EXTENSIONS OF AUGMENTED RACKS AND SURFACE RIBBON
COCYCLE INVARIANTS

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Abstract. A rack is a set with a binary operation that is right-invertible and self-distributive, properties diagrammatically corresponding to Reidemeister moves II and III, respectively. A rack is said to be an augmented rack if the operation is written by a group action. Racks and their cohomology theories have been extensively used for knot and knotted surface invariants. Similarly to group cohomology, rack 2-cocycles relate to extensions, and a natural question that arises is to characterize the extensions of augmented racks that are themselves augmented racks. In this paper, we characterize such extensions in terms of what we call fibrant and additive cohomology of racks. Simultaneous extensions of racks and groups are considered, where the respective 2-cocycles are related through a certain formula. Furthermore, we construct coloring and cocycle invariants for compact orientable surfaces with boundary in ribbon forms embedded in 3-space.

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1. Introduction

Self-distributive (binary) operations have been used since the 1980’s to construct invariants of knots and links, following the articles [13][17], where the notion of fundamental quandle was introduced, defined topologically and diagrammatically. Homology and cohomology theories of quandles were introduced, and used to construct invariants of links in 3-space, as well as knotted surfaces in 4-space [5]. These invariants are defined via certain state sum using quandle 2-cocycles, roughly described in the case of links in 3-space as follows. The initial data of the construction is a quandle \( X \), along with a 2-cocycle of \( X \) with coefficients in an abelian group \( A \). First, one defines the set of \( X \)-colorings of a fixed diagram \( D \) of a link \( L \) as the set of homomorphisms from the
fundamental quandle of $L$ (obtained through $D$) to $X$. A coloring is also regarded as an assignment of elements of $X$ to arcs of $D$, and assigned elements are called colors. For each coloring, then, one takes the Boltzmann weights of each crossing of $D$, where the 2-cocycle is evaluated, $\phi(x_\tau, y_\tau)$, where $\phi$ is a 2-cocycle and $(x_\tau, y_\tau)$ is a certain pair of colors specified at the crossing $\tau$, see Figure 1. Then for each coloring, all these weights are multiplied together over all crossings, to obtain $\prod_\tau \phi(x_\tau, y_\tau) \in A$. Upon summing over all $X$-colorings, this quantity $\sum \prod_\tau \phi(x_\tau, y_\tau)$ results to be invariant with respect to Reidemeister moves and, therefore, is independent of the choice of diagram of $D$.

![Figure 1. Positive (left) and negative (right) crossings and their colorings for binary quandles.](image)

Algebraically, quandle 2-cocycles provide certain extensions of quandles in parallel to extensions by group 2-cocycles, and there are bijective correspondences between equivalence classes of extensions and second quandle cohomology group $\mathcal{H}_2(Q)$. Relations and applications of algebraic theories of quandle extensions have been also obtained; extensions provide 2-cocycles that are used for constructing cocycle invariants, and interpretations of cocycle invariants are provided in terms of obstructions of extending colorings to extensions $\mathcal{H}_2(Q)$. Moreover, the relation between the extensions of certain quandles obtained from inner automorphisms of a group and the group itself has been studied, where homomorphisms between the cohomology groups are explicitly given.

These cocycle invariants have been generalized to invariants for framed links via racks, trivalent graphs for handlebody-links, and for surface ribbons (orientable compact surfaces with boundary in the form of ribbons) embedded in 3-space. In particular, in racks, cocycle weights are also assigned to trivalent vertices, that are group 2-cocycles, and for the purpose of defining invariants, algebraic structures and cohomology theories that have both associative and self-distributive operations in certain compatible manners have been developed.

An augmented rack is a rack $X$ with a map $\nu$ to a group $G$ acting on $X$, with certain conditions (see below). We could view augmented racks as an algebraic structure with a self-distributive operation of the rack $X$ and the group structure of $G$ intertwined. From this point of view, we provide two extension theories of augmented racks, and point out that the corresponding 2-cocycles produce invariants for surface ribbons through trivalent graph diagrams. As in racks, rack cocycles are assigned to crossings and group cocycles are assigned to trivalent vertices. The purpose of this paper is threefold: (1) define mixed cocycle conditions of cohomology groups in dimension 2 for group and rack 2-cocycles, (2) provide algebraic characterizations of these mixed cocycle conditions in terms of extensions, and (3) provide a definition of surface ribbon cocycle invariants via trivalent graphs utilizing the mixed 2-cocycles at crossings and trivalent vertices. For these goals, we focus on 2-cocycles of racks and groups, instead of aiming to formulate a general homology theory.

The paper is organized as follows. In Section 2, preliminary materials necessary to this paper are reviewed. Extensions of augmented racks and properties of their 2-cocycles are studied in Section 3.
and examples of extensions and cocycles are described. In Section 4, the construction of invariants using these extension cocycles under certain additional conditions is given.

2. Preliminaries

In this section we review materials used in this paper.

2.1. Augmented racks. A rack is a set $X$ with a binary operation $(x, y) \mapsto x * y =: S_y(x)$, where $S_y : X \to X$ is regarded as a map associated to $y$, such that $S_y$ is bijective for all $y \in X$, and $*$ is right self-distributive, $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$. It follows that $S_y$ is a rack isomorphism, and $X$ is called connected if the subgroup of the rack automorphism group generated by $\{S_y : y \in X\}$, called the inner automorphism group, acts transitively on $X$. It follows that $x \overset{*}{=} y := S_y^{-1}(x)$ is also a rack operation on $X$. A rack $X$ is called a quandle if it satisfies $x * x = x$ for all $x \in X$.

An augmented rack $(X, G)$ is a set $X$ with a right group action by a group $G$ and a map $\nu : X \to G$ satisfying the identity $\nu(x \cdot g) = g^{-1} \nu(x) g$ for all $x \in X$, $g \in G$, where the symbol $\cdot$ indicates the group action. An augmented rack has a rack operation defined by $x \overset{*}{=} y = x \cdot \nu(y)$ for $x, y \in X$. It is also said that $(X, \overset{*}{=})$ is a $G$-rack. It follows that $x \overset{*}{=} y = x \cdot \nu(y)^{-1}$.

2.2. Good involutions. Let $(X, \overset{*}{=})$ be a rack. The following definitions are from [14]. A good involution is an involution $\rho : X \to X$ that satisfies $x \overset{*}{=} \rho(y) = x \overset{*}{=} y$ and $\rho(x \overset{*}{=} y) = \rho(x)$ for all $x, y \in X$. A rack with a good involution is called a symmetric rack. If $X$ is a $G$-rack, then it is a symmetric rack with a good involution defined by $\nu(\rho(x)) = \nu(x)^{-1}$.

2.3. Group and rack 2-cocycles. In this paper we focus on 2-cocycles and corresponding extensions of groups and racks, so that we review these materials in addition to group and rack cohomology theories. References include [2],[5].

Let $G$ be a group and $A$ an abelian group. The $n$th cochain group (for the group $G$) is the set of functions $G^n \to A$ under pointwise addition, and is denoted by $C^n_G(A)$. The coboundary operator $\delta^n_G : C^n_G(A) \to C^{n+1}_G(A)$ (with trivial action on $A$) is defined by

$$(\delta^n_G f)(x_1, x_2, \ldots, x_{n+1}) = f(x_2, \ldots, x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_{i-1}, \hat{x}_i, x_i x_{i+1}, x_{i+2}, \ldots, x_{n+1}) + (-1)^{n+1} f(x_1, \ldots, x_n)$$

for $f \in C^n_G(A)$. The cocycle group, coboundary group, cohomology groups are denoted as usual by $Z^n_G(A)$, $B^n_G(A)$ and $H^n_G(A)$, respectively. For $n = 1, 2$, the differentials are formulated as

$$(\delta^n_G \zeta)(x, y) = \zeta(y) - \zeta(xy) + \zeta(x),$$
$$(\delta^n_G \eta)(x, y, z) = \eta(y, z) + \eta(x, yz) - \eta(xy, z) - \eta(x, y)$$

for $\zeta \in C^n_G(X, A)$ and $\eta \in C^n_G(X, A)$.

Let $\eta \in Z^n_G(X, A)$. Then $G \times A$ is endowed with a group structure by

$$(x, a)(y, b) := (xy, a + b + \eta(x, y))$$

for all $(x, a), (y, b) \in G \times A$. This group is called the (central) group extension of $G$ by $A$ with respect to $\eta$.

The group 2-cocycle condition has a well known diagrammatic interpretation as triangulation of a square as depicted in Figure 2. Three sides of a square are labeled by group elements, and each
triangle receives a group 2-cocycle evaluated by two sides of a triangle. As the figure shows, the
two triangulations give rise to the 2-cocycle condition, and corresponds to the associativity, that
ensures the extensions to be groups.

![Figure 2. Group 2-cocycle condition, triangulations of squares and associativity](image)

It is computed from the 2-cocycle condition \( (\delta^2 \xi) = 0 \) that a 2-cocycle satisfies \( \eta(g, e) = \eta(e, g) \) and \( \eta(g, g^{-1}) = \eta(g^{-1}, g) \) for \( g \in G \) and \( e \in G \) is the identity. These also imply that the identity of the extension \( G \times A \) is \((e, -\eta(e, e))\), and that \((g, s)^{-1} = (g^{-1}, -s - \eta(e, e))\) for \( g \in G \), \( s \in A \).

A group 2-cocycle \( \eta \) is called *normalized* if it satisfies \( \eta(e, e) = 0 \). Then it follows that \( \eta(g, e) = \eta(e, g) = \eta(g, g^{-1}) = 0 \) for all \( g \in G \). It is known that any second cohomology class has a normalized 2-cocycle representative.

Let \((X, \ast)\) be a rack and \( A \) be an abelian group, and denote the rack operation by \( \ast \). The \( n \)th *cochain group* (for the rack \( X \)) is the set of functions \( X^n \to A \) under pointwise addition, and is denoted by \( C^n_R(G, A) \). The coboundary operator \( \delta^n_R : C^n_R(X, A) \to C^{n+1}_R(X, A) \) is defined by

\[
(\delta^n_R g)(x_1, x_2, \ldots, x_{n+1}) = \sum_{i=1}^n (-1)^i [ (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) - (x_1 \ast x_i, \ldots, x_{i-1} \ast x_i, x_{i+1}, \ldots, x_{n+1}) ]
\]

for \( g \in C^n_R(X, A) \). The cocycle group, coboundary group, cohomoligy groups are denoted as usually by \( Z^n_R(X, A) \), \( B^n_R(X, A) \) and \( H^n_R(X, A) \), respectively. For \( n = 1, 2 \), the differentials are formulated as

\[
(\delta^1_R \xi)(x, y) = \xi(x) - \xi(x \ast y),
\]
\[
(\delta^2_R \phi)(x, y, z) = \phi(x, z) - \phi(x \ast y, z) - \phi(x, y) + \phi(x \ast z, y \ast z)
\]

for \( \xi \in C^1_R(X, A) \) and \( \phi \in C^2_R(X, A) \).

Let \( \phi \in Z^2_R(X, A) \). Then \( X \times A \) is endowed with a rack structure by

\[
(x, a) \ast (y, b) := (xy, a + \phi(x, y))
\]

for all \((x, a), (y, b) \in G \times A \). This rack is called the *rack extension of \( G \) by \( A \) with respect to \( \phi \)*.

A rack 2-cocycle \( \phi \in Z^2_R(X, A) \) for a symmetric rack \( X \) with a good involution \( \rho \) and a coefficient abelian group \( A \) is called *symmetric* [14] if it satisfies

\[
\phi(x, y) + \phi(\rho(x), y) = 0 \quad \text{and} \quad \phi(x, y) + \phi(x \ast y, \rho(y)) = 0
\]

for all \( x, y \in X \).
2.4. **Diagrams of surface ribbons and their moves.** In this section we review diagrams representing compact orientable surfaces with boundary embedded in 3-space (spatial surfaces with boundary). Our discussion is based on [16]. By compact surfaces with boundary, we mean surfaces that are compact and such that each component has a non-empty boundary. Such surfaces are determined by their *spines* and framing. Recall that a spine for such a surface $S$ is a trivalent graph $G$ in $S$ such that a closed regular neighborhood of $G$ in $S$ is a deformation retract of $S$. We therefore represent compact surfaces with boundary, diagrammatically, as fattened trivalent graphs in ribbon forms. We call such representations *surface ribbons*. Thus a surface ribbon is a compact orientable surface with boundary in the form of a thickened flat trivalent graph. When surface ribbons are embedded in 3-space, we consider its planar diagrams as for knot diagrams. Figure 3 (A) and (C) indicate local diagrams of such surface ribbons.

We further simplify surface ribbons to their spine, trivalent graphs, as depicted in Figure 3 (B) and (D). The ribbons are assumed to be specified by *blackboard framing*, where the arcs are fattened to ribbon forms parallel to the plane of projection, as known in framed knot diagrams. We focus on orientable surface ribbons. In this case we can assume that no half-twist occurs in diagrams, and two half-twists are represented by a small loop as in Figure 4.

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**Figure 3. Building blocks**

(A) (B) (C) (D)

**Figure 4. A loop corresponds to a full twist**

**Figure 5. Moves**
In [16], it was shown that the isotopy class of a compact orientable surface with boundary in a surface ribbon form is determined diagrammatically by the moves given in Figure 5. Moves RII, RIII and CL are the framed Reidemeister moves for framed links. Moves IY, YI and IH appear also in the study of handlebody knots in 3-space, see for instance [11]. In particular, we mention that the IH move is important in the well posedness of the diagrammatic interpretation in terms of trivalent graphs (spines), since it allows to arbitrarily desingularize higher order vertices, so that we can consider only trivalent vertices as building blocks. Matsuzaki has determined the moves for non-oriented surfaces as well [16], although we do not consider this case here.

3. Extensions of augmented racks and second cohomology groups

In this section we consider two types of extensions of augmented racks and corresponding constraints on cocycles. Equivalences of extensions are defined, and bijections to certain second cohomology groups are established.

3.1. Extensions of augmented racks with a fixed acting group. In this section we consider extensions of augmented racks with fixed acting groups.

**Definition 3.1.** Let \((X,G)\) be an augmented rack, and \(A\) an abelian group. A rack 2-cocycle \(\phi \in Z^2_R(X,A)\) is said to be additive (with respect to the second factor) if it satisfies

\[
\phi(w, x) + \phi(w \ast x, y) = \phi(w, z)
\]

for all \(w, x, y, z \in X\) such that \(\nu(x)\nu(y) = \nu(z)\).

**Lemma 3.2.** Let \((X,G)\) be an augmented rack, and let \(\xi : X \to A\) be a rack 1-cochain. Then \(\delta^1_R\xi\) is an additive 2-cocycle. Moreover, the sum of two additive cocycles is additive.

**Proof.** For \(x, y, z, w \in X\) such that \(\nu(x)\nu(y) = \nu(z)\) we have the equalities

\[
(\delta^1_R\xi)(w, x) + (\delta^1_R\xi)(w \ast x, y) = \xi(w) - \xi(w \ast x) + \xi(w \ast x) - \xi((w \ast x) \ast y)
\]

\[
= \xi(w) - \xi(w \cdot (\nu(x)\nu(y)))
\]

\[
= \xi(w) - \xi(w \ast z)
\]

\[
= (\delta^1_R\xi)(w, z).
\]

The second part of the statement is immediate. \(\square\)

As a consequence of Lemma 3.2 the following definition is well posed.

**Definition 3.3.** Let \((X,G)\) be an augmented rack, and \(A\) be an abelian group. Then, the subgroup of \(C^2_R(X,A)\) (resp. \(Z^2_R(X,A)\)) consisting of additive 2-cochains (resp. 2-cocycles) is called the additive (2-)cochain/cocycle group, and denoted by \(C^2_{R+}(X,A)\) (resp. \(Z^2_{R+}(X,A)\)). The quotient \(Z^2_{R+}(X,A)/\text{Im}(\delta^1_R)\) is called the additive (second) cohomology group of \(X\), and denoted by \(H^2_{R+}(X,A)\).

**Definition 3.4.** Let \((X,G)\) be an augmented rack, and let \(\phi \in C^2_R(X,A)\) be a rack 2-cochain with coefficients in \(A\). Then, \(\phi\) is said to be \(G\)-fibrant, or only fibrant for short, if it is constant on the fibers of \(\nu\) with respect to the second entry. In other words, for all \(x, y, z \in X\) such that \(\nu(y) = \nu(z)\) \in \(G\), it holds that \(\phi(x, y) = \phi(x, z)\). In this case, \(\phi\) induces a well defined map \(\phi : X \times \text{Im}(\nu) \to A\), which we denote by the same symbol, \(\phi(x, g) = \phi(x, y)\) where \(\nu(y) = g \in G\).

The fibrant 1-cochains are the maps \(\xi : X \to A\) that are constant on preimages of \(\nu\), i.e., for all \(x, y \in X\) such that \(\nu(x) = \nu(y)\), it holds that \(\xi(x) = \xi(y)\).
 Lemma 3.5. Let \((X,G)\) be an augmented rack, and let \(\xi : X \to A\) be a fibrant 1-cochain with coefficients in an abelian group \(A\). Then \(\delta^1 \xi\) is fibrant. Moreover, the sum of two fibrant cocycles is fibrant.

Proof. For \(x, y, z \in X\), with \(y, z \in \nu^{-1}(g)\) for some \(g \in G\), we have
\[
(\delta^1 \xi)(x, y) = \xi(x) - \xi(x * y) = \xi(x) - \xi(x * z) = (\delta^1 \xi)(x, z),
\]
if and only if \(\xi(x * y) = \xi(x * z)\), which holds true, since \(x * y = x \cdot g = x * z\). The second part of the statement is immediate. \(\square\)

Definition 3.6. Let \((X,G)\) be an augmented rack and let \(A\) be an abelian group. Then, we define the fibrant second rack cochain (resp. cocycle) group of \(X\) with coefficients in \(A\), denoted by \(C^2_{RF}(X,A)\) (resp. \(Z^2_{RF}(X,A)\)), to be the subgroup of \(C^2_R(X,A)\) (resp. \(Z^2_R(X,A)\)) that consists of fibrant rack 2-cochains (resp. cocycles). Similarly the first fibrant cochain group \(C^1_{RF}(X,A)\) is defined. The corresponding cohomology group \(Z^2_{RF}(X,A)/\text{Im}(\delta^1_{RF}(C^1_{RF}(X,A)))\) is called the fibrant second rack cohomology group, and denoted by \(H^2_{RF}(X,A)\).

Definition 3.7. Let \((X,G)\) be an augmented rack, and let \(A\) be an abelian group. Then, the fibrant-additive 2-cochains (resp. cocycles) of \((X,G)\) with coefficients in \(A\) are defined to be 2-cochains (resp. cocycles) of \(X\) that are both fibrant, and additive. They constitute a subgroup of \(C^2_R(X,A)\) (resp. \(Z^2_R(X,A)\)) which is denoted by the symbol \(C^2_{RF^+}(X,A)\) (resp. \(Z^2_{RF^+}(X,A)\)) which consists of fibrant rack 2-cochains (resp. cocycles) of \((X,G)\). The quotient of \(Z^2_{RF^+}(X,A)\) by the fibrant rack coboundaries is called the fibrant-additive cohomology group of \((X,G)\) with coefficients in \(A\), and it is denoted by the symbol \(H^2_{RF^+}(X,A) = [Z^2_{RF^+}(X,A) \cap Z^2_{RF}(X,A)]/\text{Im}(\delta^1_{RF}(C^1_{RF}(X,A)))\).

We show how fibrant-additivity relates to abelian extensions.

Proposition 3.8. Let \(X\) be a \(G\)-rack, and \(A\) an abelian group. Let \(\tilde{X} = X \times A\) be an abelian extension by \(\phi \in C^2_{RF^+}(X,A)\), where \(\phi\) is a fibrant and additive rack 2-cochain. Define \(\tilde{\nu} : \tilde{X} \to G\) by \(\tilde{\nu}(x,a) = \nu(x)\), and the action \(X \times G \to \tilde{X}\) by \((x,a) \cdot g := (x \cdot g, a + \phi(x,g))\). Then \((\tilde{X},G)\) is a \(G\)-rack.

Proof. First we note that the additivity of \(\phi\) under the assumption of being fibrant is reformulated, by setting \(\nu(x) = g\) and \(\nu(y) = h\), as \(\phi(w,g) + \phi(w \cdot g, h) = \phi(w, gh)\) from \(\phi(w,x) + \phi(w \cdot x, y) = \phi(w,z)\), since \(w \cdot x = w \cdot \nu(x) = w \cdot g\) and \(\nu(x)\nu(y) = gh = \nu(z)\).

The action defined is indeed a right \(G\)-action: for all \(w,x,y,z \in X\) such that \(\nu(s)\nu(y) = \nu(z)\), we have \([(w,d) \cdot h] \cdot h = (w \cdot g, d + \phi(w,g)) \cdot h = ([w \cdot g] \cdot h, d + \phi(w,g) + \phi(w \cdot g, h))\), which is equal to \((w,d) \cdot (gh) = (w \cdot (gh), d + \phi(x,gh))\) by the above reformulated additivity.

Then one checks the \(G\)-rack condition \(\tilde{\nu}((x,a) \cdot g) = \tilde{\nu}(x \cdot g, a + \phi(x,g)) = \nu(x \cdot g) = g^{-1}\nu(x)g = g^{-1}\tilde{\nu}(x,a)g\) as desired. \(\square\)

Definition 3.9. Let \(X\) be a \(G\)-rack, and \(A\) an abelian group, and \(\phi \in Z^2_{RF^+}(X,A)\). The \(G\)-rack \(\tilde{X} = X \times A\) defined in Proposition 3.8 by \(\phi\) is called a \(G\)-rack extension of \((X,G)\) by \(\phi\).

In Proposition 3.8, extensions are defined by 2-cochains. We show, in fact, that 2-cochains automatically are 2-cocycles. This gives a method of constructing 2-cocycles.

Proposition 3.10. Any fibrant-additive 2-cochain is a 2-cocycle: \(C^2_{RF^+}(X,A) = Z^2_{RF^+}(X,A)\).
Proof. Let $\phi \in C^2_{RF+}(X, A)$ as in Proposition 3.8. Note that the rack operation is written as
\[
(x, a) \ast (y, b) = (x, a) \cdot \tilde{\nu}(y, b) = (x, a) \cdot (\nu(y), b) = (x \cdot (\nu(y), a + \phi(x, \nu(y)) = (x \ast y, a + \phi(x, \nu(y))),
\]
which is the original rack extension by a 2-cocycle. Hence the original computation of the extension applies to obtain the rack 2-cocycle condition from the self-distributivity:
\[
((x, a) \ast (y, b)) \ast (z, c) = ((x \ast y) \ast z, a + \phi(x, y) + \phi(x \ast y, z))
\]
\[
((x, a) \ast (z, c)) \ast ((y, b) \ast (z, c)) = ((x \ast z) \ast (y \ast z), a + \phi(x, z) + \phi(x \ast z, y \ast z)),
\]
as expected. \hfill \qed

We provide constructions of examples using group extensions. Let $G$ be a group and $Q$ a union of its conjugacy classes. Then $Q$ is an augmented quandle by $\nu : Q \to G$ the inclusion, and is a conjugation subquandle of $G$.

Let $1 \to A \to \tilde{G} \overset{\nu}{\to} G \to 1$ be a central extension by $\eta \in Z^2_G(G, A)$. Let $\tilde{Q} \subset \tilde{G}$ be a union of conjugacy classes, and $Q := p(\tilde{Q})$. Then $\tilde{Q}$ is a conjugation subquandles of $\tilde{G}$. We denote the conjugation quandle operation by $\ast$ for $G$, $Q$ and $*$ for $\tilde{G}$, $\tilde{Q}$, respectively.

Lemma 3.11. Let $G$, $Q$, $\tilde{G}$, and $\tilde{Q}$ be as above. Let $s : G \to \tilde{G}$ be a set-theoretic section. Define an action of $G$ on $\tilde{Q}$ by $\tilde{x} \cdot g = s(g)^{-1}xs(g)$ for $\tilde{x} \in \tilde{X}, g \in G$. Then $\tilde{Q}$ is an augmented rack with $\tilde{\nu} := \nu \circ p$.

Proof. For $g, h \in G$, there exists $a \in A$ such that $s(g)s(h) = as(gh)$, where $A$ is regarded as a subgroup in the center $Z(\tilde{G})$. For $\tilde{x} \in \tilde{X}$ and $g, h \in G$, one computes
\[
(\tilde{x} \cdot g) \cdot h = s(h)^{-1}s(g)^{-1}\tilde{x}s(g)s(h) = (as(gh))^{-1}\tilde{x}(as(gh)) = s(gh)^{-1}\tilde{x}s(gh) = \tilde{x} \cdot (gh),
\]
so that this defined an action. One computes $\tilde{\nu}(\tilde{x} \cdot g) = \tilde{\nu}(s(g)^{-1}\tilde{x}s(g)) = \nu(g^{-1}p(\tilde{x})g)$ for $\tilde{x} \in \tilde{Q}$ and $g \in G$, as desired. \hfill \qed

Note that in the above situation, $\tilde{\nu}(\tilde{Q}) = Q$. In [8], it is proved that if $|\tilde{Q}|/|Q| = 2$, then $\tilde{Q}$ is an abelian extension of $Q$. Hence the preceding lemma gives rise to examples of Proposition 3.8. However, we do not know when the corresponding rack cocycles are additive.

Next we establish a bijection between equivalence classes of $G$-rack extensions and the second fibrant-additive cohomology group $H^2_{RF+}(X, A)$.

Definition 3.12. Let $X$ be a $G$-rack and $A$ an abelian group. Let $\tilde{X}_i = X \times A$, $i = 1, 2$, be $G$-rack extensions of a $G$-rack $X$ by $\phi_i \in Z^2_{RF+}(X, A)$ as defined in Proposition 3.8. Let $p_i : \tilde{X}_i \to X$ be the projection to the first factor. We say that $(\tilde{X}_i, p_i)$ are equivalent if there is a bijection $F : \tilde{X}_1 \to \tilde{X}_2$ such that the following diagrams commute.

\[
\begin{array}{ccc}
\tilde{X}_1 & \xrightarrow{F} & \tilde{X}_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
X & \xrightarrow{p} & X
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{X}_1 \times G & \xrightarrow{F \times id_G} & \tilde{X}_2 \\
\downarrow{F} & & \downarrow{F} \\
\tilde{X}_2 \times G & \xrightarrow{p} & \tilde{X}_2
\end{array}
\]

If such a map $F$ exists, we say that $F$ is an isomorphism of extensions, or isomorphism for short.

Theorem 3.13. Let $(X, G)$ be an augmented rack, and let $A$ be an abelian group. Then the equivalence classes of $G$-rack extensions of $(X, G)$ by $A$ are in bijective correspondence with the fibrant-additive rack cohomology group $H^2_{RF+}(X, A)$.
Proof. Let \( F : \tilde{X}_1 \to \tilde{X}_2 \) be a bijection in the definition of equivalence between extensions \( p_i : \tilde{X}_i \to X, \ i = 1, 2 \), by \( \phi_i \in Z^2_{RF}(X,A) \) as in Definition 3.12. Since \( p_1 = p_2 \circ F \) from the commutative diagram (1) in Definition 3.12, for any \((x,a) \in \tilde{X}_1 \) there exists \( \xi(x) \in A \) such that \( F(x,a) = (x, a + \xi(x)) \in \tilde{X}_2 \). This defines a function \( \xi : X \to A, \xi \in C^1_R(X,A) \). In addition, we observe that since \( (\tilde{\nu}_2 \circ F)(x,a) = \tilde{\nu}_2(x,a + \xi(x)) = \nu(x) = \tilde{\nu}_1(x,a) \), the diagram

\[
\begin{array}{ccc}
\tilde{X}_1 & \xrightarrow{F} & \tilde{X}_2 \\
\downarrow{\tilde{\nu}_1} & & \downarrow{\tilde{\nu}_2} \\
\tilde{G} & & \\
\end{array}
\]

commutes as well, establishing that \( F \) lies over \( G \). For \((x,a),(y,b) \in \tilde{X}_1 \), we have

\[
\begin{aligned}
F((x,a) \ast (y,b)) &= (x \ast y, a + \phi_1(x,y) + \xi(x \ast y)), \\
F(x,a) \ast F(y,b) &= (x, a + \xi(x)) \ast (y, b + \xi(y)) = (x \ast y, a + \xi(x) + \phi_2(x,y)),
\end{aligned}
\]

from the commutative diagram (2). Hence \( \phi_1 = \phi_2 + \delta^1_R \xi \), and we obtain \( [\phi_1] = [\phi_2] \in H^2_{RF}(X,A) \) as desired. Observe, in particular, that the previous equalities show that \( F \) is also a rack homomorphism with respect to the rack extension structures. Conversely, if \( \phi_1 = \phi_2 + \delta^1_R \xi \), then \( F((x,a)) := (x,a + \xi(x)) \) defines a desired isomorphism. \( \square \)

3.2. Simultaneous extensions of augmented racks. In this section we generalize the definition of \( G \)-rack extensions to extensions of \( X \) and \( G \) simultaneously, for augmented \( G \)-rack \( X \) by an abelian group \( A \), define appropriate cohomology, and establish a bijection between equivalence classes of such extensions and the cohomology defined.

Definition 3.14. Let \( X \) be a \( G \)-rack, \( A \) an abelian group, \( \phi \in Z^2_R(X,A) \), and \( \eta \in Z^2_G(G,A) \). We say that \( \phi \) is derived from \( \eta \), or \( \eta \)-derived, if they satisfy

\[
\phi(x,y) = -\eta(e,e) - \eta(\nu(y),\nu(y)^{-1}) + \eta(\nu(y)^{-1},\nu(x)) + \eta(\nu(y)^{-1}\nu(x),\nu(y))
\]

for all \( x, y \in X \), where \( e \in G \) is the identity. Note that the first two negative terms vanish for normalized group 2-cocycles \( \eta \).

Remark 3.15. Note that if \( \phi \) is \( \eta \)-derived in the preceding definition, then \( \phi \) is totally fibrant, in the sense that both factors depend only on the image of \( \nu \). In particular, \( \phi \) is fibrant, i.e. \( \phi \in Z^2_{RF}(X,A) \).

If \( \phi \) is totally fibrant, then it can be written as a pull-back \( \phi = \nu^* \phi \), which means \( \phi(x,y) = \bar{\phi}(\nu(x),\nu(y)) \) for some \( \bar{\phi} : G \times G \to A \).

Example 3.16. Let \((X,G)\) be an augmented rack and \( A \) an abelian group. Let \( \tilde{G} = G \times A \) be a central extension of \( G \) by a 2-cocycle \( \eta \in Z^2_G(X,A) \). Let a function \( \phi' : G \times G \to A \) be defined by

\[
\phi'(g,h) = -\eta(e,e) - \eta(h,h^{-1}) + \eta(h^{-1},g) + \eta(h^{-1}g,h),
\]

via the right-hand side of the equality in Definition 3.14. Let \( \nu = \nu^*(\phi') \) be the pull-back as described in Remark 3.15. Thus we obtain \( \eta \)-derived cocycles by pull-backs.
Proposition 3.17. Let $X$ be a $G$-rack, and $A$ an abelian group. Let $\breve{X} = X \times A$ be a rack extension by $\phi \in Z^2_R(X, A)$, and $\breve{G} = X \times A$ be a central extension of $G$ by $\eta \in Z^2_{RF_+}(G, A)$. Assume, further, that $\phi$ is additive ($\phi \in Z^2_{RF_+}(X, A)$), and $\eta$-derived (so that $\phi$ is in fact fibrant-additive, $\phi \in Z^2_{RF_+}(X, A)$).

Define $\breve{\nu} : \breve{X} \to \breve{G}$ by $\breve{\nu}(x, a) = (\nu(x), a)$, and the action $\breve{X} \times \breve{G} \to \breve{X}$ by $(x, a) \cdot (g, s) := (x \cdot g, a + \phi(x, g))$. Then $\breve{X}$ is a $\breve{G}$-rack.

Proof. The well-definedness of the action is similar to the proof of Proposition 3.8. Then one checks the $G$-rack condition:

$$\breve{\nu}((x, a) \cdot (g, s)) = \nu(x \cdot g, a + \phi(x, g)) = (\nu(x \cdot g), a + \phi(x, g)) = (g^{-1} \nu(x)g, a + \phi(x, g)),$$

and

$$(g, s)^{-1} \breve{\nu}(x, a)(g, s)$$

$$= (g^{-1}, -s - \eta(e, e) - \eta(g, g^{-1}))(\nu(x), a)(g, s)$$

$$= (g^{-1} \nu(x), a - s - \eta(e, e) - \eta(g, g^{-1}) + \eta(g^{-1}, \nu(x)))(g, s)$$

$$= (g^{-1} \nu(x)g, a - s - \eta(e, e) - \eta(g, g^{-1}) + \eta(g^{-1}, \nu(x)) + s + \eta(g^{-1} \nu(x), g))$$

so that the condition holds from the assumption that $\phi$ is $\eta$-derived.

\[\square\]

Corollary 3.18. Let $(X, G)$ be an augmented rack, and $A$ an abelian group. Let $\phi \in Z^2_{RF_+}(X, A)$ be fibrant-additive and $\eta$-derived. Then the equality

$$\phi(x, w) + \phi(y, w) + \eta(\nu(w)^{-1} \nu(x) \nu(w), \nu(w)^{-1} \nu(y) \nu(w)) = \phi(z, w) + \eta(\nu(x), \nu(y))$$

holds for all $x, y, z, w \in X$ such that $\nu(x) \nu(y) = \nu(z)$.

Proof. We use the notation in Proposition 3.17. Note that

$$\breve{\nu}(x, a) \breve{\nu}(y, b) = (\nu(x) \nu(y), a + b + \eta(\nu(x), \nu(y))).$$

If $\nu(x) \nu(y) = \nu(z)$, then we have

$$(\nu(x), a)(\nu(y), b) = (\nu(x) \nu(y), a + b + \eta(\nu(x), \nu(y))) = \breve{\nu}(z, a + b + \eta(\nu(x), \nu(y))).$$

Hence we have

$$\breve{\nu}((x, a) \ast (w, d)) \breve{\nu}((y, b) \ast (w, d))$$

$$= \nu((x, a) \cdot \nu(w, d)) \nu((y, b) \cdot \nu(w, d))$$

$$= [\nu(w, d)^{-1} \nu(x, a) \nu(w, d)] \nu(w, d)^{-1} \nu(y, b) \nu(w, d)$$

$$= \nu(w, d)^{-1} [\nu(x, a) \nu(y, b)] \nu(w, d)$$

$$= \nu((z, a + b + \eta(\nu(x), \nu(y)) \ast (w, d)))$$

$$= \nu((z, a + b + \eta(\nu(x), \nu(y)) \ast (w, d)))$$

$$= (\nu(z \ast w), a + b + \eta(\nu(x), \nu(y)) + \phi(z, w)).$$

We also obtain

$$\breve{\nu}((x, a) \ast (w, d)) \breve{\nu}((y, b) \ast (w, d))$$

$$= (\nu(x \ast w), a + \phi(x, w)) \nu(y \ast w, b + \phi(y, w))$$

$$= (\nu(x \ast w) \nu(y \ast w), a + \phi(x, w) + b + \phi(y, w) + \eta(\nu(x \ast w), \nu(y \ast w))),$$

which implies the equality as desired. \[\square\]
**Definition 3.19.** Let \((X, G)\) be an augmented rack, \(A\) be an abelian group, and \(\phi \in \mathbb{Z}_{RF+}^2(X, A)\), \(\eta \in \mathbb{Z}_G^2(X, A)\), such that \(\phi\) is \(\eta\)-derived. Then, the extension defined in Proposition 3.17, \((X \times A, G \times A)\), is called an **augmented (simultaneous) extension** of \((X, G)\) by \((\phi, \eta)\).

Next, we define an equivalence relation among augmented extensions.

**Definition 3.20.** Let \((\tilde{X}_i, \tilde{G}_i)\), for \(i = 1, 2\), denote two augmented extensions of the augmented rack \((X, G)\), where \(\tilde{X}_i = X \times A\) and \(\tilde{G}_i = G \times A\) as sets for both \(i = 1, 2\), by \((\phi_i, \eta_i)\), where \(\phi_i \in \mathbb{Z}_{RF+}^2(X_i, A)\) and \(\eta_i \in \mathbb{Z}_G^2(G, A)\), as defined in Proposition 3.17. Denote the augmentation maps by \(\tilde{\nu}_i : \tilde{X}_i \to \tilde{G}_i\).

Then, a morphism of augmented extensions is a pair of maps \(F_X : \tilde{X}_1 \to \tilde{X}_2\) and \(F_G : \tilde{G}_1 \to \tilde{G}_2\), where \(F_G\) is a group homomorphism, such that \(p_1 = p_2 \circ F_X\) and \(p_1 = p_2 \circ F_G\) with respective projections \(p_i, i = 1, 2\) in the same letter, satisfying the following diagrams commute.

\[
\begin{array}{ccc}
\tilde{X}_1 & \xrightarrow{\mathcal{F}} & \tilde{X}_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
X & & X \\
\end{array}
\quad (1) \quad \begin{array}{ccc}
\tilde{G}_1 & \xrightarrow{\mathcal{F}} & \tilde{G}_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
G & & G \\
\end{array}
\quad (2)
\]

\[
\begin{array}{ccc}
\tilde{X}_1 \times \tilde{G}_1 & \xrightarrow{\mathcal{F}_X \times \mathcal{F}_G} & \tilde{X}_1 \\
\downarrow{\mathcal{F}_X} & & \downarrow{\mathcal{F}_X} \\
\tilde{X}_2 \times \tilde{G}_2 & \xrightarrow{\mathcal{F}_G} & \tilde{X}_2 \\
\end{array}
\quad (3)
\]

An isomorphism, is a morphism where both \(F_X\) and \(F_G\) are bijections. Observe that in this case the inverses automatically satisfy the commutativity of the previous diagrams, guaranteeing that \((F_X^{-1}, F_G^{-1})\) is a morphism as well.

**Definition 3.21.** Let \((X, G)\) be an augmented rack, and let \(A\) be an abelian group. We define the **augmented rack second cohomology group** with coefficients in \(A\), denoted by \(Z_{AR}^2((X, G), A)\), to be the subgroup of the direct sum \(Z_{RF+}^2(X, A) \oplus Z_G^2(G, A)\) generated by

\[
\{ \phi(x, y) + \eta(\nu(y), \nu^{-1}) - \eta(\nu(y)^{-1}, \nu(x)) - \eta(\nu(y)^{-1}\nu(x), \nu(y)) \mid x, y \in X \}.
\]

Then we define the **augmented rack second cohomology group with coefficients in \(A\)**, denoted by \(H_{AR}^2((X, G), A)\), to be the quotient \(Z_{AR}^2((X, G), A) / \text{Im(}\delta_R^1 + \delta_G^1)\).

**Theorem 3.22.** Let \((X, G)\) be an augmented rack, and let \(A\) be an abelian group. Then the isomorphism classes of augmented extensions of \((X, G)\) by \(A\) with respect to cocycles \((\phi, \eta)\) are in bijection with the augmented rack cohomology group \(H_{AR}^2((X, G), A)\).

**Proof.** Let \((\tilde{X}_i, \tilde{G}_i)\), for \(i = 1, 2\), denote two equivalent augmented extensions of the augmented rack \((X, G)\), where \(\tilde{X}_i = X \times A\) and \(\tilde{G}_i = G \times A\) as sets for both \(i = 1, 2\), by \((\phi_i, \eta_i)\), where \(\phi_i \in \mathbb{Z}_{RF+}^2(X_i, A)\) and \(\eta_i \in \mathbb{Z}_G^2(G, A)\), and denote the augmentation maps by \(\tilde{\nu}_i : \tilde{X}_i \to \tilde{G}_i\), as in Definition 3.20 through isomorphisms \(F_X : \tilde{X}_1 \to \tilde{X}_2\) and \(F_G : \tilde{G}_1 \to \tilde{G}_2\).

By arguments similar to the proof of Theorem 3.13 from isomorphisms \(F_X\) and \(F_G\) in the commutative diagrams (1) and (2) in Definition 3.20 we have that for all \((x, a) \in \tilde{X}_1\) and \((g, b) \in \tilde{G}_1\) there exist \(\xi(x), \zeta(g) \in A\) such that \(F_X(x, a) = (x, a + \xi(x))\) and \(F_G(g, b) = (g, b + \zeta(g))\).
Commutativity of diagrams (1) and (3) implies that $F_X$ is a rack homomorphism, which combined with the fact that $F_G$ is a group homomorphism by hypothesis, implies that $[\phi_1] = [\phi_2]$ and $[\eta_1] = [\eta_2]$ through $\xi$ and $\zeta$. Since $(\tilde{X}_i, \tilde{G}_i)$ are augmented rack extensions through $(\phi_i, \eta_i)$, it also follows that $\phi_i$ is $\eta_i$-derived, for $i = 1, 2$. Hence $[(\phi_1, \eta_1)] = [(\phi_2, \eta_2)] \in H^2_{\text{AR}}((X, G), A)$. Below we check that the rest of the commutative diagrams for isomorphisms do not impose additional constraint.

One computes, for $(x, a) \in \tilde{X}_1$,

\[
\begin{align*}
(\nu_2 \circ F_X)(x, a) &= \nu_2(x, a + \xi(x)) = (\nu(x), a + \xi(x)), \\
(F_G \circ \nu_1)(x, a) &= F_G(\nu_1(x, a)) = F_G(\nu(x), a) = (\nu(x), a + \xi(x)),
\end{align*}
\]

hence (3) in Definition 3.20 commutes under the assumption.

For (4), one computes, for $(x, a) \in \tilde{X}_1$ and $(g, b) \in \tilde{G}_1$,

\[
\begin{align*}
F_X((x, a) \cdot (g, b)) &= F_X(x \cdot g, a + \phi_1(x, g)) = (x \cdot g, a + \phi_1(x, g) + \xi(x \cdot g)), \\
F_X(x, a) \cdot F_G(g, b) &= (x, a + \xi(x)) \cdot (g, b + \zeta(g)) = (x \cdot g, a + \xi(x) + \phi_2(x, g)),
\end{align*}
\]

so that we obtain $\phi_1(x, g) = \phi_2(x, g) + \xi(x) - \xi(x \cdot g) = \phi_2(x, g) + (\delta_R \xi)(x, g)$, and we obtain that $[\phi_1] = [\phi_2] \in \mathbb{Z}^2_{\text{RF}+}(X, A)$. Thus an isomorphism implies $[(\phi_1, \eta_1)] = [(\phi_2, \eta_2)] \in H^2_{\text{AR}}((X, G), A)$.

Conversely, if $[(\phi_1, \eta_1)] = [(\phi_2, \eta_2)] \in H^2_{\text{AR}}((X, G), A)$, then there are 1-cocohains $\xi \in C^1_R(X, A)$ and $\zeta \in C^1_G(G, A)$ such that $\phi_1 = \phi_2 + \delta_R \xi$ and $\eta_1 = \eta_2 + \delta G \zeta$. Define $F_X : \tilde{X}_1 \to \tilde{X}_2$ and $F_G : \tilde{G}_1 \to \tilde{G}_2$ by $F_X(x, a) = (x, a + \xi(x))$ and $F_G(g, b) = (g, b + \zeta(g))$, then it is readily checked that the diagrams (1) through (4) commute.

4. Colorings and cocycle invariants of surface ribbon diagrams by augmented racks

4.1. Colorings of trivalent graphs representing surface ribbons. Let $(X, G)$ be an augmented symmetric rack with a good involution $\rho$. Let $D$ be a graph diagram of a surface ribbon $S$, and $A(D)$ the set of arcs of $D$. Let $\tilde{D}$ denote $D$ with an orientation specified, and $\tilde{A}(\tilde{D})$ the set of directed arcs of $\tilde{D}$.

An orientation of edges are also specified by normal vectors of edges, that are obtained from the orientation of an edge by rotating it counterclockwise by 90 degrees as in Figure 6.

The sign $\epsilon(\tilde{a}, v)$ of an oriented edge $\tilde{a}$ incident to a vertex $v$ is defined, following [12], by

\[
\epsilon(\tilde{a}, v) = \begin{cases} 
1 & \text{if the orientation of } \tilde{a} \text{ points into } v, \\
-1 & \text{otherwise.}
\end{cases}
\]

In Figure 6(A), three edges are labeled by $a_i$, $i = 1, 2, 3$. In (B), all-in orientations are specified to give oriented edges $\tilde{a}_i$, and in this case, we have $\epsilon(\tilde{a}_i, v) = 1$ for $i = 1, 2, 3$. In (C), the orientations of the top two arcs are reversed to give $\tilde{a}_i'$ for $i = 1, 2$. The labels $g, h, k$ are used below.

A coloring of $\tilde{D}$ by an augmented rack $(X, G)$ with a good involution $\rho$ is a $\mathcal{C} : A(\tilde{D}) \to X$, such that the coloring conditions described below are satisfied at every crossing and vertex. In the figures, the letter assigned to each directed arc $\tilde{a}$ represents an element of $X$ that is the image $\mathcal{C}(\tilde{a}) \in X$ called a color of the directed arc $\tilde{a}$.

(C1) First, we require that if a color of an oriented arc $\tilde{a}$ is $x \in X$, then the color of the same edge with reversed orientation $\tilde{a}'$ is $\rho(x)$.

(C2) At each crossing, the coloring condition is as depicted in Figure 1.
At each vertex $v$, if the colors on the incident edges $\bar{a}, \bar{b}, \bar{c}$ in clockwise are $x, y, z \in X$, respectively, then it is required that

$$\nu(x)^{\epsilon(\bar{a}, v)} \nu(y)^{\epsilon(\bar{b}, v)} \nu(z)^{\epsilon(\bar{c}, v)} = e,$$

the identity element of $G$.

In Figure 6 (B), the oriented edges are labeled by $\nu(x) = g, \nu(y) = h$ and $\nu(z) = k$. With these orientations in the figure, we have $ghk = e$ from (C3). In (C), the orientations of the top two arcs are reversed, so that the colors are changed for $\bar{a}_1'$ and $\bar{a}_2'$ are changed to $\rho(x)$ and $\rho(y)$, respectively, and labeled by $\nu(\rho(x)) = g^{-1}$ and $\nu(\rho(y)) = h^{-1}$.

The set of colorings of $\bar{D}$ by an augmented rack $(X, G)$ with a good involution $\rho$ is simply denoted by $\text{Col}_X(\bar{D})$.

The following was mentioned in [8].

**Lemma 4.1.** Let $(X, G)$ be an augmented rack. If $X$ is finite and connected, then the fibers $\nu^{-1}(g)$ are in bijection over the image $g \in \text{Im}(\nu)$ of $\nu$.

**Proof.** Let $g, h \in \text{Im}(\nu)$, and we establish a bijection between $\nu^{-1}(g)$ and $\nu^{-1}(h)$. Let $x \in \nu^{-1}(g)$ and $y \in \nu^{-1}(h)$. Since $X$ is finite and connected, there exists a sequence $x_0, x_1, \ldots, x_n \in X$ such that $x = x_0$, $x_n = y$ and $x_{i+1} = x_i \ast x_i$ for all $i = 1, \ldots, n - 1$. Set $k = \nu(x_1) \cdots \nu(x_{n-1})$, then $y = x \cdot k$, and $h = \nu(y) = \nu(x \cdot k) = k^{-1} \nu(x)k = k^{-1}gk$. Since $\ast x_i$ is a bijection on $X$ for each $i$, $\cdot k$ defines a bijection on $X$. For any $z \in \nu^{-1}(g)$, we have $\nu(z \cdot k) = k^{-1} \nu(z)k = k^{-1}gk = h$, hence the image of $\cdot k$ is in $\nu^{-1}(h)$. Therefore $\cdot k$ defines a bijection $\nu^{-1}(g) \rightarrow \nu^{-1}(h)$.  

**Theorem 4.2.** Let $(X, G)$ be a finite augmented connected rack with a good involution $\rho$. Let $\bar{D}$ be an oriented diagram of a trivalent spatial graph. Then the set of colorings $\text{Col}_X(\bar{D})$ of $\bar{D}$ by $X$ is in bijection under each of the moves in Figure 5 and therefore, its cardinality is an invariant of surface ribbons.

**Proof.** First we show that the set of colorings is in bijection under reversal of the orientation of an edge $\bar{a}$. Let $\mathcal{C}$ be a given coloring of an oriented graph diagram $\bar{D}$. Let $\bar{a}$ be an oriented edge, and $\bar{a}'$ the same edge with reversed orientation. Then by the coloring condition (C1), if $\mathcal{C}(\bar{a}) = x$, then $\mathcal{C}(\bar{a}')$ is uniquely defined to be $\rho(x)$. For bijection, we show that this assignment, together with the colors already assigned by $\mathcal{C}$ on all the other edges satisfy the remaining conditions (C2) and (C3). The condition (C2) is shown in [12]. For the condition (C3), assume $x$ assigned to an arc whose orientation is reversed. Then $x$ is changed to $\rho(x)$, and the sign is reversed to $\epsilon(\bar{a}', v) = -\epsilon(\bar{a}, v)$, so that $\nu(x)^{\epsilon(\bar{a}, v)}$ is replaced by

$$\nu(\rho(x))^{\epsilon(\bar{a'}, v)} = [\nu(x)^{-1}]^{[-\epsilon(\bar{a}, v)]} = \nu(x)^{\epsilon(\bar{a}, v)},$$

hence, in fact, it stays unchanged. The other cases are similar.
Let us consider now the IY move of Figure 5, with a coloring $C$ in LHS of Figure 8. We again do not consider the weights represented in the picture. The color at the bottom arc specified by $C$ is $z'$. The two arcs colored by $x$ and $y$ underpass the arc $w$ and are colored by labels $x \ast w$ and $y \ast w$. From the definition of $C$ it follows that the color $z'$ of the remaining edge satisfies $\nu(z') = \nu(x \ast w) \nu(y \ast w)$.

One computes $\nu(z') = \nu(x \ast w) \nu(y \ast w) = \nu(x \cdot \nu(w)) \nu(y \cdot \nu(w)) = \nu(w)^{-1} \nu(x) \nu(y) \nu(w)$. Set $z = z'^{\ast} w$, then $z' = z \ast w$. One computes $\nu(z') = \nu(z \ast w) = \nu(z \cdot \nu(w)) = \nu(w)^{-1} \nu(z) \nu(w)$, hence we obtain $\nu(x) \nu(y) = \nu(z)$. On the RHS, given the color $z'$ at the bottom, the color $z$ in the figure is required to satisfy $z' = z \ast w$. Hence this color $z$ is uniquely determined from $z'$ and $w$, and satisfies the coloring rule (C3), yielding a unique coloring for the RHS.

Let us now consider the IH move of Figure 5. The coloring condition corresponds to associativity in $G$, because on the LHS we have $\nu(u) = \nu(x) \nu(y)$ and $\nu(u) = \nu(y) \nu(z)$, while on the RHS we have $\nu(v) = \nu(y) \nu(z)$ and $\nu(v) = \nu(x) \nu(v)$. This situation is depicted in Figure 9, where $\nu$ is abbreviated. For fixed $x, y, z, w$, the possible choices of $u$ that determine the colorings $C$ are determined by all the $u \in X$ such that $u \in \nu^{-1}(\nu(x) \nu(y))$, while on the RHS the colorings $C'$ are determined by the maps such that $v \in \nu^{-1}(\nu(y) \nu(z))$. By Lemma 4.1 these sets are in bijection. □
4.2. Cocycle invariants from $G$-rack extensions. In this section we show that the cocycle invariant can be defined using rack cocycles corresponding to $G$-rack extensions defined in Section 3.1 when the cocycle satisfies an additional requirement, which we now define.

Definition 4.3. Let $X$ denote an augmented rack with $\nu : X \to G$. Let $\phi \in Z^2_R(X, A)$ be a rack 2-cocycle of $X$ with coefficients in the abelian group $A$. Then $\phi$ is called pre-additive if for all $x, y, z \in X$ such that $\nu(x)\nu(y) = \nu(z)$, it holds that $\phi(x, w) + \phi(y, w) = \phi(z, w)$.

Remark 4.4. Direct computations show that it is not always the case that a coboundary $\phi = \delta_R \xi$ satisfies the pre-additivity.

Let $X$ denote an augmented rack with $\nu : X \to G$ with a good involution $\rho$. Let $\phi$ be a rack 2-cocycle of $X$ with coefficients in the abelian group $A$, that is fibrant-additive $\phi \in Z^2_{RF^+}(X, A)$ and pre-additive.

Let $S$ be a surface ribbon, and let us denote by $\vec{D}$ a diagram of $S$ with some orientation. For each coloring $C$ of $\vec{D}$, we define the following Boltzmann weight. At each crossing $\tau$ as in Figure 1, we set $B_{\phi}(C, \tau) = \phi(x, y)^{\sigma(\tau)}$, where $\sigma(\tau)$ denotes the sign of $\tau$ according to the convention established in Figure 1. Each vertex $v$ of $\vec{D}$ does not receive a weight.

Definition 4.5. Let $X$ be an augmented rack with $\nu : X \to G$ and with a good involution $\rho$. Let $\phi$ be a rack 2-cocycle of $X$ with coefficients in the abelian group $A$. Assume that $\phi$ is pre-additive and symmetric with respect to $\rho$. Let $S$ be a connected surface ribbon and let $\vec{D}$ denote a diagram of $S$. Then the cocycle invariant of $S$ is defined as

$$\Psi_{\phi}(S) = \sum_C \prod_\tau B_{\phi}(C, \tau)$$

where the sum runs over all the colorings, and the products are taken over all the crossings of $\vec{D}$. If $S$ consists of multiple connected components, then one defines the same partition function for each connected component, and $\Psi_{\phi}(S)$ is understood to indicate the tuples containing all the partition functions corresponding to each connected component.

Proposition 4.6. The cocycle invariant above is well defined with respect to the choice of oriented diagram $\vec{D}$. Therefore it is an invariant of the isotopy class of the surface ribbon $S$.

Proof. We check the invariance under each move as usual. The invariance under the moves that do not involve trivalent vertices are proved in the same manner as the original quandle cocycle invariant [5].
Invariance under the YI move and IY move follow from the additivity of $\phi$ and the pre-additivity condition as Figures 7 and 8 indicates, respectively, and as the relation between the moves.

For the IH move, the bijection between the set of colorings involve the possible colors for $v$ in LHS and $v$ in RHS in Figure 9 as explained in Proof of Theorem 4.2. Since all possible choices for $v$ are from the fiber $\nu^{-1}(\nu(x)\nu(y))$, the value of the cocycle evaluation involved remains unchanged, and so does for $v$ in RHS. Hence the state sum stays invariant. □

4.3. Cocycle invariants from simultaneous augmented rack extensions. In this section we define the cocycle invariant using 2-cocycles for the simultaneous extensions of augmented racks. First we recall the following.

Lemma 4.7. [5] Let $G$ be a group, $A$ an abelian group, and $\eta \in Z^2_G(G, A)$ a normalized 2-cocycle. Then $\eta$ satisfies

1. $\eta(g, h) = -\eta(gh, h^{-1}) = -\eta(g^{-1}, gh)$ for all $g, h \in G$, called the triangle symmetry.
2. $\eta(g, h) = -\eta(k, k^{-1}g) + \eta(k^{-1}g, h) + \eta(k, k^{-1}gh)$, for all $g, h, k \in G$.

In Figure 10 diagrammatic representations of the equalities (1) in Lemma 4.7 are depicted. A coherent (consecutive) orientations of two edges are evaluated for a 2-cocycle, and give an orientation of the triangle as depicted in (A). Orientation of one of these two edges are reversed in (B) and (C), which reverses the orientation of the triangle, as well as group elements to be evaluated, as depicted. The orientation reversal of the triangle corresponds to the negative sign of the evaluated cocycle.

Recall that at a vertex $v$, the sign, $\epsilon(\vec{a}, v)$, of an oriented edge incident to $v$ was defined by $\epsilon(\vec{a}, v) = 1$ if $\vec{a}$ points toward $v$, or equivalently, the orientation normal to the given direction is clockwise, and otherwise $\epsilon(\vec{a}, v) = -1$.

Let $X$ denote an augmented rack with $\nu : X \to G$ with a good involution $\rho$. Let $\phi$ be a rack 2-cocycle of $X$ with coefficients in the abelian group $A$, and let $\eta$ be a group 2-cocycles with coefficients in $A$. We further assume that $\phi$ and $\eta$ are considered in multiplicative notation, and that $\phi$ is $\eta$-derivable as defined in Definition 3.12. Let $S$ be a surface ribbon, and let us denote by $\vec{D}$ a diagram of $S$ with some orientation. For each coloring $\mathcal{C}$ of $\vec{D}$, we define the following Boltzmann weights. At each crossing $\tau$ as in Figure 4, let $x, y \in X$ be colors assigned to oriented arcs as depicted. We set $B_{\phi, \eta}(\mathcal{C}, \tau) := \phi(x, y)^{\sigma(\tau)}$, where $\sigma(\tau)$ denotes the sign of $\tau$ according to the convention established in Figure 1.

For each vertex $v$ of $\vec{D}$, let $\vec{a}_i$, $i = 1, 2, 3$, be the incident edges at $v$, labeled in clockwise order around $v$. Set $\sigma(v) = \epsilon(\vec{a}_1, v) + \epsilon(\vec{a}_2, v) + \epsilon(\vec{a}_3, v)$. Let $\mathcal{C}(a_1) = x^{\epsilon(\vec{a}_1, v)}$, $\mathcal{C}(a_2) = y^{\epsilon(\vec{a}_2, v)}$, and $\mathcal{C}(a_3) = z^{\epsilon(\vec{a}_3, v)}$. Here we take a convention that $x^{-1}$ represents $\rho(x)$ for $x \in X$. The coloring condition implies that $x^{\epsilon(\vec{a}_1, v)}y^{\epsilon(\vec{a}_i, v)}y^{\epsilon(\vec{a}_1, v)} = 1$. Then we set the weight at $v$ for a coloring $\mathcal{C}$ to be $B_{\phi, \eta}(\mathcal{C}, v) := \eta(\nu(x), \nu(y))^{\sigma(v)}$. When the edges are ordered counterclockwise, $\vec{a}_i'$, $i = 1, 2, 3$,
then the weight is defined to be $B_{\phi, \eta}(\mathcal{C}, v) := \eta(\nu(y), \nu(x))^{-\sigma(v)} = \eta(\nu(\mathcal{C}(\vec{a}')), \nu(\mathcal{C}(\vec{a})))^{-\sigma(v)}$. Results similar to the following can be found in [6,12].

**Lemma 4.8.** The weight $B_{\phi, \eta}(\mathcal{C}, v) = \eta(\nu(x), \nu(y))^{\sigma(v)}$ does not depend on the choices of orientations of the three edges incident to $v$, nor the two entries $(\vec{a}_1, \vec{a}_2)$.

**Proof.** Set $\nu(x) = g$, $\nu(y) = h$ and $\nu(z) = k$ and first assume that all signs are positive. Then the coloring condition is written as $ghk = e$, hence we have that $B_{\phi, \eta}(\mathcal{C}, v) = \eta(\mathcal{C}(\vec{a}_1), \mathcal{C}(\vec{a}_2)) = \eta(\eta(g, h))^{\sigma(v)} = \eta(g, h)$. By Lemma 4.7, we have $\eta(g, h) = \eta(\nu(x), \nu(y))^{\sigma(v)} = \eta(\nu(z), \nu(x)) = \eta(\mathcal{C}(\vec{a}_3), \mathcal{C}(\vec{a}_4))$. By cyclic symmetry we obtain that $B_{\phi, \eta}(\mathcal{C}, v)$ does not depend on the choice of $(\vec{a}_1, \vec{a}_2), (\vec{a}_2, \vec{a}_3), (\vec{a}_3, \vec{a}_1)$.

We also have $\eta(g, h) = \eta(\nu(x), \nu(y)) = \eta(k^{-1}, h^{-1})^{-1}$, and together with $\eta(g, h) = \eta(h, k)$, we obtain $\eta(k^{-1}, h^{-1})^{-1} = \eta(h, k)$ for all $h, k$. Hence we have $\eta(g, h) = \eta(h^{-1}, g^{-1})^{-1}$. If the orientations of $\vec{a}_1$ and $\vec{a}_2$ are reversed, then by reading inverse colors counterclockwise from $\vec{a}_2'$ to $\vec{a}_1'$ we obtain that

$$B_{\phi, \eta}(\mathcal{C}, v) = \eta(h^{-1}, g^{-1})^{-1} = \eta(g, h).$$

Since cyclic permutations of choices or arcs and reversing orientations of two out of three arcs generate all symmetries, we obtain the claim. \qed

One of the conditions posed to define the cocycle invariant is pre-additivity. We show that it is satisfied by $\eta$-derived rack 2-cocycles. For this purpose we identify the $\eta$-derivability to the construction given in [5].

Let $G$ be a group equipped with the conjugation rack structure, and let $A$ be an abelian group. In [5], it was shown that, for a group 2-cocycle $\eta \in Z^2_G(G, A)$, the function $\phi'(g, h) := \eta(g, h) - \eta(h, h^{-1}gh)$ is a rack 2-cocycle. When $X = G$ is the conjugation quandle, it is regarded as an augmented quandle by $\nu = \text{id}_G$. Then the $\eta$-derived 2-cocycle $\phi$ is regarded as an element of $Z^2_R(X, A)$.

**Lemma 4.9.** Let $G$ be a group, also regarded as a conjugation rack, and $A$ an abelian group as above. If $\eta \in Z^2_G(G, A)$ is normalized, then the two rack 2-cocycle constructions coincide:

$$\phi'(g, h) := \eta(g, h) - \eta(h, h^{-1}gh) = \eta(h^{-1}, g, h) = \phi(x, y).$$

**Proof.** By setting $z = y$ in Item (2) of Lemma 4.7, we have $\eta(g, h) - \eta(h, h^{-1}gh) = \eta(h^{-1}x, y) - \eta(h^{-1}g)$. By the first equality in Item (1), we have $\eta(h, h^{-1}g) = -\eta(y^{-1}, x)$, as desired. \qed

**Lemma 4.10.** Let $(X, G)$ be an augmented rack, and $A$ an abelian group. If $\phi \in Z^2_{RE}(X, A)$ is $\eta$-derived for some $\eta \in Z^2_G(G, A)$ which is normalized, then $\phi$ is additive.

**Proof.** Since $\phi$ is totally fibrant if $\phi$ is $\eta$-derived by Remark 3.15, we use group elements for the variables for $\phi$, so that we denote $\phi(x, y), x, y \in X$ by $\phi(g, h)$, where $g = \nu(x)$ and $h = \nu(y)$, $g, h \in G$. To show the additivity $\phi(k, g) + \phi(g^{-1}kg, h) = \phi(k, gh)$, we show

$$\phi(k, g) + \phi(g^{-1}kg, h) + \eta(g, h) = \phi(k, gh) + \eta(g, h).$$
By substituting the formula for $\phi'$ in Lemma 4.9 instead of defining formula of $\eta$-derivability, we compute

$$\phi(k, g) + \phi(g^{-1}kg, h) + \eta(g, h)$$

as desired. In Figure 11, the sequence of applications of 2-cocycle condition is represented. Shaded regions correspond to the squares dual to crossings, and triangles correspond to $\eta$ evaluated by group elements labeling edges of triangles. Diagrammatic representations of group 2-cocycle condition in Figure 2 and that of the triangle symmetry in Figure 10 are used in this computation.

**Figure 11.** Group 2-cocycles assigned to triangles and squares

**Definition 4.11.** Let $X$ be an augmented rack with $\nu : X \rightarrow G$. Let $\phi \in Z^2_R(X, A)$ be a rack 2-cocycle, and let $\eta \in Z^2_G(G, A)$ be a normalized group 2-cocycle, such that such that $\phi$ is $\eta$-derived. Assume further that $\phi$ is additive and symmetric with respect to $\rho$. Let $S$ be a connected surface ribbon and let $\vec{D}$ denote a diagram of $S$. Then the cocycle invariant of $S$ is defined as

$$\Psi_{\phi, \eta}(S) = \sum C \prod \tau B_{\phi, \eta}(C, \tau) \prod v B_{\phi, \eta}(C, v),$$

where the sums run over all the colorings, and the products are taken over all the crossings and vertices of $\vec{D}$. If $S$ consists of multiple connected components, then one defines the same partition function for each connected component, and $\Psi_{\phi, \eta}(S)$ is understood to indicate the tuples containing all the partition functions corresponding to each connected component.

**Proposition 4.12.** The cocycle invariant above is well defined with respect to the choice of oriented diagram $\vec{D}$. Therefore it is an invariant of the isotopy class of the surface ribbon $S$. 


Proof. The invariance under the moves that do not involve trivalent vertices are proved in the same manner as the original quandle cocycle invariant [5], and the same argument as the proof of Proposition 4.6.

Invariance under the YI move follow from the additivity of $\phi$ and the $\eta$-derivability of $\phi$ as Figures 7 and 8 indicates, respectively, and as the relation between the moves and these conditions were discussed in Section 3. In particular, additivity is shown in Lemma 4.9.

For the IH move, the bijection between the set of colorings involve the possible colors for $u$ in LHS and $v$ in RHS in Figure 9 as explained in Proof of Theorem 4.2. Since all possible choices for $u$ are from the fiber $\nu^{-1}(\nu(x)\nu(y))$, the value of the cocycle evaluation involved remains unchanged, and so does for $v$ in RHS. Hence the state sum stays invariant. □

Example 4.13. Let us consider the 2-cocycles $\phi$ constructed in Example 3.16. Then by Lemma 4.10, $\phi$ is additive. Hence by Proposition 4.12, $\phi$ defines a cocycle invariant. Thus the cocycle invariant for surface ribbons can be defined from any augmented racks through central group extensions.

Although it is desirable to investigate these invariants about the relations to other invariants and novel applications, these are beyond the scope of this paper, and it is left to future studies.

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