Resolving puzzles of massive gravity with and without violation of Lorentz symmetry

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Abstract

We perform a systematic study of various versions of massive gravity with and without violations of the Lorentz symmetry in arbitrary dimension. These theories are well known to possess very unusual properties, unfamiliar from studies of gauge and Lorentz invariant models. These peculiarities are caused by the mixing of familiar transverse fields with the revived longitudinal and pure gauge (Stueckelberg) fields and are all seen already in the quadratic approximation. They are all associated with non-trivial dispersion laws, which easily allow superluminal propagation, ghosts, tachyons and essential irrationalities. Moreover, the coefficients in front of emerging modes are small, which makes the theories essentially non-perturbative within a large Vainshtein radius. Attempts to get rid of unwanted degrees of freedom by giving them infinite masses lead to the DVZ discontinuities in the parameter (moduli) space, caused by non-permutability of different limits. Also, the condition $m_{gh} = \infty$ can not be preserved already in non-trivial gravitational backgrounds and is unstable under any other perturbations of the linearized gravity. At the same time, an \textit{a priori} healthy model of massive gravity in the quadratic approximation definitely exists: it is provided by any mass level of the Kaluza–Klein tower. It bypasses the problems because the gravity field is mixed with other fields, and this explains why such mixing helps in other models. At the same time, this can imply that the really healthy massive gravity can still require an infinite number of extra fields beyond the quadratic approximation.

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1. Introduction

A renewed interest in massive gravity [1–6] and its further modifications, involving mixing with extra light fields and/or a tiny violation of the Lorentz symmetry, is dictated by the problems of cosmology nowadays, caused by the spectacular advances of observation astronomy. These days massive gravity is one of the so-far-desperate attempts to cook up a theory which naturally explains phenomenology of the hidden energy, which is currently thought to account for over one-half of the energy density of the Universe.

Whatever their relevance for phenomenological purposes, the problems of massive gravity are of the deepest theoretical interest. Abandoning the Lorentz invariance one actually opens a Pandora’s box of hidden structures, unobservable in the massless case. They include a variety of dispersion (spectral) relations for different components of the graviton field, almost as rich as in solid state physics, with non-quadratic dispersion laws, superluminal propagation, emergency of non-trivial spatial structures (wave densities) etc.

This paper (tightly connected with [6]) is an attempt to understand in the most primitive linear algebra terms the puzzling properties of massive gravity [7–9] and those of the whole new world arising after the violation of the Lorentz and gauge symmetries, which was discovered in [10–12] and nicely reviewed recently in [13] (see also [14]). Since the reasons for this strange behavior are not the main concern of all these papers, which are more interested in enumeration of different models and their relevance to cosmological applications, our goal is to make a step in this direction. In fact, we are going to perform an analysis in the old-fashioned style of [15] and of [16], where similar puzzles of the topological massive photodynamics [17] were addressed and resolved. A posteriori it looks quite similar in spirit to the original presentations in [7] and also includes a direct generalization of the original DVZ approach to the case of Lorentz non-invariant theories and/or of the models with extra scalar fields. It is of course very close to the original papers [10–13], just our accents are different. Our interest in the problem was initially motivated by studies of massive graviton radiation [18, 19] in another class of speculative models, related to the micro rather than macro world: in the TeV scale gravity [20], where the masses are presumably of the Kaluza–Klein origin and various problems of the massive gravity are supposed to be absent (other manifestations of the TeV scale gravity are discussed in [21, 22]. For mini-black-holes in cosmic ray events see Mironov et al 2003; and for mini-black-holes in neutrino experiments see the detailed review and references in Anchordoqui et al 2003 [21]).

In the present paper, we analyze only the quadratic approximation to the Einstein–Hilbert Lagrangian (linearized gravity); thus, all the effects of field interactions, including the Vainshtein radius [8] or the Boulware–Deser modes [9] and the superluminal effects [23] in curved backgrounds are beyond the scope of this paper. As already explained in [13], they are in fact intimately related to peculiar properties manifest in the quadratic approximation, in particular to the DVZ discontinuity [7] (see also [24]).

Given a quadratic action, one immediately obtains the Born interaction between the currents

\[ \phi_a K^{ab} \phi_b + J^a \phi_a \rightarrow J^a K^{-1} J^b = \frac{J^a K_{ab} J^b}{\det K}. \]  

After the Fourier transform, the entries of \( K_{ij} \) are quadratic polynomials in the frequency \( \omega \) and the space momentum \( \vec{k} \) (with some mass terms added), and

\[ \det K = \prod_{a} \lambda_a(\omega, \vec{k}). \]
In the Lorentz invariant case, the $\tilde{k}$-dependence is of course related to the $\omega$-dependence and $\lambda_a(\omega, \tilde{k}) \rightarrow \lambda_a(k^2)$ with $k^2 = -\omega^2 + \tilde{k}^2$. Coming back to the Born interaction, one can rewrite it as

$$\mathcal{J}K^{-1}\mathcal{J} = \sum_{a,b,c} \frac{\alpha_{bc}(\tilde{k})\mathcal{J}^b\mathcal{J}^c}{\lambda_a(\omega, \tilde{k}) + i \cdot 0}$$  \hspace{1cm} (1.3)

with some ‘structure constants’ $\alpha(\tilde{k})$.

The problem of dispersion relations is basically that of the eigenvalues of $K(k)$: roughly, $\omega = \varepsilon(|\tilde{k}|)$ is the condition that some eigenvalue $\lambda(k) = 0$. However, this ‘obvious’ statement requires a more accurate formulation. The point is that $K$ is actually a quadratic form, not an operator, which means that it can always be brought to the canonical form with only $\pm 1$ and 0 at the diagonal, thus leaving no room for the quantities like $\lambda(k)$. Still, this ‘equally obvious’ counter statement is also partly misleading, because we are interested not in an isolated quadratic form, but in a family of those, defined over the moduli space of all coupling constants (in the massive gravity case, the space of all masses) $M$. This means that the sets of $\pm 1$ and 0 can change as one moves along $M$, and the degeneracy degree of the quadratic form $K(k)$ can change. Of course, this degree (a number of 0s at the diagonal) is an integer and changes abruptly, and thus is not a very nice quantity. A desire to make it smooth brings us back to a concept of $\lambda(k)$. However, in order to introduce $\lambda(k)$ one needs an additional structure, for example, a metric in the space of fields.

In application to our needs, one can introduce the ‘eigenvalues’ $\lambda(k)$ as follows: consider instead of $\Pi = 1 \frac{1}{K} J$ a more general quantity

$$\Pi(\lambda|k) = J \frac{1}{K - \lambda I} J.$$  \hspace{1cm} (1.4)

Then as a function of $\lambda$, it can be represented as a sum of the contributions of different poles

$$\Pi(\lambda|k) = \sum_{a,b,c} \frac{\alpha_{bc}(k)J_bJ_c}{\lambda_a - \lambda}.$$  \hspace{1cm} (1.5)

then $\lambda_a(k)$ are exactly the ‘eigenvalues’ that we are interested in, and our original

$$\Pi(k) = \sum_{a,b,c} \frac{\alpha_{bc}(k)J_b(\tilde{k})J_c(k)}{\lambda_a(k)}.$$  \hspace{1cm} (1.6)

The only thing that one should keep in mind is that this decomposition depends on the choice of the additional matrix (metric) $I$, which can be chosen in different ways; in particular, its normalization can in principle depend on the point of $M$. We shall actually assume that it does not depend, and clearly the physical properties do not depend on this choice, however, concrete expressions for $\lambda_a(k)$ do. It is important that the dispersion relations, i.e. the zeroes of $\lambda_a(k)$ are independent of $I$.

The introduction of $I$ is also important from another point of view. To be well defined, the Lorentzian partition function requires a distinction between the retarded and advanced correlators (Green functions) which are usually introduced by adding an infinitesimal imaginary term to the kinetic matrix $K$: the celebrated $i\epsilon$-prescription in the Feynman propagator. However, in the case of kinetic matrix this is not just $i\epsilon$, it is rather $i\epsilon IF$ with some particular matrix $IF$. If one identifies our $I$ with $IF$, then the dispersion relations are actually

$$\lambda_a(k) = i\epsilon,$$  \hspace{1cm} (1.7)
which implies that $\lambda_a(k)$ is, in fact, very different from $-\lambda_a(k)$, and this is related to the important concept of ghosts\(^4\).

When one has a family of theories, like massive gravities with different masses, one actually has the set of quantities $\lambda_a(\vec{k})$ and $\alpha_{abc}(\vec{k})$ ‘hanging’ over each point of the parameter (moduli) space, and one is interested in the change of this structure when moving around in the moduli space. What happens: different $\lambda_a$s can cross or merge; they can also go away to infinity, even more interesting are the properties of $\alpha$s. At some points of the moduli space, the symmetry of the underlying theory is enhanced, and one obtains a singularity, where limits along different directions do not coincide (this is exactly the origin of the DVZ ‘discontinuity’), so that such points should actually be blown up to resolve the singularity. All this is a typical string theory subject [25]; it is amusing that this standard set of questions unavoidably arises in the study of such seemingly innocent subject as the linearized massive gravity . . . .

Schematically, we consider different intermediate particles (components of the gravity field), which contribute in different channels in (1.3), or, putting this differently, just diagonalize the coefficients $\alpha$ w.r.t. the current indices so that the sum (1.3) becomes a sum over the channels and particles. In general, we are interested in a variety of channels and in contribution of different species to each of these channels. The whole pattern is characterized by the following data (see also [6]).

**Species: dispersion laws.** Sometimes the dispersion law is simple,

$$\omega = \pm \sqrt{c^2 \vec{k}^2 + M^2};$$

however, the coefficients $c^2$ and $M^2$ are of importance. The dispersion law with $c^2 > 1$ describes the superluminals and with $M^2 < 0$ the tachyons. The superluminals always travel faster than light and can violate the naive causality, [23, 26] (Superluminal propagation in general relativity was discussed in Dolgov et al 1998 [26]). They are sometimes also called tachyons in literature. But, physically, they are very different from the tachyons, which are signals of instabilities and do not violate causality (allow a simultaneous but uncorrelated and causally independent development of instabilities separated by space-like intervals). In fact, we shall see that there are more sophisticated dispersion laws

$$\omega = \lambda(|\vec{k}|) \neq \pm \sqrt{c^2 \vec{k}^2 + M^2};$$

however, their complexity does not increase with the increase of the dimension $d$: the relations between $\omega$ and $|\vec{k}|$ are at most quartic:

$$\omega^2 = \sqrt{P_2(\vec{k})} \pm \sqrt{P_4(\vec{k})},$$

where $P_n(\vec{k})$s are polynomials of degree $n$. In the case of this more complicated dispersion law, one defines $M^2$ as a position of the pole in $\omega^2$ at zero spatial momentum and defines the tachyon mass square as a real-valued solution to the equation $\lambda(0, \vec{k}) = 0$.

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\(^4\) The Feynman propagator implies that the particles with the dispersion relation $\omega = +\epsilon(|\vec{k}|)$ propagate forward in time, while the antiparticles with $\omega = -\epsilon(|\vec{k}|)$—backwards in time. Since $\theta(\pm t) = \frac{1}{2i} \int e^{\pm i \omega t} \frac{d\omega}{\pi}$, one has for the propagator

$$\frac{1}{2i} \left( \frac{1}{\omega - \epsilon - i\epsilon} + \frac{1}{-\omega - \epsilon - i\epsilon} \right) = \frac{1}{\omega^2 - \epsilon^2 - i\epsilon}.$$

For ghosts with the propagator $\frac{1}{\omega^2 + \epsilon^2}$, the situation is inverse: the particles propagate backward, while the antiparticles forward in time. See also [6, appendix B].
Residues. Coefficients $a^{abc}_a$ controlling the contribution of the specie $a$ to the channel $b$. It is important to distinguish if the ghost contributions appear in physical (say, spacetime transverse for the conserved currents) or unphysical channels (corresponding to the sources of pure gauge species).

Discontinuities appear when some $M^2_a \to \infty$. An accurate formulation of the problem is that one looks at the interaction in a given channel at large distances, but not as large as $\min(M^{-1}_a)$, so that all contributing species still look massless. However, if some $M_{a0} = \infty$, simply there is no such region and one observes a jump between the ‘long distance’ interactions for $M_{a0} = \infty$ and $M_{a0} = 0$. In other words, switching on a tiny mass scale is not an obligatory small effect if for some species this tiny scale is multiplied by infinity, constructed from other parameters (like $(A/B - 1)^{-1}$ in the Pauli–Fierz case below).

In the fully comfortable theory there are no tachyons, no ghosts and no superluminals. This, however, is rarely achievable in the theories of massive spins, greater than 1, if one decides to abandon the gauge invariance, and gives masses by an explicit, rather than spontaneous, breaking of the gauge invariance.

In this paper, we analyze the problem in a somewhat unusual way. Instead of defining the normal modes by passing to the Hamiltonian formalism, we directly diagonalize the kinetic matrices $K_{ij}$ in the momentum representation. The normal modes defined in this way are sometimes very convenient to deal with, especially in the case of massless theories, since despite the fact they are introduced in an explicitly Lorentz non-invariant way, the longitudinal modes are very distinguished in this case, they carry a lot of physical information; in particular, they easily distinguish between the propagating and non-propagating modes (without throwing the latter away from the spectrum as one often does by imposing constraints in the Hamiltonian formalism, which, in fact, makes deformations to adjacent points in the moduli space problematic, see [16] for an initial criticism of the standard approach). In contrast, in the massive theories, it could be more convenient to use the Lorentz invariant normal modes (the both types of modes coincide in the rest frame). However, as we discuss in section 3, using the Lorentzian modes is, basically, not correct, although sometimes it leads to the same results.

In what follows, we first start from the simplest example of electrodynamics in order to illustrate the mode-based approach and then continue in section 4 with the massive and massless gravity. One of our purposes in section 4 is to justify the above-mentioned mechanism of the DVZ discontinuity. Note that, as discussed in section 4, the DVZ jump occurs only in the specific Pauli–Fierz case; otherwise, the ghost mass tends to zero along with the graviton mass and the discontinuity is absent (though the ghost is present). Then in section 5 we consider generalizations to the Lorentz-violated gravity, where the main novelty is the occurrence of quasiparticles with various patterns of non-trivial dispersion laws, not very familiar in elementary particle physics. We also re-derive the known claim that the sufficient (but not necessary) condition for the theory to be ghost-free in this case is the vanishing mass term for the $h_{00}$-component of the graviton. At the same time, the condition of the absence of tachyons is more subtle and depends on details of the Lagrangian, including the number of dimensions. Finally in section 6, we turn to massive gravity mixed with some extra particles. The Kaluza–Klein massive gravitons belong to this class, what a priori explains why an addition of mixings can produce healthful theories of massive gravity (since one expects the Kaluza–Klein theory is free of pathologies). For other fashionable models of this kind, see [4, 11]. The last section contains a discussion of various physical consequences of the behavior obtained in previous sections.
2. A warm-up example: massive photodynamics

In this section, we consider the case of photodynamics which is much simpler than gravity and, hence, we use it to illustrate the approach of this paper. We look at various patterns of adding masses, including those breaking the Lorentz invariance.

2.1. Generalities

Photodynamics is the theory with the quadratic action

\[
\int \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - M^2 A_\mu^2 + J_\mu A^\mu \right) d^dx = \int (A^\mu (-k) K_{\mu\alpha}(k) A^\alpha(k) + A^\mu (-k) J_\mu(k)) d^d k. \tag{2.1}
\]

Our immediate task is as follows.

- To enumerate different modes, contained in the field \( A_\mu \), which propagate through the spacetime independently, without mixing, and to identify their properties.
- To find the sources of these modes.
- To decompose the Born interaction between the sources into contributions of different modes.

In this way, one can express various properties of the interaction mediated by our theory, through the properties of individual modes and identify the origins of particular types of unusual behavior.

Of course, the formal realization of this ‘program’ is nothing but an elementary linear algebra exercise with the kinetic matrix \( K_{\mu\alpha}(k) \) in the momentum representation, which, in the case of photodynamics, is simply an ordinary symmetric \( d \times d \) matrix:

\[
K_{\mu\alpha} = k_\mu k_\alpha - (k^2 + M^2) \eta_{\mu\alpha}. \tag{2.2}
\]

It can be easily diagonalized

\[
K_{\mu\alpha} = \sum_{a=1}^d \lambda_a v^{(a)}_\mu v^{(a)}_\alpha, \tag{2.3}
\]

where the \( d \) eigenvectors are

- gauging scalar \( v^{(s)}_\mu = q k_\mu \)
- longitudinal vector \( v^{(l)}_\mu \)
- transverse vector \( v^{(t)}_\mu, \quad i = 1, \ldots, d - 2. \) \tag{2.4}

‘Transverse’ and ‘longitudinal’ refer to the space rather than spacetime vectors. All these components are well defined at \( M^2 \neq 0 \), and the splitting exhibits a smooth limit in the massless limit \( M^2 \to 0 \).

The gauge degree of freedom \( A_\mu = q k_\mu \) is scalar; it does not mix with the other \( d - 1 \) degrees of freedom,

\[
k^\mu K_{\mu\alpha} = -M^2 k_\alpha \tag{2.5}
\]

(it is a particular eigenvector of \( K \), but it has a non-trivial Lagrangian and even a kinetic term for the Stueckelberg field \( q(x) \), whenever \( M^2 \neq 0 \) and the gauge invariance is broken. The \( d - 1 \) ‘physical’ degrees of freedom in their turn split into the \( 1+(d-2) \) components: the longitudinal and transverse photons with different eigenvalues and different properties.

Note that the very definition of normal modes is not Lorentz invariant (even if the theory is): they solve the equation \( K_{\mu\alpha} v^{(a)} = \lambda_v v^{(a)} \) and not \( K_{\mu\alpha} v^{(a)} = \tilde{\lambda} v^{(a)} \), i.e. the equation with
subscript $\mu$ on the lhs and superscript $\mu$ on the rhs, which manifestly breaks the Lorentz invariance. This is important for making the Lagrangian diagonal, when expressed through the normal mode: this follows from the orthogonality of matrix eigenvectors (not that in the case of the Euclidean signature there would be no difference, but the difference between the propagating and non-propagating modes is not seen there).

The Born interaction

$$\int J^\mu(-k) P_{\mu\nu}(k) J^\nu(k) \, d^4k$$

is defined in terms of the propagator, inverse of the kinetic matrix, which is well defined for $M^2 \neq 0$:

$$P_{\mu\nu} = (K^{-1})_{\mu\nu} = -\frac{\eta_{\mu\nu} + k\, k^\nu}{k^2 + M^2}. \quad (2.7)$$

Our notation will be as follows.

The Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, \ldots, +1)$.

The spacetime momentum $k_\mu = (\omega, k_\parallel, 0, \ldots, 0)$, $k^\mu = (-\omega, k_\parallel, 0, \ldots, 0)$, the spatial momentum will also be often denoted by $k$ and the frequency $\omega$ by $k_0$.

The Lorentz square of the spacetime momentum is $k^2 = -\omega^2 + k_\parallel^2 = -\omega^2 + k_i^2$.

The spacetime indices are denoted by Greek characters: $\mu, \nu = 0, 1, \ldots, d-1$, the space indices by Latin characters $i, j = 1, \ldots, d-1$, the transverse spatial indices (in coordinates where $\vec{k}$ is directed along the first axis) by $a, b = 2, \ldots, d-1$.

2.2. Massless photon

This is the ordinary massless photodynamics.

The kinetic matrix is

$$\begin{pmatrix}
\omega^2 + k^2 & \omega k_\parallel & 0 & 0 \\
\omega k_\parallel & k_\parallel^2 - k^2 & 0 & 0 \\
0 & 0 & -k^2 & \ldots \\
0 & 0 & 0 & -k^2
\end{pmatrix}
= 
\begin{pmatrix}
k_\parallel^2 & \omega k_\parallel & 0 & 0 \\
\omega k_\parallel & \omega^2 & 0 & 0 \\
0 & 0 & \omega^2 - k_\parallel^2 & \ldots \\
0 & 0 & 0 & \omega^2 - k_\parallel^2
\end{pmatrix}. \quad (2.8)$$

The eigenfunctions and eigenvalues

$$v_\parallel^\mu = \begin{pmatrix}
-\omega \\
k_\parallel \\
\vdots \\
0
\end{pmatrix}, \quad \lambda_\parallel = 0 \quad \text{gauging photon} = \text{Stueckelberg scalar}$$

$$v_\parallel^\mu = \begin{pmatrix}
k_\parallel \\
\omega \\
\vdots \\
0
\end{pmatrix}, \quad \lambda_\parallel = \omega^2 + k_\parallel^2 \quad \text{longitudinal photon} \quad (2.9)$$

$$v_\perp^\mu = \begin{pmatrix}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
1 & \ldots & 0 \\
0 & \ldots & 1
\end{pmatrix}, \quad \lambda_\perp = \omega^2 - k_\parallel^2 \quad \text{transverse photon.}$$

$d=2$ polarizations
Thus, the $d - 2$ transverse photons are just ordinary massless particles with the normal kinetic term, described by the eigenvalue $\omega^2 - k_\parallel^2$, which vanishes on-shell, when $\omega = k_\parallel$. This dispersion law allows non-vanishing values of $\omega$, which implies that once emitted such particles can propagate by ‘themselves’.

The pure gauge photon (gauging scalar or Stueckelberg scalar) has a vanishing eigenvalue; this means that it completely drops out of the action: it ‘does not exist’, or is ultralocal: it is just equal to its source.

The longitudinal photon is a non-trivial field, but the corresponding eigenvalue is positively defined and vanishes (i.e. is on shell) at a single point $\omega = k_\parallel = 0$: the dispersion law is simply $\omega = 0$. This means that this field cannot exist ‘by itself’, it is fully driven by the source. Still, the eigenvalue depends on $\omega$ and $\vec{k}$, which means that even for a point-like source the field can be spread in time and space.

Expanding the gauge field $A_\mu$ in different sorts of photons

$$A_\mu(k) = A_g(k)v_\mu^g + A_\parallel(k)v_\mu^\parallel + A_\perp(k)v_\mu^\perp$$

and substituting it into the action $\int (F_{\mu\nu}F^{\mu\nu} + A_\mu J_\mu)$, one obtains

$$A_\parallel^2 \left( \omega^2 + k_\parallel^2 \right)^2 + A_\perp^2 \left( \omega^2 - k_\perp^2 \right)^2 + A_g \left( -\omega J_0 + k_\parallel J_\parallel \right) + A_\parallel \left( k_\parallel J_0 + \omega J_\parallel \right) + A_\perp J_\perp.$$  \hfill (2.10)

Note that we work with unnormalized eigenvectors: this simplifies some formulas, though makes some other, like this one for the Lagrangian, look a little unusual.

The coupling $A_g J$ vanishes if $J_\mu$ is conserved, since in this case $\omega J_0 = k_\parallel J_\parallel$ and also the $A_\parallel J$ coupling can be rewritten as

$$A_\parallel(k_\parallel J_0 + \omega J_\parallel) = \left( \frac{\omega^2 + k_\parallel^2}{\omega} \right) A_\parallel J_0 + \left( \frac{\omega^2 + k_\parallel^2}{k_\parallel} \right) A_\parallel J_\parallel,$$

so that the action, expressed in terms of the separated variables, becomes

$$A_\parallel^2 \left( \omega^2 + k_\parallel^2 \right)^2 + A_\parallel^2 \left( \omega^2 - k_\parallel^2 \right) + \left( \frac{\omega^2 + k_\parallel^2}{k_\parallel} \right) A_\parallel J_0 + A_\parallel J_\parallel.$$  \hfill (2.11)

The Born interaction can be immediately read from this formula:

$$J_\mu P^{\mu\nu} J_\nu = \omega^2 - k_\parallel^2 = \left( J_\parallel^2 \right)^2 + \frac{J_\mu^2}{\omega^2 - k_\parallel^2} = \frac{J_\parallel^2}{\omega^2} + \left( J_\perp \right)^2.$$  \hfill (2.12)

one can again see that only the transverse photon does propagate. Of course, it can be alternatively obtained by first writing down the propagator $P^{\mu\nu}$, which satisfies

$$K_{\mu\nu} P^{\mu\nu} = \delta_\mu^\nu - \frac{g_\mu k_\nu}{k^2},$$

and is equal to

$$P^{\mu\nu} = \frac{\eta^{\mu\nu} + c k^\mu k_\nu}{\omega^2 - k_\parallel^2},$$

so that the interaction of two conserved currents, such that $k_\mu J_\mu = 0$, is

$$J_\mu P^{\mu\nu} J_\nu = \frac{J_\mu J_\mu}{\omega^2 - k_\parallel^2}.$$  \hfill (2.13)
In order to obtain (2.14) from this simple expression, one should substitute an explicit resolution of the conservation constraint for the current

\[ J_a = \begin{pmatrix} J_0 \\ J_i \\ J_\perp \end{pmatrix} = \begin{pmatrix} \frac{a k_i}{\omega} \\ a \omega \end{pmatrix}, \]  

(2.18)

where \( a = \frac{\rho}{k_i} = \frac{\rho}{\omega}. \)

Note that because of the gauge invariance (i.e. vanishing of the eigenvalue \( \lambda_g \)), the propagator is not just the inverse of the kinetic matrix, one should exclude the zero mode by putting the transverse matrix on the rhs of (2.15). Instead, the propagator (2.16) contains an unspecified coefficient \( c \), which drops out from coupling to the conserved current.

### 2.3. 3d topologically massive photodynamics [16, 17]

We begin the consideration of the massive photodynamics with the celebrated ‘intermediate’ example on the way from the massless to massive photodynamics, when the photon gets a mass, but the gauge invariance is still unbroken, and the Stueckelberg fields do not show up in the action. Despite the fact that the propagating photon is massive, there is a pole at vanishing momentum in the interaction of currents. Remarkably, this does not contradict the unitarity because of the existence of a single propagating massless mode (not a particle) [16]. Since it is not a particle, the long-range interaction is topological, namely in this case it is just the Aharonov–Bohm interaction.

The Lagrangian:

\[ -\frac{1}{2} F^\mu\nu F_{\mu\nu} + M \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}. \]  

(2.19)

The kinetic 3 \times 3 matrix:

\[ K_{\mu\nu} = \begin{pmatrix} \omega^2 + k^2 & \omega k_i & i M k_i \\ \omega k_i & k^2_i - k^2 & i M \omega \\ -i M k_i & -i M \omega & -k^2 \end{pmatrix} = \begin{pmatrix} k^2_i & \omega k_i & i M k_i \\ \omega k_i & \omega^2 & i M \omega \\ -i M k_i & -i M \omega & \omega^2 - k^2 \end{pmatrix}. \]  

(2.20)

The matrix is not fully symmetric, since the required symmetry property is \( K_{\mu\nu}(k) = K_{\nu\mu}(-k) \). Note also the appearance of \( i = \sqrt{-1} \) and that \( \epsilon^{012} = \epsilon^{120} \) because of the Minkowski metric.

The eigenfunctions and eigenvalues

\[ v_\mu^g = \begin{pmatrix} -\omega \\ k_\parallel \\ 0 \end{pmatrix} = k^\mu \quad \lambda_g = 0 \quad \text{Stueckelberg scalar} \]

\[ v_\mu^+ = \begin{pmatrix} k_i \\ \omega \\ \frac{k_i^2 - M^2}{k_i^2 + r^2} \end{pmatrix} \quad \lambda_+ = \omega^2 + r \quad \text{photon} \]  

(2.21)

\[ v_\mu^- = \begin{pmatrix} k_i \\ \omega \\ \frac{k_i^2 + M^2}{k_i^2 + r^2} \end{pmatrix} \quad \lambda_- = \omega^2 - r \quad \text{photon}, \]

where \( r^2 \equiv k_i^4 + M^2 (\omega^2 + k^2_i) \).
Note that the eigenvalues $\omega^2 \pm r$ are quite complicated; however, the dispersion relation $\omega^2 = r^2$ is equivalent to the standard one

$$\omega^2 = k_1^2 + M^2,$$

i.e. non-vanishing eigenvalues can be rewritten as

$$\omega^2 \pm r = \frac{(\omega^2 - k_1^2 - M^2)(\omega^2 + k_1^2)}{\omega^2 \mp r},$$

so that no irrationalities show up in the denominators of the propagator

$$P_{\mu\alpha} = \eta_{\mu\nu} - \frac{k\mu k\nu}{k^2 - M^2} + \frac{iM\epsilon_{\mu\nu\lambda\sigma}k^\lambda}{k^2(k^2 - M^2)}. \quad (2.24)$$

For the conserved currents with $k\mu J^\mu = 0$

$$J^\mu P_{\mu\alpha} J^\alpha = \frac{1}{\omega^2 - k_1^2 - M^2} \left( J_+^2 - \frac{2iM J_+ J_0}{\omega} + \frac{k^2}{\omega^2} J_0^2 \right) = \frac{1}{\omega^2 - k_1^2 - M^2} \left( J_+^2 - \frac{2iM J_+ J_0}{k_1} + \frac{k^2}{k_1^2} J_0^2 \right). \quad (2.25)$$

Note that inseparable products like $k_1^2(k^2 + M^2)$ appear in the denominators: this will be a typical feature of all massive gauge theories, which does not depend, as we already see, on whether the gauge invariance is preserved or not.

Not only are the modes inseparable, there is a pole at $p_\parallel = 0$, even though the particles are massive. This pole is indeed a long range interaction, and one can question how such interaction can occur in the theory of massive particles. The answer is [16] that the spectrum of the theory is not exhausted by massive particles; there is an additional single propagating mode without a gap. This is a single mode, not a particle with the dispersion law $\omega = 0$ consistent with the Lorentz invariance (see more examples of such dispersion laws in the gravity case with the broken Lorentz invariance, section 5, especially, section 5.6). Because of this, the long interaction which it describes cannot transfer the space momentum and is especially simple: it is the topological Aharonov–Bohm interaction, it exists and its effects are observable, and it can cause infrared divergencies in scattering cross-sections, just as the ordinary long-range Coulomb interaction does in low dimensions.

As we saw, the irrationality, $r$ is gone from the denominators in the current–current interactions; moreover, it is not seen in the numerators. This, however, is an illusion; the irrationalities are excluded at the price of considering non-diagonal interactions. If the interaction is diagonalized, i.e. written in terms of the independent modes (polarizations) $v^\mu$ and $\bar{v}^\mu$, the irrational $r$ is explicitly present in the formulas. This is again a standard feature of the massive gauge theories.

In fact, $v^\mu$ and $\bar{v}^\mu$ describe a left polarized massive photon with two massive degrees of freedom. There is an extra degree of freedom (not a particle), responsible for the long-range Coulomb interaction [16].

For illustrative purposes, in appendix A, we consider in detail the case of a ‘true’ massive photon.
3. Comment on the definition of normal modes

It is now a good time to illustrate our view on the normal modes [6]. We see that the physically relevant interaction (2.14) contains the two very different kinds of structures. In the transverse channel, the interaction is clearly mediated by the massless photon: there is a pole whenever this photon is on-shell, \( \omega^2 = \vec{k}^2 \). At the same time, in the temporal–longitudinal channel, the mediator is something very different: the pole of the corresponding propagator is at \( \omega = \vec{k} = 0 \) and nowhere else, i.e. it has the (real) co-dimension 2. The seeming co-dimension one pole at \( \omega = 0 \) is spurious: as \( \vec{k} \to 0 \) the numerator simultaneously tends to zero, so that the would-be pole is fully eliminated. Seemingly, there is no pole at \( \omega = 0 \) but \( \vec{k} \neq 0 \).

These properties of the interaction are perfectly encoded in the eigenvalues \( \lambda \):

- \( \lambda_\perp = 0 \) exactly on mass-shell of the transverse photon, while
- \( \lambda_\parallel = 0 \) only at \( \omega = \vec{k} = 0 \).

This is why we consider such definition of \( \lambda \)s physically relevant.

However, such definition may look somewhat non-conventional and is in fact inconvenient for other purposes. In particular, it is based on the explicitly Lorentz non-invariant definition of eigenvectors. An alternative Lorentz invariant definition, however, provides another value of \( \tilde{\lambda}_\parallel = \omega^2 - \vec{k}^2 \) and can be made consistent with the result for the Born interaction only within a sophisticated concept of a ‘propagating, but decoupling’ longitudinal photon. Our approach rather treats the longitudinal photon as totally ‘non-propagating’, in perfect accordance with (2.14).

Technically, the difference is as follows. We consider \( K_{\mu\nu} \) as an ordinary symmetric matrix and formally diagonalizes it by orthogonal transformations, i.e. by raising indices with the help of Euclidean \( \delta_{\mu\nu} \), instead of Minkowski \( \eta_{\mu\nu} \). In other words, we diagonalize the symmetric matrix

\[
K_{\mu\nu} = \begin{pmatrix}
\kappa_\parallel^2 & \omega k_\parallel \\
\omega k_\parallel & \omega^2
\end{pmatrix}
\]

instead of the asymmetric one

\[
K'_{\mu\nu} = \begin{pmatrix}
-\kappa_\parallel^2 & -\omega k_\parallel \\
\omega k_\parallel & \omega^2
\end{pmatrix}.
\]

The pure gauge (gauging) mode with the vanishing eigenvalue is of course the same in both cases, \( v_\mu^g = w_\mu^g = (\kappa_\parallel \omega) \); however, for the longitudinal mode one gets \( v_\mu^\parallel = (\kappa_\parallel \omega) \) with \( \lambda_\parallel = \omega^2 + k_\parallel^2 \) instead of the usual \( w_\mu^\parallel = (\kappa_\parallel \omega) \), with \( \lambda_\parallel = \omega^2 - k_\parallel^2 \). Therefore, our non-covariant modes are orthogonal in the ordinary linear algebra sense, i.e. w.r.t. the Euclidean metric \( \delta_{\mu\nu} \) instead of the Minkowski \( \eta_{\mu\nu} \), while the Lorentzian eigenmodes \( w_\mu^\parallel \) are orthogonal w.r.t. \( \eta_{\mu\nu} \), i.e. w.r.t. the group \( SO(d - 1, 1) \) instead of \( SO(d) \). Accordingly, the quadratic form kinetic matrix is

\[
K_{\mu\nu} = v_\mu^\parallel v_\nu^\parallel, \quad \begin{pmatrix}
\kappa_\parallel^2 & \omega k_\parallel \\
\omega k_\parallel & \omega^2
\end{pmatrix} = \begin{pmatrix}
k_\parallel \\
\omega
\end{pmatrix} \otimes (k_\parallel \omega)
\]

in terms of our Euclidean normal mode, and our two eigenmodes, used in (2.10) are just \( \delta_{\mu\nu} v_\parallel^\mu \) and \( \epsilon_{\mu\nu} v_\parallel^\mu \). Because of the different \( \lambda_\parallel \) our normal mode is clearly non-propagating, while the conventional longitudinal photon does propagate, just ‘decouples’. As we shall see in the following sections, when the mass is introduced, the gauge invariance is broken and the gauging (Stueckelberg) mode is revived, the difference gets even more pronounced: our normal modes (only one of them propagating) are mixtures of the longitudinal and the Stueckelberg modes, and both with non-trivial eigenvalues (A.11), while, with the Lorentz covariant definitions, the Stueckelberg mode is non-propagating with the eigenvalue \( M^2 \), and the longitudinal photon becomes propagating and acquires the standard dispersion relation.
\[ \omega^2 = k^2 + M^2, \] the same as the transverse modes, just it ‘no longer decouples’. Thus, the standard view is not very helpful in visualizing the actual properties of the massive (gauge violating) theory with its sophisticated propagators and the Born interactions.

There is a clear physical reason to deal with the Euclidean eigenvalues (see also [6]). Indeed, let us consider the massive 4-vector field Lagrangian

\[ -A_\mu (k^2 + m^2) A^\mu = A_\mu K_{\mu \nu} A^\nu. \] (3.1)

It is ill-defined as it contains ghosts, since the time component of the field has the wrong sign of the time derivative term. This is immediately reflected in the corresponding negative derivative \( \partial \lambda (\omega^2) / \partial \omega^2 \) of one of the Euclidean eigenvalues of \( K_{\mu \nu} \), while the Lorentz eigenvalues are all positive in this case.

To put it differently, one takes care of the sign in the quadratic action when performing the Gaussian integration in the path integral. In particular, while integrating over \( A_\mu \), one’s concern is only on the coefficients in front of \( A^\mu_2 \), which are exactly the diagonal elements of the Euclidean kinetic operator.

It deserves emphasizing that this subtle difference between the Euclidean and Lorentz eigenvalues is inessential when dealing with only the scalar sector (which is our main interest throughout this paper). Moreover, the majority of results below are independent of this difference between the Lorentz invariant and non-invariant definitions of normal modes. In particular, independent are \( \det K \), the characteristic equation and its decompositions. Hence, the dispersion laws, tachyons etc, are independent of the choice of modes, and only the ghost content could depend. More comments on this issue can be found in [6], especially in appendices A and B.

Note that the Lorentz covariant modes are easily restored from our formulas below: in the rest frame (i.e. at \( \vec{k} = 0 \)) the two choices coincide and the Lorentz transformation can be used to obtain the Lorentz covariant modes in all other frames. This is not so easy for the Lorentz non-covariant modes, and one has to write them down explicitly in all frames. However, even this advantage of the Lorentz modes disappears when one works with Lorentz non-invariant theories, as we do in the second half of this paper.

Thus, once more, we think that the correct procedure is to use the Euclidean modes; however, it is sometimes technically easier to use the Lorentzian modes, which still sometimes gives correct results (in particular, in the scalar sector). In order to give the reader a flavor of the difference between the Euclidean and Lorentzian eigenmodes, for illustrative purposes, we present in appendices A.1 and B calculations for the Lorentzian modes.

4. Lorentz invariant massive gravity in quadratic approximation

4.1. Generalities

The Einstein–Hilbert action \( \int R \sqrt{g} \, d^d x \) is highly nonlinear in the metric field and describes a pretty sophisticated interacting theory. However, all the peculiarities of massive gravity as compared to the ordinary general relativity show up already at the level of quadratic action for the small deviation \( h_{\mu \nu} \) of \( g_{\mu \nu} \) from the Minkowski background \( \eta_{\mu \nu} \). In this approximation, it is also easy to introduce a mass perturbation, which breaks the gauge (general coordinate) invariance and makes the graviton field massive. If the Lorentz symmetry \( SO(d - 1, 1) \) is preserved, the possible deviations from general relativity in the quadratic approximation are parameterized by two constants: \( A \) and \( B \).
The quadratic kinetic term $h_{\mu\nu}K_{\mu\nu,\alpha\beta}h_{\alpha\beta}$ is

\[
K_{\mu\nu,\alpha\beta} = (k_\mu k_\alpha \eta_{\beta\nu} + k_\mu k_\beta \eta_{\alpha\nu} + k_\alpha k_\gamma \eta_{\beta\mu} + k_\gamma k_\beta \eta_{\alpha\mu}) - 2(k_\mu k_\alpha \eta_{\beta\mu} + k_\alpha k_\beta \eta_{\mu\alpha}) - (k^2 + A)(\eta_{\mu\alpha} \eta_{\beta\nu} + \eta_{\nu\alpha} \eta_{\mu\beta}) + 2(k^2 + B)\eta_{\mu\nu} \eta_{\alpha\beta}.
\] (4.1)

The propagator, inverse of the kinetic matrix can be parameterized as follows:

\[
P_{\mu\nu,\alpha\beta} = a_1(k_\alpha k_\beta \eta_{\mu\nu} + k_\alpha k_\nu \eta_{\mu\beta} + k_\nu k_\beta \eta_{\mu\alpha} + k_\nu k_\alpha \eta_{\mu\beta}) + a_2 k_\mu k_\nu \eta_{\alpha\beta} + a_3 (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\nu\alpha} \eta_{\mu\beta}) + a_4 k_\mu k_\nu k_\alpha k_\beta,
\] (4.2)

and the Born interaction of two stress tensors is given by

\[
P_{\mu\nu,\alpha\beta}T^{\mu\nu}T^{\alpha\beta} = 4a_1(k_T)^2 + (a_2 + a_3)(k_T k)T + 2a_4 T_{\mu\nu}^2 + a_5 T^2 + a_6 (k_T k)^2 k_{\mu\nu} = 0 \rightarrow 2a_4 T_{\mu\nu}^2 + a_5 T^2.
\] (4.3)

Note that the Born interaction of conserved energy–momentum tensors, which involves only $a_4$ and $a_5$ coefficients does not exhaust all possible ways to observe gravity interactions: one can just radiate a graviton by the energy–momentum tensor, with the radiated field being

\[
h_{\mu\nu} = P_{\mu\nu,\alpha\beta}T^{\alpha\beta} = 2a_1[k_\mu (k_T)_\nu + k_\mu (k_T)_\nu] + a_2 k_\mu k_\nu T + a_3 \eta_{\mu\nu} (k_T k) + 2a_4 T_{\mu\nu}^2 + a_5 T^2 \rightarrow a_2 k_\mu k_\nu T + 2a_4 T_{\mu\nu}^2 + a_5 T^2.
\] (4.4)

Of course, violation of the gauge invariance (general covariance) in transition to the massive gravity implies that the stress tensor has not to be obligatorily conserved. However, this implies that violations are also present in the matter sector. If, as usual in the present day discussions, one restricts all the violations of conventional physics to the pure gravity sector, then the properties of matter are not changed, and the stress tensor $T_{\mu\nu}$ remains conserved. Technically, if one wants to break this property, a new current $J_\nu = k_\mu T_{\mu\nu}$, will appear in the formulas, which should be further split into the conserved current $\tilde{J}_\nu = J_\nu - k_\nu Q$ with $Q = \frac{1}{k^2} k_\mu T_{\mu\nu}$ and the ‘improved’, though non-local, stress tensor

\[
\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{k^2} (k^\mu \tilde{J}_\nu + k^\nu \tilde{J}_\mu + k^\mu k^\nu Q)
\] (4.5)

will be conserved.

Therefore (also in order to allow a smooth transition to the massless case), one would better consider not an inverse of $K$, but a solution of the equation $KP = E$ with the spacetime transverse rhs, explicitly accounting for the conservation of the energy–momentum tensor, which is the source of the gravity field. To make the discussion complete, we introduce an additional parameter $\alpha$ which interpolates between unity (at $\alpha = 1$) and the spacetime transverse (at $\alpha = 0$) rhs $E$:

\[
P_{\mu\nu,\rho\sigma}K^{\rho\sigma}_{\alpha\beta} = \alpha(\eta_{\mu\alpha} \eta_{\beta\nu} + \eta_{\mu\nu} \eta_{\beta\alpha}) + (1 - \alpha)(\eta'_{\mu\alpha} \eta'_{\beta\nu} + \eta'_{\mu\nu} \eta'_{\beta\alpha}),
\] (4.6)

where the transverse Minkowski symbol is defined as

\[
\eta'_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}.
\] (4.7)
Then, the coefficients in the propagator are

\[ a_1 = -\frac{1}{2} \frac{\alpha k^2 + \alpha A - A}{A - 2B} \]

\[ a_2 = \frac{A - 2B}{(k^2 + A)(A^2 - dAB + (d - 2)(B - A)k^2)} \]

\[ a_3 = -\frac{1}{2} \frac{1}{k^2(k^2 + A)(A^2 - dAB + (d - 2)(B - A)k^2)} \]

\[ a_4 = -\frac{1}{2} \frac{(A - B)k^2 + AB}{k^2(k^2 + A)(A^2 - dAB + (d - 2)(B - A)k^2)} \]

\[ a_5 = \frac{A^2(1 - \alpha)(dAB - A) + \alpha(d - 2)(2B - A)k^4 + (1 - \alpha)(dAB - A) - (d - 4)B)^2}{A^2k^2(k^2 + A)(A^2 - dAB + (d - 2)(B - A)k^2)} \]

\[ a_6 = -\frac{1}{2} \frac{(A - B)k^2 + AB}{k^2(k^2 + A)(A^2 - dAB + (d - 2)(B - A)k^2)} \]

(4.8)

4.2. Normal modes of the gravity field

After diagonalization of (4.1), we get the following decomposition of the gravity field \( h_{\mu\nu} \):

\[
\frac{d(d+1)}{2} = \frac{(d - 2)(d + 1)}{2} + \frac{d - 1}{2} \left( \frac{1}{\text{spacetime transverse}} + \frac{1}{\text{Stueckelberg vector}} \right) + 1
\]

(4.9)

The first line here describes the decomposition into the irreducible representations of \( SO(d-1) \) in the rest frame, while the second line that w.r.t. the helicity group \( SO(d - 2) \) acting in the space orthogonal (space transverse) to the space momentum \( k \). In fact, these decompositions remain relevant even if the Lorentz invariance is broken down to the spatial-rotation symmetry \( SO(d-1) \). In gauge invariant theory, the Stueckelberg fields do not show up in the Lagrangian (this also requires the sources to be transverse). When the gauge invariance is broken down by mass terms, the Stueckelberg fields acquire non-trivial kinetic terms, and they can mix with the transverse degrees of freedom.

In an arbitrary frame, the two transverse vectors: one from the massive graviton and the other Stueckelberg one get mixed. Also mixed are the four scalars. This means that the characteristic equation, which defines eigenvalues of the kinetic matrix, is an equation of degree \( d+d+1 \) in \( k_0 = \omega \), and \( \zeta \) is actually factorized:

\[
\text{Char}(\lambda) = (\lambda - \lambda_{\text{gr}})^{d+d} P_2(\lambda)^{d-2} Q_4(\lambda)
\]

\[
= (\lambda - \lambda_{\text{gr}})^{d+d} (\lambda - \lambda_{\text{vec}}^+) \lambda_{\text{vec}}^- (\lambda - \lambda_{\text{sc}}^-)^{d-2} \prod_{a=1}^4 (\lambda - \lambda_{\text{sc}}^a) = 0,
\]

(4.10)

where \( P_2 \) and \( Q_4 \) are polynomials of degree 2 and 4, respectively, and all their coefficients as well as \( \lambda_{\text{gr}} \) are quadratic functions of \( \omega \) and \( \zeta \). In other words, \( \lambda_{\text{gr}} \) is some bilinear combination of \( \omega \) and \( \zeta \), \( \lambda_{\text{vec}}^\pm = p_2 \pm \sqrt{p_4} \), where \( p_2 \) and \( p_4 \) are, respectively, quadratic and quartic in \( \omega \) and \( \zeta \), \( \lambda_{\text{sc}}^{1,2,3,4} \) are the roots of a degree four polynomial.
In the rest frame, the roots should be grouped in a different way:

\[
\text{Char}(\lambda) = (\lambda - \lambda_{\text{Gr}})^{\frac{(d-2)(d+1)}{2}} (\lambda - \lambda_{\text{vec}})^{d-1} (\lambda - \lambda_{\text{sc}}^+)(\lambda - \lambda_{\text{sc}}^-)|_{\vec{k}=0} = 0, \tag{4.11}
\]

i.e. at $\vec{k} = 0$

\[
\begin{align*}
\lambda_{\text{vec}}^+(\vec{k} = 0) &= \lambda_{\text{Gr}}(\vec{k} = 0), \\
\lambda_{\text{sc}}^{\text{sp}}(\vec{k} = 0) &= \lambda_{\text{Gr}}(\vec{k} = 0), \\
\lambda_{\text{sc}}^{\text{spS}}(\vec{k} = 0) &= \lambda_{\text{vec}}(\vec{k} = 0),
\end{align*}
\tag{4.12}
\]

where ‘spt’ and ‘sSt’ label the spatial trace $h_{ij}$ and the spatial Stueckelberg scalar $h_{ij} = k_i k_j s$, respectively. The remaining two scalars, the spacetime trace (stt) $h_{\mu\nu}$ and the secondary Stueckelberg scalar $h_{\mu\nu} = k_\mu k_\nu \sigma$, have eigenvalues which are roots of the quadratic equation

\[
\lambda_{\text{sc}}^\pm = q_2 \pm \sqrt{q_4}|_{\vec{k}=0},
\tag{4.13}
\]

where $q_n$ (and $t_n$ in (4.15) below) are degree $n$ polynomials of $\omega$.

In gauge invariant theory, all the $d$ Stueckelberg fields have vanishing eigenvalues, and one gets

\[
\text{Char}(\lambda) = \lambda^d (\lambda - \lambda_{\text{Gr}})^{\frac{(d-2)(d+1)}{2}} (\lambda - \lambda_{\text{vec}})^{d-1} (\lambda - \lambda_{\text{sc}}^+)(\lambda - \lambda_{\text{sc}}^-)|_{\text{GI}} = 0, \tag{4.14}
\]

i.e. in this case the two trace eigenvalues are roots of the quadratic equation

\[
\lambda_{\text{sc}}^\pm = t_2 \pm \sqrt{t_4}|_{\text{GI}}. \tag{4.15}
\]

Actually, gauge invariant will be only the massless gravity (where, by the way, the transition to the rest frame is not a justified operation). The interrelation between modes in various cases can be described by the following table of correspondence between decompositions of the massive graviton into the massive $SO(d)\text{-multiplets}$ (left column), into the massive $SO(d-2)\text{-multiplets}$ (middle column) and into the massless $SO(d-2)\text{-multiplets}$ (right column):

| Rest frame | Normal modes | Gauge invariant (massless) case |
|------------|--------------|-------------------------------|
| Graviton   | → Graviton   |                               |
| Massive graviton | ← Spatial trace → Spatial trace |
| Vector     | → Vector     |                               |
| Stueckelberg vector ← Stueckelberg vector → Stueckelberg $d$-vector |
| Secondary Stueckelberg scalar ← Secondary Stueckelberg scalar |
| Spacetime trace ← Spacetime trace → Spacetime trace |
It is also easy to describe the eigenvectors. In the coordinate system where $\mathbf{k} = k_1$ is directed along the first axis the corresponding modes look as follows (the matrix in the upper left corner is formed by directions 0 and 1):

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & \ldots & h_{ab} \\
0 & 0 \\
\end{pmatrix}
\]

transverse traceless graviton

\[
\begin{pmatrix}
0 & 0 & \ldots & v \\
0 & 0 & \ldots & w \\
\ldots & v & w \\
\ldots \\
\end{pmatrix}
\]

two transverse $(d-2)$-vectors $v$ and $w$

\[
\begin{pmatrix}
\alpha & \beta \\
\beta & \gamma \\
\delta \\
\delta \\
\end{pmatrix}
\]

four scalars $\alpha$, $\beta$, $\gamma$ and $\delta$.

Again these shapes remain the same even after the Lorentz symmetry violation $SO(d-1,1) \rightarrow SO(d-1)$.

### 4.3. Massless gravity

This is the conventional general relativity with $A = B = 0$.

The kinetic matrix (4.1) is actually a symmetric matrix of size $\frac{d(d+1)}{2}$ when acting in the space of normal modes:

\[
\begin{bmatrix}
00 & 01 & 11 & 0a & 1a & aa & ab & bb \\
00 & \cdot & \cdot & \cdot & \cdot & -k_{1\parallel} & -k_{1\parallel} & \cdot \\
01 & \cdot & \cdot & \cdot & \cdot & -\sqrt{2}\omega k_{1} & -\sqrt{2}\omega k_{1} & \cdot \\
11 & \cdot & \cdot & \cdot & \cdot & -\omega^2 & -\omega^2 & \cdot \\
0a & \cdot & \cdot & \cdot & \cdot & k_{1\perp} & \omega k_{1} & \cdot \\
1a & \cdot & \cdot & \cdot & \cdot & \omega k_{1} & \omega^2 & \cdot \\
aa & \cdot & \cdot & \cdot & \cdot & -k_{1\parallel} & -\sqrt{2}\omega k_{1} & -\omega^2 \\
ab & \cdot & \cdot & \cdot & \cdot & -\sqrt{2}\omega k_{1} & -\omega^2 & -\omega^2 + k_{1\parallel} \\
bb & \cdot & \cdot & \cdot & \cdot & -k_{1\parallel} & -\sqrt{2}\omega k_{1} & -\omega^2 \\
\end{bmatrix}
\] (4.16)

where we denote it by square brackets, keeping ordinary brackets for $d \times d$ matrices, like the gravity field $h_{\mu\nu}$.
Its eigenvectors \((d \times d)\) matrices and eigenvalues are

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & h_{ab}
\end{pmatrix}
\]

transverse traceless graviton

\[
\begin{pmatrix}
0 & 0 & \ldots & k_1 & \ldots \\
0 & 0 & \ldots & \omega & \ldots \\
k_1 & \omega & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

transverse \((d - 2)\)-vector

\[
\begin{pmatrix}
0 & 0 & \ldots & -\omega \\
0 & 0 & \ldots & k_|| \\
-\omega & k_|| & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

Stueckelberg \((d - 2)\)-vector

\[
\begin{pmatrix}
-k^2 & 0 & \ldots \\
0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

secondary Stueckelberg scalar

\[
\begin{pmatrix}
k^2 & \omega k & \ldots & 0 \\
\omega k & \omega^2 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \lambda_{\pm}
\end{pmatrix}
\]

two mixed traces, spatial and spacetime

with

\[
\lambda_{\pm} = (d - 3)(-\omega^2 + \vec{k}^2) \pm t_2 \pm \sqrt{t_4},
\]

\[
r^2 = (d - 1)^2 \omega^4 - 2(d^2 - 10d + 17)\omega^2 \vec{k}^2 + (d - 1)^2 \vec{\omega}^2 
\]

\[
= (d - 1)^2(\omega^2 - \vec{k}^2)^2 + 16(d - 2)\omega^2 \vec{k}^2
\]

(4.17)

In other words, in the massless (gauge invariant) case

\[
\lambda_{gr} = \omega^2 - \vec{k}^2,
\lambda_{vec}^+ = \omega^2 + \vec{k}^2,
\lambda_{vec}^- = 0
\]

(4.18)

and \(Q_4(\lambda)\) factorizes, in accordance with (4.15):

\[
Q_4(\lambda) = \lambda^2(\lambda^2 + (d - 3)(\omega^2 + \vec{k}^2)\lambda - (d - 2)(\omega^2 + \vec{k}^2)^2),
\]

so that the two Stueckelberg scalars have \(\lambda_{St} = 0\), and the two trace eigenvalues are

\[
\lambda_{tr}^\pm = \frac{1}{2}((d - 3)(-\omega^2 + \vec{k}^2) \pm \sqrt{(d - 1)^2(\omega^2 + \vec{k}^2) - 2(d^2 - 10d + 17)\omega^2 \vec{k}^2})
\]

(4.20)
4.4. Massive gravity in the rest frame

The rest frame, where \( k_1 = 0 \) is distinguished because the \( SO(d - 1) \) symmetry is fully restored in it, and there is no reason to distinguish between 1 and the other spatial directions. Accordingly, the kinetic matrix is

\[
\begin{bmatrix}
00 & 0i & ii & ij & jj \\
00 & B - A & 0 & -B & 0 & -B \\
0i & 0 & A & 0 & 0 & 0 \\
ii & -B & 0 & B - A & 0 & -\omega^2 + B \\
ij & 0 & 0 & 0 & \omega^2 - A & 0 \\
jj & -B & 0 & -\omega^2 + B & 0 & B - A \\
\end{bmatrix}
\]  

(4.21)

Its eigenvectors \((d \times d)\) matrices and eigenvalues are

\[
\omega^2 - A \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & h_{ab} & 0 \\
\end{bmatrix}
\] transverse traceless graviton

\[
\omega^2 - A \begin{bmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 1 \\
0 & 1 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\] transverse \((d - 2)\)-vector

\[
A \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\] Stueckelberg \((d - 2)\)-vector

\[
\omega^2 - A \begin{bmatrix}
0 & 0 & 0 \\
0 & d - 2 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}
\] secondary Stueckelberg scalar

\[
\lambda_{\pm}^{\text{sc}} \begin{bmatrix}
-(d - 1)B & 0 & 0 \\
0 & \lambda_{\pm}^{\text{sc}} + A - B & 0 \\
0 & 0 & \lambda_{\pm}^{\text{sc}} + A - B \\
\end{bmatrix}
\] two mixed traces, spatial and spacetime

or, if we do not distinguish between the 1 and the other directions

\[
\omega^2 - A \begin{bmatrix}
0 & 0 & \ldots & 0 \\
0 & h_{ij} & 0 & \ldots \\
\end{bmatrix}
\] traceless graviton

\[
A \begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 0 \\
\end{bmatrix}
\] Stueckelberg \((d - 1)\)-vector

\[
\lambda_{\pm}^{\text{sc}} \begin{bmatrix}
-(d - 1)B & 0 \\
0 & \lambda_{\pm}^{\text{sc}} + A - B \\
\end{bmatrix}
\] a mixture of secondary Stueckelberg scalar and spacetime trace.
In other words, in the rest frame in the massive gravity case one has the three possible values of the eigenvalues:

$$\lambda_{vec} = A$$

$$\lambda_{gr} = \omega^2 - A$$

$$\lambda_{sc}^\pm = -\frac{(d-2)\omega^2 + dB - 2A \pm \sqrt{(d-2)^2(B - \omega^2)^2 + 4(d-1)B^2}}{2}$$

two traces.

(4.24)

The masses present in the spectrum can be obtained by solving the equations $$\lambda_i(\omega^2 = m^2) = 0$$. This gives the mass of the graviton multiplet $$m^2 = A$$ and the mass of the two mixing scalars $$M^2 = \frac{A(dB - A)}{d-2(A-B)}$$ (the equation $$\lambda_{sc}^\pm = 0$$ has only one solution). Note that at any values of $$A$$ and $$B$$, one of the two eigenvalues that crosses the abscissa axis has the negative slope at the crossing point. This means that it is a ghost. One may say that, depending on relations between $$A$$ and $$B$$, $$M^2$$ can be positive or negative; in the latter case the ghost does not propagate (and is the tachyon, in fact). However, taking non-zero and large enough $$\vec{k}^2$$, one can get a positive solution to the equation $$\lambda_{sc}^\pm = 0$$ (due to the Lorentz invariance only the invariant combination $$\omega^2 - \vec{k}^2$$ enters the equation).

The only possibility to avoid the ghost is to put $$A = B$$ (the celebrated Pauli–Fierz case), when the mass of the ghost becomes infinite (the equation $$\lambda_{sc}^\pm = 0$$ has no solutions at all) and goes away from the spectrum.

Let us return now to the propagator. The coefficients in the propagator can be rewritten as

$$a_1 = \frac{1}{2m^2} \left( \frac{1 - \alpha}{k^2} - \frac{1}{k^2 + m^2} \right)$$

$$a_2 = \frac{1}{(d-1)m^2} \left( \frac{1}{k^2 + m^2} - \frac{1}{k^2 + M^2} \right)$$

$$a_3 = a_2 + \frac{(1 - \alpha)B}{A(dB - A)} \left( \frac{1}{k^2} - \frac{1}{k^2 + M^2} \right)$$

$$a_4 = -\frac{1}{2} \cdot \frac{1}{k^2 + m^2}$$

$$a_5 = \frac{1}{(d-1)(d-2)} \left( \frac{d-2}{k^2 + m^2} + \frac{1}{k^2 + M^2} \right)$$

$$a_6 = \frac{1}{m^4} \frac{d-2}{d-1} \left( \frac{1}{k^2 + M^2} - \frac{1}{k^2 + m^2} \right) + \frac{1 - \alpha}{m^4} \left( -\frac{m^2}{k^2} + \frac{(d-2)M^2 + m^2}{M^2(d-1)} \left( 1 - \frac{1}{k^2} - \frac{1}{k^2 + M^2} \right) \right).$$

Here we again encounter the two different dispersion laws: $$k^2 + m^2 = 0$$ and $$k^2 + M^2 = 0$$ with

$$m^2 = A, \quad M^2 = \frac{A(dB - A)}{(d-2)(A-B)}.$$  

(4.26)

There is also a fictitious pole at $$k^2 = 0$$ that contributes only at $$\alpha \neq 1$$ and makes the transverse part of the propagator. Therefore, it cancels with the conserved energy–momentum tensor.

As we already discussed, the $$M^2$$ mode in some channels (coefficients $$a_6$$) behaves as a ghost, i.e. enters with a minus sign as compared to the $$m^2$$ mode, but this does not happen in the channels attached to the conserved currents. It, however, enters the radiation, (4.4), the
coefficient $a_2$. Note that this coefficient cancels in the other distinguished case when both the masses are equal

$$m^2 = M^2 \quad \text{when} \quad A = 2B.$$  

(4.27)

The recipe to remove the ghost we mentioned is to bring its mass to infinity. In this Pauli–Fierz limit, the coefficients (4.25) become

\[
\begin{align*}
    &a_1 = \frac{1}{2m^2} \left( 1 - \frac{1}{k^2} - \frac{1}{k^2 + m^2} \right) \\
    &a_2 = \frac{1}{(d-1)m^2 k^2 + m^2} \\
    &a_3 = a_2 + \frac{(1 - \alpha)}{(d-1)m^2 k^2} \\
    &a_4 = -\frac{1}{2} \frac{1}{k^2 + m^2} \\
    &a_5 = \frac{1}{(d-1)(k^2 + m^2)} \\
    &a_6 = -\frac{1}{m^4} \left( \frac{d - 2}{d - 1} \right) \frac{1}{k^2 + m^2} + \frac{1}{m^4} \left( \frac{1}{k^2} + \frac{d - 2}{d - 1} \right) \left( \frac{1}{k^2} \right). 
\end{align*}
\]

(4.28)

The interaction of two stress tensors looks in this case especially simple if $\alpha = 1$:

\[
\begin{align*}
    &-\frac{1}{k^2 + m^2} \left[ (T_{\mu\nu})^2 - \frac{1}{d - 1} \left( T^{\lambda}_{\lambda} \right)^2 + \frac{2}{m^2} T^\mu_{\mu} - \frac{2}{m^2(d - 1)} (kt) T^\lambda_{\lambda} + \frac{d - 2}{m^2(d - 1)} (kt)^2 \right]. 
\end{align*}
\]

(4.29)

where $k^\mu T_{\mu\nu} = t_\nu$. For the conserved stress tensor $t = 0$.

However, beyond the quadratic approximation ghosts show up through the Boulware–Deser instability in curved backgrounds even in the Pauli–Fierz case (it is enough to consider quadratic perturbations but near the non-trivial metric $g_{\mu\nu} = \rho \eta_{\mu\nu}$).

The case of generic frame can be considered completely in the same way. In appendix B, we derive the Lorentzian eigenvalues and eigenmodes in this case.

4.5. The origin of DVZ discontinuity

What does this mean? The Newton potential, describing interaction between the $T_{00}$-components of the stress tensor can be read off from formula (4.3) and looks like

\[
U(r) \sim \frac{e^{-mr}}{r^{d-3}} = \frac{1}{(d - 1)(d - 2)} \left( (d - 2) \frac{e^{-mr}}{r^{d-3}} + \frac{e^{-Mr}}{r^{d-3}} \right). 
\]

(4.30)

with

\[
m^2 = A \quad \text{and} \quad M^2 = \frac{dAB - A^2}{(d - 2)(A - B)}. 
\]

(4.31)

Whenever

\[
A, B \to 0, \quad \text{potential} \quad U(r) \to \frac{1}{r} \left( 1 - \frac{1}{d - 2} \right) = \frac{d - 3}{d - 2} \frac{1}{r^{d-3}}, 
\]

(4.32)

which is also the massless gravity value, except for the special PF case when simultaneously

\[
\frac{A - B}{A^2} \to 0 \quad \text{and} \quad M \to \infty: \quad \text{then} \quad U(r) \to \frac{1}{r^{d-3}} \left( 1 - \frac{1}{d - 1} \right) = \frac{d - 2}{d - 1} \frac{1}{r^{d-3}}. 
\]

(4.33)
This is the DVZ discontinuity.

In other words, this is the discontinuity due to the different limits of the $a_5$ coefficient in the propagator

$$a_5 = - \frac{1}{k^2 + A} \frac{k^2 + \frac{A B}{A-B}}{(d-2)k^2 + \frac{dA^2 - 2AB}{(A-B)}} = - \frac{1}{(d-1)(d-2)k^2 + m^2 + 1} \left( \frac{d-2}{k^2 + m^2 + 1} \right)$$

\[
= \begin{cases} 
- \frac{1}{(d-1)k^2}, & A - B \to 0, \quad A \to 0 \\
- \frac{1}{(d-2)k^2}, & A, B \to 0.
\end{cases}
\]  

(4.34)

In formal terms, the story is about the limit of a function

$$\frac{ax + by}{cx + dy}$$

as $x, y \to 0$: the limit depends on the ratio $x/y$. It is important that this ambiguity is intimately related to the singularity of the function along the line $cx + dy = 0$.

In physical terms, in the PF case, the ghost has an infinite mass and decouples, but in generic situation its mass tends to zero along with graviton’s mass, and they both contribute to the Newton potential. When the masses exactly coincide (in particular, vanish), the ghost simply subtracts/adds to the graviton, but in a different way for different $a_5$-structures. In particular, it adds in $a_5$, which controls the Newton potential.

5. Abandoning Lorentz invariance

The Lorentz violation as a phenomenologically viable possibility was suggested by Kostelecky and Samuel in [27] (see also [28]) already long ago, but it acquired enormous attention quite recently, see [29] for incomplete lists of papers about its possible role in particle physics and cosmology. This paper is focused on the theoretical rather than phenomenological aspects of the Lorentz violation, which were also addressed in [29] and partly reviewed in [13].

5.1. Generalities

The Lorentz violation implies that the global symmetry group $SO(1, d-1)$ is broken down to $SO(d-1)$. Surprisingly or not, this leads to rather drastic changes in the structure of the theory, revealing all the results of breakdown of the gauge invariance which remained hidden in the Lorentz invariant case. In particular, since different reference frames are no longer equivalent, the $SO(d-1)$ symmetry of the spectrum in the rest frame is broken down to the helicity symmetry $SO(d-2)$ in all other frames, which gives rise to highly non-trivial dispersion relations for the elementary constituents (different polarizations) of the gravity field. These slightly unusual features remain obscured in the case of massive vectors and come out of the shadow only for the tensor fields.

From $h_{\mu\nu}$ and $k_{\mu}$, one can make

- 4 $h$-linear $SO(d-1)$-scalars: $h_{00}, h_{\mu\nu}, k_i h_{0\nu}, k_i k_j h_{ij}$,
- 2 $h$-linear $SO(d-1)$-vectors: $h_{0\mu}, k_i h_{ij}$ and
- 1 $h$-linear $SO(d-1)$-tensor: $h_{ij}$
and, thus, $14 = \frac{45}{4} + \frac{23}{4} + 1 = 10 + 3 + 1$ $b$-bilinear structures which contribute to the propagator $\mathcal{P}_{\mu
u,\alpha\beta}$. The current–current interaction then looks like

$$\mathcal{P}_{\mu
u,\alpha\beta} T^{\mu\nu} T^{\alpha\beta} = a_{10}(k_0 T_0) + a_{13}(k_i T_i) + a_{20} T_{00}(k_i k_j T_{ij}) + a_{23} T_{0i}(k_i k_j T_{ij}) + a_{40} T_{00} + a_{41} T_{i}^2 + b T_{00} + a_{50} T_{00} T_{ii} + a_{51} (T_{ii})^2 + c_1 T_{00}(k_i T_{ki}) + c_2 T_{0i}(k_i T_{ki}) + a_{60}(k_i T_{0i})(k_i k_j T_{ij}) + a_{61}(k_i k_j T_{ij})^2.$$  \tag{5.1}

Manifest expressions for the coefficients in the propagator $\mathcal{P}_{\mu\nu,\alpha\beta}$ are quite involved and can be found in appendix C. Our notation takes into account that each structure with coefficients $a_i$ in the Lorentz invariant case is now split into two, and $a_{k0}$, $a_{k1}$ are the two independent coefficients. The coefficient $b$ corresponds to what was a combination of $a_4$ and $a_5$, and $c_i$ correspond to new emerged structures. Note that $a_3$ is absent here, since one no longer keeps the parameter $\alpha$ (in the previous section, we saw it played no important role in massive cases) and define the propagator just by the equation $K \mathcal{P} = E$. This would correspond to $a_2 = a_3$.

Two of these structures, $[khh]^2$ and $[khh][kkh]$, do not appear in the Lagrangian, because they are more than quadratic in momenta. Other structures enter the Lagrangian with adjusted coefficients so that the Lorentz invariance is not broken in $k^2$-terms. By now, a conventional parametrization of the quadratic Lagrangian with manifest Lorentz and general covariance violation in the mass matrix is

$$K_{\mu\nu,\alpha\beta} h^{\mu\nu} h^{\alpha\beta} = (k_i k_\alpha \eta_{\beta\nu} + k_i k_\beta \eta_{\alpha\nu} + k_\alpha k_\beta \eta_{\nu\mu} + k_\alpha k_\beta \eta_{\mu\nu}) - k_i^2 (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\nu\alpha} \eta_{\mu\beta}) - 2(k_i k_\alpha \eta_{\mu\nu} + k_\alpha k_\beta \eta_{\nu\mu} + 2k_i^2 \eta_{\mu\nu} \eta_{\alpha\beta}) h^{\mu\nu} h^{\alpha\beta} + 2m_0^2 h_{00} + 4m_1^2 h_{0i} - 2m_0^2 h_{ij}^2 + 2m_2^2 h_{ii}^2 - 4m_3^2 h_{00} h_{ii}.$$  \tag{5.2}

The Lorentz invariance is restored provided

$$m_0^2 = B - A, \quad m_1^2 = m_2^2 = A, \quad m_3^2 = m_4^2 = B.$$  \tag{5.3}

The propagator (5.1) is obtained by inverting the square matrix in the action

$$\begin{array}{c|c|c|c|c|c|c|c}
\hline
00 & 01 & 11 & 0a & 1a & aa & ab & \hline
m_0^2 & m_1^2 & -m_1^2 & -k_i^2 - m_4^2 & \sqrt{\omega k_1} & \sqrt{\omega k_1} & k_i^2 + m_4^2 & \omega^2 + m_1^2 \\
0a & 1a & aa & ab & \hline
m_1^2 & -m_1^2 & k_i^2 + m_4^2 & \sqrt{\omega k_1} & \sqrt{\omega k_1} & k_i^2 + m_4^2 & \omega^2 + m_1^2 & -k_i^2 - m_4^2 \\
\hline
\end{array}$$

or, in the rest frame

$$\begin{array}{c|c|c|c|c}
\hline
00 & 0i & ii & ij & jj \hline
m_0^2 & 0 & -m_4^2 & 0 & -m_4^2 \\
0i & 0 & m_1^2 & 0 & 0 \\
ii & -m_4^2 & 0 & m_3^2 - m_2^2 & -\omega^2 + m_3^2 \\
ij & 0 & 0 & \omega^2 - m_2^2 & 0 \\
jj & -m_4^2 & 0 & -\omega^2 + m_3^2 & m_2^2 - m_3^2 \hline
\end{array}$$

For special values of the masses the theory acquires some residual gauge invariance [11]:

$$x^i \rightarrow x^i + \xi^i(x^i, t) : \quad \text{if } m_1 = m_2 = m_3 = m_4 = 0$$

$$t \rightarrow t + \xi^0(x^i, t) : \quad \text{if } m_0 = m_1 = m_4 = 0 \quad \tag{5.6}$$

$$x^i \rightarrow x^i + \xi^i(t) : \quad \text{if } m_0 = 1.$$
The PF gravity corresponds to $m_0 = 0, m_1^2 = m_2^2 = m_3^2 = m_4^2$. In general, the graviton mass is $m_2$, and the corresponding $\frac{2(d-3)}{2}$ transverse modes are always split from everything else, while the role of the other mass parameters is to control non-trivial dispersion relations for the other constituents of the gravity field and their severe (and generically inseparable) intermixing. A lot of these complexities disappear if $m_0 = 0$, and such a model exhibits only minor deviations from the intuition developed in experience with Lorentz invariant theories.

5.2. Rubakov’s approach

In the pioneering paper [10], V Rubakov made the first attempt to diagonalize the action (5.2).

He expressed $h_{\mu\nu}$ and the action through the transverse fields ($\chi_{ij}, u_{ij}, \psi, \tau$) and the additional Stueckelberg fields ($s_i, v, \sigma$) and, next, through the gauge invariant vector $w_i$ and scalars $\Phi$ and $\tau$:

\[
\begin{align*}
h_{00} &= \psi, \\
h_{0i} &= u_i + k_i v, \\
h_{ij} &= \tau \delta_{ij} + \chi_{ij} + (k_is_j + k_js_i) + k_i k_j \sigma, \\
k_i \chi_{ij} &= k_j \chi_{ij} = 0, \\
\chi_{ii} &= k_i \chi_{ii} = 0, \\
k_i s_i &= 0.
\end{align*}
\]

Since the gauge transformations $h_{\mu\nu} \rightarrow h_{\mu\nu} + k_{\mu/X_1\nu} + k_{\nu/X_1\mu}$ with $X_1 = 0, X_i = \xi_i + k_i \xi$ and $k_i \xi_i = 0$ act on these fields as

\[
\begin{align*}
\chi_{ij} &\rightarrow k_i \xi_j + k_j \xi_i, \\
u &\rightarrow v + \omega \xi_0, \\
\psi &\rightarrow \psi + 2 \omega \xi_0, \\
\tau &\rightarrow \tau + \omega \xi_0, \\
\sigma &\rightarrow \sigma + 2 \xi, \\
s_i &\rightarrow s_i + \xi_i.
\end{align*}
\]

the gauge invariant vector and scalars are

\[
\begin{align*}
w_i &= u_i - \omega s_i, \\
\Phi &= \psi - 2 \omega v + \omega^2 \sigma, \\
\tau &= \tau.
\end{align*}
\]

and the action for the conserved stress tensor $T_{\mu\nu}, k^\mu T_{\mu\nu} = 0$ is

\[
\begin{align*}
-k^2 \chi^2_{ij} - (d-1)(d-2)\omega^2 \tau^2 + 2 \vec{k}^2 \left( u_i^2 - (d-2) \Phi \tau + \frac{(d-2)(d-3)}{2} \tau^2 \right) & \\
+ \chi_{ij} T_{ij} - 2 u_i T_{0i} - \Phi T_{00} + \tau T_{ii} + m_0^2 \psi^2 + 2 m_1^2 (u_i^2 + \vec{k}^2 v^2) & \\
- m_2^2 (\chi_{ij} + (d-1) \tau^2 + 2 \vec{k}^2 (s_i^2 + \sigma \tau) + \vec{k}^4 \sigma^2) & \\
+ m_3^2 (\vec{k}^2 \sigma + (d-1) \tau)^2 - 2 m_4^2 \psi (\vec{k}^2 \sigma + (d-1) \tau). &
\end{align*}
\]

The first line is a quadratic part of the Einstein–Hilbert action, and it contains nothing but gauge invariant fields. In general relativity, the scalar $\Phi$ does not have a kinetic term and does not propagate. The second line contains gauge and Lorentz violating terms and depends on

5 In order to make a contact of these notations from [10] with notations of the paper [13]:

$\tau = 2 \psi, \ \chi_{ij} = h_{ij}^\ell \ell, \ s_i = -F_i, \ \sigma = 2 E, \ u_i = \delta_i, \ v = B, \ \psi = 2 \phi$. 

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all the fields, in particular, provides kinetic terms for the Stueckelberg fields \(s_i, v_i\) and \(\sigma\) (if \(m_0^2\) and \(m_1^2\) are both non-vanishing).

It is rather straightforward to diagonalize this Lagrangian and analyze its particular eigenvectors: modes of the gravity field. This was done in [10] under a strongly simplifying assumption of \(m_0 = 0\) (which also guarantees the absence of ghosts), the same kind of analysis in a general situation being rather tedious, and only the Stueckelberg sector was studied in [11] for \(m_0 \neq 0\). In what follows, we use a slightly different technique, which also has an advantage of being easily made algorithmic and thus allows one to make tedious calculations in a systematic way with the help of a computer. In fact, as in the previous section, we use two different ways of analyzes: both through manifest constructing the propagator and through immediate diagonalizing the kinetic matrix.

### 5.3. Particle content of the theory

The gravitational field is described by the symmetric \(d \times d\) matrix.

As in the Lorentz invariant case, the \(\frac{d(d+1)}{2}\) components of the gravitational field split into \(\frac{d(d+1)(d-2)}{2}\) components describing the gauge invariant traceless graviton, one scalar trace and \(d\) components of the gauging (Stueckelberg) vectors, which is in turn split into the spacetime transverse vector and the secondary Stueckelberg scalar, see equation \(4.9\) above. In the rest frame (which exists since the fields are massive), these \(\frac{d(d+1)(d-2)}{2}\) components are all degenerate, being related by action of the \(SO(d-1)\) symmetry group. However, if the Lorentz invariance is broken, the degeneration is lifted for \(\vec{k} \neq 0\) and the \(\frac{d(d+1)(d-2)}{2}\) components further split into the transverse graviton, a transverse vector and a scalar, which are representations of the 'helicity' group \(SO(d-2)\) acting in the hyperplane, orthogonal to \(\vec{k}\).

These degeneration properties are most concisely reflected in the determinant formula for the kinetic matrix

\[
\det K = D_{gr}^{d(d-1)} D_{vec}^{d-2} D \overset{\vec{k} = 0}{\longrightarrow} m_1^{2(d-2)} D_{gr}^{d(d-1)} D_{\sigma}, \tag{5.11}
\]

where

\[
D_{vec} = -m_1^2 k_0^2 + m_2^2 \vec{k}^2 + m_3^2 m_0^2 = m_1^4 \left( -k_0^2 + \frac{m_2^2 \vec{k}^2}{m_1^2} + m_3^2 \right), \tag{5.12}
\]

\[
D_{gr} = -k_0^2 + \vec{k}^2 + m_2^2 \tag{5.13}
\]

and \(D\) is a sophisticated expression of power 4 in \(k_0\) and \(|\vec{k}|\):

\[
D = (d - 2)m_0^2 m_1^2 k_0^2 + ((2d - 4)m_0^2 m_2^2 - (2d - 4)m_0^2 m_3^2 + (2d - 4)m_1^2 m_3^2 )
- (2d - 4)m_3^2 m_0^2 k_0^2 + ((d - 2)m_0^2 m_2^2 - (d - 2)m_1^2 m_2^2 )\vec{k}^2 + ((d - 1)m_0^2 m_1^2 )
- (d - 3)m_2^2 m_0^2 m_1^2 - (d - 1)m_3^2 m_0^2 m_1^2 )k_0^2 + ((d - 3)m_0^2 m_2^2 m_3^2 )
- (d - 3)m_2^2 m_0^2 m_1^2 - (2d - 4)m_3^2 m_0^2 m_1^2 + (d - 3)m_0^2 m_3^2 )\vec{k}^2 +
+ (d - 1)m_2^2 m_0^2 m_3^2 m_1^2 - m_3^2 m_0^2 m_1^2 - (d - 1)m_2^2 m_0^2 m_3^2 m_1^2 \tag{5.14}
\]

For \(d = 4\), \(5.14\) gives

\[
D = 2m_2^2 m_1^4 k_0^4 + (-4m_0^2 m_2^2 + 4m_0^2 m_3^2 + 4m_1^2 m_3^2 - 4m_1^4 )k_0^2 + (2m_2^2 m_3^2 - 2m_1^2 m_3^2 )\vec{k}^4
+ (-m_0^2 m_2^2 - 3m_0^2 m_3^2 + m_1^2 m_3^2 )k_0^2 + (m_0^2 m_2^2 m_3^2 - m_2^2 m_0^2 m_1^2
- 4m_2^2 m_1^2 m_3^2 )\vec{k}^2 + 3m_2^2 m_0^2 m_3^2 m_1^2 - m_3^2 m_0^2 m_1^2 - 3m_2^2 m_1^4 \tag{5.15}
\]

\[24\]

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At $\vec{k} = 0$, our $D_{vec}$ turns into $D_{gr}$, and $D$ decomposes into two factors, one of which is also $D_{gr}$:

$$D_{vec} \xrightarrow{\vec{k}=0} m_1^2 D_{gr},$$

$$D \xrightarrow{\vec{k}=0} D_{gr} D_{\alpha},$$

(5.16)

Clearly, a drastic simplification occurs also in the case of $m_0 = 0$ [10], when $D$ turns into

$$D_{vec} \xrightarrow{\vec{k}=0} m_1^2 (m_0^2 m_2^2 + (d - 1)m_3^2 + (d - 2)m_0^2 \omega^2 - (d - 1)m_3^2 m_4^2).$$

(5.16)

One more implication of (5.11) is that only the degrees of freedom from the first braces in expansion (4.9) are dynamical fields, even after violation of the Lorentz and gauge symmetries.

5.4. Normal modes for gravity fields

We manifestly describe the normal modes only in the rest frame in this case, since formulas in the moving frames become very involved and non-transparent. Note that the Lorentz transformation can no longer be used to deduce formulas in the moving frame. Therefore, we explicitly present and discuss, at least, eigenvalues in the generic frame. Note that the rest-frame analysis is already enough to see ghosts.

The rest frame, where $k_\parallel = 0$ is distinguished because the $SO(d-1)$ symmetry is fully restored in this frame, and there is no reason to distinguish between 1 and the other spatial directions.

The eigenvectors ($d \times d$ matrices) and eigenvalues of the kinetic matrix in this case are

$$\omega^2 - m_2^2 \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \lambda_{sc}
\end{pmatrix}$$

transverse traceless graviton

$$\omega^2 - m_2^2 \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 1 & \cdots \\
0 & 1 & 0 & \\
0 & 0 & \cdots & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}$$

transverse $(d-2)$-vector

$$m_1^2 \begin{pmatrix}
0 & 0 & \cdots & 1 & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}$$

Stueckelberg $(d-2)$-vector

$$m_1^2 \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$$

Stueckelberg scalar

$$\omega^2 - m_2^2 \begin{pmatrix}
0 & 0 & 0 \\
0 & d - 2 & 0 \\
0 & 0 & -1
\end{pmatrix}$$

secondary Stueckelberg scalar

$$\lambda_{sc}^\pm \begin{pmatrix}
-(d - 1)m_4^2 & 0 & 0 \\
0 & \lambda_{sc}^\pm - m_0^2 & 0 \\
0 & 0 & \lambda_{sc}^\pm - m_0^2
\end{pmatrix}$$

two mixed traces, spatial and spacetime
or, if one does not distinguish between the 1 and the other space directions,
\[\omega^2 - m_2^2 \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}\] traceless graviton
\[m_1^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\] Stueckelberg \((d - 1)\)-vector
\[\lambda^\pm_{sc} \begin{pmatrix} -(d - 1)m_4^2 & 0 \\ 0 & \lambda^\pm_{sc} - m_0^2 \end{pmatrix}\] a mixture of secondary Stueckelberg scalar and spacetime trace.

In other words, in the massive gravity one has three possible kinds of eigenvalues in the rest frame
\[\lambda_{gr} = \omega^2 - m_2^2 \]
\[\lambda_{vec} = m_1^2 \]
\[\lambda^{\pm}_{sc} = \xi + m_3^2 \pm \sqrt{\xi^2 + (d - 1)m_4^2} \]
\[\xi = \frac{(d - 1)m_1^2}{2} - m_0^2 + m_2^2 - \frac{(d - 2)\omega^2}{2} \] two traces

The last two eigenvalues, \(\lambda^{\pm}_{sc}\), describe only one propagating mass. Indeed, the zeroes of the two equations \(\lambda^{\pm}_{sc} = 0\) are encoded in the equation
\[\lambda^+_{sc}\lambda^-_{sc} = 2\xi m_0^2 + m_0^4 - (d - 1)m_4^4 = 0,\]
which is linear in \(\omega^2\). This mass is manifestly given by
\[M^2 = \frac{d - 1}{d - 2} \left( m_3^2 - \frac{m_4^2}{m_0^2} \right) - \frac{m_2^2}{d - 2}. \] (5.21)

Therefore, in this case of broken Lorentz symmetry, similar to the Lorentz invariant case, there are two propagating masses, \(M^2\) and \(m_2^2 = m_0^2\), although the dispersion laws are far more tricky in the present case, as we shall see in section 5.6 (e.g. some of these modes propagate with another speed of light). Note that the two masses coincide, \(M^2 = m_2^2\) provided
\[ (m_3^2 - \frac{m_4^2}{m_0^2})m_0^2 = m_4^4. \] (5.22)

Now, similar to the Lorentz invariant case, one immediately observes a ghost: since the derivative \(\frac{\partial \omega^2}{\partial \omega^2}|_{\lambda^2}\) is negative at that single point where \(\lambda^{\pm}_{sc}\) crosses the abscissa axis. This ghost may not propagate if the parameters are chosen so that \(M^2 < 0\). However, then one has to look at the non-zero spatial momentum to see if this crossing point \(\lambda^\pm_{sc}\) becomes positive.

As before, one may try to remove this ghost from the spectrum bringing its mass to infinity, which is equivalent to putting \(m_0 = 0\) provided \(m_2, m_3\) and \(m_4\) are finite. Therefore, \(m_0 = 0\) is, at least, a sufficient condition for the ghost-free spacetime, \([10]\)!

5.5. Characteristic polynomials

Now let us analyze modes in the general frame looking at the eigenvalues.

One of the eigenvalues corresponds to the graviton propagator and is
\[D_{gr} = -\lambda_{gr} = -\omega^2 + \vec{k}^2 + m_2^2 = k^2 + m_2^2. \] (5.23)

It describes the \(\frac{d(d-3)}{2}\) graviton modes.
The Stueckelberg and transverse vectors are described now by the $2 \times (d-2)$ eigenvalues\(^6\)
\[
\lambda_{\pm, \text{vec}} = \frac{1}{2} (\omega^2 + \vec{k}^2 + m_1^2 - m_2^2 \pm \sqrt{(\omega^2 + \vec{k}^2)^2 + 2(-\omega^2 + \vec{k}^2)(m_1^2 + m_2^2) + (m_1^2 + m_2^2)^2})
\]
\[= p_2 \pm \sqrt{p_4},
\]
which combine into the polynomial
\[
D_{\text{vec}} = -\lambda_{\text{vec}}^2 \lambda_{\text{vec}}^2 = p_4 - p_2^2 = -m_1^2 \omega^2 + m_3^2 \vec{k}^2 + m_1^2 m_2^2
\]
and are described by the characteristic polynomial
\[
P_2(\lambda) = \lambda^2 - 2p_2 \lambda - D_{\text{vec}}.
\]
The remaining four scalars combine into the quite involved quartic characteristic polynomial
\[
Q_4(\lambda) = \lambda^4 + \lambda^3 ((d-3)(\omega^2 - \vec{k}^2 - m_3^2) - m_0^2 - m_1^2 + 2m_2^2 - 2m_3^2)
\]
\[+ \lambda^2 \left[ (d-2)(\omega^2 + \vec{k}^2)^2 + \omega^2 (2m_2^2 + (d-3)(m_0^2 - m_1^2 + m_2^2 + m_3^2))
\]
\[+ \vec{k}^2(-2m_2^2 + (d-3)(m_0^2 + m_1^2 - m_2^2 + m_3^2 - 2m_4^2))
\]
\[+ ((m_0^2 m_2^2 - m_1^2 m_2^2 - 2m_2^2 m_3^2 - 2m_1^2 m_3^2 + m_1^2 m_0^2))
\]
\[+ \lambda \left[ (d-2)(\omega^4 (m_0^2 + m_1^2) + 2\omega^2 \vec{k}^2 (m_0^2 + m_2^2 + m_3^2) + \vec{k}^4 (m_1^2 - m_2^2 + m_3^2))
\]
\[+ \omega^2 ((d-1)(m_0^2 - m_1^2 + m_0^2 m_1^2 + m_0^2 m_1^2) + (d-3)(m_0^2 m_1^2 + m_0^2 m_1^2 - m_0^2 m_1^2)
\]
\[+ \vec{k}^2 ((d-2)(m_1^2 m_0^2 - m_2^2 m_0^2) + (d-3)(m_0^2 m_2^2 + m_1^2 m_0^2 - m_1^2 m_0^2)
\]
\[+ m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_0^2 + (d-1)(m_0^2 m_2^2 + m_1^2 m_2^2)
\]
\[+ m_1^2 m_0^2 + m_2^2 m_3^2 + m_3^2 m_0^2 + (d-1)(m_0^2 m_3^2 + m_1^2 m_3^2)
\]
\[+ (m_1^2 m_3^2 + m_2^2 m_3^2 + m_3^2 m_0^2))(5.28)
\]
so that $Q_4(0) = -D$. This quantity is definitely the same in the Lorentz and Euclidean modes, since this comes from the determinant of the kinetic operator in the scalar mode sector (similarly, $D_g$ and $D_{\text{vec}}$ are the same).

In the rest frame, where $k = 0$, we definitely return to formulas of the previous subsection
\[
\lambda_{g, r} = \omega^2 - m_2^2, \quad \lambda_{\text{vec}, r} = \omega^2 - m_2^2, \quad \lambda_{\text{scal}, r} = m_1^2,
\]
and $Q_4(\lambda)$ factorizes, in accordance with (4.13):
\[
Q_4(\lambda) = (\lambda - m_2^2)(\lambda - (\omega^2 - m_2^2)) \cdot (\lambda^2 - \lambda (m_0^2 - m_1^2) + (d-1)m_1^2 - (d-2)\omega^2)
\]
\[+ (d-2)\omega^2 m_0^2 - m_2^2 m_0^2 - (d-1)m_1^2 + (d-1)m_3^2 m_1^2).
\]
Therefore, one can see in the moving frame that, of the \(\frac{(d+1)(d-2)}{2}\) modes in the rest frame with $\lambda = \omega^2 - m_2^2$, \(\frac{(d-3)}{2}\) remain the graviton modes, while $d - 2$ modes along with $d - 2$ Stueckelberg modes (with $\lambda = m_1^2$) compose the $D_{\text{vec}}$, and one mode with $\lambda = \omega^2 - m_2^2$, one mode with $\lambda = m_1^2$ and the two trace modes altogether compose $D$ (or $Q_4(\lambda)$).

\[\lambda_{\pm, \text{vec}} = \frac{k^2 + m_1^2 + m_2^2 \pm \sqrt{(m_1^2 - m_2^2)^2 + 2(m_1^2 - m_2^2)(\omega^2 + \vec{k}^2) + k^4}}{2},
\] \[\text{(5.24)}
\]
One can see there are no any simplifications in this case.

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\]
5.6. Ghosts, tachyons and others

With the characteristic polynomial in hand, one can discuss the properties and peculiarities of the spectrum of the Lorentz violating gravity. Basically, there are three different important peculiarities: when there is a ghost, a tachyon and when the speed of the propagation of a mode differs from the light speed. First of all, the tensor sector is the most healthy of all these phenomena.

Superluminal propagation. One can most directly observe the superluminals in the vector sector of the theory. Indeed, the dispersion law $D_{\text{vec}} = 0$ implies that the vectors propagate with the speed $m_2^2/m_1^2$. This speed can be larger or less than the speed of graviton (which is 1), depending on the relation between masses.

However, the main peculiarities are related to the scalar sector of the theory.

Ghost. We already discussed the appearance of a ghost in the scalar sector in section 5.4 and concluded that it may have an infinite mass (i.e. disappears from the spectrum of the linearized gravity, but can easily come back beyond the quadratic approximation as the Boulware–Deser mode), provided $m_0^2 = 0$. Otherwise, there may be a ghost, at least, at $k^2 = 0$ and $M^2 \geq 0$. In other words, there will be a mode which is constant in space and would grow in time (simultaneously in the whole space). Still, this is not a ghost propagating in space.

In order to move slightly away from the $k^2 = 0$ point, i.e. to consider propagation in space, one can find a solution to (5.28) at small $k^2$. The limit of $k^2$ is smooth, and one would come to a ghost again unless $m_1 = 0$. Indeed, one expects a singularity in this latter case, since the determinant (5.11) is proportional to $m_1$ at zero momentum.

Let us make a closer inspection of this particular case. In this case, the dispersion law looks quite strange

$$D \sim \omega^2 k^2 = 0. \quad (5.31)$$

This corresponds to an infinite speed of light for the vector mode and leads to a very exotic ‘excitation’, being a carrier of constant interaction $\omega^2 = 0$. One can actually describe its slightly virtual version (small values of $\omega^2$) analytically: the corresponding eigenvalue is

$$\lambda_{\text{instant}} = \frac{2(d-2)\Delta \omega^2 k^2}{(d-2)(m_1^2 - m_2^2)k^2 - ((d-3)\Delta + 2(d-2)(m_1^2 m_2^2))\omega^2 + (d-1)m_1^2(m_1^2 m_2^2 - m_4^2)^3} + O(\omega^4), \quad (5.32)$$

where

$$\Delta \equiv m_2^2(m_1^3 - m_2^3) - m_4^3. \quad (5.33)$$

This excitation is simultaneously a ghost, $d\lambda/d\omega^2 < 0$ on shell, when $k^2$ lies in between the zeroes of the denominator in (5.32).

Thus, in the specific case of $m_1 = 0$, the ghost is related to excitations with the specific dispersion law, and is not a particle-like ghost. This is probably the reason why it was not recognized as a ghost in [11]. It would be interesting to better understand the physical implications of this excitation.

Tachyons. Now we return to our discussions of the ghost-free regime at $m_0 = 0$. In order to see if there are tachyons, one has to put $\omega = 0$ and look for real solutions of $\lambda(k) = 0$. Note that this is not the same as to look for a mode with negative mass square because of the tricky dispersion law. Indeed, the mass describes the pole of the propagator in $\omega$ at zero $k$, while the tachyon has to do with its $k$-dependence.

In order to guarantee the absence of tachyons in the vector and tensor sectors, one has to require $m_1^2 \geq 0$ and $m_2^2 \geq 0$. Then, it is again enough to look at the product of all four scalar
eigenvalues, which is \( D \). Thus, the tachyon is absent as soon as there is no real-valued solution of the equation
\[
-D(\omega^2, \vec{k}^2)|_{\omega = 0} = \rho \vec{k}^4 + \eta \vec{k}^2 + \zeta = 0,
\]
where
\[
\rho \equiv (d - 2)(m_1^2 m_2^2 - m_3^2 m_4^2)
\]
\[
\eta \equiv 2(d - 2)m_1^2 m_2^2 m_3^2 + (d - 3)(-m_1^2 m_2^2 m_3^2 + m_1^2 m_2^2 m_3^2 - m_4^2 m_1^2)
\]
\[
\zeta \equiv m_2^2 m_3^2 m_4^2 + (d - 1)(m_2^2 m_3^2 m_4^2 - m_2^2 m_3^2 m_4^2) = -(d - 2)m_0^2 m_1^2 m_2^2 M^2.
\]
The tachyon is absent either if the discriminant of (5.34) is negative,
\[
\eta^2 - 4 \rho \zeta < 0
\]
or if both the solutions \( \pm \sqrt{\eta^2 - 4 \rho \zeta} \) of the quadratic equation, (5.34), are negative:
\[
\eta > 0, \quad \rho \zeta > 0.
\]
If neither of these conditions is satisfied, there is a tachyon in the spectrum. In order to have a theory both without the ghost and the tachyon, one can put \( m_0^2 = 0 \) and require that
\[
m_2^2 > m_3^2, \quad 2(d - 2)m_2^2 > (d - 3)m_4^2, \quad m_4^2 \geq 0
\]
(5.38)
(these are the conditions obtained in [10] in \( d = 4 \)) or
\[
[2(d - 2)m_2^2 - (d - 3)m_4^2]^2 < 4(d - 1)(d - 2)m_0^2 (m_2^2 - m_4^2).
\]
We assumed here that \( m_1^2, m_2^2 \) and \( m_4^2 \) are non-zero.

One can also consider the border cases. If one of the masses \( m_1^2, m_2^2 \) or \( m_4^2 \) is zero, \( \zeta = 0 \), which means that there is a massless mode in the spectrum. Now one has to differ between different cases.

If \( m_2^2 = 0 \), there is a tachyon unless also \( m_3^2 = 0 \). In this latter case, the speed of light of the vector mode becomes zero, and the dispersion law
\[
D \sim \vec{k}^4 = 0
\]
implies the mode does not propagate in space at all.

There is also a possibility of \( m_3^2 = 0 \) that leads to a non-propagating mode as well, with the same dispersion law (5.40).

The last border case to consider is \( \rho = 0 \), i.e. \( m_1^2 = m_2^2 \). Then, the tachyon is absent if \( \eta > 0 \). This means
\[
2(d - 2)m_2^2 \geq (d - 3)m_4^2, \quad m_4^2 \geq 0.
\]
In particular, if the equality is realized in these formulas, \( \eta = 0 \) and
\[
2(d - 2)m_2^2 = (d - 3)m_4^2
\]
(5.42)
the dispersion law acquires the form
\[
2(d - 2)(m_2^2 - m_1^2)\omega^2 \vec{k}^2 - (d - 1)m_1^2 m_2^2 \omega^2 + (d - 1)m_1^2 m_2^2 m_4^2 = 0.
\]

No DVZ jump. According to our treatment of the DVZ discontinuity, it occurs if one, while removing a ghost from the spectrum via bringing its mass to infinity, simultaneously removes its contribution to the static potential, the quantity controlled by the \( \vec{k}^2 \)-dependence. In the Lorentz invariant case, these two things inevitably happens together. In contrast, in the non-Lorentz invariant case, the mass of the ghost (=\( \omega^2 \)-behaviour) and the static potential are unrelated and, therefore, the DVZ jump does not happen. Indeed, if one sends \( m_0 \) to zero, it makes the ghost mass infinite and removes it from the spectrum via canceling the coefficient in front of \( \omega^2 \) in \( D \), (5.14). At the same time, the coefficient in front of \( \vec{k}^4 \) in \( D \) persists to be non-zero in this case; hence, the asymptotics of the static potential does not change.
5.7. Dispersion relations

One of the spectacular puzzles of the massive gravity is the emergency of non-trivial dispersion relations \( \omega = \epsilon (|\vec{k}|) \). This looks puzzling because it usually does not happen in the theory of some \( N \) scalar massless fields perturbed by an arbitrary mass matrix:

\[
\sum_{a=1}^{N} \left( k^2 \phi_a^2 + J_a \phi_a \right) + \sum_{a,b=1}^{N} M_{ab} \phi_a \phi_b. \tag{5.44}
\]

In general, the kinetic and mass matrices define the two quadratic forms which can be simultaneously diagonalized, but the diagonalization of \( M_{ab} \) breaks down the diagonal form of the field-current coupling. This phenomenon is well known as the Kobayashi–Maskawa (KM) mixing in the Standard Model of elementary particles [30]. The characteristic equation defining the eigenvalues of such a quadratic Lagrangian without the currents

\[
D_M = \det \left( k^2 \delta_{ab} + M_{ab} \right) = 0 \tag{5.45}
\]

is actually a product

\[
D_M = \prod_{a=1}^{N} \left( k^2 + m_a^2 \right) = \prod_{a=1}^{N} (-\omega^2 + \epsilon_a^2 (|\vec{k}|)). \tag{5.46}
\]

so that the dispersion law is the standard relativistic one, \( \omega = \epsilon_a (|\vec{k}|) = \sqrt{k^2 + m_a^2} \).

If the Lorentz symmetry \( SO(d-1,1) \) is broken down to \( SO(d-1) \), there are in general the three different matrices:

\[
- \sum_{a=1}^{N} \omega_a^2 \phi_a^2 + \sum_{a,b=1}^{N} (N_{ab} k^2 + M_{ab}) \phi_a \phi_b \tag{5.47}
\]

and the characteristic equation, defining \( \omega (|\vec{k}|) \), is more sophisticated:

\[
D_{N,M} = \det \left( -\omega^2 \delta_{ab} + N_{ab} \overline{k}^2 + M_{ab} \right) = 0. \tag{5.48}
\]

While \( D_{N,M} \) is still a product like the last formula in (5.46), the roots \( \epsilon_a (|\vec{k}|) \) can now be highly non-trivial functions of the space momentum \( \vec{k} \). This would explain the origin of the non-trivial dispersion relations, but the problem is that, in the massive gravity, one does not introduce any non-trivial matrix \( N_{ab} \neq \delta_{ab} \): all Lorentz violation is concentrated in the mass matrix and does not affect the kinetic term. This seems to imply that nothing more than the KM mixing can occur with no severe change to the dispersion relations, but this is actually not the case, as we see in equations (5.12)–(5.14) and in section 5.6.

A resolution of the puzzle is in the concept of the Stueckelberg fields: when the mass matrix breaks some gauge symmetry, it gives rise to kinetic terms for the newly revived gauge degrees of freedom. From the point of view of the above scalar theory, this looks strange: if the mass matrix involves more fields than the kinetic matrix

\[
\sum_{a=1}^{N} \left( k^2 \phi_a^2 + J_a \phi_a \right) + \sum_{a,b=1}^{N+n} M_{ab} \phi_a \phi_b, \tag{5.49}
\]

then the extra \( n \) fields \( \phi_a \) should be represented as derivatives, \( \phi_a' = \sum_{b=1}^{n} k_b C_{ab}' \phi_b \). Then the new kinetic term involves new Stueckelberg fields \( \phi_a' \) and breaks the Lorentz invariance, and one comes back to the situation described in (5.48), where the non-trivial dispersion relations are of no surprise. The problem is that the above substitution \( \phi \to \psi \) looks made ad hoc, and it is actually justified only when \( \phi_a' \) describe pure gauge degrees of freedom.
6. Mixing with extra fields and Kaluza–Klein theory

Somewhat amusingly, the mixing of gravity with an additional field was considered already in the seminal paper [2]; however, there it was used just as a technical trick. Recently, this kind of modification of massive gravity was re-introduced in the second paper of [4]. Since then the subject has quickly attracted increasing attention. The idea is to add terms like
\[ h^{\mu\nu}k_\mu\pi_\nu + \pi - \text{squared terms} \]  
(6.1)

or
\[ h^{\mu\nu}k_\mu k_\nu\pi + \pi - \text{squared terms}, \]
(6.2)

and their Lorentz violating counterparts to the quadratic Lagrangian, thus introducing a mixing of the gravity field with something else, which is denoted by \( \pi \) in these formulas. These \( \pi \)-fields can be considered as shifted vector or scalar fields (say, the Goldstone fields describing fluctuations near the vacuum expectation values, which cause a spontaneous violation of the gauge and Lorentz invariances). There is already convincing evidence that such a mixing can substantially soften strange properties of the massive gravity and provide healthy perturbatively reliable models with massive graviton and Lorentz violations.

Of course, this conclusion is of no surprise, because such a healthy theory is well known for decades: this is nothing but the ordinary Kaluza–Klein gravity.

6.1. Example of Kaluza–Klein graviton, \( d + 1 = 5 \) compactification \( \rightarrow d = 4 \):

The Kaluza–Klein (KK) gravity is ordinary general relativity in higher \((d+m)\)-dimensional spacetime, compactified back into \( d \) dimensions. From the \( d \)-dimensional point of view, the theory looks as an infinite KK tower of fields with different masses, all interacting among themselves. However, in the quadratic approximation fields the different KK sectors (with different masses) do not interact and do not mix so that one can safely consider each sector separately. In this sector, one has the massive \( d \)-dimensional graviton, which should be completely free of any kinds of problem, even the gauge invariance (under the general coordinate transformations in \( d \) dimensions) is preserved. The question is how this can be consistent with the seemingly unavoidable pathologies of massive gravity, discussed in the previous sections. The answer is that this massive graviton is being mixed with the other KK fields of the same sector (with the same masses), and this is the simplest possible argument that the addition of extra fields can cure the massive gravity from all its potential problems.

Our task now is to analyze the massive KK gravity (i.e. a given mass level of the KK tower) in some detail in order to see how it works. We restrict consideration to one extra dimension, compactified on a circle of radius \( R_d \), and express everything in units of \( R_d \). Since even the quadratic action for the massive KK fields is not widely known, we begin with its detailed derivation.

6.1.1. Quadratic part of KK action in a given sector. Denote different components of the \((d+1)\)-dimensional graviton through \( h_{\mu\nu}, A_\mu = h_{\mu d} \) and \( \phi = h_{dd} \), where \( \mu, \nu = 0, 1, \ldots, d-1 \). The discrete momentum in the compactified direction is \( k_d = n \), \( n \) is an integer multiple of the inverse radius \( R_d \). We consider the particular sector at a given \( n \), it is not mixed with other sectors in the quadratic approximation.

The Einstein–Hilbert action in this approximation is
\[
2(kh)_{MN}^2 - k^2 h_{MN}^2 = 2(kh)h + k^2 h^2
\]
\[
\rightarrow 2(k_\mu h^{\mu\nu} + n A^\nu)^2 + 2(k_\mu A^\mu + n \phi)^2 - (k^2 + n^2)(h_{\mu\nu}^2 + 2A_\mu^2 + \phi^2)
\]
which is taken into account by the standard parametrization of the KK metric

$$-2(k_\mu k_\nu h^{\mu\nu} + 2n k_\mu A^\mu + n^2 \phi)(h + \phi) + (k^2 + n^2)(h + \phi)^2$$

$$= \left\{ 2(k h)^2 - k^2 h_{\mu\nu}^2 - 2(k h) h + k^2 h^2 \right\} + n^2 (h^2 - h_{\mu\nu})^2$$

$$+ 2(k_\mu k_\nu - k^2 \eta_{\mu\nu}) A^\mu A^\nu + 4n (k_\mu h^{\mu\nu} - k^2 h) \phi A_\nu + 2(k^2 h - (kh)) \phi,$$  \quad (6.3)

where \((kh)_M\) denotes \(k^H H_{\text{NM}}\) etc. The last line describes the \(h - \phi\) mixing, which exists even in the massless sector, at \(n = 0\).

The standard trick in the KK theory is to eliminate this mixing by the shift

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \frac{1}{d-2} \phi \eta_{\mu\nu},$$

which is taken into account by the standard parametrization of the KK metric

$$e^{-\frac{\phi}{d-2}} \left( \frac{g_{\mu\nu} + e^\phi A_\mu A_\nu}{e^\phi A_\nu e^{\phi}}, \frac{\eta_{\mu\nu}}{0} \right) + \left( \frac{h_{\mu\nu} - \frac{1}{d-2} \phi \eta_{\mu\nu}}{A_\nu} \right) \frac{1}{d-2} \phi$$

+ nonlinear terms, \quad (6.5)

with \(\phi = \frac{d-2}{d-1} \phi\). After this shift, one gets the quadratic part of the KK action in the form

$$\left\{ 2(k h)^2 - (k^2 + n^2) h_{\mu\nu}^2 - 2(k h) h + (k^2 + n^2) h^2 \right\}$$

$$+ 2(k_\mu k_\nu - k^2 \eta_{\mu\nu}) A^\mu A^\nu + 4n (k_\mu h^{\mu\nu} - k^2 h) \phi A_\nu$$

$$+ \frac{d-1}{d-2} \left( -k^2 \phi^2 + 2n \phi (2k_\mu A^\mu - nh) + \frac{d}{d-2} n^2 \phi^2 \right).$$

(6.6)

Now one has the familiar action in the massless sector \((n = 0)\), which describes the gravity plus photodynamics plus additional neutral Brans–Dicke scalar. However, for \(n \neq 0\) the action is still strange: the graviton has mass \(n\), but there is no mass term for the photon and the scalar has mass, different from \(n\) (worse than that the scalar ‘mass term’ has a wrong sign!). Instead, there is a severe mixing between all the three fields: \(h_{\mu\nu}, A_\mu\) and \(\phi\). To highlight the problem, one can rewrite the last line in (6.6) as

$$\frac{d-1}{d-2} \left( - (k^2 + n^2) \phi^2 + 2n \phi (2k_\mu A^\mu - nh) \right) + 2 \left( \frac{d-1}{d-2} n \phi \right)^2$$

\quad (6.7)

and the last term is especially strange.

Of course, the diagonalization of this action is not a big problem, being, in fact, a literal repetition of that for the massless gravity, only in the \(d + 1\) spacetime dimensions. Then, one certainly obtains \((d + 1)(d - 2)/2\) propagating modes with the mass \(n\) (which form the tensor multiplet of \((d + 1)-\)dimensional gravity with the \(d\)th component of spatial momentum equal to \(n\)), \(d + 1\) zero modes (corresponding to the Stueckelberg fields) and \(d(d + 1)/2\) non-propagating modes (which involve the longitudinal graviton). This result is guessed without any calculations, after some prejudices are thrown away. Still, before we proceed to the answer, it is instructive to analyze immediate peculiarities of the KK gravity.

6.1.2. Properties of massive KK graviton.

- The last term in the first line of the \(d\)-dimensional action (6.3) implies that the KK graviton corresponds to the PF choice \(A = B\).
- However, the gauge invariance is not broken, because of the \(h - A - \phi\) mixing. Indeed, one can easily check that (6.3) is invariant under

$$\delta h_{\mu\nu} = k_\mu \xi_\nu + k_\nu \xi_\mu, \quad \delta A_\mu = n \xi_\mu + k_\mu \xi, \quad \delta \phi = 2n \xi.$$

(6.8)

- Furthermore, there is no ghost, because this is the PF gravity.
As already mentioned, there should not be any BD instability, since the Kaluza–Klein theory is expected to be free of any pathologies, being the standard (massless) 5d Einstein gravity on a specific manifold (with one periodic coordinate).

6.1.3. Diagonalizing KK massive sector. We can now return to the problem of analyzing the KK Lagrangian (6.6). In fact, it is better to return one step back to (6.3). The peculiarity of the KK theory is that the massless \((n=0)\) and massive \((n \neq 0)\) sectors should be handled in two very different ways. While in the massless sector one makes the celebrated shift (6.4), which leads to complete separation of the massless graviton, vector and scalar fields in the kinetic matrix, in the massive sector one should make an absolutely different shift

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} + \frac{1}{n}(k_\mu A_\nu + k_\nu A_\mu) - \frac{1}{n^2} k_\mu k_\nu \phi.
\]

(6.9)

Note the two things: first, the coefficients have \(n\) in the denominator; thus, this shift could not be done in the massless sector, and second, after this shift the new field \(h_{\mu\nu}\) is not affected by the gauge transformations (6.8) at all, it should be itself considered as gauge invariant. The result of the shift (6.9) in (6.3) is spectacular: the last two lines are fully eliminated, i.e. this shift reduces the sector \(n \neq 0\) to just the single-PF massive graviton

\[
2(kh)_M^2 - k^2 h_{\mu\nu}^2 - 2(khk)h + k^2 h^2 \rightarrow
\]

\[
\left\{
\begin{aligned}
2(kh)_M^2 - k^2 h_{\mu\nu}^2 - 2(khk)h + k^2 h^2 + n^2 (h^2 - h_{\mu\nu}^2) \\
+ 2(k_\mu k_\nu - k^2 \eta_{\mu\nu}) A^\mu A^\nu + 4n(k_\mu h^{\mu\nu} - k^{\mu} h A_\nu) + (2k^2 h - (kh))\phi
\end{aligned}
\right.
\]

\[
\frac{2(kh)_M^2 - k^2 h_{\mu\nu}^2 - 2(khk)h + k^2 h^2}{2} + n^2 (h^2 - h_{\mu\nu}^2).
\]

This is of course what one could expect from the very beginning: the \(\frac{(d+1)(d-2)}{2}\) degrees of freedom of the \((d+1)\)-dimensional massless graviton can turn either into the \(\frac{(d+1)(d-2)}{2}\) modes of the massless graviton + massless vector + scalar in \(d\) dimensions in the \(n = 0\) sector or into the \(\frac{(d+1)(d-2)}{2}\) degrees of freedom of the \(d\)-dimensional massive graviton in the \(n \neq 0\) sectors. There is simply no room for anything but the massive graviton in the \(n \neq 0\) sector, thus nothing like the massive vector or scalar can exist there in addition to the massive graviton.

It is also instructive to look at the same counting from the point of view of the eigenvalues. The field \(h_{\mu\nu}\) in \(d + 1\) dimensions had the kinetic matrix with \(\frac{(d+1)(d-2)}{2}\) eigenvalues \(\lambda_i\), of which exactly \(d + 1\) were vanishing. This left the \(\frac{(d+1)(d+2)}{2} - (d + 1) = \frac{2(d+1)}{2}\) eigenvalues, as needed for the kinetic matrix of the \(d\)-dimensional symmetric matrix. We saw in section 4 that all these eigenvalues are indeed non-vanishing, though not all are associated with propagating particles.

6.1.4. Compactification of a vector field. To get a better illustration of what happened in the previous subsection, one can repeat the same trick for the photon field: consider the massless \((d + 1)\)-dimensional photon \(A_\mu\) and look at what happens to it after the compactification to \(d\) dimensions, where it turns into a \(d\)-component vector \(A_\mu\) and a scalar \(\phi\). The Lagrangian

\[
(k_M k_N - (k^2 + n^2) \delta_{MN}) A^M A^N = (k_\mu k_\nu - (k^2 + n^2)) A^\mu A^\nu + 2n(k_\mu A^\mu) \phi - k^2 \phi^2.
\]

(6.10)
In the simplest case of $d = 1$, one has in $(1 + 1) = 2$ dimensions just $(k\phi - nA)^2$ with the kinetic matrix

$$
\begin{pmatrix}
-n^2 & kn \\
kn & -k^2
\end{pmatrix}
$$

and the eigenvalues $\lambda = 0$ and $\lambda = k^2 + n^2$. From the one-dimensional point of view, one has instead a single mode $(k\phi - nA)$ with the eigenvalue $\lambda = 1$. The difference is dictated by the normalization: the usual fact for quadratic forms.

A similar phenomenon occurs for gravitons: the non-vanishing eigenvalues in $d+1$ and $d$ dimensions are in one-to-one correspondence, but do not literally coincide because of the different normalizations.

### 6.2. DVZ discontinuity

Since the KK graviton is the PF one, it is not too big a surprise that the DVZ jump occurs when the KK radius tends to infinity and the graviton mode with a given $n \neq 0$ becomes massless. However, in the KK case, the discontinuity has a simple explanation: there are additional fields, $A_\mu$ and $\phi$ and they also contribute to the interaction of the stress tensors. The discontinuity is exactly the contribution of these extra fields.

The Born interaction between the two stress tensors through the exchange of the KK graviton from the given-$n$ sector can be immediately read out from the massless case in (4.34) by making the two changes: $d \to d + 1$ and $k_d \to n$. We consider only the interaction between the conserved $d$-dimensional stress tensors, while $T_\mu d = T_{dd} = 0$ (actually this does not affect the formulas too much). The result is

$$
\frac{1}{k^2 + n^2} \left( T_{\mu\nu}^2 - \frac{1}{(d + 1) - 2} T^2 \right)
$$

and, in the massless limit $n \to 0$, (i.e. $R_d \to \infty$) one obtains

massless limit of KK gravity mode:

$$
\frac{1}{k^2} \left( T_{\mu\nu}^2 - \frac{1}{d - 1} T^2 \right)
$$

which is different from the answer for the ordinary massless gravity in $d$ dimensions,

massless gravity:

$$
\frac{1}{k^2} \left( T_{\mu\nu}^2 - \frac{1}{d - 2} T^2 \right).
$$

As we already said, this is not a big surprise, because to the KK answer the other fields are contributing. Moreover, (6.12) coincides with the contribution of the massless KK sector with $n = 0$, and there the contribution of the other fields is very simple: at $n = 0$ there is no mixing in (6.6), and the only source of corrections is that $T_{\mu\nu}$ is coupled to shifted $h_{\mu\nu} + \frac{1}{d-2} \phi \eta_{\mu\nu}$ instead of $T_{\mu\nu}$. This provides an additional contribution from the $\phi$-exchange, which is equal to

$$
+ \left( \frac{1}{d - 2} \right)^2 \frac{d - 2}{d - 1} T^2 \frac{1}{k^2} \left( T_{\mu\nu}^2 - \frac{1}{(d - 1)(d - 2)} \frac{T^2}{k^2} \right)
$$

(\text{the second factor takes into account the coefficient in front of the kinetic term for $\phi$ and should be added to the pure graviton exchange (6.13), thus changing $-\frac{1}{d-2}$ to $-\frac{1}{d-2} + \frac{1}{(d-1)(d-2)} = -\frac{1}{d-1}$ which reproduces (6.12).})

In other words, one observes that in the KK theory the current–current interaction gets contributions from three channels,

$$
\text{KK theory} = \text{graviton} + \text{ghost} + \text{scalar field},
$$

(6.15)
while in the massless gravity there are two channels

\[ \text{Massless gravity} = \text{graviton} + \text{ghost} \]  

and in the PF theory, where the ghost is removed out of the spectrum by taking its mass to infinity, there is only the graviton contribution

\[ \text{PF theory} = \text{graviton}. \]  

As we saw above, the contributions of the ghost and the scalar field exactly cancel each other so that the current–current interactions in the KK theory and in the PF theory coincide. If, however, one considers the KK theory with several compactified dimensions, i.e. with several scalar fields added, this compensation no longer takes place, and all the three interactions, (6.15), (6.16) and (6.17) are different.

7. Massive gravity within and beyond the quadratic approximation

In this section, we make very brief comments about other aspects of the massive gravity, some of which are not directly seen at the linearized level, but are, in fact, direct consequences of properties of the quadratic theory. These are pronounced and affective phenomena, but they do not add much that is new to the theoretical aspects of the problem.

7.1. Vainshtein radius \[8\]

This celebrated result (see also [37]) is often considered as a clear proof of pathology of the PF massive gravity, though the actual statement is much simpler. The new modes, revived by addition of the mass terms to the Einstein–Hilbert action, get the kinetic terms because, at least, some of them are the Stueckelberg fields. We emphasize once again that one does not modify the kinetic part of the Einstein–Hilbert action; only the masses (more generally, a non-trivial potential) are added. This means that the new kinetic terms enter with small coefficients \( m^2, m^2 k^2 s^2 \), and this means that the physical fields are \( ms \) rather than \( s \). This means that higher degree terms, say, \( m^2 s^3 = \frac{1}{m} (ms)^3 \) are actually entering with large couplings \( \sim m^{-1} \), and that the theory is actually strongly coupled; moreover, the interactions become stronger with the decrease of \( m^2 \). Interaction effects are of course falling with the distance; therefore, very far away, beyond some ‘Vainshtein radius’ the perturbative regime can be still reliable, but in general the perturbation theory does not make much sense at small and even moderate distances.

This strong coupling effect is a result of explicit breakdown of the gauge symmetry, and it is rather similar to what happens when the Higgs mass goes beyond the unitary limit in the case of spontaneously broken gauge symmetry in Yang–Mills theory. Nothing like this strong coupling phenomenon occurs in the case of gauge invariant KK massive gravity and, hence, it can be avoided when the gravity mixes with other fields.

7.2. Ghosts

The ghost is the field which enters the Lagrangian with the wrong sign in front of the time derivative term \( \dot{\phi}^2 \). What happens to it is that the field grows in time until this growth is stopped by nonlinear terms in the action or the theory gets into a different phase with another spectrum of quasiparticles. In this sense, this is a phenomenon of the same class as the previous one: the ghost makes the perturbative treatment unreliable and most often inadequate. Within this context, one often speaks about ‘negative norms’ and ‘violation of unitarity’, but this actually refers to ‘the perturbative unitarity’, implying that the actual spectrum of the theory is more
or less accurately described by the quadratic part of the Lagrangian, what is not always the case. While the occurrence of the ghost clearly means that the theory is not what we thought it would be, it does not necessarily mean that it is ill and incurable: it is just not perturbative and most probably strongly coupled.

There are three ways to deal with ghosts. The first option is to study the theory as it is (which is rarely done, see, however, [31] or the series of works about the ‘pathological’ Hamiltonian $H = xp$ [32]). The second option is to say that the growth of ghosts is slow enough to be acceptable (e.g., the ghost formally appears in the Lorentz invariant massive gravity with $A = 2B$, which corresponds to adding the cosmological term to the Einstein–Hilbert Lagrangian; what happens is that the fields grow together with the growth of the Universe itself). The third and the most popular option is to make a fine tuning of parameters in order to ‘eliminate’ ghosts, for example, to give them an infinite mass.

7.3. Boulware–Deser mode [9, 13]

The last option is exactly the one, ‘distinguishing’ the Pauli–Fierz gravity with $A = B$. However, the way in which the ghost acquires the infinite mass is somewhat special and actually unreliable. The mass is infinite because the coefficient in front of the kinetic term vanishes. As we already discussed, this is what actually makes the theory strongly coupled; moreover, this of course makes the fine tuning fully unreliable. Any minor deformation of the theory destroys the fine tuning and brings the ghost back to existence. The Boulware–Deser instability is a concrete example of this phenomenon: switching on a non-trivial background metric contributes to the coefficient in front of the kinetic term and shifts it away from zero. Since the background is arbitrary it cannot be compensated by a variation of just two adjusting constants $A$ and $B$.

Of course, nothing like this happens in ghost-free generalizations of the PF gravity such as the KK or other models with extra fields. Ghosts are absorbed into these additional fields in a universal, background independent way.

7.4. DVZ discontinuities

As we already explained, these discontinuities occur when comparing two theories with different sets of fields and, thus, are of no surprise. However, from the point of view of the concrete example of the massless gravity, it is instructive to distinguish between the two situations: when one compares it to the theory with more fields and with less fields.

The first is, for example, the case of KK gravity: in a given mass sector of the KK theory there are two more fields, $A_{\mu}$ and $\phi$, which mix with the massive graviton and also contribute to the Newton-like interaction. This additional contribution explains the difference.

The second is the Pauli–Fierz gravity, where one of the modes, contributing to the Newton interaction in the massless theory, becomes a ghost and is thrown away ‘by hands’. This explains the discontinuity. It is in no way a property of the massive gravity, rather it is a property of this artificial ‘throwing away’ prescription. The reason why the would-be ghost is allowed in the massless gravity is that it has the same mass as the other (non-ghost) degrees of freedom, and this results into decoupling of the negative norm states from the spectrum which leaves the theory perturbatively consistent (we do not speak about the UV problems of quantum gravity here). Notably, the same kind of absorption could be expected if instead of the Pauli–Fierz choice $A = B$ we put $A = 2B$ when the two masses continue to coincide, $M = m$. Then, there will be no DVZ discontinuity. This is the case of the cosmological term added to the Einstein–Hilbert action, and, because of the presence of a linear term in the
Lagrangian, one has to re-expand the latter above the non-flat AdS vacuum where the linear term cancels [33] (see also footnote 7 in [13] about this option).

7.5. Tachyons

The name tachyon refers to a superluminal propagation. However, in modern literature, it is actually used to mean something different: a mode that reflects a perturbative instability of the background around which the perturbative theory is developed. Technically, this means that there is a pole in the propagator at vanishing frequency, for example, $M^2 < 0$. Physically, this means that a phase transition of the first kind occurs from the perturbatively unstable vacuum to another one, stable, at least, perturbatively (false vacua which are separated from the real ones by potential barriers do not have tachyons in the perturbative spectra, their instability is essentially a non-perturbative phenomenon). Such a phase transition takes place spontaneously and independently in all points of the space: this can look like propagating a non-causal (superluminal) particle, but has a clear reason, and the processes taking place in different places are indeed casually unrelated. Tachyons are in no way a problem of the theory; they just signal that we treat (interpret) it in a wrong way. It is of course not always easy to find the right vacuum; this is often related to finding the correct non-perturbative formulation of the theory. The celebrated example of such a lasting study is interpretation of tachyons in string theory (for open superstring the puzzle was partly resolved by A Sen in [34], in generic string models it remains obscure).

What is the right vacuum of the Lorentz violating massive gravity with tachyons, and what at all is its adequate non-perturbative formulation is an open problem (not surprisingly given by the young age of this subject). However, one may expect that this vacuum does not have to be homogeneous, due to non-trivial dispersion laws (see section 5.6). Moreover, phenomenologically this may be not that bad, since the scale of these inhomogeneities is determined by the inverse masses that breaks the Lorentz invariance, the masses being very small. Inhomogeneities at such large scales are quite possible.

7.6. Superluminal propagation [23, 26]

We reserve the words ‘superluminal propagation’ for the phenomenon which is (at least, looks) different from the tachyons, i.e. is not related in any clear way to the perturbative instabilities and phase transitions. This is an occurrence of poles in the propagator at $\omega = ck$ with $c > 1$. Such poles are forbidden by the Lorentz invariance, but we saw that they naturally appeared in the spectrum of the massive gravity when the Lorentz invariance was violated (for example, $c^2 = m_1^2/m_2^2$). The meaning of superluminal dispersion laws remains controversial; we feel that most people find them unpleasant, referring mostly to causality arguments, which are also used against time machines, which are surely not forbidden by the laws of nature (see [22] for the most recent discussion). Quite similarly, the superluminals are unavoidably present in our theories, whether we like them or not. Whenever one perturbs a light-like dispersion law $\omega \sim k^2$ for a collection of light-like particles (two polarizations are already sufficient) in a Lorentz violating way, the eigenvalues split, and one goes above, another goes below $c = 1$: this is a fundamental law of linear algebra, well known as the level splitting in quantum mechanics. It is important that the Lorentz violation needs not be ‘fundamental’ (i.e. explicitly written down into the Lagrangian): a Lorentz violating background is already enough. The celebrated example is an emergency of superluminal photons in curved space [26], where the effective Lagrangian acquires a quantum correction (1-loop of the fermion or any other field in the
gravitational background) and becomes

\[ F_{\mu \nu} F_{\alpha \beta} \left( g^{\mu \alpha} g^{\nu \beta} + \text{const} \cdot R^{\mu \alpha \nu \beta} + \ldots \right) \]  

(7.1)

which, according to the above mentioned linear algebra theorem, unavoidably leads to the dispersion law with \( c > 1 \) for one of the photon polarizations.

Implications of superluminals are still badly understood. It is not even fully clear if they can be used to construct time machines, either big (of astrophysical scale)\(^7\) or mini (of Planckian scale) [22]. In any case, we repeat that time machines are allowed already in ordinary general relativity and, within the hypothetical TeV-gravity models [20], might even be massively created (though immediately evaporate) in particle collisions in modern accelerators [22].

Most important, our intuition about the Lorentz non-invariant theories is still very underdeveloped. We always discuss our models (in this context) from the point of view of an outside observer which believes the fundamental Lorentz invariance: one asks what happens if one looks at a superluminal from another frame (what happens is a singularity in the dispersion rule \( \omega + \beta k = c(k + \beta \omega) \) i.e. \( \omega = \frac{c - \beta}{1 - \beta c} \) at \( \beta = 1/c \) for \( c > 1 \) instead of the usual zero at \( \beta = c \) for \( c < 1 \)) and discuss whether this is good or bad. However, if one just looks at the theory per se, it is absolutely unclear what is going to be bad about it: there is simply no the Lorentz invariance, and it is unclear why one should ask at all what happens if one makes such a transformation. It is like writing down a theory with a rescaled x-coordinate and applying the Lorentz transformation with an old (not-rescaled) x. Even more important would be to examine what can be wrong with just a combination of two fields

\[ L = \frac{1}{2} \left( \phi_1^2 - c_1^2 \nabla \phi_1^2 + \phi_2^2 - c_2^2 \nabla \phi_2^2 \right) + V(\phi_1, \phi_2). \]  

(7.2)

If \( c_2 > c_1 \), the second field is a superluminal from the point of view of the first one, but nothing wrong is expected if one looks at the first field from the point of view of the second one.

7.7. Radiation of massive gravitons

As mentioned in the introduction, one of our original questions was if the ‘pathologies’ of the massive gravity could somehow affect the analysis [19] of the massive graviton radiation in the TeV-gravity models, a worry, naturally implied by the original analysis in [18]. However, the Kaluza–Klein theory (which is at the base of the Tev-gravity models) seems to be free of any problems, real or imaginary, of the generic massive gravity, and our current feeling is that one can treat the radiation of the massive Kaluza–Klein gravitons in the straightforward and naive way, as suggested [19]. The problem, however, deserves an independent analysis.

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\(^7\) See the detailed bibliography in the second paper of [22].
Appendix A. Massive photon

To give more insight into the methods we use in the paper, we illustrate them in this appendix in the case of a massive photon, both when the Lorentz symmetry is not broken and when it is broken. This case is much simpler than the corresponding gravity theories; however, the formulas contain most of peculiarities already in this case.

A.1. Massive photon

This is the theory with the action (2.1). In this subsection, we are going to consider for illustrative purposes the kinetic matrix not as in (2.2) with both lower indices, but with one lower and one upper. This leads to the Lorentzian eigenvalues, in contrast to the Euclidean ones obtained from (2.2). This issue is discussed in section 3; one can also find the comparison of the Euclidean and Lorentzian eigenvalues in the massive photon case in [6].

Thus, now the kinetic matrix is

\[
\begin{pmatrix}
-\omega^2 - (k^2 + M^2) & -\omega k_\parallel & 0 \\
\omega k_\parallel & k_\parallel^2 - (k^2 + M^2) & 0 \\
0 & 0 & -(k^2 + M^2)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\omega^2 - M^2 & -\omega k_\parallel & 0 \\
\omega k_\parallel & \omega^2 - M^2 & 0 \\
0 & 0 & \omega^2 - (k_\parallel^2 + M^2)
\end{pmatrix}
\] (A.1)

Its eigenvectors and eigenvalues are

\[
v_\parallel' = \begin{pmatrix}
-\omega \\
\frac{\sqrt{k_\parallel^2 - \omega^2}}{k_\parallel} \\
0
\end{pmatrix}
\]

\[\lambda_\parallel = -M^2 \]

the former Stueckelberg scalar

\[
v_\parallel'' = \begin{pmatrix}
-\frac{k_\parallel}{\sqrt{k_\parallel^2 - \omega^2}} \\
\frac{\sqrt{k_\parallel^2 - \omega^2}}{\sqrt{k_\parallel^2 - \omega^2}} \\
0
\end{pmatrix}
\]

\[\lambda_\parallel'' = \omega^2 - k_\parallel^2 - M^2 \]

the former longitudinal photon (a scalar)

\[
v_\perp'' = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

\[\lambda_\perp'' = \omega^2 - k_\parallel^2 - M^2 \]

transverse \((d - 2)\)-vector.

As in any linear algebra problem, the normalization of modes is not fixed, and one can rescale them by a factor. It is chosen so that the modes are (minimal possible) polynomials in \(\omega\) and \(k\).

The gauge field \(A_\mu\) is expanded in different sorts of photons

\[
A^\mu = A_\parallel v_\parallel''^\mu + A_\perp v_\perp''^\mu + A_\perp' v_\perp'\mu.
\] (A.3)

The Lagrangian, when expressed through the normal modes, becomes diagonal:

\[
-M^2 A_\parallel^2 + (M^2 - \omega^2 + k_\parallel^2) A_\parallel^2 + (\omega^2 - k_\parallel^2 - M^2) A_\perp^2 + J_\perp A_\perp + \frac{J_\parallel k_\parallel - J_0 \omega}{\sqrt{k_\parallel^2 - \omega^2}} A_\parallel + \frac{-J_0 k_\parallel + J_\parallel \omega}{\sqrt{k_\parallel^2 - \omega^2}} A_\parallel.
\] (A.4)

\[\text{Since we wish to avoid higher derivatives for the Stueckelberg fields (not because they are bad, the higher derivatives theory suffers from the most ghosts, and even this is not unavoidable [31], see [35] for a recent summary and a list of references, but simply to somehow restrict our moduli space), we do not consider the other popular model of the massive photon, with } (\partial_\mu A^\mu)^2 \text{ term (ironically, this was exactly the model analyzed by Stueckelberg in [36]).}\]
Accordingly, the Born interaction of currents is
\[ -\frac{1}{4} \frac{M^2 (J_0^2 - J_1^2 - J_2^2) - (J_0 \omega + J_1 k_1)^2}{M^2 (M^2 + k_1^2 - \omega^2)} . \]  
(A.5)

Even if the gauge invariance is broken, one keeps the currents conserved, \( k_{\mu} J^\mu = 0 \), then the Born interaction converts into
\[ -\frac{1}{4} \frac{J_0^2 (k_1 - \omega^2)}{k_1^2 (M^2 + k_1^2 - \omega^2)} - \frac{J_1^2}{(M^2 + k_1^2 - \omega^2)} \]
\[ = -\frac{1}{4} \left( \frac{J_0^2 (k_1^2 - \omega^2)}{M^2 + k_1^2 - \omega^2} - \frac{J_1^2}{M^2 + k_1^2 - \omega^2} \right) . \]
(A.6)

The propagator, the inverse of the kinetic matrix, is
\[ \begin{pmatrix}
\frac{M^2 - \omega^2}{(\omega^2 - k_1^2 - M^2)M^2} & 0 \\
\frac{-\omega k_1}{(\omega^2 - k_1^2 - M^2)M^2} & 0 \\
0 & \frac{1}{\omega^2 - (k_1^2 + M^2)}
\end{pmatrix} , \]
(A.7)
and this provides for the Born interaction of currents:
\[ -\frac{1}{4} \frac{J_0^2 (k_1^2 - \omega^2)}{k_1^2 (M^2 + k_1^2 - \omega^2)} - \frac{J_1^2}{(M^2 + k_1^2 - \omega^2)} \]
\[ = -\frac{1}{4} \left( \frac{J_0^2 (k_1^2 - \omega^2)}{M^2 + k_1^2 - \omega^2} - \frac{J_1^2}{M^2 + k_1^2 - \omega^2} \right) . \]
(A.8)

which coincides with (A.5) and (A.6). The poles at \( \omega = 0 \) and \( \kappa = 0 \) are spurious, as we already discussed in section 2.2. Moreover, in this case, there is no pole at \( \omega = k = 0 \), and thus no long-range interactions as well, as can be seen from the explicitly Lorentz invariant formula on the lhs of (A.8).

Define the propagator as a solution to \( K_{\mu\nu} P^{\alpha\nu} = \delta^{\alpha\mu} - (1 - \alpha) \frac{k_{\mu} k_{\nu}}{M^2} \), \( k^2 = -\omega^2 + k_1^2 \). Then, the smooth matching with the massless case (2.15) at \( \alpha = 0 \) is provided by the expression for the propagator
\[ P^{\mu\nu} = \begin{pmatrix}
k_1^2 M^2 & -2M^2 k_0^2 + 2k_1^2 \\
-2M^2 k_0^2 + 2k_1^2 & M^2 - 2k_0^2 k_1^2 + 2k_1^4 & 0 \\
-2M^2 k_0^2 + 2k_1^2 & M^2 - 2k_1^2 M^2 & k_0^2 (M^2 + 2k_1^2) \\
-2M^2 k_0^2 + 2k_1^2 & M^2 - 2k_1^2 k_0^2 + 2M^2 & 0 \\
0 & M^2 - 2k_1^2 M^2 & k_0^2 (2k_1^2 + M^2) \\
0 & 0 & \frac{1}{-k_0^2 + k_1^2 + M^2}
\end{pmatrix} . \]
(A.9)

A.2. Breaking Lorentz invariance

We assume that the Lorentz invariance \( SO(d - 1, 1) \) is broken down to the space rotation invariance \( SO(d - 1) \) only by mass terms in the Lagrangian [6, 38]. This means that instead of a single mass term \( M^2 A_i^0 \), there can be two, \( -M_0^2 \bar{A}_i^0 + M_0^2 \bar{A}_i^2 \), but it does not necessarily coincide with \( M_01 \). If gauge breaking terms were also added to the kinetic terms, this would immediately provide \( k^4 \) terms for the Stueckelberg fields in the Lagrangian. Without the gauge breaking, one could only write the electric and magnetic terms \( F^0_{ij} \) and \( F^2_{ij} \) with different coefficients, but this is equivalent to a time rescaling and does not really break the Lorentz invariance.
So, the kinetic matrix is

\[
\begin{pmatrix}
\omega^2 + (k^2 + M_0^2) & \omega k_0 \\
\omega k_0 & k^2 - (k^2 + M_1^2)
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
k^2 + M_0^2 \\
\omega k_0
\end{pmatrix}
\begin{pmatrix}
\omega k_0 \\
0
\end{pmatrix}
+ \begin{pmatrix}
0 \\
\omega^2 - M_1^2
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
- (k^2 + M_1^2)
\]  

Its eigenvectors and eigenvalues are

\[
\begin{pmatrix}
\frac{1}{\omega^2 + M_0^2 + M_1^2 - r} \\
0
\end{pmatrix}
\text{for } \lambda_- = \frac{1}{2}(\omega^2 + k_0^2 + M_0^2 - M_1^2 - r)
\]

\[
\begin{pmatrix}
\frac{1}{\omega^2 + M_0^2 + M_1^2 - r} \\
0
\end{pmatrix}
\text{for } \lambda_+ = \frac{1}{2}(\omega^2 + k_0^2 + M_0^2 - M_1^2 + r)
\]  

with

\[
\frac{r^2}{\omega^2 - k_0^2 - M_1^2}
\]

The gauge field $A_\mu$ is expanded in a different sort of photons:

\[
A^\mu := A_\perp \partial^\mu + A_\parallel \partial^\mu.
\]

The Lagrangian, expressed through the normal modes is diagonal:

\[
(P - Qr) A_\perp^2 + (P + Qr) A_\parallel^2 + (\omega^2 - k_0^2 - M_1^2) A_\perp^2 + J_\parallel A_\parallel
\]

\[
+ (2J_0 \omega k_0 + J_1 \omega^2 - J_1 k_0^2) - J_1 (M_0^2 + M_1^2) - J_1 (r) A_\perp
\]

\[
+ (2J_0 \omega k_0 + J_1 \omega^2 - J_1 k_0^2) - J_1 (M_0^2 + M_1^2) + J_1 (r) A_\perp
\]

with

\[
P = (\omega^2 - M_1^2) r^2,
\]

\[
Q = M_1^2 k_1^2 - 2(M_0^2 + M_1^2) \omega^2 + \omega^2 k_0^2 + M_0^2 M_1^2 + M_1^4
\]

and

\[
r^2 = (\omega^2 + k_0^2)^2 - 2(M_0^2 + M_1^2) (\omega^2 - k_1^2) + (M_0^2 + M_1^2)^2,
\]

so that $P^2 - Q^2 r^2 = 4\omega^2 k_1^2 (\omega^2 M_0^2 - k_0^2 M_1^2 + M_0^2 M_1^2) r^2$.

The Born interaction of currents is

\[
J_\perp^2 \frac{(2\omega k_0 J_0 + (\omega^2 - k_0^2 - M_0^2 - M_1^2 - r) J_1)^2}{P - Qr}
\]

\[
+ \frac{(2\omega k_0 J_0 + (\omega^2 - k_0^2 - M_0^2 - M_1^2 + r) J_1)^2}{P + Qr}
\]

If the currents are conserved, this converts into

\[
\frac{J_\perp^2}{\omega^2 - k_0^2 - M_1^2} \frac{(M_0^2 \omega^2 - M_0^2 k_1^2 - M_0^2 M_1^2) k_1^2}{(M_0^2 \omega^2 - M_0^2 k_1^2 - M_0^2 M_1^2) \omega^2}
\]

\[
= \frac{J_\perp^2 (M_0^2 \omega^2 - M_0^2 k_1^2)}{(M_0^2 \omega^2 - M_0^2 k_1^2 - M_0^2 M_1^2) \omega^2}
\]

\[
(17)
\]
The propagator, inverse of the kinetic matrix, is

\[
\begin{pmatrix}
\omega^2 - M_1^2 & -\omega k_\| & 0 \\
-M_0^2\omega^2 - M_1^2k_\|^2 & M_0^2 - M_1^2k_\|^2 & 0 \\
0 & 0 & \frac{1}{\omega^2 - (k_\|^2 + M_1^2)}
\end{pmatrix}
\]  

(A.18)

and this provides for the Born interaction of currents:

\[
J_2^2 \parallel \omega^2 - k_\|^2 - M_1^2 = J_2^2 (\omega_2^2 - M_1^2k_\|^2)
\]

(A.19)

what coincides with (A.16) and (A.17).

Appendix B. Generic frame

In this appendix, we obtain the Lorentzian eigenmodes and eigenvalues in the case of massive Lorentz invariant gravity. The kinetic matrix is now

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 \\
B - A & -B & -B & 0 & 0 & 0 \\
-\omega k_\| & \omega k_\| & \omega^2 - k_\|^2 - M_1^2 & 0 & 0 & 0 \\
-k_\|^2 + B & -\omega^2 + B & -\omega^2 + B & -\omega^2 + B \\
-\omega k_\| & \omega k_\| & \omega^2 - k_\|^2 - M_1^2 & 0 & 0 & 0 \\
-k_\|^2 - B & -\omega^2 + B & -\omega^2 + B & -\omega^2 + B \\
\end{pmatrix}
\]  

(B.1)

The Lorentz invariant eigenmodes are much simpler: they are obtained by Lorentz transformations from those in the rest frame. Accordingly, the eigenvalues are

\[
\lambda_\pm = A \\
\lambda_{gr} = \omega^2 - k_\|^2 - A \\
\lambda_{sc} = \frac{(d - 2)k_\|^2 + dB - 2A \pm \sqrt{(d - 2)^2(k_\|^2 + B)^2 + 4(d - 1)B^2}}{2}
\]

(B.2)

and the corresponding eigenmodes \(u^{\mu\nu}\) are

\[
\begin{pmatrix}
0 & 0 & \ldots & -\omega & \ldots \\
0 & 0 & \ldots & k_\| & \ldots \\
-\omega & \ldots & k_\| & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]  

\[
\begin{pmatrix}
\omega k_\| & -\omega^2 + k_\|^2 - A \\
\ldots & \ldots & \ldots \\
-\omega^2 + k_\|^2 - A & \omega k_\|
\end{pmatrix}
\]  

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\[
\begin{pmatrix}
0 & 0 & \cdots \\
0 & 0 & \cdots \\
\cdots & h_{ab} & \\
\cdots & \cdots \\
\end{pmatrix}
\quad \text{such that } \frac{\partial^2}{\partial x^a \partial x^b} = h_{ab}
\]

\[
\begin{pmatrix}
0 & 0 & \cdots \\
0 & 0 & \cdots \\
\cdots & k_{\|} & \omega \\
\cdots & \cdots \\
\end{pmatrix}
\quad \text{such that } \frac{\partial^2}{\partial x^a \partial x^b} = \omega
\]

\[
\begin{pmatrix}
\frac{\partial^2}{\partial x^a \partial x^b} = h_{ab} \\
\frac{\partial^2}{\partial x^a \partial x^b} = \omega
\end{pmatrix}
\]

\[
\begin{pmatrix}
k_{\|} & -\omega k_{\|} & \omega \\
-\omega k_{\|} & \omega^2 & \omega \\
\cdots & \cdots & \cdots \\
\end{pmatrix}
\quad \text{such that } \frac{\partial^2}{\partial x^a \partial x^b} = \frac{k_{\cdot}^2 - \omega^2}{d-2}
\]

\[
\begin{pmatrix}
\frac{\partial^2}{\partial x^a \partial x^b} = h_{ab} \\
\frac{\partial^2}{\partial x^a \partial x^b} = \omega
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda_+^2 + A + (d-2)\omega^2 & \sqrt{2}(d-2)\omega k_{\|} \\
\sqrt{2}(d-2)\omega k_{\|} & -\lambda_-^2 - A + (d-2)k_{\|}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\cdots \\
\cdots
\end{pmatrix}
\]

respectively. As usual, we assume here that the spatial momentum is directed along the first axis.

**Appendix C. Manifest expression for the propagator**

In this appendix, we write down the coefficients in the propagator in the case of Lorentz violated gravity. In four dimensions they are

\[
a_{10} = \frac{2}{D_{vec}D_{gr}} \left( \frac{(4m_1^2 + 2m_2^2m_0^2 - 4m_0^2m_2^2)\omega^4 - 4m_2^2m_0^2\omega^2k^2 + (2m_2^2 - 2m_2^2m_0^2)k^4}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]

\[
a_{11} = \frac{2}{D_{vec}D_{gr}} \left( \frac{(m_1^2m_2^2 + m_1^2m_1^2 + 2m_1^2m_0^2 - 2m_0^2m_0^2 - 2m_2^4)\omega^2 + (m_1^2m_1^2 + m_3^2m_1^2 - m_2^2m_2^2)k^2}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]

\[
a_{20} = \frac{2}{D_{vec}D_{gr}} \left( \frac{(m^2_1m_1^2 + m^2_2m_2^2 + m^2_1m_1^2 + 2m_1^2m_0^2 - 2m_0^2m_0^2 - 2m_2^4)\omega^2 + (m_1^2m_1^2 + m_3^2m_1^2 - m_2^2m_2^2)k^2}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]

\[
a_{21} = \frac{2}{D_{vec}D_{gr}} \left( \frac{(m^2_1m_1^2 + m^2_2m_2^2 + m^2_1m_1^2 + 2m_1^2m_0^2 - 2m_0^2m_0^2 - 2m_2^4)\omega^2 + (m_1^2m_1^2 + m_3^2m_1^2 - m_2^2m_2^2)k^2}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]

\[
a_{60} = \frac{2}{D_{vec}D_{gr}} \left( \frac{(m^2_1m_1^2 + m^2_2m_2^2 + m^2_1m_1^2 + 2m_1^2m_0^2 - 2m_0^2m_0^2 - 2m_2^4)\omega^2 + (m_1^2m_1^2 + m_3^2m_1^2 - m_2^2m_2^2)k^2}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]

\[
a_{60} = \frac{2}{D_{vec}D_{gr}} \left( \frac{(m^2_1m_1^2 + m^2_2m_2^2 + m^2_1m_1^2 + 2m_1^2m_0^2 - 2m_0^2m_0^2 - 2m_2^4)\omega^2 + (m_1^2m_1^2 + m_3^2m_1^2 - m_2^2m_2^2)k^2}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]

\[
a_{61} = \frac{2}{D_{vec}D_{gr}} \left( \frac{(m^2_1m_1^2 + m^2_2m_2^2 + m^2_1m_1^2 + 2m_1^2m_0^2 - 2m_0^2m_0^2 - 2m_2^4)\omega^2 + (m_1^2m_1^2 + m_3^2m_1^2 - m_2^2m_2^2)k^2}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]

\[
\frac{1}{4} \left( \frac{(4m_1^2m_1^2 + 2m_1^4 + 8m_4^2 - 8m_1^2m_1^2 - 8m_1^2m_1^2)\omega^4}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]

\[
\frac{1}{4} \left( \frac{(4m_1^2m_1^2 + 2m_1^4 + 8m_4^2 - 8m_1^2m_1^2 - 8m_1^2m_1^2)\omega^4}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]

\[
\frac{1}{4} \left( \frac{(4m_1^2m_1^2 + 2m_1^4 + 8m_4^2 - 8m_1^2m_1^2 - 8m_1^2m_1^2)\omega^4}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]

\[
\frac{1}{4} \left( \frac{(4m_1^2m_1^2 + 2m_1^4 + 8m_4^2 - 8m_1^2m_1^2 - 8m_1^2m_1^2)\omega^4}{m_1^2(-\omega^2 + \bar{k}^2 + m_1^2)(-\omega^2 + \frac{m_1^2}{m_2^2}\bar{k}^2 + m_2^2)\right) = -2\omega^2 + \bar{k}^2 + m_1^2
\]
\[ b = \frac{-2m^2}{D} \left( \omega - 4m^2 \right) \left( m_2^2 \omega - m_1^2 \right) + \left( m_2^2 \omega - m_1^2 \right)^2 \left( D - 2 \right) \omega^2. \]

\[ a_{40} = \frac{2}{D_{gr}} \left( \omega^2 + \left( m_2^2 - m_1^2 \right) \right), \]

\[ a_{41} = -\frac{1}{D_{gr}}, \]  

\[ a_{51} = -\frac{1}{D_{gr}^2} \left( \left( m_3^2 \omega^2 + \left( m_2^2 \omega - m_1^2 \right) \right) \omega^2 - \left( m_2^2 \omega - m_1^2 \right)^2 \left( D - 2 \right) \omega^2 \right) \]

\[ a_{50} = \frac{2}{D_{gr}} \left( \left( m_2^2 \omega - m_1^2 \right) \right) \omega^2 + \left( m_2^2 \omega - m_1^2 \right)^2 \left( D - 2 \right) \omega^2, \]

\[ c_{1} = \frac{-8 \omega (m_2^2 \omega^2 + \left( m_2^2 - m_1^2 \right) \omega^2 - m_2^2)}{D}, \]

\[ c_{2} = \frac{4 \omega}{D_{gr}} \left( m_1^2 - m_2^2 \right), \]

\[ c_{3} = \frac{4 \omega}{D_{gr}}. \]

Restoring the \( d \)-dependence, one can see that only a few coefficients slightly depend on the spacetime dimension:

\[ a_{10} = \frac{2}{D_{gr} \omega} \left( \left( d - 2 \right) m_1^2 + \left( d - 2 \right) m_2^2 - \left( d - 4 \right) m_0^2 \omega^2 \right) \omega^2 \]

\[ + \left( \left( d - 2 \right) m_1^2 + \left( d - 2 \right) m_2^2 \right) \left( D - 2 \right) \omega^2, \]

\[ a_{11} = \frac{-2 \left( -\omega^2 + \omega^2 \right)}{m_1^2 \left( -\omega^2 + \omega^2 \right) + \left( -\omega^2 + m_1^2 \omega^2 \right)} = -\frac{-2 \omega^2 + \omega^2 + m_1^2}{m_1^2 \left( -\omega^2 + \omega^2 \right) + \left( -\omega^2 + m_1^2 \omega^2 \right)} \]

\[ a_{20} = \frac{-2 \left( \left( d - 2 \right) m_1^2 - \left( d - 3 \right) m_2^2 + \left( d - 2 \right) m_0^2 \right)}{D}, \]

\[ a_{21} = \frac{2}{D_{gr} \omega} \left( \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]

\[ + \left( m_2^2 \omega + m_1^2 \omega + m_2^2 m_0^2 - \left( d - 2 \right) \omega^2 \right) \left( -\omega^2 + \omega^2 \right) \omega^2 \]
\[ a_{61} = \frac{1}{D_{vec}}D_{gr} \left( (2m_2^2m_0^2 + m_4^4 + 4m_4^2 - 4m_0^2m_2^2 - 4m_2^2m_0^2)(d - 2)\omega^4 + (4m_2^2m_0^2 - 4m_4^2 - 2m_0^2m_2^2 - m_2^4 - 2m_0^2m_1^2 + m_1^2m_2^2)(d - 2)\omega^2k^2 + (2m_2^2m_1^2 - m_1^2m_0^2)\right)\]
\[ + (2d - 12)m_0^2m_1^2 - (4d - 12)m_2^2m_1^2 + (4d - 12)m_2^2m_2^2 + (2d - 2)m_2^2m_3^2m_4^2 - 2m_0^2m_2^2 - (2d - 2)m_2^2m_3^2m_4^2 - (d - 3)m_2^2m_3^2m_4^2 - (d - 3)m^2m_3^2m_4^2\]
\[ + (d - 2)m_2^2m_3^2m_4^2k^2 + (2d - 6)m_2^2m_3^2m_4^2 - (2d - 6)m_2^2m_3^2m_4^2 + (d - 3)m_2^2m_3^2m_4^2\]
\[ + (d - 1)m_2^2m_3^2m_4^2 - (2d - 6)m_2^2m_3^2m_4^2 - m_3^4m_4^2)\right)\]
\[ a_{40} = 2 - \omega^2 + m_2^2,\]
\[ a_{41} = -\frac{1}{D_{gr}}\]
\[ a_{51} = \frac{-1}{D_{gr}}D \left( (m_2^2m_1^2)\omega^4 + (2m_2^2m_1^2 + 2m_2^2m_0^2 - 2m_2^2m_0^2)\omega^2k^2 + m_1^2(m_2^2 - m_0^2)k^4\right)\]
\[ + (m_2^4m_1^2 - m_2^2m_0^2m_1^2 - m_2^2m_0^2m_3^2)\omega^2 + (m_2^2m_0^2m_1^2 - 2m_2^2m_0^2m_1^2 + m_2^2m_0^2m_1^2 - m_2^2m_0^2m_1^2)\right\}
\[ a_{50} = \frac{2m_1^2(-m_2^2\omega^2 - m_2^2m_1^2 + m_2^2m_4^2)}{D},\]
\[ c_1 = -\frac{4(d - 2)\omega(m_2^2\omega^2 + (m_2^2 - m_3^2)k^2 - m_2^2m_4^2)}{D},\]
\[ c_2 = \frac{4\omega(m_4^2 + m_2^2m_0^2 - m_2^2m_0^2)}{D},\]
\[ c_3 = \frac{4\omega}{D_{vec}}.\] (C.4)

### C.1. Current–current interaction

Now one can insert explicit expressions for the coefficients of the propagator from this appendix into current–current interaction, (5.1). However, the results are very lengthy and non-transparent. Even for the conserved stress tensor $k_{\mu}T^{\mu\nu} = \omega T^{0\nu} + k_{\mu}T^{\mu\nu} = T^{\mu\nu}k_{\nu} = 0$ (5.1) turns into

\[ \mathcal{P}_{\mu\nu,ab}T^{\mu\nu}T^{\nu\delta} = (a_{10}\omega^2 + a_{20}\omega^2 + b - c_1\omega - a_{60}\omega^3 + a_{61}\omega^4)T^{00}_{00} + (a_{11}\omega^2 + a_{40} - c_3\omega)T^{01}_{01}\]
\[ + (a_{21}\omega^2 + a_{50} - c_2\omega)T_{00}T_{11} + a_{41}T_{01}^2 + a_{51}(T_{01})^2\] (C.5)
and in $d = 4$

tensor propagator: $a_{41} = -\frac{1}{D_2}$,

vector propagator: \( (a_{11} \omega^2 + a_{40} - c_3 \omega) = -\frac{2}{D_1 D_2} \left( 4 \omega^2 D_{gr} - D_{vec} + (m_1^2 - m_2^2) m_2^2 \right) \)

(C.6)

so that no essential cancellations happen in these expressions, and answers for the scalar channels look very lengthy and involved even in this case.

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