ON SYMBOLIC FACTORS OF S-ADIC SUBSHIFTS OF FINITE ALPHABET RANK

BASTIÁN ESPINOZA

Abstract. This paper studies several aspects of symbolic factors of S-adic subshifts of finite alphabet rank. First, we address a problem raised in [DDMP20] about the topological rank of symbolic factors of S-adic subshifts and prove that this rank is at most the one of the extension system, improving results from [Esp20] and [GH20]. As a consequence of our methods, we prove that finite topological rank systems are coalescent. Second, we investigate the structure of fibers \( \pi^{-1}(y) \) of factor maps \( \pi: (X,T) \to (Y,T) \) between minimal S-adic subshifts of finite alphabet rank and show that they have the same finite cardinality for all \( y \) in a residual subset of \( Y \). Finally, we prove that the number of symbolic factors (up to conjugacy) of a fixed subshift of finite topological rank is finite, thus extending a similar theorem of Durand for linearly recurrent subshifts.

1. Introduction

An ordered Bratteli diagram is an infinite directed graph \( B = (V,E,\leq) \) such that the vertex set \( V \) and the edge set \( E \) are partitioned into levels \( V = V_0 \cup V_1 \cup \ldots, \)
\[ E = E_0 \cup \ldots \] so that \( E_n \) are edges from \( V_{n+1} \) to \( V_n \), \( V_0 \) is a singleton, each \( V_n \) is finite and \( \leq \) is a partial order on \( E \) such that two edges are comparable if and only if they start at the same vertex. The order \( \leq \) can be extended to the set \( X_B \) of all infinite paths in \( B \), and the Vershik action \( V_B \) on \( X_B \) is defined when \( B \) has unique minimal and maximal infinite paths with respect to \( \leq \). We say that \( (X_B,V_B) \) is a BV representation of the Cantor system \( (X,T) \) if both are conjugate. Bratteli diagrams are a tool coming from \( C^* \)-algebras that, at the beginning of the 90', Herman et. al. [HPS92] used to study minimal Cantor systems. Their success at characterizing the strong and weak orbit equivalence for systems of this kind marked a milestone in the theory that motivated many posterior works. Some of these works focused on studying with Bratteli diagrams specific classes of systems and, as a consequence, many of the classical minimal systems have been characterized as Bratteli-Vershik systems with a specific structure. Some examples include odometers as those systems that have a BV representation with one vertex per level, substitutive subshifts as stationary BV (all levels are the same) [DHS99a], certain Toeplitz sequences as “equal row-sum” BV [GJ00], and (codings of) interval exchanges as BV where the diagram codifies a path in a Rauzy graph [GJ02]. Now, almost all of these examples share certain coarse dynamical behavior: they have

Date: December 2, 2020.
2020 Mathematics Subject Classification. Primary: 37B10; Secondary: 37B10.
Key words and phrases. S-adic subshifts, minimal Cantor systems, finite topological rank.
The first author thanks Doctoral Fellowship CONICYT-PFCHA/Doctorado Nacional/2020-21202229.
finitely many ergodic measures, are not strongly mixing, have zero entropy, are expansive, and their BV representations have a bounded number of vertices per level, among many others. It turns out that just having a BV representation with a bounded number of vertices per level (or, from now on, having finite topological rank) implies the previous properties (see, for example, [BKMS10], [DM08]). Hence, the finite topological rank class arises as a possible framework for studying minimal symbolic systems and proving general theorems.

This idea has been exploited in many works: Durand et. al., in a series of papers (being [DFM19] the last one), developed techniques from the well-known substitutive case and obtained a criteria for any BV of finite topological rank to decide if a given complex number is a continuous or measurable eigenvalue, Bezugly et. al. described in [BKMS10] the simplex of invariant measures together with natural conditions for being uniquely ergodic, Giordano et. al. bounded the rational rank of the dimension group by the topological rank ([HPS92], [GHH18]), among other works. It is important to remark that these works were inspired by or first proved in the substitutive case.

Now, since Bratteli-Vershik systems with finite topological rank at least two are conjugate to a subshift [DM08], it is interesting to try to define them directly as a subshift. This can be done by codifying the levels of the Bratteli diagram as substitutions and then iterate them to obtain a sequence of symbols defining a subshift conjugate to the initial BV system. This procedure also makes sense for arbitrary nested sequences of substitutions (called directive sequences), independently from the Bratteli diagram and the various additional properties that its codifying substitutions have. Subshifts obtained in this way are called S-adic (substitution-adic) and may be non-minimal (see for example [BSTY19]).

Although there are some open problems about finite topological rank systems depending directly on the combinatorics of the underlying Bratteli diagrams, others are more naturally stated in the S-adic setting (e.g., when dealing with endomorphisms, it is useful to have the Curtis–Hedlund–Lyndon Theorem) and, hence, there exists an interplay between S-adic subshifts and finite topological rank systems in which theorems and techniques obtained for one of these classes can sometimes be transferred to the other. The question about which is the exact relation between these classes has been recently addressed in [DDMP20] and, in particular, the authors proved:

**Theorem 1.1** ([DDMP20]). A minimal subshift \((X,T)\) has topological rank at most \(K\) if and only if it is generated by a proper, primitive and recognizable S-adic sequence of alphabet rank at most \(K\).

In this context, a fundamental question is the following:

**Question 1.2.** Are symbolic factors of finite topological rank systems of finite topological rank?

Indeed, the topological rank controls various coarse dynamical properties (number of ergodic measures, rational rank of dimension group, among others) which cannot increase after a factor map, and we also know that big subclasses of the finite topological rank class are stable under symbolic factors, such as the linearly recurrent and the non-superlineal complexity classes [DDMP20], so it is expected that this question has an affirmative answer. However, when trying to prove this
On symbolic factors of $S$-adic subshifts of finite alphabet rank

using Theorem 1.1 we realize that the naturally inherited $S$-adic structure of finite alphabet rank that a symbolic factor has is never recognizable. Moreover, this last property is crucial for many of the currently known techniques to handle finite topological rank systems (even in the substitutive case it is a deep and fundamental theorem of Mossé), so it is not clear why it would be always possible to obtain this property while keeping the alphabet rank bounded or why recognizability is not connected with a dynamical property of the system. Thus, an answer to this question seems to be fundamental to the understanding of the finite topological rank class.

This question has been recently addressed, first in [Esp20] by purely combinatorial methods, and then also in [GH20] in the BV formulation by using an abstract construction from [AEGR15]. In this work we refine both approaches and obtain, as a first consequence, the optimal answer to Question 1.2 in a more general non-minimal context:

**Theorem 1.3.** Let $(X,T)$ be an $S$-adic subshift generated by an everywhere growing and proper directive sequence of alphabet rank equal to $K$, and $\pi: (X,T) \to (Y,T)$ an aperiodic symbolic factor. Then, $(Y,T)$ is an $S$-adic subshift generated by an everywhere growing, proper and recognizable directive sequence of alphabet rank at most $K$.

In particular, the topological rank cannot increase after a factor map (Corollary 4.6) and, therefore, defines a dynamical invariant. An interesting corollary of the underlying construction of the proof of Theorem 1.3 is the coalescence property for this kind of systems, in the following stronger form:

**Corollary 1.4.** Let $(X,T)$ be an $S$-adic subshift generated by an everywhere growing and proper directive sequence of alphabet rank equal to $K$, and $(X,T) \stackrel{\pi_1}{\to} (X_1,T) \stackrel{\pi_2}{\to} \ldots \stackrel{\pi_L}{\to} (X_L,T)$ be a chain of aperiodic symbolic factors. If $L > \log_2 K$, then at least one $\pi_j$ is a conjugacy.

One of the results in [Dur00] is that factor maps between aperiodic linearly recurrent subshifts are finite to one. In particular, they are almost $k$-to-1 for some finite $k$. For finite topological rank subshifts, we prove:

**Theorem 1.5.** Let $\pi: (X,T) \to (Y,T)$ be a factor map between aperiodic minimal subshifts. Suppose that $(X,T)$ has topological rank equal to $K$. Then $\pi$ is almost $k$-to-1 for some $k \leq K$.

In [Dur00], the author proved that a subshift $X$ is linearly recurrent if and only if it is generated by a primitive and proper directive sequence $\sigma = (\sigma_n)_{n \geq 0}$ such that $|\sigma_n| \leq C$ for all $n \in \mathbb{N}$, where $C$ is a uniform constant, and that this kind of systems have finitely many aperiodic subshift factors up to conjugacy. Inspired by this result, we use ideas from the proof of Theorem 1.3 to obtain:

**Theorem 1.6.** Let $(X,T)$ be a minimal subshift of finite topological rank. Then, $(X,T)$ has at most $32^K K^{24K}$ aperiodic symbolic factors up to conjugacy.

In [GH20] it is proved that Cantor factors of finite topological rank systems are either equicontinuous or subshifts. This together with the results of this article gives a rough picture of the set of totally disconnected factors of a given finite topological rank system: they are either equicontinuous or subshifts satisfying the properties in Theorems 1.3, 1.4, 1.6 and 1.5. Now, in a topological sense, totally disconnected
factors of a given system \((X,T)\) are “maximal”, so, the natural next step in the study of finite topological rank systems is asking about the connected factors. As we have seen, the finite topological rank condition is a rigidity condition. By this reason, we think that the following question has an affirmative answer:

**Question 1.7.** Let \((X,T)\) be a minimal system of finite topological rank and \(\pi: (X,T) \to (Y,T)\) be a factor map. Suppose that \(Y\) is connected. Is it true that \((Y,T)\) is equicontinuous?

We remark that the finite topological rank class contains all minimal subshifts of non-superlinear complexity [DDMP20], but even for the much smaller class of linear complexity subshifts the author is not aware of results concerning Question 1.7.

1.1. **Organization.** In the next section we give the basic background in topological and symbolic dynamics needed in this article. Section 3 is devoted to prove some combinatorics lemmas. The main results about the topological rank of factors are stated and proved in Section 4. Next, in Section 5 we prove Theorem 1.5, which is a consequence of the so-called Critical Factorization Theorem. Finally, in Section 6 we study the problem about the number of symbolic factors and prove Theorem 1.6.

2. **Preliminaries**

All the intervals we will consider consist of integer numbers, i.e., \([a,b] = \{k \in \mathbb{Z} : a \leq k \leq b\}\) with \(a, b \in \mathbb{Z}\). For us, the set of natural numbers starts with zero, i.e., \(\mathbb{N} = \{0, 1, 2, \ldots\}\).

2.1. **Basics in topological dynamics.** A *topological dynamical system* (or just a system) is a pair \((X,T)\) where \(X\) is a compact metric space and \(T: X \to X\) is a homeomorphism of \(X\). We denote by \(\text{Orb}_T(x)\) the orbit \(\{T^n x : n \in \mathbb{Z}\}\) of \(x \in X\). A point \(x \in X\) is \(p\)-periodic if \(T^p x = x\), periodic if it is \(p\)-periodic for some \(p \geq 1\) and aperiodic otherwise. A topological dynamical system is aperiodic if any point \(x \in X\) is aperiodic, is minimal if the orbit of every point is dense in \(X\) and is Cantor if \(X\) is a Cantor space. We use the letter \(T\) to denote the action of a topological dynamical system independently of the base set \(X\). The *hyperspace* of \((X,T)\) is the system \((2^X,T)\), where \(2^X\) is the set of all closed subsets of \(X\) with the topology generated by the Hausdorff metric \(d_H(A,B) = \max(\sup_{x\in A} d(x,A), \sup_{y\in B} d(y,A))\), and \(T\) the action \(A \mapsto T(A)\).

A *factor* between the topological dynamical systems \((X,T)\) and \((Y,T)\) is a continuous function \(\pi\) from \(X\) onto \(Y\) such that \(\pi \circ T = T \circ \pi\). We use the notation \(\pi: (X,T) \to (Y,T)\) to indicate the factor. A factor map \(\pi: (X,T) \to (Y,T)\) is almost \(K\)-to-1 if \(\#\pi^{-1}(y) = K\) for all \(y\) in a residual subset of \(Y\). We say that \(\pi\) is distal if whenever \(\pi(x) = \pi(x')\) and \(x \neq x'\), we have \(\inf_{k \in \mathbb{Z}} \text{dist}(T^k x, T^k x') > 0\).

Given a system \((X,T)\), the *Ellis semigroup* \(E(X,T)\) associated to \((X,T)\) is defined as the closure of \(\{x \mapsto T^n x : n \in \mathbb{Z}\}\) in the product topology, where the semi-group operation is given by the composition of functions. On \(X\) we may consider the \(E(X,T)\)-action given by \(x \mapsto ux\). Then, the closure of the orbit under \(T\) of a point \(x \in X\) is equal to the orbit of \(x\) under \(E(X,T)\). If \(\pi: (X,T) \to (Y,T)\) is a factor between minimal systems, then \(\pi\) induces a surjective
map \( \pi^* : E(X,T) \to E(Y,T) \) which is characterized by the formula
\[
\pi(ux) = \pi^*(u)\pi(x) \quad \text{for all } u \in E(X,T) \text{ and } x \in X.
\]

If the context is clear, we will not distinguish between \( u \) and \( \pi^*(u) \). When \( u \in E(2^X, T) \) we write \( u \circ A \) instead of \( u A \). Also, we identify \( X \) with \( \{ x \} \subseteq 2^X : x \in X \}, so that the restriction map \( E(2^X, T) \to E(X,T) \) which sends \( u \in E(2^X, T) \) to the restriction \( u|_X : X \to X \) is an onto morphism of semigroups. As above, we will not distinguish between \( u \in 2^X \) and \( u|_X \). Finally, we can describe more explicitly \( u \circ A \) as follows: it is the set of all \( x \in X \) for which we can find nets \( x_\lambda \in A \) and \( m_\lambda \in \mathbb{Z} \) such that \( \lim_\lambda T^{m_\lambda} x_\lambda = x \) and \( \lim_\lambda T^{m_\lambda} = u \).

### 2.2. Basics in symbolic dynamics.

#### 2.2.1. Words and subshifts. Let \( \mathcal{A} \) be a finite set that we call alphabet. Elements in \( \mathcal{A} \) are called letters or symbols. The set of finite sequences or words of length \( \ell \in \mathbb{N} \) with letters in \( \mathcal{A} \) is denoted by \( \mathcal{A}^\ell \), the set of onesided sequences \( (x_n)_{n \in \mathbb{N}} \) in \( \mathcal{A} \) is denoted by \( \mathcal{A}^N \) and the set of twosided sequences \( (x_n)_{n \in \mathbb{Z}} \) in \( \mathcal{A} \) is denoted by \( \mathcal{A}^Z \). Also, a word \( w = a_1 \cdots a_k \in \mathcal{A}^k \) can be seen as an element of the free semigroup \( \mathcal{A}^+ \) endowed with the operation of concatenation. The integer \( \ell \) is the length of \( w \) and is denoted by \( |w| = \ell \). We use the notation \( w_{[i,j]} = a_{i+1}a_{i+2} \cdots a_j \) for \( 0 \leq i < j \leq \ell \). We write \( u \sqsubseteq v \) when \( u \) is a subword of \( v \), and sometimes say that \( u \) occurs in \( v \). The word \( w \in \mathcal{A}^+ \) is \( |w| \)-periodic, with \( u \in \mathcal{A}^+ \), if \( w \sqsubseteq u^\infty := uuu \cdots \). We define per: \( \mathcal{A}^+ \to \mathbb{N} \), where per(\( w \)) is the smallest \( p \) for which \( w \) is \( p \)-periodic. For \( \mathcal{W} \subseteq \mathcal{A}^+ \), we write \( \langle \mathcal{W} \rangle := \{ w \in \mathcal{W} : p \text{ per }(w) \} \) and \( |\mathcal{W}| := \max_{w \in \mathcal{W}} |w| \).

The shift map \( T: \mathcal{A}^Z \to \mathcal{A}^Z \) is defined by \( T((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}} \). A subshift is a topological dynamical system \( (X,T) \) where \( X \) is a closed and \( T \)-invariant subset of \( \mathcal{A}^Z \) (we consider the product topology in \( \mathcal{A}^Z \)) and \( T \) is the shift map. Classically one identifies \( (X,T) \) with \( X \), so one says that \( X \) itself is a subshift. When we say that a sequence in a subshift is periodic (resp. aperiodic), we implicitly mean that this sequence is periodic (resp. aperiodic) for the action of the shift. For \( x \in \mathcal{A}^Z \), \( \text{per}(x) \) is the smallest \( p \geq 1 \) for which \( x \) is \( p \)-periodic. The language of a subshift \( X \subseteq \mathcal{A}^Z \) is the set \( \mathcal{L}(X) \) of all words \( w \in \mathcal{A}^+ \) that occur in some \( x \in X \). We use the notations \( \mathcal{L}_\ell(X) \) and \( \mathcal{L}_{\geq \ell}(X) \) for the sets of words \( w \in \mathcal{L}(X) \) of length \( \ell \) and length at least \( \ell \), respectively. We remark that \( \mathcal{L}_1(X) \) is simply the set of letters \( a \in \mathcal{A} \) that occurs in some \( x \in X \).

The pair \((x, \tilde{x}) \in \mathcal{A}^Z \times \mathcal{A}^Z \) is right-asymptotic if there exist \( k \in \mathbb{Z} \) satisfying \( x_{(k,\infty)} = \tilde{x}_{(k,\infty)} \) and \( x_k \neq \tilde{x}_k \). If moreover \( k = 0 \), \((x, \tilde{x}) \) is a centered right-asymptotic. A right-asymptotic tail is an element \( x_{(0,\infty)} \), where \((x, \tilde{x}) \) is a centered right-asymptotic pair. We make similar definitions for left-asymptotic pairs and tails.

#### 2.2.2. Morphisms and substitutions. Let \( \mathcal{A} \) and \( \mathcal{B} \) be finite alphabets and \( \tau: \mathcal{A}^+ \to \mathcal{B}^+ \) be a morphism between the free semigroups that they define. Then, \( \tau \) extends naturally to maps from \( \mathcal{A}^N \) to itself and from \( \mathcal{A}^Z \) to itself in the obvious way by concatenation (in the case of a twosided sequence we apply \( \tau \) to positive and negative coordinates separately and we concatenate the results at coordinate zero).

We say that \( \tau \) is primitive if for every \( a \in \mathcal{A} \), all letters \( b \in \mathcal{B} \) occur in \( \tau(a) \), is \( r \)-proper, with \( r \in \mathbb{N} \), if there exist \( u, v \in \mathcal{B}^r \) such that \( \tau(a) \) starts with \( u \) and ends with \( v \) for any \( a \in \mathcal{A} \), and proper when is 1-proper. The minimum and
maximum length of \( \tau \) are, respectively, the numbers \( \langle \tau \rangle := \min_{a \in A} |\tau(a)| \) and \( |\tau| := |\tau(A)| = \max_{a \in A} |\tau(a)| \).

We observe that any map \( \tau: A \rightarrow B^+ \) can be naturally extended to a morphism (that we also denote by \( \tau \)) from \( A^+ \) to \( B^+ \) by concatenation, and we use this convention throughout the document. So, from now on, all maps between finite alphabets are considered to be morphisms between their free semigroups.

**Definition 2.1.** Let \( X \subseteq A^2 \) be a subshift and \( \sigma: A^+ \rightarrow B^+ \) be a morphism. We say that \((k,x) \in \mathbb{Z} \times X\) is a \( \sigma \)-factorization of \( y \in B^2 \) in \( X \) if \( y = T^k \sigma(x) \). If moreover \( k \in [0,|\sigma(x_0)|) \), \((k,x)\) is a centered \( \sigma \)-factorization in \( X \). The cuts of \((k,x)\) are defined by

\[
c_{\sigma,j}(k,x) = \begin{cases} 
  k & \text{if } j = 0, \\
  k + |\sigma(x_{[0,j)})| & \text{if } j > 0, \\
  k + |\sigma(x_{[j,0)})| & \text{if } j < 0.
\end{cases}
\]

We write \( C_\sigma(k,x) = \{c_{\sigma,j}(k,x) : j \in \mathbb{Z}\} \). The pair \((X,\sigma)\) is recognizable if every point \( y \in B^2 \) has at most one centered factorization, and recognizable with constant \( r \in \mathbb{N} \) if whenever \( y_{[-r,r]} = y'_{[-r,r]} \) and \((k,x), (k',x')\) are centered \( \sigma \)-factorizations of \( y, y' \in B^2 \) in \( X \), respectively, we have \((k,x_0) = (k',x'_0)\).

**Remark 2.2.** In the context of the previous definition:

(i) The point \( y \in B^2 \) has a (centered) factorization over \((X,\sigma)\) if and only if \( y \) belongs to the subshift \( Y := \bigcup_{n \in \mathbb{Z}} T^n \sigma(X) \). Hence, \((X,\sigma)\) is recognizable if and only if every \( y \in Y \) has a exactly one centered factorization in \((X,\sigma)\).

(ii) If \((k,x)\) is a factorization of \( y \in B^2 \) over \((X,\sigma)\), then \((k + |\sigma(x_{[0,j]})|, T^j x)\) is a factorization of \( y \) over \((X,\sigma)\) for \( j > 0 \) and \((k - |\sigma(x_{[-j,0]})|, T^{-j} x)\) is a factorization over \((X,\sigma)\) for \( j < 0 \). Two elements in this class of factorizations are said to be equivalent, and there is exactly one of them that is centered.

(iii) If \((X,\sigma)\) is recognizable, then it is recognizable with constant \( r \) for some \( r \in \mathbb{N} \) [DDMP20].

The behavior of recognizability under composition of morphisms is given by the following lemma.

**Lemma 2.3** ([DISTY19]). Let \( \sigma: A^* \rightarrow B^* \) and \( \tau: B^* \rightarrow C^* \) be non-erasing morphisms, \( X \subseteq A^2 \) be a subshift and \( Y = \bigcup_{k \in \mathbb{Z}} T^k \sigma(X) \). Then, \((X,\tau \sigma)\) is recognizable if and only if \((X,\sigma)\) and \((Y,\tau)\) are recognizable.

Let \( Y \subseteq B^2 \), \( Z \subseteq C^2 \) be subshifts and \( \pi: (Y,T) \rightarrow (Z,T) \) a factor map. The classical Curtis–Hedlund–Lyndon Theorem asserts that \( \pi \) has a local code, this is, a function \( \psi: A^{2r+1} \rightarrow B \), where \( r \in \mathbb{N} \), such that \( \pi(y) = (\psi(y_{[i-r,i+r]}))_{i \in \mathbb{Z}} \) for all \( y \in Y \). The integer \( r \) is called the radius of \( \pi \) (although it is nonunique). The following construction is from the folklore.

**Lemma 2.4.** Let \( \sigma: A^+ \rightarrow B^+ \) be a morphism, \( X \subseteq A^2 \), \( Z \subseteq C^2 \) be subshifts and \( Y = \bigcup_{k \in \mathbb{Z}} T^k \sigma(X) \). Suppose that \( \pi: (Y,T) \rightarrow (Z,T) \) is a factor map of radius \( r \) and that \( \sigma \) is \( r \)-proper. Then, there exists a morphism \( \tau: A^+ \rightarrow C^+ \) such that \( |\tau(a)| = |\sigma(a)| \) for any \( a \in A \), \( Z = \bigcup_{k \in \mathbb{Z}} T^k \tau(X) \) and the following diagram
Observe that and primitive. This set clearly defines a subshift that we call the \( \tau \)-adic subshift generated by \( \pi \). For 0 \( \leq \) \( \sigma \) \( \leq \) \( \tau \), we define \( \pi(\tau(x)) = \tau(x) \) for all \( x \in X \) and diagram (1) commutes. In particular, \( Z \supseteq \bigcup_{k \in \mathbb{Z}} T^k \tau(X) \). It left to prove the other inclusion. Let \( z \in Z \). Since \( \pi \) is onto, there exists a \( \sigma \)-factorization \( (k, x) \) of \( y \) in \( X \). Then, from the diagram, we have \( z = \pi(y) = \pi(T^k \tau(x)) = T^k \pi(\sigma(x)) = T^k \tau(x) \). This proves that \( Z \subseteq \bigcup_{k \in \mathbb{Z}} T^k \tau(X) \) and ends the proof.

2.2.3. \( S \)-adic subshifts. We recall the definition of an \( S \)-adic subshift as stated in [BSTY19]. A directive sequence \( \tau = (\tau_n : \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+) \) is a sequence of morphisms. For 0 \( \leq n < N \), we denote by \( \tau_{[n,N)} \) or \( \tau_{[n,N-1]} \), the morphism \( \tau_0 \circ \tau_1 \circ \cdots \circ \tau_{N-1} \). We say \( \tau \) is everywhere growing if \( \lim_{N \to +\infty} \langle \tau_{[0,N)} \rangle = +\infty \) and primitive if for any \( n \in \mathbb{N} \) there exists \( N > n \) such that \( \tau_{[n,N)} \) is primitive. Observe that \( \tau \) is everywhere growing if \( \tau \) is primitive.

For \( n \in \mathbb{N} \), we define \( X_{\tau}^{(n)} = \{ x \in \mathcal{A}_{\mathbb{Z}} : \text{for all } k \in \mathbb{N}, x_{[-k,k]} \text{ occurs in } \tau_{[n,N)}(a) \text{ for some } N > n \text{ and } a \in \mathcal{A}_N \} \).

This set clearly defines a subshift that we call the \( n \)-th level of the \( S \)-adic subshift generated by \( \tau \). We set \( X_{\tau} = X_{\tau}^{(0)} \) and simply call it the \( S \)-adic subshift generated by \( \tau \). If \( \tau \) is everywhere growing, then every \( X_{\tau}^{(n)} \), \( n \in \mathbb{N} \), is non-empty; if \( \tau \) is primitive, then \( X_{\tau}^{(n)} \) is minimal for every \( n \in \mathbb{N} \). There are non-primitive directive sequences that generate minimal subshifts.

The relation between levels of an \( S \)-adic subshift is given by the following lemma.

Lemma 2.5 ([BSTY19], Lemma 4.2). Let \( \tau = (\tau_n : \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+) \) be a directive sequence of morphisms. If 0 \( \leq n < N \) and \( x \in X_{\tau}^{(n)} \), then there exists a (centered) factorization in \( (X_{\tau}^{(N)}, \tau_{[n,N)}) \). In particular, \( X_{\tau}^{(n)} = \bigcup_{k \in \mathbb{Z}} T^k (X_{\tau}^{(N)}) \).

A morphism \( \tau : \mathcal{A}^+ \to \mathcal{B}^+ \) is weakly primitive if every letter \( b \in \mathcal{B} \) occurs in some \( \tau(a) \), with \( a \in \mathcal{A} \). A directive sequence \( \tau \) is weakly primitive if, for all \( n \geq 0 \), \( \tau_n \) is weakly primitive.

Lemma 2.6. If \( \tau \) is everywhere growing and weakly primitive, then, for all \( n \in \mathbb{N} \), all letters \( a \in \mathcal{A}_n \) occur in some \( x \in X_{\tau}^{(n)} \), this is, \( \mathcal{A}_n \subseteq \mathcal{L}(X_{\tau}^{(n)}) \).

We always can suppose that \( \tau \) is weakly primitive, in the following sense:

Lemma 2.7. Let \( \hat{\mathcal{B}}_n \) be the set of those letters \( b \in \mathcal{B}_n \) which occur in some \( \tau_{[n,N)}(a) \), \( a \in \mathcal{B}_N \), for any \( N > n \), and define \( \hat{\tau}_n \) as the restriction of \( \tau_n \) to \( \hat{\mathcal{B}}_{n+1} \). Then, \( \hat{\tau}_n \) extends to a morphism \( \hat{\tau}_n : \hat{\mathcal{B}}_{n+1}^+ \to \hat{\mathcal{B}}_n^+ \), the directive sequence \( \hat{\tau} = (\hat{\tau}_n)_{n \geq 0} \) is weakly primitive and \( X_{\hat{\tau}}^{(n)} = X_{\tau}^{(n)} \) for all \( n \in \mathbb{N} \).
We define the **alphabet rank** of a directive sequence $\tau$ as

$$AR(\tau) = \liminf_{n \to +\infty} \# A_n.$$  

A **contraction** of $\tau$ is a sequence $\bar{\tau} = (\tau_{[n_k,n_{k+1}]} : A^*_{n_{k+1}} \to A^*_{n_k})_{k \geq 0}$, where $0 = n_0 < n_1 < n_2 < \ldots$. Observe that any contraction of $\tau$ generates the same $S$-adic subshift $X_\tau$. When $\tau$ has finite alphabet rank, there exists a contraction $\tilde{\tau} = (\tilde{\tau}_{[n_k,n_{k+1}]} : A^*_{n_{k+1}} \to A^*_{n_k})_{k \geq 0}$ of $\tau$ in which $A_{n_k}$ has cardinality $AR(\tau)$ for every $k \geq 1$.

Finite alphabet rank $S$-adic subshifts are always **eventually recognizable**:

**Theorem 2.8** ([DDMP20, Theorem 3.7]). Let $\sigma$ be an everywhere growing directive sequence of alphabet rank equal to $K$. Suppose that $X_\sigma$ is aperiodic. Then, at most $\log_2 K$ levels $(X_\sigma^{(n)}, \sigma_n)$ are not recognizable.

We will also need the following property.

**Proposition 2.9** ([EM20, Theorem 3.3]). Let $(X,T)$ be an $S$-adic subshift generated by an everywhere growing directive sequence of alphabet rank $K$. Then, $X$ has at most $144 K^7$ right (resp. left) asymptotic tails.

**Proof.** Let $L = 144 K^7$. In the proof of Theorem 3.3 in [EM20] the authors show the following: if $\{(x_j, y_j) : j \in [0, L]\}$ is a collection of points $x_j, y_j \in X$ such that $(x_j)(\infty, 0) = (y_j)(\infty, 0)$ and $(x_j)_0 \neq (y_j)_0$ for any $j \in [0, L]$, then there exist $j \neq j' \in [0, L]$ satisfying $(x_j)(\infty, 0) = (x_{j'})(\infty, 0)$. In our language, this is equivalent to saying that $X$ has at most $L$ left-asymptotic tails. Since this is valid for any $S$-adic subshift generated by an everywhere growing directive sequence of alphabet rank $K$, $L$ is also an upper bound for right-asymptotic tails. $\square$

## 3. Combinatorics on words lemmas

In this section we present several combinatorial lemmas that will be used throughout the article.

### 3.1. Lowering the rank

Let $\sigma : A^+ \to B^+$ be a morphism. Following ideas from [RS97], we define the **rank** of $\sigma$ as the cardinality of the smallest set $D \subseteq B^+$ such that $\sigma(A^+) \subseteq D^+$. Equivalently, the rank is the minimum cardinality of an alphabet $C$ in a decomposition $A^+ \xrightarrow{\sigma} C^+ \xrightarrow{\pi} B^+$ such that $\sigma = pq$. In this subsection we prove Lemma 3.4 which states that in certain technical situation, the rank of the morphism $\sigma$ under consideration is small and its decomposition $\sigma = pq$ satisfies additional properties.

If $a \neq b \in A$ are different and $\bar{a}$ is a letter not in $A$, then we define $\phi_{a,b} : A^+ \to (A \setminus \{b\})^+$, $\psi_{a,b} : A^+ \to A^+$ and $\theta_{a,\bar{a}} : A^+ \to (A \cup \{\bar{a}\})^+$ by

$$\phi_{a,b}(c) = \begin{cases} c & \text{if } c \neq b, \\ a & \text{if } c = b. \end{cases} \quad \psi_{a,b}(c) = \begin{cases} c & \text{if } c \neq b, \\ ab & \text{if } c = b. \end{cases} \quad \theta_{a,\bar{a}}(c) = \begin{cases} c & \text{if } c \neq a, \\ \bar{a}a & \text{if } c = a. \end{cases}$$

Observe that these morphisms are weakly primitive.

For a morphism $\sigma : A^+ \to B^+$, we define $|\sigma|_1 = \sum_{a \in A} |\sigma(a)|$. When $u, v, w \in A^+$ satisfy $w = uv$, we say that $u$ is a prefix of $w$, $v$ a suffix of $w$ and write $v = u^{-1}w$. Recall that $1$ stands for the empty word.

**Lemma 3.1.** Let $\sigma : A^+ \to B^+$ be a morphism.
(i) If $\sigma(a) = \sigma(b)$ for some $a \neq b \in A$, then $\sigma = \sigma'\phi_{a,b}$, where $\sigma': (A \setminus \{b\})^+ \to B^+$ is the restriction of $\sigma$ to $(A \setminus \{b\})^+$.

(ii) If $\sigma(a)$ is a prefix of $\sigma(b)$ and $\sigma(a) \neq \sigma(b)$ for some $a, b \in A$, then $\sigma = \sigma'\psi_{a,b}$, where $\sigma': A^+ \to B^+$ is defined by

\[
\sigma'(c) = \begin{cases} 
\sigma(c) & \text{if } c \neq b, \\
t & \text{if } c = b.
\end{cases}
\]

(iii) If $\sigma(a) = st$ for some $s, t \in B^+, a \in A$, then $\sigma = \sigma'\theta_{a,\tilde{a}}$, where $\sigma': (A \cup \{\tilde{a}\})^+ \to B^+$ is defined by

\[
\sigma'(c) = \begin{cases} 
\sigma(c) & \text{if } c \neq a, \tilde{a}, \\
s & \text{if } c = \tilde{a}, \\
t & \text{if } c = a.
\end{cases}
\]

**Lemma 3.2.** Let $u, v \in A^+$ and $J$ be an index set. For $i \in J$, let $\sigma_i: A^+ \to B_i^+$ be a morphism and $w^i \in B_i^+$, with $|w^i| \geq |\sigma_i|_1$. Suppose one of the following conditions hold:

(I) $u, v$ start with different letters and $w^i$ is prefix of both $\sigma_i(u), \sigma_i(v)$ for any $i \in J$.

(II) $u, v$ end with different letters and $w^i$ is suffix of both $\sigma_i(u), \sigma_i(v)$ for any $i \in J$.

Further, assume that

\[
|\sigma_i(c)| = |\sigma_j(c)| \quad \text{for all } c \in A, i, j \in J.
\]

Then, there exist a weakly primitive morphism $q: A^+ \to C^+$, with $\#C < \#A$, and, for every $i \in J$, a morphism $p_i: C^+ \to B_i^+$ satisfying $\sigma_i = p_iq$.

**Proof.** We only do the case (I), as the other is similar. We fix $i \in J$. By contradiction, we assume that $u, v, \sigma_1, w^1, \sigma_2, w^2, \ldots$ are counterexamples for the case (I) of the lemma. Moreover, we suppose that $|\sigma_i|_1$ is as small as possible.

We write $u = au', v = bv'$, where $a, b \in A$. Observe that, since $w^j$ is a prefix of both $\sigma_j(au'), \sigma_j(bv')$ and $|w^j| \geq |\sigma_j|_1 \geq |\sigma_j|$, we have, for any $j \in J$, that

\[
\text{one of } \sigma_j(a), s^t\sigma_j(b) \text{ is a prefix of the other.}
\]

First, we the the case $\sigma_1(a) = \sigma_1(b)$. Then, by (4), $|\sigma_j(a)| = |\sigma_j(b)| = |\sigma_j(c)|$. This and (5) imply that $\sigma_j(a) = \sigma_j(b)$ for any $j \in J$. Thus, by (1) of Lemma 3.1 each $\sigma_j$ can be written as $\sigma_j'\phi_{a,b}$, where $\sigma_j'$ is the restriction of $\sigma_j$ to $(A \setminus \{b\})^+$. Since $\phi_{a,b}$ is weakly primitive, the conclusion of the lemma holds, contrary to our assumptions.

We conclude that $\sigma_1(a) \neq \sigma_1(b)$. Then, by (5), one of $\sigma_j(a), \sigma_j(b)$ is prefix of the other. Without loss of generality, we can suppose that $\sigma_1(a)$ is a prefix of $\sigma_1(b)$; we write $\sigma_1(b) = \sigma_1(a)t^t$ with $k' := |t^t| > 0$. By (4), we have $|\sigma_j(a)| = |\sigma_j(a)| = |\sigma_j(b)| - k' = |\sigma_j(b)| - k'$ and then, by (5), $\sigma_j(b) = \sigma_j(a)t^t$ for some $t^t \in B_i^+$ also. Thus, we can use (2) of Lemma 3.1 to write, for any $j \in J$, $\sigma_j = \sigma_j'\psi_{a,b}$, where $\sigma_j'$ is defined as in (2). Observe that $|\sigma_j'(c)| = |\sigma_j(c)| = |\sigma_i(c)|$ if $c \neq b$ and $|\sigma_j'(b)| = |\sigma_j(b)| - |\sigma_j(a)| = |\sigma_i(b)| - |\sigma_i(a)| = |\sigma_j'(b)|$, so

\[
|\sigma_j'(c)| = |\sigma_j'(c)| \quad \text{for any } c \in A, j \in J.
\]
We put \( \tilde{w} = \sigma_j(a)^{-1}w, \tilde{u} = \psi_{a,b}(u') \) and \( \tilde{v} = b\psi_{a,b}(v') \). Note that \( \psi_{a,b}(c) \) never starts with \( b \), so

\[ (7) \quad \tilde{u}, \tilde{v} \text{ start with different letters.} \]

Also,

\[ \begin{align*}
(8) \quad \tilde{w} &= \sigma_j(a)^{-1}w \leq_p \sigma_j(a)^{-1}\sigma_j(au') = \sigma_j(u') = \sigma_j'(\psi_{a,b}(u')) = \sigma_j'(\tilde{u}), \\
(9) \quad \tilde{w} &= \sigma_j(a)^{-1}w \leq_p \sigma_j(a^{-1}bv') = \sigma_j'(\psi_{a,b}(a^{-1}bv')) = \sigma_j'(b\psi_{a,b}(v')) = \sigma_j'(\tilde{v}),
\end{align*} \]

where, we recall, \( \leq_p \) stands for the prefix relation. Finally, we compute, using (4) and (2),

\[ (10) \quad |\tilde{w}| = |w| - |\sigma_i(a)| \geq |\sigma_i| - |\sigma_i(a)| = |\sigma_i'|_1, \]

Then, by (6), (7), (8), (9), and (10), we have that \( \sigma_j', \tilde{w}, \tilde{u}, \tilde{v} \) satisfy the hypothesis of case (1) of this lemma. Since \( |\sigma_i'|_1 = |\sigma_i| - |\sigma_i(a)| < |\sigma_i|_1 \), the minimality of \( |\sigma_i|_1 \) implies that there exist a weakly primitive morphism \( q': A^+ \to C^+ \), with \( \#C < \#A \), and, for every \( j \in J \), a morphism \( p_j: C^+ \to B^+_j \) satisfying \( \sigma_j' = p_jq' \). But then \( q := q'\psi_{a,b} \) is also weakly primitive and \( \sigma_j = pjq \) for every \( j \in J \), so the conclusion of the lemma holds for \( \sigma_j \), contrary to our assumptions.

\[ \tag*{\Box} \]

**Lemma 3.3.** Let \( u, v \in A^+ \), with \( a \) the first letter of \( u, \sigma: A^+ \to B^+ \) a morphism and \( w, s \in B^+ \), with \( |w| \geq |\sigma_1| + |s| \). Suppose that one of the following conditions hold:

(I) \( s \) is a strict prefix of \( \sigma(a) \), \( w \) is a prefix of both \( \sigma(u), s\sigma(v) \), and \( u, v \) start with different letters if \( s = 1 \).

(II) \( s \) is a strict suffix of \( \sigma(a) \), \( w \) is a suffix of both \( \sigma(u), s\sigma(v) \), and \( u, v \) end with different letters if \( s = 1 \).

Then, there exist morphisms \( q: A^+ \to C^+, p: C^+ \to B^+ \), such that \( \#C < \#A \), \( q \) is weakly primitive, \( |p|_1 < |\sigma|_1 \), and \( \sigma = pq \).

**Proof:** We only do the case (I), as the other is similar. So, suppose that \( s \) is a strict prefix of \( \sigma(a) \), \( w \) is a prefix of both \( \sigma(u), s\sigma(v) \), and \( u, v \) start with different letters if \( s = 1 \).

First, we assume \( s = 1 \). Then \( u, v \) start with different letters, so (I) of Lemma 3.2 can be applied (where the index set \( J \) is chosen as a singleton) to obtain a decomposition \( A^+ \xrightarrow{\sigma} C^+ \xrightarrow{p} B^+ \) such that \( q \) is weakly primitive, \( \#C < \#A \), and \( \sigma = pq \). Since \( C \) is strictly smaller than \( A \), \( |p|_1 < |\sigma|_1 \). Thus, the conclusion of the lemma holds.

It left to consider the case \( s \neq 1 \), so, we assume \( s \neq 1 \). It is convenient to write \( u = au', v = bv' \), where \( b \) is the first letter of \( v \). Since \( s \) is a strict prefix of \( \sigma(a) \), we can write \( \sigma(a) = st \), with \( t \in B^+ \). Then, as \( s \neq 1 \), we can use (3) of Lemma 3.1 to factorize \( s = s'\theta_{a,\tilde{a}} \), where \( \tilde{a} \) is a letter not in \( A \) and \( s' \) is defined as in (3). We put \( \tilde{w} = s^{-1}w, \tilde{u} = a\theta_{a,\tilde{a}}(u'), \tilde{v} = \theta_{a,\tilde{a}}(bv') \). Note that \( \theta_{a,\tilde{a}}(c) \) never starts with \( a \), so

\[ (11) \quad \tilde{u}, \tilde{v} \text{ start with different letters.} \]

Also,

\[ \begin{align*}
(12) \quad \tilde{w} &= s^{-1}w \leq_p s^{-1}\sigma(au') = \sigma'(\tilde{u})^{-1}\sigma'(\theta_{a,\tilde{a}}(au')) = \sigma'(a\theta_{a,\tilde{a}}(u')) = \sigma'(\tilde{u}), \\
(13) \quad \tilde{w} &= s^{-1}w \leq_p s^{-1}s\sigma(bv') = \sigma'(\theta_{a,\tilde{a}}(bv')) = \sigma'(\tilde{v}).
\end{align*} \]
Finally, we compute, using (3),

\[(14) \quad |\hat{w}| = |w| - |s| \geq |\sigma|_1 = |\sigma'|_1.\]

Then, by (11), (12), (13), and (14), we have that \(\sigma', \hat{w}, \hat{u}, \hat{v}\) satisfy the hypothesis of case (I) of Lemma 3.2 (with the index set \(J\) chosen as a singleton). Thus, we obtain morphisms \(q' : (A \cup \{\hat{a}\})^+ \to C^+\) and \(p : C^+ \to B^+\) such that \(\#C < \#(A \cup \{\hat{a}\})\), \(q'\) is weakly primitive and \(\sigma' = pq'\). Hence, \(\#C \leq \#A, \ q := q'\theta_{a,\hat{a}}\) is weakly primitive and \(\sigma = p \circ q\theta_{a,\hat{a}} = pq\). Moreover, since \(\theta_{a,\hat{a}}\) is not the identity function, we have \(|p|_1 < |\sigma|_1\). This ends the proof. \(\square\)

For an alphabet \(A\), let \(A^{++}\) be the set of words \(w \in A^+\) in which all letters occur. Then, \(\sigma\) is weakly primitive if and only if \(\sigma(A^{++}) \subseteq B^{++}\).

**Lemma 3.4.** Let \(\phi : A^+ \to C^+, \tau : B^+ \to C^+\) be morphisms such that \(\tau\) is \(\ell\)-proper, with \(\ell \geq |\phi|_1\), and \(\phi(A^+) \cap \tau(B^{++}) \neq \emptyset\). Then, there exist \(B^+ \xrightarrow{q} D^+ \xrightarrow{p} C^+\) such that

(i) \(\#D \leq \#A\),

(ii) \(\tau = pq\),

(iii) \(q(B^{++}) \subseteq D^{++}\),

(iv) \(q\) is proper.

**Proof.** By contradiction, we suppose that \(\phi, \tau\) are counterexamples for the lemma with \(|\phi|_1\) as small as possible.

Since \(\phi(A^+) \cap \tau(B^{++})\) is non-empty, there exist \(u = u_1 \cdots u_n \in A^+\) and \(v = v_1 \cdots v_m \in B^+\) with \(z = \phi(u) = \tau(v)\). If \(m = 1\), then, since \(v \in B^{++}\), we have \(\#B = \{v_1\}\) and the conclusion of the lemma trivially holds for \(B = \{a \in \sum : a\text{ occurs in }\tau(v_1)\}\), \(q : B^+ \to B^+\), \(v_1 \mapsto \tau(v_1)\) and \(p : B^+ \to C^+\) the inclusion map, contradicting the fact that \(\phi, \tau\) are counterexamples. Hence, \(m \geq 2\).

For \(k \in [1, m-1]\), let \(i_k \in [1, n]\) be the smallest integer for which \(|\tau(v_1 \cdots v_k)| < |\phi(u_1 \cdots u_{i_k})|\) holds. Since \(\phi(u_1) \leq |\phi|_1 \leq \ell \leq |\tau(v_1 \cdots v_k)|\), \(i_k\) is at least \(2\) and, thus, \(\phi(u_1 \cdots u_{i_k-1}) \leq |\tau(v_1 \cdots v_k)|\) by minimality of \(i_k\). Let \(\delta_k = |\tau(v_1 \cdots v_k)| - |\phi(u_1 \cdots u_{i_k-1})|\). We set \(w^{(k)} = z_{|(\tau(v_1 \cdots v_k)| - \delta_k, |\tau(v_1 \cdots v_k)| + \ell}) \in C^+\) and \(s_k = z_{|(\tau(v_1 \cdots v_k)| - \delta_k, |\tau(v_1 \cdots v_k)| + \ell}) \in C^+ \cup \{1\}\). Since \(\tau\) is \(\ell\)-proper, we can write

\[(15) \quad w^{(k)}(u) = s_k \tau(v_{(|\tau(v_1 \cdots v_k)| - \delta_k, |\tau(v_1 \cdots v_k)| + \ell)} = s_k \tau(v_{(|\phi(u_1 \cdots u_{i_k-1})|, |\tau(v_1 \cdots v_k)| + \ell)} = s_k \phi(u),\]

\[(16) \quad w^{(k)}(u) = \phi(v_{(|\tau(v_1 \cdots v_k)| - \delta_k, |\tau(v_1 \cdots v_k)| + \ell)} = \phi(v_{(|\phi(u_1 \cdots u_{i_k-1})|, |\tau(v_1 \cdots v_k)| + \ell)}) = s_k \phi(u_1 \cdots u_n).\]

We claim that for every \(k \in [1, m-1]\), \(s_k = 1\) and \(u_k = u_{i_k}\). To prove this, we suppose it is not true, this is, there exists \(k \in [1, m-1]\) such that either \(s_k \neq 1\) or \(u_k \neq u_{i_k}\). Then, by (15), (16) and \(|w| = \ell + |s| \geq |\phi|_1 + |s|\), we can use case (I) of Lemma 3.3 with \(\phi\) to obtain a decomposition \(A^+ \xrightarrow{\hat{\phi}'} \hat{A}^+ \xrightarrow{\hat{\phi}} C^+\) such that \(\#A \leq \#A, \ \phi = \hat{\phi}' \hat{\phi}\) and \(|\phi|_1 < |\hat{\phi}|_1\). Then, \(\ell \geq |\phi|_1 > |\hat{\phi}|_1\) and \((\hat{\phi}' \hat{\phi}) \cap \tau(B^{++}) \supseteq \hat{\phi}'(A^+) \cap \tau(B^{++}) = \phi(A^+) \cap \tau(B^{++}) \neq \emptyset\), so that \(\tau\) and \(\hat{\phi}' \hat{\phi}\) satisfy the hypothesis of this lemma. Thus, by the minimality of \(|\phi|_1\), there exists a decomposition \(B^+ \xrightarrow{q} D^+ \xrightarrow{p} C^+\) satisfying (i-iii) of this lemma, contradicting the fact that \(\phi\) and \(\tau\) are counterexamples for the lemma.

We conclude that for every \(k \in [1, m-1]\), \(s_k = 1\) and \(u_k = u_{i_k}\). A symmetric argument (that relies on case (II) of Lemma 3.3 instead of case (I)) shows that for every \(k \in [1, m-1]\), \(u_n = u_{i_{k-1}}\).
Since $|τ(v_k)| ≥ ℓ ≥ |φ|$, we have $i_k > i_{k-1}$. Thus, $τ(v_k) = s_{k-1}^{-1}φ(u_{i_{k-1}} ⋯ u_{i_k-1})s_k = φ(u_{i_{k-1}} ⋯ u_{i_k-1})$ for $k ∈ [2, m-1]$. $τ(v_1) = φ(u_1 ⋯ u_{i_1-1})$ and $τ(v_m) = φ(u_{i_m-1} ⋯ u_n)$. Since $v ∈ B^+$, we conclude that each $τ(a)$, $a ∈ B$, can be written as a concatenation $x_1 ⋯ x_N$, with $x_j ∈ φ(A)$, $x_1 = u_1$ and $x_N = u_n$. This induces a (maybe nonunique) decomposition $B^+ \xrightarrow{q} D_1^+ \xrightarrow{p_1} C^+$ such that (ii) $τ = p_1 q$, (i) $#D_1 ≤ #φ(u_1), ⋯, φ(u_n)$ ≤ $#A$ and (iv) $q$ is proper. If we define $D$ as the set of those letters $d ∈ D_1$ that occur in some $w ∈ q(B)$ and $p$ as the restriction of $p_1$ to $D$, then we obtain a decomposition $B^+ \xrightarrow{q} D_1^+ \xrightarrow{p} C^+$ that still satisfies (i), (ii), and (iv), but also $q(B^+) ⊆ D^+$. This ends the proof. □

3.2. Periodicity lemmas. We will also need a classical result from word combinatorics. We follow the presentation of [RS97]. Let $w ∈ A^*$ be a non-empty word. We say that $p$ is a local period of $w$ at the position $|u|$, if $w = uw$, with $u, v ≠ 1$, and there exists a word $z$, with $|z| = p$, such that one of the following conditions holds for some words $u'$ and $v'$:

\[
\begin{align*}
(i) & \quad u = u'z \text{ and } v = zv' \\
(ii) & \quad z = u'u \text{ and } v = zv' \\
(iii) & \quad u = u'z \text{ and } z = vv' \\
(iv) & \quad z = u'u = vv'.
\end{align*}
\]

Further, the local period of $w$ at the position $|u|$, in symbols $per(w, u)$, is defined as the smallest local period of $w$ at the position $u$. It follows directly from (17) that $per(w, u) ≤ per(w)$.

![Figure 1. The illustration of a local period.](image)

**Theorem 3.5** (Critical Factorization Theorem). Each non-empty word $w ∈ A^*$, with $|w| ≥ 2$, possesses at least one factorization $w = uv$, with $u, v ≠ 1$, which is critical, i.e., $per(w) = per(w, u)$.

**Lemma 3.6.** Let $A^+ \xrightarrow{σ} B^+ \xrightarrow{τ} C^+$ be morphisms. Suppose that $σ$ is primitive and $τ$ is $ℓ$-proper. Then, either $per(τσ(a)) > ℓ$ for all $a ∈ A$ or $τ(B^c)$ consists of only periodic points.

**Proof.** If $B = \{b\}$, then $τ(B^c) = \{τ(b)^2\}$. We suppose $#B ≥ 2$ and $per(τσ(a)) = p$ for some $a ∈ A$, $p ≤ ℓ$. Let $w = w_1 ⋯ w_{|w|} = σ(a)$, $c_i = |τ(w_1 ⋯ w_i)|$ for $i ∈ [1, |w|]$ and $c_0 = 0$. We claim that $c_i = 0 \mod p$ for all $i ∈ [0, |w| - 1]$. By contradiction, we suppose $c_i = rp + s$ for some $r ∈ \mathbb{N}$, $s ∈ [1, p-1]$ and $i ∈ [0, |w| - 1]$. Since $τ$ is $ℓ$-proper and $p ≤ ℓ$, we can find $u, v ∈ C^p$ such that $τ(b)$ starts with $u$ and ends with $v$ for all $b ∈ B$. We write $u = u''u''$, with $|u'| = s$, and compute $u'u'' = u = τ(w_i)_{0, |w|} = τ(w)_{c_i, c_i+p} = τ(w)_{rp+s, rp+s+p} = τ(w)_{s, s+p} = u''u''$. This implies, by the Fine and Wilf Theorem, that $u'', u'' ∈ \{z^k : k ≥ 1\}$ for some $z ∈ C^+$. But then $per(τσ(a)) = |z| < p$, contradicting that $per(τσ(a)) = p$. This proves the claim.
A symmetric argument (or the claim applied to $\text{Rev}(w)$) shows that $c_i = |w|$ mod $p$ for $i \in [1, |w|]$. Since $|w| \geq \#B \geq 2$, we conclude that $c_{|w|} = c_0 = 0$ mod $p$, thus $c_i = 0$ mod $p$ for any $i \in [0, |w|]$. This implies that $\tau(w_i) \in \{z^k : k \geq 1\}$ for all $i$, where $z = \sigma(w)_{[0, |w|]}$. Moreover, since $\sigma$ is primitive, $\{w_1, \ldots, w_{|w|}\} = B$, so we actually have $\tau(B) \subseteq \{z^k : k \geq 1\}$. This implies that $\tau(B^c) = \{\cdots z, z \cdots\}$. □

4. Rank of symbolic factors

In this section we prove Theorem 1.3. We start by introducing the concept of factor between directive sequences and, in Proposition 4.3, its relation with factors between $S$-adic subshifts. These ideas are the $S$-adic analogs of the concept of morphism between ordered Bratteli diagrams in [AEG15] and their Proposition 4.6. Although Proposition 4.3 can be deduced from Proposition 4.6 in [AEG15] by passing from directive sequences to ordered Bratteli diagrams and backwards, we consider this a little bit artificial since it is possible to provide a direct combinatorial proof; this is done in the Appendix. It is interesting to note that our proof is constructive (in contrast of the existential proof in [AEG15]) and shows some additional features that are consequence of the combinatorics on words analysis made.

Next, we use ideas from [Esp20] and [GH20] to prove Theorem 1.3. In particular, this improves the previous bounds in [Esp20] and [GH20] to, actually, the best possible one. We apply these results, in Corollary 4.6, to answer affirmatively Question 1.2. We end this section by showing, in Theorem 4.8, that Theorem 1.3 implies a strong coalescence property. It is worth noting that this last result is only possible because the bound in Theorem 1.3 is optimal.

4.1. Rank of factors of directive sequences. The following is the $S$-adic analog of the notion of morphism between ordered Bratteli diagrams in [AEG15].

Definition 4.1. Let $\sigma = (A^+_{n+1} \to A^+_{n})_{n \geq 0}$, $\tau = (B^+_{n+1} \to B^+_{n})_{n \geq 0}$ be directive sequences. A factor $\phi: \sigma \to \tau$ is a sequence of morphisms $\phi = (\phi_n: A^+_n \to B^+_n)_{n \geq 1}$ such that $\phi_n \sigma_n = \tau_n \phi_{n+1}$ and $X^{(n)}_{\tau} = \bigcup_{k \in \mathbb{Z}} T^k \phi_n(X^{(n)}_{\sigma})$ for all $n \geq 1$. As we do with directive sequences, we say that $\phi$ is proper when every $\phi_n$ is proper.

Remark 4.2.

1. Since $\sigma_n(X^{(n+1)}_{\sigma}) \subseteq X^{(n)}_{\sigma}$ is always true (and also the symmetric inclusion for $\tau$), the following diagram commutes for all $n \geq 1$ when $\phi$ is a factor:

$$
\begin{array}{ccc}
X^{(n+1)}_{\sigma} & \xrightarrow{\sigma_n} & X^{(n)}_{\sigma} \\
\downarrow{\phi_{n+1}} & & \downarrow{\phi_n} \\
X^{(n+1)}_{\tau} & \xrightarrow{\tau_n} & X^{(n)}_{\tau}
\end{array}
$$

2. If $X^{(n)}_{\tau}$ is minimal and $\bigcup_{k \in \mathbb{Z}} T^k \phi_n(X^{(n)}_{\sigma}) \subseteq X^{(n)}_{\tau}$, then $\bigcup_{k \in \mathbb{Z}} T^k \phi_n(X^{(n)}_{\sigma}) = X^{(n)}_{\tau}$, since in this case $\bigcup_{k \in \mathbb{Z}} T^k \phi_n(X^{(n)}_{\sigma})$ is a subsystem of $X^{(n)}_{\tau}$.

As we mentioned before, the following proposition is consequence of the main result in [AEG15]. We provide a combinatorial proof in the Appendix.

Proposition 4.3. Let $\sigma = (\sigma_n: A_{n+1} \to A_n)_{n \geq 0}$ be a proper, everywhere growing and weakly primitive directive sequence. Suppose that $X_{\sigma}$ is aperiodic. Then, there
exists a contraction $\sigma' = (\sigma_n)_{k \in \mathbb{N}}$ and a factor $\phi : \sigma' \to \tau$, where $\tau$ is recognizable, proper, everywhere growing, weakly primitive and generates $X_\sigma$.

The directive sequences $\sigma$, $\tau$ are equivalent if $\sigma = \nu'$, $\tau = \nu''$ for some contractions $\nu'$, $\nu''$ of a directive sequence $\nu$. Observe that equivalent directive sequences generate the same $S$-adic subshift. Next proposition is the main technical result of this section.

**Proposition 4.4.** Let $\phi : \sigma \to \tau$ be a factor between proper, everywhere growing and weakly primitive directive sequences. Then, there exist a proper factor $\psi : \sigma' \to \nu$, where

1. $\sigma'$ is a contraction of $\sigma$;
2. $\nu$ is proper, everywhere growing, equivalent to $\tau$ and satisfies $\text{AR}(\nu) \leq \text{AR}(\sigma)$;
3. $\nu$ is recognizable if $\tau$ is recognizable.

**Proof.** Let $\sigma = (A_n^+ \to A_n^+); \tau = (B_n^+ \to B_n^+); n \geq 0$. Up to contractions, we can suppose that $\#A_n = \text{AR}(\sigma)$ and $\tau_n$ is $|\phi_n|_1$-proper for all $n \geq 1$. First, we observe that $\phi_{n+1}(A_{n+1}^+) \subseteq B_{n+1}^+$. Indeed, every word $u \in L(X_\tau^{(n+1)})$ occurs in $\phi_{n+1}(v)$ for some $v \in L(X_\sigma^{(n+1)})$. Then, every $b \in B_{n+1} \subseteq L(X_\sigma^{(n+1)})$ occurs in some word from $\phi_{n+1}(L(X_\sigma^{(n+1)})) \subseteq \phi_{n+1}(A_{n+1})^+$, as desired. Using this we can write

$$\tau_n (B_{n+1}^+) \supseteq \tau_n (\phi_{n+1}(A_{n+1}^+)) = \phi_n (\sigma_n (A_{n+1}^+)) \subseteq \phi_n (A_n^+),$$

where in the middle step we used the commutativity from the definition of $\phi$. We deduce that $\tau_n (B_{n+1}^+) \cap \phi (A_n^+) \neq \emptyset$. Being $\tau_n$ a $|\phi_n|_1$-proper morphism, we can use Lemma 3.4 to find a decomposition $B_{n+1}^+ \xrightarrow{q_{n+1}} D_{n+1}^+ \xrightarrow{p_n} B_n^+$ such that (i) $\#D_{n+1} \leq \#A_n$, (ii) $\tau_n = p_n q_{n+1}$, (iii) $q_{n+1}(B_{n+1}^+) \subseteq D_{n+1}^+$, and (iv) $q_{n+1}$ is proper. We define the proper morphism $\nu_n = q_n p_n D_{n+1}^+ \to D_n^+$ and the proper sequence $\nu = (\nu_n)_{n \geq 1}$. Then, we have for all $n \geq 1$ the following commutative diagram:

$$\begin{array}{c}
A_n^+ & \xrightarrow{\phi_n} & B_n^+ & \xrightarrow{q_n} & D_n^+ \\
\sigma_n & \xrightarrow{\tau_n} & p_n & \xrightarrow{\nu_n} & \\
A_{n+1}^+ & \xrightarrow{\phi_{n+1}} & B_{n+1}^+ & \xrightarrow{q_{n+1}} & D_{n+1}^+ \\
\end{array}$$

From the diagram, we have $\nu_n \nu_{n+1} = \nu_n q_{n+1} = q_n \tau_n p_{n+1}$. Hence, $\langle \nu_{n,n+1} \rangle \geq \langle \tau_n \rangle$ and $\nu$ is everywhere growing as $\tau$ has the same property.

We claim that $\nu$ is weakly primitive. Indeed, since $\tau$ is weakly primitive, we have $\tau_n (B_{n+1}^+) \subseteq B_n^+$. This and (iii) above imply $q_{n+1} \tau_n (B_{n+1}^+) \subseteq D_{n+1}^+$, so, given $d \in D_n$, there exists $b \in B_{n+1}$ such that $d \subseteq \nu_n \tau_n (b)$. But, by the diagram, $q_n \tau_n (b) = \nu_n q_{n+1} (b) \in \nu_n (D_{n+1}^+)$, so $d$ occurs in some word from $\nu_n (D_{n+1})$. Since $d$ was arbitrary, the claim is proved. Thus $\nu$ is weakly primitive. By (i), $\#D_{n+1} \leq \#A_n$ and $\nu$ has alphabet rank at most $\text{AR}(\sigma)$. Observe that $\nu$, $\tau$ are equivalent as both are contractions of (id, $p_0, q_1, p_1, q_2, \ldots$).

We define the proper morphism $\psi_n = q_n \phi_n : A_n^+ \to D_n^+$ and the proper sequence $\psi = (\psi_n)_{n \geq 1}$. From the diagram, we see that $\sigma_n \psi_n = \psi_{n+1} \tau_n$, for all $n \geq 1$. Since $\sigma = (\sigma_n)_{n \geq 1}$ is such that $\tau_n q_n = \nu_n q_{n+1}$ and $\nu_n$ is weakly primitive for all $n \geq 1$, we can use Lemma 3.4 to deduce that $X^{(n)} = \bigcup_{k \in \mathbb{Z}} q_n (X^{(n)}_\tau) = \bigcup_{k \in \mathbb{Z}} q_n \phi_n (X^{(n)}_\sigma)$.
where in the last step we used that $\phi$ is a factor. This proves that $\psi$ is a factor and completes the proof.  

\section{Rank of factors of $S$-adic subshifts.} Now we are ready to prove Theorem \ref{thm:rank}. We recall its statement:

\begin{Theorem} \label{thm:rank} Let $\pi: (X,T) \to (Y,T)$ be a factor map between aperiodic subshifts. Suppose that $X$ is generated by the proper, everywhere growing and weakly primitive directive sequence $\sigma$ of alphabet rank $K$. Then, $Y$ is generated by a recognizable, proper, everywhere growing and weakly primitive directive sequence $\tau = (\tau_n: B^+_n \to B_n)_{n \geq 0}$ of alphabet rank at most $K$.

Moreover, there exists a contraction $\sigma'$ of $\sigma$ and a proper factor $\phi = (\phi_k)_{k \geq 1}$ from $\sigma'$ to $\tau$ such that $\pi(\sigma_0(x)) = \tau_0 \phi_1(x_0)$ for all $x \in X^{(1)}_{\sigma'}$.
\end{Theorem}

\begin{proof} Up to a contraction, we can suppose that $\sigma_0$ is $r$-proper and $\pi$ has radius $r$. Then, Lemma \ref{lem:proper} gives us a morphism $\tau: A^+_1 \to B^+$, where $B$ is the alphabet of $Y$, such that $\pi(\sigma_0(x)) = \tau(x_0)$ for all $x \in X^{(1)}_{\sigma}$. We define $\tilde{\sigma} = (\tau, \sigma_1, \sigma_2, \ldots)$. Since $\pi$ is onto, $\tau$ must be weakly primitive, so, up to a contraction, $\tilde{\sigma}$ is an everywhere growing, proper and weakly primitive directive sequence that generates $Y$. Since $Y$ is aperiodic, we can use Proposition \ref{prop:aperiodic} to find, after maybe some contractions, a factor $\phi: \tilde{\sigma} \to \tilde{\tau}$, where $\tilde{\tau}$ is a recognizable, proper, everywhere growing and weakly primitive directive sequence generating $Y$. Now, by Proposition \ref{prop:recognizable}, and after some contractions, there exists a proper factor $\phi: \tilde{\sigma} \to \tau$, where $\tau$ is recognizable, proper, everywhere growing, weakly primitive, satisfies $\text{AR}(\tau) \leq \text{AR}(\tilde{\sigma}) = \text{AR}(\sigma)$ and generates $Y$. Observe that $\tilde{\sigma}, \sigma$ are different at only finitely many coordinates, so, after more contractions, $\phi$ is a factor from $\sigma$ to $\tau$. Moreover, by the definition of $\tau$, we have $\pi(\sigma_0(x)) = \tau_0 \phi_1(x_0)$ for all $x \in X^{(1)}_{\sigma}$.
\end{proof}

\begin{Corollary} \label{cor:rank} Let $(X,T)$ be a minimal subshift of topological rank $K$ and $\pi: (X,T) \to (Y,T)$ a factor map, where $Y$ is an aperiodic subshift. Then, the topological rank of $Y$ is at most $K$.
\end{Corollary}

\begin{proof} By Theorem \ref{thm:rank}, $(X,T)$ is generated by a proper, primitive and recognizable directive sequence $\sigma$ of alphabet rank equal to $K$. In particular, $\sigma$ is everywhere growing and weakly primitive, so we can use Theorem \ref{thm:rank} to deduce that $(Y,T)$ is generated by a proper, everywhere growing, weakly primitive and recognizable directive sequence $\tau = (\tau_n: B^+_n \to B^+_n)_{n \geq 0}$ of alphabet rank at most $K$.

We claim that $X^{(n)}_{\tau}$ is minimal. Indeed, if $Y \subseteq X^{(n)}_{\tau}$ is a subshift, then $\tau_{0,n}(Y)$ is closed as $\tau_{0,n}: X^{(n)}_{\tau} \to X_{\tau}$ is continuous, so $\bigcup_{k \in Z} T^k \tau_{0,n}(Y) = \bigcup_{|k| \leq |\tau_{0,n}|} T^k \tau_{0,n}(Y)$ is a subshift in $X_{\tau}$ which, by minimality, is equal to it. Then, any point $x \in X^{(n)}_{\tau}$ has a $\tau_{0,n}$-factorization $(k,z)$ with $z \in Y$. By the recognizability property of $(X^{(n)}_{\tau}, \tau_{0,n})$, we obtain $Y = X^{(n)}_{\tau}$.

Now we are going to prove that for any $n \in N$ there exists $N > n$ such that $\tau_{n,N}$ is primitive. Let $n \in N$ and $R$ a constant of recognizability for $(X^{(n)}_{\tau}, \tau_{0,n})$. Since $X^{(n)}_{\tau}$ is minimal, there exists a constant $L$ such that two consecutive occurrences of a word $w \in B_n^{2R+1}$ in a point $x \in X^{(n)}_{\tau}$ are separated by at most $L$. Let $N > n$ be big enough so that $|\tau_{0,N}| \geq 2L$. Then, for all $a \in B_N \subseteq L(X^{(n)}_{\tau})$ and $w \in B_n^{2R+1}$, $w$ occurs in $\tau_{0,N}(a)$. Since $R$ is a recognizability constant for $(X^{(n)}_{\tau}, \tau_{0,n})$, we
deduce that any \( b \in B_n \) occurs in any \( \tau_{[n,N]}(a) \), with \( a \in B_N \). Hence, \( \tau_{[n,N]} \) is primitive.

By the claim, there exists a contraction \( \tau' \) that is primitive, proper, recognizable and of alphabet rank at most \( K \). We conclude that \((Y,T)\) has topological rank at most \( K \). \( \square \)

**Remark 4.7.** By the last corollary, the topological rank is a *topological invariant*, this is, conjugate systems have the same topological rank.

Recall that a system \((X,T)\) is coalescent if every endomorphism \( \pi: (X,T) \to (X,T) \) is an automorphism. We say that \( \sigma = (\sigma_n: A_{n+1}^+ \to A_n)_{n \geq 0} \) has *exact alphabet rank* if \( \# A_n = \text{AR}(\sigma) \) for all \( n \geq 1 \).

**Theorem 4.8.** Let \((X,T)\) be an \( S \)-adic subshift generated by a proper and everywhere growing direct sequence of alphabet rank equal to \( K \), and \( \pi_j: (X_{j+1},T) \to (X_j,T), j = 0, \ldots, L \), a chain of aperiodic symbolic factors, with \( X_L = X \). Suppose that \( \pi_j \) is not a conjugacy for all \( j \). Then \( L \leq \log_2(K) \).

**Proof.** Let \( \sigma = (\sigma_n: A_{n+1}^+ \to A_n)_{n \geq 0} \) be a proper, everywhere growing direct sequence of alphabet rank \( K \) generating \( X \). Recall that by Lemma 2.7 we can suppose that \( \sigma \) is weakly primitive, and, then, by an application of Theorem 4.5 with the identity factor \( \text{id}_X: (X,T) \to (X,T) \), that is also recognizable. After another contraction, \( \sigma \) has exact alphabet rank. Set \( L = \lfloor \log_2(K) \rfloor + 1 \). We are going to prove the following: if \( \pi_j: (X_{j+1},T) \to (X_j,T), j = 0, \ldots, L \), are aperiodic symbolic factors between aperiodic subshifts, with \( X_L = X \), then at least one \( \pi_j \) is a conjugation.

We put \( \sigma^{(L)} = (\sigma_{n,L}: A_{n+1,L}^+ \to A_{n,L}^+)_{n \geq 0} := \sigma \). By Theorem 4.5 applied to \( \pi_{L-1} \), there exists, after a contraction of \( \sigma^{(L)} \), a factor \( \phi^{(L-1)} = (\phi_{n,L-1}: A_{n+1,L-1}^+ \to A_{n,L-1}^+)_{n \geq 0} \), where \( \sigma^{(L-1)} = (\sigma_{n,L-1}: A_{n+1,L-1}^+ \to A_{n,L-1}^+)_{n \geq 0} \), is a proper, everywhere growing direct sequence proper direct sequence of alphabet rank at most \( K \) that generates \( X_{L-1} \) and satisfies \( \pi_{L-1}(\sigma_0,1(x)) = \sigma_{0,L-1} \phi_{1,L-1}(x) \), \( |\sigma_0,1(x)| = |\sigma_{0,L-1}(\phi_{1,L-1}(a))| \) for all \( x \in X^{(L)}_\sigma \) and \( a \in A_{1,L} \). After some contractions, we can suppose that \( \sigma^{(L-1)} \) has exact alphabet rank. The same procedure applies to \( \pi_{L-2} \), and so on. We obtain, for \( j = 0, \ldots, L-1 \), factors \( \phi^{(j)}: \sigma^{(j+1)} \to \sigma^{(j)} \) such that \( \sigma^{(j)} \) is proper, everywhere growing, weakly primitive, recognizable, has exact alphabet rank equal to \( K_j \leq K \), generates \( X_j \) and satisfies \( \pi_j(\sigma_{0,j+1}(x)) = \sigma_{0,j} \phi_{1,j}(x) \), \( |\sigma_{0,j+1}(a)| = |\sigma_{0,j}(\phi_{1,j}(a))| \) for all \( x \in X^{(j+1)}_\sigma \) and \( a \in A_{1,j+1} \).

We write \( X^{(n)}_j = X^{(n)}_{\sigma^{(j)}} \). Since \( \phi^{(j)} \) is a factor, we have \( X^{(1)}_j = \bigcup_{k \in \mathbb{Z}} T^k \phi_{1,j}(X^{(1)}_{j+1}) \) and the following diagram commutes:

\[
\begin{array}{cccccc}
X^{(0)}_0 & \overset{\phi_{0,0}}{\longleftarrow} & \cdots & \overset{\phi_{1,0}}{\longleftarrow} & X^{(1)}_j & \overset{\phi_{1,j}}{\longleftarrow} & X^{(1)}_{j+1} & \cdots & \overset{\phi_{1,L-1}}{\longleftarrow} & X^{(1)}_L \\
\sigma_{0,0} & \ & \sigma_{0,j} & \ & \sigma_{0,j+1} & \ & \sigma_{0,L} \\
X^{(0)}_0 & \overset{}{\longleftarrow} & \cdots & \overset{}{\longleftarrow} & X^{(0)}_j & \overset{\pi_j}{\longleftarrow} & X^{(0)}_{j+1} & \cdots & \overset{\pi_{L-1}}{\longleftarrow} & X^{(0)}_L
\end{array}
\]

We claim that if \((X^{(1)}_{j+1}, \phi_{1,j})\) is recognizable, then \( \pi_j \) is a conjugacy. Indeed, suppose that \((X^{(1)}_{j+1}, \phi_{1,j})\) is recognizable and let, for \( i = 0, 1 \), \( x^i \in X^{(1)}_{j+1} \) with \( y = \pi_j(x^i) \).
Since $X_{j+1} = \bigcup_{k \in \mathbb{Z}} T^k \sigma_{0,j+1}(X_{j+1}^{(1)})$ by Lemma 2.5, we can find a centered $\sigma_{0,j+1}$-factorization $(k^i, z^i)$ of $x^i$ in $X_{j+1}^{(1)}$. Then, using that $\sigma_{0,j} \phi_{1,j}(x) = \pi_j(\sigma_{0,j+1}(x))$ for $x \in X_{j+1}^{(1)}$, we can compute $T^{k_0} \sigma_{0,j} \phi_{1,j}(z^0) = \pi_j(x^0) = \pi_j(x^1) = T^{k^1} \sigma_{1,j} \phi_{1,j}(z^1)$, so $(k^i, z^i)$ is a $\sigma_{0,j} \phi_{1,j}$-factorization of $y$ in $X_{j+1}^{(1)}$ for $i = 0, 1$. Moreover, these are centered factorizations as $|\sigma_{0,j} \phi_{1,j}(a)| = |\sigma_{0,j+1}(a)|$ for all $a \in A_{1,j+1}$. Since $(X_j^{(1)}, \sigma_{0,j})$ is recognizable, Lemma 2.3 implies that $(X_{j+1}^{(1)}, \sigma_{1,j} \phi_{1,j})$ is recognizable, so $(k^0, z^0) = (k^1, z^1)$. Hence, $x^0 = x^1$ and $\pi$ is a conjugacy.

Now, by contradiction, we suppose that $\pi_j$ is not a conjugacy for all $j$. Then, by the claim, $(X_j^{(1)}, \phi_{1,j})$ is not recognizable. Observe that $\nu := (\phi_{1,0}, \phi_{1,1}, \ldots, \phi_{1,L-1}, \sigma_{1,L}, \sigma_{2,L}, \ldots)$ is an everywhere growing directive sequence. Since $\nu, \sigma^{(L)}$ have the same “tail”, it is clear that $X_{\nu}^{(m+L)} = X_{\nu}^{(m+1)}$ for all $m \in \mathbb{N}$. Thus, $X_{\nu}^{(L-1)} = \bigcup_{k \in \mathbb{Z}} T^k \sigma_{1,L-1}(X_{\nu}^{(L)}) = \bigcup_{k \in \mathbb{Z}} T^k \phi_{1,L-1}(X_{\nu}^{(1)}) = X_{\nu}^{(L-1)}$. Similarly, $X_{\nu}^{(j)} = X_j^{(1)}$ for any $j \in [1, L]$. But then the level $(X_{\nu}^{(j)}, \nu_j) = (X_j^{(1)}, \phi_{1,j})$ is not recognizable for all $j \in [0, L-1]$, contradicting Theorem 2.8. This ends de proof.

Corollary 4.9. Let $(X, T)$ be an $S$-adic shift generated by a proper and everywhere growing directive sequence of finite alphabet rank. Then, $(X, T)$ is coallescent.

Remark 4.10. A linearly recurrent shift of constant $C$ is generated by a primitive and proper directive sequence of alphabet rank at most $C(C + 1)^2$ ([Dur00]). In [DHS99], the authors proved the following

Theorem 4.11 ([DHS99]). For a linearly recurrent shift $X$ of constant $C$, in any chain of factors $\pi_j : (X_j, T) \to (X_{j+1}, T)$, $j = 0, \ldots, L$, with $X_0 = X$ and $L \geq (2C(2C + 1)^2)4C^2(2C + 1)^2$ there is at least one $\pi_j$ which is a conjugacy.

Thus, Theorem 4.11 is not only a generalization of this result to a much larger class of systems, but also improves the previous super-exponential constant to a logarithmic one.

5. Fibers of symbolic factors

The objective of this section is to prove Theorem 5.3 which states that factor maps $\pi : (X, T) \to (Y, T)$ between $S$-adic shifts of finite alphabet rank are always almost $k$-to-1 for some $k$ bounded by the alphabet rank of $X$. We start with some lemmas from topological dynamics.

Lemma 5.1 ([Aus88]). Let $\pi : X \to Y$ be a continuous map between compact metric spaces. Then $\pi^{-1} : Y \to 2^X$ is continuous at every point of a residual subset of $Y$.

Next lemma gives a sufficient condition for a factor map $\pi$ to be almost $k$-to-1. Recall that $E(X, T)$ stands for the Ellis semigroup of $(X, T)$.

Lemma 5.2. Let $\pi : (X, T) \to (Y, T)$ be a factor map between topological dynamical systems, with $(Y, T)$ minimal, and $K \geq 1$ an integer. Suppose that for every $y \in Y$ there exists $u \in E(2^X, T)$ such that $\# u \circ \pi^{-1}(y) \leq K$. Then, $\pi$ is almost $k$-to-1 for some $k \leq K$. 

Proof. First, we observe that by the description of $u \circ A$ in terms of nets at the end of Subsection 2.1 we have

\[(19) \quad \#u \circ A \leq \#A, \forall u \in E(2^X, T), \quad A \in 2^X.
\]

Now, by previous lemma, there exists a residual set $\tilde{Y} \subseteq Y$ of continuity points for $\pi^{-1}$. Let $y, y' \in \tilde{Y}$ be arbitrary. Since $Y$ is minimal, there exists a sequence $(n_\ell)_\ell$ such that $\lim_{\ell} T^{n_\ell} y = y'$. If $w \in E(2^X, T)$ is the limit of a convergent subnet of $(T^{n_\ell})_\ell$, then $w_{y'} = y'$. By the continuity of $\pi^{-1}$ at $y'$ and \[(19)\], we have

\[\#\pi^{-1}(y') = \#\pi^{-1}(wy) = \#w \circ \pi^{-1}(y) \leq \#\pi^{-1}(y).
\]

We deduce, by symmetry, that $\#\pi^{-1}(y) = \#\pi^{-1}(y')$. Hence, $k := \pi^{-1}(y)$ does not depend on the chosen $y \in \tilde{Y}$. To end the proof, we have to show that $k \leq K$. We fix $y \in \tilde{Y}$ and take, from the hypothesis, $u \in E(2^X, T)$ such that $\#u \circ \pi^{-1}(y) \leq K$. As above, from the minimality we can find $v \in E(2^X, T)$ such that $vuy = y$. Then, by the continuity of $\pi^{-1}$ at $y$ and \[(19)\],

\[k = \#\pi^{-1}(y) = \#\pi^{-1}(vuy) = \#(vuy) \circ \pi^{-1}(y) \leq \#u \circ \pi^{-1}(y) \leq K.
\]

\[\square\]

Let $\sigma : A^+ \to B^+$ be a morphism, $(k, x)$ a centered $\sigma$-factorization of $y \in B^\mathbb{Z}$ and $\ell \in \mathbb{Z}$. Note that there exists a unique $j \in \mathbb{Z}$ such that $\ell \in [c_{\sigma,j}(k, x), c_{\sigma,j+1}(k, x))$ (recall the notion of cut from Definition 2.1). In this context, we say that $x_j$ is the symbol of $(k, x)$ covering position $\ell$ of $y$.

**Theorem 5.3.** Let $\pi : (X, T) \to (Y, T)$ be a factor between subshifts, with $(Y, T)$ minimal and aperiodic. Suppose that $X$ is generated by a proper and everywhere growing directive sequence $\sigma$ of alphabet rank $K$. Then, $\pi$ is almost $k$-to-1 for some $k \leq K$.

**Proof.** Let $\sigma = (\sigma_n : A_{n+1} \to A_n)_{n \geq 0}$ be the directive sequence of $X$. Using Lemma 2.4 (and doing a contraction, if needed) we can suppose that $Y$ is generated by an everywhere growing directive sequence $\tau = (\tau_1, \tau_2, \ldots)$, where $\tau : A^+_1 \to B^+$, $\tau(x) = \pi(\sigma_0(x))$ if $x \in X^{(1)}_x = X^{(1)}_x$ and $\#A_n = K$ for every $n \geq 1$. We use the notation $\tau_{[0,n)} = \tau \sigma_{[1,n]}$. We are going to show that the sufficient condition of Lemma 5.2 holds.

For $y \in Y$ and $n \geq 1$, let $F_n(y)$ be the set of factorizations of $y$ in $(Y^\tau_{[0,n)}(Y^\tau_{[0,n)}))$.

**Claim.** There exists $\ell_n \in \mathbb{Z}$ and $G_n \subseteq \mathbb{Z} \times B_{n+1}$ with at most $K$ elements such that if $(k, x) \in F_n(y)$, then the symbol of $(k, x)$ covering position $\ell_n$ of $y$ is in $G_n$.

**Proof.** By compactness and aperiodicity of $Y$, there exists a big integer $L \in \mathbb{N}$ such that $\min \text{per}(L_{g \geq L}(Y)) > |\tau_{[0,n]}|$. If the claim does not hold, then, for every $\ell \in [0, L]$ we can find $K + 1$ factorizations $(x, k)$ of $y$ in $(Y^\tau_{[0,n)}(Y^\tau_{[0,n)}))$ such that their symbols covering position $\ell$ of $y$ are all different. Then, since $\#\tau_{[0,n]}(A_{n+1}) \leq K$, we can use the Pigeon Principle to find two of such factorizations, say $(k, x)$ and $(k', x')$, such that if $(c, a)$ and $(c', a')$ are their symbols covering position $\ell$ of $y$ then $a = a'$ and $c < c'$. Hence, $y_{(c, a+|\tau_{[0,n]}(a))]}, y_{(c', a'+|\tau_{[0,n]}(a))]} = \tau_{[0,n]}(a)$ and the word $y_{(c, a+|\tau_{[0,n]}(a))]}$ is $(c' - c)$-periodic. Since $\ell \in (c', c + |\tau_{[0,n]}(a))], this implies that the local period of $y_{[0,\ell)}$ at $\ell$ is at most $c' - c \leq |\tau_{[0,n]}|$. But, by Theorem 3.5, $\min \text{per}(L_{g \geq L}(Y)) \leq \text{per}(y_{[0,\ell)}), y_{[0,\ell)}) \leq |\tau_{[0,n]}|$ for some $\ell \in [0, L]$. This contradicts the definition of $L$ and the claim is proved. \[\square\]
We choose \( \tilde{F}_n(y) \subseteq F_n(y) \) such that \( \#\tilde{F}_n(y) = \#G_n \) and \( G_n \) is the set of symbols of factorizations \( (k, x) \in \tilde{F}_n(y) \) covering position \( \ell_n \) of \( y \). Let \( z \in \pi^{-1}(y) \) and \( (k, x) \) be a factorization of \( z \) over \( (X^{(n)}_\sigma, \sigma_{(0,n)}) \). Then, \( T^k\tau_{(n)}(x) = T^k\pi(\sigma_{(0,n)}(x)) = \pi(z) = y \) and \( (k, x) \) is a factorization of \( y \) over \( (Y^{(n)}_\tau, \tau_{(0,n)}) \). Thus, we can find \( (k', x') \in \tilde{F}_n(y) \) such that the symbols of \( (k, x) \) and \( (k', x') \) covering position \( \ell_n \) of \( y \) are the same; let \( (m, a) \) be this common symbol. Since \( \sigma \) is proper, we have

\[
\tilde{z}' = T^{k'}\sigma_{(0,n)}(x') \in X \text{ is the point that } (k', x') \text{ factorizes in } (X^{(n)}_\sigma, \sigma_{(0,n)}).
\]

Then, as \( \ell_n \in (m, m + |\sigma_{(0,n)}(a)|) \),

\[
\tilde{z}' = z'[m - (\sigma_{(0,n-1)}), m + |\sigma_{(0,n-1)}|] = z'[m - (\sigma_{(0,n-1)}), m + |\sigma_{(0,n-1)}|] \cdot \tilde{z}'[\ell_n - (\sigma_{(0,n-1)}, \ell_n + |\sigma_{(0,n-1)}|)].
\]

Thus, \( \text{dist}(T^{k}z, T^{k}\pi P_n(y)) \leq \exp(-|\sigma_{(0,n-1)}) \), \( P_n(y) \subseteq \pi^{-1}(y) \) is the set of all points \( T^{k}\sigma_{(0,n)}(x') \in X \) such that \( (k', x') \in \tilde{F}_n(y) \). Thus, \( d_H(T^{k}\pi \sigma_{(0,n)}(y), T^{k}\pi P_n(y)) \) converges to zero as \( n \) goes to infinity (where, we recall, \( d_H \) is the Hausdorff distance). Taking an appropriate convergent subnet \( u \in (T^{k}\pi)_{n \in \mathbb{N}} \) we obtain \( \#u \circ \pi^{-1}(y) \leq \sup_{n \in \mathbb{N}} \#G_n \leq K \). Since \( y \) was arbitrary, Lemma 5.2 can be applied. We conclude that \( \pi \) is almost \( k \)-to-1 for some \( k \leq K \).

\[\square\]

6. Number of symbolic factors

In this section we prove Theorem 1.6. In order to do this, we split the proof into 3 subsections. First, in subsection 6.2, we consider the case where the factor maps in Theorem 1.6 are non-distal and show, in Lemma 6.6, that we can reduce the problem to a similar one, but where the alphabets are smaller. Then, we prove Lemma 6.3 in subsection 6.1, which is the distal-extensions-case of Theorem 1.6. Finally, we prove Theorem 1.6 in subsection 6.3 by a repeated application of the previous lemmas.

6.1. Distal case. We start with some definitions. If \( (X,T) \) is a system, then we always gives \( X^k \) the diagonal action \( T^k \): \( T \times \cdots \times T \). If \( \pi: (X,T) \to (Y,T) \) is a factor map and \( k \geq 1 \), then we define \( R^k_\pi = \{(x^1, \ldots, x^k) \in X^k : \pi(x^1) = \cdots = \pi(x^k)\} \). Observe that \( R^k_\pi \) is a closed \( T^k \)-invariant subset of \( X^k \).

Next lemma follows from classical ideas from topological dynamics. See, for example, Theorem 6 in Chapter 10 of [Aus1988].

Lemma 6.1. Let \( \pi: (X,T) \to (Y,T) \) a distal almost \( k \)-to-1 factor between minimal systems, \( z = (z^1, \ldots, z^k) \in R^k_\pi \) and \( Z = \overline{\sigma \pi R^k_\pi}(z) \). Then, \( \pi \) is \( k \)-to-1 and \( Z \) is minimal.

We will also need the following lemma:

Lemma 6.2 (Lemma 21 in [Dur2003]). Let \( \pi_i: (X,T) \to (Y_i,T), i = 0,1 \), be two factors between aperiodic minimal systems. Suppose that \( \pi_0 \) is finite-to-1. If \( x, y \in X \) are such that \( \pi_0(x) = \pi_0(y) \) and \( \pi_1(x) = T^{i} \pi_1(y) \), then \( p = 0 \).

Lemma 6.3. Let \( (X,T) \) be a minimal subshift of topological rank \( K \) and \( J \) an index set with \( \#J > 2K(1227)^K \). Suppose that for every \( j \in J \) there exists a distal symbolic factor \( \pi_j: (X,T) \to (Y_j,T) \). Then, there are \( i \neq j \in J \) such that \( (Y_i,T) \) is conjugate to \( (Y_j,T) \).
Proof. By Theorem 5.3 each \( \pi_j \) is almost \( k_j \)-to-1 for some \( k_j \leq K \), so, by the Pigeon Principle, there exist \( J_1 \subseteq J \), with \( \#J_1 \geq \#J/K > 2(122K^7)^K \), and \( k \leq K \) satisfying \( k_j = k \) for all \( j \in J_1 \). For \( j \in J_1 \), we fix \( z^j = (z^j_1, \ldots, z^j_l) \in R^k_{\pi_j} \) with \( z^j_n \neq z^m_n \) for all \( n \neq m \). Let \( Z_j = \text{orb}_{T^{(k)}(z^j)} \) and \( \rho : X^k \to X \) be the factor map that projects onto the first coordinate. By Lemma 6.1 \( \pi_j \) is \( k \)-to-1 and \( Z_j \) minimal. Observe that if \( x = (x_1, \ldots, x_k) \in Z_j \), then

\[
\begin{align*}
(20) \quad & \{x_1, \ldots, x_k\} = \pi_j^{-1}(\pi_j(x_n)) \text{ for all } n \in [1, k], \\
(21) \quad & x_n \neq x_m \text{ for all } n, m \in [1, k].
\end{align*}
\]

Indeed, since \( Z_j \) is minimal, \( (T^{(k)})^n z \to x \) for some sequence \( (n_k) \), so,

\[
\inf_{n \neq m} \text{dist}(x_n, x_m) \geq \inf_{n \neq m, j \in \mathbb{Z}} \text{dist}(T^j z_n, T^j z_m) > 0,
\]

where in the last step is due the fact that \( \pi_j \) is distal. This implies \((21)\). For \((20)\) we first note that \( \{x_1, \ldots, x_k\} \subseteq \pi_j^{-1}(\pi_j(x_n)) \) as \( x \in R_{\pi_j} \), and then that the equality must hold since \( \#\pi_j^{-1}(\pi_j(x_n)) = k \).

Claim. There exist \( J_3 \subseteq J_1 \) with \( \#J_3 > 1 \) and \( Z_i = Z_j \) for all \( i, j \in J_3 \).

Proof. Let \( j \in J_1 \). Since \( Z_j \) is an infinite subshift, it contains an asymptotic pair \( x^j = (x^j_1, \ldots, x^j_k), \tilde{x}^j = (\tilde{x}^j_1, \ldots, \tilde{x}^j_k) \). First, we prove that for any \( n \in [1, k] \) and \( j \in J_1 \), \( x^j_n, \tilde{x}^j_n \) are asymptotic. Since \( x^j, \tilde{x}^j \) are asymptotic,

\[
(22) \quad \text{for any } n \in [1, k], \quad x^j_n, \tilde{x}^j_n \text{ are either asymptotic or equal.}
\]

We suppose, with the aim to obtain a contradiction, that \( x^j_n = \tilde{x}^j_n \neq x^j_l \) for some \( n \in [1, k] \). Then, since \( x^j, \tilde{x}^j \in R^k_{\pi_j} \), we have \( \pi_j(x^j_n) = \pi_j(x^j_l) = \pi_j(\tilde{x}^j_n) = \pi_j(\tilde{x}^j_l) \) for all \( m, l \in [1, k] \). Thus, using \((20)\), \( \{x^j_1, \ldots, x^j_k\} = \pi_j^{-1}(\pi_j(x^j_n)) = \pi_j^{-1}(\pi_j(\tilde{x}^j_n)) = \{\tilde{x}^j_1, \ldots, \tilde{x}^j_k\} \). This, \((21)\) and \( x^j \neq \tilde{x}^j \) imply that there exist \( m \neq l \in [1, k] \) with \( \tilde{x}^j_m = x^j_l \). We deduce that \( \tilde{x}^j_l = x^j_m \neq x^j_n \) and, by \((22)\), that \( x^j_n, \tilde{x}^j_n \) are asymptotic, so proximal, contradicting the distality of \( \pi_j \). We conclude that \( x^j_n, \tilde{x}^j_n \) are asymptotic for all \( n \in [1, k] \), as desired.

By the Pigeon Principle, there exists \( J_2 \subseteq J_1 \), with \( \#J_2 \geq \#J_1/2 \), such that either \( (x^j, \tilde{x}^j) \) is right asymptotic for all \( j \in J_2 \) or \( (x^j, \tilde{x}^j) \) is left asymptotic for all \( j \in J_2 \). Without loss of generality, we suppose that the first case occurs, this is, \( (x^j, \tilde{x}^j) \) is right asymptotic for all \( j \in J_2 \). Thus, for all \( j \in J_1 \) and \( n \in [1, k] \) we can find \( p^j_n \in \mathbb{Z} \) such that \( T^{p^j_n} x^j_n, T^{p^j_n} \tilde{x}^j_n \) are centered right-asymptotic. Since by Proposition 2.9 we have

\[
\#\{x_{(0,\infty)} : (x, \tilde{x}) \text{ is centered right-asymptotic}\} \leq 122K^7
\]

we can use the Pigeon Principle to find \( J_3 \subseteq J_2 \), with \( \#J_3 \geq \#J_2/(122K^7)^K \), and centered right asymptotic pairs \( x_n, \tilde{x}_n \in X \), for \( n \in [1, k] \), satisfying

\[
(23) \quad (x_n)_{(0,\infty)} = (T^{p^j_n} x^j_n)_{(0,\infty)} \text{ for any } j \in J_3 \text{ and } n \in [1, k].
\]

Now, by compacity, we can find \( s_1 < s_2 < \ldots \) such that \( y_n = \lim_{\ell \to \infty} T^{s_\ell} x_n \) exists for all \( n \in [1, k] \). Then, by \((23)\), \( y_k^n := \lim_{\ell \to \infty} T^{s_\ell} x^j_n \) exists for any \( j \in J_3 \). Moreover,

\[
(24) \quad y_n = T^{p^j_n} y^j_n \quad \text{and} \quad y^j := (y^j_1, \ldots, y^j_k) = \lim_{\ell \to \infty} (T^{(k)})^{s_\ell} x^j \in Z_j.
\]
Let \( i, j \in J_3 \). We are going to prove that \( Z_i = Z_j \). First, we observe that if \( n, m \in [1, k] \), then we can use (24) to compute

\[
\pi_i(y_n) = \pi_i(T^{p_n^i}y_n^i) = \pi_i(T^{p_n^i - p_m^i}y_m^i).
\]

Similarly,

\[
\pi_j(y_n) = \pi_j(T^{p_n^i}y_m^i) = T(\pi_i(T^{p_n^i - p_m^i}y_m^i)).
\]

Equations (25) and (26) and the fact that \( \pi_i, \pi_j \) are \( k \)-to-1 allow us to use Lemma 6.2 to deduce that \( p_n^i - p_m^j = (p_n^i - p_m^i) = 0 \). Thus, \( p := p_n^i - p_m^i = p_n^i - p_m^i \) for all \( n, m \in [1, k] \) and \( (T^k)^p y^i = y^j \). This implies that \( y^j \) lies both in \( Z_j \) and in \( (T^k)^p Z_i = Z_i \), since \( Z_i, Z_j \) are minimal.

We conclude that \( Z_i = Z_j \) for all \( j \in J_3 \). Since we have chosen \( J_3 \) so that \( \#J_3 \geq \#J_2/(22K^7)K \geq \#J_1/(2(22K^7)K > 1 \), the proof of the claim complete. \( \square \)

By the claim, there exist \( i \neq j \in J_1 \subseteq J \) with \( Z := Z_i = Z_j \). Let \( y \in Y_i \) and \( x = (x_1, \ldots, x_k) \in \rho^{-1}\pi_i^{-1}(y) \cap \mathbb{Z} \). Then, by (20), \( \pi_i^{-1}(y) = \{x_1, \ldots, x_k\} = \pi_j^{-1}(\pi_j(x_1)) \), so \( \pi_j \pi_i^{-1}(y) = \pi_j(x_1) \) and \( \pi_j \pi_i^{-1}(y) \) contains exactly one element \( y' \); we set \( \psi(y) = y' \). Thus, \( \psi \) is a map from \( Y_i \) to \( Y_j \). Since \( \pi_i^{-1}: Y_i \to 2^X \) is continuous (as \( \pi_i \) is distal, hence open) and commutes with \( T \), \( \psi \) has the same properties, so \( \psi: (Y_i, T) \to (Y_j, T) \) is a factor. Moreover, a similar construction gives a map \( \phi: Y_j \to Y_i \) which is the inverse function of \( \psi \). We conclude that \( \psi \) is bijective and, thus, a conjugacy between \( Y_i \) and \( Y_j \). \( \square \)

6.2. Non-distal case. Now we deal with non-distal factors. The starting point is the following simple lemma.

**Lemma 6.4.** Let \( \pi: (X, T) \to (Y, T) \) be a factor between minimal subshifts. Then, either \( \pi \) is distal or there exists a fiber \( \pi^{-1}(y) \) containing an asymptotic pair.

**Proof.** We suppose that \( \pi \) is not distal and let \( x \neq x' \in X \) proximal with \( \pi(x) = \pi(x') \). Since \( (x, x') \) is proximal, we can find, for any \( k \in \mathbb{N} \), a (maybe infinite) interval \( I_k = (a_k, b_k) \subseteq \mathbb{Z} \), with \( b_k - a_k = k \), such that \( x_i = x'_i \) and \( I_k \) is maximal (with respect to the inclusion) with this property. Since \( x \neq x' \), either \( a_k > -\infty \) or \( b_k < \infty \). We have two cases. First, if \( b_k = \infty \) for some \( k \in \mathbb{N} \), then \( a_k > -\infty \) and \( T^{b_k}(x, x') \in R^2_2 \) is a centered left-asymptotic pair. Now we deal with the case \( b_k < \infty \) for all \( k \in \mathbb{N} \). Then, by compactness, \( T^{b_k}(x, x') \) converges to a point \((z, z')\) for some sequence \( 0 \leq k_1 < k_2 < \ldots \). Since \( I_k \) become arbitrarily large and \( x_{k_1} \neq x'_{k_2} \) as \( I_k \) is maximal, \((z, z')\) is a centered right-asymptotic pair. Moreover, since \( R^2_2 \ni T^{b_k}(x, x') \) is closed, \((z, z') \in R^2_2 \). This completes the proof. \( \square \)

Next lemma allow us to pass from morphisms \( \sigma: X \to Y \) to factors \( \pi: X' \to Y \), where \( X' \) is defined in the same alphabet as \( X \) and has the “same” asymptotic pairs. We remark that its proof is simple, but tedious.

**Lemma 6.5.** Let \( X \subseteq \mathcal{A}^+ \) be an aperiodic subshift, \( \sigma: \mathcal{A}^+ \to \mathcal{B}^+ \) a morphism and \( Y = \bigcup_{k \in \mathbb{Z}} T^k \sigma(X) \). Define the morphism \( i_\sigma: \mathcal{A}^+ \to \mathcal{A}^+ \) by \( i_\sigma(a) = a^{\sigma(a)} \), \( a \in \mathcal{A} \) and \( X' = \bigcup_{k \in \mathbb{Z}} T^k \sigma(X) \). Then, \((X, i_\sigma)\) is recognizable, there exists a factor map \( \pi: (X', T) \to (Y, T) \) such that \( \pi(i_\sigma(x)) = \tau(x) \) for all \( x \in X \), and centered asymptotic pairs in \( X' \) are of the form \((i_\sigma(x), i_\sigma(x'))\), where \((x, x') \) is a centered asymptotic pair in \( X \).
Proof. First, we prove that \((X, i_\sigma)\) is recognizable. For this, we start by observing that
\begin{equation}
\text{(27) if } (k, x), (\tilde{k}, \tilde{x}) \text{ are centered } i_\sigma-\text{factorizations of } y \in X', \text{ then } x_0 = \tilde{x}_0.
\end{equation}
Indeed, since the factorization are centered, we have \(x_0 = i_\sigma(x_0)_k = y_0 = i_\sigma(\tilde{x}_0)_{\tilde{k}} = \tilde{x}_0\).

Let \(\Lambda\) be the set of tuples \((k, x, \tilde{k}, \tilde{x})\) such that \((k, x), (\tilde{k}, \tilde{x})\) are centered \(i_\sigma\)-factorizations of the same point. Moreover, for \(R \in \{\leq, >\}\), let \(\Lambda_R\) be the set of those \((k, x, \tilde{k}, \tilde{x})\in \Lambda\) satisfying \(k R \tilde{k}\).

Claim. If \((k, x, \tilde{k}, \tilde{x}) \in \Lambda_\leq\), then \((0, Tx, 0, T\tilde{x}) \in \Lambda_\leq\), and \(|i_\sigma(x_0)| - k + \tilde{k}, \tilde{x}, 0, Tx) \in \Lambda_\geq\)
whenever \((k, x, \tilde{k}, \tilde{x}) \in \Lambda_\geq\).

Proof. If \((k, x, \tilde{k}, \tilde{x}) \in \Lambda_\leq\), then, since \(x_0 = \tilde{x}_0\) by \((27)\), we can compute \(i_\sigma(Tx) = T^ki_\sigma(x) = T^ki_\sigma(\tilde{x}) = i_\sigma(T\tilde{x})\), so \((0, Tx, 0, T\tilde{x}) \in \Lambda_\leq\). Now we do the case \((k, x, \tilde{k}, \tilde{x}) \in \Lambda_\geq\). Let \(y = T^ki_\sigma(x) = T^ki_\sigma(\tilde{x})\). Then,
\[
T^{|i_\sigma(x_0)| - k + \tilde{k}}i_\sigma(x) = T^{|i_\sigma(x_0)| - k}y = T^{|i_\sigma(x_0)|}i_\sigma(x) = i_\sigma(Tx),
\]
so \(|i_\sigma(x_0)| - k + \tilde{k}, \tilde{x}, 0, T\tilde{x}) \in \Lambda_\geq\). This completes the proof of the claim. \(\square\)

Now we prove that \((X, i_\sigma)\) is recognizable. Let \((k, x, \tilde{k}, \tilde{x}) \in \Lambda\). First, we suppose that \(k = \tilde{k}\). Then, the claim implies that \((0, Tx, 0, T\tilde{x}) \in \Lambda_\leq\), thus, again by the claim \((0, T^2x, 0, T^2\tilde{x}) \in \Lambda_\leq\) and so on. We obtain \((0, T^n x, 0, T^n \tilde{x}) \in \Lambda_\leq\) for any \(n \geq 0\), so \((27)\) implies that \(x_n = \tilde{x}_n\) for all \(n \geq 0\). A similar argument shows that \(x_n = \tilde{x}_n\) for any \(n \leq 0\), and we obtain \((k, x) = (\tilde{k}, \tilde{x})\). Now we suppose \(k > \tilde{k}\). By the claim, \((p_1, \tilde{x}, 0, Tx) \in \Lambda_\geq\) for some \(p_1 \in \mathbb{Z}\). As before, we iterate the claim to obtain that \((p_2,Tx, 0,T\tilde{x}) \in \Lambda_\geq\), \((p_3, T\tilde{x}, 0, T^2x) \in \Lambda_\geq\) and so on. By \((27)\) applied to these quadruples, we deduce that \(x_0 = \tilde{x}_0, \tilde{x}_0 = (Tx)_0 = x_1, x_1 = (T\tilde{x})_0 = T\tilde{x}_0 = x_1, \tilde{x}_1 = (T\tilde{x})_0 = (T^2x)_0 = x_2\), etc. We conclude that \(x_n = \tilde{x}_n = x_0\) for any \(n \geq 0\). But this implies that the fixed point \(\cdots x_0, x_0, x_0, \cdots\) belongs to \(X\), contrary to the aperiodicity hypothesis. Thus, the case \(k > \tilde{k}\) does not occur. This proves that \((X, i_\sigma)\) is recognizable.

Now we define \(\pi\). Let \(x' \in X'\). Then, there exists a unique centered \(i_\sigma\)-factorization \((k, x)\) in \(X\). We set \(\pi(x') = T^k\tau(x)\). Since
\begin{equation}
|\tau(a)| = |i_\sigma(a)| \text{ for all } a \in A,
\end{equation}
it is easy to see that \(\pi\) commutes with \(T\). By \((iii)\) in Remark 2.2, \(\pi\) is continuous.

If \(y \in Y\), then by the definition of \(Y\) there exist a centered \((k, x)\) \(\tau\)-factorization of \(y\) in \(X\). By \((28)\) again, \((k, x)\) is a centered \(i_\sigma\)-factorization of \(x' := T^k i_\sigma(x)\). Thus, \(\pi(x') = y\) and \(\pi\) is onto.

It left to prove the asymptotic pairs conditions. We only prove it for right-asymptotic pairs, since the other case is similar. If \(Z\) is a subshift, let \(A(Z)\) be the set of centered right-asymptotic pairs. First, we observe that, from the definition of \(i_\sigma\), \((i_\sigma(x), i_\sigma(x')) \in A(X')\) whenever \((x, \tilde{x}) \in A(X)\). Now, let \((z, \tilde{z}) \in A(X')\) and \((k, x), (\tilde{k}, \tilde{x})\) be the unique centered \(i_\sigma\)-factorizations of \(z, \tilde{z}\) in \(X\), respectively. From \((iii)\) in Remark 2.2, \((X, i_\sigma)\) is recognizable with constant \(r\), so \((k, x), (\tilde{k}, \tilde{x})\)
eventually coincide, this is, there exist \( p, q \in \mathbb{Z} \) such that \((0, T^p x), (0, T^q \tilde{x})\) are \( i_\pi\)-factorizations equivalent to \((k, x), (\tilde{k}, \tilde{x})\), respectively (see (ii) in Remark 2.2 for definition of equivalent). By taking \( p, q \) maximal with this property, we have that \((T^p x, T^q \tilde{x})\) is centered right-asymptotic. Now, since \( z_0 \neq \tilde{z}_0 \), we must have \( p, q \leq 0 \), and since \( i_\pi(x_p)_1 = x_p \neq \tilde{x}_q = i_\pi(x_q)_1 \), we also have \( p, q \geq 0 \). Thus, \( p = q = 0 \).

Since \((0, T^p x), (0, T^q \tilde{x})\) were \( i_\pi\)-factorizations equivalent to \((k, x), (\tilde{k}, \tilde{x})\), we deduce that they are equal, this is, \( k = \tilde{k} = 0 \). We conclude that \((z, \tilde{z})\) has the form \((i_\sigma(x), i_\sigma(x'))\) for some \((x, x') \in A(X)\). This completes the proof.

**Lemma 6.6.** Let \( X_0 \subseteq C_0^\omega \) be a subshift with \( L \) centered asymptotic pairs, \( J_0 \) an index set and, for \( j \in J_0 \), \( \tau_j : C_0^+ \to B_j^+ \) a morphism. Let \( Y_j = \bigcup_{k \in \mathbb{Z}} T^k \tau_j(X_0) \) and \( X_0 \xrightarrow{i_{\pi_j}} X_{1,j} \xrightarrow{\pi_j} Y_j \) the sequence obtained from \((X_0, \tau_j)\) as in Lemma 6.5. Suppose that

1. \(|\tau_j(a)| = |\tau_j(a)|\) for all \( a \in C_0 \) and \( j \in J_0 \).
2. \( \pi_j \) is not distal for all \( j \in J_0 \).

Then, \( i := i_{\pi_j} \) and \( X'_1 := X_{1,j} \) do not depend on the choice of \( j \in J_0 \). Further, there exist a morphism \( \phi : C_0^+ \to C_1^+ \) with \( \#C_1 < \#C_0 \), a set \( J_1 \subseteq J_0 \) satisfying \( \#J_1 \geq \#J_0 / L \), and morphisms \( \tau'_j : C_1^+ \to B_j \), \( j \in J_1 \), such that \( \tau_j = \tau'_j \phi \).

**Proof.** First we observe that, since \( i_{\pi_j} \) is defined by \( i_{\pi_j}(a) = a^{i_{\pi_j}(a)} \), condition (I) implies that \( i := i_{\pi_j} \) and \( X_1 := X_{1,j} \) do not depend on the choice of \( j \).

Let \( j \in J_0 \). Since \( \pi_j \) is not distal, we can use Lemma 6.4 to obtain a centered asymptotic pair \((x^j, \tilde{x}^j) \in R^2_j\). Moreover, by Lemma 6.5, this asymptotic pair has the form \((x^j, \tilde{x}^j) = (i(z^j), i(\tilde{z}^j))\), where \((z^j, \tilde{z}^j)\) is a centered asymptotic pair in \( X_0 \). Since \( X_0 \) has \( L \) asymptotic pairs, we can use the Pigeon Principle to find \( J_1 \subseteq J_0 \), with \( \#J_1 \geq \#J_0 / L \), such that \((z, \tilde{z}) = (z^j, \tilde{z}^j)\) for all \( j \in J_1 \). Without loss of generality, we suppose that \((z, \tilde{z})\) is right-asymptotic. Now observe that, from the commutativity relation for \( \pi_j \) in Lemma 6.5, we have, for any \( j \in J_1 \),

\[
\tau_j(z) = \tau_j(z^j) = \pi_j(i(z^j)) = \pi_j(x^j) = \pi_j(\tilde{x}^j) = \pi_j(i(\tilde{z}^j)) = \tau_j(\tilde{z}^j) = \tau_j(z).
\]

Thus,

\[
\tau_j(z_0, \infty) = \tau_j(\tilde{z}_0, \infty) \quad \text{and} \quad z_0 \neq \tilde{z}_0 \quad \text{for all} \ j \in J_1.
\]

Let \( \ell := |\tau_i| \), where \( i \) is any element in \( J_1 \). Equation (29) allow us to use Lemma 3.2 with \( u := z_0(\ell), v := \tilde{z}_0(\ell), J := J_1 \) and \( w \) to find morphisms \( \phi : C_0^+ \to C_1^+, \tau'_j : C_1^+ \to B_j^+ \) such that \( \#C_1 < \#C_0 \) and \( \tau_j = \tau'_j \phi \) for all \( j \in J = J_1 \).

Since we have chosen \( J_1 \) so that \( \#J_1 \geq \#J_0 / L \), the proof is complete.

**6.3. Proof of main result.** We need a last lemma.

**Lemma 6.7.** Let \( X \subseteq A^Z \) be an aperiodic subshift with \( L \) asymptotic tails. Then, \((X, T, \#)\) has at most \( L^2 \cdot \# A^2 \) centered asymptotic pairs.

**Proof.** Let \( \Lambda_{\text{right}} \) be the set of centered right-asymptotic pairs in \( X \) and \( L_{\text{right}} = \{ x_0, \infty \} : (x, \tilde{x}) \in \Lambda_{\text{right}} \} \subseteq \{ x : \in \} \), where \( N_{\geq 1} = \{ 1, 2, \ldots \} \).

Let \( (x, \tilde{x}) \in \Lambda_{\text{right}} \). Note that \( x_0, \infty \in L_{\text{right}} \), \( K = \{ k \in \mathbb{Z} : x_k, \infty \in L_{\text{right}} \} \subseteq \{ -\infty, 0 \} \). We claim that \( K \) is finite. Otherwise, as \( \# L_{\text{right}} \) is finite, there exist \( k < k' \) and \( w \in L_{\text{right}} \) with \( x_{k, \infty} = w = x_{k', \infty} \), so that \( w \) has period \( k - k' \). But this implies that \( X \) contains a point of period \( k - k' \), contrary to the aperiodicity.
assumption. Thus, \( K \) is finite and \( k_{x,\tilde{x}} := \min K \) is a well-defined non-positive integer.

**Claim.** For any \( u, v \in L_{right}, \phi^{-1}(u, v) \) has at most \( K^2 \) elements, where \( K = \#A \).

**Proof.** By contradiction, let, for \( j \in [0, K^2], (x^j, \tilde{x}^j) \in \Lambda_{right} \) with \( \phi(x^j, \tilde{x}^j) = (u, v) \) and \( (x^i, \tilde{x}^i) \neq (x^j, \tilde{x}^j) \) for all \( i \neq j \). We put \( k_j := k_{x^j, \tilde{x}^j}, \tilde{k}_j = k_{\tilde{x}^j, x^j} \). First, we prove that \( k := k_j, \tilde{k} := \tilde{k}_j \) do not depend on the choice of \( j \). Since \( u = x^j(k_j, \infty) \), we have \( T^{-k_j}u = x^j(0, \infty) \) (where \( T \) is the shift in \( A^{[\geq 1]} \)) and, similarly, \( T^{-\tilde{k}_j}v = \tilde{x}^j(0, \infty) \), so that \( T^{-k_j}u = T^{-\tilde{k}_j}v \) for any \( j \in [0, K^2] \). Then, for any \( i, j \), we can compute \( T^{-k_j-i}v = T^{-k_j-i}v = T^{-k_i-k_j}v \) and deduce, by the aperiodicity of \( X \), that \( k_j + \tilde{k}_j = k_i + \tilde{k}_j \). Then, \( \ell := k_i - k_j = \tilde{k}_i - \tilde{k}_j \).

We suppose, with the aim to obtain a contradiction, that \( \ell > 0 \). Then \( k_i - k_j > 0 \) and, since \( x^j_{(k_j, \infty)} = u = x^j_{(k_i, \infty)} \), we have \( x^j_0 = u_{-k_j} = x^j_{k_i-k_j} = x^j_{\ell} \). Similarly, \( \tilde{x}^j_0 = \tilde{x}^j_{\ell} \). But then \( x^j_{(0, \infty)} = \tilde{x}^j_{(0, \infty)} \) implies \( x^j_0 = x^j_{\ell} = \tilde{x}^j_0 = \tilde{x}^j_{\ell} \), contrary to the fact that \( (x^j, \tilde{x}^j) \) is centered asymptotic. Thus, \( \ell \leq 0 \). A similar argument shows that \( \ell \geq 0 \) and, therefore, \( \ell = 0 \), this is, \( k_i = k_j, \tilde{k}_i = \tilde{k}_j \). Since \( i, j \) were arbitrary, we conclude that \( k := k_i, \tilde{k} := \tilde{k}_i \) do not depend on the choice of \( i \).

Now, since there are no more than \( K^2 \) pairs \((a, b) \in A \times A \) and \#\([0, K^2] \neq K^2 \), we can use the Pigeon Principle to find \( i \neq j \) such that \( (x^i_k, \tilde{x}^i_k) = (x^j_k, \tilde{x}^j_k) \). Then, since \( (x^i, \tilde{x}^i) \neq (x^j, \tilde{x}^j) \), either \( x^i \neq x^j \) or \( \tilde{x}^i \neq \tilde{x}^j \). Without loss of generality we suppose \( x^i \neq x^j \). Then, since \( x^j_{(k_j, \infty)} = u = x^j_{(k_i, \infty)} \) and \( x^i_{k_i} = x^i_{k_j} \), there exist \( l < k \) such that \( T^{l}x^i, T^{l}x^j \) are centered asymptotic and \( x^i_{(l, \infty)} \in L_{right} \), contrary to the fact that \( k = k_i \) is the smallest integer with this property. Thus, \#\( \phi^{-1}(u, v) \leq K^2 \). \( \square \)

Now, by the claim, we have \#\( \Lambda_{right} \leq \#L_{right}^2 \cdot K^2 \). We can obtain, in a similar way, the bound \#\( \Lambda_{left} \leq \#L_{left}^2 \cdot K^2 \) for centered left asymptotic pairs. Hence, the set \( \Lambda = \Lambda_{left} \cup \Lambda_{right} \) of all centered asymptotic pairs satisfies \#\( \Lambda \leq 2L^2K^2 \). This completes the proof. \( \square \)

Now we are ready to prove Theorem 6.8. We restate it for convenience.

**Theorem 6.8.** Let \((X, T)\) be a minimal subshift of topological rank \( K \). Then, \((X, T)\) has at most \( 3^{2K}K^{24K} \) aperiodic symbolic factors up to conjugacy.

**Proof.** We set \( R = 3^{2K}K^{24K} \). For \( i = 0, \ldots, R \), let \( \pi_i : (X, T) \to (Y_i, T) \) be an aperiodic symbolic factor. It is enough to show that \((Y_i, T)\) is conjugate to \((Y_j, T)\) for some \( i \neq j \). By contradiction, we suppose that there is no pair \( i \neq j \) with \((Y_i, T)\) conjugate to \((Y_j, T)\).

By Theorem 1.1 \( X \) is generated by a primitive and proper directive sequence \( \sigma = (\sigma_n : A^{\geq 1}_n \to A^+_n)_{n \in \mathbb{N}} \) of alphabet rank at most \( K \). Let \( r \in \mathbb{N} \) be such that every \( \pi_i \) has a radius \( r \) and let \( B_i \) the alphabet of \( Y_i \). After a contraction of \( \sigma \), we can suppose that \( \sigma_0 \) is \( r \)-proper and \#\( A_n \leq K \) for all \( n \geq 1 \). Then, by Lemma 2.3 there exist morphisms \( \tau_j : A^+_n \to B^+_n \) such that \( |\tau_j(a)| = |\sigma(a)| \) for all \( a \in A_1 \) and \( \pi(\sigma(a)) = \tau_j(x) \) for any \( x \in X^{(1)}_\sigma \).

We consider the following inductive procedure. First, we put \( X_0 = X^{(1)}_\sigma, C_0 = A_1, J_0 = [0, R] \) and \( \tau^0_j = \tau_j \) for \( j \in J_0 \). Suppose that \( X_n \subseteq C_n^+, J_n \) and \( \tau^j_n \) for \( j \in J_n \) are defined. Let \( X_n \uparrow_i X'^i_n \to Y_j \) be the sequence obtained from \((X_n, \tau^j_n)\) as
in Lemma 6.5. By Lemma 6.3 and the fact that no two \((Y_j, T)\) are conjugate, there are at most \(2K(144K^7)^K\) elements \(j \in J_n\) such that \(\pi_j^n\) is distal. Thus, we can find \(J'_n \subseteq J_n\), with \(#J'_n \geq #J_n - 2K(144K^7)^K\), such that \(\pi_j^n\) is not distal for any \(j \in J'_n\). Let \(L_n\) be the number of centered asymptotic pairs in \(X^n\). If \(#J'_n \leq L_n\), then the procedure stops. Otherwise, we can use Lemma 6.6 to find a morphism \(\phi_n : C_n^+ \to C_{n+1}^+\), with \(#C_{n+1} < #C_n\), a set \(J_{n+1} \subseteq J'_n\) satisfying \(#J_{n+1} \geq #J'_n / L_n\), and morphisms \(\tau_n^j : C_n^+ \to C_{n+1}^+\) such that \(\tau_n^j = \tau_{n+1}^j \phi_n\) for all \(j \in J_{n+1}\). Finally, we put \(X_{n+1} = \bigcup_{k \in \mathbb{Z}} T^k \phi_n(X_n)\).

Since \(#C_0 > #C_1 > \ldots\), there is a last \(C_N\) defined. We claim that \(N \geq K\). This would imply that \(#C_N \leq #C_0 - K = 0\), giving us a contradiction and thereby completing the proof of the theorem.

We write \(\phi_{(n,m)} = \phi_n \cdot \phi_{m-1}\), where \(\phi_{(n,m)} = \text{id}_{C_+^n}\) is the identity function when \(n = m\). Observe that, from the definitions, \(X_n = \bigcup_{k \in \mathbb{Z}} T^k \phi_{n-1}(X_{n-1}) = \bigcup_{k \in \mathbb{Z}} T^k \phi_{n-1}(X_{n-2}) = \cdots = \bigcup_{k \in \mathbb{Z}} T^k \phi_{(0,n)}(X_0) = \bigcup_{k \in \mathbb{Z}} T^k \phi_{(0,n)}(X^{(1)})\). Thus, by Lemma 2.5, the directive sequence \(\sigma_{(n)} = (\phi_{(0,n)},\sigma_1,\sigma_2,\ldots)\) generates \(X_n\). Moreover, \(\sigma_{(n)}\) is everywhere growing of alphabet rank at most \(K\) since \(\sigma\) has the same properties. This allow us to use Proposition 2.9 to deduce that \(X_n\) has at most \(2 \cdot 144K^7\) asymptotic tails, so Lemma 6.7 implies that \(L_n \leq 2K^2(288K^7)^2\). It follows that

\[
#J_{n+1} \geq (#J_n - 2K(144K^7)^K) / 2K^2(288K^7)^2 \quad \text{for any } n \in [0,N].
\]

Using this recurrence, that \(#J_0 > 3^{22K}K^{24K}\) and that \(K \geq 2\), it is routine to verify that

\[
#J'_n \geq #J_n - 2K(144K^7)^K \geq 2K^2(288K^7)^2 \geq L_n \quad \text{for all } n \in [0,N] \cap [0,K].
\]

Since the procedure stops only when \(#J'_n \leq L_n\), we deduce that \(N \geq K\), as we claimed. This ends the proof.

\[\square\]

**Remark 6.9.** In Theorem 1 of [Dur00], the author proved that linearly recurrent subshifts have finitely many aperiodic symbolic factors up to conjugacy. Since this kind of systems have finite topological rank (see Remark 4.10, Theorem 1.6) generalizes the theorem of [Dur00] to the much larger class of minimal finite topological rank subshifts.

### 7. Appendix

To prove Proposition 4.3, we start with some lemmas concerning how to construct recognizable pairs \((Z, \tau)\) for a fixed \(Y = \bigcup_{k \in \mathbb{Z}} T^k \tau(Z)\).

#### 7.1. Codings of subshifts.

If \(Y \subseteq \mathbb{B}^\mathbb{Z}\) is a subshift, \(U \subseteq Y\) and \(y \in Y\), we denote by \(\mathcal{R}_U(y)\) the set of *return times* of \(y\) to \(U\), this is, \(\mathcal{R}_U(y) = \{k \in \mathbb{Z} : T^k y \in U\}\).

**Lemma 7.1.** Let \(Y \subseteq \mathbb{B}^\mathbb{Z}\) be an aperiodic subshift. Suppose that \(U \subseteq Y\) is

(I) \(d\)-syndetic: for every \(y \in Y\) there exists \(k \in [0,d-1]\) with \(T^k y \in U\),

(II) of radius \(r\): \(U \subseteq \bigcup_{u \in A^r, v \in A^{r+1}} [u,v]\),

(III) \(\ell\)-proper: \(U \subseteq [u,u]\) for some \(u, v \in A^\ell\),

(IV) \(\rho\)-separated: \(U, Tu, \ldots, T^{\rho-1}U\) are disjoint.

Then, there exist a morphism \(\tau : C^+ \to B^+\) and a subshift \(Z \subseteq \mathbb{C}^\mathbb{Z}\) such that

(1) \(Y = \bigcup_{n \in \mathbb{Z}} T^n \tau(Z)\) and \(Z \subseteq \mathcal{L}(Y)\),

(2) \((Z, \tau)\) is recognizable with constant \(r + d\),
Remark 7.2. If $U \subseteq Y$ satisfies (III), then $U$ is min per($\mathcal{L}_{\geq r}(Y)$)-separated. Indeed, if $U \cap T^k U \neq \emptyset$ for some $k > 0$, then $[v] \cap T^k [v] \neq \emptyset$, where $v \in \mathcal{A}^f$ is such that $U \subseteq [v]$. Hence, $v$ is a periodic and $k \geq \min \text{per}(\mathcal{L}_{\geq r}(Y))$.

Proof. Let $y \in Y$. By (I), the sets $\mathcal{R}_U(y) \cap [0, \infty), \mathcal{R}_U(y) \cap (-\infty, 0]$ are infinite. Thus, we can write $\mathcal{R}_U(y) = \{\ldots k_i(y) < k_0(y) < k_1(y) \ldots \}$, with $\min \{i \in \mathbb{Z} : k_i(y) \geq 0\} = 1$. Let $\mathcal{W} = \{y(k_0(y), k_i(y)) : y \in Y, i \in \mathbb{Z}\} \subseteq B^+$. By (I), $\mathcal{W}$ is finite, so we can write $C := \{1, \ldots, \#\mathcal{W}\}$ and choose a bijection $\phi : C \to \mathcal{W}$. Then, $\phi$ extends to a morphism $\tau : C^+ \to B^+$. We define $\psi : Y \to C^\infty$ by $\psi(y) = (\phi^{-1}(y(k_0(y), k_i(y))))_{i \in \mathbb{Z}}$ and set $Z = \phi(Y)$. We are going to prove that $\tau$ and $Z$ satisfy (1-4).

Claim.

(i) If $y[-d-r,d+r] = y[-d-r,d+r]$, then $\psi(y) = 0 = \psi(y)'$, $\tau(\psi(y)) = T^{k_0(y)+1}y$, $\psi(T^jy) = \psi(y)$ for $j \in \mathbb{Z}$ and $k \in (k_j(y), k_{j+1}(y)]$.

By (I), we have $k_{i+1}(y) - k_i(y) \leq d$ for all $i \in \mathbb{Z}$ and, thus, $|k_0(y)|, |k_1(y)| \leq d$. Since $U$ has radius $r$ and $y[-d-r,d+r] = y[-d-r,d+r]'$, we deduce $k_0(y) = k_0(y)'$, $k_1(y) = k_0(y)'$, so $\psi(y)_0 = \phi^{-1}(y(k_0(y), k_1(y))) = \phi^{-1}(y(k_0(y)', k_1(y))) = \psi(y)'$. To prove (ii) we compute:

$$\tau(\psi(y)) = \tau(\cdots \phi^{-1}(y(k_{-1}(y), k_0(y))) \phi^{-1}(y(k_0(y), k_1(y))) \cdots)$$

$$= \cdots y(k_{-1}(y), k_0(y)) y(k_0(y), k_1(y)) \cdots = T^{k_0+1}y.$$

Finally, for (iii) we write

$$T^j\psi(y) = \cdots \phi^{-1}(y(k_{j-1}(y), k_j(y))) \phi^{-1}(y(k_j(y), k_{j+1}(y))) \cdots = \psi(T^ky) \text{ if } k \in (k_j(y), k_{j+1}(y)].$$

Now we prove the desired properties of $\tau$ and $Z$:

(1) From (i), we see that $\psi$ is continuous and, therefore, $Z$ is closed. By (iii), $Z$ is also shift-invariant, so, also a subshift. By (ii), $Y = \bigcup_{n \in \mathbb{Z}} T^n \tau(Z)$. The condition $C \subseteq \mathcal{L}(Y)$ follows from the definition of $\mathcal{W}$ and $\tau$.

(2) Next, we claim that the only centered $\tau$-interpretation in $Z$ of a point $y \in Y$ is $(-k_0(y) - 1, \psi(y))$. Indeed, this pair is a $\tau$-interpretation in $Z$ by (ii), and it is centered because $-k_0(y) - 1 \in [0, k_1 - k_0) = [0, |\psi(y)|]$). Let $(n, z)$ be another centered $\tau$-interpretation of $y$ in $Z$. By the definition of $Z$, there exists $y' \in Y$ with $z = \psi(y')$. Then, by (ii), $T^{n+k_0(y)+1}y' = T^n \tau(y') = y$. Since $(n, \psi(y'))$ is centered, we have $n + k_0(y') + 1 \in (k_0(y'), k_1(y')]$, so, by (iii) and the last equation, $\psi(y') = \psi(y)$. Hence, $y = T^n \tau(y') = T^n \tau(y) = T^{n+k_0(y)+1}y$, which implies $n = -(k_0(y) + 1)$ as $Y$ is aperiodic. This proves that $(-k_0(y) - 1, \psi(y))$ is the only $\tau$-interpretation of $y$ in $Z$. From this and (i) we deduce property (2).

(3) Since $U$ is $d$-syndetic, $|\tau(\psi(y))| = |y(k_0(y), k_{i+1}(y))| = k_{i+1}(y) - k_i(y) \leq d$ for $y \in Y$ and $i \in \mathbb{Z}$, so $|\tau| \leq d$. Similarly, we can obtain $|\tau| \geq \rho$ using that $U$ is $\rho$-separated. Let $u, v \in B^f$ satisfying $U \subseteq [u, v]$. Since $k_i, k_{i+1} \in \mathcal{R}_U(y)$, we have $u = y(k_0(y), k_i(y)+|u|)$, $v = y(k_0(y), k_{i+1}(y)+|u|)$, and, thus, $\tau$ is $\min(\ell, |\tau|)-proper$. In particular, it is $\min(\ell, \rho)$-proper.

(4) This follows directly from the definition of $\tau$ and $\mathcal{R}_U(y)$. 

(3) $|\tau| \leq d$, $|\tau| \geq \rho$ and $\tau$ is $\min(\rho, \ell)$-proper.
Lemma 7.3. For \( j = 0, 1 \), let \( \sigma_j : A_j^+ \to B^+ \) be a morphism and \( X_j \subseteq \mathcal{A}_j^\mathbb{Z} \) a subshift such that \( Y := \bigcup_{n \in \mathbb{Z}} T^n\sigma_j(X_j) \) and \( \mathcal{A}_j \subseteq \mathcal{L}(X_j) \). Suppose:

1. \( (X_0, \sigma_0) \) is recognizable with constant \( \ell \),
2. \( \sigma_1 \) is \( \ell \)-proper,
3. \( C_{\sigma_0}(k^0, x^0)(y) \supseteq C_{\sigma_1}(k^1, x^1)(y) \) for all \( y \in Y \) and \( \sigma_j \)-factorizations \( (k^j, x^j) \) of \( y \) in \( X_j \), \( j = 0, 1 \).

Then, there exist a proper morphism \( \nu : \mathcal{A}_1^+ \to \mathcal{A}_0^+ \) such that \( \sigma_1 = \sigma_0 \nu \) and \( X_0 = \bigcup_{k \in \mathbb{Z}} T^k \nu(X_1) \).

Proof. Since \( \sigma_1 \) is \( \ell \)-proper, we can find \( u, v \in B^\ell \) such that \( \sigma_1(a) \) starts with \( u \) and ends with \( v \) for any \( a \in A_1 \). We define \( \nu \) as follows. Let \( a \in A_1 \) and \( x \in X_1 \) such that \( a = x_0 \). Since \( \sigma_1 \) is \( \ell \)-proper, the word \( v.\sigma_1(a)u \) occurs in \( \sigma_1(x) \) at position 0. By (3), we can find \( w \in \mathcal{L}(X_0) \) with \( \sigma_1(x_0) = \sigma_0(w) \). We set \( \nu(a) = w \).

Since \( (X_0, \sigma_0) \) is recognizable with constant \( \ell \) and \( u, v \) have length \( \ell \), \( w \) depends only on \( v.\sigma_1(a)u \) and, therefore, \( \nu \) is well defined. Moreover, the recognizability implies that the first letter of \( \nu(a) \) depends only on \( v.u \), so \( \nu \) is left-proper. A symmetric argument shows that \( \nu \) is right-proper and, in conclusion, proper. From the definition we have \( \sigma_1 = \sigma_0 \nu \). Now, let \( x \in X_1 \) and \( (k, x') \) be a centered \( \sigma_0 \)-factorization of \( \sigma_1(x) \) in \( X_0 \). By (3), \( k = 0 \) and \( \sigma_1(x_j) = \sigma_0(x'_j) \) for some sequence \( \ldots < k_{-1} < k_0 < \ldots \). Hence, by the definition of \( \nu \), \( \nu(x) = x' \in X_0 \).

This argument shows that \( X_0 := \bigcup_{n \in \mathbb{Z}} T^n\sigma_0(X_1) \subseteq X_0 \). Then, \( \bigcup_{n \in \mathbb{Z}} T^n\sigma_0(X_0) = \bigcup_{n \in \mathbb{Z}} T^n\sigma_0\nu(X_1) = Y \), where in the last step we used that \( \sigma_0 \nu = \sigma_1 \). We deduce, from the recognizability property of \( (X_0, \sigma_0) \), that \( X_0 = X_0 \). This ends the proof. \( \square \)

7.2. Factors of \( S \)-adic sequences.

Lemma 7.4. Let \( \sigma = (A_n^+ \to A_m^+ \subseteq \mathcal{A}_m^\mathbb{Z})_{n \geq 0} \), \( \tau = (B_n^+ \to B_m^+ \subseteq \mathcal{B}_m^\mathbb{Z})_{n \geq 0} \) be weakly primitive, everywhere growing and proper directive sequences, and \( \phi = (\phi_n : A_n^+ \to B_n^+ \subseteq \mathcal{B}_n^\mathbb{Z})_{n \geq 1} \) such that \( \phi_0 \sigma_n = \tau_n \phi_{n+1} \) and \( \phi \) is weakly primitive for all \( n \geq 1 \). Then, \( \phi \) is a factor.

Proof. We have to prove that \( X_\tau^{(n)} = \bigcup_{k \in \mathbb{Z}} T^k \phi_n(X_\sigma^{(n)}) \) for all \( n \geq 1 \). Fix \( n \geq 1 \). Let \( y \in \bigcup_{k \in \mathbb{Z}} T^k \phi_n(X_\sigma^{(n)}) \) and \( \ell \geq 0 \). Then, since \( \sigma \) is everywhere growing and proper, there exist \( m > n \) and \( a \in A_m \) such that \( y \subseteq \phi_n \sigma_{[n,m]}(a) = \tau_{[n,m]} \phi_m(a) \), where in the last step we used the commutativity relation. Since this is valid for all \( \ell \geq 0 \), we deduce that \( y \in X_\tau^{(n)} \). The other inclusion is similar. \( \square \)

Now we are ready to prove Proposition 4.3. For convenience, we repeat its statement.

Proposition 7.5. Let \( \sigma = (\sigma_n : A_n \to A_{n-1} \subseteq \mathcal{A}_1^\mathbb{Z})_{n \geq 0} \) be a proper, everywhere growing and weakly primitive directive sequence. Suppose that \( X_\sigma \) is aperiodic. Then, there exists a contraction \( \sigma' = (\sigma_n)_{k \in \mathbb{N}} \) and a factor \( \phi : \sigma' \to \tau \), where \( \tau \) is recognizable, proper, everywhere growing, weakly primitive and generates \( X_\sigma \).

Proof. We start by observing that, since \( \sigma \) is everywhere growing and \( X_\sigma \) aperiodic, \( \lim_{n \to \infty} \rho_n = \infty \), so, up to a contraction of \( \sigma \), we can suppose that, for all \( n \geq 2 \),

\[ \text{(I''') } \sigma_{[0,n]} \text{ is } 3|\sigma_{[0,n-1]}|\text{-proper, } \quad \text{(II'') } \rho_n := \min \text{per}(\sigma_{[0,n]}(A_n)) \geq 3|\sigma_{[0,n-1]}|. \]
For $n \geq 2$, let $U_n = \bigcup_{u,v \in A_n^2} [\sigma_{[0,n)}(u,v)]$. Observe that $U_n$ is $|\sigma_{[0,n)}|$-syndetic, has radius $2|\sigma_{[0,n)}|$, is $3|\sigma_{[0,n-1)}|$-proper and, by Remark 7.2, is $p_n$-separated. Thus, by (H'), $U$ is $3|\sigma_{[0,n-1)}|$-separated. We use Lemma 7.1 with $(X^\sigma, \sigma_{[0,n)})$ to obtain a morphism $\nu_n: B^+_n \to A^+_0$ and a subshift $Y_n \subseteq B^+_n$ such that

$$(P^1_n) \ X^\sigma = \bigcup_{k \in \mathbb{Z}} T^k \nu_n(Y_n) \text{ and } B_n \subseteq \mathcal{L}(Y_n),$$

$$(P^2_n) \ \nu_n \text{ is recognizable with constant } 3|\sigma_{[0,n)}|,$$

$$(P^3_n) \ |\nu_n| \leq |\sigma_{[0,n)}|, \langle \nu_n \rangle \geq p_n, \text{ and } \nu_n \text{ is } 3|\sigma_{[0,n-1)}|\text{-proper},$$

$$(P^4_n) \ C_{\nu_n}(k,y) = R_{U_n}(x) \text{ for all } x \in X^\sigma \text{ and } \nu_n\text{-factorization } (k,y) \text{ of } x \text{ in } Y_n.$$  

We write $C_{\nu_n}(x) := C_{\nu_n}(k,y)$ if $x \in X^\sigma \text{ and } (k,y)$ is the unique $\nu_n$-factorization of $x$ in $Y_n$. Observe that $U_{n+1} \subseteq U_n$ for $n \geq 2$. Thus, $C_{\nu_{n+1}}(x) = R_{U_{n+1}}(x) \subseteq R_{U_n}(x) = C_{\nu_n}(x)$ for all $x \in X^\sigma$. This, $(P^3_n)$ and $(P^3_{n+1})$ allow us to use Lemma 7.3 with $(Y_{n+1}, \nu_{n+1})$ and $(Y_n, \nu_n)$ and find a proper morphism $\tau_n: B^+_{n+1} \to B^+_n$ such that $\nu_n \tau_n = \nu_{n+1}$ and $Y_n = \bigcup_{k \in \mathbb{Z}} T^k \tau_n(Y_{n+1}).$

We claim that $C_{\nu_n}(x) \supseteq C_{\sigma_{[0,n+1)}}(k,z)$ for all $x \in X^\sigma$, and $\sigma_{[0,n+1)}$-factorization $(k,z)$ of $x$ in $X^{(n+1)}$. Indeed, if $j \in \mathbb{Z}$, then $T^{c_{\sigma_{[0,n+1)}}(j,k)\sigma_{[n+1)}}}x \in [\sigma_{[0,n+1)}(z_j-1,z_{j+1})] \subseteq [\sigma_{[0,n)}(a,b)\sigma_{[n+1)}] \subseteq U_n$, where $a$ is the last letter of $\sigma_n(z_{j-1}) \text{ and } bc$ the first two letters of $\sigma_n(z_j \sigma_{[n+1)}(z_{j+1})$, so $c_{\sigma_{[0,n+1)}}(j,k)\sigma_{[n+1)}}(x) = C_{\nu_n}(x)$, as desired.

By the claim, $(P^2_n)$ and $(P^3_{n+1})$, we can use Lemma 7.3 with $(Y_n, \nu_n)$ and $(X^{(n+1)}, \sigma_{[0,n+1)})$ to obtain a proper morphism $\phi_n: A^+_{n+1} \to B^+_n$ such that $\sigma_{[0,n+1)} \phi_n = \nu_n$ and

$$(Y_n) = \bigcup_{k \in \mathbb{Z}} T^k \phi_n(X^{(n+1)}).$$

We define $\phi = (\phi_k)_{k \geq 2}, \tau = (\tau_1 := \nu_2, \tau_2, \tau_3, \ldots)$ and $\sigma' = (\sigma_n)_{n \geq 2}$. We claim that $\phi$ is a factor. From the commutativity relations for $\tau_n$ and $\phi_n$, we have

$$(30) \quad \nu_n \phi_n \sigma_{n+1} = \sigma_{[0,n+1)} \sigma_n = \sigma_{[0,n+2)} = \nu_{n+1} \phi_{n+1} = \nu_n \tau_n \phi_n \tau_{n+1}.$$  

In particular, $\nu_n \phi_n \sigma_{n+1}(x) = \nu_n \tau_n \phi_{n+1}(x)$ for any $x \in X^{(n+2)}$. Since $\phi_n \sigma_{n+1}(x)$ and $\tau_n \phi_{n+1}(x)$ are both elements of $Y_n$ and $(Y_n, \nu_n)$ is recognizable, we deduce that $\phi_n \sigma_{n+1}(x) = \tau_n \phi_{n+1}(x)$ for any $x \in X^{(n+2)}$. Thus, one of the words $\phi_n \sigma_{n+1}(x_0)$, $\tau_n \phi_{n+1}(x_0)$ is a prefix of the other. Since $A_{n+2} \subseteq \mathcal{L}(X^{(n+2)})$, we deduce that, for any $a \in A_{n+2}$, one of the words $\tau_n \phi_{n+1}(a)$, $\nu_n \phi_n \sigma_{n+1}(a)$ is a prefix of the other. But, by $(30)$, the words $\nu_n \tau_n \phi_{n+1}(a)$, $\nu_n \phi_n \sigma_{n+1}(a)$ have the same length, so $\phi_n \sigma_{n+1}(a)$ must be equal to $\tau_n \phi_{n+1}(a)$. This proves that $\phi_n \sigma_{n+1} = \tau_n \phi_{n+1}$. The following commutative diagram, valid for all $n \geq 2$, summarizes the construction so far:

![Diagram]

Recall that $\nu_n \tau_n = \nu_{n+1}$ for $n \geq 2$. Then, $\tau_1 \tau_2 \cdots \tau_n = \nu_{n+1}$, $\langle \tau_1 \tau_2 \cdots \tau_n \rangle \geq \langle \nu_{n+1} \rangle \geq p_n \rightarrow n \rightarrow \infty$ and $\tau$ is everywhere growing. Also, as each $\tau_n$ is proper, $\tau$ is proper. With this, we can use Lemma 7.4 to conclude that $\phi$ is a factor. Finally, using Lemma 2.3 with $(Y_n, \nu_n) = (Y_n, \tau_1 \tau_2 \cdots \tau_{n-1})$, we deduce that $(Y_n, \tau_{n-1})$ is recognizable, so $\tau$ is recognizable. This ends the proof. \qed
On symbolic factors of $S$-adic subshifts of finite alphabet rank

REFERENCES

[AEG15] Massoud Amini, George A. Elliott, and Nasser Golestani, *The category of Bratteli diagrams*, Canad. J. Math. 67 (2015), no. 5, 990–1023. MR 3391730

[Aus88] J. Auslander, *Minimal flows and their extensions*, ISSN, Elsevier Science, 1988.

[BKMS10] Sergey Bezuglyi, Jan Kwiatkowski, Konstantin Medynets, and Boris Solomyak, *Finite Rank Bratteli Diagrams: Structure of Invariant Measures*, arXiv e-prints (2010), arXiv:1003.2816.

[BSTY19] Valérie Berthé, Wolfgang Steiner, Jörg M. Thuswaldner, and Reem Yassawi, *Recognizability for sequences of morphisms*, Ergodic Theory Dynam. Systems 39 (2019), no. 11, 2896–2931. MR 4015135

[DDMP20] Sebastián Donoso, Fabien Durand, Alejandro Maass, and Samuel Petite, *Interplay between finite topological rank minimal Cantor systems, $S$-adic subshifts and their complexity*, arXiv e-prints (2020), arXiv:2003.06328.

[DFM19] Fabien Durand, Alexander Frank, and Alejandro Maass, *Eigenvalues of minimal Cantor systems*, J. Eur. Math. Soc. (JEMS) 21 (2019), no. 3, 727–775. MR 3908764

[DHS99a] F. Durand, B. Host, and C. Skau, *Substitutional dynamical systems, Bratteli diagrams and dimension groups*, Ergodic Theory Dynam. Systems 19 (1999), no. 4, 953–993. MR 1709427

[DHS99b] ———, *Substitutional dynamical systems, Bratteli diagrams and dimension groups*, Ergodic Theory Dynam. Systems 19 (1999), no. 4, 953–993. MR 1709427

[DM08] Tomasz Downarowicz and Alejandro Maass, *Finite-rank Bratteli-Vershik diagrams are expansive*, Ergodic Theory Dynam. Systems 28 (2008), no. 3, 739–747. MR 2422014

[Dur00] Fabien Durand, *Linearly recurrent subshifts have a finite number of non-periodic subshift factors*, Ergodic Theory and Dynamical Systems 20 (2000), no. 4, 1061–1078.

[EM20] Bastián Espinoza and Alejandro Maass, *On the automorphism group of minimal $S$-adic subshifts of finite alphabet rank*, arXiv e-prints (2020), arXiv:2008.05996.

[Esp20] Bastián Espinoza, *On symbolic factors of $S$-adic subshifts of finite alphabet rank*, arXiv e-prints (2020), arXiv:2008.13689.

[GH20] Nasser Golestani and Maryam Hosseini, *On Topological Rank of Factors of Cantor Minimal Systems*, arXiv e-prints (2020), arXiv:2008.04186.

[GHH18] T. Giordano, D. Handelman, and M. Hosseini, *Orbit equivalence of Cantor minimal systems and their continuous spectra*, Math. Z. 289 (2018), no. 3-4, 1199–1218. MR 3830245

[GJ00] Richard Gjerde and Ø rjan Johansen, *Bratteli-Vershik models for Cantor minimal systems: applications to Toeplitz flows*, Ergodic Theory Dynam. Systems 20 (2000), no. 6, 1687–1710. MR 1804953

[GJ02] Richard Gjerde and Orjan Johansen, *Bratteli-Vershik models for Cantor minimal systems associated to interval exchange transformations*, Math. Scand. 90 (2002), no. 1, 87–100. MR 1887096

[HPS92] Richard H. Herman, Ian F. Putnam, and Christian F. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. 3 (1992), no. 6, 827–864. MR 1194074

[RS97] G. Rozenberg and A. Salomaa, *Handbook of formal languages: Volume 1. word, language, grammar*, Handbook of Formal Languages, Springer, 1997.

DEPARTAMENTO DE INGENIERÍA MATEMÁTICA AND CENTRO DE MODELOAMIENTO MATEMÁTICO, UNIVERSIDAD DE CHILE, BEAUCHEF 851, SANTIAGO, CHILE.

Email address: bespinoza@dim.uchile.cl