Towards Efficient Maximum Likelihood Estimation of LPV-SS Models

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Abstract

How to efficiently identify multiple-input multiple-output (MIMO) linear parameter-varying (LPV) discrete-time state-space (SS) models with affine dependence on the scheduling variable still remains an open question, as identification methods proposed in the literature suffer heavily from the curse of dimensionality and/or depend on over-restrictive approximations of the measured signal behaviors. However, obtaining an SS model of the targeted system is crucial for many LPV control synthesis methods, as these synthesis tools are almost exclusively formulated for the aforementioned representation of the system dynamics. Therefore, in this paper, we tackle the problem by combining state-of-the-art LPV input-output (IO) identification methods with an LPV-IO to LPV-SS realization scheme and a maximum likelihood refinement step. The resulting modular LPV-SS identification approach achieves statical efficiency with a relatively low computational load. The method contains the following three steps: 1) estimation of the Markov coefficient sequence of the underlying system using correlation analysis or Bayesian impulse response estimation, then 2) LPV-SS realization of the estimated coefficients by using a basis reduced Ho-Kalman method, and 3) refinement of the LPV-SS model estimate from a maximum-likelihood point of view by a gradient-based or an expectation-maximization optimization methodology. The effectiveness of the full identification scheme is demonstrated by a Monte Carlo study where our proposed method is compared to existing schemes for identifying a MIMO LPV system.

Key words: System identification; Linear parameter-varying systems; State-space representations; Realization theory; Maximum likelihood estimation.

1 Introduction

The linear parameter-varying (LPV) modeling paradigm offers an attractive model class to capture nonlinear and/or time-varying systems with a parsimonious parameterization. The LPV model class preserves the linear signal relation between the inputs and outputs of the system, however, these linear relations are functions of a measurable, time-varying signal, the scheduling variable, denoted as \( p \). This scheduling signal can be any combination of inputs, measurable process states, outputs, or measurable exogenous variables and, in addition, these signals can be filtered by any arbitrary functional relation. Hence, the LPV modeling paradigm can represent both non-stationary and nonlinear behavior of a wide variety of physical or chemical processes, e.g., see [1–4].

The majority of LPV control synthesis methods are based upon the assumption that an LPV state-space (SS) model of the system is available, especially with static and affine dependence of the involved matrix coefficients on the scheduling variable \( p \), e.g., [5]. Hence, efficient identification of LPV-SS models in terms of computational load, statistical, and performance properties has intensively been researched. Conceptually, LPV identification can be performed as: i) the interpolation of local LTI models estimated from multiple experiments around fixed operating points, i.e., with constant \( p \), often referred to as the local identification setting; or ii) a direct model estimation problem, i.e., the global identification setting, which requires the experimental data with a varying \( p \) which is informative to uniquely identify the considered model parameters. Accordingly, global identification approaches include scheduling dynamics, see [4] for a detailed comparison between the two settings. In this paper, we will focus on the global setting and the identification of discrete-time models.

In the global setting, an attractive identification approach is
the minimization of the $\ell_2$-loss in terms of the prediction-error associated with the model. Approaches aiming at this objective are often called prediction-error methods (PEM). Early approaches in the PEM setting are proposed under the unrealistic assumption of full state measurements [6,7]. To overcome this assumption and to directly minimize the $\ell_2$ loss, gradient-based (GB) methodologies have been introduced, e.g., see [8–11]. Recently, an expectation-maximization (EM) algorithm has been developed for LPV-SS models [11], extending the set of GB methods. The EM method is more robust to an inaccurate initial estimate compared to the GB PEM; however, its convergence rate is much slower near the optimum [12]. Due to the nonlinear optimization associated with the EM and GB methods, their convergence to the maximum-likelihood (ML) estimate depends heavily on a proper initial seeding. Besides prediction-error identification methods, LPV grey-box [13,14] and LPV set-membership (SM) [15–17] identification approaches have been developed. Grey-box schemes require detailed knowledge of the dynamical structure of the system with only a few unknown parameters, which are often estimated by a Kalman like filtering strategy. The SM methods characterize noise and disturbances in a deterministic bounded-error compared to the stochastic description in PEM. In general, SM approaches have a significantly higher computational load compared to direct PEM and rely on convex outer-approximations. Hence, in order to achieve stochastically interpretable and computationally attractive identification of LPV-SS models, it is favorable to apply GB and EM based PEM. However, these methods require a proper initial estimate close to the global optimum (ML estimate) in order to exploit their advantageous properties (Problem 1).

To achieve initialization of direct PEM, alternative methods can be introduced that rely on realization theory by sacrificing ML properties for an estimation problem solvable via convex optimization. These methods boil down to: first identifying an LPV-IO model, with well-established methods available in the literature (e.g., see [5,18,19]); and, secondly, to execute an exact realization of the identified LPV-IO form to an LPV-SS model. However, such an exact realization will, in general, result in relations with rational, LPV-IO form to an LPV-SS model. However, such an exact realization will, in general, result in relations with rational, LPV-SS realization based on the estimated coefficients by a Ho-Kalman like method (Sec. 4), and 3) to have an ML estimate, refine the LPV-SS model by GB and/or EM. Hence, to achieve stochastically interpretable and computationally attractive SS identification methods capable of providing estimates that are sufficiently close to the global PEM optimum. Problem 2: Finding initial estimates in the region of attraction to the ML solution in a computationally attractive way.

Based on Problems 1 and 2, we can conclude that computationally and stochastically efficient identification of LPV-SS models on real-world sized problems remains still an open question. Hence, the goal of this paper is to provide a maximum likelihood identification scheme for LPV-SS models in the global, open-loop identification setting, which can provide an integrated solution for both problems. Specifically, to solve Problem 2, we propose to identify surrogate LPV finite impulse response (FIR) models via a novel computationally efficient correlation analysis (CRA) method or via an empirical MIMO Bayesian estimation technique. Then, realization of these models is accomplished via a novel basis reduced LPV Ho-Kalman scheme, which grows linearly in complexity compared to previous methods with exponential growth, which are introduced originally in [26,27]. Next, Problem 1 is solved by integrating the proposed pre-estimation methods into the GB and EM schemes to obtain an ML estimate. In addition, to improve the numerical properties of the GB method, we extend the enhanced Gauss-Newton method [10] to the LPV setting. Combining these methods results in a novel three-step approach with a modular structure, achieving both favorable computational properties and enabling ML estimation.

This paper is organized as follows: first, LPV-SS models with general noise structure are analyzed and compared with models relying on an innovation structure to highlight modeling limitations of the latter form considered in many LPV SID methods. Then, the considered LPV-SS identification problem is introduced (Sec. 2). Next, we present our modular identification method, defined in three steps: 1) estimate the FIR model of the underlying system using CRA or MIMO Bayesian estimation (Sec. 3), then 2) compute an LPV-SS realization based on the estimated coefficients by using a Ho-Kalman like method (Sec. 4), and 3) to have an ML estimate, refine the LPV-SS model by GB and/or EM optimization (Sec. 5). The contribution of this paper is to provide a detailed overview of the methods applied and to demonstrate that LPV identification of moderate sized models is possible with the proposed scheme. The efficiency of the combined approach is demonstrated by a Monte Carlo study and it is compared to existing LPV-SS identification schemes [15,25,28] (Sec. 6).
2 The LPV identification problem

2.1 Technical preliminaries

We define a random variable \( f \) as a measurable function \( f : \Omega \to \mathbb{R}^n \), which induces a probability measure \( \mathbf{P} \) on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\) with an associated Borel measurable space \( \mathcal{B}(\mathbb{R}^n) \) [29]. As such, a realization \( \nu \in \Omega \) of \( \mathbf{P} \), denoted \( \nu \sim \mathbf{P} \), defines a realization \( f \) of \( f \), i.e., \( f := \Gamma(\nu) \).

A stochastic process \( x \) is a collection of random variables \( x_t : \Omega \to \mathbb{R}^n \) indexed by the set \( t \in \mathbb{Z} \) (discrete time), given as \( x = \{x_t : t \in \mathbb{Z}\} \).

A realization \( \nu_t \in \Omega \) of the stochastic process defines a signal trajectory \( x := \{x_t(\nu_t) : t \in \mathbb{Z}\} \).

We call a stochastic process \( x \) stationary if the probability distribution of \( x_k \) and joint probability distribution of \( (x_k, \ldots, x_k+k) \) for any \( k \in \mathbb{N} \), are independent of the time index \( t \). In addition, a stationary process consisting of uncorrelated random variables with zero mean and finite variance is called a white noise process. The ring of all real meromorphic functions with finite dimensional space is denoted by \( \mathcal{A} \) and the operator \( \Theta : (\mathcal{A}, \mathcal{B}(\mathbb{R}^n)) \to \mathbb{R}^n \) denotes \( \Theta(f) = (\Theta f) \) with \( \Theta f \in \mathbb{R}^n \).

The time-shift operator is denoted by \( q \), i.e., \( qx(t) = x(t+1) \), and the set \( \{s, s+1, \cdots, v\} \subseteq \mathbb{N} \) is denoted as \( \mathbb{I}^v_s \).

2.2 The data-generating system

Consider a multiple-input multiple-output (MIMO), discrete-time linear parameter-varying data-generating system, defined by the following first-order difference equation, i.e., LPV-SS representation with general noise model:

\[
\begin{align*}
    x_{t+1} &= A(p_t)x_t + B(p_t)u_t + G(p_t)w_t, \quad (1a) \\
    y_t &= C(p_t)x_t + D(p_t)u_t + H(p_t)v_t, \quad (1b)
\end{align*}
\]

where \( x : \mathbb{Z} \to \mathbb{R}^{n_x} \) is the state variable, \( y : \mathbb{Z} \to \mathbb{R}^{n_y} \) is the output signal, \( u : \mathbb{Z} \to \mathbb{R}^{n_u} \) is the scheduling input, \( p : \mathbb{Z} \to \mathbb{R} \subseteq \mathbb{R}^{n_p} \) is the scheduling variable, subscript \( t \in \mathbb{Z} \) is the discrete time, \( w : \mathbb{Z} \to \mathbb{R}^{n_w} \), \( v : \mathbb{Z} \to \mathbb{R}^{n_v} \) are the sample path realizations of the zero-mean stationary processes:

\[
\begin{bmatrix}
    \mathbf{w}_t \\
    \mathbf{v}_t
\end{bmatrix} \sim \mathcal{N}(0, \Sigma), \quad \Sigma = \begin{bmatrix}
    \Omega & S \\
    S^\top & R
\end{bmatrix}, \quad (2)
\]

where \( \mathbf{w}_t : \Omega \to \mathbb{R}^{n_w} \), \( \mathbf{v}_t : \Omega \to \mathbb{R}^{n_v} \) are white noise processes, \( \Omega \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}, S \subseteq \mathbb{R}^{n_x \times n_v} \), and \( R \subseteq \mathbb{R}^{n_v \times n_v} \) are covariance matrices, such that \( \Sigma \) is positive definite. Furthermore, we will assume \( u, p, w, v \) to have left compact support to avoid technicalities with initial conditions. As often considered in LPV control theory, the matrix functions \( A(\cdot), \cdots, H(\cdot) \), defining the SS representation (1) are defined as affine combinations:

\[
\begin{align*}
    A(p_t) &= A_0 + \sum_{i=1}^{n_A} A_i \psi_i[p_t], \\
    B(p_t) &= B_0 + \sum_{i=1}^{n_B} B_i \psi_i[p_t], \\
    C(p_t) &= C_0 + \sum_{i=1}^{n_C} C_i \psi_i[p_t], \\
    D(p_t) &= D_0 + \sum_{i=1}^{n_D} D_i \psi_i[p_t], \\
    G(p_t) &= G_0 + \sum_{i=1}^{n_G} G_i \psi_i[p_t], \\
    H(p_t) &= H_0 + \sum_{i=1}^{n_H} H_i \psi_i[p_t],
\end{align*}
\]

where \( \psi_i[\cdot] : \mathbb{R} \to \mathbb{R} \) are bounded scalar functions on \( \mathbb{R} \) and \( \{A_i, B_i, C_i, D_i, H_i\}_{i=1}^{n_A} \) are constant, real matrices with appropriate dimensions. Additionally, for well-posedness, it is assumed that \( \{\psi_i[\cdot]\}_{i=1}^{n_A} \) are linearly independent over an appropriate function space and are normalized w.r.t. an appropriate norm or inner product [21]. Due to the freedom to consider arbitrary functions \( \psi_i[\cdot], (3) \) can capture a wide class of static nonlinearities and time-varying behaviors.

2.3 Properties of LPV-SS representations

In this section, we present some formal definitions needed for the analysis of various noise structures and required definitions for the LPV-IO to LPV-SS realization problem.

Note that, the deterministic part of (1) is governed by

\[
\begin{align*}
    x^d_{t+1} &= A(p_t)x^d_t + B(p_t)u_t, \quad (4a) \\
    y^d_t &= C(p_t)x^d_t + D(p_t)u_t. \quad (4b)
\end{align*}
\]

The solution set, i.e., the manifest behavior, of (4) is

\[
\mathcal{B}_d = \{(y^d, u, p) \in (\mathbb{R} \times \mathbb{R} \times \mathbb{R})^2 | \exists \mathbf{y}_t^d \in (\mathbb{R}^2)^v \text{ s.t. } (4) \text{ holds}\}. \quad (5)
\]

The behavior w.r.t. an LPV-SS representation \( S \) is denoted \( \mathcal{B}_d(S) \). On the other hand, the stochastic part of (1) is

\[
\begin{align*}
    x^s_{t+1} &= A(p_t)x^s_t + B(p_t)w_t, \quad (6a) \\
    y^s_t &= C(p_t)x^s_t + H(p_t)v_t. \quad (6b)
\end{align*}
\]

The manifest behavior corresponding to (1) is

\[
\mathcal{B}_s = \{(y^s, u, p) \in (\mathbb{R} \times \mathbb{R} \times \mathbb{R})^2 | \exists \mathbf{y}_t^s \text{ satisfying } (6) \text{ s.t. } (y^d, u, p) \in \mathcal{B}_d \text{ and } \forall t, \exists \nu \in \Omega \text{ for which } y^s_t = y^d_t + y^s_t(\nu)\}. \quad (7)
\]

To introduce the essential details of the deterministic realization step in Sec. 4, we momentarily neglect the stochastic process (6). To this end, we take the expectation \( \mathbb{E}\{y_t\} = y^d_t \), which is equivalent as taking \( v_t = w_t = 0 \) in (1). In this paper, we are interested in finding an LPV-SS representation with behavior \( \mathcal{B}_d \) and a minimal state dimension:

**Definition 1 (Minimal LPV-SS representations)** The LPV-SS representation \( S(1) \) is called state minimal, if there exists no other LPV-SS representation \( S(1) \) with \( n_x < n_x \) and equivalent manifest behavior \( \mathcal{B}_d(S) = \mathcal{B}_d(S') \). \quad (8)

For specific subclasses of LPV-SS representations in which the functional dependency structure of \( A(\cdot), \cdots, H(\cdot) \) is restricted, the minimal state dimension might differ [20], e.g., when comparing static, affine w.r.t. dynamic, rational dependency (e.g., see [30, Example 4.1]). Hence, for the remainder of the paper, state minimality is considered w.r.t. the static, affine dependency in (3).

**Lemma 2 (Equivalent LPV-SS representations [31])**

Given two state minimal LPV-SS representations (4) and \( S' \) with static, affine dependency (3) and equivalent state dimensions \( n_x = n_x \). The two representations \( S \) and \( S' \) are called equivalent, i.e., their associated manifest behaviors are equal \( \mathcal{B}_d(S) = \mathcal{B}_d(S') \), if and only if there exists a non-singular isomorphism \( T \in \mathbb{R}^{n_x \times n_x} \) such that

\[
A'_T = TA, \quad B'_T = TB, \quad C'_T = C, \quad D'_T = D_t,
\]
for all $i \in \{0, \ldots, n\}$. □

Lemma 2 is a special case of equivalence relation in the LPV case as the transformation matrix $T$ is independent of the scheduling signal, see [20, Def. 3.29] for the general case. Under dependency structure (3) and assumption of state minimality, the equivalence class of LPV-SS representations is completely characterized by the non-singular transformation matrix $T$, as given in Lem. 2.

We are interested in identification under open-loop conditions, hence, the underlying data-generating system is considered to be asymptotically stable:

**Definition 3 (Global asymptotic stability)** An LPV system, represented in terms of (4), is called globally asymptotically stable, if for all trajectories of $\{u_t, p_t, \xi_t\}$ satisfying (4), with $u_t \equiv 0$ for $t \geq 0$, and $p_t \in \mathbb{P}$, it holds that $\lim_{t \to \infty} |y_t| = 0$. □

### 2.4 LPV-SS noise models and the innovation form

A popular model for many subspace identification schemes is the innovation form, e.g., see [32]. Under some mild conditions, the LPV-SS representation (1) has the following equivalent innovation form:

**Lemma 4 (LPV-SS innovation form [30])** The LPV data-generating system (1) can be equivalently represented by a $p$-dependent innovation form

\[
\begin{align*}
\dot{x}_{t+1} &= A(p_t)x_t + B(p_t)u_t + K_t \xi_t, \quad (8a) \\
y_t &= C(p_t)x_t + D(p_t)u_t + \xi_t, \quad (8b)
\end{align*}
\]

where $\xi_t$ is the sample path of $\xi_t \sim \mathcal{N}(0, \Xi_t)$ and $K_t$ can be uniquely determined by

\[
K_t = [A(p_t)P_t^{-1}C_t^\top + G(p_t)S_tH_t(p_l)]^{-1}, \quad (8c)
\]

\[
P_{t+1|t} = A(p_t)P_t^{-1}A_t^\top - K_t \Xi_t K_t^\top + G(p_t)Q_t G(p_t), \quad (8d)
\]

\[
\Xi_t = C(p_t)P_t^{-1}C_t^\top + H_t(p_l)R_t H_t(p_l), \quad (8e)
\]

under the assumption that $\exists x_0 \in \Xi$ such that $x_{t_0} = 0$ and $\Xi_t$ is non-singular for all $t \in \{0, \infty\}$. □

In (8c)-(8e), the notation of $K_t$, $P_{t+1|t}$, and $\Xi_t$ is a shorthand for $K_t := \{K \circ p\}_t \in \mathbb{R}^{n_x \times m}$, $P_{t+1|t} := \{P_{t+1|t} \circ p\}_t \in \mathbb{R}^{n_x \times n_x}$, and $\Xi_t := \{\Xi \circ p\}_t \in \mathbb{R}^{n_y \times n_y}$. The subscript notation $t+1|t$ denotes that the matrix function at time $t+1$ depends on $p_t$ with $\tau \in \{t_0, \ldots, t\}$ where $t \geq t_0$.

In [30], it is shown that the setting of (1) is not equivalent to the innovation form with only a static, affine matrix function $K(p_l)$, similarly parametrized as (3). This static, affine structure is commonly used in many LPV SID methods [22,25,33]. It follows that, to guarantee state minimality of an innovation form based realization of (1), the Kalman gain $K_t$ in (8c) should have rational and dynamic dependency on $p$. However, [30] also shows that a static, affine $K(p_l)$ can approximate the general setting (8) if the state dimension is increased. In practice, we often need to restrict parameterization of $K$, e.g., to a static, affine parameterization similar to (3), to reduce complexity of the estimation method and variance of the model estimates. Hence, despite the possible increase of state order of the equivalent innovation form, the underlying complexity trade-off might be acceptable from a practical point of view.

### 2.5 Problem statement

In this paper, we are interested in identifying LPV-SS models (1) with dependency structure as in (3) to capture the process dynamics of the underlying data-generating system (1). Hence, our focus is not on identifying the noise structure $(\mathcal{H}, \mathcal{G})$, but to derive a methodology which can provide consistent estimates of (4) under the general noise structure of (1). We will also assume that the scalar functions $\{\psi[i]\}_{i=1}^{n_x}$ are known a priori. As a consequence, we are interested in estimating the parameters of (3), i.e.

\[
\Lambda_0 = \begin{bmatrix}
A_0 & \ldots & A_{n_x} & B_0 & \ldots & B_{n_x} \\
C_0 & \ldots & C_{n_x} & D_0 & \ldots & D_{n_x}
\end{bmatrix},
\]

with $\Lambda_0 \in \mathbb{R}_{n_x + n_y \times (n_x + n_y + 1 + n_x)}$. Based on these, we denote by $S(\Lambda_0)$ the original SS representation of the data-generating system $S$ with parameters $\Lambda_0$. According to Lem. 2, we aim at identifying an isomorphic $S(\Lambda)$ w.r.t. $S(\Lambda_0)$, due to the non-uniqueness of the SS representation based on the manifest behavior $\mathcal{B}(S(\Lambda_0))$. Hence, any $\Lambda$ in the following set

\[
\mathcal{Q}_0 = \left\{ \Lambda \mid \exists T \in \mathbb{R}^{n_x \times n_x} \text{ s.t. rank}(T) = n_x \right\}
\]

is considered to be a consistent estimate, where $\otimes$ is the Kronecker product. The set $\mathcal{Q}_0$ is also known as the indistinguishable parameter set $[9,34]$.

Given a data-set $D_N = \{u_t, p_t, y_t\}_{t=1}^{n}$ and the basis functions $\{\psi[i]\}_{i=1}^{n_x}$, our objective is to efficiently find, in a stochastic and computational sense, a model estimate in terms of $\hat{\Lambda}$ of the data-generating system (1) and, accordingly, the state dimension $n_x$. In addition, the proposed scheme should be consistent, i.e., $\hat{\Lambda} \to \Lambda \in \mathcal{Q}_0$ with probability one as $N \to \infty$. We will discuss these properties per individual identification step later on. In the remaining part of this paper, it is assumed that the data-generating LPV-SS system (1) with dependency structure (3) is structurally observable and structurally reachable\(^1\), i.e., the system is minimal, and that the input-scheduling signals are persistently exciting, such that the parameters are uniquely identifiable up to the indistinguishable parameter set. We will not address the identifiability problem nor we provide persistency of excitation conditions for the input and scheduling signals. A preliminary study can be found in [30, Chapter 5].

\(^1\) See [31] for a detailed discussion on structural observability and structural reachability in the LPV setting.
3 Identification of LPV impulse response models

In order to realize our objective defined in Sec. 2.5, the first step in the proposed three-step scheme is to capture the SS representation (1) or (8) by its surrogate impulse response representation. It turns out that the coefficients associated with this representation can be captured by linear regression methods. In this section, we present two identification schemes to capture the unknown parameters in a computationally efficient manner by: 1) correlation analysis (Sec. 3.2) or 2) Bayesian impulse response estimation (Sec. 3.3). The identified impulse response coefficients will be used to realize an SS form (Sec. 4).

3.1 LPV Impulse response representation

The surrogate infinite impulse response (IIR) representation is given as:

Lemma 5 (Infinite impulse response [20]) Any asymptotically stable LPV system according to Def. 3 has a convergent series expansion in terms of the pulse-basis \( \{q^{-i}\}_i=0^\infty \) given by

\[
y_{t} = \sum_{i=0}^{\infty} (h_i \circ p)_t q^{-i} u_t + y^*_t,
\]

where \( h_i \in \mathbb{R}^{n_y \times n_u} \) are the expansion coefficient functions, i.e., Markov coefficients, and \( y^*_t \) is a sample path of (6). □

The IIR of an asymptotically stable LPV-SS representation (1) reads as

\[
y_{t} = D(p_t) u_t + C(p_t) B(p_{t-1}) u_{t-1} + \underbrace{C(p_t) A(p_{t-1}) B(p_{t-2}) u_{t-2} + \ldots +}_{(h_2 \circ p)_t}
\]
\[
G(p_t) v_t + C(p_t) H(p_{t-1}) w_{t-1} + \ldots + \]
\[
y^*_t,
\]

where \( h_i \) converges to the zero function as \( i \to \infty \). The noise \( y^*_t \) in (11)-(12) is colored, as it is a combination of the IIR filtered innovation noise \( w \) and the additive output noise \( v \) of (1). Note that the process \( y^*_t \) is quasi-stationary due to the stability of the filters acting on \( v_t \) and \( w_t \) [30, Lem. 4.2]. For notional ease, define \( \psi(t) = \psi^1_t, \psi^0_t \equiv 1 \), and the signal vector \( \psi_t = [1, \psi^1_t, \ldots, \psi^{[n_s]}_t] \top \in \mathbb{R}^{n_s} \). The Markov coefficients can be written as

\[
(h_m \circ p)_t = C(p_t) A(p_{t-1}) \ldots A(p_{t-m+1}) B(p_{t-m}) = \sum_{i=0}^{n_y} \sum_{j=0}^{n_u} \sum_{k=0}^{n_w} C_i A_j \ldots A_k B_l \psi^i_t \psi^j_t \ldots \psi^l_{t-m} = C_i A_j \ldots A_k B_l \psi^i_t \psi^j_t \ldots \psi^l_{t-m} = \sum_{i=0}^{n_y} \sum_{j=0}^{n_u} \sum_{k=0}^{n_w} C_i A_j \ldots A_k B_l \psi^i_t \psi^j_t \ldots \psi^l_{t-m},
\]

where the individual products \( C_i A_j \ldots A_k B_l \) are the so-called sub-Markov parameters for \( m = 1, 2, \ldots \). The latter notation is used to denote the effect of the time-shift operator in a product form. The Markov coefficients in (12) are independent of the parametrization of the matrix functions and the particular state bases, while the sub-Markov parameters are dependent on the parametrization of the functional dependencies in (3).

3.2 Correlation analysis

The sub-Markov parameters (13) can be estimated by correlation analysis (CRA), solving the first step of the proposed identification scheme. CRA results in an estimation procedure which grows linearly in the number of data points and is used to estimate each parameter individually. Hence, the correlation based estimation method has a low computational load. CRA makes use of the stochastic property of \( u, p, w, v \), hence, in this section, \( u \) and \( p \) are assumed to be sample paths of stochastic processes \( u, p \), respectively. Note that, in such case, \( x \) and \( y \) obtained from (1) are sample paths of stochastic processes \( x, y \) which satisfy \( x_{t+1} = A(p_t) x_t + B(p_t) u_t + G(p_t) w_t \), and \( y_t = C(p_t) x_t + D(p_t) u_t + H(p_t) v_t \). Furthermore, we introduce \( \psi^i_t = \psi^i[p_t] \), \( \psi^0_t \equiv 1 \), and \( \psi_t = [1, \psi^1_t(p_t), \ldots, \psi^{[n_s]}_t(p_t)] \top \). The first step in the CRA is to define the \( k \)-dimensional cross-correlation.

Definition 6 The \( k \)-dimensional cross-correlation function for the jointly stationary signals \( (u, y, \psi) \) is defined as

\[
R_{y|u} [s_1, \ldots, s_n] = \mathbb{E}\{y_{s_1} \psi_{t-s_1} \ldots \psi_{t-s_n} (u_{t-s_n})^{\top}\},
\]

where \( s_i \) is a specific index sequence with \( s_1, \ldots, s_n \in \mathbb{R}^{n_s} \) and \( t-s_n \in \mathbb{R}^{n_s} \top \) is the time-shift associated with the specific basis index \( s_i \).

Note that the \( k \)-dimensional cross-correlation is independent of \( t \) due to the assumed joint stationarity of the signals.

Theorem 7 The sub-Markov parameters satisfy

\[
C_{s_1} A_{s_2} A_{s_3} \ldots A_{s_{n-1}} B_{s_n} = \frac{R_{y|u} [s_1, \ldots, s_n] u(0, \ldots, n-1, n-1)}{\sigma_{\psi_1}^2 \ldots \sigma_{\psi_n}^2},
\]

where \( \sigma_{\psi_i} = \mathbb{E}\{\psi^i_t\} \equiv \sigma_{\psi_i} \), and

\[
D_{s_1} = \frac{R_{y|u} [s_1] u(0, 0)}{\sigma_{\psi_1}^2},
\]

where \( s_1, \ldots, s_n \in \mathbb{R}^{n_s} \) are the specific index sequences, if the following assumptions hold:

C1 The output signal is generated by a stable LPV system (1) with dependency structure (3).

C2 The noise processes \( w, v \) are distributed as in (2).

C3 The input process \( u \) is a white noise process with finite variance \( \mathbb{E}\{u^2\} = \sigma_{\psi_0}^2 \) and is independent of \( w, v \).

C4 Each process \( \psi^i_t \) is assumed to be a white noise process with finite variance \( \sigma_{\psi_i}^2 = 1 \), \( \mathbb{E}\{\psi^i_t\} = \sigma_{\psi_i} \) for \( i = 1, \ldots, n_s \). The processes \( \psi^i_t \) are mutually independent and \( \psi^i_t \) is independent of \( u, w \), and \( v \).

PROOF. See Appendix A.
Condition C4 is not over restrictive, e.g., if each \( p^{[i]} \) is a function of \( p^{[i]} \) only, the analytic function \( \psi^{[i]} \) is odd and bounded with \( \psi^{[i]}(0) = 0 \), and it is driven by independent white noise scheduling signals \( p^{[i]} \) with finite variance, then C4 is satisfied. Note that the sub-Markov parameters in Thm. 7 do not depend on the time instant \( t \).

The individual sub-Markov coefficients in (14) and (15) are estimated by approximating the cross-correlation and variances in Thm. 7 based on a finite measured data-set \( D_N \). The variance of the involved signals is estimated by the unbiased sample variance and the \( k \)-dimensional cross-correlation is approximated via

\[
R_{y^{[1]} \ldots y^{[i]} u}(\tau_1, \ldots, \tau_j, \tau_u) = \frac{1}{N - \tau_u + 1} \sum_{t = \tau_u + 1}^{N} y^{[i]}_{t - \tau_i} \cdots y^{[j]}_{t - \tau_j} (u_t - \tau_u)^T.
\]

It is assumed that the stochastic processes \( u, \psi, x, y, w \) are such that \( \lim_{N \to \infty} R_{y^{[i]} \ldots y^{[i]} u}(\cdot) = R_{y^{[i]} u}(\cdot) \). For example, this assumption holds with probability 1 if \( u, \psi, x, y \) are jointly ergodic. Joint ergodicity has been proven in case \( \psi \) is a random binary noise and \( u \) is white noise [35].

The proposed CRA method may need a large data-set and \( N \gg \tau_u \) such that variance of (16) is low enough for an accurate parameter estimate. If the process \( y^s \) in (11) is a zero mean colored noise, e.g., under the general noise conditions of (6), the CRA estimation is known to be inefficient in the LTI setting [36], i.e., the variance of the estimated parameters does not correspond to the Cramér-Rao bound. The here derived extension to the LPV setting shows that similar statement holds. Therefore, a larger data-set is required to achieve equivalent parameter estimation variance compared to the case when \( y^s \) is a white noise with Gaussian distribution. However, an attractive feature of the method is that the sub-Markov parameters can be estimated individually and the computational complexity scales with \( O(2N + N^2 n_u n_n) \) where \( n \) is the amount of specific index sequences \( \{s_j, \ldots, s_j\} \). Hence, the problem scales linearly in \( N, n_u, n \) and quadratic in \( n_n \). We will see that for the basis reduced Ho-Kalman method only a subset of the sub-Markov parameters are needed for realization. Hence, the combination of the LPV-SS realization scheme with the CRA significantly reduces the computational demand, as identification of the full impulse response is omitted.

### 3.3 Bayesian impulse response estimation

As an alternative to CRA, the sub-Markov parameters can be estimated using a Tikhonov regression based LPV-FIR estimation procedure, where the optimal regularization matrix is determined in a Bayesian way with a Gaussian prior, i.e., Bayesian LPV-FIR estimation. In addition, the Bayesian framework allows to estimate the functional dependencies \( \psi^{[0]}(\cdot) \) in a nonparametric way [37, 38]. However, for the sake of simplicity, we consider that these functions are known a priori.

#### 3.3.1 The truncated IIR model

In the Bayesian framework, Eq. (11) is approximated by the following finite order truncation:

\[
y_t \approx \sum_{i=0}^{n_h} (h_i \odot p) u_{t-i} + y^s_t,
\]

with \( n_h > 0 \). Eq. (17) corresponds to a finite impulse response (FIR) model of (11) with order \( n_h \). Due to the convergence of \( h_i \), approximation error of (17) can be chosen arbitrary small by selecting \( n_h \). Furthermore, define

\[
\mathcal{M}_1 = [B_0 \ldots B_{n_p}], \quad \mathcal{M}_j = [A_0 \mathcal{M}_{j-1} \ldots A_n \mathcal{M}_{j-1}],
\]

Based on (13) and (17), the samples in \( D_N \) satisfy the following relationship:

\[
\mathbf{Y}_N = \mathbf{Y}_0 \mathbf{Y}_N + \mathbf{W}_N,
\]

where \( \mathbf{Y}_N = \begin{bmatrix} y_{n_h+1} \ldots y_N \end{bmatrix}, \quad \mathbf{W}_N = \begin{bmatrix} y_{n_p+1} \ldots y_M \end{bmatrix}, \quad \mathbf{Y}_0 = \begin{bmatrix} D_0 \ldots D_{n_p} \mathcal{M}_1 \ldots \mathcal{M}_{n_p} \mathcal{M}_1 \ldots \mathcal{M}_{n_p} \mathcal{M}_{n_p} \end{bmatrix}, \quad \mathbf{W}_N = \begin{bmatrix} y_{n_p+1} \ldots y_M \end{bmatrix}
\]

where \( M = N - n_h - 1 \), \( \mathbf{Y}_N \in \mathbb{R}^{n_y \times M} \) are the measured outputs, \( \mathbf{Y}_0 \in \mathbb{R}^{n_y \times M} \) is the collection of the to-be-estimated sub-Markov parameters with \( \mathbf{Y}_0 = \sum_{i=1}^{n_p+1} (1 + n_p)^{n_p} \), \( \mathbf{Y}_N \in \mathbb{R}^{n_y \times n_h} \) is the regression matrix and \( \mathbf{W}_N \in \mathbb{R}^{n_y \times n_h} \) is the stacked noise realization. The resulting output predictor of the MIMO FIR model (19) can be written as

\[
\hat{Y}_N = \Phi_N \hat{Y}_0,
\]

where \( n_{\theta} = n_y n_t, \quad \hat{Y}_N \in \mathbb{R}^{n_y M \times 1} \) is the predicted output, \( \hat{Y}_N = \hat{Y}_N^T = \mathbf{I}_{n_y} \in \mathbb{R}^{n_y M \times n_y} \), and \( \theta \in \mathbb{R}^{n_{\theta} \times 1} \). For notational reasons, also introduce \( Y_N = \text{vec}(\mathbf{Y}_N) \), \( \hat{Y}_0 = \text{vec}(\hat{Y}_0) \), and \( \mathbf{W}_N = \text{vec}(\mathbf{W}_N) \).

#### 3.3.2 Tikhonov regression based estimate

Even in the LTI case, a well-known issue in estimation of FIR models via the least-squares approach is the high variance of the estimated parameters, due to the relatively large number of parameters required to adequately represent the process dynamics. \( \ell_2 \) regularization makes it possible to control the so-called bias-variance trade-off, i.e., dramatically decrease the variance by introducing a relatively small bias on the estimates [39]. The corresponding weighted Ridge regression or Tikhonov regularization problem is given by

\[
\min_{\theta} \| \Phi_N^T \hat{Y}_0 - Y_N \|_{\mathbf{W}_N}^2 + \| \theta \|_{\mathbf{W}_\theta}^2,
\]
where \( \| x \|_W = \sqrt{x^T W x} \) denotes the \textit{weighted Euclidean norm}, hence, the first term in (21) corresponds to a weighted \( \ell_2 \) norm of the prediction-error of (17), while the second term is the weighted \( \ell_2 \) norm of \( \theta \). Both \( W_c, W_r \in \mathbb{R}^{n_a \times n_a} \) are positive semi-definite (symmetric) regularization matrices and the analytic solution of (21) is

\[
\hat{\theta}_{\text{RWLS}} = (\Phi_N W_c \Phi_N^T + W_r)^{-1} \Phi_N W_c Y_N. \tag{22}
\]

The regularization matrix \( W_r \) is chosen such that \( \Phi_N W_c \Phi_N^T + W_c \) is invertible. If \( W_r = 0, W_c = I \), and \( \gamma^* \) is a white noise process with Gaussian distribution then (22) is the least squares solution, which results in the asymptotically efficient, unbiased, ML estimate.

Analogous to CRA, if \( W_r = 0, W_c = I \), and the additive noise \( \gamma^* \) is a zero mean colored noise process, but uncorrelated with the input and scheduling signals; then the estimator is unbiased, although it is inefficient in terms of increased variance. If \( \gamma^* \) and \( u \) are correlated, then an LPV \textit{instrumental variable} (IV) estimator can be used to remove the bias, e.g., see [18].

### 3.3.3 A Bayesian way of optimizing regularization

One of the main questions with the application of regularization is how to choose the regularization matrix \( W_r \), such that an optimal bias-variance trade-off is found. A recently introduced efficient data-driven approach follows an \textit{empirical Bayes method} [40]. Let us assume in this section that the process noise is zero, i.e., \( w = 0 \) in (1). Hence, the output additive noise process \( v \) in (1) is equal to the output additive noise \( \gamma^* \) in (11) and (17) corresponds to an output error setting. Furthermore, assume that the parameter vector \( \theta_0 \) is a random variable with Gaussian distribution:

\[
\theta_0 \sim \mathcal{N}(\theta_0, P_0), \quad \theta_0 = 0,
\]

where the covariance matrix \( P_0 \) is a function of some hyper parameters \( \alpha \in \mathbb{R}^{n_\alpha} \). In the Bayesian setting, under the assumption that \( u \) and \( p \) are given realizations, \( \Phi_N \) is deterministic and, according to (19), the output vector \( Y_N \) and the parameters \( \theta_0 \) are jointly Gaussian variables:

\[
\begin{bmatrix} \theta_0 \\ Y_N \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ \Phi_N^T P_0 \Phi_N \end{bmatrix}, \begin{bmatrix} P_0 & \Phi_N^T P_0 \Phi_N + \Phi_N^T \Phi_N + \Phi_N^T \Phi_N + I_M \otimes \mathbb{R} \end{bmatrix} \right) \tag{23}
\]

with \( \mathbb{R} \) as in (2). It can be shown that the maximum \textit{posteriori} estimate and the minimal variance estimate of \( \theta_0 \) given \( Y_N \) is equivalent to the weighted regularized least squares estimate \( \hat{\theta}_{\text{RWLS}} \) (22), e.g., see [39], if the weighting and regularization matrices are chosen as

\[
W_c = I_M \otimes \mathbb{R}^{-1}, \quad W_r = P_0^{-1}. \tag{24}
\]

This connection makes it possible to create an estimate of \( \mathbb{R} \) and \( P_0 \) from data that minimizes the marginal likelihood w.r.t. (23). Notice that covariance matrix \( P_0 \), parametrized by \( \alpha \), and the noise covariance matrix \( \mathbb{R} \) satisfy

\[
Y_N \sim \mathcal{N} \left( 0, \Phi_N^T P_0 \Phi_N + I_M \otimes \mathbb{R} \right). \tag{25}
\]

Hence, the likelihood function of the observation \( Y_N \) given \( \alpha \) and \( \mathbb{R} \) can be used to arrive to their posteriori estimate:

\[
\hat{\alpha} = \arg\max \ f(Y_N|\alpha) = \arg\min \ -2 \log f(Y_N|\alpha) \]

\[
= \arg\min \ \log (\det (\Phi_N^T P_\alpha \Phi_N + I_M \otimes \mathbb{R})) + Y_N^T (\Phi_N^T P_\alpha \Phi_N + I_M \otimes \mathbb{R})^{-1} Y_N, \tag{26}
\]

where the constant terms are excluded and \( f(.) \) is the probability density function of the multivariate normal distribution. For a detailed description of pros and cons of the empirical Bayes method compared to other methods, see [41].

The choice of the parametrization of \( P_\alpha \) is of big importance as it governs the “quality” of the estimate. The matrix \( P_\alpha = \theta_0^T \theta_0 \) will give the lowest parameter mean-squared-error (MSE)\(^3\) [42]. However, the true system parameters \( \theta_0 \) are unknown. Therefore, \( P_\alpha \) is often chosen to be a parameterized kernel function to characterize an appropriate search space for an optimal choice of \( P_\alpha \). Many different kernel functions can be employed for this purpose, see [43] for a detailed discussion. For the sake of simplicity, in this paper, we aim at Ridge regression, i.e., we will use \( P_\alpha = \alpha I \). Regularized regression, in general, is know to provide estimates with a lower parameter MSE compared to non-regularized methods, like the CRA method. On the other hand, for the regularized regression, the complete model needs to be estimated, from which, as we will see later, not all parameters are necessary for realization. Consequently, the combination of regularized regression with LPV-SS realization loses computational efficiency compared to the CRA method with LPV-SS realization, but it is applicable under a much wider set of conditions (e.g., we can relax C2-C4 in Thm. 7).

### 4 A basis reduced Ho-Kalman SS realization

The aforementioned identification schemes of Sec. 3.2 and 3.3 can consistently estimate the sub-Markov parameters of (1) under mild assumptions. However, to achieve our goal; an efficient LPV-SS realization of the estimated FIR model is needed. In [21], the well-known Ho-Kalman realization scheme is extended to the LPV case for realizing LPV-SS models with static and affine dependence on the scheduling variable. However, the size of the \textit{l-step} extended observability and \( k \) step extended reachability matrices grows exponentially in \( l, k \) and grow polynomially in the scheduling dimension \( n_w \). Recently, we proposed a basis reduced Ho-Kalman scheme [26], where only the non-repetitive parts of the extended Hankel matrix are selected, which drastically decreases the computational load, compared to the full realization scheme of [21, 44]. The resulting scheme does not depend on any approximations, hence, it is an exact, deterministic realization scheme, and will be briefly explained in this section.

Given a set of sub-Markov parameters associated with the deterministic part (4). To indicate which sub-Markov parameters of the involved extended reachability, observability, and Hankel matrices are selected, we introduce a string...
of characters, called a selection, to denote the considered matrices and their order of multiplication. To define the set of considered strings of characters, introduce $[I^w_0 ]_n$ as the set of all $n$-length sequences of the form $(i_1, \ldots, i_n)$ with $i_1, \ldots, i_n \in I^w_0$. The elements of $I^w_0$ will be viewed as characters and the finite sequences of elements of $I^w_0$ will be referred to as strings. Then $[I^w_0 ]_n$ is the set of all strings containing exactly $n$ characters. The string $\alpha \in [I^w_0 ]_n$ is called a selection with $n \geq 0$ where $[I^w_0 ]_n = \{ \varepsilon \} \cup I^w_0 \cup \cup [I^w_0 ]_n$ and $\varepsilon$ denotes the empty string. Define by $\#(\alpha)$ the amount of characters of a single string. Applying a sequence $\alpha$ will give the ordering of multiplication of matrices $\{A_i\}_{i=0}^\infty$: if $\alpha = \varepsilon$, then $A_i = I$ else

$$A_\alpha = \prod_{i=1}^{\#(\alpha)} A_\alpha = A_{[\alpha]}_{1} A_{[\alpha]}_{2} \cdots A_{[\alpha]}_{\#(\alpha)},$$

(27)

where $[\alpha]_{i}$ denotes the $i$-th character of the string $\alpha$. As an example, let us define the set $[I^w_0 ]_2 = \{ \varepsilon \} \cup I^w_0 \cup [I^w_0 ]_2 = \{ 0, 0, 1, 0, 0, 1, 0, 1, 1 \}$. Take, for instance, $\alpha = 10 \in [I^w_0 ]_2$ which indicates $A_0 = A_1 A_0$. Based on this selection, the $(i,j)$-th element of a single sub-Markov parameter $C_\alpha A_\beta B_\gamma \in \mathbb{R}^{n \times n}$ is denoted by $C_\alpha^{[i]} A_\beta B_\gamma^{[j]}$ for $\alpha \in [I^w_0 ]_n$, $\beta \in I^w_0$, $\gamma \in I^w_v$. Then a selection of the extended reachability matrix is represented by

$$\varsigma = \left( \{ \alpha_1, \beta_1, j_1 \}, \ldots, \{ \alpha_n, \beta_n, j_n \} \right),$$

(28)

where $\alpha_1, \ldots, \alpha_n \in [I^w_0 ]_n$, $\beta_1, \ldots, \beta_n \in I^w_0$, and $j_1, \ldots, j_n \in I^w_v$. The length of the string $\alpha_1 \ldots \alpha_n$ may vary. Using this basis, a sub-matrix of the extended reachability matrix [45] is selected by

$$R_\varsigma = \left[ A_{\alpha_1} B_{\beta_1}^{[j_1]} A_{\alpha_2} B_{\beta_2}^{[j_2]} \ldots A_{\alpha_n} B_{\beta_n}^{[j_n]} \right],$$

(29)

where $R_\varsigma \in \mathbb{R}^{n \times n}$ and $[\beta_k]$ denotes the $k$-th column of $B_{\beta_k}$ for $k = 1, \ldots, n$. Analogously, a basis of the extended observability matrix is selected by

$$\nu = \{ (i_1, \gamma_1, \alpha_1), \ldots, (i_n, \gamma_n, \alpha_n) \},$$

(30)

where $i_1, \ldots, i_n \in [I^w_0 ]_n$, $\gamma_1, \ldots, \gamma_n \in I^w_0$, and $i_1, \ldots, i_n \in I^w_v$. Note that $\alpha_1 \ldots \alpha_n \in [I^w_0 ]_n$. The sub-matrix of the extended observability matrix as

$$O_\nu = \left[ \left( C_{\gamma_1}^{[i_1]} A_{\alpha_1} \right)^\top \cdots \left( C_{\gamma_n}^{[i_n]} A_{\alpha_n} \right)^\top \right]^\top,$$

(31)

where $O_\nu \in \mathbb{R}^{n \times n}$ and $[i_k]$ denotes the $k$-th row of $C_{\gamma_k}$ for $k = 1, \ldots, n$. The sets $\varsigma$ and $\nu$ are chosen appropriately, such that rank($R_\varsigma$) = $n_x$, rank($O_\nu$) = $n_x$, and hence rank($O_\nu R_\varsigma$) = $n_x$. If this condition is satisfied, then we call the selection $\varsigma$ and $\nu$ a basis selection. For such a basis selection $(\nu, \varsigma)$, define

$$H_{\nu \varsigma} = O_\nu R_\varsigma, \quad H_{\nu \varsigma, k} = O_\nu A_{\alpha_1} R_\varsigma,$$

(32)

$$H_{k \nu \varsigma} = O_\nu B_{\beta_1} R_\varsigma, \quad H_{k \nu \varsigma, k} = R_\varsigma C_{\gamma_1} A_{\alpha_1},$$

where $H_{\nu \varsigma} \in \mathbb{R}^{n \times n}$, $H_{\nu \varsigma, k} \in \mathbb{R}^{n \times n}$, $H_{k \nu \varsigma} \in \mathbb{R}^{n \times n}$, and $H_{k \nu \varsigma, k} \in \mathbb{R}^{n \times n}$. Note that these sub-Hankel matrices in (32) are composed of the sub-Markov parameters.

**Lemma 8** Define a column selection $\varsigma$ with $n_\varsigma = n_x$ and a row selection $\nu$ with $n_\nu \geq n_x$ such that rank($H_{\nu \varsigma}$) = $n_x$. The set of matrices

$$\hat{A}_k = H^\dagger_{\nu \varsigma} H_{\nu \varsigma, k}, \quad \hat{B}_k = H^\dagger_{\nu \varsigma} H_{\nu \varsigma, k}, \quad \hat{C}_k = H_{k \nu \varsigma},$$

(33)

for $k \in I^n_0$ give a joint minimal LPV-SS representation of $S$ in (1) with the dependency structure (3), i.e.,

$$\begin{bmatrix} \hat{A}_0 & \ldots & \hat{A}_{n_\varsigma} & \hat{B}_0 & \ldots & \hat{B}_{n_\varsigma} \end{bmatrix} \in \mathbb{Q}_0.$$ 

(34)

In (33), $H^\dagger_{\nu \varsigma}$ denotes the left pseudo inverse of $H_{\nu \varsigma}$. □

**PROOF.** As $H^\dagger_{\nu \varsigma}$ exists and $H_{\nu \varsigma}$ has full column rank, the proof is straightforward by applying the isomorphism $T = R_{\nu \varsigma}^{-1}$. ■

From the practical and numerical point of view, a reliable implementation of (33) follows by using singular value decomposition (SVD). Define a basis selection $n_\nu, n_\varsigma \geq n_x$ with rank($H_{\nu \varsigma}$) = $n_x$ and compute an economical SVD: $H_{\nu \varsigma} = U_{n_\nu} \Sigma_{n_\nu} V_{n_x}^\top$. Then a realization of $S$ is

$$\hat{A}_k = \hat{O}_\nu^\dagger H_{\nu \varsigma, k} \hat{R}_\varsigma^\dagger, \quad \hat{B}_k = \hat{O}_\nu^\dagger H_{\nu \varsigma, k},$$

$$\hat{C}_k = H_{k \nu \varsigma} \hat{R}_\varsigma^\dagger,$$

(35)

for $k \in I^n_0$ with pseudo inverses $\hat{R}_\varsigma^\dagger \equiv V_{n_x} \Sigma_{n_x}^{-1/2}$, $\hat{O}_\nu^\dagger \equiv \Sigma_{n_x}^{-1/2} U_{n_\nu}^\top$. Realization (35) gives an LPV-SS representation of $S$ in (1), i.e., $\{ \hat{A}_0, \ldots, \hat{O}_\nu \}$ satisfies (34). The proof of this methodology can be found in [26].

In case the sub-Hankel matrices (32) are filled with estimated sub-Markov parameters, the state order $n_x$ can be chosen based upon the magnitude of the singular values $\Sigma_{n_x}$, i.e., an approximate realization (e.g., see [46]). Note that the realization in (35) does not have any restrictions on the maximum amount of columns chosen $n_\varsigma \geq n_x$, compared to Lem. 8 where $n_\varsigma = n_x$. Hence, the rank-revealing property of the SVD of $H_{\nu \varsigma}$, allows to find a reliable estimate of $n_x$.

This bases reduced realization can considerably decrease the size of the Hankel matrix and, therefore, reduce the computational load, compared to the realization with the full Hankel matrix [21, Eq. (48)]. In the basis reduced realization, the SVD is only applied on a $n_\nu \times n_r$ matrix instead of a matrix with size $n_\nu \sum_{l=1}^i (1 + n_\nu)^l \times n_u \sum_{j=1}^j (1 + n_\nu)^j$ in the full realization case. Note that $n_\nu, n_r = n_x$ in the ideal case, which gives the computational lower bound that is similar to the LTI case. The amount of sub-Markov parameters in (32) is $n_\nu n_r (1 + n_\nu)^l + n_x (1 + n_\nu)^j$, which increases linearly in all parameters $n_\nu, n_r, n_x, n_y$, compared to $n_\nu \sum_{l=1}^i (1 + n_\nu)^l \times n_u \sum_{j=1}^j (1 + n_\nu)^j$, which grows exponentially with increasing $i$ and $j$ and polynomially with increasing $n_\nu$. To illustrate, the realization of a system with input/output dimension $n_\nu = n_u = 2$, state dimension $n_x = 4$, and scheduling dimension $n_\nu = 5$, the full Hankel matrix $H_{2,2}$ has 7056 elements, while the sub-Hankel matrices for $n_r = n_\nu = 10$ have only 940 elements.
5 Maximum likelihood refinement

The basis reduced Ho-Kalman realization cannot guarantee that the LPV-SS model realized from the identified sub-Markov parameters is a maximum likelihood estimate, even if the underlying approaches are capable of providing ML estimates. Hence, to reach the maximum likelihood LPV-SS model estimate, two solutions are explored for refinement: 1) the gradient-based (GB) search method, and 2) the expectation maximization (EM) algorithm. Both methods are nonlinear iterative optimization techniques and cannot be used as stand-alone methods, as they are prone to local minima. For example, [10, Table III] shows the number of failed model identification iterations for inefficient initial estimates in an LTI-SS identification problem. Hence, Step 1 and Step 2 of our proposed identification scheme, i.e., LPV impulse response estimation with LPV-SS realization, can be seen as a numerically efficient method for initializing GB or EM methods. The efficiency of this combination will be shown in Sec. 6.

5.1 Gradient based PEM

PEM methods aim at minimizing the mean-squared prediction-error criterion w.r.t. the free model parameters. For LPV-SS models, the minimization problem is nonconvex and nonunique based upon $\Omega_N$ [30]. The optimization is usually solved via a gradient-based search strategy such as a Newton or similar type of method. In this paper, the enhanced Gauss-Newton based search method of [10] is extended to the LPV case. The enhanced Gauss-Newton includes: 1) an automated strategy of regularization and SVD truncation on the Jacobian matrix to obtain a search direction, 2) an Armijo line search backtracking rule, and 3) lowering the dimension of the parameter space by using the data-driven local coordinate (DDLC) frame. The DDLC frame is the ortho-complement of an affine approximation of the indistinguishable set $\Omega_N$, around the current model parameters. Consequently, the DDLC ensures that the nonlinear optimization does not wander among parameterizations of SS models with equivalent manifest behavior. Additionally, the DDLC results in a minimal parametrization in the LTI case and, hence, the PEM optimization problem is of minimal dimension [30]. The combination of improved gradient-based search strategies and the DDLC frame increases the computational demand per iteration, however, in general, it significantly improves the convergence rate.

5.2 Expectation Maximization

The key element of the EM method is to presume the existence of a complete data-set $Z_N = (Y_N, X_N)$, which contains not only the actual observations $Y_N$, but also the missing state-sequence $X_N$. The iterative EM method identifies LPV-SS models by considering the state-sequence as the missing data. With this choice, the maximization of the ML is a joint estimation problem and is solved in an alternating manner. EM methods for the LTI case have been developed in [12, 47, 48] and an LPV extension of EM is given in [11]. We can apply [11] under the assumption that the noise structure in the data-generating system (1) is with $\mathcal{H}(p) = I$ and $G(p) = I$. We provide here a brief overview of the main steps of this algorithm. Each iteration of the EM consist of two steps: 1) the expectation, and 2) the maximization step. In the expectation step, given the current model estimate, the likelihood of the complete data-set conditional on the data observed is approximated. The likelihood, i.e., obtaining the unknown state trajectory $x_t$, can be estimated via various approaches, e.g., particle filtering [49], or Kalman filtering [50, 51]. In the example section, we provide the comparison using an implementation with the Kalman filter, the Kalman smoother, and a one-lag covariance smoother similar to [11]. In the second step (maximization step), the approximated likelihood is maximized with respect to the model parameters. As the state-sequence is known, the estimation problem becomes linear-in-the-parameters with an analytic solution. The EM method is relatively straightforward to implement and the computational load scales linearly with the data-set length. The EM algorithm usually converges rapidly in early stages, but its rate of convergence near the maximum is substantially lower than of the GB method, e.g., see [12, 50].

6 Simulation Example

In this section, the performance of the proposed three-step identification procedure is assessed on a Monte-Carlo simulation study using a randomly generated stable LPV-SS model in innovation form with scheduling independent matrix function, i.e., $K(p_i) = K$. The Monte-Carlo study shows the performance of the methods in the following cases:

1. Correlation analysis with basis reduced Ho-Kalman LPV-SS realization (without refinement step).
2. Correlation analysis with basis reduced Ho-Kalman LPV-SS realization and EM or GB refinement step.
3. Bayesian FIR estimation with basis reduced Ho-Kalman LPV-SS realization (without refinement step).
4. Bayesian FIR estimation with basis reduced Ho-Kalman LPV-SS realization and EM or GB refinement step.

The proposed procedure is compared to state-of-the-art LPV-SS identification methods, such as the predictor-based subspace identification (PB) [25], successive approximation identification algorithm (SA) [28], and the robust identification/validation method (RI) [15]. Furthermore, the estimated SS model by these approaches is refined, identical to the case of CRA and FIR, by using the estimated SS model as initialization for the EM or GB method. This shows which approach can provide better initialization for the ML step and how far the delivered models are from the ML estimate. The case study is performed on a Macbook pro laptop, late 2013 with an 2.6GHz Intel i5 processor and Matlab 2014b. For the comparison, the scripts provided by the authors of [15, 25, 28] are used.

6.1 Data-generating system and model structure

The data-generating system is randomly selected in terms of an SS model (8a)-(8b) with input-output dimensions $n_u = n_y = 2$, scheduling dimension $n_q = 5$, minimal state dimension $n_x = 4$, and affine dependence, i.e., the known basis functions are $\psi^{[i]} = p^{[i]}$ with $p^{[i]}$ denoting the $i^{th}$ element
of \( p \). The SS model represented system has a scheduling independent innovation matrix, i.e., \( K(p_t) = K \). This simplified innovation form is chosen, for the sake of fairness of the comparison, as all aforementioned methodologies are able to consistently identify this particular representation, except the EM methodology due to its different noise assumptions.

The system was constructed such that (1) and the innovation form based output substituted equation

\[
\hat{x}_{t+1} = (A(p_t) - KC(p_t))x_t + (B(p_t) - KD(p_t))u_t + Ky_t,
\]

are asymptotically input-to-state stable on the domain \( p_t \in \mathbb{P} = [-1, 1]^3 \) with a quadratic Lyapunov function defined by a constant symmetric matrix [52]. The LPV-SS model is available at [53].

6.2 Identification setting

The identification data-set is generated with a white \( u \) with uniform distribution \( u_t \sim U(-1, 1) \), and white \( p \) with random binary distribution on \((-0.9, 0.9)\), each of length \( N = 5 \cdot 10^3 \). The noise process \( \xi \) is taken as a white noise with distribution \( \xi_t \sim \mathcal{N}(0, \mathbb{V}) \) where \( \mathbb{V} \) is diagonal and it is chosen such that the signal-to-noise ratio (SNR)

\[
\text{SNR}^{[i]} = 10 \log \frac{\sum_{t=1}^{N} (y_t^{[i]} - \hat{y}_t^{[i]})^2}{\sum_{t=1}^{N} \hat{y}_t^{[i]}^2},
\]

is set for various Monte-Carlo experiments as \( \text{SNR}^{[i]} = \{40, 25, 10, 0\} \) dB for all \( i = 1, \ldots, n_y \). The \( i \)-th channel, i.e., element of the vector signal, and \( \text{SNR}^{[i]} \) is the SNR of the output \( y_t^{[i]} \). In this setting, the signals are jointly ergodic and the parameters can be consistently identified [35]. The performance of the scheme is tested on a validation data-set \( \mathcal{D}_{\text{val}} \) of length \( N_{\text{val}} = 200 \) with different excitation conditions than the estimation data-set:

\[
u_t = \begin{bmatrix} 0.5 \cos(0.035t) \\ 0.5 \sin(0.035t) \end{bmatrix} + \delta_{t,u}, \quad (36)
\]

\[
p_t^{[i]} = 0.25 - 0.05i + 0.4 \sin \left( 0.035t + \frac{2\pi t}{5} \right) + \delta_{t,p}, \quad (37)
\]

where \( \delta_{t,u} \in \mathbb{R}^{n_u}, \delta_{t,p} \in \mathbb{R} \) are element wise i.i.d. sequences with uniform distribution \( U(-0.15, 0.15) \). To study the statistical properties of the developed identification scheme, a Monte-Carlo study with \( N_{\text{MC}} = 100 \) runs is carried out, where in each run a new realization of the input, scheduling, and noise sequences are taken. In each run, all considered methods are applied on the identification data-set. The data-set is available at [53]. We will assess the performance of the CRA, FIR, and RI model estimates without refinement step by comparing the simulated output \( \hat{y} \) of the estimated model to the noise free output \( y_t \). In all other cases, the one-step-ahead predicted output \( \hat{y} \) of the estimated model is compared to the one-step-ahead predicted output of an oracle predictor (i.e., the one-step-ahead predicted output using the original data-generating system).

This dichotomy in assessing different signals is caused by the fact that the CRA, FIR, and RI do not identify a noise model, hence, the one-step-ahead predicted output is equal to the simulated output, therefore, comparing it to the noise free output \( y_t \) of the process part is more adequate. On the other hand, the remaining methods include an estimate of a noise model, thus the estimated plant and noise model are assessed by using the one-step-ahead predictor. In this case, the achieved results are compared w.r.t. the oracle, as its generated output is the maximum achievable output estimate given the data-set. The performance criterion used is the best fit rate (BFR)\(^4\)

\[
BFR = \max \left\{ 1 - \frac{1}{N_{\text{val}}} \sum_{t=1}^{N_{\text{val}}} \| y_t - \hat{y}_t \|^2, 0 \right\} \cdot 100\%, \quad (38)
\]

using \( \mathcal{D}_{\text{val}} \). In (38), \( \hat{y} \) defines the mean of the predicted/true output \( y_t \) in \( \mathcal{D}_{\text{val}} \) and \( \hat{y} \) is the simulated output of the model w.r.t. (36) and (37) in \( \mathcal{D}_{\text{val}} \). Next, we will provide a summary of the used design parameters, which are optimized to provide the highest BFR. The FIR model order is chosen as \( n_t = 2 \) with \( P_\alpha = \alpha I \). The hyperparameter \( \alpha \) is tuned by using the Bayesian MIMO formulation of [27]. In the realization step, the basis reduced Ho-Kalman scheme uses \( n_o = n_t = 10 \) bases, where the controllability matrix is spanned by \( \varsigma = \{ (\epsilon, 0, 2), (\epsilon, 1, 2), (\epsilon, 2, 1), (\epsilon, 2, 2), \ldots, (\epsilon, 5, 2) \} \) and the observability is spanned by \( \nu = \{ (1, 0, \epsilon), (2, 1, \epsilon), \ldots, (1, 4, \epsilon), (1, 5, \epsilon), (2, 5, \epsilon) \} \). The basis of the Hankel matrix is selected by using the entries of the full Hankel matrix with the largest absolute value. For the PB method, the future \( f \) and past window \( p \) are chosen as \( f = p = 3 \). For the SA method, the number of block rows in the Hankel matrix is chosen to be 4 and the iterative procedure is stopped if the 2-norm of the eigenvalues of the A0 matrix do not change more than \( 10^{-6} \) or if it exceeds 100 iterations. For the RI method, only the first 150 data samples are taken into account as the computational complexity of the problem does not allow to use all data points of \( \mathcal{D}_{\text{N}} \). For the EM method, the relative and absolute tolerance on the marginal log likelihood are chosen as \( 2 \cdot 10^{-3} \) and \( 10^6 \), respectively, with a maximum of 20 iterations. For the GB method, we use \( \beta = 10^{-4} \), \( \gamma = 0.75 \), \( \eta_{\text{min}} = 10^{-5} \), \( \epsilon_{\text{min}} = 0.001 \), \( \nu = 0.01 \), \( \epsilon = 10^{-4} \) according to the notation of [10], and a maximum of 20 iterations.

6.3 Analysis of the results

Table 1 shows the mean and the standard deviation of the BFR on \( \mathcal{D}_{\text{val}} \) and execution time of the estimation algorithms per Monte Carlo run for different \( \text{SNR}^y = \{40, 25, 10, 0\} \) dB. Similar results are obtained w.r.t. the simulation error, however, due to space limitations it is not presented. Note that the SA method does not often converge to the considered system with \( n_p = 5 \), hence, also a simulation study is done where the system to be identified had only \( n_p = 2 \) scheduling signals (SA2). In addition, remark that, the RI method only identifies \( C(\cdot), D(\cdot) \) and assumes \( A(\cdot), B(\cdot) \) to be known.

The table shows that the FIR with bases reduced realization outperforms the CRA, PB, SA, and RI methods. The CRA performs worse, because regularized methods, such as FIR,
provide estimates with lower parameter MSE by tuning the bias/variance trade-off. However, this tuning comes with an increased computational cost of approximately 4 times. The PB is outperformed by the FIR, as it needs to estimate significantly more parameters, which is a well known problem of this method [25, Table 1]. Estimation of the increased amount of parameters also results in an increased computational load for this method. On the other hand, PB can identify unstable systems as only the one-step-ahead predictor dynamics are required to be stable and it can also be used in a closed-loop identification setting. The SA method has, in many cases, problems with convergence. Presumably, this is caused by using an LTI subspace method to initialize the iterative scheme. The method has a substantially higher BFR and less convergence problems if the data-generating system has \( n_p = 2 \) instead of \( n_p = 5 \). The RI method can potentially outperform the other methods, as the \( A(\cdot), B(\cdot) \) matrix functions are a-priori known. However, the computational complexity of the RI method only allows to use a small portion of the data-set \( \mathcal{D}_N \) for estimation (in our case 150 out of 5000), hence, a large decrease in its performance is seen for lower SNRs. All performance criteria indicate that the additional refinement step, with the EM or GB method, will lead to a better estimate of the model, as expected. Only in case of the SNR\( [\text{dB}] = 0 \) dB noise scenario, the EM refinement step does not improve the estimate. In this case, the EM method is not able to converge due to the large noise contribution. The GB method outperforms the EM method in all cases. Partially, this might be caused by the additional steps to improve the numerical properties of the GB method, i.e., the automated strategy of regularization and SVD truncation of the Jacobian matrix and line search backtracking rule. Furthermore, the underlying data-generating system is not within the noise model set of the EM methodology, leading to a suboptimal filter with a lower achieved BFR compared to GB. Therefore, no fair conclusions can be drawn on the relative performance of EM w.r.t. GB based on this simulation study. In addition, we would like to highlight that the CRA and FIR are not statistically efficient under the considered noise scenario, as they do not identify a noise model. Hence, it is impressive that these methods are capable of providing efficient initializations of PEM, even under a non-idealistic noise scenario.

Summarizing, the proposed three-step approach results in a maximum-likelihood estimate with a lower computational time and higher performance compared to existing state-of-the-art LPV-SS identification approaches.

7 Conclusion

In this paper, we have presented a computationally efficient, modular three-step LPV-SS identification approach, which contains the following steps: 1) estimation of the Markov coefficient sequence using correlation analysis or a Bayesian FIR estimation, then 2) efficient LPV-SS realization by using a basis reduced Ho-Kalman method, and 3) refinement of the LPV-SS model estimate by a GB or EM optimization methodology. This three-step approach can consistently identify the underlying data-generating system. The effectiveness of the scheme has been demonstrated on a real-world sized MIMO LPV-SS model identification problem under harsh noise conditions and it has been compared to other methods. Any combination of the scheme was able to identify the system within seconds, significantly faster than its competitors while also achieving better performance.

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References

[1] M. Groot Wassink, M. Van de Wal, C. Scherer, and O. Bosgra, “LPV control for a wafer stage: beyond the theoretical solution,” Control Engineering Practice, vol. 13, no. 2, pp. 231–245, 2005.
[2] J. Veenman, C. W. Scherer, and H. Köröglu, “IQC-based LPV controller synthesis for the NASA HL-20 atmospheric re-entry vehicle,” in Proc. of the AIAA Guidance, Navigation, and Control Conf., (Chicago, IL, USA), pp. 1–16, Aug 2009.
[3] J. W. van Wingerden, I. Houtzager, F. Felici, and M. Verhaegen, “Closed-loop identification of the time-varying dynamics of variable-speed wind turbines,” Int. J. of Robust and Nonlinear Control, vol. 19, no. 1, pp. 4–21, 2009.
[4] A. A. Bachaus, R. Töth, A. Mesbah, and J. H. A. Luldagne, “A review on data-driven linear parameter-varying modeling approaches: A high-purity distillation column case study,” J. of Process Control, vol. 24, no. 4, pp. 272–285, 2014.
[5] J. Mohammadpour and C. Scherer, eds., Control of Linear Parameter Varying Systems with Applications. Springer, 2012.
[6] M. Nemani, R. Ravikanth, and B. A. Bamieh, “Identification of linear parametrically varying systems,” in Proc. of the 34th IEEE Conf. on Decision and Control, (New Orleans, LA, USA), pp. 2990–2995, Dec 1995.
[7] S. Rizvi, J. Mohammadpour, R. Töth, and N. Meskin, “An IV-SVM-based approach for identification of state-space LPV models under generic noise conditions,” in Proc. of the 54th IEEE Conf. on Decision and Control, (Osaka, Japan), pp. 7380–7385, Dec. 2015.
[8] L. H. Lee and K. P. Poolla, “Identifiability issues for parameter-varying and multidimensional linear systems,” in Proc. of the ASME Design, Engineering Technical Conf., (Sacramento, CA, USA), Sep. 1997.
[9] V. Verdult, N. Bergboer, and M. Verhaegen, “Identification of fully parameterized linear and nonlinear state-space systems by projected gradient search,” in Proc. of the 13th IFAC Symposium on System Identification, (Rotterdam, The Netherlands), pp. 737–742, Aug. 2003.
[10] A. Wills and B. Ninness, “On gradient-based search for multivariable system estimates,” IEEE Trans. on Automatic Control, vol. 53, no. 1, pp. 298–306, 2008.
[11] A. Wills and B. Ninness, “System identification of linear parameter varying state-space models,” in Linear Parameter-Varying System Identification: New Developments and Trends (P. Lopes dos Santos, T. P. Azvedo Perdicoullis, C. Novara, J. A. Ramos, and D. E. Rivera, eds.), ch. 11, pp. 295–316, World Scientific, 2011.
[12] M. W. Watson and R. F. Engle, “Alternative algorithms for the estimation of dynamic factor, mimic and varying coefficient regression models,” J. of Econometrics, vol. 23, pp. 385–400, Dec. 1983.
Table 1
The mean and the standard deviation (between parentheses) of the BFR and execution time of the estimation algorithms per Monte Carlo run for different SNR_s = \{40, 25, 10\} dB is given. The BFR is based on the one-step-ahead predicted output of the estimated model on the validation data-set except the methods with an asterisk for which it is based on the simulated output. The correlation analysis (CRA), finite impulse response (FIR) estimation, the predictor-based subspace identification (PB), successive approximation identification (SA), and the robust identification/invalidation (RI) method are used and refinement of the estimates is performed by expectation-maximization (EM) or gradient based (GB) algorithm. The SA2 indicates the results for the SA method where the system to be identified had n_y = 2 scheduling signals. For this table, N_MC = 100 Monte Carlo simulations are performed. The number in the superscript indicates how many successful trials have been achieved out of the 100 runs.

|                | BFR [%] | Time Elapsed [s] |
|----------------|---------|------------------|
|                | 40dB    | 25dB  | 10dB    | 0dB    | 25dB  | 0dB    |
| CRA*           | 81.47 (4.106) | 81.44 (4.433) | 81.14 (5.404) | 75.07 (5.823) | 2.269 (0.1481) | 2.229 (0.1519) |
| CRA + EM       | 99.71 (0.05576) | 98.80 (0.1710) | 91.31 (0.5744) | 74.13 (1.804) | 8.297 (0.2722) | 8.351 (0.2978) |
| CRA + GB       | 99.86 (0.02886) | 99.27 (0.1600) | 95.81 (0.8313) | 87.81 (2.513) | 8.626 (0.6105) | 12.94 (1.146) |
| FIR*           | 99.32 (0.1466) | 98.74 (0.2872) | 94.19 (1.346) | 83.55 (3.868) | 10.87 (0.5037) | 9.098 (0.4474) |
| FIR + EM       | 99.73 (0.05327) | 98.80 (0.1679) | 91.26 (0.5432) | 74.14 (1.788) | 16.88 (0.6055) | 15.21 (0.3635) |
| FIR + GB       | 99.86 (0.02886) | 99.27 (0.1600) | 95.81 (0.8313) | 87.81 (2.513) | 17.21 (0.8063) | 19.62 (1.236) |
| PB             | 95.90 (1.185) | 86.25 (2.761) | 75.89 (3.051) | 61.35 (8.831) | 88.47 (0.5274) | 88.43 (0.4078) |
| PB + EM        | 98.50 (0.2035) | 98.02 (0.2849) | 92.39 (1.703) | 78.80 (4.875) | 92.84 (0.5456) | 92.79 (0.4136) |
| PB + GB        | 99.79 (0.05832) | 99.27 (0.1600) | 95.81 (0.8312) | 87.57 (2.893) | 97.62 (1.317) | 100.2 (0.4811) |
| SA2            | 82.97 (12.90) | 83.83 (9.771) | 83.04 (10.70) | 73.78 (12.04) | 21.52 (8.165) | 26.86 (3.883) |
| SA2 + GB       | 97.29 (13.56) | 99.51 (0.1216) | 96.68 (2.363) | 90.49 (2.249) | 26.94 (8.126) | 34.21 (3.687) |
| SA             | 65.57 (4.225) | 65.55 (3.283) | 66.18 (3.630) | 53.24 (8.806) | 108.9 (3.442) | 107.0 (0.3076) |
| SA + GB        | 97.75 (8.483) | 99.31 (0.1306) | 94.51 (6.373) | 87.51 (2.358) | 118.0 (3.615) | 122.3 (1.793) |
| RI*            | 99.25 (0.1876) | 95.75 (0.9813) | 76.11 (5.991) | 26.66 (15.25) | 103.4 (5.261) | 108.9 (6.406) |
| RI + EM        | 99.73 (0.05359) | 98.80 (0.1710) | 91.29 (0.5411) | 74.34 (1.663) | 109.3 (5.307) | 114.8 (6.438) |
| RI + GB        | 99.86 (0.02887) | 99.27 (0.1600) | 95.81 (0.8313) | 87.81 (2.513) | 110.3 (5.352) | 121.3 (6.465) |

[13] G. Z. Angelis, System Analysis, Modelling and Control with Polytopic Linear Models. Phd thesis, Eindhoven University of Technology, 2001.

[14] P. Gáspár, Z. Szabó, and J. Bokor, “A grey-box identification of an LPV vehicle model for observer-based side-slip angle estimation,” in Proc. of the American Control Conf., (New York City, USA), pp. 2961–2965, Jul 2007.

[15] F. D. Bianchi and R. S. Sánchez-Peña, “Robust identification/validation in a LPV framework,” Int. J. of Robust and Nonlinear Control, vol. 20, no. 3, pp. 301–312, 2009.

[16] C. Novara, “SM identification of state-space LPV systems,” in Linear Parameter-Varying System Identification: New Developments and Trends (P. Lopes dos Santos, T. P. Azevedo Perdicoulis, C. Novara, J. A. Ramos, and D. E. Rivera, eds.), ch. 4, pp. 65–93, World Scientific, 2011.

[17] V. Ceronea, D. Piga, and D. Regnuto, “A convex relaxation approach to set-membership identification of LPV systems,” Automatica, vol. 49, pp. 2853–2859, 2013.

[18] V. Laurain, M. Gibson, R. Tóth, and H. Garnier, “Refined instrumental variable methods for identification of LPV Box-Jenkins models,” Automatica, vol. 46, no. 6, pp. 959–967, 2010.

[19] P. Lopes dos Santos, T.-P. Azevedo-Perdicoulis, C. Novara, J. A. Ramos, and D. E. Rivera, eds., Linear Parameter-Varying System Identification: New Developments and Trends. World Scientific, 2011.

[20] R. Tóth, Modeling and Identification of Linear Parameter-Varying Systems. Springer, 2010.

[21] R. Tóth, H. S. Abbas, and H. Werner, “On the state-space realization of LPV input-output models: practical approaches,” IEEE Trans. on Control Systems Technology, vol. 20, no. 1, pp. 139–153, 2012.

[22] F. Felici, J.-W. van Wingerden, and M. Verhaegen, “Subspace identification of MIMO LPV systems using a periodic scheduling sequence,” Automatica, vol. 43, no. 10, pp. 1684–1697, 2007.

[23] P. Lopes dos Santos, J. A. Ramos, and J. L. de Carvalho, “Identification of linear parameter-varying systems using an iterative deterministic-stochastic subspace approach,” in Proc. of the European Control Conf., (Kos, Greece), pp. 4867–4873, Jul 2007.

[24] W. E. Larimore, “Identification of nonlinear parameter-varying systems via canonical variate analysis,” in Proc. of the American Control Conf., (Washington, DC, USA), pp. 2247–2262, Jun 2013.

[25] J. W. van Wingerden and M. Verhaegen, “Subspace identification of bilinear and LPV systems for open- and closed-loop data,” Automatica, vol. 45, no. 2, pp. 372–381, 2009.

[26] P. B. Cox, R. Tóth, and M. Petreczky, “Estimation of LPV-SS models with static dependency using correlation analysis,” in Proc. of the 1st IFAC Workshop on Linear Parameter Varying Systems, (Grenoble, France), pp. 91–96, Oct. 2015.

[27] P. B. Cox and R. Tóth, “LPV state-space model identification in the Bayesian setting: A 3-step procedure,” in Proc. of the American Control Conf., (Boston, MA, USA), pp. 4604–4610, Jul. 2016.

[28] P. Lopes dos Santos, J. A. Ramos, and J. L. Martins de Carvalho, “Subspace identification of linear parameter-varying systems with innovation-type noise models driven by general inputs and a
measurable white noise time-varying parameter vector,” Int. J. of Systems Science, vol. 39, no. 9, pp. 897–911, 2008.

[29] V. I. Bogachev, Measure Theory. Springer-Verlag Berlin Heidelberg, 2007.

[30] P. B. Cox, Towards Efficient Identification of Linear Parameter-Varying State-Space Models. Phd thesis, Eindhoven University of Technology, 2018.

[31] M. Petreczky, R. Tóth, and G. Mercère, “Realization theory for LPV state-space representations with affine dependence,” IEEE Trans. on Automatic Control, vol. 62, no. 9, pp. 4667–4674, 2017.

[32] M. Verhaegen and M. Verdult, “Subspace identification of multivariable linear parameter-varying systems,” Automatica, vol. 38, no. 5, pp. 805–814, 2002.

[33] L. H. Lee and K. Poolla, “Identification of linear parameter-varying systems using nonlinear programming,” J. of Dynamic Systems, Measurement, and Control, vol. 121, no. 1, pp. 71–78, 1999.

[34] M. Petreczky and L. Bako, “On the notion of persistence of excitation for LPV state-space systems,” in Proc. of 50th IEEE Conf. on Decision and Control and European Control Conf., (Orlando, FL, USA), pp. 1840–1847, Dec. 2011.

[35] M. Verhaegen and M. Verdult, “System Identification: Theory for the User,” second ed., 1999.

[36] A. Golabi, N. Meskin, R. Tóth, and M. Mohammadpour, “Bayesian approach for estimation of LPV linear-regression models,” in Proc. of the 53rd IEEE Conference on Decision and Control, (Los Angeles, CA, USA), pp. 2555–2560, Dec. 2014.

[37] M. A. H. Darwish, P. B. Cox, G. Pillonetto, and R. Tóth, “Bayesian identification of LPV Box-Jenkins models,” in Proc. of the 54th IEEE Conf. on Decision and Control, (Osaka, Japan), pp. 66–71, Dec. 2015.

[38] L. Ljung, System Identification: Theory for the User. Springer, second ed., 1999.

[39] A. Golabi, N. Meskin, R. Tóth, and M. Mohammadpour, “Bayesian approach for estimation of LPV linear-regression models,” in Proc. of the 50th IEEE Conference on Decision and Control, (Los Angeles, CA, USA), pp. 2555–2560, Dec. 2014.

[40] L. Ljung, System Identification: Theory for the User. Springer, second ed., 1999.

[41] V. Verdult and M. Verhaegen, “Subspace identification of multivariable linear parameter-varying systems,” Automatica, vol. 38, no. 5, pp. 805–814, 2002.

[42] L. Ljung, System Identification: Theory for the User. Springer, second ed., 1999.

[43] A. Golabi, N. Meskin, R. Tóth, and M. Mohammadpour, “Bayesian approach for estimation of LPV linear-regression models,” in Proc. of the 53rd IEEE Conference on Decision and Control, (Los Angeles, CA, USA), pp. 2555–2560, Dec. 2014.

[44] M. A. H. Darwish, P. B. Cox, G. Pillonetto, and R. Tóth, “Bayesian identification of LPV Box-Jenkins models,” in Proc. of the 54th IEEE Conf. on Decision and Control, (Osaka, Japan), pp. 66–71, Dec. 2015.

[45] L. Ljung and T. Chen, “What can regularization offer for estimation of dynamical systems?” in Proc. of 11th IFAC International Workshop on Adaptation and Learning in Control and Signal Processing, (Caen, France), pp. 1–8, Jul. 2013.

[46] B. P. Carlin and T. A. Louis, Bayes and Empirical Bayes Methods for Data Analysis. London: Chapman & Hall, second ed., 1996.

[47] G. Pillonetto, F. Dinuzzo, T. Chen, G. De Nicolao, and L. Ljung, “Kernel methods in system identification, machine learning and function estimation: A survey,” Automatica, vol. 50, no. 3, pp. 657–682, 2014.

[48] Y. C. Eldar, “Uniformly improving the Cramér-Rao bound and maximum-likelihood estimation,” IEEE Trans. on Signal Processing, vol. 54, no. 8, pp. 2943–2956, 2006.

[49] T. Chen, H. Ohlsson, and L. Ljung, “On the estimation of transfer functions, regularizations and gaussian processes,” Automatica, vol. 48, no. 8, pp. 1525–1535, 2012.

[50] B. L. Ho and R. E. Kalman, “Effective construction of linear state-variable models from input/output functions,” Automatisierungstechnik, vol. 14, no. 1-12, pp. 545–548, 1966.

[51] V. Verdult, Nonlinear System Identification: a State-Space Approach. Phd thesis, University of Twente, 2002.

[52] S. Y. Kung, “A new identification and model reduction algorithm via singular value decompositions,” in Proc. of 12th Asilomar Conf. on Circuits, Systems and Computers, (Pacific Grove, CA, USA), pp. 705–714, Nov. 1978.

[53] R. H. Shumway and D. S. Stoffer, “An approach to time series smoothing and forecasting using the EM algorithm,” J. Time Series Analysis, vol. 3, pp. 253–264, Jul. 1982.

[54] S. Gibson and B. Ninness, “Robust maximum-likelihood estimation of multivariable dynamic systems,” Automatica, vol. 41, no. 10, pp. 1667–1682, 2005.

[55] T. B. Schön, A. Wills, and B. Ninness, “System identification of nonlinear state-space models,” Automatica, vol. 47, no. 1, pp. 39–49, 2011.

[56] S. Gibson, A. Wills, and B. Ninness, “Maximum-likelihood parameter estimation of bilinear systems,” IEEE Trans. on Automatic Control, vol. 50, no. 10, pp. 1581–1596, 2005.

[57] R. H. Shumway and D. S. Stoffer, Time Series Analysis and its Applications: with R Examples. Springer Texts in Statistics, third ed., 2010.

[58] C. W. Scherer, “Mixed H2/H∞ control for time-varying and linear parametrically-varying systems,” Int. J. of Robust and Nonlinear Control, vol. 6, pp. 929–952, 1996.

[59] P. Cox, R. Tóth, and M. Petreczky, “Data set of the paper ‘towards efficient maximum likelihood estimation of LPV-SS models’.” https://gitlab.com/pepinjcox/publication-appendices, 2018.

A Proof of Theorem 7

The proof is based on computing the expected value of the cross-correlation between the stationary signals y, ψ, u under the assumption that the signals are ergodic. First, the relation for the direct feed-through matrices Dτ is obtained. Let us substitute the IIR (11) for y in Rψ[y]→u(0, 0), which gives

\[ R_{\psi[y]}(0, 0) = -E\left\{ \left( D(p_t)u_t + C(p_t)B(p_{t-1})u_{t-1} + \cdots + y^*_t\psi_{[s]}^{[1]}u_t^\top \right) \right\} \]

\[ = E\left\{ \left( D_0 + \sum_{i=1}^{n_0} D_i\psi_i^{[1]} \right) u_t\psi_{[s]}^{[1]}u_t^\top \right\} + \]

\[ E\left\{ C(p_t)B(p_{t-1}) (u_{t-1})\psi_{[s]}^{[1]}u_t^\top \right\} + \cdots + E\{y^*_t\psi_{[s]}^{[1]}u_t^\top \} \]

\[ = D_{\psi_1}\sigma_{\psi_1}^{2}\Sigma_u^{2} \]  

(A.1)

Eq. (A.1) holds due to the whiteness property of the processes (u, ψ) and their independence. Also see that \( E\{y^*_t\psi_{[s]}^{[1]}u_t^\top \} = 0 \), as w, v, and ψ are assumed to be independent of u and y^* satisfies the relation given in (12), therefore, y^* is independent of u. Hence, \( E\{y^*_t\psi_{[s]}^{[1]}u_t^\top \} = E\{y^*_t\psi_{[s]}^{[1]}u_t^\top \} \). For all other sub-Markov parameters, let us consider the following formulation

\[ R_{\psi[y]}(0, 0)|_{n-1} = \]

\[ = E\left\{ \left( D(p_t)u_t + C(p_t)B(p_{t-1})u_{t-1} + \cdots + y^*_t\right) \psi_{[s]}^{[1]}u_t^\top \right\} \]

\[ = E\left\{ \left( C_1A_{s_2}\cdots A_{s_{n-1}}B_{s_n}(\psi_{[s]}^{[1]}u_{\tau-1})^2 \cdots \right) \psi_{[s]}^{[1]}u_{\tau-1}^\top \right\} + \]

\[ E\left\{ (D(p_t)u_t + \cdots + y^*_t)\psi_{[s]}^{[1]}u_{\tau-1}^\top \right\} \]

\[ = C_1A_{s_2}\cdots A_{s_{n-1}}B_{s_n}\psi_{[s]}^{[1]}u_{\tau-1}^\top \]  

(A.2)

Reordering (A.1) and (A.2) concludes the proof.
Remark 9  It is possible to get the same sub-Markov parameters with different multiplications of $\psi_{s_i}$ and corresponding shifts, e.g., $R_{\psi_{[s_1]_{0}^{s_2}u}(0,1,1)}$ gives the same sub-Markov parameter $C_{s_1}B_{s_2}$ as $R_{\psi_{[s_1]_{0}^{s_2}u}(0,1,4,4,1)}$. In scope of the estimation of these sub-Markov parameters, we impose the above given ordering to keep the multiplications with $\psi^{[n_1]}$ minimal.

□