Existence, uniqueness, and approximation of solutions of jump-diffusion SDEs with discontinuous drift
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Abstract

In this paper we study jump-diffusion stochastic differential equations (SDEs) with a discontinuous drift coefficient and a possibly degenerate diffusion coefficient. Such SDEs appear in applications such as optimal control problems in energy markets. We prove existence and uniqueness of strong solutions. In addition we study the strong convergence order of the Euler-Maruyama scheme and recover the optimal rate $1/2$.

Keywords: jump-diffusion stochastic differential equation, discontinuous drift, existence and uniqueness, Euler-Maruyama scheme, strong convergence rate
Mathematics Subject Classification (2010): 60H10, 65C30, 65C20, 65L20

1 Introduction

We consider a time-homogeneous jump-diffusion stochastic differential equation (SDE)

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \rho(X_{t-})dN_t, \quad t \in [0,T], \quad X_0 = \xi,$$

(1)

where $\xi \in \mathbb{R}$, $\mu, \sigma, \rho: \mathbb{R} \to \mathbb{R}$ are measurable functions, $T \in (0, \infty)$, $W = (W_t)_{t \in [0,T]}$ is a standard Brownian motion and $(N_t)_{t \in [0,T]}$ is a Poisson process with Borel measurable and bounded intensity $\lambda: [0,T] \to (0, \infty)$ on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ that satisfies the usual conditions.

Furthermore, let $N \in \mathbb{N}$, define the equidistant time grid $0 = t_0 < t_1 < \cdots < t_N = T$ with $t_{k+1} - t_k = \delta$ for all $k \in \{0, \ldots, N-1\}$ and denote for all $t \in [0,T]$, $t := \max\{t_k: t \geq t_k\}$. The time-continuous Euler-Maruyama (EM) scheme is given by $X^{(\delta)}_0 = \xi$ and

$$X^{(\delta)}_t = X^{(\delta)}_0 + \mu(X^{(\delta)}_0)(t - t_0) + \sigma(X^{(\delta)}_0)(W_t - W_{t_0}) + \rho(X^{(\delta)}_0)(N_t - N_{t_0}), \quad t \in (0,T].$$

(2)

In case the coefficients $\mu$, $\sigma$, and $\rho$ are Lipschitz, it is well known that SDE (1) admits a unique strong solution which can be approximated with the EM scheme at strong convergence order $1/2$.

The novelty in this work is that we allow $\mu$ to be discontinuous in a finite number of points. This is relevant for example for modelling energy prices, where jumps in the paths are a stylised fact, see, e.g., [1]. Control actions on energy markets often lead to discontinuities in the drift of the controlled process, cf., e.g., [40, 41].

We study existence and uniqueness of solutions to SDE (1) as well as numerical approximations of this solution.

In the jump-free case, SDEs with discontinuous drift have been studied intensively in recent years. For existence and uniqueness results see the classical papers [46, 43, 44, 45], as well as newer
results, where boundedness of the coefficients and non-degeneracy of the diffusion coefficient is no longer needed, see [24, 30, 21, 22]. For approximation results see [13, 14, 15, 21, 30, 22, 31, 32, 33, 12]. In the scalar case the best known results are $L^p$-order 1/2 of the EM scheme, see [20] and $L^p$-order 3/4 of a transformation-based Milstein-type scheme, see [27]. In the multidimensional setting the best known results are $L^2$-order $1/4$– of the EM scheme, see [23] and $L^2$-order $1/2$– of an adaptive EM scheme, see [29]. In the special case of additive noise the best known results are $L^2$-order $(1 + \kappa)^2$/2– assuming piecewise Sobolev-Slobodeckij-type regularity of order $\kappa \in (0, 1)$ for the drift, see [28] and [3], where they prove $L^2$-order 1/2– for the case where the drift is only bounded and $L^1$ (i.e. the case $\kappa = 0$). A lower error bound of order 1 for the pointwise $L^1$-error is proven in [10]. Lower bounds will be also studied in a forthcoming paper by Müller-Gronbach and Yaroslavtseva.

In the case of presence of jumps in the driving process, to the best of our knowledge there are no results available for SDEs with discontinuous drift so far. In the case of continuous coefficients however, the number of publications is still growing. This is due to the already mentioned fact that jumps often arise in models for energy markets, financial markets, or physical phenomena, see for example [30, 34]. The research directions cover for example classical Itô-Taylor approximations as in [10, 11, 33], construction of Runge-Kutta methods as in [2, 33], approximation of jump-diffusion SDEs under nonstandard assumptions as in [1, 3, 17, 18, 2], multilevel Monte Carlo methods for weak approximation as in [8], and asymptotically optimal approximations of solutions of such SDEs as in [20, 37, 38, 39].

The current paper consists of two main contributions: the first existence and uniqueness result for jump-diffusion SDEs with discontinuous drift and consequently the first approximation result for solutions to such SDEs. We obtain the optimal $L^2$-order 1/2 for the EM scheme.

We reach our goal by adopting ideas from the jump-free case from [21, 23, 26] and extending everything to the case of presence of jumps. An interesting by-product is an occupation time result for the EM process.

2 Preliminaries

In the following we denote by $L_f$ the Lipschitz constant of a generic function $f$, we define $\kappa: [0, T] \to (0, \infty)$ with $\kappa_t = \int_0^t \lambda(s) ds$, and we denote by $\dot{N} = (\dot{N}_t)_{t \in [0, T]}$ the compensated Poisson process, that is $\dot{N}_t = N_t - \kappa_t$ for all $t \in [0, T]$. Note that $\dot{N}$ is a square integrable (\mathcal{F}_t)_{t \in [0, T]}-martingale.

In order to define assumptions on the drift coefficient, we recall the following definition.

**Definition 2.1** ([22, Definition 2.1]). Let $I \subseteq \mathbb{R}$ be an interval and let $m \in \mathbb{N}$. We say a function $f: I \to \mathbb{R}$ is piecewise Lipschitz, if there are finitely many points $\zeta_1 < \ldots < \zeta_m \in I$ such that $f$ is Lipschitz on each of the intervals $(-\infty, \zeta_1) \cap I$, $(\zeta_m, \infty) \cap I$, and $(\zeta_k, \zeta_{k+1})$, $k = 1, \ldots, m - 1$.

**Assumption 2.1.** We assume the following on the coefficients of (1):

(ass-$\mu$) The drift coefficient $\mu: \mathbb{R} \to \mathbb{R}$ is piecewise Lipschitz with $m \in \mathbb{N}$ discontinuities in the points $\zeta_1, \ldots, \zeta_m \in \mathbb{R}$.

(ass-$\sigma$) The diffusion coefficient $\sigma: \mathbb{R} \to \mathbb{R}$ is Lipschitz and for all $k \in \{1, \ldots, m\}$, $\sigma(\zeta_k) \neq 0$.

(ass-$\rho$) The jump coefficient $\rho: \mathbb{R} \to \mathbb{R}$ is Lipschitz.

**Lemma 2.2.** Let Assumptions [22] hold. Then $\mu$, $\sigma$, and $\rho$ satisfy a linear growth condition, that is there exist constants $c_\mu, c_\sigma, c_\rho \in (0, \infty)$ such that

$$|\mu(x)| \leq c_\mu (1 + |x|), \quad |\sigma(x)| \leq c_\sigma (1 + |x|), \quad |\rho(x)| \leq c_\rho (1 + |x|).$$
Proof. We have that $|\sigma(x)| \leq |\sigma(x) - \sigma(0)| + |\sigma(0)| \leq L_\sigma |x| + |\sigma(0)|$. Setting $c_\sigma = \max\{L_\sigma, |\sigma(0)|\}$ we get $|\sigma(x)| \leq c_\sigma (1 + |x|)$. The analog estimate holds for $\rho$. For $x \in (-\infty, \zeta_1)$ we have that there exists an $\varepsilon \in (0, \infty)$ with $\zeta_1 - \varepsilon > x$. With this $|\mu(x)| \leq |\mu(x) - \mu(\zeta_1 - \varepsilon)| + |\mu(\zeta_1 - \varepsilon)| \leq L_\mu |x - (\zeta_1 - \varepsilon)| + |\mu(\zeta_1 - \varepsilon)| \leq L_\mu |x + L_\mu(\zeta_1 - \varepsilon) + |\mu(\zeta_1 - \varepsilon)|$. Setting $c_\mu^1 = \max\{L_\mu, L_\mu(\zeta_1 - \varepsilon) + |\mu(\zeta_1 - \varepsilon)|\}$ we get $|\mu(x)| \leq c_\mu^1 (1 + |x|)$. In the same way for $x \in (\zeta_m, \infty)$ there exists $c_\mu^2 \in (0, \infty)$ with $|\mu(x)| \leq c_\mu^2 (1 + |x|)$. In the compact interval $[\zeta_1, \zeta_m]$, $\mu$ is bounded by a constant $c_\mu^3 \in (0, \infty)$. Setting $c_\mu = \max\{c_\mu^1, c_\mu^2, c_\mu^3\}$ proves the lemma.

The transform

We will apply a transform $G: \mathbb{R} \to \mathbb{R}$ from [22] that has the property that the process formally defined by $Z = G(X)$ satisfies an SDE with Lipschitz coefficients and therefore has a solution by classical results, see [36], p. 255, Theorem 6.

The function $G$ is chosen so that it impacts the coefficients of the SDE (1) only locally around the points of discontinuity of the drift. This behaviour is ensured by incorporating a bump function $\phi: \mathbb{R} \to \mathbb{R}$ into $G$, which is defined by

$$
\phi(u) = \begin{cases} 
(1 + u)^3(1 - u)^3 & \text{if } |u| \leq 1, \\
0 & \text{otherwise}. 
\end{cases}
$$

With this the transform $G$ is defined by

$$
G(x) = x + \sum_{k=1}^{m} \alpha_k \phi\left(\frac{x - \zeta_k}{c}\right) (x - \zeta_k) |x - \zeta_k|,
$$

with

$$
\mathbb{R} \setminus \{0\} \ni \alpha_k = \frac{\mu(\zeta_k -) - \mu(\zeta_k +)}{2\sigma(\zeta_k)^2}, \quad k \in \{1, \ldots, m\},
$$

$$(0, \infty) \ni c < \min\left\{\frac{1}{\min_{1 \leq k \leq m} 6|\alpha_k|}, \frac{\zeta_{k+1} - \zeta_k}{2}\right\}.
$$

Note that $c$ is chosen such that for all $x \in \mathbb{R}$, $G'(x) > 0$, so that $G$ has a global inverse $G^{-1}: \mathbb{R} \to \mathbb{R}$. The transformation $G$ and its inverse $G^{-1}$ are Lipschitz and the function $G \in C^1_\mu$, that is it is continuously differentiable with bounded derivative. Furthermore, $G'$ is piecewise Lipschitz, since it is differentiable on $\mathbb{R} \setminus \{\zeta_1, \ldots, \zeta_m\}$ with bounded derivative, see [22], Lemma 3.8. Hence, $G'$ is Lipschitz, since it is piecewise Lipschitz and continuous, see [21], Lemma 2.2. These properties are proven in [22].

3 Existence and uniqueness result

We are going to prove our first main result.

**Theorem 3.1.** Let Assumption 2.1 hold. Then the SDE (1) has a unique global strong solution.

**Proof.** For all $k \in \{1, \ldots, m\}$ we introduce the abbreviation $\bar{\phi}_k(x) := \phi\left(\frac{x - \zeta_k}{c}\right) |x - \zeta_k|$. Since $G'$ is Lipschitz, we may apply the Meyer-Itô formula, which follows from [36], p. 221, Theorem 71], to $Z = G(X)$ and get

$$
dZ_t = \tilde{\mu}(Z_t) dt + \tilde{\sigma}(Z_t) dW_t + \tilde{\rho}(Z_{t-}) dN_t,
$$

(4)
where for all \( z \in \mathbb{R} \),

\[
\hat{\mu}(z) = \mu(G^{-1}(z)) + \sum_{k=1}^{m} \alpha_k(\hat{\phi}_k)'(G^{-1}(z))\mu(G^{-1}(z)) + \frac{1}{2} \sum_{k=1}^{m} \alpha_k(\hat{\phi}_k)''(G^{-1}(z))\sigma(G^{-1}(z))^2,
\]
\[
\hat{\sigma}(z) = \sigma(G^{-1}(z)) + \sum_{k=1}^{m} \alpha_k(\hat{\phi}_k)'(G^{-1}(z))\sigma(G^{-1}(z)),
\]
\[
\hat{\rho}(z) = G(G^{-1}(z) + \rho(G^{-1}(z))) - G(G^{-1}(z)) = G(G^{-1}(z) + \rho(G^{-1}(z))) - z. \tag{5}
\]

In [22] it is shown that \( \hat{\mu} \) and \( \hat{\sigma} \) are Lipschitz. The jump coefficient \( \hat{\rho} \) is Lipschitz due to the global Lipschitz continuity of \( G \) and \( G^{-1} \). Hence, the SDE for \( Z \), that is (4) with initial condition \( Z(0) = G(\xi) \), has a unique global strong solution by [36, p. 255, Theorem 6].

Now observe that \( (G^{-1})'(z) = 1/G'(G^{-1}(z)) \) is absolutely continuous since it is Lipschitz. Moreover, \( G^{-1}(Z_{t-}) = \lim_{s \to t-} G^{-1}(Z_s) = X_{t-} \), and by (4) we have

\[
Z_{t-} + \hat{\rho}(Z_{t-}) = G(X_{t-} + \rho(X_{t-})).
\]

This implies that

\[
G^{-1}(Z_{t-} + \hat{\rho}(Z_{t-})) - G^{-1}(Z_{t-}) = \rho(X_{t-}).
\]

Therefore, again by using Meyer-Itô formula

\[
dG^{-1}(Z_t) = ((G^{-1})'(Z_t)\hat{\mu}(Z_t) + \frac{1}{2}(G^{-1})''(Z_t)\hat{\sigma}(Z_t)^2)dt + (G^{-1})'(Z_t)\hat{\sigma}(Z_t)dW_t
\]
\[
+ (G^{-1}(Z_{t-} + \hat{\rho}(Z_{t-})) - G^{-1}(Z_{t-}))dN_t
\]
\[
= \mu(X_t)dt + \sigma(X_t)dW_t + \rho(X_{t-})dN_t.
\]

4 Convergence of the Euler-Maruyama method

Our convergence proof is based on a transformation trick from [21, 23] in combination with ideas from [26] for the estimation of discontinuity crossing probabilities. By extending both to the case of presence of jumps and proving an occupation time result for the EM process, this leads to the optimal convergence order 1/2.

4.1 Preparatory lemmas

In this section we present several lemmas, cf. the results for the jump-free case in [26], which we need for the proof of the main result of Section 4.

**Lemma 4.1.** Let Assumptions [27, 28] hold and let \( p \in [2, \infty) \). There exist a constant \( C^{(M)} \in (0, \infty) \) such that for all \( \delta \in (0, 1) \),

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} |X_t^{(\delta)}|^p \right] \leq C^{(M)}
\]

and such that for all \( s, t \in [0, T], s < t \),

\[
\mathbb{E}[|X_t^{(\delta)} - X_s^{(\delta)}|^p] \leq C^{(M)} \cdot |t - s|.
\]
Proof. There exists a constant \( c_1 \in (0, \infty) \) such that for all \( k \in \{0, 1, \ldots, N - 1\} \),

\[
|X_{t_{k+1}}^{(\delta)}|^p \leq c_1 \left( |X_{t_k}^{(\delta)}|^p + |\mu(X_{t_k}^{(\delta)})|^p \cdot \delta^p + |\sigma(X_{t_k}^{(\delta)})|^p \cdot |W_{t_{k+1}} - W_{t_k}|^p + |\rho(X_{t_k}^{(\delta)})|^p \cdot |N_{t_{k+1}} - N_{t_k}|^p \right).
\]

Furthermore there exists a constant \( c_2 \in (0, \infty) \) such that

\[
\mathbb{E}[|W_{t_{k+1}} - W_{t_k}|^p] \leq c_2 \delta^{p/2}
\]

and by Inequality (3.20),

\[
\mathbb{E}[|N_{t_{k+1}} - N_{t_k}|^p] \leq c_2 \delta.
\]

This and the linear growth of \( \mu \), \( \sigma \), and \( \rho \) imply the existence of a constant \( c_3 \in (0, \infty) \) such that

\[
\mathbb{E}\left[ |X_{t_{k+1}}^{(\delta)}|^p \right] \leq c_3 \left( 1 + \mathbb{E}\left[ |X_{t_k}^{(\delta)}|^p \right] \right).
\]

Since \( \mathbb{E}[|X_0|^p] < \infty \) it follows that

\[
\max_{k \in \{0, 1, \ldots, N\}} \mathbb{E}\left[ |X_{t_k}^{(\delta)}|^p \right] < \infty. \tag{7}
\]

We also have for \( s \in [0, T] \) that there exists a constant \( c_4 \in (0, \infty) \) such that

\[
\sup_{t \in [0, s]} |X_{t}^{(\delta)}|^p \leq c_4 \left( |X_0|^p + \sup_{t \in [0, s]} \left| \int_0^t \mu(X_u^{(\delta)}) du \right|^p + \sup_{t \in [0, s]} \left| \int_0^t \sigma(X_u^{(\delta)}) dW_u + \int_0^t \rho(X_u^{(\delta)}) dN_u \right|^p \right).
\]

The Burkholder-Davis-Gundy inequality, Doob’s maximal inequality for cádlág martingales, [25, Lemma 2.1], and the linear growth condition on \( \sigma, \rho \) ensure the existence of constants \( c_5, c_6 \in (0, \infty) \) such that

\[
\mathbb{E}\left[ \sup_{t \in [0, s]} \left| \int_0^t \sigma(X_u^{(\delta)}) dW_u \right|^p \right] \leq c_5 \mathbb{E}\left[ \int_0^s |\sigma(X_u^{(\delta)})|^p du \right] \leq c_6 \left( 1 + \int_0^s \mathbb{E}\left[ |X_u^{(\delta)}|^p \right] du \right)
\]

and since \( N_u = \tilde{N}_u + \kappa_u \),

\[
\mathbb{E}\left[ \sup_{t \in [0, s]} \left| \int_0^t \rho(X_u^{(\delta)}) dN_u \right|^p \right] \leq c_5 \mathbb{E}\left[ \int_0^s |\rho(X_u^{(\delta)})|^p du \right] \leq c_6 \left( 1 + \int_0^s \mathbb{E}\left[ |X_u^{(\delta)}|^p \right] du \right).
\]

Since \( \mu \) is of at most linear growth, an analogous estimate for the Lebesgue integral holds. Hence,

\[
\mathbb{E}\left[ \sup_{t \in [0, s]} |X_{t}^{(\delta)}|^p \right] \leq c_2 \left( 1 + \int_0^s \mathbb{E}\left[ |X_u^{(\delta)}|^p \right] du \right), \quad s \in [0, T]. \tag{8}
\]

By (7) we now obtain that

\[
\mathbb{E}\left[ \sup_{t \in [0, T]} |X_{t}^{(\delta)}|^p \right] < \infty. \tag{9}
\]

Equation (5) also yields

\[
\mathbb{E}\left[ \sup_{t \in [0, s]} |X_{t}^{(\delta)}|^p \right] \leq c_2 + c_2 \int_0^s \mathbb{E}\left[ \sup_{t \in [0, u]} |X_{t}^{(\delta)}|^p \right] du.
\]
Since (9) holds and the function \([0, T] \ni t \mapsto \mathbb{E} \left[ \sup_{s \in [0, t]} |X_t^{(\delta), x}| \right] p\) is Borel measurable (as a nondecreasing mapping), Gronwall’s Lemma yields the first assertion.

For the second statement note that for all \(s, t \in [0, T], s < t\), there exists a constant \(c_7 \in (0, \infty)\) so that it holds
\[
\mathbb{E} \left[ |X_t^{(\delta)} - X_s^{(\delta)}|^p \right] \leq c_7 \left( \mathbb{E} \left[ \left| \int_s^t \mu(X_u^{(\delta)}) du \right|^p \right] + \mathbb{E} \left[ \int_s^t \sigma(X_u^{(\delta)}) dW_u \right]^p \right).
\]

The Hölder inequality, the Burkholder-Davis-Gundy inequality, [23, Lemma 2.1], and the linear growth condition of the coefficients together with the first assertion yield the statement.

Note that we consider the \(L^2\)-error and not the \(L^p\)-error as in the jump-free case in [20], since due to (3) we will not get a better estimate for \(p > 2\) anyhow, cf. [7, Remark 3.14].

### 4.1.1 Estimation of the occupation time of the Euler-Maruyama process

In this subsection we need to make the dependence on the initial value explicit in the notation. For all \(x \in \mathbb{R}\) denote by \(X^x\) the unique strong solution of (11) with initial condition \(X_0^x = x\) and by \(X^{(\delta), x}\) the solution of the time-continuous version of the Euler-Maruyama scheme (2) starting at \(X_0^{(\delta), x} = x\). Note that from the proof of Lemma 4.1 it follows that there exists \(C^{(1)} \in (0, \infty)\) such that for all \(x \in \mathbb{R}, s, t \in [0, T], s < t, \delta \in (0, 1)\),
\[
\left( \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_t^{(\delta), x}|^2 \right] \right)^{1/2} \leq C^{(1)} (1 + |x|), \tag{10}
\]
\[
(\mathbb{E} \left[ |X_t^{(\delta), x} - X_s^{(\delta), x}|^2 \right])^{1/2} \leq C^{(1)} (1 + |x|)[t - s]^{1/2}. \tag{11}
\]

**Lemma 4.2.** Let Assumptions 2, 7 hold. Then there exists \(C^{(O)} \in (0, \infty)\) such that for all \(k \in \{1, \ldots, m\}, x \in \mathbb{R}, \delta \in (0, 1), \varepsilon \in (0, \infty)\) it holds
\[
\int_0^T \mathbb{P}(|X_t^{(\delta), x} - \zeta_k| \leq \varepsilon) dt \leq C^{(O)} (1 + x^2)(\varepsilon + \delta^{1/2}).
\]

**Proof.** By [42, Lemma 158] we have for all \(a \in \mathbb{R}\) that
\[
L^a_t(X^{(\delta), x}) = |X_t^{(\delta), x} - a| - |x - a| - \int_0^t \text{sgn}(X_{s-}^{(\delta), x} - a) dX_s^{(\delta), x}
\]
\[
- \int_0^t \left( |X_{s-}^{(\delta), x} + \sigma(X_{s-}^{(\delta), x}) - a| - |X_{s-}^{(\delta), x} - a| - \text{sgn}(X_{s-}^{(\delta), x} - a) \rho(X_{s-}^{(\delta), x}) \right) dN_s.
\]

Since \(L^a_t(X^{(\delta), x}) = |L^a_t(X^{(\delta), x})| \geq 0\), we get
\[
L^a_t(X^{(\delta), x}) \leq |X_t^{(\delta), x} - x| + \left| \int_0^t \text{sgn}(X_{s-}^{(\delta), x} - a) dX_s^{(\delta), x} \right|
\]
\[
+ \left| \int_0^t \left( |X_{s-}^{(\delta), x} + \sigma(X_{s-}^{(\delta), x}) - a| - |X_{s-}^{(\delta), x} - a| - \text{sgn}(X_{s-}^{(\delta), x} - a) \rho(X_{s-}^{(\delta), x}) \right) dN_s \right|. \tag{12}
\]
By Lemma 2.2 there exists \( c_1 \in (0, \infty) \) such that
\[
\left| \int_0^t \text{sgn}(X_{s-}^{(\delta, x)} - a) dX_{s-}^{(\delta, x)} \right| \leq \int_0^t \left( \| \mu(X_{s-}^{(\delta, x)}) \| + \| \lambda \|_\infty | \rho(X_{s-}^{(\delta, x)}) | \right) ds \\
+ \int_0^t \text{sgn}(X_{s-}^{(\delta, x)} - a) \sigma(X_{s-}^{(\delta, x)}) dW_s \left| + \int_0^t \text{sgn}(X_{s-}^{(\delta, x)} - a) \rho(X_{s-}^{(\delta, x)}) d\tilde{N}_s \right| \\
\leq c_1 \left( 1 + \sup_{t \in [0, T]} |X_t^{(\delta, x)}| \right) + \int_0^t \text{sgn}(X_{s-}^{(\delta, x)} - a) \rho(X_{s-}^{(\delta, x)}) d\tilde{N}_s \\
+ \int_0^t \text{sgn}(X_{s-}^{(\delta, x)} - a) \rho(X_{s-}^{(\delta, x)}) d\tilde{N}_s \tag{13}
\]
and
\[
\left| \int_0^t \left( |X_{s-}^{(\delta, x)} + \rho(X_{s+}^{(\delta, x)}) - a| - |X_{s-}^{(\delta, x)} - a| - \text{sgn}(X_{s-}^{(\delta, x)} - a) \rho(X_{s-}^{(\delta, x)}) \right) dN_s \right| \\
\leq \int_0^t \left( |X_{s-}^{(\delta, x)} + \rho(X_{s+}^{(\delta, x)}) - a| - |X_{s-}^{(\delta, x)} - a| - \text{sgn}(X_{s-}^{(\delta, x)} - a) \rho(X_{s-}^{(\delta, x)}) \right) dN_s \\
+ \int_0^t \left( |X_{s-}^{(\delta, x)} + \rho(X_{s+}^{(\delta, x)}) - a| - |X_{s-}^{(\delta, x)} - a| - \text{sgn}(X_{s-}^{(\delta, x)} - a) \rho(X_{s-}^{(\delta, x)}) \right) \lambda(s) ds \\
\leq \int_0^t \left( |X_{s-}^{(\delta, x)} + \rho(X_{s+}^{(\delta, x)}) - a| - |X_{s-}^{(\delta, x)} - a| - \text{sgn}(X_{s-}^{(\delta, x)} - a) \rho(X_{s-}^{(\delta, x)}) \right) d\tilde{N}_s \\
+ 2 \| \lambda \|_\infty \int_0^t |\rho(X_{s-}^{(\delta, x)})| ds \\
\leq \int_0^t \left( |X_{s-}^{(\delta, x)} + \rho(X_{s+}^{(\delta, x)}) - a| - |X_{s-}^{(\delta, x)} - a| - \text{sgn}(X_{s-}^{(\delta, x)} - a) \rho(X_{s-}^{(\delta, x)}) \right) d\tilde{N}_s \\
+ c_1 \left( 1 + \sup_{t \in [0, T]} |X_t^{(\delta, x)}| \right) . \tag{14}
\]

By (10) there exists \( c_2 \in (0, \infty) \) such that
\[
\mathbb{E} \left[ \left| \int_0^t \text{sgn}(X_{s-}^{(\delta, x)} - a) \sigma(X_{s-}^{(\delta, x)}) dW_s \right|^2 \right] \leq \mathbb{E} \left[ \int_0^t |\sigma(X_{s-}^{(\delta, x)})|^2 ds \right] \leq c_2 (1 + |x|)^2 , \]
\[
\mathbb{E} \left[ \left| \int_0^t \text{sgn}(X_{s-}^{(\delta, x)} - a) \rho(X_{s-}^{(\delta, x)}) d\tilde{N}_s \right|^2 \right] \leq \| \lambda \|_\infty \mathbb{E} \left[ \int_0^t |\rho(X_{s-}^{(\delta, x)})|^2 ds \right] \leq c_2 (1 + |x|)^2 ,
\]

7
and

\[
\mathbb{E}\left[\int_0^t \left| X_{s-}^{(x)} + \rho(X_{s-}^{(x)}) - a| - |X_{s-}^{(x)} - a| - \text{sgn}(X_{s-}^{(x)} - a)\rho(X_{s-}^{(x)}) \right|^2 d\tilde{N}_s \right]
\]

\[
= \mathbb{E}\left[\int_0^t \left( |X_{s-}^{(x)} + \rho(X_{s-}^{(x)}) - a| - |X_{s-}^{(x)} - a| - \text{sgn}(X_{s-}^{(x)} - a)\rho(X_{s-}^{(x)}) \right)^2 \lambda(s) ds \right]
\]

\[
\leq 2\|\lambda\|_\infty \mathbb{E}\left[\int_0^t \left| X_{s-}^{(x)} + \rho(X_{s-}^{(x)}) - a| - |X_{s-}^{(x)} - a| \right|^2 ds \right] + 2\|\lambda\|_\infty \mathbb{E}\left[\int_0^t |\rho(X_{s-}^{(x)})|^2 ds \right]
\]

\[
\leq 4\|\lambda\|_\infty \mathbb{E}\left[\int_0^t |\rho(X_{s-}^{(x)})|^2 ds \right] \leq c_2 (1 + |x|)^2.
\]

Together with (13) respectively (14) this gives that there exist constants \(c_3, c_4 \in (0, \infty)\) such that

\[
\mathbb{E}\left[\int_0^t \text{sgn}(X_{s-}^{(x)} - a) dX_s^{(x)} \right] \leq c_3 (1 + |x|) + \left( \mathbb{E}\left[\int_0^t \text{sgn}(X_{s-}^{(x)} - a) \sigma(X_{s-}^{(x)}) dW_s \right]^2 \right)^{1/2}
\]

\[
+ \left( \mathbb{E}\left[\int_0^t \text{sgn}(X_{s-}^{(x)} - a) \rho(X_{s-}^{(x)}) d\tilde{N}_s \right]^2 \right)^{1/2} \leq c_4 (1 + |x|),
\]

respectively

\[
\mathbb{E}\left[\int_0^t \left( |X_{s-}^{(x)} + \rho(X_{s-}^{(x)}) - a| - |X_{s-}^{(x)} - a| - \text{sgn}(X_{s-}^{(x)} - a)\rho(X_{s-}^{(x)}) \right) dN_s \right]
\]

\[
\leq \left( \mathbb{E}\left[\int_0^t \left( |X_{s-}^{(x)} + \rho(X_{s-}^{(x)}) - a| - |X_{s-}^{(x)} - a| - \text{sgn}(X_{s-}^{(x)} - a)\rho(X_{s-}^{(x)}) \right) d\tilde{N}_s \right]^2 \right)^{1/2}
\]

\[
+ c_3 \left(1 + \mathbb{E}\left[\sup_{t \in [0,T]} |X_t^{(x)}| \right]\right) \leq c_4 (1 + |x|).
\]

Combining these estimates with (12) shows

\[
\mathbb{E}[L^q_t(X^{(x)})] \leq c_4 (1 + |x|).
\]

Note that the continuous martingale part of the semi-martingale (2) starting at \(X_0^{(x)} = x\) is given by

\[
M_t = \int_0^t \sigma(X_{s-}^{(x)}) dW_s, \quad t \in [0, T].
\]

and its predictable quadratic variation is

\[
\langle M \rangle_t = \int_0^t |\sigma(X_{s-}^{(x)})|^2 ds, \quad t \in [0, T].
\]
Therefore, by [42, Lemma 159], we have for all $\varepsilon \in (0, \infty)$, $t \in [0, T]$ that

$$
E \left[ \int_0^t 1_{[\xi_k - \varepsilon, \xi_k + \varepsilon]}(X_s^{(d),x}) \cdot |\sigma(X_{s-}^{(d),x})|^2 ds \right] = E \left[ \int_0^t 1_{[\xi_k - \varepsilon, \xi_k + \varepsilon]}(X_s^{(d),x}) d\langle M \rangle_s \right]
$$

$$
= \int_{[0, T]} 1_{[\xi_k - \varepsilon, \xi_k + \varepsilon]}(a) \cdot E[L_{t_0}^2(X_t^{(d),x})] da \leq 2c_1(1 + |x|) \cdot \varepsilon.
$$

Since $\sigma$ is Lipschitz and of at most linear growth,

$$
|\sigma^2(X_s^{(d),x}) - \sigma^2(X_{s-}^{(d),x})| \leq |\sigma(X_s^{(d),x}) - \sigma(X_{s-}^{(d),x})| \cdot (|\sigma(X_s^{(d),x})| + |\sigma(X_{s-}^{(d),x})|)
$$

$$
\leq 2c\sigma|X_s^{(d),x} - X_{s-}^{(d),x}| \cdot (1 + \sup_{t \in [0, T]} |X_t^{(d),x}|).
$$

Hence, by the Cauchy-Schwarz inequality, (10), and (11) there exists $c_5 \in (0, \infty)$ such that

$$
E \left[ |\sigma^2(X_s^{(d),x}) - \sigma^2(X_{s-}^{(d),x})|^2 \right] \leq 2c\sigma \left( E[|X_s^{(d),x} - X_{s-}^{(d),x}|^2] \right)^{1/2} \cdot \left( E \left[ \left( 1 + \sup_{t \leq \varepsilon \xi_k + \varepsilon} |X_t^{(d),x}| \right)^2 \right] \right)^{1/2}
$$

$$
\leq 2c\sigma \left( E[|X_s^{(d),x} - X_{s-}^{(d),x}|^2] \right)^{1/2} \cdot \left( 1 + \left( E \left[ \sup_{t \in [0, T]} |X_t^{(d),x}|^2 \right] \right)^{1/2} \right) \leq 2c_5(1 + |x|)^2 \cdot |s - \xi_k|^{1/2}.
$$

Thus we have for all $t \in [0, T]$,

$$
E \left[ \int_0^t |\sigma^2(X_s^{(d),x}) - \sigma^2(X_{s-}^{(d),x})|^2 ds \right] \leq c_5(1 + |x|)^2 \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (s - t_k)^{1/2} ds \leq c_5T(1 + x^2)\delta^{1/2}. \tag{16}
$$

From the continuity of $\sigma$ and by the assumption that $\sigma(\xi_k) \neq 0$, we get that there exist $c_6, \varepsilon_0 \in (0, \infty)$ such that

$$
\inf_{\varepsilon \in (\xi_k - \varepsilon_0, \xi_k + \varepsilon_0)} \sigma^2(\varepsilon) \geq c_6.
$$

Combining this with (15) and (16) we get that there exists $c_7 \in (0, \infty)$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$
\int_0^T \mathbb{P}(|X_t^{(d),x} - \xi_k| \leq \varepsilon) dt = \frac{1}{c_6} \int_0^T c_6 1_{[\xi_k - \varepsilon, \xi_k + \varepsilon]}(X_t^{(d),x}) dt
$$

$$
\leq \frac{1}{c_6} \int_0^T 1_{[\xi_k - \varepsilon, \xi_k + \varepsilon]}(X_t^{(d),x}) \sigma^2(X_t^{(d),x}) dt
$$

$$
= \frac{1}{c_6} \int_0^T 1_{[\xi_k - \varepsilon, \xi_k + \varepsilon]}(X_t^{(d),x}) \sigma^2(X_{t-}^{(d),x}) dt
$$

$$
+ \frac{1}{c_6} \int_0^T 1_{[\xi_k - \varepsilon, \xi_k + \varepsilon]}(X_t^{(d),x})(\sigma^2(X_t^{(d),x}) - \sigma^2(X_{t-}^{(d),x})) dt
$$

$$
\leq \frac{1}{c_6} \int_0^T 1_{[\xi_k - \varepsilon, \xi_k + \varepsilon]}(X_t^{(d),x}) \sigma^2(X_{t-}^{(d),x}) dt
$$

$$
+ \frac{1}{c_6} \int_0^T |\sigma^2(X_t^{(d),x}) - \sigma^2(X_{t-}^{(d),x})| dt \leq 2c_4(1 + |x|)\varepsilon + 2c_5T(1 + x^2)\delta^{1/2} \leq c_7(1 + x^2)(\varepsilon + \delta^{1/2}).
$$
For $\varepsilon \in [\varepsilon_0, \infty)$ it trivially holds that
\[
\int_0^T \mathbb{P}(|X_t^{(\delta, x)} - \zeta_k| \leq \varepsilon) dt \leq T = \frac{T}{\varepsilon_0} \cdot \varepsilon_0 \leq \frac{T}{\varepsilon_0}(1 + \varepsilon^2)(\varepsilon + \delta^{1/2}).
\]
Choosing $C^{(O)} = \max\{c_7, \frac{T}{\varepsilon_0}\}$ closes the proof. \hfill \Box

### 4.1.2 Estimation of the discontinuity crossing probability

Note that as in [3], from now on we write $X_t$ instead of $X_{t-}$. This is vindicated by the continuity of the compensators of $W$ and $N$.

Let for all $k \in \{1, \ldots, m\}$, $t \in [0, T]$, $Z_k^t = \{\omega \in \Omega: (X_{t-}^{(\delta)}(\omega) - \zeta_k)(X_t^{(\delta)}(\omega) - \zeta_k) \leq 0\}$.

**Lemma 4.3.** Let Assumptions 2.1 hold. Let $k \in \{1, \ldots, m\}$, $\delta \in (0, 1)$ sufficiently small,
\[
\mathbb{P}(Z_k^T \cap Z_k^0) \leq C_1 \mathbb{P}(Z_k^\delta) + C_1 \cdot \int_{\mathbb{R}} \mathbb{P}\left(Z_k^\delta \cap \left\{|X_{t-}^{(\delta)}(\omega) - \zeta_k| \leq C_1 \delta^{1/2}(1 + |z|)\right\}\right) \cdot e^{-\frac{|z|^2}{2}} dz.
\]

**Proof.** For treating the Gaussian part we adopt arguments from [26, Proof of Lemma 5].

If $t = \underline{t}$, then for all $c_1 \in (0, \infty)$, $z \in \mathbb{R}$ it holds that
\[
Z_k^t = \{X_t^{(\delta)} - \zeta_k = 0\} \subseteq \left\{|X_{t-}^{(\delta)}(\omega) - \zeta_k| \leq c_1 \delta^{1/2}(1 + |z|)\right\}.
\]

So in this case, the assertion of the lemma holds for all $C_1 \geq 1/\sqrt{2\pi}$.

Now let $t > \underline{t}$ and let
\[
W_1 = \frac{W_t - W_{\underline{t}}}{\sqrt{t - \underline{t}}}, \quad W_2 = \frac{W_t - W_{t-}^{(\delta)}}{\sqrt{t - \underline{t}}}, \quad W_3 = \frac{W_{t-}^{(\delta)} - W_{\underline{t} - \delta}}{\sqrt{\delta - (t - \underline{t})}}, \quad \bar{P} = N_t - N_{\underline{t} - \delta}.
\]

Observe that $\bar{W}_1, \bar{W}_2, \bar{W}_3$ are standard normally distributed, $\bar{P}$ is Poisson distributed with parameter $\int_{\underline{t} - \delta}^{t} \lambda ds$, $\bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{P}$ are independent, $(\bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{P})$ is independent of $\mathcal{F}_s$ since $s \leq t - \delta$, and $(\bar{W}_1, \bar{W}_2)$ is independent of $\mathcal{F}_{t-}^{(\delta)}$.

Let $c_2 = \max\{c_\mu, c_\sigma, c_\rho\}$ and let $\delta$ be sufficiently small such that
\[
12c_2(1 + |\zeta_k|) \cdot \frac{1 + \sqrt{2\log(T/\delta)}}{\sqrt{T/\delta}} \leq 1/2.
\]

Then note that the following inclusion similar to [26, (20)] holds:
\[
Z_k^t \cap \{\bar{P} = 0\} \cap \left\{\max_{i \in \{1, 2, 3\}} |\bar{W}_i| \leq \sqrt{2\log(T/\delta)}\right\} \leq \{\bar{P} = 0\} \cap \left\{|X_{t-}^{(\delta)}(\omega) - \zeta_k| \leq \frac{12c_2(1 + |\zeta_k|) \cdot (1 + |\bar{W}_1| + |\bar{W}_2|)}{\sqrt{T/\delta}}\right\} \leq \left\{|X_{t-}^{(\delta)}(\omega) - \zeta_k| \leq \frac{12c_2(1 + |\zeta_k|) \cdot (1 + |\bar{W}_1| + |\bar{W}_2|)}{\sqrt{T/\delta}}\right\}.
\]

In fact, with the additional condition $\bar{P} = 0$, we are back to the jump-free case studied in [26]. The proof of [14] hence works exactly as the one for [26, (20)], which is a part of [26, Proof of Lemma 5].
We follow the first part of [26, Proof of Lemma 6].

For such that for all

Combining (18) with (19) and (20) finishes the proof.

Normally distributed random variables is normally distributed with mean 0 and variance 2.

For the first term on the right hand side of (18), we use the fact that the sum of standard Gaussian tail estimate.

Let Assumptions 2.1 hold. Let Lemma 4.4.

Using (17) we obtain

\[ P \left( Z_{k}^{s} \cap \left\{ \left| X_{k}^{(s)} - \zeta_{k} \right| \leq \frac{12c_{2}(1 + |\zeta_{k}|) \cdot (1 + |\overline{W}_{1} + |\overline{W}_{2}|)}{\sqrt{T/\delta}} \right\} \right) \]

\[ \leq \frac{2}{\pi} \int_{(0,\infty) \times (0,\infty)} \mathbb{P} \left( Z_{k}^{s} \cap \left\{ \left| X_{k}^{(s)} - \zeta_{k} \right| \leq \frac{12c_{2}(1 + |\zeta_{k}|) \cdot (1 + z_{1} + z_{2})}{\sqrt{T/\delta}} \right\} \right) e^{-\frac{(z_{1}^{2} + z_{2}^{2})}{2}} d(z_{1}, z_{2}) \]

\[ \leq \frac{2}{\pi} \int_{\mathbb{R}^{2}} \mathbb{P} \left( Z_{k}^{s} \cap \left\{ \left| X_{k}^{(s)} - \zeta_{k} \right| \leq \frac{12\sqrt{2}c_{2}(1 + |\zeta_{k}|) \cdot (1 + z_{1} + z_{2})}{\sqrt{T/\delta}} \right\} \right) e^{-\frac{z_{1}^{2}}{2}} d(z_{1}, z_{2}) \]

\[ = \frac{4}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbb{P} \left( Z_{k}^{s} \cap \left\{ \left| X_{k}^{(s)} - \zeta_{k} \right| \leq \frac{12\sqrt{2}c_{2}(1 + |\zeta_{k}|) \cdot (1 + z)}{\sqrt{T/\delta}} \right\} \right) e^{-\frac{z^{2}}{2}} dz. \]

For the first term on the right hand side of (18) we use a standard Gaussian tail estimate.

\[ \mathbb{P} \left( Z_{k}^{s} \cap \left\{ \{ \hat{P} > 0 \} \cup \left\{ \max_{i \in \{1,2,3\}} |\overline{W}_{i}| > \sqrt{2\log(T/\delta)} \right\} \right\} \right) \]

\[ = \mathbb{P} \left( Z_{k}^{s} \cap \left\{ \{ \hat{P} > 0 \} \cup \{ |\overline{W}_{1}| > \sqrt{2\log(T/\delta)} \} \cup \{ |\overline{W}_{2}| > \sqrt{2\log(T/\delta)} \} \cup \{ |\overline{W}_{3}| > \sqrt{2\log(T/\delta)} \} \right\} \right) \]

\[ \leq \mathbb{P}(Z_{k}^{s}) \cdot \left( 3 \mathbb{P}(\{ |\overline{W}_{1}| > \sqrt{2\log(T/\delta)} \}) + \mathbb{P}(\hat{P} > 0) \right) \]

\[ \leq \mathbb{P}(Z_{k}^{s}) \cdot \left( \frac{3\delta}{T \sqrt{\pi \log(T/\delta)}} + \mathbb{P}(\hat{P} > 0) \right) \]

\[ = \mathbb{P}(Z_{k}^{s}) \cdot \left( \frac{3\delta}{T \sqrt{\pi \log(T/\delta)}} + 1 - e^{-\int_{-\delta}^{\delta} \lambda_{s} ds} \right) \leq \mathbb{P}(Z_{k}^{s}) \cdot \left( \frac{3\delta}{T \sqrt{\pi \log(T/\delta)}} + 2\delta \| \lambda \|_{\infty} \right) . \]

Combining (18) with (19) and (20) finishes the proof.

Lemma 4.4. Let Assumptions 2.1 hold. Let \( s \in [0, T) \). There exists a constant \( C_{2} \in (0, \infty) \) such that for all \( k \in \{1, \ldots, m\} \), \( \delta \in (0, 1) \) sufficiently small,

\[ \int_{s}^{T} \mathbb{P}(Z_{k}^{s} \cap Z_{k}^{t}) dt \leq C_{2}\delta^{1/2} \cdot \left( \mathbb{P}(Z_{k}^{s}) + \mathbb{E} \left[ 1_{Z_{k}^{s}} \cdot (X_{k}^{(s)} - \zeta_{k})^{2} \right] \right) . \]

 Proof. We follow the first part of [26]. Proof of Lemma 6]. For \( s \geq T - \delta \),

\[ \int_{s}^{T} \mathbb{P}(Z_{k}^{s} \cap Z_{k}^{t}) dt \leq \int_{T - \delta}^{T} \mathbb{P}(Z_{k}^{s}) dt = \mathbb{P}(Z_{k}^{s}) \cdot \delta . \]

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Therefore we may assume \( s < T - \delta \) and hence \( \delta 
less T - 2\delta \). With this
\[
\int_s^T \mathbb{P}(Z_k^* \cap Z_k^d) dt = \int_s^{T+2\delta} \mathbb{P}(Z_k^* \cap Z_k^d) dt + \int_{T+2\delta}^T \mathbb{P}(Z_k^* \cap Z_k^d) dt \\
\leq \int_s^{T+2\delta} \mathbb{P}(Z_k^s) dt + \int_s^T \mathbb{P}(Z_k^s \cap Z_k^d) dt \leq 2\delta \mathbb{P}(Z_k^s) + \int_s^T \mathbb{P}(Z_k^s \cap Z_k^d) dt.
\]
Let \( \ell \in \mathbb{N} \) be such that \( t_\ell = \delta + 2\delta \). For \( t \in [s + 2\delta, T] \) it holds that \( k \geq \delta + \delta \geq s \). Hence we may apply Lemma 4.3 and obtain that there exists a constant \( c_1 \in (0, \infty) \) such that
\[
\int_s^T \mathbb{P}(Z_k^s \cap Z_k^d) dt \\
\leq c_1 \mathbb{P}(Z_k^s) + c_1 \int_{T+2\delta}^T \mathbb{P}(Z_k^s \cap \left\{ |X_{u-} - \zeta_k| \leq c_1 \delta^{1/2} (1 + |z|) \right\}) \cdot e^{-\frac{t^2}{2}} dtzdu \\
= c_1 \mathbb{P}(Z_k^s) + c_1 \sum_{\ell=1}^{N-1} \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^{T} \mathbb{P}(Z_k^s \cap \left\{ |X_{u-} - \zeta_k| \leq c_1 \delta^{1/2} (1 + |z|) \right\}) \cdot e^{-\frac{t^2}{2}} dtzdu \\
= c_1 \mathbb{P}(Z_k^s) + c_1 \sum_{\ell=1}^{N-1} \int_{t_\ell}^{T-\delta} \int_{t_\ell}^{T} \mathbb{P}(Z_k^s \cap \left\{ |X_{u-} - \zeta_k| \leq c_1 \delta^{1/2} (1 + |z|) \right\}) \cdot e^{-\frac{t^2}{2}} dtzdu \\
= c_1 \mathbb{P}(Z_k^s) + c_1 \int_{T+\delta}^T \mathbb{P}(Z_k^s \cap \left\{ |X_{u-} - \zeta_k| \leq c_1 \delta^{1/2} (1 + |z|) \right\}) du e^{-\frac{t^2}{2}} dz.
\]
The Markov property yields
\[
\int_{T+\delta}^T \mathbb{P}(Z_k^s \cap \left\{ |X_{u-} - \zeta_k| \leq c_1 \delta^{1/2} (1 + |z|) \right\}) du \\
= \mathbb{E} \left[ 1_{Z_k^s} \cdot \mathbb{E} \left[ \int_{T+\delta}^T 1 \left\{ |X_{u-} - \zeta_k| \leq c_1 \delta^{1/2} (1 + |z|) \right\} du \right| X^{(d)}_{T+\delta} = x \right].
\]
Lemma 4.2 ensures that there exists \( c_2 \in (0, \infty) \) such that
\[
\mathbb{E} \left[ \int_{T+\delta}^T 1 \left\{ |X_{u-} - \zeta_k| \leq c_1 \delta^{1/2} (1 + |z|) \right\} du \right| X^{(d)}_{T+\delta} = x \\
= \int_{0}^{T-2\delta} \mathbb{E} \left[ 1 \left\{ |X_{u-} - \zeta_k| \leq c_1 \delta^{1/2} (1 + |z|) \right\} \right| X^{(d)}_{0} = x \right] dt \\
\leq c_2 (1 + x^2)(c_1 \delta^{1/2} (1 + |z|) + \delta^{1/2}).
\]
Combining this with (21) gives
\[
\int_{\Delta + \delta}^{T - \delta} \mathbb{P}\left(Z^\delta_k \cap \left\{ |X^{(\delta)}_t - \zeta_k| \leq c_1 \delta^{1/2} (1 + |z|) \right\} \right) dt \\
\leq \mathbb{E}\left[ 1_{Z^\delta_k} \cdot c_2 (1 + (X^{(\delta)}_{\Delta + \delta})^2) (c_1 \delta^{1/2} (1 + |z|) + \delta^{1/2}) \right] \\
\leq 2c_2 (c_1 (1 + |z|) + 1) (1 + |\zeta_k|^2) \delta^{1/2} \cdot \left( \mathbb{P}(Z^\delta_k) + \mathbb{E}\left[ 1_{Z^\delta_k} \cdot (X^{(\delta)}_{\Delta + \delta} - \zeta_k)^2 \right] \right).
\]

\[
\square
\]

**Lemma 4.5.** Let Assumptions 2.1 hold. There exists a constant $C_3 \in (0, \infty)$ such that for all $k \in \{1, \ldots, m\}$,
\[
\int_0^T \mathbb{E}\left[ 1_{Z^\delta_k} \cdot (X^{(\delta)}_{\Delta + \delta} - \zeta_k)^2 \right] ds \leq C_3 \delta.
\]

**Proof.** First, note that
\[
1_{Z^\delta_k} \cdot |X^{(\delta)}_{\Delta + \delta} - \zeta_k| \leq 1_{Z^\delta_k} \cdot (|X^{(\delta)}_{\Delta + \delta} - X^{(\delta)}_s| + |X^{(\delta)}_s - \zeta_k|) \\
\leq 1_{Z^\delta_k} \cdot (|X^{(\delta)}_{\Delta + \delta} - X^{(\delta)}_s| + |X^{(\delta)}_s - X^{(\delta)}_\Delta|).
\]

With this,
\[
\int_0^T \mathbb{E}\left[ 1_{Z^\delta_k} \cdot (X^{(\delta)}_{\Delta + \delta} - \zeta_k)^2 \right] ds \leq \int_0^T \mathbb{E}\left[ (|X^{(\delta)}_{\Delta + \delta} - X^{(\delta)}_s| + |X^{(\delta)}_s - X^{(\delta)}_\Delta|)^2 \right] ds \\
= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[ (|X^{(\delta)}_{\Delta + \delta} - X^{(\delta)}_s| + |X^{(\delta)}_s - X^{(\delta)}_\Delta|)^2 \right] ds \\
\leq 2 \cdot \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \left( \mathbb{E}\left[ |X^{(\delta)}_{\Delta + \delta} - X^{(\delta)}_s|^2 \right] + \mathbb{E}\left[ |X^{(\delta)}_s - X^{(\delta)}_\Delta|^2 \right] \right) ds.
\]

Hence, Lemma 4.1 yields
\[
\int_0^T \mathbb{E}\left[ 1_{Z^\delta_k} \cdot (X^{(\delta)}_{\Delta + \delta} - \zeta_k)^2 \right] ds \leq 4TC^{(M)} \cdot \delta.
\]

\[
\square
\]

**Proposition 4.6.** Let Assumptions 2.1 hold. There exists a constant $C^{(\text{cross})} \in (0, \infty)$ such that for all $k \in \{1, \ldots, m\}$, $\delta \in (0, 1)$ sufficiently small,
\[
\mathbb{E}\left[ \left( \int_0^T 1_{\{(x,y) \in \mathbb{R}^2: (x - \zeta_k)(y - \zeta_k) \leq 0\}} (X^{(\delta)}_t, X^{(\delta)}_s) ds \right)^2 \right] \leq C^{(\text{cross})} \cdot \delta.
\]

**Proof.** By Lemma 4.1 we get that
\[
\mathbb{E}\left[ \left( \int_0^T 1_{\{(x,y) \in \mathbb{R}^2: (x - \zeta_k)(y - \zeta_k) \leq 0\}} (X^{(\delta)}_t, X^{(\delta)}_s) ds \right)^2 \right] = 2 \cdot \int_0^T \int_s^T \mathbb{P}(Z^\delta_k \cap Z^\delta_l) dt ds \\
\leq 2C_2 \delta^{1/2} \cdot \left( \int_0^T \mathbb{P}(Z^\delta_k) ds + \int_0^T \mathbb{E}\left[ 1_{Z^\delta_k} \cdot (X^{(\delta)}_{\Delta + \delta} - \zeta_k)^2 \right] ds \right).
\]

(22)
Next note that in Lemma 4.4 the only requirement on \( Z_k \) was that it is \( \mathcal{F}_s \)-measurable. Replacing \( Z_k \) by \( \Omega \) and setting \( s = 0 \) in Lemma 4.4 and applying Lemma 4.1 gives
\[
\int_0^t \mathbb{P}(Z_k^t) dt \leq C_2 \delta^{1/2} \cdot \left( 1 + \mathbb{E} \left[ |X_{2s}^{(\delta)} - \zeta_k|^2 \right] \right) \leq C_2 \delta^{1/2} \left( 1 + 2(\zeta_k)^2 + 2C(M) \right).
\]

Combining (22) with this and Lemma 4.5 yields
\[
\mathbb{E} \left[ \int_0^T 1_{\{ (x,y) \in \mathbb{R}^2 : (x-\zeta_k)(y-\xi_k) \leq 0 \}} \left( X_{2s}^{(\delta)}, X_{2s}^{(\delta)} \right) ds \right]^2 \leq 2C_2 \delta^{1/2} \cdot \left( C_2 \left( 1 + 2(\zeta_k)^2 + 2C(M) \right) \delta^{1/2} + C_3 \delta \right).
\]

This closes the proof. \( \square \)

4.2 Main result

**Theorem 4.7.** Let Assumptions 2.1 hold. Then there exists a constant \( C^{(EM)} \in (0, \infty) \) such that for all \( \delta \in (0, 1) \) sufficiently small,
\[
\left( \mathbb{E} \left[ \sup_{t \in [0,T]} |X_t - X_t^{(\delta)}|^2 \right] \right)^{1/2} \leq C^{(EM)} \delta^{1/2}.
\]

**Proof.** Let \( G \) be as in (3) and \( Z \) be as in (1). Since \( G^{-1} \) is Lipschitz,
\[
\left( \mathbb{E} \left[ \sup_{t \in [0,T]} |X_t - X_t^{(\delta)}|^2 \right] \right)^{1/2} = \left( \mathbb{E} \left[ \sup_{t \in [0,T]} |G^{-1}(Z_t) - G^{-1}(G(X_t^{(\delta)}))|^2 \right] \right)^{1/2} \leq L_{G^{-1}} \left( \mathbb{E} \left[ \sup_{t \in [0,T]} |Z_t - G(X_t^{(\delta)})|^2 \right] \right)^{1/2},
\]
and by the triangle inequality,
\[
\left( \mathbb{E} \left[ \sup_{t \in [0,T]} |Z_t - G(X_t^{(\delta)})|^2 \right] \right)^{1/2} \leq \left( \mathbb{E} \left[ \sup_{t \in [0,T]} |Z_t - Z_t^{(\delta)}|^2 \right] \right)^{1/2} + \left( \mathbb{E} \left[ \sup_{t \in [0,T]} |Z_t^{(\delta)} - G(X_t^{(\delta)})|^2 \right] \right)^{1/2}.
\]

There exists a constant \( c_1 \in (0, \infty) \) such that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |Z_t - Z_t^{(\delta)}|^2 \right] \leq c_1 \delta.
\]

Hence our task is to estimate the second term in (24). For all \( \tau \in [0, T] \) let
\[
u(\tau) := \mathbb{E} \left[ \sup_{t \in [0,\tau]} |Z_t^{(\delta)} - G(X_t^{(\delta)})|^2 \right]
\]
and let for all \( x_1, x_2 \in \mathbb{R}, \nu(x_1, x_2) = G'(x_1)\mu(x_2) + \frac{1}{2}G''(x_1)\sigma_2(x_2) \) and \( \zeta(x_1, x_2) = G(x_1 + \rho(x_2)) - G(x_1) \). By the Meyer-Itô formula, which follows from [36, Theorem 71], we obtain
\[
G(X_t^{(\delta)}) = G(X_0^{(\delta)}) + \int_0^t \nu(X_s^{(\delta)}, X_s^{(\delta)}) ds + \int_0^t G'(X_s^{(\delta)})\sigma(X_s^{(\delta)}) dW_s + \int_0^t \zeta(X_s^{(\delta)}, X_s^{(\delta)}) dN_s.
\]
This yields that
\[
    u(\tau) = E \left[ \sup_{t \in [0, \tau]} \int_0^t \hat{\mu}(Z_\omega^{(\delta)}) ds + \int_0^t \hat{\sigma}(Z_\omega^{(\delta)}) dW_s + \int_0^t \hat{\rho}(Z_\omega^{(\delta)}) dN_s 
        - \int_0^t \nu(X_\omega^{(\delta)}, X_\omega^{(\delta)}) ds - \int_0^t G'(X_\omega^{(\delta)}) \sigma(X_\omega^{(\delta)}) dW_s 
        - \int_0^t \zeta(X_\omega^{(\delta)}, X_\omega^{(\delta)}) dN_s \right]^2 
\]

Using
\[
    \nu(X_\omega^{(\delta)}, X_\omega^{(\delta)}) = \hat{\mu}(G(X_\omega^{(\delta)})) \quad \text{and} \quad \zeta(X_\omega^{(\delta)}, X_\omega^{(\delta)}) = \hat{\rho}(G(X_\omega^{(\delta)}))
\]
we get
\[
    u(\tau) \leq 6 \cdot \sum_{i=1}^6 E_i(\tau) \quad (26)
\]

with
\[
    E_1(\tau) = E \left[ \sup_{t \in [0, \tau]} \int_0^t [\hat{\mu}(Z_\omega^{(\delta)}) - \hat{\mu}(G(X_\omega^{(\delta)}))]^2 ds \right], \\
    E_2(\tau) = E \left[ \sup_{t \in [0, \tau]} \int_0^t [\hat{\sigma}(Z_\omega^{(\delta)}) - \hat{\sigma}(G(X_\omega^{(\delta)}))]^2 dW_s \right], \\
    E_3(\tau) = E \left[ \sup_{t \in [0, \tau]} \int_0^t [\hat{\rho}(Z_\omega^{(\delta)}) - \hat{\rho}(G(X_\omega^{(\delta)}))]^2 dN_s \right], \\
    E_4(\tau) = E \left[ \sup_{t \in [0, \tau]} \int_0^t [\nu(X_\omega^{(\delta)}, X_\omega^{(\delta)}) - \nu(X_\omega^{(\delta)}, X_\omega^{(\delta)})]^2 ds \right], \\
    E_5(\tau) = E \left[ \sup_{t \in [0, \tau]} \int_0^t [G'(X_\omega^{(\delta)}) \sigma(X_\omega^{(\delta)}) - G'(X_\omega^{(\delta)}) \sigma(X_\omega^{(\delta)})] dW_s \right], \\
    E_6(\tau) = E \left[ \sup_{t \in [0, \tau]} \int_0^t [\zeta(X_\omega^{(\delta)}, X_\omega^{(\delta)}) - \zeta(X_\omega^{(\delta)}, X_\omega^{(\delta)})]^2 dN_s \right].
\]

The Cauchy-Schwarz inequality yields
\[
    E_1(\tau) \leq T \cdot E \left[ \int_0^t |\hat{\mu}(Z_\omega^{(\delta)}) - \hat{\mu}(G(X_\omega^{(\delta)}))]^2 ds \right],
\]
and the Burkholder-Davis-Gundy inequality, see, e.g., [19, Lemma 3.7], yields
\[
    E_2(\tau) \leq 2 \cdot E \left[ \int_0^t |\hat{\sigma}(Z_\omega^{(\delta)}) - \hat{\sigma}(G(X_\omega^{(\delta)}))]^2 ds \right], \\
    E_5(\tau) \leq 2 \cdot E \left[ \int_0^t |G'(X_\omega^{(\delta)}) \sigma(X_\omega^{(\delta)}) - G'(X_\omega^{(\delta)}) \sigma(X_\omega^{(\delta)})|^2 dW_s \right].
\]

Finally, since \( N_s = \tilde{N}_s + \kappa_s \) we get by Doob’s maximum inequality, Itô’s isometry, and the Cauchy-Schwarz inequality that
\[
    E_3(\tau) \leq 2 \cdot E \left[ \sup_{t \in [0, \tau]} \int_0^t [\hat{\rho}(Z_\omega^{(\delta)}) - \hat{\rho}(G(X_\omega^{(\delta)}))] d\tilde{N}_s \right]^2 + 2 \cdot E \left[ \sup_{t \in [0, \tau]} \int_0^t [\hat{\rho}(Z_\omega^{(\delta)}) - \hat{\rho}(G(X_\omega^{(\delta)}))] d\kappa(s) \right]^2 
\]
\[
    \leq 8 \cdot E \left[ \int_0^t [\hat{\rho}(Z_\omega^{(\delta)}) - \hat{\rho}(G(X_\omega^{(\delta)}))] d\tilde{N}_s \right]^2 + 2 \cdot E \left[ \sup_{t \in [0, \tau]} \int_0^t [\hat{\rho}(Z_\omega^{(\delta)}) - \hat{\rho}(G(X_\omega^{(\delta)}))] d\kappa_s \right]^2 
\]
\[
    \leq 8 \cdot E \left[ \int_0^t [\hat{\rho}(Z_\omega^{(\delta)}) - \hat{\rho}(G(X_\omega^{(\delta)}))]^2 \lambda(s) ds \right] + 2T \cdot E \left[ \int_0^t [\hat{\rho}(Z_\omega^{(\delta)}) - \hat{\rho}(G(X_\omega^{(\delta)}))]^2 ds \right] 
\]
\[
    \leq 2 ||\lambda||_{\infty} (4 + T ||\lambda||_{\infty}) \cdot E \left[ \int_0^t [\hat{\rho}(Z_\omega^{(\delta)}) - \hat{\rho}(G(X_\omega^{(\delta)}))]^2 ds \right]
\]
and analogously
\[ E_6(\tau) \leq 2\|\lambda\|_\infty(4 + T\|\lambda\|_\infty) \cdot \mathbb{E} \left[ \int_0^T |\varsigma(X^{(\delta)}_s, X^{(\delta)}_s) - \varsigma(X^{(\delta)}_s, X^{(\delta)}_s)|^2 ds \right]. \]

Next, using that \( \bar{\mu}, \bar{\sigma}, \bar{\rho} \) are Lipschitz, we get that
\[
\begin{align*}
E_1(\tau) &\leq T(L_{\bar{\mu}})^2 \cdot \int_0^T \mathbb{E} \left[ |Z^{(\delta)}_s - G(X^{(\delta)}_s)|^2 \right] ds 
&\leq T(L_{\bar{\mu}})^2 \cdot \int_0^T u(s) ds, \\
E_2(\tau) &\leq 2(L_{\bar{\sigma}})^2 \cdot \int_0^T \mathbb{E} \left[ |\tilde{Z}^{(\delta)}_s - G(X^{(\delta)}_s)|^2 \right] ds 
&\leq 2(L_{\bar{\sigma}})^2 \cdot \int_0^T u(s) ds, \\
E_3(\tau) &\leq 2\|\lambda\|_\infty(4 + T\|\lambda\|_\infty)(L_{\bar{\rho}})^2 \cdot \int_0^T \mathbb{E} \left[ |Z^{(\delta)}_s - G(X^{(\delta)}_s)|^2 \right] ds 
&\leq (8\|\lambda\|_\infty + 2T\|\lambda\|_\infty)(L_{\bar{\rho}})^2 \cdot \int_0^T u(s) ds.
\end{align*}
\]

The linear growth condition on \( \sigma \), the Lipschitz continuity of \( G' \), and the fact that \( X^{(\delta)}_s \) is \( \mathcal{F}_s \)-measurable give
\[
E_5(\tau) \leq 2(c_\sigma)^2(L_{G'})^2 \cdot \int_0^T \mathbb{E} \left[ (1 + \|X^{(\delta)}_s\|)^2 \cdot \|X^{(\delta)}_s - X^{(\delta)}_s\|^2 \right] ds
\quad = 2(c_\sigma)^2(L_{G'})^2 \cdot \int_0^T \mathbb{E} \left[ (1 + \|X^{(\delta)}_s\|)^2 \cdot \mathbb{E} \left[ \|X^{(\delta)}_s - X^{(\delta)}_s\|^2 \ | \mathcal{F}_s \right] \right] ds.
\]

Let \( c_2 = \|\lambda\|_\infty \cdot \max\{\|\lambda\|_\infty, 1\} \). Since \( W_s - W_s \) and \( N_s - N_s \) are independent of \( \mathcal{F}_s \), \( X^{(\delta)}_s \) is \( \mathcal{F}_s \)-measurable, and by the linear growth condition on \( \mu, \sigma, \) and \( \rho \) we get for \( s \in [0, T] \),
\[
\begin{align*}
\mathbb{E} \left[ \|X^{(\delta)}_s - X^{(\delta)}_s\|^2 \ | \mathcal{F}_s \right] &= \mathbb{E} \left[ \|\mu(X^{(\delta)}_s)(s - s) + \sigma(X^{(\delta)}_s)(W_s - W_s) + \rho(X^{(\delta)}_s)(N_s - N_s)\|^2 \ | \mathcal{F}_s \right] \\
&\leq 3\|\mu(X^{(\delta)}_s)\|^2(s - s)^2 + 3\|\sigma(X^{(\delta)}_s)\|^2(s - s)^2 + 3\|\rho(X^{(\delta)}_s)\|^2\|\lambda\|_\infty^2(s - s)^2 + \|\lambda\|_\infty(s - s)) \\
&\leq 6((c_\mu)^2 + (c_\sigma)^2 + 2c_2(c_\rho)^2)(1 + \|X^{(\delta)}_s\|^2) \cdot |s - s|.
\end{align*}
\]

This and Lemma 4.1 yield
\[
E_5(\tau) \leq 24(c_\sigma)^2(L_{G'})^2((c_\mu)^2 + (c_\sigma)^2 + 2(c_\rho)^2\|\lambda\|_\infty^2) \cdot \int_0^T |s - s| \cdot \left( 1 + \mathbb{E} \left[ \|X^{(\delta)}_s\|^4 \right] \right) ds
\quad \leq 24T(c_\sigma)^2(L_{G'})^2((c_\mu)^2 + (c_\sigma)^2 + 2c_2(c_\rho)^2)(1 + C(M)) \cdot \delta.
\]

The Lipschitz continuity of \( G \) establishes
\[
E_6(\tau) \leq 4\|\lambda\|_\infty(4 + T\|\lambda\|_\infty) \cdot \int_0^T \mathbb{E} \left[ |G(X^{(\delta)}_s) + \rho(X^{(\delta)}_s) - G(X^{(\delta)}_s) + \rho(X^{(\delta)}_s)|^2 + \|G(X^{(\delta)}_s) - G(X^{(\delta)}_s)\|^2 \right] ds
\quad \leq 8(L_{G})^2\|\lambda\|_\infty(4 + T\|\lambda\|_\infty) \cdot \int_0^T \mathbb{E} \left[ \|X^{(\delta)}_s - X^{(\delta)}_s\|^2 \right] ds.
\]

By Lemma 4.1
\[
E_6(\tau) = 8C(M)(L_{G})^2\|\lambda\|_\infty(4 + T\|\lambda\|_\infty) \cdot \int_0^T |s - s|ds
\quad = 8C(M)T(L_{G})^2\|\lambda\|_\infty(4 + T\|\lambda\|_\infty) \cdot \delta.
\]

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For estimating $E_4(\tau)$ first note that
\[
E_4(\tau) \leq 2T \cdot \mathbb{E} \left[ \int_0^T |G'(X^{(\delta)}_s) - G'(X^{(\delta)}_s)|^2 \cdot |\mu(X^{(\delta)}_s)|^2 ds \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T G''(X^{(\delta)}_s) \sigma^2(X^{(\delta)}_s) - G''(X^{(\delta)}_s) \sigma^2(X^{(\delta)}_s) ds \right]^2.
\] (30)

Analog to the estimate of $E_5(\tau)$ above we obtain for the first term in (30),
\[
2T \cdot \mathbb{E} \left[ \int_0^T |G'(X^{(\delta)}_s) - G'(X^{(\delta)}_s)|^2 \cdot |\mu(X^{(\delta)}_s)|^2 ds \right] \leq 24T^2 (c_\mu)^2 (L_G')^2 ((c_\mu)^2 + (c_\rho)^2 + 2c_2(c_\rho)^2)(1 + C(M)) \cdot \delta.
\] (31)

The second term in (30) will be analysed in the way introduced in [26], but adapted to our setup. It is essential that we have not applied the Cauchy-Schwarz inequality to the second term in (30) for recovering the optimal convergence rate $1/2$. For this we define for all $k \in \{1, \ldots, m\}$ the sets
\[
\mathcal{Z}_k = \{(x, y) \in \mathbb{R}^2 : (x - \zeta_k)(y - \zeta_k) \leq 0)\}, \quad \mathcal{Z} = \bigcup_{\ell=1}^m \mathcal{Z}_\ell.
\]
that is the set of all $x, y \in \mathbb{R}$ which lie on different sides of a point of discontinuity of the drift. So for $(X^{(\delta)}_s, X^{(\delta)}_s) \in \mathcal{Z}^c$ we have that a Lipschitz condition is satisfied by $G''$. So for
\[
\frac{1}{2} \mathbb{E} \left[ \int_0^T G''(X^{(\delta)}_s) \sigma^2(X^{(\delta)}_s) - G''(X^{(\delta)}_s) \sigma^2(X^{(\delta)}_s) ds \right]^2
\]
\[
\leq \mathbb{E} \left[ \int_0^T \left[ G''(X^{(\delta)}_s) \sigma^2(X^{(\delta)}_s) - G''(X^{(\delta)}_s) \sigma^2(X^{(\delta)}_s) \right] \cdot 1_{\mathcal{Z}}(X^{(\delta)}_s, X^{(\delta)}_s) ds \right]^2
\]
\[
+ T \cdot \mathbb{E} \left[ \int_0^T \left\{ G''(X^{(\delta)}_s) \sigma^2(X^{(\delta)}_s) - G''(X^{(\delta)}_s) \sigma^2(X^{(\delta)}_s) \right\}^2 \cdot 1_{\mathcal{Z}^c}(X^{(\delta)}_s, X^{(\delta)}_s) ds \right]
\] (32)

we can estimate the second term similar to above, that is
\[
T \cdot \mathbb{E} \left[ \int_0^T |G''(X^{(\delta)}_s) \sigma^2(X^{(\delta)}_s) - G''(X^{(\delta)}_s) \sigma^2(X^{(\delta)}_s)|^2 \cdot 1_{\mathcal{Z}^c}(X^{(\delta)}_s, X^{(\delta)}_s) ds \right]
\]
\[
\leq T(c_\sigma)^4 (L_G')^2 \cdot \mathbb{E} \left[ \int_0^T (1 + |X^{(\delta)}_s|)^4 \cdot |X^{(\delta)}_s - X^{(\delta)}_s|^2 ds \right]
\]
\[
\leq 24T^2 (c_\sigma)^4 (L_G')^2 ((c_\sigma)^2 + (c_\rho)^2 + 2c_2(c_\rho)^2)(1 + C(M)) \cdot \delta.
\] (33)

For the first term in (32), observe that for all $k \in \{1, \ldots, m\}$, $x, y \in \mathbb{R}$,
\[
|x|1_{\mathcal{Z}_k}(x, y) \leq (|\zeta_k| + |x - \zeta_k|)1_{\mathcal{Z}_k}(x, y) \leq (|\zeta_k| + |x - y|)1_{\mathcal{Z}_k}(x, y),
\]
and hence
\[
(1 + x^2) \cdot 1_{\mathcal{Z}}(x, y) \leq 2m|x - y|^2 + (1 + 2 \max\{|\zeta_1|, |\zeta_m|\})^2 \cdot \sum_{k=1}^m 1_{\mathcal{Z}_k}(x, y).
\]
This and Lemma 4.1 yield for the first term in (32),
\[ \mathbb{E} \left[ \int_0^T \left( \frac{d}{ds}^2 (X_s^{(\delta)}) - \frac{d}{ds}^2 (X_s^{(\delta)}) \right) \cdot 1_{\mathcal{Z}}(X_s^{(\delta)}, X_s^{(\delta)}) ds \right] \]
\[ \leq 4(c_\delta)^4 \| G'' \|_\infty^2 \cdot \mathbb{E} \left[ \int_0^T (1 + |X_s^{(\delta)}|^2) \cdot 1_{\mathcal{Z}}(X_s^{(\delta)}, X_s^{(\delta)}) ds \right] \]
\[ \leq 4(c_\delta)^4 \| G'' \|_\infty^2 \cdot \mathbb{E} \left[ \int_0^T 2m |X_s^{(\delta)}| - X_s^{(\delta)} ds \right]^2 \]
\[ + (1 + 2 \max \{ |\zeta_1|, |\zeta_m| \}^2) \cdot \sum_{k=1}^m 1_{\mathcal{Z}_k}(X_s^{(\delta)}, X_s^{(\delta)}) ds \right]^2 \]
\[ \leq 32 T m(c_\delta)^4 \| G'' \|_\infty^2 \cdot \mathbb{E} \left[ \int_0^T (1 + \max \{ |\zeta_1|, |\zeta_m| \}^2) \cdot \sum_{k=1}^m 1_{\mathcal{Z}_k}(X_s^{(\delta)}, X_s^{(\delta)}) ds \right]^2 \]
\[ \leq 32 T m(c_\delta)^4 \| G'' \|_\infty^2 \cdot \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ |X_s^{(\delta)}| - X_s^{(\delta)} ds \right]^2 \]
\[ + 8m(c_\delta)^4 (1 + \max \{ |\zeta_1|, |\zeta_m| \}^2) \cdot \sum_{k=1}^m \mathbb{E} \left[ \int_0^T 1_{\mathcal{Z}_k}(X_s^{(\delta)}, X_s^{(\delta)}) ds \right]^2 \]
\[ \leq 32 T^2 mC^{(M)}(c_\delta)^4 \| G'' \|_\infty^2 \cdot \mathbb{E} \left[ \int_0^T (1 + \max \{ |\zeta_1|, |\zeta_m| \}^2) \cdot \sum_{k=1}^m 1_{\mathcal{Z}_k}(X_s^{(\delta)}, X_s^{(\delta)}) ds \right]^2 \].

Proposition 4.6 yields that
\[ \mathbb{E} \left[ \int_0^T 1_{\mathcal{Z}_k}(X_s^{(\delta)}, X_s^{(\delta)}) ds \right]^2 \leq C^{(cross)} \cdot \delta. \] (35)

Combining (30), with (31), (32), (33), (34), and (35) we get
\[ E_4(\tau) \leq \delta \cdot \left[ 24 T^2 \left( (c_\mu)^2 (L_G)^2 + (c_\rho)^4 (L_G)^4 \right) (|\zeta_1|^2 + |\zeta_m|^2)^2 + 2c_2(c_\rho)^2 \right] (1 + C^{(M)}) \]
\[ + 32 T^2 mC^{(M)}(c_\delta)^4 \| G'' \|_\infty^2 \]
\[ + 8m^2 C^{(cross)}(c_\delta)^4 (1 + \max \{ |\zeta_1|, |\zeta_m| \}^2)^2 \| G'' \|_\infty^2 \]. \] (36)

Combining (26) with the estimates (27), (28), (29), and (36) shows that there exist constants
\[ c_3, c_4 \in (0, \infty) \] such that
\[ 0 \leq u(\tau) \leq c_4 \int_0^\tau u(s) ds + c_3 \delta. \]
Applying Gronwall’s inequality yields for all \( \tau \in [0, T] \),
\[ u(\tau) \leq c_3 \exp(c_4 \tau) \cdot \delta \leq c_3 \exp(c_4 T) \cdot \delta. \]

Combining this with (23), (24), and (25) finally yields
\[ \left( \mathbb{E} \left[ \sup_{t \in [0, T]} |X_t - X_t^{(\delta)}|^2 \right] \right)^{1/2} \leq L_{G^{-1}} (c_1 \delta)^{1/2} + L_{G^{-1}} (c_3 \exp(c_4 T) \cdot \delta)^{1/2}. \]

This closes the proof. □
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