UNIFORM STABILITY ESTIMATE FOR THE
VLASOV-POISSON-BOLTZMANN SYSTEM

Hao Wang
School of Science, Wuhan Institute of Technology
Wuhan, 430072, China
School of Mathematics and Statistics, Wuhan University
Wuhan, 430072, China

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Abstract. This paper is concerned with the uniform stability estimate to the Cauchy problem of the Vlasov-Poisson-Boltzmann system. Our analysis is based on compensating function introduced by Kawashima and the standard energy method.

1. Introduction. The Vlasov-Poisson-Boltzmann system (called VPB in the sequel for simplicity) is a physical model describing the time evolution of dilute charged particles (for example, electrons) in the absence of an external magnetic field [2]. When the constant background charge density is normalized to be unit, the VPB system reads

$$\begin{align*}
\partial_t F + v \cdot \nabla_x F + \nabla_x \Phi \cdot \nabla_v F &= Q(F, F), \\
\Delta_x \Phi &= \int_{\mathbb{R}^3} F dv - 1,
\end{align*}$$

(1)

with initial data $F(0, x, v) = F_0(x, v)$. Here, $F(t, x, v)$ represent the number density function at time $t \geq 0$, with spatial coordinate $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The bilinear collision operator $Q$ with hard-sphere interaction [2] is defined by

$$Q(G, F)(t, x, v) = \int_{\mathbb{R}^3} \int_{S^2} |(v - v_s) \cdot \omega|(G'_s F' - G_s F) dv_s d\omega,$$

$$F' = F(t, x, v'), \quad F = F(t, x, v), \quad G'_s = G(t, x, v'_s), \quad G_s = G(t, x, v_s),$$

where $(v, v_s)$ and $(v', v'_s)$, denoting velocities of two particles before and after their collisions respectively, satisfy

$$v' = v - [(v - v_s) \cdot \omega] \omega, \quad v'_s = v_s + [(v - v_s) \cdot \omega] \omega, \quad \omega \in S^2.$$ 

The potential function $\Phi = \Phi(t, x)$ generating the self-consistent electric field $\nabla_x \Phi$ in (1) is coupled with $F(t, x, v)$ through the Poisson equation (2).

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We will restrict our attention to the fluctuation around the Maxwellian distribution. Let
\[ \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}. \]
be the normalized global Maxwellian. Set the perturbation \( f = f(t, x, v) \) by
\[ F = \mu + \sqrt{\mu} f. \]
Then \( f \) and \( \Phi \) satisfy the perturbed system:
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_v f - \frac{1}{2} v \cdot \nabla_x \Phi f - \nabla_x \Phi \cdot v \sqrt{\mu} &= Lf + \Gamma(f,f), \\
\Delta_x \Phi &= \int_{\mathbb{R}^3} \sqrt{\mu} f dv, \\
\text{with given initial data} \quad f(0, x, v) = f_0(x, v).
\end{align*}
Here \( L \) and the nonlinear collision term \( \Gamma \) are respectively given by
\[ Lf = \mu^{-1/2} Q(\mu, \sqrt{\mu} f) + \mu^{-1/2} Q(\sqrt{\mu} f, \mu), \]
\[ \Gamma(f,g) = \mu^{-1/2} Q(\sqrt{\mu} f, \sqrt{\mu} g). \]

For the linearized collision operator \( L \), we know that
\[ Lf = -\nu(v) f + Kf, \]
\[ \nu(v) = \int_{\mathbb{R}^3} \int_{S^2} |(v - v_*) \cdot \omega| \mu_* d\omega dv_*, \]
\[ (Kf)(v) = \int_{\mathbb{R}^3} \int_{S^2} |(v - v_*) \cdot \omega| \sqrt{\mu_*} (-\sqrt{\mu} f_* + \sqrt{\mu} f'_* + \sqrt{\mu} f''_* + \sqrt{\mu} f_*') d\omega dv_* \]
\[ = \int_{\mathbb{R}^3} K(v, v_*) f(v_*) dv_*. \]
Here, \( \nu(v) \) is called the collision frequency, which satisfies
\[ \nu(v) \sim (1 + |v|^2)^{1/4}, \]
and \( K \) is a self-adjoint compact operator in \( L^2(\mathbb{R}^3_v) \) with a real symmetric integral kernel \( K(v, v_*) \). The null space of the operator \( L \) is given by
\[ \mathcal{N} = KerL = \text{span}\{ \sqrt{\mu}, v_1\sqrt{\mu}, v_2\sqrt{\mu}, v_3\sqrt{\mu}, |v|^2 \sqrt{\mu} \}. \]
We know that, the linearized collision operator \( L \) is non-positive and, \(-L\) is locally coercive in the sense that there is a positive constant \( \delta_0 \) such that
\[ -\int_{\mathbb{R}^3} fLfdv \geq \delta_0 \int_{\mathbb{R}^3} \nu(v)(\{I - \mathbf{P}\} f)^2 dv, \quad \forall f \in D(L), \]
where \( \mathbf{P} \) denotes the orthogonal projection from \( L^2(\mathbb{R}^3_v) \) to \( \mathcal{N} \) and \( D(L) \) is the domain of \( L \) given by
\[ D(L) = \{ f \in L^2(\mathbb{R}^3_v) | \sqrt{\nu(v)} f \in L^2(\mathbb{R}^3_v) \}. \]
For any given function $f$, one can write

$$
\begin{align*}
PF &= \{a^f(t, x) + b^f(t, x) \cdot v + c^f(t, x)|v|^2\} \mu^{1/2}, \\
\frac{1}{2} f &= \int_{\mathbb{R}^3} (5 - |v|^2) \mu^{1/2} fdv, \\
b^f &= \int_{\mathbb{R}^3} \sqrt{\mu^{1/2} fdv}, \\
c^f &= \frac{1}{6} \int_{\mathbb{R}^3} (|v|^2 - 3) \mu^{1/2} fdv.
\end{align*}
$$

(7)

Thus, we have the macro-micro decomposition introduced in [13],

$$f(t, x, v) = PF(t, x, v) + \{I - P\}f(t, x, v),$$

where $PF$ and $\{I - P\}f$ are called the macroscopic component and the microscopic component of $f(t, x, v)$, respectively. As in [4, 5], for later use, one can rewrite $P$ as

$$
\begin{align*}
PF &= P_0f + P_1f, \\
P_0f &= (a^f + 3c^f)\mu^{1/2}, \\
P_1f &= \{b^f \cdot v + c^f (|v|^2 - 3)\} \mu^{1/2},
\end{align*}
$$

(8)

where $P_0$ and $P_1$ are projectors corresponding to the hyperbolic and parabolic parts of the macroscopic component, respectively.

**Theorem 1.1** ([4], [5]). Let $N \geq 4$. Then there are the equivalent energy functional $E_N(\cdot)$ and the corresponding dissipation rate $D_N(\cdot)$ defined by

$$
\begin{align*}
E_N(f)(t) &\sim \sum_{|\alpha|+|\beta| \leq N} ||\partial^\alpha_\beta f(t)||^2 + \sum_{|\alpha| \leq N} ||\partial^\alpha \nabla_\epsilon \Phi||^2, \\
D_N(f)(t) &\sim \sum_{|\alpha|+|\beta| \leq N} ||\partial^\alpha_\beta \{I - P\}f(t)||^2 + \sum_{|\alpha| \leq N-1} (||\partial^\alpha \nabla_\epsilon Pf||^2 + ||\partial^\alpha P_0f||^2),
\end{align*}
$$

(9)

such that the following holds. For $F_0(x, v) = \mu + \sqrt{\mu} f_0(x, v) \geq 0$, if there exists a sufficiently small $\delta > 0$ such that $E_N(f)(0) \leq \delta$, then the Cauchy problem (3), (4) and (5) of the VPB system admits a unique global solution $f(t, x, v)$ satisfying $F(t, x, v) = \mu + \sqrt{\mu} f(t, x, v) \geq 0$, and

$$
\frac{d}{dt} E_N(f)(t) + \lambda D_N(f)(t) \leq 0, \quad \forall t \geq 0.
$$

(10)

If we further assume

$$
\int_{\mathbb{R}^3} \sqrt{\mu} f_0 dv = 0,
$$

(11)

and

$$
||f_0||_\omega + ||f_0||_{L_1} \leq \delta,
$$

(12)

then one has

$$
||f(t)||_{H^\infty} \leq C\delta (1 + t)^{-3/4}, \quad \forall t \geq 0.
$$

(13)

**Remark 1.** In [5], the third term on the right-hand side of (10) takes the form $||\partial^\alpha (a^f + 3c^f)||$. Here we replace it by $||\partial^\alpha P_0f||$ since

$$
||\partial^\alpha (a^f + 3c^f)|| = ||\partial^\alpha P_0f||,
$$

Here we replace it by $||\partial^\alpha P_0f||$. Since
The main result about the uniform stability estimate of solutions is stated as follows.

**Theorem 1.2.** For $N \geq 5$. Let $F_0' = \mu + \sqrt{\mu f_0'} \geq 0$ and $F_0'' = \mu + \sqrt{\mu f_0''} \geq 0$. If there exists a sufficiently small $\delta > 0$ such that $\max\{E_N(f'(0)), E_N(f''(0))\} \leq \delta$, and $f_0'$, $f_0''$ satisfy (12), (13), then for the solutions $f'(t,x,v), f''(t,x,v)$ obtained in Theorem 1.1, we have the following stability estimate

$$\mathcal{E}_{N-1}(f' - f'')(t) \leq C \mathcal{E}_N(f' - f'')(0), \quad \forall t \geq 0.$$ 

Now we turn to review some former results on the VPB system. The global existence of the renormalized solutions with large initial data to the VPB system was proved by Lions [17] and this result was later extended to the case with boundary in [18]. On the other hand, the global classical solutions to the VPB system near Maxwellian was firstly established in [12] in periodic box. Since then, there have been extensive works on the global solutions to this system in the whole space $\mathbb{R}^3_x$. For the hard-sphere model, the global existence of solutions to the VPB system was proved in [20] and [5] in different function spaces, and the corresponding large-time behavior of solutions were obtained in [22] and [4], respectively. For the case of hard potentials and soft potentials, the authors obtained the global classical solutions and the optimal time rate in [6] and [9]. These literatures listed above are all under Grad’s angular cut-off assumption, readers can refer to [3, 7, 11, 15] for more information. In addition, for the VPB system with non-cutoff case, readers are referred to [8].

In [19], the authors established the uniform stability estimate for the Fokker-Planck-Boltzmann equation in the whole space by using the method of constructing the Kawashima compensating function and the standard energy method. In [5], the authors obtained the global classical solutions to the VPB system, and gained the Lyapunov inequality (11). Motivated by these works, we further consider the stability estimate for the VPB system (3), (4) and (5) in the whole space $\mathbb{R}^3_x$. As a main ingredient in the analysis, we borrow the method in [21] to construct the compensating function, which is helpful to establish a estimate on the macroscopic part. In this paper, when considering the energy estimate for $\tilde{f}$ in (25), the main difficulties come from the following terms

$$\langle \nabla_x \Phi \cdot \nabla_v f'', \tilde{f}\rangle, \quad \langle \nabla_x \Phi \cdot v f'' , \tilde{f}\rangle, \quad \langle \nabla_x \Phi' \cdot v \tilde{f} , \tilde{f}\rangle.$$ 

In order to deal with the above three terms, we also need to use the time decay of solutions to the VPB system [4] and the macroscopic balance laws (24), which can be seen in Lemma 3.4. Lastly, the nonlinearity in (25) involves an unbounded term $\nabla_x \Phi \cdot v f''$, by the high-order $v$-derivatives of $f''$, the higher order dissipation rate will appear in the right-hand side of the last energy estimate, so we can only get the low order uniform stability estimate.

The rest of this paper will be organized as follows. In Section 2, we give the definition of Kawashima’s compensating function and construct the compensating function for the linearized VPB system. In Section 3, we give some refined energy estimates and the proof of main result.

**Notation:** In this paper, $C$ denotes some positive constant (generally large) and $\lambda$ denotes some positive constant (generally small), where both $C$ and $\lambda$ may take different values at different places. In addition, $A \sim B$ means $\lambda_1 A \leq B \leq \lambda_2 A$ for two generic constants $\lambda_1 > 0$ and $\lambda_2 > 0$. 

this can be seen by the second equation of (8).
In the following, we define the mixed Lebesgue space \( Z_1 = L^2(\mathbb{R}^3; L^1(\mathbb{R}_x^3)) \) with the norm

\[
||f||_{Z_1} = \left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |f(x, v)| dv \right)^2 dx \right)^{1/2}.
\]

For an integrable function \( f : \mathbb{R}^3 \to \mathbb{R} \), its Fourier transform is defined by

\[
\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^3} e^{-2\pi i x \cdot \xi} f(x) dx, \quad x : \xi = \sum_{j=1}^{3} x_j \xi_j,
\]

for \( \xi \in \mathbb{R}^3 \), where \( i = \sqrt{-1} \in \mathbb{C} \). Next, we let \( \langle \cdot, \cdot \rangle \) denote the standard \( L^2 \) inner product in \( \mathbb{R}^3 \), with its \( L^2 \) norm given by \( |\cdot|_2 \), and let \( (\cdot,\cdot) \) denote the \( L^2 \) inner product in \( \mathbb{R}_x^3 \times \mathbb{R}_v^3 \) or in \( \mathbb{R}_x^3 \) with the \( L^2 \) norm given by \( ||\cdot|| \). We use \( ||\cdot||_{H^m} \), \( ||\cdot||_{H^{m,\alpha}} \) for the norm of the Hilbert space \( H^m(\mathbb{R}_x^3 \times \mathbb{R}_v^3) \), \( H^{m,\alpha}(\mathbb{R}_x^3) \), respectively. We also define

\[
|f|^2 = \langle \nu(v) f, f \rangle, \quad ||f||^2_2 = (\nu(v) f, f).
\]

The multi-indices \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta = (\beta_1, \beta_2, \beta_3) \) will be used to record spatial and velocity derivatives respectively. Specifically,

\[
\partial_{\beta}^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}.
\]

Similarly, the notation \( \partial^\alpha \) will be used when \( \beta = 0 \), and likewise for \( \partial_{\beta}. \) The length of \( \alpha \) is \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \). \( \alpha' \leq \alpha \) means that each component of \( \alpha' \) is not greater than that of \( \alpha \), and \( \alpha' < \alpha \) means that \( \alpha' \leq \alpha \) and \( |\alpha'| < |\alpha| \).

2. Compensating function. In this section, we will construct the compensating function for the VPB system (3)–(5), which is inspired by the two species Vlasov-Poisson-Boltzmann system in [21]. This method follows from the works of Kawashima [16] and Glassey [10] for the Boltzmann equation.

Under the Fourier transform, \( v \cdot \nabla_x f \mapsto i(v \cdot \xi) \hat{f} \). Now we regard \( v \cdot \xi \) as a linear operator in \( L^2(\mathbb{R}^3) \) from \( Ker L \) to \( \hat{W} \), which is the subspace of \( L^2(\mathbb{R}_x^3) \) spanned by the thirteen functions \( \varphi_j \mu^{1/2}, j = 1, 2, ..., 13 \), defined in the following:

\[
\hat{W} = \text{span}\{\varphi_j \mu^{1/2}| j = 1, 2, ..., 13\},
\]

where

\[
\varphi_1 = 1; \quad \varphi_{j+1} = v_j (j = 1, 2, 3); \quad \varphi_{j+4} = v_j^2 (j = 1, 2, 3);
\]

\[
\varphi_8 = v_1 v_2; \quad \varphi_9 = v_2 v_3; \quad \varphi_{10} = v_3 v_1; \quad \varphi_{j+10} = |v|^2 v_j (j = 1, 2, 3).
\]

We denote an orthonormal basis for \( \hat{W} \) by \( e_k, k = 1, 2, ..., 13 \), where

\[
[e_{1}, ..., e_{13}] = [\varphi_1 \mu^{1/2}, ..., \varphi_{13} \mu^{1/2}]_{A_{13 \times 13}}, \quad det A \neq 0.
\]

By a standard Gram-Schmidt procedure, one can obtain the orthonormal vectors \( \{e_k\}_{k=1}^{13} \) as follows:

\[
e_1 = \sqrt{\mu}; \quad e_{j+1} = v_j \sqrt{\mu} (j = 1, 2, 3);
\]

\[
e_5 = \frac{1}{\sqrt{6}} (|v|^2 - 3) \sqrt{\mu};
\]

\[
e_6 = \sum_{j=1}^3 e_{2j} (v_j^2 - 1) \sqrt{\mu}; \quad e_7 = \sum_{j=1}^3 e_{3j} (v_j^2 - 1) \sqrt{\mu};
\]

\[
e_8 = v_1 v_2 \sqrt{\mu}; \quad e_9 = v_3 v_2 \sqrt{\mu}; \quad e_{10} = v_1 v_3 \sqrt{\mu};
\]
Here, the constant vectors \((c_{i1}, c_{i2}, c_{i3})\), \(i = 2, 3\), which satisfy the condition \(\sum_{j=1}^{3} c_{ij} = 0\), will be given in Lemma 3.11.1 of [10]. It is easy to see that \(N = \text{span}\{e_1, e_2, e_3, e_4, e_5\}\).

Let \(P_2\) be the orthogonal projection from \(L^2(\mathbb{R}^3_+)\) into \(\mathbb{W}\):

\[
P_2 f = \sum_{k=1}^{13} \langle f, e_k \rangle e_k.
\]

In what follows, we consider the linearized VPB system

\[
\partial_t f + v \cdot \nabla_x f - Lf = g + \nabla_x \Phi \cdot v \sqrt{\mu},
\]

with initial data \(f(0, x, v) = f_0(x, v)\) and a source term \(g = g(t, x, v)\).

Set \(W_k = \langle f, e_k \rangle\), \(k = 1, 2, ..., 13\), and \(W = [W_1, ..., W_{13}]^T\). Then by using equation (15), we obtain

\[
\partial_t W + \sum_j V^j \partial_{x_j} W - \bar{L}W = \bar{g} + R,
\]

where \(V^j (j = 1, 2, 3)\) and \(\bar{L}\) are the symmetric matrices given by

\[
\bar{L} = \{(L(e_i), e_k)\}_{k, l=1}^{13}, \quad V(\xi) = \sum_{j=1}^{3} V^j \xi_j = \{(v \cdot \xi) e_k, e_l\}_{k, l=1}^{13},
\]

and \(\bar{g}\) is the vector component \(\langle g, e_k \rangle\). Here \(R\) denotes the remaing term which contains either the factor \(\langle I - P_2 \rangle f\) or \(\nabla_x \Phi(t, x)\).

For later use, we write \(\text{Re} z\) for the real part of \(z \in \mathbb{C}\) and

\[
W = [W_I, W_{II}]^T, \quad W_I = [W_1, ..., W_3]^T, \quad W_{II} = [W_6, ..., W_{13}]^T.
\]

With the above preparation, we introduce the following definition of compensating function for (15).

**Definition 2.1.** Let \(S(w), w \in S^2\), be a bounded linear operator on \(L^2(\mathbb{R}^3)\). \(S\) is called a compensating function for equation (15) if

(i) \(S(\cdot)\) is \(C^\infty\) on \(S^2\) with values in the space of bounded linear operators on \(L^2(\mathbb{R}^3)\), and \(S(-w) = S(w)\) for all \(w \in S^2\);

(ii) \(i S(w)\) is self-adjoint on \(L^2(\mathbb{R}^3)\) for all \(w \in S^2\);

(iii) There exists constant \(c_0 > 0\) such that

\[
\text{Re} \langle S(w)(v \cdot w)f, f \rangle - \langle Lf, f \rangle \geq c_0(\|Pf\|_2^2 + \|I - P\|f\|_2^2).
\]

Next, we give the following lemma in order to construct the compensating function of (15), which has been proved in Lemma 3.11.1 of [10].

**Lemma 2.2.** There exists \(13 \times 13\) real constant entry skew-symmetric matrices \(R^j (j = 1, 2, 3)\) such that for

\[
R(w) = \sum_{j=1}^{3} R^j w_j,
\]

we have

\[
\text{Re} \langle R(w)V(w)W, W \rangle \geq c_1\|W_I\|^2 - c_2\|W_{II}\|^2
\]

for some positive constants \(c_1, c_2\). Here \(\langle \cdot, \cdot \rangle\) is the inner product on \(\mathbb{C}^{13}\).
Now we are ready to exhibit a compensating function for \((15)\). We set \(R(w)\) in Lemma 2.2 as
\[
R(w) = \{r_{ij}(w)\}_{i,j=1}^{13}.
\]
For \(w \in S^2\) and some constant \(\beta > 0\) to be determined later, we set
\[
S(w)f = \sum_{k,l=1}^{13} \beta r_{kl}(w) \langle f, e_l \rangle e_k.
\]
\[\text{Lemma 2.3 (see [10, 21]). There exists } \beta > 0, \text{ such that } S(w) \text{ is a compensating function for } (15). \text{ Moreover, } S(w): L^2(\mathbb{R}^3) \rightarrow \tilde{W}.\]

By using the compensating function constructed above, we will gain an energy estimate on the linearized VPB system. Here we omit it for simplicity, since the following estimate can be obtained by using the same method in [21].

\[\text{Lemma 2.4. For equation } (15). \text{ By choosing } \kappa > 0 \text{ small enough, there exist } \delta_1, \delta_2 > 0 \text{ such that}\]
\[
\partial_t \{[(1 + |\xi|^2)(\Delta_x \tilde{\Phi})^2 - \kappa |\xi| |w f(\tilde{f}, \tilde{\Phi})|] + \delta_1 (1 + |\xi|^2) |(I - P) f|_v^2
+ \delta_2 |\xi|^2 |Pf|_v^2 + \delta_2 |\Delta_x \tilde{\Phi}|^2 \leq (1 + |\xi|^2) |\Re(\tilde{f}, \tilde{g}) - \Re(\tilde{\mu}, \tilde{g})| \tilde{\Phi}
+ C \sum_{l=1}^{13} |\langle \hat{g}, e_l \rangle|^2.\]

3. \textbf{Proof of the main result.} In this section, we devote ourselves to obtaining the uniform stability estimate of solutions to the VPB system \((3)\), \((4)\) and \((5)\).

Firstly, we give some lemmas to be used later.

\[\text{Lemma 3.1 (see [1]). Let } f = f(x) \in H^2(\mathbb{R}^3). \text{ Then}\]
\[\begin{align*}
(i) ||f||_{L^\infty} & \leq C ||\nabla f||_{L^2} ||\nabla^2 f||^{\frac{1}{2}} \leq C ||\nabla f||_{H^2}; \\
(ii) ||f||_{L^2} & \leq C ||\nabla f||_{L^2}; \\
(iii) ||f||_{L^q} & \leq C ||f||_{H^1}, \text{ for any } 2 \leq q \leq 6.
\end{align*}\]

\[\text{Lemma 2.2 (see [12, 14]).}\]
\[
- \langle \partial_{\beta} L f, \partial_{\beta} f \rangle \geq \frac{1}{2} ||\partial_{\beta} f||_v^2 - C_{\beta} ||f||_v^2.
\]
\[
|\langle \partial_{\beta} \Gamma(f, g), h \rangle| \leq C \sum_{\beta_1 + \beta_2 \leq \beta} (|\partial_{\beta_1} f|_2 |\partial_{\beta_2} g|_v + |\partial_{\beta_1} g|_2 |\partial_{\beta_2} f|_v) h|_v.
\]

\[\text{Lemma 3.3. Let } |\alpha| + |\beta| \leq N \text{ and } N \geq 4, \text{ for any } \eta > 0, \text{ there exists a constant } C_{\eta} > 0 \text{ such that}\]
\[
|\langle \partial^\alpha \Gamma(f, g), \partial^\alpha h \rangle| \leq \eta ||\partial^\alpha h||_v^2 + C_{\eta} [\mathcal{E}_N(f)(t) \mathcal{D}_N(g)(t) + \mathcal{E}_N(g)(t) \mathcal{D}_N(f)(t)],
\]
and
\[
|\langle \partial^\beta \Gamma(f, g), \partial^\beta (I - P) h \rangle| \leq \eta ||\partial^\beta (I - P) h||_v^2 + C_{\eta} [\mathcal{E}_N(f)(t) \mathcal{D}_N(g)(t) + \mathcal{E}_N(g)(t) \mathcal{D}_N(f)(t)].
\]

\[
\text{Proof. We just consider the estimate } (20) \text{ and the estimate } (21) \text{ can be obtained using the similar procedures}. \text{ Notice that}\]
\[
\partial^\alpha \Gamma(f, g) = \sum_{\alpha' \leq \alpha} C^\alpha_{\alpha'} \Gamma(\partial^{\alpha' - \alpha} f, \partial^{\alpha'} g)
\]
and
\[ \Gamma(f, g) = \Gamma(P_f, P_g) + \Gamma((I - P)f, P_g) + \Gamma((I - P)f, (I - P)g) + \Gamma(P_f, (I - P)g). \]

By using (19), we obtain
\[
\| (\partial^\alpha \Gamma(P_f, P_g), \partial^\alpha h) \|
\leq C \sum_{\alpha' \leq \alpha} \int_{\mathbb{R}^3} (|\partial^{\alpha - \alpha'} P f|_2 |\partial^{\alpha'} P g|_\nu + |\partial^{\alpha'} P g|_2 |\partial^{\alpha - \alpha'} P f|_\nu) |\partial^\alpha h|_\nu dx. \quad (22)
\]

We only consider the first term in the right-hand side of (22) because the second term can be estimated similarly. If $|\alpha - \alpha'| \leq N/2$, for any $\eta > 0$, it holds that
\[
\int_{\mathbb{R}^3} |\partial^{\alpha - \alpha'} P f|_2 |\partial^{\alpha'} P g|_\nu |\partial^\alpha h|_\nu dx
\leq \eta ||\partial^\alpha h||^2 + C_\eta \sup_{x \in \mathbb{R}^3} |\partial^{\alpha - \alpha'} P f|_2^2 \int_{\mathbb{R}^3} |\partial^{\alpha'} P g|_\nu^2 dx
\leq \eta ||\partial^\alpha h||^2 + C_\eta \sum_{|\alpha| \leq 1} ||\nabla_x \partial^\alpha \partial^{\alpha - \alpha'} P f||^2 ||\partial^\alpha P g||^2 _\nu
\leq \eta ||\partial^\alpha h||^2 + C_\eta ||\partial^\alpha P g||^2 _\nu \sum_{|\alpha| \leq 1} ||\nabla_x \partial^\alpha \partial^{\alpha - \alpha'} P f||^2
\leq \eta ||\partial^\alpha h||^2 + C_\eta E_N(g(t))D_N(f(t)),
\]
where we have used Lemma 3.1 and the exponential decay of $P g$ in $v$. The case of $|\alpha - \alpha'| > N/2$ is similar, and we obtain the same result above.

By using (19), we have
\[
| (\partial^\alpha \Gamma((I - P)f, (I - P)g), \partial^\alpha h) |
\leq C \sum_{\alpha' \leq \alpha} \left\{ \int_{\mathbb{R}^3} |\partial^{\alpha - \alpha'} (I - P) f|_2 |\partial^{\alpha'} (I - P) g|_\nu |\partial^\alpha h|_\nu dx + \int_{\mathbb{R}^3} |\partial^{\alpha'} (I - P) g|_2 |\partial^{\alpha - \alpha'} (I - P) f|_\nu |\partial^\alpha h|_\nu dx \right\}. \quad (23)
\]

First, we consider the first term in (23). If $|\alpha - \alpha'| \geq N/2$, then $|\alpha'| \leq N/2$, by the Sobolev imbedding, i.e. $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we obtain
\[
\int_{\mathbb{R}^3} |\partial^{\alpha - \alpha'} (I - P) f|_2 |\partial^{\alpha'} (I - P) g|_\nu |\partial^\alpha h|_\nu dx
\leq \eta ||\partial^\alpha h||^2 _\nu + C_\eta ||\partial^{\alpha - \alpha'} (I - P) f||^2 \sum_{|\alpha| \leq 2} ||\nabla^\alpha \partial^{\alpha'} (I - P) g||^2 _\nu
\leq \eta ||\partial^\alpha h||^2 + C_\eta E_N(f(t))D_N(g(t)).
\]
The case of $|\alpha - \alpha'| < N/2$ is similar, and we obtain the same result.

Next, we consider the second term in (23). If $|\alpha - \alpha'| \geq N/2$, by the Sobolev imbedding, one has
\[
\int_{\mathbb{R}^3} |\partial^{\alpha - \alpha'} (I - P) g|_2 |\partial^{\alpha'} (I - P) f|_\nu |\partial^\alpha h|_\nu dx
\leq \eta ||\partial^\alpha h||^2 + C_\eta ||\partial^{\alpha - \alpha'} (I - P) g||^2 \sum_{|\alpha| \leq 2} ||\nabla^\alpha \partial^{\alpha'} (I - P) f||^2 _\nu
Lemma 3.4. Let $N \geq 5$. For the system (25) and (26), there exist $\delta_1, \delta_2 > 0$ such that
\[
\frac{d}{dt}\left[ \sum_{|\alpha| \leq N-1} (|\partial^\alpha f| + |\partial^\alpha \nabla_x f|) - \eta \int_{\mathbb{R}^3} \int_{\mathbb{R}_+^3} \left| \nabla \cdot (\nabla \Phi) f \right|^2 dx dt \right]
+ \delta_1 \sum_{|\alpha| \leq N-1} ||\partial^\alpha (\mathbf{I} - \mathbf{P}) f||^2 + \delta_2 \sum_{1 \leq |\alpha| \leq N-1} ||\partial^\alpha \mathbf{P} f||^2 + \delta_2 \sum_{|\alpha| \leq N-2} ||\partial^\alpha \mathbf{P}_0 f||^2
\leq C \mathcal{E}_{N-1}(f)(t)(||f'||^2 + ||f''||^2 + \mathcal{D}_{N-1}(f')(t) + \mathcal{D}_{N}(f'')(t))
+ C D_{N-1}(\tilde{f})(t)(\mathcal{E}_{N-1}(f')(t) + \mathcal{E}_{N-1}(f'')(t)),
\]
where $\kappa > 0$ is a small constant.

Proof. In order to use Lemma 2.4 directly, we rewrite (25) as follows:
\[
[\partial_t + v \cdot \nabla_x - L] f = \nabla_x \Phi \cdot v \sqrt{\mu} + g,
\]
where
\[
g = \Gamma(\tilde{f}, f') + \Gamma(f'', f') - \nabla_x \Phi' \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_x f'' + \frac{1}{2} v \cdot \nabla_x \Phi f' + \frac{1}{2} v \cdot \nabla_x \Phi f''.
\]
Based on the equation above, from (17), we multiply the new result by $\alpha \sum_{|\alpha| \leq N - 1} (1 + |\xi|^2)^{N-2}$ and integrate it over $\xi$, to give

$$
\frac{d}{dt} \left[ \sum_{|\alpha| \leq N - 1} (||\partial^\alpha \hat{f}|^2 + ||\partial^\alpha \nabla_x \hat{\Phi}||^2) - \kappa \int \mathbb{R}^3 (1 + |\xi|^2)^{N-2} |\xi| (iS(w)\hat{f}, \hat{\Phi}) d\xi \right]
+ \delta_1 \sum_{|\alpha| \leq N - 1} ||\partial^\alpha (I - P) \hat{f}||^2
+ \delta_2 \sum_{1 \leq |\alpha| \leq N - 1} ||\partial^\alpha P \hat{f}||^2 + \delta_2 \int \mathbb{R}^3 (1 + |\xi|^2)^{N-2} |\Delta_x \hat{\Phi}|^2 d\xi
\leq \int \mathbb{R}^3 (1 + |\xi|^2)^{N-1} Re(\hat{f}, \hat{\Phi}) d\xi - \int \mathbb{R}^3 (1 + |\xi|^2)^{N-1} Re \langle \sqrt{\mu}, \hat{\Phi} \rangle \hat{\Phi} d\xi
+ C \sum_{l=1}^{13} \int \mathbb{R}^3 (1 + |\xi|^2)^{N-2} |\langle \hat{g}, e_l \rangle|^2 d\xi. \tag{28}
$$

By using (8) and (26), we have

$$
\int \mathbb{R}^3 (1 + |\xi|^2)^{N-2} |\Delta_x \hat{\Phi}|^2 d\xi = \sum_{|\alpha| \leq N - 2} ||\partial^\alpha \Delta_x \hat{\Phi}||^2 = \sum_{|\alpha| \leq N - 2} ||\partial^\alpha P_0 \hat{f}||^2. \tag{29}
$$

For the first term on the right-hand side of (28), we obtain

$$
\left| \int \mathbb{R}^3 (1 + |\xi|^2)^{N-1} Re(\hat{f}, \hat{\Phi}) d\xi \right|
\leq \sum_{|\alpha| \leq N - 1} \sum_{|\alpha| \leq N - 1} C \left( \left| \int \mathbb{R}^3 \langle \partial^{\alpha_1} \nabla_x \Phi', v \partial^{\alpha - \alpha_1} \hat{f} + \partial^{\alpha_1} \nabla_x \Phi \cdot v \partial^{\alpha - \alpha_1} \hat{f}', \partial^\alpha \hat{f} \rangle dx \right|
+ \left| \int \mathbb{R}^3 \langle \partial^{\alpha_1} \nabla_x \Phi' \cdot \nabla_v \partial^{\alpha - \alpha_1} \hat{f} + \partial^{\alpha_1} \nabla_x \Phi \cdot \nabla_v \partial^{\alpha - \alpha_1} \hat{f}', \partial^\alpha \hat{f} \rangle dx \right|
+ \left| \int \mathbb{R}^3 \langle \Gamma(\partial^{\alpha_1} \hat{f}, \partial^{\alpha - \alpha_1} \hat{f}', \partial^\alpha \hat{f}) + \Gamma(\partial^{\alpha_1} \hat{f}', \partial^{\alpha - \alpha_1} \hat{f}, \partial^\alpha \hat{f}) \rangle dx \right| \right)
= \sum_{|\alpha| \leq N - 1} \sum_{|\alpha| \leq N - 1} C(M_1 + M_2 + M_3). \tag{30}
$$

For $M_2$, first, when $|\alpha| = 0$, using Lemma 3.1 and integrating by parts in $v$, one has

$$
M_2 = \left| \int \mathbb{R}^3 \langle \nabla_x \Phi' \cdot \nabla_v \hat{f} + \nabla_x \Phi \cdot \nabla_v \hat{f} \rangle dx \right|
= \left| \int \mathbb{R}^3 \langle \nabla_x \Phi \cdot \nabla_v \hat{f} \rangle dx \right|
\leq \eta \sum_{|\alpha| \leq 1} ||\partial^\alpha \Delta_x \hat{\Phi}||^2 + C_0 ||f'||^2 \mathcal{E}_{N-1}(\hat{f}(t)).
$$

When $|\alpha| \geq 1$, $M_2$ is estimated by two cases.

Case 1. $\{|\alpha| \geq 0, \alpha_1 \leq \alpha\} \cap \{|\alpha| \leq |\alpha| - 2\}$. In this case, one has

$$
M_2 = \left| \int \mathbb{R}^3 \langle \partial^{\alpha_1} \nabla_x \Phi' \cdot \nabla_v \partial^{\alpha - \alpha_1} \hat{f} + \partial^{\alpha_1} \nabla_x \Phi \cdot \nabla_v \partial^{\alpha - \alpha_1} \hat{f}', \partial^\alpha \hat{f} \rangle dx \right|
$$
\begin{equation}
\leq \eta \|\partial^\alpha \tilde{f}\|_{\nu}^2 + C_\eta \sum_{|\alpha_1| \geq 1} \sup_{x \in \mathbb{R}^3} |\partial^{\alpha_1} \nabla_x \Phi'|^2 \|\nabla_x \partial^{\alpha_2} \tilde{f}\|^2 + C_\eta (\sup_{x \in \mathbb{R}^3} |\partial^{\alpha_1} \nabla_x \Phi|^2) \|\nabla_x \partial^{\alpha_2} \tilde{f}'\|^2
\end{equation}

\begin{equation}
\leq \eta \|\partial^\alpha \tilde{f}\|_{\nu}^2 + C_\eta \sum_{2 \leq |\alpha| \leq N-1} |\partial^{\alpha} \nabla_x \Phi|^2 \sum_{2 \leq |\alpha| \leq N-2} \|\nabla_x \partial^{\alpha} \tilde{f}\|^2 + C_\eta \sum_{1 \leq |\alpha| \leq N-1} |\partial^{\alpha} \nabla_x \Phi|^2 \sum_{|\alpha| \leq N-1} \|\nabla_x \partial^{\alpha} \tilde{f}'\|^2
\end{equation}

\begin{equation}
\leq \eta \|\partial^\alpha \tilde{f}\|_{\nu}^2 + C_\eta \mathcal{E}_{N-1}(f')(t)\mathcal{D}_{N-1}(\tilde{f})(t) + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)\mathcal{D}_{N-1}(f')(t),
\end{equation}

where we have used Lemma 3.1.

**Case 2.** \{ |\alpha_1| \geq 0, \alpha_1 \leq \alpha \} \cap \{ |\alpha_1| > |\alpha| - 2 \}. In this case, one has

\begin{equation}
M_2 \leq \eta \|\partial^\alpha \tilde{f}\|_{\nu}^2 + C_\eta (\sup_{x \in \mathbb{R}^3} |\nabla_x \partial^{\alpha_1} \tilde{f}'(\tilde{x})||\partial^{\alpha_1} \nabla_x \Phi|^2 \leq \eta \|\partial^\alpha \tilde{f}\|_{\nu}^2 + C_\eta \sum_{1 \leq |\alpha| \leq 3} |\partial^{\alpha} \nabla_x \tilde{\Phi}|^2 \sum_{|\alpha| \leq N-1} \|\partial^{\alpha} \nabla_x \Phi|^2 \leq \eta \|\partial^\alpha \tilde{f}\|_{\nu}^2 + C_\eta \mathcal{E}_{N-1}(f')(t)\mathcal{D}_{N-1}(\tilde{f})(t) + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)\mathcal{D}_{N-1}(f')(t).
\end{equation}

Thus, \(M_2\) is estimated by

\begin{equation}
M_2 \leq \eta \sum_{1 \leq |\alpha| \leq N-1} |\partial^{\alpha} \tilde{f}|^2 + \eta \sum_{|\alpha| \leq 1} |\partial^{\alpha} \Delta_x \tilde{\Phi}|^2 + C_\eta \mathcal{E}_{N-1}(f')(t)\mathcal{D}_{N-1}(\tilde{f})(t) + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)\mathcal{D}_{N-1}(f')(t).
\end{equation}

For \(M_1\), when \(|\alpha| \geq 1\), using the similar method as in \(M_2\), one has

\begin{equation}
M_1 \leq \eta \|\partial^\alpha \tilde{f}\|_{\nu}^2 + C_\eta \mathcal{E}_{N-1}(f')(t)\mathcal{D}_{N-1}(\tilde{f})(t) + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)\mathcal{D}_{N-1}(f')(t).
\end{equation}

While \(|\alpha| = 0\), one has

\begin{equation}
M_1 = |(\nabla_x \Phi' \cdot v \tilde{f} + \nabla_x \tilde{\Phi} \cdot v f'', \tilde{f})| \leq \|\nabla_x \Phi' \cdot v \tilde{f}\| + \|\nabla_x \tilde{\Phi} \cdot v f''\| \leq |M_{1,1}| + |M_{1,2}|.
\end{equation}

\(M_{1,2}\) is estimated by

\begin{equation}
M_{1,2} = (\nabla_x \tilde{\Phi} \cdot v f'', \tilde{f}) \leq (\nabla_x \tilde{\Phi} \cdot v f'', P \tilde{f}) + (\nabla_x \tilde{\Phi} \cdot v f'', \{I - P\} \tilde{f}) \leq \int_{\mathbb{R}^3} \|P \tilde{f}\|_{L^2} \|\nabla_x \tilde{\Phi}\|_{L^2} \|f''\|_{L^2} dv + \eta \|\{I - P\} \tilde{f}\|^2 + C_\eta \|\nabla_x \tilde{\Phi}\|^2 \sup_{x \in \mathbb{R}^3} \|f''\|^2 \leq \eta \|\nabla_x P \tilde{f}\|^2 + \eta \|\{I - P\} \tilde{f}\|^2 + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)\|f''\|^2 + \mathcal{D}_{N-1}(f')(t),
\end{equation}

where we have used Lemma 3.1. For \(M_{1,1}\), one has

\begin{equation}
M_{1,1} = (v \cdot \nabla_x \Phi', |P \tilde{f}|^2) + 2(v \cdot \nabla_x \Phi', (P \tilde{f})\{I - P\} \tilde{f}) + (v \cdot \nabla_x \Phi', |\{I - P\} \tilde{f}|^2).
\end{equation}
Here, the second and third terms on the right-hand side of (35) can be directly estimated as follows:

\[
2(v \cdot \nabla_x \Phi^t, (P \tilde{f})(I - P) \tilde{f}) \leq 2 \int_{\mathbb{R}^3} ||P \tilde{f}||_{L_2^3} ||\nabla_x \Phi^t||_{L_2^3} ||v(I - P) \tilde{f}||_{L_2^3} dv
\]

\[
\leq \eta ||(I - P) \tilde{f}||_{L_2^3}^2 + C_\eta \mathcal{E}_{N-1}(f')(t) D_{N-1}(\tilde{f})(t),
\]

(36)

\[
(v \cdot \nabla_x \Phi^t, ||(I - P) \tilde{f}||_{L_2^3}^2) \leq \eta ||(I - P) \tilde{f}||_{L_2^3}^2 + C_\eta \mathcal{E}_{N-1}(f')(t) D_{N-1}(\tilde{f})(t).
\]

For the first term on the right-hand side of (35), one can apply (7) to and take integration with respect to \(v\) to give

\[
(v \cdot \nabla_x \Phi^t, ||P \tilde{f}||_{L_2^3}^2) = (v \cdot \nabla_x \Phi^t, ||(a^f + b^f \cdot v + c^f|v|^2)\mu^{1/2}|^2)
\]

\[
= 2(\nabla_x \Phi^t, (a^f + c^f|v|^2)b^f \cdot v\mu)
\]

\[
= 2 \sum_{i,j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\partial_{x_i} \Phi^t) \mu (a^f b^f_i v_j v_i + c^f b^f_i v_j v_i |v|^2) dx dv
\]

\[
= 2 \sum_{j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\partial_{x_j} \Phi^t) \mu (a^f b^f_j v_j |v|^2 + c^f b^f_j v_j |v|^2) dx dv
\]

\[
= 2 \int_{\mathbb{R}^3} \nabla_x \Phi^t \cdot b^f (a^f + 5c^f) dx
\]

\[
= 2 \int_{\mathbb{R}^3} \nabla_x \Phi^t \cdot b^f (a^f + 3c^f) dx + 4 \int_{\mathbb{R}^3} \nabla_x \Phi^t \cdot b^f \alpha \tilde{c}^f dx.
\]

Applying Lemma 3.1, the first term on the right-hand side of (37) is estimated by

\[
2 \int_{\mathbb{R}^3} \nabla_x \Phi^t \cdot b^f (a^f + 3c^f) dx \leq C ||\nabla_x \Phi^t||_{L_2^3} ||b^f||_{L_2^3} ||a^f + 3c^f||_{L_2^3}
\]

\[
\leq \eta ||\nabla_x P \tilde{f}||^2 + C_\eta \sum_{|\alpha| \leq 1} ||\partial^\alpha \nabla_x \Phi^t||^2 ||a^f + 3c^f||^2
\]

\[
\leq \eta ||\nabla_x P \tilde{f}||^2 + C_\eta \mathcal{E}_{N-1}(f')(t) D_{N-1}(\tilde{f})(t).
\]

(38)

For the second term on the right-hand side of (37), replacing \(\nabla_x \Phi^t\) by (24) to give

\[
4 \int_{\mathbb{R}^3} \nabla_x \Phi^t \cdot b^f \alpha \tilde{c}^f dx
\]

\[
= 4 \int_{\mathbb{R}^3} (\partial_b b^f + \nabla_x (a^f + 5c^f) + \nabla_x \cdot \langle v \otimes v \sqrt{\mu}, (I - P) f' \rangle - (a^f + 3c^f) \nabla_x \Phi^t) \cdot b^f \alpha \tilde{c}^f dx
\]

\[
\leq 4(I_1 + I_2 + I_3 + I_4).
\]

Now we estimate the four terms above as follows. \(I_2, I_3\) and \(I_4\) are estimated by

\[
I_2 \leq ||b^f||_{L_2^3} ||c^f||_{L_2^3} ||\nabla_x (a^f + 5c^f)||_{L_2^3}
\]

\[
\leq \eta ||\nabla_x P \tilde{f}||^2 + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t) D_{N-1}(f')(t),
\]

\[
I_3 = \int_{\mathbb{R}^3} \nabla_x \cdot \langle v \otimes v \sqrt{\mu}, (I - P) f' \rangle \cdot b^f \alpha \tilde{c}^f dx
\]

\[
\leq \eta ||\nabla_x P \tilde{f}||^2 + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t) D_{N-1}(f')(t),
\]

and

\[
I_4 \leq ||a^f + 3c^f||_{L_2^3} ||\nabla_x \Phi^t||_{L_2^3} ||b^f||_{L_2^3} ||c^f||_{L_2^3}
\]

\[
\leq C \mathcal{E}_{N-1}(f')(t) D_{N-1}(\tilde{f})(t),
\]
For $I_1$, one has
\[
I_1 = \int_{\mathbb{R}^3} \partial_t b^f \cdot b^f c^f dx
= \frac{d}{dt} \int_{\mathbb{R}^3} b^f \cdot b^f c^f dx - \int_{\mathbb{R}^3} b^f \cdot b^f \partial_t c^f dx - \int_{\mathbb{R}^3} b^f \cdot b^f \partial_t c^f dx.
\]
(39)

As in (24)2 and (24)1, one has
\[
\partial_t b^f + \nabla_x (a^f + 5c^f) + \nabla_x \cdot (v \otimes v \sqrt{\mu}, \{I - P\} \tilde{f}) - \nabla_x \tilde{\Phi} = (a^{f''} + 3c^{f''}) \nabla_x \tilde{\Phi}
+ (a^f + 3c^f) \nabla_x \tilde{\Phi}'',
\]
\[
\partial_t c^f + \frac{1}{3} \nabla_x \cdot b^f + \frac{1}{6} \nabla_x \cdot \langle |v|^2 v \sqrt{\mu}, \{I - P\} \tilde{f} \rangle = \frac{1}{6} b^f \cdot \nabla_x \tilde{\Phi} + \frac{1}{6} b^f \cdot \nabla_x \tilde{\Phi}''.
\]

With the above two equalities, one can easily estimate the second and third terms on the right-hand side of (39) as follows:
\[
- \int_{\mathbb{R}^3} b^f \cdot \partial_t b^f c^f dx
\leq \eta \|\{I - P\} \tilde{f}\|^2 + \|\nabla_x P \tilde{f}\|^2 + C_\eta D_{N-1}(\tilde{f})(t) (E_{N-1}(f')(t) + E_{N-1}(f'')(t)),
\]
\[+ C_\eta \|f'\|^2 E_{N-1}(\tilde{f})(t),
\]
\[- \int_{\mathbb{R}^3} b^f \cdot \partial_t c^f dx
\leq \eta \|\{I - P\} \tilde{f}\|^2 + \|\nabla_x P \tilde{f}\|^2 + C_\eta D_{N-1}(\tilde{f})(t) (E_{N-1}(f')(t) + E_{N-1}(f'')(t)).
\]

Therefore, $I_1$ is estimated by
\[
I_1 \leq \frac{d}{dt} \int_{\mathbb{R}^3} b^f \cdot b^f c^f dx + \eta \|\{I - P\} \tilde{f}\|^2 + \|\nabla_x P \tilde{f}\|^2
+ C_\eta \|f'\|^2 E_{N-1}(\tilde{f})(t) + C_\eta D_{N-1}(\tilde{f})(t) (E_{N-1}(f')(t) + E_{N-1}(f'')(t)).
\]

Furthermore, with the estimates for $I_1$, $I_2$, $I_3$ and $I_4$, we have
\[
4 \int_{\mathbb{R}^3} \nabla_x \Phi' \cdot b^f c^f dx
\leq \frac{d}{dt} \int_{\mathbb{R}^3} b^f \cdot b^f c^f dx + \eta \|\{I - P\} \tilde{f}\|^2 + \|\nabla_x P \tilde{f}\|^2 + C_\eta E_{N-1}(\tilde{f})(t) (\|f'\|^2)
+ D_{N-1}(f')(t) + C_\eta D_{N-1}(\tilde{f})(t) (E_{N-1}(f')(t) + E_{N-1}(f'')(t)).
\]
(40)

Thus, when $|\alpha| = 0$, by collecting estimates (33), (34),(35), (36), (37), (38), (40), $M_1$ is estimate by
\[
M_1 \leq 4 \frac{d}{dt} \int_{\mathbb{R}^3} b^f \cdot b^f c^f dx + \eta \|\{I - P\} \tilde{f}\|^2 + \|\nabla_x P \tilde{f}\|^2 + C_\eta E_{N-1}(\tilde{f})(t) (\|f'\|^2)
+ \|f''\|^2 + D_{N-1}(f')(t) + D_{N-1}(f'')(t)
+ C_\eta D_{N-1}(\tilde{f})(t) (E_{N-1}(f')(t) + E_{N-1}(f'')(t)).
\]
(41)

For $M_3$, first, when $|\alpha| = 0$, applying (20) to give
\[
M_3 = \left| \int_{\mathbb{R}^3} \langle \Gamma(f', f') + \Gamma(f', f'), \tilde{f} \rangle dx \right|
= \left| \int_{\mathbb{R}^3} \langle \Gamma(f', f') + \Gamma(f', f'), \{I - P\} \tilde{f} \rangle dx \right|
\]
With the above two estimates, (31), (32), (41) and (30), we have

\[ (1 + \int |\nabla_x \Phi|^2 + \int |\nabla_x \Phi|^2 + \eta \| \{ I - P \} \tilde{f} \|^2 + \eta \| \{ I - P \} \tilde{f} \|^2 + \eta \) \]

Next, when |α| ≥ 1, applying (20) to give

\[ M_3 = \left| \int_{\mathbb{R}^3} \left( \Gamma(\partial^\alpha \tilde{f}, \partial^\alpha \Phi) + \Gamma(\partial^\alpha \Phi, \partial^\alpha \tilde{f}) \right) dx \right| \]

\[ \leq \eta \| \partial^\alpha \tilde{f} \|^2 + \eta \| \nabla_x \Phi \|^2 + \eta \| \{ I - P \} \tilde{f} \|^2 + \eta \]

\[ + C_\eta \mathcal{E}_{N-1}(\tilde{f})(\| f' \|^2 + \| f'' \|^2 + \| D_{N-1}(f')(t) + D_{N-1}(f''')(t) \|) \]

\[ + C_\eta \mathcal{D}_{N-1}(\tilde{f})(\| \mathcal{E}_{N-1}(f')(t) + \mathcal{E}_{N-1}(f''')(t) \|) + 4 \frac{\partial}{\partial t} \int_{\mathbb{R}^3} b' \cdot b' \tilde{f} dx. \]

With the above two estimates, (31), (32), (41) and (30), we have

\[ \left| \int_{\mathbb{R}^3} (1 + |\xi|^2)^{N-1} \text{Re}(\hat{\tilde{f}}, \hat{g}) d\xi \right| \]

\[ \leq \eta \sum_{1 \leq |\alpha| \leq N-1} \| \partial^\alpha \tilde{f} \|^2 + \eta \sum_{1 \leq |\alpha| \leq 1} \| \partial^\alpha \Delta_x \Phi \|^2 \]

\[ + C_\eta \mathcal{E}_{N-1}(\tilde{f})(\| f' \|^2 + \| f'' \|^2 + \| D_{N-1}(f')(t) + D_{N-1}(f''')(t) \|) \]

\[ + C_\eta \mathcal{D}_{N-1}(\tilde{f})(\| \mathcal{E}_{N-1}(f')(t) + \mathcal{E}_{N-1}(f''')(t) \|) + 4 \frac{\partial}{\partial t} \int_{\mathbb{R}^3} b' \cdot b' \tilde{f} dx. \]

For the second term on the right-hand side of (28). Note that

\[ \langle -\nabla_x \Phi' \cdot \nabla_v \tilde{f} + \frac{1}{2} v \cdot \nabla_x \Phi' \tilde{f}, \sqrt{\mu} \rangle = 0, \]

and

\[ \langle -\nabla_x \Phi' \cdot \nabla_v f'' + \frac{1}{2} v \cdot \nabla_x \Phi \tilde{f}', \sqrt{\mu} \rangle = 0. \]

We also know that \( \Gamma(\tilde{f}, f') + \Gamma(f'', \tilde{f}) \) is orthogonal to \( \sqrt{\mu} \). Thus, we have

\[ \int_{\mathbb{R}^3} (1 + |\xi|^2)^{N-1} \text{Re}(\sqrt{\mu}, \hat{g}) \Phi d\xi = 0. \]

For the third term on the right-hand side of (28), we get

\[ \int_{\mathbb{R}^3} (1 + |\xi|^2)^{N-2} |\langle \hat{g}, e_i \rangle|^2 d\xi \leq \sum_{|\alpha| \leq N-2} \int_{\mathbb{R}^3} |\langle \partial^\alpha g, e_i \rangle|^2 dx \]

\[ \leq C(J_1 + J_2 + J_3), \]

where

\[ J_1 = \sum_{|\alpha| \leq N-2} \int_{\mathbb{R}^3} |\langle \partial^\alpha (\Gamma(\tilde{f}, f') + \Gamma(f'', \tilde{f})), e_i \rangle|^2 dx, \]

\[ J_2 = \sum_{|\alpha| \leq N-2} \int_{\mathbb{R}^3} |\langle \partial^\alpha (-\nabla_x \Phi' \cdot \nabla_v \tilde{f} - \nabla_x \Phi' \cdot \nabla_v f''), e_i \rangle|^2 dx, \]

\[ J_3 = \sum_{|\alpha| \leq N-2} \int_{\mathbb{R}^3} |\langle \partial^\alpha \left( \frac{1}{2} v \cdot \nabla_x \Phi' \tilde{f} + \frac{1}{2} v \cdot \nabla_x \Phi f'' \right), e_i \rangle|^2 dx. \]
For $J_2$, integrating by parts to give

$$J_2 \leq C \sum_{|\alpha| \leq N-2} \sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}^3} \left| \left( -\nabla_x \partial^{\alpha_1} \Phi \cdot \nabla_v \partial^{\alpha_1} \hat{f} - \nabla_x \partial^{\alpha_1} \Phi \cdot \nabla_v \partial^{\alpha_1} f'' \right) \cdot e_i \right|^2 dx$$

$$\leq C \sum_{|\alpha| \leq N-2} \sum_{\alpha_1 \leq \alpha} \int_{\mathbb{R}^3} \left( |\nabla_x \partial^{\alpha_1} \Phi|^2 |\partial^{\alpha_1} \hat{f}_1|^2 + |\nabla_x \partial^{\alpha_1} \Phi|^2 |\partial^{\alpha_1} f''_1|^2 \right) dx.$$

As in $M_2$ with $|\alpha| \geq 1$ before, $J_2$ is bounded by

$$J_2 \leq C \mathcal{E}_{N-1}(f')(t) D_{N-1}(\hat{f})(t) + C \mathcal{E}_{N-1}(\hat{f})(t) D_{N-1}(f'')(t). \tag{45}$$

Similarly, $J_3$ is bounded by

$$J_3 \leq C \mathcal{E}_{N-1}(f')(t) D_{N-1}(\hat{f})(t) + C \mathcal{E}_{N-1}(\hat{f})(t) D_{N-1}(f'')(t). \tag{46}$$

For $J_1$, applying (19) to yield

$$J_1 \leq C \sum_{|\alpha| \leq N-2} \sum_{\alpha_1 \leq \alpha} \left\{ \int_{\mathbb{R}^3} \left( |\partial^{\alpha_1} \hat{f}_1|^2 |\partial^{\alpha_1} f''_1|^2 + |\partial^{\alpha_1} f''_1|^2 |\partial^{\alpha_1} \hat{f}_2|^2 \right) dx \right\}$$

$$+ \int_{\mathbb{R}^3} \left( |\partial^{\alpha_1} f''|^2 |\partial^{\alpha_1} \hat{f}_1|^2 + |\partial^{\alpha_1} \hat{f}_1|^2 |\partial^{\alpha_1} f''|^2 \right) dx$$

$$\leq C \sum_{|\alpha| \leq N-2} \sum_{\alpha_1 \leq \alpha} (J^1_1 + J^2_1),$$

where we have used the exponential decay of $e_i$ in $v$.

For $J^1_1$, if $|\alpha_1| \leq \frac{N-2}{2}$, it holds that

$$J^1_1 = \int_{\mathbb{R}^3} \left( |\partial^{\alpha_1} \hat{f}_1|^2 |\partial^{\alpha_1} f''_1|^2 + |\partial^{\alpha_1} f''_1|^2 |\partial^{\alpha_1} \hat{f}_2|^2 \right) dx$$

$$\leq \left( \sup_{x \in \mathbb{R}^3} |\partial^{\alpha_1} \hat{f}_1|^2 \right) \|\partial^{\alpha_1} f''\|_v^2 + \left( \sup_{x \in \mathbb{R}^3} |\partial^{\alpha_1} \hat{f}_2|^2 \right) \|\partial^{\alpha_1} f''\|_v^2$$

$$\leq C \mathcal{E}_{N-1}(\hat{f})(t) D_{N-1}(f')(t) + C \mathcal{E}_{N-1}(f')(t) D_{N-1}(\hat{f})(t).$$

If $|\alpha - \alpha_1| \leq \frac{N-2}{2}$, we obtain the same result above.

For $J^2_1$, similarly,

$$J^2_1 = \int_{\mathbb{R}^3} \left( |\partial^{\alpha_1} f''|^2 |\partial^{\alpha_1} \hat{f}_1|^2 + |\partial^{\alpha_1} \hat{f}_1|^2 |\partial^{\alpha_1} f''|^2 \right) dx$$

$$\leq C \mathcal{E}_{N-1}(f'')(t) D_{N-1}(\hat{f})(t) + C \mathcal{E}_{N-1}(\hat{f})(t) D_{N-1}(f'')(t).$$

With the above estimates, one has

$$J_1 \leq C \mathcal{E}_{N-1}(\hat{f})(t)(D_{N-1}(f')(t) + D_{N-1}(f'')(t))$$

$$+ C D_{N-1}(\hat{f})(t)(\mathcal{E}_{N-1}(f')(t) + \mathcal{E}_{N-1}(f'')(t)). \tag{47}$$

Plugging (45), (46) and (47) into (44) to get

$$\int_{\mathbb{R}^3} \left( 1 + |\xi|^2 \right)^{N-2} |\langle \hat{g}, e_i \rangle|^2 d\xi$$

$$\leq C \mathcal{E}_{N-1}(\hat{f})(t)(D_{N-1}(f')(t) + D_{N-1}(f'')(t))$$

$$+ C D_{N-1}(\hat{f})(t)(\mathcal{E}_{N-1}(f')(t) + \mathcal{E}_{N-1}(f'')(t)). \tag{48}$$

Note that

$$\|\partial^{\alpha} \hat{f}\|_v^2 \leq \|\partial^{\alpha} (I - P) \hat{f}\|_v^2 + \|\partial^{\alpha} P \hat{f}\|_v^2.$$
Therefore, (27) follows by choosing a small constant \( \eta > 0 \), and by plugging (29), (42), (43) and (48) into (28). This completes the proof of the lemma.

Now we consider the energy estimate on the pure spatial derivatives of \( \tilde{f} \) which satisfies (25).

**Lemma 3.5.** Let \( N \geq 5 \), then

\[
\frac{1}{2} \frac{d}{dt} \sum_{1 \leq |\alpha| \leq N-1} (||\partial^\alpha \tilde{f}||^2 + ||\partial^\alpha \nabla_x \tilde{\Phi}||^2) + \sum_{1 \leq |\alpha| \leq N-1} ||\partial^\alpha \{I - P\} \tilde{f}||^2_v \leq C \sum_{1 \leq |\alpha| \leq N-1} ||\partial^\alpha \tilde{f}||^2 + C \|\mathcal{E}_{N-1}(\tilde{f})(t)(D_{N-1}(f')(t) + D_N(f'')(t)) \|
\]

(49)

**Proof.** We apply \( \partial^\alpha \) on (25) with \( 1 \leq |\alpha| \leq N - 1 \) to yield

\[
\partial_t \partial^\alpha \tilde{f} + v \cdot \nabla_x \partial^\alpha \tilde{f} + \nabla_x \Phi' \cdot \nabla_x \partial^\alpha \tilde{f} + \nabla_x \Phi \cdot \partial_x \partial^\alpha f'' - \frac{1}{2} v \cdot \nabla_x \Phi(\partial^\alpha \tilde{f}) - \frac{1}{2} v \cdot \nabla_x \Phi (\partial^\alpha \tilde{f}) - \frac{1}{2} v \cdot \nabla_x \Phi (\partial^\alpha \tilde{f})
\]

Multiplying the above equation by \( \partial^\alpha \tilde{f} \) and taking integration over \( \mathbb{R}^3 \times \mathbb{R}^3 \) to give

\[
\frac{1}{2} \frac{d}{dt} ||\partial^\alpha \tilde{f}||^2 + (-v \cdot \nabla_x \partial^\alpha \tilde{\Phi} \sqrt{\mu}, \partial^\alpha \tilde{f}) + (\partial^\alpha \tilde{f}, \partial^\alpha \tilde{f}) = \sum_{i=1}^5 I_i,
\]

where

\[
I_1 = (\partial^\alpha \Gamma(\tilde{f}, f') + (\partial^\alpha \Gamma(f'', \tilde{f}), \partial^\alpha \tilde{f}),
I_2 = (-v \cdot \nabla_x \Phi \cdot \partial_x \partial^\alpha f'', \partial^\alpha \tilde{f}),
I_3 = \left( \frac{1}{2} v \cdot \nabla_x \Phi(\partial^\alpha \tilde{f}) + \frac{1}{2} v \cdot \nabla_x \Phi (\partial^\alpha \tilde{f}), \partial^\alpha \tilde{f} \right)
\]

and

\[
I_4 = \sum_{|\alpha'| \geq 1, |\alpha| \leq \alpha} C_{\alpha'}^\alpha (v \cdot \nabla_x \partial^\alpha \Phi \cdot \nabla_v \partial^\alpha \tilde{f} - \partial^\alpha \nabla_x \Phi \cdot \partial_v \partial^\alpha \tilde{f} + \partial^\alpha \tilde{f}),
I_5 = \sum_{|\alpha'| \geq 1, |\alpha| \leq \alpha} C_{\alpha'}^\alpha \left( \frac{1}{2} v \cdot \partial^\alpha \nabla_x \Phi(\partial^\alpha \tilde{f}) + \frac{1}{2} v \cdot \partial^\alpha \nabla_x \Phi (\partial^\alpha \tilde{f}), \partial^\alpha \tilde{f} \right).
\]

For the second part on the left-hand side of (50), integrating by parts, we conclude

\[
(-v \cdot \nabla_x \partial^\alpha \tilde{\Phi} \sqrt{\mu}, \partial^\alpha \tilde{f}) = (-v \cdot \nabla_x \partial^\alpha \tilde{\Phi}, \partial^\alpha v \tilde{f}) = -(\partial^\alpha \tilde{\Phi}, \partial^\alpha (a \tilde{f} + 3c \tilde{f})) = -(\partial^\alpha \tilde{\Phi}, \Delta_x \partial^\alpha \tilde{\Phi}) = \frac{d}{dt} ||\partial^\alpha \nabla_x \tilde{\Phi}||^2.
\]

Here we have used (7), (8) and (24).

Note that

\[-(L(\partial^\alpha \tilde{f}), \partial^\alpha \tilde{f}) \geq \delta_0 ||\partial^\alpha \{I - P\} \tilde{f}||^2_v.\]
From (50), we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|\partial^\alpha \tilde{f}\|_p^2 + \|\partial^\alpha \nabla_x \Phi\|_p^2 \right) + \delta_0 \|\partial^\alpha (I - P) \tilde{f}\|_p^2 \leq \sum_{i=1}^{5} I_i.
\] (51)

Now we estimate each term in the above as follows. For $I_1$, by (20), we obtain
\[
I_1 \leq \eta \|\partial^\alpha \tilde{f}\|_p^2 + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)(\mathcal{D}_{N-1}(f')(t) + \mathcal{D}_{N-1}(f'')(t)) + C_\eta \mathcal{D}_{N-1}(\tilde{f})(t)(\mathcal{E}_{N-1}(f')(t) + \mathcal{E}_{N-1}(f'')(t)).
\] (52)

For $I_2$, applying Sobolev inequality to give
\[
I_2 \leq \eta \|\partial^\alpha \tilde{f}\|_p^2 + C_\eta \sum_{|\alpha| \leq 2} \|\nabla_x \partial^\alpha \tilde{f}\|_p^2 \|\nabla_v \partial^\alpha f''\|_p^2
\leq \eta \|\partial^\alpha \tilde{f}\|_p^2 + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)\mathcal{D}_{N}(f'')(t).
\] (53)

Here we have used $\|\nabla_v \partial^\alpha f''\|_p^2 \leq C(\|\nabla_v \partial^\alpha (I - P) f''\|_p^2 + \|\partial^\alpha Pf''\|_p^2)$.

Using the similar procedure above to $I_3$, we obtain
\[
I_3 \leq \eta \|\partial^\alpha \tilde{f}\|_p^2 + C_\eta \mathcal{E}_{N-1}(\tilde{f}'')(t)\mathcal{D}_{N-1}(\tilde{f})(t) + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)\mathcal{D}_{N-1}(f'')(t). \quad (54)
\]

For $I_4$, $I_5$, similarly to the estimate of $M_2$ as in Lemma 3.4, one has
\[
I_4 \leq \eta \|\partial^\alpha \tilde{f}\|_p^2 + C_\eta \mathcal{E}_{N-1}(\tilde{f}'')(t)\mathcal{D}_{N-1}(\tilde{f})(t) + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)\mathcal{D}_{N-1}(f'')(t). \quad (55)
\]

I_5 \leq \eta \|\partial^\alpha \tilde{f}\|_p^2 + C_\eta \mathcal{E}_{N-1}(\tilde{f}'')(t)\mathcal{D}_{N-1}(\tilde{f})(t) + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)\mathcal{D}_{N-1}(f'')(t). \quad (56)

By collecting estimates (52), (53), (54), (55), (56), it follows from (51) that
\[
\frac{1}{2} \frac{d}{dt} \left( \|\partial^\alpha \tilde{f}\|_p^2 + \|\partial^\alpha \nabla_x \Phi\|_p^2 \right) + \delta_0 \|\partial^\alpha (I - P) \tilde{f}\|_p^2 \leq \eta \|\partial^\alpha \tilde{f}\|_p^2 + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t)(\mathcal{D}_{N-1}(f')(t) + \mathcal{D}_{N}(f'')(t)) + C_\eta \mathcal{D}_{N-1}(\tilde{f})(t)(\mathcal{E}_{N-1}(f')(t) + \mathcal{E}_{N-1}(f'')(t)).
\]

Note that
\[
\|\partial^\alpha \tilde{f}\|_p^2 \leq \|\partial^\alpha (I - P) \tilde{f}\|_p^2 + \|\partial^\alpha P \tilde{f}\|^2.
\]

Thus, we easily obtain the estimate (49) after taking summation over $1 \leq |\alpha| \leq N - 1$, provided that $\eta > 0$ is small enough.

Next, we consider the energy estimates on $\tilde{f}$ in (25) with mixed space and velocity derivatives in the following.

**Lemma 3.6.** Let $N \geq 5$. There are constants $\lambda, C > 0$ such that
\[
\frac{d}{dt} \sum_{k=1}^{N-1} C_k \sum_{|\alpha| = k} \|\partial^\alpha (I - P) \tilde{f}\|_p^2 + \lambda \sum_{|\alpha| \geq 1} \sum_{|\beta| \leq N - 1} \|\partial^\alpha \{I - P\} \tilde{f}\|_p^2 \leq C \sum_{\alpha \leq N-1} \|\partial^\alpha \tilde{f}\|_p^2 + C \sum_{|\alpha| \leq N-1} \|\partial^\alpha \{I - P\} \tilde{f}\|_p^2 + C \mathcal{E}_{N-1}(\tilde{f})(t)\mathcal{D}_{N-1}(f')(t) + \mathcal{D}_{N}(f'')(t) + C \mathcal{D}_{N-1}(\tilde{f})(t)(\mathcal{E}_{N-1}(f')(t) + \mathcal{E}_{N-1}(f'')(t)). \quad (57)
\]
Proof. Note that \( \{ I - P \}(\nabla_x \bar{\Phi} \cdot v\sqrt{\rho}) = 0 \). Applying \( \{ I - P \} \) to (25), one has

\[
\begin{align*}
\partial_t \{ I - P \} \hat{f} + v \cdot \nabla_x \{ I - P \} \hat{f} + \nabla_x \Phi' \cdot \nabla_v \{ I - P \} \hat{f} \\
+ \nabla_x \Phi \cdot \nabla_v \{ I - P \} f'' - L(\{ I - P \} \hat{f})
\end{align*}
\]

\[
= (\Gamma(\hat{f}, f') + \Gamma(f'', \hat{f})) - \{ I - P \}(v \cdot \nabla_x P \hat{f}) - \{ I - P \}(\nabla_x \Phi' \cdot \nabla_v \{ I - P \} \hat{f}) - \{ I - P \}(\nabla_x \Phi \cdot \nabla_v P f'' - \frac{1}{2} v \cdot \nabla_x \Phi P \hat{f}) \]

\[
- \{ I - P \}(\nabla_x \Phi \cdot \nabla_v P f'' - \frac{1}{2} v \cdot \nabla_x \Phi P f''') \]  

(58)

where the right-hand side is the summation of eight terms, and we denote them by \( S_i (1 \leq i \leq 8) \), respectively. Let \( 1 \leq k \leq N - 1 \), and fix \( \alpha, \beta \) with \( |\beta| = k \) and \( |\alpha| + |\beta| \leq N - 1 \). Apply \( \partial^a_\beta \) to (58), and multiply it by \( \partial^a_\beta \{ I - P \} \hat{f} \) and then integrate over \( \mathbb{R}^3 \times \mathbb{R}^3 \) to find

\[
\frac{1}{2} \frac{d}{dt} ||\partial^a_\beta \{ I - P \} \hat{f}||^2 = (\partial^a_\beta L(\{ I - P \} \hat{f}), \partial^a_\beta \{ I - P \} \hat{f})
\]

\[
= \sum_{i=1}^{8} (\partial^a_\beta S_i, \partial^a_\beta \{ I - P \} \hat{f}) - \sum_{|\alpha'| \geq 1, |\alpha'\alpha| \leq \alpha} C^\alpha_{\alpha'} (I^1_{\alpha'}, \partial^a_\beta \{ I - P \} \hat{f})
\]

\[
- \sum_{|\alpha'| \leq \alpha} C^\alpha_{\alpha'} (I^2_{\alpha'}, \partial^a_\beta \{ I - P \} \hat{f}) - \sum_{|\beta'\beta| \geq 1, |\beta'\beta| \leq \beta} C^\beta_{\beta'} (I_{\beta'}, \partial^a_\beta \{ I - P \} \hat{f}),
\]

(59)

where

\[
I^1_{\alpha'} = \nabla_x \partial^a_\alpha \Phi' \cdot \nabla_v \partial^a_{-\alpha'} \{ I - P \} \hat{f}, \quad I^2_{\alpha'} = \nabla_x \partial^a_\alpha \Phi \cdot \nabla_v \partial^a_{-\alpha'} \{ I - P \} f'',
\]

and

\[
I_{\beta'} = \partial^a_v \cdot \nabla_x \partial^a_{-\beta'} \{ I - P \} \hat{f}
\]

Now we estimate each term in equation (59). For the second left-hand side of (59), by (18), one has

\[
-(\partial^a_\beta L(\{ I - P \} \hat{f}), \partial^a_\beta \{ I - P \} \hat{f}) \geq \frac{1}{2} ||\partial^a_\beta \{ I - P \} \hat{f}||^2 - C ||\partial^a \{ I - P \} \hat{f}||^2. \]

(60)

The first right-hand side of (59) includes eight terms, for \( S_1 \), using (21) to yield

\[
(\partial^a_\beta S_1, \partial^a_\beta \{ I - P \} \hat{f})
\]

\[
= \left( \partial^a_\beta (\Gamma(\hat{f}, f') + \Gamma(f'', \hat{f})), \partial^a_\beta \{ I - P \} \hat{f} \right)
\]

\[
\leq \eta ||\partial^a_\beta \{ I - P \} \hat{f}||^2 + C_\eta \varepsilon_{N-1}(\hat{f})(t)(\mathcal{D}_{N-1}(f')(t) + \mathcal{D}_{N-1}(f'')(t))
\]

\[
+ C_\eta \mathcal{D}_{N-1}(\hat{f})(t)(\varepsilon_{N-1}(f')(t) + \varepsilon_{N-1}(f'')(t)).
\]

(61)
For $S_2$, it holds that

$$\begin{align*}
(\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}) \\
= - (\partial_\beta \partial_\alpha \{ I - P \} (v \cdot \nabla x \tilde{f}), \partial_\beta \partial_\alpha \{ I - P \} \tilde{f}) \\
\leq \eta ||\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}||_v^2 + C_\eta \sum_{|\alpha| \leq N-k-1} ||\partial^\alpha \nabla_x (a^\beta, b^\beta, c) ||^2 \quad (62)
\end{align*}$$

For $S_3$, it holds that

$$\begin{align*}
(\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}) \\
= - \sum_{\alpha' \leq \alpha} C_\alpha \left( \partial_\beta \{ I - P \} (\partial^\alpha \nabla_x \Phi' \cdot \nabla_v P \tilde{f}), \partial_\beta \partial_\alpha \{ I - P \} \tilde{f} \right) \\
- \sum_{\alpha' \leq \alpha} C_\alpha \left( \partial_\beta \{ I - P \} (\partial^\alpha \nabla_x \Phi' \partial^\alpha - \alpha' \tilde{f}), \partial_\beta \partial_\alpha \{ I - P \} \tilde{f} \right) \\
=N_1 + N_2.
\end{align*}$$

For $N_1$, if $|\alpha - \alpha'| \leq \frac{N-2}{2}$, one has

$$\begin{align*}
N_1 \leq \eta ||\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}||_v^2 + C_\eta \sum_{|\alpha'| \leq \alpha} C_\alpha \int_{\mathbb{R}^3} ||\partial^\alpha \nabla_x \Phi'||^2 \leq \beta \{ I - P \} (\partial^\alpha - \alpha' \nabla_v P \tilde{f})||_2^2 dx \\
\leq \eta ||\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}||_v^2 + C_\eta \sum_{|\alpha'| \leq \alpha} \int_{\mathbb{R}^3} ||\partial^\alpha \nabla_x \Phi'||^2 \leq \beta \{ I - P \} (\partial^\alpha - \alpha' \tilde{f})||_v^2 \||\partial^\alpha \nabla_v \Phi'||^2 \\
\leq \eta ||\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}||_v^2 + C_\eta \sum_{|\alpha'| \leq \alpha} \left( \sup_{x \in \mathbb{R}^3} ||\partial^\alpha - \alpha' \tilde{f}||^2 \right) \||\partial^\alpha \nabla_x \Phi'||^2 \\
\leq \eta ||\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}||_v^2 + C_\eta \sum_{|\alpha| \leq N-2} ||\partial^\alpha \nabla_x \Phi'||^2 \\
\leq \eta ||\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}||_v^2 + C_\eta \mathcal{E}_{N-1}(f')(t) \mathcal{D}_{N-1}(\tilde{f})(t),
\end{align*}$$

where we have used Lemma 3.1 and (7). The case of $|\alpha - \alpha'| > \frac{N-2}{2}$ is similar to the above procedure, and we can get the same result.

For $N_2$, similarly, one has

$$N_2 \leq \eta ||\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}||_v^2 + C_\eta \mathcal{E}_{N-1}(f')(t) \mathcal{D}_{N-1}(\tilde{f})(t).$$

With the above estimates, we obtain

$$\begin{align*}
(\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}) \leq \eta ||\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}||_v^2 + C_\eta \mathcal{E}_{N-1}(f')(t) \mathcal{D}_{N-1}(\tilde{f})(t). \quad (63)
\end{align*}$$

For $S_4$, using the means of similar analysis of $S_3$, we get

$$\begin{align*}
(\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}) \leq \eta ||\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}||_v^2 + C_\eta \mathcal{E}_{N-1}(\tilde{f})(t) \mathcal{D}_{N-1}(f')(t). \quad (64)
\end{align*}$$

$S_5$ is estimated by

$$\begin{align*}
(\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}) \\
= (\partial_\beta \partial_\alpha \{ I - P \} \tilde{f}), \partial_\beta \partial_\alpha \{ I - P \} \tilde{f} \\
\end{align*}$$
\[
\leq \eta \| \partial_\alpha^\eta (I - P) \hat{f} \|^2 + C_\eta \sum_{|\alpha| \leq N - k - 1} \| \partial_\alpha \nabla_x (I - P) \hat{f} \|^2.
\]

Using the similar analysis of \( S_3, S_6 \) and \( S_7 \) are estimated by
\[
(\partial_\alpha^\eta S_3, \partial_\alpha^\eta (I - P) \hat{f}) = \left( \partial_\alpha^\eta P (\nabla_x \Phi' \cdot \nabla_v (I - P) \hat{f} - \frac{1}{2} v \cdot \nabla_v \Phi (I - P) \hat{f}), \partial_\alpha^\eta (I - P) \hat{f} \right)
\]
\[
\leq \eta \| \partial_\alpha^\eta (I - P) \hat{f} \|^2 + C_\eta \mathcal{E}_{N-1}(f')(t) \mathcal{D}_{N-1}(\hat{f})(t),
\]
and
\[
(\partial_\alpha^\eta S_7, \partial_\alpha^\eta (I - P) \hat{f}) = \left( \partial_\alpha^\eta P (\nabla_x \Phi \cdot \nabla_v (I - P) f'' - \frac{1}{2} v \cdot \nabla_v \Phi (I - P) f''), \partial_\alpha^\eta (I - P) \hat{f} \right)
\]
\[
\leq \eta \| \partial_\alpha^\eta (I - P) \hat{f} \|^2 + C_\eta \mathcal{E}_{N-1}(\hat{f})(t) \mathcal{D}_{N-1}(f'')(t).
\]

For \( S_8 \), similarly as \( S_3 \), one has
\[
(\partial_\alpha^\eta S_8, \partial_\alpha^\eta (I - P) \hat{f})
\]
\[
= \left( \frac{1}{2} \partial_\alpha^\eta (v \cdot \nabla_v \Phi (I - P) \hat{f} + v \cdot \nabla_v \Phi (I - P) f''), \partial_\alpha^\eta (I - P) \hat{f} \right)
\]
\[
\leq \eta \| \partial_\alpha^\eta (I - P) \hat{f} \|^2 + C_\eta \mathcal{E}_{N-1}(f')(t) \mathcal{D}_{N-1}(\hat{f})(t) + C_\eta \mathcal{E}_{N-1}(\hat{f})(t) \mathcal{D}_{N-1}(f'')(t).
\]

For the remaining terms about \( I_{\alpha'}^1, I_{\alpha'}^2 \) and \( I_{\alpha'}^\rho \), we can estimate them as follows. First, for \( I_{\alpha'}^1 \), one has
\[
- \sum_{|\alpha'| \geq 1, \alpha' \leq \alpha} C^\alpha_{\alpha'} (I_{\alpha'}^1, \partial_\alpha^\eta (I - P) \hat{f})
\]
\[
= - \sum_{|\alpha'| \geq 1, \alpha' \leq \alpha} C^\alpha_{\alpha'} (\nabla_x \partial_\alpha^\eta \Phi' \cdot \nabla_v \partial_\alpha^\eta \alpha' (I - P) \hat{f}, \partial_\alpha^\eta (I - P) \hat{f})
\]
\[
\leq \eta \| \partial_\alpha^\eta (I - P) \hat{f} \|^2 + C_\eta \sum_{|\alpha'| = N - 2} \| \nabla_x \partial_\alpha^\eta \Phi' \| \sum_{|\alpha'| = N - 2} \| \nabla_v \partial_\alpha^\eta (I - P) \hat{f} \|^2
\]
\[
+ C_\eta \sum_{1 \leq |\alpha'| \leq N - 3} \| \nabla_x \partial_\alpha^\eta \Phi' \| \sum_{1 \leq |\alpha'| \leq N - 3} \| \nabla_v \partial_\alpha^\eta \alpha' (I - P) \hat{f} \|^2
\]
\[
\leq \eta \| \partial_\alpha^\eta (I - P) \hat{f} \|^2 + C_\eta \sum_{|\alpha'| = N - 2} \| \nabla_x \partial_\alpha^\eta \Phi' \|^2 \sum_{|\alpha'| = N - 2} \| \nabla_v \partial_\alpha^\eta (I - P) \hat{f} \|^2
\]
\[
+ C_\eta \sum_{2 \leq |\alpha'| \leq N - 1} \| \nabla_x \partial_\alpha^\eta \Phi' \|^2 \sum_{2 \leq |\alpha'| \leq N - 1} \| \partial_\alpha^\eta (I - P) \hat{f} \|^2
\]
\[
\leq \eta \| \partial_\alpha^\eta (I - P) \hat{f} \|^2 + C_\eta \mathcal{E}_{N-1}(f')(t) \mathcal{D}_{N-1}(\hat{f})(t).
\]
Similarly, \( I_{\alpha'}^2 \) is estimated by
\[
- \sum_{|\alpha'| \leq \alpha} C^\alpha_{\alpha'} (I_{\alpha'}^2, \partial_\alpha^\eta (I - P) \hat{f})
\]
\[
= - \sum_{|\alpha'| \leq \alpha} C^\alpha_{\alpha'} (\nabla_x \partial_\alpha^\eta \Phi' \cdot \nabla_v \partial_\alpha^\eta \alpha' (I - P) f'', \partial_\alpha^\eta (I - P) \hat{f})
\]
Therefore, by choosing a small constant $\eta > 0$, \((60)\) follows by plugging the estimates \((60)\)–\((71)\) into \((59)\), taking summation over \(|\beta| = k, |\alpha| + |\beta| \leq N - 1\) for each given $1 \leq k \leq N - 1$ and taking proper linear combination of those $N - 1$ estimates with properly chosen constants $C_k > 0 (1 \leq k \leq N - 1)$. Thus the proof of this lemma is completed.

**Proof of Theorem 1.2.** A suitable linear combination of \((27)\), \((49)\) and \((57)\) gives

\[
\frac{d}{dt} \left[ \sum_{|\alpha| \leq N - 1} \left( \| \partial^\alpha \tilde{f} \|^2 + \| \partial^\alpha \nabla_x \tilde{\Phi} \|^2 \right) - \kappa \int_{\mathbb{R}^3} (1 + |\xi|^2)^{N-2} |\xi_1| (\langle iS(w) \hat{\tilde{f}}, \hat{\tilde{f}} \rangle) d\xi + \sum_{k=1}^{N-1} C_k \sum_{|\beta| = k} \left( \| \partial^\beta (I - P) \hat{f} \|^2 - \int_{\mathbb{R}^3} b f' \cdot b' \hat{f} \, dx \right) \right]
\]

\[
+ \left[ \sum_{|\alpha| \leq N - 1} \| \partial^\alpha (I - P) \hat{f} \|^2 + \sum_{1 \leq |\alpha| \leq N - 1} \| \partial^\alpha P \hat{f} \|^2 \right]
\]

\[
+ \sum_{|\alpha| \leq N - 2} \| \partial^\alpha P_0 \hat{f} \|^2 + \sum_{|\alpha| + |\beta| \geq 1} \left( \| \partial^{\alpha+\beta} (I - P) \hat{f} \|^2 \right)
\]

\[
\leq C \mathcal{E}_{N-1}(\hat{f})(t) (\| f' \|^2 + \| f'' \|^2 + \mathcal{D}_{N-1}(f')(t) + \mathcal{D}_{N}(f'')(t)) + C \mathcal{D}_{N-1}(\hat{f})(t) (\mathcal{E}_{N-1}(f')(t) + \mathcal{E}_{N-1}(f'')(t)).
\]

On the other hand, the boundedness of the operator $S(w)$ implies

\[
\left| \int_{\mathbb{R}^3} (1 + |\xi|^2)^{N-2} |\xi_1| (\langle iS(w) \hat{\tilde{f}}, \hat{\tilde{f}} \rangle) d\xi \right| \leq C \sum_{|\alpha| \leq N - 1} \| \partial^\alpha \tilde{f} \|^2.
\]

Thus, we define two functionals as follows,
so we set $\delta$ by choosing $\delta$ equivalent to those defined in (9) and (10) respectively. Note that in Theorem 1.1, applying Theorem 1.1, we obtain

From (72) and (73), by choosing $\delta$ small enough such that

$$
\{E_{N-1}(\tilde{f})(t) + D_{N-1}(\tilde{f})(t) \leq C E_{N-1}(\tilde{f})(t) (\|f\|^2 + \|f''\|^2 + D_{N-1}(f')(t) + D_N(f'')(t))

+ C D_{N-1}(\tilde{f})(t) (E_{N-1}(f')(t) + E_{N-1}(f'')(t)).
$$

Applying Theorem 1.1, we obtain

$$
E_N(f')(t) + \lambda \int_0^t D_N(f')(s) ds \leq C E_N(f')(0),
$$

and

$$
E_N(f'')(t) + \lambda \int_0^t D_N(f'')(s) ds \leq C E_N(f'')(0),
$$

where

$$
\max\{E_N(f')(0), E_N(f'')(0)\} \leq \delta.
$$

From (72) and (73), by choosing $\delta = \delta^2 > 0$ small enough, we have

$$
\frac{d}{dt} E_{N-1}(\tilde{f})(t) + D_{N-1}(\tilde{f})(t) \leq C E_{N-1}(\tilde{f})(t) (\|f\|^2 + \|f''\|^2 + D_{N-1}(f')(t) + D_N(f'')(t)),
$$

so we set $\delta = \min\{\delta^1, \delta^2\}$ here. Furthermore, the estimate (74) implies that

$$
E_{N-1}(\tilde{f})(t) + \int_0^t D_{N-1}(\tilde{f})(s) ds \leq C E_{N-1}(\tilde{f})(0),
$$

where we have used the Gronwall inequality, (14) and the fact that

$$
\int_0^t (D_{N-1}(f')(s) + D_N(f'')(s)) ds \leq C \delta,
$$
and
\[ \int_0^t ||f'(s)||^2 ds \leq C\delta \int_0^t (1 + s)^{-3/2} ds \leq C\delta, \]
\[ \int_0^t ||f''(s)||^2 ds \leq C\delta. \]

Hence, the proof of Theorem 1.2 is completed. \( \square \)

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E-mail address: wang-h@whu.edu.cn