On the Expansion of Graphs of 0/1-Polytopes

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Abstract. The edge expansion of a graph is the minimum quotient of the number of edges in a cut and the size of the smaller one among the two node sets separated by the cut. Bounding the edge expansion from below is important for bounding the “mixing time” of a random walk on the graph from above. It has been conjectured by Mihail and Vazirani (see [3]) that the graph of every 0/1-polytope has edge expansion at least one. A proof of this (or even a weaker) conjecture would imply solutions of several long-standing open problems in the theory of randomized approximate counting. We present different techniques for bounding the edge expansion of a 0/1-polytope from below. By means of these tools we show that several classes of 0/1-polytopes indeed have graphs with edge expansion at least one. These classes include all 0/1-polytopes of dimension at most five, all simple 0/1-polytopes, all hypersimplices, all stable set polytopes, and all (perfect) matching polytopes.

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1 Introduction

In the early days of polyhedral combinatorics there was some hope that investigations of the graphs of 0/1-polytopes that are associated with certain sets of combinatorial objects might yield insights that could be exploited in designing algorithms for related combinatorial optimization problems. Certainly this hope was inspired by the success of Dantzig’s simplex algorithm for linear programming. Quite soon, people came across astonishing facts like the one that the diameter of the asymmetric traveling salesman polytope equals one for at most five cities and two for more than five cities (Padberg and Rao [22], apparently already discovered, but not...
published, in the early fifties by Kuhn [15]). This was even outperformed by the cut polytope of the complete graph on \( n \) nodes that has diameter one for all \( n \geq 2 \) (Barahona and Mahjoub [6]). Other polytopes turned out to have more complicated graphs, e.g., the stable set polytopes, for which two vertices are adjacent if and only if the symmetric difference of the corresponding stable sets induces a connected graph [8]. Another interesting example is the basis polytope of a matroid (i.e., the convex hull of the characteristic vectors of its bases), where two vertices are adjacent if and only if the corresponding bases have a symmetric difference of cardinality two (observed by Edmonds in the early 1970’s). All in all lots of interesting results on the graphs of special 0/1-polytopes have been obtained—however, usually without much impact on algorithms for related optimization problems.

Maybe the best-known result on graphs of general 0/1-polytopes is due to Naddef. He proved [20] that the graph of any \( d \)-dimensional 0/1-polytope has diameter at most \( d \), and thus, 0/1-polytopes satisfy the Hirsch conjecture (claiming that the graph of any \( d \)-dimensional polytope with \( n \) facets has diameter at most \( n - d \)). Some results on cycles of the graphs of general 0/1-polytopes have been proved as well by Naddef and Pulleyblank in the 1980’s [10, 22]. Nevertheless, the graphs of (general) 0/1-polytopes did not receive too much attention. Probably this was due to the fact that people did not see how to exploit potential knowledge on this topic with respect to algorithms for combinatorial optimization problems, where the interest in 0/1-polytopes originally came from. As for a source of general results on 0/1-polytopes we refer to [25].

The question on graphs of 0/1-polytopes treated in this paper is mainly motivated by the goal to design algorithms that generate random elements in classes of combinatorial objects, which often translates to the task of generating random vertices of 0/1-polytopes. Of course, in general this includes combinatorial optimization problems via appropriate choices of random distributions, but here, we will be more concerned with the task of drawing a vertex according to the uniform distribution. Maybe the most important motivation of generating (uniformly distributed) random elements from a set of combinatorial objects is the fact that in many cases this allows to count the number of objects approximately by a randomized algorithm. The first spectacular success of this method was Jerrum and Sinclair’s randomized approximation algorithm for computing the permanent in a certain large class of 0/1-matrices [11] (extended to arbitrary matrices with nonnegative integer entries by Jerrum, Sinclair, and Vigoda [13]).

For an introduction into the topic of randomized approximate counting and random generation see [24] or [25]. Here we briefly sketch the ideas on the example of the spanning trees of a given graph, although the exact number of spanning trees can be computed efficiently by Kirchhoff’s matrix tree theorem (see, e.g., [2, Chap. 24]).

Let \( T(G) \) be the set of spanning trees of a graph \( G \). The basic idea for counting spanning trees via generating them randomly is the following. Suppose, \( G' \) is the graph \( G \) plus an additional edge \( e' \), and assume, that we do already know a number \( \tau' \) approximating \( |T(G')| \). If we generate a large set \( T' \) of spanning trees in \( G' \) uniformly at random, and if \( \alpha \) is the fraction of those trees in \( T' \) that do not contain \( e' \), then we might hope that \( |T(G)| \) approximately equals \( \alpha \cdot \tau' \). Since
the number of spanning trees of the complete graph on $n$ nodes is well-known to be $n^{n-2}$, this suggests an iterative method to approximately compute $|T(G)|$ by a randomized algorithm.

We do not go into the details of this algorithm and its analysis, but rather turn to the question how to generate a spanning tree in a graph $G$ uniformly at random, where our exposition here is just meant to give an idea of the method as far as it is useful for understanding the motivation of the questions on 0/1-polytopes we will consider in this paper. The strategy is to perform a (finite) random walk on the set $T(G)$, meaning that one starts with an arbitrary spanning tree $T_0 \in T(G)$, slightly modifies $T_0$ randomly to a spanning tree $T_1$, slightly modifies $T_1$ randomly to $T_2$, and so on. After a certain number of steps one stops and takes the current tree as the desired random object. We do not go into the details of this algorithm and its analysis, but rather turn to the question how to generate a spanning tree in a graph $G$ uniformly at random, where our exposition here is just meant to give an idea of the method as far as it is useful for understanding the motivation of the questions on 0/1-polytopes we will consider in this paper. The strategy is to perform a (finite) random walk on the set $T(G)$, meaning that one starts with an arbitrary spanning tree $T_0 \in T(G)$, slightly modifies $T_0$ randomly to a spanning tree $T_1$, slightly modifies $T_1$ randomly to $T_2$, and so on. After a certain number of steps one stops and takes the current tree as the desired random object. The passage from $T_i$ to $T_{i+1}$ could be performed in the following way. For technical reasons, we first flip an unbiased coin in order to decide if we “do nothing” and stay at $T_{i+1} := T_i$, or if we try to get to a modified tree as described subsequently. We first choose a pair $(e, f)$ of edges of $G$ uniformly at random. If it happens that $e \notin T_i$ and $f$ lies on the cycle in $T_i \cup \{e\}$ then we proceed to $T_{i+1} := T_i \setminus \{f\} \cup \{e\}$. Otherwise, we stay at $T_{i+1} := T_i$.

Thus, we perform a random walk in the graph $G(T(G))$ that has the spanning trees of $G$ as its nodes, where two trees are connected if and only if their symmetric difference consists of two edges. All transition probabilities (i.e., for each ordered pair $T$ and $T'$ of adjacent nodes in $G(T(G))$ the probability that we proceed to $T'$ if we currently are at $T$) equal $\frac{1}{2m}$, where $m$ is the number of edges of $G$. By standard arguments (see Section 2) one can prove that the random walk will be at each spanning tree with the same probability at step $i$ if $i$ tends to infinity, no matter at which spanning tree we started. However, for algorithmic purposes it is of course important that this convergence does not happen too slow. Responsible for the speed of convergence is the edge expansion of $G(T(G))$ (see Figure 1), where the edge expansion of a graph $H = (V, E)$ is the number

$$X(H) := \min \left\{ \frac{\delta(S)}{|S| - |V \setminus S|} : S \subset V, S \neq \emptyset, V \right\}$$

(with $\delta(S)$ denoting the set of all edges with one end node in $S$ and the other one in $V \setminus S$). If $X(G(T(G)))$ is bounded by the reciprocal of a polynomial in the size of $G$, then the random walk described above converges “sufficiently fast.” Actually, it is well-known that in our case even $X(G(T(G))) \geq 1$ holds (see the remarks at the end of Section 2).

Viewing this example of generating spanning trees randomly as a prototype, one might formulate a strategy for random generation of certain combinatorial objects as follows. First, one has to choose a neighborhood structure on the objects and then, transition probabilities have to be assigned appropriately. Here, “appropriately” means (a) that the random walk should asymptotically behave according to the desired probability distribution and (b) it should do so approximately already after a small number of steps. Let us assume that the distribution we aim at is the
Figure 1. If the neighborhood structure on which a random walk is performed allows to partition the objects into two large parts with only a few connections between them, then the random walk cannot converge quickly. Fortunately, the converse of this statement is true as well.

uniform distribution. Then, provided that the neighborhood structure is (as in the example) symmetric and connected, we can achieve goal (a) always by choosing the same probability for all proper transitions. In this case, goal (b) is equivalent to choosing a neighborhood structure with a “not too small” edge expansion. Of course, in order to be able to efficiently simulate the random walk it should be also possible to draw for each object uniformly at random one of its neighboring objects. However, this will not be at our focus here.

Thus, we are faced with the task to come up with good candidates for neighborhood structures. Suppose that the set of objects we are interested in is a family of subsets of a finite set (like in the example of spanning trees). Then the graph of the associated polytope (the convex hull of the characteristic vectors of the subsets in the family) is a natural candidate, where the graph is defined by the 1-skeleton, i.e., the zero- and the one-dimensional faces. In fact, the neighborhood structure we considered in the example is given by the graph of the spanning tree polytope. Two vertices of that polytope are adjacent if and only if the symmetric difference of the corresponding spanning trees consists of two edges (since the spanning trees of some graph are the bases of a matroid, the graphic matroid defined by that graph).

As mentioned above, two vertices of a stable set polytope are adjacent if and only if the symmetric difference of the corresponding stable sets induces a connected subgraph $S$. Since matchings correspond to stable sets in the line graph, two vertices of a matching polytope thus are adjacent if and only if the symmetric difference of the corresponding matchings is connected. The same is true for perfect matching polytopes, since they are faces of matching polytopes, since they are faces of matching polytopes. Two of the most prominent random walks in combinatorics are the ones designed and analyzed by Jerrum and Sinclair [1] on the set of (near-)perfect matchings of a bipartite graph and on the set of all matchings of an arbitrary graph. While the first one lead to a randomized approximation algorithm for the permanent (for a certain class of 0/1-matrices), the second one yielded a randomized approximation algorithm for evaluating the partition function of a monomer-dimer system in statistical physics,
which is the same as the generating function of the matchings in an arbitrary graph. In both cases, the random walk was performed on a subgraph of the graph of the associated 0/1-polytope, and the crucial step was to prove that this subgraph has a large edge expansion. Another example is the random generation of 0/1-knapsack solutions (leading to a randomized approximation algorithm for counting as well) due to Morris and Sinclair [18]. The key step in their result again was to show that a certain subgraph of the graph of the 0/1-knapsack polytope has large edge expansion.

It seems to be clear from these examples that it is important to investigate the question for the edge expansion of general 0/1-polytopes (i.e., the convex hulls of arbitrary sets of points with coordinates from \{0, 1\}). Actually, it appears from a citation in a paper of Feder and Mihail [9] (which we will be concerned with in Section 4) that Mihail and Vazirani have considered this question some time ago. Feder and Mihail (and also Mihail [17]) quote them with the conjecture that the graph of every 0/1-polytope has edge expansion at least one. Of course, even a proof showing that the edge expansion of the graph of any \(d\)-dimensional 0/1-polytope is bounded by one over a polynomial in \(d\) would be very important (see also Section 5).

While this extensive introduction was intended to shed some light on the relevance of the question for expansion properties of graphs of 0/1-polytopes, the rest of the paper is meant to support the conjecture of Mihail and Vazirani by some partial results. In Section 3 we show that the conjecture indeed is true for every 0/1-polytope whose dimension does not exceed five. In Section 2 we list a few well-known facts on random walks. The main goal for this is to provide some background that is relevant for Section 3. As a side effect, the concepts treated in this introduction may become a bit more clearer. In Section 4 we present some methods for bounding the edge expansion that are especially suited for graphs of (certain) 0/1-polytopes. In particular, it will turn out that simple 0/1-polytopes, hyper-simplices, and stable set polytopes satisfy Mihail and Vazirani’s conjecture. We conclude with some remarks in Section 5.

The results presented in Sec. 3 have been obtained in joined work with Janina Werner [24].

## 2 Expansion and Eigenvalues

The aim of the present section is to explain the connection between the edge expansion of a graph and the second largest eigenvalue of a certain matrix, which will be relevant in Section 3. This connection originates in Alon’s and Milman’s work [4, 5] and was specifically adapted for our context by Aldous [3]. Our treatment closely follows Behrend’s book [7].

Let \( G = (V, E) \) be a graph (without loops or multiple edges) on \( n := |V| \) nodes. We define a random walk (i.e., transition probabilities for all edges—in both directions) on \( G \) in a canonical way. Let \( \Delta_{\text{max}} \) be the maximum degree of a vertex in \( G \). Each pair \( (v, w) \) of vertices such that \( \{v, w\} \in E \) is an edge of \( G \) receives a constant transition probability \( p_{vw} := \tau := \frac{1}{\Delta_{\text{max}}} \). If \( v \in V \) is a node of degree \( \Delta_v \), then we set \( p_{vv} := \frac{1}{2} + (\Delta_{\text{max}} - \Delta_v) \cdot \tau \). Let \( P \in \mathbb{R}^{V \times V} \) be the matrix with entries
$p_{vw} (v, w \in V)$. As defined here, $P$ is a symmetric doubly-stochastic matrix with a real spectrum $\lambda_1 = 1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n \geq 0$. Let $M \in \mathbb{R}^V$ be a matrix whose columns are eigenvectors of $P$ that form an orthonormal basis of $\mathbb{R}^V$ such that the $i$-th column is an eigenvector for the eigenvalue $\lambda_i$. In particular, the first column of $M$ is $(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$. Then we have

$$P = M \cdot \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \cdot M^T$$

(after suitably numbering the vertices of $G$).

If the row vector $\pi \in \mathbb{R}^V$ describes the probability distribution for the start vertex of the random walk, then the distribution after performing $i$ steps of our random walk is given by $\pi \cdot P^i$, i.e., by

$$\pi \cdot M \cdot \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \cdot M^T . \quad (1)$$

For $i \rightarrow \infty$ this converges to

$$\pi \cdot M \cdot \left( \begin{array}{cccc} 1 & 0 & \ldots & 0 \\ 0 & \frac{1}{n} & \ldots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{n} & \ldots & \frac{1}{n} \end{array} \right) \cdot M^T = \pi \cdot \left( \begin{array}{cccc} \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} \end{array} \right) = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right).$$

Thus, as it was intended, asymptotically the random walk will give convergence to the uniform distribution over $V$, independently of the start distribution (e.g., independent from the start vertex). Moreover, it follows from (1) that the speed of convergence is determined by the second largest eigenvalue $\lambda_2$. Intuitively it seems to be clear that the edge expansion of $G$ determines how fast the convergence happens. And, in fact, there is the following strong connection between the edge expansion and $\lambda_2$ (see [7, Theorem 11.3]).

**Theorem 1.** Let $G$ be a graph with maximum degree $\Delta_{\text{max}}$, and let $0 \leq \lambda_2 < 1$ be the second largest eigenvalue of the matrix $P$ defined as above. Then we have

$$(1 - \lambda_2) \cdot \Delta_{\text{max}} \leq \lambda'(G) \leq \sqrt{8 \cdot (1 - \lambda_2) \cdot \Delta_{\text{max}}} .$$

The original application of this theorem was, of course, to derive upper bounds on the size of $\lambda_2$ by the edge expansion, since the latter one seems to be easier to access in structural analyses than the first one. However, with respect to algorithmic issues the situation is somehow the other way around. While computing the edge expansion is NP-hard (see Theorem [3]), the second largest eigenvalue can be calculated efficiently. We will exploit this fact in the next section.
3 Small Dimensions

Aichholzer classified all 0/1-polytopes of dimension less than or equal to five up to isometries of the cube, i.e., up to flipping and permuting the coordinates [1]. Table 1 shows the number of classes for each dimension.

Table 1. The numbers of classes of 0/1-polytopes.

| Dimension | 1  | 2  | 3  | 4  | 5           |
|-----------|----|----|----|----|-------------|
| # Classes | 1  | 2  | 12 | 349| 1226525     |

Thus, in principle one can compute the edge expansion of the graph of each 0/1-polytope up to dimension five by computer. Unfortunately, the following result shows that in general, computing the edge expansion is difficult. This is well-known for some time (e.g., [14]). However, since we could not find an explicit proof in the literature, we include one here.

Theorem 2. The problem of computing $\chi(G)$ for arbitrary graphs $G$ is NP-hard.

Proof. We reduce the problem of finding a maximum (unweighted) cut in a graph (which was proved to be NP-hard by Karp [14]) to the problem of computing the edge expansion of some related graph. The proof is an extension of the proof of the NP-hardness of the equicut problem given by Garey, Johnson, and Stockmeyer [10].

Let $G = (V,E)$ be a graph with $n := |V|$ nodes. We construct a graph $G' = (V',E')$, where $V' = V \uplus W$ for some set $W$, disjoint from $V$, with $|W| = n$, and with $E'$ containing all possible edges except the ones in $E$. Thus, $G'$ has $n' = 2n$ nodes. We denote by $\delta_G(S)$ and $\delta_{G'}(S')$ the set of all edges of $G$ respectively $G'$ having precisely one end node in $S$ respectively $S'$ and define

$$
\eta_{G'}(S') := \frac{|\delta_{G'}(S')|}{\min(|S'|, |V' \setminus S'|)}.
$$

We first show that it suffices to consider node subsets of cardinality $\frac{n'}{2} = n$ in order to compute the edge expansion of $G'$. Let $S \subseteq V$ and $T \subseteq W$ be two sets of nodes of $G'$ with $k := |S| + |T| \leq n$. We have $|\delta_{G'}(S \cup T)| = k \cdot (2n - k) - |\delta_G(S)|$ and

$$
\eta_{G'}(S \cup T) = 2n - k - \frac{|\delta_G(S)|}{k}.
$$

In particular, if $k = n$ then

$$
\eta_{G'}(S \cup T) = n - \frac{|\delta_G(S)|}{n}.
$$

holds.
We claim that the right hand side of (3) is less than or equal to the right hand side of (2) for each \(1 \leq k \leq n\). Indeed, this claim is equivalent to

\[
n - k + \left( \frac{1}{n} - \frac{1}{k} \right) \cdot |\delta_G(S)| \geq 0,
\]

which follows from

\[
|\delta_G(S)| \leq |S| \cdot n \leq k \cdot n.
\]

Thus, we have (where the second equation follows from (3))

\[
\mathcal{X}(G') = \min \{ \eta_{G'}(S \cup T) : S \subseteq V, W \subseteq W, |S| + |T| = n \} = \min \left\{ n - \delta_G(S) : S \subseteq V, W \subseteq W, |S| + |T| = n \right\} = n - \max \{ |\delta_G(S)| : S \subseteq V \}.
\]

In view of Theorem 2 we decided first to calculate the lower bounds on the edge expansion provided by Theorem 1 for each 0/1-polytope of dimension four and five. And, somewhat surprising, it turned out that for none of the polytopes this bound was less than one. Thus, the conjecture of Mihail and Vazirani is true for 0/1-polytopes up to dimension five.

**Theorem 3.** The graph of each 0/1-polytope of dimension less than or equal to five has edge expansion at least one.

Figure 3 shows that in many cases the lower bound given by the second largest eigenvalue even was significantly larger than one.

![Figure 2](image_url)

**Figure 2.** The (lower) eigenvalue bounds on the edge expansion for all 1226525 five-dimensional 0/1-polytopes.
4 Flow Methods

In this section, we describe methods for proving that a graph has good edge expansion properties that are specifically suited for graphs of 0/1-polytopes. Applying these methods we will show that the conjecture of Mihail and Vazirani is true for well-known classes of 0/1-polytopes (see Corollaries 8 and 14). On the other hand it will be quite obvious that the methods are not sufficient to prove the conjecture in its whole generality.

4.1 Expansion and flows

In order to bound the edge expansion of a graph \( G = (V, E) \) from below we will construct certain flows in the (uncapacitated) network \( \mathcal{N}(G) = (V, A) \), where \( A \) contains for each edge \( \{u, v\} \in E \) both arcs \( (u, v) \) and \( (v, u) \). This strategy dates back to the method of “canonical paths” developed by Sinclair (see [23]). The extension to flows was explicitly exploited by Morris and Sinclair [18]. Feder and Mihail [9] use random canonical paths, which can equivalently be formulated in terms of flows.

The crucial idea is to construct for each ordered pair \( (s, t) \in V \times V \) a flow \( \phi_{(s,t)} : A \rightarrow \mathbb{Q}^{\geq 0} \) in the network \( \mathcal{N}(G) \) sending one unit of some commodity from \( s \) to \( t \). Let \( \phi := \sum_{(s,t) \in V \times V} \phi_{(s,t)} \) be the sum of all these flows. By

\[
\phi_{\text{max}} := \max \{ \phi(a) : a \in A \}
\]

we denote the maximal amount of \( \phi \)-flow on any arc. By construction of \( \phi \), the total amount \( \phi(S : V \setminus S) \) of \( \phi \)-flow leaving \( S \) is at least \( |S| \cdot (n - |S|) \), where \( n = |V| \). On the other hand, we have \( \phi(S : V \setminus S) \leq \phi_{\text{max}} \cdot |\delta(S)| \). This implies \( |S| \cdot (n - |S|) \leq \phi_{\text{max}} \cdot |\delta(S)| \), and hence, if \( |S| \leq \frac{n}{2} \) holds,

\[
\frac{|\delta(S)|}{|S|} \geq \frac{n}{2 \cdot \phi_{\text{max}}}. 
\]

Thus, we have proved

\[
\mathcal{X}(G) \geq \frac{n}{2 \cdot \phi_{\text{max}}}. \tag{4}
\]

In the light of inequality (4) it is clear that the task is to construct a flow \( \phi \) as above with \( \phi_{\text{max}} \) as small as possible in order to prove a strong lower bound on the edge expansion of \( G \).

4.2 Fractional wall-matchings

While the setting presented so far applies to general graphs, we now derive a method to construct \( \phi \) in the special situation where \( G \) is the graph of a 0/1-polytope. The method generalizes ideas for analyzing random walks on the bases-exchange graph of matroids due to Feder and Mihail [1].

Let \( P \subset \mathbb{R}^d \) be a 0/1-polytope. A wall of \( P \) is the intersection of \( P \) with any face of the cube \( C_d := \{ x \in \mathbb{R}^d : 0 \leq x_i \leq 1 \text{ for all } i \} \supseteq P \). Thus, the walls of \( P \)
are special faces of \( P \). Usually, we will identify a wall of \( P \) with its vertices. The faces \( F \) of \( C_d \) are in one-to-one correspondence with the vectors \( \sigma(F) \in \{0, 1, \ast\}^d \) (and vice versa) via

\[
F = \{ x \in C_d : x_i = \sigma(F)_i \text{ for all } i \text{ with } \sigma(F)_i \neq \ast \}.
\]

For a face \( F \neq C_d \) of \( C_d \) let \( \mu(F) := \min \{ i : \sigma(F)_i = \ast \} \) be the “smallest direction” of \( F \). Let \( W \) be a wall of \( P \) and let \( F \) be the inclusion minimal face of \( C_d \) with \( W = P \cap F \). The vector \( \sigma(W) := \sigma(F) \) indicates the components in which all vertices of \( W \) agree, and \( \mu(W) := \mu(F) \) is the smallest coordinate direction of any edge of \( W \). We define \( W_0 := \{ w \in W : w_{\mu(W)} = 0 \} \) and \( W_1 := \{ w \in W : w_{\mu(W)} = 1 \} \), and denote by \( B(W) \) the bipartite subgraph of \( G(P) \) induced by the two disjoint subsets \( W_0 \) and \( W_1 \) of nodes of \( G(P) \).

A wall \( W \) of \( P \) is called initial if there is some \( i \in \{0, 1, \ldots, d\} \) such that \( \sigma(W)_j \in \{0, 1\} \) for \( 1 \leq j \leq i \) and \( \sigma(W)_j = \ast \) for all \( j > i \). The following fact follows immediately from the definitions.

**Lemma 4.** For every edge \( e \) of a 0/1-polytope \( P \) there is a unique initial wall \( W \) of \( P \) such that \( e \) is an edge of \( B(W) \).

Thus the bipartite graphs associated with the initial walls of \( P \) induce a partition of the edges of \( P \).

A bipartite graph with bipartition \( L \sqcup R \) has a fractional matching if one can assign nonnegative weights to its edges such that all nodes in \( L \) have the same weighted degree, and the same does hold for all nodes in \( R \) as well (see Figure 3).

![Figure 3](image.png)

**Figure 3.** The bipartite graph on \( L \sqcup R \) has a fractional matching if and only if in the network indicated in the figure there is a (non negative) flow sending \( |L| \cdot |R| \) units of some commodity from \( l \) to \( r \). The arcs leaving \( l \) have capacities \( |R| \), the arcs entering \( r \) have capacities \( |L| \), and the arcs connecting \( L \) to \( R \) have infinite capacities.

**Observation 5.** If a bipartite graph \( B \) with bipartition \( L \sqcup R \) has a fractional matching and there is a constant amount of some commodity located in each node
in $L$ (or $R$, respectively), then one can distribute the entire amount of the commodity from $L$ to $R$ through the edges of $B$ such that each node in $R$ (or $L$, respectively) receives the same amount of the commodity.

A 0/1-polytope $P$ has fractional wall-matchings if $B(W)$ has a fractional matching for every wall $W$ of $P$. In general, the bipartite graph $B(W)$ associated to a wall $W$ of a 0/1-polytope $P$ does not necessarily have a fractional matching (see Figure 4). However, several interesting classes of 0/1-polytopes have fractional wall-matchings, as we will show below. The method to construct suitable flows $\phi$ we will describe does only work for such 0/1-polytopes. Thus, from now on we assume that $P \subset \mathbb{R}^d$ is a 0/1-polytope that has fractional wall-matchings.

Let $t \in \text{vert}(P)$ be a vertex of $P$. We will particularly be concerned with the initial walls

$$ W_i(t) := \{ w \in \text{vert}(P) : w_1 = t_1, \ldots, w_i = t_i \} \quad (i = 0, 1, \ldots, d) . $$

These walls form a flag of $P$, i.e., we have

$$ \{ t \} = W_d(t) \subseteq W_{d-1}(t) \subseteq \ldots \subseteq W_1(t) \subseteq W_0(t) = P . $$

For each $i \in \{1, \ldots, d\}$ we define $W_i(t) := W_{i-1}(t) \setminus W_i(t)$. Now we are ready to construct all flows $\phi_{(s,t)}$, $s \in V$, simultaneously in $d$ steps. Imagine a single unit of some commodity initially placed at each node. Suppose that before we perform step $i \in \{1, \ldots, d\}$ the $n$ units of the commodity are distributed uniformly among the nodes in $W_{i-1}(t)$ (as it is the case before the first step). Since we have assumed that $P$ has fractional wall-matchings we can route (see Observation 5) the amount of commodity distributed at the nodes in $W_{i-1}(t)$ through the arcs corresponding to the edges of $B(W_{i-1}(t))$ such that afterwards the $n$ units of our commodity are uniformly distributed among the nodes in $W_i(t)$. Figure 4 illustrates the construction.

For each pair $(s,t) \in V \times V$ we thus have defined a flow in the network $\mathcal{N}(G(P))$ sending one unit of some commodity from $s$ to $t$. It remains to bound the maximal flow $\phi_{\max}$ produced by $\phi := \sum_{(s,t) \in V \times V} \phi_{(s,t)}$ at any arc. Therefore,
let \( (x, y) \) be any arc of \( \mathcal{N}(G(P)) \). By Lemma 4, there is a unique initial wall \( W \) of \( P \) such that \( B(W) \) contains the edge \( \{x, y\} \) (see Figure 5). Due to symmetry reasons,

we might assume \( x \in W_0 \) and \( y \in W_1 \). Let \( A^-(x) \) and \( A^+(y) \) be the sets of out-arcs respectively in-arcs incident to \( x \) respectively \( y \) corresponding to edges of \( B(W) \). In particular, we have \( (x, y) \in A^-(x) \) and \( (x, y) \in A^+(y) \). The arcs going from \( W_0 \) to \( W_1 \) are only used by the flows \( \phi_{(s,t)} \) with \( s \notin W_1 \) and \( t \in W_1 \). Thus, the total amount of flow carried by these arcs is \( \frac{|W_0||W_1|}{|W|} \cdot n \). Consequently, precisely \( \frac{|W_1|}{|W|} \cdot n \) units of flow are sent through \( A^-(x) \) and \( \frac{|W_0|}{|W|} \cdot n \) units of flow are sent through \( A^+(y) \). Hence, \( (x, y) \) carries at most

\[
\min \left\{ \frac{|W_1|}{|W|}, \frac{|W_0|}{|W|} \right\} \cdot n \leq \frac{n}{2}
\]

units of flow. Since this holds for every arc of \( \mathcal{N}(G(P)) \), we have \( \phi_{\text{max}} \leq \frac{n}{2} \). By (4), this proves the following result.

**Theorem 6.** If \( P \) is a 0/1-polytope that has fractional wall-matchings, then \( \chi(G(P)) \geq 1 \) holds.

Thus, 0/1-polytopes that have fractional wall-matchings satisfy the conjecture of Mihail and Vazirani.
4.3 Walls with regular graphs

Let us say that a 0/1-polytope \( P \) has regular walls if the graph of every wall of \( P \) is regular, i.e., all its vertices have the same degree. It is obvious that every 0/1-polytope with regular walls has fractional wall-matchings (see Figure 7). This proves the following consequence of Theorem 6.

![Figure 7](image)

**Figure 7.** Regular walls yield fractional wall-matchings, since in each of the relevant bipartite graphs all vertices in the left shore have the same degree and the same is true for all vertices in the right shore.

**Corollary 7.** If a 0/1-polytope \( P \) has regular walls then \( X(\mathcal{G}(P)) \geq 1 \) holds.

A \( d \)-dimensional polytope \( P \) is simple if every vertex lies in precisely \( d \) facets, or, equivalently, if \( \mathcal{G}(P) \) is \( d \)-regular. The polytopes

\[
\text{conv} \left\{ v \in \{0, 1\}^d : \sum_{i=1}^d v_i = \varrho \right\} \quad (\varrho \in \{0, 1, \ldots, d\})
\]

are called hyper-simplices (they are special Knapsack polytopes).

**Corollary 8.** If a 0/1-polytope \( P \) is simple or a hyper-simplex, then \( X(\mathcal{G}(P)) \geq 1 \) holds.

**Proof.** Every face of a simple polytope is simple, and thus has a regular graph. Every wall of a hyper-simplex is a hyper-simplex, again. Since hyper-simplices obviously have a transitive automorphism group, they have regular graphs. Thus, in any of the two cases of the claim, \( X(\mathcal{G}(P)) \geq 1 \) holds by Corollary 7.

4.4 Balanced uniform 0/1-polytopes

A 0/1-polytope \( P \subset \mathbb{R}^d \) is called \( \varrho \)-uniform (\( \varrho \in \{0, 1, \ldots, d\} \)) if it is contained in the hyperplane \( \{ x \in \mathbb{R}^d : \sum_{i=1}^d x_i = \varrho \} \), i.e., if all vertices of \( P \) have precisely \( \varrho \)
ones. For instance, hyper-simplices and basis polytopes of matroids are uniform. Obviously, every wall of a uniform 0/1-polytope is uniform as well.

A 0/1-polytope \( P \subset \mathbb{R}^d \) is balanced if for every \( \sigma \in \{0, 1, \star\} \) and for each pair \( i, j \in \{1, \ldots, d\} \) with \( i \neq j \) and \( \sigma_i = \sigma_j = \star \) the relation
\[
|W_{0,0}| \cdot |W_{1,1}| \leq |W_{1,0}| \cdot |W_{0,1}|
\]
holds, where \( W \) is the wall of \( P \) defined by \( \sigma \) and \( W_{\alpha,\beta} := \{ w \in W : w_i = \alpha, w_j = \beta \} \).

If \( W_{0,1} \cup W_{1,1} \neq \emptyset \) (i.e., there is some \( w \in W \) with \( w_j = 1 \)), then (5) is equivalent to
\[
\frac{|W_{1,1}|}{|W_{0,1}| + |W_{1,1}|} \leq \frac{|W_{1,1}| + |W_{1,0}|}{|W_{0,1}| + |W_{1,1}| + |W_{0,0}| + |W_{1,0}|}.
\]
This means, that for a vertex \( w \) chosen uniformly at random from \( W \) the probability of the event \( w_i = 1 \) does not increase by conditioning on the event \( w_j = 1 \). Similarly, (5) is equivalent to the fact that for a vertex \( w \) chosen uniformly at random from \( W \) the probability of the event \( w_i = 0 \) does not increase by conditioning on the event \( w_j = 0 \).

The property of being balanced is not invariant under arbitrary symmetries of the cube. However, it is invariant under simultaneous “flipping” of all coordinates (and under arbitrary permutations of the coordinates).

**Proposition 9.** Balanced uniform 0/1-polytopes have fractional wall-matchings.

We omit the proof, which closely follows the corresponding proof on the bases-exchange graph of balanced matroids due to Feder and Mihail [9].

Proposition [9] and Theorem [6] imply the following.

**Theorem 10.** Every balanced uniform 0/1-polytope \( P \) satisfies \( X(G(P)) \geq 1 \).

A matroid \( \mathcal{M} \) on the ground set \( E \) has the **negative correlation property** if for a basis \( B \) chosen uniformly at random from the set of bases of \( \mathcal{M} \) and for every pair of elements \( e, f \in E \)
\[
\text{prob}[e \in B] \geq \text{prob}[e \in B | f \in B]
\]
holds, i.e., the probability of the event \( e \in B \) does not increase by conditioning on the event \( f \in B \). A matroid \( \mathcal{M} \) is balanced if every minor of \( \mathcal{M} \) has the negative correlation property. Regular (in particular: graphic) matroids are known to be balanced.

It is obvious that the basis polytope \( P(\mathcal{M}) := \text{conv} \{ \chi(B) : B \text{ basis of } \mathcal{M} \} \) (where \( \chi(B) \) is the characteristic vector of \( B \subseteq E \)) of a balanced matroid \( \mathcal{M} \) is uniform and balanced. Thus, Theorem \[10\] immediately yields \( X(G(P(\mathcal{M}))) \geq 1 \). Notice that the actual adjacency structure on \( P(\mathcal{M}) \) is irrelevant for this.

Hence, Theorem \[10\] generalizes the result of Feder and Mihail [9] saying that the bases-exchange graphs (where two bases are adjacent if and only if their symmetric difference has two elements) of balanced matroids have edge expansion at least one.
4.5 Cube-spanned walls

The technique described in this subsection is particularly suited for proving that 0/1-polytopes coming from certain combinatorial problems have graphs with large edge expansion (see Cor. 14). It relies on the high symmetry of the graph \( G(Q) \) of a cube \( Q \), from which one easily derives the following fact (where the antipodal vertex of some vertex \( x \) of \( Q \) is the vertex with maximum distance from \( x \) in \( G(Q) \))

**Observation 11.** For a cube \( Q \) it is possible to define for each pair \((s,t)\) of antipodal vertices a flow \( \psi_{(s,t)} \) in \( N(G(Q)) \) sending one unit of some commodity from \( s \) to \( t \) such that for the total flow \( \psi := \sum_{s,t} \psi_{(s,t)} \) one has \( \psi(a) = 1 \) for each arc \( a \) in \( N(G(Q)) \).

Let \( P \subset \mathbb{R}^d \) be any 0/1-polytope. A subset \( C \subseteq \text{vert}(P) \) of vertices of \( P \) is called an affine cube in \( P \) if \( C \) is affinely isomorphic to \( \{0,1\}^k \) for some \( k \), or, equivalently, if there is a subset \( I \subseteq \{1, \ldots, d\} \) (with \( |I| = k \)) such that the orthogonal projection of \( \mathbb{R}^d \) onto \( \mathbb{R}^I \) induces a bijection between \( C \) and \( \{0,1\}^I \). It is not hard to see that \( C \subseteq \text{vert}(P) \) is an affine cube (in \( P \)) if and only if there are 0/1-vectors \( z^{(1)}, \ldots, z^{(k)} \in \{0,1\}^d \), pairwise orthogonal to each other, such that for every \( x \in C \)

\[
C = \left\{ x \oplus \epsilon_1 z^{(1)} \oplus \cdots \oplus \epsilon_k z^{(k)} : \epsilon \in \{0,1\}^k \right\}
\]

(where \( \oplus \) denotes addition modulo two). The vertex \( x \oplus z^{(1)} \oplus \cdots \oplus z^{(k)} \) is the antipodal vertex of \( x \) in \( C \). In particular, we will be interested in affine edge-cubes in \( P \), i.e., affine cubes in \( P \) on which \( G(P) \) induces the graph of a cube.

For a subset \( A \subseteq \text{vert}(P) \) let us call the intersection of all walls of \( P \) that contain \( A \) the wall spanned by \( A \), denoted by \( W(A) \). A wall \( W \) of \( P \) is edge-cube spanned if there is an affine edge-cube \( C \) with \( W(C) = W \) (which is equivalent to the fact that each pair of antipodal vertices in \( C \) spans \( W \)). A wall \( W \) of \( P \) is uniquely edge-cube spanned if it is spanned by an affine edge-cube \( C \) in \( P \) and if it is not spanned by any other affine edge-cube \( C' \neq C \) in \( P \). In this case, we call the vertices in \( C \) the cube vertices of \( W \). See Figure 8 for examples.

![Figure 8](image-url) Three 3-dimensional walls. The first one is spanned by a cube (but not edge-cube spanned), the second one is edge-cube spanned (but not uniquely edge-cube spanned), and the third one is uniquely edge-cube spanned.
For a vertex \( x \in W \) in a wall \( W \) of \( P \) the vertex \( x^{(W)} := x \oplus t^{(W)} \) is the *mirror image* of \( x \) with respect to \( W \), where \( t^{(W)} \) is the 0/1-vector having ones precisely in those components where \( \sigma(W) \) has stars. In general, \( x^{(W)} \) needs not to be contained in \( W \). If, however, a wall \( W \) is spanned by an affine cube \( C \), then \( x^{(W)} \) is the antipodal vertex of \( x \) in \( C \) for every \( x \in C \); in particular, \( x^{(W)} \in W \).

**Lemma 12.** Let \( P \) be a 0/1-polytope, and let \( u,v \in \text{vert}(P) \), \( u \neq v \), be two distinct vertices of \( P \). There are at most \( \frac{1}{2}|\text{vert}(P)| \) walls \( W \) of \( P \) such that \( u,v \), and their mirror images with respect to \( W \) are contained in \( W \).

**Proof.** Let \( \mathcal{W}(u,v) \) be the set of all walls \( W \) of \( P \) such that \( u,v \), and their mirror images \( u^{(W)}, v^{(W)} \) are contained in \( W \). We have

\[
u^{(W)} \oplus v^{(W)} \leq t^{(W)} \quad \text{and} \quad u^{(W)} \oplus u^{(W)} = v^{(W)} \oplus v^{(W)} = t^{(W)} \tag{7}
\]

(where \( \leq \) is meant to hold component-wise). Since \( u \neq v \) we have \( u^{(W)} \neq v^{(W)} \). Thus, we can define a map \( \omega \) assigning to each \( W \in \mathcal{W}(u,v) \) the two-element subset \( \omega(W) := \{u^{(W)}, v^{(W)}\} \) of \( \text{vert}(P) \).

Suppose, for \( W,W' \in \mathcal{W}(u,v) \) we have \( x \in \omega(W) \cap \omega(W') \). After possibly interchanging the roles of \( u \) and \( v \), by (7) we have \( v \oplus x \leq u \oplus x \) and thus \( t^{(W)} = u \oplus x = t^{(W')} \), yielding \( W = W' \). Thus the images of \( \omega \) have pairwise empty intersections, which implies the lemma. \( \square \)

**Theorem 13.** Let \( P \) be a 0/1-polytope such that each pair \( s,t \in \text{vert}(P) \), \( s \neq t \), of distinct vertices \( s \) and \( t \) is a pair of antipodal cube vertices in a uniquely edge-cube spanned wall of \( P \). Then \( \mathcal{N}(\mathcal{G}(P)) \geq 1 \) holds.

**Proof.** In each affine edge-cube spanning a uniquely edge-cube spanned wall of \( P \) we construct a flow as described in Observation 11. Let \( \phi \) be the sum of all these flows. Since each pair \( s,t \in \text{vert}(P) \), \( s \neq t \), is a pair of antipodal cube vertices in a uniquely edge-cube spanned wall of \( P \), the flow \( \phi \) has the properties required in Subsection 4.1. Lemma 12 ensures that each arc \((u,v)\) in the network \( \mathcal{N}(\mathcal{G}(P)) \) is a cube-arc in at most \( \frac{n}{2} \) uniquely edge-cube spanned walls, if \( n \) is the number of vertices of \( P \). Thus we have \( \phi_{\text{max}} \leq \frac{n}{2} \), and by (4) we obtain the claim of the theorem. \( \square \)

Theorem 13 in particular yields a unified proof for the following results which appeared in 17 (where only a proof for the statement concerning the perfect matching polytope is given).

**Corollary 14.** The graphs of the stable set polytope, the matching polytope, and the perfect matching polytope associated with an arbitrary graph have edge expansion at least one.

**Proof.** Let \( G = (V,E) \) be a graph and let \( P \) be its stable set polytope. For two vertices \( s \) and \( t \) of \( P \) let \( A_s,A_t \subseteq V \) be the corresponding stable sets in \( G \), and denote
by $A^{(1)},\ldots,A^{(k)} \subseteq V$ the node sets of the connected components of the subgraph of $G$ induced by the symmetric difference of $A_s$ and $A_t$. Define $A_s^{(i)} := A^{(i)} \cap A_s$ and $A_t^{(i)} := A^{(i)} \cap A_t$. For each $\epsilon \in \{s,t\}^k$ the set
\[ S_\epsilon := (A_s \cap A_t) \cup A_s^{(1)} \cup \cdots \cup A_\epsilon^{(k)} \]
is stable in $G$. By Chvátal’s result [8] two vertices of $P$ are adjacent if (and only if) the symmetric difference of the corresponding stable sets induces a connected subgraph of $G$. Thus, the set $C$ of vertices of $P$ corresponding to $\{S_\epsilon : \epsilon \in \{s,t\}^k\}$ is an affine edge-cube in $P$, spanning the wall $W$ which is defined by the equations $x_v = 0$, $v \in V \setminus (A_s \cup A_t)$ and $x_v = 1$, $v \in A_s \cap A_t$. Clearly, $s$ and $t$ are antipodal vertices of $C$. Since all pairs of mirror images in $W$ belong to $C$ (these pairs correspond to bipartitions of the subgraph of $G$ that is induced by $A^{(1)} \cup \cdots A^{(k)}$), $W$ is uniquely edge-cube spanned by $C$. Thus, $\mathcal{X}(P) \geq 1$ by Theorem 13.

Since the matching polytope of a graph $G$ is the stable set polytope of the line graph of $G$ (having the edges of $G$ as vertices, which are adjacent if and only if the corresponding edges of $G$ have a common end node), the claim on matching polytopes follows.

Perfect matching polytopes satisfy the requirements of Theorem 13 as well; they even have the property that each pair of vertices spans a wall which is an affine edge-cube.

5 Some Remarks

The results presented in this paper support the conjecture that graphs of 0/1-polytopes inherently have good expansion properties and therefore may in principle be good candidates for defining neighborhood structures in the context of random walks. In fact, we have proved for some classes of 0/1-polytopes, including simple 0/1-polytopes, stable set polytopes, and all 0/1-polytopes up to dimension five, that their graphs have edge expansion at least one.

A proof of the conjecture that the edge expansion of the graph of any $d$-dimensional 0/1-polytope is bounded by the reciprocal of a polynomial in $d$ would have important consequences, even if this was proved only for uniform 0/1-polytopes. For instance, such a result would imply that indeed the bases-exchange graphs of arbitrary matroids have sufficiently large edge expansion in order to construct a randomized approximate counting algorithm. In particular, this would solve the open questions for randomized approximation algorithms for counting connected spanning subgraphs of a graph, forests of a prescribed size in a graph, or maximal independent subsets in a given set of vectors over $GF(2)$ (see [12]).

Therefore, one might hope that, while the concept of the graph of a 0/1-polytope has not proven to be very useful in the context of combinatorial optimization, it might have a successful revival in the context of random generation and counting of certain combinatorial objects.
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