Time discretization of the spin Calogero-Moser model and the semi-discrete matrix KP hierarchy

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Abstract

We introduce the discrete time version of the spin Calogero-Moser system. The equations of motion follow from the dynamics of poles of rational solutions to the matrix KP hierarchy with discrete time. The dynamics of poles is derived using the auxiliary linear problem for the discrete flow.

1 Introduction

In the seminal paper [1] it was discovered that the motion of poles of rational solutions to the Korteweg-de Vries and Boussinesq equations is given by dynamics of the many-body Calogero-Moser system of particles [2, 3, 4] with some additional restrictions in the phase space. In [5, 6] it was shown that in the case of the Kadomtsev-Petviashvili (KP) equation this correspondence becomes an isomorphism: poles of rational solutions to the KP equation move exactly as Calogero-Moser particles with the interaction potential $1/x^2$. This remarkable connection was further generalized to elliptic solutions in [7]. Later the correspondence between dynamics of poles of rational KP solutions and many-body integrable systems of particles on the line was extended [8] to the level of hierarchies. Namely, it was proved that the evolution of poles in $t_1 = x$ with respect to the KP time $t_k$ with $k \geq 2$ is governed by the higher Hamiltonian $H_k$ of the integrable Calogero-Moser system.

Similar approach to the rational solutions of the matrix KP hierarchy was developed in the paper [9], in which the results of [10] for the matrix KP equation were extended to the whole hierarchy (see also [11, 12, 13, 14, 15], where similar results are discussed from other perspectives and points of view). The matrix extension of the KP hierarchy

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is closely related to the so-called multi-component KP hierarchy \cite{16, 17, 18, 19}. It has been shown that the evolution of data of the rational solutions (positions of poles $x_i$ and some vectors $a^\alpha_i$, $b^\alpha_i$ parameterizing rank 1 matrix residues at the poles) is isomorphic to the spin generalization of the Calogero-Moser system known as the Gibbons-Hermsen system \cite{20}.

It is natural to expect that integrable time discretization of the Calogero-Moser system and its spin generalization can be obtained from dynamics of poles of rational solutions to semi-discrete soliton equations. “Semi” means that the time becomes discrete while the space variable $x$, with respect to which one considers pole solutions, remains continuous. At the same time, it is known that integrable discretizations of soliton equations can be regarded as belonging to the same hierarchy as their continuous counterparts. Namely, the discrete time step is equivalent to a special shift of infinitely many continuous hierarchical times. This fact lies in the basis of the method of generating discrete soliton equations developed in \cite{21}. For integrable time discretization of many-body systems see \cite{22, 23, 24, 25, 26}.

The equations of motion for the rational Calogero-Moser model in discrete time $p$ have the form \cite{22}

$$\sum_j \frac{1}{x_i(p) - x_j(p + 1)} + \sum_j \frac{1}{x_i(p) - x_j(p - 1)} = 2 \sum_{j \neq i} \frac{1}{x_i(p) - x_j(p)}.$$  \hspace{1cm} (1)

Remarkably, they coincide with the nested Bethe ansatz equations for the integrable quantum Gaudin magnet based on the algebra $gl(M)$, the discrete time variable $p$ playing the role of “level” of the nested Bethe ansatz (in this case the values of $p$ are restricted to $0, 1, \ldots, M$).

In this paper we derive equations of motion in discrete time for the spin generalization of the Calogero-Moser model. It appears that they look like equations (1) “dressed” by the spin variables $a^\alpha_i(p)$, $b^\alpha_i(p)$ associated with each particle with coordinate $x_i(p)$ ($\alpha = 1, \ldots, N$):

$$\sum_j \frac{b^\gamma_i(p + 1)b^\delta_j(p + 1)a^\beta_i(p)}{x_i(p) - x_j(p + 1)} + \sum_j \frac{b^\gamma_i(p - 1)b^\delta_j(p - 1)a^\beta_i(p)}{x_i(p) - x_j(p - 1)} = 2 \sum_{j \neq i} \frac{b^\gamma_i(p)a^\gamma_j(p)b^\beta_j(p)a^\delta_i(p)}{x_i(p) - x_j(p)},$$  \hspace{1cm} (2)

where the summation over repeated indices $\beta, \gamma$ is implied. These equations follow from the dynamics of poles of rational solutions to the discrete time matrix KP hierarchy. Their possible meaning from the point of view of Bethe ansatz is to be clarified.

The main body of the paper starts from the multi-component KP hierarchy in the bilinear formalism and subsequent specialization to the matrix KP hierarchy in section 2, where the discrete time evolution is also considered. In section 3 we deal with pole dynamics of rational solutions to the matrix KP hierarchy. The evolution of the poles and the vectors $a^\alpha_i$, $b^\alpha_i$ in discrete time is derived with the help of the corresponding linear problems for the Baker-Akhiezer function and its adjoint. In the appendix, these linear problems are derived from the basic bilinear identity for the tau-function.
2 The multi-component and matrix KP hierarchies

We start from the multi-component KP hierarchy. Our exposition follows [18, 19]. The multi-component KP hierarchy contains $N$ infinite sets of continuous time variables

$$t = \{t_1, t_2, \ldots, t_N\}, \quad t_\alpha = \{t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, \ldots\}, \quad \alpha = 1, \ldots, N$$

and $N$ discrete integer variables $s = \{s_1, s_2, \ldots, s_N\}$ called charges. They are constrained by the condition $\sum_{\alpha=1}^{N} s_\alpha = 0$. In the bilinear formalism, the dependent variable is the tau-function $\tau(s; t)$ which depends on the charges and the times. The $N$-component KP hierarchy is defined as the infinite set of bilinear equations for the tau-function that follow from the basic bilinear identity

$$\sum_{\gamma=1}^{N} \epsilon_{\alpha\gamma}(s) \epsilon_{\beta\gamma}(s') \oint_{C_\infty} dz z^{s_\gamma-s'_\gamma+\delta_{\alpha\gamma}+\delta_{\beta\gamma}-2} e^{\xi(t_{\gamma}, t_{\gamma}', z)}$$

$$\cdot \tau(s + e_\alpha - e_\gamma, t - [z^{-1}]_\gamma) \tau(s' + e_\gamma - e_\beta, t' + [z^{-1}]_\gamma) = 0, \quad \alpha, \beta = 1, \ldots, N,$$

valid for any $s, s', t, t'$. Here $e_\alpha$ is the vector whose $\alpha$th component is 1 and all other entries are equal to zero, so for $\alpha \neq \gamma$ we have $(s + e_\alpha - e_\gamma)_\beta = s_\beta + \delta_{\alpha\beta} - \delta_{\beta\gamma}$. Next, we use the following standard notation:

$$\xi(t_{\gamma}, z) = \sum_{k \geq 1} t_{\gamma,k} z^k,$$

$$(t \pm [z^{-1}]_\gamma)_{\alpha,k} = t_{\alpha,k} \pm \delta_{\alpha\gamma} \frac{z^{-k}}{k}$$

and $\epsilon_{\alpha\gamma}(s)$ is the sign factor

$$\epsilon_{\alpha\gamma}(s) = \begin{cases} (-1)^{s_\alpha + \ldots + s_\gamma} & \text{if } \alpha < \gamma \\ 1 & \text{if } \alpha = \gamma \\ -(-1)^{s_\gamma + \ldots + s_\alpha} & \text{if } \alpha > \gamma. \end{cases}$$

Obviously, for any distinct $\alpha, \beta$ it holds $\epsilon_{\alpha\beta}(s) = -\epsilon_{\beta\alpha}(s)$. The integration contour $C_\infty$ is a big circle around $\infty$.

Along with the tau-function, an important role in the theory of integrable hierarchies is played by the Baker-Akhiezer function. In the multi-component KP hierarchy, the Baker-Akhiezer function $\Psi(s, t; z)$ and its adjoint $\Psi^\dagger(s, t; z)$ are $N \times N$ matrices with components defined by the formulae:

$$\Psi_{\alpha\beta}(s, t; z) = \epsilon_{\alpha\beta}(s) \left( \frac{\tau(s + e_\alpha - e_\beta, t - [z^{-1}]_\beta)}{\tau(s; t)} \right) z^{s_\beta + \delta_{\alpha\beta} - 1} e^{\xi(t_\beta, z)},$$

$$\Psi_{\alpha\beta}^\dagger(s, t; z) = \epsilon_{\beta\alpha}(s) \left( \frac{\tau(s + e_\alpha - e_\beta, t + [z^{-1}]_\alpha)}{\tau(s; t)} \right) z^{-s_\alpha + \delta_{\alpha\beta} - 1} e^{-\xi(t_\alpha, z)},$$

(4)

(here and below $\dagger$ does not mean the Hermitian conjugation). The complex variable $z$ is called the spectral parameter. Around $z = \infty$, the Baker-Akhiezer function $\Psi$ can be
represented in the form of the series

$$\Psi_{\alpha\beta}(s, t; z) = \left(\delta_{\alpha\beta} + \sum_{k \geq 1} \frac{w_{\alpha\beta}^{(k)}(s, t)}{z^k}\right) z^{s_{\beta}} e^{\xi(t_s, z)},$$  \hspace{1cm} (5)

where \(w_{\alpha\beta}^{(k)}(s, t)\) are some matrix functions. In terms of the Baker-Akhiezer functions, the bilinear identity \((3)\) can be written as

$$\oint_{C_\infty} dz \, \Psi(s, t; z) \Psi^\dagger(s', t'; z) = 0.$$  \hspace{1cm} (6)

Another approach to the multi-component KP hierarchy is based on matrix pseudo-differential operators. The hierarchy can be understood as an infinite set of evolution equations in the times \(t\) for matrix functions of a variable \(x\). For example, the coefficients \(w_{\alpha\beta}^{(k)}\) of the Baker-Akhiezer function can be taken as such matrix functions, the evolution being \(w_{\alpha\beta}^{(k)}(x) \rightarrow w_{\alpha\beta}^{(k)}(x, t)\). In what follows we denote \(\tau(x, t), w_{\alpha\beta}^{(k)}(x, t)\) simply as \(\tau(t), w_{\alpha\beta}^{(k)}(t)\), suppressing the dependence on \(x\). Let us introduce the matrix pseudo-differential “wave operator” \(W\) with matrix elements

$$W_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{k \geq 1} w_{\alpha\beta}^{(k)}(t) \partial^{-k} x,$$  \hspace{1cm} (7)

where \(w_{\alpha\beta}^{(k)}(t)\) are the same matrix functions as in \((5)\). The Baker-Akhiezer function (at \(s = 0\)) can be written as a result of action of the wave operator to the exponential function:

$$\Psi(t; z) = W \exp\left(xzI + \sum_{\alpha=1}^{N} E_{\alpha} \xi(t_\alpha, z)\right),$$  \hspace{1cm} (8)

where \(E_{\alpha}\) is the \(N \times N\) matrix with the components \((E_{\alpha})_{\beta\gamma} = \delta_{\alpha\beta} \delta_{\alpha\gamma}\). The adjoint Baker-Akhiezer function can be written as

$$\Psi^\dagger(t; z) = \exp\left(-xzI - \sum_{\alpha=1}^{N} E_{\alpha} \xi(t_\alpha, z)\right) W^{-1}.$$  \hspace{1cm} (9)

Here the operators \(\partial_x\) entering \(W^{-1}\) act to the left (i.e., we define \(f \partial_x \equiv -\partial_x f\)).

It is proved in \([18]\) that the Baker-Akhiezer function and its adjoint satisfy the linear equations

$$\partial_{t_{\alpha,m}} \Psi(t; z) = B_{\alpha m} \Psi(t; z),$$

$$-\partial_{t_{\alpha,m}} \Psi^\dagger(t; z) = \Psi^\dagger(t; z) B_{\alpha m},$$  \hspace{1cm} (10)

where \(B_{\alpha m}\) is the differential operator

$$B_{\alpha m} = \left(W E_{\alpha} \partial^m_x W^{-1}\right)_+.$$  

The notation \((\ldots)_+\) means the differential part of a pseudo-differential operator, i.e. the sum of all terms with \(\partial^k_x\), where \(k \geq 0\). In particular, it follows from \((10)\) at \(m = 1\) that

$$\sum_{\alpha=1}^{N} \partial_{t_{\alpha,1}} \Psi(t; z) = \partial_x \Psi(t; z).$$  \hspace{1cm} (11)
so the vector field $\partial_x$ can be identified with the vector field $\sum_\alpha \partial_{t_\alpha,1}$.

The matrix KP hierarchy results from the multicomponent KP one after a restriction of the time and charge variables in the following manner:

$$t_{\alpha,m} = t_m, \quad s_\alpha = 0 \quad \text{for each } \alpha \text{ and } m.$$ The corresponding vector fields are related as $\partial_{t_m} = \sum_{\alpha=1}^N \partial_{t_{\alpha,m}}$. In what follows we omit the variables $s$ in the notation for the tau-function and the Baker-Akhiezer functions and put $s = s' = 0$ in the bilinear identity, so it acquires the form

$$\sum_{\gamma=1}^N \epsilon_{\alpha\gamma} \epsilon_{\beta\gamma} \int_{\mathbb{C}_\infty} dz \ z^{\delta_{\alpha\gamma} + \delta_{\beta\gamma} - 2e(\gamma^{-1}, z)} \tau_{\alpha\gamma}(t - [z^{-1}]) \tau_{\gamma\beta}(t' + [z^{-1}]) = 0,$$  \hspace{1cm} (12)

where

$$\tau_{\alpha\beta}(t) = \tau(e_\alpha - e_\beta; t) \quad \text{(13)}$$

and $\epsilon_{\alpha\gamma} = 1$ if $\alpha \leq \gamma$, $\epsilon_{\alpha\gamma} = -1$ if $\alpha > \gamma$. The Baker-Akhiezer function has the expansion

$$\Psi_{\alpha\beta}(t; z) = (\delta_{\alpha\beta} + w^{(1)}_{\alpha\beta}(t)z^{-1} + O(z^{-2})) e^{\xi(t, z)},$$  \hspace{1cm} (14)

where $\xi(t, z) = \sum_{k \geq 1} t_k z^k$. The coefficient $w^{(1)}_{\alpha\beta}(t)$ plays an important role in what follows. As is seen from (12),

$$w^{(1)}_{\alpha\beta}(t) = \begin{cases} \epsilon_{\alpha\beta} \frac{\tau_{\alpha\beta}(t)}{\tau(t)} & \text{if } \alpha \neq \beta \\ - \frac{\partial_{t_{\alpha,1}} \tau(t)}{\tau(t)} & \text{if } \alpha = \beta. \end{cases} \quad \text{(15)}$$

Differentiating the bilinear identity with respect to $t_m$ and putting $t' = t$ after this, we obtain the following useful corollary:

$$\frac{1}{2\pi i} \sum_{\gamma=1}^N \int_{\mathbb{C}_\infty} dz \ z^m \Psi_{\alpha\gamma}(t; z) \Psi_{\gamma\beta}^+(t; z) = -\partial_{t_m} w_{\alpha\beta}^{(1)}(t) \quad \text{(16)}$$

or, equivalently,

$$\sum_\gamma \text{res}_\infty \left( z^m \Psi_{\alpha\gamma} \Psi_{\gamma\beta}^+ \right) = -\partial_{t_m} w_{\alpha\beta}^{(1)} \quad \text{(17)}.$$

Recalling (14), we can identify $\partial_x = \partial_{t_1} = \sum_{\alpha=1}^N \partial_{t_{\alpha,1}}$. Equations (10) imply that the Baker-Akhiezer function of the matrix KP hierarchy and its adjoint satisfy the linear equations

$$\partial_{t_m} \Psi(t; z) = B_m \Psi(t; z), \quad m \geq 1,$$  \hspace{1cm} (18)

$$-\partial_{t_m} \Psi^+(t; z) = \Psi^+(t; z) B_m, \quad m \geq 1,$$

where $B_m$ is the differential operator $B_m = \left( W \partial^m_x W^{-1} \right)_+$. At $m = 1$ we have $\partial_{t_1} \Psi = \partial_x \Psi$, so the evolution in $t_1$ is a shift of the variable $x$: $w^{(k)}(x, t_1, t_2, \ldots) = w^{(k)}(x + t_1, t_2, \ldots)$.

At $m = 2$ equations (18) yield the linear problems

$$\partial_{t_2} \Psi = \partial_x^2 \Psi + V(t) \Psi, \quad \text{(19)}$$
\[-\partial_{t_2} \Psi^\dagger = \partial^2_x \Psi^\dagger + \Psi^\dagger V(t) \]

with

\[ V(t) = -2\partial_x w^{(1)}(t). \]

The discrete time evolution in the matrix KP hierarchy is defined as a shift of continuous time variables according to the rule [21]

\[ \tau^p = \tau \left( t - p \sum_{\alpha=1}^{N} [\mu^{-1}]_{\alpha} \right), \quad \Psi^p = \Psi \left( t - p \sum_{\alpha=1}^{N} [\mu^{-1}]_{\alpha}; z \right), \]

where \( p \) is the discrete time variable and \( \mu \) is a continuous parameter. Each \( \mu \) corresponds to its own discrete time flow. This hierarchy is called semi-discrete because the variable \( x \) (and time \( t_1 \)) remains continuous. One can show, using the explicit expressions of the Baker-Akhiezer functions through the tau-function and some corollaries of the bilinear identity (see the appendix) that the corresponding linear problems have the form

\[ \mu \Psi^p - \mu \Psi^{p+1} = \partial_x \Psi^p + \left( w^{(1)}(p + 1) - w^{(1)}(p) \right) \Psi^p, \]

\[ \mu \Psi^{\dagger p} - \mu \Psi^{\dagger p+1} = \partial_x \Psi^{\dagger p} + \Psi^{\dagger p} \left( w^{(1)}(p) - w^{(1)}(p - 1) \right), \]

or, in components,

\[ \mu \Psi^p_{\alpha\beta} - \mu \Psi^{p+1}_{\alpha\beta} = \partial_x \Psi^p_{\alpha\beta} + \sum_{\gamma} \left( w^{(1)}(p + 1) - w^{(1)}(p) \right) \Psi^p_{\gamma\beta}, \]

\[ \mu \Psi^{\dagger p}_{\alpha\beta} - \mu \Psi^{\dagger p+1}_{\alpha\beta} = \partial_x \Psi^{\dagger p}_{\alpha\beta} + \sum_{\gamma} \Psi^{\dagger p}_{\alpha\gamma} \left( w^{(1)}(p) - w^{(1)}(p - 1) \right). \]

### 3 Rational solutions to the matrix KP hierarchy and time discretization of the Calogero-Moser model

We are going to study solutions to the matrix KP hierarchy which are rational functions of the variable \( x \) (and, therefore, \( t_1 \)). For the rational solutions, the tau-function should be a polynomial in \( x \):

\[ \tau^p = C \prod_{i=1}^{N} (x - x_i). \]

The \( N \) roots \( x_i \) (assumed to be distinct) depend on the times \( t \) and on the discrete variable \( p \). They are going to be coordinates of particles in the spin Calogero-Moser system. Disregarding the dependence on \( t \), we will denote \( x_i = x_i(p) \). It is clear from [4] that the Baker-Akhiezer functions \( \Psi, \Psi^\dagger \) (and thus the coefficient \( w^{(1)} \)), as functions of \( x \), have simple poles at \( x = x_i \). It is shown in [9] that the residues at these poles are matrices of rank 1. Namely, we can parametrize them through some column vectors \( a_i = (a_i^1, a_i^2, \ldots, a_i^N)^T \), \( b_i = (b_i^1, b_i^2, \ldots, b_i^N)^T \) (\( T \) means transposition) as follows:

\[ \text{res}_{x = x_i} w^{(1)}_{\alpha\beta} = -a_i^\beta b_i^\alpha \quad \text{or} \quad \text{res}_{x = x_i} w^{(1)} = -a_i b_i^T. \]
Note that in [10] the form (27) was derived from some algebro-geometric reasoning using analytic properties of the Baker-Akhiezer function on the algebraic curve. For the residues of the Baker-Akhiezer functions we have [9]:

\[
\begin{align*}
\res_{x=x_i} \Psi_{\alpha\beta} &= e^{xz+\xi(t,z)}a_i^\alpha c_i^\beta, \\
\res_{x=x_i} \Psi_{\alpha\beta}^\dagger &= e^{-xz-\xi(t,z)}c_i^{*\alpha} b_i^\beta,
\end{align*}
\]

(28)

where \(c_i^\alpha, c_i^{*\alpha}\) are components of some vectors \(c_i = (c_i^1, \ldots, c_i^N)^T, c_i^{*} = (c_i^{*1}, \ldots, c_i^{*N})^T\).

The vectors \(a_i, b_i\) depend on the times \(t_k\) with \(k \geq 2\) while the vectors \(c_i, c_i^{*}\) depend on the same set of times and on \(z\). The dependence of the vectors on the discrete time will be denoted as \(a_i^\alpha = a_i^\alpha(p), b_i^\alpha = b_i^\alpha(p)\). Therefore, we have the following representation of the Baker-Akhiezer functions:

\[
\Psi_{\alpha\beta} = e^{xz+\xi(t,z)} \left( C_{\alpha\beta} + \sum_{i=1}^{N} \frac{a_i^\alpha c_i^\beta}{x-x_i} \right),
\]

(29)

\[
\Psi_{\alpha\beta}^\dagger = e^{-xz-\xi(t,z)} \left( C^{-1}_{\alpha\beta} + \sum_{i=1}^{N} \frac{c_i^{*\alpha} b_i^\beta}{x-x_i} \right),
\]

(30)

where \(C_{\alpha\beta}\) is a matrix of some \(x\)-independent coefficients. The fact that the constant term in the adjoint Baker-Akhiezer function is the inverse matrix \(C_{\alpha\beta}^{-1}\) follows from [9].

For the matrices \(w^{(1)}\) and \(V = -2\partial_x w^{(1)}\) we have

\[
w^{(1)}_{\alpha\beta} = S_{\alpha\beta} - \sum_{i=1}^{N} \frac{a_i^\alpha b_i^\beta}{x-x_i}, \quad V_{\alpha\beta} = -2 \sum_{i=1}^{N} \frac{a_i^\alpha b_i^\beta}{(x-x_i)^2},
\]

(31)

where the matrix \(S\) does not depend on \(x\). Tending \(x \to \infty\) in (17), one concludes that \(\partial_{t_m} S = 0\) for all \(m \geq 1\), so the matrix \(S\) does not depend on all the times.

Following [9, 10], we first consider the dynamics of poles with respect to the time \(t_2\). To this end, we consider the linear problems (19), (20),

\[
\partial_{t_2} \Psi_{\alpha\beta} = \partial_x^2 \Psi_{\alpha\beta} - 2 \sum_{i=1}^{N} \frac{a_i^\alpha b_i^\gamma}{(x-x_i)^2} \Psi_{\gamma\beta},
\]

\[
-\partial_{t_2} \Psi_{\alpha\beta}^\dagger = \partial_x^2 \Psi_{\alpha\beta}^\dagger - 2 \sum_{\gamma} \Psi_{\gamma\alpha}^\dagger \sum_{i=1}^{N} \frac{a_i^\gamma b_i^\beta}{(x-x_i)^2}
\]

and substitute here the pole ansatz for the Baker-Akhiezer functions. Consider first the equation for \(\Psi\). First of all, comparing the behavior of the both sides as \(x \to \infty\), we conclude that \(\partial_{t_2} C_{\alpha\beta} = 0\), so \(C_{\alpha\beta}\) does not depend on \(t_2\) (in a similar way, one can see from the higher linear problems that \(C_{\alpha\beta}\) does not depend on all the times \(t_k\)). Equating coefficients at the poles at \(x = x_i\) of different orders, we get the conditions:

- **At** \(\frac{1}{(x-x_i)^3}\): \(b_i^\gamma a_i^\gamma = 1\) or \(b_i^T a_i = 1\);

- **At** \(\frac{1}{(x-x_i)^2}\): \(a_i^\alpha c_i^\beta \bar{x}_i = -2 a_i^\alpha c_i^\beta - 2 a_i^\alpha \partial_i^\alpha - 2 \sum_{k \neq i} \frac{a_i^\alpha b_k^\gamma a_i^k c_k^\beta}{x_i-x_k}, \quad \bar{b}_i^\beta = b_i^\gamma C_{\gamma\beta};

- **At** \(\frac{1}{(x-x_i)}\): \(\partial_{t_2} (a_i^\alpha c_i^\beta) = -2 \sum_{k \neq i} \frac{a_i^k b_k^\gamma a_i^\gamma c_k^\beta}{(x_i-x_k)^2}, \quad \partial_{t_2} (a_i^\alpha c_i^\beta) = -2 \sum_{k \neq i} \frac{a_i^k b_k^\gamma a_i^\gamma c_k^\beta}{(x_i-x_k)^2}.\)
where summation over repeated Greek indices is implied and $\dot{x}_i = \partial_{t_2} x_i$. The conditions coming from the third order poles are constraints on the vectors $\mathbf{a}_i$, $\mathbf{b}_i$. The conditions coming from the second order poles can be written in the matrix form:

$$\sum_{k=1}^{N} (zI - L)_{ik} c_k^\alpha = -\tilde{b}_i^\alpha, \quad L_{ik} = -\frac{\dot{x}_i}{2} \delta_{ik} - (1 - \delta_{ik}) \frac{\tilde{b}_i^\gamma a_k^\gamma}{x_i - x_k},$$

(32)

where $I$ is the $N \times N$ unity matrix. The matrix $L$ is going to be the Lax matrix for the spin Calogero-Moser system. The conditions at the first order poles give evolution equations in the time $t_2$. They are written in detail in [9]. Similar calculations with the linear problem for $\Psi^\dagger$ lead to the same constraints $\mathbf{b}_i^T \mathbf{a}_i = 1$ and to the linear equations for the vectors $\mathbf{c}_i^*$ with the same Lax matrix $L$:

$$\sum_{k=1}^{N} c_k^* (zI - L)_{ki} = \tilde{a}_i^\alpha, \quad \tilde{a}_i^\alpha = C^{-1}_\alpha \gamma a_i^\gamma.$$  

(33)

For completeness, we give here the equations of motion in the time $t_2$ (see [9] for details):

$$\dot{a}_i^\alpha = -2 \sum_{k \neq i} \frac{b_i^\gamma a_i^\alpha b_k^\gamma}{(x_i - x_k)^2}, \quad \dot{\tilde{b}}_i^\alpha = 2 \sum_{k \neq i} \frac{b_i^\gamma a_i^\alpha b_k^\gamma}{(x_i - x_k)^2},$$

$$\ddot{x}_i = -8 \sum_{k \neq i} \frac{b_i^\gamma a_i^\gamma b_k^\gamma a_k^\gamma}{(x_i - x_k)^3}.$$  

(35)

Now we turn to the discrete time evolution. The Baker-Akhiezer functions are

$$\Psi_{\alpha\beta}^p = \left(1 - \frac{z}{\mu}\right)^p e^{xz + \xi(t,z)} \left(C_{\alpha\beta} + \sum_i \frac{a_i^\alpha(p)c_i^\beta(p)}{x - x_i(p)}\right),$$

(36)

$$\Psi_{\alpha\beta}^{tp} = \left(1 - \frac{z}{\mu}\right)^{-p} e^{-xz - \xi(t,z)} \left(C^{-1}_{\alpha\beta} + \sum_i \frac{c_i^*\alpha(p)b_i^\beta(p)}{x - x_i(p)}\right).$$  

(37)

We should substitute them into the linear problems [21], [24] and compare the coefficients at the poles at $x = x_i(p)$ and $x = x_i(p+1)$. Note that the constant term $S_{\alpha\beta}$ in $w_{\alpha\beta}^{(1)}(p)$ cancels in the combination $w_{\alpha\beta}^{(1)}(p+1) - w_{\alpha\beta}^{(1)}(p)$ because the shift $p \rightarrow p + 1$ is equivalent to a shift of times and $S_{\alpha\beta}$ does not depend on them. The cancellation of poles gives the following conditions:

- At $\frac{1}{(x - x_i(p))^2}$: $\tilde{b}_i^\gamma(p) a_i^\gamma(p) = 1$;
- At $\frac{1}{x - x_i(p+1)}$:

$$(z - \mu)a_i^\alpha(p+1)c_i^\beta(p+1) = -a_i^\alpha(p+1)\tilde{b}_i^\beta(p+1) - \sum_j a_i^\alpha(p+1)\tilde{b}_j^\gamma(p+1)a_j^\gamma(p)c_j^\beta(p)/x_i(p+1) - x_j(p);$$

- At $\frac{1}{x - x_i(p)}$:

$$(z - \mu)a_i^\alpha(p)c_i^\beta(p) + a_i^\alpha(p)\tilde{b}_i^\beta(p) - \sum_j a_i^\alpha(p+1)\tilde{b}_j^\gamma(p+1)a_j^\gamma(p)c_j^\beta(p)/x_i(p) - x_j(p+1)$$

$$+ \sum_{j \neq i} a_i^\alpha(p)b_j^\gamma(p)a_j^\gamma(p)c_j^\beta(p)/x_i(p) - x_j(p) + \sum_{j \neq i} b_i^\gamma(p)b_j^\gamma(p)a_i^\gamma(p)c_j^\beta(p)/x_i(p) - x_j(p) = 0.$$
Similar conditions follow from the linear problem for $\Psi^\dagger$:

- At $\frac{1}{x-x_i(p)^2}$:  
  \[ b_i^\gamma(p)a_i^\gamma(p) = 1; \]

- At $\frac{1}{x-x_i(p-1)}$:  
  \[
  (z - \mu)c_i^{*\alpha}(p-1)b_i^\beta(p-1) = \bar{a}_i^\alpha(p-1)b_i^\beta(p-1) + \sum_j c_j^{*\alpha}(p)b_j^\gamma(p)a_i^\gamma(p-1)b_j^\beta(p-1) \times \frac{x_i(p-1) - x_j(p)}{x_i(p) - x_j(p)};
  \]

- At $\frac{1}{x-x_i(p)}$:  
  \[
  (z - \mu)c_i^{*\alpha}(p)b_i^\beta(p) - \bar{a}_i^\alpha(p)b_i^\beta(p) + \sum_j c_j^{*\alpha}(p)b_j^\gamma(p)a_i^\gamma(p)b_j^\beta(p) \times \frac{x_i(p) - x_j(p)}{x_i(p) - x_j(p) - x_i(p)} + \sum_j c_j^{*\alpha}(p)b_j^\gamma(p)a_i^\gamma(p)b_j^\beta(p) = 0.
  \]

The condition at the second order pole is the same constraint as before. Introduce the matrices

\[
L_{ij}(p) = -\delta_{ij} \frac{\dot{x}_i(p)}{2} - (1 - \delta_{ij}) \frac{b_i^\gamma(p)a_j^\gamma(p)}{x_i(p) - x_j(p)} \tag{38}
\]

(the same Lax matrix as in (82)) and

\[
M_{ij}(p) = \frac{b_i^\gamma(p+1)a_j^\gamma(p)}{x_i(p+1) - x_j(p)},
\]

then the above conditions coming from the first order poles at $x_i(p)$ and $x_i(p \pm 1)$ can be written as

\[
\begin{align*}
  \left( z - \mu \right) c_i^{*\alpha}(p+1) &= -\bar{b}_i^\beta(p+1) - \sum_j M_{ij}(p)c_j^\beta(p) \\
  a_i^\alpha(p) \left[ \sum_j \left( z\delta_{ij} - L_{ij}(p) \right) c_j^\beta(p) + \bar{b}_i^\beta(p) \right] &= 0 \tag{40} \\
  + c_i^\beta(p) \left[ \sum_j a_j^\alpha(p+1)M_{ji}(p) + \sum_j a_j^\alpha(p)L_{ji}(p) - \mu a_i^\alpha(p) \right] &= 0,
\end{align*}
\]

\[
\begin{align*}
  \left( z - \mu \right) c_i^{*\alpha}(p-1) &= \bar{a}_i^\alpha(p-1) - \sum_j c_j^{*\alpha}(p)M_{ji}(p-1) \\
  b_i^\beta(p) \left[ \sum_j c_j^{*\alpha}(p) \left( z\delta_{ij} - L_{ji}(p) \right) - \bar{a}_i^\alpha(p) \right] &= 0 \tag{41} \\
  + c_i^{*\alpha}(p) \left[ \sum_j M_{ij}(p-1)b_j^\beta(p-1) + \sum_j L_{ij}(p)b_j^\beta(p) - \mu b_i^\beta(p) \right] &= 0.
\end{align*}
\]
The first brackets in the second equations vanish because of (32), (33). Introduce the $N$-component column vectors $C^\alpha = (c^\alpha_1, \ldots, c^\alpha_N)^T$, $C^{*\alpha} = (c^{*\alpha}_1, \ldots, c^{*\alpha}_N)^T$, $A^\alpha = (a^\alpha_1, \ldots, a^\alpha_N)^T$, $B^\alpha = (b^\alpha_1, \ldots, b^\alpha_N)^T$ and $\tilde{A}^\alpha = (\tilde{a}^\alpha_1, \ldots, \tilde{a}^\alpha_N)^T$, $\tilde{B}^\alpha = (\tilde{b}^\alpha_1, \ldots, \tilde{b}^\alpha_N)^T$. In this notation, the above equations are rewritten as linear equations for the vectors $A^\alpha$ and $B^\beta$,

$$
\begin{align*}
\text{A}^{\alpha T}(p+1)M(p) + A^{\alpha T}(p)L(p) &= \mu A^{\alpha T}(p) \\
M(p-1)B^\beta(p+1) + L(p)B^\beta(p) &= \mu B^\beta(p),
\end{align*}
$$

(42)

and linear equations for the vectors $C^\alpha$ and $C^{*\alpha}$:

$$
\begin{align*}
(z - \mu)C^\beta(p+1) &= -\tilde{B}^\beta(p+1) - M(p)C^\beta(p) \\
(z - \mu)C^\beta(p) &= -\tilde{B}^\beta(p) + L(p)C^\beta(p) - \mu C^\beta(p),
\end{align*}
$$

(43)

$$
\begin{align*}
(z - \mu)C^{*\alpha T}(p+1) &= \tilde{A}^{\alpha T}(p+1) - C^{*\alpha T}(p)M(p-1) \\
(z - \mu)C^{*\alpha T}(p) &= \tilde{A}^{\alpha T}(p) + C^{*\alpha T}(p)L(p) - \mu C^{*\alpha T}(p).
\end{align*}
$$

(44)

From equations (43) we have

$$M(p)C^\beta(p) + \left(L(p+1) - \mu I\right)C^\beta(p+1) = 0$$

(45)

while from (44) we have

$$C^{*\alpha T}(p+1)M(p) + C^{*\alpha T}(p)\left(L(p) - \mu I\right) = 0.$$  

(46)

Substituting equations (43) into (45) and equations (44) into (46), we obtain

$$M(p)\left(-\tilde{B}^\beta(p) + L(p)C^\beta(p) - \mu C^\beta(p)\right) - \left(L(p+1) - \mu I\right)\left(\tilde{B}^\beta(p+1) + M(p)C^\beta(p)\right) = 0,$$

$$\left(\tilde{A}^{\alpha T}(p) + C^{*\alpha T}(p)L(p) - \mu C^{*\alpha T}(p)\right)M(p-1)$$

$$+ \left(\tilde{A}^{\alpha T}(p-1) - C^{*\alpha T}(p)M(p-1)\right)\left(L(p-1) - \mu I\right) = 0,$$

or, after simplifying and taking into account equations (42),

$$\left(M(p)L(p) - L(p+1)M(p)\right)C^\beta(p) = 0,$$

$$C^{*\alpha T}(p+1)\left(M(p)L(p) - L(p+1)M(p)\right) = 0.$$  

This implies the consistency condition

$$L(p+1)M(p) = M(p)L(p)$$

(47)

or $L(p+1) = M(p)L(p)M^{-1}(p)$ which is the discrete Lax equation. One can show by a direct calculation that it holds provided the equations (42) are satisfied, which are equations of motion for the poles $x_i$ and components of the vectors $a_i$, $b_i$. Let us write down these equations in detail:

$$\sum_j b^\beta_i(p) a^\gamma_j(p-1) b^\beta_j(p-1) \left(x_i(p) - x_j(p-1)\right) = \frac{\dot{x}_i(p)}{2} b^\beta_i(p) + \sum_{j \neq i} b^\gamma_i(p) a^\gamma_j(p) b^\beta_j(p) \left(x_i(p) - x_j(p)\right) + \mu b^\beta_i(p),$$

(48)
\[
\sum_j a_j^\beta (p + 1) b_j^\gamma (p + 1) a_j^\gamma (p) \frac{x_j (p + 1) - x_j (p)}{x_j (p + 1) - x_i (p)} = \frac{\dot{x}_i (p)}{2} a_i^\gamma (p) + \sum_{j \neq i} \frac{a_j^\beta (p) b_j^\gamma (p) a_i^\gamma (p)}{x_j (p) - x_i (p)} + \mu a_i^\alpha (p). \tag{49}
\]

Multiply the first equation by \(a_i^\beta (p)\) and sum over \(\beta\), then multiply the second equation by \(b_i^\gamma (p)\), sum over \(\alpha\) and take into account the constraint \(b_i^\gamma a_i^\gamma = 1\). Subtracting the resulting equations, we eliminate \(\dot{x}_i (p)\) and obtain the equations of motion \([2]\):

\[
\sum_j b_j^\gamma (p) a_j^\gamma (p + 1) b_j^\beta (p + 1) a_j^\beta (p) \frac{x_j (p) - x_j (p + 1)}{x_j (p) - x_j (p)} + \sum_j b_j^\gamma (p) a_j^\gamma (p - 1) b_j^\beta (p - 1) a_j^\beta (p) \frac{x_j (p) - x_j (p - 1)}{x_j (p) - x_j (p)}
= 2 \sum_{j \neq i} \frac{b_j^\beta (p) a_j^\gamma (p) b_j^\beta (p) a_i^\beta (p)}{x_j (p) - x_j (p)}. \tag{50}
\]

Summing the resulting equations, we obtain an expression for \(\dot{x}_i (p)\):

\[
\dot{x}_i (p) = \sum_j b_j^\gamma (p) a_j^\gamma (p - 1) b_j^\gamma (p - 1) a_j^\gamma (p - 2) b_j^\beta (p - 2) \frac{x_j (p) - x_j (p - 1)}{(x_j (p) - x_i (p))^2 (x_k (p - 2) - x_j (p - 1))} + \sum_{j, k} \frac{b_j^\gamma (p) a_k^\gamma (p) b_k^\gamma (p) a_j^\gamma (p - 1) b_j^\beta (p - 1)}{(x_j (p - 1) - x_i (p))^2 (x_k (p) - x_j (p - 1))}
+ \sum_{j \neq k} \frac{b_j^\gamma (p) a_j^\gamma (p - 1) a_j^\gamma (p - 1) b_j^\beta (p - 1) + b_j^\gamma (p) b_j^\gamma (p - 1) a_j^\gamma (p - 1) a_j^\gamma (p - 1) b_j^\beta (p - 1)}{(x_j (p - 1) - x_i (p))^2 (x_i (p) - x_k (p - 1))} = 0, \tag{51}
\]

\[
\sum_j b_j^\gamma (p + 1) a_j^\gamma (p + 2) a_j^\gamma (p + 1) a_j^\gamma (p + 2) \frac{x_j (p + 1) - x_j (p)}{(x_j (p + 1) - x_i (p))^2 (x_k (p + 2) - x_j (p + 1))} + \sum_{j, k} \frac{b_j^\gamma (p + 1) a_k^\gamma (p + 1) b_k^\gamma (p) a_j^\gamma (p + 1) b_j^\beta (p + 1) a_j^\gamma (p + 1) b_j^\beta (p + 1) a_j^\gamma (p + 1)}{(x_j (p + 1) - x_i (p))^2 (x_k (p) - x_j (p + 1))}
+ \sum_{j \neq k} \frac{b_j^\gamma (p + 1) a_j^\gamma (p + 1) b_j^\gamma (p + 1) a_j^\gamma (p) a_j^\gamma (p + 1) + b_j^\gamma (p + 1) a_j^\gamma (p + 1) b_j^\gamma (p + 1) a_j^\gamma (p) a_j^\gamma (p + 1)}{(x_j (p + 1) - x_i (p))^2 (x_i (p) - x_k (p + 1))} = 0. \tag{52}
\]

At last, let us discuss the continuum limit of equations \([50], [18], [19]\). We expect that the continuous time flow \(t\) corresponding to \(p\) tends to \(t_2\). We set \(x_i (p) = \lambda p + y_i (p)\) and expand \(x_i (p \pm 1) = \pm \lambda + x_i (p) \pm \varepsilon \partial_t y_i + \frac{\varepsilon^2}{2} \partial_t^2 y_i + O(\varepsilon^3), a_i^\alpha (p \pm 1) = a_i^\alpha \pm \varepsilon \partial_t a_i^\alpha + O(\varepsilon^2), b_i^\gamma (p \pm 1) = b_i^\gamma \pm \varepsilon \partial_t b_i^\gamma + O(\varepsilon^2)\), where \(\lambda = O(\sqrt{\varepsilon})\). Separating the terms with \(j = i\) in the first line of \([50]\), we expand this equation up to the first non-vanishing order \(O(\varepsilon)\) as \(\varepsilon \to 0\) and obtain

\[
\partial_t^2 y_i = -2g \sum_{j \neq i} \frac{b_j^\gamma a_j^\gamma b_j^\beta a_i^\beta}{(y_i - y_j)^3}, \quad g = \lambda^4 / \varepsilon^2 = O(1), \tag{54}
\]
which are equations of motion for the continuous time spin Calogero-Moser system. We see that the coupling constant depends on the way of tending the time step to zero. Comparison with \((35)\) shows that one should put \(g = 4\), i.e., \(\lambda^4 = 4\varepsilon^2\), then in the limit one has \(\partial_x^2 y_i = \ddot{y}_i\).

The continuum limit of equations \((48), (49)\) is a little bit more tricky. Expanding \((48)\) as \(\lambda, \varepsilon \to 0\), we obtain:

\[
\frac{b^\beta_i}{\lambda} - \frac{\varepsilon}{\lambda^2} b^\beta_i \partial_t y_i - \frac{\varepsilon}{\lambda} b^\beta_i \partial_t a^\gamma_i b^\beta_i - \frac{\varepsilon}{\lambda} \partial_t b^\beta_i - \lambda \sum_{j \neq i} \frac{b^\gamma_j a^\alpha_i b^\beta_j}{(y_i - y_j)^2} + O(\varepsilon) = \frac{\dot{y}_i}{2} b^\beta_i + \mu b^\beta_i.
\]

Comparison of the leading terms gives \(\lambda = \mu^{-1}\). The next order terms give

\[
\partial_t y_i = -\frac{\lambda^2}{2 \varepsilon} \dot{y}_i - \lambda b^\gamma_i \partial_t a^\alpha_i + O(\varepsilon).
\]

In order to make the \(t\)-flow identical to the \(t_2\)-flow in the limit, one should require \(\lambda^2 = -2\varepsilon\) which agrees with the condition \(\lambda^4 = 4\varepsilon^2\) mentioned above. Then in the order \(O(\lambda)\) we obtain

\[
\partial_t b^\beta_i = 2 \sum_{j \neq i} \frac{b^\gamma_j a^\alpha_i b^\beta_j}{(y_i - y_j)^2},
\]

which is the second equation in \((34)\). The first one is obtained in a similar way from equation \((49)\).

### 4 Conclusion

To conclude, we have derived the discrete time equations of motion for the characteristic data of rational solutions to the semi-discrete matrix KP hierarchy – positions of \(N\) poles \(x_i\) and vectors \(a^\alpha_i, b^\alpha_i\) parametrizing rank 1 matrix residues at the poles. These equations define integrable time discretization of the spin Calogero-Moser \(N\)-particle system (which is known as the Gibbons-Hermsen system). They look like equations of motion for the discrete time spinless Calogero-Moser system “dressed” by scalar products of the vectors \(a^\alpha_i, b^\alpha_i\). The continuum limit of these equation coincides with the equations of motion for the Gibbons-Hermsen system. The main technical tool for the derivation of the discrete time equations of motion is the linear problems for the Baker-Akhiezer functions of the semi-discrete matrix KP hierarchy, which are obtained as corollaries of the basic bilinear identity for the tau-function.

It would be interesting to extend the results of this work to elliptic solutions to the semi-discrete matrix KP hierarchy. We expect that the method of the paper \([7]\) is applicable in this case and the resulting equations of motion have a similar structure, with the simple pole function \((x - x_i)^{-1}\) being replaced by the Weierstrass \(\zeta\)-function \(\zeta(x - x_i)\).
Appendix

Some corollaries of the bilinear identity

Here we list some corollaries of the basic bilinear identity \([12]\) which are used below in the appendix for the derivation of the linear problems \([21], [25]\).

Differentiating the bilinear identity \([12]\) with respect to \(t_{\gamma,1}\) and setting \(t' = t - [\mu^{-1}]_\beta\), we obtain, after calculating the residues, for any distinct \(\alpha, \beta, \gamma\):

\[
\tau_{\alpha \beta}(t - [\mu^{-1}]_\beta) \partial_{t_{\gamma,1}} \tau(t) - \tau(t) \partial_{t_{\gamma,1}} \tau_{\alpha \beta}(t - [\mu^{-1}]_\beta) + \frac{\epsilon_{\alpha \gamma} \epsilon_{\gamma \beta}}{\epsilon_{\alpha \beta}} \tau_{\alpha \gamma}(t) \tau_{\gamma \beta}(t - [\mu^{-1}]_\beta) = 0 \quad (A1)
\]

(no summation over repeated indices!). Differentiating \([12]\) with respect to \(t_{\beta,1}\) and setting \(t' = t - [\mu^{-1}]_\alpha - [\nu^{-1}]_\beta\), we obtain, for any distinct \(\alpha, \beta, \gamma\):

\[
\begin{align*}
\partial_{t_{\beta,1}} \tau_{\alpha \beta}(t - [\nu^{-1}]_\beta) & \tau(t - [\mu^{-1}]_\alpha) - \partial_{t_{\beta,1}} \tau(t - [\mu^{-1}]_\alpha) \tau_{\alpha \beta}(t - [\nu^{-1}]_\beta) \\
+ & \nu \tau_{\alpha \beta}(t - [\mu^{-1}]_\alpha) - \nu \tau_{\alpha \beta}(t - [\mu^{-1}]_\alpha - [\nu^{-1}]_\beta) = 0.
\end{align*}
\]

(A2)

In a similar way, differentiating \([12]\) with respect to \(t_{\alpha,1}\) and setting \(t' = t - [\mu^{-1}]_\alpha - [\nu^{-1}]_\beta\), we obtain, for any distinct \(\alpha, \beta, \gamma\):

\[
\begin{align*}
\partial_{t_{\alpha,1}} \tau_{\alpha \beta}(t - [\nu^{-1}]_\beta) & \tau(t - [\mu^{-1}]_\alpha) - \partial_{t_{\alpha,1}} \tau(t - [\mu^{-1}]_\alpha) \tau_{\alpha \beta}(t - [\nu^{-1}]_\beta) \\
- & \mu \tau_{\alpha \beta}(t - [\mu^{-1}]_\alpha) + \mu \tau(t) \tau_{\alpha \beta}(t - [\mu^{-1}]_\alpha - [\nu^{-1}]_\beta) = 0.
\end{align*}
\]

(A3)

Differentiating \([12]\) at \(\beta = \alpha\) with respect to \(t_{\gamma,1}\) \((\gamma \neq \alpha)\) and setting \(t' = t - [\mu^{-1}]_\alpha\), we obtain, for any distinct \(\alpha, \gamma\):

\[
\partial_{t_{\gamma,1}} \tau(t - [\mu^{-1}]_\alpha) \tau(t) - \partial_{t_{\gamma,1}} \tau(t) \tau(t - [\mu^{-1}]_\alpha) + \mu \tau_{\alpha \gamma}(t) \tau_{\gamma \alpha}(t - [\mu^{-1}]_\alpha) = 0. \quad (A4)
\]

Setting \(t' = t + [\mu^{-1}] - [\nu^{-1}]_\beta\), where we abbreviate \([\mu^{-1}] \equiv \sum_{\gamma=1}^{N} [\mu^{-1}]_{\gamma}\), we represent the bilinear identity \([12]\) at \(\alpha \neq \beta\) in the form

\[
\begin{align*}
\epsilon_{\beta \alpha} \int_{C_{\infty}} dzz^{-1} & \left(1 - \frac{z}{\mu}\right) \tau(t - [\mu^{-1}]_\alpha) \tau_{\beta \alpha}(t + [\nu^{-1}]_\beta + [z^{-1}]_\alpha) \\
+ & \epsilon_{\alpha \beta} \int_{C_{\infty}} dzz^{-1} \left(1 - \frac{z}{\mu}\right) \tau_{\alpha \beta}(t - [\mu^{-1}]_\beta) \tau(t + [\mu^{-1}] - [\nu^{-1}]_\beta + [z^{-1}]_\beta) \\
+ & \sum_{\gamma \neq \alpha, \beta} \epsilon_{\alpha \gamma} \epsilon_{\beta \gamma} \int_{C_{\infty}} dzz^{-2} \left(1 - \frac{z}{\mu}\right) \tau_{\alpha \gamma}(t - [\mu^{-1}]_{\gamma}) \tau_{\gamma \beta}(t + [\mu^{-1}] - [\nu^{-1}]_\beta + [z^{-1}]_\gamma) = 0.
\end{align*}
\]

Calculating the residues and multiplying by \(\mu\), we obtain:

\[
\begin{align*}
\epsilon_{\beta \alpha} \mu \tau(t) \tau_{\beta \alpha}(t + [\mu^{-1}] - [\nu^{-1}]_\beta) + \epsilon_{\beta \alpha} \partial_{t_{\gamma,1}} \tau\tau_{\beta \alpha}(t + [\mu^{-1}] - [\nu^{-1}]_\beta) \\
- & \epsilon_{\beta \alpha} \tau(t) \partial_{t_{\gamma,1}} \tau_{\beta \alpha}(t + [\mu^{-1}] - [\nu^{-1}]_\beta) \\
+ & \epsilon_{\alpha \beta} (\mu - \nu) \tau_{\alpha \beta}(t - [\mu^{-1}]_\beta) \tau(t + [\mu^{-1}]) + \epsilon_{\alpha \beta} \nu \tau_{\alpha \beta}(t) \tau(t + [\mu^{-1}] - [\nu^{-1}]_\beta) \\
- & \sum_{\gamma \neq \alpha, \beta} \epsilon_{\alpha \gamma} \epsilon_{\beta \gamma} \tau_{\alpha \gamma}(t) \tau_{\gamma \beta}(t + [\mu^{-1}] - [\nu^{-1}]_\beta) = 0. \quad (A5)
\end{align*}
\]
Setting \( t' = t + [\mu^{-1}] - [\nu^{-1}]_\alpha \), we represent the bilinear identity (12) at \( \beta = \alpha \) in the form
\[
\left\{ \int_{C_\infty} dz \frac{1 - \frac{z}{\mu}}{1 - \frac{z}{\nu}} \tau(t - [z^{-1}]_\alpha) \tau(t + [\mu^{-1}] - [\nu^{-1}]_\alpha + [z^{-1}]_\alpha) \right. \\
+ \sum_{\gamma \neq \alpha} \int_{C_\infty} dz z^{-2} \left( 1 - \frac{z}{\mu} \right) \tau_{\alpha \gamma}(t - [z^{-1}]_\gamma) \tau_{\gamma \alpha}(t + [\mu^{-1}] - [\nu^{-1}]_\alpha + [z^{-1}]_\gamma) = 0.
\]
Calculating the residues and multiplying by \( \mu \), we obtain:
\[
\nu(\mu - \nu) \tau(t - [\nu^{-1}]_\alpha) \tau(t + [\mu^{-1}]) - \nu(\mu - \nu) \tau(t) \tau(t + [\mu^{-1}] - [\nu^{-1}]_\alpha) \\
- \nu \partial_{t_{\alpha, 1}} \tau(t) \tau(t + [\mu^{-1}] - [\nu^{-1}]_\alpha) + \nu \tau(t) \partial_{t_{\alpha, 1}} \tau(t + [\mu^{-1}] - [\nu^{-1}]_\alpha) \\
- \sum_{\gamma \neq \alpha} \tau_{\alpha \gamma}(t) \tau_{\gamma \alpha}(t + [\mu^{-1}] - [\nu^{-1}]_\alpha) = 0.
\]
(A6)

Note that under the identification \( \tau_{\alpha \beta}^p = \tau_{\alpha \beta}(t - p[\mu^{-1}]) \) and the formal substitution \( \mu = 0 \) equations (A5) and (A6) become the corresponding bilinear equations for the matrix modified KP hierarchy [27].

**Derivation of the linear problems (24), (25)**

Here we show that the linear problems (24), (25), which are the basic tools for deriving the equations of motion for the discrete time pole dynamics, are equivalent to corollaries of the bilinear identity (12). The derivation is similar to the one given in [27].

Let us give some details of the calculations for the linear problem (24). We start from the case \( \alpha \neq \beta \). Substituting
\[
\Psi_{\alpha \beta}^p = \epsilon_{\alpha \beta} \frac{\tau_{\alpha \beta}^p(t - [z^{-1}]_\beta)}{\tau(t)} z^{\delta_{\alpha \beta} - 1} \left( 1 - \frac{z}{\mu} \right) e^{xz + \xi(t, z)}
\]
and
\[
w_{\alpha \gamma}^{(1)}(p) = \begin{cases} \\
\epsilon_{\alpha \gamma} \frac{\tau_{\alpha \gamma}^p(t)}{\tau(t)} & \text{if } \alpha \neq \gamma \\
- \frac{\partial_{t_{\alpha, 1}} \tau_{\gamma \beta}^p(t)}{\tau(t)} & \text{if } \alpha = \gamma
\end{cases}
\]
into (24), we write it in the form
\[
\mu z^{-1} \epsilon_{\alpha \beta} \frac{\tau_{\alpha \beta}^p(t - [z^{-1}]_\beta)}{\tau(t)} + (1 - \mu z^{-1}) \epsilon_{\alpha \beta} \frac{\tau_{\alpha \beta}^{p+1}(t - [z^{-1}]_\beta)}{\tau^{p+1}(t)} =
\epsilon_{\alpha \beta} \frac{\tau_{\alpha \beta}^p(t - [z^{-1}]_\beta)}{\tau(t)} + z^{-1} \epsilon_{\alpha \beta} \partial_{t_{\alpha, 1}} \left( \frac{\tau_{\alpha \beta}^p(t - [z^{-1}]_\beta)}{\tau(t)} \right) \\
+ \sum_{\gamma \neq \alpha} z^{\delta_{\gamma \beta} - 1} \epsilon_{\alpha \gamma} \epsilon_{\gamma \beta} \left( \frac{\tau_{\alpha \gamma}^{p+1}(t)}{\tau^{p+1}(t)} - \frac{\tau_{\alpha \gamma}^p(t)}{\tau(t)} \right) \frac{\tau_{\gamma \beta}^p(t - [z^{-1}]_\beta)}{\tau(t)}
\]
\[-z^{-1} \epsilon_{\alpha \beta} \left( \frac{\partial_{\alpha_1} \tau^{p+1}(t)}{\tau^{p+1}(t)} - \frac{\partial_{\alpha_1} \tau^p(t)}{\tau^p(t)} \right) \frac{\tau^p_\alpha \beta(t - [z^{-1}] \beta)}{\tau^p(t)}.\]

After some obvious transformations, separating the terms with the denominator \((\tau^p(t))^2\), we can rewrite this as
\[
-(1 - \mu z^{-1}) \epsilon_{\alpha \beta} \frac{\tau^{p+1}_\alpha \beta(t - [z^{-1}] \beta)}{\tau^{p+1}(t)} + (1 - \mu z^{-1}) \epsilon_{\alpha \beta} \frac{\tau^p_\alpha \beta(t - [z^{-1}] \beta)}{\tau^p(t)}
+ \epsilon_{\alpha \beta} \frac{\tau^{p+1}_\alpha \beta(t) \tau^p(t - [z^{-1}] \beta)}{\tau^{p+1}(t) \tau^p(t)}
+ z^{-1} \epsilon_{\alpha \beta} \frac{\partial_{\alpha_1} \tau^p(t - [z^{-1}] \beta)}{\tau^p(t)} - z^{-1} \epsilon_{\alpha \beta} \frac{\partial_{\alpha_1} \tau^{p+1}_\alpha \beta(t) \tau^p_\alpha \beta(t - [z^{-1}] \beta)}{\tau^{p+1}(t) \tau^p(t)}
+ z^{-1} \sum_{\gamma \neq \alpha, \beta} \epsilon_{\alpha \gamma} \epsilon_{\beta \gamma} \frac{\tau^{p+1}_\alpha \gamma(t) \tau^p_\gamma \beta(t - [z^{-1}] \beta)}{\tau^{p+1}(t) \tau^p(t)}
\]

Th idea is to transform the expression in the brackets \(\ldots\) using the bilinear relations (A1) and (A2). Namely, taking into account that \(\partial_i = \sum_{\gamma} \partial_{\gamma, i, 1}\), we rewrite it in the form
\[
\{ \ldots \} = -z^{-1} \tau^p_\alpha \beta(t - [z^{-1}] \beta) \partial_{\alpha_1} \tau^p(t) - \tau^p_\alpha \beta(t) \tau^p(t - [z^{-1}] \beta)
- z^{-1} \sum_{\gamma \neq \alpha, \beta} \left[ \tau^p_\alpha \beta(t - [z^{-1}] \beta) \partial_{\gamma, 1} \tau^p(t) + \frac{\epsilon_{\alpha \gamma} \epsilon_{\beta \gamma}}{\epsilon_{\alpha \beta}} \tau^p_\alpha \gamma(t) \tau^p_\gamma \beta(t - [z^{-1}] \beta) \right]
\]

and apply (A2) with \(\mu = \infty, \nu = z\) in the first line and (A1) with \(\mu = z\) in the second line. The result is
\[
\{ \ldots \} = -\tau^p(t) \left[ z^{-1} \sum_{\gamma \neq \alpha} \partial_{\gamma, 1} \tau^p_\alpha \beta(t - [z^{-1}] \beta) + \tau^p_\alpha \beta(t - [z^{-1}] \beta) \right].
\]

Substituting this back and multiplying by \(\tau^{p+1}(t) \tau^p(t)\), we obtain, after some cancellations:
\[
z \epsilon_{\alpha \beta} \tau^{p+1}_\alpha \beta(t) \tau^p(t - [z^{-1}] \beta) - z \epsilon_{\alpha \beta} \tau^p(t) \tau^{p+1}_\alpha \beta(t - [z^{-1}] \beta)
+ \mu \epsilon_{\alpha \beta} \tau^p(t) \tau^{p+1}_\alpha \beta(t - [z^{-1}] \beta) - \mu \epsilon_{\alpha \beta} \tau^{p+1}(t) \tau^p_\alpha \beta(t - [z^{-1}] \beta)
+ \epsilon_{\alpha \beta} \partial_{\alpha_1} \tau^{p+1}_\alpha \beta(t - [z^{-1}] \beta) - \epsilon_{\alpha \beta} \partial_{\alpha_1} \tau^{p+1}(t) \tau^p_\alpha \beta(t - [z^{-1}] \beta)
\]
\[
+ \sum_{\gamma \neq \alpha, \beta} \epsilon_{\alpha \gamma} \epsilon_{\beta \gamma} \tau^{p+1}_\alpha \gamma(t) \tau^p_\gamma \beta(t - [z^{-1}] \beta) = 0.
\]

Taking into account that \(\tau^p_\alpha \beta(t) = \tau_\alpha \beta \left( t - p[\mu^{-1}] \right) \), one can see that this is exactly the bilinear relation (A5), where one should put \(\nu = z\).

Let us now pass to the case \(\alpha = \beta\) in (23):
\[
\mu \frac{\tau^p(t - [z^{-1}] \alpha)}{\tau^p(t)} + (z - \mu) \frac{\tau^{p+1}(t - [z^{-1}] \alpha)}{\tau^{p+1}(t)} = z \frac{\tau^p(t - [z^{-1}] \alpha)}{\tau^p(t)} + \partial_{\alpha_1} \left( \frac{\tau^p(t - [z^{-1}] \alpha)}{\tau^p(t)} \right)
\]
\[-z^{-1} \sum_{\gamma \neq \alpha} \left( \frac{\tau_{\alpha\gamma}^{p+1}(t)}{\tau_{\gamma\alpha}^{p+1}(t) - \tau_{\alpha\gamma}^{p}(t) \tau_{\gamma\alpha}^{p}(t)} - \frac{\tau_{\alpha\gamma}^{p}(t)}{\tau_{\gamma\alpha}^{p}(t) - \tau_{\alpha\gamma}^{p}(t) \tau_{\gamma\alpha}^{p}(t)} \right) \frac{\tau_{\gamma\alpha}^{p}(t - [z^{-1}]_{\alpha})}{\tau_{\gamma\alpha}^{p}(t)} \]

\[-\left( \frac{\partial_{t_{\alpha\gamma}} \tau_{\alpha\gamma}^{p+1}(t)}{\tau_{\gamma\alpha}^{p+1}(t) - \tau_{\alpha\gamma}^{p}(t) \tau_{\gamma\alpha}^{p}(t)} - \frac{\partial_{t_{\alpha\gamma}} \tau_{\gamma\alpha}^{p}(t)}{\tau_{\gamma\alpha}^{p}(t) - \tau_{\alpha\gamma}^{p}(t) \tau_{\gamma\alpha}^{p}(t)} \right) \frac{\tau_{\gamma\alpha}^{p}(t - [z^{-1}]_{\alpha})}{\tau_{\gamma\alpha}^{p}(t)} \].

Separating the terms with the denominator \((\tau_{\gamma\alpha}^{p}(t))^2\), we rewrite this as

\[
(\mu - z) \frac{\tau_{\alpha\gamma}^{p+1}(t - [z^{-1}]_{\alpha})}{\tau_{\gamma\alpha}^{p+1}(t)} - (\mu - z) \frac{\tau_{\alpha\gamma}^{p}(t - [z^{-1}]_{\alpha})}{\tau_{\gamma\alpha}^{p}(t)} + \frac{\partial_{t_{\alpha\gamma}} \tau_{\alpha\gamma}^{p+1}(t)}{\tau_{\gamma\alpha}^{p+1}(t) - \tau_{\alpha\gamma}^{p}(t)} \tau_{\gamma\alpha}^{p}(t - [z^{-1}]_{\alpha}) \]

\[-z^{-1} \sum_{\gamma \neq \alpha} \left( \frac{\tau_{\alpha\gamma}^{p}(t)}{\tau_{\gamma\alpha}^{p+1}(t) \tau_{\gamma\alpha}^{p}(t)} - \frac{\partial_{t_{\alpha\gamma}} \tau_{\alpha\gamma}^{p+1}(t)}{\tau_{\gamma\alpha}^{p+1}(t) \tau_{\gamma\alpha}^{p}(t)} \right) \frac{\tau_{\gamma\alpha}^{p}(t - [z^{-1}]_{\alpha})}{\tau_{\gamma\alpha}^{p}(t)} \]

\[+ \frac{1}{(\tau_{\gamma\alpha}^{p}(t))^2} \left\{ \sum_{\gamma \neq \alpha} \left( z^{-1} \frac{\tau_{\alpha\gamma}^{p}(t)}{\tau_{\gamma\alpha}^{p+1}(t) \tau_{\gamma\alpha}^{p}(t)} \frac{\tau_{\alpha\gamma}^{p}(t - [z^{-1}]_{\alpha})}{\tau_{\gamma\alpha}^{p}(t)} - \tau_{\gamma\alpha}^{p}(t - [z^{-1}]_{\alpha}) \partial_{t_{\alpha\gamma}} \tau_{\gamma\alpha}^{p}(t) \right) \right\} = 0.\]

Using the 3-term relation \((A4)\), we have for the expression in the brackets \{\ldots\}:

\[\{ \ldots \} = -\tau_{\gamma\alpha}^{p}(t) \sum_{\gamma \neq \alpha} \partial_{t_{\alpha\gamma}} \tau_{\gamma\alpha}^{p}(t - [z^{-1}]_{\alpha}),\]

so, after multiplying by \(\tau_{\alpha\gamma}^{p+1}(t) \tau_{\gamma\alpha}^{p}(t)\), we obtain for the previous expression:

\[z(\mu - z) \tau_{\gamma\alpha}^{p}(t) \tau_{\alpha\gamma}^{p+1}(t - [z^{-1}]_{\alpha}) - z(\mu - z) \tau_{\alpha\gamma}^{p+1}(t) \tau_{\gamma\alpha}^{p}(t - [z^{-1}]_{\alpha}) \]

\[+ z \partial_{t_{\alpha\gamma}} \tau_{\alpha\gamma}^{p}(t - [z^{-1}]_{\alpha}) \tau_{\alpha\gamma}^{p+1}(t) - z \partial_{t_{\alpha\gamma}} \tau_{\alpha\gamma}^{p+1}(t) \tau_{\gamma\alpha}^{p}(t - [z^{-1}]_{\alpha}) \]

\[- \sum_{\gamma \neq \alpha} \frac{\tau_{\alpha\gamma}^{p+1}(t)}{\tau_{\gamma\alpha}^{p}(t) \tau_{\gamma\alpha}^{p}(t)} \frac{\tau_{\alpha\gamma}^{p}(t - [z^{-1}]_{\alpha})}{\tau_{\gamma\alpha}^{p}(t)} = 0.\]

One can see that this is exactly the bilinear relation \((A6)\), where one should put \(\nu = z\).

Equation \((25)\) for \(\Psi\) can be processed in a similar way using the bilinear relations \((A1)\), \((A3)\), \((A4)\).

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