Fans in the Theory of Real Semigroups
II. Combinatorial Theory

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March 2017

Abstract

In [DP5a] we introduced the notion of fan in the categories of real semigroups and their dual abstract real spectra and developed the algebraic theory of these structures. In this paper we develop the combinatorial theory of ARS-fans, i.e., fans in the dual category of abstract real spectra. Every ARS is a spectral space and hence carries a natural partial order called the specialization partial order. Our main result shows that the isomorphism type of a finite fan in the category ARS is entirely determined by its order of specialization. The main tools used to prove this result are: (1) Crucial use of the theory of ternary semigroups, a class of semigroups underlying that of RSs; (2) Every ARS-fan is a disjoint union of abstract order spaces (called levels); (3) Every level carries a natural involution of abstract order spaces, and (4) The notion of a standard generating system, a combinatorial tool replacing, in the context of ARSs, the (absent) tools of combinatorial geometry (matroid theory) employed in the cases of fields and of abstract order spaces.

Introduction

In [DP5a] we introduced a notion of fan in each of the dual categories RS and ARS of real semigroups, and of abstract real spectra, dubbed, respectively, RS-fans and ARS-fans. The emphasis in [DP5a] was on the algebraic theory of RS-fans. The present paper, a continuation of [DP5a], is devoted to develop the combinatorial theory of ARS-fans, i.e., fans in the category of abstract real spectra.

The Introduction to [DP5a] gives an account of the role of fans in the theories of preordered fields, of quadratic forms, and in real algebraic and analytic geometry.

Our main result in this paper is Theorem 3.11, showing that the isomorphism type of a finite ARS-fan (in the category ARS) is entirely determined by its order of specialization as a spectral space. The proof of this result relies on a combinatorial machinery that we set up in §§1,2. This machinery also gives detailed information on the structure of ARS-fans under their order of specialization.

In Section 1 we introduce the notion of level of an ARS-fan $(X, F)$. Levels are the pieces $L_I = \{h \in X \mid h^{-1}[0] = I\}$ of the partition of the character space $X$ of the real semigroup $F$, determined by the ideals $I$ of $F$. Since $F$ is a RS-fan, its ideals are necessarily prime and saturated ([DP5a], Prop. 1.6 (4), Cor. 3.10 (1)), and the family of them is totally ordered under inclusion ([DP5a], Fact 1.4). By Proposition 5.11 of [DP5a], each level is an abstract space of orders and therefore (by results from [D1], [D2] and [Li]) possesses a structure of combinatorial geometry (matroid). Further, there exist canonical AOS-morphisms linking each level to any level determined by a larger ideal (i.e., a “higher” level); Proposition 1.2 (2).

In the next §2 we exploit the combinatorial geometric structure of the levels to investigate the fine structure of ARS-fans. In Theorem 2.8 we show that multiplication of a character of

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1 For a general reference on spectral spaces, see [DST].
level $L_i$ by any pair of elements $g_1, g_2 \in X$ so that $Z(g_i) := g_i^{-1}[0] \subseteq I$ ($i = 1, 2$) defines an involution of the AOS $L_i$. These involutions are compatible with the order of specialization between levels induced by inclusion of the determining ideals (2.8(e)). Further, we prove that these involutions permute certain AOS-subfans of the levels defined by combinatorial conditions (Propositions [2.10] and [2.11]). Altogether, the results proved in this section show that the order structure of ARS-fans is subject to strong constraints, illustrated in [2.18].

To prove the isomorphism Theorem [3.11] the combinatorial machinery mentioned above is used together with the notion of a standard generating system introduced in [3.4]. This notion is a substitute for the combinatorial geometric notions existing in the context of AOSs, but absent in that of ARSs.

Preliminaries. For easy reference we state, without proof, the following simple facts proved in [DP5a] and frequently used below. The axioms defining the notion of a ternary semigroup (abbreviated TS) appear in [DP5a], Def. 1.1, and [DP1], §1, p. 100; $X_T$ denotes the set of TS-homomorphisms of a TS, $T$, into the TS $3 = \{1, 0, −1\}$ (the TS-characters of $T$).

The first Lemma gives several characterizations of the specialization order of the spectral topology on the character set of a ternary semigroup.

**Lemma 0.1** Let $T$ be a TS, and let $g, h \in X_T$. The following are equivalent:
1. $g \leadsto h$ (i.e., $h$ is an specialization of $g$).
2. $h^{-1}[1] \subseteq g^{-1}[1]$. (equivalently, $h^{-1}[-1] \subseteq g^{-1}[-1]$).
3. $g^{-1}[[0, 1]] \subseteq h^{-1}[[0, 1]]$.
4. $Z(g) \subseteq Z(h)$ and $\forall a \in G$ ($a \notin Z(h)$ $\Rightarrow$ $g(a) = h(a)$).
5. $h = h^2g$ (equivalently, $h^2 = hg$). $\square$

We also register the following algebraic characterizations of inclusion and equality of zero-sets of elements of $X_T$.

**Lemma 0.2** Let $T$ be a TS, and let $u, g, h \in X_T$. Then:
1. $Z(g) \subseteq Z(h) \iff h = hg^2$.
2. $Z(g) = Z(h) \iff g^2 = h^2$.
3. If $u \leadsto g, h$, then $Z(g) \subseteq Z(h)$ if and only if $g \leadsto h$. $\square$

**Proposition 0.3** Let $G$ be a RS-fan. Then:
1. For all elements $g, h \in X_G$ such that $g \leadsto h$ (hence $Z(g) \subseteq Z(h)$) and every ideal $I$ such that $Z(g) \subseteq I \subseteq Z(h)$ there is $f \in X_G$ such that $g \leadsto f \leadsto h$ and $Z(f) = I$.
2. For every $g \in X_G$ and every ideal $I \supseteq Z(g)$ there is a (necessarily unique) $f \in X_G$ such that $g \leadsto f$ and $Z(f) = I$.
3. For every ideal $I$ of $F$ there is an $f \in X_G$ such that $Z(f) = I$. $\square$

1 Levels of a ARS-fan

The saturated prime ideals of a real semigroup induce a partition of its character space. The pieces are called levels: the level corresponding to a saturated prime ideal $I$ of $G$ is the set of all $g \in X_G$ such that $Z(g) = I$. In the case of RS-fans, (proper) ideals —automatically prime and saturated ([DP5a], Prop. 1.6 (4), Cor. 3.10 (1))— are totally ordered under inclusion ([DP5a], Fact 1.4), a fact of much help in studying the relationship between their levels. This notion, together with that of a connected component [2.14], will be the main technical tools employed in the analysis of the fine structure of ARS-fans carried out in this paper.
Proposition 5.11 in [DP5a] shows that the levels of an ARS-fan have a canonical structure of AOS-fans (L.1(a)), that is, of fans in the category of abstract order spaces (cf. [M], §3.1, pp. 37 ff.). We prove (Proposition L.2) that inclusion of ideals induces AOS-morphisms between the corresponding levels (cf. L.1(c)). As an application we prove (Corollary L.10) that the cardinality of a finite RS-fan, \( F \), and that of its character space, \( X_F \), are related by the identity card \((F) = 2 \cdot \text{card}(X_F) + 1\), an analog for RSs of a result known to hold for reduced special groups.

Preliminaries and Notation 1.1 (a) Given a real semigroup \( G \) and a saturated prime ideal \( I \) of \( G \), we denote by \( G_I \) the RSG \((G/I) \setminus \{\pi(0)\}\), cf. [DP5a], Prop. 5.11. The congruence of \( G \) determined by the set of characters \( \mathcal{H}_I = \{h \in X_G | Z(h) = I\} \) is denoted by \( \sim_I \), cf. [DP5a], §5.C. Every character \( h \in \mathcal{H}_I \) induces a map \( \hat{h} : G_I \rightarrow \{\pm 1\} \) defined by \( \hat{h} \circ \pi_I = h \). The correspondence \( h \mapsto \hat{h} \) is a bijection between the set \( L_I(G) = \mathcal{H}_I \) and the space of orders \( X_{G_I} \) of \( G_I \). \( (L_I \) stands for “I-th level”; see item (b.ii) below). Thus, we can identify the set \( L_I(G) \subseteq X_G \) with the AOS \((X_{G_I}, G_I)\). We shall systematically use this identification in the sequel, and unambiguously refer to the AOS structure of the set \( L_I(G) \). In case \( G \) is a RS-fan, [DP5a], Prop. 5.11, shows that \( L_I(G) \) is an AOS-fan.

(b) Let \( F \) be a RS-fan.

(i) We denote by \( \text{Spec}(F) \) the set of all proper ideals of \( F \).

(ii) For \( I \in \text{Spec}(F) \) the set \( L_I(F) = \{h \in X_F | Z(h) = I\} \) is called the \( I \)-th level of \( X_F \).

(iii) For \( f \in X_F \), the depth of \( f \), denoted \( d(f) \), is the order type of the set \( \{g \in X_F | f \sim g\} \) under the order of specialization, \( \text{depthfan} \) depth of element (Since \( (X_F, \sim) \) is a root-system, the order \( \sim \) is total on this set.)

(iv) For \( I \in \text{Spec}(F) \), the order type of the set \( \{J \in \text{Spec}(F) | J \supseteq I\} \) under the (total) order of inclusion will be called the depth of \( I \), denoted \( d(I) \).

(v) The length of \( X_F \), denoted \( \ell(X_F) \), is the order type of the (totally ordered) set \( \text{Spec}(F) \).

(c) (AOS- and ARS-morphisms: [M], §2, pp. 23-24, and §6, p. 103)

(i) Let \((X, G), (Y, H)\) be ARS’s. A map \( F : X \rightarrow Y \) is an AOS-morphism iff for all \( a \in H \) there is \( b \in G \) so that \( a \circ F = b \). Here, for \( x \in G \), \( \hat{x} : X \rightarrow 3 \) denotes the map “evaluation at \( x^*\)”, \( \hat{x}(\sigma) := \sigma(x) \), for \( \sigma \in X, \) and similarly for \( H \).

(ii) The definition of an AOS-morphism is similar, with \((X, G), (Y, H)\) AOS’s, and the evaluation maps taking values in \( \{\pm 1\} \).

(d) If \( f : G \rightarrow H \) is a RS-morphism (resp. RSG-morphism), the dual map \( f^* : X_H \rightarrow X_G \) defined by \( f^*(\gamma) := \gamma \circ f \) for \( \gamma \in X_H \), is an AOS-morphism (resp., AOS-morphism).

Remarks. (a) Clearly, the union and the intersection of an inclusion chain of (proper) prime ideals in any ternary semigroup is a (proper) prime ideal. In particular, if \( F \) is a fan, the totally ordered set \( (\text{Spec}(F), \subseteq) \) is (Dedekind) complete.

Proposition 1.2 Let \( F \) be a RS-fan and let \( I \subseteq J \) be ideals of \( F \). With notation as in L.1

1. The rule \( a/J \mapsto a/I \) \((a \in F \setminus J)\) defines a homomorphism of special groups \( \iota_{JI} : F_J \rightarrow F_I \).

2. The map \( \iota_{JI} : L_I(F) \rightarrow L_J(F) \) assigning to each \( g \in L_I(F) \) the unique element \( h \in L_J(F) \) such that \( g \sim h \) is an AOS-morphism.

Proof. (1) (a) \( \iota_{JI} \) is well-defined.

We must show: \( a, b \in F \setminus J \land a \sim_J b \Rightarrow a \sim_I b \). Since \( I \subseteq J \), this is clear from Lemma 5.10 of [DP5a] which states that, for an ideal \( K \) of \( F \) and \( a, b \in F \setminus K \), \( a \sim_J b \Leftrightarrow \exists z \notin K (az = bz) \).

Clearly, we have:
(b) \( \iota_{JJ} \) is a group homomorphism sending \(-1/J\) to \(-1/I\).

Since \( F_j \) is a RSG-fan, \( \iota_{JJ} \) is automatically a homomorphism of special groups.

(2) By (1) and \( \text{(1.1)} \), the map \( \iota^*_I : X_{F_I} \to X_{F_J} \), dual to \( \iota_{JJ} \), is an AOS-morphism. The map \( \kappa_I \) is \( \kappa_I = (\varphi_I)^{-1} \circ \iota^*_I \circ \varphi_I \), where \( \varphi_I \) denotes the bijection \( g \mapsto \hat{g} (g \in L_I(F)) \), identifying \( L_I(F) \) with \( X_{F_I} \), \( \text{(1.1)} \), and similarly for \( L_J(F) \). It only remains to prove \( g \mapsto \kappa_I(g) \), for \( g \in L_I(F) \).

To ease notation, write \( h = \kappa_I(g) \). According to Lemma \( \text{(1.1)} \) we must show \( Z(g) \subseteq Z(h) \) and \( a \notin Z(h) \Rightarrow g(a) = h(a) \). The inclusion of zero-sets is \( I \subseteq J \). Let \( a \notin Z(h) = J \). Since:

\[
\varphi_J(h) = \varphi_J(\kappa_I(g)) = \iota^*_J(\varphi_I(g)) = \varphi_I(g) \circ \iota_{JJ} \quad \varphi_I(g)(a/I) = g(a) \quad \text{and} \quad \varphi_J(h)(a/J) = h(a),
\]

as required. \( \square \)

Next we prove that the depth of an ideal in a fan is the same as the depth of any element in the corresponding level; in particular, elements of the same depth belong to the same level.

**Proposition 1.3** Let \( F \) be a RS-fan. For \( f \in X_F \), we have \( d(f) = d(Z(f)) \); equivalently, the sets \( \{ g \in X_F | f \leadsto g \} \) (ordered under specialization) and \( \{ J \in \text{Spec}(F) | J \supseteq Z(f) \} \) (ordered under inclusion) are order-isomorphic.

**Proof.** To ease notation, set \( f \uparrow = \{ g \in X_F | f \leadsto g \} \) and \( I \uparrow = \{ J \in \text{Spec}(F) | J \supseteq \} \) (\( I \in \text{Spec}(F) \)). The required order isomorphism is the map \( Z : f \uparrow \to Z(f) \uparrow \) assigning to each \( g \in f \uparrow \) its zero-set. That

(a) \( Z \) is increasing, \quad and \quad (b) \( Z \) is surjective,

is clear, from \( g \leadsto h \Rightarrow Z(g) \subseteq Z(h) \) and Proposition \( \text{1.1} \) (2), respectively. That

(c) \( Z \) is injective,

follows from \( \text{1.2} \) (3). \( \square \)

A trivial variant of the proof of \( \text{1.3} \) gives:

**Proposition 1.4** Let \( F \) be a RS-fan. Given \( f_1, f_2 \in X_F \), such that \( f_1 \leadsto f_2 \), the intervals \( \{ g \in X_F | f_1 \leadsto g \leadsto f_2 \} \) (ordered under specialization) and \( \{ J \in \text{Spec}(F) | Z(f_1) \subseteq J \subseteq Z(f_2) \} \) (ordered under inclusion) are order-isomorphic. \( \square \)

The results in the next two Lemmas will be frequently used in this and in subsequent sections.

**Lemma 1.5** Let \( G \) be a RS and let \( g_1, \ldots, g_r, h \in X_G \) be so that \( \bigcup_{i=1}^r Z(g_i) \subseteq Z(h) \). For \( i = 1, \ldots, r \), let \( f_i \in X_G \) be such that \( g_i \leadsto f_i \) and \( Z(g_i) \subseteq Z(f_i) \subseteq Z(h) \). Then,

\[
(*) \quad h \cdot g_1 \cdot \ldots \cdot g_r = h \cdot f_1 \cdot \ldots \cdot f_r.
\]

Note. The products in \( (*) \) may not be in \( X_G \).

**Proof.** Obviously, \( (*) \) holds whenever \( x \in Z(h) \). If \( x \notin Z(h) \), from the assumptions we get \( x \notin \bigcup_{i=1}^r Z(g_i) \) and \( x \notin \bigcup_{i=1}^r Z(f_i) \). Since \( g_i \leadsto f_i \), we get \( g_i(x) = f_i(x) \) for \( i = 1, \ldots, r \) (Lemma \( \text{1.1} \) (4)), and \( (*) \) follows. \( \square \)

**Lemma 1.6** Let \( F \) be a RS-fan. Then,

(a) \( \quad \text{For } i = 1, \ldots, r, \text{ with } r \text{ odd, let } g_i, h_i \in X_F \text{ be such that } g_i \leadsto h_i \). Then, \( g_1 \cdot \ldots \cdot g_r \leadsto h_1 \cdot \ldots \cdot h_r. \)

(b) \( \quad \text{Let } h_1, h_2, f, g, k \in X_F \text{ be such that } f, g \leadsto h_1, k \leadsto h_2 \), and \( Z(h_1) \subseteq Z(h_2) \). Then, \( fgk \leadsto h_2. \)


Proposition 1.9

Corollary 1.10

For a finite RS-fan, since the preceding cardinality identities.

Proof. (a) For \( i = 1, \ldots, r \) we have \( h^2 = h_i g_i \) (Lemma 1.(5)). Multiplying these equalities termwise gives \((h_1 \cdots h_r)^2 = (h_1 \cdots h_r)(g_1 \cdots g_r)\), which proves the assertion.

(b) By Lemma 1.11 we must prove \( h^2 = h_2(fgk) \). Obviously, this equality holds at every \( x \in Z(h_2) \). If \( x \notin Z(h_2) \), then \( x \notin Z(h_1) \), and \( f, g \sim h_1 \) implies \( h_1(x) = f(x) = g(x) \neq 0 \); also \( k \sim h_2 \) implies \( h_2(x) = k(x) \neq 0 \), whence \( f(x)g(x) = 1 \) and \( h_2(x)k(x) = 1 \). This yields \((h_2fgk)(x) = (f(x)g(x))(h_2(x)k(x)) = 1 \). On the other hand, \((h_2(x))^2 = 1\), proving that the required identity holds at \( x \notin Z(h_2) \) as well. 

Our last result in this section, Corollary 1.10, shows that if \( F \) is a finite RS-fan and \( X_F \) its character space, then \( \text{card}(F) = 2 \cdot \text{card}(X_F) + 1 \). This identity is the analog of a well-known result relating the cardinalities of a finite RSG-fan and its space of orders (ABR, p. 75). The result follows from a more general observation, valid for RS-fans of arbitrary cardinality.

Proposition 1.7 Let \( I \subseteq J \) be consecutive ideals of a RS-fan \( F \) (with, possibly, \( J = F \)). Then,

(i) Under product induced by \( F \), \( J \setminus I \) is a group of exponent 2 with unit \( x^2 \) for any \( x \in J \setminus I \) (and distinguished element \( -1 = -x^2 \)).

(ii) The restriction of the quotient map \( \pi_I : (J \setminus I) : J \rightarrow F/I \rightarrow \{\pi_I(0)\} \) is a group isomorphism preserving the distinguished element \(-1\).

Proof. (i) Since \( I \) is prime, \( J \setminus I \) is closed under product. Given \( x, y \in J \setminus I \), we must prove \( x^2 = y^2 \) (which implies \( x^2y = y^2y \)). By the separation theorem for TSSs (DP1, Thm. 1.9, pp. 103-104) it suffices to show that \( h(x^2) = h(y^2) \) for all \( h \in X_F \). If \( J \subseteq Z(h) \), then \( h(x^2) = h(y^2) \). If \( Z(h) \subseteq I \), then \( h(x), h(y) \neq 0 \), and \( h(x^2) = h(y^2) \).

(ii) Clearly, \( \pi_I(x) \neq \pi_I(0) \), i.e., \( \pi_I(x) \in F_I \), for all \( x \in J \setminus I \), and \( \pi_I \) preserves product.

— \( \pi_I \) \((J \setminus I)\) is injective.

Suppose \( \pi_I(x) = \pi_I(y) \), i.e., \( x \sim y \), with \( x, y \in J \setminus I \). By DP5a, Lemma 5.10 (cf. proof of 1.2), \( xz = yz \) for some \( z \notin I \). To prove \( x = y \), let \( h \in X_F \). If \( J \subseteq Z(h) \), then \( h(x) = h(y) = 0 \). If \( Z(h) \subseteq I \), then \( h(z) \neq 0 \), and we get \( h(x) = h(y) \).

— \( \pi_I(x^2) = \pi_I(1) \), for \( x \in J \setminus I \).

Clear, for \( Z(h) = I \) implies \( h(x^2) = 1 \). In particular, \( \pi_I \) preserves \(-1\).

— \( \pi_I \) \((J \setminus I)\) is onto \( F_I \).

Let \( p \in F_I \); then, \( p = \pi_I(q) \) with \( q \notin I \). Taking \( z \in J \setminus I \), we have \( qz^2 \in J \setminus I \), whence \( \pi_I(qz^2) = \pi_I(q) \pi_I(z^2) = \pi_I(q) \pi_I(1) = \pi_I(q) = p \).

Notation 1.8 Let \( F \) be a finite RS-fan, and let

\[
\{0\} = I_n \supset I_{n-1} \supset \cdots \supset I_2 \supset I_1 \subset F = I_0 \]

be the set of all ideals; thus, for \( 1 \leq d \leq n \), \( I_d \) is the ideal of depth \( d \). We set \( F_d = F/I_d = (F/I_d) \setminus \{\pi_d(0)\} \), where \( \pi_d : F \rightarrow F/I_d \) denotes the canonical quotient map. We also write \( L_d \) for \( L_{I_d} \); cf. 1.11(b).

Clearly, \( F \setminus \{0\} = \bigcup_{d=1}^n (I_d-1 \setminus I_d) \) (disjoint union), whence, by 1.7 we have \( \text{card}(F) = \sum_{d=1}^n \text{card}(I_d-1 \setminus I_d) + 1 = \sum_{d=1}^n \text{card}(F_d) + 1 \). Since the levels partition \( X_F \), 1.11 yields:

Proposition 1.9 For any finite RS-fan \( F \), \( \text{card}(X_F) = \sum_{d=1}^n \text{card}(L_d) = \sum_{d=1}^n \text{card}(X_{F_d}) \).

Corollary 1.10 For a finite RS-fan, \( F \), \( \text{card}(F) = 2 \cdot \text{card}(X_F) + 1 \).

Proof. Since the \( F_d \) are finite RSG-fans (DP5a, Prop. 5.11), we know that \( \text{card}(F_d) = 2 \cdot \text{card}(X_{F_d}) \) for \( 1 \leq d \leq n \) (see ABR, p. 75). The result follows, then, from Proposition 1.7 and the preceding cardinality identities.
2 Involution of ARS-fans

Notation 2.1 In addition to the notation introduced in Definition 1.1 for \( J \subseteq I \) in \( \text{Spec}(F) \) we define the sets:
\[
S^I_J = \{ h \in L_I | \exists g \in X_F (g \sim h \land Z(g) = J) \}.
\]
\[
C^I_J = \{ h \in S^I_J | \forall g' \in X_F (g' \sim h = J \subseteq Z(g')) \}.
\]
That is, \( S^I_J \) consists of those elements of level \( I \) having predecessors of level \( J \) or lower in the specialization partial order; \( C^I_J \) is the set of elements in \( L_I \) having predecessors at level \( J \) but not lower.

Remarks 2.2 (i) For \( I \in \text{Spec}(F) \), \( S^I_{(0)} = C^I_{(0)} \), and \( S^I_I = L_I \). (Recall that \( \{0\} \) is the smallest member of \( \text{Spec}(F) \), i.e., the zero-set of the lowest level of \( X_F \).)

(ii) For \( J \subseteq I \) in \( \text{Spec}(F) \), \( S^I_J \neq \emptyset \).

Proof. Let \( g \in X_F \) be such that \( Z(g) = J \) (exists by Proposition 1.3 (3)). If \( h \) is the unique \( \sim \)-successor of \( g \) of level \( I \) (Proposition 1.3 (2)), then \( h \in S^I_J \).

(iii) For \( J \subseteq I \) in \( \text{Spec}(F) \), \( S^I_J = \text{Im}(\kappa_J) \), where \( \kappa_J : L_J(F) \longrightarrow L_I(F) \) is the AOS-morphism defined in Proposition 1.2 (2).

(iv) For \( J \subseteq I \) in \( \text{Spec}(F) \), \( S^I_J \supseteq \bigcup \{ C^I_{J'} | J' \in \text{Spec}(F) \text{ and } J' \subseteq J \} \). (Note that \( C^I_{J'} \) may be empty for some \( J' \subseteq J \).)

(v) For \( J \subseteq I \) in \( \text{Spec}(F) \), \( C^I_J = S^I_J \setminus \bigcup \{ S^I_{J'} | J' \in \text{Spec}(F) \text{ and } J' \subset J \} \).

(vi) For \( J, J' \subseteq I \) in \( \text{Spec}(F) \), \( J \neq J' \), we have \( C^I_J \cap C^I_{J'} = \emptyset \).

In order to render later arguments as transparent as possible, we recall the following simple (and well-known) facts about fans in the categories RSG and AOS.

Lemma 2.3 Let \( g : H \longrightarrow G \) be a SG-homomorphism between RSG-fans, and let \( g^* : (X_G, G) \longrightarrow (X_H, H) \) denote the AOS-morphism dual to \( g \) (cf. 1.1 (d)). Then,

(1) With representation induced by that of \( H \), \( \text{Im}(g) \) is a RSG-fan, and \( G \) is isomorphic to the extension of \( \text{Im}(g) \) by the exponent-two group \( \Delta = G/\text{Im}(g) \).

(2) \( (\text{Im}(g^*), H/\text{ker}(g)) \) is an AOS-fan.

Remarks 2.4 (a) For the definition of extension of a SG by a group of exponent two, see DMI, Ex. 1.10, p. 10.

(b) By the duality between RSGs and AOSs (DMI, Ch. 3), the dual statement holds as well: given an AOS-morphism of (AOS)-fans, \( \kappa : (X, G) \longrightarrow (Y, H) \), the assertions (1) and (2) hold with \( g := \kappa^* \) (the SG-morphism dual to \( \kappa \)), and with \( g^* = \kappa \).

Sketch of proof of 2.3. (1) The first assertion is easily checked. For the second, \( \text{Im}(g) \) is a direct summand of the group \( G \). Let \( pr : G \longrightarrow \text{Im}(g) \) be the projection onto the factor \( \text{Im}(g) \); \( pr \) is a SG-morphism (\( G \) and \( \text{Im}(g) \) are fans), and is the identity on \( \text{Im}(g) \). The isomorphism between \( G \) and \( \text{Im}(g)[\Delta] \) is \( f(a) = \langle pr(a), a/\text{Im}(g) \rangle \), for \( a \in G \).

(2) Recall that \( g^* \) is defined by composition, \( g^*(\sigma) = \sigma \circ g (\sigma \in X_G) \), see 1.1 (d), and that \( \text{Im}(g^*)^\perp = \bigcap \{ \text{ker}(\gamma) | \gamma \in \text{Im}(g^*) \} = \bigcap \{ \text{ker}(\sigma \circ g) | \sigma \in X_G \} \). Routine checking from these definitions proves that \( \text{Im}(g^*) \) is closed under product of any three members (since \( X_G \) is), and that \( \text{Im}(g^*)^\perp = \text{ker}(g) \) (since \( \bigcap \{ \text{ker}(\sigma) | \sigma \in X_G \} = \{1\} \)), whence \( \text{Im}(g^*) \subseteq X_{H/\text{ker}(g)} \).

Clearly, the map \( \overline{\gamma} : H/\text{ker}(g) \longrightarrow \text{Im}(g) \) induced by \( g \) is an SG-isomorphism. Thus, we have a commutative diagram of SG-morphisms:
\[
\begin{array}{ccc}
H \xrightarrow{g} \text{Im}(g) & \xrightarrow{\text{pr}} & G \\
\pi \downarrow & & \downarrow \text{pr}
\end{array}
\]

\[ \text{Im}(g)[\Delta] \]

It only remains to show that \( \text{Im}(g^*) \supseteq X_{H/\ker(g)} \). Any SG-character \( \gamma : H/\ker(g) \to \mathbb{Z}_2 \) can be lifted to a map \( \sigma : G \to \mathbb{Z}_2 \), via the identification of \( G \) with \( \text{Im}(g)[\Delta] \), as follows: for each \( a \in G \) there is \( b \in H \) such that \( \text{pr}(a) = g(b) \). We set \( \sigma(a) = \gamma(b/\ker(g)) = \gamma(\pi(b)) \). In terms of the diagram above, we have: \( \sigma = \gamma \circ (\overline{\pi})^{-1} \circ \text{pr} \). It follows that \( \sigma \) is a well-defined SG-morphism, i.e., \( \sigma \in X_G \), and (since \( \text{pr} \circ g = g \) and \( (\overline{\pi})^{-1} \circ g = \pi \)), \( g^*(\sigma) = \sigma \circ g = \gamma \circ \pi \). □

Lemma 2.3 together with 2.2(iii) and 1.2(2), gives:

**Corollary 2.5** Let \( F \) be a RS-fan, and let \( J \subseteq I \) be in \( \text{Spec}(F) \). The set \( S_F^I \) is an AOS-fan. Indeed, it is a sub-fan of \( L_I(F) \), when the latter is endowed with its structure of AOS-fan, as indicated in 1.4. More generally, if \( F \subseteq L_I(F) \) is an AOS-fan, the set \( \{ h \in L_I | \exists g \in F \cdot (g \rightsquigarrow h) \} \) is an AOS-subfan of \( L_I(F) \).

**Proof.** The first assertion is a special case of the second (with \( F = L_I(F) \)). For the latter, observe that \( S_F^I(F) = \kappa_{JF[F]} = \text{Im}(\kappa_{JF[F]}) \) and use Remark 2.4(b).

The following definition will have a crucial role in the sequel:

**Definition 2.6** Let \( F \) be a RS-fan, let \( g_1, g_2 \in X_F \), and fix \( I \in \text{Spec}(F) \) so that \( Z(g_1), Z(g_2) \subseteq I \). We define a map \( \varphi_{g_1,g_2}^I : L_I(F) \to L_I(F) \) as follows: for \( h \in L_I(F) \),

\[ \varphi_{g_1,g_2}^I(h) = hg_1g_2. \]

**Note.** Since \( Z(g_i) \subseteq I = Z(h) \) (\( i = 1,2 \)), we have \( Z(hg_1g_2) = I \), whence \( hg_1g_2 \in L_I \).

**Fact 2.7** With notation as in Definition 2.6, let \( J \in \text{Spec}(F) \) be such that \( Z(g_1) \cup Z(g_2) \subseteq J \subseteq I \), and for \( i = 1, 2 \), let \( g_i' \) be the unique \( \rightsquigarrow \)-successor of \( g_i \) of level \( J \). Then, \( \varphi_{g_1,g_2}^J = \varphi_{g_1',g_2'}^J \).

Thus, in 2.6 we may assume \( Z(g_i) = Z(g_2) \).

**Proof.** Lemma 1.5 shows that \( hg_1g_2 = hg_1'g_2' \), for \( h \in L_I \).

**Theorem 2.8** With notation as in Definition 2.6, we have:

(a) \( \varphi_{g_1,g_2}^J \) is an AOS-automorphism of \( L_I \).
(b) \( \varphi_{g_1,g_2}^J \) is an involution: for \( h \in L_I \), \( \varphi_{g_1,g_2}^J(\varphi_{g_1,g_2}^J(h)) = h \).
(c) For \( i = 1, 2 \), let \( h_i \) be the unique \( \rightsquigarrow \)-successor of \( g_i \) in \( L_I \). Then, \( \varphi_{g_1,g_2}^J(h_i) = h_2 \).

In particular,

(d) If \( g_1, g_2 \), have a common specialization \( h \) at some level \( I \supseteq Z(g_1), Z(g_2) \), then \( h \) is a fixed point of \( \varphi_{g_1,g_2}^J \).

(e) Let \( J \subseteq I \) be in \( \text{Spec}(F) \). Assume \( Z(g_1), Z(g_2) \subseteq J \), and let \( h_1 \in L_J, h_2 \in L_I \). Then,

\[
\begin{align*}
h_1 \rightsquigarrow h_2 & \Rightarrow \varphi_{g_1,g_2}^J(h_1) \rightsquigarrow \varphi_{g_1,g_2}^J(h_2).
\end{align*}
\]

For the proof of this Theorem we will need an improvement on 1.1(a), valid for fans but not for arbitrary RSs; namely:

**Fact 2.9** Let \( F \) be a RS-fan, and \( I \) be an ideal of \( F \). Any \( g \in X_F \) such that \( Z(g) \subseteq I \) induces an SG-character \( \hat{g} : F_I \to \mathbb{Z}_2 \), by setting \( \hat{g} \circ \pi_I = g \).
Proof. The only delicate point is well-definedness: for \( a \in F \setminus I \), \( a \sim_I 1 \Rightarrow g(a) = 1 \). By [DP5a], Lemma 5.10, \( a \sim_I 1 \) means \( az = z \) for some \( z \not\in I \) (see proof of [1.2]); then \( g(z) \neq 0 \), and taking images under \( g \) in this equality yields \( g(a) = 1 \).

Proof of Theorem 2.8. We begin by proving:

(b) For \( h \in L_1(F) \), \( \varphi^{g_1,g_2}_I(h) = h g_1^2 g_2^2 \). But \( h g_1^2 g_2^2 = h \); this is clear if \( h(x) = 0 \) \( (x \in F) \); if \( h(x) \neq 0 \), then \( g_1(x) \neq 0 \) (since \( Z(g_1) \subseteq Z(h) \)), and hence \( g_1^2(x) = 1 \) \( (i = 1, 2) \), proving the stated identity, and item (b).

(a) i) \( \varphi^{g_1,g_2}_I \) is an AOS-morphism.

Since \( F_I \) is the RSG-fan dual to \( L_I(F) \), we must show (see [1.1](c)):

(*) For every \( \alpha \in F_I \), there is \( \beta \in F_I \) such that \( \hat{\alpha} \circ \varphi^{g_1,g_2}_I = \hat{\beta} \),

where \( \hat{\alpha} : X_{F_I} \to \mathbb{Z}_2 \) denotes evaluation at \( \alpha \). We claim that \( \beta = \alpha \hat{g}_1(\alpha) \hat{g}_2(\alpha) \) does the job.
By Fact 2.9 \( \hat{g}_1(\alpha) \in \mathbb{Z}_2 \) \( (i = 1, 2) \), whence \( \beta \in F_I \). For \( h \in L_I(F) \) we have:

\[
(\hat{\alpha} \circ \varphi^{g_1,g_2}_I)(h) = \hat{\alpha}(h g_1 g_2) = h(\alpha) \hat{g}_1(\alpha) \hat{g}_2(\alpha) = h(\alpha \hat{g}_1(\alpha) \hat{g}_2(\alpha)) = h(\beta) = \hat{\beta}(h),
\]

as required. Note that (b) implies

ii) \( \varphi^{g_1,g_2}_I \) is bijective.

iii) The dual map \( (\varphi^{g_1,g_2}_I)^* : F_I \to F_I \) is also bijective.

Item (i) proves that, for \( \alpha \in F_I \), \( (\varphi^{g_1,g_2}_I)^*(\alpha) = \alpha \hat{g}_1(\alpha) \hat{g}_2(\alpha) \). For injectivity, assume \( \alpha \hat{g}_1(\alpha) \hat{g}_2(\alpha) = 1 \); if \( \hat{g}_1(\alpha) \hat{g}_2(\alpha) = 1 \), then \( \alpha = 1 \), whence (as \( \hat{g}_1 \) is a SG-character), \( \hat{g}_1(\alpha) = 1 \) \( (i = 1, 2) \), and \( \alpha = 1 \), contradiction. Thus, \( \hat{g}_1(\alpha) \hat{g}_2(\alpha) = 1 \), which entails \( \alpha = 1 \). For surjectivity, given \( F_I \), set \( \alpha = \beta \hat{g}_1(\beta) \hat{g}_2(\beta) \). Then \( \hat{g}_1(\alpha) = \hat{g}_2(\beta) \) and \( \hat{g}_2(\alpha) = \hat{1}(\beta) \), whence \( (\varphi^{g_1,g_2}_I)^*(\alpha) = \beta \).

(c) We must prove \( h_4 g_1 g_2 = h_2 \). This clearly holds at any \( x \in Z(h_1) = Z(h_2) \). If \( x \not\in Z(h_1) \) \( (i = 1, 2) \), then \( x \in Z(g_i) \); since \( g_i \not\to h_1 \), it follows \( h_1(x) = g_i(x) \neq 0 \) (Lemma 0.1(4)), and \( h_1(x) g_i(x) = 1 \); hence, \( h_1 g_1 g_2(x) = g_2(x) = h_2(x) \).

(e) Lemma 1.6(a) immediately implies \( h_2^2 = h_2 h_1 \Rightarrow (h_2 g_1 g_2)^2 = (h_2 g_1 g_2)(h_1 g_1 g_2) \).

By use of these involutions we obtain a number of regularity results concerning the order structure of ARS-fans.

**Proposition 2.10** Let \( F \) be a RS-fan. For \( J \subseteq J_1 \subseteq J_2 \subseteq I \) in \( \text{Spec}(F) \), and \( h \in S_I \) set:

\[
B^{J_1,J_2}(h) = \{ g \in S_{J_1}^F \mid g \sim_I h \}, \quad \text{and} \quad A^{J_1,J_2}(h) = \{ g \in C_{J_2}^I \mid g \sim_I h \}.
\]

Then,

(a) For \( h_1, h_2 \in S_I \), we have \( \text{card}(B^{J_1,J_2}(h_1)) = \text{card}(B^{J_1,J_2}(h_2)) \).

(b) For \( h_1, h_2 \in C_I \), we have \( \text{card}(A^{J_1,J_2}(h_1)) = \text{card}(A^{J_1,J_2}(h_2)) \).

**Remark.** The assumptions of the Proposition guarantee that the sets \( B^{J_1,J_2}(h) \) are non-empty. In fact, given \( h \in S_I \), there is \( u \sim_I h \) so that \( Z(u) \subseteq J \); set \( J' = Z(u) \). Since \( J' \subseteq J \subseteq J_2 \), \( u \) has a unique \( \sim_I \) successor \( g \) in \( L_{J_2} \). But \( u \sim g \), \( h \) and \( J_2 = Z(g) \subseteq I = Z(h) \) imply \( g \sim_I h \) (Lemma 0.2(3)). Since \( J' \subseteq J \subseteq J_1 \), we conclude that \( g \in S_{J_1}^F \), i.e., \( g \in B^{J_1,J_2}(h) \).

The sets \( A^{J_1,J_2}(h) \) may be empty for some choices of \( h \) and the \( J_i \)'s. However, if \( h \in C_I \) and \( J_1 = J \), we have \( A^{J_1,J_2}(h) \neq \emptyset \). Indeed, in this case the element \( g \in S_{J_2} \) constructed above is in \( C_{J_1} \); if \( g \in S_{J_1}^F \) for some \( J' \subseteq J \), then \( g \sim_I h \) would imply \( h \in S_{J_1}^F \), contrary to the assumption \( h \in C_I \).

**Proof of Proposition 2.10.** (a) With \( J_1, J_2 \) as in the statement, write \( B_i \) for \( B^{J_1,J_2}(h_i) \) \( (i = 1, 2) \).
1, 2). The assumption \( h_i \in S^I_j \) implies the existence of elements \( u_i \in X_F \) so that \( u_i \leadsto h_i \) and \( Z(u_i) \subseteq J \). Replacing \( u_i \) by its unique successor of level \( J \) we may assume \( Z(u_i) = J \) (see 2.7). We fix \( u_i \)'s with these properties throughout the proof, and for \( J \subseteq J' \subseteq I \) we denote by \( \varphi_{J'} \) the involution \( \varphi_{u_1^{a_1} u_2^{a_2}} \) of \( L_{J'} \) defined in 2.4.

Since the maps \( \varphi_{J'} \) are bijective, it is enough to prove \( \varphi_{J'_2}[B_1] = B_2 \). Further, since \( \varphi_{J'_2} \) is an involution it suffices just to prove the inclusion \( \subseteq \), i.e.,

\[ (*) \quad g \in S^J_{J'_1} \text{ and } g \leadsto h_1 \Rightarrow \varphi_{J'_2}(g) \leadsto h_2 \text{ and } \varphi_{J'_2}(g) \in S^J_{J'_1}. \]

(i) \( \varphi_{J'_2}(g) = g u_1 u_2 \leadsto h_2. \)

Immediate consequence of Lemma 1.6(b), since \( g, u_1 \leadsto h_1 \) and \( u_2 \leadsto h_2 \).

(ii) \( \varphi_{J'_2}(g) \in S^J_{J'_1}. \)

Since \( g \in S^J_{J'_1} \), there is \( v \leadsto g \) so that \( Z(v) = J \supseteq J = Z(u_i) \) \( (i = 1, 2) \); thus, \( v \) is in the domain of \( \varphi_{J'}(v) = \varphi_{u_1^{a_1} u_2^{a_2}}(v) \), and Theorem 2.8(e) gives \( \varphi_{J'}(v) \leadsto \varphi_{J'_2}(g) \), proving (ii) and item (a).

(b) Write \( A_i \) for \( A^{J'_1, J'_2}(h_i) \) \( (i = 1, 2) \). As above, it suffices to prove the analogue of \( (*) \):

\[ (**) \quad g \in S^J_{J'_1} \text{ and } g \leadsto h_1 \Rightarrow \varphi_{J'_2}(g) \leadsto h_2 \text{ and } \varphi_{J'_2}(g) \in S^J_{J'_1}. \]

(iii) There is no \( w \in X_F \) such that \( Z(w) \subseteq J_1 \) and \( w \leadsto \varphi_{J'_1}(g) \).

Otherwise, we would have \( w \leadsto \varphi_{J'_2}(g) \leadsto h_2 \) (the last relation holding by \( (*) \)). Since \( h_2 \in C^I_{J'_1} \), we get \( J \subseteq Z(w) \), and since \( Z(u_i) = J \), \( \varphi_{Z(w)}(w) \) is defined. Theorem 2.8(e) applied to the first of the preceding inequalities yields: \( \varphi_{Z(w)}(w) \leadsto \varphi_{J'_2}( \varphi_{J'_2}(g) ) = g \). This, together with \( \varphi_{Z(w)}(w) \in L_{Z(w)} \) and \( Z(w) \subseteq J_1 \), contradicts the assumption \( g \in C^{J'_1}_{J'_2} \), proving (iii), and item (b).

A slight variant of the argument proving Proposition 2.10 yields:

**Proposition 2.11** Let \( F \) be a RS-fan and let \( J \subseteq I \) be in \( \text{Spec}(F) \). For \( g_1, g_2 \in X_F \), such that \( Z(g_i) \subseteq J \) \( (i = 1, 2) \), the map \( \varphi_{g_1^{a_1} g_2^{a_2}} \) is a permutation of \( S^I_J \) and of \( C^I_J \).

For a RS-fan, \( F \), and \( h \in X_F \), we denote by \( P_h = \{ g \in X_F | g \leadsto h \} \) the root-system of predecessors of \( h \) under specialization. We begin by proving:

**Proposition 2.12**

1. \( P_h \) is an ARS-fan. In particular,
2. Any connected component of an ARS-fan is an ARS-fan.

**Proof.** (1) Lemma 0.1.2 shows that \( g \leadsto h \) iff \( T = h^{-1}[1] \subseteq g^{-1}[1] \). With notation as in \( M \), §6.3, p. 110, and §6.6, p. 126, the latter condition just means \( g \in U(T) \), i.e., \( P_h \) is the saturated set \( U(T) = W(T) \cap U(T^2) \). \( M \), Cor. 6.6.8, p. 126, proves that sets of this form are ARSs. Lemma 1.6(a) shows that it is closed under products of three elements, hence a fan by the results of \( DP5a \), §3.

(2) Follows from (1) by taking \( h \) to be the (unique) \( \leadsto \)-top element of the given connected component.

Continuing the analysis of (ARS)-fans of the form \( P_h \), we show:

**Theorem 2.13** Let \( F \) be a RS-fan and let \( J \subseteq I \) be in \( \text{Spec}(F) \). Let \( h_1 \in C^I_J \), \( h_2 \in S^I_J \). For \( i = 1, 2 \), we write \( P_i \) for \( P_{h_i} \). Then,

1. There is an ARS-embedding \( \varphi \) of \( P_1 \) into \( P_2 \). Further, \( \varphi[P_1] = \{ u \in P_2 | J \subseteq Z(u) \} \).
2. \( \varphi \) is an order-embedding of \( (P_1, \leadsto) \) into \( (P_2, \leadsto) \).

Theorem 2.13 Let \( F \) be a RS-fan and let \( J \subseteq I \) be in \( \text{Spec}(F) \). Let \( h_1 \in C^I_J \), \( h_2 \in S^I_J \). For \( i = 1, 2 \), we write \( P_i \) for \( P_{h_i} \). Then,

1. There is an ARS-embedding \( \varphi \) of \( P_1 \) into \( P_2 \). Further, \( \varphi[P_1] = \{ u \in P_2 | J \subseteq Z(u) \} \).
2. \( \varphi \) is an order-embedding of \( (P_1, \leadsto) \) into \( (P_2, \leadsto) \).
Proof. Our assumption on the \( h_i \)'s guarantees the existence of \( u_i, u_2 \in L_j \) so that \( u_i \sim h_i \ (i = 1, 2) \). For \( J \subseteq J' \subseteq I \) in Spec(\( F \)) let \( \varphi_{j'} \) denote the involution \( \varphi_{j'}^{u_1,u_2} \) of \( L_{j'} \) (Definition 2.6).

(1) We construct \( \varphi : P_1 \rightarrow P_2 \) by "collecting together" all the relevant maps \( \varphi_{j'} \ (J \subseteq J' \subseteq I) \): given \( g \in L_{j'} \), \( g \sim h_1 \), we set
\[
\varphi(g) = \varphi_{j'}(g).
\]
Since the levels \( L_{j'} \) are pairwise disjoint, \( \varphi \) is well-defined.

i) \( \varphi[P_1] \subseteq P_2 \).

By Theorem 2.8(e), \( g \sim h_1 \) implies \( \varphi_{j'}(g) \sim \varphi_{j'}(h_1) \). Since \( h_1 \) is the unique successor of \( u_i \) at level \( I \), 2.8(c) yields \( \varphi_{j'}^i(h_1) = h_2 \), whence \( \varphi_{j'}(g) \sim h_2 \), as required. Note this also gives \( J \subseteq J' = Z(\varphi(g)) \).

ii) \( \{ u \in P_2 \mid J \subseteq Z(u) \} \subseteq \varphi[P_1] \).

Let \( u \) be in the left-hand side, with \( J' = Z(u) \), say. Set \( v = \varphi_{j'}(u) \); then, \( \varphi_{j'}(v) = u \) (2.8(b)). By 1.6(b), \( u_1 \sim h_1 \) and \( u, u_2 \sim h_2 \) imply \( u u_1 u_2 = \varphi_{j'}(u) = v \sim h_1 \), i.e., \( v \in P_1 \). Hence \( \varphi(v) = u \in \varphi[P_1] \).

iii) \( \varphi \) is injective.

This is clear using 2.8(b), since \( Z(\varphi(g)) = Z(g) \) for \( g \in P_1 \).

iv) \( \varphi \) is an ARS-morphism.

The proof is similar to that of item (a) in Theorem 2.8. The statement to be proved is:

(\( \dagger \)) For every \( a \in F \) there is \( b \in F \) such that \( (a/T_2) \circ \varphi = b/T_1 \),

where, for \( i = 1, 2, T_i = h_i^{-1}[1], P_i = U(T_i), a/T_2 : P_2 \rightarrow 3 \) is the evaluation map: \( a/T_2(g) = \hat{g}(a/T_2) = g(a), \) for \( g \in P_2 \), and similarly for \( b/T_1 : P_1 \rightarrow 3. \) (Note that \( g \in P_2 = U(T_2) \) ensures that \( a/T_2 \) depends only on the congruence class of \( a \) modulo \( T_2 \).)

The conclusion of (\( \dagger \)) can equivalently be written as \( \varphi(g)(a/T_2) = \hat{g}(b/T_1), \) i.e., \( (u u_1 u_2)(a) \)

\( g(b). \) Since \( u (a) \in \{0, 1, -1\} \ (i = 1, 2) \), it is clear that the element \( b = a u_1(a) u_2(a) \in F \) verifies (\( \dagger \)); see 2.8(a).

(2) Since \( h_2 \in C_j^I \), the preceding construction can be performed with the roles of \( h_1 \) and \( h_2 \) reversed. Routine verification using 2.8(b) shows that the map obtained is \( \varphi^{-1} \), which then is an ARS-morphism, proving that \( \varphi \) is an ARS-isomorphism.

\( \square \)

Proposition 2.10 and Theorem 2.13 provide significant information on the structure of the connected components of ARS-fans.

Definition and Remarks 2.14 (a) Let \( (X, \preceq) \) be a root-system and let \( g_1, g_2 \in X \). Define:
\[
g_1 \equiv_C g_2 \text{ if } g_1, g_2 \text{ have a common \( \preceq \)-upper bound}.
\]
\( \equiv_C \) is an equivalence relation; its classes are called connected components of \( (X, \preceq) \).

(b) The \( \sim \)-top elements of the connected components of an ARS-fan \( (X, F) \) have all the same level, namely the level determined by the maximal ideal \( M \) of \( F \); cf. Proposition 0.3(3).

(c) Since every connected component of an ARS-fan is itself an ARS-fan 2.12(2), the zero-sets of its elements attain a lowest level, which can be explicitly determined, cf. Proposition 2.15 below. However, different components may have different lowest levels, see Corollary 2.17 \( \square \)

Notation. The sets \( L_j, S_j^I \) and \( C_j^I \) defined in 1.1 and 2.1 relativize in an obvious way to the connected components of a fan \( (X, F) \); if \( K \) is such a component and \( J \subseteq I \) are in Spec(\( F \)) we set:
\[
L_j(K) = L_j \cap K, \quad S_j^I(K) = S_j^I \cap K, \quad \text{and} \quad C_j^I(K) = C_j^I \cap K.
\]

Note that some (or all) of these sets may be empty, depending on \( I, J \) and the component \( K \). Clearly, if \( h_0 \) is the \( \sim \)-top element of \( K \), we have \( L_j(K) = \{ g \in L_j \mid g \sim h_0 \} \), and similarly
for $S^I_j(K)$ and $C^I_j(K)$. $L_i(K) \neq \emptyset$ just means that $K$ “reaches at least” the $I$-th level of $X$ (possibly lower).

\begin{proposition}
Let $K$ be a connected component of an ARS-fan $(X,F)$. Let $h_0$ be the $\sim$-top element of $K$, and let $T = h_0^{-1}[1]$. Then, the lowest level of $K$ (i.e., the smallest ideal $I$ of $F$ such that $L_i(K) \neq \emptyset$) is $I = \Gamma \cap -\Gamma$, where $\Gamma$ is the saturated subsemigroup of $F$ generated by $\text{Id}(F) \cdot T$.
\end{proposition}

Note. The subsemigroup $\text{Id}(F) \cdot T$ may not be saturated, since $\text{Id}(F) \cdot T \cap -\text{Id}(F) \cdot T$ is not, in general, an ideal; see [DP5a], Cor. 3.10 (2).

\begin{proof}
With notation as in \ref{212} we have $K = P_{h_0} = U(T) = \{ g \in X | g \upharpoonright T = 1 \}$ is the ARS $X_{F/T}$ (where $F/T = F/\sim_K$; with $\sim_K$ denoting the congruence on $F$ induced by $K$). Let $\pi_T : F \to F/T$ be the quotient map. The lowest level of $X_{F/T}$ is $\{0\}$; with $K$ identified to a subset of $X$ via the map $g \mapsto \hat{g}$ ($\hat{g} \circ \pi_T = g$), the corresponding ideal of $F$ is $\pi_T^{-1}[0] = \{ a \in F | a \sim_K 0 \}$. Then, with the ideal $I$ defined in the statement, we must prove, for $a \in F$:

\[
\forall a \in I \quad a \sim_K 0 \iff \exists a \in I.
\]

(\Rightarrow) This follows from $I \subseteq Z(g)$ for all $g \in K$. Since $g \upharpoonright T = 1$, we get $\text{Id} \cdot T \subseteq P(g) = g^{-1}[0,1]$; since $P(g)$ is a saturated subsemigroup, it becomes $\Gamma \subseteq P(g)$. Hence, $x \in I = \Gamma \cap -\Gamma$ implies $g(x) = 0$.

(\Leftarrow) Assume $a \not\sim_K 0$. In order to prove $a \not\sim_K 0$ we construct a character $g \in X$ such that $g \upharpoonright T = 1$ and $g(a) \neq 0$ (i.e., $g(a^2) = 1$). The ideal $I$ is prime and saturated ([DP5a], 3.10 (1)). Since $I = \Gamma \cap -\Gamma$, there is a saturated subsemigroup $S$ of $F$ such that $\Gamma \subseteq S$ and $S$ maximal with $S \cap -S = I$. By [DP1], Lemma 3.5, p. 114, $S \cup -S = F$, and $S$ defines a character $g \in X$ with $Z(g) = I$, by setting $g \upharpoonright (S \setminus -S) = 1$, $g \upharpoonright (-S \setminus S) = -1$ and $g \upharpoonright I = 0$. Note that we have, $I \cap a^2T = \emptyset$.

Otherwise, there is $t \in T$ such that $a^2t \in I$; since $I$ is prime and $a \not\sim I$, we get $t \in I$, contradicting $T \cap I = h_0^{-1}[1] \cap Z(h_0) = \emptyset$.

Since $a^2T \subseteq S$, (\Rightarrow) implies $-S \cap a^2T = \emptyset$, whence $g \upharpoonright a^2T = 1$ by the definition of $g$.
\end{proof}

\begin{corollary}
Let $(X,F)$ be an ARS-fan and let $K_1, K_2$ be connected components of $(X,F)$.

Then,

1. Let $I \in \text{Spec}(F)$; if $L_i(K_i) \neq \emptyset$ for $i = 1, 2$, then $\text{card}(L_i(K_i)) = \text{card}(L_i(K_i))$.

2. Let $J \subseteq J'$ be in $\text{Spec}(F)$, and assume $L_i(K_i) \neq \emptyset$ for $i = 1, 2$. Then, $\text{card}(S^I_j(K_i)) = \text{card}(S^J_j(K_i))$.
\end{corollary}

\begin{proof}
(1) follows from (2), as $L_i = S^I_i$.

(2) Fix $i \in \{1, 2\}$. Let $h_i$ be the $\sim$-top element of $K_i$. The assumption $L_i(K_i) \neq \emptyset$ implies that the sets $S^I_j(K_i) = \{ g \in S^I_j | g \sim h_i \}$ are non-empty. Now, applying Proposition 2.10(a) with $I = M = \text{the maximal ideal of } F, \ J_1 = J, \ J_2 = J'$ we have $B^J_j(h_i) = \{ g \in S^J_j | g \sim h_i \} = S^J_j(K_i)$, and the result follows.
\end{proof}

\begin{remark}
Assertion (2) of Corollary 2.16 fails, in general, if the sets $S^I_j(K_i)$ are replaced by $C^I_j(K_i)$, even if both sets $C^I_j(K_i), \ i = 1, 2$, are assumed non-empty. The snag is that $C^I_j(K_i) \neq \emptyset$ does not imply that the $\sim$-top element $h_i$ of $K_i$ belongs to $C^M_j(K_i)$, a condition required for Proposition 2.10(b) to apply. It is easy to construct counterexamples.
\end{remark}

Theorem 2.13 gives:
Corollary 2.17 Let $K_1, K_2$ be connected components of the ARS-fan $(X, F)$. Let $I_1, I_2 \in \text{Spec}(F)$ be the lowest levels of $K_1, K_2$, resp. (cf. 2.15). Then,

1. If $I_2 \subseteq I_1$, then $K_1$ endowed with the specialization order is (order-)isomorphic to the root-system obtained by deleting all levels $I \subseteq I_1$ in $K_2$.
2. If $I_1 = I_2$, then $K_1, K_2$ are order-isomorphic.

Proof. (1) Use Theorem 2.13(1) with $I = M = \text{the maximal ideal of } F$, $J = I_1$, and $h_1, h_2$ the $\sim$-top elements of $K_1, K_2$, resp. The ARS-embedding $\varphi : K_1 \rightarrow K_2$ constructed therein verifies $\varphi[K_1] = \{ u \in K_2 \mid I_1 \subseteq Z(u) \}$, which is exactly statement (1).
(2) follows from Theorem 2.13(2).

2.18 Some impossible configurations.

The preceding results show that there are strong constraints on the order structure of ARS-fans, especially when there is more than one connected component. We include a few examples to help the reader visualize the extent of those restrictions.

1. A configuration like

![Configuration](image)

contradicts Corollary 2.16(1).

2. The four-component configuration

![Configuration](image)

(where the components pairwise verify the conclusion of 2.16(2)) is also impossible: $\text{card}(S^3_4) = 3$ is not a power of 2, and hence $S^3_4$ (shown with arrows) cannot be an AOS-fan (see Corollary 2.15). However, the same configuration with $K_3$ replaced by another copy of $K_4$ does not clash with either 2.16 or 2.17.

Note. Our notation here (and below) follows the convention introduced in 1.8 for finite fans. Thus, $S^3_4$ stands for the set $S^{I_3}_{I_4}$, see 2.1 and 3.1.

3. The two-component root-system

![Configuration](image)
contradicts both Corollary 2.16 (2) (card \((S_4^3(K_1)) = 4\), but card \((S_4^3(K_2)) = 2\)) and Corollary 2.17 \((K_1\) and \(K_2\) have the same “length” but are not order-isomorphic). 

3 The specialization root-system of finite ARS-fans

In this section we shall mostly deal with finite fans in the categories \(\text{ARS}\) and \(\text{RS}\). Our main result is Theorem 3.11 —the isomorphism theorem for finite ARS-fans— which proves that, in this case, the order of specialization alone determines the isomorphism type. The proof depends on the notion of a “standard generating system” which we introduce in 3.4.

3.1 Notation and Reminder

(a) Notation 1.8 for finite (ARS- and RS-)fans is used systematically in this section, adapted in a self-explanatory way; e.g., for \(1 \leq k \leq j \leq n = \ell(X_F)\), \(L_k\) (or \(L_k(X_F)\), if necessary), will stand for the level \(L_{I_k}\), \(S_k\) for the set \(S_{I_k}\), etc.

(b) Recall that the AOSs have a combinatorial geometric (matroid) structure; it was introduced in \([D1]\) and \([D2]\) for spaces of orders of fields, and later generalized to abstract order spaces in \([L2]\). In general, ARSs do not possess such a structure. Thus, combinatorial geometric notions such as dependent set, independent set, basis, closed set, closure, dimension, etc., will always refer to the above-mentioned combinatorial geometric structure, and apply only to AOSs. For the definition and the mutual relationships, in the general context of matroid theory, of combinatorial notions such as those just mentioned, the reader is referred to \([Wh]\).

Since the combinatorial geometric structure of any AOS is isomorphic to that of a set of vectors in a (possibly infinite-dimensional) vector space over the two-element field \(\mathbb{F}_2\) with the structure induced by linear dependence (cf. \([D1]\), Thm. 3.1, p. 618), the notions above coincide with the corresponding notions over vector spaces. For example, a subset \(A \subseteq X\) of an AOS \((X, G, -1)\) \((G a group of exponent 2)\) is dependent \(g_1, \ldots, g_r \in A (r \geq 2)\), such that \(g = g_1 \cdot \ldots \cdot g_r\) (as characters of \(G\)). Since functions in \(X\) send \(-1\) to \(-1\), this functional identity can only hold if \(r\) is odd. Likewise, \(A\) is closed \(r\) is odd. Likewise, \(A\) is closed if the product of any odd number of members of \(A\) belongs to \(A\).

\(\Box\)

Warning. In this section the words \(closed\) set and \(closure\) are used only in the combinatorial geometric sense just defined.
Lemma 3.2 Let \((X, F)\) be an ARS-fan (not necessarily finite). Let \(J \subseteq I\) be in \(\text{Spec}(F)\), and let \(A \subseteq L_J\), \(B \subseteq L_I\), be sets such that:

(i) The unique \(\leadsto\) -successor in \(L_I\) of each \(g \in A\) belongs to \(B\).

(ii) Every \(h \in B\) has a unique \(\leadsto\) -predecessor in \(A\).

Then, \(A\) dependent \(\Rightarrow\) \(B\) dependent.

Proof. By assumption there are pairwise distinct elements \(g, g_1, \ldots, g_r \in A\) such that \(g = g_1 \cdot \ldots \cdot g_r\); as observed above, \(r\) is odd \(\geq 3\). Let \(h, h_1, \ldots, h_r\) be the unique successors of \(g, g_1, \ldots, g_r\), resp., in \(B\) coming from (i); thus, \(g \leadsto h\) and \(g_i \leadsto h_i\), for \(i = 1, \ldots, r\). By Lemma 3.2 (a) we have \(g = g_1 \cdot \ldots \cdot g_r \leadsto h_1 \cdot \ldots \cdot h_r\). Since \(h_1 \cdot \ldots \cdot h_r \in L_I\) \((r\) odd\) and \(g\) has a unique \(\leadsto\) -successor in \(L_I\), we get \(h = h_1 \cdot \ldots \cdot h_r\).

By assumption (ii), every element in \(A\) is the unique predecessor of an element in \(B\). Since \(g_i \neq g_j\), we get \(h_i \neq h_j\) for \(1 \leq i \neq j \leq r\); likewise, \(h_i \neq h_j\) for \(i = 1, \ldots, r\). This proves that \(h\) is the product of \(r\) distinct elements in \(B\), and hence that \(B\) is dependent. □

Proposition 3.3 (Choice of basis).

Let \((X, F)\) be a finite ARS-fan; let \(1 \leq k < n = \ell(X)\). Let \(G\) be an arbitrary AOS-subfan of \(L_{k+1} = L_{k+1}(X)\). Let \(F = \{h \in L_k\} \) Then, \(g \in G\) such that \(g \leadsto h\) is a basis of the depth-\(k\) successors of elements of \(G\) (cf. Lemma 2.8). Assume:

\((*)\) \(\forall h, h' \in F, \) card \([(g \in G \mid g \leadsto h)]\) = card \([(g \in G \mid g \leadsto h')].

Let \(B = \{h_1, \ldots, h_r\}\) be a basis of \(F\) (as an AOS), and let \(C\) be a basis of the AOS-fan \(P_h = \{g \in G \mid g \leadsto h\}\) (see Lemma 2.17 (1)). For \(i = 2, \ldots, r\), let \(g_i \in G\) be such that \(g_i \leadsto h_i\).

Then, \(C \cup \{g_2, \ldots, g_r\}\) is a basis of \(G\).

Proof. If \(r = 1\), then \(F = B = \{h_1\}\), whence \(G = \{g \in G \mid g \leadsto h_1\}\), and the result holds by the choice of \(C\). Henceforth we assume \(r \geq 2\). We observe:

— \(r = \text{card}(B) = \text{dim}(F).\) Since \(F\) is an AOS-fan, \(\text{card}(F) = 2^{r-1}.\)

— For every \(h \in F\), \(A_h = \{g \in G \mid g \leadsto h\}\) is an AOS-fan; this follows from the assumption that \(G\) is an AOS-fan, since \(A_h\) is closed under the product of any three of its elements, cf. Lemma 3.2 (b).

— \(A_h \cap A_{h'} = \emptyset\) for \(h \neq h'\) in \(F\).

By assumption (*), \(\text{card}(A_h) = \text{card}(A_{h'})\) \((= 2^{p-1},\) say\), for \(h, h' \in F\). Since \(G = \bigcup_{h \in F} A_h\), we get \(\text{card}(G) = \text{card}(F) \cdot \text{card}(A_h)\) \((\text{any } h \in F)\), and then \(\text{card}(G) = 2^{r-1} \cdot 2^{p-1} = 2^{p+r-2};\) hence \(\text{dim}(G) = p + r - 1.\) Since \(\text{card}(C \cup \{g_2, \ldots, g_r\}) = p + r - 1,\) it suffices to prove:

\((**)\) \(C \cup \{g_2, \ldots, g_r\}\) is an independent set.

Proof of (**). Assume false.

Case 1. Some \(g_{i_0}\), with \(2 \leq i_0 \leq r,\) is dependent on the rest, i.e., there are \(C' \subseteq C\) and \(J \subseteq \{2, \ldots, r\} \setminus \{i_0\}\) so that \(g_{i_0} = \prod_{c \in C \setminus C'} c \cdot \prod_{j \in J} g_j\), i.e.,

\((+)\) \(\prod_{c \in C \setminus C'} c = \prod_{J \subseteq J \cup \{i_0\}} g_j.\)

— If card \((C')\) is odd, since \(A_{h_1}\) is an AOS-fan, and hence a closed set, the left-hand side of (+) is an element \(g' \leadsto h_1\) and we have \(g' = \prod_{J \subseteq J \cup \{i_0\}} g_j = 1\). Setting \(A = \{g'\} \cup \{g_j \mid j \in J \cup \{i_0\}\} \subseteq L_{k+1}\) and \(B = \{h_1\} \cup \{h_j \mid j \in J \cup \{i_0\}\} \subseteq L_k\), the assumptions of Lemma 3.2 are met. Since \(A\) is dependent, so is \(B,\) contradicting that \(B \subseteq B\) and \(B\) is a basis of \(F,\) whence an independent set.

— If \(C' = \emptyset,\) the same argument works, yielding a contradiction.

— Assume card \((C')\) even \(> 0.\) Fix \(c_0 \in C'.\) Then card \((C' \setminus \{c_0\})\) is odd, and \(g' = \prod_{c \in C \setminus \{c_0\}} c \in L_{k+1};\) also \(g' \leadsto h_1,\) and we have:
$c_0 \cdot g' \cdot \prod_{j \in J \cup \{i_0\}} g_j = 1.$

Pick any index $j_0 \in J \cup \{i_0\}$ (so, $j_0 \geq 2$). Since $c_0 \cdot g' \sim h_1$ and $g_{j_0} \sim h_{j_0}$, Lemma 1.6(b) yields $g'_{j_0} := c_0 \cdot g_{j_0} \sim h_{j_0}$ and $g' \cdot \prod_{j \in (J \cup \{i_0\}) \setminus \{j_0\}} g_j = 1$. Hence, $A = \{g'_{j_0}\} \cup \{g_j \mid j \in (J \cup \{i_0\}) \setminus \{j_0\}\}$ is a dependent subset of $L_{k+1}$. Setting $B = \{h_j \mid j \in J \cup \{i_0\}\}$ the assumptions of Lemma 3.2 are met, and hence $B$ is also dependent, contradicting that $B \subseteq B$.

Case 2. Some $c_0 \in C$ is dependent on the rest.

Then, there are $C' \subseteq C \setminus \{c_0\}$ and $J \subseteq \{2, \ldots, r\}$ so that

$++ \quad c_0 = \prod_{c \in C'} c \cdot \prod_{j \in J} g_j.$

Note that $J \neq \emptyset$ (otherwise $C$ would be dependent). Taking $J$ minimal so that $(++)$ holds, and picking $j_0 \in J$, it follows that $c_0$ is not in the closure of $C' \cup \{g_j \mid j \in J \setminus \{j_0\}\}$ (cf. Warning, end of 3.1(b)). By the exchange property, $g_{j_0}$ is in the closure of $C' \cup \{c_0\} \cup \{g_j \mid j \in J \setminus \{j_0\}\}$, contrary to the result of Case 1.

3.4 Standard generating systems.

For any finite ARS-fan, $(X, F)$, we will construct, by induction on $k$, $1 \leq k \leq n = \ell(X)$, a class of bases $B_k$ of the AOS-fan $L_k(X)$. Each basis $B_k$ will be required to satisfy the additional condition:

(*) For $k \leq j \leq n$, $B_k \cap S^k_j$ is a basis of the AOS-fan $S^k_j$.

This additional requirement guarantees that the inductive construction of the $B_k$’s is not interrupted before the $n$-th (and last) step. The construction uses Proposition 3.3 and the results from §2 above. The set $B = \bigcup_{k=1}^n B_k$ is called a standard generating system for $(X, F)$.

Construction of standard generating systems.

Level 1. It suffices to observe that a basis $B_1$ of $L_1$ satisfying condition $(*)$ exists. Begin by choosing a basis $B_1(n)$ of the AOS-fan $S^1_n = C^1_n$ (cf. Corollary 2.5). $S^1_n$ is a closed subset (cf. Warning, end of 3.1(b)) of the (AOS)-fan $S^1_{n-1} = S^1_n \cup C^1_{n-1}$; hence, $B_1(n)$ is an independent subset of $S^1_{n-1}$; choose $B_1(n-1)$ to be a basis of $S^1_{n-1}$ extending $B_1(n)$.

In general, assume that, for $1 < j \leq n$ an increasing sequence $B_1(1) \subseteq \ldots \subseteq B_1(j)$ of independent subsets of $L_1$ has been chosen so that $B_1(\ell)$ is a basis of the AOS-fan $S^1_{\ell}$ ($1 \leq \ell \leq n$).

As above, $B_1(j)$ is an independent subset of the fan $S^1_{j-1} = S^1_j \cup C^1_{j-1}$. Let $B_1(j-1)$ be a basis of $S^1_{j-1}$ extending $B_1(j)$. Set $B_1 = B_1(1)$; by construction, $B_1 \cap S^1_j = B_1(j)$ is a basis of $S^1_j$.

Induction step. Given an integer $k$, $1 \leq k < n$, assume there exists a basis $B_k$ of $L_k$ satisfying property $(*)$; thus, for $k \leq j \leq n$, $B_k(j) = B_k \cap S^k_j$ is a basis of $S^k_j$. Further, since $S^k_n \subseteq \ldots \subseteq S^k_k = L_k$, we have $B_k(n) \subseteq \ldots \subseteq B_k(k) = B_k$. Using Proposition 3.3 we define a subset $B_{k+1}$ of $L_{k+1}$ as follows:

— Firstly, fix an element $h_0 \in B_k(n)$ (this set is non-empty because $n = \ell(X)$). Pick a basis $B_{k+1}(n, h_0)$ of the AOS-fan $\{g \in S^k_{n+1} \mid g \sim h_0\}$.

— Next, for each $h \in (B_k \cap S^k_{k+1}) \setminus \{h_0\}$ there is a maximal index $j = j(h)$, $k + 1 \leq j \leq n$, so that $h \in B_k \cap S^k_j = B_k(j)$; clearly, $h \not\in S^k_{j+1}$, whence $h \in C^k_j = S^k_j \setminus S^k_{j+1}$ (if $j = n$, then $h \in C^k_n = C^k_0$). Since $j \geq k + 1$, we have $\{g \in C^k_{j+1} \mid g \sim h\} \neq \emptyset$. Choose an element $g_h \in C^k_{j+1}$ so that $g_h \sim h$.

— Finally, set
Corollary 3.6 \[ B_{k+1} = B_{k+1}(n, h_0) \cup \{g_h | h \in (B_k \cap S_{k+1}^p) \setminus \{h_0\}\}. \]

Claim. For \( k \geq 1 \leq p \leq n \), \( B_{k+1} \cap S_{k+1}^p \) is a basis of \( S_{p}^{k+1} \).

Proof of Claim. We apply Proposition 2.10 with the following choice of parameters:

- \( \mathcal{G} = S_{p}^{k+1} \) (whence \( \mathcal{F} = S_{p}^{k} \), since \( k + 1 \leq p \));
- \( B = B_k \cap S_{p}^{k} \) (a basis of \( \mathcal{F} \));
- \( C = B_{k+1}(n, h_0) \) (a basis of \( \{g \in S_{n}^{k+1} | g \leadsto h_0\}\)).

Proposition 2.10 (a) shows that the cardinality assumption

\[
\text{card (} \{g \in S_{p}^{k+1} | g \leadsto h\} \text{)} = \text{card (} \{g \in S_{p}^{k+1} | g \leadsto h'\} \text{)}, \quad (h, h' \in S_{k}^{k})
\]

of Proposition 2.10 holds. We conclude that

\[
D := B_{k+1}(n, h_0) \cup \{g_h | h \in (B_k \cap S_{p}^{k}) \setminus \{h_0\}\}
\]

is a basis of \( S_{p}^{k+1} \). The Claim follows from:

(\( i \)) \( B_{k+1} \cap S_{p}^{k+1} = D \).

Proof of (\( i \)). Since \( B_{k+1}(n, h_0) \subseteq D \cap B_{k+1} \) (see [\(*\)], we need only prove:

(\( \subseteq \)) If \( h \in (B_k \cap S_{k+1}^p) \setminus \{h_0\} \) and \( g_h \in S_{k}^{k+1} \), then \( h \in B_k \cap S_{p}^{k} \).

This clearly follows from \( g_h \in S_{k+1}^p \), \( g_h \leadsto h \) and \( h \in S_{k+1}^k \).

(\( \supseteq \)) Since \( k + 1 \leq p \), we have \( B_k \cap S_{p}^{k} = B_k(p) \subseteq B_k(k + 1) = B_k \cap S_{k+1}^k \). On the other hand, if \( h \in (B_k \cap S_{k+1}^p) \setminus \{h_0\} \) and, as above, \( j(h) \) denotes the largest index \( j \) so that \( k + 1 \leq j \leq n \) and \( h \in B_k(j) \), we have \( p \leq j(h) \), whence \( S_{j(h)} \subseteq S_{p}^{k+1} \). By choice, \( g_h \in C_{j(h)}^{k+1} \), it follows that \( g_h \in S_{p}^{k+1} \), as required.

Remarks 3.5

(a) In general, there are many different standard generating systems for a finite ARS-fan \((X, F)\). The construction in [3.3] allows for several choices of the bases \( B_{1}(j) (1 \leq j \leq n) \) and, at each successive step, \( k \), for many choices of elements \( h_0 \in B_k(n) \), of bases \( B_{k+1}(n, h_0) \), and of elements \( g_h \in C_n^{k+1} \) under each \( h \in (B_k \cap S_{k+1}^p) \setminus \{h_0\} \). In spite of this lack of uniqueness, we shall prove below that any standard generating system determines the isomorphism type of a finite ARS-fan.

(b) Some of the sets \( C_{j}^{k} = S_{j}^k \setminus S_{j+1}^k \) may be empty. However, if \( C_{j}^{k} \neq \emptyset \), then, necessarily, \( B_k \cap C_{j}^{k} \neq \emptyset \). Indeed, if \( j = n \), then \( C_n^k \neq \emptyset \) (as \( n = \ell(X) \)) and \( C_n^k = S_n^k \) is an AOS-fan; since \( B_k \cap C_n^k \) is a basis of \( C_n^k \), it must contain at least one element. If \( j < n \), since \( S_{j+1}^k \) is a fan, it is a closed set; as it is disjoint from \( C_j^k \), then no element of \( C_j^k \) is dependent on \( S_{j+1}^k \). Hence, any basis of \( S_{j}^k = S_{j+1}^k \cup C_{j}^{k} \) must contain an element of \( C_{j}^{k} \).

Any standard generating system for a finite ARS-fan has the following property:

Corollary 3.6 Let \( B \) be a standard generating system for a finite ARS-fan \((X, F)\); let \( n = \ell(X) \), and \( 1 \leq k \leq m \leq n \). Then, for every \( g \in B_m = B \cap L_m(X) \), the unique depth-\( k \) successor of \( g \) in \( X \) belongs to \( B \) (hence \( B_k = B \cap L_k(X) \)).

Proof. By the construction in [3.3] this holds for \( m = k + 1 \). In fact, let \( g \in B_{k+1} \), \( h' \in L_k \), and \( g \leadsto h' \). By the definition of \( B_{k+1} \) and uniqueness of the successor of \( g \) in \( L_k \), if \( g \in B_{k+1}(n, h_0) \), then \( h' = h_0 \in B_k(n) \subseteq B_k \); if \( g = g_h \), with \( h \in (B_k \cap S_{k+1}^p) \setminus \{h_0\} \), we get \( h' = h \in B_k \). Then iterate.
For the proof of the Isomorphism Theorem \[\text{3.11}\] below we shall need the characterizations of ARS-morphisms between fans proved in \[\text{3.9}\] and \[\text{3.10}\] below, which, in turn, follow from the Small Representation Theorem \[\text{3.8}\].

**Definition 3.7** Let \(G, H\) be RSs, let \(X_G, X_H\) be their character spaces, and let \(Z \subseteq X_G\). A map \(F : Z ightarrow X_H\) **preserves 3-products** (in \(Z\)) iff for all \(h_1, h_2, h_3 \in Z\),

\[
h_1 h_2 h_3 \in Z \Rightarrow F(h_1 h_2 h_3) = F(h_1) F(h_2) F(h_3).
\]

**Proposition 3.8** (Small representation theorem). Let \(G\) be a RS. The following conditions are equivalent for a map \(F : X_G \rightarrow 3\):

1. \(f\) is continuous in the constructible topology of \(X_G\).
2. \(f\) preserves 3-products in \(X_G\).
3. \(f\) is represented by an element of \(G\): there is \(a \in G\) so that \(f = \hat{a}\).

**Proof.** (2) \(\Rightarrow\) (1) is clear since the evaluation maps have properties (1.a) and (1.b).

(1) \(\Rightarrow\) (2). We use the representation theorem \[\text{M}\], Cor. 8.3.6, p. 162. It suffices to check that the assumptions of this theorem as well as one of the equivalent conditions in its conclusion hold under our hypotheses in (1). In our notation, the conditions to be checked are: for \(x, y \in X_G\),

\[
\begin{align*}
&\text{(†)} \quad f(x) = 0 \text{ and } Z(x) \subseteq Z(y) \Rightarrow f(y) = 0. \\
&\text{(††)} \quad f(x) \neq 0 \text{ and } x^{-1}[0, 1] \supseteq y^{-1}[0, 1] \Rightarrow f(x) = f(y). \\
&\text{(†††)} \quad \text{For any saturated prime ideal } I \text{ of } G, \text{ either} \\
&\quad \text{(i)} \quad f[\{u \in X_G \mid Z(u) = I\}] = 0, \text{ or} \\
&\quad \text{(ii)} \quad \prod_{i=1}^4 f(x_i) = 1 \text{ for any 4-element AOS-fan } \{x_1, \ldots, x_4\} \text{ in } \{u \in X_G \mid Z(u) = I\}.
\end{align*}
\]

Condition (†) follows at once from Lemma 0.12(2) (as \(Z(x) \subseteq Z(y) \Rightarrow y = yx^2\)).

Condition (††) follows from Lemma 0.1(3),(5): \(x^{-1}[0, 1] \supseteq y^{-1}[0, 1] \Rightarrow x = x^2 y\). Since \(f(x) \neq 0 \Rightarrow f(x^2) = 1\), assumption (1.b) implies \(f(x) = f(x^2)f(y) = f(y)\).

As for (†††), if (i) does not hold, (†) implies \(f(u) \neq 0\) for all \(u \in X_G\). Let \(\{x_1, \ldots, x_4\}\) be an AOS-fan in \(u \in X_G\) such that \(Z(u) = I\). Thus, \(x_4 = x_1 x_2 x_3\) and \(f(x_4) \neq 0\) for \(i = 1, \ldots, 4\). Assumption (1.b) gives \(f(x_4) = f(x_1)f(x_2)f(x_3) \neq 0\), i.e., \(\prod_{i=1}^4 f(x_i) = 1\).

**Corollary 3.9** A map \(F : (X_1, F_1) \rightarrow (X_2, F_2)\) between ARS-fans is an ARS-morphism iff \(F\) is continuous for the constructible topology (of both source and target) and preserves 3-products in \(X_1\) (cf. \[\text{3.7}\]).

**Proof.** (\(\Leftrightarrow\)) If \(F\) has the stated properties and \(a \in F_2\), then \(\hat{a} \circ F : X_1 \rightarrow 3\) also has those properties, and, by Proposition \[\text{3.8}\] is represented by an element of \(F_1\); hence, \(F\) is an ARS-morphism (cf. \[\text{3.1}\](c.i)).

(\(\Rightarrow\)) Assume \(F\) is an ARS-morphism. For continuity it suffices to show that \(F^{-1}[V]\) is open constructible in \(X_1\) whenever \(V\) is basic open for the constructible topology of \(X_2\), i.e., of the form \(V = U(a_1, \ldots, a_n) \cap Z(a)\) with \(a, a_1, \ldots, a_n \in F_2\) (see \[\text{M}\], p. 111). By the assumption on \(F\), there are \(b, b_1, \ldots, b_n \in F_1\) such that \(\hat{a} \circ F = \hat{b}\) and \(\hat{a}_1 \circ F = \hat{b}_i\) for \(i = 1, \ldots, n\). These functional identities imply \(F^{-1}[V] = U(b_1, \ldots, b_n) \cap Z(b)\), as required.

Preservation of 3-products by \(F\) follows easily from the same property for \(\hat{a}\) and \(\hat{b}\) using the functional identity \(\hat{a} \circ F = \hat{b}\).

**Lemma 3.10** Let \((X_1, F_1), (X_2, F_2)\) be ARS-fans.
(1) For a map $F : X_1 \rightarrow X_2$ the following are equivalent:

(i) $F$ preserves 3-products in $X_1$.

(ii) a) $F$ preserves 3-products of elements of the same level: for all $I \in \text{Spec}(F_i)$ and all $h_1, h_2, h_3 \in L_I(X_1)$, $F(h_1 h_2 h_3) = F(h_1) F(h_2) F(h_3)$.

b) $F$ is monotone for the specialization order: for $g, h \in X_1$, $g \preceq h$ $\Rightarrow$ $F(g) \preceq F(h)$. ($\preceq$ denotes specialization in $X_1$).

(2) If $(X_1, F_i)$ is finite, any map verifying one of the equivalent conditions (i) or (ii) in (1) is a morphism of ARSs.

(3) If both $(X_1, F_i)$, $(X_2, F_2)$ are finite, any bijection $F : X_1 \rightarrow X_2$ verifying one of the equivalent conditions in (1) is an isomorphism of ARSs.

**Proof.** (1). (i) $\Rightarrow$ (ii). (ii.a) is a special case of (i).

(ii.b) $g \preceq h$ $\Leftrightarrow$ $h = h^2 g$ (Lemma 0.1). By (i), $F(h) = F(h)^2 F(g)$, and this equality (in $X_2$) gives $F(g) \preceq F(h)$.

(ii) $\Rightarrow$ (i). Let $h_1, h_2, h_3$ be any three elements in $X_1$; say $Z(h_3) \subseteq Z(h_2) \subseteq Z(h_1)$. Let $I = Z(h_1)$ and for $i = 2, 3$ let $h'_i$ be the unique successor of $h_i$ in $L_I(X_1)$; Lemma 1.3 shows that $h_1 h_2 h_3 = h_1 h'_2 h'_3$; then, assumption (ii.a) gives

$$F(h_1 h_2 h_3) = F(h_1) F(h'_2) F(h'_3).$$

By (ii.b) we have $F(h'_2) \preceq F(h'_1)$, $(i = 2, 3)$. Next, note that $Z(F(h'_i)) \subseteq Z(F(h_i))$ for $i = 2, 3$. In fact, since $h_1, h'_2$ belong to the same level $L_I$, $Z(h_1) = Z(h'_2)$, and Lemma 0.2(2) yields $h'_3 = h^2$. Scaling by $h_1$ gives $h_i = h_i h''_i$. Since $F$ preserves 3-products of the same level, $F(h_1) = F(h_1) F(h''_2)$, which, by 0.2(1), gives $Z(F(h''_2)) \subseteq Z(F(h_1))$. Same argument for $i = 3$. Using 1.3 again, these inclusions and $F(h'_i) \preceq F(h''_i)$, $(i = 2, 3)$ prove:

$$F(h_1) F(h'_2) F(h'_3) = F(h_1) F(h_2) F(h_3),$$

as required.

(2) follows at once from Corollary 3.3 since the continuity requirement is automatically fulfilled in this case: the constructible topology in $X_1$ is discrete.

(3) By (2) it only remains to prove that $F^{-1} : X_2 \rightarrow X_1$ preserves 3-products in $X_2$. Let $g_1, g_2, g_3 \in X_2$ and let $h_i = F^{-1}(g_i)$, $i = 1, 2, 3$. From (1.i) we have $F(h_1 h_2 h_3) = g_1 g_2 g_3$. Composing both sides of this equality with $F^{-1}$ gives the desired conclusion:

$$F^{-1}(g_1 g_2 g_3) = F^{-1}(F(h_1 h_2 h_3)) = h_1 h_2 h_3 = F^{-1}(g_1) F^{-1}(g_2) F^{-1}(g_3).$$

**Remark.** Note that any isomorphism of ARS-fans preserves depth.

**Theorem 3.11** (The isomorphism theorem for finite ARS-fans.) Let $(X_1, F_i)$, $(X_2, F_2)$ be finite ARS-fans and let $\preceq_i$, $\preceq_2$ denote their respective specialization orders. If $(X_1, \preceq_i)$ and $(X_2, \preceq_2)$ are order-isomorphic, then $X_1$ and $X_2$ are isomorphic ARSs.

**Proof.** The order-isomorphism assumption implies:

(1) $\ell(X_1) = \ell(X_2)$ ($= n$, say, fixed throughout the proof).

(2) For $1 \leq k \leq j \leq n$, $\text{card}(C^k_j(X_1)) = \text{card}(C^k_j(X_2))$.

The proof of (2) is an easy exercise. Since $C^k_\ell \cap C^\ell_\ell = \emptyset$ for $k \leq \ell \neq \ell' \leq n$ and $S^k_\ell = \bigcup_{\ell' = \ell}^n C^k_\ell$, we get:

(3) For $1 \leq k \leq j \leq n$, $\text{card}(S^k_j(X_1)) = \text{card}(S^k_j(X_2))$.

(4) For $1 \leq k < n$ and all $h \in S^k_n(X_1), h' \in S^k_n(X_2)$, we have:
\[ \text{card}\{\{g \in S_{n}^{k+1}(X_{1}) \mid g \sim_{1} h\}\} = \text{card}\{\{g' \in S_{n}^{k+1}(X_{2}) \mid g' \sim_{2} h'\}\}. \]

**Proof of (4).** Consider the two-variable formula in the language \(\{\leq\}\) of order:

\[ \varphi(x, y) := x \in S_{n}^{k+1} \land x \leq y. \]

It is left as an exercise for the reader to write a first-order formula in \(\{\leq\}\) expressing the notion \(x \in S_{n}^{k+1}\); cf. [1].

If \(\sigma\) denotes the order isomorphism between \((X_{1}, \sim_{1})\) and \((X_{2}, \sim_{2})\), for \(g, h \in X_{1}\) we have:

\[ (X_{1}, \sim_{1}) \models \varphi(g, h) \iff (X_{2}, \sim_{2}) \models \varphi(\sigma(g), \sigma(h)). \]

It follows that \(\sigma\) maps \(\{g \in S_{n}^{k+1}(X_{1}) \mid g \sim_{1} h\}\) bijectively onto \(\{g' \in S_{n}^{k+1}(X_{2}) \mid g' \sim_{2} \sigma(h)\}\).

Now, if \(h \in S_{n}^{k}(X_{1})\), then \(\sigma(h) \in S_{n}^{k}(X_{2})\). If \(h' \in S_{n}^{k}(X_{2})\), apply Proposition 2.10 with \(h_{1} = h'\) and \(h_{2} = \sigma(h)\) to conclude. \(\Box\)

Since the sets in item (4) are AOS-fans (Corollary 2.5), they have the same dimension, i.e., any bases of each of them have the same cardinality. If \(B^{1}, B^{2}\) are standard generating systems for \(X_{1}, X_{2}\), respectively, then \(B^{1} \cap S_{j}^{k}(X_{1})\) is a basis of the fan \(S_{j}^{k}(X_{1})\), for \(1 \leq k \leq j \leq n\) and \(i = 1, 2\); from (3) we get:

(5) For \(1 \leq k \leq j \leq n\), \(\text{card}(B^{1} \cap S_{j}^{k}(X_{1})) = \text{card}(B^{2} \cap S_{j}^{k}(X_{2}))\).

In particular, for \(S_{k}^{k} = L_{k}\) we obtain:

(6) If \(1 \leq k \leq n\), then \(\text{card}(B_{k}^{1}) = \text{card}(B_{k}^{2})\) (where \(B_{k}^{i} = B^{i} \cap L_{k}(X_{i})\)).

Next, we choose an arbitrary standard generating system \(B^{1}\) for \(X_{1}\). By induction on \(k\), \(1 \leq k \leq n\), we construct a standard generating system \(B^{2}\) of \(X_{2}\) (\(B^{2} = \bigcup_{k=1}^{n} B_{k}^{2}\)) and a map \(f_{k} : B_{k}^{1} \rightarrow B_{k}^{2}\) so that:

(7) i) For \(k \leq j \leq n\), \(f_{k}[B^{1} \cap S_{j}^{k}(X_{1})] = B^{2} \cap S_{j}^{k}(X_{2})\).

ii) If \(k < n\), \(g \in B_{k+1}^{1}\), \(h \in B_{k}^{1}\) and \(g \sim_{1} h\), then \(f_{k+1}(g) \sim_{2} f_{k}(h)\).

Construction of \(B^{2}\) and the maps \(f_{k}\).

Level 1. \(B_{1}^{2}\) is built as in the level 1 step in [3.3] with notation therein, \(f_{1} : B_{1}^{1} \rightarrow B_{1}^{2}\) is any bijection mapping \(B_{1}^{1}(j)\) onto \(B_{1}^{2}(j)\), for \(1 \leq j \leq n\). Such a bijection exists by (5) above \((k = 1)\).

Induction step. Assume \(B_{1}^{2}, \ldots, B_{k}^{2}\) and \(f_{1}, \ldots, f_{k}\) already constructed, so that:

- For \(1 \leq j \leq k\) and \(j \leq \ell \leq n\), \(B_{j}^{2} \cap S_{\ell}^{\ell}(X_{2})\) is a basis of the AOS-fan \(S_{\ell}^{\ell}(X_{2})\) and \(f_{j}[B_{j}^{1} \cap S_{\ell}^{\ell}(X_{1})] = B_{j}^{2} \cap S_{\ell}^{\ell}(X_{2})\).

- Condition (7.ii) holds for all \(j\) such that \(1 \leq j < k\).

The basis \(B_{k+1}^{2}\), and along with it the map \(f_{k+1}\), are defined by performing the construction of the inductive step in [3.3] with the following choice of parameters:

- If \(h_{0} \in B_{k}^{1} \cap S_{n}^{k}(X_{1})\), and \(B_{k+1}^{1}(n, h_{0})\) is a basis of the (AOS)-fan \(\{g \in S_{n}^{k+1}(X_{1}) \mid g \sim_{1} h_{0}\}\), then take \(B_{k+1}^{2}(n, f_{k}(h_{0}))\) to be a basis of the fan \(\{g' \in S_{n}^{k+1}(X_{2}) \mid g' \sim_{2} f_{k}(h_{0})\}\). This is possible since \(f_{k}(h_{0}) \in B_{k}^{2} \cap S_{n}^{k}(X_{2})\), by (7.i). Using item (4), we let \(f_{k+1}[B_{k+1}^{1}(n, h_{0})] = B_{k+1}^{2}(n, f_{k}(h_{0}))\).

- If \(g \in B_{k+1}^{1} \cap C_{j}^{k+1}(X_{1})\) with \(k+1 \leq j \leq n\), but \(g \notin B_{k+1}^{1}(n, h_{0})\), then, by the construction performed in the inductive step of [3.3] if \(h\) is the unique depth-\(k\) successor of \(g\), we have \(h \in B_{k}^{1} \cap C_{j}^{k}(X_{1})\), \(h \neq h_{0}\) and \(g = g_{h}\). In this case choose any element \(g' \sim_{2} f_{k}(h)\) such that \(g' \in C_{j}^{k+1}(X_{2})\), and set \(f_{k+1}(g) = g'\). This is possible since \(f_{k}(h) \in B_{k}^{2} \cap C_{j}^{k}(X_{2})\) (which follows
easily from (7.i). Clearly, this construction guarantees that (7.i) and (7.ii) hold for \( k + 1 \).

Note that (7.ii) implies, by iteration, its own generalization:

(7) iii) If \( 1 \leq k < m \leq n \), \( g \in B^1_m \), \( h \in B^1_k \) and \( g \rightsquigarrow h \), then \( f_m(g) \overset{\sim}{\rightarrow} f_k(h) \).

Since \( B^1_i = B^i \cap L_k(X_i) \) is a basis of the AOS-fan \( L_k(X_i) \), \( i = 1, 2 \), we get:

(8) The bijection \( f_k \) extends (uniquely) to an AOS-isomorphism \( \tilde{f}_k : L_k(X_1) \rightarrow L_k(X_2) \) mapping \( S^j_k(X_1) \) onto \( S^k_j(X_2) \), for all \( j \) such that \( k \leq j \leq n \).

Now set \( F : X_1 \rightarrow X_2 \) to be \( F = \bigcup_{k=1}^n \tilde{f}_k \). We prove:

Claim. \( F \) is an isomorphism of ARSs.

**Proof of Claim.** Since \( X_i = \bigcup_{k=1}^n L_k(X_i) \) (disjoint union) for \( i = 1, 2 \), and \( \tilde{f}_k \) maps \( L_k(X_1) \) bijectively onto \( L_k(X_2) \), we have:

(a) \( F \) is well-defined and bijective.

(b) For all \( k, 1 \leq k \leq n \), \( F \) preserves 3-products in \( L_k \).

This is clear: by (8) \( F \mid L_k(X_i) = \tilde{f}_k : L_k(X_1) \rightarrow L_k(X_2) \) is an isomorphism of AOS-fans.

(c) \( F \) is monotone for the specialization order.

Let \( g, h \in X_i \) be such that \( g \rightsquigarrow h \); say \( d(g) = m \geq d(h) = k \). We must prove \( F(g) \overset{\sim}{\rightarrow} F(h) \).

Since \( B^1_m \) generates \( L_m(X_1) \), then \( g = g_1 \cdot \ldots \cdot g_r \) with \( g_1, \ldots, g_r \in B^1_m \) and \( r \) necessarily odd (possibly \( = 1 \)). By Corollary 3.6 if \( h_i \) is the unique depth-\( k \) successor of \( g_i \), then \( h_i \in B^1_k \).

Also, \( g_i \rightsquigarrow h_i \) \( (i = 1, \ldots, r) \) implies \( g = g_1 \cdot \ldots \cdot g_r \rightsquigarrow h_1 \cdot \ldots \cdot h_r \). Since both \( h \) and \( h_1 \cdot \ldots \cdot h_r \) are successors of \( g \) of the same level \( k \), we get \( h = h_1 \cdot \ldots \cdot h_r \). As \( F \) preserves products of any odd number of elements of the same level, we have:

\[
F(g) = F(g_1) \cdot \ldots \cdot F(g_r) \quad \text{and} \quad F(h) = F(h_1) \cdot \ldots \cdot F(h_r).
\]

Since \( g_i \rightsquigarrow h_i \), \( g_i \in B^1_m \), and \( h_i \in B^1_k \), item (7.iii) yields \( F(g) = f_m(g_i) \overset{\sim}{\rightarrow} f_k(h_i) = F(h_i) \) \( (i = 1, \ldots, r) \). Then, by (8) again,

\[
F(g) = F(g_1) \cdot \ldots \cdot F(g_r) \overset{\sim}{\rightarrow} F(h_1) \cdot \ldots \cdot F(h_r) = F(h),
\]

which proves (c). The Claim follows from (a)–(c) using Lemma 3.10(3). This completes the proof of Theorem 3.11. \( \square \)

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