Research Article

A Note on $n$-Divisible Positive Definite Functions

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1. Introduction

We start with some notations and definitions. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the families of integers, non-negative integers, real, and complex numbers, respectively. In the sequel, $M(\mathbb{R})$ denotes the Banach algebra of bounded regular complex-valued Borel measures on $\mathbb{R}$ with the convolution as multiplication. $M(\mathbb{R})$ is equipped with the usual total variation norm $\| \mu \|$ of $\mu \in M(\mathbb{R})$. The Fourier-Stieltjes transform of $\mu \in M(\mathbb{R})$ is given by

$$\hat{\mu}(x) = \int_{\mathbb{R}} e^{ixt} d\mu(t).$$

(1)

A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is said to be positive definite if

$$\sum_{j,k=1}^{m} f(x_j - x_k) c_j c_k \geq 0,$$

(2)

for each $m \in \mathbb{N}$ and all $c_1, \cdots, c_m \in \mathbb{C}$, $x_1, \cdots, x_m \in \mathbb{R}$. Any such a function satisfies

$$f(-x) = f(x),$$

(3)

for all $x \in \mathbb{R}$.

Positive definite functions on groups have a long history and have many applications in probability theory and areas such as stochastic processes [1], harmonic analysis [2], potential theory [3], and spectral theory [4]. See [5] for other applications and details. The analysis of the properties of positive definite functions has vast literature, and the above list is only a small sample.

We will denote by $PD(\mathbb{R})$ the family of continuous nontrivial ($\equiv 0$) positive definite functions on $\mathbb{R}$. Note that if positive definite $f$ is continuous in a neighborhood of the origin, then it is uniformly continuous in $\mathbb{R}$ (see, e.g., [4, Corollary 1.4.10]). Bochner’s theorem gives a description of $f \in PD(\mathbb{R})$ in terms of the Fourier transform. Namely, according to this theorem (see, e.g., [6, p. 71]), a continuous function $f : \mathbb{R} \rightarrow \mathbb{C}$ is positive definite if and only if there exists a nonnegative $\mu_f \in M(\mathbb{R})$ such that $f(x) = \hat{\mu}_f(x)$, $x \in \mathbb{R}$. This statement implies, in particular, that

$$|f(x)| \leq f(0) = \| \mu_f \|,$$

(4)

for all $x \in \mathbb{R}$. If, in addition, positive $\mu_f \in M(\mathbb{R})$ satisfies $\| \mu_f \| = 1$, then on the language of probability theory, such a $\mu_f$ and the function $f(x) = \hat{\mu}_f(-x)$, $x \in \mathbb{R}$, are called a probability measure and its characteristic function, respectively (see, e.g., [7, 8], p.p. 8-9).

Recall that in the probability theory a random variable $\xi$ is called $n$-divisible for certain $n \in \mathbb{N}$, $n > 1$, if there exist independent and identically distributed random variables
\( \xi_1, \cdots, \xi_n \) such that \( \xi_1 + \cdots + \xi_n \) has the same distribution as \( \xi \). In terms of the characteristic function \( f \) of a real-valued random variable, this means that \( f \) is \( n \)-divisible if there exists \( g \in \text{PD}(\mathbb{R}) \) such that

\[
f(x) = g^n(x),
\]

for all \( x \in \mathbb{R} \). Next, \( f \in \text{PD}(\mathbb{R}) \) is said to be infinite-\( n \)-divisible if it is \( n \)-divisible for each \( n \in \mathbb{N}, \ n \geq 2 \).

In the sequel, \( \text{PD}_n(\mathbb{R}) \) and \( \text{PD}_{\text{in}}(\mathbb{R}) \) denotes the families of \( n \)-divisible and infinite divisible functions in \( \text{PD}(\mathbb{R}) \), respectively. An early overview over divisibility of distributions is given in [9]. Until very recently, the vast majority of divisible positive definite functions or divisible distributions considered in the literature are also infinite-\( n \)-divisible. Important applications of \( n \)-divisibility is in modelling, for example, bug populations in entomology [10] or in financial aspects of various insurance models [11, 12].

The motivation for our investigation are the following: (i) partly from the fact that properties of functions in \( \text{PD}_{\text{in}}(\mathbb{R}) \) has rich literature (see, e.g., [9, 13, 14]), but the \( n \)-divisible functions have been studied much less; and (ii) partly from the fact that some properties of functions from \( \text{PD}_{\text{in}}(\mathbb{R}) \) and from \( \text{PD}_n(\mathbb{R}) \) may differ in essence. One of those properties is the following: if \( f \in \text{PD}_{\text{in}}(\mathbb{R}) \), then for each \( n \in \mathbb{N}, \ n > 1 \), there is an unique \( g \in \text{PD}(\mathbb{R}) \) such that \( g^n = f \), but there is \( n \)-divisible \( f \) such that the factor \( g \) in \( g^n = f \) is generally not unique. In this paper, we study the following problems: (i) how rich can be the class \( \{g \in \text{PD}(\mathbb{R}) : g^n = f\} \); and (ii) what properties of \( f \) determine the size of \( \{g \in \text{PD}(\mathbb{R}) : g^n = f\} \). We present several precise estimates for the cardinality of this class. Also, the main results are validated via illustrative examples.

More precise, for \( n \in \mathbb{N}, \ n > 1 \) and \( f \in \text{PD}_n(\mathbb{R}) \), we wish to study the family

\[
D_n(f) = \{g \in \text{PD}(\mathbb{R}) : g^n = f\}
\]

and the quantity \( \text{card}(D_n(f)) \), i.e., the cardinality of \( D_n(f) \). Note that there exists \( f \in \text{PD}_n(\mathbb{R}) \) such that the factor \( g \) in Equation (5) will generally be not unique. It turns out that \( \text{card}(D_n(f)) \) depends on \( n \) and in some way also depends on the geometric structure of the zeros set \( N_f \) of \( f \in \text{PD}_n(\mathbb{R}) \) (see our Theorem 1 below). The essential support \( S_f \) of \( f \in \text{PD}(\mathbb{R}) \) is defined by \( S_f = \mathbb{R} \setminus N_f \). Combining Equation (3) with Equation (4), gives

\[
0 \in S_f \text{ and } -S_f = S_f.
\]

Since functions \( f \in \text{PD}(\mathbb{R}) \) are continuous on \( \mathbb{R} \), it follows that \( S_f \) is an open subset of \( \mathbb{R} \). Therefore, \( S_f \) can be represented as a finite or infinite but countable \( S_f = \bigcup_{j \in \Omega} E_j \), where \( \{E_j\}_{j \in \Sigma} \) is the family of all open connected components of \( S_f \). In the sequel, \( \text{comp}(S_f) \) denotes the cardinality of \( \Sigma \). According to relations (7), we see that either there is an \( k \in \mathbb{N} \) such that \( \text{comp}(S_f) = 2k - 1 \) or \( \text{comp}(S_f) = \infty \).

**Theorem 1.** Let \( n \in \mathbb{N}, \ n \geq 2 \), and let \( f \in \text{PD}_n(\mathbb{R}) \). Assume that

\[
\text{comp}(S_f) = 2k - 1,
\]

for some \( k \in \mathbb{N} \). Then

\[
\text{card}(D_n(f)) \leq n^{k-1}.
\]

The following theorem shows that the estimate (9) is accurate.

**Theorem 2.** For each \( n \in \mathbb{N}, \ n \geq 2 \), each \( k \in \mathbb{N} \), and any open subset \( E \) of \( \mathbb{R} \) which satisfies

\[
0 \in E, -E = E \text{ and } \text{comp}(E) = 2k - 1,
\]

there exists \( f \in \text{PD}_n(\mathbb{R}) \), such that \( S_f = E \) and

\[
\text{card}(D_n(f)) = n^{k-1}.
\]

We will present two examples of \( f \in \text{PD}_n(\mathbb{R}) \) such that Equation (11) is satisfied. In order to make the examples easier to understand, we will consider only small values of \( k \). For \( \alpha > 0 \), set

\[
A_\alpha(x) = (\max \{1 - |x|; 0\})^\alpha, \quad x \in \mathbb{R}.
\]

We note that \( A_\alpha \in \text{PD}(\mathbb{R}) \) if and only if \( \alpha \geq 1 \) (see, e.g., [15, 16, p. 282]). We start with the case where \( S_f \) is a bounded subset of \( \mathbb{R} \).

**Example 3.** For \( n \in \mathbb{N}, \ n > 1 \), let

\[
f_1(x) = A_n(x) + \frac{1}{2^{3n}}\left[\left(A_n(x - \pi - 1) + A_1(2x - 2\pi + 1)\right)^n + \left(A_1(x + \pi + 1) + A_1(x + 2\pi - 1)\right)^n\right]
\]

\[
+ \frac{1}{2^{5n}}\left[\left(A_1(x - 11) + A_1(x - 12) + A_1(x - 13) + A_1(x - 14)\right)^n + (A_1(x + 11) + A_1(x + 12) + A_1(x + 13) + A_1(x + 14)\right)^n]\right].
\]

Then \( \text{comp}(S_{f_1}) = 5 \), since

\[
S_{f_1} = (-15,-10) \cup (-2\pi,-\pi) \cup (-1,1) \cup (\pi,2\pi) \cup (10,15).
\]

Moreover, \( f_1 \in \text{PD}_n(\mathbb{R}) \) and

\[
\text{card}(D_n(f_1)) = n^2.
\]

Now let us give an example of \( f \in \text{PD}_n(\mathbb{R}) \) such that \( S_f \) has unbounded components.
Example 4. For any $a > 15$ and $n \in \mathbb{N}$, $n > 1$, let
\[
f_2(x) = f_1(x) + \frac{1}{2\pi} \left( \sum_{r=1}^{\infty} \frac{1}{r^2} \Lambda_1(x - a - r) \right)^n + \left( \sum_{r=1}^{\infty} \frac{1}{r^2} \Lambda_1(x + a + r) \right)^n,
\]
where $f_1$ was defined by Equation (13). Then comp $(S_{f_2}) = 7$, since
\[
S_{f_2} = (-\infty, -a) \cup (-15, -10) \cup (-2\pi, -\pi) \cup (-1, 1) \cup (\pi, 2\pi) \cup (10, 15) \cup (a, \infty).
\]
Moreover, $f_2 \in PD_n(\mathbb{R})$ and
\[
\text{card } (D_n(f_2)) = n^7.
\]

Theorem 5. Suppose that an open subset $E$ of $\mathbb{R}$ satisfies
\[
0 \in E, -\infty \in E \text{ and } \text{comp } (E) = \infty.
\]

Let $\{E_j\}_{j \in \Sigma}$ be the family of open connected components of $E$. Assume that there is $\sigma > 0$ such that
\[
\inf_{j \in \Sigma} \left( \sup_{a, b \in E_j} |a - b| \right) = 2\sigma.
\]

Then, for any $n \in \mathbb{N}$, $n \geq 2$, there exists $f \in PD_n(\mathbb{R})$ such that $S_f = E$ and
\[
\text{card } (D_n(f)) = \infty.
\]

2. Preliminaries and Proofs

If $\nu, \mu \in M(\mathbb{R})$, then the convolution $\nu * \mu$ is defined by
\[
\nu * \mu(E) = \int_{\mathbb{R}} \nu(E - x) \, d\mu(x)
\]
for each Borel subset $E$ of $\mathbb{R}$. Note that
\[
\widehat{\nu} * \check{\mu} = \check{\nu} \cdot \widehat{\mu}.
\]

In particular, for any $n \in \mathbb{N}$,
\[
(\mu^n) = (\check{\mu})^n,
\]
where the convolution power $\mu^n$ is defined as the $n$-fold iteration of the convolution of $\mu$ with itself.

The Lebesgue space $L^1(\mathbb{R})$ can be identified with the closed ideal in $M(\mathbb{R})$ of measures absolutely continuous with respect to the Lebesgue measure $dx$ on $\mathbb{R}$. Namely, if $\varphi \in L^1(\mathbb{R})$, then $\varphi$ is associated with the measure
\[
\mu_{\varphi}(E) = \int_E \varphi(t) \, dt
\]
for each Borel subset $E$ of $\mathbb{R}$. Hence $\hat{\varphi}(x) = \int_{\mathbb{R}} e^{-itx} \varphi(t) \, dt$.

In particular, if $\mu = \varphi(t) \, dt$, where $\varphi \in L^1(\mathbb{R})$ and $\varphi$ is such that $||\varphi||_{L^1(\mathbb{R})} = 1$ and $\varphi \geq 0$ on $\mathbb{R}$, then $\varphi$ is called the probability density function of $\mu$, or the probability density for short.

We define the inverse Fourier transform by
\[
\hat{\psi}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \psi(x) \, dx,
\]
$t \in \mathbb{R}$. Then the inversion formula $\hat{\hat{\psi}} = \psi$ holds for suitable $\psi \in L^1(\mathbb{R})$.

Proof of Theorem 1. The conditions (7) and (8) imply that there exists a sequence of real numbers
\[
0 < b_0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{k-1} < b_{k-1},
\]
such that
\[
S_j = \left( \bigcup_{j=1}^{(k-1)} E_j \right) \bigcup E_0 \bigcup \left( \bigcup_{j=1}^{k-1} E_j \right),
\]
where $E_0 = (-b_0, b_0)$, $E_j = (a_j, b_j)$, and $E_{-j} = (-b_{-j}, -a_{-j})$ for $j = 1, 2, \cdots, k-1$. Note that in Equation (27) also might be $b_{-j} = +\infty$. Let $g \in D_n(f)$. Then it is immediate that $S_j = S_g$ and
\[
|g(x)|^n = |f(x)|
\]
for all $x \in S_j$. Fix any $E_j \subset S_j$ in Equation (28). Since $E_j$ is an open connected component of $S_j$, we have there are two continuous functions $u_{j, j} : E_j \rightarrow (-\pi, \pi)$ such that
\[
f(x) = |f(x)| e^{iu_{j, j}(x)} \quad \text{and} \quad g(x) = |g(x)| e^{iu_{j, j}(x)}
\]
for all $x \in E_j$. Using the identity $g^n = f$, it follows from Equations (29) and (30) that, for each for $j \in \{-k-1, \cdots, -1\}$, there exists some integer $m_j$ in $\{0, \cdots, n-1\}$ such that
\[
n \cdot u_{j, j}(x) = u_{j, j}(x) + 2\pi m_j
\]
for all \( x \in E_j \). Therefore, for any \( x \in S_j \), we have

\[
\begin{align*}
g(x) &= \sum_{j=-(k-1)}^{k-1} |g(x)| x_{E_j}(x) e^{in\theta_j} \\
&= \sum_{j=-(k-1)}^{k-1} |f(x)|^{1/n} x_{E_j}(x) e^{i(nj+2\pi mn)/n},
\end{align*}
\]

(32)

where \( x_{E_j} \) is the indicator function of the set \( E_j \) and \( |f(x)|^{1/n} \) denotes the positive \( n \)th root of positive number \( |f(x)| \), \( x \in S_j \). We claim that \( m_0 = 0 \). Indeed, Equation (4) implies that \( f(0) \) and \( g(0) \) are positive numbers. Then Equation (30) implies that \( u_{\ell,0}(0) = u_{\ell,0}(0) = 0 \). Combining this with Equation (31), yields the claim. Next, applying the property \( \chi \) of Theorem 1 is proved.

Proof of Theorem 2. Note that as in the proof of Theorem 1, in light of Equation (10), we see that there exists a sequence of real numbers Equation (27) such that

\[
E = \left( \bigcup_{j=1}^{-(k-1)} E_j \right) \bigcup E_0 \bigcup \left( \bigcup_{j=1}^{k-1} E_j \right).
\]

(34)

Let us split our proof into two cases. First we consider the case when all \( E_j \) in Equation (34) are finite intervals. Denote by \( 2\sigma \) the minimal length of \( E_j, j = -(k-1), \ldots, k-1 \), i.e.,

\[
2\sigma = \min \left\{ 2b_0; \min_{1 \leq j \leq k-1} (b_j - a_j) \right\}. 
\]

(35)

Let \( \varphi \in PD(\mathbb{R}) \). Assume, in addition, that \( \varphi \) is real-valued on \( \mathbb{R} \) and

\[
S_0 = (-\sigma, \sigma). 
\]

(36)

For example, we can take \( \varphi(x) = \Lambda_\omega(x) \), where the truncated power function \( \Lambda_\omega \) was defined by Equation (11). Next, for each \( j \in \{1, 2, \ldots, k-1\} \), we take any sequence of real numbers \( \{\tau_{j,s}\}_{s=1}^{m(j)} \) such that

\[
a_j + \sigma = \tau_{j,1} < \tau_{j,2} < \cdots < \tau_{j,m} = b_j - \sigma. 
\]

(37)

In addition, we assume that

\[
\tau_{j,(s+1)} - \tau_{j,s} < 2\sigma 
\]

(38)

for all \( s = 1, \ldots, m(j) \). Then we define the function

\[
u_j(x) = \sum_{s=1}^{m(j)} \varphi(x - \tau_{j,s}),
\]

(39)

\( x \in \mathbb{R} \). Now Equations (37) and (38) imply that \( u_j \) is supported on \([a_j, b_j] \). The problem Equation (27) is even and positive on \( S_0 \). Therefore, the function Equation (39) and the function

\[
u_j(-x) = \sum_{s=1}^{m(j)} \varphi(x + \tau_{j,s}),
\]

(40)

are strictly positive on \( E_j \) and on \( E_j = E_j \), respectively.

Let us define the function \( u_0 \) such that it is supported on \([-b_0, b_0] = E_0 \) and \( S_0 = E_0 \). To this end, we take any sequence \( \{\theta_i\}_{i=0}^l \) of real numbers such that

\[
-b_0 + \sigma = \theta_1 < \cdots < \theta_l < \theta_0 = 0, 
\]

(41)

\[
\theta_i - \theta_{i+1} < 2\sigma,
\]

(42)

for \( i = 0, \ldots, l-1 \). Next, for an arbitrary sequence of positive numbers \( \{\omega_i\}_{i=0}^l \), we define

\[
u_0(x) = \sum_{i=0}^l \omega_i (\varphi(x - \theta_i) + \varphi(x + \theta_i)),
\]

(43)

\( x \in \mathbb{R} \). Of course, Equations (41) and (42) imply that \( u_0 \) is supported on \([-b_0, b_0] = E_0 \) and \( S_0 = E_0 \).

Finally, given any fixed sequence of positive numbers \( \{\alpha_j\}_{j=1}^{k-1} \), we set

\[
u(x) = u_0(x) + \sum_{j=1}^{k-1} \alpha_j (u_j(x) + u_j(-x)),
\]

(44)

\( x \in \mathbb{R} \).
We claim that \( u \in PD(\mathbb{R}) \) and \( S_n = E \). First, as real-valued function \( \varphi \in PD(\mathbb{R}) \) satisfies Equation (36), it follows from Equations (34)–(38) and from Equations (41)–(42) that \( u \) is continuous on \( \mathbb{R} \) and
\[
S_n = \bigcup_{j=-k}^{k} E_j = E. \tag{46}
\]
Second, since \( u \) is continuous and compactly supported, it follows that \( u \in L^1(\mathbb{R}) \). Therefore, the inverse Fourier transform of \( u \) is well-defined. Hence
\[
\hat{u}(t) = \check{u}_0(t) + \sum_{j=1}^{k} \left( \alpha_j \check{v}(t) \sum_{i=1}^{m(j)} \cos (\tau_{j,i} \cdot t) \right)
\]
\[
= 2\check{v}(t) \left[ \sum_{i=0}^{l} \alpha_i \cos (\theta_i \cdot t) + \sum_{j=1}^{k} \left( \alpha_j \sum_{i=1}^{m(j)} \cos (\tau_{j,i} \cdot t) \right) \right]
\]
\[
= 2\check{v}(t) \left[ \alpha_0 + \sum_{i=1}^{l} \alpha_i \cos (\theta_i \cdot t) + \sum_{j=1}^{k} \left( \alpha_j \sum_{i=1}^{m(j)} \cos (\tau_{j,i} \cdot t) \right) \right]. \tag{47}
\]
Let us fix the previously chosen positive numbers \( \omega_1, \ldots, \omega_k \) and \( \alpha_1, \ldots, \alpha_{k-1} \). Then we increase if necessary, the value of \( \omega_0 \) in such a way that
\[
\omega_0 > \sum_{i=1}^{l} \omega_i + \sum_{j=1}^{k-1} \alpha_j m_j. \tag{48}
\]
Bochner’s theorem shows that \( \hat{u}(t) \geq 0 \) for all \( t \in \mathbb{R} \), since \( \varphi \in PD(\mathbb{R}) \). Combining Equation (36) with (48), we see that \( \check{u} \) is nonnegative on \( \mathbb{R} \) and \( \check{u} \neq 0 \). In addition, we conclude (see, e.g., [15, p. 409]) that \( \hat{u} \in L^1(\mathbb{R}) \). Thus, applying the Fourier transform to \( \check{u} \) and using again Bohner’s theorem, we see that \( u \) is continuous nontrivial positive definite, i.e., \( u \in PD(\mathbb{R}) \). This proves our claim.

Define
\[
f(x) = u_0^n(x) + \sum_{j=1}^{k-1} \alpha_j^n \left( u_j^n(x) + u_j^n(-x) \right). \tag{49}
\]
\( x \in \mathbb{R} \). We claim that \( f \) satisfies the hypotheses of Theorem 2. Let us first prove that \( f \in PD(\mathbb{R}) \). Indeed, from Equation (27) we see that the essential support \( S_n \) of \( u \), defined by Equation (44) can be represented as the union \( \bigcup_{j=-k}^{k} E_j \) of a family of pairwise disjoint sets \( E_j = S_j, j = -(k-1) \cdots, k-1 \). Therefore,
\[
u^n(x) = u_0^n(x) + \sum_{j=1}^{k-1} \alpha_j^n \left( u_j^n(x) + u_j^n(-x) \right) \tag{50}
\]
x \( \in \mathbb{R} \). We have already proven that \( u \in PD(\mathbb{R}) \). On the other hand, it is well known that for each \( n \in \mathbb{N} \) and any \( \zeta \in PD(\mathbb{R}) \), it follows that \( \zeta^n \in PD(\mathbb{R}) \). Thus, \( u^n \in PD(\mathbb{R}) \). Combining this fact with Equation (41), we conclude that \( f \in PD(\mathbb{R}) \) and \( S_n = S_n = \bigcup_{j=-k}^{k} E_j = E \).

Second, we will prove that the function \( f \) defined by Equation (50) has the property Equation (11). To this end, let \( Z_n \) denote the group \( Z_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\} \). Given \( A = (p_1, p_2, \ldots, p_{k-1}) \in Z_n^{k-1} \), define
\[
g_A(x) = u_0(x) + \sum_{j=1}^{k-1} \alpha_j \left( e^{2\pi i p_j/n} u_j(x) + e^{-2\pi i p_j/n} u_j(-x) \right), \tag{51}
\]
x \( \in \mathbb{R} \). We claim that \( g_A \in D_n(f) \). By the same argument as before for the function \( u \) defined by Equation (44), we see that \( g_A \in L^1(\mathbb{R}) \). Therefore, \( g_A \) is well-defined and
\[
g_A(t) = 2\check{v}(t) \left[ \alpha_0 + \sum_{i=1}^{l} \alpha_i \cos (\theta_i \cdot t) + \sum_{j=1}^{k-1} \left( \alpha_j \sum_{i=1}^{m(j)} \cos (\tau_{j,i} \cdot t + \frac{2\pi p_j}{n}) \right) \right], \tag{52}
\]
t \( \in \mathbb{R} \). Now using Equation (48), we get that \( g_A(t) \geq 0 \) for all \( t \in \mathbb{R} \) and \( g_A \neq 0 \). Hence, by Bochner’s theorem it follows that \( g_A \in PD(\mathbb{R}) \). Next,
\[
g_A^n(x) = u_0^n(x) + \sum_{j=1}^{k-1} \alpha_j^n \left( \left( e^{2\pi i p_j/n} u_j(x) \right)^n + \left( e^{-2\pi i p_j/n} u_j(-x) \right)^n \right)
\]
\[
= u_0^n(x) + \sum_{j=1}^{k-1} \alpha_j^n \left( u_j^n(x) + u_j^n(-x) \right)^n. \tag{53}
\]
x \( \in \mathbb{R} \). Combining this representation with Equations (49) and (50), we see that \( g_A \in D_n(f) \), which yields our claim.

Finally, again using the fact that \( S_n \) is the union of a family of pairwise disjoint sets \( S_n, j = -(k-1), \cdots, k-1 \), we conclude from Equation (51) that \( g_A \equiv g_A \), for some \( A, A_1 \in Z_n^{k-1} \), if and only if \( A = A_1 \). This proves Equation (11) in the case where each \( E_j \) in Equation (33) is a finite interval.

Now we consider the second case with \( b_{k-1} = \infty \), i.e., if in Equation (33) we have \( E_{k-1} = (a_{k-1}, \infty) \). Using the same \( \varphi \in PD \) satisfying Equation (36), we define the functions \( u_0 \) and \( u_j, j = -(k-2), \cdots, -1, 1, \cdots, k-2 \) by Equations (43) and (39)–(40), respectively. For \( j = k-1 \), let us take an arbitrary sequence of positive numbers \( \{ \gamma_r \}_{r=1}^{\infty} \) such that
\[
\sum_{r=1}^{\infty} \gamma_r = 1. \tag{54}
\]
Then we define
\[ u_{k-1}(x) = \sum_{r=1}^{\infty} y_r \varphi(x - a_k - r\sigma), \] (55)
\[ x \in \mathbb{R}. \]
Obviously, \( \text{supp } (u_{k-1}) = E_{k-1} \) and \( S_{a_k-} = E_{k-1} \). Next, for the function \( u \), defined by Equation (44), it follows from Equation (47) that
\[ \hat{u}(t) = 2\varphi(t) \left[ \omega_0 + \sum_{j=1}^{l-1} \omega_j \cos (\theta_j t) + \sum_{j=1}^{k-2} \alpha_j \sum_{i=1}^{m(j)} \cos (\tau_{ji} t) \right] + \alpha_{k-1} \sum_{r=1}^{\infty} \gamma_r \cos ((a_{k-1} + r\sigma)t). \] (56)

Again, for fixed positive numbers \( \omega_1, \cdots, \omega_l, \alpha_1, \cdots, \alpha_{k-1} \), and \( \gamma_1, \gamma_2, \cdots \), we take the value of \( \omega_0 \) in such a way that
\[ \omega_0 > \sum_{j=1}^{l-1} \omega_j + \sum_{j=1}^{k-2} \alpha_j m_j + \alpha_{k-1}. \] (57)

Combining Equation (36) with Equation (57), we conclude from Equation (56) that \( u \) is nonnegative on \( \mathbb{R} \) and \( u \equiv 0 \). Thus, \( u \in PD(\mathbb{R}) \). Finally, we claim that the function \( f \) defined by Equation (49) also satisfies the hypotheses of Theorem 2 in our case with \( E_{k-1} = (a_{k-1}, \infty) \). The proof of this claim is exactly the same as that of the first case. Therefore, we skip the details of this proof. Theorem 2 is proved.

**Proof of Example 3.** We claim that there are \( u_0, u_1, u_2 \) and \( a_1, a_2, a_3 \) such that the function in Equation (13) coincides with the function defined by Equation (50). Indeed, set \( \varphi = \Lambda_1, j = 3, m_1 = 2, m_2 = 4 \) and
\[ \tau_{11} = \pi + 1, \tau_{12} = 2\pi - 1, \tau_{21} = 11, \tau_{22} = 12, \tau_{23} = 13, \tau_{24} = 14. \] (58)

Since \( \sigma \) defined by Equation (35) is equal now to 1, then \( \tau_{ji} \) defined above satisfies Equations (37) and (38). Next, set
\[ v_1(x) = \Lambda_1(x - \pi - 1) + \Lambda_1(x - 2\pi + 1), \]
\[ v_2(x) = \Lambda_1(x - 11) + \Lambda_1(x - 12) + \Lambda_1(x - 13) + \Lambda_1(x - 14). \] (59)

For \( v_0 = \Lambda_1 \), \( a_1 = 1/8 \) and \( a_2 = 1/16 \), let \( v \) be given by
\[ v(x) = \Lambda_1(x) + \frac{1}{8} (v_1(x) + v_1(-x)) + \frac{1}{16} (v_2(x) + v_2(-x)). \] (60)

It is easily seen that \( \Lambda_1, v_1, v_2, v_0 \) are supported on a family of pairwise disjoint sets. Therefore,
\[ v^n(x) = \Lambda_n(x) + \frac{1}{8} (v^n_1(x) + v^n_1(-x)) + \frac{1}{16} (v^n_2(x) + v^n_2(-x)), \] (61)
since \( \Lambda_1^n = \Lambda_n \). Next
\[ S_{\rho} = S_\rho = (-15, -10) \cup (2\pi, -\pi) \cup (-1, 1) \cup (\pi, 2\pi) \cup (10, 15). \] (62)

The function \( v^n \) coincides with the function in Equation (13) and is defined by the same rules as in Equation (50). Our claim is proved.

Now, it is enough to show that \( v \in PD(\mathbb{R}) \). Indeed,
\[ \hat{v}(t) = 2\Lambda_1(t) \left[ 1 + \sum_{j=1}^{2} \alpha_j \sum_{i=1}^{m(j)} \cos (\tau_{ji} t) \right] + \frac{1}{16} \left( \cos (\tau_{21} t) + \cos (\tau_{22} t) + \cos (\tau_{23} t) + \cos (\tau_{24} t) \right) \geq \Lambda_1(t) \geq 0 \] (63)
for all \( t \in \mathbb{R} \), since \( \Lambda_1 \in PD(\mathbb{R}) \). Therefore, Bochner’s theorem shows that \( v \in PD(\mathbb{R}) \).

By repeating the finally part of the proof of Theorem 2, we complete the proof of Example 3.

**Proof of Example 4.** This example concerns the case that was considered in the second part of the proof of Theorem 2, i.e., when \( S_\rho \) contains two unbounded components \( (-\infty, -a) \) and \( (a, \infty) \). Also, as in the proof of Theorem 2, is enough to show that Equations (54) and (57) are satisfied. Indeed, Equation (54) is clear, since \( \gamma_1 = 1/2 \), \( r = 1, 2, \cdots \). We conclude from Equations (13) and (16) that
\[ k = 4, \sigma = 1, m_1 = 2, m_2 = 4, \omega_0 = 1, \]
\[ \omega_1 = 0, \alpha_1 = \frac{1}{8}, \alpha_2 = \frac{1}{16} \quad \text{and} \quad \alpha_3 = \frac{1}{4}. \] (64)

Therefore, a simple calculation shows that Equation (57) is also satisfied. This completes the proof.

**Proof of Theorem 5.** We will prove this theorem using essential the same techniques as in the proof of Theorem 2. Therefore, we sketch the proof only. From Equations (19) and (20) we have that there exits an infinite sequence
\[ 0 < b_0 < a_1 < b_1 < a_2 < b_2 < \cdots < \infty \] (65)
such that

\[ E = \bigcup_{j \in \mathbb{Z}} E_j \tag{66} \]

where \( E_0 = (-b_0, b_0) \) and \( E_j = (a_j, b_j) = -E_{-j} \) for each \( j \in \mathbb{N} \). Moreover, from Equation (20) we see that

\[ \min \left\{ 2b_0 \cdot \inf_{j \in \mathbb{N}} (b_j - a_j) \right\} = 2 \sigma > 0. \tag{67} \]

Let \( \varphi \in PD(\mathbb{R}) \) be the same function satisfying Equation (36). For \( j = 0 \) and for \( j \in \mathbb{N} \), we define the functions \( u_0 \) and \( u_j \) by Equations (39) and (43), respectively. Note that the sequences \( \{\tau_j\}_{j=1}^{m(j)} \) and \( \{\theta_j\}_{j=0}^{l} \) satisfy Equations (37)–(38) and (40)–(42), respectively. For any \( j \in \mathbb{N} \), let us define

\[ \alpha_j = \frac{1}{2m(j)|E_j|}, \tag{68} \]

where \( |E_j| \) is the length of \( E_j \), i.e., \( |E_j| = \sup \{|a - b|: a, b \in E_j\} \). Note that the condition (20) guarantees that \( \{\alpha_j\}_{j \in \mathbb{N}} \) is a well-defined sequence of positive numbers. Set

\[ u(x) = u_0(x) + \sum_{j \in \mathbb{N}} \alpha_j (u_j(x) + u_j(-x)), \tag{69} \]

\( x \in \mathbb{R} \). Combining Equations (4), (39), and (43) with (68), we get

\[ |u(x)| \leq 2\varphi(0) \left[ \left( \sum_{j=0}^{l} \omega_j \right) I_{E_0}(x) + \sum_{j \in \mathbb{N}} \left( \frac{1}{2|E_j|} I_{E_j}(x) \right) \right] \tag{70} \]

for any \( x \in \mathbb{R} \). Here, \( I_D(x) \) denotes the indicator function of a subset \( D \subset \mathbb{R} \). Hence,

\[ \|\varphi\|_{L^1(\mathbb{R})} \leq 2\varphi(0) \left[ 2b_0 \sum_{j=0}^{l} \omega_j + 1 \right]. \tag{71} \]

Therefore, \( \varphi \in L^1(\mathbb{R}) \) and

\[ u(t) = 2\varphi(t) \left[ \omega_0 + \sum_{j=1}^{l} \omega_j \cos (\theta_j \cdot t) \right. \right. \tag{72} \]

\[ + \sum_{j \in \mathbb{N}} \left( \alpha_j \sum_{i=1}^{m(j)} \cos (\tau_{j,i} \cdot t) \right) \right] \]

Again, for fixed positive numbers \( \omega_1, \cdots, \omega_k \), we take the value of \( \omega_0 \) in such a way that

\[ \omega_0 > \sum_{i=1}^{l} \omega_i + \frac{1}{2\sigma}. \tag{73} \]

Combining this condition with Equation (67) and Equation (68), we conclude that the function \( \tilde{u} \) in Equation (61) is nonnegative on \( \mathbb{R} \) and \( \tilde{u} \neq 0 \). Thus, \( u \in PD(\mathbb{R}) \). Define

\[ f(x) = u_0^0(x) + \sum_{j \in \mathbb{N}} a_j^0 \left( u_j^0(x) + u_j^0(-x) \right), \tag{74} \]

\( x \in \mathbb{R} \). Next, for \( \Lambda = (p_1, p_2, \cdots) \in \mathbb{Z}^\infty \), set

\[ g_{\Lambda}(x) = u_0(x) + \sum_{j \in \mathbb{N}} a_j \left( e^{2\pi i p_j/n} u_j(x) + e^{-2\pi i p_j/n} u_j(-x) \right), \tag{75} \]

\( x \in \mathbb{R} \). We claim that: (i) \( f \in PD(\mathbb{R}) \), (ii) \( E \) defined by Equations (66) satisfies Equations (19) and (20), and (iii) \( g_{\Lambda} \equiv g_{\Lambda_1} \) for some \( \Lambda, \Lambda_1 \in \mathbb{Z}^\infty \), if and only if \( \Lambda = \Lambda_1 \). This proves Equation (21) and therefore completes the proof of our theorem.

3. Conclusion

We study the \( n \)-divisible functions in \( PD(\mathbb{R}) \), where \( PD(\mathbb{R}) \) denotes the family of continuous positive definite functions on the real line \( \mathbb{R} \). While there is rich literature on infinite-divisible functions in \( PD(\mathbb{R}) \), for an integer \( n > 1 \), properties of \( n \)-divisible functions from \( PD(\mathbb{R}) \) have been studied much less. Surprisingly, it appears that some properties of infinite-divisible and \( n \)-divisible functions \( f \in PD(\mathbb{R}) \) may differ in essence. In this paper, we examine one such property, which has not yet been discussed in detail in the literature. More precisely, if \( f \in PD(\mathbb{R}) \) is infinite-divisible, then it is well-known that, for each integer \( n > 1 \), there is an unique \( g \in PD(\mathbb{R}) \), such that \( g^n = f \). On the other hand, there is \( n \)-divisible \( f \) such that the factor \( g \) in \( g^n = f \) is generally not unique. For \( n \)-divisible \( f \in PD(\mathbb{R}) \), we study the following questions: (i) how rich can be the class \( D_n(f) = \{ g \in PD(\mathbb{R}) : g^n = f \} \); and (ii) what properties of \( f \) determine the size of \( D_n(f) \). We present several precise estimates for the cardinality of \( D_n(f) \). Also, the main results are validated via illustrative examples.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that they have no competing interests.
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