Semi-Infinite Wedges and Vertex Operators

Eugene Stern

Abstract

The level 1 highest weight modules of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_n)$ can be described as spaces of certain semi-infinite wedges. Using a $q$-antisymmetrization procedure, these semi-infinite wedges can be realized inside an infinite tensor product of evaluation modules. This realization gives rise to simple descriptions of vertex operators and (up to a scalar function) their compositions.

1 Representations of $\mathfrak{sl}_\infty$

1.1 Infinite tensors and infinite wedges

Let $V = \mathbb{C}^\infty$ be a countable-dimensional $\mathbb{C}$-vector space, with basis $\{v_i\}_{i \in \mathbb{Z}}$. The Lie algebra $\mathfrak{sl}_\infty = \mathfrak{sl}_\infty(\mathbb{C})$, consisting of infinite matrices with finitely many non-zero entries and trace 0, acts on $V$. The elements $e_i$, $f_i$, and $h_i$, where

\begin{align*}
e_i \cdot v_j &= \delta_{i,j-1} \cdot v_{j-1} \\
f_i \cdot v_j &= \delta_{i,j} \cdot v_{j+1} \\
h_i \cdot v_j &= (\delta_{i,j} - \delta_{i,j-1}) v_j,
\end{align*}

generate $\mathfrak{sl}_\infty$.

Consider the tensor product $V \otimes V \otimes V \otimes \cdots$. In this tensor product, $\mathfrak{sl}_\infty$ acts by

\begin{align*}
e_i \cdot (v_{m_1} \otimes v_{m_2} \otimes \cdots) &= \sum_{j: \ m_j = i+1} v_{m_1} \otimes \cdots \otimes v_{m_{j-1}} \otimes v_i \otimes v_{m_{j+1}} \otimes \cdots \\
f_i \cdot (v_{m_1} \otimes v_{m_2} \otimes \cdots) &= \sum_{j: \ m_j = i} v_{m_1} \otimes \cdots \otimes v_{m_{j-1}} \otimes v_{i+1} \otimes v_{m_{j+1}} \otimes \cdots \\
h_i \cdot (v_{m_1} \otimes v_{m_2} \otimes \cdots) &= (\# \{ r : m_r = i \} - \# \{ r : m_r = i + 1 \}) \cdot v_{m_1} \otimes v_{m_2} \otimes \cdots
\end{align*}

To be precise, the infinite sums on the right hand side of (4)-(6) lie in an appropriate completion of $V \otimes V \otimes V \otimes \cdots$. Moreover, the action of $h_i$ given by (6) is defined only for some tensors $v_{m_1} \otimes v_{m_2} \otimes \cdots$. One restriction that certainly suffices is that each $v_i$ should appear only finitely often among the $v_{m_j}$.

The infinite wedge product $\bigwedge^\infty V$ may be embedded inside $V \otimes V \otimes V \otimes \cdots$ by an antisymmetrization procedure. Let $S_\infty$ denote the infinite symmetric group, which is generated
by adjacent transpositions \( \sigma_i = (i \ i + 1), \ i \in \mathbb{Z}^+ \), with the usual Coxeter relations. \((S_\infty \text{ consists of bijections } \mathbb{Z}^+ \to \mathbb{Z}^+ \text{ which fix all but a finite number of elements.})\) The Bruhat length \( l(\sigma) \) of an element \( \sigma \in S_\infty \) is the length \( l \) of a minimal expression \( \sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_l} \), where the \( \sigma_{i_j} \) are adjacent transpositions. (Thus, \((-1)^l(\sigma) = \text{sgn}(\sigma)\).)

The antisymmetrization of \( v_{m_1} \otimes v_{m_2} \otimes \cdots \) is the pure wedge

\[
v_{m_1} \wedge v_{m_2} \wedge \cdots = \sum_{\sigma \in S_\infty} (v_{m_1} \otimes v_{m_2} \otimes \cdots) \cdot (-1)^{l(\sigma)} \sigma.
\]

(Here \( S_\infty \) acts on the right on infinite pure tensors in the obvious way: \( \sigma \) switches the \( i \)-th and \( i + 1 \)-st entries.) Let \( \bigwedge^\infty V \subseteq V \otimes V \otimes V \otimes \cdots \) be the span of all pure wedges.

Since \( v_{m_1} \wedge v_{m_2} \wedge \cdots \) is 0 unless the \( m_i \) are all distinct, the action of \( \mathfrak{sl}_\infty \) on the pure tensors that appear in the expansion of any pure wedge is well defined. Because the actions of \( \mathfrak{sl}_\infty \) and \( S_\infty \) commute, the action of any \( X \in \mathfrak{sl}_\infty \) on the antisymmetrization of \( v_{m_1} \otimes v_{m_2} \otimes \cdots \) will yield the antisymmetrization of \( X \cdot (v_{m_1} \otimes v_{m_2} \otimes \cdots) \). Thus the action of \( \mathfrak{sl}_\infty \) on tensors in \( V \otimes V \otimes V \otimes \cdots \) induces an action on \( \bigwedge^\infty V \).

Consider the tensor

\[
v_{(i)} = v_i \otimes v_{i-1} \otimes v_{i-2} \otimes \cdots.
\]

Denote by \( V_{(i)} \) the subspace of \( V \otimes V \otimes V \otimes \cdots \) spanned by pure tensors that are the same as \( v_{(i)} \) after finitely many terms. \( V_{(i)} \) is preserved by the action of both \( \mathfrak{sl}_\infty \) and \( S_\infty \).

The pure wedge

\[
v_{\Lambda_i} = v_i \wedge v_{i-1} \wedge v_{i-2} \wedge \cdots
\]

lies in \( V_{(i)} \). It is a highest weight vector of \( \mathfrak{sl}_\infty \) with highest weight \( \Lambda_i \), and generates the irreducible \( \mathfrak{sl}_\infty \)-module \( V_{\Lambda_i} \subseteq V_{(i)} \) with highest weight \( \Lambda_i \). (Here \( \Lambda_i \) is the fundamental weight of \( \mathfrak{sl}_\infty \) defined by the equation \( \Lambda_i(h_j) = \delta_{ij} \).) A basis for \( V_{\Lambda_i} \) is given by wedges \( v_{m_1} \wedge v_{m_2} \wedge \cdots \) (with the \( m_i \) decreasing), which are the same as \( v_{\Lambda_i} \) after finitely many terms. (See [5] or [6] for details.) Such wedges will be called \textit{semi-infinite}.

**Remark** A semi-infinite wedge \( v_{m_1} \wedge v_{m_2} \wedge \cdots \subseteq V_{\Lambda_i} \) may be represented by a Young diagram in the following way. Set \( \lambda_j = m_j - (i - j + 1) \), and notice that \( m_j > m_{j+1} \) implies that \( \lambda_j \geq \lambda_{j+1} \). Then set \( \lambda = (\lambda_1, \lambda_2, \ldots) \). After enough terms, the semi-infinite wedge becomes equal to \( v_{\Lambda_i} \), so for large enough \( j \), \( m_j = i - j + 1 \), and \( \lambda_j = 0 \). In other words, \( \lambda \) is a finite Young diagram. (For example, the highest weight vector \( v_{\Lambda_i} \) corresponds to the empty Young diagram.) This correspondence sets up an isomorphism between representations of \( \mathfrak{sl}_\infty \) on spaces of semi-infinite wedges and representations on the space of Young diagrams written down in \([4]\).

### 1.2 Vertex operators

If \( v_{m_1} \wedge v_{m_2} \wedge \cdots \in V_{\Lambda_i} \), then it is a sum of tensors all of which lie in \( V_{(i)} \). Given such a tensor \( v_{k_1} \otimes v_{k_2} \otimes v_{k_3} \otimes \cdots \in V_{(i)} \), notice that the tensor \( v_{k_2} \otimes v_{k_3} \otimes \cdots \) is an element of \( V_{(i-1)} \). This gives rise to a natural map \( \Phi_{(i)} : V_{(i)} \to V \otimes V_{(i-1)} \). Since this map is essentially the identity map, it commutes with the action of \( \mathfrak{sl}_\infty \).
Now take the wedge \( v_{m_1} \wedge v_{m_2} \wedge \cdots \) (remember, the \( m_i \) are assumed to be decreasing) and expand it as a sum of tensors. Collect together the tensors having \( v_{m_j} \) as their first term. The result is

\[
(-1)^{j-1}v_{m_j} \otimes (v_{m_1} \wedge \cdots \wedge v_{m_{j-1}} \wedge v_{m_{j+1}} \wedge \cdots).
\]

This shows that \( \Phi(i) \) maps \( V_{\Lambda_i} \) into \( V \otimes V_{\Lambda_{i-1}} \). (It is necessary to take an appropriate completion since the image of a wedge in \( V_{\Lambda_i} \) will be an infinite sum of \( v_{m_j} \)'s tensored with wedges in \( V_{\Lambda_{i-1}} \).) In particular,

\[
\Phi(i)(v_{\Lambda_i}) = v_i \otimes v_{\Lambda_{i-1}} + \sum_{j=1}^{\infty} (-1)^j v_{i-j} \otimes (v_i \wedge \cdots \wedge v_{i-(j-1)} \wedge v_{i-(j+1)} \wedge \cdots),
\]

i.e., \( v_i \) is the “matrix coefficient” corresponding to \( v_{\Lambda_{i-1}} \).

Next, consider a composition

\[
\Phi(i-(j-1))\Phi(i-(j-2)) \cdots \Phi(i) : V_{\Lambda_i} \to V \otimes \cdots \otimes V \otimes V_{\Lambda_{i-j}}.
\]

**Proposition 1.1** The matrix coefficient corresponding to \( v_{\Lambda_{i-j}} \) is \( v_i \wedge v_{i-1} \wedge \cdots \wedge v_{i-(j-1)} \).

**Proof.** Collect together terms ending in \( v_{k_1} \otimes v_{k_2} \otimes v_{k_3} \otimes \cdots \), where \( k_1, k_2, k_3, \ldots \) is a particular finite rearrangement of \( i-j, i-j-1, i-j-2, \ldots \).

## 2 Representations of \( \hat{sl}_n \)

### 2.1 Evaluation modules

The affine algebra \( \hat{sl}_n \) has a standard evaluation representation defined in the following way. Let \( E_i, F_i, \) and \( H_i, i = 0, 1, \ldots, n-1 \), be the standard Serre generators of \( \hat{sl}_n \). Consider an \( n \)-dimensional vector space with basis \( \{v_1, \ldots, v_n\} \) on which these generators act as follows:

\[
E_i \cdot v_j = \delta_{i,j-1} \cdot z^{\delta_{i,0}} \cdot v_{j-1}
\]

(9)

\[
F_i \cdot v_j = \delta_{i,j} \cdot z^{-\delta_{i,0}} \cdot v_{j+1}
\]

(10)

\[
H_i \cdot v_j = (\delta_{i,j} - \delta_{i+1,j}) \cdot v_j.
\]

(11)

The indices in expressions (9)-(11) should all be read modulo \( n \). (For instance, if \( j = 1 \), then \( v_{j-1} = v_n \).) It is easiest to regard \( z \) as a formal variable by tensoring over \( \mathbb{C} \) with the ring \( \mathbb{C}[z, z^{-1}] \). The resulting \( \hat{sl}_n \)-module is denoted by \( V(z) \).

\( V(z) \) is related to the \( \hat{sl}_\infty \)-module \( V = \mathbb{C}^\infty \) of the previous section in the following way. Identify the basis \( \{v_i\}_{i \in \mathbb{Z}} \) of \( V \) with the basis \( \{z^{j} \cdot v_i \}_{i \in \mathbb{Z}, j \in \mathbb{N}} \) of \( V(z) \) by \( z^{j} \cdot v_i = v_{i-nj} \). When \( V(z) \) is identified with \( V \) in this way, the generators \( E_i, F_i, \) and \( H_i \) of \( \hat{sl}_n \) act as infinite sums of the generators of \( \hat{sl}_\infty \):

\[
E_i = \sum_{j \equiv i \mod n} e_j \quad F_i = \sum_{j \equiv i \mod n} f_j \quad H_i = \sum_{j \equiv i \mod n} h_j
\]

(12)
2.2 The thermodynamic limit

As in the previous section, to build highest weight modules for \( \hat{\mathfrak{sl}}_n \), it is necessary to consider the infinite tensor product

\[ V_{z_1, z_2, z_3, \ldots} = V(z_1) \otimes V(z_2) \otimes V(z_3) \otimes \cdots. \]

Unfortunately, it is not possible to define an honest action of \( \hat{\mathfrak{sl}}_n \) in this tensor product. The most that can be hoped for is a formal action of the Serre generators \( E_i, F_i, \) and \( H_i \), with the understanding that when \( E_i \) and \( F_i \) act on a pure tensor, the result can be an infinite sum of pure tensors. Explicitly, let \( E_i \) and \( F_i \) act as formal sums of operators

\[
\Delta(E_i) = \sum_{j=1}^{\infty} 1 \otimes \cdots \otimes 1 \otimes E_i \otimes 1 \otimes 1 \otimes \cdots \]

(13)

\[
\Delta(F_i) = \sum_{j=1}^{\infty} 1 \otimes \cdots \otimes 1 \otimes F_i \otimes 1 \otimes 1 \otimes \cdots, \]

(14)

where the action of \( E_i \) and \( F_i \) in \( V(z_j) \) is given by (9)-(10). A highest or lowest weight vector is then just a vector killed by all the \( E_i \) or all the \( F_i \).

An analogous formula for the action of \( H_i \) will not work: if defined naively, the action of \( H_i \) on most tensors will give a divergent answer. One way to get around this is to restrict to a particular class of tensors and to use a version of (12). Namely, \( H_i \) acting on \( V(z_j) \) can be thought of as the sum over \( d \in \mathbb{Z} \) of operators \( H_i(d) \), where

\[
H_i(d) \cdot (z^d v_j) = \delta_{d,d'}(\delta_{i,j} - \delta_{i+1,j}) \cdot z^{d'} v_j. \]

(15)

Again, \( v_0 \) means \( v_n \) here. Then the action of \( H_i(d) \) in \( V_{z_1, z_2, z_3, \ldots} \) is given by the infinite sum

\[
\sum_{j=1}^{\infty} 1 \otimes \cdots \otimes 1 \otimes H_i(d) \otimes 1 \otimes 1 \otimes \cdots. \]

(16)

Now, letting \( H_i \) act on an arbitrary tensor in \( V_{z_1, z_2, z_3, \ldots} \) as

\[
H_i = \sum_{d \in \mathbb{Z}} H_i(d) \]

(17)

is still likely to give a divergent answer. However, this formula can at least be used for those tensors which eventually become periodic; i.e., they have at their tail end an infinite sequence of the form

\[
z_k^d v_n \otimes z_k^{d+1} v_{n-1} \otimes \cdots \otimes z_k^{d+n-1} v_1 \otimes z_k^{d+n} v_n \otimes z_k^{d+n+1} v_{n-1} \otimes \cdots. \]

(All but finitely many of the \( H_i(d) \) will act by 0 on such a tensor.) Only tensors of this type (which will be called semi-infinite) will be considered in the rest of this paper.

The action of the compositions \( E_i F_i \) and \( F_i E_i \) cannot be defined in \( V_{z_1, z_2, z_3, \ldots} \): even the action on a semi-infinite tensor would yield a divergent result. However, the commutator of
\(E_i\) and \(F_i\) can still be taken, in the following formal way. \(E_i\) and \(F_i\) can be viewed as sums over \(d \in \mathbb{Z}\) of operators \(E_i(d)\) and \(F_i(d)\), whose action in \(V(z)\) is given by

\[
E_i(d) \cdot (z^{d'} \cdot v_j) = \delta_{d,d'} \cdot \delta_{i,j-1} \cdot z^{d'+\delta_i,0} \cdot v_{j-1} \quad (18)
\]

\[
F_i(d) \cdot (z^{d'} \cdot v_j) = \delta_{d,d'} \cdot \delta_{i,j} \cdot z^{d'-\delta_i,0} \cdot v_{j+1}. \quad (19)
\]

\(E_i(d)\) and \(F_i(d)\) act in \(V_{z_1,z_2,z_3,\ldots}\) as sums of the actions in each component of the tensor product, in exact analogy with (16). Then the equations

\[
[E_i, F_i] = \left[ \sum_{d \in \mathbb{Z}} E_i(d), \sum_{d \in \mathbb{Z}} F_i(d) \right] = \sum_{d \in \mathbb{Z}} H_i(d) = H_i
\]

hold formally in \(V_{z_1,z_2,z_3,\ldots}\) (more properly, they hold formally in the subspace of \(V_{z_1,z_2,z_3,\ldots}\) spanned by semi-infinite tensors).

Notice that even though \(V(z)\) and all finite tensor powers of it are level 0 \(\widehat{\mathfrak{s}l}_n\)-modules, the \(H_i\) act in \(V_{z_1,z_2,z_3,\ldots}\) as though it were a level 1 \(\widehat{\mathfrak{s}l}_n\)-module. For example,

\[
H_i(\cdot z_1 v_n \otimes \cdots \otimes z_n v_1 \otimes z_{n+1}^2 v_n \otimes \cdots \otimes z_{2n}^2 v_1 \otimes \cdots) = \delta_{i,0}(\cdot z_1 v_n \otimes \cdots \otimes z_n v_1 \otimes z_{n+1}^2 v_n \otimes \cdots \otimes z_{2n}^2 v_1 \otimes \cdots),
\]

because all \(H_i(d)\) for \(i \neq 0, d \neq 1\) act by 0. So \(z_1 v_n \otimes \cdots \otimes z_n v_1 \otimes z_{n+1}^2 v_n \otimes \cdots \otimes z_{2n}^2 v_1 \otimes \cdots\) has weight \(\Lambda_0\). (In this section, \(\Lambda_i\) is the \(i\)-th fundamental weight of \(\widehat{\mathfrak{s}l}_n\), defined by \(\Lambda_i(H_j) = \delta_{i,j}\).)

It is worth pointing out that the formal action of the operators \(E_i, F_i, H_i \in \widehat{\mathfrak{s}l}_n\) in the infinite tensor product is compatible with finite tensor products. That is, for \(X = E_i, F_i,\) or \(H_i\),

\[
X \cdot (z_1^{d_1} v_{m_1} \otimes z_2^{d_2} v_{m_2} \otimes z_3^{d_3} v_{m_3} \otimes \cdots) = (X \cdot z_1^{d_1} v_{m_1}) \otimes (z_2^{d_2} v_{m_2} \otimes z_3^{d_3} v_{m_3} \otimes \cdots) \quad (20)
\]

\[
+ z_1^{d_1} v_{m_1} \otimes X \cdot (z_2^{d_2} v_{m_2} \otimes z_3^{d_3} v_{m_3} \otimes \cdots).
\]

(In (20), the action of \(X\) on the left hand side and on the second term on the right hand side are as defined above.) This allows formal manipulation of the action in \(V_{z_1,z_2,z_3,\ldots}\) as though it were an ordinary tensor product: it is legitimate to break off finitely many factors, add on finitely many factors, etc.

As before, highest weight vectors are constructed by an antisymmetrization procedure. The antisymmetrization of \(z_1^{d_1} v_{m_1} \otimes z_2^{d_2} v_{m_2} \otimes \cdots\) is

\[
z_1^{d_1} v_{m_1} \wedge z_2^{d_2} v_{m_2} \wedge \cdots = \sum_{\sigma \in S_\infty} (z_1^{d_1} v_{m_1} \otimes z_2^{d_2} v_{m_2} \otimes \cdots) \cdot (-1)^l(\sigma) \sigma. \quad (21)
\]

Here \(\sigma \in S_\infty\) acts by switching \(v_{m_i}\) and \(v_{m_{i+1}}\) and by switching variables as well. For example,

\[
(z_1^{-5} v_3 \otimes z_2^3 v_6 \otimes z_3^{-2} v_1 \otimes \cdots) \cdot \sigma_1 = z_1^3 v_6 \otimes z_2^{-5} v_3 \otimes z_3^{-2} v_1 \otimes \cdots.
\]

The subscripts on the variables have been dropped in the infinite wedge notation, but it should be understood that the antisymmetrization is a sum of terms with \(z_1\)'s in the first factor, \(z_2\)'s in the second factor, and so on.
Consider the wedge
\[ v_{\Lambda_i} = zv_i \wedge zv_{i-1} \wedge \cdots \wedge zv_1 \wedge z^2v_n \wedge \cdots \wedge z^2v_1 \wedge \cdots. \]  
(22)

Notice that this wedge is a sum of semi-infinite tensors. Let \( F_i \) denote the space spanned by wedges that are the same as \( v_{\Lambda_i} \) after finitely many terms. As in the previous section, such wedges will be called semi-infinite; they are all sums of semi-infinite tensors.

The formal action of \( E_i, F_i, \) and \( H_i \in \widehat{sl}_n \) on the semi-infinite tensors in \( V_{z_1, z_2, z_3, \ldots} \) defined above induces an honest action on \( F_i \). The wedge \( v_{\Lambda_i} \) is a highest weight vector of weight \( \Lambda_i \); it is killed by each \( E_j \) because it is killed by all the \( E_j(d) \). (Since the fundamental weights \( \Lambda_i \) are usually indexed by \( i = 0, 1, \ldots, n - 1 \), while the \( v_i \) are indexed by \( i = 1, 2, \ldots, n \), it is worth adding that \( i = 0 \) on the left hand side of (22) corresponds to \( i = n \) on the right hand side.)

The highest weight vector \( v_{\Lambda_i} \) generates an irreducible \( \widehat{sl}_n \)-module \( V_{\Lambda_i} \subset F_i \) of highest weight \( \Lambda_i \). (For a proof of irreducibility, see [3].) It is important to point out that \( V_{\Lambda_i} \) is strictly smaller than \( F_i \). For example, for \( n = 2 \) and \( i = 0 \), the wedges \( v_2 \wedge zv_1 \wedge z^2v_2 \wedge z^2v_1 \wedge \cdots \) and \( v_1 \wedge zv_2 \wedge z^2v_2 \wedge z^2v_1 \wedge \cdots \in F_{(0)} \) do not lie in \( V_{\Lambda_0} \), although their sum does, being \( F_1 F_0 \cdot v_{\Lambda_0} \).

On the other hand, \( F_{(0)} \) is a unitary \( \widehat{sl}_n \)-module (see [3]), and therefore completely reducible, so there is a projection \( p_i : F_i \to V_{\Lambda_i} \) which kills all the components of \( F_i \) except the one generated by \( v_{\Lambda_i} \). In particular, \( p_i(v_{\Lambda_i}) = v_{\Lambda_i} \).

Once again, there is a correspondence between wedges and Young diagrams. For example, if \( i = 0 \), the semi-infinite wedge
\[ v_3 \wedge v_1 \wedge zv_{n-2} \wedge zv_{n-3} \wedge \cdots \wedge zv_1 \wedge z^2v_n \wedge \cdots \wedge z^2v_1 \wedge \cdots \]
corresponds to the Young diagram \((3,2)\). In the notation of [1], the value of \( i \) (i.e., the highest weight) determines the way in which the diagram is to be colored.

### 2.3 Vertex operators

Just as in the first section, splitting off the first component of a tensor defines intertwiners; in this case,
\[ \Phi_{(i)} : F_i \to V(z) \otimes F_{(i-1)}. \]

It is worthwhile to say explicitly what this means in the cases \( i = 0 \) and \( i = 1 \). \( \Phi_{(0)} \) maps \( F_{(0)} \) into \( V(z) \otimes F_{(n-1)} \). (I.e., the indices should really be read modulo \( n \.) \( \Phi_{(1)} \) maps \( F_{(1)} \), spanned by wedges that look like \( zv_1 \wedge z^2v_n \wedge \cdots \wedge z^2v_1 \wedge \cdots \) after finitely many terms, to \( V(z) \otimes W \), where \( W \) is spanned by wedges that look like \( z^2v_n \wedge z^2v_{n-1} \wedge \cdots \wedge z^2v_1 \wedge z^3v_n \wedge \cdots \) after finitely many terms. \( W \) is evidently isomorphic to, and can be identified with, \( F_{(0)} \).

(That is, the \( \Phi_{(i)} \) cycle around after \( n \) iterations: \( \Phi_{(1)} \) is followed by \( \Phi_{(0)} \), which acts on the \( \widehat{sl}_n \)-module spanned by wedges that look like \( z^2v_n \wedge z^2v_{n-1} \wedge \cdots \wedge z^2v_1 \wedge z^3v_n \wedge \cdots \) after finitely many terms, and so on.) Each \( \Phi_{(i)} \) is an intertwiner because of (21).

**Proposition 2.1** *When the composition*
\[ \Phi_{(n-j-1)} \Phi_{(n-j-2)} \cdots \Phi_{(0)} (\Phi_{(1)} \Phi_{(2)} \cdots \Phi_{(0)})^d : F_{(0)} \to V(z_1) \otimes V(z_2) \otimes \cdots \otimes V(z_{nd+j}) \otimes F_{(n-j)} \]
acts on the highest weight vector \(v_{\Lambda_0}\), the matrix coefficient corresponding to \(v_{\Lambda_{n-j}}\) is

\[
zv_n \wedge \cdots \wedge zv_1 \wedge \cdots \wedge z^d v_n \wedge \cdots \wedge z^{d+1} v_n \wedge \cdots \wedge z^{d+1} v_{n-(j-1)}. \tag{23}
\]

The proof is as in the previous section.

Next, define the intertwiner \(\Phi(i) = (id \otimes p_{i-1}) \circ \Phi(i) \circ j_i : V_{\Lambda_i} \to V(z) \otimes V_{\Lambda_{i-1}}\). Here \(j_i : V_{\Lambda_i} \to F(i)\) is the inclusion map, and \(p_{i-1} : F(i-1) \to V_{\Lambda_{i-1}}\) is the projection. The intertwiners \(\Phi(i)\) are examples of vertex operators (see, for example, [3]). Thus, the system of \(nN\)

\[
\text{Proposition 2.2} \quad \text{When the composition} \quad (\Phi(n-j) \Phi(n-j-2) \cdots \Phi(0) \Phi(1) \Phi(2) \cdots \Phi(0))^d \quad \text{acts on the highest weight vector} \quad v_{\Lambda_0}, \quad \text{the matrix coefficient corresponding to} \quad v_{\Lambda_{n-j}} \quad \text{is a scalar multiple of} \quad (23).
\]

It is well-known that the “highest-to-highest matrix coefficients” of a composition of vertex operators are given by the Knizhnik-Zamolodchikov equations (see, for example, [3]). Thus, appropriate multiples of the wedges given by \(23\) should be solutions to these equations. As an example of this, consider an iteration \((\Phi(n-1) \cdots \Phi(1) \Phi(0))^N\). Then the KZ system is a system of \(nN\) differential equations with values in the \(\mathfrak{sl}_n\)-module \(V^\otimes nN\) \((V = \mathbb{C}^n)\):

\[
\left(\frac{\partial}{\partial z_i} - \frac{1}{n+1} \sum_{j \neq i} t_{ij} \frac{1}{z_i - z_j}\right) \cdot F = 0. \tag{26}
\]

Here \(t_{ij} = P_{ij} - \frac{1}{n} \text{id}\), where \(P_{ij}\) acts on \(V^\otimes nN\) by switching the vectors (but not the variables) in the \(i\)-th and \(j\)-th places. Then the solution is indeed a multiple of \((23)\). It is given by the following proposition, which came out of discussions with Nicolai Reshetikhin:

\[
\text{Proposition 2.3} \quad \text{The function} \quad F(z_1, \ldots, z_{nN}) = \prod_{1 \leq i < j \leq nN} (z_i - z_j)^{-1/n} \cdot (v_n \wedge v_{n-1} \wedge \cdots \wedge v_1 \wedge zv_n \wedge \cdots \wedge z^{N-1}v_n \wedge \cdots \wedge z^{N-1}v_1) = \prod_{1 \leq i < j \leq nN} (z_i - z_j)^{-1/n} z_1^{-1} \cdots z_{nN}^{-1} \cdot (zv_n \wedge zv_{n-1} \wedge \cdots \wedge zv_1 \wedge z^2v_n \wedge \cdots \wedge z^Nv_n \wedge \cdots \wedge z^Nv_1)
\]

is a solution to the \(\mathfrak{sl}_n\)-KZ system.
3 Representations of $U_{sl}$ sections. To begin with, there is a quantization of the picture laid out in the first two sections. The results have obvious analogs for iterations of vertex operators beginning with any level 1 highest weight $\hat{sl}_n$-module.

3.1 Preliminaries

The main subject of this paper is a quantization of the $sl_\infty$ given by $U_q(sl_\infty)$, which also acts on $V = \mathbb{C}^\infty$. $U_q(sl_\infty)$ is generated by elements $e_i$, $f_i$, $k_i$, and $k_i^{-1}$, $i \in \mathbb{Z}$, with relations

$$k_i k_j = k_j k_i$$
$$k_i e_i = q e_i k_i$$
$$k_i f_i = q^{-1} f_i k_i$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}$$

$U_q(sl_\infty)$ acts on $V$ as follows: the action of $e_i$ and $f_i$ is the same as in the classical case (see (34)–(36)), while $k_i$ acts as $q^{h_i}$, where the action of $h_i$ is given by (3).

There is a coproduct on $U_q(sl_\infty)$ given by

$$\Delta(k_i) = k_i \otimes k_i$$
$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i$$
$$\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i.$$  

This coproduct gives rise to the following action of $U_q(sl_\infty)$ on certain infinite pure tensors $v_{m_1} \otimes v_{m_2} \otimes \cdots$

$$e_i \cdot (v_{m_1} \otimes v_{m_2} \otimes \cdots) = \sum_{j: m_j = i + 1} q^{\#\{r: r > j, m_r = i\} - \#\{r: r > j, m_r = i + 1\}} v_{m_1} \otimes \cdots \otimes v_{m_j} \otimes v_i \otimes v_{m_{j+1}} \otimes \cdots$$

$$f_i \cdot (v_{m_1} \otimes v_{m_2} \otimes \cdots) = \sum_{j: m_j = i} q^{\#\{r: r < j, m_r = i + 1\} - \#\{r: r < j, m_r = i\}} v_{m_1} \otimes \cdots \otimes v_{m_j} \otimes v_{i+1} \otimes v_{m_{j+1}} \otimes \cdots$$

$$k_i \cdot (v_{m_1} \otimes v_{m_2} \otimes \cdots) = q^{\#\{r: m_r = i\} - \#\{r: m_r = i+1\}} v_{m_1} \otimes v_{m_2} \otimes \cdots.$$
As before, the terms on the right hand side lie in an appropriate completion of the infinite tensor product. To ensure the action is well defined, the domain is again restricted to the subspace of \( V \otimes V \otimes V \otimes \cdots \) spanned by tensors \( v_{m_1} \otimes v_{m_2} \otimes \cdots \) in which all \( v_j \) appear only finitely many times.

### 3.2 \( q \)-antisymmetrization

For \( i \in \mathbb{Z} \), consider the pure tensor

\[
v_{(i)} = v_i \otimes v_{i-1} \otimes v_{i-2} \otimes \cdots.
\]

Denote by \( V_{(i)} \) the subspace of \( V \otimes V \otimes V \otimes \cdots \) spanned by all pure tensors that are the same as \( v_{(i)} \) after finitely many terms. Notice that the action of \( U_q(\mathfrak{sl}_\infty) \) on each \( V_{(i)} \) is well defined. \( v_{(i)} \) has weight \( \Lambda_i \) with respect to the subalgebra \( U_q(\mathfrak{h}) \subseteq U_q(\mathfrak{sl}_\infty) \) generated by \( \{k_i, k_i^{-1}\}_{i \in \mathbb{Z}} \).

Consider the \( q \)-antisymmetrization of \( v_{(i)} \) given by

\[
v_{\Lambda_i} = \sum_{\sigma \in S_\infty} v_{(i)} \cdot (-q)^{l(\sigma)} \sigma.
\]  

\( v_{\Lambda_i} \) is an infinite sum of elements of \( V_{(i)} \), all of which have weight \( \Lambda_i \).

**Proposition 3.1** Every \( e_j \) acts on \( v_{\Lambda_i} \) by zero.

**Proof.** For concreteness, assume that \( i = 0 \). In this case, if \( j \geq 0 \), there is nothing to prove. If \( j < 0 \), partition \( S_\infty \) into left cosets of the subgroup \( H = \{\text{id}, \sigma_j\} \). Each such coset looks like \( \{\sigma, \sigma_j \sigma\} \), where \( l(\sigma_j \sigma) = l(\sigma) + 1 \). Group together terms in (31) in pairs corresponding to these cosets, and consider one such pair. The term corresponding to \( \sigma \) has \( v_{j+1} \) appearing to the left of \( v_j \). The term corresponding to \( \sigma_j \sigma \) looks just like the term corresponding to \( \sigma \) except that it has the opposite sign, an extra factor of \( q \), and \( v_{j+1} \) and \( v_j \) are switched, so that \( v_{j+1} \) appears to the right of \( v_j \). By (33), \( e_j \) kills the sum of these two terms. All other pairs are killed in the same way.

If \( v_{m_1} \otimes v_{m_2} \otimes \cdots \in V_{(i)} \) is such that the \( m_i \) are decreasing, set

\[
v_{m_1} \wedge q v_{m_2} \wedge q \cdots = \sum_{\sigma \in S_\infty} (v_{m_1} \otimes v_{m_2} \otimes \cdots) \cdot (-q)^{l(\sigma)} \sigma.
\]  

The \( q \)-antisymmetrized tensor \( v_{m_1} \wedge q v_{m_2} \wedge q \cdots \) will be called a \( q \)-\textit{wedge}. Let \( V_{\Lambda_i} \) denote the space spanned by the \( q \)-wedges \( v_{m_1} \wedge q v_{m_2} \wedge q \cdots \) (with the \( m_i \) decreasing), which are the same as \( v_{\Lambda_i} \) after finitely many terms. (Such \( q \)-wedges will be called \textit{semi-infinite}.)

By the above, \( v_{\Lambda_i} \) generates a highest weight \( U_q(\mathfrak{sl}_\infty) \)-submodule in \( V_{(i)} \), of highest weight \( \Lambda_i \). This module is spanned by the semi-infinite \( q \)-wedges described above, and \( U_q(\mathfrak{sl}_\infty) \) acts on these wedges in the obvious way. (Notice that \( f_j \) kills any \( q \)-wedge of the form \( \cdots \wedge q v_{j+1} \wedge q v_j \wedge q \cdots \), which means that the action of \( U_q(\mathfrak{sl}_\infty) \) cannot generate any \( q \)-wedges in which any \( v_j \) appears more than once.) This representation is irreducible because it is a \( q \)-deformation of the irreducible \( \mathfrak{sl}_\infty \)-module \( V_{\Lambda_i} \) constructed in the first section.

Identifying \( q \)-wedges with Young diagrams as before gives rise to an isomorphism of \( V_{\Lambda_i} \) with the level one highest weight \( U_q(\mathfrak{sl}_\infty) \)-modules described in [7].
3.3 Vertex operators

Intertwiners $\Phi_{(i)}: V_{\Lambda_i} \to V \otimes V_{\Lambda_{i-1}}$ can be defined exactly as in the classical case, by splitting off the first component of every tensor. The $q$-analog of the results of Section 1.2 are the following:

**Proposition 3.2** The image of the highest weight vector $v_{\Lambda_i} \in V_{\Lambda_i}$ under $\Phi_{(i)}$ is given by

$$\Phi_{(i)}(v_{\Lambda_i}) = v_i \otimes v_{\Lambda_{i-1}} + \sum_{j=1}^{\infty} (-q)^j v_{i-j} \otimes (v_i \wedge q \cdots \wedge q v_{i-(j-1)} \wedge q v_{i-(j+1)} \wedge \cdots).$$

**Proposition 3.3** Under the composition

$$\Phi_{(i-1)} \cdots \Phi_{(i-j-2)} \cdots \Phi_{(i)} : V_{\Lambda_i} \to V \otimes \cdots \otimes V \otimes V_{\Lambda_{i-j}},$$

the matrix coefficient corresponding to $v_{\Lambda_{i-j}}$ is $v_i \wedge q v_{i-1} \wedge q \cdots \wedge q v_{i-(j-1)}$.

4  Representations of $U_q(\hat{\mathfrak{sl}}_n)$

4.1 Evaluation modules

The quantum affine algebra $U_q(\hat{\mathfrak{sl}}_n)$ is an algebra generated by elements $E_i$, $F_i$, $K_i$, and $K_i^{-1}$ for $i = 0, 1, \ldots, n - 1$. These elements satisfy the analogs of relations (27)-(34), for example,

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

with the added stipulation that the indices in all the relations should be read modulo $n$.

As in the classical case, $U_q(\hat{\mathfrak{sl}}_n)$ acts on an evaluation module $V(z)$. Precisely, if $V(z)$ is the same vector space as in Section 2, then the generators of $U_q(\hat{\mathfrak{sl}}_n)$ act as follows:

$$K_i \cdot v_j = q^{(\delta(i,j) - \delta(i+1,j))} \cdot v_j,$$

$$E_i \cdot v_j = \delta(i,j-1) \cdot z^{\delta(i,0)} \cdot v_{j-1},$$

$$F_i \cdot v_j = \delta(i,j) \cdot z^{-\delta(i,0)} \cdot v_{j+1}.$$  

Here again, all the indices should be read modulo $n$. So $E_i$ and $F_i$ act as in Section 4, while $K_i$ acts as $z^{H_i}$. Under the identification $z^j \cdot v_i = v_{i-nj}$ of $V(z)$ with $C^\infty$, $E_i$, $F_i$, and $K_i$ acting in $V(z)$ can be expressed in terms of elements of $U_q(\mathfrak{sl}_\infty)$ acting in $C^\infty$:

$$E_i = \sum_{j \equiv i \mod n} e_j, \quad F_i = \sum_{j \equiv i \mod n} f_j, \quad K_i = \prod_{j \equiv i \mod n} k_j$$

The identification of these two modules should not be taken to mean that $U_q(\hat{\mathfrak{sl}}_n)$ can in general be considered to be sitting inside $U_q(\mathfrak{sl}_\infty)$. The fact that the operators on $C^\infty$ defined by equations (43) satisfy the relation (44) is a consequence of the following equation for $U_q(\mathfrak{sl}_\infty)$ acting in $C^\infty$:

$$\sum_{j \equiv i \mod n} \frac{k_j - k_j^{-1}}{q - q^{-1}} = \frac{1}{q - q^{-1}} \left( \prod_{j \equiv i \mod n} k_j - \prod_{j \equiv i \mod n} k_j^{-1} \right).$$
4.2 The thermodynamic limit

To build highest weight modules, it is again necessary to consider the infinite tensor product

\[ V_{z_1,z_2,z_3,...} = V(z_1) \otimes V(z_2) \otimes V(z_3) \otimes \cdots. \]

As in the previous section, a coproduct is needed to define how the generators of \( U_q(\widehat{\mathfrak{sl}_n}) \) act (at least formally) on an appropriate subspace of \( V_{z_1,z_2,z_3,...} \). The coproduct on \( U_q(\widehat{\mathfrak{sl}_n}) \) is analogous to the one on \( U_q(\mathfrak{sl}_\infty) \); explicitly it is given by

\[
\Delta(K_i) = K_i \otimes K_i \quad (49)
\]

\[
\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \quad (50)
\]

\[
\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i. \quad (51)
\]

Iterating this gives rise to the “infinite coproduct”

\[
\Delta^\infty(K_i) = K_i \otimes K_i \otimes K_i \otimes \cdots \quad (52)
\]

\[
\Delta^\infty(E_i) = \sum_{j=1}^\infty 1 \otimes \cdots \otimes 1 \otimes \underbrace{E_i} \otimes K_i \otimes K_i \otimes \cdots \quad (53)
\]

\[
\Delta^\infty(F_i) = \sum_{j=1}^\infty K_i^{-1} \otimes \cdots \otimes K_i^{-1} \otimes \underbrace{F_i} \otimes 1 \otimes 1 \otimes \cdots \quad (54)
\]

This coproduct should define a formal action of the operators \( K_i, E_i, \) and \( F_i \in U_q(\widehat{\mathfrak{sl}_n}) \) on semi-infinite tensors (semi-infinite meaning the same thing here as in Section 2) in \( V_{z_1,z_2,z_3,...} \). This is done exactly as in Section 2; the only new feature of the quantum case is that it is necessary to say how \( K_i \otimes K_i \otimes K_i \otimes \cdots \) acts. The only possibility is to make it act as \( q^{H_i} \), where the action of \( H_i \) in \( V_{z_1,z_2,z_3,...} \) is given by (17). For example,

\[
(K_i \otimes K_i \otimes K_i \otimes \cdots) \cdot (z_1v_n \otimes \cdots \otimes z_n v_1 \otimes z_{n+1} v_1 \otimes \cdots) = q^{\delta_{i,0}} \cdot (z_1v_n \otimes \cdots \otimes z_n v_1 \otimes z_{n+1} v_1 \otimes \cdots \otimes z_{2n} v_1 \otimes \cdots).
\]

With these definitions, the operators \( K_i, E_i, \) and \( F_i \) act formally on semi-infinite tensors in \( V_{z_1,z_2,z_3,...} \). This formal action is compatible with finite tensor products (i.e., an analog of (20) holds).

4.3 \( q \)-antisymmetrization

Some care is required to produce highest weight vectors for \( U_q(\widehat{\mathfrak{sl}_n}) \), since the naive \( q \)-antisymmetrization given by equation (11) does not work. The correct approach is to use a form of quantum Weyl duality. The symmetric group \( S_d \) has a quantum analog known as a Hecke algebra, to be denoted here by \( H_d(q^2) \). \( H_d(q^2) \) is an \( d! \)-dimensional algebra generated by elements \( T_i, i = 1, \ldots, d-1 \), satisfying the relations

\[
T_i^2 = (q^2 - 1) T_i + q^2 \quad (55)
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (56)
\]

\[
T_i T_j = T_j T_i \quad \text{if } |i - j| > 1. \quad (57)
\]
The elements \( T_i \) are \( q \)-analogues of the adjacent transpositions \( \sigma_i = (i \ i + 1) \) in the symmetric group \( S_d \).

\( H_d(q^2) \) acts on the right on the tensor product \( V(z_1) \otimes \cdots \otimes V(z_d) \) as follows. Write elements \( z_1^{d_1} v_{m_1} \otimes \cdots \otimes z_d^{d_d} v_{m_d} \) as \( (v_{m_1} \otimes \cdots \otimes v_{m_d}) \cdot z_1^{d_1} \cdots z_d^{d_d} \). \( H_d \) can act on both the tensor part and the polynomial part of such an expression. The action on the tensor part is the usual one, permuting factors:

\[
(v_{m_1} \otimes \cdots \otimes v_{m_d})^{\sigma_i} = v_{m_1} \otimes \cdots \otimes v_{m_{i+1}} \otimes v_{m_i} \otimes \cdots \otimes v_{m_d}.
\]

Similarly, the action on the polynomial part is to permute variables: if \( z = z_1^{d_1} \cdots z_d^{d_d} \), then

\[
z^{\sigma_i} = (z_1^{d_1} \cdots z_d^{d_d})^{\sigma_i} = z_1^{d_1} \cdots z_i^{d_{i+1}} z_i^{d_i} \cdots z_d^{d_d}.
\]

Then,

\[
((v_{m_1} \otimes \cdots \otimes v_{m_d}) \cdot z) \cdot (T_i) = \begin{cases} 
-q(v_{m_1} \otimes \cdots \otimes v_{m_d})^{\sigma_i} \cdot z^{\sigma_i} & \text{if } m_i < m_{i+1} \\
-(q^2 - 1)(v_{m_1} \otimes \cdots \otimes v_{m_d}) \cdot z^{\sigma_i} & \text{if } m_i = m_{i+1} \\
-q(v_{m_1} \otimes \cdots \otimes v_{m_d})^{\sigma_i} \cdot z^{\sigma_i} & \text{if } m_i > m_{i+1}
\end{cases}
\]

(58)

Remarks

1. Notice that, for example, \((v_i \otimes v_i) \cdot T_1 = -v_i \otimes v_i\); also, \((v_{i+1} \otimes v_i) \cdot T_1 = -qv_i \otimes v_{i+1}\), and \((v_1 \otimes z_2 v_n) \cdot T_1 = -q z_1 v_n \otimes v_i\). (These equations remain true if the left and right hand sides are multiplied by \((z_1 z_2)^d\) for any \(d\).) In particular, the \(T_i\)'s, rather than the \(-T_i\)'s, will be used to do \(q\)-antisymmetrization.

2. The *affine* Hecke algebra \( \hat{H}_d(q^2) \) acts on \( V(z_1) \otimes \cdots \otimes V(z_d) \) as the centralizer of the action of \( U_q(\hat{\mathfrak{s}l}_n) \) given by the coproduct in (49)-(51) (equivalently, the action of \( U_q(\hat{\mathfrak{s}l}_n) \) is given by a finite version of (52)-(54)). The action of \( H_d(q^2) \) written down above comes from regarding it as a subalgebra of \( \hat{H}_d(q^2) \) in the obvious way.

There is a chain of inclusions \( H_1(q^2) \subset H_2(q^2) \subset H_3(q^2) \subset \cdots \), so equation (58) also defines an action of the *infinite* Hecke algebra \( H_\infty(q^2) = \bigcup_{d \geq 1} H_d(q^2) \) on the thermodynamic limit \( V_{\infty,\infty,\infty,\ldots} \). (\( H_\infty(q^2) \) is generated by elements \( T_1, T_2, T_3, \ldots \) with the corresponding relations.) This action commutes with the action of \( U_q(\hat{\mathfrak{s}l}_n) \) in \( V_{\infty,\infty,\infty,\ldots} \) since any element acts in only finitely many factors.

There is a natural basis for \( H_\infty(q^2) \) made up of elements \( T_\sigma \) corresponding to \( \sigma \in S_\infty \). More precisely, if \( \sigma = \sigma_i \sigma_{i+1} \cdots \sigma_i \) is a minimal length expansion of \( \sigma \in S_\infty \) in terms of adjacent transpositions, let \( T_\sigma = T_{i_1} T_{i_2} \cdots T_{i_t} \). It is a consequence of the relations (56)-(57) that \( T_\sigma \) depends only on \( \sigma \), and not on the factorization into adjacent transpositions.

Let \( z_1^{d_1} v_{m_1} \otimes z_2^{d_2} v_{m_2} \otimes \cdots \) be a semi-infinite tensor. Define its \emph{q-antisymmetrization} to be

\[
z_1^{d_1} v_{m_1} \wedge_q z_2^{d_2} v_{m_2} \wedge_q \cdots = \sum_{\sigma \in S_\infty} (z_1^{d_1} v_{m_1} \otimes z_2^{d_2} v_{m_2} \otimes \cdots) \cdot T_\sigma.
\]

(59)
Here the action of $T_\sigma$ is given by equation (58). Again, subscripts on the variables have been dropped in infinite wedge notation. By the first remark above, (59) really is an antisymmetrization, rather than a symmetrization.

**Conjecture** The sum given by (59) converges in the power series topology. (I.e., the coefficient of each particular tensor $z_1^{i_1} v_k \otimes z_2^{i_2} v_k \otimes \cdots$ is a well-defined power series in $q$.)

The formal action of $K_i, E_i, F_i \in U_q(\hat{\mathfrak{sl}}_n)$ on semi-infinite tensors gives rise to a genuine action of $U_q(\hat{\mathfrak{sl}}_n)$ on the vector space spanned the $q$-wedges. As in Section 2, this vector space is a level 1 module. Since each $T_\sigma$ commutes with the action of $U_q(\hat{\mathfrak{sl}}_n)$, when $X \in U_q(\hat{\mathfrak{sl}}_n)$ acts on the $q$-antisymmetrization of $z_1^{d_1} v_m \otimes z_2^{d_2} v_{m_2} \otimes \cdots$, the result is the $q$-antisymmetrization of $\Delta^\infty(X) \cdot (z_1^{d_1} v_m \otimes z_2^{d_2} v_{m_2} \otimes \cdots)$.

**Proposition 4.1** The $q$-wedge

$$v_{\Lambda_i} = z v_i \wedge z v_{i-1} \wedge \cdots \wedge z v_1 \wedge z^2 v_n \wedge \cdots \wedge z^2 v_1 \wedge \cdots$$

is a highest weight vector of weight $\Lambda_i$.

**Proof.** By the above, it is enough to check that if $z_1^{d_1} v_m \otimes z_2^{d_2} v_{m_2}$ appears somewhere in a tensor, then the $q$-antisymmetrization of that tensor is zero. Grouping the basis elements of $H_\infty(q^2)$ in pairs $\{T_\sigma, T_i T_\sigma\}$ corresponding to cosets of the subgroup $H = \{1, \sigma_i\} \subset S_\infty$ reduces the problem to showing that such a vector is killed by $1 + T_\sigma_i$. This is an immediate consequence of the first remark following equation (58).

The highest weight vector $v_{\Lambda_i}$ generates a highest weight module $V_{\Lambda_i}$ of weight $\Lambda_i$ inside the space $F_{(i)}$ spanned by the $q$-wedges that are the same as $v_{\Lambda_i}$ after finitely many terms. $V_{\Lambda_i}$ is a $q$-deformation of the corresponding $\mathfrak{sl}_n$-module constructed in Section 2, and is therefore irreducible.

**Remarks**

1. The $q$-antisymmetrization for $U_q(\mathfrak{sl}_\infty)$ given by (12) is essentially a special case of the one for $U_q(\hat{\mathfrak{sl}}_n)$ given by (59), since it follows from (58) that

$$(v_{k_1} \otimes v_{k_2} \otimes \cdots) \cdot T_i = -q v_{k_1} \otimes \cdots \otimes v_{k_{i+1}} \otimes v_{k_i} \otimes v_{k_{i+2}} \otimes \cdots$$

as long as $k_i > k_{i+1}$.

2. The action of $U_q(\hat{\mathfrak{sl}}_n)$ on $F_{(i)}$ was originally constructed by Hayashi in [4].

The following lemma relating the $q$-antisymmetrizations of $z_1^{d_1} v_m \otimes z_2^{d_2} v_{m_2} \otimes \cdots$ and of $(z_1^{d_1} v_m \otimes z_2^{d_2} v_{m_2} \otimes \cdots) \cdot T_i$ will be useful in the next section:

**Lemma** For any $T_i$,

$$\sum_{\sigma \in S_\infty} (z_1^{d_1} v_m \otimes z_2^{d_2} v_{m_2} \otimes \cdots) T_i \cdot T_\sigma = q^2 \sum_{\sigma \in S_\infty} (z_1^{d_1} v_m \otimes z_2^{d_2} v_{m_2} \otimes \cdots) \cdot T_\sigma.$$

**Proof of Lemma.** As before, group the basis elements of $H_\infty(q^2)$ in pairs $\{T_\sigma, T_i T_\sigma\}$
corresponding to left cosets of \( \{\text{id}, \sigma_i\} \) in \( S_\infty \). Using the first Hecke relation \((55)\), compute

\[
T_i(T_\sigma + T_iT_\sigma) = T_iT_\sigma + ((q^2 - 1)T_i + q^2)T_\sigma = q^2(T_iT_\sigma + T_\sigma).
\]

The lemma follows.

### 4.4 Vertex operators

Splitting off the first component of every tensor defines intertwiners

\[
\Phi(i) : F(i) \to V(z) \otimes F(i-1)
\]

as in Section 2.3. Since \( F(i) \) is a \( q \)-deformation of the completely reducible \( \widehat{\mathfrak{sl}}_n \)-module \( F(i) \) from Section 2, it is also completely reducible. This makes it possible to define intertwiners \( \Phi(i) : V_\Lambda \to V(z) \otimes V_{\Lambda_{i-1}} \) by composing with the projection \( F(i-1) \to V_{\Lambda_{i-1}} \). These intertwiners satisfy the obvious \( q \)-analog of Proposition 2.2. Up to normalization, they are the vertex operators studied in \( 3 \) and \( 2 \) by means of the quantum KZ equation. Their iterations can be computed exactly as in the classical case.

As an example, take the case of \( U_q(\mathfrak{sl}_2) \) and consider the operator \( \Phi(0) : V_{\Lambda_0} \subset F(0) \to V(z) \otimes F(1) \).

**Proposition 4.2** The image of \( v_{\Lambda_0} \) under \( \Phi(0) \) is given by

\[
\Phi(0)(v_{\Lambda_0}) = \sum_{j=1}^{\infty} q^{3(j-1)} z^j v_2 \otimes (zv_1 \wedge zv_1 \wedge \cdots \wedge z^{j-1} v_1 \wedge z^j v_1 \wedge z^{j+1} v_2 \wedge \cdots) \quad (60)
\]

\[
-\sum_{j=1}^{\infty} q^{3(j-1)+1} z^j v_1 \otimes (zv_1 \wedge zv_1 \wedge \cdots \wedge z^j v_2 \wedge z^{j+1} v_2 \wedge z^{j+1} v_1 \wedge \cdots).
\]

**Proof.** The assertion states that the \( q \)-antisymmetrization of \( zv_2 \otimes zv_1 \otimes z^2 v_2 \otimes \cdots \) is given by equation \((50)\). (For simplicity of notation, the subscripts on the variables have been left off, but again it should be understood that all \( z \)'s have subscripts on them recording the factor in which they appear.) The idea of the proof is to compute the antisymmetrization using \((58)\). Each term in \((50)\) corresponds to a collection of tensors with a particular basis element \( z^j v_k \) appearing in the first component. So imagine performing the antisymmetrization by first moving into the first component whichever \( z^j v_k \) will go there: choose a particular \( z^j v_k \) (say the one in the \( r \)-th component) and apply \( T_{r-1}, T_{r-2}, \ldots, T_1 \) to move it all the way over to the left. The effect of one step in this sequence can be computed using the following formulas for the action of \( H_2(q^2) \) (with generator \( T \)) on \( V(z) \otimes V(z) \) (again, it should be understood that the \( z \)'s in the first and second components represent different \( z \)'s):

\[
(z^j v_i \otimes z^k v_i) \cdot T = -q^2 z^k v_i \otimes z^j v_i - (q^2 - 1)(z^{k-1} v_i \otimes z^{j+1} v_i + z^{k-2} v_i \otimes z^{j+2} v_i + \cdots + z^{j+1} v_i \otimes z^{k-1} v_i)
\]

\[
(z^j v_1 \otimes z^k v_2) \cdot T = -q z^k v_2 \otimes z^j v_1 - (q^2 - 1)(z^{k-1} v_1 \otimes z^{j+1} v_2 + z^{k-2} v_1 \otimes z^{j+2} v_2 + \cdots + z^{j+1} v_1 \otimes z^{k-1} v_2)
\]

\[
(z^j v_2 \otimes z^k v_1) \cdot T = -q z^k v_1 \otimes z^j v_2 - (q^2 - 1)(z^{k} v_2 \otimes z^{j} v_1 + z^{k-1} v_2 \otimes z^{j+1} v_1 + \cdots + z^{j+1} v_2 \otimes z^{k-1} v_1)
\]
These formulas follow directly from (68). Here \( j \leq k \) except in the last equation, where \( j < k \). The upshot is that after applying a sequence \( T_{r-1}, T_{r-2}, \ldots, T_1 \), the result will be
\[
\sum_{j=1}^{\infty} q^{3(j-1)} z^j v_2 \otimes (z v_2 \otimes z v_1 \otimes \cdots \otimes z^{j-1} v_1 \otimes z^j v_1 \otimes z^{j+1} v_2 \otimes \cdots)
\]  
\[= - \sum_{j=1}^{\infty} q^{3(j-1)+1} z^j v_1 \otimes (z v_2 \otimes z v_1 \otimes \cdots \otimes z^j v_2 \otimes z^{j+1} v_2 \otimes z^{j+1} v_1 \otimes \cdots) + \text{other terms},
\]  
where the other terms, which come from the terms following the first one in (61)-(63), all have to the right of the first component one of the following four sequences:
\[
z^k v_1 \otimes z^j v_1 \otimes z^{j+1} v_2 \otimes z^{j+1} v_1 \otimes \cdots \otimes z^{k-1} v_1 \otimes z^k v_2 \otimes z^k v_1 \\
z^k v_1 \otimes z^j v_2 \otimes z^j v_1 \otimes z^{j+1} v_2 \otimes z^{j+1} v_1 \otimes \cdots \otimes z^{k-1} v_1 \otimes z^k v_2 \otimes z^k v_1 \\
z^k v_2 \otimes z^j v_1 \otimes z^{j+1} v_2 \otimes z^{j+1} v_1 \otimes \cdots \otimes z^{k-1} v_2 \otimes z^{k-1} v_1 \otimes z^k v_2 \\
z^k v_2 \otimes z^j v_2 \otimes z^j v_1 \otimes z^{j+1} v_2 \otimes z^{j+1} v_1 \otimes \cdots \otimes z^{k-1} v_2 \otimes z^{k-1} v_1 \otimes z^k v_2
\]  
Here \( j \leq k \), and in the third sequence, \( j < k \).

At this point, the first component is left alone, and the other components are completely \( q \)-antisymmetrized. Antisymmetrizing the terms listed explicitly in (64) gives the answer. The other terms do not figure in the answer because strings of the form (65)-(68) antisymmetrize to 0. This can be seen by induction on the length of the string, using the lemma in the preceding section. If the rightmost \( z^k v_1 \) (or \( z^k v_2 \), as appropriate), appears in the \( r \)-th component, apply \( T_{r-1}, T_{r-2}, \ldots \) to move it over next to the \( z^k v_1 \) or \( z^k v_2 \) on the left. A term containing a \( z^k v_1 \otimes z^k v_1 \) or a \( z^k v_2 \otimes z^k v_2 \) antisymmetrizes to 0, and by the lemma, it is enough to show that the extra terms created by this procedure also antisymmetrize to 0. By (61)-(63), all these extra terms contain sequences of the form (65)-(68) of shorter length, which antisymmetrize to 0 by induction.

**Remarks**

1. The formula for \( \Phi_{(0)}(v_{\Lambda_0}) \) given by equation (61) looks (perhaps with all \( z_i \) set equal to 1) exactly like formula (A3.1) in [2] giving the expansion of \( \Phi_{(0)}(v_{\Lambda_0}) \) in terms of the upper global base for \( V_{\Lambda_1} \). This suggests that \( q \)-wedges may be related in a reasonable way to the upper global base. (Notice that the coproduct used in [2] differs by a permutation from the one given here, and the semi-infinite tensor product considered there is \( \cdots V(z_3) \otimes V(z_2) \otimes V(z_1) \). Therefore, the tensors appearing in [2] should be reversed to match the conventions of this paper.)

2. As in the classical case, solutions to appropriate \( q \)-KZ equations (for example, those listed on p. 116 of [2]) should be understood (up to scalars) as finite \( q \)-antisymmetrizations. The functions \( w_n(z) \) (see pp. 121-122 of [2]) are (the reverses of) the \( q \)-wedges \( v_2 \wedge_q v_1 \wedge_q \cdots \wedge_q z^{m-1} v_2 \wedge_q z^{m-1} v_1 \) if \( n = 2m \), and \( v_1 \wedge_q v_2 \wedge_q \cdots \wedge_q z^{m-1} v_2 \wedge_q z^{m-1} v_1 \) if \( n = 2m - 1 \). Also as in the classical case, the infinite \( q \)-antisymmetrization \( v_{\Lambda_0} \) should be nothing more than a limit of these solutions as the number of variables goes to infinity. The details will be worked out in a separate publication.
3. In [2], a tensor product \( \cdots \otimes V(z_i) \otimes V(z_{i+1}) \otimes \cdots \) which is infinite in both directions is considered. The problem is to find the ground state vector of the XXZ Hamiltonian, which spans the trivial component of \( V(\Lambda_0)^* \otimes V(\Lambda_0) \subset \cdots \otimes V(z_i) \otimes V(z_{i+1}) \otimes \cdots \). From the point of view outlined here, this vector should be a \( q \)-antisymmetrization which is infinite in both directions.

I am grateful to Nicolai Reshetikhin for introducing me to this subject and to this problem, and for many stimulating conversations. I am also grateful to Tetsuji Miwa for reading an early draft of this paper and pointing out inaccuracies.

References

[1] E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado, *Paths, Maya diagrams, and representations of \( \hat{\mathfrak{sl}}(r,\mathbb{C}) \)*, Adv. Stud. Pure Math. **19** (1989), 149-191.

[2] B. Davies, O. Foda, M. Jimbo, T. Miwa, and A. Nakayashiki, *Diagonalization of the XXZ Hamiltonian by vertex operators*, Commun. Math. Phys. **151** (1993), 89-153.

[3] I. Frenkel and N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Commun. Math. Phys. **146** (1992), 1-60.

[4] T. Hayashi, *Q-analogues of Clifford and Weyl algebras — spinor and oscillator representations of quantum enveloping algebras*, Commun. Math. Phys. **127** (1990), 129-144.

[5] V. Kac, “Infinite Dimensional Lie Algebras,” Cambridge University Press, 1990.

[6] V. Kac and A. Raina, “Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras,” World Scientific, 1987.

[7] K. Misra and T. Miwa, *Crystal base for the basic representation of \( U_q(\hat{\mathfrak{sl}}(n)) \)*, Commun. Math. Phys. **134** (1990), 79-88.

Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720
stern@math.berkeley.edu