π₁ OF SYMPLECTIC AUTOMORPHISM GROUPS AND
INVERTIBLES IN QUANTUM HOMOLOGY RINGS

PAUL SEIDEL

1. INTRODUCTION

The aim of this paper is to establish a connection between the topology of the
automorphism group of a symplectic manifold \((M, \omega)\) and the quantum prod-
uct on its homology. More precisely, we assume that \(M\) is closed and con-
ected, and consider the group \(\text{Ham}(M, \omega)\) of Hamiltonian automorphisms
with the \(C^\infty\)-topology. \(\text{Ham}(M, \omega)\) is a path-connected subgroup of the
symplectic automorphism group \(\text{Aut}(M, \omega)\); if \(H^1(M, \mathbb{R}) = 0\), it is the con-
ected component of the identity in \(\text{Aut}(M, \omega)\). We introduce a homomor-
phism \(q\) from a certain extension of the fundamental group \(\pi_1(\text{Ham}(M, \omega))\)
to the group of invertibles in the quantum homology ring of \(M\). This in-
variant can be used to detect nontrivial elements in \(\pi_1(\text{Ham}(M, \omega))\). For
example, consider \(M = S^2 \times S^2\) with the family of product structures
\(\omega_\lambda = \lambda (\omega_{S^2} \times 1) + 1 \times \omega_{S^2}, \lambda > 0\). This example has been studied by
Gromov [6], McDuff [8] and Abreu [1]. McDuff showed that for \(\lambda \neq 1\),
\(\pi_1(\text{Ham}(M, \omega))\) contains an element of infinite order. This result can be
recovered by our methods. In a less direct way, the existence of \(q\) imposes
topological restrictions on all elements of \(\pi_1(\text{Ham}(M, \omega))\). An example of
this kind of reasoning can be found in section 10; a more impor-
tant one will
appear in forthcoming work by Lalonde, McDuff and Polterovich.

To define \(q\), we will use the general relationship between loops in a topo-
logical group and bundles over \(S^2\) with this group as structure group. In
our case, a smooth map \(g : S^1 \to \text{Aut}(M, \omega)\) determines a smooth fibre
bundle \(E_g\) over \(S^2\) with a family \(\Omega_g = \{\Omega_{g,z}\}_{z \in S^2}\) of symplectic structures
on its fibres. If \(g(S^1)\) lies in \(\text{Ham}(M, \omega)\), the \(\Omega_{g,z}\) are restrictions of a closed
2-form on \(E_g\). We will call a pair \((E, \Omega)\) with this property a Hamiltonian
fibre bundle.

Witten [25] proposed to define an invariant of such a bundle \((E, \Omega)\) over \(S^2\)
with fibre \((M, \omega)\) in the following way: choose a positively oriented complex
structure \(j\) on \(S^2\) and an almost complex structure \(\tilde{J}\) on \(E\) (compatible with
the symplectic structure on each fibre) such that the projection \(\pi : E \to S^2\)

\begin{flushright}
Date: Revised version, April 30, 1997.
Supported by a TMR grant from the European Community.
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is $(\hat{J}, j)$-linear. For generic $\hat{J}$, the space $\mathcal{S}(j, \hat{J})$ of pseudoholomorphic sections of $\pi$ is a smooth finite-dimensional manifold, with a natural evaluation map $ev_z : \mathcal{S}(j, \hat{J}) \to E_z$ for $z \in S^2$. After fixing a symplectic isomorphism $i : (M, \omega) \to (E_z, \Omega_z)$ for some $z$, $ev_z$ should define a homology class in $M$. This rough description does not take into account the lack of compactness of $\mathcal{S}(j, \hat{J})$. Part of this problem is due to ‘bubbling’: since we deal with it using the method of Ruan and Tian in [16], we have to make the following Assumption $(W^+)$. $(M, \omega)$ satisfies one of the following conditions:

(a) there is a $\lambda \geq 0$ such that $\omega(A) = \lambda c_1(A)$ for all $A \in \pi_2(M)$;
(b) $c_1|\pi_2(M) = 0$;
(c) the minimal Chern number $N \geq 0$ (defined by $c_1(\pi_2(M)) = NZ$) is at least $n - 1$, where $2n = \dim M$.

Here $c_1$ stands for $c_1(TM, \omega)$. An equivalent assumption is

$$A \in \pi_2(M), 2 - n \leq c_1(A) < 0 \implies \omega(A) \leq 0.$$ 

Therefore $(W^+)$ is more restrictive than the notion of weak monotonicity (see e.g. [7]) in which $2 - n$ is replaced by $3 - n$. In particular, the Floer homology $HF_*(M, \omega)$ [7] and the quantum cup-product [16] are well-defined for manifolds which satisfy $(W^+)$. Note that all symplectic four-manifolds belong to this class. Recent work on Gromov-Witten invariants seems to indicate that the restriction $(W^+)$ might be removed at the cost of introducing rational coefficients, but this will not be pursued further here.

The other aspect of the non-compactness of $\mathcal{S}(j, \hat{J})$ is that it may have infinitely many components corresponding to different homotopy classes of sections of $E$. By separating these components, one obtains infinitely many invariants of $(E, \Omega)$. These can be arranged into a single element of the quantum homology group $QH_*(M, \omega)$, which is the homology of $M$ with coefficients in the Novikov ring $\Lambda$. This element is normalized by the choice of a section of $E$; it depends on this section only up to a certain equivalence relation, which we call $\Gamma$-equivalence. If $S$ is a $\Gamma$-equivalence class of sections, we denote the invariant obtained in this way by $Q(E_g, \Omega_g, S_{\tilde{g}})$.

The bundles $(E_g, \Omega_g)$ for a Hamiltonian loop $g$ do not come with a naturally preferred $\Gamma$-equivalence class of sections. Therefore we introduce an additional piece of data: $g$ acts on the free loop space $\Lambda M = C^\infty(S^1, M)$ by

$$(g : x)(t) = g_t(x(t)).$$

Let $\mathcal{L}M$ be the connected component of $\Lambda M$ containing the constant loops. There is an abelian covering $p : \mathcal{L}M \to \mathcal{L}M$ such that $\Gamma$-equivalence classes of sections of $E_g$ correspond naturally to lifts of the action of $g$ to $\mathcal{L}M$. Let $S_{\tilde{g}}$ be the equivalence class corresponding to a lift $\tilde{g} : \mathcal{L}M \to \mathcal{L}M$. We define

$$q(g, \tilde{g}) = Q(E_g, \Omega_g, S_{\tilde{g}}).$$
Let $G$ be the group of smooth loops $g : S^1 \to \text{Ham}(M, \omega)$ such that $g(0) = \text{Id}$, and $\tilde{G}$ the group of pairs $(g, \tilde{g})$. We will use the $C^\infty$-topology on $G$, and a topology on $\tilde{G}$ such that the homomorphism $\tilde{G} \to G$ which ‘forgets’ $\tilde{g}$ is continuous with discrete kernel.

The covering $\tilde{LM}$ was originally introduced by Hofer and Salamon in their definition of $HF^*(M, \omega)$ [7]. They set up Floer homology as a formal analogue of Novikov homology for this covering. Now the $\tilde{G}$-action on $\tilde{LM}$ commutes with the covering transformations. Pursuing the analogy, one would expect an induced action of $\tilde{G}$ on Floer homology. If we assume that $(W^\pm)$ holds, this picture is correct: there are induced maps $HF^*(g, \tilde{g}) : HF^*(M, \omega) \to HF^*(M, \omega)$ for $(g, \tilde{g}) \in \tilde{G}$. These maps are closely related to $q(g, \tilde{g})$. The relationship involves the ‘pair-of-pants’ product $\ast_{PP} : HF_*(M, \omega) \times HF_*(M, \omega) \to HF_*(M, \omega)$ and the canonical isomorphism $\Psi^+ : QH_*(M, \omega) \to HF_*(M, \omega)$ of Pidnichkin, Salamon and Schwarz [15],[18].

**Theorem 1.** For any $(g, \tilde{g}) \in \tilde{G}$ and $b \in HF_*(M, \omega)$,

$$HF_*(g, \tilde{g})(b) = \Psi^+(q(g, \tilde{g})) \ast_{PP} b.$$ 

This formula provides an alternative approach to $q$ and $Q$, and we will use it to derive several properties of these invariants. Let $\ast$ be the product on $QH_*(M, \omega)$ obtained from the quantum cup-product by Poincaré duality. We will call $\ast$ the quantum intersection product. The quantum homology ring $(QH_*(M, \omega), \ast)$ is a ring with unit and commutative in the usual graded sense.

**Corollary 2.** For any Hamiltonian fibre bundle $(E, \Omega)$ over $S^2$ with fibre $(M, \omega)$ and any $\Gamma$-equivalence class $S$ of sections of $E$, $Q(E, \Omega, S)$ is an invertible element of $(QH_*(M, \omega), \ast)$.

By definition, $Q(E, \Omega, S)$ is homogeneous and even-dimensional, that is, it lies in $QH_{2i}(M, \omega)$ for some $i \in \mathbb{Z}$. We will denote the group (with respect to $\ast$) of homogeneous even-dimensional invertible elements of $QH_*(M, \omega)$ by $QH_*(M, \omega)^\times$.

**Corollary 3.** $q(g, \tilde{g})$ depends only on $[g, \tilde{g}] \in \pi_0(\tilde{G})$, and

$$q : \pi_0(\tilde{G}) \to QH_*(M, \omega)^\times$$

is a group homomorphism.
is an exact sequence

\[ \cdots \to \Gamma \to \pi_0(\tilde{G}) \to \pi_0(G) \to 1. \]

On the other hand, for every \( \gamma \in \Gamma \) there is an element \([M] \otimes \langle \gamma \rangle \in QH_*(M, \omega)\) which is easily seen to be invertible. The map \( \tau : \Gamma \to QH_*(M, \omega)^\times \) defined in this way is an injective homomorphism.

**Proposition 4.** For \( \gamma \in \Gamma \), \( q(Id, \gamma) = \tau(\gamma) \).

It follows that there is a unique homomorphism \( \bar{q} \) such that the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\tau} & \pi_0(\tilde{G}) \\
\downarrow & & \downarrow q \\
\Gamma & \xrightarrow{\tau} & \pi_0(G) \\
\end{array}
\]

\[
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\pi_0(\tilde{G}) & \longrightarrow & \pi_0(G) \\
\downarrow q & & \downarrow \bar{q} \\
QH_*(M, \omega)^\times & \longrightarrow & QH_*(M, \omega)^\times / \tau(\Gamma) \\
\end{array}
\]

commutes. \( \bar{q} \) is more interesting for applications to symplectic geometry than \( q \) itself, since \( \pi_0(G) \cong \pi_1(\text{Ham}(M, \omega)) \).

In the body of the paper, we proceed in a different order than that described above. The next section contains the definition of \( \tilde{G} \) and of Hamiltonian fibre bundles. Section 3 is a review (without proofs) of Floer homology. Our definition is a variant of that in [7], the difference being that we allow time-dependent almost complex structures. This makes it necessary to replace weak monotonicity by the condition \((W^+)\). The following three sections contain the definition and basic properties of the maps \( HF_*(g, \tilde{g}) \). An argument similar to the use of ‘homotopies of homotopies’ in the definition of Floer homology shows that \( HF_*(g, \tilde{g}) \) depends only on \([g, \tilde{g}] \in \pi_0(\tilde{G})\). Moreover, we prove that \( HF_*(g, \tilde{g}) \) is an automorphism of \( HF_*(M, \omega) \) as a module over itself with the pair-of-pants product. This second property is considerably simpler: after a slight modification of the definition of \( \ast_{PP} \), the corresponding equation holds on the level on chain complexes. \( Q(E, \Omega, S) \) is defined in section 7, and section 8 describes the gluing argument which establishes the connection between this invariant and the maps \( HF_*(g, \tilde{g}) \). After that, we prove the results stated above. The grading of \( q(g, \tilde{g}) \) is determined by a simple invariant \( I(g, \tilde{g}) \) derived from the first Chern class. This relationship is exploited in section 10 to obtain some vanishing results for this invariant. The final section contains two explicit computations of \( q(g, \tilde{g}) \), one of which is the case of \( S^2 \times S^2 \) mentioned at the beginning.

**Acknowledgments.** Discussions with S. Agnihotri, S. Donaldson and V. Munoz were helpful in preparing this paper. I have profited from the referee’s comments on an earlier version. I am particularly indebted to L. Polterovich for explaining to me the point of view taken in section 4 and for his encouragement. Part of this paper was written during a stay at the Université de Paris-Sud (Orsay) and the Ecole Polytechnique.
2. Hamiltonian loops and fibre bundles

Throughout this paper, \((M, \omega)\) is a closed connected symplectic manifold of dimension \(2n\). We will write \(c_1\) for \(c_1(TM, \omega)\). As in the Introduction, \(\Lambda M = C^\infty(S^1, M)\) denotes the free loop space of \(M\), \(\mathcal{L}M \subset \Lambda M\) the subspace of contractible loops, \(\text{Ham}(M, \omega)\) the group of Hamiltonian automorphisms and \(G\) the group of smooth based loops in \(\text{Ham}(M, \omega)\). We use the \(C^\infty\)-topology on \(\text{Ham}(M, \omega)\) and \(G\).

**Lemma 2.1.** The canonical homomorphism \(\pi_0(G) \rightarrow \pi_1(\text{Ham}(M, \omega))\) is an isomorphism.

This follows from the fact that any continuous loop in \(\text{Ham}(M, \omega)\) can be approximated by a smooth loop. Note that it is unknown whether \(\text{Ham}(M, \omega)\) with the \(C^\infty\)-topology is locally contractible for all \((M, \omega)\) (see the discussion on p. 321 of [11]). However, there is a neighbourhood \(U \subset \text{Ham}(M, \omega)\) of the identity such that each path component of \(U\) is contractible. This weaker property is sufficient to prove Lemma 2.1.

\(G\) acts on \(\Lambda M\) by (1.2).

**Lemma 2.2.** \(g(\mathcal{L}M) = \mathcal{L}M\) for every \(g \in G\).

The proof uses an idea which is also the starting point for the definition of induced maps on Floer homology. Let \(\mathcal{H} = C^\infty(S^1 \times M, \mathbb{R})\) be the space of periodic Hamiltonians\(^1\). The perturbed action one-form \(\alpha_H\) on \(\Lambda M\) associated to \(H \in \mathcal{H}\) is

\[
\alpha_H(x)\xi = \int_{S^1} \omega(\dot{x}(t) - X_H(t, x(t)), \xi(t))dt
\]

where \(X_H\) is the time-dependent Hamiltonian vector field of \(H\). The zero set of \(\alpha_H\) consists of the 1-periodic solutions of

\[
(2.1) \quad \dot{x}(t) = X_H(t, x(t)).
\]

We say that a Hamiltonian \(K_g \in \mathcal{H}\) generates \(g \in G\) if

\[
(2.2) \quad \frac{\partial g_t}{\partial t}(y) = X_{K_g}(t, g_t(y)).
\]

**Lemma 2.3.** Let \(K_g\) be a Hamiltonian which generates \(g \in G\). For every \(H \in \mathcal{H}\), define \(H^g \in \mathcal{H}\) by

\[
H^g(t, y) = H(t, g_t(y)) - K_g(t, g_t(y)).
\]

Then \(g^*\alpha_H = \alpha_{H^g}\).

The proof is straightforward.

\(^1\)We identify \(S^1 = \mathbb{R}/\mathbb{Z}\) throughout.
Proof of Lemma 2.2. Assume that \( g(\mathcal{LM}) \) is a connected component of \( \Lambda M \) distinct from \( \mathcal{LM} \). If \( H \) is small, \( \alpha_H \) has no zeros in \( g(\mathcal{LM}) \) and by Lemma 2.3, \( \alpha_H \) has no zeros in \( \mathcal{LM} \). This contradicts the Arnol’d conjecture (now a theorem) which guarantees the existence of at least one contractible 1-periodic solution of (2.1) for every Hamiltonian. \( \Box \)

Since we will only use this for manifolds satisfying \((W^+)\), we do not really need the Arnol’d conjecture in full generality, only the versions established in [7] and [14].

The space \( \tilde{\mathcal{LM}} \) introduced in [7] is defined as follows: consider pairs \((v, x) \in C^\infty(D^2, M) \times \mathcal{LM}\) such that \( x = v|\partial D^2 \). \( \tilde{\mathcal{LM}} \) is the set of equivalence classes of such pairs with respect to the following relation: \((v_0, x_0) \sim (v_1, x_1)\) if \( x_0 = x_1 \) and \( \omega(v_0 \# \iota) = 0, c_1(v_0 \# \iota) = 0 \). Here \( v_0 \# \iota : S^2 \rightarrow M \) is the map obtained by gluing together \( v_0, v_1 \) along the boundaries. \( p : \tilde{\mathcal{LM}} \rightarrow \mathcal{LM}, p(v, x) = x \), is a covering projection for the obvious choice of topology on \( \tilde{\mathcal{LM}} \). Define \( \Gamma = \pi_2(M)/\pi_2(M)_0 \), where \( \pi_2(M)_0 \) is the subgroup of classes \( a \) such that \( \omega(a) = 0, c_1(a) = 0 \). \( \Gamma \) can also be defined as a quotient of \( \text{im}(\pi_2(M) \rightarrow H_2(M; \mathbb{Z})) \). Therefore the choice of base point for \( \pi_2(M) \) is irrelevant, and any \( A : S^2 \rightarrow M \) determines a class \([A] \in \Gamma\). Clearly \( \omega(A) \) and \( c_1(A) \) depend only on \([A]\). By an abuse of notation, we will write \( \omega(\gamma) \) and \( c_1(\gamma) \) for \( \gamma \in \Gamma \). \( \Gamma \) is the covering group of \( p \). It acts on \( \tilde{\mathcal{LM}} \) by ‘gluing in spheres’: \([A] \cdot [v, x] = [A \# v, x]\) (see [7] for a precise description).

**Lemma 2.4.** The action of any \( g \in G \) on \( \mathcal{LM} \) can be lifted to a homeomorphism of \( \tilde{\mathcal{LM}} \).

**Proof.** Since \( \tilde{\mathcal{LM}} \) is a connected covering, it is sufficient to show that the action of \( g \) on \( \mathcal{LM} \) preserves the set of smooth maps \( S^1 \hookrightarrow \mathcal{LM} \) which can be lifted to \( \tilde{\mathcal{LM}} \). Such a map is given by a \( B \in C^\infty(S^1 \times S^1, M) \) with \( \omega(B) = 0, c_1(B) = 0 \). Its image under \( g \) is given by \( B'(s, t) = g_t(B(s, t)) \). Now \( \omega(B') = \omega(B) \) because \( (B')^*\omega = B^*\omega + dt \), where \( \theta(s, t) = K_g(t, g_t(B(s, t)))dt \) for a Hamiltonian \( K_g \) as in (2.2). Similarly, \( c_1(B') = c_1(B) \) because there is an isomorphism \( D : B^*TM \rightarrow (B')^*TM \) of symplectic vector bundles, given by \( D(s, t) = Dg_t(B(s, t)) \). \( \Box \)

**Definition 2.5.** \( \tilde{G} \subset G \times \text{Homeo}(\tilde{\mathcal{LM}}) \) is the subgroup of pairs \((g, \tilde{g})\) such that \( \tilde{g} \) is a lift of the \( g \)-action on \( \mathcal{LM} \).

We give \( \tilde{G} \) the topology induced from the \( C^\infty \)-topology on \( G \) and the topology of pointwise convergence on \( \text{Homeo}(\mathcal{LM}) \). This makes \( \tilde{G} \) into a topological group, essentially because a lift \( \tilde{g} \) of a given \( g \) is determined by the image of a single point. The projection \( \tilde{G} \rightarrow G \) is onto by Lemma 2.4, and since its kernel consists of the pairs \((\text{Id}, \gamma)\) with \( \gamma \in \Gamma \), there is an exact sequence

\[
1 \rightarrow \Gamma \rightarrow \tilde{G} \rightarrow G \rightarrow 1
\]
of topological groups, with $\Gamma$ discrete.

A point $c = [v, x] \in \mathcal{LM}$ determines a preferred homotopy class of trivializations of the symplectic vector bundle $x^*(TM, \omega)$. This homotopy class consists of the maps $\tau_c : x^*TM \longrightarrow S^1 \times (\mathbb{R}^{2n}, \omega_0)$ which can be extended over $v^*TM$, and it is independent of the choice of the representative $(v, x)$ of $c$. For $(g, \tilde{g}) \in \tilde{G}$,

$$l(t) = \tau_{\tilde{g}(c)}(t)Dg(x(t))\tau_c(t)^{-1} \quad (t \in S^1)$$

is a loop in $\text{Sp}(2n, \mathbb{R})$. Up to homotopy, it does not depend on $c$ and on the trivializations. We define the ‘Maslov index’ $I(g, \tilde{g}) \in \mathbb{Z}$ by $I(g, \tilde{g}) = \deg(l)$, where $\deg : H_1(\text{Sp}(2n, \mathbb{R})) \longrightarrow \mathbb{Z}$ is the standard isomorphism induced by the determinant on $U(n) \subset \text{Sp}(2n, \mathbb{R})$.

**Lemma 2.6.** $I(g, \tilde{g})$ depends only on $[g, \tilde{g}] \in \pi_0(\tilde{G})$. The map $I : \pi_0(\tilde{G}) \longrightarrow \mathbb{Z}$ is a homomorphism, and $I(\text{Id} \circ \gamma) = c_1(\gamma)$ for all $\gamma \in \Gamma$.

We omit the proof. Because of (1.3), it follows that $I(g, \tilde{g}) \mod N$ depends only on $g$. In this way, we recover a familiar (see e.g. [24, p. 80]) invariant

$$\tilde{I} : \pi_0(\tilde{G}) = \pi_1(\text{Ham}(M, \omega)) \longrightarrow \mathbb{Z}/N\mathbb{Z}.$$

Now we turn to symplectic fibre bundles. A smooth fibre bundle $\pi : E \longrightarrow B$ together with a smooth family $\Omega = (\Omega_b)_{b \in B}$ of symplectic forms on its fibres is called a symplectic fibration; a symplectic fibre bundle is a symplectic fibration which is locally trivial. Note that $\Omega$ defines a symplectic structure on the vector bundle $TE^v = \ker(D\pi) \subset TE$.

Here we consider only the case $B = S^2$. It is convenient to think of $S^2$ as $D^+ \cup_{S^1} D^-$, where $D^+, D^-$ are closed discs. Fix a point $z_0 \in D^-$. A symplectic fibre bundle $(E, \Omega)$ over $S^2$ with a fixed isomorphism $i : (M, \omega) \longrightarrow (E_{z_0}, \Omega_{z_0})$ will be called a symplectic fibre bundle with fibre $(M, \omega)$; we will frequently omit $i$ from the notation.

Let $g$ be a smooth loop in $\text{Aut}(M, \omega)$. The ‘clutching’ construction produces a symplectic fibre bundle $(E_g, \Omega_g)$ over $S^2$ by gluing together the trivial fibre bundles $D^\pm \times (M, \omega)$ using

$$\phi_g : \partial D^+ \times M \longrightarrow \partial D^- \times M,$$

$$\phi_g(t, y) = (t, g_t(y)).$$

In an obvious way, $(E_g, \Omega_g)$ is a symplectic fibre bundle with fibre $(M, \omega)$.

**Convention 2.7.** We have identified $\partial D^+, \partial D^-$ with $S^1$ and used these identifications to glue $D^+$ and $D^-$ together. We orient $D^+, D^-$ in such a way that the map $S^1 \longrightarrow \partial D^+$ preserves orientation while the one $S^1 \longrightarrow \partial D^-$ reverses it; this induces an orientation of $S^2$.

We will use this construction only for loops with $g(0) = \text{Id}$. By a standard argument, it provides a bijection between elements of $\pi_1(\text{Aut}(M, \omega))$ and isomorphism classes of symplectic fibre bundles over $S^2$ with fibre $(M, \omega)$. 

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Definition 2.8. A symplectic fibre bundle \((E, \Omega)\) over a surface \(B\) is called Hamiltonian if there is a closed two-form \(\tilde{\Omega}\) on \(E\) such that \(\tilde{\Omega}|_{E_b} = \Omega_b\) for all \(b \in B\).

For \(B = S^2\), these are precisely the bundles corresponding to Hamiltonian loops:

Proposition 2.9. \((E_g, \Omega_g)\) is Hamiltonian iff \([g]\) lies in the subgroup \(\pi_1(\text{Ham}(M, \omega)) \subset \pi_1(\text{Aut}(M, \omega))\).

Proof. Let \(g \in G\) be a loop generated by \(K_g \in \mathcal{H}\). We denote the pullback of \(\omega\) to \(D^+ \times M\) by \(\omega^\pm\). Choose a 1-form \(\theta\) on \(D^+ \times M\) such that \(\theta(t, y) = K_g(t, g_t(y)) dt\) for \(t \in S^1\) and \(\theta\{z\} \times M = 0\) for all \(z \in D^+\). There is a continuous 2-form \(\tilde{\Omega}\) on \(E_g\) such that \(\tilde{\Omega}|_{D^+ \times M} = \omega^+ + d\theta\) and \(\tilde{\Omega}|_{D^- \times M} = \omega^-\), and for a suitable choice of \(\theta\) it is smooth. \(\tilde{\Omega}\) is closed and extends the family \(\Omega_g\), which proves that \((E_g, \Omega_g)\) is Hamiltonian.

Conversely, let \(g\) be a based loop in \(\text{Aut}(M, \omega)\) such that \((E_g, \Omega_g)\) is Hamiltonian with \(\tilde{\Omega} \in \Omega^2(E_g)\). We will use the exact sequence

\[ 1 \longrightarrow \pi_1(\text{Ham}(M, \omega)) \longrightarrow \pi_1(\text{Aut}(M, \omega)) \xrightarrow{F} H^1(M, \mathbb{R}) \]

where \(F\) is the flux homomorphism (see [2] or [12, Corollary 10.18]). Choose a point \(z_+ \in D^+\). For \(\lambda \in C^\infty(S^1, M)\), consider the maps

\[ T_\lambda : S^1 \times S^1 \longrightarrow M, \quad T_\lambda(r, t) = g_t(\lambda(r)), \]

\[ T^-_\lambda : S^1 \times S^1 \longrightarrow D^- \times M, \quad T^-_\lambda(r, t) = (z_0, g_t(\lambda(r))) \text{ and} \]

\[ T^+_\lambda : S^1 \times S^1 \longrightarrow D^+ \times M, \quad T^+_\lambda(r, t) = (z_+, \lambda(r)). \]

By definition \(\langle F(g), [\lambda] \rangle = \omega(T_\lambda) = \tilde{\Omega}(T^-_\lambda)\). Since \(T^-_\lambda\) and \(T^+_\lambda\) are homotopic in \(E_g\), \(\tilde{\Omega}(T^-_\lambda) = \tilde{\Omega}(T^+_\lambda)\); but clearly \(\tilde{\Omega}(T^+_\lambda) = 0\). Therefore \(F(g) = 0\), and by (2.3), \([g] \in \pi_1(\text{Ham}(M, \omega))\).

Let \((E, \Omega)\) be a Hamiltonian fibre bundle over \(S^2\), with \(\tilde{\Omega} \in \Omega^2(E)\) as above. We say that two continuous sections \(s_0, s_1\) of \(E\) are \(\Gamma\)-equivalent if \(\tilde{\Omega}(s_0) = \tilde{\Omega}(s_1)\) and \(c_1(TE^v, \Omega)(s_0) = c_1(TE^v, \Omega)(s_1)\). Using the exact sequence

\[ \cdots \longrightarrow \pi_2(E_{s_0}) \longrightarrow \pi_2(E) \longrightarrow \pi_2(S^2) \longrightarrow \cdots, \]

it is easy to see that this equivalence relation is independent of the choice of \(\tilde{\Omega}\). Let \(S\) be the \(\Gamma\)-equivalence class of a section \(s\). By an abuse of notation, we write \(\tilde{\Omega}(S) = \tilde{\Omega}(s)\) and \(c_1(TE^v, \Omega)(S) = c_1(TE^v, \Omega)(s)\).

Lemma 2.10. Let \((E, \Omega)\) be a Hamiltonian fibre bundle over \(S^2\) with fibre \((M, \omega)\) and \(\tilde{\Omega} \in \Omega^2(E)\) a closed extension of \(\Omega\).

(i) \(E\) admits a continuous section.
(ii) For two $\Gamma$-equivalence classes $S_0, S_1$ of sections of $E$, there is a unique $\gamma \in \Gamma$ such that

$$
\tilde{\Omega}(S_1) = \tilde{\Omega}(S_0) + \omega(\gamma),
$$

$$
c_1(TE^v, \Omega)(S_1) = c_1(TE^v, \Omega)(S_0) + c_1(\gamma).
$$

$\gamma$ is independent of the choice of $\tilde{\Omega}$.

(iii) Conversely, given a $\Gamma$-equivalence class $S_0$ and $\gamma \in \Gamma$, there is a unique $\Gamma$-equivalence class $S_1$ such that (2.5) holds.

Proof. (i) $(E, \Omega)$ is isomorphic to $(E_g, \Omega_g)$ for some $g : S^1 \to \text{Aut}(M, \omega)$. Since it is Hamiltonian, we can assume that $g \in G$. Choose a point $y \in M$; by Lemma 2.2, there is a $v \in C^\infty(D^-, M)$ such that $v(t) = g_t(y)$ for $t \in S^1 = \partial D^-$. To obtain a section of $E_g$, glue together $s^+ : D^+ \to D^+ \times M$, $s^-(z) = (z, y)$ and $s^- : D^- \to D^- \times M$, $s^-(z) = (z, v(z))$.

(ii) is an easy consequence of (2.4).

(iii) Let $i : (M, \omega) \to (E_{z_0}, \Omega_{z_0})$ be the preferred symplectic isomorphism. Take $s_0 \in S_0$ and $A : S^2 \to M$ such that $[A] = \gamma$ and $i(A(z)) = s_0(z_0)$ for some $z \in S^2$. A section $s_1$ with the desired properties can be obtained by gluing together $s_0$ and $i(A)$. The uniqueness of $S_1$ is clear from the definition.

In the situation of Lemma 2.10, we will write $\gamma = S_1 - S_0$ and $S_1 = \gamma + S_0$.

**Definition 2.11.** Let $(E, \Omega)$ be a Hamiltonian fibre bundle over $S^2$ and $S$ a $\Gamma$-equivalence class of sections of $E$. We will call $(E, \Omega, S)$ a normalized Hamiltonian fibre bundle.

This notion is closely related to the group $\tilde{G}$: for $(g, \tilde{g}) \in \tilde{G}$, choose a point $c \in \tilde{L}M$ and representatives $(v, x)$ of $c$ and $(v', x')$ of $\tilde{g}(c)$. The maps $s^+_g : D^+ \to D^+ \times M$, $s^+_g(z) = (z, v(z))$ and $s^-_g : D^- \to D^- \times M$, $s^-_g(z) = (z, v'(z))$ define a section $s_g$ of $E_g$. To be precise, $s^+_g$ is obtained from $v$ by identifying $D^+$ with the standard disc $D^2$ such that the orientation (see 2.7) is preserved; in the case of $D^-$, one uses an orientation-reversing diffeomorphism. By comparing this with the definition of $I(g, \tilde{g})$, one obtains (2.6)

$$
c_1(TE^v_g, \Omega_g)(s_g) = -I(g, \tilde{g}).
$$

**Lemma 2.12.** The $\Gamma$-equivalence class of $s_g$ is independent of the choice of $c$ and of $v, v'$.

We omit the proof. It follows that $(g, \tilde{g})$ determines a normalized Hamiltonian fibre bundle $(E_g, \Omega_g, S_g)$, where $S_g$ is the $\Gamma$-equivalence class of $s_g$. This defines a one-to-one correspondence between $\pi_0(\tilde{G})$ and isomorphism classes of normalized Hamiltonian fibre bundles with fibre $(M, \omega)$. We will only use the easier half of this correspondence:

**Lemma 2.13.** Every normalized Hamiltonian fibre bundle with fibre $(M, \omega)$ is isomorphic to $(E_g, \Omega_g, S_g)$ for some $(g, \tilde{g}) \in G$. 


3. Floer homology

From now on, we always assume that \((M, \omega)\) satisfies \((W^+)\). In \([7]\), weak monotonicity is used to show that a generic \(\omega\)-compatible almost complex structure admits no pseudo-holomorphic spheres with negative Chern number. Under the more restrictive condition \((W^+)\), this non-existence result can be extended to families of almost complex structures depending on \(\leq 3\) parameters. To make this precise, we need to introduce some notation. Let \(J = (J_b)_{b \in B}\) be a smooth family of almost complex structures on \(M\) parametrized by a manifold \(B\). \(J\) is called \(\omega\)-compatible if every \(J_b\) is \(\omega\)-compatible in the usual sense. We denote the space of \(\omega\)-compatible families by \(\mathcal{J}(B; M, \omega)\).

The parametrized moduli space \(\mathcal{M}^s(J) \subset B \times C^\infty(\mathbb{C}P^1, M)\) associated to \(J \in \mathcal{J}(B; M, \omega)\) is the space of pairs \((b, w)\) such that \(w\) is \(J_b\)-holomorphic and simple (not multiply covered). In the case of a single almost complex structure \(J\), a \(J\)-holomorphic sphere \(w\) is called regular if the linearization of the equation \(\partial J(u) = 0\) at \(w\), which is given by an operator
\[
D_J(w) : C^\infty(w^*TM) \to \Omega^0,1(w^*(TM, J)),
\]
is onto (see \([11, \text{Chapter 3}]\)). Similarly, \((b, w) \in \mathcal{M}^s(J)\) is called regular if the extended operator
\[
\hat{D}_J(b, w) : T_bB \times C^\infty(w^*TM) \to \Omega^0,1(w^*(TM, J_b)),
\]
is onto. Here \(i\) is the complex structure on \(\mathbb{C}P^1\) and \(D\) \(\hat{J}(b) : T_bB \to C^\infty(\text{End}(TM))\) is the derivative of the family \((J_b)\) at \(b\). \(\hat{J}\) itself is called regular if all \((b, w) \in \mathcal{M}^s(J)\) are regular, and the set of regular families is denoted by \(\mathcal{J}^{\text{reg}}(B; M, \omega) \subset \mathcal{J}(B; M, \omega)\).

\(\hat{D}_J(b, w)\) is a Fredholm operator of index \(2n + 2c_1(w) + \text{dim } B\). For \(k \in \mathbb{Z}\), let \(\mathcal{M}_k^s(J) \subset \mathcal{M}^s(J)\) be the subspace of pairs \((b, w)\) with \(c_1(w) = k\). By applying the implicit function theorem, one obtains

**Lemma 3.1.** If \(J \in \mathcal{J}^{\text{reg}}(B; M, \omega)\), \(\mathcal{M}_k^s(J)\) is a smooth manifold of dimension \(2n + 2k + \text{dim } B\) for all \(k\).

Assume that \(B\) is compact and choose \(J_0 \in \mathcal{J}(B; M, \omega)\). Let \(U_\delta(J_0) \subset \mathcal{J}(B; M, \omega)\) be a \(\delta\)-ball around \(J_0\) with respect to Floer’s \(C^\infty\)-norm (see \([7]\) or \([21, \text{p. 101–103}]\)).

**Theorem 3.2.** For sufficiently small \(\delta > 0\), \(\mathcal{J}^{\text{reg}}(B; M, \omega) \cap U_\delta(J_0) \subset U_\delta(J_0)\) has second category; in particular, it is \(C^\infty\)-dense.

This is a well-known result (see e.g. \([11, \text{Theorem 3.1.3}]\)). It is useful to compare its proof with that of the basic transversality theorem for pseudo-holomorphic curves, which is the special case \(B = \{pt\}:\) in that case, one
shows first that the ‘universal moduli space’

\[ \mathcal{M}^{\text{univ}} \xrightarrow{\pi} \mathcal{J}(pt; M, \omega) \]

is smooth. \( \mathcal{J}^{\text{reg}}(pt; M, \omega) \) is the set of regular values of \( \pi \), which is shown to be dense by applying the Sard-Smale theorem (we omit the details which arise from the use of \( C^\infty \)-spaces). In the general case, a family \( (J_b)_{b \in B} \) is regular iff the corresponding map \( B \rightarrow \mathcal{J}(pt; M, \omega) \) is transverse to \( \pi \). Therefore the part of the proof which uses specific properties of pseudoholomorphic curves (the smoothness of \( \mathcal{M}^{\text{univ}} \)) remains the same, but a different general result has to be used to show that generic maps \( B \rightarrow \mathcal{J}(pt; M, \omega) \) are transverse to \( \pi \).

A family \( J \) is called semi-positive if \( \mathcal{M}_k^s(J) = \emptyset \) for all \( k < 0 \); the semi-positive families form a subset \( \mathcal{J}^+(B; M, \omega) \subset \mathcal{J}(B; M, \omega) \).

**Lemma 3.3.** If \( \dim B \leq 3 \), \( \mathcal{J}^{\text{reg}}(B; M, \omega) \subset \mathcal{J}^+(B; M, \omega) \).

**Proof.** \( PSL(2, \mathbb{C}) \) acts freely on \( \mathcal{M}^s_k(J) \) for all \( k \). If \( J \in \mathcal{J}^{\text{reg}}(B; M, \omega) \), the quotient is a smooth manifold and

\[ \dim \mathcal{M}_k^s(J)/PSL(2, \mathbb{C}) \leq 2n + 2k - 3. \]

Therefore \( \mathcal{M}_k^s(J) = \emptyset \) for \( k \leq 1 - n \). But for \( 2 - n \leq k < 0 \), \( \mathcal{M}_k^s(J) = \emptyset \) by (1.1). \( \square \)

**Corollary 3.4.** If \( B \) is compact and \( \dim B \leq 3 \), \( \mathcal{J}^+(B; M, \omega) \) is \( C^\infty \)-dense in \( \mathcal{J}(B; M, \omega) \).

Later, we will also use a ‘relative’ version of Corollary 3.4, in which \( B \) is non-compact and one considers families \( J \) with a fixed behaviour outside a relatively compact open subset of \( B \). We omit the precise statement.

In contrast with the case of negative Chern number, holomorphic spheres with Chern number 0 or 1 can occur in a family \( J = (J_b)_{b \in B} \) even if \( \dim B \) is small. When defining Floer homology, special attention must be paid to them. For any \( k \geq 0 \), let \( V_k(J) \subset B \times M \) be the set of pairs \((b, y)\) such that \( y \in \text{im}(w)\) for a non-constant \( J_b\)-holomorphic sphere \( w \) with \( c_1(w) \leq k \). \( V_k(J) \) is the union of the images of the evaluation maps

\[ \mathcal{M}_j^s(J) \times_{PSL(2, \mathbb{C})} \mathbb{C}P^1 \rightarrow B \times M \]

for \( j \leq k \). For regular \( J \) and \( j \leq 0 \), the dimension of \( \mathcal{M}_j^s(J) \times_{PSL(2, \mathbb{C})} \mathbb{C}P^1 \) is \( \leq 2n + \dim B - 4 \). Therefore \( V_0(J) \) is (loosely speaking) a codimension-4 subset of \( B \times M \). Similarly, \( V_1(J) \) has codimension 2.

For any \( H \in \mathcal{H} \), the pullback of \( \alpha_H \) to \( \mathcal{L}M \) is exact: \( p^*\alpha_H = da_H \) with

\[ a_H(v, x) = -\int_{D^2} v^*\omega + \int_{S^1} H(t, x(t))dt. \]

Let \( Z(\alpha_H) \subset \mathcal{L}M \) be the zero set of \( \alpha_H \) and \( \text{Crit}(a_H) = p^{-1}(Z(\alpha_H)) \) the set of critical points of \( a_H \). \( c = [v, x] \in \text{Crit}(a_H) \) is nondegenerate iff \( x \) is a
regular zero of \( \alpha_H \), that is, if the linearization of (2.1) at \( x \) has no nontrivial solutions. We denote the Conley-Zehnder index of such a critical point by \( \mu_H(c) \in \mathbb{Z} \).

**Convention 3.5.** Let \( H \) be a time-independent Hamiltonian, \( y \in M \) a nondegenerate critical point of \( H \) and \( c = [v, x] \in \mathcal{LM} \) the critical point of \( a_H \) represented by the constant maps \( v, x \equiv y \). Our convention for \( \mu_H \) is that for small \( H \), \( \mu_H(c) \) equals the Morse index of \( y \). This differs from the convention in [17] by a constant \( n \).

Assume that all critical points of \( a_H \) are nondegenerate, and let \( \text{Crit}_k(a_H) \) be the set of critical points of index \( k \). The Floer chain group \( CF_k(H) \) is the group of formal sums

\[
\sum_{c \in \text{Crit}_k(a_H)} m_c <c>
\]

with coefficients \( m_c \in \mathbb{Z}/2 \) such that \( \{ c \in \text{Crit}_k(a_H) \mid m_c \neq 0, a_H(c) \geq C \} \) is finite for all \( C \in \mathbb{R} \).

A family \( J = (J_t)_{t \in S^1} \in \mathcal{J}(S^1; M, \omega) \) defines a Riemannian metric

\[
(\xi, \eta)_J = \int_{S^1} \omega(\xi(t), J_t\eta(t)) dt
\]
on \( \mathcal{LM} \). Let \( \nabla_J a_H \) be the gradient of \( a_H \) with respect to the pullback of \( (\cdot, \cdot)_J \) to \( \mathcal{LM} \). A smooth path \( \tilde{u} : \mathbb{R} \to \mathcal{LM} \) is a flow line of \( -\nabla_J a_H \) iff its projection to \( \mathcal{LM} \) is given by a map \( u \in C^\infty(\mathbb{R} \times S^1, M) \) such that

\[
(3.1) \quad \frac{\partial u}{\partial s} + J_t(u(s, t)) \left( \frac{\partial u}{\partial t} - X_H(t, u(s, t)) \right) = 0 \quad \text{for all } (s, t) \in \mathbb{R} \times S^1.
\]

If \( u \) is a solution of (3.1) whose energy \( E(u) = \int |\frac{\partial u}{\partial t}|^2 \) is finite, there are \( c_-, c_+ \in \text{Crit}(a_H) \) such that

\[
\lim_{s \to \pm \infty} \tilde{u}(s) = c_\pm.
\]

Moreover, if the limits are nondegenerate, the linearization of (3.1) at \( u \) is a differential operator \( D_{H, J}(u) : C^\infty(u^*TM) \to C^\infty(u^*TM) \) whose Sobolev completion

\[
D_{H, J}(u) : W^{1,p}(u^*TM) \to L^p(u^*TM)
\]

\( p \geq 2 \) is a Fredholm operator with index \( \text{ind}(u) = \mu_H(c_-) - \mu_H(c_+) \).

For \( c_-, c_+ \in \text{Crit}(a_H) \), let \( \mathcal{M}(c_-, c_+; H, J) \) be the space of solutions of (3.1) which have a lift \( \tilde{u} : \mathbb{R} \to \mathcal{LM} \) with limits \( c_-, c_+ \). The Floer differential

\[
\partial_k(H, J) : CF_k(H) \to CF_{k-1}(H)
\]
is defined by the formula

\[
(3.2) \quad \partial_k(H, J)(<c_->) = \sum_{c_+ \in \text{Crit}_{k-1}(a_H)} \#(\mathcal{M}(c_-, c_+; H, J)/\mathbb{R})<c_+>,
\]
extended in the obvious way to infinite linear combinations of the $<c_\pm>$ (from now on, this is to be understood for all similar formulae). $\mathbb{R}$ acts on $\mathcal{M}(c_-, c_+; H, J)$ by $(s_0, u)(s, t) = u(s - s_0, t)$, and $\#$ denotes counting the number of points in a set mod 2. In order for (3.2) to make sense and define the ‘right’ homomorphism, $(H, J)$ has to satisfy certain conditions.

**Definition 3.6.** $(H, J) \in \mathcal{H} \times \mathcal{J}^{\text{reg}}(S^1; M, \omega)$ is called a regular pair if

1. every $[v, x] \in \text{Crit}(a_H)$ is nondegenerate and satisfies $(t, x(t)) \notin V_1(J)$ for all $t \in S^1$;
2. every solution $u : \mathbb{R} \to \mathcal{LM}$ of (3.1) with finite energy is regular (that is, the linearization of (3.1) at $u$ is onto);
3. if in addition $\text{ind}(u) \leq 2$, $(t, u(s, t)) \notin V_0(J)$ for all $(s, t) \in \mathbb{R} \times S^1$.

These are essentially the conditions defining $\mathcal{H}^{\text{reg}}(J)$ in [7], adapted to the case of time-dependent almost complex structures. (ii) implies that $\mathcal{M}(c_-, c_+; H, J)$ is a manifold of dimension $\mu_H(c_-) - \mu_H(c_+)$ for all $c_-, c_+$. The proof of the main compactness result [7, Theorem 3.3] carries over to our situation using Corollary 3.4. It follows that if $(H, J)$ is a regular pair, the r.h.s. of (3.2) is meaningful and defines differentials $\partial_k(H, J)$ such that $\partial_{k-1}(H, J)\partial_k(H, J) = 0$.

**Definition 3.7.** The Floer homology $HF_*(H, J)$ of a regular pair $(H, J)$ is the homology of $(CF_*(H), \partial_*(H, J))$.

The existence of regular pairs is ensured by the following Theorem, which is a variant of [7, Theorems 3.1 and 3.2].

**Theorem 3.8.** The set of regular pairs is $C^\infty$-dense in $\mathcal{H} \times \mathcal{J}(S^1; M, \omega)$.

So far, we have only used families of almost complex structures parametrized by $S^1$. For such families Corollary 3.4 holds whenever $(M, \omega)$ is weakly monotone; therefore, the stronger assumption $(W^+)$ has not been necessary up to now. Two-parameter families of almost complex structures occur first in the next step, the definition of ‘continuation maps’.

A homotopy between regular pairs $(H^-, J^-)$, $(H^+, J^+)$ consists of an $H \in C^\infty(\mathbb{R} \times S^1 \times M, \mathbb{R})$ and a $J \in \mathcal{J}^{\text{reg}}(\mathbb{R} \times S^1; M, \omega)$, such that $(H(s, t, \cdot), J_{s,t})$ is equal to $(H^-(t, \cdot), J^-)$ for $s \leq -1$ and to $(H^+(t, \cdot), J^+)$ for $s \geq 1$. For $c_- \in \text{Crit}(a_{H^-})$ and $c_+ \in \text{Crit}(a_{H^+})$, let $\mathcal{M}^\Phi(c_-, c_+; H, J)$ be the space of solutions $u \in C^\infty(\mathbb{R} \times S^1, M)$ of

$$\frac{\partial u}{\partial s} + J_{s,t}(u)\left(\frac{\partial u}{\partial t} - X_H(s, t, u)\right) = 0$$

which can be lifted to paths $\tilde{u} : \mathbb{R} \to \tilde{\mathcal{LM}}$ with limits $c_-, c_+$. A homotopy $(H, J)$ is regular if every solution of (3.3) is regular (the linearization is onto) and if for those solutions with index $\leq 1$,

$(s, t, u(s, t)) \notin V_0(J)$ for all $(s, t) \in \mathbb{R} \times S^1$. 

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For a regular homotopy, the ‘continuation homomorphisms’
\[ \Phi_k(H, J) : CF_k(H^-, J^-) \to CF_k(H^+, J^+) \]
are defined by
\[ \Phi_k(H, J)(<c_->) = \sum_{c_+ \in \text{Crit}_k(a_{H^+})} \#M^\Phi(c_-, c_+; H, J)<c_+>. \]

This is a homomorphism of chain complexes and induces an isomorphism of Floer homology groups, which we denote equally by \( \Phi^* \). For fixed \((H^\pm, J^\pm)\), these isomorphisms are independent of the choice of \((H, J)\). Moreover, they are functorial with respect to composition of homotopies. Therefore we can speak of a well-defined Floer homology \( HF^*_\omega(M) \) independent of the choice of a regular pair (see [17] for a detailed discussion, which carries over to our case with minor modifications). The proof that the ‘continuation maps’ are independent of \((H, J)\) involves three-parameter families of almost complex structures, and hence uses the full strength of Corollary 3.4.

**Remark 3.9.** The definition used here leads to a Floer homology which is canonically isomorphic to that defined in [7]. The only difference is that we have allowed a wider range of perturbations. The reason for this will become clear in the next section.

Since \( a_H = p^*a_H \), Crit(\(a_H\)) is \( \Gamma \)-invariant. This induces an action of \( \Gamma \) on \( CF_\alpha(H) \) for every regular pair \((H, J)\). The action does not preserve the grading: \( \gamma \) maps \( CF_k(H) \) to \( CF_{k-2c_1(\gamma)}(H) \), because
\[ \mu_H(A\#v, x) = \mu_H(v, x) - 2c_1(A) \]
for \([v, x] \in \text{Crit}(a_H)\) and \( A : S^2 \to M \) (see [7] or [5, Proposition 5]; the difference in sign to [7] is due to the fact that they consider Floer cohomology rather than homology). To recover a proper grading, one should assign to \( \gamma \in \Gamma \) the ‘dimension’ \(-2c_1(\gamma)\). We denote the subset of elements of ‘dimension’ \( k \) by \( \Gamma_k \subset \Gamma \).

**Definition 3.10.** For \( k \in \mathbb{Z} \), let \( \Lambda_k \) be the group of formal sums
\[ \sum_{\gamma \in \Gamma_k} m_\gamma <\gamma> \]
with \( m_\gamma \in \mathbb{Z}/2 \), such that \( \{ \gamma \in \Gamma_k \mid m_\gamma \neq 0, \omega(\gamma) \leq C \} \) is finite for all \( C \in \mathbb{R} \). The multiplication
\[ <\gamma> \cdot <\gamma'> = <\gamma + \gamma'> \]
can be extended to the infinite linear combinations (3.5), and this makes \( \Lambda = \bigoplus_k \Lambda_k \) into a commutative graded ring called the Novikov ring of \((M, \omega)\).

For every \( \gamma \in \Gamma \) and \( c \in \widehat{\Lambda M} \), \( a_H(\gamma \cdot c) = a_H(c) - \omega(\gamma) \). Therefore the \( \Gamma \)-action on \( CF_\alpha(H) \) extends naturally to a graded \( \Lambda \)-module structure. Since
the differentials $\partial_*(H,J)$ and the homomorphisms $\Phi_*(H,J)$ are $\Lambda$-linear, this induces a $\Lambda$-module structure on $HF_*(H,J)$ and on $HF_*(M,\omega)$.

The quantum homology $QH_*(M,\omega)$ is the graded $\Lambda$-module defined by

$$QH_k(M,\omega) = \bigoplus_{i+j=k} H_i(M;\mathbb{Z}/2) \otimes \Lambda_j.$$ 

**Theorem 3.11.** $HF_*(M,\omega)$ is isomorphic to $QH_*(M,\omega)$ as a graded $\Lambda$-module.

This was proved by Piunikhin, Salamon and Schwarz [15] (less general versions had been obtained before by Floer [4] and Hofer-Salamon [7]); a detailed account of the proof is in preparation [18]. Their result is in fact much stronger: it provides a canonical isomorphism $\Psi^+ : QH_*(M,\omega) \rightarrow HF_*(M,\omega)$ which also relates different product structures. We will use this construction in section 8.

4. THE $\tilde{G}$-ACTION ON FLOER HOMOLOGY

We begin by considering the action of $G$ on $\mathcal{J}(S^1; M,\omega)$. For $J = (J_t)_{t \in S^1} \in \mathcal{J}(S^1; M,\omega)$ and $g \in G$, define $J^g = (J^g_t)_{t \in S^1}$ by $J^g_t = Dg_{-1}^t J_t Dg_t$.

**Lemma 4.1.** If $J \in \mathcal{J}_{\text{reg}}(S^1; M,\omega)$, $J^g \in \mathcal{J}_{\text{reg}}(S^1; M,\omega)$ for all $g \in G$.

**Proof.** $w \in C^\infty(\mathbb{C}P^1, M)$ is $J^g_t$-holomorphic iff $w' = g_t(w)$ is $J_t$-holomorphic. Since

$$\hat{D}_J(t,w')(1, \frac{\partial g_t}{\partial t}(w)) = \frac{\partial}{\partial t} \left( \frac{1}{2} Dg_t \circ dw + \frac{1}{2} J_t \circ Dg_t \circ dw \circ i \right) = \frac{1}{2} Dg_t \circ \frac{\partial J^g_t}{\partial t} \circ dw \circ i,$$

there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{R} \times C^\infty(w^*TM) & \xrightarrow{\hat{D}_{J^g}(t,w)} & \Omega^{0,1}(w^*(TM, J^g_t)) \\
\downarrow \delta & & \downarrow \delta' \\
\mathbb{R} \times C^\infty((w')^*TM) & \xrightarrow{\hat{D}_J(t,w')} & \Omega^{0,1}((w')^*(TM, J_t)).
\end{array}$$

Here $\delta(\tau, W) = (\tau, Dg_t(w)W + \tau \frac{\partial g_t}{\partial t}(w))$, and $\delta'$ maps a homomorphism $\sigma : T\mathbb{C}P^1 \rightarrow w^*TM$ to $\delta'(\sigma) = Dg_t(w) \circ \sigma$. If $w$ is simple, $\hat{D}_J(t,w')$ is onto by assumption, and since $\delta, \delta'$ are isomorphisms, $\hat{D}_{J^g}(t,w)$ is onto as well.

The same argument shows that $\hat{D}_{J^g}(t,w)$ and $\hat{D}_J(t,w')$ have the same index; therefore

$$V_k(J^g) = \{(t,y) \in S^1 \times M \mid (t,g_t(y)) \in V_k(J)\}$$ (4.1)
for all $k \geq 0$. The metric on $\mathcal{L}M$ associated to $J^g$ is
\begin{equation}
(\cdot, \cdot)_{J^g} = g^*(\cdot, \cdot)_J.
\end{equation}

**Remark 4.2.** The $G$-action does not preserve the subspace of families $(J_t)$ such that $J_t = J_0$ for all $t$. This is the reason why we have allowed $t$-dependent almost complex structures in the definition of Floer homology.

For $(H, J) \in \mathcal{H} \times \mathcal{J}(S^1; M, \omega)$ and $g \in G$, we call the pair $(H^g, J^g)$, where $H^g$ is as in Lemma 2.3, the *pullback* of $(H, J)$ by $g$. Recall that $H^g$ depends on the choice of a Hamiltonian which generates $g$. By Lemma 2.3, $\alpha_{H^g} = g^* \alpha_H$; therefore
\begin{equation}
a_{H^g} = \tilde{g}^* a_H + \text{(constant)}
\end{equation}
and $\text{Crit}(a_H) = \tilde{g}(\text{Crit}(a_{H^g}))$ for every lift $\tilde{g} : \tilde{\mathcal{L}}M \to \tilde{\mathcal{L}}M$ of the action of $g$.

**Lemma 4.3.** For all $c_-, c_+ \in \text{Crit}(a_{H^g})$, there is a bijective map
\[
\mathcal{M}(c_-, c_+; H^g, J^g)/\mathbb{R} \to \mathcal{M}(\tilde{g}(c_-), \tilde{g}(c_+); H, J)/\mathbb{R}.
\]
Moreover, if $(H, J)$ is a regular pair, $(H^g, J^g)$ is also regular.

**Proof.** Let $\tilde{u} : \mathbb{R} \to \tilde{\mathcal{L}}M$ be a smooth path whose projection to $\mathcal{L}M$ is given by $u \in C^\infty(\mathbb{R} \times S^1, M)$. Recall that $u$ satisfies (3.1) iff
\begin{equation}
\frac{d\tilde{u}}{ds} + \nabla_J a_H(\tilde{u}(s)) = 0
\end{equation}
Let $\tilde{v}(s) = \tilde{g}^{-1}(\tilde{u}(s))$. By (4.2) and (4.3), $\tilde{u}$ satisfies (4.4) iff $\tilde{v}$ is a solution of
\begin{equation}
\frac{d\tilde{v}}{ds} + \nabla_J a_{H^g}(\tilde{v}(s)) = 0.
\end{equation}
The map of solution spaces defined in this way is clearly bijective and $\mathbb{R}$-equivariant. Assuming that $(H, J)$ is regular, we now check that its pullback $(H^g, J^g)$ satisfies the conditions for a regular pair. (i) Let $c = [v, x]$ be a critical point of $a_{H^g}$, $c' = \tilde{g}(c)$ and $\xi \in C^\infty(x^*TM)$ with $D^2 a_{H^g}(c)(\xi, \cdot) = 0$. By (4.3), $D^2 a_H(c')(Dg(x)\xi, \cdot) = 0$. Since $c'$ is nondegenerate, $\xi = 0$, and therefore $c$ is also nondegenerate. The last sentence in Definition 3.6(i) follows from (4.1). (ii) can be proved using the same method as in Lemma 4.1, which also shows that a solution of (4.5) and the corresponding solution of (4.4) have the same index. Together with (4.1), this implies that $(H^g, J^g)$ satisfies condition 3.6(iii).

**Proposition 4.4.** If $c \in \text{Crit}(a_{H^g})$ is nondegenerate, $\mu_{H^g}(\tilde{g}(c)) = \mu_{H^g}(c) - 2I(g, \tilde{g})$.

Recall that $\mu_{H^g}(c)$ is defined in the following way (see [17], [7]): choose a representative $(v, x)$ of $c$ and a symplectic trivialization $\tau_c : x^*TM \to S^1 \times$
\((\mathbb{R}^{2n}, \omega_0)\) which can be extended over \(v^*TM\). Let \(\Psi_{H^s,c} : [0; 1] \rightarrow \text{Sp}(2n, \mathbb{R})\) be the path given by

\[
\Psi_{H^s,c}(t) = \tau_c(t) D\phi_{H^s}^t(x(0))\tau_c(0)^{-1},
\]

where \((\phi_{H^s}^t)_{t \in \mathbb{R}}\) is the Hamiltonian flow of \(H^s\). \(\Psi_{H^s,c}(0) = \text{Id}\), and since \(c\) is nondegenerate, \(\det(Id - \Psi_{H^s,c}(1)) \neq 0\). A path \(\Psi\) with these properties has an index \(\mu_1(\Psi) \in \mathbb{Z}\), and \(\mu_{H^s}(c)\) is defined by \(\mu_{H^s}(c) = \mu_1(\Psi_{H^s,c})\). \(\mu_1\) has the following property [5, Proposition 5]:

**Lemma 4.5.** If \(\Psi, \Psi'\) are related by \(\Psi'(t) = l(t)\Psi(t)l(0)^{-1}\) for some \(l \in C^\infty(S^1, \text{Sp}(2n, \mathbb{R}))\), \(\mu_1(\Psi') = \mu_1(\Psi) - 2\deg(l)\).

**Proof of Proposition 4.4.** Let \((v', x')\) be a representative of \(\hat{g}(c)\) and \(\tau_{\hat{g}(c)}\) a trivialization of \((x')^*TM\) which can be extended over \((v')^*TM\). \(\mu_H(\hat{g}(c))\) is defined using the path

\[
\Psi_{H, \hat{g}(c)}(t) = \tau_{\hat{g}(c)}(t) D\phi_H^t(x'(0))\tau_{\hat{g}(c)}(0)^{-1}.
\]

\(H^g\) is defined in such a way that \(\phi_H^t = g_t \phi_{H^g} g_t^{-1}\). Therefore \(\Psi_{H, \hat{g}(c)}(t) = l(t)\Psi_{H^g,c}(t)l(0)^{-1}\), where \(l(t) = \tau_{\hat{g}(c)}(t) Dg_t(x(t))\tau_c(0)^{-1}\). \(l\) is a loop in \(\text{Sp}(2n; \mathbb{R})\), and \(\deg(l) = I(g, \hat{g})\) by definition. Proposition 4.4 now follows from Lemma 4.5.

**Definition 4.6.** Let \((H, J)\) be a regular pair, \((g, \hat{g}) \in \tilde{G}\) and \((H^g, J^g)\) the pullback of \((H, J)\) by \(g\). For \(k \in \mathbb{Z}\), define an isomorphism

\[
CF_k(g, \hat{g}; H, J, H^g) : CF_k(H^g) \rightarrow CF_{k-2I(g, \hat{g})}(H)
\]

by \(CF_k(g, \hat{g}; H, J, H^g)(<c>) = <\hat{g}(c)>,\) extended in the obvious way to infinite sums of the generators \(<c>\). Because of (4.3), this respects the finiteness condition for the formal sums in \(CF_*(H)\).

In view of the definition of the differential (3.2), Lemma 4.3 says that \(CF_*(g, \hat{g}; H, J, H^g)\) is an isomorphism of chain complexes. We denote the induced isomorphisms on Floer homology by

\[
HF_k(g, \hat{g}; H, J, H^g) : HF_k(H^g, J^g) \rightarrow HF_{k-2I(g, \hat{g})}(H, J).
\]

Consider two regular pairs \((H^-, J^-), (H^+, J^+)\), two Hamiltonians \(K^-_g, K^+_g\) which generate \(g\) and the pullbacks \(((H^-)^g, (J^-)^g)\), \(((H^+)^g, (J^+)^g)\) using \(K^-_g\) and \(K^+_g\), respectively. Let \((H, J)\) be a homotopy between \((H^-, J^-)\) and \((H^+, J^+)\). Choose \(\psi \in C^\infty(\mathbb{R}, \mathbb{R})\) with \(\psi(-\infty; -1] = 0\) and \(\psi|[1; \infty) = 1\), and define \(H^g \in C^\infty(\mathbb{R} \times S^1 \times M, \mathbb{R})\) by

\[
H^g(s, t, y) = H(s, t, g_t(y)) - (1 - \psi(s))K^-_g(t, g_t(y)) - \psi(s)K^+_g(t, g_t(y)).
\]

Together with the family \(J^g\) given by \(J^g_{s, t} = Dg_t^{-1}J_s Dg_t\), this is a homotopy from \(((H^-)^g, (J^-)^g)\) to \(((H^+)^g, (J^+)^g)\).
Lemma 4.7. For all \( c_-, c_+ \in \text{Crit}(a_{(H^-)^g}), c_+ \in \text{Crit}(a_{(H^+)^g}) \), there is a bijective map

\[
\mathcal{M}^\Phi(c_-, c_+; H^g, J^g) \longrightarrow \mathcal{M}^\Phi(\tilde{g}(c_-), \tilde{g}(c_+); H, J).
\]

Moreover, if \((H, J)\) is a regular homotopy, \((H^g, J^g)\) is also regular.

We omit the proof, which is similar to that of Lemma 4.3.

Corollary 4.8. If \((H, J)\) is a regular homotopy, the diagram

\[
\begin{array}{ccc}
CF_\ast((H^-)^g, (J^-)^g) & \xrightarrow{\Phi_\ast(H^g, J^g)} & CF_\ast(H^-, J^-) \\
\downarrow & & \downarrow \\
CF_\ast((H^+)^g, (J^+)^g) & \xrightarrow{\Phi_\ast(H^g, J^g)} & CF_\ast(H^+, J^+)
\end{array}
\]

commutes.

This is proved by a straightforward computation using Lemma 4.7. It follows that for every \((g, \tilde{g}) \in \overline{G}\), there is a unique automorphism

\[
HF_\ast(g, \tilde{g}) : HF_\ast(M, \omega) \longrightarrow HF_\ast(M, \omega)
\]

independent of the choice of a regular pair. We list some properties of these maps which are clear from the definition.

Proposition 4.9. (i) \(HF_\ast(g, \tilde{g})\) is an automorphism of \(HF_\ast(M, \omega)\) as a \(\Lambda\)-module.

(ii) For \((g, \tilde{g}) = \text{Id}_{\overline{G}}\), \(HF_\ast(g, \tilde{g}) = \text{Id}_{HF_\ast(M, \omega)}\).

(iii) If \((g, \tilde{g}) = (\text{Id}, \gamma)\) for some \(\gamma \in \Gamma\), \(HF_\ast(g, \tilde{g})\) is equal to the multiplication by \(\langle \gamma \rangle \in \Lambda\).

(iv) For \((g_1, \tilde{g}_1), (g_2, \tilde{g}_2) \in \overline{G}\), \(HF_\ast(g_1g_2, \tilde{g}_1\tilde{g}_2) = HF_\ast(g_1, \tilde{g}_1)HF_\ast(g_2, \tilde{g}_2)\).

\[\square\]

5. Homotopy invariance

Let \((g_r, \tilde{g}_r)_{0 \leq r \leq 1, t \in S^1}\) be a smooth family of Hamiltonian automorphisms of \((M, \omega)\) with \(g_r, 0 = \text{Id}\) for all \(r\). This family defines a path \((g_r)_{0 \leq r \leq 1}\) in \(\overline{G}\).

Let \((g_r, \tilde{g}_r)_{0 \leq r \leq 1}\) be a smooth lift of this path to \(\overline{G}\). The aim of this section is to prove the following

Proposition 5.1. For \((g_r, \tilde{g}_r)_{0 \leq r \leq 1}\) as above,

\[
HF_\ast(g_0, \tilde{g}_0) = HF_\ast(g_1, \tilde{g}_1) : HF_\ast(M, \omega) \longrightarrow HF_\ast(M, \omega).
\]

Choose a smooth family \((K_r)_{0 \leq r \leq 1}\) of Hamiltonians such that \(K_r\) generates \(g_r\). Let \((H, J)\) be a regular pair and \((H^{g_r}, J^{g_r})\) its pullbacks (using \(K_r\)). The maps induced by \((g_0, \tilde{g}_0)\) and \((g_1, \tilde{g}_1)\) are

\[
HF_\ast(g_0, \tilde{g}_0; H, J, H^{g_0}) : HF_\ast(H^{g_0}, J^{g_0}) \longrightarrow HF_\ast(H, J),
\]

\[
HF_\ast(g_1, \tilde{g}_1; H, J, H^{g_1}) : HF_\ast(H^{g_1}, J^{g_1}) \longrightarrow HF_\ast(H, J).
\]
Proposition 5.1 says that if \((H', J')\) is a regular homotopy from \((H^{g_1}, J^{g_1})\) to \((H^{g_0}, J^{g_0})\),
\[
(5.1) \quad H_F(s, g_1, g_1; H, J, H^{g_1}) = H_F(s, g_0, g_0; H, J, H^{g_0}) \Phi_s(H', J').
\]
Because of Proposition 4.9(iv), it is sufficient to consider the case \((5.1)\) with limits \((5.3)\) such that
\[
\text{Choose } K_0 = 0. \text{ Then } (H^{g_0}, J^{g_0}) = (H, J), \text{ and } (5.1) \text{ is reduced to }
\]
\[
(5.2) \quad \Phi_s(H', J') H_F(s, g_1; H, J, H^{g_1})^{-1} = Id_{H_F(s, H, J)}.
\]
The rest of the section contains the proof of this equation.

**Definition 5.2.** Let \((H, J), (H^{g_r}, J^{g_r})\) and \((H', J')\) be as above. A deformation of homotopies compatible with them consists of a function \(H \in C^\infty([0; 1] \times \mathbb{R} \times S^1 \times M, \mathbb{R})\) and a family of almost complex structures \(J = (J_{r,s,t}) \in J_{\text{reg}}([0; 1] \times \mathbb{R} \times S^1; M, \omega)\), such that
\[
\begin{align*}
\bar{H}(r, s, t, y) &= H^{g_r}(t, y), \quad \bar{J}_{r,s,t} = J_{r}^{g_r} \quad \text{for } s \leq -1, \\
\bar{H}(r, s, t, y) &= H(t, y), \quad \bar{J}_{r,s,t} = J_t \quad \text{for } s \geq 1, \\
\bar{H}(0, s, t, y) &= H(t, y), \quad \bar{J}_{0,s,t} = J_t \\
\bar{H}(1, s, t, y) &= H'(s, t, y), \quad \bar{J}_{1,s,t} = J'_{s,t}.
\end{align*}
\]

Let \((\bar{H}, \bar{J})\) be a deformation of homotopies. Consider a pair \((r, u) \in [0; 1] \times C^\infty(\mathbb{R} \times S^1, M)\) such that
\[
(5.3) \quad \frac{\partial u}{\partial s} + \bar{J}_{r,s,t}(u(s, t)) \left( \frac{\partial u}{\partial t} - X_H(r, s, t, u(s, t)) \right) = 0.
\]

We say that \(u\) converges to \(c_-, c_+ \in \text{Crit}(a_H)\) if there is a smooth path \(\bar{u} : \mathbb{R} \to \bar{L}M\) with
\[
\lim_{s \to -\infty} \bar{u}(s) = \bar{g}_r^{-1}(c_-), \quad \lim_{s \to +\infty} \bar{u}(s) = c_+
\]
such that \(u(s, \cdot) = p(\bar{u}(s))\) for all \(s\). We denote the space of such pairs \((r, u)\) with limits \(c_\pm\) by \(M^h(c_-, c_+; \bar{H}, \bar{J})\). The linearization of \((5.3)\) at a pair \((r, u) \in M^h(c_-, c_+; \bar{H}, \bar{J})\) is given by an operator
\[
D^h_{H, J}(r, u) : \mathbb{R} \times C^\infty(u^*TM) \to C^\infty(u^*TM).
\]

Since \(c_-, c_+\) are nondegenerate critical points of \(a_H, \bar{g}_r^{-1}(c_-)\) is a nondegenerate critical point of \(a_{H^{g_r}}\) for all \(r\). It follows that the Sobolev completion \((p > 2)\)
\[
(5.4) \quad D^h_{H, J}(r, u) : \mathbb{R} \times W^{1,p}(u^*TM) \to L^p(u^*TM)
\]
is a Fredholm operator of index \(\mu_{H^{g_r}}(\bar{g}_r^{-1}(c_-)) - \mu_H(c_+) + 1\). By Lemma 4.4, \(\mu_H(c_-) = \mu_{H^{g_r}}(\bar{g}_r^{-1}(c_-)) - 2I(g_r, \bar{g}_r)\). However, since \((g_r, \bar{g}_r)\) is homotopic to the identity in \(G\), \(I(g_r, \bar{g}_r) = 0\). It follows that \(\text{ind} D^h_{H, J}(r, u) = \mu_H(c_-) - \mu_H(c_+) + 1\).
$(\tilde{H}, \tilde{J})$ is called regular if \((5.4)\) is onto for all \((r, u)\), and \((r, s, t, u(s, t)) \not\in V_0(J)\) for all \((r, s, t) \in [0; 1] \times \mathbb{R} \times S^1\) and \(u \in M^h(c_-, c_+; \tilde{H}, \tilde{J})\) such that \(\mu_H(c_-) \leq \mu_H(c_+)\). An analogue of Theorem 3.8 ensures that regular deformations of homotopies exist. Regularity implies that the spaces \(M^h(c_-, c_+; \tilde{H}, \tilde{J})\) are smooth manifolds. The boundary \(\partial M^h(c_-, c_+; \tilde{H}, \tilde{J})\) consists of solutions of \((5.3)\) with \(r = 0\) or 1. For \(r = 0\), \((5.3)\) is
\[
(5.5) \quad \frac{\partial u}{\partial s} + J_t(u) \left( \frac{\partial u}{\partial t} - X_H(t, u) \right) = 0
\]
and for \(r = 1\), it is
\[
(5.6) \quad \frac{\partial u}{\partial s} + J'_{s,t}(u) \left( \frac{\partial u}{\partial t} - X_{H'}(s, t, u) \right) = 0.
\]
Since \((H, J)\) is a regular pair and \((H', J')\) a regular homotopy, this implies that \(\partial M^h(c_-, c_+; H, J) = \emptyset\) if \(\mu_H(c_+) = \mu_H(c_-) + 1\).

**Lemma 5.3.** Let \((\tilde{H}, \tilde{J})\) be a regular deformation of homotopies and \(c_- \in \text{Crit}_{k_-}(a_H)\), \(c_+ \in \text{Crit}_{k_+}(a_H)\).

(i) If \(k_+ = k_- + 1\), \(M^h(c_-, c_+; \tilde{H}, \tilde{J})\) is a finite set.

(ii) If \(k_+ = k_-\), \(M^h(c_-, c_+; \tilde{H}, \tilde{J})\) is one-dimensional, and there is a smooth compactification \(\overline{M}^h(c_-, c_+; \tilde{H}, \tilde{J})\) whose boundary \(\partial \overline{M}^h(c_-, c_+; \tilde{H}, \tilde{J})\)

\[(5.7) \quad M^h(c_-, c_+; \tilde{H}, \tilde{J}) \times (M(c, c_+; H, J)/\mathbb{R})
\]

for all \(c \in \text{Crit}_{k_+ + 1}(a_H)\) and

\[(5.8) \quad (M(c_-, c_+; H, J)/\mathbb{R}) \times M^h(c_+, c_+; \tilde{H}, \tilde{J})
\]

for all \(c' \in \text{Crit}_{k_+ - 1}(a_H)\).

This compactification is constructed in the same way as that used in [17, Lemma 6.3]. We omit the proof. The fact that the spaces \(M^h(c_-, c_+; \tilde{H}, \tilde{J})\) of dimension \(\leq 1\) can be compactified without including limit points with ‘bubbles’ is a consequence of the regularity condition (compare [7, Theorem 5.2]). The limits \((5.8)\) arise in the following way: let \((r_m, u_m)\) be a sequence in \(M^h(c_-, c_+; \tilde{H}, \tilde{J})\) with \(\lim_m r_m = r\) and such that \(u_m\) converges uniformly on compact subsets to some \(u_\infty \in C^\infty(\mathbb{R} \times S^1, M)\). Assume that there are \(s_m \in \mathbb{R}\), \(\lim_m s_m = \infty\), such that the translates \(\hat{u}_m(s, t) = u_m(s - s_m, t)\)
converge on compact subsets to a map \(\hat{u}_\infty\) with \(\partial \hat{u}_\infty / \partial s \not\equiv 0\). \(\hat{u}_\infty\) is a solution of
\[
\frac{\partial \hat{u}_\infty}{\partial s} + J'_{s,t}(\hat{u}_\infty) \left( \frac{\partial \hat{u}_\infty}{\partial t} - X_H(t, \hat{u}_\infty) \right) = 0.
\]
Therefore \(\tilde{v}_\infty(s, t) = g_{r,t}(\hat{u}_\infty(s, t))\) defines a solution of \((5.5)\). \((\tilde{v}_\infty, (r, u_\infty))\) is the limit of the sequence \(u_m\) in \((5.8)\). The points \((5.7)\) in the compactification arise by a similar, but simpler process using translations with \(s_m \to -\infty\).
\[ h_k(\bar{H}, \bar{J}) : CF_k(H) \rightarrow CF_{k+1}(H), \]
\[ h_k(\bar{H}, \bar{J})(<c_->) = \sum_{c_+} \#\mathcal{M}^h(c_-, c_+; \bar{H}, \bar{J})<c_+>. \]

To show that this formal sum is indeed an element of \( CF_*(H, J) \), a slightly stronger version of Lemma 5.3(i) is necessary; we omit the details.

**Lemma 5.5.** For all \( k \),

\[ \partial_{k+1}(H, J)h_k(\bar{H}, \bar{J}) + h_{k-1}(\bar{H}, \bar{J})\partial_k(H, J) = \Phi_k(H', J')CF_k(g_1, \bar{g}_1; H, J, H^{g_1})^{-1} - Id_{CF_k(H)}. \]

**Proof.** By definition,

\[ \Phi_k(H', J')CF_k(g_1, \bar{g}_1; H, J, H^{g_1})^{-1}<c_-> = \sum_{c_+} \#\mathcal{M}^\Phi(\bar{g}_1^{-1}c_-, c_+; H', J')<c_+>. \]

Comparing (5.6) with (3.3), one sees that \( u \in \mathcal{M}^\Phi(\bar{g}_1^{-1}c_-, c_+; H', J') \) iff \( (1, u) \in \mathcal{M}^h(c_-, c_+; \bar{H}, \bar{J}) \). The other part of \( \partial \mathcal{M}^h(c_-, c_+; \bar{H}, \bar{J}) \) is given by solutions of (5.5) with limits \( c_+ \). If \( c_-, c_+ \) have the same Conley-Zehnder index, any regular solution of (5.5) is stationary in the sense that \( \partial u/\partial s = 0 \), and therefore this part of \( \partial \mathcal{M}^h(c_-, c_+; \bar{H}, \bar{J}) \) is empty unless \( c_- = c_+ \), in which case it contains a single point. It follows that

\[ (\Phi_k(H', J')CF_k(g_1, \bar{g}_1; H, J, H^{g_1})^{-1} - Id) <c_-> = \sum_{c_+} \#\partial \mathcal{M}^h(c_-, c_+; \bar{H}, \bar{J})<c_+> \]

(the sign is irrelevant since we use \( \mathbb{Z}/2 \)-coefficients). Similarly, the coefficients of \( <c_+> \) in \( \partial_{k+1}(H, J)h_k(\bar{H}, \bar{J})<c_-> \) and \( h_{k-1}(\bar{H}, \bar{J})\partial_k(H, J)<c_-> \) are given by the number of points in (5.7) and (5.8). The statement now follows from the fact that \( \overline{\partial \mathcal{M}^h(c_-, c_+; \bar{H}, \bar{J})} \) contains an even number of points.

**Lemma 5.5** shows that \( \Phi_*(H', J')CF_*(g_1, \bar{g}_1; H, J, H^{g_1})^{-1} \) is chain homotopic to the identity, which implies (5.2).

6. **Pair-of-pants product**

The main result of this section, Proposition 6.3, describes the behaviour of the pair-of-pants product on Floer homology under the maps \( HF_*(g, \bar{g}) \). It turns out that \( HF_*(g, \bar{g}) \) does not preserve the ring structure; rather, it is an automorphism of \( HF_*(M, \omega) \) as a module over itself.
The definition of the pair-of-pants product for weakly monotone symplectic manifolds can be found in [15] or [11, Chapter 10] (a detailed construction in the case $[\omega]\pi_2(M) = 0$ is given in [21]). We need to modify this definition slightly by enlarging the class of admissible perturbations. This would be necessary in any case to adapt it to the use of time-dependent almost complex structures; a further enlargement ensures that the relevant moduli spaces transform nicely under (a subset of) $G$. We now sketch this modified definition.

Consider the punctured surface $\Sigma = \mathbb{R} \times S^1 \setminus (0; 0)$. It has a third tubular end

$$e : \mathbb{R}^- \times S^1 \to \Sigma, \quad e(s, t) = \left( \frac{1}{4}e^{2\pi s} \cos(2\pi t), \frac{1}{4}e^{2\pi s} \sin(2\pi t) \right)$$

around the puncture, and the surface $\hat{\Sigma}$ obtained by capping off this end with a disc can be identified with $\mathbb{R} \times S^1$. For $u \in C^\infty(\Sigma, M)$ we will denote $u \circ e$ by $u^e$. Two maps $u \in C^\infty(\Sigma, M)$ and $v_0 \in C^\infty(D^2, M)$ which satisfy $\lim_{s \to -\infty} u^e(s, \cdot) = v_0|D^2$ can be glued together to a map $u\#v_0 : \hat{\Sigma} \to M$. By identifying $\hat{\Sigma} = \mathbb{R} \times S^1$, this gives a path $\mathbb{R} \to \Lambda M$. On the two remaining ends, $u\#v_0$ has the same asymptotic behaviour as $u$. In particular, if there are $x_-, x_+ \in \mathcal{L}M$ such that $\lim_{s \to \pm \infty} u(s, \cdot) = x_{\pm}$, the path $u\#v_0$ lies in $\mathcal{L}M$ and converges to $x_-, x_+$.

**Definition 6.1.** We say that $u \in C^\infty(\Sigma, M)$ converges to $c_-, c_+ \in \mathcal{L}M$ if

$$\lim_{s \to \pm \infty} u(s, \cdot) = p(c_{\pm}), \quad \lim_{s \to -\infty} u^e(s, \cdot) = p(c_0)$$

and if for $c_0 = [v_0, x_0]$, the path $u\#v_0 : \mathbb{R} \to \mathcal{L}M$ has a lift $\tilde{u}\#v_0 : \mathbb{R} \to \tilde{\mathcal{L}M}$ with limits $c_-, c_+$.

This is independent of the choice of the representative $(v_0, x_0)$ for $c_0$ and of the choices involved in the gluing.

Let $(H^-, J^-)$, $(H^0, J^0)$ and $(H^+, J^+)$ be regular pairs. Choose $J = (J_z)_{z \in \Sigma} \in \mathcal{J}^{\text{reg}}(\Sigma; M, \omega)$ and $H \in C^\infty(\Sigma \times M, \mathbb{R})$ with $H|[e([-1; 0] \times S^1) \times M] = 0$, such that $(H, J)$ agrees with one regular pair over each end of $\Sigma$, more precisely:

$$H(-s, t, y) = H^-(t, y), \quad J_{(-s,t)} = J^-_t,$$

$$H(s, t, y) = H^+(t, y), \quad J_{(s,t)} = J^+_t,$$

$$H(e(-s,t), y) = H^0(t, y), \quad J_{(e(-s,t))} = J^0_t$$

for $s \geq 2$, $t \in S^1$ and $y \in M$. The space of such $(H, J)$ will be denoted by $\mathcal{P}(H^-, J^-, H^0, J^0, H^+, J^+)$. $u \in C^\infty(\Sigma, M)$ is called $(H, J)$-holomorphic if

$$\frac{\partial u}{\partial s}(s, t) + J_{s,t}(u(s, t)) \left( \frac{\partial u}{\partial t}(s, t) - X_H(s, t, u(s, t)) \right) = 0$$
for \((s,t) \in \Sigma \setminus e((-\infty; -1] \times S^1)\) and
\[
\frac{\partial u^e}{\partial s}(s,t) + J_{e(s,t)}(u^e(s,t)) \left( \frac{\partial u^e}{\partial t}(s,t) - X_{H}(e(s,t), u^e(s,t)) \right) = 0
\]
for \((s,t) \in \mathbb{R}^- \times S^1\). For \(z = (s,t) \in e([-1;0] \times S^1)\), the first equation be written as \(\partial_{\bar{J}} u(z) = 0\) using the complex structure on \(\Sigma \cong (\mathbb{C}/i\mathbb{Z}) \setminus \{0;0\}\). Similarly, the second equation is \(\partial_{J^+(s,t)} u^e(s,t) = 0\) for \((s,t) \in [-1;0] \times S^1\). Since \(e\) is holomorphic, this implies that the two equations match up smoothly.

**Definition 6.2.** For \(c_- \in \text{Crit}(a_{H^-}), c_0 \in \text{Crit}(a_{H^0})\) and \(c_+ \in \text{Crit}(a_{H^+})\), \(\mathcal{M}^{PP}(c_-, c_0, c_+; H, J)\) is the space of \((H, J)\)-holomorphic maps which converge to \(c_-, c_0, c_+\) in the sense discussed above.

The linearization of the two equations for an \((H, J)\)-holomorphic map at \(u \in \mathcal{M}^{PP}(c_-, c_0, c_+; H, J)\) is given by single differential operator \(D_{H,J}^u(u)\) on \(\Sigma\) which becomes a Fredholm operator in suitable Sobolev spaces. Following the same method as in the previous sections, we call \((H, J)\) regular if all these operators are onto and if for all \((H, J)\)-holomorphic maps \(u\) with \(\text{ind}(D_{H,J}^u(u)) \leq 1\) and all \(z \in \Sigma\), \((z, u(z)) \notin V_0(J)\). A transversality result analogous to Theorem 3.8 shows that the set of regular \((H, J)\) is dense in \(\mathcal{P}(H^{-}, J^{-}, H^0, J^0, H^+, J^+)\). If \((H, J)\) is regular and \(\mu_{H^-}(c_-) = \mu_{H^+}(c_+) + \mu_{H^0}(c_0) - 2n\), the space \(\mathcal{M}^{PP}(c_-, c_0, c_+; H, J)\) is finite. The pair-of-pants product
\[
PP_{i,j}(H, J) : CF_i(H^-) \otimes CF_j(H^0) \to CF_{i+j-2n}(H^+)
\]
for regular \((H, J)\) is defined by
\[
PP_{i,j}(H, J)(<c_- > \otimes <c_0>) = \sum_{c_+} \# \mathcal{M}^{PP}(c_-, c_0, c_+; H, J) <c_+>.
\]
A compactness theorem shows that this formal sum lies in \(CF_{*}(H^+)\). The grading is a consequence of our convention for the Conley-Zehnder index.

The construction of the product is completed in the following steps: using the spaces \(\mathcal{M}^{PP}\) of dimension 1 and their compactifications, one shows that \(PP_* (H, J)\) is a chain homomorphism. The induced maps
\[
(6.1) \quad PP_* (H, J) : HF_*(H^-, J^-) \times HF_*(H^0, J^0) \to HF_*(H^+, J^+)
\]
are independent of the choice of \((H, J)\) (the starting point for the proof is that \(\mathcal{P}(H^-, J^-, H^0, J^0, H^+, J^+)\) is contractible). Finally, a gluing argument proves that the maps (6.1) for different regular pairs are related by continuation isomorphisms. Therefore they define a unique product
\[
*_{PP} : HF_*(M, \omega) \times HF_*(M, \omega) \to HF_*(M, \omega).
\]
Using the assumption \((W^+)\), the proofs of these properties for the product as defined in [15] can be easily adapted to our slightly different setup.
Proposition 6.3. For all \((g, \tilde{g}) \in \tilde{G}\) and \(a, b \in HF_*(M, \omega)\),
\[HF_*(g, \tilde{g})(a *_{PP} b) = HF_*(g, \tilde{g})(a) *_{PP} b.\]

Because of the homotopy invariance of \(HF_*(g, \tilde{g})\) (Proposition 5.1) it is sufficient to consider the case where \(g_t = Id_M\) for \(t \in [-\frac{1}{4}, \frac{1}{4}] \subset S^1\) (clearly, any path component of \(\tilde{G}\) contains a \((g, \tilde{g})\) with this property). Choose a Hamiltonian \(K_g\) which generates \(g\) such that \(K_g(t, \cdot) = 0\) for \(t \in [-\frac{1}{4}, \frac{1}{4}]\).

Let \(((H^\pm)^9, (J^\pm)^9)\) be the pullback of \((H^\pm, J^\pm)\) using \(K_g\). The proof of Proposition 6.3 relies on the following analogue of the ‘pullback’ of a regular pair: for \((H, J) \in \mathcal{P}(H^-, J^-, H^0, J^0, H^+, J^+)\), define \((H^9, J^9)\) by
\[J^9_{(s,t)} = Dg_t^{-1}J_{(s,t)}Dg_t, \quad H^9(s, t, y) = H(s, t, g_t(y)) - K_g(t, g_t(y))\]
for \((s, t) \in \Sigma, y \in M\). Because \(g_t = Id_M\) for \(t \in [-\frac{1}{4}, \frac{1}{4}]\) and
\[\text{im}(e) \subset (\mathbb{R} \times [-1/4; 1/4]) \setminus (0; 0) \subset \Sigma,\]
\((H^9, J^9)\) satisfies \(H^9(e(s, t), \cdot) = H(e(s, t), \cdot)\) and \(J^9_{e(s,t)} = J_{e(s,t)}\) for all \((s, t) \in \mathbb{R}^- \times S^1\). In particular, \(H^9\) vanishes on \(e([-1; 0]) \times S^1\) \(\times M\). It follows that \((H^9, J^9) \in \mathcal{P}((H^-)^9, (J^-)^9, H^0(J^0, (H^+)^9, (J^+)^9))\). The next Lemma is the analogue of Lemma 4.3:

Lemma 6.4. For all \((c_-, c_0, c_+) \in \text{Crit}(a_{(H^+)^9}) \times \text{Crit}(a_{H^0}) \times \text{Crit}(a_{(H^+)^9})\), there is a bijective map
\[\mathcal{M}^{PP}(c_-, c_0, c_+; H^9, J^9) \rightarrow \mathcal{M}^{PP}((\tilde{g}(c_-)), c_0, \tilde{g}(c_+); H, J)\]
Moreover, if \((H, J)\) is regular, so is \((H^9, J^9)\).

Proof. A straightforward computation shows that if \(u, v \in C^\infty(\Sigma, M)\) are related by \(u(s, t) = g_t(v(s, t))\), \(u\) is \((H, J)\)-holomorphic iff \(v\) is \((H^9, J^9)\)-holomorphic (note that \(v^e = u^e\), and that the second equation is the same in both cases). Now assume that \(v\) converges to \(c_-, c_0, c_+\) as defined above, and choose a representative \((v_0, x_0)\) of \(c_0\). The maps \(u#v_0, v#v_0 : \mathbb{R} \times S^1 \rightarrow M\) can be chosen such that
\[(u#v_0)(s, t) = g_t((v#v_0)(s, t)).\]
By assumption, the path \(\mathbb{R} \rightarrow \mathcal{LM}\) given by \(v#v_0\) has a lift \(\widehat{v#v_0} : \mathbb{R} \rightarrow \widehat{\mathcal{LM}}\) with limits \(c_-, c_+\). Define
\[\widehat{u#v_0}(s) = \tilde{g}(\widehat{v#v_0}(s)).\]
Clearly, \(\widehat{u#v_0}\) is a lift of \(u#v_0\) with limits \(\tilde{g}(c_-), \tilde{g}(c_+).\) Therefore \(u\) converges to \(\tilde{g}(c_-), c_0, \tilde{g}(c_+).\) The proof of regularity is similar to that of Lemma 4.1; we omit the details. \(\square\)

From Lemma 6.4 and the definition of \(PP_*(H, J)\), it follows that
\[CF_*(g, \tilde{g}; H^+, J^+, (H^+)^9)PP_*(H^9, J^9)((c_-) \otimes (c_0)) = PP_*(H, J)((\tilde{g}(c_-)) \otimes (c_0))\]
for every regular \((H,J) \in \mathcal{P}(H^-, J^-, H^0, J^0, H^+, J^+), c_- \in \text{Crit}(a_{(H^-)\#})\) and \(c_0 \in \text{Crit}(a_{H^0})\). Proposition 6.3 follows directly from this.

7. Pseuhoodomorphic sections

Let \((E, \Omega)\) be a symplectic fibre bundle over \(S^2\) with fibre \((M, \omega)\) and \(\pi : E \to S^2\) the projection. We will denote by \(\mathcal{J}(E, \Omega)\) the space of families \(J = (J_z)_{z \in S^2}\) of almost complex structures on the fibres of \(E\) such that \(J_z\) is \(\Omega_z\)-compatible for all \(z\). For \(J \in \mathcal{J}(E, \Omega)\) and \(k \in \mathbb{Z}\), let \(\mathcal{M}_k^\#(J)\) be the space of pairs \((z, w) \in S^2 \times C^\infty(\mathbb{C}P^1, E)\) such that \(w\) is a simple \(J_z\)-holomorphic curve in \(E_z\) with \(c_1(TE_z, \Omega_z) \cdot \pi(w) = k\). Because \((E, \Omega)\) is locally trivial, these spaces have the same properties as the spaces of holomorphic curves in \(M\) with respect to a two-parameter family of almost complex structures (see section 3). In particular, there is a dense subset \(\mathcal{J}^{reg}(E, \Omega) \subset \mathcal{J}(E, \Omega)\) such that\(^2\) for \(J \in \mathcal{J}^{reg}(E, \Omega)\), \(\mathcal{M}_k^\#(J) = \emptyset\) for all \(k < 0\) and \(\mathcal{M}_0^\#(J)\) is a manifold of dimension \(\dim \mathcal{M}_0^\#(J) = \dim E\). It follows that the image of the evaluation map

\[ \eta : \mathcal{M}_0^\#(J) \times_{PSL(2, \mathbb{C})} \mathbb{C}P^1 \to E \]

is a subset of codimension 4.

Following the convention of section 2, \((E, \Omega)\) is equipped with a preferred isomorphism \(i : (M, \omega) \to (E_{z_0}, \Omega_{z_0})\) for the marked point \(z_0 \in S^2\). We need to recall the transversality theory of cusp-curves; a reference is [11, Chapter 6].

**Definition 7.1.** Let \(J\) be an \(\omega\)-compatible almost complex structure on \(M\). A simple \(J\)-holomorphic cusp-curve with \(r \geq 1\) components is a

\[ v = (w_1, \ldots, w_r, t_1, \ldots, t_r, t'_1, \ldots, t'_r) \in C^\infty(\mathbb{C}P^1, M)^r \times (\mathbb{C}P^1)^{2r} \]

with the following properties:

(i) For \(i = 1 \ldots r\), \(w_i\) is a simple \(J\)-holomorphic curve;

(ii) \(\text{im}(w_i) \neq \text{im}(w_j)\) for \(i \neq j\);

(iii) \(w_i(t_i) = w_{i+1}(t'_{i+1})\) for \(i = 1 \ldots r - 1\).

The Chern number of \(v\) is defined by \(c_1(v) = \sum_i c_1(w_i)\). \(PSL(2, \mathbb{C})^r\) acts freely on the space of simple cusp-curves with \(r\) components by

\[ (a_1 \ldots a_r) \cdot (w_1, \ldots, w_r, t_1, \ldots, t_r, t'_1, \ldots, t'_r) = (w_1 \circ a_1^{-1}, \ldots, w_r \circ a_r^{-1}, a_1(t_1), \ldots, a_r(t_r), a_1(t'_1), \ldots, a_r(t'_r)). \]

Let \(\mathcal{C}_{r,k}(J)\) be the quotient of the space of cusp-curves with \(r\) components and Chern number \(k\) by this action, and

\[ \eta_1, \eta_2 : \mathcal{C}_{r,k}(J) \to M \]

\(^2\)This uses the assumption \((W^+)\); again, weak monotonicity would not be sufficient.
the maps given by \( \eta_1(v) = w_1(t'_1) \), \( \eta_2(v) = w_r(t_r) \) for \( v \) as above. For a generic \( J, \mathcal{C}_{r,k}(J) \) is a smooth manifold of dimension \( 2n + 2k - 2r \) \([11, Theorem 5.2.1(ii)]\). Let \( \mathcal{J}^{reg,20}(E,\Omega) \subset \mathcal{J}^{reg}(E,\Omega) \) be the subset of families \( J = (J_z)_{z \in S^2} \) such that \( J = Di^{-1}J_{z_0}Di \) has this regularity property; \( \mathcal{J}^{reg,20}(E,\Omega) \) is dense in \( \mathcal{J}(E,\Omega) \).

Let \( J \) be a positively oriented complex structure on \( S^2 \) and \( J \in \mathcal{J}(E,\Omega) \). We call an almost complex structure \( \hat{J} \) on \( E \) compatible with \( j \) and \( J \) if \( D\pi \circ \hat{J} = j \circ D\pi \) and \( J|E_z = J_z \) for all \( z \in S^2 \). The space of such \( \hat{J} \) will be denoted by \( \hat{\mathcal{J}}(j,J) \). If we fix a \( \hat{J}_0 \in \hat{\mathcal{J}}(j,J) \), any other such \( \hat{J} \) is of the form \( \hat{J}_0 + \theta \circ D\pi \), where \( \theta : \pi^*(TS^2,j) \to (TE^v,J) \) is a smooth \( \mathbb{C} \)-antilinear vector bundle homomorphism (recall that \( TE^v = \ker D\pi \) is the tangent bundle along the fibres); conversely, \( \hat{J}_0 + \theta \circ D\pi \in \hat{\mathcal{J}}(j,J) \) for any such \( \theta \).

For \( j, J \) as above and \( \hat{J} \in \hat{\mathcal{J}}(j,J) \), we call a smooth section \( s : S^2 \to E \) of \( \pi(j,\hat{J}) \)-holomorphic if

\[
ds \circ j = \hat{J} \circ ds.
\]

If \( E = S^2 \times M \) and \( J_z \) is independent of \( z \), this is the ‘inhomogeneous Cauchy-Riemann equation’ of \([16]\) for maps \( S^2 \to M \) with an inhomogeneous term determined by the choice of \( J \).

Let \( \mathcal{S}(j,\hat{J}) \) be the space of \( (j,\hat{J}) \)-holomorphic sections of \( E \). For every section \( s, J \) induces an almost complex structure \( s^*J \) on \( s^*TE^v \), given by \( (s^*J)_z = (J_z)_s(z) \). The linearization of (7.1) at \( s \in \mathcal{S}(j,\hat{J}) \) is a differential operator

\[
D_j(s) : C^\infty(s^*TE^v) \to \Omega^{0,1}(s^*TE^v,s^*J)
\]
on \( S^2 \). \( D_j(s) \) differs from the \( \bar{\partial} \)-operator of \( (s^*TE^v,s^*J) \) by a term of order zero. Therefore it is a Fredholm operator with index \( d(s) = 2n + 2c_1(TE^v,\Omega)(s) \). If \( D_j(s) \) is onto, \( S(j,\hat{J}) \) is a smooth \( d(s) \)-dimensional manifold near \( s \).

**Definition 7.2.** Let \( j \) be a positively oriented complex structure on \( S^2 \), \( J = (J_z)_{z \in S^2} \in \mathcal{J}^{reg,20}(E,\Omega) \) and \( \hat{J} \) the almost complex structure on \( M \) given by \( \hat{J} = Di^{-1}J_{z_0}Di \). \( \hat{J} \in \hat{\mathcal{J}}(j,J) \) is called regular if \( D_j(s) \) is onto for all \( s \in \mathcal{S}(j,\hat{J}) \), the map

\[
ev : S^2 \times \mathcal{S}(j,\hat{J}) \to E, \ ev(z,s) = s(z)
\]
is transverse to \( \eta \) and the map

\[
ev_{z_0} : \mathcal{S}(j,\hat{J}) \to M, \ ev_{z_0}(s) = i^{-1}(s(z_0))
\]
is transverse to \( \eta_1 \).

**Proposition 7.3.** The subspace \( \hat{\mathcal{J}}^{reg}(j,J) \subset \hat{\mathcal{J}}(j,J) \) of regular \( \hat{J} \) is \( C^\infty \)-dense.
Lemma 7.4. Let $J$ be an almost complex structure on $E$ such that $J \in \mathcal{J}(j, \mathcal{J})$ for some $j, \mathcal{J}$. There is a two-form $\sigma$ on $S^2$ such that $\Omega + \pi^*\sigma$ is a symplectic form on $E$ which tames $\hat{J}$. 

This is a well-known method for constructing symplectic forms, due to Thurston [23]. Lemma 7.4 makes it possible to apply Gromov’s compactness theorem to $(j, \hat{J})$-holomorphic sections, since any such section is a $\hat{J}$-holomorphic curve in $E$. It is easy to see that any ‘bubble’ in the limit of a sequence of $(j, \hat{J})$-holomorphic sections must be a $J_z$-holomorphic curve in some fibre $E_z$. As in [16], we can simplify the limits by deleting some components and replacing multiply covered curves by the underlying simple ones. If $\mathcal{J} \in \mathcal{J}^{\text{reg}}(E, \Omega)$, every $J_z$ is semi-positive, and the total Chern number does not increase during the process. The outcome can be summarized as follows:

Proof (sketch). The strategy of the proof is familiar. The tangent space of $\mathcal{J}(j, \mathcal{J})$ at any point is a subspace of the space of bundle homomorphisms $I : TE \to TE$. As explained above, it contains precisely those $I$ which can be written as $I = \theta \circ D\pi$, where $\theta : \pi^*(TS^2, j) \to (TE^v, \mathcal{J})$ is a $\mathbb{C}$-antilinear homomorphism. Let $\mathcal{U}$ be the space of such $\theta$ whose $C^\infty$-norm is finite. Consider first the condition that $D_j(s)$ is onto. The main step in the proof is to show that

$$D^{\text{univ}}(s, \hat{J}) : C^\infty(s^*TE^v) \times \mathcal{U} \to \Omega^{0,1}(s^*TE^v, s^*\mathcal{J}),$$

$$D^{\text{univ}}(s, \hat{J})(S, \theta) = D_j(s)S + \frac{1}{2}(\theta \circ D\pi) \circ ds \circ j$$

is onto for all $\hat{J} \in \mathcal{J}(j, \mathcal{J})$ and $s \in S(j, \hat{J})$. $D\pi \circ ds = Id$ and therefore $D^{\text{univ}}(s, \hat{J})(0, \theta)_\ast = \frac{1}{2}\theta(s(z)) \circ j$. Because $s : S^2 \to E$ is an embedding, this means that $D^{\text{univ}}(s, \hat{J})(0 \times \mathcal{U})$ is dense in $\Omega^{0,1}(s^*TE^v, s^*\mathcal{J})$. Since $D_j(s)$ is a Fredholm operator, it follows that $D^{\text{univ}}(s, \hat{J})$ is onto.

Now consider the condition that $\text{ev}$ is transverse to $\eta$. The proof does not use any specific properties of $\eta$; for any smooth map $c : C \to E$, the space of $\hat{J}$ such that $\text{ev}$ is transverse to $c$ is dense in $\mathcal{J}(j, \mathcal{J})$. As in [11, Section 6.1], this follows from the fact that the evaluation map on a suitable ‘universal moduli space’ is a submersion. The main step is to prove that the operator

$$T_zS^2 \times C^\infty(s^*TE^v) \times \mathcal{U} \to T_{s(z)}E \times \Omega^{0,1}(s^*TE^v, s^*\mathcal{J}),$$

$$(\xi, S, \theta) \mapsto (ds(z)\xi + S(z), D^{\text{univ}}(s, \hat{J})(S, \theta))$$

is onto for all $\hat{J} \in \mathcal{J}(j, \mathcal{J})$, $s \in S(j, \hat{J})$ and $z \in S^2$. This is slightly stronger than the surjectivity of $D^{\text{univ}}$, but it can be proved by the same argument. The proof of the transversality condition for $\text{ev}_{z_0}$ is similar. 

From now on, we assume that $(E, \Omega)$ is Hamiltonian. Let $\tilde{\Omega} \in \Omega^2(E)$ be a closed form with $\tilde{\Omega}|E_z = \Omega_z$ for all $z$.

Lemma 7.4. Let $\hat{J}$ be an almost complex structure on $E$ such that $\hat{J} \in \mathcal{J}(j, \mathcal{J})$ for some $j, \mathcal{J}$. There is a two-form $\sigma$ on $S^2$ such that $\tilde{\Omega} + \pi^*\sigma$ is a symplectic form on $E$ which tames $\hat{J}$. 

This is a well-known method for constructing symplectic forms, due to Thurston [23]. Lemma 7.4 makes it possible to apply Gromov’s compactness theorem to $(j, \hat{J})$-holomorphic sections, since any such section is a $\hat{J}$-holomorphic curve in $E$. It is easy to see that any ‘bubble’ in the limit of a sequence of $(j, \hat{J})$-holomorphic sections must be a $J_z$-holomorphic curve in some fibre $E_z$. As in [16], we can simplify the limits by deleting some components and replacing multiply covered curves by the underlying simple ones. If $\mathcal{J} \in \mathcal{J}^{\text{reg}}(E, \Omega)$, every $J_z$ is semi-positive, and the total Chern number does not increase during the process. The outcome can be summarized as follows:
Lemma 7.5. Let $j$ be a positively oriented complex structure on $S^2$, $\hat{J}$ an almost complex structure on $E$ such that $\hat{J} \in \mathcal{J}(j,J)$ for some $J \in \mathcal{J}^\text{reg}(E,\Omega)$, and $\hat{J}$ the almost complex structure on $M$ which corresponds to $J_{z_0}$ through the isomorphism $i$. Let $(s_m)_{m \in \mathbb{N}}$ be a sequence in $\mathcal{S}(j,\hat{J})$ such that $\bar{\Omega}(s_m) \leq C$ and $c_1(TE^v,\Omega)(s_m) \leq c$ for all $m$. Assume that $(s_m)$ has no convergent subsequence, and that $s_m(z_0)$ converges to $y \in E_{z_0}$. Then one of the following (not mutually exclusive) possibilities holds:

(i) there is an $s \in \mathcal{S}(j,\hat{J})$ with $c_1(TE^v,\Omega)(s) < c$ and $s(z_0) = y$;
(ii) there are $s \in \mathcal{S}(j,\hat{J})$, $z \in S^2$ and $w \in \mathcal{M}^0_0(J) \times \mathbb{CP}^1$ such that $c_1(TE^v,\Omega)(s) = c$, $s(z_0) = y$ and $s(z) = \eta(w)$;
(iii) there is an $s \in \mathcal{S}(j,\hat{J})$ and a $v \in C_{r,k}(J)$ with $k + c_1(TE^v,\Omega)(s) \leq c$, $i(\eta_1(v)) = s(z_0)$ and $i(\eta_2(v)) = y$.

(iii) is the case where bubbling occurs at $z_0$ in such a way that $y$ does not lie in the image of the principal component of the limiting cusp curve.

We will denote by $\mathcal{S}(j,\hat{J},S) \subset \mathcal{S}(j,\hat{J})$ the space of $(j,\hat{J})$-holomorphic sections of $E$ which lie in a given $\Gamma$-equivalence class $S$.

Lemma 7.6. For every $C \in \mathbb{R}$, there are only finitely many $\Gamma$-equivalence classes $S$ with $\bar{\Omega}(S) \leq C$ and $\mathcal{S}(j,\hat{J},S) \neq \emptyset$.

Proof. The stronger statements in which $\Gamma$-equivalence is replaced by homological equivalence or even homotopy are well-known (see [11, Corollary 4.4.4]) consequences of Gromov’s compactness theorem.

Proposition 7.7. If $\hat{J} \in \mathcal{J}^\text{reg}(j,J)$ for some $j$ and $J \in \mathcal{J}^\text{reg,z_0}(E,\Omega)$, the map

$$\text{ev}_{z_0} : \mathcal{S}(j,\hat{J},S) \rightarrow M, \quad \text{ev}_{z_0}(s) = i^{-1}(s(z_0))$$

is a pseudo-cycle (in the sense of [11, Section 7.1]) of dimension $d(S) = 2n + 2c_1(TE^v,\Omega)(S)$ for any $\Gamma$-equivalence class $S$.

Proof. Because $\hat{J}$ is regular, $\mathcal{S}(j,\hat{J},S)$ is a smooth manifold. Its dimension is given by the index of $D_J(s)$ for any $s \in \mathcal{S}(j,\hat{J},S)$, which is $d(S)$. It remains to show that $\text{ev}_{z_0}$ can be compactified by countably many images of manifolds of dimension $\leq d(S) - 2$. The compactification described in Lemma 7.5 has this property: in case (i), $\text{ev}_{z_0}$ lies in the image of

$$\text{ev}_{z_0} : \mathcal{S}(j,\hat{J},S') \rightarrow M$$

for some $S'$ with $c_1(TE^v,\Omega)(S') < c_1(TE^v,\Omega)(S)$. Clearly $\dim \mathcal{S}(j,\hat{J},S') \leq d(S) - 2$. Now, consider case (ii). Let $S_0$ be the space of $(j,\hat{J})$-holomorphic sections $s$ with $c_1(TE^v,\Omega)(s) = c_1(TE^v,\Omega)(S)$. The space of $(s,z,w)$ as in Lemma 7.5(ii) is the inverse image of the diagonal $\Delta \subset E \times E$ by the map

$$\text{ev} \times \eta : (E^2 \times S_0) \times (\mathcal{M}^0_0(J) \times \mathbb{CP}^1) \rightarrow E \times E.$$
\( \dim \mathcal{M}_0^\gamma(J) \times_{PSL(2, \mathbb{C})} \mathbb{C}P^1 = \dim E - 4 \), and \( S^2 \times S_0 \) is a manifold of dimension \( d(S) + 2 \). Since \( J \) is regular, \( \text{ev} \) is transverse to \( \eta \); it follows that \( (\text{ev} \times \eta)^{-1}(\Delta) \) has dimension \( d(S) - 2 \). Case (iii) is similar.

The following way of assigning a homology class to a pseudo-cycle is due to Schwarz [19].

**Lemma 7.8.** Let \( c : C \to M \) be a \( k \)-dimensional pseudo-cycle in \( M \) which is compactified by \( c_\infty : C_{k-2}^\infty \to M \). If \( h \) is a Riemannian metric on \( M \) and \( f \in C^\infty(M, \mathbb{R}) \) is a Morse function for which the Morse complex \((CM_*(f), \partial, c, c_\infty)\) is defined and such that stable manifold \( W^s(y; f, h) \) (for the negative gradient flow) of any critical point \( y \) of \( f \) is transverse to \( c, c_\infty \), the sum

\[
\sum_y \#c^{-1}(W^s(y; f, h)) <y>
\]

over all critical points \( y \) of Morse index \( k \) is a cycle in \( CM_*(f) \) (we use Morse complexes with \( \mathbb{Z}/2 \)-coefficients).

Such a pair \((f, h)\) always exists, and the homology class of the cycle in the ‘Morse homology’ [20] of \( M \) is independent of all choices. Since Morse homology is canonically isomorphic to singular homology [18], this defines a class in \( H_k(M; \mathbb{Z}/2) \) which we denote by \([c(C)]\). If \( C \) is compact, \([c(C)]\) is the fundamental class in the classical sense.

**Definition 7.9.** Let \((E, \Omega, S)\) be a normalized Hamiltonian fibre bundle over \( S^2 \) with fibre \((M, \omega)\). Let \( d = 2n + 2c_1(TE^1, \Omega)(S) \). We define

\[
Q(E, \Omega, S) = \sum_{\gamma \in \Gamma}[\text{ev}_{z_0}(S(j, \hat{J}, \gamma + S))] \otimes <\gamma> \in QH_d(M, \omega),
\]

where \( j \) is a positively oriented complex structure on \( S^2 \) and \( \hat{J} \in \mathcal{J}_{\text{reg}, 2d}(E, \Omega) \).

The notation \( \gamma + S \) was introduced in section 2. The formal sum defining \( Q(E, \Omega, S) \) lies in \( QH_*(M, \omega) \) because of Lemma 7.6. It is of degree \( d \) because the dimension of \( S(j, \hat{J}, \gamma + S) \) is \( d + 2c_1(\gamma) \) and \(<\gamma>\) has degree \(-2c_1(\gamma)\).

**Proposition 7.10.** \( Q(E, \Omega, S) \) is independent of the choice of \( j, J \) and \( \hat{J} \).

We omit the proof. It is based on the fact that cobordant pseudo-cycles determine the same homology class, and is similar to [11, Proposition 7.2.1]. Sometimes it is convenient to define \( QH_*(M, \omega) \) in terms of Morse homology as the homology of the graded tensor product \((CM_* \otimes \Lambda, \partial(f, h) \otimes \text{Id})\). An element of \( CM_* \otimes \Lambda \) is a (possibly infinite) linear combination of \(<y> \otimes <\gamma>\) for \( y \in \text{Crit}(f), \gamma \in \Gamma \). Assume that \((f, h)\) satisfies the conditions of Lemma 7.8 with respect to the pseudo-cycles \( \text{ev}_{z_0} : S(j, \hat{J}, S') \to M \) for all
$S'$, and define

$$S(j, \hat{J}, S', y) = \{ s \in S(j, \hat{J}, S') \mid \text{ev}_{20}(s) \in W^s(y; f, h) \}$$

for $y \in \text{Crit}(f)$. Then $Q(E, \Omega, S)$ is the homology class of the cycle

$$(7.2) \quad \sum_{y, \gamma} \#S(j, \hat{J}, \gamma + S, y)<y> \otimes <\gamma> \in CM_\ast(f) \otimes \Lambda,$$

where the sum is over all $(y, \gamma) \in \text{Crit}(f) \times \Gamma$ such that $i_f(y) = d + 2c_1(\gamma)$ ($i_f$ denotes the Morse index).

It is often difficult to decide whether a given $\hat{J}$ and its restriction $J = \hat{J}|TE^v$ are regular, since this depends on all holomorphic sections and the holomorphic curves in the fibres. Moreover, in many examples the most natural choice of $\hat{J}$ is not regular. However, almost complex structures which satisfy a weaker condition can be used to determine $Q(E, \Omega, S)$ partially: to compute $[\text{ev}_{20}(S(j, \hat{J}, \gamma + S))]$ for a single $\gamma$, it is sufficient that $S(j, \hat{J}, \gamma + S)$ and all its possible limits listed in Lemma 7.5 are regular. Since the contributions from those $\gamma$ with $d + 2c_1(\gamma) < 0$ or $d + 2c_1(\gamma) > 2n$ are zero, sometimes all of $Q(E, \Omega, S)$ can be computed using an almost complex structure which is not regular, as in the following case:

**Proposition 7.11.** Let $(E, J)$ be a compact complex manifold, $\pi : E \rightarrow \mathbb{CP}^1$ a holomorphic map with no critical points, $\bar{\Omega} \in \Omega^2(E)$ a closed form whose restrictions $\Omega_z = \bar{\Omega}|E_z$ are Kähler forms, and $i : (M, \omega) \rightarrow (E_{z_0}, \Omega_{z_0})$ an isomorphism for some $z_0 \in \mathbb{CP}^1$. Assume that

(i) the space $S$ of holomorphic sections $s$ of $\pi$ with $c_1(TE^v)(s) \leq 0$ is connected. In particular, all of them lie in a single $\Gamma$-equivalence class $S_0$.

(ii) For any $s \in S$, $H^{0,1}(\mathbb{CP}^1, s^*TE^v) = 0$.

(iii) Any holomorphic map $w \in C^\infty(\mathbb{CP}^1, E)$ such that $\text{im}(w) \subset E_z$ for some $z \in \mathbb{CP}^1$ satisfies $c_1(TE)(w) \geq 0$.

(iv) if $w$ is as before and not constant, and $c_1(TE)(w) + c_1(TE^v)(S_0) \leq 0$, then $s(z) \notin \text{im}(w)$ for all $s \in S$.

In that case, $S$ is a smooth compact manifold and

$$Q(E, \Omega, S_0) = (\text{ev}_{z_0})_*[S] \otimes <0>,$$

where $<0> \in \Lambda$ is the element corresponding to the trivial class in $\Gamma$.

8. A GLUING ARGUMENT

**Definition 8.1.** For $(g, \tilde{g}) \in \tilde{G}$, define

$$q(g, \tilde{g}) = Q(E_g, \Omega_g, S_{\tilde{g}}) \in QH_\ast(M, \omega).$$
By definition, \( Q(E_g, \Omega_g, S_\tilde{g}) \) is an element of degree \( 2n + 2c_1(TE_g^\omega, \Omega_g)(S_\tilde{g}) \).

Using (2.6) it follows that
\[
q(g, \tilde{g}) \in QH_{2n-2I(g, \tilde{g})}(M, \omega).
\]

Let \( e = [M] \otimes \langle 0 \rangle \) be the ‘fundamental class’ in \( QH_*(M, \omega) \). The connection between the invariant \( q \) and the \( G \)-action on Floer homology is given by

**Theorem 8.2.** For all \((g, \tilde{g}) \in \tilde{G},\)
\[
q(g, \tilde{g}) = \Psi^- HF_*(g, \tilde{g}) \Psi^+(e).
\]

Here
\[
\Psi^+: QH_*(M, \omega) \to HF_*(M, \omega), \quad \Psi^- : HF_*(M, \omega) \to QH_*(M, \omega)
\]
are the canonical homomorphisms of Piunikhin, Salamon and Schwarz [15]. They proved that \( \Psi^+, \Psi^- \) are isomorphisms and inverses of each other, and a part of their proof will serve as a model for the proof of Theorem 8.2. More precisely, we consider the proof of \( \text{Id}_{QH_*(M, \omega)} = \Psi^- \Psi^+ \) given in [15] and specialize it to the case
\[
e = \Psi^- \Psi^+(e).
\]

There are two steps in proving this: first, a gluing argument shows that \( \Psi^- \Psi^+(e) \) is equal to a certain Gromov-Witten invariant. Then one computes that the value of this invariant is again \( e \). The Gromov-Witten invariant which arises can be described in our terms as the \( Q \)-invariant of the trivial Hamiltonian fibre bundle \((S^2 \times M, \omega)\) with the \( \Gamma \)-equivalence class \( S_0 \) containing the constant sections. Therefore, what the gluing argument in [15] proves is
\[
(8.1) \quad Q(S^2 \times M, \omega, S_0) = \Psi^- \Psi^+(e).
\]

This is precisely the special case \((g, \tilde{g}) = \text{Id}_G\) of Theorem 8.2, since in that case \((E_g, \Omega_g, S_\tilde{g}) = (S^2 \times M, \omega, S_0)\) and \( HF_*(g, \tilde{g}) \) is the identity map. The analytical details of the proof of Theorem 8.2 are practically the same as in the special case (8.1); the only difference is that the gluing procedure is modified by ‘twisting’ with \( g \). Since a detailed account of the results of [15] based on the analysis in [21] is in preparation, we will only sketch the proof of Theorem 8.2.

The basic construction underlying the proof is familiar from section 2: a pair \((v^+, v^-) \in C^\infty(D^+, M) \times C^\infty(D^-, M)\) such that \( v^+|\partial D^+ = x \) and \( v^-|\partial D^- = g(x) \) for some \( x \in L \) gives rise to a section of \( E_g \). If \([v^-, g(x)] = \tilde{g}([v^+, x]) \in \tilde{LM}\), this section lies in the \( \Gamma \)-equivalence class \( S_\tilde{g} \). More generally, if \([v^-, g(x)] = \gamma \cdot \tilde{g}([v^+, x]) \) for some \( \gamma \in \Gamma \), the section lies in \((-\gamma) + S_\tilde{g}\). The sign \(-\gamma\) occurs for the following reason: to obtain the element \([v^-, g(x)] \in \tilde{LM}\), one uses a diffeomorphism from \( D^- \) to the standard disc \( D^2 \). Following Convention 2.7, this diffeomorphism should be orientation-reversing.
To adapt this construction to the framework of surfaces with tubular ends, we replace $D^+, D^-$ by $\Sigma^+ = D^+ \cup_{S^1} (\mathbb{R}^+ \times S^1)$ and $\Sigma^- = (\mathbb{R}^- \times S^1) \cup_{S^1} D^-$, with the orientations induced from the standard ones on $\mathbb{R}^\pm \times S^1$. A map $u \in C^\infty(\Sigma^+, M)$ such that $\lim_{s \to \infty} u(s, \cdot) = x$ for some $x \in LM$ can be extended continuously to the compactification $\Sigma^\infty = \Sigma^+ \cup \{\infty\} \times S^1$.

Using an orientation-preserving diffeomorphism $D^2 \to \Sigma^+$, this defines an element $c = [u, x] \in \hat{LM}$. We say that $u$ converges to $c$. There is a parallel notion for $u \in C^\infty(\Sigma^-, M)$, except that in this case the identification of $\Sigma^-$ with $D^2$ reverses the orientation.

**Definition 8.3.** For $\gamma \in \Gamma$, let $\mathcal{P}(\gamma)$ be the space of pairs $(u^+, u^-) \in C^\infty(\Sigma^+, M) \times C^\infty(\Sigma^-, M)$ such that $u^+$ converges to $c$ and $u^-$ to $(-\gamma) \cdot \tilde{g}(c)$ for some $c \in \hat{LM}$.

Fix a point $z_0 \in D^- \subset \Sigma^-$. We will denote the trivial symplectic fibre bundles $\Sigma^\pm \times (M, \omega)$ by $(E^\pm, \Omega^\pm)$, and the obvious isomorphism $(M, \omega) \to (E^-, \Omega^-)$ by $i$. For $R \geq 0$, consider $\Sigma_R^+ = D^+ \cup_{S^1} ([0; R] \times S^1) \subset \Sigma^+$, $\Sigma_R^- = ([R; \infty) \times S^1) \cup D^- \subset \Sigma^-$ and $\Sigma_R = \Sigma_R^+ \cup_{\partial \Sigma_R^+} \Sigma_R^-$, with the induced orientation. With respect to Riemannian metrics on $\Sigma^+, \Sigma^-$ whose restriction to $\mathbb{R}^\pm \times S^1$ is the standard tubular metric, $\Sigma_R$ is a surface with an increasingly long ‘neck’ as $R \to \infty$. Let $(E_R, \Omega_R)$ be the symplectic fibre bundle over $\Sigma_R$ obtained by gluing together $E^+_R = E^+|\Sigma_R^+$ and $E^-_R = E^-|\Sigma_R^-$ using $\phi_y : E^+|\partial \Sigma_R^+ \to E^-|\partial \Sigma_R^-$, $\phi_y(R, t, y) = (-R, t, g_y(y))$. The terminology introduced in section 2 applies to bundles over $\Sigma_R$ since it is homeomorphic to $S^2$. $(E_R, \Omega_R)$ is Hamiltonian; it differs from $(E_y, \Omega_y)$ only because the base is parametrized in a different way (more precisely, $(E_R, \Omega_R)$ is the pullback of $(E_y, \Omega_y)$ by an oriented diffeomorphism $\Sigma_R \to S^2$). Let $S_R$ be the $\Gamma$-equivalence class of sections of $E_R$ which corresponds to $S_y$.

For $(u^+, u^-) \in \mathcal{P}(\gamma)$, one can construct a section $u^+ \#_R u^-$ of $E_R$ for large $R$ by ‘approximate gluing’: let $x = \lim_{s \to \infty} u^+(s, \cdot) \in LM$. There is an $R_0 \geq 0$ and a family $(\tilde{u}^+_R, \tilde{u}^-_R)_{R \geq R_0}$ in $\mathcal{P}(\gamma)$ which converges uniformly to $(u^+, u^-)$ as $R \to \infty$ and such that $\tilde{u}^+_R(s, t) = x(t)$ for $s \geq R$, $\tilde{u}^-_R(s, t) = g_y(x(t))$ for $s \leq -R$. Define

$$(u^+ \#_R u^-)(z) = \begin{cases} (z, \tilde{u}^+_R(z)) & z \in \Sigma^+_R, \\ (z, \tilde{u}^-_R(z)) & z \in \Sigma^-_R. \end{cases}$$

If $R$ is large, $u^+ \#_R u^-$ lies in the $\Gamma$-equivalence class $\gamma + \Sigma_R$.

We will now introduce nonlinear $\bar{\partial}$-equations on $\Sigma^+, \Sigma^-$ and state the gluing theorem for solutions of them. Like the pair-of-pants product, these equations are part of the formalism of ‘relative Donaldson type invariants’ of [15].

Let $j^+$ be a complex structure on $\Sigma^+$ whose restriction to $\mathbb{R}^+ \times S^1$ is the standard complex structure. Choose a regular pair $(H^\infty, J^\infty)$ and $H^+ \in$
Let \( H^+ \) be a Riemannian metric on \( M \) and \( f \in C^\infty(M, \mathbb{R}) \) a Morse function. If \( c \) is a critical point of \( a_{H^{-\infty}} \) and \( y \) a critical point of \( f \), we denote by \( \mathcal{M}^{-}(c, y; H^{-\infty}, J^{-}) \) the space of solutions \( u \) of (8.3) which converge to \( c \) and with \( u(z_0) \in W^s(y; f, h) \). In the generic case, \( \mathcal{M}^{-}(c, y; H^{-\infty}, J^{-}) \) is a manifold of dimension \( \mu_{H^{-\infty}}(c) - 1 \), and the zero-dimensional spaces are again finite. For fixed \( (H^{\infty}, J^{\infty}) \) and \( (H^{-\infty}, J^{-\infty}) \), we will denote the space of all \( (H^{\infty}, J^{\infty}, H^{-\infty}, J^{-}, f, h) \) by \( \mathcal{C}(H^{\infty}, J^{\infty}, H^{-\infty}, J^{-\infty}) \).

(8.2) and (8.3) can be written in a different way using an idea of Gromov. For \((z, y) \in \Sigma^+ \times M\), let \( \nu^+(z, y) : T_z \Sigma^+ \longrightarrow T_y M \) be the \((j^+, J^+_z)\)-antilinear homomorphism given by

\[
\nu^+(z, y) = \begin{cases} 
\frac{ds \otimes J^+_z X_{H^+}(s, t, y) + dt \otimes X_{H^+}(s, t, y)}{\text{for } z = (s, t, y)} & \text{for } z \in D^+.
\end{cases}
\]

\( u \) is a solution of (8.2) iff

\[
(8.4) \quad du(z) + J^+_z \circ du(z) \circ j^+ = \nu^+(z, u(z)).
\]

One can think of \( J^+ = (J^+_z)_{z \in \Sigma^+} \) as a family of almost complex structures on the fibres of the trivial bundle \( E^+ \). Consider the almost complex structure

\[
\hat{J}^+(z, Y) = (j^+ Z, J^+_z Y + \nu^+(z, y)(j^+ Z))
\]

on \( E^+ \). \( \hat{J}^+ \{ z \} \times M = J^+_z \) for all \( z \in \Sigma^+ \), and the projection \( E^+ \longrightarrow \Sigma^+ \) is \((\hat{J}^+, J^+\text{-})\)-linear. A straightforward computation shows that \( u \) is a solution of (8.4) iff the section \( s(z) = (z, u(z)) \) of \( E^+ \) is \((\hat{J}^+, J^+\text{-})\)-holomorphic. There is an almost complex structure \( \hat{J}^- \) on \( E^- \) such that solutions of (8.3) correspond to \((j^-, \hat{J}^-)\)-holomorphic sections in the same way.
From now on, we will assume that \((H^\infty, J^\infty)\) is the pullback of \((H^{-\infty}, J^{-\infty})\) by \(g\). As a consequence, \((J^+_z)_{z \in \Sigma^+_R}\) and \((J^-_z)_{z \in \Sigma^-_R}\) can be pieced together to a smooth family \(J_R = (J_{R,z})_{z \in \Sigma_R} \in \mathcal{J}(E_R, \Omega_R)\) for any \(R \geq 2\). \(j^+|\Sigma^+_R\) and \(j^-|\Sigma^-_R\) determine a complex structure \(j_R\) on \(\Sigma_R\), and there is a unique \(J_R \in \mathcal{J}(j_R, J_R)\) such that \(J_R|E^+_R = J^+_R|E^+_R\). As in the previous section, we denote the space of \((J_R, \hat{J}_R)\)-holomorphic sections of \(E_R\) in a \(\Gamma\)-equivalence class \(S\) by \(S(j_R, \hat{J}_R, S)\), and by \(S(j_R, \hat{J}_R, S, y)\) for \(y \in \text{Crit}(f)\) the subspace of those \(s\) such that \(i^{-1}(s(z_0)) \in W^s(y; f, h)\).

For \((\gamma, y) \in \Gamma \times \text{Crit}(f)\) with

\[(8.5) \quad i_f(y) = 2n - 2I(g, \tilde{g}) + 2c_1(\gamma),\]

let \(S'(H^+, J^+, H^-, J^-, \gamma, y)\) be the disjoint union of

\[(8.6) \quad \mathcal{M}^+(c; H^+, J^+) \times \mathcal{M}^-(\langle -\gamma \rangle \cdot \tilde{g}(c), y; H^-, J^-)\]

for all \(c \in \text{Crit}(a_{H^\infty})\). Equivalently, \(S'(H^+, J^+, H^-, J^-, \gamma, y)\) is the space of \((u^+, u^-) \in \mathcal{P}(\gamma)\) such that \(s^+(z) = (z, u^+(z))\) is \((j^+, J^+)\)-holomorphic, \(s^-(z) = (z, u^-(z))\) is \((j^-, J^-)\)-holomorphic and \(i^{-1}(s(z_0)) \in W^s(y; f, h)\).

For generic \((H^+, J^+)\) and \((H^-, J^-)\), \(\dim \mathcal{M}^+(c; H^+, J^+) = 2n - \mu_{H^\infty}(c)\) and \(\dim \mathcal{M}^-(\langle -\gamma \rangle \cdot \tilde{g}(c), y; H^-, J^-) = \mu_{H^{-\infty}}(\langle -\gamma \rangle \cdot \tilde{g}(c)) - i_f(y) = \mu_{H^\infty}(c) - 2n\) (the last equality uses (8.5), (3.4) and Lemma 4.4). Therefore the product \((8.6)\) is zero-dimensional if \(\mu_{H^\infty}(c) = 2n\), and is empty otherwise. A compactness theorem shows that \(S'(H^+, J^+, H^-, J^-, \gamma, y)\) is a finite set.

We can now state the ‘gluing theorem’ which is the main step in the proof of Theorem 8.2.

**Theorem 8.4.** There is a subset of \(\mathcal{C}(H^\infty, J^\infty, H^{-\infty}, J^{-\infty})\) of second category such that if \((H^+, J^+, H^-, J^-, f, h)\) lies in this subset, the following holds: for \((\gamma, y) \in \Gamma \times \text{Crit}(f)\) satisfying (8.5), there is an \(R_0 > 2\) and a family of bijective maps

\[\#_R : S'(H^+, J^+, H^-, J^-, \gamma, y) \rightarrow S(j_R, \hat{J}_R, \gamma + S_R, y)\]

for \(R \geq R_0\).

The construction of \(\#_R\) relies on the following property of the ‘approximate gluing’ \(\#_R\) for \((u^+, u^-) \in S'(H^+, J^+, H^-, J^-, \gamma, y)\): because \(\hat{u}^+_R\) converges uniformly to \(u^+\) as \(R \to \infty\), it approximately satisfies (8.2) for large \(R\). Therefore \(\hat{s}^+_R(z) = (z, \hat{u}^+_R(z))\) is an approximately \((j^+, J^+)\)-holomorphic section of \(E^+\). Using the same argument for \(u^-\), it follows that \(\hat{s}_R = u^+_R \#_R u^-\) is an approximately \((j_R, \hat{J}_R)\)-holomorphic section of \(E_R\). More precisely,

\[\|d\hat{s}_R + \hat{J}_R \circ d\hat{s}_R \circ j_R\|_p \to 0 \quad \text{as} \quad R \to \infty.\]

Here \(\| \cdot \|_p \) is the \(L^p\)-norm on \(\Sigma_R\) with respect to a family of metrics with an increasingly long ‘neck’ as described above.
The explicit function theorem in the space of \( i \) \# \( R \). By construction, \( \# R \) is injective for large \( R \). Surjectivity is proved by a limiting argument (‘stretching the neck’). Let \( s \) be a sequence of sections with \( R_m \to \infty \), such that \( s_m|\Sigma_{R_m} \to u \) on compact subsets to sections \( s^+(z) = (z, u^+(z)) \) of \( E^+ \) and \( s^-(z) = (z, u^-(z)) \) of \( E^- \). Then \( s^\pm \) is \((j^+, J^-)\)-holomorphic. If the pair \((s^+, s^-)\) describes the ‘geometric limit’ of the sequence \( s_m \) completely, \((u^+, u^-) \in S'(H^+, J^+, H^-, J^-, \gamma, y) \) and \( s_m = u^+ \#_R u^- \) for large \( m \). Transversality and dimension arguments are used to exclude more complicated limiting behaviour.

It remains to explain how Theorem 8.2 is derived from the technical Theorem 8.4. The main point is that \( \Psi^+, \Psi^\pm \) are the ‘relative Donaldson type invariants’ associated to \( \Sigma^+, \Sigma^- \). The element \( \Psi^+(e) \in HF_{2n}(M, \omega) \) has a particularly simple description in terms of the moduli spaces \( \mathcal{M}^+ \): it is the homology class of the cycle

\[
\Psi^+(e; H^+, J^+) = \sum_{c \in Crit_2(n, a_{H^\infty})} \# \mathcal{M}^+(c; H^+, J^+) < c > \in CF_{2n}(H^\infty)
\]

for generic \((H^+, J^+)\). It follows that \( HF_*(g, \tilde{g}) \Psi^+(e) \in HF_{2n-2f(g, \tilde{g})}(M, \omega) \) is represented by

\[
\sum_c \# \mathcal{M}^+(c; H^+, J^+) < \tilde{g}(c) > \in CF_{2n-2f(g, \tilde{g})}(H^-\infty).
\]

To define \( \Psi^- \), we consider \( Q HF_*(M, \omega) \) as the homology of \((CM_*(f) \otimes \Lambda, \partial(f, h) \otimes Id)\). For generic \((H^-, J^-)\), the formula

\[
\Psi^-(H^-, J^-, f, h)(< c >) = \sum_{\gamma, y} \# \mathcal{M}^-(\gamma \cdot c, y; H^-, J^-) < y > \otimes < -\gamma >,
\]

where the sum is over all \((\gamma, y) \in \Gamma \times Crit(f)\) such that \( i_f(y) = \mu_{H^-\infty}(c) - 2c_1(\gamma) \), defines a \( \Lambda \)-linear homomorphism of chain complexes

\[
\Psi^-(H^-, J^-, f, h) : CF_*(H^-\infty) \to CM_*(f) \otimes \Lambda.
\]

\( \Psi^- \) is defined as the induced map on homology groups. Here, as in the case of \( \Psi^+ \), we have omitted several steps in the construction; we refer to [15] for a more complete description. As usual, our setup differs slightly from that in [15] because the almost complex structures may be \( z \)-dependent; however, the proofs can be easily adapted, provided always that \((W^+)\) is satisfied.
By comparing the expressions for $HF_*(g, \tilde{g})\Psi^+(e)$ and $\Psi^-$ with the definition of $\mathcal{S}'(H^+, J^+, H^-, J^-, \gamma, y)$, one obtains

$$\Psi^- HF_*(g, \tilde{g})\Psi^+(e) = \sum_{\gamma \in \Gamma} [c_\gamma] \otimes <\gamma>, $$

where $[c_\gamma] \in H_*(M; \mathbb{Z}/2)$ is the homology class of the cycle

$$c_\gamma = \sum_y \#\mathcal{S}'(H^+, J^+, H^-, J^-, \gamma, y)<y> \in CM_*(f);$$

the sum runs over all $y \in \text{Crit}(f)$ with $i_f(y) = 2n - 2I(g, \tilde{g}) + 2c_1(\gamma)$.

Clearly, $Q(E_R, \Omega_R, S_R) = Q(E_{\tilde{g}}, \Omega_g, S_{\tilde{g}})$ for all $R \geq 0$. To obtain an explicit expression for $Q(E_R, \Omega_R, S_R)$, choose $J'_R \in J_{\text{reg}, \Omega_R}(E_R, \Omega_R)$, $\tilde{J}_R \in \tilde{J}_{\text{reg}}(j_R, J'_R)$, a Riemannian metric $h'$ and a Morse function $f'$, all of which satisfy appropriate regularity conditions. By (7.2),

$$Q(E_R, \Omega_R, S_R) = \sum_{\gamma \in \Gamma} [c_{\gamma, R}] \otimes <\gamma>, $$

where $c_{\gamma, R} \in CM_*(f)$ is given by

$$c_{\gamma, R} = \sum_{y'} \#\mathcal{S}(j_R, \tilde{J}_R, \gamma + S_R, y')<y'>.$$

This time the sum is over those $y' \in \text{Crit}(f')$ such that $i_{f'}(y') = 2n - 2I(\tilde{g}, \tilde{g}) + 2c_1(\gamma)$, but by equation (2.6) the Morse indices are the same as above.

The statement of Theorem 8.2 is that $[c_{\gamma, R}] = [c_\gamma]$ for all $\gamma$. Since $[c_{\gamma, R}]$ is independent of $R$, it is sufficient to show that for each $\gamma$ there is an $R$ such that for a suitable choice of $J'_R, \tilde{J}_R, f', h', [c_{\gamma, R}] = [c_\gamma]$. Choose some $\gamma_0 \in \Gamma$. We can assume that $H^+, J^+, H^-, J^-$, $f, h$ have been chosen as in Theorem 8.4. Since $f$ has only finitely many critical points, there is an $R$ such that Theorem 8.4 holds for all $(\gamma_0, y)$. Then

$$c_{\gamma_0} = \sum_y \#\mathcal{S}(j_R, \tilde{J}_R, \gamma_0 + S_R, y)<y>.$$

and it follows that $c_{\gamma_0} = c_{\gamma_0, R}$ if we take $J'_R = J_R$, $\tilde{J}_R = \tilde{J}_R$ and $(f', h') = (f, h)$. Note that $J_R \in J(E_R, \Omega_R)$ and $\tilde{J}_R \in \tilde{J}(j_R, J_R)$ are not ‘generic’ choices because they are $s$-independent on the ‘neck’ of $\Sigma_R$. However, a final consideration shows that for large $R$, they can be used to compute the coefficient $[c_{\gamma_0, R}]$ of $Q(E_R, \Omega_R, S_R)$.

9. PROOF OF THE MAIN RESULTS

For $a_1, a_2, a_3 \in H_*(M; \mathbb{Z}/2)$ and $A \in H_3(M; \mathbb{Z})$, let $\Phi_{(A, \omega)}(a_1, a_2, a_3) \in \mathbb{Z}/2$ be the mod 2 reduction of the Gromov-Witten invariant of [16, Section 8].
It is zero unless
\begin{equation}
\dim(a_1) + \dim(a_2) + \dim(a_3) + 2c_1(A) = 4n.
\end{equation}

Intuitively, \( \tilde{\Phi}_{(A,\omega)}(a_1, a_2, a_3) \) is the number (modulo 2) of \( J \)-holomorphic spheres in the class \( A \) which meet suitable cycles representing \( a_1, a_2 \) and \( a_3 \).

Let \( a_1 *_A a_2 \in H_*(M; \mathbb{Z}/2) \) be the class defined by
\[
(a_1 *_A a_2) \cdot a_3 = \tilde{\Phi}_{(A,\omega)}(a_1, a_2, a_3)
\]
for all \( a_3 \), where \( \cdot \) is the ordinary intersection product. For \( \gamma \in \Gamma \), let \( a_1 *_{\gamma} a_2 \in H_*(M; \mathbb{Z}/2) \) be the sum of \( a_1 *_A a_2 \) over all classes \( A \) which can be represented by a smooth map \( w : S^2 \to M \) with \( [w] = \gamma \) (only finitely many terms of this sum are nonzero). The quantum intersection product \( * \) on \( QH_*(M,\omega) \) is defined by the formula
\[
(a_1 \otimes \langle \gamma_1 \rangle) * (a_2 \otimes \langle \gamma_2 \rangle) = \sum_{\gamma \in \Gamma} (a_1 *_{\gamma} a_2) \otimes \langle \gamma_1 + \gamma_2 + \gamma \rangle,
\]
extended to infinite linear combinations in the obvious way. \( * \) is a bilinear \( \Lambda \)-module map with the following properties:

(i) if \( a_1 \in QH_i(M,\omega) \) and \( a_2 \in QH_j(M,\omega) \), then \( a_1 *_A a_2 \in QH_{i+j-2n}(M,\omega) \).
(ii) \( \cdot \) is associative.
(iii) \( e = [M] \otimes \langle 0 \rangle \in QH_{2n}(M,\omega) \) is the unit of \( * \).
(iv) \( \Psi^+(a_1 * a_2) = \Psi^+(a_1) \ast_{PP} \Psi^+(a_2) \in HF_*(M,\omega) \).

(i) follows from (9.1), (ii) is due to Ruan and Tian [16]; another proof is given in [11]. (iii) is [11, Proposition 8.1.4(iii)] and (iv) is the main result of [15]. From the two last items and the fact that \( \Psi^+ \) is an isomorphism, one obtains

**Lemma 9.1.** \( u = \Psi^+(e) \) is the unit of \( (HF_*(M,\omega), \ast_{PP}) \). \( \square \)

**Proof of Theorem 1.** By Theorem 8.2, \( q(g, \tilde{g}) = \Psi^+HF_*(g, \tilde{g})(u) \). Because \( \Psi^+ \) is the inverse of \( \Psi^+ \), this can be written as \( \Psi^+(q(g, \tilde{g})) = HF_*(g, \tilde{g})(u) \). From Lemma 9.1 and Proposition 6.3, it follows that
\[
HF_*(g, \tilde{g})(b) = HF_*(g, \tilde{g})(u \ast_{PP} b) = HF_*(g, \tilde{g})(u) \ast_{PP} b = \Psi^+(q(g, \tilde{g})) \ast_{PP} b \.
\]
\( \square \)

**Proof of Proposition 4.** Proposition 4.9(iii) says that \( HF_*(Id, \gamma) \) is given by multiplication with \( \langle \gamma \rangle \in \Lambda \). Using Theorem 8.2, \( \Psi^- \Psi^+ = Id \) and the fact that \( \Psi^- \) is \( \Lambda \)-linear, we can compute
\[
q(Id, \gamma) = \Psi^-HF_*(Id, \gamma)(u) = \Psi^-(\langle \gamma \rangle u) = \langle \gamma \rangle \Psi^-(u) = \langle \gamma \rangle ([M] \otimes \langle 0 \rangle) = [M] \otimes \langle \gamma \rangle.
\]
Proof of Corollary 3. Essentially, this is a consequence of the fact that the maps $HF_*(g, \tilde{g})$ define a $\tilde{G}$-action on Floer homology (Proposition 4.9(iv)). Take $(g_1, \tilde{g}_1), (g_2, \tilde{g}_2) \in \tilde{G}$. Using Theorem 8.2 twice, we obtain

$$q(g_1 g_2, \tilde{g}_1 \tilde{g}_2) = \Psi^-(HF_*(g_1 g_2, \tilde{g}_1 \tilde{g}_2))(u)$$
$$= \Psi^-(HF_*(g_1, \tilde{g}_1)HF_*(g_2, \tilde{g}_2))(u)$$
$$= \Psi^-(HF_*(g_1, \tilde{g}_1)\Psi^+(q(g_2, \tilde{g}_2))).$$

Theorem 1 with $(g, \tilde{g}) = (g_1, \tilde{g}_1)$ and $b = \Psi^+(q(g_2, \tilde{g}_2))$ says that

$$HF_*(g_1, \tilde{g}_1)\Psi^+(q(g_2, \tilde{g}_2)) = \Psi^+(q(g_1, \tilde{g}_1)) * \Psi^+(q(g_2, \tilde{g}_2)).$$

Using property (iv) above, we conclude that

$$q(g_1 g_2, \tilde{g}_1 \tilde{g}_2) = q(g_1, \tilde{g}_1) * q(g_2, \tilde{g}_2).$$

As a special case of Proposition 4, $q(Id_{\tilde{G}}) = e$. The fact that $q(g, \tilde{g})$ depends on $(g, \tilde{g})$ only up to homotopy is a consequence of the homotopy invariance of $HF_*(g, \tilde{g})$ (Proposition 5.1) and Theorem 8.2.

Proof of Corollary 2. Because of Lemma 2.13, this is an immediate consequence of Corollary 3.

10. AN APPLICATION TO THE MASLAV INDEX

Recall that $\tilde{I} : \pi_1(\text{Ham}(M, \omega)) \longrightarrow \mathbb{Z}/N\mathbb{Z}$ ($N$ denotes the minimal Chern number) is defined by

$$\tilde{I}(g) = I(g, \tilde{g}) \mod N,$$

where $\tilde{g} : \tilde{LM} \longrightarrow \tilde{LM}$ is any lift of $g \in G$. If $M$ is simply-connected, the definition of $\tilde{I}$ is elementary; in general, it uses Lemma 2.2, which is based on the Arnol’d conjecture. Our first result is a simple consequence of the existence of the automorphisms $HF_*(g, \tilde{g})$.

Proposition 10.1. If $(M, \omega)$ satisfies $c_1|\pi_2(M) = 0$, $\tilde{I}$ is the trivial homomorphism.

Proof. In this case, $\tilde{I}$ is $\mathbb{Z}$-valued because $I(g, \tilde{g}) \in \mathbb{Z}$ is independent of the choice of $\tilde{g}$. Since $\tilde{I}$ is a homomorphism, it is sufficient to show that $\tilde{I}(g) \leq 0$ for all $g$. The assumption on $c_1$ also implies that the grading of the Novikov ring $\Lambda$ is trivial. It follows from Theorem 3.11 that $HF_0(M, \omega) \cong H_0(M; \mathbb{Z}/2) \otimes \Lambda \cong \Lambda$ and $HF_k(M, \omega) = 0$ for $k < 0$. By Proposition 4.4, $HF_*(g, \tilde{g})$ maps $HF_0(M, \omega)$ isomorphically to $HF_{-2I(g, \tilde{g})}(M, \omega)$, which is clearly impossible if $I(g, \tilde{g}) > 0$.

To obtain more general results, it is necessary to use the multiplicative structure. Consider

$$Q^+ = \bigoplus_{i < 2n} H_i(M; \mathbb{Z}/2) \otimes \Lambda \subset QH_*(M, \omega).$$
Since the degree of any element of $\Lambda$ is a multiple of $2N$, $QH_k(M,\omega) \subset Q^+$ for any $k$ such that $k \not\equiv 2n \mod 2N$.

**Lemma 10.2.** If $Q^+ \ast Q^+ \subset Q^+$, every homogeneous invertible element of $QH_*(M,\omega)$ has degree $2n + 2iN$ for some $i \in \mathbb{Z}$.

**Proof.** Assume that $x \in QH_k(M,\omega)$ is invertible and $k \not\equiv 2n \mod 2N$; then $x \in Q^+$. Since $\ast$ has degree $-2n$ and the unit $e$ lies in $QH_{2n}(M,\omega)$, the inverse $x^{-1}$ has degree $4n - k$. Clearly $4n - k \not\equiv 2n \mod 2N$, and therefore $x^{-1} \in Q^+$, $x^{-1} \ast e = e \notin Q^+$, contrary to the assumption that $Q^+ \ast Q^+ \subset Q^+$.

**Proposition 10.3.** If $(M,\omega)$ satisfies (W+) and $Q^+ \ast Q^+ \subset Q^+$, $\bar{I}$ is trivial.

**Proof.** For all $(g,\tilde{g}) \in \tilde{G}$, $q(g,\tilde{g}) \in QH_*(M,\omega)$ is invertible by Corollary 2. $q(g,\tilde{g})$ has degree $2n - 2I(g,\tilde{g})$. Using Lemma 10.2 it follows that $I(g,\tilde{g}) \equiv 0 \mod N$, and therefore $\bar{I}(g) = 0$.

By definition, $Q^+ \ast Q^+ \subset Q^+$ iff $\tilde{\Phi}_{(A,\omega)}(x_1, x_2, [pt]) = 0$ for all $A \in H_2(M;\mathbb{Z})$ and all $x_1, x_2 \in H_*(M;\mathbb{Z}/2)$ of dimension $<2n$. This is certainly true if there is an $\omega$-tame almost complex structure $J$ on $M$ and a point $y \in M$ such that no non-constant $J$-holomorphic sphere passes through $y$. A particularly simple case is when there are no non-constant $J$-holomorphic spheres at all. Then the quantum intersection product reduces to the ordinary one, that is,

$$\begin{equation}
(x_1 \otimes \langle \gamma_1 \rangle) \ast (x_2 \otimes \langle \gamma_2 \rangle) = (x_1 \cdot x_2) \otimes \langle \gamma_1 + \gamma_2 \rangle.
\end{equation}
$$

For example, if

$$\begin{equation}
(c_1 - \lambda[\omega])\pi_2(M) = 0 \text{ for some } \lambda < 0,
\end{equation}
$$
c$1(w) < 0$ for any non-constant pseudoholomorphic curve $w$. If in addition $N \geq n - 2$, there is a dense set of $J$ such that any $J$-holomorphic sphere satisfies $c_1(w) \geq 0$ and hence must be constant.

**Corollary 10.4.** If $(M^{2n},\omega)$ satisfies (10.2) and its minimal Chern number is at least $n - 1$, the homomorphism $\bar{I}$ is trivial.

Note that we have sharpened the condition on $N$ in order to fulfill (W+). By a slightly different reasoning, it seems likely that Corollary 10.4 remains true without any assumption on $N$; this is one of the motivations for extending the theory beyond the case where (W+) holds.

As a final example, consider the case where $M$ is four-dimensional. (W+) holds for all symplectic four-manifolds. For $N = 1$, $\bar{I}$ is vacuous. Assume that $N \geq 2$ (this implies that $(M,\omega)$ is minimal). In that case, it is a result of McDuff that (10.1) holds unless $(M,\omega)$ is rational or ruled. For convenience, we reproduce the proof from [13]: for generic $J$, there are no non-constant $J$-holomorphic spheres $w$ with $c_1(w) \leq 0$ because the moduli space of such curves has negative dimension. Therefore, if (10.1) does not
hold, there is a \( w \) with \( c_1(w) \geq 2 \). After perturbing \( J \), we can assume that \( w \) is immersed with transverse self-intersections [9, Proposition 1.2]. Then \((M,\omega)\) is rational or ruled by [10, Theorem 1.4].

**Corollary 10.5.** \( \tilde{T} \) is trivial for all \((M^4,\omega)\) which are not rational or ruled. \( \square \)

11. Examples

In the first example, \( M \) is the Grassmannian \( \text{Gr}_k(\mathbb{C}^m) \) with the usual symplectic structure \( \omega \) coming from the Plücker embedding. \((M,\omega)\) is a monotone symplectic manifold. The \( U(m) \)-action on \( \mathbb{C}^m \) induces a Hamiltonian \( PU(m) \)-action \( \rho \) on \( M \). Let \( g \) be the loop in \( \text{Ham}(M,\omega) \) given by \( g_t = \rho(\text{diag}(e^{2\pi it},1,\ldots,1)) \). Obviously, \([g] \in \pi_1(\text{Ham}(M,\omega))\) lies in the image of \( \rho_* : \pi_1(PU(m)) \to \pi_1(\text{Ham}(M,\omega)) \), and since \( \pi_1(PU(m)) \cong \mathbb{Z}/m \), \([g]^m = 1\).

Let \( H \) be the Hopf bundle over \( \mathbb{C}P^1 \). The fibre bundle \( E = E_g \) is the bundle of Grassmannians associated to the holomorphic vector bundle \( H^{-1} \oplus \mathbb{C}^{m-1} \):

\[
E = \text{Gr}_k(H^{-1} \oplus \mathbb{C}^{m-1}) \xrightarrow{\pi} \mathbb{C}P^1.
\]

\( H^{-1} \oplus \mathbb{C}^{m-1} \) is a subbundle of the trivial bundle \( \mathbb{C}P^1 \times \mathbb{C}^{m-1} \). This induces a holomorphic map \( f : E \to \text{Gr}_k(\mathbb{C}^{m+1}) \) which is an embedding on each fibre \( E_z \). Let \( \Omega_z \in \Omega^2(E_z) \) be the pullback of the symplectic form on \( \text{Gr}_k(\mathbb{C}^{m+1}) \) by \( f|E_z \). \((E,\Omega)\) is a Hamiltonian fibre bundle.

As for any bundle of Grassmannians associated to a vector bundle, there is a canonical \( k \)-plane bundle \( P_E \to E \). \( P_E \) is a subbundle of \( \pi^*(H^{-1} \oplus \mathbb{C}^{m-1}) \) and

\[
(11.1) \quad TE_v \cong \text{Hom}(P_E,\pi^*(H^{-1} \oplus \mathbb{C}^{m-1})/P_E).
\]

Therefore \( c_1(TE_v) = -k c_1(\pi^*H) - m c_1(P_E) \). \( P_E \) is isomorphic to \( f^*P \), where \( P \) is the canonical \( k \)-plane bundle on \( \text{Gr}_k(\mathbb{C}^{m+1}) \). It follows that

\[
(11.2) \quad \deg(s^*TE_v) = -k - m \deg(s^*f^*P)
\]

for any section \( s \) of \( E \). If \( s \) is holomorphic, \( f(s) \) is a holomorphic curve in \( \text{Gr}_k(\mathbb{C}^{m+1}) \), and since \( c_1(P) \in H^2(\text{Gr}_k(\mathbb{C}^{m+1})) \) is a negative multiple of the symplectic class, either \( \deg(s^*f^*P) \leq -1 \) or \( f(s) \) is constant. In the first case, \( \deg(s^*TE_v) > 0 \) by (11.2). In the second case, \( s \) must be one of the ‘constant’ sections

\[
s_W(z) = 0 \oplus W \in \text{Gr}_k(H_z \oplus \mathbb{C}^{m-1})
\]

for \( W \in \text{Gr}_k(\mathbb{C}^{m-1}) \). We now check that \((E,\Omega)\) satisfies the conditions of Proposition 7.11.

(i) We have shown that any holomorphic section with \( \deg(s^*TE_v) \leq 0 \) is a ‘constant’ one. The space \( S \) of such sections is certainly connected. Their \( \Gamma \)-equivalence class \( S_0 \) satisfies \( c_1(TE_v)(S_0) = -k \).
(ii) $s_\Omega^* TE^\omega = \text{Hom}(W, H^{-1} \oplus \mathbb{C}^{m-1}/W) \cong \text{Hom}(\mathbb{C}^k, H^{-1} \oplus \mathbb{C}^{m-k-1})$ is a sum of line bundles of degree 0 or $-1$, hence $H^{0,1}(\mathbb{C}P^1, s_\Omega^* TE^\omega) = 0$.

(iii) and (iv) follow from the fact that $(M, \omega)$ is monotone with minimal Chern number $N = m > k$.

$(E_{z_0}, \Omega_{z_0})$ is identified with $(M, \omega)$ by choosing an element of $H^{-1}_{z_0}$ which has length 1 for the standard Hermitian metric on $H^{-1}$. The evaluation map $ev_{z_0} : S \to M$ is an embedding whose image is $\text{Gr}_k(\mathbb{C}^{m-1}) \subset \text{Gr}_k(\mathbb{C}^m)$.

By Proposition 7.11, $Q(E, \Omega, S_0) = [\text{Gr}_k(\mathbb{C}^{m-1})] \otimes <0>$. Using the diagram (1.4), we can derive from $[g]^m = 1$ that

$$(\text{Gr}_k(\mathbb{C}^{m-1})] \otimes <0>)^m = [M] \otimes <\gamma>$$

for some $\gamma \in \Gamma$. Because of the grading, $\gamma$ must be $k$ times the standard generator of $\pi_2(M) = H_2(M ; \mathbb{Z})$. (11.3) can be verified by a direct computation, since $(QH_*(M, \omega), *)$ is known [22] [26].

Our second example concerns the rational ruled surface $M = \mathbb{F}_2$. Recall that $\mathbb{F}_r$ $(r > 0)$ is the total space of the holomorphic fibre bundle $p_r : \mathbb{P}(\mathbb{C} \oplus H^r) \to \mathbb{C}P^1$. We will use two standard facts about $\mathbb{F}_r$.

**Lemma 11.1.** For any holomorphic section $s$ of $p_r$, $\text{deg}(s^* T\mathbb{F}_r) \geq -r$. There is a unique section such that equality holds, and all others have $\text{deg}(s^* T\mathbb{F}_r) \geq -r + 2$.

**Lemma 11.2.** For every holomorphic map $w : \mathbb{C}P^1 \to \mathbb{F}_2$, $c_1(w) \geq 0$. All non-constant $w$ with $c_1(w) < 2$ are of the form $w = s_\omega \circ u$, where $u : \mathbb{C}P^1 \to \mathbb{C}P^1$ is holomorphic and $s_\omega$ is the section with $\text{deg}(s_\omega^* T\mathbb{F}_2) = -2$.

Lemma 11.1 is a special case of a theorem on irreducible curves in $\mathbb{F}_r$ [3, Proposition IV.1]. Lemma 11.2 can be derived from Lemma 11.1 by considering a map $w : \mathbb{C}P^1 \to \mathbb{F}_2$ such that $p_2w$ is not constant as a section of the pullback $(p_2w)^* T\mathbb{F}_2$.

Since $\mathbb{P}(\mathbb{C} \oplus H^2) \cong \mathbb{P}(H^{-2} \oplus \mathbb{C})$, an embedding of $H^{-2} \oplus \mathbb{C}$ into the trivial bundle $\mathbb{C}P^1 \times \mathbb{C}^3$ defines a holomorphic map $f : M \to \mathbb{C}P^2$. Let $\tau_\omega$ be the standard integral Kähler form on $\mathbb{C}P^2$. For all $\lambda > 1$, $\omega_\lambda = (\lambda - 1)p_2^* \tau_1 + f^* \tau_2$ is a Kähler form on $M$.

The action of $S^1$ on $H^2$ by multiplication induces a Hamiltonian circle action $g$ on $(M, \omega_\lambda)$. $(E, \Omega) = (E_g, \Omega_g)$ can be constructed as follows: $E = \mathbb{P}(V)$ is the bundle of projective spaces associated to the holomorphic vector bundle $V = \mathbb{C} \oplus pr_1^* H^2 \oplus pr_2^* H^{-1}$

over $\mathbb{C}P^1 \times \mathbb{C}P^1$ ($pr_1, pr_2$ are the projections from $\mathbb{C}P^1 \times \mathbb{C}P^1$ to $\mathbb{C}P^1$). $\pi : E \to \mathbb{C}P^1$ is obtained by composing $\pi_V : \mathbb{P}(V) \to \mathbb{C}P^1 \times \mathbb{C}P^1$ with $pr_2$. The fibres $E_z = \pi^{-1}(z)$ are the ruled surfaces $\mathbb{P}(\mathbb{C} \oplus H^2 \oplus H^{-1})$. Any $\xi \in H^{-1}_{z}$ with unit length for the standard Hermitian metric determines a
biholomorphic map $E_z \to M$. Since two such maps differ by an isometry of $(M, \omega_3)$, the symplectic structure $\Omega_z$ on $E_z$ obtained in this way is independent of the choice of $\xi$. $(E, \Omega)$ is a Hamiltonian fibre bundle.

A holomorphic section $s$ of $\pi$ can be decomposed into two pieces: a section $s_1 = \pi_V \circ s : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ of $pr_2$ and a section $s_2$ of

$$F = \mathbb{P}(s_1 V) \to \mathbb{C}P^1.$$

In a sense, this second piece is given by $s$ itself, using the fact that $s(z) \in \mathbb{P}(V_{s_1(z)})$ for all $z$. The decomposition leads to an exact sequence

$$(11.4) \quad 0 \to s_2^*(TF^v) \to s^*TE^v \xrightarrow{D\pi_V} s_1^*(\ker(Dpr_2)) \to 0$$

of holomorphic vector bundles over $\mathbb{C}P^1$. Since $s_1$ is a section of $pr_2$, it is given by $s_1(z) = (u(z), z)$ for some $u : \mathbb{C}P^1 \to \mathbb{C}P^1$. Clearly, $s_1^*(\ker(Dpr_2)) = u^*\mathbb{C}P^1$. Let $d$ be the degree of $u$. The fibre bundle $F = \mathbb{P}(\mathbb{C} \oplus u^*H^2 \otimes H^{-1})$ is isomorphic to $\mathbb{P}_1$ for $d = 0$ and to $\mathbb{P}_{2d-1}$ for $d > 0$. If $d > 0$, $\deg(s_2^*TF^v) \geq 1 - 2d$ by Lemma 11.1. Since $\deg(u^*\mathbb{C}P^1) = 2d$, it follows from (11.4) that $\deg(s^*TE^v) > 0$. If $d = 0$, $s_1(z) = (c, z)$ for some $c \in \mathbb{C}P^1$. The same argument as before shows that $\deg(s^*TE^v) > 0$ unless $s_2$ is the unique section of $F$ such that $\deg(s_2^*TF^v) = -1$. It is easy to write down this section explicitly. We conclude that any holomorphic section $s$ of $\pi$ with $\deg(s^*TE^v) \leq 0$ belongs to the family $\mathcal{S} = \{s_c\}_{c \in \mathbb{C}P^1}$, where

$$(11.5) \quad s_c(z) = [1 : 0] \in \mathbb{P}(\mathbb{C} \oplus H^2_c \otimes H^{-1}_z) \subset E_z.$$

These sections have $\deg(s_c^*TE^v) = -1$. For any $z \in \mathbb{C}P^1$, the evaluation map $\mathcal{S} \to E_z$ is an embedding. Its image is the curve of self-intersection 2 which corresponds to $C^+ = \mathbb{P}(\mathbb{C} \oplus 0) \subset M$ under an isomorphism $E_z \cong M$ as above. We will now verify that the conditions of Proposition 7.11 are satisfied.

(i) We have already shown that $\mathcal{S}$ is connected and that $c_1(TE^v)(S_0) = -1$ for its $\Gamma$-equivalence class $S_0$.

(ii) For $s \in \mathcal{S}$, (11.4) reduces to

$$0 \to H^{-1} \to s^*TE^v \to \mathbb{C} \to 0.$$

By the exact sequence of cohomology groups, $H^{0,1}(\mathbb{C}P^1, s^*TE^v) = 0$.

(iii) is part of Lemma 11.2.

(iv) Consider the curve $C^- = \mathbb{P}(0 \oplus H^2) \subset M$. $C^-$ is the image of the unique section $s_-$ of $pr_2$ with $\deg(s_-^*TF^v_2) = -2$. Let $w : \mathbb{C}P^1 \to E_z$ be a non-constant holomorphic map with $c_1(TE)(w) < 2$. By Lemma 11.2, the image of $w$ is mapped to $C^-$ under an isomorphism $i : E_z \cong M$ chosen as above. We have seen that $i(s(z)) \in C^+$ for all $s \in \mathcal{S}$. Since $C^+ \cap C^- = \emptyset$, it follows that $s(z) \notin \text{im}(w)$.

We obtain $Q(E, \Omega, S_0) = [C^+] \otimes \langle 0 \rangle$. 
Lemma 11.3. Let $x^+, x^- \in \pi_2(M) \cong H_2(M;\mathbb{Z})$ be the classes of $C^+, C^-$, and $\bar{x}^+, \bar{x}^- \in H_2(M;\mathbb{Z}/2)$ their reductions mod 2. Then

$$(\bar{x}^+ \otimes <0>)^2 = [M] \otimes (<\frac{1}{2}(x^+ - x^-)> - <\frac{1}{2}(x^+ + x^-)>$$

in $(QH_*(M, \omega_\lambda), *)$.

Proof. In [8] McDuff showed that $(M, \omega_\lambda)$ is symplectically isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ with the product structure $\lambda(\tau_1 \times 1) + 1 \times \tau_1$. Such an isomorphism maps $x^\pm$ to $a \pm b$, where $a = [\mathbb{C}P^1 \times pt]$ and $b = [pt \times \mathbb{C}P^1]$. Let $\bar{a}, \bar{b}$ the mod 2 reductions of these classes. The quantum intersection product on $\mathbb{C}P^1 \times \mathbb{C}P^1$ is known (see e.g. [16, Proposition 8.2 and Example 8.5]); it satisfies

$$(\bar{a} \otimes <0>)^2 = [\mathbb{C}P^1 \times \mathbb{C}P^1] \otimes <b>, \ (\bar{b} \otimes <0>)^2 = [\mathbb{C}P^1 \times \mathbb{C}P^1] \otimes <a>.$$  

Because of the $\mathbb{Z}/2$-coefficients, this implies the relation stated above. \hfill \Box

This can be used to give a proof of the following result of McDuff.

Corollary 11.4. For all $\lambda > 1$, $[g] \in \pi_1(\text{Ham}(M, \omega_\lambda))$ has infinite order.

Proof. Let $\bar{g} : \mathcal{LM} \to \mathcal{LM}$ be the lift of $g$ corresponding to the equivalence class $S_0$. By Theorem 3 and Lemma 11.3,

$$q(g^2, \bar{g}^2) = Q(E, \Omega, S_0)^2 = [M] \otimes <\frac{1}{2}(x^+ - x^-)> (<0> - <x^->)$$

and

$$q(g^{2m}, \bar{g}^{2m}) = [M] \otimes <\frac{m}{2}(x^+ - x^-)> (<0> - <x^->)^m,$$

$$q(g^{2m+1}, \bar{g}^{2m+1}) = \bar{x}^+ \otimes <\frac{m}{2}(x^+ - x^-)> (<0> - <x^->)^m$$

for all $m \geq 0$. Since $x^-$ is represented by an algebraic curve, $\omega_\lambda(x^-) > 0$ for all $\lambda > 1$ (in fact, $\omega_\lambda(x^-) = \lambda - 1$). Therefore the class of $x^-$ in $\Gamma$ has infinite order, and

$$(<0> - <x^->)^m \notin \{<\gamma> | \gamma \in \Gamma\} \subset \Lambda$$

for all $m \geq 1$. It follows that $q(g^k, \bar{g}^k) \notin \tau(\Gamma)$ for all $k > 0$. Because of the diagram (1.4), this implies that $[g^k] \in \pi_1(\text{Ham}(M, \omega))$ is nontrivial. \hfill \Box

Remark 11.5. Using again Theorem 3 and Lemma 11.3, one computes that

$$q(g^{-1}, \bar{g}^{-1}) = (\bar{x}^+ \otimes <0>)^{-1} =$$

$$= \bar{x}^+ \otimes <\frac{1}{2}(x^- - x^+)> (<0> + <x^-> + <2x^-> + \cdots).$$

In this case, a direct computation of the invariant seems to be more difficult because infinitely many moduli spaces contribute to it.
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