On the spectrum of some Bloch-Torrey vector operators

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Abstract

We consider the Bloch-Torrey operator in $L^2(I, \mathbb{R}^3)$ where $I \subseteq \mathbb{R}$. In contrast with the $L^2(I, \mathbb{R}^2)$ (as well as the $L^2(\mathbb{R}^k, \mathbb{R}^2)$) case considered in previous works. We obtain that $\mathbb{R}_+$ is in the continuous spectrum for $I = \mathbb{R}$ as well as discrete spectrum outside the real line. For a finite interval we find the left margin of the spectrum. In addition, we prove that the Bloch-Torrey operator must have an essential spectrum for a rather general setup in $\mathbb{R}^k$, and find an effective description for its domain.

1 Introduction

1.1 The Bloch-Torrey operator

We consider a simplified version of the Bloch-Torrey equation [31, Eq. (4)], that is commonly used to model Diffusion-Weighted Magnetic Resonance Imaging (DW-MRI). For infinite relaxation times and constant diffusivity it assumes the form

$$\partial_t \mathbf{m} = -\gamma \mathbf{b} \times \mathbf{m} + D \Delta \mathbf{m}. \quad (1.1)$$

This time-dependent equation describes the evolution in time of a vector field $\mathbf{m}$ on $\mathbb{R}^3$, representing the magnetization vector under the action of an external magnetic field $\mathbf{b}$.

To obtain any information on the semigroup associated with (1.1), we need to analyze the resolvent of a suitable realization of the differential operator $-D \Delta + \gamma \mathbf{b} \times$. After dilation and a change of notation we write

$$B_\epsilon(x, d_x) := -\epsilon^2 \Delta + \mathbf{b} \times. \quad (1.2)$$
In the sequel we denote the magnetization \( m \) by \( u \). We begin by considering the general problem in \( \mathbb{R}^3 \) providing first a precise definition of this spectral problem.

**Proposition 1.1.** Let \( b \in C^\infty(\mathbb{R}^3; \mathbb{R}^3) \). Then, the closure \( B_\epsilon \) of the operator \( B_\epsilon(x, dx) \) which is priori defined on \( C^\infty_0(\mathbb{R}^3, \mathbb{R}^3) \) is maximally accretive as an unbounded operator in \( L^2(\mathbb{R}^3, \mathbb{R}^3) \).

The proof is given in Section 2.

**Remark 1.2.** Note that \( B_\epsilon \) can be extended as an unbounded operator in \( L^2(\mathbb{R}^3, \mathbb{C}^3) \) which commutes with the complex conjugation. Hence its spectrum is invariant to complex conjugation.

By the Hille-Yosida theorem (see [10, Theorem 8.3.2]) there exists a continuous semi-group associated with \( B_\epsilon \) and it is therefore natural to attempt to obtain some of its properties in the sequel.

In Section 3 we focus on the case when \( b(x) \) depends only on one variable (say \( x_1 \)).

In this case, we apply a partial Fourier transform in the \( x_2 \) and \( x_3 \) direction which leads to the following family of (1D) operators depending on \( ((\xi_2, \xi_3) \in \mathbb{R}^2) \)

\[
B_\epsilon(x_1, \frac{d}{dx_1}, \xi_2, \xi_3) := -\epsilon^2 \frac{d^2}{dx_1^2} \otimes I + \begin{pmatrix}
\epsilon^2 (\xi_2^2 + \xi_3^2) & -b_3 & b_2 \\
b_3 & \epsilon^2 (\xi_2^2 + \xi_3^2) & -b_1 \\
-b_2 & b_1 & \epsilon^2 (\xi_2^2 + \xi_3^2)
\end{pmatrix}.
\]

(1.3)

The above operator (after reduction to the case \( \xi_2 = \xi_3 = 0 \)) is considered, assuming linearity of \( b \), i.e.,

\[
b(x_1) = b_0 + x_1 b_1.
\]

where \( b_0 \in \mathbb{R}^3 \) and \( b_1 \in \mathbb{R}^3 \setminus \{0\} \).

Using translation we can obtain \( b_1 \perp b_0 \). Then, after rotation and renormalization, denoting the canonical basis in \( \mathbb{R}^3 \) or \( \mathbb{C}^3 \) by \((\hat{i}_1, \hat{i}_2, \hat{i}_3)\), it is sufficient to consider the case when

\[
b_0 = b_0 \hat{i}_1 \text{ and } b_1 = \hat{i}_3.
\]

Note that \( b \) is divergence free, as is required from magnetic fields by Maxwell equations. The operator \( B_\epsilon \) becomes

\[
B_\epsilon(x, \frac{d}{dx}) := -\epsilon^2 \frac{d^2}{dx^2} - [xb_1 + b_0] \times ,
\]

(1.4)

Note that the case \( b_0 = 0 \) reduces to the two-dimensional case. More generally, if we suppose that \( b = b(x_1, x_2, x_3) \hat{i}_3 \) (though in the case of a divergence free field we get \( b = b(x_1, x_2) \hat{i}_3 \)) then the skew-symmetric matrix associated with \( b \times \) is given by

\[
\mathcal{M} = \begin{pmatrix}
0 & -b & 0 \\
b & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

(1.5)
The eigenvectors associated with $M$ are $\hat{i}_3$, $\nu$, and $\bar{\nu}$, where,
\[ \nu = \frac{1}{\sqrt{2}}(-i\hat{i}_1 + \hat{i}_2). \] (1.6)

Since $\{\nu, \bar{\nu}, \hat{i}_3\}$ form an orthonormal basis for $\mathbb{C}^3$ we may apply rotation to $-\epsilon^2\Delta + M$ to obtain in this new basis the operator
\[
\bar{B}_\epsilon := \begin{pmatrix}
-\epsilon^2\Delta + ib & 0 & 0 \\
0 & -\epsilon^2\Delta - ib & 0 \\
0 & 0 & -\epsilon^2\Delta
\end{pmatrix}.
\] (1.7)

Obviously, in this basis $-\epsilon^2\Delta + M$ can be considered as three separate scalar operators. The spectral properties of $-\epsilon^2\Delta + ib$ have been considered in [2, 4, 23]. Note that if we define $-\epsilon^2\Delta + M$ on $L^2(\mathbb{R}^3, \mathbb{R}^3)$ for $b = x_1$ we obtain that the spectrum is $\mathbb{R}_+$ (which is precisely $\sigma(-\Delta)$ on $L^2(\mathbb{R}^3)$) given that $\sigma(-\epsilon^2\Delta + ix) = \emptyset$ on $L^2(\mathbb{R}^3)$.

### 1.2 Main statements

We now present the main results of this work. In the case where $B_\epsilon$ is defined in $\mathbb{R}$ we obtain

**Theorem 1.3.** Let $B_\epsilon$ be defined by (1.4), on the domain
\[
D(B_\epsilon) = \{ u \in H^2(\mathbb{R}, \mathbb{C}^3) \mid b \times u \in L^2(\mathbb{R}, \mathbb{C}^3) \}.
\]

Then we have:

- $\Lambda \in \sigma(B_\epsilon) \iff \bar{\Lambda} \in \sigma(B_\epsilon)$.
- $\mathbb{R}_+ \subset \sigma(B_\epsilon)$.
- Let for $n \in \mathbb{N}^*$ and $\epsilon > 0$, $\kappa_n^0(\epsilon) := i + \frac{2n-1}{2}(1+i)\epsilon$. Then for any $N \in \mathbb{N}^*$ there exist positive $\epsilon_0$ and $\hat{C}$, such that for all $0 < \epsilon \leq \epsilon_0$ a sequence $\{\kappa_n(\epsilon)\}_{n=1}^N \subset \sigma(B_\epsilon)$ can be found, satisfying
  \[
  \left| \kappa_n(\epsilon) - \kappa_n^0(\epsilon) \right| \leq \hat{C}\epsilon^2, \quad \text{for } n = 1 \ldots , N. \] (1.8)
- Let $\varrho > 0$ and $\hat{R} > 0$. Let further
  \[ N_\varrho = \left[ \frac{2\varrho + 1}{2} \right], \] (1.9)
  where $[\cdot]$ denotes the integer part. Set now
  \[
  \mathcal{D}(\hat{R}, \varrho, \epsilon) = \{ \Lambda \in \mathbb{C} \setminus \bigcup_{n=1}^{N_\varrho} (B(\kappa_n^0(\epsilon), \hat{R}^2) \cup B(\bar{\kappa}_n^0(\epsilon), \hat{R}^2)) \} \cap \{ \Re \Lambda \leq \varrho \epsilon \} \cap \{ \Im \Lambda \neq 0 \}. \] (1.10)

Then, there exist positive $C$ and $\hat{R}_0 > 1$ such that for all $\hat{R}_0 < \hat{R} < \sqrt{2\epsilon}^{-1}$ and $\Lambda \in \mathcal{D}(\hat{R}, \varrho, \epsilon)$ it holds that
\[
\|(B_\epsilon - \Lambda)^{-1}\| \leq C \left( 1 + \frac{\epsilon^{2/3}}{|\Im \Lambda|^2} + \frac{1}{\hat{R}^2\epsilon} \right).
\] (1.11)
Remark 1.4. We note that for $\Lambda_r < 0$, it holds, since $B_\epsilon$ is accretive, that
\[ \| (B_\epsilon - \Lambda)^{-1} \| \leq \frac{1}{|\Lambda_r|}. \]

We now state our main result for the Dirichlet realization $B_I^\epsilon$ of the operator $B_\epsilon(x, d_x)$ (see (1.14)) in $I = (a, b)$.

**Theorem 1.5.** The domain of $B_I^\epsilon$ is given by
\[ D(B_I^\epsilon) = H^2((a, b), C^3) \cap H^1_0((a, b), C^3), \]
Furthermore, let
\[ \rho_0 = \inf_{w \in H^1_0(a, b)} \frac{I(w)}{\| [x^2 + 1]^{1/2} w \|^2_2}, \]
where
\[ I(w) = \|(xw)'\|^2_2 + \|w'\|^2_2. \]
Then
\[ \lim_{\epsilon \to 0} \epsilon^{-2} \Re \sigma(B_I^\epsilon) = \rho_0. \quad (1.12) \]

The two-dimensional setting described by (1.5) has received significant attention in the literature (cf. for instance [2, 12, 13, 9]) sometimes with time-dependent magnetic field. An example of a divergence-free magnetic field whose direction varies in space has been presented in the classical work of Torrey [31], where $b = (g x, g y, -2g z)$ for some $g \in \mathbb{R}$.

In the mathematical literature (1.2) with varying field direction has been considered in the context of vector Schrödinger operators [27, 26]. Thus, from the results in [27] we can conclude that a contraction semigroup is associated with (1.2) (an immediate conclusion of Proposition 1.1). It should be noted that the maximal domain of (1.2) is not found in [27]. In contrast, in [26] the maximal domain is found under the assumption that $|\nabla b| |b|^{-\alpha}$ is bounded in $\mathbb{R}^d$ for some $0 \leq \alpha < 1/2$. We bring more general results (in our context) in Section 2.

The rest of this contribution is arranged as follows. In the next section we address the general operator (1.2), in the context of Schrödinger operators with matrix potential. In particular we give conditions for the existence of an essential spectrum and obtain the maximal domain for a rather general setting. In Section 3 we prove Theorem 1.3. Finally, in Section 4 we prove Theorem 1.5.

### 2 Properties of Schrödinger operators with matrix-valued potentials

In this section we derive some basic properties of the operator $B_\epsilon$ given by (1.2), in settings significantly more general than that of (1.4). The analysis applies in particular to the general differential operator (1.2) and hence also to the one-dimensional operator (1.4).
2.1 A more general operator

One can generalize (1.2) even further by considering the operator
\[ P(x, dx) := -\Delta \otimes I_d + M(x), \]
(2.1)
where \( I_d \) is the identity matrix acting on \( \mathbb{R}^d \) and \( M \in C^\infty(\mathbb{R}^k, M_d(\mathbb{R})) \), where \( M_d(\mathbb{R}) \) denotes the set of all \( d \times d \) matrices with real entries. We set \( \epsilon = 1 \) as the value of \( \epsilon \) does not have any effect on the properties which we consider in this section.

Set
\[ M_s = \frac{1}{2}(M + M^t), \quad M_{as} = \frac{1}{2}(M - M^t). \]

We further assume that
\[ M_s \geq 0, \]
(2.2)
which is certainly the case in (1.2), where \( M \) is skew-symmetric.

2.2 Accretivity

In this subsection, we extend maximal accretivity results that have been established for the selfadjoint operator \(-\Delta + V\) (see [11, Theorem 6.6.2]) and also for two interesting non-selfadjoint operators: the Fokker-Planck operator [18] and the complex Schrödinger operator \(-\Delta + iV\) [17].

**Proposition 2.1.** Let \( P \), given by (2.1), be defined on \( C^\infty_0(\mathbb{R}^k, \mathbb{R}^d) \) and satisfy (2.2). Then, its closure, under the graph norm, denoted by \( \overline{P} \), is maximally accretive as an unbounded operator in \( L^2(\mathbb{R}^k, \mathbb{R}^d) \). Moreover,
\[ D(\overline{P}) \subset H^1(\mathbb{R}^k, \mathbb{R}^d). \]

**Proof.** We first observe that \( P(x, dx) \) is accretive on \( C^\infty_0(\mathbb{R}^k, \mathbb{R}^d) \). To this end it is sufficient to note that
\[ \langle Pu, u \rangle_{L^2(\mathbb{R}^k, \mathbb{R}^d)} \geq 0, \]
which holds since \( P \) is the sum of the non negative operator \((-\Delta) \otimes I_d + M_s(x)\) and the antisymmetric matrix \( M_{as}(x)\). We can then follow the proof in [17, exercise 13.7] (which refers to Theorem 13.14 and the proof of Theorem 9.15).

**Remark 2.2.** Proposition 1.1 follows as a particular case of Proposition 2.1 for \( k = d = 3 \) and \( M_s = 0 \).

2.3 Essential spectrum

If \( \dim \ker M(x) > 0 \) for all \( x \in \mathbb{R}^k \) (or for a suitable sequence of balls with centers tending to \(+\infty\)) as in the case \( M_s \equiv 0 \) we may attempt to exploit the fact that locally, in any of the directions spanning \( \ker M \), \( P \) is expected to behave like \( \Delta \) to show that its resolvent is not compact and even that \( \mathbb{R}_+ \subseteq \sigma(P) \). We begin with the following proposition, establishing non-compactness of \((P - \lambda)^{-1}\).
Proposition 2.3. Let \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{R}^k \) satisfy \( |a_n| \to +\infty \) and \( |a_n - a_m| \geq 1 \) for all \( n \neq m \). Suppose that there exists \( C > 0 \) and a unit vector field \( c(x) \) such that, for all \( n \in \mathbb{N} \) and \( x \in B(a_n, \frac{1}{2}) \),
\[
M(x)c(x) = 0 ,
\]
and
\[
|d_x^\alpha c_j(x)| \leq C, \text{ for all } \alpha \text{ s.t } 1 \leq |\alpha| \leq 2, j = 1, 2, \ldots, d . \tag{2.3a}
\]
and
\[
|d_x^\alpha c_j(x)| \to 0, \forall \alpha \text{ s.t } 1 \leq |\alpha| \leq 2, j = 1, 2, \ldots, d . \tag{2.3b}
\]
Then the resolvent of \( P \) is not compact.

Proof. One looks for an infinite orthonormal family \((n \geq N)\) in the form
\[
\Phi_n(x) := 2^d c(x)\phi(2(x - a_n)),
\]
where \( \phi \in C_0^\infty(B(a,b)) \) is of unity norm (i.e., \( \|\phi\|_{L^2} = 1 \)). Note that the above construction guarantees that \( \|\Phi_n\|_2 = 1 \).

As \( M\Phi_n \equiv 0 \), it can be easily verified that \( \{P\Phi_n\}_{n=1}^{\infty} \) is uniformly bounded. It follows that the resolvent of \( P \) cannot be compact.

Remark 2.4. If we do not assume that \( c(x) \) is a unit vector. Then \( (2.3) \) can be replaced by (assuming \( c(x) \neq 0 \) for \( |x| \geq R \)) the existence of \( C > 0 \) such that:
\[
|d_x^\alpha c_j(x)| \leq C |c(x)|, \text{ for all } \alpha \text{ s.t } 1 \leq |\alpha| \leq 2, j = 1, 2, \ldots, d , \tag{2.4}
\]
which is normally easier to verify than \( (2.3b) \).

We note that for \( (1.3) \) we have \( k = 1, d = 3, M_s = 0 \) and \( c = b \). In this case we may conclude that

Corollary 2.5. Suppose that for \( |x| \geq R \) it holds that \( b \neq 0 \) and that for some \( C > 0 \)
\[
|d_x^\alpha b_j(x)| \leq C |b(x)|, \forall \alpha \text{ s.t } 1 \leq |\alpha| \leq 2, j = 1, 2, 3 . \tag{2.5}
\]
Then the resolvent of \( B_\epsilon \), given by \( (2.3) \), is not compact.

Making slightly stronger assumptions on \( c \) we now prove the existence of an essential spectrum for \( P \).

Proposition 2.6. Let \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{R}^k \) satisfy \( |a_n| \to +\infty \) and
\[
r_n = \inf_{m \neq n} |a_n - a_m| \xrightarrow{n \to \infty} \infty .
\]

Suppose that there exists a unit vector field \( c \) and \( R > 0 \) such that, for all \( n \in \mathbb{N} \) and \( x \in B(a_n, r_n) \),
\[
M(x)c(x) = 0 , \tag{2.6a}
\]
and
\[
|d_x^\alpha c_j(x)| \xrightarrow{|x| \to \infty} 0, \forall \alpha \text{ s.t } 1 \leq |\alpha| \leq 2, j = 1, 2, \ldots, d . \tag{2.6b}
\]
Then, \( \mathbb{R}_+ \subseteq \sigma(P) \), where \( \mathbb{R}_+ := [0, +\infty) \).
Proof. One looks for an infinite orthonormal family \((n \geq N)\) in the form

\[
\Phi_n(x) := r_n^{-d} c(x) \phi(2r_n^{-1}(x - a_n)),
\]

where \(\phi \in C_0^\infty(B(0, 1))\) is of unity norm (i.e., \(\|\phi\|_{L^2} = 1\)).

Let \(\lambda \in \mathbb{R}_+\), and \(\xi \in \mathbb{R}^k\) satisfy \(|\xi|^2 = \lambda\). Let further \(\Psi_n = e^{i\xi \cdot x} \Phi_n\). As \(M \Psi_n = 0\), it can be easily verified that

\[
\|(P - \lambda) \Psi_n\|_2 \to 0.
\]

It follows that \(\lambda \in \sigma(P)\) and hence \(\mathbb{R}_+ \subseteq \sigma(P)\).

Remark 2.7. If \(c(x)\) is not assumed to be a unit vector, then (2.6c) should be replaced (assuming \(c \neq 0\)) by,

\[
|d_\alpha x c_j(x)| \leq \delta(x)|c(x)|, \quad \forall \alpha \text{ s.t } 1 \leq |\alpha| \leq 2, \ j = 1, 2, \ldots, d,
\]

where \(\delta(x) \to 0\).

Assuming a linear magnetic field (as in (1.4)) we consider a field \(b\) satisfying

\[
b = A x + f,
\]

where \(A \neq 0\) is a \(d \times k\) matrix and \(f \in \mathbb{R}^d\). Let \(w\) denote an eigenvector of \(A^T A\) corresponding to a non-zero eigenvalue (which clearly exists since \(A \neq 0\)). Choosing \(a_n = s_n w\), where \(r_n = s_n - s_{n-1} \uparrow \infty\), it can be easily verified that there exists \(C > 0\) such \(|A x| \geq C|x|\) in \(B(a_n, r_n/2)\). Consequently, \(b\) satisfies (2.7), and hence \(\mathbb{R}_+ \subseteq \sigma(B_c)\) whenever \(b\) satisfies (2.8). Note that if \(A = 0\), then, by (1.7),

\[
\sigma(P) = \mathbb{R}_+ \cup \{\mathbb{R}_+ + i|f|\} \cup \{\mathbb{R}_+ - i|f|\}.
\]

Of particular interest is the case (1.4). Here \(A = \hat{i}_3\) and hence \(\mathbb{R}_+\) is in the essential spectrum of \(B_c\).

2.4 Maximal estimates

A natural question is the effective description of \(D(P)\), which is currently defined as the closure of \(C_0^\infty(\mathbb{R}^k, \mathbb{R}^d)\) under the graph norm. While far from having an optimal result we can still determine the domain in two different cases: The first of them concerns matrices with positive symmetric part and coefficients of bounded derivatives. In the second one we assume a more general class of skew-symmetric matrices.

Proposition 2.8. Suppose that \(M\) satisfies (2.2) and suppose that there exist \(C > 0\) and \(R > 0\) such that, for \(|x| \geq R\) it holds that

\[
|d_\alpha M_{i,j}(x)| \leq C, \quad \forall i, j = 1, 2, \ldots, d,
\]

Then,

\[
D(P) = \{u \in H^2(\mathbb{R}^k, \mathbb{R}^d), \ Mu \in L^2(\mathbb{R}^k, \mathbb{R}^d)\}.
\]
Proof. Let $u \in C^\infty_0(\mathbb{R}^k, \mathbb{R}^d)$. Clearly,

$$\| \nabla u \|_2^2 + \langle u, M\alpha u \rangle = \langle u, \mathcal{P}u \rangle \leq \frac{1}{2} (\| \mathcal{P}u \|_2^2 + \| u \|_2^2) .$$

By (2.13) we can conclude that

$$\| \nabla u \|_2^2 = \langle u, \mathcal{P}u \rangle \leq \frac{1}{2} (\| \mathcal{P}u \|_2^2 + \| u \|_2^2) . \tag{2.11}$$

Next, we write

$$\| M\alpha u \|_2^2 = \langle M\alpha u, \mathcal{P}u \rangle + \langle M\alpha u, \Delta u \rangle \leq \| \mathcal{P}u \|_2 \| M\alpha u \|_2 - \langle M, \nabla u, \nabla u \rangle - \langle (\nabla M)u, \nabla u \rangle \tag{2.12}$$

$$\leq \| \mathcal{P}u \|_2 \| M\alpha u \|_2 - \langle (\nabla M)u, \nabla u \rangle .$$

We can then conclude from (2.9), (2.11) and (2.12) the existence of $C$ such that

$$\| M\alpha u \|_2 \leq C(\| \mathcal{P}u \|_2^2 + \| u \|_2^2), \forall u \in C^\infty_0(\mathbb{R}^k, \mathbb{R}^d). \tag{2.13}$$

Let $(u, \mathcal{P}u) \in [L^2(\mathbb{R}^k, \mathbb{R}^d)]^2$ and $\{u_n\}_n \subset C^\infty_0(\mathbb{R}^k, \mathbb{R}^d)$ satisfy $u_n \to u$ in the graph norm. Then, by (2.13), $\{M\alpha u_n\}_n$ is a Cauchy sequence in $L^2(\mathbb{R}^k, \mathbb{R}^d)$. Since $M\alpha u_n \to M\alpha u$ in $L^2(\Omega, \mathbb{R}^d)$ for any $\Omega \subset \mathbb{R}^k$ we may conclude that the limit of $\{M\alpha u_n\}_n$ in $L^2(\mathbb{R}^k, \mathbb{R}^d)$ is $M\alpha u \in L^2(\mathbb{R}^k, \mathbb{R}^d)$. By subtraction from $\mathcal{P}u$, we obtain $\Delta u \in L^2(\mathbb{R}^k, \mathbb{R}^d)$, thus $u \in H^2(\mathbb{R}^k, \mathbb{R}^d)$ and hence $u \in D(\mathcal{P})$. 

We note that the above result is a particular case of [26, Theorem 3.2].

We now obtain a stronger result for the case where $M$ is skew-symmetric.

**Proposition 2.9.** Suppose that $M = M_{as} \in C^2(\mathbb{R}^k, M_d(\mathbb{R}))$ is a skew symmetric matrix. Let

$$S(x) = \inf_{\lambda_j \in \sigma(M(x)) \backslash \{0\}} |\lambda_j| ,$$

and suppose that there exist $C > 0$ and $R > 0$ such that, for $|x| \geq R$ it holds that

$$|d^\alpha_{\mathbb{R}^d}M_{i,j}(x)| \leq C S(x), \forall i, j = 1, 2, \ldots, d, 1 \leq |\alpha| \leq 2 . \tag{2.14}$$

and that $\dim \ker M(x)$ is constant for $|x| \geq R$. Then,

$$D(\mathcal{P}) = \{ u \in H^2(\mathbb{R}^k, \mathbb{R}^d), M\alpha u \in L^2(\mathbb{R}^k, \mathbb{R}^d) \} . \tag{2.15}$$

**Proof.** We note that under our assumptions, all eigenvalues in $\sigma(M(x))$ are purely imaginary, $\{0\} \in \sigma(M(x))$ when $d$ is odd, and that $S(x)$ is continuous. Note further that by the condition on $\dim \ker M(x)$ we have either $M(x) = 0$ or $S(x) > 0$ for all $|x| > R$. The treatment of the first case being evident, we treat the second case. We introduce for any $x$, $\Pi_0(x)$ the projector on the kernel of $M(x)$ and $\Pi(x) := (I - \Pi_0(x))$. When $|x| \geq R$, we note that $\Pi_0(x)$ (hence $\Pi(x)$) depends smoothly on $x$ and note that we have

$$\Pi_0(x) := \frac{1}{2\pi i} \int_{\gamma} (z - M(x))^{-1} dz ,$$

where $\gamma$ is any simple closed contour in the upper half plane, and $\gamma$ misses the point $z = \lambda_j$, $\lambda_j \in \sigma(M(x)) \backslash \{0\}$. Note further that $\Pi_0(x)$ and $\Pi(x)$ are skew-symmetric and $\Pi(x)$ is bounded.

We now introduce the projector $\Pi(x)$ on the range of $M(x)$ as

$$\Pi(x) := \frac{1}{2\pi i} \int_{\gamma} (z - M(x))^{-1} dz . \tag{2.16}$$

We now note that $\Pi(x)$ is skew-symmetric and $\Pi(x)$ is bounded.
for any positively oriented circle $\gamma$ of radius strictly smaller than $S(x)$.

We now estimate $\partial_x \Pi_0(x)$ for $|x| > R$. Let $x_0 \in \mathbb{R}^d$. For any $\gamma$ of radius smaller than $S(x_0)$, we have

$$
(\partial_x \Pi_0)(x_0) = \frac{1}{2\pi i} \int_\gamma (z - M(x_0))^{-1}(\partial_x M)(x_0)(z - M(x_0))^{-1}dz
$$

Since

$$
\|(z - M(x))^{-1}\| \leq \frac{1}{\min(|z|, S(x) - |z|)},
$$

we obtain, by choosing $\gamma$ to be of radius $\frac{1}{2}S(x_0)$, with the aid of (2.14) and the fact that the length of $\gamma$ is $\pi S(x_0)$, the existence of $C > 0$ such that, for all $|x| \geq R$,

$$
\| (\partial_x \Pi_0)(x) \|_{M_d(\mathbb{R})} \leq C, \forall j = 1, \ldots, k. \tag{2.16}
$$

Similarly, for any $\gamma$ of radius $< S(x_0)$, it holds that

$$
(\partial_x, \partial_x \Pi_0)(x_0)
= \frac{1}{2\pi i} \int_\gamma (z - M(x_0))^{-1}(\partial_x, \partial_x M)(x_0)(z - M(x_0))^{-1}dz
+ \frac{1}{2\pi i} \int_\gamma (z - M(x_0))^{-1}(\partial_x M)(x_0)(z - M(x_0))^{-1}(\partial_x M)(x_0)(z - M(x_0))^{-1}dz.
$$

Using (2.16) we establish the existence of $C > 0$ such that, for $|x| \geq R$,

$$
\| (\partial_x \Pi_0)(x) \|_{M_d(\mathbb{R})} \leq C, \forall j, \ell = 1, \ldots, k. \tag{2.17}
$$

We now introduce $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi = 1$ on $B(0, R)$ and let $\tilde{\chi} = 1 - \chi$. Next, we write for any $u \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$

$$
\|Mu\|^2 = \langle Mu, Pu \rangle + \langle \chi Mu, \Delta u \rangle + \langle \tilde{\chi} Mu, \Delta u \rangle
= \langle Mu, Pu \rangle - \langle (\nabla \chi) Mu, \nabla u \rangle - \langle \chi \nabla Mu, \nabla u \rangle + \langle \tilde{\chi} Mu, \Delta u \rangle \tag{2.18}
$$

Since $M \in C^2(\mathbb{R}^d, M_d(\mathbb{R}))$ we can conclude from (2.11) that

$$
- \langle (\nabla \chi) Mu, \nabla u \rangle - \langle \chi \nabla Mu, \nabla u \rangle \leq C\|u\|^2_2 \|\nabla u\|^2_2 \leq C\|\Pi u\|^2_2 (\|Pu\|^2_2 + \|u\|^2_2) \tag{2.19}
$$

To bound the last term on the right-hand-side of (2.18) we first observe that

$$
M = M\Pi = \Pi M = \Pi M\Pi.
$$

Hence,

$$
\langle \tilde{\chi} Mu, \Delta u \rangle = \langle \tilde{\chi} M\Pi u, \Delta u \rangle = \langle \tilde{\chi} Mu, [\Pi, \Delta] u \rangle + \langle \tilde{\chi} M\Pi u, \Delta \Pi u \rangle
$$

By (2.16), (2.17), and (2.11) we have that

$$
\| [\Pi, \Delta] u \|^2 \leq C(\|u\|^2_2 + \|\nabla u\|^2_2) \leq C(\|Pu\|^2_2 + \|u\|^2_2),
$$

and hence

$$
\| \langle \tilde{\chi} Mu, [\Pi, \Delta] u \rangle \| \leq C \| Mu \|^2_2 (\| Pu \|^2_2 + \| u \|^2_2). \tag{2.20}
$$
Finally, we write
\[ \langle \tilde{\chi} M \Pi u, \Delta \Pi u \rangle = -\langle [\nabla, \tilde{\chi} M] \Pi u, \nabla \Pi u \rangle \\
= -\langle (\nabla \tilde{\chi}) M u, \nabla \Pi u \rangle - \langle \tilde{\chi}(\nabla M) u, \nabla \Pi u \rangle \]

By (2.14) and (2.16) we obtain, for any \( \eta \in (0,1) \)
\[ |\langle \tilde{\chi} M \Pi u, \Delta \Pi u \rangle| \leq \eta \|Mu\|_2^2 + C_\eta (\|\nabla u\|_2^2 + \|u\|_2^2) \]

Substituting the above (with sufficiently small \( \eta \)), together with (2.20) and (2.19) into (2.18) yields
\[ \|M u\|_2 \leq C (\|P u\|_2 + \|u\|_2), \forall u \in C_0^\infty (\mathbb{R}^k, \mathbb{R}^d). \quad (2.21) \]

We complete the proof in the same manner as in the proof of the previous proposition.

As mentioned before, in [26] the authors consider the operator
\[ P = -\text{div} Q \nabla + M, \]
for the case where there exists \( \beta \in \mathbb{R} \) such that
\[ \xi \cdot M(x)\xi \geq \beta |\xi|^2, \quad (2.22) \]
for all \( \xi \in \mathbb{R}^d \) and \( x \in \mathbb{R}^k \). In the case \( Q = I \), it is shown in [26] that when \( \nabla M \circ M^{-\gamma} \) is bounded in \( L^\infty (\mathbb{R}^d) \) for some \( 0 \leq \gamma < 1/2 \), then
\[ D(P) = \{ u \in H^2(\mathbb{R}^d) | Mu \in L^2(\mathbb{R}^d) \}. \]

(It should be mentioned that the results in [26] are stated in \( L^p \) for any \( p \in (1, \infty) \) whereas here we consider only the case \( p = 2 \).) We note that while (2.22) clearly holds in the case where \( M \) is skew-symmetric, (2.14) applies to cases where \( \nabla M \circ M^{-\gamma} \) is unbounded in \( L^\infty (\mathbb{R}^d) \) for all \( 0 \leq \gamma < 1/2 \). Thus, for instance, in the case \( d = 2 \) we may consider (see [26, Example 2.4])
\[ M = \begin{bmatrix} 0 & 1 + |x|^r \\ -(1 + |x|^r) & 0 \end{bmatrix} \]

Since for \( \nabla M \circ M^{-\gamma} \) to be bounded we must have \( \gamma \geq 1 - r^{-1} \), one can apply the results in [26] for \( r < 2 \) only, whereas (2.14) holds for \( r \geq 2 \) as well.

**Corollary 2.10.** Let \( d = k = 3 \) and \( Mu = b \times u \). Then, if there exist \( C > 0 \) and \( R > 0 \) such that for all \( |x| \geq R \) it holds that \( b(x) \neq 0 \) and
\[ |d_\alpha^r b_j(x)| \leq C |b(x)|, \forall \alpha \text{ s.t. } 1 \leq |\alpha| \leq 2, j = 1, 2, 3, \quad (2.23) \]

then
\[ D(P) = \{ u \in H^2(\mathbb{R}^3, \mathbb{R}^3), b \times u \in L^2(\mathbb{R}^3, \mathbb{R}^3) \}. \]

**Proof.** Since \( S(x) = |b(x)| \) we can easily conclude (2.9) from (2.23).
2.5 Bounded components

In this section we consider the case $d = 3$, where $M$ is the matrix associated with a vector product with $b$. Assuming that two components of $b$ are bounded, we may obtain $D(\mathcal{P})$ even in cases where the third component does not satisfy (2.23).

2.5.1 Characterization of the domain

To this end we use the results in [19], obtained for the scalar operator $-\Delta + iV(x)$.

**Proposition 2.11.** Let $\mathcal{B}_1$ denote the closure under the graph norm of (1.2) with $\epsilon = 1$, where $b = (b_1, b_2, b_3)$. Suppose that $b_1$ and $b_2$ belong to $L^\infty(\mathbb{R}^k)$. Suppose further for some $r \in \mathbb{Z}_+$ that $b_3 \in C^{r+1}(\mathbb{R}^k)$ satisfies

$$\max_{|\beta|=r+1} |D_x^\beta b_3(x)| \leq C_0 \sqrt{\sum_{|\alpha| \leq r} |D_x^\alpha b_3(x)|^2 + 1}. \quad (2.24)$$

Then

$$D(\mathcal{B}_1) = \{ u \in H^2(\mathbb{R}^k, \mathbb{R}^3), \ b \times u \in L^2(\mathbb{R}^k, \mathbb{R}^3) \}. \quad (2.25)$$

**Proof.** Let $v$ be defined by (1.3). In the basis $\{v, \bar{v}, \hat{i}_3\}$, the skew-symmetric matrix $M$ assumes the form

$$\tilde{M} := \begin{pmatrix} -ib_3 & 0 & -(b_1 - ib_2)/\sqrt{2} \\ 0 & ib_3 & -(b_1 + ib_2)/\sqrt{2} \\ (b_1 + ib_2)/\sqrt{2} & (b_1 - ib_2)/\sqrt{2} & 0 \end{pmatrix}.$$ 

and

$$\tilde{\mathcal{B}}_1 := -\Delta \otimes I_3 + \tilde{M}.$$ 

Consider $\tilde{u} \in L^2(\mathbb{R}^k, \mathbb{R}^d)$ satisfying $\tilde{\mathcal{B}}_1 \tilde{u} = \tilde{f} \in L^2(\mathbb{R}^k, \mathbb{R}^3)$. Then, it holds that

$$-\Delta \tilde{u}_1 - ib_3 \tilde{u}_1 = \tilde{g}_1,$$

$$-\Delta \tilde{u}_2 + ib_3 \tilde{u}_2 = \tilde{g}_2,$$

$$-\Delta \tilde{u}_3 = \tilde{g}_3,$$ \quad (2.26)

where

$$\tilde{g}_1 = \tilde{f}_1 + \frac{b_1 - ib_2}{\sqrt{2}} \bar{u}_3,$$

$$\tilde{g}_2 = \tilde{f}_2 + \frac{b_1 + ib_2}{\sqrt{2}} \bar{u}_3,$$

$$\tilde{g}_3 = \tilde{f}_3 - \frac{b_1 + ib_2}{\sqrt{2}} \bar{u}_1 - \frac{b_1 - ib_2}{\sqrt{2}} \bar{u}_2. \quad (2.27)$$

Clearly, $\tilde{g}_i \in L^2(\mathbb{R}^k)$ for all $i \in \{1, 2, 3\}$. By standard elliptic estimates we then conclude that $\tilde{u}_3 \in H^2(\mathbb{R}^k)$. We can then apply on the two first lines [19, Theorem 5] to conclude that

$$\|b_3 \tilde{u}_i\|_2 \leq C(\|	ilde{g}_i\|_2 + \|ar{u}_i\|_2), \quad i = 1, 2.$$

Hence $\Delta \tilde{u}_i \in L^2(\mathbb{R}^k)$ for $i = 1, 2$, and by standard elliptic estimates $\tilde{u}_i \in H^2(\mathbb{R}^k)$ for $i = 1, 2$. We can thus conclude that

$$D(\tilde{\mathcal{B}}_1) = \{ \tilde{u} \in H^2(\mathbb{R}^k, \mathbb{R}^3), \ b_3 \tilde{u}_1 \in L^2(\mathbb{R}^k), \ b_3 \tilde{u}_2 \in L^2(\mathbb{R}^k) \}. \quad (2.25)$$

An inverse transformation to the original basis establishes (2.25).
2.5.2 Resolvent estimates

We now obtain estimates for the resolvent of $\mathcal{B}_1$, using well known resolvent estimates obtained for $-\Delta + ib_3$. Better estimates are obtained in the next sections for the particular case where $b$ is given by (1.4).

The resolvent equation $(\tilde{\mathcal{B}}_1 - \lambda)\tilde{u} = \tilde{f}$, takes the form (dropping the accent in the sequel):

\[ f_1 = (-\Delta - ib_3 - \lambda)u_1 - \frac{b_1 - ib_2}{\sqrt{2}} u_3 \] (2.28a)
\[ f_2 = (-\Delta + ib_3 - \lambda)u_2 - \frac{b_1 + ib_2}{\sqrt{2}} u_3 \] (2.28b)
\[ f_3 = (-\Delta - \lambda)u_3 + \frac{b_1 + ib_2}{\sqrt{2}} u_1 + \frac{b_1 - ib_2}{\sqrt{2}} u_2. \] (2.28c)

Assuming that $\lambda \notin \sigma(-\Delta \pm ib_3)$ we write

\[ R_{\pm}(\lambda) = (-\Delta \mp ib_3 - \lambda)^{-1} \]

to obtain the following equation for $u_3$:

\[ ((-\Delta - \lambda) - cR_-(\lambda)\bar{c} - \bar{c}R_+ (\lambda)c) u_3 = f_3 - cR_+ (\lambda)f_1 - \bar{c}R_- (\lambda)f_2, \] (2.29)

where

\[ c := (b_1 + ib_2)/\sqrt{2}. \]

Assuming further $\lambda \notin \mathbb{R}^+$ (note that $b$ does not necessarily meet the condition set in Remark 2.11) we can attempt to estimate the norm of the well-defined bounded operator

\[ K_{\lambda} := \frac{1}{2} (-\Delta - \lambda)^{-1} (cR_-(\lambda)\bar{c} + \bar{c}R_+ (\lambda)c). \] (2.30)

To proceed further we need to introduce the following assumption on the resolvent of $-\Delta \pm ib_3$

**Assumption 2.12.** For a given interval $I$, there exist $s < 1$, $D_1 > 0$ and $D_2 > 0$ such that, if $\Re \lambda \in I$ and $|\Im \lambda| \geq D_1$, then

\[ \|R_{\pm}(\lambda)\| \leq D_2 |\Im \lambda|^s. \] (2.31)

**Remark 2.13.** The above bound applies for $I = (-\infty, \tau)$ for every $\tau \in \mathbb{R}$, and $b_3(x) = x_1$ (in which case $s = 0$ due to translation invariance, see [17, Proposition 14.11]) or $b_3(x) = x_1^2$ (where $s = -\frac{1}{2}$, see [17, Proposition 14.13]).

Given the above assumption we can obtain the following resolvent estimate

**Proposition 2.14.** Let $b_3 \in C^r(\mathbb{R}^k)$ satisfy assumption 2.12 for some given interval $I$. Then, there exist $C_1 > 0$ and $C_2 > 0$ such that, for all $\lambda \in \mathbb{C}$ satisfying $\Re \lambda \in I$ and $|\Im \lambda| \geq C_1$, it holds that $\lambda \notin \sigma(\mathcal{B}_1)$ and

\[ \|(\mathcal{B}_1 - \lambda)^{-1}\| \leq C_2 |\Im \lambda|^s. \] (2.31)
Proof. Given the fact that $$\|(-\Delta - \lambda)^{-1}\| \leq |\Re \lambda|^{-1}$$ we obtain from (2.30), Assumption 2.12 and the boundedness of c that

$$\|K_{\lambda}\| \leq C |\Re \lambda|^{s-1}.$$ 

Hence, for sufficiently large $$|\Re \lambda|$$, $$I + K_{\lambda}$$ is invertible. Applying $$(-\Delta - \lambda)^{-1}$$ to (2.29), however, yields

$$(I + K_{\lambda})u_3 = (-\Delta - \lambda)^{-1}(f_3 - c R_-(\lambda)f_1 - c R_+(\lambda)f_2). \quad (2.32)$$

Hence, we get

$$\|u_3\| \leq C |\Re \lambda|^{\max(s,0)-1}\|f\|. \quad (2.33)$$

From the first two lines of (2.28) we then conclude (2.31) for $$u_1$$ and $$u_2$$. \hfill \blacksquare

### 2.5.3 Point spectrum

**Proposition 2.15.** Let $$\Omega = \mathbb{C} \setminus (\mathbb{R}_+ \cup \sigma(-\Delta \pm ib_3))$$. Under the assumptions of Proposition 2.11, it holds, for any $$\lambda \in \Omega$$, that $$\lambda \in \sigma(B_1)$$ if and only if $$-1 \in \sigma(K_{\lambda})$$. Moreover, if $$-\Delta \pm ib_3$$ has a compact resolvent, then $$\lambda$$ is an eigenvalue of $$B_1$$. Finally, $$\lambda$$ is an isolated eigenvalue of $$B_1$$ of finite multiplicity.

**Proof.** Let $$\lambda \in \Omega$$. We begin by the trivial observation that if $$-1 \in \sigma(K_{\lambda})$$ then by (2.29) $$(B_1 - \lambda)^{-1}$$ must be unbounded. If $$-1 \notin \sigma(K_{\lambda})$$, we may conclude from (2.29), (b) and the boundedness of $$R_\pm(\lambda)$$ that $$\lambda \in \rho(B_1)$$.

Suppose now that $$R_\pm(\lambda)$$ is compact. Then so is $$K_\lambda$$, and hence, if $$-1 \in \sigma(K_{\lambda})$$, then $$-1$$ is an eigenvalue of $$K_\lambda$$ and $$\lambda$$ is an eigenvalue of $$B_1$$. Consider the family $$\Omega \ni \lambda \mapsto (\hat{B}_1 - \lambda) \in \mathcal{L}(D(\hat{B}_1, \mathcal{L}^2(\mathbb{R}^k, \mathbb{C}^3))).$$ By the foregoing discussion,

$$\dim \ker(\hat{B}_1 - \lambda) = \dim \ker(K_{\lambda} - 1) < \infty.$$ 

As $$\hat{B}_1^* = -\Delta \otimes I_3 + \hat{M}^*$$, we may apply the same arguments to obtain that $$\dim \ker(\hat{B}_1^* - \bar{\lambda}) < \infty$$, and hence $$\dim \text{coker}(\hat{B}_1 - \lambda) < \infty$$. We can then conclude that $$(\hat{B}_1 - \lambda)$$ is a Fredholm operator for each $$\lambda \in \Omega$$. Clearly, $$(\hat{B}_1 - \lambda)$$ is invertible for sufficiently large $$|\Re \lambda|$$ or negative $$\Re \lambda$$. Hence the index, which is constant in $$\Omega$$ is zero and we can use either [30, Proposition 2.3] or [24, Theorem 2.1] (relying on [15]), to obtain that $$(\hat{B}_1 - \lambda)^{-1}$$ is, (see [24]) a finite-meromorphic family in $$\Omega$$. The proposition is proved. \hfill \blacksquare

**Remark 2.16.** In the next sections we consider the case $$k = 1$$ and $$b_3(x) = x$$. In this case, $$\sigma(-\Delta \pm ib_3) = \emptyset$$ and $$\mathbb{R}_+ \subset \sigma(\check{B}_1)$$. By Proposition 2.13 it follows that $$\sigma(\check{B}_1) \cap (\mathbb{C} \setminus \mathbb{R}_+)$$ is discrete.

**Remark 2.17.** We note that one can establish Proposition 2.13 whenever $$\check{c} R_+(\lambda) \check{c}$$ is compact. For example, if we consider the above case where $$k = 1$$ and $$b_3(x) = x$$, we may allow for $$|c| = O(|x|^\gamma)$$ as $$|x| \to \infty$$ for some $$0 \leq \gamma < 1/2$$. In the next sections, assuming $$k = 1$$, constant $$b_1, b_2 = 0$$, and $$b_3(x) = x$$, we obtain much more precise results for $$B_c$$ in the asymptotic regime $$\epsilon \to 0$. 

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2.6 \( \tilde{B}_1 \) in the presence of boundary

Consider a bounded smooth open set \( \mathcal{O} \subset \mathbb{R}^k \) and the Dirichlet realization of \( P \) in \( \mathcal{O} \) denoted by \( P^\mathcal{O} \). For \( M \in C^2(\mathbb{R}^k, M_d(\mathbb{R})) \), \( P^\mathcal{O} \) is a bounded perturbation of \( -\Delta \) and hence \( D(P^\mathcal{O}) = H^2(\mathcal{O}, \mathbb{R}^d) \cap H^1_0(\mathcal{O}, \mathbb{R}^d) \), and \( P^\mathcal{O} \) has a compact resolvent. Nevertheless, in the presence of a small parameter \( \epsilon \), i.e. when \( P^\mathcal{O} = -\epsilon^2 \Delta + M \)

the behaviour of spectrum and the resolvent in the limit \( \epsilon \to 0 \) could probably be understood from the analysis of linearized operators acting on \( \mathbb{R}^k \). This is precisely the case in the two dimensional setting (1.7) when \( P^\mathcal{O} \) is equivalent to the Dirichlet realization of \( -\epsilon^2 \Delta + iV \) (see for instance [23, 2, 8]).

3 The (1D)-model in \( \mathbb{R} \).

In this section we consider the operator (1.4) acting on \( \mathbb{R} \).

3.1 Problem setting

In the standard basis of \( \mathbb{C}^3 \), the system (1.4) reads for \( u = (u_1, u_2, u_3) \)

\[
\mathcal{B}_\epsilon u := \begin{pmatrix}
-\epsilon^2 \frac{d^2}{dx^2} & x & 0 \\
-x & -\epsilon^2 \frac{d^2}{dx^2} & 1 \\
0 & -1 & -\epsilon^2 \frac{d^2}{dx^2}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}
\tag{3.1}
\]

It has been established in a more general context in either Proposition 2.9 or Proposition 2.14 that

Proposition 3.1. \( \mathcal{B}_\epsilon \) is a closed operator in \( L^2(\mathbb{R}, \mathbb{R}^3) \) whose domain is

\[
D(\mathcal{B}_\epsilon) = \{ u \in H^2(\mathbb{R}, \mathbb{R}^3) \ | \ [xb_1 + b_0] \times u \in L^2(\mathbb{R}, \mathbb{R}^3) \}. \tag{3.2}
\]

Let \( v \) be given by (1.6). We begin by rewriting \( \mathcal{B}_\epsilon \) in the basis \((v, \bar{v}, b_1)\) of \( \mathbb{C}^3 \).

We thus set

\[
\tilde{u}_1 = u \cdot v = \frac{1}{\sqrt{2}}(-iu_1 + u_2) \ ; \ \tilde{u}_2 = u \cdot \bar{v} = \frac{1}{\sqrt{2}}(iu_1 + u_2) \ ; \ \tilde{u}_3 = u \cdot b_1 = u_3.
\]

In this new basis the operator \( \tilde{\mathcal{B}}_\epsilon \) becomes

\[
\tilde{\mathcal{B}}_\epsilon := \begin{pmatrix}
-\epsilon^2 \frac{d^2}{dx^2} + ix & 0 & \frac{1}{\sqrt{2}} \\
0 & -\epsilon^2 \frac{d^2}{dx^2} - ix & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\epsilon^2 \frac{d^2}{dx^2}
\end{pmatrix}. \tag{3.3}
\]

We attempt to obtain resolvent estimates for the problem

\[
(\tilde{\mathcal{B}}_\epsilon - \Lambda) \tilde{u} = \epsilon^2 \tilde{f}.
\tag{3.4}
\]
Applying the transformation 
\[ x \rightarrow \varepsilon^{2/3} x \]
yields, 
\[ (\tilde{B}_\varepsilon - \lambda)\tilde{u} = \tilde{f}, \]  
(3.5)
where, with 
\[ \varepsilon = \varepsilon^{4/3} \text{ and } \lambda = \varepsilon^{-2/3} \Lambda \]  
(3.6)
\[ \tilde{B}_\varepsilon \] is given by
\[ \tilde{B}_\varepsilon(x, \frac{d}{dx}) := \left( \begin{array}{ccc} -\frac{d^2}{dx^2} + ix & 0 & \frac{1}{\sqrt{2}}\varepsilon^{-\frac{1}{2}} \\ 0 & -\frac{d^2}{dx^2} - ix & \frac{1}{\sqrt{2}}\varepsilon^{-\frac{1}{2}} \\ -\frac{1}{\sqrt{2}}\varepsilon^{-\frac{1}{2}} & -\frac{1}{\sqrt{2}}\varepsilon^{-\frac{1}{2}} & -\frac{d^2}{dx^2} \end{array} \right), \]  
(3.7)
Equivalently we may write the spectral equation in the form
\[ \begin{align*} 
(-\frac{d^2}{dx^2} + ix - \lambda)\hat{u}_1 + \frac{1}{\sqrt{2}}\varepsilon^{-\frac{1}{2}}\hat{u}_3 &= \hat{f}_1, \\
(-\frac{d^2}{dx^2} - ix - \lambda)\hat{u}_2 + \frac{1}{\sqrt{2}}\varepsilon^{-\frac{1}{2}}\hat{u}_3 &= \hat{f}_2, \\
(-\frac{d^2}{dx^2} - \lambda)\hat{u}_3 - \frac{1}{\sqrt{2}}\varepsilon^{-\frac{1}{2}}(\hat{u}_1 + \hat{u}_2) &= \hat{f}_3. 
\end{align*} \]  
(3.8)
We omit the accents of \( u_i \) and \( f_i \) \((i = 1, 2, 3)\) in the sequel.
Let the Fourier transform of \( u \) be defined in the following manner
\[ \hat{u}_i(\omega) = \mathcal{F}(u_i)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega x} u_i(x) dx \quad i = 1, 2, 3. \]
Applying \( \mathcal{F} \) to (3.8) yields
\[ \begin{align*} 
(\omega^2 - \lambda)\hat{u}_1 - \frac{d\hat{u}_1}{d\omega} + \frac{1}{\sqrt{2}}\varepsilon^{-1/2}\hat{u}_3 &= \hat{f}_1, & \text{in } \mathbb{R} \quad (3.9a) \\
(\omega^2 - \lambda)\hat{u}_2 + \frac{d\hat{u}_2}{d\omega} + \frac{1}{\sqrt{2}}\varepsilon^{-1/2}\hat{u}_3 &= \hat{f}_2, & \text{in } \mathbb{R} \quad (3.9b) \\
(\omega^2 - \lambda)\hat{u}_3 - \frac{\varepsilon^{-1/2}}{\sqrt{2}}(\hat{u}_1 + \hat{u}_2) &= \hat{f}_3, & \text{in } \mathbb{R}. \quad (3.9c) 
\end{align*} \]
We search for solutions in \( X = X_1 \times X_1 \times X_3 \), where
\[ X_1 = \{ u \in H^1(\mathbb{R}) \mid \omega^2 u \in L^2(\mathbb{R}) \} ; \quad X_3 = \{ u \in L^2(\mathbb{R}) \mid \omega^2 u \in L^2(\mathbb{R}) \}. \]  
(3.10)
We now set
\[ \hat{u}_d = \hat{u}_1 - \hat{u}_2; \quad \hat{u}_s = \hat{u}_1 + \hat{u}_2; \quad \hat{f}_d = \hat{f}_1 - \hat{f}_2; \quad \hat{f}_s = \hat{f}_1 + \hat{f}_2. \]  
(3.11)
Subtracting (3.9a) from (3.9b) yields
\[ -\frac{d\hat{u}_s}{d\omega} + (\omega^2 - \lambda)\hat{u}_d = \hat{f}_d. \]  
(3.12)
Summing up (3.9b) and (3.9a) yields with the aid of (3.9c), assuming \( \lambda \notin \mathbb{R}_+ \),
\[
\frac{d \hat{u}_d}{d \omega} - \left[ (\omega^2 - \lambda) + \frac{\varepsilon^{-1}}{\omega^2 - \lambda} \right] \hat{u}_s = -\hat{f}_s + \sqrt{2\varepsilon^{-1/2}} \frac{\hat{f}_3}{\omega^2 - \lambda}.
\]
(3.13)

Extracting \( \hat{u}_d \) from (3.12) and then substituting into (3.13) we obtain
\[
-\frac{d}{d \omega} \left( \frac{1}{\omega^2 - \lambda} \frac{d \hat{u}_s}{d \omega} \right) + \left[ (\omega^2 - \lambda) + \frac{\varepsilon^{-1}}{\omega^2 - \lambda} \right] \hat{u}_s = g,
\]
(3.14)

with
\[
g := \hat{f}_s + \varepsilon^{-1/2} \frac{\hat{f}_3}{\omega^2 - \lambda} + \frac{d}{d \omega} \left( \frac{\hat{f}_d}{\omega^2 - \lambda} \right).
\]
(3.15)

### 3.2 The case \( 0 < |\Re \lambda| < \varepsilon^{-1/2} \)

We write \( \lambda_r = \Re \lambda \) and \( \lambda_i = \Im \lambda \).

**Proposition 3.2.** For any \( 0 < \delta \leq 1/2 \), there exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \) and any triple \((u, f, \lambda)\) satisfying \( 0 < |\lambda_i| \leq (1 - 2\delta^4)^{1/2}\varepsilon^{-1/2} \) and (3.8), it holds that
\[
\|u_1 + u_2\|_2 \leq C\varepsilon^{1/2} \left( 1 + \frac{[1 + \varepsilon^{1/2}(\lambda_r)^{1/2}]}{|\lambda_i|} \right) \|f\|_2.
\]
(3.16)

**Proof.** Without any loss of generality we assume \( \Im \lambda > 0 \), as the transformation
\[
\lambda \to \bar{\lambda} ; \quad \hat{u}_s \to \bar{\hat{u}}_s ; \quad f \to \bar{f}
\]
leaves (3.14) unaltered.

We split the discussion into three different cases depending on the value of \( \lambda_r \).
We begin, however, by obtaining some identities and inequalities that are valid in all cases.

**Preliminary inequalities**

Taking the inner product, in \( L^2(\mathbb{R}) \), of (3.14) with \( \hat{u}_s \) yields, for the imaginary part
\[
\Im\left( \hat{u}'_s, \frac{\hat{u}_s'}{\omega^2 - \lambda} \right) - \lambda_i \| \hat{u}_s \|_2^2 + \varepsilon^{-1} \Im\left( \hat{u}_s, \frac{\hat{u}_s}{\omega^2 - \lambda} \right) = -\Im\langle \hat{u}_s, g \rangle,
\]
Consequently, it holds that
\[
\left\| \frac{\hat{u}'_s}{\omega^2 - \lambda} \right\|_2^2 - \| \hat{u}_s \|_2^2 + \varepsilon^{-1} \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2^2 = -\frac{1}{\lambda_i} \Im\langle \hat{u}_s, g \rangle.
\]
(3.17)

Next, we estimate the inner product \( \langle \hat{u}_s, g \rangle \) with \( g \) defined in (3.15). Clearly,
\[
\left| \langle \hat{u}_s, \frac{\hat{f}_3}{\omega^2 - \lambda} \rangle \right| \leq \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \| f_3 \|_2.
\]
Furthermore, integration by parts yields that
\[
\left| \langle \hat{u}_s, \left( \frac{\hat{f}_d}{\omega^2 - \lambda} \right) \rangle \right| \leq \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \left\| f_d \right\|_2.
\]
Hence,
\[
\left| \langle \hat{u}_s, g \rangle \right| \leq \left\| \hat{u}_s \right\|_2 \left\| f_s \right\|_2 + \left\| \frac{(\hat{u}_s)'}{\omega^2 - \lambda} \right\|_2 \left\| f_d \right\|_2 + \varepsilon^{-1/2} \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \left\| f_3 \right\|_2. \tag{3.18}
\]
By (3.18) we have
\[
\left| \langle \hat{u}_s, g \rangle \right| \leq \sqrt{2} \left\| \hat{u}_s \right\|_2^2 + \frac{1}{|\lambda_i|} \left| \langle \hat{u}_s, g \rangle \right| \left\| f \right\|_2.
\]
Taking the square then yields
\[
\left| \langle \hat{u}_s, g \rangle \right|^2 \leq 2 \left( \left\| \hat{u}_s \right\|_2^2 + \frac{1}{|\lambda_i|} \left| \langle \hat{u}_s, g \rangle \right| \right) \left\| f \right\|_2^2,
\]
which can be rewritten in the form
\[
\left( \left| \langle \hat{u}_s, g \rangle \right| - \frac{1}{\lambda_i} \left\| f \right\|_2^2 \right)^2 \leq (4 \left\| \hat{u}_s \right\|_2^2 + \frac{\left\| f \right\|_2}{\lambda_i^2}) \left\| f \right\|_2^2.
\]
Consequently,
\[
\left| \langle \hat{u}_s, g \rangle \right| \leq \left( \left\| f \right\|_2 \lambda_i + \sqrt{4 \left\| \hat{u}_s \right\|_2^2 + \frac{\left\| f \right\|_2^2}{\lambda_i^2}} \right) \left\| f \right\|_2,
\]
and hence
\[
\left| \langle \hat{u}_s, g \rangle \right| \leq 2 \left( \frac{\left\| f \right\|_2}{\lambda_i} + \left\| \hat{u}_s \right\|_2 \right) \left\| f \right\|_2. \tag{3.19}
\]
Next, taking the inner product of (3.14) from the left with \((\omega^2 - \lambda)\hat{u}_s\) yields for the real part,
\[
\Re \langle (\omega^2 - \lambda)\hat{u}_s, g \rangle = \left| (\hat{u}_s)'/2 \right|^2 + \Re \left( 2\omega \hat{u}_s, \frac{(\hat{u}_s)'}{\omega^2 - \lambda} \right) + (\varepsilon^{-1} - \lambda_i^2) \left\| \hat{u}_s \right\|_2^2 + \left( |\omega^2 - \lambda| \right) \left| \hat{u}_s \right|_2^2. \tag{3.20}
\]
Finally, by (3.15) we have that
\[
\left| \langle (\omega^2 - \lambda)\hat{u}_s, g \rangle \right| \leq \left( |\omega^2 - \lambda| \right) \left| \hat{u}_s \right|_2 \left| f_s \right|_2 + \left( \left\| \hat{u}_s \right\|_2 + \frac{2\omega}{\omega^2 - \lambda} \right) \left\| f_d \right\|_2 + \varepsilon^{-1/2} \left\| \hat{u}_s \right\|_2 \left\| f_3 \right\|_2,
\]
and since
\[ \left\| \frac{\omega \hat{u}_s}{\omega^2 - \lambda} \right\|_2 \leq (\lambda_r)^{1/2} \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 + \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2, \tag{3.21} \]
we obtain that
\[ |\langle (\omega^2 - \lambda)\hat{u}_s, g \rangle| \leq \varepsilon^{-1/2} \left\| \hat{u}_s \right\|_2 \left\| f_3 \right\|_2 + \left\| (\omega^2 - \lambda)\hat{u}_s \right\|_2 \left\| f_s \right\|_2 \]
\[ + \left( \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 + (\lambda_r)^{1/2} \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 + \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \right) \left\| f_d \right\|_2. \tag{3.22} \]
We now observe that
\[ \left\| \frac{2\omega \hat{u}_s}{\omega^2 - \lambda} \right\| \leq 2 \left\| \omega \hat{u}_s \right\|_2 \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2. \tag{3.23} \]
Set,
\[ \lambda_{r,m} = \max(\lambda_r, 1). \tag{3.24} \]
Next we write, with the aid of Cauchy's inequality,
\[ 2 \left\| \omega \hat{u}_s \right\|_2 \leq 2 \left\| \omega^2 \hat{u}_s \right\|_2 \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \]
\[ \leq \lambda_{r,m}^{-1/2} \left\| \omega^2 \hat{u}_s \right\|_2 + \lambda_{r,m}^{1/2} \left\| \hat{u}_s \right\|_2 \]
\[ \leq \lambda_{r,m}^{-1/2} \left( \left\| (\omega^2 - \lambda)\hat{u}_s \right\|_2 + \lambda_{r,m}^{-1/2} (\lambda_r) \left\| \hat{u}_s \right\|_2 \right) + \lambda_{r,m}^{1/2} \left\| \hat{u}_s \right\|_2 \]
\[ \leq \lambda_{r,m}^{-1/2} \left( \langle \omega^2 - \lambda \rangle \hat{u}_s \right) \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 + 2 \lambda_{r,m}^{1/2} \left\| \hat{u}_s \right\|_2. \tag{3.25} \]
Substituting (3.22) into (3.20) yields, with the aid of (3.23) and (3.25) that
\[ \left\| (\hat{u}_s)' \right\|_2 \leq \varepsilon^{-1/2} \left\| \hat{u}_s \right\|_2 \left\| f_3 \right\|_2 + \left\| (\omega^2 - \lambda)\hat{u}_s \right\|_2 \left\| f_s \right\|_2 \]
\[ + \left( \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 + \lambda_{r,m}^{1/2} \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \right) \left\| f_d \right\|_2. \tag{3.26} \]
By (3.17) and (3.19) we have
\[ \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \leq \left\| \hat{u}_s \right\|_2 + \frac{1}{\lambda_i} \langle \hat{u}_s, g \rangle \leq \left\| \hat{u}_s \right\|_2 + \frac{2}{\lambda_i} \left( \left\| f_3 \right\|_2 + \left\| f_s \right\|_2 \right), \]
Consequently,
\[ \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \leq \left( \left\| \hat{u}_s \right\|_2 + \frac{\left\| f_3 \right\|_2}{\lambda_i} \right)^2 + \frac{\left\| f_s \right\|_2^2}{\lambda_i^2}, \]
and hence
\[ \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \leq \left\| \hat{u}_s \right\|_2 + \frac{2}{\lambda_i} \left\| f \right\|_2. \tag{3.27} \]
By Cauchy inequality we have that for any $\alpha > 0$
\[ \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \leq \frac{1}{\sqrt{1/2}} \left( \alpha \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 + \frac{1}{\alpha} \left\| \hat{u}_s \right\|_2 \right), \]
which in particular implies
\[ \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \leq \lambda_{r,m}^{1/2} \left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 + \lambda_{r,m}^{-1/2} \left\| \hat{u}_s \right\|_2. \tag{3.28} \]
Using (3.17) and (3.19) once again we obtain that
\[
\left\| \frac{\hat{u}_s}{\omega^2 - \lambda} \right\|_2 \leq \varepsilon^{1/2} \left\| \hat{u}_s \right\|_2 + \frac{2\varepsilon^{1/2}}{\lambda_i} \| f \|_2 .
\]
(3.29)
Substituting (3.29) into (3.28) yields
\[
\left\| \frac{\hat{u}_s}{\omega^2 - \lambda^{1/2}} \right\|_2 \leq (\varepsilon^{1/2} \lambda_{r,m}^{1/2} + \lambda_{r,m}^{-1/2}) \left\| \hat{u}_s \right\|_2 + \frac{2\varepsilon^{1/2} \lambda_{r,m}^{1/2}}{\lambda_i} \| f \|_2 .
\]
(3.30)
Substituting (3.30), together with (3.29) and (3.27) into (3.26), we obtain that
\[
\| (\hat{u}_s)' \|_2 + (\varepsilon^{-1} - \lambda_i^2) \left\| \hat{u}_s \right\|_2 + (\varepsilon^2 - \lambda_r) \hat{u}_s \|_2^2 \\
\leq \varepsilon^{-1/2} \| \hat{u}_s \|_2 \| f_s \|_2 + (\omega^2 - \lambda_r) \hat{u}_s \|_2 + 2\varepsilon^{1/2} \left\| \hat{u}_s \right\|_2 \left( \left\| \hat{u}_s \right\|_2 + \frac{2}{\lambda_i} \| f \|_2 \right) \\
+ \left( \left\| \hat{u}_s \right\|_2 + (2\varepsilon^{-1/2} \lambda_{r,m}^{1/2} + \lambda_{r,m}^{-1/2}) \right) \left\| \hat{u}_s \right\|_2 + \frac{4\varepsilon^{1/2} \lambda_{r,m}^{1/2}}{\lambda_i} \| f \|_2 \| f_d \|_2 .
\]
From the above and using the condition on \( \lambda_i \), we can conclude that there exists \( C > 0 \) such that
\[
\| (\hat{u}_s)' \|_2 + (\omega^2 - \lambda_r) \hat{u}_s \|_2 \leq C \left( \lambda_{r,m}^{1/4} \left\| \hat{u}_s \right\|_2 + \left[ 1 + \frac{\varepsilon^{1/2} \lambda_{r,m}^{1/2} + \lambda_{r,m}^{-1/2}}{\lambda_i} \right] \left\| f \right\|_2 \right)
\]
which, when substituted, together with (3.29) and (3.30), into (3.22) yields
\[
\left\| (\omega^2 - \lambda) \hat{u}_s, g \right\| \leq C \left[ \left[ 1 + \frac{\varepsilon^{1/2} \lambda_{r,m}^{1/2} + \lambda_{r,m}^{-1/2}}{\lambda_i} \right] \left\| f \right\|_2 + \lambda_{r,m}^{1/4} \left\| \hat{u}_s \right\|_2 \left( \left\| f_s \right\|_2 + \| f_d \|_2 \right) \\
+ \varepsilon^{-1/2} \| \hat{u}_s \|_2 \| f_s \|_2 + \left( 2\varepsilon^{-1/2} \lambda_{r,m}^{1/2} + \lambda_{r,m}^{-1/2} \right) \| \hat{u}_s \|_2 + \frac{4\varepsilon^{1/2} \lambda_{r,m}^{1/2}}{\lambda_i} \| f \|_2 \| f_d \|_2 \right].
\]
Hence,
\[
\left\| (\omega^2 - \lambda) \hat{u}_s, g \right\| \leq C \left[ \left[ 1 + \frac{\varepsilon^{1/2} \lambda_{r,m}^{1/2} + \lambda_{r,m}^{-1/2}}{\lambda_i} \right] \| f \|_2 + \left( \lambda_{r,m}^{1/4} + \varepsilon^{-1/2} + \varepsilon^{1/2} \lambda_{r,m}^{1/2} + \lambda_{r,m}^{-1/2} \right) \left\| \hat{u}_s \right\|_2 \right] \| f \|_2
\]
(3.31)
Case 1: \( 0 \leq \lambda_r \leq \delta^2 \varepsilon^{-1/2} / 8. \)
Let \( \eta \in C_0^\infty (\mathbb{R}, [0, 1]) \), satisfy \( \bar{\eta} = \sqrt{1 - \eta^2} \in C^\infty (\mathbb{R}, [0, 1]) \) and
\[
\eta(x) = \begin{cases} 
0 & x < \frac{1}{2} \\
1 & x > 1. 
\end{cases}
\]
(3.32)
Let further
\[
\eta_e(\omega) = \eta(e^{1/4} |\omega| / \delta) \text{ and } \bar{\eta}_e(\omega) = \bar{\eta}(e^{1/4} |\omega| / \delta) .
\]
Taking the inner product, in \( L^2(\mathbb{R}) \), of (3.13) with \( \eta_e^2 \hat{u}_s \) yields, for the real part
\[
\left\| \eta_e \hat{u}_s' \right\|_2^2 + 2\Re \left\langle \eta_e \hat{u}_s, \frac{\eta_e \hat{u}_s'}{\omega^2 - \lambda} \right\rangle + \\
+ \varepsilon^{-1} \left\| \frac{\eta_e \hat{u}_s}{\omega^2 - \lambda} \right\|_2^2 + \left\| \omega^2 - \lambda \right\|_2^2 \| \hat{u}_s \|_2^2 = \Re \left\langle \eta_e \hat{u}_s, \eta_e g \right\rangle .
\]
(3.33)
As \( \lambda_r \leq \delta^2 \varepsilon^{-\frac{3}{4}} \), it follows that \( (\omega^2 - \lambda_r) \geq \delta^2 \varepsilon^{-\frac{3}{4}} \) on the support of \( \eta_r \), and hence there exists \( C > 0 \) such that

\[
\| \eta \dot{u}_s \|_2^2 \leq C \varepsilon^{\frac{1}{4}} (\varepsilon \| \dot{u}_s \|_2^2 + |\langle \eta \dot{u}_s, \eta \dot{v}_g \rangle|).
\] (3.34)

From (3.17) we can thus conclude that, for sufficiently small \( \varepsilon \),

\[
\| \eta \dot{u}_s \|_2^2 \leq (1 - \delta^4) \varepsilon^{-1} \left\| \eta \frac{\dot{u}_s}{\omega^2 - \lambda} \right\|_2^2 \leq (1 - \delta^4) \varepsilon^{-1} \left\| \frac{\dot{u}_s}{\omega^2 - \lambda} \right\|_2^2 \leq (1 - \delta^4) \| \dot{u}_s \|_2^2 + \frac{1}{\lambda_i} |\langle \dot{u}_s, g \rangle|.
\]

(3.35)

To obtain the inequality we need to use the fact that on the support of \( \eta_r \) we have, for \( \varepsilon \) sufficiently small,

\[
|\omega^2 - \lambda|^2 \leq \delta^4 \varepsilon^{-1} + (1 - 2\delta^4) \varepsilon^{-1} = (1 - \delta^4) \varepsilon^{-1}.
\]

We use (3.17) to obtain the above inequality.

Combining (3.35) with (3.34) yields, for any \( \delta \in (0, \frac{1}{2}] \), the existence of \( \varepsilon_0 > 0 \) and \( C > 0 \) such that, for \( \varepsilon \in (0, \varepsilon_0) \),

\[
\| \dot{u}_s \|_2^2 \leq C (\varepsilon^{\frac{1}{4}} |\langle \eta \dot{u}_s, \eta \dot{v}_g \rangle| + \frac{1}{\lambda_i} |\langle \dot{u}_s, g \rangle|).
\]

(3.36)

Combining the above with (3.17) yields

\[
\left\| \frac{(\dot{u}_s)}{\omega^2 - \lambda} \right\|_2^2 + \varepsilon^{-1} \left\| \frac{\dot{u}_s}{\omega^2 - \lambda} \right\|_2^2 \leq C \left( \varepsilon^{\frac{1}{4}} |\langle \eta \dot{u}_s, \eta \dot{v}_g \rangle| + \frac{1}{\lambda_i} |\langle \dot{u}_s, g \rangle| \right).
\]

(3.37)

Next, we estimate \( \langle \eta \dot{u}_s, \eta \dot{v}_g \rangle \). As in the proof of (3.19) we may write that

\[
\left| \langle \eta \dot{u}_s, \eta \dot{v}_s \rangle \right| \leq \left\| \frac{\dot{u}_s}{\omega^2 - \lambda} \right\|_2 \| f_3 \|_2.
\]

Furthermore, we have that

\[
\left| \langle \eta \dot{u}_s, \left( \frac{\dot{f}_d}{\omega^2 - \lambda} \right) \right| \leq \left\| \frac{\dot{u}_s}{\omega^2 - \lambda} \right\|_2 \| f_d \|_2 + C \varepsilon^{\frac{1}{4}} \left\| \frac{\eta' (\varepsilon^{1/4} \cdot \dot{u}_s)}{\omega^2 - \lambda} \right\|_2 \| f_d \|_2.
\]

Consequently we may write, for sufficiently small \( \varepsilon \)

\[
|\langle \eta \dot{u}_s, \eta \dot{v}_g \rangle| \leq (1 + C \varepsilon^{\frac{1}{4}}) \| \dot{u}_s \|_2 \| f_d \|_2 + \left\| \frac{\dot{u}_s}{\omega^2 - \lambda} \right\|_2 \| f_d \|_2 + 2 \varepsilon^{\frac{1}{4}} \| f_3 \|_2.
\]

(3.38)

Substituting the above together with (3.19) into (3.36) and (3.37) then yields

\[
\left\| \left( \frac{(\dot{u}_s)}{\omega^2 - \lambda} \right) \right\|_2^2 + \varepsilon^{-1} \left\| \frac{\dot{u}_s}{\omega^2 - \lambda} \right\|_2^2 + \| \dot{u}_s \|_2^2 \leq C \left( \frac{1}{\lambda_i^2} + \varepsilon \right) \| f \|_2^2 \leq \frac{C}{\lambda_i^2} \| f \|_2^2.
\]

(3.39)
where for the last inequality we have used the assumption on $|\lambda_i|$. Consequently, by (3.20), as $|\lambda_i| < (1 - 2\delta^4)^{1/2}\varepsilon^{-1/2}$, and by (3.21) it holds that

$$
\|\hat{u}_s\|^2 + \||\omega^2 - \lambda_s|\hat{u}_s\|^2 + \delta^4\varepsilon^{-1}\|\hat{u}_s\|^2 \\
\leq 2\|\hat{u}_s\|^2 \left(\lambda^{1/2}\|\hat{u}_s\|_2 + \|\frac{\hat{u}_s}{|\omega^2 - \lambda|^1/2}\|_2\right) + 2\|\langle\omega^2 - \bar{\lambda}\rangle\hat{u}_s, g\|.
$$

(3.40)

From (3.39) we get

$$
\left\|\frac{\hat{u}_s}{|\omega^2 - \lambda|^{1/2}}\right\|_2^2 \leq \left\|\frac{\hat{u}_s}{|\omega^2 - \lambda|}\right\|_2\|\hat{u}_s\|_2 \leq C\frac{\varepsilon}{\lambda_i^2} \|f\|^2
$$

and, for $0 \leq \lambda \leq \delta^2\varepsilon^{-1/8}$,

$$
\lambda_i\left\|\frac{\hat{u}_s}{|\omega^2 - \lambda|}\right\|_2^2 \leq \tilde{C}\lambda_i\frac{\varepsilon}{\lambda_i^2} \|f\|^2 \leq \tilde{C}\frac{\varepsilon}{\lambda_i^2} \|f\|^2.
$$

(3.41)

By (3.40) we now obtain that

$$
\|\hat{u}_s\|^2 + \||\omega^2 - \lambda|\hat{u}_s\|^2 + \varepsilon^{-1}\|\hat{u}_s\|^2 \leq C\left(\|\langle\omega^2 - \bar{\lambda}\rangle\hat{u}_s, g\| + \frac{\varepsilon}{\lambda_i^2} \|f\|^2\right).
$$

(3.42)

We note that (3.31) implies in Case 1

$$
\|\langle\omega^2 - \bar{\lambda}\rangle\hat{u}_s, g\| \leq C\left(1 + \frac{\varepsilon^{1/2}\lambda_i^{1/2} + \lambda^{-1/2}_{r,m}}{\lambda_i^2}\|f\|_2 + \varepsilon^{-1/2}\|\hat{u}_s\|_2\right) \|f\|_2
$$

(3.43)

Substituting (3.43) into (3.42) yields, for any $\delta > 0$, the existence of positive $C$, and $\varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_0)$, it holds that

$$
\|\hat{u}_s\|^2 + \||\omega^2 - \lambda|\hat{u}_s\|^2 + \varepsilon^{-1}\|\hat{u}_s\|^2 \leq C\left(1 + \frac{\varepsilon^{1/2}}{\lambda_i^2} + \frac{1}{\lambda_i^2}\right) \|f\|_2 + \varepsilon^{-1/2}\|\hat{u}_s\|_2\|f\|_2
$$

(3.44)

and hence

$$
\|\hat{u}_s\|_2 \leq C\varepsilon\left[1 + \frac{\varepsilon^{1/2}}{\lambda_i^2} + \frac{1}{\lambda_i^2}\right] \|f\|^2,
$$

(3.45)

establishing, thereby, (3.16) in this case.

Case 2: $\lambda_r > \delta^2\varepsilon^{-1/8}$.

We begin by the simple observation that in this case $\lambda_{r,m} = \lambda_r$. Then we use (3.25), (3.27), and (3.25) to conclude that

$$
\left|\frac{\hat{u}_s'}{\omega^2 - \lambda}\right| \leq (\lambda_r^{-1/2}\|\omega^2 - \lambda_r\|\hat{u}_s\|_2 + 2\lambda_r^{-1/2}\|\hat{u}_s\|_2) \left(\|\hat{u}_s\|_2 + \frac{C}{\lambda_i^2}\|f\|_2\right).
$$
from which we conclude that
\[
\left| \left( 2\omega \hat{u}_s, \frac{\hat{u}'_s}{\omega^2 - \lambda} \right) \right| \leq \lambda_{r}^{-3/4}\| (\omega^2 - \lambda_r) \hat{u}_s \|_2^2 \\
+ (2\lambda_r^{1/2} + \lambda_r^{-1/4}) \| \hat{u}_s \|_2^2 + \frac{C}{\lambda_i} \| \mathbf{f} \|_2 \left( 2\lambda_r^{1/2} \| \hat{u}_s \|_2 + \lambda_r^{-1/2} \| (\omega^2 - \lambda_r) \hat{u}_s \|_2 \right).
\]
Substituting the above into (3.47) then yields
\[
\| \hat{u}'_s \|_2^2 + (\varepsilon^{-1} - \lambda_i^2 - 2\lambda_r^{1/2} - \lambda_r^{-1/4}) \| \hat{u}_s \|_2^2 + (1 - \lambda_r^{-3/4}) \| (\omega^2 - \lambda_r) \hat{u}_s \|_2^2 \\
\leq \left| \left( (\omega^2 - \lambda) \hat{u}_s, g \right) \right| + \frac{C}{\lambda_i} \| \mathbf{f} \|_2 \left( 2\lambda_r^{1/2} \| \hat{u}_s \|_2 + \lambda_r^{-1/2} \| (\omega^2 - \lambda_r) \hat{u}_s \|_2 \right).
\]
From (3.46), observing that \( \varepsilon^{-1} - \lambda_i^2 \geq 0 \), we can conclude that
\[
\| (\omega^2 - \lambda_r) \hat{u}_s \|_2 \leq C \left( \| (\omega^2 - \lambda) \hat{u}_s, g \|_{\frac{1}{2}} + \lambda_r^{1/4} \| \hat{u}_s \|_2 + \frac{\lambda_r^{-1/4}}{\lambda_i} \| \mathbf{f} \|_2 + \frac{\lambda_r^{1/4}}{\lambda_i} \| \mathbf{f} \|_2 \| \hat{u}_s \|_2^2 \right).
\]
Substituting the above into (3.46) yields
\[
\| \hat{u}'_s \|_2^2 + (\varepsilon^{-1} - \lambda_i^2 - 2\lambda_r^{1/2} - \lambda_r^{-1/4}) \| \hat{u}_s \|_2^2 + (1 - \lambda_r^{-3/4}) \| (\omega^2 - \lambda_r) \hat{u}_s \|_2^2 \\
\leq \left| \left( (\omega^2 - \lambda) \hat{u}_s, g \right) \right| + \frac{C}{\lambda_i} \| \mathbf{f} \|_2 \left( 2\lambda_r^{1/2} \| \hat{u}_s \|_2 + \frac{\lambda_r^{-1}}{\lambda_i} \| \mathbf{f} \|_2 + \frac{\lambda_r^{1/4}}{\lambda_i} \| \mathbf{f} \|_2 \| \hat{u}_s \|_2^2 \right).
\]
We recall from [6, Proposition 3.1] that the ground state energy of the anharmonic oscillator
\[
- \frac{d^2}{dx^2} + \left( \frac{1}{2} \varepsilon^2 - \beta \right)^2
\]
acting on \( \mathbb{R} \), behaves as \( \beta \to +\infty \) as \( \sqrt{2} \beta \). Hence, we can conclude, after dilation, that for sufficiently small \( \varepsilon \),
\[
\| \hat{u}'_s \|_2^2 + (1 - \lambda_r^{-3/4}) \| (\omega^2 - \lambda_r) \hat{u}_s \|_2^2 \geq 2 [1 - \lambda_r^{-3/4}]^{1/2} \lambda_r^{1/2} (1 - C \lambda_r^{-1}) \| \hat{u}_s \|_2^2 \geq 2(\lambda_r^{1/2} - \lambda_r^{-1/4}) \| \hat{u}_s \|_2^2.
\]
Substituting the above into (3.47) yields
\[
(\varepsilon^{-1} - \lambda_i^2 - 3\lambda_r^{-1/4}) \| \hat{u}_s \|_2^2 \leq C \left| \left( (\omega^2 - \lambda) \hat{u}_s, g \right) \right| + \frac{C}{\lambda_i} \| \mathbf{f} \|_2 \left( \lambda_r^{-1/2} \| \hat{u}_s \|_2 + \frac{\lambda_r^{-1}}{\lambda_i} \| \mathbf{f} \|_2 \right).\]
from which we conclude (recall that \( |\lambda_i| < (1 - 2\delta^4)^{1/2} \varepsilon^{-1/2} \)) that for sufficiently small \( \varepsilon \)
\[
\| \hat{u}_s \|_2^2 \leq C \varepsilon \left( \left| \left( (\omega^2 - \lambda) \hat{u}_s, g \right) \right| + \frac{\varepsilon \lambda_r + \lambda_r^{-1}}{\lambda_i^2} \| \mathbf{f} \|_2 \right).
\]
We note that (3.31) reads in this case
\[
\left| \left( (\omega^2 - \lambda) \hat{u}_s, g \right) \right| \leq C \left[ 1 + \frac{\varepsilon^{1/2} \lambda_r^{1/2}}{\lambda_i} \right] \| \mathbf{f} \|_2 + (\varepsilon^{-1/2} + \varepsilon^{1/2} \lambda_r^{1/2}) \| \hat{u}_s \|_2 \| \mathbf{f} \|_2
\]
(3.49)
Hence, consequently, this completes the proof of the proposition. λ which is correct without any limitation on the value of \( \lambda \). Proposition 3.2.

Remark 3.3. Notice that in this third case, since \( B_e \) is accretive, we have also

\[
\| u \|_2 \leq \frac{C}{\lambda_e} \| f \|_2
\]

which is correct without any limitation on the value of \( \lambda_e \) as in the statement of Proposition 3.2.
As a consequence of Proposition 3.2, we get under the same assumptions:

**Proposition 3.4.** Let

\[ S(\varepsilon, \delta) = \{ \lambda \in \mathbb{C} \setminus \mathbb{R}_+ \mid |\Im \lambda| < (1 - \delta)\varepsilon^{-1/2} \} . \]

Then, for any \( \delta > 0 \) there exists \( \varepsilon_0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \) we have

\[ \sigma(\hat{B}_\varepsilon) \cap S(\varepsilon, \delta) = \emptyset . \]

**Proof.** Proposition 3.2 establishes boundedness of \( u_s \) only. To prove boundedness of \( u \) one needs, therefore to prove, in addition, the boundedness of \( \hat{u}_d \) and \( \hat{u}_s \) for all \( \lambda \in S(\varepsilon, \delta) \). To this end we first observe that by (3.9c) it holds that

\[ \| \hat{u}_3 \|_2 \leq C|\lambda_i|^{-1} \left( 1 + \frac{1 + \varepsilon^{1/2}(\lambda_r)_+^{1/2}}{|\lambda_i|} \right) \| f \|_2 . \]  

(3.54)

To prove boundedness of \( \hat{u}_d \) we first observe that by (3.9a)-(3.9b)

\[ (\omega^2 - \lambda)\hat{u}_d - \hat{u}'_d = \hat{f}_1 - \hat{f}_2 , \]

from which we conclude that

\[ \| \hat{u}_d \| \leq \frac{1}{|\lambda_i|} (\| \hat{u}'_d \| + \| f \|) . \]

We now observe that by (3.44), (3.47), and (3.52) we have that

\[ \| \hat{u}'_s \|_2 \leq C \left( 1 + \frac{1 + \varepsilon^{1/2}(\lambda_r)_+^{1/2}}{|\lambda_i|} \right) \| f \|_2 . \]

Combining the above yields

\[ \| u \| \leq C \left( 1 + \frac{1}{|\lambda_i|} \right) \left( 1 + \frac{1 + \varepsilon^{1/2}(\lambda_r)_+^{1/2}}{|\lambda_i|} \right) \| f \|_2 , \]

(3.55)

or, equivalently, that for any \( \lambda \in \mathbb{C} \) satisfying \( 0 < |\lambda_i| \leq (1 - 2\delta^4)^{1/2}\varepsilon^{-1/2} \) it holds that

\[ \|(\hat{B}_\varepsilon - \lambda)^{-1}\| \leq C \left( 1 + \frac{1}{|\lambda_i|} \right) \left( 1 + \frac{1 + \varepsilon^{1/2}(\lambda_r)_+^{1/2}}{|\lambda_i|} \right) . \]  

(3.56)

We can deduce from (3.56) the following bound on \( \| \hat{B}_\varepsilon - \Lambda \|^{-1} \).

**Corollary 3.5.** For any \( \Lambda \in \mathbb{C} \) satisfying \( 0 < |\Lambda_i| \leq (1 - 2\delta^4)^{1/2} \) it holds that

\[ \| \hat{B}_\varepsilon - \Lambda \|^{-1} \leq C \varepsilon^{-\frac{1}{3}} \left( 1 + \varepsilon^{\frac{1}{4}} \frac{1}{|\Lambda_i|} \right) \left( 1 + \varepsilon^{\frac{1}{4}} \frac{1 + \varepsilon^{1/3}(\lambda_r)_+^{1/2}}{|\lambda_i|} \right) . \]  

(3.57)
3.3 Point spectrum

We now establish the existence of a point spectrum for $\tilde{\mathcal{B}}_\varepsilon$ for $|\lambda_i| > \varepsilon^{-1}/2$. As before, we can assume $\lambda_i > \varepsilon^{-1}/2$. Our main result, which is included in the statement of Theorem 1.3, is:

**Proposition 3.6.** For every $k \in \mathbb{N}$ there exist $\varepsilon_k > 0$ and $R_k > 0$, such that $B(\lambda_k, R_k \varepsilon^{\frac{1}{2}}) \cap \sigma(\tilde{\mathcal{B}}_\varepsilon) \neq \emptyset$ for all $0 < \varepsilon < \varepsilon_k$, where $\lambda_k = i\varepsilon^{-\frac{1}{2}} + \frac{2k-1}{2}(1+i)\varepsilon^{\frac{1}{4}}$.

3.3.1 Preliminary reduction

We begin with the following substitution

$$\tilde{u}_s = (\omega^2 - \lambda)^{1/2} v, \quad (3.58)$$

in (3.14) (for $f = 0$) to obtain $\mathcal{M}_\lambda v = 0$ with

$$\mathcal{M}_\lambda \overset{df}{=} -\frac{d^2}{d\omega^2} + \left[(\omega^2 - \lambda)^2 + \varepsilon^{-1} + \frac{2\omega^2 + \lambda}{(\omega^2 - \lambda)^2}\right] \quad (3.59)$$

Note that the last term $\frac{2\omega^2 + \lambda}{(\omega^2 - \lambda)^2}$ is, as $\lambda_i > \varepsilon^{-1/2}$, $C^\infty$ and bounded. Hence $\mathcal{M}_\lambda$ is a bounded perturbation of the anharmonic oscillator:

$$\mathcal{M}_\lambda^0 = -\frac{d^2}{d\omega^2} + (\omega^2 - \lambda)^2 + \varepsilon^{-1}. \quad (3.60)$$

Note that $\mathcal{M}_\lambda^0$, or the anharmonic oscillator, has been intensively studied [29] (in the form $-\frac{d^2}{d\omega^2} + \omega^2 + \beta\omega^4$) and later, in the above form, in [28, 16] for real values of $\lambda$.

It has been established (see [21] and references therein) that $\mathcal{M}_\lambda$ is for all $\lambda \not\in \mathbb{R}_+$, a closed operator whose domain satisfies

$$D(\mathcal{M}_\lambda) = \{u \in L^2(\mathbb{R}) \mid \mathcal{M}_\lambda u \in L^2(\mathbb{R})\} = \{u \in H^2(\mathbb{R}) \mid \omega^4 u \in L^2(\mathbb{R})\}, \quad (3.61)$$

and is maximally accretive.

We now observe the following:

**Lemma 3.7.** For any $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$

$$0 \in \sigma(\mathcal{M}_\lambda) \iff \varepsilon^{2/3} \lambda \in \sigma(\tilde{\mathcal{B}}_\varepsilon). \quad (3.62)$$

**Proof.** We first prove that $0 \in \sigma_p(\mathcal{M}_\lambda) \iff \lambda \in \sigma_p(\tilde{\mathcal{B}}_\varepsilon)$ (the point spectrum). Suppose that $u \in D(\tilde{\mathcal{B}}_\varepsilon)$ is an eigenfunction associated with $\lambda \in \sigma_p(\tilde{\mathcal{B}}_\varepsilon)$. Then, let

$$v(\omega) = \frac{1}{(\omega^2 - \lambda)^{1/2}} \mathcal{F}(u_1 + u_2)(\omega).$$

Clearly, $v \in L^2(\mathbb{R})$ and, by the construction of $\mathcal{M}_\lambda$ we have that $\mathcal{M}_\lambda v = 0$. Consequently, $v \in D(\mathcal{M}_\lambda)$ and hence $0 \in \sigma_p(\mathcal{M}_\lambda)$. (Note that for any $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ the spectrum of $\mathcal{M}_\lambda$ is discrete.)
Conversely, suppose that $0 \in \sigma_p(M_\lambda)$ and let $v \in D(M_\lambda)$ denote an associated eigenfunction. Set
\[ \hat{u}_s = (\omega^2 - \lambda)^{1/2} v. \] (3.63)
As $v \in D(M_\lambda)$ it follows from (3.61) that $\hat{u}_s \in L^2(\mathbb{R})$. We now define, as in (3.9c) and (3.12)
\[ \hat{u}_d = \frac{1}{\omega^2 - \lambda} \frac{d \hat{u}_s}{d\omega}, \quad \hat{u}_3 = \frac{1}{\sqrt{2}} \frac{\omega}{\omega^2 - \lambda} \frac{\partial}{\partial \omega} \hat{u}_s. \] (3.64)
From (3.17) (with $g = 0$) we can conclude that $\hat{u}_d \in L^2(\mathbb{R})$. As $\hat{u}_3 \in L^2(\mathbb{R})$ we may conclude that
\[ \mathbf{u} = \begin{bmatrix} \frac{1}{2} \mathcal{F}^{-1}(\hat{u}_s + \hat{u}_d) \\ \frac{1}{2} \mathcal{F}^{-1}(\hat{u}_s - \hat{u}_d) \\ \mathcal{F}^{-1}(\hat{u}_3) \end{bmatrix} \in L^2(\mathbb{R}). \]

Consequently, since $(\check{B}_\epsilon - \lambda)\mathbf{u} = 0$ we obtain that $\mathbf{u} \in D(\check{B}_\epsilon)$ and hence $\lambda \in \sigma(\check{B}_\epsilon)$. Since $\epsilon^{2/3} \check{B}_\epsilon$ is obtained from $B_\epsilon$ via unitary dilation and rotation we obtain that $\epsilon^{2/3} \lambda \in \sigma(B_\epsilon)$. The proof of (3.62) is now completed by using Proposition 2.15 together with the fact that $(-\epsilon^2 \Delta + ix - \lambda)^{-1}$ is compact.

### 3.3.2 Heuristics and dilation

We begin by making some intuitive observations on $M_\lambda^0$. To this end we return to our initial spectral parameter $\Lambda$ (see (3.6)):
\[ \lambda = \epsilon^{-\frac{1}{2}} \Lambda, \] (3.65)
which, when substituted into (3.60), yields
\[ \check{M}_\Lambda^0 := -\frac{d^2}{d\omega^2} + \epsilon^{-1}(1 + \Lambda^2) - 2\Lambda \epsilon^{-\frac{1}{2}} \omega^2 + \omega^4. \]
Here and in the following we use the term “critical value” for any $\Lambda \in \mathbb{C}$ for which $\ker M_\lambda^0 \neq \{0\}$. The above form suggests it would be plausible to look for critical values near $\Lambda = i$ (so that $\epsilon^{-1}(1 + \Lambda^2)$ is of the same order of $2\Lambda \epsilon^{-\frac{1}{2}} \omega^2$). Hence, we set
\[ \Lambda = i + \epsilon^{\frac{3}{4}} \mu \] (3.66a)
and
\[ \omega = \epsilon^{\frac{1}{8}} \tilde{\omega} \] (3.66b)
to obtain, after multiplication by $\epsilon^{\frac{1}{2}}$,
\[ \check{M}_\mu^0 := -\frac{d^2}{d\tilde{\omega}^2} - 2i\tilde{\omega}^2 + 2i\mu + \epsilon^{\frac{3}{4}} \mu^2 - 2\mu \epsilon^{3/4} \tilde{\omega}^2 + \epsilon^{\frac{1}{2}} \tilde{\omega}^4. \] (3.66c)
Neglecting small terms (in the limit $\epsilon \to 0$) we thus expect $-2i\mu$ to be an eigenvalue of the complex harmonic oscillator $-\frac{d^2}{d\tilde{\omega}^2} - 2i\tilde{\omega}^2$ (cf. [17]). We now apply the previous rescaling (3.65) and (3.66) to $M_\lambda$ to obtain, (for convenience we return to the parameter $\epsilon = \epsilon^{\frac{3}{4}}$)
\[ \check{M}_{\mu, \epsilon} := -\frac{d^2}{d\tilde{\omega}^2} - 2i\tilde{\omega}^2 + 2i\mu + \epsilon\mu^2 - 2\mu \epsilon^{3/4} \tilde{\omega}^2 + \epsilon^{\frac{3}{4}} \tilde{\omega}^4 + \epsilon \Phi(\epsilon, \tilde{\omega}, \mu), \] (3.67)
where
\[ \Phi(\epsilon, \tilde{\omega}, \mu) := \frac{(i + 2\epsilon\tilde{\omega}^2 + \epsilon\mu)}{(-i + \epsilon^2 - \epsilon\mu)^2}. \] (3.68)

Clearly,

**Lemma 3.8.** For any \( \Lambda \in \mathbb{C} \setminus \mathbb{R}_+ \) such that \( \Lambda = i + \epsilon\mu \)
\[ 0 \in \sigma(\hat{\mathcal{M}}_{\mu,\epsilon}) \iff \Lambda \in \sigma(\mathcal{B}_{\epsilon}). \] (3.69)

Hence it remains to find critical values \( \mu \in \mathbb{C} \) such that \( \hat{\mathcal{M}}_{\mu,\epsilon} \) has a non trivial kernel.

### 3.3.3 Formal asymptotics

By (3.67), \( \hat{\mathcal{M}}_{\mu,\epsilon} \) is close to the complex harmonic oscillator. In the following we present a formal expansion relying on that intuition.

**Proposition 3.9.** For any \( k \in \mathbb{N}^* \), there exists sequences \( \{\mu_{k,\ell}\}^{\infty}_{\ell=1} \subset \mathbb{C} \) and \( \{f_{k,\ell}\}^{\infty}_{\ell=1} \subset \mathcal{S}(\mathbb{R}) \) such that, as \( \epsilon \to 0 \),
\[ \mu_k \sim \sum_{\ell>0} \epsilon^\ell \mu_{k,\ell}, \] (3.70a)
\[ f_k = \sum_{\ell\geq 0} \epsilon^\ell f_{k,\ell}, \] (3.70b)
and
\[ \hat{\mathcal{M}}_{\mu_{k,\epsilon}} f_k \sim 0. \] (3.70c)

In particular, we have
\[ \hat{\mathcal{M}}_{\mu_{k,\epsilon}} f_{k,0} = \epsilon r^{(0)}_k (\tilde{\omega}, \epsilon), \]
where \( \omega \mapsto \exp(e^{i\frac{\pi}{4}}\omega^2/\sqrt{2}) r^{(0)}_k (\omega, \epsilon) \) is polynomially bounded.

For \( \epsilon = 0 \), we have
\[ \hat{\mathcal{M}}_{\mu_{k,0}} f_{k,0} = 0. \]

**Proof.** For the leading order we have by (3.70c)
\[ \hat{\mathcal{M}}_{\mu_{k,0}} f_{k,0} = \left( -\frac{d^2}{d\omega^2} - 2i\omega^2 + 2i\mu_{k,0} \right) f_{k,0} = 0. \] (3.71)

The eigenvalues of the complex harmonic oscillator are well-known (cf. for instance [17, Proposition 14.12]). Hence,
\[ \mu_{k,0} = \frac{(2k - 1)}{\sqrt{2}} e^{i\frac{\pi}{4}}. \] (3.72)

We can normalize the corresponding eigenmode by setting
\[ \int |f_{k,0}|^2(\tilde{\omega})d\tilde{\omega} = 1. \]
The coefficient of $\epsilon$ assumes the form

$$M_{\mu_k,0}f_{k,1} = -2i\mu_{k,1}f_{k,0} - (\mu_{k,0}^2 - 2\mu_{k,0}\omega^2 + \omega^4 - 1)f_{k,0}.$$  \hspace{1cm} (3.73)

A necessary and sufficient condition for the existence of solution for (3.73) is obtained by taking the inner product with $\bar{f}_{k,0}$, yielding

$$\mu_{k,1} = \frac{\int_{\mathbb{R}}([\omega^2 - \mu_{k,0}]^2 - 1)f^2_{k,0} d\omega}{2i\int_{\mathbb{R}}f^2_{k,0} d\omega}.$$  \hspace{1cm} (3.74)

Under (3.74), there exists a unique solution $f_{k,1}$ of (3.73) if we add the condition

$$\int_{\mathbb{R}} f_{k,0}(\omega)f_{k,1}(\omega) d\omega = 0.$$

We can then continue by recursion, to prove (3.72) for any order. To prove that $\exp(e^{i\frac{\pi}{4}}\omega^2/\sqrt{2})r_k^{(0)}(\omega, \epsilon)$ is polynomially bounded we use the well-known properties of Hermite functions to conclude that

$$\left| \exp \left( \frac{1}{\sqrt{2}} \exp \left( i\frac{\pi}{4} \right) \omega^2 \right) f_{k,0} \right| \leq C|\omega|^k.$$  

Then, by direct substitution we obtain that

$$\left| \exp \left( \frac{1}{\sqrt{2}} \exp \left( i\frac{\pi}{4} \right) \omega^2 \right)r_k^0 \right| \leq C|\omega|^{k+4}.$$

We have formally established that, for sufficiently small $\epsilon$, $\sigma(B_\epsilon)$ should contain a sequence of points $\Lambda_k \sim i + \epsilon \mu_k$.

In the following, we attempt to rigorously prove these formal estimates. Two of the difficulties we face in the forthcoming rigorous analysis are:

1. It involves a non-linear spectral problem.
2. It involves the spectral analysis of a non-selfadjoint operator.

Non-self-adjointness prohibits the use of the spectral theorem to estimate error terms. To mitigate this problem we use analytic dilation so that the leading order operator, converts from the complex harmonic oscillator into the real harmonic oscillator.

### 3.3.4 Analytic dilation.

We begin by recalling from (3.59)

$$M_\lambda \overset{def}{=} -\frac{d^2}{d\omega^2} + \left[(\omega^2 - \lambda)^2 + e^{-1} + \frac{2\omega^2 + \lambda}{(\omega^2 - \lambda)^2}\right].$$

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Let $\theta \in \mathbb{R}$. We introduce the unitary dilation operator

$$u \mapsto (U(\theta)u)(x) = e^{-\theta/2}u(e^{-\theta}x).$$

(3.75)

We then define

$$\mathcal{M}_{\lambda, \theta} := U(\theta)^{-1}M_{\lambda} U(\theta) = -e^{-2\theta} \frac{d^2}{d\omega^2} + V_{\lambda, \theta},$$

(3.76)

where

$$V_{\lambda, \theta} = (e^{2\theta} \omega^2 - \lambda)^2 + \varepsilon^{-1} + \frac{2e^{2\theta} \omega^2 + \lambda}{(e^{2\theta} \omega^2 - \lambda)^2}.$$  

Similarly, we define

$$\mathcal{M}_{\lambda, \theta}^0 := -e^{-2\theta} \frac{d^2}{d\omega^2} + V_{\lambda, \theta}^0,$$

(3.77)

where

$$V_{\lambda, \theta}^0 = (e^{2\theta} \omega^2 - \lambda)^2 + \varepsilon^{-1}.$$  

$\mathcal{M}_{\lambda, \theta}^0$ can be extended to the strip $|\Im \lambda| < 2\pi$, using (3.77), as a globally quasi-elliptic operator (see [20], [22]) whose principal term (in the sense of this theory) is $-e^{-2\theta} \frac{d^2}{d\omega^2} + e^{4\theta} \omega^4$.

It is then a rather standard matter to show that its spectrum is independent of $\theta$. Hence it remains necessary to control the effect of the "perturbation term"

$$\phi(\omega, \lambda, \theta) = \frac{2e^{2\theta} \omega^2 + \lambda}{(e^{2\theta} \omega^2 - \lambda)^2}.$$  

We observe that if $|\Im \lambda| < \frac{2\pi}{8} + \frac{\pi}{2\theta}$ and $\pi/4 + \varepsilon_0 < \arg \lambda < 7\pi/4 - \varepsilon_0$ (for some $\varepsilon_0 > 0$) and $|\lambda| > 0$, then $(\omega, \lambda, \theta) \mapsto \phi(\omega, \lambda, \theta)$ remains bounded. Consequently, $\mathcal{M}_{\lambda, \theta}$ is sectorial and possesses a compact resolvent. We now use standard arguments: We observe that for $\theta \in \mathbb{R}$, since $U(\theta)$ is a unitary operator, $\sigma(\mathcal{M}_{\lambda, \theta})$ is independent of $\theta$. By analytic continuation it must also be constant for $-\frac{\pi}{8} - \frac{\pi}{2\theta} < \Im \lambda < \frac{\pi}{8} + \frac{\pi}{2}$ (see [25, Section VI.1.3], [22, Section 12] or [3]). Hence we have obtained, for $\varepsilon_0 = \frac{\pi}{8}$,

**Proposition 3.10.** Let $\lambda \in \mathbb{C} \setminus \{0\}$ satisfy $3\pi/8 < \arg \lambda < 13\pi/8$, and let $\theta$ be such that $|\Im \lambda| < \frac{\pi}{10}$. Then,

$$\sigma(\mathcal{M}_{\lambda, \theta}) = \sigma(\mathcal{M}_\lambda)$$

(3.78)

In particular, it holds that

$$0 \in \sigma(\mathcal{M}_{\lambda, \theta}) \iff 0 \in \sigma(\mathcal{M}_\lambda).$$

(3.79)

Setting $\theta = \frac{\pi}{8}$, we obtain

**Corollary 3.11.** For all $\lambda \in \mathbb{C} \setminus \{0\}$ satisfying $3\pi/8 < \arg \lambda < 13\pi/8$, it holds that

$$0 \in \sigma(\mathcal{M}_{\lambda, \frac{\pi}{8}}) \iff \varepsilon^{1/2} \lambda \in \sigma(\mathcal{B}_\varepsilon).$$

(3.80)

Finally, we apply (3.65) and (3.66) to $\mathcal{M}_{\lambda, \frac{\pi}{8}}$ to obtain (using the original parameter $\varepsilon$) the operator

$$\tilde{L}_{\mu, \varepsilon} := -\frac{d^2}{ds^2} + 2s^2 + 2e^{3i\frac{\pi}{8}} \mu + \varepsilon e^{i\frac{\pi}{8}} \mu^2 - 2i\mu \varepsilon s^2 + e^{3i\frac{\pi}{8}} \varepsilon s^4 + \varepsilon e^{i\frac{\pi}{8}} \Phi(\varepsilon, e^{i\frac{\pi}{8}} s, \mu).$$

(3.81)

We search for $\mu$ values for which $\tilde{L}_{\mu, \varepsilon}$ has a non trivial kernel.
3.3.5 Weighted estimates

**Lemma 3.12.** Let $r > 0$, $\mu \in B(0, r)$ and $(g, w)$ a pair in $L^2(\mathbb{R}) \times D(\hat{\mathcal{L}}_{\mu, \epsilon})$ such that $\hat{\mathcal{L}}_{\mu, \epsilon} w = g$. Suppose that $\|e^{i|\cdot|}g\|_2 < \infty$. Then, there exist $\epsilon_0 > 0$ and $C > 0$ such that for all $0 < \epsilon < \epsilon_0$ it holds that

$$\|e^{i|\cdot|}w\|_2 \leq C(\|w\|_2 + \|e^{i|\cdot|}g\|_2).$$

**Proof.** Let $\varphi(s) = \sqrt{1 + s^2}$ and $\varphi_n(s) = \tilde{\eta}(s/n)\sqrt{1 + s^2}$ (where $\tilde{\eta}$ is defined in (3.32)). Integration by parts yields that for any $0 < \alpha > 0$, there exists $C_\alpha(r) > 0$ such that, for any $n \geq 1$

$$\Re \left( e^{-3\pi/8} \langle e^{2\varphi_n(s)} \hat{\mathcal{L}}_{\mu, \epsilon} w, w \rangle \right) \geq \cos(3\pi/8) \left[ \|w\|_2^2 + 2\|w\|_2 + \|e^{\varphi_n} w\|_2^2 - \alpha \|e^{\varphi_n} w\|_2^2 \right] - C_\alpha \|w\|_2^2.$$

To obtain the second inequality we use the uniform boundedness of $\Phi(\epsilon, e^{i\frac{\pi}{8}}s, \mu)$ (see (3.85)). To obtain the last inequality we choose first $0 < \alpha > \cos(3\pi/8)/2$ and then $\epsilon_0$ small enough so that $C_\alpha \epsilon < \cos(3\pi/8)$ for all $0 < \epsilon < \epsilon_0$. We can now conclude that there exist $C > 0$ and $C_1 > 0$ such that

$$\|e^{\varphi_n} g\| \|e^{\varphi_n} w\| \geq C \|e^{\varphi_n} w\|^2 - C_1 \|w\|^2.$$

Finally, we can take the limit as $n \to +\infty$.

**Lemma 3.13.** Let $k \in \mathbb{N}$ and $\mu_{k,0}$ be given by (3.70). Let further $g \in L^2(\mathbb{R}, e^{i|\cdot|})$ and $(\mu, w) \in \partial B(\mu_{k,0}, r) \times D(\hat{\mathcal{L}}_{\mu, \epsilon})$ satisfy

$$\hat{\mathcal{L}}_{\mu, \epsilon} w = g.$$ (3.83)

Then, there exist positive $\epsilon_0$, $r_0$, $C_0$ and $C$ such that for all $(\epsilon, r)$ satisfying $0 < \epsilon \leq \epsilon_0$, $C_0 \epsilon \leq r \leq r_0$, it holds that

$$\|w\|_2 \leq C \frac{C}{r} (\|\epsilon^{i|\cdot|}g\|_2 + \|g\|_2).$$ (3.84)

**Proof.** Using (3.81), we rewrite (3.83) in the form

$$\left( -\frac{d^2}{ds^2} + 2s^2 + 2e^{3i\frac{\pi}{8}} \mu \right) w = g + \epsilon R(s, \epsilon, \mu) w,$$ (3.85)

where

$$R(s, \epsilon, \mu) := -2i \mu s^2 + e^{3i\frac{\pi}{8}} s^4 + e^{i\frac{\pi}{8}} \mu^2 + e^{i\frac{\pi}{8}} \Phi(\epsilon, e^{i\frac{\pi}{8}}s, \mu).$$ (3.86)

Using the spectral theorem for the harmonic oscillator, we immediately obtain that for $\epsilon$ and $r$ small enough we have

$$\|w\|_2 \leq C \frac{C}{r} (\|g\|_2 + \|w\| + \epsilon \|e^{\varphi} w\|).$$
Using Lemma 3.12 we then deduce
\[ \|w\|_2 \leq \frac{C}{r}(\|g\|_2 + \epsilon\|w\| + \epsilon\|e^\epsilon g\|). \]

Assuming that \(\epsilon/r\) is small enough (which is obtained via a suitable choice of \(C_0\)) we obtain (3.84). \(\blacksquare\)

### 3.3.6 Proof of Proposition 3.6

**Lemma 3.14.** For every \(k \in \mathbb{N}\) there exist \(\epsilon_k > 0\) and \(R_k > 0\), such that for all \(0 < \epsilon < \epsilon_k\) there exists \(\mu_k'(\epsilon) \in B(\mu_k, 0)\) for which \(\hat{L}_{\mu_k'(\epsilon)}\) has a non trivial kernel.

**Proof.** Let \(w_k\) denote the \(k\)-th normalized eigenfunction of the harmonic oscillator \(-d^2/ds^2 + 2s^2\), corresponding to the eigenvalue \((2k - 1)/\sqrt{2}\). By (3.81) we have that
\[ g_k := \hat{L}_{\mu, \epsilon}w_k = [(2k - 1)\sqrt{2} + 2e^{3i\pi/4}\mu]w_k + \epsilon R(s, \epsilon, \mu)w_k. \]

We now define
\[ \mu = \mu_{k,0} + \rho e^{i\theta}, \quad \nu(\mu, k) = [(2k - 1)\sqrt{2} + 2e^{3i\pi/4}\mu], \]
where \(\mu_{k,0}\) is defined in (3.72), to obtain
\[ \hat{L}_{\mu, \epsilon}w_k = \nu(\mu, k, \theta)w_k + \epsilon R(s, \epsilon, \mu)w_k. \quad (3.87) \]

Note that
\[ \nu(\mu, k, \theta) = 2\rho e^{3i\pi/4 + \theta} + \mathcal{O}(\epsilon). \]

Suppose, for a contradiction, that \(\hat{L}_{\mu, \epsilon}\) is invertible for all \(\mu \in B(\mu_{k,0}, r)\). Then in this ball, \(\mu \mapsto \hat{L}_{\mu}^{-1}\) is an holomorphic family of compact operators acting on \(L^2(\mathbb{R})\). By (3.87), we may write
\[ \frac{1}{\nu(\mu, k, \theta)}w_k = (\hat{L}_{\mu, \epsilon})^{-1}w_k + \frac{\epsilon}{\nu(\mu, k, \theta)}(\hat{L}_{\mu, \epsilon})^{-1}R(s, \epsilon, \mu)w_k. \quad (3.88) \]

We now take the scalar product with \(w_k\) and integrate along a circle of radius \(r/2\) centered at \(\mu_{k,0}\), assuming that
\[ r \geq C\epsilon \]
for some, sufficiently large, \(C > 0\),
\[ \int_{\partial B(\mu_{k,0}, r/2)} \langle w_k, \hat{L}_{\mu, \epsilon}w_k \rangle d\mu - \frac{1}{2}e^{-3i\pi/4} \leq \frac{C\epsilon}{r} \int_{\partial B(\mu_{k,0}, r/2)} \langle w_k, \hat{L}_{\mu, \epsilon}^{-1}Rw_k \rangle d\mu. \quad (3.89) \]

We now turn to estimate \(\|\hat{L}_{\mu, \epsilon}^{-1}Rw_k\|_2\). To this end we use (3.84) to establish that
\[ \|\hat{L}_{\mu, \epsilon}^{-1}Rw_k\|_2 \leq \frac{C}{r}(\epsilon\|e^{1/2}Rw_k\|_2 + \|Rw_k\|_2). \quad (3.90) \]
Using the well-known exponential decay of $w_k(\omega)$ as $|\omega| \to \infty$ (an Hermite function $H$) and the polynomial boundedness of $R$ we obtain
\[
\| \widehat{L}_{\mu, \epsilon}^{-1} R w_k \|_2 \leq \frac{C_k}{r}.
\] (3.91)

Finally, by the assumed holomorphy of $\widehat{L}_{\mu, \epsilon}^{-1}$,
\[
\frac{1}{2} = \left| \int_{\partial B(\mu_k, r/2)} \langle w_k, \widehat{L}_{\mu, \epsilon}^{-1} w_k \rangle \, d\mu - \frac{1}{2} e^{-i\pi/4} \right| \leq \hat{C} \frac{\epsilon}{r}.
\] (3.92)

Letting $r = R_k \epsilon$ leads to a contradiction for sufficiently large $R_k$ and $0 < \epsilon \leq \epsilon_k$.

This completes the proof of the lemma.

Remark 3.15. By (3.69) and (3.80) it follows that for $\Lambda = i + \epsilon \mu$
\[
0 \in \sigma(\widehat{L}_{\mu, \epsilon}) \iff \Lambda \in \sigma(B_{\epsilon}).
\]

By Lemma 3.14 we then obtain that for each $k \in \mathbb{N}$ there exists $R_k > 0$ and $\Lambda_k \in B(i + \mu_k, 0 \epsilon, R_k \epsilon^2)$ which belongs to $\sigma(B_{\epsilon})$. We can now easily conclude (1.8) from (3.6).

3.4 Reminder on the complex harmonic oscillator

We recall that the complex harmonic oscillator $h$, constitutes the principal part of $\mathcal{M}_{\mu, \epsilon}$. We recall that
\[
h = -\frac{d^2}{d\xi^2} - 2i\xi^2,
\] (3.93)

is defined on $D(h) = H^2(\mathbb{R}) \cap L^2(\mathbb{R}; \xi^2 \, d\xi)$, and that its spectrum is
\[
\sigma(h) := \mathbb{N}_{-\pi/4} = \{ \sqrt{2} e^{-i\pi/4}(2k - 1) \}_{k=1}^{\infty}.
\] (3.94)

Moreover, its numerical range is $C_{++} = \{ z \in \mathbb{C}, \Re z \geq 0, \Im z \leq 0 \}$.

Proposition 3.16. The following estimates for the resolvent of $h$ hold:

1. If $z \notin C_{++}$, then $z \in \rho(h) := \mathbb{C} \setminus \sigma(h)$ and
\[
\| (h - z)^{-1} \| \leq d(z, C_{++})^{-1}.
\] (3.95a)

2. For any compact set $K \in \mathbb{C}$, there exists a constant $C_K$ such that, for any $z \in K \cap \rho(h)$ we have
\[
\| (h - z)^{-1} \| \leq C_K \left( \frac{1}{d(z, \sigma(h))} + 1 \right).
\] (3.95b)

3. There exist $\delta_0 > 0$ and $B_0 > 0$ such that for all $z \in \rho(h)$ such that
\[
\Re z \geq 0 \text{ and } |\Im z| \leq \delta_0(\Re z)^3,
\] (3.95c)

it holds that
\[
\| (h - z)^{-1} \| \leq B_0 \left( \frac{1}{d(z, \sigma(h))} + \frac{1}{1 + |\Re z|^{1/3}} \right).
\] (3.95d)
Proof.
The first item is a consequence of the sectoriality of \( \mathfrak{h} \) (See [17, Remark 14.14]).
For the second item, by [17, Proposition 14.12], we may use Riesz-Schauder theory
to conclude that for sufficiently large \( N \) we have
\[
(\mathfrak{h} - z)^{-1} = \sum_{n=1}^{N} \frac{\Pi_n}{(\sqrt{2}e^{-i\pi/4}(2n-1) - z)} + T_N(z),
\]
where \( \Pi_n \) denotes the projection on the \( n \)‘th eigenfunction of \( \mathfrak{h} \) and \( T_N(z) \) is holomorphic in \( K \).
Finally, we prove the third item. To this end, let \( v \in D(\mathfrak{h}) \) and \( g \in L^2(\mathbb{R}) \) satisfy
\[
(\mathfrak{h} - z)v = g.
\]
Applying a Fourier transform to (3.93) yields
\[
\left( -2\frac{d^2}{dx^2} + i(x^2 - \Re z) \right) \hat{v} = i\hat{g} + \Im z \hat{v}. \tag{3.96}
\]
Using [17, Proposition 14.13] yields that there exists \( C > 0 \) such that for \( \Re z \geq C \)
we have
\[
\|v\|_2 \leq \frac{C}{\Re z^3} (\|g\|_2 + |\Im z||v||_2). \tag{3.97}
\]
By (3.95c) we can conclude that for sufficiently small \( \delta_0 \)
\[
\|v\|_2 \leq \frac{C}{\Re z^3} \|g\|_2. \tag{3.97}
\]
For \( \Re z \leq C \) it holds that \( |\Im z| \leq C^{-\delta_0}, \) and hence \( z \) belongs to a compact set in \( \mathbb{C} \)
and one can conclude the proof of proposition from (3.95b).

3.5 Application to resolvent estimates
We continue the analysis of the spectral properties of \( \mathcal{B}_\epsilon \) by obtaining resolvent
estimates for \( \mathcal{B}_\epsilon \) in a set in the form
\[
\mathcal{V}(\epsilon, g) := \{ \Lambda \in \mathbb{C} , \Lambda_i \geq 1/2 , \Lambda_r \leq g \epsilon \}.
\]
As \( \Lambda = i + \epsilon \mu \) it holds that
\[
\Lambda_r = \epsilon \mu_r , \Lambda_i = 1 + \epsilon \mu_i .
\]
Equivalently we may write,
\[
\Omega(\epsilon, g) = \left\{ \mu \in \mathbb{C} , \mu_i \geq -\frac{1}{2\epsilon} , \mu_r \leq \Re \epsilon \right\},
\]
where \( \mu_r = \Re \mu \) and \( \mu_i = \Im \mu \).
We begin by rewriting (3.67) in the form
\[
\mathcal{M}_{\mu, \epsilon} := -\frac{d^2}{d\omega^2} - 2i(1+\epsilon \mu_i)\omega^2 + i[2\mu_r + i\mu_i(2+\epsilon \mu_i - 2i\epsilon \mu_r)] + \epsilon (\omega^2 - \mu_r)^2 + \epsilon \Phi(\epsilon, \omega, \mu) ,
\]
We further apply the rescaling
\[ \xi = \tilde{\omega} [1 + \epsilon \mu_i]^{1/4} \] (3.98)
to obtain, after division by \([1 + \epsilon \mu_i]^{1/2}\),
\[ M_{\mu, \epsilon} = -\frac{d^2}{d\xi^2} - 2i\xi^2 - z_0 + \frac{\epsilon}{[1 + \epsilon \mu_i]^{1/2}} \left( \left[ (1 + \epsilon \mu_i)^{-1/2} \xi^2 - \mu_r \right]^2 + \Phi(\epsilon, [1 + \epsilon \mu_i]^{-1/4} \xi, \mu) \right), \] (3.99)
where
\[ z_0(\mu, \epsilon) := \frac{\mu_i (2 + \epsilon \mu_i) - 2i \mu_r (1 + \epsilon \mu_i)}{[1 + \epsilon \mu_i]^{1/2}} \] (3.100)
Note that for \( \mu \in \Omega(\epsilon, \varrho) \cap \{ \mu_i \geq 16 \varrho^3 \delta_0^{-1} \} \cap \{ |\mu_r| \leq \varrho \}, \) we have,
\[ |\Im z_0|^3 \leq 8 \varrho^3 \left( \frac{1}{\mu_i} + \epsilon \right) |\Re z_0|. \] (3.101)
We then have

**Lemma 3.17.** For any \( \varrho > 0 \), there exist \( \epsilon_0 > 0 \) and \( B_1 > 0 \) such that if \( \epsilon \in (0, \epsilon_0] \) and \( \mu \in \Omega(\epsilon, \varrho) \), then
\[ \| (h - z_0)^{-1} \| \leq B_1 \left( \frac{1}{d(z_0, \sigma(h))} + \frac{1}{1 + |\Re z_0|^{1/3}} \right). \] (3.102)

**Proof.** The lemma easily follows Proposition 3.16.

For \( \mu_i > 16 \varrho^3 \delta_0^{-1} \) it follows that \( \Re z_0 > 0 \) and that there exists \( \epsilon_0(\delta_0, \varrho) \) such that for \( 0 < \epsilon \leq \epsilon_0(\varrho, \delta_0) \), we have
\[ 8 \varrho^3 \left( \frac{1}{\mu_i} + \epsilon \right) \leq \delta_0. \]
Consequently, we may conclude (3.102) from (3.95c) in the case \( |\mu_r| \leq \varrho \). In the case \( \mu_r < -\varrho \) we use (3.95a). Hence, it remains to treat the case when \( \epsilon \mu_i \in (-\varrho, \epsilon 16 \varrho^3 \delta_0^{-1}) \). In this case \( \Im z_0 \) is bounded and \( \Re z_0 \) has the sign of \( \mu_i \). If \( \mu_i \geq -C \), for some \( C > 0 \), then we can apply (3.95b). Otherwise when \( \mu_i < -C \) we may use (3.95a) once again to complete the proof. 

**Lemma 3.18.** Let for \( R > 0 \) and \( \epsilon > 0 \),
\[ \tilde{\Omega}(\epsilon, R) = \{ z \in \mathbb{C} | d(z, \sigma(h)) \geq Re \}. \] (3.103)
Then, for all \( \varrho > 0 \), there exist \( R_0 > 0 \) and \( C > 0 \) such that, for all \( R_0 < R < \epsilon^{-1} \), \( \mu \in \Omega(\epsilon, \varrho) \), for which \( z_0(\mu, \epsilon) \in \tilde{\Omega}(\epsilon, R) \) and for every pair \( (\tilde{v}, \tilde{g}) \in D(M_{\mu, \epsilon}) \times L^2(\mathbb{R}; (\xi^2 + 1)^2 d\xi) \) satisfying
\[ M_{\mu, \epsilon} \tilde{v} = \tilde{g}, \] (3.104)
we have
\[ \| \tilde{v} \|_2 \leq \frac{C}{R\epsilon} \left( \| \tilde{g} \|_2 + \epsilon \| \xi^2 \tilde{g} \|_2 \right). \] (3.105)
Proof. The proof is based on presenting $\mathcal{M}_{\mu,\epsilon}$, given by (3.99), as a perturbation of $\mathfrak{h}$. We preliminary observe that by global estimates for the quartic oscillator (see for example Helffer-Robert [20, 21]) $\xi^6 \bar{v}, \xi^4 \bar{v}'$, and $\xi^2 \bar{v}''$ belong to $L^2$, if $\xi^2 \bar{g} \in L^2$. Using the fact that $1 + \epsilon \mu_i = \Lambda_i$ we may write

$$\mathcal{M}_{\mu,\epsilon} = \mathfrak{h} - z_0 + \epsilon \Lambda_i^{-\frac{1}{2}} (\Lambda_i^{-\frac{1}{2}} \xi^2 - \mu_r)^2 + \epsilon \Lambda_i^{-\frac{1}{2}} \Phi.$$  

(3.106)

We first attempt to estimate the effect of the perturbation term $\epsilon \Lambda_i^{-1/2} (\Lambda_i^{-\frac{1}{2}} \xi^2 - \mu_r)^2$. To this end we first observe that

$$\mathfrak{h} - z_0 = - \frac{d^2}{d\xi^2} - \Re z_0 - 2i \Lambda_i \frac{1}{2} (\Lambda_i^{-\frac{1}{2}} \xi^2 - \mu_r).$$

Thus, to estimate the perturbation term, we evaluate, having in mind that $(\Lambda_i^{-1/2} \xi^2 - \mu_r)^3 \bar{v} \in L^2$, the quantity $-\Lambda_i^{1/2} \Re \langle (\Lambda_i^{-1/2} \xi^2 - \mu_r)^3 \bar{v}, \mathcal{M}_{\mu,\epsilon} \bar{v} \rangle$ to obtain

$$- \Lambda_i^{1/2} \Re \langle (\Lambda_i^{-1/2} \xi^2 - \mu_r)^3 \bar{v}, \mathcal{M}_{\mu,\epsilon} \bar{v} \rangle = 2 \Lambda_i \Re ((\Lambda_i^{-1/2} \xi^2 - \mu_r)^2 \bar{v}, \bar{g}) - 6 \Re \langle (\Lambda_i^{-1/2} \xi^2 - \mu_r)^3 \bar{v}, \bar{v} \rangle - \epsilon \Re \langle (\Lambda_i^{-1/2} \xi^2 - \mu_r)^3 \bar{v}, \Phi(\epsilon, \Lambda_i^{-1/4} \xi, \mu) \bar{v} \rangle,$$

implying that

$$2 \Lambda_i \Re ((\Lambda_i^{-1/2} \xi^2 - \mu_r)^2 \bar{v}, \bar{g}) = \Lambda_i^{1/2} \Re ((\Lambda_i^{-1/2} \xi^2 - \mu_r)^2 \bar{v}, \bar{g}) \leq \Lambda_i^{1/2} \Re (\Lambda_i^{-1/2} \xi^2 - \mu_r)^2 \bar{v} \parallel_2 \Re (\Lambda_i^{-1/2} \xi^2 - \mu_r) \bar{g} \parallel_2$$

$$+ 6 \Re (\Lambda_i^{-1/2} \xi^2 - \mu_r)^2 \bar{v} \parallel_2 \Re \xi^2 \bar{v}',$$

$$+ \epsilon \Re (\Lambda_i^{-1/2} \xi^2 - \mu_r)^2 \bar{v} \parallel_2 \Re \Phi(\epsilon, \Lambda_i^{-1/4} \xi, \mu) (\Lambda_i^{-1/2} \xi^2 - \mu_r) \bar{v} \parallel_2.$$  

Next we attempt to bound $\Phi(\epsilon, \Lambda_i^{-1/4} \xi, \mu)$ under the assumptions of the lemma. Recall from (3.68) that

$$\Phi(\epsilon, \omega, \mu) := \frac{(i + 2 \epsilon \omega^2 + \epsilon \mu)}{(-i + \epsilon \omega^2 - \epsilon \mu)^2}.  

(3.107)$$

For all $\Lambda \in \mathcal{V}(\epsilon, \varrho)$ it holds that

$$\Re + \tau \rightarrow \frac{(2 \tau + \Lambda)}{(\tau - \Lambda)^2},$$

is uniformly bounded. Consequently, uniform boundedness of $\Phi(\epsilon, \omega, \mu)$ follows as well. Hence, there exists $C > 0$ such that

$$\Re ((\Lambda_i^{-1/2} \xi^2 - \mu_r)^2 \bar{v}, \bar{g}) \leq \frac{C}{\Lambda_i^{1/2}} (\parallel \xi \bar{v}' \parallel_2^2 + \Lambda_i \Re ((\Lambda_i^{-1/2} \xi^2 - \mu_r) \bar{g} \parallel_2^2 + \epsilon \parallel \bar{v} \parallel_2^2).  

(3.108)$$

To bound $\parallel \xi \bar{v}' \parallel_2^2$ we use the identity

$$\Re (\xi^2 \bar{v}, \bar{g}) = \parallel \xi \bar{v}' \parallel_2^2 - \parallel \bar{v} \parallel_2^2 - \Re z_0 \parallel \xi \bar{v} \parallel_2^2$$

$$+ \epsilon \Lambda_i^{-1/2} \parallel \xi (\Lambda_i^{-1/2} \xi^2 - \mu_r) \bar{v} \parallel_2^2 + \epsilon \Re \langle \xi^2 \bar{v}, \Phi(\epsilon, \Lambda_i^{-1/4} \xi, \mu) \bar{v} \rangle,$$

from which we obtain

$$\parallel \xi \bar{v}' \parallel_2^2 \leq (\Re z_0) + \parallel \xi \bar{v} \parallel_2^2 + C(\parallel \bar{v} \parallel_2^2 + \parallel \xi^2 \bar{g} \parallel_2^2).  

(3.109)$$
We now use the identity

\[ -\Re \langle \tilde{v}, \tilde{g} \rangle = 2\|\xi \tilde{v}\|^2 - 2\mu_r \Lambda_i^{1/2} \|\tilde{v}\|_2^2 - \epsilon \Lambda_i^{-1/2} \Re \left( \tilde{v}, \Phi(\epsilon, \Lambda_i^{-1/4} \xi, \mu) \tilde{v} \right), \]

to obtain

\[ 2\|\xi \tilde{v}\|^2 \leq \|\tilde{v}\|_2^2 \|\tilde{g}\|_2 + [2(\mu_r) + \Lambda_i^{1/2} + C \epsilon \Lambda_i^{-1/2}]\|\tilde{v}\|_2^2, \]

which implies, having in mind that \( \Lambda_i > \frac{1}{2} \) and \( \mu_r \leq \varrho \),

\[ \|\xi \tilde{v}\|_2^2 \leq \hat{C} \left( \Lambda_i^{3/2} \|\tilde{v}\|_2^2 + \Lambda_i^{-\frac{3}{2}} \|\tilde{g}\|_2^2 \right). \] (3.110)

Substituting the above into (3.109) yields

\[ \|\tilde{v}\|_2^2 \leq C \left( \Lambda_i^{1/2} (\Re z_0)_+ + 1 \right) \|\tilde{v}\|_2^2 + (\epsilon^2 + \epsilon^{-1/2} \Lambda_i^{1/2}) \|\tilde{g}\|_2^2, \]

which yields, upon substitution into (3.108),

\[ \|(\Lambda_i^{-1/2} \xi^2 - \mu_r)^2 \tilde{v}\|_2^2 \leq C \left( \Lambda_i^{-1} \|(\Lambda_i^{-1/2} \xi^2 + \epsilon^{-1/2}) \tilde{g}\|_2^2 + \left( \Re z_0 \right)_+ + 1 \right) \|\tilde{v}\|_2^2. \] (3.111)

To complete the proof, we observe from (3.106) applied to \( \tilde{v} \), that

\[ (h - z_0)\tilde{v} = \frac{\epsilon}{\Lambda_i^{1/2}} [(\Lambda_i^{-1/2} \xi^2 - \mu_r)^2 + \Phi(\epsilon, \Lambda_i^{-1/4} \xi, \mu)] \tilde{v} + \tilde{g}. \]

By (3.98) and (3.111) it holds that

\[ \|\tilde{v}\|_2 \leq C \left( \frac{1}{d(z_0, \sigma(h))} + \frac{1}{1 + |\Re z_0|^{1/3}} \right) \left[ \epsilon \Lambda_i^{-1/2} \left( \frac{(\Re z_0)_+ \Lambda_i^{3/2}}{1 + 1} + 1 \right)^{1/2} \|\tilde{v}\|_2 + \frac{\epsilon}{\Lambda_i} \|\Lambda_i^{-1/2} \xi^2 \tilde{g}\|_2 + \|\tilde{g}\|_2 \right]. \] (3.112)

We now show that for any \( \eta > 0 \) there exist \( \epsilon_0 \) and \( R_0 \) such that, for all \( (\epsilon, R) \in (0, \epsilon_0) \times [R_0, +\infty) \), for \( \mu \in \Omega(\varrho, \epsilon, R) \) we have

\[ \delta(\epsilon, \mu) := \left( \frac{1}{d(z_0, \sigma(h))} + \frac{1}{1 + |\Re z_0|^{1/3}} \right) \epsilon \Lambda_i^{-1/2} \left( \frac{(\Re z_0)_+ \Lambda_i^{3/2}}{1 + 1} + 1 \right)^{1/2} \leq \eta, \] (3.113)

where

\[ \Omega(\varrho, \epsilon, R) := \Omega(\varrho, \epsilon) \cap \hat{\Omega}(\epsilon, R). \]

We consider four different cases.

1. If \( \mu_i \geq 1 \), we observe that by (3.101) and the location of \( \sigma(h) \), there exists \( C_0 > 0 \) such that if \( \Re z_0 \geq C_0 \), then \( d(z_0, \sigma(h)) \geq \frac{1}{C_0} \Re z_0 \). Thus, if \( d(z_0, \sigma(h)) \leq 1 \), we obtain, using the fact that \( \Lambda_i \geq 1 \) for \( \mu_i \geq 0 \), that \( \delta(\epsilon, \mu) \leq C \epsilon \). For \( d(z_0, \sigma(h)) \geq 1 \) and \( \Re z_0 \geq C_0 \) it holds that

\[ \delta(\epsilon, \mu) \leq \hat{C} \epsilon (1 + |\Re z_0|^{3/2} \Lambda_i^{-\frac{3}{2}}). \] (3.114)

As

\[ 0 < \Re z_0 = \frac{\mu_i (2 + \epsilon \mu_i)}{[1 + \epsilon \mu_i]^{1/2}} = \frac{1}{\epsilon (\Lambda_i^2 - 1)} \Lambda_i^{-\frac{3}{2}} \leq \frac{1}{\epsilon \Lambda_i^{3/2}}, \]

we obtain that

\[ \delta(\epsilon, \mu) \leq \hat{C} \epsilon^{\frac{5}{6}}. \]
2. In the case \( \Re z_0 \leq C_0 \), we have
\[
\delta(e, \mu) \leq \hat{C} \epsilon(1 + d(z_0, \sigma(\mathfrak{h}))^{-1}) \leq C \left( \frac{\epsilon}{1 + R} \right). 
\tag{3.115}
\]

3. If \( 0 \leq \mu_i \leq 1 \) and \( d(z_0, \sigma(\mathfrak{h})) \leq 1/2 \) there exists \( C > 0 \) such that \( \Re z_0 \leq C |\Im z_0| \). Furthermore, it holds that \( \Im z_0 < 0 \) and hence \( 0 \leq \mu_r \leq \varphi \). Consequently,
\[
|\Im z_0| \leq C \varphi,
\]
and hence \( \Re z_0 \leq C \) and we may proceed as in item 2. Otherwise, if \( d(z_0, \sigma(\mathfrak{h})) \geq 1/2 \), we may invoke (3.114) and proceed as in item 1.

4. Finally, if \( \mu_i < 0 \), we have \( \Re z_0 < 0 \) and \( \frac{1}{2} \leq \Lambda_i \leq 1 \) and hence we can obtain (3.115) once again.

For sufficiently small \( \eta > 0 \) we can conclude the existence of positive \( C, \epsilon_0 \) and \( R_0 \) such that for all \( (\epsilon, R) \in (0, \epsilon_0) \times [R_0, +\infty) \) and \( \mu \in \Omega(\varphi, \epsilon, R) \) it holds that
\[
\|\tilde{v}\|_2 \leq C \left( d(z_0, \sigma(\mathfrak{h}))^{-1} + \frac{1}{|\Re z_0|^{1/4} + 1} \right) \left( \|\tilde{g}\|_2 + \epsilon \Lambda_i^{-2} \|\xi^2 \tilde{g}\|_2 \right). 
\tag{3.116}
\]

We can now easily verify (3.105). \( \blacksquare \)

Coming back to the resolvent of \( \tilde{B}_\epsilon \), we now prove:

**Proposition 3.19.** There exists \( R_0 > 0 \) and \( C > 0 \), such that, for any \( \mathfrak{f} \in H^2(\mathbb{R}, \mathbb{C}^3) \), any \((\epsilon, R)\) satisfying \( R < R < \epsilon^{-1} \), and \( \Lambda \in \mathcal{V}(\epsilon, \varphi) \) for which \( \mu \in \hat{\Omega}(\epsilon, R) \) it holds that
\[
\| (\tilde{B}_\epsilon - \Lambda)^{-1} \mathfrak{f} \|_2 \leq \frac{C}{R \epsilon^{3/2}} \left( \| \mathfrak{f} \|_2 + \epsilon^2 \| \mathfrak{f}'' \|_2 + (1 + \Lambda_2^2)^{-1/2} \| x \mathfrak{f} \|_2 \right), 
\tag{3.117}
\]
where \( \mathfrak{f}_\perp = (f_1, f_2, 0) \).

**Proof.** Consider (see (3.4)) a triple \((\tilde{u}, \mathfrak{f}, \Lambda)\) such that
\[
(\tilde{B}_\epsilon - \Lambda)\tilde{u} = \mathfrak{f} = \epsilon^\frac{2}{3} \tilde{f}.
\]
We first recall that for \( \Lambda_r \leq -1 \) we have (see Remark 3.3)
\[
\epsilon^2 \| \nabla \tilde{u} \|_2^2 - \Lambda_r \| \tilde{u} \|_2^2 = \Re(\tilde{u}, \mathfrak{f}),
\]
and hence
\[
\| \tilde{u} \|_2 \leq |\Lambda_r|^{-1} \| \mathfrak{f} \|_2.
\]
Consequently, we have for \( \epsilon^{-\frac{2}{3}} \Lambda_r = \lambda_r < -1 \)
\[
\|(\tilde{B}_\epsilon - \Lambda)^{-1} \mathfrak{f} \|_2 \leq \epsilon^{-\frac{2}{3}} \| \mathfrak{f} \|_2,
\tag{3.118}
\]
which implies (3.117) in this case.
Through the rest of this proof we assume in addition that \( \lambda_r > -1 \).

From (3.98) which reads \( \xi = \hat{\omega} \Lambda^\frac{1}{3} \), and (3.66) which reads \( \omega = \epsilon^\frac{1}{3} \hat{\omega} \), we get \( \xi = \epsilon^{-1/6} \Lambda_i^{1/4} \omega \). From (3.58) we then conclude that
\[
\lambda = \epsilon^{-2/3} \Lambda, \quad \tilde{v}(\xi) = (\omega^2 - \lambda)^{-\frac{1}{2}} \tilde{u}_s(\omega) \text{ and } \tilde{g}(\xi) = \Lambda_i^{-1/2} \epsilon^{1/3} \tilde{g}(\omega).
\]
Estimation of $\hat{u}_s$ for $\lambda_r > -1$.
Recall that $\hat{u}_s$ is introduced in (3.11), and that the definition of $g$ is given in (3.15) which reads
\[
g := \hat{f}_s + \epsilon^{\frac{-2}{3}} \frac{\hat{f}_3}{\omega^2 - \lambda} + \frac{d}{d\omega} \left( \frac{\hat{f}_d}{\omega^2 - \lambda} \right). \tag{3.119}
\]
Thus, by using (3.105) and (3.110) we obtain, for $R_0 < R < \frac{1}{\epsilon}$,
\[
\| (\epsilon^{\frac{1}{3}} \Lambda_i^{-\frac{1}{2}} \xi^2 + 1) \frac{\hat{f}_3}{\omega^2 - \lambda} \|_2 \leq \frac{C}{R\epsilon} \left( \| \hat{g} \|_2 + \epsilon \Lambda_i^{-3/2} \| \xi^2 \hat{g} \|_2 \right) + C \epsilon \Lambda_i^{-\frac{1}{2}} \| \hat{g} \| ,
\]
which implies for a new constant $C > 0$
\[
\| (\epsilon^{\frac{1}{3}} \Lambda_i^{-\frac{1}{2}} \xi^2 + 1) \frac{\hat{f}_3}{\omega^2 - \lambda} \|_2 \leq \frac{C}{R\epsilon} \left( \| \hat{g} \|_2 + \epsilon \Lambda_i^{-3/2} \| \xi^2 \hat{g} \|_2 \right) ,
\]
and hence
\[
\left\| \left( \frac{\omega^2 + 1}{\omega^2 - \lambda} \right)^{1/2} \hat{u}_s \right\|_2 \leq \frac{C}{R\epsilon} \epsilon^{-\frac{2}{3}} \Lambda_i^{-1/2} \left( \| g \|_2 + \epsilon^{2/3} \Lambda_i^{-1} \| \omega^2 g \|_2 \right). \tag{3.120}
\]
As for all $\lambda \in \mathbb{C}$ satisfying $-1 \leq \lambda_r \leq 1$ and $\lambda_i \geq 1$
\[
\left| \frac{\omega^2 + 1}{\omega^2 - \lambda} \right| \geq \frac{1}{\sqrt{2} \lambda_i} ,
\]
we obtain, for sufficiently large $R_0$ that, for $R_0 < R < \frac{1}{\epsilon}$,
\[
\| \hat{u}_s \|_2 \leq \frac{C}{R\epsilon} \left( \| g \|_2 + \epsilon^{2/3} \Lambda_i^{-1} \| \omega^2 g \|_2 \right). \tag{3.120}
\]
We now estimate of the right hand side in (3.120) using (3.119).

**Estimation of terms involving $\hat{f}_3$.**

We begin by observing that
\[
\epsilon^{-\frac{2}{3}} \left\| \frac{\hat{f}_3}{\omega^2 - \lambda} \right\|_2 \leq \epsilon^{-\frac{2}{3}} \Lambda_i^{-1} \| \hat{f}_3 \|_2 = \Lambda_i^{-1} \| \hat{f}_3 \|_2 .
\]
Next, to estimate $\Lambda_i^{-1} \left\| \frac{\omega^2 \hat{f}_3}{\omega^2 - \lambda} \right\|_2$ we observe that, for $\lambda_r \in (-1, +1)$ and $\lambda_i \geq 1$, it holds that
\[
\left| \frac{\omega^2}{\omega^2 - \lambda} \right| \leq 2 . \tag{3.121}
\]
Hence, we can conclude that
\[
\Lambda_i^{-1} \left\| \frac{\omega^2 \hat{f}_3}{\omega^2 - \lambda} \right\|_2 \leq 2 \Lambda_i^{-1} \| \hat{f}_3 \|_2 .
\]

**Estimation of terms involving $\hat{f}_d$.**
We first write
\[
\frac{d}{d\omega} \left( \frac{\hat{f}_d}{\omega^2 - \lambda} \right) = \frac{\hat{f}_d'}{\omega^2 - \lambda} - \frac{2\omega \hat{f}_d}{(\omega^2 - \lambda)^2}.
\]

For the first term on the right-hand-side we conclude that
\[
\| \frac{\hat{f}_d'}{\omega^2 - \lambda} \|_2 \leq \epsilon^2 \Lambda_i^{-1} \| \hat{f}_d' \|_2.
\]

Then, with the aid (3.121), valid for \(\lambda_i \in (-1, +1)\) and \(\lambda_i \geq 1\), we obtain that
\[
\left| \frac{2\omega}{(\omega^2 - \lambda)} \right| = \left| 2 \frac{\omega^2}{(\omega^2 - \lambda)} \right| \frac{1}{(\omega^2 - \lambda)} \leq 2\epsilon \Lambda_i^{-3/2}.
\]

Consequently,
\[
\left\| \frac{2\omega \hat{f}_d}{(\omega^2 - \lambda)^2} \right\|_2 \leq 2\epsilon \Lambda_i^{-3/2} \| \hat{f}_d \|_2.
\]

Summarizing the above yields
\[
\| g \|_2 \leq C \left[ \| \hat{f}_d \|_2 + \Lambda_i^{-1} \| \hat{f}_d' \|_2 + \epsilon^2 \Lambda_i^{-1} \| \hat{f}_d'' \|_2 + \epsilon \Lambda_i \| \hat{f}_d \|_2 \right]. 
\]

Next, using (3.121) once again yields
\[
\epsilon^2 \Lambda_i^{-1} \left\| \frac{\omega^2 \hat{f}_d'}{\omega^2 - \lambda} \right\|_2 \leq 2\epsilon \Lambda_i^{-1} \| \hat{f}_d \|_2.
\]

Finally, it holds that
\[
2\epsilon \Lambda_i^{-1} \left\| \frac{\omega^3 \hat{f}_d}{\omega^2 - \lambda} \right\|_2 \leq 2\epsilon \Lambda_i^{-3} \| \hat{f}_d \|_2.
\]

In conclusion, we have proved the existence of \(C > 0\) and \(R_0 > 0\) such that for \(R_0 < R < \frac{1}{\sqrt[3]{7}}\), \(\Lambda \in \mathcal{V}(\epsilon, \rho) \cap \{\Lambda_r > -\epsilon^{2/3}\}\) for which \(\mu \in \hat{\Omega}(\epsilon, R)\)
\[
\| \hat{u}_s \|_2 \leq \frac{C}{R\epsilon} \left[ \| (\epsilon^{2/3} \Lambda_i^{-1} - 1) \hat{f}_s \|_2 + \| \hat{f}_s' \|_2 + \epsilon^{2/3} \Lambda_i^{-1} \| \hat{f}_d' \|_2 + \epsilon \| \hat{f}_d \|_2 \right].
\]

**Estimation of \( \hat{u}_3 \).** As in the proof of Proposition 3.4 we write (3.91) in the form
\[
\hat{u}_3 = \frac{\epsilon^{-2/3}}{(\omega^2 - \lambda) \sqrt{2}} \hat{u}_s + \frac{\hat{f}_3}{\omega^2 - \lambda},
\]

which implies
\[
\| \hat{u}_3 \|_2 \leq \frac{1}{|\Lambda_i| \sqrt{2}} \| \hat{u}_s \|_2 + \epsilon^{2/3} \frac{1}{|\Lambda_i|} \| \hat{f}_3 \|_2 \leq C \Lambda_i^{-1} (\| \hat{u}_s \|_2 + \epsilon^{2/3} \| \hat{f}_3 \|_2).
\]

Hence, using the fact that \(R \leq 1\), we obtain by (3.123)
\[
\| \hat{u}_3 \|_2 \leq \frac{C}{R\epsilon} \left[ \| (\epsilon^{2/3} \Lambda_i^{-1} - 1) \hat{f}_s \|_2 + \| \hat{f}_s' \|_2 + \epsilon^{2/3} \Lambda_i^{-1} \| \hat{f}_d' \|_2 + \epsilon \| \hat{f}_d \|_2 \right].
\]

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Estimation of $\hat{u}_d$ We begin with an estimate of $\hat{u}_d'$. By (3.20) which reads
\[
\Re\langle (\omega^2 - \bar{\lambda})\hat{u}_s, g \rangle = \|\hat{u}_s\|^2 + \Re\left\langle 2\omega\hat{u}_s, \frac{\hat{u}_s'}{\omega^2 - \lambda} \right\rangle + (\varepsilon^{-1} - \lambda_i^2)\|\hat{u}_s\|^2 + \|\omega^2 - \lambda\|\hat{u}_s\|^2,
\]
we may conclude that
\[
\|\omega^2 - \bar{\lambda}\|\hat{u}_s\|g\|_2 \geq \|\hat{u}_s\|^2 - 2\|\hat{u}_s\|\|\hat{u}_s\|_2 + \left(\varepsilon^{-1} - \lambda_i^2\right)\|\hat{u}_s\|^2 + \|\omega^2 - \lambda\|\hat{u}_s\|^2.
\]
Hence,
\[
\|\hat{u}_s\|_2 \leq C\left(\|g\|^2 + \|\omega^2 - \lambda\|^2 + \lambda_i^2\|\hat{u}_s\|^2\right).
\]
Using (3.121), we get
\[
\left\|\frac{\omega(\hat{u}_s')}{\omega^2 - \lambda}\right\|_2 \leq \sqrt{2}\lambda_i^{-\frac{1}{2}}\|\hat{u}_s\|_2 \leq C\varepsilon^\frac{1}{3}\|\hat{u}_s\|_2.
\]
Consequently, we can conclude that under the assumed conditions on $\Lambda, \varepsilon, R$ it holds that
\[
\|\hat{u}_s\|_2 \leq C\left(\|g\|^2 + \lambda_i^2\|\hat{u}_s\|^2\right).
\]
We can, thus, deduce from (3.12) that
\[
\|\hat{u}_d\| \leq C\varepsilon^{2/3}\Lambda_i^{-1}\left(\|\hat{f}_d\| + \|\hat{u}_s'\|\right),
\]
which leads to
\[
\|\hat{u}_d\| \leq C\varepsilon^{2/3}\Lambda_i^{-1}\left(\|\hat{f}_d\| + \|\hat{g}\|_2\right) + C\|\hat{u}_s\|_2.
\]
Substituting (3.122) into (3.126) yields
\[
\|\hat{u}_d\| \leq C\varepsilon^{2/3}\left(\|\hat{f}_d\| + \|\hat{f}_s\|_2 + \|\hat{f}_s\|_2 + \varepsilon^{2/3}\Lambda_i^{-1}\|\hat{f}_d\|_2\right) + C\|\hat{u}_s\|_2.
\]
With the aid of (3.123) we then conclude
\[
\|\hat{u}_d\|_2 \leq \frac{C}{R\varepsilon}\left[\|\left(e^{2}\omega^2 + 1\right)\hat{f}_s\|_2 + \|\hat{f}_s\|_2 + \varepsilon^{2/3}\Lambda_i^{-1}\|\hat{f}_d\|_2 + \|\hat{f}_d\|_2\right].
\]
Combining the above with (3.123) and (3.124) then yields for $R_0 < R < \varepsilon^{-1}$
\[
\|\hat{u}\|_2 \leq \frac{C}{R\varepsilon}\left[\|\left(e^{2}\omega^2 + 1\right)\hat{f}_s\|_2 + \|\hat{f}_s\|_2 + \varepsilon^{2/3}\Lambda_i^{-1}\|\hat{f}_d\|_2 + \|\hat{f}_d\|_2\right].
\]
In term of the original variables $(u, f)$ (3.129) reads
\[
\|u\|_2 \leq \frac{C}{R\varepsilon}\left(e^2\|f''\|_2 + \|f\|_2 + \Lambda_i^{-1}\|xf\|_2\right).
\]
3.6 Proof of Theorem 1.3

We now complete the proof of Theorem 1.3. Let \( u \in D(\mathcal{B}_r) \) and \( f \in L^2(\mathbb{R}, \mathbb{C}^3) \) satisfy \((\mathcal{B}_r - \Lambda)u = f\) for some \( \Lambda = \Lambda_r + i\Lambda_i \in \mathbb{C} \). As above we consider only the case \( \Lambda_i > 0 \).

The case \( 0 < \Lambda_i < 1/2 \).

Here we have by (3.57) for \( \Lambda_r \leq \frac{1}{\epsilon} \)
\[
\| (\mathcal{B}_r - \Lambda)^{-1} \| = \| (\tilde{\mathcal{B}}_r - \Lambda)^{-1} \| \leq C\epsilon^{2/3} \left( \epsilon^{-2/3} + \frac{1}{\Lambda_i} \right)^2. \tag{3.130}
\]

The case \( \Lambda_i > 1/2 \).

Here we attempt to use (3.117), to which end we first observe that
\[
\Re \langle u, (\tilde{\mathcal{B}}_r + 1 - i\Lambda_i)u \rangle = \epsilon^2 \left\| \frac{du}{dx} \right\|^2 + \| u \|^2.
\]
It follows that
\[
\| (\tilde{\mathcal{B}}_r + 1 - i\Lambda_i)^{-1} \| + \epsilon \| (\tilde{\mathcal{B}}_r + 1 - i\Lambda_i)^{-1} \|_{\mathcal{L}(L^2, H^1)} \leq 3. \tag{3.131}
\]

Integration by parts yields that, for any \( w \in D(\tilde{\mathcal{B}}_r) \),
\[
-\Re \left( \frac{d^2 w}{dx^2}, (\tilde{\mathcal{B}}_r + 1 - i\Lambda_i)w \right) = \epsilon^2 \left\| \frac{d^2 w}{dx^2} \right\|^2 - \left\| \frac{dw}{dx} \right\|^2 - 3 \langle \frac{dw_1}{dx}, u_1 \rangle + 3 \langle \frac{dw_2}{dx}, u_2 \rangle.
\]
Consequently, we obtain that
\[
\| (\tilde{\mathcal{B}}_r + 1 - i\Lambda_i)^{-1} \|_{\mathcal{L}(L^2, H^2)} \leq \frac{C}{\epsilon^2}. \tag{3.132}
\]

Finally, recall that by (2.25)
\[
D(\tilde{\mathcal{B}}_r) = \{ u \in H^2(\mathbb{R}, \mathbb{C}^3) \mid xu_\perp \in L^2(\mathbb{R}, \mathbb{C}^3) \},
\]
where \( u_\perp = (u_1, u_2, 0) \).

For \( s > 0 \), we equip \( D(\tilde{\mathcal{B}}_r) \) with the norm
\[
\| u \|_{(B,s,\epsilon)} = \epsilon^2 \| u'' \|_2 + \epsilon \| u' \|_2 + \| u \|_2 + s^{-1} \| xu_\perp \|_2,
\]
and denote this normed space by \( D_{s,\epsilon} \).

An integration by parts yields (see (3.3))
\[
\Re \langle (\mathcal{B}_r - \Lambda)^{-1} (\tilde{\mathcal{B}}_r + 1 - i\Lambda_i)u \rangle
\]
\[
= \| xu_\perp \|_2^2 + \Lambda_i(\langle xu_1, u_1 \rangle - \langle xu_2, u_2 \rangle) + \frac{1}{\sqrt{2}} (\Re \langle x(u_2 - u_1), u_3 \rangle)
\]
\[
+ \epsilon^2 (\Re \langle u_2, u'_2 \rangle - \Re \langle u_1, u'_1 \rangle).
\]

Consequently, combining the above with (3.131), we get
\[
\Lambda_i^{-1} \| x \pi_\perp (\tilde{\mathcal{B}}_r + 1 - i\Lambda_i)^{-1} \| \leq C, \tag{3.133}
\]

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where \( \pi_\perp \) denotes the projection on the two first components.

We now choose \( s = \Lambda_i \). Combining (3.133) with (3.132) yields

\[
\|(\tilde{B}_e + 1 - i\Lambda_i)^{-1}\|_{\mathcal{L}(L^2,\mathcal{D}_{\Lambda_i})} \leq C. \tag{3.134}
\]

Let \( \Lambda \in \mathcal{V}(\epsilon, \rho) \) satisfy \( z_0(\mu, \epsilon) \in \hat{\Omega}(\epsilon, R) \) where \( z_0 \) is given by (3.100), \( \hat{\Omega} \) by (3.103), and \( R_0 \) is sufficiently large, so that (3.117) holds true for \( R_0 \leq R < \frac{1}{\epsilon} \). We begin by observing that (3.117) implies

\[
\|(\tilde{B}_e - \Lambda)^{-1}\|_{\mathcal{L}(\mathcal{D}_{\Lambda_i}, L^2)} \leq \frac{C}{R \epsilon^{5/3}}.
\]

We now use the resolvent identity

\[
(\tilde{B}_e - \Lambda)^{-1} = (\tilde{B}_e + 1 - i\Lambda_i)^{-1} + (\Lambda + 1)(\tilde{B}_e - \Lambda)^{-1}(\tilde{B}_e + 1 - i\Lambda_i)^{-1},
\]

to establish that

\[
\|(\tilde{B}_e - \Lambda)^{-1}\| \leq \|(\tilde{B}_e + 1 - i\Lambda_i)^{-1}\| + C\|(\tilde{B}_e - \Lambda)^{-1}\|\|(\tilde{B}_e + 1 - i\Lambda_i)^{-1}\|_{\mathcal{L}(L^2,\mathcal{D}_{\Lambda_i})}.
\]

We may conclude from the above that:

**Proposition 3.20.** Let \( \rho > 0 \). There exist \( R_0 > 0 \) and \( \epsilon_0 > 0 \) such that for any \( 0 < \epsilon < \epsilon_0 \), \( \Lambda \in \mathcal{V}(\epsilon, \rho) \), \( R_0 < R < \frac{1}{\epsilon} \), \( \Lambda_i > \frac{1}{2} \), and \( z_0(\mu, \epsilon) \in \hat{\Omega}(\epsilon, R) \) it holds that

\[
\|(\tilde{B}_e - \Lambda)^{-1}\| \leq C \left( 1 + \frac{1}{R \epsilon^2} \right). \tag{3.135}
\]

We finally provide a more explicit condition in guaranteeing the validity of the assumptions of proposition 3.20. We introduce for \( \tilde{R} > 0 \), \( \epsilon > 0 \)

\[
\tilde{D}^+(\epsilon, \tilde{R}) = \{ \exists \Lambda > 0, \ d(\Lambda - i, \epsilon \sigma(h^*)) > \tilde{R} \epsilon^2 \}.
\]

Note that for all \( 0 < \epsilon \) the set \( \{ \Lambda \in \mathbb{C}, \ d(\Lambda, \epsilon \sigma(h^*)) < \frac{1}{2} \epsilon \} \) is a union of disjoint disks.

**Lemma 3.21.** Let \( \rho > 0 \). There exist \( \tilde{R}_0 > 1 \) and \( \epsilon_0 > 0 \), such that for any \( \tilde{R}_0 < \tilde{R} < 1/(\sqrt{2}\epsilon) \), and \( \Lambda \in \mathcal{V}(\epsilon, \rho) \cap \mathcal{D}^+(\epsilon, \tilde{R}) \), we have that \( z_0(\mu, \epsilon) \in \hat{\Omega}(\epsilon, R) \) for all \( R \geq \sqrt{2}\tilde{R} \).

**Proof.** Let \( 1 < \tilde{R}_0 < \tilde{R} < [\epsilon \sqrt{2}]^{-1} \) and \( 0 < R \leq \sqrt{2}(\tilde{R} - 2N_\rho^2) \), where \( N_\rho \) is given by (1.9). (Note that \( R < \sqrt{2}\tilde{R} < \epsilon^{-1} \).) Suppose for a contradiction that for some \( n \leq N_\rho \), it holds that \( \Lambda \in \mathcal{V}(\epsilon, \rho) \cap \mathcal{D}^+(\epsilon, \rho, \tilde{R}) \) but

\[
d(z_0, (2n - 1)[1 - i]) \leq R \epsilon. \tag{3.137}
\]

Then, \( |\Re z_0 - (2n - 1)| \leq R \epsilon \) which can be rewritten in the form

\[
\left| \frac{\Lambda_i^2 - 1}{\Lambda_i^{1/2}} - (2n - 1)\epsilon \right| \leq R \epsilon^2. \tag{3.138}
\]
Clearly, $\Lambda_i \geq 1$, otherwise we have
\[
\left| \frac{\Lambda_i^2 - 1}{\Lambda_i^{1/2}} - (2n - 1)\epsilon \right| \geq (2n - 1)\epsilon,
\]
contradicting (3.138) as $R\epsilon < 1$.

For $\Lambda_i \geq 1$ we have, as $(\Lambda_i + 1) \geq 2\Lambda_i^{1/2}$,
\[
R\epsilon^2 \geq \frac{\Lambda_i^2 - 1}{\Lambda_i^{1/2}} + (2n - 1)\epsilon \geq 2(\Lambda_i - 1) - (2n - 1)\epsilon,
\]
and hence
\[
1 \leq \Lambda_i \leq 1 + \frac{2n - 1}{2} - \frac{R}{2} \epsilon^2.
\]
(3.139)

Returning to (3.138), we write
\[
-R\epsilon^2 \leq \frac{\Lambda_i^2 - 1}{\Lambda_i^{1/2}} - (2n - 1)\epsilon \leq \left( 2 + \frac{2n - 1}{2} \epsilon + \frac{R}{2} \epsilon^2 \right)(\Lambda_i - 1) - (2n - 1)\epsilon,
\]
which leads to (as $n \leq N_\rho$ where $N_\rho$ is defined by (1.9))
\[
\Lambda_i \geq 1 + \frac{(2n - 1)\epsilon - R\epsilon^2}{2 + N_\rho \epsilon + \frac{R}{2} \epsilon^2} \geq 1 + \frac{1}{2}(2n - 1)\epsilon - \frac{R\epsilon^2}{2}\left[ 1 - N_\rho \epsilon - \frac{R}{2} \epsilon^2 \right],
\]
and hence
\[
\Lambda_i - 1 \geq \frac{2n - 1}{2} - \frac{R}{2} \epsilon^2 + \left( \frac{2n - 1}{2} \epsilon + \frac{R}{2} \epsilon^2 \right) \leq \frac{1}{2}(2n - 1)\epsilon - \left( \frac{R}{2} + 2N_\rho \epsilon \right) \epsilon^2.
\]

Combining the above with (3.139) yields
\[
\left| \Lambda_i - 1 - \frac{2n - 1}{2} \epsilon \right| \leq \left( \frac{R}{2} + 2N_\rho \epsilon \right) \epsilon^2.
\]
(3.140)

By (3.137) we have, in addition to (3.138),
\[
|\Im z_0 + (2n - 1)| \leq R\epsilon.
\]
(3.141)

In view of (3.140) we can rewrite
\[
\left| \frac{1}{2}(2n - 1) - \mu_r \right| \leq \left( \theta N_\rho + \frac{R}{2} \right) \epsilon,
\]
or equivalently
\[
\left| \frac{1}{2}(2n - 1)\epsilon - \Lambda_r \right| \leq \left( N_\rho^2 + \frac{R}{2} \right) \epsilon^2.
\]

Combining the above with (3.140) yields
\[
d(\Lambda_i - (1 + i)\frac{2n - 1}{2} \epsilon) \leq \sqrt{2}\left( \frac{2N_\rho^2 + \frac{R}{2}}{2} \right) \epsilon^2 < \hat{R}\epsilon^2.
\]
(3.142)

A contradiction. $

Combining Lemma 3.21 with Proposition 3.20 (applied with $R = \sqrt{2}\hat{R}$) establishes (1.11) for $\Lambda_i > 1/2$. We then use (3.130) to obtain (1.11) for all $\Lambda_i > 0$, $\Lambda_r < \phi\epsilon$ and $\Lambda \in D^+(\hat{R}, \phi, \epsilon)$. Remark 3.15 and Lemma 3.14 together establish (1.8).
4 Finite intervals

As stated in Section 2.6 we expect that the behaviour of $B_\epsilon$ acting on $\mathbb{R}$, can intuitively explain the behaviour of $B_\epsilon$ on a bounded interval, in the limit $\epsilon \to 0$. In the following, we focus on the part of $\sigma(B_\epsilon)$ that tends to $\mathbb{R}_+$ (which is in the spectrum of $B_\epsilon$ on $\mathbb{R}$ by Remark 2.7).

4.1 The problem

We now define the operator $B_\epsilon^I$ whose associated differential operator is given by (4.1) in the interval $I = (a, b)$, and its domain by

$$D(B_\epsilon^I) = H^2((a, b), \mathbb{C}) \cap H_0^1((a, b), \mathbb{C}).$$

Let $(u, \Lambda) \in D(B_\epsilon^I) \times \mathbb{C}$ denote an eigenpair of $B_\epsilon^I$. Let further $L_\pm = -\epsilon^2 \frac{d^2}{dx^2} \pm ix$ be defined on $H^2((a, b) \cap H_0^1((a, b), \mathbb{C})$. Then by (3.4) (with $f = 0$) and (3.3) we may set

$$u_1 = -(L_- - \Lambda)^{-1} u_3 \quad ; \quad u_2 = -(L_+ - \Lambda)^{-1} u_3,$$

and hence by (3.4) and (3.3) we obtain

$$(\mathcal{P}_\Lambda - \Lambda) u_3 = 0, \quad (4.1a)$$

where

$$\mathcal{P}_\Lambda \overset{\text{def}}{=} -\epsilon^2 \frac{d^2}{dx^2} + \frac{1}{2} [ (L_- - \Lambda)^{-1} + (L_+ - \Lambda)^{-1}] \quad (4.1b)$$

is defined on

$$D(\mathcal{P}_\Lambda) = H^2((a, b), \mathbb{C}) \cap H_0^1((a, b), \mathbb{C}).$$

Note that $(L_\pm - \Lambda)^{-1}$ is well defined, for sufficiently small $\epsilon$, whenever $\Re \Lambda < \epsilon^{2/3} |\nu_1|/2$ where $\nu_1$ denotes the leftmost zero of Airy’s function [1].

Some intuition can be gained by considering the operator

$$\mathcal{A}_\Lambda = (L_- - \Lambda)(\mathcal{P}_\Lambda - \Lambda)(L_+ - \Lambda) =$$

$$-\epsilon^2 \left[ (L_- - \Lambda) \frac{d^2}{dx^2} (L_+ - \Lambda) + \frac{d^2}{dx^2} \right] - \Lambda [1 + (L_- - \Lambda)(L_+ - \Lambda)], \quad (4.2)$$

where $D(\mathcal{A}_\Lambda) = \{ w \in H^6((a, b), w(a) = w(b) = w''(a) = w''(b) = 0 \}$. Note that for $\Lambda \leq C\epsilon^2$ it holds that $0 \in \sigma(\mathcal{A}_\Lambda) \Leftrightarrow 0 \in \sigma(\mathcal{P}_\Lambda - \Lambda)$ since $(L_\pm - \Lambda)$ is invertible. Assuming $\Lambda = \Lambda_0 \epsilon^2$ where $\Lambda_0$ is independent of $\epsilon$ we get

$$(\epsilon^2 \mathcal{A}_\Lambda)|_{\epsilon=0} = -x \frac{d^2}{dx^2} x - \frac{d^2}{dx^2} - \Lambda_0 (1 + x^2) \quad (4.3)$$

Let $\tilde{\mathcal{A}}_\Lambda = (1 + x^2)^{-1}(\epsilon^2 \mathcal{A}_\Lambda)|_{\epsilon=0}$ which is selfadjoint on $L^2((a, b), (1 + x^2))$. By the foregoing discussion, we expect $\Lambda_0$ to be the ground state of $\tilde{\mathcal{A}}_\Lambda$. Denote the principal eigenfunction by $w_0$. We expect that $u_3 \approx (L_+ - \Lambda) w_0$ and hence, by setting $\epsilon = 0$ once again we can conclude that $u_3 \approx x w_0$. In fact, we shall rigorously corroborate this approximation in the sequel.

The above heuristical argument suggests in addition that for any $C > 0$ there exists $\epsilon_C$ such that for all $\epsilon \in (0, \epsilon_C]$, we have $\sigma(B_\epsilon^I) \cap \{ \Re \Lambda < C\epsilon^2 \} \subset \mathbb{R}_+$. Furthermore, while we do not prove that in the following, one may expect to obtain for any fixed $j \geq 1$, that $\Lambda_j \epsilon^2 + o(\epsilon^2) \in \sigma(B_\epsilon^I)$ where $\Lambda_j$ is an eigenvalue of the operator $\tilde{\mathcal{A}}_\Lambda$. 44
4.2 Upper bound for the bottom of the real spectrum

For $\Lambda \in \mathbb{R}$, $\mathcal{P}_\Lambda$ with domain $D(\mathcal{P}_\Lambda) := H^2(a,b) \cap H^1_0(a,b)$ is a selfadjoint operator on $L^2(a,b)$ with compact resolvent. Consequently, we may define for $\Lambda \in \mathbb{R}$,

$$\nu(\Lambda) = \inf_{u \in H^1_0(\{0\}) \setminus \{0\}} \frac{\langle u, (\mathcal{P}_\Lambda - \Lambda)u \rangle}{\|u\|_2^2}.$$ (4.4)

It can be easily verified that if $\nu(\Lambda) = 0$ then $\Lambda \in \sigma(B_I^\varepsilon)$.

Furthermore, if $\Lambda \in \sigma(B_I^\varepsilon) \cap \mathbb{R}$ then there exists $u \in D(\mathcal{P}_\Lambda)$ such that $(\mathcal{P}_\Lambda - \Lambda)u = 0$, and hence, from the definition of $\nu$ we also learn that $\nu(\Lambda) \leq 0$ in this case. It can be easily verified that for any $u$ in $D(\mathcal{P}_\Lambda)$ and $\Lambda \in \mathbb{R}$

$$\langle u, \mathcal{P}_\Lambda u \rangle = \varepsilon^2 \left[ \|u'\|_2^2 + \frac{1}{2}(\|w'_+\|_2^2 + \|w'_-\|_2^2) \right] - \frac{\Lambda}{2}(\|w_+\|_2^2 + \|w_-\|_2^2),$$ (4.5)

where $w_\pm = (L_\pm - \Lambda)^{-1}u$.

Before obtaining bounds on $\nu(\Lambda)$ we need the following auxiliary lemma

**Lemma 4.1.** Let $K > 0$. There exist positive $\epsilon_0$ and $C$, such that for all $0 < \varepsilon < \epsilon_0$, for any real $\Lambda \leq K\epsilon^2$, for any triple $(w_0, w_-, w_+)$ s.t. $w_0 \in H^2(a,b) \cap H^1_0(a,b)$ and $w_\pm = (L_\pm - \Lambda)^{-1}(xw_0)$, it holds that,

$$\|\tilde{w}_\pm\|_2 + \epsilon^{2/3}\|\tilde{w}'_\pm\|_2 \leq C\epsilon^{4/3}\|w_0\|_{2,2},$$ (4.6)

where $\|\cdot\|_{2,2}$ denotes the norm in $H^2(a,b)$, and

$$\tilde{w}_\pm = w_\pm \pm iw_0.$$ (4.7)

**Proof.** It can be easily verified that $\tilde{w}_\pm \in H^1_0(a,b)$ and that

$$(L_\pm - \Lambda)\tilde{w}_\pm = -i(\pm \epsilon^2 w''_0 \pm \Lambda w_0).$$

We can now establish (4.6) by using either [5, Proposition 5.2] or [23, Theorem 1.1]. Here we use the fact that $K\epsilon^2 < \epsilon^2|\nu_1|/2$ for sufficiently small $\epsilon$.

**Lemma 4.1** allows us to obtain an upper bound for $\nu(\Lambda)$.

**Lemma 4.2.** Let

$$\rho_0 = \inf_{w \in H^1_0(a,b) \setminus \{0\}} \frac{I(w)}{\|(x^2 + 1)^{1/2}w\|_2^2},$$ (4.8a)

where

$$I(w) = \|(xw)'\|_2^2 + \|w'\|_2^2.$$ (4.8b)

There exist positive $\epsilon_0$ and $r_+$ such that for all $0 < \epsilon < \epsilon_0$, one can find $\Lambda_1 \in \sigma(B_I^\varepsilon) \cap \mathbb{R}$ satisfying

$$\Lambda_1 < \rho_0 \epsilon^2 (1 + r_+ \epsilon^{2/3}).$$ (4.9)
Proof.
Note first that
\[ \rho_0 > \frac{\pi^2}{(b-a)^2} \]
since for any \( w \in H^1_0(a,b) \setminus \{0\} \)
\[ \frac{I(w)}{\|([x^2 + 1]^{1/2}w)^2\|_2^2} \geq \frac{\pi^2(\|xw\|^2_2 + \|w\|^2_2)}{(b-a)^2\|([x^2 + 1]^{1/2}w)^2\|_2^2} = \frac{\pi^2}{(b-a)^2}. \] (4.10)
Equality in (4.10) is achieved when both \( w_0 = Cxw_0 = \hat{C} \sin((x - a)/(b - a)) \) which is clearly impossible.
Let \( w_0 \) denote a minimizer of (4.8) satisfying \( \|([x^2 + 1]^{1/2}w_0)^2\|_2 = 1 \). (The proof that \( w_0 \) exists is rather standard, and is therefore omitted.) Then, \( w_0 \) must satisfy
\[ -(x^2 + 1)w_0'' - 2xw_0' - \rho_0(x^2 + 1)w_0 = 0. \]
Notice that the above balance is identical with (4.3) with \( \Lambda_0 = \rho_0 \), \( w_0 \) being the ground state of the operator \( \tilde{A}_{\Lambda} \). It can be readily verified from the above that
\[ \|w_0''\|_2 \leq \|w_0'\|_2 + \rho_0. \]
From the fact that \( w_0 \) is a minimizer of (4.8) we readily conclude that
\[ \|w_0'\|_2 \leq \rho_0^{1/2} \]
and hence
\[ \|w_0''\|_2 \leq (\rho_0^{1/2} + \rho_0). \] (4.11)
Next, we select \( \rho_0 < K \) and \( \Lambda \leq K \epsilon^2 \). As in (4.7) we set \( \tilde{w}_\pm = w_\pm + iw_0 \) and we obtain by (4.6) and (4.11) that there exist positive \( \epsilon_0 \) and \( C \) such that for all \( 0 < \epsilon < \epsilon_0 \)
\[ \|w_\pm\|_2 - \|w_0\|_2 + \|w_\pm'\|_2 - \|w_0'\|_2 \leq \|\tilde{w}_\pm\|_2 + \|\tilde{w}_\pm'\|_2 \leq C\epsilon^{2/3}. \]
Consequently, with \( u = xw_0 \),
\[ \langle u, (P_{\Lambda} - \Lambda)u \rangle \leq \epsilon^2 I(w_0) - \Lambda + r_+ \epsilon^{8/3} = \rho_0 \epsilon^2 - \Lambda + r_+ \epsilon^{8/3}. \]
It follows that for all \( K \epsilon^2 > \Lambda > \rho_0 \epsilon^2 + r_+ \epsilon^{8/3} \) we have \( \nu(\Lambda) < 0 \). By (4.5) and since for any \( w \in H^1_0(a,b) \) we have \( \|w'\|_2^2 \geq (\pi^2/(b-a)^2) \|w\|_2^2 \), it follows that \( \nu(\pi^2 \epsilon^2/(b-a)^2) > 0 \), and hence, by the continuity of \( \nu(\Lambda) \) we have that
\[ \sigma(B^I_\epsilon) \cap (\pi^2 \epsilon^2/(b-a)^2, \rho_0 \epsilon^2 + r_+ \epsilon^{8/3}) \neq \emptyset. \]
4.3 Lower bound for the bottom of the real spectrum

To obtain a lower bound for $\Lambda_1$ we rewrite \((4.15)\) in the form, with $w_\pm = (\mathcal{L}_\pm - \Lambda)^{-1}u$,

$$\langle u, (\mathcal{P}_\Lambda - \Lambda)u \rangle = \frac{1}{2} \left( |e^2 J^+_\Lambda (w_+) - \Lambda| ||w_+||^2_2 + |e^2 J^-_\Lambda (w_-) - \Lambda||w_-||^2_2 \right),$$  \hspace{1cm} \text{ (4.12a)}

where

$$||w||^2_\pm = \left| \left( - e^2 \frac{d^2}{dx^2} \pm ix - \Lambda \right) w \right|^2_2 + ||w||^2_2,$$

\text{ (4.12b)}

and

$$J^\pm_\Lambda (w) = \frac{||d}{dx} \left( - e^2 \frac{d^2}{dx^2} \pm ix - \Lambda \right) w^2 \pm ||w'||^2_2.$$  \hspace{1cm} \text{ (4.12c)}

Here we have used that

$$||u'||^2_2 = \frac{1}{2} \left[ \left| \left| \frac{d}{dx} \left( - e^2 \frac{d^2}{dx^2} + ix - \Lambda \right) w \right| \right|^2_2 + \left| \left| \frac{d}{dx} \left( - e^2 \frac{d^2}{dx^2} - ix - \Lambda \right) w \right| \right|^2_2 \right],$$

and

$$||u||^2_2 = \frac{1}{2} \left[ \left| \left| \left( - e^2 \frac{d^2}{dx^2} + ix - \Lambda \right) w \right| \right|^2_2 + \left| \left| \left( - e^2 \frac{d^2}{dx^2} - ix - \Lambda \right) w \right| \right|^2_2 \right].$$

Note that since $u \in H^1_0(a,b)$ we obtain that $w''_\pm \in H^1_0(a,b)$ as well. Hence $w_\pm$ belongs to the domain of the form $J^\pm_\Lambda$ which is defined by

$$D(J^\pm_\Lambda) = \{ w \in H^2((a,b), \mathbb{C}) \cap H^1_0(a,b) \mid w'' \in H^1_0(a,b) \}.$$  \hspace{1cm} \text{ (4.12b)}

We now define

$$\mu(\Lambda) = \inf_{w \in D(J^\pm_\Lambda) \setminus \{0\}} J^\pm_\Lambda (w).$$

As $J^\pm_\Lambda (w) = J^\mp_\Lambda (\bar{w})$ there is no need to define a different minimization problem for $J^-$. As in \((4.10)\) one can establish that $J^\pm_\Lambda (w) > \pi^2/(b-a)^2$ for all $w \in D(J^\pm_\Lambda)$. Consider then a sequence \(\{w^{(k)}\}_{k=1}^\infty \subset D(J^\pm_\Lambda)\) of unity norm, i.e., $||w^{(k)}||^2_\pm = 1$, satisfying $J^\pm_\Lambda (w^{(k)}) \leq C$ for some $C > \pi$. By considering an appropriate subsequence, we may assume that $w^{(k)}$ is weakly convergent in $D(J^\pm_\Lambda)$ and strongly convergent in $H^2(a,b) \cap H^1_0(a,b)$ denote the weak limit by $w$. By the strong convergence we must have $||w||^2_\pm = 1$. Furthermore, by the weak convergence we have

$$\left< \frac{d}{dx} \left( - e^2 \frac{d^2}{dx^2} \pm ix - \Lambda \right) w^{(k)}, \frac{d}{dx} \left( - e^2 \frac{d^2}{dx^2} \pm ix - \Lambda \right) w \right> \rightarrow \left< \frac{d}{dx} \left( - e^2 \frac{d^2}{dx^2} \pm ix - \Lambda \right) w \right>^2_2,$$

and hence

$$\left< \frac{d}{dx} \left( - e^2 \frac{d^2}{dx^2} \pm ix - \Lambda \right) w \right>^2 \leq \liminf \left< \frac{d}{dx} \left( - e^2 \frac{d^2}{dx^2} \pm ix - \Lambda \right) w^{(k)} \right>^2_2.$$

By the strong convergence

$$||w'||^2_2 = \lim_{k \rightarrow +\infty} ||(w^{(k)})'||^2_2,$$

and hence

$$\liminf J^\pm_\Lambda (w^{(k)}) \geq J^\pm_\Lambda (w).$$
If \( \{w^{(k)}\}_{k=1}^{\infty} \) is a minimizing sequence, we may immediately conclude from the above lower semicontinuity that \( w \) is a minimizer.

The foregoing discussion leads us to state the following:

**Lemma 4.3.** Let \( \rho_0 \) be defined by (4.8). There exists positive \( \epsilon \) and \( r_- \) such that for all \( 0 < \epsilon < \epsilon_0 \), and every \( \Lambda_1 \in \sigma(B^\epsilon_R) \cap \mathbb{R} \) we have

\[
\rho_0 \epsilon^2 (1 - r_- \epsilon^2) < \Lambda_1.
\]  (4.13)

**Proof.** Let \( w_1 \) denote a unity norm (i.e., \( \|w_1\|_+ = 1 \)) ground state of \( \mathcal{J}_\Lambda^+ \), associated with \( \mu(\Lambda) \) for some \( \Lambda \) in the interval \( (\pi^2 \epsilon^2 / (b - a)^2, \rho_0 \epsilon^2 + r_+ \epsilon^{8/3}) \). Clearly,

\[
- \epsilon^2 (\mathcal{L}_- - \Lambda) \frac{d^2}{dx^2}(\mathcal{L}_+ - \Lambda) w_1 - \epsilon^2 w''_1 - \mu(\Lambda)[1 + (\mathcal{L}_- - \Lambda)(\mathcal{L}_+ - \Lambda)] w_1 = 0. \]  (4.14)

Taking the inner product of (4.14) with \(-w''_1\) yields for the real part

\[
\epsilon^2 \Re \langle (\mathcal{L}_+ - \Lambda) w'_1, \frac{d^2}{dx^2}(\mathcal{L}_+ - \Lambda)w_1 \rangle + \|w'_1\|^2_2 - \mu(\Lambda)[\|w'_1\|^2_2 - \Re \langle (\mathcal{L}_+ - \Lambda) w''_1, (\mathcal{L}_+ - \Lambda) w_1 \rangle] = 0. \]  (4.15)

For the first term on the left-hand-side we write

\[
\Re \langle (\mathcal{L}_+ - \Lambda) w'_1, \frac{d^2}{dx^2}(\mathcal{L}_+ - \Lambda)w_1 \rangle = \|(\mathcal{L}_+ - \Lambda) w''_1\|^2_2 + \Re \langle (\mathcal{L}_+ - \Lambda) w''_1, 2i w'_1 \rangle.
\]

It follows that

\[
\Re \langle (\mathcal{L}_+ - \Lambda) w''_1, \frac{d^2}{dx^2}(\mathcal{L}_+ - \Lambda) w_1 \rangle \geq -\|w'_1\|^2_2 \geq -\mu(\Lambda).
\]

For the last term on the left-hand-side of (4.15) it holds that

\[
-\Re \langle (\mathcal{L}_+ - \Lambda) w''_1, (\mathcal{L}_+ - \Lambda) w_1 \rangle = \left\| \frac{d}{dx}(\mathcal{L}_+ - \Lambda) w_1 \right\|^2_2 + \Re \langle 2i w'_1, (\mathcal{L}_+ - \Lambda) w_1 \rangle,
\]

and hence (recalling that \( \|w_1\|_+ = 1 \))

\[
\Re \langle (\mathcal{L}_+ - \Lambda) w''_1, (\mathcal{L}_+ - \Lambda) w_1 \rangle \leq \mathcal{J}_\Lambda^+(w_1) + 1 = \mu(\Lambda) + 1.
\]

Consequently, we obtain from (4.15) that

\[
\|w''_1\|^2_2 \leq 2\mu(\Lambda)(\mu(\Lambda) + 1).
\]

Since for any \( \epsilon \)-independent function \( \tilde{w} \) satisfying \( \|\tilde{w}\|_+ = 1 \), \( K > 0 \), and \( \Lambda < K\epsilon^2 \), \( \mathcal{J}_\Lambda(\tilde{w}) \) is bounded, as \( \epsilon \to 0 \), there must exist a positive constant \( \hat{C} \) such that

\[
\mu(\Lambda) := \mathcal{J}_\Lambda(w_1) \leq \hat{C}.
\]

Consequently, there exists \( \hat{C} > 0 \) and \( \epsilon_0 \) such that for any \( \epsilon \in (0, \epsilon_0) \) and \( \Lambda \in \left( \frac{\pi^2}{(b-a)^2}, \epsilon^2 \rho_0 \epsilon^2 + r_+ \epsilon^{8/3} \right) \),

\[
\|w''_1\|_2 \leq \hat{C}. \]  (4.16)
Next, we write
\[
\left\| \frac{d}{dx}(-\epsilon^2 \frac{d^2}{dx^2} + ix - \Lambda)w_1 \right\|^2_2 = \epsilon^4 \|w_1^{(3)}\|^2_2 + \left\| \frac{d}{dx}(ix - \Lambda)w_1 \right\|^2_2 - 2\epsilon^2 \Re \left\langle \frac{d}{dx}(ix - \Lambda)w_1, w_1^{(3)} \right\rangle
\]  
(4.17)
For the last term on the right-hand-side we have
\[
-\Re \left\langle \frac{d}{dx}(ix - \Lambda)w_1, w_1^{(3)} \right\rangle = -\Lambda \|w_1''\|^2_2 - 2\Im \langle w_1', w_1'' \rangle.
\]
Consequently, by (4.16) we can conclude the existence of \(C > 0\), independent of \(\epsilon\), such that
\[
-2\epsilon^2 \Re \left\langle \frac{d}{dx}(ix - \Lambda)w_1, w_1^{(3)} \right\rangle \geq -C\epsilon^2.
\]
Substituting the above into (4.17) yields, for some \(C > 0\)
\[
\left\| \frac{d}{dx}(-\epsilon^2 \frac{d^2}{dx^2} + ix - \Lambda)w_1 \right\|^2_2 \geq \left\| \frac{d}{dx}(ix - \Lambda)w_1 \right\|^2_2 - C\epsilon^2 \geq \|(xw_1)'\|^2_2 - \tilde{C}\epsilon^2.
\]
Hence, by (4.12) we have
\[
\mu(\Lambda) = J^+_\Lambda(w_1) \geq \|(xw_1)\|^2_2 + \|w_1'\|^2_2 - \tilde{C}\epsilon^2 \geq \rho_0\|(x^2 + 1)^{1/2}w_1\|^2_2 - C\epsilon^2.
\]  
(4.18)
By (4.16) and the fact that \(\Lambda \in (\pi^2 \epsilon^2/(b-a)^2, \rho_0 \epsilon^2 + r_+ \epsilon^{8/3})\) it holds that
\[
1 = \|w_1\|_{+} \leq \|\sqrt{x^2 + 1}w_1\|_2 + C\epsilon^2.
\]
Substituting the above into (4.18) yields that there exists \(r_+ > 0\) such that
\[
\mu(\Lambda) \geq \rho_0 - r_+ \epsilon^2.
\]
(4.19)
We can now conclude from (4.12) that
\[
\langle u, (\mathcal{P}_\Lambda - \Lambda)u \rangle \geq \frac{1}{2} [\rho_0 \epsilon^2 - r_+ \epsilon^4 - \Lambda](\|w_+\|^2_2 + \|w_-\|^2_2),
\]
which immediately leads to (4.13).

4.4 Proof of Theorem 1.5

Let \(\Lambda = \Lambda_r + i\Lambda_i \in \mathbb{C}\) satisfy \(\Re \Lambda < \rho_0 \epsilon^2 - r_- \epsilon^4\). Let further \(u \in D(\mathcal{P}_\Lambda)\) and \(w_\pm = (\mathcal{L}_\pm - \Lambda)^{-1}u\). It can be easily verified that
\[
\Re \langle u, (\mathcal{L}_\pm - \Lambda)^{-1}u \rangle = \epsilon^2 \|w_\pm'\|^2_2 - \Lambda_r \|w_\pm\|^2_2.
\]
Consequently we may write that
\[
\Re \langle u, (\mathcal{P}_\Lambda - \Lambda)u \rangle = \frac{1}{2} \left(\epsilon^2 J^+_\Lambda(w_+) - \Lambda_r\|w_+\|^2_2 + \epsilon^2 J^-_\Lambda(w_-) - \Lambda_r\|w_-\|^2_2\right),
\]
which immediately leads to (4.13).
With the aid of (4.19) we then conclude that, for sufficiently small \( \epsilon \),

\[
\Re \langle u, (P_L - \Lambda)u \rangle \geq \frac{1}{2}(\rho_0 \epsilon^2 - \Lambda_r - r_- \epsilon^4)(\|w_+\|^2_2 + \|w_-\|^2_2),
\]

(4.20)
yielding

\[
\liminf_{\epsilon \to 0} \epsilon^{-2} \Re \sigma(B^I_\epsilon) \geq \rho_0.
\]

(4.21)

By Lemma 4.2 it holds that

\[
\limsup_{\epsilon \to 0} \epsilon^{-2} \Re \sigma(B^I_\epsilon) \leq \rho_0,
\]

which together with (4.21) achieves the proof of Theorem 1.5.

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