Quantum dynamics of open many-qubit systems strongly coupled to a quantized electromagnetic field in dissipative cavities

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(Dated: September 1, 2021)

Abstract

We study quantum dynamics of many-qubit systems strongly coupled to a quantized electromagnetic cavity mode, in the presence of decoherence and dissipation for both fermions and cavity photons. The analytic solutions are derived for a broad class of open quantum systems in Lindblad approximation. They include identical qubits, an ensemble of qubits with a broad distribution of transition frequencies, and multi-level electron systems. Compact analytic solutions for time-dependent quantum state amplitudes and observables become possible with the use of the stochastic equation of evolution for the state vector. We show that depending on the initial quantum state preparation, the systems can evolve into a rich variety of entangled states with destructive or constructive interference between the qubits. In particular, dissipation in a cavity can drive the system into the dark states completely decoupled from the cavity modes. We also find the regimes in which multi-electron systems with a broad distribution of transition frequencies couple to the quantized cavity field as a giant collective dipole.
I. INTRODUCTION

Solid-state cavity quantum electrodynamics (QED) attracted much interest as a promising platform for quantum information and quantum sensing systems; see, e.g., [1–5] for recent reviews. A typical scenario involves an ensemble of quantum emitters (ideally, two-level systems) such as quantum dots, defects in crystals, or molecules, strongly coupled to a quantized electromagnetic (EM) field in a dielectric or plasmonic cavity. We will call these quantum emitters qubits for brevity, to be distinguished from the logical qubits which may involve many two-level systems as well as photonic or mixed degrees of freedom. Several or many qubits are required for most applications. Although direct near-field coupling of qubits with each other is possible and desired for some gating protocols, in the nanophotonics context such a coupling would require deterministic placement of qubits with sub-nm accuracy, which is challenging. A simpler scenario which still permits various ways of the quantum state manipulation is the one in which the qubits are coupled only through the common cavity mode(s). This is precisely the situation considered in this paper. We derive step by step analytic solutions for the quantum dynamics of an ensemble of qubits strongly coupled to a quantized cavity mode, in the presence of decoherence and coupling to dissipative reservoirs, starting from the simplest case of \( N \) identical qubits, then including a spread of transition frequencies in two or more two-level qubits, and finally the multilevel fermionic systems with arbitrary energy level distribution in the bands. Compact analytic solutions for the quantum states and observables in strongly coupled open quantum systems become possible within the formalism of the stochastic equation for the state vector which we recently developed [6, 7].

As we show in the paper, this approach brings significant advantages as compared to the existing methods of treating open quantum systems. Indeed, the Heisenberg-Langevin formalism faces challenges when dealing with nonperturbative dynamics of strongly coupled systems as it leads to nonlinear operator-valued equations. The master equation formalism quickly becomes intractable analytically and allows only numerical treatment as the number of the degrees of freedom increases; see Appendix A for a detailed comparison. Furthermore, determining the measure of entanglement within the master equation formalism is a separate problem since convenient universal figures of merit exist only for simplest systems. In contrast, our approach gives an analytic solution for a nonperturbative evolution of a strongly
coupled system with an arbitrary number of the degrees of freedom. Furthermore, the explicit expression for the state vector makes the determination of entanglement or lack thereof much more transparent; it allows one to compare the resulting quantum state with Bell states and other quantum states popular in quantum information applications; it shows how the observables characterizing a given state gets affected by relaxation and eventually drown in an increasing noise, etc. Unlike the Wigner-Weisskopf method, our approach treats the effects of noise correctly and yields a correct equilibrium state at long times.

Of course the possibility of analytic treatment comes with its limitations. We use the rotating wave approximation (RWA) throughout the paper, which means that all frequency scales such as Rabi frequencies and detunings are much smaller than the cavity mode frequency and the optical transition frequencies in the qubits. Beyond RWA, general analytic solutions become impossible in the fully quantized and nonlinear (nonperturbative) regime of light-matter interaction between fermionic and bosonic fields. Note that some recent experiments with low-frequency transitions (e.g. microwave or terahertz) in strongly coupled multi-electron systems went beyond the RWA into the ultra strong coupling regime; e.g., [8–10].

The simplest situation is when the qubits are identical, for example if the qubit is a particular transition between two electron or vibrational states in a given molecule and the system includes several or many identical molecules. We solve it in Sec. II in the dissipationless case and then in Secs. IV and V in the presence of dissipation and decoherence. It is common knowledge [1, 4, 11] that a system of $N$ identical qubits responds to the EM field as one qubit with a “giant” collective dipole moment which scales as $\sqrt{N}$. At the same time, the qubits themselves demonstrate a much more complex dynamics. Depending on the initial excited quantum state of the qubits, they may develop either constructive or destructive interference during their evolution, which leads to either a complete energy transfer to the cavity mode or a complete suppression of such transfer. For example, a system in which only one qubit is initially excited may evolve into an entangled state of all qubits which is decoupled from the cavity field. Moreover, in the typical case relevant for the plasmonic nanocavity [12−17], when the dissipation of a cavity mode is much faster than the decoherence in a qubit, the field dissipation inevitably drives the system into such a decoupled entangled state which remains preserved until the dissipation in an ensemble of qubits kicks in. Note that the existence of entangled dark states has been recently
predicted in [18][19] by calculating the concurrence with the master equation in the Lindblad approximation.

In Secs. III and IV the quantum evolution of two qubits with arbitrary vacuum Rabi frequencies and frequency detunings from the cavity mode are described. The formation of dark states decoupled from the cavity mode due to destructive quantum interference is predicted. In Sec. V we calculate the emission spectra for an ensemble of $N$ qubits in a leaky cavity.

Finally, in Sec. VI, we solve for the evolution of the electron systems in which transition frequencies may differ by a large value, much larger than the homogeneous linewidth of an individual emitter and the linewidth of a cavity mode. This introduces an additional effective decoherence for an ensemble of qubits, in addition to the coupling of electrons and photons to their dissipative reservoirs. At the same time, different transition frequencies introduce extra degrees of freedom and add more functionality to the system. Examples of such systems include semiconductor quantum wells, quantum dots, or defects in crystals.

Since all results in the paper are in the analytic form and the plots are normalized by the single-qubit vacuum Rabi frequency, here we list typical values of the parameters in experimental solid-state nanophotonic systems that determine the strength of light-matter coupling and relaxation rates. To determine if the strong coupling regime and quantum entanglement in the electron-photon system can be achieved, one has to compare the relaxation rates with the characteristic Rabi frequency which determines the oscillations of energy between the cavity field and the qubits. The strong-coupling conditions for an ensemble of $N$ qubits are easier to implement since the effective Rabi frequency scales as $\sqrt{N}$. At the same time, placing a large number of qubits in a nanocavity may present a challenge by itself as individual quantum emitters will experience nonuniform field distribution of a cavity mode and the coupling between them is difficult to control and may lead to additional decoherence.

The parameters mentioned below were obtained from recent experimental reports and reviews such as [1][5] [8][10], [12][17], [20][24]. This paper does not attempt to provide a comprehensive overview of the rapidly growing literature on the subject.

In electron-based quantum emitters the largest oscillator strengths in the visible/near-infrared range have been observed for excitons in organic molecules, followed by perovskites and more conventional inorganic semiconductor quantum dots. The typical variation of
the dipole matrix element of the optical transition which enters the Rabi frequency is from tens of nm to a few Angstrom (in units of the electron charge). Within the same kind of a system, the dipole moment grows with increasing wavelength. The relaxation times are strongly temperature and material quality-dependent, varying from tens or hundreds of ps for electric-dipole allowed interband transitions in quantum dots at 4 K to the tens and hundreds of µs for spin qubits based on defects in semiconductors and diamond at mK temperatures. Dipole-forbidden optical transitions, e.g. dark excitons in quantum dots, can have the lifetime up to milliseconds, but their coupling to light could be too weak to reach the strong coupling regime. At room temperature the decoherence rates for the optical transitions are in the tens of meV range.

The photon decay times are longest for dielectric micro- and nano-cavities: photonic crystal cavities, nanopillars, distributed Bragg reflector mirrors, microdisk whispering gallery mode cavities, etc. Their quality factors are typically between $10^3 \sim 10^7$, corresponding to photon lifetimes from sub-ns to µs range. However, the field localization in the dielectric cavities is diffraction-limited, which limits the attainable single-qubit vacuum Rabi frequency values to hundreds of µeV \[21\] \[22\]. The effective decay rate of the eigenstates (e.g. exciton-polaritons) in dielectric cavity QED systems is typically limited by the relaxation in the fermion quantum emitter subsystem.

In plasmonic cavities, field localization on a nm and even sub-nm scale has been achieved, but the photon decay time is in the ps or even fs range and therefore, the photon losses dominate the overall decoherence rate. Still, when it comes to strong coupling at room temperature to a single quantum emitter such as a single molecule or a quantum dot, the approach utilizing plasmonic nanocavities has seen more success so far. In these systems the Rabi splitting of the order of 100-200 meV has been observed \[12\] \[17\].

II. N IDENTICAL QUBITS

Consider first a textbook problem of $N$ identical two-level systems with states $|0_j\rangle$ and $|1_j\rangle$, where $j = 1, ..., N$, with energy levels 0 and $W$. We introduce fermionic operators of annihilation and creation of an excited state $|1_j\rangle$,

$$\hat{\sigma}_j = |0_j\rangle \langle 1_j|, \quad \hat{\sigma}^{\dagger}_j = |1_j\rangle \langle 0_j|,$$  \hspace{1cm} (1)
the dipole moment operator,
\[ \hat{d} = \sum_{j=1}^{N} \left( d \hat{\sigma}^\dagger_j + d^\ast \hat{\sigma}_j \right), \tag{2} \]
and the Hamiltonian for all qubits,
\[ \hat{H}_a = W \sum_{j=1}^{N} \hat{\sigma}^\dagger_j \hat{\sigma}_j. \tag{3} \]

Here \( \hat{d} = \langle 1_j | \hat{d} | 0_j \rangle \) is the same for all \( j \). Our \( N \)-qubit system interacts with a single-mode field
\[ \hat{E} = E(r) \hat{c} + E^\ast(r) \hat{c}^\dagger, \tag{4} \]
where \( \hat{c} \) and \( \hat{c}^\dagger \) are standard annihilation and creation operators for bosonic Fock states.

The function \( E(r) \) determines the spatial structure of the electric field in a cavity. It is normalized as [25]
\[ \int_V \frac{\partial \left[ \omega^2 \varepsilon(\omega, r) \right]}{\omega \partial \omega} E^\ast(r) \hat{E}(r) d^3r = 4\pi \hbar \omega \tag{5} \]
to preserve the standard form of the field Hamiltonian,
\[ \hat{H}_{em} = \hbar \omega \left( \hat{c}^\dagger \hat{c} + \frac{1}{2} \right). \tag{6} \]

The relation between the modal frequency \( \omega \) and the function \( E(r) \) can be found by solving the boundary-value problem of the classical electrodynamics. Here \( V \) is a quantization volume and \( \varepsilon(\omega, r) \) is the dielectric function of a dispersive medium that fills the cavity.

The total Hamiltonian after adding interaction with the field is
\[ \hat{H}_{a+em} = \hat{H}_a + \hat{H}_{em} - \hat{d} \cdot (E \hat{c} + E^* \hat{c}^\dagger), \tag{7} \]
\[ \hat{H} = \hbar \omega \left( \hat{c}^\dagger \hat{c} + \frac{1}{2} \right) + \sum_{j=1}^{N} \left[ W \hat{\sigma}^\dagger_j \hat{\sigma}_j - \left( \mathbf{d} \cdot \mathbf{E} \hat{\sigma}^\dagger_j \hat{\sigma}_j + \mathbf{d}^\ast \cdot \mathbf{E}^* \hat{\sigma}_j \hat{\sigma}^\dagger_j \right) \right], \tag{8} \]
where we assumed that the field amplitude is the same at the position of all qubits. If the field frequency is close to resonance,
\[ \frac{W}{\hbar} \approx \omega \tag{9} \]
we can use the RWA Hamiltonian in which counter-rotating terms are dropped:
\[ \hat{H} = \hbar \omega \left( \hat{c}^\dagger \hat{c} + \frac{1}{2} \right) + \sum_{j=1}^{N} \left[ W \hat{\sigma}^\dagger_j \hat{\sigma}_j - \left( \mathbf{d} \cdot \mathbf{E} \hat{\sigma}^\dagger_j \hat{\sigma}_j + \mathbf{d}^\ast \cdot \mathbf{E}^* \hat{\sigma}_j \hat{\sigma}^\dagger_j \right) \right], \tag{10} \]
The Schrödinger equation with the Hamiltonian (10) is separable into low-dimensional blocks and solvable analytically for any excitation energy and for an arbitrary multiphoton state; see the Appendix A for a general treatment. However, the generation of nonclassical multiphoton states is still an experimental challenge and there is no compelling reason in presenting more general but also more cumbersome formulas for multiphoton excitations. That is why we choose the initial conditions for which the transitions to states beyond single-photon excitations are forbidden by energy conservation. These are most popular states widely used in quantum information applications, including for example Bell states and their generalizations to many-qubit systems. Then the states that can be reached as a result of evolution of the system include the ground state $|0\rangle \Pi_{j=1}^N |0_j\rangle$ and the states with energies close to the single-photon energy:

$$\Psi = C_{00} |0\rangle \Pi_{j=1}^N |0_j\rangle + C_{10} |1\rangle \Pi_{j=1}^N |0_j\rangle + \sum_{j=1}^N C_{0j} |0\rangle |1_j\rangle \Pi_{m\neq j}^N |0_m\rangle .$$

(11)

Substituting the state vector (11) into Eq. (10) leads to the equations for coefficients,

$$\dot{C}_{00} + i\frac{3}{2} \omega C_{00} = 0,$$

(12)

$$\dot{C}_{10} + i\frac{3}{2} \omega C_{10} - i\Omega_R^* \sum_{j=1}^N C_{0j} = 0,$$

(13)

$$\dot{C}_{0j} + i \left(\frac{\omega}{2} + \frac{W}{\hbar}\right) C_{0j} - i\Omega_R C_{10} = 0,$$

(14)

where the Rabi frequency $\Omega_R = \frac{d \cdot E}{\hbar}$. Let $\sum_{j=1}^N C_{0j} = F$. Then the equation for $C_{00}(t)$ is decoupled,

$$C_{00}(t) = C_{00}(0)e^{-i\frac{3}{2} \omega t};$$

(15)

and the other two equations,

$$\dot{C}_{10} + i\frac{3}{2} \omega C_{10} - i\Omega_R^* F = 0,$$

(16)

$$\dot{F} + i \left(\frac{\omega}{2} + \frac{W}{\hbar}\right) F - iN \Omega_R C_{10} = 0,$$

(17)

can be easily solved. Indeed, after making the substitution

$$\begin{pmatrix} C_{10} \\ F \end{pmatrix} = \begin{pmatrix} G_{10} \\ G_F \end{pmatrix} e^{-i\frac{3}{2} \omega t},$$

7
we obtain

\[ \dot{G}_{10} - i\Omega_R G_F = 0, \quad (18) \]
\[ \dot{G}_F + i\Delta G_F - i\Omega_R N G_{10} = 0, \quad (19) \]

where \( \Delta = \frac{W}{\hbar} - \omega. \)

This leads to the following equation for the variable \( G_{10} \) which describes the quantum state of the field,

\[ \dot{G}_{10} + i\Delta \dot{G}_{10} + N|\Omega_R|^2 G_{10} = 0, \quad (20) \]

which looks exactly like the one for a single qubit, but with an \( N \)-qubit Rabi frequency obtained by \( |\Omega_R|^2 \mapsto N|\Omega_R|^2 \). It has the solution

\[ G_{10} = A_1 e^{\Gamma_1 t} + A_2 e^{\Gamma_2 t}, \]

where the constants \( A_{1,2} \) are determined by initial conditions and

\[ \Gamma_{1,2} = -i\frac{\Delta}{2} \pm i\sqrt{\frac{\Delta^2}{4} + N|\Omega_R|^2}. \quad (21) \]

Then the solution for coefficients \( C_{0j} \) can be obtained by substituting the expression for \( C_{10} \) into Eq. (14),

\[ C_{0j}(t) = e^{-i\left(\frac{3\omega^2}{2} + \Delta\right)t} \left( C_{0j}(0) + i\Omega_R \int_0^t e^{i\Delta \tau} G_{10}(\tau) d\tau \right). \quad (22) \]

Therefore, a system of \( N \) identical qubits responds to the EM field as one qubit with a “giant” dipole moment \( \propto \sqrt{N} \). At the same time, the qubits themselves demonstrate a much more complex dynamics. Their interaction with a quantum field leads to their entanglement not only with the field but also with each other; see, e.g., [26]. These quantum correlations within the ensemble of qubits can profoundly alter the evolution of the whole system. We illustrate it with a few examples.

First, consider our system in the factorized initial state \( \Psi(0) = |1\rangle \prod_{j=1}^N |0_j\rangle \), in which the photon mode is excited, all qubits are in their ground states, and there is no entanglement between any degrees of freedom. Considering for simplicity the case \( \Delta = 0 \), the subsequent evolution of the state is given by

\[ \Psi = e^{-i\frac{3\omega^2}{2}t} \left[ \cos \left( \sqrt{N}|\Omega_R|t \right) |1\rangle \prod_{j=1}^N |0_j\rangle + \frac{i e^{i\theta}}{\sqrt{N}} \sin \left( \sqrt{N}|\Omega_R|t \right) \sum_{j=1}^N |0\rangle \prod_{m \neq j}^N |1_j\rangle \prod_{m \neq j}^N |0_m\rangle \right], \quad (23) \]
where $\theta = \text{Arg}[\Omega_R]$. At the moments of time $\sqrt{N}|\Omega_R|t = \frac{\pi}{2} + \pi M$ the qubits are entangled with each other in a Bell-type state but not entangled with the field, whereas at an arbitrary moment of time, there are entangled both with each other and with the field. The excitation energy periodically oscillates between the cavity mode and an entangled state of the qubits. If the system is open, the excitation will dissipate through relaxation in both photon and qubit subsystems.

Now consider a different factorized initial state $\Psi(0) = |0\rangle |1_j\rangle \Pi_{j=2}^N |0_j\rangle$, in which only one qubit is excited, for definiteness the one with $j = 1$. This implies that qubits can be individually addressed, for example by a beam of light or a near-field nanotip with excitation radius much smaller than the distance between the qubits.

Taking again $\Delta = 0$ for simplicity, we obtain the solution

$$
\Psi = e^{-i\frac{H}{2} t} \left[ \frac{ie^{-i\theta}}{\sqrt{N}} \sin \left( \sqrt{N}|\Omega_R|t \right) |1\rangle \Pi_{j=1}^N |0_j\rangle + \left( 1 - \frac{1 - \cos \left( \sqrt{N}|\Omega_R|t \right) }{N} \right) |0\rangle |1_1\rangle \Pi_{j=2}^N |0_j\rangle 
- \frac{1 - \cos \left( \sqrt{N}|\Omega_R|t \right) }{N} \sum_{j=2}^N |0\rangle |1_j\rangle \Pi_{m\neq j}^N |0_m\rangle \right],
$$

(24)

It follows from the time-dependent coefficients in Eq. (24) that for $N \gg 1$ the occupation of the initial state given by the second term on the right-hand side of Eq. (24) oscillates around 1 with a small amplitude $\frac{1}{N}$, whereas the occupations of all other states oscillate around zero with equally small amplitudes. This means that the probability of the energy transfer from the excited qubit to the field or to an ensemble of ground-state qubits is about $N$ times smaller than the probability that the initial state is preserved, which scales as

$$
\left| 1 - \frac{1 - \cos \left( \sqrt{N}|\Omega_R|t \right) }{N} \right|^2 \geq \left( 1 - \frac{2}{N} \right)^2.
$$

This result holds true whether or not we know which particular qubit is excited, as long as the excitation radius remains much smaller than the distance between the qubits. Contrary to famous double-slit experiments, for the qubit excitation the "which qubit" information constitutes a classical uncertainty. Indeed, imagine that we illuminate the qubits with a tightly confined beam from the far field or with a near field of the tip. The answer to the question "which qubit got excited" depends on the spatial mode structure which is determined by classical Maxwell’s equations. Therefore, the created state will have the structure $\Psi(0) = |0\rangle |1_j\rangle \Pi_{q\neq j}^N |0_q\rangle$, where the classical uncertainty in the value of the qubit number $j$ does not give rise to the entangled quantum state of the type $\Psi(0) = \frac{1}{\sqrt{N}} \sum_{j=1}^N |0\rangle |1_j\rangle \Pi_{q\neq j}^N |0_q\rangle$ (as quantum uncertainty would).
This reasoning can be generalized to the situation in which a group of \( J \) qubits fits inside the excitation radius, where \( J \ll N \). We show in Sec. IV below that if a group of \( J \ll N \) qubits is initially excited, their deexcitation will be suppressed too and the initial state will be preserved with probability \( \sim 1 - \frac{J}{N} \). Here again, the uncertainty of the type “which group is illuminated” is classical. At the same time, for a beam carrying single-photon energy there is a quantum uncertainty “which qubit inside the group will get excited”.

The presence of \( N > 1 \) qubits allows the existence of entangled qubit states that are completely decoupled from the cavity field. Indeed, for the initial condition \( C_{10} = 0 \) we can find any number of entangled qubit states satisfying the equation \(|C_{00}|^2 + \sum_{j=1}^{N} |C_{0j}|^2 = 1\), \( \sum_{j=1}^{N} C_{0j} = 0 \), i.e. the states that are decoupled from the field. This solution is of the type

\[
\begin{pmatrix}
C_{10} \\
\vdots \\
C_{0j} \\
\vdots \\
C_{0j}
\end{pmatrix} \propto e^{-i\frac{3\omega}{2}t} \begin{pmatrix}
0 \\
\vdots \\
A_{0j}e^{-i\Delta t} \\
\vdots \\
\sum_{j=1}^{N} A_{0j} = 0
\end{pmatrix},
\]

The partial and complete decoupling of qubits from the field can be illustrated by considering initial Bell-type states of the two qubits:

\[
\Psi_{\pm}(0) = |0\rangle \frac{|1_1\rangle |0_2\rangle \pm |0_1\rangle |1_2\rangle}{\sqrt{2}} \Pi_{j=3}^{N} |0_j\rangle.
\]  

(25)

Consider again the exact resonance for simplicity, \( \Delta = 0 \). If the field is not excited initially, i.e., \( C_{10}(0) = 0 \), the solution for the complex probability amplitudes takes the form

\[
C_{10}(t) = i e^{-i\theta} e^{-i\frac{3\omega}{2}t} \frac{F(0)}{\sqrt{N}} \sin \left( \sqrt{N}|\Omega_R|t \right),
\]

(26)

\[
C_{0j}(t) = e^{-i\frac{3\omega}{2}t} \left( C_{0j}(0) - \frac{F(0)}{N} \left[ 1 - \cos \left( \sqrt{N}|\Omega_R|t \right) \right] \right),
\]

(27)

where \( F(0) = \sum_{j=1}^{N} C_{0j}(0) \). In our case we have \( C_{01}(0) = \frac{1}{\sqrt{2}} \) and \( C_{02}(0) = \pm \frac{1}{\sqrt{2}} \) for \( \Psi_{\pm}(0) \) respectively; \( C_{0j>2} = 0 \). Therefore, for \( \Psi_{(-)}(0) \) we obtain \( F(0) = 0 \), i.e., \( \Psi_{(-)}(t) = e^{-i\frac{3\omega}{2}t} \Psi_{(-)}(0) \) and the state of two qubits is completely uncoupled from the field.

For \( \Psi_{(0)}(0) \) we obtain \( F(0) = \sqrt{2} \) and

\[
C_{10}(t) = i e^{-i\theta} e^{-i\frac{3\omega}{2}t} \frac{\sqrt{2}}{\sqrt{N}} \sin \left( \sqrt{N}|\Omega_R|t \right),
\]

(28)

\[
C_{0,1,2}(t) = e^{-i\frac{3\omega}{2}t} C_{0,1,2}(0) \left( 1 - \frac{2}{N} \left[ 1 - \cos \left( \sqrt{N}|\Omega_R|t \right) \right] \right),
\]

(29)
\[ C_{0,j>2}(t) = C_{0,1,2}(0) \frac{2}{N} \left[ 1 - \cos \left( \sqrt{N} |\Omega_R| t \right) \right]. \] (30)

Obviously, for \( \Psi_{(+)}(0) \) the probability amplitudes are changing but the change is small as long as \( \frac{2}{N} \ll 1. \)

To summarize, if no qubits are excited in the initial state, their ensemble interacts with a photon as one giant qubit. If the system is in the initial state in which one qubit or a small number of qubits are initially excited, the energy transfer of its excitation to other qubits and the photon mode will be slowed down by a factor of \( N. \) This will significantly slow down the relaxation of the initial excitation if the relaxation in the qubit subsystem is slower than that for the cavity mode. Moreover, there are entangled qubit states that are completely decoupled from the cavity field. For a solid state system with large \( N, \) the excitation and control of such or any other specified entangled multi-qubit states can be tricky. However, as we show below, the EM field dissipation may drive the multi-qubit system towards such states.

The monolithic solid-state systems where a very large \( N \) can be reached include, for example, intersubband transitions in doped quantum wells (QWs), excitons in semiconductor QWs or 2D semiconductors, and a 2D electron gas in a strong magnetic field. In these systems, strong and even ultra-strong coupling to a cavity field has been demonstrated, although so far with a classical EM field; e.g., \([8, 10]\). Once the quantized field regime is reached, one can imagine coupling several N-qubit systems to a common cavity field, e.g. a multi-quantum well system or excitons in different valleys of 2D semiconductors \([26]\). The control of specific single-qubit excitations in the monolithic systems is a challenge. If such a control is desirable, it could be easier to deal with a system of spatially isolated qubits in a cavity, e.g. quantum dots or molecular qubits, as proposed in \([18]\).

### III. TWO QUBITS WITH DIFFERENT TRANSITION FREQUENCIES

When the qubits are not identical, this opens more possibilities for quantum state manipulation but also makes the dynamics more complex. The conceptually simplest system includes two qubits which have different transition frequencies. Here we describe their dissipationless dynamics whereas the effects of dissipation are included in Sec. IV.

Assume that both transition frequencies are close enough to the field frequency, so that
we can still use the RWA:

\[
\hat{H} = \hbar \omega \left( \hat{c}^{\dagger} \hat{c} + \frac{1}{2} \right) + \sum_{j=1,2} \left[ W_j \hat{\sigma}_j \hat{\sigma}_j - \hbar \Omega_{Rj} \left( \hat{\sigma}_j^{\dagger} \hat{c} + \hat{c} \hat{\sigma}_j \right) \right],
\]

(31)

where \( \Omega_{R1,2} = \frac{d_{1,2} E}{\hbar} \). However, the frequency detuning values can be arbitrary with respect to Rabi frequencies. Consider only the ground state and lowest excited state with excitation energy close to single photon energy (i.e. \( n = 1 \)). The wave function including only those states is

\[
\Psi = C_{000} |0\rangle |0\rangle |0\rangle + C_{100} |1\rangle |0\rangle |0\rangle + C_{010} |0\rangle |1\rangle |0\rangle + C_{001} |0\rangle |0\rangle |1\rangle .
\]

(32)

Here the order of the terms is always \(|\text{photon}\rangle |\text{atomN1}\rangle |\text{atomN2}\rangle\).

From the Schrödinger equation we obtain

\[
\dot{C}_{000} + i \frac{\omega}{2} C_{000} = 0;
\]

(33)

\[
\dot{C}_{100} + i \frac{3}{2} \omega C_{100} = -i \left( \Omega_{R1}^{*} C_{010} + \Omega_{R2}^{*} C_{001} \right) = 0,
\]

(34)

\[
\dot{C}_{010} + i \left( \omega \frac{1}{2} + \frac{W_1}{\hbar} \right) C_{010} = 0,
\]

(35)

\[
\dot{C}_{001} + i \left( \omega \frac{1}{2} + \frac{W_2}{\hbar} \right) C_{001} = 0,
\]

(36)

One can eliminate high frequencies with

\[
\begin{pmatrix}
C_{100} \\
C_{010} \\
C_{001}
\end{pmatrix} =
\begin{pmatrix}
G_{100} \\
G_{010} \\
G_{001}
\end{pmatrix} e^{-i \omega_{3/2} t},
\]

which gives

\[
\dot{G}_{100} - i \left( \Omega_{R1}^{*} G_{010} + \Omega_{R2}^{*} G_{001} \right) = 0,
\]

(37)

\[
\dot{G}_{010} + i \Delta_1 G_{010} = 0,
\]

(38)

\[
\dot{G}_{001} + i \Delta_2 G_{001} = 0,
\]

(39)

where \( \Delta_{1,2} = \frac{W_{1,2}}{\hbar} - \omega \). The solution for coefficients can be found as

\[
\begin{pmatrix}
G_{100} \\
G_{010} \\
G_{001}
\end{pmatrix} \propto e^{\Gamma t};
\]

\[
\begin{pmatrix}
\Gamma & -i \Omega_{R1}^{*} & -i \Omega_{R2}^{*} \\
-i \Omega_{R1} & \Gamma + i \Delta_1 & 0 \\
-i \Omega_{R2} & 0 & \Gamma + i \Delta_2
\end{pmatrix}
\begin{pmatrix}
G_{100} \\
G_{010} \\
G_{001}
\end{pmatrix} = 0,
\]

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which gives an equation for eigenvalues,

$$\Gamma (\Gamma + i\Delta_1)(\Gamma + i\Delta_2) + |\Omega_{R1}|^2 (\Gamma + i\Delta_2) + |\Omega_{R2}|^2 (\Gamma + i\Delta_1) = 0 \quad (40)$$

This is a cubic equation which can be solved analytically or on a computer.

![Graph](image)

**FIG. 1:** Frequency eigenvalues from Eq. (40) as a function of detuning $\Delta_1$, when $|\Omega_{R1}| = \Omega_{R}$, $|\Omega_{R2}| = 2\Omega_{R}$, and $\Delta_2 = \Delta_1 + 5\Omega_{R}$. All frequencies are in units of $\Omega_{R}$.

Figures 1 and 2 show one example of the solution for eigenenergies and eigenstates, respectively, when $|\Omega_{R1}| = \Omega_{R}$, $|\Omega_{R2}| = 2\Omega_{R}$, and $\Delta_2 = \Delta_1 + 5\Omega_{R}$. Two anticrossing points are visible in the spectra in Fig. 1. As is clear from Fig. 2, by controlling the initial conditions for a given eigenstate and tuning through resonances with a cavity mode one can arrive at an arbitrary entangled state or transfer the excitation from one qubit to another or from any of the qubits to the photon state.

If $\Delta_1 = \Delta_2 = \Delta$, we obtain

$$(\Gamma + i\Delta)(\Gamma^2 + i\Delta\Gamma + |\Omega_{R1}|^2 + |\Omega_{R2}|^2) = 0. \quad (41)$$

In addition to the solutions of $\Gamma^2 + i\Delta\Gamma + |\Omega_{R1}|^2 + |\Omega_{R2}|^2 = 0$ denoted as states $a$ and $b$ in Figs. 3 and 4 below, we have a root $\Gamma = -i\Delta$ for any $\Omega_{R1,2}$. This solution is a dark state which does not couple to the field:

$$\begin{pmatrix} C_{100} \\ C_{010} \\ C_{001} \end{pmatrix} \propto \begin{pmatrix} 0 \\ 1 \\ -\frac{\Omega_{R1}}{\Omega_{R2}} \end{pmatrix}. \quad (42)$$
FIG. 2: Amplitudes squared of the eigenstates from Eq. (40) as a function of detuning $\Delta_1$, when $|\Omega_{R1}| = \Omega_R$, $|\Omega_{R2}| = 2\Omega_R$, and $\Delta_2 = \Delta_1 + 5\Omega_R$.

Figures 3 and 4 illustrate this example, showing eigenenergies and eigenstates for $\Delta_1 = \Delta_2 = \Delta$. Clearly, one state denoted as Dark state on the figures is decoupled from light at any detunings whereas the two bright states $a$ and $b$ interact strongly through coupling to a cavity mode. By changing the initial conditions one can store the excitation in a dark state and read it out on demand, of course within the limits imposed by decoherence.

As a final example, consider the case $\Delta_1 = 0$, $\Delta_2 = \Delta$, which gives

$$|\Omega_{R2}|^2 \Gamma + (|\Omega_{R1}|^2 + \Gamma^2) (\Gamma + i\Delta) = 0.$$  \hspace{1cm} (43)$$

In particular, when $|\Delta| \gg |\Omega_{R1,2}|$ there is an approximate analytic solution $\Gamma_1 \approx -i\Delta$, $\Gamma_{2,3} \approx \pm|\Omega_{R1}|$ which means that the field couples only to a resonant qubit.
FIG. 3: Frequency eigenvalues from Eq. (40) for $\Delta_1 = \Delta_2 = \Delta$ as a function of detuning $\Delta$, when $|\Omega_{R1}| = \Omega_R$ and $|\Omega_{R2}| = 2\Omega_R$. The dark state which is decoupled from the cavity mode at any detuning is defined in Eq. (42).

IV. DYNAMICS OF MULTI-QUBIT SYSTEMS IN THE PRESENCE OF DISSIPATION

A. The stochastic equation of evolution for the state vector

A standard way to include the effects of dissipation is based on the master equation for the density matrix $\hat{\rho}$ of the system [27],

$$\frac{d}{dt} \hat{\rho} = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho} \right] + \hat{L}(\hat{\rho}),$$

(44)

where $\hat{L}(\hat{\rho})$ is the relaxation operator. If there are $M$ states in a given basis $|\alpha\rangle$, Eq. (44) corresponds to $\frac{1}{2}M(M + 1)$ equations for the matrix elements $\rho_{\alpha\beta} = \rho_{\beta\alpha}^*$. The number of equations that need to be solved can be reduced to $M$ within the method of the stochastic equation of evolution for the state vector [6, 7]. This becomes possible if the structure of the relaxation operator permits representing the right-hand side of Eq. (44) in the form

$$-\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho} \right] + \hat{L}(\hat{\rho}) = -\frac{i}{\hbar} (\hat{H}_{eff} \hat{\rho} - \hat{\rho} \hat{H}_{eff}^\dagger) + \delta \hat{L}(\hat{\rho}),$$

(45)

where $\hat{H}_{eff} = \hat{H} + \hat{H}^{(ah)}$ is an effective non-Hermitian Hamiltonian. In the simplest case of inelastic relaxation (i.e., ignoring the elastic scattering processes leading to pure dephasing) in a system with nonequidistant spectrum and zero temperature the anti-Hermitian part $\hat{H}^{(ah)}$ of the effective Hamiltonian describes the decay of the state populations and quantum coherences, whereas the term $\delta \hat{L}(\hat{\rho})$ describes an increase in the populations of lower states due to relaxation from higher states [28, 29].
FIG. 4: Amplitudes squared of the eigenstates from Eq. (40) for \( \Delta_1 = \Delta_2 = \Delta \) as a function of detuning \( \Delta \), when \( \Omega_{R1} = \Omega_R \) and \( \Omega_{R2} = 2\Omega_R \). The dark state which is decoupled from the cavity mode at any detuning is defined in Eq. (42).

Within the Markovian models of relaxation the stochastic equation for the state vector takes the form

\[
\frac{d}{dt} |\Psi\rangle = -\frac{i}{\hbar} \hat{H}_{\text{eff}} |\Psi\rangle - \frac{i}{\hbar} |\mathcal{R}\rangle .
\]

In Eq. (46) the vector \( |\mathcal{R}\rangle \) is a stochastic Langevin source with the following statistical properties:

\[
|\mathcal{R}\rangle = 0, \quad \overline{\mathcal{R}_\alpha(t') \mathcal{R}^\dagger_\beta(t'')} = \hbar^2 \delta(t' - t'') D_{\alpha\beta}, \quad D_{\alpha\beta} = \langle \alpha | \delta \hat{L}(\hat{\rho}) | \beta \rangle \Rightarrow |\psi\rangle \langle \psi| ,
\]

the overbar (\( \cdots \)) means averaging over the noise statistics, \( \mathcal{R}_\alpha = \langle \alpha | \mathcal{R} \rangle \). The dyadics \( \mathcal{C}_\alpha \mathcal{C}^\dagger_\beta \) in Eqs. (46),(47) where \( \mathcal{C}_\alpha = \langle \alpha | \Psi \rangle \) correspond to the density matrix elements \( \rho_{\alpha\beta} \) in the master equation (see the proof in [6]).

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The observables in the method of the stochastic equation are determined as

\[ g = \langle \Psi | \hat{g} | \Psi \rangle, \]

where \( \hat{g} \) is an operator corresponding to the physical quantity \( g \). This definition differs from a standard one by an additional averaging over the noise statistics. The choice of operators \( \hat{H}^{(oh)} \) and correlators \( D_{\alpha\beta} \) should ensure the conservation of the norm of the stochastic vector, \( \langle \Psi | \Psi \rangle = 1 \), and bring the system to a physically reasonable steady state in the absence of an external perturbation.

Note that the stochastic equation of the type of Eq. (46) is conceptually similar to the method of quantum jumps which is often used in numerical Monte-Carlo simulations [28, 30]. The origin of any stochastic equation approach as opposed to the density matrix formalism is of course rooted in the Langevin formalism. An important example of using the Langevin formalism in the derivation of the fluctuation-dissipation theorem can be found in [31].

Another widely used method to include the effects of dissipation in quantum optics is the Heisenberg-Langevin approach [28, 32]. However, when applied to the dynamics of strongly coupled systems, the Heisenberg equations become nonlinear (see, e.g., [28]), whereas the stochastic equation for the state vector, Eq. (46), is always linear, which is an important advantage of this method.

The representation of the type shown in Eq. (45) is possible, in particular, for the Lindblad relaxation operator. Here we will use the Lindbladian \( \hat{L}(\hat{\rho}) \) in the case of independent dissipative reservoirs for the field and the qubits and at zero temperature. The case of arbitrary temperatures is considered in [6]. Note that for a qubit with the transition in the visible or near-IR range even a room-temperature reservoir is effectively at zero temperature.

\[ L(\hat{\rho}) = -\sum_j \left( \frac{\gamma_j}{2} (\hat{\sigma}_j^\dagger \hat{\sigma}_j \hat{\rho} + \hat{\rho} \hat{\sigma}_j^\dagger \hat{\sigma}_j - 2 \hat{\sigma}_j \hat{\rho} \hat{\sigma}_j^\dagger) \right) - \frac{\mu}{2} (\hat{c}^\dagger \hat{c} \hat{\rho} + \hat{\rho} \hat{c}^\dagger \hat{c} - 2 \hat{c} \hat{\rho} \hat{c}^\dagger), \quad (48) \]

which gives

\[ \hat{H}_{\text{eff}} = \hat{H} - i\hbar \frac{1}{2} \left( \sum_j \gamma_j \hat{\sigma}_j^\dagger \hat{\sigma}_j + \mu \hat{c}^\dagger \hat{c} \right), \quad (49) \]

\[ \delta \hat{L}(\hat{\rho}) = \sum_j \gamma_j \hat{\sigma}_j \hat{\rho} \hat{\sigma}_j^\dagger + \mu \hat{c} \hat{\rho} \hat{c}^\dagger. \quad (50) \]

Here the relaxation constants \( \mu \) and \( \gamma_i \) are determined by the cavity Q-factor and inelastic relaxation of the qubits, respectively. Elastic relaxation processes (pure dephasing) are included later in this section.
Consider state vectors $\Psi$ of the type given in Eq. (11). Assuming different detunings $\Delta_i = \frac{W_i}{\hbar} - \omega$ for each qubit in the most general case, we obtain a set of stochastic equations

$$\dot{C}_{00} + \left(\frac{i\omega}{2} + \gamma_{00}\right)C_{00} = -\frac{i}{\hbar}R_{00}; \quad (51)$$

$$\dot{C}_{10} + \left(i\frac{3\omega}{2} + \gamma_{10}\right)C_{10} - i\sum_{j=1}^{N} \Omega_{Rj}^* C_{0j} = -\frac{i}{\hbar}R_{10}, \quad (52)$$

$$\dot{C}_{0j} + \left(i\frac{3\omega}{2} + i\Delta_j + \gamma_{0j}\right)C_{0j} - i\Omega_{Rj} C_{10} = -\frac{i}{\hbar}R_{0j}, \quad (53)$$

where the relaxation constants are related to the EM field and qubit relaxation constants in the Lindbladian Eq. (48) by

$$\gamma_{00} = 0, \quad \gamma_{10} = \frac{\mu}{2}, \quad \gamma_{0j} = \frac{\gamma_j}{2}. \quad (54)$$

The noise properties are given by

$$\mathcal{R}_{\alpha\beta}^*(t') \mathcal{R}_{\beta\alpha} (t'') = \hbar^2 \delta_{\alpha\beta} \delta_{\alpha\beta} D_{\alpha\alpha,\alpha\alpha} \delta(t' - t''), \quad (55)$$

$$D_{00,00} = \sum_{j=1}^{N} \gamma_j |C_{0j}|^2 + \mu |C_{10}|^2, \quad D_{10,10} = 0, \quad D_{0j,0j} = 0. \quad (56)$$

To include elastic relaxation (pure dephasing) in the Lindbladian Eq. (48) we need to add the term [29]

$$L^{(el)}(\hat{\rho}) = -\sum_j \frac{\gamma_j^{(el)}}{2} (\hat{\sigma}_{zj} \hat{\sigma}_{zj}^\dagger \hat{\rho} + \hat{\rho} \hat{\sigma}_{zj}^\dagger \hat{\sigma}_{zj} - 2 \hat{\sigma}_{zj}^\dagger \hat{\rho} \hat{\sigma}_{zj}),$$

where $\hat{\sigma}_{zj} = \hat{\sigma}_{zj}^\dagger = |1_j \rangle \langle 1_j| - |0_j \rangle \langle 0_j|$ and $\gamma_j^{(el)}$ is an elastic relaxation constant. Recent analysis [6, 7] shows that for two-level qubits the elastic processes can be included by making the following replacements in the expressions for $\gamma_{0j}$ and $D_{0j,0j}$: $\gamma_{0j} \rightarrow \gamma_{0j} + \gamma_j^{(el)}$, $D_{0j,0j} \rightarrow D_{0j,0j} + 2\gamma_j^{(el)} |C_{0j}|^2$. These relationships lead to standard relaxation timescales of populations, $T_{1j} = \frac{1}{\gamma_j}$ and coherence, $T_{2j} = \frac{1}{2\gamma_j + \gamma_j^{(el)}}$ [29]. Therefore, including pure decoherence processes leads to corrections in the last of Eqs. (54) and the last of Eqs. (56), namely

$$\gamma_{0j} = \frac{\gamma_j}{2} + \gamma_j^{(el)}, \quad D_{0j,0j} = 2\gamma_j^{(el)} \quad (57)$$

Taking into account Eqs. (57), it is easy to show that for any set of elastic scattering rates Eqs. (51)-(53) conserve the norm:

$$\sum_{j=1}^{N} |C_{0j}|^2 + |C_{10}|^2 + |C_{00}|^2 = 1, \quad (58)$$
and Eqs. (52)-(53) preserve the following relationship which includes only the rates of inelastic relaxation:

\[
\frac{d}{dt} \left( \sum_{j=1}^{N} |C_{0j}|^2 + |C_{10}|^2 \right) = - \sum_{j=1}^{N} \gamma_j |C_{0j}|^2 - \mu |C_{10}|^2.
\] (59)

In the absence of elastic scattering one can drop the noise sources \(-\frac{i}{\hbar} \mathcal{R}_{10}\) and \(-\frac{i}{\hbar} \mathcal{R}_{0j}\) in the right hand side of Eqs. (52) and (53). Including these noise sources would lead to the appearance of the terms in the expressions for \(C_{10}\) and \(C_{0j}\) which are linear with respect to corresponding random functions. Their contribution will be zero after averaging over the noise statistics, provided the autocorrelators \(D_{10,10}\) and \(D_{0j,0j}\), and all “off-diagonal” correlators (see Eqs. (55),(56)) are equal to zero. Therefore, the equations for \(C_{10}\) and \(C_{0j}\) become simplified:

\[
\dot{C}_{10} + \left( i \frac{3\omega}{2} + \frac{\mu}{2} \right) C_{10} - i \sum_{j=1}^{N} \Omega_{Rj}^* C_{0j} = 0,
\] (60)

\[
\dot{C}_{0j} + \left( i \frac{3\omega}{2} + i \Delta_j + \frac{\gamma_j}{2} \right) C_{0j} - i \Omega_{Rj} C_{10} = 0.
\] (61)

The equation for \(C_{00}\) still contains a corresponding noise source term,

\[
\dot{C}_{00} + i \frac{\omega}{2} C_{00} = -\frac{i}{\hbar} \mathcal{R}_{00}.
\] (62)

Eqs. (60)-(62) can be considered an improvement over the Weisskopf-Wigner approximation because, unlike the latter, they conserve the norm of the state vector; see Eq. (58). We will use Eqs. (60)-(62) for further analysis to avoid overly cumbersome expressions without losing any significant physics.

**B. Two-qubit dynamics in the presence of dissipation**

In this case \(j = 1, 2\). To maintain the same notations as in Sec. III, we will seek the state vector \(\Psi\) in the form of Eq. (11). We will rewrite Eqs. (60)-(62) after adapting the notations in Eq. (11) to the case of two qubits and one photon mode: \(C_{0j=1} \Rightarrow C_{010}\), \(C_{0j=2} \Rightarrow C_{001}\), \(C_{10} \Rightarrow C_{100}\), \(C_{00} \Rightarrow C_{000}\), \(\mathcal{R}_{00} \Rightarrow \mathcal{R}_{000}\), which gives

\[
\dot{C}_{100} + \left( i \frac{3\omega}{2} + \frac{\mu}{2} \right) C_{100} - i (\Omega_{R1}^* C_{010} + \Omega_{R2}^* C_{001}) = 0,
\] (63)

\[
\dot{C}_{010} + \left( i \frac{3\omega}{2} + i \Delta_1 + \frac{\gamma_1}{2} \right) C_{010} - i \Omega_{R1} C_{100} = 0.
\] (64)
\[ \dot{C}_{001} + \left( i \frac{3 \omega}{2} + i \Delta_2 + \frac{\gamma_2}{2} \right) C_{001} - i \Omega R_2 C_{100} = 0, \]  
(65)

\[ \dot{C}_{000} + i \frac{\omega}{2} C_{000} = -\frac{i}{\hbar} \overline{R}_{000}, \]  
(66)

where

\[ \overline{R}_{000} = 0, \quad \overline{R}_{000}^*(t') \overline{R}_{000}(t'') = \hbar^2 D_{000,000} \delta(t' - t''), \quad D_{000,000} = \gamma_1 |C_{010}|^2 + \gamma_2 |C_{001}|^2 + \mu |C_{100}|^2. \]  
(67)

Making standard substitutions in Eqs. (63)-(65),

\[ \begin{pmatrix} C_{100} \\ C_{010} \\ C_{001} \end{pmatrix} = \begin{pmatrix} G_{100} \\ G_{010} \\ G_{001} \end{pmatrix} e^{-i \omega \frac{3}{2} t}, \quad G_{100} \propto e^{\Gamma t}, \]  

We obtain

\[ \begin{pmatrix} \Gamma + \frac{\mu}{2} & -i \Omega R_1 & -i \Omega R_2 \\ -i \Omega R_1 & \Gamma + i \Delta_1 + \frac{\gamma_1}{2} & 0 \\ -i \Omega R_2 & 0 & \Gamma + i \Delta_2 + \frac{\gamma_2}{2} \end{pmatrix} \begin{pmatrix} G_{100} \\ G_{010} \\ G_{001} \end{pmatrix} = 0, \]  
(68)

which gives the dispersion equation for frequency eigenvalues:

\[ (\Gamma + \frac{\mu}{2})(\Gamma + i \Delta_1 + \frac{\gamma_1}{2})(\Gamma + i \Delta_2 + \frac{\gamma_2}{2}) + |\Omega R_1|^2 (\Gamma + i \Delta_2 + \frac{\gamma_2}{2}) + |\Omega R_2|^2 (\Gamma + i \Delta_1 + \frac{\gamma_1}{2}) = 0. \]  
(69)

To avoid cumbersome algebra, we consider two important limiting cases.

1. **Very different qubits**

   This is the case when one qubit is close to resonance with a cavity mode whereas another qubit is “off-resonant”:

   \[ \left| i \Delta_2 + \frac{\gamma_2}{2} \right| \gg |\Omega R_2|, \quad \left| i \Delta_1 + \frac{\gamma_1}{2} \right| \leq |\Omega R_1|. \]  
(70)

   If inequalities (70) are satisfied, an asymptotic solution of Eq. (69) has the following roots,

   \[ \Gamma_1 \approx -i \Delta_2 - \frac{\gamma_2}{2}, \quad \Gamma_{2,3} \approx -\frac{i \Delta_1 + \frac{\gamma_1}{2} + \mu}{2} \pm i \sqrt{|\Omega R_1|^2 - \left( \frac{i \Delta_1 + \frac{\gamma_1}{2} - \mu}{2} \right)^2} \]  
(71)
up to the terms of the order of \( \frac{|\Omega_{R2}|^2}{|i \Delta_2 + \frac{\mu}{2}|^2} \) and \( \frac{|\Omega_{R1}|^2}{|i \Delta_1 + \frac{\mu}{2}|} \) respectively. For the matrix in the left-hand side of Eq. (68) the eigenvalue \( \Gamma_1 \) corresponds to the eigenstate

\[
\begin{pmatrix} G_{100} \\ G_{010} \\ G_{001} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

, whereas eigenvalues \( \Gamma_{2,3} \) correspond to the eigenstates

\[
\begin{pmatrix} G_{100} \\ G_{010} \\ G_{001} \end{pmatrix} \propto \begin{pmatrix} 1 \\ \frac{i \Omega_{R1}}{\Gamma_{2,3} + i \Delta_1 + \frac{\mu}{2}} \\ 0 \end{pmatrix}.
\]

Similarly to Sec. III, the system is split into the nonresonant qubit and a coupled system “cavity mode + resonant qubit”. In the presence of dissipation the conditions for such decoupling in Eq. (70) depend on the relaxation constants and can be satisfied even at resonance \( \Delta_1 = \Delta_2 = 0 \) if one of the qubits decays fast enough: \( \gamma_2 \gg \gamma_1, \mu, |\Omega_{R2}| \).

2. Identical qubits

Now we can put \( \Delta_1 = \Delta_2 = \Delta \) and \( \gamma_1 = \gamma_2 = \gamma \) in Eqs. (63)-(65); the values of \( \Omega_{R1} \) and \( \Omega_{R2} \) are still different, because the qubits can be located inside the cavity in places with different amplitudes of the field mode; see Eqs. (4) and (5).

The eigenvalue equation (69) takes the form

\[
(\Gamma + i \Delta + \frac{\gamma}{2}) \left[ (\Gamma + \frac{\mu}{2})(\Gamma + i \Delta + \frac{\gamma}{2}) + |\Omega_{R1}|^2 + |\Omega_{R2}|^2 \right] = 0,
\]

which has the solutions

\[
\Gamma_1 = -i \Delta - \frac{\gamma}{2}, \quad \Gamma_{2,3} = -\frac{i \Delta + \frac{\gamma}{2} + \frac{\mu}{2}}{2} \pm i \sqrt{|\Omega_{R1}|^2 + |\Omega_{R2}|^2 - \left( \frac{i \Delta + \frac{\gamma}{2} - \frac{\mu}{2}}{2} \right)^2}.
\]

One can see that \( \text{Re}[\Gamma_{2,3}] < 0 \) is always satisfied. The general solution has the form

\[
\begin{pmatrix} C_{100} \\ C_{010} \\ C_{001} \end{pmatrix} = e^{-i \frac{\Delta}{2} t} \times \begin{pmatrix} 0 \\ 1 \\ -\frac{\Omega_{R1}}{\Omega_{R2}} \end{pmatrix} + B e^{i \Gamma_1 t} \begin{pmatrix} 1 \\ \frac{i \Omega_{R1}}{\Gamma_{2,3} + i \Delta + \frac{\mu}{2}} \end{pmatrix} + C e^{i \Gamma_3 t} \begin{pmatrix} 1 \\ \frac{i \Omega_{R2}}{\Gamma_{2,3} + i \Delta + \frac{\mu}{2}} \end{pmatrix},
\]

where \( N_1 = 1 + \frac{|\Omega_{R1}|^2}{|\Omega_{R2}|^2}, \quad N_{2,3} = 1 + \frac{|\Omega_{R1}|^2}{|\Omega_{2,3} + i \Delta + \frac{\mu}{2}|} + \frac{|\Omega_{R2}|^2}{|\Omega_{2,3} + i \Delta + \frac{\mu}{2}|} \); the constants \( A, B \) and \( C \) are determined by initial conditions.

The first term in the sum in Eq. (74) corresponds to the solution in which the qubits are decoupled from the field as a result of destructive interference of their quantum states.
This interference leads to interesting consequences in the case typical for plasmonic nanocavities, when the decay of the cavity mode is much faster than the relaxation of the qubits, i.e. $\mu \gg \gamma$. At short enough time intervals $\frac{\gamma}{2}t \ll 1$, one can put $\gamma = 0$ in both Eqs. (73). In this case the eigenfrequencies are

$$
\Gamma_1 = -i\Delta, \quad \Gamma_{2,3} = -\frac{\mu}{4} - i \left( \frac{\Delta}{2} \mp \sqrt{|\Omega_{R1}|^2 + |\Omega_{R2}|^2 - \frac{(i\Delta - \frac{\mu}{2})^2}{4}} \right)
$$

Therefore, the first term in the brackets in Eq. (74) does not decay whereas other terms decay fast, with the rate $\sim \frac{\mu}{4}$. It means that at times $\frac{\mu}{4}t \gg 1$ the dissipative dynamics drives the system to the asymptotic steady state in which the qubits are in an entangled state completely decoupled from the cavity mode, whereas the EM field is in its vacuum state:

$$
\Psi = e^{-i\frac{\mu}{4}t} |0\rangle \left[ e^{-i\frac{\mu}{4}t} A \frac{|0\rangle |1\rangle |0\rangle - \frac{\Omega_{R1}}{\Omega_{R2}} |0\rangle |0\rangle |1\rangle}{\sqrt{N_1}} + C_{000} |0\rangle |0\rangle |0\rangle \right],
$$

(75)

The ground-state amplitude averaged over the noise statistics satisfies $\overline{C_{000}} = C_{000}(0)$, $|\overline{C_{000}}|^2 = 1 - |A|^2$. It is obvious from the latter equality and the conservation of the norm that $|A|^2$ determines the steady-state value of the average qubit population at times $\frac{1}{\mu} \ll t \ll \frac{1}{\gamma}$.

The expression for $|A|^2$ is

$$
|A|^2 = \frac{1 - |C_{000}(0)|^2 - |C_{100}(0)|^2}{1 + |Z|^2} \left( 1 + \frac{|\Omega_{R1}|^2}{|\Omega_{R2}|^2} \right) \left( \frac{|\Omega_{R2}|^2 - Z \frac{\Omega_{R2}}{\Omega_{R1}}}{1 + \frac{|\Omega_{R2}|^2}{|\Omega_{R1}|^2}} \right)^2,
$$

(76)

where $Z = C_{001}(0) C_{100}(0)$.

The value of $|A|^2$ reaches a maximum when $\text{Arg}[Z] = \pi - \text{Arg} \left[ \frac{\Omega_{R2}}{\Omega_{R1}} \right]$ and $|Z| = \left| \frac{\Omega_{R1}}{\Omega_{R2}} \right|$, which corresponds to $C_{001}(0) = -C_{100}(0) \frac{\Omega_{R1}}{\Omega_{R2}}$ and

$$
|A|^2 = 1 - |C_{000}(0)|^2 - |C_{100}(0)|^2.
$$

(77)

Clearly, at its maximum the total qubit occupation is equal to its initial value, i.e. it remains constant. For any other initial conditions, the initial qubit population dissipates partially, until it reaches the decoupled entangled steady state with a smaller value of $|A|^2$.

This is illustrated in Fig. 5 which shows the contour plot of the value of $|A|^2$ on the complex $Z$ plane for real $\Omega_{R2} = 2\Omega_{R1}$ and a particular set of initial conditions $C_{000}(0) = C_{100}(0) = 0$. For this particular choice of parameters, the maximum of $|A|^2$ is reached on the real axis at the point $Z = -\frac{1}{\sqrt{2}}$ as follows from the above and is shown in the figure.
FIG. 5: The contour plot of the value of $|A|^2$ on the complex $Z$ plane for $\Omega_{R2} = 2\Omega_{R1}$ and $C_{000}(0) = C_{100}(0) = 0$.

A conceptually similar effect of dissipation-driven decoupling of the dynamical system from the dissipative reservoir as a result of destructive interference was found in [7] for one qubit coupled to two EM modes in a cavity with a time-modulated parameter (modulation-induced transparency).

One can also identify the initial conditions for which a constructive interference of qubit quantum states occurs, which leads to a rapid complete energy transfer from the qubits to the cavity field followed by its decay. This set of initial conditions corresponds to the minimum value $|A|^2 = 0$ in Fig. 5. In this case the excited qubit population decays completely to the ground state due to coupling to the leaky cavity mode before the relaxation in the qubits even kicks in.

C. The formation of a decoupled entangled state of N qubits

Consider again a system of N identical qubits. We will again use Eqs. (60)-(62) with $\Delta_j = \gamma_j = 0$, $\Omega_{Rj} = \Omega_R$:

$$\dot{C}_{10} + i \left( \frac{3\omega}{2} + \frac{\mu}{2} \right) C_{10} - i\Omega_R^* \sum_{j=1}^N C_{0j} = 0,$$

$$\dot{C}_{0j} + i \frac{3\omega}{2} C_{0j} - i\Omega_R C_{10} = 0.$$
After making the same substitutions as in Sec. II, 
\[ \sum_{j=1}^{N} C_{0j} = F, \quad \begin{pmatrix} C_{10} \\ F \end{pmatrix} = \begin{pmatrix} G_{10} \\ G_{F} \end{pmatrix} e^{-i \frac{3 \omega_2}{2} t}, \quad \begin{pmatrix} G_{10} \\ G_{F} \end{pmatrix} \propto e^{\Gamma t}, \]
we obtain
\[ \begin{pmatrix} \Gamma + \mu - i \Omega_R^* \\ -i N \Omega_R \end{pmatrix} \begin{pmatrix} C_{10} \\ G_{F} \end{pmatrix} = 0, \quad (80) \]
where
\[ \Gamma^2 + \frac{\mu}{2} \Gamma + N |\Omega_R|^2 = 0. \quad (81) \]
The resulting solution for the amplitudes \( C_{10} \) and \( F \) is
\[ \begin{pmatrix} C_{10} \\ F \end{pmatrix} = e^{-(i \frac{3 \omega_2}{2} + \frac{\mu}{4}) t} \left[ \frac{A e^{i \Xi t}}{\sqrt{1 + N}} \begin{pmatrix} 1 \\ \frac{\Gamma_1 + \mu}{i \Omega_R} \end{pmatrix} + \frac{B e^{-i \Xi t}}{\sqrt{1 + N}} \begin{pmatrix} 1 \\ \frac{\Gamma_2 + \mu}{i \Omega_R} \end{pmatrix} \right], \quad (82) \]
where the constants \( A \) and \( B \) are determined by initial conditions,
\[ \Gamma_{1,2} = -\frac{\mu}{4} \pm i \Xi, \quad \Xi = \sqrt{N |\Omega_R|^2 - \frac{\mu^2}{16}}, \quad (83) \]
The expression for \( C_{10} (t) \) which follows from Eq. (82) can be substituted into Eq. (79) in order to determine the solution for the amplitudes \( C_{0j} \):
\[ C_{0j} (t) = C_{0j} (0) e^{-i \frac{3 \omega_2}{2} t} + i \Omega_R e^{-i \frac{3 \omega_2}{2} t} \int_{0}^{t} e^{i \frac{3 \omega_2}{2} \tau} C_{10} (\tau) d\tau. \quad (84) \]
The expression for \( C_{00} (t) \) can be obtained using Eq. (82),
\[ C_{00} (t) = C_{00} (0) e^{-i \frac{\omega}{2} t} + \delta C_{00}, \quad (85) \]
where
\[ \delta C_{00} = -\frac{i}{\hbar} e^{-i \frac{\omega}{2} t} \int_{0}^{t} e^{i \frac{\omega}{2} t'} \mathcal{R}_{00} (t') dt'. \quad (86) \]
Since \( \delta C_{00} = 0 \), we obtain \( |C_{00} (t)|^2 = |C_{00} (0)|^2 + |\delta C_{00}|^2 \). The easiest way to calculate the averaged probability \( |C_{00} (t)|^2 \) is to use the conservation of the norm of the state vector, Eq. (58).

At times longer than the cavity decay time, \( \mu t \to \infty \), we always have \( C_{10} = 0 \) and \( F = \sum_{j=1}^{N} C_{0j} = 0 \). However, this does not necessarily mean that \( \sum_{j=1}^{N} |C_{0j}|^2 = 0 \). In other words, an ensemble of qubits may approach a quasi-steady-state with a finite average energy which is decoupled from a dissipative cavity field. This situation is similar to the destructive interference in a system of two qubits considered in previous subsection IVB. The exact form of the asymptotic final state depends on the initial conditions. Here we consider two natural initial conditions which lead to very different evolutions of the quantum state.
1. Only the cavity mode is excited initially

\[ \Psi(0) = |1\rangle \Pi_{j=1}^N |0_j\rangle. \]  \hspace{1cm} (87)

In this case Eq. (82) gives

\[ C_{10} = e^{-(i\frac{\omega}{2} + \frac{\mu}{4})t} \left[ \cos(\Xi t) - \frac{\mu}{4\Xi} \sin(\Xi t) \right] \]  \hspace{1cm} (88)

and

\[ F = e^{-(i\frac{\omega}{2} + \frac{\mu}{4})t} \frac{\Omega_R}{\Xi} \sin(\Xi t). \]  \hspace{1cm} (89)

Substituting Eq. (88) into Eq. (84) we arrive at

\[ \Psi = e^{-(i\frac{\omega}{2} + \frac{\mu}{4})t} \left\{ \left[ \cos(\Xi t) - \frac{\mu}{4\Xi} \sin(\Xi t) \right] |1\rangle \Pi_{j=1}^N |0_j\rangle + i \frac{\Omega_R}{\Xi} \sin(\Xi t) |0\rangle \sum_{j=1}^N |1_j\rangle \Pi_{m\neq j}^N |0_m\rangle \right\} + e^{-i\frac{\omega}{2}t} C_{00} |0\rangle \Pi_{j=1}^N |0_j\rangle \]  \hspace{1cm} (90)

where

\[ C_{00} = 0, \quad |C_{00}(0)|^2 = 1 - e^{-\frac{\mu}{2}t} \left[ 1 + \frac{\mu^2}{8\Xi^2} \sin^2(\Xi t) - \frac{\mu}{2\Xi} \cos(\Xi t) \sin(\Xi t) \right]. \]  \hspace{1cm} (91)

The final state at long enough times \( \mu t \to \infty \) is factorized,

\[ \Psi(t \to \infty) = e^{-i\frac{\omega}{2}t} |0\rangle \Pi_{j=1}^N |0_j\rangle. \]  \hspace{1cm} (92)

This solution corresponds to zero probability of the excitation transfer from the cavity field to an ensemble of qubits.

2. The initial condition corresponds to the excitation of qubits whereas the cavity field is in the vacuum state

The most general initial state of this kind can be written as

\[ \Psi(0) = \sum_{j=1}^N C_{0j}(0) |0\rangle |1_j\rangle \Pi_{q\neq j}^N |0_q\rangle. \]  \hspace{1cm} (93)

In this case Eq. (82) gives

\[ C_{10} = e^{-(i\frac{\omega}{2} + \frac{\mu}{4})t} iF(0) \frac{\Omega_R}{\Xi} \sin(\Xi t) \]  \hspace{1cm} (94)
\[ F = e^{-\left(i \frac{3\omega}{2} + \frac{\mu}{2}\right)t} F(0) \left[ \cos (\Xi t) + \frac{\mu}{4\Xi} \sin (\Xi t) \right]. \]  

(95)

where \( F(0) = \sum_{j=1}^{N} C_{0j}(0) \). Substituting Eq. (94) into Eq. (84) and using the formula

\[ \int_{0}^{\infty} e^{-\frac{\mu}{4} \tau} \sin (\Xi \tau) d\tau = \frac{\Xi}{N|\Omega_R|^2} \]

for integration, we obtain the solution for the amplitudes \( C_{0j} \) at the times \( \mu t \to \infty \), but before the qubit decoherence kicks in, i.e. \( \gamma t \ll 1 \),

\[ C_{0j}(t \to \infty) = \left[ C_{0j}(0) - \frac{1}{N} \sum_{j=1}^{N} C_{0j}(0) \right] e^{-i \frac{3\omega}{2} t}. \]  

(96)

It corresponds to the vacuum state of the field and the following entangled state of the qubits,

\[ \Psi(t \to \infty) = e^{-i \frac{3\omega}{2} t} \sum_{j=1}^{N} \left[ C_{0j}(0) \left( 1 - \frac{1}{N} \right) - \frac{1}{N} \sum_{q\neq j}^{N} C_{0q}(0) \right] |0\rangle |1_j\rangle \Pi_{q\neq j}^{N} |q\rangle 
+ e^{-i \frac{3\omega}{2} t} C_{00}(0) \Pi_{j=1}^{N} |0_j\rangle \]  

(97)

where \( C_{00} = 0 \). The magnitude of \( |C_{00}(0)|^2 \) is easiest to find via the conservation of the norm of the state vector, Eq. (58):

\[ |C_{00}(0)|^2 = 1 - \sum_{j=1}^{N} |C_{0j}(0)|^2, \]

where the values of \( C_{0j} \) are given by Eq. (96).

As an example, consider an initial state in which only one qubit is excited. Although no knowledge is needed regarding which one of the qubits is excited (see the comment in Sec. II), for definiteness we will take the one with \( j = 1 \), i.e., \( C_{01}(0) = 1, C_{0j\neq1}(0) = 0 \). In this case \( C_{01}(t \to \infty) = 1 - \frac{1}{N}, C_{0j\neq1}(t \to \infty) = -\frac{1}{N}, |C_{00}(0)|^2 = \frac{1}{N} \). The state vector in Eq. (97) allows one to calculate the probabilities of three possible events:

a) Dissipation of an excited qubit due to photon emission into the decaying cavity field; the probability is \( P_{rad} = |C_{00}(0)|^2 = \frac{1}{N} \).

b) Energy transfer from the excited qubit to other qubits through their collective coupling to the EM cavity field; the probability is \( P_{tr} = (N - 1) |C_{0j\neq1}(t \to \infty)|^2 = \frac{N-1}{N^2} \).

c) An initially excited qubit preserves its initial state; the probability is \( P_{pr} = |C_{01}(t \to \infty)|^2 = \left(1 - \frac{1}{N}\right)^2 \).

To summarize, if there is only one qubit in the cavity, \( N = 1 \), then its excitation energy will be transferred to a resonant photon with probability \( P_{rad} = 1 \) and quickly dissipate. For a relatively small number of qubits \( N \geq 2 \) (but with only one qubit initially excited), there are comparable probabilities of all three events: \( P_{rad} < 1, P_{pr} \neq 0 \) and \( P_{tr} \neq 0 \). If \( N \gg 1 \) we obtain \( P_{pr} \gg P_{rad} \approx P_{tr} \approx \frac{1}{N} \), i.e. the probability for an initially excited qubit
to preserve its state is much higher than the probabilities of energy outcoupling to the field or other qubits.

Therefore, an ensemble of ground-state qubits effectively shields an excited qubit from its outcoupling to a resonant cavity field. The shielding is due to destructive interference of quantum states in an entangled state.

The same kind of screening exists for an arbitrary initial state of a relatively small group of $J$ qubits if $J \ll N$:

$$
\Psi(0) = |0\rangle \sum_{j=1}^{J} C_{0j}(0) |0\rangle |1_j\rangle \Pi_{q \neq j}^N |0_q\rangle,
$$

(98)

where $\sum_{j=1}^{J} |C_{0j}(0)|^2 = 1$. Indeed, in this case Eq. (97) has $C_{0,1 \leq j \leq J}(0) \sim \frac{1}{\sqrt{J}}$, $\frac{1}{N} \sum_{q \neq j}^{N} C_{0q}(0) = \frac{1}{N} \sum_{j=1}^{J} C_{0j}(0) \sim \frac{\sqrt{J}}{N}$, i.e., the perturbation of the initial state is small: of the order of $\frac{J}{N}$,

$$
C_{0,1 \leq j \leq J}(t \to \infty) = C_{0,1 \leq j \leq J}(0) + o\left(\frac{J}{N}\right).
$$

Similarly to the case of two qubits, one can also find an example of constructive interference of qubit quantum states which leads to a rapid complete energy transfer from the qubits to the cavity field even if $N \gg 1$. This happens for the initial state of the type

$$
\Psi(0) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |0\rangle |1_j\rangle \Pi_{q \neq j}^N |0_q\rangle,
$$

(99)

Starting from this initial condition, the state vector approaches $\Psi(t \to \infty) = |0\rangle \Pi_{j}^N |0_j\rangle$ on the timescale $\frac{\mu t}{4} \gg 1$. Note that the initial state (99) which ensures constructive interference for qubit radiation into the cavity mode corresponds to the solution for the state vector in Eq. (90) at the moments of time $\Xi t = \frac{\pi}{2} + M\pi$ if we neglect dissipation, $\mu = 0$.

V. EMISSION FROM AN ENSEMBLE OF N QUBITS IN A LEAKY CAVITY

Detecting radiation from quantum emitters placed in nanocavities is one of the most straightforward ways to study their quantum dynamics [2, 12, 15, 17, 28, 33]. The power spectrum received by the detector can be calculated as [28, 33]

$$
P(\nu) = A \cdot S(\nu),
$$

where

$$
S(\nu) = \frac{1}{\pi} \Re \int_0^{\infty} d\tau e^{i\nu\tau} \int_0^{\infty} dt K(t, \tau),
$$

(100)
\[ K = \langle \Psi (0) | \hat{c} (t) \hat{c} (t + \tau) | \Psi (0) \rangle, \]  

(101)

\( \hat{c} \) and \( \hat{c} (t) \) are Heisenberg creation and annihilation operators for the cavity field, \( \Psi (0) \) is an initial state of the system. The coefficient \( A \) is determined by the cavity design, spatial structure of the cavity field, and detector properties.

These equations indicate that to calculate the power spectrum one has to solve the Heisenberg-Langevin equations for the operators \( \hat{c} (t) \) and \( \hat{c} \) [28] and evaluate the correlator including averaging over the noise statistics, \( K \Rightarrow \langle \Psi (0) | \hat{c} \hat{c} (t + \tau) | \Psi (0) \rangle \). However, the Heisenberg-Langevin equations are nonlinear in the strong-coupling Rabi oscillations regime for a single-photon field. Therefore, it is more convenient to utilize the solution of the linear stochastic equation (46) for the state vector.

### A. Calculating the emission spectrum with the stochastic equation for the state vector

It is instructive to consider first the dissipationless problem. If \( \hat{U} (t) \) is the evolution operator of the system, then the correlator can be expressed as

\[ K = \langle \Psi (0) | \hat{U} (t) \hat{c} \hat{c} (t + \tau) \hat{U} (t + \tau) | \Psi (0) \rangle = \langle \hat{c} \hat{c} (t) | \hat{U} (t) \hat{U} (t + \tau) | \hat{c} \hat{c} (t + \tau) \rangle, \]

(102)

where \( \hat{c} \) and \( \hat{c} \) are Schrödinger’s (constant) operators. We will use the notation \( \hat{U} (t) \Rightarrow \hat{U}_{t_0} (t') \) for the evolution operator which indicates the initial moment of time \( t_0 \) and the duration of evolution \( t' = t - t_0 \). With this notation we substitute \( \hat{U} (t) \Rightarrow \hat{U}_0 (t) \), \( \hat{U} (t + \tau) \Rightarrow \hat{U}_0 (t + \tau) = \hat{U}_t (\tau) \hat{U}_0 (t) \) in Eq. [102],

\[ K = \langle \hat{c} \hat{c} (t) | \hat{U}_0 (t) \hat{U}_t (\tau) \hat{U}_0 (t) \hat{U}_t (\tau) | \hat{c} \hat{c} (t + \tau) \rangle. \]

Taking into account the unitarity condition \( \hat{U}_0 (t) \hat{U}_0 \) (t) = 1, we obtain

\[ K = \langle \hat{U}_t (\tau) \hat{c} \hat{c} (t) | \hat{c} \hat{c} (t + \tau) \rangle. \]

(103)

After introducing the notations

\[ \Psi_C (t) = \hat{c} \hat{c} (t), \quad \Phi (t, \tau) = \hat{U}_t (\tau) \Psi_C (t), \]

(104)

We arrive at
\[ K(t, \tau) = \langle \Phi(t, \tau) | \Psi_C(t + \tau) \rangle. \tag{105} \]

One can formulate these steps as a general procedure how to calculate the correlator \( K(t, \tau) \) from the solution to the Schrödinger equation:

(a) Calculate the function \( \Psi_C(t + \tau) = \hat{c}\Psi(t + \tau) \), where \( \Psi(t + \tau) \) is the solution to the Schrödinger equation at the time interval \([0, t + \tau]\) with initial condition \( |\Psi(0)\rangle \).

(b) Calculate the function \( \Phi(t, \tau) \). In order to do that, one has to solve the Schrödinger equation at the time interval \([t, t + \tau]\) with initial condition \( \Psi(t) \). To calculate \( \Psi_C(t) \), one needs to repeat step (a) over the time interval \([0, t]\) instead of \([0, t + \tau]\).

Note that a complete closed system including both the dynamical system and the reservoir has some evolution operator too, say \( \hat{U}_{t_0}(t') \), even though it is unknown to us. Nevertheless, if we arrive at Eq. (105) for the closed system, this equation would contain both dynamical and reservoir degrees of freedom. Within the Langevin approach which describes the system with stochastic equations of evolution, the averaging over the reservoir degrees of freedom is equivalent to averaging over the statistics of the noise sources \[31\]. This paradigm allows one to describe open systems without relying on the density matrix.

To summarize, one can calculate the power spectrum by solving Eq. (46) to find the functions \( \Psi_C(t) \), \( \Psi_C(t + \tau) \), and \( \Phi(t, \tau) \) which depend on the noise sources, then substitute them functions into Eq. (105) and perform averaging over the noise statistics:

\[ K(t, \tau) = \langle \Phi(t, \tau) | \Psi_C(t + \tau) \rangle. \tag{106} \]

Eq. (106) follows a general scheme of calculating the observables using the stochastic equation for the state vector: a standard procedure of solving the Schrödinger equation is supplemented by averaging over the noise statistics.

**B. Emission spectrum of an excited qubit screened by an ensemble of ground-state qubits**

Consider again an ensemble of \( N \) identical qubits in a cavity and assume that the cavity field decays much faster than the qubits. As before, we will solve for the evolution over the intermediate timescales when only the field dissipation has to be taken into account. Consider an initial state in which only one qubit is excited, corresponding to Eq. [93] in
which one should take $C_{01} = 1$ and $C_{0j \neq 1} = 0$. As usual, we seek the solution of the stochastic equation for the state vector in the form of Eq. (11). From Eq. (11) and the first of Eqs. (104) one can get

$$
\Psi_C(t) = \hat{c}\Psi(t) = C_{10}(t) |0\rangle \prod_{j=1}^{N} |0_j\rangle
$$

(107)

According to the above procedure, we need to find the solution of Eqs. (60)-(62) with initial condition (107) at the time interval $[t, t+\tau]$. One can see that Eqs. (60),(61) have a trivial zero solution, whereas Eq. (62) yields

$$
\Phi(t, \tau) = \left[ e^{-i\Xi \tau} C_{10}(t) - i\hbar^{-1} e^{-i\Xi \tau} \int_0^\tau e^{i\Xi t'} R_{00}(t + t') dt' \right] |0\rangle \prod_{j=1}^{N} |0_j\rangle
$$

(108)

Substituting Eqs. (107) and (108) into Eq. (106) and taking into account that the term linear with respect to the noise source gives zero upon averaging, we obtain

$$
K(t, \tau) = e^{i\Xi \tau} C_{10}^*(t) C_{10}(t + \tau)
$$

(109)

Using Eq. (94) for the function $C_{10}(t)$ we get

$$
K(t, \tau) = |\Omega_R|^2 \frac{1}{\Xi^2} e^{-(i\omega + \frac{\mu}{4})\tau} e^{-\frac{\mu}{4}t} \sin(\Xi t) \sin(\Xi (t + \tau))
$$

(110)

The resulting power spectrum in Eq. (100) is given by

$$
S(\nu) = \frac{8 |\Omega_R|^2}{\pi \mu (\mu^2 + 16 \Xi^2)} \Re \left( \frac{\mu - 2i(\nu - \omega)}{\frac{\mu}{4} - i(\nu - \omega)} \right)^2 + \Xi^2,
$$

which can be transformed using the second of Eqs. (83) as

$$
S(\nu) = \frac{1}{2\pi} \frac{|\Omega_R|^2}{(\nu - \omega)^2 - \left( N |\Omega_R|^2 - \frac{\mu^2}{8} \right)^2 + \frac{\mu^2}{4} \left( N |\Omega_R|^2 - \frac{\mu^2}{16} \right)}.
$$

(111)

Under the condition $\mu \ll 2 |\Omega_R| \sqrt{N}$ we can obtain

$$
S(\nu) = \frac{1}{2\pi} \frac{|\Omega_R|^2}{(\nu - \omega)^2 - N |\Omega_R|^2}^2 + \frac{\mu^2}{4} N |\Omega_R|^2,
$$

i.e., the spectrum consists of two well-resolved lines shifted with respect to $\omega$ by $\pm |\Omega_R| \sqrt{N}$, with the maximum value $S_{\text{max}} \left( \pm |\Omega_R| \sqrt{N} \right) = \frac{1}{\pi} \frac{\mu^2}{N \mu^2}$ and linewidth $\sim \frac{\mu}{2}$. The dependence $S_{\text{max}} \propto \frac{1}{N}$ reflects the destructive interference effect described in Section C: the probability of the photon emission by a qubit scales as $P_{\text{rad}} \approx \frac{1}{N}$. This behavior is illustrated in Fig. 6 which shows the emission spectra given by Eq. (111) for three different qubit numbers $N$ and the cavity decay rate $\mu/2 = \Omega_R$. 

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FIG. 6: Normalized emission spectra given by Eq. (111) for three values of \( N \) and the cavity decay rate \( \mu/2 = \Omega_R \). The height of the peaks scales as \( 1/N \).

VI. MANY-QUBIT SYSTEMS WITH DIFFERENT TRANSITION FREQUENCIES

In this section we consider an ensemble of qubits with a large spread of transition frequencies interacting with a cavity mode. This is usually the case for quantum dots where the inhomogeneous broadening is related to dispersion of their sizes. Assuming that the inhomogeneous broadening dominates, we will neglect the homogeneous broadening of the qubit-field resonance. We will still consider single-photon excitations when the state vector can be represented in the form of Eq. (11) and the coefficients satisfy Eqs. (60)-(62). If the condition \(|\Delta_j| \gg \gamma_j^2 \), \( \mu^2 \) is satisfied for most qubits, we can neglect the terms \( \gamma_j^2 C_{0j} \) and \( \mu^2 C_{10} \) in Eqs. (60), (61), which lead to homogeneous broadening. To preserve the norm of the state vector, we also need to drop the noise term in the right-hand side of Eq. (62). The resulting set of equations becomes

\[
\dot{C}_{10} + i \frac{3\omega}{2} C_{10} - i \sum_{j=1}^{N} \Omega_{Rj}^{*} C_{0j} = 0, \quad (112)
\]

\[
\dot{C}_{0j} + i \left( \frac{3\omega}{2} + i\Delta_j \right) C_{0j} - i\Omega_{Rj} C_{10} = 0, \quad (113)
\]

\[
\dot{C}_{00} + i \frac{\omega}{2} C_{00} = 0. \quad (114)
\]
Eigenfrequencies of the system \([112]-[113]\) are determined in the same way as in previous sections,

\[
\begin{pmatrix}
C_{10} \\
C_{01} \\
C_{02} \\
\vdots
\end{pmatrix} \propto e^{-i\frac{\Delta}{\delta}t},
\]

(115)

where

\[
\begin{vmatrix}
\Gamma & -i\Omega^*_R & -i\Omega^*_{R2} & \cdots \\
-i\Omega^*_{R1} & (\Gamma + i\Delta_1) & 0 & \cdots \\
-i\Omega^*_{R2} & 0 & (\Gamma + i\Delta_2) & \cdots \\
\vdots & \vdots & \ddots & \cdots
\end{vmatrix} = 0,
\]

(116)

These equations can be solved numerically. However, some analytic progress is possible if the spectrum of qubit frequencies is continuous.

A. Continuous spectrum

The formal solution of Eq. \([113]\) is

\[
G_{0j}(\tau) = G_{0j}(0) e^{-i\Delta_j t} + i\Omega^*_{Rj} e^{-i\Delta_j t} \int_0^t e^{i\Delta_j \tau} G_{10}(\tau) d\tau.
\]

(117)

Substituting it into Eq. \([112]\) gives

\[
\dot{G}_{10} = i \sum_{j=1}^{N} \Omega^*_{Rj} G_{0j}(0) e^{-i\Delta_j t} - \int_0^t \sum_{j=1}^{N} |\Omega^*_{Rj}|^2 e^{i\Delta_j (\tau-t)} G_{10}(\tau) d\tau.
\]

(118)

Now we can go from summation to integration by introducing

\[
\sum_{j=1}^{N} (\cdots) \Rightarrow \int_{-\infty}^{\infty} (\cdots) g(\Delta) d\Delta.
\]

where \(g(\Delta)\) is the spectral density of qubit states which can be presented in a normalized form as

\[
g = \frac{N}{2\delta} f\left(\frac{\Delta}{\delta}\right),
\]

(119)

where \(2\delta\) is the width of the spectrum; \(f\left(\frac{\Delta}{\delta}\right) \to 0\) when \(|\Delta| \gg \delta\) and \(\frac{1}{2\delta} \int_{-\infty}^{\infty} f\left(\frac{\Delta}{\delta}\right) d\Delta = 1.\)
Taking the initial state as $|1\rangle \Pi_{j=1}^{N} |0_j\rangle$ (i.e. one photon is present whereas all qubits are in the ground state) we obtain

$$\dot{G}_{10} = - \int_0^t \int_{-\infty}^{\infty} e^{i\Delta(t-\tau)} D(\Delta) G_{10}(\tau) d\Delta d\tau,$$

where

$$D(\Delta) = g(\Delta) |\Omega_R|^2(\Delta).$$

From Eq. (120) one can obtain

$$\dot{G}_{10} = - \int_0^t \tilde{D}(\tau - t) G_{10}(\tau) d\tau,$$

where

$$\tilde{D}(\tau) = \int_{-\infty}^{\infty} e^{i\Delta \tau} D(\Delta) d\Delta$$

is a Fourier conjugate to the spectrum $D(\Delta)$. If we introduce the average value of the Rabi frequency as $\langle |\Omega_R|^2 \rangle$, then using Eq. (119) one obtains

$$D(\Delta) = \frac{N \langle |\Omega_R|^2 \rangle}{2\delta} f \left( \frac{\Delta}{\delta} \right).$$

(123)

Note that the relationship between the characteristic width $2\delta$ of the spectrum $D(\Delta)$ and the characteristic temporal width $\Im$ of the Fourier conjugate $\tilde{D}(\tau)$ is given by a standard formula $\Im \times 2\delta \approx 2\pi$. Next, we derive approximate solutions for the complex amplitudes in various limits.

Let's assume that the function $G_{10}(t)$ varies over a certain timescale $T$, which is unknown. If the spectral width $\delta$ is large enough,

$$T\delta \gg \pi, \quad t\delta \gg \pi$$

(124)

(which implies $\Im \ll T, t$), then one can obtain from Eq. (121)

$$\dot{G}_{10} \approx -G_{10}(t) \int_{-\infty}^{0} \tilde{D}(\tau) d\tau,$$

or

$$\dot{G}_{10} = -\kappa G_{10},$$

(125)

where the absorption rate is

$$\kappa = \int_{-\infty}^{0} \tilde{D}(\tau) d\tau$$

(126)
Consider now the case when we still have $t\delta \gg \pi$, but $T\delta \ll \pi$ (i.e. $t \gg \Im \gg T$). The condition $t \gg \Im$ allows one to transform Eq. (121) as

$$
\dot{G}_{10} \approx - \int_{-\infty}^{t} \tilde{D} (\tau - t) G_{10} (\tau) d\tau.
$$

(127)

We will seek the solution to Eq. (127) in the form $G_{10} = g e^{\Gamma t}$, and, since $\Im \gg T \approx \Gamma^{-1}$, we write the right-hand side of Eq. (127) as a series in powers of $\Gamma^{-1}$ by sequential integration by parts:

$$
\int_{-\infty}^{t} \tilde{D} (\tau - t) G_{10} (\tau) d\tau = \left( \frac{\tilde{D} (0)}{\Gamma} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{d^n \tilde{D} (\tau)}{d\tau^n} \right)_{\tau=0} \frac{1}{\Gamma^{n+1}} \right) g.
$$

(128)

From Eqs. (127, 128) one can obtain the equation for $\Gamma$:

$$
\Gamma^2 + \tilde{D} (0) = \sum_{n=1}^{\infty} (-1)^n \left( \frac{d^n \tilde{D} (\tau)}{d\tau^n} \right)_{\tau=0} \frac{1}{\Gamma^n}.
$$

(129)

Taking into account that $\frac{d}{d\tau} \sim \frac{1}{\Im}$, the right-hand side of Eqs. (129) is an expansion in powers of a small parameter $(\Im \Gamma)^{-1}$. Note that $\tilde{D} (\tau)$ is a Fourier conjugate for the real spectrum $D (\Delta)$, therefore $\tilde{D} (0) = \int_{-\infty}^{\infty} D (\Delta) d\Delta$ is real. In zeroth order with respect to $(\Im \Gamma)^{-1}$ we obtain $\Gamma = \Gamma_0 = \pm i \sqrt{\tilde{D} (0)}$. Next, we substitute $\Gamma = \Gamma_0 + \eta$ into Eq. (129), where $|\Gamma_0| \gg |\eta|:

$$
\eta = \frac{\sum_{n=1}^{\infty} (-1)^n \left( \frac{d^n \tilde{D} (\tau)}{d\tau^n} \right)_{\tau=0} \frac{1}{\Gamma_0^n}}{2\Gamma_0 + \frac{1}{\Gamma_0} \sum_{n=1}^{\infty} (-1)^n n \left( \frac{d^n \tilde{D} (\tau)}{d\tau^n} \right)_{\tau=0} \frac{1}{\Gamma_0^{n+1}}}
$$

(130)

Taking into account that $\left( \frac{d^n \tilde{D} (\tau)}{d\tau^n} \right)_{\tau=0} = \int_{-\infty}^{\infty} (i\Delta)^n D (\Delta) d\Delta$, and $D (\Delta)$ is a real quantity, it is easy to see that in the numerator of Eq. (130) all terms are real, and in the denominator all terms are imaginary. Therefore, $\eta$ is imaginary, i.e. it is a frequency shift, not the decay constant. This is true in any order with respect to $(\Im \Gamma)^{-1}$. In particular, consider the largest term,

$$
\eta \approx \frac{1}{2\tilde{D} (0)} \left( \frac{d \tilde{D} (\tau)}{d\tau} \right)_{\tau=0} = i \int_{-\infty}^{\infty} \Delta D (\Delta) d\Delta
$$

(131)

This term is of the order of $\Im^{-1} \sim \delta$, i.e. of the order of the spectral bandwidth.

If we use Eq. (123), in which the function $f (\Delta / \delta)$ has a rectangular shape, we get

$$
\kappa \approx \pi N \frac{\langle \Omega_R^2 \rangle}{2\delta}, \quad |\Gamma_0| \approx \sqrt{N \langle \Omega_R^2 \rangle}.
$$

(132)

It is easy to see that the quantity $2\kappa$ corresponds exactly to the probability of the transition per unit time (from Fermi’s Golden Rule) from the state $|1\rangle \Pi_{j=1}^N |0_j\rangle$ into the continuous
spectrum of states of excited qubits. For the field-matter coupling regime described by the absorption rate Eq. (126) we have \( T \sim \kappa^{-1} \); in this case the required condition \( T\delta \gg \pi \) corresponds to \( \sqrt{N \langle |\Omega_R|^2 \rangle} \ll \frac{\delta}{\pi} \).

In the light-matter interaction regime corresponding to the frequency shift \( \sim |\Gamma_0| \), we have \( T \sim |\Gamma_0|^{-1} \); the required condition \( T\delta \ll \pi \) for this regime corresponds to \( \sqrt{N \langle |\Omega_R|^2 \rangle} \gg \frac{\delta}{\pi} \).

To summarize this part, the effect of the inhomogeneous broadening depends on the relationship between the collective Rabi frequency \( \sqrt{N \langle |\Omega_R|^2 \rangle} \) and the spread of transition frequencies \( \Delta_j = \frac{W_j}{\hbar} - \omega \). If this spread is greater than the collective Rabi frequency, the light-qubit coupling leads to the photon absorption accompanied by the transition into the continuum of excited-qubit states described by a standard perturbation theory. In the opposite case, when the collective Rabi frequency exceeds the spread of transition frequencies, the dynamics of individual qubits becomes synchronized and the photon interacts with an ensemble of qubits as with a single qubit with a giant dipole moment enhanced by the factor \( \sqrt{N} \).

**B. Electron states in quantum wells**

Here we consider a multilevel quantum-confined electron system such as electron states in a quantum well or perhaps in a quantum wire or a multilevel quantum dot. In this case optical transitions occur generally between two groups of electron energy states, for example between electron states in the conduction band and valence band. Let’s take zero energy between these two groups and denote positive energies in the conduction band as \( W_j \) (Latin indices) and negative energies in the valence band as \( -W_\alpha \) (Greek indices). The frequencies of the optical transitions are

\[
\omega_{j\alpha} = \frac{W_j + W_\alpha}{\hbar}.
\]

We won’t consider here the intraband optical transitions within each group, e.g. \( \alpha \leftrightarrow \beta \) or \( m \leftrightarrow n \), although the formalism below can be easily extended to include them.

The RWA Hamiltonian is

\[
\hat{H} = \hbar \omega \left( \hat{c}^\dagger \hat{c} + \frac{1}{2} \right) + \sum_{j=1}^{J} W_j \hat{a}_j^\dagger \hat{a}_j - \sum_{\alpha=1}^{A} W_\alpha \hat{a}_\alpha^\dagger \hat{a}_\alpha - \hbar \sum_{j=1}^{J} \sum_{\alpha=1}^{A} \left[ \Omega_{R;j\alpha} \hat{a}_\alpha^\dagger \hat{c} + \Omega_{R;j\alpha}^* \hat{a}_\alpha \hat{c}^\dagger \right],
\]

\[(134)\]
where $\Omega_{R;ja} = \frac{d_{ja} E_j}{\hbar}$.

Instead of the excitation and deexcitation operators for a qubit that are specific to a two-level system, $\hat{\sigma}^\dagger$ and $\hat{\sigma}$, it is easier to introduce standard creation and annihilation operators of the fermion states. Therefore, the states that were denoted as $|0_{ja}\rangle$ and $|1_{ja}\rangle$ when using the operators $\hat{\sigma}^\dagger$ and $\hat{\sigma}$ become $|0_j\rangle |1_{\alpha}\rangle$ and $|1_j\rangle |0_{\alpha}\rangle$ when using standard fermion operators.

We consider again lowest-energy states corresponding to zero- or single-photon excitation,

$$\Psi = C_{00} |0\rangle \Pi_{j=1} |0_j\rangle \Pi_{\alpha=1} |1_{\alpha}\rangle + C_{10} |1\rangle \Pi_{j=1} |0_j\rangle \Pi_{\alpha=1} |1_{\alpha}\rangle$$

$$+ \sum_{j,\alpha} C_{0ja} |0\rangle |1_j\rangle |0_{\alpha}\rangle \Pi_{\alpha
\neq \alpha} |1_{\beta}\rangle.$$  (135)

The equations for excited state coefficients are

$$\dot{C}_{10} + i \left( \frac{3}{2} \omega - \sum_{\alpha} \frac{W_{\alpha}}{\hbar} \right) C_{10} - i \sum_{j,\alpha} \Omega_{R;ja}^* C_{0ja} = 0,$$  (136)

$$\dot{C}_{0ja} + i \left( \frac{1}{2} \omega + \frac{W_j}{\hbar} - \sum_{\beta \neq \alpha} \frac{W_{\beta}}{\hbar} \right) C_{0ja} - i \Omega_{R;ja} C_{10} = 0,$$  (137)

Following a standard procedure, we seek the solution as

$$\begin{pmatrix} C_{10} \\ C_{0ja} \end{pmatrix} = \begin{pmatrix} G_{10} \\ G_{0ja} \end{pmatrix} e^{-i \left( \frac{3}{2} \omega - \sum_{\alpha} \frac{W_{\alpha}}{\hbar} \right) t},$$

which gives

$$\dot{G}_{10} - i \sum_{j,\alpha} \Omega_{R;ja}^* G_{0ja} = 0,$$  (138)

$$\dot{G}_{0ja} + i \Delta_{ja} G_{0ja} - i \Omega_{R;ja} G_{10} = 0,$$  (139)

where $\Delta_{ja} = \omega_{ja} - \omega$.

Their formal solution can be written as

$$G_{0ja} = G_{0ja} (0) e^{-i \Delta_{ja} t} + i \Omega_{R;ja} e^{-i \Delta_{ja} t} \int_0^t e^{i \Delta_{ja} \tau} G_{10} (\tau) d\tau,$$  (140)

$$\dot{G}_{10} = i \sum_{j,\alpha} \Omega_{R;ja}^* G_{0ja} (0) e^{-i \Delta_{ja} t} - \int_0^t \sum_{j,\alpha} |\Omega_{R;ja}|^2 G_{10} (\tau) e^{i \Delta_{ja} (\tau - t)} d\tau.$$  (141)

We assume that zero detuning $\Delta_{ja} = 0$ corresponds to some particular values of energies $W_{ja}$ and $W_{\alpha}$, and introduce the energy detunings with respect to those values:

$$\Delta_j = \frac{W_{ja} - W_j}{\hbar}, \quad \Delta_{\alpha} = \frac{W_{\alpha} - W_{\alpha}}{\hbar}.$$
In the limit of a continuous spectrum we obtain
\[ \dot{G}_{10} = i \int \int \Omega_{R} (\Delta_{J} + \Delta_{A}) g_{J} (\Delta_{J}) g_{A} (\Delta_{A}) G_{0,J,A} (0) e^{-i(\Delta_{J} + \Delta_{A})t} d\Delta_{J} d\Delta_{A} 
- \int_{0}^{t} \int \int |\Omega_{R}|^{2} (\Delta_{J} + \Delta_{A}) g_{J} (\Delta_{J}) g_{A} (\Delta_{A}) G_{10} (\tau) e^{i(\Delta_{J} + \Delta_{A})(\tau-t)} d\Delta_{J} d\Delta_{A} d\tau, \] (142)
where \( \sum_{J,A}^{N,\alpha} (\cdots) \Rightarrow \int \int (\cdots) g_{J} (\Delta_{J}) g_{A} (\Delta_{A}) d\Delta_{J} d\Delta_{A}, \Delta_{j,\alpha} \Rightarrow \Delta_{J} + \Delta_{A} \); the functions \( g_{J} (\Delta_{J}), g_{A} (\Delta_{A}) \) are corresponding spectral densities of states.

The subsequent arguments follow those in the previous section. For example, if the initial state has \( G_{0,J,A} (0) = 0 \), i.e., the electron states are not excited,
\[ \dot{G}_{10} = - \int_{0}^{t} \int \int e^{i\Delta_{J}(\tau-t)} e^{i\Delta_{A}(\tau-t)} D (\Delta_{J}, \Delta_{A}) G_{10} (\tau) d\Delta_{J} d\Delta_{A} d\tau \] (143)
where
\[ D (\Delta_{J}, \Delta_{A}) = |\Omega_{R}|^{2} (\Delta_{J} + \Delta_{A}) g_{J} (\Delta_{J}) g_{A} (\Delta_{A}). \]
From Eq. (143) one obtains
\[ \dot{G}_{10} = - \int_{0}^{t} \tilde{D} (\tau-t) G_{10} (\tau) d\tau, \] (144)
where
\[ \tilde{D} (\tau) = \tilde{D} (\tau_{1}, \tau_{2})_{\tau_{1}=\tau_{2}=\tau}, \]
\[ \tilde{D} (\tau_{1}, \tau_{2}) = \int \int e^{i\Delta_{J}\tau_{1}} e^{i\Delta_{A}\tau_{2}} D (\Delta_{J}, \Delta_{A}) d\Delta_{J} d\Delta_{A} \] (145)
is the Fourier conjugate corresponding to the 2D spectrum \( D (\Delta_{J}, \Delta_{A}) \). Introducing normalized densities of states,
\[ g_{J} (\Delta_{J}) = \frac{J}{2\delta_{J}} f_{J} \left( \frac{\Delta_{J}}{\delta_{J}} \right), \quad g_{A} (\Delta_{A}) = \frac{A}{2\delta_{A}} f_{A} \left( \frac{\Delta_{A}}{\delta_{A}} \right) \] (146)
where \( 2\delta_{J,A} \) are characteristic spectral widths, i.e.,
\[ f_{J,A} \left( \frac{\Delta_{J,A}}{\delta_{J,A}} \right) \rightarrow 0 \text{ for } |\Delta_{J,A}| \gg \delta_{J,A} \text{ and } \frac{1}{2\delta_{J,A}} \int_{-\infty}^{\infty} f_{J,A} \left( \frac{\Delta_{J,A}}{\delta_{J,A}} \right) d\Delta_{J,A} = 1, \]
and introducing some average value of \( \langle |\Omega_{R}|^{2} \rangle \), we obtain the 2D spectrum as
\[ D (\Delta_{J}, \Delta_{A}) = \frac{J A}{4\delta_{J} \delta_{A}} \langle |\Omega_{R}|^{2} \rangle f_{J} \left( \frac{\Delta_{J}}{\delta_{J}} \right) f_{A} \left( \frac{\Delta_{A}}{\delta_{A}} \right). \] (147)
The average value of the Rabi frequency \( \langle |\Omega_{R}|^{2} \rangle \) can be estimated by averaging over the matrix elements of the dipole-allowed optical transitions \( \alpha \rightarrow j \). The “width” of the 2D
spectrum $D(\Delta J, \Delta A)$ is of the order of $2\delta J \times 2\delta A$. The corresponding temporal “width” of the function $\tilde{D}(\tau) = \tilde{D}(\tau_1, \tau_2)_{\tau_1=\tau_2=\tau}$ is of the order of $\pi \delta^{-1}_{\text{max}}$, where $\delta_{\text{max}} = \max [\delta J, \delta A]$.

In what follows we repeat the same steps as in previous section, because they are based on the same equation (compare Eqs. (121) with Eq. (144)).

Consider the function $G_{10}(t)$ which varies over a certain unknown timescale $T$. In the limit of a long $T$, namely

$$T\delta_{\text{max}} \gg \pi, \quad t\delta_{\text{max}} \gg \pi,$$

Eq. (144) becomes an equation for an exponentially decaying function,

$$\dot{G}_{10} = -\kappa G_{10},$$

where the decay rate is given by

$$\kappa = \int_{-\infty}^{0} \tilde{D}(\tau) d\tau$$

As mentioned in the previous section, Eqs. (149), (150) correspond to a standard transition into the states with continuous spectrum, which is described by Fermi’s Golden Rule.

In the opposite limit of a short $T$, namely $T\delta_{\text{max}} \ll \pi$, similarly to subsection A, we obtain the regime of a giant qubit:

$$G_{10} \propto e^{\Gamma t}, \quad \Gamma \approx \pm i |\Gamma_0| + o\left(\frac{\delta_{\text{max}} |\Gamma_0|}{\pi}\right)$$

where $|\Gamma_0| = \sqrt{\tilde{D}(0)}$ and the function $\tilde{D}(\tau)$ is given by Eqs. (144), (145). Similarly to subsection A, all subsequent terms in the expansion of this solution in powers of the small parameter $\frac{\delta_{\text{max}} |\Gamma_0|}{\pi}$ are imaginary, i.e. they do not lead to the decay of Rabi oscillations.

Within the model described by Eq. (147), if we further assume the functions $f_{J,A} \left( \frac{\Delta J,A}{\delta_{J,A}} \right)$ to have a rectangular shape, we obtain the expressions (131) obtained in the previous subsection, after substituting $N \mapsto J \times A$ and $\delta \mapsto \delta_{\text{max}}$. The same substitutions should be made in the inequalities describing the conditions for various regimes:

(i) The regime of transitions to the continuum, corresponding to the standard perturbation theory, is realized when $\sqrt{JA \langle |\Omega_R|^2 \rangle} \ll \frac{\delta_{\text{max}}}{\pi}$;

(ii) The regime of a giant qubit is realized when $\sqrt{JA \langle |\Omega_R|^2 \rangle} \gg \frac{\delta_{\text{max}}}{\pi}$.

The spin-related degeneracy can be included by doubling the values of $JA$. 

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VII. CONCLUSIONS

We found analytic solutions for the quantum dynamics of many-qubit systems strongly
coupled to a quantized electromagnetic cavity mode, in the presence of decoherence and
dissipation for both fermions and cavity photons. The analytic solutions are derived for
a broad class of open quantum systems including identical qubits, an ensemble of qubits
with a broad distribution of transition frequencies, and multi-level electron systems. The
formalism is based on the stochastic equation of evolution for the state vector, within Markov
approximation for the relaxation processes and rotating wave approximation with respect to
the optical transition frequencies. This approach brings significant advantages as compared
to the existing methods of treating open quantum systems. Indeed, in the master equation
formalism the number of equations for $M$ degrees of freedom grows as $\frac{1}{2}M(M+1)$. Therefore,
with growing number of the degrees of freedom it quickly becomes intractable analytically
and has to be solved numerically. In contrast, within our approach one has to solve a
drastically smaller number of linear equations as explained in Appendix A below. This
happens not only because the number of coefficients in the state vector is much smaller than
the number of independent density matrix elements, but also because equations for the state
vector coefficients are separable into low-dimensional uncoupled blocks whereas equations for
the density matrix elements are not: they are all coupled by cross-terms. The latter is true
also for the matrix elements of the Heisenberg operators. Moreover, for a nonperturbative
dynamics of a strongly coupled system the Heisenberg equations are nonlinear operator-
valued equations even in the simplest single-qubit case.

Furthermore, the explicit expression for the state vector within our approach makes the
determination of entanglement or lack thereof much more transparent; it allows one to
compare the resulting quantum state with Bell states and other quantum states popular
in quantum information applications; it shows how the observables characterizing a given
state gets affected by relaxation and noise. Another well-known approach, the Wigner-
Weisskopf method allows one to obtain the rate of decay of an averaged state vector through
the imaginary parts of complex eigenenergies describing the relaxation of the populations
of excited states. However, it ignores the noise component of the state vector which is
determined by ground-state populations (and by excited states if the reservoir temperature
is high enough). It does not predict a correct equilibrium state at long times which is
determined by thermal and quantum noise. It does not conserve the norm as a result. Unlike the Wigner-Weisskopf method, our approach preserves the noise-averaged norm of the state vector, treats the effects of noise correctly, and yields a correct equilibrium state at long times.

We demonstrate in the analytic derivation that the interaction of an ensemble of qubits with a single-mode quantum field lead to highly unusual entangled states of practical importance, with destructive or constructive interference between the qubits depending on the initial excitation. For example, if a single excited photon interacts with an ensemble of qubits initially in their ground state, their constructive interference leads to the whole ensemble behaving as one collective dipole. However, if one or a small fraction of qubits were excited initially whereas the field was in the vacuum state, the subsequent relaxation drives the whole ensemble of qubits into an entangled dark state which is completely decoupled from the leaky cavity mode. Only a small fraction $1/N$ of the initial excitation energy is lost before the system goes into the dark state, where $N$ is the number of qubits in the ground state. This effect is due to destructive interference in the resulting entangled state of the qubits.

We find the regimes in which an inhomogeneously broadened ensemble of qubits or a multi-electron system with a broad distribution of transition frequencies couple to the quantized cavity field as a giant collective dipole. We derived the criteria when this regime of a collective dipole is destroyed by the inhomogeneous broadening of their frequencies, which is important for many realistic systems. The analytic solutions describing the evolution of the system are obtained in all limiting cases. These results are generalized beyond two-level systems to electron-hole excitations in quantum wells or other semiconductor nanostructures.

ACKNOWLEDGMENTS

This work has been supported in part by the Air Force Office for Scientific Research Grant No. FA9550-17-1-0341, National Science Foundation Award No. 1936276, and Texas A&M University through STRP, X-grant and T3-grant programs. M.T. and M.E. acknowledge the support by the Center of Excellence “Center of Photonics” funded by The Ministry of Science and Higher Education of the Russian Federation, Contract No. 075-15-2020-906.
Appendix A: Evolution of the states with arbitrary excitation energies

For an arbitrary quantum state of the N-qubit system coupled to a cavity mode the state vector can be expanded over all possible combinations of subsystems as

\[ \Psi = \sum_{n=0}^{\infty} \sum_{p=0}^{N} \sum_{\alpha_p=1} C_{np\alpha_p} |n\rangle |p,\alpha_p\rangle, \]  

(A1)

where \(|n\rangle\) is a Fock state of the boson (EM) field and \(|p,\alpha_p\rangle\) is a fermion state. Here index \(\alpha_p\) denotes different subsets of \(p\) elements out of a set of \(j = 1, 2, \ldots N\), which correspond to the excitation of \(p\) qubits out of \(N\). The total number of such subsets is determined by the binomial coefficient \(C_{pN}^p = \frac{N!}{p!(N-p)!}\). The state \(|p,\alpha_p\rangle\) can be written as

\[ |p,\alpha_p\rangle = \left( \prod_{j_p \in \alpha_p} \hat{\sigma}_{j_p}^{\dagger} \right) |0\rangle_{qub}, \]  

(A2)

where \(j_p \in \alpha_p\) are qubit numbers belonging to the subset marked by index \(\alpha_p\) and

\[ |0\rangle_{qub} = \prod_{j=1}^{N} |0\rangle_j. \]  

(A3)

As expected, the energy of the given state \(|n, p, \alpha_p\rangle\) does not depend on the index \(\alpha_p\):

\[ E_{n,p,\alpha_p} = \hbar \left( \frac{1}{2} + n \right) \omega + pW. \]  

(A4)

Substituting the state vector (A1) into the Schrödinger equation with the Hamiltonian (10) leads to a set of linear equations for the probability amplitudes \(C_{np\alpha_p}\), which is split into independent blocks corresponding to the condition

\[ n + p = m = \text{const.} \]  

(A5)

For \(m \leq N\) each block contains \(\sum_{p=0}^{m} C_{pN}^p\) equations which have the form

\[ \dot{C}_{np\alpha_p} + i \left[ \left( \frac{1}{2} + n \right) \omega + pW \right] C_{np\alpha_p} = 
\]

\[ i \left( \sqrt{m+1} \Omega_R \sum_{\alpha_{p-1}}^{p} C_{(n+1)(p-1)\alpha_{p-1}} + \sqrt{n} \Omega^*_R \sum_{\alpha_{p+1}}^{N-p} C_{(n-1)(p+1)\alpha_{p+1}} \right) = 0. \]  

(A6)

For given values of \(n\) and \(p\) satisfying Eq. (A5), Eq. (A6) corresponds to \(C_{pN}^p\) equations for different subsets \(\alpha_p\). Note that in the equation for \(C_{m0\alpha_0}\) the index \(\alpha_0\) corresponds to only one possible subset with zero energy of the fermionic subsystem: \(C_{N0}^0 = 1\), i.e., \(|0,\alpha_0\rangle = |0\rangle_{qub}\).
In the sums $\sum_a^b(\ldots)$ the lower index shows the type of a subset and the upper index shows the number of elements in the sum. Equation (A5) implies that the subsets $\alpha_p - 1$ and $\alpha_p + 1$ are related to the subset $\alpha_p$ through

$$|p, \alpha_p\rangle = \hat{\sigma}_{j(p-1)}^+ |p-1, \alpha_{p-1}\rangle; \quad |p, \alpha_p\rangle = \hat{\sigma}_{j(p+1)}^- |p+1, \alpha_{p+1}\rangle,$$

where each pair $\alpha_p, \alpha_{p-1}$ or $\alpha_p, \alpha_{p+1}$ corresponds to a certain value of the qubit index: $j_{p-1}$ or $j_{p+1}$. Each subset $\alpha_p$ corresponds to a certain finite number of subsets $\alpha_{p-1}$ or $\alpha_{p+1}$ which contribute to the summation in Eq. (A6).

For $m > N$ the number of equations in each block stops growing and is equal to $\sum_{p=0}^N \mathcal{C}_p^N$. In this case it is convenient to write down separately the equation corresponding to the state with maximum energy of the fermionic subsystem, equal to $N \times W$:

$$\dot{C}_{(m-N)\alpha_N} + i \left[ \left( \frac{1}{2} + m - N \right) \omega + N \frac{W}{\hbar} \right] C_{(m-N)\alpha_N} -$$

$$i \sqrt{m - N} \Omega \sum_{\alpha_{N-1}}^N C_{(m-N+1)(N-1)\alpha_{N-1}} = 0,$$

where the index $\alpha_N$ corresponds to the only possible subset ($\mathcal{C}_N^N = 1$) for the state $|N, \alpha_N\rangle = \prod_{j=1}^N |1_j\rangle$.

Solving the resulting set of equations in order to determine all probability amplitudes and therefore all observables is much easier with this method than with any other existing approach that we are aware of. As an example, let’s consider initial conditions which limit the number $m \leq M$ where $M > N$. In this case the density matrix equation leads to a set of coupled equations for independent matrix elements; the total number of equations is equal to

$$\frac{1}{2} \left[ \sum_{p=0}^N (M-p)\mathcal{C}_N^p \right] + 1 \sum_{p=0}^N (M-p)\mathcal{C}_N^p.$$

This number is obviously much higher than the maximum number of equations for the state vector coefficients within each block, which is equal to $\sum_{p=0}^N \mathcal{C}_N^p$.

This drastic reduction in the number of equations to solve is not only because the number of coefficients in the state vector is much smaller than the number of independent density matrix elements; it is also because equations for the state vector coefficients are separable into uncoupled blocks whereas equations for the density matrix elements are not: they are all coupled by cross-terms. The latter is true also for the matrix elements of the Heisenberg
operators. Moreover, for a strongly coupled system the Heisenberg equations are nonlinear operator-valued equations even in the simplest single-qubit case.

For the Hamiltonian (10) the rank of the problem (the size of each block) can be further reduced significantly. Indeed, Eq. (A6) can be summed over all subsets, \( \sum_{\alpha_p} C_{\alpha_p} \). This results in the following equations for the variables \( F_{np} = \sum_{\alpha_p} C_{\alpha_p} \):

\[
\dot{F}_{np} + i \left( \left( \frac{1}{2} + n \right) \omega + p \frac{W}{1} \right) F_{np} - i \left( \sqrt{n + 1} (N - p + 1) \Omega_R F_{(n+1)(p-1)} + \sqrt{n} \Omega_{R}^\ast (p + 1) F_{(n-1)(p+1)} \right) = 0,
\]

where the indices \( n \) and \( p \) are related by Eq. (A5). Therefore, the number of independent equations within each block is equal to \( m + 1 \) if \( m \leq N \) and is equal to \( N + 1 \) if \( m > N \). Each equation contains only two variables if \( n = m \) or \( n = 0 \), or \( p = M \). If none of these conditions are satisfied, then each equation contains three variables.

By far the most popular and accessible states are single-photon excitations. They are obtained from Eq. (A8) by taking \( m = 1 \). This gives rise to the second-order set of equations (16) and (17) from the main text.

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