Research Article

The Dynamics of a Stochastic SIR Epidemic Model with Nonlinear Incidence and Vertical Transmission

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In this study, we build a stochastic SIR epidemic model with vertical infection and nonlinear incidence. The influence of the fluctuation of disease transmission parameters and state variables on the dynamic behaviors of the system is the focus of our study. Through the theoretical analysis, we obtain that there exists a unique global positive solution for any positive initial value. A threshold \( R_0 \) is given. When \( R_0 < 1 \), the diseases can be extincted with probability one. When \( R_0 > 1 \), we construct a stochastic Lyapunov function to prove that the system exists an ergodic stationary distribution, which means that the disease will persist. Then, we obtain the conditions that the solution of the stochastic model fluctuates widely near the equilibria of the corresponding deterministic model. Finally, the correctness of the results is verified by numerical simulation. It is further found that the fluctuation of disease transmission parameters and infected individuals with the environment can reduce the threshold of disease outbreak, while the fluctuation of susceptible and recovered individuals has a little effect on the dynamic behavior of the system. Therefore, we can make the disease extinct by adjusting the appropriate random disturbance.

1. Introduction

At the beginning of 2020, a sudden epidemic (COVID-19) has disrupted people’s normal life. In order to curb the spread of the epidemic, the state has taken measures such as closing cities and delaying the opening of schools to protect people’s lives to the maximum extent. So far, the disease has not been completely controlled. Emerging infectious diseases have brought fear and inconvenience to people’s lives and have a great impact on the global economy. It is very important to study the spread of infectious diseases.

Dynamic modeling is an important method to study the spread of infectious diseases. In the classical epidemic model, such as the SIR model, the total population is generally divided into three categories, the number of susceptible individuals by \( S(t) \), the number of infectious individuals by \( I(t) \), and the number of permanently immune individuals by \( R(t) \). The model is as follows.

\[
\frac{dS}{dt} = A + bS(t) - \frac{\beta S(t)I(t)}{f(I)} - dS(t) + b(1 - p)I(t),
\]
\[
\frac{dI}{dt} = bpI(t) + \frac{\beta S(t)I(t)}{f(I)} - (d + \mu + \gamma)I(t),
\]
\[
\frac{dR}{dt} = bR(t) + \gamma I(t) - dR(t),
\]

where \( A \) is the recruitment rate of susceptible corresponding to immigration, \( b \) is the birth rate, \( d \) is the nature death rate, \( \mu \) is the disease induced mortality rate, \( \gamma \) is the rate of recovery from infection, a positive constant \( p (0 \leq p \leq 1) \) is the proportion of infection in the offspring of infected mothers, \( 1 - p \) is the proportion of susceptible individuals in the offspring of infected mothers, and \( \beta \) is the transmission coefficient between compartments \( S(t) \) and \( I(t) \). \( f(I) \) is
continuously differentiable function and assume $f(0) = 1$ and $f'(1) \geq 0$.

For the theoretical research of this model (1), there are a lot of literatures; interested readers can refer to the literature [1–5]. The classical ODE model can reflect the spread of disease to some extent, but in real life, the spread of infectious diseases is also affected by many random factors (as shown in [6]), for example, the unpredictability of person-to-person contact, which means that there is not necessarily uniform contact between individuals. In other words, the process of disease transmission is inevitably affected by random factors. In this way, the stochastic differential model is more suitable than the deterministic model.

There are many literatures on the stochastic differential model of infectious diseases [7–20]. Here, we mainly introduce the literature of stochastic differential models related to the SIR epidemic model in detail. In literature [8], Aaid and Omari built the SIRS stochastic differential model with parameter perturbation, vaccination of recruitment susceptible, and nonlinear incidence rate. They found that large enough random disturbances can suppress outbreaks. In 2015, Liu and Chen [7] considered a model of literature [8] and observed that the solution of the system fluctuates near the disease-free equilibrium under suitable conditions. Zhou et al. [16] studied an SIR model with the bilinear infection rate and stochastic perturbation of parameter and state variables and got the conditions of survival and stationary distribution.

Since the contact between populations and the population itself are affected by many random factors in the environment, we will introduce stochastic white noise perturbations into system (1) by two different approaches.

First of all, inspired by the literature [21], we consider a discrete time Markov chain. For a fixed time increment $\Delta t > 0$, we define the process $X^{\Delta t}(t) = (X^{\Delta t}_1(t), X^{\Delta t}_2(t), X^{\Delta t}_3(t))$ (for convenience, we replace $S, I, R$ by $X_1, X_2, X_3$) for $t = 0, \Delta t, 2\Delta t, \ldots$, where $X^{\Delta t}(0) = (S(0), I(0), R(0)) \in \mathbb{R}^3_+$ is a deterministic initial value. Let $\{V^{\Delta t}_i(k)\}_{k=0}^\infty (i = 1, 2, 3)$ be three sequences of random variables. Suppose that these variables are jointly independent and that within each sequence the variables are identically distributed, such that

$$E[V^{\Delta t}_i(k)] = 0,$$

$$E[V^{\Delta t}_i(k)]^2 = \sigma_i^2 \Delta t, \quad i = 1, 2, 3,$$

where the parameter $\sigma_i \geq 0 (i = 1, 2, 3)$ represents the intensity of the environmental white noise. We assume that $X^{\Delta t}$ grows within that time period according to the deterministic system (1) and the random amount $(V^{\Delta t}_1(k)X^{\Delta t}_1(k\Delta t), V^{\Delta t}_2(k)X^{\Delta t}_2(k\Delta t), V^{\Delta t}_3(k)X^{\Delta t}_3(k\Delta t))^T$. Specifically, for $k = 0, 1, \ldots$, we set

$$X^{\Delta t}((k + 1)\Delta t) = X^{\Delta t}_1(k\Delta t) + V^{\Delta t}_1(k)X^{\Delta t}_1(k\Delta t) + \left(A - (d - b)X^{\Delta t}_1(k\Delta t) + b(1 - p)X^{\Delta t}_2(k\Delta t) - \frac{\beta X^{\Delta t}_1(k\Delta t)X^{\Delta t}_2(k\Delta t)}{f(X^{\Delta t}_2(k\Delta t))}\right)\Delta t,$$

$$X^{\Delta t}_2((k + 1)\Delta t) = X^{\Delta t}_2(k\Delta t) + V^{\Delta t}_2(k)X^{\Delta t}_2(k\Delta t) + \left(bpX^{\Delta t}_2(k\Delta t) - (d + \mu + \gamma)X^{\Delta t}_2(k\Delta t) + \frac{\beta X^{\Delta t}_2(k\Delta t)X^{\Delta t}_1(k\Delta t)}{f(X^{\Delta t}_2(k\Delta t))}\right)\Delta t,$$

$$X^{\Delta t}_3((k + 1)\Delta t) = X^{\Delta t}_3(k\Delta t) + V^{\Delta t}_3(k)X^{\Delta t}_3(k\Delta t) + \left(\gamma X^{\Delta t}_2(k\Delta t) - (d - b)X^{\Delta t}_3(k\Delta t)\right)\Delta t.$$

Then, we will show that $X^{\Delta t}(t)$ converges weakly to a diffusion process as $\Delta t \rightarrow 0$. To determine the drift coefficients of the diffusion, first, let $p^{\Delta t}(x, dy)$ denote the transition probabilities of the homogeneous Markov chain $\{X^{\Delta t}(k\Delta t)\}_{k=0}^\infty$, that is,

$$\frac{1}{\Delta t} \int (y - s)p^{\Delta t}(x, dy) = E\left[A - (d - b)s - \frac{\beta si}{f(i)} + b(1 - p)i + \frac{s}{\Delta t}V^{\Delta t}_i(k)\right] = A - (d - b)s - \frac{\beta si}{f(i)} + b(1 - p)i,$$

for all $x = (s, i, r) \in \mathbb{R}^3_+$ and any Borel set $A \in \mathbb{R}^3_+$.
\[ \frac{1}{\Delta t} \int (y_2 - i)p_{\Delta t}(x, dy) = E\left[ b pi + \frac{\beta si}{f(i)} - (d + \mu + \gamma)i + \frac{i}{\Delta t} V^\Delta_2 (k) \right] \]

\[ = b pi + \frac{\beta si}{f(i)} - (d + \mu + \gamma)i, \]

\[ \frac{1}{\Delta t} \int (y_3 - r)p_{\Delta t}(x, dy) = E\left[ yi - (d - b)r + \frac{r}{\Delta t} V^\Delta_3 (k) \right] = yi - (d - b)r. \]

To determine the diffusion coefficients, we consider the moments

\[ g^{\Delta t}_{ij}(z) = \frac{1}{\Delta t} \int (y_i - z_i)(y_j - z_j)p_{\Delta t}(x, dy), \quad i, j = 1, 2, 3 \]

\[ \left| g^{\Delta t}_{11}(x) - \sigma^2_{1} x^2 \right| = \frac{1}{\Delta t} E \left[ \Delta t \left( A - (d - b)s - \frac{\beta si}{f(i)} + b(1 - p)i \right) + sV^\Delta_1 (k) \right]^2 - \sigma^2_{1} x^2 \]

\[ = \Delta t \left( A - (d - b)s - \frac{\beta si}{f(i)} + b(1 - p)i \right)^2. \]

Therefore,

\[ \lim_{\Delta t \to 0} \sup_{|x| \leq K} \left| g^{\Delta t}_{11}(x) - \sigma^2_{1} x^2 \right| = 0, \quad \text{for all} \quad 0 < K < \infty. \]

We can conclude that as \( \Delta t \to 0 \), \( X^\Delta t(t) \) converge weakly to the solution \( X(t) = (S(t), I(t), R(t)) \) of the following SDE:

\[ dS = \left( A - (d - b)S(t) - \frac{\beta S(t)I(t)}{f(I(t))} + b(1 - p)I(t) \right) dt + \sigma_1 S(t) dB_1(t), \]

\[ dI = \left( bpi(t) + \frac{\beta S(t)I(t)}{f(I(t))} - (d + \mu + \gamma)I(t) \right) dt + \sigma_2 I(t) dB_2(t), \]

\[ dR = (\gamma I(t) - (d - b)R(t))dt + \sigma_3 R(t) dB_3(t), \]
where $B_i(t)$, $i = 1, 2, 3$ is a standard Brownian motion.

Then, according to [23], we replace the parameter $\beta$ by $\beta + \sigma_4 dB_4(t)$. The stochastic SIR epidemic model with vertical transmission and nonlinear incidence is as follows.

\[ dS = \left( A - (d - b)S(t) - \frac{\beta S(t)I(t)}{f(I(t))} + b(1 - p)I(t) \right) dt + \sigma_1 S(t)dB_1(t) - \frac{\sigma S(t)I(t)}{f(I(t))} dB_4(t), \]

\[ dI = \left( \beta S(t)I(t) + \frac{\beta S(t)I(t)}{f(I(t))} - (d + \mu + \gamma)I(t) \right) dt + \sigma_2 I(t)dB_2(t) + \frac{\sigma S(t)I(t)}{f(I(t))} dB_4(t), \]

\[ dR = (\gamma I(t) - (d - b)R(t))dt + \sigma_3 R(t)dB_3(t), \]

where $B_i(t)$ is a standard Brownian motion and parameter $\sigma_4 \geq 0$ represents the intensity of the environmental white noise.

Then, according to [23], we replace the parameter $\beta$ by $\beta + \sigma_4 dB_4(t)$. The stochastic SIR epidemic model with vertical transmission and nonlinear incidence is as follows.

2. Existence and Uniqueness of the Global Positive Solution

In order to analyze the dynamic behavior of the system, we first need to discuss whether the solution of the system is nonnegative and global existence? In stochastic differential equations, if their coefficients satisfy local Lipschitz conditions and linear growth conditions, system (15) has a global positive solution [24]. However, the coefficients of system (15) do not satisfy linear growth conditions, which may lead to diseases exploded at a finite time. Next, we will show that the solution of system (15) does not blow up in finite time, so that the solution of system (15) will be global.

**Theorem 1.** For any given initial condition $(S(0), I(0), R(0)) \in R^3_+$, there is a unique positive solution $(S(t), I(t), R(t))$ to system (13) on $t \geq 0$, and the solution remains in $R^3_+$ with probability one, namely, $(S(t), I(t), R(t)) \in R^3_+$ for all $t \geq 0$ almost surely.

**Proof.** Since the coefficients of system (15) are locally Lipschitz continuous, for any given initial value $(S(0), I(0), R(0)) \in R^3_+$, there is a unique local solution $(S(t), I(t), R(t))$ on $t \in [0, \tau_e)$, where $\tau_e$ is the explosion time. Next, we will verify that this solution is global, i.e., $\tau_e = +\infty$ a.s. First, let $\xi_0$ be sufficiently large, such that $(S(0), I(0), R(0))$ all lie within the interval $[(1/\xi_0), \xi_0]$. For each integer $\xi \geq \xi_0$, define the stopping time as [24]

\[ \tau_\xi = \inf\{t \in [0, \tau_e) : S(t) \leq 0 or I(t) \leq 0 or R(t) \leq 0\}. \]  

Without loss of generality, let inf $\emptyset = +\infty$ ($\emptyset$ is the empty set); we have, $\tau_e \leq \tau_\xi$. If we prove that $\tau_\xi = +\infty$, a.s., then $\tau_e = +\infty$ and $(S(t), I(t), R(t)) \in R^3_+$ a.s. for all $t \geq 0$. If the statement is not true, then there is a positive constant $T$, such that

\[ P[\tau_\xi < T] > 0. \]  

Define a $C^2$ function $V: R^3_+ \rightarrow R$ by

\[ V(S(t), I(t), R(t)) = \ln(S(t)I(t)R(t)). \]  

For $\omega \in \{ \tau_\xi < T \}$ and all $t \in [0, \tau_\xi)$, by Itô’s formula, one can verify that

\[ dV(x, t) = \partial_{xx}V(x, t) + \partial_{xx}V(x, t)g(x, t)dB(t), \]
\[
\frac{dV}{dt}(S(t), I(t), R(t)) = \left( \frac{A}{S(t)} - \frac{\beta I(t)}{f(I(t))} - \frac{b(1-p)I(t)}{S(t)} + \frac{\beta S(t)}{f(I(t))} + b \right) + (d - b) \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) - \\
\frac{1}{2} \left( -\frac{\sigma_1^2 t^2}{f^2(I(t))} + \frac{\sigma_2^2 S^2(t)}{f^2(I(t))} \right) + \frac{1}{2} \left( \frac{\sigma_1^2 t^2}{f^2(I(t))} + \frac{\sigma_2^2 S^2(t)}{f^2(I(t))} \right) \right) dt + \sigma_1 dB_1(t) - \frac{I(t)}{f(I(t))} dB_4(t) \\
+ \sigma_2 dB_2(t) + \sigma_3 dB_3(t) \\
\geq \left( -\beta I(t) - 2(d - b) - (d + \mu + \gamma) - \frac{1}{2} \sigma_4^2 S(t) \right) \\
\frac{I(t)}{f(I(t))} dB_4(t) + \frac{S(t)}{f(I(t))} dB_4(t) + \sigma_3 dB_3(t). \\
\] (23)

Using \( f(0) = 1 \) and \( f'(I) \geq 0 \), the inequality (23) can be reduced to

\[
\frac{dV}{dt}(S(t), I(t), R(t)) \geq H(S(t), I(t), R(t))dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t) \\
- \frac{I(t)}{f(I(t))} dB_4(t) + \frac{S(t)}{f(I(t))} dB_4(t) + \sigma_3 dB_3(t), \\
\] (24)

here,

\[
H(S(t), I(t), R(t)) = -\beta I(t) - 2(d - b) - d \\
- \mu - \gamma - \frac{1}{2} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) \\
- \frac{1}{2} \sigma_4^2 t^2(t) + \frac{1}{2} \sigma_4^2 S(t). \\
\] (25)

Integrating both sides of the inequality (24) from 0 to \( t \) yields

\[
V(S(t), I(t), R(t)) \geq V(S(0), I(0), R(0)) \\
+ \int_0^t H(S(r), I(r), R(r))dr + \sigma_1 B_1(t) \\
+ \sigma_2 B_2(t) + \sigma_3 \int_0^t S(r) - I(r) dB_4(r) \\
+ \sigma_3 dB_3(t). \\
\] (26)

Note that some components of \( S(\tau_\ell), I(\tau_\ell), R(\tau_\ell) \) equal 0; therefore,

\[
\lim_{t \to \tau_\ell} V(S(t), I(t), R(t)) = -\infty. \\
\] (27)

Letting \( t \to \tau_\ell \) in (26), it leads to the contradiction:

\[
-\infty \geq V(S(0), I(0), R(0)) \\
+ \int_0^{\tau_\ell} H(S(r), I(r), R(r))dr + \sigma_1 B_1(\tau_\ell) \\
+ \sigma_2 B_2(\tau_\ell) + \sigma_3 \int_0^{\tau_\ell} S(r) - I(r) dB_4(r) + \sigma_3 B_3(\tau_\ell) \\
> -\infty. \\
\] (28)

Thus, \( \tau_\ell = +\infty \) a.s. The proof is completed. \( \square \)

### 3. Extinction of the Disease

For system (1), using the notations in [25], we can deduce the basic reproductive number \( R_0 \).

First, we have two vectors \( S \) and \( P' \) to represent the new infection term and remaining transfer terms, respectively:
\[
\mathcal{F} = \begin{pmatrix}
\frac{\beta SI}{f(I)} \\
0 \\
0
\end{pmatrix},
\]

\[
\mathcal{V} = \begin{pmatrix}
(d + \mu + \gamma)I - bpI \\
-bR - \gamma I + dR \\
-A - bS + \frac{\beta SI}{f(I)} + dS - b(1 - p)I
\end{pmatrix}.
\]

The infected compartment is \(I\) then,

\[
F = \frac{\beta A}{d - b},
\]

\[
V = (d + \mu + \gamma - bp).
\]

Hence, the reproduction number is given by

\[
R_0 = \rho(FV^{-1}) = \frac{\beta A}{(d - b)(d + \mu + \gamma - bp)},
\]

where \(d > 0\) and \(0 \leq p \leq 1\). If \(R_0 \leq 1\), the disease will go to extinction. If \(R_0 > 1\), the disease will persist. For system (15), we focus on whether diseases can be extincted by regulating the parameters of system (15). For the sake of simplicity, denote \(\langle x(t) \rangle = (1/t) \int_0^t x(s)ds\).

Lemma 1. Let \((S(t), I(t), R(t))\) be the solution of system (15) with initial value \((S(0), I(0), R(0)) \in \mathbb{R}_+^3\), then

\[
\lim_{t \to \infty} \frac{S(t)}{t} = 0,
\]

\[
\lim_{t \to \infty} \frac{I(t)}{t} = 0,
\]

\[
\lim_{t \to \infty} \frac{R(t)}{t} = 0, \text{ a.s.,}
\]

\[
\lim_{t \to \infty} \int_0^t S(u)dB_1(u) = 0, \text{ and}
\]

\[
\lim_{t \to \infty} \int_0^t I(u)dB_2(u) = 0, \text{ a.s.,}
\]

\[
\lim_{t \to \infty} \int_0^t R(u)dB_3(u) = 0, \text{ a.s.,}
\]

moreover,

\[
\lim_{t \to \infty} \sup \langle S(t) + I(t) + R(t) \rangle \leq \frac{A}{d - b}, \text{ a.s.}
\]

Proof. Using the method of Lemma 1 in [26], it is easy to prove the conclusion (32) and (33). Then, we prove that conclusion (34) holds. From system (15), we obtain

\[
d(S(t) + I(t) + R(t)) = (A - (d - b)(S(t) + I(t) + R(t)) - \mu I(t))dt
\]

\[
+ \sigma_1 S(t)dB_1(t) + \sigma_2 I(t)dB_2(t) + \sigma_3 R(t)dB_3(t).
\]

Integrating both sides of (35) from 0 to \(t\), then dividing by \(t\) yields

\[
\frac{S(t) + I(t) + R(t)}{t} - \frac{S(0) + I(0) + R(0)}{t} = A - (d - b)\langle S(t) + I(t) + R(t) \rangle - \mu \langle I(t) \rangle
\]

\[
+ \sigma_1 \int_0^t S(u)dB_1(u) \quad + \sigma_2 \int_0^t I(u)dB_2(u) \quad + \sigma_3 \int_0^t R(u)dB_3(u).
\]

Taking the limit of both sides of (36) and using (32) and (33), we have

\[
\lim_{t \to \infty} \sup \langle S(t) + I(t) + R(t) \rangle \leq \frac{A}{d - b}, \text{ a.s.}
\]

The proof of the lemma is completed.

Define

\[
R'_0 = \frac{1}{d - b} \left( \frac{\beta A}{d - b} - \frac{1}{2} \sigma_2^2 - \frac{1}{2} \sigma_1^2 \left( \frac{A}{d - b} \right)^2 \right)
\]

\[
= R_0 - \frac{\sigma_2^2 + \sigma_1^2 (A(d - b))^2}{2(d + \mu + \gamma - bp)}
\]

It is obvious that \(R'_0 \leq R_0\). \(\square\)
Theorem 2. Let \((S(t), I(t), R(t))\) be the solution of system (15) with initial value \((S(0), I(0), R(0)) \in R^3_+\) and suppose that one of the following conditions holds:

\(A\) \(R_0^\prime < 1\) and \(\beta > \frac{A\sigma_4^2}{d-b}\)

\(B\) \(\beta \leq \frac{A\sigma_4^2}{d-b}\) and \(\frac{\beta^2}{2\sigma_4^2} - (d + \mu + \gamma - bp) \leq 0\) and \(\sigma_4 \neq 0\),

then, \(I(t)\) will tend to zero exponentially with probability one, i.e.,

\[
\lim_{t \to \infty} \sup_t \frac{\log I(t)}{t} \leq (d + \mu + \gamma - bp)(R_0^\prime - 1)
\]

\[
< 0, \text{ a.s., if } (A) \text{ holds,}
\]

\[
\lim_{t \to \infty} \sup_t \frac{\log I(t)}{t} \leq \frac{\beta^2}{2\sigma_4^2} - (d + \mu + \gamma - bp) - \frac{1}{2}\sigma_4^2 < 0, \text{ a.s., if } (B) \text{ holds.}
\]

In addition, we also have

\[
\lim_{t \to \infty} \frac{\log R(t)}{t} \leq \min\{d - b + \frac{1}{2}\sigma_3^2, k\}, \text{ a.s.,}
\]

where \(k\) is a positive constant.

Proof. Using Itô's formula, we have

\[
\frac{\log I(t)}{t} = \frac{\log I(0)}{t} + \beta \left( \frac{S(t)}{f(I(t))} \right) - (d + \mu + \gamma - bp) - \frac{1}{2}\sigma_4^2 \left( \frac{S(t)}{f(I(t))} \right)^2
\]

\[
+ \frac{B_2(t)}{t} + \sigma_4 \int_0^t \left( S(u) / f(I(u)) \right) dB_4(u)
\]

\[
\leq \frac{\log I(0)}{t} + \beta \left( \frac{S(t)}{f(I(t))} \right) - (d + \mu + \gamma - bp) - \frac{1}{2}\sigma_4^2 \left( \frac{S(t)}{f(I(t))} \right)^2
\]

\[
+ \frac{B_2(t)}{t} + \sigma_4 \int_0^t \left( S(u) / f(I(u)) \right) dB_4(u)
\]

\[
= \frac{\log I(0)}{t} - \frac{1}{2}\sigma_4^2 \left( \frac{S(t)}{f(I(t))} \right)^2 - \left( \frac{\beta}{\sigma_4^2} \right)^2 - \frac{\beta^2}{2\sigma_4^2} - (d + \mu + \gamma - bp) - \frac{1}{2}\sigma_4^2
\]

\[
+ \frac{B_2(t)}{t} + \sigma_4 \int_0^t \left( S(u) / f(I(u)) \right) dB_4(u)
\]

Taking integrate both sides of (43) from 0 to \(t\), then dividing by \(t\), we yield
By the strong law of large numbers for martingales [24] and Lemma (32), we have
\[
\lim_{t \to \infty} \frac{B_4(t)}{t} = 0, \quad \text{a.s.}
\]
(45)
and using (45), we obtain
\[
\limsup_{t \to \infty} \frac{\log I(t)}{t} \leq \frac{\beta A}{d - b} - (d + \mu + \gamma - bp) - \frac{1}{2} \sigma_2^2
\]
\[
\leq \frac{\beta A}{d - b} - (d + \mu + \gamma - bp) - \frac{1}{2} \sigma_2^2 \frac{A^2}{d - b}
\]
\[
= (d + \mu + \gamma - bp) (R_0^\ast - 1),
\]
\[
< 0.
\]
If condition (B) holds, (44) becomes
\[
\log I(t) \geq \log I(0) + \frac{\beta^2}{2 \sigma_4^2} - \frac{1}{2} \sigma_2^2 + \sigma_2 B_2(t)
\]
\[
+ \sigma_4 \frac{\int_0^t (S(u)/f(I(u)))dB_4(u)}{t}.
\]
(48)
Furthermore, using (45), we can get
\[
\limsup_{t \to \infty} \frac{\log I(t)}{t} \leq \frac{\beta^2}{2 \sigma_4^2} - (d + \mu + \gamma - bp) - \frac{1}{2} \sigma_2^2 < 0,
\]
(49)
So far, we proved (40) and (41). From these two limits, we can get that there is a constant \(k > 0\), such that for almost all \(\omega \in \Omega\), there exists a \(T_0 = T_0(\omega) > 0\), when \(t > T_0\),
\[
I(t, \omega) \leq e^{-kt}.
\]
Without loss of generality, assume that \(I(t, \omega) \leq e^{-kt}\) for all \(t \geq 0\), and this implies
\[
\lim_{t \to \infty} I(t) = 0, \quad \text{a.s.}
\]
(50)
Solving the third equation of system (15) and denoting \(h(t) = (d - b + (1/2)\sigma_3^2)k - \sigma_3 B_2(t)\), we have
\[
R(t) = e^{-k(t)} \left[ R(0) + \int_0^t e^{b(u)} I(u)du \right]
\]
\[
\leq e^{-k(t)} R(0) + e^{-k(t)} \int_0^t e^{h(u)} ye^{-kt}du
\]
\[
= F_1 + F_2.
\]
(51)
It is obvious that
\[
\lim_{t \to \infty} \frac{\log F_1}{t} = -(d - b) - \frac{1}{2} \sigma_3^2
\]
\[
\lim_{t \to \infty} \frac{\log F_2}{t} = -k.
\]
(52)
Therefore, using (51), we have
\[
\limsup_{t \to \infty} \frac{\log R(t)}{t} \leq - \min \left\{ -d + b + 1/2 \sigma_3^2, k \right\}, \quad \text{a.s.}
\]
(53)
Furthermore, from system (15), we can obtain
\[
\frac{1}{t} \left[ S(t) + I(t) - S(0) + I(0) \right] = A - (d - b) \langle S(t) \rangle
\]
\[
- (d + \mu + \gamma - b) \langle I(t) \rangle
\]
\[
+ \sigma_1 \frac{\int_0^t S(u)dB_1(u)}{t}
\]
\[
+ \sigma_2 \frac{\int_0^t I(u)dB_2(u)}{t}.
\]
(54)
Consequently,
\[
\langle S(t) \rangle = \frac{A}{d - b} - \frac{d + \mu + \gamma - b}{d - b} \langle I(t) \rangle + \langle \psi(t) \rangle,
\]
(55)
where
\[
\psi(t) = \frac{1}{d - b} \left( \frac{S(t) + I(t) - S(0) + I(0)}{t} \right)
\]
\[
+ \sigma_1 \frac{\int_0^t S(u)dB_1(u)}{t} + \sigma_2 \frac{\int_0^t I(u)dB_2(u)}{t}.
\]
(56)
According to Lemma 1, we get
\[ \lim_{t \to \infty} \psi(t) = 0, \quad \text{a.s.} \quad (57) \]
Combining (50), (55), and (57), we can see that
\[ \lim_{t \to \infty} \langle S(t) \rangle = \frac{A}{d-b} \quad \text{a.s.} \quad (58) \]
The proof is completed. □

Remark 1. In condition (B) of Theorem 2, if \( \sigma_q = 0 \), then we can get a degenerate result of condition (A):
\[ R_0^q = \frac{1}{d + \mu + \gamma - bp} \left( \frac{A \beta}{d - b} - \frac{1}{2} \sigma_i^2 \right) = R_0 - \frac{1}{2(d + \mu + \gamma - bp)} < 1. \quad (59) \]
Thus, we have the following corollary to supplement Theorem 2.

Corollary 1. Let \( (S(t), I(t), R(t)) \) be the solution of system (15) with the initial value \( (S(0), I(0), R(0)) \in R_+^3 \). Assume that condition (59) holds, then \( I(t), R(t) \) almost surely exponentially converge to zero and \( S(t) \) in the mean almost surely converges to \( A/(d-b) \).

4. Existence of the Stationary Distribution

For the deterministic model (1), if \( R_0^q > 1 \), there was an unique globally asymptotically stable endemic equilibrium which implies the disease will be persistent. But for system (15), although there is no endemic equilibrium, we also expect to know the trend of the positive solution of the system. The trend of positive solution can be explained by the stationary distribution. If we can prove that system (15) exists a stationary distribution, it can show that the disease persists for a long time. Then, we will give a main theorem that there exists a stationary distribution for system (15) according to a well-known result from Khasminskii [27].

Let \( X(t) \) be a regular time-homogeneous Markov process in \( R^3 \) described by the stochastic differential equation:
\[ dX(t) = h(x)dt + \sum_{i=1}^k g_i(x)dB_i(t), \quad (60) \]
where \( h(x) = (h_1(x), h_2(x), \ldots, h_4(x)), \quad g_i(x) = (g^1_i(x), g^2_i(x), \ldots, g^6_i(x)), \) and \( B_i(t) (i = 1, 2, \ldots, k) \) are the standard Brownian motions defined on some probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \).

The diffusion matrix for equation (60) is defined as follows:
\[ A(x) = (a_{ij}(x)) = \sum_{i=1}^k g_i(x)g^j_i(x). \quad (61) \]

Lemma 2 (See [27]). Suppose that there exists a regular boundary domain \( U \subset R^3 \), satisfy the following properties.
(i) In the domain \( U \) and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix \( A(x) \) is bounded away from zero
(ii) If \( x \in R^d \setminus U \), the mean time \( \tau \) at which a path issuing from \( x \) reaches the set \( U \) is finite, and \( \sup_{x \in \mathbb{R}^d} E_x^\tau < \infty \) for every compact subset \( K \subset R^d \), then
\[ \mathbb{P}_x \left( \lim_{t \to \infty} \frac{1}{T} \int_0^T f(x(t))dt \right) = 1, \quad (62) \]
where \( f(x) \) is a function integrable with respect to the measure \( \pi \).

Remark 2. To validate (i), it suffices to prove that there is a positive number \( G \), such that \( \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq G|\xi|^2 \) for all \( x \in U \) and \( \xi \in R^d \) [28, 29]. To validate (ii), it suffices to prove that there exists a nonnegative \( C^2 \) function \( V \) and a neighborhood \( U \), such that for some \( c > 0 \), \( \mathcal{L}V(x) < -c \), \( x \in R^d \setminus U \) [30].

Theorem 3. Suppose that \( R_0^q > 1 \). If the conditions
\[ 0 < \theta < \min\{k_1S^2 + k_2I^2 + k_3R^2\}, \quad k_i > 0, (i = 1, 2, 3), \quad (63) \]
hold, then there exists a stationary distribution and ergodic property for system (15). Here, \( (S^*, I^*, R^*) \) is the unique endemic equilibrium of system (1), and

\[ k_1 = d - b - \sigma_1^2 - \frac{(2d + \mu + \gamma - 2b)f(I^*)\sigma_1^2I^*}{\beta}, \]
\[ k_2 = d + \mu + \gamma - 2b - \sigma_2^2, \]
\[ k_3 = d - b - \gamma - \sigma_3^2, \]
\[ \theta = \left( \sigma_1^2 + \frac{(2d + \mu + \gamma - 2b)f(I^*)\sigma_1^2I^*}{\beta} \right)S^2 + \sigma_2^2 I^* + \left( \sigma_3^2 + \frac{(2d + \mu + \gamma - 2b)f(I^*)}{2\beta} \right)R^2. \quad (64) \]
Proof. When \( R_0^* > 1 \), which means that \( R_0 > 1 \), there has an epidemic equilibrium \( E^* (S^*, I^*, R^*) \) in system (1) and \( S^*, I^*, R^* \) are satisfied with

\[
A = (d - b)S^* + \frac{\beta S^* I^*}{f(I^*)} - b(1 - p)I^* ,
\]

\[
\frac{\beta S^* I^*}{f(I^*)} = (d + \mu + \gamma - b)I^* ,
\]

\[
\gamma I^* = (d - b)R^* .
\]

Next, build a function \( V: R^3_+ \rightarrow R_+ \) by

\[
V(S(t), I(t), R(t)) = V_1(S(t), I(t), R(t)) + \frac{(2d + \mu + \gamma - 2b)f(I^*)}{\beta} V_2(S(t), I(t), R(t)) + V_3(S(t), I(t), R(t)),
\]

where

\[
V_1(S(t), I(t), R(t)) = \frac{1}{2} (S(t) - S^* + I(t) - I^*)^2 ,
\]

\[
V_2(S(t), I(t), R(t)) = I(t) - I^* - I^* \ln \frac{I(t)}{I^*} ,
\]

\[
V_3(S(t), I(t), R(t)) = \frac{1}{2} (R(t) - R^*)^2 .
\]

By computing and using \((a + b)^2 \leq 2(a^2 + b^2), f'(I) \geq 0, \) and \( ab \leq ((a^2 + b^2)/2) \), we obtain

\[
\mathcal{L}V_1 = (S(t) - S^* + I(t) - I^*) (A - (d - b)S(t) - (d + \mu + \gamma - b)I(t))
\]

\[+ \frac{1}{2} \sigma_1^2 S^2 (t) + \frac{1}{2} \sigma_2^2 I^2 (t) \]

\[= (S(t) - S^* + I(t) - I^*) (- (d - b)(S(t) - S^*) - (d + \mu + \gamma - b)(I(t) - I^*)) \]

\[+ \frac{1}{2} \sigma_1^2 (S(t) - S^* + S^*)^2 + \frac{1}{2} \sigma_2^2 (I(t) - I^* + I^*)^2 \]

\[\leq -(d - b - \sigma_1^2) (S(t) - S^*)^2 - (2d + \mu + \gamma - 2b)(S(t) - S^*) (I(t) - I^*) \]

\[
\mathcal{L}V_1 = (S(t) - S^* + I(t) - I^*) (A - (d - b)S(t) - (d + \mu + \gamma - b)I(t))
\]

\[+ \frac{1}{2} \sigma_1^2 S^2 (t) + \frac{1}{2} \sigma_2^2 I^2 (t) \]

\[= (S(t) - S^* + I(t) - I^*) (- (d - b)(S(t) - S^*) - (d + \mu + \gamma - b)(I(t) - I^*)) \]

\[+ \frac{1}{2} \sigma_1^2 (S(t) - S^* + S^*)^2 + \frac{1}{2} \sigma_2^2 (I(t) - I^* + I^*)^2 \]

\[\leq -(d - b - \sigma_1^2) (S(t) - S^*)^2 - (2d + \mu + \gamma - 2b)(S(t) - S^*) (I(t) - I^*) \]

\[-(d + \mu + \gamma - b - \sigma_2^2) (I(t) - I^*)^2 + \sigma_1^2 S^2 + \sigma_2^2 I^2 ,
\]
\[ \mathcal{L}V_2 = \left( I(t) - I^* \right) \left( \frac{\beta S(t)}{f(I(t))} - (d + \mu + \gamma + bp) \right) + \frac{1}{2} \sigma_2^2 I^* + \frac{1}{2} \sigma_2^2 I^* \frac{S^2(t)}{f^2(I(t))} \]

\[ = \left( I(t) - I^* \right) \left( \frac{\beta S(t)}{f(I(t))} - \frac{\beta S^*}{f(I^*)} \right) + \frac{1}{2} \sigma_2^2 I^* + \frac{1}{2} \sigma_2^2 I^* \frac{S^2(t)}{f^2(I(t))} \]

\[ = -\left( I(t) - I^* \right) \left( \frac{f(I(t))}{f(I^*)} \beta S(t) \frac{(I(t) - I^*)}{(I^*)} + \frac{\beta (S(t) - S^*) (I(t) - I^*)}{f(I^*)} \right) + \frac{1}{2} \sigma_2^2 I^* \]

\[ \leq \frac{\beta (S(t) - S^*) (I(t) - I^*)}{f(I^*)} + \frac{1}{2} \sigma_2^2 I^* + \sigma_2^2 I^* \frac{(S(t) - S^*)^2}{f^2(I(t))} \]

\[ \leq \frac{\beta (S(t) - S^*) (I(t) - I^*)}{f(I^*)} + \frac{1}{2} \sigma_2^2 I^* + \sigma_2^2 I^* \frac{(S(t) - S^*)^2}{f^2(I(t))} \]  

Substituting (68) into

\[ \mathcal{L}V(t) = \mathcal{L}V_1(t) + \frac{2 \left( d + \mu + \gamma - 2b \right) f(I^*)}{\beta} \mathcal{L}V_2(t) + \mathcal{L}V_3(t). \]  

(69)

leads to

\[ k_1 = d - b - \sigma_1^2 - \frac{2 \left( d + \mu + \gamma - 2b \right) f(I^*)}{\beta} \sigma_2^2 I^*. \]

\[ k_2 = d + \mu + \frac{\gamma}{2} - b - \sigma_2^2, \]

\[ k_3 = d - b - \frac{\gamma}{2} - \sigma_3^2, \]

\[ \theta = \left( \sigma_1^2 + \frac{2 \left( d + \mu + \gamma - 2b \right) f(I^*)}{\beta} \sigma_2^2 I^* \right) S^2 + \sigma_2^2 I^* \left( I^* + \frac{2 \left( d + \mu + \gamma - 2b \right) f(I^*)}{2\beta} \right) + \sigma_3^2 R^2. \]  

(71)

If

\[ 0 < \theta < \min\{k_1 S_{\sigma_1^2}, k_2 I_{\sigma_2^2}, k_3 R_{\sigma_3^2}\}, \quad k_i > 0, \quad (i = 1, 2, 3), \]

(72)

\[ U = \left\{ (S(t), I(t), R(t)) | k_1 (S(t) - S^*)^2 + k_2 (I(t) - I^*)^2 + k_3 (R(t) - R^*)^2 \leq \theta \right\}, \]

(73)

\[ k_1 (S(t) - S^*)^2 + k_2 (I(t) - I^*)^2 + k_3 (R(t) - R^*)^2 \geq \theta + c. \]  

(74)

Finally, we obtain
\[ \mathcal{L}V(S(t), I(t), R(t)) \leq -c, \quad (S(t), I(t), R(t)) \in R_3^+ \cup U. \] (75)

We verify that condition (ii) of Lemma 2 holds. Then, we will prove that condition (i) holds in Lemma (60). The diffusion matrix associated to system (15) is

\[
A = \begin{bmatrix}
\sigma_1^2 S^2 + \sigma_2^3 I^2 f^2(I) & \sigma_1^2 S^2 \frac{I^2}{f^2(I)} & 0 \\
-\sigma_1^2 S^2 \frac{I^2}{f^2(I)} & \sigma_2^2 I^2 + \sigma_3^2 I^2 f^2(I) & 0 \\
0 & 0 & \sigma_3^2 R^2
\end{bmatrix}. 
\] (76)

Let a constant \( G = \min \{\sigma_1^3 S^2, \sigma_2^3 I^2, \sigma_3^3 R^2\} > 0 \), for any \((S, I, R) \in U\), and \((\xi_1, \xi_2, \xi_3) \in R_3^+\). We can calculate that

\[
\sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j = \sigma_1^2 S^2 \xi_1^2 + \sigma_2^2 I^2 \xi_2^2 + \sigma_3^2 R^2 \xi_3^2 + \sigma_3^3 S^2 \frac{I^2}{f^2(I)} (\xi_1 - \xi_2)^2 \geq \min \{\sigma_1^3 S^2, \sigma_2^3 I^2, \sigma_3^3 R^2\} [\xi_1^2 + \xi_2^2 + \xi_3^2] = G|\xi|^2.
\] (77)

This shows that condition (i) of Lemma 2 holds. Therefore, there has a unique stationary distribution and the ergodic property of system (15). The proof is completed. \( \square \)

5. Asymptotic Behavior around the Equilibria of System (1)

In the section, we will study the asymptotic behavior of the solution of system (15) around the disease-free equilibrium and endemic equilibrium under random perturbations.

**Theorem 4.** Let \((S(t), I(t), R(t))\) be the solution of system (15) with initial value \((S(0), I(0), R(0)) \in R_3^+.\) Suppose \(R_0 \leq 1, \text{if} \sigma_1^2 < d - b, \sigma_2^2 < 2(d + \mu - bp) + \gamma, \text{and} \sigma_3^2 < 2(d - b) - \gamma \) hold, then

\[
\lim_{t \to \infty} \sup \frac{1}{t} \mathbb{E} \left[ S(u) - \frac{A}{d - b} \right]^2 + I^2(u) + R^2(u) \leq \frac{\sigma_2^2 A^2}{\lambda(d - b)^2}
\] (78)

where \( \lambda = \min \{d - b - \sigma_1^2, d + \mu + (\gamma/2) - bp - (\sigma_2^2/2), d - b - (\gamma/2) - (\sigma_3^2/2)\}. \)

**Proof.** Now, define a function \( V: R_3^+ \to R_1 \) by

\[
V(t) = V_1(t) + 2\frac{d + \mu + \gamma - b(1 + p)}{\beta} V_2(t) + V_3(t),
\] (79)

where

\[
\mathcal{L}V_1(t) = \left( S(t) - \frac{A}{d - b} + I(t) \right) (A - (d - b)S(t) - (d + \mu + \gamma - bp)I(t)) + \frac{\sigma_2^2 S^2(t) + \sigma_2^2 I^2(t)}{2}.
\]

By calculating, we obtain

\[
V_1(t) = \frac{1}{2} \left( S(t) - \frac{A}{d - b} + I(t) \right)^2,
\]

\[
V_2(t) = I(t),
\]

\[
V_3(t) = \frac{1}{2} R^2(t).
\] (80)
\[
\begin{align*}
\mathcal{L}V_1(t) &= \beta \left( \frac{S(t)}{f(I(t))} - \frac{A}{d-b} + \frac{A}{d-b} \right) I(t) - (d + \mu + \gamma - b p) I(t) \\
&\geq \beta \left( S(t) - \frac{A}{d-b} \right) I(t) + \left( \frac{\beta A}{d-b} - (d + \mu + \gamma - b p) \right) I(t) \\
&\geq \beta \left( S(t) - \frac{A}{d-b} \right) I(t), \\
\mathcal{L}V_2(t) &= \beta \left( \frac{S(t)}{f(I(t))} - \frac{A}{d-b} + \frac{A}{d-b} \right) I(t) - (d + \mu + \gamma - b p) I(t) \\
&\geq \beta \left( S(t) - \frac{A}{d-b} \right) I(t) + \sigma_2^2 \left( \frac{R^2(t)}{2} \right) \\
&\geq \beta \left( S(t) - \frac{A}{d-b} \right) I(t), \\
\mathcal{L}V_3(t) &= \gamma R(t) I(t) - (d - b) R^2(t) + \frac{\sigma_3^2 R^2(t)}{2} \leq - \left( d - b - \frac{\gamma}{2} - \frac{\sigma_3^2}{2} \right) R^2(t) + \frac{\gamma}{2} R(t).
\end{align*}
\]

Thus,

\[
\mathcal{L}V(t) = \mathcal{L}V_1(t) + \frac{2}{\beta} \frac{d + \mu + \gamma - b (1 + p)}{d - b} \mathcal{L}V_2(t) + \mathcal{L}V_3(t)
\]

\[
\begin{align*}
\gamma \left( d - b - \sigma_1^2 \right) S(t) - \frac{A}{d-b} \right)^2 - \left( d + \mu + \frac{\gamma}{2} - b p - \frac{\sigma_3^2}{2} \right) I^2(t) \\
- \left( d - b - \frac{\gamma}{2} - \frac{\sigma_3^2}{2} \right) R^2(t) + \frac{\sigma_3^2 A^2}{(d - b)^2}.
\end{align*}
\]

We calculate the derivative of \( V(t) \) along the trajectories of system (15):

\[
\begin{align*}
dV(t) &= dV_1(t) + \frac{2}{\beta} \frac{d + \mu + \gamma - b (1 + p)}{d - b} dV_2(t) + dV_3(t) \\
&= \mathcal{L}V(t) dt + \left( S(t) - \frac{A}{d-b} + I(t) \right) \sigma_1 S(t) dB_1(t) - \sigma_4 S(t) I(t) f(I(t)) dB_4(t) \\
&\quad + \sigma_2 I(t) dB_2(t) + \sigma_4 S(t) I(t) f(I(t)) dB_4(t) + \sigma_3 R^2(t) dB_3(t).
\end{align*}
\]

Integrating (83) from 0 to \( t \) on both sides, then taking expectation, we obtain

\[
EV(t) - EV(0) \leq E \int_0^t \left[ - \left( d - b - \sigma_1^2 \right) \left( S(u) - \frac{A}{d-b} \right)^2 - \left( d + \mu + \frac{\gamma}{2} - b p - \frac{\sigma_3^2}{2} \right) I^2(u) - \left( d - b - \frac{\gamma}{2} - \frac{\sigma_3^2}{2} \right) R^2(u) \right] du \]

\[
+ \frac{\sigma_3^2 A^2}{(d - b)^2} t,
\]

(84)
which implies that

$$\lim_{t \to \infty} \sup_t \frac{1}{t} E \int_0^t \left[ (d - b - \sigma_1^2) \left( S(u) - \frac{A}{d-b} \right)^2 + \left( d + \mu + \frac{\gamma}{2} - b + \frac{\sigma_1^2}{2} \right) I^2(u) + \left( d - b - \frac{\gamma}{2} - \frac{\sigma_1^2}{2} \right) R^2(u) \right] du$$

$$\leq \frac{\sigma_1^2 A^2}{(d-b)^2}.$$  \hspace{1cm} (85)

Denote \( \lambda = \min\{d - b - \sigma_1^2, d + \mu + \frac{\gamma}{2} - b - \frac{\sigma_1^2}{2}, d - b - \frac{\gamma}{2} - \frac{\sigma_1^2}{2}\} \). Finally, from (85), we have

$$\lim_{t \to \infty} \sup_t \frac{1}{t} E \int_0^t \left[ \left( S(u) - \frac{A}{d-b} \right)^2 + I^2(u) + R^2(u) \right] du \leq \frac{\sigma_1^2 A^2}{\lambda (d-b)^2}. \hspace{1cm} (86)$$

The proof is completed. \( \square \)

**Corollary 2.** From Theorem 4, when \( \sigma_i = 0, i = 1, 2, 3, 4 \), by using the method of [31], we obtain

$$\mathcal{L} V(t) \leq - (d - b) \left( S(t) - \frac{A}{d-b} \right)^2 - \left( d + \mu + \frac{\gamma}{2} - b + \frac{\sigma_1^2}{2} \right) I^2(t) - \left( d - b - \frac{\gamma}{2} - \frac{\sigma_1^2}{2} \right) R^2(t) \leq 0. \hspace{1cm} (87)$$

Thus, when \( R_0 \leq 1 \) and \( d - b > 0 \), \( 2(d + \mu - b) + \gamma > 0 \), and \( 2(d - b) > \gamma > 0 \) hold, the disease-free equilibrium \((A/d-b), 0, 0)\) of system (1) is globally asymptotically stable.

**Remark 3.** In system (15), assume \( \sigma_i = 0, i = 1, 2, 3, 4 \). Now, if \( \sigma_i = 0 \), we find that

$$\mathcal{L} V(t) \leq - (d - b) \left( S(t) - \frac{A}{d-b} \right)^2 - \left( d + \mu + \frac{\gamma}{2} - b + \frac{\sigma_1^2}{2} \right) I^2(t) - \left( d - b - \frac{\gamma}{2} - \frac{\sigma_1^2}{2} \right) R^2(t), \hspace{1cm} (88)$$

which shows that \( \mathcal{L} V(t) \) is negative definite under the conditions \( \sigma_2^2 < 2(d + \mu - b) + \gamma \) and \( \sigma_3^2 < 2(d - b) - \gamma \). So, we have the following corollary.

**Corollary 3.** Suppose that \( R_0 \leq 1 \) and \( \sigma_i = 0 \). The solution of system (15) is stochastically asymptotically stable in the large [24] as \( \sigma_2^2 < 2(d + \mu - b) + \gamma \) and \( \sigma_3^2 < 2(d - b) - \gamma \).

$$\lim_{t \to \infty} \sup_t \frac{1}{t} E \int_0^t \left[ k_1 \left( S(u) - S^* \right)^2 + k_2 \left( I(u) - I^* \right)^2 + k_3 \left( R(u) - R^* \right)^2 \right] du \leq \theta. \hspace{1cm} (89)$$

**Proof.** If \( R_0^i > 1 \), it is obvious that \( R_0 > 1 \). Applying Itô’s formula to (66), we have
\[
\begin{align*}
dV(S(t), I(t), R(t)) &= dV_1(S(t), I(t), R(t)) + \left(2d + \mu + \gamma - 2b\right)f(I^*) \frac{dV_2(S(t), I(t), R(t))}{\beta} \\
&= \mathcal{L}V(S(t), I(t), R(t)) + \left(S(t) - S^* + I(t) - I^*\right) \left(\sigma_1 S(t) dB_1(t) + \sigma_2 I(t) dB_2(t)\right) \\
&\quad + \left(I(t) - I^*\right) \left(\sigma_3 dB_2(t) + \sigma_4 \frac{S(t)}{f(I(t))} dB_4(t)\right) \\
&\quad + \left(R(t) - R^*\right) R(t) \sigma_3 dB_3(t).
\end{align*}
\]

Figure 1: The subgraphs (a)–(c) denote the numerical simulation of conditions (A) and (B) of Theorem 2 and Corollary 1, respectively.
Integrating (90) from 0 to \( t \) on both sides and from (70), we obtain

\[
V(S(t), I(t), R(t)) \geq V(S(0), I(0), R(0)) + \int_{0}^{t} \left( -k_1 (S(u) - S^*)^2 - k_2 (I(u) - I^*)^2 
- k_3 (R(u) - R^*)^2 \right) du + \theta t + M(t),
\]

\[
(91)
\]
\[ M(t) = \int_0^t \left( S(u) - S^* + I(u) - I^* \right) \left( \sigma_1 S(u) dB_1(u) + \sigma_2 I(u) dB_2(u) \right) \]
\[ + \int_0^t \left( I(u) - I^* \right) \left( \sigma_3 dB_2(u) + \sigma_4 \frac{S(u)}{f(I(u))} dB_4(u) \right) \]
\[ + \int_0^t \left( R(u) - R^* \right) R(u) \sigma_5 dB_3(u), \]

where

Figure 4: The histogram (a)–(c) of solution of the stochastic system (15) with \( R_0 > 1 \). System (15) exists a unique stationary distribution.
which is a continuous local martingale; then, taking expectation on both sides of (91) yields

\[
EV(S(t), I(t), R(t)) \geq EV(S(0), I(0), R(0)) + \int_0^t \left( -k_1(S(u) - S^*)^2 - k_2(I(u) - I^*)^2 - k_3(R(u) - R^*)^2 \right) du + \theta.
\] (93)

Finally, we obtain

\[
\lim_{t \to \infty} \sup \frac{1}{t} \int_0^t \left( k_1(S(u) - S^*)^2 + k_2(I(u) - I^*)^2 + k_3(R(u) - R^*)^2 \right) du \leq \theta.
\] (94)

This completes the proof.

Corollary 4. From Theorem 5, when \( \sigma_i = 0, i = 1, 2, 3, 4 \), by using the method of [31], we obtain

\[
\mathcal{L}V(t) \leq -(d - b)(S(t) - S^*)^2 - \left( d + \mu + \frac{y}{2} - b \right)(I(t) - I^*)^2 - \left( d - b - \frac{y}{2} \right)(R(t) - R^*)^2 \leq 0.
\] (95)

Thus, when \( R_0 > 1 \) and \( d - b > 0, 2(d + \mu - b) + y > 0, \) and \( 2(d - b) - y > 0 \) hold, the endemic equilibrium \( (S^*, I^*, R^*) \) of system (1) is globally asymptotically stable.

Remark 4. From Theorems 4 and 5, it can be seen that the solution of system (15) will oscillate around the disease-free equilibrium and endemic equilibrium of system (1) under some conditions, and the disturbance intensity is proportional to the white noise intensity. From a biological point of view, the solution of system (15) fluctuates most of the time around the disease-free equilibrium and endemic equilibrium of system (1) due to the small magnitude of the random disturbance.

6. Conclusions and Numerical Simulations

In this study, we are mainly concerned about the influence of stochastic factors on the behavior of the epidemic SIR model.
with a nonlinear incidence and vertical transmission. Here, the stochastic factors mainly include the disease transmission coefficient and state variables affected by white noise. System (15) has a unique global positive solution starting from the positive initial value. Furthermore, we give a threshold value $R^*_0$ to distinguish the persistence and extinction of diseases, which differs from the basic reproductive number $R_0$ corresponding to system (1) by a noise term. When $R^*_0 < 1$, the conditions for extinction of disease are obtained in system (15). By constructing a stochastic Lyapunov function, we prove that an ergodic stationary distribution exists in system (15) when $R^*_0 > 1$, which means that the disease will persist. Finally, we study the asymptotic properties of solution of system (15) around disease-free equilibrium and endemic equilibrium of the deterministic model (1) under certain conditions and under the existence of stationary distribution, respectively. From the biological point of view, the interference of environmental white noise may have certain influence on the stability of the biological system: the ability of population to adapt to the environment is limited. If the intensity of white noise in the environment is small enough, the stability of the population will not be damaged. If the intensity of white noise is large in the environment, it may lead to the extinction of species.

In the following part, we will give some examples to verify the theoretical results by numerical simulation. The method of numerical simulation is shown in reference [32]. For convenience, take $f(I) = 1 + mI$.

**Example 1.** Fix the parameters $A = 0.43, \beta = 0.9, d = 0.6, \mu = 0.1, \gamma = 0.2, b = 0.1, p = 0.2, \sigma_1 = 0.1, \sigma_2 = 0.5, \sigma_3 = 0.1, \sigma_4 = 1, m = 1$ in system (15). It is easy to calculate

$$R^*_0 = \frac{1}{d + \mu + \gamma - bp} \left( \frac{A \beta}{d - b} - \frac{1}{2} \sigma^2_2 - \frac{1}{2} \sigma^2_2 \left( \frac{A}{d - b} \right)^2 \right) = 0.3173 < 1, \tag{96}$$

and $0.9 = \beta > (A \sigma^2_2 / (d - b)) = 0.86$. Obviously, this set of parameters satisfies condition (A) of Theorem 2. From Figure 1(a), it can be seen that the state variable $S(t)$ fluctuates around $(A/(d - b)) = 0.86$, and $I(t), R(t)$ tend to zero with the increase of time. If we adjust $\beta = 0.8$, other parameters remain unchanged. It is obvious that $0.8 = \beta < (A \sigma^2_2 / (d - b)) = 0.86$ and

$$\frac{\beta^2}{2\sigma^2_2} - \frac{(d + \mu + \gamma - bp)}{2\sigma^2_2} = -0.6850 < 0 \tag{97}$$

Condition (B) of Theorem 2 is satisfied. As shown in Figure 1(b), $S(t), I(t), R(t)$ fluctuate around $(A/(d - b)) = 0.86, 0, 0$ with the increase of time, respectively. This proves that the conclusion of Theorem 2 is correct. Furthermore, take $\sigma_1 = 0$ and other parameters keep the same with the first set of parameters. Under this set of parameters, $R^*_0 = 0.7375 < 1$. From Figure 1(c), it can be seen that the result of Corollary 1 is true.

**Example 2.** Take $A = 0.55, \beta = 0.9, d = 0.6, \mu = 0.1, \gamma = 0.2, b = 0.1, p = 0.2, \sigma_1 = 0.1, \sigma_2 = 0.5, \sigma_3 = 0.1, \sigma_4 = 1, m = 1$.

Then, we draw the graph of the state variable $S(t), I(t), R(t)$ of system (1) and system (15) with the increase of time. Under the parameters, $R^*_0 < 1 < R_0$. When $R_0 > 1$, there is a stable endemic equilibrium for system (1). When $R^*_0 < 1$, according to Theorem 2, $S(t), I(t), R(t)$ fluctuate around $(A/(d - b)) = 1.1, 0, 0$ with the increase of time. From Figure 2, we can find that adding random perturbations to the environment can suppress the disease outbreaks.

**Example 3.** Choose the parameters $A = 2, \beta = 0.9, d = 0.6, \mu = 0.1, \gamma = 0.2, b = 0.1, p = 0.2, \sigma_1 = \sigma_2 = \sigma_3 = 0.1, \sigma_4 = 0.01, m = 1$. We can calculate that $R_0 = 4.09 > 1, R^*_0 = 4.0843 > 1$, $0.776 = \theta < \min\{k_1 \sigma^2, k_2 \sigma^2, k_3 \sigma^2\} = 0.0858$. When $R_0 > 1$, there is a stable positive equilibrium for system (1). When $R^*_0 > 1$, there exists a stationary distribution for system (15) according to Theorem 3. From Figures 3(a) and 3(b), all the conclusions are explained. Furthermore, we draw the histogram of Figure 3(a) and find that $S(t), I(t), R(t)$ obey normal distribution, as shown in Figure 4.

**Example 4.** Fix $A = 0.43, \beta = 0.9, d = 0.6, \mu = 0.1, \gamma = 0.2, b = 0.1, p = 0.2, \sigma_1 = 0.5, \sigma_2 = 0.1, \sigma_3 = 1, m = 1$. Take $\sigma_1 = 0.2, 0$, respectively. From Figures 5(a) and 5(b), it is shown that solution of system (15) is stochastically asymptotically stable in the large.

In the study, we only consider the white noise. In real life, the environmental impact may be not only white noise but also color noise, so can we consider both of them in the model? In addition, it may be more reasonable to consider the spatial factors and time delay when building the model. We leave these issues for further investigation and look forward to resolution in the future.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**

[1] S. Gao, D. Xie, D. Xie, and L. Chen, “Pulse vaccination strategy in a delayed SIR Epidemic Model with vertical transmission,” *Discrete & Continuous Dynamical Systems-B*, vol. 7, no. 1, pp. 77–86, 2007.

[2] B. S. Busenberg and K. Cooke, *Vertically Transmitted Diseases*, Springer, Berlin, Germany, 1993.
[3] Z. Ma, Y. Zhou, W. Wang, and Z. Jin, *Mathematical Modeling and Research of Infectious Disease Dynamics*, Science Press, Beijing, China, 2004.

[4] W.-m. Liu, S. A. Levin, and Y. Iwasa, "Influence of nonlinear incidence rates upon the behavior of SIRS epidemiological models," *Journal of Mathematical Biology*, vol. 23, no. 2, pp. 187–204, 1986.

[5] L. Aadil, O. Lahcen, K. Driss, and B. Aziza, "Complete global stability for an SIRS epidemic model with generalized non-linear incidence and vaccination," *Applied Mathematics and Computation*, vol. 218, no. 11, pp. 6519–6525, 2012.

[6] R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, NJ, USA, 1973.

[7] Q. Liu and Q. Chen, "Analysis of the deterministic and stochastic SIRS epidemic models with nonlinear incidence," *Physica A: Statistical Mechanics and Its Applications*, vol. 428, pp. 140–153, 2015.

[8] L. Aadil and L. Omari, "Extinction and stationary distribution of a stochastic SIRS epidemic model with non-linear incidence," *Statistics and Probability Letters*, vol. 83, no. 4, pp. 960–968, 2013.

[9] W. Guo, Q. Zhang, X. Li, and W. Wang, "Dynamic behavior of a stochastic SIRS epidemic model with media coverage," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 14, pp. 5506–5525, 2018.

[10] L. Aadil, S. Adel, E. F. Mohamed, and T. Abdessamad, "The effect of a generalized nonlinear incidence rate on the stochastic SIS epidemic model," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 1, pp. 1137–1146, 2021.

[11] C. Ji and D. Jiang, "The threshold of a non-autonomous SIRS epidemic model with stochastic perturbations," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 5, pp. 1773–1782, 2017.

[12] T. Nafeisha, B. Wen, and Z. Teng, "The stationary distribution in a class of stochastic SIRS epidemic models with non-monotonic incidence and degenerate diffusion," *Mathematics and Computers in Simulation*, vol. 182, pp. 888–912, 2021.

[13] Q. Liu, D. Jiang, N. Shi, T. Hayat, and A. Alsaedi, "The threshold of a stochastic SIS epidemic model with imperfect vaccination," *Mathematics and Computers in Simulation*, vol. 144, pp. 78–90, 2018.

[14] C. Xu, "Global threshold dynamics of a stochastic differential equation SIS model," *Journal of Mathematical Analysis and Applications*, vol. 447, no. 2, pp. 736–757, 2017.

[15] Q. Yang, D. Jiang, N. Shi, and C. Ji, "The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence," *Journal of Mathematical Analysis and Applications*, vol. 388, no. 1, pp. 248–271, 2012.

[16] Y. Zhou, W. Zhang, and S. Yuan, "Survival and stationary distribution of a SIR epidemic model with stochastic perturbations," *Applied Mathematics and Computation*, vol. 244, pp. 118–131, 2014.

[17] Y. Chen and W. C. Zhao, "Dynamical analysis of a stochastic SIRS epidemic model with saturating contact rate," *Mathematical Biosciences and Engineering: MBE*, vol. 17, no. 5, pp. 5925–5943, 2020.

[18] Y. Wang, G. Liu, and G. Liu, "Dynamics analysis of a stochastic SIRS epidemic model with nonlinear incidence rate and transfer from infectious to susceptible," *Mathematical Biosciences and Engineering*, vol. 16, no. 5, pp. 6047–6070, 2019.

[19] Q. Yang, X. Mao, and X. Mao, "Stochastic dynamics of SIRS epidemic models with random perturbation," *Mathematical Biosciences and Engineering*, vol. 11, no. 4, pp. 1003–1025, 2014.