VARIATIONAL PRINCIPLES FOR SPECTRAL RADIUS OF WEIGHTED ENDO MORPHISMS OF $C(X,D)$

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Abstract. We give formulas for the spectral radius of weighted endomorphisms $a\alpha : C(X,D) \to C(X,D)$, $a \in C(X,D)$, acting on the algebra of continuous functions from a compact Hausdorff space $X$ into a unital Banach algebra $D$. Under the assumption that $\alpha(C(X) \otimes 1) \subseteq \alpha(1)C(X) \otimes 1$, endomorphism $\alpha$ generates a partial dynamical system $(X,\varphi)$. We establish two kinds of variational principles for $r(a\alpha)$: using linear extensions of $(X,\varphi)$ and using Lyapunov exponents associated with ergodic measures for $(X,\varphi)$.

Our results are most efficient when $D = \mathcal{B}(F)$, for a Banach space $F$, and endomorphisms of $\mathcal{B}(F)$ induced by $\alpha$ are inner isometric. The established formulas can be also applied to a vast class of operators that we call abstract weighted shift operators associated with endomorphisms. In particular, our variational principles are far reaching generalizations of similar formulas obtained by Kitover, Lebedev, Latushkin, Stepin and others.

Introduction

Let $\alpha : A \to A$ be an endomorphism of a Banach algebra $A$. The study of spectra of weighted endomorphisms $a\alpha : A \to A$, $a \in A$, has a long tradition and is interesting in its own right, see, for instance, [Kit79], [Kam79], [Kam81], [JR88], where usually the case when $A$ is commutative and/or $a\alpha$ is compact is considered. Our interest in weighted endomorphisms stems from their relationship with weighted composition operators. Spectral properties of such operators play a crucial role in numerous problems in mathematical physics, ergodic theory, stochastic processes, information theory, the theory of solvability of functional differential equations, wavelet analysis etc. We refer, for example, to the books and survey articles [Wal82], [LS91], [LM94], [AL94], [KL94], [Ant96], [CL99], [ABL12].

In the case when the shift (the underlying map) is reversible, the theory of weighted composition operators was axiomatized in [AL94]. Namely, let $A$ be a Banach subalgebra of the algebra $\mathcal{B}(E)$ of bounded linear operators acting on a Banach space $E$ and let $T \in \mathcal{B}(E)$ be an invertible isometry such that $TAT^{-1} = A$. Then operators of the form $aT$, $a \in A$, are called (abstract) weighted shift operators with weight in the algebra $A$. Within this setting the formula $\alpha(a) := TaT^{-1}$, $a \in A$, defines an automorphism $\alpha : A \to A$ of $A$. It turns out that all the fundamental spectral data concerning $aT$, $a \in A$, can be efficiently phrased and analyzed in terms of the noncommutative dynamical system $(A,\alpha)$, see [AL94], [Ant96], [CL99], [ABL12]. In particular, the spectral radii of the weighted shift $aT \in \mathcal{B}(E)$ and the weighted automorphism $a\alpha \in \mathcal{B}(A)$ coincide.

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When $A$ is a commutative uniform unital Banach algebra (i.e. the Gelfand transform $A \ni a \rightarrow \hat{a} \in C(X)$ is an isometry) the spectral radius is given by the following variational principle (see [Kit79], [Leb79], and [AL94] or [Ant96]):

$$r(aT) = r(aa) = \max_{\mu \in \text{Erg}(X,\varphi)} \exp \int_X \ln |\hat{a}(x)| \, d\mu,$$

where $\varphi : X \rightarrow X$ is the homeomorphism dual to $\alpha$, and $\text{Erg}(X,\varphi)$ is the set of $\varphi$-ergodic probability measures. This covers, for instance, the situation when $E = L^p(X)$ or $E = C(X)$, $A$ is the algebra of operators of multiplication by functions in $C(X)$ and $T$ is an operator of composition with a measure preserving homeomorphism $\varphi$. Latushkine and Stepin [LS91] analyzed the case when $A = C(X,B(H))$ for a separable Hilbert space $H$ and $\alpha : A \rightarrow A$ is given by a composition with a homeomorphism $\varphi : X \rightarrow X$. This models situation of weighted composition operators acting on vector-valued spaces $E = L^p(X,H)$. Under the assumption that $a \in C(X,B(H))$ takes values in compact operators $\mathcal{K}(H) \subseteq B(H)$, they proved (see also [AL94] or [CL99]) that

$$\ln r(aT) = \ln r(aa) = \sup_{\mu \in \text{Erg}(X,\varphi)} \lambda_\mu,$$

where $\lambda_\mu$ is the maximal Lyapunov exponent appearing in Ruelle’s version of Multiplicative Ergodic Theorem [Rue82] applied to the dynamical measure system $(X,\mu,\varphi)$ and the cocycle coming from $a : X \rightarrow B(H)$.

The aim of the present paper is to give a detailed picture of the corresponding variational principles in a general irreversible situation and for more general non-commutative algebras of weights. In fact, we introduce and initiate a study of abstract weighted shifts $aT$, $a \in A$, associated with an endomorphism $\alpha : A \rightarrow A$ (Definition 2.14). In our setting $T \in \mathcal{B}(E)$ is a partial isometry on a Banach space $E$, as defined by Mbekhta [Mbe04], and $\alpha(a) = TaS$, $a \in A$, where $S \in \mathcal{B}(E)$ is a partial isometry adjoint to $T$. We note that any contractive endomorphism on a unital Banach algebra can represented in this form (Proposition 2.17). Moreover, since in this article we focus on spectral radius, our analysis boils down to the study of spectral radius of the weighted endomorphism $aa : A \rightarrow A$, as we always have $r(aT) = r(aa)$ (see Proposition 2.18). A number of concrete examples of abstract weighted shifts associated with endomorphisms were considered in [Kwa09], see also [Kwa12]. We plan to investigate their spectral properties in a forthcoming paper.

In the present article we have established a number of variational principles (VPs in short). A general scheme of relationships between them is presented on Figure 1.

We start with preliminary Sections 1 and 2 where we discuss the necessary objects and results concerning endomorphisms (i.e. irreversible and non-commutative dynamics) and weighted shift operators associated with endomorphisms. In contrast to reversible dynamics associated with automorphisms natural maps associated with endomorphisms are partial mappings, i.e. continuous maps $\varphi : \Delta \rightarrow X$ defined on a subset $\Delta \subseteq X$ of a compact space $X$ [1] In section 3 we show that natural ergodic measures $\varphi : \Delta \rightarrow X$ are the usual measures for the restriction $\varphi : \Delta_\infty \rightarrow \Delta_\infty$ to an essential domain of $\varphi$. Our main technical tool is what we call variational principle for lim sup of empirical

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1 This seemingly annoying technical detail turns out to be a friend in disguise, as when considering ‘linear extensions’ we need to deal with partial maps anyway, and we believe that keeping track of the domains is beneficial here.
averages over \((X, \varphi)\) (Theorem 3.4). It implies that for empirical averages the operations \(\lim\) and \(\sup\) do commute (Corollary 3.5). Moreover, it readily gives a generalization of formula (1) to the case of weighted endomorphisms of a commutative uniform algebra \(A\) (Theorem 5.1).

In order to deal with noncommutative Banach algebras, in Section 4, we study Lyapunov exponents associated with an operator valued function \(a : \Delta \to B(F)\) and a partial dynamical system \((X, \varphi)\). We introduce spectral exponent \(\lambda(a, \varphi)\) which is equal to \(\ln r(a\alpha)\) when \(a \in C(X, B(F))\) and \(\alpha : C(X, B(F)) \to C(X, B(F))\) is given by a composition with \(\varphi\) (Definition 4.2). We construct a continuous linear extension \((\tilde{X}, \tilde{\varphi})\) of \((X, \varphi)\) where \(\tilde{X} = X \times [B]\) and \([B]\) is a quotient of a unit ball in \(F^*\). Variational principle for \(\lim \sup\) of empirical averages applied to \((\tilde{X}, \tilde{\varphi})\) implies a generalization of (1) that expresses the spectral exponent \(\lambda(a, \varphi)\) in terms of maximum of integrals taken over the extended system \((\tilde{X}, \tilde{\varphi})\) (Theorem 4.6). By projecting measures from \((\tilde{X}, \tilde{\varphi})\) to \((X, \varphi)\), the aforementioned generalization of (1) implies a generalization of (2), which says that the spectral exponent is the maximum of measure exponents and it realizes as Lyapunov exponent in a concrete direction when passing to dual space (Theorem 4.10 and Corollary 4.13). This result has a flavor of a variational principle for topological pressure. We note that we achieved it without appealing to any of Multiplicative Ergodic Theorems, cf. Remark 4.14.

Finally, in Section 5, we apply the aforementioned results to weighted endomorphisms of \(A = C(X, D)\) where \(D\) is a unital Banach algebra. We assume that the endomorphism satisfies \(\alpha(C(X) \otimes 1) \subseteq \alpha(1)C(X) \otimes 1\), which is equivalent to assuming that \(\alpha\) is of the form

\[
\alpha(a)(x) = \begin{cases} 
\alpha_x(a(\varphi(x))), & x \in \Delta, \\
0, & x \notin \Delta,
\end{cases}
\]

where \((X, \varphi)\) is a partial dynamical system and \(\{\alpha_x\}_{x \in \Delta}\) a continuous field of endomorphisms of \(D\). So that not only the shift \(\varphi\) but also the ’twist’ \(\{\alpha_x\}_{x \in \Delta}\) is involved.
We show that the logarithm of spectral radius \( r(aa) \) is equal to a spectral exponent of a cocycle with values in \( \mathcal{B}(D) \). This leads to analogues of (1), (2) developed for general case (Theorem 5.14). The obtained formulas can be significantly improved when \( D = \mathcal{B}(F) \) and all the endomorphisms \( \{ \alpha_x \}_{x \in \Delta} \) are isometric and inner. Then for any family \( \{ T_x \}_{x \in \Delta} \subseteq \mathcal{B}(F) \) that implements endomorphisms \( \{ \alpha_x \}_{x \in \Delta} \), we may formulate generalizations of (1), (2) using cocycles associated to the function

\[
\Delta \ni x \to a(x)T_x \in \mathcal{B}(F),
\]

see Theorems 5.9 and 5.10. One may view these last theorems as main results of the paper. When \( \dim(F) < \infty \), they can be applied to any contractive endomorphism of \( C(X,B(F)) \) (Remark 5.12). In essence, all the VPs in the paper can be deduced from these theorems. One of the main difficulties in proving them is that, in general, there are cohomological obstructions implying that the map (3) is discontinuous (Remark 1.15). In fact, the special form of continuous linear extension \((\tilde{X},\tilde{\varphi})\), we constructed in Section 4 was dictated by the need of overcoming this difficulty.

En passant, we mention that any operator \( a \in \mathcal{B}(F) \) may be treated as an element in \( C(\{ x \}, \mathcal{B}(F)) \). Then our generalization of (1) gives an intriguing 'Dynamical Variational Principle' for the spectral radius of an arbitrary operator \( a \in \mathcal{B}(F) \) (Theorem 5.3), and our generalization of (2) gives an improvement of the classical Gelfand’s formula (Corollary 5.5).

1. Endomorphisms of Banach algebras

A general (irreversible) dynamics on non-commutative structures (algebras) is naturally given by means of endomorphisms. In this section we discuss the corresponding objects and facts that will be used in our further analysis.

1.1. Endomorphisms of commutative algebras and partial maps. Let \( A \) be a commutative Banach algebra with an identity. Recall that the space \( X \) of non-zero linear multiplicative functionals on \( A \) equipped with *-weak topology (induced from the dual space \( A^\ast \)) is a compact Hausdorff. It is called the maximal ideals space of \( A \), or the (Gelfand) spectrum of \( A \). The Gelfand transform is the homomorphism:

\[
A \ni a \to \hat{a} \in C(X), \quad \hat{a}(x) := x(a),
\]

where \( C(X) \) is the algebra of all complex valued continuous functions on \( X \). Recall that \( A \) is called semisimple if the Gelfand transform is a monomorphism, that is when the radical \( R(A) := \bigcap_{x \in X} \ker x \) of \( A \) is zero. We say that \( A \) is regular if it is semisimple and the functions \( \hat{a}, a \in A \), separate points from closed sets in \( X \); that is for each point \( x \in X \) and a closed subset \( F \subseteq X \) such that \( x \notin F \) there exists an element \( a \in A \) such that \( \hat{a}(x) \neq 0 \) and \( \hat{a}(F) = 0 \).

Let \( \alpha : A \to A \) be an endomorphism. Then for every \( x \in X \) the functional \( x \circ \alpha \) is linear and multiplicative. Thus the dual operator to \( \alpha \) defines the map \( \varphi \) from \( X \) to the set \( X \cup \{ 0 \} \) (the functional \( x \circ \alpha \) may be zero). Note that the restriction of this map to \( \Delta = \varphi^{-1}(X) \) takes values in \( X \) and therefore it can be considered a partial map on \( X \).

**Proposition 1.1.** For any endomorphism \( \alpha : A \to A \) of a commutative Banach algebra \( A \) there is a uniquely determined continuous map \( \varphi : \Delta \to X \) defined on a clopen (closed
and open) subset $\Delta \subseteq X$ such that

\begin{equation}
\widehat{\alpha}(a)(x) = \begin{cases} 
\hat{a}(\varphi(x)), & x \in \Delta \\
0, & x \notin \Delta
\end{cases}, \quad a \in A.
\end{equation}

Moreover, $\Delta = X$ iff $\alpha$ preserves the identity of algebra $A$.

**Proof.** Uniqueness of $\varphi : \Delta \to X$ follows from that the functions $\hat{a}, a \in A$, separate points of $X$. Since $\alpha(1)$ is the characteristic function of the set $\Delta$, it follows that $\Delta$ is clopen and it is equal to $X$ if and only if $\alpha(1) = 1$, cf. [Żel68, 20]. Thus we have only to verify the continuity of $\varphi$ (note that we do not presuppose the continuity of $\alpha$). If $A$ is a semisimple, then $\alpha$ is automatically continuous. In general the spectrum of quotient algebra $A/R(A)$, can be naturally identified with $X$. Therefore the dual map to the homomorphism $\hat{\alpha} : A \to A/R(A)$ given by $\hat{\alpha}(a) := a + R(A), a \in A$, coincides with the dual map to $\alpha$. So the continuity of $\hat{\alpha}$, [Żel68, Theorem 13.2], implies the continuity of $\varphi$. \hfill \square

**Definition 1.2.** We will call the map $\varphi : \Delta \to X$ satisfying (4) the partial map dual to the endomorphism $\alpha$. In general, by a partial dynamical system we mean a pair $(X, \varphi)$ where $X$ is compact Hausdorff space and $\varphi : \Delta \to X$ is a continuous map defined on an open set $\Delta \subseteq X$.

1.2. **Endomorphisms of $C(X, D)$.** Let $A := C(X, D)$ be the Banach algebra of continuous functions defined on a compact Hausdorff space $X$ and taking values in a Banach algebra $D$. Any endomorphism $\alpha$ of $A$ give rise to a family of endomorphisms $\{\alpha_x\}_{x \in X}$ of $D$ where

\begin{equation}
\alpha_x(a) := \alpha(1 \otimes a)(x), \quad a \in D, \ x \in X.
\end{equation}

Moreover, since $\alpha(1 \otimes a) \in C(X, D)$, the mapping $X \ni x \mapsto \alpha_x(a) \in D$ is continuous for every $a \in D$. We will call $\{\alpha_x\}_{x \in X}$ the continuous field of endomorphisms of $D$ generated by the endomorphism $\alpha$. More generally, by a continuous field of endomorphism of $D$ on $X$ we mean any family $\{\alpha_x\}_{x \in X} \subseteq \text{End}(D)$ such that $X \ni x \mapsto \alpha_x(a) \in D$ is continuous for every $a \in D$. Note that for any continuous field $\{\alpha_x\}_{x \in X} \subseteq \text{End}(D)$ the set

\begin{equation}
\Delta := \{x \in X : \alpha_x \neq 0\}
\end{equation}

is open in $X$. If, in addition, $D$ is unital, then $A$ is unital and $\Delta$ is clopen, as we have $\Delta = \{x \in X : \|\alpha(1)(x)\| = 1\}$.

**Proposition 1.3.** Let $D$ be a Banach algebra and $A = C(X, D)$. For any continuous partial map $\varphi : \Delta \to X$ (defined on a clopen set $\Delta \subseteq X$) and any continuous field of non-zero endomorphisms $\{\alpha_x\}_{x \in \Delta}$, the formula

\begin{equation}
\alpha(a)(x) = \begin{cases} 
\alpha_x(a(\varphi(x))), & x \in \Delta \\
0, & x \notin \Delta
\end{cases}, \quad a \in A,
\end{equation}

defines an endomorphism $\alpha$ of the algebra $A$. Formula (7) determines both the field of endomorphisms $\{\alpha_x\}_{x \in \Delta}$ and the partial dynamical system $(X, \varphi)$ uniquely.

Moreover, if $D$ is unital then an arbitrary endomorphism $\alpha$ of $A$ is of the form (7) if and only if

\begin{equation}
\alpha(C(X) \otimes 1) \subseteq (C(X) \otimes 1) \cdot \alpha(1),
\end{equation}
that is, when $\alpha$ "almost invariates" the algebra $C(X) \otimes 1$.

Proof. We adopt the proof of [Kwa16, Proposition 3.5]. It is obvious that the map given by (7) is multiplicative and linear. It is well defined as using the Lipschitz property of bounded linear maps we get that the map $X \times A \ni (x,a) \mapsto \alpha_x(a) \in D$ is continuous for every continuous field of endomorphisms. If $\alpha$ is given by (7), then both the field of endomorphisms and the set $\Delta$ are determined by $\alpha$ via formulae (5), (6). Now if we assume that $\alpha$ satisfies (7) with $\varphi$ replaced with a different map $\varphi' : \Delta \to X$, then there is $x \in \Delta$ such that $\varphi(x) \neq \varphi'(x)$. Take $a \in C(X)$ such that $a(\varphi(x)) = 1$ and $a(\varphi'(x)) = 0$, and let $b \in D \setminus \ker \alpha_x$. On one hand we get $\alpha(a \otimes b)(x) = a(\varphi(x))\alpha_x(b) = \alpha_x(b) \neq 0$, while on the other $\alpha(a \otimes b)(x) = a(\varphi'(x))\alpha_x(b) = 0$, a contradiction.

Assume that $D$ is unital and $\alpha$ satisfies (8). Then for every $a \in C(X)$ there exists $a' \in C(X)$ such that

$$\alpha(a \otimes 1)(x) = a'(x)\alpha_x(1), \quad x \in X.$$ 

Clearly, the function $a'$ is uniquely determined by $a$ on the set $\Delta := \{x \in X : \alpha(A)(x) \neq 0\} = \{x \in X : \alpha_x(1) \neq 0\}$. Since $||\alpha_x(1)|| \in \{0\} \cup [1, +\infty)$ (as a norm of an idempotent), the set $\Delta$ is open and compact. Now it is straightforward to see that the formula $\Phi(a) = a'|_{\Delta}$ defines an endomorphism $\Phi : C(X) \to C(\Delta) \subseteq C(X)$ whose range is $C(\Delta)$. Hence $\Phi$ generates a partial map $\varphi : \Delta \to X$, see Proposition 1.1. For every $a \in C(X)$ and $b \in D$ we have

$$\alpha(a \otimes b)(x) = \alpha(a \otimes 1)a(1 \otimes b)(x) = \Phi(a)(x)\alpha_x(b) = a(\varphi(x))\alpha_x(b) = \alpha_x(a \otimes b(\varphi(x))).$$

Thus $\alpha$ satisfies (8), by continuity and linearity. \hfill $\Box$

In the nomenclature of [Kwa16, Definition 3.3], an endomorphism $\alpha : C(X, D) \to C(X, D)$ satisfying (7) is said to be induced by a morphism $(\varphi, \{\alpha_x\}_{x \in \Delta})$ of the corresponding bundle of algebras. In accordance with Definition 1.2 we adopt the following:

**Definition 1.4.** If $\alpha : C(X, D) \to C(X, D)$ is an endomorphism of the form (7) we say that $\alpha$ generates the partial dynamical system $(X, \varphi)$.

We will show that when $D = B(F)$ for a Banach space and the endomorphisms in the associated field are inner, then $\alpha$ generates a partial dynamical system (see Proposition 1.11 below). In particular, if $\dim(F) < \infty$, then every endomorphism of $C(X, B(F))$ generates a partial dynamical system. In the infinite dimensional case, even when $F = H$ is a Hilbert space, there are *-endomorphisms of $C(X, B(H))$ that do not generate partial dynamical systems in the sense of Definition 1.4.

**Example 1.5.** Let $H$ be a Hilbert space and let $V_1, \ldots, V_n \subseteq B(H), n > 1$, be isometries with orthogonal ranges: $V_i^*V_i = 1, V_i^*V_j = 0, i \neq j$. Let $X = \{x_1, x_2, \ldots, x_n\}$ be discrete space and consider $\alpha : C(X, B(H)) \to C(X, B(H))$ given by

$$\alpha(a)(x) = \sum_{i=1}^n V_i a(x_i) V_i^*, \quad a \in C(X, B(H)), x \in X.$$

One readily sees that $\alpha$ is a *-endomorphism of $C(X, B(H))$ that does not satisfy (8).
1.3. Endomorphisms of $C(X, B(F))$ generating inner fields of endomorphisms. Throughout this subsection we fix a Banach space $F$.

**Definition 1.6.** An endomorphism $\alpha : B(F) \to B(F)$ is **inner** if there are $T, S \in B(F)$ such that $TaS = \alpha(a)$ for all $a \in B(F)$.

**Proposition 1.7.** Suppose that $\alpha : B(F) \to B(F)$ is an inner endomorphism and $T, S \in B(F)$ are such that $TaS = \alpha(a)$ for all $a \in B(F)$.

1. $\alpha$ is injective if and only if $ST = 1$;
2. if $\alpha$ is contractive then $\alpha$ is isometric if and only if $ST = 1$ and up to normalization $T$ is an isometry (i.e. for $\lambda = \|T\|^{-1}$, we have that $\lambda T$ is an isometry, and $\alpha(\cdot) = \lambda T(\cdot)\lambda^{-1}S$).

**Proof.** For $x \in F$ and $f \in F^*$ we denote by $\Theta_{x,f} \in B(F)$ the corresponding rank one operator, given by the formula $\Theta_{x,f}(y) = f(y)x$, $y \in F$.

(1). If $ST = 1$, then $S\alpha(a)T = STaST = a$ for all $a \in B(F)$, and hence $\alpha$ is injective. Conversely, assume that $ST \neq 1$. If $\alpha(1) = 0$, then $\alpha$ is not injective. Thus we may assume that $\alpha(1) \neq 0$. Then $ST \neq \gamma 1$, for every $\gamma \in \mathbb{C}$. Indeed, if $ST = \gamma 1$, then

$$\alpha(1) = \alpha(1)^2 = TSTS = \alpha(S) = \alpha(\gamma 1) = \gamma \alpha(1)$$

implies that $\gamma = 1$, which contradicts $ST \neq \gamma 1$. Thus there exists $x \in F$ such that $STx$ and $x$ are linearly independent. By Hahn-Banach theorem there exists $f \in F^*$ such that $f(x) = 1$ and $f(STx) = 0$. Then on one hand $\Theta_{x,f}$ is a projection onto the space spanned by $x \neq 0$. On the other hand, $\alpha(\Theta_{x,f}) = \Theta_{Txs,Tsf}$ is an idempotent (since $\Theta_{x,f}$ is an idempotent) and its range is contained in the space spanned by $Tx$. However, $\Theta_{Txs,Tsf}Tx = f(STx)Tx = 0$. Hence $\alpha(\Theta_{x,f}) = 0$ and therefore $\alpha$ fails to be injective.

(2). Let $\alpha$ be contractive. If $ST = 1$ and $T$ is an isometry, then for every $a \in B(F)$ we have

$$\|a\| = \|Ta\| = \|TaST\| = \|\alpha(a)T\| \leq \|\alpha(a)\| \leq \|a\|.$$ 

Hence $\alpha$ is isometric.

Conversely, assume that $\alpha$ is isometric. Then $ST = 1$ by part (1). Let us take any functional $f \in F^*$ of norm 1. Note that $S^*f \neq 0$ since $S$ is surjective. Now for each $h \in F$ we get

$$\|Th\| = \|\Theta_{Th,f}\| = \frac{\|\Theta_{Th,S^*f}\|}{\|S^*f\|} = \frac{\|\alpha(\Theta_{h,f})\|}{\|S^*f\|} = \frac{\|h\|}{\|S^*f\|}.$$ 

Thus replacing $S, T$ with $\frac{1}{\|S^*f\|}S, \|S^*f\|T$, we may assume that $T$ is an isometry. \qed

**Proposition 1.8.** For any endomorphism $\alpha : B(F) \to B(F)$ the following statements are equivalent:

1. $\alpha$ is injective and inner;
2. $\alpha(B(F)) = \alpha(1)B(F)\alpha(1)$ and there is an injective $T \in B(F)$ such that $Ta = \alpha(a)T$, for every $a \in B(F)$.

Moreover, if (2) holds then there exists a unique $S \in B(F)$ such that $\alpha(a) = TaS$ for all $a \in B(F)$. If $\alpha$ is isometric, we may choose $T$ to be isometry and then $\|S\| = 1$. 

Proof. Assume (1) and let $T, S \in \mathcal{B}(F)$ be such that $TaS = \alpha(a)$ for all $a \in \mathcal{B}(F)$. Then
\[ \alpha(SaT) = TsaTS = \alpha(1)\alpha(1), \]
which implies $\alpha(\mathcal{B}(F)) = \alpha(1)\mathcal{B}(F)\alpha(1)$ (because we always have $\alpha(B(F)) \subseteq \alpha(1)B(F)\alpha(1)$). By Proposition \textbf{1.7(1)} we have $ST = 1$. Hence $T$ is injective and for every $a \in \mathcal{B}(F)$ we get $Ta = TaST = \alpha(a)T$. Thus (1) $\Rightarrow$ (2).

Now assume (2). Note that for any $h \in F \setminus \{0\}$ we have $\mathcal{B}(F)h = F$. Hence
\[ TF = TB(F)F = \alpha(\mathcal{B}(F))TF = \alpha(1)\mathcal{B}(F)\alpha(1)TF = \alpha(1)F. \]

Since $\alpha(1) \in \mathcal{B}(F)$ is an idempotent, this implies that $TF$ is a closed complemented subspace of $F$. In particular, $T : F \to \alpha(1)F$ is a bounded invertible operator. Defining
\begin{equation}
S_h := T^{-1}(\alpha(1)h), \quad h \in F,
\end{equation}
we get $S \in \mathcal{B}(F)$ such that $TaS = \alpha(1)$, and therefore equality $Ta = \alpha(a)T$, $a \in \mathcal{B}(F)$ implies that $TaS = \alpha(a)TS = \alpha(a)\alpha(1) = \alpha(a)$. Hence $\alpha$ is inner. Clearly, we have $ST = 1$ and therefore $\alpha$ is injective by Proposition \textbf{1.7(1)}. Thus (2) $\Rightarrow$ (1).

Suppose now that equivalent conditions (1), (2) hold. Let $T$ be as in (2) and let $S \in \mathcal{B}(F)$ be any operator such that $TaS = \alpha(a)$ for all $a \in \mathcal{B}(F)$. Since $Tsa(1) = \alpha(1)\alpha(1) = \alpha(1)$, we conclude that $S_{\alpha(1)}F = T^{-1}$ (recall that $T : F \to \alpha(1)F$ is a bijection). Note also that $ST = 1$ by Proposition \textbf{1.7(1)}. Hence $S = STS = \alpha(1)$. Therefore $S$ has to be of the form (9).

If $\alpha$ is isometric, we may choose $T$ to be isometry by Proposition \textbf{1.7(2)}. Then $\|Sh\| = \|T^{-1}(\alpha(1)h)\| = \|\alpha(1)h\| \leq \|h\|$ and $\|Sh\| = \|h\|$ if $h \in \alpha(1)F$. Thus $\|S\| = 1$. \hfill $\square$

Example 1.9 (endomorphisms of $M_n(\mathbb{C})$). By Skolem–Noether theorem endomorphisms of $M_n(\mathbb{C})$ are necessarily inner automorphisms: for every non-zero endomorphism $\alpha : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ there is an invertible $T \in M_n(\mathbb{C})$ such that $\alpha(a) = TaT^{-1}$. This well known fact could also be recovered using Proposition \textbf{1.8}.

Example 1.10 ($\ast$-endomorphisms of $\mathcal{B}(H)$). Let $H$ be a separable Hilbert space. Let $\alpha : \mathcal{B}(H) \to \mathcal{B}(H)$ be a $\ast$-endomorphism. It is well known, cf. [Lac93], [BP96], that $\alpha$ is necessarily injective. Moreover, there is a number $n = 1, 2, \ldots, \infty$ called multiplicity index or Powers index for $\alpha$, and a family $\{V_i\}_{i=1}^n$ of isometries with orthogonal ranges such that
\[ \alpha(a) = \sum_{i=1}^n V_i a V_i^*, \quad a \in \mathcal{B}(H), \]
where in the case $n = \infty$ the sum is weakly convergent. It follows that a $\ast$-endomorphism of $\mathcal{B}(H)$ is inner if and only if its multiplicity index is 1 (the only if part follows from the last part of Lemma \textbf{1.7} and Proposition \textbf{2.1} below).

Endomorphism in Example 1.5 generates a field of endomorphisms of $\mathcal{B}(H)$ with Powers index $n > 1$, and thus they are not inner. This agrees with the following:

Proposition 1.11. Let $\alpha : C(X, \mathcal{B}(F)) \to C(X, \mathcal{B}(F))$ be an endomorphism and let $\{\alpha_x\}_{x \in \Delta}$ be the field of (non-zero) endomorphisms of $\mathcal{B}(F)$ generated by $\alpha$. If the endomorphisms $\{\alpha_x\}_{x \in \Delta}$ are inner, then $\alpha$ generates a partial dynamical system. More specifically, $\{\alpha_x\}_{x \in \Delta}$ are inner if and only if $\alpha$ is of the form
\[ \alpha(a)(x) = \begin{cases} T_x a(x) S_x, & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases} \quad a \in C(X, \mathcal{B}(F)), \]
where $\varphi : \Delta \to X$ is a continuous partial map and $\{T_x, S_x\}_{x \in \Delta} \subseteq \mathcal{B}(F)$. 

Proof. By Proposition 1.3, it suffices to show that $\alpha$ satisfies condition (8). To this end, let $a \in C(X)$. We need to show that there is $a' \in C(X)$ such that
\[
\alpha(a \otimes 1)(x) = a'(x)\alpha_x(1), \quad x \in X.
\]
Since we assume $\{\alpha_x\}_{x \in \Delta}$ are inner, there are operators $\{T_x, S_x\}_{x \in \Delta} \subseteq \mathcal{B}(F)$ such that $\alpha_x(a) = T_xaS_x$ for $a \in \mathcal{B}(F)$ and $x \in \Delta$. In particular, we have $\alpha_x(S_xaT_x) = \alpha_x(1)a\alpha_x(1)$. Thus putting $F_x := \alpha_x(1)F$ for each $x \in \Delta$ we may identify $\mathcal{B}(F_x)$ with $\alpha_x(1)\mathcal{B}(F)\alpha_x(1)$. Then we get $\alpha(1 \otimes \mathcal{B}(F))(x) = \mathcal{B}(F_x) \subseteq \mathcal{B}(F)$. This implies that $\alpha(a \otimes 1)(x)$ lies in the center $Z(\mathcal{B}(F_x)) = \{\lambda \cdot \alpha_x(1) : \lambda \in \mathbb{C}\}$ of the algebra $\mathcal{B}(F_x)$. Indeed, for all $b \in \mathcal{B}(F)$ we have
\[
\alpha(a \otimes 1)(x)\alpha(1 \otimes b)(x) = \alpha(a \otimes b)(x) = \alpha(1 \otimes b)(x)\alpha(a \otimes 1)(x).
\]
Hence for each $x \in \Delta$ there is a number $a'(x) \in \mathbb{C}$ such that
\[
\alpha(a \otimes 1)(x) = a'(x)\alpha_x(1).
\]
The function $a' : \Delta \rightarrow \mathbb{C}$ obtained in this way is continuous this is forced by the continuity of the maps $\Delta \ni x \mapsto \alpha(a \otimes 1)(x) = a'(x)\alpha_x(1)$ and $\Delta \ni x \mapsto \alpha_x(1))$. Putting $a' \equiv 0$ outside the clopen set $\Delta$, we get the desired function $a' \in C(X)$. \hfill \Box

Corollary 1.12. A mapping $\alpha : C(X, M_n(\mathbb{C})) \rightarrow C(X, M_n(\mathbb{C}))$ is an endomorphism if and only if it is of the form
\[
\alpha(a)(x) = \begin{cases} T_xa(\varphi(x))T_x^{-1}, & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases} \quad a \in C(X, M_n(\mathbb{C})),
\]
where $\varphi : \Delta \rightarrow X$ is a continuous partial mapping of $X$ and $\{T_x\}_{x \in \Delta} \subseteq M_n(\mathbb{C})$ is a family of invertible matrices.

Proof. Combine Proposition 1.11 and Example 1.9. \hfill \Box

Let $\{T_x, S_x\}_{x \in \Delta} \subseteq \mathcal{B}(F)$ in Proposition 1.11 and consider $\mathcal{B}(F)$ with strong operator topology. There is no a priori given universal way of choosing the operators $\{T_x, S_x\}_{x \in \Delta}$, and the mapping $X \ni x \mapsto T_x \in \mathcal{B}(F)$ may be discontinuous or even unmeasurable:

Example 1.13. Let $X = [0, 1]$ and $\alpha_x = id$, for $x \in X$. Suppose $V$ is a Vitali set (an unmeasurable subset of $[0, 1]$). Put $T_x := 1$, $S_x := 1$ for $x \in V$, and $T_x := -1, S_x := -1$ for $x \notin V$. Then $T : X \rightarrow \mathcal{B}(\mathbb{C})$ is an unmeasurable field of isometries generating continuous field of automorphisms $\alpha_x = T_x(\cdot)S_x$.

In the isometric case, the choice of $\{T_x, S_x\}$ is unique up to constants in $T$ and we may choose $x \mapsto T_x$ to be continuous locally. Thus, the obstacles to continuity of $x \mapsto T_x$ may be identified by means of cohomological data:

Lemma 1.14. Suppose that $\{\alpha_x\}_{x \in \Delta}$ is a continuous field of inner isometric endomorphisms of $\mathcal{B}(F)$. Let $\{T_x, S_x\}_{x \in \Delta}$ be such that $\alpha_x(a) = T_xaS_x$, for $a \in \mathcal{B}(F)$, and $T_x$ is an isometry, for $x \in \Delta$.

(1) The operators $\{T_x, S_x\}_{x \in \Delta}$ are determined up to constants in $T$. Namely, if $\{T'_x, S'_x\}_{x \in \Delta} \subseteq \mathcal{B}(F)$ where $T'_x$ are isometries, then $\alpha_x(a) = T'_xaS'_x$ for all $a \in \mathcal{B}(F)$ if and only if there is $\lambda \in T$ such that $T_x = \lambda T'_x$ and $S_x = \lambda S'_x$.
(2) For any $x_0 \in \Delta$ there is an open neighbourhood $U \subseteq \Delta$ of $x_0$ and numbers $\{\lambda_x\}_{x \in U} \subseteq \mathbb{T}$ such that the map $U \ni x \mapsto \lambda_x T_x h \in F$ is continuous for every $h \in F$.

Proof. (1). By Proposition 1.7(1) we have $S_x T_x = S_x^* T_x^* = 1$. Thus for every $a \in \mathcal{B}(F)$ we get
\[ S_x T_x^* a = S_x^* T_x a S_x^* T_x^* = S_x^* \alpha(a) T_x^* = S_x^* T_x a S_x T_x^* = a S_x T_x^* . \]
Hence $S_x T_x^*$ belongs to the center of $\mathcal{B}(F)$ and therefore is a multiple of the identity operator. That is $S_x T_x^* = \lambda 1$ for $\lambda \in \mathbb{C}$. This implies that $T_x^* = \lambda T_x$ (because both $T_x$, $T_x^*$ are isometries onto $\alpha_x(1) F$, and $S_x : \alpha_x(1) F \to F$ is an inverse to $T_x$). Since $\|T_x\| = \|T_x^*\| = 1$ we get $\lambda \in \mathbb{T}$.

(2). Take $x_0 \in X$ and $h_0 \in F$. Without loss of generality we may assume $\|h_0\| = 1$. Let $f \in F^*$, be such that $\|f\| = f(h_0) = 1$. Then $\Theta_{h_0} f$ is a norm one projection onto the space spanned by $h_0$. Since the map $\Delta \ni x \mapsto \alpha_x(\Theta_{h_0} f) \in \mathcal{B}(F)$ is continuous the set
\[ U := \{ x \in \Delta : \alpha_x(\Theta_{h_0} f) h_0 \neq 0 \} = \{ x \in \Delta : f(S_x h_0) \neq 0 \} \]
is open. For every $x \in U$ we put
\[ h_x := \frac{\alpha_x(\Theta_{h_0} f) h_0}{\|\alpha_x(\Theta_{h_0} f) h_0\|} = \frac{f(S_x h_0)}{|f(S_x h_0)|} T_x h_0 \]
so that we have $\|h_x\| = 1$. Note that $U \ni x \mapsto h_x \in F$ is continuous. Also the map $\Delta \times \mathcal{B}(F) \ni (x, a) \mapsto \alpha_x(a) \in \mathcal{B}(F)$ is continuous, because $\|\alpha_x\| = 1$ for $x \in \Delta$. Hence for every $h \in F$ the map
\[ U \ni x \mapsto \tilde{T}_x h := \alpha_x(\Theta_{h_0} f) h_x = \frac{\alpha_x(\Theta_{h_0} f) h_0}{\|\alpha_x(\Theta_{h_0} f) h_0\|} = \frac{f(S_x h_0)}{|f(S_x h_0)|} T_x h \]
is continuous. Thus putting $\lambda_x := \frac{f(S_x h_0)}{|f(S_x h_0)|}$ we get the assertion. \hfill \Box

Remark 1.15. Let $\{\alpha_x\}_{x \in X}$ be a continuous field of $\ast$-automorphisms of the algebra compact operators $\mathcal{K}(H)$ on an infinite dimensional (separable) Hilbert space $H$. Then a similar fact to Lemma 1.14 holds, cf. [RW98, Proposition 1.6]. That is, $\{\alpha_x\}_{x \in X}$ is implemented by a field of unitary operators $\{U_x\}_{x \in X}$, which locally can be chosen to be continuous. Therefore, $\{\alpha_x\}_{x \in X}$ extends uniquely to a continuous field of $\ast$-automorphisms $\mathcal{B}(H)$, and we may identify such fields with $C(X)$-linear automorphisms of $A := C(X, \mathcal{B}(H))$. The latter form a group that we denote by $\text{Aut}_{C(X)}(A)$. Let $\text{Inn}(A)$ be the group of inner automorphisms of $A$. Clearly, $\text{Inn}(A)$ is a subgroup of $\text{Aut}_{C(X)}(A)$, and $\alpha \in \text{Aut}_{C(X)}(A)$ can be written in the form $\alpha(a)(x) = U_x a U_x^*$, $a \in A$, for a continuous map $X \ni x \mapsto U_x \in \mathcal{B}(H)$ if and only if $\alpha \in \text{Inn}(A)$. By [RW98, Theorem 5.42] we have
\[ \text{Aut}_{C(X)}(A)/\text{Inn}(A) \cong H^2(X, \mathbb{Z}), \]
where $H^2(X, \mathbb{Z})$ is the second Čech cohomology group of $X$ with integer coefficients. Thus whenever $H^2(X, \mathbb{Z})$ is non-trivial, so for instance when $X$ is a two-dimensional sphere or a torus, there is always a continuous field of (inner) automorphisms $\{\alpha_x\}_{x \in X}$ of $\mathcal{B}(H)$ such that every field of operators $\{T_x\}_{x \in X} \in \mathcal{B}(H)$ that implements $\{\alpha_x\}_{x \in X}$ is discontinuous (then $T_x$ is necessarily a unitary and $S_x = T_x^*$).
2. Abstract weighted shift operators and weighted endomorphisms

In this section we introduce abstract weighted shift operators associated with endomorphisms. To this end, we use the notion of partial isometry acting on Banach spaces in the sense of Mbekhta. We show that spectral radii of the corresponding weighted partial isometries and weighted endomorphisms are equal. Thus the results of the present paper can be readily applied to a vast class of operators acting on Banach spaces.

2.1. Partial isometries on Banach spaces and endomorphisms. Recall that an operator $T \in B(H)$ acting on a Hilbert space $H$ is a partial isometry if it is an isometry on the orthogonal complement of its kernel. Then $(\text{Ker} T)^\perp$ is called the initial subspace and $TH$ the final subspace of $T$. Partial isometries on Hilbert spaces have a number of various well known characterisations. For instance, $T \in B(H)$ is a partial isometry iff one of the following equivalent conditions hold:

i) $T^*T$ is an orthogonal projection (onto initial subspace),
ii) $TT^*$ is an orthogonal projection (onto the final subspace),
iii) $TT^*T = T$,
iv) $T^*TT^* = T^*$.

We recall one more characterisation of partial isometries which leads to a generalization of this notion to the realm of Banach spaces.

**Proposition 2.1** ([Mbe04] 3.1, 3.3). Let $H$ be a Hilbert space. An operator $T \in B(H)$ is a partial isometry if and only if $T$ is a contraction and there exists a contraction $S \in B(H)$ which is a generalized inverse to $T$, that is $TST = T$ and $STS = S$ (then we necessarily have $S = T^*$).

**Definition 2.2** ([Mbe04]). Let $T$ be an operator on a Banach space $E$. We say that $T$ is a partial isometry if it is a contraction and there is a contraction $S \in B(E)$ such that $TST = T$, $STS = S$.

Contractions $T$ and $S$ satisfying the above relations will be called mutually adjoint partial isometries.

**Remark 2.3.**

i) A partial isometry on a Banach space can have more than one partial isometry as adjoint, see Example 2.5 below.

ii) Not every isometry on a Banach space is a partial isometry. On the other hand, there are spaces (that are not Hilbert spaces) where all isometries are partial isometries. For example, $L^p$, $1 \leq p < \infty$ are such Banach spaces, cf. [Mbe04].

The following proposition is a slightly extended version of [Mbe04] Proposition 4.2]. It gives a useful description of adjoints to a partial isometry on Banach space.

**Proposition 2.4.** Let $T \in B(E)$. The following conditions are equivalent:

i) $T$ is a partial isometry,

ii) a) the kernel $\text{Ker} T$ of operator $T$ possesses a complement $M$ such that restriction of $T$ on $M$ is an isometry,

b) there exists a contractive projection $P \in B(E)$ onto the range of operator $T$.

If conditions i), ii) are satisfied then relations

$$STE = M, \quad TS = P$$
establish a bijective correspondence between partial isometries $S$ adjoint to $T$ and pairs $(M, P)$, where $M$ is a complement to $\text{Ker} T$ and $P$ is a contractive projection onto $TE$.

**Example 2.5.** Let us consider the classical unilateral left shift operator $T_N$ acting on the space $E = \ell^p(\mathbb{N})$, $p \in [1, \infty]$ or $E = c_0(\mathbb{N})$:

$$T_N(x(1), x(2), x(3), \ldots) = (x(2), x(3), \ldots).$$

Clearly, $T_N$ is a partial isometry in the sense of Definition 2.2. The only projection onto $T_N E = E$ is the identity operator. If $E = \ell^p(\mathbb{N})$ with $p < \infty$, then the only complement to the subspace $\text{Ker} T_N$ on which the operator $T_N$ is an isometry is the subspace $M = \{x \in E : x(1) = 0\}$. Therefore in this case the only partial isometry adjoint to $T_N$ is the classical right shift

$$S_N(x(1), x(2), \ldots) = (0, x(1), x(2), \ldots).$$

In the case when $E = \ell^\infty(\mathbb{N})$ or $E = c_0(\mathbb{N})$, the situation changes. Indeed, then complements to the kernel $T_N$ on which the operator $T_N$ is an isometry can be indexed by elements of the unit ball of the dual space $E^*$:

$$M_f = \{x \in \ell^\infty(\mathbb{N}) : x(1) = f(x(2), x(3), \ldots)\}, \quad f \in E^*, \quad \|f\| \leq 1.$$ 

Hence all the partial isometries adjoint to $T_N : E \to E$ are of the form

$$S_f x = (f(x), x(1), x(2), \ldots), \quad f \in E^*, \quad \|f\| \leq 1.$$ 

Thus, if $E = c_0(\mathbb{N})$, then partial isometries adjoint to $T_N$ are indexed by probability measures on $\mathbb{N}$, while if $E = \ell^\infty(\mathbb{N})$, then they are indexed by all normalized finitely additive measures on $\mathbb{N}$.

Let us fix a pair of mutually adjoint partial isometries $T, S \in \mathcal{B}(E)$ acting on a Banach space $E$. This pair naturally defines the following two mappings on $\mathcal{B}(E)$:

$$\alpha(a) := TaS, \quad \alpha_*(a) := SaT.$$ 

It is straightforward to see that the mappings $\alpha, \alpha_* : \mathcal{B}(E) \to \mathcal{B}(E)$ are mutually adjoint partial isometries on the Banach space $\mathcal{B}(E)$. We will discuss now some natural criteria for multiplicativity of the partial isometry $\alpha$ on subsets of $\mathcal{B}(E)$. For a subset $M \subseteq \mathcal{B}(E)$ we denote by $M'$ its commutant, that is $M' := \{a \in \mathcal{B}(E) : ba = ab \text{ for each } b \in M\}$. In the Hilbert space case there is a number of conditions equivalent to multiplicativity of $\alpha$ on $M$:

**Proposition 2.6.** Let $T \in \mathcal{B}(H)$ be a partial isometry on a Hilbert space $H$ and put $\alpha(a) := TaT^*$ for $a \in \mathcal{B}(H)$. Let $M \subseteq \mathcal{B}(H)$ be a self-adjoint set, that is $M^* = M$. The following conditions are equivalent:

1. $T^* T \in M'$
2. $Ta = \alpha(a)T$ for every $a \in M$
3. $aT^* = T^* \alpha(a)$ for every $a \in M$.
4. $\alpha(ab) = \alpha(a) \alpha(b)$ for every $a, b \in M$

**Proof.** ii) and iii) are equivalent, as one is the adjoint of the other. That i) is equivalent to ii) and iii), and that they imply iv) is easy and follows from Proposition 2.9 below,
see also [LO04, Proposition 2.2]. To see that iv) implies ii) let \( a \in M \). Then
\[
\|Ta - \alpha(a)T\|^2 = \|Taa^*T - \alpha(a)Ta^*T - TaT^*\alpha(a) + \alpha(a)TT^*\alpha(a)^*\| = 0,
\]
because \( a^* \in M \) and therefore \( \alpha(aa^*) = \alpha(a)\alpha(a^*) \).

**Remark 2.7.** If \( M \) contains the identity operator \( 1 \in \mathcal{B}(H) \), the condition iv) in Proposition 2.6 implies that \( T \) is necessarily a partial isometry. Indeed, if \( T \in \mathcal{B}(H) \) is any operator such that \( \alpha(\cdot) := T(\cdot)T^* \) satisfies this condition with \( a = b = 1 \), then \( TT^* = \alpha(1) = \alpha(1 \cdot 1) = \alpha(1)\alpha(1) = (TT^*)^2 \) is an orthogonal projection, and hence \( T \) a partial isometry.

**Corollary 2.8.** Let \( A \subseteq \mathcal{B}(H) \) be a \(*\)-subalgebra and \( T \in \mathcal{B}(H) \) a partial isometry. The map \( \alpha(a) := TaT^* \), \( a \in \mathcal{B}(H) \), restricts to an endomorphism of \( A \) if and only if
\[
TAT^* \subseteq A, \quad T^*T \in A'.
\]

**Proof.** The map \( \alpha \) preserves \( A \) if and only if \( TAT^* \subseteq A \). It is multiplicative on \( A \) if and only if \( T^*T \in A' \) by Proposition 2.6.

In the general Banach space case only some implications in the above equivalences remain valid:

**Proposition 2.9.** Let \( T \in \mathcal{B}(E) \) be a partial isometry with an adjoint \( S \in \mathcal{B}(E) \), and let \( M \subseteq \mathcal{B}(H) \). Put \( \alpha(a) := TaS \) for \( a \in \mathcal{B}(E) \). The following conditions are equivalent:
\[
\begin{align*}
\text{i)} \quad & ST \in M', \\
\text{ii)} \quad & Ta = \alpha(a)T \text{ and } aS = S\alpha(a) \text{ for every } a \in M.
\end{align*}
\]
Each of these equivalent conditions imply that \( \alpha(ab) = \alpha(a)\alpha(b) \), \( a, b \in M \).

**Proof.** Assuming i), for any \( a \in M \), we have \( Ta = TSTa = TaST = \alpha(a)T \) and \( aS = aSTS = STaS = S\alpha(a) \). Hence i) \( \Rightarrow \) ii). Conversely, using ii), for any \( a \in M \), we get \( STa = S\alpha(a)T = aST \). Thus ii) \( \Rightarrow \) i). Moreover, the definition of a partial isometry along with i) imply that for \( a, b \in M \) we have \( \alpha(ab) = TabS = TSTabS = TaSTbT = \alpha(a)\alpha(b) \).

**Corollary 2.10.** Let \( A \subseteq \mathcal{B}(E) \) be an algebra, and \( T \) and \( S \) be mutually adjoint partial isometries such that
\[
(10) \quad TAS \subseteq A, \quad ST \in A'.
\]
Then the mapping \( \alpha : A \to A \) is an endomorphism of \( A \).

For a given algebra \( A \) and a partial isometry \( T \) relations (10) may be satisfied by different partial isometries \( S \) that are adjoint to \( T \), and the corresponding endomorphisms of \( A \) may be different, cf. Example 2.12 below. However, as the next statement shows, for commutative algebras and \(*\)-endomorphisms of \(*\)-algebras, \( \alpha : A \to A \) does not depend on the choice of operator \( S \) in (10).

**Proposition 2.11.** Let \( A \subseteq \mathcal{B}(E) \) be an algebra containing \( 1 \in \mathcal{B}(E) \). Let \( T \) be a partial isometry and \( S_1, S_2 \in \mathcal{B}(E) \) be partial isometries adjoint to \( T \) such that
\[
TAS_i \subseteq A, \quad S_iT \in A', \quad \text{for } i = 1, 2.
\]
Restrictions of maps $\alpha_i(\cdot) = T(\cdot)S_i$, $i = 1, 2$, to $A$ coincide, that is they generate the same endomorphism $\alpha : A \to A$ if and only if $\alpha_1(1) = \alpha_2(1)$. And $\alpha_1(1) = \alpha_2(1)$ if and only if $\alpha_1(1)\alpha_2(1) = \alpha_2(1)\alpha_1(1)$. In particular, $\alpha_1 = \alpha_2$ on $A$, whenever $A$ is commutative or when $A$ is a *-algebra and both $\alpha_i$, $i = 1, 2$, are *-preserving.

Proof. Note that for $a \in A$ we have

\[
\alpha_1(a) = \alpha_1(a \cdot 1) = T(a \cdot 1)S_1 = TS_2T(a \cdot 1)S_1 = (TaS_2)(T1S_1) = \alpha_2(a)\alpha_1(1).
\]

By symmetry we also get $\alpha_2(a) = \alpha_2(a)\alpha_2(1)$. Thus, if $\alpha_1(1) = \alpha_2(1)$ then endomorphisms $\alpha_i : A \to A$ do coincide. The foregoing relations also show that

\[
(11) \quad \alpha_1(1) = \alpha_2(1)\alpha_1(1) \quad \text{and} \quad \alpha_2(1) = \alpha_1(1)\alpha_2(1).
\]

Therefore $\alpha_1(1) = \alpha_2(1)$ if and only $\alpha_1(1)\alpha_2(1) = \alpha_2(1)\alpha_1(1)$.

Finally, assume that $A$ is a *-algebra and $\alpha_i$, $i = 1, 2$, are *-preserving. Then $\alpha_i(1) = \alpha_i(1)^*$, $i = 1, 2$. These equalities along with (11) give $\alpha_1(1) = \alpha_1(1)^* = \alpha_1(1)^*\alpha_2(1)^* = \alpha_1(1)\alpha_2(1) = \alpha_2(1)$.

\[\square\]

Example 2.12. Let $E = c(\mathbb{N})$ be the space of converging sequences with sup-norm. Then the operator

\[
T(x_1, x_2, \ldots) := (x_1, x_1, x_2, x_3, \ldots)
\]

is an isometry on $E$. Let us consider the following two contractions $S_1$ and $S_2$ on $E$:

\[
S_1(x_1, x_2, \ldots) := ((x_1 + x_2)/2, x_3, \ldots),
\]

\[
S_2(x_1, x_2, \ldots) := (x_2, x_3, \ldots).
\]

These are partial isometries adjoint to $T$, and $S_1T = S_2T = 1$. Thus by Corollary 2.10, $\alpha_1(a) := TaS_1$ and $\alpha_2(a) := TaS_2$ are endomorphisms of $A := B(E)$. These endomorphisms are different, since $\alpha_1(1) = TS_1 \neq TS_2 = \alpha_2(1)$.

Example 2.13. Let $E = \ell^1(\mathbb{N})$ and $T$ be the operator given by

\[
Tx = (x_2 - x_1, x_3, x_4, \ldots).
\]

In this situation we have the following complements of the kernel of $T$ on which $T$ is an isometry:

\[
M_\lambda = \{ x \in E : x(1) = -\lambda x(2), \quad \lambda \in [0, \infty), \quad M_\infty = \{ x \in E : x(1) = 0 \}.
\]

In addition $T$ is a surjection. Therefore $T$ is a partial isometry, and every partial isometry $S$ adjoint to $T$ is an isometry (see Proposition 2.4). These operators are of the following form

\[
S_\lambda x = \left( \frac{-1}{1 + \lambda} x(1), \frac{\lambda}{1 + \lambda} x(1), x(2), x(3), \ldots \right), \quad S_\infty x = (0, x(1), x(2), x(3), \ldots),
\]

where $\lambda \in [0, \infty)$. Clearly $TS_\lambda = 1$, $\lambda \in [0, \infty]$, and thus the mappings

\[
\alpha_\lambda(a) = TaS_\lambda, \quad a \in B(E), \quad \lambda \in [0, \infty],
\]

preserve the identity $1 \in B(E)$. Thus in view of Proposition 2.11, whenever we have a unital subalgebra $A \subseteq B(E)$ such that a pair $(T, S_\lambda)$ satisfy relations (10), the restriction of $\alpha_\lambda$ to $A$ does not depend on $\lambda$. Let us consider two situations:

i) If $A = \{1z : z \in \mathbb{C} \}$, then all the pairs $(T, S_\lambda)$, $\lambda \in [0, \infty]$ satisfy relations (10) and all the mappings $\alpha_\lambda$ define the identity endomorphism on $A$. 
ii) If $A$ is the algebra of operators of multiplication by sequences that are constant beginning from the second coordinate (that is sequences of the form $(a, b, b, b, \ldots)$, $a, b \in \mathbb{C}$) then all the mappings $\alpha_\lambda$, $\lambda \in [0, \infty]$ preserve the algebra $A$ while only the operators $\alpha_0$ and $\alpha_\infty$ are multiplicative on $A$. Moreover $\alpha_0$ and $\alpha_\infty$ define different endomorphisms on $A$:

$$\alpha_0(a, b, b, b, \ldots) = (a, b, b, b, \ldots), \quad \alpha_\infty(a, b, b, b, \ldots) = (b, b, b, b, \ldots).$$

This does not contradict Proposition 2.11 since among the pairs $(T, S_\lambda)$, $\lambda \in [0, \infty]$ only $T$ and $S_\infty$ satisfy relations (10).

2.2. Weighted shift operators on Banach spaces. Now we are in a position to formulate the definition of abstract weighted shift operators associated with endomorphisms, that generalizes [AL94, 3.1] and appears in [Kwa09]:

**Definition 2.14.** Let $E$ be a Banach space. Suppose that $A \subseteq B(E)$ is an algebra containing $1 \in B(E)$ and let $T \in B(E)$ be a partial isometry which admits an adjoint partial isometry $S$ satisfying

$$TAS \subseteq A, \quad ST \in A'.$$

So that $\alpha : A \to A$ given by $\alpha(a) := T(a)S$, $a \in A$, is an endomorphism of $A$, by Corollary 2.10. We call operators of the form

$$aT, \quad a \in A,$$

(abstract) weighted shift operators associated with the endomorphism $\alpha$. We refer to $A$ as to the algebra of weights. The role of shift is played by $\alpha$.

**Remark 2.15.**

i) Recall that if $A$ is commutative then the endomorphism $\alpha$ in this definition does not depend on the choice of $S$, and the same is true when $A$ is a $\ast$-algebra and $\alpha$ is a $\ast$-endomorphism (see Proposition 2.11). In these cases we will say that $T$ generates the endomorphism $\alpha$.

ii) If $A$ is a commutative Banach algebra then $\alpha : A \to A$ determines a partial map $\varphi$ on the spectrum $X$ of $A$ (Proposition 1.1). In this case we also say that $T$ generates the partial map $\varphi$.

**Example 2.16 (Classical weighted shift operators).** The classical bilateral weighted shifts are abstract weighted shifts in the sense of [AL94, 3.1] and therefore all the more in the sense of Definition 2.14. Let $T_N$, $S_N$ be the classical unilateral shift operators on the space $E$ of Example 2.5 and let $A \subseteq B(E)$ be the algebra of operators of multiplication by bounded sequences: $A \cong \ell^\infty(\mathbb{N})$. Then $T_N$ and $S_N$ are mutually adjoint partial isometries and

$$T_NAS_N \subseteq A, \quad S_NAT_N \subseteq A$$

and, in particular, $T_NS_N, S_NT_N \in A \subseteq A'$. Therefore the classical unilateral weighted shift operators $aT_N, aS_N, a \in A$, are abstract weighted shift operators with weights in $A \cong \ell^\infty(\mathbb{N})$ in our sense, though they are not abstract weighted shifts in the sense of [AL94, 3.1], as none of $T_N$ and $S_N$ is invertible. Note that being an abstract weighted shift depends on the algebra $A$ we consider.

A $\ast$-endomorphism $\alpha : B(H) \to B(H)$, where $H$ is a Hilbert space, is generated by a single (partial) isometry $T \in B(H)$ if and only if Power’s index of $\alpha$ is 1, see subsection 1.3. On the other hand, it is well known, see [KL13, Theorem 1.11] or
Proposition 2.17. Let $A$ be a unital Banach algebra and let $\alpha : A \rightarrow A$ be a contractive endomorphism. Then there is a Banach space $E$, a unital isometric homomorphism $\pi : A \rightarrow \mathcal{B}(E)$ and mutually adjoint partial isometries $S, T \in \mathcal{B}(E)$ such that

$\pi(\alpha(a)) = T\pi(a)S, \text{ for } a \in A \text{ and } ST \in \pi(A)'$.

Thus for each $a \in A$ the operator $\pi(a)T$ is an abstract weighted shift associated with the endomorphism (isometrically conjugated with) $\alpha$.

Proof. We consider the Banach space

$E := \{x = (x(0), x(1), \ldots) : x(n) \in \alpha^n(1)A, \text{ for } n \geq 0, \text{ and } \|x\| := \sup_{n \in \mathbb{N}} \|x(n)\| < \infty\}$.

Then the formula $(\pi(a)x) := \alpha^n(a)x(n)$ defines a unital homomorphism $\pi : A \rightarrow \mathcal{B}(E)$. Using that $\alpha$ is contractive and that $\pi(a)(1, \alpha(1), \alpha^2(1), \ldots) = (a, \alpha(a), \alpha^2(a), \ldots)$ one gets that $\|\pi(a)\| = \|a\|$ for every $a \in A$. Hence $\pi$ is isometric. It is readily check that putting

$T(x(0), x(1), \ldots) := (x(1), x(2), \ldots), \quad S(x(0), x(1), \ldots) := (0, \alpha(1)x(0), \alpha^2(1)x(1), \ldots)$,

we get the desired mutually adjoint partial isometries $S, T \in \mathcal{B}(E)$.

The main idea behind introducing abstract weighted shifts is that some of their spectral properties can be investigated in terms of the associated non-commutative dynamical system $(A, \alpha)$. In this paper we focus on spectral radii. As the next proposition shows for these spectral characteristics the relationship between $aT : E \rightarrow E$ and weighted endomorphism $\alpha a : A \rightarrow A$ is as literal as one may think. Moreover, it also tells us that the spectral radius depends only on the values of the ‘cocycle’ generated by $a$ and $\alpha$, i.e. the sequence of elements $\alpha a(\alpha), \ldots \alpha^n(\alpha), n = 1, 2, \ldots$ (cf. Subsection 1.1).

Proposition 2.18. Suppose that $aT, a \in A$, is an abstract weighted shift operator and $\alpha : A \rightarrow A$ is an associated endomorphism. Then for the spectral radius $r(aT)$ of the operator $aT$ we have

$r(aT) = r(a\alpha) = \lim_{n \rightarrow \infty} \|a\alpha(\alpha)\ldots \alpha(n)(\alpha)\|^{\frac{1}{n}}$

where $r(a\alpha)$ is the spectral radius of the weighted endomorphism $\alpha a : A \rightarrow A$ treated as an element of $\mathcal{B}(A)$. In particular, $r(aT) = r(\alpha^k(a)T) = r(\alpha^k(a)a)$ for every $k \in \mathbb{N}$.

Proof. The formula $r(a\alpha) = \lim_{n \rightarrow \infty} \|a\alpha(\alpha)\ldots \alpha(n)(\alpha)\|^{\frac{1}{n}}$ is a consequence of Gelfand’s formula $r(a\alpha) = \lim_{n \rightarrow \infty} \|(a\alpha)^n\|_\mathcal{B}(A)^{\frac{1}{n}}$, and the following two inequalities:

$\|(a\alpha)^{n+1}\|_\mathcal{B}(A) = \|a\alpha(\alpha)\ldots \alpha(n)(\alpha)\alpha^{n+1}\|_\mathcal{B}(A) \leq \|a\alpha(\alpha)\ldots \alpha(n)(\alpha)\|$
and
\[ \|a^{\alpha}(a)\| \leq \|a^{\alpha}(a)\|_B \|a\| \]

By Proposition 2.9, we have \( Ta = \alpha(a)T \), and hence by induction we have \( T^k a = \alpha^k(a)T^k \). Using this and that \( T \) is a contraction, we get
\[ \|(aT)^{n+1}\| = \|\alpha(a)\ldots\alpha^n(a)T^{n+1}\| \leq \|\alpha(a)\ldots\alpha^n(a)\|. \]

On the other hand, using that \( T \) and \( S \) are contractions we have
\[ \|a^{\alpha}(a)\| \leq \|a^{\alpha}(a)\|_B \|a\| = \sup_{\|b\| = 1} \|a^{\alpha}(a)\| T^n B^n \|a\| \]
\[ \leq \|a^{\alpha}(a)\| T^n \|a\| = \|(aT)^n\| \|a\|. \]

Thus \( r(aT) = \lim_{n \to \infty} \|(aT)^n\|^{\frac{1}{n}} = \lim_{n \to \infty} \|a^{\alpha}(a)\ldots\alpha^n(a)\|^{\frac{1}{n}} \).

Proceeding in a similar way one can show that
\[ \lim_{n \to \infty} \|a^{\alpha}(a)\ldots\alpha^{n-1}(a)\|^{\frac{1}{n}} = \lim_{n \to \infty} \|a^{\alpha}(a)\ldots\alpha^n(a)\| \),

and therefore \( r(aT) = r(\alpha(a)T) \). Thus by induction, \( r(aT) = r(\alpha^k(a)T) \) for every \( k \in \mathbb{N} \).

**Corollary 2.19.** Let \( \alpha : A \to A \) be a contractive endomorphism of a unital Banach algebra \( A \). For any \( a \in A \) and \( k \in \mathbb{N} \) we have
\[ r(\alpha(a)) = r(\alpha^k(a)\alpha) = \lim_{n \to \infty} \|a^{\alpha}(a)\ldots\alpha^n(a)\|^{\frac{1}{n}}. \]

**Proof.** By Proposition 2.17 we may view \( \alpha \) as being associated with an abstract weighted shift, and then the assertion follows from Proposition 2.18. \( \Box \)

### 3. Ergodic Measures for Partial Maps and Limits of Empirical Averages

We start by introducing some notation for the partial dynamical system \((X, \varphi)\) (cf. Definition 1.2). For \( n \in \mathbb{N} \) we denote by \( \Delta_n \) and \( \Delta_{-n} \) respectively the domain and the range of the partial map \( \varphi^n \). Namely, the sets \( \Delta_n \) are given by inductive formulae \( \Delta_0 = X, \Delta_n := \varphi^{-1}(\Delta_{n-1}) \), \( n > 0 \), and then \( \Delta_{-n} := \varphi^n(\Delta_n) \). Note that for \( n > 0 \), the sets \( \Delta_n \) are clopen while \( \Delta_{-n} \), in general, are only closed. For all \( n, m \in \mathbb{N} \) one has
\[ \varphi^n : \Delta_n \to \Delta_{-n}, \]
\[ \varphi^n(\varphi^m(x)) = \varphi^{n+m}(x), \quad x \in \Delta_{n+m}. \]

Note that if \( \varphi \) is a partial map dual to an endomorphism \( \alpha \), cf. Definition 1.2 then \( \varphi^n \) is nothing but the map dual to the endomorphism \( \alpha^n \).

**Definition 3.1.** We define the *essential domain* of the partial map \( \varphi \) as the set
\[ \Delta_{\infty} := \bigcap_{n \in \mathbb{Z}} \Delta_n. \]

Then the map \( \varphi : \Delta_{\infty} \to \Delta_{\infty} \) is everywhere defined and surjective.

\(^2\)Alternatively, one may readily adapt the proof of Proposition 2.18.
Standard definitions of invariant and ergodic measures for full maps make sense also for partial maps. Thus we define them this way. However, we could equivalently define them as the corresponding notions for the restricted full map \( \varphi : \Delta_\infty \to \Delta_\infty \).

**Definition 3.2.** Let \((X, \varphi)\) be a partial dynamical system. Let \(\mu\) be a normalized Radon measure on \(X\). We say that \(\mu\) on \(X\) is \(\varphi\)-invariant, if \(\mu(\varphi^{-1}(\omega)) = \mu(\omega)\), for every Borel \(\omega \subseteq X\). If in addition for every Borel \(\omega \subseteq X\) we have

\[
\varphi^{-1}(\omega) = \omega \implies \mu(\omega) = 0 \text{ or } \mu(\omega) = 1,
\]

we call \(\mu\) \(\varphi\)-ergodic. We denote by Inv\((X, \varphi)\) the set of all normalized \(\varphi\)-invariant Radon measures on \(X\), and by Erg\((X, \varphi)\) the measures in Inv\((X, \varphi)\) that are \(\varphi\)-ergodic.

**Lemma 3.3.** If \(\mu \in \text{Inv}(X, \varphi)\), then \(\text{supp} \mu \subseteq \Delta_\infty\). Thus one may consider \(\varphi\)-invariant (ergodic) measures for the partial map \(\varphi : \Delta \to X\) as \(\varphi\)-invariant (ergodic) measures for full map \(\varphi : \Delta_\infty \to \Delta_\infty\):

\[
\text{Inv}(X, \varphi) = \text{Inv}(\Delta_\infty, \varphi), \quad \text{Erg}(X, \varphi) = \text{Erg}(\Delta_\infty, \varphi).
\]

**Proof.** By continuity of measure it suffices to show that \(\mu(\Delta_n) = 1\) for every \(n \in \mathbb{Z}\). We do it inductively. The zero step is obvious because \(\mu(\Delta_0) = \mu(X) = 1\). However, if we have \(\mu(\Delta_{k-1}) = 1\) for some \(k > 0\), then using equality \(\Delta_k = \varphi^{-1}(\Delta_{k-1})\) and \(\varphi\)-invariance of \(\mu\) we get \(\mu(\Delta_k) = \mu(\Delta_{k-1}) = 1\). Hence \(\mu(\Delta_n) = 1\) for all \(n \geq 0\).

Now let us notice that for every \(k > 0\) we have \(\Delta_{-k} = \varphi(\Delta_{k+1} \cap \Delta)\) and therefore

\[
\mu(\Delta_{-k}) = \mu(\varphi^{-1}(\Delta_{-k})) = \mu(\varphi^{-1}(\varphi(\Delta_{-k+1} \cap \Delta_1))) \geq \mu(\Delta_{-k+1} \cap \Delta_1) = \mu(\Delta_{-k+1})
\]

where we used \(\varphi\)-invariance of \(\mu\), the inclusion \(\varphi^{-1}(\varphi(\Delta_{-k+1} \cap \Delta_1)) \supseteq \Delta_{-k+1} \cap \Delta_1\) and the above shown fact that \(\mu(\Delta_1) = 1\). Thus the assumption that \(\mu(\Delta_{-k+1}) = 1\) implies \(\mu(\Delta_{-k}) = 1\). Hence by induction we get \(\mu(\Delta_{-n}) = 1\) for all \(n \geq 0\).

The next variational principle (in the full map case) is implicit in a number of works, cf. [Leb79], [Kit79], [AL94], [Ant96]. It implies that when considering empirical averages, i.e. the sums of the form \((12)\) below, the operations \(\lim\) and \(\sup\) commute (see Corollary 3.5).

**Theorem 3.4** (Variational principle for \(\lim\sup\) of empirical averages). Let \((X, \varphi)\) be a partial dynamical system and let \(f : \Delta \to \mathbb{R}\) be a continuous and bounded from above function where \(\Delta\) is the domain of \(\varphi\) (an open subset of \(X\)). We define the corresponding empirical averages to be functions \(S_n(f) : \Delta_\infty \to \mathbb{R}\), \(n > 0\), given by

\[
(12) \quad S_n(f)(x) = \frac{1}{n}(f(x) + f(\varphi(x)) + \ldots + f(\varphi^{n-1}(x)), \quad x \in \Delta_n.
\]

Then

\[
\lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) = \max_{\mu \in \text{Inv}(\Delta_\infty, \varphi)} \int_{\Delta_\infty} f \, d\mu = \max_{\mu \in \text{Erg}(\Delta_\infty, \varphi)} \int_{\Delta_\infty} f \, d\mu,
\]

if \(\text{Erg}(\Delta_\infty, \varphi) \neq \emptyset\) and \(\lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) = -\infty\) otherwise.

**Proof.** Clearly, the sequence \(a_n := \sup_{x \in \Delta_n}(f(x) + f(\varphi(x)) + \ldots + f(\varphi^{n-1}(x))\) is sub-additive, and \(\sup_{x \in \Delta_n} S_n(f)(x) = \frac{a_n}{n}\). Hence the limit \(\lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x)\) exists.
and is equal to \( \inf_{n \in \mathbb{N}} \sup_{x \in \Delta_n} S_n(f)(x) \) (it may be \(-\infty\)). For any \( \mu \in \text{Inv}(X, \varphi) \) we have \( \int_{\Delta} f \, d\mu = \int_{\Delta} f \circ \varphi \, d\mu \) and therefore

\[
\int_{\Delta} f \, d\mu = \lim_{n \to \infty} \int_{\Delta} S_n(f) \, d\mu \leq \lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x).
\]

To construct a measure for which the converse inequality holds, we may assume that \( \lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) > -\infty \). Then there are points \( x_n \in \Delta_n \) such that \( S_n(f)(x_n) \geq \sup_{x \in \Delta_n} S_n(f)(x) - 1/n \), so that \( \lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) = \lim_{n \to \infty} S_n(f)(x_n) \). Put

\[
\nu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\varphi^i(x_n)}, \quad n \in \mathbb{N},
\]

where \( \delta_x \) is the unit measure accumulated in point \( x \in X \). By Banach–Alaoglu theorem there is a subsequence \( \nu_{n_k} \) convergent in the *-weak topology to a probability measure \( \nu \). In other words,

\[
\nu(h) = \lim_{k \to \infty} \nu_{n_k}(h), \quad h \in C(X).
\]

To prove that \( \nu \) is \( \varphi \)-invariant it suffices to show that

\[
\nu(h \circ \varphi) = \nu(h), \quad \text{for all } h \in C(X),
\]

where by \( (h \circ \varphi)(x) \) we mean \( h(\varphi(x)) \), when \( x \in \Delta \), and 0 otherwise. However, using the definition of \( \nu \) (and \( \nu_{n_k} \)) and boundedness of \( h \in C(X) \) we get

\[
\nu(h \circ \varphi) - \nu(h) = \lim_{k \to \infty} [\nu_{n_k}(h \circ \varphi) - \nu_{n_k}(h)] = \lim_{k \to \infty} \frac{1}{n_k} [h(\varphi^{n_k}(x_{n_k})) - h(x_{n_k})] = 0.
\]

Thus \( \nu \) is \( \varphi \)-invariant and in particular it supported on \( \Delta_\infty \), by Lemma 3.3.

To prove the inequality \( \lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) \leq \int_{\Delta} f \, d\mu \) note that in view of the choice of points \( x_n \) and definition of measures \( \nu_n \) we have

\[
\lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) = \lim_{k \to \infty} \nu_{n_k}(f)(x) = \lim_{k \to \infty} \nu_{n_k}(f).
\]

Even though \( f \) is continuous on \( \Delta \), we can not directly conclude that \( \lim_{k \to \infty} \nu_{n_k}(f) = \nu(f) \), as \( f \) may not be bounded from below. Nevertheless, putting for \( n \in \mathbb{N} \)

\[
f_n(x) := \begin{cases} f(x), & f(x) \geq -n \\ -n, & f(x) < -n \end{cases}
\]

we have \( f \leq f_n \) and \( f_n \in C(X) \). Hence for each \( n \in \mathbb{N} \)

\[
\lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) = \lim_{k \to \infty} \nu_{n_k}(f) \leq \lim_{k \to \infty} \nu_{n_k}(f_n) = \nu(f_n).
\]

Moreover, the functions \( f_n \) form a decreasing sequence that converges pointwise to \( f \) on \( \Delta \), which is \( \nu \)-almost everywhere. Thus \( \nu(f) = \lim_{n \to \infty} \nu(f_n) \geq \lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) \).

This concludes the proof of the equality

\[
\lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) = \max_{\mu \in \text{Inv}_{\Delta_\infty} (\varphi)} \int_{\Delta} f \, d\mu.
\]
To finish the proof we need to show that the maximum above is attained at an ergodic measure. To this end take \( \nu \in \text{Inv}(\Delta_\infty, \phi) \) such that \( \int_{\Delta_\infty} f \, d\nu = \max_{\mu \in \text{Inv}(\Delta_\infty, \phi)} \int_{\Delta_\infty} f \, d\mu \). Then for every \( \mu \in \text{Erg}(\Delta_\infty, \phi) \) we have

\[
\int_{\Delta_\infty} f \, d\nu \geq \int_{\Delta_\infty} f \, d\mu.
\]

By Choquet-Bishop-de Leeuw Theorem (see, for instance, [Phe01, page 22]), there exists a probability measure \( m \) on \( \text{Erg}(\Delta_\infty, \phi) \) such that

\[
\int_{\Delta_\infty} f \, d\nu = \int_{\text{Erg}(\Delta_\infty, \phi)} \left( \int_{\Delta_\infty} f \, d\mu \right) \, dm(\mu).
\]

The above equality and the earlier inequality imply existence of \( \mu \in \text{Erg}(\Delta_\infty, \phi) \) with \( \int_{\Delta_\infty} f \, d\nu = \int_{\Delta_\infty} f \, d\mu \). \qed

**Corollary 3.5.** Retain the notation and assumptions of Theorem 3.4. There is \( \mu \in \text{Erg}(\Delta_\infty, \phi) \) and a subset \( Y \subseteq \Delta_\infty \), \( \mu(Y) = 1 \), such that

\[
\lim_{n \to \infty} S_n(f)(y) = \lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) \quad \text{for every} \ y \in Y.
\]

In particular, \( \lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) = \max_{x \in \Delta_\infty} \lim_{n \to \infty} S_n(f)(x) \).

**Proof.** By Theorem 3.4 there is \( \mu \in \text{Erg}(\Delta_\infty, \phi) \) such that \( \lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) = \int_{\Delta_\infty} f \, d\mu \). By the Birkhoff-Khinchin ergodic theorem there exists a subset \( Y \subseteq \Delta_\infty \), \( \mu(Y) = 1 \) such that for every \( y \in Y \) the sum \( S_n(f)(y) \) converges to \( \int_{\Delta_\infty} f \, d\mu \). This gives the first part of assertion.

For the second part note that for every \( x \in \Delta_\infty \), the sequence \( a_n := (f(x) + f(\phi(x)) + \ldots + f(\phi^{n-1}(x)) \) is (sub-)additive, and therefore the limit of \( S_n(f)(x) = \frac{a_n}{n} \) exists. Moreover, we clearly have \( \sup_{x \in \Delta_\infty} \lim_{n \to \infty} S_n(f)(x) \leq \lim_{n \to \infty} \sup_{x \in \Delta_n} S_n(f)(x) \). This together with the first part of the assertion gives the desired equality. \qed

4. **Variational principles for cocycles and Lyapunov exponents**

Here we introduce the spectral exponent of an operator-valued function \( a : \Delta \to \mathcal{B}(F) \) and prove variational principles that express this exponent either in terms of a linear extension or in terms of measure Lyapunov exponents. These results will serve as fundamental instruments in the proofs of all the variational principles for spectral radius of weighted endomorphisms discussed further in Section 5.

4.1. **Cocycles, Lyapunov exponents and linear extensions.** Let us fix a partial dynamical system \((X, \phi)\) and a Banach space \( F \). Let us also fix an operator valued function \( \Delta \ni x \to a(x) \in \mathcal{B}(F) \) which is bounded in the sense that \( \sup_{x \in \Delta} ||a(x)|| < \infty \).

**Definition 4.1.** We associate to the triple \((X, \phi, a)\) two functions \( C^f_{a, \phi}, C^b_{a, \phi} : \{ (x, n) \in X \times \mathbb{N} : x \in \Delta_n \} \to \mathcal{B}(F) \) given by the formulae

\[
C^f_{a, \phi}(x, n) := a(x) \cdot a(\phi(x)) \cdot \ldots \cdot a(\phi^{n-1}(x)),
\]

\[
C^b_{a, \phi}(x, n) := a(\phi^{n-1}(x)) \cdot a(\phi(x)) \cdot a(x),
\]

where \( x \in \Delta_n, n \in \mathbb{N} \). We call \( C^f_{a, \phi} \) and \( C^b_{a, \phi} \) cocycles of \( a \) with respect to \( \phi \). We refer to \( C^f_{a, \phi} \) as a forward cocycle and to \( C^b_{a, \phi} \) as a backward cocycle.
We are interested in ergodic properties of these cocycles and the arising variational principles for them. If \( \varphi : X \to X \) is a homeomorphism we have
\[
C^f_{a,\varphi}(x, n) = C^b_{a,\varphi^{-1}}(\varphi^{n-1}(x), n), \quad C^b_{a,\varphi}(x, n) = C^f_{a,\varphi^{-1}}(\varphi^{n-1}(x), n),
\]
so we can study properties of the forward cocycle by looking at the backward cocycle, and vice versa. In general, in the irreversible case, we can relate the forward and backward cocycles by passing to adjoints. Namely, let \( F^* \) be the space dual to \( F \) and define the adjoints \( C^*_{a,\varphi} \), \( C^b_{a,\varphi} \) : \( \{(x, n) \in X \times \mathbb{N} : x \in \Delta_n\} \to \mathcal{B}(F^*) \) of \( C^f_{a,\varphi} \), \( C^b_{a,\varphi} \) by taking pointwise adjoints: \( C^*_{a,\alpha}(x, n) := C_{a,\alpha}(x, n)^* \) and \( C^*_{a,\alpha}(x, n) := C_{a,\alpha}(x, n)^* \). Similarly, define \( \Delta \ni x \mapsto a^*(x) := a(x)^* \in \mathcal{B}(F^*) \). Then we have
\[
C^f_{a,\varphi} = C^b_{a,\varphi} \quad \text{and} \quad C^b_{a,\varphi} = C^f_{a,\varphi}. \tag{13}
\]
Both \( C^b_{a,\varphi} \) and \( C^f_{a,\varphi} \) have their advantages. Ergodic properties of \( C^b_{a,\varphi} \) seem easier to calculate, while \( C^f_{a,\varphi} \) is more relevant for calculation of spectral radius:

**Definition 4.2.** Since the sequence \( a_n := \sup_{x \in \Delta_n} \ln \| C^f_{a,\varphi}(x, n) \| \) is subadditive (i.e. \( a_{m+n} \leq a_m + a_n \)), we have the following equality
\[
\lambda(a, \varphi) := \lim_{n \to \infty} \sup_{x \in \Delta_n} \frac{1}{n} \ln \| C^f_{a,\varphi}(x, n) \| = \inf_{n \in \mathbb{N}} \sup_{x \in \Delta_n} \frac{1}{n} \ln \| C^f_{a,\varphi}(x, n) \|.
\]
We call its common value \( \lambda(a, \varphi) \in [-\infty, \infty) \) the **spectral exponent** of \( a : \Delta \to \mathcal{B}(F) \) with respect to \( (X, \varphi) \).

**Remark 4.3.** Let \( D \subseteq \mathcal{B}(F) \) be Banach algebra, and let \( \alpha : C(X, D) \to C(X, D) \) be a contractive endomorphism that generates a partial dynamical system \( (X, \varphi) \). For every \( a \in D \subseteq \mathcal{B}(F) \) we define the function \( C_{a,\alpha}(x, n) \) \( X \times \mathbb{N} : x \in \Delta_n \to \mathcal{B}(F) \) by the formula
\[
C_{a,\alpha}(x, n) := a \cdot \alpha(a) \cdot \ldots \cdot \alpha^{n-1}(a)(x) \quad \text{for} \ x \in \Delta_n.
\]
It follows from Corollary 2.19 that
\[
r(aa) = \lim_{n \to \infty} \max_{x \in \Delta_n} \| C_{a,\alpha}(x, n) \|^{\frac{1}{n}}.
\]
In particular, if the field of endomorphisms \( \{\alpha_x\}_{x \in \Delta} \) generated by \( \alpha \) is trivial, i.e \( \alpha_x \equiv \text{id}_D \), \( x \in \Delta \), then \( C_{a,\alpha} = C^f_{a,\varphi} \) and therefore
\[
\ln r(aa) = \lambda(a, \varphi).
\]
This motivates Definition 4.2. In addition it also indicates why in the case when the field \( \{\alpha_x\}_{x \in \Delta} \) is non-trivial, deriving formulas for \( r(aa) \) is much harder and requires extra work and this will be our aim.

Let \( B \) be the unit ball in the dual space \( F^* \) equipped with *-weak topology. In order to deal with the potential discontinuity of \( a : \Delta \to \mathcal{B}(F) \), cf. Example 1.13 and Remark 1.15, we will consider a quotient of \( B \) by the following equivalence relation:
\[
v \sim w \iff v = \lambda w \text{ for some } \lambda \in \mathbb{T}.
\]
Note that \( v \sim w \) if and only if \( |v| = |w| \) as functions. We will write \([v] \) for the equivalence class of \( v \in B \). Thus \([v] = \{ w \in B : |v| = |w| \} \). In what follows we denote by \([B] \) the factor space \( B/\sim \).

**Lemma 4.4.** The space \([B] \) is compact and Hausdorff.
Proof. $[B]$ is compact as a continuous image of the compact space $B$. Now, take any $[v], [w] \in [B]$ with $[v] \neq [w]$. Then there is $h \in F$ such that $|v(h)| \neq |w(h)|$. We may assume that $|v(h)| < |w(h)|$. Take $\varepsilon = \frac{|w(h)| - |v(h)|}{2}$ and put

$$V := \{[u] \in [B] : |v(h)| - |u(h)| < \varepsilon\}, \quad W := \{[u] \in [B] : |w(h)| - |u(h)| < \varepsilon\}.$$ 

Then $V$ and $W$ are open neighbourhoods of $[v]$ and $[w]$ in $[B]$. Moreover, if we assume that $[u] \in V \cap W$, then there are $\lambda_v, \lambda_w \in \mathbb{T}$ such that $|\lambda_v v(h) - u(h)|, |\lambda_w w(h) - u(h)| < \varepsilon$. This implies

$$|w(h)| - |v(h)| = |\lambda_w w(h) - v(h)| \leq |\lambda_w w(h) - \lambda_v v(h)|$$

$$\leq |\lambda_v v(h) - u(h)| + |\lambda_w w(h) - u(h)| < 2\varepsilon = |w(h)| - |v(h)|,$$

a contradiction. Hence $V$ and $W$ are disjoint, and $[B]$ is Hausdorff.

By Lemma 4.4, the space $\tilde{X} := X \times [B]$ equipped with the product topology is a compact Hausdorff space. Construction described in the next definition plays the principal role in variational principles under study.

**Definition 4.5.** We say that a triple $(X, \varphi, a)$ admits a continuous linear extension $(\tilde{X}, \tilde{\varphi}, \tilde{a})$ if the set

$$\tilde{\Delta} := \{(x, [v]) \in \tilde{X} : x \in \Delta, a^*(x)v \neq 0\},$$

is open in $\tilde{X}$, and the maps $\tilde{\varphi} : \tilde{\Delta} \to \tilde{X}$ and $\tilde{a} : \tilde{\Delta} \to [0, +\infty)$ given by

$$\tilde{\varphi}(x, [v]) := \left( \varphi(x), \frac{a^*(x)v}{\|a^*(x)v\|} \right), \quad \tilde{a}(x, [v]) := \|a^*(x)v\|$$

are continuous.

Clearly, $(X, \varphi, a)$ admits a continuous extension whenever $a$ is continuous, that is when $a \in C(\Delta, \mathcal{B}(F))$. However, as it is shown further, it may occur that $(X, \varphi, a)$ may admit a continuous extension even when $a$ is not measurable.

**Theorem 4.6** (Variational principle using linear extension). Let $(X, \varphi)$ be a partial dynamical system and let $\Delta \ni x \to a(x) \in \mathcal{B}(F)$ a bounded function such that $(X, \varphi, a)$ admits the continuous linear extension $(\tilde{X}, \tilde{\varphi}, \tilde{a})$ (this holds, e.g., when $a \in C(\Delta, \mathcal{B}(F))$). Then

$$\lambda(a, \varphi) = \max_{\mu \in \text{Inv}(\tilde{\Delta}_\infty, \tilde{\varphi})} \int_{\tilde{\Delta}_\infty} \ln \tilde{a}(x, v) \, d\mu = \max_{\mu \in \text{Erg}(\tilde{\Delta}_\infty, \tilde{\varphi})} \int_{\tilde{\Delta}_\infty} \ln \tilde{a}(x, v) \, d\mu$$

where $\tilde{\Delta}_\infty$ is the essential domain of $\tilde{\varphi}$. We assume here that $\ln(0) = -\infty$ and $\lambda(a, \varphi) = -\infty$ if $\text{Erg}(\tilde{\Delta}_\infty, \tilde{\varphi}) = \emptyset$ (this is the case when $\tilde{\Delta}_\infty = \emptyset$ and all the more when $\Delta_\infty = \emptyset$).

**Proof.** Let $\tilde{\Delta}_n$ be the domain of $\tilde{\varphi}^n$. Simple calculation gives that

$$\|C_{a^*, \varphi}^b(x, n)v\| = \prod_{k=0}^{n-1} \tilde{a}(\tilde{\varphi}^k(x, [v]))$$
whenever \((x, [v]) \in \widetilde{\Delta}_n\) and \(\|C^b_{a, \varphi}(x, n)v\| = 0\) otherwise. Moreover, by (13) we have \(C^b_{a, \varphi}(x, n) = C^f_{a, \varphi}(x, n)^*\). Thus we get

\[
\lambda(a, \varphi) = \lim_{n \to \infty} \sup_{x \in \Delta_n} \frac{1}{n} \ln \|C^f_{a, \varphi}(x, n)^*v\| = \lim_{n \to \infty} \sup_{x \in \Delta_n, v \in B} \frac{1}{n} \ln \|C^f_{a, \varphi}(x, n)^*v\|
\]

where \(S_n(\ln \tilde{a})\) is the empirical average corresponding to \(\ln \tilde{a}\) and \(\tilde{\varphi}\), see (12). Thus the assertion follows from Theorem 3.4.

\[\square\]

**Remark 4.7.** Note that \(\widetilde{\Delta}_\infty \subseteq \Delta_\infty \times [S]\) where \(S\) is the unit sphere in \(F^*\), and thus \([S]\) is the projective space of \(F^*\). Lemma 3.3 implies that

\[\text{Inv}(\widetilde{\Delta}_\infty, \tilde{\varphi}) = \text{Inv}(\Delta_\infty \times [S], \tilde{\varphi}) \quad \text{and} \quad \text{Erg}(\widetilde{\Delta}_\infty, \tilde{\varphi}) = \text{Erg}(\Delta_\infty \times [S], \tilde{\varphi}).\]

In particular, in Theorem 4.6 one could replace \(\widetilde{\Delta}_\infty\) with \(\Delta_\infty \times [S]\). Moreover, the projection \(p : \Delta_\infty \times [S] \to \Delta_\infty\) onto the first coordinate induces a natural projection \(p^*\) of measures: for each Borel measure \(\tilde{\mu}\) on \(\Delta_\infty \times [S]\), \(p^*(\tilde{\mu})\) is a Borel measure on \(\Delta_\infty\) where \(p^*(\tilde{\mu})(\omega) := \tilde{\mu}(p^{-1}(\omega)), \omega \subseteq \Delta_\infty\). By definitions of \(p^*\) and \(\tilde{\varphi}\) it follows that

\[p^*(\text{Inv}(\widetilde{\Delta}_\infty, \tilde{\varphi})) \subseteq \text{Inv}(\Delta_\infty, \tilde{\varphi}) \quad \text{and} \quad p^*(\text{Erg}(\widetilde{\Delta}_\infty, \tilde{\varphi})) \subseteq \text{Erg}(\Delta_\infty, \tilde{\varphi}).\]

**Corollary 4.8.** Retain the notation and assumptions of Theorem 4.6. There is \(\tilde{\mu} \in \text{Erg}(\widetilde{\Delta}_\infty, \tilde{\varphi})\) and a subset \(Y \subseteq \widetilde{\Delta}_\infty\), \(\tilde{\mu}(Y) = 1\), such that

\[\lambda(a, \varphi) = \lim_{n \to \infty} \frac{1}{n} \ln \|C^b_{a, \varphi}(x, n)v\| \quad \text{for every} \ y \in Y.\]

**Proof.** In the last step in the proof of Theorem 4.6 instead of Theorem 3.4 apply Corollary 3.5 and use that \(\prod_{k=0}^{n-1} \tilde{a}(\varphi^k(x, [v])) = \|C^b_{a, \varphi}(x, n)v\|\), cf. (15). \[\square\]

We will define Lyapunov exponents with respect to a partial dynamical system \((X, \varphi)\) as the corresponding objects with respect to the (full) dynamical system \((\Delta_\infty, \varphi)\).

**Definition 4.9.** Let \(C : \Delta_\infty \times \mathbb{N} \to B(F)\) be a function. Lyapunov exponent of \(C\) at a point \(x \in \Delta_\infty\) in direction \(v \in F\) is given by the formula

\[\lambda(x, v) := \liminf_{n \to \infty} \frac{1}{n} \ln \|C(x, n)v\|.
\]

The maximal Lyapunov exponent of \(C\) at \(x\) is

\[\lambda_x(C) := \liminf_{n \to \infty} \frac{1}{n} \ln \|C(x, n)\|.
\]

We also put \(\lambda(C) := \liminf_{n \to \infty} \sup_{x \in \Delta_\infty} \frac{1}{n} \ln \|C(x, n)\|\).
For any function $C : \Delta_\infty \times \mathbb{N} \to \mathcal{B}(F)$, for which the corresponding Lyapunov exponents exist, we clearly have

$$\sup_{x \in \Delta_\infty} \sup_{v \in F, \|v\| = 1} \lambda_x(C, v) \leq \sup_{x \in \Delta_\infty} \lambda_x(C) \leq \lambda(C).$$

Note that the left most expression involves the least number of conditions, and hence is the easiest to calculate. The chief importance of backward cocycles associated with operator valued functions lies in that for such cocycles the three above expressions are equal. Moreover, exploiting the ergodic properties of the considered systems we may substantially decrease the domains of the suprema:

**Theorem 4.10** (Variational principle using Lyapunov exponents I). Let $(X, \varphi)$ be a partial dynamical system and let $\Delta \ni x \to (x, a(x)) \in \mathcal{B}(F)$ a bounded function such that $(X, \varphi, a)$ admits the continuous linear extension (this holds, e.g., when $a \in C(\Delta, \mathcal{B}(F))$). Then

Let $\Omega \subseteq \Delta_\infty$ be any set such that $\mu(\Omega) > 0$ for every $\mu \in \operatorname{Erg}(\Delta_\infty, \varphi)$. Then

$$\lambda(a, \varphi) = \lambda(C_{a, \varphi}^f) = \max_{x \in \Omega} \lambda_x(C_{a, \varphi}^f) = \max_{(x, v) \in \Omega \times S} \lambda_x(C_{a, \varphi}^b, v)$$

where $S$ is the unit sphere in $F^*$. \hfill \Box

**Proof.** By (13) and the definition of $\lambda(a, \varphi)$ for any $(x, v) \in \Delta_\infty \times S$ we have

$$\lambda_x(C_{a, \varphi}^b, v) \leq \lambda_x(C_{a, \varphi}^f) = \lambda_x(C_{a, \varphi}^f) \leq \lambda(C_{a, \varphi}^f) \leq \lambda(a, \varphi).$$

Thus it suffices to find $(x, v) \in \Omega \times S$ with $\lambda(a, \varphi) = \lambda_x(C_{a, \varphi}^b, v)$. To this end, take $\tilde{\mu} \in \operatorname{Erg}(\Delta_\infty, \tilde{\varphi})$ and $Y \subseteq \Delta_\infty \subseteq \Delta_\infty \times [S]$ as in Corollary 4.8. That is, we have $\tilde{\mu}(Y) = 1$ and $\lambda(a, \varphi) = \lambda_x(C_{a, \varphi}^b, v)$ for every $(x, [v]) \in Y$. Using the projection $p : \Delta_\infty \times [S] \ni (x, [v]) \to x \in \Delta_\infty$, we get $\mu := p^*(\tilde{\mu}) \in \operatorname{Erg}(\Delta_\infty, \varphi)$, cf. Remark 4.7. Since $\mu(p(Y)) = 1$ we have $p(Y) \cap \Omega \neq \emptyset$. For every $x \in p(Y) \cap \Omega$ there is $v \in S$ such that $(x, [v]) \in Y$ and therefore $\lambda(a, \varphi) = \lambda_x(C_{a, \varphi}^b, v)$.

We may rephrase the above theorem in more appealing (but slightly weaker) form, using measure exponents:

**Lemma 4.11.** Let $a$ be such that $\Delta_\infty \ni x \to \|C_{a, \varphi}(x, n)\|$ is bounded and measurable, for every $n \in \mathbb{N}$. For any $\mu \in \operatorname{Erg}(\Delta_\infty, \varphi)$ there exists a number $\lambda_\mu(a, \varphi) \in [-\infty, +\infty)$ such that

$$\lambda_\mu(a, \varphi) = \lambda_x(C_{a, \varphi}^f) = \lambda_x(C_{a, \varphi}^b) \quad \text{for } \mu\text{-almost every } x \in \Delta_\infty.$$

**Proof.** The sequence of functions $F_n(x) := \ln \|C_{a, \varphi}(x, n)\| = \ln \|C_{a, \varphi}(x, n)\|$, is subadditive in the sense that $F_{m+n} \leq F_m \circ \varphi^n + F_n$. Thus the assertion follows from Kingman’s subadditive theorem, see for instance [Rue82] Theorem A.1.

**Definition 4.12.** We call the number $\lambda_\mu(a, \varphi)$ defined in Lemma 4.11 the (maximal) measure exponent of $a$ with respect to the ergodic (partial) system $(X, \varphi, \mu)$.

**Corollary 4.13** (Variational principle using Lyapunov exponents II). Let $(X, \varphi)$ be a partial dynamical system and let $\Delta \ni x \to (x, a(x)) \in \mathcal{B}(F)$ a bounded function such that $(X, \varphi, a)$ admits the continuous linear extension (this holds, e.g., when $a \in C(\Delta, \mathcal{B}(F))$). Then

$$\lambda(a, \varphi) = \max_{\mu \in \operatorname{Erg}(\Delta_\infty, \varphi)} \lambda_\mu(a, \varphi).$$

That is, the spectral exponent is the maximum of measure exponents.
Proof. It follows from Theorem 4.10 along with Lemma 4.11.

Remark 4.14. The above variational principles are closely related with the Multiplicative Ergodic Theorem (MET). The first MET was established by Oseledets [Ose68] and various generalizations keep appearing until the present times, see [GTQ15] and references therein. All MET’s apply to backwards cocycles and require some compactness assumptions. In particular, if one wants to combine them with the formula for the spectral exponent \( \lambda(a, \varphi) \) one needs to pass to duals. For our purposes versions of MET’s in [Thi87] and [GTQ15] seem the most relevant. They imply that if \( \Delta \ni x \to a^*(x) \in \mathcal{B}(F^*) \) is asymptotically compact, and either \( F^{**} \) is separable ([GTQ15] Theorem 14]) or \( a^* \) is continuous and \( \Delta_\infty \) is a complete metric space ([Thi87] Theorem 2.3]), then for every \( \mu \in \text{Erg}(\Delta_\infty, \varphi) \) we have Oseledets filtrations of \( F^* \) for the cocycle \( C_a^{\mu}. \) That is, for every \( \mu \in \text{Erg}(\Delta_\infty, \varphi) \) there exists a decrasing sequence of numbers \( \{\lambda_i\}_{i=1}^\infty \), where \( 1 \leq r \leq \infty \), such that for \( \mu \)-almost every \( x \in \Delta_\infty \) there is a measurable filtration of closed subspaces, \( F^* = V_1(x) \supseteq V_2(x) \supseteq \cdots \supseteq V_{r+1}(x) := \cap_{i=1}^\infty V_i(x) \) satisfying \( a^*(x)(V_i(x)) \subseteq V_i(\varphi(x)) \), the codimension of \( V_i(x) \) is finite and does not depend on \( x \), \( \lambda(x)(C_a^{\mu}, v) = -\infty \) for every \( v \in V_{r+1}(x) \), and

\[
\lambda_i = \lambda(x)(C_a^{\mu}, v) \quad \text{for every } v \in V_i(x) \setminus V_{i+1}(x) \text{ and each } i < r + 1.
\]

In particular, \( \lambda_{\mu}(a) = \lambda_1 \) realizes as a Lyapunov exponent \( \lambda(x)(C_a^{\mu}, v) \) in the direction \( v \) for every \( v \in V_1(x) \setminus V_2(x) \) and \( \mu \)-almost every \( x \in \Delta_\infty \). We stress that by Theorem 4.10 the spectral exponent \( \lambda(a, \varphi) \) always realizes as a Lyapunov exponent \( \lambda(x)(C_a^{\mu}, v) \) for some direction \( v \) and some measure \( \mu \), without any compactness assumption on \( a \)!

5. Spectral radius of weighted endomorphisms of \( A = C(X, D) \)

In this section, we use VPs obtained in the previous section, to give formulae for the spectral radii \( r(a\alpha) \), \( a \in A \), under the following assumptions:

- \( A \cong C(X, D) \) where \( D \) is a unial Banach algebra;
- \( \alpha : A \to A \) is contractive and \( \alpha(C(X) \otimes 1) \subseteq \alpha(1)C(X) \otimes 1 \). Then by Proposition 1.3 \( \alpha \) generates a partial dynamical system \( (X, \varphi) \), see Definition 1.4.

As we have seen in Proposition 2.18 the spectral radius of an abstract weighted shift operator \( aT : E \to E \) coincides with the spectral radius of the associated weighted endomorphism \( a\alpha : A \to A \). In particular, the situation where \( E = L^p(X, \mathcal{B}(F)) \) is a vector-valued function space, \( T \) is a composition operator, and \( A \) consists of operators of multiplication by operator-valued functions in \( C(X, D) \), where \( D \subseteq \mathcal{B}(F) \). But the developed formulae can also be applied when \( T \) is not a priori a composition operator.

We start with two extremal cases, when \( D = \mathbb{C} \) is trivial or \( X = \{x\} \) is trivial. Then we get, respectively: the formula for \( r(a\alpha) \) where \( A \) is a commutative uniform algebra (subsection 5.1), and a ’Dynamical Variational Principle’ and an intriguing version of Gelfand’s formula for the spectral radius \( r(a) \) of an arbitrary operator \( a \in \mathcal{B}(F) \) (subsection 5.2).

In subsection 5.3 we consider the case where \( D = \mathcal{B}(F) \) and the associated field of endomorphisms \( \{\alpha_x\}_{x \in \Delta} \) consists of inner endomorphism of \( \mathcal{B}(F) \). Finally, in subsection 5.4 we derive formulas for spectral radius where \( D \) is arbitrary. We achieve this by reducing the general case to the special one treated in subsection 5.3.
5.1. Variational principle for commutative algebra of weights. Here we assume that $A$ is a uniform algebra, sometimes also called function algebra [Zel68, 30.1]. Thus $A$ is a commutative Banach algebra such that $\|a\| = r(a)$ for every $a \in A$. Equivalently, this means that the Gelfand transform on $A$ is an isometry, and we may view $A$ as a closed subalgebra of $C(X)$. The following variational principle generalizes the corresponding results in [Leb79], [Kit79] (see also [AL94, 4], [Ant96, 5] or [Kwa09]) where the situation of an automorphism is analysed. We could derive it either from Theorem 4.6 or 4.10, but it also follows directly from Theorem 3.4.

**Theorem 5.1.** Let $\alpha : A \to A$ be an endomorphism of a uniform algebra $A$. Let $(X, \varphi)$ be the partial dynamical system dual to $(A, \alpha)$, see Definition 1.2. Then for any $a \in A$ the spectral radius of the weighted endomorphism $\alpha a : A \to A$ is given by

$$r(a\alpha) = \max_{\mu \in \Inv(\Delta_\infty, \varphi)} \exp \int_{\Delta_\infty} \ln |\hat{a}(x)| d\mu = \max_{\mu \in \Erg(\Delta_\infty, \varphi)} \exp \int_{\Delta_\infty} \ln |\hat{a}(x)| d\mu,$$

where $\ln(0) = -\infty$, $\exp(-\infty) = 0$ and $r(a\alpha) = 0$ if $\Delta_\infty = \emptyset$.

**Proof.** We have $r(a\alpha) = \lim_{n \to \infty} \|a\alpha(a)\ldots a^n(a)\|^\frac{1}{n}$ by Proposition 2.18. Using this and that the Gelfand transform is isometric on $A$ we get

$$\ln r(a\alpha) = \lim_{n \to \infty} \frac{1}{n} \ln \|a\alpha(a)\ldots a^n(a)\| = \lim_{n \to \infty} \sup_{x \in \Delta_\alpha} \frac{1}{n} \sum_{k=0}^{n-1} \ln |\alpha^k(a)(x)| = \lim_{n \to \infty} \sup_{x \in \Delta_\alpha} S_n(\ln |\hat{a}|)(x).$$

Thus the assertion follows from Theorem 3.4. \hfill \Box

**Remark 5.2.** Formula (17) can be improved in the following sense. A closed subset $F \subseteq X$ is called a maximizing set for algebra $A$ if $\|a\| = \max_{x \in F} |\hat{a}(x)|$ for every $a \in A$. There exists a uniquely defined minimal maximizing set for $A$ which is called Shilov boundary and is denoted by $\partial A$. If a map $\varphi$ preserves $\partial A$, which is always the case when $\alpha : A \to A$ is an epimorphism (one can apply [Zel68, Theorem 15.3]), then $\varphi$ preserves $\partial A \cap \Delta_\infty$ and therefore

$$r(a\alpha) = \max_{\mu \in \Erg(\partial A \cap \Delta_\infty, \varphi)} \exp \int_{\partial A \cap \Delta_\infty} \ln |\hat{a}(x)| d\mu.$$

5.2. Dynamical variational principle for an arbitrary operator. Theorem 5.9 implies a theoretically interesting formula for the spectral radius of an arbitrary operator. Namely, let $F$ be a Banach space and let $a \in B(F)$ be arbitrary. Let $|B| = B/\sim$, as before, be the factor of the unit ball $B$ in $F^*$. Consider a partial mapping $\tilde{\varphi} : \Delta \to |B|$ given by

$$\tilde{\varphi}([v]) := \left( \frac{a^*(v)}{\|a^*(v)\|} \right), \quad \tilde{\Delta} := \{ [v] : a^*(v) \neq 0 \}.$$

With this notation we have the following variational principle.

**Theorem 5.3.** Let $F$ be a Banach space. Then for every $a \in B(F)$

$$r(a) = \max_{\mu \in \Inv(|B|, \tilde{\varphi})} \exp \int_{|B|} \ln \|a^*(v)\| d\mu = \max_{\mu \in \Erg(|B|, \tilde{\varphi})} \exp \int_{|B|} \ln \|a^*(v)\| d\mu$$

where $\tilde{\varphi}$ is given by (18), and $r(a) = 0$ if Erg $([B], \tilde{\varphi}) = \emptyset$. 

Proof. Note that $\ln r(a) = \lambda(a, \varphi)$ where $X := \{x\}$ is a singleton and $\varphi(x) := x$ is the identity map on $X$, cf. Remark 4.3. Hence the assertion follows from Theorem 4.6 \hfill \Box

Remark 5.4. In the situation under consideration there is no need to pass from the ball $B$ to the factor space $[B]$. Namely, for any $a \in B(F)$ we have

$$r(a) = \max_{\mu \in \text{Inv}(B, \tilde{\varphi})} \exp \int_B \ln \|a^*(v)\| \, d\mu = \max_{\mu \in \text{Erg}(B, \tilde{\varphi})} \exp \int_B \ln \|a^*(v)\| \, d\mu$$

where $\tilde{\varphi}(v) := \frac{a^*(v)}{\|a^*(v)\|}$ is defined on $\Delta := \{v : a^*(v) \neq 0\}$. The proof of formula (20) from (19) goes by comparing the ergodic measures with respect to $\tilde{\varphi}$ considered here and to that of (18), and noticing that the integrals $\int \ln \tilde{a} \, d\mu$ over the corresponding measures coincide. Taking into account Remark 4.7, one could also replace $B$ in (20) with the unit sphere $S$ in $F^*$, and $[B]$ in (19) with the projective space $[S]$.

We obtained Theorem 5.3 as a special case of Theorem 4.6. Applying in the same manner Corollary 4.13 one gets nothing but $r(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}}$. However, applying formula (16) in Theorem 4.10 gives us a slight improvement of the Gelfand’s formula:

Corollary 5.5 (Gelfand’s formula revisited). For every operator $a \in B(F)$ on a Banach space $F$ we have

$$r(a) = \max_{v \in S} \lim_{n \to \infty} \|a^n v\|^{\frac{1}{n}}$$

where $S$ is the unit sphere in the dual space $F^*$. In particular, if $F$ is a reflexive space, then there is a direction $v \in F$ such that

$$r(a) = \lim_{n \to \infty} \|a^n v\|^{\frac{1}{n}}.$$

Proof. The first part follows from Theorem 4.10 applied to the case $X := \{x\}$ and $\varphi(x) := x$. We get the second part because $r(a) = r(a^*)$, for any $a \in B(F)$ and for reflexive space $F$ one has $F \cong (F^*)^*$ and $a = (a^*)^*$.

5.3. Variational principles in the case $A = C(X, B(F))$. In this subsection we make the following standing assumptions

- $A = C(X, B(F))$ where $F$ is a Banach space;
- $\alpha : A \to A$ is an endomorphism such that the corresponding field of (non-zero) endomorphisms $\{\alpha_x\}_{x \in \Delta}$ of $B(F)$ consists of inner isometric monomorphisms.

Then by Propositions 1.11, 1.7 we have

$$\alpha(a)(x) = \begin{cases} T_x a(\varphi(x)) S_x, & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases} \quad a \in C(X, B(F)),$$

where $\varphi : \Delta \to X$ is a continuous partial map and $\{T_x, S_x\}_{x \in \Delta} \subseteq B(F)$ are such that $S_x T_x = 1$ and $T_x$ is an isometry, for every $x \in \Delta$. In this situation we may extend Remark 4.3 and express the spectral radius of $aa$ by a spectral exponent of a forward cocycle associated with $\Delta \ni x \to a(x)T_x$:

Proposition 5.6. With the above assumptions, for every $a \in A = C(X, B(F))$ we have

$$r(aa) = \lim_{n \to \infty} \sup_{x \in \Delta_n} \|C^f_{aT_x \varphi}(x, n)\|^{\frac{1}{n}} = e^{\lambda(aT, \varphi)},$$
where $aT$ denotes the function $\Delta \ni x \mapsto a(x)T_x \in B(F)$, and $C^r_{aT,\varphi}(x, n) := a(x)T_x \cdot a(\varphi(x))T_{\varphi(x)} \cdots a(\varphi^{n-1}(x))T_{\varphi^{n-1}(x)}$ is the associated forward cocycle.

Proof. We introduce the following notation: for any sequence of points $x_1, \ldots, x_n \in X$ and any field of operators $X \ni x \mapsto R_x \in B(F)$ we put $R_{x_1 x_2 \cdots x_n} := R_{x_1} R_{x_2} \cdots R_{x_n}$. Thus for every $k \in \mathbb{N}$, $a \in C(X, B(F))$ and $x \in \Delta_k$ we have

$$\alpha^k(a)(x) = T_{x \varphi(x) \cdots \varphi^{k-1}(x)} a(\varphi^k(x)) S_{\varphi^{k-1}(x) \cdots \varphi(x) x}.$$  

Note that $S_x T_x = 1$ for $x \in \Delta$ implies $S_{\varphi^n(x) \cdots \varphi(x) x}$ for every $x \in \Delta_n$. Thus putting $C_{a, \alpha}(x, n) := a \cdot \alpha(a) \cdots \alpha^{n-1}(a)(x)$ for $x \in \Delta_n$, cf. Remark 4.3, for every $x \in \Delta_n$ we get

$$C_{a, \alpha}(x, n + 1) = C_{aT, \varphi}(x, n) a(\varphi^n(x)) S_{\varphi^{n-1}(x) \cdots \varphi(x) x}.$$  

As $\{S_x, T_x\}_{x \in \Delta}$, are contractions it follows that

$$\|C_{a, \alpha}(x, n + 1)\| \leq \|a\| \cdot \|C_{aT, \varphi}(x, n)\|.$$  

On the other hand, we have

$$\|C_{aT, \varphi}(x, n + 1)\| = \|C_{aT, \varphi}(x, n) a(\varphi^n(x)) S_{\varphi^{n-1}(x) \cdots \varphi(x) x} \| \leq \|C_{aT, \varphi}(x, n) a(\varphi^n(x))\|$$  

$$= \|C_{aT, \varphi}(x, n) a(\varphi^n(x)) S_{\varphi^{n-1}(x) \cdots \varphi(x) x} T_{x \varphi(x) \cdots \varphi^{n-1}(x)} \| \leq \|C_{aT, \varphi}(x, n) a(\varphi^n(x)) S_{\varphi^{n-1}(x) \cdots \varphi(x) x} \| = \|C_{a, \alpha}(x, n + 1)\|.$$  

Combining the above inequalities we get

$$\|C_{a, \alpha}(x, n + 1)\| \leq \|a\| \cdot \|C_{aT, \varphi}(x, n)\| \leq \|a\| \cdot \|C_{a, \alpha}(x, n)\|. \quad (22)$$

However, $r(aa) = \lim_{n \to \infty} \max_{x \in \Delta_n} \|C_{a, \alpha}(x, n)\|^{+}$ by Corollary 4.3. Thus inequalities (22) give $r(aa) = \lim_{n \to \infty} \sup_{x \in \Delta_n} \|C_{aT, \varphi}(x, n)\|^{\frac{1}{n}}$. \hfill \Box

In general the map $x \mapsto aT(x) = a(x)T_x$ may be far from being continuous, cf. Example 1.13 and Remark 1.15. Nevertheless, as we show now, the triple $(X, \varphi, aT)$ admits the continuous linear extension in the sense of Definition 4.5. To this end recall that $B$ is the unit ball in the dual space $F^*$ equipped with $*$-weak topology, and $[B]$ is the factor space $B/\sim$ where $[v] = \{w \in B : |v| = |w|\}$ is the equivalence class of $v \in B$.

**Lemma 5.7.** For every $a \in A$ and $v \in F^*$, the map $\Delta \ni x \mapsto [T_x^* a(x)^* v] \in [B]$ is continuous.

**Proof.** The topology in $[B]$ is generated by sets

$$U_{w, h, \varepsilon} := \{[u] \in [B] : \exists \lambda \in T \ |w(h) - \lambda u(h)| < \varepsilon\}$$

where $w \in F^*$, $h \in F$ and $\varepsilon > 0$.

Let us take any $x_0 \in \Delta$ and suppose that $[T_{x_0}^* a(x_0)^* v] \in U_{w, h, \varepsilon}$ for some $w, h, \varepsilon$. That is, there is $\lambda \in T$ such that

$$|w(h) - \lambda v(a(x_0) T_{x_0} h)| < \varepsilon.$$  

Let $\delta_1, \delta_2 > 0$ be arbitrary. Let $V \subseteq \Delta$ be an open neighbourhood of $x_0$ such that $\|a(x) - a(x_0)\| < \delta_1$. By Lemma 1.14(2) we may find an open neighbourhood $U$ of $x_0$
contained in $V$ and numbers $\{\lambda_x\}_{x \in U} \subseteq \mathbb{T}$ such that $\|\lambda_x T_x h - T_{x_0} h\| < \delta_2$. Then for every $x \in U$ we have

$$|w(h) - \lambda \lambda_x v(a(x)T_x h)| \leq |w(h) - \lambda \lambda_x v(a(x_0)T_{x_0} h)| + |\lambda v(a(x)T_x h) - \lambda \lambda_x v(a(x)T_{x_0} h)|$$

$$+ |\lambda v(a(x)T_{x_0} h) - \lambda \lambda_x v(a(x)T_x h)|$$

$$\leq |w(h) - \lambda \lambda_x v(a(x_0)T_{x_0} h)| + \delta_1 \|v\|\|T_{x_0} h\| + \delta_2 \|v\|\|a(x)\|$$

$$< |w(h) - \lambda \lambda_x v(a(x_0)T_{x_0} h)| + \delta_1 \|v\|\|T_{x_0} h\| + \delta_2 \|v\|\|a(x_0)\| + \delta_1).$$

Clearly, we may assume that $v \neq 0$. Then we put

$$\delta_2 := \varepsilon - \frac{|w(h) - \lambda \lambda_x v(a(x_0)T_{x_0} h)|}{2\|v\|\|T_{x_0} h\|}$$

If $\|T_{x_0} h\| = 0$ this already gives $|w(h) - \lambda \lambda_x v(a(x)T_x h)| < \varepsilon$. If $\|T_{x_0} h\| \neq 0$, then putting

$$\delta_1 := \varepsilon - \frac{|w(h) - \lambda \lambda_x v(a(x_0)T_{x_0} h)|}{2\|v\|\|T_{x_0} h\|}$$

we also get $|w(h) - \lambda \lambda_x v(a(x)T_x h)| < \varepsilon$. Thus for every $x \in U$ we get $[T_x^* a(x)^* v] \in U_{w,h,\varepsilon}$. This gives the assertion.

**Lemma 5.8.** For every $a \in A$ and $v \in F^*$, the map $\Delta \ni \alpha \mapsto \|T_x^* a(x)^* v\| \in [0, \infty)$ is continuous.

**Proof.** Recall that $T_x$ is an isometry from $F$ onto $\alpha_x(1)F$ and $\alpha_x(1)$ is a norm one projection. Hence

$$\|T_x^* a(x)^* v\| = \sup_{\|h\| = 1} \|v(a(x)T_x h)\| = \sup_{\|h\| = 1} \|v(a(x)\alpha_x(1)h)\| = \|v(a(x)\alpha_x(1))^* v\|.$$ 

Thus the assertion follows from the continuity of $\Delta \ni \alpha \mapsto a(x)\alpha_x(1) \in B(F)$. 

**Theorem 5.9.** Let $A = C(X, \mathcal{B}(F))$ where $F$ is a Banach space and suppose that $\alpha : A \to A$ is an endomorphism of $A$ such that the generated field of endomorphisms $\{\alpha_x\}_{x \in \Delta}$ consists of isometric inner endomorphism of $\mathcal{B}(F)$. Then there is a dual partial dynamical system $(X, \varphi)$ and a family of isometries $\{T_x\}_{x \in \Delta} \subseteq \mathcal{B}(F)$ satisfying $T_x b = \alpha_x(b)T_x$ for $b \in \mathcal{B}(F)$. For any $a \in A$ the map $\Delta \ni x \mapsto aT_x(x) := a(x)T_x$ admits the linear extension in the sense of Definition 4.3. In particular, fixing $a \in A$ and defining

1. $\tilde{X} := X \times [S]$ where $[S]$ is a projective space of the dual Banach space $F^*$,

2. $\tilde{\Delta} := \{(x, [v]) \in \tilde{X} : x \in \Delta, T_x^* a(x)^* v \neq 0\}$, and

$$\tilde{\varphi}(x, [v]) := \left(\varphi(x), \left[\frac{T_x^* a(x)^* v}{\|T_x^* a(x)^* v\|}\right]\right), \quad \tilde{a}(x, [v]) := \|T_x^* a(x)^* v\|, \quad (x, [v]) \in \tilde{\Delta},$$

we get

$$r(\alpha a) = \max_{\mu \in \text{Inv}(\Delta_\infty, \tilde{\varphi})} \exp \int_{\Delta_\infty} \ln \tilde{a}(x, [v]) d\mu = \max_{\mu \in \text{Erg}(\tilde{\Delta}, \tilde{\varphi})} \exp \int_{\tilde{\Delta}} \ln \tilde{a}(x, [v]) d\mu$$

where $\tilde{\Delta}_\infty$ is the essential domain of $\tilde{\varphi}$.

**Proof.** Let us fix $a \in A = C(X, \mathcal{B}(F))$ and define the triple $\tilde{\Delta} := \{(x, [v]) \in \tilde{X} : x \in \Delta, T_x^* a(x)^* v \neq 0\}$ is open in $\tilde{X}$. In view of Lemmas 5.7 and 5.8 the maps $\tilde{\varphi} : \tilde{\Delta} \to \tilde{X}$ and $\tilde{a} : \tilde{\Delta} \to [0, +\infty)$ are continuous.
Hence $(\tilde{X}, \tilde{\varphi}, \tilde{a})$ is the continuous linear extension of $(X, \varphi, aT)$. By Proposition 5.6, \(r(\alpha a) = e^{\lambda(aT, \varphi)}\) where \(aT(x) = a(x)T_x\), \(x \in \Delta\). Thus applying Theorem 4.6 gives \(\lambda(aT, \varphi) = \max_{\mu \in \text{Erg}(\Delta, \varphi)} \int_{\Delta} \ln \tilde{a}(x, [v]) \, d\mu = \max_{\mu \in \text{Erg}(\Delta, \varphi)} \int_{\Delta} \ln \tilde{a}(x, [v]) \, d\mu\). In view of Remark 4.7, essential domains for the systems \((X \times [B], \tilde{\varphi})\) and \((X \times [S], \tilde{\varphi})\) are the same. This gives the assertion.

Now we are ready to describe relationships between Lyapunov exponents and spectral radius of weighted endomorphisms of \(C(X, B(H))\). The forthcoming result generalizes variational principle of Latushkin and Stepin [LS91, LS91], established in the case where \(F = H\) is a Hilbert space, \(a\) takes values in compact operators, \(\varphi\) is a homeomorphism and \(\alpha_x = \text{id}\).

**Theorem 5.10.** Let \(A = C(X, B(F))\) where \(F\) is a Banach space and suppose that \(\alpha : A \to A\) is an endomorphism of \(A\) such that the generated field of endomorphisms \(\{\alpha_x\}_{x \in \Delta}\) consists of isometric inner endomorphism of \(B(F)\). For every \(a \in A\) we have

\[
\ln r(\alpha a) = \max_{\mu \in \text{Erg}(\Delta, \varphi)} \lambda_\mu(aT, \varphi)
\]

where \((X, \varphi)\) is the partial dynamical system dual to \(\alpha\), and \(aT(x) = a(x)T_x\) for \(x \in \Delta\), where \((T_x)_{x \in \Delta} \subseteq B(F)\) a family of isometries such that \(T_x b = \alpha_x(b)T_x\) for \(b \in B(F)\). Moreover, for any set \(\Omega \subseteq \Delta_\infty\) such that \(\mu(\Omega) > 0\) for every \(\mu \in \text{Erg}(\Delta_\infty, \varphi)\), we have

\[
\ln r(\alpha a) = \max_{(x, n) \in \Omega \times S} \lambda_\chi(C_{T^* a^*} \mathbb{1}, v)
\]

where \(S\) is the unit sphere in \(F^*\), and \(\Delta \ni x \to T^* a^*(x) = T_x^* a^*(x) \in B(F^*)\).

**Proof.** By Lemmas 5.7 and 5.8 \((X, \varphi, aT)\) admits a continuous linear extension. Hence we get the assertion by Theorem 4.10 and Corollary 4.13.

**Corollary 5.11.** Retain the notation and assumptions of Theorem 5.10. In particular, let \(\Omega \subseteq \Delta_\infty\) be such that \(\mu(\Omega) > 0\) for every \(\mu \in \text{Erg}(\Delta_\infty, \varphi)\). Then

\[
r(\alpha a) = \max_{x \in \Omega} \lim_{n \to \infty} \|C^f_{aT, \varphi}(x, n)\|^{\frac{1}{n}} = \max_{x \in \Omega} \lim_{n \to \infty} \|C_{a, a}(x, n)\|^{\frac{1}{n}},
\]

where \(C_{a, a}(x, n) := a \cdot \alpha(a) \cdot \ldots \cdot \alpha^{n-1}(a)(x)\), cf. Remark 4.3.

**Proof.** The first equality follows from Theorem 5.10 and the second from (22).

**Remark 5.12.** If \(F\) is finite dimensional, then every endomorphism \(\alpha : C(X, B(F)) \to C(X, B(F))\) is of the form (21), by Corollary 1.12. Thus assumptions of Theorems 5.9 and 5.10 are satisfied whenever the induced endomorphisms \(\{\alpha_x\}_{x \in \Delta}\) are isometric (they are necessarily automorphisms). In particular, if \(F = H \cong \mathbb{C}^n\) is a finite dimensional Hilbert space, then every \(\ast\)-endomorphism \(\alpha : C(X, B(H)) \to C(X, B(H))\) satisfies the assumptions of Theorems 5.9 and 5.10. Moreover, in the Hilbert space case we have \(H^* \cong H\) and thus the above results can be phrased without passing to dual spaces.

**5.4. Variational principle in the case** \(A = C(X, D)\). In this final subsection we consider the case when \(A = C(X, D)\) and \(\{\alpha_x\}_{x \in \Delta}\) are arbitrary endomorphisms of \(D\). We show that by an adequate choice of a cocycle with values in \(B(D)\) we may reduce the general problem to the situation already treated in previous sections.

To this end, let \(\alpha\) be a contractive endomorphism of \(C(X, D)\) that generates a partial dynamical system \((X, \varphi)\). Thus \(\alpha\) is given by formula (7) where \(\{\alpha_x\}_{x \in \Delta}\) is a continuous
field of endomorphisms of $D$. We also assume that $D$ is a unital Banach algebra (the unit $1 \in D$ has norm one). This allows us to treat $D$ as a subalgebra of $B(D)$. Namely, we have an isometric homomorphism $D \ni d \mapsto \overline{d} \in B(D)$ where $\overline{d}v := dv$, for $v \in D$. We may extend this embedding to an isometric homomorphism $C(X, D) \ni b \mapsto \overline{b} \in C(X, B(D))$ where $\overline{b}(x) := b(x)$, for $x \in X$. Moreover, each $\alpha_x : D \to D$, $x \in \Delta$, is contractive and in particular $\alpha_x \in B(D)$. In fact, we may treat $\{\alpha_x\}_{x \in \Delta}$ as an element $\overline{\alpha} \in C(X, B(D))$ where

$$\overline{\alpha}(x) := \begin{cases} \alpha_x, & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases}$$

We denote by $T_\varphi : C(X, B(D)) \to C(X, B(D))$ an endomorphism associated with $(X, \varphi)$:

$$T_\varphi(b)(x) := \begin{cases} b(\varphi(x)), & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases} \quad b \in C(X, B(D)).$$

Since $\Delta$ is clopen for any $a \in D$ the map $\Delta \ni x \mapsto \alpha_x \overline{a}(\varphi(x)) \in B(D)$, which we denote by $\overline{\alpha} \cdot \overline{a} \circ \varphi$, can be treated as an element of $C(X, B(D))$. Formally, $\overline{\alpha} \cdot \overline{a} \circ \varphi = \overline{\alpha T_\varphi(\overline{a})}$.

**Proposition 5.13.** With the above notation for every $a \in C(X, D)$ we have

$$\ln r(\alpha a) = \ln r(\overline{\alpha T_\varphi(\overline{a}) T_\varphi}) = \lambda(\overline{\alpha} \cdot \overline{a} \circ \varphi, \varphi).$$

That is, the spectral radii of weighted endomorphisms $\alpha a : C(X, D) \to C(X, D)$ and $\overline{\alpha T_\varphi(\overline{a}) T_\varphi} : C(X, B(D)) \to C(X, B(D))$ do coincide, and their logarithms are equal to the spectral exponent of the function $\Delta \ni x \mapsto \alpha_x \overline{a}(\varphi(x)) \in B(D)$ with respect to $(X, \varphi)$.

**Proof.** We have $\ln r(\overline{\alpha T_\varphi(\overline{a}) T_\varphi}) = \lambda(\overline{\alpha} \cdot \overline{a} \circ \varphi, \varphi)$ by Remark 4.3. By Corollary 2.19 $r(\alpha a) = r(\alpha a) = \lim_{n \to \infty} \|(aa)^n\|^{1/n}$ where we treat $\alpha a = \alpha(\overline{a})\alpha$ as a weighted endomorphism of $C(X, D)$. By the same corollary we have $\ln r(\overline{\alpha T_\varphi(\overline{a}) T_\varphi}) = \lim_{n \to \infty} \|\overline{\alpha T_\varphi(\overline{a})} \cdot \ldots \cdot T_\varphi^{n-1}(\overline{\alpha T_\varphi(\overline{a})})\|^{1/n}$. Thus it suffices to show that, for every $n \in \mathbb{N}$,

$$\|(aa)^n\|_{C(X,D)} = \|\overline{\alpha T_\varphi(\overline{a})} \cdot \ldots \cdot T_\varphi^{n-1}(\overline{\alpha T_\varphi(\overline{a})})\|_{C(X,B(D))}.$$ 

To this end, we note that for each $x \in \Delta$ the operator $\alpha_x \overline{d}\overline{a}(\varphi(x)) \in B(D)$ acts according to the formula $[\alpha_x \overline{d}\overline{a}(\varphi(x))]d = \alpha_x \overline{d}\overline{a}(\varphi(x))d$ for $d \in D$. Having this in mind we obtain

$$\|(aa)^n\| = \sup_{b \in A, \|b\| = 1} \|(aa) \cdot (aa)(aa)b\|_A$$

$$= \sup_{b \in A, \|b\| = 1} \sup_{x \in \Delta_n} \|\alpha_x \left( a(\varphi(x))a(\varphi(x))a(\varphi^2(x)) \cdots \alpha_{x, \varphi^{n-1}}(a(\varphi^n(x))b(\varphi^n(x))) \right)\|_D$$

$$= \sup_{x \in \Delta_n} \sup_{d \in D, \|d\| = 1} \|\alpha_x \left( a(\varphi(x))a(\varphi(x))a(\varphi^2(x)) \cdots \alpha_{x, \varphi^{n-1}}(a(\varphi^n(x))d) \right)\|_D$$

$$= \sup_{x \in \Delta_n} \sup_{d \in D, \|d\| = 1} \|\left( \alpha_x a(\varphi(x)) \cdots \alpha_{x, \varphi_{n-1}(x)}a(\varphi^2(x)) \cdots \alpha_{x, \varphi_{n-1}(x)}a(\varphi^n(x)) \right) d\|_D$$

$$= \sup_{x \in \Delta_n} \|\alpha_x a(\varphi(x))a(\varphi(x))a(\varphi^2(x)) \cdots \alpha_{x, \varphi_{n-1}(x)}a(\varphi^n(x))\|_{B(D)}$$

$$= \|\overline{\alpha T_\varphi(\overline{a})} \cdot \ldots \cdot T_\varphi^{n-1}(\overline{\alpha T_\varphi(\overline{a})})\|_{C(X,B(D))}.$$  

□
The theory developed in previous sections give formulas for $r(\pi T_\alpha(\pi) T_\nu) = e^{\lambda(\pi, \pi \circ \varphi, \psi)}$, which in view of Proposition 5.13 are also formulas for $r(\alpha \alpha)$. For the sake of completeness we state them explicitly:

**Theorem 5.14.** Let $\alpha : A \to A$ be a contractive endomorphism of $A = C(X, D)$ generating a partial dynamical system $(X, \varphi)$ and let $\alpha \in A$. Let $\Omega \subseteq \Delta_\infty$ be any set such that $\mu(\Omega) > 0$ for every $\mu \in \operatorname{Erg}(\Delta_\infty, \varphi)$. Then

$$
\ln r(\alpha \alpha) = \max_{\mu \in \operatorname{Erg}(\Delta_\infty, \varphi)} \lambda_\mu(\alpha \cdot \pi \circ \varphi, \psi) = \max_{(x, v) \in \Omega \times \delta} \lambda_x(C_h^b(\pi \circ \varphi, \psi), v)
$$

where $\pi \cdot \pi \circ \varphi$ denotes the function $\Delta \ni x \to \alpha_x \tilde{a}(\varphi(x)) \in B(D)$, $(\pi \circ \varphi) \psi$ denotes the function $\Delta \ni x \to a(\varphi(x)) \alpha^*_x \in B(D^*)$ and $S$ is the unit sphere in $D^*$.

Moreover, defining ‘a linear extension’ $(\tilde{X}, \tilde{\varphi}, \tilde{a})$ of $(X, \varphi, a)$ as follows:

1. $\tilde{X} := X \times [S]$ where $[S]$ is a projective space of the dual Banach space $D^*$,
2. $\tilde{\Delta} := \{(x, v) \in \tilde{X} : x \in \Delta, \quad a(\varphi(x)) \alpha^*_x v \neq 0\}$ and

$$
\tilde{\varphi}(x, v) := \left(\varphi(x), \frac{a(\varphi(x)) \alpha^*_x v}{\|a(\varphi(x)) \alpha^*_x v\|}\right), \quad \tilde{a}(x, v) := \|a(\varphi(x)) \alpha^*_x v\|, \quad (x, v) \in \tilde{\Delta},
$$

we get

$$
r(\alpha \alpha) = \max_{\mu \in \operatorname{Inv}(\Delta_\infty, \tilde{\varphi})} \exp \int_{\Delta_\infty} \ln \tilde{a}(x, v) \, d\mu = \max_{\mu \in \operatorname{Erg}(\Delta_\infty, \tilde{\varphi})} \exp \int_{\Delta_\infty} \ln \tilde{a}(x, v) \, d\mu
$$

where $\tilde{\Delta}_\infty$ is the essential domain of $\tilde{\varphi}$.

**Proof.** In view of Proposition 5.13, the first part of assertion follows from Theorem 5.10 applied to a weighted endomorphism with a weight equal to $\pi \cdot \pi \circ \varphi$ and trivial field of endomorphisms of $B(D)$. Alternatively, one can apply Corollary 4.13, Theorem 4.10 and Remark 4.17. Similarly, to get the second part of assertion one can apply either Theorem 5.9 or Theorem 4.10.

**Remark 5.15.** Variational principles obtained in Theorem 5.14 exploit the operator algebra $B(D)$ and the dual space $D^*$ of a Banach algebra $D$. As a rule these spaces are rather bulky. For example, if $D = B(F)$ for some Banach space $F$ then $B(D) = B(B(F))$ and $D^* = B(F)^*$ which are huge in comparison to $B(F)$ and $F^*$, respectively. Thus Theorems 5.9, 5.10 are much more efficient, but we can apply them only under the assumption that the associated field $\{\alpha_x\}_{x \in \Delta}$ consists of isometric inner endomorphism of $B(F)$.

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