A characterization of BMO self-maps of a metric measure space

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Abstract

This paper studies functions of bounded mean oscillation (BMO) on metric spaces equipped with a doubling measure. The main result gives characterizations for mappings that preserve BMO. This extends the corresponding Euclidean results by Gotoh to metric measure spaces. The argument is based on a generalization of Uchiyama’s construction of certain extremal BMO-functions and John-Nirenberg’s lemma.

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1 Introduction

Let $X$ be a complete metric space equipped with a metric $d$ and a Borel regular outer measure $\mu$ satisfying the doubling condition. A locally integrable function $f : X \to \mathbb{R}$ is of bounded mean oscillation, denoted as $f \in \text{BMO}(X)$, if

$$\|f\|_* = \sup_{B} \int_{B} |f - f_{B}| \, d\mu < \infty,$$

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where the supremum is taken over all balls $B \subset X$. We discuss invariance properties of BMO-functions. More precisely, we extend a characterization of Gotoh [8, 9] of mappings that preserve BMO to the metric setting. A $\mu$-measurable map $F: X \to X$ is a BMO-map if $F^{-1}(E)$ is a $\mu$-null set for each $\mu$-null set $E \subset X$, for every $f \in \text{BMO}(X)$ the composed map $C_F(f) = f \circ F$ is in $\text{BMO}(X)$. The first condition guarantees the uniqueness of the BMO-map. Moreover, the composition operator $C_F$ is a bounded operator from $\text{BMO}(X)$ to $\text{BMO}(X)$.

The class of BMO-functions is used, for example, in harmonic analysis, partial differential equations and quasiconformal mappings. Indeed, the first invariance property for BMO-functions was obtained by Reimann [22], where he showed that a homeomorphism is a BMO-map if and only if it is quasiconformal, provided the homeomorphism is assumed to be differentiable almost everywhere. Later Astala showed in [1] that the differentiability assumption is superfluous for a suitably localized result. The advantage of the approach by Gotoh [8] is that it applies to general measurable functions and hence is a more suitable to extensions to the metric setting. The Euclidean theory for BMO-functions is well understood, but not so much in a general metric measure space. For related metric space results we refer to [3, 17, 19, 20] and also to [2, Section 3.3].

We generalize the construction of certain extremal BMO-functions by Uchiyama [26] (see also [6, Section 2]) to doubling spaces. The result is stated in Theorem 2.1 and it constitutes the first part of the present paper. In the second part, we consider characterizations of BMO-maps between doubling spaces. Our main result is stated in Theorem 3.1. The characterizations in Theorem 3.1 are along the lines of the ones due to Gotoh [8, 9].

## 2 Construction of certain BMO-functions

Throughout the paper, $X$ is a complete metric space equipped with a metric $d$ and a Borel regular outer measure $\mu$ satisfying the doubling condition. An open ball

$$B(x, r) = \{y \in X : d(y, x) < r\}, \quad x \in X, \ r > 0,$$

is simply denoted by $B$, we write $\text{rad}(B)$ for the radius of the ball $B$, and $\lambda B = \{y \in X : d(y, x) < \lambda r\}, \lambda > 0$, is the ball with the same center, but the radius dilated by the factor $\lambda$. 

In this paper, the doubling condition means that there exists a constant $c_D > 1$ such that for all $x \in X$, $0 < r < \infty$ and $y \in X$ such that $B(x, 2r) \cap B(y, r) \neq \emptyset$, we have

$$\mu(B(x, 2r)) \leq c_D \mu(B(y, r)).$$

Notice that this condition is usually required to hold only for $x = y$, but if this standard doubling condition is valid with some uniform constant $c_\mu$, then

$$\mu(B(x, 2r)) \leq \mu(B(y, 8r)) \leq c_\mu^3 \mu(B(y, r)),$$

i.e. our version of the standard doubling condition is satisfied with $c_D = c_\mu^3$. The standard doubling condition implies that if $B(x, R) \subset X$, $y \in B(x, R)$, and $0 < r \leq R < \infty$, then

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq c_\mu^{-2} \left( \frac{r}{R} \right)^{\log_2 c_\mu}.$$

We refer, for instance, to [2, Lemma 3.3].

We recall that a locally integrable function $f : X \to \mathbb{R}$ has bounded mean oscillation, denoted as $f \in BMO(X)$, if

$$\|f\|_* = \sup_B \frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu < \infty,$$

where the supremum is taken over all balls $B \subset X$. We will identify functions which only differ by a constant; we shall call $\|f\|_*$ the BMO-norm of $f$. Here both $f_B$ and the barred integral $\bar{\int}_B f\, d\mu$ denote the integral average of $f$ over a ball $B$.

The following theorem is a metric space counterpart of a construction of certain BMO-functions in Uchiyama [26] and Garnett–Jones [6].

**Theorem 2.1.** Let $\lambda > 1$ and let $E_1, \ldots, E_N$, $N \geq 2$, be $\mu$-measurable subsets of $X$ such that

$$\min_{1 \leq j \leq N} \frac{\mu(E_j \cap B)}{\mu(B)} \leq c_D^{-4\lambda} \quad (2.1)$$

for any ball $B \subset X$. Then there exist functions $\{f_j\}_{j=1}^N$ such that

$$\sum_{j=1}^N f_j(x) = 1, \quad (2.2)$$
and for each $1 \leq j \leq N$

$$0 \leq f_j(x) \leq 1,$$  \hspace{1cm} (2.3)

and

$$f_j(x) = 0 \text{ } \mu\text{-almost everywhere on } E_j,$$  \hspace{1cm} (2.4)

and moreover,

$$\|f_j\|_* \leq \frac{c_1}{\lambda}.$$  \hspace{1cm} (2.5)

Here $c_1$ is a constant that only depends on $c_D$ and $N$. Conversely, if there exists $\{f_j\}_{j=1}^N$ that satisfy (2.2)–(2.4) and

$$\|f_j\|_* \leq \frac{c_2}{\lambda}$$

holds with a sufficiently small constant $c_2$, only depending on $c_D$ and $N$, for every $1 \leq j \leq N$, then (2.1) holds.

Before the proof of the theorem, we fix some notation and state few lemmas that will be needed later. Let $q$ be a large integer, depending only on $c_D$ and $N$, such that

$$1 + Nc_D^6q \leq 2^q.$$  \hspace{1cm} (2.6)

For every $k \in \mathbb{Z}$, let $r_k = 2^{-kq}$ and let $D_k$ be a maximal set of points such that $d(x,y) \geq \frac{1}{2}r_k$ whenever $x, y \in D_k$. Let $D = \bigcup_{k \in \mathbb{Z}} D_k$. Moreover, let

$$B_k = \{B(x, r_k) : x \in D_k\}.$$  \hspace{1cm}

From the maximality of the set $D_k$ it follows that for every $k \in \mathbb{Z}$,

$$X = \bigcup_{B \in B_k} B.$$  \hspace{1cm}

We say that a function $a \in C(X)$ is adapted to a ball $B = B(x, r)$, if

$$\text{supp } a \subset B(x, 2r) \text{ and } |a(x) - a(y)| \leq \frac{d(x,y)}{r}.$$  \hspace{1cm}

For a ball $B$, we set

$$g_j(B) = \log_{c_D} \frac{\mu(B)}{\mu(E_j \cap B)}, \quad 1 \leq j \leq N.$$  \hspace{1cm} (2.7)

Let us state the following simple lemma for the function $g_j$. 
Lemma 2.2. Let $k$ be a positive integer. If $B_1 \subset B_2$ and $c_D^k \mu(B_1) \geq \mu(B_2)$ for the balls $B_1$ and $B_2$ in $X$, then

$$g_j(B_1) \geq g_j(B_2) - k.$$ 

Proof. Clearly

$$g_j(B_1) = \log_{c_D} \frac{\mu(B_1)}{\mu(B_1 \cap E_j)} \geq \log_{c_D} \frac{c_D^k \mu(B_2)}{\mu(B_2 \cap E_j)} = g_j(B_2) - k.$$  

The next result is well known for the experts, but we recall it here.

Lemma 2.3. Let $f \in \text{BMO}(X)$. Then

$$\frac{1}{2} \|f\|_* \leq \sup \left| \int_X f g \, d\mu \right| \leq \|f\|_*,$$

where the supremum is taken over all functions $g$ for which there exists a ball $B$ such that

$$\text{supp} \, g \subset B, \quad \|g\|_\infty \leq \frac{1}{\mu(B)}, \quad \text{and} \quad \int_X g \, d\mu = 0.$$

Conversely, if $f$ is a locally integrable function on $X$ and the supremum above is finite, then $f \in \text{BMO}(X)$ with the above norm estimate.

Proof. First notice that for any $g$ as above, we have

$$\left| \int_X f g \, d\mu \right| = \left| \int_X (f - f_B) g \, d\mu \right| \leq \int_B |f - f_B| \, d\mu \leq \|f\|_*.$$

This gives the upper bound.

To see the lower bound, let $\varepsilon > 0$ and let $B$ be a ball such that

$$\|f\|_* \leq \int_B |f - f_B| \, d\mu + \varepsilon.$$ 

Let $h \in L^\infty(B)$ with $\|h\|_{L^\infty(B)} \leq 1$ be a function for which

$$\int_B |f - f_B| \, d\mu = \int_B (f - f_B)h \, d\mu. \quad \text{(2.8)}$$
Since \( \int_B (f - f_B) \, d\mu = 0 \), we have
\[
\int_B |f - f_B| \, d\mu = \int_B (f - f_B)(h - h_B) \, d\mu. \tag{2.9}
\]
Define
\[
g = \frac{(h - h_B)\chi_B}{2\mu(B)}.
\]
Then
\[
\text{supp } g \subset B, \quad \|g\|_{L^\infty(B)} \leq \frac{1}{\mu(B)} \quad \text{and} \quad \int_X g \, d\mu = 0.
\]
Moreover
\[
\int_X fg \, d\mu = \frac{1}{2\mu(B)} \int_B f(h - h_B) \, d\mu
\]
\[
= \frac{1}{2\mu(B)} \int_B (f - f_B)(h - h_B) \, d\mu. \tag{2.10}
\]
By combining the equation (2.9) and (2.10) we conclude that
\[
\int_B |f - f_B| \, d\mu = 2\mu(B) \int_X fg \, d\mu
\]
and
\[
\int_X fg \, d\mu = \frac{1}{2} \int_B |f - f_B| \, d\mu \geq \frac{1}{2} \left( \|f\|_* - \varepsilon \right).
\]
The claim follows by passing \( \varepsilon \to 0 \).

The equation (2.8) together with the above inequalities also indicates that the finiteness of \( \sup |\int_X fg \, d\mu| \) implies \( f \in \text{BMO}(X) \).

The proof of the metric space version of the following John-Nirenberg lemma can be found for example in Theorem 3.15 in [2]. See also [3] and [20].

\textbf{Lemma 2.4.} Let \( B \subset X \) be a ball and \( f \in \text{BMO}(5B) \). Then for every \( \lambda > 0 \)
\[
\mu(\{x \in B : |f(x) - f_B| > \lambda\}) \leq 2\mu(B) \exp \left( -\frac{A\lambda}{\|f\|_*} \right).
\]
The positive constant \( A \) depends only on the doubling constant \( c_D \).
We are ready for the proof of the main result of this chapter.

Proof of Theorem 2.1. The necessity part of the theorem is an immediate consequence of Lemma 2.4. Fix \( \lambda > 1 \) and let \( B \) be a ball. By (2.2), there exists \( j_0 \) such that
\[
(f_{j_0})_B \geq \frac{1}{N}.
\]
Thus, by Lemma 2.4 and (2.4), we have
\[
\frac{\mu(B \cap E_{j_0})}{\mu(B)} \leq \frac{\mu(\{x \in B : |f_{j_0}(x) - (f_{j_0})_B| \geq 1/N\})}{\mu(B)} \leq 2e^{-A/(N\|f\|_*)} \leq 2 \exp\left(-\frac{A\lambda}{Nc^2}\right) \leq c_D^{-4\lambda},
\]
if \( c_2 \) is chosen to be small enough. This completes the proof of the necessity part of Theorem 2.1.

Then we consider the sufficiency. By (2.1), we have
\[
\mu\left(\bigcap_{j=1}^N E_j\right) = 0.
\]
Thus, if \( \lambda > 1 \) is smaller than a given number, then the functions
\[
f_j = \frac{\chi_{E_j^c}}{\sum_{k=1}^N \chi_{E_k^c}}, \quad 1 \leq j \leq N,
\]
satisfy the desired properties (we denote the characteristic function of a set \( A \) by \( \chi_A \)). So we may assume that \( \lambda \) is large enough.

First, we assume that
\[
E_1, \ldots, E_N \subset B_0
\]
for some \( B_0 \in \mathcal{B}_0 \). We will inductively construct the sequences of BMO functions \( \{f_{j,h}\}_{h=1}^\infty \), \( 1 \leq j \leq N \), such that
\[
\sum_{j=1}^N f_{j,h}(x) = \lambda, \quad (2.12)
\]
\[
0 \leq f_{j,h}(x) \leq \lambda, \quad (2.13)
\]
\( f_{j,h}(x) \leq g_{j}(B) \) for every \( x \in B, \) if \( B \in \mathcal{B}_h, \)

\[
\text{and} \quad \|f_{j,h}\|_* \leq c_1. \tag{2.15}
\]

If the functions \( f_{j,h} \) above have been constructed, there exists a sequence 
\( 1 \leq h_1 < h_2 < \ldots \) such that \( \{f_{j,h_k}\}_{k=1}^\infty \) converge weak* in \( L^\infty \) as \( k \to \infty, \)

since \( \|f_{j,h}\|_\infty \leq \lambda \) by (2.13). We set

\[
f_j = \text{weak}^* - \lim_{k \to \infty} \frac{f_{j,h_k}}{\lambda}, \quad 1 \leq j \leq N.
\]

Then (2.2) and (2.3) follow from (2.12) and (2.13). Let \( g \) be as in Lemma 2.3. Then

\[
\left| \int f_j g \, d\mu \right| = \frac{1}{\lambda} \left| \lim_{k \to \infty} \int f_{j,h_k} g \, d\mu \right| \leq \frac{1}{\lambda} \limsup_{k \to \infty} \|f_{j,h_k}\|_* \leq c_1. \lambda.
\]

Thus (2.5) with constant \( 2c_1 \) follows from Lemma 2.3. Since, by Lebesgue’s theorem,

\[
\lim_{r \to 0} \sup_{\substack{B \ni x \in E_j \cap B \cap \text{rad}(B) \leq r}} g_j(B) = 0
\]

for \( \mu \)-almost every \( x \in E_j \), we have by (2.14)

\[
\lim_{h \to 0} f_{j,h}(x) = 0
\]

for \( \mu \)-almost every \( x \in E_j \). Thus (2.4) follows. Hence \( \{f_j\}_{j=1}^N \) are the desired functions.

To remove the restriction (2.11), we take balls \( B_p \in \mathcal{B}_{-p}, \) \( p = 1, 2, \ldots, \)

such that \( B_{p-1} \subset B_p \) for every \( p, \) and we can construct \( f_{j,p} \) such that all other conditions are as for \( B_0, \) except that

\[
f_{j,p} = 0 \quad \text{on } E_j \cap B_p.
\]

Then there exists a sequence \( 1 \leq p_1 < p_2 \ldots \) such that \( \{f_{j,p_k}\}_{k=1}^\infty \) converge weak* in \( L^\infty \). Then

\[
f_j = \text{weak}^* - \lim_{k \to \infty} f_{j,p_k}, \quad 1 \leq j \leq N,
\]

are the desired functions.
Thus, to complete the proof Theorem 2.1 we shall construct a sequence of functions that satisfy the conditions (2.12)–(2.15). The proof is organized as follows. In Lemma 2.5, we will construct the sequence \( \{f_{j,h}\}_{h=0}^{\infty}, 1 \leq j \leq N \), and show that these functions satisfy the conditions (2.12)–(2.14). And finally, in Lemma 2.7 we show that the condition (2.15) is valid for the functions.

**Lemma 2.5.** Let \( E_1, \ldots, E_N \) satisfy (2.1) and (2.11). Then there exist \( \{f_{j,h}\} \) and \( A_{j,h} \subset B_h \) having the properties (2.12)–(2.14) and satisfying the following conditions

\[
|f_{j,h}(x) - f_{j,h}(y)| \leq 2^{(h+1)q}d(x, y), \quad (2.16)
\]

\[
A_{j,h} = \{B \in B_h : \sup_B f_{j,h-1} > g_j(B)\}, \quad (2.17)
\]

\[
f_{j,h}(x) \geq f_{j,h-1}(x) - c_D^3 q, \quad (2.18)
\]

and

\[
f_{j,h}(x) \geq f_{j,h-1}(x) \text{ for } x \not\in \bigcup_{B \in A_{j,h}} 2B. \quad (2.19)
\]

**Proof.** By (2.1), we have

\[
\max_{1 \leq j \leq N} g_j(B_0) \geq 4\lambda.
\]

Set

\[
s(B_0) = \min\{j : 1 \leq j \leq N, g_j(4B_0) \geq 4\lambda\},
\]

\[
f_{s(B_0),0} = \lambda, \quad \text{and} \quad f_{j,0} = 0 \text{ for } j \neq s(B_0).
\]

Assume now that the functions \( f_{1,k-1}, \ldots, f_{N,k-1} \) have been defined and satisfy the conditions (2.12)–(2.14), (2.16), (2.18) and (2.19). Define \( A_{j,k} \) by (2.17). For any ball \( B \), let \( b_B \) denote a function that is adapted to \( B \), \( 0 \leq b_B \leq 1 \) and \( b_B = 1 \) on \( B \). Let \( A_{j,k} = \{B_m\}_{m=1}^p \). Set \( a_{B_1} = \min\{q b_{B_1}, f_{j,k-1}\} \) and

\[
a_{B_m} = \min\left\{q b_{B_m}, f_{j,k-1} - \sum_{n=1}^{m-1} a_{B_n}\right\} \text{ for } m = 2, \ldots, p.
\]

Since the supports of \( \{b_{B_m}\} \) overlap at most \( c_D^3 \) times, the functions \( c_D^{-3}q^{-1}a_{B_m} \) are adapted to \( B_m \). Set

\[
\tilde{f}_{j,k} = f_{j,k-1} - \sum_{B \in A_{j,k}} a_B = f_{j,k-1} - v_{j,k}.
\]
Since
\[ \tilde{f}_{j,k} = \max \left\{ f_{j,k-1} - \sum_{B \in A_{j,k}} q_B, 0 \right\}, \]
we see that \( \{\tilde{f}_{j,k}\} \) satisfy (2.13), (2.18) and (2.19).

If \( B \in A_{j,k} \) and \( x \in B \), then by Lemma 2.2
\[ \tilde{f}_{j,k}(x) \leq \max\{f_{j,k-1}(x) - q, 0\} \leq \max\{g_j(\tilde{B}) - q, 0\} \leq g_j(B), \]
for every \( \tilde{B} \in B_{k-1} \) such that \( B \subset \tilde{B} \).

If \( B \in B_k \setminus A_{j,k} \) and \( x \in B \), then
\[ \tilde{f}_{j,k}(x) \leq f_{j,k-1}(x) \leq g_j(B) \]
by the definition of \( A_{j,k} \). So \( \{\tilde{f}_{j,k}\} \) satisfies (2.14). These functions do not satisfy the property (2.12), and hence we shall modify the functions further.

We set
\[ f_{j,k} = \tilde{f}_{j,k} + \sum_{B \in \bigcup_{m=1}^{N} A_{m,k}} a_B = \tilde{f}_{j,k} + w_{j,k}. \]

The modified sequence \( \{f_{j,k}\} \) satisfies (2.12). Also the conditions (2.13), (2.18), and (2.19) are met since \( a_B \geq 0 \).

Let us next look at the condition (2.14). If \( B \in B_k \) and \( w_{j,k} = 0 \) on \( B \), then
\[ f_{j,k} = \tilde{f}_{j,k} \leq g_j(B) \text{ on } B, \]
since \( \tilde{f}_{j,k} \) satisfies (2.14). If \( B \in B_k \) and \( w_{j,k} \neq 0 \) on \( B \), then, by the definition of \( w_{j,k} \), there exists a ball \( \tilde{B} \in B_k \) such that
\[ B \cap 2\tilde{B} \neq \emptyset \text{ and } g_j(4\tilde{B}) \geq 4\lambda. \]
Then \( B \subset 4\tilde{B} \). By Lemma 2.2
\[ g_j(B) \geq g_j(4\tilde{B}) - 2 \geq \lambda. \]
So by (2.13), we have
\[ f_{j,k}(x) \leq \lambda \leq g_j(B) \]
and consequently (2.14) holds.
Let us show that the condition (2.16) holds. If \( x, y \in \tilde{B} \) and \( \tilde{B} \in B_k \), then

\[
\left| (-v_{j,k}(x) + w_{j,k}(x)) - (-v_{j,k}(y) + w_{j,k}(y)) \right| \\
\leq \sum_{B \in \bigcup_{m=1}^{N} A_{m,k}} |a_B(x) - a_B(y)| \quad (2.20)
\]

Since the supports of \( \{a_B\}_{B \in \bigcup_{m=1}^{N} A_{m,k}} \) overlap at most \( N c_D^3 \) times, (2.20) is dominated by

\[
N c_D^3 \cdot c_D^3 q \cdot \frac{d(x, y)}{r_k} = N c_D^6 q 2^{qk} d(x, y).
\]

From this we conclude that

\[
|f_{j,k}(x) - f_{j,k}(y)| \leq |f_{j,k-1}(x) - f_{j,k-1}(y)| + N c_D^6 q 2^{qk} d(x, y) \\
\leq (1 + N c_D^6 q) 2^{qk} d(x, y) \leq 2^{(k+1)q} d(x, y),
\]

where we used (2.16) for \( f_{j,k-1} \), and also the inequality (2.6). \( \square \)

Lemma 2.6.

\[
f_{j,k}(x) \leq g_j(B) - \frac{1}{3} \log_2 \frac{r}{r_h} + 8 \cdot 2^q + 6
\]

for every \( x \in B = B(y, r) \) for any \( B \) such that \( r \leq 4r_h \).

Proof. There are at most \( c_D^3 \) balls in \( B_1, \ldots, B_k \) with the centers in \( D_h \) such that \( B_i \cap B \neq \emptyset \). Let

\[
\delta = \min_{1 \leq i \leq k} g_j(B_i) = g_j(B_{i_0}).
\]

By (2.14)

\[
\inf_{x \in B} f_{j,h}(x) \leq \delta,
\]

and by (2.16) we have

\[
f_{j,h}(x) \leq \delta + 2^{(h+1)q} 2r \leq \delta + 8 \cdot 2^q
\]

whenever \( x \in B \).
On the other hand,

\[ g_j(B) = \log_{c_D} \frac{\mu(B)}{\mu(B \cap E_j)} \]

\[ \geq \log_{c_D} \frac{\mu(B)}{\sum_i \mu(B_i \cap E_j)} \]

\[ \geq \log_{c_D} \frac{\mu(B)}{c^2_D \max \{ \mu(B_i \cap E_j) \}} \]

\[ = \log_{c_D} \frac{\mu(B)}{\mu(B_{i0})} + \log_{c_D} \frac{\mu(B_{i0})}{\mu(B_{i0} \cap E_j)} + \log_{c_D} \frac{1}{c^3_D} \]

\[ \geq \log_{c_D} \frac{\mu(B)}{\mu(B_{i0})} + \delta - 3 \]

\[ \geq \frac{1}{3} \log_2 \frac{r}{r_h} + \delta - 6. \]

The desired result follows from the two previous estimates. \(\square\)

We finish to proof of Theorem 2.1 by proving the following lemma.

**Lemma 2.7.** \( \| f_{j,h} \|_* \leq c_1. \)

**Proof.** Let \( B = B(x, r) \) be any ball. If \( r \leq 2^{-hq} \) then, by (2.16), we have

\[ \inf_{c \in \mathbb{R}} \int_B |f_{j,h} - c| d\mu \leq 2^q. \]  

(2.21)

If \( 0 \leq n < h \) and \( 2^{-(n+1)q} < r \leq 2^{-nq} \), let

\[ \beta_j = \int_B f_{j,n} d\mu. \]

Notice that by Lemma 2.6

\[ \beta_j \leq g_j(4B) + \frac{1}{3} q + 8 \cdot 2^q + 6. \]

(2.22)

We will show that

\[ \int_B |f_{j,h} - \beta_j| d\mu \leq C. \]

(2.23)
Let
\[
\{ x \in B : |f_{j,h}(x) - \beta_j| \geq \alpha \} = \{ x \in B : f_{j,h}(x) < \beta_j - \alpha \} \cup \{ x \in B : f_{j,h}(x) > \beta_j + \alpha \} \quad (2.24)
\]
\[
= G(B, j, \alpha) \cup H(B, j, \alpha).
\]

First, we estimate \( \mu(G(B, j, \alpha)) \). Let \( \alpha > 2^{q+1} \). Note that \( f_{j,n}(x) > \beta_j - 2^{q+1} \) on \( B \) by (2.16). So if \( x \in G(B, j, \alpha) \) then, by (2.19), there exists \( \tilde{B} \in A_{j,k} \), \( n < k \leq h \), such that \( x \in 2\tilde{B} \) and \( f_{j,k}(x) < \beta_j - \alpha \). So by (2.18), we have
\[
f_{j,k-1}(x) < \beta_j - \alpha + c_D^3 q,
\]
and by (2.16)
\[
f_{j,k-1}(y) < \beta_j - \alpha + c_D^3 q + 3
\]
for every \( y \in \tilde{B} \). Thus, by the definition of \( A_{j,k} \), we obtain
\[
g_j(\tilde{B}) < \beta_j - \alpha + c_D^3 q + 3.
\]
By the above, we can use the standard 5-covering theorem ([2, Lemma 1.7]) and take disjoint balls \( \{B_m\} \subset \bigcup_{n<k \leq h} A_{j,k} \) such that
\[
B_m \subset 4B, \quad G(B, j, \alpha) \subset \bigcup_m 5B_m,
\]
and
\[
g_j(B_m) < \beta_j - \alpha + c_D^3 q + 3. \quad (2.25)
\]
Thus
\[
\mu(G(B, j, \alpha)) \leq c_D^3 \sum_m \mu(B_m) = c_D^3 \sum_m \mu(E_j \cap B_m) c_D^{g_j(B_m)}
\]
\[
\leq C c_D^{\beta_j - \alpha} \sum_m \mu(E_j \cap B_m)
\]
\[
\leq C c_D^{g_j(4B) - \alpha} \sum_m \mu(E_j \cap B_m)
\]
\[
\leq C c_D^{g_j(4B) - \alpha} \mu(E_j \cap 4B) \leq C \mu(B) c_D^{-\alpha}.
\]

Here we used first (2.7), then (2.25), (2.22) and finally (2.7) again.
Let us then estimate the measure $\mu(H(B, j, \alpha))$. Let $\alpha > (N - 1)2^{q+1}$. Note that $\sum_{m=1}^{N} \beta_m = \lambda$ by (2.12). So if $x \in H(B, j, \alpha)$, then

$$
\sum_{1 \leq m \leq N, m \neq j} f_{m,h}(x) = \lambda - f_{j,h}(x) = \sum_{m=1}^{N} \beta_m - f_{j,h}(x)
$$

$$
= \left( \sum_{1 \leq m \leq N, m \neq j} \beta_m \right) - (f_{j,h}(x) - \beta_j)
$$

$$
< \left( \sum_{1 \leq m \leq N, m \neq j} \beta_m \right) - \alpha.
$$

Thus

$$
\sum_{1 \leq m \leq N, m \neq j} (\beta_m - f_{m,h}(x)) > \alpha.
$$

So

$$
x \in \bigcup_{1 \leq m \leq N, m \neq j} G(B, m, \alpha/(N - 1)),
$$

and consequently

$$
H(B, j, \alpha) \subset \bigcup_{1 \leq m \leq N, m \neq j} G(B, m, \alpha/(N - 1)).
$$

By (2.26), we have

$$
\mu(H(B, j, \alpha)) \leq C(N - 1)\mu(B)c^{-\alpha/(N-1)}. \quad (2.27)
$$

Thus, if $2^{-hq} \leq r \leq 1$, then (2.23) follows from (2.26) and (2.27). If $r > 1$, then put $\beta_{s(B_0)} = \lambda$ and $\beta_j = 0$ for $j \neq s(B_0)$. Then (2.23) follows from the same argument. Thus Lemma 2.7 follows from (2.21) and (2.23). □

The proof of Theorem 2.1 is now complete. □

3 Characterizations of BMO-maps

We say that a $\mu$-measurable map $F: X \to X$ is a BMO-map if
(I) $F^{-1}(E)$ is a $\mu$-null set for each $\mu$-null set $E \subset X$;

(II) for every $f \in \text{BMO}(X)$ the composed map $C_F(f) = f \circ F$ is in BMO(X).

We shall prove a metric space generalization of a theorem due to Go-toh [8, Theorem 3.1] which characterizes BMO-maps between doubling metric measure spaces. In the proof we apply Uchiyama’s construction proved in Section 2. The condition (3.1) has a similar flavor as the conditions in [7] and [16] related to invariance properties of quasiconformal mappings.

**Theorem 3.1.** Suppose that $F: X \to X$ is $\mu$-measurable. Then the following conditions are equivalent:

(i) There exist positive finite constants $K$ and $\alpha$ such that for an arbitrary pair of $\mu$-measurable subsets $E_1, E_2$ of $X$ we have

$$\sup_B \min_{k=1,2} \frac{\mu(F^{-1}(E_k) \cap B)}{\mu(B)} \leq K \left( \sup_B \min_{k=1,2} \frac{\mu(E_k \cap B)}{\mu(B)} \right)^{\alpha},$$

where the suprema are taken over all balls $B$ in $X$;

(ii) There exist constants $0 < \gamma < 1/4$ and $\lambda > 0$ such that for an arbitrary pair of $\mu$-measurable subsets $E_1, E_2$ of $X$ satisfying

$$\sup_B \min_{k=1,2} \frac{\mu(E_k \cap B)}{\mu(B)} < \lambda,$$

we have

$$\sup_B \min_{k=1,2} \frac{\mu(F^{-1}(E_k) \cap B)}{\mu(B)} < \gamma,$$

where the suprema are taken over all balls $B$ in $X$;

(iii) $F$ is a BMO-map with the operator norm of $C_F$ bounded by $CK/\alpha$, where $C$ depends only on the doubling constant.

The condition (i) readily implies the condition (ii), and hence to show the equivalence of conditions (i)–(iii), it is enough to prove implications (i)$\Rightarrow$(iii), (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i), in Propositions 3.7, 3.8, and 3.9 respectively. The Uchiyama construction of BMO functions, presented in Section 2, is used in the proof of Proposition 3.9. For the proof of the bound for the operator norm, see Proposition 3.7.
Remark 3.2. Let us comment on the condition (i).

(1) Setting $E_1 = E_2 = X$ in (3.1) it can be seen that $K \geq 1$.

(2) If (3.1) is valid for some positive $\alpha_0$ it clearly holds for all $0 < \alpha < \alpha_0$. And moreover, since the condition (3.1) is interesting mainly with small values of the exponent $\alpha$, we shall assume, without loss of generality, that $\alpha \leq 1$.

We shall next prove several lemmas on BMO functions.

Lemma 3.3. Let $f \in \text{BMO}(X)$. Then

$$
\min\{\mu(\{x \in B: f(x) \geq t\}), \mu(\{x \in B: f(x) \leq s\})\} \\
\leq 2\mu(B) \exp \left(-C \frac{t-s}{\|f\|_*}\right)
$$

for every $-\infty < s \leq t < \infty$, where $C$ is a positive constant depending on the doubling constant $c_D$.

Proof. By symmetry, we may assume that $f_B \leq (s + t)/2$. Then Lemma 2.4 implies that

$$
\mu(\{x \in B: f(x) \geq t\}) \leq \mu\left(\left\{x \in B: |f(x) - f_B| \geq \frac{t-s}{2}\right\}\right) \\
\leq 2\mu(B) \exp \left(-\frac{A(t-s)}{2\|f\|_*}\right).
$$

If $f_B \geq (s + t)/2$, we get a similar estimate for $\mu(\{x \in B: f(x) \leq s\})$. \qed

A converse of the statement in Lemma 3.3 is presented in the following.

Lemma 3.4. Let $f: X \to \mathbb{R}$ be a $\mu$-measurable function with $|f| < \infty$ $\mu$-almost everywhere in $X$. Assume there exist positive constants $C_1, C_2$ such that for every ball $B$ in $X$ we have

$$
\min\{\mu(\{x \in B: f(x) \geq t\}), \mu(\{x \in B: f(x) \leq s\})\} \\
\leq C_1\mu(B) \exp (-C_2(t-s))
$$

for every $-\infty < s \leq t < \infty$. Then $f \in \text{BMO}(X)$ and

$$
\|f\|_* \leq 4(C_1 + 1)C_2^{-1} \exp(2C_2).
$$
In the proof of Lemma 3.4 we apply the following lemma which can be found in [8, Lemma 4.5].

**Lemma 3.5.** Let \( \lambda : \mathbb{R} \to [0, 1] \) be a non-constant, non-decreasing function. Assume that there exists positive constants \( C_1, C_2 \) such that
\[
\min\{\lambda(s), 1 - \lambda(t)\} \leq C_1 \exp(-C_2(t-s))
\]
for every \(-\infty < s \leq t < \infty\). Then there exists \( t_0 \in \mathbb{R} \) such that
\[
\max\{\lambda(t_0 - t), 1 - \lambda(t_0 + t)\} \leq (C_1 + 1) \exp(2C_2) \exp(-C_2t)
\]
for each \( t \geq 0 \).

**Proof of Lemma 3.4.** We apply Lemma 3.5 by setting
\[
\lambda(t) = \frac{\mu(\{x \in B : f(x) \leq t\})}{\mu(B)}.
\]
Then by the hypothesis \( \lambda(t) \) meets the assumption in Lemma 3.5 with the same constants \( C_1 \) and \( C_2 \). Hence there exists \( t_0 \in \mathbb{R} \) such that the second inequality of Lemma 3.5 is valid for every \( t \geq 0 \). This implies that
\[
\nu(t) = \mu(\{x \in B : |f(x) - t_0| \geq t\}) \leq 2(C_1 + 1)\mu(B) \exp(2C_2) \exp(-C_2t)
\]
for every \( t \geq 0 \). We obtain
\[
\int_B |f - f_B| d\mu \leq 2 \int_B |f - t_0| d\mu = 2 \int_0^\infty \nu(t) dt \leq 4(C_1 + 1)C_2^{-1} \exp(2C_2)\mu(B)
\]
from which the claim follows.

In Euclidean spaces the following lemma is due to Strömberg [24]. A result similar to this has also been considered for nondoubling measures by Lerner in [18].

**Lemma 3.6.** Let \( f : X \to \mathbb{R} \) be \( \mu \)-measurable. Assume that there exist constants \( 0 \leq \gamma < (4c_D^3)^{-1} \), and \( \lambda > 0 \) such that for each ball \( B \) in \( X \) we have
\[
\inf_{c \in \mathbb{R}} \mu(\{x \in B : |f(x) - c| \geq \lambda\}) \leq \gamma \mu(B). \tag{3.2}
\]
Then \( f \in \text{BMO}(X) \) satisfying \( \|f\|_* \leq C\lambda \), where a positive constant \( C \) depends only on the doubling constant \( c_D \).
Proof. Let $f$ be $\mu$-measurable on $X$, and fix $\gamma$ and $\lambda$ such that the hypothesis (3.2) is satisfied for each ball in $X$. Fix a ball $B \subset X$ and let $c_0$ be the number where the infimum in (3.2) is reached. For each $m = 1, 2, \ldots$ we write

$$S_m^+ = \{ x \in B : f(x) - c_0 > m\lambda \},$$
$$S_m^- = \{ x \in B : f(x) - c_0 < -m\lambda \},$$
$$S_m = S_m^+ \cup S_m^- = \{ x \in B : |f(x) - c_0| > m\lambda \},$$
$$E_m = \{ x \in B : m\lambda < |f(x) - c_0| \leq (m+1)\lambda \},$$

and

$$E_0 = \{ x \in B : |f(x) - c_0| \leq \lambda \}.$$

Let us estimate the measure of the set $S_m^+$. First notice that $S_m^+ \subset S_{m-1}^+$. For $\mu$-almost every $x \in S_{m-1}^+$, there exists a ball $B_x = B(x, r_x)$ such that

$$\frac{1}{2c_D} \mu(B_x) < \mu(B_x \cap S_{m-1}^+) \leq \frac{1}{2} \mu(B_x) \quad (3.3)$$

and

$$\mu(B(x, r) \cap S_{m-1}^+) > \frac{1}{2} \mu(B(x, r))$$

for all $r < \frac{1}{2}r_x$; see, for example, Theorem 3.1 and Remark 3.2 in [14].

By a well known 5-covering theorem ([2, Lemma 1.7]), we can cover the set $S_{m-1}^+$ by finite or countable sequence of balls $\{B_i\}_i$ satisfying (3.3) such that the balls $\{\frac{1}{5}B_i\}_i$ are disjoint. It follows from (3.3) that the infimum in (3.2) is reached with some constant $c$ such that

$$c_0 + (m-2)\lambda \leq c \leq c_0 + m\lambda$$

in each of the balls $B_i$, and hence $c - c_0 \leq m\lambda$.

We conclude, by applying the inequality (3.2) in balls $B_i$, that

$$\mu(S_{m+1}^+) \leq \sum_i \mu(B_i \cap S_{m+1}^+) \leq \gamma \sum_i \mu(B_i) \leq c_D^3 \gamma \sum_i \mu(\frac{1}{5}B_i)$$

$$\leq 2c_D^3 \gamma \sum_i \mu(\frac{1}{5}B_i \cap S_{m-1}^+) \leq 2c_D^3 \gamma \mu(S_{m-1}^+)$$

Since $\mu(S_1^+) \leq \mu(S_1) < \gamma \mu(B)$, it follows from the previous estimate that

$$\mu(S_{2m+2}^+) \leq \mu(S_{2m+1}^+) \leq (2c_D^3 \gamma)^{m+1} \mu(B)$$
for each \( m = 1, 2, \ldots \). Since a similar estimate holds for \( S_m \), we altogether have
\[
\mu(S_m) \leq 2(2c_D^3 \gamma)^{m/2} \mu(B).
\]
We thus conclude
\[
\int_B |f - f_B| d\mu \leq \frac{2}{\mu(B)} \left( \sum_{m=0}^{\infty} \int_{E_m} |f - c_0| d\mu \right)
\leq \lambda + 2 \sum_{m=1}^{\infty} (m + 1) \frac{\mu(S_m)}{\mu(B)}
\leq \lambda \left( 1 + 2 \sum_{m=1}^{\infty} (m + 1) (2c_D^3 \gamma)^{m/2} \right)
\leq \lambda \left( 1 + 2 \sum_{m=1}^{\infty} (m + 1) 2^{-m/2} \right).
\]
Since the preceding estimate holds for any ball \( B \subset X \), the claim follows.

Proposition 3.7. [(i) \( \Rightarrow \) (iii)] Let \( F : X \to X \) be \( \mu \)-measurable and assume that there exist positive finite constants \( K \) and \( \alpha \) such that the condition (i) of Theorem 3.1 holds. Then \( F \) is a BMO-map satisfying \( \|C_F\| \leq CK/\alpha \), where \( C \) depends on the doubling constant \( c_D \).

Proof. The condition (i) implies that if \( E \) is a \( \mu \)-null subset of \( X \) then also \( \mu(F^{-1}(E)) = 0 \).

Let \( f \in \text{BMO}(X) \) and set for each \( -\infty < s \leq t < \infty \)
\[
E_1 = \{ x \in X : f(x) \leq s \} \quad \text{and} \quad E_2 = \{ x \in X : f(x) \geq t \}. \tag{3.4}
\]
It follows from Lemma 3.3 that
\[
\min\{\mu(E_1 \cap B), \mu(E_2 \cap B)\} \leq 2\mu(B) \exp\left(-C\frac{t-s}{\|f\|_*}\right)
\]
for all balls \( B \) in \( X \). The condition (i) implies
\[
\min\{\mu(F^{-1}(E_1) \cap B), \mu(F^{-1}(E_2) \cap B)\} \leq 2^\alpha K \mu(B) \exp\left(-C\frac{\alpha(t-s)}{\|f\|_*}\right)
\]

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for all balls $B$ in $X$. Since

$$F^{-1}(E_1) \cap B = \{ x \in B : (f \circ F)(x) \leq s \}$$

and

$$F^{-1}(E_2) \cap B = \{ x \in B : (f \circ F)(x) \geq t \},$$

it follows from Lemma 3.4 that $f \circ F \in \text{BMO}(X)$ and (recall that $\alpha \leq 1$, see Remark 3.2)

$$\|C_F(f)\| \leq \frac{4(2^n K + 1)\|f\|_{\ast}}{\alpha} \exp(2C\alpha/\|f\|_{\ast})$$

$$= \frac{CK\|f\|_{\ast}}{\alpha} \exp(C\alpha/\|f\|_{\ast}),$$

where $C$ is a positive constant depending on the doubling constant $c_D$. Applying the preceding estimate to $\tau f$, $\tau > 0$, and letting $\tau \to \infty$, we obtain that $\|C_F\| \leq CK/\alpha$.

**Proposition 3.8 ((ii) $\Rightarrow$ (iii)).** Let $F : X \to X$ be $\mu$-measurable and assume that there exist constants $0 < \gamma < (4c_D^3)^{-1}$ and $\lambda > 0$ such that the condition (ii) of Theorem 3.1 holds. Then $F$ is a BMO-map satisfying $\|C_F\| \leq C\lambda$, where $C$ depends on the doubling constant $c_D$ and $\gamma$.

**Proof.** The condition (ii) implies that if $E$ is a $\mu$-null subset of $X$ then also $\mu(F^{-1}(E)) = 0$.

Let $f \in \text{BMO}(X)$ and assume, without loss of generality, that $\|f\|_{\ast} = 1$. We define the sets $E_1$ and $E_2$ for each $-\infty < s < t < \infty$ as in (3.4). We apply Lemma 3.3 and obtain

$$\sup_B \min_{k=1,2} \frac{\mu(E_k \cap B)}{\mu(B)} \leq 2 \exp(-C(t-s)) < \lambda,$$

whenever $t - s \geq C_1$, where $C_1$ only depends on $\lambda$ and the constant $C$ from Lemma 3.3. Hence the condition (ii) implies that

$$\sup_B \min_{k=1,2} \frac{\mu(F^{-1}(E_k) \cap B)}{\mu(B)} < \gamma.$$

For every ball $B$ in $X$ we set

$$s_B = \sup \{ s \in \mathbb{R} : \mu(\{ x \in B : f(F(x)) \leq s \})$$

$$\leq \mu(\{ x \in B : f(F(x)) \geq s + C_1 \}) \}.$$
Since $|f(F(x))| < \infty$ for $\mu$-almost every $x \in X$, we have that $s_B \neq \pm \infty$. Hence

$$\mu(\{ x \in B : f(F(x)) \leq s_B - 1 \}) < \gamma \mu(B)$$

and

$$\mu(\{ x \in B : f(F(x)) \geq s_B + C_1 + 1 \}) < \gamma \mu(B).$$

If we set $c_B = s_B + C_1/2$ and $\tau = 1 + C_1/2$, we obtain

$$\mu(\{ x \in B : |f(F(x)) - c_B| \geq \tau \}) \leq 2\gamma \mu(B).$$

The claim follows from Lemma 3.6.

We shall apply the Uchiyama construction in the proof of the following result.

**Proposition 3.9** ((iii) $\Rightarrow$ (i)). Let $F : X \to X$ be a BMO-map. Then there exist positive constants $K$ and $\beta$, depending only on the doubling constant $c_D$, such that the condition (i) of Theorem 3.1 holds with $\alpha = \beta/\|C_F\|$. 

**Proof.** Let $E_1$ and $E_2$ be $\mu$-measurable subsets in $X$ and let $\lambda > 0$ be such that

$$c_D^{-4\lambda} = \sup \min_{B} \frac{\mu(E_k \cap B)}{\mu(B)}.$$

By Theorem 2.1 there exist the functions $f_1$ and $f_2$, both in $\text{BMO}(X)$, such that $f_1 + f_2 = 1$, $0 \leq f_k \leq 1$, $f_k = 0$ on $E_k$, and $\|f_k\|_* \leq C_1/\lambda$ for $k = 1, 2$, where a positive constant $C_1$ depends on the doubling constant $c_D$. Define for $k = 1, 2$ the composed function $g_k = f_k \circ F$. Then $g_1 + g_2 = 1$, $0 \leq g_k \leq 1$, $g_k = 0$ on $F^{-1}(E_k)$, and $\|g_k\|_* \leq C_1\|C_F\|/\lambda$ for $k = 1, 2$.

Let us fix a ball $B$ in $X$. Clearly, we may assume that $(g_1)_B \geq 1/2$. Then by Lemma 2.4 we obtain

$$\frac{\mu(F^{-1}(E_1) \cap B)}{\mu(B)} \leq \frac{\mu(\{ x \in B : |g_1(x) - (g_1)_B| \geq 1/2 \})}{\mu(B)} \leq 2 \exp(-C\lambda/\|C_F\|),$$

where $C$ is a positive constant depending on the doubling constant $c_D$. By plugging in the value of $\lambda$, we obtain

$$\sup \min_{B} \frac{\mu(F^{-1}(E_k) \cap B)}{\mu(B)} \leq 2 \left( \sup \min_{B} \frac{\mu(E_k \cap B)}{\mu(B)} \right)^{C/\|C_F\|}$$

which completes the proof. \qed
### 3.1 $A_p$-weights and BMO-maps

We close this paper by discussing the connection between Muckenhoupt $A_p$-weights and BMO-maps.

It is well known that if $\omega$ is an $A_p$-weight for some $1 \leq p < \infty$, then $\log \omega \in \text{BMO}(X)$, and on the other hand, whenever $f \in \text{BMO}(X)$, then $e^{\delta f}$ is an $A_p$-weight for some $\delta > 0$ and $1 \leq p < \infty$. We refer to [5] for this result in the Euclidean setting. It straightforward to verify that the result has its counterpart also in metric measure spaces with a doubling measure.

We can add the following condition to the list in Theorem 3.1:

(iv) For each $A_p$-weight $\omega$, with some $1 \leq p < \infty$, the composed map $\omega^\delta \circ F$ is an $A_{p'}$-weight for some positive $\delta$ and $1 \leq p' < \infty$.

In Euclidean spaces, the condition (iv) can be stated in terms of $A_\infty$-weights, see [3, Corollary 3.3], and these weights have several but equivalent characterizations. In general metric spaces $A_\infty$-weights have first been defined and studied in [25]. In this generality, however, these different conditions are not necessarily equivalent. In particular, the class of $A_\infty$-weights can be strictly larger than the union of $A_p$-weights [25]. Several characterizations for $A_\infty$-weights and their relations in doubling metric measure spaces have also been studied in [15].

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