is basically unaware of the boundary condition. Hence, in calculating the spectrum of the system with boundary conditions \( A \) and \( B \) introduced above, we may regard the system as being defined periodically on a torus of length \( 2l \) with left-movers only. The conformal towers of \( T \) are unaffected by the boundary conditions, \( A, B \). However, which conformal towers occur \textit{does} depend on these boundary conditions. We introduce the characters of the Virasoro algebra, for the various conformal towers:

\[ \chi_a(e^{-\pi \beta/l}) \equiv \sum_i e^{-\beta E^a_i(2l)} \quad (2.12) \]

where \( E^a_i(2l) \) are the energies in the \( i^{th} \) conformal tower for length \( 2l \). i.e.:

\[ E^a_i(2l) = \frac{\pi}{l} x^a_i - \frac{\pi c}{24l}, \quad (2.13) \]

where the \( x^a_i \)'s correspond to the (left) scaling dimensions of the operators in the theory and \( c \) is the conformal anomaly.

The spectrum of \( H_{AB}^l \) can only consist of some combination of these conformal towers. i.e.:

\[ Z_{AB} = \sum_a n_{AB}^a \chi_a(e^{-\pi \beta/l}), \quad (2.14) \]

where the \( n_{AB}^a \) are some non-negative integers giving the multiplicity with which the various conformal towers occur. For minimal conformal field theories, \( a \) runs over the finite set of irreducible representations of the Virasoro algebra. For more complicated theories like the ones of interest in this paper, the corresponding set is infinite, and it is not always clear what kind or representations to expect on the right hand side of this equation. We stress that we are interested in boundary conditions restricted by conformal invariance only, so characters of generalized symmetry algebras are not relevant here.

Importantly, only these multiplicities depend on the boundary conditions, not the characters, which are a property of the bulk left-moving system. Thus, a specification of all possible multiplicities, \( n_{AB}^a \) amounts to a specification of all possible boundary conditions \( A \). The problem of specifying conformally invariant
boundary conditions has been reduced to determining sets of integers, \( n^A_B \). For minimal conformal field theories, where the number of conformal towers is finite, only a finite number of integers needs to be specified.

Now let us focus on the boundary states, \(|A\rangle\). These must obey the operator condition:

\[
[T(x) - \bar{T}(x)] |A\rangle = 0 \quad (\forall x).
\]

(2.15)

(Note that, after the modular transformation, \( x \) denotes the periodic co-ordinate.) Imposing periodic boundary conditions along the boundary and Fourier transforming with respect to \( x \), this becomes:

\[
[L_n - \bar{L}_{-n}] |A\rangle = 0.
\]

(2.16)

This implies that all boundary states, \(|A\rangle\) must be linear combinations of the “Ishibashi states”:

\[
|a\rangle \equiv \sum_m |a;m\rangle \otimes |a;-m\rangle.
\]

(2.17)

Here \( m \) labels all states in the \( a^{th} \) conformal tower. The first and second factors in Eq. (2.17) refer to the left and right-moving sectors of the Hilbert Space. Here, \( a \) belongs to the set of representations of the Virasoro algebra which appear simultaneously in the right and left sector of the theory with periodic boundary conditions. It is in general a smaller set than the one appearing in Eq. (2.11). Thus we may write:

\[
|A\rangle = \sum_a |a\rangle \langle a|0\rangle |A\rangle.
\]

(2.18)

Here,

\[
|a0\rangle \equiv |a;0\rangle \otimes |a;0\rangle.
\]

(2.19)

(Note that while the states, \(|a;m\rangle \otimes |b;n\rangle\) form a complete orthonormal set, the Ishibashi states, \(|a\rangle\) do not have finite norm.) Thus, specification of boundary states is reduced to determining the matrix elements, \( \langle a0|A\rangle \). For minimal conformal field theories, there is a finite number of such matrix elements. Thus the partition function becomes:

\[
Z_{AB} = \sum_a \langle A|a0\rangle \langle a0|B\rangle \langle a|e^{-lH/\beta} |a\rangle.
\]

(2.20)

From the definition of the Ishibashi state, \(|a\rangle\) we see that:

\[
\langle a|e^{-lH/\beta} |a\rangle = \sum_m e^{-2lE_m^a(\beta)},
\]

(2.21)

the factor of 2 in the exponent arising from the equal contribution to the energy from \( T \) and \( \bar{T} \). This can be written in terms of the characters:

\[
\langle a|e^{-lH/\beta} |a\rangle = \chi_a(e^{-4\pi l/\beta}).
\]

(2.22)

We are now in a position to equate these two expressions for \( Z_{AB} \):

\[
Z_{AB} = \sum_a \langle A|a0\rangle \langle a0|B\rangle \chi_a(e^{-4\pi l/\beta}) = \sum_a n_{AD}^a \chi_a(e^{-\beta/4}).
\]

(2.23)

This equation must be true for all values of \( l/\beta \). It is very convenient to use the modular transformation of the characters:

\[
\chi_a(e^{-\beta/4}) = \sum_b S^a_b \chi_b(e^{-4\pi l/\beta}).
\]

(2.24)
Here we refer to \( S_0 \) as the matrix of modular transformations. We thus obtain a set of equations relating the multiplicities, \( n_{AB}^a \), which determine the spectrum for a pair of boundary conditions and the matrix elements \( \langle a0|A \rangle \) determining the boundary states:

\[
\sum_b S_{ab}^a n_{AB}^b = \langle A|a0\rangle\langle a0|B \rangle.
\]

We refer to these as Cardy’s equations; they basically allow a determination of the boundary states and spectrum. The problem is to find a set of boundary states \( A \) (defined by the coefficients \( \langle A|a0 \rangle \)) satisfying Cardy’s equations, such that, for any pair \( A, B \) in this set, the \( n_{AB}^b \) are non-negative integer coefficients, the identity representation appearing at most once, \( n_{AB}^0 \leq 1 \). An important point in deriving \( (2.25) \) is the linear independence of the characters. This property may actually not hold for “non-diagonal” theories. In such cases, there will generally be some other quantum number that allows one to distinguish the corresponding operators, and still prove Eq. \( (2.25) \). The study of these equations has given rise to considerable theoretical activity recently. In the case of rational conformal field theories, and with the additional key hypothesis of completeness, it has been shown that finding sets of conformally invariant boundary conditions is equivalent to finding integer valued representations of the fusion algebra. The latter problem is under reasonable control, and as a result, conformal boundary conditions for minimal models have for instance been classified in Ref. (22).

In the present paper, we are interested in a more difficult problem: in fact, since we look for boundary conditions respecting only conformal invariance, the two boson problem has the complexity of an irrational theory, even for rational values of the radius. Very little is known about this case, and we rely heavily on a not entirely systematic but highly efficient method called fusion, which is inspired by Cardy’s seminal paper and was used with success in the Kondo problem.

Generally, boundary states corresponding to trivial boundary conditions can be found by inspection. i.e., given \( n_{AA}^0 \) we can find \( \langle a|A \rangle \). We can then generate new (sometimes non-trivial) boundary states by fusion, i.e. given any conformal tower, \( c \), we can obtain a new boundary state \( |B \rangle \) and new spectrum \( n_{AB}^a \) from the “fusion rule coefficients”, \( N_{ab}^c \). These non-negative integers are defined by the operator product expansion (OPE) for (chiral) primary operators, \( \phi_a \). In general the (OPE) of \( \phi_a \) with \( \phi_b \) contains the operator \( \phi_c \) \( N_{ab}^c \) times. We denote by \( B \) the new boundary condition and boundary state, generated by fusion from the boundary condition \( A \) by fusion with the operator \( c \). The partition function, \( Z_{BD} \), with an arbitrary boundary condition \( D \) at the other end of the system, is given by the multiplicities: given by a fusion with a primary field \( c \) has multiplicities given by the fusion rule coefficients \( N_{bc}^a \) as

\[
n_{DB}^a = \sum_b N_{bc}^a n_{AD}^b,
\]

hence the name “fusion construction”. Because \( N_{bc}^a \) are non-negative integers, the new multiplicities \( n_{DB}^a \) are also non-negative integers, and thus physical. Alternatively, in the closed string channel, the new boundary state \( |B \rangle \) is defined through

\[
\langle a0|B \rangle = \langle a0|A \rangle \frac{S_a^a}{S_0},
\]

where \( 0 \) labels the conformal tower of the identity operator.

Let us now show that the boundary state \( \langle 2.27 \rangle \) gives the multiplicity \( \langle 2.26 \rangle \), i.e. they satisfy Cardy’s equation \( \langle 2.25 \rangle \). The right-hand side of Eq. \( \langle 2.25 \rangle \) becomes:

\[
\langle D|a0\rangle\langle a0|B \rangle = \langle D|a0\rangle\langle a0|A \rangle \frac{S_a^a}{S_0}.
\]

The left-hand side becomes:

\[
\sum_b S_{ab}^a n_{DB}^b = \sum_{b,d} S_{ab}^a N_{ad}^b n_{DA}^d.
\]
We now use a remarkable identity relating the matrix of modular transformations to the fusion rule coefficients, known as the Verlinde formula:

\[ \sum_b S^a_b N^b_d = \frac{S^a_c S^c_e}{S^e_0}. \]  

(2.30)

This gives:

\[ \sum_b S^a_b n_{dB} = \frac{S^a_0}{S^c_0} \sum_d S^c_d n_{DA} = \frac{S^a_0}{S^c_0} \langle D|a0\rangle \langle a0|A \rangle = \langle D|a0\rangle \langle a0|B \rangle, \]  

(2.31)

proving that fusion does indeed give a new solution of Cardy’s equations. The multiplicities, \( n_{BB}' \), are given by double fusion:

\[ n_{BB}' = \sum_{b,d} N^a_{bc} N^c_{bd} n_{AA}. \]  

(2.32)

[Recall that \( |B \rangle \) is obtained from \( |A \rangle \) by fusion with the primary operator \( c \).] It can be checked that the Cardy equation with \( A = B \) is then obeyed.

As mentioned before, the key problem in boundary CFT is the construction of a complete set of boundary states (and b.c.’s), that is the largest possible set of boundary states, \( |A_i \rangle \) such that \( Z_{A_i,A_i} \) is a physical partition function. By physical we mean that the partition function with any two boundary conditions is always given by non-negative integer multiplicities \( n^a_{A_i,A_j} \). Noting that any linear combination of physical boundary states with non-negative integer coefficients,

\[ \sum_i n_i |A_i \rangle, \]  

(2.33)

also gives a physical partition function in this sense, we see that we must impose an additional condition to eliminate such states. The lowest energy state with the same b.c. at both ends of the system is independent of the b.c. corresponding to the absolute finite size groundstate, with \( x_i = 0 \). (This follows from making a conformal mapping to the half-plane and using the fact that the identity boundary operator exists with any b.c.) We may choose to impose an additional condition that \( Z_{A_i,A_i} \) contains the zero energy state exactly once, \( n^0_{A_i,A_i} = 1 \). This eliminates linear combination states. Of course, once we have found a complete set, we may always form linear combinations with non-negative integer coefficients. Actually, the fusion construction sometimes gives such linear combination states. These can also arise from integrable flows and have physical applications in some cases. (The existence of extra dimension 0 boundary operators is often associated with first order phase transitions.)

In the case of minimal diagonal theories, conformal boundary conditions are in one to one correspondence with the set of Virasoro representations. The corresponding boundary states can be generated by fusion starting from the boundary state \( |0\rangle \) defined by its diagonal partition function, \( n^a_{00} = \delta^a_0 \).

In more complicated cases, fusion has proven an invaluable tool. One of the key advantages of the method is that one can use fusion with fields that are not in the spectrum of the bulk theory. Indeed, as already explained above, while the the boundary state (in the closed string channel) must be compatible with the bulk spectrum, the conformal towers appearing in the open string channel do not have to be in the bulk spectrum. In many cases, it is then possible to use not the fusion algebra of the original bulk theory but another one, still compatible with the modular transformation matrix, to generate boundary states. For instance, in the case of the three state \( (A_4, D_4) \) Potts model, the characters appearing in the bulk partition function are a subset of all the possible characters at \( c = \frac{4}{5} \). Using the full matrix of modular transformations and fusion algebra for the associated diagonal tetracritical Ising model \( (A_4, A_5) \) led to the discovery of the new boundary condition in Ref. (11). We will discuss in the next section how the fusion method can successfully be applied to the case of a free boson.

Before leaving this section, we should emphasize that complete sets of boundary states form compatible sets of states which may be overlapping. For example, in the Ising model the three states corresponding to spin up, spin down and free b.c.’s are a complete mutually compatible set. However, one could also
consider imposing an up or down b.c. on the dual spins in the Ising model. Such states should also be compatible with the free b.c. but not with the fixed b.c.’s on the original (non-dual) spins. This non-compatibility is presumably related to the non-local nature of the dual spin variables when expressed in terms of the original spins. This implies that one couldn’t define a physical partition function with the original spins pointing up at one boundary and the dual spins pointing up (or down) at the other. In a physical application one typically begins with a particular b.c. and then wants to find all other ones which are compatible with it. For instance, we may be interested in adding arbitrary local interactions near the boundary and finding all b.c.’s to which the system can renormalize.

B. Dirichlet Neumann, and Rotated Boundary Conditions For Free Bosons

We review the simplest conformally invariant boundary conditions, namely Dirichlet and Neumann boundary conditions and some of their generalizations, for multicomponent compactified free bosonic field theory. The Lagrangian density of the free boson theory is given by

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \tilde{\phi})^2,$$  \hfill (2.34)

where $\tilde{\phi}$ is a $c$-component boson field. We identify the boson field as

$$\tilde{\phi} \sim \phi + 2\pi \vec{a}$$  \hfill (2.35)

where $\vec{a}$ is any vector in the compactification lattice $\Gamma$, which is a $c$-dimensional Bravais lattice.

Let us consider the finite size system of length $l$, with periodic boundary conditions. The standard canonical quantization gives the canonical equal-time commutation relations

$$[\phi^j(t,x), \Pi^k(t',x')] = i\delta^{jk} \sum_m \delta(x - x' - ml),$$  \hfill (2.36)

where $\Pi = \partial_\mu \tilde{\phi}$. Together with the equation of motion $\partial_\mu \partial^\mu \phi = 0$, we obtain the mode expansion of the field $\tilde{\phi}$:

$$\tilde{\phi}(t,x) = \phi_0 + \frac{2\pi x}{l} \vec{a} + \frac{\vec{p}}{l} + \sum_{n=1}^\infty \left( \frac{\tilde{a}_{nL}}{\sqrt{4\pi n}} e^{-i \frac{2\pi}{l} (nx + l)} + \text{h.c.} \right) + \sum_{n=1}^\infty \left( \frac{\tilde{a}_{nR}}{\sqrt{4\pi n}} e^{i \frac{2\pi}{l} (nx - l)} + \text{h.c.} \right),$$  \hfill (2.37)

where $\vec{a}$ belongs to $\Gamma$ and $\vec{p}$ is the conjugate momentum of $\phi_0$. (h.c. denotes Hermitian conjugate.) Because $\phi_0$ should be identified as $\phi_0 \sim \phi_0 + 2\pi \Gamma$, the eigenvalues of $\vec{p}$, which we label $\vec{v}$, belong to the lattice $\Gamma^*$ which is the dual of $\Gamma$. This is defined by the condition:

$$\vec{a} \cdot \vec{v} \in \mathbb{Z},$$  \hfill (2.38)

for all vectors $\vec{a} \in \Gamma$ and $\vec{v} \in \Gamma^*$. Each component of the vector $\tilde{a}_{nL}$ is an annihilation operator; the $a^j_{nL}$ satisfies the commutation relation $[a^j_{nL}, a^{k\dagger}_{mL}] = \delta^{jk} \delta_{nm}$ and all the other commutators involving $a^j_{nL}$ vanish. A similar definition applies to $\tilde{a}_{nR}$, which commutes with $a^j_{nL}$ and $a^{j\dagger}_{mL}$.

Now we can decompose the field $\tilde{\phi}$ into chiral components, namely left-mover $\tilde{\phi}_L$ and right-mover $\tilde{\phi}_R$:

$$\tilde{\phi}_L(x^+) = \frac{\tilde{\phi}_0}{2} + \frac{\tilde{\phi}_0}{2} + \frac{1}{2l} (2\pi \vec{a} + \vec{p}) x^+ + \sum_{n=1}^\infty \left( \frac{\tilde{a}_{nL}}{\sqrt{4\pi n}} e^{-i \frac{2\pi}{l} nx^+} + \text{h.c.} \right),$$  \hfill (2.39)

where $x^+ = t + x$. The dual field $\tilde{\phi} = \phi_L - \phi_R$ has the mode expansion:

$$\tilde{\phi}(t,x) = \tilde{\phi}_0 + \frac{2\pi t}{l} \vec{a} + \frac{x}{l} \vec{v} + \sum_{n=1}^\infty \left( \frac{\tilde{a}_{nL}}{\sqrt{4\pi n}} e^{-i \frac{2\pi}{l} (nt + l)} + \text{h.c.} \right) - \sum_{n=1}^\infty \left( \frac{\tilde{a}_{nR}}{\sqrt{4\pi n}} e^{i \frac{2\pi}{l} (nt - l)} + \text{h.c.} \right).$$  \hfill (2.40)
The chiral component of the energy-momentum tensor is given by $T(x^+) = (\partial_+ \bar{\phi})^2$. The mode expansion of the energy-momentum tensor gives the generators $L_m$ of the Virasoro algebra:

$$T(x^+) = \frac{2\pi}{l^2} \sum_m L_m e^{-imx^+}.$$ \hfill (2.41)

Using the mode expansion (2.39), we obtain

$$L_m = \frac{1}{2} \sum_l :\bar{\alpha}_{m-l,L} \cdot \bar{\alpha}_{lL} :,$$ \hfill (2.42)

where :: denotes the normal ordering and

$$\bar{\alpha}_{nL} \equiv \left\{ \begin{array}{ll}
-i \sqrt{n} \tilde{a}_{nL} & (n > 0) \\
\frac{1}{\sqrt{4\pi}}(2\pi \bar{u} + \bar{p}) & (n = 0) \\
+i \sqrt{n} a_{nL}^d & (n < 0).
\end{array} \right.$$ \hfill (2.43)

The generators $\bar{L}_m$ for the other chirality are given by

$$\bar{L}_m = \frac{1}{2} \sum_l :\bar{\alpha}_{m-l,R} \cdot \tilde{\alpha}_{lR} :,$$ \hfill (2.44)

where

$$\bar{\alpha}_{nR} \equiv \left\{ \begin{array}{ll}
-i \sqrt{n} \tilde{a}_{nR} & (n > 0) \\
\frac{1}{\sqrt{4\pi}}(-2\pi \bar{u} + \bar{p}) & (n = 0) \\
+i \sqrt{n} a_{nR}^d & (n < 0).
\end{array} \right.$$ \hfill (2.45)

An oscillator vacuum $|\text{vac}\rangle$ of this theory, which satisfies $a_{nL/R}^j|\text{vac}\rangle = 0$, is thus characterized by two sets of zero-mode quantum numbers $\bar{u} \in \Gamma$ and $\bar{v} \in \Gamma^*$, where $\bar{v}$ is the eigenvalue of the operator $\bar{p}$. We will denote the oscillator vacuum with these quantum numbers as $|\bar{u}, \bar{v}\rangle$.

Now let us consider the possible conformally invariant b.c.’s. As was discussed in Section IIA, the conformal invariance of the b.c., Eq. (2.16), implies that these are linear combinations of the Ishibashi states, defined in Eq. (2.17). Of course, the primary states must be contained within the Hilbert space of the system (with periodic b.c.’s.) A physical boundary state must satisfy Cardy’s equation (2.25), which gives a strong constraint on the coefficients of the Ishibashi states.

The (multicomponent) free boson field theory, which we discuss in the present paper, has an infinite number of primary fields. The classification of the boundary states for this case is much more difficult than that for CFTs with a finite number of conformal towers discussed in Section IIA. In fact, at present we are far from the complete classification. Even for the single-component $c = 1$ theory, for which the Dirichlet/Neumann b.c. and their generalizations are generally believed to form the complete set, we know no complete proof of this. Here we just present some simple boundary states for a multicomponent free boson theory. Some examples of more non-trivial boundary states for $c = 2$ will be constructed using a conformal embedding in Section V.

An oscillator vacuum $|\bar{u}, \bar{v}\rangle$ is naturally a primary state with respect to the Virasoro algebra. However, there are an infinite number of Virasoro primaries which are not oscillator vacua. The most general boundary states would include linear combinations of Ishibashi states based on these complicated Virasoro primaries. Nevertheless, as we will show in the following, the simplest boundary states can be written in terms of oscillator vacua and creation operators of oscillator bosons.

A sufficient (but not necessary) condition for the free boson theory to satisfy the conformal invariance (2.16) is given by

$$(\bar{\alpha}_{nL} - \bar{\alpha}_{-nR}) |B\rangle = 0,$$ \hfill (2.46)

for all integer $n$. This corresponds to the imposition of invariance under the Heisenberg algebra, as discussed in the previous subsection, near Eq. (2.7). For $n = 0$, it reads $\bar{u}|B\rangle = 0$. It means that, to
satisfy eq. (2.46), we can use only the states built on the oscillator vacua with \( \vec{u} = 0 \). It can be shown that the condition (2.46) is satisfied by the state

\[
|\langle \vec{0}, \vec{v} \rangle \rangle = \exp \left( - \sum_n \vec{a}^\dagger_n \cdot \vec{a}^\dagger_n \right) |\langle \vec{0}, \vec{v} \rangle \rangle.
\]

(2.47)

This might be regarded as an Ishibashi state with respect to the \( U(c) \) current algebra. It is a linear combination of (an infinite number of) Ishibashi states with respect to the Virasoro algebra.

While each state (2.47) satisfies (2.46) and hence the conformal invariance, it does not satisfy Cardy’s consistency condition. Let us consider a linear combination of (2.47)

\[
|N(\vec{\phi}_0)\rangle = g_N \sum_{\vec{v}} \exp (i \vec{v} \cdot \vec{\phi}_0) |\langle \vec{0}, \vec{v} \rangle \rangle,
\]

(2.48)

for a constant \( g_N \) and constant vector \( \vec{\phi}_0 \). The diagonal partition function on the strip is given by

\[
Z_{NN}(\tilde{q}) = g_N^2 \left( \frac{1}{\eta(q)} \right)^c \sum_{\vec{q}} \tilde{q}^{2\pi \vec{u}^2},
\]

where the summation is over the entire dual lattice \( \Gamma^* \). Modular transforming to the open string channel, it reads

\[
Z_{NN}(q) = (4\pi)^{c/2} V_0(\Gamma) g_N^2 \left( \frac{1}{\eta(q)} \right)^c \sum_{\vec{u}} q^{2\pi \vec{u}^2},
\]

(2.50)

where \( V_0(\Gamma) \) is the volume of the unit cell of the compactification lattice \( \Gamma \). We note that the volume of the unit cell of the dual lattice is given by:

\[
V_0(\Gamma^*) = 1/V_0(\Gamma).
\]

(2.51)

The factor \( q^{\vec{u}^2}/(\eta(q))^2 \) is the character of the \( U(c) \) current algebra, and is a superposition of the Virasoro characters with non-negative integer coefficients. Choosing

\[
g_N = \frac{1}{(4\pi)^{c/4} \sqrt{V_0(\Gamma)}}
\]

(2.52)

the state (2.48) satisfies Cardy’s condition and thus is a physical boundary state. The constant \( g_D \) actually represents the (generally fractional) “ground-state degeneracy” due to the boundary. From the diagonal partition function (2.50) in the open string channel, we can read off the scaling dimensions of boundary operators occurring with the boundary condition. Namely, the scaling dimensions of the boundary operators are given by

\[
\Delta_N = 2\pi \vec{u}^2 + (\text{non-negative integer}),
\]

(2.53)

where \( \vec{u} \) is an element of the lattice \( \Gamma \).

The physical meaning of the boundary state \( |N(\vec{\phi}_0)\rangle \) turns out to be the Dirichlet boundary condition on \( \vec{\phi} \) namely \( \vec{\phi} = \vec{\phi}_0 \) at the boundary. This can be checked by the calculation of the partition function in the open string channel imposing this boundary condition. This b.c. is equivalent to the Neumann boundary condition on the dual field, \( \vec{d}^{-\vec{\phi}}/dx|_0 = 0 \). We label the state with respect to the dual field for convenience in the following section. We note that it is not clear whether the N boundary state is the only solution of the Cardy’s condition even within the linear combinations of the bosonic Ishibashi states (2.43).

A similar construction of the boundary state is possible, starting from the condition

\[
(\vec{\alpha}_n + \vec{\alpha}_{-n})|B\rangle = 0,
\]

(2.54)
instead of (2.46). Again we are imposing invariance under the Heisenberg algebra, Eq. (2.7). The bosonic Ishibashi state is given by

\[ |(\vec{u}, \vec{0})\rangle = \exp \left( + \sum_n \vec{a}_{nL}^\dagger \cdot \vec{a}_{nR} \right) |(\vec{u}, \vec{0})\rangle. \tag{2.55} \]

A physical boundary state is given

\[ |D(\tilde{\phi}_0)\rangle = g_D \sum_{\vec{u}} \exp \left( i \vec{u} \cdot \tilde{\phi}_0 \right) |(\vec{u}, \vec{0})\rangle, \tag{2.56} \]

where the summation is over the entire lattice \( \Gamma \) and

\[ g_D = \pi^{c/4} \sqrt{V_0(\Gamma)}. \tag{2.57} \]

The diagonal partition function reads, in the closed string channel,

\[ Z_{DD}(\tilde{q}) = g_D^2 \left( \frac{1}{\eta(\tilde{q})} \right)^c \sum_{\vec{u}} \tilde{q}^{\vec{u}^2}, \tag{2.58} \]

and in the open string channel

\[ Z_{DD}(q) = \left( \frac{1}{\eta(q)} \right)^c \sum_d q^{d^2/(2\pi)}, \tag{2.59} \]

The scaling dimensions of the boundary operators are

\[ \Delta_D = \frac{\vec{v}^2}{2\pi} + \text{(non-negative integer)}, \tag{2.60} \]

where \( \vec{v} \) is an element of the dual lattice \( \Gamma^* \). The physical meaning of this boundary state is the Neumann boundary condition for the field \( \tilde{\phi} \), or equivalently the Dirichlet boundary condition for the dual field \( \tilde{\phi} \), i.e. \( \tilde{\phi} = \tilde{\phi}_0 \) at the boundary.

In the simplest case, \( c = 1 \),

\[ \Gamma = \{ nR | n \in \mathbb{Z} \}, \quad \Gamma^* = \{ m/R | m \in \mathbb{Z} \} \tag{2.61} \]

where \( R \) is the compactification radius. In this case the degeneracies are given by:

\[ g_N = 1/|\pi^{1/4}\sqrt{2R}|, \quad g_D = \pi^{1/4}\sqrt{R}. \tag{2.62} \]

For the multicomponent (\( c > 1 \)) case, there is a simple generalization of the above construction of N and D boundary states. Since the generators of the Virasoro algebra (2.43) are bilinears in the creation/annihilation operator \( a_n \), a sufficient condition for the conformal invariance (2.16) is given by

\[ (\tilde{a}_{nL} - \mathcal{R} \tilde{a}_{-nR}) |B\rangle = 0, \tag{2.63} \]

where \( \mathcal{R} \) is an orthogonal \( c \times c \) matrix. This corresponds to another sort of covariance under the Heisenberg algebra, Eq. (2.8). This includes the N case (2.46) which corresponds to \( \mathcal{R} = 1 \) (identity matrix) and the D case (2.54) which amounts to \( \mathcal{R} = -1 \). Other choices of the orthogonal matrix gives different boundary states. For example, when \( \mathcal{R} \) is a diagonal matrix with diagonal elements +1 or −1, it represents a mixture of the D and N b.c.’s. When \( \mathcal{R} \) is a rotation matrix, it gives a b.c. corresponding to the QBM under an external magnetic field. We postpone the discussion of such a problem to a later publication.

The zero modes of the boson are restricted by the condition (2.63) with \( n = 0 \), namely

\[ 2\pi \tilde{u} + \vec{v} = \mathcal{R}(-2\pi \tilde{u} + \vec{v}). \tag{2.64} \]
Its solution is given as a linear combination of basic solutions

$$\vec{u}, \vec{v} = \sum_{n} \lambda_n (\vec{u}_n, \vec{v}_n),$$

(2.65)

where $\lambda_n$’s are integer coefficients. For such $$(\vec{u}, \vec{v})$$, $2\pi \vec{u} + \vec{v}$$ form a new lattice $2\pi \tilde{\Gamma}$. Only the oscillator vacua with these zero modes can contribute to the boundary state.

A possible boundary state is given as

$$|\mathcal{R}\rangle = g_{\mathcal{R}} \sum_{(\vec{u}, \vec{v})} |(\vec{u}, \vec{v})\rangle,$$

(2.66)

where the sum is taken over (2.65) and

$$|(\vec{u}, \vec{v})\rangle = \exp \left( \sum_{n=1}^{\infty} -a_{nL}^\dagger \mathcal{R} a_{nR}^\dagger \right) |(\vec{u}, \vec{v})\rangle.$$

(2.67)

The diagonal partition function for this state $Z_{\mathcal{R}\mathcal{R}}$ turns out to be identical to that for $Z_{\mathcal{N}\mathcal{N}}$, if the original lattice $\Gamma$ is replaced by $\tilde{\Gamma}$. The scaling dimensions of the boundary operators and $g$-factors are also easily given by this replacement.

For the multicomponent free boson, we have thus constructed an infinite variety of b.c.’s by the generalization of D/N as in eq. (2.63), for the infinite possible different choices of $\mathcal{R}$. The diagonal partition function for these b.c.’s has a structure similar to that for D or N b.c. and can be written by a non-negative linear combination of bosonic (Heisenberg) characters $q^h/\eta(q)^c$. We emphasize that we have just given some simple examples of physical boundary states for the free boson theory; we have not exhausted all the possible conformally invariant b.c.’s.

In fact, more non-trivial “non-bosonic” boundary states are possible in the multicomponent free boson, at least in some cases. In Section V we will construct such highly non-trivial boundary states and will demonstrate that they are indeed not bosonic; they cannot be understood by a generalization of D and N as described above.

C. Dirichlet and Neumann Boundary Conditions From Fusion: $c=1$ Case

The problem of finding out all possible conformal invariant boundary conditions is still an open one for all but minimal theories. Even the case of the free boson $c = 1$ is not fully understood yet. In this section, we discuss this problem further, and demonstrate how the twist field (of dimension $1/16$) allows one to connect D and N boundary conditions in some simple rational cases. This gives some justification for the fusion procedure to be implemented in the following sections in the case $c = 2$. Some of the following results are related to the more abstract approach in Ref. (31).

We thus consider a single compact free boson, corresponding to the one dimensional limit of the model reviewed in the previous section with the compactification lattice $\{nR | n \in Z\}$. We restrict to the case $R^2 = p^2 / 2\pi$, $p$ an integer. ($p' = 1$ is the $SU(2)$ symmetric point and $p' = 2$ is the square of the Ising model.) The torus partition function of the periodic boson can be expressed in terms of the generalized characters

$$K^{(M)}_\lambda = K^{(M)}_{-\lambda} = K^{(M)}_{\mu - \lambda} = \frac{1}{\eta} \sum_{n=-\infty}^{\infty} q^{(nM+\lambda)^2/2M},$$

(2.68)

as

$$Z_{\text{per}} = \sum_{\lambda=0}^{M-1} |K^{(M)}_\lambda|^2,$$

(2.69)
with \( M \equiv 2p' \). This decomposition reflects the symmetry of the periodic boson at this particular radius under a chiral algebra (traditionally denoted by \( \mathcal{A}_{p'} \)) which is larger than the Virasoro algebra, and is generated by, besides the stress energy tensor \( T \), the current \( j \) and a pair of vertex operators of integer spin \( p' \). The primary fields of the algebra \( \mathcal{A}_{p'} \) are vertex operators \( V_{\lambda}, \lambda = 0, \ldots, M - 1 \), with fusion coefficients

\[
N_{\lambda\mu}^{\nu} = \delta_{\lambda + \mu, \nu} \mod N \tag{2.70}
\]

while charge conjugation takes \( V_{\lambda} \to V_{M-\lambda} \).

For the same periodic boson, we now consider partition functions in the open string channel. Two kinds of conformal boundary conditions are known beforehand: the Dirichlet boundary conditions, where the value of the field on the boundary is fixed modulo the compactification lattice, \( \phi = \phi_0 + 2\pi n R \), and the Neumann boundary conditions, where the value of the dual field on the boundary is fixed modulo the dual compactification lattice, \( \tilde{\phi} = \phi_0 + m/R \). These boundary conditions preserve the Virasoro symmetry, but do not, in general, preserve the chiral symmetry \( \mathcal{A}_{p'} \). In fact, only the set of \( M \) Dirichlet boundary conditions with \( \phi_0 = \lambda 2\pi R, \lambda = 0, \ldots, M - 1 \) do so, with

\[
Z_{D(\lambda/2R)D(\mu/2R)} = K_{\lambda-\mu} \tag{2.71}
\]

These Dirichlet boundary conditions are presumably the only \( \mathcal{A}_{p'} \) invariant boundary conditions. Introducing the Ishibashi states \( |\lambda\rangle \rangle \) associated with the module \( \lambda \) of this algebra, the corresponding boundary states read

\[
|\tilde{\lambda}\rangle = \sum_{\mu=0}^{M-1} \frac{S_{\lambda}^\mu}{\sqrt{S_0^\mu}} |\mu\rangle\rangle = \sum_{\mu=0}^{M-1} e^{2i\pi \lambda \mu/M} M^{1/4} |\mu\rangle\rangle \tag{2.72}
\]

where the matrix of modular transformations is given by

\[
S_{\lambda}^\mu = \frac{1}{M^{1/2}} e^{2i\pi \lambda \mu/M}. \tag{2.73}
\]

The general DD partition functions \( Z_{D(\lambda/2R)D(\mu/2R)} \) can naturally be obtained by fusion from the partition function \( Z_{D(0)D(0)} \) using the fusion algebra coefficients (2.70).

It is possible to write some of the Neumann partition functions in terms of the \( K_{\lambda} \)'s. For instance

\[
Z_{N(0)N(0)} = \sum_{\lambda=0,\lambda \ even}^{M-1} K_{\lambda}^{(M)} \tag{2.74}
\]

Similarly,

\[
Z_{N(0)N(\pi/2R)} = \sum_{\lambda=1,\lambda \ odd}^{M-1} K_{\lambda}^{(M)} \tag{2.75}
\]

All the Neumann boundary conditions do however break the symmetry \( \mathcal{A}_{p'} \). This is most clearly seen by writing the Dirichlet-Neumann partition function, which is independent of the radius, and well known to be

\[
Z_{ND} = \frac{1}{4\eta} \sum_{n} q^{(2n-1)^2/16}. \tag{2.76}
\]

This partition function cannot be expressed in terms of generalized characters. The corresponding closed string partition function cannot either, but of course, it is expressible in terms of the bulk \( c = 1 \) Virasoro characters, as it should be since both \( D \) and \( N \) are possible conformal boundary conditions for the free periodic boson:
To relate D and N boundary conditions by fusion, it is necessary to relax the constraint on invariance under $A_{p'}$. Of course, what one should really do, since one is only interested in general in imposing conformal invariance on the boundary conditions, not any higher symmetry, is to simply use the fusion algebra of the $c = 1$ theory for arbitrary radius; this however poses technical problems - in particular, continuous modular transformations - which are, for the moment, insurmountable. To proceed, the empirical strategy we use is to consider “variants” of the theory, with different, finite, extended algebras, for which fusion can be implemented and, hopefully, meaningful boundary conditions for our original free boson found. The natural candidate that comes to mind is the $Z_2$ orbifold, with partition function (recall $R^2 = p'/2\pi; \; M = 2p'$)

$$Z_{orb} = \sum_{\lambda=1}^{p'-1} \left| K_{p'}^{(2p')} \right|^2 + 2 \left| \frac{1}{2} K_{p'}^{(2p')} \right|^2 + 2 \sum_{\lambda=1,3} \left| K_{\lambda}^{(8)} \right|^2 + \frac{1}{2\eta} \sum_n q^n n^2 + \left( -1 \right)^n q^n n^2$$

The operator content of this theory is quite different from that of the periodic free boson. The most obvious feature is that, due to the $\phi \rightarrow -\phi$ identification, the multiplicity two of vertex operators has now disappeared: we get instead operators with multiplicity one, the first $p' - 1$ terms, which correspond to operators which we call $\Phi_{\lambda}$, of dimension $\frac{n^2}{p'}$. The next term in the partition function stands for two degenerate operators $\Phi_{p'}^{(i)}$ of dimension $p'/4$ (the two energy operators in the Ising square case), the next for the twice degenerate twist $\sigma^{(i)}$ and excited twist $\tau^{(i)}$ operators of dimensions $1/16, 9/16$ (the spins and the tensor product of spin and energy for the Ising square case). Finally the last two terms are respectively the characters of the identity and the current (which we denote $\theta$), which is primary in the orbifold algebra. We will denote the corresponding characters (which appear as moduli squares in the foregoing formula) $K_1$ and $K_\theta$.

To proceed, observe that, presumably, the boundary states of the periodic boson and the orbifold boson are not mutually compatible in general: for instance, a Dirichlet boundary state for the orbifold $|D_o(\phi_0)\rangle$ should be a linear combination of Dirichlet boundary states for the non orbifold boson at values $\pm \phi_0$, and the normalization will make $|D_o(\phi_0)\rangle$ incompatible with $|D(\phi_0)\rangle$. There are thus various families of boundary states we can envision constructing. To start - and get quickly to the point - we consider the particular cases of $|D(\phi_0 = 0)\rangle$, $|N(\phi_0 = 0)\rangle$ for the periodic boson. The corresponding partition functions, in the open string channel, can be fully written in terms of the generalized characters of the orbifold algebra:

$$Z_{D(0)D(0)} = K_1 + K_\theta$$
$$Z_{N(0)N(0)} = K_1 + K_\theta + 2 \left[ K_2^{(2p')} + K_4^{(2p')} + \ldots + \frac{1}{2} K_p^{(2p')} \right]$$
$$Z_{DN} = K_1^{(8)} + K_3^{(8)}$$

($p'$ even). Formulas (2.79) suggest that it is now possible to connect D and N by fusion, which we now demonstrate.

To do so, we need the fusion algebra of the orbifold theory: it can be found for instance in Ref.[33] with the following results ($p'$ even):

$$\sigma^{(1)} \times \theta = \sigma^{(1)}$$
$$\sigma^{(1)} \times \sigma^{(1)} = I + \Phi_{p'}^{(1)} + \sum_{\lambda \text{ even}} \Phi_{\lambda}$$
$$\sigma^{(1)} \times \Phi_{\lambda \text{ even}} = \sigma^{(1)}$$

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The corresponding Dirichlet partition functions are

\[ \sigma^{(1)} \times \Phi_{\lambda \text{ even}} = \sigma^{(1)} \]

\[ \sigma^{(1)} \times \Phi_{\lambda' \text{ even}} = \sigma^{(1)} \]

\[ \Phi_{\lambda'}^{(i)} \times \Phi_{\lambda'}^{(i)} = I \]

\[ \Phi_{\lambda'}^{(i)} \times \Phi_{\lambda'}^{(2)} = \theta \]

\[ \Phi_{\lambda} \times \Phi_{\mu} = \Phi_{\lambda - \mu} + \Phi_{\lambda + \mu} \]

\[ \Phi_{2p' - \lambda} \times \Phi_{\lambda} = \Phi_{2\lambda} + \Phi_{\lambda}^{(1)} + \Phi_{\lambda}^{(2)} \]

(2.80)

where the fusion of \( \Phi_{\lambda} \) operators holds only for \( \mu \neq \lambda, 2p' - \lambda \), and \( \lambda \) is defined modulo \( p' \).

We now start with \( Z_{DD} \) and fuse with the \( \sigma^{(1)} \) operator. From \( \sigma^{(1)} \times I = \sigma^{(1)} \) and \( \sigma^{(1)} \times \theta = \tau^{(1)} \) we get the partition function \( Z_{ND} \), since the two terms in the last equation of (2.78) are the characters of the twist and excited twist operators respectively.

For \( p' \) even, we fuse again with \( \sigma^{(1)} \). We need the basic fusion rule \( \sigma^{(1)} \times \sigma^{(1)} = I + \phi_{\mu}^{p'} + \sum_{\lambda \text{ even}} \phi_{\lambda} \), and also the rule deduced from associativity \( \sigma^{(1)} \times \tau^{(1)} = \theta + \phi_{\mu}^{p'} + \sum_{\lambda \text{ even}} \phi_{\lambda} \); the fusion gives therefore \( Z_{NN} \). To summarize: by fusion with \( \sigma^{(1)} \) we thus found \( Z_{DD} \rightarrow Z_{ND} \rightarrow Z_{NN} \).

Just to illustrate the difference with \( p' \) odd, we remark that in the latter case, while the first fusion is the same, the next one has to be with the other twist operator, \( \sigma^{(2)} \). Using the equivalent of (2.80) in this case

\[ \sigma^{(2)} \times \sigma^{(1)} = I + \sum_{\lambda \text{ even}} \phi_{\lambda} \]

\[ \sigma^{(2)} \times \sigma^{(1)} = \theta + \sum_{\lambda \text{ even}} \phi_{\lambda} \]

(2.81)

we recover \( Z_{NN} \) indeed. This time therefore we have \( Z_{DD} \rightarrow \sigma^{(1)} \) \( Z_{ND} \rightarrow \sigma^{(2)} \) \( Z_{NN} \).

To express the boundary states, it is better now to carry out a more complete analysis within the boundary conditions of the orbifold model. We thus consider the proper orbifold Dirichlet boundary states, which are generally defined by \( (\phi_0 \neq 0, \pi R) \)

\[ |D_o(\phi_0)\rangle = \frac{1}{\sqrt{2}} [ |D(\phi_0)\rangle + |D(-\phi_0)\rangle ] \]

(2.82)

The corresponding Dirichlet partition functions are

\[ Z_{D_o(\lambda/2R)D_o(\mu/2R)} = K_{\lambda - \mu} + K_{\lambda + \mu}, \lambda \neq \mu, \lambda \neq 2p' - \mu \]

\[ = K_I + K_{\theta} + K_{2\lambda}, \lambda = \mu \]

\[ = K_{2\lambda} + \frac{1}{2} K_{p'}^{(2p')}, \lambda = 2p' - \mu \]

(2.83)

The only orbifold Neumann boundary states compatible with the orbifold algebra are the “end point” ones: \( \phi_0 = 0, \pi R \). We will discuss them later.

We now need the modular transformation matrix, which reads:

\[
S = \frac{1}{\sqrt{8p'}} \begin{pmatrix}
1 & 1 & 1 & 2 & \sqrt{p'} & \sqrt{p'} \\
1 & 1 & 1 & 2 & -\sqrt{p'} & -\sqrt{p'} \\
1 & 1 & 1 & 2 & \sqrt{p'} & -\sqrt{p'} \\
2 & 2 & 2(-1)^\lambda & 4 \cos \frac{\pi \mu}{2p'} & 0 & 0 \\
\sqrt{p'} & -\sqrt{p'} & (-1)^{i'-j'} \sqrt{p'} & 0 & \delta_{i'j'} \sqrt{2p'} & -\delta_{i'j'} \sqrt{2p'} \\
\sqrt{p'} & -\sqrt{p'} & (-1)^{i''-j} \sqrt{p'} & 0 & -\delta_{i'j'} \sqrt{2p'} & \delta_{i'j'} \sqrt{2p'} 
\end{pmatrix}
\]

(2.84)
where the conventions are that the matrix maps \( \left( I, \theta, \Phi^{(i)}_\nu, \Phi^{(j)}_\lambda, \sigma^{(i')}^{(j')}, \tau^{(i')}^{(j')} \right) \) onto the same quantities but with \( i, j \) exchanged.

After modular transformations

\[
Z_{D_o(\lambda/2R)D_o(\mu/2R)} = \frac{1}{\sqrt{2p}} \left[ 2\tilde{K}_I + 2\tilde{K}_\theta + \frac{1}{2}(-1)^{\lambda \pm \mu} \tilde{K}_{p'}^{(2p')} + 4 \sum_{\nu=1}^{\nu'} \cos \frac{\pi \lambda \nu}{2p'} \cos \frac{\pi \mu \nu}{2p'} \tilde{K}_{\nu'}^{(2p')} \right] \tag{2.85}
\]

Expressions of the various boundary states in terms of Ishibashi states of the orbifold algebra follows:

\[
|D_o(\lambda/2R)\rangle = \frac{1}{(2p')^{1/4}} \left\{ |I\rangle + \sqrt{2} |\theta\rangle + \sqrt{2}(-1)^{\lambda} \left( |\phi_{p'}^{(1)}\rangle + |\phi_{p'}^{(2)}\rangle \right) + \sqrt{2} \sum_{\nu=1}^{\nu'} \cos \frac{\pi \lambda \nu}{2p'} |\phi_{\nu'}\rangle \right\} \tag{2.86}
\]

Setting \( \lambda = 0 \) gives the periodic Dirichlet boundary state, up to a factor of \( \sqrt{2} \):

\[
|D(0)\rangle = \frac{1}{(2p')^{1/4}} \left\{ |I\rangle + |\theta\rangle + \frac{1}{\sqrt{2}} \left( |\phi_{p'}^{(1)}\rangle + |\phi_{p'}^{(2)}\rangle \right) + \sqrt{2} \sum_{\lambda=1}^{\nu'} |\phi_{\lambda}\rangle \right\} \tag{2.87}
\]

Modular transformation gives the other partition functions in the closed string channel

\[
Z_{D(0)N(0)} = \frac{1}{\sqrt{2p'}} \left[ \sqrt{p'} \tilde{K}_I - \sqrt{p'} \tilde{K}_\theta \right]
\]

\[
Z_{N(0)N(0)} = \frac{2}{\sqrt{2p'}} \left[ \tilde{K}_I + \tilde{K}_\theta + 2\frac{1}{2} \tilde{K}_{p'}^{(2p')} \right] \tag{2.88}
\]

From this, we obtain the Neumann boundary state

\[
|N(0)\rangle = \frac{1}{(2p')^{1/4}} \left\{ \sqrt{p'} |I\rangle - \sqrt{p'} |\theta\rangle + \sqrt{p'} \sqrt{2} \left( |\phi_{p'}^{(1)}\rangle - |\phi_{p'}^{(2)}\rangle \right) \right\} \tag{2.89}
\]

Observe that, because of the particular radius we have chosen, the Ishibashi states which appear in \( |N\rangle \) all appear in \( |D\rangle \) also: this is a necessary condition for \( N \) to be obtained from \( D \) by fusion, as follows from Eq. (2.27) and of course would not hold in general.

So far, the boundary states we considered are all in the periodic sector - the Hilbert space in the closed string channel is the one of the periodic boson. As discussed in Ref. (34) [see also Ref. (31)], there are also eight boundary states which involve, in addition, the twisted sector. These states are

\[
|D_o(\phi_0)\rangle = 2^{-1/2} |D(\phi_0)\rangle \pm 2^{-1/4} |D(\phi_0)_T\rangle \tag{2.90}
\]

where \( \phi_0 \) is at the fixed points of the orbifold, \( \phi_0 = 0, \pi R \), and similar Neumann like states:

\[
|N_o(\phi_0)\rangle = 2^{-1/2} |N(\tilde{\phi}_0)\rangle \pm 2^{-1/4} |N(\tilde{\phi}_0)_T\rangle \tag{2.91}
\]

with \( \tilde{\phi}_0 = 0, \pi/2R \).

The partition functions found in Ref.(34) can be written as

\[
Z_{D_o(\phi_0)\pm, D_o(\phi_0)\pm} = K_I
\]

\[
Z_{D_o(\phi_0)\pm, D_o(\phi_0)\mp} = K_\theta
\]

\[
Z_{D_o(\phi_0)\acute{\epsilon}, D_o(\pi R-\phi_o)\acute{\epsilon}} = \frac{1}{2} K_{p'}^{(2p')} \tag{2.92}
\]

Similarly for the Neumann sector one has
Finally, combining Neumann and Dirichlet gives

\[ Z_{N_o(\phi_0)\epsilon,N_o(\phi_0)\epsilon'} = K_1^{(8)} \text{, resp. } K_3^{(8)} \]  

(2.94)

where the choice depends on the signs \( \epsilon, \epsilon' \) and the values of \( \phi_0, \phi_0 \).

Corresponding boundary states can of course be written, giving rise to rather bulky expressions we will avoid here. Rather, we would like to stress that the orbifold boundary conditions are all connected by fusion indeed. Starting with the first partition function in (2.92), we get to the second one by fusion with the field \( \theta \), thanks to \( \theta \times \theta = I \), so \( D_o(\phi_0)^+ \) follows from \( D_o(\phi_0)\epsilon \) by fusion with \( \theta \). Similarly, starting again with the first partition function in (2.92) and fusing with \( \phi_p^{(i)} \) gives the third partition function using that \( \theta \times \phi_p^{(i)} = \phi_p^{(j)} \), so \( |D_o(\pi r - \phi_0)\nu_1 \rangle \) follows from \( |D_o(\pi r - \phi_0)\nu_2 \rangle \) by fusion with \( \phi_p^{(i)} \).

The Neumann and Dirichlet sectors are still related through \( \sigma^{(i)} \), or, also \( \tau^{(i)} \), using \( \theta \times \sigma^{(i)} = \tau^{(i)} \). Finally, fusion with \( \theta \) and \( \phi_p^{(i)} \) gives the remaining partition functions in the Neumann sector. As for the continuous orbifold Dirichlet and Neumann boundary conditions, they are obtained by fusion with the operators \( \phi \).

One of the lessons illustrated here, is that to obtain all possible conformal boundary conditions for the free boson theory, one needs to consider its \( Z_2 \) orbifold as well. It is not clear in general what good “variant” of the bulk theory is needed to explore all possible conformal boundary conditions for a given model. In the case \( c = 2 \) for example, we shall see that fusion with a field of dimension 1/8 allows one to go from D to N. This is a natural generalization of the \( c = 1 \) case, the 1/8 field being now a “double” twist field for the bosons \( \phi_{1,2} \). We will also see however that fusion with a field of dimension 2/5 is also necessary to explore the possible boundary states. The origin of this field in the two boson language is still mysterious. Is it some other sort of twist field?

The discovery of non trivial boundary conditions should also lead to new bulk theories. In the \( c = 1 \) case for instance, consideration of the possible D and N boundary conditions should lead one to the discovery of the \( Z_2 \) orbifold, were this theory not already known. This is also related to the existence of a \( c = 1 \) theory - the square of the Ising model - which is not a periodic boson. In the \( c = 2 \) case, the “embedding” we will use to describe partition functions with boundaries leads similarly to a bulk theory - the product of Ising, tricritical Ising and Potts - which does not seem to be any simple sort of two boson orbifold. Understanding its meaning would be crucial to generalize our work to other values of the triangular lattice parameter, \( a \).

### III. Quantum Brownian Motion and Conformal Field Theory

We first discuss the one-dimensional case. The study of Brownian motion of a classical particle described by the simple Langevin equation

\[ M \frac{d^2Q}{dt^2} + \eta \frac{dQ}{dt} + \frac{\partial V}{\partial Q} = \xi(t) \]  

(3.1)

is of course one of the basic topic in non-equilibrium statistical mechanics. In (3.1), \( \eta \) is a phenomenological friction coefficient, \( V \) an external potential, \( M \) the particle mass, and \( \xi \) is a random force that obeys

\[ \langle \xi(t) \rangle = 0 \]

\[ \langle \xi(t)\xi(t') \rangle = 2\eta T \delta(t-t') \]  

(3.2)
More recently, there has been a great deal of interest in trying to describe the quantum behavior of systems for which the classical motion would be described by the Langevin equation (3.1). The most tractable approach has been to couple the “particle” (which might actually represent something quite different, like the phase in a Josephson junction) to a bath (an environment) with an infinite number of degrees of freedom, which provides both the friction and the fluctuating force. The simplest example of that approach is the Caldeira Leggett model where the coordinate $Q$ is coupled linearly to an infinite set of harmonic oscillators, with the Hamiltonian

$$H = \frac{P^2}{2M} + V(Q) + \frac{1}{2} \sum_k \left( \frac{p_k^2}{m_k} + m_k \omega_k^2 \left[ q_k - \frac{Q \lambda_k}{m_k \omega_k} \right]^2 \right).$$

(3.3)

Here $Q$ and $P$ are the co-ordinate and momentum of the particle; $V(Q)$ is an arbitrary potential, with $V(Q + a) = V(Q)$. The exact distribution of coupling constants $\lambda_k$, masses $m_k$ and frequencies $\omega_k$ is important for the properties of the particle only through the weighted density of states

$$J(\omega) \equiv \frac{\pi}{2} \sum_k \frac{\lambda_k^2}{m_k \omega_k} \delta(\omega - \omega_k), \quad \omega > 0.$$  

(3.4)

In the limit where the number of oscillators is infinite, $J(\omega)$ becomes a continuous function. In the following we will restrict to the so called ohmic case $J(\omega) = \eta \omega$; with this choice, it can be shown that the Hamiltonian (3.3) when treated classically yields, after elimination of the bath degrees of freedom, the Langevin equation (3.1).

By a canonical transformation on the oscillator bath:

$$q_k \rightarrow \frac{\lambda_k q_k}{m_k \omega_k^2}, \quad p_k \rightarrow \frac{m_k \omega_k^2 p_k}{\lambda_k},$$

(3.5)

we may rewrite the Hamiltonian in the form:

$$H = \frac{P^2}{2M} + V(Q) + \frac{1}{2} \sum_k \left( \frac{p_k^2}{m_k} + \tilde{m}_k \tilde{\omega}_k^2 (q_k - Q \lambda_k/m_k \omega_k)^2 \right),$$

(3.6)

where:

$$\tilde{m}_k \equiv \lambda_k^2/m_k \omega_k^4, \quad \tilde{\omega}_k \equiv \omega_k, \quad \tilde{\lambda}_k \equiv \lambda_k^2/m_k \omega_k^2 = \tilde{m}_k \tilde{\omega}_k^2$$

(3.7)

are the masses, frequencies and couplings of the transformed oscillators. Henceforth, we drop the tilde notation on these quantities. The Hamiltonian of Eq. (3.4) is invariant under the translation:

$$Q \rightarrow Q + a, \quad q_k \rightarrow q_k + a.$$  

(3.8)

As the particle moves in the periodic potential it drags a cloud of other “fictitious particles” with it which don’t feel the periodic potential $V$ but are bound to the real particle by a quadratic interaction. Due to the translation symmetry of Eq. (3.8), there is a total momentum, $P_T$, which is conserved modulo $2\pi/a$:

$$P_T \equiv P + \sum p_k.$$  

(3.9)

Explicitly, a basis of eigenstates of $H$ may be chosen such that any eigenfunction gets multiplied by the phase $e^{iP_T a}$ under this translation. Without loss of generality, the crystal momentum, $P_c$, may be chosen to lie in the first Brillouin zone, $|P_c| < \pi/a$.

After the canonical transformation of Eq. (3.5) the weighted density of states becomes:

$$J(\omega) \equiv \frac{\pi}{2} \sum_k \lambda_k \omega_k \delta(\omega - \omega_k), \quad \omega > 0.$$  

(3.10)
The properties of the original particle are unaffected by a rescaling of $\omega_k$ (independently for each value of $k$) provided that $\lambda_k$ is scaled by the inverse factor such that $\lambda_k\omega_k$ is held fixed.

We note that an alternative way of writing the Hamiltonian is in terms of the total momentum $P_T$ and transformed oscillator co-ordinates,

$$q_k' \equiv q_k - Q,$$ (3.11)

which define a transformed canonical set of co-ordinates. The Hamiltonian in these co-ordinates becomes:

$$H = \frac{(P_T - \sum p_k)^2}{2M} + V(Q) + \frac{1}{2} \sum \left[ \frac{p_k^2}{m_k} + m_k \omega_k^2 q_k'^2 \right].$$ (3.12)

A convenient choice for the oscillator bath is the Fourier modes of a one-dimensional free massless boson quantum field. In order to begin with a discrete set of oscillators, we may define the field theory with vanishing boundary conditions in a box of size $2l$, so that:

$$\omega_k = k, \quad k = \pi n/2l, \quad n = 1, 2, \ldots .$$ (3.13)

Here we have set an arbitrary “velocity of light” to 1. Strictly speaking, an ultraviolet cut off must be applied, $k < \Lambda$, but the universal critical behavior will be independent of the cut off.

Note that there is a zero-frequency oscillator in the limit $l \to \infty$. The field couples to the particle at $x = 0$ and the boundaries of the box are at $x = \pm l$. The oscillator co-ordinates and momenta can be expressed in terms of the boson creation and annihilation operators as:

$$q_k = i\sqrt{k/2}(a_k - a_k^\dagger), \quad p_k = (a_k + a_k^\dagger)/\sqrt{2k}.$$ (3.14)

We may then define the field and conjugate momentum field on a fictitious one-dimensional space, $x$, completely unrelated to the physical space in which the particle moves, by:

$$\phi(x) \equiv \sqrt{1/l} \sum_k p_k \sin k(x + l), \quad \Pi(x) \equiv -\sqrt{1/l} \sum_k q_k \sin k(x + l).$$ (3.15)

We note that $\phi(x)$ corresponds to a “non-compact” boson field, so no term linear in $x$ is allowed in its mode expansion.

We see that, with this choice of oscillator bath, in order to obtain ohmic dissipation, we must choose, $\lambda_k = 2\eta/l$. From Eq. (3.7) this implies that the oscillator masses are given by:

$$m_k = \frac{\lambda_k}{\omega_k^2} = \frac{\eta}{lk^2}. \quad (3.16)$$

The fictitious particles become infinitely heavy at zero frequency!

In this case the Hamiltonian of Eq. (3.6) can be written:

$$H = \frac{P^2}{2M} + V(Q) + \frac{1}{2} \int_{-l}^l dx \left[ \left( \frac{d\phi}{dx} \right)^2 + (\Pi + \sqrt{2\eta}Q^2(x))^2 \right].$$ (3.17)

It is convenient to separate $\phi(x)$ into its even and odd components since the odd part decouples from the particle:

$$\phi_{e,o}(x) \equiv [\phi(x) \pm \phi(-x)]/\sqrt{2}. \quad (3.18)$$

The even field obeys a Neumann b.c., $d\phi_e/dx = 0$ at $x = 0$. Thus we may equivalently consider a field defined on the positive axis, $0 \leq x < l$, with a Neumann b.c. at $x = 0$ and a Dirichlet b.c. at $x = l$ coupled to the particle. This Hamiltonian is equivalent to Eq. (3.6) with:

$$\tilde{\omega}_k = k, \quad \tilde{\lambda}_k = 2\eta/l, \quad k = \pi(n + 1/2)/l, \quad n = 0, 1, 2, \ldots .$$ (3.19)

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We might expect the translational symmetry to lead to the usual band structure in which the discrete set of eigenstates of given crystal momentum depends in some smooth way on $P_c$. However, when there is an oscillator, $q_0$, of infinite mass a dramatic simplification occurs. This follows from observing that any eigenfunction of crystal momentum $P_c$, can be written in the form:

$$\Psi = e^{iP_c q_0} \psi,$$  \hspace{1cm} (3.20)

where $\psi(Q, q_k)$ is invariant under the translation of Eq. (3.8). Since $p_0$ does not appear in $H$, it follows that the periodic function, $\psi$, obeys exactly the same Schroedinger equation as the full wave-function, $\Psi$, independent of $P_c$. This implies that all eigenstates come in degenerate sets, where the degenerate members are labelled by $P_c$. Effectively we can give the infinite mass “particle”, $q_0$, all the momentum at no cost in energy. This degeneracy is strictly infinite if $Q$ is allowed to range over the entire real line. Alternatively, if we restrict $Q$ to an interval of size $N\alpha$ then we expect the eigenvalues to form nearly degenerate sets of size approximately $N\alpha$, the number of allowed momenta in the Brillouin zone. In a localized phase we can identify this degeneracy with the $N$ minima of $V$ at which the particle can become localized. However we emphasize that this degeneracy is an exact property of $H$ (at infinite $N\alpha$) and does not depend on which phase the system is in. We have dwelled on this point here because we will find that, in the limit where the oscillator frequencies form a dense set, the groundstate degeneracy can take the more general form $g\tilde{N}$ where $g$ is a universal (generally non-integer) number of $O(1)$ which does depends on which phase the system is in. This rather unusual behavior of the model is related to its essentially unphysical nature as a real model of QBM, mentioned in the Introduction. We elaborate on this point further in subsection C.

In the field theory representation, the conserved total momentum operator and transformed oscillator co-ordinates are:

$$P_T = P + \sqrt{\eta} \phi_e(0), \quad \Pi'_e(x) = \Pi_e(x) + 2\sqrt{\eta} Q\delta(x).$$  \hspace{1cm} (3.21)

The transformed Hamiltonian (keeping only the even part) is:

$$H = \frac{(P_T - \sqrt{\eta} \phi_e(0))^2}{2M} + V(Q) + \frac{1}{2} \int_0^l dx \left[ \frac{d\phi_e}{dx} \right]^2 + \Pi_e^2. \hspace{1cm} (3.22)$$

Much of the literature on QBM proceeds by integrating out the field, $\phi_e(x)$ to obtain a non-instantaneous effective action for the particle. However, we introduce a different approach here in keeping with previous analyses of quantum impurity problems using boundary CFT methods. In our approach we rather absorb the particle co-ordinate and momentum into a b.c. on the field, in the effective low energy Hamiltonian. This has the advantage that correlation functions of the field are immediately accessible. For the QBM problem this allows us to study the back-reaction of the particle on the oscillator bath. We note that this dichotomy of approaches also exists in other types of quantum impurity problems. There are two limits in which the QBM Hamiltonian reduces to a boundary sine-Gordon model: the strong and weak corrugation limits. We consider each in turn.

A. Strong Corrugation Limit

In the “strong corrugation” limit the potential is large compared to the dissipation. More precisely, we require the energy bands produced by the potential $V$, ignoring the dissipation, to have large band gaps compared to the energy scale of the dissipation, $\eta/M$. In this limit we expect that band-mixing effects due to the dissipation will be small and the wave-functions for the particle-bath system will be linear combinations of wave-functions in a particular band, $\psi_n(Q; P_c)$. Here $P_c$ is the crystal momentum and the wave-functions, $\psi_n$, are periodic:

$$\psi_n(Q + a; P_c) = \psi_n(Q; P_c).$$  \hspace{1cm} (3.23)

The solutions of the non-dissipative Schroedinger equation are $e^{iP_c Q}\psi(Q; P_c)$ and the periodic functions obey the equation:
\[
\left\{ \frac{1}{2M} \left[ -\frac{d^2}{dQ^2} - 2iP_c \frac{d}{dQ} + P_c^2 \right] + V(Q) \right\} \psi(Q; P_c) = E\psi(Q; P_c) \tag{3.24}
\]

Projecting the Hamiltonian of Eq. (3.22) into a fixed energy band reduces it to:

\[
H_n \rightarrow \epsilon_n[P_c - \sqrt{\eta}\phi_e(0)] + \int_0^l dx \frac{1}{2} \left[ \left( \frac{d\phi_e}{dx} \right)^2 + \Pi' \right]. \tag{3.25}
\]

Here \(\epsilon_n(P_c)\) is the dispersion relation of the \(n^{th}\) energy band for the non-dissipative problem. Note that \(\phi_e\) obeys a N b.c. at \(x = 0\) and a D b.c. at \(x = l\). The degeneracy discussed above corresponds to shifting \(\phi_e(x)\) by \(P_c/\sqrt{\eta}\). However, this is inconsistent with the vanishing b.c. at \(x = l\) so the degeneracy is lifted by effects of \(O(1/l)\) as expected. Upon expanding \(\epsilon_n\) in harmonics:

\[
\epsilon_n(P_c) = A_n + B_n \cos Pa + \ldots \tag{3.26}
\]

and ignoring the (less relevant) higher harmonics, we obtain the boundary sine-Gordon model:

\[
H = B \cos[aP_c - \sqrt{\eta}a\phi_e(0)] + \int_0^l dx \frac{1}{2} \left[ \left( \frac{d\phi_e}{dx} \right)^2 + \Pi' \right]. \tag{3.27}
\]

This result can be confirmed by considering a (single band) tight-binding model. Introducing operators \(c_j\) which annihilate a particle at site \(j\), the position operator is represented as:

\[
Q = \sum_j j c_j^\dagger c_j. \tag{3.28}
\]

The transformation to the total momentum operator then corresponds to a unitary transformation by the operator:

\[
U \equiv e^{-i\sqrt{2\eta}Q\phi(0)}. \tag{3.29}
\]

Explicitly:

\[
U^\dagger [\Pi(x) + \sqrt{2\eta}Q\delta(x)]U = \Pi(x). \tag{3.30}
\]

The part of the Hamiltonian containing only the particle can be diagonalized in momentum space. Defining:

\[
c_p \equiv \frac{1}{N} \sum_j c_j e^{-ipj}, \tag{3.31}
\]

the particle part of the Hamiltonian is:

\[
H_0 = \sum_p \epsilon(p) c_p^\dagger c_p. \tag{3.32}
\]

We find:

\[
U^\dagger c_p U = c_{p + \sqrt{2\eta}\phi(0)}, \tag{3.33}
\]

and hence the Hamiltonian is transformed into Eq. (3.25), using \(\sqrt{2}\phi(0) = \phi_e(0)\). In the case of nearest neighbor hopping the dispersion relation is:

\[
\epsilon(P) = -2t \cos Pa. \tag{3.34}
\]
The above derivation generalizes this result slightly to the case of several well-separated bands with arbitrary dispersion relation. Of course, for sufficiently weak dissipation and sufficiently low temperature, we will only be concerned with the lowest band.

It is convenient to define the parameter:

$$ R \equiv \frac{1}{\sqrt{\eta a}} $$

so that the Hamiltonian is invariant under the translation:

$$ \phi_e \rightarrow \phi_e + 2\pi R. $$

However, as mentioned above, the boson $\phi_e$ is not compact so we cannot identify the field under this translation. We return to this point below.

The boundary interaction has scaling dimension

$$ \Delta = \frac{\eta a^2}{2\pi} = \frac{1}{2}\pi R^2. $$

This can be verified, for example, by using the N b.c. to replace $\dot{\phi}''(0)$ by $2\phi_L(0)$ and using the standard result that the scaling dimension of $\cos \beta \phi_L$ is $\beta^2/8\pi$. The boundary interaction is relevant for $\eta a^2 < 2\pi$. For $\eta a^2 > 2\pi$, we may think of the hopping term as being irrelevant corresponding to a flow to the localized fixed point. For $\eta a^2 < 2\pi$ we expect a flow to the diffusing fixed point. Note that either increasing the distance over which the particle must hop or increasing the coupling to the heat bath promotes localization.

**B. Weak Corrugation Limit**

The momentum operator $P_T$ and co-ordinate $Q_T$ can be absorbed into the field, $\phi_e$, by transforming to a dual oscillator bath, a transformation that was introduced by Fisher and Zwerger from a different viewpoint. This is done by first diagonalizing the oscillator part of the Hamiltonian, whose even part is:

$$ H_0 \equiv \frac{\eta |\phi_e(0)|^2}{2M} + \frac{1}{2} \int_0^l dx \left[ \left( \frac{d\phi_e}{dx} \right)^2 + \Pi_e^2 \right]. $$

This quadratic Hamiltonian can be diagonalized by solving the classical Euler-Lagrange equations:

$$ \omega^2 \phi(x) = \left[ -\frac{d^2}{dx^2} + \frac{2\eta}{M} \delta(x) \right] \phi(x). $$

This is just the Schrödinger equation with a $\delta$-function potential. The odd wave-functions are unaffected by the potential and the even ones can be expressed in terms of a phase shift, $\delta(k)$:

$$ \phi_k(x) \propto \cos[k|x| + \delta(k)], $$

with frequency $\omega(k) = |k|$. [Here $k$ runs over the wave-vectors of the even modes, given in Eq. (3.19).] Substituting into Eq. (3.39) we obtain the phase shifts:

$$ \cot \delta(k) = -\frac{kM}{\eta}. $$

For small $k$, this becomes:

$$ \delta(k) \approx \frac{\pi}{2} + \frac{kM}{\eta} + O(k^2). $$
The allowed wave-vectors are given by:

$$kl + \delta(k) = \pi(n + 1/2), \quad n = 1, 2, 3, \ldots$$  (3.43)

For large \(l\) \((l \gg M/\eta)\) and small \(k\) \((k \ll \eta/M)\) we obtain:

$$k \approx \pi n/l, \quad (n = 1, 2, 3, \ldots)$$  (3.44)

The mode expansion for \(\phi_e(x)\) now takes the form:

$$\phi_e(x) = \sqrt{2} \sum_k p_k \cos[k|x| + \delta(k)],$$  (3.45)

and similarly for \(\Pi_e(x)\). [The \(p_k\)'s are the same operators, up to a minus sign, that occur in Eq. (3.13) for wave-vectors corresponding to odd values of \(n\) in Eq. (3.13).] Noting that, for small \(k\),

$$\cos(k|x| + \delta) \approx -\sin[k|(x) + M/\eta|],$$  (3.46)

we see that the effect of the \(\phi_e(0)^2\) term in \(H_0\) at low energies, is to impose, approximately a Dirichlet b.c., \(\phi_e = 0\) on the even field. (More accurately, the low energy field vanishes at \(x = -M/\eta\) but, in the low energy effective theory \(\phi(x)\) does not vary on such short wavelengths so we ignore this shift.) It is now convenient to introduce a transformed even field, \(\phi'(x)\) obeying the D. b.c. at \(x = 0\):

$$\phi'(x) = -\sqrt{2} \sum_k \sin kx p_k.$$  (3.47)

The even part of \(H_0\) may be written in terms of \(\phi'\) as simply:

$$H_0 = \frac{1}{2} \int_0^l dx \left[ \left( \frac{d\phi'}{dx} \right)^2 + \Pi'^2 \right].$$  (3.48)

We now must consider the term in the Hamiltonian \(H\) of Eq. (3.22) that couples the field to the particle:

$$H_{int} = -\sqrt{\eta/M} \phi_e(0) \Pi_T.$$  (3.49)

This may be expressed in terms of the \(p_k\) operators of Eq. (3.45) using the explicit expression for the phase shift in Eq. (3.41) as:

$$\phi_e(0) = -M \sqrt{\frac{2}{l}} \sum_k \frac{k p_k}{\sqrt{\eta^2 + (kM)^2}}.$$  (3.50)

We see that, at low energies this is given by:

$$\phi_e(0) \approx M \frac{d\phi'}{\eta dx}.$$  (3.51)

but we do not make this approximation here, proceeding to a more exact analysis. The entire Hamiltonian of Eq. (3.22) can now be written:

$$H \approx \frac{1}{2} \sum_k \left[ \left( k p_k + \sqrt{\frac{2\eta}{l(\eta^2 + (kM)^2)}} \Pi_T \right)^2 + \phi_k^2 \right] + \frac{\Pi_T^2}{2\eta^2} + V(Q).$$  (3.52)

In order to complete the square we have used a crucial identity:
\[
\frac{2\eta}{l} \left[ \frac{1}{2\eta^2} + \sum_{n=1}^{\infty} \frac{1}{\eta^2 + (\pi n M/|l|)^2} \right] = \frac{1}{M} \coth(l\eta/M) = \frac{1}{M} + O(e^{-2l\eta/M}).
\]  

(3.53)

and dropped the exponentially small term. [Note that we are being a bit cavalier and using the small \(k\) result, \(k \approx \pi n/l\) for all \(n\). It can be shown that the corrections to this approximation only affect the exponentially small term in Eq. (3.53).] It is now convenient to Fourier transform to position space, using:

\[
\frac{d\phi'}{dx} = -\sqrt{\frac{2}{l}} \sum_k k p_k \cos kx.
\]  

(3.54)

To this end it is convenient to introduce the function:

\[
f(x) = 2 \sqrt{\eta} \frac{\pi M}{\eta} K_0 \left( \frac{\eta x}{M} \right).
\]  

(3.55)

Here the sum is over \(k = \pi n/l\) with \(n = 1, 2, 3, \ldots\) \(f(x)\) vanishes exponentially at large \(x\). Up to corrections which are exponentially small in \(l\eta/M\), we may write:

\[
f(x) = \frac{2 \sqrt{\eta} l}{\pi M} K_0 \left( \frac{\eta x}{M} \right).
\]  

(3.56)

Here \(K_0\) is a modified Bessel function. We may thus rewrite \(H\) as:

\[
H = \frac{1}{2} \int_0^l dx \left\{ \left[ \frac{d\phi'}{dx} - f(x) P_T \right]^2 + \Pi'^2 \right\} + V(Q).
\]  

(3.57)

It is now possible to make a further transformation that absorbs \(P_T\) into \(\phi'\):

\[
\phi''(x) = \phi'(x) + \int_x^l dx' f(x') P_T.
\]  

(3.58)

We may now also absorb \(Q\) by going over to the dual field. This is defined, as usual by:

\[
\frac{d\phi''}{dx} = \Pi''
\]

\[
\frac{d\Pi''}{dx} = -\Pi''.
\]  

(3.59)

These equations only determine \(\phi''(x)\) up to a constant term. This may be fixed by imposing the canonical commutation relations and Euler-Lagrange equations on \(\Pi''(x)\) and \(\phi''(x)\). The finite momentum part of the dual field is:

\[
\phi''(x) = -\sqrt{\frac{2}{l}} \sum_k q_k \cos kx
\]  

(3.60)

The zero mode term in \(\phi''(x)\), which we write \(\phi_0''\), must be chosen to commute with all finite momentum modes of \(\Pi''(x)\) but have a non-zero commutator with the zero mode of \(\Pi''(x)\):

\[
\Pi_0'' = -P_T \int_0^l f(x) \frac{dx}{l} = -\frac{P_T}{l\sqrt{\eta}}
\]  

(3.61)

Using the commutator:
\[
[\phi'(x), \tilde{\phi}'(y)] = i \left[ \frac{x}{T} - \theta(x - y) \right],
\] (3.62)

which can be checked from the mode expansion of these fields, we find

\[
\tilde{\phi}''_0 = -\sqrt{\eta} \int_0^l dy f(y) \tilde{\phi}'(y) - \sqrt{\eta} Q.
\] (3.63)

Note that \(\tilde{\phi}''(x)\) obeys N b.c.’s at \(x = 0\) and \(l\). We see that \(Q\) has become (part of) the zero-mode of the dual field. The Hamiltonian can be written:

\[
H \to \frac{1}{2} \int_0^l dx \left[ \tilde{\Pi}''^2 + \left( \frac{d\tilde{\phi}''}{dx} \right)^2 \right] + V \left[ -\tilde{\phi}''(0) \sqrt{\frac{\eta}{M}} \right],
\] (3.64)

Now consider integrating out the high wave-vector components of \(\tilde{\phi}''(x)\) to obtain a low energy effective Hamiltonian. Since \(f(x)\) vanishes exponentially on the length scale \(M/\eta\), it follows that once we have reduced the wave-vector cut-off to a value much less than \(\eta/M\) we may approximate \(\tilde{\phi}''(x)\) by its value at \(x = 0\) inside the integral in the last term of Eq. (3.64), leading to the simplified expression:

\[
H \to \frac{1}{2} \int_0^l dx \left[ \tilde{\Pi}''^2 + \left( \frac{d\tilde{\phi}''}{dx} \right)^2 \right] + V \left[ -\tilde{\phi}''(0) \sqrt{\frac{\eta}{\pi}} \right].
\] (3.65)

The degeneracy mentioned above, corresponds to shifting \(\tilde{\phi}''(x)\), or equivalently \(Q\), by a constant \(na\) where \(a\) is the lattice spacing. This degeneracy is lifted at finite \(l\) as expected. Now let us consider a particular choice for \(V\):

\[
V(Q) = V_0 \cos(2\pi Q/a).
\] (3.66)

Our Hamiltonian then becomes the standard boundary sine-Gordon model:

\[
H = \frac{1}{2} \int_0^l dx \left[ \tilde{\Pi}''^2 + \left( \frac{d\tilde{\phi}''}{dx} \right)^2 \right] + V_0 \cos \left[ 2\pi R \tilde{\phi}''(0) \right],
\] (3.67)

where we have again introduced the parameter \(R\) defined in Eq. (3.35). Recall that we obtained the boundary sine-Gordon model only after integrating out high frequency modes, reducing the cut-off to \(\eta/M\). This process will, in general, introduce renormalization of the interactions. In order to be able to ignore this renormalization, the original dimensionless coupling constant must be small. We may estimate how small it needs to be by considering the renormalization of the boundary interaction in Eq. (3.67). This boundary interaction has a scaling dimension of

\[
\Delta = \frac{2\pi}{\eta a^2} = 2\pi R^2.
\] (3.68)

Thus, upon reducing the cut-off from its bare value, \(\Lambda\) to \(\eta/M\), the effective coupling is renormalized to:

\[
\frac{V_0}{\Lambda} \to \frac{V_0}{\Lambda} \left( \frac{\Lambda}{\eta/M} \right)^{1-\pi/\eta a^2}.
\] (3.69)

Requiring the renormalized dimensionless coupling constant to be small after reducing the cut off to \(\eta/M\) gives the condition:

\[
V_0 \ll (\eta/MA)^{1-\pi/\eta a^2} \Lambda.
\] (3.70)
Thus, this boundary sine-Gordon model is only obtained in the weak corrugation limit. The dimensionless parameter, $\sqrt{\eta a} = 1/R$ may take any value, however.

We note that the weak corrugation formulation of the problem is actually equivalent at $l \to \infty$, under a dual transformation, to a tight binding model interacting with a heat bath with a Lorentzian weighted density of states, $J(\omega)$. This can be seen by starting from Eq. (3.52) and making the duality transformation:

$$p_k \rightarrow -q_k/k, \quad q_k \rightarrow kp_k, \quad P_T \rightarrow -Q, \quad Q \rightarrow P_T.$$  \hfill (3.71)

The dual Hamiltonian is:

$$H \approx \frac{Q^2}{2\eta l} + V(P_T) + \frac{1}{2} \sum_k \left[ \left( q_k + \sqrt{\frac{2\eta}{l[\eta^2 + (kM)^2]}} Q \right)^2 + k^2 p_k^2 \right].$$  \hfill (3.72)

The last term has the same form as our original heat bath term in Eq. (3.3) with

$$\omega_k = k, \quad m_k = \frac{1}{k^2}, \quad \lambda_k = -\sqrt{\frac{2\eta}{l[\eta^2 + (kM)^2]}}$$  \hfill (3.73)

and hence a weighted density of states:

$$J(\omega) = \frac{\pi/2}{\sum_k \frac{2\eta k}{l[\eta^2 + (kM)^2]}} \delta(\omega - k).$$  \hfill (3.74)

Taking the limit $l \to \infty$ this gives:

$$J(\omega) = \frac{\eta \omega}{\eta^2 + \omega^2 M^2}.$$  \hfill (3.75)

This is ohmic below a cut off scale $\eta/M$:

$$J(\omega) \approx \frac{\omega}{\eta}.$$  \hfill (3.76)

Note that the dissipation strength parameter, $\eta$ has been inverted. We may also drop the first term in Eq. (3.72) in the limit $l \to \infty$ so that the particle part of the Hamiltonian is diagonal in momentum space. This is then equivalent to a tight binding model with a dispersion relation:

$$\epsilon(P_T) \rightarrow V(P_T).$$  \hfill (3.77)

This gives another way of understanding why a boundary sine-Gordon model is obtained in both the weak and strong corrugation cases. We note however, that the $P_T^2/2\eta l$ term, which gave $\Pi''(x)$ a zero mode, plays quite an important role in the following sub-section.

### C. Groundstate Degeneracy and Boson Compactification

An interesting quantity in quantum impurity problems is the zero temperature impurity entropy or its exponential, the “groundstate degeneracy”, $g$. More generally, we may define an impurity free energy by subtracting off the bulk free energy (the term proportional to $l$), then taking the length of the bulk system to infinity. The zero temperature impurity entropy is defined this way, with the $T \to 0$ limit taken after the infinite length limit. Note that if we take $T \to 0$ first, before taking $l \to \infty$, the degeneracy is necessarily integer valued since the spectrum of the finite size Hamiltonian is discrete. However, taking the limits in the opposite order, we are effectively dealing with a continuous spectrum and it is entirely possible to obtain a non-integer (even non-rational) “groundstate degeneracy” $g$, independent of system...
g has been argued to be universal and to always decrease under renormalization group flow between boundary fixed points. Thus it plays a role in boundary critical phenomena analogous to that of the conformal anomaly parameter, $c$, in two dimensional bulk critical phenomena. A new feature occurs when we consider $g$ in QBM: it turns out to be proportional to the length of the interval on which the particle is allowed to move. We emphasize that we are taking the length, $l$, of the fictitious line interval used to define the oscillator bath to $\infty$. Thus the oscillator spectrum becomes continuous. On the other hand, we are considering a long but finite physical line interval, of length $Na$ with $N \gg 1$, on which the particle moves.

To orient ourselves let us first consider the partition function in the case $\eta = 0$ where the particle is decoupled from the heat bath and the periodic potential $V$ is set to zero. We only consider the even oscillator modes, with energies $\pi(n+1/2)/l$, $n = 0, 1, 2, \ldots$. (Recall that $\phi_e$ obeys a N b.c. at $x = 0$ and a D b.c. at $x = l$ so that the allowed momenta are proportional to the half-integers.) Thus the oscillator part of the partition function is:

$$Z_{osc} = \prod_{n=0}^{\infty} \left[ 1 - e^{-\pi(n+1/2)/lT} \right]^{-1}. \quad (3.78)$$

Taking the limit $lT \gg 1$ we obtain:

$$Z_{osc} \rightarrow \frac{e^{\pi lT/6}}{\sqrt{2}}, \quad (3.79)$$

corresponding to an oscillator groundstate degeneracy of

$$g_{osc} = \frac{1}{\sqrt{2}}. \quad (3.80)$$

This must be multiplied by the partition function of the particle. To make this well-defined, we place the particle in a box of size $Na$, with periodic b.c.’s. In this case the allowed momenta are $P = 2\pi m/Na$ giving the full partition function:

$$Z = \sum_{m=-\infty}^{\infty} \exp \left[ - \left( \frac{2\pi m}{Na} \right)^2 \frac{1}{2MT} \right] e^{\pi lT/6}. \quad (3.81)$$

If we now take $T \rightarrow 0$, holding $N$ fixed, the particle has a unique groundstate with $P = 0$ so it makes no contribution to $g$. We note, however, that in the opposite limit, $(Na)^2MT \gg 1$, the partition function is proportional to $N$ since the particle may be treated classically:

$$Z \approx \frac{e^{\pi lT/6}}{\sqrt{2}} aN \int_{-\infty}^{\infty} dP e^{-P^2/2MT} = \frac{aN}{2} \sqrt{\frac{MT}{\pi}} e^{\pi lT/6}. \quad (3.82)$$

Remarkably, this factor of $N$ will persist in the partition function as we take $T \rightarrow 0$ at fixed $N$, once we couple the particle to the oscillators. If we now include the periodic potential $V$, the partition function behaves the same way at temperatures small compared to the bandwidth. This follows since we are only concerned with low momentum states in the lowest energy band, whose dispersion may be approximated by:

$$\epsilon_1(P) \approx \frac{P^2}{2M^*}, \quad (3.83)$$

for some effective mass, $M^*$.

Let us now couple the particle to the oscillator bath, $\eta > 0$, but, for the moment ignore the potential $V$. Thus we are considering a freely diffusing particle. From the mode expansion of the dual field, $\hat{\delta}^n(x)$, which obeys N b.c.’s at $x = 0, l$ with the zero mode of Eq. (3.61), or more directly from Eq. (3.52), we see that the spectrum is given by:
\[ E = \frac{P_l^2}{2l\eta} + \sum_{m=1}^{\infty} \frac{\pi m}{l} n_m. \]  

(3.84)

Here the \( n_m \)'s are the occupation numbers of the finite momentum oscillator modes. In this case the oscillator partition function is given by:

\[ Z_{osc} = \prod_{m=1}^{\infty} (1 - e^{-\pi m / lT})^{-1} \to \frac{e^{\pi lT/6}}{\sqrt{2lT}}. \]  

(3.85)

Thus the entire partition function, in the limit of large \( l \), for the freely diffusing particle is given by:

\[ Z_{diff} = e^{\pi lT/6} \frac{\sqrt{2lT}}{2\pi} \int_{-\infty}^{\infty} dP \frac{1}{2lT} e^{-P^2/2lT}. \]  

(3.86)

We have thus obtained a groundstate degeneracy:

\[ g_{diff} = \frac{\sqrt{2}}{\sqrt{\pi}} aN. \]  

(3.88)

Of course, the precise value of \( g \) depends on precisely how we have defined the oscillator bath, in particular, the boundary condition at \( x = l \) on the original field \( \varphi(x) \). However, ratios of \( g \) at different fixed points are expected to be independent of these choices, depending on the oscillators only through the parameter \( \eta \). This is related to the fact that the degeneracy may be regarded as a product of factors for each of the two boundaries of the system. Comparing this calculation to the case discussed in the previous paragraph we see that the degree of freedom associated with \( P_l \) has become infinitely massive in the limit \( l \to \infty \) so that the associated partition function remains in the classical regime all the way down to \( T = 0 \), yielding the factor of \( N \).

Now let us consider the effect of the periodic potential, beginning with the weak corrugation Hamiltonian of Eq. (3.67). This has a scaling dimension of \( 2\pi / \eta a^2 \). If the potential is irrelevant, \( \eta a^2 < 2\pi \), we expect that \( g \) remains unchanged. On the other hand, if it is relevant, \( \eta a^2 > 2\pi \), we expect a RG flow to a different fixed point. The nature of this fixed point may be deduced by assuming that \( V_0 \) flows to \( \infty \), pinning \( \tilde{\varphi}''(0) \) at its minima. On the other hand, \( \tilde{\varphi}'' \) still obeys \( N \) b.c.'s at \( x = l \). Thus its mode expansion takes the form:

\[ \tilde{\varphi}''(x) = \sqrt{\eta} am + \sqrt{\frac{2}{l}} \sum_{n=0}^{\infty} \sin[\pi(n+1/2)x/l]p_k. \]  

(3.89)

The oscillator part of \( Z \) is the same as in Eqs. (3.78), (3.79) for the case where the bath is decoupled from the particle, and the sum over \( m \) simply gives a factor of \( N \), the number of minima of the potential where the particle can get localized. Thus we obtain, in the localized phase:

\[ g_{loc} = \frac{N}{\sqrt{2}}. \]  

(3.90)

Despite all the transformations and the RG flow that went into this result the answer is intuitively obvious. The degeneracy is simply \( N \) times that of the decoupled bath, reflecting the \( N \) locations where the particle can be localized. Note that for the decoupled bath, the original field \( \varphi_\alpha \) obeys \( N \) b.c. at \( x = 0 \) and D at \( x = l \). On the other hand, at the localized fixed point the transformed dual field, \( \tilde{\varphi}'' \) obeys D b.c. at \( x = 0 \) and N at \( x = l \).
It is a useful check on our results to derive the low energy spectra using the strong corrugation form of the Hamiltonian, Eq. (3.25). In this case the simple limit is the one in which the dispersion relation \( \epsilon_n \to 0 \). This corresponds to the limit of zero hopping between sites when the particle is localized. We expect to renormalize to this fixed point whenever \( \epsilon_n \) of Eq. (3.26) is irrelevant. This has dimension \( \eta \epsilon_n^2 / 2\pi \) and so is irrelevant for \( \eta \epsilon_n^2 > 2\pi \). It is very important that the hopping term is irrelevant in the strong corrugation formulation whenever the potential is relevant in the weak corrugation formulation and vice versa. This follows since the scaling dimensions are the inverse of each other. Thus the diffusing fixed point is stable when the localized one is unstable and vice versa. This is consistent with our assumption that the coupling constant flows to \( \infty \) whenever the potential is relevant in the weak corrugation formulation and irrelevant when it is relevant. Ignoring the hopping term, \( \epsilon_n \), the partition function is just that of the decoupled oscillator bath, Eq. (3.78), multiplied by the result of integrating over \( P_{c} \). Note that the crystal momentum, \( P_{c} \) is restricted to lie in the first Brillouin zone. The number of momenta in the first zone when the particle lives on a line of length \( N a \) is \( N \), so we obtain a partition function of \( N N_{osc} \), the same partition function (and finite size spectrum) as we obtained using the weak corrugation formulation. Now let us consider the case where the hopping term is relevant. If \( \epsilon_n \) renormalizes to \( \infty \), \( \phi_c(0) \) gets pinned at one of the minima of \( \epsilon_n \) which we can take to lie at \( \phi_c(0) = (2\pi/a)(P_c + m/\sqrt{|\eta|}) \) for integer \( m \). At \( x = l \), \( \phi_c \) obeys a simple D b.c., \( \phi_c(l) = 0 \). Thus the mode expansion for \( \phi_c(x) \) contains a winding mode in this case:

\[
\phi_c(x) = \frac{(l - x) P_c + 2\pi m/a}{\sqrt{\eta}} - \sqrt{\frac{2}{l}} \sum_n \sin(\pi n x/l)p_n.
\]

For fixed \( P_{c} \), the partition function is:

\[
e^{\pi l T / 6} \frac{\sqrt{2 l T}}{e^{-(P_{c}+2\pi m/a)^2/2lT\eta}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-(P_{c}+2\pi m/a)^2/2lT\eta}.
\]

We must further sum \( P_{c} \) over the Brillouin zone. The result of the combined sums is a sum over \( P_{c} = 2\pi m/N a \) with \( m \) summed over all integers. Thus we obtain the same partition function as in Eq. (3.87), which resulted from the weak corrugation formulation and hence the same finite size spectrum.

We also note that the same ratio of \( g \) factors could be obtained using a compact boson field with D and \( N \) b.c.’s. In this case we would identify:

\[
\phi_c(x) \leftrightarrow \phi_c(x) + 2\pi R,
\]

and

\[
\tilde{\phi}''(x) \leftrightarrow \tilde{\phi}''(x) + 1/R,
\]

the smallest possible compactification radius consistent with the BSG interactions. In fact, although the original field \( \phi(x) \) is non-compact, as we remarked above, the transformed field, \( \tilde{\phi}''(x) \) can be thought of as being compact. This follows from the mode expansion of Eqs. (3.61), (3.63) once we impose periodic boundary conditions on \( Q \). Since \( Q \) is identified with \( Q + N a \) it follows from Eq. (3.63) that

\[
\tilde{\phi}''(x) \leftrightarrow \tilde{\phi}''(x) - N/R.
\]

The effective compactification radius for the dual boson is increased by a factor of \( N \). The allowed values of the zero mode of the conjugate momentum given by Eq. (3.61) become \( \Pi''_0 = -2\pi R/N \) the correct values for the periodic boson radius \( R/N \). The N b.c. that we have been discussing corresponds to an N b.c. on a periodic boson. However the D b.c. is not a simple D b.c. for the periodic boson. The boundary sine-Gordon interaction of Eq. (3.67) has \( N \) inequivalent minima. Thus \( \tilde{\phi}''(0) \) may take any of these \( N \) inequivalent values corresponding to a “multi-Dirichlet” b.c.

The corresponding N boundary state is:

\[
|N''(\phi = 0)\rangle = g_N^0 \sum_m |0, N 2\pi m/R\rangle.
\]
The D boundary state for this compactification radius is:

$$|D^n(\tilde{\phi} = \tilde{\phi}_0)\rangle = g^n_D \sum_n e^{i\tilde{\phi}_0 n R/N} |(n R/N, 0)\rangle$$  (3.98)

with

$$g^n_D = \frac{\pi^{1/4}}{\sqrt{R N}}$$  (3.99)

The multiple-Dirichlet boundary state, $|MD\rangle$, is obtained by summing $\tilde{\phi}$ over the $N$ lattice points where the particle can be localized:

$$|MD^n\rangle = \sum_{m=1}^N |D^n(\phi = m R)\rangle.$$  (3.100)

Using:

$$\sum_{m=1}^N e^{i2\pi mn/N} = N \delta_{n,Np},$$  (3.101)

we thus obtain:

$$|MD^n\rangle = \sqrt{N} |D(\tilde{\phi} = 0)\rangle,$$  (3.102)

proportional to the ordinary D boundary state with radius $R$. Thus:

$$g^2_{MD} = \sqrt{N} g_D = \pi^{1/4}\sqrt{N R}.$$  (3.103)

Note that both degeneracy factors are simply increased by a factor of $\sqrt{N}$ compared to the ordinary compact D and N values. We may use these degeneracy factors for each boundary to calculate the total degeneracy associated with the partition functions. These give us:

$$g_{\text{diff}} = (g^N)^2 = \frac{N}{\sqrt{4\pi R}},$$  (3.104)

$$g_{\text{loc}} = g^n_D g^N_{MD} = \frac{N}{\sqrt{2}}.$$  (3.105)

These are exactly the same values obtained from an explicit calculation in Eqs. (3.88) and (3.90).

We note that the infinite groundstate degeneracy, in the limit where the particle is diffusing on an infinite line, is related to the unphysical nature of the Caldeira-Legggett type model for QBM. The more physical applications of the model, to the quantum wire problem, involve compact bosons from the beginning, and do not have this strange behavior.

In general, the change in $\ln g$ due to a change in b.c.’s at one end of the system appears to be insensitive to the compactness or non-compactness of the boson. This arises from the fact that the compactness only appears when one considers properties depending on both boundaries whereas $g$ is a property of each boundary separately. This can be checked very explicitly in the boundary sine-Gordon case, where changes in $g$ can be computed perturbatively to a very large order, and matched against the thermodynamic Bethe ansatz results. In the perturbation theory, there is no dependence on the compactification at all orders.

We note that, upon imposing periodicity of the Hamiltonian under $\phi \rightarrow \phi + 2\pi R$ or $\tilde{\phi} \rightarrow \tilde{\phi} + 1/R$, the correct operator content at the D and N fixed points is given by the finite size spectrum of the ordinary
Thus in order to deduce the operator content (which is consistent with the imposed symmetry) from the finite size spectrum, it is convenient to use compact bosons of radius $R$. The main effect of the non-compactness seems to be to increase the groundstate degeneracy in the partition function by a factor of $N$, a phenomena whose origin can be traced back to the infinite mass oscillator discussed near the beginning of this section. In our discussion of the triangular lattice case we shall use bosons compactified on this lattice. Most physical quantities, such as the change in $\ln g$, are not affected by this compactification.

Note that in the case when the N (diffusing) b.c. is stable, $R > 1/\sqrt{2\pi}$, $g_N < g_D$ and vice versa. Thus the “g-theorem” is obeyed; the RG flow always serves to decrease $g$.

**IV. QUANTUM BROWNIAN MOTION ON A TRIANGULAR LATTICE**

In this section we first review the phase diagram of the boundary 3-state Potts model. We then introduce the triangular lattice QBM problem and conjecture a phase diagram by analogy with that of the Potts model. This is substantiated and studied in more detail in the following sections.

**A. Phase Diagram of the Boundary 3-State Potts Model**

The 3-state Potts model is a natural generalization of the Ising model to a discrete spin variable that takes on three equivalent values. (We sometimes refer to these as $A$, $B$ and $C$.) As such it is naturally related to QBM on a triangular lattice, as will become clear in the next subsection. Here we review the boundary Potts model phase diagram. The classical Potts model Hamiltonian contains nearest neighbor interactions of these spins such that the energy is (-1) when two neighboring spins are in the same state and zero otherwise. The critical behavior of the two dimensional Potts model is equivalent to that of a one dimensional quantum chain. In addition to the classical Potts interaction this also contains a transverse field term which permutes the spins on each site between the three states with equal amplitudes. A complete set of conformal invariant b.c.’s for the quantum Potts chain consists\footnote{11,36} of only four b.c.’s. One of these is the fixed b.c. in which the quantum spin at the end of the semi-infinite chain is constrained to take a fixed value (either $A$, $B$ or $C$). A second b.c. is a mixed b.c. where the quantum spin at the end of the chain is hopping back and forth between two of the states (eg. $A$ and $B$). A third b.c. is the free b.c. which results from simply terminating the bulk Hamiltonian without otherwise modifying it near the boundary. Finally there is a fourth b.c. whose physical interpretation is not obvious, and which we referred to as the “new” b.c.

In Ref. (\footnote{11}) a phase diagram was conjectured for the boundary Potts model in which both the transverse and longitudinal fields acting on the spin at the end of the chain was varied. As stated above, when the boundary transverse field is positive (the same sign as in the bulk at the bulk critical point), the system is at the free fixed point. If a boundary longitudinal field is now turned on which favors the $A$ state, the system renormalizes to the fixed b.c. On the other hand, if this boundary longitudinal field has the opposite sign so as to equally favor $B$ and $C$ then the system renormalizes to the mixed b.c. It should be emphasized that the mixed b.c. corresponds to the system dynamically jumping back and forth between the $B$ and $C$ states. In particular, it is invariant under the $Z_2$ sub-group of $Z_3$ which interchanges $B$ and $C$. On the other hand, the broken symmetry b.c. whose boundary state is sum of $B$ and $C$ fixed boundary states is unstable against an RG flow to the mixed b.c. If there is no longitudinal field and the boundary transverse field is negative, then the system renormalizes to the “new” b.c. The special case where both types of boundary fields vanish is a type of degenerate boundary condition for which the corresponding boundary state is a linear combination of the three different fixed boundary states.

It turns out that all of these RG flows have direct analogies in the problem of QBM on a triangular lattice, which thus provides a particular realization of the quantum Potts chain.
B. Back to Quantum Brownian Motion

We consider a natural extension of the 1D QBM model discussed in Sec. 2 to two dimensions. The Hamiltonian of Eq. (3.6) is extended by introducing a separate set of oscillators, $q_k^a$ for each component of the particle co-ordinate vector, $Q^a$, with an identical set of masses and coupling constants:

$$H = \frac{\vec{P}^2}{2M} + V(\vec{Q}) + \sum_k \left[ \frac{\vec{p}_k^2}{2m_k} + \frac{m_k\omega_k^2}{2} \left( \vec{q}_k - \frac{\vec{Q}_a}{m_k\omega_k^2} \right)^2 \right].$$  \hspace{1cm} (4.1)

In the ohmic case we may again represent the oscillators by free massless relativistic boson fields, however we now get a separate boson, $\phi^a(x)$ for each component $Q^a$. We emphasize that the bosons still live on a fictitious one dimensional line. The derivation of the boundary sine-Gordon model in the strong and weak corrugation limits carries over directly to the multi-dimensional case. In the strong corrugation limit the Hamiltonian is:

$$H_S = \frac{1}{2} \int_0^l dx \left[ \vec{\Pi}^2 + \left( \frac{d\phi^a}{dx} \right)^2 \right] + \epsilon \left[ \vec{P}_c - \sqrt{\eta} \phi^a(0) \right].$$  \hspace{1cm} (4.2)

We have dropped extraneous ′ and e notation. Here $\epsilon(\vec{P})$ is the dispersion relation (in the lowest band) for the dissipationless problem. $\vec{\phi}(x)$ obeys an N b.c. at $x = 0$ and a D b.c. at $x = l$. The c-numbers $\vec{P}_c$ can take any values in the first Brillouin zone. Similarly in the weak corrugation formulation this Hamiltonian is:

$$H_W = \frac{1}{2} \int_0^l dx \left[ \vec{\Pi}^2 + \left( \frac{d\phi^a}{dx} \right)^2 \right] + V \left[ \frac{\phi^a(0)}{\sqrt{\eta}} \right].$$  \hspace{1cm} (4.3)

(We have again dropped the cumbersome ″ notation.) $\tilde{\phi}(x)$ obeys N b.c.’s at $x = 0, l$.

The observations about boson compactness made in Sec. III carry over to the two dimensional case. We henceforth set:

$$\eta = 1,$$  \hspace{1cm} (4.4)

by a rescaling of the lattice spacing. We label the physical lattice on which the particle moves $\Gamma^*$. It is convenient to impose periodic b.c.’s on the particle co-ordinate, $\vec{Q}$, so that we identify:

$$\vec{Q} \leftrightarrow \vec{Q} + N\vec{a},$$ \hspace{1cm} (4.5)

where $\vec{a}$ is any vector in $\Gamma^*$ and $N$ is a large integer. It follows from the derivation in Sec. III of the boundary sine-Gordon model for QBM that $\phi$ may be regarded as compactified on this “coarse lattice” with spacing bigger by the factor of $N$. Equivalently, we may regard $\phi$ as being compactified on the lattice $\Gamma^*$ with an extra factor of $N$ appearing in the degeneracy, $g$.

We wish to consider a model of QBM in a periodic potential with hexagonal symmetry. The tight-binding model that we consider is a simple generalization of the one introduced in Ref. (7). We consider a triangular lattice generated by the vectors:

$$\vec{a}_1 = (a, 0), \quad \vec{a}_2 = (a/2, \sqrt{3}a/2).$$  \hspace{1cm} (4.6)

This model exhibits interesting behavior for any value of the dimensionless parameter $a$ but in this paper we focus primarily on the value:

$$a^2 = 4\pi/3,$$  \hspace{1cm} (4.7)
since, only then, can it be mapped into the Potts model. The simplest form of the tight-binding Hamiltonian, before adding dissipation, is:

\[ H = -t \sum_{\langle i,j \rangle} c_i^\dagger c_j, \]  

(4.8)

where the sum is over nearest neighbors.

In order to understand the connection with the 3-state Potts model it is useful to focus on the symmetry transformations that leave fixed the center of one of the triangles on the lattice. These consist of three-fold rotations and reflections, referred to as 3m in the international crystal nomenclature. There is actually a larger point group symmetry, 6mm, when one considers transformations that hold fixed a lattice point. However, this is not relevant for our purposes. It is convenient to decompose the lattice into three sublattices, \( A, B \) and \( C \) such that the nearest neighbours of a point in one sub-lattice are in the other two, as drawn in Fig. (2). Note that a \( 2\pi/3 \) rotation of the lattice maps each point on the \( A \) sublattice into a point on the \( B \) sublattice and similarly \( B \rightarrow C \) and \( C \rightarrow A \). Similarly a reflection about a line passing through \( A \) points interchanges all \( B \) points with \( C \) points. Hence we may think of this point group symmetry as the permutation group on three objects, \( S^3 \). This is the symmetry of the 3-state Potts model. We will also consider an on-site potential which assigns different energies to the \( A, B \) and \( C \) sub-lattices, \( v_A, v_B \) and \( v_C \), thus breaking the \( S^3 \) symmetry.

![FIG. 2. Triangular lattice with A,B and C sublattices marked. Note that the A points also form a triangular lattice with spacing \( \sqrt{3}a \) and orientation rotated by 90°. The B and C points form a honeycomb lattice.](image)

To map out the phase diagram let us begin with the strong corrugation formulation in the case \( t > 0 \) and all \( v_i = 0 \). \( t \) is relevant for this lattice spacing, (of dimension \( x = 2/3 \)) corresponding to the boundary sine-Gordon Hamiltonian:

\[ H_S = H_0 + H_{int} \]  

(4.9)

where

\[
    H_0 = \frac{1}{2} \int_0^t dx \left[ \Pi^2 + \left( \frac{d\phi}{dx} \right)^2 \right],
\]

\[
    H_{int} = -t \left[ \cos \sqrt{2\pi} \sqrt{\frac{2}{3}} \phi_1 + \cos \sqrt{2\pi} \left( \sqrt{\frac{1}{6}} \phi_1 + \sqrt{\frac{1}{2}} \phi_2 \right) + \cos \sqrt{2\pi} \left( \sqrt{\frac{1}{6}} \phi_1 - \sqrt{\frac{1}{2}} \phi_2 \right) \right].
\]

(4.10)
(The fields are all evaluated at $x = 0$ but we suppress this argument. We have set $\vec{P}_C$ to 0.) We expect $t$ to induce an RG flow to the perfect mobility phase, corresponding to a D boundary condition, $\phi(0) = 0$. This fixed point is stable as we can see by considering the lowest dimension potential term with the full symmetry of the lattice, in the weak corrugation formulation, which is:

$$H_W = H_0 + H_{int}$$

where

$$H_0 = \frac{1}{2} \int_0^1 dx \left[ \tilde{\Pi}^2 + \left( \frac{d\phi}{dx} \right)^2 \right]$$

$$H_{int} = -v \left[ \cos \sqrt{4\pi} \phi_2 + \cos \sqrt{\pi} \left( \phi_2 + \sqrt{3} \phi_1 \right) + \cos \sqrt{\pi} \left( \phi_2 - \sqrt{3} \phi_1 \right) \right].$$

This has dimension 2 and is irrelevant. Now consider turning on a potential $v_A < 0$, which favors the $A$ sub-lattice. We do this in the most symmetric possible way, preserving a $Z_3$ symmetry of rotation about a lattice point, as well as a mirror symmetry about a lattice link, the 3m subgroup of the original 6mm point group.

Choosing the origin to lie on the $A$ sub-lattice, this potential is:

$$H \rightarrow H - v_A \left[ \cos \sqrt{2\pi} \sqrt{\frac{2}{3}} \phi_1 + \cos \sqrt{2\pi} \left( \sqrt{\frac{1}{6}} \phi_1 + \sqrt{\frac{1}{2}} \phi_2 \right) + \cos \sqrt{2\pi} \left( \sqrt{\frac{1}{6}} \phi_1 - \sqrt{\frac{1}{2}} \phi_2 \right) \right]$$

This operator has dimension 2/3 at the perfect mobility fixed point, which corresponds to a D b.c. on $\phi$ or equivalently, an N b.c. on $\phi$; it is relevant. Thus, for $v_A > 0$, we expect a flow to a localized fixed point, corresponding to a D b.c. on $\phi$. The stability of this fixed point can be checked by observing that, if $v_A$ flows to infinity, then only hopping between $A$ sub-lattice points is possible. The particle gets localized on one of the $A$ sub-lattice sites. This is a stable fixed point since the intra-sublattice is now $\sqrt{3}a$ and hence the hopping term gives:

$$H_{int} = -t \left[ \cos \sqrt{4\pi} \phi_2 + \cos \sqrt{\pi} \left( \phi_2 + \sqrt{3} \phi_1 \right) + \cos \sqrt{\pi} \left( \phi_2 - \sqrt{3} \phi_1 \right) \right].$$

of dimension 2 which is irrelevant. Note that the dimension of the nearest neighbor hopping term is 2/3, smaller by a factor of 3 due to the reduced distance of the hop and hence relevant. The effect of the symmetry breaking term, $v_A$, is to stabilize a localized phase on one of the sublattices.

The analogy with the Potts model is quite transparent. The perfect mobility fixed point corresponds to the free b.c. in the Potts model. The potential $v_A$ corresponds to a longitudinal boundary field which produces a flow to the fixed (A) b.c. Now consider the case $v_A < 0$. If $v_A \rightarrow -\infty$, the particle is localized on one of the $B$ or $C$ sub-lattice sites. Together, these two sub-lattices define a honeycomb lattice. However, for finite negative $v_A$, this is not a stable fixed point. This can be seen by considering hopping on this honeycomb lattice. Since the nearest neighbor distance is again $a$, this hopping term has dimension 2/3 and is relevant. Thus, following the logic of Yi and Kane, there must be an intermediate mobility fixed point. In the Potts model we add a boundary field which favors either the $B$ or $C$ state. This leads to a flow to the mixed b.c. of the boundary Potts model. Thus we see that the new fixed point found by Yi and Kane corresponds physically to the mixed boundary fixed point in the Potts model. In the Potts model we think of the boundary spin as fluctuating back and forth between $B$ and $C$ states. In the QBM problem we think of the particle as hopping back and forth between $B$ and $C$ sublattices. As can be shown explicitly, this state has an intermediate mobility. We refer to this as the “Y” fixed point, after Yi-Kane.

In fact, we can discover yet another intermediate fixed point in the QBM problem by pursuing this analogy further. We now set the symmetry breaking potentials $v_i$, to 0 but consider a negative hopping term, $t < 0$. This would seem to correspond to a negative transverse boundary field in the Potts model. This was argued to lead to a flow to another fixed point, unimaginatively referred to as the “new” fixed
point in Ref. (11). Thus we expect a new fixed point to occur in the QBM problem with $t < 0$. We refer to this as the “W” fixed point. Such a new fixed point is possible due to the lack of particle-hole symmetry of this model. For $t > 0$ the lowest energy single-particle state has crystal momentum $\vec{p} = 0$. However, for $t < 0$ the lowest energy single-particle states occur at two inequivalent crystal momenta: $\vec{p} = \pm (2\pi/3a, \pi/\sqrt{3}a)$. These two lowest energy states for the particle are dual to the $B$ and $C$ sub-lattices in the mixed phase. In fact, duality in QBM is easily understood, simply corresponding to:

$$\vec{\phi} \leftrightarrow \tilde{\vec{\phi}}.$$  \hspace{1cm} (4.15)

Comparing Eqs. (4.10) and (4.13) we see that the flow from localized to perfect mobility is dual to the flow from perfect mobility to localized on the $A$ sub-lattice, induced by a positive $t$ or $v_A$ respectively. Similarly, the flow from localized to mixed is dual to the flow from perfect mobility to new, induced by a negative $t$ or $v_A$ respectively.

If both transverse and longitudinal boundary fields are set to 0 in the Potts model we get a phase referred to as $A + B + C$ signifying that the boundary spin remains fixed in any of its 3 possible states. The corresponding boundary state is a sum of three boundary states corresponding to the 3 possible fixed b.c.’s. Turning on a positive transverse field produces a flow to the free b.c. whereas a negative transverse field produces a flow to the new b.c. The analogue in QBM is to start with $t = 0$, corresponding to the particle being localized at any lattice site ($A$, $B$ or $C$). We may, if we wish, include an irrelevant intra-sublattice hopping, of dimension 2. Now turning on $t$ induces a flow to either perfect mobility or new fixed points. This phase diagram is summarized schematically in Fig. (3). Note that, since only localized behavior is possible on the line $t = 0$, two copies of the $Y$ fixed point must occur, as drawn. Also note that, although the flow away from the “localized on $A$, $B$ or $C$” fixed point depends on the sign of $t$, that from the “localized on $B$ or $C$” fixed point does not. This is consistent with the fact that the sign of the hopping term cannot be changed by a redefinition of the sign of the electron operators for a triangular lattice but can be for a honeycomb lattice.

We also comment briefly on the model with general values of $a$. The strong corrugation Hamiltonian of Eq. (4.10) becomes:

![FIG. 3. Phase diagram and RG flows of the triangular lattice QBM model with hopping strength $t$ and potential $v_A$. Neumann and Dirichlet b.c.’s are imposed on the dual fields, $\tilde{\vec{\phi}}$. As pointed out in Ref. (7) and discussed in Sec. VII, the “localized on $B$ or $C$” phase corresponds to the weak coupling fixed point in the 3-channel Kondo problem.](image-url)
\[ H_{\text{int}} = -t \left[ \cos \alpha \phi_1 + \cos \alpha \left( \frac{\phi_1}{2} + \frac{\sqrt{3} \phi_2}{2} \right) + \cos \alpha \left( \frac{\phi_1}{2} - \frac{\sqrt{3} \phi_2}{2} \right) \right] \]

(4.16)

and the symmetry breaking weak corrugation term in Eq. (4.13) becomes:

\[ H \rightarrow H - v_A \left[ \cos \frac{4\pi \tilde{\phi}_1}{3a} + \cos \left( \frac{2\pi \tilde{\phi}_1}{3a} + \frac{2\pi \tilde{\phi}_2}{\sqrt{3}a} \right) + \cos \left( \frac{2\pi \tilde{\phi}_1}{3a} - \frac{2\pi \tilde{\phi}_2}{\sqrt{3}a} \right) \right] \]

(4.17)

The scaling dimension of the hopping term at the (symmetric) localized fixed point and of the symmetry breaking potential at the N fixed point are:

\[ \Delta_D = \frac{a^2}{2\pi} \]

(4.18)

\[ \Delta_N = \frac{8\pi}{9a^2}. \]

(4.19)

Thus the hopping term is relevant for \( a^2 < 2\pi \) and \( v_A \) is relevant for \( a^2 > \frac{8\pi}{9} \) indicating that the perfect diffusion fixed point is unstable in this region. For \( v_A > 0 \) we again expect a flow to the fixed point where the particle gets localized on the A sublattice. The scaling dimension of the intra-sublattice hopping term is \( \frac{3a^2}{2\pi} \) so this localized phase is stable in this region and we expect an RG flow from diffusing to “localized on A”. On the other hand, if \( v_A < 0 \) we should consider the stability of the “localized on B or C” phase. The scaling dimension of the inter-sublattice hopping term (between the B and C sub-lattices) is \( \frac{a^2}{2\pi} \), so neither diffusing nor localized phases are stable over the range:

\[ \frac{8\pi}{9} < a^2 < 2\pi. \]

(4.20)

In this entire range we expect a non-trivial fixed point. This can be studied using an \( \epsilon \) expansion for \( a^2 = 8\pi/9 + \epsilon \) or \( a^2 = 2\pi - \epsilon \) for \( 0 < \epsilon \ll 1 \). It can be solved exactly at \( a^2 = \frac{4\pi}{3} \) as we show in the next section. In general, however, it remains an unsolved problem.

It is also instructive to calculate the degeneracy \( (g) \) factors at the diffusing \((v_A = 0)\) and “localized on A” phases. We may do this using the results in Sec. IIB for compact bosons, taking into account the connection of the QBM problem with compact bosons explained in Sec. III and earlier in this section. We assume that the space on which the particle moves is periodic with

\[ \vec{Q} \leftrightarrow \vec{Q} + N\vec{a}_i, \]

(4.21)

where the primitive vectors \( \vec{a}_i \) are given by Eq. (4.6). The unit cell area for the fine lattice with spacing \( a \) is:

\[ V_0(\Gamma^*) = \sqrt{3}a^2/2. \]

(4.22)

(Note that this is twice the area of a triangle on the lattice; each triangle contains 1/2 point or three points, each of which is shared with six triangles.) Thus the degeneracy factor for the D fixed point where the particle is localized on any of A B or C sublattices is:

\[ g_{3D} = \frac{N\sqrt{2\pi}}{3^{1/4}a}. \]

(4.23)

This is the result for the compactified boson with dual lattice \( \Gamma^* \) from Eq. (2.57) multiplied by the factor of \( N \). The degeneracy for the state where the particle is localized on the A sub-lattice only is:

\[ g_D = g_{3D}/3, \]

(4.24)

and that for the state where the particle is localized on the B or C sub-lattices is:

\[ g_{DD} = (2/3)g_{3D}. \]

(4.25)

These factors of 1/3 and 2/3 can be understood as simply corresponding to the number of sites on which the particle can be localized. Alternatively, we may think of the D fixed point as being the same as 3D
except that the lattice constant \( a \) is increased by a factor of \( \sqrt{3} \) and the factor \( \mathcal{N} \), defined in Eq. (4.5), is decreased by a factor of \( \sqrt{3} \) in order that the area on which the particle moves remains fixed. The degeneracy factor for the \( N \) fixed point is:

\[
\mathcal{g}_N = \frac{\mathcal{N}a^3}{2\sqrt{2}\pi}.
\]  

(4.26)

This is obtained from Eq. (2.52) for a compact boson, multiplied by the factor of \( \mathcal{N} \). Note that the total degeneracy for the phase where the particle is localized on \( A \), \( B \) or \( C \) sub-lattices is:

\[
\mathcal{g}_\text{loc} = \mathcal{g}_D \mathcal{g}_N = \frac{\mathcal{N}^2}{2}.
\]  

(4.27)

Following the discussion in Sec. III for the one-dimensional case, we see that this is simply the number of lattice sites times the degeneracy for the decoupled oscillator both. (The factor of \( 1/\sqrt{2} \) that occurs in the one-dimensional case gets squared in two dimensions.) We also see that the ratio between fully localized and freely diffusing fixed points is:

\[
\frac{\mathcal{g}_D}{\mathcal{g}_N} = \frac{4\pi}{3\sqrt{3}a^2}.
\]  

(4.28)

Thus the flow from fully localized to freely diffusing is consistent with the \( g \)-theorem only for:

\[
a^2 < \frac{4\pi}{\sqrt{3}}.
\]  

(4.29)

On the other hand the ratio of \( g \)-factors between the “localized on \( A \)” and freely diffusing fixed points is:

\[
\frac{\mathcal{g}_D}{\mathcal{g}_N} = \frac{4\pi}{3\sqrt{3}a^2},
\]  

so the flow between freely diffusing and localized on \( A \) is consistent with the \( g \)-theorem for:

\[
a^2 > \frac{4\pi}{3\sqrt{3}}.
\]  

(4.31)

It is instructive to consider a more general model with even less symmetry than that of Eq. (4.13). The model of Eq. (4.13) in which we have changed the energy on the \( A \) sites, relative to \( B \) and \( C \) sites still has a 3m point group symmetry when we consider transformations holding fixed a lattice point (\( A \), \( B \) or \( C \)). The less symmetric model that we now consider has no rotational symmetry whatsoever. This model is obtained from Eq. (4.13) (weak corrugation formulation) by letting the coefficients of the 3 terms be different:

\[
H_W = H_0 - v_1 \cos \sqrt{2\pi} \left\{ \frac{2}{3} \phi_1 - v_2 \cos \sqrt{2\pi} \left( \sqrt{\frac{1}{6}} \phi_1 + \sqrt{\frac{1}{2}} \phi_2 \right) - v_3 \cos \sqrt{2\pi} \left( \sqrt{\frac{1}{6}} \phi_1 - \sqrt{\frac{1}{2}} \phi_2 \right) \right\}.
\]  

(4.32)

The behavior of this model becomes obvious in the case \( v_1 \neq 0, v_2 = v_3 = 0 \). In this case the (relevant) potential only depends on the co-ordinate \( Q_1 \). Thus we expect the motion of the particle to be localized in the 1-direction but freely diffusing in the 2-direction. Now there is only 1 minimum per unit cell for either sign of \( v_1 \). We may think of the particle as diffusing freely along vertical lines on the lattice. To check the stability of this fixed point we should consider the hopping process between vertical lines. Since the horizontal distance between points on the \( A \) sub-lattice is 1/2 times the lattice spacing (of the \( A \) sub-lattice) we expect this hopping term to have dimension 1/2 [smaller by 1/4 than the dimension 2 hopping in Eq. (4.14)]. Thus this fixed point is unstable.
V. CONSTRUCTION OF BOUNDARY STATES VIA A CONFORMAL EMBEDDING

A. Conformal embedding

Let us consider the QBM on the triangular lattice discussed in Section IV, considering the weak corrugation Hamiltonian of Eq. (4.17). Because the symmetry among the sublattices A, B and C is broken by the applied potential, the generic translation symmetry of the model is the invariance under translations which do not interchange the sublattices. We can impose this symmetry on the system, so that we can use the compactified formulation. The compactification lattice \( \Gamma^* \) is thus a triangular lattice of nearest neighbor distance \( \sqrt{3}a \). Namely, we introduce the compactification

\[
\frac{\phantom{i}^*}{\phi} \sim \phi + \Gamma^*,
\]

where \( \Gamma^* \) is a triangular lattice generated by

\[
(0, \sqrt{3}a), \quad (3a/2, \sqrt{3}a/2).
\]

Its dual \( \Gamma \) is generated by \((2/3a, 0)\) and \((1/3a, 1/\sqrt{3}a)\). The allowed vertex operators made from \( \frac{\phantom{i}^*}{\phi} \) are of the form \( \exp (iv \cdot \frac{\phantom{i}^*}{\phi}) \), where \( v \) is an element of the lattice \( \Gamma^* \). As mentioned in Sec. IIIC, the operator content upon compactification corresponds to the possible boundary perturbations given the imposed symmetry. Note that the Hamiltonian of Eq. (4.17) is the most general one, up to less relevant operators, respecting the 3m point group symmetry.

For \( v_A = 0 \), the corresponding boundary condition is Neumann on \( \frac{\phantom{i}^*}{\phi} \), while it is Dirichlet on \( \frac{\phantom{i}^*}{\phi} \) for \( v_A \to \infty \). However, there are other nontrivial boundary conditions which are neither Dirichlet nor Neumann, as discussed in Sec. IV. They correspond to nontrivial phases in QBM. Our problem, then, is to construct and to classify the possible boundary states of the \( c = 2 \) CFT. Possible boundary conditions may be restricted by higher symmetries, such as current conservation at the boundary. However, in the case of QBM, apparently the only requirement (at the RG fixed point) is the conformal invariance on the boundary and Cardy’s consistency conditions. Even if there are extra symmetries in the bulk, they are not necessarily respected at the boundary. This allows some extra boundary conditions which would be forbidden if the higher symmetry is imposed on the boundary.

Unfortunately, at present we have no systematic understanding of the boundary states in an irrational CFT, which has an infinite number of primary fields. While we do not know the solution to this fundamental problem, in this paper we analyze some nontrivial boundary conditions for the special compactification radius \( a^2 = 4\pi/3 \). In this case, the irrational \( c = 2 \) CFT admits a conformal embedding in terms of rational CFTs. For such a rational CFT, we can invoke the fusion construction of boundary states, to find non-trivial boundary states for the \( c = 2 \) CFT.

This value of \( a^2 \) is actually the point where Yi and Kane obtained the exact value of the mobility at a nontrivial fixed point by a slightly different approach. Our approach leads to a somewhat more systematic description of the boundary states, and several additional results including novel boundary states.

The present \( c = 2 \) boundary CFT admits a conformal embedding \( c = 2 = 1/2 + 7/10 + 4/5 = (\text{Ising}) + (\text{Tricritical Ising}) + (\text{Potts}) \). Namely, the partition functions on the strip can be expressed (in the open string channel), as

\[
Z_{DD}(g) = (x^4_0 x^T_0 + x^4_{1/2} x^T_{3/2})(x^P_0 + x^P_3) + (x^4_0 x^T_{3/5} + x^4_{1/2} x^T_{11/10})(x^P_{2/5} + x^P_{7/5}),
\]

\[
Z_{NN}(g) = (x^4_0 x^T_0 + x^4_{1/2} x^T_{3/2})(x^P_0 + x^P_3 + 2x^P_{2/3}) + (x^4_0 x^T_{3/5} + x^4_{1/2} x^T_{11/10})(2x^P_{1/15} + x^P_{2/5} + x^P_{7/5}),
\]

where \( x^4_{h,T,P} \) is the Virasoro character of the weight \( h \) for Ising, Tricritical Ising and Potts model, respectively. (The argument \( q \) is omitted.) The two amplitudes have the same factors for Ising and
Tricritical Ising sector; the only difference is in the Potts sector. We emphasize that these are the partition functions for compact bosons compactified on the lattice, $\Gamma^*$ of Eq. (5.2). They do not quite correspond to the partition functions occurring in the QBM problem. Nonetheless, as explained in Sec IIIC, these are the partition functions which are related to the boundary operator content. We remark that, since the 0 and 3 and also 2/5 and 7/5 partition functions only occur added together in Eqs. (5.3) and (5.4), $Z_{DD}$ and $Z_{NN}$ can be expressed in terms of the W-characters of the 3-state Potts model, invariant under W-symmetry.

In fact, we have no proof of these identities. However, we have verified, using MATHEMATICA, that the first several (40 to 100) terms in the series expansion agree exactly. In the following, we assume this conformal embedding. Because of several additional pieces of evidences, we believe it is indeed correct. Namely, as we will show later, we can construct several reasonable boundary states based on the embedding.

We also note that, the torus partition function of the $c = 2$ theory, even at this particular compactification, apparently can not be written in terms of Ising, tricritical Ising and Potts models. We are in a somewhat strange situation that the boundary CFT admits the conformal embedding while the bulk CFT does not. Although this is difficult to understand, it does not pose a real problem in our construction. The fusion construction is a multiplication by constant factors of each Ishibashi state appearing in the reference boundary state, Eq. (2.27). These constant factors are given by the matrix elements of the matrix of modular transformations. Thus, by construction, the obtained boundary states is a linear combination of the Ishibashi states. They are well contained in the Hilbert space of the theory, as long as the original reference state is a physical boundary state.

In fact, there is an alternative conformal embedding, also involving the Potts model, in which the Ising and tri-critical Ising models are replaced by the $Z_3^{(5)}$ conformal field theory (A detailed review of $Z_3^{(p)}$ theory is given in Ref. [7]). This may be regarded as a sort of tri-critical 3-state Potts model. Again there is a W-algebra and only W characters occur in the conformal embedding. This alternative conformal embedding is obtained from Eqs. (5.3) and (5.4) by replacing the Ising and tricritical Ising characters by the $Z_3^{(5)}$ characters. Here we use the (conjectured) identity

\[
\chi^I_0 \chi^T_0 + \chi^I_{1/2} \chi^T_{3/2} = \chi^5_0 + 2\chi^5_2, \tag{5.5}
\]

\[
\chi^I_0 \chi^T_{3/5} + \chi^I_{1/2} \chi^T_{1/10} = 2\chi^5_{3/5} + \chi^5_{8/5}. \tag{5.6}
\]

Here $\chi^5$ refers to characters in the $Z_3^{5}$ theory. We note that, again, this identity is only verified up to finite order with MATHEMATICA. The equivalence between the $Z_3^{(5)}$ model and the product of Ising and tri-critical Ising models is only a partial equivalence. Certain sums of conformal characters in the two theories are presumably equal but there is not a complete equivalence of all characters.

Further insight into the conformal embeddings can be obtained by rewriting the interaction term in Eq. (4.17) in a more symmetric way, first by reexpressing everything in terms of chiral components, using $\tilde{\phi}_L = \phi_R$ at the fixed point, and by introducing the new chiral fields $\Phi_j$. 

40
\[ \Phi_1 = \frac{\sqrt{5}}{2} \Phi_{1L} \]
\[ \Phi_2 = -\frac{4\pi}{30} \left( \phi_{1L} + \sqrt{3}\phi_{2L} \right) \]
\[ \Phi_3 = -\frac{4\pi}{30} \left( \phi_{1L} - \sqrt{3}\phi_{2L} \right) \]

with propagators
\[ \langle \Phi_i(z)\Phi_i(w) \rangle = -\frac{16\pi}{5} \ln(z-w) \]
\[ \langle \Phi_i(z)\Phi_j(w) \rangle = \frac{8\pi}{15} \ln(z-w), \quad i \neq j. \]  

(5.7)

The (weak corrugation) interaction term in Eq. (4.17) now reads, recalling \( a^2 = 4\pi/3 \),
\[ H_{int} = -v_A (\cos \Phi_1 + \cos \Phi_2 + \cos \Phi_3) = -(v_A/2)(\Psi_1 + \Psi_1^\dagger) \]

where:
\[ \Psi_1 = \sum_{k=1}^{3} e^{i\Phi_k}. \]  

(5.9)

(5.10)

\( \Psi_1 \) corresponds to the fundamental \( Z_3 \) parafermion field of the Potts model. (This generalizes the representation of a Majorana fermion-\( Z_2 \) parafermion- as \( \cos \Phi \).)

The relation between the \( c = 2 \) free bosonic theory and the Potts model is not straightforward. It can be formulated in terms of cosets beginning with \( SU(3)_1 \times SU(3)_1 \). This conformal field theory arises from bosonization of critical \( SU(3) \) “spin” chains (the two factors of \( SU(3)_1 \) arising from left and right movers) and the associated boundary critical phenomena is closely related to that of the triangular lattice QBM problem. It will be discussed in a later paper. It is natural to associate the \( SU(3)_2 \) CFT with the diagonal \( SU(3) \) symmetry of this model. The remaining coset, \( SU(3)_1 \times SU(3)_1 / SU(3)_2 \), with \( c = 4/5 \), gives the 3-state Potts model. This is also equivalent to \( SU(2)^3/U(1) \). Yi and Kane first discussed the non-trivial critical behavior of the the triangular lattice QBM problem using results from the 3-channel \( SU(2) \) Kondo problem, which is, in turn, related to the \( SU(2)^3 \) CFT. The second factor in the conformal embedding, \( SU(3)_2 \) can be further factorized into two \( U(1) \) CFT’s ([the maximal abelian sub-algebra of \( SU(3) \]) and a \( c=6/5 \) coset, \( SU(3)_2 / U(1)^2 \) which is naturally regarded as the \( Z_3^2 \) CFT. Since the \( SU(3)_1 \) CFT, with \( c=2 \), is actually equivalent to two free bosons, this conformal embedding essentially expresses 2 free bosons in terms of the Potts model and the \( Z_3^2 \) CFT. One can certainly write the stress energy tensor for 2 free bosons as \( T = T_1 + T_2 \) where:
\[ T_1 = \frac{1}{5} \left[ \frac{1}{2} \sum_{j=1}^{3} (\partial \Phi_j)^2 + \sum_{j<k} e^{i(\Phi_j - \Phi_k)} \right] \]
\[ T_2 = \frac{1}{5} \left[ \frac{3}{4} \sum_{j=1}^{3} (\partial \Phi_j)^2 - \sum_{j<k} e^{i(\Phi_j - \Phi_k)} \right] \]

(5.11)

such that the short distance expansion of \( T_1 \) with \( T_2 \) is trivial. Here \( \partial \) denotes \( \partial/\partial z \) where \( z = \tau + ix \) and \( \tau \) is imaginary time. \( T_1 \) is a stress energy tensor with central charge \( c_1 = \frac{4}{5} \) of the coset \( SU(3)_1 \times SU(3)_1 / SU(3)_2 \) or \( SU(2)_3 / U(1) \), and similarly for \( T_2 \) with \( c_2 = \frac{6}{5} \) of the \( SU(3)_2 / U(1)^2 \) coset. The parafermion \( \Psi_1 \) turns out to be primary both for \( T_1 \) and \( T_1 + T_2 \), but it is not the case for most other operators of interest. Some of the most crucial operators in the Potts model, like the field with \( \Phi_1 \), do not have any known representation in the two boson theory, neither as vertex operators nor generalized twist fields.

It follows that, strictly speaking, we do not have a conformal embedding. As mentioned before, the situation is reminiscent of what happens at the Ising square point of the \( c = 1 \) theory: while \( c = 1 = \frac{1}{2} + \frac{1}{2} \), and a decomposition of the \( c = 2 \) stress energy tensor analogous to (5.11) exists, the periodic boson
theory cannot be considered as the product of two Ising theories; only the $\mathbb{Z}_2$ orbifold can. In the absence of an understanding of the orbifold, one would nevertheless observe that, if the torus partition function of the periodic boson cannot be expressed in terms of Ising model characters, boundary partition functions can:

\[
Z_{NN} = (\chi_0^f + \chi_{1/2}^f)^2 \\
Z_{DD} = \chi_0^{f^2} + \chi_{1/2}^{f^2} \\
Z_{ND} = \chi_{1/16}^f (\chi_0^f + \chi_{1/2}^f).
\]  

(5.12)

Again, the existence of a “boundary embedding” in our $c = 2$ case suggests that the situation is similar, and that there is some sort of orbifold version of the bosonic theory for which there would be a genuine conformal embedding. Nevertheless, since the perturbation in (4.17) makes sense as a pure Potts operator, it is natural to expect it to induce a flow purely in the Potts sector of the theory. Thus we come to the important conclusion that the interaction term in Eq. (4.17) is purely a Potts operator. In fact, it is precisely the same boundary operator which corresponds to a boundary magnetic field, $h\delta_{i,1}$ in the Potts model where $\sigma = 1, 2, 3$ labels the Potts variable. The case $\nu_A < 0$ then corresponds to $h < 0$, inducing a flow from free to fixed boundary conditions in the Potts model, while the case $\nu_A > 0$ corresponds to $h > 0$, and should instead induce a flow from free to mixed. This verifies the handwaving arguments of Sec. IV relating QBM to the Potts model. Note however that this connection is only established for the special choice of lattice spacing $a^2 = 4\pi/3$.

We also consider the more general QBM model of Eq. (4.32) in which the remaining 3m point group symmetry is broken. Upon using the conformal embedding, we now expect that products of Potts with $Z_{3}^{(5)}$ (or Ising $\times$ tri-critical Ising) operators will appear in the Hamiltonian.

In the remainder of this section we use the fusion technique to extract information about the boundary critical points of the QBM model. By considering fusion in the Potts sector we can study all the fixed points that occur in the 3m symmetric model. We expect that we have obtained all fixed points with this symmetry yielding the phase diagram of Fig. 3. On the other hand, by considering fusion in the other sectors we obtain a collection of additional fixed points whose properties we understand in less detail. In Sec. VI we discuss the integrability of some of the RG flows between fixed points and thereby confirm the corresponding ratios of $g$-factors.

**B. Fusion in the Potts sector**

The Potts sector in the first term of the amplitudes $\chi_0^p + \chi_3^p$ and $\chi_0^p + \chi_3^p + 2\chi_2^p$ are exactly the fixed-fixed and free-free amplitudes in the Potts model. It suggests that the Dirichlet and the Neumann boundary conditions correspond to the free and fixed boundary conditions of the Potts model, respectively.

In the (pure) Potts model, we can construct the free boundary state from the fixed boundary state by the fusion with the weight-1/8 primary operator $O_{44}$ (which is absent in the bulk spectrum of the Potts.) In fact, the fusion with the same operator in the Potts sector in our conformal embedding gives the Neumann b.c. from the Dirichlet b.c. The Potts sector in the first term of (5.3) is the free-free amplitude of the Potts model, and is thus transformed to the fixed-fixed amplitude by double fusion:

\[
\chi_0^p + \chi_3^p \rightarrow \chi_{1/8}^p + \chi_{3/8}^p \rightarrow \chi_0^p + \chi_3^p + 2\chi_2^p.
\]

(5.13)

where each $\rightarrow$ means the fusion with 1/8. On the other hand, the Potts sector of the second term of (5.3) is transformed as:

\[
\chi_{2/5}^p + \chi_{7/5}^p \rightarrow \chi_{1/40}^p + \chi_{21/40}^p \rightarrow 2\chi_{1/15}^p + \chi_{2/5}^p + \chi_{7/5}^p.
\]

(5.14)

Thus, by double fusion with 1/8 in the Potts sector, the Neumann-Neumann amplitude (5.4) is obtained from the Dirichlet-Dirichlet amplitude (5.3).

In this fusion construction of the Neumann boundary state, the ratio of the ground-state degeneracy $g_N/g_D$ is given by the ratio of modular $S$-matrix elements of the Potts model. Thus it automatically
agrees with the ratio of Potts degeneracy \( g_{\text{free}} / g_{\text{fixed}} \). The correspondence between Dirichlet/Neumann boundary conditions in the \( c = 2 \) theory and fixed/free boundary conditions in the Potts model is also consistent with the integrable field theory approach taken in Section VI.

In a similar way, we can construct another boundary state for the \( c = 2 \) theory, which corresponds to the mixed boundary condition of the (pure) Potts model. In the Potts model, the mixed boundary condition is obtained from the fixed boundary condition by fusion with the 2/5-operator (\( c \)). Thus we attempt a fusion with the same operator in the Potts sector to construct a boundary state from the Dirichlet boundary state. The Potts sector of the first term in (5.3) is transformed as:

\[
\chi_0^P + \chi_3^P \rightarrow \chi_0^P + \chi_3^P + \chi_2^P + \chi_7^P,
\]

which is the same as the transformation of fixed-fixed amplitude to mixed-mixed one in the Potts model. That of the second term is transformed as:

\[
\chi_2^P + \chi_7^P \rightarrow \chi_0^P + \chi_3^P + \chi_2^P + \chi_7^P \rightarrow \chi_0^P + \chi_3^P + \chi_2^P + 2\chi_7^P.
\]

Thus, the amplitude for the nontrivial boundary state (Y-state) is given by

\[
Z_{YY}(q) = (\chi_0^P + \chi_1^P/2\chi_3^P/2)(\chi_0^P + \chi_3^P + \chi_2^P + \chi_7^P) + (\chi_0^P + \chi_3^P + \chi_2^P + 2\chi_7^P),
\]

where the argument \( q \) is omitted. By the modular transformation to the “closed string” channel, it is expressed as

\[
Z_{YY}(\tilde{q}) = \frac{3 + \sqrt{3}}{4\sqrt{3}}[(\chi_0^P + \chi_1^P/2\chi_3^P/2)\chi_0^P + \chi_3^P + \chi_2^P + 2\chi_7^P) + \frac{3 - \sqrt{3}}{4\sqrt{3}}[(\chi_0^P + \chi_1^P/2\chi_3^P/2)\chi_0^P + \chi_3^P + \chi_2^P + 2\chi_7^P),
\]

where the omitted argument is now \( \tilde{q} \). The ground-state degeneracy for this state is \( g_Y = \sqrt{(3 + \sqrt{3})/(4\sqrt{3})} = 2\cos(\pi/5)/\sqrt{2\sqrt{3}} \). Again, by construction, the ratio of the degeneracy is equal to corresponding one in the pure Potts model:

\[
\frac{g_N}{g_Y} = \frac{g_{\text{free}}}{g_{\text{mixed}}} = \frac{\sqrt{3}}{2\cos \pi/5}.
\]

It turns out that this Y-state is identical to the nontrivial fixed point found by Yi and Kao using a different mapping. This will be confirmed by the calculation of mobility, and also by a physical consideration on the relation to the Potts model.

It has been known that the Potts model admits fixed, mixed and free boundary conditions as conformally invariant boundary conditions. Recently, a new boundary condition was found in the Potts model and related to the mixed one by the duality transformation. The new boundary state is obtained by fusion with the operator \( O_{22} \) of dimension 1/40 from the fixed boundary state. Applying the fusion with the same operator to the Dirichlet-Dirichlet amplitude, we obtain a new boundary state, which we label \( W \), for the present problem:

\[
Z_{WW}(q) = (\chi_0^P + \chi_1^P/2\chi_3^P/2)(\chi_0^P + \chi_3^P + \chi_2^P + 2\chi_7^P + 2\chi_1^P) + (\chi_0^P + \chi_1^P/2\chi_3^P/2)(\chi_0^P + \chi_3^P + \chi_2^P + 2\chi_7^P + 4\chi_1^P). \tag{5.20}
\]

This is another nontrivial boundary state, which corresponds to the Potts “new” boundary state. The ground-state degeneracy of the \( W \)-state is given by \( g_W = 2\cos(\pi/5)\sqrt{3}/2 \).

Thus, there is a corresponding boundary state in the present problem for each boundary state in the Potts model. The ratios of \( g \)-factors between the boundary states are identical to the corresponding ratios.
in the Potts model, by construction. Moreover, more boundary states can be constructed in the Potts model by forming superposition of several boundary states. Such boundary states contain dimension-zero boundary operator(s) other than identity. Although they are often unphysical due to their instability, they can be relevant for some physical situations with a first-order transition.

In the present problem, the potential minima form a triangular lattice isomorphic to $\Gamma$ for $v_A > 0$. However, for negative $v_A$, the potential minima form a hexagonal lattice, which has two points within each unit cell of $\Gamma$. In the limit $v_A \to -\infty$, the corresponding boundary condition is Dirichlet, but the boundary field is pinned to one of two inequivalent minima. This can be represented by “Double Dirichlet” boundary state, which is the superposition of two Dirichlet boundary states for the two minima. Considering the correspondence Dirichlet ↔ fixed and Neumann ↔ free, it would be natural if the double Dirichlet corresponds to the sum of two fixed, say $A$ and $B$ boundary states in the Potts model. Indeed, the double Dirichlet amplitude can be expressed as follows:

$$Z_{DD,DD}(q) = (\chi^T_0 \chi_0^T + \chi_{1/2}^T \chi_{3/2}^T)(2\chi_0^P + 2\chi_3^P + 2\chi_{2/3}^P) + (\chi^T_0 \chi_{3/5} + \chi_{1/2}^T \chi_{1/10})(2\chi_{1/15}^P + 2\chi_{2/5}^P), \quad (5.21)$$

where the Potts part of the first term is the amplitude for the superposed state $A + B$.

To summarize, based on the conformal embedding, we find several boundary states for the $c = 2$ problem constructed by fusion in Potts sector as follows. There is a $c = 2$ boundary state corresponding to each Potts boundary state.

| $c = 2$ model | Potts model $g$-factor (for $c = 2$) |
|---------------|-------------------------------------|
| Dirichlet     | Fixed ($A$)                         | $1/\sqrt{2\sqrt{3}}$ |
| Neumann       | Free                                | $\sqrt{3}/2$ |
| Y(i-Kane)     | Mixed ($AB$)                        | $2 \cos(\pi/5)/\sqrt{2\sqrt{3}}$ |
| Double Dirichlet A + B |                         | $\sqrt{2/\sqrt{3}}$ |
| W             | New                                 | $2 \cos(\pi/5)\sqrt{3/2}$ |

C. Fusion in Ising/Tricritical Ising sector

It is also possible to construct other boundary states by fusion in Ising or tricritical-Ising sectors. The $U$-state is obtained from $D$-state by fusion with either $1/16$ operator of Ising or $7/16$ operator of Tricritical Ising. Its diagonal partition function is given by

$$Z_{UU}(q) = (\chi^T_0 + \chi^T_{1/2})(\chi^P_0 + \chi^P_{3/2})(\chi^P_0 + \chi^P_{3/5} + \chi^P_{3/5})(\chi^P_{1/10} + \chi^P_{3/15} + \chi^P_{3/5})], \quad (5.22)$$

with the degeneracy $g_U = 1/3^{1/4}$. Similarly, $S$-state is obtained by fusion with the same operators from the $N$-state.

$$Z_{SS}(q) = (\chi^T_0 + \chi^T_{1/2})(\chi^P_0 + \chi^P_{3/2})(\chi^P_0 + \chi^P_{3/5} + 2\chi^P_{3/3} + \chi^P_{3/3})(\chi^P_{1/10} + \chi^P_{3/15} + \chi^P_{3/5}], \quad (5.23)$$

with the degeneracy $g_S = 3^{1/4}$. It turns out that these states represent mixtures of Dirichlet and Neumann: imposing Dirichlet in one component and Neumann in the other.

On the other hand, another boundary state, the $T$-state is obtained from $D$-state by fusion with $3/80$ operator of Tricritical Ising. The diagonal amplitude is

$$Z_{TT}(q) = (\chi^T_0 + \chi^T_{1/2})(\chi^P_0 + \chi^P_{1/10} + \chi^P_{3/5} + \chi^P_{3/3})(\chi^P_1 + (\chi^T_0 + 2\chi^T_{1/10} + 2\chi^T_{3/5} + \chi^T_{3/3})\chi^P_1], \quad (5.24)$$

with the degeneracy $g_T = (7 + 3\sqrt{5}/6)^{1/4}$. Yet another state, $R$, is found by fusion with the same operator from $N$-state.
which has the degeneracy \( g_R = [3(7 + 3\sqrt{5})]/2 \)^{1/4}. The calculation of the mobility, which will be given later, suggests that they are mixtures of \( Y \) and \( W \), namely taking the dual of one component of the bosons in \( Y \) or \( W \) boundary states.

**D. Fusion in the \( Z_3^{(5)} \) sector**

We can apply similar lines of arguments as in the previous section to the \( Z_3^{(5)} \) sector in the conformal embedding of the \( c = 2 \) model. As we will see, we obtain new boundary states as well as those found using the Potts sector. Starting from the Dirichlet boundary state, we have tried fusion with all primary fields in the \( Z_3^{(5)} \) model. As a result, we found new boundary states, besides the same ones obtained by Potts fusion. For simplicity, here we focus on the boundary states with single dimension-0 boundary operator.

A new boundary state \( F \) is obtained by fusion with \( Z_3^{(5)} \) operators of dimension 1/9, 7/9 or 13/9 from the \( D \) boundary state:

\[
Z_{FF} = \chi_I(\chi_0^5 + 2\chi_2^5 + 3\chi_{1/2}^5) + \chi_5^5(3\chi_{1/3}^5 + \chi_{1/2}^5 + \chi_{1/10}^5).
\]

This turns out to represent free diffusion under a magnetic field\(^{[2]} \). While a detailed discussion of this identification will be given in a later publication, we will see a signature of the magnetic field, namely the Hall effect, in the next subsection.

Another one, \( X \) is obtained from the \( D \) by fusion with an operator of dimension 2/45, 17/45 or 32/45. The amplitude in the open string channel is given by

\[
Z_{XX}(q) = \chi_I(\chi_0^5 + 2\chi_{3/5}^5 + \chi_{8/5}^5 + 2\chi_2^5 + 3\chi_{1/2}^5 + 3\chi_{1/10}^5) + \chi_5^5(3\chi_{3/5}^5 + 2\chi_{8/5}^5 + 2\chi_2^5 + 3\chi_{1/2}^5 + 6\chi_{1/10}^5).
\]

By construction, the ground-state degeneracy \( g_X \) satisfies

\[
\frac{g_X}{g_D} = \frac{S_c^0}{S_0^0}
\]

where \( S \) is the matrix modular transformations of the \( Z_3^{(5)} \) model and 0 and \( c \) represents the identity operator and the operator used in the fusion. \( S_0^0 = \sqrt{(5 - \sqrt{5})}/6 \) and \( S_c^0 = \sqrt{(5 + \sqrt{5})}/10 \) gives \( g_X = 4\cos(\pi/5)g_D = 2g_Y \). Fusion with an operator of dimension 3/5 or 8/5 gives the same \( Y \)-state previously obtained by fusion in Potts sector.

On the other hand, starting from the Neumann boundary state, we obtain two additional boundary states \( V \) and \( Z \) by fusion in the \( Z_3^{(5)} \) sector. No other states are found by fusion in \( Z_3^{(5)} \) on other known states. To summarize, there are four new boundary states \( F, X, V \) and \( Z \) obtained by fusion in \( Z_3^{(5)} \) sector. As we will see later, these four states exhibits finite Hall mobility, and are presumably related to RG fixed points of QBM under a magnetic field.

**E. Mobility and nature of the boundary states**

Now let us calculate the mobility in the QBM problem. It helps us to identify the physical natures of the constructed boundary states. The mobility is defined as the linear response coefficient of the velocity.
of the particle to the external force. The external $\vec{F}$ acting on the particle is represented by the additional term to the Hamiltonian, $-\vec{F} \cdot \vec{Q}$. The linear response of the velocity of the particle to this force is given by the Kubo formula

$$\langle \frac{dQ^a}{dt} \rangle = -F^b \int dt \langle \frac{dQ^a(0)}{dt} \rangle \langle \frac{dQ^b}{dt} \rangle (t). \quad (5.28)$$

Thus the static mobility tensor can be expressed as

$$\mu_{ab} = \lim_{\omega \to 0} \frac{\pi}{\omega} \left( \frac{dQ^a}{dt}(\omega) \frac{dQ^b}{dt}(-\omega) \right). \quad (5.29)$$

Using the weak corrugation formulation we can identify $\vec{Q}$ with $\vec{\phi}(0)$, and the velocity operator $d\vec{Q}/dt$ with $d\vec{\phi}(0)/dt$, which corresponds to the current operators. Thus the mobility is deduced from the two-point boundary correlation function of the current $j_a = J_a + \bar{J}_a \ (a = x,y)$, where $J_a$ and $\bar{J}_a$ are $a$-component of the holomorphic and antiholomorphic currents. The mobility corresponds to the conductance in the problem of a tunneling between quantum wires, which can be also mapped to essentially the same boundary CFT problem. Note that we now have a two-component boson, and thus the current has two components.

The time dependence of the correlation function is basically insensitive to the boundary, although some coefficients do depend on the boundary state. As in Ref. \[42\], the $J_a, \bar{J}_a$ and $J_b, \bar{J}_b$ correlations are insensitive to the boundary conditions, and can be fixed by an appropriate normalization of the current operator. On the other hand, while the $J_a, \bar{J}_a$ correlation obeys always the same power law, the amplitude depends on the boundary condition. The amplitude $A_{ab}$ is determined by the boundary state as

$$A_{ab} = \frac{\langle 0|J^a_{a,1}\bar{J}^b_{b,1}|B\rangle}{\langle B|B\rangle}, \quad (5.30)$$

where $|0\rangle$ is the ground state, $|B\rangle$ is the boundary state under consideration, and $J^a_{a,1}$ ($\bar{J}^b_{b,1}$) is the level-1 annihilation part of the holomorphic (antiholomorphic) current. By a suitable normalization of the current operator, the mobility $\mu_{ab}$ can be written as

$$\mu_{ab} = \frac{\delta_{ab} + A_{ab}}{2}. \quad (5.31)$$

Since the N boundary condition corresponds to the delocalized phase, we normalize the mobility in the N phase to be unity, fixing $A^N_{ab} = \delta_{ab}$ for the Neumann boundary condition. On the other hand, the D boundary condition corresponds to the localized phase and the mobility should vanish, thus $A^D_{ab} = -\delta_{ab}$. In our conformal embedding approach, the two level-1 states are identified as

$$J_{x,-1}|0\rangle = |0\rangle_T|3/5\rangle_T|2/5\rangle_P, \quad (5.32)$$
$$J_{y,-1}|0\rangle = |1/2\rangle_T|1/10\rangle_T|2/5\rangle_P, \quad (5.33)$$

since they are the only two combination of Virasoro primaries with weight 1.

The coefficients $A_{ab}$ of these states in the boundary state are changed by fusion. The changes are given by the modular $S$-matrix elements. Thus we can calculate the coefficients and consequently the mobility of the new boundary state constructed by fusion. Since the both states contains the same Potts primary state $|2/5\rangle_P$, fusion in the Potts sector gives $A_{ab} = A\delta_{ab}$. Thus the mobility is also diagonal and isotropic: $\mu_{ab} = \mu\delta_{ab}$.

The $N$-state is obtained by fusion with the operator of dimension 1/8 in the Potts sector from the $D$-state, and the amplitudes is given by

$$\frac{A^N}{A^D} = \frac{S^2_{1/8}}{S^2_0} \frac{S^0}{S^0_{1/8}} = -1, \quad (5.34)$$

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where $S$ is the matrix of modular transformations for the Potts model. This is indeed consistent with the physical consideration $\mu^D = 0$ and $\mu^N = 1$.

Similarly, the Y-state is obtained by fusion with 2/5-operator in the Potts sector and thus the amplitude for the Y-state $A_Y$ is given by

$$ A_Y = \frac{S^{2/5}_{2/5}}{S^0_{0} S^0_{2/5}} = -\frac{3 - \sqrt{5}}{2}. \quad (5.35) $$

The mobility $\mu$ is then

$$ \mu^Y = \frac{(5 - \sqrt{5})}{4} = 2 \sin^2 \frac{\pi}{5}. \quad (5.36) $$

This agrees with the result obtained for the nontrivial fixed point by Yi and Kane, implying that our Y-state is identical to their fixed point obtained by a different mapping.

The W-state is obtained by fusion with 1/40-operator in the Potts sector:

$$ A_W = \frac{S^{2/5}_{1/40}}{S^0_{0} S^0_{1/40}} = \frac{3 - \sqrt{5}}{2}. \quad (5.37) $$

Thus $\mu^W = (\sqrt{5} - 1)/4$. This is equal to $1 - \mu^Y$; this is related to the duality between the Y-state and W-state.

Next let us consider the fusion in the Ising/tricritical Ising sector. Since the two components of the currents are related to the Ising/tricritical Ising sector in an asymmetric manner, in general the amplitude $A_{ab}$ is asymmetric ($A_{xx} \neq A_{yy}$) although it is diagonal ($A_{xy} = 0$). For example, the U-state is obtained from D-state by fusion with the 1/16 primary in the Ising sector. Thus, the coefficients are given by

$$ A_{xx}^U = \frac{S^0_{1/16}}{S^0_{0} S^0_{1/16}} A_{xx}^D = -1, \quad (5.38) $$

$$ A_{yy}^U = \frac{S^{1/2}_{1/16}}{S^0_{0} S^0_{1/16}} A_{yy}^D = 1, \quad (5.39) $$

where $S$ represents the matrix of modular transformations of the Ising model. As a result, the mobility in the U-state is anisotropic: $\mu^U_{xx} = 0, \mu^U_{yy} = 1$ ($\mu^U_{yx} = 0$). This represents perfect mobility in $y$-direction and complete localization in the orthogonal $x$-direction. This would naturally correspond to a “mixed” b.c. of D and N. Indeed, this is the case as we demonstrate in the following.

A similar calculation gives the mobility for the S-state as $\mu^S_{xx} = 1, \mu^S_{yy} = 0$ ($\mu^S_{yx} = 0$). One might think that U and S are equivalent upon a space rotation, because their mobilities are the same if we exchange $x$ and $y$ directions. However, they are not equivalent because the underlying lattice $\Gamma$ is not invariant under $\pi/2$ rotations. The inequivalence can be also seen in the following construction of the mixed D/N boundary state.

As discussed in Section IIIB, the mixed D/N state can be constructed from (2.63) with an orthogonal matrix $R$. Let us choose

$$ R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.40) $$

This would correspond to D (localized) in $x$-direction and N (free diffusion) in $y$, as in the U-state. Then the allowed zero modes are given by the integer-coefficient linear combinations of

$$ [\vec{u}_1, \vec{v}_1] = \left[ \begin{array}{c} 1 \\ \sqrt{3}\pi \end{array} \right], \quad (5.41) $$

$$ [\vec{u}_2, \vec{v}_2] = \left[ \begin{array}{c} (0, 0, 0) \\ (0, \sqrt{2}\pi) \end{array} \right]. \quad (5.42) $$

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Thus, the new lattice $\tilde{\Gamma}$ introduced in Section IIIB is a rectangular lattice with the unit cell of the size $\frac{1}{\sqrt{3\pi}} \times \frac{1}{\sqrt{\pi}}$. The diagonal amplitude for this state is given by

$$Z(q) = \left( \frac{1}{\eta(q)} \right)^2 \sum_{\vec{v} \in \tilde{\Gamma}^*} q^{\vec{v}^2/(2\pi)}. \tag{5.43}$$

This actually agrees with the diagonal amplitude $Z_{UU}(q)$, implying that the U-state is identical to the constructed mixed D/N state. Of course, the $g$-factor also agrees: $g = \sqrt{\pi V_0(\tilde{\Gamma})} = 3^{-1/4} = g_U$.

On the other hand, the S-state would correspond to

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.44}$$

In this case, the bases for the allowed zero-modes is given by

$$[\vec{u}_1, \vec{v}_1] = \left[ (0, \frac{1}{\sqrt{\pi}}), (0, 0) \right], \hspace{1cm} [\vec{u}_2, \vec{v}_2] = \left[ (0, 0), (2\sqrt{3\pi}, 0) \right]. \tag{5.45}$$

This is inequivalent to the previous case (5.40), because the lattice $\Gamma$ is not invariant under a rotation by $\pi/2$. Now the new lattice $\tilde{\Gamma}$ is again a rectangular lattice, but with a different size $1/\sqrt{3\pi} \times \sqrt{3/\pi}$. This gives $g = \sqrt{\pi V_0(\tilde{\Gamma})} = 3^{1/4} = g_S$. Moreover, its diagonal partition function agrees with $Z_{SS}$, implying that the S-state is in fact the mixed D/N state.

We can construct other mixed D/N states, by considering other (infinitely many) possible rotation matrices $R$. The fusion construction based on the present conformal embedding does not give those states. It does not mean that these states do not exist. It rather means that, only a part of the possible b.c.'s can be accessible by the conformal embedding which reduces the number of conformal towers effectively. In general, we do not know how far we can reach by a particular conformal embedding. From physical considerations, however, we think that the fixed points appearing in the QBM model introduced in Section IV have been exhausted.

The T-state is obtained from the D-state by fusion with $3/80$ primary in the tricritical Ising sector. The coefficients are given by

$$A_{T\ xx} = \frac{S_{3/80}^{3/5}}{S_0^{3/5} S_{3/80}^{0}} A_{D\ xx} = \frac{3 - \sqrt{5}}{2}, \tag{5.47}$$

$$A_{T\ yy} = \frac{S_{3/80}^{1/10}}{S_0^{1/10} S_{3/80}^{0}} A_{D\ yy} = \frac{-3 + \sqrt{5}}{2}. \tag{5.48}$$

Thus the mobility is given by $\mu_{T\ xx}^x = \mu^Y$ and $\mu_{T\ yy}^y = \mu^W = 1 - \mu^Y$. The T-state seems to be a mixture of Y and W boundary states.

By a similar calculation, we find the mobility in the R-state as $\mu_{R\ xx}^x = \mu^W$, $\mu_{R\ yy}^y = \mu^Y$. Namely, the mobility in the R-state is equivalent to the T-state after $\pi/2$ rotation in the $xy$-plane. They are again not equivalent, presumably reflecting the fact that the lattice $\Gamma$ is not invariant. $T$- and $R$- states are apparently given by “taking dual” of Y or W for only one component of the boson. However, we do not know how they can be realized in text of QBM. Possibly they correspond to some anisotropic QBM. Below we summarize the properties of all the boundary states obtained by fusion in Potts, Ising or tricritical Ising sectors.
In the embedding using $Z_3^{(5)}$, the level-1 “current” states corresponds to the product state $|3\rangle_5|2\rangle_P$, where $|3\rangle_5$ is the primary state of the $Z_3^{(5)}$ theory with weight $3/5$. In fact, there are two such primary states in the $Z_3^{(5)}$ theory, which are complex conjugates. Let us denote one of them as $|\bar{3}\rangle_5$. In this approach, the natural identification would be

\[ \frac{1}{2}(J_{x,-1} + iJ_{y,-1})|0\rangle = |\bar{3}\rangle_5|\bar{2}\rangle_P, \quad (5.49) \]

\[ \frac{1}{2}(J_{x,-1} - iJ_{y,-1})|0\rangle = |\bar{3}^*\rangle_5|\bar{2}\rangle_P. \quad (5.50) \]

Assuming this correspondence, for a boundary state constructed from the D-state by fusion with the $Z_3^{(5)}$ operator $\phi$, we find

\[ A_{ab} = -\delta_{ab}\text{Re}A - \epsilon_{ab}\text{Im}A, \quad (5.51) \]

where $\epsilon_{ab}$ is the antisymmetric tensor $\epsilon_{xy} = -\epsilon_{yx} = 1, \epsilon_{xx} = \epsilon_{yy} = 0$. Here $A$ is defined as

\[ A = \frac{S_3^{3/5} S_0^*}{S_3^{3/5} S_0^{3/5}}, \quad (5.52) \]

where $S_j^i$ is the matrix of modular transformations of the $Z_3^{(5)}$ theory. Thus, when the matrix of modular transformations acquires an imaginary part, there is a non-vanishing off-diagonal mobility. This corresponds to the Hall effect. Physically, such an effect is expected in QBM under a magnetic field.

The $Y$-state is obtained by fusion with the $8/5$-operator in the $Z_3^{(5)}$ sector. Thus

\[ A_Y = \frac{S_{8/5}^{3/5} S_0^*}{S_{8/5}^{3/5} S_0^{8/5}} = -\frac{3 - \sqrt{5}}{2}. \quad (5.53) \]

This gives the same result as we have obtained by fusion in the Potts sector. The off-diagonal mobility vanishes in this case.

On the other hand, a similar calculation on the F-state reads

\[ A_F = \frac{S_{1/9}^{3/5} S_0^*}{S_0^{1/9} S_{1/9}^{3/5}} = -\frac{1 - \sqrt{3}i}{2}. \quad (5.54) \]

This complex amplitude gives a non-vanishing off-diagonal mobility:

\[ \mu_{xx} = \mu_{yy} = \frac{3}{4}, \quad \mu_{xy} = -\mu_{yx} = \frac{\sqrt{3}}{4}. \quad (5.55) \]
By a fusion with the conjugate operators, we can also obtain the boundary state with opposite sign of the off-diagonal mobility.

The other three states $X, V$ and $Z$ also exhibit the non-vanishing off-diagonal mobility. Thus they should also correspond to a critical behavior under a magnetic field. The mobility in these states is summarized as follows.

| Boundary state | $g$-factor | $\mu_{xx}$ | $\mu_{xy}$ |
|----------------|------------|----------|-----------|
| $F$            | $\sqrt{\frac{2}{3}}$ | $\frac{3}{4}$ | $\pm \frac{\sqrt{3}}{2}$ |
| $X$            | $[\frac{14}{3} + 2\sqrt{5}]^{1/4}$ | $\frac{1+\sqrt{5}}{\sqrt{2}} \pm 3\sqrt{\frac{3}{2}}-\frac{\sqrt{15}}{4}$ |
| $V$            | $[6(7 + 3\sqrt{5})]^{1/4}$ | $\frac{7-\sqrt{5}}{\sqrt{3}} \pm 3\sqrt{\frac{3}{2}}-\frac{\sqrt{15}}{4}$ |
| $Z$            | $\sqrt{2/3}$ | $\frac{1}{4}$ | $\pm \frac{\sqrt{3}}{4}$ |

**F. Structure of the non-trivial boundary state**

We have obtained the explicit expression of the amplitude for the boundary state $Y$. It gives some insight into the structure of the nontrivial boundary state.

In general, a conformally invariant boundary state should be a linear combination of Ishibashi states. Each Ishibashi state is constructed from a spinless primary field that appears in the bulk theory. Thus, the closed string channel amplitude (5.18) must be consistent with the bulk spectrum.

In $c = 1$ free boson theory with a generic compactification radius, each bosonic Fock space build on a vacuum corresponds to an irreducible representation of the Virasoro algebra (except for zero winding-number sector.) However, in $c = 2$, a bosonic Fock space is always reducible and contains an infinite number of irreducible representations of the Virasoro algebra. Thus, there are infinitely more Ishibashi states other than those built on a bosonic vacuum. Nevertheless, the $c = 2$ Dirichlet and Neumann boundary states have simple expressions in terms of bosons, and their (self-)amplitudes are written as $\sum h \tilde{q}^h/\eta(\tilde{q})^2$. Roughly speaking, the factor $1/\eta(\tilde{q})^2$ means the boundary state is made of whole bosonic Fock space.

However, the non-trivial boundary state is not made of such “bosonic” boundary states. This can be seen as follows. Expressing the diagonal amplitude (5.18) as a sum of the character $\tilde{q}^h/\eta(\tilde{q})^2$ of the Heisenberg algebra,

$$Z_{YY}(\tilde{q}) = \left( \frac{g_Y}{\eta(\tilde{q})} \right)^2 \left[ 1 + (21 - 9\sqrt{5})\tilde{q}^{1/6} + \frac{9(3 - \sqrt{5})}{2} \tilde{q}^{1/2} + (16 - 6\sqrt{5})\tilde{q}^{2/3} + (5 - 3\sqrt{5})\tilde{q} + 9(3 - \sqrt{5})\tilde{q}^{7/6} + 3(7 - 3\sqrt{5})\tilde{q}^{3/2} + (5 - 3\sqrt{5})\tilde{q}^{5/3} + \ldots \right].$$

This involves negative coefficients at least for $\tilde{q}$ and $\tilde{q}^{5/3}$. While any given partition function can be written as an infinite sum of the Heisenberg character, the coefficients reveal the nature of the boundary state. If the boundary state is a linear combination of the bosonic boundary states, all the coefficients must be positive.

Thus, this is a highly nontrivial boundary state which cannot be constructed by a generalization such as eq. (2.63) or as in Ref. (30) of the Dirichlet or Neumann. To our knowledge, this is the first proof that such a non-bosonic boundary state does exist in a free boson field theory. Similar non-triviality can be shown for W,R,T,X and V states.

On the other hand, since the boundary state is a linear combination of Ishibashi states, the diagonal amplitude must be a linear combination of $c = 2$ Virasoro characters with positive coefficients. In fact, the amplitude (5.18) is expressed as

$$4\sqrt{3}Z_{YY}(\tilde{q}) = (3 + \sqrt{5})\chi_0^2 + 6(3 - \sqrt{5})\chi_{1/6}^2 + 18\chi_{1/2}^2 + (18 - 2\sqrt{5})\chi_{2/3}^2 + \ldots,$$

(5.57)
with positive coefficients, where the \( c = 2 \) Virasoro characters for weight \( h \) are given by \( \chi^2_h = q^{h-1/12} \prod_{n=1}^{\infty} (1 - q^n) \) for \( h > 0 \) and \( \chi^2_h = (q^{-1/12} - q^{11/12}) \prod_{n=1}^{\infty} (1 - q^n) \) for \( h = 0 \). Moreover, all the primary weights correspond to those of spinless fields in the bulk spectrum. These are guaranteed by the fusion construction.

Thus this nontrivial boundary state is certainly a consistent boundary state for the \( c = 2 \) free boson theory, although it appears to be very complicated in terms of the bosons. A possible direction to extend the construction of the nontrivial state for different lattices \( \Gamma^* \) is to write down the boundary state in the bosonic language and then guess a generalization of the boundary state for other lattices. However, so far we have been unable to develop this approach.

VI. INTEGRABLE FIELD THEORY ANALYSIS

A. Integrable flows away from the perfect mobility (Neumann) fixed point

Here we consider the Hamiltonian of Eq. (4.32) for small \( v_1, v_2, v_3 \). This model turns out to be integrable for general values of \( v_1, v_2, v_3 \) as discussed in detail in Ref. (44). This integrability is, in a distant sense, related to the existence of the conformal embedding; more precisely, it is the underlying parafermionic algebra satisfied by the vertex operators of (4.17) at that particular radius that ensures the existence of non local conserved currents.44

For general values of \( v_1, v_2, v_3 \) the scattering theory describing (4.32) is simply made up of three left moving and three right moving massless particles with mass parameters \( m_1, m_2, m_3 \), energy and momenta being parametrized by rapidities \( e = \pm p = m_j e^{\theta} \). Scattering between left and right movers is trivial since the theory is scale invariant in the bulk, while scattering among left or among right movers is described by pure CDD factors: \( S_{k,k+1} = i \tanh \left( \frac{\theta}{2} - \frac{i\pi}{4} \right) \), \( S_{1,3} = 1 \). Only the first particle then scatters non trivially on the impurity, with a boundary R matrix again given by a CDD factor, \( R = i \tanh \left( \frac{\theta - \theta_B}{2} - \frac{i\pi}{4} \right) \). Here, \( \theta_B \) parametrizes the impurity energy scale \( T_B \propto e^{\theta_B} \propto v_3^3 \). The exact dependence of \( T_B \), as well as the dependence of the mass parameters upon the \( v_i \)'s are complicated and will not be needed in the following, except for special cases.

The main result of the integrable approach is that the impurity free energy can be computed via the thermodynamic Bethe ansatz (TBA). If we parametrize the filling fractions of the particles by pseudo energies \( \epsilon_j = \frac{1}{1+e^{\epsilon_j/T}} \), the latter are solutions of a set of integral equations

\[
\epsilon_j = m_j e^{\theta} - T \sum_k N_{jk} \int \frac{d\theta'}{2\pi \cosh(\theta - \theta')} \frac{1}{\cosh(\theta - \theta')} \ln \left( 1 + e^{-\epsilon_k(\theta')/T} \right),
\]

(6.1)

Here, \( j, k \) take values 1, 2, 3 and can be represented as the nodes of an \( A_3 \) Dynkin diagram with incidence matrix \( N_{jk} \) (crossed nodes indicate the existence of a source term in the TBA equations)

\[
\begin{array}{ccc}
m_1 & m_2 & m_3 \\
\otimes & \otimes & \otimes
\end{array}
\]

The impurity free energy reads then

\[
F_{imp} = -T \int \frac{d\theta}{2\pi \cosh(\theta - \theta_B)} \ln \left( 1 + e^{-\epsilon_1(\theta)/T} \right).
\]

(6.2)

Formula (1.2) gives us quick access to the change of boundary entropies between the UV fixed point (Neumann boundary conditions, \( T/T_B = \infty \)) and the possible infrared (IR) fixed points (\( T/T_B = 0 \)) our model can flow to. Rather than entropies, we will use the ground state degeneracy, \( g = e^{S/T} \). The change of g-factor is then neatly expressed as
\[
\frac{g_{UV}}{g_{IR}} = \left( \frac{1 + e^{-\epsilon_1(-\infty)/T}}{1 + e^{\epsilon_1(-\infty)/T}} \right)^{1/2}.
\]

(6.3)

When \( \theta \to -\infty \), the source terms just drop from the equations, the \( \epsilon \)'s go to constants that obey the system, setting \( x_j = e^{-\epsilon_j(-\infty)/T} \),

\[
x_j = \prod_k (1 + x_k)^{N_{jk}/2}.
\]

This is solved right away by \( x_1 = x_3 = 2, x_2 = 3 \).

Setting similarly \( y_j = e^{\epsilon_j(\infty)/T} \), the system obeyed by the \( y_j \)'s depends on the source terms: different possibilities can arise. If \( m_1 \neq 0 \), the source term for \( \epsilon_1 \) diverges, and therefore \( y_1 = 0 \). If \( m_1 = 0 \) and \( m_2 \neq 0 \), in the IR the system is simply \( \epsilon_1 = 0 \) so \( y_1 = 1 \). Finally, if \( m_1 = 0 \) and \( m_2 = 0 \), the IR system is

\[
y_1 = (1 + y_1)^{1/2}
\]

with solution \( y_1 = \frac{4 + \sqrt{5}}{2} \). We thus obtain the possible ratios of degeneracies

\[
\frac{g_{UV}}{g_{IR}} = \sqrt{3}, \quad m_1 \neq 0
\]

\[
\frac{g_{UV}}{g_{IR}} = \frac{\sqrt{3}}{2}, \quad m_1 = 0, m_2 \neq 0
\]

\[
\frac{g_{UV}}{g_{IR}} = \frac{\sqrt{3}}{2 \cos \frac{\pi}{5}}, \quad m_1 = m_2 = 0
\]

(6.4)

with \( 2 \cos \frac{\pi}{5} = \sqrt{\frac{3 + \sqrt{5}}{2}} \).

We obtained the first ratio in Secs. III and IV when discussing the flow from N boundary conditions to D boundary conditions (i.e. freely diffusing to localized) in the case \( v_1 = v_2 = v_3 > 0 \). For general \( v_i \), the potential of Eq. (4.32) has a unique minimum (within each unit cell) and the TBA equations in the first case describe the flow from N to D in the general case where the three-fold symmetry is broken. This is confirmed by the analysis of the central charge associated with (6.4) which gives generically \( c = 2 \), hence indicating a flow within the whole two boson theory. (i.e. not purely within the Potts sector.) The third ratio was also obtained previously when we discussed the flow from N to Y in the case \( v_1 = v_2 = v_3 < 0 \).

Finally the second ratio is somewhat trivial: it corresponds to a flow from Neumann to Dirichlet boundary conditions for the field \( \tilde{\phi}_1 \) while the field \( \tilde{\phi}_2 \) remains at the Neumann fixed point. This corresponds to the \( U \) boundary state discussed in Sec. IV. This is obtained by setting \( v_2 = v_3 = 0 \), while \( v_1 \) can take either sign. The analysis of the central charge associated with (6.4) gives \( c = 1 \), confirming that the flow takes place in the \( c = 1 \) sector only.

B. \( W_3 \) integrable flows

Integrable flows can be used to explore more thoroughly the phase space and the possible fixed points in our problem.

The integrability of the flow from free to fixed or mixed boundary conditions in the Potts model, and therefore, from Neumann to Dirichlet or the Y fixed point in the \( c = 2 \) theory, is ultimately related with the fact that the \( Z_3 \) parafermionic theory (the three state Potts model) can be represented as the coset \( SU(2)/U(1) \). The field \( \Psi_1 \) is then the “adjoint” operator in this construction.\(^1\) We observe that this

\(^1\)In a general coset construction, one usually calls adjoint the operator obtained by taking the identity representation for the algebras in the numerator, and the adjoint for the algebra in the denominator.
theory can also be represented as the coset \( SU(3) \times SU(3)_2 \). The adjoint operator for this coset now has dimension \( \Delta = \frac{5}{2} \), and physically corresponds to the energy operator of the Potts model. It can be shown to define an integrable perturbation that preserves the \( W_3 \) symmetry: more precisely, there is a natural deformation of the current \( W_3 \) that is conserved, at least perturbatively, away from the UV fixed point, if this fixed point itself preserves \( W_3 \) symmetry. It is natural to assume that this extends all the way to the IR fixed point, which therefore should have \( W_3 \) symmetry too.

In our problem, the operator with \( \Delta = \frac{5}{2} \) is present in the spectrum for mixed boundary conditions in the Potts model, or, equivalently, at the \( Y \) fixed point in the \( c = 2 \) theory. In the Potts model it corresponds to applying a magnetic field at the mixed \( (AB) \) fixed point which breaks the remaining \( Z_2 \) symmetry, and favors the \( A \) state. This corresponds to a flow from mixed to fixed. In the QBM problem we expect this operator to produce an RG flow from \( Y \) to localized fixed points; if \( Y \) corresponds to the particle being on \( A \) and \( B \) sub-lattice then the flow is to a state where the particle is localized on an \( A \) site. This RG flow can also be studied using integrability.

For the more general coset \( SU(3)_k \times SU(3)_{k+1} \), perturbed by the operator of weight \( \Delta = 1 - \frac{3}{k+4} \), which preserves \( W_3 \) symmetry, \( S \) matrices and \( R \) matrices are known\(^6\), and the impurity free energy can be computed once again using the thermodynamic Bethe ansatz. It reads

\[
F_{\text{imp}} = -T \int \frac{d \theta}{2 \pi} \left[ G_1(\theta - \theta_B) \ln \left( 1 + e^{-\epsilon_{11}/T} \right) + G_2(\theta - \theta_B) \ln \left( 1 + e^{-\epsilon_{21}/T} \right) \right].
\]

Here, as before, \( T_B \) is an impurity energy scale, \( T_B \propto e^{\theta_B} \), \( G_1 \) and \( G_2 \) are known kernels, and the \( \epsilon \) are pseudo energies, solutions of the equations:

\[
\epsilon_{ij} = \delta_{ij} m e^\theta - T \sum_k N_{ij,ik} G_1 \ln \left( 1 + e^{-\epsilon_{1k}/T} \right) - T \sum_l N_{ij,jl} G_2 \ln \left( 1 + e^{\epsilon_{lj}/T} \right).
\]

Here, \( ij \) are line and column labels on the following diagram (the cross on the \((1,1)\) node stands for the mass term in \((6.6)\))

\[\begin{array}{ccccc}
&  &  &  & \\
&  &  &  & \\
&  &  &  & \\
&  &  &  & \\
&  &  &  & \\
&  &  &  & \\
1 & \cdots & \cdots & \cdots & k \\
&  &  &  & \\
&  &  &  & \\
&  &  &  & \\
&  &  &  & \\
&  &  &  & \\
\end{array}\]

\( i = 1, 2, j = 1, \ldots, k \), and \( N_{ij,kl} \) its incidence matrix.

We will not need the exact form of the kernels in what follows, and shall concentrate on the ground state degeneracies in the UV and in the IR. From the previous formulas, we find the simple result (the \( y_{j1} \) all vanish here)

\[
g_{UV}/g_{IR} = \frac{1}{2} \ln \left( 1 + x_{11} \right) + \ln \left( 1 + x_{21} \right),
\]

where \( x_{ij} = e^{-\epsilon_{ij}/T} \) is solution of the \( \theta \to -\infty \) limit of the TBA system:

\[
x_{ij} = \prod_k (1 + x_{ik})^{-N_{ij,ik}/2} \prod_l \left( 1 + \frac{1}{x_{lj}} \right)^{-N_{ij,lj}/2}.
\]

For the case \( k = 1 \) of interest here, the system reduces to \( x = x_{11} = x_{21}, x = \frac{\sqrt{5}}{\sqrt{1+x}} \) or \( x = 2 \cos \frac{\pi}{5} \).

Hence

\[
g_{UV}/g_{IR} = 2 \cos \frac{\pi}{5},
\]
This is the ratio $g_{\text{mixed}}/g_{\text{fixed}}$ in the Potts model, or the ratio $g_Y/g_D$ for the two boson system. We thus expect that perturbation by the operator $\Delta = \frac{7}{2}$ does not lead to a new fixed point, but induces a flow from Y to D in the two boson system. Let us stress that this is a rather abstract statement: although we know the operator with $\Delta = \frac{7}{2}$ is present in the spectrum (because of the partition functions analysis), we are not able to represent it in terms of bosonic operators, since we do not have an entirely clear picture of what the Y boundary conditions mean for the bosons.

VII. GENERALIZATION TO HIGHER DIMENSIONS AND RELATIONS WITH THE KONDO MODEL

A. Generalization to higher dimensions

Many of our results can be generalized to the case of $n - 1$ bosons, corresponding to QBM on an $(n - 1)$-dimensional lattice. Yi and Kane found a generalization of the Y fixed point for all $n > 3$ using a mapping onto the $n$-channel $SU(2)$, overscreened Kondo model. They also observed that the 3-dimensional $(n = 4)$ case corresponds to QBM on a diamond lattice. The conformal embedding is now naturally understood in terms of an $SU_n$ $U$-abelian sub-algebra, $n \geq 3$-dimensional $(n = 4)$ case corresponds to QBM on a diamond lattice. The conformal embedding is now

\[ \frac{n - 1}{n + 2} = \frac{2(n - 1)}{n + 2} \]

The first component of the embedding is the theory of $Z_n$ parafermions, that can also be considered as the coset $SU(2)/U(1)$, with $c = \frac{2(n - 1)}{n + 2}$. The fundamental parafermion in this theory has weight $\frac{n - 1}{n}$, and can be bosonized using a system of $n$ bosons satisfying (for the chiral components)

\[ \langle \Phi_i(z) \Phi_i(w) \rangle = -2 \frac{n - 1}{n} \ln(z - w) \]

\[ \langle \Phi_i(z) \Phi_j(w) \rangle = \frac{4}{n} \ln(z - w), \quad i \neq j, \]

as

\[ \Psi_i = \sum_{k=1}^{n} e^{i \Phi_k}. \]

The embedding then follows from the decomposition $T = T_1 + T_2$, where

\[ T_1 = \frac{1}{n + 2} \left[ -\frac{1}{2} \sum_{j=1}^{n} (\partial \Phi_j)^2 + \sum_{j \neq k} e^{i(\Phi_j - \Phi_k)} \right] \]

\[ T_2 = \frac{1}{n + 2} \left[ -\frac{n}{4} \sum_{j=1}^{n} (\partial \Phi_j)^2 - \sum_{j \neq k} e^{i(\Phi_j - \Phi_k)} \right] \]

such that the short distance expansion of $T_1$ with $T_2$ is trivial, $T_1$ is a stress energy tensor with central charge $c_1 = \frac{2(n - 1)}{n + 2}$ of the coset $SU(n_1) \times SU(n_2)/SU(n_2)$ or $SU(2)_n/U(1)$, and similarly for $T_2$ with $c_2 = \frac{n(n - 1)}{n + 2}$ of the $SU(n_2)/U(1)^{n-1}$ coset. $T = T_1 + T_2$ of course is the stress energy tensor of the initial $(n - 1)$ boson theory.

It turns out that the $n - 1$ boson theory with boundary perturbation

\[ -\sum_{j=1}^{n} v_j \cos \Phi_j, \]

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is integrable\[^2\] the particular case where all the \(v_j\)'s are equal corresponding again to a perturbation of the form \(\Psi_1 + \Psi_1^\dagger\). The impurity free energy reads, generalizing (6.2)

\[
F_{\text{imp}} = -T \int \frac{d\theta}{2\pi} \frac{1}{\cosh(\theta - \theta_B)} \ln \left(1 + e^{-\epsilon_n(\theta)/T}\right),
\]

(7.6)

where the \(\epsilon\) satisfy the same TBA equations as (6.1), but the incidence matrix is the one of the following diagram

Here again, \(T_B \propto v_n^0\), and the \(m_j\) are dimensionless numbers functions of the couplings \(v_j\).

In the UV, the system of equations for the \(x\)'s is solved easily with

\[
x_j = \left(\frac{j}{n-1}\right)^2 - 1, \quad j = 1, \ldots, n-2;
\]

\[
x_{n-1} = x_n = n - 1.
\]

If \(m_n\) does not vanish, one has \(y_n = 0\), and then we obtain the ratio

\[
\frac{g_{UV}}{g_{IR}} = \sqrt{n}.
\]

(7.7)

This generalizes the first ratio in (6.4), and corresponds to a flow from Neumann to Dirichlet boundary conditions in the \(n-1\) bosons problem, where at the D fixed point, the field lies on a \(n-1\) dimensional lattice generalizing \(\Gamma^*\).

When \(m_n\) vanishes, the ratio of degeneracies can take various values depending on which of the other \(m_j\)'s vanish. A particularly interesting case is when all \(m_j\) but \(m_{n-1}\) vanish. This corresponds to a perturbation with all \(a_j = 0, j \neq 1\), and \(a_1 > 0\). In that case, one gets a ratio

\[
\frac{g_{UV}}{g_{IR}} = \frac{\sqrt{n}}{2\cos\frac{\pi}{n+2}},
\]

(7.8)

generalizing the case \(n = 3\); we denote the corresponding fixed point by \(Y_n\). These fixed points were discovered by Yi and Kane\[^3\].

Let us discuss the other cases. If \(m_{n-1}\) still does not vanish, and in addition some of the other \(m_j\)'s don’t vanish \((j \leq n-2)\), the ratio is of the form

\[
\frac{g_{UV}}{g_{IR}} = \frac{\sqrt{n}}{2\cos\frac{\pi}{n+2}},
\]

indicating a flow to a fixed point encountered previously for a lower value of \(n, Y_{n-k}\). When \(m_{n-1}\) does vanish, while some of the other \(m_j\)'s don’t \((j \leq n-2)\), one gets a ratio of the form

\[
\frac{g_{UV}}{g_{IR}} = \sqrt{\frac{n}{n-k}}.
\]

This corresponds to an IR fixed point where \(k\) components of the field have D boundary conditions, the other ones having N. We see therefore that, as in the \(n = 3\) case, no other fixed point is reached, besides the various \(D,N\) combinations, and the \(Y_k\) fixed points.

As before, the perturbation of N boundary conditions by the adjoint operator in the coset \(\frac{SU(2n)}{U(1)}\) does not preserve the \(W_n\) symmetry. A perturbation preserving this symmetry is the one with the adjoint in the coset picture \(\frac{SU(n)_{1}\times SU(n)_{1}}{SU(n)_{2}}\). The perturbing field now has dimension \(\Delta = 1 - \frac{n}{n+2}\), and the perturbation is integrable. The boundary free energy reads as in (6.3), with now a sum over \(n-1\) pseudo energies. The TBA diagram is as in the \(SU(3)\) case, but the “base” is the Dynkin diagram of \(SU(n)\), ie \(A_{n-1}\) instead of \(A_2\). For more general cosets \(\frac{SU(n)_{1}\times SU(n)_{1}}{SU(n)_{k+1}}\), the diagram looks as follows.
with the obvious generalization for the equations (6.6) and (6.5) that the row labels now run from 1 to \( n - 1 \). In particular the ratio of boundary entropies in the UV and IR now reads

\[
\frac{g_{\text{UV}}}{g_{\text{IR}}} = \frac{1}{2} \sum_{i=1}^{n-1} \ln (1 + x_{i1}).
\]  (7.9)

For \( k = 1 \) we have

\[
x_{i1} = \prod_j \left( 1 + \frac{1}{x_{j1}} \right)^{-N_{i,1}/2}.
\]  (7.10)

The solution of this system is

\[
\frac{g_{\text{UV}}}{g_{\text{IR}}} = \frac{g_Y}{g_D} = 2 \cos \frac{\pi}{n+2}.
\]  (7.11)

This corresponds to a flow from \( Y_n \) to \( D \) in the \( n - 1 \) bosons language, generalizing (6.9).

As in the \( n = 3 \) case, the key ingredient to understand the phase diagram is the set of boundary conditions and flows in the \( Z_n \) parafermions theory \( SU(2)_n/U(1) \). These theories have known lattice model or quantum spin chains realizations, and it is possible to show, using results in Ref. (49), that the \( N \) and \( D \) fixed points correspond to free and fixed boundary conditions, while the operator \( \Psi_1 + \Psi_1^\dagger \) is the most relevant order parameter, and corresponds again to a longitudinal boundary field. The generalization of the mixed boundary conditions for the \( (Z_3) \) Potts model to the \( Z_n \) case has not been investigated, to our knowledge.

B. Relation to the Kondo problem

The emergence of parafermions is also closely related with the \( n \)-channel Kondo problem. Recall that this problem involves originally \( 2n \) Dirac fermions, which can be bosonized in terms of the current algebras \( SU(2)_n \times SU(n)/U(1) \). Only the spin currents interact with the boundary spin, leading to a flow entirely within the \( SU(2)_n \) sector, from the weak coupling fixed point in the UV to the non Fermi liquid Kondo fixed point in the IR. The ratio of degeneracy factors is

\[
\frac{g_{\text{UV}}}{g_{\text{IR}}} = 2 \cos \frac{\pi}{n+2}.
\]  (7.11)

As is well known, the strong coupling Kondo fixed point does not depend on the anisotropy, and can be reached as well starting from an anisotropic Kondo interaction. There is a particularly simple choice of this anisotropy which corresponds to the Toulouse limit in the \( n = 1 \) case, or the Emery Kivelson limit in the \( n = 2 \) case, where an additional \( U(1) \) decouples, and the flow takes place entirely within the \( SU(2)_n/U(1) \) sector, ie the parafermionic theory. The perturbation Hamiltonian then reads

\[
H_{\text{int}} = v \left( \Psi_1^\dagger \sigma^- + \Psi_1 \sigma^+ \right).
\]  (7.12)

One can then argue from the integrable analysis that the IR fixed point - the strong coupling Kondo fixed point - coincides with the \( Y_n \) fixed point identified previously, the generalization of the “mixed” boundary conditions of the Potts model to the \( Z_n \) case. The UV fixed point - the weakly coupled Kondo fixed point - does not have a simple interpretation in terms of the \( Z_n \) variables. For the case \( n = 3 \), it coincides with the “localized on B or C” fixed point discussed in section IV. This correspondance was used extensively by Yi and Kane.\[\text{[56]}\]
C. Another relation to the Kondo problem: a remark on the marginal case

Returning to the two boson model \((n = 3)\), we note that the case \(a^2 = \frac{8\pi}{9}\) can also be studied by integrability techniques, using an unexpected mapping onto the four-channel Kondo model. To do this, recall that it is possible to represent the currents of the \(SU(2)_q\) algebra (that has \(c = 2\)) with two bosons as follows:

\[
\begin{align*}
\mathcal{J}_x &= \sigma_x \cos(\sqrt{2\pi} \tilde{\phi}_{1L} + \sqrt{6\pi} \tilde{\phi}_{2L}) \\
\mathcal{J}_y &= \sigma_y \cos(\sqrt{2\pi} \tilde{\phi}_{1L} - \sqrt{6\pi} \tilde{\phi}_{2L}) \\
\mathcal{J}_z &= \sigma_z \cos(\sqrt{8\pi} \tilde{\phi}_{1L}). \\
\end{align*}
\]

(7.13)

A similar representation was used by Fabrizio and Gogolin recently; these authors however did not explicitly consider the \(\sigma\) operators in their representation (cocycles), which will play a crucial role in what follows, though it doesn’t affect their results. The four channel anisotropic Kondo problem has a boundary interaction

\[
H_{\text{int}} = J_{\perp} [\mathcal{J}_x \mathcal{J}_x(0) + \mathcal{J}_y \mathcal{J}_y(0)] + J_{/}/\mathcal{J}_z \mathcal{J}_z(0),
\]

(7.14)

and is integrable for any value of the anisotropy.

Now let us use our bosonization. The boundary action reads then, using the boundary conditions \(\tilde{\phi}_{1L} = \tilde{\phi}_{1R}\) in the UV to introduce the non chiral fields,

\[
J_{\perp} \left[ \tau_x \sigma_x \cos(\sqrt{\frac{\pi}{2}} \tilde{\phi}_1 + \sqrt{\frac{3\pi}{2}} \tilde{\phi}_2)(0) + \tau_y \sigma_y \cos(\sqrt{\frac{\pi}{2}} \tilde{\phi}_1 - \sqrt{\frac{3\pi}{2}} \tilde{\phi}_2)(0) \right] + J_{/}/\tau_z \sigma_z \cos(\sqrt{2\pi} \tilde{\phi}_1(0)).
\]

(7.15)

Consider now the expansion of the boundary free energy in powers of \(J_{\perp}, J_{/}/\). for every insertion of \(\cos(\sqrt{2\pi} \tilde{\phi}_1)\), we need one insertion of \(\cos(\sqrt{\frac{\pi}{2}} \tilde{\phi}_1 + \sqrt{\frac{3\pi}{2}} \tilde{\phi}_2)\) and one of \(\cos(\sqrt{\frac{\pi}{2}} \tilde{\phi}_1 - \sqrt{\frac{3\pi}{2}} \tilde{\phi}_2)\), contributing a term \(\tau_x \sigma_x \tau_y \sigma_y \tau_z \sigma_z = -1\) to the spin part. It follows that all the spin terms actually disappear, and that the boundary free energy is the same as the one of the model \((7.13)\) with no spin variables:

\[
H_{\text{int}} = -2J_{\perp} \cos(\frac{\pi}{2} \tilde{\phi}_1 \cos(\frac{3\pi}{2} \tilde{\phi}_2) - J_{/}/\cos(\sqrt{2\pi} \tilde{\phi}_1).
\]

(7.16)

This is nothing but the weak corrugation Hamiltonian of section 4 (eq. (4.17)), this time for \(a^2 = \frac{8\pi}{9}\), where the perturbation is marginal.

Using the TBA for the anisotropic Kondo model, the impurity free energy is simply expressed as

\[
F_{\text{imp}} = -T \int \frac{d\theta}{2\pi \cosh(\theta - \theta_B)} \ln \left(1 + e^{-\epsilon_1(\theta)/T}\right),
\]

(7.17)

where \(T_B \propto e^{\theta_B}\) is the Kondo temperature, and \(\epsilon_1\) is a pseudo energy, solution of the TBA system

\[
\epsilon_j = m \delta_{4,j} e^{\theta} - T \sum_k N_{jk} \int \frac{d\theta'}{2\pi \cosh(\theta - \theta')} \ln \left(1 + e^{-\epsilon_k(\theta')/T}\right).
\]

(7.18)

Here, the labels \(k\) run over the nodes of the diagram

\[
\begin{array}{cccccccc}
1 & 2 & 4 & t-3 & t-1 & t-2 & t-1 & t-2 & t
\end{array}
\]

\[\text{H.S. thanks P. Fendley and N. Warner for discussions on this point.}\]
and we have restricted to the simplest values of the anisotropy such that $\frac{t}{t_1} = 1 - J_{/ /}. $

Provided $J_{/ /} \geq 0$, the perturbation generates a flow (the situation here is quite different from the one boson theory, where the boundary cosine interaction is truly marginal). Independently of the value of $J_{/ /}$, one finds $x_1 = e^{-\epsilon(-\infty)/T} = 3$, while $x_1 = e^{-\epsilon(\infty)/T} = 2$. The ratio $\frac{g_{UV}}{g_{IR}} = \frac{2}{\sqrt{3}}$, the well known value for the 4 channel Kondo model. This coincides with the ratio $\frac{g_{N}}{g_{D}} (4.30)$ for our values of the coupling constant. For $J_{/ /} \leq 0$ (the equivalent of $v_A < 0$ in section 4), meanwhile, the Kondo perturbation is irrelevant, and no flow is generated. Hence, we conclude that for $\Delta_N = 1 (4.19)$, there is no new fixed point, and presumably the $Y$ fixed point becomes identical with the $N$ one in the limit $a^2 = \frac{2\pi}{9}$. An $\epsilon$-expansion around this marginal point was developed by Kane and Fisher.

VIII. CONCLUSION

We have established the general connection between a simple model for QBM and boundary CFT with considerable attention paid to the issue of boson compactness. For a simple Hamiltonian we have conjectured a complete phase diagram, which is analogous to that of the boundary Potts model. These results were obtained by using a sort of “boundary embedding” involving the Potts model and the fusion approach to generating boundary states. We have also mentioned other fixed points which occur with more general Hamiltonians and which arise from fusion in the other sectors of the conformal embedding. We have discussed the integrability of some of the RG flows.

We note that the $Y$ b.c. is more dynamical or “quantum” than $D$ or $N$. Within the weak corrugation formulation, while the potential term $v_A$ of Eq. (1.17) is relevant, it does not simply “pin” the boson fields at one of its minima; rather they fluctuate between the two inequivalent minima. This is only possible for a range of lattice spacing, $8\pi/9 < a^2 < 2\pi$ where a semi-classical analysis fails.

We have given an explicit demonstration that other, highly non-trivial b.c.’s are possible for 2 free periodic bosons besides $D$ and $N$ and their variants. However, since this demonstration rests on the conformal embedding and fusion, and since we don’t have a conformal embedding for general values of the radius parameter, $a$, our construction only works at one special value of $a$. A more straightforward understanding of these b.c.’s, directly in terms of the bosons eludes us. Thus this general problem, known since 1992 remains open.

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