Unified description of resonance and decay phenomena

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Abstract

In the Feshbach projection operator formalism, resonance as well as decay phenomena are described by means of the complex eigenvalues and eigenfunctions of the non-Hermitian Hamilton operator $H_{\text{eff}}$ that appears in an intermediate stage of the formalism. The formalism can be applied for the description of isolated resonances as well as for resonances in the overlapping regime. Time asymmetry is related to the time operator which is a part of $H_{\text{eff}}$. An expression for the decay rates of resonance states is derived. For isolated resonance states $\lambda$, this expression gives the fundamental relation $\tau_\lambda = \hbar/\Gamma_\lambda$ between life time and width of a resonance state. A similar relation holds for the average values obtained for narrow resonances superposed by a smooth background term. In the cross over between these two cases (regime of overlapping resonances), the decay rate decreases monotonously as a function of increasing time.
I. INTRODUCTION

An old problem of standard quantum mechanics is the relation between the life time $\tau_\lambda$ of a resonance state and its width $\Gamma_\lambda$ that determines the Breit-Wigner energy distribution around the energy of this state in a scattering process. The problem arises, on the one hand, from the fact that the states of standard quantum mechanics are discrete, i.e. they are not coupled to the scattering states of the continuum and have an infinite long life time. An exception are Gamow states being decaying states with a finite life time. They have however a single-particle structure and do not correspond to the resonance states with more complicated many-particle structure and longer life time. Moreover, they do not exist in Hilbert space quantum mechanics. On the other hand, the spectroscopic information such as position $E_\lambda$ in energy and width $\Gamma_\lambda$ of a resonance state is obtained, usually, from the poles of the $S$ matrix, i.e. by continuing the $S$ matrix into the complex energy plane. The spectroscopic information obtained in this manner, can be considered to be reliable only for long-lived resonance states the widths of which are small. Thus, the relation between $\tau_\lambda$ and $\Gamma_\lambda$ of a resonance state can not be derived convincingly in standard quantum mechanics.

The present experimental situation is more convincing. A few years ago, high precision measurements have been performed \cite{1, 2} in which life times and decay widths of resonance states have been measured independently from one another. The experimental values confirm the relation

$$\tau_\lambda = \frac{\hbar}{\Gamma_\lambda}$$

(1)

to a high degree of accuracy. Recently, a violation of the exponential decay law at long times has been found experimentally in luminescence decays of many dissolved organic materials after pulsed laser excitation over more than 20 life times \cite{3}. The turnover into the non-exponential decay regime takes place sharply (in double-logarithmic scale) at long times.

As has been shown by Bohm et al. \cite{4, 5} (see also the recent papers \cite{6, 7}), a consistent mathematical theory with unification of resonance and decay phenomena according to the relation (1) leads to time asymmetry of quantum mechanics. While in the standard quantum theory the set of states and the set of observables are mathematically identified and described by the same Hilbert space, this is not so in the theory by Bohm et al. Instead, the Hardy space is introduced with time asymmetrical boundary conditions for time symmetric dynamical equations. According to this theory, time asymmetry is inherent in the dynamics
of a quantum system. One distinguishes between prepared states and detected observables: the observables can be detected only after the states are prepared. In the framework of this theory, a simple description of decaying states can be given for Gamow states. The description of resonance states with more complicated structure remains however a problem.

Another possibility to unify resonance and decay phenomena is to consider a non-Hermitian Hamilton operator instead of the Hermitian Hamilton operator basic of standard quantum mechanics. The eigenvalues of a non-Hermitian operator are complex and provide therefore the life times of resonance states in a natural manner via the relation (1). Numerical studies have been performed by using two different methods. One of these methods is based on the Feshbach projection operator (FPO) technique [8] while the other one uses complex scaling [9].

Using the FPO technique, the decay rates of resonance states are calculated some years ago for isolated as well as for overlapping resonance states. In these calculations, the resonance states are described by the eigenstates of a non-Hermitian Hamilton operator. While the decay rates are constant in time as long as the resonance states do not overlap, they start to oscillate in the neighborhood of branch points in the complex plane where the eigenvalues of two resonance states coincide. These oscillations are caused by the fact that the decay rate at the considered energy $E$ is determined not only by the life time of one individual resonance state but also by that of the neighboring one with which it overlaps. That means, the decay rate of an individual resonance state is ill defined in the regime of overlapping resonance states. Accordingly to this result, it holds $\Xi_\lambda(t) = \Phi_\lambda^*(t)$ for the dual basis vectors $\Xi_\lambda$ and $\Phi_\lambda$ of a complex symmetric non-Hermitian Hamilton operator in the neighborhood of the branch point (exceptional point) only at the time $t = t_0$ [11]. Taking into account the contributions of all overlapping resonance states at the energy $E$, the decay rate ceases to oscillate and decreases smoothly as a function of time [10].

Complex scaling is used in [12] in order to calculate time dependent observables. In this paper, the time-asymmetry problem in non-Hermitian quantum mechanics is discussed. It is argued that the non-Hermitian description of the system is valid on time scales that are long enough to regard only the localized part of the wave packet after the scattered part has left the interaction region. In the numerical calculations, isolated resonances are considered, for which the decay rate does not oscillate.

In other approaches, the quantum measurement and its relation to the arrow of time is
considered, see e.g. Schulman [13]. Recently, the passage time distribution for a spread-out quantum particle to transverse a specific region is calculated using a special quantum model for the detector involved [14]. In this model, the coupling between a collection of spins and their environment is enhanced by the detection of the particle. Such a model is not in contradiction with the results of the FPO method, according to which an enhancement of the coupling between system and environment will, generally, be generated in the regime of overlapping resonance states due to the alignment of the resonance states to the states of the environment [15, 16]. In the FPO formalism, the enhancement of the coupling is, however, independent of whether or not the particle is detected. It is an internal property of an open quantum system.

In the present paper, it will be shown that preparation and detection of resonance states are both involved also in the theory based on the FPO method. Thus, this method embodies the time asymmetry of quantum mechanics studied by Bohm et al. [4, 5]. It is involved in the non-Hermitian part of the Hamilton operator $H_{\text{eff}}$ that describes the resonance states in the FPO formalism and contains the time operator. Moreover, the FPO formalism allows also to study time asymmetry in a system with narrow resonance states (different from the Gamow states) and of resonance states near decay thresholds and in the overlapping regime.

In Sect. II of the present paper, the basic equations of the FPO formalism are given, including the expression for the resonance part of the $S$ matrix. The peculiarities of the FPO formalism are summarized and contrasted with the standard formalism in Sect. III. In the following Sect. IV, an expression for the time dependent decay rate is given that holds true for isolated as well as for overlapping resonance states. In Sect. V, the decay rate in the regime of overlapping resonances is considered in detail. The results are summarized in the last section.

II. BASIC RELATIONS OF THE FESHBACH PROJECTION OPERATOR FORMALISM

In the FPO formalism [17], the full function space is divided into two subspaces: the $Q$ subspace contains all wave functions that are localized inside the system and vanish outside of it while the wave functions of the $P$ subspace are extended up to infinity and vanish inside the system, see [8]. The wave functions of the two subspaces can be obtained by
standard methods: the $Q$ subspace is described by the Hermitian Hamilton operator $H_B$ that characterizes the closed system with discrete states, while the $P$ subspace is described by the Hermitian Hamilton operator $H_C$ that contains the continuum of scattering wave functions. In the FPO formalism, the closed system (defined by the Hamilton operator $H_B$) will be opened by coupling the wave functions of the $Q$ subspace to those of the $P$ subspace under the assumption $P + Q = 1$. Due to this coupling, the discrete states of the closed system pass into resonance states of the open system. The resonance states have, in general, a finite life time.

The basic equation of the FPO formalism

$$ (H - E) \Psi^E_C = 0 $$

has to be solved in the whole function space $P + Q$. It contains the decay of the subsystem localized in the $Q$ subspace, into the surrounding $P$ subspace where the decay products will be detected. The excitation of the states localized in the $Q$ subspace may take place via one of the channels $C$ included in (2) or by another process described by

$$ (H - E) \Psi^E_F = F, $$

where the inhomogeneity (source term) $F$ on the right-hand side of (3) describes the excitation of the state $\Psi^E_C$ by a process different from scattering. It may describe, e.g., the Coulomb excitation of nuclear states in photo-nuclear reactions, see [18]. In this case, $F = H_{\text{int}} \phi_T$ where $H_{\text{int}}$ is the interaction of the electromagnetic field with the target ground state $\phi_T$. Eq. (3) may describe also the excitation of an optically prepared sample of ultra-cold atoms. In the present paper, the value $F$ will not be specified. It is $F = 0$ only in the scattering process. It should be underlined however that both equations (2) and (3) are defined in the whole function space defined by $P + Q = 1$. Therefore, the Hamilton operator $H$ appearing in these equations is Hermitian.

In solving (2) and (3) in the function space $P + Q = 1$ by using the FPO technique, the non-hermitian Hamilton operator

$$ H_{\text{eff}} = H_B + \sum_C V_{BC} \frac{1}{E^+ - H_C} V_{CB} $$

appears which contains $H_B$ as well as an additional non-hermitian term that describes the coupling of the resonance states via the common environment. Here $V_{BC}$, $V_{CB}$ stand for the
coupling matrix elements between the eigenstates of $H_B$ and the environment \[8\] that may consist of different continua $C$. The operator $H_{\text{eff}}$ is symmetric, 

$$(H_{\text{eff}} - z\lambda) \phi_{\lambda} = 0,$$  

its eigenvalues $z\lambda$ and eigenfunctions $\phi_{\lambda}$ are complex. The eigenvalues provide not only the energies of the resonance states but also their widths. The eigenfunctions are biorthogonal. For details see \[8\].

The eigenvalues and eigenfunctions of $H_B$ contain the interaction $u$ of the discrete states which is given by the nondiagonal matrix elements of $H_B$. This interaction is of standard type in closed systems and may be called therefore internal interaction. The eigenvalues and eigenfunctions of $H_{\text{eff}}$ contain additionally the interaction $v$ of the resonance states via the common continuum ($v$ is used here instead of the concrete matrix elements of the second term of $H_{\text{eff}}$). This part of interaction is, formally, of second order and may be called external interaction. While $u$ and $\text{Re}(v)$ cause level repulsion in energy, $\text{Im}(v)$ is responsible for the bifurcation of the widths of the resonance states (resonance trapping). The phenomenon of resonance trapping has been proven experimentally in microwave cavities \[19\].

Since the effective Hamilton operator \[4\] depends explicitly on energy $E$, so do its eigenvalues $z\lambda$ and eigenfunctions $\phi_{\lambda}$. Far from thresholds, the energy dependence is weak, as a rule, in an energy interval of the order of magnitude of the width of the resonance state. The solutions of the fixed-point equations $E_{\lambda} = \text{Re}(z\lambda)\big|_{E=E_{\lambda}}$ and of $\Gamma_{\lambda} = -2 \text{Im}(z\lambda)\big|_{E=E_{\lambda}}$ are numbers that coincide with the poles of the $S$ matrix. In the FPO formalism, however, it is not necessary to consider the poles of the $S$ matrix since the spectroscopic information on the system follows directly from the complex eigenvalues $z\lambda$ of $H_{\text{eff}}$. Moreover, in the physical observables related to the $S$ matrix the eigenvalues $z\lambda$ with their full energy dependence are involved, see \[16\]. Due to this fact, information on the vicinity (in energy) of the considered resonance states such as the position of decay thresholds and of neighboring resonance states is involved in the $S$ matrix and can be received. Such an information can not be obtained from the poles of the $S$ matrix being (energy-independent) numbers.

In contrast to the (parametric) trajectories of the eigenvalues of a Hermitian Hamilton operator, those of a non-Hermitian one may cross. The crossing points are branch points in the complex energy plane (called exceptional points in the mathematical literature). Physically, they are responsible for the avoided level crossing phenomenon appearing in
their vicinity. More precisely: in approaching the branch points under different conditions, we have level repulsion (together with widths equilibration) or widths bifurcation (together with level attraction). For details see [8].

The eigenfunctions $\phi_\lambda$ of $H_{\text{eff}}$ are complex and biorthogonal,

$$\langle \phi_\lambda^* | \phi_{\lambda'} \rangle = \delta_{\lambda,\lambda'}$$

with the consequence that $|8|

$$\langle \phi_\lambda | \phi_\lambda \rangle \equiv A_\lambda \geq 1 \quad (7)$$

$$B_\lambda' \equiv \langle \phi_\lambda | \phi_{\lambda' \neq \lambda} \rangle = -B_\lambda \equiv -\langle \phi_{\lambda' \neq \lambda} | \phi_\lambda \rangle$$

$$|B_\lambda'| \geq 0 \quad . \quad (8)$$

The solution of (2) reads $[8]$

$$|\Psi E_C \rangle = |\xi E_C \rangle + \sum_\lambda \mid \Omega C _\lambda \rangle \langle \Omega C _\lambda^* | F \rangle = \frac{\langle \phi_\lambda^* | V | \xi E_C \rangle}{E - z_\lambda}$$

where

$$|\Omega C _\lambda \rangle = \left(1 + \frac{1}{E_+ - H_C}V_{CB}\right)|\phi_\lambda \rangle$$

is the wave function of the resonance state $\lambda$ and the $\xi E_C$ are the (coupled) scattering wave functions of the continuum into which the system is embedded. According to [8, 18], the eigenfunctions $\phi_\lambda$ of the non-Hermitian Hamilton operator $H_{\text{eff}}$ give the main contribution to the scattering wave function $\Psi E_C$ in the interior of the system,

$$|\Psi E_C \rangle \rightarrow |\hat{\Psi} E_C \rangle = \sum_\lambda c_{E_C \lambda}^E |\phi_\lambda \rangle ; \quad c_{E_C \lambda}^E = \frac{\langle \phi_\lambda^* | V | \xi E_C \rangle}{E - z_\lambda}$$

and

$$\langle \Psi E_C | \rightarrow \langle \hat{\Psi} E_C | = \sum_\lambda ^E c_{E_C \lambda}^* \langle \phi_\lambda^{\text{left}} | = \sum_\lambda ^E c_{E_C \lambda}^* \langle \phi_\lambda^{\text{right}} | .$$

The weight factors $c_{E_C \lambda}^E$ contain the decay of the states $\lambda$ at the energy $E$. The solution of (3) is $[8, 18]$
\begin{equation}
\langle \xi^E | V | \phi_{\lambda} \rangle = \sum_{\lambda} c^{E*}_{F,\lambda} \langle \phi^*_{\lambda} | V | \xi^E \rangle \end{equation}

This expression shows immediately that the resonance phenomena (described by the $S$ matrix) are determined by the decay properties of the resonance states (described by the complex eigenvalues $z_\lambda$ and eigenfunctions $\phi_\lambda$ of the non-Hermitian Hamilton operator $H_{\text{eff}}$). Thus, the FPO formalism provides a unified description of resonance and decay phenomena. The expression (16) shows however also that, generally, the energy dependence of the eigenvalues $z_\lambda$ and eigenfunctions $\phi_\lambda$ of $H_{\text{eff}}$ causes deviations from the Breit-Wigner resonance line shape and from the exponential decay law. The deviations become important for isolated resonance states in the long-time scale due to the fact that the decay thresholds lie at a finite energy [20]. This result agrees qualitatively with experimental data [3]. At high level density, deviations appear even in the short-time scale due to the mutual influence of neighbored resonance states, see Sect. V and [16].

III. PECULIARITIES OF THE FESHBACH PROJECTION OPERATOR FORMALISM

The main advantages of the FPO formalism consist in the following.

(i) The spectroscopic information on the resonance states is obtained directly from the complex eigenvalues $z_\lambda$ and eigenfunctions $\phi_\lambda$ of the non-Hermitian Hamilton operator $H_{\text{eff}}$. The $z_\lambda$ and $\phi_\lambda$ are energy dependent functions, generally, and contain the influence of neighboring resonance states as well as of decay thresholds onto the considered state $\lambda$. This
energy dependence allows to describe decay and resonance phenomena also in the very neighborhood of decay thresholds and in the regime of overlapping resonances. Since also the coupling coefficients between system and continuum depend on energy, the unitarity of the $S$ matrix is guaranteed, see e.g. [21].

(ii) The resonance states are directly related to the discrete states of a closed system described by standard quantum mechanics (with the Hermitian Hamilton operator $H_B$). They are generated by opening the system, i.e. by coupling the discrete states to the environment of scattering states by means of the second term of the Hamilton operator $H_{\text{eff}}$. Therefore, they are realistic (long-lived many-particle) states of an open quantum system.

(iii) In the FPO formalism it is not necessary to consider the poles of the $S$ matrix. Therefore, additional mathematical problems in the neighborhood of branch points (exceptional points) in the complex plane are avoided.

(iv) The phases of the eigenfunctions $\phi_\lambda$ of $H_{\text{eff}}$ are not rigid in the vicinity of a branch point. This fact allows spectroscopic reordering processes in the system under the influence of the scattering wave functions of the environment into which the system is embedded.

These features are involved in all present-day [22] calculations performed on the basis of the FPO formalism. In numerical studies, the main problem arises from the definition of the two subspaces $Q$ and $P$ such that it is meaningful for spectroscopic studies (see the discussion of this point in the reviews [8]). The basic idea is the following: $H_B$ describes the closed system (localized in the interior of the system) which becomes open when embedded into the environment of the extended scattering wave functions described by $H_C$. Therefore, all values characteristic of resonance states can be traced back to the corresponding values of discrete states by controlling the coupling to the continuum. That means with $v \to 0$, the transition from resonance states (described by the non-Hermitian $H_{\text{eff}}$) to discrete states (described by the Hermitian $H_B$) can be controlled.

Another peculiarity of the FPO formalism is the existence of a time operator which is the residuum of the non-Hermitian Hamilton operator $H_{\text{eff}}$. The life time $\tau_\lambda$ of a resonance state follows from the eigenvalue $z_\lambda$ of $H_{\text{eff}}$ in the same manner as the energy $E_\lambda$ of this state. Both values are fundamentally different from the time $t$ and the energy $E$. They characterize the states $\lambda$ while $t$ and $E$ appear as general parameters. In the closed system with the Hermitian Hamilton operator $H_B$, only the energies $E_B$ of the states can be determined. The eigenvalues are real and the widths are zero, $\Gamma_B = 0$. Due to the coupling to the
continuum, energy shifts $E_\lambda - E_B$ of the states appear as well as the finite life times $\tau_\lambda \propto (\Gamma_\lambda - \Gamma_B)^{-1} = \Gamma_\lambda^{-1}$ of the resonance states. Both, the energy shifts and the finite life times, follow from the second term of the non-Hermitian operator $H_{\text{eff}}$ [see Eq. (4)]. Usually, the numbers $E_\lambda$ and $\Gamma_\lambda$ can be obtained directly from the $z_\lambda$. Only in the case the $z_\lambda$ are strongly dependent on energy, the corresponding fixed-point equations have to be solved.

The energies $E_\lambda$ and life times $\tau_\lambda$ of the resonance states $\lambda$ of an open quantum system are bounded from below (see [16] for the discussion of the brachistochrone problem in open quantum systems). Mathematically, the existence of the time operator entails the time asymmetry involved in the FPO formalism.

IV. TIME DEPENDENT VALUES

The time dependent Schrödinger equation reads

$$H_{\text{eff}} \dot{\Psi}^E(t) = i \hbar \frac{\partial}{\partial t} \Psi^E(t).$$  \hspace{1cm} (17)

The right solutions may be represented, according to (11), by an ensemble of resonance states $\lambda$ that describes the decay of the system at the energy $E$,

$$|\dot{\Psi}^E (\text{right}) (t)\rangle = e^{-iH_{\text{eff}}t/\hbar} |\dot{\Psi}^E (\text{right}) (t_0)\rangle = \sum_\lambda e^{-iz_\lambda t/\hbar} c_{\lambda 0} |\phi_\lambda^{(\text{right})}\rangle$$  \hspace{1cm} (18)

with $|\phi_\lambda^{(\text{right})}\rangle = |\phi_\lambda\rangle$ and $c_{\lambda 0} = \langle \phi_\lambda^* | V | \xi^E_C \rangle / (E - z_\lambda)$. The $z_\lambda$ and $\phi_\lambda$ are the (energy dependent) eigenvalues and eigenfunctions of the time-independent Hamilton operator $H_{\text{eff}}$, Eq. (4), while the $\xi^E_C$ are the scattering wave functions of the environment. The left solution of (17) reads

$$\langle \Psi^E (\text{left}) (t) | = \langle \Psi^E (\text{left}) (t_0) | e^{iH_{\text{eff}}t/\hbar}$$

$$= \sum_\lambda \langle \phi_\lambda^{(\text{left})} | d_{\lambda t} e^{iz_\lambda t/\hbar}$$  \hspace{1cm} (19)

with $\langle \phi_\lambda^{(\text{left})} | = \langle \phi_\lambda^* |$ and $d_{\lambda t} = c_{\lambda 0} = \langle \xi^E_C | V | \phi_\lambda\rangle / (E - z_\lambda^*)$ or $d_{\lambda t} = c_{\lambda E}^* (t) = \langle F^* (t) Q | \phi_\lambda \rangle / (E - z_\lambda^*)$ according to (12) and (15), respectively. It describes the excitation of the system at the energy $E$. The source term is, generally, time dependent: $F = F(t)$.

By means of (18) and (19) the population probability

$$\langle \dot{\Psi}^E (\text{left}) (t) | \dot{\Psi}^E (\text{right}) (t) \rangle = \sum_\lambda c_{\lambda 0} d_{\lambda t} e^{-\Gamma_\lambda t/\hbar}$$  \hspace{1cm} (20)
at the energy $E$ can be defined. The decay rate reads

$$k_{gr}(t) = -\frac{\partial}{\partial t} \ln \langle \hat{\Psi}_E^{(\text{left})}(t) | \hat{\Psi}_E^{(\text{right})}(t) \rangle = \frac{1}{\hbar} \sum_\lambda \Gamma_\lambda c_{\lambda 0} d_{\lambda t} e^{-\Gamma_\lambda t/\hbar}.$$  \hfill (21)

For an isolated resonance state $\lambda$, (21) passes into the standard expression

$$k_{gr}(t) \to k_\lambda = \frac{\Gamma_\lambda}{\hbar}.$$  \hfill (22)

The value $k_\lambda$ is constant in time and corresponds to (1) with $\tau_\lambda = 1/k_\lambda$. It describes the idealized case of an exponential decay law and, according to (16), a Breit-Wigner resonance in the cross section. Generally, deviations from the exponential decay law and from the Breit-Wigner line shape appear under the influence of neighboring resonance states and (or) of decay thresholds (see e.g. [8]). Also the background term appearing in most reactions may cause deviations from the ideal exponential decay law.

The excitation process may occur on a much shorter time scale than the decay process. In such a case, the function $d_{\lambda t}$ will be a step-like function at $t = t_0$. It is possible therefore to study the pure decay process starting at the time $t_0$. When, in other cases, $\langle \phi_\lambda^* | QF(t) \rangle$ is constant for $t > t_0$ (or increases with $t$ in a certain time interval), excitation and decay take place at the same time in this time interval. This is the case also for the scattering process corresponding to $F = 0$. Eq. (21) describes the decay rate also in the regime of overlapping resonances. For numerical results see Ref. [10]. The overlapping and mutual influence of resonance states is maximal at the branch points in the complex plane where two eigenvalues $z_\lambda$ and $z_{\lambda'}$ of the effective Hamilton operator $H_{\text{eff}}$ coalesce. Nevertheless, the decay rate is everywhere smooth as can be seen also directly from (21). This result coincides with the general statement according to which all observable quantities behave smoothly at singular points.

The expressions (18) and (19) are valid only when (11) holds, i.e. at times $t$ at which the wave functions $\Psi^E$ have a localized part in the interior of the system at the energy $E$ so that the representation (11) is meaningful at this energy. According to (18) and (19), this is the case for times $t \geq t_0$ where $t_0$ is a finite value. Without loss of generality, it can be chosen $t_0 = 0$. The quantum system described in the framework of the FPO formalism is therefore time asymmetric. The time asymmetry is involved in the non-Hermitian part of
the Hamilton operator $H_{\text{eff}}$ which contains the time operator. This can be seen also from the expression (20) for the population probability.

The consideration of only the time interval $0 \leq t \leq \infty$ in (17) is related to the fact that the decay of a resonance state (at the energy $E$ of the system) starts at a finite time (say $t_0 = 0$) at which the system can be considered to be excited, i.e. (11) is meaningful at this energy. This fact agrees with the concept of a semigroup description introduced in [4], which distinguishes between prepared and measured states. In our formalism, the decaying (measured) states are described by the eigenvalues and eigenfunctions of the non-Hermitian Hamilton operator $H_{\text{eff}}$ involved in the $|\hat{\Psi}^E\rangle$, Eq. (11). The preparation of the resonance states is described by the $\langle \hat{\Psi}^E |$, Eq. (15). It may be very different for different reactions.

The decay properties of the resonance states can be studied best when their excitation takes place in a time interval that is very short as compared to the life time $\tau_\lambda$ of the resonance states. In such a case, the time $t_0 = 0$ is well defined and no perturbation of the decay process by the still continuing excitation process will take place. In [7], such a situation is studied in single ion experiments. The results demonstrate the beginning of time for a decaying state. That means, they prove the time asymmetry in quantum physics.

V. DECAY RATES IN THE REGIME OF OVERLAPPING RESONANCE STATES

In the regime of overlapping resonance states, spectroscopic reordering processes take place [8]. Most interesting is the phenomenon of width bifurcation and the loss of the phase rigidity of the wave functions of the resonance states under the influence of the branch points in the complex energy plane. For a detailed study of the last phenomenon (loss of phase rigidity) see [16].

The decay rate $k_{\text{gr}}$, Eq. (21), contains the widths $\Gamma_\lambda$ of the individual resonance states $\lambda$. When the resonance states overlap, it is however difficult to receive information on the decay rates $k_\lambda(t)$ of the individual states. The reason is that (21) contains also the contributions from all the neighboring states $\lambda' \neq \lambda$ at the energy $E$. In order to get $k_\lambda(t)$, one has to consider

$$H_{\text{eff}} \phi_\lambda(t) = i \hbar \frac{\partial}{\partial t} \phi_\lambda(t)$$  (23)
instead of (17) with
\[
|\phi_\lambda(t)\rangle = e^{-iH_{\text{eff}}t/\hbar} |\phi_\lambda(t_0)\rangle \\
= e^{-iz_\lambda t/\hbar} c_\lambda  |\phi_\lambda\rangle + \sum_{\lambda' \neq \lambda} e^{-iz_{\lambda'} t/\hbar} c_{\lambda'0} |\phi_{\lambda'}\rangle
\]
(24)
and
\[
\langle \phi_\lambda(t) | = \langle \phi_\lambda(0) | e^{iH_{\text{eff}}t/\hbar} \\
= \langle \phi_\lambda | d_\lambda t e^{iz_\lambda t/\hbar} + \sum_{\lambda' \neq \lambda} \langle \phi_{\lambda'} | d_{\lambda'} t e^{iz_{\lambda'} t/\hbar}.
\]
(25)
The functions $\phi_\lambda$ are biorthogonal, see (6), (7) and (8). It follows
\[
\langle \phi_\lambda(t) | \phi_\lambda(t) \rangle = c_\lambda d_\lambda t e^{-\Gamma_\lambda t/\hbar} \langle \phi_\lambda | \phi_\lambda \rangle + \sum_{\lambda' \neq \lambda} e^{-(\Gamma_\lambda + \Gamma_{\lambda'}) t/2\hbar} \left( c_{\lambda0} d_{\lambda' t} e^{i(E_{\lambda'} - E_\lambda) t/\hbar} - c_{\lambda'0} d_{\lambda t} e^{i(E_\lambda - E_{\lambda'}) t/\hbar} \right) \langle \phi_{\lambda'} | \phi_\lambda \rangle
\]
(26)
and
\[
k_\lambda(t) = -\frac{\partial}{\partial t} \ln \langle \phi_\lambda(t) | \phi_\lambda(t) \rangle \text{ contains oscillating terms in the overlapping regime at the energy } E \text{ considered. These oscillating terms correspond to the fact that, at a certain energy } E \text{ of the system, an individual level is ill defined because of its overlapping with other levels. The oscillations vanish by summing over the contributions from all individual states and considering the wave functions } \Psi^E \text{ that are solutions of (2) with the Hermitian Hamilton operator } H, \text{ see (20).}
\]
The thus, the oscillations of the decay rates $k_\lambda(t)$ of the individual resonance states illustrate in a direct manner how neighboring resonance states influence one another. The physically relevant expressions for the decay rate are, however, the $k_{\text{gr}}(t)$. For numerical results obtained for both values, $k_{\text{gr}}(t)$ and $k_\lambda(t)$, in some special cases in the neighborhood of branch points in the complex plane see [10].

Another interesting problem is the saturation of the average decay rate $k_{\text{av}}$ in the regime of strongly overlapping resonances. According to the bottleneck picture of the transition state theory, it starts at a certain critical value of bound-continuum coupling [23]. This saturation is caused by widths bifurcation (resonance trapping [8]) occurring in the neighborhood of the branch points in the complex plane [24]. Widths bifurcation creates long-lived resonance states together with a few short-lived resonance states. The definition of an average life time of the resonance states is meaningful therefore only for either the long-lived states or the short-lived ones. The long-lived (trapped) resonance states are almost decoupled from the
continuum of decay channels. Their widths $\Gamma_\lambda$ saturate therefore with increasing bound-continuum coupling. The $\Gamma_\lambda$ are almost the same for all the different states $\lambda$, see [8], i.e. $\Gamma_{av} \approx \Gamma_\lambda$ for all long-lived trapped resonance states. It follows therefore

$$k_{av} \approx \Gamma_{av}/\hbar$$  \hspace{1cm} (27)

from (21). According to the average width $\Gamma_{av}$, the average life time of the long-lived states can be defined by $\tau_{av} = 1/k_{av}$. Then (27) is equivalent to the basic equation (11). That means, the basic relation between life times and decay widths of resonance states holds not only for isolated resonance states [see Eq. (22)], but also for narrow resonance states superposed by a smooth background (originating from a few short-lived resonance states [8, 15]). In the last case, the relation holds for the average values $\Gamma_{av}$ and $\tau_{av}$.

VI. SUMMARY

As has been discussed in this paper, the FPO formalism is characterized by two Hamilton operators: the Hermitian $H$ and the non-Hermitian $H_{eff}$. The non-Hermitian operator $H_{eff}$, Eq. (4), appears only at an intermediate stage of the FPO formalism. It is characteristic of the subsystem localized in a certain space region and opened by coupling it to the surrounding subspace of extended scattering states. It contains the time operator. The observables related to the whole system, such as the resonance structure of the scattering process, are described by the wave functions $\Psi^E$ that are solutions of the equations (2) and (3), respectively, with the Hermitian Hamilton operator $H$. The unitarity of the $S$ matrix is guaranteed at all energies.

Although $H_{eff}$ is an operator appearing only at an intermediate stage, it causes fundamental phenomena involved in the FPO formalism. The $\Psi^E$ can be represented in the set of eigenfunctions $\phi_\lambda$ of $H_{eff}$ which are biorthogonal, see Sect. II. This nontrivial representation causes, among others, the time asymmetry involved in the FPO formalism: the asymmetry rests on the fact that only localized states decay, i.e. states for which the representation (11) is meaningful. This fact corresponds to the formulation by Bohm et al. [4] that the states have to be prepared before they can be registered. Furthermore, the appearance of the non-Hermitian Hamilton operator $H_{eff}$ in the FPO formalism guarantees the unified description of resonance and decay phenomena. Its eigenvalues $z_\lambda$ describe, on the one hand, the reso-
nance phenomena involved in the resonance part of the $S$ matrix, Eq. (16). On the other hand, the decay of the states $\lambda$ lying at the energy $\text{Re}(z_\lambda)$ is determined by $\text{Im}(z_\lambda)$. Thus, the resonance phenomena are directly related to the decay properties of the system. This result being in accordance with longtime experience, as well as with the time asymmetry can not be obtained in standard quantum mechanics with Hermitian Hamilton operators in the Hilbert space.

As a result of the study on the basis of the FPO formalism, we state that the basic relation (1) between life time and decay width of resonance states holds not only at low level density (where the resonances are well separated from one another) but also at high level density (where trapped long-lived resonance states are superposed by a smooth background). In these cases the decay rate $k_\lambda(t)$ of the individual state $\lambda$ and the average decay rate $k_{av}(t)$ of the long-lived trapped resonance states, respectively, is (almost) constant in time. Deviations may occur due to the position of thresholds in the neighborhood. In the overlapping regime however, the mutual influence of the different resonance states onto each other causes a time dependence of the decay rate: $k_{gr}(t)$ decreases monotonously with increasing $t$ according to (21). In the case the decay rate is constant in time, the decay occurs according to an exponential law while the decay takes place according to a non-exponential law when the resonances are not well separated from one another and the decay rate depends on time.

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