The Zeros of Orthogonal Polynomials and Markov–Bernstein Inequalities for Jacobi-Exponential Weights on $(-1,1)$

1. Introduction and Results

Let $w$ be a weight in $I = (a, b)$, $-\infty < a < b < \infty$, for which the moment problem possesses an unique solution. $P_n$ stands for the set of polynomials of degree at most $n$. $\| \cdot \|_{L_p(t)}$ is an usual (weighted) $L_p(I)$ (quasi) norm on interval $I$.

Assume that $W = e^{Q(x)}$ where $Q: (-1,1) \to [0,\infty)$ is continuous. $W$ is an exponential weight on $I$. Also, let $0 < p < \infty$, $a \leq t_r < t_{r-1} < \cdots < t_1 < b$, $p_i > -1/p$, $i = 1, 2, \ldots, r$, and

$$U(x) = \prod_{i=1}^{r} |x - t_i|^{p_i},$$

where $U$ is a generalized Jacobi weight on $I$. The combination $WU$ is called a Jacobi-exponential weight on $I$. This paper deals with the zeros of orthogonal polynomials and Markov–Bernstein inequalities for the Jacobi-Exponential weight $WU$ on $(-1,1)$. In addition, Markov–Bernstein inequalities for the weight $WU$ are also obtained.

Definition 1 (see [1], Definition 1.7, p. 14). Given $c$, $t \geq 0$ and a non-negative Borel measure $\nu$ with compact support in $C$ and total mass $\leq t$, we say that

$$P(z) = c \exp \left( \int \ln |z - s| \, d\nu(s) \right),$$

is an exponential of a potential of mass $\leq t$. We denote the set of all such $P$ by $\mathcal{P}_t$.

We note that for $P \in P_n$, $|P| \in \mathcal{P}_t$, $t \geq n$.

Definition 2 (see [1], p. 19). Let $w$ be a weight in $I$. For $0 < p < \infty$, generalized Christoffel functions with respect to $w$ for $z \in C$ are defined by

$$\lambda_{p,n}(w; z) = \inf_{P \in \mathcal{P}_n} \left( \frac{\|Pw\|_{L_p(t)}}{|P(z)|} \right)^{1/p}.$$

For $p = \infty$, generalized Christoffel functions with respect to $w$ for $z \in C$ are defined by

$$\lambda_{\infty,n}(w; z) = \inf_{P \in \mathcal{P}_n} \frac{\|Pw\|_{L_\infty(t)}}{|P(z)|}.$$

Moreover, for the classical Christoffel function $\lambda_n(w^2; x)$ with respect to $w^2$, we have

$$\lambda_n(w^2; x) = \inf_{P \in \mathcal{P}_{n+1}} \left( \frac{\int |Pw(t)|^2 \, dt}{\int |P(t)|^2 \, dt} \right) = \lambda_{2,n-1}(w; x).$$
A function \( f: (c, d) \to (0, \infty) \) is said to be *quasi-increasing* (or *quasidecreasing*) if there exists \( C > 0 \) such that \( f(x) \leq (or \geq) C f(y), c < x \leq y < d \).

**Definition 3.** (see [1], pp. 10–12). Let \( a < 0 < b \). Assume that \( W = e^{-Q} \) where \( Q: I \to [0, \infty) \) satisfies the following properties:

(a) \( Q' \in C(I) \) and \( Q(0) = 0 \).

(b) \( Q' \) is nondecreasing in \( I \).

(c) \( \lim_{x \to a^{-}} Q(x) = \lim_{x \to b^{-}} Q(x) = \infty \).

(d) The function
\[
T(x) = \frac{xQ'(x)}{Q(x)}, \quad x \neq 0,
\]
is quasidecreasing in \((a, 0)\) and quasi-increasing in \((0, b)\), respectively. Moreover, \( T(x) \geq 1, x \in I \setminus \{0\} \).

(e) There exists \( \epsilon_0 \in (0, 1) \) such that for \( y \in I \setminus \{0\} \),
\[
T(y) \sim T\left(y \left[1 - \frac{\epsilon_0}{T(y)}\right]\right).
\]

Then, we write \( W \in \mathcal{F} \).

\[
\Delta_t = \Delta_t(Q) = [a_t, a_t],
\]
\[
\delta_t = \delta_t(Q) = \frac{1}{2} (a_t + |a_t|),
\]
\[
\eta_{st} = \eta_{st}(Q) = \left[\sqrt{\frac{2}{\delta_t}} T(a_{st}) \right]^{-2/3},
\]
\[
\varphi_t(x) = \varphi_t(Q; x) = \begin{cases} 
\frac{|x - a_{-t}||x - a_{+t}|}{\sqrt{(|x - a_{-t}| + |a_{-t}| \eta)} (|x - a_t| + a_t \eta)} & x \in [a_t, a_t] \\
\varphi_t(a_t) & x \in (a_t, b) \\
\varphi_t(a_{-t}) & x \in (a_{-t}, a_{-t})
\end{cases}
\]
\[
I_{L_s} = I_{L_s}(Q) = [a_{1/2} + L \eta, a_t(1 + L \eta)], \quad L > 0,
\]
\[
K_{L_s} = K_{L_s}(Q) = [-1 + L(1 + a_{-t}), 1 - L(1 - a_t)], \quad L > 1.
\]

In 1994 and 2001, Levin and Lubinsky [1, 2] discussed orthogonal polynomials for exponential weights \( W^2 \) on \([-1, 1]\) and \((a, b)\), \( a < 0 < b \), respectively. Then, they [3, 4] dealt with exponential weights \( x^{2n}W(x)^2 \), \( a > -1/2 \), in \([0, b]\). Kasuga and Sakai [5] considered generalized Freud weights \( x^{2n}W(x)^2 \) in \((\infty, \infty)\). Recently, we discussed generalized Jacobi-exponential weights \( UW \) and subsequently dealt with a particular case \((1 - x^2) e^{-Q(x)} \) on \((-1, 1)\) in [9].

(f) Furthermore, assume that there exist \( C, \epsilon_1 > 0 \) such that for all \( x \in I_1 \setminus \{0\} \),
\[
\int_{x - \epsilon_1}^{x} \frac{|Q(s) - Q(x)|}{|s - x|^{1/2}} ds \leq C |Q'(x)| \frac{T(x)}{|x|^{1/2}}.
\]
polynomials for generalized Jacobi-exponential weights in the case \(-1 = t_r < t_{r-1} < \cdots < t_1 < t_0 = 1\).

Mastroianni and Totik in [10] gave the estimates of the spacing of zeros for doubling weights; in general, however, Jacobi-exponential weights \(UW\) are not doubling weights, so our main result (Theorem 4) cannot follow from it. The distribution of the zeros of orthogonal polynomials plays an important role in weighted approximation, for example, Mastroianni and Notarangelo [11, 12] applied the zeros for exponential weight on the function \(\frac{\phi_1(x)}{\Phi_n(x)U_n(x)^pW(x)^p}\), needed:

\[
\Delta_{\ast} = \Delta_{\ast}(Q^\ast),
\]

\[
\phi_{\ast} = \phi_{\ast}(Q^\ast; x),
\]

\[
\rho^\ast = \rho(U^\ast) = p_1 + p_r - 2p_0 + \sum_{i=2}^{r-1} \max[p_i, 0],
\]

\[
\mathcal{L}_{\lambda} = \mathcal{L}_{\lambda}(Q^\ast),
\]

\[
\mathcal{U}_{\ast} = \mathcal{U}_{\ast}(x) = \prod_{i=1}^{r} \left[ |x - t_i| + \frac{1}{1 + t_i} \right]^{p_i},
\]

\[
\mathcal{U}_{\ast}^\ast = \left[ \left( 1 - x^2 \right)^{1/2} + \frac{1}{n} - 2p_0 \right] \mathcal{U}_{\ast}(x).
\]

In all that follows, \(\mathcal{I}\) denotes the open interval \((-1, 1)\).

**Theorem 1** (see [7], Theorem 1.7). Let \(W \in \mathcal{F} (\text{Lip}(1/2))\) and \(0 < p < \infty\). Assume that

\[
x \overset{\text{lim}}{\longrightarrow} 0, \quad x < \frac{\Lambda - 1}{2\Lambda |p_0|}, \quad p_0 \neq 0,
\]

and for some constant \(\mu\) satisfying

\[
2\lambda |p_0| < \mu < 1 - \frac{1}{\Lambda},
\]

the function \(Q\) is nondecreasing in \(\mathcal{I}\).

(a) Then there exist \(n_0 > 0\) such that for \(n \geq n_0\) and \(x \in \mathcal{I}_{\lambda, \ast}\) with \(L > 0\), the relation

\[
\lambda_p (xU^\ast; x) \sim \phi_{n_0}^\ast(x) \mathcal{U}_{\ast}(x)^p W(x)^p.
\]

uniformly holds.

(b) Furthermore, there exists \(n_1 > 0\) such that for \(n \geq n_1\) and \(x \in \mathcal{I}\), the relation

\[
\lambda_p (UW; x) \geq C \phi_{n_1}^\ast(x) \mathcal{U}_{\ast}(x)^p W^\ast(x)^p,
\]

uniformly holds.

By specializing to \(p = 2\) of Theorem 1, we obtain estimates for the classical Christoffel functions.

**Corollary 1.** Assume that the conditions of Theorem 1 hold.

(a) Then, there exist \(n_0 > 0\) such that for \(n \geq n_0\) and \(x \in \mathcal{I}_{\lambda, \ast}\) with \(L > 0\), the relation

\[
\lambda_n (UW; x) \sim \phi_{n_0}^\ast(x) \mathcal{U}_{\ast}(x)^2 W(x)^2,
\]

uniformly holds.

(b) Furthermore, if \(p_0 \leq 0\), there exist \(C, n_1 > 0\) such that for \(n \geq n_1\) and \(x \in \mathcal{I}\), the relation

\[
\lambda_n (UW; x) \geq C \phi_{n_1}^\ast(x) \mathcal{U}_{\ast}(x)^2 W^2(x)^2,
\]

uniformly holds.

Our results will mainly center on the zeros of orthogonal polynomials for Jacobi-exponential weights \(UW\) and Markov–Bernstein inequalities.

**Theorem 2.** Let \(W = e^{-Q^\ast(x)}\), where \(Q: \mathcal{I} \rightarrow [0, \infty)\) is convex with \(Q(a+) = Q(b-) = 0\) and \(Q(x) > Q(0) = 0, x \in \mathcal{I}\). Let \(0 < p < \infty\), \(P \in \mathcal{P}_{r, p, 2}\), \(p_0 \geq 0, i = 2, \ldots, r - 1\). Assume that relation (16) is valid and \(\mathcal{Q}\) is nondecreasing in \(\mathcal{I}\). Then,

\[
\|PUW\|_{L_p(\mathcal{I}, \mathcal{U}_{\ast}(x))} < \|PUW\|_{L_p(\mathcal{I}, \mathcal{U}_{\ast}(x))} \leq \|PUW\|_{L_p(\mathcal{I}, \mathcal{U}_{\ast}(x))}.
\]

In particular, this holds for not identically vanishing polynomials \(P\) of degree \(\leq t - p^\ast - (2/p)\). For \(p = \infty\), (22) holds with \(< \) replaced by \(\leq\).

**Theorem 3.** Let \(0 < p \leq \infty\) and \(p_i \geq 0, i = 2, \ldots, r - 1\). Assume that relation (16) is valid and \(\mathcal{Q}\) is nondecreasing in \(\mathcal{I}\).

(a) Let \(W \in \mathcal{F} (\text{Lip}(1/2))\). Then, for \(t \geq 1\) and \(P \in \mathcal{P}_{t, p, 2}\),

\[
\|PUW\|_{L_p(\mathcal{I}, \mathcal{U}_{\ast}(x))} \leq C \|PUW\|_{L_p(\mathcal{I}, \mathcal{U}_{\ast}(x))}.
\]

(b) Let \(0 < \alpha < 1\) and \(W \in \mathcal{F} (\text{Dini})\). Then, for \(t \geq 1\) and \(P \in \mathcal{P}_{t, p, 2}\),

\[
\|PUW\|_{L_p(\mathcal{I}, \mathcal{U}_{\ast}(x))} \leq C \|PUW\|_{L_p(\mathcal{I}, \mathcal{U}_{\ast}(x))}.
\]
Theorem 4. Let $W \in \mathcal{F}(\text{Lip}(1/2))$ and $p_i > (-1/2), 1 \leq i \leq r$. Assume that relation (16) is valid, $Q$ is nondecreasing in $I$, and
\[ \phi_i^* (x) = O(1), \quad t \rightarrow \infty. \] (25)
(a) Then, for large enough $n$ and $1 \leq k \leq n - 1$,
\[ x_{kn} - x_{k+1,n} \leq c \phi_n^* (x_{kn}). \] (26)
(b) Furthermore, if $r = 2$, then for large enough $n$ and $1 \leq k \leq n - 1$,
\[ x_{kn} - x_{k+1,n} \sim \phi_n^* (x_{kn}). \] (27)

Remark 1. By [7], (Lemma 2.12), for zeros $x_{kn}, x_{k+1,n} \in K_{L,n}$ with $L > 1$, the statement (a) of Theorem 4 is valid and $\leq$ can be replaced by $\sim$.

Theorem 5. Assume that the assumptions of Theorem 2 hold. Then,
\[ a_{n,p^{*} - 1/2}^* < x_{mn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < a_{n+1,p^{*} + 1/2}. \] (28)

Theorem 6. Let $W \in \mathcal{F}(\text{Lip}(1/2))$. Assume that relation (16) is valid and $Q$ is nondecreasing in $I$.
(a) Then,
\[ x_{kn} \geq a_n^* (1 - c \eta_n^*), \]
\[ x_{mn} \leq a_n (1 - c \eta_{n-1}^*). \] (29)

(b) Furthermore, if $p_i \geq 0, i = 2, \ldots, r - 1$, then for large enough $n$,
\[ 1 - \frac{x_{kn}}{a_n} \sim \eta_n^*, \]
\[ 1 - \frac{x_{mn}}{a_n} \sim \eta_{n-1}^*. \] (30)

We prove Theorems 2–4 and Theorem 6 in Section 3, but first we need some auxiliary lemmas and the proofs of Corollary 1 and Theorem 5, which are presented in Section 2.

2. Auxiliary Lemmas

Lemma 1 (see [1], Theorem 4.1, p. 95). Let $W = e^{-Q(x)}$, where $Q: I \rightarrow (0, \infty)$ is convex with $Q(a) = Q(b) = \infty$ and $Q(x) > Q(0) = 0, x \in \Gamma(0)$. Let $0 < p \leq \infty$ and $P \in \mathcal{F}_{-1/2} (\text{Lip}(1/2))$. Then,
\[ \|PW\|_{L_p(\Delta_0)} < \|PW\|_{L_r(\Delta_0)}. \] (31)

In particular, this holds for not identically vanishing polynomials $P$ of degree $\leq t - 2/p$. For $p = \infty$, (31) holds with $<$ replaced by $\leq$.

Lemma 2 (see [1], Theorem 10.1, p. 293). Let $0 < p \leq \infty$.
(a) Let $W \in \mathcal{F}(\text{Lip}(1/2))$. Then, for $t \geq 1$ and $P \in \mathcal{F}_t$,
\[ \|PW\|_{L_p(\Delta_0)} \leq C \|PW\|_{L_r(\Delta_0)} \] (32)
(b) Let $W \in \mathcal{F}(\text{Dini})$ and $0 < \alpha < 1$. Then, for $t \geq 1$ and $P \in \mathcal{F}_t$,
\[ \|PW\|_{L_p(\Delta_0)} \leq C \|PW\|_{L_r(\Delta_0)} \] (33)

Lemma 3 (see [7], Lemma 2.13). Let $I = (-1, 1)$ and $W \in \mathcal{F}(\text{Lip}(1/2))$. Assume that (16) is valid and $Q$ is nondecreasing in $I$. Then, $W^* \in \mathcal{F}(\text{Lip}(1/2))$.

Lemma 4. For fixed index $k, 1 \leq k \leq n - 1$, let $I_k = [x_{k+1,n}, x_{kn}]$. Let $j, 1 \leq j \leq r$, satisfy
\[ \min_{x \in I_k} |x - t_j| = \min_{x \in I_k} |x - t_{j+1}|. \] (34)

Then,
\[ \prod_{j \in I} |x_{kn} - t_j|^p \sim \prod_{j \in I} \left( |x_{kn} - t_j| + \frac{1}{n} \right)^p, \]
\[ \sim \prod_{j \in I} |x - t_j|^p, \quad x \in I_k, k = k, k + 1. \] (35)

Proof. Following the argument in the proof of Lemma 2.5 in [6], we get (35) by replacing $\delta/n$ with $1/n$. \hfill \Box

Lemma 5. Let $W \in \mathcal{F}$ and (25) be valid. Then, there exists $t_0 > 0$ such that for $t > t_0$ and for each index $j$, $2 \leq j \leq r - 1$,
\[ |x - t_j| \sim t - j \sim t - j + \frac{\delta}{t} \sim |x - t_{j+1}| + \phi_i (x), \] (36)
holds uniformly for $x \in I$.

Proof. By (1.55) in [1], we see that there exists $t_0 > 0$ such that for $t > t_0$, $|a_k(x)| \sim 1$, so we have $\delta_n \sim 1$ for $t > t_0$. Also, notice that $-1 < t_{j-1} < \cdots < t_{j-1} < 1$, and (36) follows from Lemma 2.7 in [6]. \hfill \Box

Lemma 6. Let $W \in \mathcal{F}(\text{Lip}(1/2))$. Assume that relation (16) is valid and $Q$ is nondecreasing in $I$. Then, there exists $L > 0$ such that for $t$ large enough,
\[ a_{t+1/2}^* \leq a_i^* (1 + L \eta_i^*). \] (37)

Proof. By Lemma 3.11(a) in [1], for $t > 0$,
\[ \frac{a_i}{a_i^*} - 1 \sim -1 \leq T(a_i^*) \frac{\rho^* (1/2)}{t}. \] (38)

Fix $t_0 = (\rho^* + 1/2)^3$; for $t \geq t_0$, we have
\[ \frac{1}{T(A_i^*)} \frac{\rho^* + 1/2}{t} \leq t^{-2/3} T^{-1} (a_i^*). \] (39)

On the other hand, using Definition 2 of $\eta_i^*$, we obtain
\[ \eta^*_n \geq C T^{-1} (a_t^*) t^{-2/3}, \]  
(40)
as \( a^*_t \sim \delta^*_t \) and \( T(a^*_t) > 1 \).

Thus, by (38), for large enough \( t \),

\[ \frac{a_t^{*n+1/2}}{a_t^*} - 1 \leq L \eta^*_n. \]  
(41)

This yields (37).

Since the last lemma is based on the results of Corollary 1 and Theorem 5, we present the proofs of Corollary 1 and Theorem 5 first.

Proof of Corollary 1. It is the special case of Theorem 1 when \( p = 2 \) we use (5) and the relation \( \varphi^*_n \sim \varphi^*_{n-1} \) in I from Lemma 9.7 [1]. We also see that \( \mathcal{U}_n \sim \mathcal{U}_{n-1} \) for \( n \) large enough.

Proof of Theorem 5. By Lemma 3, \( W^* \) satisfies the conditions of \( W \). For \( U^* = \{x + \{1|p - 3| \} x - \{1|p - 3| \} x \} \), \( i \) we have \( p - p_i \geq 0, p - p_i \geq 0 \). Meanwhile, \( p_i \geq 0, i = 2, \ldots, r - 1 \), so (28) follows directly from Theorem 1.9 in [6].

Lemma 7. Let \( W \in \mathcal{F} (\text{Lip} (1/2)) \) and \( r = 2 \). Assume that relation (17) is valid and \( \mathcal{Q} \) is nondecreasing in \( I \). Let \( \varphi^*_{jn} \in \mathcal{P}_{n+1} \), be the fundamental polynomials of Lagrange Interpolation at the zeros \( p^*_{jn} \) (\( (UW)^2, x \)) satisfying \( \varphi^*_{jn} (x_{kp}) = \delta_{kp} \). Then, for each index \( j \), \( 1 \leq j \leq n \) and large enough \( n \),

\[ |\ell_{jn} WU|^*(x) (WU)^{-1}(x_{jn}) \]

\[ + |\ell_{j+1,n} WU|^*(x) (WU)^{-1}(x_{j+1,n}) \leq C, \quad x \in I_j. \]

Proof. Notice that

\[ \ell_{jn} (x) = \frac{\mathcal{K}_n(x, x_{jn})}{\mathcal{K}_n(x, x_{jn})}, \]  
(43)

where \( \mathcal{K}_n(x, t) = \sum_{k=0}^n p_k(x) p_k(t) \) is the \( n \)th reproducing kernel function. Applying the Cauchy–Schwarz inequality to \( \mathcal{K}_n(x, t) \), we obtain

\[ |\ell_{jn} WU|^*(x) (WU)^{-1}(x_{jn}) \leq \left( \frac{\mathcal{K}_n(x, x_{jn}) (WU)^2(x)}{\mathcal{K}_n(x, x_{jn}) (WU)^2(x_{jn})} \right)^{1/2} \]

\[ = \left( \frac{\lambda_n^{-1} (WU)^2, x (WU)^2(x)}{\lambda_n^{-1} (WU)^2, x_{jn} (WU)^2(x_{jn})} \right)^{1/2}. \]  
(44)

By Lemma 6 and (28), we see \( I_j \subset \mathcal{I}_{1,ln}, 1 \leq j \leq n \). Now applying the Christoffel function bounds of Corollary 1 (a) and (b), it follows from the above relation that

\[ |\ell_{jn} WU|^*(x) (WU)^{-1}(x_{jn}) \]

\[ \leq C \left( \frac{\varphi^*_n (x_{jn})}{\varphi^*_n (x)} \right)^{1/2} \frac{\mathcal{U}_n (x)(x) - 2U^2(x)}{(U_n(x)(x))^2 (WU)^2(x_{jn})}, \quad x \in I_j. \]  
(45)

According to the definition of \( W^* \),

\[ W^*(x) = (1 - x^2)^{p_0} W(x), \]

and then

\[ U_n(x) W^*(x) = (1 - x^2)^{p_1} \left( 1 - x^2 \right)^{1/2} \left( 1 - x^2 \right)^{1/2} + \frac{1}{n} \right) - 2p_0 \mathcal{U}_n (x) W(x), \]  
(46)

which by (2.23) in [7] for \( x \in \mathcal{I}_{1,ln} \) gives

\[ U_n(x) W^*(x) \sim \mathcal{U}_n(x) W(x). \]  
(47)

It follows from (48) that for large enough \( n \),

\[ |\ell_{jn} WU|^*(x) (WU)^{-1}(x_{jn}) \leq C \left( \frac{\varphi^*_n (x_{jn})}{\varphi^*_n (x)} \right)^{1/2} \left( \mathcal{U}_n (x)(x) - 2U^2(x) \right) \]

\[ \leq C \left( \frac{\varphi^*_n (x_{jn})}{\varphi^*_n (x)} \right)^{1/2}, \quad x \in I_j, \]  
(49)

as when \( r = 2 \),

\[ \left( \mathcal{U}_n (x)(x) - 2U^2(x) \right) \sim 1. \]  
(50)

Further, applying Theorem 5.7(b) in [1], we conclude for \( x \in I_j \),

\[ \varphi^*_n (x_{jn}) \sim \varphi^*_n (x), \]  
(51)

so that

\[ |\ell_{jn} WU|^*(x) (WU)^{-1}(x_{jn}) \leq C, \quad x \in I_j, \]  
(52)

and with a similar discussion, we also have

\[ |\ell_{j+1,n} WU|^*(x) (WU)^{-1}(x_{j+1,n}) \leq C, \quad x \in I_j, \]  
(53)

This proves (42).

3. Proof of Theorems

3.1. Proof of Theorem 2. It is easy to check that \( Q^*: I \rightarrow [0, \infty) \) is convex with \( Q^* (a^+ + b^-) = Q^* (b^- - a^+) = \infty \) and \( Q^* (x) > Q^* (0) = 0 \), \( x \in \mathcal{I}[0] \), so by considering Lemma 3, \( W^* \) satisfies the assumptions about \( W \). Furthermore, for \( P \in \mathcal{P}_{t - p^* - (2/p)} \),
\[ PU^* \in \mathcal{Q}_{r-2/p}. \] (54)

Observe that
\[ U(x)W(x) = U^*(x)W^*(x). \] (55)

Then, applying Lemma 1, we obtain the results.

### 3.2. Proof of Theorem 3

(a) By Lemma 3, \( W^* \in \mathcal{Q}(\text{Lip}(1/2)) \). For \( P \in \mathcal{P}_{r-2/p} \), we have \( PU^* \in \mathcal{Q}_r \). Thus, by (55), relation (23) follows from (32).

(b) If \( W^* \in \mathcal{Q}(\text{Dini}) \), then with the similar discussion as (a) and using (33), we prove that the statement of (b) is valid. So, it is necessary to prove that if \( W \in \mathcal{Q}(\text{Dini}) \), then \( W^* \in \mathcal{Q}(\text{Dini}) \).

The properties of (a) – (c) in Definition 3 hold for \( W^* \) if \( W \in \mathcal{Q}(\text{Dini}) \) because of the same argument as in the proof of Lemma 2.13 in [7] since properties of (a)–(c) in Definition 3 are the same for both \( \mathcal{Q}(\text{Lip}(1/2)) \) and \( \mathcal{Q}(\text{Dini}) \). We will prove that the property of (f) in Definition 3 also holds for \( W^* \).

By (2.38) in [7], we have
\[
S := \int_{x_*}^{x+\epsilon x/[(1-\rho)T(x)]} \frac{Q''(s) - Q''(x)}{s-x} ds \\
\leq \int_{x_*}^{x+\epsilon x/[(1-\rho)T(x)]} \frac{Q''(s) - Q''(x)}{s-x} ds \\
= \left[ \right]
\]

According to Definition 3 (f),
\[ S_1 \leq C|Q'(x)|. \] (57)

### 3.3. Proof of Theorem 4

(a) The proof is similar to Theorem 1.7 in [6], but we provide the details with modification. Denote by \( \{\tilde{\xi}_{kn}\}_{k=1}^{n} \) the fundamental polynomials of Lagrange interpolation at the zeros \( \{x_{kn}\}_{k=1}^{n} \) of the orthogonal polynomials \( p_n(WU^2, x) \) for the weight \( WU^2 \).

Recall (5); the infimum is actually attained when we take \( P \) to be \( \tilde{\xi}_{kn} \in \mathcal{P}_{n-1} \) satisfying \( \tilde{\xi}_{kn}(x_m) = \delta_{kj} \). So, a classical Gauss quadrature formula for the weight \( WU^2 \) is
\[ \lambda_n(WU^2; x_{kn}) = \int_{1}^{2\sigma_{kn}(WU)^2}. \] (65)

By Lemma 11.8 in [1], (pp. 320–321) and relation (55), we infer that
\[
\lambda_n(WU^2; x_{kn}) = \int_{1}^{2\sigma_{kn}(WU)^2, x_{kn}} - x_{kn})^{-2} + \lambda_n(WU^2; x_{k+1,n})W^*(x_{k+1,n})^{-2} \\
= \int_{1}^{2\sigma_{kn}(WU)^2, x_{kn}} - x_{kn})^{-2} + \lambda_n(WU^2; x_{k+1,n})W^*(x_{k+1,n})^{-2} \\
\cdot W^*(t)^2U^*(t)^2 dt \\
\geq \int_{x_{k+1,n}}^{x_{kn}} \left[ \lambda_n(WU^2; x_{kn})^{-2} + \lambda_n(WU^2; x_{k+1,n})W^*(x_{k+1,n})^{-2} \right] \\
\cdot W^*(t)^2U^*(t)^2 dt \\
\geq \frac{1}{2} \int_{x_{k+1,n}}^{x_{kn}} U^*(t)^2 dt. \] (66)
On the other hand, according to Lemma 6 and Theorem 5, $I_k \subset I^{*}_{j/n}$, so that by (20),

$$\lambda_n((WU)^2;x_kn) W^*(x_kn)^2 + \lambda_n((WU)^2;x_{k+1,n}) W^*(x_{k+1,n})^{-2} \leq c\left[\varphi_n^*(x_kn) U_n^*(x_kn)^2 + \varphi_n^*(x_{k+1,n}) U_n^*(x_{k+1,n})^2\right].$$

(67)

Let $j = j(k)$ be $j$ defined by (34); then by (51) and Lemma 4, we get

$$c\varphi_n^*(x_kn)\left[U_n^*(x_kn)^2 + U_n^*(x_{k+1,n})^2\right] \geq \int_{x_{k+1,n}}^{x_{k,n}} U^*(t)^2 \, dt \sim \int_{x_{k+1,n}}^{x_{k,n}} |t - t_j|^2 p_j^2 \, dt \prod_{i=1, i \neq j}^r |x_{k,n} - t_i|^2 p_i^2,$$

(68)

where $p_j^* = p_j^{* (k)}$, $p_*^j = p_j - p_0$, and $p_r^* = p_r - p_0$. Also, we have

$$U_n^*(x_kn)^2 + U_n^*(x_{k+1,n})^2 \sim \left[\left|x_{k,n} - t_j\right|^2 + \frac{1}{n}\right] + \left[\left|x_{k+1,n} - t_j\right|^2 + \frac{1}{n}\right]^{2p_j^2} \cdot \prod_{i=1, i \neq j}^r \left|x_{k,n} - t_i\right|^{2p_i^2},$$

(69)

and by (35), we further get

$$U_n^*(x_kn)^2 + U_n^*(x_{k+1,n})^2 \sim \left[\left|x_{k,n} - t_j\right|^2 + \frac{1}{n}\right] + \left[\left|x_{k+1,n} - t_j\right|^2 + \frac{1}{n}\right]^{2p_j^2} \cdot \prod_{i=1, i \neq j}^r \left|x_{k,n} - t_i\right|^{2p_i^2}. $$

(70)

By (68) and (70), we get the following relation after simplifying by $\prod_{i=1, i \neq j}^r |x_{k,n} - t_j|^{2p_j^2}$:

$$\int_{x_{k+1,n}}^{x_{k,n}} |t - t_j|^{2p_j^2} \, dt \leq c\varphi_n^*(x_kn) \left[\left|x_{k,n} - t_j\right| + \frac{1}{n}\right] + \left[\left|x_{k+1,n} - t_j\right| + \frac{1}{n}\right]^{2p_j^2} \right].$$

(71)

In fact, for $x, y \in I_k$, using (2.8) in [6] and following the argument in the proof of Lemma 5 in [6], we can obtain $(1 - x)^{p_j^2 - p_k^2} \sim (1 - x + (1/n)^{p_j^2 - p_k^2} \sim (1 - y)^{p_j^2 - p_k^2} \sim (1 + y + (1/n)^{p_j^2 - p_k^2}$ and $(1 + x)^{p_j^2 - p_k^2} \sim (1 + x + (1/n)^{p_j^2 - p_k^2} \sim (1 + y)^{p_j^2 - p_k^2}$, so (71) can be written as

$$\int_{x_{k+1,n}}^{x_{k,n}} |t - t_j|^{2p_j^2} \, dt \leq c\varphi_n^*(x_kn) \left[\left|x_{k,n} - t_j\right| + \frac{1}{n}\right] + \left[\left|x_{k+1,n} - t_j\right| + \frac{1}{n}\right]^{2p_j^2} \right].$$

(72)

where $2 \leq j \leq r - 1$.

Further, by (36),

$$\int_{x_{k+1,n}}^{x_{k,n}} |t - t_j|^{2p_j^2} \, dt \leq c\varphi_n^*(x_kn) \left[\left|x_{k,n} - t_j\right| + \varphi_n^*(x_kn)\right]^{2p_j} + \left[\left|x_{k+1,n} - t_j\right| + \varphi_n^*(x_kn)\right]^{2p_j} \right].$$

(73)

By calculation from (73), we get

$$\frac{1}{2p_j + 1} \left|x_{k,n} - t_j\right|^{2p_j + 1} + \sigma \left|x_{k+1,n} - t_j\right|^{2p_j + 1}$$

$$= \int_{x_{k+1,n}}^{x_{k,n}} |t - t_j|^{2p_j} \, dt \leq c\varphi_n^*(x_kn) \left[\left|x_{k,n} - t_j\right| + \varphi_n^*(x_kn)\right]^{2p_j} + \left[\left|x_{k+1,n} - t_j\right| + \varphi_n^*(x_kn)\right]^{2p_j} \right],$$

where

$$\sigma = \begin{cases} 1, & t_j \in I_k, \\ -1, & t_j \notin I_k. \end{cases}$$

(75)

We distinguish two cases.

**Case 1.** $p_j \geq 0$. By Lemma 2.6 in [6], we assert that if $p \geq 0, B_0 \geq A_0 \geq 0, C_0 \geq 0, \sigma = \pm 1$, and $B_0 + \sigma A_0 \leq C_0 \leq C_0 [(B_0 + C_0)^p + (A_0 + C_0)^p]$, then $B_0 + \sigma A_0 \leq C_0$.

Using this inequality, it follows from (74) that

$$x_{k,n} - x_{k+1,n} \leq c\varphi_n^*(x_kn).$$

(76)

**Case 2.** $-1/2 < p_j < 0$. By (74),

$$\frac{1}{2p_j + 1} \left|x_{k,n} - t_j\right|^{2p_j + 1} + \sigma \left|x_{k+1,n} - t_j\right|^{2p_j + 1}$$

$$= \int_{x_{k+1,n}}^{x_{k,n}} |t - t_j|^{2p_j} \, dt \leq c\varphi_n^*(x_kn) \min\left\{\varphi_n^*(x_kn)^{2p_j}, \left|x_{k+1,n} - t_j\right|^{2p_j}, \left|x_{k,n} - t_j\right|^{2p_j}\right\}.$$ 

(77)

**Case 2.1.** $t_j \in I_k$. Inequality (77) gives

$$x_{k,n} - t_j \leq c\varphi_n^*(x_kn)^{2p_j + 1}, \ k = k, k + 1,$$

which yields (76).

**Case 2.2.** $t_j \notin I_k$. In this case, we distinguish two subcases. Suppose without loss of generality that $x_{k+1,n} > t_j$.

**Case 2.2.1.** If $|x_{k+1,n} - t_j| \geq 2c_0\varphi_n^*(x_kn)$, where $c_0$ is given by (77), then
which by (77) gives
\[
(x_{k+1,n} - t_j)^{2p} \leq \left(1 - |p_j|\right)^{-1} (x_{kn} - t_j)^{2p} \leq 2(x_{kn} - t_j)^{2p}, 
\]
(80)

On the other hand, by (77)–(80),
\[
c_0 \Phi_n^* (x_{kn}) (x_{k+1,n} - t_j)^{2p_j} 
\geq \int_{x_{k+1,n}}^{x_{kn}} (t - t_j)^{2p_j} dt \geq (x_{kn} - t_j)^{2p_j} (x_{kn} - x_{k+1,n}) 
\geq \frac{1}{2} (x_{kn} - t_j)^{2p_j} (x_{kn} - x_{k+1,n}), 
\]
and hence (76) follows.

Case 2.2.2. \(|x_{k+1,n} - t_j| < 2c_0 \Phi_n^* (x_{kn})\). By (77),
\[
c_0 \Phi_n^* (x_{kn})^{2p_j+1} 
\geq \frac{1}{2p_j + 1} \left( (x_{kn} - t_j)^{2p_j+1} - (x_{k+1,n} - t_j)^{2p_j+1} \right) 
\geq \frac{1}{2p_j + 1} \left( (x_{kn} - t_j)^{2p_j+1} - 2c_0 \Phi_n^* (x_{kn})^{2p_j+1} \right), 
\]
(82)

So, \(x_{kn} - t_j \leq c_0 \Phi_n^* (x_{kn})\) and (76) follows.

(b) Now, let us prove (27). We must prove that for some constant \(c > 0\) and \(n\) large enough, we have
\[
x_{kn} - x_{k+1,n} \geq c \Phi_n^* (x_{kn}), \quad k = 1, 2, \ldots, n - 1. 
\]
(83)

First, by our Markov–Bernstein inequality (23) and Lemma 7, we have that
\[
\| (\ell_{kn} W U)' \|_{L_{\infty}(0)} \| (W U)^{-1} (x_{kn}) \|_{L_{\infty}(I)} \leq C \| \ell_{kn} W U \|_{L_{\infty}(0)} \| (W U)^{-1} (x_{kn}) \|_{L_{\infty}(I)}, 
\]
(84)

Then, by the mean value theorem, for some \(\xi\) between \(x_{kn}\) and \(x_{k+1,n}\),
\[
1 = (\ell_{kn} W U)' (x_{kn}) (W U)^{-1} (x_{kn}) - (\ell_{kn} W U)' (x_{k+1,n}) (W U)^{-1} (x_{kn}) 
\geq (\ell_{kn} W U)' (\xi) (W U)^{-1} (x_{kn}) (x_{kn} - x_{k+1,n}) 
\leq C (\Phi_n^*)^{-1} (\xi) (x_{kn} - x_{k+1,n}). 
\]
(85)

Thus, by (51), we get the lower bound and finish the proof of (b).

3.4. Proof of Theorem 6. By Lemma 3, \(W^* \in \mathcal{F} (\text{Lip}(1/2))\).
Then, following the argument in the proof of Theorem 5, the statements of Theorem 6 follow directly from Theorem 1.10 in [6].

Data Availability
No data were used to support this study.

Conflicts of Interest
The author declares that there are no conflicts of interest.

Acknowledgments
This research was supported in part by the National Natural Science Foundation of China (no. 11626060) and Scientific Research Fund of Fujian Provincial Education Department (no. JAT160172).

References
[1] A. L. Levin and D. S. Lubinsky, Orthogonal Polynomials for Exponential Weights, Springer, New York, NY, USA, 2001.
[2] A. L. Levin and D. S. Lubinsky, “Christoffel functions and orthogonal polynomials for exponential weights on \([-1, 1]\),” Memoirs of the American Mathematical Society, vol. 11, no. 535, 1994.
[3] A. L. Levin and D. S. Lubinsky, “Orthogonal polynomials for exponential weights \(x^{p_j}e^{-Q(x)}\) on \([0, d]\),” Approximation Theory, vol. 134, pp. 199–256, 2005.
[4] A. L. Levin and D. S. Lubinsky, “Orthogonal polynomials for exponential weights \(x^{p_j}e^{-Q(x)}\) on \([0, d]\),” Approximation Theory, vol. 139, pp. 107–143, 2006.
[5] T. Kasuga and R. Sakai, “Orthor normal polynomials with generalized Freud-type weights,” Journal of Approximation Theory, vol. 121, no. 1, pp. 13–53, 2003.
[6] R. Liu and Y. G. Shi, “The zeros of orthogonal polynomials for Jacobi-exponential weights, abstract and applied analysis,” Article ID 386359, 17 pages, 2012.
[7] R. Liu and Y. G. Shi, “Generalized christoffel functions for Jacobi-exponential weights on \([-1, 1]\),” Acta Mathematica Hungarica, vol. 148, no. 1, pp. 17–42, 2016.
[8] Y. G. Shi, “Generalized Christoffel functions for Jacobi-exponential weights,” Acta Mathematica Hungarica, vol. 140, no. 1-2, pp. 71–89, 2013.
[9] Y. G. Shi, "Orthogonal polynomials for Jacobi-exponential weights \((1-x)^\rho e^{-Q(x)}\) on \((-1,1)\)," \((-1,1)\) Acta Mathematica Hungarica, vol. 140, no. 4, pp. 363–376, 2013.

[10] G. Mastroianni and V. Totik, "Uniform spacing of zeros of orthogonal polynomials," Constructive Approximation, vol. 32, no. 2, pp. 181–192, 2010.

[11] G. Mastroianni and I. Notarangelo, "Lagrange interpolation with exponential weights on \((-1,1)\)," Approximation Theory, vol. 167, pp. 65–93, 2013.

[12] G. Mastroianni and I. Notarangelo, "Lagrange interpolation at Pollaczek-Laguerre zeros on the real semiaxis," Journal of Approximation Theory, vol. 245, pp. 83–100, 2019.