Vortex driven phase transition in Topologically Massive QED

Yuichi Hoshino

Kushiro National College of Technology,

Otanoshike Nishi-2-32-1, Kushiro City, Hokkaido 084-0916, Japan

Abstract

There is chiral like symmetry for 4-component massless fermion in (2+1)-dimensional gauge theory. Since QED$_3$ with Chern-Simons term contains vortex solution for vector potential, one may expect vortex driven phase transition as Kosterlitz-Thouless type where chiral condensate is washed away at zero temperature. To study this possibility, we evaluate the fermion propagator by Dyson-Schwinger equation numerically and spectral function analytically in the Landau gauge. For quenched case we adopt Ball-Chiu vertex to keep gauge invariance of the results. The critical value of topological mass, above which chiral condensate washed away, turned out to be $O(10^{-2})e^2$ at least for weak coupling in both cases.
I. INTRODUCTION

Dynamical effects caused by Chern-Simons terms have been known in various places. For example, it is well-known to explain the quantum Hall effects, so-called statistics transmutation [1]. On the other hand, vortex has an important role to wash away long-range order in XY model or condensate of (2+1)-dimensional model for superfluid helium film [2, 3]. These are known as Kosterlitz-Thouless transition. However, its detailed dynamics...
have not been known. It is an interesting problem to study the dynamics of the above phase transition. In this sense QED$_3$ with Chern-Simons term gives us an example to show an dynamical effects of vortex on chiral condensate. QED$_3$ with Chern-Simons term has a vortex solution for vector potential by solving Maxwell equation with charged particle [4]. Therefore the situation is similar to the isolated vortex inside superfluid at high temperature. We can examine the phase transition by solving Dyson-Schwinger equation for the fermion self-energy for chiral condensate and its destruction by vortices. In this model Chern-Simons term is absorbed to parity odd part of the gauge boson propagator and gauge boson acquires a mass. Pure QED part of gauge boson contributes to the condensation of $\epsilon \overline{\epsilon}$, while the latter may wash away the condensate at zero temperature. These are the main goals of our analyses. In 4-dimensional representation of spinor we have chiral symmetry for massless fermion. If it breaks dynamically we have two kinds of mass as chiral symmetry breaking and parity violation. For infinitesimal value of the topological mass it has been pointed out by K.I.Kondo and P.Maris that chiral symmetry restores and parity violating phase remains within $1/N$ expansion and nonlocal gauge of Dyson-Schwinger equation [5], [6], [7]. Here we consider weak coupling and gauge covariant approximation which satisfy Ward-Takahashi relation at first. After that we examine the $1/N$ expansion in the Landau gauge. In the final section we evaluate the spectral function of the fermion propagator [8], [9]. This method is helpful to determine the infrared behaviour or one particle singularity of the propagator in the existence of massless boson as photon, which has been known in QED$_{3+1}$. Since QED$_{2+1}$ is super renormalizable we get a short distance behaviour of the fermion propagator too. As a result we show the effect of vortex on the lowest order spectral function of parity even scalar part of the propagator. So that we find the critical value of topological mass above which chiral condensate is washed away. Fortunately this value coincides with that obtained by numerical analysis of Dyson-Schwinger equations.

II. TOPOLOGICALLY MASSIVE QED

The Lagrangian density of Topologically Massive QED$_3$ is written [4].

\[
L = \bar{\psi} (i \gamma \cdot (\partial - ieA) - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\mu}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho - \frac{1}{2\eta} (\partial \cdot A)^2.
\] (1)
This system is characterized by equations of motion

\[ (i\gamma \cdot (\partial - ieA) - m)\psi(x) = 0, \]
\[ \partial_{\mu}F^{\mu\nu} + \frac{\mu}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = J^{\nu}. \]

Under the gauge transformation

\[ \psi \rightarrow e^{i\Omega}\psi, \]
\[ A_{\mu} \rightarrow A_{\mu} + \frac{1}{e}\partial_{\mu}\Omega, \]

Lagrangian is invariant but the gauge fixing term changes by a total derivative

\[ L_{g} \rightarrow L_{g} + \partial_{\alpha}(\frac{\mu}{4e}\epsilon^{\alpha\mu\nu}F_{\mu\nu}\Omega). \]

However action \( \int d^{3}xL(x) \) is gauge invariant. It is easily seen by partial integration.

### A. vortex solution

There exists a vortex solution of vector potential in Topologically Massive QED\(_{3} \) [4]. It has been known by solving equation of motion at large \( |x| \)

\[ \partial_{\mu}F^{\mu\nu} + \mu\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = J^{\nu}. \]

We separate the above equation into time and space components

\[ \nabla \cdot E - \mu B = \rho, \]
\[ \epsilon^{ij}\partial_{j}B - \frac{\partial E^{i}}{\partial t} + \mu\epsilon^{ij}E^{j} = J^{i}, \]

where \( B = -1/2\epsilon^{ij}F_{ij} = \epsilon^{ij}\partial_{i}A_{j}, E^{i} = F^{i0} \). Using dual field strength

\[ *F^{\mu} = \frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta}, \]
\[ F^{\mu\nu} = \epsilon^{\mu\nu\alpha}F_{\alpha}, \]

relativistic wave equation for electric and magnetic fields are derived and its solution is given. Here we review that part shortly. Equation of motion in dual field strength is written

\[ \partial_{\mu}\epsilon^{\mu\nu\gamma\delta}F_{\gamma} + \mu*F^{\nu} = J^{\nu}. \]
Multiply $-\epsilon_{\alpha\beta\nu}$ and taking trace we obtain

$$\partial_\mu(g_\alpha^\mu g_\beta^\gamma - g_\beta^\mu g_\alpha^\gamma)F_\gamma - \frac{\mu}{2}\epsilon_{\alpha\beta\nu}\epsilon^{\nu\alpha\beta}F_{ab} = -\epsilon_{\alpha\beta\nu}J^\nu,$$

(13)

and

$$\partial^aF_\beta - \partial^\beta F_a - \mu F_{a\beta} = -\epsilon_{a\beta\nu}J^\nu.\quad (14)$$

Taking divergence of the above equation, when the current is conserved $\partial^aF_a = 0$, we have

$$\Box^aF_\beta - \mu \partial_\alpha F_{a\beta} = -\epsilon_{a\beta\nu}\partial_\alpha J^\nu.$$  

(15)

Using equation of motion

$$\partial_\alpha F_{a\beta} = -\frac{\mu}{2}\epsilon_{\beta\gamma\delta}F^{\gamma\delta} + J_\beta$$

$$= -\frac{\mu}{2}\epsilon_{\beta\gamma\delta}\epsilon^{\gamma\delta\alpha*}F_\alpha + J_\beta$$

$$= -\frac{\mu}{2}2\delta_\alpha^*F_\alpha + J_\beta$$

$$= -\mu^*F_\beta + J_\beta,$$

(16)

twice, we get

$$\Box^*F_\beta - \mu \partial_\alpha F_{a\beta} = -\epsilon_{a\beta\nu}\partial_\alpha J^\nu.$$  

(15)

Solution of dual field strength is given

$$^*F_\mu = \frac{\mu J_\mu - \epsilon_{\mu\alpha\nu}\partial_\nu J_\alpha}{\Box + \mu^2},$$

(19)

where

$$\Box + \mu^2\Delta(x) = -i\delta^3(x),$$

(20)

$$\Delta(x) = \frac{-i}{\Box + \mu^2}\delta^3(x),$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ik\cdot x}}{\mu^2 - k^2 + i\epsilon} = \frac{e^{-\mu\sqrt{r^2 - t^2}}}{4\pi i \sqrt{r^2 - t^2} + i\epsilon}.$$  

(21)

In position space we have the solution of wave equation with source $J(y)$

$$^*F_\mu(x) = \int d^3y\Delta(x-y)(\mu J_\mu(y) - \epsilon_{\mu\alpha\nu}\partial^\alpha J_\nu(y))$$

$$= \int d^3y(\Delta(x-y)\mu g_{\mu\nu} + \epsilon_{\mu\alpha\nu}(\partial^\alpha\Delta(x-y))J_\nu(y)).$$

(22)

(23)
Here we neglect the homogeneous solution. We have shown here that both electric and magnetic fields are massive and short range which decreases exponentially \( \exp(-\mu r)/4\pi r \). Let us return to the equation of motion. Surface integral of first equation at large radius yields

\[
\int \nabla \cdot \mathbf{E} \, dS = \int E_n dr = 2\pi R E_n, \tag{24}
\]

\[
2\pi R E_n - \mu \int B dS = \int \rho dS = Q. \tag{25}
\]

Radial component of electric field \( E_n(R) \) decrease as \( \exp(-\mu R)/4\pi R \) for large \( R \). We can safely neglect the first term at spatial infinity. The \( -\int dr B \), is time-independent which follows from conservation of \( *F^\mu \), and proportional to total charge. Static solution of the vector potential is given for classical field

\[
-\mu \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int dS j^0(r) = Q, \tag{26}
\]

\[
-\mu \oint \mathbf{A} \cdot d\mathbf{l} = -\mu \oint \nabla \theta \cdot d\mathbf{l} = -2\pi \mu \theta = Q, \tag{27}
\]

\[
\mathbf{A}(x)|_{x|\to\infty} \rightarrow -\frac{Q}{2\pi \mu} \nabla \arctan\left(\frac{y}{x}\right). \tag{28}
\]

That is even though magnetic field is short range, magnetic potential is long range. Therefore we find that if QED action contains Chern-Simons term and charged particle, we have a vortex solution for vector potential as an Aharonov-Bohm effect. The above vortex solution with quantization of \( \mu \) is well known in quantum Hall systems which explains quantization of Hall conductance and the possibility of statistics transmutation. Hereafter we will examine possibility of wash away condensate of chiral order parameter in QED\(_{2+1}\) with massless four component fermion by the existence of vortex.

**B. Chiral symmetry and Ward-Takahashi relation**

Here we consider the dynamical effects as mass generation in the presence of Chern-Simons term. If the fermion is massless \( m = 0 \), L has \( U(2) \) symmetry generated by \( \{ I, \gamma_3, \gamma_5, \tau \}, \{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu\nu} \) for \( \mu, \nu = (0, 1, 2) \), \( \gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \gamma_{1,2} = -i \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \tau = -i[\gamma_3, \gamma_5]/2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \gamma_3 \)
and $\gamma_5$ act as chiral transformation
\[
\psi(x) \rightarrow e^{i\gamma_3 \theta_3} \psi(x), \\
\bar{\psi}(x) \rightarrow e^{i\gamma_5 \theta_5} \bar{\psi}(x).
\] (29)

Scalar density is mixed by above transformation
\[
\bar{\psi}(x)\psi(x) \rightarrow \cos(2\theta_{3,5}) \bar{\psi}(x)\psi(x) + i \sin(2\theta_{3,5}) \bar{\psi}(x)\gamma_{3,5} \psi(x), \\
\bar{\psi}(x)\tau\psi(x) \rightarrow \bar{\psi}(x)\tau\psi(x).
\] (30)

Dynamical mass generation breaks $U(2)$ symmetry down to $U(1)_I \times U(1)_\tau$ which is generated by $\{I, \tau\}$ \cite{10}. There exists discrete parity transformation.
\[
P\psi(t, x, y) P^{-1} \rightarrow i\gamma^1 \gamma^3 \psi(t, -x, y).
\] (31)

Parity even and odd mass are transformed
\[
P m_e \bar{\psi} \psi P^{-1} \rightarrow m_e \bar{\psi} \psi, \\
P m_o \bar{\psi} \tau \psi P^{-1} \rightarrow -m_o \bar{\psi} \tau \psi.
\] (32) (33)

Both of these mass terms are invariant under $U(1)_I \times U(1)_\tau$ transformation.
\[
\psi(x) \rightarrow e^{i\theta} \psi(x), \\
\bar{\psi}(x) \rightarrow e^{i\tau \theta} \bar{\psi}(x).
\] (34)

We find the eigenvalue of the free particle Hamiltonian
\[
H = \gamma^0 \left( \gamma^i p^i + m_e I + m_o \tau \right)
\] (35)
as $E^2 = p^2 + m_{\pm}^2$, $m_{\pm} = m_e \pm m_o$, where $i = 1, 2, I$ is a 4 $\times$ 4 unit matrix and $\tau$ is a operator defined above. Two kinds of mass are written by
\[
m_e I + m_o \tau = \begin{pmatrix} m_+ & 0 \\ 0 & m_- \end{pmatrix} = \begin{pmatrix} m_e + m_o & 0 \\ 0 & m_e - m_o \end{pmatrix}.
\] (36)

So that we may split 4-component spinor into upper and lower components by projection operator
\[
\psi_\pm = \chi_\pm \psi = \chi_\pm \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \chi_\pm = \frac{1 \pm \tau}{2}, \chi_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \chi_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\] (37)
From the Lagrangian

\[ L = \bar{\psi}_+(i\gamma \cdot \partial - m_+)\psi_+ + \bar{\psi}_-(i\gamma \cdot \partial - m_-)\psi_- \] (38)

we have the free propagator

\[ S_0(p) = -\frac{\gamma \cdot p + m_+}{p^2 - m_+^2} \chi_+ - \frac{\gamma \cdot p + m_-}{p^2 - m_-^2} \chi_- \] (39)

Since Chern-Simons term induces parity odd fermion mass, it is convenient to introduce parity odd mass from the beginnings and choose the basis of the eigenvalue of the Hamiltonian. Chiral representation is enough for this purposes. In our 4-component spinor representation parity violating mass \( \bar{\psi} \gamma \psi \) is a pseudo scalar and current \( J^{3,5}_\mu = \bar{\psi} \gamma_3 \gamma_5 \gamma \psi \) transform as an vector, which mix each other under parity and charge conjugation transformation with arbitrary phase. Starting from Dirac equation

\[ (\gamma \cdot \partial + m_0)\psi = ie\gamma \cdot A \psi, \bar{\psi}(\gamma \cdot \partial - m_0) = -ie\bar{\psi} \gamma \cdot A, \] (40)

combination of above two equation, we obtain

\[ \partial_\lambda J^{3,5}_\lambda = 2m_0 J^{3,5} = D, \] (41)

where

\[ J^{3,5}_\lambda = \bar{\psi} \gamma_{3,5} \gamma_\lambda \psi, J^{3,5} = \bar{\psi} \gamma_{3,5} \psi. \] (42)

A more general proof by Ward-Takahashi-relation of the vector current,

\[ \partial_\mu T(J_{3,5\mu}(x)\psi(y)\bar{\psi}(z)) = D + \delta^{(3)}(x - y)\delta(y)\bar{\psi}(z) + \delta^{(3)}(x - z)\psi(y)\bar{\psi}(z) \]

\[ = D + \gamma_{3,5} S_F(x - z)\delta^{(3)}(x - y) + \delta^{(3)}(x - z)S_F(y - z)\gamma_{3,5} \] (43)

by operator of symmetry transformation,

\[ \delta(x_0 - y_0)[J_{3,5\mu}(x), \psi(y)] = \delta \psi(y)\delta^{(3)}(x - y) = \gamma_{3,5} \psi(y)\delta^{(3)}(x - y), \] (44)

\[ \delta(x_0 - y_0)[J_{3,5\mu}(x), \bar{\psi}(y)] = \delta \bar{\psi}(y)\delta^{(3)}(x - y) = \bar{\psi}(y)\gamma_{3,5} \delta^{(3)}(x - y). \] (45)

In terms of the vertex function it is written

\[ (p - q)_\mu \Gamma_{3,5\mu}(p, q) = 2m_0 \Gamma_{3,5}(p, q) + \gamma_{3,5} S_F^{-1}(q) + S_F^{-1}(p)\gamma_{3,5}. \] (46)
\[ \Gamma_{3,5\mu}(p, q) = \int d^3 x d^3 y e^{i(p\cdot y - q\cdot x)} \langle T J_{3,5\mu}(0) \psi(y) \overline{\psi}(x) \rangle_T, \]
\[ \Gamma_{3,5}(p, q) = \int d^3 x d^3 y e^{i(p\cdot y - q\cdot x)} \langle T J_{3,5}(0) \psi(y) \overline{\psi}(x) \rangle_T, \]

where the subscript \( T \) means truncation of the fermion propagator. For the proof of Ward-Takahashi relation in terms of integral equation for \( \Gamma_{3,5,\mu,5\mu} \), see [11]. For vanishing bare mass \( m_0 \), current conservation is spontaneously broken by parity even mass generation

\[ \lim_{p\to q} (p - q)^\mu \Gamma_{3,5\mu}(p, q) = \{ \gamma_{3,5}, S_F^{-1}(p) \} \neq 0. \]  

(47)

In other form it is well known

\[ \partial_\mu T(J_{5\mu}(x) \overline{\psi}(y) \gamma_5 \psi(y)) = -2\overline{\psi}(x) \psi(x). \]  

(48)

Taking vacuum expectation value of both sides of equation, non-vanishing of the r.h.s indicates the existence of massless scalar NG boson in the vertex \( \Gamma_{3,5\mu}(p, q) \), of the form

\[ \frac{i f_\pi(p - q)\mu X_\pi(P = 0, p - q)}{(p - q)^2}, \]  

(49)

where \( X_\pi \) is a Bethe-Salpeter amplitude for total momentum \( P = 0 \). It is equal to scalar part of propagator by Ward-Takahashi relation. If the right hand side is finite and we have two order parameter \( \langle \overline{\psi} \psi \rangle, \langle \overline{\psi} \tau \psi \rangle \) of chiral symmetry breaking and parity violation. Dyson-Schwinger equation is familiar to obtain non-perturbative propagator and dynamical mass. Therefore we set up the Dyson-Schwinger equation for the electron propagator in this representation.

### III. DYSON-SCHWINGER EQUATION

#### A. quenched case

Propagator of fermion and gauge boson are given

\[ S(p) = \frac{i}{A(p) \gamma \cdot (p - B(p))} \rightarrow \frac{i}{p^2 A_\mu^2(p) - B^2_\mu(p)} \chi_+ + \frac{i}{p^2 A_{-\mu}^2(p) - B_{-\mu}^2(p)} \chi_-, \]  

(50)

\[ D_{\mu\nu}(k) = \frac{1}{i} \left[ \frac{g_{\mu\nu} - k_\mu k_\nu / (k^2 + i\epsilon) - i\mu\nu_{\rho\sigma} k_\rho / (k^2 + i\epsilon)}{k^2 - \mu^2 + i\epsilon} + \xi \frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2} \right]. \]  

(51)

The quenched Schwinger-Dyson equation for the self-energy \( \Sigma(p) \) is written

\[ -i\Sigma(p) = ((-i\epsilon)^2 \int \frac{d^3 q}{(2\pi)^3} \gamma_\mu S(q) \Gamma_\nu(p, q) D_{\mu\nu}(k), \]  

(52)

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where \( k = q - p \). In this normalization \( \Sigma \) is real in the region \( p^2 < 0 \). We separate the vector and scalar part of the propagator by taking the trace of \( \Sigma \). Trace formulae for \( (4 \times 4) \) \( \gamma \) matrices are reduced to \((2 \times 2)\) ones with projection operator \( \chi_\pm \)

\[
tr(\chi_\pm) = 2, \quad tr(\gamma_\gamma \gamma_\nu \chi_\pm) = 2g_{\mu\nu}, \quad tr(\gamma_\mu \gamma_\nu \gamma_\rho \chi_\pm) = \mp 2i\epsilon_{\mu\nu\rho}.
\]

In Chiral representation parity violating mass and vector part are different sign for up and down component. We have the coupled integral equation for \( A_\pm(p) \) and \( B_\pm(p) \) in the Landau gauge by taking trace

\[
i(S^{-1} - S_0^{-1}) = ((A(p) - 1)\gamma \cdot p - B(p)) = i\Sigma(p),
\]

\[
2(A(p) - 1)p^2 = tr(i\gamma \cdot p\Sigma(p)), -2B(p) = tr(\Sigma(p)).
\]

To keep gauge invariance of the action we adopt the gauge covariant approximation by introducing Ball-Chiu vertex ansatz for longitudinal part[9].

\[
\Gamma_{\mu}^{BC}(p, q) = \Gamma_{\mu}^{T}(p, q) + \frac{A(p) + A(q)}{2}\gamma_\mu + \frac{A(p) - A(q)}{2(p^2 - q^2)}\gamma \cdot (p + q)(p + q)_\mu
\]

\[
- \frac{B(p) - B(q)}{p^2 - q^2}(p + q)_\mu,
\]

\[
(p - q)_\mu \Gamma_{\mu}^{BC}(p, q) = i(S^{-1}(q) - S^{-1}(p))
\]

\[
= A(p)\gamma \cdot p - B(p) - (A(q)\gamma \cdot q - B(q)).
\]

In this case the Dyson-Schwinger equations are following equations in the Landau gauge

\[
B(p)_\pm = \frac{e^2}{4\pi^2} \int \frac{d^3q}{q^2A(q)\pm + B(q)\pm^2(k^2 + \mu^2)}[(A(p)_\pm + A(q)_\pm)(B(q)_\pm \mp \mu A(q)_\pm)\left(\frac{1}{2} - \frac{p^2 - q^2}{2k^2}\right)]
\]

\[
+ \{\Delta A_\pm(B(q)_\pm \mp \frac{\mu}{2}A(q)_\pm) - \Delta B_\pm A(q)_\pm\}((p^2 + q^2) - \frac{k^2}{2} - \frac{(p^2 - q^2)^2}{2k^2})],
\]

\[
A(p)_\pm = 1 + \frac{e^2}{4\pi^2p^2} \int \frac{d^3q}{q^2A(q)\pm + B(q)\pm^2}(\frac{\mu}{2}B(q)_\pm - A(q)_\pm)\left(\frac{1}{2} + \frac{p^2 - q^2}{2k^2}\right)
\]

\[
+ A(q)_\pm\left(\frac{(p^2 - q^2)}{4k^2} - \frac{k^2}{4}\right) + \{\Delta A_\pm(\frac{\mu}{2}B(q)_\pm - A(q)_\pm)\left(\frac{1}{2} + \frac{p^2 - q^2}{2k^2}\right)
\]

\[- \Delta B_\pm(\mp \mu A(q)_\pm + B(q)_\pm)\}((p^2 + q^2) - \frac{k^2}{2} - \frac{(p^2 - q^2)^2}{2k^2})].
\]
where \( k = q - p \),
\[
\Delta A = \frac{A(p) - A(q)}{p^2 - q^2}, \quad \Delta B = \frac{B(p) - B(q)}{p^2 - q^2}.
\] (62)

Angular integral formulae are given in Appendices A.

For bare vertex we set \( \Gamma_\mu(p, q) = \gamma_\mu \). Recently we obtained the numerical solutions with this approximation[6,7]. It is said that the vortex destroys condensate in condensed matter physics. At bare vertex or inclusion the BC vertex we find \( \mu_{cr} \approx 0.01e^2 \) which is the same order of magnitude with Raya et.al[6] in FIG.1. Simple vertex correction as \( \gamma_\mu \rightarrow A(p)\gamma_\mu \) yields \( \mu_{cr} = 8 \cdot 10^{-3}e^2 \).

Numerically some instability arise from \( \Delta BB(q) \) term in \( A \) equation. This term is mass changing and compete the reduction of mass by parity violating mass term. In pure QED this term is sensitive for low momentum and infrared cut-off of momentum integration. The effects on vacuum expectation value is relatively large. If we demand that the mass \( B \) vanishes at the transition point and neglect this term, we find the clear transition as the bare vertex case with the same critical point. We can determine it by introducing renormalization of the topological mass as
\[
\mu_R = \mu + \frac{e^2}{2\pi} \theta(\mu - \mu_{cr}),
\] (63)
where \( \mu_{cr} \) is determined self-consistently in FIG.2. By this analysis critical point \( \mu_{cr} \) in the bare vertex case turned to be the true one. Renormalization of topological mass is shown in Appendix. At least for low energy we can safely neglect \( B\Delta B \) terms and find clear phase

FIG. 1: chiral order parameter \( \langle \bar{\psi}\psi \rangle \) for bare vertex with \( \mu = 10^{-2.3+0.1n}e^2, n = 0..7 \).
FIG. 2: chiral order parameter $\langle \psi \psi \rangle$ for $\mu = (10^{-2.3+0.1n} + \theta(n - 3)/2\pi)e^2$ with BC vertex

FIG. 3: parity violating order parameter $\langle \bar{\psi}\tau\psi \rangle$ for $\mu = (10^{-2.3+0.1n} + \theta(n - 3)/2\pi)e^2$ with BC vertex.

transition. In FIG. 3 we see the parity violating order parameter $\langle \bar{\psi}\tau\psi \rangle$ as a function of topological mass. Near the critical point $\mu_{cr}, \langle \bar{\psi}\tau\psi \rangle$ is lower than $\langle \bar{\psi}\psi \rangle$ which is shown in Fig. 4 and first order phase transition occurs.
FIG. 4: Parity even and odd order parameter $\langle \psi \psi \rangle, \langle \bar{\psi} \tau \psi \rangle$ for $\mu = (10^{-2.3+0.1n} + \theta(n - 3)/2\pi)e^2$ with BC vertex.

FIG. 5: $\langle \psi \psi \rangle$ for bare vertex with one massless flavour loop correction. $\mu = (0.008 + 0.0005 \cdot (n - 1))e^2$.

B. effects of vacuum polarization

Using free vector propagator,

$$D^0_{\mu\nu}(p) = \frac{-i(g_{\mu\nu} - p_{\mu}p_{\nu}/p^2 - i\mu\epsilon_{\mu\nu\alpha}p^\alpha/p^2)}{p^2 - \mu^2 + i\epsilon} - i\xi p_{\mu}p_{\nu}/p^4,$$  \hspace{1cm} (64)
FIG. 6: $\langle \bar{\psi} \psi \rangle$ for bare vertex with two massless flavour loop correction with $\mu = (0.005 + 0.0005 \cdot (n - 1))e^2$.

we shall calculate one-loop corrections to the vector propagator

$$(D'^{-1})_{\mu\nu} = (D_0^{-1})_{\mu\nu} - i\Pi_{\mu\nu}, \quad (65)$$

and set up the Dyson-Schwinger equation with bare vertex

$$S^{-1} = S_0^{-1} + i\Sigma,$$

$$\Sigma(p) = -ie^2 \int \frac{d^3k}{(2\pi)^3} \gamma^\mu S(p + k)\gamma^\nu D'_{\mu\nu}(k). \quad (66)$$

One-loop vacuum polarization is evaluated for fermion mass $m_e \neq 0, m_o = 0$ with Pauli-Villars or dimensional reguralization to remove gauge noninvariant cut-off dependent term and Parity-odd Chern-Simons term is not induced here. See appendix and [4].

$$\Pi_{\mu\nu}(k) \equiv ie^2 N \int d^3p Tr(\gamma_\mu \gamma_\nu \frac{1}{p - m} \gamma_\gamma \frac{1}{(p - k) - m}),$$

$$= -e^2 N T_{\mu\nu}^\mu \frac{1}{8\pi} [\sqrt{k^2} + \frac{4m^2}{\sqrt{k^2}} \ln(\frac{2m + \sqrt{k^2}}{2m - \sqrt{k^2}}) - 4m], \quad (67)$$

$$T_{\mu\nu} = (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}), \quad d^3p = \frac{d^3p}{(2\pi)^3}, \quad (68)$$
Polarization function $\Pi(k)$ is

$$
\Pi(k) = -\frac{e^2}{8\pi} N[(\sqrt{k^2} + 4m^2) \ln(\frac{2m + \sqrt{k^2}}{2m - \sqrt{k^2}}) - 4m],
$$

$$
= -\frac{e^2}{8} N\sqrt{-k^2}(k^2 > 0, m = 0),
$$

$$
= -\frac{e^2}{6\pi m} k^2 + O(k^4/(k^2/m \ll 1)). \tag{69}
$$

For $m = 0$ case

$$
\Pi_{\mu\nu}(k) = -\frac{e^2 N}{8} (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \sqrt{-k^2}, \tag{70}
$$

gives a non-perturbative correction to the vector propagator for low energy in the chiral symmetric phase. From the relation

$$
(D^{-1})_{\mu\nu} D_{\nu\rho} = g_{\mu\rho}, \tag{71}
$$

we have the inverse of the bare photon propagator

$$
(D^{-1})_{\mu\nu} = (p^2 g_{\mu\nu} - p_\mu p_\nu + i\mu\epsilon_{\mu\nu\alpha\rho}p^\alpha) + i\frac{p_\mu p_\nu}{\xi}. \tag{72}
$$

Adding the vacuum polarization of massless loop

$$
(D^{-1})_{\mu\nu} = i(p^2 g_{\mu\nu} - p_\mu p_\nu + i\mu\epsilon_{\mu\nu\alpha\rho}p^\alpha) + i\frac{p_\mu p_\nu}{\xi}
$$

$$
= i(p^2 + \frac{e^2 N}{8} \sqrt{-p^2})(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) + i\mu\epsilon_{\mu\nu\alpha\rho}p^\alpha] + i\frac{p_\mu p_\nu}{\xi}, \tag{73}
$$

we obtain the full propagator

$$
D'_{\mu\nu} = -i\frac{(p^2 - \pi(p^2))(g_{\mu\nu} - p_\mu p_\nu/p^2) - i\mu\epsilon_{\mu\nu\alpha\rho}p^\rho}{(p^2 - \pi(p^2))^2 - \mu^2 p^2} - i\frac{p_\mu p_\nu}{p^4}. \tag{74}
$$

In Euclid space it has the form

$$
D'_{\mu\nu} = -i\frac{A(-p^2)(\delta_{\mu\nu} - p_\mu p_\nu/p^2) - \mu\epsilon_{\mu\nu\alpha\rho}p^\rho}{A(-p^2)^2 + \mu^2 p^2} - i\frac{p_\mu p_\nu}{p^4}, \tag{75}
$$

where

$$
A(-p^2) = (p^2 + \frac{e^2 N}{8} \sqrt{-p^2}). \tag{76}
$$

Including vacuum polarization for photon the Dyson-Schwinger equation has the following form

$$
\Sigma(p^2) = e^2 \int \frac{d^3 q}{(2\pi)^3} \gamma_\mu S(q) \gamma_\nu D'_{\mu\nu}(k). \tag{77}
$$
In Euclid space with $\xi = 0$ gauge, we have

$$B_{\pm}(p) = \frac{e^2}{4\pi^3} \int d^3q \left[ \frac{(k^2 + \pi(k^2))B_{\pm}(q)}{[A_{\pm}^2(q)q^2 + B_{\pm}^2(q)](k^2 + \pi(k^2))^2 + \mu^2k^2} \right] \mp \frac{\mu A_{\pm}(q)(q \cdot k)}{[A_{\pm}^2(q)q^2 + B_{\pm}(q)](k^2 + \pi(k^2))^2 + \mu^2k^2]}, \quad (78)$$

$$A_{\pm}(p) = 1 + \frac{e^2}{4\pi^3p^2} \int d^3q \left[ \frac{(k^2 + \pi(k^2))A_{\pm}(q)(p \cdot k)(q \cdot k)}{[A_{\pm}^2(q)q^2 + B_{\pm}^2(q)](k^2 + \pi(k^2))^2 + \mu^2k^2} \right] \mp \frac{\mu B_{\pm}(q)(p \cdot k)}{[A_{\pm}^2(q)q^2 + B_{\pm}^2(q)](k^2 + \pi(k^2))^2 + \mu^2k^2}]. \quad (79)$$

Here we apply the Ball-Chiu vertex as in the quenched case. Kondo and Maris applied $1/N$ approximation to solve the equation. However in the gauge covariant approximation which satisfy Ward-Takahashi relation has been shown that chiral order parameter in quenched and unquenched case are same at $N = 1$ for weak coupling for $\mu = 0$ case. So we choose covariant gauge and weak coupling. It is easy to take BC vertex and improve the Dyson-Schwinger equation as in the quenched case. But we have not enough memories to run PC. So that we only show the massless loop correction in the Landau gauge case. In this case critical value $\mu_{cr}$ in unquenched case is approximately the same value for quenched case. For example we find numerically $\mu_{cr} \sim 0.01e^2$ for $N = 1$ and $\mu_{cr} \sim 0.008e^2$ for $N = 2$. At the critical point we find that $m_e = m_o$, $B_{\pm}(p) = 0$ and $m_e$ vanishes above the critical point. This is the destruction mechanism of superfluidity by vortex in our model. For $N \geq 2$ case we have very small values of order parameter for small topological mass. For large topological mass order parameter changes its sign at some value of topological mass. There may be a strong coupling phase for $N \geq 2$ at least for small topological mass, where vacuum expectation value $\langle \bar{\psi}\psi \rangle$ vanishes. For these cases $1/N$ expansion may be a good way to study the phase structure for strong coupling region in our model. A famous critical number of flavour which was derived in the linearized Schwinger-Dyson equation has been know as $N_c = 32/\pi^2$ above which the chiral symmetry is restored. The results of $1/N$ expansion and the phase structure derived by KI.Kondo and P.Maris may be realized[5].

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IV. ANALYSIS BY SPECTRAL FUNCTION

A. definition of spectral function

In this section we would like to determine critical value of topological mass above which chiral condensate is washed away theoretically. First we notice that the spectral function for the propagator as one of the possibility citeOPS. In three dimension, absence of ultraviolet divergences is important. If we know only infrared behaviour or the leading logarithm of infrared divergence near the mass shell, it is possible to determine the whole region of the propagator in position space by the anomalous dimension, which is supplied by lowest ordered spectral function. By this method we find that only short distance behaviour of the propagator is modified and we have a finite chiral condensate for pure QED\textsubscript{2+1} [9]. If we choose soft-photon exponentiation to include all orders of soft-photon emission by electron, its spectral function may be written as $e^F$, where $F$ is a model independent spectral function of the lowest order in the coupling constant for pure QED\textsubscript{2+1}

$$\rho(x) = e^{F(|x|)},$$

$$S_F(x) = S^0_F(x)\rho(x).$$ (81)

Here we consider the fermion spectral function. The vacuum expectation value of the anticommutator has the form [13]

$$iS'(x, y) = \langle 0 | \{ \psi(x), \overline{\psi}(y) \} | 0 \rangle$$

$$= \sum_n [\langle 0 | \psi(0) | n \rangle \langle n | \overline{\psi}(0) | 0 \rangle e^{-ip_n \cdot (x - y)} + \langle 0 | \overline{\psi}(0) | n \rangle \langle n | \psi(0) | 0 \rangle e^{ip_n \cdot (x - y)}].$$ (82)

We introduce the spectral amplitude by grouping together in the sum over $n$ all states of given three-momentum $q$

$$\rho_{\alpha\beta}(q) = (2\pi)^2 \sum_n \delta^{(3)}(p_n - q) \langle 0 | \psi_\alpha(0) | n \rangle \langle n | \overline{\psi}_\beta(0) | 0 \rangle$$ (83)

and set out to construct its general form from invariance arguments. $\rho(q)$ is a $4 \times 4$ matrix and may be expanded in terms of 16 linearly independent products of $\gamma$ matrices. Under the assumptions of Lorentz invariance and Parity transformation it reduces to the form

$$\rho(q)_{\alpha\beta} = \rho_1(q)\gamma \cdot q + \rho_2(q)\delta_{\alpha\beta}.$$ (84)
Second term in (82) can be related directly to (83) with the aid of PCT invariance of the vacuum [13, 14]. Parity, Charge conjugation and Time reversal transformation are defined in our representation of \( \gamma \) matrices

\[
P\psi(t, x, y)P^{-1} = i\gamma^1\gamma^3\psi(t, -x, y),
\]

\[
PA^0(t, x, y)P^{-1} = A^0(t, -x, y),
\]

\[
PA^1(t, x, y)P^{-1} = -A^1(t, -x, y),
\]

\[
PA^2(t, x, y)P^{-1} = A^2(t, -x, y),
\]

\[
P\overline{\psi}(t, x, y)\gamma^1\psi(t, x, y)P^{-1} = -\overline{\psi}(t, -x, y)\gamma^1\psi(t, -x, y)
\]

\[
C\psi(x)C^{-1} = C\gamma^0\psi^* = C\overline{\psi}^T = \psi_c,
\]

\[
C = \gamma^2, C^{-1}\gamma^\mu C = -\gamma^\mu T,
\]

\[
CA^\mu(t, x, y)C^{-1} = -A^\mu(t, x, y),
\]

\[
T\psi_\alpha(t, x, y)T^{-1} = T_{\alpha\beta}\psi_\beta(-t, x, y), T = -i\gamma^2\gamma^3,
\]

\[
TA^0(t, x, y)T^{-1} = A^0(-t, x, y),
\]

\[
TA^{1,2}(t, x, y)T^{-1} = -A^{1,2}(-t, x, y).
\]

Effects of PCT transformation on \( \overline{\psi}(y)\psi(x) \) is

\[
PCT\psi_\alpha^A(t, x, y)T^{-1}C^{-1}P^{-1} = -\gamma^1_{\gamma\alpha}\overline{\psi}^\gamma(t, -x, y),
\]

\[
PCT\psi_\beta^B(t, x, y)T^{-1}C^{-1}P^{-1} = -\gamma^1_{\lambda\beta}\psi_\lambda(t, -x, y),
\]

\[
PCT\overline{\psi}_\alpha^A(t', x', y')\psi_\beta^B(t, x, y)T^{-1}C^{-1}P^{-1} = \gamma^1_{\alpha\gamma}\psi_\gamma^B(t, -x, y)\overline{\psi}_\lambda^A(t', -x', y')\gamma^1_{\lambda\beta}.
\]

Inserting (98) along with (83) into (82) and using \( \gamma^{2T} = -\gamma^2 \), we obtain finally

\[
is^\alpha_{\alpha\beta}(x - y) = \int \frac{d^3q}{(2\pi)^3}\theta(q_0)[(\gamma \cdot q\rho_1(q^2) + \rho_2(q^2))^2_{\alpha\beta}e^{-iq(x-y)}
\]

\[
+ \{\gamma^1(\gamma \cdot q\rho_1(q^2) + \rho_2(q^2))^1_{\alpha\beta}e^{iq(x'-y')}
\]

\[
= \int \frac{d^3q}{(2\pi)^3}\theta(q_0)[\rho_1(q^2)i\gamma \cdot \partial_x + \rho_2(q^2)]_{\alpha\beta}(e^{-iq(x-y)} - e^{iq(x'-y')})
\]

\[
= \int \frac{d^3q}{(2\pi)^3}[\theta(q_0)\gamma \cdot q\rho_1(q^2) + \epsilon(q_0)\rho_2(q^2)]_{\alpha\beta}e^{-iq(x-y)},
\]
where \( x' = (-t, -x_1, x_2), (\gamma^\dagger)^2 = -1 \). Since \( \rho \) vanishes for space-like \( q^2 \), we may also write this as an integral over mass spectrum by introducing

\[
\rho(q^2) = \int_0^\infty \rho(s) \delta(q^2 - s) ds. \tag{100}
\]

We find

\[
iS'(x - y) = - \int ds [i\rho_1(s) \gamma \cdot \partial + \rho_2(s)] i\Delta(x - y; \sqrt{s})
= \int ds \{ \rho_1(s) iS(x - y; \sqrt{s}) + [\sqrt{s}\rho_1(s) - \rho_2(s)] i\Delta(x - y; \sqrt{s}) \} \tag{101}
\]

where invariant \( \Delta \) function is given

\[
i\Delta'(x, y) \equiv \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \tag{102}
= \sum_n \langle 0 | \phi(0) | n \rangle \langle n | \phi(0) | 0 \rangle (e^{-iP_n(x-y)} - e^{iP_n(x-y)})
= \frac{1}{(2\pi)^2} \int d^3q \rho(q^2) \theta(q_0) (e^{-iq(x-y)} - e^{iq(x-y)}) \tag{103}
= \frac{1}{(2\pi)^2} \int_0^\infty ds \rho(s) \int d^3q \delta(q^2 - s) \epsilon(q_0) e^{-iq(x-y)}
= \int_0^\infty ds \rho(s) i\Delta(x - y, \sqrt{s}). \tag{104}
\]

The above spectral representation goes through unchanged for the vacuum expectation value of the time-ordered product of Dirac field; it is necessary only to replace the \( iS \) and \( i\Delta \) by the Feynman propagator \( S_F \) and \( \Delta_F \). If we know the matrix element \( \langle 0 | \psi_\alpha(0) | n \rangle \), we can determine the spectral function \( \rho_1 \) and \( \rho_2 \). Perturbative \( O(\alpha^2) \) spectral function can be obtained by the usual definition

\[
\rho^{(2)}(p^2) = \int \frac{d^3x e^{-ip \cdot x}}{(2\pi)^3} \int \frac{d^3r}{(2\pi)^2} e^{ir \cdot x} \int \frac{d^2k}{(2\pi)^2} e^{ik \cdot x} \langle 0 | \psi(0) | r, k \rangle \langle r, k | \bar{\psi}(0) | 0 \rangle
= \int \frac{d^3x e^{-ip \cdot x}}{(2\pi)^3} \int \frac{d^3r}{(2\pi)^2} e^{ir \cdot x} \int \frac{d^2k}{(2\pi)^2} e^{ik \cdot x} \sum_{\lambda, \sigma} T_1 \mathcal{T}_1. \tag{105}
\]

In perturbation theory one photon emission matrix element is given

\[
T_1 = \langle 0 | \psi(0) | r, k \rangle \simeq \langle 0^\text{in} | T[U(\infty, -\infty) \psi^\text{in}(0)] | r; \text{in} \rangle
= -i \int 0^\text{in} | T[\psi^\text{in}(0), e \int d^3x \bar{\psi}^\text{in}(x) \gamma_\mu \psi^\text{in}(x) A^\text{in}_\mu(x)] | r; \text{in} \rangle \tag{106}
= -i e \int d^3x S^0_F(0 - x) \gamma_\mu \langle 0 | \psi^\text{in}(x) | r \rangle \langle 0 | A^\text{in}_\mu(x) | k \rangle
= \frac{-ie}{(r + k) - m + i\epsilon} \gamma_\mu \epsilon^{\mu}_S(k) U_S(r) \sqrt{m/E_r} \frac{1}{\sqrt{2k_0}}. \tag{107}
\]
For the evaluation of \( \rho(p^2) \), if we integrate \( x \) first, we obtain \( \delta^{(3)}(k + r - p) \) for energy-momentum conservation. In our case first we integrate \( k \). After that we exponentiate the function \( F \) and integrate \( r \) in the non-perturbative case. At that stage the results are position dependent. Finally we obtain the spectral function in position space \( \rho(x) \) for infinite number of photon emission

\[
\rho(x) = \int \frac{d^2re^{ir\cdot x}}{(2\pi)^2} \times \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int \frac{d^2k}{(2\pi)^2} \theta(k_0) \delta(k^2) e^{ik\cdot x} \sum_{\lambda} \delta^{(3)}(p - r - \sum_{i=1}^{\infty} k_i) T_n \bar{T}_n, \right)
\]

(108)

where the notation \( f(k)_0 = 1, f(k)_n = \prod_{i=1}^{n} f(k_i) \) is used and \( T_n \bar{T}_n \) may be replaced to \((T_1 \bar{T}_1)^n\) in our approximation. The polarization sum for gauge boson is given

\[
\sum_{\lambda} e^{\mu}_\lambda(k) e^{\nu}_\lambda(k) = -[g^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} - i\mu \epsilon^{\mu\nu\rho} \frac{k^\rho}{k^2}].
\]

(109)

Since we do not fix the numbers of photon \( n \) we sum up from \( n = 0 \) equals zero to infinity. The function

\[
F = \int \frac{d^2k}{(2\pi)^2} \sum_{\lambda,S} T_1 \bar{T}_1 e^{ik\cdot x},
\]

(110)

is a phase space integral of one photon intermediate state which may be exponentiated as \( e^F \). We use the trace formula to get

\[
\sum_{\lambda,S} T_1 \bar{T}_1 = \gamma \cdot r A(r) + B(r).
\]

(111)

\[
A(r) = \frac{1}{4m^2} Tr(r \cdot \gamma \sum_{\lambda,S} T_1 \bar{T}_1),
\]

\[
B(r) = \frac{1}{4} Tr(\sum_{\lambda,S} T_1 \bar{T}_1).
\]

(112)

To order \( e^2 \) we have model independent spectral function for 4-component fermion

\[
F(\mu|x|) = -e^2 \int \frac{d^3k}{(2\pi)^2} e^{-ik\cdot x} \theta(k_0) \frac{\gamma \cdot p + m}{2m} \frac{1}{(r \cdot k)^2} \delta(k^2 - \mu^2) \frac{m^2}{(r \cdot k)^2} + \frac{1}{r \cdot k} - (\xi - 1) \frac{\partial}{\partial k^2} \delta(k^2 - \mu^2)].
\]

(113)

Hereafter we keep in mind the projection operator of positive energy \((\gamma \cdot p + m)/2m\) for the propagator. From this form \( \rho_2 = m \rho_1 \) in our approximation. To obtain the explicit form of
the function $F$, we use parameter trick

$$\lim_{\epsilon \to 0} \int_0^\infty d\alpha e^{-\alpha(\epsilon - i k \cdot r)} = \frac{i}{k \cdot r},$$  \hspace{1cm} (114)$$

$$\lim_{\epsilon \to 0} \int_0^\infty d\alpha e^{-\alpha(\epsilon - i k \cdot r)} = -\frac{1}{(k \cdot r)^2},$$  \hspace{1cm} (115)$$

combined with positive frequency part of the propagator with bare mass $\mu$

$$D^+(x) = \int \frac{d^3k}{i(2\pi)^2} \theta(k^0)\delta(k^2 - \mu^2)e^{ik \cdot x}$$

$$= \frac{1}{i(2\pi)^2} \int_0^\infty \frac{2\pi kdJ_0(k|x|)}{2\sqrt{k^2 + \mu^2}} = \frac{e^{-\mu|x|}}{4\pi i |x|},$$  \hspace{1cm} (116)$$

we obtain the function $F$ symbolically as

$$F = ie^2m^2 \int_0^\infty d\alpha D^+(x + \alpha r) - e^2 \int_0^\infty d\alpha D^+(x + \alpha r) - ie^2(\xi - 1) \frac{\partial}{\partial \mu^2} D^+(x, \mu^2)$$

$$= ie^2m^2F_1(x) - e^2F_2(x) + e^2(\xi - 1)F_L.$$  \hspace{1cm} (117)$$

We get each terms

$$F_2 = \int \frac{d^3k}{(2\pi)^2} \theta(k^0)\delta(k^2 - \mu^2)e^{ik \cdot x} \frac{1}{r \cdot k}$$

$$= \lim_{\epsilon \to 0} \int_0^\infty d\alpha e^{-\alpha(\epsilon - i k \cdot r)} \int \frac{d^3k}{i(2\pi)^2} \theta(k^0)\delta(k^2 - \mu^2)e^{ik \cdot x}$$

$$= \int_0^\infty d\alpha D^+(x + \alpha r) = \frac{E_1(\mu|x|)}{4\pi m},$$  \hspace{1cm} (118)$$

$$F_1 = -\int \frac{d^3k}{(2\pi)^2} \theta(k^0)\delta(k^2 - \mu^2)e^{ik \cdot x} \frac{1}{(r \cdot k)^2}$$

$$= \lim_{\epsilon \to 0} \int_0^\infty d\alpha e^{-\alpha(\epsilon - i k \cdot r)} \frac{d^3k}{(2\pi)^2} \theta(k^0)\delta(k^2 - \mu^2)e^{ik \cdot x}$$

$$= \int_0^\infty d\alpha D^+(x + \alpha r) = \frac{\exp(-\mu|x|) - \mu|x|E_1(\mu|x|)}{4\pi m^2 \mu i},$$  \hspace{1cm} (119)$$

$$F_L = -i \frac{\partial}{\partial \mu^2} \frac{e^{-\mu|x|}}{4\pi i |x|} = -\frac{1}{8\pi \mu} \frac{\partial}{\partial \mu} \left( \frac{e^{-\mu|x|}}{|x|} \right)$$

$$= \frac{1}{8\pi} \exp(-\mu|x|),$$  \hspace{1cm} (120)$$

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt, (|\text{arg } z| < \pi)$$  \hspace{1cm} (121)$$

$$E_1(\mu|x|) \sim -\gamma - \ln(\mu|x|), |\mu|x| \ll 1,$$  \hspace{1cm} (122)$$

$$E_1(\mu|x|) \sim \frac{\exp(-\mu|x|)}{\mu|x|} (1 - \frac{1}{\mu|x|} + \frac{2}{(\mu|x|)^2}) (\mu|x| \gg 1),$$  \hspace{1cm} (123)$$
where $r^2 = m^2$. For short distance we have

$$F \simeq \frac{e^2}{4\pi m} (\gamma + \ln(\mu|x|)) + \frac{(\xi + 1)e^2}{8\pi} \frac{1}{\mu}. \quad (124)$$

Above spectral function contains linear infrared divergent term proportional to $1/\mu$. This term depends on the gauge parameter. So we choose $\xi = -1$ gauge to drop it. Other terms are independent of $\xi$. In 4-dimension well-known infrared behaviour of charged particle is reproduced in this way. In QED$_3$ we have used this technique to determine a short-distance behaviour of the propagator. In the long distance region propagator behaves as free one for finite $\mu$. If we set anomalous dimension which is a coefficient of $\ln(\mu|x|)$ to be unity $e^2/4\pi m = 1$, we obtain the propagator at short distance as

$$S_F(x) = -\frac{(i\gamma \cdot \partial + m)e^{-m|x|}(\mu|x|)}{4\pi|x|}e^{\gamma}$$

$$= -(i\gamma \cdot \partial + m)\frac{\mu e^{\gamma}}{4\pi}e^{-m|x|}, \quad (125)$$

which shows condensation $-iTr(S_F(x)) = \text{finite}$. In this case the trace means positive energy part only. For massless gauge boson vacuum expectation value is infrared cut-off dependent. If we set $\mu = m = e^2/4\pi$, we have $\langle \bar{\psi}\psi \rangle = e^4/32\pi^2$ which is very close to the numerical value by solving Dyson-Schwinger equation in Euclid momentum space. In our topologically massive QED, there are two order parameters $\langle \bar{\psi}\tau\psi \rangle, \langle \bar{\psi}\tau\psi \rangle$ that have non-vanishing value for small topological mass.

### B. contribution of Chern-Simons term

Just above the critical point there exists only parity violating mass with non-vanishing order parameter $\langle \bar{\psi}\tau\psi \rangle$. So that we may expect that the chiral symmetry breaking mass vanishes and parity violating condensate remains at the critical point. We show this is the case in evaluating the lowest order spectral function with Chern-Simons term. In the case of parity violation, we may include parity violating part of the spectral function $\overline{\tau}\gamma^\mu(\gamma_\mu \tau)$. Since we adopt the chiral representation of the propagator in solving the Dyson-Schwinger equation, we will determined spectral functions in chiral representation too. Contribution of Chern-Simons term for the spectral function in chiral representation is given in the Appendix B

$$F_{CS}^+ = -e^2 \int \frac{d^3k}{(2\pi)^2} e^{ik_0x} \theta(k_0) \tau[\gamma \cdot r \{(-\frac{1}{\mu} + \frac{\mu}{m^2})\frac{1}{8r \cdot k} - \frac{\mu}{8(r \cdot k)^2} + \frac{\mu}{m} \frac{1}{8r \cdot k} + \frac{m\mu}{4(r \cdot k)^2}\}]\delta(k^2 - \mu^2). \quad (126)$$
After the integration we obtain the correction of wave function renormalization by Chern-Simons term \((1/r \cdot k)\) term as \(F_2\)

\[
F^+(\mu|x|) = (-\frac{\gamma \cdot r + m}{2m} + \frac{\gamma \cdot r}{m} \frac{e^2}{32\pi m} - \frac{e^2\mu}{32\pi m^2})E_1(\mu|x|),
\]

for \(S_+\). At short distance we take into account only \(E_1(\mu|x|) \sim -\gamma - \ln(\mu|x|)\). If we exponentiate \(F(\mu|x|)\) as \(e^F\) using \((\gamma \cdot r)^2/m^2 = 1\), have

\[
e^{A\gamma \cdot r/m \ln(\mu|x|)} = \cosh(A \ln(\mu|x|)) + \frac{\gamma \cdot r}{m} \sinh(A \ln(\mu|x|))
\]

\[
= \frac{\gamma \cdot r + m}{2m}(\mu|x|)^A + \frac{m - \gamma \cdot r}{2m}(\mu|x|)^{-A}.
\]

Final form is written as

\[
e^F = \frac{\gamma \cdot r + m}{2m}(\mu|x|)^A \exp(\gamma + \frac{e^2\mu}{32\pi m^2} \ln(\mu|x|)), A = \frac{e^2}{8\pi m} - \frac{e^2}{32\pi m^2},
\]

for short distance. In chiral representation if \(A = 1\), we have non vanishing condensation of \(\langle \bar{\psi}\psi \rangle_+\). Otherwise \(\langle \bar{\psi}\psi \rangle_+ = 0\). Below the critical value of the topological mass \(\mu \leq \mu_{cr}\) there are two kind of mass with different anomalous dimension. In that case we cannot separate them in the chiral representation. However at the \(\mu = \mu_{cr}\) there is only parity odd mass and its condensate. So that we set wave function renormalization \(A = 0\), other contribution is only logarithmic. Therefore we set \(e^2/8\pi m = 1\) and have \(e^2/32\pi \mu = 1\) for critical value of \(\mu\). In this case we have only parity violating order parameter \(\langle \bar{\psi}\tau\psi \rangle\) above the critical point. This is just the desirable form to give parity odd order parameter \(\langle \bar{\psi}\tau\psi \rangle\). So that at the critical point chiral order parameter vanishes and it undergoes into parity violating phase. So we conclude that the critical value of the topological mass is \(\mu_{cr} = e^2/32\pi \simeq .10^{-2}e^2\) which is totally consistent with the value in our numerical analysis of Dyson-Schwinger equation for quenched case. This transition is the same as Kosterlitz-Thouless type at finite temperature where single vortex excitation destroys the superfluidity. So that the Topological Massive QED\(_3\) is the same with QED\(_3\) with single vortex at zero temperature except for small topological mass. In the statistical model, behaviour of the propagator near the critical point is studied with renormalization group equation for vortex number (chemical potential) and the inverse temperature\([2,3]\). In our model spectral function provide us anomalous dimension and we can determine the ultraviolet behaviour and critical point in the existence of topological mass. Fortunately our Dyson-Schwinger equation successfully determines the structure of the propagator and dynamical mass near the critical region.
V. SUMMARY

In this work we studied the dynamics of Kosterlitz-Thouless type transition in Topologically Massive Abelian Gauge Theory in three dimensional space-time. In this model equation of motion contain vortex solution for the vector potential. We showed that there exists a critical value of the topological mass above which chiral condensate is washed away for four component fermion. This phenomenon turned out to be gauge invariant by the choice of BC vertex in solving Dyson-Schwinger equation for fermion self-energy. In the analysis of spectral function we showed it modify the short distance behaviour of the propagator in position space. In that case anomalous dimension control the order parameter of chiral condensate in the absence of topological mass. However if we add Chern-Simons term to the Lagrangian parity odd part of the gauge boson propagator is destructive to parity even part. In this way lowest order spectral function vanishes at the critical value of topological mass and the chiral condensate is washed away. Our next step is to evaluate critical temperature by solving Dyson-Schwinger equation or spectral function, temperature dependence of specific heat and compare them with the experiment.

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VI. APPENDICES

A. Angular integral

1. Quenched case

\[
I_0(p, q) = \int_{-1}^{1} \frac{d \cos \theta}{k^2 + \mu^2} \ln \frac{(p - q)^2 + \mu^2}{(p + q)^2 + \mu^2},
\]
\[
I_1(p, q) = \int_{-1}^{1} \frac{d \cos \theta}{k^2 + \mu^2} \frac{(p^2 + q^2) - \frac{k^2}{2} - (p^2 - q^2)^2}{2},
\]
\[
I_2(p, q) = \int_{-1}^{1} \frac{d \cos \theta}{k^2 + \mu^2} \left( \frac{1}{2} \pm \frac{p^2 - q^2}{2k^2} \right)
= -\frac{1}{4pq} \ln \frac{(p - q)^2 + \mu^2}{(p + q)^2 + \mu^2} \pm \frac{p^2 - q^2}{4\mu^2pq} \ln \frac{1 + \mu^2/(p - q)^2}{1 + \mu^2/(p + q)^2},
\]
\[
I_3(p, q) = \int_{-1}^{1} \frac{d \cos \theta}{k^2 + \mu^2} \left( \frac{p^2 - q^2)^2}{4k^2} - \frac{k^2}{4} \right)
= \frac{(p^2 - q^2)^2}{8\mu^2pq} \ln \frac{1 + \mu^2/(p - q)^2}{1 + \mu^2/(p + q)^2} - \frac{1}{2} - \frac{\mu^2}{8pq} \ln \frac{(p - q)^2 + \mu^2}{(p + q)^2 + \mu^2}.
\]

After angular integral we have the following coupled integral equation

\[
B(p) = \frac{e^2}{4\pi^2} \int_{0}^{\infty} dq q^2 \ln \frac{(A(p) + A(q))(B(q) + I_0[p, q]) + \mu A(q) I_2(p, q)}{A(q) + A(p)} I_1(p, q),
\]
\[
A(p) = 1 + \frac{e^2}{4\pi^2pq^2} \int_{0}^{\infty} dq q^2 \ln \frac{(A(p) + A(q))(B(q) + I_2(p, q) + A(q) I_3(p, q))}{B(q) + A(p)} I_1(p, q).
\]

2. Unquenched case

For unquenched case, to evaluate angular integral we may use complex number to represent parity even and odd piece of the photon propagator

\[
\text{Re} \left( \frac{1}{k^2 + \pi(k^2) + i\mu\sqrt{k^2}} \right) = \frac{k^2 + \pi(k^2)}{(k^2 + \pi(k^2))^2 + \mu^2k^2},
\]
\[
\text{Im} \left( \frac{1}{k^2 + \pi(k^2) + i\mu\sqrt{k^2}} \right) = \frac{-\mu\sqrt{k^2}}{(k^2 + \pi(k^2))^2 + \mu^2k^2}.
\]
It has been known that the integral kernel is a logarithmic function as

$$K(p, q) \propto \ln\left(\frac{|p - q| + c}{p + q + c}\right)$$  \hspace{1cm} (137)

where $c = e^2 N/8$. So that the angular integral with finite topological mass $\mu$ may be a analytic continuation from the case $\mu = 0$ to $\mu \neq 0$.

$$K(p, q) \propto \ln\left(\frac{|p - q| + c + i\mu}{p + q + c + i\mu}\right).$$  \hspace{1cm} (138)

For example

$$J_0(p, q) = \int_{-1}^{1} \frac{dt}{(p^2 + q^2 - 2pqt) + (c + i\mu)\sqrt{p^2 + q^2 - 2pqt}} = \frac{1}{pq} \ln\left(\frac{p + q + c + i\mu}{|p - q| + c + i\mu}\right).$$  \hspace{1cm} (139)

The integration of the type

$$-\mu \int_{-1}^{1} \frac{d\cos \theta \cdot q \cdot k}{(k^2 + ck)^2 + \mu^2 k^2} = \text{Im} \int_{-1}^{1} \frac{d\cos \theta \cdot q \cdot (q - p)}{((p - q)^2 + (c + i\mu)\sqrt{(p - q)^2})\sqrt{(p - q)^2}}$$

is rewritten

$$J_2(p, q) = -\int_{-1}^{1} \text{Im}\left(\frac{dt}{(p^2 + q^2 - 2pqt) + (c + i\mu)\sqrt{p^2 + q^2 - 2pqt}}\right)(p^2 - q^2) - (p - q)^2 \quad 2 \sqrt{p^2 + q^2 - pqt}
= -\frac{1}{2pq} \text{Im}(p + q - |p - q| - (c + i\mu)\ln\left(\frac{p + q + c + i\mu}{|p - q| + c + i\mu}\right))
- \frac{p^2 - q^2}{2pq} \text{Im}\left(\frac{1}{c + i\mu} \ln\left(\frac{p + q}{|p - q| + c + i\mu}\right)\right).$$  \hspace{1cm} (141)

In the same way

$$-\mu \int_{-1}^{1} \frac{d\cos \theta \cdot p \cdot k}{(k^2 + ck)^2 + \mu^2 k^2} = \text{Im} \int_{-1}^{1} \frac{d\cos \theta \cdot p \cdot (q - p)}{((p - q)^2 + (c + i\mu)\sqrt{(p - q)^2})\sqrt{(p - q)^2}}$$

$p \cdot (q - p) = -(p^2 - q^2)/2 - (p - q)^2/2$ leads

$$J_2(p, q) = -\int_{-1}^{1} \text{Im}\left(\frac{dt}{(p^2 + q^2 - 2pqt) + (c + i\mu)\sqrt{p^2 + q^2 - 2pqt}}\right)(p^2 - q^2) + (p - q)^2 \quad 2 \sqrt{p^2 + q^2 - pqt}
= -\frac{1}{2pq} \text{Im}(p + q - |p - q| - (c + i\mu)\ln\left(\frac{p + q + c + i\mu}{|p - q| + c + i\mu}\right))
+ \frac{p^2 - q^2}{2pq} \text{Im}\left(\frac{1}{c + i\mu} \ln\left(\frac{p + q}{|p - q| + c + i\mu}\right)\right).$$  \hspace{1cm} (143)

Since

$$\frac{(p \cdot k)(q \cdot k)}{k^2} = -\frac{(p - q)^2}{4} + \frac{(p^2 - q^2)^2}{4(p - q)^2}$$

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\[ J_3(p, q) = \text{Re} \int_{-1}^{1} \frac{d\cos\theta}{(p - q)^2 + (c + i\mu)\sqrt{(p - q)^2}} \frac{(p \cdot k)(q \cdot k)}{k^2} \]

\[ = -\frac{1}{4pq} \text{Re}[2pq + (c + i\mu)(|p - q| - (p + q)) + (c + i\mu)^2 \ln\left(\frac{p + q + c + i\mu}{|p - q| + c + i\mu}\right)] \]

\[ -(p + q)^2|p - q| + (p - q)^2(p + q) + \frac{(p^2 - q^2)^2}{(c + i\mu)^2} \ln\left(\frac{(p + q)(|p - q| + c + i\mu)}{|p - q|(p + q + c + i\mu)}\right) ] \]

(144)

The Dyson-Schwinger equations are rewritten as

\[ B_{\pm}(p) = \frac{e^2}{2\pi^2} \int q^2 dq \left[ \frac{B_{\pm}(q)J_0(p,q)}{A_{\pm}^2(q)q^2 + B_{\pm}^2(q)} \pm \frac{A_{\pm}(q)J_2(p,q)}{A_{\pm}^2(q)q^2 + B_{\pm}^2(q)} \right], \]

\[ A_{\pm}(p) = 1 + \frac{e^2}{2\pi^2p^2} \int q^2 dq \left[ \frac{A_{\pm}(q)J_3(p,q)}{A_{\pm}^2(q)q^2 + B_{\pm}^2(q)} \pm \frac{B_{\pm}(q)J_2(p,q)}{A_{\pm}^2(q)q^2 + B_{\pm}^2(q)} \right]. \] (145)

B. evaluation of vacuum polarization

In a two-dimensional representation, the trace of products of four \( \gamma \)-matrices are:

\[ tr(I_2) = 2, \]

\[ tr(\gamma^\mu) = 0, \]

\[ tr(\gamma^\mu\gamma^\nu) = 2g^{\mu\nu}, \]

\[ tr(\gamma^\mu\gamma^\nu\gamma^\rho) = -ie^{\mu\nu\rho}, \]

\[ tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 2(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \] (146)

\[ \Pi_{\mu\nu}(k) \equiv -e^2 \int \frac{d^2p}{(2\pi)^2 i} Tr(\gamma_\mu \frac{1}{\gamma \cdot p - m} \gamma_\nu \frac{1}{\gamma \cdot (p - k) - m}). \] (147)

Substituting the chiral representation of the propagator into \( \Pi_{\mu\nu}(k) \)

\[ S(p) = \frac{i}{\gamma \cdot p - m + i\epsilon} \rightarrow i\frac{\gamma \cdot p + m_+}{p^2 - m_+^2} \chi_+ + i\frac{\gamma \cdot p + m_-}{p^2 - m_-^2} \chi_-, \] (148)

\[ Tr(\gamma_\mu \frac{\gamma \cdot p + m_+}{p^2 - m_+^2} \chi_+ + \frac{\gamma \cdot p + m_-}{p^2 - m_-^2} \chi_-)\gamma_\nu \frac{\gamma \cdot (p - k) + m_+}{(p - k)^2 - m_+^2} \chi_+ + \frac{\gamma \cdot (p - k) + m_-}{(p - k)^2 - m_-^2} \chi_-) \]

\[ = Tr(\gamma_\mu \frac{\gamma \cdot p + m_+}{p^2 - m_+^2} \chi_+ \gamma_\nu \frac{\gamma \cdot (p - k) + m_+}{(p - k)^2 - m_+^2} \chi_+) + Tr(\gamma_\mu \frac{\gamma \cdot p + m_-}{p^2 - m_-^2} \chi_- \gamma_\nu \frac{\gamma \cdot (p - k) + m_-}{(p - k)^2 - m_-^2} \chi_-), \] (149)
\[ Tr(\gamma_\mu(\gamma \cdot p + m_+)\chi_+ \gamma_\nu(\gamma \cdot (p - k) + m_+)\chi_+) \]
\[ = 2(p_\mu(p - k)_\nu + p_\nu(p - k)_\mu - g_{\mu\nu}(p \cdot (p - k) - m_+^2) + im_+\epsilon_{\mu\nu\rho}p^\rho - im_+\epsilon_{\mu\nu\rho}(p - k)^\rho). \] (150)

\[ Tr(\gamma_\mu(\gamma \cdot p + m_-)\chi_- \gamma_\nu(\gamma \cdot (p - k) + m_-)\chi_-) \]
\[ = 2(p_\mu(p - k)_\nu + p_\nu(p - k)_\mu - g_{\mu\nu}(p \cdot (p - k) - m_-^2) - im_-\epsilon_{\mu\nu\rho}p^\rho + im_-\epsilon_{\mu\nu\rho}(p - k)^\rho). \] (151)

\[ \Pi_{\mu\nu}(k) = -e^2 \int_0^1 dx \int \frac{d^3p'}{(2\pi)^3} \frac{2N_{\mu\nu}}{(-p'^2 + K)^2}; \] (152)

where

\[ K = -k^2x(1 - x) - m^2, p' = p - k(1 - x), \] (153)

\[ N_{\mu\nu} = [-m^2 + x(1 - x)k^2]\delta_{\mu\nu} - \frac{1}{3}p'^2\delta_{\mu\nu} - 2k_\mu k_\nu x(1 - x). \] (154)

Regulating the ultraviolet divergence by cut-off

\[ \int_0^\Lambda \frac{p'^4dp'}{(-p'^2 + K)^2i} = -\frac{3}{2} \arctan\left(\frac{\Lambda}{K}\right)K + \Lambda + O\left(\frac{1}{\Lambda}\right), \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p'^2 + K)^2} = \frac{1}{8\pi/\sqrt{K}}. \] (155)

we obtain the vacuum polarization tensor for two-component fermion

\[ \Pi_{\mu\nu}(-k^2) = -e^2 \int_0^1 dx \int \frac{d^3p'}{(2\pi)^3} \frac{2N_{\mu\nu}}{(p'^2 + k^2x(1 - x) + m^2)^2} \]
\[ = \frac{-4e^2}{8\pi}(\delta_{\mu\nu}k^2 - k_\mu k_\nu) \int_0^1 dx \frac{x(1 - x)}{\sqrt{m^2 + k^2x(1 - x)}} + \frac{e^2\Lambda}{3\pi^2}\delta_{\mu\nu} \]
\[ = \frac{-e^2}{8\pi}(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})[(\sqrt{-k^2} + 4m^2/\sqrt{-k^2}) \arctan(\sqrt{-k^2}/2m) + 2m] + \frac{e^2\Lambda}{3\pi^2}\delta_{\mu\nu}. \] (156)

In Minkowski space we have

\[ \Pi_{\mu\nu}(k^2) = \frac{e^2}{16\pi}(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})((\sqrt{k^2} + 4m^2/\sqrt{k^2}) \ln(\frac{2m - \sqrt{k^2}}{2m + \sqrt{k^2}} + 4m) - \frac{e^2\Lambda}{3\pi^2}g_{\mu\nu}. \] (158)

From the above expression, vacuum polarization tensor for chiral representation of fermion is given

\[ \Pi_{\mu\nu}(p) = -\frac{2e^2}{3\pi^2}\Lambda g_{\mu\nu} + T_{\mu\nu} \frac{e^2}{8\pi}p^2 \left( \int_{2|m_+|}^\infty \frac{da(1 + 4m^2/a^2)}{p^2 - a^2 + i\epsilon} + \int_{2|m_-|}^\infty \frac{da(1 + 4m^2/a^2)}{p^2 - a^2 + i\epsilon} \right) \]
\[ + i\epsilon_{\mu\nu\rho}p^\rho \frac{e^2}{2\pi}(m_+ \int_{2|m_+|}^\infty \frac{da}{p^2 - a^2 + i\epsilon} - m_- \int_{2|m_-|}^\infty \frac{da}{p^2 - a^2 + i\epsilon}). \] (159)
Here we derive the contributions of Chern-Simons term to the spectral function. For scalar and vector parts are given by trace subsections:

\[ B(r, k)_+ = \frac{e^2}{16 m (r \cdot k)^2} tr((r + k) \cdot \gamma + m) \gamma^\mu (r \cdot \gamma + m) \gamma^\nu ((r + k) \cdot \gamma + m) i \epsilon_{\mu \nu \rho} \frac{k^\rho}{\mu^2}, \] (160)

\[ A(r, k)_+ = \frac{e^2}{16 m^3 (r \cdot k)^2} tr(r \cdot \gamma ((r + k) \cdot \gamma + m) \gamma^\mu (r \cdot \gamma + m) \gamma^\nu ((r + k) \cdot \gamma + m) i \epsilon_{\mu \nu \rho} \frac{k^\rho}{\mu^2}, \] (161)

where \( \gamma \) matrices are \( 4 \times 4 \). First we evaluate \( B(r, k) \).

\[
tr((r + k) \cdot \gamma + m)^2 \chi^\gamma \gamma^\mu (r \cdot \gamma + m) \gamma^\nu i \epsilon_{\mu \nu \rho} \frac{k^\rho}{\mu} \\
= tr([2(m^2 + r \cdot k) + k^2 + 2 m(r + k) \cdot \gamma] \chi^\gamma \gamma^\mu (r \cdot \gamma + m) \gamma^\nu i \epsilon_{\mu \nu \rho} \frac{k^\rho}{\mu} \\
= [(2(m^2 + r \cdot k) + k^2)r_\sigma tr(\chi^\gamma \gamma^\mu \gamma^\rho \gamma^\nu) + 2 m^2 (r + k)_\sigma tr(\chi^\gamma \gamma^\rho \gamma^\nu)] i \epsilon_{\mu \nu \rho} \frac{k^\rho}{\mu}.
\] (162)

\[
B(r, k) = - \frac{4(2(m^2 + r \cdot k) + \mu^2)r \cdot k}{32(m \cdot k)^2 \mu} + \frac{8 m^2 (r \cdot k + \mu^2)}{32(m \cdot k)^2 \mu} \\
= - \frac{4 \mu r \cdot k}{32(m \cdot k)^2 \mu} - \frac{1}{4 m \mu} + \frac{8 m^2 \mu}{32(m \cdot k)^2 \mu} \\
= - \frac{\mu}{8 m (r \cdot k)} - \frac{1}{4 m \mu} + \frac{\mu m}{4(r \cdot k)^2}.
\] (163)

Next we evaluate \( A(r, k) \).

\[
tr(r \cdot \gamma ((r + k) \cdot \gamma + m) \gamma^\mu (r \cdot \gamma + m) \gamma^\nu ((r + k) \cdot \gamma + m)) \\
= tr((r + k) \cdot \gamma + m) r \cdot \gamma ((r + k) \cdot \gamma + m) \chi^\gamma \gamma^\mu (r \cdot \gamma + m) \gamma^\nu \\
= tr((r + k) \cdot \gamma + m) (m^2 + r \cdot k + m r \cdot \gamma) \chi^\gamma \gamma^\mu (r \cdot \gamma + m) \gamma^\nu \\
= tr((2m(m^2 + r \cdot k) + (2m^2 + r \cdot k)r \cdot \gamma + (m^2 + r \cdot k)k \cdot \gamma) \chi^\gamma \gamma^\mu (r \cdot \gamma + m) \gamma^\nu \\
= 2 m^2 + r \cdot k)r_\sigma tr(\chi^\gamma \gamma^\rho \gamma^\nu) + m(2m^2 + r \cdot k)r_\sigma tr(\chi^\gamma \gamma^\rho \gamma^\nu) + m(m^2 + r \cdot k)k_\sigma tr(\chi^\gamma \gamma^\rho \gamma^\nu) \\
= 4 m^2 + r \cdot k)r_\sigma \epsilon_{\sigma \mu \nu} - 2 m(2m^2 + r \cdot k)r_\sigma \epsilon_{\sigma \mu \nu} - 2 m(m^2 + r \cdot k)k_\sigma \epsilon_{\sigma \mu \nu}.
\] (164)

\[
A(r, k) = \frac{1}{32 m^3 (r \cdot k)^2} tr(r \cdot \gamma ((r + k) \cdot \gamma + m) \gamma^\mu (r \cdot \gamma + m) \gamma^\nu ((r + k) \cdot \gamma + m)) i \epsilon_{\mu \nu \rho} \frac{k^\rho}{\mu} \\
= - \frac{1}{32 m^3 (r \cdot k)^2} (8 m^2 + r \cdot k) \frac{r \cdot k}{\mu} + 4 m(2m^2 + r \cdot k) \frac{r \cdot k}{\mu} + 4 m(m^2 + r \cdot k) \mu \\
= - \frac{1}{8 m^2 \mu} - \frac{\mu}{8(r \cdot k)^2} + \frac{\mu}{8 m^2 (r \cdot k)}.
\] (165)