Macroscopic Quantum Tunneling of a Fluxon in a Long Josephson Junction

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Macroscopic quantum tunneling (MQT) for a single fluxon moving along a long Josephson junction is studied theoretically. To introduce a fluxon-pinning force, we consider inhomogeneities made by modifying thickness of an insulating layer locally. Two different situations are studied: one is the quantum tunneling from a metastable state caused by a single inhomogeneity, and the other is the quantum tunneling in a two-state system made by two inhomogeneities. In the quantum tunneling from a metastable state, the decay rate is estimated within the WKB approximation. Dissipation effects on a fluxon dynamics are taken into account by the Caldeira-Leggett theory. We propose a device to observe quantum tunneling of a fluxon experimentally. Required experimental resolutions to observe MQT of a fluxon seem attainable within the presently available micro-fabrication technique. For the two-state system, we study quantum resonance between two stable states, i.e., macroscopic quantum coherence (MQC). From the estimate for dissipation coefficients due to quasiparticle tunneling, the observation of MQC appears to be possible within the Caldeira-Leggett theory.

KEYWORDS: macroscopic quantum tunneling, fluxon, long Josephson junction, dissipation, Caldeira-Leggett theory, sine-Gordon equation, macroscopic quantum coherence

§1. Introduction

It is commonly recognized that the sine-Gordon equation plays an outstanding role in many physical problems. One of the most important applications is a long Josephson junction (LJJ). When the junction width is taken large enough in one direction (defined as x-direction), a phase difference $\phi$ of superconductors across the junction may have spatial
dependence in the $x$-direction. It is believed that dynamics of the phase difference $\phi$ is well described by a classical equation\(^\text{2)
}
\[ \phi_{tt} - \phi_{xx} + \sin \phi + \alpha \phi_t - \beta \phi_{xxt} + f = 0. \] (1.1)

Here $x$ and $t$ are measured in units of the Josephson penetration length $\lambda_J$ and of the inverse Josephson plasma frequency $\omega_p^{-1}$, respectively. The dissipation coefficient $\alpha$ is related to quasiparticle tunneling through the oxide barrier, and $\beta$ is related to the normal current of quasiparticle parallel to the junction. The external current $f$ is assumed spatially uniform.

It is known that solitons in the form of fluxons propagate along the junction following the classical equation (1.1). Experimentally, fluxons in LJJs were first observed indirectly by zero-field steps on the current voltage (I-V) characteristics of the junction\(^\text{2)
}). The zero-field steps are well explained by fluxon propagation governed by eq. (1.1) with repeated reflections at the open ends of the junction\(^\text{3)
}). Since then, development of experimental techniques has made it possible to directly observe profiles of separate fluxons and a space-time pattern of their interaction\(^\text{4,5)
}.

These experiments are basically explained by ‘classical’ theories, and there is no clear experimental indication so far which needs ‘quantum’ theories to explain its results. One might argue that it is natural to expect an essentially classical motion for a single fluxon of the size of micrometer. However, we show in this paper that quantum effects can indeed show up in this extended object. Recently, Hermon et al. discussed the quantum dynamics of a single fluxon in a long circular Josephson junction\(^\text{6)
}). In a subsequent paper\(^\text{7)
} they discussed a possible fluxon interference experiment. However in those papers, decoherence effects due to couplings to the environment, i.e., dissipation has been neglected. From the study in dissipative two-level systems\(^\text{8)
} it is expected that the quantum effects are strongly suppressed by the dissipation so that in real experiments, the observation of the quantum effects proposed by Hermon et al. may be more difficult than in the ideal case without the dissipation.

In this paper, we propose other experiments to observe quantum effects of a single fluxon. The effect which we deal with here is quantum tunneling of a fluxon. Quantum tunneling phenomenon is one of the most typical ones which cannot be explained by classical theories. Because a fluxon is a macroscopic object with a length scale of micrometer, the quantum tunneling of a fluxon can be recognized as the macroscopic quantum tunneling (MQT). The macroscopic tunneling phenomena have already been studied in some other systems. For example, much attention has been paid on MQT of the phase of current-biased junctions and the dissipation on it. MQT in this system has been observed experimentally\(^\text{9,10)
} and it has been claimed that the tunneling rate agrees with the value predicted by the Caldeira-Leggett theory\(^\text{11)
} within a phenomenological treatment of the dissipation.
To consider the quantum tunneling of a fluxon, we introduce structural inhomogeneities which capture a fluxon at a certain point in a Josephson junction line. Such inhomogeneities have been studied by McLaughlin and Scott in a pioneering paper. In that paper, they have considered interaction between a fluxon and a microresister (i.e., a narrow region where critical current density $j_c$ is reduced to $j'_c$) or a microshort (i.e., an narrow region with a enhanced critical current density). Schematic drawings of these inhomogeneities are shown in Fig. 1. McLaughlin and Scott have proposed a model of LJJ with a local inhomogeneity in the form of the equation of motion

$$\phi_{tt} - \phi_{xx} + \sin \phi + \alpha \phi_t - \beta \phi_{xxt} + f - \varepsilon \delta(x) \sin \phi = 0. \quad (1.2)$$

The last term in the left hand side represents a local change of the critical current density at $x = 0$. The normalized strength of the inhomogeneity, $\varepsilon$ is defined by $\varepsilon = (j_c - j'_c)b/j_c \lambda_J$. The cases $\varepsilon > 0$ and $\varepsilon < 0$ correspond to a microresister and a microshort, respectively. As we show later, when $\varepsilon > 0$ (i.e., a microresister), the fluxon is attracted to the microresister and captured there if the fluxon does not have enough kinetic energy. In other words, the microresister plays a role of a pinning potential for the fluxon. We only deal with the case $\varepsilon > 0$, i.e., microresisters. Throughout this paper, we assume $b \ll \lambda_J$ (i.e. $\varepsilon \ll 1$) so that the soliton size is larger than $b$ to justify the description by $\delta(x)$ in (1.2) for the microresister.

In this paper, we study the quantum tunneling of a fluxon in two situations. First, we consider the situation that there exists only one microresister in the junction. When the external current $f$ is applied on the junction, the fluxon-pinning state becomes metastable. We study quantum tunneling of a single fluxon from the metastable state. The decay rate
is estimated within the framework of the WKB approximation by taking proper values of experimental parameters. We also propose experimental design to observe quantum tunneling of a fluxon, and study conditions to allow the observation in this device. Next, we study the quantum tunneling between two stable states made by two microresisters. In this situation, we pursue possibility to observe resonance between two localized levels, i.e., macroscopic quantum coherence. In both cases, we consider dissipation effects using the Caldeira-Leggett theory.

The paper is organized as follows. In §2, we introduce a model Hamiltonian for fluxons in the analogy to the Caldeira-Leggett theory, and derive the effective action which describes dynamics of a fluxon. In §3, we summarize controllable experimental parameters, and set up them to the values which are accessible in practical experiments. Some quantities characteristic of LJJs, are also estimated. In §4, we consider quantum tunneling of a single fluxon from a metastable state, and in §5, we study quantum tunneling of a single fluxon in a two-state system made by two microresisters. Concluding remarks are given in §6.

§2. Formulation

2.1 Classical equation of motion for a single fluxon

In this subsection, we derive a classical equation of motion for a fluxon. For this purpose, we analyze the equation (1.2) with the assumption that parameters, $\alpha$, $\beta$, $f$ and $\varepsilon$ are all small. In this case, eq. (1.2) can be considered as a perturbed sine-Gordon equation. The soliton solution of the unperturbed sine-Gordon equation takes the form of a kink,

$$\phi_0(x, t; u) = 4 \arctan \left( \exp \left( \frac{x - q(t)}{1 - u^2} \right) \right),$$

(2.1)

which corresponds to a fluxon in LJJs. Here $q(t) = ut$ is the center coordinate of the kink, and $u$ is the kink velocity ($|u| < 1$). The velocity is normalized by the light velocity, $\bar{c} = \lambda J \omega_p$. In this paper, we only discuss the nonrelativistic limit $|u| \ll 1$.

Based on the kink solution (2.1), the classical perturbation theory can be applied to eq. (1.2). McLaughlin and Scott have shown that the perturbations only affect dynamics of the center coordinate $q(t)$ and do not change the form of the kink within the framework of the lowest approximation. The equation of motion for $q(t)$ is obtained as

$$m \ddot{q} + m \left( \alpha + \frac{\beta}{3} \right) \dot{q} + \frac{\partial V(q)}{\partial q} = 0,$$

(2.2)

in the nonrelativistic limit. Here $m$ is the classical soliton mass and identically equals to 8. Note that the damping strength working on the dynamics of a fluxon is related to $\alpha$ and $\beta$ which are the coefficients of the dissipative terms in the field equation (1.2) for $\phi(x, t)$. The
effective potential $V(q)$ for a fluxon is given by

$$V(q) = -2\pi fq - \frac{2\varepsilon}{\cosh^2 q}. \quad (2.3)$$

The first term is a driving force due to the external current, and the second term is a pinning potential caused by the microresister located at the origin.

### 2.2 The Caldeira-Leggett theory

As shown in eq. (2.2), the classical equation of motion for a fluxon includes a damping term due to the fact that the phase difference $\phi$ is a macroscopic variable. To deal with dissipation effects on MQT of a fluxon phenomenologically, the Caldeira-Leggett model is introduced in this subsection. We provide harmonic oscillators coupled to the macroscopic variable $q$, and write the Hamiltonian as

$$H = \frac{p^2}{2m} + V(q) + \sum_j \frac{1}{2} \left[ \frac{p_j^2}{m_j} + m_j\omega_j^2 \left( x_j - \frac{c_j}{m_j\omega_j} q \right)^2 \right]. \quad (2.4)$$

Here $p = m\dot{q}$ is a momentum variable conjugate to $q$, whereas $q_j$ and $p_j = m_j\dot{q}_j$ are a coordinate and a momentum of harmonic oscillators, respectively. The Caldeira-Leggett model has succeeded in explaining experiments of junctions phenomenologically. Here, we extend this treatment to LJJs. The reservoir parameters, $m_j$, $\omega_j$ and $c_j$ are characterized by a spectral function

$$J(\omega) = \frac{\pi}{2} \sum_j \frac{c_j^2}{m_j\omega_j} \delta(\omega - \omega_j). \quad (2.5)$$

We choose the spectral function as

$$J(\omega) = m \left( \alpha + \frac{\beta}{3} \right) \omega. \quad (2.6)$$

Then, the classical equation (2.2) is derived from the Hamiltonian (2.4) by eliminating the reservoir degrees of freedom.

From this model, we obtain the partition function $Z$ after integrating out the reservoir degrees of freedom as

$$Z = Z_R \oint Dq(\tau) \exp \left( -\frac{1}{g^2} S_{\text{eff}}[q(\tau)] \right). \quad (2.7)$$

Here $Z_R$ is the partition function of the unperturbed reservoir, and $g^2$ is the normalized Planck constant defined by $g^2 = \hbar\omega_0/E_0$ where $E_0$ is the energy scale in the LJJ. The effective action $S_{\text{eff}}$ is calculated as

$$S_{\text{eff}}[q(\tau)] = \int_0^{1/T} d\tau \left( \frac{1}{2} m\dot{q}^2 + V(q) \right) + \frac{1}{2} \int_0^{1/T} d\tau \int_0^\tau d\tau' K(\tau - \tau') \left( q(\tau) - q(\tau') \right)^2, \quad (2.8)$$
where $T$ is a temperature normalized by $\hbar \omega_p/k_B$. Here, $k_B$ is Boltzmann’s constant. The kernel $K(\tau)$ is given by

$$K(\tau) = \frac{1}{\pi} \int_0^\infty m \left( \alpha + \frac{\beta}{3} \right) \omega D_\omega(\tau) \, d\omega, \quad (2.9)$$

$$D_\omega(\tau) = \frac{\cosh \left[ \omega \left( \frac{1}{2T} - |\tau| \right) \right]}{\sinh (\omega/2T)}. \quad (2.10)$$

In the later sections, we make estimate of the tunneling rate based on the effective action (2.8).

The derivation of the effective action (2.8) is, however, quite intuitive. In the above derivation, infinite degrees of freedom of the field $\phi$ is first reduced to only one degree of freedom, i.e., the center coordinate $q$ in the equation of motion (2.2). Then, quantization for $q$ is performed. If the quantum fluctuation around a fluxon affects quantum tunneling, the effective action (2.8) may take a different form, because the fluctuation of the field is eliminated in the derivation of (2.2) before the quantization. In the next subsection, we check the validity of the effective action (2.8) by applying the Caldeira-Leggett formalism directly to the equation of motion for the field variable, eq. (1.2). The readers who are not interested in details of theoretical derivation may skip the next subsection.

2.3 Another derivation of the effective action

In this subsection, the effective action (2.8) is derived in another way. The plan is as follows. First, quantization for the field variable $\phi$ is performed. The dissipation effects are taken into account by constructing a model Hamiltonian on the analogy of the Caldeira-Leggett theory. Then, the semiclassical theory is applied to the model with the path integral method. Through the perturbation expansion for the parameters, $\alpha$, $\beta$, $f$, and $\varepsilon$, we show that the effective action (2.8) is reproduced within the lowest order approximation.

First, we derive the Hamiltonian which yields classical field equation (1.2). In the absence of dissipation, the Hamiltonian is easily obtained as

$$H_\alpha = \int dx \left[ \frac{\phi_x^2}{2} + \frac{\phi_t^2}{2} + (1 - \cos \phi) + f \phi - \varepsilon \delta(x)(1 - \cos \phi) \right]. \quad (2.11)$$

In the dissipative case, the dissipative terms, $-\alpha \phi_t$ and $\beta \phi_{xx}$, are obtained by introducing two kinds of harmonic oscillators coupled linearly via the field variables $\phi(x,t)$ and $\phi_x(x,t)$ as follows:

$$H_\alpha = \int dx \sum_j \left[ \frac{p_j^2(x)}{2m_j} + \frac{m_j \omega_j^2}{2} \left( q_j(x) - \frac{c_j}{m_j \omega_j^2} \phi(x) \right)^2 \right], \quad (2.12)$$
\[ H_\beta = \int dx \sum_j \left[ \frac{p_j^2(x)}{2m'_j} + \frac{m'_j \omega_j^2}{2} \left( q'_j(x) - \frac{c'_j}{m'_j \omega_j^2} \phi_x(x) \right)^2 \right]. \] (2.13)

These oscillators constitute a proper heat bath causing the dissipation.

The reservoir is characterized by the spectral functions \( J_\alpha(\omega) \) and \( J_\beta(\omega) \) defined by

\[ J_\alpha(\omega) = \pi \frac{c_j^2}{m_j \omega_j} \delta(\omega - \omega_j), \] (2.14)
\[ J_\beta(\omega) = \pi \frac{c'_j^2}{m'_j \omega'_j} \delta(\omega - \omega'_j). \] (2.15)

In order to produce the dissipative terms, \(-\alpha \phi_t\) and \(\beta \phi_{xxt}\), we choose the spectral functions as

\[ J_\alpha(\omega) = \alpha \omega, \] (2.16)
\[ J_\beta(\omega) = \beta \omega. \] (2.17)

The total Hamiltonian is given by

\[ H = H_S + H_\alpha + H_\beta. \] (2.18)

From this Hamiltonian, the perturbed sine-Gordon equation (1.2) is derived in the classical limit. The details of the derivation is given in Appendix A. The dissipative sine-Gordon system with the Hamiltonian \( H_\alpha \) has been used in the study of charge density wave and long Josephson junctions. It should be noted that the dissipation term \( \beta \phi_{xxt} \) in (1.2) is expressed in a simple way by the coupling to the derivative \( \phi_x \).

After integrating out the the reservoir degrees of freedom, we obtain the partition function \( Z \) as

\[ Z = Z_R^{(\alpha)} Z_R^{(\beta)} \oint \mathcal{D} \phi \exp \left( -\frac{1}{g^2} S_{\text{eff}}[\phi(x, \tau)] \right). \] (2.19)

Here \( Z_R^{(\alpha)} \) and \( Z_R^{(\beta)} \) are partition functions of the unperturbed harmonic oscillations, and \( g^2 \) is the normalized Plank constant defined in § 2.2. The effective action \( S_{\text{eff}} \) is given by

\[ S_{\text{eff}} = S_S + S_\alpha + S_\beta, \] (2.20)

where

\[ S_S = \int_0^{1/T} d\tau \int dx \left( \frac{1}{2} \phi_x^2 + \frac{1}{2} \phi_t^2 + (1 - \cos \phi) + f \phi - \varepsilon \delta(x)(1 - \cos \phi) \right), \] (2.21)
\[ S_\alpha = \frac{1}{2} \int_0^{1/T} d\tau \int_0^\tau d\tau' \int dx K_\alpha(\tau - \tau') \left( \phi(x, \tau) - \phi(x, \tau') \right)^2, \] (2.22)
\[ S_\beta = \frac{1}{2} \int_0^{1/T} d\tau \int_0^\tau d\tau' \int dx K_\beta(\tau - \tau') \left( \phi_x(x, \tau) - \phi_x(x, \tau') \right)^2. \] (2.23)
The kernels, $K_{\alpha}(\tau)$ and $K_{\beta}(\tau)$ are given by

\[
K_{\alpha}(\tau) = \frac{1}{\pi} \int_0^\infty \alpha \omega D_\omega(\tau) d\omega, 
\]

(2.24)

\[
K_{\beta}(\tau) = \frac{1}{\pi} \int_0^\infty \beta \omega D_\omega(\tau) d\omega, 
\]

(2.25)

where $D_\omega(\tau)$ is defined by (2.10).

Here we derive an effective theory of a single fluxon by approximating the quantization of the field theory defined by (2.21)-(2.23). For this purpose, we consider the semiclassical theory for the sine-Gordon equation. The semiclassical quantization of the field theory has been studied extensively in the context of high-energy physics in 1970’s.\cite{16,17} In the semiclassical method, the classical soliton solution is regarded as the ground state of a Fock space called the one-soliton sector. This sector is completely separated from the sector containing no soliton, because of the topological stability of the soliton. The states of the one-soliton sector are constructed by a perturbative expansion in $g^2$. This expansion is valid as long as $g^2$ is small. As shown in §3, $g^2$ is estimated as $g^2 < 10^{-2}$ in proper choices of experimental parameters. Hence, the semiclassical approach can be applied. In this subsection, we only explain the outline of this approach. The details of the semiclassical calculation is given in Appendix [B].

First we consider the unperturbed sine-Gordon theory. The partition function is given by

\[
Z_0 = \oint D\phi(x, \tau) \exp \left( - \frac{1}{g^2} S_0[\phi(x, \tau)] \right). 
\]

(2.26)

Here $S_0$ is the Euclidean action,

\[
S_0 = \int d\tau \int dx \left( \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi_x^2 + (1 - \cos \phi) \right), 
\]

(2.27)

where $\dot{\phi} = \frac{\partial \phi}{\partial t}$. This field theory possesses a nontrivial stationary solution

\[
\phi_0(x - q) = 4 \arctan \left[ \exp(x - q) \right], 
\]

(2.28)

where $q$ is the center coordinate of the soliton. This new variable $q$ is called a collective coordinate in the field theory, and regarded as a dynamical variable. In the semiclassical approach, we only consider the paths around the stationary solution because they predominantly contribute to the partition function (2.26). The deviation around the stationary solution is denoted by $\eta$ as

\[
\phi(x, \tau) = \phi_0(x - q(\tau)) + g\eta(x - q(\tau), \tau). 
\]

(2.29)

In terms of LJJ’s, the collective coordinate $q(\tau)$ describes dynamics of a single fluxon, and the rest degrees of freedom, $\eta(x, \tau)$ represent the fluctuation around the fluxon. The perturbation
expansion of the action (2.27) in terms of η is nothing but that in terms of $g^2$. In the lowest approximation, we obtain ‘plasmon’ excitations by the quantization of the quadratic parts for η in the action. The plasmon excitations have a energy gap of $\hbar \omega_p$. Since the energy gap is estimated as several Kelvin, the plasmon excitation can be neglected at low enough temperatures of the order of mK. The higher order terms of η in the action only contribute to the partition function as the higher order of $g^2$. The semiclassical expansion of $g^2$ needs careful treatments especially in the treatment of the collective coordinate $q$. (See Appendix B.) We only write the result:

$$Z_0 \approx \int Dq(\tau) \exp \left( -\frac{1}{g^2} \int d\tau \left( m + \frac{m}{2} \dot{q}^2 \right) \right).$$

(2.30)

From this result, we find that the fluxon behaves like a free particle with the mass $m = 8$. It is a natural result because there is no force which modifies the velocity of the soliton.

Next we consider the perturbed sine-Gordon equation (1.1). The perturbation part of the action is given by

$$S_{\text{ext}}[\phi(x, \tau)] = \int_0^{1/T} d\tau \int dx f(\phi),$$

(2.31)

$$S_{\text{pin}}[\phi(x, \tau)] = \int_0^{1/T} d\tau \int dx \epsilon \delta(x)(1 - \cos \phi),$$

(2.32)

in addition to (2.22) and (2.23). Here we assume that the perturbation parameters, α, β, f, ε are all small. Then, we take the perturbation expansion in terms of these small terms in addition to the expansion in $g^2$. From straightforward calculation, it is shown that the lowest order contribution of the perturbation expansion is obtained only by substituting the soliton solution (2.28) to the perturbative actions. In other words, the perturbations do not modify the waveform of the soliton in the lowest order.

First, we consider the external current term (2.31). Substitution of (2.28) gives

$$S'_{\text{ext}}[q(\tau)] = S_{\text{ext}}[\phi_0(x - q(\tau))] - S_{\text{ext}}[\phi_0(x)]$$

$$= \int_0^{1/T} d\tau (-2\pi f q).$$

(2.33)

Here the prime implies the effective action for the variable $q(\tau)$. We subtracted the constant $S_{\text{ext}}[\phi_0(x)]$, because we have chosen the origin of the potential energy at $q = 0$.

The pinning potential is obtained by substituting (2.28) to (2.32),

$$S'_{\text{pin}}[q(\tau)] = S_{\text{pin}}[\phi_0(x - q(\tau))]$$

$$= \int_0^{1/T} d\tau \frac{-2\epsilon}{\cosh^2 q}.$$  

(2.34)

Finally, we consider the dissipation described by (2.23) and (2.22). The effective action is obtained as

$$S'_{\alpha}[q(\tau)] = S_{\alpha}[\phi_0(x - q(\tau))].$$
\[ S'_\beta[q(\tau)] = S''_{\beta}[\phi_0(x - q(\tau))] \]

\[ = \frac{1}{2} \int_0^{1/T} d\tau \int_0^\tau d\tau' \int dx K_\beta(\tau - \tau') \left( \phi'_0(x - q(\tau)) - \phi'_0(x - q(\tau')) \right)^2 \]

where \( \phi'_0 = \frac{\partial \phi_0}{\partial x} \). Since the integration over \( x \) is difficult to carry analytically, we assume that the paths of the collective coordinate \( q(\tau) \) satisfy

\[ |q(\tau) - q(\tau')| \ll 1 \tag{2.37} \]

for all \( \tau, \tau' \). This condition is well satisfied in the case we consider in this paper. Using the approximations

\[ \phi_0(x - q(\tau)) - \phi_0(x - q(\tau')) \simeq -\phi'_0(x - q(\tau)) \left( q(\tau) - q(\tau') \right) \tag{2.38} \]

\[ \phi'_0(x - q(\tau)) - \phi'_0(x - q(\tau')) \simeq -\phi''_0(x - q(\tau)) \left( q(\tau) - q(\tau') \right) \tag{2.39} \]

and the identities

\[ \int dx \phi'^2_0 = 8 \equiv m \quad \text{and} \quad \int dx \phi''_0^2 = \frac{m}{3} \tag{2.40} \]

we obtain from (2.35) and (2.36)

\[ S'_{\alpha}[q(\tau)] = \frac{1}{2} \int_0^{1/T} d\tau \int_0^\tau d\tau' mK_\alpha(\tau - \tau') \left( q(\tau) - q(\tau') \right)^2 \tag{2.41} \]

\[ S'_{\beta}[q(\tau)] = \frac{1}{2} \int_0^{1/T} d\tau \int_0^\tau d\tau' \frac{m}{3} K_\beta(\tau - \tau') \left( q(\tau) - q(\tau') \right)^2 \tag{2.42} \]

We summarize the total effective action of a fluxon. The partition function is given by

\[ Z = \oint Dq(\tau) \exp \left( -\frac{1}{g^2} S'_{\text{eff}}[q(\tau)] \right) \tag{2.43} \]

where

\[ S'_{\text{eff}}[q(\tau)] = \int d\tau \left( \frac{1}{2} m\dot{q}^2 + V(q) \right) + \frac{1}{2} \int d\tau \int d\tau' K(\tau - \tau') \left( q(\tau) - q'(\tau) \right)^2 \tag{2.44} \]

\[ V(q) = -2\pi f q - \frac{2\varepsilon}{\cosh^2(q)} \tag{2.45} \]

The kernel \( K(\tau) \) is given by

\[ K(\tau) = \frac{1}{\pi} \int \frac{D\omega}{m \left( \alpha + \frac{\beta}{3} \right)} D\omega(\tau) \]
where $D_{\omega}(\tau)$ is defined in eq. (2.10). The effective action obtained here agrees with one given in eq. (2.8), and it has been shown that more intuitive derivation in § 2.2 is justified. One might feel that this result is trivial. It is, however, nontrivial because the fluxon has a finite size in space. In fact, if the condition (2.37) does not hold, the action has a different form from the simple action (2.44).

§3. Experimental Parameters

In this section, we summarize controllable experimental parameters relevant to the quantum tunneling of a fluxon. A layout of a LJJ is shown in Fig. 2, and experimentally parameters realizable within the available technique are given in Table I. The values given in the table are indeed typical ones in actual experiments. The width of the Josephson junction $W$ is now a controllable variable and scaled by micrometer.

First, we associate the experimental parameters with the parameters in the normalized sine-Gordon equation (1.1). In the derivation of (1.1), the normalized length, time and energy are introduced. In the original unit, the equation of motion without inhomogeneities is written as

$$\omega_p^{-2}\phi_{tt} - \lambda_J^2\phi_{xx} + \sin\phi + \alpha\omega_p^{-1}\phi_t - \beta\omega_p^{-1}\lambda_J^2\phi_{xxt} + j/j_c = 0.$$  (3.1)

Here, $j$ is the external current density applied to the junction.

The length scale in the LJJ is given by the Josephson penetration length $\lambda_J$, which may be varied in the range from 20 $[\mu m]$ to 200 $[\mu m]$ by controlling the critical current density $j_c$. In the present estimate, $\lambda_J$ is obtained as

$$\lambda_J = \left( \frac{\Phi_0}{4\pi\mu_0L_Jj_c} \right)^{1/2} \sim 27 [\mu m].$$  (3.2)
Table I. Experimental parameters used for an estimate.

| Parameters                                      | Estimated value       |
|------------------------------------------------|-----------------------|
| $d$ : Thickness of the junction                 | 2 [nm]                |
| $\varepsilon_r$ : Relative dielectric constant  | 8                     |
| $j_c$ : Critical current density                | $3 \times 10^6$ [A/m$^2$] |
| $W$ : Width of the junction                     | $W$ [$\mu$m] (variable) |
| $\lambda_L$ : London penetration length         | 60 [nm]               |

Here $\Phi_0 = h/2e$ is the unit flux and $\mu_0 (= 4\pi \times 10^{-7}$ [Hm$^{-1}$]) is the permeability of the vacuum. The size of a fluxon is characterized by $\lambda_J$ and the junction length $L$ must be taken larger than $\lambda_J$ so that at least a single fluxon exists in the junction.

The time is scaled by $\omega_p^{-1}$ where $\omega_p$ is the plasma frequency, which is estimated from the value given in Table I as

$$\omega_p = \left( \frac{2\pi j_c d}{\Phi_0 \varepsilon_r \varepsilon_0} \right)^{1/2} \sim 5.1 \times 10^{11}$ [1/s],

(3.3)

where $\varepsilon_0 (= 8.85 \times 10^{-12}$ [Fm$^{-1}$]) is the dielectric constant of the vacuum. Note that $\omega_p$ is independent of $W$. From this estimate, the plasmon excitation gap is estimated as

$$\hbar\omega_p \sim 3.9$ [K].

(3.4)

Well below this temperature, the plasmon excitations can be neglected.

The energy is measured by a unit energy $E_0$,

$$E_0 = \frac{\Phi_0}{2\pi} j_c W \lambda_J \sim 1.93 \times 10^3 W$ [K].

(3.5)

From this, the energy of a single fluxon is obtained,

$$8E_0 \sim 1.55 \times 10^4 W$ [K]

(3.6)

where the factor 8 comes from the dimensionless mass of a fluxon. The energy of a single fluxon is so large that nucleation of soliton-antisoliton pairs can be neglected at sufficiently low temperatures.
The normalized Planck constant, $g^2$ is given as

$$\frac{\hbar \omega_p}{E_0} = \frac{16\pi}{137} \left( \frac{2\lambda d}{W\varepsilon_t} \right)^{1/2}. \quad (3.7)$$

Note that $g^2$ is independent of the critical current $j_c$. In the present estimate, the value of $g^2$ becomes

$$g^2 \sim 2.0 \times 10^{-3} \frac{W}{W}. \quad (3.8)$$

When $W \sim 1[\mu m]$, we get small value, $g^2 \sim 0.002$. Hence, the semiclassical approach explained in §2.3 is applicable.

The mass of a fluxon is estimated as

$$m_f = \frac{8E_0}{\bar{c}^2} \sim 1.25 \times 10^{-3} W m_e, \quad (3.9)$$

where $m_e$ is the electron mass and $\bar{c}$ is defined by $\bar{c} = \lambda_j \omega_p$. Since we find that the fluxon mass is remarkably small ($\sim 10^{-3} m_e$ if we take $W$ as $1[\mu m]$), we can expect substantial quantum effects of fluxons, in spite of the large fluxon size and the small normalized Planck constant. Moreover, it should be noted that $m_f$ is proportional to the junction width $W$. Hence, the fluxon mass can be controlled by changing the value of $W$.

Finally, we estimate the dissipation coefficients, $\alpha$ and $\beta$. The coefficient $\alpha$ is related to the quasiparticle resistance per area $r_{qp}$ as $\alpha = 1/r_{qp} C' \omega_p$. Here, $C' = \varepsilon_r \varepsilon_0 / d$ is the capacitance per unit area. This relation has been obtained experimentally by Pedersen and Welner.\(^{18}\) The quasiparticle resistance $r_{qp}$ obtained below the gap voltage is strongly enhanced at low temperatures. Hence, the dissipation coefficient $\alpha$ becomes very small at sufficiently low temperatures of mK order.

To make a comparison, we introduce a dissipation coefficient $\alpha_n$ defined by $\alpha_n = 1/r_n C' \omega_p$. Here, $r_n$ is the normal resistance obtained above the gap voltage. Since $r_n$ is much smaller than $r_{qp}$, the dissipation coefficient $\alpha_n$ gives the upper limit for $\alpha$, i.e., $\alpha \ll \alpha_n$. Then, $r_n$ can be associated with $j_c$ as\(^{23}\)

$$j_c r_n = \frac{\pi \Delta_0}{2e}, \quad (3.10)$$

at sufficiently low temperatures ($k_B T \ll \Delta_0$). Here $\Delta_0$ is the gap of the superconductor at zero temperature. From (3.3) and (3.10), we obtain

$$\alpha_n = \frac{\hbar \omega_p}{\pi \Delta_0}, \quad (3.11)$$

which gives a good estimate for $\alpha_n$. Using $\Delta_0 = 14[K]$ and (3.3), we have $\alpha_n = 0.088$. Thus, we evaluate $\alpha \ll 0.088$.

The dissipation coefficient $\beta$ also originates from quasiparticle current. It has been observed that the ratio between $\alpha$ and $\beta$ is independent of the temperature.\(^{21, 22}\) Hence, it is expected
that $\beta$ is also strongly suppressed at sufficiently low temperatures. We estimate the upper bound for $\beta$ as $\beta \ll 0.01$, which is an experimentally determined value by Davidson et al.\cite{21, 22} at relatively high temperatures ($\sim 4$ [K]). This evaluation for $\beta$ is rough, because the experimental situation in Ref.\cite{21} is rather different from ones employed in this paper. The dissipation amplitude $\alpha + \beta/3$ is, however, not changed drastically even if $\beta$ is enlarged several times, because $\alpha$ is expected to be dominant in the dissipation amplitude.

On the basis of the values given in this section, we estimate the tunneling rate in the later sections.

§4. Tunneling from a Metastable State

4.1 Tunneling rate

In this section, we consider the quantum tunneling from a metastable state made by a single microresister in a LJJ. First, we formulate the decay rate using the Langer’s WKB method.\cite{19, 20} The potential form $V(q)$ is given in (2.45) as

$$V(q) = -2\pi f q - \frac{2\varepsilon}{\cosh^2 q}. \quad (4.1)$$

The first term is the driving force due to the external current, and the second term is the pinning potential caused by a single microresister. If $f$ is small, the potential $V(q)$ has a metastable state at $q = q_0$ defined by $V'(q_0) = 0$. However, if $f$ is increased and takes a critical value $f_c$, the metastable state disappears. The critical value $f_c$ is calculated as

$$f_c = \frac{4\varepsilon}{3\sqrt{3\pi}} \approx 0.245\varepsilon(\ll 1). \quad (4.2)$$

When the external current $f$ is taken as $f = f_c - \eta$ ($\eta > 0$), the potential energy has a barrier $V_0$ as shown in Fig. 3. To observe quantum tunneling in the laboratory, $V_0$ must be small. Hence, we assume $\eta \ll f_c$. The potential is approximated by the quadratic-plus-cubic potential around a metastable state as

$$V(x) = -\frac{16\varepsilon}{9\sqrt{3}} x^3 + \frac{32\pi\varepsilon \eta}{3\sqrt{3}} x^2, \quad (4.3)$$

where the origin of the coordinate $x$ is located at a metastable state ($q = q_0 + x$). From this approximated potential, the potential barrier height $V_0$ is calculated as

$$V_0 = \sqrt{\frac{8\pi^3 \eta^3}{\sqrt{3}\varepsilon}} \approx 12.0\varepsilon^{-1/2} \eta^{3/2}. \quad (4.4)$$

The frequency of small oscillation around the metastable minimum defined by $\omega_0 = (V''(0)/m)^{1/2}$ is also obtained as

$$\omega_0 = \left(\frac{2\pi\varepsilon \eta}{3\sqrt{3}}\right)^{1/4} \approx 1.05\varepsilon^{1/4} \eta^{1/4}, \quad (4.5)$$
Fig. 3. The potential form $V(q)$ for a fluxon (a) when the external current $f$ is taken as the critical value $f_c$, and (b) when $f$ is taken as a slightly smaller value $f = f_c - \eta(\eta > 0)$ than $f_c$.

where the fluxon mass $m = 8$ has been substituted. Note that $V_0$ and $\omega_0$ are normalized by $E_0$ and the plasma frequency $\omega_p$, respectively. For reference, we consider the exit point $x_0(> 0)$ defined by $V(x_0) = 0$. It is easily calculated as

$$x_0 = \sqrt{\frac{9\sqrt{3}\eta}{8\pi \varepsilon}} \cong 0.79\eta^{1/2}\varepsilon^{-1/2}. \quad (4.6)$$

The quantum decay rate from the metastable state at $T = 0$ can be calculated by applying Langer’s method to the effective action (2.8). The tunneling rate $\Gamma$ takes the form

$$\Gamma = A \exp(-B). \quad (4.7)$$

The exponent $B$ is determined by the action of the nontrivial path $q_B(\tau)$ which minimizes the effective action (2.8). This path is called a bounce solution. As far as $x_0 \ll 1$, $q_B(\tau)$ satisfies

$$|q_B(\tau) - q_B(\tau')| \ll 1 \quad (4.8)$$

for all $\tau$, $\tau'$. Therefore, the assumption (2.37) is satisfied. The explicit forms of $A$ and $B$ are obtained only for the limiting cases. In actual experiments, the damping coefficient $\alpha + \beta/3$ in eq. (2.2) is small ($\ll 0.1$ in the present situation). Then, the estimate in the weak damping limit at $T = 0$ is obtained by a perturbative treatment as

$$A = \sqrt{60\omega_0\omega_p} \left( \frac{B}{2\pi} \right)^{1/2} (1 + O(a)), \quad (4.9)$$

$$B = \frac{36V_0E_0}{5\hbar\omega_0\omega_p} \left( 1 + 1.74a + O(a^2) \right) \sim \frac{82.2\eta^{5/4}\varepsilon^{-3/4}}{g^2} (1 + 1.74a + O(a^2)) \quad (4.10)$$
where \( a = (\alpha + \beta/3)/2\omega_0 \). Since the predominant effect of the dissipation appears in the
exponent of the decay rate, the correction for the prefactor due to the dissipation is neglected
in this paper.

Since \( g^2 \) takes a small value as given in \( \S \), the exponent \( B \) takes a fairly large value. To
observe MQT of a fluxon in a time scale of the laboratory, we need to take a value of the
parameter \( \eta \) small to increase the decay rate. For example, if we take \( \eta = 5 \times 10^{-4}, \varepsilon = 0.1 \),
then the values of \( V_0 \) and \( \bar{h}\omega_0 \) are estimated from (4.4) and (4.3) as
\[
V_0 E_0 \sim 0.82 W [K] \tag{4.11}
\]
\[
\bar{h}\omega_0 \omega_p \sim 0.34 [K] \tag{4.12}
\]
in the original unit. We note that \( \eta = 5 \times 10^{-4} \) and \( \varepsilon = 0.1 \) are not difficult to realize
experimentally. When \( W = 1[\mu m] \) and dissipation effects are neglected (\( \alpha = \beta = 0 \)), the
tunneling rate \( \Gamma \) is estimated from (4.7) with (4.9) and (4.10) as
\[
\Gamma \sim 2 \times 10^4 [1/s], \tag{4.13}
\]
which is large enough to observe the MQT in the laboratory.

4.2 Experimental design to observe MQT of a fluxon

In this subsection, we propose experimental apparatus to observe the macroscopic tun-
neling of a single fluxon efficiently. A schematic configuration of the considered Josephson
junction is shown in Fig. 4. Two superconducting cylinders are separated by a thin insulating
layer, where one fluxon is captured. The fluxon is accelerated by an externally driven current
\( I \), which is assumed to be spatially-uniform. A single microresister is made by thickening
the insulating layer locally. When \( I = 0 \), the fluxon is captured at the microresister. The
critical current of the junction is denoted by \( I_0 \).

Experiments on the junctions with a similar topological geometry have already been per-
dformed. In these experiments, circular LJJs with the width \( W \sim 10[\mu m] \) have been
fabricated. Further, by using scanning electron microscopy, it has been possible to introduce
individual fluxons into such a system.

The method to observe MQT of a fluxon may be very similar to the experiment technique
used by Voss and Webb.\textsuperscript{9} When the external current \( f = I/I_0 \) is small, the fluxon stays at
a metastable state caused by the microresister. When the current \( f \) approaches the critical
current \( f_c \) given by (4.2), the metastable state vanishes and the fluxon becomes free to move.
The depinning of a fluxon may occur before \( f \) becomes \( f_c \) because of the quantum tunneling.
After the depinning, moving fluxons generate a voltage across the junction. Therefore, we
can probe the depinning of a fluxon by observing an I-V curve of the junction (See Fig. 5).
Since the tunneling is a stochastic process, the switching current to voltage states has a
Fig. 4. A schematic configuration of the proposed experimental apparatus. An external current $I$ is applied to the circular LJJ. Only one fluxon is captured through the junction. A microresister is made in the junction.

distribution. By applying an alternating current, a lot of fluxon-depinning events can be observed. (The moving fluxon is captured in repeated intervals which satisfy $|f| \ll f_c$.) The probability distribution of the depinning current $P(f)$ is related to the tunneling rate $\Gamma(f)$ as

$$P(f) df = \Gamma(f) \left( 1 - \int_0^f P(f') df' \right) dt. \quad (4.14)$$

By solving this equation for $P(f)$, we obtain

$$P(f) = \left| \frac{df}{dt} \right|^{-1} \Gamma(f) \exp \left( - \left| \frac{df}{dt} \right|^{-1} \int_0^f \Gamma(f') df' \right), \quad (4.15)$$

where $|df/dt|$ is a sweep rate of the external current, which must be small compared with $\omega_p$. Thus, we may obtain the tunneling rate $\Gamma(f)$ by measuring $P(f)$ experimentally.

4.3 Simulation of the distribution of depinning current

To observe MQT of a fluxon in the experiments proposed in the previous subsection, several experimental requirements must be satisfied. Among them, requirements for two physical
Fig. 5. The expected I-V curve of a circular LJJ. Depinning of a fluxon can be measured by switching to voltage states. The averaged switching current and the distribution width are denoted with $f - \langle \eta \rangle$ and $\Delta \eta$, respectively. The tunneling rate $\Gamma(f)$ is obtained by the form of the distribution.

quantities are important. One is the current accuracy to measure $\Delta \eta$, and the other is the temperature $T$.

To examine whether the observation of MQT is possible, we first estimate $\Delta \eta$ by using the estimated values given in § 3. We perform numerical integration of (4.15) to obtain $P(f)$. The tunneling rate $\Gamma(f)$ is obtained by substituting (3.8) to (4.9) and (4.10) in the original unit as

$$
\Gamma(f) \simeq 3.1B^{1/2}\omega_0\omega_p \exp(-B), \tag{4.16}
$$

$$
B \sim 4.1 \times 10^4 \eta^{5/4} \varepsilon^{-3/4} W(1 + 1.74a), \tag{4.17}
$$

$$
\omega_0 \simeq 1.05 \varepsilon^{1/4} \eta^{1/4}, \tag{4.18}
$$

where $\omega_p$ is estimated in eq. (3.3), and $a = (\alpha + \beta/3)/2\omega_0$ is assumed to be small. Here $\varepsilon$ is taken as 0.1 throughout this paper, which may be realized by thickening the insulator layer locally with zero critical current density in a range of $0.1\lambda_j \sim 2.7 [\mu m]$, which can be designed in the present available technique. For convenience, we take a serrated wave as an alternating current form as shown in Fig. 6. The amplitude of the current is taken as twice of $f_c$ and the frequency is taken as $1/T = 15 [1/s]$ to be compared with the experiment.
Fig. 6. The form of the external current used in the estimate. The amplitude and the frequency of the alternating current is taken as $2f_c$ and $1/T = 15[1/s]$, respectively.

Numerical calculation of (4.15) for $W = 1[\mu m]$ and $\varepsilon = 0.1$ is performed for two cases; we call the first a strongly dissipative case ($\alpha + \beta/3 = 0.091$), and the second a dissipationless case ($\alpha = \beta = 0$). Since the dissipation coefficient $\alpha + \beta/3$ becomes very small at low temperatures in actual experiments, we expect that the distribution of the depinning current is almost the same as dissipationless case. To study dissipation effects, however, we also consider the strongly dissipative case, in which the damping coefficients are taken as their upper bound estimated in §3. The value of $\Delta \eta$ in the strongly dissipative case gives the lower bound for $\Delta \eta$ in actual experiments.

The obtained distribution is shown in Fig. 7. From the distribution, we obtain $\Delta \eta = 2.0 \times 10^{-5}$, $\langle \eta \rangle = 2.6 \times 10^{-4}$ for strongly dissipative case, and $\Delta \eta = 3.1 \times 10^{-5}$, $\langle \eta \rangle = 4.9 \times 10^{-4}$ for dissipationless case. We find that $\Delta \eta$ is suppressed by dissipation effects. We also find that dissipation effects on measurement of the quantum tunneling are not small but modest. For the strongly dissipative case, necessary accuracy of the current measurement is given by $\Delta \eta/f_c \sim 8 \times 10^{-4}$. On the other hand, in the experiment performed by Voss and Webb, $\Delta I/I_0 < 2 \times 10^{-3}$ has already been realized. Here $\Delta I$ is the distribution width of the switching currents to the voltage states, and $I_0$ is the critical current. Therefore, if circular LJJ$s$ with $W = 1[\mu m]$ can be fabricated, accuracy of the current measurement to observe $\Delta \eta$ seems attainable in the present available techniques.

If we change the junction width $W$, the distribution width $\Delta \eta$ takes a different value. The $W$-dependence of $\Delta \eta$ is shown in Fig. 8 for both strongly dissipative case and dissipationless case. From this dependence, we find that when the junction width $W$ is taken larger, more
accurate measurement for the current is needed. On the contrary, when $W$ is taken smaller, $\Delta \eta$ becomes large, and the observation of MQT of a fluxon becomes easier. In practical experiments, we should take a proper value of $W$ so that the measurement of $\Delta \eta$ is possible. By fitting the estimated data, we obtain $\Delta \eta \propto W^{-0.787}$ for dissipationless case. The value of the power ($\sim -0.8$) originates from the fact that the exponent is proportional to $\eta^{5/4} W$. For strongly dissipative case, fitting gives $\Delta \eta \propto W^{-0.863}$. As seen in Fig. 8, dissipation always suppresses $\Delta \eta$, and the suppression becomes large with the increase of $W$. This suppression is explained by the enhance of the normalized damping strength $a = (\alpha + \beta/3)/2\omega_0$ due to the decrease of $\omega_0 \sim \langle \eta \rangle^{1/4}$.

Next, we consider finite temperature effects. At high temperatures, thermally-activated decay occurs. The decay rate is formulated as

$$\Gamma_{\text{th}} = \frac{\omega_0}{2\pi} \exp \left( -\frac{V_0}{k_B T} \right),$$

where $V_0$ is the energy barrier. Here $V_0$ and the temperature $T$ are measured in the original unit. It is expected that there exists a crossover temperature $T^*$ which separates the thermal activated region and the quantum tunneling region. Below $T^*$, the decay rate becomes
Fig. 8. The theoretical $W$-dependence of $\Delta \eta$ with and without damping for $j_c = 3 \times 10^6 [A/m^2]$, $f_c = 0.0245$. Fitting gives $\Delta \eta \propto W^{-0.863}$ for the strongly dissipative case, and $\Delta \eta \propto W^{-0.787}$ for the dissipationless case.

independent of the temperature. The crossover temperature $T^*$ is defined as:

$$T^* = \frac{\hbar \omega_b}{2\pi k_B} \left((1 + \gamma^2)^{1/2} - \gamma\right),$$

where $\gamma = \alpha + \beta/3$, and $\omega_b$ is the frequency at the top of the potential barrier. For the quadratic-plus-cubic potential, $\omega_b$ always agrees with $\omega_0$, which can be controlled by the external current $f = f_c - \eta$. We assume that $T^*$ is averaged by the current sweep to the value at $f = f_c - \langle \eta \rangle$. Then, we obtain the crossover temperature for $W = 1 [\mu m]$ as $T^* \sim 54 [mK]$ for dissipationless cases, and as $T^* \sim 42 [mK]$ for strongly dissipative case. Here, it appears that the suppression of $T^*$ due to damping is not so large.

To observe MQT of a fluxon, the junction must be cooled to low enough temperatures below $T^*$. We show in Fig. 9 theoretical widths $\Delta \eta$ expected from the pure thermal activation by eq. (4.19) as well as from the pure MQT with and without damping by eq. (4.16) with (4.17) for $W = 1 [\mu m]$. The expected temperature-dependences of $\Delta \eta$ are also shown by the bold solid curve when both effects coexist. From this calculation, it seems that the crossover temperature $T^*$ estimated by (4.20) gives a proper criterion.

Finally, we consider the case that the experimental parameters given in Table I are modified. Since the accurate measurement of the parameters, $\lambda_L$, $\varepsilon_r$, and $d$ is difficult, we only
Fig. 9. The theoretical estimate for temperature-dependence of $\Delta \eta$ for the pure thermal activation and the pure MQT tunneling with and without damping. The behaviors of $\Delta \eta$ for the case with both effects are also shown by the bold thick curves. The crossover temperature $T^*$ obtained by (4.20) is 42 [mK] for the strongly dissipative case, and 54 [mK] for the dissipationless case.

Consider $j_c$-control. The critical current density $j_c$ can be changed in a wide range by controlling the thickness of the oxide barrier \[24\]. The $j_c$-dependence of the estimated quantities is given from (3.2), (3.3) and (3.5) by

\[
\lambda_3 \propto j_c^{-1/2}, \\
E_0 \propto j_c^{1/2}, \\
\omega_p \propto j_c^{1/2}.
\]

(4.21) \hspace{1cm} (4.22) \hspace{1cm} (4.23)

Since $g^2 = \hbar \omega_p/E_0$ is independent of $j_c$, we cannot control quantum fluctuations by $j_c$. Instead of it, the damping parameter $\alpha$ can be controlled by $j_c$. When the gap of a superconductor $\Delta_0$ is fixed, $j_c$-dependence of $\alpha$ is obtained from (3.3) and (3.11) as

\[
\alpha \propto j_c^{1/2}.
\]

(4.24)

According to this relation, the dissipation effects get important with the increase of $j_c$. The $j_c$-dependence of $\Delta \eta$ for $W = 1 [\mu m]$ with and without damping is shown in Fig. 10. The widths $\Delta \eta$ are almost independent of $j_c$ in the dissipationless case, while $\Delta \eta$ is suppressed in the strongly dissipative case with the increase of $j_c$.

From (4.23), the crossover temperature $T^* \propto \omega_p$ is also controlled by $j_c$. The $j_c$-dependence of $T^*$ calculated from (1.20) with and without damping for $W = 1 [\mu m]$ is shown in Fig. 11.
Fitting the data without damping gives $T^* \propto j_c^{0.506}$, which is nearly the value expected from (4.23). The deviation originates from the small change in $\langle \eta \rangle$. In the strongly dissipative case, $T^*$ is suppressed, and the suppression from the dissipationless case grows with the increase of $j_c$, because the damping effects become important through (4.24). At any rate, however, the absolute value of the crossover temperature $T^*$ is enhanced with the increase of $j_c$. Hence, $j_c$ should be taken large to observe MQT of a fluxon more easily in the range of $j_c$ given in Fig. 11.

§5. Tunneling in a Two-State System

In this section, we study a two-state system made by two microresistors. In classical mechanics, a fluxon may stay at either of two stable states at zero temperature. In quantum mechanics, however, quantum tunneling through the energy barrier is possible. When the energy levels of the ground state at each well is the same and dissipation is neglected, the quantum tunneling makes an energy splitting and generates oscillation of a fluxon between the two wells retaining the coherence. This macroscopic effects is called quantum macroscopic coherence (MQC).

In the last ten years, MQC has been investigated theoretically in detail. MQC is the key phenomenon to clarify the validity of quantum mechanics at macroscopic level. MQC is far more sensitive to environmental suppression than MQT because a coupling between the relevant macroscopic variable and its environment rapidly destroys the phase coherence between two states, while we need to retain phase coherence for much longer time for MQC...
than MQT. Thus MQC is in general much harder to observe. Several experiments have been proposed to observe MQC on current biased Josephson junctions and SQUIDs. However, to date, there is no evidence for MQC in spite of substantial effort. Here, we mainly focus on dissipation effects on MQC in LJJs, and pursue possibility to observe MQC in this system.

When there exists no external current \( (f = 0) \), we obtain the potential made by two microresisters as

\[
V(q) = -\frac{2\varepsilon}{\cosh^2(q - l/2)} - \frac{2\varepsilon}{\cosh^2(q + l/2)},
\]

where the origin of \( q \) is taken at the midpoint of the resistors, and the distance between the resistors is denoted with \( l \). The potential \( V(q) \) has only one stable state for small \( l \), while the potential has a double well structure, when \( l > l_0 \). The critical length \( l_0 \) is given by

\[
l_0 = \ln \left( \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right) \approx 1.317.
\]

We only consider the case that the potential barrier is small. In this situation, we may assume \( l = l_0 + a \) with \( a \ll 1 \). The potential term can be expanded around \( q = 0 \) as

\[
V(q) = \frac{32}{27}\varepsilon q^4 - \frac{16}{3\sqrt{3}}\varepsilon aq^2.
\]

From this potential, the position of the stable states is obtained as \( q = \pm q_0 \) where

\[
q_0 = \sqrt{\frac{3\sqrt{3}}{4}}a \approx 1.14a^{1/2}.
\]
When \( a \ll 1 \), we obtain \( q_0 \ll 1 \) and the expansion in the form of the potential (5.3) is valid around the two stable states.

The amplitude of the dissipation is determined by a dimensionless quantity \( K \) as

\[
K = m \left( \alpha + \frac{\beta}{3} \right) \left( \frac{2q_0}{2\pi g^2} \right)^2.
\]  
(5.5)

Here \( m = 8 \) is the mass of a fluxon. By using the estimated values of \( g^2 \), we estimate \( K \) as

\[
K \sim 3.3 \times 10^3 \left( \alpha + \frac{\beta}{3} \right) aW.
\]  
(5.6)

Note that \( K \) is independent of details of the potential form and determined by the distance between potential minima \( 2q_0 \) and the width of the junction \( W \).

To estimate \( K \), we assume that the ratio between \( \alpha \) and \( \beta \) is independent of temperatures. Then, because \( \beta \) is not dominant in the dissipation amplitude \( \alpha + \beta/3 \) at relatively high temperatures as is observed experimentally, we expect that \( \beta \) is not dominant also at low temperatures. Hence we neglect the \( \beta \)-term, and estimate the value of \( K \) roughly as

\[
K \sim 3 \times 10^2 aW \frac{\alpha}{\alpha_n} = 3 \times 10^2 aW \frac{r_n}{r_{qp}}.
\]  
(5.7)

Here, we inserted the dissipation coefficient \( \alpha_n = 0.088 \) estimated in §3. The ratio between the normal resistance \( r_n \) and the quasiparticle resistance \( r_{qp} \) can be obtained experimentally.

The condition to observe MQC is given as

\[
K \ll 1 \quad \text{and} \quad k_B T \ll \hbar \Gamma / K.
\]  
(5.8)

If \( a = 0.1, W = 1, \) and \( r_n/r_{qp} = 10^{-3} \) can be realized, the estimated value of \( K(\sim 0.03) \) seems to satisfy the first conditions in (5.8). Although the estimate of the ratio \( r_n/r_{qp} = 10^{-3} \) is very rough, it seems to be attainable at low temperatures of mK order. Hence, we expect that the observation of MQC in LJJs appears to be possible within the Caldeira Leggett theory.

§6. Concluding Remarks

In this paper, we have studied two kinds of quantum tunneling for a single fluxon. We have found that the quantum tunneling from a metastable state may be observed at low temperatures (mK order), and that the junction width should be as small as possible. If the junction width is taken as \( \sim 1[\mu \text{m}] \), required accuracy of current measurement seems attainable in the laboratory. We have also found that the observation of MQC in a two-state system in LJJs appears to be possible, because dissipation due to the quasiparticle tunneling is strongly suppressed at low temperatures. It should be noted, however, that the damping
amplitude may increase by other dissipation sources. In the experiment on SQUIDs,\textsuperscript{28} the observed damping amplitude is much larger than one estimated by quasiparticle resistance. In addition to MQC, we expect other macroscopic quantum effects characteristic in a two-state system. For example, it may be possible to observe the incoherent tunneling and the population inversion between quantum states in LJJs as observed in SQUIDs.\textsuperscript{28,29}

The most characteristic feature of LJJs is that the phase difference $\phi$ has a spatial dependence. In other words, $\phi$ is a field variable with infinite degrees of freedom. This opens the possibility of studying combined effects of macroscopic quantum phenomena and many-body effects beyond the phenomenological theory by Caldeira and Leggett. In this paper, however, dynamics of only one degree of freedom, i.e., the center position of the fluxon has been considered as the first attempt. Other infinite degrees of freedom of the field appears in the form of plasmons. They, however, do not play an important role in quantum dynamics of a fluxon because of two reasons: (i) plasmon excitation has an energy gap, and plasmons are suppressed at low temperatures; (ii) in the lowest order of $g^2$, plasmons are decoupled from the fluxon. Hence, within the present study, characteristic features of the field with infinite degrees of freedom do not appear. To study many-body effects of the field variable, we must consider different situations. Fortunately, it is rather easy to devise LJJs compared with other systems described by field theories. New and rich quantum phenomena may appear by such devices, and it will be a challenging subject to study the interplay of many-body interaction and the macroscopic quantum effects.

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Appendix A: Derivation of the equation of motion

In this appendix, we derive the classical equation of motion (1.2) from the model Hamiltonian (2.18) with (2.11)-(2.13) within the classical mechanics. First, we write the Hamilton equations,

\[ \dot{\Pi} = -\frac{\delta H}{\delta \phi(x)}, \]  
\[ \dot{p}_j = -\frac{\delta H}{\delta q_j(x)}, \]  
\[ \dot{p}_{j}' = -\frac{\delta H}{\delta q_{j}'(x)}. \]  

(A.1)  
(A.2)  
(A.3)
Here $\Pi = \dot{\phi}$ is a momentum conjugate to $\phi$. By substituting the Hamiltonians (2.11)-(2.13) into these equations, we obtain series of equations,

\begin{align*}
\phi_{tt} - \phi_{xx} + \sin \phi + f - \varepsilon \delta(x) \sin \phi + \frac{\delta H_\alpha}{\delta \phi(x)} + \frac{\delta H_\beta}{\delta \phi(x)} &= 0, \\
m_j \ddot{q}_j + m_j \omega_j^2 q_j - c_j \phi &= 0, \\
m'_j \ddot{q}'_j + m'_j \omega_j'^2 q'_j - c'_j \phi_x &= 0.
\end{align*}

(A.4)

where

\begin{align*}
\frac{\delta H_\alpha}{\delta \phi(x)} &= - \sum_j c_j \left( q_j - \frac{c_j^2}{m_j \omega_j^2} \phi \right), \\
\frac{\delta H_\beta}{\delta \phi(x)} &= \sum_j c'_j \left( q'_{j,x} - \frac{c_j'^2}{m_j' \omega_j'^2} \phi_{xx} \right).
\end{align*}

(A.7)

Note that the sign is changed in (A.8) from the integration by part. The Fourier transformations of $q_j$, $q'_j$ and $\phi$ are denoted by

\begin{align*}
\phi(x, t) &= \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \tilde{\phi}(k, \omega) e^{ikx - i\omega t}, \\
q_j(x, t) &= \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \tilde{q}_j(k, \omega) e^{ikx - i\omega t}, \\
q'_j(x, t) &= \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \tilde{q}'_j(k, \omega) e^{ikx - i\omega t}.
\end{align*}

(A.9)

(A.10)

(A.11)

Fourier transforms of (A.5)-(A.8) and elimination of $\tilde{q}_j$ and $\tilde{q}'_j$ give

\begin{align*}
\tilde{\delta H_\alpha} &= - i \omega \tilde{\alpha}(\omega) \tilde{\phi}(k, \omega), \\
\tilde{\delta H_\beta} &= - i k^2 \omega \tilde{\beta}(\omega) \tilde{\phi}(k, \omega),
\end{align*}

(A.12)

(A.13)

where

\begin{align*}
\tilde{\alpha}(\omega) &= - \frac{2i\omega}{\pi} \int_0^\infty d\omega' \frac{J_\alpha(\omega')}{\omega'(\omega'^2 - \omega^2 - i0^+)} , \\
\tilde{\beta}(\omega) &= - \frac{2i\omega}{\pi} \int_0^\infty d\omega' \frac{J_\beta(\omega')}{\omega'(\omega'^2 - \omega^2 - i0^+)}.
\end{align*}

(A.14)

(A.15)

Here, the spectral functions $J_\alpha(\omega)$ and $J_\beta(\omega)$ are defined by (2.14)-(2.15). The positions of poles are taken off the real axis to satisfy the causality. When we choose the spectral functions as $J_\alpha(\omega) = \alpha \omega$, $J_\beta(\omega) = \beta \omega$, we find that $\tilde{\alpha}(\omega)$ and $\tilde{\beta}(\omega)$ acquire real parts as $\tilde{\alpha}'(\omega) = \alpha$, $\tilde{\beta}'(\omega) = \beta$. Finally the inverse Fourier transforms of (A.12) and (A.13) give dissipative terms in the classical perturbed sine-Gordon equation (1.2).
Appendix B: Details of the Semiclassical Theory

In this appendix, the semiclassical calculation is presented in detail. Here, we adopt a path integral method used by Gervais et al.\textsuperscript{30} The functional integral method obscures problems of quantum operator ordering.\textsuperscript{31} Actually in the correct formalism based on operator canonical transformations,\textsuperscript{32} additional terms, which is absent in the straightforward functional approach, appear in the expansion. The extra terms are, however, irrelevant to the present calculation because the correction appears only in higher order of $g^2$.

First we consider the unperturbed sine-Gordon equation given by (2.26) and (2.27). We introduce a new field variable $\phi = g \varphi$ to perform the expansion of $g^2$ easily. The partition function can be modified as

$$Z_0 = \int \mathcal{D}\varphi(x, \tau)\mathcal{D}\Pi(x, \tau) \exp \left[ i \int d\tau \int dx \Pi \dot{\varphi} - \int d\tau \int dx \left( \frac{1}{2} \Pi^2 + \frac{1}{2} \varphi^2 + \frac{1}{g^2} (1 - \cos g\varphi) \right) \right], \quad (B.1)$$

where $\Pi$ is a field momentum variable conjugate to $\varphi$.

The stationary nontrivial solutions takes the form

$$\varphi_0(x - q) = \frac{4}{g} \arctan \left[ \exp(x - q) \right], \quad (B.2)$$

where $q$ is the center coordinate of the soliton. Then, we describe the field $\varphi$ as follows:

$$\varphi(x, \tau) = \varphi_0(x - q(\tau)) + \eta(x - q(\tau), \tau). \quad (B.3)$$

Since $\varphi$ is a quantum field, it has an infinite number of degrees of freedom in the functional integral. Thus, $\eta(x - q(\tau), \tau)$ also has an infinite number of quantum degrees of freedom, and $\varphi_0(x - q(\tau))$ contains only one quantum degree of freedom in $q(\tau)$. In order to keep the number of degrees of freedom, we should set a subsidiary condition for $\eta$,

$$\int d\rho \varphi'_0(\rho) \eta(\rho, \tau) = 0. \quad (B.4)$$

Here $\rho = x - q(\tau)$ is a soliton-fixed coordinate, and the prime denotes derivative with respect to $x$. In a proper way, we can define momentum variables $p(\tau)$, $\pi(\rho, \tau)$ conjugate to $q(\tau)$, $\eta(\rho, \tau)$. The partition function is modified as

$$Z = \int \mathcal{D}p(\tau)\mathcal{D}q(\tau)\mathcal{D}\pi(\rho, \tau)\mathcal{D}\eta(\rho, \tau) \delta(\int d\rho \varphi'_0(\rho) \eta(\rho, \tau)) \delta(\int d\rho \varphi'_0(\rho) \pi(\rho, \tau)) \exp \left( i \int d\tau p \dot{q} + i \int d\tau \int d\rho \pi \dot{\eta} - \int d\tau H \right), \quad (B.5)$$

where

$$H = M_0 + \frac{(p + \int d\rho \pi \eta')^2}{2M_0(1 + \xi/M_0)^2} + \frac{1}{2} \int d\rho \left( \pi^2 + \eta^2 + (1 - \frac{2}{\cosh^2 \rho}) \eta^2 \right)$$
Here \( M_0 = 8/g^2 \), and
\[
\xi = \int d\rho \varphi'_0(\rho) \eta'(\rho, \tau).
\]

We divide the Hamiltonian into three parts,
\[
H_1 = M_0 + \frac{p^2}{2M_0},
H_2 = \int d\rho \frac{1}{2} \left( \pi^2 + \eta^2 + (1 - \frac{2}{\cosh^2 \rho}) \eta^2 \right),
H_3 = H - H_1 - H_2.
\]

Here \( H_1 \) describes fluxon dynamics in the lowest approximation. The quadratic part \( H_2 \) is easily quantized, and gives plasmon excitations. These excitations have a gap of \( \hbar \omega_p \) in the original unit. Taking \( H_1 + H_2 \) as the unperturbed part, we can perform the traditional perturbation expansion for \( H_3 \), which includes only higher order terms than quadratic. This expansion for \( g^2 \ll 1 \) is obtained by Gervais et al.\(^3\) in detail. The result is
\[
Z_0 = \int \mathcal{D}q(\tau) \mathcal{D}p(\tau) \exp \left[ i \int d\tau p(\tau) \dot{q}(\tau) - \int d\tau \left( \frac{m}{g^2} + \frac{g^2 p(\tau)^2}{2m} + \Delta E[p(\tau)] \right) \right],
\]
where \( m \) is the soliton mass which equals identically to 8, and \( Z_p \) is the partition function of unperturbed plasmons, which is irrelevant to the tunneling rate. The quantum correction \( \Delta E \) can be calculated in a systematic way using the traditional perturbation theory represented by familiar graphical diagrams. As shown in eq. (B.11), the lowest order approximation for \( g^2 \) is nonrelativistic. The Lorentz-invariant form is obtained by summing up all relevant diagrams.\(^3\) The result is \( \Delta E \sim \mathcal{O}(g^2) \), and there exists no correction to the term proportional to \( p^2 g^2 \) in \( \Delta E \).

When \( g^2 \) is small, \( \Delta E \) can be neglected. After integrating (B.11) over \( p \), we obtain
\[
Z_0 = \int \mathcal{D}q(\tau) \exp \left( -\frac{1}{g^2} \int d\tau \left( m + \frac{m}{2} \dot{q}(\tau)^2 \right) \right).
\]

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\[ V = \frac{I}{I_0} \]

\[ \Delta \eta \]

\[ f = \frac{l}{l_0} \]

\[ f_c - \langle \eta \rangle \]
With damping

Without damping

\[ P(f) \, df \]

\[ f = \frac{l}{l_0} \]
\[ \log_{10}(\Delta \eta/f_c) \]

\[ \log_{10} W [\mu m] \]

Without damping

With damping
Thermal activation with damping vs. without damping.
With damping
Without damping