The transition to Fulde-Ferrel-Larkin-Ovchinnikov (FFLO) superfluid phases [1,2] has given rise to much work, but a full theoretical understanding of these inhomogeneous phases has not yet been reached. This is of high practical importance since experiment relies heavily on theory when it needs to test if these phases have actually been observed. Indeed beside their basic theoretical interest which, in addition to superconductivity and its coexistence with magnetism [3], ranges from ultracold gases [4] to quark matter as it might occur in the core of neutron stars [5,6], FFLO phases are also of very high applied interest, because they correspond to the superconducting phases which should appear in superconductors with very high critical fields in the vicinity of the transition to the normal state. A main strategy to observe these transitions in superconductors is to get rid of the orbital currents, responsible for the standard disappearance of the superconducting phase at low critical fields, and reach the regime of Pauli paramagnetic limitation. This can be done in clean quasi two-dimensional systems such as organic compounds made of widely separated conducting planes, or in high $T_c$ compounds. In these cases hopping between planes is very strongly inhibited. So when a strong magnetic field is applied parallel to the planes, the resulting orbital currents perpendicular to the planes are very weak and there is basically no orbital pair breaking effect which leaves open the possibility of transitions to FFLO phases at much higher fields. Nevertheless the small interplane coupling is necessary to suppress phase fluctuations and produce phase which can be properly described by a mean field theory. Indeed observations of this transition have been claimed quite recently in such organic compounds as $\kappa$-(BEDT-TTF)$_2$Cu(NCS)$_2$ [7,8] or $\lambda$-(BETS- TTF)$_2$GaCl$_4$ or FeCl$_4$ [9,10].

Actually when the field is not exactly parallel to the planes there are small orbital currents and associated vortices appear. The physics of the evolution from this vortex state to the pure FFLO phase is quite remarkable, both in the linear [11–13,3] and in the non linear regime [14] since it involves a series of vortex states phases corresponding to Landau levels with increasing quantum number. Compared to this complex structure the limiting 2D FFLO phase sounds simple. Indeed the FFLO transition in 2D systems is believed to be second order and in particular Burkhart and Rainer [15] have studied in details the transition to a planar phase, where the order parameter $\Delta(\mathbf{r})$ is a simple $\cos(\mathbf{q}.\mathbf{r})$ at the transition. This phase has been found by Larkin and Ovchinnikov [2] to be the best one in 3D at $T = 0$ for a second order phase transition. And in 3D it is also found to be the preferred one in the vicinity of the tricritical point and below [16–18], although in this case the transition turns out to be first order (except at very low temperature). However it is not clear that this simple order parameter is always the best one since, as first explored by Larkin and Ovchinnikov, it is in competition with any superposition of plane waves, provided that their wavevectors have all the same modulus.

In this paper we explore the low temperature range in 2D for the pure FFLO phase transition in the simple isotropic case and show that surprisingly there is a cascade of the second order transitions toward order parameters with ever increasing complexity. In principle there is an infinite number of phases. Therefore the pure FFLO phase display a structure just as rich as the vortex phases that we mentioned above, and the interplay between both promises to be of exceptional interest. A first step in this direction has been made recently by Shimahara [19] who, instead of simple cosine, found a transition toward a superposition of up to six plane waves when looking for more complex 'cristalline' structures. We find agreement with his results. However we will see that cristallinity does not give the proper understanding of these phases. Instead we show that, when the temperature is lowered toward $T = 0$, the cascade of transitions toward order parameters with an ever increasing number of plane waves is due to the singular nature of the $T = 0$ limit, which is already apparent in the $T = 0$ wavevector dependence of the second order term in the free energy expansion in powers of the order parameter.

Our study follows the general framework set up by Larkin and Ovchinnikov [2] and used by Shimahara [19] in the 2D case. The free energy difference $\Omega = \Omega_s − \Omega_n$ between the superconducting and the normal state is ex-
expanded to fourth order in powers of the order parameter \( \Delta(\mathbf{r}) = \sum \Delta_{\mathbf{q}} \exp(i\mathbf{q} \cdot \mathbf{r}) \). The second order contribution to \( \Omega \) is \( N_0 \sum q \Omega_2(q, \bar{\mu}, T) \Delta_{\mathbf{q}}^2 \), where \( N_0 \) is the single spin density of states at the Fermi surface and \( \bar{\mu} = (\mu_t - \mu_\downarrow)/2 \) is half the chemical potential difference between the two spin populations. At the FFLO transition we have \( \Omega_2(q, \bar{\mu}, T) = 0 \). Naturally the actual transition corresponds to the largest possible \( \bar{\mu} \) at fixed \( T \). It is easily seen [2] that the corresponding wavevector \( \bar{q} \) for the order parameter is obtained by minimizing:

\[
I(\bar{q}, \bar{\mu}, T) = 2\pi T \text{Im} \sum_{n=0}^{\infty} \frac{1}{\sqrt{(i\omega_n)^2 - \bar{\mu}^2 q^2}} - \frac{1}{i\omega_n} \tag{1}
\]

with respect to \( \bar{q} \), where we have introduced the dimensionless wavevector \( \bar{q} = qv_F/2\bar{\mu} \) and \( \bar{\omega}_n = \omega_n - i\bar{\mu} \) with \( \omega_n = \pi T(2n + 1) \) being the Matsubara frequency. This can be rewritten as an integral over real frequency. At \( T = 0 \) one finds explicitly \( I(\bar{q}, \bar{\mu}, T) = \text{Re} \ln(1 + \sqrt{1 - q^2}) - \ln 2 \), which is minimum for \( \bar{q} = 1 \), in agreement with Shimahara [20] and Burkhardt and Rainer [15]. We note that the location \( \bar{q} = 1 \) of the minimum is a singular point for \( I \) since \( I = \ln(\bar{q}/2) \) for \( \bar{q} > 1 \), so \( I \) is continuous while its derivative is discontinuous for \( \bar{q} = 1 \). For \( T \neq 0 \) but low enough temperature, the condition giving the minimum can be expanded in powers of \( t^{1/2} \) where \( t = T/\bar{\mu} \) is a reduced temperature. We have found that expanding \( \bar{q} - 1 \) up to \( t^2 \) gives a result in excellent agreement with a straightforward numerical evaluation at \( t < 0.1 \). Here (and similarly below) we do not give the somewhat lengthy full expression [21], but we omit the last two terms and display only the lowest order correction which is:

\[
\bar{q} - 1 = \frac{t}{2} \ln \left( \frac{\pi}{2t} \right) \tag{2}
\]

The second order term in \( \Omega \) fixes only the modulus, given by Eq.(2), of the \( N \) wavevectors entering the plane wave expansion of the order parameter. This leaves open all the various possible order parameters arising from any combinations of these plane waves. The selected state will correspond to the lowest fourth order term, which is given [2] in general by:

\[
\frac{N_0}{2} \sum_{i,j} (2 - \delta_{q_i,q_j}) |\Delta_{q_i}|^2 |\Delta_{q_j}|^2 J(\alpha_{q_i},\alpha_{q_j}) \tag{3}
\]

\[+(1 - \delta_{q_i,q_j} - \delta_{-q_i,-q_j}) \Delta_{q_i} \Delta_{-q_i} \Delta_{q_j} \Delta_{-q_j} \tilde{J}(\alpha_{q_i},\alpha_{q_j}) \]

where \( \alpha_{q_i,q_j} \) is the angle between \( q_i \) and \( q_j \). Just as for the second order term \( J(\alpha) \) and \( \tilde{J}(\alpha) \) can be expressed [21] as frequency integrals, expanded at low temperature \( t \ll 1 \) and evaluated for the optimal value of \( \bar{q} \) (given by Eq.(2) to leading order). As for the optimal \( \bar{q} \) we have carried this expansion up to second order in \( t^{1/2} \). If we restrict ourselves to leading order as in Eq.(2) we find for example for \( J(\alpha) \):

\[
16t\bar{\mu}^2 J(\alpha) = \frac{\pi}{\alpha \cosh^2[(1/4) \ln \frac{\bar{\mu}}{\mu} - \beta^2/2]} - \frac{8}{\sqrt{\pi}(1 + \cos(\alpha/2))} \int_0^\infty dv \exp(-v^2) \frac{1}{v^2 + \beta^2} \tag{4}
\]

where we have set \( \beta^2 = (1 - \cos(\alpha/2))/t \). The result up to second order in \( t^{1/2} \) is not much more complicated. A similar result [21] is found for \( \tilde{J}(\alpha) \).

![Fig. 1.](image-url)

**FIG. 1.** \( J(\alpha) \) for values 0.01 and 0.05 of the reduced temperature \( t = T/(\bar{q} \bar{\mu}) = t/\bar{q} \) (with essentially \( \bar{q} \approx 1 \) in all this range). The lines are calculated using our low temperature expansion for \( J(\alpha) \), while the points correspond to the direct numerical summation over Matsubara frequencies. The dashed line in the insert (which displays the same results with different scale) is the asymptotic behaviour for \( J(\alpha) \) in the \( T \to 0 \) limit Eq.(5).

Plots of \( J(\alpha) \) for various low temperatures can be seen in Fig. 1, either obtained from these low temperature expressions or from exact calculations of \( J(\alpha) \) obtained by direct numerical summation over Matsubara frequencies after analytical angular averaging. We see that, at this level of accuracy, both agree remarkably well up to rather high temperatures which shows that our low temperature expansion is quite under control. Now Fig.1 shows that \( J(\alpha) \) has at low \( T \) a quite remarkably structured behaviour which can be easily understood from some limiting cases. First let us take the limit \( T \to 0 \) at fixed \( \alpha \). This implies \( \beta^2 \to \infty \), the first term in Eq.(4) goes to zero and the easy integration leads to:

\[
J(\alpha) = -\frac{1}{4\bar{\mu}^2 \sin^2(\alpha/2)} \tag{5}
\]

On the other hand if at fixed \( T \) we take the limit \( \alpha \to 0 \), we have \( \beta^2 \to 0 \). The limiting behaviour of the integral is \( \pi/(2\beta) - \pi/2 \), the dominant divergent contributions from the two terms in Eq.(4) cancel out giving:
which goes naturally to infinity for $T \to 0$. From these two limiting situations we can understand that, for most of the $\alpha$ range, $J(\alpha)$ is negative as it can be seen from Eq.(5) and it goes to large negative values when $\alpha$ gets very small. On the other hand for $\alpha = 0$ or very small $J(\alpha)$ is positive and very large, as results from Eq.(6) (surprisingly $J(\alpha)$ starts first to increase strongly before going down to very negative values).

We have made the same kind of treatment [21] for $\tilde{J}(\alpha)$. The limit $T \to 0$ with fixed $\alpha$ leads to $\tilde{J}(\alpha) = -\pi/4 \sin \alpha$ while in the limit $\alpha \to 0$ at fixed $T$ we find $\tilde{J}(\alpha) = -1/4$. These two cases make reasonable that $\tilde{J}(\alpha)$ is always negative, as we find. However, just as for $J(\alpha)$, one finds also a singular behaviour at small $\alpha$. While for $\beta \approx \alpha/(8t)^{1/2} \gg 1$ and $\tilde{J}(\alpha)$ diverges as $-\pi/4 \alpha$, it goes to the finite value $-1/4$ for $\alpha = 0$. Nevertheless the divergent behaviour in $\alpha^{-1}$ is weaker than the one in $\alpha^{-2}$ found for $J(\alpha)$ and similarly $\tilde{J}(0) \gg |\tilde{J}(0)|$ at low $T$. So $J(\alpha)$ will play the dominant role and at first we omit $\tilde{J}(\alpha)$ from our considerations.

Let us now consider which is the most favorable order parameter. We will always find that the fourth order term is positive, so we have to make it as small as possible in order to minimize the free energy. We note first that there is unequivocally a strongly positive contribution, proportional to $J(0)$, coming from the $N$ terms $\mathbf{q}_\alpha = \mathbf{q}_i$ in the first sum of Eq.(3). However their unfavorable effect is relatively reduced when $N$ increases since there are $N^2$ terms in this sum. Since we clearly have to avoid other positive contributions of this kind, we want to forbid the angle domain $0 < \alpha < \alpha_0$ where we can take $\alpha_0$ as giving $J(\alpha_0) = 0$, which is very close to the (strongly negative) minimum of $J(\alpha)$ as it can be seen on Fig. 1. So the angle between any two different wavevectors $\mathbf{q}_i$ and $\mathbf{q}_j$ should be at least $\alpha_0$. On the other hand it is of interest to have angles close to $\alpha_0$ since they give strongly negative contributions to $\Omega$. Since it is better to have the number of wavevectors $N$ as large as possible, we come from symmetry reasons to the conclusion that we have to choose the wavevectors $\mathbf{q}_i$ angularly equally spaced with the angle between two nearest wavevectors as close as possible to $\alpha_0$. Now $\alpha_0 \to 0$ when $T \to 0$. Therefore we have indeed a cascade of transitions corresponding to order parameters with an ever increasing number $N$ of wavevectors when $T \to 0$. This singular behaviour is clearly linked to the fact that it is not possible, in particular for $J(\alpha)$, to perform an expansion in the limit $T \to 0$, i.e. this limit is singular.

More precisely it is quite a difficult problem to handle this minimization from scratch, and for example to prove that the minimum $\Omega$ is not obtained for a completely disordered situations. However symmetry makes such a result quite unlikely. On the other hand if we take for granted that the wavevectors are equally spaced angularly it is easy to prove that the weight $|\Delta_{\mathbf{q}_i}|$ are all equal, because this problem can be mapped on a tight binding Hamiltonian on a ring. Conversely if we assume that all these weights are equal the problem is equivalent to find the equilibrium position of atoms on a ring with repulsive short range interaction and attractive long range interaction. We expect the equilibrium to be a crystalline structure corresponding in our case to equally spaced wavevectors.

$$J(0) = \frac{1}{4\tilde{V}^2} \frac{1}{t}$$

$$(6)$$

FIG. 2. $G_0^{-1}$ as a function of temperature for various plane waves number $N$ (for clarity only solutions up to $N = 14$ are presented). The arrows give the values of the successive transitions. In the upper panel the dashed lines are the exact Matsubara summations. The full lines in the upper panel, as well as all the lines in the lower panel, are from the low temperature expansion for $J(\alpha)$ and $\tilde{J}(\alpha)$. In the insert the full dots are the exact temperature locations of the transitions together with the corresponding values of $N$ (the empty dots just give $N - 2$). The full line gives the low temperature evaluation of these transitions from the solution of a simple transcendental equation, and the almost undistinguishable dashed line comes from the explicit first iteration solution of this equation.
In the low temperature limit \((t \ll 1)\), we can easily be more quantitative and write simple analytical answers for our problem. Indeed the result is dominated by the small \(\alpha\) behaviour of \(J(\alpha)\). In this range Eq.(4) becomes:

\[
\mu^2 J(\alpha) \approx \frac{\pi}{16t \alpha \cosh^2 X} - \frac{1}{\alpha^2} \tag{7}
\]

where \(X = x - (1/4) \ln(\pi/2t)\) with \(x = \alpha^2/16t\). The second term is the asymptotic form Eq.(5). The zero of \(J(\alpha)\) we are looking for is found for large \(X\) and satisfies \(\alpha^2 = (k/\sqrt{2})x^{1/4}\) with \(k = (2\pi^3/\ell^2)^{1/4}\), which is easily solved iteratively for large \(k\). The leading order \(x = \ln(k/\sqrt{2})\) gives \(\alpha_0 = (8t \ln[(\pi^2/2)^{1/2}/t])^{1/2}\). The first order iteration is \(x = \ln(k/\sqrt{2}) + 1/4 \ln[\ln(k/\sqrt{2})]\). Then the optimum value of \(N\) is given by the integer value of \(2\pi/\alpha_0\), which conversely fixes the temperature at which the system switches from \(N-2\) to \(N\) wavevectors. In the insert of Fig.3. one sees that the first iteration gives results almost undistinguishable from the exact solution of the above transcendental equation for \(x\).

Exact results can be obtained by handling numerically the various steps. The minimization of the free energy with respect to \(|\Delta_\mathbf{q}|\) gives \(\Omega/N_0 = -\Omega_2^2(q, \mu, T)/G_2(N)\) with:

\[
NG_2(N) = 2J(0) + 4 \sum_{n=1}^{N-1} J(2\pi n/N) \tag{8}
\]

where we have omitted the \(\bar{J}\) terms for simplicity. One sees in Fig. 3 the results (including the \(\bar{J}\) terms) for \(G_2^{-1}(N)\), i.e. essentially the free energy, for various values of \(N\). The stable state has the highest \(G_2^{-1}(N)\). The cascade of phase transitions is clearly seen in the lower panel, corresponding to the lower temperature range. The arrows show the successive transition temperatures. The results from straight numerical Matsubara summation can not be distinguished from those obtained from our low \(T\) analytical expression for \(J(\alpha)\) and \(\bar{J}(\alpha)\), except for the higher temperature, as seen in the upper panel. The critical temperatures for switching from \(N-2\) to \(N\) are shown as dots in the insert of Fig. 3. It can be seen that our above asymptotic calculation for these temperatures, shown as full curves, are actually quite good.

At low temperature we can make an asymptotic calculation of \(G_2(N)\). As seen on Fig. 1 \(J(\alpha)\) switches rapidly above \(\alpha_0\) to its large angle asymptotic behaviour. Moreover at low \(T\) the number of plane waves \(N\) is large and in Eq.(8) the sum is dominated by the small angles terms, which leads [21] to:

\[
\mu^2 G_2(N) \approx \frac{1}{2Nt} - \frac{N}{3} \tag{9}
\]

This expression corresponds in Fig. 3 to the rising part, on the low temperature side, of \(G_2^{-1}(N)\). On the other hand the downturn is due to contributions from the positive part of \(J(\alpha)\) which can not be evaluated so easily.

When we substitute in Eq.(9) the value \(N = 2\pi/\alpha_0\) for the optimum plane wave number we have:

\[
\mu^2 G_2(N) = \frac{\alpha_0}{4\pi t} \left(1 - \frac{8\pi^2 t}{3\alpha_0^2}\right) \tag{10}
\]

Since \(8t/\alpha_0^2 \sim 1/\ln(1/t)\) from our above evaluation, we find that \(G_2(N)\) is always be positive in the low temperature range, which means that the transition stays always second order. When we take into account the contribution of the \(\bar{J}\) terms in Eq. 3 that we had omitted so far, one gets an additional \(-\ln N\) contribution in Eq.(9), which arises only when \(N\) is even because for equally spaced wavevectors, the \(\bar{J}\) terms contribute only in this case since they require opposite wavevectors. This contribution is just enough to systematically tilt the balance in favor of even \(N\). In this respect the phase \(N = 3\) found by Shimahara [19] appears as an exception. Otherwise our cascade of phase transitions contains only order parameters with an even number of plane waves, and the order parameter is just a real sum of cosines, with still a degeneracy due to the freedom of choosing the phases in these cosines.

We note that, although the transitions between the various FFLO phases and the normal phase are second order, we expect the transitions between different FFLO phases in the superfluid domain to be first order since one can not go continuously from a given order parameter to the next one. Finally we have seen that these successive transitions are directly due to the singularity which occurs in 2D at \(T = 0\). This singularity itself arises because the two Fermi circles corresponding to opposite spins come just in contact when one applies the shift corresponding to the wavevector \(q\) of the FFLO phase. Therefore this singularity is a general feature of 2D physics and we expect it to give rise to similar consequences in more realistic and more complex models describing actual physical systems.

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