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$p$th Moment Stability of a Stationary Solution for a Reaction Diffusion System with Distributed Delays

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Abstract: In this paper, the Sobolev embedding theorem, Holder inequality, the Lebesgue control convergence theorem, the operator norm estimation technique, and critical point theory are employed to prove the existence of nontrivial stationary solution for $p$-Laplacian diffusion system with distributed delays. Furthermore, by giving the definition of $p$th moment stability, the authors use the Lyapunov functional method and Kamke function to derive the stability of nontrivial stationary solution. Moreover, a numerical example illuminates the effectiveness of the proposed methods. Finally, an interesting further thought is put forward, which is conducive to the in-depth study of the problem.

Keywords: large-scale variational methods; Lyapunov functional; $p$th moment stability

1. Introduction

It is well known that, in practical engineering, electrons inevitably diffuse in the inhomogeneous electromagnetic field. In addition, hence, the stability analysis of the reaction-diffusion system has become a hot topic [1–14]. In recent decades, many authors, such as Linshan Wang, Qiankun Song and Jinde Cao, have studied the stability of Laplacian reaction-diffusion neural networks with time delay, and achieved fruitful results in Laplacian diffusion systems [7–14]. On the other hand, $p$-Laplacian diffusion systems have also been widely studied.

In 2018, the asymptotic behavior of a $p$-Laplacian reaction-diffusion dynamic system is studied in [15]:

$$\frac{\partial u}{\partial t} - a(lu)\Delta p u = f(u) + h(t). \quad (1)$$

However, in practical engineering, the time delay is unavoidable, which may lead to chaos and instability of the system [16–24]. Thus, in this paper, we are to investigate the delayed $p$-Laplacian reaction-diffusion dynamic system. In addition, similar research has already begun. In [24], the $p$-Laplacian diffusion was firstly introduced in a time-delay dynamical system. Inspired by some methods of [24], the authors in [25] employed an impulsive differential inequality lemma to further study the time-delay neural networks with pulse perturbation. In [16], Ruofeng Rao and Shouming Zhong employed the Ekeland variational principle and Lyapunov stability theory to derive a globally...
exponential $p$th moment stability criterion for a Markovian jumping T-S fuzzy diffusion system with time-delays:

$$\begin{equation}
\frac{d u_i(t,x)}{dt} = D_i \text{div} \left( |\nabla u_i|^{p-2} \nabla u_i \right) - b_i(u_i) + \sum_{j=1}^{N} \rho_i(\omega(t)) \left( c_{ij}(r(t)) f_i(u_i(t,x)) + d_{ij}(r(t)) g_i(u_i(t-\tau_i(t),x)) \right) + I_i, \quad i \in \mathcal{N}, t > 0, x \in \Omega, u_i(t,x) = 0, \quad i \in \mathcal{N}, t > 0, x \in \partial \Omega. \tag{2}
\end{equation}$$

On the other hand, fuzzy logic theory has shown to be an appealing and efficient approach to deal with the analysis and synthesis problems for complex nonlinear systems. Among various kinds of fuzzy methods, Takagi–Sugeno (T–S) fuzzy models provide a successful method to describe certain complex nonlinear systems using some local linear subsystems [26–31].

Motivated by some ideas and methods in [16,32–40,46], we are to investigate the stability of a class of $p$-Laplacian diffusion T–S fuzzy system via variational methods that are different from those of existing literature related to reaction–diffusion fuzzy systems.

2. Preliminaries

Consider the following $p$-Laplacian system with time delays:

**Fuzzy rule $j$:** IF $\omega_1(t)$ is $\mu_{j1}$ and $\cdots$ $\omega_m(t)$ is $\mu_{jm}$ THEN

$$\begin{equation}
\frac{\partial u_i(t,x)}{\partial t} = a_i(t,x) \Delta_p u_i(t,x) - b_i(u_i(t,x)) + c_{ij} f_i(u_i(t,x)) + d_{ij} \int_{\tau_i(t)}^{t} f_i(u_i(s,x)) ds + I_i, \quad i \in \mathcal{N}, t > 0, x \in \Omega, u_i(\theta, x) = \phi_i(\theta, x), \quad (\theta, x) \in [-\tau, 0] \times \Omega, i \in \mathcal{N}, u_i(t,x) = 0, \quad (t,x) \in [-\tau, +\infty) \times \partial \Omega, i \in \mathcal{N}, \tag{3}
\end{equation}$$

where $\mathcal{N} \triangleq \{1, 2, \cdots, n\}$ is a finite index set, and $\Omega \subseteq \mathbb{R}^n$ is a bounded set with a smooth boundary $\partial \Omega$. $\{\mu_{jk} (j = 1, 2, \cdots, N; k = 1, 2, \cdots, m)\}$ is a fuzzy set, $\omega_k(t)$ represents a premise variable, $m$ is the number of premise variables, and $N$ is the number of IF-THEN rules. $a_i(t,x) u_i, \cdots, u_n$ represents a diffusion operator. Time delay $\tau(t)$ satisfies $\tau(t) \in [0, \tau]$, where $\tau > 0$ is a constant. $b_i$ represents the behavior function, dependent on $t$ and $x$. $c_{ij}$ and $d_{ij}$ denote the strength of state links between neurons, $f_i$ denotes the activation function of the neurons at $i$, and $I_i$ denotes the external input of the neurons at $i$. Assume in this paper that $I_i$ is a bounded quantity.

In view of a standard fuzzy inference method, the system (3) can be inferred as follows:

$$\begin{equation}
\frac{\partial u_i}{\partial t} = a_i(t,x) \Delta_p u_i(t,x) - b_i(u_i(t,x)) + \sum_{j=1}^{N} \rho_i(\omega(t)) \left( c_{ij} f_i(u_i(t,x)) + d_{ij} \int_{\tau_i(t)}^{t} f_i(u_i(s,x)) ds \right) + I_i, \quad t > 0, x \in \Omega, i \in \mathcal{N}, u_i(\theta, x) = \phi_i(\theta, x), \quad (\theta, x) \in [-\tau, 0] \times \Omega, i \in \mathcal{N}, u_i(t,x) = 0, \quad (t,x) \in [-\tau, +\infty) \times \partial \Omega, i \in \mathcal{N}, \tag{4}
\end{equation}$$

where $\omega(t) = (\omega_1(t), \cdots, \omega_m(t))$, and $Y_i(\omega(t))$ is the membership function corresponding to rule $j$.

In addition, $\rho_i(\omega(t)) = \frac{Y_i(\omega(t))}{\sum_{i=1}^{N} Y_i(\omega(t))}$ with $\sum_{j=1}^{N} \rho_j(\omega(t)) = 1$ and $\rho_j(\omega(t)) \geq 0$.

In this paper, we assume
(H1) for any given $q \in (1, p)$,
\[ \lim_{|r| \to \infty} \frac{b_i(r)}{r^{q-1}} = 0 = \lim_{|r| \to \infty} \frac{f_i(r)}{r^{q-1}}; \]

(H2) for any given $i \in \mathcal{N}$, there is the corresponding positive number $a_i$ such that
\[ a_i = \inf_{t,x,u} a_i(t, x, u_1, u_2, \ldots, u_n) > 0, \]
where $u = (u_1, u_2, \ldots, u_n);

(H3) for any given $i \in \mathcal{N}$, there is the corresponding positive number $\mathcal{F}_i$ such that
\[ |f_i(s) - f_i(r)| \leq \mathcal{F}_i |s - r|, \quad \forall s, r \in R; \]

(H4) for any given $i \in \mathcal{N}$, there is the corresponding positive number $\bar{b}_i$ such that
\[ \inf_{r \in R} b_i'(r) \geq \bar{b}_i > 0. \]

**Definition 1.** The nontrivial solution $\xi^*(x) = (\xi^*_1(x), \ldots, \xi^*_n(x))^T$ of the system (4) is said to be $p$th moment stable if, for any given $\varepsilon > 0$ and any given initial value $\phi = (\phi_1, \ldots, \phi_n)^T$, there exists $\delta(\varepsilon, \phi) > 0$ for $\phi$ with $\sum_{i=1}^n \|\phi_i - \xi^*_i\|_T < \delta(\varepsilon, \phi)$ such that
\[ \sum_{i=1}^n \left( \int_{\Omega} |u_i(t, \phi, x) - \xi^*_i(x)|^p dx \right)^{\frac{1}{p}} < \varepsilon, \quad \forall t \geq t_0, \]

where $\|\phi_i - \xi_i^*\|_T = \sup_{-\tau \leq t \leq 0} \left( \int_{\Omega} |\phi_i(t, x) - \xi^*_i(x)|^p dx \right)^{\frac{1}{p}}$.

Here, the above definition mainly imitates the Definition 7.1 of [41] (Chapter 1).

3. Main Result

**Theorem 1.** If the conditions (H1) and (H2) hold, there is a nontrivial stationary solution for fuzzy system (4). If, in addition, the conditions (H3) and (H4) and the following condition hold:
\[ \bar{b}_i > \mathcal{F}_i \sum_{j=1}^N (|c_{ij}| + \tau |d_{ij}|), \quad \forall i \in \mathcal{N}, \tag{5} \]
then, for any given $p > 1$, the nontrivial stationary solution is $p$th moment stable.

**Proof.** Denote $a_i(t) = a_i(t, u_1, \ldots, u_n)$. We shall complete the proof after two steps. □

**Step 1.** We claim that there is a nontrivial stationary solution for fuzzy system (4). Indeed, for any given $i \in \mathcal{N}$, we consider the following functional:
\[ I_i(\xi_i(x)) = \int_{\Omega} \left[ \frac{1}{p} a_i(t) |\nabla \xi_i(x)|^p + B_i(\xi_i(x)) \right] dx, \]

where $B_i(\xi_i) = \int_0^T b_i(r) dr$, and $\mathcal{F}_i(\xi_i) = \int_0^T f(r) dr$.

Below, we prove $I_i \in C^1(W_{0}^{1,p} (\Omega), R)$. Indeed, let
\[ I_i(\xi_i(x)) = \Lambda_i(\xi_i(x)) + \Gamma_i(\xi_i(x)) \]
with
\[ \Gamma_i(\xi_i(x)) = \int_{\Omega} \left[ B_i(\xi_i(x)) - \sum_{j=1}^{N} \rho_j(\omega(t)) \left( c_{ij} + d_{ij}(t) \right) F_i(\xi_i(x)) - f_i(\xi_i(x)) \right] dx \]
and
\[ \Lambda_i(\xi_i(x)) = \frac{1}{p} \int_{\Omega} |a_i(t)| \| \nabla \xi_i(x) \|^p dx. \]

For convenience, we denote
\[ g(x, \xi_i) = b_i(\xi_i(x)) - \sum_{j=1}^{N} \rho_j(\omega(t)) \left( c_{ij} + d_{ij}(t) \right) F_i(\xi_i(x)) - f_i(\xi_i(x)) \]
and
\[ G(x, \xi_i) = \int_{0}^{\xi_i} g(x, \xi) dt = B_i(\xi_i(x)) - \sum_{j=1}^{N} \rho_j(\omega(t)) \left( c_{ij} + d_{ij}(t) \right) F_i(\xi_i(x)) - f_i(\xi_i(x)). \]

Combining the continuity hypothesis and (H1) results in
\[ |g(x, \xi_i)| \leq C_0 + M_0 |\xi_i|^{q-1}, \]
where both \( C_0 > 0 \) and \( M_0 > 0 \) are constants.

Let \( v \in W_0^{1,p}(\Omega) \) with \( \|v\| < r \), for the functional \( \Gamma_i(\xi_i) = \int_{\Omega} G(x, \xi_i(x)) dx \), we can see it from the differential mean value theorem that the Gateaux differential of \( \Gamma_i(\xi_i) \) is
\[ D\Gamma_i(\xi_i,v) = \lim_{s \to 0} \frac{1}{s} \int_{\Omega} g(x, \xi_i + sv)v \ dx, \]
where \( \theta \in [0, 1] \). Now, we try to employ the Lebesgue control convergence theorem to deal with the limit. For \( |s| \leq 1 \), the Young inequality derives
\[ |g(x, \xi_i + sv)v| \leq \left( C_0 + M_0 |\xi_i + sv|^{q-1} \right) |v| \]
\[ \leq \frac{q-1}{q} \left( C_0 + M_0 |\xi_i + sv|^{q-1} \right)^{\frac{q}{q-1}} + \frac{A}{q} |v|^q \]
\[ \leq 2 M_0^{\frac{q}{q-1}} \left( C_0^{\frac{q}{q-1}} + M_0^{\frac{q}{q-1}} |\xi_i|^{q-1} + |v|^q \right) + \frac{A}{q} |v|^q, \]
in which the function \( 2 M_0^{\frac{q}{q-1}} \left( C_0^{\frac{q}{q-1}} + M_0^{\frac{q}{q-1}} |\xi_i|^{q-1} + |v|^q \right) + \frac{A}{q} |v|^q \) is a Lebesgue Integrable function due to Sobolev embedding theorem [42]. In addition, then the Lebesgue control convergence theorem yields
\[ D\Gamma_i(\xi_i,v) = \int_{\Omega} \lim_{s \to 0} g(x, \xi_i + sv)v \ dx = \int_{\Omega} g(x, \xi_i)v \ dx. \]

It is not difficult to prove that \( D\Gamma_i(\xi_i,v) \) is linear on \( v \). In fact,
\[ D\Gamma_i(\xi_i,k_1u + k_2v) = \int_{\Omega} g(x, \xi_i)(k_1u + k_2v)v \ dx = k_1 D\Gamma_i(\xi_i,u) + k_2 D\Gamma_i(\xi_i,v). \]

On the other hand, it follows from \( |g(x, \xi_i)| \leq C_0 + M_0 |\xi_i|^{q-1} \) that the operator \( \mathcal{G} : \xi_i \to g(x, \xi_i) \) is the bounded continuous operator of \( L^q(\Omega) \to L^{q-1}(\Omega) \). In addition, the Sobolev embedding theorem yields
\[ |D\Gamma_i(\xi_i,v)| \leq \|g(x, \xi_i)\|_{L^{q-1}} \|v\|_{L^q} \leq C \|g(x, \xi_i)\|_{L^{q-1}} \|v\|, \]
where \( \|v\| = \left( \int_\Omega |\nabla v|^p dx \right)^{\frac{1}{p}} \). Hence, \( \|D\Gamma_i(\xi_i)\| \leq C\|g(x, \xi_i)\|_{L^{\frac{p}{p-1}}} \), which implies that

\[
D\Gamma_i(\xi_i, v) = \langle D\Gamma_i(\xi_i), v \rangle, \quad \forall \xi_i, v \in W_0^{1,p}(\Omega).
\]

Below, we shall show that the Gateaux derivative \( D\Gamma_i(\xi_i) \) is continuous on \( \xi_i \). In fact,

\[
|\langle D\Gamma_i(\xi_i) - D\Gamma(u), v \rangle| \leq C\|g(x, \xi_i) - g(x, u)\|_{L^{\frac{p}{p-1}}} \|v\|
\]

which implies the norm of the operator

\[
\|D\Gamma_i(\xi_i) - D\Gamma(u)\| \leq C\|g(x, \xi_i) - g(x, u)\|_{L^{\frac{p}{p-1}}}.
\]

This means the operator \( \Sigma : g(x, \xi_i) \to D\Gamma_i(\xi_i) \) is continuous on \( L^{\frac{p}{p-1}} \to (W_0^{1,p})^* \). In addition, the Sobolev embedding theorem yields that the operator \( \mathcal{I} : \xi_i \to \xi_i \) is continuous on \( W_0^{1,p} \to L^q(\Omega) \). In addition, hence, \( D\Gamma_i = \Sigma \circ \mathcal{I} \circ \mathcal{I} : \xi_i \to D\Gamma_i(\xi_i) \) is the continuous operator of \( W_0^{1,p} \to (W_0^{1,p})^* \). Thereby, \( \Gamma_i \) is Frechet differentiable at any \( \xi_i \in W_0^{1,p}(\Omega) \). In addition, \( \Gamma_i \in C^1(W_0^{1,p}(\Omega), R) \).

In addition, it follows from \([43,44]\) and the condition of \( a_i(t) \) that \( \Lambda_i \in C^1(W_0^{1,p}(\Omega), R) \). Therefore, we have proved \( I_i \in C^1(W_0^{1,p}(\Omega), R) \).

If, for any given \( i \in \mathcal{N} \), the critical point of \( I_i \) exists, say \( \xi_i^* = (\xi_1^*, \xi_2^*, \cdots, \xi_n^*) \), a nontrivial stationary solution for the fuzzy system, where \( \xi_i^* = \xi_i^*(x) \). Below, we shall prove the existence of the critical point. In fact, the condition \((H1)\) yields

\[
|b_i(r)| + |f_i(r)| \leq C_1 + |r|^{q-1}, \quad \forall r \in R,
\]

where \( C_1 > 0 \). By the arbitrariness of \( q \), we select a suitable constant \( q \in (1, p) \) such that

\[
|f_i(r)| \leq C_1|r| + \frac{1}{q}|r|^q, \quad \forall r \in R.
\]

Letting \( C_1 \) be big enough, we can prove

\[
|b_i(r)| \leq C_1|r| + \frac{1}{q}|r|^q, \quad \forall r \in R.
\]

In fact, combining the continuity of \( b_i \) and the hypothesis \((H1)\) yields that there exists \( \bar{C}_1 > 0 \) such that

\[
|b_i(r)| \leq \bar{C}_1|r| + \frac{1}{q}|r|^q, \quad \forall r \in R,
\]

which deduces the inequality \((8)\) due to the big \( C_1 \).

From the Sobolev embedding theorem, we know that there are \( C_2, C_3 > 0 \) such that

\[
I_i(\xi_i) \geq \frac{1}{p}\|\xi_i\|_p^p - C_2\|\xi_i\|_q^q - C_3\|\xi_i\|,
\]

where \( \|\xi_i\| = \left( \int_\Omega |\nabla \xi_i|^p dx \right)^{\frac{1}{p}} \). In addition, \((3.5)\) and \( 1 < q < p \) yield that the lower bound of \( I_i \) exists. We shall prove that \( I_i \) is coercive on \( W_0^{1,p}(\Omega) \). Due to \( \|\eta_k\| \to \infty \) and \( I_i(\eta_k) \leq C \), \((9)\) leads to a contradiction. Hence, \( I_i \) must be coercive on \( W_0^{1,p}(\Omega) \). In addition, hence, there exists the constant

\[
c_i = \inf_{v \in W_0^{1,p}(\Omega)} I_i(v),
\]
from which there exists a minimization sequence \( \{\eta_k\} \subset W_0^{1,p}(\Omega) \) such that \( I_i(\eta_k) \to c_i \) whenever \( k \to \infty \). In addition, we know from the coercivity and the lower bound of the functional \( I_i \) that \( c_i \) is the global minimum of \( I_i \) on the Sobolev space \( W_0^{1,p}(\Omega) \). Moreover, if \( \eta_k \to \xi_i^* \in W_0^{1,p}(\Omega) \), then \( I_i(\xi_i^*) = c_i \) with \( I_i'(\xi_i^*) = 0 \), and \( \xi_i^* \) must be in the bounded subset of \( W_0^{1,p}(\Omega) \). Thus, the minimization sequence \( \{\eta_k\} \subset W_0^{1,p}(\Omega) \) satisfies \( I_i(\eta_k) \to c_i \) and \( I_i'(\eta_k) \to 0 \). In the inequality (9), let \( \xi_i = \eta_k \). Then, (9) yields that \( \{\eta_k\}_{k=1}^\infty \) is bounded on \( W_0^{1,p}(\Omega) \). Now, we claim that there exists \( \xi_i^* \in W_0^{1,p}(\Omega) \) with \( \eta_k \to \xi_i^* \). Indeed, we define the operators \( \mathfrak{A}_i, \mathfrak{B}_i : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^* \) as follows:

\[
< \mathfrak{A}_i(v), w > = a_i(t) \int_\Omega |\nabla v|^p - 2 \nabla v \nabla w dx, \quad \forall \, v, w \in W_0^{1,p}(\Omega)
\]  

and

\[
< \mathfrak{B}_i(v), w > = \int_\Omega \left[ b_i(v) - \sum_{j=1}^N \rho_j(\omega(t)) \left( c_{ij} f_i(v) + d_{ij} \tau(t) f_i(v) \right) - J_i \right] w dx, \quad \forall \, v, w \in W_0^{1,p}(\Omega).
\]  

It follows from (H1) that there are positive numbers \( C_4, C_5 \) such that

\[
\left| b_i(r) - \sum_{j=1}^N \rho_j(\omega(t)) \left( c_{ij} f_i(r) + d_{ij} \tau(t) f_i(r) \right) - J_i \right| \leq C_4 + C_5 |r|^{q-1}, \quad \forall \, r \in R.
\]

From [4,5], the operators \( \mathfrak{A}_i \) and \( \mathfrak{A}_i^{-1} \) are continuous, and \( \mathfrak{B}_i \) is compact operator. On one hand,

\[
< I_i(\eta_k), v > = < \mathfrak{A}_i(\eta_k), v > + < \mathfrak{B}_i(\eta_k), v >, \quad \forall \, v \in W_0^{1,p}(\Omega),
\]

and

\[
\mathfrak{A}_i(\eta_k) + \mathfrak{B}_i(\eta_k) \to 0, \quad k \to \infty.
\]

On the other hand, \( \{\eta_k\}_{k=1}^\infty \) is bounded on the space \( W_0^{1,p}(\Omega) \). Thus, it follows by the reflexivity of \( W_0^{1,p}(\Omega) \) that there is a subsequence of \( \{\eta_k\}_{k=1}^\infty \) that is weak convergent, say, \( \{\eta_k\}_{k=1}^\infty \). Since \( \mathfrak{B} \) is compact operator, there is a subsequence of \( \{\eta_k\}_{k=1}^\infty \) such that \( \{\mathfrak{B}_i(\eta_k)\}_{k=1}^\infty \) is convergent. In addition, \( \{\mathfrak{A}_i(\eta_k)\}_{k=1}^\infty \) owns a convergent subsequence. Moreover, the continuity of \( \mathfrak{A}^{-1} \) yields that \( \{\eta_k\}_{k=1}^\infty \) owns a convergent subsequence, say, \( \eta_k \to \xi_i^* \in W_0^{1,p}(\Omega) \). Hence, we have proved \( I_i(\xi_i^*) = c_i \) and \( I_i'(\xi_i^*) = 0 \). By the arbitrariness of \( i \), we have also proved that there is a nontrivial stationary solution \( \xi^* = (\xi_1^*, \xi_2^*, \cdots, \xi_n^*) \) for the system (4).

**Step 2.** To prove that \( \xi_i^* \) is \( p \)th moment stable.

Consider the Lyapunov–Krasovskii functional \( \mathcal{V}_i = \mathcal{V}_{i1} + \mathcal{V}_{i2} \), where

\[
\mathcal{V}_{i1} = \int_\Omega |u_i(t, x) - \xi_i^*(x)|^p dx,
\]

\[
\mathcal{V}_{i2} = p \mathcal{F}_i \sum_{j=1}^N |d_{ij}| \int_\Omega \left( \int_{t-j}^t dz \int_{t-z}^t |u_i(u, x) - \xi_i^*(x)|^{p-1} |u_i(s, x) - \xi_i^*(x)| ds \right) dx,
\]

then the conditions (H2)–(H4), boundary value condition, variational method, and Young inequality deduce
\[
\frac{dV_t}{dt} = p \int_\Omega [u_i(t,x) - \xi^*_i(x)]^p \left[ \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( (|\nabla u|^p - 2 \frac{\partial u}{\partial x_j}) - (|\nabla \xi^*_i|^p - 2 \frac{\partial \xi^*_i}{\partial x_j}) \right) \right] \nonumber \\
- \left( b_i(u_i - b_i(\xi^*_i)) + \sum_{j=1}^n p_j(\omega(t)) \left( c_{ij}(f_i(u_i)) - f_i(\xi^*_i) \right) + d_{ij} \int_{-t}^t (f_i(u_i(s,x)) - f_i(\xi^*_i(s))) \, ds \right) \, dx \nonumber \\
\leq - b_i p \int_\Omega |u_i(t,x) - \xi^*_i(x)|^p \, dx + p F_i \int_\Omega |u_i(t,x) - \xi^*_i(x)|^{p-1} |u_i(t+s,x) - \xi^*_i(x)| \, ds \nonumber \\
\text{and} \quad \left| d_{ij} \right| \int_{-t}^t |u_i(s,x) - \xi^*_i(x)| \, dx \nonumber \\
= - \left( b_i p - p F_i \sum_{j=1}^N |c_{ij}| + \tau |d_{ij}| \right) \int_\Omega |u_i(t,x) - \xi^*_i(x)|^p \, dx \leq 0, \quad \forall t \geq 0, \; i \in N. 
\]

(16)

(15) yields

\[
\frac{dV_i}{dt} = p F_i \sum_{j=1}^N |d_{ij}| \int_\Omega \left( \int_{-t}^0 |u_i(t,x) - \xi^*_i(x)|^p \, ds - \int_{-t}^0 |u_i(t,x) - \xi^*_i(x)|^{p-1} |u_i(t+s,x) - \xi^*_i(x)| \, ds \right) \, dx 
\]

\[
= p F_i \sum_{j=1}^N |d_{ij}| \int_\Omega \left( \tau |u_i(t,x) - \xi^*_i(x)|^p - \int_{-t}^0 |u_i(t,x) - \xi^*_i(x)|^{p-1} |u_i(t+s,x) - \xi^*_i(x)| \, ds \right) \, dx 
\]

\[
= p F_i \sum_{j=1}^N |d_{ij}| \int_\Omega \left( \tau |u_i(t,x) - \xi^*_i(x)|^p - \int_{-t}^t |u_i(t,x) - \xi^*_i(x)|^{p-1} |u_i(s,x) - \xi^*_i(x)| \, ds \right) \, dx 
\]

Therefore,

\[
\frac{dV_i}{dt} \leq - \left( b_i p - p F_i \sum_{j=1}^N |c_{ij}| + \tau |d_{ij}| \right) \int_\Omega |u_i(t,x) - \xi^*_i(x)|^p \, dx \leq 0, \quad \forall t \geq 0, \; i \in N. 
\]

Thus, \( \xi^*_i \) is \( p \)-th moment stable due to the definition of \( V_i \) and [41] (Theorem 2.1).

4. Numerical Example

In fuzzy system (2.2), the number of IF-THEN rules is \( N = 2 \). Let \( n = 2 \), and for \( i = 1, 2 \), \( f_i(r) = i \sin r \), \( b_i(r) = 5i r \), then (H1) holds for \( a_i(1, x_2, 2, u_1, u_2) = \frac{i + 3}{30} [2 + \sin(x_1 x_2 + u_1 + u_2)] \). In addition, then \( g_i = \frac{i + 3}{30} > 0 \), i.e., the condition (H2) is fulfilled. Hence, Theorem 1 yields that there exists a nontrivial stationary solution \( \xi^*_i = (\xi^*_1(x_1, x_2), \xi^*_2(x_1, x_2)) \). In addition, letting \( F_i = i \), the condition (H3) holds for \( b_i = 5i > 0 \), and so the condition (H4) holds too. Let \( \tau(t) = 2 + \cos t \), then \( \tau = 3 \). Letting \( c_{ij} = \frac{i + 3}{30} \), \( d_{ij} = \frac{i}{30} \), the direct computation leads to

\[
b_j = 5i > i \sum_{j=1}^2 \frac{i + j + 3j}{30} = F_i \sum_{j=1}^N (|c_{ij}| + \tau |d_{ij}|), \quad \forall i \in \{1, 2\}, 
\]

i.e.,

\[
b_1 = 5 > \frac{14}{30} = \sum_{j=1}^2 \frac{1 + j + 3j}{30} = F_1 \sum_{j=1}^N (|c_{1j}| + \tau |d_{1j}|) 
\]

and

\[
b_2 = 10 > \frac{5}{3} = \sum_{j=1}^2 \frac{2 + j + 6j}{30} = F_2 \sum_{j=1}^N (|c_{2j}| + \tau |d_{2j}|), 
\]

and so condition (5) is fulfilled. It follows from Theorem 1 that, for any given \( p > 1 \), the nontrivial stationary solution \( \xi^*_i = (\xi^*_1(x_1, x_2), \xi^*_2(x_1, x_2)) \) is \( p \)-th moment stable.
Remark 1. In this numerical example, \( \tau = 3 \) is the bigger upper bound of time delays, which shows the effectiveness of the proposed methods.

5. Conclusions and Further Considerations

Mainly inspired by some methods and ideas of literature [16,27,40] related to \( p \)-Laplace, we employed the critical point theory, variational technique, and Lyapunov functional method to derive the existence theorem of \( p \)th moment stable non-trivial stationary solutions for a class of \( p \)-Laplacian reaction–diffusion delay systems. The theorem holds for all \( p > 1 \), and the methods used in this paper are different from those in previous related literature to some extent. For example, we proved \( L \in C^1(W_0^{1,p}(\Omega), R) \) while it was not proved in related literature. A numerical example illustrates the effectiveness of the proposed methods. In [45] (Theorem 4.3), Ruofeng Rao and Shouming Zhong proposed a stability criterion for the delayed feedback financial system, in which the pulse effect occurs at a long time (see [45] (Remark 7) for details). How can the impulse control method be used to derive the stability criterion for the \( p \)-Laplacian diffusion system (4)? This is an interesting problem.

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