An explicit example of HQET at one-loop order of perturbation theory

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As an explicit example of HQET, we construct correlation functions containing a heavy-light axial current, both in the static approximation and in full QCD. This enables us to investigate the size of finite mass effects. As we use the lattice regularisation, we can also check the mass dependence of discretisation errors.

1. CORRELATION FUNCTIONS

The axial current of one light and one heavy quark flavour is phenomenologically interesting, as the decay constant $f_B$ of the B-meson is defined by a matrix element of such a current. This matrix element requires a non-perturbative treatment, i.e. a lattice calculation. Since the bottom quark is too heavy to be included directly in Monte Carlo simulations on today’s computers, the use of effective theories like HQET is a necessity. In a recent paper \cite{1}, we tested HQET for a special correlation function, and studied the discretisation errors connected with the heavy quark mass.

The operators under investigation are the axial current

\begin{equation}
A_0(x) = \tilde{\psi}_1(x) \gamma_0 \gamma_5 \psi_2(x)
\end{equation}

of two relativistic quark flavours $\psi_1$ and $\psi_2$, and the static-light axial current

\begin{equation}
A_0^{\text{stat}}(x) = \tilde{\psi}_1(x) \gamma_0 \gamma_5 \psi_4(x),
\end{equation}

where the static quark field satisfies

\begin{equation}
P_+ \psi_h = \psi_h, \quad P_+ = \frac{1}{2} (1 + \gamma_0),
\end{equation}

and its dynamics is determined by the Eichten-Hill action \cite{3}

\begin{equation}
S_h = a^4 \sum_x \tilde{\psi}_h(x) \nabla_0^2 \psi_h(x),
\end{equation}

The correlation functions we use are defined in a finite space-time volume $L \times L \times L \times L$ with Schrödinger functional boundary conditions, i.e.

- The gauge field is periodic in space.
- No spatial boundary conditions need to be specified for the static quark field.
- The relativistic quark fields are periodic in space up to a phase $\theta$, e.g. (with unit vector $k$)

\begin{equation}
\psi_1(x + \hat{k}) = e^{i\theta} \psi_1(x).
\end{equation}

- Dirichlet boundary conditions in time.

The space-like link variables on the time boundaries are chosen as $U_\theta(x)|_{x_0 = 0,L} = 1$. Correlation functions can contain derivatives $\partial_\mu, \partial_\mu^\perp, \partial_\mu^\parallel, \partial_\mu^\perp$, with respect to the boundary quark fields (see \cite{3} for their definition).

We apply the Symanzik improvement programme \cite{4} to the axial currents, defining improved operators $A_0$ and $A_0^{\text{stat,1}}$, as well as to the action \cite{3}. We use the correlation functions

\begin{align}
f_4^1(x_0) &= \frac{-a^6}{2} \sum_{y,z} \langle A_0^4(x_0) \tilde{\zeta}_2(y) \gamma_5 \zeta_1(z) \rangle, \quad (6) \\
f_1 &= \frac{-a^{12}}{2L^6} \sum_{u,v,y,z} \langle \tilde{\zeta}_1(u) \gamma_5 \zeta_2(v) \times \tilde{\zeta}_2(y) \gamma_5 \zeta_1(z) \rangle, \quad (7) \\
f_{A_0,1}^{\text{stat,1}}(x_0) &= \frac{-a^6}{2} \sum_{y,z} \langle A_0^{\text{stat,1}}(x_0) \times \tilde{\zeta}_h(y) \gamma_5 \zeta_1(z) \rangle, \quad (8) \\
f_{\zeta_1}^{\text{stat}} &= \frac{-a^{12}}{2L^6} \sum_{u,v,y,z} \langle \tilde{\zeta}_1(u) \gamma_5 \zeta_1(v) \times \tilde{\zeta}_h(y) \gamma_5 \zeta_1(z) \rangle. \quad (9)
\end{align}
2. TESTING HQET

Our aim is to compare HQET with full QCD in the continuum limit, which requires renormalisation. A renormalised coupling is defined by matching to the \( \overline{\text{MS}} \) scheme,

\[
\frac{g_{\text{MS}}^2}{\mu^2} = g_0^2 + O(g_0^4),
\]

where \( g_0 \) is the bare lattice coupling. The quark masses are also renormalised by matching to the \( \overline{\text{MS}} \) scheme. The \( \overline{\text{MS}} \) mass \( m_{1,\overline{\text{MS}}} \) of the first quark flavour is chosen to be zero, while we introduce the dimensionless parameter \( z = L m_{2,\overline{\text{MS}}} (\mu = 1/L) \)

\[
z = L m_{2,\overline{\text{MS}}} (\mu = 1/L)
\]

to parametrise the mass of the second quark flavour. Taking the continuum limit means \( a/L \to 0 \) while keeping \( z \) fixed. We form the ratios

\[
X_I(L/a) = \frac{f_1^{\text{stat},I} (L/2)}{\sqrt{f_1^{\text{stat}}}},
\]

\[
Y_I(z, L/a) = \frac{f_1^{\text{stat},I} (L/2)}{\sqrt{f_1^{\text{stat}}}},
\]

in which the boundary field renormalisation constants and a divergence due to the heavy quark self energy cancel. These ratios are expanded as

\[
X_I = X_I^{(0)} + X_I^{(1)} g_0^2 + O(g_0^4),
\]

and analogously for \( Y_I \). We introduce a renormalisation scheme for the light-light axial current by demanding that the renormalised ratio \( Y_{CA} \) satisfies the current algebra relations at \( z = 0 \). The static-light axial current requires a scale dependent renormalisation, and we define a renormalised ratio \( X_{\text{match}}(\mu) \) by imposing the matching condition

\[
X_{\text{match}}(m_{2,\overline{\text{MS}}}) = Y_C A(z, L/a)
\]

\[
+ O(1/z) + O((a/L)^2).
\]

The renormalised ratios are expanded in powers of the coupling analogous to equation (14).

At tree level, the continuum limits \( X_I^{(0)} \) and \( Y_I^{(0)}(z) \) can be calculated analytically. \( Y_I^{(0)}(z) \) contains terms of the form \( e^{-z^2} \), which are not analytic in \( 1/z \). However, the analytic part can be expanded; the results at different orders are shown in figure 1. For \( z > 2 \) the expansion provides a good approximation, and the non-analytic part does not seem to be significant.

![Figure 1. The ratio \( Y_I^{(0)} \) (solid curves) and the expansion of its analytic part at order \( 1/z \) (long dashes), \( 1/z^2 \) (short dashes), and \( 1/z^3 \) (dotted curves).](image)

At one-loop level, HQET predicts that

\[
X^{(1)}_{\text{match}}(m_{2,\overline{\text{MS}}}) = X^{(1)}_I (L/a)
\]

\[
+ \left\{ B^\text{stat}_A - \gamma_0 \ln(z a/L) \right\} X^{(0)}_I
\]

\[
+ O((a/L)^2),
\]

with \( \gamma_0 = -1/4 \pi^2 \). This means that \( B^\text{stat}_A \) can be obtained as the \( 1/z \to 0 \) limit of

\[
B^\text{stat}_A(z) = \gamma_0 \ln(z a/L)
\]

\[
+ \frac{1}{X^{(0)}_I} \left\{ Y^{(1)}_{CA}(z) - X^{(1)}_I (L/a) \right\},
\]

where it is understood that the continuum limit is taken on the right hand side. \( B^\text{stat}_A \) is shown for two different values of the phase \( \theta \) in figure 2, where fits of the expected form

\[
B^\text{stat}_A + \hat{f} \frac{1}{z} + \tilde{f} \frac{1}{z^2} \ln(z) + O(1/z^2)
\]

are also shown. The extrapolation results are
Figure 2. $\hat{B}^{\text{stat}}_A(z)$ in the continuum limit. The dotted curves are fits to the filled symbols, of the form predicted by HQET.

\begin{align}
B^{\text{stat}}_A(\theta = 0.0) &= -0.136(3), \\
B^{\text{stat}}_A(\theta = 0.5) &= -0.137(1).
\end{align}

This agrees with a previous result [9], where matrix elements between quark states were used. Turning the argument around, this means that using the $B^{\text{stat}}_A$ value from [9], HQET describes the heavy quark limit of QCD, independent from the chosen correlation functions. To our knowledge, this is the first time that this fact has been established in an explicit example.

3. DISCRETISATION ERRORS

A further interesting point to study is the quark mass dependence of discretisation errors. We discover that for not too large masses, the discretisation errors both at tree-level and at one-loop order are roughly proportional to $(am^2_{\overline{MS}})^2$. However, for $am^2_{\overline{MS}} > 1/4$, the behaviour of discretisation errors changes completely. This is illustrated in figure 3, where $B^{\text{stat}}_A$ is shown both in the continuum limit and at finite lattice spacing $L/a = 32$. A closer look shows that this behaviour originates from a breakdown of $O(a)$ improvement when the quark mass in lattice units becomes too large, as it was previously observed in a perturbative analysis of the Schrödinger functional coupling [10]. This means that extrapolations in the quark mass should be carried out in the continuum limit rather than at finite lattice spacing.

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