CHARACTER SHEAVES AND GENERALIZATIONS

G. Lusztig

Dedicated to I. M. Gelfand on the occasion of his 90th birthday

1. Let $k$ be an algebraic closure of a finite field $\mathbb{F}_q$. Let $G = GL_n(k)$. The group $G(\mathbb{F}_q) = GL_n(\mathbb{F}_q)$ can be regarded as the fixed point set of the Frobenius map $F : G \to G, (g_{ij}) \mapsto (g_{ij}^q)$. Let $\overline{\mathbb{Q}}_l$ be an algebraic closure of the field of $l$-adic numbers, where $l$ is a prime number invertible in $k$. The characters of irreducible representations of $G(\mathbb{F}_q)$ over an algebraically closed field of characteristic 0, which we take to be $\overline{\mathbb{Q}}_l$, have been determined explicitly by J.A.Green [G]. The theory of character sheaves [L2] tries to produce some geometric objects over $G$ from which the irreducible characters of $G(\mathbb{F}_q)$ can be deduced for any $q$. This allows us to unify the representation theories of $G(\mathbb{F}_q)$ for various $q$. The geometric objects needed in the theory are provided by intersection cohomology.

Let $X$ be an algebraic variety over $k$, let $X_0$ be a locally closed irreducible, smooth subvariety of $X$ and let $\mathcal{E}$ be a local system over $X_0$ (we say "local system" instead of "$\overline{\mathbb{Q}}_l$-local system"). Deligne, Goersky and MacPherson attach to this datum a canonical object $IC(\overline{X}_0, \mathcal{E})$ (intersection cohomology complex) in the derived category $D(X)$ of $\overline{\mathbb{Q}}_l$-sheaves on $X$; this is a complex of sheaves which extends $\mathcal{E}$ to $X$ (by 0 outside the closure $\overline{X}_0$ of $X_0$) in the most economical possible way so that local Poicaré duality is satisfied. We say that $IC(\overline{X}_0, \mathcal{E})$ is irreducible if $\mathcal{E}$ is irreducible.

Now take $X = G$ and take $X_0 = G_{rs}$ to be the set of regular semisimple elements in $G$. Let $T$ be the group of diagonal matrices in $G$. For any integer $m \geq 1$ invertible in $k$ we have an unramified $n!/m^n$-fold covering

$$\pi_m : \{(g, t, xT) \in G_{rs} \times T \times G/T; x^{-1}gx = t^m\} \to G_{rs}, \quad (g, t, xT) \mapsto g.$$ An irreducible local system $\mathcal{E}$ on $G_{rs}$ is said to be admissible if it is a direct summand of the local system $\pi_m^!\overline{\mathbb{Q}}_l$ for some $m$ as above. The character sheaves on $G$ are the complexes $IC(G, \mathcal{E})$ for various admissible local systems $\mathcal{E}$ on $G_{rs}$.

We show how the irreducible characters of $G(\mathbb{F}_q)$ can be recovered from character sheaves on $G$. If $A$ is a character sheaf on $G$ then its inverse image $F^*A$ under $F$ is again a character sheaf. There are only finitely many $A$ (up to isomorphism) such that $F^*A$ is isomorphic to $A$. For any such $A$ we choose an isomorphism

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φ \colon F^* A \sim \to A \text{ and we form the characteristic function } \chi_{A,\phi} : G(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l \text{ whose value at } g \text{ is the alternating sum of traces of } \phi \text{ on the stalks at } g \text{ of the cohomology sheaves of } A. \text{ Now } \phi \text{ is unique up to a non-zero scalar hence } \chi_{A,\phi} \text{ is unique up to a non-zero scalar. It turns out that}

(a) \chi_{A,\phi} \text{ is (up to a non-zero scalar) the character of an irreducible representation of } G(\mathbf{F}_q) \text{ and } A \mapsto \chi_{A,\phi} \text{ gives a bijection between the set of (isomorphism classes of) character sheaves on } G \text{ that are isomorphic to their inverse image under } F \text{ and the irreducible characters of } G(\mathbf{F}_q).

(This result is essentially contained in [L1,L3].) The main content of this result is that the (rather complicated) values of an irreducible character of } G(\mathbf{F}_q) \text{ are governed by a geometric principle, namely by the procedure which gives the intersection cohomology extension of a local system.}

2. More generally, assume that } G \text{ is a connected reductive algebraic group over } k. \text{ The definition of the } IC(G, \mathcal{E}) \text{ given above for } GL_n \text{ makes sense also in the general case. The complexes on } G \text{ obtained in this way form the class of uniform character sheaves on } G. \text{ Consider now a fixed } \mathbf{F}_q \text{-rational structure on } G \text{ with Frobenius map } F : G \to G. \text{ The analogue of property 1(a) does not hold in general for } (G, F). \text{ It is still true that the characteristic functions of the uniform character sheaves that are isomorphic to their inverse image under } F \text{ are linearly independent class functions } G(\mathbf{F}_q) \to \mathbf{Q}_l. \text{ However they do not form a basis of the space of class functions. Moreover they are in general not irreducible characters of } G(\mathbf{F}_q) \text{ (up to a scalar); rather, each of them is a linear combination with known coefficients of a "small" number of irreducible characters of } G(\mathbf{F}_q) \text{ (where "small" means "bounded independently of } q\)"}; \text{ this result is essentially contained in [L1,L3].}

It turns out that the class of uniform character sheaves can be naturally enlarged to a larger class of complexes on } G.

For any parabolic } P \text{ of } G, \text{ } U_P \text{ denotes the unipotent radical of } P. \text{ For a Borel } B \text{ in } G, \text{ the images under } c^B : G \to G/U_B \text{ of the double cosets } BwB \text{ form a partition } G/U_B = \bigcup_w (BwB/U_B).

An irreducible intersection cohomology complex } A \in \mathcal{D}(G) \text{ is said to be a character sheaf on } G \text{ if it is } G\text{-equivariant and if for some/any Borel } B \text{ in } G, \text{ } c^B_* A \text{ has the following property:}

(*) \text{ any cohomology sheaf of this complex restricted to any } BwB/U_B \text{ is a local system with finite monodromy of order invertible in } k.

Then any uniform character sheaf on } G \text{ is a character sheaf on } G. \text{ For } G = GL_n \text{ the converse is also true, but for general } G \text{ this is not so.}

Consider again a fixed } \mathbf{F}_q \text{-rational structure on } G \text{ with Frobenius map } F : G \to G. \text{ The following partial analogue of property 1(a) holds (under a mild restriction on the characteristic of } k).

(a) The characteristic functions of the various character sheaves } A \text{ on } G \text{ (up to isomorphism) such that } F^* A \sim A \text{ form a basis of the vector space of class functions } G(\mathbf{F}_q) \to \mathbf{Q}_l.
3. We now fix a parabolic $P$ of $G$. For any Borel $B$ of $P$ let $\tilde{c}^B : G/U_P \to G/U_B$ be the obvious map. Now $P$ acts on $G/U_P$ by conjugation.

An irreducible intersection cohomology complex $A \in D(G/U_P)$ is said to be a parabolic character sheaf if it is $P$-equivariant and if for some/any Borel $B$ in $P$, $\tilde{c}^B_* A$ has property 2(*). When $P = G$, we recover the definition of character sheaves on $G$.

Consider now a fixed $F_q$-rational structure on $G$ with Frobenius map $F : G \to G$ such that $P$ is defined over $F_q$. Then $G/U_P$ has a natural $F_q$-rational structure with Frobenius map $F$. The following generalization of 2(a) holds (under a mild restriction on the characteristic of $k$).

(a) The characteristic functions of the various parabolic character sheaves $A$ on $G/U_P$ (up to isomorphism) such that $F^* A \xrightarrow{\sim} A$ form a basis of the vector space $V$ of $P(F_q)$-invariant functions $G(F_q)/U_P(F_q) \to \bar{Q}_l$.

The proof is given in [L5]. It relies on a generalization of property 2(a) to not necessarily connected reductive groups which will be contained in the series [L6].

If $h : G(F_q) \to \bar{Q}_l$ is the characteristic function of a character sheaf as in 2(a) then by summing $h$ over the fibres of $G(F_q) \to G(F_q)/U_P(F_q)$ we obtain a function $\bar{h} \in V$. It turns out that each function $\bar{h}$ is a linear combination of a "small" number of elements in the basis of $V$ described above. (The fact such a basis of $V$ exists is not apriori obvious.)

The parabolic character sheaves on $G/U_P$ are expected to be a necessary ingredient in establishing the conjectural geometric interpretation of Hecke algebras with unequal parameters given in [L4].

4. In this section $G$ denotes an abelian group with a given family $\mathfrak{F}$ of automorphisms such that

- (i) if $F \in \mathfrak{F}$ and $n \in \mathbb{Z}_{>0}$, then $F^n \in \mathfrak{F}$;
- (ii) if $F \in \mathfrak{F}$, $F' \in \mathfrak{F}$ then there exist $n, n' \in \mathbb{Z}_{>0}$ such that $F^n = F'^{n'}$;
- (iii) for any $F \in \mathfrak{F}$, the map $G \to G, x \mapsto F(x)x^{-1}$ is surjective with finite kernel.

For $F \in \mathfrak{F}$ and $n \in \mathbb{Z}_{>0}$, the homomorphism

$$N_{F^n/F} : G \to G, x \mapsto xF(x) \ldots F^{n-1}(x),$$

restricts to a surjective homomorphism $G^{F^n} \to G^F$. (If $y \in G^F$ we can find $z \in G$ with $y = F^n(z)z^{-1}$, by (i),(iii). We set $x = F(z)z^{-1}$. Then $x \in G^{F^n}$ and $N_{F^n/F}(x) = y$.) Let $X$ be the set of pairs $(F, \psi)$ where $F \in \mathfrak{F}$ and $\psi \in \text{Hom}(G^F, \bar{Q}_l^*)$. Consider the equivalence relation on $X$ generated by $(F, \psi) \sim (F^n, \psi \circ N_{F^n/F})$. Let $G^* = \text{Hom}(G^F, \bar{Q}_l^*)$. We define a group structure on $G^*$. We consider two elements of $G^*$; we represent them in the form $(F, \psi), (F', \psi')$ where $F = F'$ (using (ii)) and we define their product as the equivalence class of $(F, \psi\psi')$; one checks that this product is independent of the choices. This makes $G^*$ into an abelian group. The unit element is the equivalence class of $(F, 1)$ for any $F \in \mathfrak{F}$. For $F \in \mathfrak{F}$ we define an automorphism $F^* : G^* \to G^*$ by sending an element of $G^*$ represented by $(F^n, \psi)$ with $n \in$
Consider the local system \( E \) on \( G \) (group.) Then properties 4(i)-4(iii) are satisfied for \( \mathcal{G} \) the map \( \text{Hom}(G^F, \bar{Q}_l^+) \rightarrow G^* \), \( \psi \mapsto (F, \psi) \) is

(a) a group isomorphism of \( \text{Hom}(G^F, \bar{Q}_l^+) \) onto the subgroup \( (G^*)^F \) of \( G^* \).

(This follows from the surjectivity of \( N_{F^n/F} : G^{F^n} \rightarrow G^F \).)

5. Assume now that \( G \) is an abelian, connected (affine) algebraic group over \( k \). We define the notion of character sheaf on \( G \).

Let \( \mathfrak{F} \) be the set of Frobenius maps \( F : G \rightarrow G \) for various rational structures on \( G \) over a finite subfield of \( k \). (These maps are automorphisms of \( G \) as an abstract group.) Then properties 4(i)-4(iii) are satisfied for \( (G, \mathfrak{F}) \) hence the abelian group \( G^* \) is defined as in §4. We will give an interpretation of \( G^* \) in terms of local systems on \( G \). Let \( F \in \mathfrak{F} \). Let \( L : G \rightarrow G \) be the Lang map \( x \mapsto F(x)x^{-1} \). Consider the local system \( E = L_! \bar{Q}_l \) on \( G \). Its stalk at \( y \in G \) is the vector space \( E_y \) consisting of all functions \( f : L^{-1}(y) \rightarrow \bar{Q}_l \). We have \( E_y = \bigoplus_{\psi \in \text{Hom}(G^F, \bar{Q}_l^+)} E^\psi_y \)

where

\[
E^\psi_y = \{ f \in E_y ; f(zx) = \psi(z)f(x) \quad \forall z \in G^F, x \in L^{-1}(y) \}
\]

We have a canonical direct sum decomposition \( E = \bigoplus \psi E^\psi \) where \( E^\psi \) is a local system of rank 1 on \( G \) whose stalk at \( y \in G \) is \( E^\psi_y \) (\( \psi \) as above). There is a unique isomorphism of local systems \( \phi : F^*E^\psi \rightarrow E^\psi \) which induces identity on the stalk at 1. This induces for any \( y \in G \) the isomorphism \( E^\psi_{F(y)} \rightarrow E^\psi_y \) given by \( f \mapsto f' \) where \( f'(x) = f(F(x)) \). If \( y \in G^F \), this isomorphism is multiplication by \( \psi(y) \).

Thus, the characteristic function \( \chi_{E^\psi, \phi} : G^F \rightarrow \bar{Q}_l \) is the character \( \psi \).

Let \( n \in \mathbb{Z}_{\geq 0} \). Let \( L' : G \rightarrow G \) be the map \( x \mapsto F^n(x)x^{-1} \). Consider the local system \( E' = L'_! \bar{Q}_l \) on \( G \). Its stalk at \( y \in G \) is the vector space \( E'_y \) consisting of all functions \( f' : L'^{-1}(y) \rightarrow \bar{Q}_l \). We define \( E_y \rightarrow E'_y \) by \( f \mapsto f' \) where \( f'(x) = f(N_{F^n/F}x) \) (note that \( N_{F^n/F}(L'^{-1}(y)) \subset L^{-1}(y) \)). This is induced by a morphism of local systems \( E \rightarrow E' \) which restricts to an isomorphism \( E^\psi \rightarrow E'^{\psi'} \) where \( \psi' = \psi \circ N_{F^n/F} \in \text{Hom}(G^{F^n}, \bar{Q}_l^+) \).

From the definitions we see that, if \( \psi, \psi' \in \text{Hom}(G^F, \bar{Q}_l^+) \) then for any \( y \in G \) we have an isomorphism \( E^\psi_y \otimes E^\psi'_y \rightarrow E^{\psi \psi'}_y \) given by multiplication of functions on \( L^{-1}(y) \). This comes from an isomorphism of local systems \( E^\psi \otimes E^\psi' \rightarrow E^{\psi \psi'} \).

A character sheaf on \( G \) is by definition a local system of rank 1 on \( G \) of the form \( E^\psi \) for some \( (F, \psi) \) as above. Let \( \mathcal{S}(G) \) be the set of isomorphism classes of character sheaves on \( G \). Then \( \mathcal{S}(G) \) is an abelian group under tensor product. The arguments above show that \( (F, \psi) \mapsto E^\psi \) defines a (surjective) group homomorphism \( G^* \rightarrow \mathcal{S}(G) \). This is in fact an isomorphism. (It is enough to show that, if \( (F, \psi) \) is as above and \( \psi' \in \text{Hom}(G^F, \bar{Q}_l^+) \) is such that the local systems \( E^\psi, E^{\psi'} \) are isomorphic, then \( \psi = \psi' \). As we have seen earlier, each of \( E^\psi, E^{\psi'} \) has a unique isomorphism \( \phi, \phi' \) with its inverse image under \( F : G \rightarrow G \) which induces the identity at the stalk at 1. Then we must have \( \chi_{E^\psi, \phi} = \chi_{E^{\psi'}, \phi'} \) hence \( \psi = \psi' \). Note that for \( F \in \mathfrak{F} \), the map \( F^* : G^* \rightarrow G^* \) corresponds under the isomorphism
$G^* \xrightarrow{\sim} S(G)$ to the map $S(G) \to S(G)$ given by inverse image under $F$. Using this and 4(a), we see that, for $F \in \mathcal{F}$, the map $\text{Hom}(G^F, \mathbb{Q}_l^\ast) \to S(G), \psi \mapsto E^\psi$ is a group isomorphism of $\text{Hom}(G^F, \mathbb{Q}_l^\ast)$ onto the subgroup of $S(G)$ consisting of all character sheaves on $G$ that are isomorphic to their inverse image under $F$. We see that in this case the analogue of 1(a) holds.

From the definitions, we see that,

(a) if $\mathcal{L}_1 \in S(G)$ and $m : G \times G \to G$ is the multiplication map then $m^* \mathcal{L}_1 = \mathcal{L}_1 \otimes \mathcal{L}_1$.

In the case where $G = k$, our definition of character sheaves on $G$ reduces to that of the Artin-Schreier local systems on $k$.

6. In this section we assume that $G$ is a unipotent algebraic group over $k$ of "exponential type" that is, such that the exponential map from Lie $G$ to $G$ is well defined (and an isomorphism of varieties.) In this case we can define character sheaves on $G$ using Kirillov theory. Namely, for each $G$-orbit in the dual of Lie $G$ we consider the local system $\mathcal{E} \in S(k)$, $\mathcal{E} \neq \mathbb{Q}_l$ extended by 0 on the complement of the orbit. Taking the Fourier-Deligne transform we obtain (up to shift) an irreducible intersection cohomology complex on Lie $G$ (since the orbit is smooth and closed, by Kostant-Rosenlicht). We can view it as an intersection cohomology complex on $G$ via the exponential map. The complexes on $G$ thus obtained are by definition the character sheaves of $G$. Using Kirillov theory (see [K]) we see that in this case the analogue of 1(a) holds.

Assume, for example, that $G$ is the group of all matrices

$$[a, b, c] = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with entries in $k$ and that $2^{-1} \in k$. Consider the following intersection cohomology complexes on $G$:

(i) the complex which on the centre \{(0, b, 0); b \in k\} is the local system $\mathcal{E} \in S(k)$, $\mathcal{E} \neq \mathbb{Q}_l$ extended by 0 to the whole of $G$;

(ii) the local system $f^* \mathcal{E}$ where $f[a, b, c] = (a, c)$ and $\mathcal{E} \in S(k^2)$.

The complexes (i),(ii) are the character sheaves of $G$.

7. In this section we assume that $G$ is a connected unipotent algebraic group over $k$ (not necessarily of exponential type). We expect that in this case there is again a notion of character sheaf on $G$ such that over a finite field, the characteristic functions of character sheaves form a basis of the space of class functions and each characteristic function of a character sheaf is a linear combination of a "small" number of irreducible characters. Thus here the situation should be similar to that for a general connected reductive group rather than that for $GL_n$. We illustrate this in one example. Assume that $k$ has characteristic 2. Let $G$ be the group
consisting of all matrices of the form

\[
\begin{pmatrix}
1 & a & b & c \\
0 & 1 & d & b + ad \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

with entries in \( k \); we also write \([a, b, c, d] \) instead of the matrix above. (This group can be regarded as the unipotent radical of a Borel in \( Sp_4(k) \).)

Let \( \mathcal{E}_0 \in S(k) \) be the local system on \( k \) associated in §5 to \( F_q \) and to the homomorphism \( \psi_0 : F_q \to \mathbb{Q}_l^+ \) (composition of the trace \( F_q \to F_2 \) and the unique injective homomorphism \( F_2 \to \mathbb{Q}_l^+ \)).

Consider the following intersection cohomology complexes on \( G \):

(i) the complex which on the centre \([0, b, c, 0]; (b, c) \in k^2 \) is the local system \( \mathcal{E} \in S(k^2), \mathcal{E} \neq Q_1 \) (see §5) extended by 0 to the whole of \( G \);

(ii) the complex which on \([a_0, b, c, 0]; (b, c) \in k^2 \) (with \( a_0 \in k^* \) fixed) is the local system \( pr_2^* \mathcal{E} \) where \( \mathcal{E} \in S(k), \mathcal{E} \neq Q_1 \) (see §5) extended by 0 to the whole of \( G \);

(iii) the complex which on \([0, b, c, d_0]; (b, c) \in k^2 \) (with \( d_0 \in k^* \) fixed) is the local system \( f^* \mathcal{E}_0 \) where \( f[0, b, c, d_0] = \alpha b + \alpha^2 d_0 c \) (with \( \alpha \in k^* \) fixed) extended by 0 to the whole of \( G \);

(iv) the complex which on \([a_0, b, c, d_0]; (b, c) \in k^2 \) (with \( a_0, d_0 \in k^* \) fixed) is the local system \( f^* \mathcal{E}_0 \) where \( f[a_0, b, c, d_0] = a_0^{-2} d_0^{-1} c \) extended by 0 to the whole of \( G \);

(v) the local system \( f^* \mathcal{E} \) on \( G \) where \( f[a, b, c, d] = (a, d) \in k^2 \) and \( \mathcal{E} \in S(k^2) \).

By definition, the character sheaves on \( G \) are the complexes in (i)-(v) above. Note that there are infinitely many subvarieties of \( G \) which appear as supports of character sheaves (this in contrast with the case of reductive groups). There is a symmetry that exchanges the character sheaves of type (ii) with those of type (iii). Namely, define \( \xi : G \to G \) by

\[
[a, b, c, d] \mapsto [d, c + ab + a^2 d, b^2 + dc + abd, a^2].
\]

Then \( \xi \) is a homomorphism whose square is \([a, b, c, d] \mapsto [a^2, b^2, c^2, d^2] \); moreover, \( \xi^* \) interchanges the sets (ii) and (iii) and it leaves stable each of the sets (i), (iv) and (v).

Now \( G \) has an obvious \( F_q \)-structure with Frobenius map \( F : G \to G \). We describe the irreducible characters of \( G(F_q) \).

(I) We have \( q^2 \) one dimensional characters \( U \to \mathbb{Q}_l^+ \) of the form \([a, b, c, d] \mapsto \psi_0(xa + yd) \) (one for each \( x, y \in F_q \)).

(II) We have \( q - 1 \) irreducible characters of degree \( q \) of the form \([0, b, c, 0] \mapsto q \psi_0(xb) \) (all other elements are mapped to 0), one for each \( x \in F_q - \{0\} \).

(III) We have \( q - 1 \) irreducible characters of degree \( q \) of the form \([0, b, c, 0] \mapsto q \psi_0(xc) \) (all other elements are mapped to 0), one for each \( x \in F_q - \{0\} \).
(IV) We have $4(q - 1)^2$ irreducible characters of degree $q/2$, one for each quadruple $(a_0, d_0, e_1, e_2)$ where

$a_0 \in F_q, d_0 \in \mathbb{F}_q^*, e_1 \in \text{Hom}(\{0, a_0\}, \pm 1), e_2 \in \text{Hom}(\{0, d_0\}, \pm 1),$

namely

$$[a, b, c, d] \mapsto (q/2)e_1(a)e_2(d)\psi_0(a_0^{-2}d_0^{-1}(ba + ba_0 + c)),$$

if $a \in \{0, a_0\}, d \in \{0, d_0\}$; all other elements are sent to 0.

A character of type (II) is obtained by inducing from the subgroup $\{[a, b, c, d] \in G(F_q); d = 0\}$ the one dimensional character $[a, b, c, 0] \mapsto \psi_0(xb)$ where $x \in F_q - \{0\}$. A character of type (III) is obtained by inducing from the commutative subgroup $\{[a, b, c, d] \in G(F_q); a = 0\}$ the one dimensional character $[0, b, c, d] \mapsto \psi_0(xc)$ where $x \in F_q - \{0\}$. A character of type (IV) is obtained by inducing from the subgroup $\{[a, b, c, d] \in G(F_q); a \neq 0\}$ where $a \in F_q - \{0\}$; the one dimensional character $[a, b, c, d] \mapsto \epsilon_1(a)\psi_0(fd + a_0^{-2}d_0^{-1}(ba + ba_0 + c))$ where $f \in F_q$ is chosen so that $\psi_0(fd_0) = \epsilon_2(d_0)$ (the induced character does not depend on the choice of $f$).

Consider the matrix expressing the characteristic functions of character sheaves $A$ such that $F^*A \cong A$ (suitably normalized) in terms of irreducible characters of $G(F_q)$. This matrix is square and a direct sum of diagonal blocks of size $1 \times 1$ (with entry 1) or $4 \times 4$ with entries $\pm 1/2$, representing the Fourier transform over a two dimensional symplectic $\mathbb{F}_2$-vector space. There are $(q - 1)^2$ blocks of size $4 \times 4$ involving the irreducible characters of type IV.

We see that, in our case, the character sheaves have the desired properties. We also note that in our case, $G(F_q)$ has some irreducible character whose degree is not a power of $q$ (but $q/2$) in contrast with what happens in the situation in §6.

8. Let $\epsilon$ be an indeterminate. For $r \geq 2$ let $A_r = k[\epsilon]/(\epsilon^r)$. Let $G = GL_n(A_r)$. Let $B$ (resp. $T$) be the group of upper triangular (resp. diagonal) matrices in $G$. Then $G$ is in a natural way a connected affine algebraic group over $k$ of dimension $n^2r$ and $B, T$ are closed subgroups of $G$. On $G$ we have a natural $\mathbb{F}_q$-structure with Frobenius map $F : G \to G, (g_{ij}) \mapsto (g_{ij}^{(q)})$ where for $a_0, a_1, \ldots, a_{r-1}$ in $k$ we set $\left(a_0 + a_1\epsilon + \cdots + a_{r-1}\epsilon^{r-1}\right)^{(q)} = a_0^q + a_1^q\epsilon + \cdots + a_{r-1}^q\epsilon^{r-1}$. The fixed point set of $F : G \to G$ is $GL_n(\mathbb{F}_q[\epsilon]/(\epsilon^r))$. For $i \neq j$ in $[1, n]$, we consider the homomorphism $f_{ij} : k \to T$ which takes $x \in k$ to the diagonal matrix with $ii$-entry equal to $1 + \epsilon^{r-1}x$, $jj$-entry equal to $1 - \epsilon^r - x$ and other diagonal entries equal to 1. Since $T$ is connected and commutative, the group $S(T)$ is defined (see §5). Let $L \in S(T)$. We will assume that $L$ is regular in the following sense: for any $i \neq j$ in $[1, n]$, $f_{ij}^*L$ is not isomorphic to $Q_i$.

Let $\pi : B \to T$ be the obvious homomorphism. Consider the diagram

$$G \overset{a}{\leftarrow} Y \overset{b}{\to} T$$

where

$$Y = \{(g, xB) \in G \times G/B; x^{-1}gx \in B\}, a(g, xB) = g, b(g, xB) = \pi(x^{-1}gx).$$
Then \( b^*\mathcal{L} \) is a local system on \( Y \) and we may consider the complex \( a_i b^*\mathcal{L} \) on \( G \).

As in \( \S 5 \), we can find an integer \( m_0 > 0 \) such that, for any \( m \in \mathcal{M} = \{m_0, 2m_0, 3m_0, \ldots \} \), \( \mathcal{L} \) is associated to \( (\mathbb{F}_{q^m}, \psi_m) \) where \( \psi_m \in \text{Hom}(T^{F^m}, Q_l^\ast) \). We can regard \( \psi_m \) as a character \( B(\mathbb{F}_{q^m}) \to Q_l^\ast \) via \( \pi : B \to T \); inducing this from \( B(\mathbb{F}_{q^m}) \) to \( G(\mathbb{F}_{q^m}) \) we obtain a representation of \( G(\mathbb{F}_{q^m}) \) whose character is denoted by \( c_m \). It is easy to see (using the regularity of \( \mathcal{L} \)) that this character is irreducible.

For \( m \in \mathcal{M} \), there is a unique isomorphism \( (F^m)^\ast \mathcal{L} \iso \mathcal{L} \) of local systems on \( T \) which induces the identity on the stalk of \( \mathcal{L} \) at 1. This induces an isomorphism \( (F^m)^\ast (b^* \mathcal{L}) \iso b^* \mathcal{L} \) (where \( F : Y \to Y \) is \( (g, xB) \mapsto (F(g), F(x)B) \)) and an isomorphism \( (F^m)^\ast (a_i b^* \mathcal{L}) \iso a_i b^* \mathcal{L} \) in \( \mathcal{D}(G) \). Let \( \chi_m : G^{F_m} \to Q_l^\ast \) be the characteristic function of \( a_i b^* \mathcal{L} \) with respect to this isomorphism. From the definitions we see that \( \chi_m = c_m \). This shows that \( a_i b^* \mathcal{L} \) behaves like a character sheaf except for the fact that it is not clear that it is an intersection cohomology complex.

We conjecture that:

(a) if \( \mathcal{L} \) is regular then \( a_i b^* \mathcal{L} \) is an intersection cohomology complex on \( G \).

(The conjecture also makes sense and is expected to be true when \( GL_n \) is replaced by any reductive group, and \( G \) by the corresponding group over \( A_r \).) Thus one can expect that there is a theory of character sheaves for \( G \), as far as generic principal series representations and their twisted forms is concerned. But one cannot expect a complete theory of character sheaves in this case (see \( \S 13 \)).

In \( \S 9-\S 12 \) we prove the conjecture in the special case where \( G = GL_2(k) \) and \( r = 2 \).

9. Let \( \mathcal{A} = \mathcal{A}_2 = k[\epsilon]/(\epsilon^2) \). Let \( V \) be a free \( \mathcal{A} \)-module of rank 2. Let \( G \) be the group of automorphisms of the \( \mathcal{A} \)-module \( V \). This is the group of all automorphisms of the 4-dimensional \( k \)-vector space \( V \) that commute with the map \( \epsilon : V \to V \) given by the \( \mathcal{A} \)-module structure. Hence \( G \) is an algebraic group of dimension 8 over \( k \). Let \( \mathcal{G} \) be the set of all pairs \( (g, V_2) \) where \( g \in G \) and \( V_2 \) is a free \( \mathcal{A} \)-submodule of \( V \) of rank 1 such that \( gV_2 = V_2 \). For \( k = 1, 2 \), let \( X_k \) be the set of all \( \mathcal{A} \)-submodules of \( V \) that have dimension \( k \) as a \( k \)-vector space. Let \( \tilde{G} \) be the set of all triples \( (g, V_1, V_2) \) where \( g \in G \), \( V_1 \subset X_1, V_2 \subset X_2, V_1 \subset V_2, gV_1 = V_1, gV_2 = V_2 \) and the scalars by which \( g \) acts on \( V_1 \) and \( V_2/V_1 \) coincide. We can regard \( \mathcal{G} \) as a subset of \( \tilde{G} \) by \( (g, V_2) \mapsto (g, \epsilon V_2, V_2) \). Note that \( \tilde{G} \) is naturally an algebraic variety over \( k \) and \( 0\tilde{G} \) is an open subset of \( \tilde{G} \).

The group of units \( \mathcal{A}' \) of \( \mathcal{A} \) is an algebraic group isomorphic to \( k^* \times k \). Hence \( S(\mathcal{A}') \) is defined. Let \( \mathcal{L}_1 \in S(\mathcal{S}'), \mathcal{L}_2 \in S(\mathcal{S}') \). Let \( \mathcal{L} = \mathcal{L}_1 \boxtimes \mathcal{L}_2 \in S(\mathcal{A'} \times \mathcal{A}) \), \( \mathcal{E} = \mathcal{L}_2 \boxtimes \mathcal{L}_1^* \in S(\mathcal{A}') \). Define \( f : 0\tilde{G} \to \mathcal{A'} \times \mathcal{A'} \) by \( f(g, V_2) = (\alpha_1, \alpha_2) \) where \( \alpha_1 \in \mathcal{A}' \) is given by \( g v = \alpha_1 v \) for \( v \in V_2 \) and \( \alpha_2 \in \mathcal{A}' \) is given by \( g v' = \alpha_2 v' \) for \( v' \in V/V_2 \). Let \( \tilde{\mathcal{L}} = f^*(\mathcal{L}_1 \boxtimes \mathcal{L}_2) \), a local system on \( 0\tilde{G} \). Define \( f_i : 0\tilde{G} \to \mathcal{A}' \) (\( i = 1, 2 \)) by \( f_1(g, V_2) = \alpha_1 \alpha_2, f_2(g, V_2) = \alpha_1 \) where \( \alpha_1, \alpha_2 \) are as above. Then \( \tilde{\mathcal{L}} = f_1^* \mathcal{L}_1 \boxtimes f_2^* \mathcal{L}_2 \). (We use \( 5(a) \).)

We shall assume that \( \mathcal{L} \) is regular in the following sense: the restriction of \( \mathcal{E} \) to
the subgroup $T = \{1 + \epsilon c; c \in \mathbb{k}\}$ of $\mathcal{A}'$ is not isomorphic to $\mathbb{Q}_l$.

**Lemma 10.** (a) $\tilde{G}$ is an irreducible, smooth variety and $\tilde{G} - 0\tilde{G}$ is a smooth irreducible hypersurface in $\tilde{G}$.

(b) We have $IC(\tilde{G}, \tilde{L})|_{\tilde{G} - 0\tilde{G}} = 0$.

Note that $f_1 : 0\tilde{G} \to \mathcal{A}'$ extends to the whole of $\tilde{G}$ by $f_1(g, V_1, V_2) = \det_A(g : V \to V)$. Hence $f_1^*\mathcal{L}_1$ extends to a local system on $\tilde{G}$ and we have $IC(\tilde{G}, \tilde{L}) = f_1^*\mathcal{L}_1 \otimes IC(\tilde{G}, f_2^*\mathcal{E})$. Hence to prove (b) it is enough to show that $IC(\tilde{G}, f_2^*\mathcal{E})$ is zero on $\tilde{G} - 0\tilde{G}$.

Let $Z$ (resp. $H$) be the fibre of the second projection $\tilde{G} \to X_1$ (resp. $\tilde{G} - 0\tilde{G} \to X_1$) at $V_1 \in X_1$. Since $\mathcal{G}$ acts transitively on $X_1$ it is enough to show that $Z$ is smooth, irreducible, $H$ is a smooth, irreducible hypersurface in $Z$ and $IC(Z, f_2^*\mathcal{E})$ is zero on $H$ (the restriction of $f_2$ to $Z$ is denoted again by $f_2$).

Let $e_1, e_2$ be a basis of $V$ such that $V_1 = k e_1$. The subspaces $V_2 \subset V_2$ such that $V_1 \subset V_2$ are exactly the subspaces $V_2 z', z'' = k e_1 + k(z'e_1 + z'' e_2)$ where $(z', z'') \in \mathbb{k}^2 - \{0\}$. An element $g \in \mathcal{G}$ is of the form

$$g e_1 = a_0 e_1 + b_0 e_2 + a_1 e_1 + b_1 e_2,$$

$$g e_2 = c_0 e_1 + d_0 e_2 + c_1 e_1 + d_1 e_2$$

where $a_i, b_i, c_i, d_i \in \mathbb{k}$ satisfy $a_0 d_0 - b_0 c_0 \neq 0$.

The condition that $ge e_1 \in ke e_1$ is $b_0 = 0$. The condition that $g V_2 z', z'' = V_2 z', z''$ is that $z'b_1 + z'' d_0 = a_0 z''$ if $z' \neq 0$ (no condition if $z' = 0$). The condition that the scalars by which $g$ acts on $V_1$ and $V_2 z', z'' / V_1$ coincide is $a_0 = d_0$ if $z' = 0$ (no condition if $z' \neq 0$).

We see that we may identify $Z$ with

$$\{(a_0, c_0, d_0, a_1, b_1, c_1, d_1; z', z'') \in \mathbb{k}^7 \times (\mathbb{k}^2 - \{0\})/\mathbb{k}^*;$$

$$a_0 \neq 0, d_0 \neq 0, z'b_1 = z''(a_0 - d_0)\}$$

and $H$ with the subset defined by $z' = 0$. In this description it is clear that $Z$ is irreducible, smooth and $H$ is a smooth, irreducible hypersurface in $Z$. The function $f_2$ takes a point with $z' \neq 0$ to $a_0 + \epsilon(a_1 + z'' z' c_0)$. To prove the statement on intersection cohomology we may replace $Z$ by the open subset $z'' \neq 0$ containing $H$. Thus we may replace $Z$ by

$$Z_1 = \{(a_0, c_0, d_0, a_1, b_1, c_1, d_1; z) \in \mathbb{k}^7 \times \mathbb{k}; a_0 \neq 0, d_0 \neq 0, zb_1 = a_0 - d_0\}$$

and $H$ by the subset defined by $z = 0$. The function $f_2$ is defined on $Z_1 - H$ by

$$a_0 + \epsilon(a_1 + z^{-1} c_0) = (a_0 + \epsilon a_1)(1 + \epsilon z^{-1} c_0 a_0^{-1}).$$

Thus $f_2 = f_3 f_4$ where $f_3$ (resp. $f_4$) is defined on $Z_1 - H$ by $a_0 + \epsilon a_1$ (resp. $1 + \epsilon z^{-1} c_0 a_0^{-1}$). Hence $f_2^*\mathcal{E} = f_3^*\mathcal{E} \otimes f_4^*\mathcal{E}$. Now $f_3$ extends to $Z_1$ hence $f_3^*\mathcal{E}$ extends
to a local system on \( Z_1 \). We have \( IC(Z_1, f_2^*\mathcal{E} \otimes f_4^*\mathcal{E}) = f_2^0\mathcal{E} \otimes IC(Z_1, f_4^*\mathcal{E}) \). It is enough to show that \( IC(Z_1, f_4^*\mathcal{E}) \) is zero on \( H \). We make the change of variable \( c = c_0a_0^{-1} \). Then \( Z_1 \) becomes

\[
Z_1 = \{(a_0, c, a_1, b_1, c_1, d_1; z) \in k^7 \times k; a_0 \neq 0, a_0 - zb_1 \neq 0\},
\]

\( H \) is the subset defined by \( z = 0 \) and \( f_4 : Z_1 - H \to \mathcal{A}' \) is given by \( 1 + \varepsilon z^{-1}c \). Let \( \tilde{Z}_1 = \{(a_0, c, a_1, b_1, c_1, d_1; z) \in k^7 \times k \} \) and let \( H_1 \) be the subset of \( \tilde{Z}_1 \) defined by \( z = 0 \). Then \( Z_1 \) is open in \( \tilde{Z}_1 \) and \( f_4 \) is well defined on \( \tilde{Z}_1 - H_1 \) by \( 1 + \varepsilon z^{-1}c \).

Hence \( f_4^*\mathcal{E} \) is well defined on \( \tilde{Z}_1 - H_1 \). It is enough to show that \( IC(\tilde{Z}_1, f_4^*\mathcal{E}) \) is zero on \( H_1 \). Let \( H' = \{(c, z) \in k^2; z = 0\} \) and define \( f' : k^2 - H' \to \mathcal{A}' \) by \( f'(c, z) = 1 + \varepsilon z^{-1}c \). It is enough to show that \( IC(k^2, f'^*\mathcal{E}) \) is zero on \( H' \). Let \( P \) be the projective line associate to \( k^2 \). Then \( H' \) defines a point \( x_0 \in P \). Since \( f' \) is constant on lines, it defines a map \( h : P - \{x_0\} \to \mathcal{A}' \). Since \( P \) is 1-dimensional we have \( IC(P, h^*\mathcal{E}) = \mathcal{F} \) where \( \mathcal{F} \) is a constructible sheaf on \( P \) whose restriction to \( P - \{x_0\} \) is \( h^*\mathcal{E} \). It is enough to show that

- \((c)\) the stalk of \( \mathcal{F} \) at \( x_0 \) is 0;
- \((d)\) \( H^i(P, \mathcal{F}) = 0 \) for \( i = 0, 1 \).

(Indeed, \((c)\) implies that \( IC(k^2, f'^*\mathcal{E}) \) is zero at \((c, 0)\) with \( c \neq 0 \) and \((d)\) implies that \( IC(k^2, f'^*\mathcal{E}) \) is zero at \((0, 0)\).)

Consider the standard \( F_q \)-rational structures on \( k^2, X, \mathcal{A}' \) and let \( F \) be the corresponding Frobenius map. We may assume that \( \mathcal{E} \) is associated as in §5 to \((F_q, \psi)\) where \( \psi \in \text{Hom}(\mathcal{A}', \bar{Q}_l^e) \). For any \( m \in \mathbb{Z}_{\geq 0} \) there is a unique isomorphism \( \phi_m : (F^m)^*\mathcal{E} \isom \mathcal{E} \) which induces the identity on the stalk of \( \mathcal{E} \) at \( 1 \). The characteristic function of \( \mathcal{E} \) with respect to this isomorphism is \( a' \mapsto \psi(N_{F_m/F}(a')) \), \( a' \in \mathcal{A}'_{/F_m} \). Since, by assumption, \( \mathcal{E}|_T \) is not isomorphic to \( \bar{Q}_l^e \), \( \psi|_T \) is not the trivial character. Hence \( \psi \circ N_{F_m/F} : \mathcal{A}'_{/F_m} \to \bar{Q}_l^e \) is non-trivial on \( T_{/F_m} \). Now \( \phi_m \) induces an isomorphism \( \phi'_m : (F^m)^*h^*\mathcal{E} \isom h^*\mathcal{E} \). We show that

\[
\sum_{x \in P \setminus \{x_0\}} \text{tr}(\phi'_m, (h^*\mathcal{E})_x) = 0.
\]

An equivalent statement is:

\[
\sum_{(c, z) \in (F_q)^m \times \bar{F}_{q^m}^*} (\psi \circ N_{F_m/F}) (1 + \varepsilon c^{-1}z) = 0,
\]

which follows from the fact that \( \psi \circ N_{F_m/F} : \mathcal{A}'_{/F_m} \to \bar{Q}_l^e \) is non-trivial on \( T_{/F_m} \).

Introducing \((e)\) in the trace formula for Frobenius, we see that

\[
\sum_{i=0}^2 (-1)^i \text{tr}(\phi'_m, H^i(P, \mathcal{F})) = \text{tr}(\phi'_m, \mathcal{F}_{x_0})
\]

where \( \mathcal{F}_{x_0} \) is the talk of \( \mathcal{F} \) at \( x_0 \) and \( \phi'_m \) is in fact equal to \( \phi_1' m \) (for \( m = 1, 2, 3, \ldots \)). By Deligne’s purity theorem, \( H^i(P, \mathcal{F}) \) together with \( \phi'_1 \) is pure of weight \( i \); by Gabber’s theorem [BBD], \( \mathcal{F}_{x_0} \) together with \( \phi'_1 \) is mixed of weight \( \leq 0 \). Hence from \((f)\) we deduce that \( H^1(P, \mathcal{F}) = 0, H^2(P, \mathcal{F}) = 0 \) and \( \dim H^0(P, \mathcal{F}) = \dim \mathcal{F}_{x_0} \). By the hard Lefschetz theorem [BBD] we have \( \dim H^0(P, \mathcal{F}) = \dim H^2(P, \mathcal{F}) \). It follows that \( H^0(P, \mathcal{F}) = 0 \) hence \( \mathcal{F}_{x_0} = 0 \). This proves \((c),(d)\). The lemma is proved.

**Lemma 11.** Define \( \rho : \tilde{G} \to G \) by \( (g, V_2) \mapsto g \). Let \( K = \rho_\tilde{L} \). Let \( G_0 \) be the open dense subset of \( G \) consisting of all \( g \in G \) such that \( g : eV \to eV \) is regular,
semisimple. Let $\rho_0 : \rho^{-1}(G_0) \to G_0$ be the restriction of $\rho$. Then $\rho_0! \mathcal{L}$ is a local system on $G_0$. We have $\dim \text{supp} H^i K < \dim G - i$ for any $i > 0$.

The first assertion of the lemma follows from the fact that $\rho_0$ is a double covering. To prove the second assertion it is enough to show that, for $i > 0$, the set $G_i$ consisting of the points $g \in G$ such that $\dim \rho^{-1}(g) = i$ and $\oplus_j H^j_\mathbb{C}(\rho^{-1}(g), \mathcal{L}) \neq 0$ has codimension $> 2i$ in $G$.

Consider the fibre $\rho^{-1}(g)$ for $g \in G$. We may assume that, with respect to a suitable $A$-basis of $V$, $g$ can be represented as an upper triangular matrix $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, c$ in $A$ and $b \in A$. (Otherwise, $\rho^{-1}(g)$ is empty.) There are five cases:

Case 1. $a - d \in A'$. Then $\rho^{-1}(g)$ consists of two points.

Case 2. $a - d \in \epsilon A, b \in A'$. Then $\rho^{-1}(g)$ is an affine line.

Case 3. $a - d \in \epsilon A - \{0\}, b \in \epsilon A$. Then $\rho^{-1}(g)$ is a disjoint union of two affine lines.

Case 4. $a = d, b \in \epsilon A - \{0\}$. Then $\rho^{-1}(g)$ is an affine line.

Case 5. $a = d, b = 0$. Then $\rho^{-1}(g)$ is an affine line bundle over a projective line.

In case 2, we may identify $\rho^{-1}(g), \mathcal{L}|_{\rho^{-1}(g)}$ with $P - \{x_0\}, F|_{P - \{x_0\}}$ in the proof of Lemma 10. Then the argument in that proof shows that $H^j_\mathbb{C}(\rho^{-1}(g), \mathcal{L}) = 0$ for all $j$. We see that $G_1$ consists of all $g$ as in case 3 and 4, hence $G_1$ has codimension 3 in $G$. We see that $G_2$ consists of all $g$ as in case 5, hence $G_2$ has codimension 6 in $G$. The lemma is proved. Note that without the assumption that $\mathcal{L}$ is regular, the last assertion of the lemma would not hold (there would be a violation coming from $g$ in case 2.)

12. We show:

(a) 

$$\rho_1! \mathcal{L} = IC(G, \rho_0! \mathcal{L}).$$

Define $\bar{\rho} : \tilde{G} \to G$ by $\bar{\rho}(g, V_1, V_2) = g$. Clearly, $\bar{\rho}$ is proper. Let $j : \mathcal{G} \to G$ be the inclusion. We have $\rho = \bar{\rho} \circ j$ hence $\rho_1! \mathcal{L} = \bar{\rho}_1! (j_1! \mathcal{L})$. By Lemma 10, we have $j_1! \mathcal{L} = IC(\mathcal{G}, \mathcal{L})$ hence $\rho_1! \mathcal{L} = \bar{\rho}_1! IC(\mathcal{G}, \mathcal{L})$. Since $\bar{\rho}$ is proper, $\bar{\rho}_1!$ commutes with the Verdier duality $\mathcal{D}$. Hence $\mathcal{D}(\rho_1! \mathcal{L}) = \bar{\rho}_1! \mathcal{D} IC(\mathcal{G}, \mathcal{L})$. Hence $\mathcal{D}(\rho_1! \mathcal{L})$ equals $\bar{\rho}_1 IC(\tilde{G}, \mathcal{L}^*)$ up to a shift. Now the same argument that shows $j_1! \mathcal{L} = IC(\mathcal{G}, \mathcal{L})$ shows also $j_1\mathcal{L}^* = IC(\mathcal{G}, \mathcal{L}^*)$. Hence, up to shift, $\mathcal{D}(\rho_1! \mathcal{L})$ equals $\bar{\rho}_1 j_1\mathcal{L}^* = \rho_1! \mathcal{L}^*$. Now the argument in Lemma 12 can also be applied to $\mathcal{L}^*$ instead of $\mathcal{L}$ and yields $\dim \text{supp} H^i \rho_1! \mathcal{L}^* < \dim G - i$ for any $i > 0$. Thus, $\rho_1! \mathcal{L}$ satisfies the defining properties of $IC(G, \rho_0! \mathcal{L})$ hence it is equal to it. This proves (a).

We see that conjecture 8(a) holds for $n = 2, r = 2$.

13. If $G$ is a connected affine algebraic group over $k$ which is neither reductive nor nilpotent, one cannot expect to have a complete theory character sheaves for $G$. Assume for example that $G$ is the group of all matrices

$$[a, b] = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$
with entries in \( k \). The group \( G(F_q) \) (for the obvious \( F_q \)-rational structure) has \( (q - 1) \) one dimensional representations and one \((q - 1)\)-dimensional irreducible representation. The character of a one dimensional representation can be realized in terms of an intersection cohomology complex (a local system on \( G \)), but that of the \((q - 1)\) dimensional irreducible representation appears as a difference of two intersection cohomology complexes, one given by the local system \( \overline{Q}_l \) on the unipotent radical of \( G \) and one supported by the unit element of \( G \). A similar phenomenon occurs for \( G \) as in §9 and for a \((q^2 - 1)\)-dimensional irreducible representation of \( G(F_q) \).

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DEPARTMENT OF MATHEMATICS, M.I.T., CAMBRIDGE, MA 02139