Quantized Affine Lie Algebras and Diagonalization of Braid Generators

M.D. Gould and Y.-Z. Zhang

Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia

Abstract:

Let $U_q(\hat{G})$ be a quantized affine Lie algebra. It is proven that the universal R-matrix $R$ of $U_q(\hat{G})$ satisfies the celebrated conjugation relation $R^\dagger = TR$ with $T$ the usual twist map. As applications, braid generators are shown to be diagonalizable on arbitrary tensor product modules of integrable irreducible highest weight $U_q(\hat{G})$-module and a spectral decomposition formula for the braid generators is obtained which is the generalization of Reshetikhin’s and Gould’s forms to the present affine case. Casimir invariants are constructed and their eigenvalues computed by means of the spectral decomposition formula. As a by-product, an interesting identity is found.
1 Introduction

Braid generators for quantum (super)groups are shown to be diagonalizable on any tensor product module of irreducible highest weight (IHW) modules of the quantum (super)groups, in both case of multiplicity-free [1] and with multiplicity [2]. Such a diagonalized form is seen to be very useful in computing the quantum group invariants such as link polynomials. Here we continue the development and show some similar results for quantized affine Lie algebras $U_q(\hat{G})$.

After recalling, in section 2, some fundamentals on $U_q(\hat{G})$, we in section 3 show that the universal $R$-matrix of $U_q(\hat{G})$ satisfies a conjugation relation. In section 4 we show that braid generators are diagonalizable on tensor product modules of integrable IHW $U_q(\hat{G})$-module, and obtain a spectral decomposition formula of the braid generators which is the generalization of Reshetikhin’s and Gould’s forms [1] [2] to our affine case. Using this spectral decomposition formula we construct, in section 5, a family of Casimir invariants and compute their eigenvalues on integrable IHW modules; the eigenvalues are absolutely covergent for $|q| > 1$. Interestingly enough, we obtain an identity which bears similarity to the power series expansion of a function in its absolutely convergent region. Section 6 is devoted to some remarks.

2 Fundamentals

Let $A = (a_{ij})_{0 \leq i,j \leq r}$ be a symmetrizable, generalized Cartan matrix in the sense of Kac[3]. Let $\hat{G}$ denote the affine algebra associated with the corresponding symmetric Cartan matrix $A_{\text{sym}} = (a_{ij}^{\text{sym}}) = (\alpha_i, \alpha_j), \ i, j = 0, 1, ... r$, $r$ is the rank of the corresponding finite-dimensional simple Lie algebra. The quantum algebra $U_q(\hat{G})$ is defined to be a Hopf algebra with generators: \{e_i, f_i, q^{h_i} (i = 0, 1, ..., r), q^d\} and relations [4] [5],

\[
q^h q^{h'} = q^{h+h'} (h, h' = h_i (i = 0, 1, ..., r), d)
\]

\[
q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i
\]

\[
[e_i, f_j] = \delta_{ij} q^{h_i} - q^{-h_i} \over q - q^{-1}
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = 0 \quad (i \neq j)
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0 \quad (i \neq j)
\]

where

\[
e_i^{(k)} = \frac{e_i^k}{[k]_q!}, \quad f_i^{(k)} = \frac{f_i^k}{[k]_q!}, \quad [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]_q! = [k]_q[k-1]_q \cdots [1]_q
\]

The Cartan subalgebra (CSA) of $\hat{G}$ is generated by \{h_i, i = 0, 1, ..., r; d\}. However, we will choose as the CSA of $U_q(\hat{G})$

\[
\mathcal{H} = \mathcal{H}_0 \bigoplus \mathbb{C}c \bigoplus \mathbb{C}d
\]
respectively, so that \( (ab)^\dagger \) will use the notations which extend uniquely to an algebra anti-automorphism and an anti-involution on all of \( G \) satisfying, among others, \( y^\dagger = y \) and corresponding quantum representations have the same weight spectrum. Following the usual convention, we denote the weight of a representation by \( \Lambda \equiv (\lambda, \kappa, \tau) \), where \( \lambda \in \mathcal{H}_0^* \subset \mathcal{H}^* \) is a weight of \( G \) and \( \kappa = \Lambda(c) \), \( \tau = \Lambda(d) \). The non-degenerate form \( \langle , \rangle \) on \( \mathcal{H}^* \) is defined by \[ (\Lambda, \Lambda') = (\lambda, \lambda') + \kappa' \tau' + \kappa \tau \] for \( \Lambda' = (\lambda', \kappa', \tau') \). With these notations we have \( \rho = (\rho_0, 0, 0) + g(0, 1, 0) \)
where $\rho_0$ is the half sum of positive roots of $G$ and $2g = (\psi, \psi + 2\rho_0)$.

We call
\[
D_q[\Lambda] = \text{tr}(\pi_\Lambda(q^{2\rho}))
\]
the $q$-dimension of the integrable IIHW representation $\pi_\Lambda$: explicitly
\[
D_q[\{\lambda, \kappa, \tau\}] = q^{2\tau}\tilde{D}_q[\{\lambda, \kappa, 0\}],
\]
\[
\tilde{D}_q[\{\lambda, \kappa, 0\}] = D_q^0[\lambda] \prod_{t=1}^{\infty} \left( \frac{1 - q^{-2t(\kappa + g)}}{1 - q^{-2tg}} \right)^r \prod_{\alpha \in \Phi_0} \prod_{t=1}^{\infty} \frac{1 - q^{-2t(\rho_0, \alpha) - 2t(\kappa + g)}}{1 - q^{-2t(\rho_0, \alpha) - 2tg}}
\]
with
\[
D_q^0[\lambda] = \prod_{\alpha \in \Phi_+^o} \frac{q^{-(\lambda + \rho_0, \alpha)} - q^{-(\rho_0, \alpha)}}{q^{-(\alpha, \alpha)} - q^{-(\rho_0, \alpha)}}
\]
where $\Phi_0$ and $\Phi_0^+$ denote the set of roots and positive roots of $G$, respectively. Note that the $q$-dimension \eqref{eq:q_dim} is absolutely convergent for $|q| > 1$. Following the similar lines as in \cite{9}, we can show

**Proposition 1**: Let $V(\Lambda)$ be an IIHW $U_q(\hat{G})$-module with highest weight $\Lambda \in D^+$, $D^+$ satands for the set of all dominant integral weights. If the operator $\Gamma \in U_q(\hat{G}) \otimes \text{End}V(\Lambda)$ satisfies $\Delta_\Lambda(a)\Gamma = \Gamma\Delta_\Lambda(a), \forall a \in U_q(\hat{G})$, where $\Delta_\Lambda = (I \otimes \pi_{\Lambda})\Delta$, then
\[
C = (I \otimes \text{tr})\{[I \otimes \pi_\Lambda(q^{2\rho})]\Gamma\}
\]
belongs to the center of $U_q(\hat{G})$, i.e. $C$ is a Casimir invariant of $U_q(\hat{G})$.

## 3 Conjugation Relation

In this section we show that the universal R-matrix $R$ satisfies a conjugation relation. We illustrate the proof for the nowtwisted case, $U_q(\hat{G}) = U_q(\mathcal{G}^{(1)})$ and the same arguments are valid for the twisted case.

To begin with, let $\hat{G} = sl(2)^{(1)}$. Fix a normal ordering in the positive root system $\Delta_+$ of $sl(2)^{(1)}$:
\[
\alpha, \alpha + \delta, ..., \alpha + n\delta, ..., \delta, 2\delta, ..., m\delta, ..., \beta + l\delta, ..., \beta
\]
where $\alpha$ and $\beta$ are simple roots and $l, m, n \geq 0$; $\delta = \alpha + \beta$ is the minimal positive imaginary root. Khoroshkin and Tolstoy show \cite{11} that the universal R-matrix for $U_q(sl(2)^{(1)})$ may be written as
\[
R = \left( \prod_{n \geq 0} \exp_{q_\alpha}((q - q^{-1})(E_{\alpha + n\delta} \otimes F_{\alpha + n\delta})) \right) \cdot \exp \left( \sum_{n > 0} n[q_n^{-1}(q_\alpha - q^{-1})(E_{n\delta} \otimes F_{n\delta})] \right) \cdot \left( \prod_{n \geq 0} \exp_{q_\alpha}((q - q^{-1})(E_{\beta + n\delta} \otimes F_{\beta + n\delta})) \right) \cdot q^{h_\alpha \otimes h_\alpha + c \otimes d + d \otimes c}
\]
where \( c = h_{\alpha} + h_{\beta} \) and the order in the product \( (13) \) coincides with the chosen normal order \( (14) \); Cartan-Weyl generators \( E_{\gamma}, \ F_{\gamma} = \theta(E_{\gamma}), \ \gamma \in \Delta_+ \) of \( U_q(sl(2)^{(1)}) \) are defined by

\[
E_{\alpha} = e_{\alpha} q^{-h_{\alpha}/2}, \quad E_{\beta} = e_{\beta} q^{-h_{\beta}/2}, \quad F_{\alpha} = q^{h_{\alpha}/2} f_{\alpha}, \quad F_{\beta} = q^{h_{\beta}/2} f_{\beta}
\]

\[
\tilde{E}_{\delta} = [(\alpha, \alpha)]_{q}^{-1} [E_{\alpha}, E_{\beta}]_q \quad E_{\alpha+n\delta} = (-1)^n \left( \text{ad} \tilde{E}_{\delta} \right)^n E_{\alpha}
\]

\[
E_{\beta+n\delta} = \left( \text{ad} \tilde{E}_{\delta} \right)^n E_{\beta}, \ldots, \quad \tilde{E}_{n\delta} = (\alpha, \alpha)]_{q}^{-1} [E_{\alpha+(n-1)\delta}, E_{\beta}]_q
\]

and

\[
\tilde{E}^{n\delta} = \sum_{k_1 p_1 + \ldots + k_m p_m = n} \frac{\left( q^{(\alpha, \alpha)} - q^{-(\alpha, \alpha)} \right)^{\sum_i p_i - 1}}{p_1! \ldots p_m!} (E_{k_1 \delta})^{p_1} \ldots (E_{k_m \delta})^{p_m}
\]

By means of the following relations shown in \( [1] \)

\[
S(E_{\alpha+n\delta}) = -q^{n(\alpha, \beta)} E_{\alpha+n\delta}, \quad S(E_{\beta+n\delta}) = -q^{n(\alpha, \beta)} E_{\beta+n\delta}
\]

\[
S(F_{\alpha+n\delta}) = -q^{-n(\alpha, \beta)} E_{\alpha+n\delta}, \quad S(F_{\beta+n\delta}) = -q^{-n(\alpha, \beta)} E_{\beta+n\delta}
\]

\[
S(\tilde{E}^{n\delta}) = -q^{n(\alpha, \beta)} \tilde{E}^{n\delta}, \quad S(E_{n\delta}) = -q^{n(\alpha, \beta)} E_{n\delta}
\]

\[
S(F_{n\delta}) = -q^{-n(\alpha, \beta)} F_{n\delta}, \quad S(F_{n\delta}) = -q^{-n(\alpha, \beta)} F_{n\delta}
\]

We now prove the following

**Proposition 2:** The universal \( R \)-matrix for \( U_q(sl(2)^{(1)}) \) satisfies the conjugation relation: \( R^\dagger = R^T \)

**Proof:** The universal \( R \)-matrix \( (13) \) can be written as

\[
R = \sum_{l, n, k} A_{l, n, k}(q) (E_{\alpha})^{l_0} \ldots (E_{\alpha+N\delta})^{l_N} \ldots (E_{\delta})^{n_1} \ldots (E_{L\delta})^{n_L} \ldots
\]

\[
\cdot (E_{\beta+M\delta})^{k_M} \ldots (E_{\beta})^{k_0} \otimes (F_{\alpha})^{l_0} \ldots (F_{\alpha+N\delta})^{l_N} \ldots (F_{\delta})^{n_1} \ldots (F_{L\delta})^{n_L} \ldots
\]

\[
\cdot \ldots (F_{\beta+M\delta})^{k_M} \ldots (F_{\beta})^{k_0} \otimes q^{\frac{1}{2}}h_{\alpha} \otimes h_{\alpha} + c \otimes d + d \otimes c
\]

where \( \{l\} = \{l_0, l_1, \ldots, l_N, \ldots\} \), \( \{n\} = \{n_1, n_2, \ldots, n_L, \ldots\} \), \( \{k\} = \{k_0, k_1, \ldots, k_M, \ldots\} \); the constants \( A_{l, n, k}(q) \) are given by

\[
A_{l, n, k}(q) = \frac{(q - q^{-1})_{l_0 + l_1 + \ldots + l_N + \ldots} (q - q^{-1})_{k_0 + k_1 + \ldots + k_M + \ldots}}{(l_0)_{q_0} \ldots (l_N)_{q_0} \ldots (k_0)_{q_0} \ldots (k_M)_{q_0} \ldots}
\]

\[
\cdot \frac{1^{n_1} \ldots L^{n_L} \ldots (q - q^{-1})_{n_1 + n_2 + \ldots + n_L + \ldots}}{[1]_{q_0}^{n_1} \ldots [L]_{q_0}^{n_L} \ldots n_1! \ldots n_L! \ldots}
\]

From \( (13) \) we deduce

\[
(S \otimes S) R^\dagger = \sum_{l, n, k} A_{l, n, k}(q) S(E_{\alpha})^{l_0} \ldots S(E_{\alpha+N\delta})^{l_N} \ldots S(E_{\delta})^{n_1} \ldots S(E_{L\delta})^{n_L} \ldots
\]

\[
\cdot S(E_{\beta+M\delta})^{k_M} \ldots S(E_{\beta})^{k_0} \otimes S(F_{\alpha})^{l_0} \ldots S(F_{\alpha+N\delta})^{l_N} \ldots S(F_{\delta})^{n_1} \ldots S(F_{L\delta})^{n_L}
\]

\[
\ldots \ldots S(F_{\beta+M\delta})^{k_M} \ldots S(F_{\beta})^{k_0} \otimes q^{\frac{1}{2}}h_{\alpha} \otimes h_{\alpha} + c \otimes d + d \otimes c
\]
which gives, with the help of (18)

\[(S \otimes S)R^\dagger = R^T\]  \hspace{1cm} (22)

Hence by (5)

\[R^\dagger = (S^{-1} \otimes S^{-1})R^T = R^T,\]  \hspace{1cm} (23)
as required. \hspace{1cm} \Box

Next we come to general case: \(\hat{G} = G^{(1)}\). Fix some order in the positive root system \(\Delta_+\) of \(G^{(1)}\), which satisfies an additional condition,

\[\alpha + n\delta \leq k\delta \leq (\delta - \beta) + l\delta\]  \hspace{1cm} (24)

where \(\alpha, \beta \in \Delta^0_+, \Delta^0_+\) is the positive root system of \(G\); \(k, l, n \geq 0\) and \(\delta\) is the minimal positive imaginary root. Then the universal \(R\)-matrix \(U_q(G^{(1)})\) may be written in the following form \([11]\),

\[
R = \left(\prod_{\gamma \in \Delta^\leq_+, \gamma < \delta} \exp_{q\gamma} \left(\frac{q^{-1}}{C_{\gamma}(q)} E_{\gamma} \otimes F_{\gamma}\right)\right) \cdot \exp \left(\sum_{n>0} \sum_{i,j=1}^r C_{ij}^n(q)(q - q^{-1})(E_{n\delta}^{(i)} \otimes F_{n\delta}^{(j)})\right) \cdot \left(\prod_{\gamma \in \Delta^\leq_+, \gamma \geq \delta} \exp_{q\gamma} \left(\frac{q^{-1}}{C_{\gamma}(q)} E_{\gamma} \otimes F_{\gamma}\right)\right) \cdot q^{\sum_{i,j=1}^r (\delta_{\gamma \gamma})^i j h_i \otimes h_j + c \otimes d + d \otimes c}
\]  \hspace{1cm} (25)

where \(c = h_0 + h_\psi, \psi\) is the highest root of \(G\) and Cartan-Weyl generators \(E_{\gamma}\) and \(F_{\gamma} = \theta(E_{\gamma})\), \(\gamma \in \Delta_+\) are defined similarly as to \([16, 17, 11]\); \((C_{ij}^n(q))\), \(i, j = 1, 2, \ldots, r\), is the inverse of the matrix \((B_{ij}^n(q))\), \(i, j = 1, 2, \ldots, r\) with

\[
B_{ij}^n(q) = (-1)^{n(1-\delta_{ij})} q^{-n} \left(\frac{q_i^n - q_j^n}{q_i - q_j} q - q^{-1}\right)^{-1}, \quad q_{ij} = q^{(\alpha_i, \alpha_j)}, \quad q_i \equiv q_{\alpha_i}
\]  \hspace{1cm} (26)

and \(C_{\gamma}(q)\) is a normalizing constant defined by

\[
[E_{\gamma}, F_{\gamma}] = \frac{C_{\gamma}(q)}{q - q^{-1}} \left(q^{h_\gamma} - q^{-h_\gamma}\right), \quad \gamma \in \Delta^\leq_+
\]  \hspace{1cm} (27)

The order in the product of the \(R\)-matrix coincides with the chosen normal ordering \([24]\) in \(\Delta_+\). One can show \([3]\) that for any \(\alpha \in \Delta^0_+\),

\[
S(E_{\alpha+n\delta}^\dagger) = -q^{(\alpha, \alpha-2\rho)/2-n(\delta, \rho)} E_{\alpha+n\delta}
\]

\[
S(F_{\alpha+n\delta}^\dagger) = -q^{-h(\alpha, \alpha-2\rho)/2+n(\delta, \rho)} E_{\alpha+n\delta}
\]

\[
S(E_{\delta-\alpha+n\delta}^\dagger) = -q^{(\delta, \delta-\alpha-2\rho)/2-n(\delta, \rho)} E_{\delta-\alpha+n\delta}
\]

\[
S(F_{\delta-\alpha+n\delta}^\dagger) = -q^{-h(\delta, \delta-\alpha-2\rho)/2+n(\delta, \rho)} E_{\delta-\alpha+n\delta}
\]

\[
S(E_{n\delta}^{(i)\dagger}) = -q^{n(\delta, \rho)} E_{n\delta}^{(i)}, \quad S(F_{n\delta}^{(i)\dagger}) = -q^{n(\delta, \rho)} E_{n\delta}^{(i)}
\]

\[
S(E_{n\delta}^{(i)}) = -q^{n(\delta, \rho)} F_{n\delta}^{(i)}, \quad S(F_{n\delta}^{(i)}) = -q^{n(\delta, \rho)} E_{n\delta}^{(i)}
\]  \hspace{1cm} (28)
We are now in a position to state

**Proposition 3:** The universal $R$-matrix for $U_q(G^{(1)})$ satisfies the conjugation relation: $R^\dagger = R^T$

**Proof:** This is proven exactly as for proposition 2 by means of (28). □

**Remark 1:** Proposition 2 and 3 are also valid for twisted quantized affine Lie algebras.

### 4 Evaluation of Braid Generators

It is a well established fact that for quasitriangular Hopf algebras, there exists a distinguished element [4][10]

$$u = \sum_t S(b_t)a_t$$

(29)

where $a_t$ and $b_t$ are coordinates of the universal $R$-matrix

$$R = \sum_t a_t \otimes b_t$$

(30)

One can show that $u$ has inverse

$$u^{-1} = \sum_t S^{-2}(b_t)a_t$$

(31)

and satisfies

$$S^2(a) = uau^{-1}, \quad \forall a \in U_q(\hat{G})$$

$$\Delta(u) = (u \otimes u)(R^T R)^{-1}$$

(32)

where $R^T = T(R)$. It is easy to check that $v = uq^{-2h_\rho}$ belongs to the center of $U_q(\hat{G})$ and satisfies

$$\Delta(v) = (v \otimes v)(R^T R)^{-1}$$

(33)

Moreover, on an irreducible representation with highest weight $\Lambda \equiv (\lambda, \kappa, \tau) \in D^+$, the Casimir operator $v$ takes the eigenvalue

$$\chi_\Lambda = q^{-(\lambda, \lambda+2\rho)}$$

(34)

Let $V \equiv V(\lambda, \kappa, 0)$ (where without loss generality we have set $\tau = 0$) and $P$ be the permutation operator on $V \otimes V$ defined by $P(|\mu > \otimes |\nu >) = |\nu > \otimes |\mu >, \forall |\mu >, |\nu > \in V$ and Let

$$\sigma = PR \quad \in \text{End}(V \otimes V)$$

(35)

Here and in what follows we regard elements of $U_q(\hat{G})$ as operators on $V$. Then (3) is equivalent to

$$\sigma \Delta(a) = \Delta(a)\sigma \quad \forall a \in U_q(\hat{G})$$

(36)

which means that $\sigma$ is an operator obeying the condition of proposition 1. Furthermore,

**Proposition 4:** $\sigma$ is self-adjoint and thus may be diagonalized.
**Proof:** It follows from propositions 2, 3 and remark 1:

\[
\sigma^\dagger = (PR)^\dagger = R^\dagger P = R^TP = P \cdot PR^T P = PR = \sigma \quad \Box 
\]  

(37)

Recall that \( \lim_{q \to 1} \sigma = P \) and \( P \) is diagonalizable on \( V \otimes V \) with eigenvalues \( \pm 1 \). Following [2], we define the subspaces

\[
W_\pm = \{ w \in V \otimes V \mid \lim_{q \to 1} \sigma w = \pm w \} 
\]

(38)

Since \( \sigma \) is self-adjoint we may clearly write

\[
V \otimes V = W_+ \bigoplus W_-
\]

(39)

Let \( P[\pm] \) denote the projection operators defined by

\[
P[\pm](V(\lambda, \kappa, 0) \otimes V(\lambda, \kappa, 0)) = W_\pm
\]

(40)

Since \( \sigma \) is an \( U_q(\hat{G}) \)-invariant each subspace \( W_\pm \) determines a \( U_q(\hat{G}) \)-module and \( P[\pm] \) commute with the action of \( U_q(\hat{G}) \). On the other hand, it is shown [4] that the tensor product \( V \otimes V \) of integrable \( U_q(\hat{G}) \)-module \( V(\lambda, \kappa, 0) \) is completely reducible and the irreducible components are integrable highest weight representations. This means that we may decompose the tensor product \( V \otimes V \) according to

\[
V(\lambda, \kappa, 0) \otimes V(\lambda, \kappa, 0) = \bigoplus_{(\mu, 2\kappa, -s) \in D^+} \bigoplus_{s \geq 0} \bar{V}(\mu, 2\kappa, -s) \quad (41)
\]

where the sum on \( \mu \) is finite and

\[
\bar{V}(\mu, 2\kappa, -s) = V(\mu, 2\kappa, -s) \bigoplus \cdots \bigoplus V(\mu, 2\kappa, -s) \quad (m_{\mu,s} \text{ terms})
\]

(42)

with \( m_{\mu,s} \) being the multiplicity of the module \( V(\mu, 2\kappa, -s) \) in the tensor product decomposition (41). Let \( P[\mu, s] \) be the central projections:

\[
P[\mu, s](V(\lambda, \kappa, 0) \otimes V(\lambda, \kappa, 0)) = \bar{V}(\mu, 2\kappa, -s)
\]

(43)

We then deduce the following decompositions:

\[
W_\pm = \bigoplus_{(\mu, 2\kappa, -s) \in D^+} \bigoplus_{s \geq 0} \bar{V}_\pm(\mu, 2\kappa, -s)
\]

(44)

where

\[
\bar{V}_\pm(\mu, 2\kappa, -s) = P[\pm]\bar{V}(\mu, 2\kappa, -s)
\]

(45)

Let

\[
P[\mu, s; \pm] = P[\mu, s]P[\pm] = P[\pm]P[\mu, s]
\]

(46)

then

\[
\bar{V}_\pm(\mu, 2\kappa, -s) = P[\mu, s; \pm](V(\lambda, \kappa, 0) \otimes V(\lambda, \kappa, 0)) = P[\mu, s]W_\pm
\]

(47)
We now observe
\[ \sigma^2 = PRP \cdot R = R^T R = (v \otimes v) \Delta(v^{-1}) \] (48)
where we have used (33) in the last step. It then follows from (34) that on the submodule
\[ \bar{V}_\mu(\mu, 2\kappa, -s) \] the \( \sigma^2 \) has the eigenvalue
\[ \chi(\mu, 2\kappa, -s)(\sigma^2) = q^{(\mu, \mu + 2\rho_0) - 2(2\kappa + g)} \] (49)
Since \( \sigma \) is self-adjoint the above equation implies
\[ \chi(\mu, 2\kappa, -s)(\sigma) = \pm q^{(\mu, \mu + 2\rho_0)/2 - (2\kappa + g)} \] (50)
on \( \bar{V}_\mu(\mu, 2\kappa, -s) \), respectively. We thus arrive at the following spectral decomposition formula for \( \sigma \) and its powers:
\[ \sigma_l = q^{-l(\lambda, \mu + 2\rho_0)} \sum_{(\mu, 2\kappa, -s) \in D^+} \sum_{s=0}^{\infty} q^{l(\mu, \mu + 2\rho_0)/2 - sl(2\kappa + g)} \cdot \left( P[+] + (-1)^l P[-] \right) P[\mu, s], \quad l \in \mathbb{Z} \] (51)
This representation of the braid generators are the generalization of the ones in [1] to the case at hand.

It should be pointed that that although \( \sigma \) and its eigenvalue are bounded above for \(|q| > 1\), \( \sigma^{-1} \) and its eigenvalue are not.

5 Casimir Operators

It follows from (33) and proposition 1 that operators
\[ C^{(l)} = (I \otimes \text{tr})\{[I \otimes \pi_A(q^{2h_v})]\sigma^l\}, \quad l \in \mathbb{Z}^+ \] (52)
are Casimir invariants acting on \( V(\lambda) \). Their eigenvalues, denoted \( c^{(l)}_A \), are given by

**Proposition 5:**
\[ c^{(l)}_A = q^{-l(\lambda, \mu + 2\rho_0)} \sum_{(\mu, 2\kappa, -s) \in D^+} \sum_{s=0}^{\infty} (m_{\mu, s}^+ + (-1)^l m_{\mu, s}^-) q^{l(\mu, \mu + 2\rho_0)/2 - sl(2\kappa + g)} \frac{D_q[\mu, 2\kappa, -s]}{D_q[\lambda, \kappa, 0]}, \quad l \in \mathbb{Z}^+ \] (53)
where \( m_{\mu, s}^\pm \in \mathbb{Z}^+ \) are the mutliplicity of \( V(\mu, 2\kappa, -s) \) in \( W_\pm \), respectively, so that \( m_{\mu, s} = m_{\mu, s}^+ + m_{\mu, s}^- \). The r.h.s. of (53) is absolutely covergent for \(|q| > 1\).

**Proof:** This can be proven as in [2] with the help of the decomposition (51). \( \square \)

On the other hand, for \( l = 1 \) we can prove by direct computation
\[ C^{(1)} = (I \otimes \text{tr})\{[I \otimes \pi_A(q^{2h_v})]\sigma\} = c^{(1)}_A I \] (54)
with
\[ c^{(1)}_\Lambda = q^{\Lambda, \Lambda+2 \rho} \] (55)

This is done as follows. Let \(|\Lambda|>\) denote the highest weight vector of \(V(\Lambda)\) with the highest weight \(\Lambda\). Consider
\[ < \Lambda | q^{2h_\rho} C^{(1)} | \Lambda >= q^{2(\Lambda, \rho)} c^{(1)}_\Lambda \] (56)

then by (55) and (54)
the l.h.s. = \[ \sum_\nu < \Lambda \otimes \nu | \Delta(q^{2h_\rho}) P_\Sigma l a_t \otimes b_t | \Lambda \otimes \nu >= \sum_\nu \otimes \lambda |q^{2h_\rho} a_t \otimes q^{2h_\rho} b_t | \Lambda \otimes \nu > \]
= \[ \sum_\Lambda < \lambda | q^{4h_\rho} q^{-2h_\rho} b_t q^{2h_\rho} a_t | \Lambda > \]
= \[ q^{4(\Lambda, \rho)} < \Lambda | S^{-2} (b_t) a_t | \Lambda >= q^{4(\Lambda, \rho)} < \Lambda | u^{-1} | \Lambda >= q^{2(\Lambda, \rho)} < \Lambda | u^{-2} q^{-2h_\rho} | \Lambda > \]
= \[ q^{2(\Lambda, \rho)} < \Lambda | v^{-1} | \Lambda >= q^{2(\Lambda, \rho)} + (\Lambda, \Lambda+2 \rho) \] (57)

where use has been made of (34) and
\[ q^{-2h_\rho} b_t q^{2h_\rho} = S^{-2}(b_t) \] (58)

By comparing (54) with (53), we obtain the interesting identity
\[ q^{2(\lambda, \lambda+2 \rho)} = \sum_{(\mu, 2\kappa, -s) \in D^+} \sum_{s=0}^\infty (m^+_{\mu, s} - m^-_{\mu, s}) q^{(\mu, \mu+2 \rho)} / 2^{-s(2\kappa+g)} D_q[(\mu, 2\kappa, -s)] D_q[(\lambda, \kappa, 0)] \] (59)

Note that the r.h.s of the above relation is absolutely convergent for \(|q| > 1\). Eq.(59) bears similarity to the power series expansion of a function in its absolutely convergent region.

### 6 Remarks

In ref.11 the spectral decomposition formulae similar to (51) are applied to define link polynomials explicitly for a general irreducible representation of a quantum (super)group. However in the case at hand care must be taken in order to avoid divergence issues since the inverse of the braid generator is unbounded above. It would be of interest to apply the results above to determine generalized link polynomials associated with the integrable IHW representations of the present affine case. Work towards this goal is under investigation.

**Acknowledgements:**

The financial support from the Australian Research Council is gratefully acknowledged.
References

[1] N. Reshetikhin, *Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links: I, II*, preprints LOMI E-4-87, E-17-87;

[2] M. D. Gould, *Lett. Math. Phys.* 24 (1992) 183; J. R. Links, M. D. Gould and R. B. Zhang, *Quantum supergroups, link polynomials and representation of the braid generator*, The University of Queensland preprint, 1992

[3] V. G. Kac, *Infinite dimensional Lie algebras*, Prog. Math. 44, Birkhäuser, Boston/Basel/Stuttgart, 1983

[4] V. G. Drinfeld, *Proc. ICM, Berkeley* 1 (1986) 798

[5] M. Jimbo, *Lett. Math. Phys.* 10 (1985) 63, and *ibid* 11 (1986) 247; *Topics from representations of $U_q(\mathfrak{g})$ – a introductory guide for physicists*, Nankai Lectures, 1991, in: Quantum Groups and Quantum Integrable Systems, eds. M.-L. Ge, (World Scientific, 1992)

[6] M. Rosso, *Commun. Math. Phys.* 117 (1989) 581; G. Lusztig, *Adv. Math.* 70 (1988) 237

[7] Y.-Z. Zhang and M. D. Gould, *Unitarity and complete reducibility of certain modules over quantized affine Lie algebras and On universal R-matrix for quantized nontwisted rank 3 affine Lie algebras*, The University of Queensland preprints, UQMATH-93-02, hepth/9303096 and UQMATH-93-01, hepth/9303095

[8] P. Goddard and D. Olive, *Int. J. Mod. Phys.* A1 (1986) 303

[9] R. B. Zhang, M. D. Gould and A. J. Bracken, *Commun. Math. Phys.* 137 (1991) 13; M. D. Gould, R. B. Zhang and A. J. Bracken, *J. Math. Phys.* 32 (1991) 2298

[10] I. B. Frenkel and N. Reshetikhin, *Commun. Math. Phys.* 146 (1992) 1

[11] S. M. Khoroshkin and U. N. Tolstoy, *The universal R-matrix for quantized nontwisted affine Lie algebras*, in: *Proc. 4th Workshop, Obniusk*, 1990, to appear in Funkz. Analyz. i ego Pril.