A weak compactness theorem of the Donaldson–Thomas instantons on compact Kähler threefolds

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Abstract

In [Ta], we introduced a gauge-theoretic equation on symplectic 6-manifolds, which is a version of the Hermitian–Einstein equation perturbed by Higgs fields, and called it a Donaldson–Thomas equation, to analytically approach the Donaldson–Thomas invariants. In this article, we consider the equation on compact Kähler threefolds, and study some of the analytic properties of solutions to them, using analytic methods in higher-dimensional Yang–Mills theory developed by Nakajima [N1], [N2] and Tian [Ti] with some additional arguments concerning an extra nonlinear term coming from the Higgs fields. We prove that a sequence of solutions to the Donaldson–Thomas equation of a unitary vector bundle over a compact Kähler threefold has a converging subsequence outside a closed subset whose real two-dimensional Hausdorff measure is finite, provided that the $L^2$-norms of the Higgs fields are uniformly bounded. We also prove an $n/2$-compactness theorem of solutions to the equations on compact Kähler threefolds.

1 Introduction

The Donaldson–Thomas invariant is a deformation invariant of Calabi–Yau threefolds, which was constructed by Thomas [Th] from the moduli space of (semi-)stable sheaves by using algebraic geometry techniques. There are further generalization of this by Joyce and Song [JS] and Kontsevich and Soibelman [KS1], [KS2]. These fit into programmes by Donaldson and Thomas [DT] and Donaldson and Segal [DS], and many outcomes of them were made in both Mathematics and Physics.

We approach these invariants by using an analysis, aiming at revealing more symmetry and structures in the theory of the Donaldson–Thomas invariants. In [Ta], we introduced perturbed Hermitian-Einstein equations on
symplectic 6-manifolds, which we called Donaldson–Thomas equations, with a solution called Donaldson–Thomas instanton, to analytically approach the Donaldson–Thomas invariants, studied the infinitesimal deformation and the Kuranishi model of the moduli space of the Donaldson–Thomas instantons, and described the moduli space as a symplectic quotient by using a moment map for the action of gauge group. We also introduced a stability condition which ought to produce a Hitchin–Kobayashi-type correspondence for the Donaldson–Thomas instanton on Kähler threefolds.

In this article, to make the argument simple in some sense, we assume that the underlying manifold is a compact Kähler threefold, and look into the analytic aspect of the Donaldson–Thomas (D–T) instantons, especially, bubbling phenomena of them at the initial phase. Bubbling phenomena of Yang–Mills fields were first studied by Uhlenbeck [U1], [U2] (see also [W]), and later by Nakajima [N1], [N2]. Tian [T1] further analysed them by using geometric measure theoretic methods developed by Lin [Li]. We use these methods with some additional arguments concerning an extra nonlinear term coming from the Higgs fields to analyse the Donaldson–Thomas instantons on compact Kähler threefolds.

The equations on compact Kähler threefolds. Let \( Z \) be a compact Kähler threefold with Kähler form \( \omega \), and let \( E \) be a unitary vector bundle over \( Z \) of rank \( r \). A complex structure on \( Z \) gives the splitting of the space of the complexified two forms as \( \Lambda^2 \otimes \mathbb{C} = \Lambda^{1,1} \oplus \Lambda^{2,0} \oplus \Lambda^{0,2} \), and \( \Lambda^{1,1} \) further decomposes into \( \mathbb{C} \langle \omega \rangle \oplus \Lambda^{0,1}_0 \).

We consider the following equations for a connection \( A \) of \( E \), and an \( u(E) \)-valued \((0,3)\)-form \( u \) on \( Z \):

\[
F^0_{A} = 0, \quad \partial^*_A u = 0, \quad (1.1)
\]

\[
F^1_{A} \wedge \omega^2 + [u, \bar{u}] + i \frac{\lambda(E)}{3} Id_E \omega^3 = 0, \quad (1.2)
\]

where \( \lambda(E) \) is a constant defined by \( \lambda(E) := 6\pi(c_1(E) \cdot [\omega]^2)/r[\omega]^3 \). We call these equations the Donaldson–Thomas equations, and we call a solution \( (A, u) \) to these equations a Donaldson–Thomas instanton (or a D–T instanton for short).

We remark that, in the Kähler case, the Weitzenböck formula (3.2) implies that the Higgs field \( u \) is absent for \( c_1(Z) > 0 \), and covariantly constant for \( c_1(Z) = 0 \). This property of the Higgs field is similar to the Hitchin pair [Hi]; so this article may virtually concern compact Kähler threefolds of general type. However, we expect that this might provide a model in some
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sense for attacking problems on compact symplectic 6-manifolds with $c_1 = 0$ because of strong similarities between these two geometries.

As mentioned above, using the methods by [N1], [N2], [Ti] with some additional arguments on the Higgs fields, we prove a weak compactness theorem of the Donaldson–Thomas instantons on compact Kähler threefolds.

**Theorem 1.1.** Let $Z$ be a Kähler threefold, and let $E$ be a unitary vector bundle over $Z$. Let $\{(A_n, u_n)\}$ be a sequence of $D$–$T$ instantons of $E$. We assume that $\int_Z |u_n|^2 dV_g$ are uniformly bounded. Then there exists a subsequence $\{(A_{n_j}, u_{n_j})\}$ of $\{(A_n, u_n)\}$, a closed subset $S$ of $Z$ whose real two-dimensional Hausdorff measure is finite, and a sequence of gauge transformations $\{\sigma_j\}$ over $Z \setminus S$ such that $\{\sigma_j^*(A_{n_j}, u_{n_j})\}$ converges to a $D$–$T$ instanton over $Z \setminus S$.

In addition, following [Z1], [Z2], we also prove an $n/2$-compactness theorem of the Donaldson–Thomas instantons on compact Kähler threefolds.

**Theorem 1.2.** Let $\{(A_n, u_n)\}$ be a sequence of $D$–$T$ instantons of a unitary vector bundle $E$ over a compact Kähler threefold $Z$ with $\int_Z |F_{A_n}|^2 dV_g \leq C$, where $C > 0$ is a uniform constant. We assume that $\int_Z |u_n|^2 dV_g$ are also uniformly bounded. Then there exists a sequence of gauge transformations $\{\sigma_j\}$ and a subsequence $\{(A_{n_j}, u_{n_j})\}$ of $\{(A_n, u_n)\}$ such that $\{\sigma_j^*(A_{n_j}, u_{n_j})\}$ converges to a smooth $D$–$T$ instanton of $E$ over $Z$.

On the assumption on the uniform bound on the $L^2$-norm of $u$.

As in the case of the Hitchin pair [Hi], there is a circle action on the moduli space of $D$–$T$ instantons by $(A, u) \mapsto (A, e^{i\theta}u)$ when the underlying manifold is a Kähler threefold. The action is Hamiltonian, and the moment map with respect to the action is given by $||u||_{L^2}^2$. For the Hitchin pair, Hausel [Ha] introduced a compactification of the moduli space of the Hitchin pairs by using the symplectic cut developed by Lerman [Le]. We pursue an analogy of this to compactify the moduli space of $D$–$T$ instantons in the direction of the Higgs fields; so assuming $L^2$-bound for $u$ should fit into the context.

The organization of this article is as follows. In Section 2, we prove a monotonicity formula for the Donaldson–Thomas instantons on compact Kähler threefolds. In Section 3, we derive a bound on $u$, and prove an $\varepsilon$-regularity theorem for the Donaldson–Thomas instantons on compact Kähler threefolds. The proof of Theorem 1.1 is given in Section 4. In Section 5, we prove the $n/2$-compactness theorem (Theorem 1.2) of the Donaldson–Thomas instantons on compact Kähler threefolds.
Notation. Throughout this article, $C, C'$, and $C''$ are positive constants, but they can be different each time they occur.

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2 Monotonicity formula

Let $Z$ be a compact Kähler threefold with Kähler metric $g$, and let $E$ be a unitary vector bundle over $Z$ of rank $r$. We fix a Hermitian metric on $E$. Our starting point is the following identity:

$$
\int_Z \left\{ |F_{A,0}^2 + \bar{\partial}_A u|^2 + \frac{1}{2} |F_{A,1}^1 \wedge \omega^2 + [u, \bar{u}] + i \frac{\lambda(E)}{3} Id_E \omega^3|^2 \right\} dV_g
$$

$$
+ (2c_2(E) - c_1(E)^2) \cdot [\omega] + \frac{3 (c_1(E) \cdot [\omega]^2)^2}{2r [\omega]^2}
$$

$$
= \int_Z \left\{ \frac{1}{2} (|F_A|^2 + |\bar{D}_A u|^2 + |[u, \bar{u}]|^2)
$$

$$
+ \langle F_{A,1}^1 \wedge \omega^2 + i \frac{\lambda(E)}{3} Id_E \omega^3, [u, \bar{u}] \rangle \right\} dV_g,
$$

where $\bar{D}_A u = \bar{\partial}_A u + \partial_A \bar{u}$. Here we used the Bianchi identity $\bar{\partial}_A F_{A,0}^2 = 0$ to deduce $\int_Y \langle F_{A,0}^2, \bar{\partial}_A u \rangle dV_g = 0$. We put

$$
L(A, u) = \int_Z \left\{ \frac{1}{2} (|F_A|^2 + |\bar{D}_A u|^2 + |[u, \bar{u}]|^2)
$$

$$
+ \langle F_{A,1}^1 \wedge \omega^2 + i \frac{\lambda(E)}{3} Id_E \omega^3, [u, \bar{u}] \rangle \right\} dV_g.
$$

If $(A, u)$ is a D–T instanton, $L(A, u)$ becomes

$$
L(A, u) = \frac{1}{2} \int_Z \left\{ |F_A|^2 - |[u, \bar{u}]|^2 \right\} dV_g
$$

$$
= (2c_2(E) - c_1(E)^2) \cdot [\omega] + \frac{3 (c_1(E) \cdot [\omega]^2)^2}{2r [\omega]^2},
$$
where $F_A^\perp$ is the $\Lambda^{1,1}$-component of $F_A$. One can see also from this that the $L^2$-norm of the curvature $F_A$ is bounded by the topological constant and $\int_Z |[u, \bar{u}]|^2 dV$, if $(A, u)$ is a D–T instanton. Hence, from Proposition 3.1, it is bounded if $\|u\|_{L^2}$ is bounded.

In this section, we prove the following monotonicity formula for the Donaldson–Thomas instantons on Kähler threefolds.

**Proposition 2.1.** Let $(A, u)$ be a D–T instanton of a unitary vector bundle $E$ over a compact Kähler threefold $Z$. Then, for any $z \in Z$, there exists a positive constant $r_z$ such that for any $0 < \sigma < \rho < r_z$, the following holds:

\[
\frac{1}{\rho^2} e^{a\rho^2} \int_{B_{\rho}(z)} m(A, u) dV - \frac{1}{\sigma^2} e^{a\sigma^2} \int_{B_{\sigma}(z)} m(A, u) dV \\
\geq \int_{\sigma}^{\rho} 8\tau^{-3} e^{a\tau^2} \int_{B_{\tau}(z)} |[u, \bar{u}]|^2 dV \, d\tau \\
+ \int_{B_{\rho}(z) \setminus B_{\sigma}(z)} r^{-2} e^{a\tau^2} \left\{ 4 \left| \frac{\partial}{\partial r} [F_A^\perp] \right|^2 - 12 \left| \frac{\partial}{\partial r} |[u, \bar{u}]| \right|^2 \right\} dV, \tag{2.1}
\]

where $m(A, u) := |F_A^\perp|^2 - |[u, \bar{u}]|^2$, and $a$ is a constant which depends only on $Z$.

**Proof.** The proof goes along almost the same line as that of [Ti, Th. 2.1.1, 2.1.2] (see also [P, Th. 1]). Thus, we describe it rather sketchily.

First, as in [Ti, pp. 208], we consider a one-parameter family of diffeomorphisms $\{\varphi_t\}_{|t| < \infty}$ of $Z$ with $\varphi_0 = id_Z$. We fix a connection $A_0$, and denote by $D$ its covariant derivative. For $(A, u) \in \mathcal{A}(E) \times \Omega^{0,3}(Z, u(E))$, where $\mathcal{A}(E)$ is the space of connections of $E$, we define a one-parameter family $\{(A_t, u_t)\}$ in the following way. Let $\tau_t^0$ be the parallel transport of $E$ associated to $A_0$ along the path $\varphi_t(z)_{0 \leq t \leq t}$, where $z \in Z$. We define a family of connections $A_t$ by defining its covariant derivative as $D_X^s := (\tau_t^0)^{-1} (D_{d\varphi_t(X)} (\tau_t^0(s)))$, where $X \in TZ$ and $s \in \Gamma(Z, E)$. Then the curvature of $A_t$ is written as $F_{A_t}(X, Y) = (\tau_t^0)^{-1} \cdot F_A(d\varphi_t(X), d\varphi_t(Y)) \cdot \tau_t^0$, where $X, Y \in TZ$. We also define $u_t$ by $\varphi_t^* u$. We now assume that $(A, u)$ is a D–T instanton. Then, the
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same computation as in [Ti, pp. 209] yields

\[ 0 = \frac{d}{dt} L(A_t, u_t) \bigg|_{t=0} \]

\[ = -\frac{1}{2} \int_Z \left( |F_A^+|^2 \text{div} X + 4 \sum \langle F_A^+([X, e_i], e_j), F_A^+(e_i, e_j) \rangle \right) dV_g \]

\[ - \frac{1}{2} \int_Z \left( |\overline{u, u}|^2 \text{div} X \right. \]

\[ \left. + 12 \sum \langle [u, \bar{u}]([X, e_i], e_{i_2}, \ldots, e_{i_6}), [u, \bar{u}](e_{i_1}, \ldots, e_{i_6}) \rangle \right) dV_g, \]

where \( \{e_i\} \) is an orthonormal basis of \( T_z Z \) at \( z \in Z \), and \( X = \beta \frac{d\phi}{\partial t} \bigg|_{t=0} \).

Furthermore, using \( [X, e_i] = \nabla_X e_i - \nabla_{e_i} X \) and \( g(\nabla_X e_i, e_k) = -g(\nabla_X e_k, e_i) \),
we get

\[ 0 = \int_Z \left( |F_A^+|^2 \text{div} X - 4 \sum \langle F_A^+(\nabla_{e_i} X, e_j), F_A^+(e_i, e_j) \rangle \right) dV_g \]

\[ - \int_Z \left( |\overline{u, u}|^2 \text{div} X \right. \]

\[ \left. - 12 \sum \langle [u, \bar{u}](\nabla_{e_i} X, e_{i_2}, \ldots, e_{i_6}), [u, \bar{u}](e_{i_1}, \ldots, e_{i_6}) \rangle \right) dV_g, \]

This is a version of the first variation formula for the Donaldson–Thomas instantons on a compact Kähler threefold, from which we deduce, for example, the monotonicity formula.

Next, for \( z \in Z \), we take a positive number \( r_z \) so that the following holds: there are normal coordinates \( \zeta = (\zeta_1, \ldots, \zeta_6) \) with \( z = (0, \ldots, 0) \) in the geodesic ball \( B_{r_z}(z) \) of \( Z \) with respect to the metric \( g \), and

\[ |g_{ij} - \delta_{ij}| \leq c(z)(|\zeta_1|^2 + \cdots + |\zeta_6|^2), \quad |dg_{ij}| \leq c(z)\sqrt{|\zeta_1|^2 + \cdots + |\zeta_6|^2} \]

for some constant \( c(z) > 0 \) which depends only on \( r_z \) and the curvature of \( g \). We then denote by \( r = r(\zeta) \) the distance function from \( z \), and by \( \phi \) a positive function on the unit sphere \( S^5 \), and define a cut-off vector field \( X \) by \( X(\zeta) := \xi(r)\phi(\zeta/r) r \frac{\partial}{\partial r} \), where \( \xi \) is a smooth function with compact support in \( B_{r_z}(z) \). We now take an orthonormal basis \( \{\partial_{\zeta_1}, e_2, \ldots, e_6\} \) around \( z \). Then, as in [Ti, pp. 211], \( \nabla_{\zeta} \frac{\partial}{\partial r} = 0, \nabla_{\zeta} X = (\xi r + \xi)\phi(\zeta/r) \frac{\partial}{\partial r}, \nabla_{e_i} X = \xi r e_i \phi \frac{\partial}{\partial r} + \xi \phi \sum_{j=1}^6 b_{ij} e_j (2 \leq i \leq 6), |b_{ij} - \delta_{ij}| \leq C'(c(z))r^2 \), and \( C' \) is a
positive constant. Plugging these into (2.2), we obtain
\[
\int_Z |F_A|^2 (\xi' + 2\xi + C'c(z)r^2\xi) \phi dV_g - \int_Z [u, \bar{u}]^2 (\xi' + 6\xi + C'c(z)r^2\xi) \phi dV_g
\]
\[
= 4 \int_Z \left\{ \xi' \phi \left| \frac{\partial}{\partial r} [F_A^+] \right|^2 + \xi r \left( \frac{\partial}{\partial r} [F_A^+, \nabla \phi] [F_A^+] \right) \right\}
\]
\[
- 12 \int_Z \left\{ \xi' \phi \left| \frac{\partial}{\partial r} [u, \bar{u}] \right|^2 + \xi r \left( \frac{\partial}{\partial r} [u, \bar{u}], \nabla \phi [u, \bar{u}] \right) \right\}.
\]
(2.3)

Let \( \chi(r) \) be a function which is smooth and satisfies \( \chi(r) = 1 \) for \( r \in [0, 1] \), \( \chi(r) = 0 \) for \( r \in [1 + \varepsilon, \infty) \) where \( \varepsilon > 0 \), and \( \chi'(r) \leq 0 \). We choose \( \xi(r) = \xi_r(r) := \chi(r/\rho) \) for \( r \in [\sigma, \rho] \). Then we obtain \( \tau \frac{\partial}{\partial r} (\xi_r(r)) = -r \xi'_r(r) \).

From this with (2.3), we get
\[
\frac{\partial}{\partial \tau} \left( \tau^{-2} e^{\alpha r^2} \int_Z \xi_r \phi |F_A^+|^2 \right) - \tau^4 \frac{\partial}{\partial \tau} \left( \tau^{-6} e^{\alpha r^2} \int_Z \xi_r \phi |[u, \bar{u}]|^2 \right)
\]
\[
\geq 4\tau^{-2} e^{\alpha r^2} \frac{\partial}{\partial \tau} \left( \int_Z \xi_r \phi \left| \frac{\partial}{\partial r} [F_A^+] \right|^2 \right) - 12\tau^{-2} e^{\alpha r^2} \frac{\partial}{\partial \tau} \left( \int_Z \xi_r \phi \left| \frac{\partial}{\partial r} [u, \bar{u}] \right|^2 \right)
\]
\[
- 4\tau^{-3} e^{\alpha r} \int_Z \xi_r \phi \left( \frac{\partial}{\partial r} [F_A^+, \nabla \phi] [F_A^+] \right)
\]
\[
+ 12\tau^{-3} e^{\alpha r} \int_Z \xi_r \phi \left( \frac{\partial}{\partial r} [u, \bar{u}], \nabla \phi [u, \bar{u}] \right),
\]
where \( a \) is a positive constant with \( a \geq C'c(z) \). Then, integrating this on \( \tau \) and letting \( \varepsilon \) go to zero, we obtain
\[
\rho^{-2} e^{\alpha r^2} \int_{B_\rho(z)} \phi m(A, u) dV_g - \sigma^{-2} e^{\alpha r^2} \int_{B_\sigma(z)} \phi m(A, u) dV_g
\]
\[
\geq \int_{B_\rho(z) \setminus B_\sigma(z)} r^{-2} e^{\alpha r^2} \phi \left( 4 \left| \frac{\partial}{\partial r} [F_A^+] \right|^2 - 12 \left| \frac{\partial}{\partial r} [u, \bar{u}] \right|^2 \right) dV_g
\]
\[
- 4 \int_\sigma \tau^{-3} e^{\alpha r} \int_{B_r(z)} r \left| \nabla [F_A^+] \right| \nabla \phi [F_A^+] dV_g d\tau
\]
\[
- 12 \int_\sigma \tau^{-3} e^{\alpha r} \int_{B_r(z)} r \left| \nabla [u, \bar{u}] \right| \nabla \phi [u, \bar{u}] dV_g d\tau
\]
\[
+ \int_\sigma 8\tau^{-3} e^{\alpha r^2} \int_{B_r(z)} \phi [u, \bar{u}]^2 dV_g d\tau.
\]
Then taking \( \phi \equiv 1 \) gives Proposition 2.1. \( \square \)
3 Estimates

3.1 A bound on \( u \)

In this subsection, we derive a bound on \( u \). We use ideas by Mares in the study of the Vafa–Witten equations in his Ph.D thesis [Ma].

**Proposition 3.1.** Let \((A, u)\) be a D–T instanton of a unitary vector bundle \( E \) over a compact Kähler threefold \( Z \). Then we have
\[
||u||_{L^\infty} \leq C ||u||_{L^2},
\]  
where \( C > 0 \) is a positive constant which depends only on \( Z \).

**Proof.** This is basically a rephrasing of [Ma, Th. 3.1.1]. First, we use the following form of the Weitzenböck formula (see [LM, Th. 8.17]):
\[
\bar{\partial}_A \partial_A u = \nabla^*_A \nabla_A u + su + [i\Lambda F_A^1, u],
\]  
where \( s \) is the scalar curvature of the metric \( g \), and \( \Lambda = (\wedge \omega)^* \). Then, by using Eqs. (1.1) and (1.2), we get
\[
\langle u, \nabla^*_A \nabla_A u \rangle = -s ||u||^2 - ||u, \bar{u}||^2.
\]  
Hence we obtain \( \langle u, \nabla^*_A \nabla_A u \rangle \leq C ||u||^2 \). Then we invoke [Ma, Th. 3.1.2] to deduce that \( ||u||_{L^\infty} \leq C ||u||_{L^2} \).

From Proposition 3.1, we immediately get the following.

**Corollary 3.2.** Let \((A, u)\) be a D–T instanton of a unitary vector bundle \( E \) over a compact Kähler threefold \( Z \). Assume a bound on the \( L^2 \)-norm of \( u \). Then for any \( z \in Z \) and \( \varepsilon > 0 \), there exists a number \( r_0 > 0 \) such that for all \( 0 < r \leq r_0 \) we have \( \frac{1}{r^2} \int_{B_r(z)} ||u, \bar{u}||^2 dV_g \leq \varepsilon \).

3.2 Curvature estimate

In this subsection, we prove an \( \varepsilon \)-regularity theorem for the Donaldson–Thomas instantons on compact Kähler threefolds.

**Proposition 3.3.** Let \((A, u)\) be a D–T instanton of a unitary vector bundle \( E \) over a compact Kähler threefold \( Z \). Then there exist constants \( \varepsilon > 0 \) and \( C > 0 \) which depend only on \( Z \) such that for any \( z \in Z \) and \( 0 < r < r_z \), where \( r_z \) is the constant in Proposition 2.1, if \( \frac{1}{r^2} \int_{B_r(z)} |F_A|^2 dV_g \leq \varepsilon \) and \( \int_{B_r(z)} ||u||^2 dV_g < \varepsilon \) then
\[
|F_A|(z) \leq \frac{C}{r^2} \left( \frac{1}{r^2} \int_{B_r(z)} |F_A|^2 dV_g \right)^{1/2} + \frac{C \varepsilon}{r^2}.
\]
Proof. The proof also goes almost identically to that of [Ti, Th. 2.2.1] except that we have the extra nonlinear term coming from the Higgs fields.

Lemma 3.4. Let \((A, u)\) be a D–T instanton of a unitary vector bundle \(E\) over a compact Kähler threefold \(Z\). Let \(z \in Z\) and \(0 < r < r_z\). Suppose that \(\int_{B_r(z)} |u|^2 \, dv_g \leq \varepsilon\). Then the following holds:
\[
\Delta |F_A| \geq -C\varepsilon - C'|F_A| - C''|F_A|^2,
\]
where \(C, C', C'' > 0\) are constants which depend only on \(Z\).

Proof. We use the following form of the Weitzenböck formula:
\[
\nabla_A^* \nabla_A \varphi = \Delta_A \varphi + R(g)\# \varphi + F_A \ast \varphi,
\]
where \(\varphi \in \Omega^p(Z, u(E)), R(g)\) is the Riemannian curvature of \(g\), and \(\#\) and \(\ast\) are multi-linear maps (see [Ti, pp. 214] for explicit expressions). Then we get
\[
\Delta |F_A|^2 = 2|\nabla_AF_A|^2 - 2\langle \nabla_A^* \nabla_AF_A, F_A \rangle
\]
\[
= 2|\nabla_AF_A|^2 - 2\langle D_A D_A^* F_A + R(g)\# F_A + F_A \ast F_A, F_A \rangle
\]
\[
\geq 2|\nabla_AF_A|^2 - C\|u, \bar{u}\|_{L^\infty} |F_A| - C'|F_A|^2 - C''|F_A|^3,
\]
where we used the equation \((1 + \ast) \wedge \omega) F_A = \Lambda^2 [u, \bar{u}],\) the Bianchi identity \(D_A F_A = 0,\) and Proposition 3.1. Thus, we get \(\Delta |F_A| \geq -C\varepsilon - C'|F_A| - C''|F_A|^2.\)

Next, we put \(f(\rho) := (r - 2\rho)^2 \sup_{x \in B_{\rho}(z)} |F_A|(x)\), where \(\rho \in [0, r/2]\).

This function is continuous; thus it attains its maximum at some \(\rho_0 \in [0, r/2]\).

Lemma 3.5. Let \(z \in Z\) and \(0 < r < r_z\). Suppose that \(\frac{1}{r^2} \int_{B_r(z)} |F_A|^2 \, dv_g \leq \varepsilon\) and \(\int_{B_r(z)} |u|^2 \, dv_g \leq \varepsilon\) for \(\varepsilon > 0\) sufficiently small. Then \(f(\rho_0) \leq 64\).

Proof. We put \(b = \sup_{x \in B_{\rho_0}(z)} |F_A|(x) = |F_A|(x_0)\), and take \(\sigma = (r - 2\rho_0)/4.\) Then, we get
\[
\sup_{x \in B_{\rho_0}(z)} |F_A|(x) \leq \sup_{x \in B_{\rho_0 + \sigma}(z)} |F_A|(x)
\]
\[
\leq \frac{(r - 2\rho_0)^2}{(r - 2\rho_0 - 2\sigma)^2} \sup_{x \in B_{\rho_0}(z)} |F_A|(x) = 4b.
\]
We now suppose for a contradiction that \( f(\rho_0) > 64 \). Clearly we have \( \sigma \sqrt{b} \geq 2 \). We then take \( \ell = \max\{b,1/r_0^2\} \), where \( r_0 \) is the constant in Corollary 3.2. We define a new metric by \( \tilde{g} := g \), and rescale \((A,u)\) as \((A,\tilde{u}) = (A,\ell^2 u)\). Then \((A,\tilde{u})\) is a D–T instanton with respect to the metric \( \tilde{g} \), and we get \( |F_A|_{\tilde{g}} = \ell^{-1}|F_A| \), where \( |F_A|_{\tilde{g}} \) is the energy density with respect to \( \tilde{g} \). We then obtain \( \sup_{z \in B_2(x_0,\tilde{g})} |F_A|_{\tilde{g}}(z) \leq 4 \).

On the other hand, from (3.3), we have \( \Delta_{\tilde{g}}|F_A|_{\tilde{g}} \geq -C\varepsilon - C'|F_A|_{\tilde{g}} - C''|F_A|_{\tilde{g}}^2 \). Thus, we get \( \Delta_{\tilde{g}}|F_A|_{\tilde{g}} \geq -C\varepsilon - C'|F_A|_{\tilde{g}} \) on \( B_2(x_0,\tilde{g}) \). Hence, by the mean value theorem (see e.g. [GT, Th. 9.20]), we obtain

\[
1 + C\varepsilon = |F_A|_{\tilde{g}}(x) + C\varepsilon \leq C \left( \int_{B_1(x_0,\tilde{g})} |F_A|_{\tilde{g}}^2 dV_{\tilde{g}} \right)^{\frac{1}{2}} + C'\varepsilon.
\]

Furthermore, from Proposition 2.1 and Corollary 3.2, we get

\[
\int_{B_1(x_0,\tilde{g})} |F_A|_{\tilde{g}}^2 dV_{\tilde{g}} = \left( \sqrt{\ell} \right)^2 \int_{B_{\frac{1}{\ell^2}}(x_0,g)} |F_A|^2 dV_g
\]

\[
= \left( \sqrt{\ell} \right)^2 \int_{B_{\frac{1}{\ell}}(x_0,g)} m(A,u) dV_g + 3 \left( \sqrt{\ell} \right)^2 \int_{B_{\frac{1}{\ell^2}}(x_0,g)} ||u,\tilde{u}||^2 dV_g
\]

\[
\leq \frac{1}{(r/2)^2} e^{(r/4)^2} \int_{B_{\frac{1}{r^2}}(x_0,g)} m(A,u) dV_g + 3 \left( \sqrt{\ell} \right)^2 \int_{B_{\frac{1}{\ell^2}}(x_0,g)} ||u,\tilde{u}||^2 dV_g
\]

\[
\leq 2^2 e^{(a/4)r^2} \varepsilon + 3\varepsilon.
\]

Hence, we obtain \( 1 \leq C \left( 2^2 e^{(a/4)r^2} \varepsilon + 3\varepsilon \right)^{\frac{1}{2}} + C'\varepsilon \), but this contradicts the assumption that \( \varepsilon \) is sufficiently small. Thus Lemma 3.5 holds.

From Lemma 3.5, we obtain \( \sup_{r \in B_1/\ell^2(z)} r^2 |F_A| \leq 4 f(\rho_0) \leq C \). We again define a new metric by \( g' := r^{-2} g \), and rescale \((A,u)\) as \((A,u') = (A,r^{-2} u)\). Then we get \( |F_A|_{g'} \leq C \). Combining this with (3.3), we obtain \( \Delta_{g'}|F_A|_{g'} \geq -C\varepsilon - C'|F_A|_{g'} \). Hence the mean value theorem again implies Proposition 3.3.

From Proposition 3.3 and a result by Uhlenbeck [U1, Th. 2.7], we get the following.

**Corollary 3.6.** There exist constants \( \varepsilon > 0 \), \( C > 0 \), and \( r_\varepsilon > 0 \) such that for any \( z \in Z \) and \( 0 < r < r_\varepsilon \), if \((A,u)\) is a D–T instanton over \( B_r(z) \) with \( r^{-2} \int_{B_r(z)} |F_A|^2 dV_g \leq \varepsilon \), then there exists a gauge transformation \( \sigma \) over \( B_r(z) \) such that \( d^* \sigma (A) = 0 \) and \( ||\sigma(A)||_{L^\infty(B_r(z))} \leq C ||F_{\sigma(A)}||_{L^\infty(B_r(z))} \).
4 A weak convergence.

In this section, using the results in Sections 2 and 3, we prove Theorem 1.1. We basically follow the proof of a previous result for Yang–Mills connections by Nakajima [N2] (see also [Ti, Prop. 3.1.2]).

First, we take ε as in Corollary 3.6, and consider a set

$$S = \bigcap_{\delta > r > 0} \left\{ z \in Z : \liminf_{n \to \infty} \frac{1}{r^2} \int_{B_r(z)} \left\{ |F_{A_n}^\perp|^2 - ||[u_n, \bar{u}_n]|^2 \right\} dV_g \geq \varepsilon \right\},$$

where δ is the injective radius of (Z, g). One can easily check that this S is closed.

**Lemma 4.1.** The real two-dimensional Hausdorff measure of S is finite.

**Proof.** Let K be a compact subset of Z. We take a covering \{B_{2\delta}(z_\alpha)\} (\alpha = 1, \ldots, N) of S ∩ K, where z_\alpha ∈ S ∩ K, and \( B_\delta(z_\alpha) \cap B_\delta(z_\beta) = \emptyset \) for \( \alpha \neq \beta \). Then for sufficiently large n, \( r^{-2} \int_{B_r(z_\alpha)} m(A_n, u_n) dV_g \geq \frac{C}{\varepsilon} \) for \( \alpha = 1, \ldots, N \).

Thus, we obtain

$$\sum_{\alpha} r^2 \leq \frac{2}{\varepsilon} \sum_{\alpha} \int_{B_{2\delta}(z_\alpha)} m(A_n, u_n) dV_g \leq \frac{2}{\varepsilon} \int_Z m(A_n, u_n) dV_g \leq \frac{C}{\varepsilon}.$$

Hence, the real two-dimensional Hausdorff measure of \( K \cap S \) is finite.

We next take a point \( z \in Z \setminus S \). By the definition of the set S, we can find a number \( N \in \mathbb{N} \) and a radius \( r' > 0 \) such that \( \frac{1}{r^2} \int_{B_r(z)} m(A_n, u_n) dV \leq \varepsilon \) for any \( 0 < r < r' \) and \( n \geq N \). We take \( r \leq \min \{ r', r_0, r_\varepsilon \} \). From Corollary 3.2, we have \( \frac{1}{r^2} \int_{B_r(z)} ||[u_n, \bar{u}_n]|^2 dV_g \leq \varepsilon \) for all \( n \in \mathbb{N} \). Hence we get

$$\frac{1}{r^2} \int_{B_r(z)} |F_{A_n}|^2 dV \leq \frac{2}{r^2} \int_{B_r(z)} ||[u_n, \bar{u}_n]|^2 dV_g + \varepsilon \leq 3\varepsilon.$$

Therefore, from Corollary 3.6, there exists a Coulomb gauge \( \sigma_n \) such that d*\( \sigma_n(A_n) = 0 \) on \( B_r(z) \) with \( ||\sigma_n(A_n)||_{L^\infty(B_r(z))} \leq C||F_{\sigma_n(A_n)}||_{L^\infty(B_r(z))}. \)

Since Eqs. (1.1) and (1.2) are gauge invariant, each \( (\sigma_n(A_n), \sigma_n(u_n)) \) satisfies the Donaldson–Thomas equations. Furthermore, Eqs. (1.1) and (1.2) with d*\( \sigma(A) = 0 \) form an elliptic system; thus, by standard elliptic theory, we also get uniform bounds on the derivatives of \( (\sigma_n(A_n), \sigma_n(u_n)) \). Hence, there exists a subsequence which converges to a D–T instanton on \( B_{r/2}(z) \) in smooth topology.

We then patch the gauges above together by using arguments in [DK, §4.4.2] (see also [U2, §3]) to get a sequence of gauge transformations \( \sigma_j \) on
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Z \setminus S and a subsequence \{(A_{n_j}, u_{n_j})\} such that \(\sigma_j(A_{n_j}, u_{n_j})\) converges to a D–T instanton on \(Z \setminus S\). We omit the detail here, since it becomes a formal repetition.

5 A convergence

In this section, we prove Theorem 1.2, which can be thought of as an \(n/2\)-compactness theorem for the Donaldson–Thomas instantons on compact Kähler threefolds. This sort of analysis for the Yang–Mills and the coupled Yang–Mills fields were studied by Sibner [S], and the convergence results for them were obtained by Zhang [Z1], [Z2]. The proof goes by analysing the singular set of a limit D–T instanton in the same way as in [Z1], [Z2].

Convergence of measures and the structure of singular sets. First, from Proposition 3.3 and the Hölder inequality, we immediately obtain the following.

**Corollary 5.1.** Let \((A, u)\) be a D–T instanton of a unitary vector bundle \(E\) over a compact Kähler threefold \(Z\). Then there exist constants \(\varepsilon > 0\) and \(C > 0\) such that for any \(z \in Z\) and \(0 < r < r_z\), if \(\int_{B_r(z)} |F_A|^3 \, dV_g \leq \varepsilon\) and \(\int_{B_r(z)} |u|^2 \, dV_g \leq \varepsilon\), then

\[
|F_A|(z) \leq C \left( \int_{B_r(z)} |F_A|^3 \, dV_g \right)^{\frac{1}{3}} + \frac{C \varepsilon}{r^2}.
\]

We then prove the following reproduction of [Z2, Th. 4.2] (see also [Ti, Th. 4.2.3]).

**Proposition 5.2.** Let \(\{(A_n, u_n)\}\) be a sequence of D–T instantons of a unitary vector bundle \(E\) over a compact Kähler threefold \(Z\). We assume that \(\int_Z |F_{A_n}|^3 dV_g \leq C\), and that \(\int_Z |u_n|^2 dV_g\) are also uniformly bounded. Then, there exist a subsequence \(\{(A_k, u_k)\}\), a sequence of gauge transformations \(\{\sigma_k\}\), and a finite set of points \(T = \{z_\alpha\}_{\alpha=1}^\ell \subset Z\) such that \(\sigma_k(A_k, u_k)\) converges to a D–T instanton \((A, u)\) over \(Z \setminus T\).

Moreover, for each \(\alpha = 1, \ldots, \ell\) there exists a positive constant \(\theta_\alpha > 0\) such that

\[
|F_{A_k}|^3 dV_g \rightarrow |F_A|^3 dV_g + \sum_{\alpha=1}^\ell \theta_\alpha \delta_{z_\alpha}.
\]
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weakly in the sense of the Radon measure, where \( \delta_{z_\alpha} \) is the Dirac measure at \( z_\alpha \).

**Proof.** We follow the proof of [Z2, Th. 4.2]. First we put

\[
T := \bigcap_{\delta > r > 0} \left\{ z \in Z \mid \liminf_{n \to \infty} \int_{B_r(z)} |F_{A_n}|^3 dV_g \geq \varepsilon \right\}.
\]

One can easily check that the set \( T \) is closed, and can prove the following two lemmas in the same way as in Theorem 1.1.

**Lemma 5.3.** The zero-dimensional Hausdorff measure of \( T \) is finite.

**Lemma 5.4.** There exist a subsequence \( \{(A_k, u_k)\} \) of \( \{(A_n, u_n)\} \) and a sequence of gauge transformations \( \{\sigma_k\} \) such that \( \sigma_k(A_k, u_k) \) converges to \( (A, u) \) outside \( Z \setminus T \).

We then consider the Radon measures \( \mu_k := |F_{A_k}|^3 dV_g \). By taking a subsequence if necessary, \( \mu_k \) weakly converges to a Radon measure \( \mu \) on \( Z \).

We write \( \mu = |F_A|^3 dV_g + \nu \), where \( \nu \) is a non-negative Radon measure on \( Z \).

Since the support of \( \nu \) is in \( T \), we write \( \nu = \sum \theta_\alpha \delta_{y_\alpha} \), where \( \theta_\alpha \geq 0 \). In fact, \( \theta_\alpha \) is positive, because, by using a cut-off function \( \chi \in C^\infty(Z) \) with \( \chi(z) = 1 \) on \( B_r(z_\alpha) \) and \( \chi(z) = 0 \) on \( Z \setminus B_{2r}(z_\alpha) \), where \( B_{2r}(z_\alpha) \) is a geodesic ball of radius \( 2r \) with centre at \( z_\alpha \) so that \( T \cap B_{2r}(z_\alpha) = \{z_\alpha\} \), we obtain

\[
\varepsilon \leq \liminf_{k \to \infty} \int_{B_r(z_\alpha)} |F_{A_k}|^3 dV_g \leq \lim_{k \to \infty} \int_{B_r(z_\alpha)} \chi |F_{A_k}|^3 dV_g \leq \theta_\alpha + \int_{B_{2r}(z_\alpha)} |F_A|^3 dV_g.
\]

Thus, by taking \( r \to 0 \), we get \( \theta_\alpha \geq \varepsilon > 0 \).

We next take normal coordinates \( (\zeta_1, \ldots, \zeta_6) \) around \( z_\alpha \), and denote by \( B(\zeta, \rho) \) an open ball of radius \( \rho \) with centre at \( \zeta \) in the normal coordinates. Imitating [Z2, (4.9)], we define a function

\[
\tilde{E}(k, \rho) := \sup_{\zeta \in B(0, r)} \int_{\exp_{z_\alpha}(B(\zeta, \rho))} |F_{A_k}|^3 dV_g
\]

for \( 0 \leq \rho \leq r \). The function \( \tilde{E}(k, \rho) \) is continuous and non-decreasing in \( \rho \) and \( \tilde{E}(k, 0) = 0 \).

From the definition of \( T \), we have \( \tilde{E}(k, r) \geq \int_{B_r(z_\alpha)} |F_{A_k}|^3 dV_g \geq \frac{3\varepsilon}{4} \) for \( k \) sufficiently large. Since \( \tilde{E}(k, r) \) is continuous, there exist \( 0 < \rho_k < r \)
and \( \zeta_k \in \overline{B(0,r)} \) such that \( \tilde{E}(k, \rho_k) = \int_{\exp_{z_{\alpha}}(B(\zeta_k, \rho_k))} |F_{A_k}|^3 dV_g = \frac{\varepsilon}{2} \). Since \( T \cap B_{2r}(z_{\alpha}) = \{ z_{\alpha} \} \), \( \rho_k \to 0, \eta_k \to 0 \) as \( k \to \infty \).

We now define \( A_{k, \rho_k} := \tau_{\rho_k}^* \exp_{z_{\alpha}}^* A_k, \ u_{k, \rho_k} := \rho_k^{-2}\tau_{\rho_k}^* \exp_{z_{\alpha}}^* u_k \), where \( \tau_{\rho_k}(v) := \zeta_k + \rho_k v \) for \( v \in T_{z_{\alpha}} Z \). Then, \((A_{k, \rho_k}, u_{k, \rho_k})\) satisfies the Donaldson–Thomas equations on \( T_{z_{\alpha}} Z \) with respect to a metric \( g_k := \rho_k^{-2}\tau_{\rho_k}^* \exp_{z_{\alpha}}^* g \). Moreover, we have \( \int_{T_{z_{\alpha}} Z} |F_{A_{k, \rho_k}}|^3 dV_{g_k} = \int_{B_{2r}(z_{\alpha})} |F_{A_k}|^3 dV_g \leq C \).

The following, a replication of [Z2, Th. 4.3], also holds for the Donaldson–Thomas instantons case.

**Proposition 5.5.** Let \((A_k, u_k)\) and \( T \) be as in Proposition 5.2, and let \( z_{\alpha} \in T \). Then there exists a subsequence of \( \{(A_{k, \rho_k}, u_{k, \rho_k})\} \), which converges to a smooth D–T instanton \((B, v)\) of the trivial bundle over \((T_{z_{\alpha}} Z, g_{z_{\alpha}})\) with \( |F_B| \neq 0 \) and \( \int_{T_{z_{\alpha}} Z} |F_B|^3 dV_{g_{z_{\alpha}}} \leq \theta_\alpha \).

**Proof.** The proof is formally the same as that of [Z2, Th. 4.3]. We have

\[
\tilde{E}(k, \rho_k) = \int_{B(0,1)} |F_{A_{k, \rho_k}}|^3 dV_g = \frac{\varepsilon}{2}.
\]

Thus, from Corollary 5.1, if \( \varepsilon \) is small, we get \( |F_{A_{k, \rho_k}}| \leq C \varepsilon^{\frac{1}{4}} + \eta_k \varepsilon \). Hence, there exist a subsequence \((A_{k', \rho_{k'}}, u_{k', \rho_{k'}})\) and a sequence of gauge transformations \( \{\sigma_{k'}\} \) such that \( \sigma_{k'}(A_{k', \rho_{k'}}, u_{k', \rho_{k'}}) \) converges to a D–T instanton \((B, v)\) on \((T_{z_{\alpha}} Z, g_{z_{\alpha}})\) \(\cong (C^3, g_0)\). From (5.1), we have \( \int_{B(0,1)} |F_B|^3 dV_{g_{z_{\alpha}}} = \frac{\varepsilon}{2} \).

Thus, \( |F_B| \neq 0 \). Also by Fatou’s lemma,

\[
\int_{T_{z_{\alpha}} Z} |F_B|^3 dV_{g_{z_{\alpha}}} \leq \liminf_{k' \to \infty} \int_{T_{z_{\alpha}} Z} |F_{A_{k'}}|^3 dV_{g_{k'}} \leq \theta_\alpha + \int_{B_{2r}(z_{\alpha})} |F_{A_k}|^3 dV_g.
\]

Thus, taking \( r \to 0 \), we obtain \( \int_{T_{z_{\alpha}} Z} |F_B|^3 dV_{g_{z_{\alpha}}} \leq \theta_\alpha \).

**Proof of Theorem 1.2.** From Proposition 5.2, we can find a subsequence \( \{(A_k, u_k)\} \) of \( \{(A_n, u_n)\} \) and a sequence of gauge transformations \( \{\sigma_k\} \) such that \( \sigma_k(A_k, u_k) \) converges to a D–T instanton over \( Z \setminus T \), where \( T \) is a finite set of points. If \( T \neq \emptyset \), then by using Proposition 5.5, we can construct a D–T instanton \((B, v)\) on \( C^3 \) with \( \int_{C^3} |F_B|^3 dV_{g_0} < C \) and \( |F_B| \neq 0 \). On the other hand, from the Weitzenböck formula (3.2), \( u \) vanishes on \((C^3, g_0)\), namely,
D–T instantons are just Hermitian–Einstein connections on \((\mathbb{C}^3, g_0)\). Thus we get a Hermitian–Einstein connection on \((\mathbb{C}^3, \omega_0)\) with \(\int_{\mathbb{C}^3} |F_A|^2 dV_{g_0} \leq C\) and \(F_A \neq 0\). However, this contradicts the following result by Zhang [Z1].

**Theorem 5.6 ([Z1] Theorem 3.3).** If \(A\) is a Hermitian–Einstein connection over \((\mathbb{C}^3, \omega_0)\) with \(\int_{\mathbb{C}^3} |F_A|^3 dV_{g_0} \leq C\), where \(g_0\) is the standard metric on \(\mathbb{C}^3\), then \(F_A \equiv 0\).

Thus, \(T = \emptyset\). This proves Theorem 1.2. \(\square\)

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