THE BAGGER-LAMBERT MODEL AND TYPE IIA STRING THEORY

JACK MORAVA

ABSTRACT. Notes from a talk at the Workshop on Geometry, Topology, and Physics at the University of Pittsburgh, 14-15 May 2014

We conjecture the existence of a ‘compactified’ version of Fukaya’s homology for symplectic manifolds, which carries a canonical 2-Gerstenhaber algebra structure. This may help to understand the 2-Lie algebra structure involved in models [2] for interacting D-branes.

1. INTRODUCTION

This tentative sketch was inspired by a very interesting recent review article [2] about multiple D-branes in M-theory. There I learned, for example, that:

- the low-energy (‘supergravity’) limit of M-theory involves geometric objects: ambient space-time, branes, . . . ; but

- M-theory is thought to have no coupling constants, and hence has no natural candidate for a perturbative approximation.

- Nevertheless, it is thought to specialize to the classical string theories (type I, IIA, IIB, . . . ) in the right circumstances.

The Bagger-Lambert model (which is currently apparently not particularly fashionable) aspires to accommodate wrapping of branes and related phenomena, which may lack any very familiar interpretation in geometric topology: branes, as they appear in M-theory, are analytic objects, which makes intuitions about issues such as transversality problematic.

The supergravity limit of the BL model is formulated in terms of Lie 2-algebras [HW], involving a triple bracket ⟨−, −, −⟩ satisfying an analog of the Jacobi identity. Together with a more classical graded-commutative algebra, this defines a 2-analog of a Gerstenhaber algebra.
The Bagger-Lambert-Gustavsson action is

\[
\text{something like an 11D supergravity Lagrangian} + \\
\text{interaction terms like } |(X, X, X)|^2.
\]

In interesting examples the Lie 2-algebra structures can be reinterpreted in terms of classical Lie algebras, resulting in more familiar-looking models involving Chern-Simons-type lagrangians for (perhaps unexpected) combinations of gauge groups.

The realization that many classical 2D conformal field theories (eg associated to free loopspaces of manifolds) have Gerstenhaber algebra structures arising from the homology of an action of the topologists’ little 2-disks operad [12 §7.4] profoundly affected the later developments in such theories [14 §2.4].

**The aim of this talk** is to show how certain generalized Type IIA (ie symplectic Fukaya) models manifest natural Gerstenhaber 2-algebra structures (arising from certain underlying operad actions), and to suggest that this may reflect an action of some kind of homotopy Gerstenhaber 2-algebra structure on the underlying algebras of differential forms on the space-time background.

\section*{2. Gromov-Witten invariants in Type IIB string theory and (small) quantum cohomology}

\subsection*{2.1} The Deligne-Knudsen-Mumford moduli stack $\overline{M}_{g,n}(\mathbb{C})$ of genus $g$ stable (ie with 0-dimensional Lie algebra of automorphisms) complex algebraic curves, marked with $n$ distinct smooth points ($n \geq 3$ if $g = 0$, $\geq 1$ if $g = 1$), has a canonical stratification indexed by certain abstract weighted (‘modular’) connected graphs (with $n$ external vertices). When $g = 0$ these are (abstract, unrooted) trees.

Gluing two such curves together at chosen marked points defines morphisms

\[
\overline{M}_{g,n+1} \times \overline{M}_{h,1+m} \to \overline{M}_{g+h,n+m}
\]

which make the collection \{\$\overline{M}_{g,n}\$\} into a generalized (‘modular’) operad; but this sketch will be concerned with the classical operad defined by the subcollection \{\$\overline{M}_{0,n+1}\$\} ($n \geq 2$) and its suitably associative maps

\[
\overline{M}_{0,n+1} \times \prod_{1 \leq k \leq n} \overline{M}_{0,i_k+1} \to \overline{M}_{0,\sum i_k+1}
\]
corresponding to the grafting of rooted trees). These spaces have torsion-free homology; moreover
1) forgetting to distinguish a marked point makes both of these (generalized) operads into cyclic operads [10], and
2) without new ideas about \( \overline{M}_{0,2} \), these operads are non-unital [16].

2.2 Now suppose that \( V \) is a complex projective smooth algebraic variety, 'convex' in a certain sense; then there are [BM . . . ] moduli stacks (or orbifolds) \( \overline{M}_{g,n}(V) \) of stable (ie with 0-dimensional Lie algebras of automorphisms [13]) curves
\[
\phi : C_{g,n} \rightarrow V
\]
in \( V \), together with gluing maps generalizing the case \( V = \text{pt} \) above, as well as evaluation morphisms
\[
GW : \overline{M}_{g,n}(V) \rightarrow \overline{M}_{g,n} \times V^n.
\]
The Gromov-Witten invariant
\[
GW_{g,n}(V) \in H_*(\overline{M}_{g,n} \times V^n, \Lambda)
\]
is the cohomology class (with coefficients in the rational group ring \( \Lambda = \mathbb{Q}[H_2(V,\mathbb{Z})] \), perhaps completed) defined by this (locally algebraic) cycle, with its components weighted by their degrees \( d(\phi) \in H_2(V,\mathbb{Z}) \).

2.3 It will often be useful below to interpret a map \( A \rightarrow X \times Y \) as a geometric correspondence \( A : X \rightarrow Y \). In the category of smooth compact oriented manifolds and maps, Poincaré duality defines an associated homomorphism
\[
[A] : H^*(X) \rightarrow H^{*+d}(Y), \ d:=\dim Y - \dim A
\]
of cohomology groups (with coefficients in, say, a ring allowing a Kühneth isomorphism); details of such constructions are summarized below in an appendix. Duality then allows us to interpret Gromov-Witten invariants as elements \( GW_{g,n+1}(V) \) in
\[
\text{Hom}^1_\Lambda(H_*(\overline{M}_{g,n+1}), \text{Hom}(H^*(V)^{\otimes n}, H^*(V)))
\]
(whence cohomology, from now on, has coefficients in the Novikov ring \( \Lambda \)).

In a Cartesian closed category
\[
\text{End}_n(X) := \text{Maps}(X^n, X)
\]
defines the endomorphism operad of \( X \), and the associativity properties of pointwise gluing imply that
\[
GW_{0,*} : H_*(\overline{M}_{0,\bullet+1}, \Lambda) \rightarrow \text{End}_\Lambda^*(H^*(V, \Lambda))
\]
is a morphism of operads: thus \( H^*(V, \Lambda) \) becomes an algebra over the (small) quantum cohomology operad \( \{H_*(\overline{M}_{0,\bullet+1}, \Lambda)\} \).
2.4 As an application, Manin’s polycommutative product \[9\]

\[v \ast_t w := \sum_{n \geq 1} \mathbb{M}_{0,n+2}(v \otimes w \otimes t \otimes \cdots \otimes t)\]

\((t, v, w \in H^*(V, \Lambda))\), with \(t\) repeated \(n\) times in the product on the right) makes the cohomology of \(V\) into a Frobenius manifold. Taking higher genus terms into account defines something like a Frobenius manifold structure on the complex cobordism of \(V\) [17].

3. Type IIA strings, Fukaya’s category, and Devadoss’s mosaic operad

At this point we start over, now with \((V, \omega)\) a compact symplectic manifold:

3.1 Definition A Lagrangian polygon (cf [FO3])

\[\mathbb{L} := \langle L_1, \ldots, L_n \rangle\]

in \(V\) consists of

- oriented Lagrangian submanifolds \(L_1, \ldots, L_n\) of \(V\) (cyclically ordered for convenience), such that \(L_i\) intersects \(L_{i+1}\) transversally,
- a pseudoholomorphic map \(F : D \to V\) from the closed two-disk to \(V\), together with a choice \(\{z_1, \ldots, z_n\} \subset \partial D = \mathbb{P}_1(\mathbb{R})\) of \(n\) distinct points on the boundary of the disk, such that \(F(z_i) \in L_i \cap L_{i+1}\),
- such that \(F\) maps the interval \(I_k := [z_i, z_{i+1}] \subset \partial D\) to \(L_i\),
- satisfying a relative spin condition:

\[w_2(T_{L_i}) \in \text{image } [H^2(V, \mathbb{Z}_2) \to H^2(L_i, \mathbb{Z}_2)].\]

A morphism \(\mathbb{L} \to \mathbb{L}'\) of Lagrangian polygons is a commutative diagram

\[
\begin{array}{ccc}
D & \xrightarrow{\Phi} & D' \\
\downarrow F & & \downarrow F' \\
V & \xrightarrow{\Phi'} & V
\end{array}
\]

in which \(\Phi\) preserves \(\omega\), while \(\phi\) is holomorphic, taking the boundary decomposition of \(\mathbb{L}\) to that of \(\mathbb{L}'\). There is then a topological stack \(\mathbb{A}_n(V)\) of such Lagrangian \(n\)-gons in \(V\), with invertible maps of such polygons as morphisms.

3.2 Remarks:

1) There is an implicit action of the dihedral group of order \(2n\) on \(\mathbb{A}_n\): the cyclic group of order \(n\) acts by shifting the labels on the \(L_k\)’s, while reversing their order takes the category \(\mathbb{A}_n\) into itself, perhaps reversing its orientation.
2) Fukaya defines an algebra structure on the free \( \mathbb{Z} \)-module generated by equivalence classes \([L]\) of oriented Lagrangian submanifolds of \( V \), with product
\[
[L_i] \cdot [L_j] = \sum \Psi_{ij}^k[L_k],
\]
where the coefficients \( \Psi_{ij}^k := \#\langle L_i, L_j, L_k \rangle \) count the number of Lagrangian triangles bounded by the indicated Lagrangians – under the expectation that the space of such things is a zero-dimensional oriented manifold.

3) Note that we can integrate \( \omega \) over \( L \) to obtain its area \( \omega(L) \).

4) It will simplify notation below to write \( \{I_*\} \) for the ordered partition \( I_1, \ldots, I_k \) of the projective line into intervals.

### 3.3 Definition

The locus \( \overline{M}_{0,n}(\mathbb{R}) \subset \overline{M}_{0,n}(\mathbb{C}) \) of real points on the moduli stack of genus zero curves marked with \( n \) distinct smooth points can be identified with a compactification
\[
\text{Config}^n(\mathbb{P}_1(\mathbb{R}))//\text{PSl}_2(\mathbb{R}) := \overline{M}_{0,n}(\mathbb{R})
\]
of the quotient of the space of distinct \( n \)-tuples on the real projective line, under projective equivalence. Its elements can be regarded as (possibly decomposed) hyperbolic \( n \)-gons in the Poincaré disk, with geodesic boundaries, having all vertices on \( \mathbb{P}_1(\mathbb{R}) \) and one in particular at \( \infty \). The collection \( \{\overline{M}_{0,\bullet+1}(\mathbb{R})\} \) defines Devadoss’s \textit{mosaic} operad [5]; relaxing the choice of vertex at infinity makes it a cyclic operad.

### 3.4.1 Conjecture, cf [18, 19]:

Under reasonable hypotheses on \( V \), there are completions
\[
\mathbb{A}_n(V) \subset \overline{\mathbb{A}}_n(V)
\]
constructed by adjoining strata of decomposed Lagrangian polygons, indexed by \textbf{planar} trees, together with maps of these polygons to \( V \) which are holomorphic on the interiors of its components, and continuous on their boundaries. Evaluation defines (Fredholm) maps
\[
\overline{\mathbb{A}}_n(V) \to \overline{M}_{0,n}(\mathbb{R}) \times V^{\{I_*\}}
\]
of topological groupoids; where
\[
V^{\{I_*\}} = \prod_{1 \leq k \leq n} V^{I_k}
\]
(note that the space \( V^{I_k} \) of free maps of the interval \( I_k \) to \( V \) is homotopy equivalent to \( V \) itself).
Moreover, these maps satisfy an associativity condition, which requires some abbreviation to display:

\[
\begin{align*}
\mathcal{F}_{n+1} \times X^{I(i_{n+1})_1} \prod_i \mathcal{F}_{i+1} & \longrightarrow \mathcal{A}_{i_{n+1}} \\
M \times \prod_i X^{I(i_{i+1})_1} X & \longrightarrow \mathcal{M}_{0, \sum_{i_{n+1}}(\mathbb{R})} \times X'
\end{align*}
\]

where

\[
M := \mathcal{M}_{0,n+1}(\mathbb{R}) \times \prod_i \mathcal{M}_{0,i_{n+1}}(\mathbb{R}),
\]
and

\[
X' := \prod_i X^{I(i_{i+1})_1} \times X^{I(n+1)}_{n+1}.
\]

Roughly speaking, then, we have geometric correspondences

\[
\mathcal{A}_{n+1} : \mathcal{M}_{0, n+1}(\mathbb{R}) \longrightarrow \text{End}_*(X')
\]

which define an \(\{H_*(\mathcal{M}_{0, n+1}(\mathbb{R}), \tilde{\Lambda})\}\)-algebra structure on \(H^*(V, \tilde{\Lambda})\): where now

\[
\tilde{\Lambda} := \mathbb{Q}\{\{q\}\}[t]
\]

is an algebra over a field of Puiseux series in \(q = \exp(h)\), with components of \(\mathcal{F}_{n+1}\) weighted by \(\exp(\omega(L) h) t^n\).

### 4. Hochschild homology of \(A_\infty\) ringspectra

#### 4.1 The fundamental geometric fact about the moduli spaces \(\{\mathcal{M}_{0, n}(\mathbb{R})\}\) is that they are aspherical. They are tessellated

\[
\Sigma_{n+1} \times D_{n+1} K_n \longrightarrow \mathcal{M}_{0, n+1}(\mathbb{R})
\]

by Stasheff associahedra \(K_n\), defining a piecewise negatively curved metric which implies them to be spaces of type \(K(\pi, 1)\); thus, for example, \(\mathcal{M}_{0, 5}(\mathbb{R})\) is Kepler’s Great Dodecahedron.

Work of Etinghof, Henriques, Kamnitzer and Rains [6 Theorem 2.14] shows that algebras over the (unital) operad \(\{H_*(\mathcal{M}_{0, n+1}(\mathbb{R}), \mathbb{Q})\}\) are rational 2-Gerstenhaber algebras. On the other hand,

\[
H_*(\mathcal{M}_{0, n+1}(\mathbb{R}), \mathbb{Z}_2) \cong H_{n/2}(\mathcal{M}_{0, n+1}(\mathbb{C}), \mathbb{Z}_2).
\]

When \(n > 3\) Devadoss’s spaces are non-orientable, and the homology of their orientation covers is not yet understood.

#### 4.2 The action of the dihedral group \(D_{n+1}\) appeared in §3.2 above. By regarding \(\Sigma_{n+1}\) as \(\Sigma_n \cdot C_{n+1}\), the presentation above defines the structure

\[
\Sigma_n \times K_n \rightarrow (\Sigma_n \cdot C_{n+1}) \times D_{n+1} K_n \rightarrow \mathcal{M}_{0, n+1}(\mathbb{R})
\]
of an $A_\infty$ space on the collection $\{\overline{M}_{0,0+1}(\mathbb{R})\}$, permitting us to interpret $H^*(V, \Lambda)$ as an $A_\infty$ algebra (but unitality (cf §2.1.2, [15 §5.5.7]) deserves further attention . . . ).

Angeltveit [1] has defined a generalized Hochschild homology for $A_\infty$ ring spectra. I will close by noting that the resulting

\[ \text{HH}_*(H^*(V, \Lambda)) \]

seems related in interesting ways to the symplectic cohomology of $V$ defined recently by Ganatra [8]. The mod two analog of this construction, and possible variants defined using Devadoss’s orientation covers of $\overline{M}_{0,0+1}(\mathbb{R})$, have yet to be considered.

5. Appendix on conventions

**Correspondences** (re §2.3)

\[ A : X \to Y \iff A \to X \times Y , \]
\[ B : Y \to Z \iff B \to Y \times Z , \]

\[ \Rightarrow \]
\[ A \times_Y B \to X \times Z \]
defined by

\[
\begin{array}{ccc}
A & \times_Y B \\
\downarrow & \downarrow & \downarrow \\
A & \times_Y Z & B \\
\downarrow & \downarrow & \downarrow \\
X \times Y & X \times Z & Y \times Z \\
\downarrow & \downarrow & \downarrow \\
Y & & \\
\end{array}
\]

Note, the category of correspondences is **self-dual**.

**Example** Hecke correspondences are defined by morphisms

\[ G \to H \times K \]
of (finite?) groups . . .

Note, if the objects involved are Poincaré-duality objects, then a correspondence $A : X \to Y$ defines (assuming a Künneth formula)

\[ [A] \in H^*(X \times Y) \cong \text{Hom}(H^*(X), H^*(Y)) \]
satisfying $[A \times_Y B] = [A] \circ [B]$. 
Indexing conventions (re §3.4): In fiber products of the form

\[ \overline{A_{n+1}} \times_{\mathbb{R}} \overline{A_{1+m}} \]

the parametrization of the boundary segment \([z_{n}, z_{n+1}]\) on the left is identified (via the action of the projective group) with the parametrization of the segment \([z_{1}, z_{2}]\) on the right: thus the iterated fiber product on the top left of the diagrams involves identifications over a product of the form

\[ \prod_{1 \leq k \leq n} X^{I(i_k+1)}_1 , \]

with \(I(i_k+1)\) being the first interval in the partition \(\{I(i_k+1)_+\}\) of \(\mathbb{P}_1(\mathbb{R})\) defined by the polygon \(\mathbb{L}_{i_k+1}\). In §3.4.2, the terms in the fiber products have been regrouped for readability, using a telescoping product identification of the form

\[ \text{pt} \times \prod X^{I(i_+)}_1 \prod X^{I(i_+)}_1 \cong \prod X^{I(i_+)}_1 . \]

Note also that composition of correspondences involves maps of the form

\((X \times Y) \times_Y (Y \times Z) \cong X \times Y \times Z \to X \times Z\)

which ‘cancel’ paired copies of \(Y\). After taking cohomology, these cancellations corresponds to applications of the trace map

\[ H^*(Y, k) \otimes H^*(Y, k) \to k . \]

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THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218
E-mail address: jack@math.jhu.edu