INVERSE LIMITS OF MACAULAY’S INVERSE SYSTEMS

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Abstract. Generalizing a result of Masuti and the second author, we describe inverse limits of Macaulay’s inverse systems for Cohen–Macaulay factor algebras of formal power series or polynomial rings over an infinite field. On the way we find a strictness result for filtrations defined by regular sequences. It generalizes both a lemma of Uli Walther and the Rees isomorphism.

Introduction

Let $K$ be a field, and let $P$ be either the (standard graded) polynomial ring $K[x_1, \ldots, x_n]$ or the formal power series ring $K[[x_1, \ldots, x_n]]$ (with trivial grading). The injective hull $E$ of $K$ over $P_0$ defines a duality $\sim^\vee = \sim^* \Hom_P(\sim, E)$ between Artinian and finitely generated (graded) $P$-modules. In particular, this yields an antiisomorphism $\sim^\perp$ of the lattices of (graded) ideals $I \triangleleft P$ and (graded) $P$-submodules $W$ of $D = P^\vee$ (see (2.1)). The ideals $I$ for which $P/I$ is Artinian correspond to finitely generated submodules $W = I^\perp$. In this case and for the polynomial ring the correspondence was proved by Macaulay at the beginning of the 20th century. The dual $I^\perp$ is known as the inverse system of $I$ (see [Mac94]).

Around 1960 Macaulay’s correspondence turned out to be a special case of Matlis duality (see [Mat58; Gab60]). Later it was rediscovered and further developed (see for instance [Ems78; Iar94; Ger96; GS98; Kle07; CI12]). Recent applications concern the $n$-factorial conjecture (see [Hai94]), Waring’s problem (see [Ger96]), the geometry of the punctual Hilbert scheme of Gorenstein schemes (see [IK99]), the analytic classification of Artinian Gorenstein rings (see [ER12]), the cactus rank (see [RS13]), and the Kaplansky–Serre problem (see [RŞ14]).

Elias and Rossi (see [ER17]) described the first generalization of Macaulay’s inverse system in the positive-dimensional case. Their result which applies to Gorenstein algebras was extended by Masuti and the second author (see [MT18]) to the case of level algebras. We give
a more conceptual description of their construction in terms of inverse limits. We show how to drop the level-hypothesis by identifying the various socles in the inverse system (see Corollary 1.9). This fact is deduced from a general strictness result for filtrations defined by regular sequences, which generalizes both a lemma of Walther (see [Wal17, Lem. 6.5]) and the Rees isomorphism. The full generality of this result is not needed for our application.

Our main result (see Theorem 2.8) gives an explicit description of inverse limits of Macaulay’s inverse systems obtained by dividing out powers of a linear regular sequence. It applies to (graded) Cohen–Macaulay factor algebras of formal power series (or polynomial) rings over an infinite field (see Lemma 2.2 for a more intrinsic description of these types of algebras).

1. Strict filtrations by regular sequences

We underline vectors and denote (component-wise) residue classes by an overline. We apply maps component-wise to vectors. All rings considered are commutative unitary. By an $R$-sequence we mean a regular sequence in $R$.

Let $R$ be a ring. Any ideal $I \triangleleft R$ defines an exhaustive decreasing filtration

$$R = I^0 \supset I \supset I^2 \supset I^3 \supset \cdots$$

on $R$ denoted by $I^*$. It is called separated if $\bigcap_{k \in \mathbb{N}} I^k = 0$. The $I$-order of $p \in R$ is

$$\text{ord}_I(p) = \max \{ k \in \mathbb{N} \mid p \in I^k \} \in \mathbb{N} \cup \{ \infty \}.$$ 

We abbreviate $R_I := R/I$. The associated $I$-graded ring

$$\text{gr}_I R = \bigoplus_{l \in \mathbb{N}} I^l/I^{l+1}$$

is a homogeneous graded $R_I$-algebra. There is an $I$-symbol map

$$\sigma_I: R \setminus \bigcap_{k \in \mathbb{N}} I^k \to \text{gr}_I R, \quad p \mapsto p \in \text{gr}_I^{\text{ord}_I(p)} R = I^{\text{ord}_I(p)}/I^{\text{ord}_I(p)+1}.$$ 

Any ideal $J \triangleleft R$ induces a filtration $\text{gr}_I J^*$ on $\text{gr}_I R$, where

$$\text{gr}_I^l J^k = (J^k \cap I^l + I^{l+1})/I^{l+1} \subset \text{gr}_I^l R,$$

$$\text{gr}_I J^k = \bigoplus_{l \in \mathbb{N}} \text{gr}_I^l J^k \subset \text{gr}_I R.$$ 

We refer to any filtration induced by $J$ as a $J$-filtration. If $J$ is generated by $f = f_1, \ldots, f_r \in R$, then we use $f$ as a shortcut for the $J$-filtration.

Remark 1.1. Let $\sigma_J(p), \sigma_J(q) \in \text{gr}_J R$. Then

$$\text{ord}_J(pq) \geq \text{ord}_J p + \text{ord}_J q,$$
with equality equivalent to \( pq \notin I^{\text{ord}_g p + \text{ord}_g q + 1} \). It follows that
\[
\sigma(g)(pq) = \begin{cases} 
\sigma(g)(pq) & \text{if } \text{ord}_g(pq) = \text{ord}_g p + \text{ord}_g q, \\
0 & \text{otherwise}. 
\end{cases}
\]

Let \( g = (g_1, \ldots, g_s) \in R^s \) and denote by \( Y = (Y_1, \ldots, Y_s) \) corresponding indeterminates of degree 1.

**Theorem 1.2 (Rees).** The Rees map of graded \( R_\underline{g} \)-algebras
\[
(1.2) \quad \varphi_\underline{g}: R_\underline{g}[Y] \longrightarrow \text{gr}_g R, \\
Y_i \longrightarrow \sigma(g_\underline{i}),
\]
is an isomorphism if \( g \) is an \( R \)-sequence (see [BH93, Thm. 1.1.8]). □

**Remark 1.3.** If \( P \in R[Y] \) such that \( \text{ord}_g(P(g)) = l \), then, using Remark 1.1,
\[
\sigma(g)(P(g)) = \overrightarrow{P}(\sigma(g)) = \varphi_g(\overrightarrow{P}),
\]
where \( P \rightarrow \overrightarrow{P} \) under \( R[Y] \rightarrow R_\underline{g}[Y] \).

**Remark 1.4.** If (1.2) is an isomorphism, then the components of \( \sigma_\underline{g}(g) \) are regular on \( \text{gr}_g R \). With Remark 1.1, it follows that
\[
\sigma_\underline{g}(pg^m) = \sigma_\underline{g}(p)\sigma_\underline{g}(g^m) = \sigma_\underline{g}(p)\sigma_\underline{g}(g^m),
\]
for all \( \sigma_\underline{g}(p) \in \text{gr}_g R \) and \( m \in \mathbb{N}^s \). By Theorem 1.2, this holds if \( g \) is an \( R \)-sequence.

Let \( f = (f_1, \ldots, f_r) \in R^r \), and set
\[
h = (h_1, \ldots, h_t) = f, \underline{g} \in R^r \times R^s = R^t.
\]
Denote by \( X = (X_1, \ldots, X_r) \) indeterminates of degree 1 corresponding to \( f \), and set
\[
Z = (Z_1, \ldots, Z_t) = X, Y.
\]
For \( i \in \{1, \ldots, u\} \), let
\[
0 \neq \underline{m}_i = (\underline{k}_i, \underline{l}_i) \in \mathbb{N}^r \times \mathbb{N}^s = \mathbb{N}^t
\]
be the rows of a matrix
\[
(1.3) \quad M = (KL).
\]
We denote by \( \underline{h}^M, \underline{f}^K, \underline{g}^L \in R^s \) the vectors with respective entries \( \underline{h}^M, \underline{f}^K, \underline{g}^L \in R \). Consider the \( R \)-linear map
\[
\underline{h}^M: R^u \longrightarrow R, \\
\underline{e}_i \longrightarrow \underline{h}^M \underline{e}_i,
\]
with image $\langle h^M \rangle$. Assigning degrees $\deg r_i = |l_i|$ to the generators defines a $g$-filtration $\bigoplus_{i=1}^n \langle g \rangle^{r_i} - i$ on $R$ and turns $h^M$ into a filtered map. It fits into a commutative diagram of filtered $R$-linear maps

$$
\begin{array}{ccc}
(R^n, \bigoplus_{i=1}^n \langle g \rangle^{r_i - i}) & \xrightarrow{h^M} & (R, \langle g \rangle^*) \\
\downarrow & & \downarrow \\
(\langle h^M \rangle, \sum_{i=1}^n \langle g \rangle^{r_i - i} h_i^M) & \xrightarrow{k^M} & (\langle h^M \rangle, \langle g \rangle^* \cap \langle h^M \rangle).
\end{array}
$$

The bottom map is the identity of $\langle h^M \rangle$, but its source and target carry respectively the image and preimage filtration from the source and target of the (horizontal) map $k^M$. If it identifies the two filtrations, then $h^M$ is said to be $g$-strict. The vertical maps are $g$-strict by construction.

The following proposition gives a generalized Rees isomorphism.

**Proposition 1.5.** Suppose that both $h = f, g$ and $\underline{g}, \underline{f}$ are $R$-sequences, and that the $\underline{f}$-filtration on $R_{\underline{g}}$ is separated and complete. Then $h^M$ is $g$-strict. In particular, the Rees map (1.2) induces an isomorphism of graded $R_{\underline{g}}$-algebras

$$
R_{\underline{g}}[Y]/\langle \underline{f}^k Y^L \rangle \xrightarrow{\varphi_{\underline{g}}} \text{gr}_{\underline{g}}(R)/\langle \sigma_{\underline{g}}(h^M) \rangle \xrightarrow{\cong} \text{gr}_{\underline{g}}(R/\langle h^M \rangle),
$$

where $\underline{f}^k Y^L$ denotes the vector with entries $\underline{f}^k Y^L$.

The proof of Proposition 1.5 relies on the following lemma proved by Uli Walther for $k = 1$ (see [Wal17, Lem. 6.5]). He assumes that $R$ is a domain, and that $g, f$ is an $R$-sequence in every order. However, his proof needs only that $\underline{f}, \underline{g}$ is an $R$-sequence.

**Lemma 1.6.** Suppose that $\underline{g}$ and $\underline{f}, \underline{g}$ are $R$-sequences. Let $P \in R[Y]_l$ such that $P(\underline{g}) \in \langle \underline{f} \rangle^k$. Then $P(\underline{g}) = Q(\underline{g})$ for some $Q \in \langle \underline{f} \rangle^k[Y]_l$. In particular, the Rees map (1.2) induces a filtered isomorphism

$$
\varphi_{\underline{g}} : (R_{\underline{g}}[Y], \langle \underline{f} \rangle^* [Y] \xrightarrow{\left(\text{gr}_{\underline{g}} R, \text{gr}_{\underline{g}} \langle \underline{f} \rangle^* \right)}.
$$

**Proof.** We proceed by induction on $k$. The claim is vacuous for $k = 0$. By Walther’s lemma (see [Wal17, Lem. 6.5]), we may assume that $P = \sum_i P_i f_i \in \langle \underline{f} \rangle[Y]_l$ with $P_i \in R[Y]_l$, and hence $P(\underline{g}) = \sum_i P_i(\underline{g}) f_i \in \langle \underline{f} \rangle^k$. The following proof of Proposition 1.5 relies only on the particular claim, which reduces to the Rees isomorphism in case $r = 0$. Applying Proposition 1.5 with $r = 0$, $\underline{f}$ playing the role of $\underline{g}$ and $\underline{m} = e$, it follows that $P_i(\underline{g}) \in \langle \underline{f} \rangle^{k-1}$. By induction hypothesis, $P_i(\underline{g}) = Q_i(\underline{g})$ for
some \( Q_i \in \langle f \rangle^{k-1} \mid Y \mid_l \). Then \( P(g) = Q(g) \) for \( Q = \sum_i Q_i f_i \in \langle f \rangle^k \mid Y \mid_l \). This proves the first claim.

Suppose now that \( g \) is an \( R \)-sequence. Then \( \varphi_g \) is an isomorphism by Theorem 1.2. Clearly \( \varphi_g(\langle f^\bullet \mid Y \mid \rangle) \subset \text{gr}_g \langle f^\bullet \rangle \) (see (1.1)). For the converse inclusion, take \( \sigma_g(x) \in \text{gr}_g^l \langle f \rangle^k \mid Y \mid_l \). Then \( x = P(g) \in \langle f \rangle^k \mid Y \mid_l \) for some \( P \in R[Y]_l \). By the first claim, we may assume that \( P \in \langle f \rangle^k \mid Y \mid_l \). Then \( R[Y] \rightarrow R_g[Y] \) maps \( P \mapsto P \) with \( y = \sigma_g(P) = \varphi_g(P) \) by Remark 1.3. This shows that \( \varphi_g(\langle f^k \mid Y \mid \rangle) = \text{gr}_g^l \langle f \rangle^k \) and the particular claim follows. □

A second ingredient of the proof of Proposition 1.5 is the following general relation of strict and graded exactness.

**Lemma 1.7.** Let \( A \) be a filtered ring, and let

\[
C : \quad N' \underset{\alpha'}{\longrightarrow} N \underset{\alpha}{\longrightarrow} N^u
\]

be a filtered complex of \( A \)-modules with associated graded complex \( \text{gr} C \).

(a) If \( C \) is strict exact, then \( \text{gr} C \) is exact (see [Sjö73, Lem. 1.(a)]).

(b) If \( \text{gr} C \) is exact and the filtration on \( N \) is exhaustive, then \( \alpha \) is strict (see [Sjö73, Lem. 1.(b)]).

(c) If the filtration on \( N' \) is complete and the filtration on \( N \) is exhaustive and separated, then \( C \) is strict exact if and only if \( \text{gr} C \) is exact (see [Sjö73, Lem. 1.(e)]).

Proof of Proposition 1.5. Set \( U = \{(i, j) \mid 1 \leq i < j \leq u \} \) and consider the \( R \)-linear map

\[
R^U \longrightarrow R^u,
\]

\[
\xi_{i,j} \longmapsto f^{m_i - \min\{m_i, m_j\}} \xi_{i,j} - f^{m_j - \min\{m_i, m_j\}} \xi_{j,i},
\]

where \( \min \) denotes the component-wise minimum. Assign to the generators bidegrees

\[
\deg(\xi_i) = (|L_i|, |K_i|), \quad \deg(\xi_{i,j}) = (|L_i| + |L_j|, |K_i| + |K_j|).
\]

With component-wise \( f \)- and \( g \)-filtrations,

\[
C : \quad R^U \longrightarrow R^u \xrightarrow{h^M} R
\]

becomes a bifiltered complex of free \( R \)-modules. By Lemma 1.7.(b), it suffices to show that the \( g \)-graded complex \( \text{gr}_g C \) is exact. This can be checked on graded pieces. By Lemma 1.7.(c), it suffices to show that the \( f \)-filtration \( \text{gr}_f \langle f^\bullet \rangle^* \) is separated and complete on each graded piece of \( \text{gr}_g C \), and that the associated graded complex \( \text{gr}_f \text{gr}_g C \) is exact.
By Lemma 1.6, the Rees isomorphism \( \varphi_\mathcal{g}: R_\mathcal{g}[Y] \to \text{gr} R \) identifies the induced \( f\)-filtration \( \text{gr} \langle f \rangle \) on \( \text{gr} R \) with the \( \mathcal{f} \)-filtration on the coefficient ring \( R_\mathcal{g} \). We denote by \( \text{gr}_f \varphi_\mathcal{g} \) the associated graded isomorphism. The graded pieces of \( \text{gr} R \), and hence of \( \text{gr} C \), are finite direct sums of \( R_\mathcal{g} \). By hypothesis, the \( f\)-filtration is separated and complete on each summand. It follows that the induced \( f\)-filtration is separated and complete on each graded piece of \( \text{gr} C \).

By Theorem 1.2 applied to the regular \( R_\mathcal{g}\)-sequence \( \mathcal{f} \) and Lemma 1.6, there is a bigraded isomorphism of \( R_h\)-algebras

\[
R_h[Z] \cong (R_h)[Z][Y] \longrightarrow \text{gr}_\mathcal{f}(R_h)[Y] \cong \text{gr}_f(\text{gr}_\mathcal{g} R). \tag{1.4}
\]

Let now \( m = (k, l) \in \mathbb{N}^* \times (\mathbb{N}^* \times \mathbb{N}^*) \). Note that \( \mathcal{f}^k \neq 0 \) by Remark 1.4 applied to the \( R_\mathcal{g}\)-sequence \( \mathcal{f} \). By \( R_\mathcal{g}\)-linearity of \( \varphi_\mathcal{g} \) and Remark 1.4,

\[
\varphi_\mathcal{g}(\mathcal{f}^k Y^l) = \mathcal{f}^k \varphi_\mathcal{g}(Y^l) = \sigma_\mathcal{g}(f^k) \sigma_\mathcal{g}(g^l) = \sigma_\mathcal{g}(f^k g^l) = \sigma_\mathcal{g}(h^m). \tag{1.5}
\]

It follows that isomorphism (1.4) maps

\[
Z^m = X^k Y^l \longrightarrow \sigma_\mathcal{f}(\mathcal{f}^k Y^l) \longrightarrow \sigma_\mathcal{f}(\varphi_\mathcal{g}(\mathcal{f}^k Y^l)) = \sigma_\mathcal{f}(\sigma_\mathcal{g}(h^m)).
\]

The isomorphism (1.4) thus turns \( \text{gr}_f \text{gr}_\mathcal{g} C \) into the exact complex (see [Eis95, Lem. 15.1])

\[
\begin{array}{ccccccc}
R_\mathcal{g}[Z]^U & \longrightarrow & R_\mathcal{g}[Z]^u & \longrightarrow & R_\mathcal{g}[Z], \\
\epsilon_{i,j} & \longrightarrow & Z_i^m - \min\{m, m\} e_j & \longrightarrow & Z_i^m - \min\{m, m\} e_j, \\
& & & \longrightarrow & Z_i^m,
\end{array}
\]

which proves the first claim.

With \( R/\langle h^M \rangle \) equipped with the image \( \mathcal{g}\)-filtration,

\[
R^u \longrightarrow R/\langle h^M \rangle \longrightarrow 0
\]

is an exact complex of \( \mathcal{g}\)-strict \( R\)-linear maps. Then the corresponding \( \mathcal{g}\)-graded complex is exact by Lemma 1.7.(a), and hence

\[
\text{gr}_\mathcal{g}(R/\langle h^M \rangle) \cong \text{gr}_\mathcal{g}(R)/\text{gr}_\mathcal{g}(h^M) \cong \text{gr}_\mathcal{g}(R)/\langle \sigma(h^M) \rangle.
\]

With (1.5), this proves the particular claim. \( \square \)

We now specialize to the case where \( g = h \) and \( R \) is Noetherian \( * \) local graded with \( * \) maximal ideal \( m_R \). Denote the (homogeneous) socle of \( R \) by

\[
soc R = \text{ann}_R m_R. \tag{1.6}
\]
We assume that \( h \) has homogeneous components making \( \langle h \rangle \triangleleft R \) a graded ideal. Then \( \langle h \rangle \subset m_R \) and the filtration \( \langle h \rangle^* \) is separated by Krull intersection theorem (see [Nor53, §3.1, Thm. 1]) and Nakayama lemma (see [BH93, Ex. 1.5.24]). With \( R \) also \( \gr_h R = R_h \) is Noetherian *local graded with *maximal ideal \( m_{R_h} = m_R / \langle h \rangle \). The \( R_h \)-algebra \( \gr_h R \) is now bigraded with unique bigraded maximal ideal

\[
\text{(1.7)} \quad m_{\gr_h R} = m_{R_h} + \langle h \rangle.
\]

For any bigraded algebra \( S \) with unique bigraded maximal ideal \( m_S \), we define the (bihomogeneous) socle as in (1.6).

The rows of the matrix \( M \) from (1.3) generate a monoid ideal

\[
\text{(1.8)} \quad \text{ann}_{\gr_h R} \left( \frac{R[Z]}{\langle Z^M \rangle} \right) \cong \bigoplus_{m \in \soc M} R Z^m = R^{\soc M}.
\]

By Dickson’s lemma (see [Dic13]), every monoid ideal is finitely generated.

**Definition 1.8.** Let \( \mathcal{M} \subset \mathbb{N}^t \) be a monoid ideal. By the socle of \( \mathcal{M} \) we mean the subset

\[
\soc M = \{ m \in \mathbb{N}^t \mid m + (\mathbb{N}^t \setminus \{0\}) \subset \mathcal{M} \} \subset \mathbb{N}^t.
\]

For \( d \in \mathbb{N} \) we write \( \soc_d \mathcal{M} = \{ m \in \soc M \mid |m| = d \} \).

By definition,

\[
\text{(1.9)} \quad \soc \left( \frac{R[Z]}{\langle Z^M \rangle} \right) = \bigoplus_{m \in \soc M} \text{soc}(R) Z^m \cong \text{soc}(R)^{\soc M}.
\]

**Corollary 1.9.** Suppose that \( R \) is a Noetherian *local graded ring, and that \( h \) a (component-wise) homogeneous \( R \)-sequence. Then the symbol map (extended by zero)

\[
R / \langle h \rangle^M \longrightarrow \gr_h R / \langle h \rangle^M,
\]

\[
\sigma_h(x) \longrightarrow \sigma_h(x),
\]

identifies socles.

**Proof.** Proposition 1.5 yields an isomorphism of free \( R_h \)-modules

\[
\text{(1.10)} \quad \text{ann}_{\gr_h(R)} / \langle \sigma_h(h)^M \rangle \left( \sigma_h(h) \right) \cong \bigoplus_{m \in \soc M} \text{ann}_{\gr_h(R)} / \langle h \rangle^M \left( \sigma_h(h) \right) \cong \bigoplus_{m \in \soc M} \sigma_h(\langle h \rangle^m).
\]
Since $\sigma_{\langle h \rangle} \in \mathfrak{m}_\langle \langle h \rangle \rangle$ by (1.7), it follows that

$$\text{soc}(\text{gr}_\langle h \rangle(\langle h \rangle^M)) \subset \bigoplus_{m \in \text{soc} M} R_{\langle h \rangle} \sigma_{\langle h \rangle}(\langle h \rangle^m).$$

Moreover, the $R_{\langle h \rangle}$-linear surjection

$$R/\langle h \rangle^M \supset \bigoplus_{m \in \text{soc} M} R_{\langle h \rangle} \bar{h}^m \twoheadrightarrow \bigoplus_{m \in \text{soc} M} R_{\langle h \rangle} \sigma_{\langle h \rangle}(\langle h \rangle^m) \simeq R_{\langle h \rangle} \text{soc}^M$$

onto the free $R_{\langle h \rangle}$-module must be an isomorphism, and hence

$$R/\langle h \rangle^M \supset \bigoplus_{m \in \text{soc} M} R_{\langle h \rangle} \bar{h}^m \simeq \bigoplus_{m \in \text{soc} M} R_{\langle h \rangle} \sigma_{\langle h \rangle}(\langle h \rangle^m) \subset \text{gr}_\langle h \rangle(\langle h \rangle^M)$$

is an isomorphism of free $R_{\langle h \rangle}$-modules. The action of the respective graded and bigraded maximal ideal on these modules reduces to that of $\mathfrak{m}_{R_{\langle h \rangle}}$. Therefore, it remains to show that

$$\text{soc}(\langle h \rangle^M) \subset \bigoplus_{m \in \text{soc} M} R_{\langle h \rangle} \bar{h}^m.$$

To this end, let $0 \neq \bar{x} \in \text{soc}(\langle h \rangle^M)$ of $\langle h \rangle$-order $d = \text{ord}_{\langle h \rangle}(\bar{x})$. In particular, $x\langle h \rangle \subset \langle h \rangle^M$ since $h \in \mathfrak{m}_R$. By Remark 1.4 and Proposition 1.5, taking symbols yields

$$\sigma_{\langle h \rangle}(x)\sigma_{\langle h \rangle}(h) = \sigma_{\langle h \rangle}(xh) \in \text{gr}_{\langle h \rangle} \langle h \rangle^M = \langle \sigma_{\langle h \rangle}(h)^M \rangle.$$

Then, by (1.10), $\sigma_{\langle h \rangle}(\bar{x}) \in \text{ann}_{\text{gr}_{\langle h \rangle}(\langle h \rangle^M)}(\sigma_{\langle h \rangle}(h))$ can be written as

$$\sigma_{\langle h \rangle}(\bar{x}) = \sum_{m \in \text{soc} M} x_m \sigma_{\langle h \rangle}(\bar{h}^m) \in \text{gr}_{\langle h \rangle}(R/\langle h \rangle^M),$$

where $x_m \in R_{\langle h \rangle}$. With $x' = x - \sum_{m \in \text{soc} M} x_m h^m$, this means that

$$\bar{x'} = \bar{x} - \sum_{m \in \text{soc} M} x_m \bar{h}^m \in \langle \bar{h} \rangle^{d+1} \triangleleft R/\langle h \rangle^M,$$

and hence $\text{ord}_{\langle h \rangle}(\bar{x'}) > d = \text{ord}_{\langle h \rangle}(\bar{x})$ if $\bar{x'} \neq 0$. By (1.8), $Z^m/\langle Z \rangle \subset \langle h \rangle^M$. Substituting $Z = \bar{h}$ gives $h^m \langle h \rangle \subset \langle h \rangle^M$, and hence $x'/\langle h \rangle \subset \langle h \rangle^M$. Since $\text{soc} M$ is finite, iterating yields

$$\bar{x} = \sum_{m \in \text{soc} M} x_m \bar{h}^m.$$

The remaining inclusion follows. □
2. Macaulay’s inverse system

Let $P$ be a Noetherian local graded ring with maximal ideal $m_P$. Then $P_0$ is Noetherian local with maximal ideal $m_{P_0} = (m_P)_0$. We assume that $P$ is complete, which means that $P_0$ is complete. A $P$-module $M$ is called Artinian if every descending chain of graded $P$-submodules is stationary. Using Nakayama lemma (see [BH93, Ex. 1.5.24]), we define its socle degree to be the nilpotency index

$$\text{socdeg } M = \inf \{ k \in \mathbb{Z} \mid m_P^k M = 0 \} - 1 \in \mathbb{N} \cup \{-\infty\}.$$ 

Denote by $E_{P_0}(P_0/m_{P_0})$ the injective hull of the residue field of $P_0$.

**Theorem 2.1** (Graded Matlis duality). The dualizing functor

$$-^\vee = ^* \text{Hom}_{P_0}( -, E_{P_0}(P_0/m_{P_0}))$$

defines an antiequivalence between the categories of Artinian and finitely generated graded $P$-modules (see [BH93, Thm. 3.6.17]).

With $D = P^\vee$, the functor $-^\vee$ induces an antiisomorphism of lattices

\begin{align*}
(I \trianglelefteq P \text{ graded ideal}) & \longleftrightarrow \{ W \subset D \text{ graded } P\text{-module} \}, \\
I & \longmapsto I^\perp = (P/I)^\vee, \\
(D/W)^\vee & = W^\perp,
\end{align*}

where $P/I$ is Artinian if and only if $W$ is finitely generated.

Let $K$ be field, and let $x = x_1, \ldots, x_n$ be indeterminates, where $n \in \mathbb{N} \setminus \{0\}$. Denote by $P$ either the (standard graded) polynomial ring $K[x]$ or the formal power series ring $K[[x]]$, both with $m_P = \langle x \rangle$. In both cases, $D$ identifies as a $K$-vector space with a polynomial ring $K[X]$ in indeterminates $X = X_1, \ldots, X_n$ with $P$-module structure given by (see [Eli18, Thm. 2.3.2])

\begin{equation}
X^\underline{u} \cdot X^\underline{m} = \begin{cases} 
X^{\underline{m} - \underline{u}} & \text{if } m \geq u, \\
0 & \text{otherwise.} 
\end{cases}
\end{equation}

Note that $(m_P^k)^\perp = D_{<k} = \bigoplus_{j=0}^{k-1} D_j$ with $\dim_K D_{<k} < \infty$, for all $k \in \mathbb{N}$. With (2.1), it follows that

\begin{equation}
\max \deg I^\perp = \text{socdeg}(P/I), \quad \dim_K I^\perp < \infty,
\end{equation}

if $P/I$ is Artinian.

In the case where $K$ is infinite and $P/I$ is Cohen–Macaulay, the following lemma is the starting point for our explicit description of $I^\perp$.

**Lemma 2.2.** Suppose that $R$ is a Noetherian complete local homogeneous graded algebra with coefficient field $K$. Then $R \cong P/I$ where $I \subset K[y][z] = P$ with $P_0 = K[y]$ and indeterminates $z$ of degree 1. Suppose that $P = K[x]$ or $P = K[[x]]$, $K$ is infinite, and that $R$ is
Cohen–Macaulay of dimension $d$. Then, after a $K$-linear change of coordinates, $x = y, z$ and $z$ maps to an $R$-sequence of length $d$.

Proof. By hypothesis, $R_0$ is Noetherian and $R_1$ is a finite $R_0$-module (see [BH93, Prop. 1.5.4]). Then $R_0 \cong K[y]/I_0$ by Cohen structure theorem and the first claim follows. Suppose now that $P = K[x]$ or $P = K[y]$. If $K$ is infinite and $\text{grade}(m_R, R) > 0$, then the $K$-vector space $(\mathfrak{T})_R \cong m_R/m_R^2$ is not the finite union of proper subspaces $\bigcup_{p \in \text{Ass}_R} (p + m_R^2)/m_R^2$. Then some $K$-linear combination of $\mathfrak{T}$ is regular on $R$ and the second claim follows by induction (see [BH93, Prop. 1.5.12]). \hfill $\square$

Let $d \in \{0, \ldots, n\}$ and partition $x = y, z$ into sets of indeterminates $y = y_1, \ldots, y_{n-d}$ and $z = z_1, \ldots, z_d$. Partition $X = Y, Z$ correspondingly into sets of indeterminates $Y = Y_1, \ldots, Y_{n-d}$ and $Z = Z_1, \ldots, Z_d$. The indeterminates $X, Y, Z$ are not related to the ones denoted by the same symbols in §1. Consider the inverse system over $\mathbb{N}^d$ defined by

$$\begin{align*}
u &\mapsto D, \quad \nu \leq m \mapsto z^{m-\nu} \in \text{End}_P(D) \\
\text{with limit } \varprojlim D &= K[Y][Z].
\end{align*}$$

Notation 2.3. Consider the $P$-submodules

$$V^{j,k}_{m} = \langle x^k \mid |k| \leq |m| + k, k = (l, n), n_j < m_j - 1 \rangle_p \subset D$$

where $j \in \{1, \ldots, d\}$, $k \in \mathbb{N}$ and $m \in \mathbb{N}^d$.

Remark 2.4. By definition, $V^{j,k}_{m}$ is an intersection of $P$-modules

$$V^{j,k}_{m} = \langle x^k \mid |k| \leq |m| + k \rangle_p \cap \langle Y^d Z^{m} \mid n_j < m_j - 1 \rangle_p$$

and applying the lattice antiisomorphism (2.1) yields

$$(V^{j,k}_{m})^\perp = \langle x^k \mid |k| \leq |m| + k \rangle_p^\perp \cap \langle Y^d Z^{m} \mid n_j < m_j - 1 \rangle_p^\perp = \langle x \rangle^{m+k+1} + \langle z_m \rangle^{m_j-1}.$$ 

Definition 2.5. Let $d \in \{0, \ldots, n\}$, and let $H \subset \varprojlim D$ be a finite $K$-vector subspace. Denote by $H_m$ its image in the copy of $D$ assigned to $m \in \mathbb{N}^d$ and consider the $P$-submodule

$$(2.4) \quad W_m = \langle H_m \rangle_p \subset D.$$

We call $H$ a limit inverse system of dimension $\dim H = d$, type $H = r \in \mathbb{N} \setminus \{0\}$ and socle degree $\text{socdeg} H = s \in \mathbb{N}$ if

(a) $\dim_K H = r$,
(b) $\min \{m \in \mathbb{N}^d \mid H_m \neq 0\} = 1$,
(c) $\max \text{deg} H_m = |m| + s - d$ and
(d) $W_m \cap V_{m}^{j,d-s} \subset W_{m-s}$,

for all $m \in \mathbb{N}^d$ and $j \in \{1, \ldots, d\}$. We consider $H \simeq H'$ as equivalent if $W_m = W'_m$, for all $m \in \mathbb{N}^d$.

Remark 2.6.
(a) Condition 2.5.(b) implies that for all \( \underline{m} \in \mathbb{N}^d \) and \( i \in \{1, \ldots, d\} \),
\[ z_i^{m_i} \cdot H_{\underline{m}} = 0, \]
and hence \( \max \text{deg}_{\underline{m}} H_{\underline{m}} = m_i - 1 \). In particular,
\[ \max \text{deg}_{\underline{m}} H_{\underline{m}} = |\underline{m}| - d. \]

(b) Condition 2.5.(c) can be substituted by \( \max \text{deg} H_1 = s \) and
\[ \max \text{deg} H_{\underline{m}} \leq |\underline{m}| + s - d, \]
for all \( \underline{m} \in \mathbb{N}^d \).

For any \( I \triangleleft P \) and \( \underline{m} \in \mathbb{N}^d \), we set
\[ (2.5) \quad I_{\underline{m}} = I + \langle z_1^{m_1}, \ldots, z_d^{m_d} \rangle \triangleleft P, \quad R_{\underline{m}} = P/I_{\underline{m}}. \]

Lemma 2.7. Any \( I \triangleleft P \) can be recovered from (2.5) as
\[ I = \bigcap_{\underline{n} \in \mathbb{N}^d} I_{\underline{n}}. \]

Proof. This is a consequence of Krull intersection theorem. \( \square \)

Theorem 2.8. Let \( d \in \{0, \ldots, n\} \), \( r \in \mathbb{N} \setminus \{0\} \) and \( s \in \mathbb{N} \). Then there
is a bijection between
(a) the set of (graded) ideals \( I \triangleleft P \) such that \( R_{\underline{m}} = P/I_{\underline{m}} \)
is Cohen–Macaulay, \( \dim R = d \), \( z = z_1, \ldots, z_d \) maps to an \( R \)-sequence,
\[ \text{type} R = r \] and \( \text{socdeg} R_1 = s \)
(b) the set of limit inverse systems \( H \subset \lim_{\leftarrow} D \) with \( \dim H = d \),
\[ \text{type} H = r \] and \( \text{socdeg} H = s \) modulo equivalence.

The map from (a) to (b) is defined by setting (see \( (2.5) \))
\[ (2.6) \quad W_{\underline{m}} = I_{\underline{m}}^1 = R_{\underline{m}}^1 \subset D \]
and taking \( H \subset \lim_{\leftarrow} \mathbb{N}^d W_{\underline{n}} \) the image of a \( \mathbb{K} \)-linear section of the canonical surjection
\[ (2.7) \quad \lim_{\leftarrow} D \supset \lim_{\leftarrow} \mathbb{N}^d W_{\underline{n}} \longrightarrow \lim_{\leftarrow} \mathbb{N}^d (W_{\underline{n}} \otimes \mathbb{K}) \cong W_1^1 \otimes \mathbb{K} \cong \text{soc}(R_1)^\vee, \]
where the inverse systems are defined by \( \underline{n} \leq \underline{m} \mapsto z_{\underline{m} - \underline{n}} \). The map
from (b) to (a) is defined by setting (see \( (2.4) \))
\[ (2.8) \quad I = \bigcap_{\underline{n} \in \mathbb{N}^d} W_{\underline{n}}^1. \]

Lemma 2.9. (a) Let \( I \) be in the set 2.8.(a) and \( W_{\underline{m}} \) as in \( (2.6) \). Then \( R_{\underline{m}} \) is Artinian, and hence \( \dim_{\mathbb{K}} W_{\underline{m}} < \infty \).
(b) Let \( H \) in the set 2.8.(b) and \( W_{\underline{m}} \) as in \( (2.4) \). Then
\[ \max \text{deg} W_{\underline{m}} = |\underline{m}| + s - d, \]
and hence \( \dim_{\mathbb{K}} W_{\underline{m}} < \infty \).

Proof. (a) Since \( (z) = \sqrt{\langle z_1^{m_1}, \ldots, z_d^{m_d} \rangle} \) and \( z \) maps to an \( R \)-sequence of
length \( d = \dim R \), \( \dim R_{\underline{m}} = \dim R_1 = 0 \). Then \( R_{\underline{m}} \) is Artinian by
Hopkins theorem, and hence \( \dim_{\mathbb{K}} W_{\underline{m}} < \infty \) by \( (2.3) \).
Lemma 2.10. Let $I$ be in the set 2.8.(a) and $W_m$ as in (2.6).

(a) There is a canonical surjection (2.7).

Let $H \subset \lim_{\leftarrow n \in \mathbb{N}} W_n$ be the image of a $K$-linear section of the surjection (2.7).

(b) The $P$-module $W_m$ is minimally generated by $H_m$, for all $m \in \mathbb{N}^d$.

In particular, (2.4) holds true.

(c) The $K$-vector space $H$ is in the set 2.8.(b).

Proof. In the following, $\underline{n}, \underline{m} \in \mathbb{N}^d$ with $\underline{n} \leq \underline{m}$.

(a) Consider the surjection of direct systems represented by

\[
\begin{array}{ccc}
R & \xrightarrow{z_{\underline{n}}} & R \\
\downarrow & & \downarrow \\
R_{\underline{n}} & \xrightarrow{z_{\underline{m}-\underline{n}}} & R_{\underline{m}}.
\end{array}
\]

Applying $-^\vee$ yields an inclusion of inverse systems represented by

\[
\begin{array}{ccc}
D & \xleftarrow{z_{\underline{n}}} & D \\
\downarrow & & \downarrow \\
W_{\underline{n}} & \xleftarrow{z_{\underline{m}-\underline{n}}} & W_{\underline{m}}.
\end{array}
\]

Left-exactness of the inverse limit then yields the inclusion in (2.7).

We now apply §1 with $g = h = z$ and $M$ the matrix with diagonal $m \in \mathbb{N}^d$. Then $\soc M = \{m - 1\}$ in Definition 1.8.

By Proposition 1.5, Corollary 1.9 and (1.9), the bottom map in (2.9) identifies (homogeneous) socles. This yields an inclusion of direct systems represented by

\[
\begin{array}{ccc}
R_{\underline{n}} & \xrightarrow{z_{\underline{m}-\underline{n}}} & R_{\underline{m}} \\
\downarrow & & \downarrow \\
\soc R_{\underline{n}} & \xrightarrow{=} & \soc R_{\underline{m}}.
\end{array}
\]

By Lemma 2.9.(a), $R_{\underline{m}}$ is Artinian, and hence (see [Eli18, Prop. 2.4.3])

\[
\soc(R_{\underline{m}})^\vee \cong I_{\underline{m}}^\perp / m_P \cdot I^\perp_{\underline{m}} \cong W_{\underline{m}} \otimes K.
\]

With $\underline{m} = \underline{1}$, this is the second isomorphism in (2.7).
Applying (2.11) to the bottom row of (2.10), this yields a trivial inverse system represented by
\[
soc(R_m) \cong soc(R_m) \\
W_m \otimes K \cong W_m \otimes K,
\]
and hence the first isomorphism in (2.7).

Consider the short exact sequence of inverse systems represented by (2.12)
\[
0 \to \mathfrak{m}_P \cdot W_m \to W_m \to W_m \otimes K \to 0.
\]
Since \(\dim_K W_m < \infty\) by Lemma 2.9.(a), the left inverse system in (2.12) satisfies the Mittag–Leffler condition. Therefore, the inverse limit preserves exactness when applied to (2.12). This yields the surjection in (2.7).

(b) Any \(K\)-linear section \(\sigma\) of (2.7) with image \(H\) fits into a diagram
\[
\begin{array}{ccc}
\lim_{m \in \mathbb{N}^d} W_m & \xrightarrow{\cong} & \lim_{m \in \mathbb{N}^d} (W_m \otimes K) \\
H_m & \xrightarrow{\cong} & W_m \\
& \xrightarrow{\cong} & W_m \otimes K
\end{array}
\]
and the claim follows by Nakayama lemma (see [BH93, Ex. 1.5.24]). \(\square\)

(c) By construction, \(H \cong H_1 \cong soc(R_1)\). Using that \(-^\vee\) preserves length, this gives condition 2.5.(a) (see [BH93, Lem. 1.2.19]),
\[
(2.13) \quad \dim_K H = \dim_K H_1 = \dim_K soc R_1 = \operatorname{type} R = r.
\]
For \(m \neq 1\), \(R_m = 0\), and hence \(H_m \subset W_m = 0\). With (2.13), condition 2.5.(b) follows. Part (b) with \(m = 1\) gives \(\langle H_1 \rangle = W_1 = I_1^\perp\), and hence \(\max \deg H_1 = \operatorname{socdeg} R_1 = s\) by (2.2) and (2.3). Condition 2.5.(c) follows by (2.2). By (2.1) and Remark 2.4,
\[
(W_m \cap V_i^{j,s-d})^\perp = W_m^\perp + (V_i^{j,s-d})^\perp
\]
\[
= I + \langle z_1^{m_1}, \ldots, z_d^{m_d} \rangle + \langle x \rangle^{m_j+s-d+1} + \langle z_j^{m_j-1} \rangle
\]
\[
\supset I + \langle z_1^{m_1}, \ldots, z_j^{m_j-1}, \ldots, z_d^{m_d} \rangle = W_{m-e_j}^\perp.
\]
Condition 2.5.(d) follows with (2.1).

**Lemma 2.11.** Let \(H\) be in the set 2.8.(b) and \(I\) as in (2.8).
(a) There is an equality \(I_m = W_m^\perp\) for all \(m \in \mathbb{N}^d\).
(b) The sequence \(\sigma\) maps to an \(R\)-sequence.
(c) The ring \(R = P/I\) is Cohen–Macaulay with \(\dim R = d\).
Proof.  
(a) Let \( j \in \{1, \ldots, d\} \) and \( m \in \mathbb{N}^d \). By Lemma 2.9.(b), \( W_{m + e_j} \subset D \) is a finitely generated (graded) \( P \)-submodule. Then \( P/W_{m + e_j} \) is Artinian by (2.1). By (2.3) and Lemma 2.9.(b), 
\[
\text{socdeg}(P/W_{m + e_j}) = \max \deg W_{m + e_j} = \lvert m + e_j \rvert + s - d,
\]
and hence 
\[
W_{m + e_j} \supset \langle x \rangle |m + e_j| + s - d + 1.
\]
Using Definition 2.5.(d), (2.1) and Remark 2.4, it follows that 
\[
W_{m} \subset (W_{m + e_j} \cap V_{m + e_j})^\perp = W_{m + e_j} + \langle x \rangle |m + e_j| + s - d + 1 + \langle z_j \rangle.
\]
This already implies that (see [MT18, Prop. 10, Claim 1]) 
\[
W_{m} \subset I + \langle z_1^{m_1}, \ldots, z_d^{m_d} \rangle.
\]
The opposite inclusion holds true since \( I \subset W_{m} \) by definition and \( z_i^{m_i} \cdot H_m = 0 \), for all \( i \in \{1, \ldots, d\} \), by Remark 2.6.(a). 

(b) By dualizing surjections \( \mathbb{A}^2 \twoheadrightarrow W_m \rightarrow W_n \) for suitable \( m, n \in \mathbb{N}^d \) with \( n \leq m \), one shows that \( z \) maps to a weak \( R \)-sequence (see [MT18, Prop. 10, Claim 2]). Since \( W_1 \neq 0 \) by Definition 2.5.(b), \( I_1 = W_1^\perp \neq R \) by part (a) with \( m = 1 \) and (2.1). Thus, \( R_1 \neq 0 \) and \( z \) maps to an \( R \)-sequence.

(c) By part (a) with \( m = 1 \), the ring \( R_1 \) is Artinian, and hence \( \dim R_1 = 0 \) by Hopkins theorem. With (b), it follows that \( R_{mR} \) and hence \( R \) is Cohen–Macaulay with \( \dim R = d \) (see [BH93, Ex. 2.1.27.(c)]). 

\[ \square \]

Proof of Theorem 2.8. This follows from (2.1), Lemmas 2.7, 2.10 and 2.11. 

\[ \square \]

Example 2.12. Let us consider the irreducible algebroid curve 
\[ R = \mathbb{C}[t^6, t^7, t^{11}, t^{13}]. \]
Note that \( R \) is not quasi-homogeneous. We write \( R \cong P/I \) where 
\[
P = \mathbb{C}[x, y, z, w],
\]
\[
I = \langle w - xy, yz - x^3, xz^2 - y^4, z^3 - x^2y^3, y^5 - x^4z \rangle.
\]
The element \( x \in P \) maps to the regular element \( t^6 \in R \). It can be checked that type \( R = 2 \) and \( P/(I + \langle x \rangle) \) has Hilbert–Samuel function
It follows that $R$ is not level. Using SINGULAR (see [Dec+18]), we compute the socles of $R_m$ (see (2.5)) up to $m = 3$:

$$\text{soc } R_1 = \langle x^2, y^3 \rangle,$$
$$\text{soc } R_2 = \langle x^2, y^3 \rangle,$$
$$\text{soc } R_3 = \langle x^2, z^3 \rangle.$$ 

They fit into the commutative diagram (see (2.10))

$$\begin{array}{ccc}
R_1 & \xrightarrow{x} & R_2 \xrightarrow{x} R_3 \\
\parallel & & \parallel \\
\text{soc } R_1 & \cong & \text{soc } R_2 \cong \text{soc } R_3.
\end{array}$$

Using a SINGULAR library by Elias (see [Eli15]), we compute the limit inverse system $H$ associated to $I$ by Theorem 2.8 up to $m = 7$:

$$H_1 = \langle y^3, z^2 \rangle_K,$$
$$H_2 = \langle xy^3 + y^2w, xz^2 + y^4 \rangle_K,$$
$$H_3 = \langle x^2y^3 + xy^2w + yz^2 + x^2z^2 + yx^4 + y^3w \rangle_K,$$
$$H_4 = \langle x^3y^3 + x^2y^2w + xzw^2 + x^2z^3 + y^4z + w^3, x^3z^2 + x^2y^4 + xy^3w + y^3z^2 + w^2 \rangle_K,$$
$$H_5 = \langle x^4y^3 + x^3y^2w + x^2yw^2 + x^2z^3 + x^4z + x^3zw + y^3z^2w + y^2z^2 + y^4, x^5z + x^4y^4 + x^3y^3w + x^2yz^3 + x^2y^2w^2 + xzw^2 + xy^3w + xy^5z + y^4z^2w + w^4 \rangle_K,$$
$$H_6 = \langle x^6y^3 + x^5y^2w + x^4z^3 + x^4yw^2 + x^3y^4z + x^3w^3 + x^2y^4zw + xy^2zw^2 + xy^4z^2w + z^3w + yzw^3 + y^5z^2, x^6z^2 + x^5y^4 + x^4y^3w + x^3yz^3 + x^3y^2w^2 + x^2z^3w + x^2yw^3 + x^2y^5z + xy^4zw + xzw^4 + y^2z^4 + y^3zw^2 \rangle_K.$$ 

**References**

[BH93] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*. Vol. 39. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993, pp. xii+403. isbn: 0-521-41068-1.

[CI12] Y. H. Cho and A. Iarrobino. “Inverse systems of zero dimensional schemes in $P^n$”. In: *J. Algebra* 366 (2012), pp. 42–77. isbn: 0021-8693. doi: 10.1016/j.jalgebra.2012.04.032.

[Dec+18] Wolfram Decker et al. *SINGULAR — A computer algebra system for polynomial computations*. Version 4-1-1, 2018.

[Dic13] Leonard Eugene Dickson. “Finiteness of the Odd Perfect and Primitive Abundant Numbers with $n$ Distinct Prime Factors”. In: *Amer. J. Math.* 35.4 (1913), pp. 413–422. isbn: 0002-9327. doi: 10.2307/2370405.
[Eis95] David Eisenbud. *Commutative algebra with a view toward algebraic geometry*. Vol. 150. Graduate Texts in Mathematics. New York: Springer-Verlag, 1995, pp. xvi+785. ISBN: 0-387-94269-6.

[Eli15] Juan Elias. *Inverse-syst.lib*, Singular library for computing Macaulay's inverse systems. 2015. arXiv: 1501.01786.

[Eli18] Juan Elias. “Inverse systems of local rings”. In: *Commutative Algebra and its Interactions to Algebraic Geometry*. Ed. by Nguyen Tu Cuong, Le Tuan Hoa, and Ngo Viet Trung. Vol. 2210. Lecture Notes in Mathematics. Springer, Cham, 2018, pp. 119–163. DOI: 10.1007/978-3-319-75565-6_2.

[Ems78] J. Emsalem. “Géométrie des points épais”. In: *Bull. Soc. Math. France* 106.4 (1978), pp. 399–416. ISSN: 0037-9484.

[ER12] J. Elias and M. E. Rossi. “Isomorphism classes of short Gorenstein local rings via Macaulay’s inverse system”. In: *Trans. Amer. Math. Soc.* 364.9 (2012), pp. 4589–4604. ISSN: 0002-9947.

[ER17] J. Elias and M. E. Rossi. “The structure of the inverse system of Gorenstein $k$-algebras”. In: *Adv. Math.* 314 (2017), pp. 306–327. ISSN: 0001-8708.

[Gab60] Pierre Gabriel. “Objets injectifs dans les catégories abéliennes”. In: *Algèbre et théorie des nombres*. Vol. 12. Sém. Dubreil. no. 2, exp. 17. Faculté des Sciences de Paris. Secrétariat mathématique, Paris, 1960, pp. 1–32.

[Ger96] Anthony V. Geramita. “Inverse systems of fat points: Waring’s problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals”. In: *The Curves Seminar at Queen’s, Vol. X (Kingston, ON, 1995)*. Vol. 102. Queen’s Papers in Pure and Appl. Math. Queen’s Univ., Kingston, ON, 1996, pp. 2–114.

[GS98] A. V. Geramita and H. K. Schenck. “Fat points, inverse systems, and piecewise polynomial functions”. In: *J. Algebra* 204.1 (1998), pp. 116–128. ISSN: 0021-8693. DOI: 10.1006/jabr.1997.7361.

[Hai94] Mark D. Haiman. “Conjectures on the quotient ring by diagonal invariants”. In: *J. Algebraic Combin.* 3.1 (1994), pp. 17–76. ISSN: 0925-9899.

[Iar94] Anthony A. Iarrobino. “Associated graded algebra of a Gorenstein Artin algebra”. In: *Mem. Amer. Math. Soc.* 107.514 (1994), pp. viii+115. ISSN: 0065-9266.

[iK99] Anthony Iarrobino and Vassil Kanev. *Power sums, Gorenstein algebras, and determinantal loci*. Vol. 1721. Lecture Notes in Mathematics. Appendix C by Iarrobino and Steven L. Kleiman. Springer-Verlag, Berlin, 1999, pp. xxxii+345. ISBN: 3-540-66766-0.
[Kle07] Jan O. Kleppe. “Families of Artinian and one-dimensional algebras”. In: *J. Algebra* 311.2 (2007), pp. 665–701. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2006.11.019.

[Mac94] F. S. Macaulay. *The algebraic theory of modular systems*. Cambridge Mathematical Library. Revised reprint of the 1916 original, With an introduction by Paul Roberts. Cambridge University Press, Cambridge, 1994, pp. xxxii+112. ISBN: 0-521-45562-6.

[Mat58] Eben Matlis. “Injective modules over Noetherian rings”. In: *Pacific J. Math.* 8 (1958), pp. 511–528. ISSN: 0030-8730.

[MT18] Shreedevi K. Masuti and Laura Tozzo. “The structure of the inverse system of level K-algebras”. In: *Collect. Math.* 69.3 (2018), pp. 451–477. ISSN: 0010-0757. DOI: 10.1007/s13348-018-0212-3.

[Nor53] D. G. Northcott. *Ideal theory*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 42. Cambridge, at the University Press, 1953, pp. viii+111.

[RS13] Kristian Ranestad and Frank-Olaf Schreyer. “The variety of polar simplices”. In: *Doc. Math.* 18 (2013), pp. 469–505. ISBN: 1431-0635.

[RŠ14] Maria Evelina Rossi and Liana M. Šega. “Poincaré series of modules over compressed Gorenstein local rings”. In: *Adv. Math.* 259 (2014), pp. 421–447. ISSN: 0001-8708.

[Sjö73] Gunnar Sjödin. “On filtered modules and their associated graded modules”. In: *Math. Scand.* 33 (1973), 229–249 (1974). ISSN: 0025-5521. DOI: 10.7146/math.scand.a-11486.

[Wal17] Uli Walther. “The Jacobian module, the Milnor fiber, and the $D$-module generated by $f^s$”. In: *Invent. Math.* 207.3 (2017), pp. 1239–1287. ISSN: 0020-9910.

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