Error Correcting Codes

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Abstract Here we present some revised arguments to a randomized algorithm proposed by Sudan to find the polynomials of bounded degree agreeing on a dense fraction of a set of points in $\mathbb{F}^2$ for some field $\mathbb{F}$.

1 Introduction

Here we will discuss some concepts in the field of error-correcting codes.

Definition 1. Given $\Sigma$ a collection of symbols, and $x, y \in \Sigma^n$. We define the hamming distance between $x$ and $y$ denoted as $HD(x, y)$ as $|\{i \in [n] : (x)_i \neq (y)_i\}|$. That is, the number of indices at which $x$ and $y$ differ.

Example 1. Given $\Sigma = \{0, 1, 2\}$ and $x = “201”$, and $y = “222”$. We have that $HD(x, y) = 1$.

Definition 2. Let $\Sigma$ be a collection of symbols and $n, k, \delta \in \mathbb{Z}$. We say $C \subset \Sigma^n$ is a $[n, k, \delta]$ code if $|C| = |\Sigma|^k$, and $\forall x, y \in C, HD(x, y) \geq \delta$.

Definition 3. Let $\Sigma$ be a collection of symbols and $C$ a $[n, k, \delta]$ code. If $\tau \in \mathbb{Z} : 2\tau + 1 \leq \delta$, then we say $C$ is a $\tau$ error correcting code.

Definition 4. Let $F$ be a finite field of cardinality $n$. Let $C$ be a $[n, d + 1, n - d]$ code of an alphabet $\Sigma$, we say $C$ is a Reed-Solomon Code if $C = \{p(0) | p(w) \cdots | p(w^{|F| - 1}) : p(x) \in F[x], \text{deg}(p) \leq d\}$. Here $|$ denotes string concatenation, and $w \in F$, is a generator of $F^*$.

Definition 5. We will refer the maximum-likelihood decoding problem as the following task: Given a $[n, k, \delta]$ code, a string $s \in \Sigma^n$, a string in $c \in C$ such that $HD(s, c) \leq HD(s, x)$, $\forall x \in C$. We will refer to the list decoding problem as: Given a string $s \in \Sigma^n$, a $[n, k, \delta]$ code, and a parameter $\tau \in \mathbb{N}$, return all $c \in C$ such that $HD(s, c) \leq \tau$.

2 Algorithm

Remark 1. We will present a randomized algorithm by Sudan-’96, for the following problem: Given a field $F$, $\{(x_i, y_i)\}_{i=1}^n \subset F^2$ and parameters $t, d \in \mathbb{N}$, find all $f(x) \in F[x]$ such that $|\{i \in [n] : f(x_i) = y_i\}| \geq t$, and $\text{deg}(f) \leq d$.

Remark 2. We define concept of weighted degree which will be relevant to the randomized algorithm to be presented. Given $(w_x, w_y) \in \mathbb{Z}^2$ which we will call weights and a bivariate monomial in $x, y, c_{ij}x^iy^j$, we say the weighted degree of such a monomial is $i \cdot w_x + j \cdot w_y$. Given a bivariate polynomial $Q(x, y) \in F[x, y]$, we say the weighted degree of $Q(x, y) = \sum_{i,j} c_{ij}x^iy^j$ to be the maximum of the weighted degrees of its monomials.

Algorithm 1. Define the following randomized algorithm: Let $\{(x_i, y_i)\}_{i \in [n]}$, $d, t \in \mathbb{N}$ be inputs to the algorithm, and $m, l$ parameters to be determined to optimize the algorithm. Then:

- Find a $P(x, y) \in F[x, y]$ such that $P(x, y)$ has weighted degree with weights $(1, d)$ at most $m + l \cdot d$, $P(x, y)$ is not identically zero, and $P(x, y)$ vanishes on $\{(x_i, y_i)\}_{i \in [n]}$. That is, $P(x_i, y_i) = 0, \forall i \in [n]$.
• Factor \( P(x, y) \) into irreducible polynomials in \( F[x, y] \). (2)

• Check all functions \( f(x) \in F[x] \) of degree at most \( d \), such that \( (y - f(x)) | P(x, y) \), and \( f(x_i) = y_i \) for at least \( t \) distinct choices of \( i \in [n] \). (3)

Remark 3. We will justify that this algorithm runs in polynomial time.

Proposition 1. The polynomial as described in step (1) can be found in polynomial time, with respect to the size of the field, if such a polynomial exists.

Proof. By the conditions imposed by the weighted degree constraint, we can write \( P(x, y) \in F[x, y] \) as \( \sum_{j=0}^{m} \sum_{i=0}^{(l-1)d} c_{ij} x^i y^j \), because \( j \leq l \), \( i \leq m + (l - j)d \) implies that \( (i, j) \cdot (1, d) \leq (m + (l - j)d, j) \cdot (1, d) = m + ld \), which is the weighted degree of \( P(x, y) \). To find the polynomial which satisfies the conditions in (1), we require that \( \sum_{j=0}^{l} \sum_{i=0}^{m+ld} c_{ij} x^i y^j = 0 \), \( \forall k \in [n] \). Let \( |F| = N \). Using a brute force approach to determine the appropriate values of \( c_{ij} \), we can obtain a solution in \( O(n \cdot N^{(m+ld)l}) \), which is polynomial in \( N \) for fixed parameters \( m, l, d \). However, this can be solved in polynomial time with respect to the number of constraints \( n \).

Proposition 2. If the parameters \( m, l \) are such that \( (m+1)(l+1) + d\binom{l+1}{2} > n \), then a function \( F(x, y) \in F[x, y] \) as described in (1) exists.

Proof. Let \( \eta = (m+1)(l+1) + d\binom{l+1}{2} \). Note that if \( F(x, y) \) is defined as \( F(x, y) = \sum_{j=0}^{n} \sum_{i=0}^{m+ld} c_{ij} x^i y^j \), then there are \( \eta \) many \( x_{ij} \)'s. To find the polynomial \( F(x, y) \), we need to solve the system \( A\vec{x} = 0 \), where \( \vec{x} \) represents the \( c_{ij} \), and \( A \) has dimensions \( n \times \eta \). So this amounts to finding the null space of \( A \). Under the assumption that \( \eta > n \), we have that \( \dim(N(A)) \geq 1 \), where \( N(A) \) is the null space of \( A \). Hence, we may choose a \( y \in N(A) \setminus \{0\} \) to obtain the desired \( c_{ij} \)'s.

Proposition 3. If \( F(x, y) \in F[x, y] \) satisfies (1), and \( f(x) \in F[x] \) satisfies \( \{|i \in [n]: f(x_i) = y_i\}| \geq t \), and \( t > m + ld \), then \( y - f(x) \) divides \( P(x, y) \).

Remark 4. Let \( f(x) \in F[x] \). Denote the condition \( \{|i \in [n]: f(x_i) = y_i\}| \geq t \) as (*), and say \( f(x) \) satisfies (*) should it be the case.

Proof. Let \( f(x) \in F[x] \) satisfy (*). We claim that \( P(x, f(x)) \) is identically zero. Since \( P(x, y) \) has \( (1, d) \) weighted degree at most \( m + ld \), we have that \( P(x, f(x)) \) (as a uni-variate polynomial) has degree at most \( m + ld \) since \( f(x) \) has degree at most \( d \). However \( P(x, f(x)) = 0 \) whenever \( x = x_i \), for some \( i \in [n] \). If \( f(x) \) satisfies (*), then there are at least \( t \) zeros. Under the assumption that \( t > m + ld \), we have that the number of roots of \( P(x, f(x)) \) is greater than its degree, so \( P(x, f(x)) \equiv 0 \). Consider \( P(x, y) = P_t(x) = \sum_{j=0}^{n} \sum_{i=0}^{m+ld} c_{ij} x^i y^j \). Since \( P_t(f(x)) = 0 \), we have that \( P_t(y) \) has a root \( f(x) \). By the division algorithm, \( (y - f(x)) \) divides \( P_t(y) = P(x, y) \), which is the claim.

Remark 5. It remains to choose the parameters \( m, l \) such that \( t > m + ld \) and \( (m+1)(l+1) + d\binom{l+1}{2} > n \). We can rephrase the condition to be \( (m+1)(l+1) + d\binom{l+1}{2} \geq n + 1 \). Observe that this condition yields \( m \geq \frac{n+1-d\binom{l+1}{2}}{l+1} - 1 \). Suppose that we want \( t \geq m + ld + 1 \) \( \implies \) \( t \geq \frac{n+1-d\binom{l+1}{2}}{l+1} + ld = \frac{n+1}{l+1} + dl = \frac{n+1}{l+1} + \frac{d}{2} \). To find the minimum of this function with respect to \( l \), we perform a first derivative which yields \( -\frac{(n+1)}{(l+1)^2} + \frac{d}{2} = 0 \implies \frac{d}{2} = \sqrt{\frac{2(n+1)}{l+1}} - 1 \). Substituting the expression for \( l \) in for the expression on \( m \), we obtain that \( m \geq \frac{n+1-d\binom{l+1}{2}}{l+1} - 1 = \frac{n+1}{l+1} - \frac{d}{2} - 1 = \frac{2(n+1)}{l+1} - \frac{d}{2} - 1 \). This yields for the condition on \( t \) that \( t \geq m + ld + 1 \geq \frac{d}{2} + d \cdot \left( \sqrt{\frac{2(n+1)}{l+1}} - 1 \right) = \frac{d}{2} + \sqrt{2(n+1)d} - d = \sqrt{2(n+1)d} - \frac{d}{2} \). This will allow us to make the following claim, which follows from the previous propositions.

Corollary 1. Given a field \( F \) and a set of points \( \{(x_i, y_i)\}_{i \in [n]} \subset F^2 \), and parameters \( l, t \in \mathbb{N} \) such that \( t \geq d \cdot \left( \sqrt{\frac{2(n+1)}{d}} - \frac{d}{4} \right) \), then there is a polynomial time algorithm in \( n \) which finds all polynomials \( f(x) \in F[x] \) which satisfy (*), and have degree at most \( d \).
Proof. Setting \( m = \left\lfloor \frac{d}{2} \right\rfloor - 1 \) gives and \( l = \left\lfloor \frac{2(n+1)}{d} \right\rfloor - 1 \) gives that \( (m+1)(l+1) + d\left(\frac{l+1}{2}\right) \geq n+1 \). By proposition 2, a function \( P(x, y) \) not identically zero satisfying that \( P(x_i, y_i) = 0, \forall i \in [n] \) exists. Under the assumption that \( t \geq d \cdot \left\lfloor \frac{2(n+1)}{d} \right\rfloor - \left\lfloor \frac{d}{2} \right\rfloor > m + ld \), we have that \( y - f(x) \) divides \( P(x, y) \) should such an \( f(x) \in F[x] \) satisfy (*). By step 3 in the algorithm, \( f(x) \) will be reported as output. \( \square \)

**Definition 6.** Denote the tuple of \( k \) variables \( (x_1, \ldots, x_k) = \bar{x} \). Let \( F \) be a field, \( H \subset F \), and \( g : H^k \to F \). Given parameters \( t, d \) and \( \epsilon \) as in Remark 6, output all polynomials \( f \) of degree at most \( d \) such that \( |\{\bar{x} \in H^k : f(\bar{x}) = g(\bar{x})\}| \geq t \). Here define the degree of \( f \) to be the maximum degree of its monomials.

**Remark 6.** Let \( f(x) \in F[\bar{x}] \). We say \( f(x) \) satisfies the condition (*) if \( |\{\bar{x} \in H^k : f(\bar{x}) = g(\bar{x})\}| \geq t \).

**Definition 7.** Generalize the definition of weight to an \( n \)-variate polynomial. First, the \( \omega(1, \ldots, n) \) weighted degree of a monomial \( \prod_{i=1}^n x_i^{d_i} \) is defined to be \( \sum_{i=1}^n d_i \). Define the \( \omega(1, \ldots, n) \) weighted degree of an \( n \)-variate polynomial to be the maximum of the weighted degrees of its monomials (which have non-zero coefficients).

**Algorithm 2.** Define the following algorithm. Let \( F, H, k, t, d, \epsilon \) as in definition 6, and \( m, l \in \mathbb{N} \) be parameters to be determined.

- Find a \( P(x_1, \ldots, x_k, y) \in F[x_1, \ldots, x_k, y] \) such that \( P(\bar{x}, y) \) has weighted degree with weights \( (1, \ldots, 1, d) \) at most \( m + l \cdot d \), \( P(\bar{x}, y) \) is not identically zero, and \( P(\bar{x}, y) \) vanishes on \( \{ \bar{x}, g(\bar{x}) \} : \bar{x} \in H^k \). That is, \( P(\bar{x}, g(\bar{x})) = 0, \forall \bar{x} \in H^k \). 

- Factor \( P(\bar{x}, y) \) into irreducible polynomials in \( F[\bar{x}, y] \). (2)

- Check all functions \( f(\bar{x}) \in F[\bar{x}] \) of degree at most \( d \), such that \( y - f(\bar{x}) \mid P(\bar{x}, y) \), and \( f(\bar{x}) = g(\bar{x}) \) for at least \( t \) distinct choices of \( \bar{x} \in H^k \). (3)

**Remark 7.** This is more or less a generalization of the uni-variate case. We will state conditions for the existence of \( P(\bar{x}, y) \), and show that a polynomial \( f(\bar{x}) \in F[\bar{x}] \) satisfying (*) will be such that \( y - f(\bar{x}) \mid P(\bar{x}, y) \). Let \( |H| = h \).

**Proposition 4.** If \( m + ld \geq k(h-1) \), then a non-trivial polynomial \( P(\bar{x}, y) \in F[\bar{x}, y] \) vanishing on \( S = \{(\bar{x}, g(\bar{x})) \in F^{k+1} : \bar{x} \in H^k \} \) exists.

**Proof.** We want to show that the number of monomials of a \( k + 1 \) variate weighted degree polynomial is greater than \( |H|^k \). Then we can apply a similar argument for there being a non-trivial solution to the system of linear equations \( A\bar{x} = 0 \) for finding the coefficients of such a polynomial \( P(\bar{x}, y) \). We observe that a polynomial of \((1, \ldots, 1, d) \) weighted degree \( m + ld \) contains \( \sum_{j=0}^l (\frac{m+(l-j)d+k}{k}) \) monomials. This is because \( P(\bar{x}, y) = \sum_{j=0}^l P_j(\bar{x})y^j \), where \( P_j \) has total degree at most \( m + ld - jd = m + (l-j)d \). Hence, let \( M(\bar{Q}) \) denote the number of distinct monomials of a polynomial \( \bar{Q} \). Then \( M(P) = \sum_{j=0}^l M(P_j) = \sum_{j=0}^l (\frac{m+(l-j)d+k}{k}) \), which implies the claim. Now we would like \( M(P) > h^k \). To do this, we provide some lower bounds. Observe that \( M(P) = \sum_{j=0}^l (\frac{m+(l-j)d+k}{k}) \geq \sum_{j=0}^l (\frac{m+(l-j)d+k}{k}) \geq (\frac{m+ld+k}{k})^k \geq \frac{(k(h-1)+k)^k}{h^k} \), which proves the proposition. \( \square \)

**Proposition 5.** If \( t \geq (m + ld)h^{-k-1} \), where \( t \) is the number of agreements of a \( k \)-variate polynomial \( f \) on the set \( S = \{(\bar{x}, g(\bar{x})) \in F^{k+1} : \bar{x} \in H^k \} \), then \( y - f(\bar{x}) \mid P(\bar{x}, y) \).

**Proof.** We observe that \( \theta_f(\bar{x}) = P(\bar{x}, f(\bar{x})) \) is a \( k \)-variate polynomial of total degree \( m + ld \). Let \( Z(\bar{Q}, \bar{S}) = \{ \bar{x} \in S : Q(\bar{x}) = 0 \} \). By the Schwartz-Zippel Lemma, if \( |Z(\theta_f, H^k)| > \deg(\theta_f) \cdot |H|^{k-1} \), then \( \theta_f = P(\bar{x}, f(\bar{x})) \equiv 0 \). But \( t = |Z(\theta_f, H^k)| > \deg(\theta_f) \cdot |H|^{k-1} = (m + ld)h^{-k-1} \) by assumption so \( P(\bar{x}, f(\bar{x})) \equiv 0 \), and \( y - f(\bar{x}) \mid P(\bar{x}, y) \), since it is a root of \( P(\bar{x}, y) \). \( \square \)

**Lemma 1.** (Schwartz-Zippel) Let \( p(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n] \) be a polynomial of total degree \( d \) that is not equivalently 0. Let \( |S| < F \) be an arbitrary finite subset of the field \( F \). Then \( \mathbb{P}_{x \in_R S^d}[p(x) = 0] \leq \frac{d}{|S|} \).
Proof. We proceed by induction. Considering the uni-variate case yields that $\mathbb{P}_{x \in R} [p(x) = 0] \leq \frac{d}{|S|}$. This is true because $p$ has degree $d$ and since $p$ is not identically 0 we have that there are at most $d$ roots in $F$, and hence there are at most $d$ roots of $p$ in a finite subset $S \subset F$. Let $k = \deg_{x_n}(p)$. Then we may write $p(x) = x_n^k q(x_1, \ldots, x_{n-1}) + r(x_1, \ldots, x_n)$, where $q(x_1, \ldots, x_{n-1})$ has total degree at most $d - k$ and $r(x_1, \ldots, x_n)$ has $x_n$ degree strictly less than $k$. For a $\vec{x} \in R \ S^k$, we have that

$$\mathbb{P}[p(\vec{x}) = 0] = \mathbb{P}[p(\vec{x}) = 0 \mid q(\vec{x}) \neq 0] \mathbb{P}[q(\vec{x}) \neq 0] + \mathbb{P}[p(\vec{x}) = 0 \mid q(\vec{x}) = 0] \mathbb{P}[q(\vec{x}) = 0]$$

by Bayes Formula. But $\mathbb{P}[p = 0] \leq \mathbb{P}[q = 0] + \mathbb{P}[p = 0 \mid q \neq 0] \leq \frac{d - k}{|S|} + \frac{k}{|S|} = \frac{d}{|S|}$, by the inductive hypothesis. This completes the proof. 

Remark 8. There is some subtlety to the fact that $\mathbb{P}[p = 0 \mid q \neq 0] \leq \frac{k}{|S|}$. This is because if we are given that $q \neq 0$, then $p(x_1, x_2, \ldots, x_n)$ considered in that regard becomes a uni-variate polynomial of degree $k$ in the variable $x_n$ that is not identically 0, as we are fixing that $x_1, \ldots, x_{n-1}$, and we then may apply the inductive hypothesis. The application of the Schwartz-Zippel Lemma as presented here to proposition 5 is that the contrapositive of the statement of lemma 1 is sufficient to deduce that $\theta_I = 0$ as it has more than $\deg(f) \cdot h^{k-1}$ roots on $H^k$.

Theorem 1. If the parameters $d, t, |H| = h, k \in \mathbb{N}$, are such that $\frac{t}{dh^{k-1}} > \frac{k(h-1)}{d}$, and the open interval $\left(\frac{k(h-1)}{d}, \frac{t}{dh^{k-1}}\right)$ contains a positive integer, then Algorithm 2 as described above outputs all the desired polynomials.

Proof. Note that to obtain the non-trivial polynomial $P(\vec{x}, y) \in F[x_1, \ldots, x_k, y]$ which vanishes on $S$, we need by proposition 4 that $m + ld \geq k(h - 1)$. Similarly, to ensure that $f(\vec{x}) \in F[x_1, \ldots, x_k]$ of degree at most $d$ having at least $t$ agreements on $S$ satisfies that $y - f(x) \mid P(\vec{x}, y)$, we require that $(m + ld) < \frac{k}{h^{k-1}}$. Hence, we want $m, l \in \mathbb{N}$ such that $\frac{k(h-1)}{d} > m + ld > k(h - 1)$. Considering simpler case of setting $m = 0$, and finding the appropriate $l$, we obtain that such an $l$ exists precisely when $I = \left(\frac{k(h-1)}{d}, \frac{t}{dh^{k-1}}\right)$ contains a positive integer. Assuming that the given parameters are such that $l$ exists, by proposition 4, we obtain a non-zero polynomial vanishing on $S$. and by proposition 5, we have that a polynomial $f \in F[\vec{x}]$ of degree at most $d$ having at least $t$ agreements on $S$ will have $y - f$ divide $P(\vec{x}, y)$. Hence, Algorithm 2 will return the desired polynomials.

3 References

- M. Sudan, Decoding of Reed-Solomon Codes Beyond the Error Correction Bound. [http://people.csail.mit.edu/madhu/papers/1996/reads-journ.pdf](http://people.csail.mit.edu/madhu/papers/1996/reads-journ.pdf)