CHAOTIC THRESHOLD OF A CLASS OF HYBRID PIECEWISE-SMOOTH SYSTEM BY AN IMPULSIVE EFFECT VIA MELNIKOV-TYPE FUNCTION

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Abstract. In this paper, we study the chaotic behavior of a class of hybrid piecewise-smooth system incorporated into an impulsive effect (HPSS-IE) under a periodic perturbation. More precisely, we assume that the unperturbed system with a homoclinic orbit, it transversally jumps across the first switching manifold by an impulsive stimulation and continuously crosses the second switching manifold. Then the corresponding Melnikov-type function is derived. Based on the new Melnikov-type function, the bifurcation and chaotic threshold of the perturbed HPSS-IE are analyzed. Furthermore, numerical simulations are precisely demonstrated through a concrete example. The results indicate that it is an extension work of previous references.

1. Introduction. For the past few decades, nonlinear dynamical systems described by discontinuous (non smooth) differential equations have been a hot topic among scholars. These systems have been extensively used to model phenomena in many fields such as physical problems and medicine biology [8, 14, 12, 15, 26], etc. Additionally, bifurcation and chaotic dynamical behavior represent an even greater interest in nonlinear dynamical system [7, 17, 19, 36, 25, 48, 49, 43, 44, 45, 46, 47].

As is well known, homoclinic bifurcation leads to chaos in smooth system is associated with the existence of a simple zero of the corresponding Melnikov function. More precisely, when an autonomous differential equation has a homoclinic orbit with a small amplitude, periodic perturbation. If there has a simple zero in the corresponding Melnikov function, then the perturbed stable and unstable manifolds would be broken and intersect transversally. It implies that there exists chaos in the sense of Smale horseshoes (see [18, 50]). To investigate this kind of
problems, Melnikov method [35] is a very powerful tool. In recent years, Melnikov method that has been extended and improved through some heuristic works [2, 3, 4, 5, 6, 9, 11, 24, 22, 33, 34, 40].

In [23], the non-smooth Melnikov function for discontinuous planar system was derived by Kukučka. Then, Li and his cooperators did a series of meaningful works to extend Melnikov function to discontinuous time-periodic planar (hybrid) systems with simple perturbation techniques (e.g. see [27, 29, 28, 30, 31, 32]). They considered a piecewise smooth planar system under periodic perturbation with one switching manifold, and obtained the Melnikov-type function to detect the distance between the stable and unstable manifolds of a perturbed homoclinic orbit in [32]. Whereafter, they [28] improved the Melnikov function to a general planar piecewise-smooth system which possibly is non-zero trace and defined a small reset map to describe the instantaneous impact rule. Very recently, they considered the non-smooth oscillators with multiple switching manifolds, a good result obtained in (see [30]) that the total energy difference between the left and right Hamilton functions at two intersection points with each switch manifold was zero. On the other hand, Tian et al. investigated the bifurcation and chaos of duffing system with instantaneous impulsive effect (jump discontinuities), the corresponding Melnikov-type function is given in [38]. It can be seen that the non-smooth Melnikov function is quite different with smooth case, it may be in the form of piecewise integration (see [30, eq. (39)]) or generate additional terms (see [23, eq. (57)], [32, eq. (28)] and [38, eq. (31)]). This is mainly due to the non-smooth condition at switching manifold or sudden change by instantaneous impulsive effect.

It is noticed that the non-instantaneous impulsive effect has a jump at the impulsive point and still remains active on a finite time interval after starting at an arbitrary impulsive point (see [20]). Motivated by the works on the non-instantaneous impulsive differential equations (e.g. see [1, 10, 13, 16, 21, 37, 39, 42, 41, 51]) and aforementioned works. In this paper, we study the chaotic behaviour of HPSS-IE under a periodic perturbation. The unperturbed system is described as follows: (i) we suppose the unperturbed piecewise hybrid system to possess a homoclinic orbit, it receives a stimulation such as an impulsive effect on the first switching manifold, it may cause an instantaneous sudden change and transversely jump across the first switching manifold. (ii) Then it is continuously changing at the second switching manifold after remaining active for a finite time interval (one can refer to see Fig. 1(a)). If this system is subjected to a periodic perturbation, the mentioned Melnikov-type functions (such as [23, eq. (57)], [30, eq. (39)], [32, eq. (28)] and [38, eq. (31)]) are unfit to deal with this issue of both pulse-induced jump discontinuities and piecewise-smooth planar hybrid system. We see that Li et al. [30] had studied global dynamics of a class of hybrid piecewise-smooth system with four switching manifolds, they analyzed the right piecewise-defined homoclinic orbit transversally and continuously crosses two switching manifolds. Different from them, this work studies the right piecewise-smooth homoclinic orbit transversally jumps across the first switching manifold by an impulsive effect. It may lead us to obtain different Melnikov-type function and results.

Therefore, in this work, we aim to extend the Melnikov-type function into a jump discontinuity and piecewise smooth hybrid system.

The work is arranged as follows. In Sec. 2, we introduce the model and some basic assumptions. The Sec. 3 is devoted to derive the Melnikov-type function to a class of HPSS-IE under a periodic perturbation. An example and numerical
simulations are constructed to apply the derived Melnikov-type function to study the chaotic behaviour of the system in Sec. 4. Finally, we give a conclusion.

2. Description of the model. In this section, a class of HPSS-IE is studied. We assume that the unperturbed system has a homoclinic orbit to the hyperbolic saddle point. Furthermore, the orbit transversally jumps across the first switching manifold and continuously crosses the second switching manifold (as we present in introduction). Meanwhile, this system is perturbed by a damping term and non-autonomous periodic function. “A jump discontinuity and piecewise smooth system” Melnikov-type function is derived to investigate this phenomenon. Some basic assumptions and ideas are taken from [30, 32, 38]. The system is described by:

$$
\begin{align*}
\dot{x}(t) &= f^-(x) + \varepsilon q^-(x, \dot{x}, t), \quad x \in V_-, \\
\dot{x}(t) &= f^N(x) + \varepsilon q^N(x, \dot{x}, t), \quad x \in V_N, \\
\dot{x}(t) &= f^+(x) + \varepsilon q^+(x, \dot{x}, t), \quad x \in V_+, \\
\Delta x \mid_{x=\Gamma_0, \Gamma_1} &= 0, \\
\Delta \dot{x} \mid_{x=\Gamma_0} &\neq 0, \Delta \dot{x} \mid_{x=\Gamma_1} = 0,
\end{align*}
$$

(1)

where \( \dot{x} = \frac{dx}{dt}, \dot{\varepsilon} = \frac{d^2x}{dt^2}, x \in \mathbb{R}, t \in \mathbb{R} \) and \( 0 < \varepsilon \ll 1 \) (sufficiently small). Denote that \( \dot{x} = y \). Then the state-space \( \mathbb{R}^2 \) is divided into three open, disjoint regions \( (V_-, V_N \text{ and } V_+) \) by two switching manifolds \( \Gamma_0 \) and \( \Gamma_1 \). Namely, \( \mathbb{R}^2 = V_- \cup \Gamma_0 \cup V_N \cup \Gamma_1 \cup V_+ \). For convenience, we define three regions and two switching manifolds as follows:

$$
\begin{align*}
V_- &= \{(x, y) \in \mathbb{R}^2 \mid x < x_0\}, \\
V_N &= \{(x, y) \in \mathbb{R}^2 \mid x_0 < x < x_1\}, \\
V_+ &= \{(x, y) \in \mathbb{R}^2 \mid x > x_1\}, \\
\Gamma_0^+ &= \{(x, y) \in \mathbb{R}^2 \mid x = x_0, y > 0\}, \\
\Gamma_0^- &= \{(x, y) \in \mathbb{R}^2 \mid x = x_0, y < 0\}, \\
\Gamma_1^+ &= \{(x, y) \in \mathbb{R}^2 \mid x = x_1, y > 0\}, \\
\Gamma_1^- &= \{(x, y) \in \mathbb{R}^2 \mid x = x_1, y < 0\}.
\end{align*}
$$

(2)

Obviously, for the switching manifold, we have \( \Gamma_0 = \Gamma_0^+ \cup \Gamma_0^- \) and \( \Gamma_1 = \Gamma_1^+ \cup \Gamma_1^- \). Suppose that the functions \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( q : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \), they are both sufficiently differentiable \((C^r, r \geq 2)\) on the corresponding region. \( q \) is \( \tilde{T} \) \((\tilde{T} = \frac{2\pi}{\omega})\) periodic in \( t \). We also let \( \dot{x}^+ \mid_{x=x_0, x_1} \) be the right-hand and left-hand limit of \( \dot{x} \) at \( x = x_0 \) and \( x = x_1 \). Similarly, \( \dot{x}^\pm \mid_{x=x_0, x_1} \) represent the right-hand and left-hand limit of \( \dot{x} \) at \( x = x_0 \) and \( x = x_1 \), respectively. It yields

$$
\begin{align*}
\Delta x \mid_{x=\Gamma_0, \Gamma_1} &= \Delta x \mid_{x=x_0, x_1} = x^+ \mid_{x=x_0, x_1} - x^- \mid_{x=x_0, x_1}, \\
\Delta y \mid_{x=\Gamma_0, \Gamma_1} &= \Delta y \mid_{x=x_0, x_1} = \dot{x}^+ \mid_{x=x_0, x_1} - \dot{x}^- \mid_{x=x_0, x_1}, \\
\dot{x}^+ \mid_{x=x_0} &= (1 - \varepsilon \eta)\alpha \dot{x}^- \mid_{x=x_0},
\end{align*}
$$

(3)

where \( \alpha \) is a constant of the unperturbed system that instantaneous change by an impulsive excitation and \((1-\varepsilon \eta)\alpha\) denotes the instantaneous change of the perturbed system by an impulsive excitation.

Let

$$
\begin{align*}
Z &= \begin{pmatrix} x \\ y \end{pmatrix}, \quad P^\pm(Z) = \begin{pmatrix} f^\pm(x) \\ y \end{pmatrix}, \quad Q^\pm(Z,t) = \begin{pmatrix} 0 \\ q^\pm(Z,t) \end{pmatrix}, \\
P^N(Z) = \begin{pmatrix} y \\ f^N(x) \end{pmatrix}, \quad Q^N(Z,t) = \begin{pmatrix} 0 \\ q^N(Z,t) \end{pmatrix}.
\end{align*}
$$
Then system (1) is equivalent to
\[
\begin{align*}
\dot{Z} &= P^{-}(Z) + \varepsilon Q^{-}(Z, t), \quad x < x_0, \\
\dot{Z} &= P^{N}(Z) + \varepsilon Q^{N}(Z, t), \quad x_0 < x < x_1, \\
\dot{Z} &= P^{+}(Z) + \varepsilon Q^{+}(Z, t), \quad x > x_1, \\
\Delta x \mid_{x=x_0, x_1} &= 0, \\
\Delta y \mid_{x=x_0} \neq 0, \Delta y \mid_{x=x_1} &= 0.
\end{align*}
\]

For \( \varepsilon = 0 \), we can obtain the unperturbed system of (4) as follows:
\[
\begin{align*}
\dot{Z} &= P^{-}(Z), \quad x < x_0, \\
Z &= P^{N}(Z), \quad x_0 < x < x_1, \\
\dot{Z} &= P^{+}(Z), \quad x > x_1, \\
\Delta x \mid_{x=x_0, x_1} &= 0, \\
\Delta y \mid_{x=x_0} \neq 0, \Delta y \mid_{x=x_1} &= 0.
\end{align*}
\]

We assume that system (5) is a piecewise Hamilton system. It admits a hyperbolic equilibrium \( p_0 \in V_- \) and a piecewise smooth homoclinic orbit \( Z_h(t) = (x_h(t), y_h(t))^T \). The expression of homoclinic orbit \( Z_h(t) \) is described by:

\[
Z_h(t) = \begin{cases}
Z_h^1(t), & t \leq t_a^1, \\
Z_h^2(t), & t \geq t_a^2, \\
Z_h^3(t), & t_a^1 \leq t \leq t_b^2, \\
Z_h^4(t), & t_a^1 \leq t \leq t_b^2, \\
Z_h^5(t), & t_c^2 \leq t \leq t_c^2, \\
Z_h^6(t), & t_c^2 \leq t \leq t_d^2.
\end{cases}
\]

where \( 0 < t_a^1 < t_b^1 < t_d^1 < t_d^2 \), \( Z_h^{1,2,-}(t) \in V^- \) for \( t \leq t_a^1 \) or \( t \geq t_d^2 \), \( Z_h^{1,2,N}(t) \in V^N \) for \( t_a^1 \leq t \leq t_b^2 \) or \( t_c^2 \leq t \leq t_d^2 \) and \( Z_h^5(t) \in V^+ \) for \( t_d^2 \leq t \leq t_c^2 \). Meanwhile, we have the following relations:

\[
\begin{align*}
Z_h^1(t_a^1) &= (x_h^1(t_a^1), y_h^1(t_a^1))^T = (x_0, y_0)^T \in \Gamma_0^+, \\
Z_h^2(t_b^2) &= (x_h^2(t_b^2), y_h^2(t_b^2))^T = (x_1, y_1)^T \in \Gamma_1^+, \\
Z_h^3(t_c^2) &= (x_h^3(t_c^2), y_h^3(t_c^2))^T = (x_1, -y_1)^T \in \Gamma_1^-, \\
Z_h^4(t_d^2) &= (x_h^4(t_d^2), y_h^4(t_d^2))^T = (x_0, -y_0)^T \in \Gamma_0^-, \\
Z_h^5(t_d^2) &= (x_h^5(t_d^2), y_h^5(t_d^2))^T = (x_0, y_0)^T \in \Gamma_0^+.
\end{align*}
\]

Clearly, \( t_a^1 \) and \( t_d^2 \) represent the times at which the unperturbed orbit \( Z_h(t) \) reaches or leaves the first switching manifold \( x_0 \). And \( t_b^2 \) and \( t_c^2 \) are the times at which the unperturbed orbit \( Z_h(t) \) reaches or leaves the second switching manifold \( x_1 \) (see Fig. 1(b)). For convenience, we suppose the unperturbed system (5) satisfy

\[
\begin{align*}
[n \cdot P^-(Z_h^{1,2,-}(t_a^1))] \cdot [n \cdot P^N(Z_h^{1,2,N}(t_d^1))] > 0, \\
[n \cdot P^-(Z_h^{1,2,N}(t_b^2))] \cdot [n \cdot P^+(Z_h^{1,2,2}(t_c^2))] > 0, \\
[n \cdot P^+(Z_h^{1,2,N}(t_c^2))] \cdot [n \cdot P^N(Z_h^{1,2,N}(t_d^2))] > 0, \\
[n \cdot P^+(Z_h^{1,2,N}(t_d^2))] \cdot [n \cdot P^-(Z_h^{1,2,N}(t_a^1))] > 0,
\end{align*}
\]

where \( n \) is the normal vectors of the \( \Gamma_0 \) and \( \Gamma_1 \). That is \( n_0(x, y) = n_1(x, y) = (1, 0) \). Thus, the homoclinic orbit passes clockwise through two switching manifolds \( \Gamma_0 \) and \( \Gamma_1 \) (see Fig. 1(b)).
For the sake of calculation in following section, let $f^-(x_0) = f^N(x_0) = 0$ and $f^+(x_1) = f^+(x_1) = 0$.

Fig. 1 Piecewise smooth homoclinic orbit of the unperturbed system (5).

Then system (5) can be rewritten by

$$
\begin{align*}
\dot{Z} &= JDZ H^-(Z), \quad x < x_0, \\
\dot{Z} &= JDZ H^N(Z), \quad x_0 < x < x_1, \\
\dot{Z} &= JDZ H^+(Z), \quad x > x_1,
\end{align*}
$$

(7)

where $H(Z) = \begin{cases} H^-(Z), & x < x_0 \\ H^N(Z), & x_0 < x < x_1 \\ H^+(Z), & x > x_1 \end{cases}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $DZ$ represents the partial derivatives with respect to $Z$, $JDZ H^\pm(Z) = P^\pm(Z)$ and $JDZ H^N(Z) = P^N(Z)$.

As we see, system (7) can be regarded as a type of HPSS-IE. It can be explained that when the homoclinic orbit transversely jumps across the first switching manifold $x = x_0$ by an instantaneous impulsive stimulation. Then it continuously changes at the second switching manifold $x = x_1$ after remaining active on a finite time interval ($x_0 < x < x_1$). Essentially, system (7) is a hybrid system made of an instantaneous impulsive effect and a piecewise smooth system.

3. Melnikov method analysis of the perturbed system (4). In this section, in order to detect the chaotic threshold of HPSS-IE under a periodic perturbation, the corresponding Melnikov-type function is derived.

The system (4) can be suspended into three dimensional phase space as follows:

$$
\begin{align*}
\dot{Z} &= P^-(Z) + \varepsilon Q^-(Z, t), \quad x < x_0, \\
\dot{Z} &= P^N(Z) + \varepsilon Q^N(Z, t), \quad x_0 < x < x_1, \\
\dot{Z} &= P^+(Z) + \varepsilon Q^+(Z, t), \quad x > x_1, \\
\dot{\theta} &= 1, \\
\Delta x |_{x=x_0,x_1} &= 0, \\
\Delta y |_{x=x_0} &\neq 0, \Delta y |_{x=x_1} = 0,
\end{align*}
$$

(8)
where \( \theta = t \pmod{\bar{T}} \in S^1 \), and the three dimensional phase space reads \((Z, \theta) \in \mathbb{R}^2 \times S^1\). By aforementioned assumptions in section 2, the suspended system (8) has a hyperbolic periodic orbit \( \varphi_0 = \{(p_0, \theta) \mid p_0 \in V^-, \theta \in S^1\} \) for \( \varepsilon = 0 \). It possesses stable and unstable manifolds expressed by \( W^u(\varphi_0) \) and \( W^s(\varphi_0) \), respectively. For \( 0 < \varepsilon \ll 1 \) (sufficiently small), the suspended system (8) admits a hyperbolic periodic orbit \( \varphi_\varepsilon(t) = \{(p_\varepsilon, \theta) \mid p_\varepsilon = p_0 + O(\varepsilon)\} \). \( \varphi_\varepsilon(t) \) would possess stable and unstable manifolds expressed by \( W^u(\varphi_\varepsilon(t)) \) and \( W^s(\varphi_\varepsilon(t)) \), which are \( \varepsilon \) close to \( W^u(\varphi_0) \) and \( W^s(\varphi_0) \) respectively (see [18, Lemmas 4.5.1 and 4.5.2]).

To detect the distance of the stable manifold \( W^s(\varphi_\varepsilon(t)) \) and unstable manifold \( W^u(\varphi_\varepsilon(t)) \) on an proper section is the main purpose in this section. Define that a Poincaré map \( p^\varepsilon_{t_0} : \Sigma_{t_0} \rightarrow \Sigma_{t_0} \), which \( \Sigma_{t_0} = \{(Z, \theta) \in \mathbb{R}^2 \times S^1 \mid \theta = t_0 \in [0, \bar{T}]\} \). By the Poincaré map \( p^\varepsilon_{t_0} \), we consider \( p_\varepsilon \) to be a hyperbolic fix point which possesses one-dimensional \( W^s(p_\varepsilon(t_0)) \) (stable manifold) and \( W^u(p_\varepsilon(t_0)) \) (unstable manifold) on the section \( \Sigma_{t_0} \).

\( Z^u_{h,\varepsilon}(t + t_0, t_0, \varepsilon) \) and \( Z^s_{h,\varepsilon}(t + t_0, t_0, \varepsilon) \) denote that the unique trajectories of the perturbed system (4) that lie in unstable manifold \( W^u(p_\varepsilon(t_0)) \) and stable manifold \( W^s(p_\varepsilon(t_0)) \). They are \( \varepsilon \) close to unperturbed orbit \( Z^u(s_\varepsilon)(t) \), respectively. In the plane \( \Sigma_{t_0} \), let \( L \) be a line segment which is perpendicular to the unperturbed orbit \( Z_h(t) \) at \( t = 0 \). In fact, the unstable manifold \( W^u(p_\varepsilon(t_0)) \) and stable manifold \( W^s(p_\varepsilon(t_0)) \) of system (4) would be five cases shown in Fig. 2. These five cases can be identical. Without loss of generality, we take Fig. 2(c) as the research object.

As shown in Fig. 2(c), the perturbed stable and unstable manifolds would be broken and intersect in section \( V^+ \). Assume that the two points \( Z^u_{h,\varepsilon}(t_0, t_0, \varepsilon) \) and \( Z^s_{h,\varepsilon}(t_0, t_0, \varepsilon) \) are the intersection of the perturbed stable manifold \( Z^u_{h,\varepsilon}(t + t_0, t_0, \varepsilon) \) and unstable manifold \( Z^s_{h,\varepsilon}(t + t_0, t_0, \varepsilon) \) in the section \( V^+ \).
For $0 < \varepsilon \ll 1$ (sufficiently small) and each $t_0 \in \mathbb{R}$, the perturbed orbit $Z_{h,\varepsilon}(t + t_0, t_0, \varepsilon)$ can be presented by:

$$Z_{h,\varepsilon}(t + t_0, t_0, \varepsilon) = \begin{cases} 
Z_{h,\varepsilon}^u(t + t_0, t_0, \varepsilon) = & Z_{h,\varepsilon}^{u,1,-}(t + t_0, t_0, \varepsilon), \quad t \leq t_{a,\varepsilon}^-,
Z_{h,\varepsilon}^{u,1,+}(t + t_0, t_0, \varepsilon), \quad t_{a,\varepsilon}^- \leq t \leq t_{b,\varepsilon}^-,
Z_{h,\varepsilon}(t + t_0, t_0, \varepsilon), \quad t_{b,\varepsilon}^- \leq t \leq t_0,
Z_{h,\varepsilon}^{u,2,+}(t + t_0, t_0, \varepsilon), \quad t \leq t_{c,\varepsilon}^-,
Z_{h,\varepsilon}^{u,2,-}(t + t_0, t_0, \varepsilon), \quad t \geq t_{d,\varepsilon}^-,
\end{cases}$$

and

$$Z_{h,\varepsilon}^s(t + t_0, t_0, \varepsilon) = \begin{cases} 
Z_{h,\varepsilon}^{s,1,-}(t + t_0, t_0, \varepsilon) = Z_{h,\varepsilon}^{s,1,-}(t + t_0, t_0) + O(\varepsilon^2),
Z_{h,\varepsilon}^{s,1,N}(t + t_0, t_0, \varepsilon) = Z_{h,\varepsilon}^{s,1,N}(t + t_0, t_0) + O(\varepsilon^2),
Z_{h,\varepsilon}^{s,1,+}(t + t_0, t_0, \varepsilon) = Z_{h,\varepsilon}^{s,1,+}(t + t_0, t_0) + O(\varepsilon^2),
Z_{h,\varepsilon}^{s,2,+}(t + t_0, t_0, \varepsilon) = Z_{h,\varepsilon}^{s,2,+}(t + t_0, t_0) + O(\varepsilon^2),
Z_{h,\varepsilon}^{s,2,N}(t + t_0, t_0, \varepsilon) = Z_{h,\varepsilon}^{s,2,N}(t + t_0, t_0) + O(\varepsilon^2),
Z_{h,\varepsilon}^{s,2,-}(t + t_0, t_0, \varepsilon) = Z_{h,\varepsilon}^{s,2,-}(t + t_0, t_0) + O(\varepsilon^2).
\end{cases}$$

Moreover,

$$Z_{h,\varepsilon}^{u,1,-}(t_{a,\varepsilon}^- + t_0, t_0, \varepsilon) \neq Z_{h,\varepsilon}^{u,1,N}(t_{a,\varepsilon}^- + t_0, t_0, \varepsilon),
Z_{h,\varepsilon}^{u,1,N}(t_{a,\varepsilon}^- + t_0, t_0, \varepsilon) = Z_{h,\varepsilon}^{u,1,+}(t_{b,\varepsilon}^- + t_0, t_0, \varepsilon),
Z_{h,\varepsilon}^{u,1,+}(t_{b,\varepsilon}^- + t_0, t_0, \varepsilon) = Z_{h,\varepsilon}^{u,2,N}(t_{c,\varepsilon}^- + t_0, t_0, \varepsilon),
Z_{h,\varepsilon}^{u,2,N}(t_{d,\varepsilon}^- + t_0, t_0, \varepsilon) \neq Z_{h,\varepsilon}^{u,2,-}(t_{d,\varepsilon}^- + t_0, t_0, \varepsilon),$$

where $t_{a,\varepsilon}^\pm$ and $t_{d,\varepsilon}^\pm$ represent the times at which the perturbed orbit $Z_{h,\varepsilon}(t + t_0, t_0, \varepsilon)$ reaches or leaves the first switching manifold $x_0$. And $t_{b,\varepsilon}^\pm$ and $t_{c,\varepsilon}^\pm$ are the times at which the perturbed orbit $Z_{h,\varepsilon}(t + t_0, t_0, \varepsilon)$ reaches or leaves the second switching manifold $x_1$ (see Fig. 3).

Fig. 3 The stable and unstable manifolds of perturbed homoclinic orbit for system (4).
Moreover, it is easy to obtain $t_{a,\varepsilon}^-, t_{b,\varepsilon}^-, t_{c,\varepsilon}^-$ and $t_{d,\varepsilon}^+$ can be expanded at $\varepsilon = 0$ as:

\[
\begin{align*}
 t_{a,\varepsilon}^+ &= t_{a,1}^+ + \varepsilon t_{a,2}^+ + O(\varepsilon^2), \\
 t_{b,\varepsilon}^+ &= t_{b,1}^+ + \varepsilon t_{b,2}^+ + O(\varepsilon^2), \\
 t_{c,\varepsilon}^+ &= t_{c,1}^+ + \varepsilon t_{c,2}^+ + O(\varepsilon^2), \\
 t_{d,\varepsilon}^+ &= t_{d,1}^+ + \varepsilon t_{d,2}^+ + O(\varepsilon^2).
\end{align*}
\]

(12)

Moreover, it is easy to obtain $t_{a,\varepsilon}^-, t_{b,\varepsilon}^-, t_{c,\varepsilon}^-$ and $t_{d,\varepsilon}^-$ are:

\[
\Delta_1(u(s),\pm)(t + t_0, t_0) = \varepsilon P^\pm(Z_h(t)) \wedge Z_1(u(s),\pm)(t + t_0, t_0),
\]

(13)

\[
\Delta_1(u(s),N)(t + t_0, t_0) = \varepsilon P^N(Z_h(t)) \wedge Z_1(u(s),N)(t + t_0, t_0).
\]

(14)

In the smooth case, it implies

\[
\Delta_1(u(s),\pm)(t + t_0, t_0) = \varepsilon P^\pm(Z_h(t)) \wedge Q^\pm(Z_h(t), t + t_0),
\]

(15)

\[
\Delta_1(u(s),N)(t + t_0, t_0) = \varepsilon P^N(Z_h(t)) \wedge Q^N(Z_h(t), t + t_0),
\]

(16)

where "\wedge" represents a operator defined by $f \wedge g = f_1 g_2 - f_2 g_1$ for any $f = (f_1, f_2)^T$, $g = (g_1, g_2)^T$.

Now, let us define a new Hamiltonian function to measure the distance between the stable and unstable manifolds at two points $Z_{h,\varepsilon}^{u,1,+}(t_0, t_0, \varepsilon)$ and $Z_{h,\varepsilon}^{u,2,+}(t_0, t_0, \varepsilon)$ for system (4) as follows:

\[
H_2(t_0) = H_2[Z_{h,\varepsilon}^{u,1,+}(t_0, t_0, \varepsilon)] - H_0[Z_{h,\varepsilon}^{u,2,+}(t_0, t_0, \varepsilon)]
\]

\[
= \Delta_1^{u,+}(t_0, t_0) - \Delta_1^{u,-}(t_0, t_0) + O(\varepsilon^2)
\]

\[
= \Delta_1^{u,+}(t_0, t_0) - \Delta_1^{u,+}(t_0 + t_0^+, t_0) + \Delta_1^{u,+}(t_0 + t_0^+, t_0)
\]

\[
- \Delta_1^{u,N}(t_0 + t_0^+, t_0 + t_0^+, t_0 - \Delta_1^{u,N}(t_0 + t_0^+, t_0 + t_0^+, t_0)
\]

\[
+ \Delta_1^{u,N}(t_0 + t_0^+, t_0) - \Delta_1^{u,-}(t_0 + t_0^+, t_0) + \Delta_1^{u,-}(t_0 + t_0^+, t_0)
\]

\[
- \Delta_1^{u,-}(t_0 + t_0^+, t_0)
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + O(\varepsilon^2),
\]

(17)

where

\[
I_1 = \Delta_1^{u,N}(t_0 + t_0^+, t_0) - \Delta_1^{u,N}(t_0 + t_0^+, t_0)
\]

\[
= \varepsilon \int_{t_0^+}^{t_0^+} P^N(Z_{h,\varepsilon}^{1,N}(\tau)) \wedge Q^N(Z_{h,\varepsilon}^{1,N}(\tau), \tau + t_0) d\tau + O(\varepsilon^2).
\]


\[ I_2 = \Delta_i^{u_-(t_0 + t_a^-, t_0)} - \Delta_i^{u_-(\infty, t_0)} = \varepsilon \int_{-\infty}^{-\infty} P^- (Z_h^{1-} (\tau)) \wedge Q^- (Z_h^{1-} (\tau), \tau + t_0) d\tau + \mathcal{O}(\varepsilon^2), \]
\[ I_3 = \Delta_i^{u^-}(+\infty, t_0) - \Delta_i^{u^-}(t_0 + t_a^+, t_0) = \varepsilon \int_{t_a^+}^{+\infty} P^- (Z_h^{2-} (\tau)) \wedge Q^- (Z_h^{2-} (\tau), \tau + t_0) d\tau + \mathcal{O}(\varepsilon^2), \]
\[ I_4 = \Delta_i^{s-N}(t_0 + t_d^-, t_0) - \Delta_i^{s-N}(t_0 + t_d^+, t_0) = \varepsilon \int_{t_d^-}^{t_d^+} P^N (Z_h^{2,N} (\tau)) \wedge Q^N (Z_h^{2,N} (\tau), \tau + t_0) d\tau + \mathcal{O}(\varepsilon^2), \]
\[ I_5 = \Delta_i^{u^+}(t_0, t_0) - \Delta_i^{u^-}(t_0 + t_b^+, t_0) + \Delta_i^{s^+}(t_0 + t_c^-, t_0) - \Delta_i^{s^+}(t_0, t_0) = \varepsilon \int_{t_b^+}^{t_c^-} P^+ (Z_h^1 (\tau)) \wedge Q^+ (Z_h^1 (\tau), \tau + t_0) d\tau + \mathcal{O}(\varepsilon^2), \]
\[ J_1 = \Delta_i^{u,N}(t_0 + t_a^+, t_0) - \Delta_i^{u^-}(t_0 + t_a^-, t_0) + \Delta_i^{s^+}(t_0 + t_d^-, t_0) - \Delta_i^{s^+}(t_0 + t_d^+, t_0) = \varepsilon \int_{t_d^-}^{t_d^+} P^+ (Z_h^1 (\tau)) \wedge Q^+ (Z_h^1 (\tau), \tau + t_0) d\tau + \mathcal{O}(\varepsilon^2), \]
\[ J_2 = \Delta_i^{u^+}(t_0 + t_b^+, t_0) - \Delta_i^{u^+}(t_0 + t_b^-, t_0) + \Delta_i^{s-N}(t_0 + t_c^-, t_0) - \Delta_i^{s^+}(t_0 + t_c^+, t_0) = \varepsilon \int_{t_b^-}^{t_c^+} P^+ (Z_h^1 (\tau)) \wedge Q^+ (Z_h^1 (\tau), \tau + t_0) d\tau + \mathcal{O}(\varepsilon^2). \]

By using the definition of Hamilton functions, according to [30, Theorem 3], we have \( J_2 = 0 \). For calculations of \( I_i \) (\( i = 1, \cdots, 5 \)), one can refer to [30, Theorems 1 and 2] for detail. And \( J_1 \) is of the form

\[ J_1 = -4\varepsilon \eta \int_0^{x_0} f^{-}(x) dx + \varepsilon \left( \frac{1}{\alpha^2} - 1 \right) \left[ \int_{t_b^-}^{t_b^+} P^+ (Z_h^1 (\tau)) \wedge Q^+ (Z_h^1 (\tau), \tau + t_0) d\tau + \int_{t_c^-}^{t_c^+} P^+ (Z_h^1 (\tau)) \wedge Q^+ (Z_h^1 (\tau), \tau + t_0) d\tau + \int_{t_d^-}^{t_d^+} P^+ (Z_h^1 (\tau)) \wedge Q^+ (Z_h^1 (\tau), \tau + t_0) d\tau \right] + \mathcal{O}(\varepsilon^2). \]

Note that the biggest difference between our work and [30] is the calculations of \( J_1 \) since the jump discontinuities at first switching manifold. Hence, we give the calculations of \( J_1 \) in the Appendix (A1) for detail.

Substituting \( I_i \) (\( i = 1, \cdots, 5 \)), \( J_1 \) and \( J_2 \) into (17), we obtain

\[ H_\varepsilon(t_0) = \varepsilon \left[ \int_{-\infty}^{t_a^-} P^- (Z_h^{1-} (\tau)) \wedge Q^- (Z_h^{1-} (\tau), \tau + t_0) d\tau + \frac{1}{\alpha^2} \int_{t_b^-}^{t_b^+} P^+ (Z_h^1 (\tau)) \wedge Q^+ (Z_h^1 (\tau), \tau + t_0) d\tau + \frac{1}{\alpha^2} \int_{t_c^-}^{t_c^+} P^+ (Z_h^1 (\tau)) \wedge Q^+ (Z_h^1 (\tau), \tau + t_0) d\tau + \frac{1}{\alpha^2} \int_{t_d^-}^{t_d^+} P^+ (Z_h^1 (\tau)) \wedge Q^+ (Z_h^1 (\tau), \tau + t_0) d\tau \right] + \mathcal{O}(\varepsilon^2). \]
Naturally, the resulting first-order Melnikov-type function of HPSS-IE is of the form:

\[
M(t_0) = \int_{-\infty}^{t_0^-} P^{-}(Z_h^{-1}(\tau)) \wedge Q^{-}(Z_h^{-1}(\tau), \tau + t_0) d\tau \\
+ \frac{1}{\alpha^2} \int_{t_0^+}^{t_0^-} P^N(Z_h^{1,N}(\tau)) \wedge Q^N(Z_h^{1,N}(\tau), \tau + t_0) d\tau \\
+ \frac{1}{\alpha^2} \int_{t_0^+}^{t_0^+} P^+(Z_h^+(\tau)) \wedge Q^+(Z_h^+(\tau), \tau + t_0) d\tau \\
+ \frac{1}{\alpha^2} \int_{+\infty}^{t_0^+} P^N(Z_h^{2,N}(\tau)) \wedge Q^N(Z_h^{2,N}(\tau), \tau + t_0) d\tau \\
+ \int_{t_0^+}^{t_0^+} P^-(Z_h^{-2}(\tau)) \wedge Q^-(Z_h^{-2}(\tau), \tau + t_0) d\tau \\
- 4\eta \int_0^{x_0} f^-(x) dx.
\]

Note that (21) can be used to detect the homoclinic tangency into a jump discontinuity and piecewise smooth hybrid differential system. It extends the Melnikov-type function to HPSS-IE.

**Remark 1.** If \(\alpha = 1\) and \(\eta = 0\), i.e., instantaneous impulsive effect vanishes, HPSS-IE reduces to a three piecewise-smooth system with a homoclinic orbit transversally and continuously crosses two switching manifolds, then the obtained formula \(M(t_0)\) in (21) reduces to the \(M(t_0)\) in [30, eq. (39)].

**Remark 2.** If \(t_a^+ = t_b^-\) and \(t_c^+ = t_d^-\), i.e., HPSS-IE reduces to a piecewise smooth system by an impulsive effect with one switching manifold, the obtained formula \(M(t_0)\) in (21) reduces to the \(M(t_0)\) in [38, eq. (31)].

Thus, our work is an extension of [30, 38].

4. **An example and numerical simulation.** In this section, an example is used to study the chaos dynamics of HPSS-IE under a periodic perturbation. The system can be written by

\[
\begin{align*}
\frac{dx}{dt} &= y, \quad \frac{dy}{dt} = x - x^3 + \varepsilon(-\mu y + f_0 \cos \omega t), \quad x < 1, \\
\frac{dx}{dt} &= y, \quad \frac{dy}{dt} = x + \varepsilon(-\mu y + f_0 \cos \omega t), \quad 1 < x < \frac{3}{2}, \\
\frac{dx}{dt} &= y, \quad \frac{dy}{dt} = -x + \frac{3}{2} + \varepsilon(-\mu y + f_0 \cos \omega t), \quad x > \frac{3}{2}, \\
\Delta x \mid_{x=1/2} &= 0, \\
\Delta y \mid_{x=1} &\neq 0, \Delta y \mid_{x=3/2} = 0.
\end{align*}
\]

For \(\varepsilon = 0\), the unperturbed system takes the form:

\[
\begin{align*}
\frac{dx}{dt} &= y, \quad \frac{dy}{dt} = x - x^3, \quad x < 1, \\
\frac{dx}{dt} &= y, \quad \frac{dy}{dt} = x, \quad 1 < x < \frac{3}{2}, \\
\frac{dx}{dt} &= y, \quad \frac{dy}{dt} = -x + \frac{3}{2}, \quad x > \frac{3}{2}, \\
\Delta x \mid_{x=1/2} &= 0, \\
\Delta y \mid_{x=1} &\neq 0, \Delta y \mid_{x=3/2} = 0.
\end{align*}
\]
The corresponding Hamilton function of the unperturbed system (23) is

\[
H(Z) = \begin{cases} 
H^{-}(x, y) = \frac{1}{2} y^2 - \frac{1}{3} x^2 + \frac{1}{3} x^4, & x < 1, \\
H^{+}(x, y) = \frac{1}{2} y^2 - \frac{1}{3} x^2, & 1 < x < \frac{3}{2}, \\
H^{+}(x, y) = \frac{1}{2} y^2 + \frac{1}{3} x^2 - \frac{3}{2} x, & x > \frac{3}{2}.
\end{cases}
\]

(24)

Clearly, \( p_0 = (0, 0) \) is a saddle point of the unperturbed system (23), and a piecewise homoclinic solution of \( p_0 \) is formulated as

\[
\begin{pmatrix} 
x_h(t) \\
y_h(t)
\end{pmatrix} = \begin{cases} 
(\sqrt{2}\sech(t - t_1) - \sqrt{2}\sech(t - t_1)\tanh(t - t_1))T, & t \leq t_a^-, \\
(\exp(t - t_2) - \exp(t - t_2))^T, & t_a^- \leq x \leq t_b^-, \\
\left(\frac{3}{2} + \frac{3}{2}\cos(t - t_3) - \frac{3}{2}\sin(t - t_3)\right)^T, & t_b^- \leq x \leq t_c^-, \\
(\exp(t_4 - t) - \exp(t_4 - t))^T, & t_c^- \leq x \leq t_d^-, \\
\left(\sqrt{2}\sech(t - t_5) - \sqrt{2}\sech(t - t_5)\tanh(t - t_5)\right)^T, & t \geq t_d^+,
\end{cases}
\]

(25)

where \( t_1 = 1.70042207, t_2 = 0.8186686196, t_3 = 2.794930055, t_4 = 4.771191490, t_5 = 3.889817903, \) initial value \( x_h(0) = 0.5 \) and \( y_h(0) = 0.4677071732. \) It passes the first switching manifold \( x_0 = 1 \) at times \( t_a^- = 0.8186686196 \) and \( t_d^+ = 4.771191490. \) And the second switching manifold \( x_1 = \frac{3}{2} \) at times \( t_b^- = 1.224133728 \) and \( t_c^+ = 4.365726382, \) respectively.

Therefore, according to (21), the Melnikov-type function of system (22) is given by

\[
M(t_0) = \int_{-\infty}^{t_a^-} y_h(\tau)[-\mu y_h(\tau) + f_0 \cos(\tau + t_0)]d\tau \\
+ \frac{1}{\alpha^2} \int_{t_b^-}^{t_c^-} y_h(\tau)[-\mu y_h(\tau) + f_0 \cos(\tau + t_0)]d\tau \\
+ \frac{1}{\alpha^2} \int_{t_c^-}^{t_d^-} y_h(\tau)[-\mu y_h(\tau) + f_0 \cos(\tau + t_0)]d\tau \\
+ \frac{1}{\alpha^2} \int_{t_d^-}^{+\infty} y_h(\tau)[-\mu y_h(\tau) + f_0 \cos(\tau + t_0)]d\tau \\
- 4\eta \int_0^1 (x - x^3)dx
\]

= \( f_0\sqrt{A^2 + B^2}\cos(\omega t_0 + \varphi) - \mu C - \eta, \)

(26)

where \( \tan \varphi = \frac{\xi}{\eta}, \) taking \( \alpha = \sqrt{2}, \) then we obtain the expressions of \( A, B \) and \( C, \) one can see the Appendix (A2).

After above analysis, we have the following theorem:

**Theorem 4.1.** If and only if the following inequality holds:

\[
f_0 \geq \left| \frac{\mu C + \eta}{\sqrt{A^2 + B^2}} \right|,
\]

such that \( M(t_0) = 0 \) has a simple zero \( t_0. \)

Letting \( \varepsilon = 0.01, \eta = 0.1 \) and \( \mu = 2, 3, 4, \) respectively. The threshold curves of chaos \( f_0 - \omega \) for system (22) are shown in Fig. 4(a). Moreover, we choose \( \mu = 2, \omega = 1.2, \varepsilon = 0.01, \) initial value \( x(0) = 0.5 \) and \( y(0) = 0.4677071732, \) then the parameter \( f_0 \) bifurcation diagram for system (22) is plotted in Fig. 4(b).
Next, numerical simulations are employed to verify the accuracy of the qualitative analysis for system (22). Firstly, taking initial value $x(0) = 0.5$ and $y(0) = 0.46701732$, $\mu = 2$, $\omega = 1.2$, $\eta = 0.1$, $\varepsilon = 0.01$, $f_0 = 20$, and $f_0 = 30$, respectively. It is shown that the dynamic behaviour of system (22) trends to periodic motions as time increases (see Fig. 5 and Fig. 6).
Then, set $f_0 = 40$ and $f_0 = 50$, others parameters keep the same with Fig. 4. Obviously, these parameters can satisfy the conditions of Theorem 1 such that the chaotic behaviour of system (22) may occur (see Fig. 7 and Fig. 8).

![Phase portraits and Poincaré section](image1.png)

Fig. 7 The phase portraits, Poincaré section and time history curves of system (22), taking $f_0 = 40$, $\omega = 1.2$, $\mu = 2$ and $\varepsilon = 0.01$. 

![Time history curves](image2.png)
Appendix.

5. Conclusion. In this paper we extend the Melnikov-type function to a class of HPSS-IE under a periodic perturbation. We first consider the unperturbed system which admits a homoclinic orbit to the hyperbolic saddle point. The orbit transversally jumps across the first switching manifold and continuously crosses the second switching manifold, respectively. Then the system is perturbed by a damping term and non-autonomous periodic function. Based on [30, 32, 38], the corresponding switching manifold, respectively. HPSS-IE under a periodic perturbation. We first consider the unperturbed system (21) generating some additional terms caused by sudden change with impulsive effect and discontinuity switching manifold. Our results extend the existing literature. At last, through an example, the correctness of our conclusion is verified by numerical simulations.

6. Appendix. (A1) The calculations of $J_1$ is as follows, we divides two steps to proceed.

(i) Firstly, we define that

$$\Delta_{\varepsilon} = P^+(Z_h(t)) \wedge Z_{h,\varepsilon}^+(t + t_0, t_0, \varepsilon),$$

$$\Delta_{\varepsilon} = P^0(Z_h(t)) \wedge Z_{h,\varepsilon}^0(t + t_0, t_0, \varepsilon).$$

By (6), we know that $[y_h^1(t_a^-)]^2 = [y_h^2(t_a^-)]^2 = (y_a)^2, [y_h^1(t_a^+)]^2 = [y_h^2(t_a^+)]^2 = (y_a)^2$. And

$$f^-(x_h^1(t_a^-)) = f^+(x_h^1(t_a^-)) = f^-(x_0) = f^+(x_0) = 0,$$

$$f^N(x_h^1(t_a^-)) = f^N(x_h^1(t_a^+)) = f^N(x_0) = 0.$$ 

Then

$$\Delta_{\varepsilon} = P^+(Z_h(t)) \wedge Z_{h,\varepsilon}^+(t + t_0, t_0, \varepsilon) - P^-(Z_h(t)) \wedge Z_{h,\varepsilon}^-(t + t_0, t_0, \varepsilon) = P^N(Z_h(t)) \wedge Z_{h,\varepsilon}^N(t + t_0, t_0, \varepsilon) + O(\varepsilon^2)$$

Fig. 8 The phase portraits, Poincaré section and time history curves of system (22), taking $f_0 = 50$, $\omega = 1.2$, $\mu = 2$ and $\varepsilon = 0.01$. 

(b) Time history curves of $(t, x)$
(c) Time history curves of $(t, y)$
By [52] and [38, eq. (16)], it gives

\[
P^N(Z_{h,x}^N(t_a^+)) \wedge [Z_{h,x}^{1,N}(t_a^+) + \varepsilon Z_{h,x}^{1,1,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)}\]

\[-P^N(Z_{h,x}^N(t_a^+)) \wedge [Z_{h,x}^{1,1,N}(t_a^+) + \varepsilon Z_{h,x}^{1,1,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)}\]

\[+P^N(Z_{h,x}^N(t_a^+)) \wedge [Z_{h,x}^{1,2,N}(t_a^+) + \varepsilon Z_{h,x}^{1,1,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)}\]

\[-P^N(Z_{h,x}^N(t_a^+)) \wedge [Z_{h,x}^{1,2,N}(t_a^+) + \varepsilon Z_{h,x}^{1,2,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)}\]

\[= P^N(Z_{h,x}^{1,N}(t_a^+)) \wedge [Z_{h,x}^{1,1,N}(t_a^+) + \varepsilon P^N(Z_{h,x}^{1,N}(t_a^+)) \wedge Z_{h,x}^{2,1,N}(t_0 + t_a^+, t_0)\]

\[-P^N(Z_{h,x}^{1,N}(t_a^+)) \wedge [Z_{h,x}^{1,2,N}(t_a^+) + \varepsilon P^N(Z_{h,x}^{1,N}(t_a^+)) \wedge Z_{h,x}^{2,2,N}(t_0 + t_a^+, t_0)\]

\[+P^N(Z_{h,x}^{1,N}(t_a^+)) \wedge [Z_{h,x}^{1,2,N}(t_a^+) + \varepsilon P^N(Z_{h,x}^{1,N}(t_a^+)) \wedge Z_{h,x}^{2,1,N}(t_0 + t_a^+, t_0)\]

\[= [y_{h,x}^{1,N}(t_a^+)]^2 - f_N(x_{h,x}^{1,N}(t_a^+))x_{h,x}^{1,N}(t_a^+) - [y_{h,x}^{1,N}(t_a^+)]^2 + f_N(x_{h,x}^{1,N}(t_a^+))x_{h,x}^{1,N}(t_a^+)\]

\[+\varepsilon P^N(Z_{h,x}^{1,N}(t_a^+)) \wedge Z_{h,x}^{1,1,N}(t_0 + t_a^+, t_0) + P^N(Z_{h,x}^{1,N}(t_a^+)) \wedge Z_{h,x}^{1,2,N}(t_0 + t_a^+, t_0)\]

\[+P^N(Z_{h,x}^{1,N}(t_a^+)) \wedge Z_{h,x}^{1,2,N}(t_0 + t_a^+, t_0) + \varepsilon P^N(Z_{h,x}^{1,N}(t_a^+)) \wedge Z_{h,x}^{1,2,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)\]

\[= \Delta_{h,x}^{u,N}(t_0 + t_a^+, t_0) - \Delta_{h,x}^{u,N}(t_0 + t_a^+, t_0) + \Delta_{h,x}^{u,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)\]

\[(30)\]

(ii) From [38, eq. (27)], we write that

\[J_1 = \Delta_{h,x}^{u,N}(t_0 + t_a^+, t_0) - \Delta_{h,x}^{u,N}(t_0 + t_a^+, t_0)\]

\[+ \Delta_{h,x}^{u,N}(t_0 + t_a^+, t_0) - \Delta_{h,x}^{u,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)\]

\[= \Delta_{h,x}^{u,N}(t_0 + t_a^+, t_0) - O(\varepsilon^2)\]

\[(31)\]

By [52] and [38, eq. (16)], it gives

\[Z_{h,x}^{u,1,N}(t_0 + t_a^+, t_0, \varepsilon) = Z_{h,x}^{u,1,N}(t_0 + t_a^+, t_0, \varepsilon) + \varepsilon \phi_{a_1}^+ P^N(Z_{h,x}^{u,1,N}(t_0 + t_a^+, t_0, \varepsilon)) + O(\varepsilon^2)\]

\[Z_{h,x}^{u,2,N}(t_0 + t_a^+, t_0, \varepsilon) = Z_{h,x}^{u,2,N}(t_0 + t_a^+, t_0, \varepsilon) + \varepsilon \phi_{a_1} P^N(Z_{h,x}^{u,2,N}(t_0 + t_a^+, t_0, \varepsilon)) + O(\varepsilon^2)\]

\[(32)\]

Using (10), (29) and (32), we have

\[y_{h,x}^{u,1,N}(t_0 + t_a^+, t_0, \varepsilon) = y_{h,x}^{u,1,N}(t_a^+) + \varepsilon \phi_{a_1} f_N(x_{h,x}^{u,1,N}(t_a^+)) + \varepsilon y_{h,x}^{u,1,N}(t_a^+, t_0) + O(\varepsilon^2)\]

\[= y_{h,x}^{u,1,N}(t_a^+) + \varepsilon \phi_{a_1} f_N(x_0) + \varepsilon y_{h,x}^{u,1,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)\]

\[= y_{h,x}^{u,1,N}(t_a^+) + \varepsilon y_{h,x}^{u,1,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)\]

\[y_{h,x}^{u,2,N}(t_0 + t_a^+, t_0, \varepsilon) = y_{h,x}^{u,2,N}(t_a^+) + \varepsilon \phi_{a_1} f_N(x_{h,x}^{u,2,N}(t_a^+)) + \varepsilon y_{h,x}^{u,2,N}(t_a^+, t_0) + O(\varepsilon^2)\]

\[= y_{h,x}^{u,2,N}(t_a^+) + \varepsilon \phi_{a_1} f_N(x_0) + \varepsilon y_{h,x}^{u,2,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)\]

\[= y_{h,x}^{u,2,N}(t_a^+) + \varepsilon y_{h,x}^{u,2,N}(t_0 + t_a^+, t_0) + O(\varepsilon^2)\]

\[(33)\]

According to the third equation of (3), we obtain

\[y_{h,x}^{u,1,N}(t_a^+) = (1 - \varepsilon \eta) \alpha y_{h,x}^{u,2,N}(t_a^+), y_{h,x}^{u,2,N}(t_a^+) = (1 - \varepsilon \eta) \phi_{a_1} y_{h,x}^{u,2,N}(t_a^+), y_{h,x}^{u,1,N}(t_0 + t_a^+, t_0, \varepsilon) = (1 - \varepsilon \eta) \alpha y_{h,x}^{u,2,N}(t_0 + t_a^+, t_0, \varepsilon), y_{h,x}^{u,2,N}(t_0 + t_a^+, t_0, \varepsilon) = (1 - \varepsilon \eta) \phi_{a_1} y_{h,x}^{u,2,N}(t_0 + t_a^+, t_0, \varepsilon)\]

\[(34)\]
Substituting (33) and (34) into (31), it is not difficult to obtain

\[
J_1 = (1 - \frac{1}{\alpha^2}) y_{h}^{1, N}(t_+^+) y_{h}^{1, N}(t_+^+) + \varepsilon y_{h}^{1, N}(t_+^+) y_{h}^{1, N}(t_+^+) + \varepsilon y_{h}^{1, N}(t_+^+) y_{h}^{1, N}(t_+^+) + O(\varepsilon^2)
\]

\[
= -\varepsilon \int_{t_+^+}^{t_+^+} y_{h}^{1, N}(t_+^+) y_{h}^{1, N}(t_+^+) + \varepsilon \int_{t_+^+}^{t_+^+} y_{h}^{1, N}(t_+^+) y_{h}^{1, N}(t_+^+) + \varepsilon y_{h}^{1, N}(t_+^+) y_{h}^{1, N}(t_+^+) + O(\varepsilon^2)
\]

\[
= -\varepsilon \int_{t_+^+}^{t_+^+} y_{h}^{1, N}(t_+^+) y_{h}^{1, N}(t_+^+) + \varepsilon \int_{t_+^+}^{t_+^+} y_{h}^{1, N}(t_+^+) y_{h}^{1, N}(t_+^+) + \varepsilon y_{h}^{1, N}(t_+^+) y_{h}^{1, N}(t_+^+) + O(\varepsilon^2)
\]

\[
\int_{t_+^+}^{t_+^+} P^N(Z_h^{2, N}(\tau)) \wedge Q^N(Z_h^{2, N}(\tau), \tau + t_0) d\tau
\]

\[
= -4\varepsilon \int_{0}^{\infty} f^-(x) dx + \varepsilon \left(\frac{1}{\alpha^2} - 1\right) J_2
\]

\[
= \int_{t_+^+}^{t_+^+} P^N(Z_h^{2, N}(\tau)) \wedge Q^N(Z_h^{2, N}(\tau), \tau + t_0) d\tau
\]

\[
= \varepsilon \left(\frac{1}{\alpha^2} - 1\right) \int_{t_+^+}^{t_+^+} P^N(Z_h^{2, N}(\tau)) \wedge Q^N(Z_h^{2, N}(\tau), \tau + t_0) d\tau
\]

\[
-4\varepsilon \int_{0}^{\infty} f^-(x) dx + O(\varepsilon^2)
\]

(A2) The expressions of A, B and C are as follows:

\[
A = \int_{-\infty}^{t_+^+} y_h(\tau) \cos \omega \tau d\tau + \frac{1}{\alpha^2} \int_{t_+^+}^{t_+^+} y_h(\tau) \cos \omega \tau d\tau + \frac{1}{\alpha^2} \int_{t_+^+}^{t_+^+} y_h(\tau) \cos \omega \tau d\tau
\]

\[
+ \frac{1}{\alpha^2} \int_{t_+^+}^{t_+^+} y_h(\tau) \cos \omega \tau d\tau + \int_{t_+^+}^{+\infty} y_h(\tau) \cos \omega \tau d\tau
\]
\[
\begin{align*}
&= \int_0^1 \left[ \cos(\text{arcsech} \frac{x}{\sqrt{2}} + t_1) - \cos(\text{arcsech} \frac{x}{\sqrt{2}} + t_4) \right] dx \\
&+ \frac{1}{2} \int_{t_5}^{t_6} \exp(\tau - t_2) \cos \omega \tau d\tau - \frac{1}{2} \int_{t_4}^{t_5} \exp(\tau - t_4) \cos \omega \tau d\tau \\
&+ \frac{1}{2} \int_{t_3}^{t_4} \cos \omega \tau d\tau \\
&+ \int_{t_2}^{t_3} \cos \omega \tau d\tau
\end{align*}
\]

\[
B = \int_{t_6}^{t_7} y_h(\tau) \sin \omega \tau d\tau + \frac{1}{\alpha^2} \int_{t_5}^{t_6} y_h(\tau) \sin \omega \tau d\tau + \frac{1}{\alpha^2} \int_{t_4}^{t_5} y_h(\tau) \sin \omega \tau d\tau
\]

\[
= \int_0^1 \left[ \sin(\text{arcsech} \frac{x}{\sqrt{2}} + t_1) - \sin(\text{arcsech} \frac{x}{\sqrt{2}} + t_4) \right] dx \\
+ \frac{1}{2} \int_{t_5}^{t_6} \exp(\tau - t_2) \sin \omega \tau d\tau - \frac{1}{2} \int_{t_4}^{t_5} \exp(\tau - t_4) \sin \omega \tau d\tau \\
+ \int_{t_3}^{t_4} \sin(\text{arcsech} \frac{x}{\sqrt{2}}(x - \frac{3}{2})) dx
\]

\[
C = \int_{-\infty}^{t_6} [y_h(\tau)]^2 d\tau + \frac{1}{\alpha^2} \int_{t_5}^{t_6} [y_h(\tau)]^2 d\tau + \frac{1}{\alpha^2} \int_{t_4}^{t_5} [y_h(\tau)]^2 d\tau
\]

\[
= \int_{-\infty}^{t_6} [2 \text{sech}(\tau - t_1) \tanh(\tau - t_1)]^2 d\tau + \frac{1}{2} \int_{t_5}^{t_6} [\exp(\tau - t_2)]^2 d\tau \\
+ \frac{1}{2} \int_{t_4}^{t_5} [\exp(\tau - t_4)]^2 d\tau + \int_{t_3}^{t_4} [2 \text{sech}(\tau - t_3) \tanh(\tau - t_3)]^2 d\tau
\]

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