Term structure modelling with overnight rates beyond stochastic continuity

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The LIBOR reform

- London Interbank Offered Rate (LIBOR), computed as the trimmed average of rates reported by a panel of banks, for five currencies (CHF, EUR, GBP, JPY, USD) and seven tenors (1D, 1W, 1M, 2M, 3M, 6M, 1Y).
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- July 2017: *The future of LIBOR* speech by Andrew Bailey (FCA): announcement of LIBOR discontinuation after 2021.

- Transition towards **transaction-based overnight rates** as benchmark rates. ARRC, June 2017: Secured Overnight Funding Rate (SOFR) in the US.
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- Transition towards transaction-based overnight rates as benchmark rates. ARRC, June 2017: Secured Overnight Funding Rate (SOFR) in the US.
- FCA, March 2021: cessation of LIBOR on 31/12/2021. Complete discontinuation after June 2023.
- May 2021: *Life after LIBOR* speech by Andrew Bailey:
  
  “transition to the most robust overnight rates, underpinned by deep underlying markets, will support a stronger more transparent financial system and ultimately benefit all market participants”.
Adoption of overnight rates

Source: ISDA-Clarus RFR adoption indicator, February 2023.
Trades in SOFR

Chart 5: SOFR Trade Count by Product (thousands)

Source: DTCC SDR
Alternative risk-free rates

- Alternative risk-free rates (RFRs) are (nearly) risk-free overnight rates;
- SOFR (US), SONIA (UK), TONA (JP), SARON (CH), €STR (EU);

Being risk-free, RFRs reflect the current level of policy rates: as documented by Backwell and Hayes (2022), most of the variation in SONIA over the years 2016-2020 occurs in correspondence to the meeting dates of the Monetary Policy Committee of the Bank of England. The meeting dates follow a predetermined calendar. RFRs are prone to upward/downward spikes at regulatory reporting dates: SOFR is on average 20.25 bps higher at quarter-ends compared to other dates (source: Klingler and Syrstad (2021), period: 08/2014 - 12/2019). These facts bring evidence of the presence of stochastic discontinuities: new information arriving at pre-determined dates that affects the level of the rates.
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SOFR behavior: spikes and hikes

SOFR time series from 01/01/2018 until 12/12/2022 (source: Refinitiv).
SOFR behavior: spikes and hikes

- Let us consider the spike observed on 17/09/2019. According to Anbil et al. (2020):

  *Strains in money markets in September seem to have originated from routine market events, including a corporate tax payment date and Treasury coupon settlement. The outsized and unexpected moves in money market rates were amplified by a number of factors.*

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- Presence of stochastic discontinuities in the RFR dynamics. This phenomenon is playing an important role in recent works:
  - Andersen and Bang (2020): spikes in the SOFR dynamics, both at totally inaccessible times and at anticipated times.
  - Gellert and Schlögl (2021): a diffusive HJM model for instantaneous forward rates, with jumps/spikes at fixed times in the short rate, inspired by SOFR.
  - Brace et al. (2022): diffusive HJM model with stochastic volatility.
  - Backwell and Hayes (2022): a short-rate model for the SONIA rate, based on a pure jump process with expected and unexpected jumps times.
  - Fontana et al. (2020): multi-curve framework with stochastic discontinuities.
A quick overview of the literature on RFR modelling

- General aspects of the Libor reform: Henrard (2019), Piterbarg (2020), Klingler and Syrstad (2021), Baig and Winters (2022).
- Mercurio (2018): short rate model for SOFR, adding a deterministic spread to the OIS rate.
- Lyashenko and Mercurio (2019): one of the first and most influential contributions, extending the classical Libor market model.
- Macrina and Skovmand (2020): rational model driven by an affine process.
- Willems (2020): extended SABR model applied to caplet pricing.
- Extensions of the Hull-White model: Hofman (2020), Turfus (2020), Hasegawa (2021), Xu (2022).
- Fontana (2023): general affine models for RFRs and pricing formulae.
- Skov and Skovmand (2021), Skov and Skovmand (2022): multi-factor Gaussian models for SOFR futures.
- Rutkowski and Bickersteth (2021): Vasiček model for SOFR, discussing pricing and hedging in the presence of funding costs and collateralization.
Outline

1. Numéraire, backward-looking and forward-looking rates;
2. an extended HJM framework;
3. the affine semimartingale setup;
4. an extended Hull-White model;
5. hedging problems.
The RFR numéraire

- We consider a continuous-time RFR process $\rho = (\rho_t)_{t \geq 0}$. In line with empirical evidence, $\rho$ is allowed to have expected and unexpected jumps.
- The numéraire $S^0$ asset:

$$S^0_t = \exp \left( \int_{(0,t]} \rho_u \eta(du) \right),$$

where $\eta(du) = du + \sum_{n \in \mathbb{N}} \delta_{\{t_n\}}(du)$.
- The set $\mathcal{T} := \{t_n; n \in \mathbb{N}\}$ of roll-over dates, at which $S^0$ is expected to jump.
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- The set $\mathcal{T} := \{t_n; n \in \mathbb{N}\}$ of roll-over dates, at which $S^0$ is expected to jump.
- Depending on the specification of $\rho$ and $\eta$, this setup includes:
  - classical short-rate approach (corresponding to $\mathcal{T} = \emptyset$);
  - discretely updated bank account at overnight frequency:
    \[ S^0_t = \prod_{t_{n+1} \leq t} \left( 1 + r_{t_n} (t_{n+1} - t_n) \right), \]
    where $r_{t_n}$ is the overnight rate for the time interval $[t_n, t_{n+1}]$.
- Denote by $P(t, T)$ the zero-coupon bond (ZCB) price at $t$ for maturity $T$. 
Backward-looking and forward-looking rates

- LIBOR rates are term rates: how to use RFRs to replace them?

\[ R(S, T) = \prod_{n \in \mathbb{N}} (S, T)(1 \frac{t_n}{P(t_n, t_{n+1})} - 1), \]

where \( \mathbb{N}(S, T) := \{ n \in \mathbb{N} : S \leq t_n \text{ and } t_{n+1} \leq T \} \).

According to the ISDA protocol, \( R(S, T) \) is chosen as the LIBOR fallback, up to an additive spread determined from historical data. This rate is backward-looking, since its value is only known at \( T \).

Forward-looking rate \( F(S, T) \): rate \( K \) such that the single-period swap (SPS) delivering \( (T - S)(R(S, T) - K) \) at maturity \( T \) has zero value at time \( S \).

CME Term SOFR and Refinitiv Term SONIA are forward-looking rates. 12/29/2021: ARRC endorsed CME term SOFR as forward-looking rate. The use of term SOFR for derivatives is currently restricted by ARRC, but there is increasing market pressure for derivatives referencing term SOFR.
Backward-looking and forward-looking rates

- LIBOR rates are term rates: how to use RFRs to replace them?
- The setting-in-arrears rate $R(S, T)$ is

$$R(S, T) := \frac{1}{T - S} \prod_{n \in N(S, T)} \left( \frac{1}{P(t_n, t_{n+1})} - 1 \right),$$

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Forward term rates

As in Lyashenko and Mercurio (2019), we can consider two types of forward rates:

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Consequence of the above definitions:

$$F(t, S, T) = R(t, S, T), \quad \text{for all } t \in [0, S].$$

The forward-looking forward rate $F(t, S, T)$ stops evolving at time $S$, while the backward-looking forward rate $R(t, S, T)$ continues to evolve until time $T$, with

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⇒ Forward-looking and backward-looking forward rates can be consolidated into a single process $R(\cdot, S, T)$. We call this process the **forward term rate**.
Forward term rates

Payoff $1 + (T - S)R(S, T)$ at maturity $T$ can be statically replicated as follows:
- buy-and-hold strategy in one ZCB with maturity $S$;
- at time $S$, invest 1 in a roll-over strategy remunerated at the overnight rate.

This implies the following (classical) representation of forward term rates:

$$R(t, S, T) = \frac{1}{T - S} \left( \frac{P(t, S)}{P(t, T)} - 1 \right),$$

where we extend ZCB bond prices beyond maturity by setting

$$P(t, S) = \frac{P(t, t_n(t))}{P(t_{n(t)-1}, t_{n(t)})} \prod_{n \in N(S,t)} \frac{1}{P(t_n, t_{n+1})}, \quad \text{for } t > S,$$

with $n(t) := \inf\{ n \in \mathbb{N} : t_n \geq t \}$. 
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with $n(t) := \inf\{n \in \mathbb{N} : t_n \geq t\}$.

Similarly to classical (single-curve) interest rate models, the family of ZCB prices $\{P(\cdot, T); T > 0\}$ constitutes the fundamental basis of a term structure model.
An extended HJM framework

We start by specifying \textbf{ZCB prices} as follows:

\[
P(t, T) = \exp \left( - \int_{(t, T]} f(t, u) \eta(du) \right),
\]

where \( \eta(dt) = dt + \sum_{n \in \mathbb{N}} \delta_{\{t_n\}}(dt) \) and we assume that

\[
f(t, T) = f(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \varphi(s, T) dW_s + V(t, T),
\]

with \( W \) a \( d \)-dim. Brownian motion and \( V(\cdot, T) \) a pure jump process such that

\[
\{ \Delta V(\cdot, T) \neq 0 \} \subseteq \Omega \times S,
\]

where \( S = \{s_1, \ldots, s_M\} \).

The set \( S \) contains expected jump dates, i.e., dates at which the RFR \( \rho \) and forward term rates are expected to exhibit jumps.

Remarks:

- Lévy-type jumps can be included;
- we do not exclude the case \( S \cap \mathcal{T} \neq \emptyset \);
- \( S \) can be generalized to a countable family of predictable times.
Martingale representation property

The representation of instantaneous forward rates implicitly uses the following.

**Assumption**

There exists a family \((\xi_1, \ldots, \xi_M)\) of random variables such that \(\xi_i\) is \(\mathcal{F}_{s_i}\)-measurable, for all \(i = 1, \ldots, M\), and every local martingale \(N = (N_t)_{t \geq 0}\) can be represented as

\[
N = N_0 + \int_0^\cdot \theta_t dW_t + \sum_{i=1}^M f_i(\xi_i) \mathbf{1}_{[s_i, +\infty[},
\]

where \(f_i(\cdot) : \Omega \times X \to \mathbb{R}\) is a \((\mathcal{F}_{s_i^-} \otimes \mathcal{B}_X)\)-measurable function such that

\[
E[f_i(\xi_i)|\mathcal{F}_{s_i^-}] = 0 \quad \text{a.s.}
\]

We denote by \(\mathcal{H}\) the space of all such functions \(f = (f_1, \ldots, f_M)\).
Technical assumptions

The following conditions hold a.s.:

(i) the initial forward curve \( T \to f(0, T) \) is \( (\mathcal{F}_0 \otimes \mathcal{B}_{\mathbb{R}_+}) \)-measurable, real-valued and satisfies \( \int_0^T |f(0, u)| du < +\infty \), for all \( T > 0 \);

(ii) the drift process \( \alpha : \Omega \times \mathbb{R}^2 \to \mathbb{R} \) is progressively measurable, satisfies \( \alpha(t, T) = 0 \) for \( T < t \), and

\[
\int_0^T \int_0^u |\alpha(s, u)| ds \eta(du) < +\infty, \quad \text{for all } T > 0;
\]

(iii) the volatility process \( \varphi : \Omega \times \mathbb{R}_+^d \to \mathbb{R}^d \) is progressively measurable and satisfies \( \varphi(t, T) = 0 \) for \( T < t \), and

\[
\sum_{i=1}^d \int_0^T \left( \int_0^u |\varphi^i(s, u)|^2 ds \right)^{1/2} \eta(du) < +\infty, \quad \text{for all } T > 0;
\]

(iv) the stochastic discontinuity process \( V(\cdot, T) \) satisfies \( \int_0^T |\Delta V(s, u)| du < +\infty \) for all \( s \in S \) and \( \Delta V(t, T) = 0 \) for \( T < t \).
An extended HJM framework

Objective: characterize when $Q$ is a risk-neutral measure, i.e., $S^0$-denominated ZCB prices are local martingales under $Q$. This ensures absence of arbitrage in the sense of *no asymptotic free lunch with vanishing risk* (NAFLVR, see Cuchiero et al. (2016)), with respect to the numéraire $S^0$. 
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As a preliminary to the next theorem, we define

$$
\bar{\alpha}(t, T) := \int_{[t, T]} \alpha(t, u) \eta(du),
$$

$$
\bar{\varphi}(t, T) := \int_{[t, T]} \varphi(t, u) \eta(du),
$$

$$
\bar{V}(t, T) := \int_{[t, T]} \Delta V(t, u) \eta(du).
$$
HJM-type conditions

Theorem

Q is a risk-neutral measure if and only if (some integrability properties hold) and the following four conditions are satisfied:

(i) \[ f(t, t) = \rho_t, \]

(ii) \[ \bar{\alpha}(t, T) = \frac{1}{2} \| \bar{\varphi}(t, T) \|^2 \]

(iii) for every \( j = 1, \ldots, N \) it holds that

\[ f(t_j-, t_j) = \rho_{t_j-} - \log \left( E \left[ e^{-\Delta \rho_{t_j}} | \mathcal{F}_{t_j-} \right] \right), \]

(iv) for every \( i = 1, \ldots, M \) it holds that

\[ E \left[ e^{-\Delta \rho_{s_i} \delta \tau(s_i)} \left( e^{-\int_{(s_i, \tau]} \Delta V(s_i, u) \eta(du)} - 1 \right) | \mathcal{F}_{s_i-} \right] = 0. \]

Remark: if \( S \cap \mathcal{T} = \emptyset \), then conditions (i) and (iii) can be jointly written as

\[ f(t, t) = \rho_t, \quad \eta(dt) \otimes dQ \text{-a.e.} \]
Example: a Cheyette-type model

An extension of the Cheyette model with stochastic discontinuities:

- for simplicity, no roll-over dates \((T = \emptyset)\), so that \(S^0 = \exp(\int_0^\cdot r_u du)\);
- forward rates are specified as follows:

\[
f(t, T) = f(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \varphi(s, T) dW_s + \sum_{s_i \leq t} (\alpha_i(T) + \xi_i g_i(T)),
\]

with independent \(\xi_i \sim \mathcal{N}(\mu_i, \sigma^2_i)\), for \(i = 1, \ldots, M\);
- separable volatility structure (one factor, for illustration):

\[
\varphi(t, T) = \frac{a(T)}{a(t)} b(t) \quad \text{and} \quad g_i(T) = a(T) B_i.
\]

- Under this volatility structure, it holds that

\[
f(t, T) = f(0, T) + \frac{a(T)}{a(t)} X_t + U(t, T),
\]

where \(X\) is a mean-reverting Gaussian Markov process with mean-reversion speed \(\partial_t \log(a(t))\), diffusion coefficient \(b\) and jumps at dates \(\{s_1, \ldots, s_T\}\), and \(U(t, T)\) is a deterministic function.
The presence of expected jump times requires an extension of affine processes: affine semimartingales generalize affine processes by allowing for jumps at fixed times with possibly state-dependent jump sizes (see Keller-Ressel et al. (2019)).

An affine semimartingale $X = (X_t)_{t \geq 0}$ taking values in $\mathbb{R}^m_+ \times \mathbb{R}^n$ satisfies

$$E \left[ e^{\langle u, X_T \rangle} \mid F_t \right] = \exp \left( \phi_t(T, u) + \langle \psi_t(T, u), X_t \rangle \right),$$

for all $u \in \mathcal{U} = \mathbb{C}^m \times i\mathbb{R}^n$, where the functions $\phi_t(T, u)$ and $\psi_t(T, u)$ satisfy generalized Riccati equations.
The affine framework

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for all \( u \in U = \mathbb{C}_+^m \times i\mathbb{R}^n \), where the functions \( \phi_t(T, u) \) and \( \psi_t(T, u) \) satisfy generalized Riccati equations.

Short-rate approach: let the RFR be given by

\[
\rho_t = \ell(t) + \langle \Lambda, X_t \rangle,
\]

for all \( t \geq 0 \), where the function \( \ell \) fits the initially observed term structure.

Proposition

The joint process \((X, \int_0^t \rho_u \eta(du))\) is an affine semimartingale.

- Similar to the enlargement of the state-space approach of Duffie et al. (2003).
- Fourier-based methods for pricing a variety of interest rate derivatives.
An example: an extended Hull-White model

Assume that $\rho = (\rho_t)_{t \geq 0}$ satisfies

$$d\rho_t = (\alpha(t) + \beta \rho_t)\,dt + \sigma dW_t + dJ_t,$$

where $J$ is a pure jump process independent of $W$:

$$J = \sum_{i=1}^{M} \xi^i 1_{[s_i, +\infty[},$$

In the Gaussian case (i.e., $(\xi_i)_{i=1,\ldots,M}$ independent and Gaussian):

- explicit formula for ZCB prices;
- Black-type formula for post-Libor caplets/floorlets.
Hedging with stochastic discontinuities

- Stochastic discontinuities induce market incompleteness.
- We therefore make use of the concept of local risk-minimization.
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Recall that, in our setup, every local martingale $N$ can be represented as

$$N = N_0 + \int_0^t \theta_s dW_s + \sum_{i=1}^M f_i(\xi_i)1_{[s_i, +\infty[}.$$

Suppose that the market contains a single risky asset with price process $X = X_0 + A + M$, where $A$ is a predictable process of finite variation, and $M = \int_0^t \eta_s dW_s + \sum_{i=1}^\infty \omega_i(\xi_i)1_{[s_i, +\infty[}$ is a square-integrable martingale.

For instance, $X$ can represent the price process of a SOFR future contract (currently the most liquid SOFR product).
Hedging with stochastic discontinuities

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Recall that, in our setup, every local martingale $N$ can be represented as

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- Suppose that the market contains a single risky asset with price process

$$X = X_0 + A + M,$$

where

- $A$ is predictable process of finite-variation,
- $M = \int_0^\cdot \eta_t dW_t + \sum_{s_i \leq \cdot} w_i(\xi_i)$ is a square-integrable martingale.

- For instance, $X$ can represent the price process of a SOFR future contract (currently the most liquid SOFR product).
Hedging with stochastic discontinuities

Let $H \in L^2$ be an $\mathcal{F}_T$-measurable payoff. We denote by $\Theta$ the set of predictable processes $\zeta$ such that $E[\int_0^T \zeta_u^2 d\langle M \rangle_u + (\int_0^T \zeta_u dA_u)^2] < +\infty$.

Definition

- We call $H$-admissible strategy a pair $\varphi = (\zeta, V)$, where $\zeta = (\zeta_t)_{t \in [0, T]} \in \Theta$ and $V = (V_t)_{t \in [0, T]}$ is an adapted process such that $V_T = H$ a.s.
- We say that an $H$-admissible strategy $\varphi = (\zeta, V)$ is locally risk-minimizing if the associated cost process

$$C_t(\varphi) := V_t - \int_0^t \zeta_u dX_u,$$

for $t \in [0, T]$, is a square-integrable martingale strongly orthogonal to $M$.

Remarks:

- $\zeta_t$ and $V_t$ represent respectively the positions held in the traded security and the portfolio value at time $t$, for all $t \in [0, T]$;
- if $X$ satisfies the so-called structure condition, the above definition is equivalent to the original definition of Schweizer (1991).
Hedging with stochastic discontinuities

- By absence of arbitrage, there exists a predictable process $\lambda$ such that $A = \int_0^\cdot \lambda_u \, d\langle M \rangle_u$. In particular, this implies that

$$\Delta A_{s_i} = \lambda_{s_i} E[(\Delta M_{s_i})^2 | \mathcal{F}_{s_i-}] \text{ a.s., for all } i = 1, \ldots, M.$$  

- Assume that $\hat{Z} := E(-\int_0^\cdot \lambda_u dM_u)$ is a strictly positive square-integrable martingale and define the minimal martingale measure by $d\hat{Q} = \hat{Z}_T dQ$.
Hedging with stochastic discontinuities

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  \[
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  \]

- Assume that $\hat{Z} := \mathcal{E}(-\int_0^\cdot \lambda_u dM_u)$ is a strictly positive square-integrable martingale and define the minimal martingale measure by $d\hat{Q} = \hat{Z}_T dQ$.

- We can then define the $\hat{Q}$-martingale $\hat{H} = (\hat{H}_t)_{t \in [0, T]}$ by
  \[
  \hat{H}_t := \hat{E}[H|\mathcal{F}_t], \quad \text{ for all } t \in [0, T],
  \]
  where we denote by $\hat{E}$ the expectation with respect to $\hat{Q}$.
Hedging with stochastic discontinuities

- By absence of arbitrage, there exists a predictable process $\lambda$ such that $A = \int_0^\cdot \lambda_u \, d\langle M \rangle_u$. In particular, this implies that
  \[
  \Delta A_{s_i} = \lambda_{s_i} E[(\Delta M_{s_i})^2 | F_{s_i-}] \text{ a.s., for all } i = 1, \ldots, M.
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- Assume that $\hat{Z} := \mathcal{E}(-\int_0^\cdot \lambda_u dM_u)$ is a strictly positive square-integrable martingale and define the minimal martingale measure by $d\hat{Q} = \hat{Z}_T \, dQ$.

- We can then define the $\hat{Q}$-martingale $\hat{H} = (\hat{H}_t)_{t \in [0, T]}$ by
  \[
  \hat{H}_t := \hat{E}[H | F_t], \quad \text{for all } t \in [0, T],
  \]
  where we denote by $\hat{E}$ the expectation with respect to $\hat{Q}$.

- By Bayes’ formula, $\hat{H} = N/\hat{Z}$, with $N_t := E[\hat{Z}_T H | F_t]$, for all $t \in [0, T]$.

- As a consequence of the martingale representation assumption, we have that
  \[
  N = N_0 + \int_0^\cdot \theta_u \, dW_u + \sum_{s_i \leq t} \Delta N_{s_i}.
  \]
Hedging with stochastic discontinuities

**Proposition**

Let $H$ be an $\mathcal{F}_T$-measurable random variable such that $\sup_{t \in [0, T]} \hat{H}_t \in L^2$.

Define the predictable process

$$\zeta^H_t := (\hat{Z}_t^{-1} \eta_t^{-1} \theta_t + \hat{H}_t - \lambda_t) \delta_{S^c}(t) + \frac{E[\Delta \hat{H}_t \Delta M_t | \mathcal{F}_t^{-}]}{E[(\Delta M_t)^2 | \mathcal{F}_t^{-}]} \delta_S(t).$$

If $\zeta^H \in \Theta$, then the strategy $\varphi^H = (\zeta^H, V^H)$ is locally risk-minimizing, where $V^H_t = \hat{H}_t$, for all $t \in [0, T]$.

**Remarks:**

- perfect replication at all times $t \in [0, T] \setminus S$, when the only active source of randomness is the Brownian motion $W$;
- at the discontinuity dates $S = \{s_1, \ldots, s_M\}$, the strategy $\zeta^H_{s_i}$ is determined by a linear regression of $\Delta \hat{H}_{s_i}$ onto $\Delta X_{s_i}$, conditionally on $\mathcal{F}_{s_i^{-}}$:

$$\zeta^H_{s_i} = \frac{\text{Cov}(\Delta \hat{H}_{s_i}, \Delta X_{s_i} | \mathcal{F}_{s_i^{-}})}{\text{Var}(\Delta X_{s_i} | \mathcal{F}_{s_i^{-}})},$$

- In the paper, explicit formula for the locally risk-minimizing strategy of a SOFR term caplet with respect to a SOFR future.
Thank you for your attention

For more information:

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