OPTIMAL INVESTMENT AND DERIVATIVE DEMAND UNDER PRICE IMPACT

MICHAEL ANTHROPELOS, SCOTT ROBERTSON, AND KONSTANTINOS SPILOPOULOS

ABSTRACT. This paper studies the effects of price impact upon optimal investment, as well as the pricing of, and demand for, derivative contracts. Assuming market makers have exponential preferences, we show for general utility functions that a large investor’s optimal investment problem with price impact can be re-expressed as a constrained optimization problem in fictitious market without price impact. While typically the (random) constraint set is neither closed nor convex, in several important cases of interest, the constraint is non-binding. In these instances, we explicitly identify optimal demands for derivative contracts, and state three notions of an arbitrage free price. Due to price impact, even if a price is not arbitrage free, the resulting arbitrage opportunity only exists for limited position sizes, and might be ignored because of hedging considerations. Lastly, in a segmented market where large investors interact with local market makers, we show equilibrium positions in derivative contracts are inversely proportional to the market makers’ representative risk aversion. Thus, large positions endogenously arise either as market makers approach risk neutrality, or as the number of market makers becomes large.

1. INTRODUCTION

In this paper, we study the effects of price impact upon optimal investment, as well as the pricing of, and optimal demand for, derivative contracts. Our goal is to understand how price impact in the underlying hedging strategies affects prices and demand for contingent claims, and at what extent the arbitrage opportunities, if they exist, can be exploited. In particular, we wish to see if taking into account the impact, it can ever be optimal to own large positions in derivatives. In the spirit of [4], where for models without price impact, large positions endogenously arise in conjunction with either vanishing hedging errors, risk aversion or transactions costs, we want to examine whether optimal demands become large as the price impact vanishes.

Rather than specifying the price impact of trades exogenously, as in [6, 13, 18, 28, 49, 39, 47, 43] amongst others, we follow the financial economics paradigm, where the impact is determined endogenously. We use the impact models first developed in [25, 50], and subsequently extended by [8, 9]. Here, market makers compete when quoting prices for a given demand process $Q$. The competition causes market makers to form Pareto optimal allocations, and cash balances which preserve their expected utilities. The cash balance process then determines a wealth process for the

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large investor. The problem of hedging claims in this setting was considered in the recent articles [20, 45]. In the current paper, as a natural counterpart, we consider the continuous time optimal investment problem with random endowment (the corresponding static problem is studied in [3]).

The contribution of this paper is fourfold. Firstly, assuming exponential preferences for the market makers, we show the large investor’s optimal investment problem in the price impact model can be re-expressed as a constrained optimization problem in fictitious market without price impact. This result is valid with general utility functions on the real line for the large investor (see Theorem 3.1). An indicative example is when the large investor’s utility is exponential; here, the corresponding constrained optimal investment problem in the fictitious market is under power utility. However, the constraint set varies with both time and scenario, and typically is neither closed nor convex, as the examples in Section 3.1 show. As such, much of the standard theory for solving the optimal investment problem does not apply (c.f. [13]). Adding to the difficulty is that the (if finite) boundary of the constraint set may correspond to an infinite investor demand in the price impact model (see Section 3.2 for a more detailed discussion). In order to rule out this situation, care must be taken when selecting the terminal payoff of the traded asset, and its relationship to the endowments of both the large investor and market makers.

Secondly, we show that in many cases of interest, such as when endowments are portfolios of the traded assets, the constraint is non-binding, and in fact, optimal strategies are static. In other words, whenever endowments are securitized and there is no exogenous order flow to the market makers by other large investors or “noise traders”, the initial optimal order puts the large investor and the market makers to a Pareto-optimal situation.

Thirdly, we study the problem of optimal demand on derivative contracts under the presence of price impact on the underlying market. For this, using the results of [4], we show that even taking price impact into account, optimal demand for derivative securities may be large, as it is inversely proportional to the (representative) market maker risk aversion. Thus, positions become large as the risk aversion vanishes. The latter situation occurs naturally as the number of market makers increases since the representative risk tolerance is the sum of the individual risk tolerances. Thus, our results support the observed position sizes of over-the-counter derivatives (even when price impact is present), provided that either there are a large number of market makers, or that market makers are sufficiently close to risk-neutrality.

In relation to the later, we use the notion of Partial Equilibrium Price Quantities (PEPQ) as introduced in [5], where two large investors engage in an OTC transaction amongst themselves, at a price and quantity which is utility-optimal for each. However, as we deal with price impact, care must be taken when describing exactly whom the large investors are trading with in order to hedge their risks. We take the perspective of a segmented economy (c.f. [10, 41]) where each large investor trades in his “local” market with local market maker and local traded assets. Then, the investors additionally trade with each other on a separate claim, according to the PEPQ formulation. Segmentation avoids the large investors from hedging demands with the same market maker (and hence having to declare a cash-balance sharing rule), and enables private trading of
the derivative rather than embedding the derivative into the previously existing traded assets. It is in this segmented setting that optimal demands become large inversely proportional to market makers’ risk aversion (in exactly the same manner as [4]).

Lastly, we clarify what it means for a derivative price to be “arbitrage free”, when there is price impact on the underlying market. We provide three such notions, which all coincide in the frictionless case, but which in this price impact model, yield strikingly different arbitrage free price ranges. Indeed, even when the random constraint of the fictitious market is non-binding, the three definitions yield arbitrage free price ranges from a singleton, to the maximal interval of the essential infimum to essential supremum. Furthermore, we show that even when the price of a derivative gives rise to an arbitrage opportunity for the large investor, the gain from the arbitrage is “limited”, in the sense that it can be exploited only for small enough positions in the derivative. This is because of the price impact: when the large investor wants to cover the open position that gives the arbitrage by trading with the market makers, he changes their inventory and hence their pricing rules. Therefore, when the open position on the derivative is rather large, after a certain point the hedging becomes very expensive and the arbitrage vanishes. This means, in particular, that if the large investor wants to buy/sell the derivative for hedging reasons, he may ignore the arbitrage opportunity when the benefits from hedging outweigh the limited gain from arbitrage. This comes in sharp contrast with the notion of arbitrage in the market without price impact.

The rest of the paper is organized as follows. In Section 2, we describe the price impact model and define the large investor’s optimal investment problem. In Section 3, we identify the large investor’s optimal investment problem with that in a fictitious model with no price impact, but with constrained trading and utility fields. In Section 4, we specify to when the large investor has exponential preferences, presenting an equivalent representation of the value function as that of the constrained optimization problem for an exponential investor in a basis risk model. Imposing an assumption that guarantees that the constraint is non-binding, we solve the optimal investment problem and provide a number of concrete examples (where the assumption holds). Section 5 is dedicated to derivative demand and pricing. After defining three notions of an arbitrage free price, we show the price ranges for each notion and establish the corresponding optimal demand problem. In Section 5.2 we show optimal large positions arise endogenously as the price impact parameter vanishes, and in Section 5.3 we endogenize the traded derivative price using the PEPQ framework.

2. The model

We begin with a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) which supports a standard \(d\)-dimensional Brownian motion \(B\). We denote by \(\mathbb{F}\) the \(\mathbb{P}\)-augmentation of the natural filtration for \(B\), so \(\mathbb{F}\) satisfies the usual conditions of right continuity and saturation at 0 of all \(\mathbb{P}\)-null sets.

We consider a finite time horizon \(T > 0\), and a finite number of securities with terminal payoff denoted by the \(k\)-dimensional, \(\mathcal{F}_T\) measurable random vector \(\Psi\) (throughout, we shall also restrict \(\mathbb{F}\) to \([0, T]\)). Transactions on the securities are made through a collection of market makers, who quote prices based upon the demand. For the market makers’ pricing rule, we follow the models
introduced in \[25, 50\] and further developed in \([8, 9]\). We suppose the market makers are expected utility maximizers, have random endowments, and for a given order \(q \in \mathbb{R}^k\), (aggregately) ask a price \(X(q) \in \mathbb{R}\) according to two conditions: (i) the total order, price and random endowment are distributed between market makers in a Pareto-optimal way; and (ii) each market maker remains at indifference (each one’s expected utility remains unchanged after the transaction). Every time an order is satisfied by the market makers, their inventory and hence their pricing rule changes. In continuous time (see \([9]\)), both the order flow \(Q = \{Q_t\}_{t \leq T}\) and the cash balance \(X(Q) = \{X_t(Q)\}_{t \leq T}\) are adapted stochastic processes, and \(X_t(Q)\) is determined at each \(t \leq T\) by the aforementioned two conditions.

In this setting, we further assume market makers have exponential preferences. This allows us to consider hereafter a single (representative) market maker. More precisely, it is well-known (see \([10, 11, 52]\)) the Pareto-optimal sharing problem of exponential utility maximizers can be written as an optimization problem of a representative agent, whose utility is again exponential, and whose risk tolerance and endowment are, respectively, the sum of individual market makers’ risk tolerances and endowments. Furthermore, condition (ii) implies the cash balance asked by the market makers is the indifference price of the representative agent (see Theorem 3.1 of \([8]\) and Theorem 4.9 of \([9]\)).

**Remark 2.1.** The assumption of market makers’ indifference pricing is economically reasonable (c.f. \([50]\)). It is justified not only by the intense competition among market makers (see the related discussion in \([3]\) and \([8]\)), but also by the well-documented fact that market makers are willing to offer better prices to large investors (see \([2, 22]\) and the references therein). In our setting, “better price” means each market maker asks for the minimum price which compensates him for the position he opens (i.e. his indifference value).

The (representative) market maker’s endowment is (an \(\mathcal{F}_T\) measurable random variable) \(\Sigma_0\), while \(\gamma > 0\) is his absolute risk aversion (that is, \(1/\gamma\) is his risk tolerance). We make the standing assumption on \(\Sigma_0\), \(\gamma\) and \(\Psi\) (c.f. \([9\text{, Assumption 2.4}], [7\text{, equation (15)}]\)).

**Assumption 2.2.** For all \(q > 0\), \(E\left[e^{-\gamma \Sigma_0 + q|\Psi|}\right] < \infty\).

Besides the security vector \(\Psi\), we suppose a money market account is available for trading at an exogenously determined price that, without loss of generality, we normalize to 1. In this market, there is a large investor who wishes to trade \(\Psi\) with the market maker in order to reduce the risk induced by his (\(\mathcal{F}_T\) measurable) endowment \(\Sigma_1\). When the market maker satisfies an order submitted by the large investor, his inventory and pricing rule change, resulting in a price impact.

The large investor has preferences described by a general utility function \(U\) defined on the real line, which is \(C^2\), strictly concave, and satisfies the conditions of reasonable asymptotic elasticity (c.f. \([46]\)).

To gain intuition, first consider the single period case. As in \([8]\), we take the market maker’s perspective regarding the demand’s sign, and assume the large investor submits an order of \(-q\)
units for Ψ to the market maker. As discussed above, the market maker quotes a cash balance \( X(q) \) to remain utility indifferent (in expectation) to the order. As the market maker’s endowment is now \( \Sigma_0 + X(q) + q'\Psi \), \( X(q) \) must satisfy

\[
E \left[ -e^{-\gamma(\Sigma_0 + X(q) + q'\Psi)} \right] = E \left[ -e^{-\gamma\Sigma_0} \right]; \quad \implies X(q) = \frac{1}{\gamma} \log \left( \frac{E \left[ e^{-\gamma\Sigma_0 - q'\Psi} \right]}{E \left[ e^{-\gamma\Sigma_0} \right]} \right).
\]

In other words, \( X(q) \) is the seller’s indifference price (or certainty equivalent) of \(-q\) units of Ψ under risk aversion \( \gamma \) and endowment \( \Sigma_0 \). At the terminal time \( T \), the large investor’s wealth is

\[
V(q) = - (X(q) + q'\Psi).
\]

In continuous time (see [9]), an order flow is submitted by the large investor, in the form of an adapted stochastic process \( Q \), and the cash balance \( X(Q) \) is an induced stochastic process. In particular, for each \( t \in [0, T] \), \( Q_t \) is the cumulative number of securities sold by the large investor to the market maker, while \( X_t(Q) \) denotes the cumulative cash amount paid by the investor to market maker. As in the static case, \( X(Q) \) is determined through the principle of market maker indifference.

For a given \( Q \), we denote by \( V(Q) = \{V_t(Q)\}_{t \leq T} \) the large investor’s gains process. In particular, at time \( t \leq T \), \( V_t(Q) \) represents the cash amount the large investor will receive if he sells his cumulative order (see [9, Equation (4.19)]). For example, in analogy to (2.2), at the terminal time \( V_T(Q) = - (X_T(Q) + Q_T'\Psi) \).

For the analysis that follows, we need to introduce some further notation and definitions. In view of Assumption 2.2 for all \( q \in \mathbb{R}^k \) the process

\[
N_t(q) := E \left[ e^{-\gamma\Sigma_0 - q'\Psi} | \mathcal{F}_t \right]; \quad t \leq T,
\]

is a strictly positive martingale, and hence by predictable representation, we may write

\[
\frac{N_t(q)}{N_0(q)} = \mathcal{E} \left( \int_0^t H_s(q)'dB_s \right); \quad t \leq T,
\]

for some predictable process \( H(q) \) such that \( \int_0^T |H_t(q)|^2 dt < \infty \) a.s.. The map \((t, \omega, q) \rightarrow H_t(q)(\omega)\) is \( \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^k) \) measurable, where \( \mathcal{P}(\mathbb{F}) \) is the \( \mathbb{F} \)-predictable sigma-algebra, and in fact, \( H(q) \) is regular in \( q \) as described in [7, Lemma 5.5], as well as in [30]. Define the class of processes

\[
A_{PI} := \left\{ Q \in \mathcal{P}(\mathbb{F}) \mid \int_0^T |H_t(Q_t)|^2 dt < \infty \text{ a.s.} \right\}.
\]

Using [7, Theorem 3.2, Section 5.2], [9, Theorem 4.9], we obtain the following representation for \( V(Q) \) in terms of \( H(Q) \). The proof of Lemma 2.3 is in Appendix A below.

**Lemma 2.3.** Let Assumption 2.2 hold. Then, for \( Q \in A_{PI} \), \( V(Q) \) is well defined with

\[
V_t(Q) = \frac{1}{\gamma} \int_0^t (H_s(Q_s) - H_s(0))'(dB_s - H_s(0)ds) - \frac{1}{2\gamma} \int_0^t |H_s(Q_s) - H_s(0)|^2 ds; \quad t \leq T.
\]
In view of Lemma 2.3, we may define the large investor’s value function, for a given initial capital \( x \) and endowment \( \Sigma_1 \). To ensure the value function is well defined, we assume:

**Assumption 2.4.** \( \mathbb{E} \left[ (U(x + \Sigma_1))^+ \right] < \infty \) for all \( x \in \mathbb{R} \).

With this assumption, the large investor’s value function is defined as:

\[
(2.7) \quad u(x; \Sigma_1) := \sup_{Q \in \mathcal{A}_T} \mathbb{E} \left[ U \left( x + V_T(Q) + \Sigma_1 \right) \right].
\]

### 3. A Constrained Optimal Investment Problem with No Price Impact

Presently, we identify \( u(x; \Sigma_1) \) with the value function for an optimal investment problem in a fictitious model with no price impact, but (i) with trading restricted to a random constraint set, and (ii) where the investor’s preferences are described by a utility field.

As above, the money market account is set to 1. There are \( d \) tradeable assets with an exogenously given price process \( S \) evolving according to

\[
(3.1) \quad \frac{dS_t}{S_t} = \lambda_t dt + dB_t; \quad t \leq T,
\]

for a (to-be-determined) predictable \( d \)-dimensional process \( \lambda \), such that \( \int_0^T |\lambda_t|^2 dt < \infty \), a.s.. Provided the requisite integrability, \( \lambda \) is the unique market price of risk and, by construction, there is a unique measure \( Q_0 \) on \( \mathcal{F}_T \) under which \( S \) is a true martingale. \( Q_0 \) has density

\[
(3.2) \quad \frac{dQ_0}{dP} \bigg|_{\mathcal{F}_T} = \mathcal{E} \left( -\int_0^T \lambda_t dB_t \right)_T.
\]

Throughout, we will write \( \mathbb{E}^0 \) when taking expectations with respect to \( Q_0 \). Self-financing trading strategies are denoted by \( \pi \), where \( \pi^i_t \) is the proportion of wealth invested in \( S^i \) at \( t \), \( i = 1, ..., d \). We assume \( \pi \in \mathcal{P}(\mathbb{F}) \) is such that \( \int_0^T |\pi^2_t| dt < \infty \) a.s. (with further assumptions below). The wealth process induced by \( \pi \) has dynamics

\[
\frac{dX_t(\pi)}{X_t(\pi)} = \pi^i_t (\lambda_t dt + dB_t); \quad t \leq T,
\]

so that, with initial wealth \( X_0 = e^{\gamma x} \), the terminal wealth is

\[
(3.3) \quad X_T(\pi) = \exp \left( \gamma x + \int_0^T \pi^i_t (dB_t + \lambda_t dt) - \frac{1}{2} \int_0^T |\pi^2_t|^2 dt \right).
\]

Here is the key observation: define \( \lambda := -H(0) \), assume \( \pi = H(Q) - H(0) \), and compare (3.3) with (2.6). We see

\[
(3.4) \quad X_T(\pi) = e^{x + V_T(Q)}; \quad \Rightarrow \quad x + V_T(Q) = \frac{1}{\gamma} \log(X_T(\pi)).
\]

In other words, there is a direct connection between the wealth process (induced by \( \pi \)) in the fictitious market (3.1) and the gains process (induced by \( Q \)) in the market with price impact. The requirement \( \pi^i_t = H_t(Q^i_t) - H_t(0) \) is similar to the invertibility condition in [20, Condition 3.7], and
the bounded-ness condition in [45, Proposition 3.5]. Note from (2.3) and (2.4) and \( \lambda = -H(0) \), \( Q_0 \) has terminal density

\[
\frac{dQ_0}{dP} \bigg|_{F_T} = e^{-\gamma \Sigma_0} \mathbb{E} \left[ e^{-\gamma \Sigma_0} \right] = \mathbb{E} \left( \int_0^T H_t(0)' dB_t \right)_T.
\]

Clearly, for \( Q \in A_{PI} \) we can construct \( \pi \). To go in the reverse direction, we must have for \( \text{Leb}_{[0,T]} \) almost every \( t \leq T \) that \( \mathbb{P} \) almost surely \( \pi_t \) lies in the random constraint set \( K_t^0 \), where

\[
K_t := \left\{ H_t(q) \mid q \in \mathbb{R}^k \right\}, \quad K_t^0 := \left\{ H_t(q) - H_t(0) \mid q \in \mathbb{R}^k \right\}.
\]

Therefore, we define the acceptable strategies

\[
A := \left\{ \pi \in \mathcal{P}(\mathcal{F}) \mid \int_0^T |\pi_t|^2 dt < \infty \text{ a.s.} \right\}, \quad A_C := \left\{ \pi \in A \mid \pi \in K^0_0, \text{Leb}_{[0,T]} \times \mathbb{P} \text{ a.s.} \right\}.
\]

In [7, Lemma 5.5] the map \( q \rightarrow H(q) \) was shown to be continuous (in fact, the map is much more regular: see [30]). Thus, by the Kuratowski-Ryll-Nardzewski measurable selection theorem for any \( \theta \in \mathcal{P}(\mathcal{F}) \) such that \( \theta_t \in K_t \) a.s. for \( t \leq T \), we can select \( Q \in \mathcal{P}(\mathcal{F}) \) so that \( H(Q) = \theta \), \( \text{Leb}_{[0,T]} \times \mathbb{P} \) a.s.

Having establishing the above connection, the next step is to find the appropriate optimization problem in the fictitious market corresponding to (2.7). To this end, recall the utility function \( U \in C^2(\mathbb{R}) \) which is strictly increasing, concave and satisfies the reasonable asymptotic elasticity conditions. Define the utility field \( \tilde{U}(w,\omega) : (0, \infty) \times \Omega \) by

\[
\tilde{U}(w,\omega) := U \left( \frac{1}{\gamma} \log(w) + \Sigma_1(\omega) \right).
\]

It is easy to verify that for \( \omega \) fixed, \( \tilde{U}(w) \) is strictly increasing, concave, and satisfies the asymptotic elasticity conditions in [31]. Using \( \tilde{U} \) in the fictitious market, we define (with a slight abuse of notation) the value functions

\[
\tilde{u}(x; \Sigma_1) := \sup_{\pi \in A} \mathbb{E} \left[ \tilde{U} \left( X_T(\pi), \Sigma_1 \right) \mid X_0 = e^{\gamma x} \right],
\]

as well as

\[
\tilde{u}_C(x; \Sigma_1) := \sup_{\pi \in A_C} \mathbb{E} \left[ \tilde{U} \left( X_T(\pi), \Sigma_1 \right) \mid X_0 = e^{\gamma x} \right].
\]

Based on the above discussion, we have the following result:

**Theorem 3.1.** Let Assumptions 2.2 and 2.4 hold, and for \( x \in \mathbb{R} \) let \( u(x; \Sigma_1) \) be from (2.7). Then

\[
u(x; \Sigma_1) = \tilde{u}_C(x; \Sigma_1) \leq \tilde{u}(x; \Sigma_1).
\]
3.1. Discussion on the constrained problem. The unconstrained utility maximization problem for random endowments and utility fields has been well studied. For endowments with utility functions on \((0, \infty)\) we highlight \([26]\), while for utility fields, we pay particular attention to \([37]\), which contains necessary and sufficient conditions for existence of optimal policies, in the general incomplete setting. In the present setting, the analysis greatly simplifies as the unconstrained market is complete. Indeed, we have the immediate identification

\[
\tilde{u}(x; \Sigma_1) = \sup_{\xi \in L^0(\mathcal{F}_T)} \left\{ \mathbb{E}[U(x + \xi + \Sigma_1)] \mid \mathbb{E}^0 \left[ e^{\gamma \xi} \right] \leq 1 \right\},
\]

since \(X_T(\pi) = e^{\gamma(\xi + x)}\), for \(\pi \in \mathcal{P}(\mathbb{F})\) satisfying \(\mathcal{E} \left( \int_0^T \pi_t'(dB_t - H_t(0)dt) \right)_T = e^{\gamma \xi} / \mathbb{E}^0 \left[ e^{\gamma \xi} \right].\)

The utility maximization problem with random constraints is also by now standard. However, most of the literature (c.f. \([13], [29]\)), assumes the constraint set is both closed and convex a.s. for \(t \leq T\). Closedness is used to obtain an optimal policy when the constraint binds, and convexity is used when constructing the auxiliary markets with no constraints. Unfortunately, in the present situation, the random set \(K_t^o\) of \([3, 6]\) may be neither closed nor convex. Indeed, we now provide three related examples: first, when there is no constraint as \(K_t = \mathbb{R}^d\); second, when \(K_t\) is not closed; and third, when \(K_t\) is not convex.

Example 3.2. Bachelier model: no constraint. Similarly to \([9, \text{Example 4.11}]\), let \(k = d\), \(\Sigma_0 = \int_0^T f_t' dB_t\) and \(\Psi = \int_0^T \psi_t dB_t\), where \(f \in L^2([0, T]; \mathbb{R}^d)\) and \(\psi \in L^2([0, T]; \mathbb{R}^{d \times d})\). Assume \(\psi_t\) is invertible, and the map \(t \to \psi_t^{-1}\) is continuous. Clearly, Assumption 2.2 holds, and \(N(q)\) from \([2, 3]\) evaluates to

\[
N_t(q) = e^{\frac{1}{2} \gamma^2 \int_0^T |f_s + \psi_s q|^2 ds} \mathcal{E} \left( -\gamma \int_0^T (f_s + \psi_s q)' dB_s \right); \quad t \leq T.
\]

Thus,

\[
H_t(q) = -\gamma (f_t + \psi_t q) \quad \text{and} \quad H_t(q) - H_t(0) = -\gamma \psi_t q.
\]

Hence, \(K_t = K_t^o = \mathbb{R}^d\) with \(\pi_t = H_t(Q_t) - H_t(0) \iff Q_t = -\gamma \psi_t^{-1} \pi_t.\)

Example 3.3. Digital claim: a non-closed set. Consider when \(k = d = 1\), \(\Sigma_0 = 0\), and \(\Psi = 1_{B_T \geq 0}\). Here we have on \(\{B_t = b\}\) and for \(\tau = T - t:\)

\[
K_t = K_t^o = \frac{1}{\sqrt{\tau}} \varphi \left( \frac{b}{\sqrt{\tau}} \right) \times \left( \frac{1}{\Phi \left( \frac{b}{\sqrt{\tau}} \right)}, \frac{1}{1 - \Phi \left( \frac{b}{\sqrt{\tau}} \right)} \right),
\]

where \(\Phi\) is the cumulative function of the standard normal distribution, and \(\varphi\) its probability density function. The left endpoint above occurs as \(q \downarrow -\infty\), while the right endpoint occurs as \(q \uparrow \infty\). To verify \([3, 13]\), we note for \(\Sigma_0 = 0\), \(\Psi = f(B_T)\) with \(f\) bounded and measurable, it follows that \(H_t(q) = -\partial_q v(t, B_t; q)\), where \(v(t, b; q) := -\log \left( \mathbb{E} \left[ e^{-\gamma q f(B_T)} \mid B_t = b \right] \right)\). Indeed, this can be shown using Itô’s formula and the transition density for \(B\). In this instance

\[
v(t, b; q) = -\log \left( e^{-\gamma q} + \Phi \left( -\frac{b}{\sqrt{T - t}} \right) (1 - e^{-\gamma q}) \right).
\]
hence the constraint is effectively absent (i.e. non-binding). In case $k = k_2$, it is easy to see the problem falls outside the scope of most of the existing literature. In addition, in the case of $A$, define $p = \Phi(b_2)$. Then, for $q = q_2$, make the right hand side of the second equality in (3.14) strictly positive. Then, for $p_1$ near $b_1/\tau$ we have that $(p_1, p_2) \notin K^0_\tau$ while for $|p_1|$ large enough, $(p_1, p_2) \in K^0_\tau$. This is shown in Figure 1. In the interior of the shaded region, there are two solutions $(q_1^+, q_2^+), (q_1^-, q_2^-)$ to $(H_1(q), H_2(q)) = (p_1, p_2)$. Along the boundary, but not where the lines cross, there is a unique solution to $(H_1(q), H_2(q)) = (p_1, p_2)$, and where the lines cross there is an uncountable family of solutions to $(H_1(q), H_2(q)) = (p_1, p_2)$. This shows the constraint set is closed, but also that typically (depending upon the endowment $\Sigma_1$) there will not be a unique optimal demand policy for the large investor’s optimal investment problem.

Example 3.4. Two dimensional digital and linear claim: a closed but non-convex set. Consider when $k = d = 2$, $\Sigma_0 = 0$ and $\Psi = \left(B_1^1, B_2^1 \mid B_2^2 \geq 0\right)$. Here, with $\tau = T - t$, and on $\{B_1^1 = b_1, B_2^2 = b_2\}$ define $A = \Phi(-b_2/\sqrt{\tau})$ and $C = \sqrt{\tau}/\phi(-b_2/\sqrt{\tau})$. Then, $K_\tau = K_\tau^0 = \{(p_1, p_2)\}$ where $(p_1, p_2) \in \mathbb{R}^2$ must satisfy

$$\frac{1}{(1 - A)C} < p_2 < \frac{1}{AC}; \quad (\tau p_1 - b_1)^2 \geq 2\tau (1 - 2(1 - ACp_2)(1 - A)) \log \left(\frac{1 + (1 - A)Cp_2}{1 - ACp_2}\right).$$

It is easy to see $K_\tau^0$ is not convex. Indeed, for any fixed $t, b_1, b_2$ choose $p_2$ close enough to $1/(AC)$ to make the right hand side of the second equality in (3.14) strictly positive. Then, for $p_1$ near $b_1/\tau$ we have that $(p_1, p_2) \notin K_\tau^0$ while for $|p_1|$ large enough, $(p_1, p_2) \in K_\tau^0$. This is shown in Figure 1. In the interior of the shaded region, there are two solutions $(q_1^+, q_2^+), (q_1^-, q_2^-)$ to $(H_1(q), H_2(q)) = (p_1, p_2)$. Along the boundary, but not where the lines cross, there is a unique solution to $(H_1(q), H_2(q)) = (p_1, p_2)$, and where the lines cross there is an uncountable family of solutions to $(H_1(q), H_2(q)) = (p_1, p_2)$. This shows the constraint set is closed, but also that typically (depending upon the endowment $\Sigma_1$) there will not be a unique optimal demand policy for the large investor’s optimal investment problem.

3.2. The dichotomy. In view of the above examples, there is a dichotomy when solving the optimal investment problem in this class of price impact models. Indeed, under very mild conditions upon $U$ and $\Sigma_1$ (see for instance [58] and the references therein), there will be an optimal policy $\hat{\pi}$ to the unconstrained problem in (3.9). Then, in the constrained case, we have two alternatives:

I. $\hat{\pi} \in K^0$, $\operatorname{Leb}_{[0,T]} \times \mathbb{P}$ almost surely.

II. There exists a set $(a, b) \times E \in [0, T] \times \mathcal{F}_T$ with $a < b$ and $\mathbb{P}[E] > 0$ s.t. $\hat{\pi}_t \notin K^0_{\tau} \text{ on } E \times (a, b)$.

In case I., the constrained problem admits the same answer as the unconstrained problem, and hence the constraint is effectively absent (i.e. non-binding). In case II., the optimal investment problem falls outside the scope of most of the existing literature. In addition, in the case of II., one may end up with a strange situation where the large investor is induced to demand an infinite
number of shares from the market maker, and the market maker, at least formally, is willing to quote a finite cash balance for this demand. For example, such a situation can occur in the one-dimensional setting of Example 3.3 by setting \( \hat{\pi} \) to be, e.g., the left endpoint in (3.13) for \( t \in (a, b) \).

To avoid such pathologies, we restrict attention hereafter to case \( I \).

### 4. Exponential Preferences

The usefulness of Theorem 3.1 is highlighted when we specify the large investor’s utility function to be exponential. Here, it turns out that exponential utility in the price impact model is connected to power utility in the fictitious constrained market. To proceed, we denote by \( \alpha > 0 \) the absolute risk aversion of the large investor and specify Assumption 2.4 to Assumption 4.1.

Assumption 4.1 allows us to employ the usual “change of measure” trick (c.f. [15]) on the endowment \( \Sigma_1 \). Define the measure \( \tilde{P} \) on \( \mathcal{F}_T \) by

\[
\frac{d\tilde{P}}{dP}\bigg|_{\mathcal{F}_T} = \frac{e^{-\alpha \Sigma_1}}{E[e^{-\alpha \Sigma_1}]} = \mathcal{E}\left(\int_0^T \phi_t dB_t\right)_T,
\]

where by predictable representation, \( \phi \in \mathcal{P}(\mathcal{F}) \) is such that \( \int_0^T |\phi_t|^2 dt < \infty \) a.s. From Girsanov’s theorem, \( \tilde{B} := B - \int_0^T \phi_t dt \) is a \( \tilde{P} \) Brownian motion on \([0, T]\). Lastly, write \( \tilde{E} \) when taking expectations with respect to \( \tilde{P} \). With this notation we present:

**Proposition 4.2.** Let Assumptions 2.2 and 4.1 hold and recall \( \tilde{u} \) and \( \tilde{u}_C \) from (3.9) and (3.10) respectively. Then, taking \( U(x) = -e^{-\alpha x} \), we have

\[
\tilde{u}(0; \Sigma_1) = \frac{\alpha}{\gamma} E\left[e^{-\alpha \Sigma_1}\right] \left( \sup_{\pi \in \mathcal{A}} \tilde{E}\left[\frac{1}{\tilde{p}} (X_T(\pi))^p \mid X_0 = 1 \right] \right),
\]

\[
\tilde{u}_C(0; \Sigma_1) = \frac{\alpha}{\gamma} E\left[e^{-\alpha \Sigma_1}\right] \left( \sup_{\pi \in \mathcal{A}_C} \tilde{E}\left[\frac{1}{\tilde{p}} (X_T(\pi))^p \mid X_0 = 1 \right] \right),
\]

where \( p := -\alpha/\gamma. \)

**Proof.** Assumption 4.1 clearly implies Assumption 2.4 when \( U(x) = -e^{-\alpha x} \). Next, the value function identifications readily follow from (3.4), since

\[
E\left[-e^{-\alpha \left(\frac{1}{\gamma} \log(X_T(\pi)) + \Sigma_1\right)} \mid X_0 = 1 \right] = E\left[-e^{-\alpha \Sigma_1} X_T(\hat{\pi})^{-\alpha/\gamma} \mid X_0 = 1 \right];
\]

\[
= \frac{\alpha}{\gamma} E\left[e^{-\alpha \Sigma_1}\right] \tilde{E}\left[\frac{1}{\tilde{p}} (X_T(\pi))^p \mid X_0 = 1 \right].
\]

\[\square\]
4.1. **A basis risk identification.** Before solving the optimal investment problem for exponential preferences, we present an alternate identification of the value functions in (3.9) and (3.10), in terms of the value functions from a fictitious *basis risk model* with random trading constraints. The purpose of this identification is to highlight how, even when there is essentially no constraints i.e. \( K_t = \mathbb{R}^d \), we may view the market maker risk aversion \( \gamma \) as a measure of market incompleteness.

Extend the probability space to support a \( d \)-dimensional Brownian motion \( W \) independent of \( B \), and denote by \( \mathcal{F}_{W,B} \) the \( \mathbb{P} \)-augmentation of \((B,W)\)'s natural filtration (recall \( \mathcal{F} \) is the \( \mathbb{P} \)-augmentation of \( B \)'s natural filtration). Next, define the following parameter which plays the role of “correlation” in the fictitious market:

\[
\rho := \sqrt{\frac{\alpha}{\alpha + \gamma}} \in (0, 1); \quad \bar{\rho} := \sqrt{1 - \rho^2} = \sqrt{\frac{\gamma}{\alpha + \gamma}}.
\]

As before, the money market is set to 1. Instead of (3.1), we consider a fictitious market with \( d \) tradeable assets with price process \( S \) evolving as

\[
\frac{dS_t}{S_t} = \rho (dB_t - H_t(0)dt) + \bar{\rho}dW_t; \quad t \leq T.
\]

These dynamics correspond to the basis risk model (see among others [36]), and \( \rho \) is the instantaneous correlation of the assets’ returns with the shocks coming from \( B \). As \( |\rho| < 1 \), even absent trading constraints, the presence of shocks coming from \( W \) makes the market incomplete. However, as \( \gamma \downarrow 0 \) the shocks from \( W \) become independent of the tradeable assets, meaning that \( \mathcal{F}_T \) measurable claims may be hedged. Therefore, \( \gamma \) is a measure of market incompleteness.

In this market, the trading strategies are denoted by \( \theta \), which represent the monetary positions in \( S \) (not the proportion of wealth). In the same spirit as Section 3 and in the view of the sets (3.6) and (3.7), for the constrained basis risk model, the class of acceptable strategies is

\[
A_{br,C} := \left\{ \theta \in \mathcal{P}(\mathbb{F}) \mid \int_0^T \theta_t^2 dt < \infty \text{ a.s.}, \; \gamma \rho \theta \in K^0, \; \text{Leb}_{[0,T]} \times \mathbb{P} \text{ a.s.} \right\}.
\]

Note that we restrict to \( \mathbb{F} \) predictable strategies since \( W \) is absent from the large investor’s optimal investment problem. Next, define the unconstrained class

\[
A_{br} := \left\{ \theta \in \mathcal{P}(\mathbb{F}^W) \mid \int_0^T \theta_t^2 dt < \infty \text{ a.s.} \right\}.
\]

For initial capital \( x \in \mathbb{R} \), and \( \theta \) in \( A_{br} \) or in \( A_{br,W} \) the wealth process \( X(\theta) \) has dynamics

\[
dX_t(\theta) = \theta'_t \rho (dB_t - H_t(0) dt) + \bar{\rho} \theta'_t dW_t; \quad t \leq T.
\]

We then define the value functions

\[
\tilde{u}_{br}(x; \Sigma_1) := \sup_{\theta \in A_{br}} \mathbb{E} \left[ -e^{-\alpha(X_T(\theta)+\Sigma_1)} \right];
\]

\[
\tilde{u}_{br,C}(x; \Sigma_1) := \sup_{\theta \in A_{br,C}} \mathbb{E} \left[ -e^{-\alpha(X_T(\theta)+\Sigma_1)} \right],
\]

and present the following identification:
Proposition 4.3. Let Assumptions 2.2 and 4.1 hold. Then

\[ \tilde{u}(0; \Sigma_1) = \tilde{u}_{br}(0; \Sigma_1); \quad \tilde{u}_C(0; \Sigma_1) = \tilde{u}_{br,C}(0; \Sigma_1). \]  

Furthermore, it suffices to optimize over \( \mathcal{F} \) predictable strategies to obtain \( \tilde{u}_{br} \).

Proof. Let \( \pi \) be in either \( A_C \) or \( A \). From (3.3) and (3.8)

\[ \log \left( -\tilde{U}(X_T(\pi), \Sigma_1) \right) = -\frac{\alpha}{\gamma} \int_0^T \pi_t'(dB_t - H_t(0)dt) + \frac{\alpha}{2\gamma} \int_0^T |\pi_t|^2 dt - \alpha \Sigma_1. \]

Next, let \( \theta \) be in either \( A_{br} \) or \( A_{br,C} \) but \( \mathcal{F} \) predictable. Since \( H(0) \) is also \( \mathcal{F} \) predictable, and \( \Sigma_1 \) is \( \mathcal{F}_T \) measurable, we have from (4.7) that

\[ \log \left( E \left[ e^{-\alpha X_T(\theta) - \alpha \Sigma_1} \mathcal{F}_T \right] \right) = -\alpha \int_0^T \theta_t'(dB_t - H_t(0)dt) + \frac{1}{2} \int_0^T \alpha^2 \rho^2 \theta_t' \theta_t dt - \alpha \Sigma_1. \]

To match the first integral we need \( \pi_t = \gamma \rho \theta_t \), but this is precisely possible when \( \pi \in A, \theta \in A_{br} \) or \( \pi \in A_C, \theta \in A_{br,C} \). Using this identification, the second integral gives

\[ \frac{\alpha^2 \rho^2 \theta_t'}{\alpha + \gamma} \theta_t = \frac{\alpha^2 \gamma}{\alpha} \theta_t' \theta_t = \alpha \gamma \rho^2 \theta_t' \theta_t = \frac{\alpha}{\gamma} \pi_t' \pi_t; \quad t \leq T, \]

and hence the terms are matched and the second equality in (4.9) is proved. As for the first equality, by the above, it will hold if the value function \( \tilde{u}_{br}(0; \Sigma_1) \) is attained using \( \mathcal{F} \) predictable strategies. But, this has been shown in a number of papers: see [16, 17, 24, 42, 51] amongst others. \( \square \)

4.2. Solving the optimal investment problem. We now return to the power identification (Proposition 4.2). We keep the Assumptions 2.2, 4.1 in force, and notice that Hölder’s inequality ensures

\[ E \left[ e^{-\beta (\Sigma_1 + \Sigma_0)} \right] < \infty; \quad \beta := \frac{\alpha \gamma}{\alpha + \gamma}, \]

so there exists \( M \in \mathcal{P}(\mathcal{F}) \) with \( \int_0^T |M_t|^2 dt < \infty \) a.s. which satisfies

\[ \frac{e^{-\beta (\Sigma_1 + \Sigma_0)}}{E \left[ e^{-\beta (\Sigma_1 + \Sigma_0)} \right]} = \mathcal{E} \left( \int_0^T M_t dB_t \right)_T. \]

In view of the dichotomy concerns discussed in Section 3.2, we make the following assumption on \( \Sigma_0, \Psi \) and \( \Sigma_1 \), which ensures the constraint \( \check{\pi} \in \mathcal{K}^0 \) is non-binding:

Assumption 4.4. \( M \in \mathcal{K}, \text{Leb}_{[0,T]} \times \mathbb{P} \) almost surely.

Assumption 4.4 states an implicit connection between endowments \( \Sigma_0 \) and \( \Sigma_1 \) and the tradeable assets \( \Psi \), and as will be shown in Section 4.3 holds in many cases of practical interest. Hence the solutions to the constrained and unconstrained optimization problems coincide. In particular, we have the following result:
Proposition 4.5. Let Assumptions 2.2, 4.1 and 4.4 hold. Then, for each of the optimal investment problems in (4.2), the optimal trading strategy $\hat{\pi} \in \mathcal{A}_C$ is

$$\hat{\pi} = M - H(0),$$

and the value functions (2.7), (4.2) and (4.8) all coincide, taking the explicit value

$$-E \left[ e^{-\gamma \Sigma_0} \right]^{-\frac{p}{\gamma}} \times E \left[ e^{-\beta (\Sigma_0 + \Sigma_1)} \right]^\frac{p}{\beta}. $$

Proof. In light of Propositions 4.2 and 4.3 it suffices to show both $\tilde{u}_C(0; \Sigma_1)$ and $\tilde{u}(0; \Sigma_1)$ take the value in (4.13). From (3.5), (4.1), the density of the (unique) martingale measure $Q_0$ with respect to $\tilde{P}$ on $\mathcal{F}_T$ is

$$\tilde{Z}_T = \frac{dQ_0}{dP} \bigg|_{\mathcal{F}_T} = \frac{dQ_0}{dP} \bigg|_{\mathcal{F}_T} \times \frac{dP}{d\tilde{P}} \bigg|_{\mathcal{F}_T} = \frac{E \left[ e^{-\alpha \Sigma_1} \right]}{E \left[ e^{-\gamma \Sigma_0} \right]} e^{\alpha \Sigma_1 - \gamma \Sigma_0}. $$

Recall from Proposition 4.2 that $p = -\alpha/\gamma$ and $q := p/(p - 1) = \alpha/ (\alpha + \gamma)$. A straight-forward calculation shows

$$\tilde{Z}_T^q = \left( \frac{E \left[ e^{-\alpha \Sigma_1} \right]}{E \left[ e^{-\gamma \Sigma_0} \right]} \right)^{\frac{\alpha}{\alpha + \gamma}} \frac{e^{-\beta (\Sigma_0 + \Sigma_1)}}{e^{-\alpha \Sigma_1}}. $$

Thus, $\tilde{E} \left[ Z_T^q \right] < \infty$ and by the standard complete market duality theory for power utility with initial wealth $e^{\gamma \times 0} = 1$ (c.f. [29, 23, Lemma 5]) we obtain by explicit computation

$$\tilde{u}(0, \Sigma_1) = \frac{\alpha}{\gamma} E \left[ e^{-\alpha \Sigma_1} \right] \left( \frac{1}{p} \tilde{E} \left[ \tilde{Z}_T^q \right]^{1-p} \right) = -E \left[ e^{-\gamma \Sigma_0} \right]^{-\frac{p}{\gamma}} \times E \left[ e^{-\beta (\Sigma_0 + \Sigma_1)} \right]^\frac{p}{\beta}. $$

As for $\hat{\pi}$, the (sufficient, c.f. [23, Lemma 5]) first order conditions for optimality are

$$X_T(\hat{\pi}) = \frac{\tilde{Z}_T^q}{\tilde{E} \left[ \tilde{Z}_T^q \right]} \frac{1}{\tilde{Z}_T}. $$

For a given strategy $\pi$ we have

$$\log \left( X_T(\pi) \right) = \int_0^T \pi_t'(dB_t - H_t(0))dt - \frac{1}{2} \int_0^T |\pi_t|^2 dt. $$

Using (3.5), (4.11), (4.14) and (4.15), the log of the right hand side in (4.16) reduces to

$$\int_0^T (M_t - H_t(0))^t (dB_t - H_t(0))dt - \frac{1}{2} \int_0^T |M_t - H_t(0)|^2 dt. $$

From here, it is clear $\hat{\pi} = M - H(0)$. Assumption 4.4 implies $\hat{\pi} \in \mathcal{A}_C$, completing the proof.

Corollary 4.6. Under the assumptions of Proposition 4.5, for each of the optimal investment problems in (4.8), the optimal trading strategy is $\hat{\theta} = \hat{\pi}/(\gamma \rho) = (M - H(0)) / (\gamma \rho)$, so that $\hat{\theta} \in \mathcal{A}_{br}$. 

□
4.3. Examples. Assumption 4.4 may at first glance seem artificial and overly restrictive. However, there are concrete examples of interest where Assumption 4.4 holds, and as a consequence Proposition 4.5 applies. One category of examples correspond to when $K^o = \mathbb{R}^d$, $\text{Leb}_{[0,T]} \times \mathbb{P}$ almost surely: such is the Example 3.2. We now provide additional examples.

Example 4.7. Markov models. Assume for sufficiently smooth functions, $\Sigma_0 = \Sigma_0(Y_T)$ and $\Psi = \Psi(Y_T)$ where $Y$ is a diffusion with dynamics

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dB_t; \quad Y_0 = y_0.$$  

Here $Y$ is understood as an economic factor. Letting $F(y; q) := -\gamma \Sigma_0(y) - \gamma q^\prime \Psi(y)$, we notice, using Markov property, that

$$N_t(q) = v(t, Y_t; q); \quad v(t, y; q) := \mathbb{E} \left[ e^{F(Y_T; q)} | Y_t = y \right],$$

and for $q$ fixed, $v(t, y; q)$ solves the Cauchy problem

$$v_t + \mathcal{L}v = 0, \text{ on } [0, T) \times \mathbb{R}^d; \quad v(T, \cdot) = e^{F(\cdot; q)}, \text{ on } \mathbb{R}^k,$$

where $\mathcal{L}$ is the infinitesimal generator of $Y$. Assume the coefficients $b, \sigma$, and the terminal condition $F$ are regular enough so there exists a $u \in C^{1,2,0}$ that solves the above Cauchy problem (see for example [19] for general regularity results). Using Itô’s formula

$$dN_t(q) = \nabla_y v(t, Y_t; q)\sigma(Y_t)dB_t = v(t, Y_t; q)\nabla_y v(t, Y_t; q)\sigma(Y_t)dB_t$$

$$= N_t(q)\nabla_y \log v(t, Y_t; q)\sigma(Y_t)dB_t.$$  

Thus, $H_t(q) = \sigma(Y_t)'\nabla_y \log v(t, Y_t; q)$, and validity of Assumption 4.4 depends on smoothness and monotonicity properties of the map $q \mapsto \nabla_y \log v(t, Y_t; q)$. As a matter of fact, Theorem 3 of [20] shows when $d = k = 1$; $b(t, y) = b(t); \sigma(t, y) = \sigma(t); \Sigma_0$ and $\Psi$ are of linear growth (i.e. $|\Sigma_0(y)| + |\Psi(y)| \leq K(1 + |y|)$); and $\Psi$ strictly monotone in $\mathbb{R}$, then $\mathcal{K}_t = \mathcal{K}_t^o = \mathbb{R}$. Interestingly, [20] also provides an example where $\Psi$ is the payoff of a European call (which is not strictly monotone) and shows in this case $\mathcal{K}_t \neq \mathbb{R}$.

As previously mentioned, Assumption 4.4 states an implicit relation between $\Sigma_0$, $\Sigma_1$, and $\Psi$. Typically, $\Psi$ is assumed linear in $\Sigma_0, \Sigma_1$ (see [11, 21, 52] as well as the more recent [33, 44]). In other words, agents securitize their risky positions, and through trading they achieve a mutually beneficial risk reduction. Assuming the endowments are portfolios of the tradeable securities essentially implies each agent has securitized his endowment, making it available for trading. The following examples are related to this (typical) setting, and verify Assumption 4.4.

Example 4.8. Endowments as portfolios of $\Psi$ and an independent component. Let $d = k + 1$ and write $\mathcal{B} = (B^1, ..., B^k)$. For $k_0, k_1 \in \mathbb{R}^k$, assume

$$\Sigma_0 = k_0' \Psi + Y_0; \quad \Sigma_1 = k_1' \Psi + Y_1,$$
where \( \Psi \) is \( \mathcal{F}_T^E \) measurable and \( Y_0, Y_1 \) are \( \mathcal{F}_T^{k+1} \) measurable. \( Y_0 \) and \( Y_1 \) can be interpreted as the idiosyncratic components of the respective endowments; while the positions in \( \Psi \) are the part invested in the tradeable asset (for the market maker, the latter can be thought as the aggregate inventory). Notice if \( Y_0 = Y_1 = 0 \) then both the large investor and the market maker have endowments which are portfolios of \( \Psi \) (such as in [44]). As \( (Y_0, Y_1) \) and \( \Psi \) are independent given \( \mathcal{F}_t \) for all \( t \leq T \), it follows that for \( q \in \mathbb{R}^k \)

\[
N_t(q) = \mathbb{E} \left[ e^{-\gamma \Sigma_0 - \gamma q^\prime \Psi} \big| \mathcal{F}_t \right] = \mathbb{E} \left[ e^{-(\gamma k_0 + q^\prime)\Psi} \big| \mathcal{F}_t \right] \mathbb{E} \left[ e^{-\gamma Y_0} \big| \mathcal{F}_t \right]; \quad t \leq T.
\]

In (2.4), write \( H(q) = (\overline{H}(q), H^{k+1}(q)) \). We see

\[
\mathcal{E} \left( \int_0^T \overline{H}_u(q) \cdot dB_u \right)_T = \frac{e^{-\gamma (k_0 + q^\prime)\Psi}}{\mathbb{E} \left[ e^{-\gamma (k_0 + q^\prime)\Psi} \right]}; \quad \mathcal{E} \left( \int_0^T H^{k+1}_t(q) dB^{k+1}_t \right)_T = \frac{e^{-\gamma Y_0}}{\mathbb{E} \left[ e^{-\gamma Y_0} \right]}.
\]

As \( H^{k+1} \) does not depend upon \( q \)

\[
\mathcal{K}_t = \left( \{\overline{H}_t(q)\}_{q \in \mathbb{R}^k}, H^{k+1}_t(0) \right); \quad \mathcal{K}_t^0 = \left( \{\overline{H}_t(q)\}_{q \in \mathbb{R}^k}, 0 \right).
\]

Next, recall \( M = (\overline{M}, M^{k+1}) \) from (4.10), (4.11). Due to the independence

\[
\mathcal{E} \left( \int_0^T M'_t dB_t \right)_T = \frac{e^{-\beta (k_0 + q^\prime)\Psi}}{\mathbb{E} \left[ e^{-\beta (k_0 + q^\prime)\Psi} \right]} \frac{e^{-\beta (Y_0 + Y_1)}}{\mathbb{E} \left[ e^{-\beta (Y_0 + Y_1)} \right]} = \mathcal{E} \left( \int_0^T \overline{M}_t dB_t \right)_T \mathcal{E} \left( \int_0^T M^{k+1}_t dB^{k+1}_t \right)_T.
\]

Since

\[
-\beta (\Sigma_0 + \Sigma_1) = -\gamma \Sigma_0 - \gamma \frac{\alpha \Sigma_1 - \gamma \Sigma_0}{\alpha + \gamma},
\]

we see that \( \overline{M} = \overline{H}((\alpha k_1 - \gamma k_0)/(\alpha + \gamma)) \), but we can only have \( M^{k+1} = H^{k+1}(0) \) (i.e. Assumption 4.4) when \( \alpha Y_1 = \gamma Y_0 \). Absent this condition, it fails. However, the constraint \( \pi \in \mathcal{K}^0 \) means we cannot invest in \( S^{k+1} \), while the optimal demand in \( \overline{\mathcal{K}} \) is \( \overline{M} - \overline{H}(0) \). Thus, for the large investor, the static position \( \hat{Q}_t = (\alpha k_1 - \gamma k_0)/(\alpha + \gamma), t \leq T \) is optimal, and there is no need for dynamic trading. Indeed, the optimal order puts the market maker and the large investor at a Pareto optimal situation (see [10]), meaning that there is no other mutually beneficial transaction to be made between them. In other words, if there is no other exogenous randomness in the market, such as noise traders or other investors’ orders to market makers, their pricing remains stable and large investor will not submit any other order up to the terminal time.

**Example 4.9. Bachelier model revisited.** Our last example shows dynamic optimal trading strategies are possible. We extend the Bachelier model of Example 3.2 by additionally setting \( \Sigma_1 = \int_0^T g'_t dB_t \) where \( g \in L^2([0, T]; \mathbb{R}^d) \). From Example 3.2 we have \( \mathcal{K}_t^0 = \mathbb{R}^d \); so Assumption 4.4 holds.

Recall from (3.12), \( H(q) - H(0) = -\gamma q^\prime \). On the other hand, we readily see that \( M \) and \( \hat{\pi} \) from (4.11), (4.12) respectively are

\[
M = -\beta (f + g); \quad \hat{\pi} = M - H(0) = \frac{\gamma}{\alpha + \gamma} (\gamma f - \alpha g).
\]
Therefore, $\hat{Q} = (1/(\alpha + \gamma))\psi^{-1}(\alpha g - \gamma f)$. Note that the optimal demand is deterministic, but not necessarily static as it was in Example 4.8. However, when $f = \psi k_0$ and $g = \psi k_1$ (endowments are portfolios of $\Psi$) we recover again static optimal positions.

5. Contingent Claim Pricing and Endogenously Rising Large Positions

While the pricing of contingent claims (or derivative securities) is well studied in frictionless complete or incomplete markets, studies on the corresponding problem in the presence of price impact are scarce (see for instance [35] and [12] on option pricing under market with price impact concerns). The section works toward that direction from the large investor’s perspective. We consider a claim with $F_T$ measurable payoff $h$. In contrast to the tradeable securities with terminal payoff $\Psi$, $h$ is not necessarily traded through the market makers. As explained in greater detail below, the large investor buys or sells $h$ from some other investor, and then hedges his position by dynamically trading with the market makers. As the latter trading induces price impact, the traditional hedging and pricing arguments must be changed.

Recall $Q_0$ from (3.5), and consolidate Assumptions 2.2, 4.1, 4.4 by enforcing

Assumption 5.1. $E\left[e^{-\gamma \Sigma_0 + p|\Psi|}\right] < \infty$, $E\left[e^{-\gamma \Sigma_0 + p|h|}\right] < \infty$ and $E\left[e^{-\alpha \Sigma_1 + p|h|}\right] < \infty$ for all $p \geq 0$.

Furthermore, $K^0 = \mathbb{R}^d$, Leb$_{[0,T]} \times \mathbb{P}$ a.s.

5.1. Arbitrage free prices. Assumption 5.1 implies the large investor can hedge the claim $uh$, for any $u \in \mathbb{R}$. Indeed, as $E^0 [e^{\gamma uh}] < \infty$, there is a demand process $Q \in A_{PI}$ and a (per unit) initial capital $\pi_t(u)$ such that $uh(\pi_t(u)) + V_T(Q) = uh$ a.s.. Using martingale representation and (2.6)

\begin{equation}
\frac{e^{\gamma uh}}{E^0 [e^{\gamma uh}]} = E \left( \int_0^T \pi_t'(dB_t^0 - H_t(0)dt) \right)_T = e^{\gamma V_T(Q)},
\end{equation}

through the identification $\pi_t = H_t(Q_t) - H_t(0)$. The required initial capital is

\begin{equation}
\pi_t(u) := \frac{1}{\gamma u} \log \left(E^0 [e^{\gamma uh}] \right),
\end{equation}

which is exactly the market maker’s indifference (per unit) value for selling $u$ units of $h$. As transactions on $h$ are not necessarily made through the market makers, we want to identify prices for $h$ which preclude arbitrage by trading with the market makers. Despite the completeness implied by $K^0 = \mathbb{R}^d$, due to price impact, there need not be only one “arbitrage free” price for $uh$. In fact, we presently develop three notions of an arbitrage free price, which coincide in the frictionless case, but give strikingly different answers considering price impact.

To develop the first two notions, we see from (5.2) that in contrast to the frictionless case, the replicating capital for $-uh$ is not $-u\pi_t(u)$. Rather, it is $-u\pi_t(u)$ where

\begin{equation}
\beta_t(u) := -\frac{1}{\gamma u} \log \left(E^0 [e^{-\gamma uh}] \right),
\end{equation}

is the market maker’s indifference (per unit) value for buying $u$ units of claim $h$. Jensen’s inequality implies $\beta_t(u) \leq \pi_t(u)$, which simply verifies the fact that buying value (bid) is less than the
corresponding selling value (ask). In fact, the latter inequality is strict if \( h \) is not a.s. constant (see [5] Proposition 3.6).

We first define a strong notion of an arbitrage free price for \( h \), consistent with that in [18]:

**Definition 5.2.** \( p \) is an arbitrage free for all positions in \( h \), provided for all \( Q \in \mathcal{A}_{PI} \) and \( u \in \mathbb{R} \), if \( up + V_T(Q) - uh \geq 0 \) a.s., then \( up + V_T(Q) - uh = 0 \) a.s..

The reasoning behind Definition 5.2 is clear: we rule out the large investor being able to start with nothing, and for some \( u \in \mathbb{R} \), trade \( u \) units of \( h \) at 0 for a per-unit price \( p \), trade upon price impact with market maker \( \Psi \) over \( [0, T] \), and cover his position with positive probability of gain.

Due to the non-linearity of price impact, the above definition is strong as it rules out arbitrages for all position sizes \( u \) simultaneously. A weaker notion of an arbitrage free price is:

**Definition 5.3.** \( p \) is an arbitrage free at the level \( u > 0 \) in \( h \) provided

(a) For all \( Q \in \mathcal{A}_{PI} \), if \( up + V_T(Q) - uh \geq 0 \) a.s., then \( up + V_T(Q) - uh = 0 \) a.s..

(b) For all \( Q \in \mathcal{A}_{PI} \), if \( -up + V_T(Q) + uh \geq 0 \) a.s., then \( -up + V_T(Q) + uh = 0 \) a.s.

In other words, an arbitrage free price \( p \) for at the level \( u > 0 \) rules out arbitrages for either buying or selling \( u \). Based on these two definitions we have:

**Proposition 5.4.** Let Assumption [5.2] hold. Then,

i) The range of arbitrage free prices for \( h \) in the sense of Definition 5.2 is the singleton \( \mathbb{E}^0[h] \).

ii) For any fixed \( u > 0 \), the range of arbitrage free prices for \( h \) in the sense of Definition 5.3 is the closed interval \( [\underline{h}(u), \overline{h}(u)] \) from (5.2) and (5.3) respectively.

**Proof.** We first prove the statement regarding Definition 5.3. To this end, fix \( u > 0 \), \( p \in \mathbb{R} \) and \( Q \in \mathcal{A}_{PI} \). Set \( W(Q) = up + V_T(Q) - uh \) and note

\[
\frac{\mathbb{E}^0[e^{\gamma(uh+W(Q))}]}{\mathbb{E}^0[e^{\gamma uh}]} = e^{\gamma up} \frac{\mathbb{E}^0[e^{\gamma V_T(Q)}]}{\mathbb{E}^0[e^{\gamma uh}]} \leq e^{\gamma up} \mathbb{E}^0[e^{\gamma(uh-W(Q))}] = e^{\gamma u(p-\overline{h}(u))},
\]

where the inequality used (2.6). Therefore, if \( p \leq \overline{h}(u) \) then \( p \) satisfies part (a) in Definition 5.3. If \( p > \overline{h}(u) \), we observe for \( Q \in \mathcal{A}_{PI} \) from (5.1)

\[
W(Q) = up + V_T(Q) - uh = u(p - \overline{h}(u)),
\]

so \( p \) does not satisfy part (a) of Definition 5.3. This gives the upper bound. For the lower bound, now set \( \bar{W}(Q) = -up + V_T(Q) + uh \). We then have

\[
\frac{\mathbb{E}^0[e^{\gamma(-uh+\bar{W}(Q))}]}{\mathbb{E}^0[e^{-uh}]} = e^{-\gamma up} \frac{\mathbb{E}^0[e^{\gamma V_T(Q)}]}{\mathbb{E}^0[e^{-uh}]} \leq e^{-\gamma up} \mathbb{E}^0[e^{-\gamma(uh-W(Q))}] = e^{-\gamma u(p-\underline{h}(u))}.
\]

Therefore if \( p \geq \underline{h}(u) \), then \( p \) satisfies part (b) in Definition 5.3. If \( p < \underline{h}(u) \), we observe for \( Q \in \mathcal{A}_{PI} \) from (5.1) with \( -h \) replacing \( h \)

\[
\bar{W}(Q) = -up + V_T(Q) + uh = u(-p + \underline{h}(u)),
\]
so that \( p \) does not satisfy part (b) of Definition 5.3. This gives the lower bound, finishing the result for Definition 5.3.

For item i), we note that if \( p \) is arbitrage free in the sense of Definition 5.2, then for all \( u > 0 \), \( p \) is arbitrage free in the sense of Definition 5.3, and hence for all \( u > 0 \) we must have

\[
- \frac{1}{\gamma u} \log \left( \mathbb{E}^0 \left[ e^{-\gamma uh} \right] \right) \leq p \leq \frac{1}{\gamma u} \log \left( \mathbb{E}^0 \left[ e^{\gamma uh} \right] \right).
\]

Hölder’s inequality shows the function on the left side above is decreasing in \( u > 0 \) and the function on the right is increasing in \( u \). Furthermore, by the dominated convergence theorem, the limit as \( u \downarrow 0 \) of each of these functions is \( \mathbb{E}^0 [h] \). Thus, \( \mathbb{E}^0 [h] \) is arbitrage free in the sense of Definition 5.3 for all \( u > 0 \). Also, since for all \( Q \in \mathcal{A}_{P1} \), \( \mathbb{E} \left[ e^{\gamma V_T(Q)} \right] \leq 1 \), we see \( \mathbb{E}^0 [h] \) is also (vacuously) arbitrage free at \( u = 0 \). Finally, by the symmetry of Definition 5.3 we see \( \mathbb{E}^0 [h] \) is also the only arbitrage free price in the sense of Definition 5.2 even for \( u < 0 \). \( \square \)

Remark 5.5. According to Proposition 5.4, if the large investor finds an ask (respectively bid) price for \( u > 0 \) units of \( h \) that is lower (resp. higher) than \( h(u) \) (resp. \( \overline{h}(u) \)), an arbitrage opportunity arises, by buying (resp. selling) \( u \) units of \( h \) and then hedging by dynamically trading \( \Psi \) with the market makers. On the other hand, if the only price of \( h \) the large investor can find is \( \mathbb{E}^0 [h] \), there is no arbitrage opportunity for any position \( u \in \mathbb{R} \). Note, also, that both of the above definitions can be generalized to include arbitrage prices for each time \( t < T \).

Since \( h(u) \) (resp. \( \overline{h}(u) \)) is decreasing (resp. increasing) in \( u > 0 \), we immediately see:

**Corollary 5.6.** Let Assumption 5.1 hold. If \( p \) is an arbitrage price in the sense of Definition 5.3 for \( u > 0 \), then \( p \) is an arbitrage free price (in the sense of Definition 5.3) for all \( u' \geq u \).

Remark 5.7. Corollary 5.6 implies for any price \( p \neq \mathbb{E}^0 [h] \), the induced arbitrage opportunity is “limited”, in the sense that there is a maximum number of units on the claim at which the arbitrage can be exploited. In other words, due to price impact, the arbitrage vanishes when the investor takes larger positions on the claim. Intuitively, large positions on the claim require large hedging positions in \( \Psi \). This changes the market maker’s inventory, and the resultant pricing rule is to the detriment of the large investor. This is in sharp contrast with arbitrage in markets without price impact, which can be exploited for arbitrarily large positions.

Definitions 5.2 and 5.3 establish two notions of arbitrage which do not take into account the position, or preferences, of the large investor. This is an important omission, because the whole discussion on price impact is dedicated to investors large enough to create price impact. Since arbitrages are not scalable, the range of arbitrage free prices (in Definition 5.3) and especially the singleton (in Definition 5.2) are quite narrow. We now define a third, weaker, notion which states that \( p \) is an arbitrage free price if, given the possibility to purchase arbitrary quantities of \( h \) for \( p \), the investor’s optimal demand is finite. To state it, for a given initial capital \( x \), utility function \( U \), and endowment \( \Sigma_1 \), recall the value function \( u(x; \Sigma_1) \) from (2.7).
Definition 5.8. $p$ is a utility-demand based arbitrage free price for $h$ if for every $\{u_n\}_{n \in \mathbb{N}}$ such that
\begin{equation}
\lim_{n \to \infty} u(x - pu_n, \Sigma_1 + u_n h) = \sup_{u \in \mathbb{R}} u(-pu; \Sigma_1 + uh),
\end{equation}
we have $\sup_n |u_n| < \infty$.

For exponential preferences, the following result shows the range of demand-based arbitrage free prices is maximal. This is in stark contrast to the frictionless case (see [27]).

Proposition 5.9. Let Assumption 5.1 hold, and assume the large investor’s utility is exponential with risk aversion $\alpha$. The range of utility-demand based arbitrage free prices is $(\text{essinf } h, \text{esssup } h)$. For $p$ within this range, the optimal demand is the unique solution $u = u(p)$ to
\begin{equation}
p = \frac{E[h e^{-\beta(\Sigma_0 + \Sigma_1 + uh)}]}{E[e^{-\beta(\Sigma_0 + \Sigma_1 + uh)}]}.
\end{equation}

Proof. Using Proposition 4.5 with large investor endowment $\Sigma_1 + uh$, we see
\begin{equation}
u(-pu; \Sigma_1 + uh) = -e^{\alpha pu} E[e^{-\gamma \Sigma_0} - \frac{\alpha}{\gamma} \times E[e^{-\beta(\Sigma_0 + \Sigma_1 + uh)}]]^{\frac{\alpha}{\beta}}.
\end{equation}
Thus, the optimization problem is to minimize
\begin{equation}
p u + \frac{1}{\beta} \log \left( E\left[e^{-\beta(\Sigma_0 + \Sigma_1 + uh)}\right]\right),
\end{equation}
on $u \in \mathbb{R}$. Under the given integrability assumptions the above function is both smooth and strictly convex in $u$ (provided $h$ is not a.s. constant). The first order conditions for optimality are (5.5), and that the right hand side of (5.5) goes to $\text{essinf } (h)$ (resp. $\text{esssup } (h)$) as $u$ goes to $\infty$ (resp. $-\infty$) has been shown in [42].

Propositions 5.4 and 5.9 imply there are arbitrage prices in the sense of Definitions 5.2, 5.3, at which the large investor’s optimal demand is finite. Indeed, this holds for all $p \neq E^0 [h]$. However, due to price impact, the arbitrage gain is limited (c.f. Remark 5.7). On the other hand, the large investor may want to trade the claim for hedging purposes. If, for instance, the claim is negatively correlated with his endowment, he has motive to take a long position (even if $p > E^0 [h]$, meaning that the arbitrage is provided by a short position). When the benefits from hedging exceed the (limited) arbitrage, the optimal demand does not exploit it. The following example reinforces this point.

Example 5.10. We revisit the Bachelier model of Examples 3.2 and 4.9 and assume $h = \int_0^T y'_t dB_t$, where $y \in L^2([0, T]; \mathbb{R}^d)$. Assumption 5.1 readily holds and hence any position $u$ in $h$ can be hedged. Simple calculations show $E^0 [h] = -\gamma \int_0^T y'_t f_t dt$, along with
\begin{equation}
\bar{h}(u) = E^0 [h] + \frac{1}{2} \gamma u \int_0^T y'_t y_t dt; \quad h(u) = E^0 [h] - \frac{1}{2} \gamma u \int_0^T y'_t y_t dt.
\end{equation}
The (linear) optimal demand function $p \rightarrow \hat{u}(p)$ from (5.8) is
\begin{equation}
\hat{u}(p) = -\frac{p + \beta \int_0^T y'_t(f_t + g_t) dt}{\beta \int_0^T y'_t y_t dt} = -\frac{p + \alpha \text{Cov}[h, V_T(\hat{Q}(0)) + \Sigma_1]}{\beta \text{Var}[h]},
\end{equation}
where $\hat{Q}(0)$ is the optimal order flow with (endowment $\Sigma_1$ but) no claim, and the last equality follows using Example 4.9 at $\Sigma_1 = \int_0^T g'_t dB_t$. Next, a direct calculation specifying Example 4.9 to $\Sigma_1 = \int_0^T g'_t dB_t + \hat{u}(p)h$ shows for the optimal strategy $\hat{Q} = \hat{Q}(\hat{u}(p))$
\begin{equation}
-p\hat{u}(p) + V_T(\hat{Q}) + \hat{u}(p)h = \text{Constant} + \frac{1}{\alpha + \gamma} \int_0^T (\gamma f_t + \gamma \hat{u}(p) y_t - \alpha g_t)' dB_t.
\end{equation}
Thus, the large investor does not utilize an arbitrage strategy (save for the pathological case when
$\gamma f_t + \gamma \hat{u}(p)y_t - \alpha g_t \equiv 0$ which can only happen if $a) \gamma f_t - \alpha g_t \propto y_t$ and $b)$ the traded price is $p = -\alpha \int_0^T y'_t g_t dt$.

Now, from (5.8) we readily get that $\hat{u}(p) > 0$ if and only if $p < -\alpha \text{Cov}[h, V_T(\hat{Q}(0)) + \Sigma_1]$. This is intuitive because if $h$ is negatively correlated with the large investor’s (optimal) terminal wealth absent the claim, then he has a hedging-related incentive to purchase $h$, and will take long positions at higher prices. Furthermore, if $\hat{u}(p) > 0$, then, as shown in the proof of Proposition 5.4, in order to rule out arbitrages of the form $-p\hat{u}(p) + V(Q)_T + ph$ we must have $p \geq h(\hat{u}(p))$.

Writing $p = -\alpha \text{Cov}[h, V(\hat{Q}(0))_T + \Sigma_1] - \epsilon$ for some $\epsilon > 0$ we find (see again Example 4.9)
\begin{equation}
-p - h(\hat{u}(p)) = \frac{\gamma - \alpha}{2\alpha} \epsilon + \gamma \text{Cov}[h, V_T(\hat{Q}(0))].
\end{equation}
For the sake of clarity, assume $\gamma = \alpha$. Here, arbitrages arise (but are not used by the large investor) when $\text{Cov}[h, V_T(\hat{Q}(0))] < 0$. In this instance, the large investor’s hedging demands outweigh the desire to incorporate the arbitrage.

We thus see arbitrage arguments may not be sufficient to uniquely price, or “value”, $h$, even when it can be hedged. One way to obtain unique value is through the large investor’s utility indifference valuation. Define the (per unit, bid) indifference value $p^I$ for $u$ units of $h$ through the balance equation
\begin{equation}
u(x - up^I; \Sigma_1 + uh) = u(x; \Sigma_1).
\end{equation}
For exponential preferences, $p^I$ is independent of $x$, and hence we write $p^I(u; \Sigma_1)$. Furthermore, from Proposition 4.5 we obtain the explicit formula
\begin{equation}
p^I(u; \Sigma_1) = -\frac{1}{\alpha u} \log \left( \frac{u(0; \Sigma_1 + uh)}{u(0; \Sigma_1)} \right) = -\frac{1}{\beta u} \log \left( \frac{\mathbb{E} \left[ e^{-\beta (\Sigma_0 + \Sigma_1 + uh)} \right]}{\mathbb{E} \left[ e^{-\beta (\Sigma_0 + \Sigma_1)} \right]} \right).
\end{equation}
Note that $p^I$ depends on the large investor’s endowment and risk aversion. As such, it need not be arbitrage free in the sense of Definitions 5.2, 5.3 however, it is always arbitrage free in the sense of Definition 5.8.
5.2. **Endogenously rising large positions.** Using the results of Section 4.2 together with the general theory developed in [4], we now show, for fixed large investor preferences, that large positions arise *endogenously* as the market maker’s risk aversion \( \gamma \) vanishes. There are two main situations consistent with \( \gamma \to 0 \). First, when the number of market makers increases, as this implies growth in the aggregate risk tolerance \( 1/\gamma \). Second, as an approximation to market maker risk-neutrality. In fact, assuming risk neutral market makers is common in micro-structure literature (c.f. [34] and the references therein).

Through the basis-risk lens of Section 4, the market maker’s risk aversion is a measure of market incompleteness. Indeed, \( \gamma \to 0 \) implies the correlation \( \rho \) of (4.3) tends to 1. Remarkably, despite the large investor’s price impact, it is possible for large positions in \( h \) to endogenously arise as \( \gamma \to 0 \). More precisely, let Assumption 5.1 hold and fix the large investor risk aversion \( \alpha \). To use notation consistent with [4], for \( n \in \mathbb{N} \) set

\[
(5.10) \quad r_n := \alpha + \gamma_n, \frac{\gamma_n}{\alpha},
\]

which is the aggregate risk tolerance of market makers and large investor and consider the case where \( \gamma_n \to 0 \) so \( r_n \to \infty \). From (5.9) (note, \( \beta_n = 1/r_n \)), we see along the rate \( u = \ell r_n, \ell \in \mathbb{R} \backslash \{0\} \)

\[
(5.11) \quad p^I(\ell r_n; \Sigma_1) = -\frac{1}{\ell} \log \left( \frac{\mathbb{E}[e^{-\ell h - \beta_n(\Sigma_0 + \Sigma_1)}]}{\mathbb{E}[e^{-\beta_n(\Sigma_0 + \Sigma_1)}]} \right).
\]

Then, as established in [4, Section 6.2], the dominated convergence theorem yields that as \( n \uparrow \infty \)

\[
(5.12) \quad p^\infty(\ell) := \lim_{n \to \infty} p^I(\ell r_n; \Sigma_1) = -\frac{1}{\ell} \log \left( \mathbb{E}[e^{-\ell h}] \right).
\]

**Remark 5.11.** Consider when there are \( n \to \infty \) market makers with identical risk aversions \( \gamma > 0 \) and endowments \( \Sigma_0 \). In this case, the representative market maker’s risk aversion is \( \gamma_n = \gamma/n \) and the aggregate endowment is \( \Sigma_n = n\Sigma_0 = (\gamma/\gamma_n)\Sigma_0 \). In other words, even as \( \gamma_n \to 0 \), \( \gamma_n \Sigma_n = \gamma \Sigma \neq 0 \), and hence (representative) market maker’s risk neutrality does not necessarily correspond to zero endowment. Here, the limit in (5.12) is

\[
(5.13) \quad \lim_{n \to \infty} p^I(\ell r_n; \Sigma_1) = -\frac{1}{\ell} \log \left( \frac{\mathbb{E}[e^{-\ell h - \gamma \Sigma_0}]}{\mathbb{E}[e^{-\gamma \Sigma_0}]} \right).
\]

More generally, our analysis can handle when the market maker (as well as the large investor) endowment grows at the rate \( r_n \), and this assumption is reasonable in the large market maker case. Indeed, from the basis risk model dynamics (1.4), the limit in (5.13) is the unique arbitrage free price in the \( \rho = 1 \) market with filtration \( \mathbb{F} \).

To connect limiting indifference values with endogenous large positions, consider the optimal purchase problem of Definition 5.8. For \( p \in (\text{essinf}(h), \text{esssup}(h)) \), and \( n \in \mathbb{N} \) fixed, Proposition 5.9 states that there is unique optimal demand \( \hat{u}_n(p) \). Furthermore, the convergence in (5.12), together with the continuity of \( \ell \to p^\infty(\ell) \) at \( \ell = 0 \), verifies Assumption 3.3 of [4]. Thus, by [4]
Theorems 4.3, 4.4, as $\gamma_n \to 0$ it holds for all $p \neq p^\infty(0)$ that
\[
0 < \liminf_{n \to \infty} \frac{|\hat{u}_n(p)|}{r_n} \leq \limsup_{n \to \infty} \frac{|\hat{u}_n(p)|}{r_n} < \infty.
\]
Thus, optimal positions become large at the rate $r_n$. Additionally, unless $h$ is a.s. constant, the results of \[4\] also show that since the map $\ell \mapsto \ell p^\infty(0)$ is strictly convex.

Recall $p^\infty(0) = E[h]$ (or $p^\infty(0) = E[he^{-\gamma \Sigma}] / E[e^{-\gamma \Sigma}]$ if the endowment does not vanish as in Remark 5.11). Therefore, whenever $p \neq p^\infty(0)$, the optimal demand increases in magnitude to $\infty$ exactly at the rate $r_n$. The price $p^\infty(0)$ is the limit of the arbitrage-free prices $E^0_n[h]$ in the sense of Definition 5.2 (as well as Definition 5.3 for $u$ fixed as $n \to \infty$), so $p \neq p^\infty(0)$ corresponds to the large investor obtaining a very advantageous price as the market makers approach risk neutrality. Lastly, \[4\] verifies the above conclusions for a sequence of traded prices $\{p_n\}_{n \in \mathbb{N}}$ provided $p_n \to p \neq p^\infty(0)$: there is no need for the traded price to be fixed for each $n$.

5.3. Partial Equilibrium Price Quantities. As shown above, large positions endogenously arise when $p \neq p^\infty(0)$, and when the market maker is approximately risk neutral. Furthermore, for $n$ large enough, $p$ is not arbitrage free in the sense of either Definition 5.2 or 5.3. Clearly, it is necessary to justify such a price is asked or offered by some other investor (recall the large investor does not trade $h$ with the market maker).

We justify this assumption in a segmented markets setting. Segmented market models are common in the literature (see \[40, 41\]), where the large investors’ role is played by so-called “arbitrageurs” and the market makers are competitive, price-takers investors. In our setting, there are two markets ($A$ and $B$) with different market maker endowments $\Sigma^B_0$, $\Sigma^A_0$ and tradeable security payoffs $\Psi^A$, $\Psi^B$. By “segmented”, we mean the respective market makers do not trade with one another. There are two large investors with respective endowments $\Sigma^A_1$, $\Sigma^B_1$, and utility functions $U_A$, $U_B$. Each large investor trades (with price impact) in $\Psi^i$, $i \in \{A, B\}$ with his respective “local” market maker. The large investors can also trade the claim $h$ with one another, in the form of an bilateral over-the-counter (OTC) transaction (similarly to the bilateral trade between arbitrageurs in \[40, 41\]). The motivation for this setting is that market makers in a specific product/market are not necessarily involved in other products/markets (e.g. those with different transaction costs, regulation, geography, security contracting, etc.).

To determine the optimal traded price $p$ and quantity $u$, we use the notion of Partial Equilibrium Price Quantity (PEPQ) of \[5\]. We say a pair $(p^*, u^*) \in \mathbb{R}^2$ is a PEPQ provided
\[
(5.14) \quad u^* \in \arg\max_{u \in \mathbb{R}} \{u_A(x_A - p^* u, \Sigma_A + uh)\} \bigcap \arg\max_{u \in \mathbb{R}} \{u_B(x_B + p^* u, \Sigma_B - uh)\},
\]
where $u_A$ and $u_B$ are the investors’ value functions defined as (2.7) (see also (5.4)). As such, a PEPQ consists of a price at which the optimal demand of one investor coincides with the negative optimal demand of the other (i.e. their bilateral market clears out).
For exponential utilities, the problem of finding such equilibrium pairs simplifies. Indeed, from (5.9) we conclude \((p^*, u^*)\) is a PEPQ provided
\[
(5.15) \quad u^* \in \text{argmax}_{u \in \mathbb{R}} \{ u p^{I,A}(u, \Sigma_1) - p^* u \} \bigcap \text{argmax}_{u \in \mathbb{R}} \{ p^* u - u p^{I,B}(-u, \Sigma_1) \}.
\]

Based on [5, Theorem 5.8], we get the following:

**Proposition 5.12.** Let Assumption 5.1 hold for both markets and define
\[
\beta^i := \frac{\gamma_i \alpha_i}{\gamma_i + \alpha_i}, \quad \text{for } i \in \{A, B\}.
\]

If \(\beta_A(\Sigma_0^A + \Sigma_1^A) - \beta_B(\Sigma_0^B + \Sigma_1^B)\) is not constant, there is a unique PEPQ \((p^*, u^*) \in (\text{essinf}(h), \text{esssup}(h))\times \mathbb{R} \). In fact
\[
u^* = \text{argmax}_{u \in \mathbb{R}} \{ u p^{I,A}(u; \Sigma_1^A) + u p^{I,B}(-u; \Sigma_1^B) \},
\]
and
\[
(5.16) \quad p^* = \frac{\mathbb{E} \left[ h e^{-\beta_A(\Sigma_0^A + \Sigma_1^A + u^* h)} \right]}{\mathbb{E} \left[ e^{-\beta_A(\Sigma_0^A + \Sigma_1^A + u^* h)} \right]}
= \frac{\mathbb{E} \left[ h e^{-\beta_B(\Sigma_0^B + \Sigma_1^B - u^* h)} \right]}{\mathbb{E} \left[ e^{-\beta_B(\Sigma_0^B + \Sigma_1^B - u^* h)} \right]}.
\]

In other words, the partial equilibrium price \(p^*\) is the marginal valuation of both large investors when their endowments include the equilibrium quantity \(u^*\).

**Example 5.13.** For \(i \in \{A, B\}\), we consider the Bachelier model of Examples 3.2, 4.9 and 5.10, noting that \(h = \int_0^T y_t dB_t\) is common for both investors. Indifference valuation (5.9) yields for \(i \in \{A, B\}\)
\[
p^{I,i}(u; \Sigma_i^1) = -\frac{\beta^i}{2} \int_0^T (u y_t + 2(f_t^i + g_t^i))' y_t dt.
\]

From Proposition 5.12, the unique equilibrium quantity is
\[
(5.17) \quad u^* = \frac{\int_0^T (\beta_B(f_t^B + g_t^B) - \beta_A(f_t^A + g_t^A))' y_t dt}{(\beta_A + \beta_B) \int_0^T y_t' y_t dt}.
\]

The equilibrium price is
\[
(5.18) \quad p^* = -\Gamma \int_0^T (f_t^A + g_t^A + f_t^B + g_t^B)' y_t dt, \quad \Gamma := \frac{\beta_A \beta_B}{\beta_A + \beta_B},
\]
so that
\[
(5.19) \quad \frac{1}{\Gamma} = \frac{1}{\alpha_A} + \frac{1}{\alpha_B} + \frac{1}{\gamma_A} + \frac{1}{\gamma_B}
\]
is the aggregate risk tolerance of the market makers and large investors. It is clear there is no reason for \(p^*\) to be the unique (for the respective markets) arbitrage free price in the sense of Definition 5.2 (c.f. Remarks 5.5 and 5.7). In particular, if endowments \(\tilde{\Sigma}_A\) and \(\tilde{\Sigma}_B\) are sufficiently different, the equilibrium quantity is large which essentially means that none of the investors exploits any arbitrage when trading with market makers. In these cases, the mutual benefits from risk sharing outweigh the limited arbitrage gains (a sharp difference of the models of segmented markets with
no price impact, where the investors could create and mutually exploit arbitrage opportunities (see for instance [40] and the references therein).

5.4. Large Partial Equilibrium Quantities. We saw in Section 5.2 that each individual investor (whose risk aversion $\alpha^i$, $i \in \{A, B\}$ is fixed) will endogenously own a large position in the contingent claim $h$ provided a) the market maker risk aversion $\gamma^i \approx 0$ and b) the traded price $p$ is strictly away from $E^{0,i}[h]$, where $E^{0,i}[h]$ is the unique arbitrage free price in the sense of Definition 5.2. In this section, we verify the above assumptions in two stylized, but representative situations in the PEPQ framework. First, when one large investor has an existing large position in the claim, and second when each market has a large number of market makers.

Example 5.14. Investor $B$ in the large claim regime, market makers near risk neutrality. Assume $\gamma^A, \gamma^B \to 0$, and the market makers’ endowments do not scale with the respective risk tolerance so that $\Sigma^A_0, \Sigma^B_0$ are fixed. For simplicity we assume investor $A$ has no endowment and investor $B$ has an endowment of $u = \ell/\gamma^B h, \ell \neq 0$, so he is already in the large claim regime. Equilibrium pricing formula (5.16) specifies to

$$p^* = \frac{\mathbb{E}\left[he^{-\frac{\gamma^A \alpha^A}{\gamma^A + \alpha^A}(\Sigma^A_0 + u^* h)}\right]}{\mathbb{E}\left[e^{-\frac{\gamma^A \alpha^A}{\gamma^A + \alpha^A}(\Sigma^A_0 + u^* h)}\right]} = \frac{\mathbb{E}\left[he^{-\frac{\gamma^B \alpha^B}{\gamma^B + \alpha^B}(\Sigma^B_0 + (\frac{\ell}{\gamma^B} - u^*) h)}\right]}{\mathbb{E}\left[e^{-\frac{\gamma^B \alpha^B}{\gamma^B + \alpha^B}(\Sigma^B_0 + (\frac{\ell}{\gamma^B} - u^*) h)}\right]}.$$

As $\gamma^A, \gamma^B \to 0$, assume $\gamma^A/\gamma^B \to \delta \in (0, \infty)$. If $p^* \to p^\infty(0) = \mathbb{E}[h]$ then necessarily (due to the strict monotonicity of the Esscher transform) the equation for $B$ implies $u^* \approx \ell/\gamma^B$. However, by considering the equation for $A$, this in turn implies $p^* = \mathbb{E}[he^{-\delta h}] / \mathbb{E}[e^{-\delta h}] \neq \mathbb{E}[h]$. Thus, it must be that $p^*$ does not limit to $p^\infty(0)$, and large positions endogenously arise for $A$. In fact, it is easy to see that $u^* \approx \ell/(\gamma^A + \gamma^B)$.

Example 5.15. Large number of market makers. This example emphasizes that, under the presence of price impact, the large (equilibrium) positions on a claim could arise even when the large investors’ initial endowments do not contain any position on the claim. As in Remark 5.11, we assume for each $n$ there are $n$ market makers with risk aversion $\gamma^j$ and endowment $\Sigma^j_0$ (in each local market $i \in \{A, B\}$). We let each large investor have fixed endowment, and in fact without loss of generality (for this example), set $\Sigma^i_0 = 0$. Here, (5.16) specifies to

$$p^* = \frac{\mathbb{E}\left[he^{-\gamma^A \alpha^A/n}(\Sigma_0^A + u^*_n h)\right]}{\mathbb{E}\left[e^{-\gamma^A \alpha^A/n}(\Sigma_0^A + u^*_n h)\right]} = \frac{\mathbb{E}\left[he^{-\gamma^B \alpha^B/n}(\Sigma_0^B + u^*_n h)\right]}{\mathbb{E}\left[e^{-\gamma^B \alpha^B/n}(\Sigma_0^B + u^*_n h)\right]}.$$

Now, assume optimal positions are dominated by $n$ in that $|u^*_n|/n \to 0$. As $n \to \infty$, this implies

$$\frac{\mathbb{E}\left[he^{-\gamma^A \Sigma^A_0}/\Sigma^A_0\right]}{\mathbb{E}\left[e^{-\gamma^A \Sigma^A_0}/\Sigma^A_0\right]} = \frac{\mathbb{E}\left[he^{-\gamma^B \Sigma^B_0}/\Sigma^B_0\right]}{\mathbb{E}\left[e^{-\gamma^B \Sigma^B_0}/\Sigma^B_0\right]}.$$
Therefore, we see that except in the highly particular case when the above equality holds, large positions will “spontaneously” (i.e. when neither large investor owns a position in \( h \) ahead of time) arise as the number of market makers becomes large.

**Appendix A. Optimal investment problem for large investor**

*Proof of Lemma 2.3.* From [7, Theorem 3.2] and [9, Theorem 4.9] we know that under Assumption 2.2 the gains process is well defined for all locally bounded predictable strategies \( Q \). It remains to obtain the explicit expression for \( V \) in (2.6). Using the notation of [7, 9], for \((v, x, q) \in (0, \infty) \times \mathbb{R} \times \mathbb{R}^d\) define \( \Sigma(x, q) := \Sigma_0 + x + q' \Psi \), \( r(v, x) := -ve^{-\gamma x} \), \( F_t(v, x, q) := \mathbb{E} \left[ r(v, \Sigma(x, q)) | \mathcal{F}_t \right] \) and

\[
G_t(u, y, q) := \sup_{v > 0} \inf_{x \in \mathbb{R}} (uv + xy - F_t(v, x, q)) .
\]

Direct calculation yields

\[
(A.1) \quad F_t(v, x, q) = -ve^{-\gamma x} N_t(q); \quad G_t(u, y, q) = \frac{y}{\gamma} \log \left( \frac{N_t(q)}{-u} \right) .
\]

Following the terminology of [9, Equations (3.19) and (4.16)], we define the market maker’s expected utility process \( \{U_t\}_{t \leq T} \) by \( U_t = U_t(v, x, q) = (\partial/\partial v) F_t(v, x, q) \) for \( t \leq T \). From (2.4) and (A.1), we readily get that \( U_t \) will solve the affine stochastic differential equation

\[
\frac{dU_t}{U_t} = H_t(Q_t)' dB_t; \quad t \leq T .
\]

which admits the explicit strong solution

\[
U_t = U_0 \mathcal{E} \left( \int_0^\cdot H_s(Q_s)' dB_s \right) \bigg|_t; \quad U_0 = \mathbb{E} \left[ -e^{-\gamma \Sigma_0} \right] = -N_0(0) ,
\]

provided \( Q \in \mathcal{A}_{PI} \). Continuing and slightly abusing the notation, [9, Equation (4.19)] implies the gains process \( V_t(Q) \) is

\[
V_t(Q) = -G_t(U_t, 1, 0) = -\frac{1}{\gamma} \log \left( \frac{N_t(0)}{-U_t(Q)} \right) ;
\]

\[
= -\frac{1}{\gamma} \log \left( \frac{N_t(0)}{N_0(0)} \right) + \frac{1}{\gamma} \int_0^t H_s(Q_s)' dB_s - \frac{1}{2\gamma} \int_0^t |H_s(Q_s)|^2 ds .
\]

Now, from (2.4) we see that

\[
\frac{N_t(0)}{N_0(0)} = \mathcal{E} \left( \int_0^\cdot H_s(0)' dB_s \right) \bigg|_t ,
\]

which implies

\[
V_t(Q) = \frac{1}{\gamma} \int_0^t (H_s(Q_s) - H_s(0))' dB_s - \frac{1}{2\gamma} \int_0^t (|H_s(Q_s)|^2 - |H_s(0)|^2) ds .
\]

The latter expression coincides with (2.6), finishing the proof. \( \square \)
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Department of Banking and Financial Management, University of Piraeus, Piraeus, Greece
E-mail address: anthropel@unipi.gr

Questrom School of Business, Boston University, Boston, MA 02215
E-mail address: scottrob@bu.edu

Department of Mathematics & Statistics, Boston University, Boston, MA 02215
E-mail address: kspiliop@math.bu.edu