Non-Extreme Black Holes of Five Dimensional
N=2 AdS Supergravity

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Abstract

We derive and analyse the full set of equations of motion for non-extreme static black holes (including examples with the spatial curvatures $k = -1$ and $k = 0$) in D=5 N=2 gauged supergravity by employing the techniques of “very special geometry”. These solutions turn out to differ from those in the ungauged supergravity only in the non-extremality function, which has an additional term (proportional to the gauge coupling $g$), responsible for the appearance of naked singularities in the BPS-saturated limit. We derive an explicit solution for the STU model of gauged supergravity which is incidentally also a solution of D=5 N=4 and N=8 gauged supergravity. This solution is specified by three charges, the asymptotic negative cosmological constant (minimum of the potential) and a non-extremality parameter. While its BPS-saturated limit has a naked singularity, we find a lower bound on the non-extremality parameter (or equivalently on the ADM mass) for which the non-extreme solutions are regular. When this bound is saturated the extreme (non-supersymmetric) solution has zero Hawking temperature and finite entropy. Analogous qualitative features are expected to emerge for black hole solutions in D = 4 gauged supergravity as well.

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1 Introduction

Recently, there has been renewed interest in gauged supergravity theories in various dimensions. It is motivated by the fact that the ground state of these theories is anti-deSitter (AdS) space-time and thus they may have implications for the recently proposed AdS/CFT correspondence \[1, 2, 3\], which implies an equivalence of Type IIB string theory (or M-theory) on anti-deSitter (AdS) space-time and the conformal field theory (CFT) on the boundary of this space.

Specifically, Type IIB string theory on $AdS_5 \times S^5$ is conjectured \[4\] to be dual to D=4 $N=4$ superconformal Yang-Mills theory in the infinite t’Hooft coupling limit $g_{YM}^2 N \to \infty$. It is of special interest to address cases with less than 32 conserved supercharges and thus lower or no supersymmetry, in order to shed light on the nature of the correspondence there. Supergravity vacua with less supersymmetry may have an interpretation on the CFT side as an expansion of the theory around non-zero vacuum expectation value of certain operators. (Solutions with no supersymmetry could also be viewed as excitations above the ground state of the theory.)

One set of non-trivial gravitational backgrounds which preserve only part of the symmetry are BPS-saturated solutions, e.g., BPS-saturated black holes. Unfortunately, D=5 static BPS-saturated black holes \[5\] of gauged supergravity have naked singularities and thus, their singular geometry indicates ill defined properties of the theory at small distances on the gravity side \[6\].

The purpose of this paper is to explore non-extreme static black hole solutions of D=5 gauged supergravity. One of the motivation of this study is to shed light on the geometry of these solutions, in particular their singularity structure. In particular we would like to explore the range of ADM mass parameters for which the horizons are present and in turn determine their thermodynamic features. These features could potentially provide an insight into dynamics of Yang-Mills theories with broken supersymmetry.

Since the standard D=5 black holes have a spherical $S_3$-symmetry, these black holes may act as a gravitational background for the dual Yang-Mills theory with the global geometry of $R \times S_3$. (It may be important to replace the $S_3$ by a more general three-dimensional space with constant curvature $k$, i.e. along with the ordinary static black holes with $k = 1$, static solutions with $k = -1$ and $k = 0$ may also be of interest.) Interestingly, now the (charged) solutions have an extreme limit (with a vanishing Hawking temperature) which does not coincide with the BPS-saturated limit. So, they may serve as a non-supersymmetric gravity background at zero temperature. This situation is similar to the four-dimensional case, where charged black holes of gauged supergravity also allow for a zero-temperature non-supersymmetric limit \[7\].

Within this more general setting we address such static black holes, with $k = \pm 1, \ 0$. After briefly reviewing D=5 $N=2$ gauged supergravity theory in Section 2 we derive BPS-saturated topological black holes in gauged supergravity, also with naked singularities, were obtained in \[8, 9\].
the equations of motion for the specific field Ansätze in Section 3. (Note that a subclass of solutions of \( N = 2 \) supergravity are actually also solutions of supergravity theories with more, i.e. \( N = 4 \) or \( N = 8 \) supersymmetries.) In Section 4 we write an explicit solution for the case of a special prepotential, which is a gauged version of the three charge solution of ordinary \( N = 4,8 \) supergravity \([3], [9]\). (For equal charges a gauged solution has been discussed in \([10]\).) For \( k = 1 \) solutions we specifically identify the lower bound on the non-extremality parameter (or equivalently the ADM mass) which ensures that these solutions have regular horizons and further discuss their thermodynamic features.

\section{D=5 \( N=2 \) gauged supergravity}

In the context of M-theory, the theory of five-dimensional \( N = 2 \) supergravity coupled to abelian vector supermultiplets arise by compactifying eleven-dimensional supergravity, the low-energy theory of M-theory, on a Calabi-Yau three-folds \([1, 12]\). The massless spectrum of the theory contains \( (h_{(1,1)} - 1) \) vector multiplets with real scalar components, and thus \( h_{(1,1)} \) vector bosons (the additional vector boson is the graviphoton). The theory also contains \( h_{(2,1)} + 1 \) hypermultiplets, where \( h_{(1,1)} \) and \( h_{(2,1)} \), are the Calabi-Yau Hodge numbers. The anti-de Sitter supergravity can be obtained by gauging the \( U(1) \) subgroup of the \( SU(2) \) automorphism group of the \( N = 2 \) supersymmetry algebra, which breaks \( SU(2) \) down to the \( U(1) \) group. The gauging is achieved by introducing a linear combination of the abelian vector fields already present in the ungauged theory, i.e. \( A_\mu = V_i A^I_\mu \), with a coupling constant \( g \). The coupling of the fermi-fields to the \( U(1) \) vector field breaks supersymmetry, and in order to preserve \( N = 2 \) supersymmetry, one has to introduce gauge-invariant \( g \)-dependent terms. In a bosonic background, these additional terms give a scalar potential \([13]\).

The bosonic part of the effective gauged supersymmetric \( N = 2 \) Lagrangian which describes the coupling of vector multiplets to supergravity is given by

\begin{equation}
\begin{align*}
& e^{-1} \mathcal{L} = \frac{1}{2} R + g^2 V - \frac{1}{4} G_{IJ} F^I_{\mu \nu} F^{\mu \nu J} - \frac{1}{2} G_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + \frac{e^{-1}}{48} e^{\mu \nu \rho \sigma \lambda} C_{IJK} F^I_{\mu \nu} F^J_{\rho \sigma} A^K_\lambda ,
\end{align*}
\end{equation}

with the space-time indices \( (\mu, \nu) = 0, 1, \cdots, 4 \) have \((-1, 1, \cdots, 1)\) signature, \( R \) is the scalar curvature, \( F^I_{\mu \nu} \) are the Abelian field-strength tensors and \( e = \sqrt{-g} \) is the determinant of the Fünfbein \( e^a_m \), \( V \) is the potential given by

\begin{equation}
V(X) = V_i V_j \left( 6 X^I X^J - \frac{9}{2} G^{ij} \partial_i X^I \partial_j X^J \right)
\end{equation}

where \( X^I \) represent the real scalar fields which have to satisfy the constraint

\begin{equation}
\mathcal{V} = \frac{1}{6} C_{IJK} X^I X^J X^K = 1 .
\end{equation}
Also:

\[ G_{IJ} = -\frac{1}{2} \partial_i \partial_J \log V \bigg|_{V=1}, \quad G_{iJ} = \partial_i X^I \partial_J X^J G_{IJ} \bigg|_{V=1}, \quad (4) \]

where \( \partial_i \) refers to a partial derivative with respect to the scalar field \( \phi^i \). The physical quantities in (4) can all be expressed in terms of the homogeneous cubic polynomial \( V \) which defines a “very special geometry” \( [14] \).

Further useful relations are

\[ \partial_i X^I = -\frac{2}{3} G_{IJ} \partial_J X^J, \quad X^I = \frac{2}{3} G_{IJK} X^K. \quad (5) \]

It is worth pointing out that for Calabi-Yau compactification, \( V \) is the intersection form, \( X^I \) and \( X_I = \frac{1}{6} C_{IJK} X^K \) correspond to the size of the two- and four-cycles and \( C_{IJK} \) are the intersection numbers of the Calabi-Yau threefold.

Using the relationship (which can be proven within techniques of very special geometry)

\[ G^{ij} \partial_i X^I \partial_j X^J = G^{IJ} - \frac{2}{3} X^I X^J, \quad (6) \]

the potential can also be written as

\[ V(X) = 9 V_I V_J \left( X^I X^J - \frac{1}{2} G^{IJ} \right). \quad (7) \]

### 3 The Equations Of Motion

Before turning to the equations of motion we discuss the Ansatz for the non-extreme solution. For the standard static black holes the geometry at a given radius is \( R \times S^3 \) where the three-sphere becomes the horizon at \( r = r_H \). However in the context of CFT/AdS correspondence it may be useful to discuss a more general class of solutions, where the \( S^3 \) is replaced by a three-dimensional space with constant curvature \( k \). (In D=4 this replacement has been discussed for topological black holes \( [15] \), where the horizon is a genus \( g \) Riemann surface.) To be specific we consider a hypersurface given by the equation

\[ X_1^2 + X_2^2 + X_3^2 + \frac{k}{|k|} X_4^2 = \frac{1}{k}, \quad (8) \]

i.e. for \( k = 1 \) it is a \( S_3 \), for \( k = -1 \) it is a pseudo-sphere and for \( k = 0 \) it is a flat space. Introducing angular coordinates, the metric becomes

\[ d\Omega_{3,k} = d\chi^2 + \left( \frac{\sin \sqrt{k} \chi}{\sqrt{k}} \right)^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \quad (9) \]

Taking this spherical part, our Ansatz for the metric and the scalars reads

\[ ds^2 = -e^{-4U} f dt^2 + e^{2U} \left( \frac{dr^2}{f} + r^2 d\Omega_{3,k} \right), \quad X_I = \frac{1}{3} e^{-2U} H_I. \quad (10) \]
Motivated by the form of $f$ in the case of the BPS-saturated solution \([4]\), i.e. $f_{BPS} = 1 + g^2 r^2 e^{6U}$, and the form for the non-extreme solutions in the ungauged case, i.e. $f = k - \mu/r^2$, we take the following Ansatz for the function $f$:

$$f = k - \frac{\mu}{r^2} + g^2 r^2 e^{6U} ,$$ \hspace{1cm} (11)

where $\mu$ is the non-extremality parameter. On the other hand the Ansatz for the $U$-function and the scalars $X^I$ remains the same as for the extreme case and it is given by a set of harmonic functions:

$$H_I = h_I + \frac{q_I}{r^2} .$$ \hspace{1cm} (12)

Note that the choice to express the above Ansätze (11) for $f$ and (10) for $X_I$ in terms of harmonic functions $H_I$ (12) is special and corresponds to solutions with a special form of the prepotential $V$ \([3]\), i.e. “toroidal”-type compactifications. (Note that the discussion of the Einstein equations in Section 3.2 relies heavily on this form of the Ansätze.) In general $H_I$ need not be harmonic and thus the derived equations of motion for the harmonic form of $H_I$ need not have a consistent solution (See a discussion at the end of Section 3).

### 3.1 Gauge field equations

In solving the equations of motion we start with the gauge field equation, which reads

$$\frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} G_{IJ} F^{J \mu \nu} \right) \sim \epsilon^{\alpha \beta \gamma \lambda} F_{\alpha \beta}^J F_{\gamma \lambda}^K C_{IJK} .$$ \hspace{1cm} (13)

Since we are considering only electrically charged and non-rotating solutions ($F_{mn} = 0$, $(m, n) = (1 \cdots 4)$) the right-hand-side (rhs) vanishes. Thus,

$$\frac{1}{\sqrt{g}} \partial_r \left( \sqrt{g} G_{IJ} g^{00} g^{rr} F_{\cdot r0}^J \right) = \frac{1}{r^3} \partial_r \left( r^3 e^{4U} G_{IJ} F_{\cdot r0}^J \right) = 0 ,$$ \hspace{1cm} (14)

which is solved by

$$F_{\cdot r0}^J = -\frac{\sqrt{k}}{2} e^{-4U} G^{JI} \partial_r \tilde{H}_I ,$$ \hspace{1cm} (15)

where

$$\tilde{H}_I = 1 + \frac{\tilde{q}_I}{r^2} ,$$ \hspace{1cm} (16)

is a new set of harmonic functions with parameters $\tilde{q}_I$ corresponding to the physical electric charges. Note that in the extreme limit the $\tilde{H}_I$’s turn out to coincide with the $H_I$’s, introduced in (12). We have chosen the coefficient in front of the rhs of (15) in order to get the known extreme solution with $k = 1$. The appearance of $\sqrt{k}$ will be motivated below, namely, the Einstein equations can be solved in the extreme-case and for $k = 0, -1$ only when the coefficient on the rhs of (15) is chosen to be proportional to $\sqrt{k}$. (c.f. eq. (27)).
The appearance of the harmonic function in (15) seems to indicate that the generalization to a multi-center solution is straightforward. This, however, is not the case. The Bianchi identity is solved only, if the solution depends on \( r \) only, or otherwise for \( \tilde{H}_I = H_I \), which corresponds the extreme case.

### 3.2 Einstein equations

When expressed only in terms of the Ricci tensor the Einstein equation becomes

\[
R_{\mu\nu} = \left( F_{\mu\nu}^2 - \frac{1}{6} g_{\mu\nu} F^2 \right) + \partial_\mu X^I \partial_\nu X^J G_{IJ} - \frac{2}{3} g^2 V(X) g_{\mu\nu},
\]

with \( F_{\mu\nu}^2 \equiv G_{IJ} F_{\mu\nu}^I F_J^J \) and we have used \( \partial_\mu \partial_i \partial_\nu \partial_j G_{ij} = \partial_\mu X^I \partial_\nu X^J G_{IJ} \). First we consider the combination that determines the \( f \)-function

\[
R_0^0 + 2R_\theta^\theta = -2g^2 V(X).
\]

Calculating the Ricci tensor for our metric Ansatz, one finds

\[
g^{-2}(R_0^0 + 2R_\theta^\theta) = -\exp\left[ -\frac{2}{3} (X h)^2 - |h|^2 \right].
\]

(Primes refer to derivatives with respect to \( r \)). Therefore, the dependence on \( \mu \) and the spatial curvature \( k \) drop out, thus confirming the correct dependence of \( f \) (14) on \( \mu \). Note that in the special case of \( U = 0 \) the scalars are constant and the potential becomes constant \( V = 6 \), corresponding to the \( AdS \) vacuum.

From (10) one obtains that \( e^{2U} = \frac{1}{3} X^I H_I \) (recall \( X^I X_I = 1 \)) and using the relation \( H_I \partial_i X^I \sim X_I \partial_i X^I = 0 \) we find

\[
(e^{2U})' = \frac{1}{r} \left[ \frac{2}{3} (X h) - 2e^{2U} \right],
\]

with \( (X h) \equiv X^I h_I \). Similarly,

\[
(X^I)' = e^{-2U} \frac{1}{r} \left[ \frac{2}{3} X^I (X h) - G^{IJ} h_J \right].
\]

Using these relations we get

\[
g^{-2}(R_0^0 + 2R_\theta^\theta) = -e^{-2U} \left[ 2(X h)^2 - |h|^2 \right] = -2h_I h_J (X^I X^J - \frac{1}{2} G^{IJ})
\]

where \( |h|^2 \equiv G^{IJ} h_I h_J \). Comparing the form of the potential \( V \) given in (7) with the rhs of (22), we precisely reproduce the equation (18), providing the following relationship between the constant parts \( h_I \) of the harmonic functions \( H_I \) (12) and the expansion coefficients for the gauge field \( A_\mu = V_I A^I_\mu \) is satisfied:

\[
h_I = 3V_I.
\]
where we have used $F$.

The $R_i$ (Note there is no summation over the index $r$) on the other hand for $R_r^r$ we obtain:

$$R_r^r = -e^{-2U} (k - \frac{\mu}{r^2}) (U'' + \frac{3}{r} U') - e^{-2U} (k - \frac{\mu}{r^2}) U' - \frac{2}{3} g^2 V,$$

$$(24)$$

(Note there is no summation over the index $i$, which refers only to the angular components.) Interestingly, all the dependence on the gauge potential ($\propto g^2$) drops out of the Einstein equations (17) and thus these equations are identical to those obtained for the non-extreme black holes of ungauged $N = 2$ supergravity.

As next step we consider the $R_0^0$ component of Einstein equations, which becomes

$$R_0^0 + \frac{2}{3} g^2 V = \frac{1}{3} F^2 \equiv \frac{1}{3} G_{IJ} F^{I}_{\mu} F^{J}_{\mu} = \frac{k}{6} e^{-6U} G^{IJ} \partial_r \tilde{H}_I \partial_r \tilde{H}_J ,$$

$$(25)$$

where we have used $F_0^2 = \frac{1}{6} g_0^2 \partial_r \tilde{H}_I \partial_r \tilde{H}_J$ (since we have only electric fields) and we have inserted the form of the gauge field (15). In order to simplify the analysis we can employ the fact that the form of the $U$ function is the same as in the extreme case (i.e. when $\tilde{H}_I = H_I$, $\mu = 0$), and thus it solves the following equation

$$2 e^{-2U} (U'' + \frac{3}{r} U') = \frac{1}{2} e^{-6U} G^{IJ} \partial_r H_I \partial_r H_J .$$

$$(26)$$

Hence, the $R_0^0$ equation becomes

$$2 e^{-2U} (k - \frac{\mu}{r^2}) U' = \frac{1}{6} e^{-6U} G^{IJ} \left[ (k - \frac{\mu}{r^2}) \partial_r H_I \partial_r H_J - k \partial_r \tilde{H}_I \partial_r \tilde{H}_J \right] .$$

$$(27)$$

Introducing $\tilde{\mu} \equiv \frac{\mu}{k}$ (for $k \neq 0$) in (27) yields:

$$2 (e^{6U})' \frac{\tilde{\mu}}{r^3} = \frac{1}{2} e^{2U} G^{IJ} \left[ (1 - \frac{\tilde{\mu}}{r^2}) \partial_r H_I \partial_r H_J - \partial_r \tilde{H}_I \partial_r \tilde{H}_J \right] .$$

$$(28)$$

On the other hand for $k = 0$ we obtain:

$$2 (e^{6U})' \frac{\mu}{r^3} = \frac{1}{2} e^{2U} G^{IJ} \left( -\frac{\mu}{r^2} \right) \partial_r H_I \partial_r H_J .$$

$$(29)$$

The $R_r^r$ part of the Einstein equations:

$$R_r^r = \frac{1}{3} F^2 + \partial_r X^I \partial^r X^J G_{IJ} - \frac{2}{3} g^2 V .$$

$$(30)$$
can be cast in the same form as (28). Namely, again using the relations (20) and (21) we cast the scalar kinetic part in the following form:

\[
\partial_r X^I \partial_r X^J G_{IJ} = \frac{1}{r^2} e^{-6U} \left[ - \frac{2}{3} (X h)^2 + |h|^2 \right] \left( k - \frac{\mu}{r^2} + g^2 r^2 e^{6U} \right),
\]

and

\[
U'' + 2(U')^2 + \frac{3}{r} U' = e^{-4U} \left[ \frac{2}{3} (X h)^2 - |h|^2 \right].
\]

Inserting (31) and (32) in (30) we arrive at the same equation as that for the \( R^0_0 \) component of the Einstein equations (28).

Due to the symmetry of the \( R^i_i \) (angular) components of the Einstein equations, the rest of \( R^i_i \) components are redundant, and thus the analysis of the Einstein equations is complete.

### 3.3 Scalar equations

The scalar field equation reads

\[
\frac{1}{\sqrt{g}} \partial_{\mu} \left( \sqrt{g} g^{\mu\nu} G_{ij} \partial_{\nu} \phi^j \right) = \frac{1}{4} \partial_i G_{JK} \left[ F^J_{\mu\nu} F^K_{\mu\nu} - \frac{1}{k} F^J_{\mu\nu} F^K_{\mu\nu} \right],
\]

where \( \partial_i \) refers to a partial derivative with respect to the scalar fields \( \phi^i \). The left-hand-side (lhs) of (33) becomes

\[
e^{-2U} \left[ r^3 (k - \frac{\mu}{r^2} + g^2 r^2 e^{6U}) \right] G_{ij} \partial_r \phi^j.
\]

We again employ the fact, that the scalar fields are independent of \( k \), \( \mu \) and \( g \), i.e. they have the same form as in the known extreme case for \( k = 1 \). Therefore, multiplying the extreme equation (i.e. taking \( \tilde{H}_I = H_I \) in the field strengths) with \( (k - \frac{\mu}{r^2}) \) and subtracting it from both sides, only the term \( \propto (k - \frac{\mu}{r^2}) \) survives and we find

\[
e^{-2U} (k - \frac{\mu}{r^2})' G_{ij} \partial_r \phi^j
\]

\[
= \frac{1}{4} \partial_i G_{JK} \left[ F^J_{\mu\nu} F^K_{\mu\nu} - \frac{1}{k} F^J_{\mu\nu} F^K_{\mu\nu} \right]_{\tilde{H}_I = H_I}
\]

\[
= \frac{1}{8} e^{-6U} \partial_i G^{JK} \left[ (1 - \frac{\mu}{r^2}) \partial_r \tilde{H}_I \partial_r H_J - \partial_r \tilde{H}_I \partial_r H_J \right].
\]

Again, introducing \( \tilde{\mu} \equiv \frac{\mu}{k} \) the equation (35) takes the following form:

\[
2 e^{6U} \frac{\mu}{r^3} G_{ij} \partial_r \phi^j = \frac{1}{8} e^{2U} \partial_i G^{JK} \left[ (1 - \frac{\mu}{r^2}) \partial_r \tilde{H}_I \partial_r H_J - \partial_r \tilde{H}_I \partial_r H_J \right].
\]

while for \( k = 0 \) eq. (35) can be more more conveniently written as:

\[
2 e^{6U} \frac{\mu}{r^3} G_{ij} \partial_r \phi^j = \frac{1}{8} e^{2U} \partial_i G^{JK} \left( - \frac{\mu}{r^2} \right) \partial_r H_I \partial_r H_J.
\]
In conclusion, we have analysed all the equations of motion of the Lagrangian (1) with the static Ansätze (10), (12) for the metric and the scalar fields. The analysis of the gauge field equations in Subsection 3.1. introduced the harmonic functions \( \tilde{H}_I \) (c.f. (15)), the analysis of Einstein equations in Subsection 3.2 confirmed the Ansatz (11) for the function \( f \) in (14). In addition it yielded one additional constraint given by equation (28). The study of the scalar equations in Subsection 3.3 yields one more set of equations (36). Thus, solving these equations will fix the remaining parameters in the harmonic functions \( (H_I, \tilde{H}_I) \). Note that up to the replacement of \( \mu \to \tilde{\mu} \) both (28) and (36) are the same as in the ungauged case!

In general, the equations of motion, i.e. (28) and (36) (or (29) and (37)), cannot be solved in terms of the harmonic function Ansätze (12), only and one should instead regard functions \( H_I \), which determine \( \phi^i \) and \( e^{2U} \), as general functions, not necessarily harmonic. In this case these equations become coupled second order differential equations including a damping term (proportional to the first derivative) and a potential (coming from the field strengths); see also [16].

On the other hand, for a specific choice of \( \mathcal{V} \) it is possible to find an explicit solution in terms of harmonic functions, in particular for analogs of the “toroidal”-type compactifications discussed for the ungauged cases in [17], [18]. This is the three-charge configuration with no self-intersections, i.e. only \( C_{123} \neq 0 \) in (3). In this case the solution for \( U \) and \( \phi^i \) can be expressed in terms of harmonic functions \( H_I \), and the two sets of harmonic functions \( H_I \) and \( \tilde{H}_I \) are proportional to each other by a constant matrix [16]. In the following we will analyse in detail a specific example in this subclass, the \( STU \) model.

4 Discussion of a special solution

As an example we consider the \( STU \) model which has only one intersection number \( C_{123} = 1 \) nonzero. This model can be embedded into gauged \( N = 4 \) and \( N = 8 \) supergravity as well. In the following we shall derive the explicit solution and its properties.

4.1 Solution for the \( STU \) model

This model is given by the prepotential

\[
\mathcal{V} = STU = 1
\]

Taking \( S = X^1 \), \( T = X^2 \) and \( U = X^3 \) one gets for \( e^{6U} \) and the matrix \( G^{IJ} \)

\[
e^{6U} = H_1 H_2 H_3 \quad , \quad G^{IJ} = 2 \begin{pmatrix} S^2 & T^2 \\ U^2 & \end{pmatrix}
\]

(39)
Considering $S$ as the dependent field, i.e. $S = 1/TU$ we find

$$G_{ij} = \begin{pmatrix} \frac{1}{T^2} & \frac{1}{TU} \\ \frac{1}{TU} & \frac{1}{U^2} \end{pmatrix} \quad \text{and} \quad G^{ij} = \frac{4}{3} \begin{pmatrix} T^2 & -TU \\ -TU & U^2 \end{pmatrix}. \quad (40)$$

For this case the potential reads (assuming $h_I = (1, 1, 1)$ and thus $V_I = \frac{1}{3}$, c.f. (23)):

$$V(T, U) = 2\left(\frac{1}{U} + \frac{1}{T} + TU\right) \quad (41)$$

with the minimum $V_{\text{min}}(T = U = 1) = 6$ which is reached in the asymptotic vacuum with cosmological constant given by $g^2 V_{\text{min}}$. The Ansätze (10), along with (15) and (11), yield the following explicit form for the fields:

$$ds^2 = -(H_1H_2H_3)^{-2/3}f dt^2 + (H_1H_2H_3)^{1/3}\left(f^{-1}dr^2 + r^2d\Omega_{3,k}\right), \quad f = k - \frac{4}{r^2} + g^2 r^2 H_1H_2H_3, \quad X^I = H_I^{-1}(H_1H_2H_3)^{1/3}, \quad F_{r0}^I = -\frac{\sqrt{k}}{2}(H_I)^{-2}\partial_r H_I, \quad (42)$$

where $k$ determines the spatial curvature of $d\Omega_{3,k}$. Notice, for $k = 0$ the gauge fields vanish, but the scalars remain non-trivial. Finally $(q_I, \tilde{q}_I)$ are fixed by the equations (28) and (35) and one finds

$$q_I = \tilde{\mu} \sinh^2 \beta_I, \quad \tilde{q}_I = \tilde{\mu} \sinh \beta_I \cosh \beta_I \quad (\tilde{\mu} \equiv \frac{\mu}{k}), \quad (43)$$

which are the same expressions as in the ungauged case (since the equations are the same).

Note also, that for the extreme case ($\mu = 0$) with $k = 0$ and equal charges, i.e. $\beta_1 = \beta_2 = \beta_3$ ($H_1 = H_2 = H_3$) we find exactly the $AdS_5$ part of the $D3$-brane!

In the following subsections we turn to the discussion of the global space-time structure and thermodynamics of these solutions. We will restrict ourselves to the case of $k = +1$, only; the global structure for $k = -1, 0$ is very different and will be discussed elsewhere.

### 4.2 ADM mass

In order to determine the ADM mass we will follow a procedure given by Horowitz/Myers [18] (a generalization of the Nester’s procedure for asymptotically non-flat space-time). First by defining a new radial coordinate

$$\rho^2 \equiv r^2(H_1H_2H_3)^{1/3}, \quad (44)$$

the metric (42) can be written as:

$$ds^2 = -e^{-2V}dt^2 + e^{2W}d\rho^2 + \rho^2 d\Omega_{3,k}, \quad (45)$$
where:

\[ e^{-2V} = f(H_1 H_2 H_3)^{-2/3}, \quad e^{2W} = f^{-1}(H_1 H_2 H_3)^{1/3} \left( \frac{dr}{d\rho} \right)^2, \]  

(46)

Then, the ADM mass of the system is defined as the following surface integral at radial infinity:

\[ M_{ADM} = -\frac{1}{8\pi G} \int_{\partial M} N(K - K_0), \]  

(47)

where \( N = e^{-V} \) is the norm of the time-like Killing vector and \( K \) is the extrinsic curvature. In our case it is given by \( K = n^r \partial_r A \sim e^{-W} \rho^2 \), where \( A \) is the asymptotic area and \( n^r = e^{-W} \) is the normal vector. \( K_0 \) corresponds to \( K \) defined in the same (reference) non-flat background but without any matter fields.

Carrying out the procedure for our particular case we arrive at the following result:

\[ M_{ADM} = q_1 + q_2 + q_3 + \frac{3}{2} \mu, \]  

(48)

where we have taken the Newton’s constant \( G = \frac{\pi}{4} \).

### 4.3 Condition for the existence of horizons

We now turn to the discussion of the global space-time structure of the solution. In particular horizons appear at zeros of the function \( f \) (or \( e^{-2V} \) in (45)). Hence, we have to look for solutions of the following, effectively cubic equation for \( x \equiv r^2 \):

\[ x^2 f = g^2 \left( x^3 + Ax^2 - Bx + q_1 q_2 q_3 \right) = 0, \]  

(49)

with

\[ A \equiv \sum_{i=1}^{3} q_i + \frac{1}{g^2}, \quad B \equiv \frac{\mu}{g^2} - \sum_{i>j=1}^{3} q_i q_j > 0. \]  

(50)

Note, a necessary condition for having at least one zero of (49) for \( x > 0 \) is \( B > 0 \). The extrema of (49) \((x^2 f)(x)' = 0\) are at

\[ x_{\pm} = \frac{1}{3} A (1 \pm y), \quad y = 1 + z \equiv \sqrt{1 + \frac{3B}{A^2}} > 1. \]  

(51)

Thus discarding the extremum \( x_- < 0 \), a sufficient constraint to have at least one horizon is that:

\[ x_+^2 f(x_+) \leq 0, \]  

(52)

or equivalently (employing (50), (51)) and (52)):

\[ -2z^3 - 3z^2 + C \leq 0, \quad C \equiv \left( \frac{3}{A} \right)^3 \prod_{i=1}^{3} q_i \leq 1, \]  

(53)
with equality sign corresponding to the case of coinciding inner and outer horizons. Introducing \( \varphi = \arccos(z - \frac{1}{2}) \), the inequality (53) becomes:

\[
\cos 3\varphi = \cos \left[ 3 \arccos \left( 1 + \frac{3B}{A^2} - \frac{1}{2} \right) \right] \geq 2C - 1 ,
\]

(54)

where \( A, B \) and \( C \) are given in (50) and (53). It is straightforward to transcribe this inequality as a lower bound on the value of \( B \) or equivalently of \( \mu \). For the well-defined classical solution the charges \( q_i \) and the gauge coupling \( g \) are assumed to be in the range \( q_i > 1 > g^2 \). The bound on \( z \) becomes especially explicit in the two limiting cases (i) \( g^2q_i \ll 1 \) and thus \( C \ll 1 \) and (ii) \( g^2q_i \gg 1 \) and \( q_i \sim q_2 \sim q_3 \), and thus \( C - 1 \ll 1 \):

\[
z \geq z_{\text{crit}} = \sqrt{\frac{C}{3}} (1 + \mathcal{O}(C)) , \quad C \ll 1 ,
\]

(55)

\[
z \geq z_{\text{crit}} = \frac{1}{2} + \frac{2}{9} (C - 1) + \mathcal{O}((C - 1)^2) , \quad C - 1 \ll 1 .
\]

In the case of \( C \ll 1 \), i.e. \( g^2q_i \ll 1 \), one then obtains the following explicit bound:

\[
\mu \geq \mu_{\text{crit}} = 2 \sqrt{g^2 \prod_{i=1}^{3} q_i + g^2 \sum_{i>j}^{3} q_i q_j + \mathcal{O}((g^2 q_i)^{3/2} q_i)} \quad g^2q_i \ll 1 ,
\]

(56)

while the second limit \( C \sim 1 \), i.e. \( q_i \sim q \), \( g^2q \gg 1 \), corresponds to the following bound:

\[
\mu \geq \mu_{\text{crit}} = \frac{27}{4} g^2 q^2 + \frac{5}{2} q + \mathcal{O}(g^{-2}) , \quad q_i \sim q , \quad g^2q \gg 1 .
\]

(57)

Choosing \( \mu \) large enough in order to comply with the inequality (54) (and more explicitly, with (56) and (57) in the case of special limits) ensures that the \( f \)-function has two positive and one negative zero and can be written in the form:

\[
f = \frac{g^2(r^2 + r_0^2)(r^2 - r_+^2)}{r^4} (r^2 - r_+^2) ,
\]

(58)

where \( r_\pm \) denote respectively the outer and inner horizons. In the extreme limit \( r_+^2 \rightarrow r_+^2 \rightarrow x_+ \) the two horizons coincide and \( \mu \) saturates the lower bound, i.e. \( \mu = \mu_{\text{crit}} \), as discussed above. (For the discussion of an equivalent bound for four-dimensional charged black holes with constant negative cosmological constant we refer to \([19]\).)

### 4.4 Bekenstein-Hawking entropy and Hawking temperature

The Bekenstein-Hawking entropy is specified by the area of the outer horizon, and thus it is a valid concept for the black holes with regular horizons. In particular we are interested

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*The case (i) can be approximated by \( \cos 3\varphi = -\cos(3\varphi + \pi) = -1 + \frac{1}{2}(3\varphi + \pi)^2 \pm ... \) and therefore \( \varphi = -\frac{\pi}{3} + \frac{2}{3}\sqrt{C} \pm ... \), while the (ii) can be approximated by \( \cos 3\varphi = 1 + \frac{2}{3}\varphi^2 \pm ... = 2C - 1 \) and therefore \( \varphi = \frac{2}{3}\sqrt{C - 1} \pm ... \).*
in the entropy of the solutions that saturate the bound on \( \mu \), i.e. those with the inner and outer horizon coinciding.

In case (i) with \( g^2 q_i \ll 1 \) the expression can be cast, by using (55), (56) and (10), into the following explicit form:

\[
S_{\text{crit}} = \frac{A_{r_+}}{4G} = 2\pi \sqrt{\prod_{i=1}^{3} q_i \left( 1 + \mathcal{O}((g^2 q_i)^{1/2}) \right)} , \quad g^2 q_i \ll 1 .
\]  

(59)

The expression for the entropy resembles very closely that for the BPS-saturated black holes in the ungauged supergravity case [21], except that the parameters \( q_i \) are related to the physical charges \( \tilde{q}_i \) through equations (43). Notice, that the radius of the AdS space scales with the inverse gauge coupling \( g \) and therefore this limit \( (g^2 q_i \ll 1) \) corresponds to the leading order term in the large \( N \) expansion \((N \sim 1/g^2)\) which, interestingly, is independent of \( g \).

In case (ii) the entropy assumes, using (55), (57) and (10), the following form:

\[
S_{\text{crit}} = 2\pi \sqrt{\frac{27}{8} q^3 \left( 1 + \mathcal{O}((g^2 q)^{-1}) \right)} , \quad q_i \sim q , \quad g^2 q \gg 1 .
\]  

(60)

Note the new numerical factors in this entropy. (It would of course be interesting to obtain an explicit form for the entropy \( S_{\text{crit}} \) for the whole range of \( g^2 q_i \) values.)

As usual, the Hawking temperature is determined by the periodicity of the Euclidean time. The \((r, t)\)-part of the metric in the Euclidean time \( \tau \) is conformally equivalent to

\[
dr^2 + e^{-6U} f^2 d\tau^2 .
\]  

(61)

For \( r^2 \simeq r_+^2 \) we have

\[
e^{-3U} f = \frac{2g^2 (r_+^2 + r_0^2)(r_+^2 - r_-^2)}{\sqrt{(r_+^2 + q_1)(r_+^2 + q_2)(r_+^2 + q_3)}} (r - r_+) .
\]  

(62)

Therefore in order to cancel the conical singularity the periodicity

\[
\tau \sim \tau + \frac{1}{T_H}
\]  

implies that the Hawking temperature is given by

\[
T_H = \frac{g^2 (r_+^2 + r_0^2)(r_+^2 - r_-^2)}{\pi \sqrt{(r_+^2 + q_1)(r_+^2 + q_2)(r_+^2 + q_3)}}
\]  

(64)

Thus, if both horizons coincide \( r_- = r_+ \) the Hawking temperature vanishes, but the solution does not coincide with the BPS-saturated solution [4]. This result is analogous to the “cold” AdS black holes discussed in [7] in the context of \( D = 4 \) gauged supergravity.
Notice, for an uncharged black hole the situation is completely different. In this case $H_1 = H_2 = H_3 = 1$ and one finds for the $f$-function
\[
f = \frac{g^2}{r^2} \left( r^2 - r_+^2 \right) \left( r^2 - r_-^2 \right)
\]
with $r^2_{\pm} = \frac{1}{2g^2} (-1 \pm \sqrt{1 + 4g^2\mu})$. In this case the Hawking temperature becomes $T_H = \frac{g^2}{2\pi r^+} (r^2_+ - r^2_-)$ and since $r^2_-$ is negative it can never vanish. Instead, it diverges for $\mu \to 0, \infty$ and has a minimum at $\mu = \frac{3}{g^2}$: $T_H^{\text{min}} = \frac{\sqrt{2}g}{\pi r^+}$. Therefore, at this temperature the black hole is in thermal equilibrium with the thermal radiation (see discussion in [20]) and it gives a lower bound for the black hole size $r^+_{\text{min}} = 1/\sqrt{2g^2}$.

5 Conclusion

In this paper we investigated charged black holes of $D = 5$ gauged $N=2$ supergravity, by deriving the complete set of equations of motion (with a specific Ansatz for the fields in the theory). In order to keep these solutions as general as possible we considered static solution with the spatial geometry not only of a three-sphere $S_3$ ($k = 1$), as it would be natural for a static black hole, but we also included examples of Einstein spaces with constant spatial curvatures $k = -1$ and $k = 0$.

The Ansätze for the metric and the scalar fields are a natural generalization of the solutions for the ungauged supergravity. The main difference appears in an additional term in the non-extremality function, which is due to the gauging of the theory, and it is a proportional to the gauge coupling $g^2$ (see (11)). We showed, that in the Einstein and scalar field equations this additional term is precisely compensated by the contribution from the (gauged) potential, thus rendering the form of these equations to be the same as in the ungauged case. Therefore the static spherically symmetric solutions of the ungauged supergravity can be promoted to solutions of the gauged supergravity by adding to the non-extremality function the specific term proportional to $g^2$ (see (11)). However, this additional term has important consequences for the global space-time structure of these solutions; in the BPS-saturated (supersymmetric) limit the solutions have naked singularities! As a consequence, there is a lower bound on the non-extremality parameter $\mu$ (or equivalently the ADM mass), determined by the condition that the Hawking temperature vanishes, i.e. the outer and inner horizon coincide.

We demonstrated these results on a representative example of the $STU$ model, i.e. a three-charge black hole solution with two scalar fields (which incidentally is also a solution of $D = 5$ gauged $N = 4$ and $N = 8$ supergravity theory). In this case, the solution can be expressed completely in terms of harmonic functions. (For a general prepotential, as discussed at the end of Section 3 it is not possible to solve the equations of motion in terms of harmonic functions only.) In particular we found the explicit form of the solution, derived the $ADM$ mass and found the explicit bounds on the non-extremality parameter that ensures regular horizons. When the bound is saturated the solution is
not supersymmetric, i.e. $\mu = \mu_{\text{crit}} \neq 0$, however the Hawking temperature vanishes. For this limit we calculated also the entropy for the solution with $k = 1$, which in the case of small $g^2$ (i.e. large $N$ limit on the CFT side) assumes a form that resembles that of the corresponding BPS-saturated solution in the ungauged supergravity. This example may serve as an interesting gravity background for the study of the AdS/CFT correspondence.

The analysis presented here for the case of $D = 5$ gauged $N = 2$ supergravity has a natural generalization to the case of non-extreme black hole solutions of $D = 4$ gauged supergravity. The same qualitative changes between the solutions in the gauged and ungauged cases [14] are expected to take place. Namely, only the non-extremality function is expected to be modified by a specific term proportional to $g^2$, while other fields would satisfy the same equations of motion as in the ungauged case. The global space-time is expected to changed accordingly; the regular solutions again have to satisfy a lower bound on the non-extremality parameter $\mu \geq \mu_{\text{crit}}$. A representative solution to demonstrate these phenomena explicitly would again be within the $STU$ model, corresponding to the four-charge static black hole solutions.

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