CENTRAL OPEN SETS TILINGS

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Abstract. We introduce a method for constructing collections of subsets of $\mathbb{R}^n$, using an iterated function system, a set $T$, and a cost function. We refer to these collections as tilings. The special case where $T$ is the central open set of an iterated function system that obeys the open set condition is emphasized. The notion of the central open set associated with an iterated function system of similitudes, introduced in 2005 by Bandt, Hung, and Rao, is reviewed. A practical method for calculating pictures of central open sets is described. Some general properties and examples of the tilings are presented.

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1. Introduction

The goal of this paper is to describe a simple method for producing a wide range of tilings (in a generalized sense). The method uses an iterated function system (IFS) acting on $\mathbb{R}^n$, together with a set $T$, and a cost function $c$. Figure 1 illustrates part of such a tiling where $T$ is the central open set of an IFS. The possible structures of the new tilings are diverse, yet all are handled with the same underlying mathematical device. The formalism yields examples in analysis, geometry, and dynamics; it leads to extensive and rich families of variants of the self-similar tilings introduced in [10, 11]. Our methods are based on

![Figure 1](image.png)

Figure 1. See Example 5.1. The left panel illustrates part of an unbounded central open set tiling. The right panel shows some of the ways in which the four prototiles (see inset box) can meet. See also Figures 5 and 6.
addresses associated with IFSs and mappings from these addresses into tilings and tiling spaces.

Our main examples use the central open set of the attractor of an IFS to provide the shapes of the tiles. In this case the tilings have properties that suggest they may be used to model patterns that arise naturally. For example, such tiles have self-similar features and may touch, and there may also be gaps that repeat at different scales. Type “mudcracks” into a search engine to see illustrations of seemingly related natural patterns.

In this paper we use the following lexicon. We use the word *tiling* to mean a collection of closed subsets of $\mathbb{R}^n$, and a *tile* is a member of the collection. This is more general than the standard definitions, see Grunbaum and Sheppard [15]. We say that two tiles meet if their intersection is non-empty, and we say that they touch if their intersection is non-empty and contains no interior points. The support of a tiling is the union of its tiles. Two tilings meet if the union of the tiles in their intersection is a tiling whose support is the intersection of the supports of the two tilings. A set of prototiles for a tiling is a set of tiles such that each member of the tiling is related to a prototile by an isometry. The set of isometries may be restricted to translations. We say that two tiles have the same shape if they are related by an isometry. If two tiles are related by an isometry we say that they are copies of one another. We say that a tiling is commensurate if the sizes of all its tiles belong to a geometrical progression; otherwise the tiling is incommensurate; see also [22].

A patch of a tiling is the set of its tiles that have non-empty intersection with a finite set, typically a disk or rectangle.

In this paper we are concerned with the situation where all tiles have the same shape. But the theory is readily generalized to multiple shapes, by using graph-directed IFS, as in [13].

2. Iterated function systems and central open sets

Here we review the notion of an IFS of similitudes and of a central open set introduced in 2005 by Bandt, Hung, and Rao, [4].

Roughly following [4], let $F = \{\mathbb{R}^n; f_1, f_2 \ldots f_m\}$ denote a collection of contractive similitudes, that is $f_i : \mathbb{R}^n \to \mathbb{R}^n$ with

$$|f_i(x) - f_i(y)| = \lambda_i|x - y|$$

for all $x, y \in \mathbb{R}^n$ where the $\lambda_i \in (0, 1)$ are the contraction factors and $|\cdot|$ denotes the Euclidean norm. We refer to $F$ as an iterated function system. By slight abuse of notation we use the same symbol $F$ to denote the mapping from sets to sets $F : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ defined by $F(S) = \{f_i(x) : x \in S, i = 1, \ldots, m\}$. It is well-known that there exists a unique non-empty compact set $A \subset \mathbb{R}^n$ such that

$$A = F(A) = \bigcup_{i=1}^m f_i(A)$$

where $f_i(A) = \{f_i(x) | x \in A\}$, [16].

The set $A$ is called the *attractor* of the iterated function system $F$, because

$$\lim_{k \to \infty} F^k(\{x\}) = A$$

for all $x \in \mathbb{R}^n$,
where convergence is with respect to the Hausdorff metric on $\mathbb{R}^n$, and $F^k$ is the function $F$ composed with itself $k$ times. We say that the basin of $A$ is $\mathbb{R}^n$. The fast basin [5, 6] of $A$ is a subset of the basin defined by

$$B = \{x \in \mathbb{R}^n | F^k(\{x\}) \cap A \neq \emptyset, \text{ some } k \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of positive integers. The fast basin is the set of points such that some finite orbit meets $A$. Fast basins are related to but distinct from the fractal blow-ups introduced by Strichartz [24] and the macro-fractals introduced by Banakh and Novosad [2]. We will use $B \setminus A$ in calculations in Section 4.

Roughly quoting [4], the attractor $A$ is the union of smaller copies of itself, $A_i = f_i(A)$, where each $A_i$ consists of smaller copies $A_{ij} = f_i(f_j(A))$, and so on. For any positive integer $k$, we can consider the set $\Sigma^k$ of words $i = i_1 \ldots i_k$ from the alphabet $\Sigma = \{1, 2, \ldots m\}$. Writing $f_i = f_i f_{i_2} \ldots f_{i_k}$ and $A_i = f_i(A)$ we have

$$A = \bigcup \{f_i(A)|i \in \Sigma^k\}.$$
Figure 2. Examples of patches of three unbounded tilings. The leftmost two images are related to an IFS whose attractor is a square. The rightmost image is related to an IFS whose attractor is a Sierpinski triangle. Two of the panels use $T = L$, a decorated leaf-shaped set, see Example 3.2.

where $d(x, Y) = \inf\{|x - y| \mid y \in Y\}$. Bandt et al. [4] prove the following theorem and its elegant corollary.

**Theorem 2.1.** If the OSC holds, then the central open set $C$ is a feasible open set. If the OSC does not hold then $C$ is empty.

**Corollary 2.2.** The OSC holds if and only if $A$ is not contained in $\overline{H}$.

It is an unanswered question as to whether or not it is true that the OSC holds if and only if $A \neq A \cap \overline{H}$. The latter was claimed by M. Moran [19], but his proof contains a gap [4].

Questions relating to the existence of, and the structure of, a feasible open set for a given IFS are very subtle, see for example [25].

3. Diverse tilings derived from an IFS and a cost function

In this Section we describe a general construction of tilings using an IFS $F$, a set $T \subset \mathbb{R}^n$, and a cost function $c$ defined below. The resulting tilings may have overlapping tiles with non-empty interiors. Such tilings might be used to model fallen leaves carpeting a forest floor, duckweed on the surface of a pond, cracks in dried mud, or to design patterns for wallpaper.

Let $F$ be a IFS consisting of at least two distinct similitudes, and let $T$ be a closed subset of $\mathbb{R}^n$. For convenience we suppose that $A \cap T \neq \emptyset$, but this is not necessary. We use sets of similitudes, which are subsets of $\mathcal{N}$ determined by the cost function and are applied to $T$, to form collections of scaled and translated and possibly flipped (i.e. turned upside down, in the two-dimensional case) copies of $T$, with possible overlaps.

Let $\mathbb{H}(\mathbb{R}^n)$ be the closed bounded subsets of $\mathbb{R}^n$ equipped with the spherical Hausdorff metric, see [12, 13]. This metric $d$ is defined as follows. Let $P(x)$ be the stereographic projection of $x \in \mathbb{H}(\mathbb{R}^n)$ onto the $(n + 1)$ dimensional sphere tangent to $\mathbb{R}^n$ at the origin. Then $d(x, y)$ is the Hausdorff distance between $P(x)$ and $P(y)$ using the round metric $d_R$ on the sphere. Let $d_H(X, Y)$ be the Hausdorff distance between sets of subsets $X$ and $Y$ of $\mathbb{R}^n$ calculated using $d_R$. Let $\mathbb{H}(\mathbb{H}(\mathbb{R}^n))$ be the collections of subsets of $\mathbb{H}(\mathbb{R}^n)$ that are closed with respect to $d_H$. 
For each $i \in \{1, \ldots, M\}$ write $\lambda_i = s^{a_i}$ where $s = \max\{\lambda_i | i = 1, \ldots, m\}$ and assign a cost $c_i > 0$ to the map $f_i$. For example we may choose $c_i = a_i$. For $i = i_1 i_2 \cdots \in \Sigma^\infty$, write $i[k] = i_1 \ldots i_k \in \Sigma^*$, and $i[0] = \emptyset$. Define a cost function $c : \{\emptyset\} \cup \Sigma^* \to (0, \infty)$ by

$$c(i[k]) = c_{i_1} + c_{i_2} + \cdots + c_{i_k}, \quad c(\emptyset) = 0.$$  

Define a mapping $\Pi_T : \{\emptyset\} \cup \Sigma^* \cup \Sigma^\infty \to \mathcal{H}(\mathbb{H}(\mathbb{R}^n))$ by

$$\Pi_T(i[k]) = f_{-i(k)}(\{f_{j(l)}(T) | j \in \Sigma^\infty, l \in \mathbb{N}, c(j[l] - 1) \leq c(i[k]) < c(j[l])\}),$$

$$f_{-i(k)} := f_{i_1}^{-1} \cdots f_{i_k}^{-1}, \quad f_{j(l)} = f_{j_1} \cdots f_{j_l},$$

$$\Pi_T(i) = \bigcup_{k=1}^\infty \Pi_T(i[k]) \text{ for all } i \in \Sigma^\infty, \Pi_T(\emptyset) = \{f_1(T), f_2(T), \ldots, f_m(T)\}.$$  

$\Pi_T$ is well-defined because $\{\Pi_T(i[k]) | k = 1, 2, \ldots\}$ is a nested increasing sequence of collections of sets:  

$$\Pi_T(i[0]) \subset \Pi_T(i[1]) \subset \Pi_T(i[2]) \ldots$$

for all $i \in \Sigma^\infty$. This is true because

$$\Pi_T(i[k + 1]) = f_{-i(k+1)}(\{f_{j(l)}(T) | j \in \Sigma^\infty, l \in \mathbb{N}, c(j[l] - 1) \leq c(i[k + 1]) < c(j[l])\})$$

$$\supset f_{-i(k+1)}(\{f_{j(l)}(T) | j \in \Sigma^\infty, l \in \mathbb{N}, c(j[l] - 1) \leq c(i[k + 1]) < c(j[l]), j_1 = i_{k+1}\})$$

$$= f_{i_1}^{-1} \cdots f_{i_k}^{-1} (f_{i_{k+1}} f_{j_2} \cdots f_{j_l})(T) | j \in \Sigma^\infty, l \in \mathbb{N}, c(j[l] - 1) \leq c(i[k]) < c(j[l])$$

$$= \Pi_T(i[k]).$$

Equation (3.1) is the key mathematical device in this paper. In general $\Pi_T(i)$ is a collection of subsets of $\mathbb{R}^n$, that we call tiles. These tiles are translated, scaled, maybe flipped and/or rotated, copies of $T$. They may be overlapping and the support of the tiling $\Pi_T(i)$, namely $\bigcup \{t \in \Pi_T(i)\} \subset \mathbb{R}^n$ may be complicated.

If $F$ obeys the OSC and $T \subset \overline{C}$, then distinct sets of the form $f_{-i(k)} f_{j(l)}(T)$ in the tiling $\Pi_T(i)$ are non-overlapping; that is, the interiors of the intersections of distinct tiles are empty.

Denote the range of $\Pi_T : \Sigma^\infty \to \mathcal{H}(\mathbb{H}(\mathbb{R}^n))$ by $\mathcal{T}_T = \{\Pi_T(i) | i \in \Sigma^\infty\}$. As a consequence of properties of, and structures associated with, the shift map $\sigma : \Sigma^\infty \to \Sigma^\infty$, much can be said, along the lines of [12, 13], about continuity properties of $\Pi_T : \Sigma^\infty \to \mathcal{T}_T$ with respect to the metric $d_H$ defined above, dynamics, invariant measures, and ergodic properties associated with mappings that take $\mathcal{T}_T$ into itself, such as certain inflation and deflation operations.

**Example 3.1.** Let $F = \{\mathbb{R}^1; f_1, f_2\}, f_1(x) = \frac{x}{2}, f_2(x) = \frac{x+1}{2}$. Then $A = [0, 1]$. Choosing $T = [-\frac{1}{3}, \frac{4}{3}]$, we find $\Pi_{[-\frac{1}{3}, \frac{4}{3}]}(T) = \{[-\frac{1}{6} + \frac{n-1}{2}, \frac{1}{6} + \frac{2}{3}] | n = 1, 2, \ldots\}$. That is, $\Pi_{[-\frac{1}{3}, \frac{4}{3}]}(T)$ is a collection of overlapping closed intervals whose union is $[-\frac{1}{6}, \infty)$.

**Example 3.2.** (i) The leftmost panel in Figure 2 illustrates part of a tiling $\Pi_A(T)$ where the IFS of four similitudes each with scaling factor 0.5, attractor $A = [0, 1] \times [0, 1]$, and the cost function is defined by $c_1 = 1, c_2 = 1.3, c_3 = 1.5, c_4 = 2$. (ii) The middle panel illustrates the same part, but of $\Pi_L(T)$ where $L$ is the support of a leaf picture. In this case the tiles have been decorated by a picture of a leaf. (iii) The rightmost panel is related
to an IFS of three maps, whose attractor is a Sierpinski triangle, and the cost function specified by \(c_1 = c_2 = c_3 = 1\), and the same set \(L\).

For a two-dimensional affine transformation \(f: \mathbb{R}^2 \to \mathbb{R}^2\) we write
\[
f = \begin{bmatrix} a & b & e \\ c & d & g \end{bmatrix}
\]
for \(f(x, y) = (ax + by + e, cx + dy + g)\)
where \(a, b, c, d, e, g \in \mathbb{R}\).

**Example 3.3.** In the special case where \(c_i = a_i \in \mathbb{N}\) for all \(i = 1, \ldots m\), we call the sets
\[
T_k = s^{-k} \{ f(i|l)(A) | c(i|l) - 1 \leq k < c(i|l) \}, T_0 = F(A),
\]
canonical tilings. They play a natural role in fractal tilings [13], and in connecting them to algebraic geometry [1]. Two sequences of canonical tilings are illustrated in Figure 3. The IFSs may be deduced from \(T_0 = \{ f_1(A), f_2(A) \}\). The IFS for the top sequence is
\[
F = \{ \mathbb{R}^2; f_1, f_2 \}
\]
where
\[
f_1 = \begin{bmatrix} 0 & s & 0 \\ -s & 0 & s \end{bmatrix}, f_2 = \begin{bmatrix} -s^2 & 0 & 1 \\ 0 & s^2 & 0 \end{bmatrix}
\]
with \(s + s^2 = 1, s > 0\), and for the lower sequence
\[
f_1 = \begin{bmatrix} s & 0 & 1 \\ 0 & s & 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 & s^2 & 0 \\ s^2 & 0 & 0 \end{bmatrix}
\]
with \(s + s^4 = 1, s > 0\). In these two cases, and others like them, when the cost function is defined by \(c_i = i\), there is a simple relationship between the canonical tilings and the tilings \(\Pi_A(\cdot, \cdot)\), namely
\[
\Pi_A(i|k) = f_{-c(i|k)} s^{c(i|k)} T_{c(i|k)}, \text{ where } c(i|k) = i_1 + i_2 + \ldots + i_k,
\]
for all \(i\) and \(k\). That is, \(\Pi_A(i|k)\) is isometric to \(T_{c(i|k)}\), see [13].

4. **Central open set tilings**

In this Section we assume that \(F\) obeys the OSC, and consider the two special cases \(T = \mathcal{A}\) and \(T = \Omega\) in the mapping \(\Pi_T: \emptyset \cup \Sigma^* \cup \Sigma^\infty \to \mathbb{H}(\mathbb{H}(\mathbb{R}^n))\). For consistency with [13] we define
\[
\Pi = \Pi_A \text{ and } \Xi = \Pi_{\Omega}.
\]
These tilings are of particular interest to us. Here’s why. If \(A\) has nonempty interior, so that \(\overline{A} = A\), these tilings may be examples of conventional self-similar tilings as defined by [15], or tilings with fractal boundaries [17, 18]. But they are more general because they may be tilings with infinitely many incommensurate tile sizes. The case \(T = \Omega\) is special because it seems to be an extreme case: we conjecture that if \(T\) is chosen to be a closed set that contains \(\Omega\), then \(\Pi_T(i)\) contains overlapping tiles.

The mapping \(\Pi = \Pi_A\), and the fractal tilings it generates when \(c_i = a_i\), were introduced and studied in [9, 10, 11]. We refer to \(\Pi(i)\) as a fractal tiling, and we refer to sets of the form \(f_{-c(i|k)}^{-1} f_{j|l}^{-1}(A)\) as fractal tiles. Note that fractal tiles may have empty interiors. They have non-empty interiors when \(A\) has non-empty interior. The relationship of the address
Example 4.2. Let $F = \{f_1, f_2\}$, $f_1(x) = \frac{x}{2}$, $f_2(x) = \frac{x+1}{2}$. Then $A = [0, 1]$, $C = (0, 1)$ and we find $\Pi(\bar{T}) = \Xi(\bar{I}) = \left\{[\frac{n-1}{2}, \frac{n}{2}] \subset \mathbb{R} | n \in \mathbb{N}\right\}$, and if the tail of $i \in \Sigma$ is neither $\bar{T} = 11...$ nor $\bar{I} = 12...$, then $\Pi(i) = \Xi(i) = \left\{[\frac{n-1}{2}, \frac{n}{2}] | n \in \mathbb{Z}\right\}$. If the tail of $i \in \Sigma$ is either $\bar{T} = 11...$ or $\bar{I} = 22...$, then $\Pi(i)$ is also tiling by half unit intervals, but the support is either $(k, \infty)$ or $(-\infty, k]$ for some $k \in \mathbb{Z}$.

Example 4.2. Let $F = \{f_1, f_2\}$, $f_1(x, y) = (\frac{x}{2}, \frac{y}{2})$, $f_2(x, y) = (\frac{x+1}{2}, \frac{y}{2})$. The attractor is $A = [0, 1] \times \{0\}$, and the central open set is $C = (0, 1) \times (-\infty, \infty)$ is unbounded. If the tail of $i \in \Sigma$ is neither $\bar{T} = 11...$ nor $\bar{I} = 12...$, then $A = [0, 1] \times \{0\}$, $\Pi(i) = \left\{[\frac{n}{2}, \frac{n+1}{2}] \times \{0\} \subset \mathbb{R}^2 | n \in \mathbb{Z}\right\}$, and $\Xi(i) = \left\{[\frac{n}{2}, \frac{n+1}{2}] \times (-\infty, \infty) : n \in \mathbb{Z}\right\}$.

Example 4.3. Let $F = \{f_1, f_2, f_3\}$ where

$$f_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, f_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, f_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{4} \end{bmatrix}.$$ 

Then the attractor is a Sierpinski triange and the central open set is a hexagon, see [4]. See Figure 4.

In Theorem 4.4 below we establish some properties of the collection of tilings $T_T$. We use the following terminology. We say that $i \in \Sigma^n$ is disjunctive, when given any finite word $j|p = j_1j_2...j_p$, there is $k \in \mathbb{N}$ such that $i_k+1...i_{k+p} = j_1j_2...j_p$. We say that $i \in \Sigma^\infty$ is reversible when $A$ has non-empty interior $A^\circ$ and there exists $k < l$ such that
Figure 4. From left to right: (i) part of the support of a central open set tiling; (ii) a patch of the same central open set tiling; (iii) same patch, also showing the underlying fractal tiles. See Example 4.3.

\[ f_l f_{l-1} \ldots f_k(A) \subset A^c, \] the interior of \( A \). If \( A \) has non-empty interior, then disjunctive is a special case of reversible, \([8, 9]\), but disjunctiveness is much easier to check than reversibility.

**Theorem 4.4.** Let \( F = \{\mathbb{R}^n; f_1, \ldots f_m\} \) be an IFS of contractive similitudes. Let \( T \subset \mathbb{R}^n \) be closed and let \( c \) be a cost function. Then \( \Pi_T(i|k) \) and \( \Pi_T(i) \) are well-defined collections of closed subsets of \( \mathbb{R}^n \), (i.e. they are tilings), and Equation (3.1) holds for all \( k \in \mathbb{N} \) and all \( i \in \Sigma^\infty \).

If \( F \) obeys the OSC, then the following statements are true.

(i) For all \( i \in \Sigma^\infty \) the interiors of the tiles that comprise \( \Xi(i) = \Pi_C(i) \) are disjoint.

(ii) The interior of \( f_{i|k}^{-1} f_{j|l}(C) \) is the central open set for the iterated function system \( SFS^{-1} \) where \( S = f_{i|k}^{-1} f_{j|l}(C) \) is a similitude. In this sense all tiles in \( \Xi(i) \) are central open sets.

(iii) In \( \mathbb{R}^2 \), if \( A \) is a polygon and \( c_i = a_i \in \mathbb{N} \) for all \( i \in \{1, \ldots m\} \), then \( \Xi(i) = \Pi(i) \).

(iv) In general the tiling \( \Pi_T(i) \) is incommensurate both as defined here and in the sense of [22]. The tilings \( \Pi_T(i) \), in particular \( \Pi(i) \) and \( \Xi(i) \), are commensurate when \( c_i = a_i \in \mathbb{N} \) for all \( i = 1, 2, \ldots, m \).

(v) Let \( i, j \in \Sigma^\infty \), \( p, q \in \mathbb{N} \), and the cost function \( c \), be such that \( \sigma^p i = \sigma^q j \), \( c(i|p) = c(j|q) \) and \( c_l = a_l \) for all \( l \in \{1, 2, \ldots m\} \). Then

\[ \Pi_T(i) = E \Pi_T(j) \]

for all \( T \), where \( E = f_{i_1}^{-1} \ldots f_{i_p}^{-1} f_{j_1} \ldots f_{j_3} \).

(vi) If \( i \in \Sigma^\infty \) is reversible and \( A \) has non-empty interior, then the support of \( \Pi(i) \), namely \( \bigcup\{t \in \Pi_A(i)\} \), is \( \mathbb{R}^n \).

**Proof.** The initial assertion follows at once from Equation (3.1). This generalizes, in the case of a single vertex, a core result in [13].

(i) We need to show that for all \( i \in \Sigma^\infty \) the interiors of the tiles that comprise \( \Xi(i) = \Pi_C(i) \) are disjoint. Suppose

\[ f_{i|k}^{-1} f_{j|l}(C) \cap f_{i|p}^{-1} f_{t|q}(C) \neq \emptyset \]
for some $k, p, l, q \in \mathbb{N}$, $j, t \in \Sigma^\infty$ such that

$$c(j|l - 1) \leq c(i|k) < c(j|l) \text{ and } c(t|q - 1) \leq c(i|p) < c(t|q).$$

We can assume $k < p$, $i_k \neq j_1$ and $i_p \neq t_1$. It follows that

$$f_{j_1} \cdots f_{j_l}(C) \cap f_{i_k}^{-1} \cdots f_{i_p}^{-1} f_{t_1} \cdots f_{t_q}(C) \neq \emptyset.$$
Figure 6. Examples of some tiling patterns that can be formed using the prototiles in Figure 7, when the tiles are allowed to touch only as in Figure 7. The arrows show directions in which part of the tiling may be repeated periodically.

This implies

\[ f_{i_p} \cdots f_{i_{k+1}} f_{j_1} \cdots f_{j_l}(C) \cap f_{t_1} \cdots f_{t_q}(C) \neq \emptyset. \]

This implies

\[ f_{i_p}(C) \cap f_{t_1}(C) \neq \emptyset, \]

where \( i_p \neq t_1 \) which contradicts the fact that \( C \) obeys the OSC.

(ii) This is an exercise in change of coordinates. We show that the interior of \( f_{(i|k)}^{-1} f_{(j|l)}(C) \) is the central open set for the iterated function system \( SFS^{-1} \) where \( S = f_{(i|k)}^{-1} f_{(j|l)} \) is a similitude.

Let \( S \) be the similitude \( f_{(i|k)}^{-1} f_{(j|l)} \). Then \( A' := f_{(i|k)}^{-1} f_{(j|l)}(C) = SA \) is the attractor of the IFS \( F' = SFS^{-1} := \{ \mathbb{R}^n; Sf_iS^{-1} | i = 1, 2, \ldots m \} \). The neighbor maps of \( F' \) are
Figure 7. Left column: The four prototiles, with respect to translations, in Figures 1, 5, 6. The bottom row illustrates the points of contact in every central open set tiling $\Xi(i), i \in \Sigma^\infty$. The remaining boxes illustrate all of the allowed combinations between pairs of tiles, indexed by row and column.

$$\{Sf_{p}^{-1}f_{q}S^{-1}| p_1 \neq q_1, p, q \in \Sigma^*\},$$ so the central open set for $F'$ is

$$C' = \{x' \in \mathbb{R}^m | d(x', A') < d(x, C')\}$$

$$= \{x' \in \mathbb{R}^m | d(x', SA) < d(x', SC)\}$$

$$= \{Sx \in \mathbb{R}^m | d(Sx, SA) < d(Sx, SC)\}$$

$$= S\{x \in \mathbb{R}^m | d(x, A) < d(x, C)\} = SC$$

where in the penultimate step we have used the fact that $S$ is a similitude.

(iii) This follows from [10], which considers the case of self-similar polygonal tilings, upon noting that if $A$ has non-empty interior and obeys the OSC then $A = \overline{C}$ is a polygon.

(iv) If $c_i = a_i \in \mathbb{N}$ for all $i$, then it is readily seen that each tile in $\Pi(i)$ is a copy of $T$ scaled by $s^a$ for some $a \in \{1, 2, \ldots, a_{\text{max}}\}$ where $a_{\text{max}} = \max\{a_1, a_2, \ldots, a_m\}$.

(v) Since $i_{p+1}i_{p+2}\ldots = j_{q+1}j_{q+2}\ldots$ it follows that

$$\Pi_T(\sigma^p i) = \Pi_T(\sigma^q j)$$
where \( \sigma : \Sigma^\infty \to \Sigma^\infty \) is the shift operator. Since \( c(i|p) = c(j|q) \), we have
\[
\{ f_{j_1} \cdots f_{j_k}(T) | j' \in \Sigma^\infty, l \in \mathbb{N}, c(j'|l - 1) \leq c(i|k) < c(j'|l) \} = \\
\{ f_{j_1} \cdots f_{j_k}(T) | j' \in \Sigma^\infty, l \in \mathbb{N}, c(j'|l - 1) \leq c(j|q) < c(j'|l) \}.
\]
The result now follows from
\[
f_{i_p} \cdots f_{i_1} \bigcup_{k \geq p} \{ f_{i_1}^{-1} \cdots f_{i_k}^{-1} \{ f_{j_1} \cdots f_{j_k}(T) | j' \in \Sigma^\infty, l \in \mathbb{N}, c(j|l - 1) \leq c(i|k) < c(j|l) \} \} = \\
f_{j_q} \cdots f_{j_1} \bigcup_{k \geq q} \{ f_{j_1}^{-1} \cdots f_{j_k}^{-1} \{ f_{j_1} \cdots f_{j_k}(T) | j' \in \Sigma^\infty, l \in \mathbb{N}, c(j|l - 1) \leq c(i|k) < c(j|l) \} \}.
\]

(vi) This follows similar lines to [9, 13] and is omitted here. The argument there rests on the observation that, for reversible addresses \( i \), the support of the tiling \( \Pi(|i|k + l) \) is contained in the interior of the tiling \( \Pi(|i|k) \) for large enough \( l \).

When the IFS \( F \) is rigid and \( c_i = a_j \in \mathbb{N} \) for all \( i = 1, 2, \ldots, m \), a converse of (v) in the Theorem is true. We say that \( F \) is rigid (with respect to translations) when the statement “\( T_k \) meets \( ET_i \)” for any \( k, l \in \{ 1, 2, \ldots, \max\{a_i\} \} \) and any translation \( E \), implies “\( T_k \) is contained in \( ET_i \) or vice-versa”. Both examples in Figure 2 are rigid. The only way that a rigid tiling \( \Pi(i) \) can meet a translation of another rigid tiling \( \Pi(j) \) is when Equation 4.1 holds. Such tilings cannot be periodic and have interesting properties, see [13] and references.

5. Calculations and examples

In this Section we present examples, including ones which show how we calculate approximations to central open sets.

**Example 5.1.** We consider the IFS \( \{ \mathbb{R}^2 ; f_1, f_2 \} \) where
\[
f_1 = \begin{bmatrix} s & 0 & 1 \\ 0 & s & 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 & s^2 & 0 \\ s^2 & 0 & 0 \end{bmatrix}
\]
with \( s + s^4 = 1, s > 0 \). This IFS was mentioned in Example 3.3 and its attractor is illustrated at the bottom left in Figure 3. In the top image in Figure 5 we illustrate the central open set and suggest how it was approximated. In this example and others the attractor \( A \) and parts of the fast basin \( B \) were calculated by using random iteration [7]. The portion of \( B \setminus A \) closest to \( A \) was assumed to be the union of the sets \( f_i^{-1}f_j(A) \) for \( i_1 \neq j_1 \) and \( i, j \in \Sigma^4 \), also computed by random iteration. In order to estimate points on \( \partial C = \partial \Sigma \), circles that appeared to touch both \( B \setminus A \) and \( A \) were constructed. Calculations and constructions were performed on digital images of resolution \( 1024 \times 1024 \). The bottom image in Figure 5 illustrates \( \Sigma(11111111111111) \). This picture was constructed by starting from a computed image of \( \Pi(11111111111111) \) which is a translation of the canonical tiling \( T_{13} \). Figure 6 illustrates four tiling patterns constructed using the tiling rules in Figure 7. The arrows point in directions in which a part of the pattern could be repeated periodically.
Example 5.2. We consider the IFS $F = \{\mathbb{R}^2; f_1, f_2, f_3\}$ defined by
\[ f_1 = \begin{bmatrix} .85 & -.05 & .53842 \\ .05 & .85 & -.15789 \end{bmatrix}, f_2 = \begin{bmatrix} .17 & .22 & .195909 \\ -.22 & .17 & .776364 \end{bmatrix}, f_3 = \begin{bmatrix} -.17 & -.22 & .805 \\ -.22 & .17 & .776364 \end{bmatrix}. \]
The attractor is illustrated in green in the central zone of the top left panel in Figure 8. The attractor is totally disconnected although the image makes it appear to have connected components, because of digitization effects. The red set in the top left panel is an approximation to the relevant part of $B \setminus A$ and was calculated in the same way as in Example 5.1. A close-up is shown on the right, illustrating how we estimated the central open set. To make pictures of some tilings we chose $c_1 = 1$ and $c_2 = c_3 = 8$. The scaling factor for $f_1$ is $s_1 = 0.851 \ldots$ while the scaling factors for $f_2$ and $f_3$ are $s_2 = s_3 = 0.278 \ldots$ so $s_1^8 \approx s_2 = s_3$, but $s_1^8 \neq s_2$. Thus $\Pi(i)$ and $\Xi(i)$ incommensurate tilings for any $i \in \Sigma^\infty$. The bottom left image illustrates the tiling $\Xi(1111111)$ surrounded by $f_1^{-8} \partial C$. It illustrates the relationship between $\partial C$ and some tiles. We observed that there appeared to be eight different tile sizes in any square patch of $\Pi(i)$ digitized at resolution $2048 \times 2048$, when keeping $s_1^{-8} A$ to be roughly the size of the viewing window. This accords with a comment in [4] regarding a result of Schief [21]: “There exists an integer $N$ such that at most $N$ incomparable pieces $A_1 (= f_j(A))$ of size $\geq \varepsilon$ can intersect the $\varepsilon$-neighborhood (sic) of a piece $A_1$ of diameter $\varepsilon$.” (The sets $A_{i_1 \ldots i_n}$ and $A_{i_1 \ldots i_m}$, referred to in the quote as “pieces”, are said to be incomparable if there exists no $k_1 \ldots k_p$ such that $j_1 \ldots j_n = i_1 \ldots i_m k_1 \ldots k_p$ or $i_1 \ldots i_m = j_1 \ldots j_n k_1 \ldots k_p$.) An example of part an unbounded tiling $\Pi(i)$ is illustrated at the bottom right in Figure 8.

**Example 5.3.** In Figure 9 we illustrate the calculation of the central open set for an IFS involving two maps. The method is the same as described in connection with Figures 8 and 5. Here the IFS is close to the one illustrated in the top row of Figure 3, see Example 3.3. Each map here is slightly more contractive and rotated by a small amount. The IFS here is $\{\mathbb{R}^2; f_1, f_2\}$ where

$$f_1 = \begin{bmatrix} -.02447 & .777910 & 0 \\ -.77791 & -.02447 & .78615 \end{bmatrix}, \quad f_2 = \begin{bmatrix} .61156 & -.019221 & -.019221 \\ -.019221 & .61156 & 0 \end{bmatrix}$$

As in Example 5.2 this numerical model is only approximately scaling, but we treat it as though it is by choosing $c_1 = 1$ and $c_2 = 2$. We find this example interesting because it suggests natural situations involving cracks.

**Example 5.4.** See Figure 10. The IFS here is $\{\mathbb{R}^2; f_1, f_2\}$ where

$$f_1 = \begin{bmatrix} .6413 & -.3283 & .3231 \\ .3283 & .6413 & -.133 \end{bmatrix}, \quad f_2 = \begin{bmatrix} -.2362 & .4620 & .8052 \\ .4620 & .2362 & .5093 \end{bmatrix}$$

One can see how $\overline{C}$ is approximately tiled by $f_1(\overline{C})$ and $f_2(\overline{C})$ in the left image. That is, one can see the relationship between $\overline{C}$ and $\{f_1(\overline{C}), f_2(\overline{C})\}$. Note that the actual attractor may not obey the OSC, our estimated “central open set” $\overline{C}$ may not obey the OSC, and the tiling $\Pi_{\overline{C}}(i)$ may be overlapping.

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Figure 9. See Example 5.3. Top left illustrates the attractor of an IFS of two maps (green), and its fast basin minus the attractor (red). The top right hand image illustrates how the boundary, shown in yellow, of the central open set $C$ was estimated. The lower left image shows a patch of $\Pi(i)$, specifically the tiles which meet a (dark) square. The lower right image shows the corresponding patch of of $\Xi(i)$.

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Figure 10. See Example 5.4. The left image shows an approximate central open set $\tilde{C}$ and corresponding fast basin. The sets $f_1(\tilde{C})$ and $f_2(\tilde{C})$ are shown to fit neatly inside $\tilde{C}$, and are surrounded by an approximation to $B \setminus A$. It may well be true that the attractor is overlapping and there is no central open set. Nonetheless, a tiling $\Pi_{\tilde{C}}(i)$, a patch of which is illustrated on the right, is only slightly overlapping.

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