NODAL GEOMETRY AND TOPOLOGY OF LOW ENERGY EIGENFUNCTIONS

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Abstract. We investigate various aspects of the nodal geometry and topology of Laplace eigenfunctions on compact Riemannian manifolds and bounded Euclidean domains, with particular emphasis on the low frequency regime. This includes geometry and topology of nodal sets, particularly in and around the area of the Payne property, opening angle estimates, (fundamental) spectral gaps etc., and behaviour of all of the above under small scale perturbations. We aim to highlight interesting aspects of spectral theory and nodal phenomena tied to ground state/low energy eigenfunctions, as opposed to asymptotic results.

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1. Introduction and preliminaries

Let $(M, g)$ be a compact Riemannian manifold. Consider the eigenvalue equation

$$-\Delta \varphi = \lambda \varphi,$$

(1.1)
where $\Delta$ is the Laplace-Beltrami operator given by (using the Einstein summation convention)
\[
\Delta f = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j f \right),
\]
where $|g|$ is the determinant of the metric tensor $g_{ij}$. In the Euclidean space, this reduces to the usual $\Delta = \partial_1^2 + \cdots + \partial_n^2$. Observe that we are using the analyst’s sign convention for the Laplacian, namely that $-\Delta$ is positive semidefinite.

If $M$ has a boundary, we will consider either the Dirichlet boundary condition
\[
\varphi(x) = 0, \; x \in \partial M, \tag{1.2}
\]
or the Neumann boundary condition
\[
\partial_\eta \varphi(x) = 0, \; x \in \partial M, \tag{1.3}
\]
where $\eta$ denotes the outward pointing unit normal on $\partial M$.

Recall that if $M$ has a reasonably regular boundary, $-\Delta_g$ has a discrete spectrum
\[
0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \to \infty,
\]
repeated with multiplicity with corresponding (real-valued $L^2$ normalized) eigenfunctions $\varphi_k$. Also, let $n_{\varphi_\lambda} = \{x \in M : \varphi_\lambda(x) = 0\}$ denote the nodal set of the eigenfunction $\varphi_\lambda$. Sometimes for ease of notation, we will also denote the nodal set by $n(\varphi_\lambda)$. Recall that any connected component of $M \setminus n_{\varphi_\lambda}$ is known as a nodal domain of the eigenfunction $\varphi_\lambda$ denoted by $\Omega_\lambda$. These are domains where the eigenfunction is not sign-changing (this follows from the maximum principle).

In this paper, we study the nodal topology and geometry associated to Laplace eigenfunctions, for both curved and flat domains, and with both Dirichlet and/or Neumann boundary conditions. Most of our results are related to the low frequency regime (low energy eigenfunctions), though some will also apply to the high frequency asymptotics, but no result is purely asymptotic in nature. We mainly use tools from elliptic PDEs in conjunction with perturbation theoretic methods of classical analysis.

1.1. Notational convention. When two quantities $X$ and $Y$ satisfy $X \leq c_1 Y \; (X \geq c_2 Y)$ for constants $c_1, c_2$ dependent on the geometry $(M, g)$, we write $X \lesssim_{(M, g)} Y$ (respectively $X \gtrsim_{(M, g)} Y$). Unless otherwise mentioned, these constants will in particular be independent of eigenvalues $\lambda$. Throughout the text, the quantity $\frac{1}{\sqrt{\lambda}}$ is referred to as the wavelength and any quantity (e.g. distance) is said to be of sub-wavelength (super-wavelength) order if it is $\lesssim_{(M, g)} \frac{1}{\sqrt{\lambda}}$ (respectively $\gtrsim_{(M, g)} \frac{1}{\sqrt{\lambda}}$).

First we recall a few basic facts and results that we would need in the sequel.

1.2. Characterization of eigenvalues. We note that the Sobolev space $H^1(M)$ can be defined as the completion of $C^\infty(M)$ with respect to the inner product
\[
\langle f, g \rangle_{H^1} := \langle f, g \rangle_{L^2(M)} + \langle \nabla f, \nabla g \rangle_{L^2(M)} \tag{1.4}
\]
where $f, g \in C^\infty(M)$. Next, we introduce a bilinear form in $H^1(M)$. Consider the bilinear form on $C^\infty(M) \times C^\infty(M)$,
\[
D(f, g) := \langle \nabla f, \nabla g \rangle_{L^2(M)}.
\]
Since $H^1(M)$ is the completion of $C^\infty(M)$ in the induced norm, given $f, g \in H^1(M)$, there exists sequence $\{f_i\}, \{g_i\} \in C^\infty(M)$ converging to $f, g$ in $H^1$ norm. Then we can define the bilinear form on $H^1(M) \times H^1(M)$ as
\[
D(f, g) = \lim_{i \to \infty} \langle \nabla f_i, \nabla g_i \rangle_{L^2(M)}. \tag{1.5}
\]
Now we have the following characterization of Laplacian eigenvalues.
Theorem 1.1. For any \( k \in \mathbb{N} \), let \( \{ \varphi_1, \varphi_2, \ldots, \varphi_{k-1} \} \) be the first \( k-1 \) orthonormal eigenfunctions. Then for any \( f \in H^1(M) \), \( f \not= 0 \) such that
\[
\langle f, \varphi_1 \rangle = \cdots = \langle f, \varphi_{k-1} \rangle = 0
\]
we have
\[
\lambda_k \leq \frac{D(f, f)}{\|f\|^2}.
\] (1.6)
Moreover, the equality holds if and only if \( f \) is an eigenfunction of \( \lambda_k \).

Note that for characterising the Dirichlet-Laplacian eigenvalues, the admissible function class in the above theorem is \( H^1_0(M) \), which is the completion of \( C_0^\infty(M) \) with respect to the induced norm from the above defined inner product. But, for Neumann boundary condition and manifolds without boundary, we use the above characterisation as it is.

Moreover, we also note that, in case of a boundaryless manifold or a manifold with Neumann boundary condition we have that \( \lambda_1 = 0 \) and for a manifold with Dirichlet boundary, we have that \( \lambda_1 > 0 \). This follows from the above characterisation.

1.3. Eigenfunctions and Fourier synthesis. In a certain sense, the study of eigenfunctions of the Laplace-Beltrami operator is the analogue of Fourier analysis in the setting of compact Riemannian manifolds. Recall that the Laplace eigenequation is the standing state for a variety of partial differential equations modelling physical phenomena like heat diffusion, wave propagation or Schrödinger problems. Below, we note down this well-known method of “Fourier synthesis”:

\[
\begin{align*}
\text{Heat equation} & \quad \partial_t u - \Delta u = 0 \quad u(t, x) = e^{-\lambda t} \varphi(x) \quad (1.7) \\
\text{Wave equation} & \quad \partial_t^2 u - \Delta u = 0 \quad u(t, x) = e^{i\sqrt{\lambda} t} \varphi(x) \quad (1.8) \\
\text{Schrödinger equation} & \quad i\partial_t u - \Delta u = 0 \quad u(t, x) = e^{i\lambda t} \varphi(x) \quad (1.9)
\end{align*}
\]

Further, note that \( u(t, x) = e^{\sqrt{\lambda} t} \varphi(x) \) solves the harmonic equation \( \partial_t^2 u - \Delta u = 0 \) on \( \mathbb{R} \times M \). In the interest of completeness, we include one last useful heuristic: if one considers the eigenequation (1.1) on metric balls of radius \( \sqrt{\lambda} \) and rescale to a ball of radius 1, it produces an “almost harmonic” function (see Section 2 of [Ma] for more details).

A motivational perspective of the study of Laplace eigenfunctions then comes from quantum mechanics (via the correspondence with Schrödinger operators), where the \( L^2 \)-normalized eigenfunctions induce a probability density \( \varphi^2(x) dz \), i.e., the probability density of a particle of energy \( \lambda \) to be at \( x \in M \). Another physical (real-life) motivation of studying the eigenfunctions, dated back to the late 18th century, is based on the acoustics experiments done by E. Chladni which were in turn inspired from the observations of R. Hooke in the late 17th century. But what is surprising (at least to the present authors!) is that the earliest observation of these vibration patterns were made by G. Galileo in early 17th century. We quote below, from Dialogues concerning two new sciences, his observation about the vibration patterns on a brass plate.

“As I was scraping a brass plate with a sharp iron chisel in order to remove some spots from it and was running the chisel rather rapidly over it, I once or twice, during many strokes, heard the plate emit a rather strong and clear whistling sound; on looking at the plate more carefully, I noticed a long row of fine streaks parallel and equidistant from one another. Scraping with the chisel over and over again, I noticed that it was only when the plate emitted this hissing noise that any marks were left upon it; when the scraping was not accompanied by this sibilant noise there was not the least trace of such marks. I noted also that the marks made when the tones were higher were closer together; but when the tones were deeper, they were farther apart.”

Of similar essence, the experiments of Chladni consist of drawing a bow over a piece of metal plate whose surface is lightly covered with sand. When resonating, the plate is divided into regions that vibrate in opposite directions causing the sand to accumulate on parts with no vibration.
The study of the patterns formed by these sand particles (Chladni figures) were of great interest which led to the study of nodal sets and nodal domains. Below we discuss few properties of the Laplace eigenfunctions.

1.4. Elementary facts about eigenvalues and eigenfunctions. We now collect some elementary facts about the eigenvalues of the Laplace-Beltrami operator. All of these might not be directly required in the rest of the paper, but are implicit in the general thinking process.

One well known global property of the eigenfunctions is the following theorem which gives an upper bound on the number of nodal domains corresponding to the \( k \)th eigenfunction \( \varphi_k \).

**Theorem 1.2** (Courant’s nodal domain theorem). The number of nodal domains of \( \varphi_k \) can be at most \( k \). In other words, the total number of connected components of \( M \setminus \mathcal{N}_{\varphi_k} \) is strictly less than \( k + 1 \).

**Remark 1.3.** \( \varphi_1 \) is always non sign changing. In the case of Laplace eigenfunctions, this can be easily observed by replacing \( \varphi_1 \) by \( |\varphi_1| \), which is non-negative and using the variational characterisation (1.6) above.

**Remark 1.4.** The multiplicity of \( \lambda_1 \) is always 1 i.e. \( \lambda_1 \) is simple (for a manifold without boundary, \( \lambda_1 = 0 \) corresponding to the constant eigenfunctions). If not, then \( \varphi_2 \) has a constant sign, from the previous remark. This contradicts the fact that \( \langle \varphi_1, \varphi_2 \rangle = 0 \), since \( \varphi_1 \) has a constant sign as well. As a result, \( \lambda_1 \) is characterized as being the only eigenvalue with eigenfunction of constant sign. For significantly more general operators (like Schrödinger operators), the result is still true, but this requires the use of the Krein-Rutman theorem.

**Remark 1.5.** The above two remarks imply that \( \varphi_2 \) has exactly two nodal domains. Moreover, any \( \varphi_k \) has at least two nodal domains for \( k \geq 2 \).

We include a small lemma here, which is somewhat obvious, but we did not find it explicitly mentioned in the literature.

**Lemma 1.6.** Let \( \Omega \subseteq M \) be a domain. Let \( \varphi_1 \) be the ground state Dirichlet eigenfunction of \( \Omega \) and \( \varphi_2 \) be a second Dirichlet eigenfunction of \( \Omega \). Then there exists a point \( x \in \Omega \) such that \( \varphi_1(x) = \varphi_2(x) \).

**Proof.** It is clear that \( v(t, x) := e^{-\lambda_1 t} \varphi_1 - e^{-\lambda_2 t} \varphi_2 \) satisfies the heat equation \( (\partial_t - \Delta)v = 0 \) with Dirichlet boundary conditions, and that \( v(t, x) \to 0 \) as \( t \to \infty \). Using the parabolic maximum principle, it is clear that at \( t = 0 \), \( \sup_{\Omega} v(t, x) \leq 0 \), which means that there is some point \( x \in \Omega \) where \( \varphi_2(x) \geq \varphi_1(x) \). Since the reverse inequality holds on the nodal set of \( \varphi_2 \), the above claim follows from the intermediate value property. \( \square \)

**Theorem 1.7** (Domain Monotonicity). Suppose \( \Omega_1 \subseteq \Omega_2 \subseteq M \). Then their fundamental Dirichlet eigenvalues satisfy

\[
\lambda_1(\Omega_2) \leq \lambda_1(\Omega_1),
\]

and the above inequality is strict if the set \( \Omega_2 \setminus \Omega_1 \) has positive capacity.

**Theorem 1.8** (Wavelength density). For any \( (M, g) \), there exists a constant \( C > 0 \) (depending on \( g \)) such that every ball of radius bigger that \( C/\sqrt{\lambda} \) intersects with the nodal set corresponding to \( \varphi_{\lambda} \).

Note that the above theorem tells us that the nodal domains cannot be too “fat” which leads to a natural question: How “fat” or “thin” can a nodal domain be?\(^1\) One way of interpreting the “thickness” of a domain is to measure the size of the largest ball that can be inscribed inside that domain. The inner radius estimates of nodal domains have been studied for years with fascinating results. We state the following result which is the culmination of several works:

\(^1\)Admittedly, the question becomes more interesting when we investigate this question for higher (asymptotic) eigenfunctions.
Furthermore, let $x_0$ be a maximum point of $\varphi_\lambda$ on $\Omega_\lambda$. Then the ball of inner radius can be centered at $x_0$.

We end this segment with the following.

1.5. Overview of the paper. Here we take the space to list some salient results of this paper.

First, we discuss the stability of topological properties of the first Dirichlet nodal set (or, nodal set for any of the second Dirichlet eigenfunctions) under small perturbations. Some of these results are natural extensions to our previous work in [MS]. Among other results, we prove that satisfying the strong Payne property (or not satisfying) are both open conditions in a one-parameter family of perturbations of a given bounded domain $\Omega \subseteq \mathbb{R}^n$: this is Proposition 2.5 below. Observe that the results hold in all dimensions, and not restricted to the plane only.

After a general laying of foundations, we discuss behaviour of the second Dirichlet eigenfunction in the connector or “handle” region of thin dumbbell domains (for a proper definition, see Subsection 2.3 below), namely the behaviour of the nodal set and mass concentration properties of the eigenfunction. In particular, we check the validity of the Payne property (see Definition 2.1 below). We quote the result:

Theorem 1.10. Consider two bounded domains $\Omega_1$, $\Omega_2 \subset \mathbb{R}^n$ ($n \geq 2$) with $C^2$ boundary and a one parameter family of smooth dumbbells $\Omega_\epsilon$ whose connector widths go to zero as $\epsilon \to 0$. Let $\lambda_2, \lambda_2^{\Omega_i}$ denote the second eigenvalues of $\Omega_i$, $\Omega_\epsilon$ corresponding to eigenfunctions $\varphi_2^{\Omega_i}$, $\varphi_{2,\epsilon}$ respectively, $i = 1, 2$. Assume that the connector does not intersect $\mathcal{N}(\varphi_2^{\Omega_i}) \cap \partial \Omega_i$ and $\Omega_i$ do not have the same first or second Dirichlet eigenvalues.

Then, for sufficiently small $\epsilon$, $\mathcal{N}(\varphi_{2,\epsilon})$ does not enter one of the subdomains $\Omega_i$. Moreover, if $\lambda_2 \to \lambda_2^{\Omega_i}$ for some $i = 1, 2$ and $\Omega_i$ satisfies the strong Payne property, then for sufficiently small $\epsilon > 0$, $\Omega_\epsilon$ satisfies the strong Payne property as well.

Next, we start a discussion about negative results surrounding the Payne property. We first describe the basic starting space for the two-dimensional counterexample in [HHN] and the higher dimensional counterexample in [F]. In Subsection 2.4 we give a new counterexample to the Payne property in higher dimensions. Our example is a perturbation of the base domain provided by Fournais, and initially has the merit of being simply connected as well. We then indicate how to jazz it up to get a domain which violates the Payne property and has prescribed topological complexity. This is again, a perturbation argument based on previous constructions in [MS]. We believe that conceptually our example might be slightly simpler than the counterexample in [Ke].

Next, we investigate the interconnection of multiplicity of eigenvalues and the topology of the second nodal set. We begin by checking that results in [Do,Gi] implying topological complexity of the domain from “detachment” or “non-intersection” of nodal sets of eigenfunctions still hold true for Euclidean domains with appropriate boundary conditions. Further, we show that for a broad class of non-convex domains, the multiplicity of the second Dirichlet eigenvalue is $\leq 2$. Here is the statement of the latter:

Theorem 1.11. Consider a bounded simply-connected domain $\Omega \subseteq \mathbb{R}^2$ satisfying the following:

- the boundary $\partial \Omega$ contains exactly two distinct points $P, Q$ dividing $\partial \Omega$ into two components $\Gamma_j, j = 1, 2$ such that the outward unit normal at $P$ (respectively, $Q$) is in the direction of the vector joining $(0,0)$ to $P$ (respectively, vector joining $Q$ to $(0,0)$).
• at every point \((x, y) \in \Gamma_1\), the outward normal \(\eta\) makes an acute angle with \((-y, x)\) and at every point \((x, y) \in \Gamma_2\), the outward normal \(\eta\) makes an obtuse angle with \((-y, x)\).

In such domains, the multiplicity of the second Dirichlet eigenvalue is at most 2.

**Remark 1.12.** Observe that the above theorem includes in particular domains which are convex in one direction, when the origin is taken arbitrarily far from the domain (point at infinity).

Next, using ideas outlined previously in [MS], we prove an extension of an old result of Stern (see [Stc, BH]). Recall that a domain is called \(k\)-connected if it has \(k + 1\) boundary components. Also, given an eigenfunction \(\varphi\), let \(\mu(\varphi)\) denote the number of nodal domains corresponding to \(\varphi\). Then, we have the following result.

**Theorem 1.13.** Suppose we are given any \(k\)-connected domain \(D \subset \mathbb{R}^2\) and a natural number \(N\). Then we can construct a \(k\)-connected domain \(\Omega\) for which there are at least \(N\) distinct eigenvalues \(\lambda_j\) for which \(\mu(\varphi_j) = 2\). \(\Omega\) can be constructed so that all but one of the boundary components of \(\Omega\) and \(D\) are homothetic (see Figure 9 below).

Now, we begin investigation into some geometric properties of the nodal set of low energy eigenfunctions. Our discussion is mainly centred around the angular properties of the nodal set where it meets the boundary. First, we have the following for the case of the nodal set intersecting the boundary (the interior case has already been addressed in [GM1]):

**Theorem 1.14.** Suppose \(\mathcal{N}_\varphi\) intersects the boundary of the manifold \(\partial M\) at a point \(p\). When \(\dim M = 3\), the nodal set \(\mathcal{N}_\varphi\) at \(p\) satisfies an interior cone condition (see Definition 3.3 below) with angle \(\gtrsim \frac{1}{\sqrt{\lambda}}\). When \(\dim M = 4\), \(\mathcal{N}_\varphi\) at \(p\) satisfies an interior cone condition with angle \(\gtrsim \frac{1}{\lambda^{3/2}}\). Lastly, when \(\dim M \geq 5\), \(\mathcal{N}_\varphi\) at \(p\) satisfies an interior cone condition with angle \(\gtrsim \frac{1}{\lambda}\).

Now, we give a sharpened version of Theorem 1.14 above about the angle of intersection at the points of nodal sets provided the nodal set is locally the intersection of two hypersurfaces.

**Theorem 1.15.** Let \(p \in M\), where \(M\) is a compact manifold of dimension \(n \geq 3\). Let \(p\) lie at the intersection of two nodal hypersurfaces \(M_1, M_2\). Let \(\eta_1, \eta_2 \in S^{n-1}\) be two unit normal vectors to \(M_1\) and \(M_2\) at \(p\). If the order of vanishing of \(\varphi_\lambda\) at \(p\) is \(n_0\), then the angle between \(M_1\) and \(M_2\) at \(p\), \(\arccos(\eta_1, \eta_2) \in \mathbb{P}\), where

\[
P = \left\{ \frac{p}{q} : q = 1, 2, \cdots n_0, p = 0, 1, \cdots, q \right\}.
\]

At last, we come full circle, and all the topological and geometric ideas developed till now culminate in the following corollary:

**Corollary 1.16.** Let \(\Omega \subset \mathbb{R}^2\) be a convex domain with \(C^2\)-boundary. \(\Omega\) satisfies the strong Payne property.

As is well-known, this result is due to Melas (see [M]). However, our proof is somewhat different (and in our opinion, substantially simpler). Our investigation reveals that any proposed solution to Payne conjecture has a natural local and a global component. Namely, the local component is the angle estimate of the nodal set at the boundary, which we address in all dimensions (via Theorem 1.15 above). The global component is proving that non-negative Neumann boundary data of the second Dirichlet eigenfunction implies simplicity of the second Dirichlet eigenvalue. The latter is an old idea from [Lin], and is not known (to our knowledge) beyond convex domains. We note in passing that Corollary 1.16 follows in turn from Theorem 3.11 below. In particular, this raises the natural question an affirmative answer to which should settle the Payne conjecture in the settings described above:

**Question 1.17.** Given a domain \(\Omega \subseteq \mathbb{R}^n\) or a closed manifold \(M\), and let \(\mathcal{M}(\Omega)\) and \(\mathcal{M}(M)\) denote the moduli space of all smooth perturbations of \(\Omega\) and all Riemannian metrics on \(M\) respectively. Are these moduli spaces path connected?
For more details, see subsection 3.3 below.

In the last section of our paper, we start focussing more on low energy spectral theory (properties of eigenvalues) with the help of perturbation theoretic tools. To set up our methods, we first give a proof of a well-known result of Uhlenbeck that generic smooth perturbations of a domain have simple Dirichlet spectrum (this is Theorem 4.1 below). The ideas involved in our proof are based on [GS] and are well-known by now, though we have not seen this exact proof in literature. Finally, we start investigating the question whether the fundamental gap $\lambda_2 - \lambda_1$ is attained on convex domains. Here is what we prove: let $\mathcal{C}$ denote the class of $C^2$-convex planar domains and let $\mathfrak{C}$ denote the class of small $C^2$-perturbations of $C^2$-convex planar domains. Let $\mathfrak{P}$ stand for either class of domains. Then, we have that

**Theorem 1.18.** Let $\Omega \in \mathfrak{P}$ with diameter $D = 1$ and inner radius $\rho$. There exists a universal constant $C \ll 1$ such that if $\rho \leq C$, $\Omega$ cannot be the minimiser of the fundamental gap functional in $\mathfrak{P}$.

The proof uses a significant portion of the ideas on perturbation theory developed so far in Section 4. We are unable yet to get a result in the full class of all convex domains. However, the popular belief in the community seems that the fundamental gap is not saturated in the class of all convex domains, and any infimising sequence for $\lambda_2 - \lambda_1$ (under the normalisation $D = 1$) should degenerate to a line segment. This indicates that the “correct regime” to look for in the search for minimisers is the class of narrow convex domains.

The paper is interspersed with open questions/speculations/conjectures which we believe to be of interest, at any rate to the present authors!

2. **Topology of low energy nodal sets**

A celebrated conjecture in [P] states that for a bounded domain $\Omega \subseteq \mathbb{R}^2$ the second eigenfunction of the Laplacian with zero boundary condition does not have a closed nodal line i.e., it cannot happen that the nodal set is an embedded circle in $\Omega$ which does not intersect the boundary $\partial \Omega$. We will refer this conjecture as the Payne conjecture or the nodal line conjecture throughout our text.

The second nodal domains represent a 2-partition of $\Omega$ minimising the spectral energy, that is

$$\lambda_2(\Omega) = \inf \{ \max \{ \lambda_1(\Omega_1), \lambda_1(\Omega_2) \} : \Omega_1, \Omega_2 \subset \Omega \text{ open}, \Omega_1 \cap \Omega_2 = \emptyset, \Omega_1 \cup \Omega_2 = \Omega \}$$

where the infimum is attained only when $\Omega_1, \Omega_2$ are the nodal domains of some $\varphi_2$. One idea behind the above conjecture is that it would be suboptimal from the perspective of energy minimisation to have one nodal domain concentrated somewhere in the interior of $\Omega$, with the other occupying its boundary.

Liboff in [Li] conjectured that the nodal surface of the first excited state of a three-dimensional convex domain intersects its boundary in a single simple closed curve. The conjecture is analogous to that of Payne in dimension 3.

2.1. **Some previous work on topology of first nodal sets.** In this section we mention some progress made on the above conjecture in a chronological order.

[P1] addressed the conjecture provided the domain $\Omega \subseteq \mathbb{R}^2$ is symmetric with respect to one line and convex with respect to the direction vertical to this line. [Lin] following a similar approach proved the conjecture provided the domain $\Omega \subseteq \mathbb{R}^2$ is smooth, convex and invariant under a rotation with angle $2\pi p/q$, where $p$ and $q$ are positive integers. Both the proofs rely heavily on the symmetry of domain. In [LN], Lin and Ni provided a counter-example of the nodal domain conjecture for the Dirichlet Schrodinger eigenvalue problem. For each $n \geq 2$, they construct a radially symmetric potential $V$ in a ball so that the nodal domain conjecture is violated. In 1991, Jerison proved in [J] that the conjecture is true for long thin convex sets in $\mathbb{R}^2$. More specifically, there is an absolute constant $C$ such that given a convex domain $\Omega \subseteq \mathbb{R}^2$ with $\frac{\text{diam}(\Omega)}{\text{inrad}(\Omega)} \geq C$, we have that the nodal set corresponding to the second eigenfunction intersects...
the boundary at exactly two points. Here \( \text{inrad}(\Omega) \) denotes the radius of the largest ball that can be inscribed in \( \Omega \) and \( \text{diam}(\Omega) \) denotes the diameter. In the following year, Melas relaxed the condition of “long and thin” in [M] and proved the conjecture for any bounded convex domain \( \Omega \) in \( \mathbb{R}^2 \) with \( C^\infty \) boundary.

To the extent of our knowledge, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and Nadirashvili in [HHN] provided the first counter-example of the Payne conjecture in \( \mathbb{R}^2 \) for the case of Dirichlet Laplacian. We outline the basic idea of their construction in Subsection 2.4 below. We mention is passing that boundedness of the domain is crucial for results of the Payne type (see [FKL]).

Regarding the topological properties of the second nodal set in higher dimensions, Jerison [J2] extended his result for long and thin convex sets in higher dimensions (see also related follow up work in [Da, FK2, KT]). Fournais in [F] extended the result of [HHN] in higher dimensions and proved that the first nodal set does not intersect the boundary (we outline his construction in Subsection 2.4 below). The domain constructed by Fournais was not topologically simple, which was later addressed in [Ke]. Recently Kiwan, in [Ki], proved the nodal domain conjecture for domains which are of the form \( A \setminus B \) where \( A \) and \( B \) have sufficient symmetry and convexity.

2.2. \textbf{Payne property and perturbation of domains.} Let \( \varphi \) be a Dirichlet eigenfunction for a bounded domain \( \Omega \subset \mathbb{R}^n \). For any \( p \in \partial \Omega, \ p \in \mathcal{N}_\varphi \) if and only if \( \frac{\partial \varphi}{\partial n} = 0 \), where \( \eta \) denotes the outward normal at \( p \). The proof for dimension \( n = 2 \) is covered in Lemma 1.2 of [Lin], and one can check that a similar proof is true in higher dimensions as well. Let \( x = (x_1, \ldots, x_n) =: (x', x_n) \), and let the domain \( \Omega \) be tangent to the \( x'-\)hyperplane at the origin. If \( \frac{\partial \varphi}{\partial n}(0) \neq 0 \), then by the implicit function theorem, \( x_n \) is uniquely solvable as a function of \( x' \) in a neighbourhood of 0, which means that the only zeros of \( \varphi \) near the origin occur on \( \partial \Omega \). The converse case is addressed by a variant of the Hopf boundary principle (see Lemma H of [GNN]).

Consider any Dirichlet eigenfunction \( \varphi \) whose nodal set \( \mathcal{N}_\varphi \) divides \( \Omega \) into exactly two nodal domains. In particular, any first nodal set (nodal set corresponding to some second eigenfunction) always divides the domain into exactly two components. Then we have the following three cases.

- (SP) If \( \frac{\partial \varphi}{\partial \eta} \) changes sign on the boundary then \( \mathcal{N}_\varphi \cap \partial \Omega \neq \emptyset \) and \( \mathcal{N}_\varphi \) divides at least one component of \( \partial \Omega \) into exactly two components.
- (WP) If \( \frac{\partial \varphi}{\partial \eta} \geq 0 \) (w.l.o.g.) on the boundary, then \( \mathcal{N}_\varphi \cap \partial \Omega \neq \emptyset \) but \( \partial \Omega \setminus \mathcal{N}_\varphi \) has same number of connected components as \( \partial \Omega \).
- (NP) If \( \frac{\partial \varphi}{\partial \eta} > 0 \) (w.l.o.g.) on the boundary, then \( \mathcal{N}_\varphi \cap \partial \Omega = \emptyset \).

\textbf{Definition 2.1.} We say that any second eigenfunction \( \varphi \) satisfies the Payne property if the nodal set of \( \varphi \) intersects the boundary \( \partial \Omega \), that is either (SP) or (WP) is true. We say that \( \varphi \) satisfies the strong Payne property if only (SP) is true. Also, we say that \( \Omega \) satisfies the (strong) Payne property if every second eigenfunction \( \varphi \) of \( \Omega \) satisfies the (strong) Payne property.

Now, suppose \( \Omega_t \) is a one-parameter family of domains such that for all \( k = 1, \cdots \), we have

\[ \lim_{t \to 0} \lambda_k(\Omega_t) = \lambda_k(\Omega_{t_0}). \tag{2.1} \]

Assume additionally that for each \( k \), we have the following \( C^\infty \)-convergence\(^2\) of eigenfunctions \( \varphi_{k,t} \in C^\infty(\Omega_t) \), where \( \varphi_{k,t} \) is defined as the \( k \)-th Dirichlet eigenfunction of \( \Omega_t \):

\[ \lim_{t \to t_0} \varphi_{k,t} = \varphi_{k,t_0} \text{ on } \tilde{\Omega} := \{ x : \exists \varepsilon_x > 0 \text{ such that } x \in \bigcap_{t_0 - \varepsilon, t_0 + \varepsilon} \Omega_t \forall \varepsilon < \varepsilon_x \}. \tag{2.2} \]

All the eigenfunctions involved are assumed to be \( L^2 \)-normalized, i.e.,

\[ \| \varphi_{k,t} \|_{L^2(\Omega_t)} = \| \varphi_{k,t_0} \|_{L^2(\Omega_{t_0})} = 1. \]

\(^2\)Actually, it turns out that for almost all of our applications, only \( C^0 \)-convergence would suffice.
Lemma 2.2. Let \( N(\varphi_{k,t}) \) and \( N(\varphi_{k,t_0}) \) denote the nodal sets corresponding to \( \varphi_{k,t} \) and \( \varphi_{k,t_0} \) respectively. Consider a sequence of points \( \{x_i\} \) such that for each \( i \), \( x_i \in N(\varphi_{k,t_i}) \cap \tilde{\Omega} \). If the limit \( x \) of \( \{x_i\} \) exists, then \( x \in N(\varphi_{k,t_0}) \).

Observe that there is no requirement of convexity or boundary regularity.

Proof. Denote \( \tilde{\varphi}_{k,t} = \varphi_{k,t} |_{\tilde{\Omega}} \). We apply the convergence \( \varphi_{k,t} \to \varphi_{k,t_0} \) in \( C_0(\tilde{\Omega}) \) to obtain,

\[
|\varphi_{k,t_0}(x_i)| = |\varphi_{k,t_0}(x_i) - \tilde{\varphi}_{k,t_i}(x_i)| \leq \|\varphi_{k,t_0} - \tilde{\varphi}_{k,t_i}\|_{C_0(\tilde{\Omega})} \to 0 \quad \text{as} \quad i \to \infty.
\]

Now, \( \varphi_{k,t_0} \) being a continuous function, \( x_i \to x \) implies \( \varphi_{k,t_0}(x_i) \to \varphi_{k,t_0}(x) \). Therefore, \( \varphi_{k,t_0}(x) = 0 \) i.e. \( x \in N(\varphi_{k,t_0}) \). \( \square \)

Note that even \( C^\infty \)-convergence of eigenfunctions is not strong enough to give convergence of nodal sets in the Hausdorff sense (or any other appropriate sense). The problem is, in the limit the nodal set can become topologically non-generic (for example, developing a node). As a trivial example, consider the functions \( x^2y^2 + \epsilon^2 \), which converge to \( x^2y^2 \) and develop a non-trivial nodal set in the limit. More specifically, if \( k \)-th eigenfunctions \( \varphi_{k,\epsilon} \) of \( \Omega_\epsilon \) converge to eigenfunction \( \varphi_{k,0} \) of \( \Omega_0 \), the nodal sets \( N(\varphi_{k,\epsilon}) \) do not necessarily converge to \( N(\varphi_{k,0}) \). Now we have the following

Lemma 2.3. If \( N(\varphi_{2,t_0}) \subset \tilde{\Omega} \), then for \( t \) close enough to \( t_0 \), the nodal set \( \varphi_{k,t} \) is fully contained inside \( \tilde{\Omega} \).

Proof. The proof is slightly long, but follows by routine modifications to the proof of Lemma 3.6 of [MS], hence we skip the details. \( \square \)

Next, we recall the following convergence theorem from [HP].

Theorem 2.4 (Theorem 2.2.25, [HP]). Let \( K_n \) be a sequence of compact sets contained in a fixed compact set \( B \). Then there exist a compact set \( K \) contained in \( B \) and a subsequence \( K_{n_k} \) that converges in the sense of Hausdorff to \( K \) as \( k \to \infty \).

As a consequence, we have the following

Proposition 2.5. Satisfying the property (SP) or (NP) is an open condition.

Proof. By Lemma 2.3, we know that \( N(\varphi_{2,\epsilon}) \) is eventually inside \( \tilde{\Omega} \subset \Omega_{t_0} \). By a straightforward topological argument, if \( N(\varphi_{2,\epsilon}) \) does not intersect \( \partial\tilde{\Omega} \), then it is an embedded hypersurface with possible “lower dimensional” singularities. By precompactness in Hausdorff metric as explained in Theorem 2.4, one can extract a subsequence called \( N(\varphi_{2,\epsilon_i}) \), which converges to a set \( X \subset \Omega_{t_0} \) in the Hausdorff metric. By Lemma 2.2, we already know that \( X \subset N(\varphi_{2,t_0}) \). It follows that for \( i \) large enough, \( N(\varphi_{2,\epsilon_i}) \) is within any \( \delta \)-tubular neighbourhood of \( N(\varphi_{2,t_0}) \).

Figure 1. Property (NP) is an open condition
Let \( n(\varphi_{2,t_0}) \) satisfy \((NP)\). Then \( n(\varphi_{2,t_0}) \) does not intersect the boundary \( \partial \tilde{\Omega} \). For small enough \( \delta \), the \( \delta \)-tubular neighbourhood of \( n(\varphi_{2,t_0}) \) does not intersect \( \partial \tilde{\Omega} \). This implies that given such a \( \delta \), for large enough \( i \), \( n(\varphi_{2,\epsilon_i}) \) does not intersect \( \partial \tilde{\Omega} \). More specifically, \( n(\varphi_{2,\epsilon_i}) \cap \Omega_{\epsilon_i} = \emptyset \).

Now assume that \( n(\varphi_{2,t_0}) \) satisfies \((SP)\). If possible, let \((SP)\) is not an open condition, that is there exists a subsequence \( \{k\} \subset \{i\} \) such that \( n(\varphi_{2,\epsilon_k}) \) does not satisfy \((SP)\). This means that one of nodal domains of the second Dirichlet eigenfunction of \( \Omega_{\epsilon_k} \) is within any \( \delta \)-tubular neighbourhood of \( n(\varphi_{2,t_0}) \) and the volume of such a tubular neighbourhood is going to 0 as \( \delta \searrow 0 \).

Figure 2. If property \((SP)\) is not an open condition

This will contradict the Faber-Krahn inequality (or the inner radius estimate for the second nodal domain of \( \Omega_{\epsilon_k} \)), and imply that for large enough \( i \), \( n(\varphi_{2,\epsilon_i}) \) intersects the boundary. Moreover, if the first nodal set is a submanifold, then using Thom’s isotopy theorem (see [Ab], Section 20.2) one can conclude that for large enough \( i \), \( n(\varphi_{2,\epsilon_i}) \) is diffeomorphic to \( n(\varphi_{2,t_0}) \). This is precisely the case in dimension 2, by Theorem 1.1 of [M]. □

We finish this section with the following

**Remark 2.6.** Let \( \Omega \subset \mathbb{R}^n \) be a domain which can be realised as a one-parameter family of real-analytic perturbations of the ball. Let the unit ball be denoted by \( \Omega_0 \) and \( \Omega_1 = \Omega \). Then \( \{t \in [0,1] : \Omega_t \text{satisfies} (SP)\} \) is an open set.

2.3. **Domains with narrow connector: Payne and mass concentration properties.** As is already pointed out, by the work in [M], the strong Payne property is known to hold on convex domains. By further work in [MS], it is also known to hold on domains obtained from small perturbations of convex domains. Somehow a natural approach would be to investigate the validity of the conjecture on domains which are in some sense both “very far” from being convex, or being small perturbations thereof. A natural class of such domains would be the so-called dumbbell domains.

Consider two bounded disjoint open sets \( \Omega_1 \) and \( \Omega_2 \) in \( \mathbb{R}^n, n \geq 2 \) and let \( Q \) be a line segment joining \( \Omega_1 \) and \( \Omega_2 \). For some small enough fixed \( \epsilon > 0 \), consider

\[
\Omega_\epsilon = \Omega_1 \cup \Omega_2 \cup \left( \bigcup_{x \in Q} B(x, \epsilon) \right).
\]

That is, the dumbbell domain \( \Omega_\epsilon \) is obtained by joining \( \Omega_1 \) and \( \Omega_2 \) with a connector \( Q_\epsilon := \Omega_\epsilon \setminus (\Omega_1 \cup \Omega_2) \) of width \( \epsilon \). Then Theorem 2.3.20 of [He] says that

\[
\lambda_k(\Omega_\epsilon) \to \lambda_k(\Omega_1 \cup \Omega_2) \quad \text{as} \quad \epsilon \to 0.
\]

Let \( \Lambda_i \) denote the spectrum of \( \Omega_i \) \( (i = 1, 2) \), and \( \varphi_{\Omega_i}^k \) denote the \( k \)-th eigenfunction of \( \Omega_i \) corresponding to the eigenvalue \( \lambda_{\Omega_i}^k \). Considering the Dirichlet boundary condition on \( \Omega_\epsilon \), it is known from [JK] (also see [GN]) that if \( \Lambda_1 \cap \Lambda_2 = \emptyset \) then each eigenfunction \( \varphi_{k,\epsilon} \) on the
domain \( \Omega \) approaches in \( L^2 \)-norm to an eigenfunction \( \varphi_{k,0} := \varphi_k^{(0)} \) which is fully localised in one subdomain \( \Omega \) and zero in the other. The fact that the spectra of \( \Omega_1 \) and \( \Omega_2 \) do not intersect is important for localisation of the eigenfunctions to exactly one subdomain \( \Omega \).

We are interested in looking at the nodal sets of the second eigenfunctions of these dumbbell domains with narrow connectors. We now begin proving Theorem 1.10.

**Proof.** We know that each eigenvalue of the Dirichlet Laplacian on \( \Omega \) converges to some eigenvalue of \( \Omega_1 \) or \( \Omega_2 \). Without loss of generality, we can rearrange \( \Lambda_1 \cup \Lambda_2 \) in the following two ways.

Case I: \( \lambda_1^{\Omega_1} < \lambda_2^{\Omega_1} < \lambda_j^{\Omega_1} \leq \cdots \) for some \( i = 1, 2, \) and \( j \geq 2 \).

Case II: \( \lambda_1^{\Omega_2} < \lambda_2^{\Omega_2} < \lambda_j^{\Omega_2} \leq \cdots \), for some \( i = 1, 2, \) and \( j \in \mathbb{N} \). In general, we label the above arrangement as \( \lambda_{1,0} \leq \lambda_{2,0} \leq \cdots \).

Consider a straight line \( \Gamma \) (see diagram below) such that \( \Gamma \cap Q_\epsilon \) divides every \( \Omega \) into two components such that \( \Omega_1 \) and \( \Omega_2 \) are contained in each of the two components. Also, consider a sequence \( \{ \epsilon_i \} \to 0 \) and redefine \( \varphi_{2,\epsilon}, \varphi_{2,0} \) on \( \mathbb{R}^n \) as

\[
\varphi_{2,\epsilon_i} = \begin{cases} \varphi_{2,\epsilon_i} & \text{on } \Omega_{\epsilon_i}, \\ 0, & \text{otherwise.} \end{cases}
\]

and

\[
\varphi_{2,0} = \begin{cases} \varphi_1^{(0)} & \text{(or, } \varphi_1^{(0)} \text{) on } \Omega_2 \text{ (or, } \Omega_1 \text{ respectively),} \\ 0, & \text{otherwise.} \end{cases}
\]

Then from the discussion above we have that \( \| \varphi_{2,\epsilon_i} - \varphi_{2,0} \|_{L^2(\mathbb{R}^n)} \to 0 \) as \( \epsilon_i \to 0 \). We first prove that \( \varphi_{2,\epsilon} \to \varphi_{2,0} \) in \( C^0(\Omega_1) \) assuming that \( \varphi_{2,0} \) is localised in \( \Omega_1 \).

Consider

\[
(\Delta + \lambda_{2,0})[\varphi_{2,\epsilon_i} - \varphi_{2,0}] = (\lambda_2 - \lambda_{2,\epsilon_i})\varphi_{2,\epsilon_i},
\]

and

\[
(\Delta + \lambda_{2,\epsilon})\varphi_{2,\epsilon_i} = 0
\]
on \( \Omega_{\epsilon_i} \). Note that \( \varphi_{2,0} = 0 \) and \( \varphi_{2,\epsilon_i} = 0 \) on \( \partial \Omega_{\epsilon_i} \). Now applying Theorem 8.15 of [GT] in the above two equations consecutively, we have that for some \( q > 2 \) and \( \nu > \sup\{\lambda_{2,0}, \lambda_{2,\epsilon_i}\} \),

\[
\| \varphi_{2,\epsilon_i} - \varphi_{2,0} \|_{L^\infty(\Omega_{\epsilon_i})} \leq \| \varphi_{2,\epsilon_i} - \varphi_{2,0} \|_{L^\infty(\Omega_\epsilon)}
\]

\[
\leq C \left( \| \varphi_{2,\epsilon_i} - \varphi_{2,0} \|_{L^2(\Omega_{\epsilon_i})} + \| (\lambda_2 - \lambda_{2,\epsilon_i})\varphi_{2,\epsilon_i} \|_{L^{q/2}(\Omega_{\epsilon_i})} \right)
\]

\[
\leq C \left( \| \varphi_{2,\epsilon_i} - \varphi_{2,0} \|_{L^2(\Omega_{\epsilon_i})} + \frac{C'}{\epsilon_i} \lambda_2 \| \varphi_{2,\epsilon_i} \|_{L^2(\Omega_{\epsilon_i})} \right)
\]

\[
\leq C \left( \| \varphi_{2,\epsilon_i} - \varphi_{2,0} \|_{L^2(\mathbb{R}^n)} + \frac{C'}{\epsilon_i} \| \varphi_{2,\epsilon_i} \|_{L^2(\mathbb{R}^n)} \right),
\]

where \( C, C' \) depends on \( q, \nu, \) and \( |\Omega_{\epsilon_i}| \). For each \( i, |\Omega_{\epsilon_i}| \) is uniformly bounded which implies that the constants on the right hand are independent of \( i \).

Now using \( \lambda_{2,\epsilon_i} \to \lambda_2 \) and \( \| \varphi_{2,\epsilon_i} - \varphi_{2,0} \|_{L^2(\mathbb{R}^n)} \to 0 \) we have that as \( i \to \infty \)

\[
\| \varphi_{2,\epsilon_i} - \varphi_{2,0} \|_{L^\infty(\Omega_{\epsilon_i})} \to 0.
\]

Considering Case I, i.e. \( \lambda_2 = \lambda_2^{\Omega_2} \) and \( \varphi_{2,\epsilon_i} \to \varphi_{1,\Omega_2} \) in \( L^2(\Omega_2) \), \( \varphi_{1,\Omega_2} \) does not change sign in \( \Omega_2 \). Without loss of generality, assume that \( u_1^{(0)} > 0 \) in \( \Omega_2 \). Also, \( N(\varphi_{2,\epsilon}) \) divides \( \Omega \) into two components. Since \( \varphi_{1,\Omega_2} > 0 \) in \( \Omega_2 \) and \( \| \varphi_{2,\epsilon_i} - \varphi_{2,0} \|_{C^0(\Omega_2)} \to 0 \), it is clear that for small enough \( \epsilon, N(\varphi_{2,\epsilon}) \) lies completely in \( \Omega_\epsilon \setminus \Omega_2 \).

In Case II, we have \( \lambda_2 = \lambda_2^{\Omega_1} \) with \( \varphi_{2,\epsilon_i} \to \varphi_{1,\Omega_1}^{(0)} \) in \( L^2(\Omega_1) \). Recall that we have assumed that \( Q_\epsilon \) is away from \( N(\varphi_{1,\Omega_1}^{(0)}) \) (\( i = 1, 2 \)), that is, \( Q_\epsilon \) does not intersect \( N(\varphi_{1,\Omega_1}^{(0)}) \) for any \( \epsilon > 0 \).

Consider a hypersurface \( \Gamma' \subset \Omega_1 \) such that \( \Gamma' \) divides every \( \Omega_\epsilon \) into two components and \( N(\varphi_{1,\Omega_1}^{(0)}) \) lies in one fixed component \( \Omega' \) and \( Q_\epsilon \) is completely contained in the other component for every \( \epsilon > 0 \) small enough. If possible, let \( N(\varphi_{2,\epsilon_i}) \) intersects the connector \( Q_\epsilon \), for every \( \epsilon > 0 \), and \( N(\varphi_{2,\epsilon_i}) \cap \Gamma' = \{ p_i \} \in \Omega_1 \) and \( p \in \Omega_1 \) be the limit of a subsequence \( \{ p_j \} \subset \{ p_i \} \). From
Lemma 2.2, we have that \( p \in N(\varphi_{\Omega_1}^{\Omega_2}) \) which is a contradiction since our choice of \( \Gamma' \) ensures that \( p \in \Gamma' \) is away from \( N(\varphi_{\Omega_1}^{\Omega_2}) \cap \partial \Omega_1 \). So there exists \( \epsilon_0 > 0 \) such that \( N(\varphi_{\Omega_1}^{\Omega_2}, \epsilon) \subset \Omega^0 \) for any \( \epsilon < \epsilon_0 \).

\[ \text{Figure 3. Behaviour of nodal line as } \epsilon \to 0 \]

Now we would like to show that under Case II, if \( \Omega_1 \) satisfies the strong Payne property then, for small enough \( \epsilon \), so does \( \Omega_\epsilon \). Using Theorem 2.4 one can extract a subsequence called \( N(\varphi_{\epsilon}^{\Omega_2}) \) which converges to a set \( X \subset \Omega_1 \) in the Hausdorff metric and by Lemma 2.2 we know that \( X \subset N(\varphi_{\Omega_1}^{\Omega_2}) \). Now following the argument as in Proposition 2.5 concludes the proof. \( \square \)

Now we would like to address a mass concentration question in the connector of the dumbbell, following the general line of enquiry in [BD,vdBB]. [GM3] also establishes \( L^\infty \) estimates in domains with long narrow tubes.

**Proposition 2.7.** Consider a dumbbell domain \( \Omega_\epsilon \subset \mathbb{R}^n \), where \( n \geq 2 \) (see diagram below). Let \( Q(z) := \Omega_\epsilon \cap \{ x \in \mathbb{R}^d : x_1 = z \} \) be the cross-section of \( \Omega_\epsilon \) at \( x_1 = z \in \mathbb{R} \) by a hyperplane perpendicular to the coordinate axis \( x_1 \) and let \( \mu(z) \) be the first eigenvalue of the Laplace operator in \( Q(z) \), with Dirichlet boundary condition on \( \partial Q(z) \) and \( \mu \) = inf \{\( \mu(z) : z \in (z_0, z_2) \)\}. If \( \lambda_{2,0} < \mu \) then,

\[
\|\varphi_{\epsilon}^{\Omega_2}\|_{L^2(Q)} \leq \frac{\sqrt{2}}{\sqrt{\mu - \lambda_{2,0}}} (1 - e^{-\frac{1}{2}\sqrt{\mu - \lambda_{2,0}}} \|\varphi_{\epsilon}^{\Omega_2}\|_{L^2(Q(0))}^2).
\]

(2.3)

\[ \text{Figure 4. Mass concentration in the connector} \]

**Proof.** We use the following result from [DNG]:

**Theorem 2.8.** Let \( z_1 = \inf \{ z \in \mathbb{R} : Q(z) \neq \emptyset \}, \quad z_2 = \sup \{ z \in \mathbb{R} : Q(z) \neq \emptyset \} \). Fix \( z_0 \in (z_1, z_2) \). Let \( \varphi \) be a Dirichlet-Laplacian eigenfunction in \( \Omega \), and \( \lambda \) the associate eigenvalue. If \( \lambda < \mu \), then

\[
\|\varphi\|_{L^2(Q(z))} \leq \|\varphi\|_{L^2(Q(z_0))} e^{-\beta \sqrt{\mu - \lambda}(z-z_0)}, \quad z > z_0.
\]

(2.4)

with \( \beta = 1/\sqrt{2} \).
Without loss of generality, assume that \( \varphi_{2,\epsilon} \) is localised on \( \Omega_2 \) as \( \epsilon \to 0 \). Following the convention in [DNG], choosing \( z_0 = 0 \), we call \( \Omega_2 \) as the ‘basic’ domain and \( \Omega \setminus \Omega_2 \) the ‘branch’. Note that in our case of dumbbells, without loss of generality, \( \mu \) is attained for some \( z \) for which \( Q(z) \in \Omega_1 \). Note that if \( \epsilon' < \epsilon \), then \( \lambda_{2,\epsilon'} > \lambda_{2,\epsilon} \), which implies \( \lambda_{2,\epsilon} \) monotonically converges to \( \lambda_{2,0} \). From the assumption that \( \lambda_{2,0} < \mu \) (this assumption puts a restriction on the “fatness” of the dumbbells that we can consider), for small enough \( \epsilon \), \( \lambda_{2,\epsilon} < \mu \).

Now, applying Theorem 2.8 for \( u_{2,\epsilon} \), we have
\[
\|\varphi_{2,\epsilon}\|_{L^2(Q(z))} \leq \|\varphi_{2,\epsilon}\|_{L^2(\Omega(0))} e^{-\beta(\sqrt{\mu - \lambda_{2,\epsilon}})z}, z > 0.
\]
From assumption, \( \lambda_{2,0} < \mu \) and \( \lambda_{2,\epsilon} < \lambda_{2,0} \) implies \( \mu - \lambda_{2,\epsilon} > \mu - \lambda_{2,0} \). Using this, we can rewrite the above inequality as
\[
\|\lambda_{2,\epsilon}\|_{L^2(Q(z))} \leq \|\lambda_{2,\epsilon}\|_{L^2(\Omega(0))} e^{-\beta(\sqrt{\mu - \lambda_{2,0}})z}, z > 0. \tag{2.5}
\]
Integrating both sides of the above inequality from \( z = 0 \) to \( z = 1 \), we have,
\[
\|\varphi_{2,\epsilon}\|_{L^2(Q^*)}^2 = \int_0^1 \|\varphi_{2,\epsilon}\|_{L^2(Q(z))}^2 dz \leq \int_0^1 \|\varphi_{2,\epsilon}\|_{L^2(\Omega(0))}^2 e^{-\beta(\sqrt{\mu - \lambda_{2,0}})z} dz \tag{2.6}
\]
\[
\leq C(\Omega_1, \Omega_2) \|\varphi_{2,\epsilon}\|_{L^2(\Omega(0))}^2. \tag{2.7}
\]
where \( C(\Omega_1, \Omega_2) = \frac{1}{\beta\sqrt{\mu - \lambda_{2,0}}} \left( 1 - e^{-\beta(\sqrt{\mu - \lambda_{2,0}})1} \right) \).

2.4. Counterexample of Payne property for simply connected domains in higher dimensions. We begin by discussing the counter-example as given in [HHN]. First we choose two concentric balls \( B_{R_1} \) and \( B_{R_2} \) in \( \mathbb{R}^2 \) such that
\[
\lambda_1(B_{R_1}) < \lambda_1(B_{R_2} \setminus B_{R_1}) < \lambda_2(B_{R_1}).
\]
Next, we carve out holes into \( \partial(B_{R_2} \setminus B_{R_1}) \cap \partial B_{R_1} \). Let \( N \in \mathbb{N} \) and \( \epsilon < \pi/N \). The domain \( \Omega_{N,\epsilon} \) is defined as
\[
\Omega_{N,\epsilon} = B_{R_1} \cup (B_{R_2} \setminus B_{R_1}) \cup \left( \bigcup_{j=0}^{N-1} \left\{ x \in \mathbb{R}^2 : r = R_1, \omega \in \left( \frac{2\pi j}{N} - \epsilon, \frac{2\pi j}{N} + \epsilon \right) \right\} \right). \tag{2.8}
\]
Then the first nodal line does not intersect the boundary for sufficiently large \( N \) and small \( \epsilon \).

![Figure 5. N=4 (HHN, Figure 1)](image)

In higher dimensions, the domain constructed by Fournais was motivated from the example in [HHN] described above and defined as follows:
\[
\Omega_\epsilon = B_{R_1} \cup (B_{R_2} \setminus B_{R_1}) \cup \left( \bigcup_{i=1}^{N} B(x_i, \epsilon) \right), \tag{2.9}
\]
where \( B_R \) is a ball of radius \( R \) centered at 0, and \( x_1, \ldots, x_N \in S_{R_1}^{n-1} \) are chosen in such a way that the “patches” \( B(x_i, \epsilon) \cap S_{R_1}^{n-1} \) are evenly distributed over \( S_{R_1}^{n-1} \), the sphere with center at 0.
and radius $R_1$. For convenience, moving forward we will refer to the sphere $S^{n-1}_{R_1}$ as $S$. Also, $R_1$ and $R_2$ are chosen such that

$$\lambda_1(B_{R_1}) < \lambda_1(B_{R_2} \setminus B_{R_1}) < \lambda_2(B_{R_1}).$$

Then for small enough $\epsilon$ the second eigenfunction $\varphi_{2,\epsilon}$ satisfies

$$\mathcal{H}(\varphi_{2,\epsilon}) \cap \partial \Omega = \emptyset.$$

The main idea in [F] is to prove that for small enough $\delta > 0$, there is $\epsilon > 0$ such that $\varphi_{2,\epsilon}(x) > 0$ on $|x| = R_1 - \delta$. Then using various assumptions made during the construction along with certain topological restrictions of the first nodal set $\mathcal{H}(\varphi_{2,\epsilon})$, one concludes that $\mathcal{H}(\varphi_{2,\epsilon})$ is contained inside $B_{R_1 - \delta}$.

Note that the domain $\Omega_\epsilon$ described above is not simply-connected. Our goal in this section is to produce a simply connected domain whose nodal set does not intersect the boundary.

Let $\Omega_0 := \Omega_\epsilon$, where $\Omega_\epsilon$ is the above described domain of Fournais for which the nodal set is contained in $B_{R_1 - \delta}$. Throughout the rest of the proof, the above $\epsilon$ and $\delta$ will remain fixed. From $\Omega_0$ we can construct simply connected domains by adding $(n-1)$-dimensional “tunnels” or “strips” $T_\eta$ along $S$ in between the “patches” $B(x_i, \epsilon)$ such that every patch is connected to the neighbouring patches by tunnels (see the figure below). Our idea is to make these tunnels narrow enough so that the nodal set of $\Omega_0$ does not get sufficiently perturbed.

**Figure 6.** Topologically simple counterexample to Payne

Let $\eta > 0$. For any $i, j \in \{1, \ldots, N\}$ ($i \neq j$) let $p_{ij}(t) : [0, 1] \rightarrow S$ be a path between $x_i$ and $x_j$ along $S$ such that the length of $p_{ij}$ is $\text{dist}_S(x_i, x_j)$, the geodesic distance between $x_i$ and $x_j$ on $S$. Let $t_0$ and $t_1$ be such that $p_{ij}(t_0) \in \partial B(x_i, \epsilon)$ and $p_{ij}(t_1) \in \partial B(x_j, \epsilon)$. Now consider the path segment $P_{ij} = [p_{ij}(t_0), p_{ij}(t_1)]$. Let $\tau_{ij}(\eta)$ denote the $\eta$-tubular neighbourhood of $P_{ij}$.

Define the tunnel $T^{i,j}_\eta := S \cap \tau_{ij}(\eta)$. Let there be $k_N$ tunnels in total. Denote

$$T_{\eta} := \bigcup_{i=1}^{k_N} T_{\eta}^{i,j}.$$  

Now, we define a family of domains $\Omega_{\eta}$ as

$$\Omega_{\eta} := B_{R_1} \cup (B_{R_2} \setminus B_{R_1}) \cup \bigcup_{i=1}^{N} B(x_i, \epsilon) \cup T_{\eta} = \Omega_0 \cup T_{\eta}.$$

It is easy to check that $\Omega_{\eta}$ is simply connected and given any sequence $\eta_\ell \searrow 0$ there is a subsequence $\{\eta_j\} \subseteq \{\eta_\ell\}$ such that $\Omega_{\eta_j}$ converges to $\Omega_0$ in Hausdorff metric. Now we check that the perturbation to the nodal set is controlled.

Let $\varphi_{j,\eta}, \varphi_{j,0}$ denote the eigenfunction corresponding to eigenvalues $\lambda_{j,\eta}, \lambda_{j,0}$ of the Dirichlet-Laplacian $-\Delta_{\eta}, -\Delta_0$. We assume that the eigenfunctions are $L^2$-normalized. Also note that, from our assumption, we have that $\varphi_{2,0} > 0$ in $\Omega_0 \setminus B_{R_1 - \delta}$. 
Let \( \{ \eta_p \} \not\to 0 \) be any strictly monotonically decreasing sequence and \( X_p := B_{R_2} \setminus \Omega_{\eta_p} \). Note that \( \{ X_p \} \) is a increasing family of compact sets. Define

\[
P_p := \bigcup_{k \geq p} X_k \quad \text{and} \quad Q_p := \bigcap_{k \geq p} X_k.
\]

Using the convention from [Sto], we have that \( P_p \not\to X = \lim X_p \) and \( Q_p \not\to X = \overline{\lim} X_p \) where \( X = B_{R_2} \setminus \Omega'_0 \) and \( \Omega'_0 := \Omega_0 \cup (\cup P_j) \). Also, for any \( p, m \in \mathbb{N}, X_p \triangle X_m \subset T_{\eta_m} \) which has finite capacity and \( \text{cap}(\Omega \cap \Omega') = 0 \). Then using Theorem 2.2 of [Sto], we have that, 

\[-\Delta_{\eta_p} \text{ converges to } -\Delta_0 \text{ as } p \to \infty \text{ in norm resolvent sense} \]

(recall that if \( \{ T_p \}_{n=1}^\infty \) and \( T \) are unbounded self-adjoint operators, then \( T_p \to T \) in norm resolvent sense means that for some \( z \in \mathbb{C} \setminus \mathbb{R}, \|(zI - T_p)^{-1} - (zI - T)^{-1}\| \to 0 \) as \( p \to \infty \)). In particular, for \( \lambda_{2,0} \) there exists a sequence \( \eta_p \to 0 \) such that

\[
\lambda_{2,\eta_p} \to \lambda_{2,0}.
\]

Redefining \( \varphi_{2,\eta_p}, \varphi_{2,0} \) by 0 on \( \mathbb{R}^n \setminus \Omega_{\eta_p}, \mathbb{R}^n \setminus \Omega \) we also have that \( \varphi_{2,\eta_p} \to \varphi_{2,0} \) in \( L^2(\mathbb{R}^n) \).

We know that \( \mathcal{N}(\varphi_{2,0}) \) is completely contained inside \( B_{R_1-\delta} \). Now we would like to show that \( \varphi_{2,\eta_p} \to \varphi_{2,0} \) in \( C^0(B_{R_1-\delta}) \).

Consider

\[
(\Delta + \lambda_{2,0})|\varphi_{2,\eta_p} - \varphi_{2,0}| = (\lambda_{2,0} - \lambda_{2,\eta_p})|\varphi_{2,\eta_p},
\]

and

\[
(\Delta + \lambda_{2,\eta_p})\varphi_{2,\eta_p} = 0
\]
on \( \Omega_{\eta_p} \). Note that \( \varphi_{2,0} \geq 0 \) and \( \varphi_{2,\eta_p} = 0 \) on \( \partial \Omega_{\eta_p} \). Now using Theorem 8.15 of [GT] consecutively on the above equations as done in the proof of Theorem 1.10 we have that for some \( q > n \) and \( \nu > \text{sup}\{\lambda_{2,0}, \lambda_{2,\eta_p}\} \),

\[
\|\varphi_{2,\eta_p} - \varphi_{2,0}\|_{L^\infty(B_{R_1-\delta})} \leq C \left( \|\varphi_{2,\eta_p} - \varphi_{2,0}\|_{L^2(\mathbb{R}^n)} + C'|\lambda_{2,0} - \lambda_{2,\eta_p}| \cdot \|\varphi_{2,\eta_p}\|_{L^2(\mathbb{R}^n)} \right), \quad (2.10)
\]

where \( C, C' \) depends on \( n, q, \nu, \) and \( |\Omega_{\eta_p}| \). For each \( p, |\Omega_{\eta_p}| \) is uniformly bounded which implies that the constants on the right hand are independent of \( p \). Now using \( \lambda_{2,\eta_p} \to \lambda_{2,0} \) and \( \|\varphi_{2,\eta_p} - \varphi_{2,0}\|_{L^2(\mathbb{R}^n)} \to 0 \) we have that as \( p \to \infty \)

\[
\|\varphi_{2,\eta_p} - \varphi_{2,0}\|_{L^\infty(B_{R_1-\delta})} \to 0.
\]

which gives our desired \( C^0(B_{R_1-\delta}) \) convergence.

Finally, using Lemma 2.2 and Proposition 2.5 we know that \( \mathcal{N}(\varphi_{2,\eta_0}) \) converges to \( \mathcal{N}(\varphi_{2,0}) \). So, for sufficiently large \( n_0 \in \mathbb{N} \) we have,\n
\[
\mathcal{N}(\varphi_{2,\eta_{n_0}}) \subset B_{R_1-\delta}.
\]

In other words, we have a simply connected domain \( \Omega_{\eta_{n_0}} \in \mathbb{R}^n(n \geq 3) \) for which

\[
\mathcal{N}(\varphi_{2,\eta_{n_0}}) \cap \partial \Omega_{\eta_{n_0}} = \emptyset.
\]

**Remark 2.9.** We have constructed a simply-connected version of Fournais’ counterexample. However, one can further tweak our construction to design counterexamples with prescribed topology. The way to do it is by taking the connected sum of \( \Omega_{\eta_{n_0}} \) with a manifold \( M' \) of prescribed topology, and taking on \( \Omega_{\eta_{n_0}} \# M' \) the piecewise metric

\[
g = \left\{ \begin{array}{ll}
g|_{\Omega_{\eta_{n_0}}} & \varepsilon g|_{M'},
\end{array} \right. \quad (2.11)
\]
mollifying it and then take \( \varepsilon \) arbitrarily small. The details of such a construction are in [MS], and we skip the details.
2.5. Eigenvalue multiplicity, nodal intersection/detachment and topology. As a consequence of Lemma 2.2, we have the following

**Corollary 2.10.** If \( \Omega_t \) is a one-parameter family of perturbations of \( \Omega_0 \), and if \( \varphi_t \to \varphi_0 \), then for \( t \) small enough, the number of nodal domains of \( \varphi_0 \) are at least as many as the number of nodal domains of \( \varphi_t \).

The above corollary implies that the number of nodal domains cannot go down in the limit. We observe that this is not true in general. As an example, consider the family of functions \( f_\varepsilon(x,y) = (x^2y^2 - \varepsilon^2)(x^2 + y^2 - 1) \). For small enough non-zero \( \varepsilon \), \( f_\varepsilon \) has 9 nodal domains, one of which disappears in the limit \( \varepsilon \to 0 \).

**Proposition 2.11.** If \( \Omega \subseteq \mathbb{R}^n \) is a simply connected domain, and \( \varphi \) and \( \psi \) are two eigenfunctions corresponding to the same eigenvalue, then every connected component of \( N_\psi \) intersects \( N_\varphi \) at least one point.

**Proof.** Let \( \Omega_\varphi \) be a nodal domain for \( \varphi \). Then,
\[
\int_{\Omega_\varphi} \Delta \varphi \psi - \varphi \Delta \psi = \int_{\partial \Omega_\varphi} \frac{\partial \varphi}{\partial \eta} \psi - \frac{\partial \psi}{\partial \eta} \varphi
= \int_{\partial \Omega_\varphi} \frac{\partial \varphi}{\partial \eta} \psi = 0,
\]
which implies the sign change of \( \psi \) on \( \partial \Omega_\varphi \), and hence the result. Observe that we have used the fact that on a simply connected domain, \( \partial \Omega_\varphi \) has to be connected. \( \square \)

Observe that the above is not true when the domain has more complicated topology. We refer the reader to [Do] where, following earlier work in [Gi], it is proved that on a closed Riemannian manifold \( M \), if there exist two eigenfunctions for the same eigenvalue which do not simultaneously vanish, then \( H_1(M) \neq 0 \). An illustrative example could be that of a torus (which is a surface of revolution in \( \mathbb{R}^3 \)), where the nodal set can be “pushed” or translated using the rotational isometry. However, it is an interesting question whether such a result could be true on Euclidean domains. The argument in Donnelly is too pretty not to mention, and we add to it our observation that his argument would also extend to Euclidean domains with Neumann boundary conditions.

**Proposition 2.12.** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded domain with Neumann boundary conditions. Suppose as before that \( \varphi \) and \( \psi \) are two eigenfunctions corresponding to the same eigenvalue \( \mu \). If their nodal sets do not intersect, then \( H_1(\Omega) \neq 0 \).

**Proof.** The proof is essentially a minor observation on the proof in [Do]. Let \( u := \varphi + i\psi \), and \( X := \text{Im} \frac{\nabla u}{u} \). By our assumption, the denominator is never zero. On calculation, it can be checked that the flow of the vector field \( Y := \frac{X}{\sqrt{1+|X|^2}} \) leaves invariant the finite measure \( \frac{|\psi|^2}{\sqrt{1+|X|^2}} dV \). On a closed manifold, this is a complete flow. On a manifold with boundary to use the same idea, we need the Neumann boundary condition, as then the gradient of the eigenfunctions is tangential to the boundary. Now, by Poincaré recurrence, there is an integral curve \( \alpha \) of the vector field \( Y \) which returns arbitrarily close to its starting point. It can be completed to a closed path \( \alpha \) such that \( \int_\alpha \text{Im} \frac{du}{u} \geq 1 \), which means that both \( \varphi(\alpha) \) and \( \psi(\alpha) \) are not zero homologous in \( C^* \).

This raises the natural question for the corresponding case of Dirichlet boundary conditions. It is again a quick observation that if \( M \) is a compact manifold with totally geodesic boundary \( \partial M \), then the double \( \tilde{M} \) of \( M \) can be constructed with \( C^2 \) Riemannian metric (see [Mo]) where the Dirichlet eigenfunctions can be continued by “reflection about the boundary”. Clearly in this case, obvious modifications of the argument from [Do] will still work. However, it is not clear how to proceed in the case of a Euclidean domain, where the corresponding statement still seems intuitively true.
So, for the case of Dirichlet boundary conditions, instead of taking the approach of [Do], it seems fruitful to revert back to the original approach of Gichev. A careful scan of the ideas in [Gi] reveals the following:

**Proposition 2.13.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and consider the Laplacian on \( \Omega \) with the Dirichlet boundary condition. If \( \varphi \) and \( \psi \) are two eigenfunctions corresponding to the same eigenvalue \( \lambda \) whose nodal sets do not intersect, then \( H^1(\Omega) \neq 0 \).

The proof is verbatim similar to the one in [Gi], with obvious modifications to allow for the boundary, and we skip it.

**Remark 2.14.** In light of the assumptions imposed on all the results of this section, it is natural to wonder when the Laplace-Beltrami operator of a closed manifold has repeated spectrum. Firstly, it is a well-known fact that generic spaces have simple spectrum. This is a well-known transversality phenomenon investigated in [Al,U] (we also give our own proof in Section 4 below). On the other hand, it is a well-known heuristic (by now folklore) that the presence of symmetries of the space \( M \) leads to repetitions in the spectrum. An explicit proof of this heuristic using a variant of the Peter-Weyl argument has been recorded in [Ta]. The main claim is that the presence of a non-commutative group \( G \) of isometries of the space will lead to infinitely many repeated eigenvalues of the Laplace-Beltrami operator.

**Remark 2.15.** It is clear that on a simply connected domain, satisfaction of the strong Payne property implies that the multiplicity of the second Dirichlet eigenvalue is at most two. Suppose there are three eigenfunctions \( \varphi_j, j = 1, 2, 3 \) corresponding to \( \lambda_2 \). Then picking any two points \( p, q \in \partial \Omega \), one can find an eigenfunction \( \psi_{pq} = \sum_j \alpha_j \varphi_j \) such that \( \psi \) intersects \( \partial \Omega \) exactly at \( p, q \). Now, consider a sequence \( p_n, q_n \) approaching a common point \( o \in \partial \Omega \). Then in the limit one gets an eigenfunction \( \psi \) which intersects \( \partial \Omega \) at exactly \( o \), and vanishes to order at least 2 there, which would be a contradiction of the strong Payne property.

With that in place, we now begin proving Theorem 1.11.

**Proof.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded simply connected domain. Let \( (0,0) \notin \Omega \) be a point such that the boundary \( \partial \Omega \) contains exactly two distinct points \( P, Q \) dividing \( \partial \Omega \) into two components \( \Gamma_j, j = 1, 2 \) such that the unit outward normal at \( P \) is in the direction to the vector joining \( (0,0) \) to \( P \) and the unit outward normal (blue arrows in Figure 7) at \( Q \) is opposite to the vector joining \( (0,0) \) to \( Q \) (green arrows in Figure 7). In particular, for the points \( P = (P_1, P_2) \) and \( Q = (Q_1, Q_2) \) we have

\[
\left< \eta_P, \frac{(P_1, P_2)}{|P|} \right> = 1, \quad \text{and} \quad \left< \eta_Q, \frac{(Q_1, Q_2)}{|Q|} \right> = -1.
\]

Considering the rotational vector field

\[ X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \]

from our second assumption we have that, \( \left< \eta(x,y), X_{(x,y)} \right> > 0 \) (respectively, \( < 0 \)) when \( (x,y) \in \Gamma_1 \) (respectively, \( \Gamma_2 \)), and \( \left< \eta_P, X_P \right> = \left< \eta_Q, X_Q \right> = 0 \).

Suppose to the contrary, that the multiplicity of \( \lambda_2 \) is at least 3. Up to forming linear combinations, let \( \psi \) be a second eigenfunction whose nodal set intersects \( \partial \Omega \) at \( P, Q \). Also, given \( O \in \partial \Omega \), one can find a second eigenfunction \( \varphi \) whose second nodal set intersects \( \partial \Omega \) at \( O \), as in Remark 2.15 (see diagram below).
Figure 7. A non-convex domain satisfying the assumptions of Theorem 1.11

One can check that $[\Delta, X] = 0$, which gives us that

$$
\int_{\Omega} \varphi \Delta(X \psi) - X \psi \Delta \varphi = \int_{\Omega} \varphi X \Delta \psi - X \psi \Delta \varphi
= \int_{\Omega} -\lambda_2 \varphi X \psi - X \psi \Delta \varphi
= \int_{\Omega} \Delta \varphi X \psi - \Delta \varphi X \psi
= 0.
$$

Then we have

$$
0 = \int_{\Omega} \varphi \Delta(X \psi) - X \psi \Delta \varphi = \int_{\partial \Omega} X \psi \frac{\partial \varphi}{\partial \eta} = \int_{\partial \Omega} \langle X, \eta \rangle \frac{\partial \varphi}{\partial \eta}.
$$

Since $\langle X, \eta \rangle \frac{\partial \varphi}{\partial \eta}$ does not change sign on $\partial \Omega$, $\frac{\partial \varphi}{\partial \eta}$ must which leads to a contradiction.

2.6. Multi-connected planar domains and results of Lewy and Stern. In [Le], Lewy proved an interesting lower estimate on the number of domains in which the nodal lines of spherical harmonics divide the sphere.

**Theorem 2.16** (Lewy). Let $k \in \mathbb{N}$ be odd. Then there is a spherical harmonic of degree $k$ with exactly two nodal domains. Let $k \in \mathbb{N}$ be even. Then there is a spherical harmonic of degree $k$ with exactly three nodal domains.

Extending the above result a bit further, the present authors show in [MS] that on any closed surface of genus $\gamma \geq 1$, we can find a metric such that it has arbitrarily many eigenfunctions with 2 nodal domains.

We prove a similar result for multi-connected domains in $\mathbb{R}^2$. First we look at the following result of Stern from her thesis [Ste].

**Theorem 2.17** (Stern). For the square $[0, \pi] \times [0, \pi] \subset \mathbb{R}^2$, there exists a sequence $\{\varphi_m\}$ of Dirichlet eigenfunctions associated with the eigenvalues $\lambda_{2m,1} = 4m^2 + 1, m \geq 1$, such that $\varphi_m$ has exactly two nodal domains.

Let $S$ denote the square $[0, \pi] \times [0, \pi]$. For $p, q \in \mathbb{N} \cup \{0\}$, denote

$$
\varphi_{p,q} = \sin(px) \sin(qx)
$$

as the eigenfunction corresponding to eigenvalue $\lambda_{p,q} := p^2 + q^2$ of $S$. Consider the family of eigenfunctions $\varphi_{2m,1} + c\varphi_{1,2m}$ corresponding to eigenvalue $4m^2 + 1$. Stern claimed that $\varphi_{2m,1} + \varphi_{1,2m}$ is given by Figure 8(A).

---

For a chronology and proper accreditation of the results of Lewy and Stern, see the discussion in the paper [BH].
As $\epsilon$ moves away from 1 all the nodal crossings disappear at the same time and the nodal sets look like Figure 8 of [St]. So for every $m \in \mathbb{N}$, there exist $\varphi_m = \varphi_{2m,1} + \epsilon(m)\varphi_{1,2m}$ such that $\varphi_m$ has exactly two nodal domains. Stern in [St] gives a geometric idea of the proof but missing few details which were later completed by Bérard and Helffer in [BH].

Now we start proving Theorem 1.13, which is an extension of Theorem 2.17 above.

**Proof.** Consider any $k$-connected domain $D \subset \mathbb{R}^2$. Scaling $D$ by $\epsilon > 0$ we call it $\epsilon D$. Choosing $\epsilon$ such that

$$\frac{\lambda_1(D)}{4(N+1)^2 + 1} < \epsilon^2 < \frac{\lambda_1(D)}{4N^2 + 1},$$

we have

$$\lambda_{2m,1} < \lambda_1(\epsilon D) \quad \text{for all} \quad m \leq N, \quad \text{and} \quad \lambda_1(\epsilon D) < \lambda_{2(N+1),1}. \quad (2.12)$$

Now fixing $\epsilon$, we construct a family of dumbbells $\Omega_\delta$ as described in Subsection 2.3 with $\Omega_1 = S$ and $\Omega_2 = \epsilon D$ (see Figure 9).

Using (2.12), arranging the eigenvalues of $S$ and $D$ in non-decreasing order, we have

$$\lambda_{1,1} < \lambda_{2,1} \leq \cdots \leq \lambda_{4,1} \leq \cdots \leq \lambda_{2N,1} < \lambda_1(\epsilon D) \leq \cdots .$$

Let $p_1, p_2, \ldots, p_N$ be the indices corresponding to the eigenvalues $\lambda_{2,1}, \lambda_{4,1}, \cdots \lambda_{2N,1}$ respectively. Let $\varphi_{p_j,\epsilon}$ denote the $p_j$-th eigenfunction of $\Omega_\delta$ corresponding to eigenvalue $\lambda_{p_j,\delta}$. From the limiting conditions as described in Subsection 2.3 we have that $\lambda_{p_j,\delta} \to \lambda_{2j,1}$ and $\varphi_{p_j,\delta} \to \varphi_{p_j,0}$ as $\delta \to 0$ where

$$\varphi_{p_j,0} = \begin{cases} \varphi_j & \text{on } \Omega_1 = S, \\ 0 & \text{otherwise}. \end{cases}$$
Then we have that for each \( j \in \{1, \cdots, N\} \), there exists \( \delta_j \) such that for all \( \delta < \delta_j \), \( N_{\varphi_{p_j, \delta}} \) is completely contained within a \( \delta_j \) neighbourhood of \( N_{\varphi_j} \). As discussed by Stern, \( \varphi_{p_j, 0}|_S = \varphi_j \) satisfies \((SP)\) on \( S \) and divides \( S \) into two nodal domains (see Figure 8.B). Following the techniques in Theorem 1.10 and Proposition 2.5, we have that \( N_{\varphi_{p_j, \delta}} \) is diffeomorphic to \( N_{\varphi_j} \) and hence divides \( \Omega_{\delta_j} \) into exactly two parts.

Now choosing \( \Omega = \Omega_\delta \) where \( \delta = \min\{\delta_1, \ldots, \delta_N\} \) does the job. \(\Box\)

3. Geometric properties of nodal sets

3.1. Opening angles nodal domains in the interior and the boundary. It was shown by Melas [M] that the nodal domain for the second Dirichlet eigenfunction which intersects the boundary \( \partial \Omega \) cannot have an “opening angle” of 0 or \( \pi \) at the point of intersection. Here, we provide a generalisation of this result from a different perspective, one that was introduced in [GM1].

3.1.1. Interior cone conditions. In dimension \( n = 2 \), a well-known result of Cheng [Ch] says the following (see also [St] for a proof using Brownian motion):

**Theorem 3.1.** For a compact Riemannian surface \( M \), the nodal set \( N_{\varphi_\lambda} \) satisfies an interior cone condition with opening angle \( \alpha \gtrsim \frac{1}{\sqrt{\lambda}} \).

Furthermore, in dimension 2, the nodal lines form an equiangular system at a singular point of the nodal set. The idea behind Cheng’s proof is the following: using a local power series expansion due to Bers (see Theorem 3.4 below), near any point of vanishing the eigenfunction “looks like” a homogeneous harmonic polynomial whose degree matches the order of vanishing at that point. If the order of vanishing is \( k \), then in two dimensions such a function would be a linear combination of \( r^k \cos k\theta \) and \( r^k \sin k\theta \). This gives an equiangular nodal junction.

![Figure 10. Four equiangular “rays” from p](image)

The situation is significantly more complicated in higher dimensions. Setting \( \dim M \geq 3 \), we discuss the question whether at the singular points of the nodal set \( N_{\varphi} \), the nodal set can have arbitrarily small opening angles, or even “cusp”-like situations, or the nodal set has to self-intersect “sufficiently transversally”. We observe that in dimension \( n \geq 3 \) the nodal sets satisfies an appropriate “interior cone condition”, and give an estimate on the opening angle of such a cone in terms of the eigenvalue \( \lambda \).

Now, in order to properly state or interpret such a result, one needs to define the concept of “opening angle” in dimension \( n \geq 3 \). We start by defining precisely the notion of tangent directions in our setting.

**Definition 3.2.** Let \( \Omega_\lambda \) be a nodal domain and \( x \in \partial \Omega_\lambda \), which means that \( \varphi_\lambda(x) = 0 \). Consider a sequence \( x_n \in N_{\varphi} \) such that \( x_n \to x \). Let us assume that in normal coordinates around \( x \),
\[ x_n = \exp(r_nv_n), \text{ where } r_n \text{ are non-negative real numbers, and } v_n \in S(T_{x_0}M), \text{ the unit sphere in } T_{x_0}M. \] Then, we define the space of tangent directions at \( x \), denoted by \( S_xN_\varphi \) as

\[ S_xN_\varphi = \{ v \in S(T_{x}M) : v = \lim v_n, \text{ where } x_n \in N_\varphi, x_n \to x \}. \tag{3.1} \]

Observe that there are more well-studied variants of the above definition, for example, as due to Clarke or Bouligand (for more details, see \[ \square \)). With that in place, we now give the following definition of “opening angle”.

**Definition 3.3.** We say that the nodal domain \( \Omega_\lambda \) satisfies an interior cone condition with opening angle \( \alpha \) at \( x \in N_\varphi \subset \partial \Omega_\lambda \) in , if any connected component of \( S(T_{x}M) \setminus S_x\partial N_\varphi \) has an inscribed ball of radius \( \gtrsim \alpha \).

We will use Bers scaling of eigenfunctions near zeros (see \[ \square \)). We quote the version as appeared in \[ \square \], Section 3.11.

**Theorem 3.4** (Bers). Assume that \( \varphi_\lambda \) vanishes to order \( k \) at \( x_0 \). Let \( \varphi_\lambda(x) = \varphi_k(x) + \varphi_{k+1}(x) + \ldots \) denote the Taylor expansion of \( \varphi_\lambda \) into homogeneous terms in normal coordinates \( x \) centered at \( x_0 \). Then \( \varphi_k(x) \) is a Euclidean harmonic homogeneous polynomial of degree \( k \).

We also use the following inradius estimate for real analytic metrics (see \[ \square \]).

**Theorem 3.5.** Let \( (M,g) \) be a real-analytic closed manifold of dimension at least 3. If \( \Omega_\lambda \) is a nodal domain corresponding to the eigenfunction \( \varphi_\lambda \), then there exist constants \( \lambda_0, c_1 \) and \( c_2 \) which depend only on \( (M,g) \), such that

\[ \frac{c_1}{\lambda} \leq \text{inrad}(\Omega_\lambda) \leq \frac{c_2}{\sqrt{\lambda}}, \lambda \geq \lambda_0. \tag{3.2} \]

Since the statement of Theorem \[ \square \] is asymptotic in nature, we need to justify that if \( \lambda < \lambda_0 \), a nodal domain corresponding to \( \lambda \) will still satisfy \( \text{inrad}(\Omega_\lambda) \geq \frac{c_2}{\sqrt{\lambda}} \) for some constant \( c_3 \). This follows from the inradius estimates of Mangoubi in \[ \square \], which hold for all frequencies. Consequently, we can assume that every nodal domain \( \Omega \) on \( S^n \) corresponding to the spherical harmonic \( \varphi_k(x) \), as in Theorem \[ \square \] has inradius \( \gtrsim \frac{1}{\sqrt{x}} \).

Now we start proving Theorem \[ \square \].

**Proof.** Since the eigenvalue \( -\Delta \varphi_\lambda = \lambda \varphi_\lambda \) is satisfied at \( p \), one can check that the proof of Theorem \[ \square \] above still works at \( p \), for all \( p \in M \cup \partial M \). We observe that Theorem \[ \square \] applies to spherical harmonics, and in particular the function \( \exp^*(\varphi_k) \), restricted to \( S(T_{x_0}M) \), where \( \varphi_k(x) \) is the homogeneous harmonic polynomial given by expanding \( \varphi \) at \( p \) in terms of \( x \in M \cup \partial M \) given by Theorem \[ \square \]. Also, a nodal domain for any spherical harmonic on \( S^2 \) (respectively, \( S^3 \)) corresponding to eigenvalue \( \lambda \) has inradius \( \sim \frac{1}{\sqrt{x}} \) (respectively, \( \gtrsim \frac{1}{x^{1/3}} \)).

With that in place, it suffices to prove that

\[ S_{x_0}N_\varphi \subseteq S_{x_0}N_{\varphi_k}. \tag{3.3} \]

Now by definition, \( v \in S_{x_0}N_\varphi \) if there exists a sequence \( x_n \in N_\varphi \) such that \( x_n \to x_0, x_n = \exp(r_nv_n) \), where \( r_n \) are positive real numbers and \( v_n \in S(T_{x_0}M) \), and \( v_n \to v \).

This gives us,

\[ 0 = \varphi_\lambda(x_n) = \varphi_\lambda(r_n\exp v_n) \]
\[ = r_n^k\varphi_k(\exp v_n) + \sum_{m>k} r_n^m\varphi_m(\exp v_n) \]
\[ = \varphi_k(\exp v_n) + \sum_{m>k} r_n^{m-k}\varphi_m(\exp v_n) \]
\[ \to \varphi_k(\exp v), \text{ as } n \to \infty. \]

Observing that \( \varphi_k(x) \) is homogeneous, this proves \eqref{3.3}. \[ \square \]
Observe that Theorem 1.14 above tells us that the following two situations in Figure 11 can never happen at the boundary for the nodal set of any eigenfunction (there is nothing specific about the second eigenfunction).

![Figure 11. Impermisssible angle of intersections for any bounded domain in $\mathbb{R}^2$](image)

**Remark 3.6.** In $\dim M = 2$, since any point $p \in \partial M$ satisfies the eigenequation $-\Delta \varphi = \lambda \varphi$, the local expansion of Bers is true on the boundary as well. Then using the above ideas of Cheng on the boundary, we have that if $p \in \partial M$ has $k^{\text{th}}$ order of vanishing then $H_\varphi$ forms an equiangular junction at $p$ with respect to the tangent at $p$.

### 3.2. More precise estimates on opening angles

We are now going to investigate in more detail the angle between two nodal hypersurfaces at a point of intersection. In some sense, our results here are going to be higher dimensional analogues of Cheng’s result outlined in Remark 3.6.

Very interestingly, such problems have been investigated from a completely different viewpoint in classical Fourier analysis, namely, the existence of Heisenberg uniqueness pairs. For the sake of completeness, we include a basic definition here:

**Definition 3.7.** Let $M \subseteq \mathbb{R}^n$ be a manifold and $\Sigma \subseteq \mathbb{R}^n$ be a set. We say that $(M, \Sigma)$ is a Heisenberg uniqueness pair if the only finite measure $\mu$ supported on $M$ with Fourier transform vanishing on $\Sigma$ is $\mu = 0$.

As a typical example of the kind of result from HUP that motivates us, we quote the following result:

**Theorem 3.8 (F-BGJ).** Let $n \geq 2$ and $\Omega$ be a domain in $\mathbb{R}^n$ with $0 \in \Omega$. Let $\theta_1, \theta_2 \in S^{n-1}$ be such that $\arccos(\theta_1, \theta_2) \in \pi \mathbb{Q}$. Let $k \in \mathbb{R}$ and let $u$ be a solution of the Laplace–Helmholtz equation on $\Omega$:

$$\Delta u + k^2 u = 0.$$ 

If $u$ satisfies one of the following boundary conditions

$$u = 0 \text{ on } (\theta_1^+ \cap \Omega) \cup (\theta_2^+ \cap \Omega)$$

or

$$u = 0 \text{ on } \theta_1^+ \cap \Omega \text{ and } \partial_{\eta} u = (\theta_2^+ \cap \Omega),$$

then $u \equiv 0$.

As is clear (and also mentioned by the authors in F-BGJ), in dimension $n = 2$ the above theorem follows from results in Ch. We also observe that Theorem 1.14 above also shows that the angle between $\theta_j, j = 1, 2$ cannot be arbitrarily small (depending on the eigenvalue).
Before proceeding with the proof proper, we take the space to make a few cursory comments about the curvature of nodal sets, which are also of independent interest. In dimension $n = 2$, the geodesic curvature of the nodal set $N$ can be calculated as

$$k_g = \frac{1}{2} \nabla_\eta \log |\nabla \varphi|^2,$$

where $\eta$ is the unit normal to the nodal line. In higher dimensions, since the nodal set $N$ is given implicitly as the vanishing set of an eigenfunction $\varphi$, it is well-known that the mean curvature of $N$ is given by the formula

$$K_M = -\nabla \left( \frac{\nabla \varphi}{|\nabla \varphi|} \right),$$

see [Go] for example. On computation it is clear that the mean curvature on the nodal set vanishes if and only if

$$\sum_i \frac{\partial \varphi}{\partial x_i} \cdot \frac{\partial}{\partial \eta} \left( \frac{\partial \varphi}{\partial x_i} \right) = 0,$$

where $\eta$ is the unit normal to the nodal set $N$.

In higher dimensions, the situation is slightly more complicated, and from (3.6) we have the following:

**Proposition 3.9.** The nodal set $N_\varphi$ of a Laplace eigenfunction $\varphi$ satisfies one of the following:

1. The mean curvature of $N_\varphi$ is non-zero or
2. $\partial^2_\eta \varphi = 0$ on an open dense set of $N_\varphi$, where $\eta$ represents the normal to the nodal set.

**Proof.** The proof is based on a direct computation using (3.4) and (3.5), and is straightforward.

Now we begin proving Theorem 3.15. Our proof is a modification of ideas in [F-BGJ].

**Proof.** Recall, we would like to show that if the order of vanishing of $\varphi_\lambda$ at $p$ is $n_0$, then the angle between $M_1$ and $M_2$ at $p$, $\arccos \langle \eta_1, \eta_2 \rangle \in P$, where $p$ lies in the intersection of two nodal hypersurfaces $M_1$ and $M_2$, and

$$P = \left\{ \begin{array}{c} \delta q : q = 1, 2, \cdots n_0, p = 0, 1, \cdots, q \end{array} \right\}.$$

If possible, let $\arccos \langle \eta_1, \eta_2 \rangle \notin P$ where $\eta_i$ is a unit normal to $M_1$ at $p$. Without loss of generality, we can assume that $p = 0$, the origin in $\mathbb{R}^n$. Consider the spherical coordinates in $\mathbb{R}^n$, $(r, \theta, \varphi)$ where $r \geq 0$, $\theta := (\theta_1, \cdots, \theta_{n-2}) \in [0, \pi)^{n-2}$, $\varphi \in [0, 2\pi)$.

It is known that, $\{Y_\alpha : \alpha \in S := \mathbb{N}_0^{n-2} \times \mathbb{Z}\}$ forms a basis of spherical harmonics where

$$Y_\alpha(r, \theta, \varphi) = r^{\alpha_0} \exp (i\alpha_{n-1} \varphi) \tilde{Y}_\alpha(\theta),$$

with $\tilde{Y}_\alpha(\theta) := \prod_{i=1}^{n-2} (\sin \theta_{n-i})^{\alpha_{i+1}} C^{\gamma_i}_{\alpha_i}(\cos \theta_i)$, and $|\alpha| = \alpha_0 + \alpha_{i+1} + \cdots + \alpha_{n-1}$, $\gamma_i = |\alpha| + 1 + (n - i - 1)/2$, where $C^{\gamma_i}_{\alpha_i}$ are the Gegenbauer polynomials. From the orthogonality of $C^{\gamma_i}_{\alpha_i}$, for each $n \geq 1$, notice that the set

$$\left\{ \tilde{Y}(\beta, m) : (\beta, m) \in \mathbb{N}_0^{n-1}, |\beta| + m = n \right\}, \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

is linearly independent.

Consider $V_\epsilon$ be an open ball around 0. Then $M_i \cap V_\epsilon$ can be parametrized in polar coordinates as

$$M_i \cap V_\epsilon = \left\{ (r, \theta, \psi_i(r, \theta)) : 0 \leq r < \epsilon, \theta \in [0, \pi)^{n-2} \right\},$$

where $\psi_i(r, \theta) \in S^1$ and $\psi_i$’s are smooth functions.

Defining $\varphi_i(\theta) := \lim_{r \to \theta} \psi_i(r, \theta)$, from our assumption $\arccos \langle \eta_1, \eta_2 \rangle \notin P$, it follows that $\varphi_1 - \varphi_2 \notin P$. 

**LOW ENERGY LAPLACE EIGENFUNCTIONS**
\[
\varphi_\lambda(r, \theta, \psi(r, \theta)) = r^{n_0} \sum_{m=-n_0}^{n_0} \left( \sum_{|\beta|+|m|=n_0} c_{\beta,m} \tilde{Y}_{\beta,m}(\theta) \right) e^{im\varphi_1} + o(r^{n_0}).
\]
(3.7)

Since, \(\varphi_\lambda = 0\) on \(M_i \cap V_i\), as \(r \to 0\), it follows that,
\[
\sum_{m=-n_0}^{n_0} \left( \sum_{|\beta|+|m|=n_0} c_{\beta,m} \tilde{Y}_{\beta,m}(\theta) \right) e^{im\varphi_1} = 0,
\]
that is
\[
\sum_{|\beta|=n_0} c_{\beta,0} \tilde{Y}_{\beta,0}(\theta) + \sum_{m=1}^{n_0} \left( \sum_{|\beta|+m=n_0} (c_{\beta,m}e^{im\varphi_1} + c_{\beta,-m}e^{-im\varphi_1}) \tilde{Y}_{\beta,m}(\theta) \right) = 0
\]
(3.8)

Since \(\{\tilde{Y}_{(\beta,m)}\}\) is linearly independent, \(c_{\beta,0} = 0\) whenever \(|\beta| = n_0\) and for each \(m = 1, 2, \ldots, n_0\), we have \(n_0\) system of equations
\[
c_{\beta,m}e^{im\varphi_1} + c_{\beta,-m}e^{-im\varphi_1} = 0,
\]
\[
c_{\beta,m}e^{im\varphi_2} + c_{\beta,-m}e^{-im\varphi_2} = 0.
\]
Notice that, the determinant of each of the above systems is \(2i \sin m(\varphi_1 - \varphi_2), m = 1, 2, \ldots, n_0\). If
\[
(\varphi_1 - \varphi_2) \notin \left\{ \frac{p}{q} \pi : q = 1, 2, \ldots, n_0, p = 0, 1, \ldots, q \right\},
\]
then each of above \(n_0\) determinants is non-zero, which forces each \(c_{\beta,m} = c_{\beta,-m} = 0\), which implies that the coefficient of \(r^{n_0}\) is zero. But this contradicts the fact that \(\varphi_\lambda\) has \(n_0\) order of vanishing at 0. So,
\[
\arccos (\eta_1, \eta_2) \in P.
\]

\[\square\]

**Remark 3.10.** Recall the celebrated result of [DF] that any \(\lambda\)-eigenfunction \(\varphi_\lambda\) vanishes to at most order \(c(M, g)\sqrt{\lambda}\) for any point in \(M\). Also recall, from [HS] that, the nodal set contains smooth \((n-1)\) dimensional submanifolds having finite \((n-1)\)-dimensional measure in each compact subset of \(\Omega\) and a closed countable \((n-2)\)-rectifiable set. Then, using our result above, we have that whenever two nodal hypersurfaces intersect, the admissible angles between such intersecting hypersurfaces is from the set
\[
P = \left\{ \frac{p}{q} \pi : q = 1, 2, \ldots, \lfloor \sqrt{\lambda} \rfloor, p = 0, 1, \ldots, q \right\}.
\]
(3.9)

Also, using Theorem [1.14] or its interior case as in [GMI], we can rule out the cases when \(p = 0, q\) for every \(q = 1, 2, \ldots, \sqrt{\lambda}\). Then one sees that the minimum angle (in the sense of Theorem [1.15]) between two nodal hypersurfaces is \(\frac{1}{\sqrt{\lambda}}\).

To sum up the discussion so far in this section: consider a point \(x \in N_\varphi\). Then the opening angle of a nodal domain at \(x\) will in general be given by Definition [3.3]. However, if \(x\) happens to lie at the intersection of some nodal hypersurfaces, then the angle between any pair of such nodal hypersurfaces will come from the set \(P\) in (3.9).
3.3. Payne property, eigenvalue multiplicity and genericity of planar simply connected domains. It now seems that the Payne conjecture neatly splits into a local component and a global component. The local aspect deals with the angle at the point of intersection of the first nodal set with the boundary and the global aspect depends on the multiplicity of the second eigenvalue. For a collection of domains $\mathcal{D}$, consider the following two properties.

(L) For any $D \in \mathcal{D}$, if $N_{\varphi_2}$ intersects the boundary $\partial D$, then the possible angle at the point of intersection(s) is finitely many.

(G) If $\frac{\partial \varphi_2}{\partial \eta} \geq 0$ on the boundary $\partial D$ then $\lambda_2(D)$ is simple for every $D \in \mathcal{D}$, where $\eta$ denotes the outward normal to the boundary.

From what has gone above, for any bounded domain $\Omega \subset \mathbb{R}^2$ with $C^2$-boundary, we have that (L) is true. More specifically, if $N_{\varphi_2}$ intersects the boundary $\partial \Omega$ at exactly one point then the angle at the point of intersection is $\pi/3$ and if it intersects at exactly two points then the nodal set is normal to the boundary.

So, proving strong Payne property for simply connected domain with $C^2$ boundary in $\mathbb{R}^2$ boils down to the following

Theorem 3.11. Let $\mathcal{D}$ be a collection of bounded domains with $C^2$-boundary that satisfy (G) and let $\Omega, \Omega' \in \mathcal{D}$ be such that there is a one-parameter family of smooth deformations $D_t \in \mathcal{D}$ with $D_0 = \Omega$ and $D_1 = \Omega'$.

If $\Omega'$ satisfies the strong Payne property so does $\Omega$.

Proof. From the hypothesis of our statement we have that $D_1 = \Omega'$ satisfies the strong Payne property. If possible, let $\Omega$ does not satisfy (SP). Let

$$t_0 = \sup\{t \in [0, 1) : D_t \text{ does not satisfy (SP)}\}.$$
Let \( \varphi_t \) denote a normalized second eigenfunction corresponding to the second eigenvalue \( \lambda_t \) of \( D_t \). There exists a sequence \( \{t_i\} \to t_0 \) (it can be a constant sequence as well) such that \( \frac{\partial \varphi_t}{\partial \eta} \geq 0 \) on \( \partial D_{t_i} \) for all \( i \). Also, consider any sequence \( \{t'_i\} \setminus t_0 \). Note that \( \varphi_{t'_i} \) must intersect the boundary \( \partial D_{t'_i} \) exactly twice. Then using standard elliptic estimates, there exists subsequences \( \varphi_{t_j} \) and \( \varphi'_{t'_j} \) converging to \( \varphi_0 \) and \( \varphi'_0 \) respectively where \( \varphi_0 \) and \( \varphi'_0 \) are second eigenfunctions of \( D_{t_0} \).

From the limiting conditions, it is clear that \( \frac{\partial u_0}{\partial \eta} \geq 0 \) on \( \partial D_{t_0} \) and \( \varphi'_0 \) intersects \( \partial D_{t_0} \) at least once. Since \( D \) satisfies \( G \), we have that \( u_0 = u'_0 \). Combining all of these we have that the nodal set of \( \varphi_0 \) intersects \( \partial D_{t_0} \) at exactly one point. From the discussion above in Remark 3.6 for \( t = t_0 \), the angle at the point of intersection is \( \pi/3 \), whereas for \( t > t_0 \), the angle at each point of intersection at the boundary is \( \pi/2 \). No angle in between is possible, which contradicts continuity.

Although the above proof is for dimension 2, we have the same observation for higher dimensions as well. Now we give a quick proof of Corollary 1.16.

**Proof.** Note that from Theorem 2.4 of [Lin], \( G \) is true for convex domains in \( \mathbb{R}^2 \). Let

\[ \mathcal{D} = \{ D \subset \mathbb{R}^2 : D \text{ is convex and } \partial D \text{ is smooth} \}. \]

Take \( \Omega' = B(0,1) \), ball of radius 1 centered at 0, and \( \Omega = D \in \mathcal{D} \) and consider the linear deformation from \( \Omega \) to \( \Omega' \).

Note that in Theorem 3.11 we can make do without the condition \( G \) if we assume that for all \( t \in [0,1] \), \( D_t \) has simple second eigenvalue. From the fact that generic spaces have simple spectrum (see [Al],[U]) we have the heuristic that “almost all” simply connected domains will have simple spectrum which would allow us to perturb any simply connected domain to a convex domain is such a way that at all deformation stages the second eigenvalue is simple. But such an endeavour would entail a significantly deeper understanding of the domains with repeated spectrum in the moduli space of all domains. To wit, one needs to investigate the path connectedness properties of the residual set of domains of simple spectra coming from tranversality considerations.

4. Perturbation theory, low energy phenomena and spectral gaps

We study variation of Dirichlet Laplace spectrum and corresponding Laplace eigenfunctions with \( C^2 \) variations of bounded Euclidean domains, particularly those domains \( \Omega \) which are critical points of \( \lambda_1 - \lambda_2 \), where \( i \geq 2 \).

Let the smooth deformation space of \( \Omega \) be given by a Banach manifold \( \mathcal{B} \). First, we prove the following:

**Theorem 4.1.** The set of points inside \( \mathcal{B} \) (each represented by a perturbation of our starting domain \( \Omega \)) such that the Dirichlet Laplacian has simple spectrum is a residual set.

We note that Theorem 4.1 is not new, for example see Example 3, Section 4 of [U]. But we give our own proof, based on the perturbation formalism of [GS].

Results of the nature of Theorem 4.1 are ultimately based on transversality phenomena (as illustrated in [U]). Loosely speaking, they can be considered infinite dimensional analogues of the following statement: generically, all symmetric matrices have non-repeated eigenvalues. At a more basic level, an equivalent statement is the fact that single variable polynomials generically have non-repeated roots.

4.1. **Proof of Theorem 4.1.** The topic of variation of spectra under perturbation has a long history starting with the analytic perturbation theory of Kato (see [Ka]). In the case of rather generic families of elliptic operators, see pioneering work in [A] and [U]. In case the perturbation is non-generic, such results have been recently studied in, for example, [HJ1],[HJ2],[Mu] etc. In this note, we give a slight variant of a proof for Theorem 4.1. To set up the stage, we start.
by considering a bounded domain $\Omega \subset \mathbb{R}^n$, and consider a vector field $V$ defined on $\mathbb{R}^n$, whose coordinates we denote by $(V_1, ..., V_n)$, and whose regularity we assume to be $C^2$ for immediate purposes. Now, consider the perturbation of the domain $\Omega$ along the vector field $V$ to the domain $\Omega_\varepsilon$, defined by $\{x^\varepsilon = x + \varepsilon V : x \in \Omega\}$. We wish to study the variation of the eigenequation

$$-\Delta \varphi = \lambda \varphi$$

(4.1)

along the parameter $\varepsilon$. However, to fit the language of perturbation theory, instead of dealing with a one-parameter family of domains, it is much more convenient to pull back all $\Omega_\varepsilon$ to the original domain $\Omega$, so that we get a one-parameter family of elliptic PDEs on $\Omega$ whose coefficients are dependent on $\varepsilon$. This lands us in the familiar framework of a family of self-adjoint operators with common domain of definition varying over a Banach manifold. Upon computation following [GS] (see Sections 4, 5 and particularly pp. 299, Section 6), we see that the eigenequation

$$-\Delta_\varepsilon \varphi_\varepsilon = \lambda_\varepsilon \varphi_\varepsilon$$

on $\Omega_\varepsilon$, when pulled back to $\Omega$, becomes

$$A_\varepsilon u = \sum_{j,k} -\partial_k (J \beta_{kj} \partial_j u) = \lambda_\varepsilon Ju,$$

(4.2)

with the Dirichlet boundary condition being preserved, and where $J$ is the determinant of the Jacobian matrix of the transformation $x \mapsto x^\varepsilon$, and $\beta_{jk} = \sum l \frac{\partial x_j}{\partial x_l} \frac{\partial x_k}{\partial x_l}$. To see this, write

$$(A_\varepsilon u, v)_{L^2(\Omega)} = (-\Delta_\varepsilon u^\varepsilon, v^\varepsilon)_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} -\Delta_\varepsilon u^\varepsilon v^\varepsilon dx^\varepsilon = \sum_k \int_{\Omega_\varepsilon} \partial_{x^\varepsilon_k} u^\varepsilon \partial_{x^\varepsilon_k} v^\varepsilon dx^\varepsilon = \sum_{i,j,k} \int_{\Omega} \partial_{x_i} \frac{\partial x_j}{\partial x_k} \partial_{x_i} \frac{\partial x_j}{\partial x_k} J dx$$

$$= \int_{\Omega} -\partial_{x_i} (J \frac{\partial x_i}{\partial x_k} \frac{\partial x_j}{\partial x_k} \partial_{x_j} u) v dx.$$

The main idea behind the computation is that since

$$\frac{\partial x^\varepsilon_j}{\partial x_i} = \delta_{ij} + \varepsilon \frac{\partial V_j}{\partial x_i}$$

up to first order errors in $\varepsilon$, we can write that

$$\frac{\partial x^\varepsilon_j}{\partial x^\varepsilon_k} = \delta_{jk} - \varepsilon \left( \frac{\partial V_j}{\partial x_k} + \frac{\partial V_k}{\partial x_j} \right) + O(\varepsilon),$$

whereas $J$ can be expressed as

$$J = 1 + \varepsilon \left( \sum_j \frac{\partial V_j}{\partial x_j} \right) + O(\varepsilon),$$

and

$$\beta_{jk} = \delta_{jk} - \varepsilon \left( \frac{\partial V_j}{\partial x_k} + \frac{\partial V_k}{\partial x_j} \right) + O(\varepsilon),$$

and $\beta_{ik}$ has a power series expansion

$$\beta_{ik} = \delta_{ik} + \sum_{i=1}^{\infty} \varepsilon^i \beta_{ij}^i,$$

where $\beta_{ij}^1 = -\left( \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right)$, as mentioned before. For details on the above, see [GS], Section 6.

All told then, the perturbation $A_\varepsilon$ can be expressed as

$$A_\varepsilon u = -\Delta u + \varepsilon \left( \sum_{j,k} \partial_k ((\partial_k V_j + \partial_j V_k) \partial_j u) + \frac{1}{2} \sum_j \partial_j u \Delta (V_j) \right) + O(\varepsilon).$$

(4.3)
Now we bring in the Sard-Smale transversality formalism used by Uhlenbeck. We first quote the theorem:

**Theorem 4.2.** Let $\Phi : H \times B \to E$ be a $C^k$ map, where $H, B$ and $E$ are Banach manifolds with $H$ and $E$ separable. If $0$ is a regular value of $\Phi$ and $\Phi_0 := \Phi(., b)$ is a Fredholm map of index $< k$, then the set $\{ b \in B : 0 \text{ is a regular value of } \Phi_0 \}$ is residual in $B$.

Here, we wish to check Theorem 4.2 for our domain perturbations in the particular setting that $H = E = \mathcal{D}(\Delta)$ and $B$ is the collection of parameters for domain perturbation. For starters, all $A_\varepsilon$ are self-adjoint by the Kato-Rellich theorem, being relatively bounded perturbations of $A_0 = \Delta_\Omega$. Also, ellipticity in such cases implies the Fredholm property, as is well-known.

Now, if $0$ is not a regular value as above, we have that for all perturbations given by $V$ and $\varepsilon$ small, we have Laplace eigenfunctions $\varphi, \psi$ ($\psi$ corresponding to the eigenvalue $\lambda$) such that

\[
-\int_\Omega \sum_{jk} ((\partial_k V_j + \partial_j V_k) \partial_j \varphi \partial_k \psi) + \frac{\varphi \psi}{2} \Delta \left( \sum_j \partial_j V_j \right) = -\int_{\partial \Omega} \frac{\partial \varphi}{\partial \eta} \frac{\partial \psi}{\partial \eta} \left( \sum_k V_k \eta_k \right) = 0.
\]

This basically means that by Holmgren’s uniqueness theorem, $\varphi$ and $\psi$ are identically zero, establishing our claim.

### 4.2. Discussion on some variants of Theorem 4.1

Above we discussed the generic spectral simplicity of Euclidean domains in $\mathbb{R}^n$. However, it is a valid question to ask what kind of generic properties the spectrum has provided the word “generic” is constrained on a much smaller moduli space. For example, we refer the readers to recent work in [HJ1,HJ2]. Here, we outline a short result to illustrate the line of thought in these later works. Consider the family $\mathcal{F}$ of all domains $\Omega$ in the plane whose boundary can be written as $\partial \Omega = \delta \cup C$, where

1. $\delta$ is a disjoint collection of straight line segments and $C$ is a disjoint collection of strictly curved real analytic line segments, and
2. $\delta \cap C$ is a finite collection of points.

**Claim 4.3.** The subcollection of domains in $\mathcal{F}$ which have simple Dirichlet spectrum is a residual set.

**Proof.** We would like to show that given any $k$, the subfamily of $\mathcal{F}$ whose first $k$ Dirichlet eigenvalues are non-repeated form a residual set. Since the intersection of countably many residual sets is residual, the sub-family of $\mathcal{F}$ that has simple Dirichlet spectrum is also residual.

Since any two members $\Omega_1, \Omega_2$ of $\mathcal{F}$ which satisfy $|\delta_1| = |\delta_2|$ and $|C_1| = |C_2|$ can be joined by a one-parameter family of real-analytic maps, and the Rayleigh quotient varies analytically under analytic perturbations, one sees immediately that given $i < j \leq m$, it is enough to find one $\Omega$ in this family for which $\lambda_i \neq \lambda_j$.

Now, take any such member of this family, and find a rectangle $R$ with simple Dirichlet spectrum (it is easy to check that for a rectangle with side lengths $l_1, l_2$ this happens if and only if $(\frac{l_2}{l_1})^2 \notin \mathbb{Q}$). Now, start by inscribing $\Omega$ inside $R$ (by scaling if necessary), and consider a one-parameter family of analytic perturbations $\Omega_t$ which converge to $R$. By spectral convergence, it is clear that for any given $m$, there exists a large enough $t$ such that the first $m$ Dirichlet eigenvalues of $\Omega_t$ are non-repeated.

$$\square$$

### 4.3. Fundamental gap, narrow convex domains and small perturbations

Now we take a look at the problem of minimising the fundamental gap. Recall the main result from [AC]: for any convex domain $\Omega \subset \mathbb{R}^n$,

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{D^2},$$

where $D = \text{diam } \Omega$. Now, the following question is natural:
Question 4.4. Is the above inequality saturated by some domain?

The popular belief in the community seems that it is not, and any infimising sequence for $\lambda_2 - \lambda_1$ (under the normalisation $D = 1$) should degenerate to a line segment. In particular, the correct regime to look for in the search for minimisers is the class of narrow convex domains. This problem seems quite difficult, as standard precompactness ideas (e.g., see recent work in [MTV, KL]) do not apply directly. Also, it is quite resistant to perturbative techniques, as generic perturbations (even small ones) might destroy convexity. In addition, the problem seems quite sensitive to the class of domains: it might demonstrate a markedly different behaviour if the overall class of domains is changed, for example see [LR].

Recall that $C$ denotes the class of $C^2$-convex planar domains and $\mathcal{F}$ denotes the class of small $C^2$-perturbations of $C^2$-convex planar domains. Also, $\mathcal{F}$ stand for either class of domains. Now we begin proving Theorem 1.18. We finish the proof in two steps. First, we prove the following:

Theorem 4.5. Let $\Omega \in \mathcal{F}$ with diameter $D = 1$ and inner radius $\rho$ which minimises the fundamental gap functional $\lambda_2 - \lambda_1$ in $\mathcal{F}$. There exists a universal constant $C \ll 1$ such that if $\rho \leq C$, then $\lambda_2(\Omega)$ is not simple.

Proof. Recall the Hadamard formula (see Section 2.5.2 of [He]) which expresses the evolution of Laplace spectrum with respect to perturbation of a domain $\Omega$ by a vector field $V$:

$$\lambda_k'(0) = -\int_{\partial \Omega} \left( \frac{\partial \varphi}{\partial \eta} \right)^2 V \cdot \eta \, dS. \quad (4.4)$$

Suppose $\lambda_i, \lambda_j$ are Dirichlet eigenvalues of $\Omega$ with corresponding eigenfunctions $\varphi_i, \varphi_j$ respectively. If $\lambda_j - \lambda_i$ considered as a function of domains has a critical point at a domain $\Omega$, then we must have that

$$(\lambda_j - \lambda_i)'(0) = -\int_{\partial \Omega} \left( \left( \frac{\partial \varphi_j}{\partial \eta} \right)^2 - \left( \frac{\partial \varphi_i}{\partial \eta} \right)^2 \right) V \cdot \eta \, dS, \quad (4.5)$$

for all perturbation vector fields $V$. Now we specify to the special case $j = 2, i = 1$, and the above calculation with the Hadamard formula holds true under the assumption that $\lambda_2$ is simple.

Now, consider a small perturbation vector field $V$ such that $V = 0$ at $x_1, x_2$ (see diagram below) and $V \cdot \eta$ is non-sign-changing away from $x_1, x_2$. Note that the aforementioned $V$ constrains the diameter to be fixed along the perturbation. Then, this implies that

$$\frac{\partial \varphi_1}{\partial \eta} = \frac{\partial \varphi_2}{\partial \eta} \quad \text{on} \quad \partial \Omega \quad \text{away from} \quad \{x_1, x_2\}.$$

Using the maximum principles and the Hopf Lemma, we find that without loss of generality, $\frac{\partial \varphi_1}{\partial \eta} > 0$ on $\partial \Omega$. Moreover, since convex domains or their small perturbations satisfy the strong Payne property (as proved in [M, MS]), using Lemma 1.2 of [Lin] there exist exactly two points $p_1, p_2 \in \partial \Omega$ such that

$$\varphi_2(p_1) = \varphi_2(p_2) = 0, \quad \text{and} \quad \frac{\partial \varphi_2}{\partial \eta}(p_1) = \frac{\partial \varphi_2}{\partial \eta}(p_2) = 0.$$

Figure 14. Small perturbation of a narrow convex domain

From work in [J1, J2] (and perturbation arguments based on [MS]), it is known that $p_1, p_2$ cannot be near $x_1, x_2$ once the domain $\Omega$ is long and narrow enough (which is encoded in the
statement by the universal constant $C$). This is a contradiction since

$$0 = \left| \frac{\partial \varphi_2}{\partial \eta} (p_i) \right| = \left| \frac{\partial \varphi_1}{\partial \eta} (p_i) \right| > 0.$$  

\hfill \Box

**Question 4.6.** The following interesting question comes up in connection to the proof of the last theorem. On a domain $\Omega$, can there be two Dirichlet eigenfunctions (corresponding to different eigenvalues) such that they also have the same Neumann data? One is tempted to speculate that such an event should not happen unless $\Omega$ is a ball. As pointed out by Antoine Henrot, the first and sixth eigenfunctions on the planar disc are both radially symmetric, so they can be scaled to have the same Neumann data. This is in turn related to the famous Schiffer conjecture.

We augment the above result by the following observation.

**Theorem 4.7.** Let $\Omega \subset \mathbb{R}^n$ be a $C^2$-domain. Assume that $\Omega$ has a multiple eigenvalue of the Dirichlet Laplacian

$$\lambda_{k+1}(\Omega) = \lambda_{k+2}(\Omega) = \ldots = \lambda_{k+m}(\Omega).$$

Then for each fixed $1 \leq l \leq m$ there exists a deformation field $\Omega_t$ passing through $\Omega_0 := \Omega$ generated by a $C^2$-vector field $V$ such that for small enough $t$,

$$\lambda_{k+1}(\Omega_t) < \lambda_{k+1}(\Omega_0), \ldots, \lambda_{k+l}(\Omega_t) < \lambda_{k+l}(\Omega_0),$$

and

$$\lambda_{k+l+1}(\Omega_t) > \lambda_{k+l+1}(\Omega_0), \ldots, \lambda_{k+m}(\Omega_t) > \lambda_{k+m}(\Omega_0).$$

Furthermore, we can ensure that

$$|\Omega_0| = |\Omega_t|.$$  

Suppose $\Omega_0 \in \mathcal{P}$ is a long narrow domain as in Theorem 4.5 above. Then, we can additionally ensure that $\Omega_t \in \mathcal{P}$ and

$$\text{diam}(\Omega_0) = \text{diam}(\Omega_t).$$

The proof is based on some ideas in Lemma 1 of [HO].

**Proof.** We begin by observing that because of the existence of multiple eigenvalues, $\lambda_{k+p}(\Omega_t)$ is not differentiable at $t = 0$ in the usual Frechet sense, but there is a nice formula giving directional derivatives, in the sense of the limit

$$\frac{\lambda_{k+p}(\Omega_t) - \lambda_{k+p}(\Omega_0)}{t}$$

as $t \to 0$. Such directional derivatives are precisely the eigenvalues of the $m \times m$-matrix

$$M = \left( - \int_{\partial \Omega} \frac{\partial u_i}{\partial \eta} \frac{\partial u_j}{\partial \eta} V.\eta \, d\sigma \right), \quad p = 1, 2, \ldots, m, \quad (4.6)$$

where $u_i, 1 \leq i \leq m$ denotes the eigenspace for the repeated eigenvalue $\lambda_{k+1}$.

Let us consider points $A_1, A_2, \ldots, A_m \in \partial \Omega$ the choice of which will be explained below. Also, consider a deformation vector field $V$ such that $V.\eta = 1$ in a $\varepsilon$-neighbourhood of $A_1, A_2, \ldots, A_l$ and $V.\eta = -1$ in a $\varepsilon$-neighbourhood of $A_{l+1}, \ldots, A_m$, and regularized in a $2\varepsilon$-neighbourhood around each such point maintaining $|\Omega_0| = |\Omega_t|$. To preserve the diameter also, one just needs to choose the points $A_j$ sufficiently away from $x_1, x_2$ (see the figure above).

By (4.6) above, it suffices to prove that the symmetric matrix $M$ has signature $(l, m - l)$. When $\varepsilon \to 0$, $M$ converges to the matrix

$$M = \left( - \sum_{k=1}^{l} \frac{\partial u_i}{\partial \eta} (A_k) \frac{\partial u_j}{\partial \eta} (A_k) + \sum_{k=l+1}^{m} \frac{\partial u_i}{\partial \eta} (A_k) \frac{\partial u_j}{\partial \eta} (A_k) \right).$$

Consider the column vectors $v_{A_k} := (\frac{\partial u_1}{\partial \eta} (A_k), \ldots, \frac{\partial u_m}{\partial \eta} (A_k))^T$. Note that $M = V \cdot W$, where

$$V = (v_{A_1}, \ldots, v_{A_m}) \quad \text{and} \quad W = (-v_{A_1}, \ldots, -v_{A_l}, v_{A_{l+1}}, \ldots, v_{A_m})^T.$$
It is enough to ensure that the vectors \( \{v_{Ak} : k = 1, \ldots, m\} \) are linearly independent. Then the signature of \( M \) is \((l, m - l)\).

Without loss of generality, let there exist fixed \( \{A_1, \ldots, A_{m-1}\} \) away from \( x_1, x_2 \) such that \( \{v_{Ak} : k = 1, \ldots, m - 1\} \) are linearly independent. Let \( v_{Ak} := (\alpha_{ik})^T = (\frac{\partial m}{\partial \eta}(A_k), \ldots, \frac{\partial m}{\partial \eta}(A_k))^T \) for \( k = 1, \ldots, m - 1 \). If possible, let for any choice of \( A_m \not \in \{A_1, \ldots, A_{m-1}, x_1, x_2\} \) there exist constants \( C_1(A_m), \ldots, C_{m-1}(A_m) \) not all zero such that

\[
v_{Am} = C_1(A_m)v_{A_1} + \cdots + C_{m-1}(A_m)v_{A_{m-1}}.
\]

This gives a system of \( m \) linear equations with \( m - 1 \) variables

\[
\frac{\partial u_k}{\partial \eta}(A_m) = \alpha_{1k}C_1(A_m) + \cdots + \alpha_{(m-1)k}C_{m-1}(A_m), \quad k = 1, \ldots, m.
\]

Solving for \( C_1(A_m), \ldots, C_{m-1}(A_m) \) using the last \( m - 1 \) equations and replacing in the first equation, on an open set \( S \subset \partial \Omega \) away from \( x_1, x_2 \), we have that

\[
\sum_{p=1}^{m} c_p \frac{\partial u_p}{\partial \eta} = \frac{\partial}{\partial \eta} \left( \sum_{p=1}^{m} c_p u_p \right) = 0 \quad \text{on } S.
\]

This is a contradiction from Hölmgren’s uniqueness theorem.

The following question seems interesting:

**Question 4.8.** Could we also ensure that \( \lambda_{k+l}(\Omega_t) = \lambda_{k+l}(\Omega_0) < \lambda_{k+l+1}(\Omega_0) \) at the expense of changing the volume of \( \Omega_t \)?

Now, if we observe the way that the vector field was chosen in the proof of Theorem 4.7, it is clear that if one wants to fix only the diameter, one can choose such a \( V \) easily such that \( \lambda_1(\Omega_t) > \lambda_1(\Omega_0) \) for small enough \( t \). This reduces the gap even further, contradicting that \( \Omega_0 \) is a minimiser.

Finally, putting Theorems 4.5 and 4.7 together, we conclude the proof of Theorem 1.18

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