Fermions and gravitational gyrotropy

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Abstract. In conventional general relativity without torsion, high-frequency gravitational waves couple to the chiral number density of spin one-half quanta: the polarization of the waves is rotated by $2\pi N_5\ell_{Pl}^2$, where $N_5$ is the chiral column density and $\ell_{Pl}$ is the Planck length. This means that if a primordial distribution of gravitational waves with E-E or B-B correlations passed through a chiral density of fermions in the very early Universe, an E-B correlation in the waves, and one in the cosmic microwave background, would be generated. Less obviously but more primitively, the condition Albrecht called ‘cosmic coherence’ would be violated, changing the restrictions on the class of admissible cosmological gravitational waves.

Keywords: fermions, chiral effects, gravitational waves, gyrotropy, gravitational memory
1 Introduction

Two, presumably deeply interfused, puzzles in cosmology are the origin of leptons and the emergence of the chiral character of their spectrum and interactions which is familiar at ordinary scales. These must develop in the very early Universe, but just how early is unclear. We do not know the physical mechanisms involved, and the only model-independent constraint on the energy-scales is that they must be beyond those which have been substantially explored in collider experiments. In view of this, it is worthwhile considering general properties of fermions in cosmological contexts.

I describe here some remarkable effects involving fermions and gravitational waves, which should on ordinary scales be tiny but could be significant in cosmology. These effects grow out of a striking coupling of gravitational radiation to chiral physics: a chiral current \( j^5_a \) (of right- minus left-handed spin one-half particles) rotates the polarization of the waves by an angle

\[
2\pi\ell_{Pl}^2 N_5, \tag{1.1}
\]

where \( \ell_{Pl} \) is the Planck length and

\[
N_5 = \int j^5_a \, d\gamma^a \tag{1.2}
\]
is the *chiral column density* along the null geodesic $\gamma$ giving the geometric-optics trajectory of the wave. This rotation may be called *gravitational gyrotropy*, in parallel with the corresponding optical effect.

Three points should be made immediately. Most importantly, the results here are predictions of *conventional* general relativity, without the addition of any torsion term or other modification. Second, the effect described here does not, of itself, introduce any explicit chiral symmetry breaking. Rather, it correlates chiral asymmetries in the fermi field with those in the propagation of gravitational radiation. Finally, while such a gyrotropy could be caused by a circular birefringence of gravitational waves, that is, from the two circular polarizations propagating at different speeds, that is not what happens here. The signal speed is the same — the speed of light — for the two polarizations. What is going on is not a refractive effect but a twisting one.

Although in principle this rotation should occur very generally, the Planck-area prefactor makes its magnitude unobservably small in virtually all circumstances considered in contemporary astrophysical models. For instance, even for a supernova emitting $\sim 10^{54}$ neutrinos from a $\sim 10$ km radius, the chiral column density is $\sim 10^{41} \text{cm}^{-2}$, some twenty-five orders of magnitude below the Planck scale. Similarly, even if all the dark matter in the Universe were in $\sim 10\text{keV}$ sterile neutrinos, the corresponding number density would be $\sim 10^2 \text{cm}^{-3}$, so that the present induced mean cosmic rotation rate would be $\sim 10^{-37}/\text{Gyr}$. (Of course, dark matter is not uniformly distributed, but it is very hard to see how a density enhancement could be plausibly large enough to compensate for the smallness of this figure.)

For this reason, I focus here on the possibility that there was at some period in the very early Universe a significant chiral number density through which primordial gravitational waves passed. Such situations are not contemplated by most contemporary models, but are possible in light of our ignorance. The expression (1.1) is so simple and natural that it can reasonably be taken to motivate such work, and indeed is plausibly a hint of important chiral physics involving gravity in the very early Universe.

Because $j_5^a$ is parity-odd, the rotations will interconvert E- and B-polarizations of gravitational waves, and a characteristic prediction of this effect would be the creation of E-B correlations from E-E or B-B ones. There would be a corresponding correlation on the cosmic microwave background, because the electromagnetic polarization tensors respond directly to the gravitational waves.

There is another consequence which is less obvious but more primitive. In most conventional models, there are important restrictions, not just on the probability distributions, but on the subspace of admissible data for primordial gravitational waves. These restrictions appear most explicitly in inflationary arguments, where the waves are supposed to be squeezed into a particular subspace, which then serves as an initial-data space for waves in a hot big bang. Albrecht [1] called this restriction *cosmic coherence*. It is used very strongly in the standard models of the fluctuations in the cosmic microwave background, where the gravitational waves’ influence is expressed by transfer functions responding to the waves’ initial values (but not their time-derivatives).

It turns out that the form of the gravitational gyrotropy breaks this coherence. That is, initially coherent waves, propagating through a chiral density, will decohere. Put another way, if we examine the waves at this later point, and imagine running them back in time through a Universe without a chiral density, they would appear not to have come from the squeezed subspace. This difference would in principle be detectable if the gravitational waves could be measured. It also will affect the cosmic microwave background: working out the
details of this requires the complementary transfer functions mentioned above.

I briefly discuss one more consequence in principle of the gyrotropy. This is that it provides a mechanism for the generation B-mode Bondi shear whenever there are memory effects; this B-mode shear can be thought of as the specific (per unit mass) spin angular momentum of gravitational radiation emitted from a system [2].

The organization of the paper deserves a little comment.

Evidently special properties of fermi fields underlie all the effects discussed here, and evidently what must enter (since the fields couple to gravity via Einstein’s equation) is the fermionic stress–energy, that is, certain tensorial formulas derived from spinorial expressions. However, once these results are in hand the spinors are not explicitly needed. Because of this, and because such derivations are rather technical and specialized, all of the explicitly spinorial computations are placed in an appendix, which can be omitted by the reader willing to simply accept its results. For readers interested in the technicalities, it is worth noting that the computation is done by two-spinor techniques, which are here more efficient than four-spinors and gamma matrices. The relation between the approaches is given explicitly. The appendix can be read after section 2.2.

Section 2 explains the mechanism driving the rotation (making use of the results of the appendix). The details of the computations are not used subsequently, so the reader interested primarily in the results can safely skim this. However, this is where the distinction between gyrotropy and birefringence is explained.

In section 3, I apply the results to a simple \( k = 0 \) Friedmann–Robertson–Walker cosmological model, where primordial gravitational waves encounter a chiral density in the very early Universe. Section 4 gives a summary and discussion.

Conventions. The metric has signature \(+−−−\); tensorial indices are small Latin letters; the indices \( j, k \) are reserved for three-tensors. The curvatures obey \( [\nabla_a, \nabla_b] \nu^d = R_{abc} \nu^d \), \( R = R_{ab} b^a \) and Einstein’s equation is \( G_{ac} = R_{ac} - (1/2)Rg_{ac} = -8\pi GT_{ac} \), with \( G \) being Newton’s constant, the stress–energy being \( T_{ac} \) (and any cosmological constant is absorbed in \( T_{ac} \)). Factors of \( c \), the speed of light, are suppressed in most places.

The tensorial conventions all accord with those in Penrose and Rindler [3, 4]. Spinors only appear explicitly in the appendix, and conventions for them are discussed there.

2 Mechanism

In this section the rotation of the polarization of gravitational waves will be derived. While some of the calculations are lengthy and technical, the main idea is straightforward.

While in the most general circumstances there is no simple way to isolate the wavelike degrees of freedom of a gravitational system from others, in many situations it is legitimate to model the waves as (relatively) rapidly changing linear perturbations on a given background. If we consider a system with matter as well as gravity, it will be described by a set of coupled equations:

\[
\begin{align*}
G_{ac} &= -8\pi GT_{ac} \\
\text{equations of motion for the matter}
\end{align*} \right\},
\]

where \( G_{ac} \) is the Einstein tensor and \( T_{ac} \) is the stress–energy (including any cosmological constant term). The first-order perturbations of this will have the form

\[
\begin{align*}
\delta G_{ac} &= -8\pi G\delta T_{ac} \\
\text{first-order perturbed matter equations of motion}
\end{align*} \right\}.
\]
In general these are coupled. Here $\delta G_{ac}$ is a second-order wave operator acting on the metric perturbation $\delta g_{ac}$. The quantity $\delta T_{ac}$ in general contains two sorts of terms: there are those $\delta_{\text{matter}} T_{ac}$ coming from perturbations of the matter fields figuring in $T_{ac}$, and there are those $\delta_{\text{g}} T_{ac}$ coming from the dependence of $T_{ac}$ on the metric. It is these second sorts of terms which are responsible for the novel effects.

What is special about fermions is that their stress–energies involve first derivatives of the metric. (By contrast, minimally coupled scalar fields’, and electromagnetic and Yang–Mills fields’, stress–energies only depend on the metric algebraically.) This means that the right-hand side of the top line of (2.2) contributes first-derivative terms to the wave equation for $\delta g_{ac}$. Since the highest- (second-) order terms are unaffected, the characteristics of the system are null hypersurfaces, as usual, and the wave profiles can be taken (exploiting the gauge freedom) to be transverse in the usual sense. Since the characteristics are null, the waves propagate at the speed of light. However, the transport equations, which govern the propagation of those profiles along the null geodesics ruling the characteristics, are affected by the first-order terms. The coefficient of the first-order term involves the chiral current $j^5_a$.

One can see even from this outline that what is essential here are the fermion ‘kinetic’ terms in the stress–energy, as those are the ones depending explicitly on the metric’s derivatives. The different sorts of mass terms do not enter explicitly in this argument. (They contribute subdominant corrections to the geometric-optics approximation.)

We have not so far restricted the matter terms, and one could imagine different first-order perturbations of the matter (compatible with the high-frequency character of the metric perturbations). However, we assume further that the matter is unperturbed except for its response to the gravitational waves. (Formally speaking, the matter perturbations evolve via retarded Green’s functions; equivalently, the matter perturbations vanish in the distant past.) To make this retardation explicit, we should formally work with wave-packets. However, we will not do so; we shall only use Fourier modes and simply bear in mind that really one must integrate over a family of them.

The distinction between gyrotropy and birefringence can be made at this point. Birefringence generally refers to different indices of refraction for different polarizations; mathematically, this corresponds to polarization-dependent highest-order terms in the propagation equation, which govern the speed of the wave. A circular birefringence thus will evidently give rise to a rotation of polarization, that is, a gyrotropy. However, we see that the gyrotropy here does not come about for this reason, because the highest-order terms are unchanged. Gyrotropy without birefringence is possible because the waves are transverse.

### 2.1 First-order perturbations of geometry

We consider linear perturbations of an arbitrary ‘background’ space–time with metric $g_{ab}$. If this is perturbed by $\delta g_{ab} = h_{ab}$, the change in the connection is

$$\delta \Gamma^{d}_{bc} = \frac{1}{2} \left[ \nabla_{b} h^{d}_{c} + \nabla_{c} h^{d}_{b} - \nabla^{d} h_{bc} \right].$$

This gives a change

$$\delta R_{ac} = R_{p(a} h^{p}_{c)} - R_{(a|b|c)} h^{b}_{q} h^{q}_{a} + \frac{1}{2} \left( \nabla_{a} \nabla_{c} h - 2 \nabla_{(c} \nabla_{|b|} h_{a)}^{b} + \nabla^{2} h_{ac} \right),$$

---

1 The discussion has been given here, as is common, in terms of the equations on the background describing the perturbations. This is not entirely satisfactory, as the propagation of the waves should really be treated in a gauge-invariant manner. An invariant way of phrasing the statements is that the full gravity–plus–fermion system is a quasilinear one whose highest-order term is set by the metric (and not its derivatives).
in the Ricci tensor and
\[ \delta G_{ab} = R_{p(a}h_{b)c}^p - R_{(a(b)c)d}h^{pb} + \frac{1}{2} \left( \nabla_a \nabla_c h - 2\nabla_c \nabla \left( \frac{1}{2} \nabla h_a \right)^b + \nabla^2 h_{ac} \right) \]
\[ - \frac{1}{2} (h_{ac} R - g_{ac} R_{pq} h_{pq}) + \frac{1}{2} g_{ac} \left( -\nabla^2 h + \nabla^p \nabla^q h_{pq} \right) \]
(2.5)
in the Einstein tensor, where \( h = h_a^a \). (Here vertical strokes set off postscripts not included in the symmetrizations.)

2.2 High-frequency expansion

We now make the high-frequency expansion. This will be done in an invariant way, although it will depend on a choice of a foliating family of smooth null hypersurfaces, which will be the wave-fronts. Let these hypersurfaces be the level sets of a function \( u \) (increasing towards the future, and having dimension time). Then \( l_a = \nabla_a u \) is the null normal, and one has \( l^b \nabla_b l^a = 0 \), so \( l^a \) is the parallel-propagated null tangent to the null geodesics ruling the hypersurfaces.

It is convenient to complete \( l^a \) to a normalized, oriented, null tetrad \( l^a, m^a, \bar{m}^a, n^a \), where
\[ l^a n_a = -m^a \bar{m}_a = 1 \]
(2.6)
(and all other inner products among them vanish). The orientation is such that, at any point we could choose a standard basis with
\[ l^a = 2^{-1/2} (\partial_t + \partial_z) \]
\[ n^a = 2^{-1/2} (\partial_t - \partial_z) \]
\[ m^a = 2^{-1/2} (\partial_x - i \partial_y) \]
(2.7)
(2.8)
(2.9)
We may choose this basis to be transported parallel along \( l^a \). (It is not quite uniquely determined; one could perform a null rotation about \( l^a \). Note that the \( x, y, z \) figuring here could not generally be comoving coordinates in a cosmological application.)

We put
\[ \delta g_{ab} \sim e^{-i\omega u} \left( h_{ab}^{(0)} + \frac{h_{ab}^{(1)}}{-i\omega} + \frac{h_{ab}^{(2)}}{(-i\omega)^2} + \cdots \right) + \text{conjugate}; \]
(2.10)
we are interested in this in the limit of large \( \omega \). We remark that utilizing a covector field
\[ \xi_a \sim e^{-i\omega u} \left( \xi_a^{(0)} + \cdots \right) + \text{conjugate}, \]
(2.11)
we may make gauge transformations
\[ 2\nabla_a (\xi_b) \sim 2e^{-i\omega u} \left( -i\omega \xi_a^{(0)} + \nabla (\xi_a^{(0)}) + l_a \xi_b^{(1)} + \cdots \right) + \text{conjugate}, \]
(2.12)
and thus by taking \( \xi_a^{(0)} = 0 \) and adjusting \( \xi_a^{(1)} \), we may eliminate any term in \( h_{ab}^{(0)} \) proportional to a symmetrization of \( l_a \) with another covector.

We now expand the Einstein tensor and the stress–energy in inverse powers of \( \omega \). For the Einstein tensor, we get
\[ \delta G_{ab} = e^{-i\omega u} \left( G_{ab}^{(-2)} (-i\omega)^2 + G_{ab}^{(-1)} (-i\omega) + \cdots \right) + \text{conjugate}. \]
(2.13)
The stress–energy is treated in the appendix.
2.2.1 The order \((-i\omega)^2\) term

As noted above, the order \((-i\omega)^2\) term for the perturbed stress–energy vanishes, and then the leading term in eq. (2.13) gives an algebraic restriction on \(h_{ab}^{(0)}\). Precisely, we find

\[
\delta G_{ac}^{(-2)} = \frac{1}{2} \left( l_a l_c h^{(0)} - 2l_a l_b h_c^{(0)b} + g_{ac}(p^p h_{pq}^{(0)}) \right)
\]

must vanish, and it is an algebraic exercise to verify this forces \(h_{ab}^{(0)}\) to be transverse and trace-free (modulo gauge). We may therefore discard gauge terms and write

\[
h_{ab}^{(0)} = \phi m_a m_b + \psi \overline{m}_a \overline{m}_b,
\]

for some functions \(\phi, \psi\). (Here \(\phi\) and \(\psi\) are not scalar metric perturbations, but the spin-weighted coefficients of tensor perturbations.)

Here note that, in terms of the basis (2.7)–(2.9) (which, recall, is not comoving), we have

\[
m_a m_b = \frac{1}{2} [(dx dx - dy dy) - i(dx dy + dy dx)]
\]

expressed the polarization in terms of the usual \(e_+ = (dx dx - dy dy)\) and \(e_\times = (dx dy + dy dx)\). (This is the left circular polarization tensor in the sense of Misner, Thorne and Wheeler [5], but would be the gravitational analog of right circular optical polarization in the sense of Jackson [6]. The contribution of the \(e^{-i\omega \phi} m_a m_b\) term to the Weyl tensor is left-handed in the usual particle-physics sense, and that of \(e^{-i\omega \psi} \overline{m}_a \overline{m}_b\) is right-handed.)

2.2.2 The order \((-i\omega)\) term

The next, order \((-i\omega)\), term in the expansion is more complicated. The equation

\[
G_{ac}^{(-1)} = -8\pi GT_{ac}^{(-1)}
\]

is a system of ten scalar equations involving \(h_{ac}^{(0)}\) and \(h_{ac}^{(1)}\). We shall in this section only consider the left-hand side.

Explicitly, we have

\[
G_{ac}^{(-1)} = \frac{1}{2} \left( \nabla_a (l_c h^{(0)}) + l_a \nabla_c h^{(0)} - 2(\nabla_a l_b h_c^{(0)b} + l_a \nabla_b h_c^{(0)b}) + \nabla^p (l_p h_{ac}^{(0)}) + l_p \nabla^p h_{ac}^{(0)} \right)
\]

\[
+ \frac{1}{2} g_{ac} (\nabla_p (p^p h^{(0)}) - l_p \nabla^p h^{(0)} + \nabla^p (l^p h_{pq}^{(0)}) + p^p \nabla q h_{pq}^{(0)})
\]

\[
+ \frac{1}{2} \left( l_a l_c h^{(1)} - 2l_a l_b h_c^{(1)b} + g_{ac}(p^p h_{pq}^{(1)}) \right).
\]

Note that the last line here has the same form as eq. (2.14), but with \(h_{ab}^{(1)}\) replacing \(h_{ab}^{(0)}\). This means that, if \(T_{ab}^{(-1)}\) were known, one could solve certain components of eq. (2.17) algebraically for \(h_{ab}^{(1)}\) (modulo gauge) in terms of \(h_{ab}^{(0)}\) and \(T_{ab}^{(-1)}\). In fact, it is not hard to check, given the form (2.15) of \(h_{ab}^{(0)}\), that the only terms in (2.18) that cannot be influenced by the choice of \(h_{ab}^{(1)}\) are those proportional to \(m_a m_c\) and \(\overline{m}_a \overline{m}_c\). These will turn out to give us the transport equations.

We have, then, for the left-hand sides of the transport equations,

\[
\overline{m}^a \overline{m}^c G_{ac}^{(-1)} = l \cdot \nabla \phi + (1/2)(\nabla \cdot l) \phi
\]

\[
m_a m_c G_{ac}^{(-1)} = l \cdot \nabla \psi + (1/2)(\nabla \cdot l) \psi.
\]
2.3 The transport equations

We may now assemble our results to get the transport equations. These are the $m^am^a$ and $\overline{m}^a\overline{m}^a$ components of the next-to-leading order term in the high-frequency expansion of the Einstein equation $G_{ac}^{(-1)} = -\pi GT_{ac}^{(-1)}$. These components of the Einstein tensor were derived in the previous subsection; of the the stress–energy, in the appendix.

The transport equations are:

$$l \cdot \nabla \phi + (1/2)(\nabla \cdot l) \phi = +4\pi G l \cdot j^5 \phi \quad (2.21)$$

$$l \cdot \nabla \psi + (1/2)(\nabla \cdot l) \psi = -4\pi G l \cdot j^5 \psi. \quad (2.22)$$

We have thus, for the evolution of the perturbations along the geodesic,

$$\phi(\gamma(s)) = \phi(\gamma(s_0)) \exp \int_{s_0}^{s} \left[-(1/2)\nabla \cdot l + 4\pi G l \cdot j^5\right] ds \quad (2.23)$$

$$\psi(\gamma(s)) = \psi(\gamma(s_0)) \exp \int_{s_0}^{s} \left[-(1/2)\nabla \cdot l - 4\pi G l \cdot j^5\right] ds. \quad (2.24)$$

In these, the real factor $\nabla \cdot l$ gives the usual amplification or attenuation due to the convergence or divergence of the null geodesics. However, it is the phase factors which are of interest here.

Thus the gravitational wave would have the coefficients of its polarization tensors $m_a m_a$, $\overline{m}_a \overline{m}_a$ rotated by $4\pi G \int j^5_a d\gamma^a$ (writing now $l^a ds = d\gamma^a$), which is the effect a rotation by $2\pi G \int j^5_a d\gamma^a$, looking along the axis of propagation into the oncoming wave, would have. Of course, one should bear in mind that only the total metric $g_{ab} + \delta g_{ab}$ is observable, so it is typically the differences in rotations, from one geodesic to another, which are of significance.

3 Gravitational gyrotropy in cosmology

In this section, the foregoing results are applied to perturbations of a $k = 0$ Friedmann–Robertson–Walker space–time. We will suppose that very early in the Universe, a stochastic distribution on such perturbations existed, with no chiral current or other material contribution to the waves’ transport equations. However, we will suppose that at some point, later but still very early, a chiral current developed, and then was rapidly diluted by the expansion of the Universe. Thus the chiral current will be approximated by a delta-function time-dependence. For simplicity and clarity, we work only to first order in the chiral current.

It is straightforward to adapt the analysis developed in the previous section to this situation. (Because, however, it was developed in the context of general space–times, there are some minor differences of convention in how terms are grouped compared to those common in cosmology.)

3.1 Preliminaries

We will now use comoving coordinates $x^j$, and a positive-definite standard Euclidean three-metric $\varepsilon_{jk}$ used to raise and lower Euclidean indices; for most purposes we use a three-vector notation. We usually use a conformal time $\eta$, although in some places physical time $t$ is more natural. The metric is then

$$ds^2 = a(\eta)^2 \left(d\eta^2 - \varepsilon_{jk} dx^j dx^k\right) \quad (3.1)$$

$$= dt^2 - a^2 \varepsilon_{jk} dx^j dx^k. \quad (3.2)$$
For the perturbations, we will take the circular frequency \( \omega = k \geq 0 \) and the phase function

\[
 ku = k(\eta - \hat{k} \cdot \mathbf{x}),
\]

where \( \mathbf{k} = k\hat{k} \) is the spatial wave-vector (hats indicate comoving unit vectors). Then the tangent covector is \( l_a = \nabla_a u = d\eta - \hat{k} \cdot dx \) and \( l^a = a^{-2} \left( \partial_\eta + \hat{k} \cdot \nabla \right) \).

We now take up the polarizations. We will have

\[
 m_a m_b = \frac{1}{2} a^2 (u_j u_k - v_j v_k - i(u_j v_k + v_j u_k)) dx^j dx^k
\]

for suitable constant unit vectors \( u, v \) (with \( u, v, \hat{k} \) forming a Euclidean right-handed comoving-orthonormal triad). Thus we could write

\[
 m_a m_b = \frac{1}{2} a^2 (\epsilon^a_{ab} - i\epsilon^a_{ab}),
\]

in terms of the plus and cross polarizations adapted to the comoving frame.

There is a slight subtlety here, which is the dependence of the polarization tensors on the vector \( \mathbf{k} \) and its consequent behavior under the inversion \( \mathbf{k} \rightarrow -\mathbf{k} \), which will be important shortly. On one hand, there must be a dependence (for the polarization is orthogonal to \( \mathbf{k} \)); on the other, there is, for each \( \mathbf{k} \), the freedom to make a phase rotation in \( m_a \). There is no uniform choice in the literature for this. We follow Hu and White [7], who choose \( u \) proportional to \( \partial_\theta \) and \( v \) proportional to \( -\partial_\phi \) on the sphere, which gives

\[
 m_a m_b(-\mathbf{k}) = m_a m_b(\mathbf{k}),
\]

and makes \( \epsilon^a_{ab}(\mathbf{k}) \) parity-even and \( \epsilon^a_{ab}(\mathbf{k}) \) parity-odd. It should be emphasized that this is simply a matter of convention, and that one could equally well include a minus sign (or direction-dependent phase factor). What will matter in the end is, of course, the products of the polarization tensors with the mode functions. However, just for this reason, this convention does very much affect the parities of those mode functions.

We will model the chiral current by a delta-function in time:

\[
 j_5^a = t^a n_5(\mathbf{x}) \delta(t - t_5),
\]

so that \( n_5(\mathbf{x}) \) is the impulse (with respect to physical time) of chiral density (with respect to physical volume), acting at physical time \( t_5 \). With \( \eta_5 \) the corresponding conformal time, the impulse with respect to physical time of chiral density with respect to comoving volume would be \( (a(\eta_5))^3 n_5(\mathbf{x}) \). It is worth noting that the chiral current need not be timelike (and its causal character could be a function of position). The case where it is proportional to \( t^a \) is just the simplest to work out.

### 3.2 Transport effects

We now turn to the integration of the transport equations, (2.23), (2.24). We have \( \nabla_a t^a = 2a'/a^3 \) (with the prime denoting differentiation with respect to \( \eta \)) but \( ds = a^2 d\eta \), and so

\[
 \exp -\frac{1}{2} \int \nabla_a t^a ds = a^{-1}
\]
for the transport factor accounting for the divergence of the congruence. We may therefore take, in the era before a chiral current arises,

\[ \phi(k, \eta) = a^{-1}(\eta) \lambda(k), \quad \psi(k, \eta) = a^{-1}(\eta) \rho(k), \quad (3.9) \]

where \( \lambda \) and \( \rho \), depending only on \( k \), may be regarded as giving the initial left- and right-circularly polarized densities of gravitational waves.

We now take up the effects of the waves having passed through the chiral density. From eqs. (2.23), (2.24), we get first-order changes in \( \lambda \) and \( \rho \) given by

\[ \delta_5 \lambda = + \left( 4 \pi i G \int l \cdot j_5 \, ds \right) \lambda \]

\[ \delta_5 \rho = - \left( 4 \pi i G \int l \cdot j_5 \, ds \right) \rho. \quad (3.10) \]

We may write \( \int l \cdot j_5 \, ds = \int j_5 \cdot d\gamma \), where \( \gamma \) is a null geodesic with tangent \( l^a \); the second form of the integral is manifestly independent of the parameterization of the geodesic. If \( (\eta, x) \) are the coordinates of a point in space–time, then the null geodesic starting from that point and directed along \( l^a \) will be \( \gamma(u) = (\eta + u, x + u \hat{k}) \) in a convenient, but not affine, parameterization. Then

\[ \int_{-\infty}^{0} n_5(x + u \hat{k}) \delta(t - t_5) \, du = n_5(x + (\eta_5 - \eta) \hat{k}), \quad (3.12) \]

where we have used \( a \, du = a \, d\eta = dt \) along \( \gamma \).

The form of the results at this point is quite simple, but this is in part because they are in a mixed direct space–Fourier space representation, the argument of \( n_5 \) involving both the comoving coordinate \( x \) and the Fourier direction \( \hat{k} \). To move further, and to connect with other work, we complete the passage to Fourier space.

3.3 Fourier space

The leading, order-\((-i\omega)^0\), term in the metric perturbation is

\[ \delta g_{ab} \sim 4 \pi i \int d^3 k e^{-i(k\eta - k \cdot x)} \left( \phi(k, \eta) m_a m_b + \psi(k, \eta) \bar{m}_a \bar{m}_b \right) + \text{conjugate} \]

\[ = \int d^3 k \left[ e^{-i k \eta} \phi(k, \eta) + e^{i k \eta} \phi(-k, \eta) \right] m_a m_b(k) 
+ \left( e^{-i k \eta} \psi(k, \eta) + e^{i k \eta} \psi(-k, \eta) \right) \bar{m}_a \bar{m}_b(k) \right] \]

\[ \quad \text{(3.13)} \]

where the term in square brackets gives the spatial Fourier transform, as usual. In the era before the current, this form applies with \( \phi(k, \eta), \psi(k, \eta) \) given by eq. (3.9). After that, however, there will be an additional contribution to eq. (3.13) given by

\[ \delta_5 g_{ab} \sim 4 \pi i \int d^3 k e^{-i(k\eta - k \cdot x)} a^{-1} \left( \lambda(k)n_5(x + (\eta_5 - \eta) \hat{k}) m_a m_b 
- \rho(k)n_5(x + (\eta_5 - \eta) \hat{k}) \bar{m}_a \bar{m}_b \right) + \text{conjugate}. \quad (3.14) \]

The idea now will be to Fourier transform this with respect to the spatial variables, in order to derive a scattering formula for the gravitational waves in wave-vector space. In principle,


there are various technical complications which can arise in such a calculation (related to
gauge freedom), but we shall see that in the high-frequency limit there is a simple result.

The spatial Fourier transform is

\[
(2\pi)^{-3} \int \delta_3 \gamma_{ab} e^{-ik \cdot x} d^3 x = 4\pi i a^{-1} \int d^3 k [ e^{-i k \cdot n_5 (\hat{k} - k)} e^{ik \cdot (\hat{k} - k)(\eta_s - \eta)} (\lambda(k) m_a m_b - \rho(k) \overline{m_a \overline{m_b}}) + e^{ik \cdot \overline{n}_5 (\hat{k} + k)} e^{-i k \cdot (\hat{k} + k)(\eta_s - \eta)} (\lambda(k) m_a \overline{m_b} - \overline{\rho(k) m_a m_b}) ]
\]

\[
= 4\pi i a^{-1} \int d^3 k [ e^{-i k \cdot \hat{n}_5 (\hat{k} - k)} e^{i k \cdot (\hat{k} - k)(\eta_s - \eta) + i k \cdot \eta_s} (\lambda(k) m_a m_b - \rho(k) \overline{m_a \overline{m_b}}) + e^{i k \cdot \overline{n}_5 (\hat{k} + k)} e^{-i k \cdot (\hat{k} + k)(\eta_s - \eta) - i k \cdot \eta_s} (\lambda(k) m_a \overline{m_b} - \overline{\rho(k) m_a m_b}) ]
\]

\[
= 4\pi i a^{-1} \int d^3 k [ e^{-i k \cdot \hat{n}_5 (\hat{k} - k)} e^{i (\hat{k} \cdot \overline{k}) (\eta_s - \eta) + i (\overline{k} \cdot k) \eta_s} (\lambda(k) m_a m_b - \rho(k) \overline{m_a \overline{m_b}}) + e^{i k \cdot \overline{n}_5 (\hat{k} + k)} e^{i (\overline{k} \cdot k) (\eta_s - \eta) - i (\overline{k} \cdot \overline{k}) \eta_s} (\lambda(k) m_a \overline{m_b} - \overline{\rho(k) m_a m_b}) ]
\]

where \( \hat{n}_5 \) is the Fourier transform of \( n_5 \). In each case, here, the polarization factors \( m_a m_b \), \( \overline{m_a \overline{m_b}} \) are taken at \( \hat{k} \).

Eq. (3.15) represents a single Fourier mode of the perturbation, and should therefore have the same form as the factor in square brackets in eq. (3.13), for some different \( \phi \) and \( \psi \) and with the variable \( \hat{k} \) in eq. (3.15) corresponding to \( \hat{k} \) in eq. (3.13). This does not appear to be the case, for two reasons. First, since in this regime there is no chiral current, one would think that the \( \eta \)-dependence within the square brackets would simply be through factors \( e^{\pm i k \eta} \) (modulo gauge issues), but other \( \eta \)-dependent factors appear as well. Second, the polarizations \( m_a m_b(\hat{k}) \), \( \overline{m_a \overline{m_b}(\hat{k})} \) in eq. (3.15) ought to be recast in terms of \( m_a m_b(\hat{k}) \), \( \overline{m_a \overline{m_b}(\hat{k})} \) (also modulo gauge). These two concerns do not cancel each other out. The resolution is different.

The issue is that we are applying the high-frequency limit not to a single wave but to a distribution of them weighted by \( \lambda(k) \) and \( \rho(k) \), and some sort of uniformity in the expansion must be assumed. This enters in the expressions above in the factors \( \hat{n}_5 (\hat{k} \mp \hat{k}) \). Since the basic assumption of the high-frequency limit is that the gravitational waves vary more rapidly than do quantities like the chiral density, the Fourier transform \( \hat{n}_5 \) cannot be supported significantly for very large wave-vectors; it must fall off. But since \( k \) is very large in the high-frequency expansion, we must have \( \hat{k} \) relatively close to \( \pm k \).

To proceed quantitatively, we will assume \( \| \hat{k} \mp k \| \) is \( O(1) \) in \( \| k \| \) where \( \hat{n}_5 (\hat{k} \mp k) \) has significant support. Then (taking the upper sign, for simplicity) we have

\[
\| \hat{k} \| = \| k + (\hat{k} - k) \| = \| k \| + \hat{k} \cdot (\hat{k} - k) + O(\| k \|^2) = \hat{k} \cdot \hat{k} + O(\| k \|^2)
\]

(3.16)

\[
\| \hat{k} \| - \hat{k} \cdot \hat{k} = O(\| k \|^{-1})
\]

(3.17)

\[
\| \hat{k} - \hat{k} \| = \sqrt{2 - 2 \hat{k} \cdot \hat{k}} = O(\| k \|^{-1}),
\]

(3.18)

where the last line follows from the first. The first two lines allow us to simplify the exponents in eq. (3.15); the third justifies neglecting, to leading order, discrepancies in polarizations.
relative to $k$ versus $\hat{k}$. With these simplifications, the formula (3.15) becomes

$$
(2\pi)^{-3} \int \delta_{5a} \delta_{5b} e^{-ik \cdot x} d^3x = 4\pi i a^{-1} \int d^3k \left[ e^{-ik\eta_5 (\hat{k} - k)} e^{i(k-h)\eta_5} (\lambda(k)m_a m_b - \rho(k)\overline{m}_a \overline{m}_b) + e^{ik\eta_5 (\hat{k} + k)} e^{i(k-h)\eta_5} (\lambda(k)\overline{m}_a m_b - \rho(k)\overline{m}_a m_b) \right].
$$

(3.19)

Thus the effect is (appropriately relabeling)

$$
\delta_5 \lambda(k) = 4\pi i \int d^3\hat{k} \hat{n}_5 (k - \hat{k}) e^{i(k-h)\eta_5} \lambda(\hat{k})
$$

(3.20)

$$
\delta_5 \rho(k) = -4\pi i \int d^3\hat{k} \hat{n}_5 (k - \hat{k}) e^{i(k-h)\eta_5} \rho(\hat{k}).
$$

(3.21)

Evidently the effect of the chiral density is to smear out the wave-packets defined by the distributions $\lambda(k), \rho(k)$ by an amount $\sim n_5(x - (\eta - \eta_5))\hat{k}$.

This smearing-out would contribute a correlation between different Fourier modes of each chirality (but no correlations of opposite chiralities).

### 3.4 Effects on primordial waves

Much contemporary cosmology supposes that at some point in the very early Universe a stochastic background of gravitational waves was generated. Such waves, if they subsequently passed through a chiral density, should be subject to the effects described above. In particular, if (as is usually supposed) the primordial waves were not chirally correlated (no E–B correlations), passage through a chiral density would result in an E–B correlation developing, for it would mean that a wave which later appeared as an E–mode originated with some B–components. This would in turn give rise to an E–B correlation in the cosmic microwave background.

However, this correlation cannot, quite, be computed from the standard formalism in use today. This is because there is another consequence, arguably more primitive than gyrotropy, of the waves’ passage through the chiral density. T is a violation of what Albrecht called cosmic coherence, the set of restrictions which the allowed perturbations are supposed to obey. These restrictions are the ones determined by squeezing in inflationary theories. They are sometimes called initial conditions, although they are really initial only for hot big bang era; in this sense, we may say that the effective initial conditions are changed.

#### 3.4.1 Initial data and ‘cosmic coherence’

To make this clear and see how to proceed, it will be helpful to connect with some of the standard notation figuring in treatments of the CMB. We introduce mode functions $H^{(\pm 2)}$ compatible with the notation of Hu and White [7]:

$$
H^{(2)}(k, \eta) = \sqrt{2/3} \left( e^{-ik\eta} \psi(k, \eta) + e^{ik\eta} \overline{\psi}(-k, \eta) \right),
$$

(3.22)

$$
H^{(-2)}(k, \eta) = \sqrt{2/3} \left( e^{-ik\eta} \phi(k, \eta) + e^{ik\eta} \overline{\phi}(-k, \eta) \right).
$$

(3.23)

These are to be understood as stochastic variables. (Hu and White did not work explicitly with stochastic variables, instead using mode functions of the scalars $(k, \eta)$ and probability distributions; the explicit representation is due to Seljak and Zaldarriaga [8], although not in quite this notation.)
In inflationary theory, during the inflationary period the modes freeze (and are not wavelike gravitational excitations), and then begin to change again after inflation ends. This condition may be expressed as

$$\partial_\eta H^{(\pm 2)}(\mathbf{k}, \eta) \bigg|_{\eta = \eta_e} = 0,$$  \hspace{1cm} (3.24)

where $\eta_e$ is the conformal time at which inflation ends. This is a very strong constraint on the data for the waves (and not just on the probability distribution). It restricts them to a particular Lagrange subspace. I shall show shortly that if the waves subsequently pass through a chiral density, the emerging wave cannot generally be produced by a wave satisfying (3.24) and not passing through a chiral region. However, first a little more discussion of this condition is in order.

When inflation ends, some sort of pre-heating and reheating is supposed to occur, giving a transition to a radiation-dominated expansion, at, say $\eta_{\text{rad}}$. In the transition period $\eta_e \leq \eta \leq \eta_{\text{rad}}$, in principle, the gravitational waves could be significantly affected by the precise nature of the space–time through which they pass: there is some evolution taking the data $(H^{(\pm 2)}(\mathbf{k}, \eta_e), \partial_\eta H^{(\pm 2)}(\mathbf{k}, \eta_e))$ at $\eta_e$ to those at $\eta_{\text{rad}}$, which depends on the physics of the transition.

In practice, it is usually assumed that the data at $\eta_{\text{rad}}$ can be set by a limiting case of eq. (3.24). That is, one works with a radiation-dominated model beginning at conformal time $\eta_{\text{rad}} = 0$, and takes the initial data for the gravitational waves to be constrained by

$$\lim_{\eta \downarrow 0} \partial_\eta H^{(\pm 2)}(\mathbf{k}, \eta) = 0.$$  \hspace{1cm} (3.25)

This assumption is used very strongly in contemporary modeling, and is the justification for working with transfer functions giving the response of temperature and polarization to the (limiting) initial data for the value of the tensor metric perturbation and ignoring the transfer functions for the response to the (limiting) initial value for its time-derivative.

We will need to investigate the consequences of abandoning this assumption. To do that we must look at the singularity $\eta \downarrow 0$ a bit more closely, but at the same time it is important to remember that we do not regard the hot big bang model as applying literally in this domain. Really the model is physically applicable for $\eta$ larger than some positive value, and we extend the model backwards to $\eta \downarrow 0$ simply as a mathematical convenience.

In fact, in the radiation-dominated era the high-frequency expansion is a good one, and so we have $\phi(\mathbf{k}, \eta) = \lambda(\mathbf{k})/a(\eta)$, $\psi(\mathbf{k}, \eta) = \rho(\mathbf{k})/a(\eta)$ with $a(\eta) \simeq \eta/2$, and this means that the mode functions (3.22), (3.23) will diverge as $\eta \downarrow 0$ unless the restriction (3.25) holds. Were we to take this model as physics all the way back to $\eta = 0$, we could interpret this restriction as a consistency condition, for divergent perturbations as $\eta \downarrow 0$ would not be allowed. However, because other physics is supposed to take over for small positive $\eta$, this argument does not apply. We must admit data violating (3.25).

Because the modes generally diverge as $\eta^{-1}$ as $\eta \downarrow 0$, it is easiest to work with $\eta H^{(\pm 2)}(\mathbf{k}, \eta)$,
and identify the initial data

\[
\alpha^{(2)}(k) = \lim_{\eta \to 0} \eta H^{(2)}(k, \eta) = 2\sqrt{2/3}(\rho(k) + \overline{\rho}(-k))
\]

\[
\beta^{(2)}(k) = \lim_{\eta \to 0} \eta \partial_\eta H^{(2)}(k, \eta) = -4\sqrt{2/3}ik(\rho(k) - \overline{\rho}(-k))
\]

\[
\alpha^{(-2)}(k) = 2\sqrt{2/3}(\lambda(k) + \overline{\lambda}(-k))
\]

\[
\beta^{(-2)}(k) = -8\sqrt{2/3}ik\lambda(k).
\]

Note that knowledge of these functions is equivalent to knowledge of \(\lambda\) and \(\rho\). The usual restriction (implication of 'cosmic coherence') is that \(\alpha^{(\pm 2)}\) should vanish; the data are then taken to be \(\beta^{(\pm 2)}\), and these accord with the limiting values of \(H^{(\pm 2)}\) in the case \(\alpha^{(\pm 2)} = 0\). However, we shall want to consider possible non-zero values of \(\alpha^{(\pm 2)}\).

### 3.4.2 The effects of a chiral density

We are now in a position to set up the computation for the effects of primordial waves passing through a chiral density.

We assume that the waves initially do satisfy the cosmic coherence condition. Then in terms of the initial data \(\alpha^{(\pm 2)}, \beta^{(\pm 2)}\) of the previous subsubsection, we have

\[
\alpha^{(\pm 2)} = 0
\]

\[
\beta^{(2)}(k) = -8\sqrt{2/3}ik\rho(k)
\]

\[
\beta^{(-2)}(k) = -8\sqrt{2/3}ik\lambda(k).
\]

However, the waves then pass through the chiral density, which induces changes \(\delta_5\lambda, \delta_5\rho\) according to eqs. (3.20), (3.21). These perturbations will lead to changes

\[
\delta_5\alpha^{(2)} = 2\sqrt{2/3}(\delta_5\rho(k) + \delta_5\overline{\rho}(-k))
\]

\[
= 2\sqrt{2/3}(-4\pi i) \left( \int d^3\hat{k} \hat{n}_5(k - \hat{k}) e^{i(k - \hat{k})\eta_5} \rho(k) - \int d^3\hat{k} \hat{n}_5(-k - \hat{k}) e^{-i(k - \hat{k})\eta_5}\overline{\rho}(\hat{k}) \right)
\]

\[
= 2\pi \int d^3\hat{k} \hat{n}_5(k - \hat{k}) \cos((k - \hat{k})\eta_5)\beta^{(2)}(\hat{k})/\hat{k}
\]

\[
\delta_5\alpha^{(-2)} = -2\pi \int d^3\hat{k} \hat{n}_5(k - \hat{k}) \cos((k - \hat{k})\eta_5)\beta^{(-2)}(\hat{k})/\hat{k}
\]

\[
\delta_5\beta^{(\pm 2)} = \pm 4\pi i k \int d^3\hat{k} \hat{n}_5(k - \hat{k}) \sin((k - \hat{k})\eta_5)\beta^{(\pm 2)}(\hat{k})/\hat{k}.
\]

To work out the effects of the waves on the cosmic microwave background, we must introduce transfer functions corresponding to the data \(\alpha^{(\pm 2)}\). If we let \(\Delta_X^{(\beta\pm 2)}(k)\) be the usual transfer function of type \(X\) (temperature, E- or B-mode polarization) and multipole \(l\), we
may set $\Delta^X_{l\alpha(\pm 2)}(k)$ as the corresponding transfer function for the $\alpha^{(\pm 2)}$ data. Then $X$ will be a stochastic variable

$$
\int d^3k \left( \sum_{\pm} \Delta^X_{l\alpha(\pm 2)}(\alpha^{(\pm 2)}(k) + \delta_5\alpha^{(\pm 2)}(k)) + \sum_{\pm} \Delta^X_{l\beta(\pm 2)}(\beta^{(\pm 2)}(k) + \delta_5\beta^{(\pm 2)}(k)) \right),
$$

and the expectations may be computed from this.

To compute the E-B correlation explicitly, let us assume that before the passage through the chiral density the primordial waves have $\alpha^{(\pm 2)} = 0$, and as usual $\beta^{(\pm 2)}$ with zero means and

$$
\langle \beta^{(\pm 2)}(k)\beta^{(\pm 2)}(\hat{k}) \rangle = \mathcal{P}(k)\delta(k + \hat{k})
$$

$$
\langle \beta^{(2)}(k)\beta^{(-2)}(\hat{k}) \rangle = 0.
$$

Because the E–B correlation would vanish in the absence of chiral effects, and because we work only to first order in those effects, and the terms with opposite spin-weights remain uncorrelated, we have

$$
C^E_{lB} = (2l + 1)^{-1} \sum_{\pm} \int d^3k d^3\hat{k} \times \left( \Delta^E_{l\alpha(\pm 2)} \Delta^B_{l\beta(\pm 2)} \delta_5\alpha^{(\pm 2)}(\hat{k}) + \Delta^E_{l\beta(\pm 2)} \Delta^B_{l\alpha(\pm 2)} \beta^{(\pm 2)}(k)\delta_5\alpha^{(\pm 2)}(\hat{k}) \right) + \Delta^E_{l\beta(\pm 2)} \delta_5\alpha^{(\pm 2)}(\hat{k}) \beta^{(\pm 2)}(k)\beta^{(\pm 2)}(\hat{k}) + \Delta^B_{l\beta(\pm 2)} \beta^{(\pm 2)}(k)\delta_5\alpha^{(\pm 2)}(\hat{k}) \beta^{(\pm 2)}(k).
$$

The expectations occurring follow from eqs. (3.33)–(3.35), (3.37), (3.38)

$$
\langle \delta_5\alpha^{(\pm 2)}(k)\beta^{(\pm 2)}(\hat{k}) \rangle = 2\pi\hat{\eta}_5(k + \hat{k}) \cos((k - \hat{k})\eta_5) \hat{k}^{-1}\mathcal{P}(\hat{k})
$$

$$
\langle \beta^{(2)}(k)\beta^{(\pm 2)}(\hat{k}) \rangle = \pm 2\pi\hat{\eta}_5(k + \hat{k}) \cos((k - \hat{k})\eta_5) k^{-1}\mathcal{P}(k)
$$

$$
\langle \delta_5\alpha^{(\pm 2)}(k)\beta^{(\pm 2)}(\hat{k}) \rangle = \pm 4\pi\hat{\eta}_5(-\hat{k} + k) \sin((k - \hat{k})\eta_5) \hat{k}^{-1}\mathcal{P}(\hat{k})
$$

$$
\langle \beta^{(2)}(k)\delta_5\beta^{(\pm 2)}(\hat{k}) \rangle = \pm 4\pi\hat{\eta}_5(k + \hat{k}) \sin((k - \hat{k})\eta_5) k^{-1}\mathcal{P}(k).
$$

These equations, together with eq. (3.39), give the E-B correlation. (Note that only the average of $\hat{\eta}_5(k + \hat{k})$ over the directions of $\hat{k}$ and $k$, for given magnitudes, will enter.)

The most striking thing about the E-B correlation found here is that it is nonlocal in Fourier space, in contrast to the standard formulas for the effects of gravitational waves. This nonlocality is not really very surprising, however, because, roughly speaking, the waves are scattering off the chiral density, and this has the effect of smearing them.

4 Discussion

The main result here is a remarkably clean formula for the gyrotropy of gravitational waves by a chiral column density $N_5$: a wave’s polarization is rotated by $2\pi\ell_5^2 N_5$. This is striking enough to suggest that there are important further connections between chiral physics and gravity, at very high energies or in the very early Universe. Of course, such connections have been suggested for a long time; what is different here is that the primary effect does not rely on postulating any exotic physics: it is a consequence of conventional general relativity.
This novel effect grows out of the fact that usual spin one-half stress–energy has a term depending on derivatives of the metric — unlike the behavior of other conventional elementary fields. This meant that the equations governing the transport of gravitational wave profiles along their geometric-optics trajectories respond to the presence of fermions. That coupling turns out to be to the chiral current \( j_{a}^{5} \), but with a Planck-area prefactor. Thus the effects involved are tiny unless very large chiral column densities \( N_{5} \) can be achieved.

Because \( j_{a}^{5} \) is parity-odd, this chiral coupling will interconvert E- and B-mode gravitational waves, and in particular a distribution of waves which initially had E-E or B-B correlations would develop E-B ones would develop them after passing through a chiral density. These would in principle be directly observable, if we were able to detect the gravitational waves. Because the couplings of primordial gravitational waves to the cosmic microwave background preserve spin-weights, they would produce E-B correlations in the CMB as well.

However, the passage of the gravitational waves through a chiral density will give other effects as well; in particular, it will disturb the ‘cosmic coherence’ which determines the subspace of initial conditions for gravitational waves within big-bang models. After the waves pass through the chiral density, they will acquire components in a complementary subspace. This means in particular that in order to compute the effects on the CMB we require transfer functions responding to those complementary degrees of freedom. With those, a formula for the correlation \( C_{EB}^{l} \) was derived.

It is worthwhile noting that gravitational gyrotropy has implications, in principle, for gravitational memory effects. These effects, which go back to Bondi [9] and have become best known through the paper of Christodoulou [10], concern a sort of gauge mismatch between two quiescent regimes at null infinity for an isolated system, bracketing an emission of gravitational radiation. The mismatch is characterized by the E-mode difference in the Bondi shears of the two regimes. However, it has long been known that in principle there may be a B-mode contribution as well. What the work here shows is that, if the waves pass through a chiral density on their way outwards, there will be contributions to the B-mode changes in consequence of the E-mode ones. The B-mode effects, unlike the E-mode ones, cannot be absorbed in changes of gauge; they contribute to the spin angular momentum of the gravitational radiation [2].

Acknowledgments

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A Perturbing a fermi field and its stress–energy

This appendix depends on some of the formulas derived in section 2.2.

A.1 Notation and conventions

The conventions for space–time and two-spinors are those of Penrose and Rindler [3, 4]. These books do not use four-component spinors in their analysis, and their conventions for two-component spinors and space–time are compatible with much other work. These books do give, in notes (see [3], p. 221 and [4], p. 460), formulas for converting to Dirac spinors,
but we shall not use those conventions, because they are adapted to an opposite sign for \( \gamma_a \gamma_b + \gamma_b \gamma_a \) from what is generally used in the quantum-field-theoretic literature. With our choices, our conventions for four-spinors conform to those of Schweber [14].

The space–time metric is \( g_{ab} \); it has signature \( + − − − \). The curvature tensors satisfy

\[
[\nabla_a, \nabla_b]v^d = R_{abc}^\, d v^c, \quad R_{aac} = R_{bac}, \quad R = R_a^a, \quad G_{ab} = R_{ab} - (1/2)Rg_{ab}.
\]

The volume form is \( \epsilon_{abcd} \) and we have \( \epsilon_{txyz} = +1 \) in a right-handed future-pointing orthonormal basis.

Two-spinors are denoted by symbols like \( \kappa^A \), conjugate spinors as \( \lambda_A' \). The conjugation map is denoted by an overbar:

\[
\kappa^A \rightarrow \overline{\kappa}^A.
\]

Spin-space is equipped with a non-degenerate skew form \( \epsilon^{AB} \), whose conjugate is denoted \( \epsilon^{A'B'} \).

Spinor and vector indices are related through the Infeld–van der Waerden symbols \( \sigma_a^{AA'} \), so that we put

\[
v^{AA'} = v^a \sigma_a^{AA'}, \quad \text{etc.}
\]

We have

\[
g_{ab} = \sigma_a^{AA'} \sigma_b^{BB'} \epsilon_{A'B'} \epsilon^{AB}.
\]

The spinor form of the alternating tensor \( \epsilon_{abcd} \) is

\[
i \epsilon^{AC} \epsilon^{BD} \epsilon^{A'\prime D'} - i \epsilon^{AD} \epsilon^{BC} \epsilon^{A'\prime C'}.
\]

A Dirac spinor will be represented by a pair of two-spinors:

\[
\psi = \begin{bmatrix} \psi^Q \\ \overline{\psi}_Q \end{bmatrix}.
\]

The Dirac symbols are given by

\[
\gamma_a = \sqrt{2} \begin{bmatrix} 0 & \sigma_a^{P'Q} \\ \sigma_a^{PQ} & 0 \end{bmatrix}
\]

and satisfy \( \gamma_a \gamma_b + \gamma_b \gamma_a = 2g_{ab} \). As noted above, these conventions for the relation between two- and four-component spinors, and for the Dirac gammas, differ from those of Penrose and Rindler. With the present conventions, we have an exact correspondence between the standard basis described on pp. 120–125 of Penrose and Rindler [3] or pp. 6–8 of [4] and the Weyl basis on p. 79 of Schweber [14].

The Dirac adjoint is

\[
\overline{\psi} = \begin{bmatrix} \overline{\psi}_Q' \\ \overline{\psi}^Q \end{bmatrix},
\]

and the Dirac current is thus

\[
j_{AA'} = \overline{\psi} \gamma_{AA'} \psi = \sqrt{2} (\overline{\psi}_{A'} \psi_A + \overline{\psi}_A \psi_{A'}) .
\]

More correctly, we should treat \( \psi \) as a fermi field operator and use the antisymmetrized form

\[
(1/2) [\overline{\psi}_{AA'}^{\prime}, \psi] = 2^{-1/2} (\overline{\psi}_{A'} \psi_A + \overline{\psi}_A \psi_{A'} - \psi_A \overline{\psi}_{A'} - \psi_{A'} \overline{\psi}_A) .
\]

This gives as usual the current of fermions minus antifermions.

We take

\[
\gamma_5 = \frac{1}{24} \epsilon_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}.
\]

(Many authors use \(-i\) times this.) The projectors to the right- (respectively, left-) handed spinors are

\[
(1/2) \begin{bmatrix} 1 & i \gamma_5 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

We define the chiral current as

\[
j^5_{\alpha} = -i \overline{\psi} \gamma_5 \gamma_{\alpha} \psi
\]

\[
= \overline{\psi} \gamma_{\alpha} [(1/2)(1 - i\gamma_5) - (1/2)(1 + i\gamma_5)] \psi
\]

\[
= \sqrt{2} (\overline{\psi}_{A'} \psi_A + \overline{\psi}_A \psi_{A'}) .
\]
which gives the right- minus left-handed fermions. Again, strictly speaking, we should consider an antisymmetrized version of this.

Finally, we note that the charge conjugate of a spinor is

\[
\psi_c = \begin{bmatrix}
0 & \epsilon^Q R' \\
\epsilon_{QR} & 0
\end{bmatrix}
\begin{bmatrix}
\bar{\psi}' \\
\bar{\psi}'_R
\end{bmatrix}
= \begin{bmatrix}
\bar{\psi}' Q' \\
\bar{\psi}'_Q
\end{bmatrix}.
\]  

(A.9)

(The usual basis-dependent formulas in terms of \(\gamma_2\) being replaced by the matrix with epsilons.)

We turn now to the perturbations of the fermi field and its stress–energy. We must begin by discussing the perturbation of the spin structure; with this in hand, we can consider how the field and the stress–energy respond.

### A.2 Change in the spin structure

When we consider a first-order perturbation \(\delta g_{ab} = h_{ab}\) of the metric, we must fix a convention for how to perturb the spin-structure. The natural choice is to take

\[
\delta \sigma_a^{AA'} = \frac{1}{2} h_a^{AA'}.
\]  

(A.10)

(Here we follow the usual practice of raising and lowering indices with respect to the background quantities — although, since \(h_{ab}\) is first-order, the ambiguity in the raising of its spinor indices is irrelevant.) With this choice, the skew spinor \(\epsilon_{AB}\) is preserved.

The covariant derivative operator must change as well, so that the conditions

\[
\nabla_a \sigma_b^{BB'} = 0, \quad \nabla_a \epsilon_{BC} = 0
\]  

(A.11)

preserved (to first order). After some algebra, one finds that the perturbation is

\[
(\delta \nabla_a) \kappa^Q = \gamma_a P Q \kappa^P,
\]  

(A.12)

where

\[
\gamma_{aBC} = \frac{1}{2} \nabla_{(B|Q|} h^{|C) Q a},
\]  

(A.13)

as one can verify by checking eq. (A.11). (Note that this gamma is not a Dirac symbol and in fact has a different index structure.)

We now examine this in the high-frequency limit. There we have

\[
\gamma_{aBC} = -\frac{i \omega}{2} \left( e^{-i \omega \phi} - e^{i \omega \psi} \right) m_a \alpha_B \alpha_C.
\]  

(A.14)

### A.3 Response of the fermi field

Now let us consider the change \(\delta \psi\) in the spinor field which accompanies the change in the metric. For definiteness we take the spinor to satisfy the Dirac equation

\[
(-i \gamma^a \nabla_a + m) \psi = 0,
\]  

(A.15)
although we shall see soon that only the kinetic term will matter for us. We first rewrite the equation in two-spinor form:

\[-i\sqrt{2} \nabla AA' \psi_A + m\psi_A' = 0\]
\[-i\sqrt{2} \nabla AA' \psi_A' + m\psi_A = 0\]  \hspace{1cm} (A.16)

Then the first-order perturbation is the system

\[-i\sqrt{2} \nabla AA' \delta \psi_A + m\delta \psi_A' = -i\sqrt{2} \gamma AA' \epsilon_{BB'} \psi_B \]
\[-i\sqrt{2} \nabla AA' \delta \psi_A' + m\delta \psi_A = i\sqrt{2} \gamma AA' \epsilon_{BB'} \psi_B'\]  \hspace{1cm} (A.17)

Note that the right-hand sides vanish to order \((-i\omega)\) (using eq. (A.14)). In the high-frequency limit, we will have then

\[\delta \psi \sim O((-i\omega)^{-1})\]  \hspace{1cm} (A.18)

since in this limit the integration involved in solving eq. (A.17) dominates.

### A.4 Change in the stress–energy

Now let us examine the change in the stress–energy. The stress–energy for a Dirac field is

\[T_{ab} = \frac{i}{2} \bar{\psi} \gamma_a \nabla B \psi - \frac{i}{2} \nabla (\bar{\psi} \gamma_b \psi)\]
\[= \frac{i}{\sqrt{2}} \left\{ \bar{\psi}_B \nabla a \psi_B + \bar{\psi}_B \nabla a \psi_B - \nabla a \bar{\psi}_B \psi_B - \nabla a \bar{\psi}_B \psi_B \right\} \bigg|_{\text{sym} a \leftrightarrow b}, \]  \hspace{1cm} (A.19)

where in the last line the symmetrization is indicated at the very end, to avoid cluttering the equation. There are two sorts of terms contributing to the change in this: those which come from altering the derivative operators (which we write as \(\delta_g T_{ab}\)), and those which come from the perturbations of the fermi field (written as \(\delta \psi T_{ab}\)). We have just seen that the fermi field’s change is of order \(\delta \psi \sim O((i\omega)^{-1})\), so \(\delta \psi T_{ab}\), which involves first derivatives of \(\delta \psi\), is of order \((i\omega)^0\), and will not contribute to \(T_{ab}(-1)\). We therefore concentrate on \(\delta_g T_{ab}\).

For the variation owing to the change in derivative operators, we have

\[\delta_g T_{ab} = \frac{i}{\sqrt{2}} \left\{ -\bar{\psi}_B \gamma_a B C' \psi_C - \bar{\psi}_B \gamma_a B C' \psi_C + \gamma_a B' C \psi_C \bar{\psi}_B + \gamma_a B' C \psi_C \bar{\psi}_B \right\} \bigg|_{\text{sym} a \leftrightarrow b}\]
\[= \frac{i}{\sqrt{2}} \left( \gamma_a B' C' \epsilon_{BC'} - \gamma_a B' C' \epsilon_{BC} \right) \left( -\bar{\psi}_C \psi_C + \bar{\psi}_C \psi_C \right) \bigg|_{\text{sym} a \leftrightarrow b}\]
\[= -\sqrt{2} \gamma_{ab}^{*} \epsilon_{BC'} \left( \bar{\psi}_C \psi_C - \bar{\psi}_C \psi_C \right) \bigg|_{\text{sym} a \leftrightarrow b}\]
\[= -\gamma_{ab}^{*} \epsilon_{BC'} \bigg|_{\text{sym} a \leftrightarrow b}, \]  \hspace{1cm} (A.20)

where we have put

\[\gamma_{ab}^{*} = \gamma_{ab} \epsilon_{BC'} + \gamma_{ab} \epsilon_{BC} \]  \hspace{1cm} (A.21)

and

\[\gamma_{ab} = \frac{1}{2} \epsilon_{apq} \gamma_{apq} \]  \hspace{1cm} (A.22)

is its Hodge dual; in spinor form on the last indices

\[\gamma_{ab}^{*} = -i \gamma_{ab} \epsilon_{BC'} + i \gamma_{ab} \epsilon_{BC} \]  \hspace{1cm} (A.23)
We recall that we will be interested in the component $m^a m^b \delta_{g T}^{ab}$. Using eq. (A.14), we have
\[
\bar{m}^a m^b \gamma_{abc} = \frac{\omega}{2} (e^{-i\omega u} \phi - e^{i\omega u} \psi) l_c,
\]
and so
\[
\bar{m}^a m^b \delta_{g T}^{ab} = -\frac{\omega}{2} (e^{-i\omega u} \phi - e^{i\omega u} \psi) l_c j_5.
\] (A.25)
Thus we have
\[
\bar{m}^a m^b T_{ab}^{(-1)} = -\frac{i}{2} j_a j_5 \phi
\] (A.26)
\[
\bar{m}^a m^b T_{ab}^{(-1)} = +\frac{i}{2} j_a j_5 \psi.
\] (A.27)

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