Parametric invariance

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We examine the development of the concept of parametric invariance in classical mechanics, quantum mechanics, statistical mechanics, and thermodynamics, and particularly its relation to entropy. The parametric invariance was used by Ehrenfest as a principle related to the quantization rules of the old quantum mechanics. It was also considered by Rayleigh in the determination of pressure caused by vibration, and the general approach we follow here is based on his. Specific calculation of invariants in classical and quantum mechanics are determined. The Hertz invariant, which is a volume in phase space, is extended to the case of a variable number of particles. We show that the slow parametric change leads to the adiabatic process, allowing the definition of entropy as a parametric invariance.

I. INTRODUCTION

When a mechanical system is under the influence of a disturbance caused by a time variation of one of its parameters, we expect its properties to change. However, it was found that there are some properties that remain invariant if the parameter changes very slowly. It is customary to trace the origin of this type of invariance to the Solvay Congress held in Brussels in 1911 [1]. During the discussion that followed the Einstein lecture, Lorentz remembered a conversation he had with Einstein sometime earlier. In this conversation, he asked him how the energy of a simple pendulum varies when its length is shortened by holding the string between two fingers and sliding down. Einstein replied that, if the length of the pendulum is changed in an infinitely slow manner, the energy varies in proportion to the frequency of oscillations. In other words, the ratio between energy and frequency remains invariant.

This concept of invariance appeared more consistently in the writings of Ehrenfest [2–6] on the formulation of the theory of quanta, later called old quantum theory [2, 7, 8]. He used these invariants to formulate his procedure for obtaining quantized states. To this end, he introduced in 1913 [9, 10] a hypothesis according to which the allowed motions of a system are transformed into allowed motions if the system is affected by a reversible adiabatic change. In a paper published in the following year, Einstein used the Ehrenfest hypothesis, and called it the adiabatic hypothesis [11]. The invariant quantities resulting from a reversible adiabatic change Ehrenfest called adiabatic invariants in a paper of 1914 [12]. In that same paper Ehrenfest explained what he meant by a reversible adiabatic change: it is an influence on the system in which the parameters change in an infinitely slow way.

In spite of the explanation given by Ehrenfest, the influence of the infinitely slow change of a parameter became associated to the term adiabatic. Jeans considered the term not particularly a happy one [15]. Accordingly, we find it more appropriate to name it by its definition and not by its consequences, and call it a slow parametric action, which in addition avoids the reference to any thermodynamic meaning. The invariant that results from this action we call parametric invariant. We reserve the term adiabatic invariant for a thermodynamic quantity that is constant along a slow adiabatic process, an example of which is the well known Poisson relation between pressure and volume for an ideal gas.

Here we review the concept of parametric invariance through the critical analysis of its evolution and how it is treated in classical mechanics and in quantum mechanics as well as its relation with thermodynamics particularly with entropy. The parametric invariance is included as a subject of classical mechanics [16, 22], usually connected with the technique of action-angle variables. It is treated in quantum mechanics [23–26], kinetic theory and statistical mechanics [27–29], dynamics of charged particles [30–33], and has been applied to specific problems by several authors [34–39], particularly in modern computer calculations to determine entropy and free energy by the method of adiabatic switching [40, 41].

II. EHRENFEST PRINCIPLE

Ehrenfest enunciated his hypothesis in 1916 in the following terms [12–14]: If a system be affected in a reversible adiabatic way, allowed motions are transformed into allowed motions. In the paper of 1914, Einstein stated it in the following terms [11]: With reversible adiabatic changes of a parameter, every quantum-theoretically possible state changes over into another possible state. In both statements, reversible adiabatic change is to be understood as a slow variation of a parameter.

In his paper of 1916 [12, 13], Ehrenfest considered a periodic system and showed that the time integral of twice the kinetic energy $K$ over a period,

$$I = \int 2K\,dt,$$

is a parametric invariant. Defining the time average $\overline{K}$ of the kinetic energy by

$$\overline{K} = \nu \int K\,dt,$$
where \( \nu \) is the frequency, the inverse of the period, the invariant is equivalent to \( 2K/\nu \). For a harmonic oscillator the energy \( E \) is twice the kinetic energy and the invariant becomes \( E/\nu \).

Ehrenfest had already presented the invariant (1) in his publication of 1913 but now he provided a demonstration through the use of the Lagrange analytical theory. Considering that the kinetic energy \( K \) of a system with many degrees of freedom is a quadratic form in the variables \( \dot{q}_k \), the time derivative of the coordinates \( q_k \), the Euler theorem on homogeneous functions allows us to write

\[
K = \frac{1}{2} \sum_k p_k \dot{q}_k, \quad (3)
\]

where \( p_k = \partial K/\partial \dot{q}_k \) is the momentum conjugate to \( q_k \). Using this expression Ehrenfest writes the integral (1) in the form

\[
I = \sum_k \int p_k dq_k. \quad (4)
\]

The geometrical interpretation of this expression was given by Ehrenfest as follows. In the phase space, the representative point of the system describes a closed curve which projects closed curves on each one of the planes \((q_k, p_k)\). Each term

\[
I_k = \int p_k dq_k \quad (5)
\]

of the sum in (4) represents the area of each one of the projected closed curves.

In 1915, Wilson [42] and by Sommerfeld [43], independently postulated the quantization rule by the use of phase integrals,

\[
\int p_i dq_i = n_i h, \quad (6)
\]

where \( n_i \) is an integer number and \( h \) is the Planck constant. These phase integrals were shown by Schwarzschild [44] and by Epstein [45, 46] to emerge when it is possible to separate variables by using the Hamilton-Jacobi theory to systems in which the variables can be separable. The Wilson-Sommerfeld rule is then applied to each pair of these separable canonically conjugate variables, called action and angle variables by Schwarzschild [44].

At the end of his paper of 1916, Ehrenfest asked himself whether the phase integral (6) appearing in the papers of Schwarzschild and Epstein could also be an invariant. The demonstration that indeed each one of these phase integrals is an invariant was shown by Burgers in 1916 [47, 48]. According to Burgers, if the momentum \( p_k \) in the integral (5) depends only on \( q_k \) then \( I_k \) is an invariant. Notice that Ehrenfest had showed that the sum of integrals of the type (5) is an invariant, nothing being said about each one of them.

In 1913 Bohr proposed his atomic model based on the assumptions that the electron describes stationary orbits around the nucleus [49]. Bohr assumed that the frequency \( \nu \) of the radiation emitted is half the frequency of revolution of the electron and that the amount of energy emitted is \( h \nu \) times an integer \( n \). From these assumptions he obtained the binding energy \( E \) of the electron as

\[
E = \frac{2\pi^2 me^4}{h^2 n^2}, \quad (7)
\]

where \( e \) is the charge of the electron and \( m \) its mass.

Another fundamental assumption made by Bohr was as follows. When the electron passes from one stationary orbit to another, the loss of energy in the form of radiation is equal to \( h \nu \).

As a way of justifying the stationarity of the orbits, Bohr employed the Ehrenfest hypothesis, which he named the principle of mechanical transformability, and appeared in 1918 in his paper on the quantum theory of line-spectra [50]. Bohr explains that this name indicate in a more direct way the content of the principle. This reference on the Bohr paper of 1918 turned the Ehrenfest hypothesis widely known but at the same time it became closely linked to Bohr’s work [6]. The same can be said of the Burger’s paper on the invariance of the phase integrals [6].

Although the Ehrenfest principle explained the permanence of a system in a given state, it did not explain why the states are discretized. Neither did the Wilson-Sommerfeld rule as it was introduced as a postulate. The explanation came with the emergence of quantum mechanics around 1925 which replaced classical mechanics in the explanation of the motion at the microscopic level. Within quantum mechanics, the Wilson-Sommerfeld rule was found to be valid at higher quantum numbers. As to the Ehrenfest principle, the works of Born [51], Fermi and Persico [52], and Born and Fock [53] turned it into a theorem of quantum mechanics [5, 6].

### III. WAVE MECHANICS

The quantization of the electronic orbits of the hydrogen atom used by Bohr and the quantization rule used by Sommerfeld explained accurately the spectrum of the hydrogen including the fine structure of the hydrogen lines. In spite of its successful explanation of the spectrum of atoms and several problems in atomic physics, the quantum physics up to 1925 was a collection of quantum rules without a unifying principle [2].

In 1925, two quantum theories were proposed which were latter shown to be equivalent. Heisenberg proposed a matrix theory [54] and Schrödinger [55, 56] proposed a wave theory. The point of departure of the Schrödinger theory was the relation between the wave theory of light and geometric optics [2]. Hamilton had shown that there is an analogy between the principle of least action of
mechanics and the Fermat principle of geometric optics. The principle of least action is
\[ \delta \int 2K \, dt = 0, \] (8)
where \( K \) is the kinetic energy and the minimization of the action is subject to trajectories where the energy \( T + V \) is conserved, and can be written in the form
\[ \delta \int \sqrt{2m(E-V)} \, ds = 0. \] (9)
The Fermat principle of geometric optics is
\[ \delta \int \frac{ds}{v} = 0, \] (10)
where \( v \) is the velocity of light. Thus the Fermat principle can be regarded as the principle of least action where \( v^{-1} \) plays the role of the integrand of (9) [21]. As there is a wave theory of light, which reduces to geometric optics for small wavelength, the Schrödinger theory is understood as a wave theory that reduces to the mechanics.

The wave representation of a quantum theory by Schrödinger was suggested by de Broglie who associated a wave to the motion of a particle which he called wave phase [57]. According to de Broglie, the wavelength \( \Lambda \) of the wave associated to a particle of momentum \( p \) is given by \( p = h/\lambda \), where \( h \) is the Planck constant. The use of wave naturally leads to quantization. For instance, the possible states of a standing wave are the normal modes of vibration, and the possible values of the wavelengths of a standing wave forms a discretized set of values. More generally, the quantization comes from the fact that the solution of the wave equation naturally result in the solution of an eigenvalue problem, as stated by Schrödinger in the title of his paper on wave mechanics.

In the first part of his paper on wave mechanics, Schrödinger introduced the time independent equation for an electron under the action of the inverse square force,
\[ \nabla^2 \psi + \frac{2m}{K^2} (E + \frac{e^2}{r}) \psi = 0, \] (11)
where \( K \), according to Schrödinger, must have the value \( K = h/2\pi \) so that the discrete spectrum corresponds to the Balmer series. In the second paper, he considered the one-dimensional oscillator whose equation he wrote in the abbreviated form
\[ \frac{d^2 \psi}{dx^2} + \left( \frac{a}{\sqrt{b}} - x^2 \right) \psi = 0, \] (12)
and determined the proper values of \( a/\sqrt{b} \), which are 1,3,5,... by the use of the known solution of equation (12) in terms of Hermite orthogonal functions. From this result the allowed energies of the oscillator are \( E = \hbar v(n + 1/2) \). In the forth part, Schrödinger postulates that the wave equation is a first order in time and writes
\[ \nabla^2 \psi - \frac{8\pi^2}{\hbar^2} V \psi \mp \frac{4\pi i}{\hbar} \frac{\partial \psi}{\partial t} = 0. \] (13)

In the year following the publication of the wave theory by Schrödinger there appears an approximation method proposed independently by Wentzel [58], by Brillouin [59] and by Kramers [60]. This approximation correspond to a perturbation expansion in powers of the Planck constant. The zero order approximation gives the classical result. The first order results in the Wilson-Sommerfeld rule of the old quantum mechanics. The approximation is obtained by writing the wave function in the form [23]
\[ \psi = Ae^{iS/\hbar}, \] (14)
where \( S \) does not depend on \( h \) and \( A \) is independent of time and is an expansion in powers of \( h \). Replacing it in the time independent Schrödinger equation,
\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V \psi = E \psi, \] (15)
the equation containing only terms of order zero in \( h \) is
\[ \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V = E, \] (16)
and the equation coming from terms of first order in \( h \) is
\[ \frac{\partial A}{\partial x} \frac{\partial S}{\partial x} + A \frac{\partial^2 S}{\partial x^2} = 0. \] (17)

The equation (17) can be integrated with the result \( A^2 \) proportional to the reciprocal of \( (\partial S/\partial x) \). It is now left to solve the equation (16). We consider two cases according to the sign of \( E - V \). If \( E \geq V \), we define
\[ \kappa = \sqrt{2m(E-V)}, \] (18)
and the solutions of the equations (16) and (17) are
\[ S = \pm \int \kappa(x) \, dx, \quad A = \frac{1}{\sqrt{\kappa(x)}}, \] (19)
If \( E < V \), we define
\[ \gamma = 2m(V - E), \] (20)
and the solutions are
\[ S = \pm i \int \gamma(x) \, dx, \quad A = \frac{1}{\sqrt{\gamma(x)}}. \] (21)

Let us suppose that the first condition occurs when \( a \leq x \leq b \), where \( V(a) = E \) and \( V(b) = E \), are the classical turning points. The connection of the solutions at the turning points leads to the condition [23]
\[ \int_a^b \kappa \, dx = h\pi(n + \frac{1}{2}). \] (22)
The ring is constrained to move vertically and as it moves the length BC of the pendulum changes although the total length ABC remains constant. The problem is to determine the force that tends to move the ring upwards as the pendulum swings.

The ring is acted by two vertical forces, one of them is upward and equal to the tension $\Gamma$ of the string and the other is downward and equal to $\Gamma \cos \theta$ where $\theta$ is the angle BCD. The net upward force on the ring is thus $F = \Gamma(1 - \cos \theta)$. Now the potential energy of the pendulum is $V = P\ell(1 - \cos \theta)$ where $P$ is the weight of the bob, and $\ell$ is the length BC of the pendulum. Considering that for small oscillations $\Gamma$ is approximately equal to $P$, one finds $F = V/\ell$. As the mean value of the potential energy is one half of the total energy $E$ of the pendulum, Rayleigh concludes that the upward mean force $F$ on the ring is

$$F = \frac{E}{2\ell}. \quad (24)$$

As the work done on the ring is equal to decrease in the energy of the pendulum, then $dE = -F\ell$, and $dE = -Ed\ell/2\ell$ which by integration gives $E = a/\sqrt{\ell}$, where $a$ is a constant. Although, Rayleigh did not mention it explicitly, it follows from this result that the quantity

$$I = E\sqrt{\ell} \quad (25)$$

is an invariant quantity when the length of the pendulum changes slowly with time. As the frequency of oscillation $\omega$ of a simple pendulum executing small oscillations is inversely proportional to $\sqrt{\ell}$ it follows that the quantity

$$I = \frac{E}{\omega} \quad (26)$$

is an invariant as well.

Rayleigh also treated in the same paper the problem of the force exerted by a vibrating stretched string on the points where it is attached. One end of the stretched string is fixed and the other is allowed to move by the use of a ring as shown in figure 2. The position of the ring determines the length of the vibrating string, which we denote by $\ell$. Rayleigh argues that the mean force $F$ acting on the ring is related to the total energy $E$ of the vibrating string by

$$F = \frac{E}{\ell}. \quad (27)$$

and is thus equal to the energy per unit length. Again, the work done on the ring equals the decrease in energy, $dE = -F\ell = -Ed\ell/\ell$ and, after integration, $E = a/\ell$ where $a$ is a constant, and now $E$ is inversely proportional to $\ell$. Thus,

$$I = E\ell \quad (28)$$

is invariant.

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**IV. RAYLEIGH APPROACH**

**A. Pendulum of variable length**

In a paper of 1902 [61], Rayleigh analyzed a simple pendulum with its string being varied very slowly. Motivated by the theoretical demonstration of the radiation pressure by Maxwell and its experimental confirmation by Lebedev, Rayleigh inquired whether any other kinds of vibration, such as sound vibrations, would also cause pressure. To answer this question, he posed the problem of finding the force acted by a vibrating pendulum on its pivot when its length changes slowly and continuously with time.

The length of the pendulum is changed by the use of a ring through which passes the string, as shown in figure [54x-1879], as shown in figure 1: (a) The pendulum with variable length. In the set up employed by Rayleigh the point A is fixed and the ring B moves vertically causing the variation of the length BC of the pendulum. In the set up used by Bossut and Lecornu, the ring B remains immobile and the point A moves vertically. (b) The forces acting on the ring when it is free to move upward, where $\Gamma$ is the tension of the string, $\Gamma_1 = \Gamma \cos \theta$, and $\Gamma_2 = \Gamma \sin \theta$. The net force is upward and equals $T(1 - \cos \theta)$.

Considering that the classical momentum $p$ is $\kappa$ or $-\kappa$, we may write this condition as

$$\oint p\, dx = h(n + \frac{1}{2}), \quad (23)$$

which is the Wilson-Sommerfeld rule except for the 1/2 term.

As the phase integral in the left hand side of equation (23) is invariant, it follows that the quantum number $n$ is an invariant. That is, if a parameter of the system is slowly varying in time, it remains in the same state with the same quantum number. Nevertheless, a demonstration of invariance of the quantum state was provided for the new quantum mechanics, without referring to the phase integral. In 1926, one year after the introduction of the quantum wave by Schrödinger, a demonstration of the invariance in quantum mechanics was given by Born [51] and by Fermi and Persico [52]. Two years later, a more general demonstration was provided by Born and Fock [53].

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**FIG. 1:** (a) The pendulum with variable length. In the set up employed by Rayleigh the point A is fixed and the ring B move vertically causing the variation of the length BC of the pendulum. In the set up used by Bossut and Lecornu, the ring B remains immobile and the point A moves vertically. (b) The forces acting on the ring when it is free to move upward, where $\Gamma$ is the tension of the string, $\Gamma_1 = \Gamma \cos \theta$, and $\Gamma_2 = \Gamma \sin \theta$. The net force is upward and equals $T(1 - \cos \theta)$.
B. Invariance

The main result of the approach by Rayleigh can be stated as follows. Let us consider a periodic system and the force $f$ acted by the system on the environment at a point which moves as a result of the change of a parameter $\lambda$. As the system is periodic the force $f$ oscillates in time but its time average $F$ over one cycle is nonzero. If the parameter changes slowly $F$ varies slowly and so does the energy $E$ of the system. These two quantities are related by

$$\frac{dE}{d\lambda} = -F,$$

if the parameter changes very slowly with time. To show this result we proceed as follows.

We consider a system with several degrees of freedom described by the Lagrangian $L = K - V$, where $K$ is the kinetic energy and $V$ is the potential energy. The Lagrangian $L(q, \dot{q}, \lambda)$ depends on the coordinates $q_i$ and velocities $\dot{q}_i$, which we are denoting collectively by $q$ and $\dot{q}$, respectively, and on a parameter $\lambda$ which depends on time. The Lagrange equations of motion are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

If we define the momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

then the equations of motion are written as

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i},$$

and the differential of $L$ as

$$dL = \sum_i p_i \, dq_i + \sum_i p_i \, dq_i + \frac{\partial L}{\partial \lambda} \, dt.$$

If we perform the Legendre transformation

$$H = \sum_i p_i \dot{q}_i - L,$$

then

$$dH = \sum_i \dot{q}_i \, dp_i - \sum_i \dot{p}_i \, dq_i - \frac{\partial L}{\partial \lambda} \, dt,$$

from which follows the Hamilton equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

where $H(q, p, \lambda)$ is the Hamiltonian, a function of the coordinates and momenta, and depends on time through the parameter $\lambda$. It also follows that

$$\frac{\partial H}{\partial \lambda} = \frac{dL}{dt},$$

where the first derivative is determined at constant $q$ and $p$ whereas the second at constant $q$ and $\dot{q}$.

The Hamiltonian $H$ equals the total energy $K + V$ if $K$ is a quadratic form in the velocities. Its variation with time is

$$\frac{dH}{dt} = \frac{\partial H}{\partial \lambda},$$

and as it depends on time through the parameter $\lambda$, it is not conserved. Defining the function $f$ by

$$f = -\frac{\partial H}{\partial \lambda},$$

the equation (38) can be written as

$$\frac{dH}{dt} = -cf,$$

where $c = d\lambda/dt$ is the rate of variation of the parameter. Now we proceed as follows. We divide the time axis into intervals equal to the cycle period $T$. The integration of the equation (40) over one cycle beginning at time $t_0$ and ending at time $t_1$ gives

$$\frac{\Delta H}{T} = -\frac{1}{T} \int_{t_0}^{t_1} cf(q, p, \lambda) \, dt,$$

where

$$\Delta H = H(q_1, p_1, \lambda_1) - H(q_0, p_0, \lambda_0),$$

and the subindexes refer to the beginning and ending of the interval.

We use an approximation in which the trajectory in phase space is replaced by a trajectory $(x(t), y(t))$ which is the solution of the equations of motion by considering that the parameter $\lambda$ is kept unchanged and equal to its value at the beginning at the interval. The variation of $H$ becomes

$$\Delta H = H(x_1, y_1, \lambda_1) - H(x_0, y_0, \lambda_0).$$

The variation $\Delta \lambda = \lambda_1 - \lambda_0$ of the parameter in the interval is equal to $\Delta \lambda = Tc$ and $c$ is small because the
parameter varies slowly with time. Therefore \( \Delta H / T = c \Delta H / \Delta \lambda \) can be approximated by

\[
\frac{c}{d\lambda} H(x, y, \lambda),
\]

(44)
calculated at the beginning of the interval.

Using again the same approximation, we replace \( q \) by \( x(t) \), and \( p \) by \( y(t) \) in the right hand side of (41). Defining

\[
F = \frac{1}{T} \int_{t_0}^{t_1} f(x, y, \lambda) dt,
\]

(45)
the equation (41) becomes

\[
\frac{dE}{d\lambda} = -F,
\]

(46)
where \( c \) was assumed \( c \) to be constant during the cycle. In this equation, \( E \) is equal to \( H(x, y, \lambda) \) and depends only on \( \lambda \) because it is conserved along one cycle, and the quantity \( F \) is understood as the time average of \( f \) along one cycle and also depends only on \( \lambda \). Since \( F \) is a function of \( \lambda \), this relation is understood as a differential equation in \( \lambda \). Its solution gives the explicit dependence of \( E \) on \( \lambda \), and on time since \( \lambda \) is a given function of time.

Let us use the notation

\[
\overline{f} = \frac{1}{T} \int_{t_0}^{t_0 + T} f dt,
\]

(47)
for the time average of \( f \) over one cycle of period \( T \). The function \( f \) depends on the parameter \( \lambda \), which is considered to be fixed. The main result can then be written in the abbreviated form

\[
\frac{dE}{d\lambda} = \frac{\partial H}{\partial \lambda},
\]

(48)
and \( E = \overline{H} \). Notice that, as \( H \) is conserved if \( \lambda \) is fixed, then \( E \) coincides with \( E = H \) during one period and this time average is immaterial.

C. Examples

Let us apply this approach to the Rayleigh pendulum. The Lagrangian function is given by

\[
L = \frac{1}{2} ml^2 \dot{\theta}^2 - \frac{1}{2} mg \ell \dot{\theta}^2.
\]

(49)
To determine \( f = -\partial H / \lambda \), we observe that using the equality (57), it can also be determined by \( f = \partial L / \partial \lambda \). Deriving \( L \) with respect to \( \ell \), we find

\[
f = ml \dot{\theta}^2 - \frac{1}{2} mg \ell \dot{\theta}^2.
\]

(50)
The energy \( E \) is given by

\[
E = \frac{1}{2} ml^2 \dot{\theta}^2 + \frac{1}{2} mg \ell \dot{\theta}^2.
\]

(51)
The equation of motion, keeping the parameter \( \ell \) unchanged is

\[
\ddot{\theta} = -g \theta,
\]

(52)
whose solution is \( \theta = c \cos \omega t \), where \( \omega = \sqrt{g / \ell} \). Replacing the solution in the expression for \( H \) and \( f \), and taking the time average over one cycle, we obtain

\[
E = \frac{1}{2} mgc^2 \ell, \quad \quad F = \frac{1}{4} mge^2, \quad \quad (53)
\]

which gives \( F = E / 2 \ell \), the result (24). The integration of \( dE / d\ell = -F \) gives the result \( E \sqrt{\ell} \) an invariant, or \( E / \omega \) an invariant, results already found.

For a particle of mass \( m \) bounded to a spring of coefficient \( k \), the Lagrangian function is

\[
L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2, \quad \quad (54)
\]

and the energy is

\[
E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} kx^2. \quad \quad (55)
\]
The equation of motion is \( m \ddot{x} = -kx \) whose solution is \( x = c \cos \omega t \), where \( \omega = \sqrt{k/m} \). Replacing these results in the expression for the energy, \( E = ka^2 / 2 \).

Let us suppose that the spring coefficient is the varying parameter. Then \( f = -x^2 / 2 \) and \( F = -a^2 / 4 \). Therefore, \( F = -E / 2k \) and the integration of \( dE / dk = -F \) gives \( E \sqrt{\ell} \) as an invariant, or \( E / \omega \).

Suppose now that the mass is the varying parameter. Then \( f = \dot{x}^2 / 2 \) and \( F = ka^2 / 4 \). Therefore \( F = E / 2m \) and the integration of \( dE / dm = -F \) gives \( E \sqrt{m} \) as an invariant, or \( E / \omega \).

Another example consists of a free particle of mass \( m \) that moves with speed \( v \) between two walls that are a distance \( \ell \) apart. The mean force \( F \) on the wall is the change of its momentum \( 2mv \) divided by the time \( 2\ell / v \) between two collisions, that is, \( F = mv \ell / \ell \). Considering that the kinetic energy is \( E = mv^2 / 2 \) one finds \( F = 2E / \ell \) and the integration of \( dE / dm = -F \) gives \( E \ell^2 / 2 \) as an invariant.

Let us consider now the vibrating stretched string. Denoting by \( u \) the transverse displacement \( PQ \) of the string at the point \( Q \) and by \( x \) the distance from this point to the fixed end \( A \), as shown in figure 2, the equation of motion for \( u \) is the wave equation

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
\]

(56)
where \( c \) is the velocity of the wave and equal to \( \sqrt{\Gamma / \mu} \), where \( \Gamma \) is the tension of the string and \( \mu \) is the mass per unit length. The wave equation is understood as the equation of motion associated to the Lagrangian \( L = T - V \) where \( K \) is the kinetic energy,

\[
K = \frac{\mu}{2} \int_0^\ell \left( \frac{\partial u}{\partial t} \right)^2 dx,
\]

(57)
and \( V \) is the potential energy,
\[
V = \frac{\Gamma}{2} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 dx. \tag{58}
\]
The energy is \( H = T + V \).

With the purpose of revealing the explicit dependence on \( \ell \), we change the variable of integration from \( x \) to \( y = x/\ell \), with the result
\[
K = \frac{\mu}{2} \int_0^1 \left( \frac{\partial u}{\partial t} \right)^2 dy, \tag{59}
\]
and \( V \) is the potential energy,
\[
V = \frac{\Gamma}{2\ell} \int_0^1 \left( \frac{\partial u}{\partial y} \right)^2 dy. \tag{60}
\]
Using these expressions on \( L = K - V \), we determine \( f = \partial L/\partial \ell \), which gives the following result \( f = H/\ell \), where \( H = K + V \). Therefore \( F = E/\ell \), which is the result \(27\), and, using the relation \( dE/d\ell = -F \), we find that \( E = A/\ell \) and \( E\ell \) is an invariant as already found.

The solution of the wave equation for vibrations between the fixed ends at \( x = 0 \) and \( x = \ell \) is the standing wave
\[
u = A \sin kx \cos \omega t, \tag{61}
\]
where \( k = n\pi/\ell \) and \( n \) is an integer, and \( \omega = kc \). Replacing these results into the expression for \( K \) and \( V \) and taking the time average over one cycle, we find
\[
E = \frac{\mu A^2 c^4 n^2 \pi^2}{4\ell} \quad F = \frac{\mu A^2 c^4 n^2 \pi^2}{4\ell^2}, \tag{62}
\]
showing that indeed \( E = a/\ell \) and \( F = E/\ell \).

\section{EHRENFEST INVARIANT}

\subsection{Invariance}

To demonstrate the invariance of expression \(11\) or its equivalent form \(13\) we proceed as follows. We start by writing \( F \), given by \(13\), in the form
\[
F = -\frac{1}{T} \int_0^T \frac{\partial H}{\partial \lambda} dt, \tag{63}
\]
where we have used the expression \(39\) for \( f \), and we are considering the variables \( \lambda \) and \( t \) independent of each other. Recalling that \( E = H \), we obtain
\[
\frac{dE}{d\lambda} = \frac{1}{T} \sum_i \int_0^T \left( \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial \lambda} + \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial \lambda} + \frac{\partial H}{\partial \lambda} \right) dt. \tag{64}
\]
Replacing it in the equation \(16\), written in the form
\[
\frac{dE}{d\lambda} + F = 0, \tag{65}
\]
we find
\[
\sum_i \int_0^T \left( \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial \lambda} + \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial \lambda} \right) dt = 0. \tag{66}
\]
Using the equations of motion \(46\), we find
\[
\sum_i \oint \left( \frac{\partial p_i}{\partial \lambda} dq_i - \frac{\partial q_i}{\partial \lambda} dp_i \right) = 0, \tag{67}
\]
which can be written in the form
\[
\frac{di}{d\lambda} = 0, \tag{68}
\]
where
\[
I = \frac{1}{2} \sum_i \oint (p_i dq_i - q_i dp_i). \tag{69}
\]
An integration by parts gives
\[
I = \sum_i \oint p_i dq_i. \tag{70}
\]
Since \( di/d\lambda = 0 \), it follows that \( I \) is indeed an invariant. We recall that this expression can also be written as
\[
I = \int_0^T 2K dt, \tag{71}
\]
where \( K \) is the kinetic energy and \( T \) is the period of the cycle.

The demonstration just carried out shows that the invariant \(10\) is a sum terms of the type
\[
I_i = \oint p_i dq_i. \tag{72}
\]
It does not say whether or not each term \( I_i \) is an invariant. However, if the momentum \( p_i \) in this integral \(72\) depends only on \( q_i \), which is a statement that the pair of variables \( (q_i, p_i) \) is separable from the others, then \( I_i \) is an invariant, which is the result obtained by Burgers. To show this result, it suffices to write \(72\) as
\[
I_i = \int p_i dq_i dt. \tag{73}
\]
In this form we see that the integrand is twice the kinetic energy of a system with one degree of freedom, as no other variables are involved. But this is the total kinetic energy of a system with one degree of freedom and is thus the Ehrenfest invariant. It is worth mentioning however that the separation of variables may only occur if an appropriate transformation of variables is performed.
B. Systems with one and two degrees of freedom

The Ehrenfest invariant for a system with one degree of freedom reduces to the phase integral

$$I = \oint pdq,$$

(74)

where \( p \) is the momentum conjugate to \( q \). The general form of a Lagrangian describing a conservative system with one degree of freedom is

$$L = \frac{1}{2}m\dot{q}^2 - V,$$

(75)

where \( m \) might depend on \( q \), and \( V \) is a function of \( q \) only, and \( p = \partial L/\partial \dot{q} \). As the energy is conserved we write \( E = H(q,p) \) which describes a closed curve on the phase space. Solving this equation for \( p \) and replacing the result in the phase integral, we find

$$I = 2\int \sqrt{2m(E-V)}dq,$$

(76)

where the integral is performed in the interval between the two points of return.

For a particle of mass \( m \) under the action of a harmonic force, the energy is given by

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = E,$$

(77)

where \( \omega \) is the frequency of oscillation. This equation describes in the phase space an ellipse of semi-axis equal to \( \sqrt{2mE} \) and \( \sqrt{2E/m\omega^2} \). The phase integral is the area of this ellipse and equals \( 2\pi E/\omega \), which is thus an invariant.

Another example is given by a free particle that moves along an axis and collides with walls that are a distance \( \ell \) apart. The integral (74) becomes equal to \( 2p\ell \) which is thus an invariant. Taking into account that the kinetic energy of the particle is \( E = p^2/2m \), it follows that \( E\ell^2 \) is an invariant as the wall moves slowly.

Let us consider now a particle of mass \( m \) moving in a plane under a central force. Using polar coordinates, the Lagrangian is given by

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V,$$

(78)

where \( V \) is the potential that depends on \( r \) but not on \( \theta \). The momenta conjugate to the \( r \) and \( \theta \) are, respectively, \( p_r = m\dot{r} \) and \( p_\theta = mr^2\dot{\theta} \).

One equation of motion is

$$\frac{dp_r}{dt} = mr\ddot{\theta} + f,$$

(79)

where \( f = -dV/dr \) is the centripetal force. The other equation of motion is \( dp_\theta/dt = 0 \) from which follows that the angular momentum \( p_\theta = mr^2\dot{\theta} \) is constant. Denoting by \( a \) this constant, then \( \dot{\theta} = a/mr^2 \) which replaced in the equation of motion for \( r \), gives

$$\frac{dp_r}{dt} = \frac{a^2}{mr^3} + f.$$

(80)

This equation tell us that the pair of variables \( (r,p_r) \) is separable and that

$$I_1 = \oint p_r dr$$

(81)

is an invariant.

To determine \( I_1 \) explicitly, we multiply (80) by \( p_r/m = \dot{r} \) and integrate in time to find

$$\frac{p_r^2}{2m} + V + \frac{a^2}{2mr^2} = E,$$

(82)

where \( E \) is a constant. Solving for \( p_r \) and replacing in the integral \( I_1 \), we get

$$I_1 = 2\int \sqrt{2m(E-V) - (a^2/r^2)} \, dr.$$

(83)

We remark that the integral

$$I_2 = \oint p_\theta d\theta$$

(84)

is also an invariant, in fact it is a constant, \( I_2 = 2\pi a \).

C. Particle under a central force

We wish to determine now the invariants of a system with three degrees of freedom corresponding to a particle under the action of a central force as is the case of the Kepler problem where this force is proportional to inverse of the square of the distance [16, 17, 19, 62]. In spherical coordinates, the variables are separable and for each pair of conjugate variables there corresponds an invariant of the form (74). Using spherical coordinates \( r, \theta, \phi \), with origin at the center of force, the kinetic energy is given by

$$K = \frac{m}{2}r^2 + \frac{m}{2}r^2\dot{\theta}^2 + \frac{m}{2}r^2\dot{\phi}^2 \sin^2 \theta,$$

(85)

and the Lagrangian is \( L = T - V \) where \( V(r) \) is the potential energy that depends on \( r \) only.

The conjugate momenta are

$$p_0 = mr\dot{r},$$

(86)

$$p_1 = mr^2\dot{\theta},$$

(87)

$$p_2 = mr^2\sin^2 \theta \dot{\phi},$$

(88)

and the equations of motion are

$$\frac{dp_0}{dt} = mr\ddot{\theta} + m\theta \ddot{\phi}^2 + f,$$

(89)
\[ \frac{dp_1}{dt} = mr^2 \dot{\phi}^2 \sin \theta \cos \theta, \]  
(90)

\[ \frac{dp_2}{dt} = 0, \]  
(91)

where \( f = -dV/dr \) is the central force that depends on \( r \) only. From the last equation it follows that \( p_2 \) is constant. Setting this constant equal to \( a \), then \( \dot{\phi} = a/mr^2 \sin^2 \theta \), which replaced in the equation of motion for \( p_1 \) gives

\[ \frac{dp_1}{dt} = \frac{a^2 \cos \theta}{mr^2 \sin^3 \theta} \]  
(92)

Multiplying this equation by \( p_1 = mr^2 \dot{\phi} \) and integrating in time, we find

\[ p_1^2 + \frac{a^2}{\sin^2 \theta} = b^2, \]  
(93)

where \( b \) is a constant. Therefore \( p_1 \) in the integral

\[ I_1 = \int p_1 d\theta \]  
(94)

depends only on \( \theta \) and is an invariant.

Let us write equation (89) in the form

\[ \frac{dp_0}{dt} = \frac{p_1^2}{mr^3} + \frac{p_2^2}{mr^3 \sin^2 \theta} + f. \]  
(95)

Replacing the result (93) in this equation and bearing in mind that \( p_2 = a \), we find

\[ \frac{dp_0}{dt} = \frac{b^2}{mr^3} + f. \]  
(96)

Multiplying by \( p_0/m = \dot{r} \) and integrating in time, we find

\[ \frac{p_0^2}{2m} + V + \frac{b^2}{2mr^2} = E, \]  
(97)

where \( E \) is a constant. Since \( p_0 \) depends only on \( r \), it follows that

\[ I_0 = \int p_0 dr \]  
(98)

is an invariant. Solving equation (77) for \( p_0 \) and replacing in this integral we reach the result

\[ I_0 = 2 \int \sqrt{2m(E - V) - (b^2/r^2)} \, dr. \]  
(99)

We remark that the motion of a particle under a central force is restricted to the plane defined by the velocity and the center of force. Therefore the problem could be reduced two a system with two degrees of freedom like we have done previously. However, there might be parameters that could remove the motion from this plane. In this case it is necessary to consider the problem in three dimensions as we have just done.

D. Particle on a magnetic field

Let us consider the motion of a particle of mass \( m \) and charge \( e \) in a uniform magnetic field \( B \) which is parallel to the \( z \) axis. The Lagrangian \( L \) in cylindrical coordinates is given by

\[ L = \frac{1}{2} m(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) + \frac{1}{2} eBr^2 \dot{\theta}. \]  
(100)

The momenta conjugate to \( r, \theta \) and \( z \) are, respectively,

\[ p_0 = mr \dot{r}, \]  
(101)

\[ p_1 = mr^2 \dot{\theta} + \frac{1}{2} eBr^2, \]  
(102)

\[ p_2 = m \dot{z}, \]  
(103)

and the equations of motion are

\[ \frac{dp_0}{dt} = mr \ddot{r} + eBr\dot{\theta}, \]  
(104)

\[ dp_1/dt = 0, \]  
and \[ dp_2/dt = 0. \]  
From these two last equations, it follows that \( \dot{z} = b \), a constant, and that

\[ m\dot{r}^2 + \frac{1}{2} eBr^2 = a, \]  
(105)

where \( a \) is another constant, or

\[ \dot{\theta} = \frac{a}{mr^2} - \frac{eB}{2m}. \]  
(106)

Replacing this last result in the equation of motion for \( p_0 \) we find

\[ \frac{dp_0}{dt} = \frac{a^2}{mr^3} - \frac{e^2 B^2 r}{4m}. \]  
(107)

Multiplying this equation by \( p_0/m = \dot{r} \) and integrating in time, we find

\[ \frac{p_0^2}{2m} + \frac{a^2}{2mr^2} + \frac{e^2 B^2 r^2}{8m} = E, \]  
(108)

where \( E \) is a constant. As \( p_0 \) depends only on \( r \), it follows that the phase integral

\[ I_0 = \int p_0 dr \]  
(109)

is an invariant.

VI. Dynamic Approach

The approaches to the mechanical problem of parametric action treated up to now involve an approximation in which the parameter is held constant while the system
completes a full cycle. This is the case of the Rayleigh approach just presented as well as that of Ehrenfest. We may say that the parameter varies in time in steps, the time of each step being equal to the period of a cycle in which the parameter is held constant. In other word, the parameter as a function of time looks like a staircase. Nevertheless, these approaches give correct results in the limit of infinitely slow variation of the parameter.

In the following, the problem is treat without considering the parameter fixed in a cycle but still considering that the variation of the parameter is slow. In other word, the parameter varies continuously in time rather than increasing by steps as was the case of the Rayleigh and of the Ehrenfest approaches.

We wish to determine the properties of a system in the regime of slow parametric action which is defined as follows. Let a parameter $\lambda$ varies linearly in time, that is, $\lambda = \lambda_0 (1 + \varepsilon t)$ where $\varepsilon$ is small. This regime is defined for times smaller that $1/\varepsilon$ and a quantity is an invariant if it varies little in this interval [18, 20].

A. Pendulum of variable length

The problem of the pendulum with variable length was treated by Lecornu in 1895 [64] and previously by Bossut in 1778 [65], although they did not draw the relevant conclusion of Rayleigh concerning the relation between energy and the length of the pendulum. Bossut imagined the oscillations of an unguided bucket during its ascent in a mine well. Bearing mind the figure 1, the problem is the oscillations of an unguided bucket with velocity $v_1$. The potential energy is $V = mg(\ell_0 - \ell - \ell h t)$. Notice that the second term is the potential energy related to the vertical motion of the bob.

From this expression we reach the equation of motion as

$$\ell \ddot{\theta} + 2h \dot{\theta} = -g \theta,$$

valid for small oscillations. Changing variable from $t$ to $t = \ell_0 + ht$, this equation becomes

$$\ell \frac{d^2 \theta}{dt^2} + \frac{2 \dot{\theta}}{\ell} + \frac{g}{\ell^2} \theta = 0,$$

which is the equation derived by Lecornu [64].

It is convenient to define the variable $s = \ell / g$ or $s = at + b$, where $a = h / g$ and $b = \ell_0 / g$, from which we may write the equation of motion as

$$\frac{d^2 \theta}{ds^2} + \frac{2 \dot{\theta}}{ds} + \frac{\theta}{a^2} = 0.$$

Performing the change of variables defined by $z = 2\sqrt{s}/a$ and $\phi = z \theta$ we reach the equation

$$\frac{d^2 \phi}{dz^2} + z \frac{d\phi}{dz} + (z^2 - 1)\phi = 0.$$
In this form, we see that the solutions are the Bessel functions of first order $J_1(z)$ and $Y_1(z)$ [68], that is,

$$\phi = A_1J_1(z) + A_2Y_1(z),$$  \hspace{1cm} (123)

where $A_1$ and $A_2$ are constant.

$$\theta = \frac{1}{z}[A_1J_1(z) + A_2Y_1(z)],$$  \hspace{1cm} (124)

which gives $\theta$ as a function of $t$ if we recall that $z = 2\sqrt{s}/a$ and that $s = at + b$.

As we wish to get the solution for a very slow variation of the length of the pendulum, which means that $a$ is very small, it suffices to consider the solution for large values of $z$. For as $z = 2\sqrt{s}/a$ and considering a finite value of $s = at + b$, $z$ will increase as $1/a$. Therefore, we use the asymptotic expression of the Bessel functions [68], as did Trutkov and Fock [66], namely

$$J_1(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{\pi}{4}\right),$$  \hspace{1cm} (125)

$$Y_1(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\pi}{4}\right).$$  \hspace{1cm} (126)

The solution can thus be written as

$$\theta = cs^{-3/4} \cos\left(\frac{2}{a}\sqrt{s} - \frac{\pi}{4}\right).$$  \hspace{1cm} (127)

The energy $E$ of the pendulum is the first part of the kinetic energy given by (115) plus the first part of the potential energy given by (116),

$$E = \frac{m}{2} \left(\dot{u}^2 + \ell \theta^2\right),$$  \hspace{1cm} (128)

which can be written as

$$E = \frac{m}{2} \left[ a^2 s^2 \left(\frac{d\theta}{ds}\right)^2 + s \theta^2 \right].$$  \hspace{1cm} (129)

Replacing the solution (127) in this equation, we reach the following expression for the energy

$$E = \frac{mgc^2}{\sqrt{s}} = \frac{mgc^2}{2} \sqrt{\frac{g}{\ell}},$$  \hspace{1cm} (130)

where we have neglected terms of order equal or larger that $a$. That is, the energy of the pendulum is proportional to the inverse of $\sqrt{\ell}$, the Rayleigh relation. Bearing in mind that the frequency is $\omega = \sqrt{g/\ell}$, we may write

$$E = \frac{mgc^2}{2} \omega,$$  \hspace{1cm} (131)

and $E/\omega$ is an adiabatic invariant.

In the treatment that we have just given to the pendulum with variable length, we have used the set up employed by Bossut and Lecornu, which corresponds to keep the ring of figure II fixed, while the point A moves vertically. In the original set up of Rayleigh, the point A is kept fixed while the ring B moves vertically. In this case the origin of the axis $y$ should be placed at the point A, which is fixed, rather than at the point B, which moves. The relation between $y$ and the angle $\theta$ becomes

$$y = \ell_0 - \ell + \ell \cos \theta.$$  \hspace{1cm} (132)

The axis $x$ remains the same and given by (111).

It is straightforward to show that, for small oscillations and for $\ell = \ell_0 + b t$, the equation of motion for $\theta$ for the Rayleigh set up is identical to the Bossut set up, given by equation (119). The kinetic and potential energies for the Rayleigh set up are given by the first parts of equation (115) and (116), respectively, since for the Rayleigh set up there is no vertical net translation, the total energy being given by (129), with the results (130) and (131).

**B. Harmonic oscillator of variable frequency**

Let us consider a harmonic oscillator along the $u$ axis with variable mass $m$ and variable spring coefficient $k$. The equation of motion is

$$\frac{d}{dt}(m \dot{u}) = -ku.$$  \hspace{1cm} (133)

If the mass varies linearly in time, $m = m_0 + \mu t$, and $k$ is constant the equation of motion reduces to

$$m \dot{u} + \mu \ddot{u} = -ku.$$  \hspace{1cm} (134)

This equation is identical to the equation (119) and can thus be solved in like manner.

We consider now that $m$ is constant and that the spring coefficient $k$ varies with time $\mu t$. In this case the equation of motion reduces to

$$\ddot{u} = -su,$$  \hspace{1cm} (135)

where $s = k/m$. From now on we suppose $k$ varies linearly with time, that is, $s = b + at$ where $a$ is a small quantity. Changing variable from $t$ to $s$ we get

$$a^2 \frac{d^2 u}{ds^2} = -su.$$  \hspace{1cm} (136)

Making another change of variable from $s$ to $z = a^{-2/3}s$, we reach the following equation

$$\frac{d^2 u}{dz^2} = -zu.$$  \hspace{1cm} (137)

The solutions of this equation are the Airy functions $Ai(-z)$ and $Bi(-z)$ [68],

$$u = c_1 Ai(-z) + c_2 Bi(-z),$$  \hspace{1cm} (138)

where $c_1$ and $c_2$ are constants. As we wish to get the solution for a very slow variation of the spring coefficient,
which means that \( a \) is small, and bearing in mind that \( z = s/a^{2/3} \), it suffices to consider the solutions for large values of \( z \). The asymptotic forms of the Airy functions are

\[
\text{Ai}(z) = \frac{1}{\sqrt{\pi}} z^{-1/4} \cos\left(\frac{2}{3} z^{3/2} - \frac{\pi}{4}\right),
\]

(139)

\[
\text{Bi}(z) = -\frac{1}{\sqrt{\pi}} z^{-1/4} \sin\left(\frac{2}{3} z^{3/2} - \frac{\pi}{4}\right).
\]

(140)

The solution can thus be written as

\[
u = cs^{-1/4} \cos\left(\frac{2}{3a} z^{3/2} + c_0\right),
\]

(141)

where \( c \) and \( c_0 \) are constants and we recall that \( s \) is a function of time, \( s = b + at \).

The energy \( E \) of the harmonic oscillator is

\[
E = \frac{m}{2} \dot{u}^2 + \frac{k}{2} u^2.
\]

(142)

From this expression and using the asymptotic solution, we find

\[
E = \frac{mc^2}{2} \sqrt{s},
\]

(143)

where we have neglected terms of the order equal or greater than \( a/s \). Recalling that \( \sqrt{s} \) can be understood as the frequency \( \omega = \sqrt{k/m} = \sqrt{s} \), it follows that the energy of a harmonic oscillator of variable frequency is proportional to the frequency, or that the ratio \( E/\omega \) is an invariant.

### C. General time dependence

We ask whether an expression of the type

\[
u = r(t) \cos(\theta(t)),
\]

(144)

could be the solution of the equation of motion for the harmonic oscillator of variable spring coefficient. If we replace the expression (144) in the equation (135) we find that it is indeed a solution as long as the following equations involving \( r \), \( \theta \) and \( s \) are satisfied \[63]\:

\[
2r\dot{\omega} + r\ddot{\omega} = 0,
\]

(145)

\[
\dot{r} - r\omega^2 + sr = 0,
\]

(146)

where \( \dot{\theta} = \omega \). The solution of the first equation gives

\[
r = c \omega^{-1/2},
\]

(147)

where \( c \) is an arbitrary constant and

\[
s = \omega^2 - \frac{\ddot{r}}{r}.
\]

(148)

Therefore, given \( \omega \) as a function of time, we determine \( r \) and then \( s \). By the integration of \( \dot{\theta} = \omega \), we determine \( \theta \).

Replacing the solution (144) into the expression (142), and bearing in mind that \( k/m = s \), we find

\[
E = \frac{mc^2}{2} \omega,
\]

(149)

where we have neglected terms of the order equal or larger than \( \omega/\omega \). Again we find that \( E/\omega \) is an invariant.

A simplification arises if we suppose that \( \omega \) is a finite function of \( at + b \) where \( a \) is a small quantity. In this case \( \dot{r}/r \) will be of the order \( a^2 \) and can be neglected in the expression (148), which reduces to

\[
s = \omega^2.
\]

(150)

A possible solutions for the dependence of \( \omega \) with time is

\[
\omega = at + b,
\]

(151)

which gives

\[
\theta = \frac{1}{2} at^2 + bt + c_0, \quad r = c(at + b)^{-1/2},
\]

(152)

and, using (150),

\[
s = (at + b)^2.
\]

(153)

Another solution is

\[
\omega = (at + b)^{1/2},
\]

(154)

which gives

\[
\theta = \frac{2}{3a}(at + b)^{3/2} + c_0, \quad r = c(at + b)^{-1/4},
\]

(155)

and, using (150),

\[
s = at + b.
\]

(156)

This solution is identified with that given by equation (141) if we recall that in (141), \( s \) equals \( at + b \). Yet another solution is

\[
\omega = (at + b)^{-1/2},
\]

(157)

which gives

\[
\theta = \frac{2}{a}(at + b)^{1/2} + c_0, \quad r = c(at + b)^{1/4},
\]

(158)

and, using (150),

\[
s = \frac{1}{at + b}.
\]

(159)

This solution is identified with that given by equation (127) if we recall that in equation (127) \( s \) equals \( at + b \).
D. Vibrating string of variable length

Let us denote by $\phi$ the transverse displacement PQ of the string at the point Q and by $x$ the distance from this point to the fixed end A, as shown in figure 2. The equation of motion for a uniform string is the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2},$$  \hfill (160)

where $c$ is the velocity of the wave and equal to $\sqrt{\Gamma/\mu}$ where $\Gamma$ is the tension of the string and $\mu$ is the mass per unit length. The vibration occurs only for $0 \leq x \leq \ell$ where $\ell$ is the distance of the ring B to the fixed end A. The boundary conditions are $\phi = 0$ for $x = 0$ and $x = \ell$. We wish to solve this equation as the length changes with time and determine the energy $E$ of the vibrating string which is given by

$$E = \frac{\mu}{2} \int_0^\ell \left( \frac{\partial \phi}{\partial t} \right)^2 dx + \frac{\Gamma}{2} \int_0^\ell \left( \frac{\partial \phi}{\partial x} \right)^2 dx.$$  \hfill (161)

A closed solution of the wave equation can be obtained for the following time dependence of the string length $\ell = \ell_0/(1 + \epsilon t)$.

To solve the wave equation as $\ell$ varies with time we proceed as follows. We assume a solution of the type

$$\phi = e^{i(\alpha x^2 + \theta)} \sin kx,$$  \hfill (162)

where $\alpha$ is a constant and $\theta$ is a function of $t$. The coefficient $k$ is chosen to be equal to $n\pi/\ell$ where $n$ is an integer so that $\phi$ vanishes at $x = 0$ and $x = \ell$, as desired. As $\ell$ depends on time, so does $k$, that is,

$$k = \frac{n\pi}{\ell} = \frac{n\pi}{\ell_0}(1 + \epsilon t).$$  \hfill (163)

Replacing the solution into the wave equation and bearing in mind that $k$ depends on time, we find $\alpha = k \epsilon c/2$, where $k_0 = n\pi/\ell_0$, and $\theta = kc$, which by integration gives

$$\theta = kc(t + \frac{1}{2} \epsilon t^2).$$  \hfill (164)

The other solution corresponds to the complex conjugate of this expression. Since $\phi$ is real, we sum the two solutions to obtain

$$\phi = A \cos \left( \frac{k_0 \epsilon x^2}{2c} + k_0 ct + \frac{k_0 \epsilon \epsilon t^2}{2} \right) \sin kx.$$  \hfill (165)

Replacing this result in the expression (161) for the energy $E$, we obtain the result (164) found before.

VII. QUANTUM MECHANICS

A. Parametric invariance

Born and Fock [52], in their paper of 1928 stated the invariance in the following terms. If the system was in a certain state described by a certain quantum number, the probability to change the state, by a slow variation of a parameter, is infinitely small, in spite of the fact that the change in the energy levels be of finite amount. They considered a discrete and a non-degenerate spectrum of energies except for the accidental degeneracy due to crossing of two energy eigenvalues. Demonstration of the invariance with less restrictions was given by Kato in 1950 [72]. Other demonstrations and discussions of invariance in quantum mechanics are found in several papers [73] [78] and books on quantum mechanics [23] [26].

In the following, we show that if the variation of a parameter is infinitely slow the system remains in the same quantum state. To this end we use an approach analogous to the one we employed above for the classical case. We start by consider a system described by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H}_\lambda \psi,$$  \hfill (166)

where $\psi$ is the wave function and $\mathcal{H}_\lambda$ is the Hamiltonian operator, which depends on a parameter $\lambda$ which depends on time.

Let us consider the following quantity

$$H = \langle \psi | \mathcal{H}_\lambda | \psi \rangle,$$  \hfill (167)

where $\psi$ is a solution of the equation (166).

As the Hamiltonian depends explicitly on time through the parameter, $H$ is not conserved. Its variation in time is

$$\frac{dH}{dt} = \langle \psi | \frac{\partial \mathcal{H}_\lambda}{\partial \lambda} | \psi \rangle,$$  \hfill (168)

which we write in the form

$$\frac{dH}{dt} = -c \langle \psi | \mathcal{F}_\lambda | \psi \rangle,$$  \hfill (169)

where $\mathcal{F}$ is the operator

$$\mathcal{F}_\lambda = -\frac{\partial \mathcal{H}_\lambda}{\partial \lambda},$$  \hfill (170)

and $c = d\lambda/dt$.

Now let $\phi_\lambda$ be a solution of the Schrödinger equation (166) with the condition that the parameter $\lambda$ is kept unchanged. We define the following quantities

$$E = \langle \phi_\lambda | \mathcal{H}_\lambda | \phi_\lambda \rangle,$$  \hfill (171)

and

$$F = \frac{1}{T} \int_0^T \langle \phi_\lambda | \mathcal{F}_\lambda | \phi_\lambda \rangle.$$  \hfill (172)

In accordance with the reasoning given above, for the classical case, the approximation amounts to replace $\dot{\psi}$ by $\phi_\lambda$. The resulting equation is

$$\frac{dE}{d\lambda} = -F.$$  \hfill (173)
We now write this equation in the form
\[
\int_0^T \left( \frac{d}{dt} (|\phi_\lambda \rangle \langle H_\lambda | \phi_\lambda \rangle) - \langle \phi_\lambda | F_\lambda | \phi_\lambda \rangle \right) dt = 0. \tag{174}
\]

The equation (174) can be written in the following equivalent form
\[
\int_0^T \left( \frac{\partial}{\partial \lambda} (|\phi_\lambda \rangle \langle H_\lambda | \phi_\lambda \rangle) + \langle \phi_\lambda | H_\lambda \frac{\partial}{\partial \lambda} | \phi_\lambda \rangle \right) dt = 0. \tag{175}
\]

Let us now denote by \( \phi_{\lambda n} \) and \( E_{\lambda n} \) the eigenfunctions and eigenvalues of \( H_\lambda \), where \( \lambda \) is considered to be fixed. We may then expand \( \phi_\lambda \),
\[
\phi_\lambda = \sum_n a_{\lambda n} \phi_{\lambda n}, \tag{176}
\]
We may also expand the derivative of \( \phi_\lambda \) with respect to \( \lambda \),
\[
\frac{\partial \phi_\lambda}{\partial \lambda} = \sum_n b_{\lambda n} \phi_{\lambda n}. \tag{177}
\]
Replacing these expansions in equation (175), we find
\[
\sum_n \int_0^T E_{\lambda n} (a_{\lambda n} b_{\lambda n}^* + a_{\lambda n}^* b_{\lambda n}) dt = 0, \tag{178}
\]
where we have assumed that the eigenfunctions are orthonormalized. A solution of this equation corresponds to the case where the coefficients are all zero except one of them, in which case the expression between parentheses vanishes. Therefore if the system is initially in a certain state, say state \( \phi_{\lambda n} \), it remains in this state as \( \lambda \) is varied slowly. In other words, the quantum number \( n \) is invariant.

### B. Electron on a rotating field

Let us consider the evolution of the spin of an electron in a rotating magnetic field \[25,77\]. The \( x \) and \( y \) components of the magnetic field are \( B \cos \theta \) and \( B \sin \theta \) where \( \theta \) is the time dependent parameter, \( \theta = \omega t \). In the representation where the component \( z \) of the electron spin is diagonal, the Hamiltonian is given by the square matrix
\[
H = \mu B (\sigma_x \cos \theta + \sigma_y \sin \theta), \tag{179}
\]
where \( \mu = eB/2m \) is the Bohr magneton and \( \sigma_x \) and \( \sigma_y \) are the Pauli matrix.

We define by \( \chi_+ \) and \( \chi_- \) the basis where the Pauli matrix \( \sigma_z \) is diagonal, which are the column matrices with elements 1 and 0, and 0 and 1, respectively. The eigenvectors and eigenvalues of \( H \) are
\[
\phi_1 = \frac{1}{\sqrt{2}} (e^{-i\theta/2} \chi_+ + e^{i\theta/2} \chi_-), \quad \varepsilon_1 = \mu B, \tag{180}
\]
\[
\phi_2 = \frac{1}{\sqrt{2}} (e^{-i\theta/2} \chi_+ - e^{i\theta/2} \chi_-), \quad \varepsilon_2 = -\mu B, \tag{181}
\]
It is useful to know that
\[
\frac{d\phi_1}{dt} = -\frac{i\omega}{2} \phi_2, \tag{182}
\]
\[
\frac{d\phi_2}{dt} = -\frac{i\omega}{2} \phi_1. \tag{183}
\]

The Schrödinger equation is
\[
i\hbar \frac{d\chi}{dt} = H\chi, \tag{184}
\]
where \( \chi \) is the spinor. Writing \( \chi = x\phi_1 + y\phi_2 \), the Schrödinger equation becomes
\[
i\frac{dx}{dt} + \frac{\omega}{2} y = \frac{\omega_c}{2} x, \tag{185}
\]
\[
i\frac{dy}{dt} + \frac{\omega}{2} x = -\frac{\omega_c}{2} y, \tag{186}
\]
where \( \omega_c = 2\mu B/\hbar = eB/m \) is the cyclotron frequency. The solution for the case \( x = 1 \) and \( y = 0 \) for \( t = 0 \) is
\[
x = \cos \gamma t - \frac{i\omega_c}{2\gamma} \sin \gamma t, \tag{187}
\]
\[
y = \frac{i\omega}{2\gamma} \sin \gamma t, \tag{188}
\]
where
\[
\gamma = \frac{1}{2} \sqrt{\omega^2 + \omega_c^2}. \tag{189}
\]

We remark that \( |x|^2 + |y|^2 = 1 \) so that \( \chi \) is normalized. The probability of the system to be found in the state \( \phi_1 \) and \( \phi_2 \) are respectively
\[
|x|^2 = \cos^2 \gamma t + \frac{\omega_c^2}{4\gamma^2} \sin^2 \gamma t, \tag{190}
\]
\[
|y|^2 = \frac{\omega^2}{4\gamma^2} \sin^2 \gamma t. \tag{191}
\]

In the regime \( \omega/\omega_c \ll 1 \), the probability to change from state 1 to state 2 is very small, of the order of \( (\omega/\omega_c)^2 \), and we recall that \( \omega \) is the rate of change of the parameter \( \theta = \omega t \).

### C. Square well with a moving wall

We consider a particle confined in a one-dimensional box which is equivalent to the motion of a particle under an infinite square well potential. One wall of the box is
fixed and the other moves linearly. Denoting by $\ell$ the length of the box, we assume that $\ell = \ell_0(1 + \varepsilon t)$. A closed solution of the Schrödinger equation can be found for this case and in fact a solution was given by Doescher and Rice [79]. In the following we present the solution for this problem.

The Schrödinger equation to be solved is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2},$$  \hspace{1cm} (192)

with the boundary condition that $\psi$ vanishes at $x = 0$ and at $x = \ell$.

We assume a solution of the type

$$\psi = \sqrt{\frac{2}{\ell}} e^{i(\alpha x^2 + \theta)} \sin kx,$$  \hspace{1cm} (193)

where $\alpha$ and $\theta$ are functions of $t$, and $k = n\pi/\ell$ so that $\psi$ vanishes at $x = 0$ and $x = \ell$, as required. Replacing this expression in the Schrödinger equation, we find

$$\alpha = \frac{m}{2\hbar} \frac{\varepsilon}{1 + \varepsilon t},$$  \hspace{1cm} (194)

$$\dot{\theta} = -\frac{\hbar}{2m} k^2,$$  \hspace{1cm} (195)

which integrated gives

$$\theta = -\frac{\hbar k_0^2 t}{2m(1 + \varepsilon t)}.$$  \hspace{1cm} (196)

where $k_0 = n\pi/\ell_0$.

Replacing the above results in the equation (193), we reach the following expression

$$\psi = \sqrt{\frac{2}{\ell}} \exp\{ \frac{i m^2 \varepsilon x^2 - \hbar^2 k_0^2 t}{2m(1 + \varepsilon t)} \} \sin kx.$$  \hspace{1cm} (197)

**D. Hamiltonian obeying a scaling relation**

A closed solution can also be provided when the Hamiltonian $\mathcal{H}_\Lambda(x)$ that depends on a parameter $\Lambda$ obeys the scaling relation

$$\mathcal{H}_\Lambda(x) = \lambda^b \mathcal{H}(\xi), \quad \xi = \lambda^a x,$$  \hspace{1cm} (198)

where $\mathcal{H}(\xi)$ only on $\xi$ but not on $\lambda$. The eigenfunctions $\phi(\xi)$ and the eigenvalues $E$ of $\mathcal{H}(\xi)$ are related to the eigenfunctions $\phi_\Lambda(x)$ and eigenvalues $E_\Lambda$ of the original Hamiltonian by

$$E_\Lambda = \lambda^b E,$$  \hspace{1cm} (199)

$$\phi_\Lambda(x) = \lambda^{a/2} \phi(\xi).$$  \hspace{1cm} (200)

This last relation follows from the normalization of the eigenfunctions.

A scaling relation of this type is obeyed by the Hamiltonian describing a particle in a box, in which case $a = -1$ and $b = -2$, and $\lambda$ is the length of the box. It is also obeyed by the Hamiltonian of the harmonic oscillator in which case $a = 1/2$ and $b = 1$, and $\lambda$ is the frequency of the oscillation.

We consider the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H}\psi,$$  \hspace{1cm} (201)

where we are omitting the index $\lambda$ in the Hamiltonian $\mathcal{H}(x)$, which depends on time through the parameter $\lambda$. The Hamiltonian operator

$$\mathcal{H} = \mathcal{K} + \mathcal{V}$$  \hspace{1cm} (202)

is the sum of the kinetic energy operator $\mathcal{K}$ and $\mathcal{V}$ is the potential energy. In the position representation, which we use here, $\mathcal{V}$ is a multiplying operator, that is, a function of $x$.

Let $\phi(x)$ be one of the eigenfunctions of $\mathcal{H}$ and $E$ the corresponding eigenvalue, that is,

$$\mathcal{H}\phi = E\phi,$$  \hspace{1cm} (203)

Again, we are omitting the index $\lambda$ in the eigenfunctions and eigenvalues but they contain the parameter $\lambda$ and thus depend on time through this parameter. We wish to solve the Schrödinger equation with the initial condition such that the wavefunction at $t = 0$ is one of the eigenfunctions of the Hamiltonian. Let us consider the following wave function

$$\psi_0 = e^{i\theta} \phi,$$  \hspace{1cm} (204)

where $\theta$ is given by

$$\theta = -\frac{1}{\hbar} \int_0^t E dt',$$  \hspace{1cm} (205)

which is in accordance with the initial condition. If we replace it in the Schrödinger equation, we see that it is not a solution because the term $i\hbar \partial \phi / \partial t$ does not cancel out. We assume then the following form for the solution

$$\psi = e^{i(\theta + u)} \phi,$$  \hspace{1cm} (206)

where $\theta$ is the dynamic phase given by (205) and $u$ is a function to be found. We look for a real solution for $u$ so that $\psi$ will differ from $\phi$ by a phase factor which means that the system remains in the state described by $\phi$. In addition the wavefunction $\psi$ is normalized because $\phi$ is normalized.

Replacing the expression (206) in the Schrödinger equation (201), we get the following equation

$$i\hbar \frac{\partial \phi}{\partial t} - \hbar \frac{\partial u}{\partial t} \phi = e^{-iu} \mathcal{H}(e^{iu} \phi) - E\phi.$$  \hspace{1cm} (207)

Taking into account that $\mathcal{V}$ is a multiplying operator, which is just a function, the right hand side becomes

$$e^{-iu} \mathcal{K}(e^{iu} \phi) + \mathcal{V}\phi - E\phi,$$  \hspace{1cm} (208)
and the first term of this expression is

$$-\frac{\hbar^2}{2m} \left[ \frac{\partial^2 u}{\partial x^2} \phi - \left( \frac{\partial u}{\partial x} \right)^2 \phi + 2i\frac{\partial u}{\partial x} \frac{\partial \phi}{\partial x} \right] + K\phi. \quad (209)$$

Replacing these results into equation (207) we get

$$i\frac{\partial \phi}{\partial t} - \frac{\hbar}{2m} \frac{\partial^2 \phi}{\partial x^2} = 0,$$  
(210)

which is an equation for $\phi$ as is known.

As we are looking for a real $u$, its imaginary part should vanish and the real and imaginary parts of equation (210) become

$$-\frac{\partial u}{\partial t} = \frac{\hbar}{2m} \frac{\partial^2 u}{\partial x^2}, \quad (211)$$

$$\frac{\partial \phi}{\partial t} = -\frac{\hbar}{2m} \left( \frac{\partial^2 u}{\partial x^2} \phi + 2i\frac{\partial u}{\partial x} \frac{\partial \phi}{\partial x} \right), \quad (212)$$

where we have taken into account that $\phi$ is real, a choice that is always possible to accomplish because $\mathcal{H}$ is Hermitian. As we impose that the imaginary part of $u$ vanishes, this quantity should solve both the equations (211) and (212).

The first equation (211) can be solved by the separation of variables. Assuming that $u(t, x) = \alpha(t)z(x)$ and replacing it in (207) we get

$$-\frac{1}{\alpha^2} \frac{\partial \alpha}{\partial t} = \frac{\hbar}{2m} \frac{1}{z} \left( \frac{\partial z}{\partial x} \right)^2. \quad (213)$$

The left and right hand side should be a constant that we choose to be equal to the unity, that is,

$$-\frac{\partial \alpha}{\partial t} = \alpha^2, \quad \frac{\hbar}{2m} \frac{1}{z} \left( \frac{\partial z}{\partial x} \right)^2 = z. \quad (214)$$

Integrating,

$$\alpha = \frac{\varepsilon}{1 + \varepsilon t}, \quad z = \frac{m}{2\hbar} x^2, \quad (215)$$

where $\varepsilon$ is a constant of integration. Replacing these results in the second equation (212), it becomes

$$2(1 + \varepsilon t) \frac{\partial \phi}{\partial t} = -\varepsilon \left( \phi + 2x \frac{\partial \phi}{\partial x} \right). \quad (216)$$

Using the scaling laws for $\phi$, we find the equalities

$$\frac{x}{\phi} \frac{\partial \phi}{\partial x} = \frac{\xi}{\phi} \frac{\partial \phi}{\partial \xi}, \quad (217)$$

$$\frac{1}{\phi} \frac{\partial \phi}{\partial t} = \frac{a}{2\lambda} \left( 1 + 2 \frac{\xi}{\phi} \frac{\partial \phi}{\partial \xi} \right) \frac{d\lambda}{dt}. \quad (218)$$

Replacing these relations in equation (216) we see that it becomes satisfied as long as

$$\frac{d\lambda}{dt} = -\lambda \varepsilon \frac{a}{a(1 + \varepsilon t)}. \quad (219)$$

The integration of this equation gives

$$\lambda = \lambda_0 (1 + \varepsilon t)^{-1/a}, \quad (220)$$

where $\lambda_0$ is a constant of integration, which is the value of the parameter at $t = 0$. Therefore, $u$ solves both equation under the condition (220). In other words, if the parameter $\lambda$ depends on time in accordance with (220), $u = az$ given by (220) is real and solves both equations (211) and (212).

We may draw the following conclusions from the above results. If the parameter $\lambda$ varies according to relation (220) and if the scaling (198) is fulfilled, then the wave function given by equation (206) is an exact solution and $u$ is real, that is, it is indeed a phase, given by

$$u = \frac{m \varepsilon x^2}{2\hbar (1 + \varepsilon t)}. \quad (221)$$

The solution (206) is valid for any value of the parameter $\varepsilon$, and, of course, remains a solution when $\varepsilon$ is small, which characterizes a slow variation of the parameter $\alpha$ because, according to equation (219) $d\lambda/dt$ is proportional to $\varepsilon$.

E. Harmonic oscillator of variable frequency

The quantum harmonic oscillator with variable frequency was treated by Husimi in 1953 [80]. The Hamiltonian is

$$\mathcal{H} = \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2, \quad (222)$$

where the frequency $\omega$ depends on time. The eigenfunctions $\phi_{kn}$ are

$$\phi_n(x) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}, \quad (223)$$

where $H_n(\xi)$ are the Hermite polynomials and $\xi = x/\sqrt{m\omega/\hbar}$. The corresponding eigenvalues are

$$E_n = \hbar \omega (n + \frac{1}{2}), \quad (224)$$

where $n = 0, 1, 2 \ldots$ We wish to solve the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H} \psi, \quad (225)$$

with the initial condition that the initial state is one of the eigenstates, say the eigenstate with the quantum number $n$. 

Writing the Hamiltonian in the form
\[ H = \hbar \omega \left( -\frac{1}{2} \frac{\partial^2}{\partial \xi^2} + \frac{1}{2} \xi^2 \right), \] (226)
it becomes manifest that it obeys the scaling relation \[ \text{[198]} \] with \( a = 1/2 \) and \( b = 1 \) and \( \omega \) playing the role of the parameter \( \lambda \). It is clear also that the eigenfunctions and eigenvalues are in accordance with the scaling relations \[ \text{[199]} \] and \[ \text{[200]} \].

From the results obtained above, a closed solution for the time dependent Schrödinger equation can be obtained for the following time dependence of the frequency
\[ \omega = \frac{\omega_0}{(1 + \varepsilon t)^2}, \] (227)
The solution is
\[ \psi = e^{i(\theta + \omega_0) \phi_n}, \] (228)
where \( \phi_n \) is one of the eigenfunctions and
\[ \theta = -\omega_0 (n + \frac{1}{2}) (t + \varepsilon t^2 + \frac{1}{3} \varepsilon^2 t^3), \] (229)
and
\[ u = \frac{\varepsilon m x^2}{2 \hbar (1 + \varepsilon t)}. \] (230)

F. Raising and lowering operators

We solve again the harmonic oscillator but now we use a representation in terms of the lowering and raising operators defined by
\[ a = \sqrt{\frac{m \omega}{2 \hbar}} + i \frac{p}{\sqrt{2 m \hbar \omega}}, \] (231)
\[ a^\dagger = \sqrt{\frac{m \omega}{2 \hbar}} - i \frac{p}{\sqrt{2 m \hbar \omega}}, \] (232)
where \( p = -\hbar \partial / \partial x \) is the momentum operator. They hold the relations \( a |n\rangle = \sqrt{n} |n\rangle \) and \( a^\dagger |n\rangle = \sqrt{n + 1} |n\rangle \), and fulfills the commutation relation \([a, a^\dagger] = 1\).

In terms of these lowering and raising operators, the Hamiltonian
\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \] (233)
of the harmonic oscillator becomes
\[ H = \hbar \omega (a^\dagger a + \frac{1}{2}). \] (234)
The eigenvectors of \( H \) are \( |n\rangle \), that is,
\[ H |n\rangle = E_n |n\rangle, \] (235)
and the eigenvalues are \( E_n = \hbar \omega (n + 1/2) \).

We wish to solve the Schrödinger equation
\[ i \hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle, \] (236)
considering that the frequency \( \omega \) is a time dependent parameter and that at \( t = 0 \) the oscillator is in one of its eigenstates.

As \( \omega \) is a time dependent parameter the operators \( a, a^\dagger \), and the state vectors \(|n\rangle\) depend on time through \( \omega \). It is thus convenient to determine their variation with \( \omega \). From the definitions given by \[ \text{[231]} \] and \[ \text{[232]} \], it follows that \( a \) and \( a^\dagger \) vary with \( \omega \) according to
\[ \frac{\partial a}{\partial \omega} = \frac{1}{2 \omega} a, \quad \frac{\partial a^\dagger}{\partial \omega} = \frac{1}{2 \omega} a^\dagger. \] (237)
From these relations we find
\[ \frac{d H}{d \omega} = \frac{\partial H}{\partial \omega} = m \omega x^2. \] (238)
From \[ \text{[223]} \] we establish the following variation of the eigenvectors with \( \omega \),
\[ \frac{d}{d \omega} |n\rangle = \frac{1}{4 \omega} (aa^\dagger - a^\dagger a) |n\rangle = \frac{i}{4 \hbar \omega} (xp - px)|n\rangle. \] (239)
To find the solution of the Schrödinger equation we consider the following state vector
\[ |\psi\rangle = e^{i(\theta + \alpha z)} |n\rangle, \] (240)where
\[ \theta = -\frac{1}{\hbar} \int_0^t E_n dt', \] (241)and \( \alpha \) is a time dependent scalar, and \( z = (m/2 \hbar) x^2 \), \( x^2 \) being the position operator squared. Replacing \(|\psi\rangle\) into the Schrödinger equation, we find
\[ -\frac{d\alpha}{dt} m \frac{d}{dt} x^2 |n\rangle + i \hbar \frac{d}{dt} |n\rangle = e^{-i \alpha z} H e^{i \alpha z} |n\rangle - E_n |n\rangle. \] (242)
Now, recalling that \( z \) is proportional to \( x^2 \),
\[ e^{-i \alpha z} H e^{i \alpha z} = \frac{1}{2m} e^{-i \alpha z} p^2 e^{i \alpha z} + \frac{1}{2} m \omega^2 x^2 = \]
\[ = H + \frac{\alpha}{2} (xp + px) + \frac{\alpha^2 m}{2} x^2, \] (243)
and the Schrödinger equation becomes
\[ -\frac{d\alpha}{dt} m \frac{d}{dt} x^2 |n\rangle + i \hbar \frac{d}{dt} \frac{d\omega}{dt} |n\rangle = \]
\[ = \frac{\alpha}{2} (xp + px)|n\rangle + \frac{\alpha^2 m}{2} x^2 |n\rangle. \] (244)
A solution of this equation occurs if
\[
\frac{d\alpha}{dt} = -\alpha^2,
\] (245)
from which follows the result
\[
\alpha = \frac{\varepsilon}{1 + \varepsilon t},
\] (246)
and if
\[
\frac{d\omega}{dt} = -2\alpha\omega,
\] (247)
where we have taken into account the relation (239). Solving this equation one finds the dependence of \(\omega\) with time,
\[
\omega = \frac{\omega_0}{(1 + \varepsilon t)^2}.
\] (248)
We conclude that (240) is an exact solution if \(\omega\) depends on time according to (248).

\section*{VIII. STATISTICAL MECHANICS}

\subsection*{A. Hertz invariant}

Let \(\mathcal{H}(q,p,\lambda)\) be the Hamiltonian of a system with \(n\) degrees of freedom where we are denoting by \(q\) and \(p\) the collection of coordinates \(q_1,\ldots,q_n\) and momenta \(p_1,\ldots,p_n\), and \(\lambda\) a parameter. In his treatise on statistical mechanics \cite{81}, Gibbs introduced the following integral
\[
\Phi = \int_{\mathcal{H} \leq E} dqdp,
\] (249)
which is the hypervolume of the region of the phase space enclosed by a hypersurface of constant energy \(E\). He also defines the quantity
\[
\Omega = \frac{\partial \Phi}{\partial E}.
\] (250)

Using the step function \(\vartheta(x)\), which is equal to zero or the unity according to whether the argument \(x\) is negative or positive, then \(\Phi\) can be written as
\[
\Phi = \int \vartheta(E - \mathcal{H}(\lambda)) dqdp.
\] (251)
Taking into account that the derivative of the step function is the Dirac delta function \(\delta(x)\), then \(\Omega\) can be written as
\[
\Omega = \int \delta(E - \mathcal{H}(\lambda)) dqdp.
\] (252)
Gibbs considers two possible forms for the entropy of a system. One of them is
\[
S = k \ln \Omega,
\] (253)
which is the form widely used, and the other is
\[
S' = k \ln \Phi.
\] (254)
He states that each of them has its advantage but the first form is a little more simple than the second and if simplicity is a criterion, the first form is preferable. Nevertheless, the two forms differ little from one another when the number of degrees of freedom is large, a feature that was recognized by Gibbs \cite{81}. Another appeal for the use of the first form is that \(\Omega\) occurs as the normalization of the Gibbs microcanonical distribution \cite{81}, given by
\[
\rho = \frac{1}{\Omega} \delta(E - \mathcal{H}(\lambda)).
\] (255)

In accordance with Clausius, who introduced the concept of thermodynamic entropy, this quantity is constant along a reversible adiabatic process, understood as a process carried out without the exchange of heat, and slow enough so that the system can be considered to be in equilibrium. If \(S\) and \(S'\) are to be interpreted as the thermodynamic entropy, then \(\Phi\) and \(\Omega\) should be constant along a reversible adiabatic process. The question is thus how to define a mechanical procedure that results in a reversible adiabatic process, without referring to heat. This question was implicitly answered by Paul Hertz in a paper of 1910 \cite{82}. In this paper he used a procedure that coincides with what we are calling slow parametric action to show that \(\Phi\) is constant. He assumed that the system has an external coordinate that is changed by external intervention, and that the energy is a function of \(q\), \(p\) and \(a\). The variables \(q\) and \(p\) vary according to the equations of motion and the parameter \(a\) is subject to our arbitrariness.

The approach used by Hertz was based on a hint contained in the Gibbs treatise on statistical mechanics \cite{2}. According to Gibbs \cite{81}, “the entropy of a body is not (sensibly) affected by mechanical action, during which the body is at each instant (sensibly) in a state of thermodynamic equilibrium”, which “may usually be attained by a sufficiently slow variation of the external coordinates”. The slow variation of the parameter was implicit in the Hertz use of the Gibbs microcanonical distribution. If the variation of the parameter is slow, the system remains in equilibrium and the Gibbs distribution, which is understood to be valid for system in equilibrium, can be used. Therefore, we may say that the use of the Gibbs distribution for two distinct values of the parameter, resulting from its variation with time, means that the variation is implicitly slow. The invariance of \(\Phi\) is thus a parametric invariance, although Hertz did not use this terminology but simply stated that \(\Phi\) is constant.

The demonstration of the invariance of \(\Phi\) is contained in some books on statistical mechanics \cite{27,29} and has been considered by some authors \cite{83,87}. A demonstration based on the Hertz paper was provided by de Koning and Antonelli \cite{41}. We demonstrate the invariance of \(\Phi\)
as follows. The variation in energy of a system described by a time dependent Hamiltonian is

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \quad (256)$$

If the Hamiltonian depends on time through a parameter $\lambda$ which varies slowly with time, then in accordance with (48), the right-hand side of this equation is replaced by its time average and it becomes

$$\frac{dE}{d\lambda} = \frac{\partial H}{\partial \lambda}, \quad (257)$$

where $E = \langle H \rangle$. The time average is then replaced by the average in probability,

$$\frac{dE}{d\lambda} = \langle \frac{\partial H}{\partial \lambda} \rangle, \quad (258)$$

and $\rho$ is the Gibbs microcanonical distribution, and

$$E = \langle H \rangle = \int H \rho dq dp. \quad (260)$$

The invariance of $\Phi$ means that $\Phi(E',\lambda')$ equals $\Phi(E,\lambda)$ when the parameter changes slowly from $\lambda$ to $\lambda'$ causing a change of energy from $E$ to $E'$. Considering small differences, we write

$$\frac{d\Phi}{d\lambda} = \frac{\partial \Phi}{\partial \lambda} + \frac{\partial \Phi}{\partial E} \frac{dE}{d\lambda}, \quad (261)$$

and the invariance means that $d\Phi/d\lambda$ should vanish. Using the relation (254), the equation (261) can be written as

$$\frac{d\Phi}{d\lambda} = \frac{\partial \Phi}{\partial \lambda} + \Omega \frac{dE}{d\lambda}. \quad (262)$$

We observe that, using the definition (255) of the microcanonical probability distribution, the equation (268) can be written as

$$\Omega \frac{dE}{d\lambda} = \int \frac{\partial H}{\partial \lambda} \delta(E - H) dq dp. \quad (263)$$

We also observe that, using the expression (251), the derivative of $\Phi$ with respect to $\lambda$ is

$$\frac{\partial \Phi}{\partial \lambda} = - \int \frac{\partial H}{\partial \lambda} \delta(E - H) dq dp. \quad (264)$$

From these two relations, it becomes manifest that the right hand side of (262) vanishes and so does $d\Phi/d\lambda$ as desired.

Let us calculate $\Phi$ for a system of $N$ particles confined in a container of volume $V$. The number of degrees of freedom is $3N$. The particles do not interact and the energy of the system is the kinetic energy

$$\mathcal{H} = \sum_i \frac{p_i^2}{2m}. \quad (265)$$

where $p_i$ denotes a component of each one of the particles. The quantity $\Phi$ is

$$\Phi = V^N \int_{E \leq \mathcal{H}} d^3N p, \quad (266)$$

the integral being equal to the volume of a sphere of radius $\sqrt{2mE}$ in a space of dimension $3N$, that is,

$$\Phi = V^N \frac{(2\pi mE)^{3N/2}}{(3N/2)!}, \quad (267)$$

and

$$\Omega = \frac{3N}{2E} \Phi. \quad (268)$$

B. Several parameters

Let us generalize the above result for several parameters that we denote by $\lambda_i$. The variation in time of the Hamiltonian is

$$\frac{dH}{dt} = \sum_i \frac{\partial H}{\partial \lambda_i} \frac{d\lambda_i}{dt}. \quad (269)$$

Considering that the parameter vary slowly in time this equation is replaced by

$$dE = \sum_i \frac{\partial H}{\partial \lambda_i} d\lambda_i. \quad (270)$$

The time averages are in turn replaced by averages on probability,

$$dE = \sum_i \langle \frac{\partial H}{\partial \lambda_i} \rangle d\lambda_i. \quad (271)$$

We write this equation in the form

$$dE = - \sum_i F_i d\lambda_i, \quad (272)$$

where

$$F_i = -\langle \frac{\partial H}{\partial \lambda_i} \rangle = - \int \frac{\partial H}{\partial \lambda_i} \rho dq dp, \quad (273)$$

and $\rho$ is the Gibbs microcanonical distribution

$$\rho = \frac{1}{\Omega} \delta(E - \mathcal{H}), \quad (274)$$

where

$$\Omega = \int \delta(E - \mathcal{H}) dq dp, \quad (275)$$
and depends on $\lambda_i$. Defining the integral
\[ \Phi = \int \vartheta(E - \mathcal{H}) d\varrho, \] (276)
that depends on $\lambda_i$, we see that $F_i$ can be written as
\[ F_i = \frac{1}{\Omega} \frac{\partial \Phi}{\partial \lambda_i}. \] (277)
Replacing this results in equation (272), and taking into account that
\[ \Omega = \frac{\partial \Phi}{\partial E}, \] (278)
we find
\[ \frac{\partial \Phi}{\partial E} dE + \sum_i \frac{\partial \Phi}{\partial \lambda_i} d\lambda_i = 0. \] (279)
But the left-hand side of this equation is the differential $d\Phi$. Since it equals zero, it follows that $\Phi$ is constant as we vary $E$ and the parameters $\lambda_i$.

C. Variable number of particles

The integral $\Phi$ given by the equation (249) was shown to be an invariant when a parameter changes slowly. However, the variation of the parameter did not change the number of particles. Here wish to show that the expression that is invariant when the number $n$ of particles changes is $\Phi_n = \Phi/n!$ with the factor $1/n!$.

Let $\mathcal{H}_n$ be the Hamiltonian function corresponding to a system of $n$ particles. We use the notation $x_i$ for the positions and momenta of the $i$-th particle so that the Hamiltonian $\mathcal{H}_n$ depends on the variables from $x_1$ to $x_n$ and is invariant under the permutation of the $n$ particles.

We now consider the removal of a particle from the system. To simulate this procedure we suppose that at $t = 0$ a particle is chosen at random and it its velocity is slowly reduced and its interaction with the other particles is slowly decreased. After an interval of time $\Delta t$, its velocity has vanished and it does not interact with other particles anymore. We suppose that this procedure is carried out by means of a parameter $\lambda$ that takes the value $\lambda_0$ at $t = 0$ and the value $\lambda_1$ at $t = \Delta t$. During this procedure the Hamiltonian $\mathcal{H}$ that describes the system is not invariant by permutation of all the particles. If we chose the particle $n$ to be taken out, $\mathcal{H}$ is invariant under the permutation of the remaining $n - 1$ particles.

Supposing that $\lambda$ varies slowly, then the variation of the energy $E$ with $\lambda$ is given by
\[ \frac{dE}{d\lambda} = \frac{\partial \mathcal{H}}{\partial \lambda}, \] (280)
where the average is calculated using the probability distribution
\[ \rho = \frac{1}{\Omega} \delta(E - \mathcal{H}), \] (281)
where
\[ \Omega = \int \delta(E - \mathcal{H}) d^n x, \] (282)
and depends on $E$ and $\lambda$. It is convenient to define the integral
\[ \Phi = \int \vartheta(E - \mathcal{H}) d^n x, \] (283)
so that
\[ \frac{\partial \Phi}{\partial \lambda} = - \int \frac{\partial \mathcal{H}}{\partial \lambda} \delta(E - \mathcal{H}) d^n x, \] (284)

From which we obtain after dividing by $\Omega$
\[ \frac{dE}{d\lambda} = \frac{1}{\Omega} \frac{\partial \Phi}{\partial \lambda}. \] (285)

Replacing (284) into this equation, we reach the following relation
\[ \frac{dE}{d\lambda} = - \frac{\partial \Phi}{\partial \lambda} \frac{\partial \mathcal{H}}{\partial \lambda} = \left( \frac{\partial E}{\partial \lambda} \right) _\Phi. \] (286)

This relation tell us that $E$ and $\lambda$ varies in such a way that $\Phi(E, \lambda)$ is invariant.

If $E$ is the energy when $\lambda = \lambda_0$ and $E'$ when $\lambda = \lambda_1$ then $\Phi(E, \lambda_0) = \Phi(E', \lambda_1)$. Let us define $\Phi_n^*$ by
\[ \Phi_n^* = \int \vartheta(E - \mathcal{H}_n) d^n x \] (288)
then $\Phi(E, \lambda_0) = \Phi_n^*(E)$ and $\Phi(E', \lambda_1) = n \Phi_{n-1}^*(E')$, the factor $n$ coming from the existence of $n$ possibility of removing a particle from the system, and the invariance is written as
\[ \Phi_n^*(E) = n \Phi_{n-1}^*(E') \] (289)

Defining $\Phi_n = \Phi_n^*/n!$ then the invariance becomes
\[ \Phi_n(E) = \Phi_{n-1}(E') \] (290)

From these relation it follows that
\[ \Phi_n(E) = \frac{1}{n!} \int \vartheta(E - \mathcal{H}_n) d^n x \] (291)
is invariant when one varies $E$ and $n$.

For a system of $N$ noninteracting particles we use the result (267) to find
\[ \Phi_N = \frac{V^N (2\pi m E)^{3N/2}}{N! (3N/2)!}. \] (292)
D. Quantum systems

Here we use the occupation number representation to describe the dynamics of a quantum system. To this end we introduce the operator \( a_i^\dagger \) which creates a particle in the single particle state \( \psi_i \) and the operator \( a_i \) which annihilates a particle in state \( \psi_i \). The number operator is \( a_i^\dagger a_i \) and the total number operator is

\[
N = \sum_i a_i^\dagger a_i, \tag{293}
\]

We assume the following form for the Hamiltonian

\[
H = \sum_{ij} K_{ij} a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_l a_k, \tag{294}
\]

where \( K_{ij} = \langle \phi_i | K | \phi_j \rangle \) and \( V_{ijkl} = \langle \phi_i | \phi_j | V | \phi_k | \phi_l \rangle \), and \( K \) and \( V \) are the kinetic and potential energy. We remark that \( N \) commutes with \( H \). If \( H \) does not depend explicitly on time, \( H \) is conserved and so is \( N \).

Now we wish to treat the case where the number of particle varies with time, for instance, by introducing or removing particles from the system. This is accomplished by assuming that the creation and annihilation operators depends on a parameter \( \lambda \) that varies with time. We start with the equation

\[
dE/d\lambda = \text{Tr} \frac{\partial H}{\partial \lambda} \rho, \tag{295}
\]

where \( \rho \) is the density operator, which we assume to be given by

\[
\rho = \frac{1}{\Omega} \delta(E - H), \tag{296}
\]

and

\[
\Omega = \text{Tr} \delta(E - H). \tag{297}
\]

In an explicitly form,

\[
dE/d\lambda = \frac{1}{\Omega} \text{Tr} \frac{\partial H}{\partial \lambda} \delta(E - H). \tag{298}
\]

Defining

\[
\Phi = \text{Tr} \delta(E - H), \tag{299}
\]

we find the relations

\[
\frac{\partial \Phi}{\partial E} = \Omega, \tag{300}
\]

\[
\frac{\partial \Phi}{\partial \lambda} = -\text{Tr} \frac{\partial H}{\partial \lambda} \delta(E - H), \tag{301}
\]

from which follows

\[
\frac{dE}{d\lambda} = -\frac{\partial \Phi/\partial \lambda}{\partial \Phi/\partial E} = \left( \frac{\partial E}{\partial \lambda} \right)_\Phi. \tag{302}
\]

Therefore as one varies \( E \) and \( \lambda \), \( \Phi \) remains invariant.

Let us suppose that as \( \lambda \) varies, assuming values \( \lambda_1, \lambda_2 \), and so on, the number of particles in the system are, respectively, \( n_1, n_2, \) and so on, and the energy are \( E_1, E_2, \) and so on. If we define

\[
\Phi_\star_n(E) = \text{Tr}_n \delta(E - H), \tag{303}
\]

where the trace is taken within the subspace of states with \( n \) particles, then \( \Phi(E_i, \lambda_i) = \Phi_{n_i}(E_i) \). Therefore the invariance \( \Phi(E, \lambda) \) when one varies \( E \) and \( \lambda \) is equivalent to the invariance of \( \Phi_\star_n(E) \) when one varies the energy and the number of particles.

The interpretation of \( \Phi_\star_n(E) \) is very simple. If we use a basis of eigenvectors \( |\phi_i^{(n)}\rangle \) of \( H \) belonging to the Hilbert space with \( n \) particles, with the associate eigenvalue \( E_i^{(n)} \), then

\[
\Phi_\star_n = \sum_i \delta(E - E_i^{(n)}). \tag{304}
\]

Thus the invariant \( \Phi_\star_n \) is the number of quantum states with \( n \) particles with eigenvectors less or equal to \( E \).

It is worth taking the classical limit of (303). To this end, we consider that the system is enclosed in a cubic vessel of size \( L \) and use the eigenfunctions of the momentum

\[
\phi_n(x) = \frac{1}{\sqrt{L}} e^{ikx}, \tag{305}
\]

where \( k = 2\pi n \). After expressing \( \Phi \) in terms of the basis of plane waves, we may take the classical limit. The result is

\[
\Phi_\star_N = \frac{1}{N!} \int \delta(E - H) d^nx d^np \frac{p}{h^n}, \tag{306}
\]

where \( H \) is the classical Hamiltonian, \( N \) is the number of particles, \( n \) is the number of degrees of freedom, and \( h \) is the Planck constant. Except for the factor \( h^n \), this coincides with (291).

For a system of \( N \) noninteracting particles we use the result (292) to find

\[
\Phi_\star_N = \frac{V^N}{N!} \left( \frac{2\pi mE}{h^2} \right)^{3N/2}. \tag{307}
\]

E. Canonical distribution

Let us suppose that instead of the Gibbs microcanonical distribution the system is described by the Gibbs canonical distribution

\[
\rho = \frac{1}{Z} e^{-\beta H}, \tag{308}
\]

where \( \beta \) is a parameter, \( H \) is the Hamiltonian that does not depend on \( \beta \) but depend on a parameter \( \lambda \), and

\[
Z = \int e^{-\beta H} dq dp \tag{309}
\]
We wish to determine the invariant which is analogous to that of Hertz.

The probability distribution $P(E)$ of the energies is the marginal probability distribution obtained from $\rho(q,p)$ and is given by

$$P(E) = \frac{1}{Z}\Omega e^{-\beta E}, \quad (310)$$

where

$$\Omega = \int \delta(E - \mathcal{H})dqdp, \quad (311)$$

which depends on $E$ and $\lambda$, and

$$Z = \int e^{-\beta E}\Omega dE, \quad (312)$$

which depends on $\beta$ and $\lambda$.

If the parameter $\lambda$ is varying slowly in time we use equation (258)

$$\frac{dU}{d\lambda} = \langle \frac{\partial \mathcal{H}}{\partial \lambda} \rangle, \quad (313)$$

but now the average on the right-hand side is taken over the canonical distribution $\rho$,

$$\langle \frac{\partial \mathcal{H}}{\partial \lambda} \rangle = \int \frac{\partial \mathcal{H}}{\partial \lambda} \rho dqdp. \quad (314)$$

The equation (313) determines the relation between $\beta$ and $\lambda$ along a parametric slow action. The invariant $\Psi$ that we wish to find will then be a function of $\beta$ and $\lambda$.

The right-hand side of (314) is written in a more convenient form as

$$\frac{1}{Z} \int \frac{\partial \mathcal{H}}{\partial \lambda} \delta(E - \mathcal{H}) e^{-\beta E} dqdp dE, \quad (315)$$

which is equal to

$$- \frac{1}{Z} \int \frac{\partial \Phi}{\partial \lambda} e^{-\beta E} dE, \quad (316)$$

where

$$\Phi = \int \delta(E - \mathcal{H}) dqdp, \quad (317)$$

and depends on $E$ and $\lambda$.

The average energy $U$ is written as

$$U = \frac{1}{Z} \int E e^{-\beta E}\Omega dE, \quad (318)$$

and can be obtained from $Z$ by

$$U = - \frac{1}{Z} \frac{\partial Z}{\partial \beta}. \quad (319)$$

Using the above results, the equation (313) becomes

$$\frac{dU}{d\lambda} = - \frac{1}{Z} \int \frac{\partial \Phi}{\partial \lambda} e^{-\beta E} dE. \quad (320)$$

The invariant $\Psi$, which depends on $\beta$ and $\lambda$, is such that if $\Psi(\beta,\lambda)$ is held constant then $\beta$ is connected to $\lambda$ in such a way that (320) is fulfilled. It is equivalent to say that, in equation (320),

$$\frac{d\beta}{d\lambda} = - \frac{\partial \Psi}{\partial \lambda} \frac{\partial \Psi}{\partial \beta}. \quad (321)$$

For comparison with equation (320), we write this equation in the form

$$\frac{\partial \Psi}{\partial \beta} \frac{d\beta}{d\lambda} = - \frac{\partial \Psi}{\partial \lambda}. \quad (322)$$

We show now that the following form,

$$\Psi = e^{\beta U} Z, \quad (323)$$

is invariant, where $Z$ is given by (312), and is equivalent to the expression

$$Z = \beta \int \Phi e^{-\beta E} dE, \quad (324)$$

obtained by an integration by parts. To this end we calculated its derivative with respect to $\beta$ and $\lambda$. Recalling that $\Phi$ does not depend on $\beta$, and using the result (319), we find

$$\frac{\partial \Psi}{\partial \beta} = \beta U \Psi. \quad (325)$$

The other derivative is

$$\frac{\partial \Psi}{\partial \lambda} = \left( \beta \frac{\partial U}{\partial \lambda} + \frac{1}{Z} \int \frac{\partial \Phi}{\partial \lambda} e^{-\beta E} dE \right) \Psi. \quad (326)$$

Replacing the two derivatives in equation (322), we get

$$\frac{dU}{d\lambda} = - \frac{1}{Z} \int \frac{\partial \Phi}{\partial \lambda} e^{-\beta E} dE, \quad (327)$$

which is equivalent to equation (320), and shows that $\Psi$ is an invariant.

For a system of $N$ noninteracting particles we replace the result (307) in equation (324) to find, after integration in $E$,

$$Z = \frac{V^N}{N!} \left( \frac{2\pi m}{\beta \hbar^2} \right)^{3N/2}. \quad (328)$$

Taking into account that $U = 3N/2\beta$ we obtain the invariant $\Psi$,

$$\Psi = e^{3N/2} \frac{V^N}{N!} \left( \frac{2\pi m}{\beta \hbar^2} \right)^{3N/2}. \quad (329)$$

IX. THERMODYNAMICS

A. Adiabatic invariants

An adiabatic process is understood as a process which does not involve the exchange of heat. This type of
process played an important role in the development of the theory of heat \[89, 90\]. Laplace for instance proposed that the variations of pressure in the propagation of sound in air are adiabatic processes. Carnot announced his fundamental principle by means of a cyclic process involving two isothermal and two adiabatic subprocesses. Such a cyclic process was also used by Clausius when he introduced the laws of thermodynamics. The term adiabatic was coined by Rankine in 1859 to refer to processes without the intervention of heat.

An adiabatic process may be a slow process or a very fast one as long as the system is enclosed by adiabatic walls or is in isolation. If the walls are not perfectly adiabatic, the process can still be adiabatic if it occurs very rapidly. If an air pump is compressed quickly, there is no time for heat to be exchanged with the surroundings and the process is adiabatic. In this case, the temperature of the air inside the pump increases. Just after the process is adiabatic, it was known that the quantity $\gamma$ remains constant along an adiabatic process \[91–93\]. But before Clausius, in 1865 he called entropy that remains constant along an adiabatic process.

Along an adiabatic process we may ask whether there are state functions that are invariant along this process. If the process is slow enough, this question was answered by Clausius when he introduced in 1854 a quantity, which in 1865 he called entropy that remains constant along a slow adiabatic process \[91\] \[93\]. But before Clausius, it was known that the quantity $pV^\gamma$, remains constant when an ideal gas undergoes a slow adiabatic process. Here, $p$ is the pressure, $V$ is the volume and $\gamma$ is the ratio of the two types of specific heats.

The result that $pV^\gamma$ is a constant was obtained theoretically by Poisson in 1823 \[94\] by using the equation of state of an ideal gas, $pV$ proportional to the temperature $T$, and the assumption that $\gamma$ is constant. The starting point of his derivation is the equation

$$\frac{\gamma dV}{dT} + \frac{dp}{\partial p/\partial T} = 0, \quad (330)$$

valid along an adiabatic process, which for an ideal gas becomes $pdV + Vdp$. Considering that $\gamma$ is constant, the integration of this equation gives $pV^{\gamma}$ equal to a constant. Although the equation \[330\] was derived by Poisson by assuming that heat is a state function, nevertheless it remains valid within thermodynamics. In fact, the invariance of $pV^{\gamma}$ was derived by Clausius in 1850 within the realm of thermodynamics \[92, 95\].

The derivation of the adiabatic invariant $pV^\gamma$, either by Poisson or by Clausius, involved thermal properties that depended explicitly on the temperature. However, as an adiabatic process involves no heat we may presume that any adiabatic invariant could be derived without referring to thermal properties. That is, a derivation carried out within the realm of mechanics by considering a parametric slow process. To show that this is indeed the case for the Poisson adiabatic invariant, we follow the reasoning of Rayleigh, which is summarized by equation \[29\].

If the volume $V$ of a gas enclosed in a vessel is varied slowly, then in accordance with the equation \[29\], the variation of the energy with the volume is

$$\frac{dE}{dV} = -p, \quad (331)$$

where $p$ is the pressure of the gas. For a system of noninteracting molecules, it follows from the laws of mechanics that the pressure $p$ is two-thirds of the kinetic energy $E$,

$$p = \frac{2E}{3V}. \quad (332)$$

A result derived by Krönig \[96\], by Clausius \[97, 98\], and by Maxwell \[99\] within the kinetic theory. Considering a simple gas with only translational degrees of freedom, $E$ is the total energy. Replacing the result \[332\] into \[331\], we find by integration that $EV^{2/3}$ is constant, from which follows that $pV^{5/3}$ is an adiabatic invariant.

A similar invariant is obtained for the radiation. In his treatise on electricity and magnetism of 1873 \[100\], Maxwell showed that the pressure of radiation is one-third of the density of energy, or

$$p = \frac{E}{3V}. \quad (333)$$

Replacing this result into \[331\], we find by integration that $EV^{1/3}$ from which follows that $pV^{4/3}$ is an adiabatic invariant.

It is worth mentioning that in the derivation of the Stefan-Boltzmann law carried out by Boltzmann in a paper of 1884 \[101\], he made use of the Maxwell relation \[333\] along with thermodynamic reasoning. A derivation of this law by the use of the invariant just obtained is as follows. As the entropy $S$ is an invariant we may consider it as a function of the invariant $EV^{1/3}$. Assuming that it is a homogeneous function of $E$ and $V$, it follows that $S$ is proportional to $V^{1/4}E^{3/4}$. The temperature is obtained by $1/T = \partial S/\partial E$ from which follows that $E = aVT^4$. Replacing this result in equation \[333\], we find $p$ to be proportional to $T^4$, which is a statement of the Stefan-Boltzmann law.

**B. Boltzmann and Gibbs entropy**

The concept of entropy was introduced by Clausius as a quantity that is constant along a slow adiabatic process and that increases in an irreversible process. These properties of the entropy are a brief statement of the second law of thermodynamics introduced by Clausius. The definition of entropy that emerges from the Boltzmann writings is laid down on the second property, related to the increase of entropy, and can be found in his book on the theory of gases \[102, 103\].
In a paper of 1872 Boltzmann proved that the quantity \[ H = \int f \ln f \, d^3r \, d^3v \] never increases, where \( f \) is the one-particle probability distribution. The negative of \( H \) was understood by Boltzmann as proportional to the entropy. To demonstrate this result, which comprises the Boltzmann H-theorem, Boltzmann used the transport equation that he introduced in the same paper. Replacing the Maxwell distribution in the expression for \( H \), he obtained the following expression for the entropy of an ideal gas \[ \ln V \left( \frac{4\pi T}{3m} \right)^{\frac{3}{2}} + \frac{3}{2} N, \] where \( N \) is the number of molecules, \( V \) the volume, \( m \) the mass of a molecule, and \( T \) is the mean kinetic energy.

Later on, in 1877, Boltzmann related entropy to probability, which he stated in the following terms [105, 106]. In most cases the initial state of a system is a very improbable one and the system has the tendency to reach more probable states, those of thermal equilibrium. If we apply this to the second law, we can identify that quantity, which is usually called entropy, with the probability of the state in question. For an ideal gas of \( n \) molecules, Boltzmann finds the relation between entropy and probability as follows [105, 106]. The number of complexities that corresponds to a given repartition \( n_0, n_1, n_2 \), and so on, of the \( n \) molecules into the kinetic energies \( 0, \varepsilon, 2\varepsilon \), and so on, is the permutation number
\[ P = \frac{n!}{n_0!n_1!n_2!\ldots} \]
The most probable state corresponds to the maximum of \( P \), or equivalently to the maximum of \( \ln P \). Considering that \( n_i \) are large numbers we may use the approximation \( \ln n! = n \ln n - n \) to find
\[ \ln P = n \ln n - \sum n_i \ln n_i. \]
Taking the continuous limit of the energy, the second term of this expression yields the quantity \(-H\), which is identified as the entropy. Boltzmann formulates a general principle relating \( P \) and entropy in following terms [105, 106]. The measure of permutability of all bodies will always grow in the course of the state changes and can at most remain constant as long as all bodies are in thermal equilibrium.

In a paper of 1901 on the radiation formula, Planck translated the Boltzmann relation between entropy and probability in the following terms [107]
\[ S = k \ln W, \] with the exception of an additive constant, where \( W \) is the probability (Wahrscheinlichkeit in the original paper) that the \( N \) resonators have a total energy \( U \), and \( k \) is one of the two constants of nature introduced by Planck [108], which is the Boltzmann constant. Following Boltzmann, Planck determines the number of complexities which he says is proportional to \( W \). Although, Planck speaks of \( W \) as a probability, the quantity is in fact understood in formula (338) as the reciprocal of the probability.

Gibbs considered two types of entropy, given by (255) and (254). The first form is similar to (338) in the sense that both are proportional to the logarithm of the number of states of a system with a fixed energy. Of the two types of entropy, Gibbs opted for the first. These two types of entropy considered by Gibbs refer to the microcanonical distribution. He also introduced a form of the entropy for systems described by the canonical distribution. In this case Gibbs defined entropy as the average of the index of probability with the sign reversed. As the index of probability is the logarithm of the probability, the Gibbs canonical entropy is
\[ S_G = -k \int \rho \ln \rho \, dq \, dp. \]
Although Gibbs speaks of average, in fact the entropy given by (339) is not an average of a state function, as \( \ln \rho \) is not properly a state function.

It is worth comparing the expression (339) with that related to \( H \) given by equation (334). If we consider a system of \( N \) particles, the formula (334) gives the expression
\[ S_B = -N \int f \ln f \, d^3r \, d^3v. \]
If the particles do not interact, both expression (339) and (340) give the same value. If the particles interact, \( S_B \) is distinct from \( S_G \).

As we have seen above, the invariance of the second form of the microcanonical entropy, given (254), was the concern of Paul Hertz, who demonstrated the invariance in a paper published in 1910 [82]. The invariance of the entropy was also the concern of Einstein. In a paper of 1914, he considered the entropy of a quantum system with a discretized spectrum of energies which depended on a parameter [11]. He used the Planck formula (338), relating the entropy with the number \( Z \) of possible quantum states, and asked whether the entropy remains valid when the states of the system varies under the change of the parameter. Using the Ehrenfest principle he concluded that \( Z \) is indeed an invariant under the slow parametric action and so is the entropy.

C. Work and heat

The law of conservation of energy tells that the increase of the energy of a system equals the work done on the system. Thermodynamics distinguishes two types of work. One of them is the heat \( Q \), sometimes called
internal work, and the other is the external work \( W \), of simply work. The variation \( \Delta U \) of the energy is thus \( \Delta U = Q + W \). In thermodynamics, the distinction is provided by the introduction of adiabatic walls. If a system is enclosed by adiabatic walls, there is no heat involved and the increase in energy is the external work. However, we are faced here with a circular reasoning and another way of distinguish heat and work is necessary [109].

The distinction between the two types of work is obtained by considering the connection of a system with the environment. One of them is the ordinary interaction, that we call dynamic connection, in which the state of the system varies by virtue of the connection of the dynamic variables of the system with those of the environment. The other is the parametric connection in which the state of the system varies by virtue of the variation of a parameter. A more precise distinction is provided by considering the external forces acting on the system. In the dynamic connection, the external forces depend on the dynamic variables of the environment but not on the parameters. In the parametric connection, the external forces depend on the parameter but not on the dynamic variables of the environment.

These two types of connection with the environment, allows us to mechanically distinguish \( Q \) and \( W \). If the system is only parametrically connected with the environment, there is no heat involved, \( Q = 0 \), and the change of energy of the system is the work \( W \) caused by the variation of the parameter. If the parameter is held constant and the system is dynamically connected with the environment, there is no work related to the variation of the parameter, \( W = 0 \), and the variation in energy is the work \( Q \) related to the dynamic variables, which is understood as heat.

If the system is connected to the surrounding only through the parametric action, the resulting process is thus an adiabatic process, in the thermodynamic sense. The idea of adiabatic process as a purely mechanical process is implicit in the works of Clausius and Boltzmann on the kinetic theory as they considered a thermodynamic system as a mechanical system. According to Jammer [2], the idea of an adiabatic process as related with the slow variation of a parameter is to be found in the works of Helmholtz and Heinrich Hertz, who tried to identify a parameter of the system to a cyclic variable. As we have seen above a precise understanding of an adiabatic process as the result of a variation of a parameter was implicit in the paper of Paul Hertz, who developed this concept from a hint given by Gibbs. It appears that it was on this understanding that Einstein called the Ehrenfest hypothesis about parametric invariants the adiabatic hypothesis, and that Ehrenfest called the parametric invariant the adiabatic invariant.

If the system is connected to the surrounding only through the parametric action, the resulting process is an adiabatic process no matter if the resulting process is slow or not. The distinction between a slow and a rapid process is that in the slow process the system remains in equilibrium or rather near equilibrium as we are speaking of a process evolving in time. Processes of this type are usually called quasi-static processes, and for that reason sometimes the slow parametric action is named quasi-static. We think that this is also inappropriate due to the existence of quasi-static processes, such as the isothermal process, which are not adiabatic. Thus we may say that the slow parametric action results in a quasi-static adiabatic process or a slow adiabatic process.

### D. Equilibrium thermodynamics

The existence of parametric invariance in classical and quantum mechanics is the crucial feature that allows the construction of a continuous sequence of equilibrium thermodynamic states when a parameter is changed slowly in time, as may happen when work is done upon a system. This is understood as follows. If a system in equilibrium is perturbed during a finite interval of time \( \Delta t \), it will be found in a nonequilibrium state at the end of the perturbation, and we have to wait another finite interval of time \( \tau \) for the system to reach equilibrium again. Suppose that the perturbation is carried out by the variation of a parameter \( \lambda \) which changes by \( \Delta \lambda = c \Delta t \) during the interval \( \Delta t \), where \( c \) is the rate at which the parameter is changed. If \( c \) is small enough, the adiabatic invariance guarantees that \( \tau \) becomes negligible and the system remains in equilibrium after the finite change \( \Delta \lambda \) of the parameter.

In accordance with the understanding concerning heat and work, we write the variation of the energy during a certain small interval of time when the parameters \( \lambda_i \) is slowly varying in time as

\[
dU = dQ + dW,
\]

where \( dW \) is the work performed by the system. As the parameter are slowly varying in time, the resulting process is adiabatic, \( dQ = 0 \) and \( dW = dU \). Using the expression (272) for \( dU \), we write

\[
dW = -\sum_i F_i d\lambda_i,
\]

where

\[
F_i = \frac{\partial H}{\partial \lambda_i},
\]

and \( H \) is the Hamiltonian of the system. The time average is in turn replaced by the average

\[
F_i = -\langle \frac{\partial H}{\partial \lambda_i} \rangle,
\]

over a Gibbs probability distribution \( \rho \). The energy \( U \) is understood as the average \( U = \langle H \rangle \).

If we consider a slow variation of the parameter, this process will corresponds to lines that lay down on a surface on the space spanned by the parameters \( \lambda_i \) and by
the energy $E$. Along any lines there is no exchange of heat and the parametric invariant $\Phi$ is constant. That is, each surface is characterized by a certain value of $\Phi$. The energy can then be considered as function of $\Phi$ and of the parameters, so that

$$dU = Ad\Phi - \sum_i F_i d\lambda_i,$$

(345)

which compared to (344) tell us that

$$dQ = Ad\Phi.$$  

(346)

The relation (346) is in accordance with the Clausius relation $dQ = TdS$. However, it does not mean necessarily that the entropy $S$ is equal or proportional to $\Phi$. It says that $S$ is a function of $\Phi$. To determine the specific function, we must introduce a condition. We may suppose that $S$ scales as the size of the system. Taking into account that $\Phi$ scales with the size of the system to the $n$-th power, where $n$ is the number of degrees of freedom, we may place $\ln \Phi$ as proportional to $S$ and write

$$S = k \ln \Phi,$$  

(347)

where $k$ is a constant with the same physical dimension of the entropy.

Having established the relation between the entropy $S$ and the invariant $\Phi$, we may determine the temperature $T$ by its thermodynamic definition $1/T = \partial S/\partial E$. In the case of a system described by the Gibbs microcanonical distribution, $\Phi$ is given by (251), and we find

$$\frac{1}{kT} = \frac{\Omega}{\Phi},$$  

(348)

where $\Omega$ is given by (252) and is related to $\Phi$ by $\Omega = \partial \Phi/\partial E$.

### E. Canonical distribution

If a system is in contact with the environment only through a parametric connection, the Hamiltonian $\mathcal{H}$ is a function of the internal variables, which are the dynamic variables $q$ and $p$ of the system. The external forces depend on the parameters and on the internal variables but not on the external variables, which are the variables of the environment. In this case the energy of the system is the average of $\mathcal{H}$ which is a function of the internal variables.

Now let us consider the case where the environment is subject to the dynamic connection. In this case the external forces are functions of both the internal and external variables and the energy of the system will have a term that is related to the external variables. If the system is nearly equilibrium the treatment given by the statistical mechanics is to replace the dynamic description by a description in terms of a probabilistic description given for example by the Gibbs canonical description. In this description the probability distribution is a function of a Hamiltonian which is understood as the Hamiltonian $\mathcal{H}$ that one obtains by removing the terms describing the dynamic interaction with the environment. This does not mean that the interaction has been neglected. In fact, it is taken into account through the probability distribution itself, such as the Gibbs canonical distribution which describes the interaction with a heat reservoir at a given temperature. Therefore, the energy $U$ that enters equation (341) is the average of $\mathcal{H}$ which depends only on the internal dynamic variables.

In the present case of the dynamic connection, in which the system is described by the Gibbs canonical distribution the parametric invariant is $\Psi$ given by equation (323),

$$\Psi = e^{\beta U} Z,$$  

(349)

Again the usual choice is a logarithm relation

$$S = k \ln \Psi = k \beta U + k \ln Z,$$  

(350)

which gives an entropy that increases with the size of the system. Deriving this relation with respect to $\beta$ and using relation (319), and comparing with the Clausius relation $dU = TdS$, one finds $\partial S/\partial \beta = k \beta U/\partial \beta$, which gives the relation $\beta = 1/kT$ between the parameter $\beta$ and $T$.

What is the relation between the invariant $\Psi$ and the Gibbs expression for the entropy

$$S_G = -k \int \rho \ln \rho dq dp.$$  

(351)

If we replace the distribution (308) in this equation we find that it equals $k \ln \Psi$, that is, it is an invariant and coincides with the entropy (350) that scales with the size of the system.

### X. CONCLUSION

We have argued that the slow variation of a parameter of a system results in a slow adiabatic process. Thus a parametric invariant such as the Hertz invariant is an adiabatic invariant. As the entropy is a thermodynamic variable that characterizes a reversible adiabatic process, understood as slow adiabatic process, it is constant along this process and can be understood as a parametric invariant. As the entropy grows with the size of the system, it is defined as the logarithm of the Hertz invariant.

The Hertz invariant $\Phi$ is the volume in phase space of the region enclosed by the surface of constant energy. It remains constant when a parameter is slowly changed with time. We have extended this result to the case where the number of particles varies with time with the conclusion that $\Phi^* = \Phi/N!$ is the invariant. We have also considered the quantum version of the Hertz invariant for the case of the variation of a parameter as well as the
variation of the number of particles. In this last case, the classical limit results in the quantity $\Phi^* = \Phi/N$!

The parametric invariance allows the distinction between heat and work. This is provided by considering the two types of connections of the system with the environment that we have called dynamical and parametric connection. If only the second is present and the parameter is varied, the resulting process is adiabatic. If in addition the variation of the parameter is slow, the entropy characterizes this process and can thus be associated to the parametric invariant.

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