SELF-ADJOINT BOUNDARY-VALUE PROBLEMS OF AUTOMORPHIC FORMS

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Abstract. We apply some ideas of Bombieri and Garrett to construct natural self-adjoint operators on spaces of automorphic forms whose only possible discrete spectrum is \( \lambda_s \) for \( s \) in a subset of on-line zeros of an \( L \)-function, appearing as a compact period of cuspidal-data Eisenstein series on \( GL_4 \). These ideas have their origins in results of Hejhal and Colin de Verdière. In parallel with the \( GL(2) \) case, the corresponding pair-correlation and triple-correlation results limit the fraction of on-the-line zeros that can appear in this fashion.

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1. Introduction

We apply the spectral theory of automorphic forms to the study of zeros of \( L \)-functions. A refined version of the spectral theory of automorphic forms plausibly has bearing on zeros of automorphic \( L \)-functions and other periods. This is powerfully illustrated by the following example, which is a much simpler analogue of our present result. In 1977, H. Haas [Haas 1977] attempted to numerically compute eigenvalues \( \lambda_w \) of the invariant Laplacian

\[
\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]

on \( SL_2(\mathbb{Z}) \backslash \mathfrak{H} \), parametrized as \( \lambda_w = w(w-1) \). Haas listed the \( w \)-values, intending to solve the differential equation

\[(\Delta - \lambda_w)u = 0\]

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H. Stark and D. Hejhal [Hejhal 1981] observed zeros of $\zeta$ and of an $L$-function on the list. This suggested an approach to the Riemann Hypothesis, hoping that zeros $\omega$ of $\zeta$ would be in bijection with eigenvalues $\lambda_\omega = \omega(w-1)$ of $\Delta$. Since a suitable version of $\Delta$ is a self-adjoint, non-positive operator, these eigenvalues would necessarily be non-positive also, forcing either $\text{Re}(\omega) = \frac{1}{2}$ or $\omega \in [0,1]$.

Hejhal attempted to reproduce Haas’ list with more careful computations, but the zeros failed to appear on Hejhal’s list. Hejhal realized that Haas had solved the inhomogeneous equation
\[(\Delta - \lambda_\omega)u = \delta_\omega^{afc}\]
allowing a multiple of an automorphic Dirac $\delta_\omega^{afc}$ on the right hand side. Here $\omega$ is a cube root of unity, and $\delta_\omega^{afc}(f) = f(\omega)$ for an $SL_2(\mathbb{Z})$-automorphic waveform $f$.

However, since solutions $u_\omega$ of $(\Delta - \lambda)u = \delta_\omega^{afc}$ are not genuine eigenfunctions of the Laplacian, this no longer implied non-positivity of the eigenvalues.

The natural question was whether the Laplacian could be modified so as to exhibit a fundamental solution as a legitimate eigenfunction for the perturbed operator. That is, one would want a variant $\Delta'$ for which
\[(\Delta' - \lambda_\omega)u_\omega = 0 \iff (\Delta - \lambda_\omega)u_\omega = C \cdot \delta_\omega^{afc}\]

Because of Y. Colin de Verdière’s argument for meromorphic continuation of Eisenstein series [CdV 1981], it was anticipated that $\Delta' = \Delta^{Fr}$ would be a fruitful choice for the Friedrichs extension of a suitably chosen restriction. $\Delta^{Fr}$ is self-adjoint, and therefore symmetric. This gave glimpses of progress toward the Riemann hypothesis.

Friedrichs extensions have the desired properties and they played an essential role in another story, namely Colin de Verdière’s meromorphic continuation of Eisenstein series, though there, the distribution that appeared was the evaluation of constant term at height $y = a$. There, the spaces of interest were the orthogonal complements $L^2(\Gamma \backslash \mathfrak{H})_a$ to the spaces of pseudo-Eisenstein series with test function data supported on $[a, \infty)$. $\Delta_a$ was $\Delta$ with domain $C^\infty_c(\Gamma \backslash \mathfrak{H})$ and constant term vanishing above height $y = a$. $\Delta^{Fr}$ was the Friedrichs extension of $\Delta_a$ to a self-adjoint operator on $L^2(\Gamma \backslash \mathfrak{H})_a$. In this way, a Friedrichs extension attached to the distribution on $\Gamma \backslash \mathfrak{H}$ given by
\[T_a(f) = \langle c_p f \rangle (ia)\]
has all eigenfunctions inside a $+1$-index global automorphic Sobolev space, defined as the completion of $C^\infty_c(\Gamma \backslash \mathfrak{H})$ with respect to the $+1$-Sobolev norm
\[|f|_{H^1} = \left(\langle (1 - \Delta)f, f \rangle\right)^{\frac{1}{2}}\]
The Dirac $\delta$ on a two-dimensional manifold lies in a global Sobolev space $H^{-1-\epsilon}$ with index $-1 - \epsilon$ for all $\epsilon > 0$, but not in $H^{-1}$, so by elliptic regularity, a fundamental solution lies in the $+1-\epsilon$-Sobolev space. This implies that a fundamental solution could not be an eigenfunction for any Friedrichs extension of a restriction of $\Delta$ described by boundary conditions.

The automorphic Dirac $\delta_\omega^{afc}$ is an example of a period functional. Periods of automorphic forms have been studied extensively: after all, Mellin transforms of cuspforms are noncompact periods. Hecke and Maass were aware of Eisenstein series periods: in effect, Hecke treated finite sums over Heegner points attached to negative fundamental discriminants, and Maass treated compact geodesic periods.
attached to positive fundamental discriminants. A simple example is given by

\[ E_s(i) = \frac{\zeta_Q(i)(s)}{\zeta_Q(2s)} \]

More generally, let \( \ell \) a quadratic field extension of a global field \( k \) of characteristic not 2. Let \( G = GL_2(k) \), and let \( H \) be a copy of \( \ell^x \) inside \( G \). The period of an Eisenstein series \( E_s = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g) \) along \( H \) is defined by the compactly-supported integral

\[ \text{period of } E_s \text{ along } H = \int_{Z_A H_k \backslash H_A} E_s \]

Via Iwasawa-Tate integrals,

\[ \int_{Z_A H_k \backslash H_A} E_s = \frac{\xi_\ell(s)}{\xi_k(2s)} \]

Noncompact periods have been studied extensively. Let \( G \) be a reductive group over a number field \( F \), and let \( H \subset G \) be a subgroup obtained as the fixed point set of an involution \( \theta \). [Jacquet-Lapid-Rogowski 1997] studied the period integral

\[ \Pi^H(\varphi) = \int_{H(F) \backslash H(A)} \varphi(h) \, dh \]

The authors use a regularization procedure and a relative trace formula to obtain an Euler product for \( \Pi(E) \), where \( E \) is an Eisenstein series.

This paper examines the discrete spectrum of a Friedrichs extension \( \tilde{\Delta}_\theta \) associated to a compactly-supported \( GL_4(\mathbb{Z}) \)-invariant distribution \( \tilde{\theta} \) on \( G = GL(4) \), whose projection \( \theta \) to the subspace of \( L^2(GL_4(\mathbb{Z}) \backslash GL_4(\mathbb{R})/O_4(\mathbb{R})) \) spanned by 2, 2 pseudo-Eisenstein series with fixed cuspidal data \( f \) and \( \mathcal{f} \) and the residue of this Eisenstein series, a Speh form. This distribution lies in the \(-1\) index Sobolev space. We prove that the parameters \( w \) of the discrete spectrum \( \lambda_w = w(w - 1) \), if any, of \( \tilde{\Delta}_\theta \) interlace with the zeros of the constant term of the 2, 2 Eisenstein series \( E_{P, f}^P \), where \( f \) is a \( GL(2) \) cuspidal form. Such spacing is too regular to be compatible with the corresponding pair-correlation and triple-correlation conjectures, and this powerfully constrains the number of zeros \( w \) of \( \theta E_{1-w} \) appearing in the discrete spectrum of \( \tilde{\Delta}_\theta \). In particular, the discrete spectrum is presumably sparse.

2. Spectral Theory

We follow [Langlands 1976], [MW 1990], [MW 1989], and [Garrett 2012]. Fix, once and for all, \( K_\infty = O_4(\mathbb{R}) \), and \( K_v = GL_4(\mathbb{Z}_v) \) for non-archimedean places \( v \). Let \( \mathfrak{z} \) be the center of the enveloping algebra of \( G_\infty = GL_4(\mathbb{R}) \).

**Definition 1.** Given a parabolic \( P \) in \( G = GL_4 \) and a function \( f \) on \( Z_A G_k \backslash G_A \), the constant term of \( f \) along \( P \) is

\[ c_P f(g) = \int_{N_k \backslash N_A} f(n g) \, dn \]

where \( N \) is the unipotent radical of \( P \).

We will let \( k = \mathbb{Q} \) throughout. An automorphic form is a *cuspform* if, for all parabolics \( P \), the constant term along \( P \) is zero. This is the Gelfand condition (in the weak sense). Since the right \( G_A \)-action commutes with taking constant
terms, the space of functions $L^2_{\text{cusp}}(Z_h G_k \backslash G_h)$ satisfying the Gelfand condition is $G_h$-stable, and so is a sub-representation of $L^2(Z_h G_k \backslash G_h)$. We note that there are non-$K$-finite vectors in $L^2(Z_h G_k \backslash G_h)$. R. Godement, A. Selberg, I. Gelfand and I. I. Piatetski-Shapiro showed that integral operators attached to test functions on $L^2_{\text{cusp}}(Z_h G_k \backslash G_h)$ are compact. Specifically, for $\varphi \in C_\infty^c(G_h)$ which is right $K$-invariant, the operator 

$$f \to \varphi \cdot f$$

gives a compact operator from $L^2_{\text{cusp}}(Z_h G_k \backslash G_h)$ to itself. Here 

$$(\varphi \cdot f)(y) = \int_{Z_h G_k \backslash G_h} \varphi(x) \cdot f(yx) \, dx$$

By the spectral theorem for compact operators, this sub-representation decomposes into a direct sum of irreducibles, each with finite multiplicity. The remainder of $L^2$ is decomposed as follows.

We classify non-cuspidal automorphic forms according to their cuspidal support, i.e. the smallest parabolics on which they have non-zero constant term. In $GL(4)$ there are four associate classes of proper parabolic subgroups. There is $P^1 = GL_1$, $P^{2,1,1}$, $P^{1,2,1}$, $P^{1,1,2}$, the maximal proper parabolic subgroups $P^{3,1}$, $P^{1,3}$ and $P^{2,2}$, and the standard minimal parabolic subgroup $P^{1,1,1,1}$.

**Definition 2.** A pseudo-Eisenstein series is a function of the form 

$$\Psi_\varphi(g) = \sum_{\gamma \in P \backslash G_k} \varphi(\gamma \cdot g)$$

where $\varphi$ is a continuous function on $Z_h N_h M_k \backslash G_h$ with cuspidal data on the Levi component.

For example, given the 2,2 parabolic, the function out of which the pseudo-Eisenstein series is constructed is 

$$\varphi_{\phi, f_1 \otimes f_2}(\begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \gamma) = \phi\left(\frac{\det A}{\det D}\right)^2 \cdot f_1(A) \cdot f_2(D)$$

where $\phi$ is a compactly-supported, smooth function on $\mathbb{R}$ and $f_1$ and $f_2$ are cuspforms on $GL_2$ with trivial central character. For the 3,1 parabolic, consider the function 

$$\varphi_{\phi, f_1 \otimes f_2}(\begin{pmatrix} A & * \\ 0 & d \end{pmatrix} \gamma) = \phi\left(\frac{\det A}{\det D}\right)^2 \cdot f_1(A)$$

where $A \in GL_3$ and $f_1$ is a cuspform on $GL_3$. For the 2,1,1 parabolic, let 

$$\varphi_{f, \phi_1 \cdot \phi_2}(\begin{pmatrix} A & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \gamma) = f(A) \cdot \phi_1\left(\frac{\det A}{b^2}\right) \cdot \phi_2\left(\frac{\det A}{c^2}\right)$$

The 1,1,1-pseudo-Eisenstein series is discussed later.

**Proposition 1.** In the following, abbreviate $\varphi_{\phi, f_1 \otimes f_2}$ by $\varphi$. For any square-integrable automorphic form $f$ and any pseudo-Eisenstein series $\Psi_\varphi^P$, with $P$ a parabolic subgroup,

$$\langle f, \Psi_\varphi^P \rangle_{Z_h G_k \backslash G_h} = \langle cP f, \varphi \rangle_{Z_h N_h^P M_k \backslash G_h}$$
Proof. The proof involves a standard unwinding argument. Let $N^P$ and $M^P$ denote the unipotent radical and Levi component of $P$, respectively. Observe that

$$\langle f, \Psi_P \rangle_{Z_A G_k \backslash G_k} = \int_{Z_A G_k \backslash G_k} f(g) \cdot \overline{\Psi_P(g)} \, dg = \int_{Z_A G_k \backslash G_k} f(g) \left( \sum_{\gamma \in P_k \backslash G_k} \overline{\varphi(\gamma \cdot g)} \right) \, dg$$

This is

$$= \int_{Z_A P_k \backslash G_k} f(g) \varphi(g) \, dg = \int_{Z_A N_k M_k \backslash G_k} f(g) \varphi(g) \, dg$$

$$= \int_{Z_A N_k M_k \backslash G_k} \left( \int_{N_k \backslash N_k} f(n g) \, dn \right) \varphi(g) \, dg$$

$$= \langle cP f, \varphi \rangle_{Z_A N_k^P M_k^P \backslash G_k}$$

□

From this adjointness relation, we have the following

**Corollary 1.** A square-integrable automorphic form is a cuspform if and only if it is orthogonal to all pseudo-Eisenstein series.

Since the critical issues arise at the archimedean place, we consider the real Lie group. To this end, let $G = \text{PGL}_4(\mathbb{R})$, $\Gamma = \text{PGL}_4(\mathbb{Z})$.

**Definition 3.** The standard minimal parabolic $B$ is defined as the subgroup

$$B = P^{1,1,1,1}$$

of upper-triangular matrices, with standard Levi component $A$, unipotent radical $N$, and Weyl group $W$, the latter represented by permutation matrices.

Let $A^+$ be the image in $G$ of positive diagonal matrices. Consider characters on $B$ of the form

$$\chi = \chi_s : \left( \begin{array}{cccc} a_1 & * & * & * \\ 0 & a_2 & * & * \\ 0 & 0 & a_3 & * \\ 0 & 0 & 0 & a_4 \end{array} \right) = |a_1|^{s_1} \cdot |a_2|^{s_2} \cdot |a_3|^{s_3} \cdot |a_4|^{s_4}$$

For the character to descend to $\text{PGL}_n$, necessarily $s_1 + s_2 + s_3 + s_4 = 0$.

**Definition 4.** The standard spherical vector is

$$\varphi^{sph}(pk) = \chi_s(p)$$

and the spherical Eisenstein series is

$$E_s(g) = \sum_{\gamma \in B \cap \Gamma} \varphi^{sph}(\gamma \cdot g)$$

The spherical Eisenstein series is convergent for $\text{Re}(s) \gg 1$ and meromorphically continued to an entire function of $s$ as in [Langlands 544, Appendix 1]. The function $f \to c_B f(g)$ is left $N(B \cap \Gamma)$-invariant.
Recall that for $\varphi \in C_c^\infty(N(B \cap \Gamma) \backslash G)^K \approx C_c^\infty(A^+)$, letting $\langle , \rangle_X$ be the pairing of distributions and test functions on a space $X$, the pseudo-Eisenstein series $\Psi_\varphi(g)$ enters the adjunction relation

$$\langle c_B f, \varphi \rangle_{N(B \cap \Gamma) \backslash G} = \langle f, \Psi_\varphi \rangle_{\Gamma \backslash G}$$

That is, $\varphi \rightarrow \Psi_\varphi$ is adjoint to $f \rightarrow c_B f$. Then $c_B f = 0$ is equivalent to $\langle f, \Psi_\varphi \rangle_{\Gamma \backslash G} = 0$ for all $\varphi$.

**Proposition 2.** The pseudo-Eisenstein series $\Psi_\varphi$ admits a $W$-symmetric expansion as an integral of Eisenstein series. That is,

$$\Psi_\varphi = \frac{1}{|W|} \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + ia} E_s \cdot (\Psi_\varphi, E_{2\rho - s})_{\Gamma \backslash G} \, ds$$

**Proof.** To decompose the pseudo-Eisenstein series $\Psi_\varphi$ as an integral of minimal-parabolic Eisenstein series, begin with Fourier transform on the Lie algebra $a \approx \mathbb{R}^{n-1}$ of $A^+$. Let $\langle , \rangle : a^* \times a \rightarrow \mathbb{R}$ be the $\mathbb{R}$-bilinear pairing of $a$ with its $\mathbb{R}$-linear dual $a^*$. For $f \in C_c^\infty(a)$, the Fourier transform is

$$\hat{f}(\xi) = \int_a e^{-i(x, \xi)} f(x) \, dx$$

Fourier inversion is

$$f(x) = \frac{1}{(2\pi)^{\dim a}} \int_{a^*} e^{i(x, \xi)} \hat{f}(\xi) \, d\xi$$

Let $\exp : a \rightarrow A^+$ be the Lie algebra exponential, and $\log : A^+ \rightarrow a$ the inverse. Given $\varphi \in C_c^\infty(A^+)$, let $f = \varphi \circ \exp$ be the corresponding function in $C_c^\infty(a)$. The (multiple) Mellin transform $M\varphi$ of $\varphi$ is the Fourier transform of $f$:

$$M\varphi(i\xi) = \hat{f}(\xi)$$

Mellin inversion is Fourier inversion in these coordinates:

$$\varphi(\exp x) = f(x) = \frac{1}{(2\pi)^{\dim a}} \int_{a^*} e^{i(x, \xi)} \hat{f}(\xi) \, d\xi = \frac{1}{(2\pi)^{\dim a}} \int_{a^*} e^{i(x, \xi)} M\varphi(i\xi) \, d\xi$$

Extend the pairing $\langle , \rangle$ on $a^* \times a$ to a $\mathbb{C}$-bilinear pairing on the complexification. Use the convention

$$(\exp)^{ix} = e^{i(x, x)}$$

With $a = \exp x \in A^+$, Mellin inversion is

$$\varphi(a) = \frac{1}{(2\pi)^{\dim a}} \int_{a^*} a^{ix} M\varphi(i\xi) \, d\xi = \frac{1}{(2\pi i)^\dim a} \int_{a^*} a^{x^*} M\varphi(s) \, ds$$

With this notation, the Mellin transform itself is

$$M\varphi(s) = \int_{A^+} a^{-s} \varphi(a) \, da$$

Since $\varphi$ is a test function, its Fourier-Mellin transform is entire on $a^* \otimes_{\mathbb{R}} \mathbb{C}$. Thus, for any $\sigma \in a^*$, Mellin inversion can be written

$$\varphi(a) = \frac{1}{(2\pi i)^{\dim a}} \int_{\sigma + ia^*} a^{s} M\varphi(s) \, ds$$
Identifying $N(B \cap \Gamma) \backslash G/K \approx A^+$, let $g \to a(g)$ be the function that picks out the $A^+$ component in an Iwasawa decomposition $G = NA^+K$. For $\sigma \in a^+$ suitable for convergence, the following rearrangement is legitimate,

$$\Psi_\varphi(g) = \sum_{\gamma \in (B^0 \cap \Gamma) \setminus \Gamma} \varphi(a(\gamma \circ g)) = \sum_{\gamma \in (B^0 \cap \Gamma) \setminus \Gamma} \frac{1}{(2\pi i)^{\dim a}} \int_{\sigma + i\mathbb{R}^+} a(\gamma g)^s. \mathcal{M} \varphi(s) \, ds$$

$$= \frac{1}{(2\pi i)^{\dim a}} \int_{\sigma + i\mathbb{R}^+} \left( \sum_{\gamma \in (B^0 \cap \Gamma) \setminus \Gamma} a(\gamma g)^s \right). \mathcal{M} \varphi(s) \, ds = \frac{1}{(2\pi i)^{\dim a}} \int_{\sigma + i\mathbb{R}^+} E_s(g). \mathcal{M} \varphi(s) \, ds$$

This does express the pseudo-Eisenstein series as a superposition of Eisenstein series, as desired. However, the coefficients $\mathcal{M} \varphi$ are not expressed in terms of $\Psi_\varphi$ itself. This is rectified as follows. Letting $\rho$ denote the half-sum of positive roots, \[
\langle f, E_s \rangle_{\Gamma \backslash G} = \int_{\Gamma \backslash G} f(g) E_s(g) \, dg = \int_{B \cap \Gamma \backslash G} f(g) a(g)^s \, dg \\
= \int_{N(B \cap \Gamma) \backslash G} \int_{N \cap \Gamma} f(n g) a(n g)^s \, dg = \int_{N(B \cap \Gamma) \backslash G} c_B f(g) a(g)^s \, dg \\
= \int_{A^+} c_B f(a) a^s \frac{da}{a^{2s}} = \int_{A^+} c_B f(a) a^{-(2s - \rho)} \, da = \mathcal{M} c_B f(2\rho - s)
\]

That is, with $f = \Psi_\varphi$,

$$\langle \Psi_\varphi, E_s \rangle_{\Gamma \backslash G} = \mathcal{M} c_B \Psi_\varphi(2\rho - s)$$

On the other hand, a similar unwinding of the pseudo-Eisenstein series, and the recollection of the constant term $c_B E_s$, gives

$$\langle \Psi_\varphi, E_s \rangle_{\Gamma \backslash G} = \int_{B \cap \Gamma \backslash G} \varphi(g) E_s(g) \, dg = \int_{N(B \cap \Gamma) \backslash G} \int_{N \cap \Gamma} \varphi(n g) E_s(n g) \, dg \\
= \int_{N(B \cap \Gamma) \backslash G} \varphi(g) c_B E_s(g) \, dg = \int_{A^+} \varphi(a) c_B E_s(a) \frac{da}{a^{2s}} \\
= \int_{A^+} \varphi(a) \sum_w c_w(s) a^{w \cdot s} \frac{da}{a^{2s}} \\
= \sum_w c_w(s) \int_{A^+} \varphi(a) a^{-(2s - w \cdot s)} \, da = \sum_w c_w(s). \mathcal{M} \varphi(2\rho - w \cdot s)$$

Combining these,

$$\mathcal{M} c_B \Psi_\varphi(2\rho - s) = \langle \Psi_\varphi, E_s \rangle_{\Gamma \backslash G} = \sum_w c_w(s). \mathcal{M} \varphi(2\rho - w \cdot s)$$

Replacing $s$ by $2\rho - s$, noting that $2\rho - w \cdot (2\rho - s) = w \cdot s$,

$$\mathcal{M} c_B \Psi_\varphi(s) = \sum_w c_w(2\rho - s). \mathcal{M} \varphi(w \cdot s)$$

To convert the expression

$$\Psi_\varphi(g) = \frac{1}{(2\pi i)^{\dim a}} \int_{\sigma + i\mathbb{R}^+} E_s(g). \mathcal{M} \varphi(s) \, ds$$
into a $W$-symmetric expression, to obtain an expression in terms of $c_B \Psi_\varphi$, we must use the functional equations of $E_s$. However, $\sigma + i a^*$ is $W$-stable only for $\sigma = \rho$. Thus, the integral over $\sigma + i a^*$ must be viewed as an iterated contour integral, and moved to $\rho + i a^*$.

$$
\Psi_\varphi = \frac{1}{|W|} \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + i a^*} E_{w-s} \mathcal{M} \varphi(w \cdot s) \, ds
$$

$$
= \frac{1}{|W|} \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + i a^*} E_s \left( \sum_w \frac{1}{c_w(s)} \mathcal{M} \varphi(w \cdot s) \right) \, ds
$$

On $\rho + i a^*$, we have $\frac{1}{c_w(s)} = c_w(2\rho - s)$. Therefore,

$$
\sum_w \frac{1}{c_w(s)} \mathcal{M} \varphi(w \cdot s) = \sum_w c_w(2\rho - s) \mathcal{M} \varphi(w \cdot s) = \mathcal{M} c_B \Psi_\varphi(s)
$$

This gives the desired spectral decomposition,

$$
\Psi_\varphi = \frac{1}{|W|} \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + i a^*} \mathcal{M} \Psi_\varphi(s) \, ds
$$

$$
= \frac{1}{|W|} \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + i a^*} E_s \cdot \langle \Psi_\varphi, E_{2\rho-s} \rangle \Gamma \backslash G \, ds
$$

\[\square\]

**Proposition 3.** The map $f \to (s \to \langle f, E_s \rangle)$ is an inner-product-preserving map from the Hilbert-space span of the pseudo-Eisenstein series to its image in $L^2(\rho + i a^*)$.

**Proof.** Let $f \in C_c^\infty(\Gamma \backslash G)$, $\varphi \in C_c^\infty(N \backslash G)$, and assume $\Psi_\varphi$ is orthogonal to residues of $E_s$ above $\rho$. Using the expression for $\Psi_\varphi$ in terms of Eisenstein series,

$$
\langle \Psi_\varphi, f \rangle = \frac{1}{|W|} \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + i a^*} \langle \Psi_\varphi, E_{2\rho-s} \rangle \cdot E_s ds, f \rangle
$$

$$
= \frac{1}{|W|} \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + i a^*} \langle \Psi_\varphi, E_{2\rho-s} \rangle \cdot \langle E_s, f \rangle \, ds
$$

\[\square\]

The map

$$
\Psi_\varphi \to \langle \Psi_\varphi, E_{2\rho-s} \rangle
$$

with $s = \rho + it$ and $t \in a^*$, produces functions

$$
u(t) = \langle \Psi_\varphi, E_{\rho-it} \rangle
$$

satisfying

$$
u(w t) = \langle \Psi_\varphi, E_{2\rho-w-s} \rangle = \langle \Psi_\varphi, E_{w-2\rho-s} \rangle = \langle \Psi_\varphi, \frac{E_{2\rho-s}}{c_w(2\rho-s)} \rangle
$$

$$
= c_w(s) \cdot u(t) \quad \text{for all} \ w \in W
$$

since

$$
c_w(2\rho - s) = \frac{1}{c_w(s)} = \frac{1}{c_w(s)}
$$

on $\rho + i a^*$.

**Proposition 4.** Any $u \in L^2(\rho + i a^*)$ satisfying $u(w t) = c_w(s) \cdot u(t)$ for all $w \in W$ is in the image.
Proof. First, for compactly-supported \( u \) meeting this condition, we claim
\[
\Psi_u = \frac{1}{|W|} \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + ia^*} u(t) \cdot E_{\rho + it} \, dt \neq 0
\]
It suffices to show \( c_B \Psi_u \) is not 0. With \( s = \rho + it \), the relation implies \( u(t) E_{2\rho-s} \) is invariant by \( W \). Let
\[
C = \{ t \in a^* : \langle t, \alpha \rangle > 0 \text{ for all simple } \alpha > 0 \}
\]
be the positive Weyl chamber in \( a^* \), where \( \langle , \rangle \) is the Killing form transported to \( a^* \) by duality. Then
\[
\Psi_u = \frac{1}{|W|} \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + ia^*} u(t) \cdot E_s \, dt = \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + iC} u(t) \cdot E_s \, dt
\]
Since \( u(tw) = u(t) \cdot c_w(\rho + it) \), the constant term of \( \Psi_u \) is
\[
c_B \Psi_u = \frac{1}{(2\pi i)^{\dim a}} \int_{\rho + ia^*} u(t) \cdot a^* \, dt
\]
This Fourier transform does not vanish for non-vanishing \( u \). \( \square \)

Given \( G = GL_4(\mathbb{R}) \), \( \Gamma = GL_4(\mathbb{Z}) \), and \( K = O_4(\mathbb{R}) \), it is necessary to invoke the complete spectral decomposition of \( L^2(\Gamma \backslash G/K) \), that cusps and cuspidal data Eisenstein series attached to non-minimal parabolic Eisenstein series attached to non-minimal parabolics, and their \( L^2 \) residues, as well as the minimal-parabolic pseudo-Eisenstein series, span \( L^2(\Gamma \backslash G/K) \). And we must demonstrate the orthogonality of integrals of minimal-parabolic Eisenstein series to all other spectral components.

We now decompose the pseudo-Eisenstein series with cuspidal data. We carry this out for the 3, 1 pseudo-Eisenstein series, 2, 2 pseudo-Eisenstein series, and 2, 1, 1 pseudo-Eisenstein series with cuspidal data. This follows a similar pattern as the spectral decomposition. Let \( P = P^{3,1} \). We decompose \( P^{3,1} \) and \( P^{1,3} \) pseudo-Eisenstein series with cuspidal support. The data for a \( P \) pseudo-Eisenstein series is smooth, compactly-supported, and left \( Z_k M_k \backslash N_k \)-invariant. For now, we assume that the data is spherical, i.e. right \( K \)-invariant. This means that the function is determined by its behavior on \( Z_k M_k \backslash N_k \). In contrast to the minimal parabolic case, this is not a product of copies of \( GL_1 \), so we cannot simply use the \( GL_1 \) spectral theory (Mellin inversion) to accomplish the decomposition. Instead, this quotient is isomorphic to \( GL_3(k) \backslash GL_3(k) \), so we will use the spectral theory for \( GL_3 \). If \( \eta \) is the data for a \( P^{3,1} \) pseudo-Eisenstein series \( \Psi_\eta \), we can write \( \eta \) as a tensor product \( \eta = f \otimes \mu \) on
\[
Z_{GL_3(k)} GL_3(k) \backslash GL_3(k) \cdot Z_{GL_3(k)} \backslash Z_{GL_3(k)}
\]
Saying that the data is cuspidal means that \( f \) is a cusp form. Similarly, the data \( \varphi = \varphi_{F,s} \) for a \( P^{2,1} \)-Eisenstein series is the tensor product of a \( GL_3 \) cusp form \( F \) and a character \( \chi_s = |.|^s \) on \( GL_1 \). We show that \( \Psi_{f,\eta} \) is the superposition of Eisenstein series \( E_{F,s} \), where \( F \) ranges over an orthonormal basis of cusp forms and \( s \) is on the critical line.
Proposition 5. The pseudo-Eisenstein series $\Psi_{f,\eta}$ admits a spectral decomposition

$$
\Psi_{f,\eta} = \sum_F \int_s \langle \Psi_{f,\eta}, E_{F,s} \rangle \cdot E_{F,s} \, ds
$$

where the sum is over spherical cuspforms $F$ on $GL_3(k) \backslash GL_3(A)$.

Proof. Using the spectral expansions of $f$ and $\eta$,

$$
\eta = f \otimes \eta = \left( \sum_{\text{cims } F} \langle f, F \rangle \cdot \int_s \langle \mu, \chi_s \rangle \cdot \chi_s \, ds \right) = \sum_{\text{cims } F} \int_s \langle \eta_{f,\mu}, \varphi_{F,s} \rangle \cdot \varphi_{F,s} \, ds
$$

So the pseudo-Eisenstein series can be re-expressed as a superposition of Eisenstein series

$$
\Psi_{f,\eta}(g) = \sum_{\gamma \in P_k \backslash G_k} \eta_{f,\mu}(\gamma g)
$$

$$
= \sum_{\gamma \in P_k \backslash G_k} \sum_{\text{cims } F} \int_s \langle \eta_{f,\mu}, \varphi_{F,s} \rangle \cdot \varphi_{F,s}(\gamma g) \, ds
$$

$$
= \sum_{\text{cims } F} \int_s \langle \eta_{f,\mu}, \varphi_{F,s} \rangle \cdot \sum_{\gamma \in P_k \backslash G_k} \varphi_{F,s}(\gamma g) \, ds
$$

$$
= \sum_{\text{cims } F} \int_s \langle \eta_{f,\mu}, \varphi_{F,s} \rangle \cdot E_{F,s} \, ds
$$

The coefficient $\langle \eta, \varphi \rangle_{GL_3}$ is the same as the pairing $\langle \Psi_{\eta}, E_{\varphi} \rangle_{GL_4}$, since

$$
\langle \Psi_{\eta}, E_{\varphi} \rangle = \langle c_P(\Psi_{\eta}), \varphi \rangle = \langle \eta, \varphi \rangle
$$

So the spectral decomposition is

$$
\Psi_{f,\eta} = \sum_{\text{cims } F} \int_s \langle \Psi_{f,\eta}, E_{F,s} \rangle \cdot E_{F,s} \, ds
$$

It now remains to show that pseudo-Eisenstein series for the associate parabolic, $Q = P^{1,3}$ can also be decomposed into superpositions of $P$-Eisenstein series. Notice that in the decomposition above, when we decomposed $P$-pseudo-Eisenstein series into genuine $P$-Eisenstein series, we did not use the functional equation to fold up the integral, as in the case of minimal parabolic pseudo-Eisenstein series. For maximal parabolic Eisenstein series, the functional equation does not relate the Eisenstein series to itself, but rather the Eisenstein series of the associate parabolic. We will use this functional equation to obtain the decomposition of associate parabolic pseudo-Eisenstein series. The functional equation is

$$
E_{F,s}^Q = b_{F,s} \cdot E_{F,1-s}^P
$$

where $b_{F,s}$ is a meromorphic function that appears in the computation of the constant term along $P$ of the $Q$-Eisenstein series.

Proposition 6. The pseudo-Eisenstein series $\Psi_{f,\mu}^Q$ admits a spectral decomposition

$$
\Psi_{f,\mu}^Q = \sum_F \int_s \langle \Psi_{f,\mu}^Q, E_{F,1-s}^P \rangle \cdot \lvert b_{F,1-s} \rvert^2 \cdot E_{F,1-s}^P
$$

where $F$ ranges over an orthonormal basis of cuspforms.
Proof. We consider a $Q$-pseudo-Eisenstein series $\Psi^Q_{f,\mu}$ with cuspidal data. By the same arguments used above to obtain the decomposition of $P$-pseudo-Eisenstein series, we can decompose $\Psi^Q_{f,\mu}$ into a superposition of $Q$-Eisenstein series,

$$\Psi^Q_{f,\mu}(g) = \sum_{\text{cims }F} \int_s \langle \eta_f, \varphi_{F,s} \rangle \cdot E^Q_{F,s}(g)$$

Now using the functional equation,

$$\Psi^Q_{f,\mu}(g) = \sum_{\text{cims }F} \int_s \langle \Psi^Q_{f,\mu}, b_{F,s} \cdot E^P_{F,1-s} \rangle \cdot b_{F,s} \cdot E^P_{F,1-s}$$

$$= \sum_{\text{cims }F} \int_s \langle \Psi^Q_{f,\mu}, E^P_{F,1-s} \rangle \cdot |b_{F,s}|^2 \cdot E^P_{F,1-s}$$

giving the proposition. \(\square\)

So we have a decomposition of $Q$-pseudo-Eisenstein series (with cuspidal data) into a $P$-Eisenstein series (with cuspidal data). In order to use the functional equation we did have to move some contours, but in this case there are no poles, so we did not pick up any residues. Likewise, if $\eta$ is the data for a $P^{2,1,1}$ pseudo-Eisenstein series $\Psi_\eta$, we can write $\eta$ as a tensor product $\eta = f \otimes \mu_1 \otimes \mu_2$ on

$$Z_{GL_4(k)} \backslash Z_{GL_2(k)} \times Z_{GL_4(k)} \times Z_{GL_1(k)}$$

Similarly, the data $\varphi = \varphi_{F,s_1,s_2}$ for a $P^{2,1,1}$-Eisenstein series is the tensor product of a $GL_2$ cuspform and characters $\chi_{s_1}$ and $\chi_{s_2}$ on $GL_1$. We show that $\Psi_{f,\mu}$ is the superposition of Eisenstein series $E_{F,s_1,s_2}$ where $F$ ranges over an orthonormal basis of cusp forms and $s_1$ and $s_2$ are on the vertical line.

**Proposition 7.** The $2,1,1$ pseudo-Eisenstein series $\Psi_{f,\mu_1,\mu_2}$ admits a spectral expansion

$$\Psi_{f,\mu_1,\mu_2} = \sum_{F} \int_{s_1} \int_{s_2} \langle \eta_{f,\mu_1,\mu_2}, \varphi_{F,s_1,s_2} \rangle \cdot E_{F,s_1,s_2}$$

where $F$ ranges over an orthonormal basis of cuspforms.

**Proof.** Using the spectral expansions of $f$ and $\mu$,

$$\eta = f \otimes \mu_1 \otimes \mu_2 = \sum_{\text{cims }F} \langle f, F \rangle \cdot \left( \int_{s_1} \langle \mu_1, \chi_{s_1} \rangle \cdot \chi_{s_1} \, ds_1 \right) \cdot \left( \int_{s_2} \langle \mu_2, \chi_{s_2} \rangle \cdot \chi_{s_2} \, ds_2 \right)$$

$$= \sum_{\text{cims }F} \int_{s_1} \int_{s_2} \langle \eta_f, \mu_1, \mu_2, \varphi_{F,s_1,s_2} \rangle \cdot \varphi_{F,s_1,s_2} \, ds_1 \, ds_2$$
Therefore, the pseudo-Eisenstein series can be re-expressed as a (double) superposition of Eisenstein series.

\[ \Psi_{f,\mu_1,\mu_2} = \sum_{\gamma \in P_k \setminus G_k} \eta_{f,\mu_1,\mu_2}(\gamma g) \]

\[ = \sum_{\gamma \in P_k \setminus G_k} \sum_{\text{cfsms } F} \int_{s_1} \int_{s_2} \langle \eta_{f,\mu_1,\mu_2}, \varphi_{F,s_1,s_2} \rangle \cdot \varphi_{F,s_1,s_2}(\gamma g) \, ds_1 \, ds_2 \]

\[ = \sum_{\text{cfsms } F} \int_{s_1} \int_{s_2} \langle \eta_{f,\mu_1,\mu_2}, \varphi_{F,s_1,s_2} \rangle \sum_{\gamma \in P_k \setminus G_k} \varphi_{F,s_1,s_2}(\gamma g) \, ds_1 \, ds_2 \]

\[ = \sum_{\text{cfsms } F} \int_{s_1} \int_{s_2} \langle \eta_{f,\mu_1,\mu_2}, \varphi_{F,s_1,s_2} \rangle \cdot E_{F,s_1,s_2}(g) \]

Finally, if \( \eta \) is the data for a \( P^{2,2} \) pseudo-Eisenstein series \( \Psi_{\eta} \), we can write

\[ \eta_{f,g,\mu} = f \otimes g \otimes \mu \]
on Z_{GL_4}(\mathbb{A})/Z_{GL_2}(\mathbb{A}) \times Z_{GL_2}(\mathbb{A})

where \( f \) and \( g \) are cuspforms, and \( \mu \) is a compactly-supported smooth function on \( GL(1) \). Similarly, the data \( \varphi = \varphi_{f_1,f_2,s} \) for a \( P^{2,2} \)-Eisenstein series is the tensor product of \( GL(2) \) cuspforms \( f_1 \) and \( f_2 \) and a character \( \chi_s \).

**Proposition 8.** The \( 2,2 \) pseudo-Eisenstein series \( \Psi_{\eta} \) has a spectral expansion in terms of \( 2,2 \) Eisenstein series

\[ \Psi_{\eta} = \sum_{F_1,F_2} \int_{s} \langle \eta_{f,g,\mu}, \varphi_{F_1,F_2,s} \rangle E_{F_1,F_2,s} \, ds \]

where \( F_1 \) and \( F_2 \) are cuspforms on \( GL(2) \).

**Proof.** Writing

\[ \eta = f \otimes g \otimes \mu = \left( \sum_{\text{cfsms } F} \langle f,F \rangle \cdot F \right) \left( \sum_{\text{cfsms } F} \langle g,F \rangle \cdot F \right) \left( \int_s \langle \mu,\chi_s \rangle \cdot \chi_s \right) \]

\[ = \sum_{\text{cfsms } F_1,F_2} \int_s \langle \eta_{f,g,\mu}, \varphi_{F_1,F_2,s} \rangle \cdot \varphi_{F_1,F_2,s} \, ds \]

As before, the corresponding pseudo-Eisenstein series will unwind

\[ \Psi_{\eta} = \sum_{\gamma \in P_k \setminus G_k} \eta_{f,g,\mu}(\gamma g) = \sum_{\text{cfsms } F_1,F_2} \int_{s} \langle \eta_{f,g,\mu}, \varphi_{F_1,F_2,s} \rangle \cdot E_{F_1,F_2,s} \, ds \]

Recall the construction of \( 2,2 \) pseudo-Eisenstein series. Let \( \phi \in C_c^\infty(\mathbb{R}) \) and let \( f \) be a spherical cuspform on \( GL_2 \) with trivial central character. Let

\[ \varphi\left( \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \right) = \phi\left( \frac{|\det A|}{|\det D|} \right) \cdot f(A) \cdot T(D) \]
extending by right $K$-invariance to be made spherical. Define the $P^{2,2}$ pseudo-Eisenstein series by
\[ \Psi_\varphi(g) = \sum_{\gamma \in \mathcal{P}_k \setminus G_k} \varphi(\gamma g) \]
We recall the construction of $2,1,1$ pseudo-Eisenstein series. Let $f$ be a spherical cuspform on $GL_2(\mathbb{A}) \setminus GL_2(\mathbb{A})$, and let $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R})$. Let
\[ \varphi_{f,\phi_1,\phi_2} \left( \begin{pmatrix} A & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \right) = f(A) \cdot \phi_1(\frac{\det A}{b^2}) \cdot \phi_2(\frac{\det A}{c^2}) \]
The $2,1,1$ pseudo-Eisenstein series with this data is
\[ \Psi_\varphi = \sum_{\gamma \in \mathcal{P}_k \setminus G_k} \varphi_{f,\phi_1,\phi_2}(\gamma g) \]

**Proposition 9.** The pseudo-Eisenstein series $\Psi_\varphi^{2,2}$ is orthogonal to all other pseudo-Eisenstein series in Sob$(+1)$.

**Proof.** Recall by [MW p.100] that
\[ \langle \Psi_\varphi^{2,2}, \Psi_\psi^{2,1,1} \rangle_{L^2} = 0 \]
Let us now check that they’re also orthogonal in the $+1$-Sobolev space. Note that
\[ \langle \Psi_\varphi^{2,2}, \Psi_\psi^{2,1,1} \rangle_{+1} = \langle \Psi_\varphi^{2,2}, \Psi_\psi^{2,1,1} \rangle_{L^2} + \langle \Delta \Psi_\varphi^{2,2}, \Psi_\psi^{2,1,1} \rangle_{L^2} \]
Since the first summand is zero, it suffices to prove that the second is zero. To this end, we rewrite the Casimir operator
\[ \Omega = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 \]
where
\[ \Omega_1 = \frac{1}{2} H_{1,2}^2 + E_{1,2}E_{2,1} + E_{2,1}E_{1,2} \]
and
\[ \Omega_2 = \frac{1}{2} H_{3,4}^2 + E_{3,4}E_{4,3} + E_{4,3}E_{3,4} \]
while
\[ \Omega_3 = \frac{1}{4} H_{1,2,3,4}^2 \]
We let $\Omega_4$ be the remaining terms appearing in the expression of Casimir. We prove that application of $\Omega$ to $\Psi_\varphi$ produces another function in the span of $2,2$ pseudo-Eisenstein series. Being in the span of $2,2$ pseudo-Eisenstein series renders $\Omega \Psi_\varphi$ orthogonal to all other non-associate pseudo-Eisenstein series. We will prove that when restricted to $G/K$, $\Omega_1$ acts as the $SL_2$-Laplacian on the cuspform $\tilde{f}$, $\Omega_2$ acts as the $SL_2$-Laplacian on $\tilde{f}$, while $\Omega_3$ acts as a second derivative on the test function. Indeed, let
\[ \Omega_1 = \frac{1}{2} H_{1,2}^2 + E_{1,2}E_{2,1} + E_{2,1}E_{1,2} \]
where $H_{1,2} = \text{diag}(1, -1, 0, 0)$ and $E_{i,j}$ is the matrix with 1 in the $ij$th position and 0’s elsewhere. We check how $H_{1,2}$ acts on smooth functions on $\varphi$. Let
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad D = \begin{pmatrix} f & g \\ h & i \end{pmatrix} \]
Observe that

\[ H_{1,2} \cdot \varphi\left( \begin{pmatrix} A & \ast \\ 0 & D \end{pmatrix} \right) = \frac{d}{dt} \bigg|_{t=0} \varphi\left( \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & f & g \\ 0 & 0 & h & i \end{pmatrix} , \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \]

This is

\[ \frac{d}{dt} \bigg|_{t=0} \varphi\left( \begin{pmatrix} ae^t & be^t & 0 & 0 \\ ce^t & de^t & 0 & 0 \\ 0 & 0 & f & g \\ 0 & 0 & h & i \end{pmatrix} \right) = \frac{d}{dt} \bigg|_{t=0} \varphi\left( \frac{\text{det} A}{\text{det} D} \right)^2 \cdot f\left( \begin{pmatrix} ae^t & be^{-t} \\ ce^t & de^{-t} \end{pmatrix} \right) \cdot f(D) \]

Use Iwasawa coordinates on the upper left hand \( GL(2) \) block of the Levi component, namely

\[ n_{x_i} = \begin{pmatrix} 1 & x_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad m_{y_1} = \begin{pmatrix} \sqrt{y_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{y_1}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

As in the discussion for \( SL_2(\mathbb{R}) \),

\[ (H_{1,2} f)(n_{x_i} m_{y_1}) = 2y_1 \frac{\partial}{\partial y_1} f(n_{x_i} m_{y_1}) \]

Therefore, letting \( \Delta_1 \) be \( \Omega_1 \) restricted to \( G/K \), we see that the effect of \( \Delta_1 \) on the cuspform \( f \) is just

\[ \Delta_1(f) = y_1 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1} \right) f = \lambda_f \cdot f \]

Therefore,

\[ \Delta_1(\varphi_{\phi,f,T}) = \varphi_{\phi,\lambda_f,T} = \lambda_f \cdot \varphi_{\phi,f,T} \]

A similar argument which uses \( H_{3,4}, E_{3,4} \) and \( E_{4,3} \) as the standard basis in the lower right \( 2 \times 2 \) block, shows that, for \( \Delta_2 \) the restriction of \( \Omega_2 \) to smooth functions on \( G/K \),

\[ \Delta_2(\varphi_{\phi,f,T}) = \varphi_{\phi,f,T} = \lambda_T \varphi_{\phi,\lambda_f,T} \]

It remains to check the effect of \( \Omega_3 = \frac{1}{4} H^2_{1,2,3,4} \). Observe that

\[ H_{1,2,3,4} \varphi\left( \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & f & g \\ 0 & 0 & h & i \end{pmatrix} \right) = \frac{d}{dt} \bigg|_{t=0} \varphi\left( \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & f & g \\ 0 & 0 & h & i \end{pmatrix} , \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix} \right) \]

Yet this is just

\[ \frac{d}{dt} \bigg|_{t=0} \varphi\left( \begin{pmatrix} ae^t & be^t & 0 & 0 \\ ce^t & de^t & 0 & 0 \\ 0 & 0 & fe^{-t} & ge^{-t} \\ 0 & 0 & he^{-t} & ie^{-t} \end{pmatrix} \right) \]

Which gives

\[ = \frac{d}{dt} \bigg|_{t=0} \varphi\left( \begin{pmatrix} ae^t & be^t \\ ce^t & de^t \end{pmatrix} \right) \cdot T\left( \begin{pmatrix} fe^{-t} & ge^{-t} \\ he^{-t} & ie^{-t} \end{pmatrix} \right) = 2 \cdot \varphi' \cdot f(A) \cdot T(D) \]
since both $f$ and $\mathcal{J}$ have trivial central character. Therefore, the effect of $\frac{1}{4}H_{1,2,3,4}$ as a differential operator on $\varphi_{\phi, f, \mathcal{J}}$ is

$$\frac{1}{4}H_{1,2,3,4} \cdot \varphi_{\phi, f, \mathcal{J}} = \varphi_{\phi''', f, \mathcal{J}}$$

That is,

$$\Delta_3 \varphi_{\phi, f, \mathcal{J}} = \varphi_{\phi''', f, \mathcal{J}}$$

Together the effect of the three differential operators is

$$(\Delta_1 + \Delta_2 + \Delta_3) \varphi_{\phi, f, \mathcal{J}} = \varphi_{(\lambda f + \lambda \mathcal{M}) + \phi'''', f, \mathcal{J}}$$

Therefore,

$$(\Delta_1 + \Delta_2 + \Delta_3) (\Psi \varphi_{\phi, f, \mathcal{J}}) = \Psi \varphi_{(\lambda f + \lambda \mathcal{M}) + \phi'''', f, \mathcal{J}}$$

The operator $\Delta_4$ acts by 0 on the vector $\varphi_{\phi, f, \mathcal{J}}$. Therefore,

$$\Delta \Psi \varphi_{\phi, f, \mathcal{J}} = \Psi \varphi_{(\lambda f + \lambda \mathcal{M}) + \phi'''', f, \mathcal{J}}$$

The function

$$\Psi \varphi_{(\lambda f + \lambda \mathcal{M}) + \phi'''', f, \mathcal{J}}$$

is another $2, 2$ pseudo-Eisenstein series because $(\lambda f + \lambda \mathcal{M}) + \phi'''$ is another function in $C^\infty_c(\mathbb{R})$, so [MW, p.100] applies again to give

$$\langle \Psi \varphi_{(\lambda f + \lambda \mathcal{M}) + \phi'''', f, \mathcal{J}}, \psi^{2,1,1} \rangle_{L^2} = 0$$

Therefore,

$$\langle \Delta \Psi \varphi_{\phi, f, \mathcal{J}}, \psi^{2,1,1} \rangle_{L^2} = 0$$

proving that the pseudo-Eisenstein series are orthogonal in the $+1$-index Sobolev space. An inductive argument shows that they are orthogonal in every Sobolev space.

An analogous argument shows that $2, 2$ pseudo-Eisenstein series are orthogonal to $3, 1$ pseudo-Eisenstein series, as well as $1, 1, 1$ pseudo-Eisenstein series. □

We turn our attention to the $3, 1$-Eisenstein series.

**Proposition 10.** $3, 1$ pseudo-Eisenstein series are orthogonal to all other (non-associate) pseudo-Eisenstein series in Sob$(+1)$.

**Proof.** We review the construction of $3, 1$ pseudo-Eisenstein series with cuspidal and test function data. Let $f_1$ be a spherical cuspform on $GL_3(k) \backslash GL_3(\mathbb{A})$ and $\phi \in C^\infty_c(\mathbb{R})$. Consider the vector

$$\varphi_{f_1, \phi}(\begin{pmatrix} A & * \\ 0 & d \end{pmatrix}) = f(A) \cdot \phi(\frac{\det A}{d^3})$$

Working in $GL_4$ consider the element

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}_4(\mathbb{R})$$
We determine the effect of $H_1$ as a differential operator on $\varphi_{f,\phi}$. To this end, let

$$n_{x_1x_2x_3} = \begin{pmatrix} 1 & x_1 & x_2 & 0 \\ 0 & 1 & x_3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad m_{y_1y_2y_3y_4} = \begin{pmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & y_4 \end{pmatrix}$$

Then

$$H_1 \cdot \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4}) = \frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4}) \left( \begin{array}{c} e^t \\ 0 \\ 0 \\ 0 \end{array} \right)$$

This is

$$\frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4}) = y_1 \frac{\partial}{\partial y_1} \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4})$$

Therefore,

$$H_1 \cdot \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4}) = y_1 \frac{\partial}{\partial y_1} \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4})$$

The effect of $H_2$ and $H_3$ is computed similarly. That is

$$H_2 \cdot \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4}) = y_2 \frac{\partial}{\partial y_2} \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4})$$

while

$$H_3 \cdot \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4}) = y_3 \frac{\partial}{\partial y_3} \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4})$$

With notation as before, we determine the effect of $E_{1,2}$ as a differential operator. Observe that

$$E_{1,2} \cdot \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4}) = \frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4}) \left( \begin{array}{c} 1 \\ t \end{array} \right)$$

This is just

$$\frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi}(n_{x_1+ytx_2x_3}m_{y_1y_2y_3y_4}) = y_1 \frac{\partial}{\partial x_1} \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4})$$

Therefore, the effect of $E_{1,2}$ is $y_1 \frac{\partial}{\partial x_1}$, and $E_1$ differentiates only the cuspform $f$. Similar arguments show that the effect of $E_{1,3}$ as a differential operator is

$$E_{1,3} \rightarrow y_2 \frac{\partial}{\partial x_2}$$

and

$$E_{2,3} \rightarrow y_3 \frac{\partial}{\partial x_3}$$

Observe that $E_{1,4}$, $E_{2,4}$, and $E_{3,4}$ act by 0 on $\varphi_{f,\phi}$. We prove this for $E_{1,4}$, the argument being identical for $E_{2,4}$ and $E_{3,4}$. Note

$$E_{1,4} \cdot \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4}) = \frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi}(n_{x_1x_2x_3}m_{y_1y_2y_3y_4}) \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ t \end{array} \right)$$
Let \( \Omega \) be a test function datum, its effect as a differential operator on \( \phi \) irreducible unramified principal series generated by \( f \) with the test function datum. Since \( \Omega \) \( \in \) to the trace pairing, define an element \( \Omega_2 \) where \( \Omega_2 \) is a test function datum. If \( \{X_i\} \) is a basis of \( gl_4(\mathbb{R}) \) and \( \{X_i\} \) is the dual basis relative to the trace pairing, define an element \( \Omega \in U_g \) by

\[
\Omega = \sum_i X_i X_i^* \]

Let \( \Omega_1 \) be the element of \( Z_{gl_1} \) given by

\[
\Omega_1 = \frac{1}{2} H_1^2 + \frac{1}{2} H_2^2 + \frac{1}{2} H_3^2 + E_{1,2} E_{2,1} + E_{1,3} E_{3,1} + E_{2,3} + E_{3,2}
\]

As shown above, this element differentiates the cuspidal-data, and does not interact with the test function datum. Since \( \Omega_1 \in Z_{gl_1} \), it acts by a scalar \( \lambda_f \) on the irreducible unramified principal series generated by \( f \). Then,

\[
\Omega = \Omega_1 + H_4 + \Omega_2
\]

where \( \Omega_2 = \Omega - \Omega_1 - H_4 \). Since \( \Omega_2 \) interacts with neither the cuspidal data nor the test function data, its effect as a differential operator on \( \varphi_f, \phi \) will be 0. Note that \( \Omega_1 \cdot \varphi_f, \phi = \varphi_{\lambda_f} f, \phi \), while \( H_4 \cdot \varphi_f, \phi = \varphi_f, \phi' \). Therefore,

\[
\Omega \varphi_f, \phi = \varphi_f, (\lambda_f + \phi')
\]

producing another 3, 1 pseudo-Eisenstein series, which is orthogonal to the 2, 1, 1 pseudo-Eisenstein series, 1, 1, 1 pseudo-Eisenstein series, and 2, 2 pseudo-Eisenstein series, by [MW, p.100].

\[ \square \]

Finally, we consider 2, 1, 1 pseudo-Eisenstein series. Let \( X_1, X_2, \ldots, X_n \) is a basis for \( gl_4(\mathbb{R}) \), with dual basis \( X_1^*, X_2^*, \ldots, X_n^* \) relative to the trace pairing. Let \( \Omega = \sum_i X_i \cdot X_i^* \in Z_g \), and let \( \Delta \) be \( \Omega \) descended to \( G/K \). We will show that application of \( \Delta \) to a 2, 1, 1 pseudo-Eisenstein series made with cuspidal data \( f \) and test functions \( \phi_1, \phi_2 \) produces another 2, 1, 1 pseudo-Eisenstein series. This
will prove that 2,1,1 pseudo-Eisenstein series are orthogonal to all other (non-associate) pseudo-Eisenstein series by [MW, p.100]. We recall the construction of 2,1,1 pseudo-Eisenstein series. Let \( f \) be a spherical cuspform on \( GL_2(k) \backslash GL_2(\mathbb{A}) \), and let \( \phi_1, \phi_2 \in C_c^\infty(\mathbb{R}) \). Let

\[
\varphi_{f,\phi_1,\phi_2}(\begin{pmatrix} A & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}) = f(A) \cdot \phi_1(\frac{\det A}{b^2}) \cdot \phi_2(\frac{\det A}{c^2})
\]

The 2,1,1 pseudo-Eisenstein series with this data is

\[
\Psi_\varphi = \sum_{\gamma \in P_\kappa \backslash G_k} \varphi_{f,\phi_1,\phi_2}(\gamma g)
\]

**Proposition 11.** The 2,1,1 pseudo-Eisenstein series \( \Psi_\varphi \) is orthogonal to all other (non-associate) pseudo-Eisenstein series in \( \text{Sob}(+1) \).

**Proof.** We consider basis elements of the Lie algebra \( \mathfrak{g}t_4(\mathbb{R}) \). Let \( E_{ij} \) be as before. Let \( H_i \) be the matrix with 1 on the \( i \)th diagonal entry and 0’s elsewhere. We consider the effect of the \( H_i \)’s as differential operators on \( \varphi_{f,\phi_1,\phi_2} \). It will be convenient to use an Iwasawa decomposition on the \( GL_2 \) block in the upper left hand corner. We will be considering right \( K \)-invariant functions, so \( \varphi \) is determined by its effect on \( n_x m_{y_1y_2} \) where

\[
n_x = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

and

\[
m_{y_1y_2} = \begin{pmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

We calculate \( H_1 \)’s effect on \( \varphi_{f,\phi_1,\phi_2}(n_x m_{y_1y_2}) \). Note that

\[
H_1 \cdot \varphi(n_x m_{y_1y_2}) = \frac{d}{dt} \bigg|_{t=0} \varphi(n_x m_{y_1e^t y_2}) = y_1 \frac{\partial}{\partial y_1} \varphi(n_x m_{y_1y_2})
\]

Similarly,

\[
H_2 \cdot \varphi(n_x m_{y_1y_2}) = y_2 \frac{\partial}{\partial y_2} \varphi(n_x m_{y_1y_2})
\]

Therefore, \( H_1 \) and \( H_2 \) differentiate the cuspform \( f \), and leave the functions \( \phi_1 \) and \( \phi_2 \) as they are. As before,

\[
E_{1,2} \cdot \varphi(n_x m_{y_1y_2}) = y_1 \frac{\partial}{\partial x} \varphi(n_x m_{y_1y_2})
\]

Let us consider the effect of \( H_3 \) as a differential operator on \( \varphi \). Observe that

\[
H_3 \cdot \varphi_{f,\phi_1,\phi_2}(n_x m_{y_1y_2y_3y_4}) = \frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi_1,\phi_2}(n_x m_{y_1y_2y_3e^t y_4})
\]

This is

\[
\frac{d}{dt} \bigg|_{t=0} f(A) \phi_1(\frac{\det A}{y_3^2} e^{-2t}) \phi_2(\frac{\det A}{y_4^2}) = -2 f(A) \cdot \phi_1'(\frac{\det A}{y_3^2}) \phi_2'(\frac{\det A}{y_4^2})
\]

Therefore,

\[
H_3 \cdot \varphi_{f,\phi_1,\phi_2}(n_x m_{y_1y_2y_3y_4}) = \varphi_{f,\phi_1,\phi_2,-2\phi_1',\phi_2}
\]

Similarly,

\[
H_4 \cdot \varphi_{f,\phi_1,\phi_2}(n_x m_{y_1y_2y_3y_4}) = \varphi_{f,\phi_1,-2\phi_2'}
\]
Observe that $E_{1,3}$ acts as 0 on $\varphi_{f,\phi_1,\phi_2}$. Indeed,

\[
E_{1,3} \cdot \varphi_{f,\phi_1,\phi_2} = \frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi_1,\phi_2}(\begin{pmatrix} z_1 & z_2 & 0 & 0 \\ z_3 & z_4 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}) (\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix})
\]

This is just

\[
\frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi_1,\phi_2}(\begin{pmatrix} z_1 & z_2 & 0 & 0 \\ z_3 & z_4 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}) = 0
\]

The effect of $E_{1,4}$ is computed similarly. Observe

\[
E_{1,4} \cdot \varphi_{f,\phi_1,\phi_2} = \frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi_1,\phi_2}(\begin{pmatrix} z_1 & z_2 & 0 & 0 \\ z_3 & z_4 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}) (\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix})
\]

Which is

\[
\frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi_1,\phi_2}(\begin{pmatrix} z_1 & z_2 & 0 & 0 \\ z_3 & z_4 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}) = 0
\]

The elements $E_{3,1}$, $E_{3,2}$, $E_{4,1}$ and $E_{4,2}$ also act as 0. To see that $E_{3,4}$ acts by 0, note

\[
E_{3,4} \cdot \varphi_{f,\phi_1,\phi_2} = \frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi_1,\phi_2}(\begin{pmatrix} z_1 & z_2 & 0 & 0 \\ z_3 & z_4 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}) (\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix})
\]

Which is

\[
\frac{d}{dt} \bigg|_{t=0} \varphi_{f,\phi_1,\phi_2}(\begin{pmatrix} z_1 & z_2 & 0 & 0 \\ z_3 & z_4 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}) = 0 = \frac{d}{dt} \bigg|_{t=0} f(A) \cdot \phi_1 (\frac{\det A}{y^2}) \phi_2 (\frac{\det A}{c^2}) = 0
\]

Likewise, $E_{4,3}$ acts by 0 as a differential operator. The terms which contribute non-trivially to the effect of the $PGL_4(\mathbb{R})$-Laplacian are

\[
(H_1^2 + H_2^2 + E_{1,2}E_{2,1} + E_{2,1}E_{1,2}) + H_3^2 + H_4^2
\]

the parenthetical expression acts by a scalar $\lambda_f$ on the cuspform $f$. That is,

\[
(H_1^2 + H_2^2 + E_{1,2}E_{2,1} + E_{2,1}E_{1,2}) \varphi_{f,\phi_1,\phi_2} = \varphi_{f,\phi_1,\phi_2}
\]

since $H_1^2 + H_2^2 + E_{1,2}E_{2,1} + E_{2,1}E_{1,2}$ is the Laplacian on $PGL_2(\mathbb{R})$. The remaining two terms in expression act as follows:

\[
H_3^2 \varphi_{f,\phi_1,\phi_2} = \varphi_{f,4\phi_1''}, \phi_2
\]

Therefore,

\[
(H_1^2 + H_2^2 + E_{1,2}E_{2,1} + E_{2,1}E_{1,2} + H_3^2 + H_4^2) \varphi_{f,\phi_1,\phi_2} = \varphi_{f,\phi_1,\phi_2} + \varphi_{f,4\phi_1''}, \phi_2 + \varphi_{f,4\phi_2''}
\]

Therefore, with $\Delta$ the $PGL_4(\mathbb{R})$-Laplacian,

\[
\Delta \varphi = \Psi_{f,\phi_1,\phi_2} + \Psi_{f,4\phi_1''}, \phi_2 + \Psi_{f,4\phi_2''}
\]
is again in the vector space spanned by $2, 1, 1$ pseudo-Eisenstein series, so is orthogonal to all other non-associate pseudo-Eisenstein series in $L^2$, as claimed. \qed

We review Maass-Selberg relations and the theory of the constant term for $GL_4$, as in [Harish-Chandra, p.75], [MW, p.100-101] and [Garrett 2011a]. Let $P = P^{2,2}$ be the standard, maximal parabolic subgroup

$$P^{2,2} = \begin{pmatrix} GL_2 & * \\ 0 & GL_2 \end{pmatrix}$$

with unipotent radical $N^P$ and standard Levi component $M^P$. The parabolic $P$ is self-associate. Let $f$ be an everywhere spherical cuspform on $GL_2(k) \backslash GL_2(A)$ with trivial central character and let $\varphi$ be the vector

$$\varphi(nmk) = \varphi_{s,f}(nmk) = |\det m_1|^{2s} |\det m_2|^{-2s} \cdot f(m_1) \cdot \overline{f}(m_2)$$

where

$$m = \begin{pmatrix} m_1 & * \\ 0 & m_2 \end{pmatrix}$$

with $m_1, m_2$ in $GL_2$, so that $m$ is in the standard Levi component $M$ of the parabolic subgroup $P$, $n \in N^P$ its unipotent radical, $k \in K$, and $| \cdot |$ is the idele norm.

**Definition 5.** The spherical Eisenstein series is

$$E_{s,f}^P(g) = E_{s,f}(g) = \sum_{\gamma \in P \backslash G_k} \varphi_{s,f}^P(\gamma \cdot g) \text{ for } \text{Re}(s) \gg 1$$

For $\text{Re}(s)$ sufficiently large, this series converges absolutely and uniformly on compacta. We define truncation operators. For a standard maximal proper parabolic $P = P^{2,2}$ as above, for $g = nmk$ with

$$m = \begin{pmatrix} m_1 & * \\ 0 & m_2 \end{pmatrix}$$

as above, $n \in N^P$ and $k \in O(4)$ define the spherical function

$$h^P(g) = h^P(pk) = \frac{|\det m_1|^2}{|\det m_2|^2} = \delta^P(nm) = \delta^P(m)$$

where $\delta^P$ is the modular function on $P$. For fixed large real $T$, the $T$-tail of the $P$-constant term of a left $N^P_k$-invariant function $F$

$$c^T_F(g) = \begin{cases} c_P F(g) & : h^P(g) \geq T \\ 0 & : h^P(g) \leq T \end{cases}$$

**Definition 6.** The truncation operator is

$$\Lambda^T E^P_\varphi = E^P_\varphi - E^P(c^T_P E^P_\varphi)$$

where

$$E^P_\varphi(g) = \sum_{\gamma \in P \backslash G_k} \varphi(\gamma g)$$

These are square-integrable, by the theory of the constant term([MW, pp.18-40], [Harish-Chandra]). The Maass-Selberg relations describe their inner product as follows. The inner product

$$\langle \Lambda^T E^P_\varphi, \Lambda^T E^P_\psi \rangle$$
of truncations $\Lambda^T E^P_\varphi$ and $\Lambda^T E^P_\psi$ of two Eisenstein series $E^P_\varphi$ and $E^P_\psi$ attached to cuspidal-data $\varphi$, $\psi$ on maximal proper parabolics $P = P^{2,2}$ is given as follows. The term $c_s$ refers to the quotient of Rankin-Selberg L-functions appearing in the constant term $c_P E^P_\varphi$. That is,
\[
c_s = \frac{L(2s - 1, \pi \otimes \pi')}{L(2s, \pi \otimes \pi')}
\]
as in [Langlands 544, Section 4] where $\pi$ is locally everywhere an unramified principal series isomorphic to the representation generated by the cuspform $f$ locally.

**Proposition 12. Maass-Selberg relations**

\[
\langle \Lambda^T E^P_{g_1}, \Lambda^T E^P_{g_2} \rangle = \langle g_1, g_2 \rangle \left( \frac{T^{s+\tau-1}}{s+\tau-1} + \frac{T^{s+(1-\tau)-1}}{s+(1-\tau)-1} \right)
\]

\[
+ \langle g_1^w, g_2^w \rangle c_{g_1}^w c_{g_2}^w \left( \frac{T^{(1-s)+\tau-1}}{s+(1-\tau)-1} + \frac{T^{(1-s)+(1-\tau)-1}}{(1-\tau)-1} \right)
\]

Following [M-W pp.18-40], an important consequence of the Maass-Selberg relations is that for a maximal, proper, self-associate parabolic $P$ in $GL_n$, on the half-plane $\text{Re}(s) \geq \frac{n}{2}$ the only possible poles are on the real line, and only occur if $\langle f, f^w \rangle \neq 0$. In that case, any pole is simple, and the residue is square-integrable. In particular, taking $f = f_o \times f_o$

\[
(\text{Res}_{s_n} E^P_{\varphi}, \text{Res}_{s_n} E^P_{\psi}) = \langle f_o, f_o \rangle^2 \cdot \text{Res}_{s_n} c_{s_n}^\varphi
\]
as in [Harish-Chandra, p.75]. The group $GL_4$ gives the first instance of non-constant, non-cuspidal contribution to the discrete spectrum; the residues of the Eisenstein series at its poles give Speh forms. Recall ([Langlands 544] Section 4, though he uses a different normalization), that the constant term is equal to

\[
| \frac{\det A}{\det D} |^s \cdot f(A) \cdot \overline{f(D)} + | \frac{\det A}{\det D} |^{1-s} \cdot \frac{\Lambda(2s - 1, \pi \otimes \pi')}{\Lambda(2s, \pi \otimes \pi')} \cdot \overline{f(A)} \cdot f(D)
\]
The $L$-function appearing in the numerator necessarily has a residue at the unique pole in the right half-plane. This residue of the Eisenstein series at this pole is the Speh form [Jacquet] attached to a $GL(2)$ cuspform $f$, and is in $L^2$.

We now compute the $2, 2$ constant term of the $2, 2$ Eisenstein series with cuspidal data $f$ and $\overline{f}$. Let $P = P^{2,2}$ be the self-associate standard parabolic in $G = GL_4$ with Levi component $GL_2 \times GL_2$. Let $f_1$ and $f_2$ be spherical cuspsforms on $GL_2(k) \backslash GL_2(\mathbb{A})$. Define the spherical vector

\[
\varphi_{s,f_1,f_2}(\begin{pmatrix} A & * \\ 0 & D \end{pmatrix}) = \left| \frac{\det A}{\det D} \right|^s \cdot f_1(A) \cdot f_2(D)
\]

and then extending to $G_\mathbb{A}$ by right $K_\mathbb{A}$-invariance and $Z_\mathbb{A}$-invariance everywhere locally. Define cuspidal-data Eisenstein series for $\text{Re}(s) \gg 1$ by

\[
E^P_{s,f_1,f_2}(g) = \sum_{\gamma \in \mathcal{P}_s \backslash G_\mathbb{A}} \varphi_{s,f_1,f_2}(\gamma g)
\]

**Proposition 13.** The $P$-constant term of the $P$-Eisenstein series $E^P_{s,f_1,f_2}(g)$ is given by

\[
c_P E^P_{s,f_1,f_2}(g) = \left| \frac{\det A}{\det D} \right|^s \cdot f_1(A) \cdot f_2(D) + \left| \frac{\det A}{\det D} \right|^{1-s} \cdot f_1(A) \cdot f_2(D) \cdot \frac{L(\pi_1 \otimes \pi_2, 2s - 1)}{L(\pi_1 \otimes \pi_2, 2s)}
\]
where \( \pi_1 \) is the \( G_k \)-representation generated by \( f_1 \) and \( \pi_2 \) is the \( G_k \)-representation generated by \( f_2 \).

**Proof.** The constant term of \( E_{s,f_1,f_2} \) along \( P \) is given by

\[
c_P E_{s,f_1,f_2}^P(g) = \int_{N_k \backslash N_k} E_{s,f_1,f_2}^P(n g) \, dn = \sum_{\xi \in P_k \backslash G_k / N_k} \int_{\xi^{-1} P_k \xi \cap N_k \backslash N_k} \varphi_{s,f_1,f_2}(\xi g) \, dn
\]

The double coset space \( P \backslash G / N \) surjects to \( W P \backslash W / W \) which has three double coset representatives, two of which give a nonzero contribution. The identity coset contributes a volume, which we will compute later. The nontrivial representative is \( \xi = \sigma_2 \sigma_3 \sigma_1 \sigma_2 \). Observe that \( \xi \cdot P_k \cdot \xi^{-1} \cap N_k = \{1\} \) so that

\[
c_P E_{s,f_1,f_2}^P(g) = \int_{N_k \backslash N_k} \varphi_{s,f_1,f_2}(n g) \, dn + \int_{N_k} \varphi_{s,f_1,f_2}(\xi g) \, dn
\]

To compute the contribution of the integral

\[
\int_{N_k} \varphi_{s,f_1,f_2}(\xi g) \, dn
\]

we must re-express the Eisenstein series representation-theoretically. To this end, let \( \pi_{f_1} = \otimes_{v} \pi_{f_1,v} \) be the representation of \( G_k \) generated by \( f_1 \) and let \( \pi_{f_2} = \otimes_{v} \pi_{f_2,v} \) be the \( G_k \)-representation generated by \( f_2 \). For places \( v \) outside a finite set \( S \), fix isomorphisms

\[
j_v : \text{Ind}_{f_1,v} \rightarrow \pi_{f_1,v}
\]

and

\[
l_v : \text{Ind}_{f_2,v} \rightarrow \pi_{f_2,v}
\]

Their tensor product \( j_v \otimes l_v \) is a representation of the Levi \( M = GL_2 \otimes GL_2 \). Extend representations of Levi components trivially to parabolics. A \( \pi_f \)-valued Eisenstein series is formed by a convergent sum

\[
E_{\pi}^P = \sum_{\gamma \in P_k \backslash G_k} \varphi \circ \gamma
\]

Let \( T = \otimes_v T_v : \varphi \rightarrow \int_{N_k} \varphi(\xi \gamma g) \, dn \). We have a chain of intertwinings
The intertwinings completely determined by computing its effect on the canonical spherical vector. The advantage of this set-up is that for $v$ outside the finite set $S$, the minimal parabolic unramified principal series has a canonical spherical vector, namely that

$$\bigotimes_{v \in S} \text{Ind}^{G_v}_{P_v}( (\pi_{f_1,v} \otimes \pi_{f_2,v}) v^s_{P_v} ) \otimes \bigotimes_{v \notin S} \text{Ind}^{G_v}_{B_v}( (\chi_{f_1,v} \otimes \chi_{f_2,v}) v^{s,s,s,-3s}_{B_v} )$$

iterated induction

$$\bigotimes_{P_v} \text{Ind}^{G_v}_{P_v}( (\pi_{f_1,v} \otimes \pi_{f_2,v}) v^s_{P_v} ) \otimes \bigotimes_{P_v} \text{Ind}^{G_v}_{P_v}( (\pi_{f_1,v} \otimes \pi_{f_2,v}) v^{s}_{P_v} )$$

iterated induction

$$\bigotimes_{1 \otimes (j_v \otimes \iota_v)}$$

$$\bigotimes_{1 \otimes (j_v \otimes \iota_v)}$$

$$\bigotimes_{1 \otimes (j_v \otimes \iota_v)}$$

The advantage of this set-up is that for $v$ outside the finite set $S$, the minimal parabolic unramified principal series has a canonical spherical vector, namely that spherical vector taking value 1 at $1 \in G_v$. Therefore the isomorphism $T_v$ can be completely determined by computing its effect on the canonical spherical vector. The intertwinings $T_v$ among minimal-parabolic principal series can be factored as compositions of similar intertwining operators attached to reflections corresponding to positive simple roots, each of which is completely determined by its effect on the canonical spherical vector in the unramified principal series. The simple reflection intertwinings' effect on the normalized spherical functions reduce to $GL_2$ computations.

Thus, with simple reflections

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and with corresponding root subgroups

$$N_{\sigma_1} = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad N_{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad N_{\sigma_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
The simple-reflection intertwinings

\[ S_{\sigma_1}f(g) = \int_{N_{\sigma_1}} f(\sigma_1 ng) \, dn \quad S_{\sigma_2}f(g) = \int_{N_{\sigma_2}} f(\sigma_2 ng) \, dn \]

\[ S_{\sigma_3}f(g) = \int_{N_{\sigma_3}} f(\sigma_3 ng) \, dn \]

are instrumental because we wish to compute the effect of

\[ S_{\sigma_2} \circ S_{\sigma_3} \circ S_{\sigma_1} \circ S_{\sigma_2} \]

on the normalized spherical vector in the unramified minimal-parabolic principal series \( I(s_1, s_2, s_3, s_4) \). Furthermore, \( S_{\sigma_\tau} = S_{\sigma} \circ S_{\tau} \)

Therefore, we must understand the effect of the individual \( S_{\sigma_i} \)'s. Recall that \( S_{\sigma_2} : I(s_1, s_2, s_3, s_4) \to I(s_1, s_3 + 1, s_2 - 1, s_4) \)

Similarly,

\[ S_{\sigma_1} : I(s_1, s_2, s_3, s_4) \to I(s_2 + 1, s_1 - 1, s_3, s_4) \]

and

\[ S_{\sigma_3} : I(s_1, s_2, s_3, s_4) \to I(s_1, s_2, s_4 + 1, s_3 - 1) \]

The normalized spherical function \( f^0 \in I(s_1, s_2, s_3, s_4) \) is mapped by \( S_{\sigma_1} \) to a multiple of the normalized spherical function in \( I(s_2 + 1, s_1 - 1, s_3, s_4) \). The constant is

\[ S_{\sigma_1}f^0(1) = \int f^0(\sigma_1 \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) \, dx = \int f^0(\begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) \, dx \]

Using the Iwasawa decomposition for \( GL_2(k_v) \), we show that this calculation reduces to a \( GL_2 \) calculation. Indeed, there is \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in the maximal compact of \( GL_2(k_v) \) such that

\[ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \]

Therefore,

\[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

From this, it follows that the constant \( S_{\sigma_1}f^0(1) \) with

\[ S_{\sigma_1} : I(s_1, s_2, s_3, s_4) \to I(s_2 + 1, s_1 - 1, s_3, s_4) \]

is the same as the constant in the intertwining from \( I(s_1, s_2) \to I(s_2 + 1, s_1 - 1) \) of \( GL_2 \) principal series, namely

\[ \varphi^0 \left( \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right) \, dx \]
where \( \varphi^0 \) is the normalized spherical vector in the \( GL_2 \) principal series. A similar argument applies to the other intertwining operators attached to other simple reflections. We recall the \( GL_2 \) computation below. At absolutely unramified finite places, \( \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) \in K_v = GL_2(\sigma_v) \) for \( x \leq 1 \). For \( x > 1 \),

\[
\left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -\frac{1}{x} \\ x & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & -\frac{1}{x} \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)
\]

Thus, with local parameter \( \varpi \) and residue field cardinality \( q \), since the measure of \( \{ x \in k_v : |x| = q^r \} \)

is \( (q-1)q^{r-1} \), we see that

\[
\int_{k_v} \varphi^0 \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) dx = \int_{|x| \leq 1} 1 dx + \int_{|x| > 1} \varphi^0 \left( \begin{array}{cc} 1 & -\frac{1}{x} \\ 0 & 1 \end{array} \right) dx
\]

This is

\[
1 + (1-q) \sum_{r \geq 1} q^r (1-s_{1+s_2}) = \frac{\zeta_v(s_1 - s_2 - 1)}{\zeta_v(s_1 - s_2)}
\]

with the Iwasawa-Tate unramified local zeta integral \( \zeta_v(s) \).

Using this \( GL_2 \) reduction, we see that

\( S_{\sigma_2} : I(s_1, s_2, s_3, s_4) \to I(s_1, s_3 + 1, s_2 - 1, s_4) \)

and maps the normalized spherical vector in \( I(s_1, s_2, s_3, s_4) \) to

\[
\frac{\zeta_v(s_2 - s_3 - 1)}{\zeta_v(s_2 - s_3)}
\]

times the normalized spherical function in \( I(s_1, s_3 + 1, s_2 - 1, s_4) \). Then

\( S_{\sigma_1} : I(s_1, s_3 + 1, s_2 - 1, s_4) \to I(s_3 + 2, s_1 - 1, s_2 - 1, s_4) \)

and sends the normalized spherical function in \( I(s_1, s_3 + 1, s_2 - 1, s_4) \) to

\[
\frac{\zeta_v(s_1 - s_3 - 2)}{\zeta_v(s_1 - s_3 - 1)}
\]

times the normalized spherical function in \( I(s_3 + 2, s_1 - 1, s_2 - 1, s_4) \). Then \( S_{\sigma_3} \)

maps the normalized spherical vector in \( I(s_3 + 2, s_1 - 1, s_2 - 1, s_4) \) to

\[
\frac{\zeta_v(s_2 - s_4 - 2)}{\zeta_v(s_2 - s_4 - 1)}
\]

times the normalized spherical vector in \( I(s_3 + 2, s_1 - 1, s_4 + 1, s_2 - 2) \). Finally,

\( S_{\sigma_2} : I(s_3 + 2, s_1 - 1, s_4 + 1, s_2 - 2) \to I(s_3 + 2, s_4 + 2, s_1 - 2, s_2 - 2) \) and sends the normalized spherical function in \( I(s_3 + 2, s_1 - 1, s_4 + 1, s_2 - 2) \) to

\[
\frac{\zeta_v(s_1 - s_4 - 3)}{\zeta_v(s_1 - s_4 - 2)}
\]

times the normalized spherical function in \( I(s_3 + 2, s_1 - 2, s_2 - 2) \). Altogether,

\( S_{\sigma_2} \circ S_{\sigma_3} \circ S_{\sigma_1} \circ S_{\sigma_2} \) maps the normalized spherical vector in \( I(s_1, s_2, s_3, s_4) \) to

\[
\frac{\zeta_v(s_2 - s_3 - 1)}{\zeta_v(s_2 - s_3)} \cdot \frac{\zeta_v(s_1 - s_3 - 2)}{\zeta_v(s_1 - s_3 - 1)} \cdot \frac{\zeta_v(s_3 - s_4 - 2)}{\zeta_v(s_2 - s_4 - 1)} \cdot \frac{\zeta_v(s_1 - s_4 - 3)}{\zeta_v(s_1 - s_4 - 2)}
\]

times the normalized spherical vector in the unramified principal series

\( I(s_3 + 2, s_4 + 2, s_1 - 2, s_2 - 2) \)
For \((s_1, s_2, s_3, s_4) = (s + s_{f_1}, s - s_{f_1}, -s + s_{f_2}, -s - s_{f_2})\) we get
\[
\frac{\zeta_v(s - s_{f_1} - (-s + s_{f_2}) - 1) \cdot \zeta_v(s + s_{f_1} - (-s + s_{f_2}) - 2)}{\zeta_v(s - s_{f_1} - (-s + s_{f_2})) \cdot \zeta_v(s + s_{f_1} - (-s + s_{f_2}) - 1)}
\]
the Rankin Selberg L-function
\[
\frac{L(\pi_1 \otimes \pi_2, 2s - 1)}{L(\pi_1 \otimes \pi_2, 2s)}
\]

3. Global Automorphic Sobolev Spaces

We recall basic ideas about global automorphic Sobolev spaces. For example, see Decelles [2011b], [Grubb], and [Garrett 2010]. Consider the group \(G = GL(4)\) defined over a number field \(k\). At each place \(v\), let \(K_v\) be the standard maximal compact subgroup of the \(v\)-adic points \(G_v\) of \(G\). That is, \(K_v = GL_4(\mathcal{O}_v)\) for nonarchimedean places \(v\) where \(\mathcal{O}_v\) denotes the local ring of integers, and \(K_v = O_4(\mathbb{R})\) for \(v\) real and \(K = U(n)\) for \(v\) complex. Consider the space \(C^\infty_c(\mathbb{A}_k^G/G_k, \omega)\) where \(\omega\) is a trivial central character. We define positive index global archimedean spherical automorphic Sobolev spaces as right \(K_\infty\)-invariant subspaces of completions of \(C^\infty_c(\mathbb{A}_k^G/G_k, \omega)\) with respect to a topology induced by norms associated to the Casimir operator \(\Omega\). The operator \(\Omega\) acts on the archimedean component \(f \in C^\infty_c(\mathbb{A}_k^G/G_k, \omega)\) by taking derivatives in the archimedean component. The norm \(\|f\|_{\ell}\) on \(C^\infty_c(\mathbb{A}_k^G/G_k, \omega)^K\) is
\[
\|f\|_{\ell} = \langle (1 - \Omega)^{\ell} f, f \rangle^{\frac{1}{2}}
\]
where \(\langle , \rangle\) gives the norm on \(L^2(\mathbb{A}_k^G/G_k, \omega)\), induces a topology on the space \(C^\infty_c(\mathbb{A}_k^G/G_k, \omega)^K\).

**Definition 7.** The completion \(H^\ell(\mathbb{A}_k^G/G_k, \omega)\) is the \(\ell\)-th global automorphic Sobolev space.

\(H^\ell(\mathbb{A}_k^G/G_k, \omega)\) is a Hilbert space with respect to this topology.

**Definition 8.** For \(\ell > 0\), the Sobolev space \(H^{-\ell}(\mathbb{A}_k^G/G_k, \omega)\) is the Hilbert space dual of \(H^\ell(\mathbb{A}_k^G/G_k, \omega)\).

Since the space of test functions is a dense subspace of \(H^{\ell}(\mathbb{A}_k^G/G_k, \omega)\) with \(\ell > 0\), dualizing gives an inclusion of \(H^{-\ell}(\mathbb{A}_k^G/G_k, \omega)\) into the space of distributions. The adjoints of the dense inclusions \(H^{\ell} \rightarrow H^{\ell-1}\) are inclusions
\[
H^{-\ell+1}(\mathbb{A}_k^G/G_k, \omega) \rightarrow H^{-\ell}(\mathbb{A}_k^G/G_k, \omega)
\]

4. Pre-trace formula estimates on compact periods

We give a standard argument. See, for example, [Iwaniec] and [Garrett 2010]. Set \(k = \mathbb{Q}\) throughout. Let \(\Theta\) be a \(k\)-subgroup of \(G\). Let \(|\Theta| = (\mathbb{A}_k \cap \Theta)\Theta_k \Theta_k\) and \(|G| = Z_{\mathbb{A}_k^G}/G_k/K_\infty\). For smooth \(f\) on \(Z_{\mathbb{A}_k^G}/G_k\), define the \(|\Theta|\)-period of \(f\) to be
\[
f_{\Theta, x} = \int_{[\Theta]} f(hx) \, dh
\]
Similarly, with \( \phi \) an automorphic form on \( \Theta_G \), the \([\Theta], x, \phi\)-period of \( f \) is

\[
(f, \overline{\phi}) = \int_{[\Theta]} \overline{\phi}(h) \cdot f(hx) \, dh
\]

For finite places \( v \), fix a compact open subgroup \( K_v \) of \( G_v \) such that at almost all places, \( K_v \) is the standard maximal compact subgroup of \( G_v \), and let \( K_{\text{fin}} = \prod_v K_v \). Let \( K = K_{\infty} \cdot K_{\text{fin}} \).

**Proposition 14.** The distribution given by integration along a compact quotient \( \Theta_G \) lies in \( H^{-s}(Z_G G_k \backslash G_k) \) for all

\[
s > \frac{\dim(G_{\infty}/K_{\infty}) - \dim(\Theta_{\infty}/K_{\Theta_{\infty}})}{2}
\]

**Proof.** Consider smooth \( f \) on \( Z_G G_k \backslash G_k \) generating unramified principal series at archimedean places. The usual action of compactly-supported measures \( \eta \) on suitable \( f \) on \( G_k \backslash G_k / K_{\infty} \) is given by

\[
(\eta \cdot f)(x) = \int_{G_k} \eta(g) f(xg) \, dg
\]

The \( \Theta_k \backslash \Theta_k x \)-period of \( \eta \cdot f \) admits a useful rearrangement

\[
(\eta \cdot f)_{\Theta_k \backslash \Theta_k} = \int_{Z_\Theta \Theta \backslash \Theta_k} (\eta \cdot f)(hx) \, dh = \int_{[\Theta]} \int_{G_k} \eta(g) f(hxg) \, dg \, dh
\]

\[
= \int_{[\Theta]} \int_{G_k} \eta(x^{-1}h^{-1}g) \, dg \, dh = \int_{[\Theta]} \int_{[G]} \sum_{\gamma \in G_k} \eta(x^{-1}h^{-1}\gamma g) f(g) \, dg \, dh
\]

\[
= \int_{[G]} f(g) \left( \int_{[\Theta]} \sum_{\gamma \in G_k} \eta(x^{-1}h^{-1}\gamma g) \, dh \right) \, dg
\]

Denote the inner sum and integral by \( q(g) = q_{\Theta \backslash \Theta}(g) \). For \( \eta \) a left and right \( K_{\text{fin}} \)-invariant measure, for \( f \) a spherical vector in a copy of a principal series, \( \eta \cdot f \) will be \( K_{\text{fin}} \)-invariant. Since the spherical vector in an irreducible representation is unique (up to scalar), \( \eta \cdot f = \lambda_f(\eta) \cdot f \) for some constant \( \lambda_f(\eta) \). Let \( \eta_{\infty} \) be the characteristic function of a shrinking ball \( B_\epsilon \) in \( G_\infty / K_\infty \) of geodesic radius \( \epsilon > 0 \) and at each finite place \( v \), let \( \eta_v \) be the characteristic function of \( K_v \). The ball \( B_\epsilon \) has \( v \)-adic components in \( K_v \) for almost all \( v \), and archimedean component lying within a ball of radius \( \epsilon \). Identify \( B_\epsilon \) with its pre-image \( B_\epsilon \cdot K_v \) in \( G_v \). Here, we make use of a \( G_\infty \)-invariant metric

\[
d(x, y) = \nu(x^{-1}y)
\]

on \( G_\infty / K_\infty \) where

\[
\nu(g) = \log \sup(|g|, |g^{-1}|)
\]

Here \( |\cdot| \) is the operator norm on the group \( G_v \) given by

\[
|T| = \sup_{u \leq 1} \|Tu\|
\]
Let \( \eta = \otimes_v \eta_v \). The action of such \( \eta \) changes the period by the eigenvalue. To see this, observe that

\[
(\eta_v \cdot f)(x) = \int_{K_v} \eta_v(k) f(gk) \, dk = \int_{K_v} f(g) \, dk = \text{vol}(K_v) \cdot f(g)
\]

Also, \( \eta_\infty \cdot f \) will be a spherical vector. Since the spherical vector is unique up to a constant multiple, \( \eta_\infty \cdot f = \lambda_\infty \cdot f \) for some scalar \( \lambda_\infty \). Therefore,

\[
(\eta \cdot f)_{\Theta,x} = \lambda_f(\eta) \cdot \text{vol}(K_{\text{fin}}) \cdot f_{\Theta,x}
\]

An upper bound for the \( L^2(Z_\kappa G_k \backslash G_k, \omega) \) norm of \( \eta \), and a lower bound for \( \lambda_f(\eta) \) contingent on restrictions on the spectral parameter of \( f \), yield, by Bessel’s inequality, an upper bound for a sum-and-integral of periods \( (f, \phi)_{\Theta,x} \) as follows. Estimate the \( L^2 \) norm of \( \eta \):

\[
\int_{[G]} |q(g)|^2 \, dg = \int_G \int_G \int_G \sum_{\gamma \in \mathcal{G}_k} \sum_{\gamma_2 \in \mathcal{G}_k} \eta(x^{-1} h^{-1} \gamma \gamma_2) \eta(x^{-1} h_2^{-1} \gamma_2) \, dh \, dg
\]

\[
= \int_{G_k} \int_{\Theta} \int_{\Theta} \sum_{\gamma \in \mathcal{G}_k} \eta(x^{-1} h^{-1} \gamma) \eta(x^{-1} h_2^{-1} \gamma) \, dh \, dg
\]

With \( C \) a large enough compact subset of \( \Theta_k \) to surject to \( |\Theta| = (Z_\kappa \cap \Theta) \Theta_k \backslash \Theta_k \),

\[
\int_{[G]} |q(g)|^2 \leq \int_{G_k} \int_C \int_C \sum_{\gamma \in \mathcal{G}_k} |\eta(x^{-1} h^{-1} \gamma)|^2 \cdot |\eta(x^{-1} h_2^{-1} \gamma)| \, dh \, dg
\]

The set

\[
\Phi = \Phi_{H,x,\eta}
\]

\[
= \{ \gamma \in G_k : \eta(x^{-1} h_2^{-1} \gamma) \eta(x^{-1} h_2^{-1} \gamma) \neq 0 \text{ for some } h, h_2 \in C \text{ and } g \in G_k \}
\]

\[
= \{ \gamma \in G_k : \gamma \in C x B, g^{-1}, g \in C x B \} \subset G_k \cap C x B \cdot (C x B)^{-1}
\]

the last set in the sequence above is the intersection of a closed, discrete set with a compact set, so is finite, and can only shrink as \( \epsilon \to 0^+ \).

For \( K_0 \) a compact open subgroup in the finite adele part \( G_0 \) of \( G_k \), a ball of archimedean radius \( \epsilon \) is the product \( B_\epsilon \times K_0 \). Here \( B_\epsilon \) is the inverse image in \( G_\infty \) of the geodesic ball of radius \( \epsilon \) in \( G_\infty / K_\infty \). For each \( \gamma \in \Phi \), for each \( h \in C \), \( \eta(x^{-1} h^{-1} \gamma) \neq 0 \) only for \( g \) in a ball in \( X = G_k / K_\infty \) of radius \( \epsilon \), with volume dominated by \( \epsilon^{\dim X} \). Thus,

\[
\int_{G_k \backslash G_k} |q(g)|^2 \, dg \ll \int_C \epsilon^{\dim X + \dim Y} \, dh \ll \epsilon^{\dim X + \dim Y}
\]

By automorphic Plancherel, with \( \eta \) as above,

\[
\sum_{\text{cfm} F} |\lambda_F(\eta)|^2 \cdot |(\eta \cdot F, \phi)|^2 + \ldots \ll \epsilon^{\dim X + \dim Y}
\]

Next, we give a bound on the spectral data to give a non-trivial lower bound for \( \lambda_f(\eta) \). Left and right \( K \)-invariant \( \eta \) necessarily gives \( \eta \cdot f = \lambda_f(\eta) \cdot f \), since up to scalars \( f \) is the unique spherical vector in the irreducible representation \( f \) generates. This is an intrinsic representation-theoretic relation, because an isomorphism of
principal series sends a spherical vector in the first representation to a constant multiple of the spherical vector in the second representation. That is, if
\[ \varphi : V \to W \]
is an isomorphism of representations, and \( f_1 \in V \) and \( f_2 \in W \) are the unique spherical vectors, then
\[ \varphi(f_1) = c \cdot f_2 \]
for a constant \( c \). To see this, observe that
\[ k \cdot \varphi(f_1) = \varphi(k \cdot f_1) = \varphi(f_1) \]
Therefore, \( \varphi(f_1) \) is indeed invariant under the \( K \)-action, so is the spherical vector in the representation \( V_2 \). Then a calculation gives
\[ \lambda_{f_1}(\eta) \cdot \varphi(f_1) = \eta \cdot \varphi(f_1) = c \cdot \lambda_{f_2}(\eta) = \lambda_{f_2}(\eta) \cdot c \cdot f_2 = \lambda_{f_2}(\eta) \cdot \varphi(f_1) \]
so that \( \lambda_{f_1} = \lambda_{f_2} \), as claimed. Therefore, the eigenvalue \( \lambda_f(\eta) \) can be computed in the usual model of the principal series at an archimedean place, as
\[ \eta \cdot \varphi_s^0(1) = \lambda_f(\eta) \]
for \( \varphi_s^0 \) the normalized spherical vector for \( s \in a^* \otimes_{\mathbb{R}} \mathbb{C} \), and \( \varphi^0(1) = 1 \). Thus,
\[ \lambda_f(\eta) = (\eta \cdot \varphi_s^0(1)) = \int_{G_0} \eta(g) \cdot \varphi_s^0(g) \, dg = \int_{B_0} \varphi_s^0(g) \, dg \]
Let \( P^+ \) be the connected component of the identity in the standard minimal parabolic. The Jacobian of the map \( P^+ \times K \to G_{\mathbb{R}} \) is non-vanishing at 1, and \( \varphi^0(1) = 1 \), so a suitable bound of \( \epsilon \) on the spectral parameter \( s \in a^* \otimes_{\mathbb{R}} \mathbb{C} \) will keep \( \varphi_s^0(g) \) near 1 on \( B_\epsilon \). In the example of \( GL_n(\mathbb{R}) \) with \( \varphi_s^0 \) the usual spherical vector, bounds of the form \( |s_j| \ll \frac{1}{\epsilon} \) assure that \( \Re \varphi_s^0(g) \geq \frac{\epsilon}{2} \) on \( B_\epsilon \), which prevents cancellation in the real part of \( \varphi_s^0(g) \) for \( g \in B_\epsilon \), so
\[ |\lambda_f(\eta)| = \left| \int_{B_\epsilon} \varphi_s^0(g) \, dg \right| \gg \int_{B_\epsilon} \Re \varphi_s^0(g) \, dg \gg \int_{B_\epsilon} \frac{1}{2} \, dg \gg \epsilon^{\dim X} \]
Combining the upper bound on \( |q|_{L^2}^2 \) with its lower bound on eigenvalues \( t_F(t_F - 1) \), letting \( T = \frac{1}{\epsilon} \),
\[ (\epsilon^{\dim X})^2 \times \left( \sum_{\text{c.f.m } F : |t_F| \leq T} |F_{\Theta_k \setminus \Theta_k}|^2 + \ldots \right) \ll \epsilon^{\dim X + \dim Y} \]
so
\[ \sum_{\text{c.f.m } F : |t_F| \ll T} |F_{\Theta_k \setminus \Theta_k}|^2 + \ldots \ll T^{\dim X - \dim Y} \]
Similarly,
\[ \sum_{\text{c.f.m } F : |t_F| \ll T} |(\eta \cdot F, \phi)|^2 + \ldots \ll T^{\dim X - \dim Y} \]
\( \square \)
5. Casimir Eigenvalue

Let $G = SL_4(\mathbb{R})$ and $I(s_1, s_2, s_3, s_4)$ a minimal-parabolic principal series. Let

$\mathfrak{g} = \mathfrak{sl}_4$

be the Lie algebra of $G$. For $i \neq j$, let $E_{i,j}$ be the matrix with 1 in the $(i,j)$-th position and 0 elsewhere. Let $H_{i,j}$ be the matrix with 1 in the $(i,i)$-th position and $-1$ in the $(j,j)$-th position. Observe that $H_{i,i+1}$ span the Cartan subalgebra $\mathfrak{h}$ and the $E_{i,j}$ for $i \neq j$ span the rest of the Lie algebra. Assume without loss of generality that $i < j$. We have the bracket relations

$[E_{i,j}, E_{j,i}] = H_{i,j}$

As before, the Casimir element is given by

$\Omega = \frac{1}{2}H_{1,2}^2 + \frac{1}{2}H_{2,3}^2 + \frac{1}{2}H_{3,4}^2 + \sum_{i,j} E_{i,j}E_{i,j} + E_{i,j}E_{j,i}$

Rearranging, this gives

$\Omega = \frac{1}{2}H_{1,2}^2 + \frac{1}{2}H_{2,3}^2 + \frac{1}{2}H_{3,4}^2 + \sum_{i,j} 2E_{j,i}E_{i,j} + H_{i,j}$

The lie algebra $\mathfrak{g}$ acts on $C^\infty(G)$ by

$X \cdot f(g) = \frac{d}{dt}|_{t=0} f(e^{tE})$

The product $E_{j,i}E_{i,j}$ act by 0, so Casimir is simply

$\Omega = \frac{1}{2}H_{1,2}^2 - (s_1 - s_2) + \frac{1}{4}(s_1 + s_2 - s_3 - s_4)^2 - (s_2 - s_3) + \frac{1}{2}(s_3 - s_4)^2 - (s_3 - s_4)$

Proposition 15. The Casimir operator acts on $I(s_1, s_2, s_3, s_4)$ by the scalar

$\frac{1}{2}(s_1 - s_2)^2 - (s_1 - s_2) + \frac{1}{4}(s_1 + s_2 - s_3 - s_4)^2 - (s_2 - s_3) + \frac{1}{2}(s_3 - s_4)^2 - (s_3 - s_4)$

Proof. Let us see how $H_{1,2}$ acts on $I(s_1, s_2, s_3, s_4)$. Note that

$e^{tH_{1,2}} = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Therefore

$\frac{d}{dt}|_{t=0} f( \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}) = \frac{d}{dt}|_{t=0} \chi( \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} )$

$= \frac{d}{dt}|_{t=0} e^{t s_1} \cdot e^{-t s_2}$
This is just \((s_1 - s_2)\). Likewise, we see that \(H_{i,j}\) will act on \(I_s\) by \(s_i - s_j\). Therefore, the Casimir operator will act by

\[
\frac{1}{2}(s_1 - s_2)^2 - (s_1 - s_2) + \frac{1}{2}(s_2 - s_3)^2 - (s_2 - s_3) + \frac{1}{2}(s_3 - s_4)^2 - (s_3 - s_4) + (s_1 - s_4) + (s_1 - s_3) + (s_2 - s_4)
\]

Let \(G = GL_4\) and \(I(s_1, s_2, s_3, s_4)\) a minimal-parabolic principal series. Let \(g = gl_4\) be the Lie algebra of \(G\). For \(i \neq j\), let \(E_{ij}\) be the matrix with 1 in the \((i,j)\)-th position and 0 elsewhere. Let \(H_{ij}\) be the matrix with 1 in the \((i,i)\)-th position and \(-1\) in the \((j,j)\)-th position and let \(H_{1234} = \text{diag}(1, 1, -1, -1)\). Observe that \(H_{1,i+1}\) span the Cartan subalgebra \(\mathfrak{h}\) and the \(E_{ij}\) for \(i \neq j\) span the rest of the Lie algebra. Assume without loss of generality that \(i < j\). We have the bracket relations

\[
[E_{ij}, E_{ji}] = H_{ij}
\]

As before, the Casimir element is given by

\[
\Omega = \frac{1}{2}H_{12}^2 + \frac{1}{4}H_{1234}^2 + \frac{1}{2}H_{34}^2 + (\sum_{ji} E_{ij}E_{ji} + E_{ji}E_{ij})
\]

Rearranging, this gives

\[
\Omega = \frac{1}{2}H_{12}^2 + \frac{1}{4}H_{1234}^2 + \frac{1}{2}H_{34}^2 + (\sum_{ij} 2E_{ij}E_{ji} - H_{ij})
\]

The Lie algebra \(g\) acts on \(C^\infty(G)\) by

\[
X \cdot f(g) = \left. \frac{d}{dt} f(ge^{tx}) \right|_{t=0}
\]

The product \(E_{ij}E_{ji}\) act by 0, so Casimir is simply

\[
\Omega = \left(\frac{1}{2}H_{12}^2 - H_{1,2}\right) + \left(\frac{1}{4}H_{1234}^2 - H_{23}\right) + \left(\frac{1}{2}H_{34}^2 - H_{34}\right) - H_{14} - H_{13} - H_{24}
\]

As an example computation, let us see how \(H_{12}\) acts on \(I(s_1, s_2, s_3, s_4)\). Note that

\[
e^{tH_{12}} = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

Therefore

\[
\left. \frac{d}{dt} f\right|_{t=0} \left( \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = \left. \frac{d}{dt} \right|_{t=0} \chi \left( \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)
\]

\[
= \left. \frac{d}{dt} \right|_{t=0} e^{ts_1} \cdot e^{-ts_2}
\]
This is just \((s_1 - s_2)\). Likewise, we see that \(H_{ij}\) will act on \(I_s\) by \(s_i - s_j\). Therefore, the Casimir operator will act by
\[
\frac{1}{2}(s_1 - s_2)^2 - (s_1 - s_2) + \frac{1}{4}(s_1 + s_2 - s_3 - s_4)^2 - (s_2 - s_3) + \frac{1}{2}(s_3 - s_4)^2 - (s_3 - s_4) - (s_1 - s_4) - (s_1 - s_3) - (s_2 - s_4)
\]

\[
\ell
\]

Letting \(s_1 = s + sf\), \(s_2 = -s + sf\), \(s_3 = s - sf\), \(s_4 = -s - sf\), we see that \((s_1 - s_2) = 2s\), \((s_2 - s_3) = -2s + 2sf\), \((s_3 - s_4) = 2s\), \((s_1 - s_4) = 2s + 2sf\), \((s_1 - s_3) = 2sf\), \((s_2 - s_4) = 2sf\), and finally \((s_1 + s_2 - s_3 - s_4) = 4sf\). Putting all this into the above expression for Casimir’s action gives that Casimir acts by
\[
\lambda_{s,f} = 4s^2 + 4s^2_f - 8sf - 4s
\]

Observe that
\[
\lambda_{s,f} - \lambda_{w,f} = 4(s(s - 1) - w(w - 1))
\]

\section{Friedrichs Self-Adjoint Extensions and Complex Conjugation Maps}

We review the result due to Friedrichs that a densely-defined, symmetric, semi-bounded operator admits a canonical self-adjoint extension with a useful characterization. We follow [Grubb], [Garrett 2011c], [Friedrichs 1935a] and [Friedrichs 1935b].

Let \(T\) be a densely defined, symmetric, unbounded operator on a Hilbert space \(V\), with domain \(D\). Assume further, that \(T\) is semi-bounded from below in the sense that
\[
\|u\|^2 \leq \langle u, Tu \rangle \quad \text{for all } u \in D.
\]

Let \(\langle x, y \rangle_1 = \langle Tx, y \rangle\) on \(D\). Let \(V_1\) be the completion of \(D\) with respect to the new inner product. The operator \(T\) remains symmetric for \(\langle , \rangle_1\). That is,
\[
\langle Tx, y \rangle_1 = \langle x, Ty \rangle_1
\]

for \(x, y \in D\). By Riesz-Fischer, for \(y \in V\), the continuous linear functional
\[
f(x) = \langle x, y \rangle
\]
can be written
\[
f(x) = \langle x, y' \rangle_1
\]
for a unique \(y' \in V\). Set
\[
T_{Fr}^{-1} y = y'
\]

That is, the inverse \(T_{Fr}^{-1}\) of the Friedrichs extension \(T_{Fr}\) of \(T\) is an everywhere-defined map
\[
T_{Fr}^{-1} : V \to V_1
\]
continuous for the \(\langle , \rangle_1\) topology on \(V_1\), characterized by
\[
\langle Tx, T_{Fr}^{-1} y \rangle = \langle x, y \rangle
\]
We will prove that, given \(\theta \in V_{-1}\) and \(T\theta = T|_{ker\theta}\), the Friedrichs extension \(\tilde{T}\theta\) has the feature that
\[
\tilde{T}_\theta u = f \quad \text{for } u \in V_1, f \in V
\]
exactly when

\[ T_\theta u = f + c \cdot \theta \text{ for some } c \in \mathbb{C} \]

Define a \textit{conjugation map} on \( V \) to be a complex-conjugate-linear automorphism \( j : V \to V \) with \( \langle jx, jy \rangle = \langle y, x \rangle \) and \( j^2 = 1 \). A conjugation map is equivalent to a complex-linear isomorphism

\[ \Lambda : V \to V^* \]

of \( V \) with its complex-linear dual, via Riesz-Fischer, by

\[ \Lambda(y)(x) = \langle x, jy \rangle = \langle y, x \rangle \]

Assume \( j \) stabilizes \( D \) and that \( T(jx) = jTx \) for \( x \in D \). Then \( j \) respects \( \langle \cdot, \cdot \rangle_1 \):

\[ \langle jx, jy \rangle_1 = \langle y, Tx \rangle = \langle y, x \rangle_1 \]

for \( x, y \in D \). Also, \( j \) commutes with \( T_{Fr} \):

\[ \langle x, T_{Fr}^{-1} jy \rangle_1 = \langle x, jy \rangle = \langle y, jx \rangle = \langle x, jT_{Fr}^{-1}y \rangle_1 \]

for \( x \in V_1 \) and \( y \in V \). Let \( V_{-1} \) be the complex-linear dual of \( V_1 \). We have \( V_1 \subset V \subset V_{-1} \). By design,

\[ T : D \to V \subset V_{-1} \]

is continuous when \( V \) has the subspace topology from \( V_{-1} \):

\[ |Ty|_{-1} = \sup_{|x|_1 \leq 1} |\Lambda(Ty)(x)| = \sup \langle x, jTy \rangle = |\langle x, jTy \rangle| \leq \sup |x_1||y_1| = |y|_1 \]

by Cauchy-Schwarz-Bunyakowsky. Thus the map \( T : D \to V \) extends by continuity to an everywhere-defined, continuous map

\[ T^\#: V_1 \to V_{-1} \]

by

\[ (T^\# y)(x) = \langle x, jy \rangle_1 \]

Further, \( T^\# : V_1 \to V_{-1} \) agrees with \( T_{Fr} : D_1 \to V \) on the domain \( D_1 = BV \) of \( T_{Fr} \), since

\[ (T^\# y)(x) = \langle x, jy \rangle_1 = \langle Tx, jy \rangle = \langle Tx, T_{Fr}^{-1} T_{Fr} jy \rangle = \langle T_{Fr}^{-1} Tx, T_{Fr} jy \rangle \]

which is

\[ \langle x, T_{Fr} jy \rangle = \Lambda(T_{Fr} y)(x) \text{ for } x \in D \text{ and } y \in D_1 \]

This follows since \( T_{Fr} \) extends \( T \), and noting the density of \( D \) in \( V \).

The following were presented as heuristics in [CdV 1982/1983] and treated more formally by Garrett in [Garrett 2011a]. We give complete proofs.

**Theorem 1.** The domain of \( T_{Fr} \) is \( D_1 = \{ u \in V_1 : T^\# u \in V \} \).

**Proof.** \( T^\# u = f \in V \) implies that

\[ \langle x, ju \rangle_1 = \langle T^\# u(x) \rangle = \Lambda(T^\# u)(x) = \Lambda(f)(x) = \langle x, jf \rangle \text{ for all } x \in V_1 \]

By the characterization of the Friedrichs extension, \( T_{Fr}(ju) = jf \). Since \( T_{Fr} \) commutes with \( j \), we have \( T_{Fr} u = f \). \( \square \)
Extend the complex conjugation $j$ to $V_{-1}$ by $(j\lambda)(x) = \overline{\lambda(jx)}$ for $x \in V_1$, and write
\[
(x,\theta)_{V_1 \times V_{-1}} = (j\theta)(x) = \overline{(\theta(jx))} \quad (\text{for } x \in V_1 \text{ and } \theta \in V_{-1})
\]
For $\theta \in V_{-1}$,
\[
\theta^\perp = \{ x \in V_1 : (x,\theta)_{V_1 \times V_{-1}} = 0 \}
\]
is a closed co-dimension-one subspace of $V_1$ in the $\langle , \rangle_1$-topology. Assume $\theta \notin V$. This implies density of $\theta^\perp$ in $V$ in the $\langle , \rangle$-topology.

**Theorem 2.** The Friedrichs extension $T_\theta = (T|_{\theta^\perp})_{Fr}$ of the restriction $T|_{\theta^\perp}$ of $T$ to $D \cap \theta^\perp$ has the property that $T_\theta u = f$ for $u \in V_1$ and $f \in V$ exactly when
\[
T^\# u = f + c\theta
\]
for some $c \in C$. Letting $D_1$ be the domain of $T_{Fr}$, the domain of $T_\theta$ is
\[
domain T_\theta = \{ x \in V_1 : (x,\theta)_{V_1 \times V_{-1}} = 0, T^\# x \in V + C \cdot \theta \}
\]
Proof. $T^\# u = f + c \cdot \theta$ is equivalent to
\[
\langle x, ju \rangle_1 = T^\#(u)(x) = (f + c \cdot \theta)(x) = \langle x,jf \rangle \quad (\text{for all } x \in \theta^\perp).
\]
This gives $\langle x, ju \rangle_1 = \langle x, jf \rangle$. The topology on $\theta^\perp$ is the restriction of the $\langle , \rangle_1$-topology of $V_1$, while $\theta^\perp$ is dense in $V$ in the $\langle , \rangle$-topology. Thus, $ju = T_\theta^{-1}jf$ by the characterization of the Friedrichs extension of $T_{\theta^\perp}$. Then $u = T_\theta^{-1}f$, since $j$ commutes with $T$. \qed

Given an everywhere-defined map $\tilde{T}^{-1} : V \to V_1$, characterized by
\[
\langle Tx, \tilde{T}^{-1}y \rangle = \langle x, y \rangle \quad (\text{for } x \in D, y \in V)
\]
we review the proof that given $\theta \in V_{-1}$ and $T_\theta = T|_{\ker \theta}$, the Friedrichs extension $\tilde{T}_\theta$ has the feature that
\[
\tilde{T}_\theta u = f \quad \text{for } u \in V_1, f \in V
\]
exactly when
\[
T_\theta u = f + c \cdot \theta \quad \text{for some } c \in C
\]
Observe that $T_\theta u = f + c \cdot \theta$ is equivalent to
\[
\langle x, u \rangle_1 = \langle x, Tu \rangle = \langle x, f + c \cdot \theta \rangle_{V_1 \times V_{-1}} = \langle x, f \rangle_{V_1 \times V_{-1}} \iff \tilde{T}_\theta u = f
\]
where the second equality follows from restricting in the first argument and extending in the second.

7. Moment bounds assumptions

We will need to assume a moment bound to know that the projected distribution is in the desired Sobolev space. This assumption is far weaker than Lindelof, but highly non-trivial.

**Proposition 16.** For a degree $n$ $L$-function $L(s)$ with suitable analytic continuation and functional equation, a second-moment bound
\[
\int_0^T |L\left(\frac{1}{2} + it\right)|^2 dt \ll T^A
\]
implies a pointwise bound

\[ L(\sigma_o + it, f) \ll_{\sigma_o, \epsilon} (1 + |t|)^{\frac{1}{2} + \epsilon} \quad (\text{for every } \epsilon > 0) \]

**Proof.** The proof of this is a standard argument, as follows. Fix \( \sigma_o > \frac{1}{2} \). For \( 0 < t_o \in \mathbb{R} \), let \( s_o = \sigma_o + it_o \). Let \( R \) be a rectangle in \( \mathbb{C} \) with vertices \( \frac{1}{2} \pm iT \) and \( 2 \pm iT \) for \( T > t_o \). By Cauchy’s Theorem

\[
L(s_o, f)^2 = \frac{1}{2\pi i} \int_R e^{(s-s_o)^2} \cdot L(s, f)^2 \, ds
\]

Since the L-function has polynomial vertical growth, we can push the top and bottom of \( R \) to \( \infty \), giving

\[
L(s_o)^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\frac{1}{2} - \sigma_o + it - t_o)^2} \cdot L(\frac{1}{2} + it)^2 \, dt + O(1)
\]

The part of the integral where \( |t - t_o| \geq t_o \) is visibly \( \ll_n, \sigma_o \ e^{-t_o} \):

\[
|e^{(\frac{1}{2} - \sigma_o + it - t_o)^2}| = e^{(\frac{1}{2} - \sigma_o)^2 - (t-t_o)^2} \ll_{\sigma_o} e^{\frac{t_o^2}{2}} \cdot e^{-\frac{(t-t_o)^2}{2}} \ll e^{-t_o}
\]

for \( |t - t_o| \geq t_o \). Squaring the convexity bound for \( L(\frac{1}{2} + it) \) gives

\[
|L(\frac{1}{2} + it)|^2 \ll |t|^{\frac{1}{2} + \epsilon} \quad (\text{for all } \epsilon > 0)
\]

Thus

\[
\int_{2t_o}^{\infty} e^{(\frac{1}{2} - \sigma_o + it - t_o)^2} \cdot L(\frac{1}{2} + it)^2 \, dt \ll_{\sigma_o} e^{\frac{t_o^2}{2}} \int_{2t_o}^{\infty} e^{-\frac{(t-t_o)^2}{2}} \cdot t^{\frac{1}{2} + \epsilon} \ll e^{-t_o}
\]

The other half of the tail, where \( t < 0 \), is estimated similarly. For \( 0 < t < 2t_o \), use the assumed moment estimate and the trivial estimate

\[
\frac{e^{(\frac{1}{2} - \sigma_o + it - t_o)^2}}{(\frac{1}{2} - \sigma_o + i(t - t_o))} \ll_{\sigma_o} e^{(\frac{1}{2} - \sigma_o)^2 - (t-t_o)^2} \ll_{\sigma_o} 1
\]

Then

\[
\int_{0}^{2t_o} e^{(\frac{1}{2} - \sigma_o + it - t_o)^2} \cdot L(\frac{1}{2} + it)^2 \, dt \ll_{\sigma_o} \int_{0}^{2t_o} |L(\frac{1}{2} + it)|^2 \, dt \ll t_o^A
\]

Thus,

\[
L(s_o)^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\frac{1}{2} + it - s_o)^2} \cdot L(\frac{1}{2} + it, f)^2 \, dt + O(1) \ll_{n, \sigma_o} t_o^A
\]

Then a standard convexity argument [Lang, p.263] gives the asserted \( |t_o|^{\frac{1}{2} + \epsilon} \) on \( \sigma_o = \frac{1}{2} \) for all \( \epsilon > 0 \). \( \square \)
8. Local automorphic Sobolev spaces

A notion of local automorphic Sobolev spaces $H^s_{lafc}$ defined in terms of global automorphic Sobolev spaces $H^s_{gafc}$ is necessary to discuss the meromorphic continuation of solutions $u = u_w$ to differential equations $(\Delta - \lambda_w)u = \theta$ for compactly-supported automorphic distributions $\theta$. We want a continuous embedding of global automorphic Sobolev spaces into local automorphic Sobolev spaces. This will follow immediately from the description, below. Second, compactly-supported distributions $\theta \in H^{-s}_{gafc}$ should extend to continuous linear functionals in $H^{-s}_{lafc}$. A convenient corollary is that such $\theta$ moves inside integrals appearing in a spectral decomposition/synthesis of automorphic forms lying in global automorphic Sobolev spaces. Finally, we want automorphic test functions to be dense in the local automorphic Sobolev spaces.

The necessity of the introduction of larger spaces than global automorphic Sobolev spaces is apparent already in the simplest situations. On $\Gamma \backslash \mathfrak{H}$, with $\Gamma = SL_2(\mathbb{Z})$, when $\theta \in H^{-1-\epsilon}_{gafc}$ is an automorphic Dirac $\delta_{afc}$ at $z_0 \in \Gamma \backslash \mathfrak{H}$, the spectral expansion in $\text{Re}(w) > \frac{1}{2}$ for a solution $u_w$ to that differential equation yields $u_w \in H^{1-\epsilon}_{gafc}$, but the meromorphic continuation to $\text{Re}(w) = \frac{1}{2}$ and then to $\text{Re}(w) < \frac{1}{2}$ includes an Eisenstein series $E_w$ which lies in no global automorphic Sobolev space. That $E_w$ lies in local automorphic Sobolev space $H^\infty_{lafc}$ is immediate from the smoothness of $E_w$ and the definition of the local spaces, below.

We describe local automorphic Sobolev spaces. Given a global automorphic Sobolev norm $|.|_s$, the corresponding local automorphic Sobolev norms, indexed by automorphic test functions $\varphi$, are given by

$$f \to |f|_{s,\varphi} = |\varphi \cdot f|_s \quad \text{for } f \text{ smooth automorphic}$$

**Definition 9.** The $s$-th local automorphic Sobolev space is given by

$$H^s_{lafc}(X) = \text{quasi-completion of } C^\infty_c(X) \text{ with respect to these semi-norms}$$

By definition, $C^\infty_c(X)$ is dense in $H^s_{lafc}(X)$. Continuity of the embedding of the global automorphic Sobolev spaces into the local uses integration by parts. The Lie algebra $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ where $\mathfrak{k}$ is the Lie algebra of the maximal compact subgroup $K$ and $\mathfrak{s}$ is the algebra of symmetric matrices. Choose an orthonormal basis $\{x_i\}$ for $\mathfrak{s}$ with respect to the Killing form $\langle \cdot, \cdot \rangle$. Define the gradient

$$\nabla = \sum_i X_{x_i} \otimes x_i$$

where $X_{x_i}$ is the differential operator given by $X_{x_i}f(g) = \frac{\partial}{\partial t}|_{t=0}f(g \cdot e^{tx_i})$. Observe that in the universal enveloping algebra

$$\nabla f \cdot \nabla F = (\sum_i X_{x_i}f \otimes x_i) \cdot (\sum_j X_{x_j}F \otimes x_j) = \sum_i X_{x_i}f \cdot X_{x_j}F$$

where the product is the Killing form on $\mathfrak{s}$.

**Proposition 17.** For $f, F \in C^\infty_c(\Gamma \backslash G)$, we have the integration-by-parts formula

$$\int_{\Gamma \backslash G} (-\Delta f) F = \int_{\Gamma \backslash G} \nabla f \cdot \nabla F$$
Proof. Letting $X = \Gamma \backslash G$, consider the integral
\[
\int_X \frac{\partial}{\partial t} f(g \cdot e^{tx_i}) \frac{\partial}{\partial t} F(g \cdot e^{tx_i}) \, dg
\]
Let $u = \frac{\partial}{\partial t} f(g \cdot e^{tx_i})$ and $dv = \frac{\partial^2}{\partial t^2} f(g \cdot e^{tx_i})$. Then, using the compact support of $f$ and its derivatives, we get
\[
\int_X \frac{\partial}{\partial t_1} f(g \cdot e^{tx_i_1}) \frac{\partial}{\partial t_2} F(g \cdot e^{tx_i_2}) \, dg = \int_X - \frac{\partial^2}{\partial t_1 \partial t_2} f(g \cdot e^{tx_i_1}) F(g) \, dg
\]
Taking limits as $t_1$ and $t_2$ approach 0 gives the integration-by-parts formula
\[
\int_X X_{x_1} f \cdot X_{x_2} F = \int_X (-X_{x_1} f) \cdot F
\]
and
\[
\int_X (-\Delta f) \cdot F = \int_X \nabla f \cdot \nabla F
\]

Now we can compare the local automorphic Sobolev +1-norm to the global automorphic Sobolev +1-norm as follows:

**Proposition 18.** Every local automorphic Sobolev +1-norm is dominated by the global automorphic Sobolev +1-norm.

**Proof.**
\[
|f|_{H_1}^\varphi = |\varphi f|_{H_1}^2 = \int_X (1 - \Delta)(\varphi f) \overline{\varphi f} = \int_X \nabla (\varphi f) \cdot \nabla (\overline{\varphi f}) + \int_X \varphi f \cdot \varphi f
\]
This is
\[
\int_X (f \nabla \varphi + \varphi \nabla f) \cdot (f \overline{\nabla \varphi} + \overline{\varphi \nabla f}) + |\varphi f|_{L^2}^2
\]
\[
= \int_X f^2 ||\nabla \varphi||^2 + \int_X (f \overline{\nabla \varphi} \cdot \nabla f + \varphi \overline{\nabla \varphi} \nabla f) + |\varphi f|_{L^2}^2
\]
The first and last summands are dominated by $(C_1 + C_2)|f|_{L^2}^2$ where $C_1 = \sup ||\varphi||$ and $C_2 = \sup ||\nabla \varphi||$. For the middle term, we use Cauchy-Schwarz and a constant bigger than $2 \cdot ||\varphi|| \cdot ||\nabla \varphi||$
\[
(f \overline{\nabla \varphi} \cdot \nabla f + \varphi \overline{\nabla \varphi} \nabla f) \leq \int_X 2f|f||\nabla f|| ||\nabla \varphi|| \ll \int_X |f||\nabla f||
\]
\[
\leq (\int_X |f|^2)^{\frac{1}{2}} (\int_X ||\nabla f||^2)^{\frac{1}{2}}
\]
\[
= |f|_{L^2} \cdot (\int_M (-\Delta f) \cdot f)^{\frac{1}{2}} \leq |f|_{L^2} \cdot (\int_M (1 - \Delta) f \cdot f)^{\frac{1}{2}} = |f|_{L^2} \cdot |f|_{H^1} \leq |f|_{H^1}^2
\]
That is, with an implied constant independent of $f$,
\[
|\varphi f|_{H^1} \ll |f|_{H^1}
\]

**Proposition 19.** There is a continuous map
\[
H_1^{1} \Gamma \rightarrow H_1^{1}
\]
Proof: The previous result proves continuity of $H^1_{gal} \to H^{1, \varphi}$ for every automorphic test function $\varphi$. Since $H^1_{laf}$ is the projective limit of the $H^{1, \varphi}$ over all automorphic test functions $\varphi$, the universal property of the projective limit guarantees that there must be a continuous map $H^1_{gal} \to H^1_{laf}$. \hfill \square

9. Main Theorem: Characterization and Sparsity of discrete spectrum

Recall the construction of 2, 2 pseudo-Eisenstein series. Let $\phi \in C_\infty^\infty(\mathbb{R})$ and let $f$ be a spherical cuspform on $GL_2(k) \backslash GL_2(\mathbb{A}_k)$ with trivial central character. Let

$$\varphi\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\right) = \phi\left(\left|\frac{\det A}{\det D}\right|^2 \right) \cdot f(A) \cdot \overline{f}(D)$$

extending by right $K$-invariance to be made spherical. Define the $P^{2,2}$ pseudo-Eisenstein series by

$$\Psi_\varphi(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g)$$

Given $g = \left(\begin{array}{cc} A & b \\ 0 & D \end{array}\right)$, let $h = h(g) = \left|\frac{\det A}{\det D}\right|^2$ be the height of $g$. The spectral decomposition for $\theta$ in a global automorphic Sobolev space $H^{-s}$ is

$$\bar{\theta} = \sum_{F_1 \text{ cfm } GL_2} \langle \bar{\theta}, F_1 \rangle \cdot F_1 + \sum_{F_2 \text{ cfm } GL_2} \langle \bar{\theta}, \Upsilon_{F_2} \rangle \cdot \Upsilon_{F_2}$$

$$+ \sum_{F_3, F_4 \text{ cfm } GL_2} \int_{\frac{i}{2} - i \infty}^{\frac{i}{2} + i \infty} \langle \bar{\theta}, E_{F_3, F_4, s} \rangle \cdot E_{F_3, F_4, s}^2 \, ds$$

$$+ \sum_{F_3 \text{ cfm } GL_2} \int_{\frac{1}{2} - i \infty}^{\frac{1}{2} + i \infty} \langle \bar{\theta}, E_{F_3, s}^3 \rangle \cdot E_{F_3, s}^3 \, ds + \sum_{F_4 \text{ cfm } GL_2} \int_{\rho - i \infty}^{\rho + i \infty} \langle \bar{\theta}, E_{F_4, \lambda}^2 \rangle \cdot E_{F_4, \lambda}^2 \, d\lambda$$

$$+ \int_{\rho + i \lambda_{\text{min}}} \langle \bar{\theta}, E_{\lambda} \rangle \cdot E_{\lambda} \, d\lambda$$

where $F$ and $F'$ are cuspsforms on $GL(2)$ and the $\Upsilon_F$’s are Speh forms. We are interested in the subspace $V$ of $L^2(\mathbb{A}_k, G_k \backslash G_k)$ spanned by 2, 2 pseudo-Eisenstein series with fixed cuspidal datum $f$ and $\overline{f}$, where $f$ is everywhere locally spherical. Let $D_{a,f}$ be the subspace of $V$ consisting of the $L^2$-closure of the span of 2, 2 pseudo-Eisenstein series with fixed cuspidal datum $f$ and $\overline{f}$ with test function $\varphi$ supported on $h(g) < a$ and whose constant terms have support on $b(g) < a$.

Let $\Delta_a$ be $\Delta$ restricted to $D_{a,f}$, and let $\bar{\Delta}_a$ be the Friedrichs extension of $\Delta_a$ to a self-adjoint (unbounded) operator. By construction, the domain of $\bar{\Delta}_a$ is contained in a Sobolev space $\Phi^{+1}_a$, defined as the completion of $D_{a,f}$ with respect to the $+1$-Sobolev norm $\langle f, f \rangle_1 = \langle (1 - \Delta)f, f \rangle_2$. We recall [M-W, 141-143], and [Garrett 2014] the

**Theorem 3.** The inclusion $\Phi^{+1}_a \to \Phi_a$, from $\Phi_a$ with its finer topology, is compact, so that the space $\Phi_a$ decomposes discretely.
Indeed, let $L^2_\eta$ be the subspace of $L^2(PGL_4\backslash PGL_4(\mathbb{R})/O_4(\mathbb{R}))$ with all constant terms vanishing above given fixed heights, specified by a real-valued function $\eta$ on simple positive roots described below. By its construction, the resolvent of the Friedrichs extension maps continuously from $L^2$ to the automorphic Sobolev space $H^1 = H^1(PGL_4(\mathbb{Z})\backslash PGL_4(\mathbb{R})/O_4(\mathbb{R}))$ with its finer topology. Letting

$$H^1_\eta = H^1 \cap L^2_\eta$$

with the topology of $H^1$, it suffices to show that the injection

$$H^1_\eta \to L^1_\eta$$

is compact. To prove this compactness, we show that the image of the unit ball of $H^1_\eta$ is totally bounded in $L^2_\eta$.

Let $A$ be the standard maximal torus consisting of diagonal elements of $GL_4$, $Z$ the center of $G$, and $K = O_4(\mathbb{R})$. Let $A^+$ be the subgroup of $A_{\mathbb{R}}$ with positive diagonal entries, and let $Z^+ = Z_{\mathbb{R}} \cap A^+$. A standard choice of positive simple roots is

$$\Phi = \{\alpha_i(a) = \frac{a_i}{a_{i+1}} \mid i = 1, \ldots, r-1\}$$

where $a$ is the matrix

$$a = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

Let $N^\text{min}$ be the unipotent radical of the standard minimal parabolic $P^\text{min}$ consisting of upper-triangular elements of $G$. For $g \in G_{\mathbb{R}}$, let $g = n_g a_g k_g$ be the corresponding Iwasawa decomposition with respect to $P^\text{min}$. From basic reduction theory, the quotient $Z_{\mathbb{R}} G_{\mathbb{Z}} \backslash G_{\mathbb{R}}$ is covered by the Siegel set

$$\mathcal{S} = N^\text{min}_{\mathbb{Z}} \backslash N^\text{min}_{\mathbb{R}} \cdot Z^+ A^+_0 \cdot K = Z^+ N^\text{min}_{\mathbb{Z}} \backslash \{g \in G : \alpha(a_g) \geq \frac{\sqrt{3}}{2} \text{ for all } \alpha \in \Phi\}$$

Further, there is an absolute constraint so that

$$\int_{\mathcal{S}} |f| \ll \int_{Z_{\mathbb{R}} G_{\mathbb{Z}} \backslash G_{\mathbb{R}}} |f|$$

for all $f$. For a non-negative real-valued function $\eta$ on the set of simple roots, let

$$X^\alpha_\eta = \{g \in \mathcal{S} : \alpha(a_g) \geq \eta(\alpha)\}$$

for $\alpha \in \Phi$. Let

$$C_\eta = \{g \in \mathcal{S} : \alpha(a_g) \leq \eta(\alpha) \text{ for all } \alpha \in \Phi\}$$

This is a compact set, and

$$\mathcal{S} = C_\eta \cup \bigcup_{\alpha \in \Phi} X^\alpha_\eta$$

For $\alpha \in \Phi$, let $P^\alpha$ be the standard maximal proper parabolic whose unipotent radical $N^\alpha$ has Lie algebra $n^\alpha$ including the $\alpha^{th}$ root space. That is, for $\alpha(a) = \frac{a_i}{a_{i+1}}$, the Levi component $M^\alpha$ of $P^\alpha$ is $GL_i \times GL_{4-i}$. As before, let $(c Pf)(g)$ denote the constant term along a parabolic $P$ of a function $f$ on $G_{\mathbb{Z}} \backslash G_{\mathbb{R}}$. For $P = P^\alpha$, write $c^\alpha = c_P$. For a non-negative real-valued function $\eta$ on the set of simple roots, the
Lemma 1. Fix a positive simple roots. We grant ourselves that we can control smooth cut-off functions: a Euclidean box, whose opposite faces can be identified to form a flat d-torus where the supremum is taken over \( f \) of smooth functions supported on the compact \( \mathbb{C} L \) card\( \Phi \) open balls in \( \varphi \) functions. The flat Laplacian and the Laplacian inherited from \( G \) \( \varphi \) for this covering. The functions \( (\cdot) \) values between \( \varphi \) the Friedrichs self-adjoint extension.

Proposition 20. The Friedrichs self-adjoint extension \( \tilde{\Delta}_\eta \) of the restriction of the symmetric operator \( \Delta \) to test functions in \( L^2_\eta \) has compact resolvent, and thus has purely discrete spectrum.

Proof. Let \( A^\star_\eta = \{ \alpha \in A : \alpha(a) \geq \frac{\sqrt{3}}{2} : \text{for all } \alpha \in \Phi \} \)

We grant ourselves that we can control smooth cut-off functions:

**Lemma 1.** Fix a positive simple roots \( \alpha \). Given \( \mu \geq \eta(\alpha) + 1 \), there are smooth functions \( \varphi^\alpha_\mu \) for \( \alpha \in \Phi \) and \( \varphi^0_\mu \) such that: all these functions are real-valued, taking values between 0 and 1, \( \varphi^0_\mu \) is supported in \( C_{\mu+1} \), and \( \varphi^0_\mu \) is supported in \( X^\mu_\eta \), and \( \varphi^0_\mu + \sum_\alpha \varphi^\alpha_\mu = 1 \). Further, there is a bound \( C \) uniform in \( \mu \geq \eta(\alpha) + 1 \), such that \( |f \cdot \varphi^\alpha_\mu|_{H^1} \leq C \cdot |f|_{H^1} \), and

\[
|f \cdot \varphi^\alpha_\mu|_{H^1} \leq C \cdot |f|_{H^1}
\]

for all \( \mu \geq \eta(\alpha) + 1 \).

Then the key point is

**Claim 1.** For \( \alpha \in \Phi \),

\[
\lim_{\mu \to \infty} \left( \sup_\eta \frac{|f|_{L^2}}{|f|_{H^1}} \right) = 0
\]

where the supremum is taken over \( f \in H^1_\eta \) and support(\( f \)) \( \subset X^\mu_\mu \).

Temporarily grant the claim. To prove total boundedness of \( H^1_\eta \to L^2_\eta \), given \( \epsilon > 0 \), take \( \mu \geq \eta(\alpha) + 1 \) for all \( \alpha \in \Phi \), large enough so that \( f \cdot \varphi^0_\mu \), \( |f|_{L^2} \leq \epsilon \), for all \( f \in H^1_\eta \), with \( |f|_{H^1} \leq 1 \). This covers the images \( \{ f \cdot \varphi^\alpha_\mu : f \in H^1_\eta \} \) with \( \alpha \in \Phi \) with card\( \Phi \) open balls in \( L^2 \) of radius \( \epsilon \). The remaining part \( \{ f \cdot \varphi^0_\mu : f \in H^1_\eta \} \) consists of smooth functions supported on the compact \( C_\mu \). The latter can be covered by finitely-many coordinate patches \( \phi_i : U_i \to \mathbb{R}^d \). Take smooth cut-off functions \( \varphi \) for this covering. The functions \( (f \cdot \varphi_i) \circ \phi_i^{-1} \) on \( \mathbb{R}^d \) have support strictly inside a Euclidean box, whose opposite faces can be identified to form a flat d-torus \( \mathbb{T}^d \).

The flat Laplacian and the Laplacian inherited from \( G \) admit uniform comparison on each \( \phi(U_i) \), so the \( H^1(\mathbb{T}^d) \)-norm of \( (f \cdot \varphi) \circ \phi_i^{-1} \) is uniformly bounded by the \( H^1 \)-norm. The classical Rellich lemma asserts compactness of

\[
H^1(\mathbb{T}^d) \to L^2(\mathbb{T}^d)
\]

By restriction, this gives the compactness of each \( H^1 \cdot \varphi_i \to L^2 \). A finite sum of compact maps is compact, so \( H^1 \cdot \varphi^0_\mu \to L^2 \) is compact. In particular, the image
of the unit ball from $H^1$ admits a cover by finitely-many $\epsilon$-balls for any $\epsilon > 0$. Combining these finitely-many $\epsilon$-balls with the card($\Phi$) balls covers the image of $H^1_\eta$ in $L^2_\eta$ by finitely-many $\epsilon$-balls, proving that $H^1_\eta \to L^2$ is compact.

It remains to prove the claim. Fix $\alpha = \alpha_i \in \Phi$, and $f \in H^1_\eta$ with support inside $X^\alpha_{mu}$ for $\mu \gg \eta(\alpha)$. Let $N = N^\alpha$, $P = P^\alpha$, and let $M = M^\alpha$ be the standard Levi component of $P$. Use exponential coordinates

$$n_x = \begin{pmatrix} 1_i & x \\ 0 & 1_{4-i} \end{pmatrix}$$

In effect, the coordinate $x$ is in the Lie algebra $n$ of $N_R$. Let $\Lambda \subset n$ be the lattice which exponentiates to $N^\alpha$. Given $\eta$ the natural inner product $\langle \cdot , \cdot \rangle$ invariant under the (Adjoint) action of $M_R \cap K$ that makes root spaces mutually orthogonal. Fix a nontrivial character $\psi$ on $\mathbb{R}/\mathbb{Z}$. We have the Fourier expansion

$$f(n_xm) = \sum_{\xi \in \Lambda'} \psi(x, \xi) \hat{f}_{\xi}(m)$$

with $n \in N_R$, $m \in M_R$, and $\Lambda'$ is the dual lattice to $\Lambda$ in $n$ with respect to $\langle \cdot , \cdot \rangle$, and

$$\hat{f}_{\xi}(m) = \int_{n \backslash \Lambda} \psi(x, \xi) f(n_xm) \, dx$$

Let $\Delta^n$ be the flat Laplacian on $n$ associated to the inner product $\langle \cdot , \cdot \rangle$ normalized so that

$$\Delta^n \psi(x, \xi) = -\langle \xi, \xi \rangle \cdot \psi(x, \xi)$$

Let $U = M \cap N^\text{min}$. Abbreviating $A_u = \text{Ad}u$,

$$|f|^2_{L^2} \leq \int \int_{\mathbb{R}} |f|^2 = \int_{Z^+ \backslash A_u^+} \int_{\Phi \backslash U_R} \int_{A_u^{-1} \Lambda \backslash n} |f(un_xa)|^2 \, dx \, du \, da$$

with Haar measures $dx$, $du$, $da$, and where $\delta$ is the modular function of $P_R$. Using the Fourier expansion,

$$f(un_xa) = f(un_xu^{-1} \cdot ua) = \sum_{\xi \in \Lambda'} \psi(A_u x, \xi) \cdot \hat{f}_{\xi}(ua)$$

$$= \sum_{\xi \in \Lambda'} \psi(x, A_u^* \xi) \cdot \hat{f}_{\xi}(ua)$$

Then

$$-\Delta^n f(un_xa) = \sum_{\xi \in \Lambda'} \langle A_u^* \xi, A_u^* \xi \rangle \cdot \psi(x, A_u^* \xi) \cdot \hat{f}_{\xi}(ua)$$

The compact quotient $U^\alpha_R \backslash U_R$ has a compact set $R$ of representatives in $U_R$, so there is a uniform lower bound for $0 \neq \xi \in \Lambda'$:

$$0 < b \leq \inf_{u \in R} \inf_{\xi \in \Lambda'} \langle A_u^* \xi, A_u^* \xi \rangle$$

By Plancherel applied to the Fourier expansion in $x$, using the hypothesis that $\hat{f}_0 = 0$ in $X^\alpha_{mu}$,

$$\int_{A_u^{-1} \Lambda \backslash n} |f(un_xa)|^2 \, dx = \int_{A_u^{-1} \Lambda \backslash n} |f(un_xu^{-1} \cdot ua)|^2 \, dx = \sum_{\xi \in \Lambda'} |\hat{f}_{\xi}(ua)|^2$$

$$\leq b^{-1} \sum_{\xi \in \Lambda'} \langle A_u^* \xi, A_u^* \xi \rangle \cdot |\hat{f}_{\xi}(ua)|^2 = \sum_{\xi \in \Lambda'} -\Delta^n \hat{f}_{\xi}(ua) \cdot \hat{f}(ua)$$
Next, we compare $\Delta^n$ to the invariant Laplacian $\Delta$. Let $\mathfrak{g}$ be the Lie algebra of $G_{\mathbb{R}}$, with non-degenerate invariant pairing

$$\langle u, v \rangle = \text{trace}(uv)$$

The Cartan involution $v \to v^\theta$ has +1 eigenspace the Lie algebra $\mathfrak{k}$ of $K$, and $-1$ eigenspace $\mathfrak{s}$, the space of symmetric matrices.

Let $\Phi^N$ be the set of positive roots $\beta$ whose root space $\mathfrak{g}_\beta$ appears in $\mathfrak{n}$. For each $\beta \in \Phi^N$, take $x_\beta \in \mathfrak{g}_\beta$ such that $x_\beta + x_\beta^0 \in \mathfrak{s}$, $x_\beta - x_\beta^0 \in \mathfrak{k}$, and $\langle x_\beta, x_\beta^0 \rangle = 1$: for $\beta(a) = \frac{a}{a_j}$ with $i < j$, $x_\beta$ has a single non-zero entry, at the $ij$th place. Let

$$\Omega' = \sum_{\beta \in \Phi^N} (x_\beta x_\beta^0 + x_\beta^0 x_\beta)$$

Let $\Omega'' \in U\mathfrak{g}$ be the Casimir element for the Lie algebra $\mathfrak{m}$ of $M_{\mathbb{R}}$, normalized so that Casimir for $\mathfrak{g}$ is the sum $\Omega = \Omega' + \Omega''$. We rewrite $\Omega'$ to fit the Iwasawa coordinates: for each $\beta$,

$$x_\beta x_\beta^0 + x_\beta^0 x_\beta = 2x_\beta x_\beta^0 + [x_\beta, x_\beta] = 2x_\beta^2 - 2x_\beta(x_\beta - x_\beta^0) + [x_\beta^0, x_\beta] \in 2x_\beta^2 + [x_\beta, x_\beta] + \mathfrak{k}$$

Therefore,

$$\Omega' = \sum_{\beta \in \Phi^N} 2x_\beta^2 + [x_\beta, x_\beta] \mod \mathfrak{k}$$

The commutators $[x_\beta^0, x_\beta] \in \mathfrak{m}$. In the coordinates $un_x a$ with $U\mathfrak{g}$ acting on the right, $x_\beta \in \mathfrak{n}$ is acted on by $a$ before translating $x$, by

$$un_x a \cdot e^{tx} = un_x \cdot e^{t \beta(a) \cdot x} a = un_{x + \beta(a) x} a$$

That is, $x_\beta$ acts by $\beta(a) \cdot \frac{\partial}{\partial x_\beta}$.

For two symmetric operators $S, T$ on a not-necessarily-complete inner product space $V$, write $S \leq T$ when

$$\langle Sv, v \rangle \leq \langle Tv, v \rangle$$

for all $v \in V$. We say that a symmetric operator $T$ is non-negative when $0 \leq T$. Since $a \in A_0^+$, there is an absolute constant so that $\alpha(a) \geq \mu$ implies $\beta(a) \gg \mu$.

Thus,

$$-\Delta^n = - \sum_{\beta \in \Phi^N} \frac{\partial^2}{\partial x_\beta^2} \ll \frac{1}{\mu^2} \left( - \sum_{\beta \in \Phi^N} x_\beta^2 \right)$$

on $C_c(\mathbb{X}_\mu^K)$ with the $L^2$ inner product. We claim that

$$- \sum_{\beta \in \Phi^N} [x_\beta^0, x_\beta] - \Omega'' \geq 0$$
on $C_c^\infty(X^\alpha_\mu)^K$. From this, it would follow that

$$-\Delta^n \ll \frac{1}{\mu^2} \cdot \left( - \sum_{\beta \in \Phi^N} x_\beta^2 \right) \leq \frac{1}{\mu^2} \cdot \left( - \sum_{\beta \in \Phi^N} x_\beta^2 - \sum_{\beta \in \Phi^N} [x_\beta, x_\beta] - \Omega'' \right) = \frac{1}{\mu^2} \cdot (-\Delta)$$

Then, for $f \in H^1_0$ with support in $X^\alpha_\mu$ we would have

$$|f|_{L^2}^2 \ll \int_{\mathbb{F}} -\Delta^n f \cdot \mathcal{T} \ll \frac{1}{\mu^2} \int_{\mathbb{F}} -\Delta f \cdot \mathcal{T} \ll \frac{1}{\mu^2} \int_{Z_0 \mathbb{G} \backslash \mathbb{G}} -\Delta f \cdot \mathcal{T} \ll \frac{1}{\mu^2} \cdot |f|_{H^1}^2$$

Taking $\mu$ large makes this small. Since we can do the smooth cutting-off to affect the $H^1$ norm only up to a uniform constant, this would complete the proof of total boundedness of the image in $L^2$ of the unit ball from $H^1_0$.

To prove the claimed nonnegativity of $T = -\sum_{\beta \in \Phi^N} [x_\beta, x_\beta] - \Omega''$, exploit the Fourier expansion along $N$ and the fact that $x \in \mathfrak{n}$ does not appear in $T$: noting that the order of coordinates $n_x u$ differs from that above,

$$\int_{Z^+ \mathbb{A}^+_3} \int_{U_\mathbb{A} \backslash U_\mathbb{A}} \int_{\Lambda \backslash \mathbb{A}} T f(n_x u a) \overline{T(n_x u a)} \ dx \ du \ \frac{da}{\delta(a)}$$

$$= \int_{Z^+ \mathbb{A}^+_3} \int_{U_\mathbb{A} \backslash U_\mathbb{A}} \int_{\Lambda \backslash \mathbb{A}} T \left( \sum_\xi \psi(x, \xi) \hat{f}(u a) \right) \left( \sum_\xi \overline{\psi(x, \xi')} \hat{f}(u a) \right) \ dx \ du \ \frac{da}{\delta(a)}$$

Only the diagonal summands survive the integration in $x \in \mathfrak{n}$, and the exponentials cancel, so this is

$$\int_{Z^+ \mathbb{A}^+_3} \int_{U_\mathbb{A} \backslash U_\mathbb{A}} \sum_\xi T \hat{f}(u a) \cdot \overline{T(u a)} \ dx \ du \ \frac{da}{\delta(a)}$$

Let $F_\xi$ be a left-$N_\mathbb{R}$-invariant function taking the same values as $\hat{f}_\xi$ on $U_\mathbb{R} A^+_3 K$, defined by

$$F_\xi(n_x u a k) = \hat{f}_\xi(u a k)$$

for $n_x \in N$, $u \in U$, $a \in A^+$, $k \in K$. Since $T$ does not involve $\mathfrak{n}$ and since $F_\xi$ is left $N_\mathbb{R}$-invariant,

$$T \hat{f}_\xi(u a) = T F_\xi(n_x u a) = -\Delta F_\xi(n_x u a)$$

and then

$$\int_{Z^+ \mathbb{A}^+_3} \int_{U_\mathbb{A} \backslash U_\mathbb{A}} \sum_\xi T \hat{f}(u a) \cdot T \hat{f}(u a) \ dx \ du \ \frac{da}{\delta(a)} = \int_{Z^+ \mathbb{A}^+_3} \int_{U_\mathbb{A} \backslash U_\mathbb{A}} \sum_\xi -\Delta F_\xi(u a) \cdot \overline{T}(u a) \ dx \ du \ \frac{da}{\delta(a)}$$

The individual summands are not left-$U_\mathbb{Z}$-invariant. Since $\hat{f}_\xi(\gamma g) = \hat{F}_{A^+ \xi}(g)$ for $\gamma$ normalizing $\mathfrak{n}$, we can group $\xi \in \Lambda'$ by $U_\mathbb{Z}$ orbits to obtain $U_\mathbb{Z}$ subsums and then unwind. Pick a representative $\omega$ for each orbit $[\omega]$, and let $U_\omega$ be the isotropy subgroup of $\omega$ in $U_\mathbb{Z}$, so

$$\int_{U_\mathbb{Z} \backslash U_\mathbb{R}} \sum_\xi -\Delta F_\xi(u a) \cdot \overline{T}(u a) \ dx \ du = \sum_{[\omega]} \int_{U_\mathbb{Z} \backslash U_\mathbb{R}} \sum_{\xi \in [\omega]} -\Delta F_\xi(u a) \cdot \overline{T}(u a) \ dx \ du$$

$$= \sum_{[\omega]} \int_{U_\mathbb{Z} \backslash U_\mathbb{R}} \sum_{\gamma \in U_\omega \backslash U_\mathbb{Z}} -\Delta F_{A^+ \omega}(u a) \cdot \overline{T}(u a) \ dx \ du = \sum_{[\omega]} \int_{U_\omega \backslash U_\mathbb{R}} -\Delta F_\omega(u a) \cdot \overline{T}(u a) \ dx \ du$$
Then
\[
\int_{Z^+ \setminus A_0^+} \int_{U_\mathbb{K} \setminus U_\mathbb{R}} \sum_{\xi} -\Delta F_\xi(ua) \cdot F_\xi(ua) \, du = \sum_{\omega} \int_{Z^+ \setminus A_0^+} \int_{U_\mathbb{K} \setminus U_\mathbb{R}} -\Delta F_\omega(ua) \cdot F_\omega(ua) \, du \frac{da}{\delta(a)}
\]

Since \(-\Delta\) is a non-negative operator on functions on every quotient \(Z^+ U_\mathbb{K} \setminus G_\mathbb{K}/K\) of \(G_\mathbb{R}/K\), each double integral is non-negative, proving that \(T\) is non-negative.

This completes the proof that \(H^1_n \rightarrow L^2_n\) is compact, and thus, that the Friedrichs extension of the restriction of \(\Delta\) to test functions in \(L^2_n\) has purely discrete spectrum. □

Since the pseudo-Eisenstein series appearing in the spectral decomposition are orthogonal to all other automorphic forms appearing in the spectral expansion in every Sobolev space, we can speak of the projection \(\theta\) of the period distribution \(\tilde{\theta}\) to the subspace \(V\) of \(L^2(Z_k G_k \setminus G_k)\). That is,
\[
\theta = \langle \tilde{\theta}, \Psi_f \rangle \cdot \Psi_f + \frac{1}{4\pi i} \int_{\mathbb{H} \setminus \infty} \langle \tilde{\theta}, E_{f, J_{1,s}} \rangle \cdot E_{f, J_{1,s}}
\]
where \(\langle , \rangle\) is the pairing of distributions with functions. To check \(\theta\) is well-defined, we must check that, for every square-integrable automorphic form \(f\) not in the \(L^2\)-span of \(2, 2\) pseudo-Eisenstein series, we have
\[
\langle \theta, f \rangle = 0
\]
To this end, let us check it for \(3, 1\) pseudo-Eisenstein series \(\Psi_{f_1, \phi_1}\) with cuspidal data \(f_1\) and test function data \(\phi_1\). Then
\[
\langle \theta, \Psi_{f_1, \phi_1} \rangle = \langle \tilde{\theta}, \Psi_{f_1, \phi_1} \rangle \cdot \Psi_{f_1, \phi_1} + \langle \tilde{\theta}, \Psi_{f_2, \phi_2}^2 \cdot \Psi_{f_3, \phi_3}^3 \rangle \cdot \Psi_{f_1, \phi_1}
\]
This is
\[
\langle \tilde{\theta}, \Psi_{f_1, \phi_1} \rangle \cdot \Psi_{f_1, \phi_1} + \langle \tilde{\theta}, \Psi_{f_2, \phi_2}^2 \cdot \Psi_{f_3, \phi_3}^3 \rangle = 0
\]
The Speh form \(\Psi_f\) is a \(\Delta\)-eigenfunction. Furthermore, it is orthogonal to \(3, 1\) pseudo-Eisenstein series in \(L^2\). Indeed, using the adjunction relation,
\[
\langle \Psi_{f_1, \phi_1} \Psi_{f_2, \phi_2}^3 \rangle = \langle \tilde{\theta}, \Psi_{f_1, \phi_1} \rangle
\]
Since the \(3, 1\) constant term of the Speh form \(\Psi_f\) is zero, the above is zero. Therefore, the Speh form \(\Psi_f\) is orthogonal to \(3, 1\) pseudo-Eisenstein series. Since \(2, 2\) pseudo-Eisenstein series are orthogonal to \(3, 1\) pseudo-Eisenstein series, we conclude that
\[
\langle \theta, \Psi_{f_1, \phi_1} \rangle = \langle \tilde{\theta}, \Psi_{f_1, \phi_1} \rangle \cdot \Psi_{f_1, \phi_1} + \langle \tilde{\theta}, \Psi_{f_2, \phi_2}^2 \cdot \Psi_{f_3, \phi_3}^3 \rangle \cdot \Psi_{f_1, \phi_1} = 0
\]
We now prove that for a \(2, 1, 1\) pseudo-Eisenstein series \(\Psi_{f_2, \phi_2, \phi_3}\) with cuspidal data \(f_2\) and test functions \(\phi_2\) and \(\phi_3\), that
\[
\langle \theta, \Psi_{f_2, \phi_2, \phi_3} \rangle = 0
\]
As before, this is just
\[
\langle \tilde{\theta}, \Psi_{f_2, \phi_2, \phi_3} \rangle \cdot \Psi_{f_2, \phi_2, \phi_3} + \langle \tilde{\theta}, \Psi_{f_2, \phi_2, \phi_3}^2 \cdot \Psi_{f_3, \phi_3}^2 \rangle \cdot \Psi_{f_2, \phi_2, \phi_3} = 0
\]
The second term is zero, because the pseudo-Eisenstein series are orthogonal. The first term gives zero. Indeed
\[ \langle \Upsilon_f, \Psi^{2,1,1}_{\phi_{f_2, \phi_3}} \rangle = \langle c_{2,1,1} \Upsilon_f, \varphi_{f_2, \varphi_3} \rangle = 0 \]
since the 2,1,1 constant term of the Speh form $\Upsilon_f$ is zero.

Let $\Delta \theta$ be $\Delta$ with domain $\ker \theta \cap V$. We will show that parameters for the discrete spectrum $\lambda_{s,f} = s_f(s_f - 2) + s(s - 1)$ (if any) of the Friedrichs extension $\tilde{\Delta}_\theta$ are contained in the zero-set of the $L$-function appearing in the period.

To legitimize applying the distribution $\theta$ to cuspidal-data Eisenstein series $E_{f, \Upsilon, s}$ requires discussion of local automorphic Sobolev spaces. Recall that $\theta$ is in the $-1$ global automorphic Sobolev space, so is in the $-1$ local automorphic Sobolev space. As $E_{f, \Upsilon, s}$ is in the $+1$ local automorphic Sobolev space, we can apply $\theta$ to it.

**Theorem 4.** For $\Re(w) = \frac{1}{2}$, if the equation $(\Delta - \lambda_{w,f})u = \theta$ has a solution $u \in V$, then $\theta E_{f, \Upsilon, w} = 0$. Conversely, if $\theta E_{f, \Upsilon, w} = 0$ for $\Re(w) = \frac{1}{2}$, then there is a solution to that equation in $V$, and the solution is unique with spectral expansion
\[ u = \theta(\Upsilon_f) \cdot \Upsilon_f + \frac{1}{4\pi i} \int \left( \frac{\theta E_{f, \Upsilon, 1-s}}{\lambda_{s,f} - \lambda_{w,f}} \right) E_{f, \Upsilon, s} \, ds \]
convergent in $V^+$

**Proof.** The condition $\theta \in V_{-1}$ is that
\[ \int \frac{|\theta E_{f, \Upsilon, 1-s}|^2}{1 + t^2} dt < \infty \]
Thus, $u \in V_{+1}$, and $u$ has a spectral expansion of the form
\[ u = A_f \cdot \Upsilon_f + \frac{1}{4\pi i} \int \left( \frac{\theta E_{f, \Upsilon, 1-s}}{\lambda_{s,f} - \lambda_{w,f}} \right) E_{f, \Upsilon, s} \, ds \]
with $t \to A_{\frac{1}{2} + it}$ in $L^2(\mathbb{R})$. The distribution $\theta$ has spectral expansion in $V_{-1}$,
\[ \theta = \theta(\Upsilon_f) \cdot \Upsilon_f + \frac{1}{4\pi i} \int \left( \frac{\theta E_{f, \Upsilon, 1-s}}{\lambda_{s,f} - \lambda_{w,f}} \right) E_{f, \Upsilon, s} \, ds \]

We describe the vector-valued weak integrals of [Gelfand 1936] and [Pettis 1938] and summarize the key results. We follow [Bourbaki 1963].

**Definition 10.** For $X, \mu$ a measure space and $V$ a locally convex, quasi-complete topological vector space, a Gelfand-Pettis (or weak) integral is a vector-valued integral $C^0_b(X, V) \to V$ denoted $f \to I_f$ such that for all $\alpha \in V^*$, we have
\[ \alpha(I_f) = \int_X \alpha \circ f \, d\mu \]
where the latter is the usual scalar-valued Lebesgue integral.

**Proposition 21.** Hilbert, Banach, Frechet, and LF spaces together with their weak duals are locally convex, quasi-complete topological vector spaces.

**Proposition 22.** Gelfand-Pettis integrals exist and are unique.
Proposition 23. Any continuous linear operator between locally convex, quasi-complete topological vector spaces \( T : V \to W \) commutes with the Gelfand-Pettis integral:

\[
T(I_f) = I_T f
\]

Note that \( E_{f,T,s} \) lies in a local automorphic Sobolev space. By the Gelfand-Pettis theory, if \( T : V \to W \) is a continuous linear map of locally convex topological vector spaces, where convex hulls of compact sets in \( V \) have compact closures and if \( f \) is a continuous, compactly-supported \( V \)-valued function on a finite measure space \( X \), then the \( W \)-valued function \( T \circ f \) has a Gelfand-Pettis integral, and

\[
T \left( \int_X f \right) = \int_X T \circ f
\]

Let \( V = H^1_{laut}(X) \). Note that \( V \) is a locally convex, quasi-complete topological vector space since it is the completion of \( C_0^\infty(X) \) with respect to a family of seminorms. Given a compactly-supported distribution \( \theta \in H^{-1}_{laut}(X) \), \( \theta \) extends to a continuous linear functional \( \theta \in H^{-1}_{laut}(X) \), by section 7. Since \( \theta \) is a continuous mapping \( \theta : H^{-1}_{laut}(X) \to \mathbb{C} \), given a continuous, compactly-supported \( H^1_{laut}(X) \)-valued function \( f \),

\[
\theta \int_X f = \int_X \theta \circ f
\]

Gelfand-Pettis theory allows us to move \( \theta \) inside the integral. Thus

\[
(\lambda_{\gamma_f} - \lambda_w) A_f = \theta(\gamma_f)
\]

and

\[
(\lambda_{s,f} - \lambda_{w,f}) \cdot A_s = \theta E_{f,T,1-s}
\]

The latter equality holds at least in the sense of locally integrable functions. Letting \( w = \frac{1}{2} + i \tau \), by Cauchy-Schwarz-Bunyakowsky, for any \( \epsilon > 0 \),

\[
\int_{\tau - \epsilon}^{\tau + \epsilon} |\theta E_{f,T,\frac{1}{2}+it}|^2 dt = \int_{\tau - \epsilon}^{\tau + \epsilon} |(\lambda_{\frac{1}{2}+it,f} - \lambda_{\frac{1}{2}+it,f} A_{\frac{1}{2}+it}|^2 dt
\]

Using \( s = \frac{1}{2} + it \) and rewriting the difference of eigenvalues gives us equality of the above with

\[
\int_{\tau - \epsilon}^{\tau + \epsilon} |(t-\tau)(t-1+\tau) A_{\frac{1}{2}+it}|^2 dt \leq \int_{\tau - \epsilon}^{\tau + \epsilon} |t-\tau|^2 dt \cdot \int_{\tau - \epsilon}^{\tau + \epsilon} |(t-i+\tau) A_{\frac{1}{2}+it}|^2 dt \ll \epsilon^3
\]

The function

\[
t \to \theta E_{f,T,\frac{1}{2}+it}
\]

is continuous, in fact

\[
s \to \theta E_{f,T,s}
\]

is meromorphic, since \( \theta \) is compactly supported (see [Grothendieck 1954] and [Garrett 2011 e]), so

\[
\theta E_{f,T,1-w} = 0
\]

Conversely, when \( \theta E_{1-w} = 0 \), the function

\[
t \to \frac{\theta E_{f,T,\frac{1}{2}-it}}{(\lambda_{\frac{1}{2}+it} - \lambda_w)}
\]
is continuous and square-integrable, assuring $H^1$-convergence of the integral
\[ u = \frac{\theta(\Upsilon_f) \cdot \Upsilon_f}{\lambda_{\Upsilon_f} - \lambda_{w,f}} + \frac{1}{4 \pi i} \int_{(\mathbb{H})} \frac{\theta E_{f,\Upsilon_{1-s}} \cdot E_{f,\Upsilon_{s}}}{(\lambda_{s,f} - \lambda_{w,f})} ds \]
this spectral expansion produces a solution of the differential equation. Any solution in $V^{+1}$ admits such an expansion, and the coefficients are uniquely determined, giving uniqueness. \□

Let $X_a = \{ A, D \in GL_2 : |\text{det}A||^2 = a \}$. Let $H$ be the subgroup of $GL_2 \times GL_2$ consisting of pairs $(B, C)$ so that $|\text{det}B \cdot \text{det}C| = 1$. The group $H$ acts simply transitively on $X_a$, so $X_a$ has an $H$-invariant measure. Fix $GL_2$ cuspforms $f_1$ and $f_2$ and define

\[ \eta_a F = \int_{Z_h \setminus H_k \setminus X_a} c_p(F(a)) \cdot f_1(A) \cdot f_2(D) \, dx \]

**Proposition 24.** Take $\text{Re}(w) = \frac{1}{2}$. For $a \gg 1$ such that the support of $\tilde{\theta}$ is below $h = a$, the constant term $c_p u$ of a solution $u \in V^{+1}$ to $(\Delta - \lambda_{w,f})u = \theta$ vanishes for height $h \geq a$.

**Proof.** Let $\eta_{a,f_1 \otimes f_2}$ be the functional above. This functional is in $H^{-\frac{\gamma}{2} - \epsilon}$ for all $\epsilon > 0$. Thus, for $u \in H^{+1}$,

\[ \eta_{a,f_1 \otimes f_2} u = \eta_{a,f_1 \otimes f_2} \left( \frac{\theta(\Upsilon_f) \cdot \Upsilon_f}{(\lambda_{\Upsilon_f} - \lambda_{w,f})(1,1)} + \frac{1}{4 \pi i} \int_{(\mathbb{H})} \frac{\theta E_{f,\Upsilon_{1-s}} \cdot \eta_{a,f_1 \otimes f_2} E_{f,\Upsilon_{s}}}{(\lambda_{s,f} - \lambda_{w,f})} ds \right) \]

We can break up the integral into two tails and a truncated finite part. The truncated finite part is a continuous, compactly-supported integral of functions in a local automorphic Sobolev space, so Gelfand-Pettis theory allows us to move compactly-supported distributions inside the integral. The tails are spectral expansions of functions in $H^{+1}$, and since $H^{+1}$ embeds into a local automorphic Sobolev space, the Gelfand-Pettis theory applies there also, allowing us to move the distribution inside the integral.

\[ \frac{\theta(\Upsilon_f) \cdot \eta_{a,f_1 \otimes f_2} (\Upsilon_f)}{(\lambda_{\Upsilon_f} - \lambda_{w,f})} + \frac{1}{4 \pi i} \int_{(\mathbb{H})} \frac{\eta_{a,f_1 \otimes f_2} E_{f,\Upsilon_{1-s}} \cdot \eta_{a,f_1 \otimes f_2} E_{f,\Upsilon_{s}}}{(\lambda_{s,f} - \lambda_{w,f})} ds \]

This is

\[ \theta \left( \frac{\eta_{a,f_1 \otimes f_2} (\Upsilon_f) \cdot \Upsilon_f}{(\lambda_{\Upsilon_f} - \lambda_{w,f})} + \frac{1}{4 \pi i} \int_{(\mathbb{H})} \frac{\eta_{a,f_1 \otimes f_2} E_{f,\Upsilon_{1-s}} \cdot \eta_{a,f_1 \otimes f_2} E_{f,\Upsilon_{s}}}{(\lambda_{s,f} - \lambda_{w,f})} ds \right) \]

which is

\[ \theta \left( \frac{\eta_{a,f_1 \otimes f_2} (\Upsilon_f) \cdot \Upsilon_f}{(\lambda_{\Upsilon_f} - \lambda_{w,f})} + \frac{1}{4 \pi i} \int_{(\mathbb{H})} \frac{C(a^{1-s} + c_{1-s} \alpha^s)}{(\lambda_{s,f} - \lambda_{w,f})} \cdot E_{f,\Upsilon_{1-s}} ds \right) \]

where

\[ C = \int_{Z_h \setminus H_k \setminus X_a} f(A) \cdot \overline{f}(D) \cdot f_1(A) \cdot f_2(D) \, dx \]
Since $\theta$ has compact support below $h = a$, the last integral need be evaluated only for $h \leq a$. Using the functional equation

$$c_{1-s} E_{f,T,s} = E_{f,T,1-s}$$

we see

$$\int_{\frac{1}{2}} I_{(s)} \left( \frac{c_{1-s} a^s}{(\lambda_{s,f} - \lambda_{w,f})} \right) \cdot E_{f,T,s} \, ds = \int_{\frac{1}{2}} I_{(s)} \left( \frac{a^{1-s}}{(\lambda_{s,f} - \lambda_{w,f})} \right) \cdot E_{f,T,s} \, ds$$

by changing variables. Thus, for $g$ with $h(g) \leq a$, the integral can be evaluated by residues of vector-valued holomorphic functions as in [Grothendieck] and [Garrett 2011 c].

$$\theta \left( \frac{\eta_{f_1 \otimes f_2}(T_f) \cdot T_f}{(\lambda_{T} - \lambda_{w,f})} \right) + \frac{1}{4\pi i} \int_{\frac{1}{2}} \frac{C(a^{1-s} + c_{1-s} a^s)}{(\lambda_{s,f} - \lambda_{w,f})} \cdot E_{f,T,1-s} \, ds$$

$$= \theta \left( \frac{\eta_{f_1 \otimes f_2}(T_f) \cdot T_f}{(\lambda_{T} - \lambda_{w,f})} \right) + \frac{1}{2\pi i} \int_{\frac{1}{2}} \frac{C(a^{1-s})}{(\lambda_{s,f} - \lambda_{w,f})} \cdot E_{f,T,s} \, ds$$

Consider the integral

$$\int_{\frac{1}{2}} \frac{a^{1-s} \theta E_{f,T,s}}{(\lambda_{s,f} - \lambda_{w,f})} \, ds$$

With $s = \alpha + iT$, consider a rectangle with vertices $\frac{1}{2} \pm iT$ and $T \pm iT$. Let $\gamma_1$ be the line segment from $\frac{1}{2} + iT$ to $T + iT$. Let $\gamma_2$ be the line segment from $T + iT$ to $T - iT$, and let $\gamma_3$ be the line segment from $T - iT$ to $\frac{1}{2} - iT$. We invoke our assumed subconvexity bound $\theta E_{f,T,s} \ll |s|^{1-\epsilon}$. Then we get an estimate

$$\left| \int_{\gamma_1} \frac{a^{1-s} \cdot \theta E_{f,T,s}}{\lambda_{s,f} - \lambda_{w,f}} \, ds \right| \ll \frac{a^{1-s} \cdot |s|^{1-\epsilon}}{|\lambda_{s,f} - \lambda_{w,f}|} \cdot (T - \frac{1}{2})$$

since $\gamma_1$ has length $T - \frac{1}{2}$. Then,

$$\frac{a^{1-s} \cdot |s|^{1-\epsilon}}{|\lambda_{s,f} - \lambda_{w,f}|} \cdot (T - \frac{1}{2}) \leq \frac{a^{1-s} \cdot |s|^{1-\epsilon}}{|\lambda_{s,f} - \lambda_{w,f}|} \cdot (|s| - \frac{1}{2}) \to 0$$

as $T \to \infty$, since the denominator is a degree 2 polynomial in $s$, while the numerator is a polynomial of degree $2 - \epsilon$. Likewise, for the curve $\gamma_2$, we get an estimate

$$\left| \int_{\gamma_2} \frac{a^{1-s} \cdot \theta E_{f,T,s}}{\lambda_{s,f} - \lambda_{w,f}} \, ds \right| \ll \frac{a^{1-s} \cdot |s|^{1-\epsilon}}{|\lambda_{s,f} - \lambda_{w,f}|} \cdot (2T)$$

since $\gamma_1$ has length $2T$. Then,

$$\frac{a^{1-s} \cdot |s|^{1-\epsilon}}{|\lambda_{s,f} - \lambda_{w,f}|} \cdot (T - \frac{1}{2}) \leq \frac{a^{1-s} \cdot |s|^{1-\epsilon}}{|\lambda_{s,f} - \lambda_{w,f}|} \cdot (|s|) \to 0$$

as $T \to \infty$, since the denominator is a degree 2 polynomial in $s$, while the numerator is a polynomial of degree $2 - \epsilon$. A similar argument shows that the integrals along $\gamma_2$ and $\gamma_3$ go to 0 as $T \to 0$. Therefore, the original integral

$$\int_{\frac{1}{2}} \frac{a^{1-s} \theta E_{f,T,s}}{(\lambda_{s,f} - \lambda_{w,f})} \, ds = -2\pi i \text{(sum of residues in the right half-plane)}$$
This implies
\[
\frac{1}{2\pi i} \int_{\frac{1}{2}} a^{1-s} \cdot C \cdot \theta E_{f,T,s} \, ds = -\text{(sum of residues in the right half-plane)}
\]
The Eisenstein series \( E_{f,T,s} \) has a simple pole at \( s = 1 \) ([MW] and [Garrett 2011 f]), with residue
\[
\frac{\eta_{a,f_1 \otimes f_2}(\Upsilon_f) \cdot \Upsilon_f}{(\lambda_{\Upsilon_f} - \lambda_{w,f})} \]
Therefore \( \theta E_{f,T,s} \) has residue at \( s = 1 \) given by
\[
\theta \left( \frac{\eta_{a,f_1 \otimes f_2}(\Upsilon_f) \cdot \Upsilon_f}{(\lambda_{\Upsilon_f} - \lambda_{w,f})} \right)
\]
Thus,
\[
\frac{1}{2\pi i} \int_{\frac{1}{2}} a^{1-s} \cdot C \cdot \theta E_{f,T,s} \, ds = -\theta \left( \frac{\eta_{a,f_1 \otimes f_2}(\Upsilon_f) \cdot \Upsilon_f}{(\lambda_{\Upsilon_f} - \lambda_{w,f})} \right) + \frac{a^{1-w}}{1-2w} \cdot C \cdot \theta E_{f,T,1-w}
\]
Returning to the original equation,
\[
\theta \left( \frac{\eta_{a,f_1 \otimes f_2}(\Upsilon_f) \cdot \Upsilon_f}{(\lambda_{\Upsilon_f} - \lambda_{w,f})} + \frac{1}{2\pi i} \int_{\frac{1}{2}} \frac{C(a^{1-s})}{(\lambda_{\Upsilon_f} - \lambda_{w,f})} \cdot \theta E_{f,T,s} \, ds \right) = \frac{a^{1-w}}{1-2w} \cdot C \cdot \theta E_{1-w,f,T}
\]
Since \( \theta E_{1-w,f,T} = 0 \), we are done.

Recall that \( \Phi_a \) decomposes discretely, with (square-integrable) eigenfunctions consisting of truncated Eisenstein series \( \wedge^a E_{s_j,f,T} \) of Eisenstein series for \( s_j \) such that
\[
a^s \cdot f(A) \cdot \overline{T}(D) + a^{1-s} \cdot c_s \cdot \overline{T}(A) \cdot f(D) = 0
\]
where \( (A, D) \in X_a \), and finitely-many other eigenfunctions. In fact, these truncations are in \( H^{\frac{3}{2} - \epsilon} \) for every \( \epsilon > 0 \), since they are solutions to the differential equation \( (\Delta - \lambda_{w,f})u = \eta_{a,f_1 \otimes f_2} \). There are finitely-many other eigenfunctions in addition to these truncated Eisenstein series.

Let \( S \) denote the operator \( S = 1 - \Delta_a \) with dense domain in \( \Phi_a^{+1} \) as before. Then \( S \) is an unbounded, symmetric, densely-defined operator. We have the continuous injections
\[
\Phi_a^{+1} \to \Phi_a \to \Phi_a^{-1}
\]
Then \( S \) extends by continuity to \( S^\#: \Phi^1_a \to \Phi^1_a \). Since we have the natural inclusion
\[
j : \Phi^1_a \to H^{+1}
\]
taking adjoints produces an inclusion
\[
j^*: H^{-1} \to \Phi^{-1}_a
\]
Let \( j^\#_\theta \) denote the image of \( \theta \) under this mapping. Then we can solve the differential equation
\[
(S^\# - \lambda_w)u = j^\#_\theta
\]
because \( j^\#_\theta \in \Phi_a^{-1} \).
Proposition 25. Take $a \gg 1$ such that the (compact) support of $\theta$ is below height $a$. If necessary, adjust $a$ so that $\theta E_{s_j} \neq 0$ for any $s_j$ such that

$$a^{s_j} \cdot f(A) \cdot \overline{\mathcal{J}}(D) + a^{1-s_j} \cdot c_{s_j} \cdot \overline{\mathcal{J}}(A) \cdot f(D) = 0$$

where $(A,D) \in X_a$. For $w$ not among the $s_j$, the equation $(S^\# - \lambda_{w,f})v = j_0^\theta$ has a unique solution $v_w \in V \cap \Phi_a$, this solution lies in $H^{+1}$, and has spectral expansion

$$v_w = \sum_j \frac{\theta E_{f,j,s_j} \left( \lambda_{s_j,f} - \lambda_{w,f} \right) \wedge^a E_{f,j,s_j}}{\| \wedge^a E_{f,j,s_j} \|^2}$$

Proof. As before, any solution is in $H^{+1}$, since $\theta \in H^{-1}$. The solution $v \in V \cap \Phi_a$ has an expansion in terms of the orthogonal bases $\wedge^a E_{s_j,f,j}$.

$$v_w = \sum_j A_j \frac{\wedge^a E_{s_j,f,j}}{\| \wedge^a E_{s_j,f,j} \|}$$

Thus,

$$j_0^\theta = (S^\# - \lambda_{w,f})v_w = \sum_j (\lambda_{s_j,f} - \lambda_{w,f})A_j \frac{\wedge^a E_{f,j,s_j}}{\| \wedge^a E_{f,j,s_j} \|}$$

Indeed, since the compact support of $\theta$ is below $h = a$, the projection $\theta$ to $V$ is in the $H^{-1}$ completion of $V \cap \Phi_a$. Therefore, the expansion of $j_0^\theta$ in terms of truncated Eisenstein series must be

$$j_0^\theta = \sum_j \frac{\theta E_{f,j,s_j} \wedge^a E_{f,j,s_j}}{\| \wedge^a E_{f,j,s_j} \|^2}$$

noting that $\theta E_{f,j,s_j} = \theta \wedge^a E_{f,j,s_j}$. Thus, the coefficients $A_j$ are uniquely determined, also giving uniqueness.

\[\square\]

Proposition 26. Solutions $w$ to the equation $\theta v_w = 0$ all lie on $(\frac{1}{2} + i \mathbb{R}) \cup [0,1]$, and there is exactly one such between each pair $s_j, s_{j+1}$ of adjacent solutions of

$$| \frac{\det A}{\det D} |^s \cdot \frac{\det A}{\det D} (1-s) \cdot \frac{\Lambda(2s-1, \pi \otimes \pi')}{\Lambda(2s, \pi \otimes \pi')} = 0.$$

Proof. Using the expansion of $v_w$ in $H^{+1}$ in terms of the truncated Eisenstein series, and that of $\theta \in H^{-1}$ in those terms,

$$\theta v_w = \sum_j \frac{|\theta E_{1-s_j,f,j}|^2}{\| \wedge^a E_{s_j,f,j} \|^2}$$

Since every $\lambda_{s_j,f}$ is real, for $\lambda_{w,f} \notin \mathbb{R}$, the imaginary part of $\theta v_w$ is easily seen to be nonzero, thus $\theta v_w \neq 0$. Thus, any solution lies in $(\frac{1}{2} + i \mathbb{R}) \cup \mathbb{R}$. For $\lambda_{w,f} > 0$, all the (infinitely-many) summands are nonnegative real, so the sum can not be 0. Therefore $w \in (\frac{1}{2} + i \mathbb{R}) \cup [0,1]$.

Take $\text{Re}(w) = \frac{1}{2}$ with $\lambda_{s_{j+1},f} < \lambda_{w,f} < \lambda_{s_j,f}$. Note that $\theta v_w \in \mathbb{R}$ for such $w$. For $w$ on the vertical line segment between $s_j$ and $s_{j+1}$, all summands but the $j^{th}$ and $(j+1)^{th}$ are bounded. As $w \to s_j$, $0 < \lambda_{s_j,f} - \lambda_{w,f} \to 0^+$ and $\lambda_{s_{j+1},f} - \lambda_{w,f}$ is bounded. As $w \to s_{j+1}$, $0 > \lambda_{s_{j+1},f} - \lambda_{w,f} \to 0^-$ and $\lambda_{s_j} - \lambda_{w,f}$ is bounded. Since $w \to v_w$ is a holomorphic $H^{+1}$-valued function, $\theta v_w$ is continuous.
By the intermediate value theorem, there is at least one \( w \) between \( s_j \) and \( s_{j+1} \) with \( \theta v_w = 0 \).

To see that there is at most one \( w \) giving \( \theta v_w = 0 \) between each adjacent pair \( s_j, s_{j+1} \) again use holomorphy of \( w \rightarrow v_w \), and take the derivative in \( w \):

\[
\frac{\partial}{\partial w} \theta v_w = \sum_j \frac{|\theta E_{1-s_j,f}|^2 \cdot (2w-1)}{\lambda_{s_j,f} - \lambda_{w,f}} \cdot \left\| \Lambda^a E_{s_j,f} \right\|^2
\]

Everything is positive real except the purely imaginary \( 2w-1 \), because, in fact, the height \( a \) was adjusted so that no \( \theta E_{1-s_j,f} \) vanishes. That is, away from poles, the derivative is non-vanishing, so all zeros are simple.

Returning to the proof of the theorem: suppose \( u \in V \) such that \( (S^# - \lambda_w)u = j^*_w \) with \( \text{Re}(w) = \frac{1}{2} \). For \( u \) to be an eigenfunction for \( \tilde{\Delta}_g \) requires \( \theta u = 0 \) by the nature of the Friedrichs extension.

From above, \( \eta_a u \) vanishes above a height \( a \) depending on the compact support of \( \tilde{\theta} \). Thus, \( u \in V \cap \Phi_a \), so \( u \) must be the solution \( v_w \) expressed as a linear combination of truncated Eisenstein series, and \( \theta v_w = 0 \). Since there is at most one \( w \) giving \( \theta v_w = 0 \) between any two adjacent roots \( s_j \) of

\[
\left| \frac{\det A}{\det D} \right|^s + \left| \frac{\det A}{\det D} \right|^{1-s} \cdot \frac{\Lambda(2s-1, \pi \otimes \pi')}{\Lambda(2s, \pi \otimes \pi')} = 0
\]

giving the constraint.

\[\square\]

10. L-function background

Recall that the \( 2,2 \) constant term of the \( 2,2 \) Eisenstein series with fixed cuspidal, everywhere-spherical data \( f \) and \( \overline{f} \) at height \( h = a \) is

\[
a^s + c_s a^{1-s}
\]

where

\[
c_s = \frac{\Lambda(2(1-s), f \otimes \overline{f})}{\Lambda(2s, f \otimes \overline{f})}
\]

A standard argument principle computation shows that the number of zeros of \( a^s + c_s a^{1-s} \) with imaginary parts between 0 and \( T > 0 \) is

\[
N(T) = \frac{T}{\pi} \log\left( \frac{T}{2\pi e} + T \log a + O(\log T) \right)
\]

All zeros of \( a^s + c_s a^{1-s} \) are on \( \text{Re}(s) = \frac{1}{2} \) for \( a \geq 1 \). Recall ([Iwaniec-Kowalski, p.115]) that

\[
\log L(1+iu, f \otimes \overline{f}) - \log L(1+it, f \otimes \overline{f}) = O\left( \frac{\log t}{\log \log t} \right) \cdot (u-t)
\]

for \( u \geq t \).

**Lemma 2.** The gaps between consecutive zeros of \( a^s + c_s a^{1-s} \) at height greater than or equal to \( T \) are

\[
\frac{\pi}{\log T} + O\left( \frac{1}{\log \log T} \right)
\]
Proof. The condition for the vanishing of \(a^s + c_s a^{1-s}\) can be rewritten as

\[
\frac{\Lambda(2s, f \otimes \mathcal{J})}{\Lambda(2(1-s), f \otimes \mathcal{J})} = -1
\]

where

\[
\Lambda(s, f \otimes f) = \frac{\pi^{1-s}}{2} \Gamma\left(\frac{s + \mu - \nu}{2}\right)\Gamma\left(\frac{s - \mu + \nu}{2}\right)\Gamma\left(\frac{s - \mu - \nu}{2}\right) \Lambda(2s, f \otimes f)
\]

where \(\mu\) is the parameter for the principal series \(I_\mu\) generated by \(f\), while \(\nu\) is the parameter for the principal series generated by \(\mathcal{J}\). Therefore, with \(s\) on the critical line, we have

\[
-1 = \frac{\Gamma\left(\frac{1+2it+\mu-\nu}{2}\right)\Gamma\left(\frac{1+2it-\mu-\nu}{2}\right)\Gamma\left(\frac{1+2it+\mu+\nu}{2}\right)\Gamma\left(\frac{1+2it-\mu+\nu}{2}\right)}{\Gamma\left(\frac{1-2it+\mu-\nu}{2}\right)\Gamma\left(\frac{1-2it-\mu-\nu}{2}\right)\Gamma\left(\frac{1-2it+\mu+\nu}{2}\right)\Gamma\left(\frac{1-2it-\mu+\nu}{2}\right)}
\]

All the factors on the right-hand side are of absolute value 1. The count of zeros as \(t = \text{Im}(s)\) moves from 0 to \(T\) is the number of times the right-hand side assumes the value \(-1\). Regularity is entailed by upper and lower bounds for the derivative of the logarithm of that right-hand side, for large \(t\). Observe that

\[
\frac{d}{dt}\text{Im} \log \frac{\Gamma(a + it)}{\Gamma(a - it)} = 2 \frac{d}{dt}\text{Im} \log \Gamma(a + it)
\]

From the Stirling asymptotic,

\[
\text{log} \Gamma(s) = (s - \frac{1}{2})\text{log} s - s + \frac{1}{2}\text{log} 2\pi + O\left(\frac{1}{s}\right)
\]

in \(\text{Re}(s) \geq \delta > 0\). From this, we have

\[
\text{log} \Gamma(a + it) = it\text{log} (a + it) - (a + it) + \frac{1}{2}\text{log} 2\pi + O\left(\frac{1}{a + it}\right)
\]

\[
= it\left(i(\pi + O\left(\frac{1}{t}\right)) + \text{log} t + O\left(\frac{1}{t^2}\right)\right) - (a + it) + \frac{1}{2\pi}\text{log} 2\pi + O\left(\frac{1}{a + it}\right)
\]

Therefore,

\[
\text{Im} \log \Gamma(a + it) = t\log t - t + O\left(\frac{1}{t}\right)
\]

Consider, for \(0 < \delta \ll t\),

\[
\text{Im} \log (a+i(t+\delta)) - \text{Im} \log (a+it) = ((t+\delta)\text{log} (t+\delta) - (t+\delta)) - (t\log t - t) + O\left(\frac{1}{t}\right)
\]

Which is

\[
\delta \log t - (t+\delta)\frac{\delta}{t} - \delta + O\left(\frac{1}{t}\right) = \delta \log t - 2\delta + O\left(\frac{1}{t}\right)
\]

In particular, for \(0 < \delta \leq \frac{1}{\log t}\),

\[
\text{Im} \log (a+i(t+\delta)) - \text{Im} \log (a+it) = \delta \log t + O\left(\frac{1}{\log t}\right)
\]

Let

\[
f(t) = \frac{\Gamma\left(\frac{1+2it+\mu-\nu}{2}\right)\Gamma\left(\frac{1+2it-\mu-\nu}{2}\right)\Gamma\left(\frac{1+2it+\mu+\nu}{2}\right)\Gamma\left(\frac{1+2it-\mu+\nu}{2}\right)}{\Gamma\left(\frac{1-2it+\mu-\nu}{2}\right)\Gamma\left(\frac{1-2it-\mu-\nu}{2}\right)\Gamma\left(\frac{1-2it+\mu+\nu}{2}\right)\Gamma\left(\frac{1-2it-\mu+\nu}{2}\right)}
\]

Then using the calculation above,

\[
\text{Im} \log f(t + \delta) - \text{Im} \log f(t) = 4\delta \log t + O\left(\frac{1}{\log t}\right)
\]
The result on $L(1 + it, f \otimes f \otimes f \otimes f)$ quoted above gives
\[ \log L(1 + 2it + 2(t + \delta), f \otimes f \otimes f \otimes f) - \log L(1 + 2it, f \otimes f \otimes f \otimes f) = O\left( \frac{\log t}{\log \log t} \right) \]
Therefore,
\[ \text{Im} \log \Lambda(1 + 2i(t + \delta), f \otimes f \otimes f \otimes f) - \text{Im} \log \Lambda(1 + 2it, f \otimes f \otimes f \otimes f) = 4\delta \log t + O\left( \frac{\log t}{\log \log t} \right) \cdot \delta \]
The presence of the 4 being due to the four factors of $\Gamma$ appearing. Thus, if $t$ gives a 0 of the constant term, the next $t' = t + \delta$ giving a zero of the constant term must be such that
\[ 4\delta \log t + O\left( \frac{\log t}{\log \log t} \right) \cdot \delta \geq 2\pi \]
On the other hand, when that inequality is satisfied, then the unit circle will have been traversed, and a zero of the constant term occurs.

Since periods of automorphic forms produce $L$-functions, it is anticipated that $\theta E_s$ will produce a self-adjoint, degree 4 $L$-function, with a corresponding pair-correlation conjecture. That is, given $\epsilon > 0$, there are many pairs of zeros of $\theta E_s$ within $\epsilon$ of each other. The previous section exhibits the zeros $w$ of $\theta E_s$ as parameters of the discrete spectrum of $\tilde{\Delta}_s$. Since parameters of the discrete spectra interlace with the zeros $s_j$ of $a^s + c_s a^{1-s}$, and these are regularly spaced by the argument above, the discrete spectrum is presumably sparse.

11. Appendix I: Harmonic Analysis on $GL_3$

Given a parabolic $P$ in $G = GL_3$ and a function $f$ on $Z_\mathbb{A} G_k \backslash G_\mathbb{A}$, the constant term of $f$ along $P$ is
\[ c_P f(g) = \int_{N_k \backslash N_\mathbb{A}} f(ng) \, dn \]
where $N$ is the unipotent radical of $P$. An automorphic form satisfies the Gelfand condition if, for all maximal parabolics $P$, the constant term along $P$ is zero. If such a function is also $z$-finite, and $K$-finite, it is called a cuspidal. Since the right $G_\mathbb{A}$-action commutes with taking constant terms, the space of functions meeting Gelfand’s condition is $G_\mathbb{A}$-stable, so is a sub-representation of $L^2(Z_\mathbb{A} G_k \backslash G_\mathbb{A})$. Godement, Selberg, and Piatetski-Shapiro showed that integral operators on this space are compact. Specifically, for $\varphi \in C^\infty_c(G)$, the operator $f \rightarrow \varphi \cdot f$ gives a compact operator from $L^2_{\text{cusp}}(Z_\mathbb{A} G_k \backslash G_\mathbb{A})$ to itself. Here,
\[ (\varphi \cdot f)(y) = \int_{Z_\mathbb{A} G_k \backslash G_\mathbb{A}} \varphi(x) \cdot f(yx) \, dx \]
By the spectral theorem for compact operators, this sub-representation decomposes into a direct sum of irreducibles, each appearing with finite multiplicity. To decompose the remainder of $L^2$ demands an understanding of the continuous spectrum, consisting of pseudo-Eisenstein series. We classify non-cuspidal automorphic forms according to their cuspidal support, the smallest parabolic on which they have a nonzero constant term. In $GL_3$, there are three conjugacy classes of proper parabolic subgroups. We will consider the standard parabolic subgroups $P^3 = GL_3$, $P^{2,1}$ and $P^{1,2}$ the maximal parabolics, and $P^{1,1,1}$ the minimal parabolic.
Given the 2,1 parabolic, define a smooth, compactly-supported function \( \varphi \) by

\[
\varphi\left(\begin{pmatrix} A & * \\ 0 & d \end{pmatrix}\right) = \varphi_{\varphi, f_1}\left(\begin{pmatrix} A & * \\ 0 & d \end{pmatrix}\right) = \phi\left(\frac{\det A}{d^2}\right) \cdot f_1(A)
\]

where \( f_1 \) is a GL\(_2\)-cuspform and \( \phi \) is a compactly-supported smooth function. The pseudo-Eisenstein series attached to \( \varphi \) is the function

\[
\Psi_{\varphi}^{2,1}(g) = \sum_{P_k \backslash G_k} \varphi(\gamma g)
\]

Likewise, given the 2,1 parabolic, define a function \( \psi \) by

\[
\psi\left(\begin{pmatrix} a & * \\ 0 & D \end{pmatrix}\right) = \psi_{\varphi, f_2}\left(\begin{pmatrix} a & * \\ 0 & D \end{pmatrix}\right) = \phi\left(\frac{a^2}{\det D}\right) \cdot f_2(D)
\]

again \( \phi_2 \) is a compactly-supported smooth function and \( f_2 \) is a cuspform on GL\(_2\).

Finally, given the 1,1,1 parabolic, define a function \( \psi \) by

\[
\psi\left(\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix}\right) = \psi_{\varphi, f_2}\left(\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix}\right) = g_1\left(\frac{a}{b}\right) \cdot g_2\left(\frac{b}{c}\right)
\]

where \( g_1 \) and \( g_2 \) are compactly-supported smooth functions. Then,

\[
\Psi_{\varphi}(g) = \sum_{\gamma \in P_k \backslash G_k} \psi(\gamma \cdot g)
\]

Next, we exhibit the spaces spanned by non-associate pseudo-Eisenstein series as the orthogonal complement to \( L^2 \) cuspforms.

**Proposition 27.** For any square-integrable automorphic form \( f \) and any pseudo-Eisenstein series \( \Psi_{\varphi}^P \), with \( P \) a parabolic subgroup

\[
\langle f, \Psi_{\varphi}^P \rangle_{Z_k G_k \backslash G_k} = \langle cP f, \varphi \rangle_{Z_k N_k^P M_k^P \backslash G_k}
\]

**Proof.** The proof involves a standard unwinding argument. Observe that

\[
\langle f, \Psi_{\varphi}^P \rangle_{Z_k G_k \backslash G_k} = \int_{Z_k G_k \backslash G_k} f(g) \cdot \overline{\Psi_{\varphi}^P(g)} \, dg = \int_{Z_k G_k \backslash G_k} f(g) \left( \sum_{\gamma \in P_k \backslash G_k} \overline{\varphi(\gamma \cdot g)} \right) \, dg
\]

This is

\[
= \int_{Z_k P_k \backslash G_k} f(g) \varphi(g) \, dg = \int_{Z_k N_k M_k \backslash G_k} f(g) \varphi(g) \, dg
\]

\[
= \int_{Z_k N_k M_k \backslash G_k} \int_{N_k \backslash N_k} f(ng) \overline{\varphi(ng)} \, dn \, dg
\]

\[
= \int_{Z_k N_k M_k \backslash G_k} \left( \int_{N_k \backslash N_k} f(ng) \, dn \right) \overline{\varphi(g)} \, dg
\]

\[
= \langle cP f, \varphi \rangle_{Z_k N_k^P M_k^P \backslash G_k}
\]

\( \square \)
The space spanned by $P_{1,2}$ pseudo-Eisenstein series is the same as the space spanned by $P_{2,1}$ pseudo-Eisenstein series. More generally, pseudo-Eisenstein series of associate parabolics span the same space. The $L^2$ decomposition is that $L^2(Z_k G_k \backslash G_k)$ decomposes as the direct sum of cuspforms together with the spaces spanned by the minimal parabolic pseudo-Eisenstein series and 2,1 pseudo-Eisenstein series with cuspidal data. Following the $GL_2$ case, we will decompose the pseudo-Eisenstein series into genuine Eisenstein series. There are several kinds of Eisenstein series in $GL_3$. For a parabolic $P$, the $P$-Eisenstein series is

$$E_\lambda = \sum_{\gamma \in P \backslash G_k} f_\lambda(\gamma g)$$

where $f_\lambda$ is a spherical vector in a representation $\lambda$ of $M^P$, extended to a $P$-representation by left $N$-invariance, and induced up to $G$. One of the chief ingredients in the spectral decomposition for $GL_2$ pseudo-Eisenstein series was that the Levi component was a product of copies of $GL_1$, allowing us to reduce to the spectral theory for $GL_1$. Unfortunately, this is no longer true for non-minimal parabolic pseudo-Eisenstein series, because the Levi component contains a copy of $GL_2$. Therefore, we will first decompose the minimal parabolic pseudo-Eisenstein series. To this end, we need the functional equation of the Eisenstein series. Because of the increase in dimension, there is more than one functional equation. The symmetries of the Eisenstein series can be described in terms of the action of the Weyl group $W$ on the standard maximal torus $A$, on its Lie algebra $a$, and the dual space $i a^*$. We describe the constant term and the functional equations of the Eisenstein series and use them in the spectral decomposition. For $GL_n$ the standard maximal torus $A$ is the product of $m$ copies of $GL_1$, and representations of $A$ are products of representations of $GL_1$; in the unramified case, these representations are just $y \rightarrow y^{s_i}$, for complex $s_i$. The Weyl group $W$ can be identified with the group of permutation matrices in $GL_n$. It acts on $A$ by permuting the copies of $GL_1$, and it acts on the dual in the canonical way, permuting the $s_i$, in the unramified case. We give a preliminary sketch of the constant term and functional equation of the Eisenstein series, with details to be filled in later. The constant term of the Eisenstein series (along the minimal parabolic) has the form

$$c P(E_\lambda) = \sum_{w \in W} c_w(\lambda) \cdot w_\lambda$$

where $w_\lambda$ is the image of $\lambda$ under the action of $w$ and $c_w(\lambda)$ is a constant depending on $w$ and $\lambda$ with $c_1(\lambda) = 1$. The Eisenstein series has functional equations

$$c_w(\lambda) \cdot E_\lambda = E_{w \lambda} \text{ for all } w \in W$$

We start the decomposition of $\Psi_\varphi$ by using the spectral expansion of its data $\varphi$. Recall that $\varphi$ is left $N_k$-invariant, so it is essentially a function on the Levi component, which is a product of copies of $k^\times \backslash \mathbb{J}$. By Fujisaki’s lemma, this is the product of a ray with a compact abelian group. We assume that the compact abelian group is trivial. So spectrally decomposing $\varphi$ is a higher-dimensional version of Mellin inversion.

$$\varphi = \int \langle \varphi, \lambda \rangle \cdot \lambda \, d\lambda$$
Winding up gives

\[ \Psi_\varphi(g) = \int_{i\sigma} \langle \varphi, \lambda \rangle \cdot E_\lambda(g) \, d\lambda \]

In order for this decomposition to be valid, the parameters of \( \lambda \) must have \( \text{Re}(s) \gg 1 \). However, in order to use the symmetries of the functional equations, we need the parameters to be on the critical line. In moving the contours, we pick up some residues, which are mercifully constants. Breaking up the dual space according to Weyl chambers and changing variables,

\[ \Psi_\varphi(g) - (\text{residues}) = \sum_{w \in W} \int_{1\text{st Weyl chamber}} \langle \varphi, w_\lambda \rangle \cdot c_w(\lambda) \cdot E_\lambda(g) \, d\lambda \]

Now using the functional equations,

\[ \Psi_\varphi(g) - (\text{residues}) = \sum_{w \in W} \int_{(1)} \langle \varphi, w_\lambda \rangle \cdot c_w(\lambda) \cdot E_\lambda(g) \, d\lambda \]

This is

\[ \int_{(1)} \sum_{w \in W} \langle \varphi, c_w(\lambda) w_\lambda \rangle \cdot E_\lambda(g) \, d\lambda \]

We recognize the constant term of the Eisenstein series and apply the adjointness relation

\[ \sum_{w \in W} \langle \varphi, c_w(\lambda) w_\lambda \rangle = \langle \varphi, c_p E_\lambda \rangle = \langle \Psi_\varphi, E_\lambda \rangle \]

So we have

\[ \Psi_\varphi(g) = \sum_{(1)} \langle \Psi_\varphi, E_\lambda \rangle \cdot E_\lambda(g) \, d\lambda + \text{residues} \]

Our next goal is to show that the remaining automorphic forms, namely those with cuspidal support \( P^{1,2} \) or \( P^{2,1} \) can be written as superpositions of genuine \( P^{2,1} \) Eisenstein series. To do this it suffices to decompose \( P^{2,1} \) and \( P^{1,2} \) pseudo-Eisenstein series with cuspidal support. Let \( P = P^{1,2} \) and \( Q = P^{2,1} \). We start by looking more carefully at pseudo-Eisenstein series with cuspidal data. The data for a \( P \) pseudo-Eisenstein series is smooth, compactly-supported, and left \( Z_k M_k^P \cdot N_k^P \)-invariant. Assume the data is spherical. Then the function is determined by its behavior on \( Z_k M_k^P \cdot N_k^P \). In contrast to the minimal parabolic case, this is not a product of copies of \( GL_1 \), so we can not use the \( GL_1 \) spectral theory of Mellin inversion to establish the decomposition. Instead the quotient is isomorphic to \( GL_2(k) \setminus G \), so we will use the spectral theory for \( GL_2 \). If \( \eta \) is the data for a \( P^{2,1} \) pseudo-Eisenstein series \( \Psi_\eta \), we can write \( \eta \) as a tensor product \( \eta = f \times \nu \) on \( Z_{GL_2(k)} \setminus GL_2(k) \cdot Z_{GL_2(k)} \setminus Z_{GL_2(k)} \)

Saying that the data is cuspidal means that \( f \) is a cuspidal form. Similarly the data \( \varphi = \varphi_{F,s} \) for a \( P^{2,1} \)-Eisenstein series is the tensor product of a \( GL_2 \) cusp form \( F \) and a character \( \lambda_s = |.|^s \) on \( GL_1 \). We show that \( \Psi_{f,\nu} \) is the superposition of Eisenstein series \( E_{F,s} \) where \( F \) ranges over an orthonormal basis of cuspforms and \( s \) is on a vertical line.

Using the spectral expansions of \( f \) and \( \nu \),

\[ \eta = f \otimes \nu = \left( \sum_{\text{cfs}} f \cdot F \right) \cdot \left( \int_{s} \nu \cdot \langle \lambda_s \rangle \cdot \lambda_s \, ds \right) = \sum_{\text{cfs}} \int_{s} \langle \eta_{f,\nu}, \varphi_{F,s} \rangle \cdot \varphi_{F,s} \, ds \]
So the pseudo-Eisenstein series can be re-expressed as a superposition of Eisenstein series

\[ \Psi_{f,\nu}(g) = \sum_{c \in \mathcal{M}} \int_{s} \langle \eta_{f,\nu}, \varphi_{F,s} \rangle \cdot E_{F,s}(g) \, ds \]

In fact the coefficient \( \langle \eta, \varphi \rangle_{GL_2} \) is the same as the pairing \( \langle \Psi_{\eta}, E_{\varphi} \rangle_{GL_3} \), since

\[ \langle \Psi_{\eta}, E_{\varphi} \rangle = \langle c_{P}(\Psi_{\eta}), \varphi \rangle = \langle \eta, \nu \rangle \]

So the spectral expansion is

\[ \Psi_{f,\nu} = \sum_{c \in \mathcal{M}} \int_{s} \langle \Psi_{f,\nu}, E_{F,s} \rangle \cdot E_{F,s}(g) \, ds \]

So far, we have not had to shift the line of integration to the critical line \( \frac{1}{2} + i\mathbb{R} \). It now remains to show that pseudo-Eisenstein series for the associate parabolic \( Q = P^{1,2} \) can also be decomposed into superpositions of \( P \)-Eisenstein series. For maximal parabolic pseudo-Eisenstein series, the functional equation does not relate the Eisenstein series to itself but rather to the Eisenstein series of the associate parabolic. We will use this functional equation to obtain the decomposition of associate parabolic pseudo-Eisenstein series. The functional equation is

\[ E_{Q}^{F,s} = b_{F,s} \cdot E_{P,F,1-s}^{P} \]

where \( b_{f,s} \) is a meromorphic function that appears in the computation of the constant term along \( P \) of the \( Q \)-Eisenstein series.

We consider a \( Q \)-pseudo-Eisenstein series \( \Psi_{f,\nu}^{Q} \) with cuspidal data. By the same arguments used above to obtain the decomposition of \( P \)-pseudo-Eisenstein series, we can decompose \( \Psi_{f,\nu}^{Q} \) into a superposition of \( Q \)-Eisenstein series

\[ \Psi_{f,\nu}^{Q}(g) = \sum_{c \in \mathcal{M}} \int_{s} \langle \Psi_{f,\nu}^{Q}, \varphi_{F,s} \rangle \cdot E_{F,s}^{Q}(g) \]

Now using the functional equation,

\[ \Psi_{f,\nu}^{Q}(g) = \sum_{c \in \mathcal{M}} \int_{s} \langle \Psi_{f,\nu}^{Q}, b_{F,s} \cdot E_{F,1-s}^{P} \rangle \cdot b_{F,s} \cdot E_{F,1-s}^{P} \]

So we have a decomposition of \( Q \)-pseudo-Eisenstein series (with cuspidal data) into \( P \)-Eisenstein series (with cuspidal data). In order to use the functional equation we moved some contours, but there are no poles, so no residues are acquired.

We have described the spectral decomposition of \( L^{2}(Z_{k}G_{k}\backslash G_{k}) \) as the direct sum/integral of irreducibles. Any automorphic form \( \xi \) can be written as

\[ \xi = \sum_{GL_{3}} \langle \xi, f \rangle_{f} + \sum_{GL_{2}} \int_{s} \langle \xi, E_{F,s}^{2,1} \rangle \cdot E_{F,s}^{2,1} + \int_{(1)} \langle \xi, E_{\lambda}^{1,1,1} \rangle \cdot E_{\lambda}^{1,1,1} \, d\lambda + \frac{\langle \xi, 1 \rangle}{(1,1)} \]

This converges in \( L^{2} \).
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