Reallocation Problems with Minimum Completion Time

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Abstract
Reallocation scheduling is one of the most fundamental problems in various areas such as supply chain management, logistics, and transportation science. In this paper, we introduce the reallocation problem that models the scheduling in which products are with fixed cost, non-fungible, and reallocated in parallel, and comprehensively study the complexity of the problem under various settings of the transition time, product size, and capacities. We show that the problem can be solved in polynomial time for a fundamental setting where the product size and transition time are both uniform. We also show that the feasibility of the problem is NP-complete even for little more general settings, which implies that no polynomial-time algorithm constructs a feasible schedule of the problem unless P=NP. We then consider the relaxation of the problem, which we call the capacity augmentation, and derive a reallocation schedule feasible with the augmentation such that the completion time is at most the optimal of the original problem. When the warehouse capacity is sufficiently large, we design constant-factor approximation algorithms under all the settings. We also show the relationship between the reallocation problem and the bin packing problem when the warehouse and carry-in capacities are sufficiently large.

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1 Introduction

Problem setting: Suppose that there are several warehouses that store many products (or items). Some of the products are already stored at the designated warehouses, and the others are stored at tentative warehouses. Such temporarily stored products should be reallocated to designated warehouses. Namely, each product \( p \) at the tentative warehouse \( s(p) \) is required to be reallocated to the designated warehouse \( t(p) \). To reallocate product \( p \), it takes a certain length of time \( \tau(p) \), called transit time of \( p \). Each product \( p \) also has size \( size(p) \), and each warehouse has three kinds of capacities, that is, (1) the capacity of warehouse itself, (2) carry-in size capacity, and (3) carry-out size capacity. Capacity (1) restricts the total size of products stored in a warehouse at each moment. Capacities (2) and (3) restrict the total size of products that are simultaneously carried in and out, respectively. In this setting, we consider the problem of finding a reallocation schedule with minimum completion time.

As an illustrative example of our problem, let us consider the following scenario. There are
6 products $p_1, p_2, p_3, p_4, p_5$ and $p_6$ of sizes $1, 3, 5, 6, 3$ and $4$, respectively. Two warehouses $W_1$ and $W_2$ have the warehouse capacities $20$ and $10$, carry-in capacities $6$ and $5$, and carry-out capacities $5$ and $5$, respectively. The transit time satisfies $\tau(p_i) = 1$ for every $i$ except $i = 6$, and $\tau(p_6) = 2$. Initially, products $p_1, p_2, p_3$ and $p_4$ are stored in $W_1$, which satisfies the warehouse capacity constraint, since their total size $15$ is smaller than the warehouse capacity $20$ of $W_1$. Products $p_5$ and $p_6$ are initially stored in $W_2$, which also satisfies the warehouse capacity constraint. Suppose that $p_1, p_2$ and $p_3$ are designated to be stored in $W_2$, whereas $p_4$, $p_5$ and $p_6$ are designated to be stored in $W_1$. Note that all products can be stored in the designated warehouses, since the designated allocation satisfies the warehouse capacity constraint. We also note that $p_1, p_2$ and $p_3$ cannot be moved simultaneously, since the carry-out capacity constraint is violated. Figure 1 depicts the initial and target configurations of this example.

![Figure 1](image_url)

In this example, we need to move $p_1, p_2$ and $p_3$ to $W_2$, which should be done separately due to carry-out constraint $5$. Thus we consider to move $p_1$ and $p_2$ at time $0$ and then move $p_3$ at time $1$, for example. In such a case, we also need to move either $p_5$ or $p_6$ to $W_1$ at time $0$, otherwise it violates capacity $10$ of warehouse $W_2$ at time $1$. Thus a possible schedule is to carry out $p_1$ and $p_2$ from $W_1$, and $p_5$ from $W_2$ at time $0$, and then carry out $p_3$ from $W_1$ and $p_6$ from $W_2$ at time $1$. The completion time is $3$ in this case, because $\tau(p_6) = 2$ and thus $p_6$ reaches $W_1$ at time $3$. One may consider that it is better to carry out $p_5$ instead of $p_5$ at time $0$ and then carry out $p_6$ at time $1$, but it violates carry-in constraint $6$ of $W_1$ at time $2$. By this observation, we can see that the minimum completion time is $3$ in this example.

**Applications and related work:** Reallocation scheduling is one of the most fundamental problems in various areas such as supply chain management, logistics, and transportation science. Many models and variants of reallocation have been studied from both theoretical and practical viewpoints. In fact, our problem setting is initiated by an industry-academia joint project of Advanced Mathematical Science for Mobility Society by Toyota Motor Corporation and Kyoto University [1]. The reallocation models and their variants are categorized by the following aspects: (1) reallocation cost, (2) fungibility of products, and (3) parallel/sequential execution. Our problem assumes that (1) a cost (i.e., transit time) of reallocating a product is given in advance, (2) products are not fungible, and (3) reallocations are done in a parallel way, but many other settings are possible.

For example, the dial-a-ride problem is regarded as the vehicle routing problem for reallocation [3], which designs vehicle routes and schedules for customers who request the pick-up and drop-off points. In the dial-a-ride problem, (1) the cost of reallocation
depends on the routes, (2) customers are not fungible of course, (3) reallocation is done in a sequential way for one vehicle. The bike-sharing rebalancing problem is to schedule trucks for re-distributing shared bikes with the objective of minimizing total cost [4]. In the bike-sharing rebalancing problem, (1) the cost of reallocation also depends on the routes of trucks and (3) reallocation is done in a sequential way for one truck again, but (2) shared bikes are fungible, i.e., the desired bike distribution is created without distinguishing between bikes.

In these problems, the costs arise from rather transportation of vehicles for delivery than reallocation of products. To investigate the nature of reallocation itself, it might be reasonable to assume that a cost (i.e., transit time) of reallocating a product is given in advance. Miwa and Ito [12, 13, 14] focus on reallocation scheduling under the setting: (1) a cost of reallocating products is uniform, (2) products are not fungible, and (3) reallocations are done in a sequential way, which is different from ours. Although they do not deal with non-uniform transition time, several intractability results as well as polynomial solvability ones were obtained [12, 13, 14]. A problem similar to the model of Miwa and Ito was also considered in [8], in the context of fund circulation. In [6, 7], Hayakawa pointed out the importance of controlling a payment ordering among banks in terms of the stability of economics, and introduced the problem to find a payment ordering that minimizes extra money to put in order to make payments without shortage, which can be viewed as a reallocation scheduling problem. Here the extra money corresponds to vacant spaces for the reallocation. From such a viewpoint, the computational complexity of the fund circulation was investigated in [8].

Our contribution: In this paper, we consider the reallocation scheduling in which products are with fixed cost, non-fungible, and reallocated in parallel, which we simply call the reallocation problem. We investigate the computational complexity of the reallocation problem under various scenarios. We first see the most basic scenario where both the product size and transition time are uniform. In this scenario, we present an $O(mn \log m)$-time algorithm to find a reallocation schedule with minimum completion time, where $m$ and $n$ denote the number of products and warehouses, respectively. The algorithm utilizes a cycle decomposition for the so-called demand graph. When the product size is only uniform, the problem turns out to be NP-hard, and we show that the algorithm for the basic scenario above provides a reallocation schedule whose completion time is at most twice the optimal, if one of the carry-out and carry-in capacities is sufficiently large in addition. Here we note that the carry-in and carry-out capacity constraints are always satisfied if the carry-in and carry-out capacities are sufficiently large, respectively. For more general scenarios, even the feasibility is NP-complete for very restricted settings: (1) we have only two warehouses and (2) the transit time and the product size satisfy $\tau(p) = 1$ and $\text{size}(p) \in \{1, 2\}$ for every product $p$. Due to the hardness of the feasibility, no polynomial-time algorithm can construct a feasible schedule of the problem unless P=NP. Instead, we admit the relaxation of capacity constraints, which we call capacity augmentation. Namely, we augment the original capacities with additional ones, and try to find a reallocation schedule such that it is feasible with the augmented instance and has small completion time. By utilizing the bi-criteria algorithm schemes for the generalized assignment problem [15] and the 2-sided placement problem [11], we can find in polynomial time a reallocation schedule such that (i) the completion time is at most the minimum completion time for the original problem, (ii) warehouses can be stored at most twice of the original capacities (i.e., warehouse capacity $c(w)$ is augmented with $c(w)$ for each warehouse $w$), and (iii) carry-out/in capacities are enlarged to the original ones plus the largest and second largest carried-out/in product sizes (i.e., the carry-out and
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carry-in capacities for each warehouse \( w \) are respectively augmented with \( \sigma^+_i(w) + \sigma^-_i(w) \) and \( \sigma^+_j(w) + \sigma^-_j(w) \), where \( \sigma^+_i(w) \) and \( \sigma^-_i(w) \) denote the \( i \)-th largest size of products that are initially and finally stored in warehouse \( w \), respectively. In the scenario when the carry-in (resp., carry-out) capacity is sufficiently large, condition (iii) is strengthened into the original carry-out (resp., carry-in) capacity plus the largest size of carry-out (resp., carry-in) product size. We remark that if both carry-in and carry-out capacities are sufficiently large, the problem turns out to be trivial, where an optimal reallocation schedule can be obtained by sending all products in time 0. Table 1 summarizes the results above, i.e., for general warehouse capacity.

**Table 1** The summary of results for general warehouse capacity.

| Product size (\( size(p) \)) | Transition time (\( \tau(p) \)) | Capacity | Complexity & approximability | Capacity augmentation |
|-------------------------------|-------------------------------|----------|-------------------------------|----------------------|
| uniform                       | uniform                       | general  | \( O(m \log m) \)             | ———                 |
| uniform                       | general \((+\infty)\)         | general  | 2-apx                         | ———                 |
| uniform                       | general                       | general  | NPC                           | ———                 |
| general                       | general                       | \(+\infty\)         | \( O(m + n) \)                | ———                 |
| general                       | uniform                       | \(+\infty\)         | NPC                           | \( c = 2\overline{c} \) |
| general                       | general                       | \(+\infty\)         | NPC                           | \( \overline{c} = 2\overline{c} \) |
| general                       | general                       | \(+\infty\)         | NPC                           | \( \overline{c} = 2\overline{c} \) |

*: “NPC” stands for NP-completeness of the feasibility of the reallocation problem and “NPH” stands for NP-hardness for the reallocation problem.

†: \( \overline{c} \), \( d^+ \), and \( d^- \) respectively denote augmented warehouse, carry-out, and carry-in capacities.

We finally consider the scenario when all warehouses have sufficiently large capacities, where the summary of our results can be found in Table 2. In this setting, we propose a 6-approximation algorithm for our problem that transforms a relaxed schedule above into a feasible schedule such that the completion time is at most 6 times of the minimum completion time. In the setting when at least one of carry-in and carry-out capacities is sufficiently large in addition, we present a 7/4-approximation algorithm that employs as a subroutine the first-fit-decreasing algorithm for the bin packing problem. If we further assume that the transition time is uniform, the approximation ratio is improved to 3/2, which is best possible in the setting. This follows from the fact that the problem in this setting is essentially equivalent to the bin packing problem. We also show that the reallocation problem can be solved in polynomial time, if the product size is uniform and at least one of the carry-in and carry-out capacities are sufficiently large in addition.
Table 2 The summary of results for sufficiently large warehouse capacity.

| product size (size(p)) | transition time (τ(p)) | complexity & approximability* |
|------------------------|------------------------|-------------------------------|
| uniform                | general                | O(n + m log m) [Th. 5.1]        |
| uniform                | general                | NPH [Th. 6.7]                  |
| general                | uniform                | NPH [Th. 6.5]                  |
| general                | general                | NPH [Th. 6.5]                  |
| general                | general                | NPH [Th. 6.5]                  |
| general                | general                | NPH [Th. 6.5]                  |

*: “NPH” stands for NP-hardness for the reallocation problem.

The rest of the paper is organized as follows. In Section 2, we give formal definitions and basic observations. Section 3 presents a polynomial-time algorithm for the setting where both size and transition time of products are uniform. Section 4 considers the most general setting; we introduce a notion of capacity augmentation, and present a polynomial-time algorithm under augmented capacities. In Section 5, we consider the setting where the capacities of warehouses are sufficiently large, and present constant-factor approximation algorithms for the setting. Section 6 shows hardness results in various settings.

2 Preliminaries

We first define the reallocation problem. We also introduce demand graphs of the problem, which plays a key role in designing an efficient algorithm for the reallocation problem with uniform product size and transit time.

Let $P$ be a set of products (or items), and let $W$ be a set of warehouses, where $m = |P|$ and $n = |W|$. Each product $p \in P$ has size $\text{size}(p) \in \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of nonnegative reals. It is initially stored in a source warehouse $s(p) \in W$, required to be reallocated to a sink warehouse $t(p) \in W$, and its reallocation from $s(p)$ to $t(p)$ takes transit time $\tau(p)$ ($>0$). Here we assume that $s(p) \neq t(p)$ for all $p \in P$. Namely, if it is sent from $s(p)$ at time $\theta$, it reaches $t(p)$ at time $\theta + \tau(p)$. Each warehouse $w \in W$ has capacity $c(w) \in \mathbb{R}_+$, which represents the upper bound of the total size of products stored in $w$ at any time. Moreover it has carry-out and carry-in capacities $d^+(w), d^-(w) \in \mathbb{R}_+$, which respectively represent the upper bounds of the total size of products allowed to be sent from and be received at $w$ at every time.

We consider reallocation schedules of given products in the discrete-time model, which means that for any product $p \in P$, sending time $\theta_p$ and transit time $\tau(p)$ are nonnegative and positive integers, respectively. For a warehouse $w \in W$, let $P^+(w)$ and $P^-(w)$ respectively denote the sets of products $p \in P$ initially and finally stored at $w$, i.e., $P^+(w) = \{p \in P \mid s(p) = w\}$ and $P^-(w) = \{p \in P \mid t(p) = w\}$. A reallocation schedule is a mapping $\varphi : P \rightarrow \mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of nonnegative integers, and is called feasible if it
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satisfies the following three conditions:

(i) At each time $\theta \in \mathbb{Z}_+$, the total size of products departing from $w$ is at most $d^+(w)$ for every warehouse $w \in W$, i.e., $\sum_{p \in P^+(w), \tau(p) = \theta} \text{size}(p) \leq d^+(w)$.

(ii) At each time $\theta \in \mathbb{Z}_+$, the total size of products arriving at $w$ is at most $d^-(w)$ for every warehouse $w \in W$, i.e., $\sum_{p \in P^-(w), \tau(p) = \theta} \text{size}(p) \leq d^-(w)$.

(iii) At each time $\theta \in \mathbb{Z}_+$, the total size of products located in $w$ is at most $c(w)$ for every warehouse $w \in W$, i.e., $\sum_{p \in P(w), \tau(p) \leq \theta} \text{size}(p) \leq c(w)$.

Constraints (i) and (ii) are respectively called carry-out and carry-in capacity constraints, and Constraint (iii) is called warehouse capacity constraint. In this paper, we assume that $\text{size}(p) \leq d^+(w)$ for all $p \in P^+(w)$, $\text{size}(p) \leq d^-(w)$ for all $p \in P^-(w)$, $\sum_{p \in P(w)} \text{size}(p) \leq c(w)$, and $\sum_{p \in P^-(w)} \text{size}(p) \leq c(w)$, since they are clearly necessary conditions for the feasibility and can be checked in linear time. Our problem, called the reallocation problem, is to compute a feasible reallocation schedule with the minimum completion time. Here the completion time $T$ of a reallocation schedule $\varphi$ is defined as $T = \max_{p \in P} \{\tau(p) + \rho(p)\}$.

We here remark that if both carry-in and carry-out capacities are sufficiently large, then the problem is trivial and an optimal reallocation schedule can be obtained by sending all products in time 0, i.e., letting $\varphi(p) = 0$ for all products $p \in P$. Indeed, in this case, the carry-in and carry-out capacity constraints are always satisfied. The warehouse capacity constraints are also satisfied since every warehouse $w \in W$ has no product in $P^+(w)$ in it at each time $\theta > 0$ and satisfies $\sum_{p \in P^-(w)} \text{size}(p) \leq c(w)$ by assumption.

**Theorem 2.1.** The reallocation problem can be solved in $O(m + n)$ time if $d^+ = d^- \equiv \infty$.

Before ending this section, let us fix some notation on graphs and define demand graphs, which frequently appear in the subsequent sections. An undirected or directed graph $G$ is an ordered pair of its vertex set $V(G)$ and edge set $E(G)$ and is denoted by $G = (V(G), E(G))$, or simply $G = (V, E)$. In an undirected graph $G$, the *degree* of a vertex $v$, denoted by $\delta_G(v)$, is the number of edges incident to $v$. In a directed graph $G$, the *out-degree* and *in-degree* of a vertex $v$ are respectively defined as $\delta_G^+(v) = \left|\{(w, v) \in E(G) \mid w \in V(G)\}\right|$ and $\delta_G^-(v) = \left|\{(u, v) \in E(G) \mid u \in V(G)\}\right|$. We denote by $\Delta^+(G)$ and $\Delta^-(G)$ the maximum out-degree and in-degree of a digraph $G$, respectively. Given a set $W$ of warehouses and a set $P$ of products, we represent the demand relationship for products as a directed multigraph $G = (W, \{(s(p), t(p)) \mid p \in P\})$, which is called the *demand graph* of the reallocation problem.

### 3 Uniform product size and transit time

In this section, we consider the reallocation problem with the basic scenario in which product size and transit time are both uniform. We show that by using a cycle decomposition of the demand graph, the reallocation problem can be solved in polynomial time. More precisely, we have the following result.

**Theorem 3.1.** The reallocation problem with uniform product size and uniform transit time can be solved in $O(mn \log m)$ time.

Here we recall that $m = |P|$ and $n = |W|$. Before proving the theorem, let us provide a general lower bound on the minimum completion time. For a warehouse $w \in W$, let

$$\rho(w) = \max\left\{\frac{\sum_{p \in P^+(w)} \text{size}(p)}{d^+(w)}, \frac{\sum_{p \in P^-(w)} \text{size}(p)}{d^-(w)}\right\},$$

$$\rho_{\max} = \max_{w \in W} \rho(w).$$
Then we have the following lemma.

**Lemma 3.2.** For the reallocation problem, the minimum completion time is at least \( \rho_{\text{max}} + \min_{p \in P} \{ \tau(p) \} - 1 \).

**Proof.** By carry-out capacity constraints, for any warehouse \( w \in W \), we need at least \( \lceil \sum_{p \in P^+} \text{size}(p) \rceil \) steps to send all the products \( p \in P^+(w) \). Thus the minimum completion time is at least \( \lceil \sum_{w \in W} \frac{\sum_{p \in P^+} \text{size}(p)}{d^+(w)} \rceil + \min_{p \in P} \{ \tau(p) \} - 1 \). Similarly, by carry-in capacity constraints, the minimum completion time is at least \( \lceil \sum_{w \in W} \frac{\sum_{p \in P^-} \text{size}(p)}{d^-(w)} \rceil + \min_{p \in P} \{ \tau(p) \} - 1 \), which proves the lemma.

Note that the completion time might be far from the lower bound in Lemma 3.2. However, we below show that it matches the lower bound \( \rho_{\text{max}} + \mu - 1 \) if product size and transit time are both uniform, where \( \mu \) denotes the uniform transit time, i.e., \( \mu = \tau(p) \) for all \( p \in P \).

In the rest of this section, we assume without loss of generality that

\[
\text{size}(p) = 1
\]

for all products \( p \in P \), and all functions \( d^+, d^- \), and \( c \) are integral. This is because an instance equivalent to the original one is obtained by using \( d^+(w) = \lceil \frac{d^+(w)}{c} \rceil \), \( d^-(w) = \lceil \frac{d^-(w)}{c} \rceil \), and \( c(w) = \lceil \frac{c(w)}{c} \rceil \) for all \( w \in W \), if \( \text{size}(p) = k \) for all \( p \in P \). Note that in this case we have \( \rho(w) = \max \{ \lceil \frac{|P^+(w)|}{d^+(w)} \rceil, \lceil \frac{|P^-(w)|}{d^-(w)} \rceil \} \) for all \( w \in W \). Thus we have \( \rho_{\text{max}} \leq \mu \).

Let us now present a simple but important observation of feasible reallocation schedules. For two positive integers \( a \) and \( b \) with \( a \leq b \), let \( [a, b) = \{ a, a + 1, \ldots, b \} \). Let \( Q = \{ q_1, q_2, \ldots, q_\ell \} \) be a set of products which forms a simple cycle in the demand graph \( G = (W, E) \), i.e., \( Q \) satisfies \( t(q_i) = s(q_{i+1}) \) for \( i \in [1, \ell] \) and \( s(q_i) \neq s(q_j) \) for any distinct \( i \) and \( j \), where \( q_{\ell+1} \) is defined as \( q_1 \). Then we claim that all products in \( Q \) can be sent simultaneously.

Consider a situation where all products in \( Q \) depart at time \( \theta \) (and no other product departs at time \( \theta \)). By our assumption that \( \text{size}(p) \leq d^+(w) \) for every product \( p \in P^+(w) \) and \( \text{size}(p) \leq d^-(w) \) for every product \( p \in P^-(w) \) as mentioned in Section 3, the carry-out and carry-in capacity constraints are satisfied. Moreover, \( t(q_i) \) has a room for \( q_i \)'s arrival at time \( \theta + \mu \), since \( q_{i+1} \) leaves from \( s(q_{i+1}) \) at time \( \theta \). Thus warehouse capacity constraints are also satisfied, which implies the claim. More generally, if a set \( Q \) of products can be partitioned into vertex-disjoint simple cycles, then they can be sent simultaneously.

Based on this observation, we construct an efficient algorithm for the reallocation problem when product size and transit time are both uniform. In order to explain it smoothly, we first consider the following subcase:

\[
d^+(w) = d^-(w) = 1 \text{ and } |P^+(w)| = |P^-(w)| \text{ for every warehouse } w \in W.
\]

We show that \( P \) can be partitioned into \( \rho_{\text{max}} \) sets \( P_i \) \((i \in [1, \rho_{\text{max}}])\), each of which forms vertex-disjoint simple cycles in the demand graph. This implies the existence of a feasible reallocation schedule with the completion time \( \rho_{\text{max}} + \mu - 1 \). By Lemma 3.2, we can see that it is an optimal schedule. Note that \( \rho_{\text{max}} = \max_{w \in W} |P^+(w)| \) by (2).

Let \( H \) be the bipartite graph obtained from the demand graph \( G = (W, E) \) by creating two copies \( w_1 \) and \( w_2 \) of each vertex \( w \in W \) and adding an edge \((w_1, v_2)\) for each \((w, v) \in E\); namely, \( V(H) = W_1 \cup W_2 \) and \( E(H) = \{ (w_1, v_2) \mid (w, v) \in E \} \), where \( W_1 = \{ w_i \mid w \in W \} \) for \( i = 1, 2 \). By assumption (2), \( \rho_{\text{max}} \) represents the maximum degree \( \Delta(H) \) of \( H \) and \( \delta_H(w_1) = \delta_H(w_2) \) holds for all \( w \in W \). Let us further modify the graph \( H \). Let \( H^* \) be the
bipartite graph obtained from $H$ by adding $\rho_{\max} - \delta^+_G(w)$ multiple edges $(w_1, w_2)$ for every $w \in W$. Note that $H^*$ is $\rho_{\max}$-regular, where a graph is called $d$-regular if every vertex has degree $d$. It is known that the edge set of a $d$-regular bipartite graph can be partitioned into $d$ perfect matchings \cite{10}. Moreover, we have the following lemma.

\textbf{Lemma 3.3.} Let $M^*$ be a perfect matching in $H^*$. Then $M^*$ corresponds to vertex-disjoint simple cycles in the demand graph $G$.

\textbf{Proof.} For a perfect matching $M^*$ in $H^*$, let $M = M^* \cap E(H)$ and let $H[M]$ be the subgraph of $H$ induced by $M$. Then we note that $M$ is a matching of $H$ such that the degree of $w_i$ with respect to $H[M]$ is equal to that of $w_2$ for any $w \in W$. Hence, $M^*$ corresponds to vertex-disjoint simple cycles in the demand graph $G$.

The next lemma follows from Lemma 3.3 and the argument before it.

\textbf{Lemma 3.4.} If product set $P$ satisfies $|P^+(w)| = |P^-(w)|$ for every warehouse $w \in W$, then it can be partitioned into sets $P_i$ ($i \in \{1, \max_{w \in W} |P^+(w)|\}$) such that each $P_i$ forms vertex-disjoint simple cycles in the demand graph.

Consequently, if (2) is satisfied, by setting $\varphi(p) = i - 1$ if $p \in P_i$, we obtain an optimal reallocation schedule $\varphi$, whose completion time is equal to $\rho_{\max} + \mu - 1$.

We next consider a slightly generalized case in which for every warehouse $w \in W$, $d^+(w) = d^-(w) = 1$ holds, but $|P^+(w)| = |P^-(w)|$ does not necessarily hold. Let $W^+$ and $W^-$ respectively denote the sets of warehouses $w$ with $|P^+(w)| > |P^-(w)|$ and $|P^+(w)| < |P^-(w)|$. Then we add extra products $p$ such that $s(p) \in W^-$, $t(p) \in W^+$, and $\text{size}(p) = 1$, until $|P^+(w)| = |P^-(w)|$ holds for all $w \in W$. Note that it can be done by arbitrarily pairing warehouses in $W^-$ and $W^+$, since $\sum_{w \in W} |P^+(w)| = \sum_{w \in W} |P^-(w)|$. We also claim that every warehouse has enough vacancy for such extra products. Indeed, let $\hat{P}$ denote the resulting product set. Then we have $|\hat{P}^+(w)| = |\hat{P}^-(w)| \leq \max\{|P^+(w)|, |P^-(w)|\}$, which implies that warehouse capacity constraints at the initial time and the last time are satisfied if all products are sent correctly. Moreover, we can see that the schedule of $\hat{P}$ for the previous case provides a schedule of $P$ that satisfies warehouse capacity constraints.

We finally consider the general case. Here we show that a schedule with the completion time $\rho_{\max} + \mu - 1$ can be obtained by reducing it to the case in which $d^+ = d^- = 1$. For a warehouse $w \in W$, let $h_w^+ = \lceil \frac{|P^+(w)|}{\rho_{\max}} \rceil$, $h_w^- = \lceil \frac{|P^-(w)|}{\rho_{\max}} \rceil$, and $h_w = \max\{h_w^+, h_w^-, \rho_{\max}\}$. By definition, we have $h_w^+ \leq d^+(w)$ and $h_w^- \leq d^-(w)$. For each warehouse $w \in W$, we construct $h_w$ many warehouses $w_i$ ($i \in [1, h_w]$) such that $d^- (w_i) = d^-(w) = 1$, $c(w_i) = \rho_{\max}$ if $i \in [1, h_w^+]$, and $c(w_{h_w}) = c(w) - \rho_{\max}(h_w - 1)$. Let $\hat{W}$ denote the resulting set of warehouses. We then modify source and sink warehouses $s$ and $t$ in such a way that

- For every product $p$ with $s(p) = w$, let $\hat{s}(p) = w_i$ for some $i \in [1, h_w^+]$.
- For every product $p$ with $t(p) = w$, let $\hat{t}(p) = w_i$ for some $i \in [1, h_w^-]$.
- $|\hat{P}^+(w_i)| = \rho_{\max}$ for every $i \in [1, h_w^+]$ and $|\hat{P}^+(w_{h_w})| = |\hat{P}^+(w)| - \rho_{\max}(h_w^+ - 1)$.
- $|\hat{P}^-(w_i)| = \rho_{\max}$ for every $i \in [1, h_w^-]$, and $|\hat{P}^-(w_{h_w})| = |\hat{P}^-(w)| - \rho_{\max}(h_w - 1)$.

Here $\hat{s}$ and $\hat{t}$ respectively represent the resulting source and sink warehouses, $\hat{P}^+(x) = \{p \in P \mid \hat{s}(p) = x\}$, and $\hat{P}^-(x) = \{p \in P \mid \hat{t}(p) = x\}$. Note that this can be done by numbering products in $P^+(w)$ (resp., $P^-(w)$) as $p_j$, $j \in [1, |P^+(w)|]$ (resp., $1, |P^-(w)|$), and letting $\hat{s}(p_j) = w_j/\rho_{\max}$ (resp., $\hat{t}(p_j) = w_j/\rho_{\max}$). Then by applying the discussion in the above case, we can obtain a schedule $\hat{\varphi}$ with the completion time $\rho_{\max} + \mu - 1$, since $|\hat{P}^+(w)|, |\hat{P}^-(w)| \leq \rho_{\max}$ holds for all $w \in W$. It is not difficult to see that this reallocation schedule $\hat{\varphi}$ is also feasible with the original problem instance. By Lemma 3.2, it is an optimal
reallocation schedule. We describe the whole procedure mentioned above as Algorithm **Uniform**.

We then analyze the time complexity of the scheduling algorithm. Let \( \tilde{P}_1 \) be the set of products obtained from \( \tilde{P} \) by adding extra products according to the above second case. Note that \( |\tilde{P}_1| \leq 2m \). Let \( \tilde{G} \) be the demand graph for the reallocation problem with \( \tilde{W} \) and \( \tilde{P}_1 \). Note that we have \( \delta_{\tilde{G}}^{-}(w) = \delta_{\tilde{G}}^{-}(w) \) for all \( w \in \tilde{W} \) and \( \Delta^-(\tilde{G}) = \Delta^{-}(\tilde{G}) = \rho_{\text{max}} \). Let \( \tilde{H} \) and \( \tilde{H}^* \) be the undirected bipartite graph and the undirected \( \rho_{\text{max}} \)-regular bipartite graph obtained from \( \tilde{G} \) according to the discussion immediately before Lemma 3.3 respectively. Note that \( |E(\tilde{H})| = |\tilde{P}_1| \leq 2m \) and that \( |E(\tilde{H}^*)| - |E(\tilde{H})| \leq \rho_{\text{max}} n \) since the possible \( i \in [1, h_w] \) with \( \delta_{\tilde{G}}^{\pm}(w_i) < \rho_{\text{max}} \) is only \( h_w \) for each \( w \in \tilde{W} \). A family of \( \rho_{\text{max}} \) perfect matchings in \( \tilde{H}^* \) can be obtained by computing a minimum edge-coloring of \( \tilde{H}^* \) in \( O(|E(\tilde{H}^*)| \log \rho_{\text{max}}) = O((m + n\rho_{\text{max}}) \log \rho_{\text{max}}) \) time \([2]\). Consequently, we have Theorem 3.1

**Algorithm Uniform**\((W, P, d^+, d^-, c, \text{size}, \tau)\)

**Input:** An instance of the reallocation problem with uniform size and uniform \( \tau \).

**Output:** A schedule for \( \tilde{P} \) with the completion time \( \rho_{\text{max}} + \mu - 1 \), where \( \rho_{\text{max}} = \max_{w \in \tilde{W}} \{\left\lceil \frac{\sum_{p \in P^+(w)} \text{size}(p)}{d^+(w)} \right\rceil, \left\lfloor \frac{\sum_{p \in P^-(w)} \text{size}(p)}{d^-(w)} \right\rfloor\}\).

1. for \( w \in \tilde{W} \) do
2. Divide \( w \) into \( d_w \) warehouses \( w_i, i \in [1, d_w] \).
3. Number products in \( P^+(w) \) as \( p_j, j \in [1, |P^+(w)|] \), and let \( s(p_j) = w_{\lfloor \frac{j}{\rho_{\text{max}}} \rfloor} \) for every \( p_j \in P^+(w) \).
4. Number products in \( P^-(w) \) as \( p_j, j \in [1, |P^-(w)|] \) and let \( t(p_j) = w_{\lfloor \frac{j}{\rho_{\text{max}}} \rfloor} \) for every \( p_j \in P^-(w) \).

{Denote the resulting set of warehouses and products by \( \tilde{W} \) and \( \tilde{P} \), respectively.}
5. while \( \exists i \in \tilde{W} \) with \( |P^+(w)| \neq |\tilde{P}^+(w)| \) do
6. Add to \( \tilde{P} \) an extra product \( p \) with \( s(p) = u \) and \( t(p) = v \) for some \( u, v \in \tilde{W} \) with \( |\tilde{P}^+(u)| < |\tilde{P}^+(w)| \) and \( |\tilde{P}^+(v)| > |\tilde{P}^+(w)| \).
7. Compute a partition \( \{P_i \mid i \in [1, \rho_{\text{max}}]\} \) of \( P \) such that in the corresponding demand graph \( G = (\tilde{W}, E_{\tilde{P}}) \), the set of directed edges in \( E_{\tilde{P}} \) corresponding to \( P_i \) induces a family of vertex-disjoint simple cycles in \( G \).
8. Let \( \varphi(p) = i - 1 \) for all \( p \in P_i, i \in [1, \rho_{\text{max}}] \), and output \( \{\varphi(p) \mid p \in P\} \).

We finally remark that **Uniform** delivers a 2-approximate schedule for the case in which product size is uniform and \( d^- \) is sufficiently large, even if \( \tau \) and \( d^+ \) are both general. Let \( \varphi \) be the schedule obtained by **Uniform**. Then, \( \varphi \) satisfies the warehouse capacity constraints, since it is based on a cycle decomposition. Also, by construction and \( d^- \equiv +\infty \), the carry-out and carry-in capacity constraints are satisfied. This means that \( \varphi \) is feasible. Observe that the completion time of \( \varphi \) is at most \( \rho_{\text{max}} + \max_{p \in P} \tau(p) \). Since \( \rho_{\text{max}} \) and \( \max_{p \in P} \tau(p) \) are both lower bounds on the minimum completion time, it follows that \( \varphi \) is a 2-approximate schedule. Therefore, we have the following theorem, where the case of \( d^+ \equiv +\infty \) can be treated similarly.

**Theorem 3.5.** The reallocation problem with uniform product size and \( d^- \equiv +\infty \) (or \( d^+ \equiv +\infty \)) is 2-approximable in \( O(mn \log m) \) time.
4 Approximation Algorithms for General Cases

As shown later in Theorem 6.1, it is NP-complete to decide whether a reallocation schedule is feasible. Hence, we need to replace some of the hard constraints with soft constraints. In this paper, we consider capacity augmentations. Namely, we relax three capacity constraints: carry-out, carry-in, and warehouse capacity constraints. For a warehouse \( w \in W \), let \( \sigma_1^+(w) \) and \( \sigma_2^+(w) \) respectively denote the largest and the second largest size of products in \( P^+(w) \), and similarly \( \sigma_1^-(w) \) and \( \sigma_2^-(w) \) respectively denote the largest and the second largest size of products in \( P^-(w) \).

Then we obtain the following result.

**Theorem 4.1.** (i) We can compute in polynomial time a schedule for the reallocation problem such that the completion time is at most the minimum completion time and it is feasible with the capacity augmentation in which for every warehouse \( w \in W \),

(a) the carry-out capacity \( d^+(w) \) is augmented by \( \sigma_1^+(w) \) and \( \sigma_2^+(w) \),

(b) the carry-in capacity \( d^-(w) \) is augmented by \( \sigma_1^-(w) \) and \( \sigma_2^-(w) \), and

(c) the warehouse capacity \( c(w) \) is augmented by \( c(w) \).

(ii) If the carry-in (resp., carry-out) capacity is sufficiently large, i.e., \( d^- \equiv \infty \) (resp., \( d^+ \equiv \infty \)), we can compute in polynomial time a schedule for the reallocation problem such that the completion time is at most the minimum completion time and it is feasible with the capacity augmentation of (c) and (d) for every warehouse \( w \in W \), where

(d) the carry-out (resp., carry-in) capacity is augmented by \( \sigma_1^+(w) \) (resp., \( \sigma_1^-(w) \)).

Here, for example, the statement (a) denotes that \( \sum_{p \in P^+(w) : \phi(p) = \theta} \text{size}(p) \leq d^+(w) + \sigma_1^+(w) + \sigma_2^+(w) \) for each time \( \theta \).

By the hardness result in Theorem 6.1, we cannot obtain a feasible schedule for the reallocation problem, unless \( \text{P} = \text{NP} \). However, by Theorem 4.1, if we take capacity augmentation appropriately, we can compute a feasible schedule with the augmentation whose objective value is not worse than the optimal value of the original problem.

We remark that the statement (c) always holds by the assumption in Section 2 that the total size of products in \( P^+(w) \) (resp., \( P^-(w) \)) is at most \( c(w) \). Hence, we do not consider the warehouse capacity constraints in the subsequent discussion of this section.

We first show the general case (i). For a given integer \( T \) as an upper bound of the completion time, let us represent an integer linear system formulation of the feasibility with \( T \) of the reallocation problem.

\[
\sum_{p \in P^+(w)} \text{size}(p)x_{p\theta} \leq d^+(w) \quad \forall w \in W, \forall \theta \in [0, T - 1] \tag{3}
\]

\[
\sum_{p \in P^-(w)} \text{size}(p)x_{p,\theta - \tau(p)} \leq d^-(w) \quad \forall w \in W, \forall \theta \in [1, T] \tag{4}
\]

\[
\sum_{\theta = 0}^{T - \tau(p)} x_{p\theta} = 1 \quad \forall p \in P \tag{5}
\]

\[
x_{p\theta} \in \{0, 1\} \quad \forall p \in P, \forall \theta \in [0, T - \tau(p)] \tag{6}
\]

Here \( x_{p\theta} \) denotes the indicator variable for a product \( p \in P \) and departure time \( \theta \in [0, T - 1] \); namely, \( x_{p\theta} \) takes 1 if a product \( p \) departs at time \( \theta \), and 0 otherwise. Note that \( x_{p\theta} \) with
\(\theta > T - \tau(p)\) is not defined, since \(p\) needs to arrive at \(t(p)\) by time \(T\). Inequalities (3) and (4) respectively correspond to carry-out and carry-in constraints. Equality (5) ensures that every product \(p \in P\) is sent exactly once by time \(T - \tau(p)\). Note that the minimum \(T\) for which the integer linear system is feasible is the optimal completion time for the reallocation problem.

This formulation (3)-(6) is regarded as a problem of finding a feasible solution of the 2-sided placement problem \([11]\), which is a generalization of the generalized assignment problem.

Let \(N\) denote a set of jobs, and let \(M_1\) and \(M_2\) be two disjoint sets of machines, where each job \(k\) in \(N\) is assigned to two machines \(i \in M_1\) and \(j \in M_2\). Let \(F\) denote the set of possible assignments \(F = \{(i,j,k) \in M_1 \times M_2 \times N \mid k\) can be assigned to \(i\) and \(j\}\}. Each machine \(\ell \in M_1 \cup M_2\) has the resource capacity \(d(\ell)\), and for an assignment \((i,j,k) \in F\), the amount \(s_1(i,j,k)\) (resp., \(s_2(i,j,k)\)) of resources of a machine \(i\) (resp., \(j\)) is used if a job \(k\) is assigned to machines \(i \in M_1\) and \(j \in M_2\). The 2-sided placement problem can be formulated as (7)-(11):

\[
\text{minimize } \sum_{(i,j,k) \in F} c_{ijk}x_{ijk} \quad (7)
\]

subject to

\[
\sum_{j,k:(i,j,k) \in F} s_1(i,j,k)x_{ijk} \leq d(i) \quad \forall i \in M_1 \quad (8)
\]

\[
\sum_{i,k:(i,j,k) \in F} s_2(i,j,k)x_{ijk} \leq d(j) \quad \forall j \in M_2 \quad (9)
\]

\[
\sum_{i,j:(i,j,k) \in F} x_{ijk} = 1 \quad \forall k \in N \quad (10)
\]

\[
x_{ijk} \in \{0,1\} \quad \forall (i,j,k) \in F. \quad (11)
\]

Here \(x_{ijk}\) and \(c_{ijk}\) respectively denote the indicator variable and the assignment cost for an assignment \((i,j,k) \in F\). Constraints (8) and (9) ensure that the total amount of resources needed for the assignment to a machine \(\ell\) is at most the resource capacity \(d(\ell)\).

The feasibility of the reallocation problem formulated by (3)-(6) can be transformed into the one of the 2-sided placement problem (7)-(11) as follows.

- every warehouse \(w \in W\) at every time \(\theta\) corresponds to both types of machines, denoted by \(m_1(w,\theta) \in M_1\) and \(m_2(w,\theta) \in M_2\), and the resource capacities of \(m_1(w,\theta)\) and \(m_2(w,\theta)\) are respectively defined as \(d^+(w)\) and \(d^-(w)\).
- every product \(p \in P\) corresponds to a job. Define \(F = \{(i,j,p) \mid i = m_1(s(p),\theta), j = m_2(t(p),\theta+\tau(p)) \mid \text{for } p \in P, \theta \in [0,T - \tau(p)]\}\}. For \((i,j,p) \in F\), let \(s_1(i,j,p) = s_2(i,j,p) = \text{size}(p)\).

Korupolu et al. \([11]\) proposed a polynomial-time algorithm, based on an iterative approximation method, for the capacity augmentation of the 2-sided placement problem. For \(i \in M_1\), let \(s_{i}^{\max} = \max_{j,k:(i,j,k) \in F} s_1(i,j,k)\), and for \(j \in M_2\), let \(s_{j}^{\max} = \max_{i,k:(i,j,k) \in F} s_2(i,j,k)\).

\textbf{Theorem 4.2 (11).} For the 2-sided placement problem (7)-(11), there exists a polynomial-time algorithm for finding an assignment of jobs in \(N\) to machines in \(M_1 \cup M_2\) whose cost is at most the optimal if the resource capacities for \(i \in M_1\) and \(j \in M_2\) are respectively augmented with \(2s_{i}^{\max}\) and \(2s_{j}^{\max}\).

Algorithm \textsc{Iterative}(\(W, P, d^+, d^-, \text{size}, \tau, T\)) describes the algorithm of Korupolu et al. for the formulation (3)-(6).
Proof. As shown in [11], \textsc{Iterative}(W, P, \textcolor{red}{d^+}, \textcolor{red}{d^-}, \textcolor{blue}{\textit{size}}, \tau, T) works whenever the linear relaxation of a given formulation (3)–(6) is feasible.

For a schedule \(\varphi\) obtained by \textsc{Iterative} with \(T\), let \(P^+(w, \theta) \subseteq P\) (resp., \(P^-(w, \theta)\)) be the set of products departing from (resp., arriving at) a warehouse \(w \in W\) at time \(\theta \in [0, T]\). Let \(p^+_1(w, \theta)\) and \(p^+_2(w, \theta)\) respectively denote the products in \(P^+(w, \theta)\) with the largest and second largest size, and let \(p^-_1(w, \theta)\) and \(p^-_2(w, \theta)\) respectively denote the products in \(P^-(w, \theta)\) with the largest and second largest size.

The following lemma gives an upper bound for the capacity augmentation. Note that it slightly improves the statement of Theorem 4.2 but it is necessary to obtain the results in the next sections.

\begin{lemma}
\textsc{Algorithm} \textsc{Iterative}(W, P, \textcolor{red}{d^+}, \textcolor{red}{d^-}, \textcolor{blue}{\textit{size}}, \tau, T) outputs a schedule that satisfies the following two conditions.
\begin{enumerate}
\item \(\sum_{p \in P^+(w, \theta)} \text{size}(p) \leq d^+(w) + \text{size}(p^+_1(w, \theta)) + \text{size}(p^+_2(w, \theta))\).
\item \(\sum_{p \in P^-(w, \theta)} \text{size}(p) \leq d^-(w) + \text{size}(p^-_1(w, \theta)) + \text{size}(p^-_2(w, \theta))\).
\end{enumerate}
\end{lemma}

\begin{proof}
We only prove (i), since (ii) can be shown similarly. We assume that the corresponding carry-out capacity constraint is removed during the \(\gamma\)th iteration of the while-loop in \textsc{Iterative}(W, P, \textcolor{red}{d^+}, \textcolor{red}{d^-}, \textcolor{blue}{\textit{size}}, \tau, T). Note that such an iteration must exist, since otherwise the carry-out capacity constraint \(\sum_{p \in P^+(w, \theta)} \text{size}(p) \leq d^+(w)\) is satisfied, which implies (i).

Let \(x^\gamma_{p\theta}\)'s denote the values of the LP computed in the \(\gamma\)th iteration. For \(i = 0, 1\), let \(Q_i\) be the set of products \(p \in P^+(w)\) such that the value of \(x^\gamma_{p\theta}\) has been fixed to \(i\) by the \(\gamma\)th iteration, and \(d_1 = \sum_{p \in Q_1} \text{size}(p)\). Note that \(d_1 + \sum_{p \in P^+(w) \setminus (Q_0 \cup Q_1)} \text{size}(p) x^\gamma_{p\theta} \leq d^+(w)\) and \(P^+(w, \theta) \subseteq P^+(w) \setminus Q_0\). By \(P^+(w, \theta) \subseteq P^+(w) \setminus Q_0\), we have

\[ d_1 + \sum_{p \in P^+(w, \theta) \setminus Q_1} \text{size}(p) x^\gamma_{p\theta} \leq d^+(w) \]  

(12)

Since the constraint is removed in the \(\gamma\)th iteration, we also have \(\sum_{p \in P^+(w, \theta) \setminus (Q_0 \cup Q_1)} (1 - x^\gamma_{p\theta}) \leq 2\), which again by \(P^+(w, \theta) \subseteq P^+(w) \setminus Q_0\) implies

\[ \sum_{p \in P^+(w, \theta) \setminus Q_1} (1 - x^\gamma_{p\theta}) \leq 2. \]  

(13)
\end{proof}
Observe that
\[
\sum_{p \in P^+(w, \theta)} \text{size}(p) = d_1 + \sum_{p \in P^+(w, \theta) \setminus Q_1} \text{size}(p)
\]
\[
= d_1 + \sum_{p \in P^+(w, \theta) \setminus Q_1} \text{size}(p)x_{p\theta}^* + \sum_{p \in P^+(w, \theta) \setminus Q_1} \text{size}(p)(1 - x_{p\theta}^*)
\]
\[
\leq d^+(w) + \sum_{p \in P^+(w, \theta) \setminus Q_1} \text{size}(p)(1 - x_{p\theta}^*),
\]
(14)
where the last inequality follows from [12]. Let \(p_1\) and \(p_2\) denote the products in \(P^+(w, \theta) \setminus Q_1\) with the largest and second largest sizes, respectively. Then we have
\[
\sum_{p \in P^+(w, \theta) \setminus Q_1} \text{size}(p)(1 - x_{p\theta}^*) = (\text{size}(p_1) - \text{size}(p_2))(1 - x_{p_1\theta}^*) + \text{size}(p_2)(1 - x_{p_1\theta}^*)
\]
\[
+ \sum_{p \in P^+(w, \theta) \setminus (Q_1 \cup \{p_1\})} \text{size}(p)(1 - x_{p\theta}^*)
\]
\[
\leq (\text{size}(p_1) - \text{size}(p_2))
\]
\[
+ \text{size}(p_2) \sum_{p \in P^+(w, \theta) \setminus Q_1} (1 - x_{p\theta}^*)
\]
\[
\leq \text{size}(p_1) + \text{size}(p_2)
\]
\[
\leq \text{size}(p_1^*(w, \theta)) + \text{size}(p_2^*(w, \theta)).
\]
(15)
Here the second inequality follows from [13], and the first and third ones follow from the definitions of \(p_1\) and \(\text{size}(p_1^*(w, \theta))\), respectively. By (14) and (15), we obtain the property (i).

Let \(T_{\text{min}}\) be the minimum \(T\) for which the linear relaxation of formulation [3]–[6] is feasible, and let \(\varphi\) be a schedule obtained by \text{Iterative} with \(T = T_{\text{min}}\). It is easy to see that \(T_{\text{min}}\) is the completion time of \(\varphi\), which is at most the minimum completion time of the original reallocation problem. Moreover, Lemma 4.3 implies that \(\varphi\) is feasible with the capacity augmentation mentioned in Theorem 4.1.

We thus remain to show that (I) \(T_{\text{min}}\) can be computed in polynomial time and (II) the size of linear relaxation with \(T = T_{\text{min}}\) in Algorithm \text{Iterative} is bounded by a polynomial in the input size. The following lemma proves both (I) and (II). We therefore obtain Theorem 4.1 (i).

**Lemma 4.4.** A linear relaxation of formulation [3]–[6] with \(T = T_{\text{min}}\) has a feasible solution \(x_{p\theta}^*\) \((p \in P, \theta \in [0, T_{\text{min}} - \tau(p)]\) such that \(x_{p\theta}^* = 0\) for all \(p \in P\) and all \(\theta \geq 2m\).

**Proof.** Suppose that \(x_{q\eta}^* > 0\) holds for some \(q \in P\) and \(\eta \geq 2m\). Note that we have \(\sum_{p \in P: s(p)=s(q)} \sum_{\theta \leq 2m-1} x_{p\theta}^* \leq m - x_{q\eta}^*,\) i.e., at most \(m - x_{q\eta}^*\) many products are sent from \(s(q)\) during the time interval \([0, 2m - 1]\). Thus warehouse \(s(q)\) has room to send at least
\[
f = \sum_{\theta \in [0, 2m-1]} (1 - \sum_{p \in P: s(p)=s(q)} x_{p\theta}) \geq 2m - (m - x_{q\eta}^*) = m + x_{q\eta}^*
\]
many products during the time interval \([0, 2m - 1]\), where we note that at least one product can be sent at any time. On the other hand, since
\[
\sum_{\theta \in [0, 2m-1]} \sum_{p \in P: t(p) = t(q)} x_{p(\theta+\tau(q)-\tau(p))} \leq m,
\]

warehouse \( t(q) \) has room to receive at least \( f - m \geq x_{q,\eta}^* \) many products during the time interval corresponding to the room of \( s(q) \). Therefore, by sending \( x_{q,\eta}^* \) portion of product \( q \) during the time interval \([0, 2m - 1]\), we can obtain a feasible solution \( x_{\rho,\theta}^* \) such that \([\{(p, \theta) \mid p \in P, \theta \geq 2m, x_{\rho,\theta}^* > 0\}] < \{(p, \theta) \mid p \in P, \theta \geq 2m, x_{\rho,\theta}^* > 0\} \). By repeatedly applying this procedure, we obtain a desired feasible solution of the linear relaxation (3)–(6) with \( T = T_{\text{min}} \).

Finally we consider the case of \( d^- \equiv +\infty \), since the case of \( d^+ \equiv +\infty \) can be treated similarly. In this case, we have no carry-in capacity constraints, and our feasibility problem can be represented by the generalized assignment problem, that is, the problem to assign jobs in \( N \) to machines in \( M_1 \) (i.e., the formulation (7), (8), (10), and (11) with \(|M_2| = 1\)). Therefore, the following result for the generalized assignment problem can be directly used to obtain Theorem 4.1 (ii).

**Theorem 4.5.** For the generalized assignment problem (i.e., (7), (8), (10), and (11) with \(|M_2| = 1\)), there exists a polynomial-time algorithm for finding an assignment of jobs in \( N \) to machines in \( M_1 \) whose assignment cost is at most the optimal, if capacity constraints for \( i \in M_1 \) are augmented with \( s_{i,\text{max}}^\ast \).

## 5 Problem with No Warehouse Capacity Constraint

In this section, we consider the reallocation problem with \( c \equiv +\infty \), i.e., the case where every warehouse has a sufficiently large capacity. As observed later in Theorem 6.5, the reallocation problem with \( c \equiv +\infty \) is still strongly NP-hard. On the other hand, in contrast to the general cases, we can construct constant-factor algorithms for the problem without relaxing any constraint.

**Theorem 5.1.** (i) The reallocation problem can be solved in \( O(n + m \log m) \) time if \( c \equiv +\infty \) and \( d^- \equiv +\infty \) (or \( d^+ \equiv +\infty \)) and product size is uniform.

(ii) The reallocation problem is \( 3/2 \)-approximable if \( c \equiv +\infty \) and \( d^- \equiv +\infty \) (or \( d^+ \equiv +\infty \)) and \( \tau \) is uniform.

(iii) The reallocation problem is \( 7/4 \)-approximable if \( c \equiv +\infty \) and \( d^- \equiv +\infty \) (or \( d^+ \equiv +\infty \)).

(iv) The reallocation problem is \( 4 \)-approximable if \( c \equiv +\infty \) and product size is uniform.

(v) The reallocation problem is \( 6 \)-approximable if \( c \equiv +\infty \).

We also show in Theorem 6.6 that the problem is inapproximable within a ratio of \( 3/2 - \varepsilon \) for any \( \varepsilon > 0 \). This implies that the approximation ratio of Theorem 5.1 (ii) is optimal.

### 5.1 Case of \( d^- \equiv +\infty \)

In this section, we consider the case of \( d^- \equiv +\infty \) and prove Theorem 5.1 (i)–(iii), where the case of \( d^+ \equiv +\infty \) can be treated similarly. Since \( \Delta^- \equiv +\infty \) and \( c \equiv +\infty \), we have only to consider a schedule for \( P^+(w) \) independently for each warehouse \( w \in W \).

In what follows, we will show that schedules for \( P^+(w) \) which attain approximation ratios of Theorem 5.1 (i)–(iii) can be found in polynomial time for every \( w \in W \).

We consider a schedule for \( P^+(w) \); namely, we consider an instance \( I_{\text{RP}} = (W', P^+(w), d^+(w), \infty, \infty, \text{size}, \tau) \) of the reallocation problem, where \( W' = \{w\} \cup \{p \mid p \in P^+(w)\} \) and we regard \( \text{size} \) and \( \tau \) as those restricted to \( P^+(w) \). Then, we need to partition \( P^+(w) \) into sets of products whose total size is bounded by the carry-out capacity \( d^+(w) \). Based on this observation, we can see that the problem consisting of \( I_{\text{RP}} \)’s has a similar structure to
We construct an algorithm to solve the Binpacking problem defined below. We construct approximation algorithms corresponding to Theorem 5.1(i)–(iii) by using the ones for Binpacking as subroutines.

**Problem Binpacking**

Instance: \( (I, \text{size}_{BP}, d) \) : A set \( I \) of items, a function \( \text{size}_{BP} : I \rightarrow \mathbb{R}_+ \), and a bin with capacity \( d \in \mathbb{R}_+ \).

Output: A packing of all items in \( I \) with the minimum number of bins, i.e., a partition \( \mathcal{J} \) of \( I \) with the minimum \(|\mathcal{J}|\) such that for every \( J \in \mathcal{J} \), the total size of items in \( J \) is at most \( d \).

We construct from \( \mathcal{I}_{BP} \) an instance \( \mathcal{I}_{BP} = (I, \text{size}_{BP}, d) \) of Binpacking as follows. For each product \( p_i \in P^+(w) \), we create an item \( i \) with \( \text{size}_{BP}(i) = \text{size}(p_i) \); denote the resulting set of items by \( I \). Let \( d = d^+(w) \) as the capacity of a bin. Then note that a subset \( J \) of \( I \) can be packed into one bin if and only if the corresponding set \( \{p_i \mid i \in J\} \) of products can depart from \( w \) simultaneously, since the carry-out capacity constraint for \( w \) is satisfied. Hence, it is not difficult to see that \( I \) can be packed into \( k \) bins if and only if any product in \( P \) can be sent from \( w \) by time \( k - 1 \), by mapping a set of items in \( \ell \) into a set of products departing from \( w \) at time \( \ell - 1 \). Let \( \text{opt}(w) \) be the minimum completion time of a schedule for \( \mathcal{I}_{RA} \) and \( \text{opt}_{BP}(w) \) be the minimum number of bins for \( \mathcal{I}_{BP} \). We then have the following inequality:

\[
\text{opt}(w) \geq \text{opt}_{BP}(w) - 1 + \min_{p \in P^+(w)} \tau(p). \tag{16}
\]

We first consider the case where \( \tau \) is uniform, i.e., \( \tau(p) = \mu \) for all \( p \in P \). It was shown in [16] that the so-called First-Fit Decreasing (FFD) algorithm delivers in \( O(|I| \log |I|) \) time a feasible solution \( \mathcal{J} = \{J_1, \ldots, J_k\} \) for \( \mathcal{I}_{BP} \) with \( k \leq \frac{3}{2} \text{opt}_{BP}(w) \), where \( J_\ell \) denotes the set of items packed in the \( \ell \)th bin for \( \ell \in [1, k] \). Let \( \varphi \) be the schedule for \( \mathcal{I}_{RA} \) such that \( \varphi(p) = \ell - 1 \) for every product \( p \in P^+(w) \) corresponding to an item in \( J_\ell \). Its completion time is \( k - 1 + \mu \leq \frac{3}{2} \text{opt}_{BP}(w) - 1 + \mu \leq \frac{3}{2} \text{opt}(w) \) by (16). This proves Theorem 5.1(ii).

We next consider the case where \( \tau \) is general. We sort items in \( I \) in such a way that the corresponding products satisfy \( \tau(p_1) \geq \tau(p_2) \geq \cdots \geq \tau(p_n) \). According to this order, we apply the so-called First-Fit (FF) algorithm to \( \mathcal{I}_{BP} \) to obtain a feasible solution \( \mathcal{J} = \{J_1, \ldots, J_k\} \) for \( \mathcal{I}_{BP} \), where \( J_\ell \) denotes the set of items packed in the \( \ell \)th bin for \( \ell \in [1, k] \). Here, the FF algorithm packs each item, one by one, into the bin with the lowest possible index, while opening a new bin if necessary. It was shown in [16] that \( k \leq \frac{3}{4} \text{opt}_{BP}(w) \), while \( k = \text{opt}_{BP}(w) \) clearly holds when the size is uniform. Define \( \alpha \) by 1 if the size is uniform, and \( \frac{3}{4} \) otherwise.

Let \( \varphi \) be the schedule for \( \mathcal{I}_{RA} \) such that \( \varphi(p) = \ell - 1 \) for every product \( p \in P^+(w) \) corresponding to an item in \( J_\ell \). We then claim that the completion time \( T_w \) for \( \varphi \) satisfies \( T_w \leq \alpha \text{opt}(w) \). Since the time complexity of the algorithm is dominated by sorting items in \( I \), it can be implemented in \( O(|I| \log |I|) \) time. Hence the following claim proves Theorem 5.1(ii) and (iii).

\(\triangleright\) Claim 5.2. \( T_w \leq \alpha \text{opt}(w) \).

Proof. Let \( i \in I \) be an item such that the corresponding product \( p \in P^+(w) \) arrives at \( t(p) \) at time \( T_w \). Assume that \( i \) is the \( j \)th item in \( I \) and the FF algorithm puts \( i \) in the \( \ell \)th bin. Let \( I_j \) be the set of the first \( j \) items in \( I \). By the assumption, we have \( T_w = \ell - 1 + \tau(p) \). Let \( \varphi_i \) be the schedule obtained from \( \varphi \) by restricting the product set \( P \) to those corresponding to \( I_j \). We can see that \( \varphi_i \) is the schedule obtained by the FF algorithm for \( I_j \) and \( T_w \) is also
the completion time for \( \varphi_i \). Let \( \text{opt}_{\text{BP},i} \) be the optimal value for \( I_{\text{BP}} \) restricted to \( I_j \), and let \( \text{opt}_{\text{RA},i}(w) \) be the optimal value for \( I_{\text{RA}} \) restricted to the product set corresponding to \( I_j \). Note that \( \ell \leq \text{opt}_{\text{BP},i} \). Since \( \ell \) is sorted as above, \( \text{(10)} \) implies that \( \text{opt}_{\text{RA},i}(w) \geq \text{opt}_{\text{BP},i} - 1 + \tau(p) \). Therefore, we have \( T_w = \ell - 1 + \tau(p) \leq \text{opt}_{\text{BP},i} - 1 + \tau(p) \leq \alpha(\text{opt}_{\text{BP},i} - 1 + \tau(p)) \leq \alpha\text{opt}_{\text{RA},i}(w) \leq \alpha\text{opt}(w) \), which completes the proof of the claim.

### 5.2 General cases

In this section, we prove Theorem 5.1 (iv) and (v). Let us first show Theorem 5.1 (v) by converting a schedule \( \varphi \) for the capacity augmentation obtained by algorithm \textsc{Iterative} into a 6-approximate schedule of the original reallocation problem. For this schedule \( \varphi \), let \( T_{\text{min}}, P^+(w, \theta), P^-(w, \theta), p^+_i(w, \theta) \), and \( p^-_i(w, \theta) \), \( i = 1, 2 \), be defined as Section 4. We first claim that a 9-approximate schedule can be easily obtained from \( \varphi \).

Let us partition \( P \) into three sets \( P_1 = \{ p \in P \mid p = p^+_i(s(p), \varphi(p)) \}, P_2 = \{ p \in P \mid p = p^+_i(s(p), \varphi(p)) \}, \text{ and } P_3 = P \setminus (P_1 \cup P_2). \) We further partition \( P_{\alpha} (\alpha = 1, 2, 3) \) into 3 sets \( P_{\alpha,1} = \{ p \in P_{\alpha} \mid p = p^+_i(t(p), \varphi(p) + \tau(p)) \}, P_{\alpha,2} = \{ p \in P_{\alpha} \mid p = p^-_i(t(p), \varphi(p) + \tau(p)) \}, \text{ and } P_{\alpha,3} = P_{\alpha} \setminus (P_{\alpha,1} \cup P_{\alpha,2}). \) We construct a schedule \( \psi \) such that \( \psi(p) = (3\alpha + \beta - 4)T_{\text{min}} + \varphi(p) \) if \( p \in P_{\alpha,3}. \) By definition, the schedule \( \psi \) sends any product \( p \in P_{\alpha,3} \) from \( s(p) \) to \( t(p) \) during the time interval \( [(3\alpha + \beta - 4)T_{\text{min}}(3\alpha + \beta - 3)T_{\text{min}}]. \) By this, if two products \( p \) and \( p' \) satisfy \( \psi(p) = \psi(p') + 1 \) or \( \psi(p) + \tau(p) = \psi(p') + \tau(p') \), they belong to the same set \( P_{\alpha,3}. \) This together with Lemma 4.3 implies that \( \psi \) is feasible with the original reallocation problem. Since the completion time for \( \psi \) is at most \( 9T_{\text{min}} \), by Theorem 4.1 (i), we can conclude that \( \psi \) is a 9-approximate feasible schedule.

In order to improve the approximation ratio, we need a more careful treatment for modifying a schedule \( \varphi \) for the capacity augmentation given by Theorem 4.1 (i). More precisely, we convert \( \varphi \) to a feasible schedule with the completion time \( 6T_{\text{min}} \), by giving the following three feasible schedules (a)-(c). Here for the schedule \( \varphi, T_{\text{min}}, P^+(w, \theta) \) and \( P^-(w, \theta) \) are defined in Section 4.

Products \( p^+_i(w, \theta) \) and \( p^-_i(w, \theta) \) \( i = 1, 2 \) are defined similarly as in Section 4. Let \( Q_1 = \{ p^+_i(w, \theta), p^-_i(w, \theta) \mid w \in W, \theta \in [0, T_{\text{min}}] \}. \) If more than one product in \( P^+(w, \theta) \) (resp., \( P^-(w, \theta) \)) has the same \( \text{th} \) largest size, \( Q_1 \) is not determined uniquely. In this case, we choose \( p^+_i(w, \theta) \) (resp., \( p^-_i(w, \theta) \)) so that \( Q_1 \) is (incursion-wise) minimal. Note that such \( Q_1 \) and \( Q_2 \) can be computed in polynomial time.

(a) A feasible schedule \( \psi_1 \) with the completion time \( 3T_{\text{min}} \) for a product set \( P_1 = Q_1. \)
(b) A feasible schedule \( \psi_2 \) with the completion time \( 2T_{\text{min}} \) for a product set \( P_2 = Q_2 \setminus Q_1. \)
(c) A feasible schedule \( \psi_3 \) with the completion time \( T_{\text{min}} \) for a product set \( P_3 = P \setminus (P_1 \cup P_2). \)

Note that \( \{P_1, P_2, P_3\} \) is a partition of \( P, \) and hence (a), (b), and (c) imply Theorem 5.1 (v), since a desired schedule \( \psi^* \) can be obtained by \( \psi^*(p) = \psi_1(p) \) if \( p \in P_1, \psi_2(p) + 3T_{\text{min}} \) if \( p \in P_2, \) and \( \psi_3(p) + 5T_{\text{min}} \) if \( p \in P_3. \)

In order to show these statements, let us construct an undirected bipartite graph \( H_0 = (\bar{W}_1 \cup \bar{W}_2, E_0) \) as follows. Recall that \( H = (W_1 \cup W_2, E(H)) \) is an undirected bipartite graph obtained from the demand graph \( G = (W, E_P) \) defined before Lemma 4.3. For \( i = 1, 2 \), we replace every \( w \in W_i \) with its \( T_{\text{min}} + 1 \) copies \( w_{i, \theta}, \theta \in [0, T_{\text{min}}] \); we denote by \( \bar{W}_i \) the resulting set of vertices. For every product \( p \in P, \) we replace the corresponding edge \( (s(p), t(p)) \in E(H) \) with an undirected edge \( (s(p), t(p)), (t(p), t(p) + \tau(p)) \) which connects two vertices corresponding to its departure and arrival time in \( \varphi \); we denote by \( E_0 \) the resulting set of edges. For simplicity, in the rest of this section, we identify products \( p \) in \( P \) with edges
\[ e_p = (s(p), \varphi(p), t(p), \varphi(p) + \tau(p)) \in E_0. \] For example, we write \( \text{size}(e) \) instead of \( \text{size}(p) \) if an edge \( e \) corresponds to a product \( p \).

For (a), we can see the following property on \( \sum R \).

\[ \text{Lemma 5.3.} \ A \text{ graph } H_1 = (\tilde{W}_1 \cup \tilde{W}_2, P_1) \text{ is a forest.} \]

\textbf{Proof.} Assuming a contrary that \( H_1 \) contains a cycle \( C \), we derive a contradiction. We claim that all the edges in \( C \) have the same size. Let \( V(C) = \{ v_1, v_2, \ldots, v_k \} \), and we assume without loss of generality that \( v_1 = w_1, \theta \in \tilde{W}_1 \) and \( (v_1, v_2) = p_1^+(w, \theta) \). Then we have \( (v_2, v_3) = p_1^-(w, \theta') \) and \( (v_2, v_3) = p_1^+(w, \theta') \) because otherwise we can show that there exists an edge in \( C \) that is neither \( p_1^+(w, \theta) \) nor \( p_1^-(w, \theta) \) for some \( w \in \tilde{W}_1 \). This means that \( \text{size}((v_1, v_2)) \leq \text{size}((v_2, v_3)) \). Similarly, we have \( (v_3, v_4) = p_1^+(w, \theta') \) for \( v_3 = w_2, \theta' \in \tilde{W}_1 \). By repeated applying this argument, we obtain \( \text{size}((v_1, v_2)) \leq \text{size}((v_2, v_3)) \leq \cdots \leq \text{size}((v_{k-1}, v_k)) \).

However, this contradicts the minimality of \( P_1 \) because the set obtained by removing an edge in \( C \) from \( Q_1 \) still satisfies the requirement of \( Q_1 \), a contradiction.

\[ \text{Lemma 5.4.} \ A \text{ graph } H_1 = (\tilde{W}_1 \cup \tilde{W}_2, P_1) \text{ is a forest.} \]

\textbf{Proof.} By Lemma 4.3, we can observe that for every \( w \in \tilde{W}_1 \cup \tilde{W}_2, P_1(w) \) can be partitioned into three sets \( R_i(w) \) \( (i = 1, 2, 3) \) such that the total size of edges in \( R_i(w) \) (namely, \( \sum_{e \in R_i(w)} \text{size}(e) \)) is at most \( d^+(w) \) (resp., \( d^-(w) \)) for every vertex \( w \in \tilde{W}_1 \) (resp., \( w \in \tilde{W}_2 \)).

Based on this, we prove the lemma by giving an algorithm for partitioning \( P_1 \) into three feasible sets \( F_i, i = 1, 2, 3 \). First, we regard each component \( X \) in \( H_1 \) as a rooted tree with root \( r_X \) for a vertex \( r_X \in V(X) \) chosen arbitrarily. We initially let \( F_i := \emptyset \) for \( i = 1, 2, 3 \), and repeat the following procedure for every vertex \( w \in V(H_1) \) from the root to leaves in a top-down way:

If \( w = r_X \) for some \( X \), update \( F_i := F_i \cup R_i(r_X) \) for \( i = 1, 2, 3 \). Otherwise, without loss of generality, assume that for the parent \( v \) of \( w \), \( (v, w) \in R_i(w) \) and it is contained in the current \( F_i \). Update \( F_i := F_i \cup R_i(v) \) for \( i = 1, 2, 3 \).

The resulting sets \( F_1, F_2, \) and \( F_3 \) are feasible, and can be computed in polynomial time. Moreover, by setting \( \psi(p) = \varphi(p) + (i - 1)T_{\text{min}} \) if \( p \in F_i \), we obtain a feasible schedule with the completion time at most \( 3T_{\text{min}} \).

For (b), let \( H_2 = (\tilde{W}_1 \cup \tilde{W}_2, P_2) \). Similarly to the discussion above, we can conclude that \( H_2 \) is a forest and \( P_2 \) can be partitioned into two feasible sets, since \( P_2 \) is disjoint from \( Q_1 \).

\[ \text{Lemma 5.5.} \ A \text{ graph } H_1 = (\tilde{W}_1 \cup \tilde{W}_2, P_1) \text{ is a forest.} \]

\textbf{Proof.} By Lemma 4.3, we can observe that for every \( w \in \tilde{W}_1 \cup \tilde{W}_2, P_1(w) \) can be partitioned into three sets \( R_i(w) \) \( (i = 1, 2, 3) \) such that the total size of edges in \( R_i(w) \) (namely, \( \sum_{e \in R_i(w)} \text{size}(e) \)) is at most \( d^+(w) \) (resp., \( d^-(w) \)) for every vertex \( w \in \tilde{W}_1 \) (resp., \( w \in \tilde{W}_2 \)).

Based on this, we prove the lemma by giving an algorithm for partitioning \( P_1 \) into three feasible sets \( F_i, i = 1, 2, 3 \). First, we regard each component \( X \) in \( H_1 \) as a rooted tree with root \( r_X \) for a vertex \( r_X \in V(X) \) chosen arbitrarily. We initially let \( F_i := \emptyset \) for \( i = 1, 2, 3 \), and repeat the following procedure for every vertex \( w \in V(H_1) \) from the root to leaves in a top-down way:

If \( w = r_X \) for some \( X \), update \( F_i := F_i \cup R_i(r_X) \) for \( i = 1, 2, 3 \). Otherwise, without loss of generality, assume that for the parent \( v \) of \( w \), \( (v, w) \in R_i(w) \) and it is contained in the current \( F_i \). Update \( F_i := F_i \cup R_i(v) \) for \( i = 1, 2, 3 \).

The resulting sets \( F_1, F_2, \) and \( F_3 \) are feasible, and can be computed in polynomial time. Moreover, by setting \( \psi(p) = \varphi(p) + (i - 1)T_{\text{min}} \) if \( p \in F_i \), we obtain a feasible schedule with the completion time at most \( 3T_{\text{min}} \).

For (b), let \( H_2 = (\tilde{W}_1 \cup \tilde{W}_2, P_2) \). Similarly to the discussion above, we can conclude that \( H_2 \) is a forest and \( P_2 \) can be partitioned into two feasible sets, since \( P_2 \) is disjoint from \( Q_1 \).
As for (c), it is not difficult to see that \( P_3 \) itself is feasible by Lemma \( \text{4.3} \) since \( P_3 \) is disjoint from \( Q_1 \) and \( Q_2 \).

**Lemma 5.6.** We can compute in polynomial time a feasible schedule of \( P_3 \) with the completion time at most \( T_{\min} \).

From Lemmas \( \text{5.4}, \text{5.5} \) and \( \text{5.6} \), a 6-approximate feasible schedule of \( P \) can be found in polynomial time, which proves Theorem \( \text{5.1}(v) \).

Finally, we can show that if the product size is uniform, then the reallocation problem is 4-approximable as shown in Lemma \( \text{5.7} \), which proves Theorem \( \text{5.1}(iv) \).

**Lemma 5.7.** If the product size is uniform, then \( E_0 \) can be partitioned into four feasible sets in polynomial time.

**Proof.** Similarly to the discussion in Section \( \text{3} \), we assume without loss of generality that \( \text{size}(p) = 1 \) for all products \( p \in P \), and both of \( d^+ \) and \( d^- \) are integral. It follows from Lemma \( \text{4.3} \) that

\[
\delta_{H_0}(w) \leq d^+(w) + 2 \quad (\text{resp., } d^-(w) + 2)
\]

holds for all \( w \in \hat{W}_1 \) (resp., \( \hat{W}_2 \)). Here we recall that \( \delta_{H_0}(w) \) is the degree of \( w \) in \( H_0 \). In what follows, we construct four feasible sets \( E_1, E_2, E_3, \) and \( E_4 \) which forms a partition of \( E_0 \).

Let \( E_1 \subseteq E_0 \) be a maximal feasible set of edges, and \( H_1 = (\hat{W}_1 \cup \hat{W}_2, E_0 \setminus E_1) \). We then claim that no two vertices of degree at least three are adjacent in \( H_1 \). Indeed, if \( H_1 \) would have an edge \( (v_i, v_j) \) with \( \delta_{H_1}(v_i) \geq 3, i = 1, 2 \), then \( E_1 \cup \{(v_1, v_2)\} \) would be feasible by \( \text{17} \), contradicting the maximality of \( E_1 \). Let \( E_2 \) be a set of edges in \( E_0 \setminus E_1 \) obtained by arbitrarily choosing one edge from \((E_0 \setminus E_1)(w)\) for each vertex \( w \) with \( \delta_{H}(w) \geq 3 \). Then it follows from the claim that \( E_2 \) is a matching of \( H_1 \), and hence \( E_2 \) is feasible. Let us then partition \( F = E_0 \setminus (E_1 \cup E_2) \) into two sets \( E_3 \) and \( E_4 \) such that \( E_3 \cap F(w), E_4 \cap F(w) \neq \emptyset \) for each vertex \( w \) with \( |F(w)| \geq 2 \). Namely, \( F(w) \) is partitioned into two nonempty sets by \( E_3 \) and \( E_4 \) if it contains at least two edges.

We first show that such sets \( E_3 \) and \( E_4 \) are both feasible. By symmetry, we only show the feasibility of \( E_3 \). Let \( w \) be a vertex in \( \hat{W}_1 \) which is incident to at least two edges in \( E_3 \). Then \( E_4 \) contains at least one edge incident to \( w \) by the definition of \( E_3 \) and \( E_4 \). Also note that by \(|E_0 \setminus E_1)(w)| \geq |F(w)| \geq 3 \), \( E_2 \) contains an edge incident to \( w \). Hence we have \(|E_3(w)| \leq \delta_{H_0}(w) - 2 \), which is at most \( d^-(w) \) by \( \text{17} \). Similarly, we can see that \(|E_3(w)| \leq d^-(w) \) for any \( w \in \hat{W_2} \). Therefore, \( E_3 \) is feasible.

We next show that such sets \( E_3 \) and \( E_4 \) can be found in the following manner:

(i) Initialize \( J = F \) and \( F_3, F_4 := \emptyset \).

(ii) While \( J \) contains a cycle \( C = e_1, e_2, \ldots, e_t \), update \( F_3 := F_3 \cup \{e_{2k-1} \mid k \in [1, t/2]\} \), \( F_4 := F_4 \cup \{e_{2k} \mid k \in [1, t/2]\} \), and \( J := J \setminus C \).

(iii) For each component \( X \) of the forest \((\hat{W}_1 \cup \hat{W}_2, J) \), arbitrarily take a vertex \( x \) of degree one as a root of \( X \), and regard \( X \) as a rooted directed tree. Let \( F'_3 \) denote the set of edges in \( F \) corresponding to directed edges from \( \hat{W}_1 \) to \( \hat{W}_2 \) in the directed trees, and let \( F'_4 = J \setminus F'_3 \). Let \( E_3 := F_3 \cup F'_3 \) and \( E_4 := F_4 \cup F'_4 \).

Note that every cycle in (ii) consists of an even number of edges since \( F \) forms a bipartite graph. Hence if a vertex \( w \) appears in some cycle in (ii), the property of \( F_3 \cap F(w), F_4 \cap F(w) \neq \emptyset \) is satisfied. For the other vertices \( w \) with \( |F(w)| \geq 2 \), let \( J^* \) be the set \( J \) in (iii). Then (iii)
constructs sets $F'_3$ and $F'_4$ such that $F'_3 \cap J^*(w), F'_4 \cap J^*(w) \neq \emptyset$. Therefore, the resulting $E_3 = F_3 \cup F'_3$ and $E_4 = F_4 \cup F'_4$ satisfy the desired property.

Since all the sets $E_1$, $E_2$, $E_3$, and $E_4$ can be computed in polynomial time, the proof is completed. ▷

6 Intractability Results

In this section, we investigate the intractability of the reallocation problem. In Section 6.1, we show that deciding whether the reallocation problem with $|W| = 2$ or $\text{size} \in \{1, 2\}$ is feasible or not is strongly NP-complete. It follows that the feasibility is para-NP-complete parameterized by $|W|$ or the number of types of products. In Section 6.2, we consider the case of uniform product size is strongly NP-hard, and that the problem is inapproximable within a ratio of $3/2 - \varepsilon$ for any $\varepsilon > 0$. We also show that even the case of uniform product size and transit time is polynomially solvable as shown in Section 3.

6.1 General cases

We first show that even if $|W| = 2$ and $d^- = +\infty$, the feasibility of the reallocation problem is strongly NP-complete by a reduction from 3-PARTITION, which is known to be strongly NP-complete [5], p.224.

Problem 3-PARTITION

Instance: $(\{x_1, x_2, \ldots, x_{3m}\}, B)$: A set of $3m$ positive integers $x_1, x_2, \ldots, x_{3m}$, and an integer $B$ such that $\sum_{i \in [3m]} x_i = mB$ and $B/4 < x_i < B/2$ for each $i \in [1, 3m]$.

Question: Is there a partition $\{X_1, X_2, \ldots, X_m\}$ of $[1, 3m]$ such that $\sum_{i \in X_j} x_i = B$ for each $j \in [1, m]$?

Theorem 6.1. It is strongly NP-complete to decide whether the reallocation problem is feasible even if we have $|W| = 2$, $d^- \equiv +\infty$ (resp., $d^+ \equiv +\infty$), and $\tau \equiv 1$, and $d^+$ (resp., $d^-$) and $c$ are both uniform.

Proof. We here show only the case where $d^+$ is uniform and $d^- \equiv +\infty$. The case where $d^-$ is uniform and $d^+ \equiv +\infty$ can be treated similarly. Take an instance $I_{3\text{PART}} = (\{x_1, x_2, \ldots, x_{3m}\}, B)$ of 3-PARTITION such that $x_i$ is polynomial in $m$ for $i \in [1, 3m]$. From the $I_{3\text{PART}}$, we construct an instance $I_{\text{RP}} = (W, P, d^+, d^-, c, \tau)$ of the reallocation problem as follows. Let $W = \{w_1, w_2\}$, $d^+(w_1) = d^+(w_2) = B$, $d^-(w_1) = d^-(w_2) = +\infty$, and $c(w_1) = c(w_2) = mB$. Let $P_1$ be the set of $3m$ products $p_i$, $i \in [1, 3m]$, such that every product $p_i \in P_1$ satisfies $s(p_i) = w_1$, $t(p_i) = w_2$, and $\text{size}(p_i) = x_i$. Let $P_2$ be the set of $m$ products such that every product $p \in P_2$ satisfies $s(p) = w_2$, $t(p) = w_1$, and $\text{size}(p) = B$, and $P = P_1 \cup P_2$. Let $\tau(p) = 1$ for all $p \in P$. Note that $I_{\text{RP}}$ can be constructed from $I_{3\text{PART}}$ in polynomial time. Obviously, it is only possible to exchange 3 products in $P_1$ with the total size $B$ for one product in $P_2$ at each time because $B/4 < \text{size}(p_i) < B/2$ for every $i \in [1, 3m]$. It follows that there exists a feasible schedule for the reallocation problem if and only if $I_{3\text{PART}}$ is a yes-instance of 3-PARTITION. Thus, the theorem is proved. ▷

Moreover, we show that even if $\text{size}(p) \in \{1, 2\}$ for all $p \in P$, then the feasibility of the reallocation problem is strongly NP-complete. Let $I_{\text{RP}} = (W, P, d^+, d^-, c, \tau)$ be the instance of the reallocation problem defined in the proof of Theorem 6.1. We will convert
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\[\text{Figure 2}\] Illustration of a graph \(G_p\) for a directed edge \(e = (u, v)\) corresponding to \(p \in P\) with \(\text{size}(p) = 5\). We have \(c(w) = 2\) for every vertex \(w \in W_{u,p} \cup W_{v,p} \cup \{r_{u,p}, r_{v,p}\}\), drawn as a black circle, while \(c(w) = 1\) for all \(w \in U_p \cup V_p\), drawn as a white circle. For every directed edge in \(E' \cup \{(r_{u,p}, r_{v,p})\}\), drawn as a bold arrow, its size is two, while the size of all the other directed edges is one.

\(I_{RP}\) into an equivalent instance of the reallocation problem with \(\text{size} \in \{1, 2\}\) in a similar way to [133, Theorem 3.2].

Let \(G = (W, E_p)\) be the demand graph of \(I_{RP}\). Consider a directed edge \(e = (u, v) \in E_p\) corresponding to a product \(p \in P\) where \(\{u, v\} = \{w_1, w_2\}\); note that if \(u = w_1\) and \(v = w_2\) (resp., \(u = w_2\) and \(v = w_1\)), then \(p \in P_1\) (resp., \(P_2\)) satisfies \(\text{size}(p) = x_i\) for some \(i \in [1, 3m]\) (resp., \(\text{size}(p) = B\)). We first create a set \(U_p \cup V_p\) of \(2x\) new vertices with \(|U_p| = |V_p| = x\), where \(x = \text{size}(p)\) (note that \(x (= \text{size}(p))\) is an integer). Let \(T_{u,p}\) (resp., \(T_{v,p}\)) be an in-tree (resp., out-tree) obtained by introducing some new vertices and directed edges so that \(U_p\) (resp., \(V_p\)) is the set of leaves and the in-degree (resp., out-degree) of every vertex not in \(U_p\) (resp., \(V_p\)) is exactly two, where a directed tree is called an in-tree (resp., out-tree) if the in-degree (resp., out-degree) of every vertex except its root is exactly one. Note that such a tree \(T_{u,p}\) (resp., \(T_{v,p}\)) can be constructed by pairing two vertices with out-degree (resp., in-degree) zero from leaves to the root. We denote the root of \(T_{u,p}\) (resp., \(T_{v,p}\)) by \(r_{u,p}\) (resp., \(r_{v,p}\)). We then construct the graph \(G_p\) in the following manner:

(a) We add a directed edge from \(u\) to every vertex in \(U_p\), a directed edge from every vertex in \(V_p\) to \(v\), and a directed edge \((r_{u,p}, r_{v,p})\).

(b) We divide every vertex \(w \in V(T_{u,p}) \cup V(T_{v,p}) \setminus (U_p \cup V_p \cup \{r_{u,p}, r_{v,p}\})\) into two vertices \(w'\) and \(w''\), replace every directed edge entering \(w\) with one entering \(w'\), replace every directed edge leaving \(w\) with one leaving \(w''\), and add a directed edge \((w', w'')\).

For simplicity, we refer to a product corresponding to a directed edge \(e' \in E(G_p)\), its size, and its transit time as a product \(e'\), the size of \(e'\), and the transit time of \(e'\), respectively. Let \(W_{u,p}\) (resp., \(W_{v,p}\)) denote the set of vertices generated in (b) by dividing every vertex \(w \in V(T_{u,p}) \setminus (U_p \cup \{r_{u,p}\})\) (resp., \(V(T_{v,p}) \setminus (V_p \cup \{r_{v,p}\})\)), and \(E'\) denote the set of the directed edges added in (b). Let \(\text{size}(e') = 2\) for every product \(e' \in E' \cup \{(r_{u,p}, r_{v,p})\}\) and \(\text{size}(e') = 1\) for all the other products \(e' \in E(G_p)\) \(\setminus (E' \cup \{(r_{u,p}, r_{v,p})\})\). Let \(c(w) = 2\) for all \(w \in W_{u,p} \cup W_{v,p} \cup \{r_{u,p}, r_{v,p}\}\) and \(c(w) = 1\) for all \(w \in U_p \cup V_p\). Let \(d^+(w) = B\) and \(d^-(w) = \infty\) for all \(w \in V(G_p) \setminus \{u, v\}\). Let \(\tau(e) = 1\) for all \(e \in E(G_p)\). Figure 2 shows an example of \(G_p\) for a directed edge \(e = (u, v)\) corresponding to \(p \in P\) with \(\text{size}(p) = 5\).

The following lemma implies that the graph \(G_p\) plays the same role as \(e = (u, v) \in E_p\) corresponding to \(p\), where \(E_u\) denotes the set of edges incident to vertices in \(U_p \cup W_{u,p} \cup \{u, r_{u,p}\}\).

\[\text{Lemma 6.2.}\] If \(v\) has vacancy at least \(x\), then all products in \(E(G_p)\) can depart simultane-
ousley. If \( v \) has vacancy less than \( x \), then no product in \( E_u \) can depart.

**Proof.** The former case clearly holds. Consider the latter case. Since \( v \) has vacancy less than \( x \), there exists a vertex \( w' \in V_p \) such that \((w', v)\) cannot depart. Here notice that from construction of \( G_p \),

\[
every \text{product } e' \in E(G_p) \setminus \{(w, v) \mid w \in V_p\} \text{ can arrive at } t(e') \text{ only if all products initially located in } t(e') \text{ have departed.} \tag{18}
\]

Indeed, (i) for every \( e' \in E' \), we have \( c(t(e')) = 2 \) and (ii) for every \( e' \in E(G_p) \setminus (E' \cup \{(w, v) \mid w \in V_p\}) \), either \( c(t(e')) = 1 \) holds or some product in \( E' \) is initially located at \( t(e') \) and \( c(t(e')) = 2 \). By \( [18] \), every product in \( E(G_p) \) on the path from \( r_{u,p} \) to \( w' \) cannot also depart. Since \((r_{u,p}, r_{v,p})\) cannot depart, it follows again by \( [18] \) that no product in \( E_u \setminus \{(r_{u,p}, r_{v,p})\} \) can depart.

Let \( G' \) be the graph obtained from \( G \) by replacing each edge in \( E_p \) corresponding to \( p \in P \) with \( G_p \), and \( \mathcal{I}_{RP} \) be the corresponding instance of the reallocation problem. By Lemma \( 6.2 \) we can observe that at each time, it is only possible to carry out products in \( E(G_{p_1}) \cup E(G_{p_2}) \cup E(G_{p_3}) \) with \( p_i \in P_1 \) for \( i \in \{1, 2, 3\} \), \( p_4 \in P_2 \), and \( \sum_{i=1}^{3} \text{size}(p_i) = \text{size}(p_4) = B \). Thus, \( \mathcal{I}_{RP} \) is equivalent to \( \mathcal{I}_{RP} \). Note that the size of \( \mathcal{I}_{RP} \) is polynomial in \( m \), since every \( x_i \), \( i \in \{1, 3m\} \), is polynomial in \( m \). Hence, \( \mathcal{I}_{RP} \) can be constructed from \( \mathcal{I}_{RP} \) in polynomial time. It follows that the feasibility of \( \mathcal{I}_{RP} \) is also strongly NP-complete.

**Theorem 6.3.** It is strongly NP-complete to decide whether the reallocation problem is feasible even if we have \( d^- \equiv +\infty \) (resp., \( d^+ \equiv +\infty \)), \( \text{size} \in \{1, 2\} \), and \( \tau \equiv 1 \), and \( d^+ \) (resp., \( d^- \)) is uniform.

We finally remark as a corollary of Theorems \( 6.1 \) and \( 6.3 \) that we have the following results about the para-NP-completeness.

**Corollary 6.4.** Deciding whether the reallocation problem is feasible is para-NP-complete parameterized by each of \( |W| \) and the number of types of products.

### 6.2 Case of \( c \equiv +\infty \)

We can observe that even the case of \( c \equiv \infty \) is strongly NP-hard. For two instances \( \mathcal{I}_{3\text{-PART}} \) of 3-PARTITION and \( \mathcal{I}_{RP} \) of the reallocation problem in the proof of Theorem \( 6.1 \) it is not difficult to see that there exists a schedule for \( \mathcal{I}_{RP} \) whose completion time is at most \( m \) if and only if \( \mathcal{I}_{3\text{-PART}} \) is a yes-instance of 3-PARTITION. This follows since for completing the reallocation of all products by time \( m \), it is only possible to exchange 3 products in \( P_1 \) with the total size \( B \) for one product in \( P_2 \) at each time. Note that these arguments need the carry-out/carry-in capacity constraints but not the warehouse capacity constraints. Also, note that we can easily obtain a feasible solution for \( \mathcal{I}_{RP} \) since every warehouse has a sufficiently large capacity. Hence, we have the following theorem.

**Theorem 6.5.** The reallocation problem is strongly NP-hard even if we have \( |W| = 2 \), \( d^- \equiv +\infty \) (resp., \( d^+ \equiv +\infty \)), \( c \equiv +\infty \) and \( \tau \equiv 1 \) and \( d^+ \) (resp., \( d^- \)) and \( c \) are both uniform. Hence, it is para-NP-hard parameterized by \( |W| \) even if \( c \equiv +\infty \).

We next show the inapproximability of the problem by a reduction from BINPACKING, which is known to be inapproximable within a ratio of \( 3/2 - \varepsilon \) for any \( \varepsilon > 0 \) (e.g., see \([10]\)).

Take an instance \( \mathcal{I}_{BP} = (I = \{i \in [1, |I|]\}, \text{size}_{BP}, d) \) of BINPACKING. In an opposite way to Section \( 5.1 \) we construct from the \( \mathcal{I}_{BP} \) an instance \( \mathcal{I}_{RP} = (W, P, d^+, d^-, \infty, \text{size}, \tau) \) of the
reallocation problem as follows. Let \( W = \{ u \} \cup \{ w_i \mid i \in [1, |I|] \} \), \( d^+(w) = d \) and \( d^-(w) = \infty \) for all \( w \in W \). For every item \( i \in I \), we create a product \( p_i \) with \( s(p_i) = u \), \( t(p_i) = w_i \), and \( \text{size}(p_i) = \text{size}_{BP}(i) \); denote the resulting set of products by \( P \). Let \( \tau(p) = 1 \) for all \( p \in P \).

Similarly to the observations in Section 5.1, \( I \) can be packed into \( k \) bins if and only if the reallocation of all products in \( P \) can be completed at time \( k \). Thus, we have the following theorem, where we note that \( I_{RP} \) can be constructed from \( I_{BP} \) in polynomial time and that the case of \( d^+ = \infty \) can be treated similarly.

**Theorem 6.6.** The reallocation problem is inapproximable within a ratio of \( 3/2 - \varepsilon \) for any \( \varepsilon > 0 \) in polynomial time unless \( P = \text{NP} \), even in the case where \( d^+ \) is uniform, \( d^- = \infty \) (or \( d^+ = \infty, d^- \) is uniform), \( c = +\infty \), and \( \tau \equiv 1 \).

We finally show that the case of uniform product size is strongly NP-hard, in contrast to the case of uniform product size and transit time is polynomially solvable as shown in Section 3. Namely, we have the following theorem.

**Theorem 6.7.** The reallocation problem is strongly NP-hard even if \( |W| = 2 \), all of \( d^+, d^- \), and product size are uniform, and \( c = +\infty \).

We prove this theorem by a reduction from the problem so-called Two-Machine Flowshop with Delays (TMFD) (e.g., see [17]). In Problem TMFD, we are given two machines \( M_1 \) and \( M_2 \), and a set \( J \) of jobs. Every job \( j \in J \) consists of two operations with an intermediate delay \( \ell_j \in \mathbb{Z}_+ \); the first (resp., second) operation is executed by \( M_1 \) (resp., \( M_2 \)) and the time interval between the completion time of the first one and the starting time of the second one is exactly \( \ell_j \). Processing the first (resp., second) operation of job \( j \) takes \( p_{1j} \) (resp., \( p_{2j} \)), where \( p_{1j} \) is a positive integer. It follows that the completion time of job \( j \) starting the first operation at time \( \varphi(j) \) is \( \varphi(j) + p_{1j} + \ell_j + p_{2j} \). Each machine can process at most one job at any time. The objective of TMFD is to find a schedule of all jobs in \( J \) whose completion time, i.e., \( \max_{j \in J} \{ \varphi(j) + p_{1j} + \ell_j + p_{2j} \} \) is minimized. It was shown that TMFD is strongly NP-hard even if \( p_{1j} = p_{2j} = 1 \) for all \( j \in J \) [17].

**Theorem 6.8 ([17]).** Problem TMFD is strongly NP-hard even if \( p_{1j} = p_{2j} = 1 \) for all jobs \( j \in J \).

Take an instance \( I_{TMFD} = (M_1, M_2, J, \{ \ell_j \mid j \in J \}) \) of Problem TMFD such that \( p_{1j} = p_{2j} = 1 \) for all \( j \in J \) and each of \( \ell_j \) is polynomial in \( |J| \). From the \( I_{TMFD} \), we construct an instance \( I_{RP} = (W, P, d^+, d^-, c, \text{size}, \tau) \) of the reallocation problem as follows. Let \( W = \{ w_1, w_2 \} \), \( d^+(w_1) = d^+(w_2) = d^-(w_1) = d^-(w_2) = 1 \), and \( c(w_1) = c(w_2) = \infty \). Let \( P \) be the set of products \( p_j \), \( j \in [1, |J|] \), such that every product \( p_j \in P \) satisfies \( s(p_j) = w_1 \), \( t(p_j) = w_2 \), \( \text{size}(p_j) = 1 \), and \( \tau(p_j) = \ell_j + 1 \). Note that \( I_{RP} \) can be constructed from \( I_{TMFD} \) in polynomial time. For proving Theorem 6.7, we will show that there exists a schedule for \( I_{TMFD} \) whose completion time is at most \( T \) if and only if there exists a schedule for \( I_{RP} \) whose completion time is at most \( T + 1 \).

Assume that there exists a schedule \( \varphi' \) for \( I_{TMFD} \) whose completion time is at most \( T \); let \( \varphi'(j) \) denote the time when job \( j \in J \) starts the first operation in the schedule \( \varphi' \). Then, job \( j \) starts the second operation at time \( \varphi'(j) + \ell_j + 1 \) by \( p_{1j} = 1 \). Since each machine can process at most one job at any time, we have

\[
\varphi'(j) \neq \varphi'(j') \quad \text{and} \quad \varphi'(j) + \ell_j + 1 \neq \varphi'(j') + \ell_j' + 1
\]  
for every two distinct jobs \( j, j' \in J \). Note that \( \max_{j \in J} \{ \varphi'(j) + \ell_j + 2 \} \leq T \) by \( p_{2j} = 1 \). Let \( \varphi \) be the schedule for \( I_{RP} \) such that \( \varphi(p_j) = \varphi'(j) \) for \( p_j \in P \). Then, product \( p_j \) arrives at \( w_2 \)}
at time $\varphi'(j) + \ell_j + 1$ by $\tau(p_j) = \ell_j + 1$. By \cite{9} and $\text{size}(p_j) = 1$, $\varphi$ satisfies the carry-out and carry-in capacity constraints. By $c(w_1) = c(w_2) = \infty$, it follows that $\varphi$ is feasible. The completion time for $\varphi$ is $\max_{p_j \in P}\{\varphi'(j) + \ell_j + 1\} \leq T - 1$.

Assume that there exists a schedule $\varphi$ for $\mathcal{I}_{RP}$ whose completion time is at most $T - 1$. Then since $d^-(w_1) = d^-(w_2) = 1$ and $\text{size}(p_j) = 1$ and $\tau(p_j) = \ell_j + 1$ for $p_j \in P$, it follows by the carry-out and carry-in capacity constraints that

$$\varphi(p_j) \neq \varphi(p_{j'}) \quad \text{and} \quad \varphi(p_j) + \ell_j + 1 \neq \varphi(p_{j'}) + \ell_{j'} + 1$$

(20)

for every two distinct products $p_j, p_{j'} \in P$. Note that $\max_{p_j \in P}\{\varphi(p_j) + \ell_j + 1\} \leq T - 1$.

Let $\varphi'$ be the schedule for $\mathcal{I}_{TMFD}$ such that job $j \in J$ starts the first operation at time $\varphi(p_j)$. Then, job $j$ starts the second operation at time $\varphi(p_j) + \ell_j + 1$ and completes it at time $\varphi(p_j) + \ell_j + 2$. Since each machine processes at most one job at any time in $\varphi'$ by \cite{10} and $p_{j1} = p_{j2} = 1$ for all $j \in J$, it follows that $\varphi'$ is a feasible schedule for $\mathcal{I}_{TMFD}$. The completion time for $\varphi'$ is $\max_{j \in J}\{\varphi(p_j) + \ell_j + 2\} \leq T$.

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