Radial Deformations and Cavitation in Riemannian Manifolds with Applications to Membrane Shells

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Abstract  This study is a geometric version of Ball’s work, Philos. Trans. Roy. Soc. London Ser. A 306 (1982), no. 1496, 557-611. Radial deformations in Riemannian manifolds are singular solutions to some nonlinear equations given by constitutive functions and radial curvatures. A geodesic spherical cavity forms at the center of a geodesic ball in tension by means of given surface tractions or displacements. The existence of such solutions depends on the growth properties of the constitutive functions and the radial curvatures.

Some close relationships are shown among radial curvature, the constitutive functions, and the behavior of bifurcation of a singular solution from a trivial solution. In the incompressible case the bifurcation depends on the local properties of the radial curvature near the geodesic ball center but the bifurcation in compressible case is determined by the global properties of the radial curvatures.

A cavity forms at the center of a membrane shell of isotropic material placed in tension by means of given boundary tractions or displacements when the Riemannian manifold under question is a surface of $\mathbb{R}^3$ with the induced metric. In addition, cavitation at the center of ellipsoids of $\mathbb{R}^n$ is also described if the Riemannian manifold under question is $(\mathbb{R}^n, g)$ where $g(x)$ are symmetric, positive matrices for $x \in \mathbb{R}^n$.

Keywords  radial curvature, cavitation, exponential map, membrane shell

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1 Introduction

The present paper is a geometric version of Ball’s work [2].

We investigate a class of singular solutions to the problems in which a hole forms in the center of a geodesic ball in a state of tension on a Riemannian manifold. This phenomenon of hole formation is said to be cavitation by a terminology commonly used in the special case of an elastic fluid.

The study of cavitation in the Euclidean space was initiated by Ball in the fundamental paper [2]. The work of Ball was, in part, motivated by the work of [3] and subsequently developed by many authors (see, e.g., [7, 8, 10, 11, 12, 13, 14, 15] and the review article [5]).

Let $(M, g)$ be a n-dimensional Riemannian manifold. Consider a body having strain energy $W$ in which the body occupies the open subset $\Omega$ of $M$. In a typical deformation in which a particle $x \in M$ is displaced to $u(x) \in M$ the strain energy is given by

$$E(u) = \int_{\Omega} W(du)dg. \quad (1.1)$$

The equilibrium equations of the body with zero body force are the Euler-Lagrange equations for the integral above. Solutions to these equilibrium equations are said to be equilibrium solutions.
Let \( o \in M \) be given and let \( \exp_o : M_o \to M \) be the exponential map. Radial deformations of a geodesic ball \( \Omega = \{ \exp_o \rho v \mid 0 \leq \rho < 1, v \in M_o, |v| = 1 \} \) have the form

\[
 u(x) = \exp_o \varphi(\rho)v \quad \text{for} \quad x = \exp_o \rho v \in \Omega.
\]  

Radial curvatures play a key role in the structure of radial deformations above. To get equilibrium equations for (1.2) from (1.1), we consider the case where radial curvatures on the geodesic sphere centered at \( o \) with radius \( t > 0 \) are the same, denoted by \( \kappa(t) \), for which we say \((M,g,o)\) is a model ([4]). We also assume that \( W(F) \) can be expressed as a symmetric function \( \Phi(v_1, \cdots, v_n) \) of the eigenvalues of \((F^TF)^{1/2}\). For the incompressible case the only kinematically admissible deformations of the form (1.2) are given by

\[
 \varphi(\rho) = \sigma^{-1}(\sigma(\rho) + \sigma(A)) \quad \text{for} \quad \rho \geq 0,
\]  

where \( A = \varphi(0) \geq 0 \) and function \( \sigma \) is defined by

\[
 \sigma(t) = \int_0^t f^{n-1}(s)ds \quad \text{for} \quad t \geq 0,
\]  

where \( f \) is the solution to problem

\[
 f''(t) + \kappa(t)f(t) = 0 \quad \text{for} \quad t > 0; \quad f(0) = 0, \quad f'(0) = 1.
\]  

For the compressible case \( \varphi \) has to satisfy the radial equilibrium equation (Theorem 2.1)

\[
 [f^{n-1}(\rho)\Phi_1(\rho)]_\rho = (n-1)f^{n-2}(\rho)f' \circ \varphi(\rho)\Phi_2(\rho) \quad \text{for} \quad x \in \Omega, \quad \rho(x) > 0,
\]  

where

\[
 \Phi_i(\rho) = \Phi_{v_i}(\varphi(\rho), \tau(\rho), \cdots, \tau(\rho)), \quad \tau(\rho) = \frac{f \circ \varphi(\rho)}{f(\rho)} \quad \text{for} \quad \rho > 0, \quad i = 1, 2,
\]  

where \( f \) is the solution to problem (1.5). Equation (1.6) expresses the close relationship among radial curvature \( \kappa \), constitutive function \( W \) and radial deformation \( \varphi \). Cavitation is equivalent to proving existence of solutions to problem (1.6) such that \( \varphi(0) > 0 \).

Consider existence of equilibrium solutions in the incompressible case. The assumption on the radial curvature such that formula (1.3) makes sense is the following

\[
 \int_0^{\max\{1,\varphi(1)\}} s\kappa_+(s)ds \leq 1,
\]  

where \( \kappa_+(s) = \max\{0, \kappa(s)\} \). We show (Theorem 2.3) that under assumption (1.7) if \( A > 0 \) then (1.3) generates an equilibrium solution if and only if

\[
 \frac{v^{n-1}}{(vn-1)^2} \bar{\Phi}'(v) \in L^1(\delta, \infty) \quad \text{for} \quad \delta > 1,
\]  

where

where \( \hat{\Phi}(v) = \Phi(v^{n-1}, v, \cdots, v) \). Conditions (1.8) are the same as in [2] in the Euclidean space. Let \( P \) be the radial component of the Piola-Kirchhoff stress at \( \rho = 1 \) and let \( T \) be the Cauchy stress. Under assumptions (1.7) and (1.8) and with the choice \( T(0) = 0 \) a critical value \( P_{cr} \) of \( P \) for solutions of bifurcation is given by

\[
P_{cr} = \int_{1}^{\infty} \frac{1}{v^{n-1}} \hat{\Phi}'(v) dv,
\]

which is again the same as in [2] in the Euclidean space.

In the compressible case the establishment of existence of cavitating equilibrium solutions is much more complicated. We use some similar assumptions on the growth properties of the constitutive function \( W \) as in [2] to analyze equilibrium solutions. The displacement boundary value problem in which \( \varphi(1) = \lambda > 0 \) is specified is concerned. A solution to problem (1.6) is said to be regular if \( \varphi(0) = 0 \). Since there are no explicit formulas for regular equilibrium solutions in general, we establish some estimates of regular equilibrium solutions from below and above (Theorems 4.1 and 4.2) under the radial curvature assumption (1.7). Using these estimates for regular equilibrium solutions and under the radial curvature assumptions

\[
\int_{0}^{\infty} s\kappa_+(s) ds \leq 1, \quad \int_{0}^{\infty} s\kappa_-(s) ds < \infty,
\]

where \( \kappa_-(s) = \max\{0, -\kappa\} \), we show (Theorem 4.8) that for \( \lambda \) large enough there is a unique radial minimizer \( \varphi \) of \( E \) with \( \varphi(1) = \lambda \) and \( \varphi(0) > 0 \), which is a stable cavitating equilibrium solution.

One of the direct applications of the analysis here is the cavitation problem of membrane shells. Let \( M \) be a surface in \( \mathbb{R}^3 \) with the induced metric \( g \). Suppose that the middle surface of a shell is a bounded open set \( \Omega \subset M \). First we show (Proposition 5.1) that the \( \Gamma \)-limit membrane shell, given in [6], takes the form (1.1) if all deformations of the middle are confined in \( M \). So we assume that the membrane shells have their stored energies in the form (1.1) to study their cavitation problems. In particular, the following surfaces of revolution are concerned:

\[
M = \{ (x, \psi(r)) \in \mathbb{R}^3 \mid x = (x_1, x_2) \in \mathbb{R}^2, \ r = |x| \},
\]

where \( \psi \) is a \( C^2 \) function on \([0, \infty)\). Then \((M, g, o)\) is a model where \( g \) is the induced metric of \( M \) from \( \mathbb{R}^3 \) and \( o = (0, 0, \psi(0)) \). The radial curvature is given by

\[
\kappa(t) = \frac{\psi'((\zeta(t))\psi''(\zeta(t)))}{\zeta(t)(1 + \psi'^2(\zeta(t))^2)} \quad \text{for} \quad t \geq 0,
\]

where function \( \zeta(t) \) is defined by equation

\[
t = \int_{0}^{\zeta(t)} \sqrt{1 + \psi'^2(s)} ds \quad \text{for} \quad t \geq 0.
\]
In the incompressible case a critical value of the radial component of the Piola-Kirchhoff stress $P$ at $\rho = 1$ for solutions of bifurcation is given by (1.9) under the assumptions (1.7), (1.8) and $T(0) = 0$. In the compressible case when $\varphi(1) = \lambda$ is large enough the stable solution is cavitating ($\varphi(0) > 0$) under the assumptions (1.10), $T(0) = 0$ and the growth assumptions of $W$.

Consider the Riemannian manifold $(\mathbb{R}^n, g)$ where $g = G(x)$ are symmetric and positive matrices for $x \in \mathbb{R}^n$. The radial deformation theory of Sections 2-4 describes that a a hole forms in the center of an ellipsoid in a state of tension in Section 6.

2 Equilibrium Equations for Radial Deformations on a Model

We make some preparations for our problems. Let $(M, g)$ be a $n$-dimensional Riemannian manifold with an orientation and let $\Omega \subset M$ be an open set. A map $u : \Omega \to M$ is said to be a deformation. Let $u : \Omega \to M$ be a deformation. We define the deformation gradient $d \! u$ of $u$ as a bilinear functional on $M_{u(x)} \times M_x$ by

$$d \! u(Y, X) = \langle Y, u_\ast X \rangle \circ u(x) \quad \text{for} \quad Y \in M_{u(x)}, \quad X \in M_x, \quad x \in \Omega,$$

where $\langle \cdot, \cdot \rangle = g$ is the Riemannian metric.

Let $W : M_{+}^{n \times n} \to \mathbb{R}$ be a function where $M_{+}^{n \times n}$ is the set of real $n \times n$ matrices with positive determinant. We denote by $\text{SO}(n)$ the special orthogonal group on $\mathbb{R}^n$. We assume that

$$W(F) = W(QF) = W(FQ) \quad \text{for} \quad F \in M_{+}^{n \times n}, \quad Q \in \text{SO}(n). \quad (2.2)$$

Let $x \in M$ be given. Let $\{e_i\}$ and $\{E_i\}$ be orthonormal bases of $(M_x, g(x))$ and $(M_{u(x)}, g \circ u(x))$ with the positive orientation, respectively. We define

$$W(du) = W(F), \quad (2.3)$$

where

$$F = \left(du(E_i, e_j)\right). \quad (2.4)$$

We have the following.

**Lemma 2.1** Let $W$ satisfy (2.2). Then the definition of (2.3) is independent of the selections of $\{e_i\}$ and $\{E_i\}$.

**Proof** Let $\{\hat{e}_i\}$ and $\{\hat{E}_i\}$ be the different selections. Let

$$\hat{e}_i = \sum_{j=1}^{n} q_{ij} e_j, \quad \hat{E}_i = \sum_{j=1}^{n} r_{ij} E_j.$$
Then
\[ \langle \hat{E}_i, \mathbf{u}, \hat{e}_j \rangle \circ \mathbf{u}(x) = \sum_{kl} r_{ik} (E_k, \mathbf{u} \ast e_l) q_{jl}, \]
that is,
\[ \left( d\mathbf{u}(\hat{E}_i, \hat{e}_j) \right) = R \left( d\mathbf{u}(E_i, e_j) \right) Q^T, \]
where \( R = \left( r_{ij} \right) \) and \( Q = \left( q_{ij} \right) \) are in \( \text{SO}(n) \). Then the lemma follows from (2.2). □

**Remark 2.1** Let \( M = \mathbb{R}^n \) with the Euclidean metric and \( \mathbf{u} = (u_1, \ldots, u_n) \). Then
\[ \left\langle \frac{\partial}{\partial x_i}, \mathbf{u} \ast \frac{\partial}{\partial x_j} \right\rangle = \frac{\partial u_i}{\partial x_j}. \]
In a typical deformation in which the point \( x \in \Omega \) is displaced to \( \mathbf{u}(x) \in M \) the energy of \( \mathbf{u} \) is given by
\[ E(\mathbf{u}) = \int_{\Omega} W(d\mathbf{u}) dg, \quad (2.5) \]
where \( dg \) is the volume element of \( M \) in the metric \( g \).

**Proposition 2.1** Let \( x = (x_1, \ldots, x_n) \) be a local coordinate system on \( (M, g) \) and
\[ g = \sum_{ij=1}^n g_{ij}(x) dx_i dx_j. \]
Let \( G = \left( g_{ij} \right) \). Then the equilibrium equations are the Euler-Lagrange equations for (2.5)
\[ \sum_{ij} \frac{\partial}{\partial x_l} \left( \frac{\partial W(d\mathbf{u}(x))}{\partial F_{ij}} \right) \alpha_{pi}(\mathbf{u}(x)) \alpha^{jl}(x) \sqrt{\det G(x)} = 0 \quad \text{for} \quad 1 \leq p \leq n, \quad (2.6) \]
where
\[ \left( \alpha_{ij} \right) = G^{1/2}, \quad \left( \alpha^{ij} \right) = G^{-1/2}. \]

**Proof** Let \( \mathbf{v} : M \to M \) be a deformation and let
\[ e_i(x) = \sum_{j=1}^n \alpha^{ij}(x) \partial x_j \quad \text{for} \quad 1 \leq i \leq n. \]
Then \( e_1, \ldots, e_n \) form an orthonormal basis of \( M_x \) and \( E_1, \ldots, E_n \) is an orthonormal basis of \( M_{\mathbf{v}(x)} \), where \( E_i = e_i(\mathbf{v}(x)) \) for \( 1 \leq i \leq n \). Moreover, it follows that
\begin{align*}
\langle E_i, \mathbf{v} \ast e_j \rangle &= \sum_{kl} \alpha^{ik}(\mathbf{v}(x)) \langle \partial x_k, \mathbf{v} \ast \partial x_l \rangle \alpha^{jl}(x) \\
&= \sum_{kli} \alpha^{ik}(\mathbf{v}(x)) g_{kh}(\mathbf{v}(x)) \alpha^{jl}(x) \frac{\partial v_h}{\partial x_l} \\
&= \sum_{lhi} \alpha_{ih}(\mathbf{v}(x)) \alpha^{jl}(x) \frac{\partial v_h(x)}{\partial x_l}.
\end{align*}
Let \( I(\varepsilon) = E(u + \varepsilon v) \).

We have
\[
I'(0) = \sum_{ij} \int_{\Omega} \frac{\partial W}{\partial F_{ij}} \langle E_i, v \rangle \sqrt{\det G(x)} dx
= \sum_{ij} \int_{\Omega} \left\{ \frac{\partial}{\partial x_l} \left( \frac{\partial W}{\partial F_{ij}} \alpha_{ih} (v(x)) \alpha^{jl} (x) \sqrt{\det G(x)} v_h \right) \right\} dx.
\]

Equations (2.6) follow. \( \square \)

**Remark 2.2** Let \( M = \mathbb{R}^n \) with the Euclidean metric. Equations (2.6) become
\[
\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[ \frac{\partial W(du)}{\partial F_{ij}} \right] = 0 \quad \text{for} \quad 1 \leq i \leq n.
\]

**Remark 2.3** For a general deformation \( u \), the problem (2.6) may be very complicated. Let
\[
W(F) = \frac{1}{2} |F|^2 \quad \text{for} \quad F \in M_{n \times n}^{+}.
\]

Then equations (2.6) are
\[
\Delta u_p + \sum_{ijkl=1}^{n} g^{ij}(x) \Gamma^p_{jk}(u(x)) \frac{\partial u_k}{\partial x_i} \frac{\partial u_l}{\partial x_j} = 0 \quad \text{for} \quad 1 \leq p \leq n,
\]
where \( \Delta \) is the Laplacian on \( M \) in the metric \( g \) and \( \Gamma^k_{ij} \) are the Christoffel symbols. Solutions of the above equations are called harmonic maps, see Lemma 8.1.1 in [9].

We are interested in radical deformations that are introduced below.

Let \( o \in M \) be fixed and let \( \exp_o : M_o \to M \) be the exponential map at the point \( o \) in the metric \( g \). For any \( x \in M \), there exists a pair \((\rho, v)\) with \( \rho \geq 0 \) and such that
\[
x = \exp_o \rho v \quad (2.7)
\]
where \( v = v(x) \in S_o \) and \( S_o \) is the unit sphere of \((M_o, g(o))\). Let \( d(x, y) \) be the distance function from \( x \) to \( y \) in the metric \( g \). Then \( \rho = d(o, x) \).

**Definition 2.1** A map \( u : M \to M \) is said to be a radical deformation with respect to \( o \in M \) if there is a function \( \varphi : [0, \infty) \to \mathbb{R} \) such that
\[
u(x) = \exp_o \varphi(\rho) v \quad \text{for} \quad x = \exp_o \rho v \in M. \quad (2.8)
\]
We shall solve the problem (2.6) when \( u \) is a radical deformations under appropriate assumptions on the constitutive function \( W \) and on the geometric properties of the metric \( g \). For this end, we need to computer \( du \) first.

Let \( \mathcal{X}(M) \) be all vector fields on \( M \). Let \( D \) be the Levi-Civita connection of the metric \( g \). Let \( X \) and \( Y \) be vector fields on \( M \). The curvature operator is a map \( R_{XY} : \mathcal{X}(M) \to \mathcal{X}(M) \), given by

\[
R_{XY}Z = -D_XD_YZ + D_YD_XZ + D_{[X,Y]}Z
\]

for all \( Z \in \mathcal{X}(M) \), where \([\cdot,\cdot]\) is the Lie bracket product. Let \( \gamma(t) \) be a geodesic with \( |\dot{\gamma}(t)| = 1 \) initiating from the point \( o \). A vector field \( J : [0, \infty) \to M \gamma \) is called a Jacobi field along \( \gamma \) if

\[
\ddot{J}(t) + R_{\gamma(t)}J \dot{\gamma} = 0 \quad \text{for} \quad t \geq 0.
\]

In addition, a Jacobi field \( J \) is said to be normal if

\[
\langle J(t), \dot{\gamma}(t) \rangle = 0 \quad \text{for} \quad t \geq 0.
\]

**Proposition 2.2** Let \( u \) be a radical deformation and let \( \rho = \rho(x) \) be the distance function in the metric \( g \) from \( x \in M \) to \( o \). Then

\[
u_\ast D\rho(x) = \varphi'(\rho)D\rho(\varphi(x)) \quad \text{for} \quad x \in M, \quad x \neq o.
\]

(2.9)

Let \( J(t) \) be a normal Jacobi field along the geodesic \( \gamma(t) \) with \( J(0) = 0 \). Then

\[
u_\ast J(t) = J(\varphi(t)) \quad \text{for} \quad t \in \mathbb{R}.
\]

(2.10)

**Proof** Formula (2.9) follows from expression (2.8).

Let \( \gamma(t) = \exp_o tv \) where \( v \in M_o \) with \( |v| = 1 \). Since \( \langle v, \dot{J}(0) \rangle = 0 \), there is a curve \( \sigma : [0, 1] \to M_o \) such that

\[
|\sigma(s)| = 1, \quad \sigma(0) = v, \quad \dot{\sigma}(0) = \dot{J}(0).
\]

Let

\[
\alpha(t, s) = \exp_o t\sigma(s) \quad \text{for} \quad (t, s) \in [0, \infty) \times [0, 1].
\]

Then

\[
J(t) = \alpha_\ast(t, 0) = t \exp_o \dot{J}(0) \quad \text{for} \quad t \in [0, \infty).
\]

By definition, we have

\[
u(\alpha(t, s)) = \exp_o \varphi(t)\sigma(s) \quad \text{for} \quad (t, s) \in [0, \infty) \times [0, 1],
\]

which yields

\[
u_\ast J(t) = \varphi(t) \exp_{o_\ast} \dot{J}(0) = J(\varphi(t)).
\]
For any \( v \in M_o \) with \( |v| = 1 \), there is a unique \( t_0(v) > 0 \) (or \( t_0(v) = \infty \)) such that the normal geodesic \( \gamma(t) = \exp_o tv \) is the shortest on the interval \([0, t_0)\). Let

\[
C(o) = \{ \ t_0(v)v \mid v \in M_o, \ |v| = 1 \ \}, \quad \Sigma(o) = \{ \ tv \mid v \in M_o, \ |v| = 1, \ 0 \leq t < t_0(v) \}.
\]

The set \( \exp_o C(o) \subset M \) is said to be the cut locus of \( o \) and the set \( \exp_o \Sigma(o) \subset M \) is called the interior of the cut locus of \( o \). Then

\[
M = \exp_o \Sigma(o) \cap \exp_o C(o).
\]

Furthermore, \( \exp_o : \Sigma(o) \to \exp_o \Sigma(o) \) is a diffeomorphism and \( C(o) \) is a zero measure set on \( M_o \). Then \( \exp_o C(o) \) is a zero measure set on \( M \) since it is the image of the zero measure set \( C(o) \), that is, \( \exp_o \Sigma(o) \) is \( M \) minus a zero measure set.

Let \( \psi : M_o \to M_o \) be a linear operator. We define a map \( \Psi : \exp_o \Sigma(o) \to \exp_o \Sigma(o) \) by

\[
\Psi(x) = \exp_o \rho \psi v \quad \text{for} \quad x = \exp_o \rho v \in \exp_o \Sigma(o).
\] (2.11)

**Definition 2.2** Let \( o \in M \) be fixed. The triple \((M, g, o)\) is said to be a model if for every linear isometry \( \psi : M_o \to M_o \), \( \Psi : \exp_o \Sigma(o) \to \exp_o \Sigma(o) \) is an isometry.

**Remark 2.4** A conception of models is introduced in \([4]\). Definition 2.2 above is weaker than that in \([4]\). If \( \exp_o : M \to M \) is a diffeomorphism, they are the same.

For any \( v \in S_o \), \( \gamma(t) = \exp_o tv \) is a normal geodesic initiating from the point \( o \). The radial curvature tensor along \( \gamma(t) \) is a tensor field of order two, given by

\[
R\left(\dot{\gamma}(t), X, \dot{\gamma}(t), Y\right) = \langle R(\dot{\gamma}(t)X), Y \rangle \quad \text{for} \quad X, Y \in M_{\gamma(t)}, \quad t \geq 0.
\]

A model is characterized by its radial curvature. We have

**Proposition 2.3** \((M, g, o)\) is a model if and only if there is a function \( \kappa \) on \([0, \infty)\) such that

\[
R\left(\dot{\gamma}(t), X, \dot{\gamma}(t), Y\right) = \kappa(t)\langle X, Y \rangle, \quad X, Y \in M_{\gamma(t)},
\] (2.12)

with \( \langle X, \dot{\gamma}(t) \rangle = 0 \) and \( \langle Y, \dot{\gamma}(t) \rangle = 0 \) for all \( t > 0 \), where \( \gamma(t) = \exp_o tv \in \exp_o \Sigma(o) \), for all \( v \in M_o \) with \( |v| = 1 \).

**Proof** Let \((M, g, o)\) be a model and let \( \rho > 0 \) be given. Let \( \gamma_i(t) = \exp_o tv_i \) with \( v_i \in S_0 \) for \( i = 1, 2 \). Let \( X_i \) be in \( M_{\gamma_i(\rho)} \) with \( \langle X_i, \dot{\gamma}_i(\rho) \rangle = 0 \) and \( |X_i| = 1 \), respectively, for \( i = 1, 2 \). Let \( z_i \) be in \( S_0 \) such that

\[
z_i(\rho) = X_i.
\]
where \( z_i(t) \) are the parallel translations of \( z_i \) along \( \gamma_i \) such that \( z_i(0) = z_i \), respectively, for \( i = 1, 2 \). Then
\[
\langle z_i, v_i \rangle = 0 \quad \text{for} \quad i = 1, 2.
\]

Suppose \( \psi : M_o \to M_o \) is a linear isometry such that \( \psi v_1 = v_2 \) and \( \psi z_1 = z_2 \). Then \( \Psi \), given by (2.11), is an isometry from \( \exp_o \Sigma(o) \) to \( \exp_o \Sigma(o) \). Then \( \Psi(\gamma_1(t)) = \gamma_2(t) \) and \( \Psi_*z_1(t) = z_2(t) \). We obtain
\[
\mathbf{R} \left( \dot{\gamma}_2(\rho), X_2, \dot{\gamma}_2(\rho), X_2 \right) (\gamma_2(\rho)) = \mathbf{R} \left( \Psi_*\dot{\gamma}_1(\rho), \Psi_*X_1, \Psi_*\dot{\gamma}_1(\rho), \Psi_*X_1 \right) (\Psi(\gamma_1(\rho)))
= \mathbf{R} \left( \dot{\gamma}_1(\rho), X_1, \dot{\gamma}_1(\rho), X_1 \right) (\gamma_1(\rho)).
\]

Thus, the radial curvatures are the same on the geodesic sphere \( \mathbf{S}(\rho) \), centered at \( o \) and with radii \( \rho \). Formula (2.12) follows with \( \kappa \) being the radial curvature.

Conversely, suppose (2.12) holds. Let \( \psi : M_o \to M_o \) be a linear isometry. Let \( x = \exp_o pv \in \exp_o \Sigma(o) \) be given. Let \( X_i \) be in \( M_x \) with \( |X_i| = 1 \) for \( i = 1, 2 \). Suppose \( E_i(t) \) are the parallel translations of some \( v_i \in S_o \) along \( \gamma(t) = \exp_o tv \) such that \( E_i(\rho) = X_i \), respectively, for \( i = 1, 2 \). Let
\[
\alpha_i(t,s) = \exp_ot(v + sv_i) \quad \text{for} \quad t \geq 0, \ s \in \mathbb{R}, \ i = 1, 2.
\]

Thus, \( J_i(t) = t \exp_{\alpha_i} v_i \) are Jacobi fields along \( \gamma(t) = \exp_o tv \). By formula (2.12) and \( J_i'(0) = v_i \), we obtain
\[
J_i(t) = f(t)E_i(t) \quad \text{for} \quad t \geq 0, \ i = 1, 2, \quad (2.13)
\]
where \( f \) is the solution to problem (2.15) later. In addition the formulas \( \Psi(\alpha_i(t,s)) = \exp_o t\psi(v + sv_i) \) imply
\[
\Psi_* J_i(t) = t \exp_{\alpha_i} \psi v_i
\]
are also Jacobi fields along \( \psi(\gamma(t)) = \exp_o t\psi v_i \) for \( i = 1, 2 \). By (2.12) again,
\[
\Psi_* J_i(t) = f(t)\hat{E}_i(t), \quad (2.14)
\]
where \( \hat{E}_i(t) \) are parallel translation vector fields along \( \Psi(\gamma(t)) \), respectively, for \( i = 1, 2 \), such that
\[
\hat{E}_i(0) = \psi v_i \quad \text{for} \quad i = 1, 2.
\]
It follows from (2.13) and (2.14) that
\[
\langle \Psi_* X_1, \Psi_* X_2 \rangle = \langle \Psi_* E_1(\rho), \Psi_* E_2(\rho) \rangle = \langle \hat{E}_1(0), \hat{E}_2(0) \rangle = \langle \psi v_1, \psi v_2 \rangle = \langle v_1, v_2 \rangle = \langle X_1, X_2 \rangle.
\]
Thus, \( \Psi : \exp_o \Sigma(o) \to \exp_o \Sigma(o) \) is an isometry. \( \square \)

**Remark 2.5** Formula (2.12) means that a model has the same radial curvature on a geodesic sphere \( \mathbf{S}(t) \), centered at \( o \) with radii \( t > 0 \).
Let \((M, g, o)\) be a model and let the radial curvature \(\kappa\) be given by (2.12). Consider problem
\[
\begin{cases}
  f''(t) + \kappa(t)f(t) = 0, & t > 0, \\
  f(0) = 0, & f'(0) = 1.
\end{cases}
\]  
(2.15)
Let \(f\) be the solution to problem (2.15). Then
\[
f(t) = t - \int_0^t (t - s)\kappa(s)f(s)ds,
\]  
(2.16)
which yields
\[
\lim_{t \to 0^+} \frac{f(t)}{t} = 1.
\]  
(2.17)
For our problems here, we need \(f\) and \(f'\) are all positive functions, for which the following is introduced. Let the radial curvature \(\kappa\) be given in (2.12). Let
\[
\mu_{\pm}(\lambda) = \int_0^\lambda s\kappa_{\pm}(s)ds \quad \text{for} \quad \lambda > 0,
\]  
(2.18)
where
\[
\kappa_+(s) = \max\{\kappa(s), 0\}, \quad \kappa_-(s) = \max\{-\kappa(s), 0\} \quad \text{for} \quad s \geq 0.
\]
We have ([4])

**Proposition 2.4** If
\[
\mu_+(\lambda) \leq 1,
\]  
(2.19)
then there exists \(0 < \mu_0(\lambda) \leq 1\) such that
\[
\mu_0(\lambda)\rho \leq f(\rho) \leq e^{\mu_-(\lambda)}\rho \quad \text{for} \quad \rho \in [0, \lambda],
\]  
(2.20)
and
\[
\mu_0(\lambda) \leq f'(\rho) \leq e^{\mu_-(\lambda)} \quad \text{for} \quad \rho \in [0, \lambda],
\]  
(2.21)
where \(f\) is the solution to problem (2.15).

**Proof** Let \(p_+\) and \(p_-\) solve the problems
\[
\begin{cases}
  p''_+ + \kappa_+p_+ = 0 & \text{for} \quad \rho > 0, \\
  p_+(0) = 0, & p'_+(0) = 0
\end{cases}
\]  
(2.22)
and
\[
\begin{cases}
  p''_- - \kappa_-p_- = 0 & \text{for} \quad \rho > 0, \\
  p_-(0) = 0, & p'_-(0) = 0
\end{cases}
\]  
(2.23)
respectively.
Let \(\eta(\rho) = p_-/p'_-\) for \(\rho > 0\). By (2.23) we have
\[
\eta' = 1 - \kappa_-\eta \leq 1, \quad \text{and, then} \quad \eta \leq \rho \quad \text{for} \quad \rho \geq 0,
\]
since \( \eta(0) = 0 \). It follows that

\[
\frac{p''}{p'} = \kappa_- \eta \leq \kappa_-(\rho) \rho \quad \text{for} \quad \rho \geq 0.
\]

Integrating the above inequality over \((0, \rho)\) yields

\[
1 \leq p'_-(\rho) \leq e^{\mu_-(\lambda)} \quad \text{for} \quad \rho \in [0, \lambda],
\]

(2.24)

which implies

\[
\rho \leq p_-(\rho) \leq e^{\mu_-(\lambda)} \rho \quad \text{for} \quad \rho \in [0, \lambda].
\]

(2.25)

On the other hand, from (2.22) we obtain

\[
\mu_0(\lambda) \leq p'_+(\rho) = 1 - \int_0^\rho \kappa_+(s)p_+(s)ds \leq 1,
\]

(2.26)

and then

\[
\mu_0(\lambda) \rho \leq p_+(\rho) \leq \rho \quad \text{for} \quad \rho \in [0, \lambda],
\]

(2.27)

where \( \mu_0(\lambda) = \min_{0 \leq \rho \leq \lambda} p'_+(\lambda) \leq 1 \). Moreover, we claim that assumption (2.19) implies that \( \mu_0(\lambda) > 0 \). Otherwise, if there were a point \( \rho_0 \in [0, \lambda] \) such that \( p'_+(\rho_0) = 0 \), then by (2.27)

\[
1 = \int_0^{\rho_0} \kappa_+(s)p_+(s)ds < \mu_+(\lambda) \leq 1,
\]

a contradiction.

By a comparison argument for ordinary differential equations we obtain

\[
\frac{p'_+}{p_+} \leq \frac{f'}{f} \leq \frac{p'_-}{p_-} \quad \text{for} \quad \rho \in [0, \lambda].
\]

(2.28)

Integrating the above inequalities over \((\varepsilon, \rho)\) gives

\[
p_+ \frac{f(\varepsilon)}{p_+(\varepsilon)} \leq f \leq \frac{f(\varepsilon)}{p_-(\varepsilon)}p_- \quad \text{for} \quad 0 < \varepsilon \leq \rho \leq \lambda.
\]

Letting \( \varepsilon \to 0^+ \) in the above inequalities we have

\[
p_+ \leq f \leq p_- \quad \text{for} \quad \rho \in [0, \lambda],
\]

(2.29)

since

\[
\lim_{\rho \to 0^+} \frac{f(\rho)}{\rho} = \lim_{\rho \to 0^+} \frac{p_+(\rho)}{\rho} = 1.
\]

Then (2.20) follows from (2.29), (2.27), and (2.25).

Finally, (2.21) follows from (2.28), (2.29), (2.24), and (2.26).

The following result is immediate from Proposition 2.4 which is an improvement of Lemmas 4.5 and 4.6 in [4].
Proposition 2.5 If
\[ \mu_+(\infty) \leq 1, \quad \mu_-(\infty) < \infty, \] (2.30)
then there are \(0 < \mu_0 \leq \mu_1 < \infty\) such that
\[ \mu_0 \leq f' \leq \mu_1 \quad \text{and} \quad \mu_0 \rho \leq f \leq \mu_1 \rho \quad \text{for} \quad \rho \in [0, \infty). \] (2.31)

Let \(\rho = \rho(x)\) be the distance function in the metric \(g\) from \(o\) to \(x \in M\). We recollect some properties of a model from [4] in the following.

Proposition 2.6 Let \((M, g, o)\) be a model. Then
(i) Any Jacobi field \(J(t)\) is in the form
\[ J(t) = f(t)E(t) \quad \text{for} \quad t > 0, \]
where \(E(t)\) is the parallel translation.
(ii) The Hessian of the distance function \(\rho\) is given by
\[ D^2 \rho = \frac{f'(\rho)}{f(\rho)}(g - D\rho \otimes D\rho) \quad \text{for} \quad \rho(x) > 0, \quad x \in \Sigma(o). \]
(iii) In the geodesic polar coordinates the metric \(g\) has the expression:
\[ g = d\rho^2 + f^2(\rho)d\theta^2 \quad \text{for} \quad \rho > 0, \quad x \in \Sigma(o). \] (2.32)

Proposition 2.7 Let \((M, g, o)\) be a model. Let \(u\) be a radical deformation given by (2.8). Then
\[ (du) = \text{diag} \left( \varphi'(\rho), \tau(\rho), \cdots, \tau(\rho) \right) \quad \text{for} \quad x \in \Sigma(o), \] (2.33)
where
\[ \tau(\rho) = \frac{f \circ \varphi(\rho)}{f(\rho)} \quad \text{for} \quad x \in \Sigma(o). \] (2.34)

**Proof** Let \( x = \exp o v \) where \( v \in M_o \) with \(|v| = 1\). Let \( \{E_i(t)\} \) be the parallel translation orthonormal basis of \( M_{\gamma(t)} \) along \( \gamma(t) = \exp_o tv \) with \( E_1 = v \). Then
\[ E_1(t) = D\rho(\gamma(t)) \quad \text{for} \quad t > 0. \]

By Proposition 2.2 and Proposition 2.6 (i), we have
\[ \langle E_1(\varphi(\rho)), u_*, E_1(\rho) \rangle = \langle D\rho(\varphi(\rho)), u_*, D\rho(x) \rangle = \varphi'(\rho), \]
\[ \langle E_1(\varphi(\rho)), u_*, E_j(\rho) \rangle = 0 \quad \text{for} \quad 2 \leq j \leq n, \]
\[ \langle E_i(\varphi(\rho)), u_*, E_j(\rho) \rangle = \langle E_i(\varphi(\rho)), \frac{1}{f(\rho)}u_*, J_j(\rho) \rangle = \frac{f \circ \varphi(\rho)}{f(\rho)} \delta_{ij}, \]
for \( 2 \leq i, j \leq n \). \( \square \)
2.1 Compressible Case

Let \( u : M \to M \) be a differentiable map. In order to compute equilibrium equations to energy (2.5), we consider the vector bundle \( \zeta = u^{-1}TM \) over the base manifold \((M, g)\), induced by the map \( u \),
\[
\zeta = \bigcup_{x \in M} M_{u(x)}.
\] (2.35)
The projection map \( \pi : \zeta \to M \) is given by
\[
\pi(x, Y) = x \quad \text{for} \quad x \in M, \quad Y \in M_{u(x)}.
\]

In local coordinates a section \( H \) of \( \zeta \) is in the form
\[
H(x) = \sum_{i=1}^{n} h_i(x) \partial_{x_i} \big|_{u(x)} \quad \text{for} \quad x \in M,
\] (2.36)
where \( h_i \in C^\infty(M) \) for all \( i \). Denote by \( \Gamma(\zeta) \) all sections of \( \zeta \). The connection \( D : \mathcal{X}(M) \times \Gamma(\zeta) \to \Gamma(\zeta) \), induced by the metric \( g \), is given by
\[
D_X H = \sum_{i=1}^{n} [X(h_i) \partial_{x_i} \big|_{u(x)} + h_i(x)(D_{u(x)}X \partial_{x_i}) \circ u],
\] (2.37)
where \( X \in \mathcal{X}(M) \) and \( H \in \Gamma(\zeta) \). Furthermore, for \( H_1, H_2 \in \zeta \) and \( X \in \mathcal{X}(M) \), we have
\[
X(H_1, H_2) = \langle D_X H_1, H_2 \rangle + \langle H_1, D_X H_2 \rangle.
\]

For \( H \in \Gamma(\zeta) \), we set
\[
v(t)(x) = \exp_{u(x)} tH(x) \quad \text{for} \quad t \geq 0, \quad x \in M,
\] (2.38)
where \( \exp_{u(x)} : M_{u(x)} \to M \) is the exponential map in the metric \( g \) at \( u(x) \) along the vector \( H(x) \in M_{u(x)} \) for each \( x \in M \). For each \( x \) fixed, \( v(t) \) is a geodesic on \( M \) initiating from \( u(x) \) in the metric \( g \) and
\[
v(0)(x) = u(x), \quad \dot{v}(0)(x) = H(x) \in M_{u(x)} \quad \text{for} \quad x \in M.
\]

\( v(t) \) is said to be a variation of \( u \) (Chapter 8 of [9]).

Let \( x \in M \) be given. For \( e \in M_x \), \( v_*(t)e \) is a vector field along the geodesic \( v(t)(x) \).
We have

**Lemma 2.2** Let \( u : M \to M \) be a differentiable map and let \( v(t) \) be a variation of \( u \), given by (2.38) for \( t \in (-\varepsilon, \varepsilon) \). Then
\[
D_{\dot{v}(0)} v_*(e) = D_e H \quad \text{for} \quad e \in M_x, \quad x \in M.
\] (2.39)
**Proof** We do a computation in local coordinates \( x = (x_1, \cdots, x_n) \). Let \( H \in \Gamma(\zeta) \) be given by (2.36). Let \( e = \sum_{i=1}^n \alpha_i \partial_{x_i} \) and

\[
v(t)(x) = (v_1(t,x), \cdots, v_n(t,x)) \quad \text{for} \quad (t,x) \in (-\varepsilon, \varepsilon) \times M,\]

where

\[
(v_1(0,x), \cdots, v_n(0,x)) = u(x), \quad \dot{v}(0)x = H(x) \quad \text{for} \quad x \in M.
\]

Then

\[
\dot{v}(t) = \sum_{i=1}^n \dot{v}_i(t) \partial_{x_i}|_v(t), \quad v_*(t) \partial_{x_j} = \sum_{i=1}^n v_{ix_j} \partial_{x_i}|_v(t),
\]

where \( \dot{v}_i(0) = h_i(x) \) and \( v_*(0) \partial_{x_i} = u_* \partial_{x_i} \) for \( 1 \leq i \leq n \). We have

\[
D_{\dot{v}(0)} v_* e = D_{\dot{v}(0)} \left[ \sum_i \left( \sum_j \alpha_j v_{ix_j} \right) \partial_{x_i}|_v(t) \right]
\]

\[
= \sum_i \left[ \sum_j \dot{v}_j(0) \partial_{x_j}|_v(t) + \sum_j \alpha_j \dot{v}_{ix_j} \sum_k h_k (D_{\partial_{x_k}} \partial_{x_i})|_v(x) \right]
\]

\[
= \sum_i \left[ \sum_j \alpha_j \dot{h}_{ix_j}(0) + \sum_{ik} v_{ix_j} h_k \Gamma^l_{ki} \partial_{x_i}|_v(x) \right]
\]

\[
= \sum_i \left[ \sum_j \alpha_j \dot{h}_{ix_j} + \sum_{ik} u_{ix_j} h_k \Gamma^l_{ki} \partial_{x_i}|_v(x) \right].
\] (2.40)

It follows from (2.40) that

\[
D_e H = \sum_i \left[ v_i \partial_{x_i}|_v(x) + h_i D_{u_*} \partial_{x_i} \right]
\]

\[
= \sum_i \alpha_j \dot{h}_{ix_j} \partial_{x_i}|_v(x) + h_i \sum_k u_{kx_j} (D_{\partial_{x_k}} \partial_{x_i})|_v(x)
\]

\[
= D_{\dot{v}(0)} v_* e.
\]

\[\square\]

Let \( e_1, \cdots, e_n \) be an orthonormal basis of \( M_o \). Consider the usual pole coordinates \((\rho, \theta)\) on the Euclidean space \( \mathbb{R}^n \), given by

\[
\begin{aligned}
\begin{cases}
  z_1 = \rho \cos \theta_1, \\
  z_2 = \rho \sin \theta_1 \cos \theta_2 \\
  \cdots \\
  z_{n-1} = \rho \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_{n-1} \\
  z_n = \rho \sin \theta_1 \cdots \sin \theta_{n-2} \sin \theta_{n-1},
\end{cases}
\end{aligned}
\] (2.41)

where \( \theta = (\theta_1, \cdots, \theta_{n-1}) \) and

\[
0 \leq \theta_i \leq \pi, \quad \cdots, \quad 0 \leq \theta_{n-2} \leq \pi, \quad 0 \leq \theta_{n-1} \leq 2\pi.
\]
Let
\[
\pi(\theta) = \frac{1}{\rho} \sum_{i=1}^{n} z_i e_i.
\]  
(2.42)

We have
\[
\lim_{\rho \to 0^+} \int_{S_{o}} \langle X, D\rho \rangle d\theta = \lim_{\rho \to 0^+} \int_{S_{o}} \langle X, \exp_{o^*} \pi(\theta) \rangle d\theta = \int_{S_{o}} \langle X(o), \pi(\theta) \rangle d\theta = 0,
\]
and, thus,
\[
\lim_{\rho \to 0} \int_{S_{o}} \frac{1}{\rho} \langle X, D\rho \rangle d\theta = \lim_{\rho \to 0^+} \int_{S_{o}} \frac{\langle X, D\rho \rangle - \langle X(o), \pi(\theta) \rangle(o)}{\rho} d\theta = \int_{S_{o}} \langle D\pi(\theta) X, \pi(\theta) \rangle d\theta.
\]  
(2.43)

Let the function \( W : M_{+}^{n \times n} \to \mathbb{R} \) satisfy assumption (2.2). It is well known ([16]) that there exists a symmetric function \( \Phi : \mathbb{R}^{n} \to \mathbb{R}, \mathbb{R}^{n} = \{ c = (c_1, \cdots, c_n) \in \mathbb{R}^n, c_i > 0 \text{ for } 1 \leq i \leq n \} \), such that
\[
W(F) = \Phi(v_1, \cdots, v_n) \quad \text{for all } F \in M_{+}^{n \times n},
\]  
(2.44)

where \( v_1, \cdots, v_n \) denote the singular values (or principal stretches) of \( F \) (i.e., the eigenvalues of \( (F^*F)^{1/2} \)). It is known ([1]) that \( W \in C^r(M_{+}^{n \times n}) \) if and only if \( \Phi \in C^r(\mathbb{R}^n_{++}) \) for \( r = 0, 1, 2 \) or \( \infty \). We write \( \Phi_i = \frac{\partial \Phi}{\partial v_i} \), etc. If \( W \in C^1(M_{+}^{n \times n}) \) and if \( F = \text{diag}(v_1, \cdots, v_n) \), \( v_i > 0 \), then
\[
\frac{\partial W}{\partial F}(F) = \text{diag}(\Phi_1, \cdots, \Phi_n),
\]  
(2.45)

where \( \Phi_i = \Phi_i(v_1, \cdots, v_n) \) for all \( i \). Moreover, the symmetry of the function \( \Phi \) implies
\[
\Phi_i(v_1, v, \cdots, v) = \Phi_i(v, v, \cdots, v) \quad \text{for } 2 \leq i \leq n.
\]  
(2.46)

Let \( u : \Omega \to M \) be a deformation. We say that \( u \in W^{1,p}(\Omega, M) \) if
\[
\int_{\Omega} |F|^p dg < \infty,
\]
where \( F \) is given by (2.4) and \( 1 \leq p < \infty \). Moreover, we define
\[
\det du(x) = \det F \quad \text{for } x \in \Omega.
\]

Let \( \Omega = B \) be the unit geodesic ball centered at \( o \). Let \( H \in \Gamma(\zeta) \) be a section. Then the differential of \( H \) in the connection \( D \) can be defined by
\[
DH(x) = \left( \langle E_i, D_{e_j}H \rangle \right) \quad \text{for } x \in M,
\]
where \( \{e_i\} \) and \( \{E_i\} \) are orthonormal bases of \( M_x \) and \( M_u \), respectively. We say that a deformation \( u \in W^{1,1}(B, M) \) is an equilibrium solution if \( \det du > 0 \), a.e. \( x \in B \), and \( \frac{\partial W}{\partial x} \in L^1(B) \) for \( 1 \leq i, j \leq n \), and

\[
\int_B \langle DFW, DH \rangle dg = 0 \quad \text{for all} \quad H \in C_0^\infty(B, \Gamma(\zeta)).
\]

**Remark 2.6** In general the vector bundle \( \Gamma(\zeta) \), given by (2.35), may depend on the deformation \( u \). Then does \( DH \). If \( M = \mathbb{R}^n \) with the Euclidean metric, then \( M_{u(x)} = M_x = \mathbb{R}^n \) and a section \( H \) and its differential \( DH \) are independent of the deformation.

**Theorem 2.1** Let \((M, g, o)\) be a model with \( \mu_+(1) \leq 1 \). Let \( u \) be a radical deformation given by (2.8) such that \( \mu_+(\varphi(1)) \leq 1 \). Then \( u \) is an equilibrium solution to problem (2.6) if and only if \( \varphi \in W^{1,1}(0,1), \varphi'(\rho) > 0 \) a.e. \( \rho \in (0,1] \), \( f^{n-1} \Phi_1, f^{n-1} \Phi_2 \in L^1(0,1) \), and

\[
[f^{n-1}(\rho)\Phi_1(\rho)]_\rho = (n-1)f^{n-2}(\rho)f' \circ \varphi(\rho)\Phi_2(\rho) \quad \text{for} \quad x \in \Omega, \quad \rho(x) > 0, \quad (2.47)
\]

where

\[
\Phi_i(\rho) = \Phi_i(\varphi', \tau, \ldots, \tau) \quad \text{for} \quad \rho > 0, \quad i = 1, 2,
\]

and \( \tau \) is given by (2.34).

**Proof** Let \( H \) be a section of \( \Gamma(\zeta) \) with a compact support such that \( \text{supp } H \subset B \) and let \( v(t) = \exp u(x) \cdot tH(x) \) be a variation of \( u \). Let \( x = \exp o \cdot tv \in \Sigma(o) \) be given, where \( v \in M_o \) with \( |v| = 1 \). Let \( \{E_i\} \) be an orthonormal basis of \( M_o \) with \( E_1 = v \). We transport \( \{E_i\} \) along the geodesic \( \gamma(t) = \exp o \cdot tv \) parallelly to obtain the orthonormal bases \( \{E_i(t)\} \) of \( M_o(t) \) for \( t \geq 0 \). Next, we transport parallelly the orthonormal basis \( \{E_i(\varphi(\rho))\} \) of \( M_{u(x)} \) along the geodesic \( v(t) = \exp u(x) \cdot tH \) to have the orthonormal bases \( \{\hat{E}_i(t)\} \) of \( M_{\nu(t)} \) for which

\[
\hat{E}_i(0) = E_i(\varphi(\rho)) \quad \text{for} \quad 1 \leq i \leq n. \quad (2.48)
\]

In particular,

\[
\hat{E}_1(0) = E_1(\varphi(\rho)) = D\rho(\varphi(\rho)). \quad (2.49)
\]

By Propositions 2.2 and Proposition 2.6 (i), we have

\[
u_*E_1(\rho) = \varphi'(\rho)D\rho(\varphi(\rho)), \quad \nu_*E_i(\rho) = \tau(\rho)E_i(\varphi(\rho)) \quad \text{for} \quad 2 \leq i \leq n. \quad (2.50)
\]

Let

\[
F(t) = \langle (\hat{E}_i(t), \nu_*(t)E_j(\rho)) \rangle.
\]

By Proposition 2.7,

\[
F(0) = \text{diag} \left( \varphi', \tau, \ldots, \tau \right),
\]

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since \(v_\ast(0) = u_\ast\). We obtain, by Lemma 2.2, (2.48), (2.45), (2.46) and (2.49),

\[
\frac{d}{dt}W(dv)|_{t=0} = \sum_{ij=1}^{n} \frac{\partial W}{\partial F_{ij}}(F(0)) \frac{\partial \langle \hat{E}_i(t), v_\ast E_j(\rho) \rangle}{\partial t} |_{t=0} \\
= \sum_{ij=1}^{n} \frac{\partial W}{\partial F_{ij}}(F(0)) \langle \hat{E}_i(0), D_{E_j(\rho)H} \rangle \circ u(x) \\
= \Phi_1(\rho)\langle D\rho, D\rho H \rangle \circ u(x) + \sum_{i=2}^{n} \Phi_i(E_\ast(\varphi(\rho)), D_{E_i(\rho)H} \circ u(x) \\
= \frac{\partial}{\partial \rho} [\Phi_1(\rho)\langle H, D\rho \rangle] - \frac{\partial}{\partial \rho} [\Phi_1(\rho)\langle H, D\rho \rangle] \\
+ \sum_{i=2}^{n} \langle E_i(\varphi(\rho)), D_{E_i(\rho)H} \circ u(x) \rangle. \\
(2.51)
\]

Next, let

\[H = h_0(x)D\rho|_{u(x)} + \tilde{H},\]

where

\[\tilde{H} = \sum_{j=1}^{n-1} h_j(x)\partial_\theta_j|_{u(x)}\]

and \((\rho, \theta)\) are the geodesic polar coordinates on \(M\) initiating from \(o\) in the metric \(g\). It follows from (2.52) that \(\langle H, D\rho \rangle = h_0(x)\) for \(x \in M\), i.e., \(\langle H, D\rho \rangle\) is a function on \(M\), which is independent of deformation \(u\).

Using (2.50) and Proposition 2.6 (ii), we obtain

\[
\langle E_i(\varphi(\rho)), D_{E_i(\rho)H} \rangle = \langle E_i(\varphi(\rho)), E_i(\rho)h_0(x)D\rho|_{u(x)} + h_0(x)D_{u,E_i(\rho)D\rho} + D_{E_i(\rho)H} \rangle \\
= \langle H, D\rho \rangle \tau(\rho)D^2\rho(E_i(\varphi(\rho)), E_i(\varphi(\rho))) + \langle E_i(\varphi(\rho)), \tilde{D}_{E_i(\rho)H} \rangle \\
= \frac{\langle H, D\rho \rangle f^\ast \circ \varphi(\rho)}{f(\rho)} + \langle E_i(\varphi(\rho)), \tilde{D}_{E_i(\rho)H} \rangle, \\
(2.53)
\]

for \(2 \leq i \leq n\), where \(\tilde{D}\) is the induced connection of \(S(\varphi(\rho))\) from the metric \(g\) and where \(S(\varphi(\rho))\) denotes the geodesic sphere with radii \(\varphi(\rho)\) centered at \(o\).

Let functions \(p_j\) on \(S(\varphi(\rho))\) be defined by

\[p_j(u(x)) = h_j(x) \quad \text{for} \quad u(x) \in S(\varphi(\rho)), \quad 1 \leq j \leq n - 1.\]

Using (2.37), we obtain

\[
\tilde{D}_{E_i(\rho)H}|_{u(x)} = \sum_{j=1}^{n-1} \left[ E_i(h_j(x))\partial_\theta_j|_{u(x)} + h_j(x)D_{u,E_i(\rho)\partial_\theta_j \circ u(x)} \right] \\
= \tau \sum_{j=1}^{n} \left[ E_i(p_j)(u)\partial_\theta_j|_{u} + p_j(u)\tilde{D}_{E_j(\varphi(\rho))\partial_\theta_j \circ u} \right].
\]

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Thus,

\[
\sum_{i=2}^{n} \langle E_i(\varphi(\rho)), D_{E_i(\rho)} \tilde{H} \rangle \circ u(x) = \tau \tilde{\text{div}} \tilde{H}_0 \circ u(x),
\]

(2.54)

where \( \tilde{\text{div}} \) is the divergence of the induced metric on \( S(\varphi(\rho)) \) from the metric \( g \) and \( \tilde{H}_0 \) is a vector field on \( S(\varphi(\rho)) \), given by

\[
\tilde{H}_0 = \sum_{j=1}^{n-1} p_i(u) \partial \theta_j \bigg|_{u(x)} \text{ for } u(x) \in S(\varphi(\rho)).
\]

(2.55)

By inserting (2.53) and (2.54) into (2.51), we have

\[
\frac{d}{dt} W(dv)_{t=0} = \frac{\partial}{\partial \rho}[\Phi_1(\rho) \langle H, D\rho \rangle] + [(n-1)\frac{f'}{f} \circ \varphi(\rho) - \Phi_2(\rho) - \frac{\partial}{\partial \rho} \Phi_1(\rho)] \langle H, D\rho \rangle
\]

\[
+ \tau(\rho) \Phi_2(\rho) (\tilde{\text{div}} \tilde{H}_0) \circ u(x).
\]

(2.56)

Using (2.32) and (2.56), we obtain

\[
\int_B \langle D_F W, DH \rangle dg = \int_B \frac{d}{dt} W(dv)_{t=0} \bigg|_{t=0} d\sigma = \int_B \frac{d}{dt} W(dv)_{t=0} f^{n-1}(\rho) d\rho d\theta
\]

\[
= \int_0^1 \left\{ (n-1) f^{n-2}(\rho) f' \circ \varphi(\rho) \Phi_2(\rho) - [f^{n-1}(\rho) \Phi_1(\rho)] \right\} \rho \int_{S_0} \langle H, D\rho \rangle d\theta
\]

\[
+ \int_0^1 \tau^{2-n} \Phi_2(\rho) d\rho \int_{S(\varphi(\rho))} \tilde{\text{div}} \tilde{H}_0 d\tilde{g}
\]

\[
+ \int_{S_0} (f^{n-1} \Phi_1(\rho) \langle H, D\rho \rangle) d\theta \bigg|_{\rho=1} - \lim_{\varepsilon \to 0} f^{n-1}(\varepsilon) \Phi_1(\varepsilon) \int_{S_0} \langle H, D\rho \rangle d\theta.
\]

(2.57)

where \( \tilde{g} \) is the induced metric on \( S(\varphi(\rho)) \) from the metric \( g \) and \( S_0 \subset M_0 \) is the unite sphere of \( M_0 \).

Since \((S(\varphi(\rho)), \tilde{g})\) is a compact manifold without a boundary, the second integral in the right hand side of (2.57) is zero.

Let radical deformation \( u \) be an equilibrium solution. We take \( H \in \Gamma(\zeta) \) with \( \text{supp } H \subset B \setminus \{o\} \). Then the last two integrals in the right hand (2.57) are zero. Thus, equation (2.47) follows from (2.57).

Conversely, suppose that \( f^{n-1} \Phi_1, f^{n-1} \Phi_2 \in L^1(0,1) \) and that equation (2.47) is true. It follows from equation (2.47) and the relation (2.57) that for any \( H \in C_0^{\infty}(B, \Gamma(\zeta)) \)

\[
\int_B \langle D_F W, DH \rangle dg = - \lim_{\varepsilon \to 0} f^n(\varepsilon) \Phi_1(\varepsilon) \int_{S_0} \frac{\langle H, D\rho \rangle}{\varepsilon} d\theta.
\]

(2.58)

Since \( f^{n-1} \Phi_1, f^{n-1} \Phi_2 \in L^1(B) \), it follows from (2.21) and (2.47) that

\[
(f^n \Phi_1)_\rho = f' f^{n-1} \Phi_1 + (n-1) f' \circ \varphi f^{n-1} \Phi_2 \in L^1(0,1).
\]

Thus, \( \lim_{\rho \to 0^+} f^n \Phi_1 = 0 \). By (2.58) and (2.43), \( u \) is an equilibrium solution. 

\[\square\]
**Remark 2.7** The above theorem is Theorem 4.2 in [2] for the isotropic materials if $M = \mathbb{R}^n$ with the Euclidean metric. In that case $f(\rho) = \rho$ and

$$\left(du\right) = \left(\varphi'(\rho), \frac{\varphi'(\rho)}{\rho}, \ldots, \frac{\varphi'(\rho)}{\rho}\right).$$

Equation (2.47) becomes

$$[\rho^{n-1}\Phi_1(\rho)]_\rho = (n-1)\rho^{n-2}\Phi_2(\rho).$$

Throughout this paper a function $\varphi \in W^{1,1}(0,1)$ is said to be an equilibrium solution to problem (2.47) if $\varphi$ satisfies equation (2.47) and $\varphi'(\rho) > 0$ for $\rho \in (0,1]$, and is such that $f^{n-1}\Phi_1, f^{n-1}\Phi_2 \in L^1(0,1)$. By a similar argument in [2], we have

**Theorem 2.2** Let

$$\Phi_{11}(v_1, v_2, \cdots, v_2) > 0 \quad \text{for} \quad v_1 > 0, \ v_2 > 0.$$ 

If $\varphi \in W^{1,1}(0,1)$ is an equilibrium solution to problem (2.47), then $\varphi \in C^1(0,1)$.

### 2.2 Incompressible Case

A map $u : M \to M$ is said to be incompressible if, for any element of volume $\omega$ of $M_{u(x)}$, $u^*\omega$ is an element of volume of $M_x$ for $x \in M$. Let $(M, g, o)$ be a model and let $u$ be a radical deformation given by (2.8). Let $x = \exp_o \rho v \in \Sigma(o)$ with $v \in M_0$ and $|v| = 1$. Let $\{e_i\}$ be an orthonormal basis of $M_x$ with the positive orientation such that $e_1 = D\rho(x)$. Let $\omega$ be a volume element of $M_{u(x)}$. By Proposition 2.6 (i), we have

$$u^*\omega(e_1, e_2, \cdots, e_n) = \omega(u_*e_1, u_*e_2, \cdots, u_*e_n) = \varphi'(\rho)\tau^{n-1}(\rho),$$

where $\tau$ is given by (2.34). Thus, $u$ is incompressible if and only if

$$\varphi'(\rho)f^{n-1} \circ \varphi(\rho) = f^{n-1}(\rho) \quad \text{for} \quad \rho > 0.$$ 

We define

$$\sigma(t) = \int_0^t f^{n-1}(s)ds \quad \text{for} \quad t \geq 0. \quad (2.59)$$

Then only possible such deformations satisfy

$$\sigma(\varphi(\rho)) = \int_0^\rho f^{n-1}(s)ds + \eta, \quad (2.60)$$

where $\eta$ is a constant. To get $\varphi$ from the above equation, we need some assumptions on the radical curvature.

Let conditions (2.30) hold true. It follows from (2.60) that for an incompressible deformation

$$\varphi(\rho) = \sigma^{-1}\left(\sigma(\rho) + \sigma(A)\right), \quad (2.61)$$

where $A = \varphi(0)$. 

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Lemma 2.3 Let \((M,g,o)\) be a model with \(\mu_+(1) \leq 1\). Then the radical deformations with \(\mu_+(\varphi(1)) \leq 1\), which are incompressible, belong to \(W^{1,p}(B,M)\) for \(1 \leq p < n\).

Proof We need to prove
\[
\int_0^1 \left[ \varphi^2(\rho) + (n-1)\tau^2(\rho) \right]^{p/2} f^{n-1}(\rho) d\rho < \infty \quad \text{for} \quad 1 \leq p < n.
\] (2.62)

If \(\varphi(0) = 0\), then \(\varphi(\rho) = \rho\). The above estimate is trivial. Let us assume that \(\varphi(0) = A > 0\). Then estimate (2.62) follows from (2.17). \(\square\)

Let \(u : M \rightarrow M\) be a deformation. The determinant of \(du\) is given by
\[
det du(x) = det \left( (E_i, u^*_e) \right) \quad \text{for} \quad x \in M,
\]
where \(\{e_i\}\) and \(\{E_i\}\) are orthonormal bases of \(M_x\) and \(Mu(x)\), respectively.

Let \(W\) be a constitutive function satisfying (2.2) and let \(B\) be the unit geodesic ball centered at \(o\). The equilibrium equations for incompressible radial deformations are the Euler-Lagrange equations for the functional
\[
I(u) = \int_B \{W(du) - p(x)(\det du - 1)\} dg,
\]
where the pressure \(p(x)\) is a Lagrange multiplier corresponding to the constraint of incompressibility. Then a deformation \(u : B \rightarrow M\) is said to be an equilibrium solution (\([2]\)) with corresponding measurable pressure \(p(x)\) if \(\det u = 1\) a.e. in \(B\),
\[
\partial W(du)/\partial F_{ij} - p(x)(\text{adj} du)_{ij} \in L^1(B) \quad \text{for} \quad 1 \leq i, j \leq n,
\]
and
\[
\int_B (D_F W - pD_F \det du, DH) dg = 0 \quad \text{for} \quad H \in C^\infty_0(B, \Gamma(\zeta)).
\]

Let
\[
\hat{\Phi}(v) = \Phi(v^{1-n}, v, \cdots, v) \quad \text{for} \quad v > 0.
\]
The following theorem is Theorem 4.3 in \([2]\) if \(M = \mathbb{R}^n\) is the Euclidean space.

Theorem 2.3 Let \((M,g,o)\) be a model with \(\mu_+(1) \leq 1\). The radical deformation (2.61) with \(A > 0\) and \(\mu_+(\varphi(1)) \leq 1\) is an equilibrium solution if and only if
\[
\frac{\tau^{n-1}}{(\tau^n - 1)^2} \hat{\Phi}'(\tau) \in L^1(\delta, \infty) \quad \text{for} \quad \delta > 1.
\] (2.63)

In this case the corresponding pressure is given by
\[
p = \int_\rho \frac{f' \circ \varphi(\rho)}{f \circ \varphi(\rho)} \tau^{2-n}(\rho) \hat{\Phi}'(\tau(\rho)) d\rho + \tau^{1-n}(\rho) \Phi_1(\rho) + c,
\] (2.64)

where \(\Phi_1(\rho) = \Phi_1(\tau^{1-n}, \tau, \cdots, \tau), \tau = \tau(\rho)\) is given by (2.34), and \(c\) is a constant.
Proof Let \( p \) be defined by (2.64). It is easy to check that \( X = \Phi_1 - pr^{n-1} \) satisfies the equation
\[
X_\rho + (n - 1)(\frac{f'f}{f} - \frac{\varphi'}{\varphi} \circ \varphi X - \frac{f'}{f} \circ \varphi \Phi'(\tau)) = 0 \quad \text{for} \quad \rho > 0, \tag{2.65}
\]
where \( p \) is given by (2.64).

Let \( H \) be a section of \( \Gamma(\zeta) \) with a compact support on \( B \) and let \( v(t) \) be a variation of \( u \), given by (2.38). Let \( x = \exp_o \rho v \in \Sigma(o) \) be given, where \( v \in M_\rho \) with \( |v| = 1 \). Let \( \{E_i\} \) be an orthonormal basis of \( M_\rho \) with \( E_1 = v \). We transport \( \{E_i\} \) along the geodesic \( \gamma(t) = \exp_o tv \) paralled to obtain the orthonormal bases \( \{E_i(t)\} \) of \( M_{\gamma(t)} \) for \( t \geq 0 \). Next, we transport parallelly the orthonormal basis \( \{E_i(\varphi(\rho))\} \) of \( M_{\psi(x)} \) along the geodesic \( \psi(t) = \exp_{\psi(x)} tH \) to have the orthonormal bases \( \{E_i(t)\} \) of \( M_{\psi(t)} \) for \( t \geq 0 \) such that the relations (2.48), (2.49) and (2.50) hold.

Denote
\[
P_i(t) = \left( \langle \hat{E}_i(0), v_*(t)E_1 \rangle, \cdots, \langle \hat{E}_i(t), v_*(t)E_n \rangle \right)^T \quad \text{for} \quad 1 \leq i \leq n.
\]
By (2.48), (2.39), (2.50), (2.52), (2.53) and (2.54), we have
\[
\left. \frac{\partial P_i}{\partial t} \right|_{t=0} = \left( \langle \hat{E}_i(0), D_{v_*(t)}vE_1 \rangle, \cdots, \langle \hat{E}_i(t), D_{v_*(t)}vE_n \rangle \right)^T = \left( \langle E_i \circ \varphi(\rho), D_{E_1}H \rangle, \cdots, \langle E_i \circ \varphi(\rho), D_{E_n}H \rangle \right)^T \quad \text{for} \quad 1 \leq i \leq n
\]
and
\[
\left. \frac{\partial \det d\psi}{\partial t} \right|_{t=0} = \sum_{k=1}^n \det \left( P_1(0), \cdots, \hat{P}_k(0), \cdots, P_n(0) \right)
\]
\[
= \tau^{n-1} \langle D\rho, D_D\rho H \rangle(u(x)) + \tau^{n-2} \varphi' \frac{n}{\tau} \sum_{i=2}^n \langle E_i \circ \varphi, D_{E_i(\rho)}H \rangle
\]
\[
= \tau^{n-1} \langle D\rho, D_D\rho H \rangle(u(x)) + (n - 1) \left. \frac{f'}{f} \circ \varphi \langle H, D\rho \rangle \right|_{u(x)} + \text{div} \tilde{H}_0 \bigg|_{u(x)}, \tag{2.66}
\]
where \( \tilde{H}_0 \) is a vector field on \( S(\varphi) \), given by (2.55). From (2.56), (2.66) and (2.65), a similar argument as in the proof of Theorem 2.1 yields
\[
\left. \frac{\partial I(\psi)}{\partial t} \right|_{t=0} = \int_{S(o)} f^{n-1}(\Phi_1 - pr^{n-1}) \langle H, D\rho \rangle d\theta \bigg|_{t=0}^1
\]
\[
- \int_0^1 f^{n-1} \left\{ X_\rho + (n - 1) \left( \frac{f'}{f} - \frac{\varphi'}{\varphi} \circ \varphi X - \frac{f'}{f} \circ \varphi \Phi'(\tau) \right) \right\} \int_{S(o)} \langle H, D\rho \rangle d\rho d\theta
\]
\[
= \int_{S(o)} (\Phi_1(1) - p(1) \tau^{n-1}(1)) \langle H, D\rho \rangle dS
\]
\[
- \lim_{\epsilon \to 0} \int f^{n-1}(\varepsilon) (\Phi_1 - pr^{n-1}) \int_{S(o)} \langle H, D\rho \rangle d\theta, \tag{2.67}
\]
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where $S$ is the unit geodesic sphere centered at $o$ and $p$ is given by (2.64).

Let a radical deformation $u$ be an equilibrium solution where $\varphi$ is given by (2.61) with $\varphi(0) = A > 0$. The hypothesis that $\partial W(du)/\partial F_{ij} - p(x)(\text{adj} \, du)_{ij} \in L^1(B)$ for $1 \leq i, j \leq n$ imply that

$$f^{n-1}(\Phi_1 - p \tau^{-n-1}), \quad f^{n-1}(\Phi_2 - p \tau^{-n-2} \varphi') \in L^1(0, 1),$$

(2.68)

and hence

$$f^{n-1}[\Phi_2 - \tau^{-n} \Phi_1] \in L^1(0, 1).$$

(2.69)

Since $f(0) = 0$ and $f'(0) = 1$, we take $1 > \rho_0 > 0$ such that

$$\frac{f^n(A)}{f^n(\rho)} > \sup_{0 \leq \rho \leq \rho_0} \frac{f' \circ \varphi(\rho)}{f'(\rho)} \quad \text{for} \quad 0 \leq \rho \leq \rho_0.$$

We have

$$(n-1) \int_0^1 f^{n-1}[\Phi_2 - \tau^{-n} \Phi_1] d\rho = \int_0^1 f^{n-1} \Phi'(\tau) d\rho$$

$$= \int_{\tau(\rho_0)}^{\infty} \frac{f^n(\tau^{-n}) - f' \circ \varphi}{f'(\tau) - f' \circ \varphi} \Phi'(\tau) d\tau + \int_{\rho_0}^1 f^{n-1} \Phi'(\tau) d\rho.$$ 

(2.70)

Since $f^n = f^n \circ \varphi \tau^{-n}$, it follows from (2.70) that the relation (2.69) holds if and only if the relation (2.63) is true.

Conversely, let assumption (2.63) hold and $p$ be given by (2.64). We prove that $u$ is an equilibrium solution. Similar arguments as in the proof of Theorem 4.3 in [2] show that the relations (2.68) hold, that is,

$$\partial W(du)/\partial F_{ij} - p(x)(\text{adj} \, du)_{ij} \in L^1(B) \quad \text{for} \quad 1 \leq i, j \leq n.$$

Next, using (2.68) and (2.65), we deduce

$$\lim_{\varepsilon \to 0} f^n(\varepsilon)(\Phi_1 - p \tau^{-n-1}) = 0.$$ 

Thus, by (2.67) and (2.43), we obtain

$$\left. \frac{\partial I(v)}{\partial t} \right|_{t=0} = 0.$$

$\square$

3 Cavitation in the Incompressible Case

Let the radial curvature $\kappa$ satisfy

$$\mu_+ (\delta_0) \leq 1 \quad \text{for some} \quad \delta_0 > 1.$$ 

(3.1)

Then estimates (2.20) and (2.21) hold.

We need the following.
Lemma 3.1 (i) Let $1 \geq \rho_1 > 0$ be given such that
\begin{equation}
\kappa(\rho)f^2(\rho) + nf^2(\rho) > 0 \quad \text{for} \quad 0 \leq \rho \leq \rho_1.
\end{equation}
We fix $0 < \rho_0 \leq \rho_1$ such that $\sigma(\rho_0) < \rho_1$. Then, for all $0 < A \leq b$, we have
\begin{equation}
\tau'(\rho) < 0 \quad \text{for all} \quad 0 < \rho \leq \rho_0,
\end{equation}
where
\begin{equation}
b = \min\{\sigma^{-1}(\rho_1 - \sigma(\rho_0)), \sigma^{-1}((\delta_0) - \sigma(1))\}.
\end{equation}

(ii) For $0 < A \leq b$ and $\tau \in [\tau(\rho_1), \infty)$ given, we solve $f \circ \varphi = \tau f$ and $\sigma(\varphi) = \sigma(\rho) + \sigma(A)$ together to have $\varphi = \varphi(A, \tau)$ and $\rho = \rho(A, \tau)$. Then
\begin{equation}
\varphi_A = \frac{\tau f' f^{n-1}(A)}{f^n - f^{n-1}(\tau f - f' \circ \varphi)}, \quad \rho_A = \frac{f' \circ \varphi f^{n-1}(A)}{f^n - f^{n-1}(\tau f - f' \circ \varphi)},
\end{equation}
for $0 < A \leq b$ and $\tau \in [\tau(\rho_0), \infty)$. Moreover, there are $c_1 \geq c_0 > 0$ such that
\begin{equation}
\frac{c_0 A(\tau - 1)}{(\tau^n - 1)^{1/n}} \leq \varphi - \rho \leq \frac{c_1 A(\tau - 1)}{(\tau^n - 1)^{1/n}},
\end{equation}
for all $(A, \tau) \in (0, b] \times [\tau(\rho_0), \infty)$.

(iii) For $\tau > 1$ given,
\begin{equation}
\lim_{A \to 0^+} \frac{A}{\rho} = (\tau^n - 1)^{1/n}, \quad \lim_{A \to 0^+} \frac{A}{\varphi} = \tau^{-1}(\tau^n - 1)^{1/n}.
\end{equation}

Proof (i) Since
\begin{equation}
\tau' = \frac{f' \circ \varphi - f' \tau^n}{f^n - f^{n-1}} \quad \text{for} \quad 0 < \rho \leq 1,
\end{equation}
$\tau' < 0$ if and only if
\begin{equation}
\frac{f'}{f^n} > \frac{f' \circ \varphi}{f^n \circ \varphi}.
\end{equation}
On the other hand, condition (3.2) implies
\begin{equation}
\left(\frac{f'}{f^n}\right)' = -\kappa f^2 + nf^2 < 0 \quad \text{for} \quad 0 < \rho \leq \rho_1.
\end{equation}
Thus, $f'/f^n$ is strictly decreasing for $\rho \in (0, \rho_1]$. For $0 < \rho \leq \rho_0$ and $0 < A \leq \sigma^{-1}(\rho_1 - \sigma(\rho_0))$, we have
\begin{equation}
\rho < \varphi(\rho) \leq \rho_1,
\end{equation}
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which yields $f'/f^n > f' \circ \varphi/f^n \circ \varphi$.

(ii) We differentiate $f \circ \varphi = \tau f$ and $\sigma(\varphi) = \sigma(\rho) + \sigma(A)$, respectively, with respect to the variable $A$, and obtain (3.4).

It follows from (2.61) that
\[
\int_{\rho}^{\varphi} f^{n-1} ds = \int_{0}^{A} f^{n-1} ds.
\]

Estimates (2.20) and (2.21) yield
\[
c_0(\varphi^n - \rho^n) \leq A^n \leq c_1(\varphi^n - \rho^n),
\]
for some $c_1 \geq c_0 > 0$. Since
\[
\tau^n - 1 = \frac{1}{f^n}[f^n \circ \varphi - f^n] = \frac{n}{f^n} \int_{\rho}^{\varphi} f^{n-1} f' ds,
\]
using (2.20) and (2.21), we have
\[
\frac{c_0}{\rho^n}(\varphi^n - \rho^n) \leq \tau^n - 1 \leq \frac{c_1}{\rho^n}(\varphi^n - \rho^n),
\]
for some $c_1 \geq c_0 > 0$. Thus, (3.5) follow from (3.9) and (3.10). Similar arguments yield (3.6) and (3.7).

(iii) By (3.5), for $\tau > 1$ given,
\[
\frac{1}{c_1}(\tau^n - 1)^{1/n} \leq A/\rho \leq \frac{1}{c_0}(\tau^n - 1)^{1/n}
\]
for $A > 0$ small. Let $a = \lim_{A \to 0^+} A/\rho$. Using (3.5) and (3.4), we have
\[
a = \lim_{A \to 0^+} \frac{f(A)}{f(\rho)} = \lim_{A \to 0^+} \frac{f'(A)}{f'(\rho)\rho A} = \frac{\tau^n - 1}{a^{n-1}},
\]
which yields $a = (\tau^n - 1)^{1/n}$. A similar computation gives the second formula. \qed

**Lemma 3.2** Let $\rho_0 \in (0, 1]$ be given in Lemma 3.1 and let $\delta > 1$ be given. Suppose $b > 0$ is given such that $1 < \tau(\rho_0) \leq \delta$ for all $0 < A \leq b$. Let
\[
p(A, \tau) = \frac{f' \circ \varphi(\tau - 1)}{f' \tau^n - f' \circ \varphi} \quad \text{for} \quad (A, \tau) \in (0, b] \times [\tau(\rho_0), \delta]. \quad (3.11)
\]

Then
\[
\min\{a_0, \frac{1}{n\delta}\} \leq p(A, \tau) \leq \max\{a_1, \frac{1}{n}\} \quad \text{for} \quad (A, \tau) \in (0, b] \times [\tau(\rho_0), \delta], \quad (3.12)
\]

where
\[
a_0 = \inf_{0 \leq \rho \leq \rho_0} \frac{f'^2}{\kappa f^2 + nf'^2}, \quad a_1 = \sup_{0 \leq \rho \leq \rho_0} \frac{f'^2}{\kappa f^2 + nf'^2}.
\]
Proof Let 
\[ a = \inf_{0 < A \leq b, \tau(\rho_0) \leq \delta} p(A, \tau). \]
Suppose \( A_k \to 0+ \) and \( \tau_k \in [\tau(\rho_0), \delta] \) such that
\[ p(A_k, \tau_k) \to a \quad \text{as} \quad k \to \infty. \]
We may suppose \( \tau_k \to \tau_0 \) where \( \tau_0 \in [1, \delta] \). We may also assume \( \rho(A_k, \tau_k) \to \hat{\rho}_0 \) where \( \hat{\rho}_0 \in [0, \rho_0] \).

Since \( \tau - 1 = \int_0^\tau f'(s)ds/f \) and \( f'/f' \circ \varphi - 1 = \int_0^\tau \kappa ds/f' \circ \varphi \), we have
\[ p(A, \tau) = \frac{1}{f \int_0^\tau \kappa ds/f' \circ \varphi} \]
(3.13)

Thus, we obtain
\[ a = \frac{f'^2(\hat{\rho}_0)}{\kappa(\hat{\rho}_0) f'^2(\hat{\rho}_0) \tau_0^n + (\tau_0^{n-1} + \cdots + 1)} \]
If \( \tau_0 > 1 \), by Lemma 3.1, \( \hat{\rho}_0 = 0 \) and \( a \geq 1/(n\delta) \). If \( \tau_0 = 1 \), then \( a \geq a_0 \). Similar arguments yield the right hand side of (3.12). \( \square \)

Let the function \( \Phi(v_1, \cdots, v_n) \) be given by (2.44). It is said that the Baker-Ericksen inequalities hold if
\[ \frac{v_i \Phi_i - v_j \Phi_j}{v_i - v_j} \geq 0 \quad \text{for} \quad i \neq j, \quad v_i \neq v_j. \]
We further assume that \( W \) is bounded below. By Theorem 2.3, the radical deformation \( \varphi \), given by (2.61), with \( \varphi(0) = A > 0 \), is an equilibrium solution if and only if
\[ \frac{v^n}{(v^n - 1)^2} \hat{\Phi}'(v) \in L^1(\delta, \infty) \quad \text{for} \quad \delta > 1. \]
(3.14)
Let \( p \) be given by (2.64). Then the radical component of the Cauchy stress tensor is given by
\[ T = \tau^{1-n} \Phi_1 - p \quad \text{for} \quad \rho > 0. \]
Let \( \rho_0 \in (0, 1] \) be given in Lemma 3.1. It follows from (2.64) and Lemma 3.1 (ii) that
\[ T(0) = \lim_{\rho \to 0^+} T \text{ exists if and only if the integral} \]
\[ \int_0^\infty \frac{f' \circ \varphi}{f' \tau^1 - f' \circ \varphi} \hat{\Phi}'(\tau)d\tau \]
(3.15)
converges.

The total stored energy of the deformation is given by
\[ E(A) = \omega_n \int_0^1 f'^{n-1}(\rho) \Phi(\rho) d\rho, \]
(3.16)
where \( \omega_n \) is the area of the unit sphere \( S_0 \) in \( M_0 \) and \( \Phi(\rho) = \Phi(\varphi'(\rho), \tau(\rho), \cdots, \tau(\rho)) \). We define \( E(0) = \omega_n \sigma(1) \Phi(1) \).
Proposition 3.1 Let \((M,g,o)\) be a model with \(\mu_+(1) \leq 1\) and let (3.14) hold. Let (2.61) be an equilibrium solution with \(A > 0\) and \(\mu_+(\varphi(1)) \leq 1\). Then

(i) Then \(T(0)\) exists and is finite if and only if \(E(A) < \infty\).

(ii) Let \(\Phi\) satisfy the Baker-Ericksen inequalities. Then \(T\) is an increasing function in \(\rho > 0\).

Proof Using equation (2.65), we have

\[
T'(\rho) = \frac{f' \circ \varphi}{f} \tau^{1-n} \hat{\varphi}'(\tau) = (n-1) \frac{f' \circ \varphi}{f \tau^{2n-1}} (\tau^n - 1) \frac{\tau \Phi_2 - \tau^{1-n} \Phi_1}{\tau - \tau^{1-n}} \geq 0,
\]

that is, (ii) is true. Using the formula above, we obtain

\[
\left[f^n \Phi\right]' = nf' f^{n-1} \Phi + f^n \hat{\varphi}'(\tau) = nf' f^{n-1} \Phi + (f^n - \frac{f' f^n \circ \varphi}{f' \circ \varphi}) T'.
\]

Thus,

\[
n \int_0^1 f' f^{n-1} \Phi d\rho = f^n(1) \Phi(1) - f^n(\rho) \Phi(\rho) + \int_0^1 \left( \frac{f' f^n \circ \varphi}{f' \circ \varphi} - f^n \right) T' d\rho
\]

\[
= f^n(1) \Phi(1) - f^n(\rho) \Phi(\rho) + \left( \frac{f' f^n \circ \varphi}{f' \circ \varphi} - f^n \right) T\bigg|_{\rho=1}
\]

\[
-(\frac{f' f^n \circ \varphi}{f' \circ \varphi} - f^n) T + \int_0^1 \left( \frac{f' f^n \circ \varphi}{f' \circ \varphi} - f^n \right) T d\rho,
\]

for \(0 \leq \rho \leq 1\). Using (2.15), we have

\[
\lim_{\rho \to 0^+} \frac{1}{f(\rho)} \left( \frac{f' f^n \circ \varphi}{f' \circ \varphi} - f^n \right) = -\kappa(a) f^n(A).
\]

(3.18)

In addition, it follows from (2.64) that

\[
|T| \leq \int_0^1 f' \circ \varphi \tau^{1-n} |\hat{\varphi}'(\tau)| d\rho = \int_0^1 f' \circ \varphi \tau^{1-n} |\hat{\varphi}'(\tau)| d\rho
\]

\[
+ \int_{\tau(\rho_0)}^\infty \frac{f \circ \varphi f' \circ \varphi}{\tau(f' \tau^n - f' \circ \varphi)} |\hat{\varphi}'(\tau)| d\tau.
\]

Thus, by (3.14), \(|T|\) is bounded above. Therefore the last integral in the right hand side of (3.17) converges by (3.18). Moreover, \(-f^n \Phi\) is also bounded above since \(\Phi\) is bounded below. Thus, the left hand side of (3.17) exists and is finite if and only if \(T(0)\) exists and is finite. Then (i) follows by (2.21).

We consider the case when there is a force acting on the unit geodesic sphere \(S\). A function on \(M\) is said to be a force density. We say that \(u \in W^{1,1}(B,M)\) is an equilibrium solution to the boundary value problem with corresponding pressure \(p\) if \(\det du = 1\) a.e. in \(B\),

\[
\partial W(du)/\partial F_{ij} - p(x)(\adj du)_{ij} \in L^1(B)\quad \text{for} \quad 1 \leq i, j \leq n.
\]
\[ \int_B (D_TW - pD_T \det du, DH)dg - \int_S (Dq, H)dS = 0 \quad \text{for} \quad H \in C^\infty(\overline{B}, \Gamma(\zeta)), \]

where \( q \) is a given force density which is a differentiable function on \( M \). They are the Euler-Lagrange equations for the functional

\[ I_1(u) = \int_B \{ W(du) - p(\det du - 1) \}dg - \int_S q(u)dS. \]

A force density \( q \) is said to be *radical with respect to* \( o \) if there is \( \hat{q} \in C^1(\mathbb{R}) \) such that

\[ q(x) = \hat{q}(\rho(x)) \quad \text{for} \quad x \in M. \]

In particular, we take

\[ q(x) = P\rho(x) \quad \text{for} \quad x \in M, \]

where \( P \in \mathbb{R} \) is a constant.

It is easy to check that the identity deformation \( \varphi(\rho) = \rho \) is an equilibrium solution to the boundary value problem with the corresponding pressure

\[ p = \Phi_1(1) - P. \]

Let all the assumptions in Theorem 2.3 hold. Let \( \varphi \) be given by (2.61) with \( A > 0 \). By similar arguments as in the proof of Theorem 2.3 that \( \varphi \) is an equilibrium solution to the above boundary value problem if and only if it is an equilibrium solution with the corresponding pressure

\[ p = \tau^{1-n}\Phi_1 - T, \]

where

\[ T = \frac{P}{\tau^{n-1}(1)} - \int_0^1 \frac{f' \circ \varphi}{f \circ \varphi} \tau^{-n-2}\hat{\Phi}'(\tau)d\rho \quad (3.19) \]

for \( \rho > 0 \) and \( \lambda > 0 \). As a natural boundary condition given in [2] to get a unique solution for the corresponding \( \lambda \in \mathbb{R} \), we assume that

\[ T(0) = 0. \quad (3.20) \]

The total energy of the deformation (2.61) with \( \varphi(0) = A \geq 0 \) is given by

\[ I(A) = \int_B W(du)dg - P \int_S \rho(u)dS = E(A) - \omega_n f^{n-1}(1)P\varphi(1), \quad (3.21) \]

where \( E(A) \) is given by (3.16).

By (3.19) the possible values of \( A \) such that (3.20) is satisfied are the roots of the equation

\[ P = \chi(A) \quad (3.22) \]
where
\[ \chi(A) = \tau^{n-1}(1) \int_0^1 \frac{f' \circ \varphi}{f' \circ \varphi} \tau^{2-n} \hat{\Phi}'(\tau) d\rho. \] (3.23)

Thus, the bifurcation from the trivial solution is governed by the behavior of \( \chi(A) \) as \( A \to 0^+ \).

Let
\[ \hat{I}(A) = \int_0^1 f^{n-1}(\rho) \Phi(\rho) d\rho. \]

Let \( \rho_0 \in (0,1] \) be given by Lemma 3.1. Since
\[ \hat{I}(A) = \int_1^\rho_0 f^{n-1}(\rho) \Phi(\rho) d\rho + \int_0^\rho_0 \sigma(\rho) \hat{\Phi}'(\tau) d\rho \]
using the formula in (3.4) we have
\[ \hat{I}'(A) = \int_1^\rho_0 f^{n-1}(\rho) \Phi(\tau(\rho)) d\rho + \int_0^\rho_0 \sigma(\rho) \hat{\Phi}'(\tau) d\rho + \int_0^\rho_0 \sigma(\rho) \hat{\Phi}'(\tau) d\rho, \] (3.24)

which yields
\[ I'(A) = \omega_n f^{n-1}(A) \tau^{1-n}(1)[\chi(A) - P] \quad \text{for} \quad A \geq 0. \] (3.25)

We suppose
\[ \frac{\hat{\Phi}'(\tau)}{\tau^{n-1}} \in L^1(\delta, \infty) \quad \text{for} \quad \delta > 1. \] (3.26)

Let \( \hat{\Phi}(v) \) be twice differentiable at \( v = 1 \). Thus, (3.26) and \( \hat{\Phi}'(1) = 0 \) imply
\[ \frac{\hat{\Phi}'(\tau)}{\tau^{n-1}} \in L^1(1, \infty). \]

Using (5), we have
\[ \left| \int_{\tau(\rho_0)}^\infty \sigma(\rho(A,\tau)) \hat{\Phi}'(\tau) d\tau \right| \leq cA^n \int_1^{\infty} \frac{1}{\tau^{n-1}} |\hat{\Phi}'(\tau)| d\tau. \]

By (3.24) and (3.21), we obtain
\[ \lim_{A \to 0^+} I(A) = \omega_n [\sigma(1) \Phi(1) - f^{n-1}(1)P] = I(0). \] (3.27)

**Lemma 3.3** Let \( \chi \) be given by (3.23) and let \( \hat{\Phi}(v) \) be twice differentiable at \( v = 1 \). Then
\[ \lim_{A \to 0^+} \chi(A) = \int_1^\infty \frac{1}{\tau^{n-1}} |\hat{\Phi}'(\tau)| d\tau. \] (3.28)
Proof Let \( \rho_0 \in (0, 1] \) be given in Lemma 3.1. We have
\[
\int_0^1 \frac{f' \circ \varphi}{f \circ \varphi} \tau^{2-n} \dot{\Phi}'(\tau)d\rho = \int_0^1 \frac{f' \circ \varphi}{f \circ \varphi} \tau^{2-n} \dot{\Phi}'(\tau)d\rho + \int_{\tau(\rho_0)}^\infty \frac{f' \circ \varphi}{f' \tau^n - f' \circ \varphi} \dot{\Phi}'(\tau)d\tau, \tag{3.29}
\]
for \( 0 < A \leq \sigma^{-1}(\rho_1 - \sigma(\rho_0)) \). Since \( \Phi'(1) = 0 \), the first integral in the right hand side of (3.29) goes to zero as \( A \to 0 + \).

Since
\[
\dot{\Phi}'(\tau)/(\tau - 1) \to \Phi''(1) \quad \text{as} \quad \tau \to 0 +,
\]
we fixed \( \delta_0 \geq \delta > 1 \) such that \( |\dot{\Phi}'(\tau)/(\tau - 1)| \leq |\Phi''(1)| + 1 \) for all \( 1 \leq \tau \leq \delta \). By Lemma 3.2,
\[
\int_0^\delta \frac{f' \circ \varphi}{f' \tau^n - f' \circ \varphi} \dot{\Phi}'(\tau)d\tau \leq c[|\Phi''(1)| + 1],
\]
for \( A > 0 \) small. Thus, (3.28) follows from the dominated convergence theorem. \( \square \)

Let
\[
P_{cr} = \int_1^\infty \frac{1}{\tau^n - 1} \dot{\Phi}'(\tau)d\tau.
\]
The physical meaning of \( P_{cr} \) is given in [3], [2]: (3.25), (3.27) and Lemma 3.3 show that the trivial solution \( A = 0 \) is a local minimum (resp. local maximum) if \( P < P_{cr} \) (resp. \( P > P_{cr} \)). The following proposition illustrates the close relations among the radial curvatures, the constitutive function \( \dot{\Phi}(v) \), and the behavior of \( \chi(A) \) as \( A \to 0 + \).

Proposition 3.2 Let \( \dot{\Phi}(v) \) be twice differentiable at \( v = 1 \). Then \( \chi'(0) = 0 \). Furthermore, the following holds.

(i) Let \( \kappa \) satisfy the Baker-Ericksen inequalities with \( \dot{\Phi}''(1) > 0 \). Suppose there is \( \varepsilon \in (0, 1] \) such that the radial curvature \( \kappa \) is a constant \( \kappa_0 \) for all \( \rho \in (0, \varepsilon] \). If \( \kappa_0 = 0 \), then
\[
P_{cr} - \frac{1}{2}[1 + \frac{f^n(1)}{f'(1)} \int_0^1 \frac{\kappa}{f} \dot{\Phi}''(1) > 0 \quad (\text{resp.} \quad < 0) \tag{3.30}
\]
implies \( \chi'(A) > 0 \) (resp. \( < 0 \)) for \( A \) small. If \( \kappa_0 \neq 0 \), then
\[
\kappa_0 > 0 \quad (\text{resp.} \quad < 0) \tag{3.31}
\]
implies \( \chi'(A) < 0 \) (resp. \( > 0 \)) for \( A > 0 \) small.

(ii) Let \( \kappa \) be differentiable. Let \( \kappa(0) \neq 0 \). Then
\[
\kappa(0) \int_1^\infty \frac{(\tau^2 - 1)\tau^n}{(\tau^n - 1)^2(1+n/\rho)} \dot{\Phi}'(\tau)d\tau < 0 \quad (\text{resp.} \quad > 0) \tag{3.32}
\]
implies \( \chi'(A) > 0 \) (resp. \( < 0 \)) for \( A > 0 \) small. In addition, if there is some \( \varepsilon \in (0, 1] \) such that \( \kappa = 0 \) for \( \rho \in (0, \varepsilon) \), then
\[
P_{cr} - \frac{1}{n(n-1)}[1 + \frac{f^n(1)}{f'(1)} \int_0^1 \frac{\kappa}{f^n-1} \dot{\Phi}''(1) > 0 \quad (\text{resp.} \quad < 0) \tag{3.33}
\]
implies \( \chi'(A) > 0 \) (resp. \( < 0 \)) for \( A > 0 \) small.
Proof Let \( \rho_0 \in (0,1] \) be given in Lemma 3.1. Using (3.29) and (3.4), we have
\[
\chi'(A) = (n-1)\frac{\tau A(1)}{\tau(1)} \chi(A) + \tau^{n-1}(1) I_1 - f' \circ \varphi \tau^{n-2} \hat{\Phi}'(\tau) d\rho \\
+ \int_{\rho_0}^{1} I_1 \hat{\Phi}''(\tau) \tau A d\rho + \tau^{n-1}(1) \left| -p(A, \tau) \frac{\hat{\Phi}'(\tau)}{\tau - 1} \right|_{\tau = \tau(\rho_0)}^{\tau A(\rho_0)} \\
+ f^{n-1}(A) \int_{\tau(\rho_0)}^{\infty} I_2 \hat{\Phi}'(\tau) d\tau, \tag{3.34}
\]
where
\[
I_1 = \frac{f' \circ \varphi \tau^{n-2}}{f \circ \varphi}, \quad I_2 = \frac{(\kappa f^2 \circ \varphi - \kappa \circ \varphi f''(\tau^2)) \tau^n}{f^{n-2}(f' \tau^n - f' \circ \varphi)^3},
\]
and \( p(A, \tau) \) is given by (3.11). Clearly, all the terms in the right hand side of (3.34) go to zero if the last term converges to zero as \( A \to 0 + \).

Let \( \delta > 1 \) be given. It follows from (3.5) that
\[
A \leq \frac{\rho_0}{c_0} (\tau^n - 1)^{1/n} \leq c(\tau - 1)^{1/n} \quad \text{for} \quad (A, \tau) \in (0, b) \times [\tau(\rho_0), \delta]. \tag{3.35}
\]
Using (3.5), (3.7), (3.12), and (3.35), we have
\[
|I_2| = \left| \frac{-f'^2 \circ \varphi \int_0^\rho \kappa'(s) ds + \kappa \circ \varphi [f'^2 \circ \varphi (1 - \tau^2) + 2 \tau^2 \int_0^\rho \kappa f' ds]}{f^{n-2} f'^2 \circ \varphi (\tau - 1)^3} \right| p^3 \tau^n \\
\leq c \frac{A^{(\tau - 1)^{-1/n} + \tau - 1}}{A^{n-2(\tau - 1)^{1+2/n}}} \leq c \frac{A^{1-\alpha}}{(\tau - 1)^{(2-\alpha)/n}} \frac{\hat{\Phi}'(\tau)}{\tau - 1} \quad \text{for} \quad (A, \tau) \in (0, b) \times [\tau(\rho_0), \delta]. \tag{3.36}
\]
Thus, by (3.35) again,
\[
f^{n-1}(A)|I_2| \leq c \frac{A^{1-\alpha}}{(\tau - 1)^{(2-\alpha)/n}} \frac{\hat{\Phi}'(\tau)}{\tau - 1} \quad \text{for} \quad (A, \tau) \in (0, b) \times [\tau(\rho_0), \delta]
\]
and for \( 0 \in [0,1] \). It follows from (3.34) and (3.36) that \( \chi'(0) = \lim_{A \to 0^+} \chi'(A) = 0 \).

(i) We may assume that \( 0 < \rho_0 < \varepsilon \) is small enough such that \( \varphi \leq \varepsilon \) when \( \rho \in (0, \rho_0] \).

Let \( \kappa_0 = 0 \). Then \( f'(\rho) = 1, f(\rho) = \rho_0, \) and \( I_2 = 0 \). In this case we have, by (3.34) and (3.13),
\[
\lim_{A \to 0^+} \frac{\chi'(A)}{f(A)} = \frac{f'(1)}{f^2(1)} P_{cr} + \lim_{A \to 0^+} \int_{\rho_0}^{1} \frac{f' \circ \varphi \int_0^\rho \hat{\Phi}''(\tau) d\rho}{\int_0^\rho \int_0^\rho \hat{\Phi}''(\tau) d\rho} \frac{p(A, \tau) f' \circ \varphi \hat{\Phi}'(\tau)}{f \circ \varphi (\tau - 1)_{\tau = \tau(\rho_0)}} \\
= \frac{f'(1)}{f^2(1)} P_{cr} + \left( \int_{\rho_0}^{1} \int_0^\rho \int_0^\rho \hat{\Phi}''(\tau) d\rho - \frac{1}{2\rho_0^2} \right) \hat{\Phi}''(1) \\
= \frac{f'(1)}{f^2(1)} P_{cr} - \frac{1}{2} \left[ \frac{f'(1)}{f^2(1)} \right] + \int_{\varepsilon}^{1} \frac{1}{f} d\rho \hat{\Phi}''(1).
\]
Thus, the case \( \kappa_0 = 0 \) follows.

Let \( \kappa_0 \neq 0 \). Let \( q = f'/f \circ \varphi \). By (3.2), \( (f'/f)' = -(\kappa f^2 + f'^2)/f^2 < 0 \) for \( \rho \in (0, \rho_0] \).

Thus,
\[
q^2 \tau^2 > 1 \quad \text{for} \quad (A, \tau) \in (0, b) \times [\tau(\rho_0), \infty).
\]
Let $\delta > 1$ be fixed such that

$$\frac{1}{\tau - 1} \Phi'(\tau) \geq \frac{1}{2} \Phi''(1) \quad \text{for} \quad \tau \in [1, \delta].$$

We have

$$\left| \frac{(q^2 - 1)\tau^2}{\tau - 1} + \tau + 1 \right| \geq 2 - \frac{2\kappa_0 \int f_\rho f f'ds}{f^2 \circ \varphi \int f' ds} \tau^2 \geq 2 - c|\kappa_0|f^2(\rho_0),$$

for $(A, \tau) \in (0, b] \times [\tau(\rho_0), \delta]$. We assume that $\rho_0 \in (0, \varepsilon]$ is also such that $2 - c|\kappa_0|f^2(\rho_0) > 0$.

Thus, by (3.12), we obtain

$$-\frac{1}{\kappa_0} f^2 \Phi' = \frac{(q^2 \tau^2 - 1)\tau^2 p(A, \tau)}{f' \circ \varphi (\tau - 1)^2} \Phi' - \frac{c_0}{\tau - 1} \Phi''(1) \quad \text{for} \quad (A, \tau) \in (0, b] \times [\tau(\rho_0), \delta],$$

which yields, by (3.34),

$$\lim_{A \to 0^+} \frac{\chi'(A)}{f(A)} = \begin{cases} -\infty & \text{if} \quad \kappa_0 > 0; \\ +\infty & \text{if} \quad \kappa_0 < 0. \end{cases}$$

Thus, the case $\kappa_0 \neq 0$ follows.

(ii) Since $n \geq 3$, (3.36) with $\alpha = 0$ and (3.26) imply the integral

$$\int_{\tau(\rho_0)} f^{n-2}(A) I_2 \Phi'(\tau) d\tau$$

converges as $A \to 0^+$. Using (3.34), (3.5), (3.6), and (3.8), we obtain

$$\lim_{A \to 0^+} \frac{\chi'(A)}{f(A)} = \kappa(o) \int_1^\infty \frac{(1 - \tau^2)\tau^n}{(\tau^n - 1)^{2(1+1/n)}} d\tau,$$

which gives (3.32). Finally, a similar computation as in (i) for the case $\kappa_0 = 0$ yields (3.33).

**Remark 3.1** If $\Phi$ satisfies the Baker-Ericksen inequalities, then (3.32) is equivalent to

$$\kappa(o) < 0 \quad (\text{resp.} \quad > 0).$$

Let $P > 0$. If $A_0$ is a root of (3.22), that is,

$$P = \chi(A_0),$$

then from (3.25)

$$I''(A_0) = \omega_n f^{n-1}(A_0) \tau^{1-n}(1) \chi'(A_0).$$

Let $\Phi$ satisfy the Baker-Ericksen inequalities and let $n \geq 3$. From (3.32), $A_0$ is a local minimum (resp. local maximum) of $I$ if $\kappa(o) < 0$ (resp. $\kappa(o) > 0$) (for $A_0$ small).
4 Cavitation in the Compressible Case

By Theorem 2.1, an equilibrium solution $\varphi$ satisfies the equation

$$[f^{n-1}(\rho)\Phi_1(\rho)]_\rho = (n-1)f^{n-2}(\rho)f' \circ \varphi(\rho)\Phi_2(\rho) \quad \text{for} \quad x \in \Omega, \quad \rho(x) > 0, \tag{4.1}$$

where

$$\Phi_1(\rho) = \Phi_1(\varphi', \tau, \cdots, \tau), \quad \Phi_2(\rho) = \Phi_2(\varphi', \tau, \cdots, \tau), \quad \tau(\rho) = \frac{f \circ \varphi(\rho)}{f(\rho)}.$$ 

Let $\varphi$ be a solution to problem (4.1) with $\varphi' > 0$ and $\varphi > 0$ on $(0, 1]$. We define

$$T(\rho) = \tau^{n-1}(\rho)\Phi_1(\rho) \quad \text{for} \quad \rho \in (0, 1], \tag{4.2}$$

which is the radial component of the Cauchy stress ([2]). By (4.1),

$$T'(\rho) = (n-1)f' \circ \varphi \tau^{1-n}(\tau\Phi_2 - \varphi'\Phi_1) \quad \text{for} \quad \rho \in (0, 1]. \tag{4.3}$$

It follows from (4.3) that

**Proposition 4.1** Let $\varphi$ be a solution of (4.1) with $\varphi'(\rho) > 0$ for all $\rho \in (0, 1]$. If the Baker-Ericksen inequalities hold, then

$$T'(\rho)[\varphi'(\rho) - \tau(\rho)] \leq 0 \quad \text{for} \quad \rho \in (0, 1]. \tag{4.4}$$

Let

$$\tilde{T}(\rho) = \Phi(\rho) - \varphi'(\rho)\Phi_1(\rho), \tag{4.5}$$

where $\Phi(\rho) = \Phi(\varphi', \tau, \cdots, \tau)$. $\tilde{T}$ is said to be the radial component of the inverse Cauchy stress ([2]). We obtain by (4.1) and (4.3)

$$\tilde{T}'(\rho) = -\frac{f' \circ \varphi}{f' \circ \varphi} \tau^n T' \quad \text{for} \quad \rho \in (0, 1]. \tag{4.6}$$

It follows (4.6), (4.3) and (4.1) that

$$\{f^n[\Phi - (\varphi' - \tau)\Phi_1]\}' = [f^n \tilde{T} + f \circ \varphi f^{n-1}\Phi_1]' = nf'f^{n-1}\tilde{T} + f^n \tilde{T}'$$

$$+ f' \circ \varphi f^{n-1}\Phi_1 + f \circ \varphi (f^{n-1}\Phi_1)'$$

$$= nf'f^{n-1}\Phi + (f' \circ \varphi - f')f^{n-1}[\varphi'\Phi_1 + (n-1)\tau\Phi_2]. \tag{4.7}$$

If $\kappa = 0$ for $\rho \in (0, 1]$, then $f' = 1$, $f = \rho$, and (4.7) becomes

$$\{\rho^n[\Phi - (\varphi' - \tau)\Phi_1]\}' = n\rho^{n-1}\Phi, \quad \tau = \frac{\varphi'}{\rho},$$

which is the radial version of the conservation law ([2]).
4.1 Constitutive Assumptions

Throughout this paper unless otherwise stated we assume the class of constitutive functions $W(F) = \Phi(v_1, \cdots, v_n)$ have the form $(n \geq 2)$

$$\Phi(v_1, \cdots, v_n) = \sum_{i=1}^{n} \phi(v_i) + h(v_1 \cdots v_n), \quad (4.8)$$

where functions $\phi$ and $h$ satisfy the following assumptions:

(A 1) $h : (0, \infty) \to IR$ is $C^2$ and strictly convex;

(A 2) $\lim_{v \to 0^+} h(v) = \lim_{v \to \infty} \frac{h(v)}{v} = +\infty$;

(A 3) $h$ satisfies

$$\lim_{v \to \infty} \frac{v h'(v)}{h(v)} > 1,$$

and let

$$\theta(s) = \lim_{v \to \infty} \frac{h(sv)}{h(v)} \quad \text{for} \quad s \in (0, \infty),$$

and we assume that $\theta : (0, \infty) \to (0, \infty)$ is continuous;

(A 4) $\phi : (0, \infty) \to (0, \infty)$ is $C^2$ and convex;

(A 5) $v\phi'(v)$ is increasing on $(0, \infty)$;

(A 6) Let $t_0 \geq 0$ be such that $\phi'(t_0) = 0$. Let

$$q_1(s) = \sup_{v>t_0} \frac{\phi'(v)}{\phi'(sv)} \quad \text{for} \quad s > 1; \quad q_0(s) = \inf_{v>t_0/s} \frac{\phi'(v)}{\phi'(sv)} \quad \text{for} \quad s \in (0, 1].$$

We assume that $q_1 \in C^1[1, \infty)$ and $q_0 \in C^1(0, 1]$ satisfy

$$\lim_{s \to \infty} q_1(s) = 0, \quad \lim_{s \to 0^+} q_0(s) = \infty, \quad (4.9)$$

$$q_1'(s) < 0 \quad \text{for} \quad s \in [1, \infty) \quad \text{and} \quad q_0'(s) < 0 \quad \text{for} \quad s \in (0, 1], \quad (4.10)$$

respectively.

(A 7) there are $\delta_0 > 0$ and $\delta_1 > 0$ such that if $|s - 1| < \delta_0$ then

$$|\phi'(sv)| \leq \delta_1 \frac{\phi'(v)}{v} \quad \text{for all} \quad v > 0;$$

(A 8) $\phi(v) \leq \delta_2(1 + v^\alpha + v^{-\beta})$ for all $v > 0$, where $\delta_2 > 0, 0 < \alpha < n$, and $0 \leq \beta < 1 + 1/(n - 1)$;

(A 9) $\lim_{v \to \infty} \phi(v) = +\infty$.

**Remark 4.1** (A 1), (A 2), (A 4), (A 5), (A 7), (A 8), and (A 9) are given in [2]. In addition, it is easy to check that $q_1$ and $q_0$ are decreasing on $[1, \infty)$ and $(0, 1]$, respectively, and $q_1(1) = q_0(1) = 1$. 

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Example 4.1 Let
\[ \phi(v) = \mu(v^\alpha - n) + \frac{\nu}{v^\beta}, \]
where \( \mu > 0, \nu \geq 0, 1 < \alpha < n, \) and \( 0 \leq \beta < 1 + 1/(n-1). \) Let
\[ h(v) = H(v) - n, \]
where \( H : (0, \infty) \to \mathbb{R}^+ \) is a \( C^3 \) function and satisfies \( \lim_{\delta \to 0+} H(\delta) = \infty, \) \( H''(\delta) > 0 \) for all \( \delta > 0, \) and
\[ H(\delta) = k(\delta - 1 - k^{-1})^2 \quad \text{for} \quad \delta \geq 1/2, \]
where \( k > 1. \)

Clearly, \((A4), (A5), (A7), (A8),\) and \((A9)\) are true for \( \phi. \) We check \((A6). \) Since
\[ \frac{\phi'(v)}{\phi'(sv)} = \frac{h_1(v)}{h_2(v)}, \]
where
\[ h_1(v) = \mu s^\alpha - \frac{\beta \nu}{v^{\alpha+\beta}}, \quad h_2(v) = \mu s^\alpha - 1 - \frac{\beta \nu}{s^{\beta+1} v^{\alpha+\beta}}, \]

\[ h_1'(v) h_2(v) - h_2'(v) h_1(v) = \frac{\beta(\alpha + \beta) \nu}{v^{\alpha+\beta+1}} (s^{\alpha-1} - \frac{1}{s^{\alpha+\beta}}), \]
we have
\[ q_1(s) = \lim_{v \to \infty} \frac{h_1(v)}{h_2(sv)} = \frac{1}{s^{\alpha-1}} \quad \text{for} \quad s \in [1, \infty); \quad q_0(s) = \frac{1}{s^{\alpha-1}} \quad \text{for} \quad s \in (0, 1]. \]

Thus, \((A6) \) holds.

\((A1)\) and \((A2)\) automatically hold for \( h. \) \((A3)\) is also true since
\[ \lim_{v \to \infty} \frac{v h'(v)}{h(v)} = 2, \quad \theta(s) = s^2 \quad \text{for} \quad s \in (0, \infty). \]

4.2 Equilibrium Solutions

Let \( \lambda > 0 \) be given. It follows from (4.1) and (4.8) that equilibrium solutions are given by problem
\[
\begin{cases}
\varphi(1) = \lambda, \\
f[\varphi''(\varphi') + h''(\varphi'^n - 1)\varphi'^2] = (n-1)[f' \circ \varphi' \varphi' + f' \varphi''] \quad (4.11)
\end{cases}
\]

\( -(n-1)(f' \circ \varphi' \varphi' + f' \varphi') h''(\varphi'^n - 1) \varphi'^{2n-3} \quad \text{for} \quad \rho \in (0, 1). \]

Regular Equilibrium Solutions An equilibrium solution \( \varphi \in C^1(0, 1) \) to problem (4.11) is said to be regular if \( \varphi(0) = \lim_{\rho \to 0^+} \varphi(\rho) = 0. \) In the case of the Euclidean space there is a unique regular equilibrium solution \( \varphi = \lambda \rho \) which plays an important role in the analysis ([2, 10]). We derive some properties of regular equilibrium solutions.
We assume that \( \lambda > 0 \) is given such that the radial curvature satisfies
\[
\mu_+ (\max\{\lambda, 1\}) = \int_0^{\max\{\lambda, 1\}} s\kappa_+(s)ds \leq 1.
\] (4.12)

By Proposition 2.4
\[
f'(\rho) > 0 \quad \text{for} \quad \rho \in [0, \max\{\lambda, 1\}].
\]

Let
\[
b_0(\rho) = \min_{0 \leq s \leq \rho} f'(s), \quad b_1(\rho) = \max_{0 \leq s \leq \rho} f'(s) \quad \text{for} \quad \rho \in [0, \max\{\lambda, 1\}].
\]

Set
\[
\alpha_1(\rho, s) = \max \left\{ b_1(\rho)/b_0(s), \frac{q_1^{-1}(b_0(\rho)/b_1(s))}{q_0^{-1}(b_1(\rho)/b_0(s))} \right\},
\]
\[
\alpha_0(\rho, s) = \min \left\{ b_0(\rho)/b_1(s), \frac{q_0^{-1}(b_1(\rho)/b_0(s))}{q_1^{-1}(b_0(\rho)/b_1(s))} \right\}
\]
for \((\rho, s) \in [0, 1] \times [0, \lambda]\), where \( q_0 \) and \( q_1 \) are given by (A 6).

Then the following lemma is immediate.

**Lemma 4.1** Let \( \mu_+ (\max\{\lambda, 1\}) \leq 1 \) and let \((\rho, s) \in [0, 1] \times [0, \lambda]\) be given. \(\alpha_1(\cdot, s)\) and \(\alpha_1(\rho, \cdot)\) are increasing. \(\alpha_0(\cdot, s)\) and \(\alpha_0(\rho, \cdot)\) are decreasing.

We have

**Theorem 4.1** Let \( \mu_+ (\max\{\lambda, 1\}) \leq 1 \). Let (A1), (A4), (A5), and (A6) hold. If \( \varphi \in C^1(0, 1) \) is a regular equilibrium solution to problem (4.11), then
\[
\alpha_0(\rho, \varphi(\rho))\tau(\rho) \leq \varphi'(\rho) \leq \alpha_1(\rho, \varphi(\rho))\tau(\rho) \quad \text{for} \quad \rho \in (0, 1).
\] (4.13)

**Proof** First, we prove that the right hand side of the inequalities (4.13) holds true.

We suppose for a contradiction that there is \( \rho_0 \in (0, 1) \) such that
\[
\varphi'(\rho_0) > \alpha_1(\rho_0, \varphi(\rho_0))\tau(\rho_0).
\] (4.14)

Since \(\alpha_1(\rho_0, \varphi(\rho_0)) \geq b_1(\rho_0)/b_0 \circ \varphi(\rho_0)\), by (4.14) we have
\[
f'\tau' = f' \circ \varphi \varphi' - f'\tau > (b_0\alpha_1 - b_1)\tau \geq 0 \quad \text{at} \quad \rho = \rho_0.
\] (4.15)

Next, we claim
\[
f' \circ \varphi \varphi'(\tau) - f'\varphi'(\rho) \leq 0 \quad \text{at} \quad \rho = \rho_0.
\] (4.16)

We assume that \(\varphi'(\tau(\rho_0)) \leq 0\). Since \(\varphi'\) is increasing, \(\alpha_1(\rho_0, \varphi(\rho_0))f' \circ \varphi(\rho_0) \geq f'(\rho_0)\), and \(\alpha_1(\rho_0, \varphi(\rho_0)) \geq 1\), it follows from (4.14) and (A 5) that
\[
\begin{align*}
f' \circ \varphi \varphi'(\tau) - f'\varphi'(\rho_0) & \leq f' \circ \varphi \varphi'(\tau) - f'\varphi'(\alpha_1\tau) \\
& = \left[f' \circ \varphi - \frac{f'}{\alpha_1}\right]\varphi'(\tau) + \frac{f'}{\alpha_1\tau} [\tau \varphi'(\tau) - \alpha_1\tau \varphi'(\alpha_1\tau)] \leq 0 \quad \text{at} \quad \rho = \rho_0.
\end{align*}
\]
Let $\varphi'(\tau) > 0$. Let

$$t_1 = q_1^{-1}\left(\frac{b_0(\rho_0)}{b_1 \circ \varphi(\rho_0)}\right).$$

Then $q_1(t_1) = b_0(\rho_0)/b_1 \circ \varphi(\rho_0)$, that is,

$$b_1 \circ \varphi(\rho_0)\varphi'(\tau(\rho_0)) \leq b_0(\rho_0)\varphi'(t_1\tau(\rho_0)),$$

which implies that (4.16) holds true since $\alpha_1(\rho_0, \varphi(\rho_0)) \geq t_1$.

From (4.16), (4.15), and (4.11) we obtain

$$\varphi''(\rho_0) < 0. \tag{4.17}$$

By (4.15) and (4.17) there is the smallest number $\rho_1 \in [0, \rho_0)$ such that

$$\tau' > 0 \quad \text{and} \quad \varphi''(\rho) < 0 \quad \text{for} \quad \rho \in (\rho_1, \rho_0). \tag{4.18}$$

We claim $\rho_1 = 0$. If $\rho_1 > 0$, then by (4.18) and (4.14)

$$\varphi'(\rho_1) > \varphi'(\rho_0) > \alpha_1(\rho_0, \varphi(\rho_0))\tau(\rho_0) \geq \alpha_1(\rho_1, \varphi(\rho_1))\tau(\rho_1),$$

since $\alpha_1(\rho_0, \varphi(\rho_0)) \geq \alpha_1(\rho_1, \varphi(\rho_1))$ by $\rho_0 \geq \rho_1$ and $\varphi(\rho_0) \geq \varphi(\rho_1)$. Using the same arguments as for (4.15) and (4.16), we obtain

$$\tau'(\rho_1) > 0, \quad \varphi''(\rho_1) < 0,$$

reaching a contradiction.

It follows from (4.18) with $\rho_1 = 0$ that $\varphi'(\rho) > \varphi'(\rho_0)$ for $\rho \in (0, \rho_0)$ and

$$\varphi(\rho) < \varphi(\rho_0) - \varphi'(\rho_0)\rho_0 + \varphi'(\rho_0)\rho \quad \text{for} \quad \rho \in (0, \rho_0). \tag{4.19}$$

Moreover, (4.14) implies that

$$\frac{\rho_0\varphi'(\rho_0)}{\varphi(\rho_0)} > \alpha_1(\rho_0, \varphi(\rho_0))\frac{\rho_0}{f(\rho_0)} \frac{f \circ \varphi(\rho_0)}{\varphi(\rho_0)} \geq \alpha_1(\rho_0, \varphi(\rho_0))\frac{b_0 \circ \varphi(\rho_0)}{b_1(\rho_0)} \geq 1, \tag{4.20}$$

since $\rho f_0(\rho) \leq f(\rho) \leq b_1(\rho)\rho_0$ for all $\rho \geq 0$.

From (4.19) and (4.20) we obtain $\varphi(0) < 0$ which is a contradiction again the assumption (4.14).

Similar arguments prove the left hand side of the inequalities (4.13). \qed

Actually, from the proof of Theorem 4.1 we have shown that

**Corollary 4.1** Let $\mu_+ (\max\{\lambda, 1\}) \leq 1$. Let (A1), (A4), (A5), and (A6) hold. Let $\varphi \in C^{1}(0, 1]$ be an equilibrium solution to problem (4.11). Then

$$\varphi'(\rho) \leq \alpha_1(\rho, \varphi(\rho))\tau(\rho) \quad \text{for} \quad \rho \in (0, 1].$$
Furthermore, if there is \( \rho_0 \in (0, 1] \) such that
\[
\varphi'(\rho_0) < \alpha_0(\rho_0, \varphi(\rho_0))\tau(\rho_0),
\]
then \( \varphi(0) > 0 \).

Let \( \kappa \equiv 0 \). Then \( f = \rho, b_0 = b_1 = 1, \) and
\[
\alpha_0(\rho, s) = \alpha_1(\rho, s) = 1 \quad \text{for} \quad \rho, s \in [0, \infty).
\]
It follows from (4.11) and (4.13) that a regular equilibrium solution \( \varphi \) satisfies
\[
\begin{aligned}
\varphi(1) &= \lambda, \\
\varphi' &= \frac{\varphi(\rho)}{\rho} \quad \text{for} \quad \rho \in (0, 1).
\end{aligned}
\] (4.21)

Since problem (4.21) has the unique solution \( \varphi = \lambda \rho \), we have

**Corollary 4.2** ([10]) Let \( \kappa \equiv 0 \) and let \((A1), (A4), (A5),\) and \((A6)\) hold. Then there exists a unique regular equilibrium solution \( \varphi = \lambda \rho \) to problem (4.11) with \( \varphi(1) = \lambda \).

Let \( 0 < \mu_0 \leq \mu_1 < \infty \) be given by (2.31). It follows from (2.31) that
\[
\frac{\mu_0}{\mu_1} \leq \frac{b_0(\rho)}{b_1(\rho)} \leq 1 \leq \frac{b_1(\rho)}{b_0(\rho)} \leq \frac{\mu_1}{\mu_0} \quad \text{for} \quad \rho, s \in [0, \infty),
\]
which also imply by \((A5)\) that
\[
q_0^{-1} \left( \frac{\mu_1}{\mu_0} \right) \leq q_0^{-1} \left( \frac{b_1(\rho)}{b_0(\rho)} \right) \leq 1 \leq q_1^{-1} \left( \frac{b_0(\rho)}{b_1(\rho)} \right) \leq q_1^{-1} \left( \frac{\mu_0}{\mu_1} \right) \quad \text{for} \quad \rho, s \in [0, \infty).
\]

It follows from Theorem 4.8 and Proposition 2.5 that

**Corollary 4.3** Let \( \mu_+ (\infty) \leq 1 \) and \( \mu_- (\infty) < \infty \). Let \((A1), (A4), (A5),\) and \((A6)\) hold. Let \( \varphi \in C^1(0, 1] \) be a regular equilibrium solution to problem (4.11) with \( \varphi(1) = \lambda \).

Then
\[
\eta_0 \tau(\rho) \leq \varphi'(\rho) \leq \eta_1 \tau(\rho) \quad \text{for} \quad \rho \in (0, 1],
\] (4.22)

where
\[
\eta_0 = \min \left\{ \frac{\mu_0}{\mu_1}, q_0^{-1} \left( \frac{\mu_1}{\mu_0} \right) \right\}, \quad \eta_1 = \max \left\{ \frac{\mu_1}{\mu_0}, q_1^{-1} \left( \frac{\mu_0}{\mu_1} \right) \right\}.
\]

**Corollary 4.4** Let all the assumptions in Corollary 4.3 hold. If \( \varphi \in C^1(0, 1] \) is a regular equilibrium solution to problem (4.11) with \( \varphi(1) = \lambda \), then
\[
\lambda \rho c_1 \leq \varphi(\rho) \leq \lambda \rho c_0 \quad \text{for} \quad \rho \in (0, 1], \quad \lambda > 0,
\] (4.23)

where
\[
c_0 = \eta_0 \frac{\mu_0}{\mu_1}, \quad c_1 = \eta_1 \frac{\mu_1}{\mu_0}.
\]
Proof Using (2.31) and (4.22), we have
\[
\frac{\varphi'}{\varphi} \leq \eta_1 \frac{f \circ \varphi}{\varphi} \frac{1}{f \circ \varphi} \leq \frac{c_1}{\rho}
\]
for \( \rho \in (0, 1] \),
which yields the left hand side of the inequalities (4.23). A similar argument proves the right hand side of the inequalities (4.23). \( \square \)

Remark 4.2 If \( \kappa(s) = 0 \) for all \( s \geq 0 \), then \( \mu_0 = \mu_1 = \eta_0 = \eta_1 = 1 \), and (4.23) means \( \varphi(\rho) = \lambda \rho \).

Theorem 4.2 Let \( \mu_+ (\max\{\lambda, 1\}) \leq 1 \). Let (A1), (A4), (A5), and (A6) hold. Let \( \varphi \in C^1(0, 1] \) be a regular equilibrium solution to problem (4.11) with \( \varphi(1) = \lambda \). Then

(i) there are constants \( \rho_0 \in (0, 1] \), \( c_0 > 0 \), and \( c_1 > 0 \) such that
\[
c_0 \rho \leq \varphi(\rho) \leq c_1 \rho \quad \text{for all} \quad 0 \leq \rho \leq \rho_0;
\]

(ii) the limit \( \lim_{\rho \to 0^+} \varphi(\rho)/\rho = \varphi'(0) \) exists;

(iii) there are constants \( \lambda_0 > 0 \), \( c_0 > 0 \), and \( c_1 > 0 \) such that
\[
c_0 \lambda \leq \varphi'(0) \leq c_1 \lambda \quad \text{for all} \quad \lambda \in (0, \lambda_0].
\]

Proof (i) First, we prove the right hand side of inequalities (4.24). We fix \( 0 < \rho_0 \leq 1 \) small such that
\[
e^{\mu_+ \circ \varphi(\rho_0)} \mu_+ \circ \varphi(\rho_0) < 1,
\]
which is possible since \( \varphi(0) = 0 \).

Step 1 Let
\[
\alpha(\rho) = \frac{b_1(\rho)}{b_0 \circ \varphi(\rho)} \quad \text{for} \quad \rho \in (0, \rho_0].
\]
We shall estimate
\[
\int_\rho^{\rho_0} \frac{\alpha(s)}{\varphi(s)} \tau(s) ds \quad \text{for} \quad \rho \in (0, \rho_0].
\]

Since by Proposition 2.4
\[
b_0(\rho) = 1 - \sup_{0 \leq s \leq \rho} \int_0^s \kappa f ds \geq \frac{1}{2} - \int_0^{\rho} \kappa f ds \geq 1 - \int_0^{\rho} \mu_+(\rho) \quad \text{for} \quad \rho \in (0, 1],
\]
it follows from (4.26) that
\[
b_0 \circ \varphi(\rho) \geq b_0 \circ \varphi(\rho_0) > 0 \quad \text{for} \quad 0 \leq \rho \leq \rho_0.
\]
Moreover, we have
\[
b_1(\rho) - b_0 \circ \varphi(\rho) = \sup_{0 \leq s \leq \varphi(\rho)} \int_0^s \kappa f ds - \inf_{0 \leq s \leq \rho} \int_0^s \kappa f ds
\]
\[
\leq \int_0^{\varphi(\rho)} \kappa_+ f ds + \int_0^\rho \kappa_- f ds \leq c(\rho_0, \varphi(\rho_0)) \left[ f \circ \varphi(\rho) + f(\rho) \right]
\]
(4.28)
for $0 \leq \rho \leq \rho_0$, where
\[ c(t, s) = \int_0^{\max\{t, s\}} |\kappa(\zeta)|d\zeta. \] (4.29)

Using (4.13), (4.27), and (4.28) we have
\[ \frac{\alpha(\rho) - 1}{\varphi(\rho)} \tau(\rho) = \frac{\alpha(\rho) - 1}{\varphi(\rho)} f \circ \varphi(\rho) \leq \frac{c(\rho_0, \varphi(\rho_0))}{b_0 \circ \varphi(\rho_0)} (\tau + 1)b_1 \circ \varphi(\rho_0) \]
\[ \leq \frac{c(\rho_0, \varphi(\rho_0))b_1 \circ \varphi(\rho_0)}{\alpha_0(\rho_0, \varphi(\rho_0))} \varphi'(\rho) + \frac{c(\rho_0, \varphi(\rho_0))b_1 \circ \varphi(\rho_0)}{b_0 \circ \varphi(\rho_0)} \quad \text{for } 0 \leq \rho \leq \rho_0, \] (4.30)

since $f \circ \varphi \leq (b_1 \circ \varphi)\varphi$.

In addition we have
\[ \frac{f \circ \varphi(\rho)}{\varphi(\rho)} - 1 = -\int_0^{\varphi(\rho)} k f ds + \frac{1}{\varphi(\rho)} \int_0^{\varphi(\rho)} s k f ds \]
\[ \leq 2c(\rho_0, \varphi(\rho_0))f \circ \varphi(\rho) \quad \text{for } 0 \leq \rho \leq \rho_0, \]

which implies by (4.13) that
\[ \left[ \frac{f \circ \varphi(\rho)}{\varphi(\rho)} - 1 \right] \frac{1}{f(\rho)} \leq \frac{2c(\rho_0, \varphi(\rho_0))}{\alpha_0(\rho_0, \varphi(\rho_0))} \varphi'(\rho) \quad \text{for } 0 < \rho \leq \rho_0. \] (4.31)

Furthermore,
\[ \frac{1}{f(\rho)} - \frac{1}{\rho} = \frac{1}{f(\rho)} \left[ \int_0^\rho k(1 - \frac{s}{\rho})f ds \right] \leq 2c(\rho_0, \varphi(\rho_0)) \quad \text{for } 0 < \rho \leq \rho_0. \] (4.32)

From (4.30), (4.31), and (4.32), we obtain
\[ \frac{\alpha(\rho)}{\varphi(\rho)} \tau(\rho) = \frac{\alpha(\rho) - 1}{\varphi(\rho)} \tau(\rho) + \left[ \frac{f \circ \varphi(\rho)}{\varphi(\rho)} - 1 \right] \frac{1}{f} + \left( \frac{1}{f(\rho)} - \frac{1}{\rho} \right) + \frac{1}{\rho} \]
\[ \leq c(\rho_0, \varphi(\rho_0)) \left\{ \frac{b_1 \circ \varphi(\rho_0)}{b_0 \circ \varphi(\rho_0)} + 2 \right\} \left[ \frac{\varphi'(\rho)}{\alpha_0(\rho_0, \varphi(\rho_0))} \right] + \frac{1}{\rho} \]
for $0 < \rho \leq \rho_0$, which yields
\[ \int_\rho^{\rho_0} \frac{\alpha(s)}{\varphi(s)} \tau(s) ds \leq \eta(\rho_0, \varphi(\rho_0)) + \ln \frac{\rho_0}{\rho} \quad \text{for } 0 < \rho \leq \rho_0, \] (4.33)

where
\[ \eta(t, s) = c(t, s) \frac{b_1(s)}{b_0(s)} + 2 \left[ \frac{s}{\alpha_0(t, s)} + t \right]. \] (4.34)

**Step 2** Let
\[ \alpha * (\rho) = \frac{b_0(\rho)}{b_1 \circ \varphi(\rho)} \quad \text{for } 0 \leq \rho \leq \rho_0. \]

By (4.10) we have
\[ q_1^{-1}(\alpha * (\rho)) - 1 \leq \frac{1}{c * (\rho_0, \varphi(\rho_0))} \left[ 1 - \alpha * (\rho) \right] \quad \text{for } 0 \leq \rho \leq \rho_0, \]
where
\[ c \ast (t, s) = \inf_{b_0(t)/b_1(s) \leq 1} |q_1'(q_1^{-1}(\zeta))|. \] (4.35)

Similar arguments as in Step 1 give the estimate
\[ \int_\rho^{\rho_0} q_1^{-1}(\alpha \ast (s)) \frac{f \circ \varphi(s)}{f(s) \varphi(s)} ds \leq \eta \ast (\rho_0, \varphi(\rho_0)) + \ln \frac{\rho_0}{\rho} \text{ for } 0 < \rho \leq \rho_0, \] (4.36)

where
\[ \eta \ast (t, s) = c(t, s) \left[ b_1(s) + 2 \right] \left[ \frac{s}{\alpha_0(t, s)} + \tau \right]. \]

Step 3 Using (4.13), (4.33), and (4.36) we obtain
\[ \varphi(\rho) \geq e^{-\max\{\eta(\rho_0, \varphi(\rho_0)), \eta \ast (\rho_0, \varphi(\rho_0))\} \frac{\varphi(\rho_0)}{\rho}} \rho \text{ for } 0 \leq \rho \leq \rho_0. \] (4.37)

Similar arguments prove the left hand side of inequalities (4.24).

(ii) From (4.37) we have
\[ \lim_{\rho \to 0^+} \frac{\varphi(\rho)}{\rho} \geq e^{-\max\{\eta(\rho_0, \varphi(\rho_0)), \eta \ast (\rho_0, \varphi(\rho_0))\} \frac{\varphi(\rho_0)}{\rho_0}} \] (4.38)

for \( \rho_0 \in (0, 1] \) such that (4.26) holds. It follows from (4.38) that
\[ \lim_{\rho \to 0^+} \frac{\varphi(\rho)}{\rho} \geq \lim_{\rho \to 0^+} \frac{\varphi(\rho)}{\rho} \]

since \( \lim_{\rho \to 0^+} \eta(\rho, \varphi(\rho)) = \lim_{\rho \to 0^+} \eta \ast (\rho, \varphi(\rho)) = 0 \), which is what we need.

(iii) Let \( \lambda_0 > 0 \) be small such that
\[ e^{\mu_-(\lambda_0) \mu_+(\lambda_0)} < 1. \]

Let \( 0 < \lambda \leq \lambda_0 \). Then \( \rho_0 \) in (4.26) can be taken as 1. It follows from (4.37) that
\[ \varphi'(0) \geq e^{-\max\{\eta(1, \lambda), \eta \ast (1, \lambda)\} \lambda} \text{ for } \lambda \in (0, \lambda_0]. \]

Since \( \eta(1, \lambda) \) and \( \eta \ast (1, \lambda) \) are bounded on \([0, \lambda]\), we have shown that the left hand side of inequalities (4.25). Similar arguments show that the right hand side of (4.25) holds.

Remark 4.3 In (4.24) constants \( c_0 \) and \( c_0 \) may be depend on the number \( \lambda \).

Proposition 4.2 Let \( \mu_+(\max\{\lambda, 1\}) \leq 1 \). Let (A1), (A2), (A4), (A5), (A6), (A7) and (A8) hold. Let \( \varphi \in C^1(0, 1] \) be an equilibrium solution to problem (4.11) with \( \varphi(1) = \lambda \). Then
\[(i) \sup_{0 < \rho \leq 1} \varphi'(\rho) < \infty; \]
\[(ii) \text{ the limit } \lim_{\rho \to 0^+} T(\rho) \text{ exists and is finite.} \]
Proof If $\varphi$ is regular, then (i) and (ii) follow from Theorems 4.2 and 4.1. Next, we assume that $\varphi(0) > 0$. Using Corollary 4.1, (A 7), (A 8), and (4.3) we obtain

$$|T'(\rho)| \leq c \tau^{1-n}[1+\tau^\alpha + (\alpha_1 \tau)^\alpha] \leq c \tau^{1+\alpha-n} \leq \frac{c}{\rho^{1+\alpha-n}}$$

for $\rho \in (0,1)$, which yields

$$|T(\rho)| \leq |T(1)| + c \int_\rho^1 \frac{1}{\rho^{1+\alpha-n}} d\rho \leq |T(1)| + c$$

for $\rho \in (0,1)$ (4.39) since $1 + \alpha - n < 1$.

To prove (i) we suppose for a contradiction that there is a sequence $\{\rho_j\} \subset (0,1]$ such that $\rho_j \to 0$ and

$$j \leq \varphi'(\rho_j) \leq \alpha_1(\rho_j, \varphi(\rho_j))\tau(\rho_j) \quad \text{for} \quad j \geq 1,$$

where the right hand side of the above inequalities is from Corollary 4.1. It follows from (A 6) and (A 7) that

$$\tau^{1-n}(\rho_j)|\varphi'(\varphi(\rho_j))| \leq c \tau^{1-n}(\rho_j)[1 + \varphi'(\rho_j)] \leq c[\tau^{1-n}(\rho_j) + \tau^{\alpha-n}(\rho_j)],$$

and then $\lim_{j \to \infty} \tau^{1-n}(\rho_j)\varphi'(\varphi(\rho_j)) = 0$ is true. By (A 2) we have

$$\lim_{j \to \infty} T(\rho_j) = \lim_{j \to \infty} h'(\varphi(\rho_j)\tau^{n-1}(\rho_j)) = +\infty,$$

contradicting with (4.39).

By (i) and $\varphi(0) > 0$ there is some $\rho_0 \in (0,1]$ such that

$$\varphi'(\rho) < \tau(\rho) \quad \text{for} \quad \rho \in (0,\rho_0].$$

Then $T'(\rho) \geq 0$ for $\rho \in (0,\rho_0]$ by (4.3), which implies that (ii) is true. \qed

Cavitating Equilibrium Solutions An equilibrium solution $\varphi \in C^1(0,1]$ to problem (4.11) is said to be cavitating if

(i) $\varphi(0) > 0$, and

(ii) $\lim_{\rho \to 0^+} T(\rho) = 0$.

Ball [2] has shown that when $\lambda > 0$ is small there is no cavitating equilibrium solution in the case of the Euclidean space. We present some similar results.

Proposition 4.3 Let $\mu_+(\max\{\lambda, 1\}) \leq 1$. Let (A1), (A2), (A4), (A5), (A6), and (A7) hold. Suppose that (A8) holds with $\beta = 0$. If $\varphi \in C^1(0,1]$ is a cavitating equilibrium solution to problem (4.11), then

$$\lim_{\rho \to 0^+} \varphi' \tau^{n-1} = \varpi,$$

where $\varpi > 0$ is given by $h'(\varpi) = 0$. 

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Proof By Corollary 4.1, (A 7), and (A 8) with $\beta = 0$, we have $\lim_{\rho \to 0^+} \tau^{n-1} \phi'(\varphi') = 0$. Then $0 = \lim_{\rho \to 0^+} T(\rho) = \lim_{\rho \to 0^+} h'(\varphi' \tau^{n-1})$. (4.40) follows from (A 1).

Theorem 4.3 Let $\mu_+(1) \leq 1$ and let there be $\varepsilon \in (0, 1]$ such that

$$\kappa(s) = 0 \quad \text{for} \quad s \in (0, \varepsilon). \quad (4.41)$$

Let (A 1), (A 2) and (A 4) hold. Then there is no cavitating equilibrium solution when $\lambda > 0$ is small.

Proof It follows from (4.41) that

$$f'(\rho) = 1, \quad f(\rho) = \rho \quad \text{for} \quad \rho \in [0, \varepsilon]. \quad (4.42)$$

Then the left hand side of the equation in (4.11) equals

$$(n - 1)[\phi'(\tau) - \phi'(\varphi') - (\varphi' - \tau)h''(\varphi' \tau^{n-1})\varphi' \tau^{2n-3}] \quad (4.43)$$

for $\rho \in (0, \varepsilon]$ when $\varphi(1) = \lambda < \varepsilon$, where

$$\tau(\rho) = \frac{\varphi'(\rho)}{\rho} \quad \text{for} \quad \rho \in (0, \varepsilon].$$

By (A 2) and (A 4), without loss of generality we assume that $\lambda_0 > 0$ is small such that

$$\Phi_1(\frac{\lambda_0}{\varepsilon}, \cdots, \frac{\lambda_0}{\varepsilon}) = \phi'(\frac{\lambda_0}{\varepsilon}) + h'(\frac{\lambda_0}{\varepsilon})^n < 0. \quad (4.44)$$

Let $\varphi \in C^1(0, 1]$ be a cavitating equilibrium solution with $\varphi(1) = \lambda < \lambda_0$ and we derive a contradiction below. We claim

$$\varphi'(\rho) < \tau(\rho) \quad \text{for all} \quad \rho \in (0, \varepsilon). \quad (4.45)$$

In fact, if there were some $\rho_0 \in (0, \varepsilon]$ such that $\varphi'(\rho_0) = \tau(\rho_0)$, it is easy to check by (4.43) and the uniqueness theorem that

$$\varphi(\rho) = \varphi'(\rho_0)\rho \quad \text{for all} \quad \rho \in (0, \rho_0],$$

which contradicts $\varphi(0) > 0$. Next, by (4.45), (4.3) and (4.44), we obtain

$$T(\rho) < T(\varepsilon) = \tau^{1-n}(\varepsilon)\Phi_1(\varphi'(\varepsilon), \tau(\varepsilon), \cdots, \tau(\varepsilon))$$

$$< \tau^{1-n}(\varepsilon)\Phi_1(\tau(\varepsilon), \tau(\varepsilon), \cdots, \tau(\varepsilon)) < 0 \quad \text{for} \quad \rho \in (0, \varepsilon),$$

that contradicts $T(0) = 0$. \qed
Theorem 4.4 Let $\mu_+ (1) \leq 1$ and let there be $\varepsilon \in (0,1]$ such that
\[ \kappa(s) < 0 \quad \text{for} \quad s \in (0, \varepsilon). \] (4.46)

Let $(A1), (A2), (A4), (A5), (A6), (A7),$ and $(A8)$ hold. Further suppose that
\begin{enumerate}
  \item $\phi'(1) + h'(1) = 0$;
  \item $\phi'(s) - h''(sn)s^{2n-1} < 0$ for $s > 1$.
\end{enumerate}
Then there is no cavitating equilibrium solution when $0 < \lambda < \varepsilon$.

Proof We suppose for a contradiction that $\varphi \in C^1(0,1]$ is a cavitating equilibrium solution with $\varphi(1) = \lambda < \varepsilon$. By Proposition 4.2 there is the largest number $\rho_0 \in (0,1]$ such that $\varphi'(\rho) < \tau(\rho)$ for all $\rho \in (0, \rho_0)$. (4.47)

Let $\rho_0 \geq \varepsilon$. Then
\[ \tau(\rho_0) \leq \frac{f(\lambda)}{f(\varepsilon)} < 1, \]
and therefore there is some $\rho_1 \in (0, \rho_0)$ such that $\tau(\rho_1) = 1$. The relations (4.3) and (4.47) implies that
\[ T(\rho) \leq T(\rho_1) = \tau^{1-n}(\rho_1)\Phi_1(\varphi'(\rho_1), 1, \ldots, 1) < \tau^{1-n}(\rho_1)\Phi_1(1, \ldots, 1) = 0 \] (4.48)
for all $\rho \in (0, \rho_1)$, which contradicts with $T(0) = 0$.

Let $0 < \rho_0 < \varepsilon$. By (4.47) we must have
\[ \varphi'(\rho_0) = \tau(\rho_0). \] (4.49)

If $\tau(\rho_0) \leq 1$, we get a contradiction as in (4.48). We assume that $\tau(\rho_0) > 1$. Then
\[ \rho_0 < \varphi(\rho_0) < \lambda < \varepsilon. \] (4.50)

By (4.50) and (4.46), we obtain
\[ f' \circ \varphi(\rho_0) = f'(\rho_0) - \int_{\rho_0}^{\varphi(\rho_0)} \kappa(s)f(s)ds > f'(\rho_0). \] (4.51)

Thus, it follows from (4.11), (4.49), (4.51) and (ii) that
\[ \text{the left hand side of the equation (4.11)} \]
\[ = (n - 1)(f' \circ \varphi - f')(\varphi'(\tau) - h''(\tau^n)\tau^{2(n-1)}) < 0 \quad \text{at} \quad \rho = \rho_0, \]

that is,
\[ \varphi''(\rho_0) < 0. \] (4.52)
In addition, we have, by (4.51) and (4.49),
\[ f(\rho_0)\tau'(\rho_0) = [f'(\varphi(\rho_0)) - f(\rho_0)]\tau(\rho_0) > 0. \] (4.53)
Thus, by (4.52) and (4.53), we obtain
\[ \varphi'(<\rho) = \varphi'(\rho_0) = \tau(\rho_0) > \tau(\rho) \text{ for } \rho < \rho_0 \text{ and near } \rho_0, \]
contradicting the definition of \( \rho_0 \). \( \square \)

**Remark 4.4** If \( \kappa(o) < 0 \), assumption (4.46) is true.

**Remark 4.5** Assumption (i) means that \( \Phi_i(1, \cdots, 1) = 0 \) for \( 1 \leq i \leq n \) so that the undeformed configuration is a natural state. In addition, if
\[ \phi(v) = \frac{1}{v^\beta} + v^\alpha, \quad h(v) = \frac{2\alpha}{v} + \frac{\alpha + \beta}{2}v^2, \quad \beta > 0, \quad \alpha > 1, \]
then (i) and (ii) hold.

### 4.3 Energy Minimizers

We seek to minimize
\[ I(\varphi) = \int_0^1 f^{n-1}(\rho)\Phi(\rho)d\rho \] (4.54)
among radial deformations \( \varphi \) such that \( \varphi(0) \geq 0 \) and \( \varphi(\rho) \) is increasing. For \( \lambda > 0 \) given, let
\[ \mathcal{A}_\lambda = \left\{ \varphi \in W^{1,1}(0, 1) \mid \varphi(0) \geq 0, \varphi' > 0 \text{ a.e., and } I(\varphi) < \infty \right\}. \]
Consider the minimization problem
\[ \inf_{\varphi \in \mathcal{A}_\lambda} I(\varphi). \] (4.55)

Let
\[ u(p) = \sigma \circ \varphi(\rho), \]
where \( \sigma \) is given by (2.59) and \( p = \sigma(\rho) \). Then
\[ \varphi(\rho) = \sigma^{-1} \circ u(p), \quad \rho = \sigma^{-1}(p), \]
\[ \varphi'(\rho) = \frac{u'(p)}{\sigma' \circ \varphi(\rho)}p'(\rho) = \frac{f^{n-1} \circ \sigma^{-1}(p)}{f^{n-1} \circ \sigma^{-1}(u)}u'(p), \quad \tau(\rho) = \frac{f \circ \sigma^{-1}(u)}{f \circ \sigma^{-1}(p)}, \]
and
\[ u'(p) = \sigma' \circ \varphi(\rho) \varphi'(\rho) \frac{\partial \rho}{\partial p} = \varphi'(\rho) \tau^{n-1}(p). \]
Let
\[ \varphi(p, u, q) = \phi\left(\frac{f^{n-1} \circ \sigma^{-1}(p)}{f^{n-1} \circ \sigma^{-1}(u)}q + (n-1)\phi\left(\frac{f \circ \sigma^{-1}(u)}{f \circ \sigma^{-1}(p)}\right) + h(q), \right. \]
\[ J(u) = \int_0^{\sigma(1)} \varphi(p, u, u') dp. \]

Since
\[ I(\varphi) = J(u), \]
the minimization problem (4.55) is now equivalent to minimizing \( J(u) \) on the set
\[ \{ u \in W^{1,1}(0, \sigma(1)) \mid u(\sigma(1)) = \sigma(\lambda), u(0) \geq 0, u'(p) > 0 \text{ a.e., and } J(u) < \infty \}. \]

A similar argument as in [2] yields

**Theorem 4.5** Suppose that (A1), (A2), (A4), and (A11) hold. \( I \) attains an absolute minimum on \( \mathcal{A}_\lambda \).

Following the proof of Theorem 7.3 in [2], we obtain the following.

**Theorem 4.6** Let (A1), (A2), (A4), (A7), and (A9) hold. Let \( \varphi \) be a minimizer of \( I \) on \( \mathcal{A}_\lambda \). Then \( \varphi \in C^1(0,1] \), \( \varphi'(\rho) > 0 \) for all \( \rho \in (0,1] \), \( f^{n-1}\Phi_1 \in C^1(0,1] \) and (4.11) holds for all \( \rho \in (0,1] \). If \( \varphi(0) > 0 \), then \( f^{n-2}\Phi_2 \in L^1(0,1) \) and
\[ \lim_{\rho \to 0+} T(\rho) = 0, \]
where \( T \) is the radial component of the Cauchy stress, given by (4.2).

Next, we have the following.

**Theorem 4.7** Let \( \mu_+ (\max \{ \lambda, \lambda \}) \leq 1 \). Let (A1), (A2), (A4), (A5), (A6), (A7) and (A8) hold. Let \( \varphi \in C^1(0,1] \) be an equilibrium solution to problem (4.11) with \( \varphi(1) = \lambda \). Then
\[ I(\varphi) < \infty. \]

**Proof** By using (4.8) and (4.3), we have
\[ \varphi'\Phi_1 + (n-1)\tau\Phi_2 = (n-1)[\tau\varphi'(\tau) - \varphi'\varphi'(\varphi')] + n\varphi'\tau^{n-1}T = \frac{f \circ \varphi}{f' \circ \varphi} \tau^{n-1}T' + n\varphi'\tau^{n-1}T \quad \text{for } \rho \in (0,1). \]

Thus,
\[ f^{n-1}[\varphi'\Phi_1 + (n-1)\tau\Phi_2] = \frac{1}{f' \circ \varphi} \left( f^n \circ \varphi T \right)' \quad \text{for } \rho \in (0,1). \]

Now, using (4.7) and (4.57), we obtain, by integration by parts,
\[ n \int_\rho^1 f' f^{n-1} \Phi d\rho + f^n(\rho) \Phi(\rho) = \left\{ f^n[\Phi - (\varphi' - \tau)\Phi_1] + \frac{f' - f' \circ \varphi}{f' \circ \varphi} f^n \circ \varphi T \right\} |_{\rho=1} + \int_\rho^1 \frac{f' \circ \varphi(\rho) - f' (\rho)}{f' \circ \varphi(\rho)} f^n \circ \varphi T(\rho) + f^n(\rho)[\varphi'(\rho) - \tau(\rho)]\Phi_1(\rho) + \int_\rho^1 \frac{k \circ \varphi \varphi f' - k f' \circ \varphi}{f'^2 \circ \varphi} f^n \circ \varphi T d\rho \quad \text{for } \rho \in (0,1). \]
Next, by Proposition 4.2, the term
\[ f^n(\rho)[\varphi'(\rho) - \tau(\rho)]\Phi_1(\rho) = f^{n-1} \circ \varphi(\rho)[f(\rho)\varphi'(\rho) - f \circ \varphi(\rho)]T(\rho) \]
converges as \( \rho \) goes to 0+ and the limit is finite. Moreover, by Proposition 2.4,
\[ \mu_0(1) \leq f'(\rho) \leq e^{\mu_1(1)}, \quad \mu_0(\lambda) \leq f' \circ \varphi(\rho) \leq e^{\mu_1(\lambda)} \quad \text{for} \quad \rho \in (0, 1]. \]
Thus, the second term and the last term in the right hand side of (4.58) converge as \( \rho \) goes to 0+ and their limits are finite by Proposition 4.2. The proof is complete. \( \square \)

### 4.4 Cavitation

We now derive a cavitating theorem.

**Theorem 4.8** Let \( \mu_+ (\infty) \leq 1 \) and \( \mu_- (\infty) < \infty \). Suppose that (A1)-(A9) hold. For any \( \lambda \) sufficiently large a minimizer of \( I \) on \( A_\lambda \) is a cavitating equilibrium solution.

**Proof** By Theorem 4.6 a minimizer of \( I \) on \( A_\lambda \) is an equilibrium solution. We suppose for a contradiction that \( \varphi \in C^1(0, 1) \) is a regular equilibrium solution with \( \varphi(1) = \lambda \) where \( \lambda \) is sufficiently large. For \( \varepsilon \in (0, 1) \), let
\[ \varphi_\varepsilon(\rho) = \varphi\{\sigma^{-1}[(1 - \varepsilon)\sigma(\rho) + \varepsilon\sigma(1)]\}. \]
Then
\[ \varphi_\varepsilon(1) = \lambda, \quad \varphi_\varepsilon(0) = \varphi(\sigma^{-1}(\varepsilon\sigma(1))) > 0. \]
Thus, \( \varphi_\varepsilon \in A_\lambda \) for all \( \varepsilon \in (0, 1) \). We will show that for \( \lambda \) sufficiently large and \( \varepsilon \in (0, 1) \) small
\[ I(\varphi_\varepsilon) < I(\varphi), \quad (4.59) \]
contradicting that \( \varphi \) is a minimizer of \( I \) on \( A_\lambda \).

**Step 1** Let
\[ p = \sigma^{-1}[(1 - \varepsilon)\sigma(\rho) + \varepsilon\sigma(1)], \quad \tau_\varepsilon = \frac{f \circ \varphi_\varepsilon(\rho)}{f(\rho)}, \quad \text{for} \quad \rho \in (0, 1]. \]
Simple computations yield
\[ p' = (1 - \varepsilon)\frac{f^{n-1}}{f^{n-1} \circ p}, \quad \varphi_\varepsilon'(\rho) = (1 - \varepsilon)\varphi' \circ p(\rho) \frac{f^{n-1}(\rho)}{f^{n-1} \circ p(\rho)}, \quad (4.60) \]
\[ \varphi_\varepsilon'(1) = (1 - \varepsilon)\varphi'(1), \quad \tau_\varepsilon = \tau \circ p \frac{f \circ p}{f}, \quad \varphi_\varepsilon \tau_\varepsilon p^{n-1} = (1 - \varepsilon)\varphi' \circ p r^{n-1} \circ p. \quad (4.61) \]
In addition, (A 1) gives
\[ h(t) \geq h(s) + (t - s)h'(s) \quad \text{for all} \quad t, s \in (0, \infty). \]
By using the formulas above, we obtain
\[
(1 - \varepsilon) \int_0^1 h(\varphi' \tau^{n-1}) f^{n-1} \, d\rho = \int_{p(0)}^1 h[(1-\varepsilon)\varphi' \tau^{n-1}] f^{n-1} \, d\rho
\]
\[
\leq \int_{p(0)}^1 h(\varphi' \tau^{n-1}) f^{n-1} \, d\rho - \varepsilon \int_{p(0)}^1 \varphi' \tau^{n-1} h'[1(1-\varepsilon)\varphi' \tau^{n-1}] f^{n-1} \, d\rho,
\]
which yields
\[
\int_0^1 h(\varphi' \tau^{n-1}) f^{n-1} \, d\rho \geq \varepsilon \int_{p(0)}^1 h(\varphi' \tau^{n-1}) f^{n-1} \, d\rho + (1 - \varepsilon) \int_0^1 h(\varphi' \tau^{n-1}) f^{n-1} \, d\rho
\]
\[+ \varepsilon (1 - \eta) \int_{p(0)}^1 \varphi' \tau^{n-1} h'[1(1-\varepsilon)\varphi' \tau^{n-1}] f^{n-1} \, d\rho
\]
\[+ \varepsilon \eta \int_{p(0)}^1 \varphi' \tau^{n-1} h'[1(1-\varepsilon)\varphi' \tau^{n-1}] f^{n-1} \, d\rho, \tag{4.62}
\]
for any \( \eta > 0 \) small, since \( \varepsilon < 1 \).

**Step 2** By (A 3), there are constants \( \varpi \) and \( N(\varpi) > 0 \) such that
\[
\varpi > 1, \quad x h'(x) \geq \varpi h(x), \quad \text{for all} \quad x \geq N(\varpi). \tag{4.63}
\]
Since \( \theta \) is continuous, we take \( N(\varepsilon) \geq N(\varpi) \) such that
\[
h[(1-\varepsilon)x] \geq [\theta(1-\varepsilon) - \varepsilon] h(x) \quad \text{for all} \quad x \geq N(\varepsilon). \tag{4.64}
\]
Next, for \( \varpi > 1 \) being given in (4.63), we fix \( \eta > 0 \) and \( \varepsilon > 0 \) small such that
\[
\frac{(1 - \eta)\varpi}{1 - \varepsilon} [\theta(1-\varepsilon) - \varepsilon] \geq 1, \tag{4.65}
\]
that is possible because \( \lim_{\varepsilon \to 0} \theta(1-\varepsilon) = 1 \).

By using (4.63)-(4.65), we obtain that, if \( \varphi' \tau^{n-1} \geq N(\varepsilon)/(1 - \varepsilon) \) for all \( \rho \in [p(0), 1] \), then
\[
(1 - \eta) \int_{p(0)}^1 \varphi' \tau^{n-1} h'[1(1-\varepsilon)\varphi' \tau^{n-1}] f^{n-1} \, d\rho \geq \frac{\varpi}{1 - \varepsilon} \int_{p(0)}^1 h[(1-\varepsilon)\varphi' \tau^{n-1}] f^{n-1} \, d\rho
\]
\[
\geq \frac{(1 - \eta)\varpi}{1 - \varepsilon} [\theta(1-\varepsilon) - \varepsilon] \int_{p(0)}^1 h(\varphi' \tau^{n-1}) f^{n-1} \, d\rho \geq \int_{p(0)}^1 h(\varphi' \tau^{n-1}) f^{n-1} \, d\rho. \tag{4.66}
\]
Inserting (4.66) into (4.62) yields
\[
\int_0^1 h(\varphi' \tau^{n-1}) f^{n-1} \, d\rho \geq \int_0^1 h(\varphi' \tau^{n-1}) f^{n-1} \, d\rho
\]
\[+ \frac{\varepsilon \eta}{1 - \varepsilon} \int_{p(0)}^1 \varphi' \tau^{n-1} h'[1(1-\varepsilon)\varphi' \tau^{n-1}] f^{n-1} \, d\rho, \tag{4.67}
\]
if the following condition holds
\[
\varphi' \tau^{n-1} \geq \frac{N(\varepsilon)}{1 - \varepsilon} \quad \text{for all} \quad \rho \in [p(0), 1]. \tag{4.68}
\]
\[\text{\textbf{48}}\]
Step 3 Since \( \varphi \) is regular, by Corollaries 4.3, 4.4 and Proposition 2.5, we obtain
\[
\frac{\lambda \mu_0}{\mu_1} c^{\alpha - 1} \leq \tau(\rho) \leq \frac{\lambda \mu_1}{\mu_0} c^{\alpha - 1}, \quad \frac{\eta \mu_0}{\mu_1} c^{\alpha - 1} \leq \varphi'(\rho) \leq \frac{\eta \mu_1}{\mu_0} c^{\alpha - 1},
\]
for all \( \rho \in (0, 1] \).

By (A 8) and (4.69), we have
\[
\phi(\varphi') + (n - 1)\phi(\tau) \leq c(p(0))(1 + \lambda^\alpha), \quad \text{for } \lambda \geq 1, \ \rho \in [p(0), 1],
\]
where constant \( c(p(0)) > 0 \) depends on \( p(0) \) but is independent of \( \lambda \geq 1 \). Next, by (4.60), (4.61) and (4.69), we obtain
\[
\lambda(1 - \varepsilon) \frac{\eta \mu_0}{\mu_1} f^{\alpha - 1}(0) \frac{f^{\alpha - 1}(1)}{f^{\alpha - 1}(\rho)} \leq \varphi'(\rho) \leq \lambda(1 - \varepsilon) \frac{\eta \mu_1}{\mu_0} f^{\alpha - 1}(0) \frac{f^{\alpha - 1}(1)}{f^{\alpha - 1}(\rho)}
\]
for all \( \rho \in (0, 1] \), and
\[
\lambda p^{\alpha - 1}(0) \frac{\mu_0 f \circ p(0)}{\mu_1 f(\rho)} \leq \tau(\rho) \leq \lambda p^{\alpha - 1}(0) \frac{f(1)}{f(\rho)} \quad \text{for all } \rho \in (0, 1]
\]
and
\[
\varphi'(\rho) \tau^{\alpha - 1}(\rho) \geq \lambda^n c(p(0)) \quad \text{for } \rho \in [p(0), 1].
\]

It follows from (A 8), (4.71) and (4.72) that
\[
\phi(\varphi') + (n - 1)\phi(\tau(\rho)) \leq c(p(0)) \left(1 + \lambda^\alpha \left[1 + \frac{1}{f(\rho)}\right] + \frac{1}{f(\rho)}\right) + \frac{1}{f(\rho)}
\]
for all \( \lambda \geq 1 \) and \( \rho \in (0, 1] \).

By using (A 4), (4.67), (A 8) and (4.71)-(4.74), we obtain
\[
\begin{align*}
I(\varphi) & \geq \int_{p(0)}^{1} [\phi(\varphi') + (n - 1)\phi(\tau)] f^{\alpha - 1} \rho + \int_{0}^{1} h(\varphi') \tau^{\alpha - 1} f^{\alpha - 1} \rho \\
& \geq I(\varphi(\rho)) + \int_{p(0)}^{1} [\phi(\varphi') + (n - 1)\phi(\tau)] f^{\alpha - 1} \rho \\
& \quad - \int_{0}^{1} [\phi(\varphi') + (n - 1)\phi(\tau)] f^{\alpha - 1} \rho \\
& \quad + \frac{\varepsilon \eta}{1 - \varepsilon} \int_{p(0)}^{1} \varphi' \tau^{\alpha - 1} h'(1 - \varepsilon) \varphi' \tau^{\alpha - 1} f^{\alpha - 1} \rho \\
& \geq I(\varphi(\rho)) + \frac{\varepsilon \eta}{1 - \varepsilon} \lambda^n c(p(0)) h'[1 - \varepsilon \lambda^n c(p(0))] \int_{p(0)}^{1} f^{\alpha - 1} \rho d\rho \\
& \quad - c(p(0))(1 + \lambda^\alpha),
\end{align*}
\]
when
\[
\lambda \geq \max \left\{ \left( \frac{N(\varepsilon)}{c(p(0))(1 - \varepsilon)} \right)^{1/n}, \ 1 \right\}.
\]

The proof is complete. \( \square \)
5 Cavitation for Membrane Shells

The nonlinear shell membrane energy is obtained in [6] by the $\Gamma-$ limit of the sequence of three-dimensional energies. Here we will show that such membrane energies may take a form of (2.5) where $M$ is a surface in $\mathbb{R}^3$ and $g$ is the induced metric of $M$ from $\mathbb{R}^3$.

We consider a homogeneous elastic material with stored-energy function $\hat{W} : M_{+}^{3 \times 3} \to \mathbb{R}$. Suppose that $\hat{W}$ is frame-indifferent and isotropic, that is,

$$\hat{W}(F) = \hat{W}(QFR) \quad \text{for} \quad F \in M_{+}^{3 \times 3}, \quad Q, R \in \text{SO}(3). \quad (5.1)$$

We assume that a middle surface $S$ is a bounded, connected open set of a $C^2$ surface in $\mathbb{R}^3$ and let $N$ be the normal field of $S$. For $h > 0$ given, we consider the set $\Omega_h$ defined by

$$\Omega_h = \{ x + sN(x) \mid x \in S, \ |s| < h \}.$$ 

This set is the reference configuration of a shell with thickness $2h$. Let $u : \Omega_h \to \mathbb{R}^3$ be a deformation of the shell. Then the stored energy is

$$E_h(u) = \int_{\Omega_h} \hat{W}(\nabla u(x))dx = \int_{-h}^{h} \int_{S} \hat{W}(\nabla u(x + sN(x)))A(x, s)dgds,$$

where $g$ is the induced metric on $S$ from $\mathbb{R}^3$ and

$$A(x, s) = 1 + sH + s^2\kappa,$$

$H/2$ is the mean curvature, and $\kappa$ is the Gaussian curvature in $S$.

For a deformation $u \in L^p(\Omega_1, \mathbb{R}^3)$, we define $u_h \in L^p(\Omega_h, \mathbb{R}^3)$ by

$$u_h(x + shN(x)) = u(x + sN(x)) \quad \text{for} \quad x + shN(x) \in \Omega_h. \quad (5.2)$$

We define the rescaled energies by

$$I_h(u) = \frac{1}{h}E_h(u_h) = \int_{-1}^{1} \int_{S} \hat{W}(\nabla u_h(x + hsN(x)))A(x, sh)dgds,$$

for $u \in L^p(\Omega_1, \mathbb{R}^3)$ and $h > 0$ small.

Let $x \in S$ be fixed. Let $e_1, e_2$ be an orthonormal basis of $S_x$ such that $e_1, e_2, e_3$ is an orthonormal basis of $\mathbb{R}^3$ with positive orientation, where $e_3 = N(x)$, to satisfy

$$\hat{D}_{e_i}N = \kappa_i e_i \quad \text{at} \quad x \quad \text{for} \quad i = 1, 2, \quad (5.3)$$

where $\hat{D}$ is the connection of the Euclidean space $\mathbb{R}^3$ and $\kappa_i$ are the eigenvalues of the second fundamental form of $S$ at $x$. Thus, $H(x) = \kappa_1 + \kappa_2$ and $\kappa(x) = \kappa_1\kappa_2$.

It follows from (5.2) that

$$(1 + sh\kappa_i)u_{hs}e_i = (1 + s\kappa_i)u_se_i, \quad huu_{hs}e_3 = uu_3. \quad (5.4)$$
By (5.1) and (5.4), we obtain
\[
\hat{W}(\nabla u_h(x + hN(x))) = \hat{W}(u_{h*}e_1 | u_{h*}e_2 | u_{h*}e_3) = \hat{W} \left( \frac{1 + s\kappa_1}{1 + sh\kappa_1} u_*e_1 | \frac{1 + s\kappa_2}{1 + sh\kappa_2} u_*e_2 | \frac{1}{h} u_*e_3 \right).
\] (5.5)

In particular, if we let
\[
\varphi^h(x + sN(x)) = \varphi(x) + shw(x) \quad \text{for} \quad \varphi, w \in W^{1,p}(S, \mathbb{R}^3),
\]
then
\[
\hat{W}(\nabla \varphi^h(x + sN)) = \hat{W} \left( \frac{\varphi_*e_1 + shw_*e_1}{1 + sh\kappa_1} \left| \frac{\varphi_*e_2 + shw_*e_2}{1 + sh\kappa_2} \right| w \right).
\]

Let \(\hat{W}_0(F_1, F_2) = \min_{z \in \mathbb{R}^3} \hat{W}(F_1|F_2|z)\) for \(F_1, F_2 \in \mathbb{R}^3\).

Let \(Q\hat{W}_0 = \sup\{ Z : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}, Z \text{ quasiconvex}, Z \leq \hat{W}_0 \}\) be the quasiconvex envelope of \(\hat{W}_0\).

We introduce the space
\[
V_M = \{ \varphi \in W^{1,p}(\Omega_1, \mathbb{R}^3) \mid \varphi_*N = 0 \text{ for } x \in S \},
\] (5.6)
for which we call the space of membrane displacements.

By similar arguments as in [6], we may have

**Theorem 5.1** ([6]) Let all the assumptions on the function \(\hat{W}\) in [6] hold. Further suppose (5.1) is true. Then the sequence \(I_h\) \(\Gamma\)-converges for the strong topology of \(L^p(\Omega_1, \mathbb{R}^3)\) when \(h \to 0\). For \(\varphi \in L^p(\Omega_1, \mathbb{R}^3)\), it’s \(\Gamma\)-limit is given by
\[
I_0(\varphi) = \begin{cases} 
2 \int_S Q\hat{W}_0(\varphi_*e_1, \varphi_*e_2)dg & \text{if } \varphi \in V_M, \\
+\infty & \text{otherwise}.
\end{cases}
\] (5.7)

Next, we shall reformulate the \(\Gamma\)-limit energy (5.7) to relate it to the energy formula (2.5).

Let us introduce a function \(W_0 : M_+^{2 \times 2} \rightarrow \mathbb{R}\) by
\[
W_0(F) = \min_{z \in \mathbb{R}^3} \hat{W} \left( \begin{array}{c} F \\ Z_1 \\ 0 \\ z_3 \end{array} \right),
\]
where \(z = (z_1, z_2, z_3)\) and \(Z_1 = (z_1, z_2)^T\). It is easy to check that
\[
W_0(QFR) = W_0(F) \quad \text{for} \quad F \in M_+^{2 \times 2}, \quad Q, R \in SO(2).
\] (5.8)

Let \(W = \sup\{ Z : M^{2 \times 2} \rightarrow \mathbb{R}, Z \text{ quasiconvex}, Z \leq W_0 \}\) be the quasiconvex envelope of \(W_0\). From Dacorogna’s representation formula for the quasiconvex envelope of \(W_0\), we obtain
\[
W(F) = \frac{1}{\pi} \chi \inf_{\chi \in W_0^{1,\infty}(D, \mathbb{R}^2)} \int_D W_0(F + \nabla \chi)dy,
\]

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where \( D \) is the unit disc in \( \mathbb{R}^2 \). The formula above yields, by (5.8),

\[
W(QFR) = W(F) \quad \text{for} \quad F \in M_{+}^{2 \times 2}, \quad Q, R \in \text{SO}(2). \tag{5.9}
\]

For \( \varphi \in C^1(S, \mathbb{R}^3) \), consider a \( C^1 \) surface given by

\[
S_\varphi = \{ \varphi(x) \mid x \in S \}.
\]

Denote the induced metric of \( S_\varphi \) from \( \mathbb{R}^3 \) by \( g_\varphi = \langle \cdot, \cdot \rangle \circ \varphi \).

We have the following.

**Proposition 5.1** Let the functional \( I_0 \) be given by Theorem 5.1. Then

\[
I_0(\varphi) = 2 \int_S W(d\varphi(x)) dg \quad \text{for} \quad \varphi \in C^1(S, \mathbb{R}^3), \tag{5.10}
\]

where

\[
d\varphi(x) = \left( (E_i, \varphi^* e_j)(\varphi(x)) \right)_{2 \times 2},
\]

and, \( e_1, e_2 \) and \( E_1, E_2 \) are positively orientated orthonormal bases of \( S_x \) and \( (S_\varphi)_x \), respectively.

**Proof** Let \( E_3 \) be the normal field of \( S_\varphi \). Then \( \langle E_3, \varphi^* e_i \rangle = 0 \) for \( x \in S \) and \( i = 1, 2 \). Since \( \varphi^* e_i = \sum_{j=1}^3 \langle E_j, \varphi^* e_i \rangle(\varphi(x))E_j \) for \( 1 \leq j \leq 3 \), i.e.,

\[
\left( \varphi^* e_1 \mid \varphi^* e_2 \mid z \right) = \left( E_1 \mid E_2 \mid E_3 \right) \begin{pmatrix}
\langle E_1, \varphi^* e_1 \rangle & \langle E_1, \varphi^* e_2 \rangle & \langle E_1, z \rangle \\
\langle E_2, \varphi^* e_1 \rangle & \langle E_2, \varphi^* e_2 \rangle & \langle E_2, z \rangle \\
0 & 0 & \langle E_3, z \rangle
\end{pmatrix},
\]

for all \( z \in \mathbb{R}^3 \), we obtain, by (5.1),

\[
\hat{W}_0(\varphi^* e_1, \varphi^* e_2) = W_0(d\varphi) \quad \text{for} \quad x \in S.
\]

The proof is complete. \( \square \)

Next, let us assume that \( M \subset \mathbb{R}^3 \) is a \( C^2 \) surface with the induced metric \( g \). Suppose a middle surface of a shell \( S \) is a bounded, open set of \( M \). We assume that all deformations of \( S \) are confined in \( M \). For such a deformation \( u \), we assume that the stored energy is given

\[
E(u) = \int_S W(du) dg, \tag{5.11}
\]

where \( W : M_+^{2 \times 2} \to \mathbb{R} \) satisfies (5.9).

**Membrane Shells of Revolution** Let \( \psi \) be a \( C^2 \) function on \([0, \infty)\) with \( \psi'(0) = 0 \). Consider a surface of revolution given by

\[
M = \{ (x, \psi(r)) \in \mathbb{R}^3 \mid x = (x_1, x_2) \in \mathbb{R}^2, \ r = |x| \}.
\]

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The Gaussian curvature is
\[ \kappa(p) = \frac{\psi'(r)\psi''(r)}{r(1 + \psi'^2(r))^2} \quad \text{for} \quad p = (x, \psi(r)) \in M. \]

The normal field is
\[ N(p) = \frac{1}{\sqrt{1 + \psi'^2(r)}}(-\frac{\psi'(r)}{r}, x). \]

Let \( o = (0, \psi(0)) \in M \) be fixed. Then \( M_o = \mathbb{R}^2 \). Let \( \zeta(t) \) be defined by the equation
\[ t = \int_0^{\zeta(t)} \sqrt{1 + \psi'^2(s)} \, ds \quad \text{for} \quad t \geq 0. \] (5.12)

Let
\[ \gamma(t) = \left( \zeta(t)v, \psi(\zeta(t)) \right) \quad \text{for} \quad t \in \mathbb{R}, \]
where \( v = (v_1, v_2) \in \mathbb{R}^2 \) with \( v_1^2 + v_2^2 = 1 \).

**Lemma 5.1** \( \gamma(t) \) is a normal geodesic such that
\[ \gamma(0) = o, \quad \dot{\gamma}(0) = v. \]

**Proof** Let \( D \) denote the connection of the induced metric \( g \) of surface \( M \). Then
\[ D_{\dot{\gamma}(t)} \dot{\gamma} = \dot{\gamma}(t) - \langle \dot{\gamma}(t), N(\gamma(t)) \rangle N(\gamma(t)) = 0 \quad \text{for} \quad t \geq 0, \]
which prove the lemma. \( \square \)

It follows from Lemma 5.1 that
\[ \kappa(t) = \kappa(\gamma(t)) = \frac{\psi'(\zeta(t))\psi''(\zeta(t))}{\zeta(t)(1 + \psi'^2(\zeta(t)))^2} \quad \text{for} \quad t \geq 0, \] (5.13)
where \( \zeta \) is given by (5.12).

In addition, we have the following.

**Proposition 5.2** \((M, g, o)\) is a model.

**Proof** Since \( n = 2 \), we have
\[ R(\dot{\gamma}(t), X, \dot{\gamma}(t), X) = \kappa(\gamma(t))|X|^2 \quad \text{for all} \quad X \in M_{\gamma(t)}, \quad \langle X, \dot{\gamma}(t) \rangle = 0, \]
where \( R(\cdot, \cdot, \cdot, \cdot) \) is the curvature tensor, which imply that formula (2.12) holds true. By Proposition 2.3, the proof is complete. \( \square \)

Let a middle surface \( B \) of a membrane shell be the unit geodesic disc in \((M, g)\) centered at the point \( o \), i.e.,
\[ B = \left\{ \left( \zeta(t)v, \psi(\zeta(t)) \right) \mid v \in \mathbb{R}^2, \ 0 \leq t < 1 \right\}. \]
The radial deformations are given by

\[ u(p) = (\zeta \circ \varphi(p)v, \psi \circ \zeta \circ \varphi(p)) \quad \text{for} \quad p = (\zeta(p)v, \psi(\zeta(p))) \]

where \( \varphi \) is a function on \([0, \infty)\). Thus, all the theorems, corollaries, and propositions in Sections 2-4 hold true for radial deformations \( u \). We do not repeat them here.

To end this section, we present two examples which verify the assumptions on the radial curvature in Proposition 3.2 and Theorem 4.8, respectively.

**Example 5.1** Let \( \varepsilon > 0 \) be given and let \( \psi_0 \in C^\infty_0(0, \infty) \) be such that

\[ \psi_0(t) = 0 \quad \text{for} \quad 0 \leq t \leq \varepsilon; \quad \psi_0(t) = 1 \quad \text{for} \quad t \geq 2\varepsilon. \]

Let

\[ \psi_a(t) = \kappa_0 + \frac{a\psi_0(t)}{1 + t} \quad \text{for} \quad t \geq 0, \]

where \( \kappa_0 < 0 \) is a constant and \( a > 0 \) is such that

\[ \int_0^\infty s(\kappa_a)_+(s)ds = \int_0^{2\varepsilon} s(\kappa_a)_+(s)ds \leq 1. \]

Consider the incompressible case. Let \( \Phi \) satisfy the Baker-Ericksen inequalities with \( \hat{\Phi}'(1) > 0 \) and such that (3.26) is true. Let

\[ I(A) = \int_B W(du)dg - P \int_S \rho(u) dS. \]

It follows from Proposition 3.2 (i) that, for \( \lambda > 0 \) small and \( P = \chi(A) \), \( A \) is a local minimizer of \( I(A) \).

**Example 5.2** Let

\[ \psi(s) = a \log(1 + s^2) \quad \text{for} \quad s \geq 0, \]

where \( a > 0 \) is a constant. If \( 0 < a \leq 1/\sqrt{2} \), then

\[ \int_0^\infty t\kappa_+(t)dt \leq 1, \quad \int_0^\infty t\kappa_-(t)dt < \infty. \]

By (5.13), we have

\[ \kappa(t) = 4a^2 \frac{1 - \zeta^4(t)}{[1 + (4a^2 + 2)\zeta^2(t) + \zeta^4(t)]^2} \quad \text{for} \quad t \geq 0. \]

Let \( t_0 > 0 \) be given by \( \zeta(t_0) = 1 \). Thus,

\[ \int_0^\infty t\kappa_+(t)dt = 4a^2 \int_0^{t_0} \frac{t[1 - \zeta^4(t)]}{[1 + (4a^2 + 2)\zeta^2(t) + \zeta^4(t)]^2}dt \]

\[ \leq 4a^2 \int_0^1 \frac{1 - \zeta^2}{1 + (4a^2 + 2)\zeta^2}d\zeta + \zeta^4d\zeta \leq 4a^2 \int_0^1 \frac{1 - \zeta^2}{1 + (1 + \zeta^2)^2}d\zeta = 2a^2. \]

Similar arguments yield the second estimate in (5.14).

Consider the compressible case. Let \( W \) be given by (4.8) with \( n = 2 \) such that (A1)-(A9) hold true. It follows from Theorem 4.8 that, for any \( \lambda \) sufficiently large, a minimizer of \( I \) on \( A_\lambda \) is a cavitating equilibrium solution, where \( I \) is given by (4.54).
6 Cavitation for Ellipsoids in $\mathbb{R}^n$

Let

$$g(x) = G(x)$$

be a symmetric, $C^2$, and positively definite matrix for each $x \in \mathbb{R}^n$ and regard the pair $(\mathbb{R}^n, g)$ as a Riemannian manifold. The study in the previous sections describes radial deformations of a ball-like body if we apply it to the Riemannian manifold $(\mathbb{R}^n, g)$. The existence of corresponding cavitating equilibrium solutions depends on the geometric properties of the metric $g$ and on the growth properties of the constitutive function $W$ together.

We denote the metric $g$ on $\mathbb{R}^n$ by

$$g_{ij}(x) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_g = \langle G(x) \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$$

for $x \in \mathbb{R}^n$. Then under the natural coordinates $x = (x_1, \cdots, x_n)$

$$g_{ij}(x) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_g = \langle G(x) \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$$

for $x \in \mathbb{R}^n$.

Consider a body which occupies the open subset $\Omega$ of $\mathbb{R}^n$. A map $u : \Omega \to \mathbb{R}^n$ is said to be a deformation of the body $\Omega$.

**Theorem 6.1** Let $W : M_+^{n \times n} \to \mathbb{R}$ be a constitutive function and satisfy (2.2). Then

$$W(du) = W(G^{1/2}(u(x))\nabla u(x)G^{-1/2}(x)),$$  \hspace{1cm} (6.1)

where $\nabla u$ is the gradient matrix of the map $u$ in the Euclidean space $\mathbb{R}^n$.

**Proof** Let $x \in \mathbb{R}^n$ be given. Let $\{e_i\}$ and $\{E_i\}$ be orthonormal bases of $(\mathbb{R}_x^n, g(x))$ and $(\mathbb{R}_u^n, g \circ u(x))$ with positive orientation, respectively.

Let

$$e_i = \sum_{j=1}^n \alpha_{ij} \partial x_j|_x, \quad E_i = \sum_{j=1}^n \beta_{ij} \partial x_j|_{u(x)}.$$ 

Then the relations,

$$\delta_{ij} = \sum_{k=1}^n \alpha_{ik} \langle \partial x_k, e_j \rangle_g, \quad \text{for} \quad 1 \leq i, k \leq n,$$

imply that

$$\left(\alpha_{ij}\right)\left(\langle \partial x_i, e_j \rangle_g\right) = I.$$  \hspace{1cm} (6.2)
On the other hand, the relations,
\[ \langle e_i, \partial_{x_j} \rangle_g = \sum_{k=1}^{n} \alpha_{ik} \langle \partial_{x_k}, \partial_{x_j} \rangle_g = \sum_{k=1}^{n} \alpha_{ik}g_{kj}(x), \quad \text{for } 1 \leq i, j \leq n, \]
yield
\[ \left( \langle e_i, \partial_{x_j} \rangle_g \right) = \left( \alpha_{ij} \right)G(x), \quad (6.3) \]
where \( G(x) = \left( g_{ij}(x) \right) \).

It follows from formulas (6.2) and (6.3) that
\[ \left( \alpha_{ij} \right)G(x)\left( \alpha_{ij} \right)^T = I, \]
where the superscript "T" denotes the transpose. A similar computation gives
\[ \left( \beta_{ij} \right)G(u(x))\left( \beta_{ij} \right)^T = I. \]

Noting the relations
\[ u_\ast \partial_{x_i} = \sum_{k=1}^{n} u_{kx_i} \partial_{x_k}|u(x), \]
we obtain
\[ d\mathbf{u}(E_i, e_j) = \langle E_i, u_\ast e_j \rangle_g \circ \mathbf{u}(x) = \sum_{klp} \beta_{ik}g_{kp}(u(x))u_{px_l}\alpha_{jl}, \]
that is,
\[ \left( d\mathbf{u}(E_i, e_j) \right) = \left( \beta_{ij} \right)G(u(x))\nabla \mathbf{u}(x)\left( \alpha_{ij} \right)^T = QG^{1/2}(u(x))\nabla u(x)G^{-1/2}(x)R, \]
where
\[ Q = \left( \beta_{ij} \right)G^{1/2}(u(x)), \quad R = G^{1/2}(x)\left( \alpha_{ij} \right)^T, \]
belong to \( \text{SO}(n) \). Thus, formula (6.1) follows from assumption (2.2). \( \square \)

**Total Stored Energy** Let \( dg \) denote the volume element of \( \mathbb{R}^n \) in the metric \( g \).
Then
\[ dg = \det^{1/2}G(x)dx \quad \text{for } x \in \mathbb{R}^n \]
where \( dx \) is the volume element of \( \mathbb{R}^n \) in the Euclidean metric. Let \( W : M_n^{n \times n} \to \mathbb{R} \) be a constitutive function and satisfy (2.2). By Theorem 6.1, in a typical deformation in which the particle \( x \in \Omega \) is displaced to \( \mathbf{u}(x) \in \mathbb{R}^n \) energy (2.5) becomes
\[ E(\mathbf{u}) = \int_{\Omega} \hat{W}\left( x, \mathbf{u}(x), \nabla \mathbf{u}(x) \right)dx, \quad (6.4) \]
where
\[ \hat{W}(x,y,F) = W(G^{1/2}(y)FG^{-1/2}(x))\det^{1/2}G(x), \quad (6.5) \]
for \( x, y \in \mathbb{R}^n \) and \( F \in M_{++}^{n \times n} \), is the total constitutive law. Then
\[
\hat{W}(x, y, F) = \Phi \left( v_1(x, y, F), \ldots, v_n(x, y, F) \right) \frac{1}{\sqrt{\det G(x)}},
\]
where \( v_1(x, y, F), \ldots, v_n(x, y, F) \) denote the singular values of \( G^{1/2}(x)FG^{-1/2}(y) \) for \( x, y \in \mathbb{R}^n \) and \( F \in M_{++}^{n \times n} \). Thus, formula (6.4) is composed by the constitutive function \( W \) and the matrices \( G(x) \) together.

**Example 6.1** Introduce a metric on \( \mathbb{R}^n \) by
\[
g = e^{2a(r)}I \quad \text{for} \quad r = |x| \in \mathbb{R}^n,
\]
where \( a(s) \) is a \( C^2 \) function on \([0, \infty)\) and \( I \) is the unit matrix. Suppose a constitutive function \( W = \Phi \) is given by (4.8). Then the energy density (6.5) is
\[
\hat{W}(x, y, F) = W(q(x, y)F)e^{a(|x|)} = \sum_{i=1}^{n} \phi \left( q(x, y)v_i \right) e^{na(|x|)} + h \left( q^n(x, y)v_1 \cdots v_n \right) e^{na(|x|)}
\]
for \( x, y \in \mathbb{R}^n \), where
\[
q(x, y) = \frac{e^{a(|y|)}}{e^{a(|x|)}}
\]
and \( v_1, \ldots, v_n \) denote the singular values of \( F \) for \( F \in M_{++}^{n \times n} \).

We take \( o \) to be the origin 0 in \( \mathbb{R}^n \) to consider what conditions on \( g \) are needed for \((\mathbb{R}^n, g, o)\) being a model.

**Proposition 6.1** Let \( G(x) \) satisfy
\[
G(x)x = b(\eta(x))Ax \quad \text{for} \quad x \in \mathbb{R}^n,
\]
where \( \eta(x) = \sqrt{\langle Ax, x \rangle} \), \( A \) is a symmetric, positive, and constant matrix, and \( b \) is a positive \( C^2 \) function on \([0, \infty)\) with \( b(0) = 1 \). Then
(i) \( (\mathbb{R}^n, g, o) \) is a model;
(ii) A geodesic ball of \((\mathbb{R}^n, g)\) centered at \( o \) is an ellipsoids of the Euclidean space \( \mathbb{R}^n \).

**Proof** (i) We introduce one more metric \( g_1 \) on \( \mathbb{R}^n \) by
\[
g_A(X, Y) = \langle AX, Y \rangle \quad \text{for} \quad X, Y \in \mathbb{R}_x^n, \ x \in \mathbb{R}^n.
\]
Clearly, (6.7) implies \( G(0) = g_A \). We now have three metrics on \( \mathbb{R}^n \), that are \( g, g_A \), and the Euclidean metric \( \langle \cdot, \cdot \rangle \).

Since \( A \) is constant, for \( x \in \mathbb{R}^n \) given, \( x \neq 0 \), the curve
\[
\alpha(t) = t \frac{x}{\eta(x)} \quad \text{for} \quad t \geq 0,
\]
for \( x \neq 0 \),
is a normal geodesic in \((\mathbb{R}^n, g_A)\) initiating at \(o\). Thus, \(\eta(x)\) is the distance function of \((\mathbb{R}^n, g_A)\) from \(x \in \mathbb{R}^n\) to \(o\) and

\[
\nabla_A \eta = \frac{x}{\eta} \quad \text{for} \quad x \in \mathbb{R}^n, \ x \neq 0,
\]

where \(\nabla_A\) denotes the Levi-Civita connection of \((\mathbb{R}^n, g_A)\).

Let \(\sigma\) be the solution to problem

\[
\sigma'(t) = b^{-1/2}(\sigma(t)) \quad \text{for} \quad t > 0, \quad \sigma(0) = 0.
\]

Set

\[
\gamma(t) = \sigma(t)X(x) \quad \text{for} \quad t > 0, \tag{6.8}
\]

where \(X(x) = \nabla_A \eta\). Next, we prove that \(\gamma(t)\) is a normal geodesic of \((\mathbb{R}^n, g)\).

Let \(D\) be the Levi-Civita connection of \((\mathbb{R}^n, g)\). We compute \(D_X X\). Let \(x_0 \in \mathbb{R}^n\) be given. Let \(Z\) be a constant vector satisfying \(\langle x_0, Z \rangle_{g(x_0)} = 0\). Then

\[
\langle (\nabla_A \eta)(x_0), Z \rangle_A = \frac{1}{\eta}(Ax_0, Z) = \frac{1}{\eta(x_0)b(\eta(x_0))}g(x_0)x_0, Z) = 0,
\]

\[
[X(x_0), Z] = (\nabla_A)_{X(x_0)}Z - (\nabla_A)_{Z} \nabla_A \eta = -\frac{1}{\eta(x_0)}Z \quad \text{(since} \ A \text{is constant)},
\]

\[
\langle X, Z \rangle_g = \langle G(x)\frac{x}{\eta}, Z \rangle = b(\eta)\langle \nabla_A \eta, Z \rangle_A.
\]

We have

\[
\langle D_X X, Z \rangle_{g(x_0)} = X \langle X, Z \rangle_g - \langle X, D_X X \rangle_g = X(b)\langle \nabla_A \eta(x_0), Z \rangle_A - \langle X, D_Z X \rangle_g - \langle X, [X, Z] \rangle_g
\]

\[
= -\frac{1}{2}Z|X|^2_g - \frac{1}{2}Z(b)|\nabla_A \eta|^2_A = -\frac{b'(\eta)}{2}\langle \nabla_A \eta, Z \rangle_A = 0,
\]

for all \(Z \in \mathbb{R}^n\) satisfying \(\langle x_0, Z \rangle_{g(x_0)} = 0\). Thus, we obtain

\[
D_{\gamma_0(t)}X = \langle D_X X, \frac{x_0}{g(x_0)} \rangle_{g(x_0)}\frac{x_0}{g(x_0)} = \frac{b'(\eta)}{2b(\eta)}X(x_0).
\]

It follows that

\[
D_{\gamma_0(t)}\gamma_0 = \dot{\sigma}(t)X + \sigma^2(t)D_X X = [\dot{\sigma}(t) + \frac{b'(\eta)}{2b(\eta)}]X(x_0) = 0,
\]

where \(\gamma_0(t) = \sigma(t)\frac{x_0}{\eta(x_0)}\).

Let \(\rho\) be the distance function from \(x \in \mathbb{R}^n\) to \(o\) in the metric \(g\). By (6.8), we obtain

\[
\rho(x) = \sigma^{-1}(\eta(x)) \quad \text{for} \quad x \in \mathbb{R}^n,
\]

and

\[
\exp_o \frac{x}{\eta(x)} = \sigma(t)\frac{x}{\eta(x)} \quad \text{for} \quad t \geq 0, \ x \in \mathbb{R}^n. \tag{6.9}
\]
Let $\psi : (\mathbb{R}^n, g_A) \to (\mathbb{R}^n, g_A)$ be a linear isometry. Then the operator $\Psi$ of (2.11) is given by

$$\Psi(x) = \psi x \quad \text{for} \quad x \in \mathbb{R}^n,$$

which imply that $\Psi : \exp_o \Sigma(o) \to \exp_o \Sigma(o)$ is an isometry.

(ii) Let $B(t)$ be the geodesic ball centered at $o$ with radius $t > 0$. It follows from (6.9) that

$$B(t) = \left\{ x \in \mathbb{R}^n \big| \sqrt{\langle Ax, y \rangle} < \sigma(t), 0 \leq s < t \right\},$$

that are ellipsoids in $\mathbb{R}^n$.

Remark 6.1 If $A = \text{diag} \left\{ \frac{1}{a_1}, \ldots, \frac{1}{a_n} \right\}$, then the geodesic balls (6.10) are

$$B(t) = \left\{ y \in \mathbb{R}^n \big| \sum_{i=1}^{n} \frac{y_i^2}{a_i^2} < \sigma^2(t) \right\}$$

and the geodesic spheres are

$$\left\{ y \in \mathbb{R}^n \big| \sum_{i=1}^{n} \frac{y_i^2}{a_i^2} = \sigma^2(t) \right\} \quad \text{for} \quad t > 0.$$

Remark 6.2 Let

$$\Gamma_{ij}(x) = x_i \partial x_j - x_j \partial x_i \quad x \in \mathbb{R}^n, \quad 1 \leq i, j \leq n.$$

Let $A$ be a constant matrix. Then matrices

$$G(x) = b_1(\eta)A + b_2(\eta)Ax \otimes Ax + \sum_{ij} b_{ij}(x) \Gamma_{ij}(x) \otimes \Gamma_{ij}(x) \quad \text{for} \quad x \in \mathbb{R}^n$$

meet conditions (6.7) since

$$G(x)x = [b_1(\eta) + b_2(\eta)\eta^2]Ax \quad \text{for} \quad x \in \mathbb{R}^n.$$

Let $G(x)$ satisfy assumptions (6.7). Then radial deformations are given by

$$u(x) = \sigma \circ \varphi(\rho) \frac{x}{\eta(x)} \quad \rho = \sigma^{-1}(\eta(x)), \ x \in \mathbb{R}^n.$$

Thus, all the theorems, corollaries, and propositions in Sections 2-4 hold true for radial deformations above if we apply them to the model $(\mathbb{R}^n, g, o)$.

Finally, let us consider some situations for which the radial curvature assumptions in Theorem 4.8 hold. Let $\kappa$ be a $C^1$ function in $[0, \infty)$ such that

$$\int_0^\infty s\kappa_+(s)ds \leq 1, \quad \int_0^\infty s\kappa_-(s)ds < \infty,$$

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where $\kappa_+ = \max\{0, \kappa\}$ and $\kappa_- = \min\{0, -\kappa\}$. Suppose $f$ is the solution to problem

$$f''(t) + \kappa(t)f(t) = 0 \quad \text{for} \quad t > 0; \quad f(0) = 0, \quad f'(0) = 1.$$ 

By similar arguments as in the proof of Proposition 4.2 in [4], we obtain the following.

**Proposition 6.2** Let $A$ be a symmetric, positive, and constant matrix. Let

$$G(x) = \frac{1}{f^2(\eta)} A + \frac{1}{\eta^2} \left[ 1 - \frac{f^2(\eta)}{\eta^2} \right] Ax \otimes Ax \quad \text{for} \quad x \in \mathbb{R}^n,$$

where $\eta = \sqrt{\langle Ax, x \rangle}$. Then

(i) $G(x)$ are symmetric and positive for all $x \in \mathbb{R}^n$.

(ii) $(\mathbb{R}^n, g, o)$ is a model.

(iii) The radial curvature is $\kappa(t)$ for $t \geq 0$.

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