ERGODIC INEQUALITIES OF
THREE POPULATION GENETIC MODELS

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ABSTRACT. In this article, three models are considered, they are the infinitely-
many-neutral-alleles model [4], infinite dimensional diffusion associated with
two-parameter Poisson-Dirichlet distribution [10] and the infinitely-many-alleles
model with symmetric dominance [6]. The new representations of the tran-
sition transition densities are obtained for the first two models. Lastly, the
ergodic inequalities of these three models are provided.

1. Introduction

Fleming-Viot process is the most general model in population genetics, it can
include various evolutionary forces in a single model, such as mutations, selections.
Let $E$ be the type space, and $\mathcal{P}(E)$ be the set of probability measures on $E$, then
Fleming-Viot process $Z_t$ is a $\mathcal{P}(E)$-valued diffusion process, with generator,
$$AF_f(\mu) = \sum_{1 \leq i < j \leq m} (\Phi_{ij}^{(m)} f, \mu^{m-1}) - (f, \mu^m) + \langle B^{(m)} f, \mu^m \rangle$$
$$+ 2\bar{\sigma} \sum_{i=1}^{m} (\langle K_i^{(m)} f, \mu^{m+2} \rangle - \langle f, \mu^m \rangle) + \bar{\sigma} m \langle f, \mu^m \rangle,$$

where $\mu \in \mathcal{P}(E)$ and $f \in B(E^m)$. $\Phi_{ij}^{(m)} f$ is called sampling operator, which replace
the $j$th variable of $f$ by the $i$th variable. $B f$ is called mutation operator, which
generates a Feller semigroup $\{T_t, t \geq 0\}$ defined by transition probability $P(t, x, dy)$,
and $B^{(m)}$ is the generator of semigroup
$$T_m(t)f = \int_E \cdots \int_E f(y_1, \ldots, y_m) P(t, x_1, dy_1) \cdots P(t, x_m, dy_m).$$

$K_i^{(m)}$ is called selection operator defined
$$K_i^{(m)} f = \frac{\sigma + \sigma(x_i, x_{m+1}) - \sigma(x_{m+1}, x_{m+2})}{2\sigma} f(x_1, \ldots, x_m).$$

$\sigma(x, y)$ is a symmetric function and called relative fitness of genotype $\{x, y\}$. $\sigma$ is
defined to be $\sup_{x, y, z} |\sigma(x, y) - \sigma(y, z)|$. For more comprehensive introduction to
Fleming-Viot process, please refer to survey paper [5].

If the mutation operator $B$ of Fleming-Viot process $Z_t$ is of the form
$$B f(x) = \frac{\theta}{2} \int_E (f(y) - f(x)) \nu_0(dx), \quad \theta > 0, \nu_0 \in \mathcal{P}(E),$$

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then $\forall t > 0$, $Z_t$ is almost surely of purely atomic measure. Denote the totality of purely atomic measures by $\mathcal{P}_a$. For $\mu \in \mathcal{P}_a$, if we consider the decreasing arrangement of the atomic mass of $\mu$, then we will end up with $(x_1, x_2, \cdots)$, which consists of a set
\[
\bar{\mathcal{V}}_\infty = \left\{(x_1, x_2, \cdots) \mid x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} x_i \leq 1\right\}.
\]
We can define an atomic mapping $\rho : \mathcal{P}(E) \to \bar{\mathcal{V}}_\infty$ by mapping $\mu$ to its decreasingly ordered atomic vector $(x_1, x_2, \cdots)$. Therefore, $\rho(Z_t)$ is $\bar{\mathcal{V}}_\infty$-valued process. We call Fleming-Viot process labelled model and its atomic process $\rho(Z_t)$ unlabelled model.

If there are only random sampling and mutations involved, then $\rho(Z_t)$ is the infinitely-many-neutral-alleles model [4], denoted by $X_t$, the generator of which is
\[
G = \frac{1}{2} \sum_{i,j=1}^{\infty} x_i (\delta_{i,j} - x_j) \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\theta}{2} \sum_{i=1}^{\infty} x_i \frac{\partial}{\partial x_i}, x \in \bar{\mathcal{V}}_\infty.
\]

If we include selection as well, then the unlabelled model is usually non-Markovian.

In [9], the transition density function of $\rho(Z_t)$ is obtained. In this paper, we reorganize the transition density functions of $\rho(Z_t)$ and $X_t$, and new representations of the transition density functions of $\rho(Z_t)$ and $X_t$ are obtained respectively.
The associated transition probabilities resembles the structure of transition probabilities in neutral Fleming-Viot process. This can actually shed some light to the construction of corresponding labelled model of $X_{\theta,\alpha}^t$.

Furthermore, the ergodic inequalities of $Z_t$ and $X_t$ are both available, but similar ergodic inequalities of $X_{\theta,\alpha}^t$ and $X_{\sigma}^t$ are still missing. In this article, we have obtained the ergodic inequality of $X_{\theta,\alpha}^t$ and $X_{\sigma}^t$. Especially, for $\theta > 0$, $X_{\theta,\alpha}^t$ and $X_t$ share the exactly the same ergodic inequality. Lastly, the ergodic inequality of $X_{\sigma}^t$ is stronger than the ergodic theorem stated in [6].

This paper is organized as follows: in section 2, we will talk about the transition density functions of $X_{\theta,\alpha}^t$ and $X_{\sigma}^t$. In section 3, the ergodic inequalities of them will also be discussed.

2. TRANSITION DENSITY FUNCTIONS OF $X_{\theta,\alpha}^t$ AND $X_t$

In [4] and [9], the explicit transition densities of $X_{\theta,\alpha}^t$ and $X_t$ are obtained respectively through eigen expansion. By making use of these known transition densities, we get new representations.

Theorem 2.1. $X_t$ has the following transition density

$$ p(t, x, y) = d_0^\theta(t) + d_1^\theta(t) + \sum_{n=2}^{\infty} d_n^\theta(t)p_n(x, y), $$

where

$$ d_n^\theta(t) = \sum_{m=2}^{\infty} \frac{2m + \theta - 1}{m!} (-1)^{m-n} \binom{m}{n} (n + \theta)\binom{m}{m-n} e^{-\lambda_m t}, n \geq 1. $$

$$ d_0^\theta(t) = 1 - \sum_{m=1}^{\infty} \frac{2m + \theta - 1}{m!} (-1)^{m-1} \theta \binom{m}{m-1} e^{-\lambda_m t}. $$

$$ p_n(x, y) = \sum_{|\eta| = n} p_\eta(x)p_\eta(y) \int p_\eta dPD(\theta), \quad \eta = (\eta_1, \ldots, \eta_l) \text{ is a partition of } n. $$

$p_\eta(x)$ is the continuous extension of

$$ \frac{n!}{\eta_1! \cdots \eta_l! a_1! \cdots a_n!} \sum_{i_1, \ldots, i_l \neq k} x_{i_1}^{\eta_1} \cdots x_{i_l}^{\eta_l}. $$

Define $\nu_n(x, dy) = p_n(x, y)PD(\theta)(dy)$, then the transition probability of $X_t$ is

$$ P(t, x, dy) = \left( d_0^\theta(t) + d_1^\theta(t) \right) PD(\theta)(dy) + \sum_{n=2}^{\infty} d_n^\theta(t)\nu_n(x, dy). $$

Proof. The transition density of $X_t$ is

$$ p(t, x, y) = 1 + \sum_{m=2}^{\infty} e^{-\lambda_m t} Q_m(x, y), \quad \lambda_m = \frac{m(m + \theta - 1)}{2}, $$

where

$$ Q_m(x, y) = \frac{2m + \theta - 1}{m!} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} (n + \theta)\binom{m}{m-n} p_n(x, y). $$
Then by Fubini’s theorem, we can rearrange $p(t, x, y)$ by switching the order of summation.

$$p(t, x, y) = 1 + \sum_{m=2}^{\infty} e^{-\lambda t} \sum_{n=2}^{m} \frac{2m + \theta - 1}{m!} (-1)^{m-n} \left( \frac{m}{n} \right) (n + \theta)_{(m-1)} p_n(x, y)$$

$$+ \frac{2m + \theta - 1}{m!} (-1)^{m-1} (\theta + 1)_{(m-1)} m p_1(x, y) + \frac{2m + \theta - 1}{m!} (-1)^{m} \theta_{(m-1)} p_0(x, y)$$

( for $p_1(x, y), p_0(x, y) = 1$, we have )

$$= 1 + \sum_{m=2}^{\infty} e^{-\lambda t} \sum_{n=2}^{m} \frac{2m + \theta - 1}{m!} (-1)^{m-n} \left( \frac{m}{n} \right) (n + \theta)_{(m-1)} p_n(x, y)$$

$$+ \sum_{m=2}^{\infty} e^{-\lambda t} \left( \frac{2m + \theta - 1}{m!} (-1)^{m-1} (\theta + 1)_{(m-1)} m + \frac{2m + \theta - 1}{m!} (-1)^{m} \theta_{(m-1)} \right)$$

$$= 1 + \sum_{m=2}^{\infty} e^{-\lambda t} \sum_{n=2}^{m} \frac{2m + \theta - 1}{m!} (-1)^{m-n} \left( \frac{m}{n} \right) (n + \theta)_{(m-1)} p_n(x, y)$$

$$+ \sum_{m=2}^{\infty} e^{-\lambda t} \frac{2m + \theta - 1}{m!} (-1)^{m-1} \left[ m(\theta + 1)_{(m-1)} - \theta_{(m-1)} \right]$$

since when $m = 1, m(\theta + 1)_{(m-1)} - \theta_{(m-1)} = 0$, then we have

$$= 1 - \sum_{m=1}^{\infty} e^{-\lambda t} \frac{2m + \theta - 1}{m!} (-1)^{m-1} \theta_{(m-1)}$$

$$+ \sum_{m=1}^{\infty} e^{-\lambda t} \frac{2m + \theta - 1}{m!} (-1)^{m-1} m(\theta + 1)_{(m-1)}$$

$$+ \sum_{m=2}^{\infty} e^{-\lambda t} \frac{2m + \theta - 1}{m!} (-1)^{m-n} \left( \frac{m}{n} \right) (n + \theta)_{(m-1)} p_n(x, y)$$

$$= d_0^0(t) + d_1^0(t) + \sum_{m=2}^{\infty} e^{-\lambda t} \sum_{n=2}^{m} \frac{2m + \theta - 1}{m!} (-1)^{m-n} \left( \frac{m}{n} \right) (n + \theta)_{(m-1)} p_n(x, y).$$

Switching the order of summation, we have

$$p(t, x, y) = d_0^0(t) + d_1^0(t) + \sum_{n=2}^{+\infty} d_n^0(t) p_n(x, y).$$

\[\square\]

**Theorem 2.2.** $X_t^{\theta, \alpha}$ has the following transition density

$$p^{\theta, \alpha}(t, x, y) = d_0^0(t) + d_1^0(t) + \sum_{n=2}^{\infty} d_n^0(t) p^{\theta, \alpha}_n(x, y),$$

where $d_n^0(t), n \geq 0$, are defined in theorem 2.1 and

$$p^{\theta, \alpha}_n(x, y) = \sum_{|\eta| = n} \frac{p_\eta(x)p_\eta(y)}{\int p_\eta dP_D(\theta, \alpha)}, \eta = (\eta_1, \ldots, \eta_k) \text{ is a partition of } n.$$
Define $\nu_n(x, dy) = p_n(x, y)PD(\theta, \alpha)(dy)$, then the transition probability of $X_t$ is

$$P^{\theta, \alpha}(t, x, dy) = \left( d_n^\theta(t) + d_n^\alpha(t) \right) PD(\theta, \alpha)(dy) + \sum_{n=2}^\infty d_n^\theta(t) \nu_n(x, dy). \tag{1}$$

**Proof.** The proof of this theorem is quite similar to theorem 2.1, thereby omitted. □

**Remark 2.1.** Since $X_t$ has an entrance boundary $\nabla_\infty - \nabla_\infty$, i.e. $X_t$ will immediately moves into $\nabla_\infty$ and never exits regardless of its starting point. Similarly, we can show the similar result for $X^{\theta, \alpha}_t$ informed by S.N. Ethier.

For both $X_t$ and $X^{\theta, \alpha}_t$, the structures of transition probability are so similar. They even share the coefficients $d_n^\theta(t), n \geq 0$, which is the entrance of the ancestral process discussed by Simon Tavaré in [11]. But Tavaré constructed this process only when $\theta > 0$. In fact, if we collapse the state 0 and 1, and relabel it as 1, this is essentially Kingman coalescence with mutation. We can generalize this structure to the case where $\theta > -1$.

**Proposition 2.1.** For $\theta > -1$, we have

$$e^{-\lambda_n t} \leq \sum_{k=n}^\infty d_k^\theta(t) \leq \frac{(n + \theta)(n)}{n} e^{-\lambda_n t}.$$  

In particular, when $n = 2$, we know

$$\sum_{k=2}^\infty d_k^\theta(t) \leq \frac{(2 + \theta)(3 + \theta)}{2} e^{-(\theta + 1)t}. \tag{2}$$

**Proof.** Consider a pure-death Markov chain $B_t$ in $\{1, 2, \cdots, m\}$ with Q matrix,

$$Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \lambda_2 & -\lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_3 & -\lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_m & -\lambda_m \end{pmatrix}$$

where $\lambda_k = \frac{k(k+\theta-1)}{2}$, $k \geq 2$. Running the similar arguments in Theorem 4.3 in [7], we will be able to find all the left eigenvectors and right eigenvectors of $Q$. Denote the matrix consisting of left eigenvectors by $U = (u_{ij})$ and the matrix consisting of right eigenvectors by $V = (v_{ij})$, where

$$u_{ij} = \begin{cases} \delta_{ij} & i = 1 \\ 0 & j > i > 1 \\ (-1)^{j-i} \frac{j!}{i!} \frac{(j+\theta)(i-1)!}{(i+\theta)(j-1)!} & j \leq i, i > 1, \end{cases}$$

and

$$v_{ij} = \begin{cases} 1 & j = 1 \\ 0 & j > i \\ \frac{j!}{i!} \frac{(j+\theta)(\theta)}{(i+\theta)(\theta)} & 1 < j \leq i. \end{cases}$$

Note that the row vectors of $U$ are left eigenvectors of $Q$ and the column vectors of $V$ are the right eigenvectors of $Q$. Similarly, we can also show that $UV = I$.
and $Q$ is diagonalized as $V \Lambda U$, where $\Lambda = \text{diag}\{0, -\lambda_2, \cdots, -\lambda_m\}$. Therefore, the transition matrix $P_t$ is

$$P_t = e^{tQ} = Ve^{\Lambda}U.$$  

By direct computation, we know, for $2 \leq n \leq m$,

$$P_{mn}(t) = \sum_{k=n}^{m} (-1)^{k-n} \binom{m}{k} \binom{k}{n} \binom{\theta + k}{\theta + n} \binom{\theta + k}{\theta + k} e^{-\lambda_k t}.$$  

Letting $m \to +\infty$, we have $d^\theta_n(t) = \lim_{m \to \infty} P_{mn}(t)$.

The remaining arguments are essentially due to Tavaré. By the martingale argument in chapter 6 of [1], we know

$$Z_n(t) = \frac{e^{\lambda_n t(B_t|_n)}}{(B_t + \theta)_{(n)}}.$$  

because $e^{-\lambda_n t}$ is one eigenvalue of $P_t$ and $(0, 0, \cdots, 0, \frac{n[n]}{(n + \theta)_{(n)}}, \cdots, \frac{k[n]}{(k + \theta)_{(n)}}, \cdots, \frac{m[n]}{(m + \theta)_{(n)}})^T$ is the corresponding eigenvector. So

$$EZ_n(t) = Z_n(0) = \frac{m[n]}{(m + \theta)_{(n)}}.$$  

Since, for $n \leq k \leq m$,

$$\frac{n[n]}{(n + \theta)_{(n)}} \leq \frac{k[n]}{(k + \theta)_{(n)}} \leq \frac{m[n]}{(m + \theta)_{(n)}},$$  

and

$$e^{-\lambda_n t} m[n] \frac{k[n]}{(m + \theta)_{(n)}} = e^{-\lambda_n t} EZ_n(t) = \sum_{k=n}^{m} \frac{k[n]}{(k + \theta)_{(n)}} P_{mk}(t),$$  

we have

$$\frac{n[n]}{(n + \theta)_{(n)}} P(B_t \geq n|B_0 = m) \leq e^{-\lambda_n t} \frac{m[n]}{(m + \theta)_{(n)}} P(B_t \geq n|B_0 = m) \leq \frac{m[n]}{(m + \theta)_{(n)}} P(B_t \geq n|B_0 = m).$$  

Thus, we have

$$e^{-\lambda_n t} \leq P(B_t \geq n|B_0 = m) \leq \frac{(n + \theta)_{(n)}}{n[n]} e^{-\lambda_n t}.$$  

Letting $m \to \infty$, we have

$$e^{-\lambda_n t} \leq \sum_{k=n}^{\infty} d^\theta_k(t) \leq \frac{(n + \theta)_{(n)}}{n[n]} e^{-\lambda_n t}.$$  

□
3. Ergodic Inequalities

By making use of the transition probability \( \Pi \) and the tail probability estimation (2), we can easily get the following ergodic inequality of \( X_t^\theta,\alpha \).

**Theorem 3.1.** For \( X_t^\theta,\alpha \), we have the ergodic inequality

\[
\sup_{x \in \mathcal{V}_\infty} \| P^{\theta,\alpha}(t, x, \cdot) - \Pi D(\theta, \alpha)(\cdot) \|_{\text{var}} \leq \frac{(2 + \theta)(3 + \theta)}{2} \exp\{-\theta(1 + t)\}, t \geq 0.
\]

**Proof.**

\[
\| P^{\theta,\alpha}(t, x, \cdot) - \Pi D(\theta, \alpha)(\cdot) \|_{\text{var}} \leq \sup_{A \in \mathcal{B}} | P^{\theta,\alpha}(t, x, A) - \Pi D(\theta, \alpha)(A) |
\]

\[
= | (d_n^0(t) + d_n^1(t)) \Pi D(\theta, \alpha)(A) + \sum_{n=2}^{\infty} d_n^0(t) \nu_n^{\theta,\alpha}(A) - \Pi D(\theta, \alpha)(A) |
\]

\[
\leq \sum_{n=2}^{\infty} d_n^0(t) | \nu_n^{\theta,\alpha}(A) - \Pi D(\theta, \alpha)(A) |
\]

\[
\leq \sum_{n=2}^{\infty} d_n^0(t) \leq \frac{(\theta + 2)(\theta + 3)}{2} e^{-(\theta + 1)t}.
\]

\[\square\]

**Proposition 3.1.** The transition densities \( p_\sigma(t, x, y) \) of \( X_t^\sigma \) is also ultra-bounded, i.e.

\[ p_\sigma(t, x, y) \leq \frac{1}{C_\sigma} e^{\sigma(1 + \theta)t + \sigma^2 + 3|\sigma|t \log t}. \]

**Proof.** This estimation can be easily obtained from (4.17) in \([3]\) and theorem 3.3 in \([9]\). \[\square\]

Since the one-parameter selective model is absolutely continuous with respect to one-parameter neutral model, \( \mathcal{V}_\infty - \mathcal{V}_\infty \) should also serve as an entrance boundary. Hence we can change the value of the density function \( p_\sigma(t, x, y) \) when \( x \) or \( y \) is in \( \mathcal{V}_\infty - \mathcal{V}_\infty \). Therefore, \( p_\sigma(t, x, y) \) can be chosen to be the continuous extension of \( p_\sigma(t, x, y) \) on \( \mathcal{V}_\infty \times \mathcal{V}_\infty \). Moreover, \( p_\sigma(t, x, y) \) is symmetric for \( X_t^\sigma \) is reversible. By proposition 3.1, the Poincaré inequality of \( X_t^\sigma \) also holds. It, therefore, guarantees the \( L_2 \)-exponential convergence. By running the argument in theorem 8.8 in \([1]\), we can also get the following ergodic inequality.

**Theorem 3.2.** For \( X_t^\sigma \), \( \exists K(\theta, \sigma) \), such that

\[
\sup_{x \in \mathcal{V}_\infty} \| P^\sigma(t, x, \cdot) - \pi_\sigma(\cdot) \|_{\text{var}} \leq K(\theta, \sigma) \exp\{-\text{gap}(G_\sigma)t\}, t \geq 0.
\]

**Proof.** We are going to run the argument in theorem 8.8 in \([1]\). Since

\[
P^\sigma(t, x, \cdot) = \int_{\mathcal{V}_\infty} P^\sigma(t - s, z, \cdot) P^\sigma(s, x, dz),
\]

and define \( \mu^\sigma(\cdot) = P_\sigma(X_t^\sigma \in \cdot) \), we have

\[
P^\sigma(t, x, \cdot) = \mu^\sigma P_{\tau - t}(\cdot).
\]
Therefore,
\[
\|P^\sigma(t, x, \cdot) - \pi(\cdot)\|_{\text{var}} = \|\mu^\sigma P^\sigma_{t-s}(\cdot) - \pi(\cdot)\|_{\text{var}}.
\]
By part (1) in theorem 8.8 in [1], we have, \(\forall t \geq s\),
\[
\|P^\sigma(t, x, \cdot) - \pi(\cdot)\|_{\text{var}} \leq \left\| \frac{d\mu^\sigma}{d\pi^\sigma} - 1 \right\|_2 e^{-(t-s)\operatorname{gap}(G_\sigma)}
= \sqrt{\int p(s, x, y)^2 \pi^\sigma(dy) - 1} e^{-\operatorname{gap}(G_\sigma) t}.
\]
Therefore, for \(t \geq \frac{1}{2}\), we have
\[
\|P^\sigma(t, x, \cdot) - \pi(\cdot)\|_{\text{var}} \leq \sqrt{\int p(s, x, y)^2 \pi^\sigma(dy) - 1} e^{-\operatorname{gap}(G_\sigma) t}.
\]
If we choose \(s = \frac{1}{2}\), then by Proposition 3.1, the constant
\[
K'(\theta, \sigma) = \sqrt{2ce^{\sigma(1+\theta)+\sigma^2+3|\sigma|e^{\frac{1}{2}\operatorname{gap}(G_\sigma)}}} \geq \sqrt{\int p(s, x, y)^2 \pi^\sigma(dy) - 1} e^{\frac{1}{2}\operatorname{gap}(G_\sigma)}.
\]
Then we have
\[
\sup_{x \in \mathcal{X}} \left\| P^\sigma(t, x, \cdot) - \pi^\sigma(\cdot) \right\|_{\text{var}} \leq K'(\theta, \sigma) \exp\{-\operatorname{gap}(G_\sigma) t\}, \forall t \geq \frac{1}{2}.
\]
Moreover,
\[
\sup_{x \in \mathcal{X}} \left\| P^\sigma(t, x, \cdot) - \pi^\sigma(\cdot) \right\|_{\text{var}} \leq 1, \quad \forall t \geq 0.
\]
Thus \(\forall t \in [0, \frac{1}{2}]\), if we choose \(K''(\theta, \sigma)\) such that
\[
K''(\theta, \sigma) e^{-\operatorname{gap}(G_\sigma)/2} \geq 1,
\]
then \(\forall t \in [0, \frac{1}{2}]\),
\[
\sup_{x \in \mathcal{X}} \left\| P^\sigma(t, x, \cdot) - \pi^\sigma(\cdot) \right\|_{\text{var}} \leq 1 \leq K''(\theta, \sigma) e^{-\operatorname{gap}(G_\sigma)/2} \leq K''(\theta, \sigma) \exp\{-\operatorname{gap}(G_\sigma) t\}.
\]
Therefore, choosing \(K(\theta, \sigma) = \max\{K'(\theta, \sigma), K''(\theta, \sigma)\}\), we have
\[
\sup_{x \in \mathcal{X}} \left\| P^\sigma(t, x, \cdot) - \pi^\sigma(\cdot) \right\|_{\text{var}} \leq K(\theta, \sigma) \exp\{-\operatorname{gap}(G_\sigma) t\}.
\]

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