1. Introduction

Symmetric tilings are an ancient, ubiquitous, and beautiful motif in decoration. Perhaps the most famous examples are found at the Alhambra in Granada. Of course, a physical tiling on a wall or floor ends at the corners of the room. This is unfortunate; in our mind the tiling goes on forever. This raises the question of how we can capture an infinite tiling in a finite space.

When we try to wrap up something infinite, into something finite, we often have to accept certain distortions of geometry. The goal of this note is to use modern tools to explore an old solution to this problem: we give examples of conformally correct tilings of surfaces in three-space.
2. Euclidean tilings

Perhaps the most familiar tiling is a brick tiling of a wall or pathway. Each brick meets six identical bricks, and the pattern repeats in a regular way. To add some flavour to our tilings, we will insist on triangular tiles instead of rectangles. Further, we will require that the triangles alternate in colour between black and white; each black triangle is the mirror image of any white triangle. We fondly recall our teachers’ words: “The sum of the angles of a triangle is 180 degrees.” This, and our desire for mirror symmetry, implies that the possible angles for our triangle are (90, 45, 45), (90, 60, 30), or (60, 60, 60). We call these the (2, 4, 4) triangle, the (2, 3, 6) triangle, and the (3, 3, 3) triangle respectively. In Figure 2 we have drawn a tiling by copies of the (2, 4, 4) triangle. To explain these names, we examine the vertices of the tiling (that is, any point where a pair of lines cross). There are three kinds of vertex: those where two black triangles meet, those where four meet in a left windmill, and those where four meet in a right windmill.

It may amuse you take a few minutes to draw a tiling by black and white (2, 3, 6) triangles, following the same rules.

3. Wrapping up

In fact, the tiling in Figure 2 is not infinite; it ends rather abruptly, to avoid covering its caption. This is an obvious defect: it would be more pleasing if the tiling had no boundary edge. Since the tiling cannot continue forever, we instead must find a way to wrap it around something.

So, cut Figure 2 out of the page, to get a tiled square. Glue the top edge of the square to the bottom, forming a cylinder, as shown in Figure 3a. This gets rid of two of the four edges of the square. However, by bending the square to get a cylinder we gave up something. In the page all lines of the tiling are straight; in the cylinder some lines become helices and some become circles. Nonetheless, the lengths of sides of triangles are unchanged when we measure distance along the cylinder. Likewise, the angles of the triangles remain the same.

The next step of the wrapping process is much more difficult; we want to glue the right boundary of the cylinder to the left. It is an interesting (but futile) exercise to try to do this second gluing with a paper cylinder. If we instead use some rubbery material, we get the torus (the surface of a bagel) shown in Figure 3b. This solves our problem: the tiling is now finite but without boundary edges.
Again, to get something we had to give up something. As they lie in the torus (Figure 3b), the lengths of the sides of the triangles are badly distorted: sides near the hole are squeezed while those far from the hole are stretched. This is unavoidable for the following reason. Near the hole the torus is negatively curved – a small part of it is saddle-shaped. Far from the hole the torus is positively curved – a small part looks like a small part of a sphere. And a \((2,4,4)\) tiling with correct lengths cannot lie snugly on a surface where it has either positive or negative curvature.

However, if we are careful, some of the geometry of the tiling will survive. The vertical and horizontal lines of the tiling have become circles that go around and go through the torus, respectively. These two families meet everywhere at right angles, just as in the original tiling. The diagonal lines, that were helices in the cylinder, have become \textit{Villarceau circles}. As first observed by Schöelcher [5], the Villarceau circles meet everywhere with a constant angle.

If we stretch the torus to look like a bicycle tire then the Villarceau angle decreases towards zero. If we fatten the torus then the angle increases to 180 degrees. So, as in the story of Goldilocks and the Three Bears, there is a size, somewhere in between, which is just right! At this special size, the Villarceau angle is 90 degrees. Since the torus has a reflection symmetry, at this special size the Villarceau circles and the vertical circles meet at an angle of 45 degrees. Thus the angles at the corners of the triangles are exactly the same as in the tiling of the plane. When this happens we say that the tiling is \textit{conformally correct}.

Now is a good time to eat a few oranges. Once you are refreshed, you might want to draw a conformally correct tiling by black and white \((2,3,4)\) triangles.

\section{4. Hyperbolic tilings}

The three triangles \((2,4,4)\), \((2,3,6)\), and \((3,3,3)\) are the only ones giving tilings of the flat, or \textit{euclidean}, plane. If we wish to find further examples we must learn to ignore our teachers; we must consider the possibility of a triangle where the sum of the angles is less than 180 degrees. These properly live in the hyperbolic, or \textit{lobachevskian}, plane. In Figure 4 we see a tiling of the hyperbolic plane by identical (mirrored) black and white \((2,4,6)\) triangles.

You, gentle reader, have already noticed the obvious problem – the supposedly identical triangles are not identical at all. The lengths of edges shrink as the triangles march outwards. This is an unavoidable feature of hyperbolic geometry, first proved by Hilbert [4]: there is no picture of the hyperbolic plane in euclidean three-space which faithfully represents lengths. Thus Figure 4 is not a tiling of the hyperbolic plane but is instead a \textit{model} of the tiling.
We must content ourselves with the fact that this model is again conformally correct: the angles are exact. Consequently, the number of black triangles about any vertex is either two, four, or six.

You may want to count the number of triangles at a fixed number of triangle-sized steps away from the centre of the (2, 4, 6) tiling. This behaves very differently from the similar count in the (2, 4, 4) tiling.

5. Wrapping the hyperbolic plane

In 2015, the Pearl Conard Art Gallery at the Ohio State University put on an exhibition of mathematical art. One of the organisers, Gary Kennedy, asked us to contribute a sculpture based on the Chmutov surface, in honour of Professor Sergei Chmutov. To understand this surface, there is a necessary algebraic definition:

\[ F(x, y, z) = 8(x^4 + y^4 + z^4) - 8(x^2 + y^2 + z^2) + 3. \]

For any real number \( c \) we take \( S(c) \) to be all of the points \( (x, y, z) \) in three-space solving the equation \( F(x, y, z) = c \). We call these contour surfaces. The contour surfaces for \( c = 1 \), \( c = 0 \), and \( c = -1 \) are shown in Figure 5. At \( c = 1 \) the contour has six nodes (sharp points at which the surface is not smooth) corresponding to the faces of a cube; at \( c = -1 \) the contour has twelve nodes corresponding to the edges of the same cube. This surface was first studied by Chmutov and Hirzebruch [1].

![Figure 5](image)

**Figure 5.** Above: half of the surfaces \( S(1) \) (in blue), \( S(0) \) (in green), and \( S(-1) \) (in red). Below left: the half surfaces arranged in space together, inspired by Curtis [2]. Below right: the same, drawn with greater transparency.

![Figure 6](image)

**Figure 6.** Parts of the surfaces \( S(0) \) (in green), \( S(-0.2411...) \) (in yellow), and \( S(-0.5) \) (in orange).
Our first idea for a sculpture was to wrap a hyperbolic tiling about the contour \( S(0) \); since the surface has more than one hole, no spherical or euclidean tiling will work. To decide which hyperbolic tiling to use, first note that every contour \( S(c) \) has the same symmetries as the cube. If we cut \( S(c) \) along all symmetry planes of the cube, as shown in Figure 6a, we find a patch \( Q(c) \) – a four-sided polygon curving in space. See Figure 6b. By reflecting the patch around, we can recover the whole surface. At its four corners the patch \( Q(c) \) has angles 60, 60, 90, and 90 degrees.

According to the uniformisation theorem [3] there is a unique way to lay \( Q(c) \) out in the hyperbolic plane with the correct angles at its corners. The program Confoo [7, 6] calculates how to do this. The boundary edges of the patches \( Q(c) \) then receive hyperbolic lengths. If the vertical sides were exactly half the length of the upper side, then we could divide \( Q(c) \) into four \((2, 4, 6)\) triangles, two black and two white. Unfortunately, for \( Q(0) \) this is not the case! In Figure 6b we see that the green rectangle is too wide. If we increase \( c \) from 0, we get to the blue surface in Figure 5, which pinches off six nodes at the faces of the cube. For \( Q(-0.5) \), the orange rectangle is too tall. If we continue to decrease \( c \) from \(-0.5\), we get to the red surface in Figure 5, which pinches off twelve nodes at the edges of the cube. The surface gets too skinny near the faces in one direction, and too skinny near the edges in the other.

So we can again apply the Goldilocks principle. Using binary search we find that \( c \approx -0.2411 \) gives a patch \( Q(c) \) which is just right. The corresponding yellow rectangle shown in Figure 6c has exactly the same shape as the four black and white triangles just above the centre of the \((2, 4, 6)\) tiling in Figure 4. After a significant further amount of work, we produced the sculpture shown in Figure 1.

It is amusing to count the number of triangles in the tiling. Is there a faster way than counting the triangles one at a time?

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Figure 7. A montage of renders of a black and white tiling of the Conformal Chmutov sculpture.

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