SOME COMPANIONS OF PERTURBED OSTROWSKI TYPE INEQUALITIES
FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE BOUNDED AND
APPLICATIONS

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Abstract. In this paper we establish some companions of perturbed Ostrowski type integral inequalities for functions whose second derivatives are bounded. Some applications to composite quadrature rules, and to probability density functions are also given.

1. Introduction

In 1938, Ostrowski [21] established the following interesting integral inequality for differentiable mappings with bounded derivatives:

Theorem 1.1. Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) whose derivative is bounded on \((a, b)\) and denote \( \|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty \). Then for all \( x \in [a, b] \) we have

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b - a)^2} \right] (b - a) \|f'\|_\infty, \tag{1.1}
\]

The constant \( \frac{1}{4} \) is sharp in the sense that it cannot be replaced by a smaller one.

In [13], Guessab and Schmeisser proved the following companion of Ostrowski’s inequality:

Theorem 1.2. Let \( f : [a, b] \to \mathbb{R} \) be satisfying the Lipschitz condition, i.e., \( |f(t) - f(s)| \leq M|t - s| \). Then for all \( x \in [a, b] \) we have

\[
\left| \frac{f(x) + f(a + b - x)}{2} - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{8} + 2 \left( \frac{a + b}{b - a} \right)^2 \right] (b - a) M. \tag{1.2}
\]

The constant \( \frac{1}{8} \) is sharp in the sense that it cannot be replaced by a smaller one. In (1.2), the point \( x = \frac{a+b}{2} \) gives the best estimator.

Motivated by [13], Dragomir [8] proved some companions of Ostrowski’s inequality, as follows:

Theorem 1.3. Let \( f : [a, b] \to \mathbb{R} \) be an absolutely continuous mapping on \([a, b]\). Then the following inequalities

\[
\left| \frac{f(x) + f(a + b - x)}{2} - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \begin{cases}
\left[ \frac{1}{8} + 2 \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b - a) \|f'\|_\infty, & f' \in L^\infty[a, b], \\
\frac{2^{1/q} \left( \frac{a+b}{b-a} \right)^{q+1} + \left( \frac{a+b}{b-a} \right)^{q+1} }{\left( q+2 \right)^{q+2}} \left[ \frac{1}{p} + \frac{1}{q} = 1 \right] \|f'\|_p, & f' \in L^p[a, b], \\
\left[ \frac{1}{4} + \frac{x - \frac{a+b}{2}}{b-a} \right] \|f'\|_1, & f' \in L^1[a, b]
\end{cases}
\]

hold for all \( x \in [a, \frac{a+b}{2}] \).

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Recently, Alomari [1] studied the companion of Ostrowski inequality [12] for differentiable bounded mappings. In [19], Liu established some companions of an Ostrowski type integral inequality for functions whose first derivatives are absolutely continuous and second derivatives belong to $L^p$ ($1 \leq p \leq \infty$) spaces.

**Theorem 1.4.** Let $f : [a, b] \to \mathbb{R}$ be such that $f'$ is absolutely continuous on $[a, b]$ and $f'' \in L^\infty[a, b]$. Then for all $x \in [a, \frac{a+b}{2}]$ we have

$$
\left| \frac{f(x) + f(a + b - x)}{2} - \left( x - \frac{3a + b}{4} \right) \frac{f'(x) - f'(a + b - x)}{2} - \frac{1}{b - a} \int_a^b f(t)dt \right|
\leq \left[ \frac{1}{90} + \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 \right] (b - a)^2 \| f'' \|_\infty.
$$

(1.3)

The constant $\frac{1}{90}$ is sharp in the sense that it cannot be replaced by a smaller one.

For other related results, the reader may be refer to [2, 3, 4, 7, 9, 11, 14, 15, 16, 17, 18, 22, 23, 24, 25, 26, 27, 28] and the references therein.

The main aim of this paper is to establish some companions of perturbed Ostrowski type integral inequalities for functions whose second derivatives are bounded (Theorem 2.1-2.5). Some applications to composite quadrature rules, and to probability density functions are also given.

## 2. Main results

To prove our main results, we need the following lemmas.

**Lemma 2.1.** [19] Let $f : [a, b] \to \mathbb{R}$ be such that $f'$ is absolutely continuous on $[a, b]$. Denote by $K(x, t) : [a, b] \to \mathbb{R}$ the kernel given by

$$
K(x, t) = \begin{cases} 
\frac{1}{2} (t - a)^2, & t \in [a, x], \\
\frac{1}{2} \left( t - \frac{a + b}{2} \right)^2, & t \in (x, a + b - x), \\
\frac{1}{2} (t - b)^2, & t \in (a + b - x, b],
\end{cases}
$$

(2.1)

then the identity

$$
\frac{1}{b - a} \int_a^b K(x, t)f''(t)dt = \frac{1}{b - a} \int_a^b f(t)dt - \frac{f(x) + f(a + b - x)}{2} + \left( x - \frac{3a + b}{4} \right) \frac{f'(x) - f'(a + b - x)}{2}
$$

(2.2)

holds.

**Lemma 2.2.** [12] Grüss inequality] Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\phi \leq f(t) \leq \Phi$ and $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a, b]$, where $\phi, \Phi, \gamma$ and $\Gamma$ are constants. Then we have

$$
\left| \frac{1}{b - a} \int_a^b f(t)g(t)dt - \frac{1}{b - a} \int_a^b f(t)dt \cdot \frac{1}{b - a} \int_a^b g(t)dt \right| \leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma).
$$

(2.3)
**Theorem 2.1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be such that \( f' \) is absolutely continuous on \([a, b]\). If \( f'' \in L^1[a, b] \) and \( \gamma \leq f''(x) \leq \Gamma, \forall x \in [a, b] \), then for all \( x \in \left[a, \frac{a+b}{2}\right] \) we have

\[
\left| \frac{f(x) + f(a + b - x)}{2} - \frac{f'(a + b - x)}{2} \left( x - \frac{3a + b}{4} \right) \right| \\
+ \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right] - \frac{1}{b - a} \int_a^b f(t) dt \\
\leq \frac{1}{8} (\Gamma - \gamma) \left[ \frac{b - a}{4} + \left| x - \frac{3a + b}{4} \right| \right]^2. 
\]  

(2.4)

**Proof.** It is clear that for all \( t \in [a, b] \) and \( x \in \left[a, \frac{a+b}{2}\right] \), we have

\[
0 \leq K(x, t) \leq \max \left\{ \frac{1}{2} (x - a)^2, \frac{1}{2} \left( \frac{a + b}{2} - x \right)^2 \right\} \\
= \frac{1}{4} \left\{ (x - a)^2 + \left( \frac{a + b}{2} - x \right)^2 \right\} + (x - a)^2 - \left( \frac{a + b}{2} - x \right)^2 \\
= \frac{1}{2} \left[ \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{16} + \frac{b-a}{2} \left| x - \frac{3a + b}{4} \right| \right] \\
= \frac{1}{2} \left[ \frac{b - a}{4} + \left| x - \frac{3a + b}{4} \right| \right]^2. 
\]

Applying Lemma 2.2 to the functions \( K(x, \cdot) \) and \( f''(\cdot) \), we get

\[
\left| \frac{1}{b - a} \int_a^b K(x, t) f''(t) dt - \frac{1}{b - a} \int_a^b K(x, t) dt \right| - \frac{1}{b - a} \int_a^b f''(t) dt \\
\leq \frac{1}{8} (\Gamma - \gamma) \left[ \frac{b - a}{4} + \left| x - \frac{3a + b}{4} \right| \right]^2 
\]  

for all \( x \in \left[a, \frac{a+b}{2}\right] \). By a simple calculation, we obtain

\[
\frac{1}{b - a} \int_a^b f''(t) dt = \frac{f'(b) - f'(a)}{b - a} 
\]  

and

\[
\frac{1}{b - a} \int_a^b K(x, t) dt = \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96}. 
\]

Combining (2.2), (2.5), and (2.6), we obtain (2.4) as required. \(\square\)

**Corollary 2.1.** In the inequality (2.4), choose

1. \( x = \frac{3a+b}{4} \), we get

\[
\left| \frac{f(3a+b)}{2} + \frac{f'\left(\frac{3a+b}{4}\right)}{b-a} \left( x - \frac{3a + b}{4} \right) \right| \\
+ \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right] - \frac{1}{b - a} \int_a^b f(t) dt \\
\leq \frac{1}{128} (\Gamma - \gamma)(b-a)^2. 
\]  

(2.8)

2. \( x = a \), we get

\[
\left| \frac{f(a)}{2} - \frac{f'(b) - f'(a)}{b - a} \left( x - \frac{3a + b}{4} \right) \right| \\
- \frac{1}{b - a} \int_a^b f(t) dt \\
\leq \frac{1}{32} (\Gamma - \gamma)(b-a)^2, 
\]

which is better than \[ Corollary 2.3 \] since a smaller estimator is given here.
We also have
\[
f\left(\frac{a+b}{2}\right) + \frac{f'(b) - f'(a)}{b-a} \left(\frac{(b-a)^2}{24} - \frac{1}{b-a} \int_a^b f(t)dt\right) \leq \frac{1}{32} (\Gamma - \gamma)(b-a)^2,
\]
which is the inequality given in \([5]\) Corollary 2.2.

**Corollary 2.2.** Let \( f \) as in Theorem 2.2. Additionally, if \( f \) is symmetric about \( x = \frac{a+b}{2} \), i.e., \( f(a + b - x) = f(x) \), then we have
\[
\left| f(x) - \left( x - \frac{3a+b}{4} \right) f'(x) + \frac{f'(b) - f'(a)}{b-a} \left[ \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{(b-a)^2}{96} \right] - \frac{1}{b-a} \int_a^b f(t)dt \right| \\
\leq \frac{1}{8} (\Gamma - \gamma) \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right|^2 \right]^2
\]
for all \( x \in [a, \frac{a+b}{2}] \).

**Theorem 2.2.** Let \( f : [a, b] \to \mathbb{R} \) be such that \( f' \) is absolutely continuous on \([a, b]\). If \( f'' \in L^1[a,b] \) and \( \gamma \leq f''(x) \leq \Gamma, \forall x \in [a,b] \), then for all \( x \in [a, \frac{a+b}{2}] \) we have
\[
\left| f(x) + f(a + b - x) \right| \leq \left( x - \frac{3a+b}{4} \right) f'(x) + \frac{f'(b) - f'(a)}{b-a} \left[ \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{(b-a)^2}{96} \right] - \frac{1}{b-a} \int_a^b f(t)dt \\
\leq \frac{\Gamma + \gamma}{2} \left[ \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{(b-a)^2}{96} \right] - C \left[ \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{(b-a)^2}{96} \right].
\]

**Proof.** From (2.2) and (2.7), we have
\[
\frac{1}{b-a} \int_a^b K(x,t)|f''(t) - C|dt \\
= \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(x) + f(a + b - x)}{2} \\
+ \left( x - \frac{3a+b}{4} \right) f'(x) - f'(a + b - x) - C \left[ \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{(b-a)^2}{96} \right].
\]
Let \( C = \frac{\Gamma + \gamma}{2} \), we get
\[
\left| f(x) + f(a + b - x) \right| \leq \left( x - \frac{3a+b}{4} \right) f'(x) - f'(a + b - x) \\
+ \frac{\Gamma + \gamma}{2} \left[ \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{(b-a)^2}{96} \right] - \frac{1}{b-a} \int_a^b f(t)dt \\
\leq \max_{t \in [a,b]} |f''(t) - C| \frac{1}{b-a} \int_a^b |K(x,t)|dt.
\]
We also have
\[
\max_{t \in [a,b]} |f''(t) - C| \leq \frac{\Gamma - \gamma}{2}
\]
and
\[
\frac{1}{b-a} \int_a^b |K(x,t)|dt = \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{(b-a)^2}{96}.
\]
Corollary 2.3. In the inequality (2.9), choose

(1) \( x = \frac{3a + b}{4} \), we get

\[
(2.13) \quad \left| \frac{f(a) + f(b)}{2} - \frac{f'(b) - f'(a)}{8} \frac{(b-a)^2}{2} + \frac{\Gamma + \gamma (b-a)^2}{24} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{192} (\Gamma - \gamma)(b-a)^2.
\]

(2) \( x = a \), we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{f'(b) - f'(a)}{b-a} \frac{(b-a)^2}{8} + \frac{\Gamma + \gamma (b-a)^2}{24} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{48} (\Gamma - \gamma)(b-a)^2.
\]

(3) \( x = \frac{a + b}{2} \), we get

\[
\left| \frac{f(a + b)}{2} + \frac{\Gamma + \gamma (b-a)^2}{24} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{48} (\Gamma - \gamma)(b-a)^2.
\]

Corollary 2.4. Let \( f \) as in Theorem 2.2. Additionally, if \( f \) is symmetric about \( x = \frac{a + b}{2} \), then we have

\[
\left| f(x) - \left( x - \frac{3a + b}{4} \right) f'(x) + \frac{\Gamma + \gamma}{2} \left( \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{\Gamma - \gamma}{2} \left( \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right),
\]

for all \( x \in \left[a, \frac{a + b}{2}\right] \).

Theorem 2.3. Let \( f : [a, b] \to \mathbb{R} \) be such that \( f' \) is absolutely continuous on \([a, b]\). If \( f'' \in L^1[a, b] \) and \( \gamma \leq f''(x) \leq \Gamma, \forall x \in [a, b] \), then for all \( x \in \left[a, \frac{a + b}{2}\right] \) we have

\[
(2.14) \quad \left| \frac{f(x) + f(a + b - x)}{2} - \left( x - \frac{3a + b}{4} \right) f'(x) - f'(a + b - x) \right|
\]

\[
+ \frac{f''(b) - f''(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq (S - \gamma) \left[ \frac{(b-a)^2}{48} + \frac{b-a}{4} \left| x - \frac{3a + b}{4} \right| \right]
\]

and

\[
\left| \frac{f(x) + f(a + b - x)}{2} - \left( x - \frac{3a + b}{4} \right) f'(x) - f'(a + b - x) \right|
\]

\[
+ \frac{f''(b) - f''(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq (\Gamma - S) \left[ \frac{(b-a)^2}{48} + \frac{b-a}{4} \left| x - \frac{3a + b}{4} \right| \right],
\]

where \( S = (f''(b) - f''(a))/(b - a) \).
Proof. From (2.2), (2.6), (2.7), it follows that
\[
\frac{1}{b-a} \int_a^b K(x,t) f''(t) dt - \frac{1}{(b-a)^2} \int_a^b K(x,t) dt \int_a^b f''(t) dt
\]
\[
= \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(x) + f(a + b - x)}{2} + \left( x - \frac{3a + b}{4} \right) f'(x) - f'(a + b - x)
\]
\[
(2.16)
\]
\[
- \frac{f'(b) - f'(a)}{b-a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right].
\]
We denote
\[
(2.17)
R_a(x) = \frac{1}{b-a} \int_a^b K(x,t) f''(t) dt - \frac{1}{(b-a)^2} \int_a^b K(x,t) dt \int_a^b f''(t) dt.
\]
If \( C \in \mathbb{R} \) is an arbitrary constant, then we have
\[
(2.18)
R_a(x) = \frac{1}{b-a} \int_a^b (f''(t) - C) \left[ K(x,t) - \frac{1}{b-a} \int_a^b K(x,s) ds \right] dt.
\]
Furthermore, we have
\[
(2.19)\quad |R_a(x)| \leq \frac{1}{b-a} \max_{t \in [a,b]} \left| K(x,t) - \frac{1}{b-a} \int_a^b K(x,s) ds \right| \int_a^b |f''(t) - C| dt.
\]
To compute
\[
\max_{t \in [a,b]} \left| K(x,t) - \frac{1}{b-a} \int_a^b K(x,s) ds \right|
\]
\[
= \max \left\{ \left| \frac{1}{2}(x-a)^2 - \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right|, \right.
\]
\[
\left. \left| \frac{1}{2} \left( \frac{a+b}{2} - x \right)^2 - \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right|, \right.
\]
\[
\left. \left| \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right| \right\}
\]
\[
(2.20)
= \max \left\{ \frac{b-a}{24} |6x - 5a - b|, \frac{b-a}{12} |3x - 2a - b|, \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96} \right\}
\]
we denote
\[
y_1 = \frac{b-a}{24} |6x - 5a - b|, \quad y_2 = \frac{b-a}{12} |3x - 2a - b|, \quad y_3 = \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b-a)^2}{96}.
\]
A direct computation gives that
\[
(2.21)
\begin{cases}
y_2 \geq \max\{y_1, y_3\}, & x \in \left[ a, \frac{3a+b}{4} \right], \\
y_1 > \max\{y_2, y_3\}, & x \in \left( \frac{3a+b}{4}, \frac{a+b}{2} \right].
\end{cases}
\]
Therefore, we get
\[
(2.22)\quad \max_{t \in [a,b]} \left| K(x,t) - \frac{1}{b-a} \int_a^b K(x,s) ds \right| = \max \{ y_1, y_2 \} = \left[ \frac{(b-a)^2}{48} + \frac{b-a}{4} \left| x - \frac{3a+b}{4} \right| \right].
\]
We also have
\[
(2.23)\quad \int_a^b |f''(t) - \gamma| dt = (S - \gamma)(b-a)
\]
Corollary 2.5. Under the assumptions of Theorem 2.3, choose 
\[ (2.24) \int_a^b |f''(t) - \Gamma| dt = (\Gamma - S)(b - a). \]
Therefore, we obtain \((2.14)\) and \((2.15)\) by using \((2.16)\) and choosing \(C = \gamma\) and \(C = \Gamma\) in \((2.19)\), respectively. □

Corollary 2.6. Let \(f\) as in Theorem 2.3. Additionally, if \(f\) is symmetric about \(x = \frac{a+b}{2}\), then for all \(x \in [a, \frac{a+b}{2}]\) we have
\[
\left| f\left( \frac{a+b}{2} \right) \right| + f'\left( \frac{a+b}{2} \right) \left| \frac{f'(x) - f'(a)}{b-a} \right|^2 - \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{12} (\Gamma - S)(b - a)^2.
\]
\[
\left| f\left( \frac{a+b}{2} \right) \right| + f'\left( \frac{a+b}{2} \right) \left| \frac{f'(x) - f'(a)}{b-a} \right|^2 - \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{12} (\Gamma - S)(b - a)^3.
\]

Theorem 2.4. Let \(f : [a,b] \to \mathbb{R}\) be a thrice continuously differentiable mapping in \((a,b)\) with \(f'' \in L^2[a,b]\). Then for all \(x \in [a, \frac{a+b}{2}]\) we have
We also have

\[
\frac{f(x) + f(a + b - x)}{2} - \left( x - \frac{3a + b}{4} \right) \frac{f'(x) - f'(a + b - x)}{2} + \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right] - \frac{1}{b - a} \int_a^b f(t) dt
\]

\[
(2.28)
\]

\[
\frac{1}{\pi} \|f''\|_2 \left\{ \frac{1}{320} (a + b - 2x)^5 + \frac{1}{10} (x - a)^5 - (b - a) \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right] \right\}^{1/2}.
\]

Proof. Let \( R_n(x) \) be defined by (2.17). From (2.16), we get

\[
R_n(x) = \frac{1}{b - a} \int_a^b f(t) dt - \frac{f(x) + f(a + b - x)}{2} + \left( x - \frac{3a + b}{4} \right) \frac{f'(x) - f'(a + b - x)}{2} - \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right].
\]

(2.29)

If we choose \( C = f''((a + b)/2) \) in (2.18) and use the Cauchy inequality, then we get

\[
|R_n(x)| \leq \frac{1}{b - a} \int_a^b f''(t) dt - f'' \left( \frac{a + b}{2} \right) \left[ K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) ds \right] dt
\]

\[
\leq \frac{1}{b - a} \left[ \int_a^b \left( f''(t) - f'' \left( \frac{a + b}{2} \right) \right)^2 dt \right]^{1/2} \left[ \int_a^b \left( K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) ds \right)^2 dt \right]^{1/2}.
\]

We can use the Diaz-Metcalf inequality to get

\[
\int_a^b \left( f''(t) - f'' \left( \frac{a + b}{2} \right) \right)^2 dt \leq \frac{(b - a)^2}{\pi^2} \|f''\|^2_2.
\]

We also have

\[
\int_a^b \left( K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) ds \right)^2 dt
\]

\[
= \int_a^b K(x, t)^2 dt - (b - a) \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right]^2
\]

(2.30)

Therefore, using the above relations, we obtain (2.27). \( \square \)

Corollary 2.7. Under the assumptions of Theorem 2.4, choose

(1) \( x = \frac{3a + b}{4} \), we have

\[
(2.31)
\]

(2) \( x = a \), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{f'(b) - f'(a)}{b - a} \frac{(b - a)^2}{96} - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{(b - a)^{5/2}}{48\pi\sqrt{5}} \|f''\|_2.
\]

(2) \( x = a \), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{f'(b) - f'(a)}{b - a} \frac{(b - a)^2}{12} - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{(b - a)^{5/2}}{12\pi\sqrt{5}} \|f''\|_2.
\]
(3) $x = \frac{a + b}{4}$, we have
\[
\left| f\left(\frac{a + b}{2}\right) + \frac{f'(b) - f'(a)}{b - a} \frac{(b-a)^2}{24} - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^{5/2}}{12\sqrt{5}} \|f''\|_2.
\]

Corollary 2.8. Let $f$ as in Theorem 2.3. Additionally, if $f$ is symmetric about $x = \frac{a + b}{4}$, i.e., $f(a + b - x) = f(x)$, then for all $x \in [a, a+b]$ we have
\[
\frac{1}{\pi} \|f''\|_2 \leq \frac{1}{4} \left( \frac{a + b}{2} \right)^5 + \frac{1}{10} \left( x - a \right)^5 - (b-a) \left[ \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{1}{96} \right]^2 \right\}^{1/2}.
\]

Theorem 2.5. Let $f : [a, b] \to \mathbb{R}$ be such that $f'$ is absolutely continuous on $[a, b]$ with $f'' \in L^2[a, b]$. Then for all $x \in [a, a+b]$ we have

\[
\left| f(x) + \frac{f(a) + f(b)}{2} \left( x - \frac{3a+b}{4} \right) f'(x) + \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{1}{96} \right] - \frac{1}{b - a} \int_a^b f(t) dt \right|
\]

(2.32) $\leq \frac{\sqrt{\sigma(f''')}}{b - a} \left\{ \frac{1}{320} \left( a + b - 2x \right)^5 + \frac{1}{10} \left( x - a \right)^5 - (b-a) \left[ \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{1}{96} \right]^2 \right\}^{1/2},$

where $\sigma(f'')$ is defined by

(2.33) $\sigma(f'') = \|f''\|_2^2 - \frac{(f'(b) - f'(a))^2}{b-a} = \|f''\|_2^2 - S^2(b-a)$

and $S$ is defined in Theorem 2.3.

Proof. Let $R_a(x)$ be defined by (2.17). If we choose $C = \frac{1}{\sigma_a} \int_a^b f''(s) ds$ in (2.18) and use the Cauchy inequality and (2.30), then we get

\[
|R_a(x)| \leq \frac{1}{b - a} \left| \int_a^b f''(t) dt - \frac{1}{b - a} \int_a^b f''(s) ds \right| K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) ds dt
\]

(2.34) $\left[ \frac{1}{b - a} \left[ \int_a^b \left( f''(t) - \frac{1}{b - a} \int_a^b f''(s) ds \right)^2 dt \right]^{1/2} \left[ \int_a^b \left( K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) ds \right)^2 dt \right]^{1/2} \right]$.

\[
\leq \sqrt{\sigma(f'')} \frac{1}{b - a} \left\{ \frac{1}{320} \left( a + b - 2x \right)^5 + \frac{1}{10} \left( x - a \right)^5 - (b-a) \left[ \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 + \frac{1}{96} \right]^2 \right\}^{1/2}.
\]

\[
\square
\]

Corollary 2.9. Under the assumptions of Theorem 2.8, choose

(1) $x = \frac{3a+b}{4}$, we have

(2.34) $\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+b}{4}\right)}{2} + \frac{f'(b) - f'(a)}{b - a} \frac{(b-a)^2}{96} - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^{3/2}}{48\sqrt{5}} \sqrt{\sigma(f'')}.$
Theorem 3.1. The following result holds.

Applying inequality (2.8) to the intervals $[a, b]$.

Proof.\[
\frac{f(a) + f(b)}{2} - \frac{f'(b) - f'(a)}{b - a} \frac{b - a}{12} - \frac{1}{b - a} \int_a^b f(t) dt \leq \frac{(b - a)^{3/2}}{12\sqrt{b}} \sqrt{\sigma(f''(x))}.
\]

Corollary 2.10. Let $f$ as in Theorem 2.8. Additionally, if $f$ is symmetric about $x = a + b$, then for all $x \in [a, \frac{a+b}{2}]$ we have

\[
\left| f(x) - \left( x - \frac{3a + b}{4} \right) f'(x) + \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right] - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{\sqrt{\sigma(f'')}}{b - a} \left\{ \frac{1}{320} (a - b - 2x)^5 + \frac{1}{10} (x - a)^5 - \left( \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right)^2 \right\}^{1/2}.
\]

3. Application to Composite Quadrature Rules

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partition of the interval $[a, b]$ and $h_i = x_{i+1} - x_i$ ($i = 0, 1, 2, \ldots, n - 1$).

Consider the perturbed composite quadrature rules

\[(3.1) \quad Q^1_n(I_n, f) = \frac{1}{2} \sum_{i=0}^{n-1} \left[ f \left( \frac{3x_i + x_{i+1}}{4} \right) + f \left( \frac{x_i + 3x_{i+1}}{4} \right) \right] h_i + \sum_{i=0}^{n-1} \frac{f'(x_{i+1}) - f'(x_i)}{96} h_i^2 \]

and

\[(3.2) \quad Q^2_n(I_n, f) = \frac{1}{2} \sum_{i=0}^{n-1} \left[ f \left( \frac{3x_i + x_{i+1}}{4} \right) + f \left( \frac{x_i + 3x_{i+1}}{4} \right) \right] h_i + \frac{\Gamma + \gamma}{192} \sum_{i=0}^{n-1} h_i^3.\]

The following result holds.

Theorem 3.1. Let $f : [a, b] \to \mathbb{R}$ be such that $f'$ is absolutely continuous on $[a, b]$. If $f'' \in L^1[a, b]$ and $\gamma \leq f''(x) \leq \Gamma, \forall x \in [a, b]$, then for all $x \in [a, \frac{a+b}{2}]$ we have

\[
\int_a^b f(t) dt = Q^1_n(I_n, f) + R^1_n(I_n, f),
\]

where $Q^1_n(I_n, f)$ is defined by formula (3.1), and the remainder $R^1_n(I_n, f)$ satisfies the estimate

\[(3.3) \quad |R^1_n(I_n, f)| \leq \frac{\Gamma - \gamma}{128} \sum_{i=0}^{n-1} h_i^3.\]

Proof. Applying inequality (2.8) to the intervals $[x_i, x_{i+1}]$, then we get

\[
\left| \int_{x_i}^{x_{i+1}} f(t) dt - \frac{1}{2} \left[ f \left( \frac{3x_i + x_{i+1}}{4} \right) + f \left( \frac{x_i + 3x_{i+1}}{4} \right) \right] h_i - \frac{f'(x_{i+1}) - f'(x_i)}{96} h_i^2 \right| \leq \frac{\Gamma - \gamma}{128} h_i^3
\]

for $i = 0, 1, 2, \ldots, n - 1$. Now summing over $i$ from 0 to $n - 1$ and using the triangle inequality, we get (3.3).
Theorem 3.2. Let \( f : [a, b] \to \mathbb{R} \) be such that \( f' \) is absolutely continuous on \([a, b]\). If \( f'' \in L^1[a, b] \) and \( \gamma \leq f''(x) \leq \Gamma, \forall x \in [a, b] \), then for all \( x \in \left[ a, \frac{a+b}{2} \right] \) we have
\[
\int_{a}^{b} f(t) dt = Q_n^2(I_n, f) + R_n^2(I_n, f),
\]
where \( Q_n^2(I_n, f) \) is defined by formula (3.2), and the remainder \( R_n^2(I_n, f) \) satisfies the estimate
\[
|R_n^2(I_n, f)| \leq \frac{\Gamma - \gamma}{192} \sum_{i=0}^{n} h_i^3.
\]

Proof. Applying inequality (2.13) to the intervals \([x_i, x_{i+1}]\), then we get
\[
\left| \int_{x_i}^{x_{i+1}} f(t) dt - \frac{1}{2} \left[ f \left( \frac{3x_i + x_{i+1}}{4} \right) + f \left( \frac{x_i + 3x_{i+1}}{4} \right) \right] h_i \right| 
\leq \frac{\Gamma - \gamma}{192} h_i^3
\]
for \( i = 0, 1, 2, \ldots, n - 1 \). Now summing over \( i \) from 0 to \( n - 1 \) and using the triangle inequality, we get (3.4).

Theorem 3.3. Let \( f : [a, b] \to \mathbb{R} \) be such that \( f' \) is absolutely continuous on \([a, b]\). If \( f'' \in L^1[a, b] \) and \( \gamma \leq f''(x) \leq \Gamma, \forall x \in [a, b] \), then for all \( x \in \left[ a, \frac{a+b}{2} \right] \) we have
\[
\int_{a}^{b} f(t) dt = Q_n^1(I_n, f) + R_n^1(I_n, f),
\]
where \( Q_n^1(I_n, f) \) is defined by formula (3.4), and the remainder \( R_n^1(I_n, f) \) satisfies the estimate
\[
|R_n^1(I_n, f)| \leq \frac{1}{48} (S - \gamma) \sum_{i=0}^{n} h_i^3
\]
and
\[
|R_n^1(I_n, f)| \leq \frac{1}{48} (S - \gamma) \sum_{i=0}^{n} h_i^3.
\]

Proof. Applying inequality (2.25) and (2.26) to the intervals \([x_i, x_{i+1}]\), then we get
\[
\left| \int_{x_i}^{x_{i+1}} f(t) dt - \frac{1}{2} \left[ f \left( \frac{3x_i + x_{i+1}}{4} \right) + f \left( \frac{x_i + 3x_{i+1}}{4} \right) \right] h_i \right| 
\leq \frac{1}{48} (S - \gamma) h_i^3
\]
and
\[
\left| \int_{x_i}^{x_{i+1}} f(t) dt - \frac{1}{2} \left[ f \left( \frac{3x_i + x_{i+1}}{4} \right) + f \left( \frac{x_i + 3x_{i+1}}{4} \right) \right] h_i \right| 
\leq \frac{1}{48} (S - \gamma) h_i^3,
\]
for \( i = 0, 1, 2, \ldots, n - 1 \). Now summing over \( i \) from 0 to \( n - 1 \) and using the triangle inequality, we get (3.5) and (3.6).

Theorem 3.4. Let \( f : [a, b] \to \mathbb{R} \) be a thrice continuously differentiable mapping in \((a, b)\) with \( f''' \in L^2[a, b] \). Then for all \( x \in \left[ a, \frac{a+b}{2} \right] \) we have
\[
\int_{a}^{b} f(t) dt = Q_n^3(I_n, f) + R_n^3(I_n, f),
\]
where \( Q^1_n(I_n, f) \) is defined by formula (3.1), and the remainder \( R^1_n(I_n, f) \) satisfies the estimate

\[
|R^1_n(I_n, f)| \leq \frac{\|f''\|_2}{48\pi\sqrt{5}} \sum_{i=0}^{n-1} h_i^{7/2}.
\]

\( (3.7) \)

**Proof.** Applying inequality (2.34) to the intervals \([x_i, x_{i+1}]\), then we get

\[
\left| \int_{x_i}^{x_{i+1}} f(t) dt - \frac{1}{2} \left[ f \left( \frac{3x_i + x_{i+1}}{4} \right) + f \left( \frac{x_i + 3x_{i+1}}{4} \right) \right] h_i - \frac{f'(x_{i+1}) - f'(x_i)}{96} h_i^2 \right| \\
\leq \frac{h_i^{7/2}}{48\pi\sqrt{5}} \|f''\|_2,
\]

for \( i = 0, 1, 2, \ldots, n - 1 \). Now summing over \( i \) from 0 to \( n - 1 \) and using the triangle inequality, we get (3.7).

\( \square \)

**Theorem 3.5.** Let \( f : [a, b] \to \mathbb{R} \) be such that \( f' \) is absolutely continuous on \([a, b]\) with \( f'' \in L^2[a, b] \). Then for all \( x \in [a, \frac{a+b}{2}] \) we have

\[
\int_a^b f(t) dt = Q^1_n(I_n, f) + R^1_n(I_n, f),
\]

where \( Q^1_n(I_n, f) \) is defined by formula (3.1), and the remainder \( R^1_n(I_n, f) \) satisfies the estimate

\[
|R^1_n(I_n, f)| \leq \frac{\sqrt{\sigma(f'')}}{48\pi\sqrt{5}} \sum_{i=0}^{n-1} h_i^{5/2}.
\]

\( (3.8) \)

**Proof.** Applying inequality (2.34) to the intervals \([x_i, x_{i+1}]\), then we get

\[
\left| \int_{x_i}^{x_{i+1}} f(t) dt - \frac{1}{2} \left[ f \left( \frac{3x_i + x_{i+1}}{4} \right) + f \left( \frac{x_i + 3x_{i+1}}{4} \right) \right] h_i - \frac{f'(x_{i+1}) - f'(x_i)}{96} h_i^2 \right| \\
\leq \frac{h_i^{5/2}}{48\pi\sqrt{5}} \sqrt{\sigma(f''(x))},
\]

for \( i = 0, 1, 2, \ldots, n - 1 \). Now summing over \( i \) from 0 to \( n - 1 \) and using the triangle inequality, we get (3.8).

\( \square \)

In order to compare the error between these two methods and composite trapezoidal formula, we apply the above methods and composite trapezoidal formula to specific examples. For this purpose consider the following Table:

**Table 1. Numerical results**

| \( f(x) \) | \( n \) | \( [a, b] \) | \( \int_a^b f(x) dx \) | \( T_n \) | Error of \( T_n \) | \( Q^1_n \) | \( Q^2_n \) | Error of \( Q^1_n \) and \( Q^2_n \) |
|---|---|---|---|---|---|---|---|---|
| \( \cos x - x \) | 20 | \([0, \frac{\pi}{2}]\) | -0.233701 | -0.234215 | 5.14E-4 | -0.233636 | -0.233636 | 6.5E-5 |
| \( e^{2x} \cos(x^2) \) | 20 | \([0, 1]\) | -1.176887 | -1.181466 | 4.57E-3 | -1.176316 | -1.176316 | 5.71E-4 |
| \( \frac{1}{x^2 + 2x + 3} \) | 10 | \([0, 1]\) | 0.241549 | 0.241393 | 1.56E-4 | 0.241569 | 0.241569 | 2E-5 |
| \( \tan(x^2 + x) \) | 20 | \([0, \frac{\pi}{2}]\) | 0.654999 | 0.655127 | 1.28E-4 | 0.654983 | 0.654983 | 7E-6 |
| \( \ln(x^2 + 1) \) | 20 | \([-1, 1]\) | 0.527887 | 0.529554 | 1.667E-3 | 0.527679 | 0.527679 | 2.08E-4 |
4. Application to probability density functions

Now, let $X$ be a random variable taking values in the finite interval $[a, b]$, with the probability density function $f : [a, b] \to [0, 1]$ and with the cumulative distribution function

$$F(x) = \Pr(X \leq x) = \int_a^x f(t)dt.$$ 

The following results hold:

**Theorem 4.1.** With the assumptions of Theorem 2.1, we have

$$\frac{1}{2} \left[ F(x) + F(a + b - x) \right] - \left( x - \frac{3a + b}{4} \right) \frac{f(x) - f(a + b - x)}{2}$$

$$+ \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right] - \frac{b - E(X)}{b - a}$$

$$\leq \frac{1}{8} (\Gamma - \gamma) \left[ \frac{b - a}{4} + \left| x - \frac{3a + b}{4} \right| \right]^2.$$  

(4.1)

for all $x \in [a, \frac{a + b}{2}]$, where $E(X)$ is the expectation of $X$.

**Proof.** By (2.4) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)dt,$$

we obtain (4.1). \hfill \Box

**Corollary 4.1.** Under the assumptions of Theorem 4.1 with $x = \frac{3a + b}{4}$, we have

$$\frac{1}{2} \left[ F \left( \frac{3a + b}{4} \right) + F \left( \frac{a + 3b}{4} \right) \right] + \frac{b - a}{96} \left[ f'(b) - f'(a) \right] - \frac{b - E(x)}{b - a} \leq \frac{\Gamma - \gamma}{128} (b - a)^2.$$  

(4.2)

**Theorem 4.2.** With the assumptions of Theorem 2.2, we have

$$\frac{1}{2} \left[ F(x) + F(a + b - x) \right] - \left( x - \frac{3a + b}{4} \right) \frac{f(x) - f(a + b - x)}{2}$$

$$+ \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right] - \frac{b - E(X)}{b - a}$$

$$\leq \frac{\Gamma - \gamma}{2} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right]$$

(4.3)

for all $x \in [a, \frac{a + b}{4}]$, where $E(X)$ is the expectation of $X$.

**Proof.** By (2.9) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)dt,$$

we obtain (4.3). \hfill \Box

**Corollary 4.2.** Under the assumptions of Theorem 4.2 with $x = \frac{a + b}{4}$, we have

$$\frac{1}{2} \left[ F \left( \frac{3a + b}{4} \right) + F \left( \frac{a + 3b}{4} \right) \right] + \frac{\Gamma + \gamma}{192} (b - a)^2 - \frac{b - E(x)}{b - a} \leq \frac{\Gamma - \gamma}{192} (b - a)^2.$$  

(4.4)
Theorem 4.3. With the assumptions of Theorem 2.3, we have
\[
\left| \frac{1}{2} [F(x) + F(a + b - x)] - \left( x - \frac{3a + b}{4} \right) \frac{f'(x) - f'(a + b - x)}{2} + \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} (x - \frac{3a + b}{4})^2 + \frac{(b - a)^2}{96} \right] \frac{b - E(X)}{b - a} \right| \\
\leq (S - \gamma) \left[ \frac{(b - a)^2}{48} + \frac{b - a}{4} \left| x - \frac{3a + b}{4} \right| \right]
\]
and
\[
\left| \frac{1}{2} [F(x) + F(a + b - x)] - \left( x - \frac{3a + b}{4} \right) \frac{f'(x) - f'(a + b - x)}{2} + \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} (x - \frac{3a + b}{4})^2 + \frac{(b - a)^2}{96} \right] \frac{b - E(X)}{b - a} \right| \\
\leq (\Gamma - S) \left[ \frac{(b - a)^2}{48} + \frac{b - a}{4} \left| x - \frac{3a + b}{4} \right| \right]
\]
for all \( x \in [a, \frac{a + b}{2}] \), where \( E(X) \) is the expectation of \( X \).

Proof. By (2.14) and (2.15) on choosing \( f = F \) and taking into account
\[
E(X) = \int_a^b tdF(t) = b - \int_a^b F(t)dt,
\]
we obtain (4.5) and (4.6).

Corollary 4.3. Under the assumptions of Theorem 4.3 with \( x = \frac{3a + b}{4} \), we have
\[
\left| \frac{1}{2} \left[ F \left( \frac{3a + b}{4} \right) + F \left( \frac{a + 3b}{4} \right) \right] + \frac{b - a}{96} [f'(b) - f'(a)] - \frac{b - E(x)}{b - a} \right| \leq \frac{1}{48} (S - \gamma)(b - a)^2
\]
and
\[
\left| \frac{1}{2} \left[ F \left( \frac{3a + b}{4} \right) + F \left( \frac{a + 3b}{4} \right) \right] + \frac{b - a}{96} [f'(b) - f'(a)] - \frac{b - E(x)}{b - a} \right| \leq \frac{1}{48} (\Gamma - S)(b - a)^2.
\]

Theorem 4.4. With the assumptions of Theorem 2.4, we have
\[
\left| \frac{1}{2} [F(x) + F(a + b - x)] - \left( x - \frac{3a + b}{4} \right) \frac{f'(x) - f'(a + b - x)}{2} + \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} (x - \frac{3a + b}{4})^2 + \frac{(b - a)^2}{96} \right] \frac{b - E(X)}{b - a} \right| \\
\leq \frac{1}{\pi} \| f'' \|_2 \left\{ \frac{1}{320} (a + b - 2x)^5 + \frac{1}{10} (x - a)^5 - (b - a) \left[ \frac{1}{2} (x - \frac{3a + b}{4})^2 + \frac{(b - a)^2}{96} \right]^2 \right\}^{1/2}
\]
for all \( x \in [a, \frac{a + b}{2}] \), where \( E(X) \) is the expectation of \( X \).

Proof. By (2.27) on choosing \( f = F \) and taking into account
\[
E(X) = \int_a^b tdF(t) = b - \int_a^b F(t)dt,
\]
we obtain (4.9).
Corollary 4.4. Under the assumptions of Theorem 4.4 with 
\( x = \frac{3a + b}{4} \), we have

\[
\left(4.10\right) \quad \left| \frac{1}{2} \left[ F\left( \frac{3a + b}{4} \right) + F\left( \frac{a + 3b}{4} \right) \right] + \frac{b - a}{96} \left[ f'(b) - f'(a) \right] - \frac{b - E(x)}{b - a} \right| \leq \frac{(b - a)^{5/2}}{48\pi \sqrt{5}} \| f'' \|_2.
\]

Theorem 4.5. With the assumptions of Theorem 2.5 we have

\[
\left| \frac{1}{2} \left[ F(x) + F(a + b - x) - \left( x - \frac{3a + b}{4} \right) f'(x) - \frac{f'(a + b - x)}{2} \right] + \frac{f'(b) - f'(a)}{b - a} \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right] \frac{b - E(X)}{b - a} \right|
\]

\[
\leq \sqrt{\pi (f''')} \left\{ \frac{1}{320} (a + b - 2x)^5 + \frac{1}{10} (x - a)^5 - (b - a) \left[ \frac{1}{2} \left( x - \frac{3a + b}{4} \right)^2 + \frac{(b - a)^2}{96} \right]^2 \right\}^{1/2}
\]

for all 
\( x \in \left[ a, \frac{a + b}{2} \right] \), where 
\( E(X) \) is the expectation of 
\( X \).

Proof. By (2.32) on choosing 
\( f = F \) and taking into account

\[
E(X) = \int_a^b t dF(t) = b - \int_a^b F(t) dt,
\]

we obtain (4.11). \( \square \)

Corollary 4.5. Under the assumptions of Theorem 4.4 with 
\( x = \frac{3a + b}{4} \), we have

\[
\left(4.12\right) \quad \left| \frac{1}{2} \left[ F\left( \frac{3a + b}{4} \right) + F\left( \frac{a + 3b}{4} \right) \right] + \frac{b - a}{96} \left[ f'(b) - f'(a) \right] - \frac{b - E(x)}{b - a} \right| \leq \frac{(b - a)^{3/2}}{48\sqrt{5}} \sqrt{\sigma(f''')}.
\]

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