On Nilpotent Multipliers of some Verbal Products of Groups

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Abstract

The paper is devoted to finding a homomorphic image for the $c$-nilpotent multiplier of the verbal product of a family of groups with respect to a variety $\mathcal{V}$ when $\mathcal{V} \subseteq \mathcal{N}_c$ or $\mathcal{N}_c \subseteq \mathcal{V}$. Also a structure of the $c$-nilpotent multiplier of a special case of the verbal product, the nilpotent product, of cyclic groups is given. In fact, we present an explicit formula for the $c$-nilpotent multiplier of the $n$th nilpotent product of the group $G = \mathbb{Z}^n * \ldots * \mathbb{Z}^n * \mathbb{Z}_{r_1}^n * \ldots * \mathbb{Z}_{r_t}$, where $r_{i+1}$ divides $r_i$ for all $i, 1 \leq i \leq t - 1$, and $(p, r_1) = 1$ for any prime $p$ less than or equal to $n + c$, for all positive integers $n, c$.

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1 Introduction and Motivation

Let $G = F/R$ be a free presentation of a group $G$. Then the Baer invariant of $G$ with respect to the variety $\mathcal{N}_c$ of nilpotent groups of class at most $c \geq 1$, denoted by $\mathcal{N}_cM(G)$, is defined to be

$$\mathcal{N}_cM(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, cF]}.$$ 

$\mathcal{N}_cM(G)$ is also called the $c$-nilpotent multiplier of $G$. Clearly if $c = 1$, then $\mathcal{N}_c = A$ is the variety of all abelian groups and the Baer invariant of $G$ with respect to this variety is

$$M(G) = \frac{R \cap F'}{[R, F]},$$

which is the well-known Schur multiplier of $G$.

It is important to find structures for the Schur multiplier and its generalization, the $c$-nilpotent multiplier, of some famous products of groups. Determining these Baer invariants of a given group is known to be very useful for the classification of groups into isoclinism classes (see [1]).

In 1907, Schur [17], using a representation method, found a structure for the Schur multiplier of a direct product of two groups. Also, Wiegold [19] obtained the same result by some properties of covering groups. In 1979 Moghaddam [13] found a formula for the $c$-nilpotent multiplier of a direct product of two groups, where $c + 1$ is a prime number or 4. Also, in 1998 Ellis [2] extended the formula for all $c \geq 1$. In 1997 the second author and Moghaddam [10] presented an explicit formula for the $c$-nilpotent multiplier of a finite abelian group for any $c \geq 1$. It is known that the direct product is a special case of the nilpotent product and we know that regular and verbal products are generalizations of the nilpotent product.

In 1972, Haebich [6] found a formula for the Schur multiplier of a regular product of a family of groups. Then the second author [8] extended the result
to find a homomorphic image with a structure similar to Haebich’s type for
the \(c\)-nilpotent multiplier of a nilpotent product of a family of groups.

In section two, we extend the above result and find a homomorphic image
for the \(c\)-nilpotent multiplier of a verbal product of a family of groups with
respect to a variety \(\mathcal{V}\) when \(\mathcal{V} \subseteq \mathcal{N}_{c}\) or \(\mathcal{N}_{c} \subseteq \mathcal{V}\).

A special case of the verbal product of groups whose nilpotent multiplier
has been studied more than others is the nilpotent product of cyclic groups.
In 1992, Gupta and Moghaddam [5] calculated the \(c\)-nilpotent multiplier
of the nilpotent dihedral group of class \(n\), i.e. \(G_n \cong \mathbb{Z}_2^n \ast \mathbb{Z}_2\). (Note that in
2001 Ellis [3] remarked that there is a slip in the statement and gave the
correct one.) In 2003, Moghaddam, the second author and Kayvanfar [14]
extended the previous result and calculated the \(c\)-nilpotent multiplier of the
\(n\)th nilpotent product of cyclic groups for \(n = 2, 3, 4\) under some conditions.
Also, the second author and Parvizi [11, 12] presented structures for some
Baer invariants of a free nilpotent group that is the nilpotent product of
infinite cyclic groups. Finally the authors and Mohammadzadeh [9] obtained
an explicit formula for the \(c\)-nilpotent multiplier of the \(n\)th nilpotent product
of some cyclic groups \(G = \mathbb{Z}^n \ast \ldots \ast \mathbb{Z}^n \ast \mathbb{Z}_{r_1}^n \ast \ldots \ast \mathbb{Z}_{r_t}^n\), where \(r_{i+1}\) divides \(r_i\)
for all \(i, 1 \leq i \leq t - 1\), for \(c \geq n\) such that \((p, r_1) = 1\) for any prime \(p\) less
than or equal to \(n\).

In section three, we give an explicit formula for the \(c\)-nilpotent multiplier
of the above group \(G\) when \((p, r_1) = 1\) for any prime \(p\) less than or equal to
\(n + c\), for all positive integers \(c, n\).

2 Verbal products

A group \(G\) is said to be a regular product of its subgroups \(A_i, i \in I\), where \(I\)
is an ordered set, if the following two conditions hold:
i) $G = \langle A_i | i \in I \rangle$;

ii) $A_i \cap \hat{A}_i = 1$ for all $i \in I$, where $\hat{A}_i = \langle A_j | j \in I, j \neq i \rangle$.

**Definition 2.1.** Consider the map

$$
\psi : \prod_{i \in I} A_i \to \prod_{i \in I} A_i
$$

$$a_1 a_2 \ldots a_n \mapsto (a_1, a_2, \ldots, a_n),
$$

which is a natural map from the free product of $\{A_i\}_{i \in I}$ on to the direct product of $\{A_i\}_{i \in I}$. Clearly its kernel is the normal closure of

$$\langle [A_i, A_j] | i, j \in I, i \neq j \rangle$$

in the free product $A = \prod_{i \in I} A_i$. It is denoted by $[A^i]_i$ and called the Cartesian subgroup of the free product (see [16] for the properties of cartesian subgroups).

The following theorem gives a characterization of a regular product.

**Theorem 2.2** (Golovin 1956 [4]). Suppose that a group $G$ is generated by a family $\{A_i | i \in I\}$ of its subgroups, where $I$ is an ordered set. Then $G$ is a regular product of the $A_i$ if and only if every element of $G$ can be written uniquely as a product

$$a_1 a_2 \ldots a_n u,$$

where $1 \neq a_i \in A_{\lambda_i}$, $\lambda_1 < \ldots < \lambda_n$ and $u \in [A^G_i] = \langle [A^G_i, A^G_j] | i, j \in I, i \neq j \rangle$.

**Definition 2.3.** Let $\mathcal{V}$ be a variety of groups defined by a set of laws $V$. Then the verbal product of a family of groups $\{A_i\}_{i \in I}$ associated with the variety $\mathcal{V}$ is defined to be

$$\mathcal{V} \prod_{i \in I} A_i = \frac{\prod_{i \in I} A_i}{V(A) \cap [A^A_i]}.$$
The verbal product is also known as varietal product or simply $\mathcal{V}$-product. If $\mathcal{V}$ is the variety of all groups, then the corresponding verbal product is the free product; if $\mathcal{V} = \mathcal{A}$ is the variety of all abelian groups, then the verbal product is the direct product and if $\mathcal{V} = \mathcal{N}_c$ is the variety of all nilpotent groups of class at most $c$, then the verbal product will be the nilpotent product.

Let $\{A_i | i \in I\}$ be a family of groups and

$$1 \to R_i \to F_i \xrightarrow{\theta_i} A_i \to 1$$

be a free presentation for $A_i$. We denote by $\theta$ the natural homomorphism from the free product $F = \prod_{i \in I}^* F_i$ onto $A = \prod_{i \in I}^* A_i$ induced by the $\theta_i$. Also we assume that the group $G$ is the verbal product of $\{A_i\}_{i \in I}$ associated with the variety $\mathcal{V}$. If $\psi$ is the natural homomorphism from $A$ onto $G$ induced by the identity map on each $A_i$, then we have the sequence

$$F = \prod_{i \in I}^* F_i \xrightarrow{\theta} A = \prod_{i \in I}^* A_i \xrightarrow{\psi} G = \mathcal{V} \prod_{i \in I} A_i \to 1.$$

The following notation will be used throughout this section.

**Notation 2.4.**

i) $D_1 = \prod_{i \neq j}^*[R_i, F_j]^F$;

ii) $E_c = D_1 \cap \gamma_{c+1}(F)$;

iii) $D_c = \prod_{i \neq j \text{ s.t. } \mu_i \neq 1}^*[R_i, F_{\mu_1}, \ldots, F_{\mu_c}]^F$;

iv) $K_v = V(F) \cap [F_1^F]$;

v) $K_c = \gamma_{c+1}(F) \cap [F_1^F]$.

Let $H_v$ be the kernel of $\psi_v$ and $R$ be the kernel of $\psi_v \circ \theta$. It is clear that $R$ is actually the inverse image of $H_v$ in $F$ under $\theta$, where $H_v = V(A) \cap [A_1^A]$ by the definition of the verbal product. Put $H_c = \gamma_{c+1}(A) \cap [A_1^A]$, then an immediate consequence is the following lemma.
Lemma 2.5. With the above notation we have
i) \( \theta(K_v) = H_v \) and \( \theta(K_c) = H_c \);
ii) \( G = F/R \) and \( R = \prod_{i \in I} R_i^F K_v = (\prod_{i \in I} R_i) D_1 K_v \).

Proof. (i) This follows from the definition of \( \theta \).
(ii) It is easy to see that \( \ker \theta = \prod_{i \in I} R_i^F \). On the other hand, since \( \theta(K_v) = \ker \psi_v \), we have \( R = (\ker \theta) K_v = \prod_{i \in I} R_i^F K_v \). Also for all \( r \in R_i \) and \( f \in F \), \( r^f = r[r, f] \). This implies that \( \prod_{i \in I} R_i^F = \prod_{i \in I} R_i[R_i, F] \). Since \([R_i, F_i] \subseteq R_i\), \( \prod_{i \in I} R_i^F = \prod_{i \in I} R_i D_1 \).

We now prove some lemmas to compute the \( c \)-nilpotent multiplier of \( G \).

Lemma 2.6. Keeping the above notation we have
i) \( [R, cF] = (\prod_{i \in I} [R_i, cF_i]) D_c[K_v, cF] \).
ii) If \( V(F) \subseteq \gamma_{c+1}(F) \), then \( R \cap \gamma_{c+1}(F) = \prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) E_c K_v \).
iii) If \( \gamma_{c+1}(F) \subseteq V(F) \), then \( R \cap \gamma_{c+1}(F) = \prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) E_c K_c \).

Proof. i) \[
[R, cF] = \prod_{i \in I} R_i^F K_v, cF \\
= \prod_{i \in I} [R_i, cF] F[K_v, cF] \\
= (\prod_{i \in I} [R_i, cF_i]) D_c[K_v, cF].
\]
ii) Let \( g \in R \cap \gamma_{c+1}(F) \). Then \( g = r_{\lambda_1} \ldots r_{\lambda_l} dk \) by Lemma 2.5, where \( r_{\lambda_i} \in R_{\lambda_i}, d \in D_1 \) and \( k \in K_v \). Now consider the natural homomorphism
\[
\varphi : F = \prod_{i \in I}^* F_i \to \prod_{i \in I} F_i.
\]
Since \( g \in \gamma_{c+1}(F) \), \( \varphi(g) = (r_{\lambda_1}, \ldots, r_{\lambda_l}) \in \gamma_{c+1}(\prod_{i \in I}^* F_i) = \prod_{i \in I}^* \gamma_{c+1}(F_i). \) Therefore \( r_{\lambda_i} \in \gamma_{c+1}(F_{\lambda_i}) \cap R_{\lambda_i} \) and then \( dk \in \gamma_{c+1}(F) \cap [F_i^F] \). Now since
$k \in V(F) \subseteq \gamma_{c+1}(F)$, we have $d \in \gamma_{c+1}(F) \cap D_1 = E_c$ and so the result follows.

iii) Since $K_c \subseteq K_v$, $\prod_{i \in I}(R_i \cap \gamma_{c+1}(F_i))E_cK_c \subseteq R \cap \gamma_{c+1}(F)$. For the reverse inclusion, similar to part (i), $dk \in \gamma_{c+1}(F) \cap [F^F_i]$. Therefore $R \cap \gamma_{c+1}(F) \subseteq \prod_{i \in I}(R_i \cap \gamma_{c+1}(F_i))K_c$. Now the inclusion $E_c \subseteq K_c$ shows that the equality (iii) holds. □

Lemma 2.7. With the above notation, let $\varphi_c : F \to F/E_c$ be the natural homomorphism. Then $\varphi_c(\prod_{i \in I}(R_i \cap \gamma_{c+1}(F_i))K_v)$ is the direct product of its subgroups $\varphi_c(K_v)$ and $\varphi_c(R_i \cap \gamma_{c+1}(F_i))$, $i \in I$.

Proof. The Three Subgroups Lemma shows that

$$[R_i \cap \gamma_{c+1}(F_i), K_v] \subseteq E_c \quad \text{for all } i \in I$$

and

$$[R_i \cap \gamma_{c+1}(F_i), R_j \cap \gamma_{c+1}(F_j)] \subseteq E_c \quad \text{for all } i, j \in I, i \neq j.$$

So we have

$$[\varphi_c(R_i \cap \gamma_{c+1}(F_i)), \varphi_c(K_v)] = 1 \quad \text{for all } i \in I$$

and

$$[\varphi_c(R_i \cap \gamma_{c+1}(F_i)), \varphi_c(R_j \cap \gamma_{c+1}(F_j))] = 1 \quad \text{for all } i, j \in I, i \neq j.$$

Moreover, by Theorem 2.2 we conclude that

$$\varphi_c(R_i \cap \gamma_{c+1}(F_i)) \cap \left(\prod_{i \neq j} \varphi_c(R_j \cap \gamma_{c+1}(F_j))\varphi_c(K_v)\right) = 1.$$

Now the result follows by the definition of the direct product. □
Lemma 2.8. With the previous notation,

i) If $V(F) \subseteq \gamma_{c+1}(F)$, then $\varphi_c(K_v)/\varphi_c([K_v, cF]) \cong H_v/[H_v, cA]$.

ii) If $\gamma_{c+1}(F) \subseteq V(F)$, then $\varphi_c(K_c)/\varphi_c([K_c, cF]) \cong H_c/[H_c, cA]$.

Proof. i) If $V(F) \subseteq \gamma_{c+1}(F)$, then

$$\frac{\varphi_c(K_v)}{\varphi_c([K_v, cF])} \cong \frac{K_v E_c}{[K_v, cF] E_c} \cong \frac{K_v}{K_v \cap [K_v, cF] E_c}.$$ 

On the other hand

$$\frac{\theta(K_v)}{\theta([K_v, cF])} \cong \frac{K_v \ker \theta}{[K_v, cF] \ker \theta} \cong \frac{K_v}{K_v \cap [K_v, cF] \ker \theta} \cong \frac{K_v}{K_v \cap [K_v, cF] D_1 \prod_i R_i}.$$ 

Now Theorem 2.2 and definition of $E_c$ imply that

$$\frac{\theta(K_v)}{\theta([K_v, cF])} \cong \frac{K_v}{K_v \cap [K_v, cF] E_c}.$$ 

Therefore by Lemma 2.5, we conclude that

$$\frac{\varphi_c(K_v)}{\varphi_c([K_v, cF])} \cong \frac{\theta(K_v)}{\theta([K_v, cF])} \cong \frac{H_v}{[H_v, cA]}.$$ 

ii) The proof is similar to (i).

Now we are ready to state and prove the main result of this section.

Theorem 2.9. With the above notation,

i) If $\mathcal{N}_c \subseteq \mathcal{V}$, then $\prod_{i \in I}^\times \mathcal{N}_c M(A_i) \times H_v/[H_v, cA]$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$, and if $\mathcal{V} \prod_{i \in I} A_i$ is finite, then the above structure is isomorphic to a subgroup of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$.

ii) If $\mathcal{V} \subseteq \mathcal{N}_c$, then $\prod_{i \in I}^\times \mathcal{N}_c M(A_i) \times H_v/[H_v, cA]$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$, and if $\mathcal{V} \prod_{i \in I} A_i$ is finite, then the above structure is isomorphic to a subgroup of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$.
Proof. i) By Lemma 2.6 (i),(ii)

\[
N_c M (\prod_{i \in I} A_i) \cong \frac{R \cap \gamma_{c+1}(F)}{[R, cF]} \cong \frac{\prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) E_c K_v}{\prod_{i \in I} [R_i, cF_i] D_c [K_v, cF]}.
\]

Therefore there is a natural epimorphism from \(N_c M (\prod_{i \in I} A_i)\) to

\[
\prod_{i \in I} \left( \frac{R_i \cap \gamma_{c+1}(F_i)}{[R_i, cF_i]} \right) \cong \prod_{i \in I} \left( \frac{\gamma_{c+1}(F_i)}{[R_i, cF_i]} \right) E_c K_v.
\]

Lemma 2.7 and the fact that \(\varphi_c([K_v, cF]) \subseteq \varphi_c(K_v)\) and \(\varphi_c([R_i, cF_i]) \subseteq \varphi_c(R_i \cap \gamma_{c+1}(F_i))\) imply that

\[
\varphi_c(\prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) K_v) \cong \prod_{i \in I} \varphi_c(R_i \cap \gamma_{c+1}(F_i)) \cong \prod_{i \in I} \varphi_c([R_i, cF_i]) \varphi_c(K_v) \varphi_c([K_v, cF]).
\]

It is straightforward to see that

\[
\frac{\varphi_c(R_i \cap \gamma_{c+1}(F_i))}{\varphi_c([R_i, cF_i])} \cong \frac{R_i \cap \gamma_{c+1}(F_i)}{[R_i, cF_i]}
\]

by Theorem 2.2. Therefore, the result holds by Lemma 2.8 (i).

ii) By an argument similar to (i), we obtain the result. □

We need the following lemma whose proof is straightforward.

Lemma 2.10. Let \(\{A_i \mid i \in I\}\) be a family of groups. Put \(A = \prod_{i \in I}^* A_i\). Then for all integers \(m \geq 2\),

\[
\gamma_m(A) = \prod_{i \in I} \gamma_m(A_i)(\gamma_m(A) \cap [A_i^A]).
\]

In particular if the \(A_i\) are cyclic, then \(\gamma_m(A) = \gamma_m(A) \cap [A_i^A]\).

The following corollary is an interesting consequence of Theorem 2.9 for cyclic groups.

Corollary 2.11. Let \(\{A_i \mid i \in I\}\) be a family of cyclic groups. Then
i) If $N_c \subseteq V$, then $N_cM(V\prod_{i\in I}A_i) \cong H_v/[H_v, cA]$. Moreover if $V \subseteq N_{2c}$, then $V(\prod_{i\in I}^\times A_i)$ is a homomorphic image of $N_cM(V\prod_{i\in I}A_i)$.

ii) If $V \subseteq N_c$, then $N_cM(V\prod_{i\in I}A_i) \cong H_v/[H_v, cA]$. Moreover if $N_m \subseteq V$, then $\gamma_{c+1}(\prod_{i\in I}^\ast A_i)$ is a homomorphic image of $N_cM(V\prod_{i\in I}A_i)$.

**Proof.** i) Since the $A_i$ are cyclic groups and the $R_i$ have no commutators, it is concluded that $D_c = E_c$. So the epimorphism in the proof of Theorem 2.9, is actually an isomorphism. Also $N_cM(A_i) = 1$, therefore $N_cM(V\prod_{i\in I}A_i) \cong H_v/[H_v, cA]$. Now suppose $N_c \subseteq V \subseteq N_{2c}$. The inclusion $V(A) \subseteq \gamma_{c+1}(A)$ and Lemma 2.10 imply that $V(A) \subseteq [A_i^4]$ and thus $H_v = V(A) \cap [A_i^4] = V(A)$. So we have $N_cM(V\prod_{i\in I}A_i) = V(A)/[V(A), cA]$ and hence $V(A)/\gamma_{2c+1}(A)$ is a homomorphic image of $N_cM(V\prod_{i\in I}A_i)$. On the other hand since $V \subseteq N_{2c}$, we have $V(A)/\gamma_{2c+1}(A) = V(A)/\gamma_{2c+1}(A) = V(\prod_{i\in I}^\times A_i)$. This completes the proof.

ii) An argument similar to (i), shows that $N_cM(V\prod_{i\in I}A_i) \cong H_v/[H_v, cA]$. Now since $N_m \subseteq V \subseteq N_c$, $\gamma_{c+1}(A)/\gamma_{m+c+1}(A)$ is a homomorphic image of $N_cM(V\prod_{i\in I}A_i)$ and also

$$
\frac{\gamma_{c+1}(A)}{\gamma_{m+c+1}(A)} = \gamma_{c+1}(\frac{A}{\gamma_{m+c+1}(A)}) = \gamma_{c+1}(\prod_{i\in I} A_i).
$$

Hence the result follows. □

**Remark 2.12.** Let $\{A_i| i \in I\}$ be a family of groups.

i) If $V$ is the variety of trivial groups, then Theorem 2.9 implies that $\prod_{i\in I}^\times N_cM(A_i)$ is a homomorphic image of $N_cM(\prod_{i\in I}^\ast A_i)$. In particular $M(\prod_{i\in I}^\ast A_i) = \prod_{i\in I}^\times M(A_i)$ which is a result of Miller [15].

ii) If $V$ is the variety of nilpotent groups of class at most $n$, $N_n$, then main results of the second author [8] are obtained by Theorem 2.9 and corollary 2.11.
3 Nilpotent Products of Cyclic Groups

In this section we use a result of the previous section and find a structure for the \(c\)-nilpotent multiplier of the group \(G = \mathbb{Z}^n * \ldots * \mathbb{Z}^n \mathbb{Z}_{r_1}^n * \ldots * \mathbb{Z}_{r_t}^n\), where \(r_{i+1}\) divides \(r_i\) for all \(i\), \(1 \leq i \leq t - 1\), such that \((p, r_1) = 1\) for any prime \(p\) less than or equal to \(n + c\). The proof relies on basic commutators \([7]\) and related results. We recall that the number of basic commutators of weight \(c\) on \(n\) generators, denoted by \(\chi_c(n)\), is determined by Witt formula \([7]\). Also, M. Hall proved that if \(F\) is the free group on free generators \(x_1, x_2, \ldots, x_r\) and \(c_1, \ldots, c_t\) are basic commutators of weight \(1, 2, \ldots, n\), on \(x_1, \ldots, x_r\), then an arbitrary element \(f\) of \(F\) has a unique representation,

\[
f = c_1^{\beta_1} c_2^{\beta_2} \ldots c_t^{\beta_t} \mod \gamma_{n+1}(F).
\]

In particular the basic commutators of weight \(n\) provide a basis for the free abelian group \(\gamma_n(F)/\gamma_{n+1}(F)\) (see \([7]\)).

The following theorem represents the elements of some nilpotent products of cyclic groups in terms of basic commutators.

**Theorem 3.1** ([18]). Let \(A_1, \ldots, A_t\) be cyclic groups of order \(\alpha_1, \ldots, \alpha_t\) respectively, where if \(A_i\) is infinite cyclic, then \(\alpha_i = 0\). Let \(a_i\) generate \(A_i\) and let \(G = A_1^n * \ldots * A_t^n\), where \(n\) is greater than or equal to \(2\). Suppose that all the primes appearing in the factorizations of the \(\alpha_i\) are greater than or equal to \(n\) and \(u_1, u_2, \ldots, u_t\), are basic commutators of weight less than \(n\), on the letters \(a_1, \ldots, a_t\). Put \(N_i = \alpha_{i_j}\) if \(u_i = a_{i_j}\) of weight 1, and

\[N_i = \gcd(\alpha_{i_1}, \ldots, \alpha_{i_k})\]

if \(a_{i_j}, 1 \leq j \leq k\), appears in \(u_i\). Then every element \(g\) of \(G\) can be uniquely expressed as

\[g = \prod u_i^{m_i},\]
where the $m_i$ are integers modulo $N_i$ (by $gcd$ we mean the greatest common divisor).

The following theorem is an interesting consequence of Corollary 2.11.

**Theorem 3.2.** Let $\{A_i | i \in I\}$ be a family of cyclic groups. Then

i) if $n \geq c$, then $\mathcal{N}_c M (\prod_{i \in I}^n A_i) \cong \gamma_{n+1}(\prod_{i \in I} A_i)$;

ii) if $c \geq n$, then $\mathcal{N}_c M (\prod_{i \in I}^n A_i) \cong \gamma_{c+1}(\prod_{i \in I} A_i)$.

**Proof.** i) Put $V = \mathcal{N}_n$ in Corollary 2.11 and deduce that

\[ \mathcal{N}_c M (\prod_{i \in I}^n A_i) \cong H_n / [H_n, cA]. \]

On the other hand by Lemma 2.10, $H_n = \gamma_{n+1}(A) \cap [A^A] = \gamma_{n+1}(A)$. Therefore

\[ \mathcal{N}_c M (\prod_{i \in I}^n A_i) \cong \frac{\gamma_{n+1}(A)}{[\gamma_{n+1}(A), cA]} = \gamma_{n+1}(\frac{A}{\gamma_{n+c+1}(A)}) = \gamma_{n+1}(\prod_{i \in I} A_i). \]

ii) The result follows as for (i). $\square$

Now, we are in a position to state and prove the main result of this section.

**Theorem 3.3.** Let $G = A_1 \ast \ldots \ast A_{m+t}$ be the $n$th nilpotent product of cyclic groups such that $A_i \cong Z$ for $1 \leq i \leq m$ and $A_{m+j} \cong Z_{r_j}$ and $r_{j+1} | r_j$ for all $1 \leq j \leq t - 1$. If $(p, r_1) = 1$ for any prime $p$ less than or equal to $n+c$, then

i) if $n \geq c$, then $\mathcal{N}_c M (G) \cong Z^{(g_0)} \oplus Z_{r_1}^{(g_1-g_0)} \oplus \ldots \oplus Z_{r_t}^{(g_t-g_{t-1})}$;

ii) if $c \geq n$, then $\mathcal{N}_c M (G) \cong Z^{(f_0)} \oplus Z_{r_1}^{(f_1-f_0)} \oplus \ldots \oplus Z_{r_t}^{(f_t-f_{t-1})}$,

where $f_k = \sum_{i=1}^n \chi_{c+i}(m+k)$ and $g_k = \sum_{i=1}^c \chi_{n+i}(m+k)$ for $0 \leq k \leq t$ and $Z_{r}^{(d)}$ denotes the direct sum of $d$ copies of the cyclic group $Z_r$. 

12
**Proof.** i) If \( n \geq c \), then by Theorem 3.2, it is enough to find the structure of \( \gamma_{n+1}(\prod_{i \in I} A_i) \). Suppose that \( a_i \) generates \( A_i \) and \( F \) is the free group generated by \( a_1, ..., a_{m+t} \). Let \( B \) be the set of all basic commutators of weight \( 1, 2, ..., c + n \) on the letters \( a_1, ..., a_{m+t} \). Now define

\[
D = \{ u^N \mid u \in B \text{ and } N_i = gcd(\alpha_{i_1}, ..., \alpha_{i_k}) \text{ if } a_{i_j} \text{ appears in } u \text{ for } 1 \leq j \leq k \}.
\]

Then Theorem 3.1 implies that \( \prod_{i \in I} A_i = F/\langle D \rangle \gamma_{c+n+1}(F) \) and so

\[
\gamma_{n+1}(\prod_{i \in I} A_i) = \gamma_{n+1}(\frac{F}{\langle D \rangle \gamma_{c+n+1}(F)})
\]

\[
= \gamma_{n+1}(\frac{F}{\langle D \rangle \gamma_{c+n+1}(F) \cap \gamma_{n+1}(F)})
\]

\[
\approx \frac{\gamma_{n+1}(F) / \gamma_{c+n+1}(F)}{(\langle D \rangle \cap \gamma_{n+1}(F)) / \gamma_{c+n+1}(F)}.
\]

It can be deduced from Hall Theorem that \( \gamma_{n+1}(F) / \gamma_{c+n+1}(F) \) is a free abelian group with a basis \( \overline{B}_1 = \{ u^\gamma_{c+n+1}(F) \mid u \in B_1 \} \), where \( B_1 \) is the set of all basic commutators of weight \( n + 1, ..., c + n \) on \( a_1, ..., a_{m+t} \). Also, the uniqueness of the presentation of elements implies that the abelian group \( (\langle D \rangle \cap \gamma_{n+1}(F)) / \gamma_{c+n+1}(F) \) is free with a basis

\[
\overline{E} = \{ u^\gamma_{c+n+1}(F) \mid u \in D \cap \overline{B}_1 = \bigcup_{j=1}^{t} D_j \},
\]

where \( D_j \) is the set of all \( u^r \), such that \( u \) is a basic commutator of weight \( n + 1, ..., c + n \) on \( a_1, ..., a_{m+j} \) such that \( a_{m+j} \) appears in \( u \). Also we have

\[
|D_j| = \sum_{i=1}^{c} \chi_{n+i}(m+j) - \chi_{n+i}(m+j-1) = g_j - g_{j-1}.
\]

This completes the proof.

ii) The proof is similar to (i). □
Note that the authors with F. Mohammadzadeh [9] by a different method presented a similar structure for $\mathcal{N}_cM(G)$, for $c \geq n$ with a weaker condition $(p, r_1) = 1$ for any prime $p$ less than or equal to $n$.

**Remark 3.4.** The condition $r_{j+1} \mid r_j$, in the above theorem, simplifies the structure of the $c$-nilpotent multiplier of $G$ and gives a clear formula. One can use the above method and find the structure of $\mathcal{N}_cM(G)$ without the condition $r_{j+1} \mid r_j$, but with a more complex formula. For example, for a simple case if $G = \mathbb{Z}_r^n \ast \mathbb{Z}_s$ where $(p, r) = (p, s) = 1$ for any prime $p$ less than or equal to $n + c$ and $(r, s) = d$, then

i) if $n \geq c$, then $\mathcal{N}_cM(G) \cong \mathbb{Z}_d^{(\sum_{i=1}^c \chi_{n+i}(2))}$;

ii) if $c \geq n$, then $\mathcal{N}_cM(G) \cong \mathbb{Z}_d^{(\sum_{i=1}^n \chi_{c+i}(2))}$.

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