Strongly order continuous operators on Riesz spaces

by

AKBAR BAHRAMNEZHAD AND KAZEM HAGHNEJAD AZAR

Department of Mathematic, University of Mohaghegh Ardabili, Ardabil, Iran
E-mail: bahrainmehad@uma.ac.ir

Abstract

In this paper we introduce two new classes of operators that we call strongly order continuous and strongly σ-order continuous operators. An operator $T : E \to F$ between two Riesz spaces is said to be strongly order continuous (resp. strongly σ-order continuous), if $x_{\alpha} \xrightarrow{uo} 0$ (resp. $x_{\alpha} \xrightarrow{\sigma} 0$) in $E$ implies $Tx_{\alpha} \xrightarrow{\sigma} 0$ (resp. $Tx_{\alpha} \xrightarrow{uo} 0$) in $F$. We give some conditions under which order continuity will be equivalent to strongly order continuity of operators on Riesz spaces. We show that the collection of all so-continuous linear functionals on a Riesz space $E$ is a band of $E^\sim$.

Key Words: Riesz space, order convergence, unbounded order convergence, strongly order continuous operator.

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1 Introduction

The concept of unbounded order convergence or $uo$-convergence was introduced in [8] and is proposed firstly in [3]. It has recently been intensively studied in several papers [4, 5, 6]. Recall that a net $(x_{\alpha})_{\alpha \in A}$ in a Riesz space $E$ is order convergent (or, $o$-convergent for short) to $x \in E$, denoted by $x_{\alpha} \xrightarrow{o} x$ whenever there exists another net $(y_{\beta})_{\beta \in B}$ in $E$ such that $y_{\beta} \downarrow 0$ and that for every $\beta \in B$, there exists $\alpha_0 \in A$ such that $|x_{\alpha} - x| \leq y_{\beta}$ for all $\alpha \geq \alpha_0$.

A net $(x_{\alpha})$ in a Riesz space $E$ is unbounded order convergent (or, $uo$-convergent for short) to $x \in E$ if $|x_{\alpha} - x| \land u \xrightarrow{o} 0$ for all $u \in E^+$. We denote this convergence by $x_{\alpha} \xrightarrow{uo} x$ and write that $x_{\alpha}$ $uo$-convergent to $x$. This is an analogue of pointwise convergence in function spaces.

Let $\mathbb{R}^A$ be the Riesz space of all real-valued functions on a non-empty set $A$, equipped with the pointwise order. It is easily seen that a net $(x_{\alpha})$ in $\mathbb{R}^A$ $uo$-converges to $x \in \mathbb{R}^A$ if and only if it converges pointwise to $x$. For instance in $c_0$ and $\ell_p(1 \leq p < \infty)$, $uo$-convergence of nets is the same as coordinate-wise convergence. Assume that $(\Omega, \Sigma, \mu)$ is a measure space and let $E = L_p(\mu)$ for some $1 \leq p < \infty$. Then $uo$-convergence of sequences in $L_p(\mu)$ is the same as almost everywhere convergence. In [9], Wickstead characterized the spaces in which $uo$-convergence of nets implies $uo$-convergence and vice versa. In [4], Gao characterized the space $E$ such that in its dual space $E^*$, $uo$-convergence implies $w^*$-convergence and vice versa. He also characterized the spaces in whose dual space simultaneous $uo$- and $w^*$-convergence imply weak/norm convergence. We show that the collection of all order bounded strongly order continuous linear functionals on a Riesz space $E$ is a band of $E^\sim$ [Theorem 2.7]. For unexplained terminology and facts on Banach lattices and positive operators, we refer the reader to [11, 2]. Let us start with the definition.

Definition 1. An operator $T : E \to F$ between two Riesz spaces is said to be:
1. Strongly order continuous (or, so-continuous for short), if \( x_\alpha \xrightarrow{uo} 0 \) in \( E \) implies \( Tx_\alpha \xrightarrow{uo} 0 \) in \( F \).

2. Strongly \( \sigma \)-order continuous (or, \( s\sigma_o \)-continuous for short), if \( x_n \xrightarrow{uo} 0 \) in \( E \) implies \( Tx_n \xrightarrow{uo} 0 \) in \( F \).

The collection of all so-continuous operators of \( L_b(E, F) \) will be denoted by \( L_{so}(E, F) \), that is
\[
L_{so}(E, F) := \{ T \in L_b(E, F) : T \text{ is so-continuous} \}.
\]
Similarly, \( L_{s\sigma_o}(E, F) \) will denote the collection of all order bounded operators from \( E \) to \( F \) that are \( s\sigma_o \)-continuous. That is,
\[
L_{s\sigma_o}(E, F) := \{ T \in L_b(E, F) : T \text{ is } s\sigma_o \text{-continuous} \}.
\]
Clearly, we have \( L_{so}(E, F) \subset L_{s\sigma_o}(E, F) \) and \( L_{so}(E, F) \) and \( L_{s\sigma_o}(E, F) \) are both vector subspaces of \( L_b(E, F) \).

Recall that an operator \( T : E \to F \) between two Riesz spaces is said to be order continuous (resp. \( \sigma \)-order continuous) if \( x_\alpha \xrightarrow{uo} 0 \) (resp. \( x_n \xrightarrow{uo} 0 \)) in \( E \) implies \( Tx_\alpha \xrightarrow{uo} 0 \) (resp. \( Tx_n \xrightarrow{uo} 0 \)) in \( F \). The collection of all order continuous operators of \( L_b(E, F) \) will be denoted by \( L_o(E, F) \), that is
\[
L_o(E, F) := \{ T \in L_b(E, F) : T \text{ is order continuous} \}.
\]
Similarly, \( L_c(E, F) \) will denote the collection of all order bounded operators from \( E \) to \( F \) that are \( \sigma \)-order continuous. That is,
\[
L_c(E, F) := \{ T \in L_b(E, F) : T \text{ is } \sigma \text{-order continuous} \}.
\]
Note that every so-continuous (resp. \( s\sigma_o \)-continuous) operator is order (resp. \( \sigma \)-order) continuous. The converse is not true in general. For example the identity operator \( I : c_0 \to c_0 \) is order continuous but is not so-continuous. Indeed, the standard basis sequence of \( c_0 \) is \( uo \)-converges to 0 but is not order convergent.

2 Main Results

Lemma 1. (\cite{5} Lemma 3.1). In a Riesz space we have the following:

1. If \( x_\alpha \xrightarrow{uo} x \) and \( x_\alpha \xrightarrow{uo} y \), then \( x = y \). In other hands, unbounded order limits are uniquely determined.

2. If \( x_\alpha \xrightarrow{uo} x \), \( y_\alpha \xrightarrow{uo} y \) and \( k, r \) are real numbers, then \( kx_\alpha + ry_\alpha \xrightarrow{uo} kx + ry \). Furthermore if \( x_\alpha \xrightarrow{uo} x \) and \( y_\alpha \xrightarrow{uo} y \), then \( x_\alpha \lor y_\alpha \xrightarrow{uo} x \lor y \) and \( x_\alpha \land y_\alpha \xrightarrow{uo} x \land y \). In particular \( x^+_\alpha \xrightarrow{uo} x^+ \), \( x^-_\alpha \xrightarrow{uo} x^- \) and \( |x_\alpha| \xrightarrow{uo} |x| \).

3. If \( x_\alpha \xrightarrow{uo} x \) and \( x_\alpha \geq y \) holds for all \( \alpha \), then \( x \geq y \).
Note that the \( uo \)-convergence in a Riesz space \( E \) does not necessarily correspond to a topology on \( E \). For example, let \( E = c \), the Banach lattice of real valued convergent sequences. Put \( x_n = \sum_{k=1}^{n} e_k \), where \((e_n)\) is the standard basis. Then \((x_n)\) is \( uo \)-convergent to \( x = (1, 1, 1, ...) \), but it is not norm convergent.

**Proposition 1.** Let \( E, F \) be Riesz spaces such that \( E \) is finite-dimensional. Then \( L_{uo}(E, F) = L_n(E, F) \) and \( L_{s\sigma}(E, F) = L_c(E, F) \).

**Proof.** Follows immediately if we observe that in a finite-dimensional Riesz space order convergence is equivalent to \( uo \)-convergence. \( \square \)

Recall that a Riesz space is said to be \( \sigma \)-laterally complete if every disjoint sequence has a supremum. For a set \( A \), \( \mathbb{R}^A \) is an example of \( \sigma \)-laterally complete Riesz space.

**Proposition 2.** Let \( E, F, G \) be Riesz spaces. Then we have the following:

1. If \( T \in L_{uo}(E, F) \) and \( S \in L_n(F, G) \), then \( ST \in L_{uo}(E, G) \). As a consequence, \( L_{uo}(E) \) is a left ideal for \( L_n(E) \). Similarly, \( L_{uo}(E) \) is a left ideal for \( L_c(E) \).

2. If \( E \) is \( \sigma \)-Dedekind complete and \( \sigma \)-laterally complete and \( S \in L_\mathcal{D}(E, F) \) and \( T \in L_{s\sigma}(F, G) \), then \( TS \in L_{s\sigma}(E, G) \). In this case, \( L_{s\sigma}(E, F) = L_c(E, F) \).

**Proof.**

1. Let \((x_\alpha)\) be a net in \( E \) such that \( x_\alpha \xrightarrow{uo} 0 \). By assumption, \( Tx_\alpha \xrightarrow{o} 0 \). So, \( STx_\alpha \xrightarrow{uo} 0 \). Hence, \( ST \in L_{uo}(E, G) \).

2. Let \( E \) be a \( \sigma \)-Dedekind complete and \( \sigma \)-laterally complete Riesz space. By Theorem 3.9 of [6], we see that a sequence \((x_n)\) in \( E \) is \( uo \)-null if and only if it is order null. So, if \((x_n)\) be a sequence in \( E \) such that \( x_n \xrightarrow{uo} 0 \), then \( x_n \xrightarrow{o} 0 \). Thus, \( Sx_n \xrightarrow{o} 0 \) and then \( TSx_n \xrightarrow{o} 0 \). Hence, \( TS \in L_{s\sigma}(E, G) \). Clearly, we have \( L_{s\sigma}(E, F) = L_c(E, F) \). This ends the proof. \( \square \)

Let \( T : E \rightarrow F \) be a positive operator between Riesz spaces. We say that an operator \( S : E \rightarrow F \) is dominated by \( T \) (or that \( T \) dominates \( S \)) whenever \( |Sx| \leq T|x| \) holds for each \( x \in E \).

**Proposition 3.** If a positive \( so \)-continuous operator \( T : E \rightarrow F \) dominates \( S \), then \( S \) is \( so \)-continuous.

**Proof.** Let \( T : E \rightarrow F \) be a positive \( so \)-continuous operator between Riesz spaces such that \( T \) dominates \( S : E \rightarrow F \) and let \( x_\alpha \xrightarrow{uo} 0 \) in \( E \). By part (2) of Lemma 1, \( |x_\alpha| \xrightarrow{uo} 0 \). So, by assumption, \( T|x_\alpha| \xrightarrow{o} 0 \) and from the inequality \( |Sx| \leq T|x| \) and part (2) of Lemma 1 again, we have \( Sx_\alpha \xrightarrow{o} 0 \). Hence, \( S \) is \( so \)-continuous. \( \square \)
For an operator $T : E \to F$ between two Riesz spaces we shall say that its modulus $|T|$ exists (or that $T$ possesses a modulus) whenever $|T| := T \lor (-T)$ exists in the sense that $|T|$ is the supremum of the set $\{-T, T\}$ in $L(E, F)$. If $E$ and $F$ are Riesz spaces with $F$ Dedekind complete, then every order bounded operator $T : E \to F$ possesses a modulus [2 Theorem 1.18]. From this discussion it follows that when $E$ and $F$ are Riesz spaces with $F$ Dedekind complete, then each order bounded operator $T : E \to F$ satisfies

$$T^+(x) = \sup \{Ty : 0 \leq y \leq x\}, \ 	ext{and}$$

$$T^-(x) = \sup \{-Ty : 0 \leq y \leq x\}$$

for each $x \in E^+$.

**Theorem 1.** For an order bounded linear functional $f$ on a Riesz space $E$ the following statements are equivalent.

1. $f$ is so-continuous.
2. $f^+$ and $f^-$ are both so-continuous.
3. $|f|$ is so-continuous.

**Proof.** (1) $\Rightarrow$ (2) By Lemma 1, we may assume that $(x_\alpha) \subset E^+$. Let $x_\alpha \overset{\text{uo}}{\to} 0$ and let $(r_\alpha)$ be a net in $\mathbb{R}$ such that $r_\alpha \downarrow 0$. In view of $f^+ x = \sup \{fy : 0 \leq y \leq x\}$, for each $\alpha$ there exists a net $(y_\alpha)$ in $E$ with $0 \leq y_\alpha \leq x_\alpha$ such that $f^+ x_\alpha - r_\alpha \leq fy_\alpha$. So, $f^+ x_\alpha \leq fy_\alpha + r_\alpha$. Since $x_\alpha \overset{\text{uo}}{\to} 0$, we have $y_\alpha \overset{\text{uo}}{\to} 0$. Thus, by assumption, $fy_\alpha \overset{\alpha}{\to} 0$. It follows from $f^+ x_\alpha \leq (fy_\alpha + r_\alpha) \overset{\alpha}{\to} 0$ that $f^+ x_\alpha \overset{\alpha}{\to} 0$. Hence, $f^+$ is so-continuous. Now, as $f^- = (-f)^+$, we conclude that $f^-$ is also so-continuous.

(2) $\Rightarrow$ (3) Follows from the identity $|f| = f^+ + f^-$.

(3) $\Rightarrow$ (1) Follows immediately from Proposition 3 by observing that $|f|$ dominates $f$. \hfill \Box

**Remark 1.** One can easily formulate by himself the analogue of Theorem 1 for so-continuous operators.

Recall that a subset $A$ of a Riesz space is said to be order closed whenever $(x_\alpha) \subset A$ and $x_\alpha \overset{\alpha}{\to} x$ imply $x \in A$. An order closed ideal is referred to as a band. Thus, an ideal $A$ is a band if and only if $(x_\alpha) \subset A$ and $0 \leq x_\alpha \uparrow x$ imply $x \in A$. In the next theorem we show that $L_{so}(E, \mathbb{R})$ and $L_{s\sigma o}(E, \mathbb{R})$ are both bands of $E^\sim$. The details follow.

**Theorem 2.** If $E$ is a Riesz space, then $L_{so}(E, \mathbb{R})$ and $L_{s\sigma o}(E, \mathbb{R})$ are both bands of $E^\sim$.

**Proof.** We only show that $L_{so}(E, \mathbb{R})$ is a band of $E^\sim$. That $L_{s\sigma o}(E, \mathbb{R})$ is a band can be proven in a similar manner. Note first that if $|g| \leq |f|$ holds in $E^\sim$ with $f \in L_{so}(E, \mathbb{R})$, then from Theorem 1 it follows that $g \in L_{so}(E, \mathbb{R})$. That is $L_{so}(E, \mathbb{R})$ is an ideal of $E^\sim$. To see that the ideal $L_{so}(E, \mathbb{R})$ is a band, let $0 \leq f_\lambda \uparrow f$ in $E^\sim$ with $(f_\lambda) \subset L_{so}(E, \mathbb{R})$, and let $0 \leq x_\alpha \overset{\text{uo}}{\to} 0$ in $E$. Then for each fixed $\lambda$ we have

$$0 \leq f(x_\alpha) = ((f - f_\lambda)(x_\alpha) + f_\lambda(x_\alpha)) \overset{\alpha}{\to} 0.$$

So, $f(x_\alpha) \overset{\alpha}{\to} 0$. Thus, $f \in L_{so}(E, \mathbb{R})$, and the proof is finished. \hfill \Box
Problem 1. Can we find a so-continuous operator $T : E \to F$ between Riesz spaces whose modulus is not so-continuous?

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