BIRATIONAL SPACES

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Abstract. In this paper we construct the category of birational spaces as the category in which the relative Riemann-Zariski spaces of Tem11 are naturally included. Furthermore we develop an analogue of Raynaud’s theory. We prove that the category of quasi-compact and quasi-separated birational spaces is naturally equivalent to the localization of the category of pairs of quasi-compact and quasi-separated schemes with an affine schematically dominant morphism between them localized with respect to relative blow ups and relative normalizations.

1. Introduction

In the 1930’s and 1940’s Oscar Zariski studied the problem of resolution of singularities for varieties of characteristic zero. He introduced the notion of the Riemann-Zariski space of a finitely generated field extension $k \subset K$, denoted $RZ_K(k)$. This is the space of all valuations on $K/k$ of dimension zero. Later he showed that the Riemann-Zariski space can be obtained as the projective limit of all projective models of $K/k$ [Zar44].

Temkin introduced a relative notion, the relative Riemann-Zariski space, $RZ_Y(X)$ for a separated morphism of quasi-compact and quasi-separated schemes $f : Y \to X$. He defined $RZ_Y(X)$ as the projective limit of the underlying topological spaces, of all the $Y$-modifications of $X$ Tem10Tem11.

Temkin showed that $RZ_Y(X)$ is isomorphic to the space consisting of unbounded $X$-valuations on $Y$ equipped with a suitable topology.

Our first aim in this paper is to provide a categorical approach to RZ spaces through the valuation point of view. Our approach is to first define for given rings $A \to B$ an affinoid birational space $Val(B, A)$ of unbounded $A$-valuations on $B$. Then general birational spaces $Val(Y, X)$ are glued from affinoid ones along affinoid subdomains.

We restrict our study only to the case of affine, schematically dominant morphisms $f : Y \to X$ of quasi-compact and quasi-separated schemes. However this is essentially the same as assuming that $f : Y \to X$ is a separated morphism: by Temkin’s decomposition theorem Tem11 Theorem 1.1.3] any separated morphisms $f : Y \to X$ of quasi-compact and quasi-separated schemes factors as $Y \xrightarrow{j} Z \to X$ where $j : Y \to Z$ is an affine, schematically dominant morphism and $Z \to X$ is proper. It will become clear from the construction that $Val(Y, X) = Val(Y, Z')$ by the valuative criterion for properness, so our results hold for separated morphisms.

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1Zariski originally called it the Riemann manifold [Zar40]. Later Nagata offered the name Zariski-Riemann space Nag02 to avoid confusion with the Riemann manifold of differential geometry. In Tem11 Temkin calls this the Riemann-Zariski space and we follow suite.
Our second aim is to develop an analogue of Raynaud’s theory. Let $R$ be a valuation ring of Krull dimension $1$, complete with respect to the $J$-adic topology generated by a principal ideal $J = (\pi) \subset R$ where $\pi$ is some non-zero element of the maximal ideal of $R$, and $K$ the fraction field of $R$. It is then possible to talk about the category of admissible formal $R$-schemes. On the other hand it is also possible to talk about the category of rigid $K$-spaces. It was Raynaud [Ray74] who suggested to view rigid spaces entirely within the framework of formal schemes. Elaborating the ideas of Raynaud, it is proved in [BL93] that the category of admissible formal $R$-schemes, localized with respect to class of admissible formal blow ups, is naturally equivalent to the category of rigid $K$-spaces which are quasi-compact and quasi-separated.

We will show that the localization of the category of pairs of quasi-compact and quasi-separated schemes with an affine, schematically dominant morphism between them localized with respect to relative blow ups and relative normalizations is naturally equivalent to the category of quasi-compact and quasi-separated birational spaces.

Let $A \subset B$ be commutative rings with unit. We define spaces of pairs of rings $\text{Spa}(B, A)$, and affinoid birational spaces $\text{Val}(B, A)$ which is our main interest in Section 2. We study some of their topological properties and endow $\text{Val}(B, A)$ with two sheaves of rings $\mathcal{O}_{\text{Val}(B, A)} \subset \mathcal{M}_{\text{Val}(B, A)}$ both making $\text{Val}(B, A)$ a locally ringed space. The main highlight of Section 3 is the proof that the functor $\text{Val}$ gives rise to an anti-equivalence from the localization of the category of pairs of rings with respect to relative normalizations to the category of affinoid birational spaces. Also in Section 3 we globalize the construction by introducing the notion of a general birational space. These are topological spaces equipped with a pair of sheaves such that the space is locally ringed with respect to both sheaves and is locally isomorphic to $\text{Val}(A, B)$. Finally, in Section 5 we prove that the localization of the category of pairs of quasi-compact and quasi-separated schemes with an affine schematically dominant morphism between them localized with respect to relative blow ups and relative normalizations is naturally equivalent to the category of quasi-compact and quasi-separated birational spaces. For the last step, Section 4 is dedicated to the further development of the theory of relative blow ups, and, in particular, prove the universal property of relative blow ups.

2. Construction of the Space $\text{Val}(B, A)$

Throughout all rings are assumed to be commutative with unity.

2.1. Valuations on Rings. In this Subsection we fix terminology and collect general known facts about valuations.

Given a totally ordered abelian group $\Gamma$ (written multiplicatively), we extend $\Gamma$ to a totally ordered monoid $\Gamma \cup \{0\}$ by the rules

$$0 \cdot \gamma = \gamma \cdot 0 = 0 \text{ and } 0 < \gamma \quad \forall \gamma \in \Gamma.$$ 

Definition 2.1.1. Let $B$ be a ring and $\Gamma$ a totally ordered group. A valuation $v$ on $B$ is a map $v : B \rightarrow \Gamma \cup \{0\}$ satisfying the conditions

- $v(1) = 1$
- $v(xy) = v(x)v(y) \quad \forall x, y \in B$
- $v(x + y) \leq \max\{v(x), v(y)\} \quad \forall x, y \in B.$
Note that $p = \ker v = \{ b \in B \mid v(b) = 0 \}$ is a prime ideal in $B$.

We furthermore assume that $\Gamma$ is generated, as an abelian group, by $v(B - p)$.

**Remark 2.1.2.** When $B$ is a field the above definition coincides with the classical definition of a valuation with the value group written multiplicatively.

Let $v$ be a valuation on $B$ with kernel $p$. Denote the residue field of $p$ by $k(p)$.

We obtain a diagram

$$
\begin{array}{ccc}
B & \overset{v}{\longrightarrow} & \Gamma \cup \{0\} \\
\downarrow & & \downarrow \\
B/ p & \overset{\bar{v}}{\longrightarrow} & k(p) \\
\end{array}
$$

where $\bar{v} : k(p) \rightarrow \Gamma \cup \{0\}$ is a valuation on $k(p)$ induced by $v$. On the other hand a prime ideal $p \in \text{Spec } B$ and a valuation $\bar{v}$ on the residue field $k(p)$ uniquely determine a valuation $v$ on $B$ with kernel $p$ by setting

$$
v(b) = \begin{cases}
\bar{v}(\bar{b}) & \text{if } b \notin p \\
0 & \text{if } b \in p
\end{cases}
$$

where $\bar{b}$ is the image of $b$ in $k(p)$. Hence giving a valuation $v$ on $B$ is equivalent to giving a prime ideal $p$ and a valuation $\bar{v}$ on the residue field $k(p)$.

Two valuations $v_1, v_2$ on $B$ are said to be equivalent if $\ker v_1 = \ker v_2 = p$ and the induced valuations $\bar{v}_1, \bar{v}_2$ on $k(p)$ are equivalent in the classical sense.

We will identify equivalent valuations.

With this convention a valuation $v$ on $B$ with kernel $p$ uniquely defines a valuation ring contained in $k(p)$ by

$$R_v = \{ x \in k(p) \mid \bar{v}(x) \leq 1 \}.$$

Hence a valuation $v$ on $B$ is equivalent to a diagram

$$
\begin{array}{ccc}
B & \overset{v}{\longrightarrow} & k(p) \\
\downarrow & & \downarrow \\
R_v & & \\
\end{array}
$$

**Definition 2.1.3.** Let $B$ be a ring, $A$ a subring and $v$ a valuation on $B$. We call $v$ an $A$-valuation on $B$ if $v(a) \leq 1$ for every $a \in A$.

Assume $v$ is an $A$-valuation with kernel $p$. Set $q = p \cap A$. From the condition $v(a) \leq 1 \ \forall a \in A$ we obtain a commutative diagram

$$
\begin{array}{ccc}
R_v & \overset{\sim}{\longrightarrow} & k(p) \\
\downarrow & & \downarrow \\
R_v \cap k(q) & \overset{\sim}{\longrightarrow} & k(q) \\
\end{array}
$$

\[\text{i.e. they have the same valuation ring or, equivalently, there is an order preserving group isomorphism between their images compatible with the valuations.}\]
We conclude that every $A$-valuation $v$ on $B$ uniquely defines a commutative diagram

$$
\begin{array}{c}
B \\
\downarrow \Phi \\
A
\end{array} \rightarrow
\begin{array}{c}
k(p) \\
\uparrow \\
R_v.
\end{array}
$$

Conversely any such diagram defines an $A$-valuation $v$ on $B$ and we are justified in identifying the $A$-valuation $v$ on $B$ with the 3-tuple $(p, R_v, \Phi)$.

2.2. The Auxiliary Space $\text{Spa}(B, A)$. For completeness and consistency of notation we collect here results regarding valuation spectra. The main reference of this subsection is [Hub93].

Definition 2.2.1. For any pair of rings $A \subset B$ we set

$$\text{Spa}(B, A) = \{A\text{-valuations on } B\}.$$ 

Fix a pair of rings $A \subset B$.

We provide $\text{Spa}(B, A)$ with a topology. For any $a, b \in B$ set

$$U_{a,b} = \{v \in \text{Spa}(B, A) \mid v(a) \leq v(b) \neq 0\}.$$ 

The topology is the one generated by the sub-basis $\{U_{a,b}\}_{a,b \in B}$.

Given another pair of rings $A' \subset B'$ and a homomorphism of rings $\varphi : B \to B'$ that satisfies $\varphi(A) \subset A'$, composition with $\varphi$ gives rise to the pull back map

$$\varphi^* : \text{Spa}(B', A') \to \text{Spa}(B, A).$$

Specifically given an $A'$-valuation $v = (p, R_v, \Phi) \in \text{Spa}(B', A')$, then $v \circ \varphi$ is a valuation on $B$. Since $\varphi(A) \subset A'$, $v \circ \varphi$ is an $A$-valuation. So indeed $\varphi^*(v) = v \circ \varphi \in \text{Spa}(B, A)$. Clearly its kernel is $\varphi^{-1}(p)$. Now, we have a commutative diagram

$$
\begin{array}{c}
B' \\
\downarrow \varphi \\
B \\
\downarrow \phi \\
A'
\end{array} \rightarrow
\begin{array}{c}
k(p) \\
\uparrow \\
k(\varphi^{-1}(p))
\end{array}
$$

$$
\begin{array}{c}
A \\
\downarrow \varphi^*(\Phi) \\
\uparrow \\
R_v
\end{array} \rightarrow
\begin{array}{c}
A' \\
\downarrow \Phi \\
\uparrow \\
R_{v \circ \varphi}
\end{array}
$$

It is clear that $R_{v \circ \varphi} = R_v \cap k(\varphi^{-1}(p))$ and that the ring map $\varphi^*(\Phi)$ is completely determined by $\Phi$ and $\varphi$. To conclude, $\varphi^*$ takes the point $(p, R_v, \Phi) \in \text{Spa}(B', A')$ to the point $(\varphi^{-1}(p), R_v \cap k(\varphi^{-1}(p)), \varphi^*(\Phi)) \in \text{Spa}(B, A)$.

Clearly $U_{\varphi^{-1}(a), \varphi^{-1}(b)} = \varphi^{-1}(U_{a,b})$ for any $a, b \in B$. We obtain:

Lemma 2.2.2. Let $A \subset B$ and $A' \subset B'$ be rings. For a homomorphism $\varphi : B \to B'$ satisfying $\varphi(A) \subset A'$ the pull back map $\varphi^* : \text{Spa}(B', A') \to \text{Spa}(B, A)$ is continuous.
Let \( b, a_1, \ldots, a_n \in B \) and assume that \( b, a_1, \ldots, a_n \) generate the unit ideal. Set \( B' = B_b \). Denote the canonical map \( B \to B' \) by \( \varphi_b \), and set \( A' = \varphi_b(A)[\frac{a_1}{b}, \ldots, \frac{a_n}{b}] \).

Obviously \( A' \subset B' \) so \( \text{Spa}(B', A') \) is defined. We also obtain a commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\varphi_b} & B' \\
\downarrow & & \downarrow \\
A & \to & A'
\end{array}
\]

which gives rise to the pull back map

\[
\varphi_b^* : \text{Spa}(B', A') \to \text{Spa}(B, A).
\]

In this case the pull back map is injective: if \( v = (p, R_v, \Phi) \in \text{Spa}(B, A) \) is in the image and \( v' = (p', R_{v'}, \Phi') \in \text{Spa}(B', A') \) maps to \( v \), then necessarily \( b \notin p \) and \( p' = p B' \). Hence \( k(p') = k(p) \), from which follows that \( R_{v'} = R_v \) and we have the diagram

\[
\begin{array}{ccc}
\text{Spec} k(p) & \xrightarrow{\Phi'} & \text{Spec} A' \\
\downarrow \quad & & \downarrow \quad \Phi \\
\text{Spec} R_v & \to & \text{Spec} A.
\end{array}
\]

Since \( \text{Spec} A' \to \text{Spec} A \) is separated, \( \Phi' \) is unique by the valuative criterion for separateness, so \( v' = (p', R_{v'}, \Phi') \) is unique.

For such \( A' \subset B' \) we regard \( \text{Spa}(B', A') \) as a subset of \( \text{Spa}(B, A) \).

Consider \( v' \in \text{Spa}(B', A') \) as an element of \( \text{Spa}(B, A) \). As \( \frac{a_i}{b} \in A' \) for all \( i = 1, \ldots, n \), we see that \( v'(a_i) \leq v'(b) \neq 0 \) for every \( i = 1, \ldots, n \).

Hence \( \text{Spa}(B', A') \subset \{ v \in \text{Spa}(B, A) \mid v(a_i) \leq v(b) \quad \forall \ 1 \leq i \leq n \} \).

Conversely, let \( v' \in \{ v \in \text{Spa}(B, A) \mid v(a_i) \leq v(b) \quad \forall \ 1 \leq i \leq n \} \). Since \( (b, a_1, \ldots, a_n) = B \) there are \( c_0, c_1, \ldots, c_n \) in \( B \) such that \( 1 = c_0 b + c_1 a_1 + \ldots + c_n a_n \).

Applying \( v' \) we obtain

\[
1 = v'(1) = v'(c_0 b + c_1 a_1 + \ldots + c_n a_n) \leq \\
\leq \max\{v'(c_0 b), v'(c_1 a_1), \ldots, v'(c_n a_n)\} \leq v'(b) \max\{v'(c_i)\}
\]

so necessarily \( v'(b) \neq 0 \). Hence we can extend \( v' \) to a valuation on \( B' \) by \( v'(\frac{b'}{b}) = \frac{v'(b')}{v'(b)} \) for any \( b' \in B \). Since \( v'(c) \leq 1 \quad \forall \ c \in A \) we have \( v'(A') \leq 1 \) so \( v' \in \text{Spa}(B', A') \).

It follows that \( \{ v \in \text{Spa}(B, A) \mid v(a_i) \leq v(b) \quad \forall \ 1 \leq i \leq n \} \subset \text{Spa}(B', A') \), hence we have equality. Furthermore we have

\[
\text{Spa}(B', A') = \{ v \in \text{Spa}(B, A) \mid v(a_i) \leq v(b) \quad \forall \ 1 \leq i \leq n \} = \cap_{i=1}^n U_{a_i, b}.
\]

Hence \( \text{Spa}(B', A') \) is an open subset of \( \text{Spa}(B, A) \). We obtain:

**Lemma 2.2.3.** Let \( b, a_1, \ldots, a_n \in B \) and assume that \( b, a_1, \ldots, a_n \) generate the unit ideal. Then

\[
\text{Spa} \left( B_b, \varphi_B(A) \left[ \frac{a_1}{b}, \ldots, \frac{a_n}{b} \right] \right) = \{ v \in \text{Spa}(B, A) \mid v(a_i) \leq v(b) \quad \forall \ 1 \leq i \leq n \}
\]

and \( \text{Spa} \left( B_b, \varphi_B(A) \left[ \frac{a_1}{b}, \ldots, \frac{a_n}{b} \right] \right) \) is an open subset of \( \text{Spa}(B, A) \).

**Definition 2.2.4.** We call such a set a *rational domain* of \( \text{Spa}(B, A) \) and denote it by \( \mathcal{R}(\{a_1, \ldots, a_n\}/b) \).
Remark 2.2.5. Let $a_0, \ldots, a_n, a'_0, \ldots, a'_m \in B$ such that both $a_0, \ldots, a_n$ and $a'_0, \ldots, a'_m$ generate the unit ideal. By Lemma 2.2.3

\[
\mathcal{R}\left(\{a_i\}_i^n / a_0\right) = \{v \in \text{Spa}(B, A) \mid v(a_i) \leq v(a_0) \quad \forall 1 \leq i \leq n\},
\]

\[
\mathcal{R}\left(\{a'_j\}_j^m / a'_0\right) = \{v \in \text{Spa}(B, A) \mid v(a'_i) \leq v(a'_0) \quad \forall 1 \leq j \leq m\}
\]

and

\[
\mathcal{R}\left(\{a_i a'_j\}_i,j / a_0 a'_0\right) = \\
\{v \in \text{Spa}(B, A) \mid v(a_i a'_j) \leq v(a_0 a'_0) \quad \forall 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}.
\]

Hence

\[
\mathcal{R}\left(\{a_i\}_i^n / a_0\right) \cap \mathcal{R}\left(\{a'_j\}_j^m / a'_0\right) = \mathcal{R}\left(\{a_i a'_j\}_i,j / a_0 a'_0\right).
\]

Remark 2.2.6. Let $b \in B$. If $b$ is nilpotent there is some $n > 0$ such that $b^n = 0$. For any valuation $v$ we have $0 = v(b^n) = v(b) \cdot v(b) = 0$. From this it follows that for any $a_1, \ldots, a_n \in B$ such that $(b, a_1, \ldots, a_n) = B$ we have $\mathcal{R}(\{a_1, \ldots, a_n\} / b) = 0$. If $b$ is not nilpotent, there is a prime ideal $p$ not containing $b$. Now for any $a_1, \ldots, a_n \in B$ such that $(b, a_1, \ldots, a_n) = B$ the rational domain $\mathcal{R}(\{a_1, \ldots, a_n\} / b)$ contains the trivial valuation of $k(p)$. Concluding, we have

\[
\mathcal{R}(\{a_1, \ldots, a_n\} / b) = 0 \iff b \text{ is a nilpotent element}.
\]

By a rational covering we mean the open cover defined by some $a_1, \ldots, a_n \in B$ generating the unit ideal, that is the rational domains \( \{\mathcal{R}(\{a_i\}_i^n / a_0)\}_i \).

In [Hub93], Huber defines the valuation spectrum of a ring $B$

\[
\text{Spv}(B) = \{\text{valuations on } B\}.
\]

He provides it with the topology generated by the sub-basis consisting of sets of the form $\{v : v(a) \leq v(b) \neq 0\}$ for all $a, b \in B$. Huber proves in [Hub93, 2.2] that $\text{Spv}(B)$ is a spectral space. Clearly our $\text{Spa}(B, A)$ is a subspace of Huber’s $\text{Spv}(B)$.

Lemma 2.2.7. The topological space $\text{Spa}(B, A)$ is spectral. In particular it is quasi-compact and $T_0$.

Proof. Since a closed subspace of a spectral space is again spectral it is enough to show that $\text{Spa}(B, A)$ is closed in $\text{Spv}(B)$. Following Huber’s argument we just need to show that the set of binary relations $\{\phi(\text{Spv}B) \mid a, \forall a \in A \text{ closed in } \phi(\text{Spv}B) \}$.

2.3. The Space $\text{Val}(B, A)$. We say that a valuation $v : B \to \Gamma \cup \{0\}$ is bounded if there is an element $\gamma \in \Gamma$ such that $v(b) < \gamma$ for every $b \in B$.

Definition 2.3.1. For any pair of rings $A \subset B$ we set

\[
\text{Val}(B, A) = \{v \in \text{Spa}(B, A) \mid v \text{ is unbounded}\}
\]

with the induced subspace topology from $\text{Spa}(B, A)$.
As a subspace of a $T_0$ space, $\text{Val}(B, A)$ is also a $T_0$ space.

For a valuation $v$ on $B$ with abelian group $\Gamma$ we denote by $c_{\Gamma_v}$ the convex subgroup of $\Gamma$ generated by $\{ v(b) \mid b \in B \ 1 \leq v(b) \}$. For any convex subgroup $\Lambda$ we define a map $v' : B \to \Lambda \cup \{0\}$ by $v'(b) = \begin{cases} v(b) & \text{if } v(b) \in \Lambda \\ 0 & \text{if } v(b) \notin \Lambda \end{cases}$. It is easily seen that $v'$ is a valuation on $B$ if and only if $c_{\Gamma_v} \subset \Lambda$.

The valuation $v'$ obtained in this way is called a primary specialization of $v$ associated with $\Lambda$ [HM94 §1.2]. Note that $\ker v \subset \ker v'$. It is easy to see that a valuation $v$ is not bounded if and only if it has no primary specialization other than itself.

**Lemma 2.3.2.** Let $A \subset B$ and $A' \subset B'$ be rings, and $\varphi : B \to B'$ a homomorphism satisfying $\varphi(A) \subset A'$. Assume $v, w \in \text{Spa}(B', A')$ such that $w$ is a primary specialization of $v$. Then $\varphi^*(w)$ is a primary specialization of $\varphi^*(v)$.

**Proof.** Assume that $v : B' \to \Gamma \cup \{0\}$ and that $\Lambda$ is the convex subgroup of $\Gamma$ associated with $w$. Then for any $b' \in B'$ we have $w(b') = \begin{cases} v(b') & \text{if } v(b') \in \Lambda \\ 0 & \text{if } v(b') \notin \Lambda \end{cases}$. It now follows that for any $b \in B$ we have $w(\varphi(b)) = \begin{cases} v(\varphi(b)) & \text{if } v(\varphi(b)) \in \Lambda \\ 0 & \text{if } v(\varphi(b)) \notin \Lambda \end{cases}$.

For $v \in \text{Spa}(B, A)$, let $P_v$ be the subset of all primary specializations of $v$. Primary specialization induces a partial order on $P_v$ by the rule $u \leq w$ if $u$ is a primary specialization of $w$ for $u, w \in P_v$.

**Proposition 2.3.3.** For any $v \in \text{Spa}(B, A)$, the set $P_v$ of primary specializations of $v$ is totally ordered and has a minimal element.

**Proof.** Let $v : B \to \Gamma \cup \{0\}$ be a valuation on $B$. Let $w : B \to \Lambda \cup \{0\}$ and $u : B \to \Delta \cup \{0\}$ be two distinct primary specializations of $v$. We may regard $\Lambda$ and $\Delta$ as convex subgroups of $\Gamma$, so one is contained in the other. As both $w$ and $u$ are primary specialization of $v$, both $\Lambda$ and $\Delta$ contain $c_{\Gamma_v}$. Assume $\Delta \subset \Lambda$. We want to show that $u$ is a primary specialization of $w$, i.e. $u(b) = \begin{cases} w(b) & \text{if } w(b) \in \Delta \\ 0 & \text{if } w(b) \notin \Delta \end{cases}$.

For any $b \in B$ if $w(b) > 1$ then $v(b) = w(b)$, hence $c_{\Gamma_w} \subset c_{\Gamma_v}$. Conversely if $v(b) > 1$ then since $c_{\Gamma_v} \subset \Lambda$ we have $w(b) = v(b)$, hence $c_{\Gamma_w} = c_{\Gamma_v}$. It follows that $\Delta$ is a convex subgroup of $\Lambda$ containing $c_{\Gamma_w}$.

For any $b \in B$, if $w(b) \in \Delta$ then $w(b) = v(b) \in \Delta$. It follows that $u(b) = v(b) = w(b)$. If $w(b) \notin \Delta$ then either $w(b) = 0$ or $0 \neq w(b) \in \Lambda$. If $w(b) = 0$ then $v(b) \in \Lambda$, hence $v(b) \in \Delta$ and $w(b) = 0$. If $w(b) \neq 0$ then $w(b) = v(b) \notin \Delta$, so $u(b) = 0$.

The minimal element of $P_v$ is the primary specialization associated with $c_{\Gamma_v}$. □

Next we give an algebraic criterion for a valuation $v \in \text{Spa}(B, A)$ to be in $\text{Val}(B, A)$.

**Lemma 2.3.4.** Let $v = (p, R_v, \Phi) \in \text{Spa}(B, A)$. Then $v = (p, R_v, \Phi) \in \text{Val}(B, A)$ if and only if the canonical map $B \otimes_A R_v \to k(p)$ is surjective.

**Proof.** Assume that $B \otimes_A R_v \to k(p)$ is surjective. Since we assume that $\Gamma$ is generated by the image of $B - p$, for any $1 < \gamma \in \Gamma$ there is $0 \neq f \in k(p)$ satisfying
\[ \gamma = \bar{v}(f). \] Let \( b_i \in B, r_i \in R_v \) for \( i = 1, \ldots, n \) such that \( \sum b_i \otimes r_i \in B \otimes_A R_v \) maps to \( f \) in \( k(p) \). Assume \( v(b_1) = \max\{v(b_i)\} \). Denote the image of \( b_i \) in \( k(p) \) by \( \bar{b}_i \).  

Now, as \( \bar{v}(r_i) \leq 1 \), we have

\[ \gamma = \bar{v}(f) = \bar{v}(\sum \bar{b}_i r_i) \leq \max\{\bar{v}(\bar{b}_i r_i)\} \leq \max\{\bar{v}(\bar{b}_i)\} = v(b_1). \]

Hence \( \gamma \) does not bound \( v \).

Conversely, for \( v = (p, R_v, \Phi) \in \text{Val}(B, A) \) we have a diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\bar{v}} & B \otimes_A R_v \\
\downarrow & & \downarrow \\
A & \xrightarrow{\phi} & R_v.
\end{array}
\]

For any \( f \in k(p) \), if \( \bar{v}(f) \leq 1 \) then \( f \in R_v \) and \( 1 \otimes f \in B \otimes_A R_v \) maps to \( f \in k(p) \). Assume \( \bar{v}(f) > 1 \). As \( v \in \text{Val}(B, A) \) we see that \( \Gamma = c\Gamma_v \), so there exists \( d \in B \) satisfying \( \bar{v}(f) \leq v(d) \). It follows that \( f/\bar{d} \in R_v \) where \( \bar{d} \) is the image of \( d \) in \( k(p) \) and \( d \otimes f/\bar{d} \in B \otimes_A R_v \) maps to \( f \in k(p) \).

**Remark 2.3.5.** Since for any \( A \subset B \) and \( R_v \) we always have

\[ B \otimes_Z R_v \xrightarrow{\phi} B \otimes_A R_v, \]

we can replace in the above lemma \( B \otimes_A R_v \) with \( B \otimes_Z R_v \).

**Remark 2.3.6.** Equivalently we can say that \( v \) is in \( \text{Val}(B, A) \) if and only if

\[ \text{Spec } k(p) \rightarrow \text{Spec } B \times_{\text{Spec } A} \text{Spec } R_v, \]

or equivalently

\[ \text{Spec } k(p) \rightarrow \text{Spec } B \times \text{Spec } R_v, \]

is a closed immersion.

As we have seen, given another pair \( A' \subset B' \) and a homomorphism

\[
\begin{array}{ccc}
B & \xrightarrow{\phi} & B' \\
\downarrow & & \downarrow \\
A & \xrightarrow{\phi} & A'
\end{array}
\]

composition with \( \phi \) induces a map \( \phi^* : \text{Spa}(B', A') \rightarrow \text{Spa}(B, A) \). However \( \phi^* \) does not necessarily restrict to a map \( \text{Val}(B', A') \rightarrow \text{Val}(B, A) \).

**Lemma 2.3.7.** Let

\[
\begin{array}{ccc}
B & \xrightarrow{\phi} & B' \\
\downarrow & & \downarrow \\
A & \xrightarrow{\phi} & A'
\end{array}
\]

as above. If the induced homomorphism \( B \otimes_A A' \rightarrow B' \) is integral then composition with \( \phi \) induces a map \( \phi^* : \text{Val}(B', A') \rightarrow \text{Val}(B, A) \).
Proof. For any \( v = (p, R_v, \Phi) \in Val(B', A') \) set
\[
\varphi^*(v) = v \circ \varphi = (\varphi^{-1}(p), R_v \circ \varphi, \varphi^*(\Phi)) = (\varphi^*(p), \varphi^*(R_v), \varphi^*(\Phi)).
\]
We know that \( \varphi^*(v) \in Spa(B, A) \). In order to show that \( \varphi^*(v) \in Val(B, A) \), by Lemma 2.3.4, we need to show that \( B \otimes_A \varphi^*(R_v) \to k(\varphi^*(p)) \) is surjective.
The homomorphism \( \varphi \) gives rise to the diagram
\[
\begin{array}{ccc}
B' & \xrightarrow{\varphi} & k(p) \\
\downarrow \varphi & & \downarrow k(\varphi^*(p)) \\
B & \xrightarrow{k(\varphi^*(p))} & \Phi \\
\downarrow \varphi & & \downarrow k(\varphi^*(p)) \\
A' & \xrightarrow{\Phi} & R_v \\
\downarrow \varphi^*(\Phi) & & \downarrow k(\varphi^*(p)) \\
A & \xrightarrow{\varphi^*(\Phi)} & R_{\varphi^*(v)}
\end{array}
\]
from which we see that there is a diagram
\[
\begin{array}{ccc}
B' \otimes_{A'} R_v & \xrightarrow{k(p)} & k(p) \\
\downarrow & & \downarrow \\
B \otimes_A R_{\varphi^*(v)} & \xrightarrow{k(\varphi^*(p))} & k(\varphi^*(p)).
\end{array}
\]
The upper horizontal arrow is surjective by Lemma 2.3.4.

For any \( \alpha \in k(\varphi^*(p)) \), if \( \varphi^*(v)(\alpha) \leq 1 \) then \( \alpha \) is already in \( R_{\varphi^*(v)} \) (recall that \( \varphi^*(v) \) is the induced valuation on \( k(\varphi^*(p)) \)).

If \( \varphi^*(v)(\alpha) > 1 \) then there is \( b' \in B' \) such that \( \varphi^*(v)(\alpha) \leq v(b') \) since \( \varphi(\alpha) \in k(p) \) and \( v \) is in \( Val(B', A') \). Since \( B \otimes_A A' \to B' \) is integral we have \( x_0, \ldots, x_{n-1} \in Im (B \otimes_A A' \to B') \) such that \( b^n + x_{n-1}b^{n-1} + \ldots + x_0 = 0 \). As \( v(0) = 0 \) there is some \( 0 \leq i \leq n-1 \) such that \( v(b^n) \leq v(x_ib^n) \). It follows that \( v(b') \leq v(b')^{n-i} \leq v(x_i) \).

Now, there are \( a_1, \ldots, a_m \in A' \) and \( b_1, \ldots, b_m \in B \) such that \( \sum a_j \otimes b_j \) maps to \( x_i \), so \( v(b') \leq \max \{v(a_j) \cdot v \circ \varphi(b_j)\} \leq \max \{v \circ \varphi(b_j)\} \). The last inequality is due to the fact that \( v(a) \leq 1 \) for every \( a \in A' \). Choose \( b \in \{b_1, \ldots, b_m\} \) such that \( \varphi^*(v)(b) = v \circ \varphi(b) = \max \{v \circ \varphi(b_j)\} \). Now we have \( \varphi^*(v)(\alpha) \leq \varphi^*(v)(b) \). Denoting the image of \( b \) in \( k(\varphi^*(p)) \) by \( \tilde{b} \), we have \( \varphi^*(v)(\alpha) \leq \varphi^*(v)(\tilde{b}) \) or in other words \( \varphi^*(v)(\tilde{b}) \leq 1 \), hence \( \tilde{b} \in R_{\varphi^*(v)} \) and \( b \otimes \tilde{b} \in B \otimes_A R_{\varphi^*(v)} \) maps to \( \alpha \).

\[\square\]

2.4. Rational Domains. Set \( \mathcal{X} = Val(B, A) \).

For \( b, a_1, \ldots, a_n \in B \) generating the unit ideal we defined a rational domain in \( Spa(B, A) \) as
\[
\mathcal{R}(\{a_1, \ldots, a_n\}/b) = \{v \in Spa(B, A) | v(a_i) \leq v(b) \}.
\]
We call the set \( \mathcal{R}(\{a_1, \ldots, a_n\}/b) \cap \mathcal{X} \) a rational domain in \( \mathcal{X} \) and denote it by \( \mathcal{X}(\{a_1, \ldots, a_n\}/b) \).

Obviously \( \mathcal{X}(\{a_1, \ldots, a_n\}/b) = Val(B_b, \varphi_b(A) [\frac{a_1}{b}, \ldots, \frac{a_n}{b}]) \).
Proposition 2.4.1. The rational domains of \( \mathfrak{X} \) form a basis for the topology.

Proof. Let \( w \in \mathfrak{X} \) and \( U \) an open neighbourhood of \( w \) in \( \operatorname{Spa}(B, A) \) (i.e. \( U \cap \mathfrak{X} \) is an open neighbourhood of \( w \) in \( \mathfrak{X} \)). By the definition of the topology there is a natural number \( N \) and \( a_i, b_i \in B \) such that \( v(a_i) \leq v(b_i) \neq 0 \) for each \( i = 1, \ldots, N \) such that \( w \in \bigcap_i U_{a_i, b_i} \subset U \). By taking the products \( \prod c_i \) where \( c_i \in \{a_i, b_i\} \) and \( b = \prod b_i \), replacing \( N \) with a suitable natural number, the \( a_i \)-s with the above products and shrinking \( U \), we may assume that we have \( a_1, \ldots, a_N, b \in B \) satisfying \( w \in \cap_i U_{a_i, b_i} = U \).

As \( w(b) \neq 0 \) and \( w \in \mathfrak{X} \) we see that \( w(b) \in \mathfrak{d} \Gamma_w \). Since \( w(b)^{-1} \) is not a bound of \( w \), there exists \( d \in B \) such that \( w(b)^{-1} \leq w(d) \). It follows that \( 1 = w(1) \leq w(db) \).

\[
w \in U \cap U_{1, db} = \{ v \in \operatorname{Spa}(B, A) \mid v(a_i) \leq v(b) \neq 0, 1 \leq v(db) \} = \mathcal{R}\{(da_1, \ldots, da_n, 1)/db\}.
\]

Hence \( w \in \mathcal{R}\{(da_1, \ldots, da_n, 1)/db\} \cap \mathfrak{X} = \mathfrak{X}(\{(da_1, \ldots, da_n, 1)/db\}) \subset U \cap \mathfrak{X} \).

It remains to show that the rational domains satisfy the intersection condition of a basis. However it follows from Remark 2.2.5 that

\[
\mathfrak{X}(\{(a_i)^{n_i}_{i=1}/a_0\}) \cap \mathfrak{X}(\{(a_i')^{m_i}_{j=1}/a_0'\}) = \mathfrak{X}(\{(a_i a_j')^{n_i m_j}_{i,j=1}/a_0 a_0'\}).
\]

\[\Box\]

In [Tem11], Temkin defines the semi-valuation ring \( S_v \) for a valuation \( v = (p, R_v, \Phi) \) on \( B \). It is the pre-image of the valuation ring \( R_v \) in the local ring \( B_p \). The valuation on \( B \) induces a valuation on \( S_v \). We call \( B_p \) the semi-fraction ring of \( S_v \).

We briefly recall several properties of a semi-valuation ring (for details see [Tem11 §2]).

**Remark 2.4.2.** If \( S_v \) is a semi-valuation ring with semi-fraction ring \( B_p \) then

(i) the maximal ideal \( p B_p \) of \( B_p \) is contained in \( S_v \).

(ii) considering \( v \) as a valuation on \( B_p \) or \( S_v \) we have \( \ker v = p B_p \).

(iii) \( S_v / \ker v = B_p \).

(iv) \( S_v / \ker v = R_v \), in particular \( S_v \) is a local ring.

(v) for any pair \( g, h \in S_v \) such that \( v(g) \leq v(h) \neq 0 \) we have \( g \in hS_v \).

(vi) for any co-prime \( g, h \in B_p \) (i.e. \( gB_p + hB_p = B_p \)), either \( g \in hS_v \) or \( h \in gS_v \).

(vii) the converse of (vi) is also true: for a pair of rings \( C \subset D \), if for any two co-prime elements \( g, h \in D \) either \( g \in hC \) or \( h \in gC \) then there exists a valuation on \( D \) such that \( C \) is a semi-valuation ring of \( v \) and \( D \) is its semi-fraction ring.

**Corollary 2.4.3.** Let \( v = (p, R_v, \Phi) \in \operatorname{Spa}(B, A) \). Then \( v = (p, R_v, \Phi) \in \operatorname{Val}(B, A) \) if and only if the canonical map \( B \otimes_A S_v \rightarrow k(p) \) is surjective.

**Proof.** As \( S_v \) is the pull back of \( R_v \) in \( B_p \), the canonical map \( B \otimes_A S_v \rightarrow k(p) \) factors through \( B \otimes_A R_v \). Since \( S_v / p = R_v \), the ring map \( B \otimes_A S_v \rightarrow B \otimes_A R_v \) is always surjective. Hence \( B \otimes_A S_v \rightarrow k(p) \) is surjective if and only if \( B \otimes_A R_v \rightarrow k(p) \) is surjective. Now the result follows from Lemma 2.3.3. \( \Box \)

Let us study how semi-valuation rings behave under pullback.
Remark 2.4.4. Let $A \subset B$ and $A' \subset B'$ be rings and $\varphi : B \to B'$ a ring homomorphism such that

$$
\begin{array}{c}
\xymatrix{
B \ar[r]^-\varphi & B' \\
A \ar[u] & A' \ar[u]
}
\end{array}
$$

commutes. Consider the map $\varphi^* : Spa(B', A') \to Spa(B, A)$. Let $v = (p, R_v, \Phi) \in Spa(B', A')$, then $\varphi^*(v) = v \circ \varphi$ and $R_{\varphi^*(v)} = R_v \cap k(\varphi^{-1}(p))$. We have a diagram

$$
\begin{array}{c}
\xymatrix{
B' \ar[rr]^-{B_{\varphi^{-1}(p)}} & & k(\varphi^*(p)) \\
\ar[rr]^-{B_{\varphi^*(v)}} & & R_{\varphi^*(v)} \\
S_v \ar[u] & & R_v \ar[u] \\
S_{\varphi^*(v)} \ar[u] & & \ar[u]
}
\end{array}
$$

from which we see that $B_{\varphi^{-1}(p)} \to B_p'$ restricts to a local homomorphism $S_{\varphi^*(v)} \to S_v$ of semi-valuation rings.

Given two valuations $v = (p, R_v, \Phi), w = (q, R_w, \Psi) \in Spa(B, A)$ with $p \subset q$, we would like to know if $w$ a primary specialization of $v$.

Assume $v : B \to \Gamma \cup \{0\}$ and $w : B \to \Delta \cup \{0\}$. Then it follows that $\Gamma = v(B_p^\times)$ and $\Delta = w(B_q^\times)$. Since $p \subset q$ we have a canonical homomorphism $B_q \to B_p$.

Lemma 2.4.5. With the above notation, $w$ is a primary specialization of $v$ if and only if the canonical homomorphism $B_q \to B_p$ restricts to a local homomorphism $S_w \to S_v$ of semi-valuation rings.

Proof. If $w$ is a primary specialization of $v$, then $\Delta$ is a convex subgroup of $\Gamma$

containing $\cap \Gamma_v$ and $w(b) = \begin{cases} 
v(b) & \text{if } v(b) \in \Delta, \\
0 & \text{if } v(b) \notin \Delta.
\end{cases}$

For any $x \in S_w \subset B_q$ there are $b, s \in B$, $s \notin q = \ker w$ such that $x = \frac{b}{s}$. Since $x \in S_w$ we have $w(b) \leq w(s) \neq 0$. Again from $p \subset q$ we see that $s \notin p = \ker v$ and by the definition of a primary specialisation $w(s) = v(s)$. If $w(b) \neq 0$ then, again $w(b) = v(b)$. If $w(b) = 0$ then $v(b) \notin \Delta$. Since $\cap \Gamma_v \subset \Delta$ then $v(b) < \Delta$, in particular $v(b) < v(s)$. In either case we have $v(b) \leq v(s)$ so the image of $x = \frac{b}{s}$ in $B_p$ is in $S_v$. Furthermore if $x$ is in the maximal ideal of $S_w$ i.e. $w(x) < 1$ then the same is true for its image in $S_v$, meaning that the homomorphism $B_q \to B_p$ restricts to a local homomorphism $S_w \to S_v$.

Conversely assume that we have a diagram

$$
\begin{array}{c}
\xymatrix{
B_q \ar[r] & B_p \\
S_w \ar[u] & S_v \ar[u]
}
\end{array}
$$
with the bottom arrow a local homomorphism. Define \( \alpha : \Delta \to \Gamma \) by sending \( w(x) \in \Delta \) (\( x \in B_q^\times \)) to \( v(x) \in \Gamma \). Let \( x_1, x_2 \in B_q^\times \) such that \( w(x_1) = w(x_2) \). All elements of \( B_q^\times \) are invertible so \( x_1x_2^{-1} \) is also in \( B_q^\times \). Obviously \( w(x_1x_2^{-1}) = 1 \) so \( x_1x_2^{-1} \in S_w \). Since \( S_w \to S_v \) is a local homomorphism we also have \( v(x_1)x_2^{-1} = 1 \). Hence \( v(x_1) = v(x_2) \) and \( \alpha \) is well defined. Since \( B_q^\times \) is a multiplicative subset of \( B_q \) and valuations are multiplicative, \( \alpha \) is also multiplicative. As \( 1 = w(1) = v(1) \) we get that \( \alpha(1) = 1 \) i.e. \( \alpha \) is a group homomorphism.

Note that if \( x = \frac{a}{b} \in B_q - S_w \) then its image under \( B_q \to B_p \) lies in \( B_p - S_v \), since if \( w(x) > 1 \) and \( v(x) \leq 1 \) then in particular \( x^{-1} \in S_w \) and \( w(x^{-1}) < 1 \). By the locality of the homomorphism we have \( v(x^{-1}) < 1 \). But this is impossible since it would imply that \( 1 = v(1) = v(xx^{-1}) < 1 \). Now if \( x \in B_q^\times \) with \( v(x) = 1 \) i.e. its image under \( B_q \to B_p \) has value 1, then \( x \) is already in \( S_w \) and by locality of the homomorphism we have \( v(x) = 1 \). Hence \( \alpha \) is an injection and we may regard \( \Delta \) as a subgroup of \( \Gamma \).

It is now clear that \( w(b) = \begin{cases} v(b) & \text{if } v(b) \in \Delta \\ 0 & \text{if } v(b) \notin \Delta \end{cases} \). \( \square \)

**Observation 2.4.6.** Let \( U = \mathfrak{X}(\{a_1, \ldots, a_n\}/b) \) be a rational domain. Then for any valuation \( v \in U \) we have \( 1 \leq v(b) \). For if not then \( \varphi(b) \in S_v \subset B_p \) with \( \varphi : B \to B_p \). It then follows that the ideal generated by \( \varphi(b) \) is a proper ideal of the semi-valuation ring \( S_v \). As \( v(a_i) \leq v(b) \) we have \( \varphi(a_i) \in S_v \). Furthermore \( \varphi(a_i) \in \varphi(b)S_v \) by Remark 2.4.2 (c). But \( b, a_1, \ldots, a_n \in B \) generate the unit ideal so \( \varphi(b), \varphi(a_1), \ldots, \varphi(a_n) \in B_p \) generate the unit ideal which is a contradiction.

**Theorem 2.4.7 (Transitivity of Rational Domains).** Let \( \mathfrak{X}' \) be a rational domain in \( \mathfrak{X} \) and \( \mathfrak{X}'' \) a rational domain in \( \mathfrak{X}' \). Then \( \mathfrak{X}'' \) is a rational domain in \( \mathfrak{X} \).

**Proof.**

**Case 1.** \( \mathfrak{X}' = \mathfrak{X}(\{a_i\}_{i=1}^n/1) \) and \( \mathfrak{X}'' = \mathfrak{X}'(\{b_j\}_{j=1}^m/1) \)

Then we have \( a_1, \ldots, a_n \in B \) and

\[
\mathfrak{X}' = \{ v \in \mathfrak{X} \mid v(a_i) \leq 1 \} = Val(B', A')
\]

with \( B' = B \) and \( A' = A[a_1, \ldots, a_n] \). We also have \( b_1, \ldots, b_m \in B' \) and

\[
\mathfrak{X}'' = \mathfrak{X}'(\{b_j\}_{j=1}^m/1) = Val(B'', A'')
\]

with \( B'' = B' = B \) and \( A'' = A[a_1, \ldots, a_n][b_1, \ldots, b_m] \). We see that \( \mathfrak{X}'' = \{ v \in \mathfrak{X} \mid v(a_i) \leq 1 \text{ and } v(b_j) \leq 1 \} = \mathfrak{X}(\{a_i\} \cup \{b_j\}/1) \).

**Case 2.** \( \mathfrak{X}' = \mathfrak{X}(\{a_i\}_{i=1}^n/b) \) and \( \mathfrak{X}'' = \mathfrak{X}'(\{1\}/h) \)

Now \( \mathfrak{X}' = \{ v \in \mathfrak{X} \mid v(a_i) \leq v(b) \} = Val(B', A') \) with \( B' = B \) and \( A' = \varphi_b(A)[\frac{a_1}{b}, \ldots, \frac{a_n}{b}] \). Also \( \mathfrak{X}'' = \{ v \in \mathfrak{X} \mid 1 \leq v(h) \} \) for \( h \in B_b \). There exists \( g \in B \) such that \( \varphi_g(b) = b^k h \) for some \( k \geq 0 \). Note that

\[
\mathfrak{X}'' = \mathfrak{X}' \cap \{ v \in \mathfrak{X} \mid v(b^k) \leq v(g) \}
\]

As we saw, if \( v \in \mathfrak{X}' \) then \( 1 \leq v(b) \). So \( 1 \leq v(b^k) \) as well. Then clearly \( 1 \leq v(g) \) for any \( v \in \mathfrak{X}'' \). Thus we can write

\[
\mathfrak{X}'' = \mathfrak{X}(\{a_i\}_{i=1}^n/b) \cap \mathfrak{X}(\{b^k, 1\}/g),
\]

so \( \mathfrak{X}'' \) is an intersection of two rational domains which is, as we already saw, a rational domain.
Case 3. $\mathcal{X}' = \mathcal{X} \left( \{a_i\}_{i=1}^n / b \right)$ and $\mathcal{X}'' = \mathcal{X}' / b$

Again $\mathcal{X}' = \{ v \in \mathcal{X} \mid v(a_i) \leq v(b) \} = Val(B', A')$, but now $\mathcal{X}'' = \{ v \in \mathcal{X}' \mid v(h) \leq 1 \}$ for $h \in B_0$. Again there exists $g \in B$ such that $\varphi_0(g) = b^k h$ for some $k \geq 0$. Now

$$\mathcal{X}'' = \mathcal{X}' \cap \{ v \in \mathcal{X} \mid v(g) \leq v(b^k) \},$$

which can be rewritten as

$$\mathcal{X}'' = \mathcal{X} \left( \{a_i\}_{i=1}^n / b \right) \cap \mathcal{X} \left( \{g, 1\} / b^k \right).$$

Case 4. $\mathcal{X}' = \mathcal{X} \left( \{a_i\}_{i=1}^n / b \right)$ and $\mathcal{X}'' = \mathcal{X}' \left( \{h_j\}_{j=1}^m / f \right)$

Finally we have $\mathcal{X}' = Val(B', A')$ and $\mathcal{X}'' = \mathcal{X}' \left( \{h_j\}_{j=1}^m / f \right)$ for $f, h_1, \ldots, h_m \in B'$ generating the unit ideal. As we saw

$$\mathcal{X}'' = \{ v \in \mathcal{X}' \mid v(h_j) \leq v(f) \} = \{ v \in \mathcal{X}' \mid v(h_j) \leq v(f) \text{ and } 1 \leq v(f) \} \subset \mathcal{X}' \{1\}/f.$$

Furthermore $f$ is a unit in $B'_f$ so $f^{-1}h_1, \ldots, f^{-1}h_m$ are elements in $B'_f$. Denoting $\mathcal{X}' \{1\}/f$ by $\mathcal{X}'(3)$ we have

$$\mathcal{X}'' = \mathcal{X}'(3) \left( \{f^{-1}h_j\}_{j=1}^m / 1 \right) = \cap_j \mathcal{X}'(3) \left( \{f^{-1}h_j\}/1 \right)$$

and repeated application of the previous cases gives the result. \qed

There is an obvious retraction $r : Spa(B, A) \to Val(B, A)$ given by sending every valuation $v$ to its minimal primary specialization.

**Proposition 2.4.8.** $r : Spa(B, A) \to Val(B, A)$ is continuous.

**Proof.** Let $U$ be an open subset of $\mathcal{X} = Val(B, A)$. As the rational domains form a basis for the topology it is enough to consider the case when $U$ is a rational domain of $Val(B, A)$. Let $a_1, \ldots, a_n, b \in B$ generating the unit ideal. Set

$$U = \{ v \in Val(B, A) \mid v(a_i) \leq v(b), i = 1, \ldots, n \} = \mathcal{X}(\{a_1, \ldots, a_n\}/b)$$

and

$$W = \{ v \in Spa(B, A) \mid v(a_i) \leq v(b) \} = R(\{a_1, \ldots, a_n\}/b).$$

We claim that $r^{-1}(U) = W$.

Obviously $r^{-1}(U) \subset W$. Let $w$ be a valuation in $W$ with value group $\Gamma$. If $c\Gamma_w = \Gamma$ then $r(w) = w$, that is, $w \in Val(B, A)$ so $w \in W \cap Val(B, A) = U$. If $c\Gamma_w \not\subset \Gamma$ then $r(w)(a_i) \leq r(w)(b)$ since $w(a_i) \leq w(b)$. It remains to show that $r(w)(b) \neq 0$. If $r(w)(b) = 0$ then $w(b) < c\Gamma_w$ and so $w(a_i) < c\Gamma_w$ for every $i = 1, \ldots, n$. There are $c_0, c_1, \ldots, c_n \in B$ such that $1 = bc_0 + a_1c_1 + \cdots + a_nc_n$ and $w(1) = 1 \in c\Gamma_w$. By convexity of $c\Gamma_w$ there is some $i$ such that $w(a_i)c_i \in c\Gamma_w$. Thus $w(a_i)c_i \leq w(bc_i) \in c\Gamma_w$, so $r(w)(bc_i) \neq 0$. But since $r(w)(b) = 0$ we also get $r(w)(bc_i) = 0$, which is a contradiction. We conclude that $r(w)(b) \neq 0$. \qed

**Corollary 2.4.9.** $Val(B, A)$ is quasi-compact and quasi-separated.

**Proof.** As $Spa(B, A)$ is a quasi-compact space by Lemma 2.2.4 and the retraction $r : Spa(B, A) \to Val(B, A)$ is continuous, $Val(B, A)$ is quasi-compact. Any rational domain can be viewed as $Val(B', A')$ for suitable rings $A' \subset B'$, hence any rational domain is quasi-compact. As we saw in Proposition 2.4.1, the intersection of two rational domains is again a rational domain so in particular it is quasi-compact.
Since the rational domains form a basis of the topology, any quasi-compact open subset of \( \text{Val}(B, A) \) can be viewed as a finite union of rational domains. Now the intersection of any two quasi-compact open subsets of \( \text{Val}(B, A) \) is also a finite union of rational domains, thus quasi-compact so \( \text{Val}(B, A) \) is quasi-separated. \( \square \)

**Example 2.4.10 (An Affine Scheme).** Consider \( \text{Val}(B, B) \).

Let \( v = (p, R_v, \Phi) \in \text{Val}(B, B) = X \), then \( v \) is an unbounded valuation on \( B \) such that \( v(B) \leq 1 \). The only way this could be is if \( v \) is a trivial valuation (i.e. \( \Gamma = \{1\} \)). Hence there is a \( 1 \rightarrow 1 \) correspondent between points of \( \text{Val}(B, B) \) and prime ideals of \( B \), that is points of \( \text{Spec} B \). As for the topology:

\[
\mathcal{X}(\{a_1, \ldots, a_n\}/b) = \text{Val}(B_b, \varphi_b(B) \left[ \frac{a_1}{b}, \ldots, \frac{a_n}{b} \right]) = \text{Val}(B_b, B_b) \leftrightarrow \text{Spec} B_b = D(b)
\]

So there is a homeomorphism \( \text{Val}(B, B) \cong \text{Spec} B \).

For later use we define two canonical maps of topological spaces

\[\sigma: \text{Spec} B \to X \text{ and } \tau: X \to \text{Spec} A.\]

For \( p \in \text{Spec} B \) we set \( \sigma(p) \) to be the trivial valuation on \( k(p) \), which is indeed in \( X = \text{Val}(B, A) \) since its valuation ring is \( k(p) \) and \( B \otimes_A k(p) \to k(p) \) is indeed surjective. For \( v = (p, R_v, \Phi) \in X \) we denote the maximal ideal of \( R_v \) by \( m_v \) and set \( \tau(v) = \Phi^{-1}(m_v) \in \text{Spec} A \).

**Proposition 2.4.11.** (1) The composition \( \tau \circ \sigma: \text{Spec} B \to \text{Spec} A \) is the morphism corresponding to the inclusion of rings \( A \subset B \).

(2) \( \sigma \) is continuous and injective.

(3) \( \tau \) is continuous and surjective.

**Proof.** (1) Given a prime \( p \) in \( B \), \( \sigma(p) = (p, k(p), \Phi) \). The maximal ideal of the valuation ring \( k(p) \) is the zero ideal, so

\[\tau(\sigma(p)) = \ker \Phi = \ker (A \to B \to k(p)) = p \cap A.\]

(2) Let \( U = \mathcal{X}(\{a_1, \ldots, a_n\}/b) \) be a rational domain in \( X \) and \( D(b) \) a basic open set in \( \text{Spec} B \). For any \( p \in D(b) \) we have \( \sigma(p)(b) = 1 \) and \( \sigma(p)(a_i) = 0 \) or \( 1 \), so \( \sigma(D(b)) \subset \mathcal{X}(\{a_1, \ldots, a_n\}/b) \).

Conversely if \( v \in \mathcal{X}(\{a_1, \ldots, a_n\}/b) \) and there is \( p \in \text{Spec} B \) that maps to \( v \) then we must have \( \ker(v) = p \) and \( v \) is trivial on \( k(p) \). Hence \( \sigma \) is injective and \( \sigma^{-1}(\mathcal{X}(\{a_1, \ldots, a_n\}/b)) = D(b) \).

(3) For a basic open set \( D(a) \subset \text{Spec} A \) and \( v = (p, R_v, \Phi) \in X \), then \( \tau(v) \in D(a) \) is the same as \( \Phi(\{a_1, \ldots, a_n\}/b) = m_v \) or equivalently \( v(a) \geq 1 \). In this case \( v(a) = 1 \) (since \( v(A) \leq 1 \)) and

\[v \in \mathcal{X}(\{1\}/a) = \{ w \in \text{Val}(B, A) \mid w(a) = 1 \}\]

so \( \tau^{-1}(D(a)) = \mathcal{X}(\{1\}/a) \).

As for surjectivity, first consider a maximal ideal \( q \in \text{Spec} A \). Since the morphism \( \text{Spec} B \to \text{Spec} A \) is schematically dominant there is \( p'' \in \text{Spec} B \), such that \( A \cap p'' \subset q \). Take \( p \) to be a maximal prime of \( B \) with this property. The image \( i(A) \) of \( A \) under \( i: B \to k(p) \) is a subring of \( k(p) \). The extended ideal \( i(q)i(A) \) is a proper ideal, since if \( 1 \in i(q)i(A) \) then there are \( a_1, \ldots, a_n \in q \) and \( b_1, \ldots, b_k \in A \) such that \( 1 - \sum a_i b_i \in \ker(i) = p \). As \( 1 - \sum a_i b_i \in A \) we have \( 1 - \sum a_i b_i \in A \cap p \subset q \), but since \( \sum a_i b_i \in q \) we get that \( 1 \in q \) which is a contradiction. By \([ZS50], \text{VI 4.4}\)
there is a valuation ring $R_v$ of $k(p)$ containing $i(A)$ such that its maximal ideal $m_v$ contains $i(q)i(A)$. This gives us a valuation $v = (p, R_v, \Phi = i|A) \in \text{Spa}(B, A)$. By the retraction we obtain a valuation $v' = r(v) \in \text{Val}(B, A)$ with kernel $p' \supset p$. If $p \neq p'$ then $q$ is strictly contained in $A \cap p'$, by the choice of $p$. Since $q$ is a maximal ideal we have $1 \in A \cap p'$ which is a contradiction. Hence $p' = p$ and $v' = v$ that is $v = (p, R_v, \Phi) \in \text{Val}(B, A)$. Now $\tau(v) = \Phi^{-1}(m_v) \supset q$ but since $q$ is maximal we have $\tau(v) = q$.

Now, let $q \in \text{Spec}A$ be some prime. Then $q A_q$ is the maximal ideal of $A_q$, and since $A_q$ is flat over $A$, we have $A_q \subset B_q = B \otimes_A A_q$. By the previous case there is a valuation $v \in \text{Val}(B_q, A_q)$ that $\tau : \text{Val}(B_q, A_q) \to \text{Spec}A_q$ maps to $q A_q$. The canonical homomorphisms

$$
\begin{array}{ccc}
B & \longrightarrow & B_q \\
\uparrow & & \uparrow \\
A & \longrightarrow & A_q
\end{array}
$$

induces by Lemma 2.3.7 a morphism $\text{Val}(B_q, A_q) \to \mathfrak{X}$. As $q A_q \in \text{Spec}A_q$ is pulled back to $q \in \text{Spec}A$, the image of $v$ in $\text{Val}(B, A)$ is mapped by $\tau$ to $q$. □

**Remark 2.4.12.** From the proof we see that for any $a \in A$ we have

$$
\tau^{-1}(D(a)) = \text{Val}(B_a, A_a) = \mathfrak{X}((\{1\})/a),
$$

and for every rational domain $\mathfrak{X}((\{a_1, \ldots, a_n\})/b) \subset \mathfrak{X}$ we have

$$
\sigma^{-1}(\mathfrak{X}((\{a_1, \ldots, a_n\})/b)) = D(b).
$$

**Remark 2.4.13.** A point $v = (p, R_v, \Phi) \in \mathfrak{X}$ is (as said before) a diagram

$$
\begin{array}{ccc}
B & \longrightarrow & k(p) \\
\uparrow & & \uparrow \\
A & \Phi & \longrightarrow & R_v.
\end{array}
$$

Also there is a unique semi-valuation ring $S_v$ associated to the point (namely the pull-back of $R_v$ to $B_p$), and the diagram factors through the pair $S_v \subset B_p$ i.e.

$$
\begin{array}{ccc}
B & \longrightarrow & B_p & \longrightarrow & k(p) \\
\uparrow & & \uparrow & & \uparrow \\
A & \longrightarrow & S_v & \longrightarrow & R_v.
\end{array}
$$

Since the maximal ideal of the valuation ring $R_v$ is pulled back to the maximal ideal of the semi-valuation ring $S_v$, we may rephrase the definition of $\tau$ as the pull-back of the maximal ideal of the semi-valuation ring $S_v$ to $A$. Equivalently we can say that $\tau(v)$ is the image of the unique closed point of $\text{Spec}S_v$ under the map $\text{Spec}S_v \to \text{Spec}A$.

2.5. **Rational Covering.** Any open cover of $\mathfrak{X}$ can be refined to a cover of $\mathfrak{X}$ consisting of rational domains, since the rational domains form a basis for the topology. Furthermore there is a finite sub-cover of $\mathfrak{X}$ consisting of rational domains, as $\mathfrak{X}$ is quasi-compact. Next we show that we can always refine this cover to a rational covering, that is there are elements $T = \{a_1, \ldots, a_N\} \subset B$ generating the
unit ideal such that the rational domains \( \{ X(T/a_j) \}_{1 \leq j \leq N} \) form a refinement of the finite sub-cover.

**Proposition 2.5.1.** Any finite open cover of \( Val(B,A) \) consisting of rational domains can be refined to a rational covering.

**Proof.** Let \( \{ U_i \}_{i=1}^N \) be a finite open cover of \( X = Val(B,A) \) consisting of rational domains, i.e. for every \( i = 1, \ldots, N \) we have \( a_i^{(i)}, \ldots, a_n^{(i)} \in B \) generating the unit ideal and

\[
U_i = X(\{ a_j^{(i)} \}_{1 \leq j \leq n_i}/a_1^{(i)}) = \{ v | v(a_j^{(i)}) \leq v(a_1^{(i)}) \} \quad j = 1, \ldots, n_i \}. 
\]

For every \( 1 \leq k \leq n_i \) denote \( V_{i,k} = X(\{ a_j^{(i)} \}_{1 \leq j \leq n_i}/a_k^{(i)}) \). Note that \( V_{i,1} = U_i \) and \( X = \bigcup_{k=1}^{n_i} V_{i,k} \) for each \( i = 1, \ldots, N \).

Set

\[
I = \{(r_1, \ldots, r_N) \in \mathbb{N}^N | 1 \leq r_i \leq n_i \} \quad i = 1, \ldots, N 
\]

and

\[
I' = \{(r_1, \ldots, r_N) \in I | r_i = 1 \text{ for some } i \}.
\]

For \( (r_1, \ldots, r_N) \in I \) we denote

\[
V_{(r_1, \ldots, r_N)} = \bigcap_{1 \leq i \leq N} V_{r_i,r_i} \quad \text{and} \quad a_{(r_1, \ldots, r_N)} = a_1^{(1)} \cdot a_2^{(2)} \cdots a_N^{(N)} .
\]

Note that \( V_{(r_1, \ldots, r_N)} = \{ v \in X | v(a_1) \leq v(a_{(r_1, \ldots, r_N)}) \forall \alpha \in I \} \).

We claim that for any \( (r_1, \ldots, r_N) \in I' \) we have

\[
V_{(r_1, \ldots, r_N)} = \{ v \in X | v(a_1) \leq v(a_{(r_1, \ldots, r_N)}) \forall \alpha \in I' \}.
\]

Given \( (r_1, \ldots, r_N) \in I' \), by definition we have

\[
V_{(r_1, \ldots, r_N)} \subset \{ v \in X | v(a_1) \leq v(a_{(r_1, \ldots, r_N)}) \forall \alpha \in I' \}.
\]

Conversely, if \( w \in \{ v \in X | v(a_1) \leq v(a_{(r_1, \ldots, r_N)}) \forall \alpha \in I' \} \), then \( w \in U_{i_0} \) for some \( i_0 \) since \( \{ U_i \}_{i=1}^N \) is a cover of \( X \). For simplicity we assume that \( i_0 = 1 \), so \( w(a_k^{(1)}) \leq w(a_1^{(1)}) \) for every \( 1 \leq k \leq n_1 \).

Now for any \( (j_1, \ldots, j_N) \in I - I' \) we have

\[
w(a_{(j_1, \ldots, j_N)}) = w(a_1^{(j_1)} \cdot a_2^{(j_2)} \cdots a_N^{(j_N)}) \leq w(a_1^{(1)} \cdot a_2^{(2)} \cdots a_N^{(N)}) = w(a_{(1,2,\ldots,J_N)}).
\]

As \( (1, j_2, \ldots, j_N) \in I' \) by assumption we have \( w(a_{(1,j_2, \ldots, J_N)}) \leq w(a_{(r_1, \ldots, r_N)}) \). It follows that

\[
w(a_{(j_1, \ldots, J_N)})) \leq w(a_{(1,j_2,\ldots,j_N)}) \leq w(a_{(r_1, \ldots, r_N)}),
\]

hence \( w \in V_{(r_1, \ldots, r_N)} \).

Finally note that

- \( \{ a_\alpha \}_{\alpha \in \Gamma} \) generate the unit ideal of \( B \).
- \( V_{\alpha} = X(\{ a_\beta \}_{\beta \in \Gamma}/a_\alpha) \) for every \( \alpha \in \Gamma \).
- \( X = \bigcup_{\alpha \in \Gamma} V_{\alpha} \).
- for \( (r_1, \ldots, r_N) \in I' \) if \( r_i = 1 \) then \( V_{(r_1, \ldots, r_N)} \subset U_i \).

\( \square \)

2.6. Sheaves on \( Val(B,A) \).
2.6.1. \( M_X \) and \( O_X \). We now define two sheaves on \( \mathfrak{X} = Val(B, A) \), both making \( \mathfrak{X} \) a locally ringed space.

**Notation.**

- For a pair of rings \( C \subset D \) we denote the integral closure of \( C \) in \( D \) by \( \text{Nor}_D C \).
- For quasi-compact quasi-separated schemes \( Y, X \) and an affine morphism \( f : Y \to X \), we denote the integral closure of \( O_X \) in \( f_\ast O_Y \) by \( \text{Nor}_f O_Y (O_X) \) or \( \text{Nor}_Y (O_X) \).
- In the situation above we denote \( \text{Nor}_Y X = \text{Spec}_X(\text{Nor}_Y O_X) \) and the canonical morphism \( \text{Nor}_Y X \to X \) by \( \nu \).

**Lemma 2.6.1.** Let \( A \subset B \) be rings. Denote \( X = \text{Spec}A \) and \( Y = \text{Spec}B \). Then \( X = Val(B, A) = Val(B, \text{Nor}_B A) \) and the canonical map \( \tau : \mathfrak{X} \to X \) factors through \( \text{Nor}_Y X \).

**Proof.** The canonical morphism 

\[
\nu : \text{Nor}_Y X \to X
\]

is an integral, hence universally closed and separated. Thus for any valuation \( v = (p, R_v, \Phi) \in \mathfrak{X} \) we obtain a diagram by the valuative criterion (abusing notation and denoting by \( \Phi \) both \( A \to R_v \) and the induced morphism \( \text{Spec}R_v \to \text{Spec}A \))

\[
\begin{array}{ccc}
\text{Spec}k(p) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}R_v & \longrightarrow & \text{Nor}_Y X
\end{array}
\]

That is, we obtain a unique \( (p, R_v, \Phi') \in \text{Spa}(B, \text{Nor}_B A) \). As \( v \in \mathfrak{X} \) the morphism \( \text{Spec}k(p) \to Y \times \text{Spec}R_v \) is a closed immersion by Remark 2.3.6. It follows, again from Remark 2.3.6, that \( (p, R_v, \Phi') \in Val(B, \text{Nor}_B A) \). As the lifting of \( \Phi \) is unique we have \( \mathfrak{X} = Val(B, \text{Nor}_B A) \).

It is clear that the diagram of topological spaces

\[
\begin{array}{ccc}
Val(B, \text{Nor}_B A) & \longrightarrow & \text{Nor}_Y X \\
\downarrow & & \downarrow \\
Val(B, A) & \longrightarrow & X
\end{array}
\]

commutes. \( \square \)

As the rational domains form a basis for the topology of \( \mathfrak{X} \) it is enough to define the sheaves only over the rational domains. Let \( U = \mathfrak{X}((a_1, \ldots, a_n)/b) = Val(B', A') \) where, as before, \( B' = B_b \) and \( A' = \phi_b(A)[a_1/b, \ldots, a_n/b] \). We define two presheaves \( M_X \) and \( O_X \) on the rational domains of \( \mathfrak{X} \) by the rules

\[
U \mapsto M_X(U) = B' \quad \quad U \mapsto O_X(U) = \text{Nor}_B A'.
\]

Clearly \( O_X(U) \subset M_X(U) \).

**Theorem 2.6.2.** With the above notation, the presheaves \( M_X \) and \( O_X \) are sheaves on the rational domains of \( \mathfrak{X} \).
Proof. Denote $Y = \text{Spec} B$. We know that the sets $\{D(b)\}_{b \in B}$ form a base for the topology of $Y$. Recall that we defined a map $\sigma : Y \to \mathfrak{X}$ such that for every rational domain $\mathfrak{X} \langle \{a_1, \ldots, a_n\}/b \rangle \subset \mathfrak{X}$ we have $\sigma^{-1}(\mathfrak{X} \langle \{a_1, \ldots, a_n\}/b \rangle) = D(b)$ (Remark 2.4.12). Now, by the definition of $\mathcal{M}_X$, for every rational domain $U \subset \mathfrak{X}$ we have an isomorphism of rings

$$B_b = \mathcal{M}_X(U) \simeq \sigma_* \mathcal{O}_{\text{Spec} B}(U) = \mathcal{O}_{\text{Spec} B}(D(b)) = B_b.$$ 

Let $V \subset U \subset \mathfrak{X}$ be two rational domains. Suppose $U = \mathfrak{X} \langle \{a_1, \ldots, a_n\}/b \rangle$ and $V = \mathfrak{X} \langle \{f_1, \ldots, f_m\}/g \rangle$. Then $D(g) = \sigma^{-1}(V)$ and $D(b) = \sigma^{-1}(U)$. For any $p \in D(g)$ we have $\sigma(p) \in V \subset U$, hence $p \in D(b)$. In other words we have a diagram

$$
\begin{array}{ccc}
D(g) & \longrightarrow & D(b) \\
\downarrow & & \downarrow \sigma \\
V & \longrightarrow & U \\
\end{array}
$$

From the diagram we see that the restrictions of $\mathcal{M}_X$ commute with the restrictions of $\mathcal{O}_{\text{Spec} B}$. We conclude that we have an isomorphism of presheaves between $\mathcal{M}_X$ and $\sigma_* \mathcal{O}_{\text{Spec} B}$ as presheaves on the rational domains. Since $\mathcal{O}_{\text{Spec} B}$ is a sheaf on $Y$, its restriction to a basic of the topology of $Y$ is also a sheaf. It follows that $\mathcal{M}_X$ is a sheaf on the rational domains of $\mathfrak{X}$.

Let $U$ be a rational domain in $\mathfrak{X}$ and $\{V_i\}$ an open covering of $U$ consisting of rational domains. Let $s_i \in \mathcal{O}_X(V_i)$ be sections satisfying $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for every pair $i, j$. We already know that $\mathcal{M}_X$ is a sheaf so there is a unique element $s \in \mathcal{M}_X(U)$ such that $s_i|_{V_i} = s_i$ for each $i$. We want to show that $s$ is in $\mathcal{O}_X(U)$. We may assume that $U = \mathfrak{X}$ and that $\{V_i\}$ is a rational covering corresponding to $b_1, \ldots, b_r$, that is $b_1, \ldots, b_r \in B$, none of which are nilpotent, generating the unit ideal of $B$ and $V_i = \mathfrak{X} \langle \{b_j\}_{j \in b_i} \rangle$.

We denote $Y = \text{Spec} B$, $X = \text{Spec} A$, $B_i = B_{b_i}$, $A_i = \varphi_i(A) \langle \{b_{i,j}\}_{j \in b_i} \rangle$ (where $\varphi_i$ is the canonical homomorphism $B \to B_i$), $Y_i = \text{Spec} B_i$ and $X'_i = \text{Spec} A_i$. Then $V_i = \text{Val}(B_i, A_i)$.

Now, $E = \bigoplus_{i \geq 0} A b_i$ is a finite $A$-module contained in $B$. Using the multiplication in $B$, we define $E^d$ as the image of $E \otimes_B E$ under the map $B \otimes_B E \to B$. Then $E^d$ is also a finite $A$-module contained in $B$ for any $d \geq 1$. Denoting $E^0 = A$ we obtain a graded $A$-algebra $E' = \bigoplus_{d \geq 0} E^d$ and a morphism $X' = \text{Proj} (E') \to X$. The affine charts of $X'$ are given by $\text{Spec} A'_i$, where $A'_i$ is the zero grading part of the localization $E'_i$. Clearly $A'_i \subset B_i$. Denoting $I_d = \ker (E^d \to B \to B_i)$, we have $A'_i = \lim_{d \to \infty} b_i^{-d} \cdot E^d/I_d$ which is exactly $A_i$. This means we have open immersions

$$
\begin{array}{ccc}
Y_i & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X'_i & \longrightarrow & X' \\
\end{array}
$$

As $Y = \bigcup Y_i$ the schematically dominant morphisms $Y_i \to X'_i$ glue to a schematically dominant morphism $Y \to X'$ over $X$. 

Furthermore, for each $i$ we have a commutative diagram

![Diagram](https://example.com/diagram.png)

Denoting the normalization $X'' = \text{Nor}_Y X'$ and taking the canonical morphism $\nu : X'' \rightarrow X'$, then by Lemma 2.6.1 we have a diagram over $X$

![Diagram](https://example.com/diagram.png)

We denote $X''_i = \nu^{-1}(X'_i)$. Now, by construction $s_i \in \mathcal{O}_{X''}(X''_i) = \tau^* \mathcal{O}_X(V_i)$ and $s_i|_{X''_i \cap X''_j} = s_j|_{X''_i \cap X''_j}$ for every pair $i, j$. Since $\mathcal{O}_{X''}$ is a sheaf, they glue to a section $s \in \mathcal{O}_{X''}(X'') = \tau^* \mathcal{O}_X(X)$. □

Note that the above construction yields the same topological space and the same sheaves for $A \subset B$ and for $\text{Nor}_B A \subset B$.

2.6.2. The stalks.

**Proposition 2.6.3.** For any point $v = (p, R_v, \Phi) \in X$, the stalk $\mathcal{M}_X, v$ of the sheaf $\mathcal{M}_X$ is isomorphic to $B_p$ and the stalk $\mathcal{O}_X, v$ of the sheaf $\mathcal{O}_X$ is isomorphic to the semi-valuation ring $S_v$.

**Proof.** Fix a point $v = (p, R_v, \Phi) \in X$. The inclusion of sheaves $\mathcal{O}_X \subset \mathcal{M}_X$ gives an inclusion of stalks $\mathcal{O}_X, v \subset \mathcal{M}_X, v$.

By the definition of a semi-valuation ring we have a diagram

![Diagram](https://example.com/diagram.png)

For any rational domain $v \in W = X(\{a_i\}/b) = \text{Val}(B_1, A_1)$ (we may assume that $A_1$ integrally closed in $B_1$) we have a unique factorization

![Diagram](https://example.com/diagram.png)
Taking direct limits we get a unique diagram for the stalks

\[
\begin{array}{ccc}
\mathcal{M}_{X,v} & \xrightarrow{B_p} & k(p) \\
\downarrow & & \downarrow \\
\mathcal{O}_{X,v} & \xrightarrow{S_v} & R_v
\end{array}
\]

and \( v \) induces a valuation in \( \text{Val} (\mathcal{M}_{X,v}, \mathcal{O}_{X,v}) \).

Let \( \gamma, \eta \) be co-prime elements in \( \mathcal{M}_{X,v} \), i.e. there are elements \( \rho, \tau \in \mathcal{M}_{X,v} \) such that \( \gamma \rho + \eta \tau = 1 \). It follows from Proposition 2.4.1 that the intersection of a finite number of rational domains is again a rational domain. Hence there is a rational domain \( U = \text{Val}(B', A') \) with \( g, h, r, t \in \mathcal{M}_{X}(U) = B' \) such that \( g, h, r, t \) are representatives of \( \gamma, \eta, \rho, \tau \) respectively. Then \( gr + ht \) is a representative of \( 1 \in \mathcal{M}_{X,v} \). So there is a rational domain \( V = \text{Val}(B'', A'') \subset U \) such that

\[
gr + ht|_V = 1 \in \mathcal{M}_{X}(V) = B''.
\]

Then we get that \( g|_V, h|_V \in \mathcal{M}_{X}(V) = B'' \) are representatives of \( \gamma, \eta \) and are co-prime. Furthermore \( v \) induces (canonically) a valuation on \( B'' \) which has the same valuation ring as \( v \). By the transitivity of rational domains we may assume that \( V = \text{Val}(B, A) \) and that \( g, h \in B \) are co-prime and are representatives of \( \gamma, \eta \in \mathcal{M}_{X,v} \) respectively. If \( v(g) \leq v(h) \) then \( \text{Val} \left( B_h, \varphi_h(A) \left[ \frac{g}{h} \right] \right) \) is a rational domain and

\[
v \in \text{Val} \left( B_h, \varphi_h(A) \left[ \frac{g}{h} \right] \right) \subset \text{Val}(B, A).
\]

Hence \( v(\gamma/\eta) \leq 1 \), so \( \gamma \in \eta \mathcal{O}_{X,v} \). Conversely if \( v(g) \geq v(h) \) then by the same reasoning \( \eta \in \gamma \mathcal{O}_{X,v} \). By Remark 2.4.2 (vi) \( \mathcal{O}_{X,v} \) is a semi-valuation ring and \( \mathcal{M}_{X,v} \) its semi-fraction ring. It now follows, also from Remark 2.4.2 that for \( m = \ker v \) in \( \mathcal{O}_{X,v} \) we have \( \mathcal{M}_{X,v} = (\mathcal{O}_{X,v})_m \) and \( \mathcal{O}_{X,v}/m = R_v \), the valuation ring of \( v \) in \( k(p) \). Hence \( \mathcal{O}_{X,v} = S_v \) and \( \mathcal{M}_{X,v} = B_p \).

\[
\square
\]

**Remark 2.6.4.** For any point \( p \in \text{Spec} B \) the stalks of the point \( \sigma(p) \in X \) are

\[
\mathcal{M}_{X,\sigma(p)} = \mathcal{O}_{X,\sigma(p)} = B_p.
\]

### 3. Birational Spaces

**Definition 3.1.1.** (i) A pair of rings \( (B, A) \) is a ring \( B \) and a sub-ring \( A \).

(ii) A homomorphism of pairs of rings \( \varphi : (B, A) \to (B', A') \) is a ring homomorphism \( \varphi : B \to B' \) such that \( \varphi(A) \subset A' \).

(iii) A homomorphism of pairs of rings \( \varphi : (B, A) \to (B', A') \) is called adic if the induced homomorphism \( B \otimes_A A' \to B' \) is integral.

(iv) The relative normalization of a pair of rings \( (B, A) \) is the induced pair of rings \( (B, \text{Nor}_B A) \) together with a canonical homomorphism of pairs

\[
\nu = \text{id}_B : (B, A) \to (B, \text{Nor}_B A).
\]

(v) A pair of schemes \( (Y \xrightarrow{f} X) \) or \( (Y, X) \) is a pair of quasi-compact, quasi-separated schemes \( Y, X \) together with an affine and schematically dominant morphism \( f : Y \to X \).
(vi) A morphism of pairs of schemes \( g : (Y', X') \to (Y, X) \) is a pair of morphisms \( g = (g_Y, g_X) \) forming a commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{g_Y} & Y \\
\downarrow{f'} & & \downarrow{f} \\
X' & \xrightarrow{g_X} & X.
\end{array}
\]

(vii) A morphism of pairs of schemes \( g : (Y', X') \to (Y, X) \) is called adic if the induced morphism of schemes

\[ Y' \to Y \times_X X' \]

is integral.

(viii) The relative normalization of a pair of schemes \((Y, X)\) is the induced pair of schemes \((Y, \text{Nor}_Y X)\) together with a canonical morphism of pairs

\[ \nu = (\text{id}_Y, \nu_X) : (Y, \text{Nor}_Y X) \to (Y, X), \]

where \( \text{Nor}_Y X = \text{Spec}_X (\text{Nor}_{O_Y} O_X) \) and \( \nu_X \) is the canonical morphism \( \text{Nor}_Y X \to X \).

We denote the category of pairs of rings with their morphisms by \( \text{pa-Ring} \) and the category of pairs of schemes with their morphisms by \( \text{pa-Sch} \).

**Definition 3.1.2.** Let \((Y \xrightarrow{f} X)\) be a pair of schemes. Given an open (affine) subscheme \( X' \subset X \) its preimage \( Y' = f^{-1}(X') \) is an open (affine) subscheme of \( Y' \). The restriction of \( f \) to \( Y' \) makes \((Y', X')\) a pair of schemes. We call \((Y', X')\) an open (affine) sub-pair of schemes. An affine covering of the pair \((Y, X)\) is a collection of open sub-pairs \( \{(Y_i, X_i)\} \) such that \( \{X_i\} \) are affine and cover \( X \) (then necessarily their preimages \( \{Y_i\} \) are affine and cover \( Y \)).

**Lemma 3.1.3.** Assume the elements \( b, a_1, \ldots, a_n \in B \) generate the unit ideal. Set \( B' = B_b \). Let \( \varphi_b : B \to B_b \) be the canonical map and denote \( A' = \varphi_b (A)[\frac{a_1}{b}, \ldots, \frac{a_n}{b}] \). Then the homomorphism of pairs of rings \( \varphi_b : (B, A) \to (B', A') \) is adic.

**Proof.** Since \( b, a_1, \ldots, a_n \) generate the unit ideal of \( B \)

\[ B \otimes_A A' = \varphi_b (B) \left[ \frac{a_1}{b}, \ldots, \frac{a_n}{b} \right] = B'. \]

\( \square \)

**Lemma 3.1.4.**

(i) Composition of adic morphisms of pairs of schemes is adic.

(ii) Let \( g : (Y', X') \to (Y, X) \) be an adic morphism of pairs of schemes and \((V, U)\) an open sub-pair of \((Y, X)\). Then the restriction of \( g|_{(g_Y^{-1}(V), g_X^{-1}(U))} : (g_Y^{-1}(V), g_X^{-1}(U)) \to (V, U) \) is adic.

(iii) Let \( g : (Y', X') \to (Y, X) \) be a morphism of pairs of schemes and \( \{(V_i, U_i)\} \) an affine covering of \((Y, X)\). If all the restrictions \( (g_Y^{-1}(V_i), g_X^{-1}(U_i)) \to (V_i, U_i) \) are adic, then \( g \) is adic.

**Proof.**

(i) Let \( g : (Y', X') \to (Y, X) \) and \( h : (Y'', X'') \to (Y', X') \) be adic morphisms of pairs of schemes. Then \( g \circ h : (Y'', X'') \to (Y, X) \) is a morphisms of pairs of
schemes associated to the diagram

\[
\begin{array}{ccc}
Y'' & \xrightarrow{h} & Y' \\
\downarrow & & \downarrow \\
X'' & \xrightarrow{g} & X.
\end{array}
\]

We want to show that the induced morphism \( Y'' \to Y \times_X X'' \) is integral.

This morphism factors as \( Y'' \to Y' \times_X Y'' \to Y \times_X X'' \). As \( h \) is adic the first arrow is integral. Now, \( g \) is also adic, so \( Y' \to Y \times_X X' \) is also integral. Taking the base change of this morphism by the morphism \( Y \times_X X'' \to Y \times_X X' \) we get that \( Y' \times_X Y'' \to Y \times_X X'' \) is integral. Hence the composition \( Y'' \to Y \times_X X'' \) is indeed integral.

(ii) We denote \( U'' = g_X^{-1}(U) \subset X' \) and \( V' = f'^{-1}(U') = g_Y^{-1}(V) \subset Y' \). Now \( V \times_U U'' = V \times_X X' \) is an open subset of \( Y \times_X X'' \) and we have a pull back diagram

\[
\begin{array}{ccc}
V' & \to & V \times_X X'
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & & \downarrow
\end{array}
\]

\[
\begin{array}{ccc}
Y'' & \to & Y \times_X X'.
\end{array}
\]

Since the bottom arrow is integral so is the top arrow.

(iii) For each \( i \) denote \( U'_i \) and \( V'_i \) as in (ii) and \( g_i \) the restriction of \( g \) to \( (V'_i, U'_i) \to (V_i, U_i) \). Assume that \( g_i : (V'_i, U'_i) \to (V_i, U_i) \) is adic for each \( i \). Then for each \( i \) the map \( V'_i \to V_i \times_U U'_i = V_i \times_X X' \) is integral. Now \( Y' = \cup V'_i \) and \( Y \times_X X' = \cup(V_i \times_X X') \). Since the property of being integral is local on the base we have that \( Y' \to Y \times_X X' \) is integral. \( \square \)

3.2. The bir Functor.

**Definition 3.2.1.** (i) A pair-ringed space \( (\mathcal{X}, \mathcal{M}_X, \mathcal{O}_X) \) is a topological space \( \mathcal{X} \) together with a sheaf of pairs of rings \( (\mathcal{M}_X, \mathcal{O}_X) \) such that both \( (\mathcal{X}, \mathcal{M}_X) \) and \( (\mathcal{X}, \mathcal{O}_X) \) are ringed spaces.

(ii) A morphism of pair-ringed spaces

\[
(h, h^i) : (\mathcal{X}, \mathcal{M}_X, \mathcal{O}_X) \to (\mathcal{Y}, \mathcal{M}_Y, \mathcal{O}_Y)
\]

is a continuous map \( h : \mathcal{X} \to \mathcal{Y} \) together with a morphism of sheaves of pairs of rings

\[
h^i : (\mathcal{M}_Y, \mathcal{O}_Y) \to (h_* \mathcal{M}_X, h_* \mathcal{O}_X)
\]

such that both

\[
(h, h^i) : (\mathcal{X}, \mathcal{M}_X) \to (\mathcal{Y}, \mathcal{M}_Y) \text{ and } (h, h^i) : (\mathcal{X}, \mathcal{O}_X) \to (\mathcal{Y}, \mathcal{O}_Y)
\]

are morphisms of ringed spaces.

In Section 2 we constructed a pair-ringed space \( (\mathcal{X}, \mathcal{M}_X, \mathcal{O}_X) \) from a pair of rings \((B, A)\), namely \( Val(B, A) \). From now on for any pair of rings \((B, A)\) by \( Val(B, A) \) we mean the pair-ringed space \( (Val(B, A), \mathcal{M}_{Val(B,A)}, \mathcal{O}_{Val(B,A)}) \).

**Definition 3.2.2.** (i) An affinoid birational space is a pair-ringed space \( (\mathcal{X}, \mathcal{M}_X, \mathcal{O}_X) \) isomorphic to \( Val(B, A) \) for some pair of rings \((B, A)\).

(ii) A pair-ringed space \( (\mathcal{X}, \mathcal{M}_X, \mathcal{O}_X) \) is a birational space if every point \( x \in \mathcal{X} \) has an open neighbourhood \( U \) such that the induced subspace \( (U, \mathcal{M}_X|_U, \mathcal{O}_X|_U) \) is an affinoid birational space.
(iii) A morphism of birational spaces

\[(h, h^\#) : (X, M_X, O_X) \to (\mathcal{Y}, M_{\mathcal{Y}}, O_{\mathcal{Y}})\]

is a morphism of pair-ringed spaces such that \((h^\#) : (X, O_X) \to (\mathcal{Y}, O_{\mathcal{Y}})\)
is a map of locally ringed spaces (but not necessarily \((h, h^\#) : (X, M_X) \to (\mathcal{Y}, M_{\mathcal{Y}}))\); see Example 3.2.8 below).

We denote the category of affinoid birational spaces with their morphisms by \(af\)-Birat and the category of birational spaces with their morphisms by Birat.

**Example 3.2.3.** Given a ring \(B\) we have a homeomorphism of topological spaces

\[\text{Val}(B, B) \cong \text{Spec} B\] (Example 2.4.10). From Remark 2.6.4 it is clear that considered as an affinoid birational space \(\text{Val}(B, B)\) is exactly \((\text{Spec} B, O_{\text{Spec} B}, O_{\text{Spec} B})\).

A scheme is locally isomorphic to an affine scheme so we obtain

**Corollary 3.2.4.**

1. Any scheme \((X, O_X)\) can be viewed as a birational space

\[\text{Val}(X, X) = (X, O_X, O_X)\].

2. Any pair of schemes \((Y, X)\) induces a birational space \(\text{Val}(Y, X)\).

**Remark 3.2.5.**

1. For a pair of schemes \((Y, X)\), the points of \(\text{Val}(Y, X)\) are 3-tuples \((y, R, \Phi)\) such that \(y\) is a point in \(Y\), \(R\) is a valuation ring of the residue field \(k(y)\) and \(\Phi\) is a morphism of schemes \(\text{Spec} R \to X\) making the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & \text{Spec} k(y) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Phi} & \text{Spec} R
\end{array}
\]

commute, and \(\text{Spec} k(y) \to Y \times_X \text{Spec} R\) is a closed immersion [Tem11 §3].

2. Any affine covering \(\{Y_i, X_i\}\) of \((Y, X)\) gives rise to a covering of the birational space \(\text{Val}(Y, X)\) consisting of affinoid birational spaces \(\{\text{Val}(Y_i, X_i)\}\).

**Example 3.2.6.** For a finitely generated field extension \(K/k\) there is an obvious natural map, homeomorphic onto its image, from the Zariski-Riemann space \(RZ(K/k)\) as defined by Zariski to our \(\text{Val}(K, k)\) [ZS60 Chapter VI §17]. Furthermore \(\text{Val}(K, k)\) consists of the image of \(RZ(K/k)\) together with the trivial valuation on \(K\) which is a generic point. At a valuation \(v\) the stalk is the pair of rings \((K, R_v)\).

**Theorem 3.2.7.** There is a contra-variant functor \(\text{bir}\) from the category \(\text{pa-Rings}\) of pairs of rings to the category af-Birat of affinoid birational spaces.

**Proof.** We already saw the construction of an affinoid birational space \(\text{Val}(B, A)\) from a pair of rings \((B, A)\), we set \((B, A)_{\text{bir}} = \text{Val}(B, A)\).

For two pairs of rings \((B_1, A_1), (B_2, A_2)\) and a homomorphism of pairs of rings \(\varphi : (B_1, A_1) \to (B_2, A_2)\) we define the map of topological spaces \(\varphi_{\text{bir}}\) by the composition

\[
\begin{array}{ccc}
\text{Spa}(B_2, A_2) & \xrightarrow{\varphi^*} & \text{Spa}(B_1, A_1) \\
\downarrow & & \downarrow \varphi \\
\text{Val}(B_2, A_2) & \xrightarrow{\varphi_{\text{bir}}} & \text{Val}(B_1, A_1)
\end{array}
\]
where $\varphi^*$ is the pull back map defined in section 2.2 and $r$ is the retraction defined in section 2.4.

We saw that both $\varphi^*$ (Lemma 2.2.2) and the retraction (Lemma 2.4.8) and are continuous so $\varphi_{bir}$ is continuous.

For $v = (p, R_v, \Phi) \in Val(B_2, A_2)$ we have
\[
\varphi^*(v) = v \circ \varphi \in Spa(B_1, A_1)
\]
\[
\varphi_{bir}(v) = r(v \circ \varphi) = w = (q, R_w, \Psi) \in Val(B_1, A_1).
\]

Since $w$ is a primary specialization of the pullback valuation $\varphi^*(v) = v \circ \varphi$ there is a natural homomorphism of the stalks
\[
\varphi_{bir} : (B_2, A_2)_{bir} \to (B_1, A_1)_{bir}.
\]

It is obvious that $bir$ respects identity homomorphisms. As for composition, let
\[
(B_1, A_1) \xrightarrow{\psi_{bir}} (B_2, A_2) \xrightarrow{\varphi_{bir}} (B_3, A_3)
\]
be homomorphisms of pairs of rings. For $v = (p, R_v, \Phi) \in Val(B_3, A_3)$ we have
\[
(\psi \circ \varphi)^*(v) = \varphi^*(\psi^*(v))
\]
as elements of $Spa(B_1, A_1)$. By construction $(\psi \circ \varphi)_{bir}(v)$ is a primary specialization of $\varphi^*(\psi^*(v))$. Also, $\psi_{bir}(v)$ is a primary specialization of $\psi^*(v)$ as elements of $Spa(B_2, A_2)$, and $\varphi_{bir}(\psi_{bir}(v))$ is a primary specialization of $\varphi^*(\psi_{bir}(v))$ as elements of $Spa(B_1, A_1)$. It follows from Lemma 2.3.2 that $(\psi \circ \varphi)_{bir}(v)$ is a primary specialization of $\varphi^*(\psi^*(v))$. Hence both $\psi_{bir}(\psi_{bir}(v))$ and $(\psi \circ \varphi)_{bir}(v)$ are primary specializations of $(\psi \circ \varphi)^*(v)$. They are also both minimal primary specializations, since they are elements of $Val(B_1, A_1)$. By Proposition 2.3.3 we have $\varphi_{bir}(\psi_{bir}(v)) = (\psi \circ \varphi)_{bir}(v)$.

Concluding, we obtained a functor
\[
bir : pa-Rings^{op} \to af-Birat.
\]

The following example shows that the homomorphism on the stalks of $\mathcal{M}$ can indeed be not local.

**Example 3.2.8.** Let $K$ be an algebraically closed field. Consider $A = A' = B = K[T]$ and $B' = K[T, T^{-1}]$. Let $\varphi : (B, A) \to (B', A')$ be the obvious map. Clearly $\varphi$ is not adic. Passing to birational spaces, we have
\[
\varphi_{bir} : \mathfrak{X}' = Val(K[T, T^{-1}], K[T]) \to Val(K[T], K[T]) = \mathfrak{X}.
\]

As we saw in Example 3.2.3 $\mathfrak{X} = (A^1, O_{A^1}, O_{A^1})$. Let $\eta$ be the generic point of $A^1$ and of $Spec K[T, T^{-1}] \subset A^1$. Let $v$ be the valuation in $\mathfrak{X}'$ corresponding to the valuation ring $R_v = K[T](T) \subset K(T) = k(\eta)$. It is indeed in $\mathfrak{X}'$ as $K[T, T^{-1}] \otimes K[T]_{(T)} \to K(T)$ is surjective. Since $K[T] \otimes K[T]_{(T)} \to k(\eta) = K(T)$ is not surjective the pullback $v \circ \varphi$ is not in $\mathfrak{X}$. Its primary specialization $w$ is the trivial
valuation on \( k(p) = K \) for the ideal \( p = (T) \). The stalks are \( \mathcal{M}_{X',w} = K(T) \) and \( \mathcal{M}_{X,w} = K[T]_{(T)} \). The induced homomorphism of stalks is the obvious injection \( K[T]_{(T)} \to K(T) \) which is not a local homomorphism.

**Lemma 3.2.9.** Let \( (B, A), (B', A') \) be two pairs of rings and let \( \varphi : (B, A) \to (B', A') \) be a homomorphism of pairs. Then the diagram of topological spaces

\[
\begin{array}{c}
\text{Spec} B' \xrightarrow{\varphi^*} \text{Spec} B \\
\downarrow \varphi' \quad \quad \quad \downarrow \sigma \\
\text{Val}(B', A') \xrightarrow{\varphi_{\text{bir}}} \text{Val}(B, A) \\
\downarrow \varphi' \quad \quad \quad \downarrow \tau \\
\text{Spec} A' \xrightarrow{\varphi^*|_{\text{Spec} A'}} \text{Spec} A
\end{array}
\]

commutes.

**Proof.** For \( p \in \text{Spec} B' \), \( \sigma \circ \varphi^*(p) \in \text{Val}(B, A) \) is the trivial valuation on the residue field \( k(\varphi^{-1}(p)) \). Also \( \sigma'(p) \) is the trivial valuation on the residue field \( k(p) \). The homomorphism \( \varphi : B \to B' \) induces an injection \( k(\varphi^{-1}(p)) \to k(p) \), so the composition \( \varphi_{\text{bir}} \circ \sigma'(p) \) is just the trivial valuation on the residue field \( k(\varphi^{-1}(p)) \).

Now, given \( v = (p, R_v, \Phi) \in \text{Val}(B', A') \) let \( \varphi_{\text{bir}}(v) = w = (w, R_w, \Psi) \) Denote by \( m_v \) the maximal ideal of \( S_v \) and by \( m_w \) the maximal ideal of \( S_w \). The homomorphisms \( \Phi : A' \to R_v \) and \( \Psi : A \to R_w \) factor through \( S_v \) and \( S_w \) respectively, denote these by \( \Phi' : A' \to S_v \) and \( \Psi' : A \to S_w \). By Remark 2.4.13 we have \( \tau'(v) = \Phi'^{-1}(m_v) \) and \( \tau(\varphi_{\text{bir}}(v)) = \Psi'^{-1}(m_w) \). Since \( S_v \to S_w \) is local we have \( \varphi^{-1}(\Phi'^{-1}(m_v)) \cap A = \Psi'^{-1}(m_w) \). Hence

\[
\tau(\varphi_{\text{bir}}(v)) = \Psi'^{-1}(m_w) = \varphi^{-1}(\Phi'^{-1}(m_v)) \cap A = \varphi^*|_{\text{Spec} A'}(\tau'(v)).
\]

Given two pairs of rings \( (B, A), (B', A') \) and a morphism of affinoid birational spaces \( h : \text{Val}(B, A) \to \text{Val}(B', A') \), by taking global sections we get a homomorphism of rings \( \varphi \) which makes a commutative diagram

\[
\begin{array}{c}
B' \xrightarrow{\varphi} B \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\text{Nor}_{B'} A' \xrightarrow{\text{Nor}_{B} A'} \text{Nor}_{B} A.
\end{array}
\]

Composition with the inclusion \( A' \subset \text{Nor}_{B} A' \) gives a morphism of pairs \( \varphi : (B', A') \to (B, \text{Nor}_{B} A) \). Since \( \text{Val}(B, \text{Nor}_{B} A) = \text{Val}(B, A) \) by applying the bir functor we obtain another morphism of affinoid birational spaces \( \varphi_{\text{bir}} : \text{Val}(B, A) \to \text{Val}(B', A') \).

**Theorem 3.2.10.** The bir functor is an anti-equivalence of the category of pairs of rings \( \text{pa-Rings} \) localized at the class of relative normalizations and the category of affinoid birational spaces \( \text{af-Birat} \).
Proof. Denote the class of relative normalization homomorphisms by $M$. By Lemma 2.6.1 $\text{bir}$ factors through $\text{pa-Rings}_M$ and by definition of $\text{af-Birat}$ it is essentially surjective. It remains to show that $\text{bir} : \text{pa-Rings}_M \rightarrow \text{af-Birat}$ is full and faithful.

We start by proving fullness. Let $((B', A'), (B, A))$ be two pairs of rings and $h$ a morphism of affinoid birational spaces $h : (B, A)_{\text{bir}} \rightarrow (B', A')_{\text{bir}}$. We may assume that $A'$ and $A$ are integrally closed in $B'$ and $B$ respectively.

Taking global sections we get a diagram

$$
\begin{array}{ccc}
B' & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \\
A' & \xrightarrow{\psi} & A
\end{array}
$$

i.e. $\varphi$ is a homomorphism of pairs of rings $\varphi : (B', A') \rightarrow (B, A)$. We claim that $\varphi_{\text{bir}} = h$.

Given a valuation $v = (p, R_v, \Phi) \in \text{Val}(B, A)$ we denote $h(v) = w = (q, R_w, \Psi) \in \text{Val}(B', A')$. Passing to stalks we have a diagram of pairs of rings

$$
\begin{array}{ccc}
(B', A') & \xrightarrow{\varphi} & (B, A) \\
\downarrow & & \downarrow \\
(B'_q, S_w) & \xrightarrow{h_v^1} & (B_p, S_v).
\end{array}
$$

Denote by $n = p B_p$ the maximal ideal of $B_p$, $n' = q B'_q$ the maximal ideal of $B'_q$, $m_v$ the maximal ideal of $S_v$ and $m_w$ the maximal ideal of $S_w$. Since $B'_q$ is a local ring $h_v^{-1}(n) \subset n'$. Pulling back to $B'$ we see that $\varphi^{-1}(p) \subset q$. Hence the bottom line of the above diagram can be factored as

$$
\begin{array}{ccc}
B'_q & \xrightarrow{\varphi^{-1}(p)} & (B'_q)_{h_v^{-1}(n)} \\
\downarrow & & \downarrow \\
S_w & \xrightarrow{S_v \circ \varphi} & S_v.
\end{array}
$$

By Remark 2.4.4 and the definition of morphisms of birational spaces we see that the bottom left arrow is a local homomorphism. By Lemma 2.4.3 $w$ is a primary specialisation of the pullback valuation $v \circ \varphi$. As $w$ is already in $\text{Val}(B', A')$ it has no primary specialisation other than itself. By Proposition 2.3.3 the primary specialisations of $v \circ \varphi$ are linearly ordered and we conclude that $r(v \circ \varphi) = w$ where $r$ is the retraction, or in other words $\varphi_{\text{bir}}(v) = h(v)$.

As for faithfulness, given two homomorphisms of pairs of rings $\varphi, \psi : (B', A') \rightarrow (B, A)$ such that $\varphi_{\text{bir}} = \psi_{\text{bir}}$ we obtain a diagram

$$
\begin{array}{ccc}
\text{Spec}B & \xrightarrow{\varphi^*} & \text{Spec}B' \\
\downarrow \sigma & & \downarrow \sigma' \\
\text{Val}(B, A) & \xrightarrow{\varphi_{\text{bir}}} & \text{Val}(B', A')
\end{array}
$$

for the pullback valuation $\varphi^*$. The pullback valuation $\varphi^* \circ \sigma = \sigma' \circ \psi^*$ is then a pullback valuation from $\text{Val}(B', A')$ to $\text{Val}(B, A)$ given by $\psi_{\text{bir}}$, hence $\sigma' = \psi^* \circ \sigma$ is a pullback valuation from $\text{Val}(B, A)$ to $\text{Val}(B', A')$. The retraction $r$ is then the retraction $r = \phi_{\text{bir}}^* \circ \sigma_{\text{af}}$ from $\text{Val}(B', A')$ to $\text{Val}(B, A)$. Since $\varphi_{\text{bir}}(v) = h(v)$ we have $\sigma' = \psi_{\text{bir}}^* \circ \sigma_{\text{af}}$ from $\text{Val}(B, A)$ to $\text{Val}(B', A')$. Hence $\sigma_{\text{af}} = \sigma' \circ \psi_{\text{bir}} = \phi_{\text{bir}}^* \circ \sigma$, and by Proposition 2.3.3 the primary specialisations of $v \circ \varphi$ are linearly ordered and we conclude that $r(v \circ \varphi) = w$ where $r$ is the retraction, or in other words $\varphi_{\text{bir}}(v) = h(v)$.

As for faithfulness, given two homomorphisms of pairs of rings $\varphi, \psi : (B', A') \rightarrow (B, A)$ such that $\varphi_{\text{bir}} = \psi_{\text{bir}}$ we obtain a diagram

$$
\begin{array}{ccc}
\text{Spec}B & \xrightarrow{\varphi^*} & \text{Spec}B' \\
\downarrow \sigma & & \downarrow \sigma' \\
\text{Val}(B, A) & \xrightarrow{\varphi_{\text{bir}}} & \text{Val}(B', A')
\end{array}
$$

for the pullback valuation $\varphi^*$. The pullback valuation $\varphi^* \circ \sigma = \sigma' \circ \psi^*$ is then a pullback valuation from $\text{Val}(B', A')$ to $\text{Val}(B, A)$ given by $\psi_{\text{bir}}$, hence $\sigma' = \psi^* \circ \sigma$ is a pullback valuation from $\text{Val}(B, A)$ to $\text{Val}(B', A')$. The retraction $r$ is then the retraction $r = \phi_{\text{bir}}^* \circ \sigma_{\text{af}}$ from $\text{Val}(B', A')$ to $\text{Val}(B, A)$. Since $\varphi_{\text{bir}}(v) = h(v)$ we have $\sigma' = \psi_{\text{bir}}^* \circ \sigma_{\text{af}}$ from $\text{Val}(B, A)$ to $\text{Val}(B', A')$. Hence $\sigma_{\text{af}} = \sigma' \circ \psi_{\text{bir}} = \phi_{\text{bir}}^* \circ \sigma$, and by Proposition 2.3.3 the primary specialisations of $v \circ \varphi$ are linearly ordered and we conclude that $r(v \circ \varphi) = w$ where $r$ is the retraction, or in other words $\varphi_{\text{bir}}(v) = h(v)$.
It follows from Proposition \[2.4.11\] [2], Remark \[2.6.4\] the definition of a morphism of birational spaces and Lemma \[3.2.9\] that $\varphi^* = \psi^*$ as morphisms of schemes $\text{Spec} B \to \text{Spec} B'$, and hence $\varphi = \psi$ as homomorphisms of rings $B' \to B$.

If furthermore $A'$ and $A$ are integrally closed in $B'$ and $B$ respectively, then we also have that $\varphi = \psi$ as homomorphisms of pairs of rings $(B', A') \to (B, A)$. \[
\]

As an immediate corollary we obtain

**Corollary 3.2.11.** Let $(B', A'), (B, A)$ be two pairs of rings and $h$ an isomorphism of affinoid birational spaces $h : (B, A)_{\text{bir}} \xrightarrow{\sim} (B', A')_{\text{bir}}$. Then $B \cong B'$ and $\text{Nor}_B A = \text{Nor}_B A'$ as subrings of $B$.

Next we want to characterize adic morphisms.

**Proposition 3.2.12.** A homomorphism of pairs of rings $\varphi : (B_1, A_1) \to (B_2, A_2)$ is adic if and only if the induced morphism of rings

\[
(\varphi_{\text{bir}}^*, \varphi_{\text{bir}}^* ) : (\text{Val}(B_2, A_2), \mathcal{M}_{\text{Val}(B_2, A_2)}) \to (\text{Val}(B_1, A_1), \mathcal{M}_{\text{Val}(B_1, A_1)})
\]

is a morphism of locally ringed spaces.

**Proof.** If $\varphi : (B_1, A_1) \to (B_2, A_2)$ is an adic homomorphism, then by Lemma \[2.3.7\] the pullback of every valuation in $\text{Val}(B_2, A_2)$ is already in $\text{Val}(B_1, A_1)$. Hence $\varphi_{\text{bir}}$ sends $v \in \text{Val}(B_2, A_2)$ to $v \circ \varphi \in \text{Val}(B_1, A_1)$. The induced homomorphism of stalks is the canonical map

\[
\begin{array}{ccc}
B_1 & \xrightarrow{\sim} & B_2 \\
\downarrow & & \downarrow \\
S_v & \xrightarrow{\sim} & S_v
\end{array}
\]

and both the top and bottom arrows are local homomorphisms.

For the opposite direction, denote $\mathfrak{X}_1 = \text{Val}(B_1, A_1)$ and $\mathfrak{X}_2 = \text{Val}(B_2, A_2)$ and assume that the morphism of ringed spaces $(\varphi_{\text{bir}}^*, \varphi_{\text{bir}}^* ) : (\mathfrak{X}_2, \mathcal{M}_{\mathfrak{X}_2}) \to (\mathfrak{X}_1, \mathcal{M}_{\mathfrak{X}_1})$ is a morphism of locally ringed spaces. This is the same as saying that the pull back morphism $\varphi^* : \text{Spa}(B_2, A_2) \to \text{Spa}(B_1, A_1)$ restricts to a morphism $\mathfrak{X}_2 \to \mathfrak{X}_1$.

We want to show that $B_1 \otimes_{A_1} A_2 \to B_2$ is integral.

Let $b \in B_2$. We want to show that $b$ is integral over $B_1 \otimes_{A_1} A_2$. If $b \in \text{nil}(B_2)$ there is nothing to prove. Else, there is a prime $p$ of $B_2$ such that $b \notin p$. So there is some $v \in \mathfrak{X}_2$ (with kernel $p$) such that $v(b) \neq 0$. By assumption, the $A_1$-valuation $\varphi^*(v) = v \circ \varphi$ on $B_1$ is not bounded so there exists some $b' \in B_1$ such that $v(b) \leq v \circ \varphi(b')$.

Denote by $\hat{b}$ the image of $b$ in the localization $(B_2)_{v}(\varphi(v))$, and by $\hat{b}'$ the image of $b'$ in $(B_1)_{v'}$. For the induced valuation on $(B_2)_{v}(\varphi(v))$ we have $0 < v(\hat{b}) \leq v \circ \varphi(\hat{b}')$. It follows that for $b'' = \frac{1}{\varphi(\hat{b}')}$ \in $(B_2)_{v}(\varphi(v))$ we have $v(b'') \leq 1$.

Denote the image of $A_1$ in $(B_1)_{v'}$ by $A'_1$ and the image of $A_2$ in $(B_2)_{\varphi(v)}$ by $A'_2$.

It is enough to show that $b''$ is integral over $(B_1)_{v'} \otimes_{A'_1} A'_2$. Hence we may assume that $b \in B_2$, $b \notin \text{nil}(B_2)$ and there is some $v \in \mathfrak{X}_2$ with $p = \text{ker}(v)$ such that $0 < v(b) \leq 1$, and show that $b$ is integral over $B_1 \otimes_{A_1} A_2$. It is enough to show that $b$ is integral over $A_2$. 

Denote the image of $b$ in the localization $(B_2)_{\mathfrak{p}}$ by $a$. As $0 < v(b) \leq 1$, $a$ is actually the semi-valuation ring $S_v$ of $v$. By Proposition 2.6.3 there are elements $g, f_1, \ldots, f_r$ in $B_2$ generating the unit ideal and a rational domain $U = \mathcal{X}_2(\{f_1, \ldots, f_r\}/g)$ such that

$$b' = b|_U \in \mathcal{O}_{\mathcal{X}_2}(U) = \text{Nor}_{B_2}(A'_2) = \frac{A'_2}{(f_1, \ldots, f_r)}$$

that maps to $a$ via the canonical map $\mathcal{O}_{\mathcal{X}_2}(U) \to \mathcal{O}_{\mathcal{X}_2,v} = S_v$, where $A'_2$ is the image of $A_2$ in $(B_2)_g$. Again, by replacing $B_2$ with $(B_2)_g$ and $A_2$ with $A'_2$, we may assume that

$$b \in \mathcal{O}_{\mathcal{X}_2}(\mathcal{X}_2(\{f_1, \ldots, f_r\}/1)) = \text{Nor}_{B_2}(A_2[f_1, \ldots, f_r])$$

That is, $b$ is in $B_2$ and integral over $A_2[f_1, \ldots, f_r]$. So there are $c_0, \ldots, c_s \in A_2[f_1, \ldots, f_r]$ such that $(b)^{s+1} + c_0(b)^s + \ldots + c_0 = 0$. Let $c_{s+1} = 1$ and set $M = \sum_{k=0}^s \sum_{j=0}^{s+1} A_2 c_j b^k$. Now, $M$ is a finite $A_2$-module contained in $B_2$. We have $bM \subset M$ by the integral relation. As $1 \in M$, we see that $\text{Ann}_{B_2}(M) = 0$. Thus $b$ is integral \cite{ZS58} Chapter V §1.

**Definition 3.2.13.** Given a birational space $\mathcal{X}$ and a pair of schemes $(Y, X)$. If $(Y, X)_{\text{bir}} = \mathcal{X}$ we say that $(Y, X)$ is a scheme model of $\mathcal{X}$. Given another scheme model $(Y', X')$ of $\mathcal{X}$, if there is a morphism of pairs of schemes $g : (Y', X') \to (Y, X)$ such that $g_{\text{bir}}$ is the identity we say that $(Y', X')$ dominates $(Y, X)$.

4. **Relative Blow Ups**

4.1. **Construction of a Relative Blow Up.** Let $(Y \xrightarrow{f} X)$ be a pair of schemes. Then $f_* \mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_X$-algebra via $f^* : \mathcal{O}_X \to f_* \mathcal{O}_Y$. The homomorphism $f^*$ is an injection since $f$ is schematically dominant. Let $\mathcal{E}$ be a finite quasi-coherent $\mathcal{O}_X$-module contained in $f_* \mathcal{O}_Y$ and containing $f^*(\mathcal{O}_X)$. Then $f^*$ induces a homomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_Y \to \mathcal{E}$. Pulling back to $Y$ we obtain a homomorphism of $\mathcal{O}_Y$-modules $\mathcal{O}_Y \to f^*(\mathcal{E})$. Assume that we have an isomorphism of $\mathcal{O}_Y$-modules $\varepsilon : \mathcal{O}_Y \cong f^*(\mathcal{E})$ (not the homomorphism induced by $f^*$ described above). Using the multiplication of the $\mathcal{O}_X$-algebra $f_* \mathcal{O}_Y$ we define the product $\mathcal{O}_X$-module $\mathcal{E}^d$ as the image of $\mathcal{E}^\otimes d$ under $f_* \mathcal{O}_Y^\otimes d \to f_* \mathcal{O}_Y$, with the tensor over $\mathcal{O}_X$. Now, the graded $\mathcal{O}_X$-module $\mathcal{E}' = \oplus_{d \geq 0} \mathcal{E}^d$ (taking $\mathcal{E}^0 = \mathcal{O}_X$) is quasi-coherent and has a structure of a graded $\mathcal{O}_X$-algebra. We set $X_{\mathcal{E}} = \text{Proj}_X(\mathcal{E}')$. The construction gives a natural morphism $\pi_\mathcal{E} : X_{\mathcal{E}} \to X$ which is projective. We also obtain an injection $\mathcal{E}' \to f_* \mathcal{O}_Y$ which gives rise to a natural morphism of $X$-schemes

$$Y \cong \text{Spec}_X(f_* \mathcal{O}_Y) \to \text{Spec}_X(\mathcal{E}')$$

which is affine and schematically dominant.

Let $s$ be the homogeneous element of degree 1 in $\mathcal{E}'$ corresponding to $1 \in \mathcal{O}_X(X)$ i.e. $s = f^0(1) \in \mathcal{E}'(X)$. We obtain $(X_{\mathcal{E}})_s = \text{Spec}_X((\mathcal{E}'/(s - 1) \mathcal{E}'))$ and an affine, schematically dominant open immersion $(X_{\mathcal{E}})_s \to X_{\mathcal{E}}$ \cite{GD70} II§3. Considered as an element of $f_* \mathcal{O}_Y(X)$, $s = 1$. So $(X_{\mathcal{E}})_s = \text{Spec}_X(\mathcal{E}')$. Composing we get an affine dominant morphism $f_\mathcal{E} : Y \to X_{\mathcal{E}}$. In other words we have constructed a pair of schemes $(Y \xrightarrow{f_\mathcal{E}} X_{\mathcal{E}})$ with a morphism of pairs of schemes

$$g_{\mathcal{E}} = (\text{id}_Y, \pi_\mathcal{E}) : (Y, X_{\mathcal{E}}) \to (Y, X).$$
Lemma 4.1.2. Let \( A \subset B \) be as above. Then \((Y, X)_{\text{bir}} = (Y, X)_{\text{bir}}\).

Proof. The question is local on \( X \) so we assume that \( X = \text{Spec}A, Y = \text{Spec}B, A \subset B \) and \( E = \sum_{i=0}^{n} A_{i} \) a finite \( A \)-module contained in \( B \) and containing \( A \). Denote \( E' = \oplus_{d \geq 0} E^{d} \) and \( \mathfrak{x} = \text{Val}(B, A) = (Y, X)_{\text{bir}} \). The relative blow up is \( Y \to X_{E} = \text{Proj}E' \). By functoriality of \( \text{bir} \) we have a continuous map \((Y, X_{E})_{\text{bir}} \to \mathfrak{x}\). Clearly \( X_{E} \) is proper over \( X \) so by the valuative criterion of properness we are done. \( \square \)

4.2. Properties of Relative Blow Ups. Continuing the above discussion, let \((Y', f' \to X')\) be another pair and \( h : (Y', X') \to (Y, X) \) a morphism of pairs i.e.

\[
\begin{array}{ccc}
Y' & \xrightarrow{h_Y} & Y \\
\downarrow{f'} & & \downarrow{f} \\
X' & \xrightarrow{h_X} & X.
\end{array}
\]

Then \( h_{Y}^{\ast}(E) \) is a finite quasi-coherent \( h_{Y}^{\ast}(\mathcal{O}_{X}) = \mathcal{O}_{X'} \)-module. The inclusion \( E \subset f_{\ast}\mathcal{O}_{Y} \) induces a homomorphism \( h_{Y}^{\ast}(E) \to h_{Y}^{\ast}(f_{\ast}\mathcal{O}_{Y}) \) of sheaves on \( X' \). The morphism of schemes \( h_{Y} : Y' \to Y \) is equipped with a homomorphism of sheaves \( \mathcal{O}_{Y} \to h_{Y,\ast}\mathcal{O}_{Y'} \) on \( Y \). Pushing forward to \( X \) and then pulling back to \( X' \) we get a homomorphism of sheaves on \( X' \), \( h_{X}^{\ast}f_{\ast}\mathcal{O}_{Y} \to h_{X}^{\ast}(f \circ h_{Y})_{\ast}\mathcal{O}_{Y'} \). As \((f \circ h_{Y}), \mathcal{O}_{Y'} = (h_{X} \circ f'_{\ast})_{\ast}\mathcal{O}_{Y'} \) as sheaves on \( X \), we obtain a homomorphism of sheaves on \( X' \)

\[
h_{X}^{\ast}(E) \to h_{X}^{\ast}f_{\ast}\mathcal{O}_{Y} \to h_{X}^{\ast}(f \circ h_{Y})_{\ast}\mathcal{O}_{Y'} \to f'_{\ast}\mathcal{O}_{Y'}.
\]

Definition 4.2.1. We call the image of the above morphism the inverse image module of \( E \) (with respect to the morphism of pairs \( h \)) and denote it \( h^{-1}(E) \).

It is a finite quasi-coherent \( \mathcal{O}_{X'} \)-module contained in \( f'_{\ast}\mathcal{O}_{Y'} \).

We also have

\[
\mathcal{O}_{X} \xrightarrow{f_{\ast}} f_{\ast}(\mathcal{O}_{X}) \subset E \subset f_{\ast}\mathcal{O}_{Y}.
\]

Applying \( h_{X}^{\ast} \) we get homomorphisms of \( \mathcal{O}_{X'} \)-modules

\[
h_{X}^{\ast}\mathcal{O}_{X} \xrightarrow{f_{\ast}} h_{X}^{\ast}f_{\ast}(\mathcal{O}_{X}) \to h_{X}^{\ast}E \to h_{X}^{\ast}f_{\ast}\mathcal{O}_{Y}.
\]

Using the isomorphism \( h_{X}^{\ast}(\mathcal{O}_{X}) \xrightarrow{\sim} \mathcal{O}_{X'} \) we obtain

\[
\begin{array}{ccc}
h_{X}^{\ast}\mathcal{O}_{X} & \xrightarrow{h_{Y}^{\ast}f_{\ast}(\mathcal{O}_{X})} & h_{X}^{\ast}E \\
\downarrow{h_{X}^{\ast}f_{\ast}} & & \downarrow{h_{X}^{\ast}f_{\ast}} \\
\mathcal{O}_{X'} & \xrightarrow{h_{X}^{\ast}f_{\ast}\mathcal{O}_{Y}} & h_{X}^{\ast}f_{\ast}\mathcal{O}_{Y}.
\end{array}
\]
that is, the homomorphism $O_{X'} \to f'_*O_Y$, factors through $h^{-1}(\mathcal{E})$. As $f' : Y' \to X'$ is schematically dominant, $h^{-1}(\mathcal{E})$ contains the image of $O_{X'}$.

**Proposition 4.2.2.** Let $X, Y, \mathcal{E}$ be as above and $g_\mathcal{E} : (Y, X_\mathcal{E}) \to (Y, X)$ the relative blow up. Then the inverse image module $g_\mathcal{E}^{-1}(\mathcal{E})$ is an invertible sheaf on $X_\mathcal{E}$.

**Proof.** We have $g_\mathcal{E} : (Y, X_\mathcal{E}) \to (Y, X)$

\[
\begin{array}{ccc}
Y & \xrightarrow{id} & Y \\
\downarrow{f_\mathcal{E}} & & \downarrow{f} \\
X_\mathcal{E} & \xrightarrow{\pi_\mathcal{E}} & X.
\end{array}
\]

The inverse image module $g_\mathcal{E}^{-1}(\mathcal{E})$ is the image of $\pi_\mathcal{E}^*(\mathcal{E})$ in $f_\mathcal{E}^*O_Y$ (substituting $h_Y = id_Y$ and $h_X = \pi_\mathcal{E}$ in the discussion above). Again the question is local on $X$ so we assume that $X = SpecA$, $Y = SpecB$, $A \subset B$ and $E = \sum_{i=0}^n A_b_i$ such that $b_0, \ldots, b_n \in B$ generate the unit ideal.

Denote $A_i = \varphi_i(A)[\left(\frac{b_i}{m_i}\right)] \subset B_{b_i}$ (i.e. the canonical homomorphism $B \to B_{b_i}$). Then $X_\mathcal{E} = Proj(\oplus_{d \geq 0} E^d)$. As we saw in Theorem 2.6.2 the affine charts are $X_{\mathcal{E}, i} = Spec(A_i)$. Denote by $E_i$ the image of $E \otimes_A A_i$ in $B_{b_i}$ under the map induced by multiplication. Since $b_i / b_i' \in A_i$ we see that $b_i = b_i / b_i' \cdot b_i' \in E_i$. So $E_i$ is generated by the single element $b_i' \over A_i$. In other words, multiplication by $b_i'$ gives an isomorphism of modules $O_{X_\mathcal{E}}|_{X_{\mathcal{E}, i}} \xrightarrow{\sim} g_\mathcal{E}^{-1}(\mathcal{E})|_{X_{\mathcal{E}, i}}$. □

**Proposition 4.2.3** (Universal property). Let $(Y, X)$ be a pair of schemes with $\mathcal{E}$ as above. Let $(Y', X')$ be another pair with a morphism $h = (h_Y, h_X) : (Y', X') \to (Y, X)$. If $L = h^{-1}(\mathcal{E})$ is invertible on $X'$ then $h$ factors uniquely through $g_\mathcal{E}$.

**Proof.** As this is a local question, assume $X = SpecA$ and $Y = SpecB$ with $A \subset B$. Then $\mathcal{E}(X) = E = \sum_{i=0}^n A_b_i$ and $b_0, \ldots, b_n \in B$ generate the unit ideal. The graded homomorphism

\[
\delta^d : A[T_0, \ldots, T_n] \to \oplus_{d \geq 0} E^d \\
T_i \mapsto b_i \in E
\]

gives rise to a closed immersion $\delta : X_\mathcal{E} \to \mathbb{P}^n_A$ with $\delta^*O(1) = \pi_\mathcal{E}^*(\mathcal{E})$, with $\pi_\mathcal{E} : X_\mathcal{E} \to X$ as in Proposition 1.2.2. Denote by $s_0, \ldots, s_n$ the global sections in $\Gamma(X', f'_*O_{Y'})$ corresponding to $b_0, \ldots, b_n$ via

\[
h_X^*(\mathcal{E}) \to h_X^*f_*O_Y \to h_X^*(f \circ h_Y)_*O_{Y'} \to f'_*O_{Y'}.
\]

Then $s_0, \ldots, s_n$ generate the invertible $O_{Y'}$-module $L$. Hence they induce a unique morphism $\psi : X' \to \mathbb{P}^n_A$ with $L \cong \psi^*O(1)$. If $F \in A[T_0, \ldots, T_n]$ is a homogeneous polynomial of degree $d$ such that $F(b_0, \ldots, b_n) = 0 \in B$ (i.e. $F \in \ker\delta^d$) then $F(s_0, \ldots, s_n) = 0 \in \Gamma(X', L^\otimes d)$. So $\psi$ factors through $X_\mathcal{E}$ and $h_X = \pi_\mathcal{E} \circ \psi'$ where $\psi' : X' \to X_\mathcal{E}$ is given by

\[
\begin{array}{ccc}
X' & \xrightarrow{\psi} & \mathbb{P}^n_A \\
\downarrow{\psi'^{-1}} & & \downarrow{\delta} \\
X_\mathcal{E} & \xrightarrow{\pi_\mathcal{E}} & X.
\end{array}
\]
Taking $h' = (h_Y, \psi')$ we factor $h$ as

$$
(Y', X') \xrightarrow{h} (Y, X_E) \xrightarrow{g_E} (Y, X)
$$

which shows the existence.

Now, we have

$$
L = h^{-1}(\mathcal{E}) = (g_E \circ h')^{-1}(\mathcal{E}) = h'^{-1}(\mathcal{O}_{X_E}(1))
$$

So we have a surjective homomorphism of $X'$-modules

$$
\psi'^* \mathcal{O}_{X_E}(1) = h'^* \mathcal{O}_{X_E}(1) \rightarrow h'^{-1}(\mathcal{O}_{X_E}(1)) = L.
$$

Since both $\psi'^* \mathcal{O}_{X_E}(1)$ and $L$ are invertible this homomorphism is an isomorphism, and we conclude that $h'$ is unique. □

**Lemma 4.2.4.** Let $(Y, X)$ be a pair of schemes with $\mathcal{E}$ as above. Then there is a natural isomorphism of pairs of schemes

$$(Y, \text{Nor}_Y(X_E)) \simeq (Y, \text{Nor}_Y(X)_{\text{Nor}_Y(\mathcal{E})}).$$

**Proof.** The general case immediately follows from the affine case. The affine case follows from the facts that $S^{-1}\text{Nor}_B A = \text{Nor}_{S^{-1}B}(S^{-1}A)$ and $(\text{Nor}_B A)[t] = \text{Nor}_{B[t]}(A[t])$, where $(B, A)$ is a pair of rings and $S \subset A$ is a multiplicatively closed subset. □

Assume two finite, quasi-coherent $\mathcal{O}_X$-modules $\mathcal{E}'$ and $\mathcal{E}''$ on the pair of schemes $(Y, X)$, contained in $f_j \mathcal{O}_Y$ and containing the image of $\mathcal{O}_X$, together with isomorphisms of $\mathcal{O}_Y$-modules $\varepsilon' : \mathcal{O}_Y \simeq f^*(\mathcal{E}')$ and $\varepsilon'' : \mathcal{O}_Y \simeq f^*(\mathcal{E}'')$. Then $\mathcal{E} = \mathcal{E}' \cdot \mathcal{E}''$ is also a finite, quasi-coherent $\mathcal{O}_X$-module contained in $f_j \mathcal{O}_Y$ and containing the image of $\mathcal{O}_X$. Note that $\mathcal{E}$ is the image of $\mathcal{E}' \otimes_{\mathcal{O}_Y} \mathcal{E}''$ under $f_j \mathcal{O}_Y \otimes_{\mathcal{O}_X} f_k \mathcal{O}_Y \rightarrow f_j \mathcal{O}_Y$. Taking $\mathcal{E}$ to be the isomorphism obtained by the composition of isomorphisms

$$
\mathcal{O}_Y \simeq \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y \xrightarrow{\varepsilon' \otimes \varepsilon''} f^*(\mathcal{E}') \otimes_{\mathcal{O}_Y} f^*(\mathcal{E}'') \xrightarrow{f^*(\varepsilon')} f^*(\mathcal{E}),
$$

we can form the relative blow up $(Y, X_E)$. By Proposition [4.2.2] the inverse image module $g^{-1}_{E}(\mathcal{E}) = g^{-1}_{E}(\mathcal{E}') \cdot g^{-1}_{E}(\mathcal{E}'')$ is an invertible sheaf on $X_E$. Hence $g^{-1}_{E}(\mathcal{E}')$ and $g^{-1}_{E}(\mathcal{E}'')$ are also invertible on $X_E$. By the universal property of the relative blow up $(Y, X_E) \rightarrow (Y, X)$ factors through both $(Y, X_{E'}) \rightarrow (Y, X)$ and $(Y, X_{E''}) \rightarrow (Y, X)$.

**Lemma 4.2.5.** Let $(Y, X)$ be a pair of schemes. Assume that $X'$ is an open sub-scheme of $X$. Denote $Y' = f^{-1}(X')$. Then a relative blow up of $(Y', X')$ extends to a relative blow up of $(Y, X)$.

**Proof.** [Tem11] Corollary 3.4.4] □

From the Lemma and the paragraph above it we obtain:

**Corollary 4.2.6.** Let $(Y, X)$ be a pair of schemes with open sub-pairs of schemes $(Y_1, X_1), \ldots, (Y_n, X_n)$. For each $i = 1, \ldots, n$ let $(Y_i, X_i) \rightarrow (Y_i, X_i)$ be relative blow up. Then:
(1) each relative blow up $((Y_i, X_{iE_i}) \to (Y_i, X_i))$ extends to a relative blow up $(Y, X_{E'}) \to (Y, X)$.

(2) there is a relative blow up $(Y, X_E) \to (Y, X)$ which factors through each $(Y_i, X_{iE'}) \to (Y_i, X_i)$.

5. Birational Spaces in Terms of Pairs of Schemes

In this section we show that the $\text{bir}$ functor provides an equivalence of categories between the localization of the category of pairs of schemes, with respect to the class of relative blow-ups and relative normalizations, and the category of quasi-compact and quasi-separated birational spaces.

First we need to show that the functor $\text{bir}$ takes relative blow-ups and relative normalizations to isomorphisms. Lemma 4.1.2 gives the result for relative blow-ups. As for relative normalizations, given a pair of schemes $(Y, X)$, note that for every affine $U \subset X$ we have $(\text{Nor}_Y O_X)(U) = \text{Nor}_{f^*}(O_Y)(U)$ for $f: Y \to X$. From Lemma 2.6.1 we get the result.

5.1. Faithfulness.

**Theorem 5.1.1** ($\text{bir}$ is Faithful). Let $(Y \xrightarrow{f} X)$ and $(Y' \xrightarrow{f'} X')$ be pairs of schemes such that $X = \text{Nor}_Y X$ and $X' = \text{Nor}_{Y'} X'$. Denote $\mathfrak{x} = (Y, X)_{\text{bir}}$ and $\mathfrak{x}' = (Y', X')_{\text{bir}}$. Let $g_1, g_2 : (Y, X) \to (Y', X')$ be morphisms of pairs of schemes. If $g_1,_{\text{bir}} = g_2,_{\text{bir}}$ as morphisms of the birational spaces $\mathfrak{x} \to \mathfrak{x}'$. Then $g_1 = g_2$.

**Proof.** Denote $g_1,_{\text{bir}} = g_2,_{\text{bir}} = h$.

We have a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{g_1, Y} & Y' \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
\mathfrak{x} & \xrightarrow{h} & \mathfrak{x}' \\
\downarrow{\tau} & & \downarrow{\tau} \\
X & \xrightarrow{g_1, X} & X'.
\end{array}
\]

It follows from Proposition 2.4.11 and Lemma 3.2.9 that $g_1$ and $g_2$ agree on the underlying topological spaces (both $Y$ and $X$). Denote the topological part of $g_1$ and $g_2$ by $g = (g_Y, g_X)$. Furthermore, as in the faithfulness part of Theorem 3.2.10 $g_1, Y$ and $g_2, Y$ agree as morphisms of schemes.

Let $\{ (Y_i, X_i') \}_{i \in I}$ be an open affine covering of $(Y', X')$. Denote the topological pull back of each $(Y_i', X_i')$ through $g$ by $(Y_i, X_i)$. For every $i \in I$ there is an open affine covering $\{ (Y_i'^j, X_i'^j) \}_{j \in J_i}$ of $(Y_i', X_i')$. It is enough to show that the restrictions $g_1, X_{i|X_{i,j}}$ and $g_2, X_{i|X_{i,j}}$ agree as morphisms of schemes $X_{i,j} \to X'_i$ for each $j \in J_i$ and $i \in I$. Hence we may assume that $(Y', X')$ and $(Y, X)$ are affine pairs. This was already proved in the faithfulness part of Theorem 3.2.10.

5.2. Fullness. By combining [Tem11, Lemma 3.4.6] and our Remark 2.4.12 we obtain
Lemma 5.2.1. Given a quasi-compact open subspace $\mathcal{U} \subset \mathcal{X} = \text{Val}(Y, X)$, there exists a relative blow up $(Y, X_{\mathcal{E}}) \rightarrow (Y, X)$ and an open subscheme $U$ of $X_{\mathcal{E}}$ such that $\mathcal{U} = \tau^{-1}(U) = \text{Val}(f_{\mathcal{E}}^{-1}(U), U)$.

Taken together with Corollary 4.2.6, we immediately obtain

Corollary 5.2.2. Let $(Y, X)$ be a pair of schemes and let $\Omega$ be a finite family of quasi-compact open subspaces of the associated birational space $(Y, X)_{\text{bir}}$. Then there is relative blow up $(Y, X_{\mathcal{E}}) \rightarrow (Y, X)$ together with a family $\Omega'$ of open subschemes of $X_{\mathcal{E}}$ such that the associated family $\Omega'_{\text{bir}}$ coincides with $\Omega$. Furthermore if $\Omega$ covers $(Y, X)_{\text{bir}}$, the family $\Omega'$ covers $X_{\mathcal{E}}$.

We are now ready to prove fullness.

Theorem 5.2.3 (bir is Full). Let $(Y, X)$ and $(Y', X')$ be pairs of schemes and let $h : (Y', X'_{\mathcal{E}}) \rightarrow (Y, X_{\mathcal{E}})$ be a morphism of birational spaces. Then there exist a relative blow up $(Y', X'_{\mathcal{E}}') \rightarrow (Y', X')$ and a morphism of pairs $k : (Y', N_{Y', X'_{\mathcal{E}}}) \rightarrow (Y, X)$ such that $k_{\text{bir}} = h \circ g_{\text{bir}}$, where $g$ is the morphism $(Y', N_{Y', X'_{\mathcal{E}}}) \rightarrow (Y', X_{\mathcal{E}})$.

In particular $h$ is isomorphic to $k_{\text{bir}}$.

Proof. As bir factors through the localized category, by Lemma 4.2.9 we may replace $(Y', X')$ with $(Y', N_{Y', X'_{\mathcal{E}}})$. So the morphism $g$ of the statement is just the relative blow up $(Y', X_{\mathcal{E}}') \rightarrow (Y', X')$.

Consider the affine case. Let $(B, A)$ and $(B', A')$ be two pairs of rings. By Theorem 3.2.10 the morphism between the associated birational spaces $h : (B', A'_{\mathcal{E}}) \rightarrow (B, A)_{\text{bir}}$ is given by a morphism of the pairs of rings $(B, A) \rightarrow (B', N_{B', A'} A') = (B', A')$. We see that the required relative blow up is just the identity.

For the general case, denote $\mathcal{X} = (Y, X)_{\text{bir}}$ and $\mathcal{X}' = (Y', X')_{\text{bir}}$. An affine covering $(\{Y_i, X_i\})$ of $(Y, X)$ gives an affine covering $(\{X_i\})$ of $\mathcal{X}$. Each preimage $h^{-1}(\mathcal{X}_i)$ can also be covered by finitely many open affinoid birational subspaces. Hence by Corollary 4.2.2 refining the coverings in a suitable way and replacing $(Y', X')$ with a suitable relative blow up, we may assume that we have coverings $(\{Y_i, X_i\})$ of $\mathcal{X}$ and $(\{X'_i\})$ of $\mathcal{X}'$ consisting of finitely many open affinoid birational subspaces such that $h(\mathcal{X}_i) \subset \mathcal{X}_i$ for all $i$ and both are represented by affine open coverings $(\{Y_i, X_i\})$ of $(Y, X)$ and $(\{Y'_i, X'_i\})$ of $(Y', X')$.

By the affine case we obtain for every $i$ a relative blow-up $g_i : (Y'_i, X'_{i, \mathcal{E}}) \rightarrow (Y'_i, X'_i)$ and morphism $k_i : (Y'_i, X'_{i, \mathcal{E}}) \rightarrow (Y_i, X_i)$ satisfying $k_i_{, \text{bir}} = h|_{\mathcal{X}_i} \circ g_i_{, \text{bir}}$. By Corollary 4.2.7 there is some relative blow-up $g : (Y', X'_{\mathcal{E}}) \rightarrow (Y', X')$ such that all the pairs $(Y'_i, X'_{i, \mathcal{E}})$ are open sub-pairs of $(Y', X'_{\mathcal{E}})$ and form a covering. It follows from Theorem 5.1.1 that we can glue the $k_i$ and get a morphism $k : (Y', X'_{\mathcal{E}}) \rightarrow (Y, X)$ such that $k_{, \text{bir}} = h \circ g_{, \text{bir}}$. $\square$

Corollary 5.2.4. Let the pairs of schemes $(Y, X)$ and $(Y', X')$ both be scheme models for the same birational space $\mathcal{X}$. Then there is another scheme model $(Y'', X'')$ of $\mathcal{X}$ that dominates both via relative blow ups and perhaps a relative normalization.

Proof. Again we replace $(Y, X)$ and $(Y', X')$ with $(Y, N_{Y, X})$ and $(Y', N_{Y', X'})$ respectively.

We have an isomorphism $h : (Y', X'_{\text{bir}}) \rightarrow (Y, X)_{\text{bir}}$. As $Y$ is embedded in the subset of $\mathcal{X} = (Y, X)_{\text{bir}}$ of points $v$ such that $\mathcal{M}_{\mathcal{X}, v} = O_{\mathcal{X}, v}$, its scheme structure
is determined by \((X, \mathcal{M}_X)\). The same is true for \(Y'\), so \(h\) induces an isomorphism 
\(Y \cong Y'\). We assume that \(Y' = Y\).

Applying Theorem 5.2.3 we obtain a relative blow up \(g : (Y, X'_F) \to (Y', X')\) and 
a morphism of pairs \(k : (Y', X'_F) \to (Y, X)\) such that \(k \circ h = g\), in particular 
\(k \circ h : (Y', X'_F) \to (Y, X)\) is also an isomorphism. Using Theorem 5.2.3 again 
for \(k^{-1} \circ h\) we obtain a relative blow up \(j : (Y, X_F) \to (Y, X)\) and a morphism of pairs 
\(q : (Y, X_F) \to (Y', X'_F)\) such that 
\(q \circ g = k^{-1} \circ j\).

![Diagram](attachment:image.png)

As \(j = k \circ q : (Y, X_F) \to (Y, X)\) is a relative blow up, so is \(q : (Y, X_F) \to (Y', X'_F)\). Thus 
the composition of relative blow ups \(g \circ q : (Y, X_F) \to (Y', X')\) is also a relative blow up. So \((Y, X_F)\) is the required scheme model.

\[\square\]

### 5.3. Essential Surjectivity.

**Theorem 5.3.1** (bir is Essentially Surjective). Every quasi-compact and quasi-separated birational space \(X\) has a scheme model.

**Proof.** Consider a quasi-compact and quasi-separated birational space \(X\). We want 
to show that there is a pair of schemes \((Y, X)\) satisfying \((Y, X) \cong X\). We proceed 
by induction on the number of open birational spaces which cover \(X\) and have 
scheme models. As \(X\) is quasi-compact, it is enough to consider only the case of an 
affinoid covering consisting of two subspaces.

Assume that \(X\) is covered by two quasi-compact open subspaces \(U_1\) and \(U_2\), 
which admit scheme models \((V_1, U_1)\) and \((V_2, U_2)\). Set \(\mathfrak{W} = U_1 \cap U_2\). Since \(X\) is 
quasi-separated, an application of Corollary 5.2.2 shows that, after blowing-up, we 
may assume that the open immersions \(\mathfrak{W} \subset U_1\) and \(\mathfrak{W} \subset U_2\) are represented by 
open immersions of sub-pairs \((T', W') \subset (V_1, U_1)\) and \((T'', W'') \subset (V_2, U_2)\). Now, 
using Corollary 5.2.4 we can dominate the scheme models \((T', W')\) and \((T'', W'')\) 
by a third scheme model \((T, W)\) of \(\mathfrak{W}\). Using Corollary 5.2.4 we extend the corres-
ponding blow-ups to \((V_1, U_1)\) and \((V_2, U_2)\), so we may view \((T'', W'')\) 
as an open sub-pair of \((V_1, U_1)\) and \((V_2, U_2)\). Gluing both along \(W\) yields the required scheme 
model \((Y, X)\) of \(X\).

\[\square\]

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