SOME RESULTS ON LOCALLY ANALYTIC SOCLE FOR $\text{GL}_n(\mathbb{Q}_p)$

YIWEN DING

Abstract. We study some closed rigid subspaces of the eigenvarieties, constructed using the Jacquet-Emerton functor for parabolic non-Borel subgroups. As an application (and motivation), we prove some new results on Breuil's locally analytic socle conjecture for $\text{GL}_n(\mathbb{Q}_p)$.

Contents

1. Introduction 1
2. Locally analytic representations and Jacquet-Emerton functors 5
  2.1. The BGG category $\mathcal{O}^P$ and the representations $\mathcal{F}_P(M, \pi)$ 5
  2.2. Jacquet-Emerton functors 7
  2.3. Adjunction formulas 9
  2.4. Structure of $J_{B,(P,\lambda_0)}(V)$ 11
3. Eigenvarieties and closed subspaces 15
  3.1. Notations and preliminaries 15
  3.2. Eigenvarieties 18
  3.3. Families of Galois representations 22
4. Local-global compatibility 25
References 27

1. Introduction

This note is devoted to prove some new results on Breuil’s locally analytic socle conjecture for $\text{GL}_n(\mathbb{Q}_p)$. We recall the conjecture, summarize some results and sketch the proof in $\text{GL}_3(\mathbb{Q}_p)$ case in the introduction.

Let $F$ be a quadratic imaginary extension of $\mathbb{Q}$ with $p$ split in $F$, and we fix a place $u$ above $p$; let $G$ be a definite unitary group over $\mathbb{Q}$ associated to $F/\mathbb{Q}$ which is split at $p$. Let $E$ be a finite extension of $\mathbb{Q}_p$ sufficiently large, $U^p$ a compact open subgroup of $G(\mathbb{A}^\infty)$, put

$$\hat{S}(U^p, E) := \{ f : G(\mathbb{Q}) \backslash G(\mathbb{A}^\infty) / U^p \to E \mid f \text{ is continuous} \},$$

The author is supported by EPSRC grant EP/L025485/1.
which is a Banach space over $E$ equipped with a continuous action of $G(\mathbb{Q}_p) \cong \text{GL}_3(\mathbb{Q}_p)$ (this isomorphism depends on the choice of $u$), and a continuous action of (commutative) Hecke algebra $\mathcal{H}^p$ outside $p$. The action of $\mathcal{H}^p$ commutes with that of $\text{GL}_3(\mathbb{Q}_p)$. Let $\rho$ be a continuous representation of $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ over $E$ associated to automorphic forms of $G$, and we associate to $\rho$ a maximal ideal of $\mathcal{H}^p$ (shrinking $\mathcal{H}^p$ if needed). Suppose $(\hat{S}(U^p, E)_{\text{alg}})^{m_\rho} \neq 0$, where “$\text{alg}$” denotes the locally algebraic vectors for $\text{GL}_3(\mathbb{Q}_p)$, $(\cdot)^{m_\rho}$ denotes the maximal $E$-vector space on which $\mathcal{H}^p$ acts via $\mathcal{H}^p \to \mathcal{H}^p/\mathfrak{m}_\rho$. Put

$$\hat{\Pi}(\rho) := \hat{S}(U^p, E)^{m_\rho},$$

which is an admissible unitary Banach representation of $\text{GL}_3(\mathbb{Q}_p)$, and is supposed to be (a direct sum) of the right representation corresponding to $\rho_p := \rho|_{\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})} \cong \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ in $p$-adic Langlands programme ([4]). The structure of $\hat{\Pi}(\rho)$ is still quite mysterious. In [5], Breuil made a conjecture on the locally analytic socle of $\hat{\Pi}(\rho)$, which we recall now.

Suppose $\rho_p$ is crystalline and very regular (cf. Def.3.19). Let $h := (h_1, h_2, h_3) \in \mathbb{Z}^3$ (with $h_1 < h_2 < h_3$) be the Hodge-Tate weights of $\rho_p$, $\hat{\varphi} := (\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3) \in E^3$ be the (ordered) eigenvalues of the crystalline Frobenius $\varphi$ on $D_{\text{rig}}(\rho_p)$. For any $w \in S_3$, put $w(\hat{\varphi}) := (\hat{\varphi}_w^{-1}(1), \hat{\varphi}_w^{-1}(2), \hat{\varphi}_w^{-1}(3))$, which is called a refinement for $\rho_p$. The local Langlands correspondence thus associates to $\rho_p$ a locally algebraic representation of $\text{GL}_3(\mathbb{Q}_p)$ over $E$:

$$C(1, w) := \left( \text{Ind}_{\text{alg}}^{\text{GL}_3(\mathbb{Q}_p)} \psi_w \delta_B^{-1} \right) \otimes E \mathcal{L}(\lambda)$$

where $\lambda := (-h_1, 1 - h_2, 2 - h_3)$ is a dominant weight for $T(\mathbb{Q}_p)$ (the group of diagonal matrices of $\text{GL}_3(\mathbb{Q}_p)$), $\mathcal{L}(\lambda)$ denotes the irreducible algebraic representation of $\text{GL}_3(\mathbb{Q}_p)$ with highest weight $\lambda$, $\psi_w := \text{unr}(\phi_{w(1)}^{-1}) \otimes \text{unr}(\phi_{w(2)}^{-1}) \otimes \text{unr}(\phi_{w(3)}^{-1})$, and $\delta_B = \text{unr}(p^{-2}) \otimes 1 \otimes \text{unr}(p^2)$. Since $\rho_p$ is very regular, the representation $C(1, w)$ are all isomorphic and irreducible. For each refinement $w(\hat{\varphi})$ of $\rho_p$, one can get a triangulation of $D_{\text{rig}}(\rho_p)$ of parameter $\delta = (\delta_1, \delta_2, \delta_3)$

$$\delta = \left( \text{unr}(\phi_{w(1)}^{-1}) x^{-h_{\text{alg}}(w)^{-1}(1)}, \text{unr}(\phi_{w(2)}^{-1}) x^{-h_{\text{alg}}(w)^{-1}(2)}, \text{unr}(\phi_{w(3)}^{-1}) x^{-h_{\text{alg}}(w)^{-1}(3)} \right)$$

with $w_{\text{alg}}(w) \in S_3$ (determined by $w$ and $\rho_p$). Recall the refinement is called non-critical, if $w_{\text{alg}}(w) = 1$. For each pair $(w_{\text{alg}}, w) \in S_3 \times S_3$, Breuil defined an irreducible locally analytic representation $C(w_{\text{alg}}, w)$ (cf. (32)), and conjectured

**Conjecture 1.1** ([5], [6, Conj.5.3]). For $(w_{\text{alg}}, w) \in S_3 \times S_3$, $C(w_{\text{alg}}, w)$ is a subrepresentation of $\hat{\Pi}(\rho)$ if and only if $w_{\text{alg}} \leq w_{\text{alg}}(w)$ for the Bruhat ordering.

Roughly speaking, $\text{soc} \hat{\Pi}(\rho)$ would (conjecturally) measure the criticalness of $\rho_p$. In [6], Breuil proved some results were also obtained in [3])

**Theorem 1.2** (cf. [6, Thm.1.2]). (1) If $C(w_{\text{alg}}, w)$ is a subrepresentation of $\hat{\Pi}(\rho)$, then $w_{\text{alg}} \leq w_{\text{alg}}(w)$.

(2) If $w_{\text{alg}}(w) \neq 1$, then there exists $w_{\text{alg}} \neq 1$, such that $C(w_{\text{alg}}, w)$ is a subrepresentation of $\hat{\Pi}(\rho)$.

In particular, when $\text{lg} w_{\text{alg}}(w) \leq 1$, the conjecture 1.1 was proved. In the $\text{GL}_2$ case, one needs to put more global hypothesis to get Thm.1.2 (2), and in general one only gets a weaker version of Thm.1.2 (1) (cf. [6, Thm.1.2]). The main result of this note (in $\text{GL}_3(\mathbb{Q}_p)$ case) is the following theorem which improves Thm.1.2 (2).

**Theorem 1.3** (cf. Thm.4.4, Cor.4.5). Let $s \in S_3$ be a simple reflection (i.e. $s \in \Delta := \{(12), (23)\}$), then $C(s, w)$ is a subrepresentation of $\hat{\Pi}(\rho)$ if and only if $s \leq w_{\text{alg}}(w)$. In particular, if $\text{lg} w_{\text{alg}}(w) \geq 2$, then $\oplus_{s \in \Delta} C(s, w)$ is a subrepresentation of $\hat{\Pi}(\rho)$. 

2
Let’s remark that the general GLₙ(Qₚ) case is more subtle (beside the global hypothesis), but we do prove that if lg walg(φ) ≥ 2, there exist more than one walg such that C(walg, φ) ⊆ P(φ) (see Cor. 4.6).

The proof of Thm. 1.3 follows the same strategy of [6, i.e. using results on the geometry of the eigenvariety (due to Bergdall [2], Chenevier [10]) and locally analytic representation theory (adjunction formulas due to Breuil [6]) to prove the existence of companion points (on the eigenvariety), which would correspond to irreducible components of soc P(φ). While, a key idea in this note, inspired by the adjunction formula [6, Thm.4.3] (see also [6, Rem.9.11 (ii)]), is to locate some companion points by considering closed subspaces of the eigenvariety constructed via the Jacquet-Emerton functor for parabolic non-Borel subgroups. In fact, such closed subspaces were already constructed by Hill and Loeffler [20] (although their motivation was rather different from ours).

We sketch the proof of Thm.1.3 for s = (23) and discuss some intermediate results. We keep the notations.

Jacquet-Emerton functors. Let \( P \supset B \) with the Levi subgroup \( L_P = GL_2 \times GL_1 \) (the case \( s = (12) \) would use the other maximal parabolic proper subgroup). Denote by \( \mathcal{L}_P(\lambda) \) the irreducible algebraic representation of \( L_P \) with highest weight \( \lambda \), for an admissible locally analytic representation \( V \) of \( GL_3(Q_p) \), we put (cf. [20], and §1.3)

\[
J_{B,(P,\lambda)}(V) := J_{B \cap L_P}((J_P(V) \otimes_E \mathcal{L}_P(\lambda))_{\infty} \otimes_E \mathcal{L}_P(\lambda))
\]

where \( ^{\text{\"(\cdot)_{\infty}\}} \) denotes the smooth vectors for \( GL_2(Q_p) \) which acts on \( J_P(V) \otimes_E \mathcal{L}_P(\lambda) \) via \( GL_2 \leftarrow GL_2 \times GL_1 \cong L_P, \mathcal{L}_P(\lambda) \) denotes the algebraic dual of \( \mathcal{L}_P(\lambda) \). In fact, \( J_{B,(P,\lambda)}(V) \) is a closed subrepresentation of \( T(Q_p) \) of the usual Jacquet-Emerton module \( J_B(V) \), and thus is an essentially admissible locally analytic representation of \( T(Q_p) \). We would use the subfunctor \( J_{B,(P,\lambda)}(\cdot) \) of \( J_B(\cdot) \) to construct a closed subspace of the eigenvariety. The adjunction property for \( J_{B,(P,\lambda)}(\cdot) \), which we discuss below, would allow us to get some nice properties of such closed subspace (cf. Thm.1.4, 1.5).

Adjunction formulas. Suppose \( V \) is moreover very strongly admissible (cf. [18, Def.0.12]), we have an adjunction formula (cf. Thm.2.15) (obtained by combining Breuil’s adjunction formula [6, Thm.4.3] and the adjunction formula for the classical Jacquet functor):

\[
(1) \quad \text{Hom}_{GL_3(Q_p)} \left( \mathcal{F}_{\mathcal{L}_P}^D \left( (U(g) \otimes_U \mathcal{L}_P(\lambda))^{\vee}, (\text{Ind}_{E_3(Q_p)}^{GL_3(Q_p)} \psi \otimes_E \delta^{-1}_{B})^{\infty} \right), V \right) \xrightarrow{\sim} \text{Hom}_{T(Q_p)} \left( \psi \otimes_E \chi, J_{B,(P,\lambda)}(V) \right),
\]

where \( \psi \) is a finite length smooth representation of \( T(Q_p) \) over \( E \), \( \chi \) denotes the algebraic character of \( T(Q_p) \) with weight \( \lambda \) and we refer to §2.1 for the representations \( \mathcal{F}_{\mathcal{L}_P}^D(\cdot, \cdot) \) etc.; meanwhile, recall that for \( J_B(V) \), by [6, Thm.4.3], one has

\[
(2) \quad \text{Hom}_{GL_3(Q_p)} \left( \mathcal{F}_{\mathcal{L}_P}^D \left( (U(g) \otimes_U (-\lambda))^{\vee}, \psi \otimes_E \delta^{-1}_{B} \right), V \right) \xrightarrow{\sim} \text{Hom}_{T(Q_p)} \left( \psi \otimes_E \chi, J_B(V) \right).
\]

Note the generalized Verma module \( U(g) \otimes_U \mathcal{L}_P(\lambda) \) has 2 irreducible components, while the Verma module \( U(g) \otimes_U (-\lambda) \) has 6 irreducible components (with \( U(g) \otimes_U \mathcal{L}_P(\lambda) \) as a quotient), consequently the locally analytic representation (in (1))

\[
(3) \quad \mathcal{F}_{\mathcal{L}_P}^D \left( (U(g) \otimes_U \mathcal{L}_P(\lambda))^{\vee}, (\text{Ind}_{E_3(Q_p)}^{GL_3(Q_p)} \psi \otimes_E \delta^{-1}_{B})^{\infty} \right),
\]

as a quotient of \( \mathcal{F}_{\mathcal{L}_P}^D \left( (U(g) \otimes_U (-\lambda))^{\vee}, \psi \otimes_E \delta^{-1}_{B} \right) \), has much fewer irreducible components. For example, when \( \psi = \psi_w \), then the representation in (3) has the form (where the line denotes an extension)

\[
(4) \quad C(s, w) \rightarrow C(1, w);
\]
while, \( \mathcal{F}_B^G \left( \left( U(\mathfrak{g}) \otimes U(\mathfrak{g}) (-\lambda) \right) \mathcal{V}, \psi_w \delta_B^{-1} \right) \) has the form

\[
\begin{align*}
C(s's, w) & \quad C(s, w) \\
C(s's, w) & \quad C(1, w) \\
C(s', w) & \quad C(s, w)
\end{align*}
\]

where \( s' \) denotes the simple reflection different from \( s \). The adjunction property (1) is somehow the key point of this note.

**Eigenvariety and closed subspaces.** Consider \( J_{B,(P,\lambda)}(\widehat{\mathcal{S}}(K^p, U)_{an}) \), which is an essentially admissible locally analytic representation of \( T(\mathbb{Q}_p) \) equipped moreover with a continuous action of \( \mathcal{H}^p \). Following Emerton, one gets a rigid space \( \mathcal{V}(P, \lambda) \) over \( E \) such that there exists a bijection

\[
\mathcal{V}(P, \lambda)(\overline{E}) \sim \left\{ (\chi, h) \in \widehat{T}(\overline{E}) \times \text{Spec} \mathcal{H}^p(\overline{E}) \mid (J_{B,(P,\lambda)}(\widehat{\mathcal{S}}(K^p, U)_{an}) \otimes_E \overline{E})^{T(\mathbb{Q}_p)=\chi, \mathcal{H}^p=h} \neq 0 \right\}.
\]

Indeed, \( \mathcal{V}(P, \lambda) \) is a closed subspace of the eigenvariety \( \mathcal{V} \) (constructed from \( J_B(\widehat{S}(U^p, E)_{an}) \)). By the definition of \( J_{B,(P,\lambda)}(\cdot) \), one easily sees \((\chi_1 \otimes \chi_2 \otimes \chi_3, h) \in \mathcal{V}(P, \lambda)(\overline{E}) \) implies that \( wt(\chi_1) = -h_1, \ wt(\chi_2) = 1 - h_2 \). However, the rigid space \( \mathcal{V}(P, \lambda) \) would be more subtle than the closed subspace, denoted by \( \mathcal{V}(\lambda) \), of \( \mathcal{V} \) lying above the corresponding weight space. One has

**Theorem 1.4** (cf. Thm.3.10). The classical points are Zariski-dense in \( \mathcal{V}(P, \lambda) \).

Thus one can view (the reduced subspace of) \( \mathcal{V}(P, \lambda) \) as the Zariski closure of the classical points in \( \mathcal{V}(\lambda) \). To prove Thm.1.4, one needs a classicality criterion for closed points in \( \mathcal{V}(P, \lambda) \) proved in §3.2.1 using (1), which is stronger than the general classicality criterion for the points in \( \mathcal{V} \) (e.g. see [10, §4.7.3]). Indeed, the general classicality criterion for points in \( \mathcal{V} \) seems not enough for the density of classical points in \( \mathcal{V}(P, \lambda) \) since certain weights for \( \mathcal{V}(P, \lambda) \) are fixed.

**Some étaleness results.** Denote by \( h_\rho : \mathcal{H}^p \to \mathcal{H}^p/\mathfrak{m}_\rho \to E \), one can associate to \((\rho, w)\) a classical point \( z_{\rho, w} := (\chi_w, h_\rho) \) of \( \mathcal{V}(P, \lambda) \), where \( \chi_w = \psi_w \chi_\lambda \). Consider the composition \( \kappa : \mathcal{V}(P, \lambda) \to \widehat{T} \xrightarrow{p_3} \widehat{\mathbb{Q}_p^\times} \to \mathbb{Z}_p^\times \), where \( p_3 \) denotes the projection to the third factor. One has (compare the second statement with [11, Thm.4.8])

**Theorem 1.5** (cf. Prop.3.20, Thm.3.22 and the proof of Thm.4.4). If \( C(s, w) \) is not a subrepresentation of \( \Pi(\rho)_{an} \), then \( \kappa \) is étale at \( z_{\rho, w} \). Consequently (by Thm.1.2 (1)), if \( w_{\text{alg}}(w) \in \{1, (12)\} \) (which is in fact the Weyl group of \( L_P \)), then \( \kappa \) is étale at \( z_{\rho, w} \).

To prove this theorem, as in [6] (e.g. see the proof of [6, Thm.9.10]), we use the adjunction formula (1) and the same arguments in the proof of [11, Thm.4.8]. Indeed, applying the adjunction formula (1) to the generalized eigenspace \( J_{B,(P,\lambda)}(\widehat{\mathcal{S}}(U^p, E)_{an})[T(\mathbb{Q}_p) = \chi_w, \mathcal{H}^p = h_\rho] \) then using (4), it’s not difficult to show that if \( C(s, w) \) is not a subrepresentation of \( \Pi(\rho) \), then

\[
J_{B,(P,\lambda)}(\widehat{\mathcal{S}}(U^p, E)_{an})[T(\mathbb{Q}_p) = \chi_w, \mathcal{H}^p = h_\rho] = J_{B,(P,\lambda)}(\widehat{\mathcal{S}}(U^p, E)_{\text{alg}})[T(\mathbb{Q}_p) = \chi_w, \mathcal{H}^p = h_\rho]
\]

(such equality might be viewed as an infinitesimal classicality result), from which one can deduce the étaleness result for \( \mathcal{V}(\lambda, P) \) by the same argument as in the proof of [11, Thm.4.8].

Conversely, we have the following due to Bergdall (cf. [2, Thm.B]).

**Theorem 1.6** (cf. Thm.3.23). If \( w_{\text{alg}}(w) \notin \{1, (12)\} \), then \( \kappa \) is not étale at \( z_{\rho, w} \).
Combing Thm.1.5 and Thm.1.6, Thm.1.3 thus follows. We refer to the body of the text for more detailed and more precise statements (with slightly different notations).

Acknowledgement. The debt that this work owes to [6] is clear, and I also would like to thank Christophe Breuil for answering my questions.

2. Locally analytic representations and Jacquet-Emerton functors

Let $G$ be a split connected reductive algebraic group over $\mathbb{Q}_p$, $T$ be a split maximal torus of $G$, $Z_G \subseteq T$ the center of $G$, $B$ a Borel subgroup of $G$ containing $T$, $P \supseteq B$ a parabolic subgroup of $G$ with $L_P$ the Levi subgroup and $N_P$ the nilpotent radical, thus $T \cong L_B$ and put $N := N_B$. Let $g$, $t$, $\mathfrak{z}_G$, $b$, $\mathfrak{p}$, $\mathfrak{l}_P$, $\mathfrak{n}_P$, $n$ denote the associated Lie algebras over $\mathbb{Q}_p$ of $G$, $T$, $Z_G$, $B$, $P$, $L_P$, $N_P$, $N$ respectively. Let $E$ be a finite extension of $\mathbb{Q}_p$, with $\mathcal{O}_E$ the ring of integers and $\varpi_E$ a uniformizer of $\mathcal{O}_E$.

2.1. The BGG category $\mathcal{O}^p$ and the representations $\mathcal{F}_P(M, \pi)$

Definition 2.1 ([21, §9.3]). Let $\mathcal{O}^p$ be the full subcategory of the category of linear representations of $\mathfrak{g}$ on $E$-vector spaces made out of representations $M$ such that:

1. $M$ is a finite type $U(\mathfrak{g}) \otimes_{\mathbb{Q}_p} E$-module;
2. $M|_{U(\mathfrak{p})}$ is a direct sum of irreducible algebraic $U(\mathfrak{p}) \otimes_{\mathbb{Q}_p} E$-modules;
3. for all $v \in M$, the $E$-vector space $(U(\mathfrak{n}_P) \otimes_{\mathbb{Q}_p} E)v$ is finite dimensional.

One has (cf. [21, §1.1, §1.2, §9.3])

Theorem 2.2. (1) The category $\mathcal{O}^p$ is abelian, closed under submodules, quotients and finite direct sums.

(2) For $P_1 \subset P_2$ two parabolic subgroups of $G$, $\mathcal{O}^{p_2}$ is a full subcategory of $\mathcal{O}^{p_1}$.

(3) Let $W$ be an irreducible algebraic representation of $L_P$, the generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ lies in $\mathcal{O}^p$, and admits a unique irreducible quotient denoted by $M(\lambda)$ where $\lambda$ is the highest weight of $W$.

Denote by $\Phi$ the root system of $G$, $\Delta$ the set of simple roots with respect to $B$ and $\Phi^+$ (resp. $\Phi^-$) the set of positive (resp. negative) roots. Any parabolic subgroup $P$ containing $B$ corresponds thus to a set of positive simple roots denoted by $\Delta_P$, which is the simple roots of $L_P$ with respect to $B \cap L_P$ (thus $\Delta_B = \emptyset$). Consider the $E$-vector space $t^* := \text{Hom}_{\mathbb{Q}_p}(t, E)$. Any element in $t^*$ is a weight of $t$ (or a weight for $G$). Any root $\alpha \in \Phi$ can be viewed as a weight still denoted by $\alpha$, and one has a natural embedding $\oplus_{\alpha \in \Delta} E\alpha \rightarrow t^*$. In fact, one has $\cap_{\alpha \in \Phi} \text{Ker}(\alpha) = \mathfrak{z}_G$, thus a natural decomposition $t^* \cong (\oplus_{\alpha \in \Delta} E\alpha) \oplus \mathfrak{z}_G^\circ$. Recall the root space $\oplus_{\alpha \in \Delta} E\alpha$ is equipped with a natural inner product (using Killing form) which extends to $t^*$ by putting $\langle \alpha, z \rangle = 0$ for all $z \in \mathfrak{z}_G^\circ$. Let $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$. We call a weight $\lambda$ is $P$-dominant if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Delta_P$, $\lambda$ is called dominant if $\lambda$ is $G$-dominant.

Let $\lambda \in t^*$, denote by $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda$ the Verma module with highest weight $\lambda$, denote by $\mathscr{L}(\lambda)$ the unique simple quotient of $M(\lambda)$. By [21, Thm.1.3], every simple object in $\mathcal{O}^p$ is isomorphic to a such $\mathscr{L}(\lambda)$. Note if $\lambda$ is dominant, $\mathscr{L}(\lambda)$ is thus the unique finite dimensional algebraic representation of $G$ with highest weight $\lambda$.

Proposition 2.3 (cf. [21, p.185]). Let $\lambda \in t^*$, $\mathscr{L}(\lambda) \in \mathcal{O}^p$ lies in $\mathcal{O}^p$ if and only if $\lambda$ is $P$-dominant. In particular, there exists a unique maximal parabolic $P$ of $G$ containing $B$ such that $\mathscr{L}(\lambda) \in \mathcal{O}^p$.

Following Orlik-Strauch [24], one can associate a locally analytic representation $\mathcal{F}_P^G(M, \pi)$ of $G(\mathbb{Q}_p)$ with $M \in \mathcal{O}^p$ and $\pi$ being a finite length smooth admissible representation of $L_P(\mathbb{Q}_p)$. We recall the constructions and some properties.
For a Hausdorff locally convex topological vector space $V$ over $E$, denote by $C^{la}(G(\mathbb{Q}_p), V)$ the $E$-vector space of locally analytic functions of $G(\mathbb{Q}_p)$ with values in $V$, equipped with the finest locally convex topology. This space is equipped with a right regular locally analytic $G(\mathbb{Q}_p)$-action $g(f)(g') = f(g' g)$.

While this space can also be equipped with another locally analytic $G(\mathbb{Q}_p)$-action given by $(g \cdot f)(g') = f(g^{-1} g')$, which induces by derivation a continuous $g$-action

$$\frac{d}{dt} f(\exp(-t g))|_{t=0}$$

for $x \in g, f \in C^{la}(G(\mathbb{Q}_p), V)$ and $g \in G(\mathbb{Q}_p)$.

Let $M \in \mathcal{O}^p$, and $W \subseteq M$ be a finite dimensional algebraic representation of $p$ over $E$, by Def.2.1, enlarging $W$, we can (and do) assume $W$ generates $M$ over $U(g) \otimes_{\mathbb{Q}_p} E$. One has thus an exact sequence in the category $\mathcal{O}^p$

$$0 \to \text{Ker}(\phi) \to U(g) \otimes_{U(p)} W \xrightarrow{\phi} M \to 0.$$ 

The $p$-action on $W$ can be lifted to a unique algebraic $P(\mathbb{Q}_p)$-action (cf. [24, Lem.3.2]). Denote by $W' := \text{Hom}_E(W, E)$ with $(p(f))(v) := f(p^{-1} v)$ for $p \in P(\mathbb{Q}_p)$ and $v \in W$.

Let $\pi$ be a finite length smooth admissible representation of $L_P(\mathbb{Q}_p)$ over $E$, each element of $U(g) \otimes_{\mathbb{Q}_p} W$ would induce a continuous morphism

$$C^{la}(G(\mathbb{Q}_p), W' \otimes_E \pi) \to C^{la}(G(\mathbb{Q}_p), \pi)$$

with $(x \cdot g)(f) := [x \to (x \cdot f)(g)(v)]$, where $x \in U(g), v \in W', f \in C^{la}(G(\mathbb{Q}_p), W' \otimes_E \pi)$, $x \cdot f$ denotes the $g$-action on $C^{la}(G(\mathbb{Q}_p), W' \otimes_E \pi)$ as in (6). Consider $(\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi)^{an} \subseteq C^{la}(G(\mathbb{Q}_p), W' \otimes_E \pi)$, by restriction, one gets thus an $E$-linear map

$$U(g) \otimes_{U(p)} W \to \text{Hom}_E \left( (\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi)^{an}, C^{la}(G(\mathbb{Q}_p), \pi) \right).$$

One can check this map factors through $U(g) \otimes_{U(p)} W$ (cf. [5, Lem.2.1]). Following [24, §4], put (where $X \cdot f$ is given via (7))

$$\mathcal{F}^G_{P}(M, \pi) := \left\{ f \in (\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi)^{an}, X \cdot f = 0, \forall X \in \text{Ker}(\phi) \right\},$$

which can be checked to be a closed subrepresentation of $(\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W' \otimes_E \pi)^{an}$ and independent of the choice of $W$ (thus only depends on $M$ and $\pi$, cf. [24, Prop.4.5]).

**Theorem 2.4 ([24, Thm]).** Keep the above notations.

1. $\mathcal{F}^G_{P}(M, \pi)$ is non-zero if and only if $M$ and $\pi$ are non-zero.

2. The functor $(M, \pi) \mapsto \mathcal{F}^G_{P}(M, \pi)$ (covariant on $\pi$, and contravariant on $M$) is exact in both arguments.

3. Let $Q \supset P$ be another parabolic subgroup and assume $M$ lies in $\mathcal{O}^a \subset \mathcal{O}^p$, then

$$\mathcal{F}^G_{P}(M, \pi) \cong \mathcal{F}^G_{Q}(M, (\text{Ind}_{P(\mathbb{Q}_p)}^{L_Q(\mathbb{Q}_p)} \circ L_Q(\mathbb{Q}_p) \pi)^\infty),$$

where $(\cdot)^\infty$ denotes the usual smooth parabolic induction.

4. Suppose the following hypothesis

   - if $\Phi$ has irreducible components of type $B$, $C$ or $F_4$, then $p$ is odd,
   - if $\Phi$ has irreducible components of type $G_2$, then $p > 3$;

if $M$ and $\pi$ are irreducible and $P$ is the maximal parabolic subgroup of $M$, then $\mathcal{F}^G_{P}(M, \pi)$ is an irreducible representation of $G(\mathbb{Q}_p)$.
Remark 2.5. Keep the notations of Thm. 2.4 and the hypothesis in (4), if the representation \( \text{Ind}_{P(Q_p)}^{G(Q_p)} \pi \) is irreducible, we see the irreducible components of \( \mathcal{F}_P^G(M, \pi) \) are exactly \( \mathcal{F}_P^G(M_i, \pi) \) where the \( M_i \)'s are the irreducible components of \( M \) in \( \mathcal{O}^P \).

2.2. Jacquet-Emerton functors. Let \( V \) be an essentially admissible locally analytic representation of \( G(Q_p) \) over \( E \) (cf. [15, Def.6.4.9]). Let \( N_p^0 \) be an open compact subgroup of \( N_P(Q_p) \),

\[
L_P(Q_p)^+ := \{ g \in L_P(Q_p) \mid gN_P^0g^{-1} \subseteq N_P^0 \},
\]

The closed subspace \( V^{N_p^0} \) (of vectors fixed by \( N_p^0 \)) is equipped with a natural \( L_P(Q_p)^+ \)-action given by (cf. [17, §3.4])

\[
\pi(g)(v) := \sum_{n \in N_p^0/gN_p^0g^{-1}} (ng)v, \quad v \in V^{N_p^0}, \quad g \in L_P(Q_p)^+.
\]

Following Emerton ([17, Def.3.2.1, 3.4.5]), put

\[
J_P(V) := (V^{N_p^0})_{\text{fs}} := \mathcal{L}(Z_{L_P(Q_p)})(O(Z_{L_P}), V^{N_p^0})
\]

where “\( \mathcal{L} \)” signifies continuous linear maps, “fs” signifies finite slope, \( Z_H \) denotes the center of \( H \) for an algebraic group \( H \), \( Z_{L_P}(Q_p)^+ := Z_{L_P}(Q_p) \cap L_P(Q_p)^+ \). \( Z \) denotes the rigid space over \( E \) parameterizing locally analytic characters of \( Z(Q_p) \) for a commutative algebraic group \( Z \) over \( Q_p \) and \( O(Z) \) denotes the global sections of \( Z \). Roughly speaking, \( J_P(V) \) is the maximal subspace of \( V^{N_p^0} \) on which the \( Z_{L_P}(Q_p)^+ \)-action extends canonically to a locally analytic \( Z_{L_P}(Q_p)^+ \)-action. Since \( L_P(Q_p)^+Z_{L_P}(Q_p) = L_P(Q_p) \), \( J_P(V) \) is equipped with a natural action of \( L_P(Q_p) \).

Theorem 2.6 ([17, Prop.3.4.11, Thm.4.2.32]). Keep the above notation, \( J_P(V) \) is independent of the choice of \( N_p^0 \), and is an essentially admissible locally \( Q_p \)-analytic representation of \( L_P(Q_p) \).

Theorem 2.7 ([20, Thm.5.3]). Let \( P_1 \subset P_2 \) be two parabolic subgroups of \( G \), \( V \) be an essentially admissible locally analytic representation of \( G(Q_p) \), then one has a natural isomorphism of \( L_{P_1}(Q_p)(L_{P_2}(Q_p)) \)-representations

\[
J_{L_{P_1}(Q_p), L_{P_2}(Q_p)}(J_{P_2}(V)) \cong J_{P_1}(V).
\]

2.2.1. A digression: locally algebraic vectors. Denote by \( G^D \) the derived subgroup of \( G \), and \( g^D \) the Lie algebra of \( G^D \) over \( Q_p \). Since \( G \) is reductive, \( G^D \) is semisimple, and we have a local isomorphism \( Z_G \times G^D \cong G \). Let \( T_{GD} := T \cap G^D \), which is thus a maximal split torus of \( G^D \), let \( t_{GD} \) denote the Lie algebra of \( T_{GD} \). The inclusion \( t_{GD} \hookrightarrow t \) induces a projection \( \text{Hom}_E(t, E) \rightarrow \text{Hom}_E(t_{GD}, E) \). In fact, we have an isomorphism \( t \cong t_{GD} \times \bar{Z}_G \), and \( \oplus_{\alpha \in A} E\alpha \cong \text{Hom}_E(t_{GD}, E) \). Thus a weight \( \lambda \) of \( t \) is dominant if and only if its restriction to \( t_{GD} \) is dominant.

For a locally \( Q_p \)-analytic representation \( V \) of \( G(Q_p) \) over \( E \), denote by \( V_0 \) the \( E \)-vector space generated by the vectors fixed by \( g^D \), in other words, the smooth vectors for \( G^D \). It’s straightforward to check \( V_0 \) is stable under the \( G(Q_p) \)-action and is a closed subrepresentation of \( V \). Let \( \lambda_0 \) be a dominant weight for \( G^D \), \( \lambda \) a weight for \( G \) above \( \lambda_0 \) (which is thus also dominant). Put

\[
V_{\lambda_0} := \{ V \otimes_E \mathcal{L}(\lambda)' \} \otimes_E \mathcal{L}(\lambda),
\]

where \( \mathcal{L}(\lambda)' \) denotes the dual algebraic representation of \( \mathcal{L}(\lambda) \). And we would use \( -\lambda \) to denote the highest weight of \( \mathcal{L}(\lambda)' \).

Lemma 2.8. Keep the above notation, \( V_{\lambda_0} \) is a closed subrepresentation of \( V \) of \( G(Q_p) \) and is independent of the choice of \( \lambda \), in other words, \( V_{\lambda_0} \) only depends on \( \lambda_0 \).
Proof. One has a natural $G(\mathbb{Q}_p)$-invariant map

$$
(V \otimes_E \mathcal{L}(\lambda'))_0 \otimes_E \mathcal{L}(\lambda) \rightarrow V, \, v \otimes w' \otimes w \mapsto w'(w)v.
$$

Moreover, by [15, Prop.4.2.4] (applied to $G = G^D(\mathbb{Q}_p)$), we know this is injective. Let $\mu$ be another dominant weight which restricts to $\lambda_0$, then $\mathcal{L}(\mu)$ would differ from $\mathcal{L}(\lambda)$ by certain determinant twist, from which the second part easily follows.

In particular, if $V$ is essentially admissible, so is $V_{\lambda_0}$ (cf. [15, Prop.6.4.11]).

**Lemma 2.9.** Let $\lambda$ be a dominant weight for $G$, $\pi$ be a smooth representation of $G(\mathbb{Q}_p)$ over $E$, $V$ a locally analytic representation of $G(\mathbb{Q}_p)$ smooth for the $G^D(\mathbb{Q}_p)$-action, then the following map

$$
\text{Hom}_{G(Q_p)}(\pi, V) \rightarrow \text{Hom}_{G(Q_p)} \left( \pi \otimes_E \mathcal{L}(\lambda), V \otimes_E \mathcal{L}(\lambda) \right), \, f \mapsto f \otimes \text{id},
$$

is bijective.

**Proof.** A $G(\mathbb{Q}_p)$-invariant map $g : \pi \otimes E \mathcal{L}(\lambda) \rightarrow V \otimes E \mathcal{L}(\lambda)$ induces

$$
\pi \rightarrow \pi \otimes E \mathcal{L}(\lambda) \otimes E \mathcal{L}(\lambda') \rightarrow V \otimes E \mathcal{L}(\lambda) \otimes E \mathcal{L}(\lambda');
$$

since $\pi$ is smooth, this map factors through in particular

$$
\pi \rightarrow (V \otimes E \mathcal{L}(\lambda) \otimes E \mathcal{L}(\lambda'))_0 \cong V
$$

where the last isomorphism follows from the isomorphism above [15, Prop.4.2.4]. One easily sees this gives an inverse (up to scalars) of (10).

2.2.2. Subfunctors. Return to the situation before §2.2.1 (in particular, $V$ is an essentially admissible locally analytic representation of $G(\mathbb{Q}_p)$ over $E$), and let $P_1 \subseteq P_2$ be two parabolic subgroups of $G$ containing $B$. Let $\lambda_0$ be a dominant weight for $L^P_{P_2}$ (the derived subgroup of $L_{P_2}$), for an essentially admissible locally analytic representation $V$ of $G(\mathbb{Q}_p)$, put (cf. §2.2.1)

$$
J_{P_1,(P_2,\lambda_0)}(V) := J_{P_1 \cap L_{P_2}}(J_{P_2}(V)_{\lambda_0}).
$$

By the left exactness of the functor $J_{P_1}(\cdot)$ and Lem.2.8, $J_{P_1,(P_2,\lambda_0)}(V)$ is a closed subrepresentation of $J_{P_1}(V) = J_{P_1 \cap L_{P_2}}(J_{P_2}(V))$. By Thm.2.6, 2.7 and Lem.2.8, one has

**Corollary 2.10.** $J_{P_1,(P_2,\lambda_0)}(V)$ is an essentially admissible locally analytic representation of $L_{P_2}(Q_p)$.

**Lemma 2.11.** Keep the above notation, let $\lambda$ be a dominant weight (for $L_{P_2}$) above $\lambda_0$, $\mathcal{L}_2(\lambda)$ the irreducible algebraic representation of $L_{P_2}$ with highest weight $\lambda$ we have

$$
J_{P_1,(P_2,\lambda)}(V) \xrightarrow{\sim} J_{P_1 \cap L_{P_2}}((J_{P_2}(V) \otimes_E \mathcal{L}_2(\lambda'))_0) \otimes_E \mathcal{L}_2(\lambda)^{N_{P_1 \cap L_{P_2}}}.\n$$

**Proof.** It’s sufficient to prove

$$
\left( (J_{P_2}(V) \otimes E \mathcal{L}_2(\lambda'))_0 \otimes E \mathcal{L}_2(\lambda) \right)^{N_{P_1 \cap L_{P_2}}} \equiv \left( (J_{P_2}(V) \otimes E \mathcal{L}_2(\lambda'))_0 \otimes E \mathcal{L}_2(\lambda) \right)^{N_{P_1 \cap L_{P_2}}}.\n$$

Since the action of $N_{P_1 \cap L_{P_2}}$ is smooth on $(J_{P_2}(V) \otimes E \mathcal{L}_2(\lambda'))_0$, thus we have

$$
\left( (J_{P_2}(V) \otimes E \mathcal{L}_2(\lambda'))_0 \otimes E \mathcal{L}_2(\lambda) \right)^{N_{P_1 \cap L_{P_2}}} \subseteq \left( (J_{P_2}(V) \otimes E \mathcal{L}_2(\lambda'))_0 \otimes E \mathcal{L}_2(\lambda) \right)^{n_{P_1 \cap L_{P_2}}}
$$

$$
\subseteq (J_{P_2}(V) \otimes E \mathcal{L}_2(\lambda'))_0 \otimes E \mathcal{L}_2(\lambda)^{N_{P_1 \cap L_{P_2}}} = (J_{P_2}(V) \otimes E \mathcal{L}_2(\lambda'))_0 \otimes E \mathcal{L}_2(\lambda)^{N_{P_1 \cap L_{P_2}}},\n$$

where $n_{P_1 \cap L_{P_2}}$ denotes the Lie algebra of $N_{P_1 \cap L_{P_2}}$, from which we easily deduce that the left side is contained in the right side in (11). The other direction is trivial, and the lemma follows. \qed
Keep the above notation, and denote by \( \mathcal{L}_1(\lambda) \) the irreducible algebraic representation of \( L_{P_1}(Q_p) \) with highest weight \( \lambda \), thus \( \mathcal{L}_1(\lambda) \cong \mathcal{L}_2(\lambda)^{N_{P_1} \cap L_{P_2}} \). Note that the action of \( L_{P_1}(Q_p) \cap L_{P_2}^0(\mathbb{Q}_p) \) on \( J_{P_1 \cap L_{P_2}} ((J_{P_2}(V) \otimes_E \mathcal{L}_2(\lambda))^0) \) is smooth, \( J_{P_1 \cap L_{P_2}}(V) \) is a locally algebraic representation of \( L_{P_1}^0(Q_p) \) (of type \( \mathcal{L}_1(\lambda) \)). Let \( H \) be an open compact subgroup of \( \left( Z_{L_{P_1}} \cap L_{P_2}^0 \right)(Q_p) \), \( \chi \) be a smooth character of \( H \) over \( E \), and put

\[
J_{P_1 \cap L_{P_2}}^H(\lambda)(V) := J_{P_1 \cap L_{P_2}} ((J_{P_2}(V) \otimes_E \mathcal{L}_2(\lambda))^0)^{H=\chi} \otimes_E \mathcal{L}_1(\lambda) \rightarrow J_{P_1 \cap L_{P_2}}(\lambda)(V),
\]

which is also an essentially admissible locally analytic representation of \( L_{P_1}(Q_p) \).

### 2.3. Adjunction formulas

In this section, we deduce from [6, Thm.4.3] an adjunction formula for the functor \( J_{P_1 \cap L_{P_2}}(\cdot) \), which would play a crucial role in our study below of certain closed rigid subspaces of the eigenvarieties.

Denote by \( \delta_i \) the modulus character of \( P_i(Q_p) \) for \( i = 1, 2 \), \( \delta_{12} \) the modulus character of \( P_1(Q_p) \cap L_{P_2}(Q_p) \) (where \( P_1 \cap L_{P_2} \) is viewed as a parabolic subgroup of \( L_{P_2} \)). Note the character \( \delta_i \) factors through \( L_{P_1} \), and \( \delta_{12} \) factors through \( L_{P_1} \). We have \( \delta_{12}|_{L_{P_1}} \delta_{12}|_{L_{P_1}} = \delta_1|_{L_{P_1}} \).

Let \( W \) be an irreducible algebraic representation of \( L_{P_1} \) of highest weight \( \mu \), \( \pi \) be a finite length smooth admissible representation of \( L_{P_1}(Q_p) \), suppose there exists a non-zero \( L_{P_1}(Q_p) \)-invariant map: \( \pi \otimes_E W \rightarrow J_{P_1 \cap L_{P_2}}(\lambda)(V) \). Then one has

**Lemma 2.12.** The weight \( \mu \) is \( P_2 \)-dominant and restricts to \( \lambda_0 \).

**Proof.** By considering the action of the Lie algebra of \( L_{P_1}(Q_p) \cap L_{P_2}^0(\mathbb{Q}_p) \) and Lem.2.11, there exists a \( P_2 \)-dominant weight \( \lambda \) above \( \lambda_0 \) such that \( W|_{L_{P_1}(Q_p) \cap L_{P_2}^0(\mathbb{Q}_p)} \cong \mathcal{L}_1(\lambda)|_{L_{P_1}(Q_p) \cap L_{P_2}^0(\mathbb{Q}_p)} \). Since \( L_{P_1}^0 \subseteq L_{P_1} \cap L_{P_2}^0 \), we see \( W \) differs from \( \mathcal{L}_1(\lambda) \) by certain determinantal twist \( \det_1 \). Since \( \det_1 \) is trivial on \( L_{P_1}(Q_p) \cap L_{P_2}^0(\mathbb{Q}_p) \) (thus trivial on \( (T \cap L_{P_2}^0)(\mathbb{Q}_p) \)), the weight \( \det_1 \) is \( P_2 \)-dominant. The lemma follows. \( \square \)

For a parabolic subgroup \( P \) of \( G \), we use \( \overline{P} \) to denote the parabolic subgroup of \( G \) opposite to \( P \).

**Proposition 2.13.** Let \( \lambda \) be a dominant weight for \( L_{P_1} \), which restricts to \( \lambda_0 \), \( \pi \) be a finite length smooth admissible representation of \( L_{P_1}(Q_p) \), \( V \) be an essentially admissible locally analytic representation of \( L_{P_2}(Q_p) \), then one has a natural bijection

\[
\text{Hom}_{L_{P_2}(Q_p)} \left( \left( \text{Ind}_{\overline{T}(Q_p) \cap L_{P_2}(Q_p)}^{L_{P_2}(Q_p)} \pi \otimes_E \mathcal{L}_2(\lambda), V_{\lambda_0} \right) \right) \rightarrow \text{Hom}_{L_{P_1}(Q_p)} \left( \pi \otimes_E \mathcal{L}_1(\lambda), J_{P_1 \cap L_{P_2}}(V_{\lambda_0}) \right).
\]

**Proof.** This proposition follows from the adjunction property for classical Jacquet functors. Recall \( V_{\lambda_0} \cong (V \otimes_E \mathcal{L}_2(\lambda))^0 \otimes_E \mathcal{L}_2(\lambda) \). By Lem.2.9, 2.11, one has

\[
\text{Hom}_{L_{P_1}(Q_p)} \left( \pi, J_{P_1 \cap L_{P_2}} \left( (V \otimes_E \mathcal{L}_2(\lambda))^0 \right) \right) \rightarrow \text{Hom}_{L_{P_1}(Q_p)} \left( \pi \otimes_E \mathcal{L}_1(\lambda), J_{P_1 \cap L_{P_2}}(V_{\lambda_0}) \right).
\]

Since \( \pi \) is smooth and of finite length, \( Z_{L_{P_1}}(Q_p) \) (thus \( Z_{L_{P_1}}(Q_p) \)) acts on \( \pi \) via certain finite dimensional representation, in other words, there exists an ideal \( I \) of \( \text{End} \left( \mathcal{L}_2(\lambda) \right) \) such that \( \pi = \pi' \) (where \( \pi \) is viewed as an \( \text{End} \left( \mathcal{L}_2(\lambda) \right) \)-module). So

\[
\text{Hom}_{L_{P_1}(Q_p)} \left( \pi, J_{P_1 \cap L_{P_2}} \left( (V \otimes_E \mathcal{L}_2(\lambda))^0 \right) \right) \rightarrow \text{Hom}_{L_{P_1}(Q_p)} \left( \pi, J_{P_1 \cap L_{P_2}} \left( (V \otimes_E \mathcal{L}_2(\lambda))^0 \right) \right).
\]
On the other hand, by [15, Prop.6.4.13] and [26, Thm.6.6], $(V \otimes_E \mathcal{L}_2(\lambda)'_0^\delta)$ is an admissible smooth representation of $L_{P_2}(\mathbb{Q}_p)$. In this case, the Jacquet-Emerton functor coincides with the classical Jacquet functor (cf. [17, §4.3]), and one has a bijection

$$(15) \quad \text{Hom}_{L_{P_2}(\mathbb{Q}_p)} \left( \left( \text{Ind}_{P_1(\mathbb{Q}_p) \cap L_{P_2}(\mathbb{Q}_p)}^{L_{P_2}(\mathbb{Q}_p)} \pi \otimes_E \delta_{12}^{-1} \right)^\infty \otimes_E \mathcal{L}_2(\lambda)'_0^\delta, (V \otimes_E \mathcal{L}_2(\lambda)'_0^\delta) \right) \sim \text{Hom}_{L_{P_1}(\mathbb{Q}_p)} \left( \pi, J_{P_1 \cap L_{P_2}} \left( (V \otimes_E \mathcal{L}_2(\lambda)'_0^\delta) \right) \right),$$

Note one can remove "$\mathcal{T}$" on either side since the corresponding set would not change. Again by Lem.2.9, one has a bijection

$$(16) \quad \text{Hom}_{L_{P_2}(\mathbb{Q}_p)} \left( \left( \text{Ind}_{P_1(\mathbb{Q}_p) \cap L_{P_2}(\mathbb{Q}_p)}^{L_{P_2}(\mathbb{Q}_p)} \pi \otimes_E \delta_{12}^{-1} \right)^\infty \otimes_E \mathcal{L}_2(\lambda)'_0, (V \otimes_E \mathcal{L}_2(\lambda)'_0) \right) \sim \text{Hom}_{L_{P_2}(\mathbb{Q}_p)} \left( \left( \text{Ind}_{P_1(\mathbb{Q}_p) \cap L_{P_2}(\mathbb{Q}_p)}^{L_{P_2}(\mathbb{Q}_p)} \pi \otimes_E \delta_{12}^{-1} \right)^\infty \otimes_E \mathcal{L}_2(\lambda), (V \otimes_E \mathcal{L}_2(\lambda)'_0 \otimes_E \mathcal{L}_2(\lambda)) \right).$$

Putting (14) (15) (16) together, the proposition follows. \hfill \square

**Remark 2.14.** Note the left set of (13) won’t change if $V_{\lambda_0}$ is replaced by $V$ since any non-zero $L_{P_2}(\mathbb{Q}_p)$-invariant map

$$(\text{Ind}_{P_1(\mathbb{Q}_p) \cap L_{P_2}(\mathbb{Q}_p)}^{L_{P_2}(\mathbb{Q}_p)} \pi \otimes_E \delta_{12}^{-1})^\infty \otimes_E \mathcal{L}_2(\lambda) \rightarrow V$$

factors automatically through $V_{\lambda_0}$. However, the set on the right side is rather subtle, indeed, the natural injection

$$\text{Hom}_{L_{P_1}(\mathbb{Q}_p)} \left( \pi \otimes_E \mathcal{L}_1(\lambda), J_{P_1 \cap L_{P_2}} (V_{\lambda_0}) \right) \hookrightarrow \text{Hom}_{L_{P_1}(\mathbb{Q}_p)} \left( \pi \otimes_E \mathcal{L}_1(\lambda), J_{P_1 \cap L_{P_2}} (V) \right)$$

is not bijective in general.

**Theorem 2.15.** Let $V$ be a very strongly admissible locally $\mathbb{Q}_p$-analytic representation of $G(\mathbb{Q}_p)$ (resp. [18, Def.0.12]), $\lambda$ a dominant weight for $L_{P_2}$ above $\lambda_0$, $\pi$ a finite length smooth admissible representation of $L_{P_1}(\mathbb{Q}_p)$, thus there exists a natural bijection

$$(17) \quad \text{Hom}_{G(\mathbb{Q}_p)} \left( \left( \text{Ind}_{P_1(\mathbb{Q}_p) \cap L_{P_2}(\mathbb{Q}_p)}^{L_{P_2}(\mathbb{Q}_p)} \pi \otimes_E \delta_{12}^{-1} \right)^\infty \otimes_E \mathcal{L}_2(\lambda, J_{P_1 \cap L_{P_2}} (V)), V \right) \sim \text{Hom}_{L_{P_1}(\mathbb{Q}_p)} \left( \pi \otimes_E \mathcal{L}_1(\lambda), J_{P_1 \cap L_{P_2}} (V) \right),$$

where $(\cdot)^\vee$ denotes the dual in the category $\mathcal{O}^\vee$ (cf. [21, Chap.3]).

**Proof.** By Prop.2.13 (applied to $V = J_{P_2}(V)$), one has a bijection

$$\text{Hom}_{L_{P_2}(\mathbb{Q}_p)} \left( \left( \text{Ind}_{P_1(\mathbb{Q}_p) \cap L_{P_2}(\mathbb{Q}_p)}^{L_{P_2}(\mathbb{Q}_p)} \pi \otimes_E \delta_{12}^{-1} \right)^\infty \otimes_E \mathcal{L}_2(\lambda), J_{P_2}(V) \right) \sim \text{Hom}_{L_{P_1}(\mathbb{Q}_p)} \left( \pi \otimes_E \mathcal{L}_1(\lambda, J_{P_1 \cap L_{P_2}} (J_{P_2}(V) \lambda_0) \right),$$

and the left set would not change if $J_{P_2}(V) \lambda_0$ is replaced by $J_{P_2}(V)$ (see Rem.2.14). Since $V$ is very strongly admissible, by [6, Thm.4.3], one has

$$\text{Hom}_{G(L)} \left( \left( \text{Ind}_{P_1(\mathbb{Q}_p) \cap L_{P_2}(\mathbb{Q}_p)}^{L_{P_2}(\mathbb{Q}_p)} \pi \otimes_E \delta_{12}^{-1} \right)^\infty \otimes_E \mathcal{L}_2(\lambda), J_{P_2}(V) \right) \sim \text{Hom}_{L_{P_2}(\mathbb{Q}_p)} \left( \left( \text{Ind}_{P_1(\mathbb{Q}_p) \cap L_{P_2}(\mathbb{Q}_p)}^{L_{P_2}(\mathbb{Q}_p)} \pi \otimes_E \delta_{12}^{-1} \right)^\infty \otimes_E \mathcal{L}_2(\lambda, J_{P_2}(V) \right).$$

Since $\delta_{12}|_{L_{P_2}(\mathbb{Q}_p)} = \delta_{12}|_{L_{P_1}(\mathbb{Q}_p) \delta_2}|_{L_{P_1}(\mathbb{Q}_p)}$, the theorem follows. \hfill \square

**Remark 2.16.** Keep the notations in Thm.2.15. By [6, Thm.4.3], one has a bijection

$$(18) \quad \text{Hom}_{G(\mathbb{Q}_p)} \left( \left( \text{Ind}_{P_1(\mathbb{Q}_p) \cap L_{P_2}(\mathbb{Q}_p)}^{L_{P_2}(\mathbb{Q}_p)} \pi \otimes_E \delta_{12}^{-1} \right)^\infty \otimes_E \mathcal{L}_2(\lambda), V \right) \sim \text{Hom}_{L_{P_1}(\mathbb{Q}_p)} \left( \pi \otimes_E \mathcal{L}_1(\lambda), J_{P_1}(V) \right).$$
Denote by \( M_1(\lambda) := U(\mathfrak{g}) \otimes \mathcal{L}(\lambda)' \). Note \( M_2(\lambda) \) is a quotient of \( M_1(\lambda) \) in the category \( \mathcal{O}^{\mathfrak{p}_1} \) (cf. [21, §9]), from which we deduce (by Thm. 2.4 (2) and (3)) the representation

\[
F_{\mathcal{P}_2}^G \left( \left( U(\mathfrak{g}) \otimes \mathcal{L}(\lambda)' \right)^\vee, \left( \text{Ind}^G_{\mathcal{P}_2(Q_p)}(\mathcal{L}(\lambda)') \otimes E \delta_2^{-1} \right) \right) \]

is a quotient of \( F_{\mathcal{P}_1}^G \left( \left( U(\mathfrak{g}) \otimes \mathcal{L}(\lambda)' \right)^\vee, \pi \otimes E \delta_1^{-1} \right) \). In particular, when \( P_1 \neq P_2 \), the right set of (18) might be strictly bigger than the right set of (17). This subtlety is somehow the key point of this note.

2.4. Structure of \( J_{B,(P,\lambda_0)}(V) \). Let \( H \) be an open compact uniform prop-p-subgroup of \( G(Q_p) \) such that \( H \) is a normal subgroup of the maximal open compact subgroup of \( G(Q_p) \). Let \( V \) be a locally \( Q_p \)-analytic representation of \( G(Q_p) \) over \( E \) such that \( V|_H \cong C^0(H,E)^{\oplus r} \) for \( r \in \mathbb{Z}_{\geq 1} \). This section is devoted to the structure \( J_{B,(P,\lambda_0)}(V) \). In fact, the results in this section were already obtained in [20], while we reformulate them in a way that suits our context.

For any parabolic subgroup \( P' \) of \( G \), let \( N_{P'} := H \cap N_{P'} \), \( L_{P'} := H \cap L_{P'} \), put \( L_{P'}(Q_p)^+ \) as (8) with respect to \( N_{P'} \). Note \( T \cong L_B \), and put \( N := N_B, N := N_B, N := N_B, N := N_B, T := L_B \) and \( T(Q_p)^+ := L_B(Q_p)^+ \).

Suppose the natural multiplication induces a homomorphism

\[
N^o \times L^o \times N^o \xrightarrow{\sim} H
\]

for \( * \in \{P, B\} \). Denote by \( L_{P,o}^D := L_{P}(Q_p) \cap L_{p}^o, Z_{L,p}^o := Z_{L,p}(Q_p) \cap L_{p}^o \), and suppose the following natural map is an isomorphism of \( p \)-adic analytic groups

\[
L_{P,o}^D \times Z_{L,p}^o \xrightarrow{\sim} L_{P,o}^D.
\]

Let \( \lambda \) be a dominant weight for \( G \) above \( \lambda_0 \), we use \( \mathcal{L}_P(\lambda) \) (resp. \( \mathcal{L}_G(\lambda) \)) to denote the irreducible algebraic representation of \( P \) (resp. \( G \)) over \( E \) with highest weight \( \lambda \). Recall

\[
J_{B,(P,\lambda_0)}(V) \cong \left( \left( \left( V_{\mathcal{E}}^{N_p} \otimes E \mathcal{L}_P(\lambda)' \right) \otimes E \mathcal{L}_P(\lambda) \right)^{N_{B,L_P}} \right),
\]

where \( N_{B,L_P} := N_{B \cap L_P} \cap H \), the first “fs” is with respect to \( Z_{L,p}(Q_p)^+ \), and second one to \( T(Q_p)^+ = Z_{L,B}(Q_p)^+ \). Note \( Z_{L,B}(Q_p)^+ \subseteq T(Q_p)^+ \).

**Lemma 2.17.** \( J_{B,(P,\lambda_0)}(V) \cong \left( \left( \left( V_{\mathcal{E}}^{N_p} \otimes E \mathcal{L}_P(\lambda)' \right) \otimes E \mathcal{L}_P(\lambda) \right)^{N_{B,L_P}} \right), \) where \( \left( \left( V_{\mathcal{E}}^{N_p} \otimes E \mathcal{L}_P(\lambda)' \right) \otimes E \mathcal{L}_P(\lambda) \right)^{N_{B,L_P}} = \lim_{U} (V_{\mathcal{E}}^{N_p} \otimes E \mathcal{L}_P(\lambda)' \otimes E \mathcal{L}_P(\lambda)) \) with \( U \) running through the open compact subgroups of \( L_{P,o}^D \).

**Proof.** Since \( V_{\mathcal{E}}^{N_p} \) is a closed subspace of \( V^{N_p} \), we have

\[
J_{B,(P,\lambda_0)}(V) \hookrightarrow \left( \left( V_{\mathcal{E}}^{N_p} \otimes E \mathcal{L}_P(\lambda)' \right) \otimes E \mathcal{L}_P(\lambda) \right)^{N_{B,L_P}} \]

We prove this map is bijective. Since \( \mathcal{L}_P(\lambda)^{N_{B,L_P}} \cong \chi_\lambda \) (where \( \chi_\lambda \) denotes the algebraic character of \( T \) with weight \( \lambda \)), we reduce to prove

\[
\left( \left( V_{\mathcal{E}}^{N_p} \otimes E \mathcal{L}_P(\lambda)' \right) \otimes E \mathcal{L}_P(\lambda) \right)^{N_{B,L_P}} \rightarrow \left( \left( V_{\mathcal{E}}^{N_p} \otimes E \mathcal{L}_P(\lambda)' \right) \otimes E \mathcal{L}_P(\lambda) \right)^{N_{B,L_P}}
\]

is bijective. Since \( \mathcal{L}_P(\lambda)' \cong (\mathcal{L}_G(\lambda)' \otimes E \mathcal{L}_P(\lambda)')_{\text{fs}} \), thus \( \left( V_{\mathcal{E}}^{N_p} \otimes E \mathcal{L}_P(\lambda)' \right) \otimes E \mathcal{L}_P(\lambda) \) with \( * \in \{\text{fs}, \emptyset\} \). We reduce to prove the natural injection

\[
\left( \left( V_{\mathcal{E}}^{N_p} \right) \right)_{\text{fs}} \rightarrow \left( \left( V_{\mathcal{E}}^{N_p} \right) \right)_{\text{fs}}
\]

is bijective for any essentially admissible locally analytic representation \( W \) of \( G(Q_p) \). First, note \( \left( V_{\mathcal{E}}^{N_p} \right) \) since the operation \( (\cdot)_{\text{fs}} \) depends only on the action of \( L_{P,o}^D \) and thus commutes
with \((\cdot)_{fs}\) (cf. [17, Prop.3.2.11]). Consider the inclusion map, \(((W^N_{P})_{0})_{fs}^{N_{B} \cap \lambda P} \hookrightarrow (W^N_{P})_{0}^{*}\) by the universal property [17, Prop.3.2.1 (ii)], we see this map factors through \(((W^N_{P})_{0})_{fs}^{N_{B} \cap \lambda P} \hookrightarrow ((W^N_{P})_{0})_{fs} \cong (W^N_{fs})_{0}^{*}\), whose image is thus contained in \(((W^N_{fs})_{0}^{*})_{fs}^{N_{B} \cap \lambda P}\). This gives an inverse of \((21)\).

By this lemma (and the proof),

\[ J_{B,(P,\lambda_{0})}(V) \cong \left( (V^{N_{P}} \otimes_{E} \mathcal{L}_{P}(\lambda)^{0})_{0} \otimes_{E} \mathcal{L}_{P}(\lambda) \right)_{fs}^{N_{B} \cap \lambda P} \cong \left( (V^{N_{P}} \otimes_{E} \mathcal{L}_{P}(\lambda)^{0})_{0} \otimes_{E} \mathcal{L}_{P}(\lambda) \right)_{fs}^{N_{B} \cap \lambda P} \otimes_{E} \chi_{\lambda}, \]

in the following, we would consider \(\left( (V^{N_{P}} \otimes_{E} \mathcal{L}_{P}(\lambda)^{0})_{0} \otimes_{E} \chi_{\lambda} \right)\), which only differs from \(J_{B,(P,\lambda_{0})}(V)\) by the twist \(\chi_{\lambda}\). The following lemma is well known.

**Lemma 2.18.** Let \(W\) be an irreducible algebraic representation of \(G\) over \(E\), then one has an isomorphism of \(H\)-representations \(C^{la}(H, E) \otimes_{E} W \cong C^{la}(H, E)^{\oplus \dim_{E} W}\), where \(H\) acts on the left object via diagonal action.

**Proof.** We include a proof for the convenience of the reader. We first prove \(C(H, E) \otimes_{E} W \cong C(H, E)^{\oplus \dim_{E} W}\). Since \(H\) is pro-\(p\), \((O_{E}/\varpi_{E}^{n})[[H]]\) is a complete local algebra, any finitely generated projective \((O_{E}/\varpi_{E}^{n})[[H]]\)-module is isomorphic to \((O_{E}/\varpi_{E}^{n})[[H]]^{\oplus r}\) for \(r \in \mathbb{Z}_{\geq 1}\). Thus by dualizing, any smooth admissible \(H\)-representation over \(O_{E}/\varpi_{E}^{n}\) which is moreover injective, is isomorphic to \(C(H, O_{E}/\varpi_{E}^{n})^{\oplus r}\) for \(r \in \mathbb{Z}_{\geq 1}\). Let \(W_{0}\) be a \(H\)-invariant \(O_{E}\)-lattice of \(W\), let \(V_{n} := C(H, O_{E}/\varpi_{E}^{n}) \otimes_{O_{E}/\varpi_{E}^{n}} W_{0}/\varpi_{E}^{n}\), we claim \(V_{n}\) is an injective object in the category \(\mathcal{G}_{n}\) of smooth admissible \(H\)-representations over \(O_{E}/\varpi_{E}^{n}\).

Indeed, for any \(M \in \mathcal{G}_{n}\), we have

\[ \text{Hom}_{O_{E}/\varpi_{E}^{n}} \left( M, C(H, O_{E}/\varpi_{E}^{n}) \right) \cong C(H, O_{E}/\varpi_{E}^{n}) \otimes_{O_{E}/\varpi_{E}^{n}} M^{\vee} \cong C(H, M^{\vee}), \]

which is moreover \(H\)-invariant, where the \(H\)-action on the left object is given by \(h(f)(m) := h^{-1}(f(h(m)))\), on the middle one is via diagonal action, and on \(C(H, M^{\vee})\) is given \(h(f)(h') = h^{-1}f(h'h),\) thus one has

\[ \text{Hom}_{H} \left( M, C(H, O_{E}/\varpi_{E}^{n}) \right) \cong M^{\vee}, \quad f \mapsto f(1). \]

Consequently,

\[ \text{Hom}_{H} \left( M, V_{n} \right) = \text{Hom}_{H} \left( M \otimes_{O_{E}/\varpi_{E}^{n}} (W_{0}/\varpi_{E}^{n})^{\vee}, C(H, O_{E}/\varpi_{E}^{n}) \right), \]

since the functor \(\otimes_{O_{E}/\varpi_{E}^{n}} (W_{0}/\varpi_{E}^{n})^{\vee}\) is exact, we deduce \(V_{n}\) is injective from the injectivity of \(C(H, O_{E}/\varpi_{E}^{n})\).

Thus there exists \(r \in \mathbb{Z}_{\geq 1}\), such that \(V_{n} \cong C(H, O_{E}/\varpi_{E}^{n})^{\oplus r}\). On the other hand, let \(H'\) be an open compact subgroup of \(H\) which acts trivially on \(W_{0}/\varpi_{E}^{n}\), and we have thus \((C(H, O_{E}/\varpi_{E}^{n})^{H'})^{\oplus r} \cong V_{n}^{H'} \cong (C(H, O_{E}/\varpi_{E}^{n})^{H'})^{\oplus r} \otimes_{O_{E}/\varpi_{E}^{n}} W_{0}/\varpi_{E}^{n}\) (note these are all finite sets), from which we see \(r = \dim_{E} W\). By taking projective limit on \(n\) and tensoring with \(E\), we get \(C(H, E) \otimes_{E} W \cong C(H, E)^{\oplus \dim_{E} W}\).

By [17, Prop.3.6.15], \((C(H, E) \otimes_{E} W)_{an} \cong C^{la}(H, E) \otimes_{E} W\), the lemma follows.

For a \(p\)-adic analytic group \(H'\), denote by \(\mathcal{D}(H', E) := C^{la}(H', E)_{\mathbb{A}}\) the \(E\)-algebra of distributions of \(H'\), which is a Fréchet-Stein algebra when \(H'\) is compact. Let \(\mathcal{C}^{\infty}(H', E) \hookrightarrow C^{la}(H', E)\) be the (closed) subspace of smooth functions, i.e. functions killed by the Lie algebra action. Put \(\mathcal{D}^{\infty}(H', E) := C^{\infty}(H', E)_{\mathbb{A}}\) which is a closed quotient of \(\mathcal{D}(H', E)\) and thus is also a Fréchet-Stein algebra when \(H'\) is compact. In fact, one has an isomorphism of topological algebraic distributions \(\mathcal{D}^{\infty}(H', E) \cong \lim_{U \subseteq H' \subseteq E[H'/U]} \mathcal{D}^{\infty}(H'/U)\) with \(U\) running over open compact normal subgroups of \(H'\) (cf. [25, §2]).

Let \(T_{P} \triangleq T(Q_{p}) \cap L_{P}^{\mathbb{A}}\). By (20), \(Z_{L_{P}}^{\mathbb{A}} \times L_{P}^{\mathbb{A}} \hookrightarrow L_{P}^{\mathbb{A}}\), thus

\[ T^{\circ} = T(Q_{p}) \cap L_{P}^{\mathbb{A}} \cong Z_{L_{P}}^{\mathbb{A}} \times (L_{P}^{\mathbb{A}} \cap T(Q_{p})) = Z_{L_{P}}^{\mathbb{A}} \times T_{P}. \]
This isomorphism induces thus an isomorphism of Fréchet-Stein algebras
\[ \mathcal{D}(T^*_p, E) \hat{\otimes}_E \mathcal{D}(Z_{L_p}^p, E) \cong \mathcal{D}(T^*, E). \]

Put
\[ \mathcal{D}'(T^*, E) := \mathcal{D}^\infty(T^*_p, E) \hat{\otimes}_E \mathcal{D}(Z_{L_p}^p, E) \]
which is thus a quotient of \( \mathcal{D}(T^*, E) \) and is also a Fréchet-Stein algebra.

Recall for a compact uniform prop-p-group \( H' \), and for \( \frac{1}{p} < r < 1 \), \( \mathcal{D}(H', E) \) is equipped with a multiplicative norm \( \| \cdot \|_r \) ([26, §4]). As in loc. cit., denote by \( \mathcal{D}_r(H', E) \) the completion of \( \mathcal{D}(H', E) \) via \( \| \cdot \|_r \), which is thus a Banach \( E \)-algebra. We can also define a bigger Banach \( E \)-algebra \( \mathcal{D}_{<r}(H', E) \) ([26, p.161]). One has \( \mathcal{D}(H', E) \cong \lim_{\to r} \mathcal{D}_r(H', E) \cong \lim_{\to r} \mathcal{D}_{<r}(H', E) \). For \( n \in \mathbb{Z}_{\geq 1} \), put \( r_n := \frac{1}{p^n-1} \).
For a locally analytic representation \( W \) of \( H' \), let \( W^{(n)} \) denote the subrepresentation generated by \( r_n \)-analytic vectors as in [13, §0.3]. Put \( \mathcal{C}^{(n)}(H', E) := \mathcal{C}^{la}(H', E)^{(n)} \). By definition, one has \( \mathcal{C}^{(n)}(H', E)^{(n)} \cong \mathcal{D}_{<r_n}(H', E) \).

Since \( H \) is uniform, by the isomorphisms (19) (20), so are the groups \( T^*, L^*_p, N^*_p, N^r_p, N^o, \mathcal{N}^o, T^*_p, L^r_p, Z_{L_p}^o \). For \( \frac{1}{p} < r < 1 \), put \( \mathcal{D}^\infty_r(T^*_p, E) \) to be the image of \( \mathcal{D}_r(T^*_p, E) \) via the projection \( \mathcal{D}(T^*_p, E) \to \mathcal{D}^\infty(T^*_p, E) \) for \( s \in \{r, \ldots, r_n \} \). Put \( A_n := \mathcal{D}^\infty_r(T^*_p, E) := \mathcal{D}^\infty_r(T^*_p, E) \hat{\otimes}_E \mathcal{D}_r(Z_{L_p}^p, E) \). Let \( z \in T(\mathbb{Q}_p)^{+} \) such that \( T(\mathbb{Q}_p) \) is generated by \( z^{-1} \) and \( T(\mathbb{Q}_p)^{+} \) by multiplication.

**Proposition 2.19.** For all \( n \in \mathbb{Z}_{\geq 1} \), there exists an orthonormalisable \( A_n \)-module \( M_n \) such that

1. \( M_n \) is equipped with a compact \( A_n \)-linear operator \( z_n \);
2. there exist continuous \( A_n \)-linear maps \( \alpha_n : M_n \to M_{n+1} \otimes A_{n+1} \) and \( \beta_n : M_{n+1} \otimes A_{n+1} \to M_n \) such that \( \beta_n \circ \alpha_n = z_n, \alpha_n \circ \beta_n = z_{n+1} \otimes 1_{A_n} \);
3. one has an isomorphism of \( \mathcal{D}'(T^*, E) \)-modules:

\[ \lim_{\to n} M_n \cong ((\mathcal{V}L_{1}^n \otimes_{E} \mathcal{L}(H^\vee))^0_{N^o \cap p})^\vee \]

(where the maps in the projective system are given by the composition of \( \beta_n \) and the natural map \( M_{n+1} \to M_{n+1} \otimes A_{n+1} \)), which commutes with the action \( \{z_{n}\} \) on the left and \( \pi_z \) on the right (cf. (9)).

In summary, one has the following commutative diagram

\[
\begin{array}{ccccccccc}
\left( (\mathcal{V}L_{1}^n \otimes_{E} \mathcal{L}(H^\vee))^0_{N^o \cap p} \right)^\vee & \xrightarrow{\pi_z} & M_{n+1} & \xrightarrow{\beta_n} & M_n \\
\downarrow & & \downarrow z_{n+1} \otimes id & & \downarrow z_n \\
\left( (\mathcal{V}L_{1}^n \otimes_{E} \mathcal{L}(H^\vee))^0_{N^o \cap p} \right)^\vee & \xrightarrow{\pi_z} & M_{n+1} & \xrightarrow{\beta_n} & M_n \\
\end{array}
\]

**Proof.** We use the argument of [7, Prop.5.3], which is rather a variation of the arguments in [17, §4.2]. One has \( \mathcal{V}L_{1}^n \otimes_{E} \mathcal{L}(H^\vee) \cong (V \otimes_{E} \mathcal{L}(H^\vee))^0_{N^o} \). Denote by \( \Pi := V \otimes_{E} \mathcal{L}(H^\vee) \), by Lem.2.18, \( \Pi^{\oplus r} \cong \mathcal{C}^{la}(H, E)^{(r)} \) for some \( r \in \mathbb{Z}_{\geq 1} \). Denote by \( \Pi_{H}^{(n)} \) the \( r_n \)-analytic vectors for the \( H \)-action (cf. [13, §0.3]), thus \( \Pi_{H}^{(n)} \cong \mathcal{C}^{(n)}(H, E)^{(r)} \). By the isomorphism (19), one has

\[
\Pi_{H}^{(n)} \cong \left( \mathcal{C}^{(n)}(N^r_p, E) \otimes_{E} \mathcal{C}^{(n)}(L^r_p, E) \otimes_{E} \mathcal{C}^{(n)}(N^o_p, E) \right)^{(r)}.
\]

Thus
\[
\left( \Pi_{H}^{(n)} \right)^{N^r_p} \cong \left( \mathcal{C}^{(n)}(N^r_p, E) \otimes_{E} \mathcal{C}^{(n)}(L^r_p, E) \otimes_{E} \mathcal{C}^{(n)}(N^o_p, E) \right)^{(r)} \rightarrow \left( \mathcal{C}^{(n)}(N^r_p, E) \otimes_{E} \mathcal{C}^{(n)}(L^r_p, E) \otimes_{E} \mathcal{C}^{(n)}(Z_{L_p}^r, E) \right)^{(r)}.
\]
Put $C^{\infty,n}(H', E) := C^{n}(H', E) \cap C^{\infty}(H', E)$, for a compact prop-$p$ uniform $p$-adic analytic group $H'$. One gets

\[
\left( (\Pi^{(n)}_{H})_{0} \right)^{N_{\overline{B} \otimes L_{P}}} \cong \left( C^{(\infty)}(N_{\overline{B} \otimes L_{P}}^{\circ}, E) \otimes_{E} C^{\infty}(n)(L_{P}^{\circ}, E) \right)^{\otimes r},
\]

and thus

\[
\left( (\Pi^{(n)}_{H})_{0} \right)^{N_{\overline{B} \otimes L_{P}}} \cong \left( C^{(\infty)}(N_{\overline{B} \otimes L_{P}}^{\circ}, E) \otimes_{E} C^{\infty}(n)(N_{\overline{B} \otimes L_{P}}^{\circ}, E) \right)^{\otimes r} \rightarrow \left( C^{(\infty)}(N_{\overline{B} \otimes L_{P}}^{\circ}, E) \otimes_{E} C^{\infty}(n)(N_{\overline{B} \otimes L_{P}}^{\circ}, E) \right)^{\otimes r},
\]

Note the strong dual of $C^{(\infty)}(Z_{L_{P}}^{\circ}, E) \otimes_{E} C^{\infty}(n)(T_{P}^{\circ}, E)$ is $D_{<\gamma}(Z_{L_{P}}^{\circ}, E) \otimes_{E} D_{<\gamma}^{\infty}(T_{P}^{\circ}, E)$. Put thus

\[
M_{n} := \left( (\Pi^{(n)}_{H})_{0} \right)^{N_{\overline{B} \otimes L_{P}}} \otimes_{D_{<\gamma}(Z_{L_{P}}^{\circ}, E) \otimes_{E} D_{<\gamma}^{\infty}(T_{P}^{\circ}, E)} A_{n}.
\]

Indeed, since $\left( (\Pi^{(n)}_{H})_{0} \right)^{N_{\overline{B} \otimes L_{P}}} \otimes_{D_{<\gamma}(Z_{L_{P}}^{\circ}, E) \otimes_{E} D_{<\gamma}^{\infty}(T_{P}^{\circ}, E)}$ is obviously an orthonormalisable $D_{<\gamma}(Z_{L_{P}}^{\circ}, E) \otimes_{E} D_{<\gamma}^{\infty}(T_{P}^{\circ}, E)$-module, we see $M_{n}$ is an orthonormalisable $A_{n}$-module. The existence of the maps $\alpha_{n}, \beta_{n}$ follows by the same arguments as in the proof of [7, Prop.5.3].

Let $\chi_{0}$ be a smooth character of $T_{P}^{\circ}$ over $E$, let $N(\chi_{0}) \in Z_{\geq 1}$ such that $\chi_{0} \in C^{(N(\chi_{0}))(T_{P}^{\circ}, E) \rightarrow E}$. Note $\chi_{0}$ corresponds to a maximal ideal of $D_{<\gamma}(T_{P}^{\circ}, E)$, and thus induces a projection $D_{<\gamma}^{\infty}(T_{P}^{\circ}, E) \rightarrow E$ for all $n \geq N(\chi_{0})$. Put $M_{n}^{T_{P}^{\circ}=\chi_{0}} := M_{n} \otimes_{D_{<\gamma}(T_{P}^{\circ}, E), \chi_{0}} E$, $B_{n} := D_{<\gamma}(Z_{L_{P}}^{\circ}, E)$, thus $M_{n}^{T_{P}^{\circ}=\chi_{0}}$ is an orthonormalisable $B_{n}$-module. Note the $z_{n}$-action on $M_{n}$ induces a compact $B_{n}$-linear action $z_{n}$-action on $M_{n}^{T_{P}^{\circ}=\chi_{0}}$. By the above proposition, one has

**Corollary 2.20.** One has an isomorphism

\[
\left( (V^{n_{P}} \otimes_{E} L_{P}(\lambda)^{\prime})_{0} \right)^{N_{\overline{B} \otimes L_{P}}^{T_{P}^{\circ}=\chi_{0}}} \cong \lim_{n \geq N(\chi_{0})} M_{n}^{T_{P}^{\circ}=\chi_{0}},
\]

and the following diagram commutes

\[
\begin{array}{ccc}
\left( (V^{n_{P}} \otimes_{E} L_{P}(\lambda)^{\prime})_{0} \right)^{N_{\overline{B} \otimes L_{P}}^{T_{P}^{\circ}=\chi_{0}}} & \longrightarrow & \cdots \longrightarrow M_{n+1}^{T_{P}^{\circ}=\chi_{0}} \otimes_{B_{n+1}} B_{n} \beta_{n} M_{n}^{T_{P}^{\circ}=\chi_{0}} \\
\left| \pi_{z} \right| & & \left| z_{n+1} \right| \downarrow \beta_{n} \downarrow \alpha_{n} \downarrow \left| z_{n} \right|
\end{array}
\]

Keep the above notation, put

\[
J_{B_{n}(P, \lambda)}^{T_{P}^{\circ}=\chi_{0}}(V) := \left( (V^{n_{P}} \otimes_{E} L_{P}(\lambda)^{\prime})_{0} \right)^{N_{\overline{B} \otimes L_{P}}^{T_{P}^{\circ}=\chi_{0}}} \otimes_{E} \chi_{0},
\]

which is thus a closed subrepresentation (hence also essentially admissible) of $J_{B_{n}(P, \lambda)}^{T_{P}^{\circ}=\chi_{0}}(V)$.

For a topologically finitely generated abelian group $Z$, denote by $\tilde{Z}$ the rigid space over $E$ parameterizing locally analytic characters of $Z$. By [15, Prop.6.4.6], if $Z$ is moreover compact, one has a natural isomorphism $O(\tilde{Z}) \cong D(Z, E)$ (where for a rigid analytic space $\tilde{X}$, we use $O(\tilde{X})$ to denote the global sections on $\tilde{X}$). By definition ([15, Def.6.4.9]), there exists an equivalence of categories of the category of coadmissible $O(\tilde{Z})$-modules and that of essentially admissible locally analytic representations of $Z$. 


In particular, the strong dual of $J_{B,\{P,\lambda_0\}}(V)$ (resp. $J_{B,\{P,\lambda_0\}}^{T,\chi_0}(V)$) corresponds to a coherent sheaf $\mathcal{M}_{\lambda_0}(V)$ (resp. $\mathcal{M}_{\lambda_0}^{X_0}(V)$) over $\widehat{T} := \overline{T(\mathbb{Q}_p)}$, with

$$\mathcal{M}_{\lambda_0}(V)(\widehat{T}) \simto J_{B,\{P,\lambda_0\}}(V)_b$$

(resp. $\mathcal{M}_{\lambda_0}^{X_0}(V)(\widehat{T}) \simto J_{B,\{P,\lambda_0\}}^{T,\chi_0}(V)_b$).

The character $\chi_{\lambda}$ induces an isomorphism of rigid spaces:

\begin{equation}
\chi_{\lambda} : \widehat{T} \rightarrow \widehat{T}, \quad \chi \mapsto \chi \chi_{\lambda}.
\end{equation}

One easily sees the coherent sheaf $\chi_{\lambda}^*(\mathcal{M}_{\lambda_0}(V))$ (resp. $\chi_{\lambda}^{*}(\mathcal{M}_{\lambda_0}^{X_0}(V))$) corresponds to the $T(\mathbb{Q}_p)$-representation

$$((V^{N_p} \otimes E L_p(\lambda)i)_0)_{\text{fs}}^{N_{\text{fin}}\otimes L_p, T_p = \chi_0} \quad \text{(resp. } ((V^{N_p} \otimes E L_p(\lambda)i)_0)_{\text{fs}}^{N_{\text{fin}}\otimes L_p, T_p = \chi_0})$$

One has

$$\kappa : \widehat{T} \rightarrow \widehat{T}_p \times \mathbb{G}_m \rightarrow \widehat{T}_p \times Z_{L_p}^\times \times \mathbb{G}_m$$

where the first projection maps $\chi$ to $((\chi|_{T_p}, \chi(i))$ for any $\chi \in \widehat{T(E)}$. The character $\chi_0$ corresponds to an $E$-point of $\widehat{T}_p$. The affinoids $\{\text{Spn } B_n\}_{n \in \mathbb{Z}_{\geq 2}}$ form an admissible covering of $Z_{L_p}^\times$. We view $M_{\lambda_0}^{X_0}$ as a $B_n[X]$ module with $X$ acting on $M_n$ via $z_n$, and put $M_{\lambda_0}^{X_0} := M_n^{X_0} \otimes_{E[X]} E\{\{X, X^{-1}\}$}. By [17, Prop.2.2.6], $M_{\lambda_0}^{X_0}$ is a coadmissible $B_n\{\{X, X^{-1}\}$-module, and thus corresponds to a coherent sheaf $\mathcal{M}_{\lambda_0}^{X_0}$ over $\text{Spn } B_n \times \mathbb{G}_m$. By [17, Prop.2.1.9], one has $M_{\lambda_0+1}^{X_0} \otimes_{B_n+1(\{X, X^{-1}\})} B_n\{\{X, X^{-1}\}$ over $M_{\lambda_0}^{X_0}$.

Thus $\{\mathcal{M}_{\lambda_0}^{X_0}\}_{n}$ glues to a coherent sheaf $\mathcal{M}_{\lambda_0}^{X_0}$ over $\widehat{T}_p \times \mathbb{G}_m$. By the isomorphism

\begin{equation}
\lim_{n \geq N(\lambda_0)} M_{\text{fs}}^{X_0} \otimes_{\text{Spn}(A)} ((V^{N_p} \otimes E L_p(\lambda)i)_0)_{\text{fs}}^{N_{\text{fin}}\otimes L_p, T_p = \chi_0},
\end{equation}

we see $\kappa_*(\chi_{\lambda}^*(\mathcal{M}_{\lambda_0}^{X_0}(V)))$ is a coherent sheaf over $\widehat{T}_p \times Z_{L_p}^\times \times \mathbb{G}_m$ (since its global section is no other than the module in (25)), supported on the closed subspace $Z_{L_p}^\times \times \mathbb{G}_m \hookrightarrow \widehat{T}_p \times Z_{L_p}^\times \times \mathbb{G}_m$, $(\chi, x) \mapsto (\chi_0, \chi, x)$ (and we use $\chi_0$ to denote this closed embedding), moreover, $\chi_0 \kappa_*(\chi_{\lambda}^*(\mathcal{M}_{\lambda_0}^{X_0}(V))) \simto \mathcal{M}_{\lambda_0}^{X_0}$.

Recall for a coherent sheaf $\mathcal{M}$ over a rigid analytic space $\mathfrak{X}$, the support $\text{Supp}(\mathcal{M})$ of $\mathcal{M}$ is defined to be the closed rigid subspace of $\mathfrak{X}$ such that for any affinoid open $\text{Spn } A$ of $\mathfrak{X}$, $\text{Supp}(\mathcal{M}) \times \text{Spn } A \cong \text{Spn } (A/I)$ where $I := \{a \in A \mid am = 0, \forall m \in \text{Supp}(\text{Spn}(A))\}$. Denote by $F_n(X) \in B_n\{\{X, X^{-1}\}$ the characteristic power series of the compact operator $z_n$ on $M_{\lambda_0}^{X_0}$, in [7, Lem.3.10] (see also [14, Prop.5.A.6]), one can prove the support of the coherent sheaf $\mathcal{M}_{\lambda_0}^{X_0}$ on $B_n\{\{X, X^{-1}\}$ is $\text{Spn } F_n(X^{-1})$. By [10, Prop.6.4.2], $\dim \text{Supp } \mathcal{M}_{\lambda_0}^{X_0} = \dim Z_{L_p}^\times$. Moreover, by [12, Prop.5.8], there exists an admissible covering $\{U_{n,i}\}_i$ of affinoid opens of $\text{Spn } F_n(X^{-1})$, such that the image of $U_{n,i}$ via the projection $\text{Spn } B_n \times \mathbb{G}_m \rightarrow \text{Spn } B_n$ is an affinoid open, denoted by $\text{Spn } B_{n,i}$, in $\text{Spn } B_n$, and the sections of $\mathcal{M}_{\lambda_0}^{X_0}(U_{n,i})$ form a finite projective $B_{n,i}$-module. Since (24) is an isomorphism, we deduce from the above discussion:

**Corollary 2.21.** (1) The support of $\mathcal{M}_{\lambda_0}^{X_0}(V)$ is equidimensional of dimension $\dim Z_{L_p}^\times$.

(2) There exists an admissible covering $\{U_i\}$ of $\text{Supp } \mathcal{M}_{\lambda_0}^{X_0}(V)$, such that

- the image $\mathcal{V}_i$ of $U_i$ via the composition $\text{Supp } \mathcal{M}_{\lambda_0}^{X_0}(V) \hookrightarrow \mathcal{V} \rightarrow Z_{L_p}^\times$ is an affinoid open in $Z_{L_p}^\times$.

- the sections $\mathcal{M}_{\lambda_0}^{X_0}(V)$ over $U_i$ form a finite projective $\mathcal{O}(\mathcal{V}_i)$-module.

3. Eigenvarieties and closed subspaces

3.1. Notations and preliminaries. We fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Let $F^+$ be a totally real number field, $F$ a quadratic imaginary extension of $F^+$, denote by $c$ the unique non-trivial element of $\text{Gal}(F/F^+)$. We suppose $p$ is split in $F$, thus $p$ is split in $F^+$, and for any place $v$ of $F^+$ above $p$, $v$ is split in $F$. Denote by $\Sigma_p$ the places of $F^+$ above $p$. 15
Let $G$ be a definite quasi-split unitary group over $F^+$ associated to $F/F^+$, thus $G \times_{F^+} F \cong \text{GL}_n/F$, and $G(F^+ \otimes_Q \mathbb{R})$ is compact. Note one has an isomorphism $G(F^+ \otimes_Q \mathbb{Q}_p) \cong \prod_{v \in \Sigma_p} G(F_v^+) \cong \prod_{v \in \Sigma_p} \text{GL}_n(\mathbb{Q}_p)$ (where the last isomorphism depends on the choice of the places of $F$ above each $v \in \Sigma_p$).

Let $U^p$ be an open compact subgroup of $G(\mathbb{A}_{F,+}^{p,\infty})$ with the form $U^p = \prod_{v \mid p} U_v$, put

$$\widehat{S}(U^p, E) := \left\{ f : G(F^+) \setminus G(\mathbb{A}_{F,+}^{p,\infty})/U^p \rightarrow E \mid f \text{ is continuous} \right\}.$$ 

Since $G(F^+ \otimes_Q \mathbb{R})$ is compact, $G(F^+) \setminus G(\mathbb{A}_{F,+}^{p,\infty})/U^p$ is a profinite set. We see $\widehat{S}(U^p, E)$ is a Banach space over $E$ with the norm defined by the (completed) $O_E$-lattice

$$\widehat{S}(U^p, O_E) := \left\{ f : G(F^+) \setminus G(\mathbb{A}_{F,+}^{p,\infty})/U^p \rightarrow O_E \mid f \text{ is continuous} \right\}.$$ 

Moreover, $\widehat{S}(U^p, E)$ is equipped with a continuous action of $G(F^+ \otimes_Q \mathbb{Q}_p)$ given by $(gf)(g') = f(g'g)$ for $f \in \widehat{S}(U^p, E)$, $g \in G(F^+ \otimes_Q \mathbb{Q}_p)$, $g' \in G(\mathbb{A}_{F,+}^{p,\infty})$. The lattice $\widehat{S}(U^p, O_E)$ is stable by this action, thus the Banach representation $\widehat{S}(U^p, E)$ of $G(F^+ \otimes_Q \mathbb{Q}_p)$ is unitary.

**Lemma 3.1.** Let $H$ be a compact open subgroup of $G(F^+ \otimes_Q \mathbb{Q}_p)$ such that $G(F^+) \cap (U^p H) = \{1\}$, then there exists $r \in \mathbb{Z}_{\geq 1}$ such that

$$\widehat{S}(U^p, E)|_H \cong C(H, E)^{\oplus r}.$$ 

Thus, $\widehat{S}(U^p, E)$ is a unitary admissible Banach representation of $G(F^+ \otimes_Q \mathbb{Q}_p)$ over $E$.

**Proof.** Let $S \subseteq G(\mathbb{A}_{F,+}^{p,\infty})$ be a finite representative set of the finite set $G(F^+) \setminus G(\mathbb{A}_{F,+}^{p,\infty})/(U^p H)$. We have a $H$-invariant decomposition

$$\sqcup_{s \in S} (sH) \cong G(F^+) \setminus G(\mathbb{A}_{F,+}^{p,\infty})/U^p,$$

where any element of $H$ is viewed as an element of $G(\mathbb{A}_{F,+}^{p,\infty})$ via $G(F^+ \otimes_Q \mathbb{Q}_p) \hookrightarrow G(\mathbb{A}_{F,+}^{p,\infty})$. Indeed, the surjectivity follows from the fact $S$ is a representative set of $G(F^+) \setminus G(\mathbb{A}_{F,+}^{p,\infty})/(U^p H)$, while the injectivity is from the fact $G(F^+) \cap (U^p H) = \{1\}$. The lemma follows. \hfill $\Box$

Let $S(U^p)$ be the set of primes $v$ of $F^+$ satisfying

- $v \nmid p$, and $v$ is split in $F$;
- $U_v$ is a maximal compact open subgroup of $G(F_v^+)$. 

Let $\mathcal{H}_0^p$ be the spherical Hecke algebra $O_E[\prod_{v \in S(U^p)} G(F_v^+)/\prod_{v \in S(U^p)} U_v]$. Indeed, for $v \in S(U^p)$, let $\tilde{v}$ be a finite place above $v$, which would induce an isomorphism $\iota_{G,\tilde{v}} : G(F_v^+) \cong \text{GL}_n(F_{\tilde{v}})$. For $1 \leq i \leq n$, let

$$T_v^{(i)} := \left[ U_{v^i}^{-1} G_{\tilde{v},w} \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi_{\tilde{v}}^{-i} \cdot 1 \end{array} \right) U_v \right],$$

where $\varpi_{\tilde{v}}$ is a uniformizer of $F_{\tilde{v}}$. Then the Hecke algebra $O_E[\prod_{v \in S(U^p)} G(F_v^+)/\prod_{v \in S(U^p)} U_v]$ is the $O_E$-polynomial algebra generated by $T_v^{(i)}$ for $1 \leq i \leq n$. Moreover, if we denote by $\tilde{v}^{(i)}$ the another place over $v$, then $T_v^{(i)} = (T_{v^{(i)}})^{-1} T_{v^{(i-1)}}$. Let $S$ be a finite set of places of $F^+$ containing the places $v$ where $U_v$ is ramified, such that $S \cap \Sigma_p = \emptyset$, $S \cap S(U^p) = \emptyset$ and $U_v$ is maximal hyperspecial for $v \notin S \cup \Sigma_p$. Let $\mathcal{H}^{S,p}$ be the commutative spherical Hecke algebra $O_E[\prod_{v \notin S \cup \Sigma_p} G(F_v^+)/\prod_{v \notin S \cup \Sigma_p} U_v]$, thus $\mathcal{H}_0^p \subset \mathcal{H}^{S,p}$. Note $\mathcal{H}^{S,p}$ acts naturally on $\widehat{S}(U^p, E)$, and the action commutes with $G(F^+ \otimes_Q \mathbb{Q}_p)$.

Recall the automorphic representations of $G(\mathbb{A}_{F,+}^{p,\infty})$ are the irreducible constituents of the $\mathbb{C}$-vector space of functions $f : G(F^+) \setminus G(\mathbb{A}_{F,+}^{p,\infty}) \rightarrow \mathbb{C}$, which are

- $C^\infty$ when restricted to $G(F^+ \otimes_Q \mathbb{R})$,
- locally constant when restricted to $G(\mathbb{A}_{F,+}^{p,\infty})$,
\[ G(F^+ \otimes_{Q} \mathbb{R})\)-finite,

where \( G(\mathbb{A}_{F^+}) \) acts on this space via right translation. An automorphic representation \( \pi \) is isomorphic to \( \pi_{\infty} \otimes_{\mathbb{C}} \pi_{\infty} \) where \( \pi_{\infty} = W_{\infty} \) is an irreducible algebraic representation of \( G(F^+ \otimes_{Q} \mathbb{R}) \) over \( \mathbb{C} \) and \( \pi_{\infty} \cong \text{Hom}_{G(F^+ \otimes_{Q} \mathbb{R})}(W_{\infty}, \pi) \cong \otimes_{v} \pi_{v} \) is an irreducible smooth representation of \( G(\mathbb{A}_{F^+}) \). The algebraic representation \( W_{\infty} \) is defined over \( \mathbb{Q} \) via \( \iota_{\infty} \), and we denote by \( W_{p} \), its base change to \( \mathbb{Q}_{p} \), which is thus an irreducible algebraic representation of \( G(F^+ \otimes_{Q} \mathbb{Q}_{p}) \) over \( \mathbb{Q}_{p} \). Via the decomposition \( G(F^+ \otimes_{Q} \mathbb{Q}_{p}) \overset{\sim}{\rightarrow} \prod_{v \in \Sigma_{p}} G(F_{v}^{+}) \), one has \( W_{p} \cong \otimes_{v \in \Sigma_{p}} W_{v} \) where \( W_{v} \) is an irreducible algebraic representation of \( G(F_{v}^{+}) \). One can also prove \( \pi_{\infty} \) is defined over a number field via \( \iota_{\infty} \) (e.g. see [1, §6.2.3]). Denote by \( \pi_{\infty}^{p} := \otimes_{v \mid p} \pi_{v} \), thus \( \pi \cong \pi_{\infty} \otimes_{\mathbb{Q}} \pi_{p} \). Let \( m(\pi) \in \mathbb{Z}_{\geq 1} \) be the multiplicity of \( \pi \) in the space of functions as above.

**Proposition 3.2** ([6, Prop.5.1]). One has a \( (G(F^+ \otimes_{Q} \mathbb{Q}_{p}) \times \mathcal{H}^{S,p}-\text{invariant isomorphism}) \)
\[ \hat{S}(U^{p}, E)_{\text{alg}} \otimes_{E} \mathbb{Q}_{p} \cong \bigoplus_{\pi} \left( (\pi_{\infty})^{p} \otimes_{\mathbb{Q}} (\pi_{p} \otimes_{\mathbb{Q}} W_{p}) \right)^{\oplus m(\pi)}, \]

where \( \hat{S}(U^{p}, E)_{\text{alg}} \) denotes the locally algebraic subrepresentation of \( \hat{S}(U^{p}, E) \), \( \pi \cong \pi_{\infty} \otimes_{\mathbb{C}} \pi_{\infty} \) runs through the automorphic representation of \( G(\mathbb{A}_{F^+}) \) and \( W_{p} \) is associated to \( \pi_{\infty} \) as above.

We fix a place \( u \) of \( F^{+} \) above \( p \), and \( \bar{u} \mid u \), thus we fix an isomorphism \( i_{G,\bar{u}} : G(F^{+}_{u}) \rightarrow GL_{n}(\mathbb{Q}_{p}) \). Let \( W_{u}^{p} \) be an irreducible algebraic representation of \( \prod_{v \mid p, v \neq u} G(F^{+}_{v}) \) over \( E \), \( U^{u}_{p} = \prod_{v \mid p, v \neq u} U_{v} \) be a maximal compact open subgroup of \( \prod_{v \mid p, v \neq u} G(F^{+}_{v}) \), and put \( U^{u} := U^{p} U^{u}_{p} \). Put \( \hat{S}(U^{u}, W_{u}^{p}) := (\hat{S}(U^{p}, E) \otimes_{E} W_{u}^{p})^{U^{u}}_{p} \), which is an admissible unitary Banach representation of \( G(F^{+}_{u}) \) over \( E \), and is equipped with an acting of \( \mathcal{H}^{S,u} \), the \( \mathcal{O}_{E}\)-algebra generated by \( \mathcal{H}^{S,p} \) and the spherical Hecke algebra \( \mathcal{O}_{E}[G(F^{+}_{v})/U_{v}] \) for \( v \mid p, v \neq u \). Moreover, the action of \( \mathcal{H}^{S,u} \) commutes with that of \( GL_{n}(\mathbb{Q}_{p}) \). By Lem.3.1, one has

**Lemma 3.3.** Let \( H \) be an open compact subgroup of \( GL_{n}(\mathbb{Q}_{p}) \) (viewed as a subgroup of \( G(F^{+}_{u}) \) via \( i_{G,\bar{u}} \)) such that \( (HU^{u}) \cap G(F^{+}) = 1 \), then there exists \( r \in \mathbb{Z}_{\geq 1} \) such that
\[ \hat{S}(U^{u}, W_{u}^{p})_{|U} \rightarrow \mathcal{C}(H,E)^{\oplus r}. \]

**Proof.** Applying Lem.3.1 to the group \( H \times U^{u}_{p} \), one gets
\[ \hat{S}(U^{p}, E)_{|H \times U^{u}_{p}} \cong \mathcal{C}(H \times U^{u}_{p}, E)^{\oplus r} \cong \mathcal{C}(H,E)^{\oplus r} \otimes_{E} \mathcal{C}(U^{u}_{p}, E). \]

By definition, one has
\[ \hat{S}(U^{u}, W_{u}^{p})_{|H} \cong \mathcal{C}(H,E)^{\oplus r} \otimes_{E} (\mathcal{C}(U^{u}_{p}, E) \otimes_{E} W_{u}^{p})^{U^{u}_{p}}, \]

since \( (\mathcal{C}(U^{u}_{p}, E) \otimes_{E} W_{u}^{p})^{U^{u}_{p}} \) is a finite dimensional \( E \)-vector space, the lemma follows. \( \square \)

Let \( B \) be the Borel subgroup of \( GL_{n} \) of upper triangular matrices, \( \Phi \) the root system of \( GL_{n} \), \( \Delta \) the set of simple roots with respect to \( B \) and \( \Phi^{\vee} \) (resp. \( \Phi^{-} \)) the set of positive (resp. negative) roots, and \( \mathcal{W} \) the Weyl group which is generated by the simple reflections \( s_{\alpha} \) for all \( \alpha \in \Delta \). Each subset \( I \subset \Delta \) defines a root system \( \Phi_{I} \subset \Phi \) with positive roots \( \Phi^{+}_{I} \), negative roots \( \Phi^{-}_{I} \), and Weyl group \( \mathcal{W}_{I} \subset \mathcal{W} \) generated by \( s_{\alpha} \) for \( \alpha \in I \) (put \( \mathcal{W}_{\emptyset} = \{1\} \)). Let \( P_{I} \) be the parabolic subgroup associated to \( \Delta_{I} \), denote by \( L_{I} \) the Levi subgroup of \( P_{I} \), \( N_{I} \) the nilpotent radical of \( P_{I} \), \( T_{I} \) the opposite parabolic of \( P_{I} \), \( T_{I} \) the nilpotent radical of \( T_{I} \). Thus \( B = P_{0}, G = P_{\emptyset} \), put \( T := L_{\emptyset}, N := N_{\emptyset} \) etc. Denote by \( g, b, p_{I}, l_{I}, n_{I}, \mathcal{P}_{I}, \mathcal{W}_{I}, t, n \) the associated Lie algebras of \( GL_{n}, B, P_{I}, L_{I}, N_{I}, T_{I}, N_{I}, T, N \) respectively.

Let \( \lambda \in t^{\vee} := \text{Hom}_{\mathbb{Q}_{p}}(t, E) \) be a weight, thus there exist \( k_{\lambda,i} \in \mathbb{N} \) for \( i = 1, \cdots, n \) such that \( \lambda((x_{i})_{i=1,\cdots,n}) = \sum_{i=1}^{n} k_{\lambda,i} x_{i} \). Moreover, \( \lambda \) is dominant if and only if \( k_{\lambda,1} - k_{\lambda,i+1} \in \mathbb{Z}_{\geq 0} \) for \( i = 1, \cdots, n-1 \).
1. We enumerate the simple roots with \{1, \ldots , n - 1\} such that \(a_j\) gives the weight \((x_i)_{i=1,\ldots ,n} \mapsto x_j - x_{j+1}\), and fix a bijection \(\Delta \overset{\sim}{\to} \{1, \ldots , n - 1\}\). Any subset \(I\) of \(\Delta\) can be viewed as a subset of \(\{1, \ldots , n - 1\}\), and a weight \(\lambda\) is \(P_I\)-dominant if and only if \(k_{\lambda,i} - k_{\lambda,i+1} \in \mathbb{Z}_{\geq 0}\) for all \(i \in I\). A weight \(\lambda\) is called integral if \(k_{\lambda,i} \in \mathbb{Z}\) for all \(i = 1, \ldots , n\).

Let \(I \subset \Delta\), by the highest weight theory, for any \(P_I\)-dominant integral weight \(\lambda\), there exists a unique irreducible finite dimensional algebraic representation \(L_I(\lambda)\) of \(L_I\) with highest weight \(\lambda\). This gives an one-to-one bijection between the irreducible finite dimensional algebraic representations of \(L_I\) and the \(I\)-dominant integral weights. Put \(\mathcal{L}(\lambda):= \mathcal{L}_\Delta(\lambda)\). For general \(\lambda\), denote by \(\mathcal{L}(\lambda) \in \mathcal{O}_\mathbf{B}\) to be the unique simple quotient of the Verma module \(U(g) \otimes_{U(\mathbf{G})} \lambda\).

The Weyl group acts naturally on \(t^\ast\) by \(s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha\) for \(\alpha \in \Delta\). We would use frequently the dot action given by \(w \cdot \lambda := w(\lambda + \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha) - \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha\). Note that \(s_\alpha \cdot \lambda = s_\alpha(\lambda) - \alpha\) for \(\alpha \in \Delta\) (since \(s_\alpha\) permutes the set \(\Phi^+\) \(\setminus\) \{\alpha\} and sends \(\alpha\) to \(-\alpha\)). And we have \(k_{w \cdot \lambda,i} = k_{\lambda,w^{-1}(i)} - (w^{-1}(i) - i)\).

For a locally analytic character \(\chi\) of \(T(\mathbb{Q}_p)\) over \(E\), denote by \(\text{wt}(\chi) \in t^\ast\) the corresponding weight. The character \(\chi\) is called locally algebraic (resp. dominant) if \(\text{wt}(\chi)\) is integral (resp. dominant). For an integral weight \(\lambda\), denote by \(\chi_\lambda := z^{k_{\lambda,1}} \otimes \cdots \otimes z^{k_{\lambda,n}}\).

For \(1 \leq i \leq n\), denote by \(\beta_i\) the cocharacter \(G_m \to T, x \mapsto (x, \ldots , x, 1, \ldots , 1)\).

Let \(V\) be an \(E\)-vector space equipped with an \(E\)-linear action of \(A\) (with \(A\) a set of operators), \(\chi\) a system of eigenvalues of \(A\), denote by \(V^{A=\chi}\) the \(\chi\)-eigenspace, \(V[A = \chi]\) the generalized \(\chi\)-eigenspace.

### 3.2. Eigenvarieties.

Let \(m \in \mathbb{Z}_{>0}\) such that \(\{x^i = 1\} \cap (1 + p^m \mathbb{Z}_p) = \{1\}\) for \(i = 1, \ldots , n\). Let \(H\) be the compact open subgroup generated by \(\{g \in \text{SL}_n(\mathbb{Z}_p) \mid g \equiv 1 \pmod{p^m}\}\) and \(\{g \in \text{GL}_n(\mathbb{Z}_p) \mid g \equiv 1 \pmod{p^m}\}\). For any algebraic subgroup \(M\) of GL\(_n\), denote by \(M^o := M(\mathbb{Q}_p)^o H\). Note the isomorphism (20) holds for any parabolic subgroup of GL\(_n\). For \(I \subset \Delta\), denote by \(L_I(\mathbb{Q}_p)^+\) the subgroup of \(L_I(\mathbb{Q}_p)\) defined as in (8) with respect to \(N^o_I\) and put \(Z_{L_I}(\mathbb{Q}_p)^+ := Z_{L_I}(\mathbb{Q}_p) \cap L_I(\mathbb{Q}_p)^+\).

We identify GL\(_n(\mathbb{Q}_p)\) and \(G(\mathbb{F}_p^+)^{\iota_{G,\hat{a}}}\) consider the locally analytic representation \(\tilde{S}(U_u, W^u_p)\) of GL\(_n(\mathbb{Q}_p)\), which is admissible and dense in \(\tilde{S}(U_u, W^u_p)\) (cf. [26, Thm.7.1]). Let \(I \subset \Delta\), \(\lambda_0\) be a dominant weight for SL\(_n\), \(\chi_0\) a smooth character of \(T^o_I := H \cap T(\mathbb{Q}_p) \cap L^o_I(\mathbb{Q}_p)\), applying the functor \(J_{B,I}(P_I, \lambda_0)(\cdot)\) (cf. (12)), we get an essentially admissible locally analytic representation of \(T(\mathbb{Q}_p)\) over \(E\):

\[
J_{B,I}(P_I, \lambda_0)(\tilde{S}(U_u, W^u_p))
\]

which is equipped with a continuous action of \(\mathcal{H}^{S,u}\) commuting with \(T(\mathbb{Q}_p)\). By definition (of essentially admissible locally analytic representations), we associate to \(J_{B,I}(P_I, \lambda_0)(\tilde{S}(U_u, W^u_p))\) a coherent sheaf \(N_{T,\lambda_0}(U_u, W^u_p)\) over \(\hat{T}\), equipped moreover with an \(\mathcal{O}(\hat{T})\)-linear action of \(\mathcal{H}^{S,u}\). By Emerton’s method ([16, §2.3]), we can construct an eigenvariety from the triplet \(\{N_{T,\lambda_0}(U_u, W^u_p), \hat{T}, \mathcal{H}^{S,u}\}\):

**Theorem 3.4.** There exists a rigid analytic space \(\mathcal{V}_{T,\lambda_0}^{U_u, W^u_p}(U_u, W^u_p)\) over \(E\), together with a finite morphism of rigid spaces

\[
i : \mathcal{V}_{T,\lambda_0}^{U_u, W^u_p}(U_u, W^u_p) \longrightarrow \hat{T}
\]

and a morphism of \(E\)-algebras (compatible with \(i\)) with dense image

\[
\mathcal{O}(\hat{T}) \otimes_{\mathcal{O}_E} \mathcal{H}^{S,u} \longrightarrow \mathcal{O}(\mathcal{V}_{T,\lambda_0}^{U_u, W^u_p}(U_u, W^u_p)),
\]

satisfying that
(1) a closed point $z$ of $\mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p)$ is uniquely determined by its image $\chi$ in $\overline{T(E)}$ and the induced morphism $h: H^{S,u} \to E$, called a system of eigenvalues of $H^{S,u}$, hence a such $z$ would be denoted by $(\chi, h)$;

(2) for a finite extension $L$ of $E$, $(\chi, h) \in \mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p)(L)$ if and only if the corresponding eigenspace

$$\left( J_{B,(P_i),\lambda_0}^{T_i,\lambda_0} \right) (\overline{S(U^u, W^u_p)_{\text{an}}}) \otimes_E L \right)^{T(\mathbb{Q}_p)} = \chi, H^{S,u} = h$$

is non-zero;

(3) there exists a coherent sheaf $\mathcal{M}_{I,\lambda_0}^{X_0}(U^u, W^u_p)$ over $\mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p)$ such that $i_*\mathcal{M}_{I,\lambda_0}^{X_0}(U^u, W^u_p) \cong N_{I,\lambda_0}^{X_0}(U^u, W^u_p)$ and that for an $L$-point $z = (\chi, h)$, the special fiber $z^*\mathcal{M}_{I,\lambda_0}(U^u, W^u_p)$ is naturally dual to the (finite dimensional) $L$-vector space

$$\left( J_{B,(P_i),\lambda_0}^{T_i,\lambda_0} \right) (\overline{S(U^u, W^u_p)_{\text{an}}}) \otimes_E L \right)^{T(\mathbb{Q}_p)} = \chi, H^{S,u} = h$$

Remark 3.8. Denote by $\chi_{\lambda_0}$ the algebraic character of $T(\mathbb{Q}_p) \cap SL_n(\mathbb{Q}_p)$ with weight $\lambda_0$, by definition, if $(\chi, \lambda) \in \mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p)(\overline{E})$, then $\chi \chi_{\lambda_0}$ is a smooth character of $T(\mathbb{Q}_p) \cap L^T(\mathbb{Q}_p)$ (moreover, $\chi \chi_{\lambda_0}|_{T^0} = \chi_0$). From which we deduce $k_{wt(\chi),i} - k_{wt(\chi),i+1} = k_{\lambda_0, i} - k_{\lambda_0, i+1}$ if $i, i+1 \in I$ (since $\mathfrak{s} \cong \mathfrak{sl}_n \oplus \mathfrak{j}$, we view $\lambda_0$ as a dominant weight of $\mathfrak{t}$ which equals 0 when restricted to $\mathfrak{j}$).

By Lem.3.1 and the results in §2.4, one can construct $\mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p)$ by using Buzzard’s equivariant machine ([9]). Indeed, let $A_i := \begin{pmatrix} p^i & 0 & \cdots & 0 \\ 0 & p^{i-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in GL_{i+1}(\mathbb{Q}_p), U'^{(i)} := \begin{pmatrix} A_i & 0 \\ 0 & 1 \end{pmatrix}$ for $1 \leq i \leq n-1$, $S_p := p \cdot 1_n, R_p \subset T(\mathbb{Z}_p)$ be a (finite) representative set of $T(\mathbb{Z}_p)/T^0$. For $m \in \mathbb{Z}_{\geq 1}$ big enough, let $B_m$ be the affinable algebra as in §2.4, thus $\{\text{Spm } B_m\}_m$ form an admissible covering of $\overline{Z_{0,L^1}}$. Let $M_{m}^{T^0,\chi_0}$ be the orthonormalisable $B_m$-modules as in Cor.2.20, which is equipped with a continuous action of $H^{S}$, the commutative $\mathcal{O}_K$-algebra generated by $H^{S,u}$ and $U'^{(i)}$ for $1 \leq i \leq n-1$, $S_p$ and the elements in $R_p$. Moreover, the action of $U'^{(i-1)}$ corresponds to the operator $z_m$ in Cor.2.20 and thus is compact. We apply results of [9] to $\{M_{m}^{T^0,\chi_0}, H^{S,u}, U'^{(i-1)}\}$ for each $m$, and glue them; the resulted rigid space is just $\mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p)$.

Moreover, the action of $T^0_i$ (via $\chi_0$), $Z_{0,L^1}^0$ (note $T^0 \cong T_i^0 \times Z_{0,L^1}^0$), $U'^{(i)}$ for $1 \leq i \leq n-1$, $S_p$ and $R_p$ gives rise to the morphism of $\mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p)$ over $\overline{T}$. One has thus (cf. [10, Prop.6.4.2])

**Proposition 3.6.** (1) The rigid analytic space $\mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p)$ is equidimensional of dimension $n - |I|$.

(2) Let $\kappa$ denote the composition $\mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p) \to \overline{T} \to \overline{Z_{0,L^1}^0}$, thus for any closed point $z$ of $\mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p)$ there exists an affine open neighborhood $U$ of $z$, such that $\kappa(U)$ is an affine open in $\overline{Z_{0,L^1}^0}$ and that $\kappa: U \to \kappa(U)$ is finite and surjective when restricted to any irreducible component of $U$.

Note that if $P_1 = B$, one can easily check $J_{B,(P_i),\lambda_0}^{T_i,\lambda_0}(\overline{S(U^u, W^u_p)_{\text{an}}}) \cong J_B(\overline{S(U^u, W^u_p)_{\text{an}}})$, denote by $U(U^u, W^u_p)$ the corresponding equivariant variety, which was well studied in [10]. For general case, since $J_{B,(P_i),\lambda_0}^{T_i,\lambda_0}(\overline{S(U^u, W^u_p)_{\text{an}}})$ is a closed subrepresentation of $J_B(\overline{S(U^u, W^u_p)_{\text{an}}})$, one has

**Proposition 3.7.** One has a natural closed embedding

$$\mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p) \hookrightarrow \mathcal{V}(U^u, W^u_p), (\chi, h) \mapsto (\chi, h).$$

**Remark 3.8.** Consider the natural morphism $\mathcal{V}(U^u, W^u_p) \to T^0 \to T^0_i \times \overline{Z_{0,L^1}^0}$, we view $\chi_0 \chi_{\lambda_0}|_{T^0_i}$ as a closed point of $\overline{T^0_i}$, and put $\mathcal{V}_{I,\lambda_0}^{X_0}(U^u, W^u_p)' := \mathcal{V}(U^u, W^u_p) \times_{\overline{T^0_i} \times \overline{Z_{0,L^1}^0}} \{\chi_0 \chi_{\lambda_0}|_{T^0_i} \times Z_{0,L^1}^0\}$, which is thus a
closed rigid subspace of $\mathcal{V}(U^u, W_p^u)$ containing $\mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$ (cf. Rem.3.5). Let’s remark that in general $\mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$ is more subtle than $\mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$ (see Rem.3.11, 3.24 below).

**Definition 3.9.** Let $L$ be a finite extension of $E$, $z = (\chi, \eta) \in \mathcal{V}(U^u, W_p^u)(L)$ is called classical, if

$$J_B(\tilde{S}(U^u, W_p^u)_{\text{alg}} \otimes E L)^{\mathcal{V}(Q_p) = \chi, H^{k, u} = \eta} \neq 0;$$

For a closed point $z$ of $\mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$, $z$ is called classical, if $z$ is a classical point of $\mathcal{V}(U^u, W_p^u)$ via the closed embedding (26).

Note $z = (\chi, \eta)$ being classical implies that $wt(\chi)$ is dominant. Let $Z_{cl}$ denote the set of classical points in $\mathcal{V}(U^u, W_p^u)$, it’s known that $Z_{cl}$ is Zariski-dense in $\mathcal{V}(U^u, W_p^u)$ and accumulates over the points $z = (\chi, \eta)$ with $\chi$ locally algebraic (cf. [10, §6.4.5]).

**Theorem 3.10.** The set $Z_{cl} \cap \mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)(E)$ is Zariski-dense in $\mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$ and accumulates over the points $(\chi, \eta) \in \mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$ with $\chi$ locally algebraic.

**Remark 3.11.** By Thm.3.10, $\mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$ (the reduced subspace of $\mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$) is actually the Zariski-closure of the classical points in $\mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$ (cf. Rem.3.8).

**Proof of Thm.3.10.** By Prop.3.6 (and the discussion above it), and some standard arguments as in [10, Prop.6.2.7, 6.4.6], Thm.3.10 follows from the classicality criterion Thm.3.12 below (see also Rem.3.13, the $\{wt(\chi_j')\}_{j \in \{1, \ldots, r\}}$ in Rem.3.13 gives the weight of $\chi|Z^+_L$). Note $\{\chi(\eta_j(p))\}$ (see §3.2.1 below for notations) is locally constant on $\mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$.

3.2.1. A result of classicality. Let $I \subset \{1, \ldots, n-1\}$ (where we identify the latter set with $\Delta$ as in §3.1), one can get a partition of the ordered set $\{1, \ldots, n\} = S_1 \sqcup \cdots \sqcup S_r$ such that $L_I = GL_{|S_1|} \times \cdots \times GL_{|S_r|}$. For $j \in \{1, \ldots, r\}$, put $a_j := |S_1| + \cdots + |S_j|$, and $a_0 = 0$. One has $\beta_{a_j}(p) \in Z_{L_I}(Q_p)^+$ for $j \in \{1, \ldots, r\}$. The main result of this section is (compare with [10, Prop.4.7.4])

**Theorem 3.12.** Let $L$ be a finite extension of $E$, $z = (\chi, \eta)$ be an $L$-point of $\mathcal{V}_{1,\lambda_0}^{\infty}(U^u, W_p^u)$ with $\chi$ locally algebraic and dominant, if (where $v(\cdot)$ denotes the $p$-adic valuation normalized by $v(p) = 1$)

$$v\left(\chi(\beta_{a_j}(p))\delta_B^{-1}(\beta_{a_j}(p))\right) \leq k_{wt(\chi), a_j} - k_{wt(\chi), a_{j+1}} + 1, \forall 1 \leq j \leq r - 1$$

then the point $z$ is classical.

**Remark 3.13.** By Rem.3.5, $k_{wt(\chi), a_j + 1} - k_{wt(\chi), a_{j+1}} = k_{\chi_0, a_j} - k_{\chi_0, a_{j+1}}$ for $0 \leq j \leq r - 1$. It’s easy to see $wt(\chi)$ is determined by $k_{wt(\chi), a_j}$ for $j \in \{1, \ldots, r\}$ and $\lambda_0$. Moreover, consider the restriction $\chi|Z_{L_I} = \chi_1' \otimes \cdots \chi_r'$ as a character of $\prod_{j=1}^r Q_p^\times$, with $\chi_i' = \chi_{a_{i-1}+1} \cdots \chi_{a_i}$, thus

$$wt(\chi_j') = k_{wt(\chi), a_j}|S_j| + \sum_{i=a_j+1}^{a_j+1} (k_{\lambda_0, i} - k_{\lambda_0, a_j}).$$

For $j \in \{1, \ldots, r\}$, denote by $N_j := \sum_{i=a_{j-1}+1}^{a_j} (k_{\lambda_0, i} - k_{\lambda_0, a_j})$, the numerical criterion in (27) can be reformulated as

$$v\left(\chi(\beta_{a_j}(p))\delta_B^{-1}(\beta_{a_j}(p))\right) \leq \frac{wt(\chi_j') - N_j}{|S_j|} - \frac{wt(\chi_{j+1}') - N_{j+1}}{|S_{j+1}|} + (k_{\lambda_0, a_j} - k_{\lambda_0, a_{j+1}}) + 1$$

for $j \in \{1, \cdots, r - 1\}$.

The theorem follows easily from the following proposition
Proposition 3.14. Keep the above notation, if \((27)\) holds, any vector \(v\) in the generalized eigenspace
\[
J_{B,(P_I,\lambda_0)}((\hat{S}(U^u, W_p^u))_{\text{an}} \otimes_E L)^{T^\chi} \cong [T(Q_p) = \chi, \mathcal{H}^{S,u} = \mathfrak{h}] 
\]
is locally algebraic, i.e. \(v \in J_{B,(P_I,\lambda_0)}((\hat{S}(U^u, W_p^u))_{\text{alg}} \otimes_E L)\).

Without loss of generality, we assume \(L = E\), since \(\chi\) is locally algebraic and has dominant weight, we can write \(\chi = \chi_{\text{wt}(\chi)} \psi\) where \(\psi\) is a smooth character of \(T(Q_p)\) over \(E\). For any non-zero vector
\[
v \in J_{B,(P_I,\lambda_0)}((\hat{S}(U^u, W_p^u))_{\text{an}})^{T^\chi} \cong [T(Q_p) = \chi, \mathcal{H}^{S,u} = \mathfrak{h}],
\]
the \(T(Q_p)\) subrepresentation generated by \(v\) is isomorphic to \(\chi_{\text{wt}(\chi)} \otimes_E \pi_\psi\) with \(\pi_\psi\) a finite length smooth representation of \(T(Q_p)\), whose irreducible components are all \(\psi\). By the adjunction formula Thm.2.15, the injection \(\chi_{\text{wt}(\chi)} \otimes_E \pi_\psi \hookrightarrow J_{B,(P_I,\lambda_0)}((\hat{S}(U^u, W_p^u))_{\text{an}})\) induces a non-zero map
\[
F_{\mathfrak{p}_I}^{\text{GLn}}((U(g) \otimes U(\mathfrak{g}))_I \otimes \mathcal{O} \otimes \mathcal{O}^+ \otimes \mathcal{O}^+)^{\psi \otimes E \delta_B^{-1}} \longrightarrow (\hat{S}(U^u, W_p^u))_{\text{an}}.
\]
Recall any irreducible component of \((U(g) \otimes U(\mathfrak{g}))_I \otimes \mathcal{O} \otimes \mathcal{O}^+ \otimes \mathcal{O}^+\) has the form \(\mathcal{L}(w \cdot (-\text{wt}(\chi)))\) with \(w \cdot (-\text{wt}(\chi))\) being \(P_I\)-dominant (recall \(-\text{wt}(\chi)\) denotes the highest weight of \(\mathcal{L}(\text{wt}(\chi))\)). Denote by \(\pi_I := (\text{Ind}_{\mathfrak{p}_I}^{\mathfrak{p}_I} \otimes E \delta_B^{-1})\) for simplicity, and note \(\pi_I\) has the central character \(\psi \delta_B^{-1}\).

Let \(w \in \mathcal{W}\) such that \(w \cdot (-\text{wt}(\chi))\) is \(P_I\)-dominant, let \(I_w \supseteq I\) such that \(P_{I_w}\) is the maximal parabolic subgroup for \(\mathcal{L}(w \cdot (-\text{wt}(\chi)))\). Let \(\pi_w\) be an irreducible component of \((\text{Ind}_{\mathfrak{p}_I}^{\mathfrak{p}_I} \otimes E \delta_B^{-1})\), which also has the central character \(\psi \delta_B^{-1}\). We would show

Lemma 3.15. Keep the above notation, and suppose the conditions in \((27)\) hold, then the irreducible representation \(F_{\mathfrak{p}_w}^{\text{GLn}}(\mathcal{L}(w \cdot (-\text{wt}(\chi))), \pi_w)\) does not admit any invariant lattice. Consequently, \(F_{\mathfrak{p}_w}^{\text{GLn}}(\mathcal{L}(w \cdot (-\text{wt}(\chi))), \pi_w)\) cannot be a subrepresentation of \((\hat{S}(U^u, W_p^u))_{\text{an}}\) (since \((\hat{S}(U^u, W_p^u)\) is unitary).

Proof of Prop.3.14. By Thm.2.4 and the discussion that precedes Lem.3.15, any irreducible component of
\[
F_{\mathfrak{p}_w}^{\text{GLn}}((U(g) \otimes U(\mathfrak{g}))_I \otimes \mathcal{O} \otimes \mathcal{O}^+ \otimes \mathcal{O}^+)^{\psi \otimes E \delta_B^{-1}} \longrightarrow (\hat{S}(U^u, W_p^u))_{\text{an}}
\]
has the form \(F_{\mathfrak{p}_w}^{\text{GLn}}(\mathcal{L}(w \cdot (-\text{wt}(\chi))), \pi_w)\) (with the above notations). By Lem.3.15, we see any non-zero map as in \((28)\) factors through
\[
F_{\mathfrak{p}_w}^{\text{GLn}}(\mathcal{L}(\text{wt}(\chi)), \pi_w \otimes E \delta_B^{-1}) \cong (\text{Ind}_{\mathfrak{p}_w}^{\mathfrak{p}_w} \otimes E \delta_B^{-1}) \otimes E \mathcal{L}(\text{wt}(\chi)),
\]
and thus has image in \((\hat{S}(U^u, W_p^u))_{\text{alg}}\). \qed

Proof of Lem.3.15. By [5, Cor.3.5], if \(F_{\mathfrak{p}_w}^{\text{GLn}}(\mathcal{L}(w \cdot (-\text{wt}(\chi))), \pi_w)\) admits an invariant lattice, then
\[
\chi_{w \cdot \text{wt}(\chi)}(z)\psi(z)\delta_B^{-1}(z) \in \mathcal{O}_E
\]
for all \(z \in Z_{L_{I_w}}(Q_p)^{+}\). Note \(\chi_{w \cdot \text{wt}(\chi)}\psi\delta_B^{-1} = \chi_{w \cdot \text{wt}(\chi) - \text{wt}(\chi)}\chi\delta_B^{-1}\). By the lemma below (applied to \(\lambda = \text{wt}(\chi), P = P_{I_w}\)), there exists \(1 \leq j_w \leq \delta_B^{-1}\) such that \(\beta_{a,j_w}(p) \in Z_{L_{I_w}}(Q_p)^{+}\) (indeed, since \(P_I \subseteq P_{I_w}\), for \(1 \leq a \leq n\), if \(\beta_0(p) \in Z_{L_{I_w}}(Q_p)^{+}\), one gets \(a = a_j\) for some \(1 \leq j \leq \delta_B^{-1}\), and \(\langle w \cdot \text{wt}(\chi) - \text{wt}(\chi), \beta_{a,j_w} \rangle \leq k_{\text{wt}(\chi),a,j_w} - 1\). Thus \(v(\chi_{w \cdot \text{wt}(\chi)}(\beta_{a,j_w}(p))) \leq k_{\text{wt}(\chi),a,j_w} - 1\). Consequently,
\[
v(\chi_{w \cdot \text{wt}(\chi)}(\beta_{a,j_w}(p))) \geq k_{\text{wt}(\chi),a,j_w} - k_{\text{wt}(\chi),a,j_w} + 1 + 1,
\]
a contradiction with \((27)\). \qed
Lemma 3.16. Let $P \supseteq B$ be a parabolic subgroup of $\text{GL}_n$, $\lambda$ be a dominant weight of $\mathfrak{t}$, $w \in \mathfrak{w}$, $w \neq 1$ such that $w \cdot \lambda$ is $P$-dominant, then there exists $1 \leq a \leq n$, such that the cocharacter $\beta_ a$ satisfies $\beta_ a(p) \in Z_{L_P}(Q_P)^+$ and $(w \cdot \lambda - \beta_ a) \leq k_{\lambda,a+1} - k_{\lambda,a} - 1$.

Proof. Consider the partition of the ordered set $\{1, \cdots , n\} = S_{P,1} \sqcup \cdots \sqcup S_{P,s}$ such that $L_P = \text{GL}_{|S_{P,1}|} \times \cdots \times \text{GL}_{|S_{P,s}|}$. Since $w \cdot \lambda$ is $P$-dominant, $k_{w,\lambda,i} \geq k_{w,\lambda,i+1}$ if $i,i+1 \in S_{P,j}$ for some $j \in \{1, \cdots , s\}$. Since $w \neq 1$, there exists $i$ such that $w^{−1}(i) \neq i$. Let $i_0 \in \{1, \cdots , n\}$ be the smallest number such that $w^{−1}(i_0) \neq i_0$ (thus $i_0 < w^{−1}(i_0)$), let $j_0 \in \{1, \cdots , r\}$ such that $i_0 \in S_{P,j_0}$. Note $k_{w,\lambda,i_0} = k_{\lambda,w^{−1}(i_0)} - (w^{−1}(i_0) - i_0) < k_{\lambda,i}$, for $i_0 < i \leq w^{−1}(i_0)$.

If $i \in S_{P,j_0}$, $i \geq i_0$, we claim $w^{−1}(i) \geq w^{−1}(i_0) + i - i_0$. Indeed, if $i_0 + 1 \in S_{P,j_0}$, and if $w^{−1}(i_0 + 1) < w^{−1}(i_0)$ then $k_{w,\lambda,i_0 + 1} = k_{\lambda,w^{−1}(i_0 + 1)} - (w^{−1}(i_0 + 1) - i_0 - 1) \geq k_{\lambda,w^{−1}(i_0)} - (w^{−1}(i_0 + 1) - i_0 - 1) > k_{w,\lambda,i_0}$ which contradicts to the fact $w \cdot \lambda$ is $P$-dominant. The claim follows then by induction on $i - i_0$.

We has thus $k_{w,\lambda,i} < k_{\lambda,i}$ for $i \geq i_0$, $i \in S_{P,j_0}$, let $a := |S_{P,1}| + \cdots + |S_{P,j_0}|$, one has

$$
(w \cdot \lambda - \beta_ a) = \sum_{i \geq i_0, i \in S_{P,j_0}} (k_{w,\lambda,i} - k_{\lambda,i}) = \sum_{i \geq i_0, i \in S_{P,j_0}} (k_{\lambda,w^{−1}(i)} - (w^{−1}(i) - i) - k_{\lambda,i}) \\
\leq \sum_{i \geq i_0, i \in S_{P,j_0}} (k_{\lambda,w^{−1}(i)} - 1 - k_{\lambda,i}) \leq k_{\lambda,a} - 1 \leq k_{\lambda,a+1} - k_{\lambda,a} - 1.
$$

The lemma follows.

\[\square\]

3.3. Families of Galois representations. Let $\rho$ be an $n$-dimensional continuous representation of $\text{Gal}(\overline{F}/F)$ over $E$ satisfying

- $\rho^e \cong \rho^c \otimes_{E} \varepsilon^{1-n}$ where $\rho^c(g) := \rho(cgc)$ for all $g \in \text{Gal}(\overline{F}/F)$, $c$ is the unique non-trivial element in $\text{Gal}(F/F^+)$, $\varepsilon$ denotes the cyclotomic character;
- $\rho$ is unramified for all but finitely many places of $F$, and $\rho$ is unramified for places of $F$ lying above places in $S(U^p)$.

Note by Chebotarev density theorem, $\rho$ is determined by $\rho(\text{Frob}_\ell)$ for $\ell | v$, $v \in S(U^p)$. One can associate to $\rho$ a maximal ideal $m_\rho$ of $\mathcal{H}_0^{S_p} \otimes_{\mathcal{O}_E} \mathcal{E}$ generated by elements $\{-1\}^j \text{Norm}(\bar{v})^{\frac{\text{Norm}(\bar{v})}{T_\ell}} - a_\ell^{(j)}\}$ where $j \in \{1, \cdots , n\}$, $\bar{v} \in S(U^p)$, Norm$(\bar{v})$ is the cardinality of the residue field of $F_{\ell}^\circ$, $X^n + a_\ell^{(1)}X^{n-1} + \cdots + a_\ell^{(n-1)}X + a_\ell^{(n)} \in \mathcal{E}[X]$ is the characteristic polynomial of $\rho(\text{Frob}_\ell)$ with $\text{Frob}_\ell$ a geometric Frobenius at $\bar{v}$. Put

$$
\begin{align*}
\hat{\Pi}(\rho) : = \hat{\mathcal{S}}(U^n, W^u)^{m_\rho}
\end{align*}
$$

the subspace of $\hat{\mathcal{S}}(U^n, W^u)$ annihilated by $m_\rho$, which is thus an admissible unitary Banach representation of $G(F^p_\ell) \cong \text{GL}_n(Q_\ell)$ (via $\iota_{G,\ell}$). The Galois representation $\rho$ is called modular if $\hat{\Pi}(\rho)_{\text{alg}} \neq 0$, in other word, if $\rho$ is associated to certain automorphic representation of $G$ (cf. Prop.3.2); $\rho$ is called promodular if $\hat{\Pi}(\rho) \neq 0$.

Recall to any closed point $z = (\chi, h)$ of $\mathcal{V}(U^n, W^u)$ (thus of $\mathcal{V}_{1,\lambda_0}^{\chi_0}(U^n, W^u)$), one can associate a continuous absolutely semi-simple representation $\rho_z : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{E}) \to \text{GL}_n(k(z))$ where $k(z)$ denotes the residue field at $z$. Suppose moreover

- $\chi$ is locally algebraic,
- $\text{wt}(\chi) = w \cdot \lambda$ for certain integral dominant weight $\lambda$;

22
let \((\psi_1, \cdots, \psi_n) := \chi \chi_{\text{et}}^{-1}\) by global triangulation theory \([22, 23]\) applied to \(V(U^u, W_p^u)\), the restriction \(\rho_{z, \tilde{a}} := \rho_{z}\big|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}\) is trianguline of parameter \((\delta_1, \cdots, \delta_n)\), (e.g. see \([5, \S 7]\)) \(\big(\text{i.e. the } (\varphi, \Gamma)\)-module \(D_{\text{rig}}(\rho_{z, \tilde{a}})\) over \(\mathcal{R}_L\) admits a filtration \(\{\text{Fil}^j\}_{j=0, \cdots, n}\) with \(\text{Fil}^0 D_{\text{rig}}(\rho_{z, \tilde{a}}) = 0\), \(\text{Fil}^n D_{\text{rig}}(\rho_{z, \tilde{a}}) = D_{\text{rig}}(\rho_{z, \tilde{a}})\), and \(\text{gr}^i D_{\text{rig}}(\rho_{z, \tilde{a}}) := \text{Fil}^i / \text{Fil}^{i-1} D_{\text{rig}}(\rho_{z, \tilde{a}}) \cong \mathcal{R}_L(\delta_i)\) is a rank 1 \((\varphi, \Gamma)\)-module over \(\mathcal{R}_L\) associated to \(\delta_i\)), where \(\delta_i : \mathbb{Q}_p^+ \to L^x\) are continuous characters given by

\[
(\delta_1, \cdots, \delta_i, \cdots, \delta_n) = (\psi_1 \text{ unr}(p^{n-1}), \cdots, \psi_i \text{ unr}(p^{n+1-2i}), \cdots, \psi_n \text{ unr}(p^{1-n})) \chi_{w(z, \lambda)},
\]

with \(w(z) \in \mathcal{W}\).

We define an order on \(n\)-tuples in \(\mathbb{Z}^n\): \((k_1, \cdots, k_n) \leq (k_1', \cdots, k_n')\) if \(\sum_{j=1}^i k_j \leq \sum_{j=1}^i k_j'\) for all \(1 \leq i \leq n\), and define an action of \(\mathcal{W} \cong S_n\) on \(\mathbb{Z}^n\) by \(w(k_1, \cdots, k_n) = (k_{w^{-1}(1)}, \cdots, k_{w^{-1}(n)})\). Let \(h(z) := (k_{1,1}, \cdots, -(k_{1,i} + 1 - i), \cdots, -(k_{i,0} + 1 - n))\) which are in fact the Hodge-Tate weights of \(\rho_{z, \tilde{a}}\) (where we choose the convention that the Hodge-Tate weight of the cyclotomic character \(\varepsilon = -1\), and is anti-dominant, i.e. strictly increasing.

**Proposition 3.17** \([6, \text{Prop.9.2}]\). *Keep the above notation and assumption, one has \(w(h(z)) \leq w(z)\) for the Bruhat’s ordering (cf. \([6, \text{Rem.9.4 (2)}]\)).*

**Remark 3.18.** *Note that, conjecturally, one should have \(w \leq w(z)\) by the Bruhat’s ordering (cf. \([6, \text{Rem.9.4 (2)}]\)).*

**Definition 3.19** \([8, \text{Def.2.10}]\). *Let \(\rho_p\) be a crystalline representation of \(\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\) of dimension \(n\) over \(E\), we say \(\rho_p\) is very regular if*

- \(\rho_p\) has distinct Hodge-Tate weights,
- the eigenvalues \(\{\phi_1, \cdots, \phi_n\}\) of the crystalline Frobenius on \(D_{\text{cris}}(\rho_p)\) satisfies \(\phi_i \phi_j^{-1} \neq 1, p\), for \(i \neq j\),
- \(\phi_1 \phi_2 \cdots \phi_i\) is a simple eigenvalue of the crystalline Frobenius on \(\wedge^i_p D_{\text{cris}}(\rho_p)\) for \(1 \leq i \leq n\).*

The following proposition follows from the same argument as in \([11, \text{Thm.4.8, Thm.4.10}]\).

**Proposition 3.20.** *Let \(z = (\chi, h)\) be a classical point of \(V_{I, \lambda_0}^{\chi_0}(U^u, W_p^u)\) satisfying that \(\rho_z\) is absolutely irreducible, \(\rho_{z, \tilde{a}}\) is crystalline and very regular for all \(\tilde{v}|p\). Suppose \(n \leq 3\), or \(F/F^+\) unramified, \(G\) quasi-split at all finite places, \(U_v\) maximal hyperspecial at all inert places. If*

\[
J_{B,(P_1, \lambda_0)} \bigg( \tilde{S}(U^u, W_p^u)_{\text{alg}} \otimes E L \bigg)^{T_{\text{cris}} = \chi[T(\mathbb{Q}_p) = \chi, \mathcal{H}^{S,u} = h]}
\]

\[
\sim J_{B,(P_1, \lambda_0)} \bigg( \tilde{S}(U^u, W_p^u)_{\text{an}} \otimes E L \bigg)^{T_{\text{cris}} = \chi[T(\mathbb{Q}_p) = \chi, \mathcal{H}^{S,u} = h]},
\]

*then the map \(\kappa : V_{I, \lambda_0}^{\chi_0}(U^u, W_p^u) \to \mathcal{Z}_{L, I}^{\chi_0}\) is étale at \(z\).*

**Proof.** Indeed, by the discussion above Prop.3.6, one can reduce to the same situation as in the beginning of the proof of \([11, \text{Thm.4.8}]\). Then Prop.3.14 ensures there exists a set of classical points satisfying (31) which accumulates over \(z\) (as in \([11, (4.19)]\)). The hypothesis on \(z\) gives the property \([11, (4.20)]\). This proposition then follows from the multiplicity one result as in the proof of \([11, \text{Thm.4.8, Thm.4.10}]\). \(\square\)

**Lemma 3.21.** *Let \(w, w' \in \mathcal{W}, h = (h_1, \cdots, h_n) \in \mathbb{Z}^n, h_i < h_{i+1}\) for \(i = 1, \cdots, n-1\), suppose \(w(h) \geq w'(h)\), then if \(w \in \mathcal{W}_I\), so is \(w'\).*

**Proof.** If \(w' \notin \mathcal{W}_I\), let \(i \in \{1, \cdots, n\}\) be the smallest number such that \((w')^{-1}(i)\) and \(i\) do not belong to the same partition defined as in the beginning of §3.2.1, let \(i_0 \in \{1, \cdots, r\}\) such that \(i \in S_{i_0}\) (see the
beginning of §3.2.1, let \( a := |S_1| + \cdots + |S_0| \), one sees easily (where the first equality follows from \( w \in \mathcal{W}_I \))
\[
\sum_{i=1}^a h_{w^{-1}(i)} = \sum_{i=1}^a h_i < \sum_{i=1}^a h_{(w')}^{-1}(i),
\]
a contradiction. \( \square \)

The following theorem generalizes [11, Thm.4.8,Thm.4.10].

**Theorem 3.22.** Let \( z = (\chi, h) \) be a classical point of \( \Psi_{I,\lambda_0}^{X_0}(U^u, W^u_p) \) satisfying that \( \rho_z \) is absolutely irreducible, \( \rho_z, \bar{\rho}_z \) is crystalline and very regular for all \( \overline{\nu} \mid p \). Suppose \( n \leq 3 \), or \( F/F^+ \) unramified, \( G \) quasi-split at all finite places, \( U_v \), maximal hyperspecial at all inert places. If \( w(z) \in \mathcal{W}_I \) (cf. (30)), then \( \Psi_{I,\lambda_0}^{X_0}(U^u, W^u_p) \) is étale over \( \widehat{Z}_{L_I}^{\nu} \) at \( z \).

**Proof.** By Prop.3.20, it’s sufficient to show that if \( w(z) \in \mathcal{W}_I \), then any vectors in the generalized eigenspace
\[
J_{B,(P_I,\lambda_0)}(\widehat{S}(U^u, W^u_p)_{an} \otimes \mathcal{O} L)^{\pi_{w}} = \chi [T(\mathcal{O}_p) = \chi, \mathcal{H}^{S,u} = h]
\]
is locally algebraic (cf. Prop.3.14). And as in the proof of Prop.3.14 (and we use the notations there), it’s sufficient to prove that \( \mathcal{F}_{GL}^{\nu}(\mathcal{L}(-w \cdot \text{wt}(\chi), \pi_w)) \) (see Lem.3.15) can not be a subrepresentation of \( \widehat{S}(U^P, E)[\mathcal{H}^{S,u} = h] \) if \( w \neq 1 \) and \( w \cdot \text{wt}(\chi) \) is \( P_I \)-dominant. Suppose there exist \( w \neq 1 \), \( w \cdot \text{wt}(\chi) \) being \( P_I \)-dominant and an injection
\[
\mathcal{F}_{GL}^{\nu}(\mathcal{L}(-w \cdot \text{wt}(\chi), \pi_w)) \hookrightarrow \widehat{S}(U^u, W^u_p)_{an}[\mathcal{H}^{S,u} = h].
\]
Applying the Jacquet-Emerton functor \( J_B(\cdot) \), by [5, Cor.3.4], we get a closed point \( z' = (\chi', h) \in \mathcal{V}(U^u, W^u_p) \) with \( \chi' = \chi \chi_{\nu} \cdot \chi \) (see also the proof of [6, Thm.9.3]). By Prop.3.17, \( w(h(z')) \leq w(z')(h(z')) \), note \( w(z') = w(z) \cdot h(z) = h(z) \), and thus by Lem.3.21 \( w \in \mathcal{W}_I \), which contradicts the fact that \( w \cdot \text{wt}(\chi) \) is \( P_I \)-dominant and \( w \neq 1 \). \( \square \)

Conversely, we have the following result which follows directly from results of Bergdall ([2])

**Theorem 3.23.** Let \( z = (\chi, h) \) be a classical point of \( \Psi_{I,\lambda_0}^{X_0}(U^u, W^u_p) \) satisfying that \( \rho_z \) is absolutely irreducible, and that \( \rho_z, \bar{\rho}_z \) is crystalline and very regular. Suppose \( w(z) \notin \mathcal{W}_I \), then \( \Psi_{I,\lambda_0}^{X_0}(U^u, W^u_p) \) is not étale over \( \widehat{Z}_{L_I}^{\nu} \) at \( z \).

**Proof.** Denote by \( \mathcal{V} := \Psi_{I,\lambda_0}^{X_0}(U^u, W^u_p) \), \( \mathcal{W} := \widehat{Z}_{L_I}^{\nu} \) for simplicity. Consider the tangent map \( \nabla_z : T_{V,z} \to T_{F,\chi} \), and let
\[
X := \{ (x_i) \in T_{F,\chi} \cong k(z)^n | x_i = x_{i'}, \text{ for } i, i' \in S_j, j \in \{1, \ldots, r\} \}
\]
We have \( \text{Im}(\nabla_z) \subseteq X \) (e.g. see Rem.3.11). The natural map \( T_{F,\chi} \to T_{W,\nu(z)} \) thus induces an isomorphism \( X \cong T_{W,\nu(z)} \). Since \( w(z) \notin \mathcal{W}_I \), there exists \( i \in \{1, \ldots, n\} \) such that \( (w(z))^{-1}(i) \) and \( i \) do not belong to the same partition defined by \( I \) (cf. §3.2.1). By [2, Thm.B], \( \text{Im}(\nabla_z)_i = \text{Im}(\nabla_z)_{w(z)^{-1}(i)} \), from which we deduce \( \nabla_z \) is not surjective onto \( X \), the theorem follows. \( \square \)

**Remark 3.24.** The two above theorems also imply that in general the rigid space \( \Psi_{I,\lambda_0}^{X_0}(U^u, W^u_p)' \) (cf. Rem.3.8) is different from \( \Psi_{I,\lambda_0}^{X_0}(U^u, W^u_p) \). For example, applying Thm.3.23 to \( \mathcal{V}(U^u, W^u_p) \), if \( z \) is a classical point with \( w(z) \neq 1 \), then \( \mathcal{V}(U^u, W^u_p) \) is not étale over \( \widehat{Z}_{L_I}^{\nu} \) at \( z \) for any \( I \subseteq \Delta \); however, if there exists \( I \subseteq \Delta \) such that \( w(z) \in \mathcal{W}_I \), then \( \Psi_{I,\lambda_0}^{X_0}(U^u, W^u_p) \) should be étale over \( \widehat{Z}_{L_I}^{\nu} \) at \( z \).
4. Local-global compatibility

Let $\rho_p$ be an $n$-dimensional very regular crystalline representation of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over $E$. Let $h = (h_1, \ldots, h_n)$ be the Hodge-Tate weights of $\rho_p$ (with $h_1 < h_2 < \cdots < h_n$), $(\phi_1, \phi_2, \ldots, \phi_n)$ be the eigenvalues of the crystalline Frobenius on $D_{\text{cris}}(\rho_p)$. Recall $\rho_p$ admits $n!$ triangulations (which are also called refinements) parameterized by $\mathcal{W} \cong S_n$. Indeed, for $w \in \mathcal{W}$, one has a triangulation of $\rho_p$ of parameter

$$
\chi_{w^\text{alg}(w)}(\lambda).
$$

for some $w^\text{alg}(w) \in \mathcal{W}$ (uniquely determined by $w$ and $\rho_p$), where $\lambda$ is the dominant weight of $t$ with $k_{\lambda,i} = -h_i + i$ for $i = 1, \ldots, n$. Let $\psi_{w,i} := \text{unr}(\phi_{w-i+1}^{p^{n-1}}), \psi_w := \psi_{w,1} \otimes \cdots \otimes \psi_{w,n}$. Note by the assumption on $\{\phi_i\}$, the smooth representations $\pi := \left(\text{Ind}_{B^{p\mathfrak{m}}(\mathfrak{p})}^{B(\mathfrak{p})} \psi_w \delta^{-1}_B\right)^{\infty}$ of $\text{GL}_n(\mathbb{Q}_p)$ are irreducible and isomorphic to each other (recall $\delta_B = \text{unr}(p^{1-n}) \otimes \cdots \otimes \text{unr}(p^{2n-1}) \otimes \cdots \otimes \text{unr}(p^n)$). Following Breuil ([5], §6), for $(w^\text{alg}, w) \in S_n \times S_n$, put (note the notation is slight different from that in [6, §6], indeed, the dominant weight $\lambda$ that we use differs from “$\lambda_{\text{BU}}$” in loc. cit. by $n - 1$)

$$
C(w^\text{alg}, w) := F_{\overline{\mathbb{Q}}}^{\text{GL}n}(L(w^\text{alg}, -(\lambda)), \psi_w \delta_B^{(-1)})
$$

which is irreducible by Thm.2.4 (4).

Let $\rho$ be an $n$-dimensional continuous representation of $\text{Gal}(\overline{F}/F)$ over $E$ as in the beginning of §3.3. Suppose $\rho_{\bar{u}}$ is crystalline and very regular for all $\bar{u} | p$. Let $h_{\bar{u}} := (h_{\bar{u},1}, \ldots, h_{\bar{u},n})$ denote the Hodge-Tate weights of $\rho_{\bar{u}}, (\phi_{\bar{u},1}, \cdots, \phi_{\bar{u},n})$ eigenvalues of crystalline Frobenius on $D_{\text{cris}}(\rho_{\bar{u}})$ . Let $\lambda_{\bar{u}}$ be the dominant weight of $t$ with $k_{\lambda,i} = -h_{\bar{u},i} + i - 1$ for $i = 1, \cdots, n$. We associate as above to $\rho_{\bar{u}}$ locally analytic representations $\{C(w_{\bar{u}}^\text{alg}, w_{\bar{u}})\}_{(w_{\bar{u}}^\text{alg}, w_{\bar{u}}) \in S_n \times S_n}$. Suppose moreover $\tilde{\Pi}(\rho)^{\text{alg}} \neq 0$.

**Conjecture 4.1** ([5], [6, Conj.5.3]). Keep the above notation and assumption, then

$$
\text{Hom}_{\text{GL}_n(\mathbb{Q}_p)} \left( C(w_{\bar{u}}^\text{alg}, w_{\bar{u}}), \tilde{\Pi}(\rho)_{\text{an}} \right) \neq 0
$$

if and only if $w_{\bar{u}}^\text{alg} \leq w_{\bar{u}}^\text{alg}(w_{\bar{u}})$, where the $\text{GL}_n(\mathbb{Q}_p)$ acts on $\tilde{\Pi}(\rho)$ via $i_{G_{\bar{u}}}$.

**Remark 4.2.** By [6, Prop.5.4], the conjecture is in fact independent of the choice of $\bar{u}$, i.e. if Conj.4.1 holds for $\bar{u}$ then it holds for $\bar{u}^c$.

Recall

**Theorem 4.3** ([6]). Keep the notation and assumption as in Conj.4.1.

1. If $\text{Hom}_{\text{GL}_n(\mathbb{Q}_p)} \left( C(w_{\bar{u}}^\text{alg}, w_{\bar{u}}), \tilde{\Pi}(\rho)_{\text{an}} \right) \neq 0$, then $w_{\bar{u}}^\text{alg}(h_{\bar{u}}) \leq w_{\bar{u}}^\text{alg}(w_{\bar{u}})(h_{\bar{u}})$. If moreover $n < 3$ or $\text{lg}(w_{\bar{u}}^\text{alg}(w_{\bar{u}})) \leq 2$, then $w_{\bar{u}}^\text{alg} \leq w_{\bar{u}}^\text{alg}(w_{\bar{u}})$.

2. Suppose $n \leq 3$, or $F/F^+$ unramified, $G$ quasi-split at all finite places and $U_v$ maximal hyperspecial at all inert places, for $w_{\bar{u}} \in S_n$, if $w_{\bar{u}}^\text{alg}(w_{\bar{u}}) \neq 1$, then there exists $w_{\bar{u}}^\text{alg} \in S_n \setminus \{1\}$, such that

$$
\text{Hom}_{\text{GL}_n(\mathbb{Q}_p)} \left( C(w_{\bar{u}}^\text{alg}, w_{\bar{u}}), \tilde{\Pi}(\rho)_{\text{an}} \right) \neq 0.
$$

In particular, when $\text{lg}(w_{\bar{u}}^\text{alg}(w_{\bar{u}})) \leq 1$, Conj.4.1 was proved (under the global hypothesis as in Thm.4.3). The following theorem is the main result of this note, which improves Thm.4.3 (2).

**Theorem 4.4.** Keep the notation and assumption as in Conj.4.1. Suppose $n \leq 3$, or $F/F^+$ unramified, $G$ quasi-split at all finite places and $U_v$ maximal hyperspecial at all inert places. Let $I \subset \Delta$, for $w_{\bar{u}} \in S_n$, if $w_{\bar{u}}^\text{alg}(w_{\bar{u}}) \notin \mathcal{W}_I$, then there exists $w_{\bar{u}}^\text{alg} \in S_n$ satisfying

- $w_{\bar{u}}^\text{alg} \neq 1$,
• \( \mathcal{L}(w_α^{\text{alg}} \cdot (-λ_α)) \) is an irreducible component of the generalized Verma module \( U(\mathfrak{g}) \otimes U(\pi) \mathcal{L}_I(-λ_α) \)

  (which implies in particular \( w_α^{\text{alg}} \cdot (-λ_α) \) is \( P_I \)-dominant),

such that

\[
\text{Hom}_{GL_n(\mathbb{Q}_p)} \left( C(w_α^{\text{alg}}, w_α), \hat{\Pi}(\rho)_{\text{an}} \right) \neq 0.
\]

Proof. Since \( \hat{\Pi}(\rho)_{\text{alg}} \neq 0 \), we associate to \( \rho \) a system of Hecke eigenvalues \( \mathfrak{h}_\rho : \mathcal{H}^{S,u} \to E \). Indeed, by Prop.3.2 and (29), there exists an automorphic representation \( \pi = \pi_{\infty} \otimes \pi_{\infty} \), with \( \mathcal{H}^{S,p} \) acting on \( (\pi_{\infty})^{U_\infty} \) by \( \mathcal{H}_0^{S,p} / \mathfrak{m}_u \) (since \( \pi_{\infty} \) is defined over a number field, by enlarging \( E \), we assume \( \pi_{\infty} \) is defined over \( E \)). Thus \( \mathcal{H}^{S,u} \) acts on \( (\pi_{\infty})^{U_\infty} \) via a morphism \( \mathfrak{h}_\rho : \mathcal{H}^{S,u} \to E \). Let \( \lambda_0 := \lambda_\alpha|_{\mathfrak{p}} \), for each \( w \in S_\alpha \),

we get an \( E \)-point \( z_w = (\chi_w, \mathfrak{h}_\rho) \) in \( V := V_{1, \lambda_0}(U_\infty, W_p) \) with \( \chi_w = ψ_w \chi_λ \) (where “1” denotes the trivial character). Note, with the notation in Thm.3.23, \( w_α^{\text{alg}}(w) \) is just \( w(z_w) \). By Thm.3.23, if \( w_α^{\text{alg}}(w) \notin \mathcal{W}_I \), then \( V \) is not étale over \( \mathbb{Z}_p^I \). Then by Prop.3.20, the natural injection

\[
J_{B,(P_I, λ_0)}(\mathcal{S}(U^I, W_p)_{\text{alg}}) \otimes \mathcal{L}(\mathcal{Q}_p) = χ_w, \mathcal{H}^{S,u} = \mathfrak{h}_\rho]
\]

is not surjective. The theorem follows by applying the adjunction formula Thm.2.15 to the \( (\mathcal{Q}_p) \)-representation \( J_{B,(P_I, λ_0)}(\mathcal{S}(U^I, W_p)_{\text{alg}}) \otimes \mathcal{L}(\mathcal{Q}_p) = χ_w, \mathcal{H}^{S,u} = \mathfrak{h}_\rho) \).

In particular, when \( n = 3 \), taking \( P_I \) to be a maximal proper parabolic subgroup of \( GL_3 \), then “\( w_α^{\text{alg}} \)” in the theorem is equal to the simple reflection \( s \notin \mathcal{W}_I \) if exists. Indeed, in this case, one has an exact sequence (cf. [21, §9.5])

\[
0 \to \mathcal{L}(s \cdot (-λ_0)) \to U(\mathfrak{g}) \otimes U(\pi) \mathcal{L}_I(-λ_0) \to \mathcal{L}(-λ_0) \to 0.
\]

Thus putting Thm.4.3 (1) and Thm.4.4 together, one gets

Corollary 4.5. Suppose \( n = 3 \), let \( α \in Δ \), then \( s_α ≤ w_α^{\text{alg}}(w) \) if and only if \( C(s_α, w) \) is a subrepresentation of \( \hat{\Pi}(\rho)_{\text{an}} \). In particular, if \( lg(w_α^{\text{alg}}(w)) \geq 2 \), then \( \oplus_{α ∈ Δ} C(s_α, w) \) is a subrepresentation of \( \hat{\Pi}(\rho)_{\text{an}} \).

In the general setting, let \( \mathcal{P} \) denote the set of maximal proper parabolic subgroups \( P_I \) of \( GL_n \) such that \( w_α^{\text{alg}}(w) \notin \mathcal{W}_I \), thus \( |\mathcal{P}| = \{α ∈ Δ \mid s_α ≤ w_α^{\text{alg}}(w)\} \). For each \( P_I \in \mathcal{P} \), by Thm.4.4 one gets a subrepresentation \( C(w_α^{\text{alg}}, w) \) of \( \hat{\Pi}(\rho)_{\text{an}} \) (where \( w_α^{\text{alg}} \) is the “\( w_α^{\text{alg}} \)” in Thm.4.4 applied to \( P_I \)). Note that these \( w_α^{\text{alg}} \) are distinct since for \( w \notin \mathcal{W} \) if \( w \cdot (-λ_0) \) is dominant for two different maximal proper parabolic subgroups, then \( w \cdot (-λ_0) \) is dominant, and hence \( w = 1 \). By [5, Lem.6.2], the locally analytic representations \( C(w_α^{\text{alg}}, w) \) are distinct for different \( P_I \). Thus one has

Corollary 4.6. Keep the above notation and the assumption in Thm.4.4, \( \oplus_{P_I ∈ \mathcal{P}} C(w_α^{\text{alg}}, w) \) is a subrepresentation of \( \hat{\Pi}(\rho)_{\text{an}} \).

Indeed, when \( n ≥ 4 \), for a maximal proper parabolic subgroup \( P \) of \( GL_n \), the generalized Verma module \( U(\mathfrak{g}) \otimes U(\pi) \mathcal{L}_I(-λ_0) \) might be more complicated, consequently, in general Thm.4.4 could not explicate the “\( w_α^{\text{alg}} \)” in Cor.4.6 (unlike the \( GL_3(\mathbb{Q}_p) \) case as in Cor.4.5). We end this note by an example for \( GL_4(\mathbb{Q}_p) \).

Example 4.7. Suppose \( n = 4 \), \( λ_α = 0 \), we identify the set of simple roots \( Δ \) with \( \{1, 2, 3\} \) as in §3.1, thus \( L_{\{1,2\}} = GL_2 \times GL_1 \), \( L_{\{2,3\}} = GL_1 \times GL_3 \) and \( L_{\{1,3\}} = GL_2 \times GL_2 \). Denote by \( s_i ∈ S_ι \), \( i ∈ \{1, 2, 3\} \) the corresponding simple reflection, let \( ι ⊆ \{1, 2, 3\} \), \( |ι| = 2 \), \( i^ι ∈ \{1, 2, 3\} \), \( i^ι \notin ι \), and denote by \( S_ι \) the set

26
of irreducible composants of the generalized Verma module $U(g) \otimes U(\mathfrak{g}) L_i(0)$ (which all have multiplicity one by [19, Thm.8.4]). By loc. cit., one has

$$S_i = \begin{cases} \{ \mathcal{L}(0), \mathcal{L}(s_i \cdot 0) \} & I \in \{\{1, 2\}, \{2, 3\}\} \\ \{ \mathcal{L}(0), \mathcal{L}(s_2 \cdot 0), \mathcal{L}(s_2 s_3 s_1 s_2) \cdot 0) \} & I = \{3\} \end{cases}$$

Thus as in $GL_2(\mathbb{Q}_p)$ case, one has

- let $i = 1$ or $3$, then $s_i \leq w_{alg}(w_{\tilde{u}})$ if and only if $C(s_i, w_{\tilde{u}})$ is a subrepresentation of $\tilde{\Pi}(\rho)_{an}$.
- if $w_{alg}(w_{\tilde{u}}) \not\geq s_2 s_3 s_1 s_2$, then $s_i \leq w_{alg}(w_{\tilde{u}})$ if and only if $C(s_i, w_{\tilde{u}})$ is a subrepresentation of $\tilde{\Pi}(\rho)_{an}$ for $i \in \{1, 2, 3\}$.

However, if $w_{alg}(w_{\tilde{u}}) \geq s_2 s_3 s_1 s_2$, the author does not know how to see the (conjectured) injection $C(s_2, w_{\tilde{u}}) \hookrightarrow \tilde{\Pi}(\rho)_{an}$.

References

[1] Bellache J., Chenevier G., Families of Galois representations and Selmer groups, Astrique 324, (2009).
[2] Bergdall J., Chenevier G., Families of Galois representations and Selmer groups, arXiv preprint arXiv:1410.3412, (2014).
[3] Bergdall J., Chojecki P., Ordinary representations and companion points for $U(3)$ in the indecomposable case, arXiv preprint arXiv:1405.3026 (2014).
[4] Breuil C., The emerging $p$-adic Langlands programme, Proceedings of I.C.M.2010, Vol II, 203-230.
[5] Breuil C., Vers le socle localement analytique pour $GL_n$, I, to appear in Annales de l’Institut Fourier, (2013).
[6] Breuil C., Vers le socle localement analytique pour $GL_n$, II, Math. Annalen. 361, (2015), 741-785.
[7] Breuil C., Hellmann E., Schraen B., Une interprétation modulaire de la variété trianguline, preprint, arXiv:1411.7260 (2014).
[8] Breuil C., Hellmann E., Schraen B., Smoothness and Classicality on eigenvarieties, preprint arXiv:1510.01222, (2015).
[9] Buzzard K., Eigenvarieties, London mathematical society lecture note series 320, (2007), P.59.
[10] Chenevier G., Familles $p$-adiques de formes automorphes pour $GL_n$, J. reine angew. Math, 570, (2004), 143-217.
[11] Chenevier G., On the infinite fern of Galois representations of unitary type, Ann. Sci. Éc. Norm. Supér.(4), 44(6), (2011), 963-1019.
[12] Coleman R., $p$-adic Banach spaces and families of modular forms. Invent. Math. 127(3), (1997), 417-479.
[13] Colmez P., Dospinescu G., Complétés universels de représentations de $GL_2(\mathbb{Q}_p)$, Algebra & Number Theory, 8(6), (2014), 1447-1519.
[14] Ding Y., Formes modulatres $p$-adiques sur les courbes de Shimura unitaires et compatibilité local-global, thesis, available at: https://sites.google.com/site/yiwendingmath/thesis.pdf.
[15] Emerton M., Locally analytic vectors in representations of locally $p$-adic analytic groups, to appear in Memoirs of the Amer. Math. Soc., (2004).
[16] Emerton M., On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms, Invent. Math. 164, (2006), 1-84.
[17] Emerton M., Jacquet Modules of locally analytic representations of $p$-adic reductive groups I. constructions and first properties, Ann. Sci. É.N.S.39, no. 5, (2006), 775-839.
[18] Emerton M., Jacquet modules of locally analytic representations of $p$-adic reductive groups II. The relation to parabolic induction, J. Institut Math. Jussieu, (2007).
[19] Enright T., Shelton B., Categories of highest weight modules: applications to classical Hermitian symmetric pairs. Vol. 367. American Mathematical Soc., 1987.
[20] Hill R., Loeffler D., Emerton’s Jacquet functors for non-Borel parabolic subgroups, Documenta Mathematica, 16, (2011), 1-31.
[21] Humphreys J. E., Representations of semisimple Lie algebras in the BGG category O, American Mathematical Soc., (2008).
[22] Kedlaya K., Pottharst J., Xiao L., Cohomology of arithmetic families of $(\phi, \Gamma)$-modules, to appear in J. of the Amer. Math. Soc., (2012).
[23] Liu R., Triangulation of refined families, arXiv preprint arXiv:1202.2188, (2012).
[24] Orlik S., Strauch M., *On Jordan-Hölder series of some locally analytic representations*, Journal of the American Mathematical Society, 28(1), 99-157.

[25] Schneider P., Teitelbaum J., *U(g)-finite locally analytic representations*, Represent. Theory 5, (2001), 111-128.

[26] Schneider P., Teitelbaum J., *Algebras of p-adic distributions and admissible representations*, Invent. math. 153, (2003), 145-196.

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON

E-mail address: y.ding@imperial.ac.uk