Tachyon condensation and ‘bounce’ in the D1-D5 system

Oleg Lunin\textsuperscript{1}, Samir D. Mathur\textsuperscript{2}, I.Y. Park\textsuperscript{2} and Ashish Saxena\textsuperscript{2}

\textsuperscript{1}School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA
\textsuperscript{2}Department of Physics, The Ohio State University, Columbus, OH 43210, USA

Abstract

We construct supergravity solutions dual to microstates of the D1-D3-D5 system with nonzero B field moduli. Just like the D1-D5 solutions in hep-th/0109154 these solutions are generically nonsingular everywhere, with the ‘throat’ closing smoothly near $r=0$. We write expressions relating the asymptotic supergravity fields to the integral brane charges. We study the infall of a D1 brane down the throat of the geometries. This test brane ‘bounces’ off the smooth end for generic initial conditions. The details of the bounce depend on both the choice of D1-D3-D5 microstate and the direction of approach of the infalling D1 brane. In the dual field theory description we see that the tachyon mode starts to condense, but the tachyon bounces back up the potential hill without reaching the deepest point of the potential.
1 Introduction

1.1 Motivation

The idea of AdS/CFT duality [1] suggests that we make a radical change in our notion of matter and spacetime. Consider a collection of branes placed in asymptotically flat space. The spacetime near the branes will be deformed; let us choose the branes such that the metric is $AdS_m \times S^n$ in the near horizon limit. In Einstein’s picture of gravity we have matter (the branes) near $r = 0$, and a consequent metrical deformation near $r = 0$. But the AdS/CFT correspondence says that the branes are dual to the near horizon geometry. This suggests that if we talk about the branes as well as the near horizon AdS geometry then we are ‘double-counting’. In particular if we follow the curved metric down to the vicinity of $r = 0$ then we should not find the branes there. Is this conclusion correct, and can we explicitly observe this absence of branes in the full string theory solution?

This question was addressed in [2] where we considered the D1-D5 system [3, 4]. Consider type IIB string theory in flat space, and compactify 5 spatial directions on $T^4 \times S^1$. We wrap $n_1$ D1 branes on the $S^1$ and $n_5$ D5 branes on $T^4 \times S^1$. The near horizon geometry is $AdS_3 \times S^3 \times T^4$.

The D1-D5 brane system is in the Ramond (R) sector, and it has $\sim e^{2\sqrt{2\pi} \sqrt{n_1 n_5}}$ degenerate ground states. It was found that these different ‘matter ground states’ corresponded to different dual geometries – each geometry was flat at infinity and had a throat that was locally approximately $AdS_3 \times S^3 \times T^4$, but the throat ended in a shape that was different for different microstates. Further, in the CFT we can compute a time period $\Delta t_{CFT}$ for a pair of excitations on the CFT state to travel once around the ‘effective string’ describing the D1-D5 bound state. This time was found to exactly equal the time $\Delta t_{SUGRA}$ taken for a supergravity quantum to travel down the throat and reflect back up from the end. It was crucial that the quantum in the latter computation did not encounter any ‘matter’ at the end of the throat where it could be trapped for a further length of time. Thus all the properties of the ‘matter state’ were encoded in the depth and shape of the ‘throat’ in the geometry dual to the state.

Different geometries in the above system were characterized by different shapes of a ‘singular curve’ at the end of the throat. One might think that this singular curve was somehow the location of the D1-D5 branes that made up the geometry, but it was shown [2] that the singularity here was ‘mild’ in the sense that all incoming waves reflected off the singularity instead of entering it and getting ‘trapped’. It was observed in [5] that this ‘mild singularity’ was in fact just a coordinate singularity in the generic solution, similar to the singularity at the core of a Kaluza-Klein monopole [6]. Thus the geometries dual to the different D1-D5 states are in fact generically completely nonsingular, and singularities that arise in degenerate cases are just those that occur when two Kaluza-Klein monopoles approach each other. This makes the generic case similar to a special system studied in [7, 8] where it was found that the maximally rotating D1-D5 system was described by
a nonsingular geometry. We rename the ‘singular curve’ of [2] the ‘central curve’ of the geometry, since points on this curve are the centers of Kaluza-Klein monopoles which expand out in the directions transverse to the curve.

The above results have yielded significant progress in our understanding of how the $AdS/CFT$ correspondence works. We would now like to further explore the nature of this correspondence, by investigating other phenomena in the field theory and observing their dual behavior in the string solution. The D1-D5 system has a finite dimensional moduli space. The gravity solutions constructed in [2] were obtained for a special subspace of this moduli space, where all gauge potentials on the $T^4$ were set to zero. For this special subspace we have the property that the D1-D5 system is ‘threshold bound’; thus we can separate away some D1 and D5 branes away from the D1-D5 bound state at no cost in energy.

At generic points in moduli space, however, there is a binding energy that prevents such a separation. If we start with a set of branes that are not bound to the remainder, then we expect to get a tachyonic open string mode on the system, and condensation of this tachyon would lower the energy and yield the actual bound state. To study the gravity dual of this phenomenon, we need to construct supergravity solutions describing the D1-D5 bound state at values of the moduli where the binding energy is nonzero. We construct a class of such solutions in this paper, and then use these to study the evolution of the tachyon in the dual field theory.

1.2 The solutions

We construct geometries with the following properties:

(a) Spacetime is compactified on $T^4 \times S^1$. Let the $T^4$ be rectangular, and parametrized by coordinates $z_1, z_2, z_3, z_4$.

(b) We have $n_1$ D1 branes wrapped on $S^1$, $n_5$ D5 branes wrapped on $T^4 \times S^1$.

(c) We have $n_{12}$ D3 branes wrapped on $S^1$ and the directions $z_1, z_2$ of $T^4$, and $n_{34}$ D3 branes wrapped on $S^1$ and the directions $z_3, z_4$ of $T^4$.

(d) We have a nonzero value at infinity for $b_{12} \equiv B_{z_1 z_2}, b_{34} \equiv B_{z_3 z_4}$, where $B_{\mu \nu}$ is the NS-NS 2-form gauge field.

(e) The generic geometry of the family is smooth everywhere, including the interior near $r = 0$.

Solutions of having the properties (a)-(d) were constructed in [9, 10, 11]. But the harmonic functions involved in the solutions had $1/r^2$ singularities. We follow the procedure of [12, 10] to construct geometries with the additional property (e), and this is done by starting with the solutions of [2] which, as mentioned above, are generically smooth in the interior. It was argued in [2] that true bound states of the D1-D5 system have
throats that end due to the nonzero size of the D1-D5 bound state, and the naive solution written using harmonic functions with $\frac{1}{r^2}$ singularity did not represent any configuration of the actual D1-D5 system.

After finding these solutions, we compute their mass and charges from their asymptotic behavior. We derive general expressions to count the numbers of different kinds of branes from the asymptotic values of the gravity fields. We thus find the mass $M_{\text{sugra}}(n_1, n_5, n_{12}, n_{34}, b_{12}, b_{34})$ of the supergravity solution as a function of the numbers of branes present in the bound state.

We next look at the field theory description of the bound state of branes for the same brane charges and moduli, and write down the mass $M_{\text{CFT}}(n_1, n_5, n_{12}, n_{34}, b_{12}, b_{34})$ expected for the bound state if we assume that the state is BPS. We then perform a computation along the lines of [13] to show that there is indeed a supersymmetric bound state with these charges, at least at the level of the classical brane action. We find $M_{\text{CFT}} = M_{\text{sugra}}$, as expected.

1.3 Tachyon condensation and bounce

We next turn to the computation that we wish to pursue– the infall of a D1 brane towards the D1-D3-D5 bound state. When there were no D3 branes and $B$ was zero then the D1 brane felt no force of attraction towards the D1-D5 bound state, since the D1-D5 system was threshold bound. With $B \neq 0$ we will find an attractive force on the D1 brane, and this ‘test brane’ starts to fall down the throat of the supergravity solution. In the dual ‘brane’ description we expect that this infall is described by a process of tachyon condensation [14].

A similar computation (for small values of $B$) was performed in [11], but there the throat was an infinite one, and the D1 brane proceeded down this throat without returning. But we have argued that the correct duals of D1-D3-D5 bound states have throats that are closed at the end, and this leads us to the phenomenon that we wish to study. What happens to the infalling D1 brane when it reaches the end of the throat?

We find that generically the D1 brane ‘bounces off’ the end of the throat. The details of the bounce and subsequent evolution depend on the choice of D1-D3-D5 bound state (which determined the shape of the end of the throat) as well as the direction of infall of the D1 brane. For a special class of initial conditions the D1 brane settles down, as $t \to \infty$, to a point on the ‘central curve’ of the geometry mentioned above, and becomes in the process an ‘ordinary graviton’ (as opposed to a giant graviton) traveling at the speed of light.

\footnote{For earlier work on tachyon condensation see for example [15].}
2 Constructing the supergravity solutions

We start with the D1-D5 solutions constructed in [2]. Let us briefly recall how these solutions were found. We can map the D1-D5 system, by a sequence of S,T dualities, to the FP system, where we have a fundamental string (F) wrapped \( n_5 \) times on the \( S^1 \) and \( n_1 \) units of momentum charge (P) also along the \( S^1 \). The bound state of these charges has the F string in the form of a single multiply wound string, and all the momentum P is carried by traveling waves on the string. The supergravity solution for such a multiply wound string can be constructed [16], [2], by superposing the harmonic functions arising from different strands [17, 18]. In carrying the P charge the F string is forced to bend in the transverse directions, and the bound state thus acquired a nonzero size. The fact that many profiles of the F string carry the same P leads (after quantization) to the large degeneracy \( \sim e^{2\sqrt{2}\pi \sqrt{n_1 n_5}} \) of ground states. The classical solutions for the FP bound states are parametrized by the transverse displacement profile \( \vec{F}(v) \) of the F string. Undoing the S,T dualities we obtain the D1-D5 geometries, still parametrized by this profile function:

\[
\begin{align*}
    ds^2 &= \sqrt{\frac{H}{1+K}} \left[ -(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + \sqrt{\frac{1+K}{H}} d\vec{x} d\vec{x} \\
    e^{2\Phi} &= H(1+K) \\
    C^{(2)}_{ij} &= -\frac{K}{1+K}, \quad C^{(2)}_{ij} = -\frac{A_i}{1+K}, \quad C^{(2)}_{ii} = \frac{B_i}{1+K} \\
    C^{(2)}_{ij} &= C_{ij} + \frac{A_i B_j - B_i A_j}{1+K}
\end{align*}
\]

where \( H^{-1}, K \) and \( A_i \) are given in terms of the string profile:

\[
\begin{align*}
    H^{-1} &= 1 + \frac{Q_5}{L} \int_0^L \frac{dv}{|x - \vec{F}|^2}, \quad K = \frac{Q_5}{L} \int_0^L \frac{\hat{F}^2 dv}{|x - \vec{F}|^2}, \quad A_i = -\frac{Q_5}{L} \int_0^L \hat{F}_i dv \\
    B_{z_1 z_2} &= b_{12}, \quad B_{z_3 z_4} = b_{34}
\end{align*}
\]

This makes \( B \neq 0 \) at infinity, but also adds D3 branes to the system in such a way that the overall mass is unchanged. To obtain another independent combination of \( B \) and
D3 charge we follow the procedure of [12, 10]2: We do a T-duality along $z_1$, perform a rotation in the $z_1 - z_2$ plane, and then T-dualize again in $z_1'$ - this gives nonvanishing D3 charge and a non-constant $B_{z_1 z_2}$ (which vanishes at infinity). A similar procedure is performed using the coordinates $z_3, z_4$. The steps of this calculation are given in Appendix A, and we summarize the result here:

$$ds^2 = \sqrt{\frac{H}{1 + K}} \left[ -(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + \sqrt{\frac{1 + K}{H}} d\bar{x} d\bar{x}$$

$$+ \sqrt{\frac{H}{1 + K}} \left[ h_1^{-1}(dz_1^2 + dz_2^2) + h_1^{-1}(dz_3^2 + dz_4^2) \right]$$

$$e^{2\Phi} = \frac{1}{h_1 h_2} H(1 + K), \quad B_{z_1 z_2} = \frac{\sin \theta_1 \cos \theta_1}{h_1}(1 - (1 + K)H) + b_{12}, \quad (2.7)$$

$$B_{z_3 z_4} = \frac{\sin \theta_2 \cos \theta_2}{h_2}(1 - (1 + K)H) + b_{34}, \quad (2.8)$$

$$C_{ty}^{(2)} = -\frac{K \cos \theta_1 \cos \theta_2}{1 + K} C_{ty}^{(1)} = -\frac{A_i \cos \theta_1 \cos \theta_2}{1 + K}, \quad C_{ti}^{(2)} = \frac{B_i \cos \theta_1 \cos \theta_2}{1 + K}$$

$$C_{ij}^{(2)} = \cos \theta_1 \cos \theta_2 \left(C_{ij} + \frac{A_i B_j - B_i A_j}{1 + K}\right), \quad (2.10)$$

$$C_{ty34}^{(4)} = -\frac{K H}{h_2} \cos \theta_1 \sin \theta_2, \quad C_{ty12}^{(4)} = -\frac{K H}{h_1} \cos \theta_2 \sin \theta_1,$$

$$C_{iy34}^{(4)} = -\frac{H A_i}{h_2} \cos \theta_1 \sin \theta_2, \quad C_{iy12}^{(4)} = -\frac{H A_i}{h_1} \cos \theta_2 \sin \theta_1,$$

$$C_{it34}^{(4)} = -\frac{H B_i}{h_2} \cos \theta_1 \sin \theta_2, \quad C_{it12}^{(4)} = -\frac{H B_i}{h_1} \cos \theta_2 \sin \theta_1,$$

$$C_{ij34}^{(4)} = \frac{H(1 + K)}{h_2} \cos \theta_1 \sin \theta_2 \left(C_{ij} + \frac{A_i B_j - B_i A_j}{1 + K}\right),$$

$$C_{ij12}^{(4)} = \frac{H(1 + K)}{h_1} \cos \theta_2 \sin \theta_1 \left(C_{ij} + \frac{A_i B_j - B_i A_j}{1 + K}\right),$$

$$C_{ty1234}^{(6)} = -\frac{H^2 K(1 + K) \sin \theta_1 \sin \theta_2}{h_1 h_2}, \quad C_{ty1234}^{(6)} = -\frac{H^2(1 + K) A_i \sin \theta_1 \sin \theta_2}{h_1 h_2},$$

$$C_{it1234}^{(6)} = -\frac{H^2(1 + K) B_i \sin \theta_1 \sin \theta_2}{h_1 h_2},$$

$$C_{ij1234}^{(6)} = \frac{H^2(1 + K)^2 \sin \theta_1 \sin \theta_2}{h_1 h_2} \left(C_{ij} + \frac{A_i B_j - B_i A_j}{1 + K}\right),$$

where

$$h_i = \cos^2 \theta_i + \sin^2 \theta_i (1 + K)H, \quad i = 1, 2$$

and $b_{12}, b_{34}$ are the values of $B$ at $r \to \infty.$

\(^2\)For earlier applications of the similar methods to constructing supergravity solutions see [19].
3 Mass and charges of the solution

In this section we study the asymptotic behavior of the solutions (2.7) and derive their mass and their charges. The charges are to be expressed as integers that give the numbers of D1, D3 and D5 branes in the configuration. The extraction of these integers is complicated by the fact that the field $B$ is nonzero at infinity; a $p+2$-form field strength contributes not only to the count of $p$-branes but also to branes of other dimensionalities when $B \neq 0$. Thus we begin with a derivation of the relevant field-charge relations for the theory, and then compute the charges for our solution.

3.1 Field equations

Let us begin with the action for type IIB supergravity in the absence of sources. We use the notation of [20]:

$$S_{IIB} = \frac{1}{(2\pi)^7\alpha'^4} \int \left\{ d^{10}x \sqrt{-G} e^{-2\phi} \left[ R + 4(\nabla\phi)^2 - \frac{1}{12} (H^{(3)})^2 \right] ight. \
- \left. \frac{1}{2} \ast F^{(3)} \wedge F^{(3)} - \frac{1}{2} \ast dC^{(0)} \wedge dC^{(0)} - \frac{1}{4} \ast F^{(5)} \wedge F^{(5)} \right\} \
+ \frac{1}{2(2\pi)^7\alpha'^4} \int \left( C^{(4)} + \frac{1}{2} B^{(2)} \wedge C^{(2)} \right) \wedge G^{(3)} \wedge H^{(3)}.$$  \hspace{1cm} (3.1)

Here

$$G^{(3)} = dC^{(2)}, \quad F^{(3)} = dC^{(2)} + C^{(0)} H^{(3)}, \quad F^{(5)} = dC^{(4)} + H^{(3)} \wedge C^{(2)} \hspace{1cm} (3.2)$$

In this convention the four form $C^{(4)}$ is invariant under the gauge transformation of $B^{(2)}$, while under the gauge transformation of the two form $\delta C^{(2)}$ it transforms as

$$\delta C^{(4)} = -B^{(2)} \wedge \delta C^{(2)} \hspace{1cm} (3.3)$$

For our solutions $C^{(0)} = 0$, and we assume the vanishing of $C^{(0)}$ in the equations below. We find the equations of motion following from the action (3.1) by taking the variations with respect to $C^{(2)}$ and $C^{(4)}$:

$$d^* F^{(5)} + H^{(3)} \wedge F^{(3)} = 0 \hspace{1cm} (3.4)$$

$$d^* F^{(3)} - H^{(3)} \wedge F^{(5)} = 0 \hspace{1cm} (3.5)$$

These equations should be supplemented by the Bianchi identity:

$$dF^{(3)} = 0 \hspace{1cm} (3.6)$$

and the self duality condition $F^{(5)} = \ast F^{(5)}$.

In the presence of sources we get a Chern-Simons coupling between the RR gauge fields and the currents $j^{(p+1)}$ describing D-branes.

$$S_{CS} = \frac{g}{2\pi\alpha'} \int \exp(B + 2\pi\alpha' F) \wedge \sum_{p=2}^6 C^{(p)} \wedge \left[ \ast j^{(2)} + \frac{1}{(2\pi)^2\alpha'} \ast j^{(4)} + \frac{1}{(2\pi)^4\alpha'^2} \ast j^{(6)} \right]$$

7
In this normalization when \( F = 0 \) the number of branes is given by integrating the corresponding current over an appropriate cycle:

\[
n_k = \int * j^{(k+1)}
\]

(3.7)

(The \( n_k \) are integers in the full quantum theory, while the value of \( B \) is a continuous variable [21].) The equations of motion are modified to

\[
-\frac{1}{4\pi^2g\alpha'}dF^{(3)} = * j^{(6)}
\]

\[
\frac{1}{4\pi^2g\alpha'^2} \left[ d^* F^{(5)} + H^{(3)} \wedge F^{(3)} \right] = (2\pi)^2 * j^{(4)} + \frac{1}{\alpha'}(B + 2\pi\alpha' F) \wedge * j^{(6)}
\]

(3.8)

\[
\frac{1}{4\pi^2g\alpha'^3} \left[ d^* F^{(3)} - H^{(3)} \wedge F^{(5)} \right] = (2\pi)^4 * j^{(2)} + \frac{(2\pi)^2}{\alpha'}(B + 2\pi\alpha' F) \wedge * j^{(4)}
\]

\[
+ \frac{1}{2\alpha'}(B + 2\pi\alpha' F)^2 \wedge * j^{(6)}
\]

Assuming that there are no sources for the NS two form field (i.e. \( dH^{(3)} = 0 \)), we rewrite (3.8) in a form which is more convenient for the charge computation in supergravity:

\[
-\frac{1}{4\pi^2\alpha' g}dF^{(3)} = * j^{(6)}
\]

\[
\frac{1}{(4\pi^2\alpha')^2 g}d \left[ * F^{(5)} + B^{(2)} \wedge F^{(3)} \right] = * j^{(4)} + \frac{F}{2\pi} \wedge * j^{(6)}
\]

(3.9)

\[
\frac{1}{(4\pi^2\alpha')^3 g}d \left[ * F^{(3)} - B^{(2)} \wedge F^{(5)} - \frac{1}{2} B^{(2)} \wedge B^{(2)} \wedge F^{(3)} \right] = * j^{(2)} + \frac{F}{2\pi} \wedge * j^{(4)}
\]

\[
+ \frac{1}{2} \left( \frac{F}{2\pi} \right)^2 \wedge * j^{(6)}
\]

In deriving the last equation we have used the relation

\[
H^{(3)} \wedge F^{(5)} = d(B^{(2)} \wedge F^{(5)}) - B^{(2)} \wedge (dF^{(5)} + H^{(3)} \wedge F^{(3)})
\]

\[
+ \frac{1}{2} d(B^{(2)} \wedge B^{(2)} \wedge F^{(3)}) - \frac{1}{2} B^{(2)} \wedge B^{(2)} \wedge dF^{(3)}
\]

(3.10)

as well first two equations in (3.8).

### 3.2 Obtaining charges from the field strengths

From (3.9) we read off the integer charges in terms of the field strengths

\[
n_5 = -\frac{1}{4\pi^2\alpha' g} \int_{S^3} F^{(3)}
\]

(3.11)
\[ n_1 = \frac{1}{(4\pi^2\alpha')^3g} \int_{S^3 \times T^4} \left[ *F^{(3)} - B^{(2)} \wedge F^{(5)} - \frac{1}{2} B^{(2)} \wedge B^{(2)} \wedge F^{(3)} \right] \]  
(3.12)

\[ n_{12} = \frac{1}{(4\pi^2\alpha')^2g} \int_{S^3 \times T_3 \times T_4} (F^{(5)} + B^{(2)} \wedge F^{(3)}) \]  
(3.13)

\[ n_{34} = \frac{1}{(4\pi^2\alpha')^2g} \int_{S^3 \times T_1 \times T_2} (F^{(5)} + B^{(2)} \wedge F^{(3)}) \]  
(3.14)

Here \( T_i, i = 1, 2, 3, 4 \) are the four different cycles of \( T^4 \), and \( n_{12} \) for example gives the D3 branes wrapped on the cycles \( T_1, T_2 \).

Define

\[ k_5 = -\frac{1}{4\pi^2\alpha'g} \int_{S^3} F^{(3)} \]  
(3.15)

\[ k_1 = \frac{1}{(4\pi^2\alpha')^3g} \int_{S^3 \times T^4} *F^{(3)} \]  
(3.16)

\[ k_{ij} = \epsilon_{ijkl} \frac{1}{(4\pi^2\alpha')^2g} \int_{S^3 \times T_k \times T_l} F^{(5)} \]  
(3.17)

It is convenient to introduce

\[ b_{ij} \equiv \frac{1}{L_i L_j} \int_{T_i \times T_j} B \]  
(3.18)

where \( L_i \) is the length of the \( i \)-th direction on the torus. We also define

\[ V_{ij} \equiv \frac{L_i L_j}{(2\pi)^2}, \quad V \equiv V_{12} V_{34}. \]  
(3.19)

Then we get

\[ n_5 = k_5 \]
\[ n_{12} = k_{12} - \frac{b_{34} V_{34}}{\alpha'} k_5, \quad n_{34} = k_{34} - \frac{b_{12} V_{12}}{\alpha'} k_5, \]  
(3.20)

\[ n_1 = k_1 - \frac{b_{12} V_{12}}{\alpha'} k_{12} - \frac{b_{34} V_{34}}{\alpha'} k_{34} + \frac{b_{12} b_{34} V}{(\alpha')^2} k_5 \]

### 3.3 Asymptotic behavior of the solution

As \( r \to \infty \) the solution (2.7) has the following behavior for the fields

\[ ds^2 = -dt^2 + dy^2 + d\tilde{x}^2 + d\tilde{y}^2 + dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 \]  
(3.21)

\[ e^{2\Phi} = 1, \quad B_{12} = \sin \theta_1 \cos \theta_1 \frac{Q_5 - Q_1}{r^2}, \quad B_{34} = \sin \theta_2 \cos \theta_2 \frac{Q_5 - Q_1}{r^2}, \]

\[ C^{(2)}_{t_y} = -\frac{Q_1}{r^2} \cos \theta_1 \cos \theta_2, \quad C^{(2)}_{y} = -A_i \cos \theta_1 \cos \theta_2, \quad C^{(2)}_{t_i} = B_i \cos \theta_1 \cos \theta_2, \]
\[ C^{(2)}_{ij} = \cos \theta_1 \cos \theta_2 C_{ij} \]
\[ C^{(4)}_{ty34} = -\frac{Q_1}{r^2} \cos \theta_1 \sin \theta_2, \quad C^{(4)}_{ty12} = -\frac{Q_1}{r^2} \cos \theta_2 \sin \theta_1, \]
\[ C^{(4)}_{ty34} = -A_i \cos \theta_1 \sin \theta_2, \quad C^{(4)}_{ty12} = -A_i \cos \theta_2 \sin \theta_1, \]
\[ C^{(4)}_{it34} = \cos \theta_1 \sin \theta_2 C_{ij}, \quad C^{(4)}_{it12} = \cos \theta_2 \sin \theta_1 C_{ij}, \]
\[ C^{(4)}_{ty1234} = -\frac{Q_1}{r^2} \sin \theta_1 \sin \theta_2, \quad C^{(4)}_{ty1234} = -A_i \sin \theta_1 \sin \theta_2, \]
\[ C^{(4)}_{it1234} = -B_i \sin \theta_1 \sin \theta_2, \quad C^{(4)}_{it1234} = \sin \theta_1 \sin \theta_2 C_{ij} \]

Recall that the field strengths at infinity are to be constructed from the potentials in such a way that each field strength contains the information of both the relevant electric charges as well as their magnetic duals. In computing the duals of forms we use the convention

\[ \epsilon_{tyijkl1234} = \epsilon_{ijkl} \] (3.23)
where \( i, j, k, l \) are the four noncompact directions \( x_1 \) and \( 1, 2, 3, 4 \) are directions on the \( T^4 \). For the D1 and D5 charges we need to compute \( F^{(3)} \); since we are working at infinity we can drop nonlinear terms and we get

\[ F^{(3)} = dC^{(2)} - \ast dC^{(6)} \] (3.24)

(The relative sign of the two terms on the RHS is determined by performing two T-dualities of the 5-form field strength, which is assumed to satisfy \( F^{(5)} = \ast F^{(5)} \).)

At leading order we find

\[
F^{(3)} = -Q_1 \cos \theta_1 \cos \theta_2 \left( \frac{1}{r^2} \right) \wedge dt \wedge dy + \cos \theta_1 \cos \theta_2 dC + * \left[ -Q_1 \sin \theta_1 \sin \theta_2 \left( \frac{1}{r^2} \right) \wedge dt \wedge dy \wedge dV + \sin \theta_1 \sin \theta_2 dC \wedge dV \right] \] (3.25)

Note that at \( r \to \infty \)

\[ \ast dC = -Q_5 d \left( \frac{1}{r^2} \right) \wedge dt \wedge dy \wedge dV \] (3.26)

where \( dV = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \). We choose the orientation of the \( S^3 \) at infinity to be given by the choice of sign in the following relation

\[ \int_{S^3} dC = -4\pi^2 Q_5 \] (3.27)

Then we arrive at the result

\[
F^{(3)} = -(Q_1 \cos \theta_1 \cos \theta_2 + Q_5 \sin \theta_1 \sin \theta_2) \left( \frac{1}{r^2} \right) \wedge dt \wedge dy \]
\[ + \frac{1}{Q_5} (Q_5 \cos \theta_1 \cos \theta_2 + Q_1 \sin \theta_1 \sin \theta_2) dC, \] (3.28)
\[ *F^{(3)} = -\frac{1}{Q_5} (Q_1 \cos \theta_1 \cos \theta_2 + Q_5 \sin \theta_1 \sin \theta_2) dC \wedge dV \]
\[ + (Q_5 \cos \theta_1 \cos \theta_2 + Q_1 \sin \theta_1 \sin \theta_2) d \left( \frac{1}{r^2} \right) \wedge dt \wedge dy \wedge dV \]  
(3.29)

Substituting this in the expressions for the charges we find:

\[ k_5 = \frac{1}{g(\alpha')^2} (Q_5 \cos \theta_1 \cos \theta_2 + Q_1 \sin \theta_1 \sin \theta_2) \]  
(3.30)

\[ k_1 = \frac{V}{g(\alpha')^3} (Q_1 \cos \theta_1 \cos \theta_2 + Q_5 \sin \theta_1 \sin \theta_2) \]  
(3.31)

Let us now evaluate the numbers of the three branes. First we have to find the self–dual field strength. We define:

\[ F^{(5)} = dC^{(4)} + *dC^{(4)} \]  
(3.32)

At leading order we find:

\[ dC^{(4)} = -Q_1 d \left( \frac{1}{r^2} \right) (\cos \theta_1 \sin \theta_2 dV_{34} + \cos \theta_2 \sin \theta_1 dV_{12}) \wedge dt \wedge dy \]
\[ + dC(\cos \theta_1 \sin \theta_2 dV_{34} + \cos \theta_2 \sin \theta_1 dV_{12}) \]  
(3.33)

\[ F^{(5)} = -d \left( \frac{1}{r^2} \right) dV_{34} (Q_1 \cos \theta_1 \sin \theta_2 - Q_5 \cos \theta_2 \sin \theta_1) \wedge dt \wedge dy \]
\[ -d \left( \frac{1}{r^2} \right) dV_{12} (Q_1 \cos \theta_2 \sin \theta_1 - Q_5 \cos \theta_1 \sin \theta_2) \wedge dt \wedge dy \]
\[ + \frac{1}{Q_5} dCdV_{34} (Q_5 \cos \theta_1 \sin \theta_2 - Q_1 \cos \theta_2 \sin \theta_1) \]
\[ + \frac{1}{Q_5} dCdV_{12} (Q_5 \cos \theta_2 \sin \theta_1 - Q_1 \cos \theta_1 \sin \theta_2) \]  
(3.34)

Integrating these expressions we get

\[ k_{12} = -\frac{V_{34}}{g(\alpha')^2} (Q_5 \cos \theta_1 \sin \theta_2 - Q_1 \cos \theta_2 \sin \theta_1) \]
\[ k_{34} = -\frac{V_{12}}{g(\alpha')^2} (Q_5 \cos \theta_2 \sin \theta_1 - Q_1 \cos \theta_1 \sin \theta_2) \]  
(3.35)

### 3.4 Mass in terms of charges

From the asymptotic behavior of \( g_{tt} \) we find the mass of the solution

\[ M = \frac{\pi}{4G_5} (Q_1 + Q_5) = \frac{RV}{g^2(\alpha')^4} (Q_1 + Q_5) \]  
(3.36)
We wish to relate this mass to the number of branes. We observe that
\[ V_{12}k_{12} - V_{34}k_{34} = \frac{V_{12}V_{34}}{g\alpha'^2} \sin(\theta_1 - \theta_2)(Q_1 + Q_5) \] (3.37)
\[ k_5 + \frac{k_1(\alpha')^2}{V} = \frac{1}{g\alpha'}(Q_1 + Q_5) \cos(\theta_1 - \theta_2) \] (3.38)
Thus
\[ M^2 = \left( \frac{RV}{g\alpha'^3} \right)^2 \left[ (k_5 + \frac{k_1(\alpha')^2}{V})^2 + \frac{(V_{12}k_{12} - V_{34}k_{34})^2}{V^2} \right] \] (3.39)

We express the quantities \( k_1, k_5, k_{12}, k_{34} \) in terms of the numbers of different kinds of branes by the relations inverse to (3.20):
\[ k_5 = n_5 \] (3.40)
\[ k_{12} = n_{12} + \frac{b_{34}V_{34}}{\alpha'}n_5, \quad k_{34} = n_{34} + \frac{b_{12}V_{12}}{\alpha'}n_5, \] (3.41)
\[ k_1 = n_1 + \frac{b_{12}V_{12}}{\alpha'}n_{12} + \frac{b_{34}V_{34}}{\alpha'}n_{34} + \frac{b_{12}b_{34}V}{(\alpha')^2}n_5 \] (3.42)

### 4 Mass of the D-brane state

We have constructed above the gravity dual of a D1-D3-D5 bound state with \( B \neq 0 \). In this section we compute the mass expected of such a state starting from D-brane physics. If we perform a T-duality along the \( S^1 \) directions of \( T^4 \times S^1 \), then we get a \( D0 - D2 - D4 \) bound state; we study the latter since the supercharges are easier to write for the IIA theory which can in turn be written as a reduction of 11-dimensional M theory.

Consider a D0-D2 system. If the D0 brane is not bound to the D2 brane, the system is not supersymmetric; the supersymmetries preserved by the D2 brane and the D0 brane are different. If however we allow the D0 brane to ‘dissolve’ into the D2 brane, then the mass is lowered, and the bound state is a supersymmetric (1/2 BPS) configuration. The geometries we have constructed are expected to be duals of the bound state of the D1-D3-D5 branes, so we are interested in the masses of such ‘dissolved’ configurations.

If we know that the bound state is supersymmetric, then we can deduce the mass from the charges. Let the D0, D2, D4 charges be described by \( Z, Z_{ij}, Z_{ijkl} \). Then we write \[22, 23\]
\[ \Gamma \epsilon \equiv Z\Gamma_{0s} + \frac{1}{2!}Z^{ij}\Gamma_{0ij} + \frac{1}{4!}Z^{ijkl}\Gamma_{0ijkl} = M \epsilon \] (4.1)
Here the index \( s \) represents the compact 11—direction of M-theory, while the other indices take values along the compact torus. Requiring a solution \( \epsilon \neq 0 \) gives \( M \). This computation is standard, and for our case of interest we reproduce the details in Appendix C. T-dualizing to the D1-D3-D5 system we get
\[ M^2 = \frac{R^2}{\alpha'^2} \frac{1}{g^2\alpha'} \left( (l_1l_2n_{12} \mp l_3l_4n_{34} + n_5v(b_{34} \mp b_{12}))^2 + \right. \]
\[(n_1 + v(b_{12}b_{34} \pm 1)n_5 + l_1l_2b_{12}n_{12} + l_3l_4b_{34}n_{34})^2)\]  

(4.2)

where \(l_i = \frac{L_i}{2\pi\sqrt{\alpha'}}\) are dimensionless parameters expressing the lengths \(L_i\) of the rectangular torus \(T^4\), \(v = l_1l_2l_3l_4\) and the various \(n\)'s are the number of respective branes. In the above expression we choose the upper signs if \(k_1k_5 - k_12k_{34} > 0\) and the lower signs if \(k_1k_5 - k_12k_{34} < 0\). We see that this expression for the mass agrees with (3.39). (Note that \(V_{12} = \alpha' l_1l_2, V_{34} = \alpha' l_3l_4\). Also note that in our supergravity computation we have chosen a definite sign for the charges at the outset; this choice corresponds to the upper signs in (4.2).

As mentioned above, this would be the mass of the bound state if we knew that the state was supersymmetric. But the question of whether the bound state is supersymmetric or not is a dynamical question in the theory, and this dynamical information is not contained in the starting step (4.1) of the above computation. To determine whether the branes are actually expected to form a supersymmetric bound state we follow the approach used in [13]. In this approach we start with a collection of \(D4\) branes, represent the other (dissolved) branes in terms of field strengths \(F\) on the \(D4\) branes, with the assumption that \(F\) can be taken as a diagonal \(U(n_4)\) matrix. Within this class of \(F\) we check if there is a supersymmetric configuration; if there is, then at least in a classical approximation to brane physics we would establish that the bound state is supersymmetric.

We take a rectangular torus as above and let \(b_{12}, b_{34}\) be nonzero. We have \(D0\) and \(D4\) branes as well as \(D2\) branes along the 12 and 34 directions. Since all quantities depend only on \(B + 2\pi\alpha'F\) we can set \(B = 0\) and absorb its effect into \(F\) with no loss of generality (the discreteness of \(F\) is not visible in this classical analysis). We label the \(D4\) branes by an index \(i\), with \(i = 1 \ldots n_4\). The \(D4\) branes carry field strengths \(F_{12}^{(i)}, F_{34}^{(i)}\). Define the vectors

\[\vec{V}_1 = \{F_{12}^{(1)}, \ldots, F_{12}^{(n_4)}\}\]  

(4.3)

\[\vec{V}_2 = \{F_{34}^{(1)}, \ldots, F_{34}^{(n_4)}\}\]  

(4.4)

\[\vec{V}_0 = \{1, \ldots 1\}, \quad V_0 \cdot V_0 = n_4\]  

(4.5)

We have the constraints

\[\sum_i F_{12}^{(i)} = \vec{V}_1 \cdot \vec{V}_0 = \left(\frac{2\pi}{L_1L_2}\right) n_{34}\]  

(4.6)

\[\sum_i F_{34}^{(i)} = \vec{V}_2 \cdot \vec{V}_0 = \left(\frac{2\pi}{L_3L_4}\right) n_{12}\]  

(4.7)

\[\sum_i F_{12}^{(i)}F_{34}^{(i)} = \vec{V}_1 \cdot \vec{V}_2 = \left(\frac{(2\pi)^2}{L_1L_2L_3L_4}\right) n_0\]  

(4.8)

Lastly, we have the requirement that the different \(D4\) branes be supersymmetric with respect to each other. We can use either the Yang-Mills action to describe the branes.
or the (more exact) DBI action. In the Yang-Mills limit the supersymmetry condition is

\[ F^{(1)}_{12} \pm F^{(1)}_{34} = F^{(2)}_{12} \pm F^{(2)}_{34} = \ldots = F^{(n_4)}_{12} \pm F^{(n_4)}_{34} \]  

(4.9)

where the signs ± must be chosen to be all + or all −. With the DBI action we define

\[ f^{(i)}_{12} = \tan^{-1} \left( \frac{F^{(i)}_{12}}{2\pi\alpha'} \right), \quad f^{(i)}_{34} = \tan^{-1} \left( \frac{F^{(i)}_{34}}{2\pi\alpha'} \right) \]  

(4.10)

and then the supersymmetry preservation condition is \[ \text{(4.11)} \]

Consider the Yang-Mills approximation. The constraint \( \text{(4.9)} \) can be written as

\[ \mathbf{V}_1 \pm \mathbf{V}_2 = c \mathbf{V}_0 \]  

(4.12)

If we can solve \( \text{(4.6), (4.7), (4.8), (4.12)} \) for real vectors \( \mathbf{V}_i \) then we have a supersymmetric configuration, and \( \text{(4.2)} \) would give the mass of the bound state.

The conditions \( \text{(4.6), (4.7)} \) are immediately solved by writing

\[ \mathbf{V}_1 = c_1 \mathbf{V}_0 + \mathbf{V}^\perp_1, \quad c_1 = \frac{1}{n_4} \left( \frac{2\pi}{L_1 L_2} \right) n_{34} \]  

(4.13)

\[ \mathbf{V}_2 = c_2 \mathbf{V}_0 + \mathbf{V}^\perp_2, \quad c_2 = \frac{1}{n_4} \left( \frac{2\pi}{L_3 L_4} \right) n_{12} \]  

(4.14)

where \( \mathbf{V}^\perp_1 \cdot \mathbf{V}_0 = \mathbf{V}^\perp_2 \cdot \mathbf{V}_0 = 0 \). The condition \( \text{(4.12)} \) gives \( \mathbf{V}^\perp_1 = \mp \mathbf{V}^\perp_2 \), and then \( \text{(4.8)} \) gives

\[ |\mathbf{V}^\perp_1|^2 = \pm \frac{1}{n_4} \left( \frac{2\pi}{L_1 L_2 L_3 L_4} \right)^2 (n_{12} n_{34} - n_0 n_{4}) \]  

(4.15)

If \( (n_{12} n_{34} - n_0 n_{4}) \geq 0 \) we take the + sign in \( \text{(4.9)} \), and then we can choose any \( \mathbf{V}^\perp_1 \) with real entries and length given by \( \text{(4.15)} \) to get a supersymmetric configuration. If \( (n_{12} n_{34} - n_0 n_{4}) \leq 0 \) then we can take the − sign and again get a supersymmetric configuration.\(^3\)

If we work instead with the DBI action for the branes then the analysis is slightly more complicated due to the nonlinearity in \( F \) of the supersymmetry condition \( \text{(4.11)} \). But we reach a similar conclusion, and the details are presented in Appendix D.

\(^3\)The above computations assume \( n_4 > 1 \). We expect supersymmetric states also for \( n_4 = 1 \), but the state is not described by a constant field \( F \). For \( n_4 > 1 \) we have in general many choices for the vector \( \mathbf{V}^\perp_1 \), and these choices reflect the presence of a moduli space of supersymmetric configurations. Only a small subspace of this moduli space is captured by the constant configurations however; the generic configurations are described by deformations of instanton configurations.
Figure 1: Supergravity and field theory descriptions of the absorption of a massless quantum by the D1-D5 bound state.

5 Tachyon condensation

We now consider a D1 brane winding around the $S^1$ which we parametrized by the coordinate $y$. If we take a D1-D5 system with $B = 0$ then there is no force between this ‘test brane’ and the D1-D5 bound state, and the test brane can sit at any distance $r$ from the bound state without feeling any force. If we let $B \neq 0$ then the D1-D5 system is *not* threshold bound, and there is an attractive force on the test D1 brane.

To outline the physics we expect let us first recall the results of [2] where we let a massless quantum fall into the D1-D5 ‘throat’. In Fig.1 we show the supergravity description where we have the metrics (2.1), and drawn below that the dual brane description; the branes sit in flat spacetime. In Fig.1(a) the quantum is outside the throat of the geometry, and correspondingly outside the branes in the dual picture. In Fig.1(b) the quantum enters the supergravity throat (with some probability $P_{\text{sugra}}$). In the dual picture it gets absorbed by the brane and converted to a set of left and right moving vibrations; the probability for this absorption process $P_{CFT}$ is found to satisfy $P_{CFT} = P_{sugra}$. In Fig.1(c) we find that the supergravity quantum travels down the throat, and in the field theory the two excitations separate away from each other. In Fig.1(d) the supergravity

\footnote{For a discussion of D-brane couplings when $B \neq 0$ and their field theory duals see for example [20].}
quantum reflects off the end and returns to the start of the throat in a time \( \Delta t_{\text{sugra}} \); in
the dual brane picture the left and right moving excitations travel around the branes and re-collide in a time \( \Delta t_{\text{CFT}} = \Delta t_{\text{sugra}} \).

In Fig.2 we picture the corresponding process when the infalling object is a D1 brane.

- In Fig.2(a) the D1 brane is far outside the throat, and in the dual picture it is well-separated from the D1-D5 bound state. In the gravity picture the potential energy can be found by the DBI action of the D1 brane. In the brane picture the potential between the D1 brane and the D1-D5 system can be found by a 1-loop computation in the open string channel \([27, 28]\) which, at these long distances, gets contributions from only the lowest few modes in the closed string channel.\(^5\) (These potentials are just the long distance supergravity attraction between the test brane and the bound state.)

- In Fig.2(b) the D1 brane is at the start of the supergravity throat; this is the ‘intermediate region’ which connects flat space to the locally \( AdS_3 \times S^3 \times T^4 \) geometry. The DBI action of the supergravity description yields a complicated potential function in this region. The force computation of \([27]\) gets contributions from all string modes, and one must also include multiloop processes. The D1 brane in the ‘brane description’ is now at \( r \approx 0 \).

- In Fig.2(c) we see that in the supergravity description the D1 brane continues deeper into the ‘throat’ of the geometry, where the potential energy of the D1 brane continues to decrease. In the dual description we expect that lowest open string mode – the tachyon – begins to condense, thus lowering the energy.

- In Fig.2(d) we see that the D1 brane reaches the end of the throat, and ‘bounces back’. This is the new aspect of the problem and the one that we wish to study, and it arises from our starting observation \([2]\) that the D1-D5 bound states are described by closed throats rather than an infinite throat singular at \( r = 0 \). The supergravity picture implies that in the dual brane description the tachyon, after going down the potential well to some depth, climbs back up the well (in some other direction of field space). We will find that the details of the ‘bounce’ (in the supergravity picture) depends on the choice of Ramond ground state of the D1-D5 system, as well as the direction of infall of the D1 brane toward the bound state.

Before proceeding, we briefly compare our computation with \([29, 31]\). In \([29]\) Seiberg and Witten considered a D1 brane winding around the angular direction of \( AdS_3 \), and computed the potential energy needed to expand this string to different radii \( r \). In this case \( B = 0 \), but the potential is not flat. To see the relation with our computation, note that the computation of \([29]\) was in global \( AdS_3 \), which is dual to the NS sector of the D1-D5 system. In our computation the D1 brane is studied in the Ramond sector. The two sectors are related in supergravity by a coordinate transformation, which is

\(^5\)One end of the open strings is at the D1-D5 bound state, where the boundary conditions are complicated and depend on the choice of the Ramond ground state of the D1-D5 system.
A D1 brane that is static in the R sector is rotating on the $S^3$ with unit velocity in the NS sector. Such a rotating D1 brane in $AdS_3 \times S^3$ is a ‘giant graviton’ \cite{griffiths} which feels no potential against radial expansion.\footnote{Giant gravitons in $AdS_3 \times S^3$ and their interpretation in the CFT were studied in \cite{griffiths}.}

In the Ramond sector we can get an attractive potential if we let $B \neq 0$, and that is the case that we are studying. Such a case was also studied in \cite{griffiths}. But in \cite{griffiths} the supergravity geometry at small $r$ was an infinite throat (the Poincare patch with identification $y \to y + 2\pi R$), and the D1 brane fell in towards $r = 0$ without returning.

In our study we encounter a throat similar to the one in \cite{griffiths}, but we also have an end to the throat. For a special case of the geometry (the maximally rotating configuration) the geometry at the end looks (after the spectral flow coordinate transformation) just like global $AdS_3$ times $S^3$. The fact our D1 brane moves in the 6-dimensional space (instead of just in the $AdS_3$) implies that for generic initial conditions the D1 brane does not self-intersect during the ‘bounce’.

5.1 The tachyon potential

Let us assume that the profile function $F_i(v)$ in (2.4) satisfies $\dot{F}_i \dot{F}_i = \text{constant}$. Further, let us take $Q_1 = Q_5$. Then we see from (2.4) that

$$H(1 + K) = 1$$

(5.1)
and the solution \((2.7)\) simplifies to

\[
\begin{align*}
  ds^2 &= H \left[ -(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + H^{-1} d\tilde{x} d\tilde{x} + dz \ dz \\
  e^{2\Phi} &= 1, \quad B_{z_1 z_2} = b_{12}, \quad B_{z_3 z_4} = b_{34}, \\
  C^{(2)} &= \cos \theta_1 \cos \theta_2 M^{(2)}, \quad C^{(6)} = \sin \theta_1 \sin \theta_2 M^{(2)} \land dv, \\
  C^{(4)} &= M^{(2)} \land [\cos \theta_1 \sin \theta_2 dz_3 \land dz_4 + \cos \theta_2 \sin \theta_1 dz_1 \land dz_2]
\end{align*}
\] (5.2)

where we introduced

\[
M^{(2)} = H (dt - A_i dx^i) \land (dy + B_i dx^i) + C_{ij} dx^i dx^j. \quad (5.4)
\]

(Note that we have shifted \(M^{(2)}\) by a constant 2-form, which amounts to constant shifts in the RR gauge fields. The shift of \(C^{(2)}\) must be accompanied by a shift \((3.3)\) in \(C^{(4)}\), but for the present case \(B\) is a constant field, and the shift induced in \(C^{(4)}\) is also by a constant form which gives no field strength.) With the restrictions we have chosen on the solution we have \(H = 0\); though this will not be the case for more general solutions we do not expect any of the essential physics of the infall to be different.

To find the potential felt by the D1 brane we must dualize \(C^{(6)}\) (using the relation \((3.24)\); this gives an extra contribution to \(C^{(2)}\)

\[
\tilde{C}^{(2)} = \sin \theta_1 \sin \theta_2 M^{(2)}
\] (5.5)

and the total RR 2-form becomes:

\[
\tilde{C}^{(2)} = \cos(\theta_1 - \theta_2) M^{(2)}
\] (5.6)

We assume that the D1-brane is in a wavefunction that is uniform on the \(T^4\), and the \(T^4\) plays no further part in the analysis. The D1 brane wraps the direction \(y\), and we consider only its center-of-mass motion – i.e., we set to zero all vibrations of the D1 brane. The action is

\[
S = -T_1 \int d^2 \xi e^{-\Phi} \sqrt{-\det(G_{ab} + B_{ab})} + T_1 \int \tilde{C}^{(2)}
\] (5.7)

We choose the static gauge

\[
t = \tau, \quad y = R \sigma
\] (5.8)

where the dynamics is described by \(x^i = x^i(\tau)\). We denote the derivatives with respect to the worldvolume variables \(\xi_0 = \tau\) and \(\xi_1 = \sigma\) by a dot and a prime respectively.

Then the action \((5.7)\) becomes:

\[
S = -T_1 R \int d\tau d\sigma H \sqrt{1 - A_i \dot{x}^i} - H^{-2} \dot{x}^2 + T_1 R \int d\tau d\sigma H \cos(\theta_1 - \theta_2)
\] (5.9)

From this action we construct a worldsheet Hamiltonian which gives the energy of the configuration

\[
E = 2\pi T_1 RH \left[ \frac{1 - A_i \dot{x}^i}{\sqrt{1 - A_i \dot{x}^i}^2 - H^{-2} \dot{x}^2} - \cos(\theta_1 - \theta_2) \right] - 4\pi T_1 R \sin^2 \frac{\theta_1 - \theta_2}{2}
\] (5.10)
where we have added a constant so as to make the energy vanish when the D1 brane is at rest at spatial infinity.

To find the effective potential felt by the D1 brane we set $\dot{x} = 0$ in (5.10), getting

$$V(x) = -4\pi T_1 R (1 - H) \sin^2 \frac{\theta_1 - \theta_2}{2}$$

(5.11)

Looking at the definition of $H^{-1}$ in (2.4), we note that $0 \leq H < 1$ everywhere, and $H = 0$ only for points $x_i$ that are on the curve $x_i = F_i(v)$ for some $v$. As explained in the introduction, we term this curve the ‘central curve’ of the geometry. The minimum of the potential is reached at all points this central curve, and we find that the value of this minimum is universal, depending only on the charges and $B$ field moduli but not on the profile $F_i(v)$:

$$V_{\text{min}} = -4\pi T_1 R \sin^2 \frac{\theta_1 - \theta_2}{2}$$

(5.12)

We will see however that as far as the dynamics of the tachyon is concerned, a D1 brane falling in on a generic trajectory does not reach the minimum of the potential, but reflects back after reaching some other value of the potential.

The fact that the minimum (5.12) does not depend on $F_i(v)$ is expected from the fact that the binding energy of the D1 brane to the D1-D3-D5 bound state is given only in terms of the charges and moduli. The potential energy (5.12) is just the binding energy of the D1 brane in the D1-D3-D5 bound state. To see this we fix and denote the energy of the bound state as $M(n_1, n_{12}, n_{34}, n_5)$. Then from (3.39) we find

$$M(n_1 + 1, n_{12}, n_{34}, n_5) - M(n_1, n_{12}, n_{34}, n_5) \approx \frac{dM}{dn_1} = \frac{dM}{dk_1}$$

$$= \left( \frac{RV}{g\alpha'} \right)^2 \frac{1}{M} \left( k_5 + \frac{k_1 (\alpha')^2}{V} \right) \frac{(\alpha')^2}{V} = \frac{R}{g\alpha'} \cos(\theta_1 - \theta_2)$$

(5.13)

(at the last step we also used (3.36) and (3.38)). Then the binding energy is

$$\delta E \equiv M(n_1 + 1, n_{12}, n_{34}, n_5) - M(n_1, n_{12}, n_{34}, n_5) - M(1, 0, 0, 0) = \frac{2R}{g\alpha'} \sin^2 \frac{\theta_1 - \theta_2}{2}$$

which agrees exactly with (5.12) (the tension of the D1 brane is $T_1 = 1/(2\pi\alpha'g)$).

We see that the test D1 brane experiences no potential in the supergravity description if $\theta_1 = \theta_2$. From (3.35) we see that this is equivalent to $\frac{k_{12}}{V_{34}} = \frac{k_{34}}{V_{12}}$. In the ‘brane’ computation we find that the test D1 brane experiences no potential if $b_{12} + 2\pi\alpha'F_{12} = b_{34} + 2\pi\alpha'F_{34}$. Using (3.20) we see that the latter condition is the same as the condition $\frac{k_{12}}{V_{34}} = \frac{k_{34}}{V_{12}}$.

5.2 Motion of an infalling D1 brane

To explicitly find the motion of the D1 brane we take a further subset of the above configurations – those where $F_i(v)$ describes a circle in the $x_i$ space. These configurations
were studied in [7][8] and were also used in [2] to study a massless quantum falling down the throat. These configurations have $F_i F_i = \text{constant}$, and here we further choose $Q_1 = Q_5 \equiv Q$. The harmonic functions (2.4) are

\[ H^{-1} = 1 + \frac{Q}{f_0}, \quad K = \frac{Q}{f_0}, \quad A_\phi = -\frac{Q a \sin^2 \theta}{f_0}, \quad B_\psi = -\frac{Q a \cos^2 \theta}{f_0}, \quad (5.14) \]

where

\[ f_0 \equiv r^2 + a^2 \cos^2 \theta \quad (5.15) \]

The flat metric on the $x_i$ space has been written in terms of new coordinates

\[ dx_i dx_i \equiv f_0 \left( \frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + (a^2 + r^2) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\psi^2 \quad (5.16) \]

and the complete metric is

\[
\begin{align*}
\frac{ds^2}{1 + Q f_0} & = \left[ -\left( dt + \frac{Q a \sin^2 \theta}{f_0} d\phi \right)^2 + \left( dy - \frac{Q a \cos^2 \theta}{f_0} d\psi \right)^2 \right]^{1/2} \\
& + \left( 1 + \frac{Q}{f_0} \right) \left[ f_0 \left( \frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + (a^2 + r^2) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\psi^2 \right]^{1/2} + dz dz \\
\end{align*}
\]

(5.17)

From the symmetry of the solution we see that we can set $\dot{\psi} = \dot{\phi} = 0$ for the motion of the D1 brane. The action (5.9) for such configurations becomes:

\[
S = -T_1 R \int d\tau d\sigma H \left[ 1 - \left( 1 + \frac{Q}{f_0} \right)^2 f_0 \left( \frac{r^2}{r^2 + a^2} + \dot{\theta}^2 \right) \right]^{1/2} + T_1 R \int d\tau d\sigma H \cos(\theta_1 - \theta_2) \\
\]

(5.18)

We further note that the Lagrangian depends on $\theta$ only through the combination $f_0 = r^2 + a^2 \cos^2 \theta$. Thus the derivative $\delta L/\delta \theta$ vanishes for $\theta = 0$ and $\theta = \pi/2$. If $\theta$ is set to either of these values then it stays constant, and we get a 1-dimensional problem in the variable $r$. We analyze these two cases as they give us two physically opposite limits out of the generic set of trajectories.

**i) Trajectory with $\theta = 0$.** Consider the case where the total energy (5.10) is zero. We get

\[
\dot{r} = \pm \left( 1 + \frac{Q}{r^2 + a^2} \right)^{-1} \left[ 1 - \left( 1 + 2 \sin^2 \frac{\theta_1 - \theta_2}{2} \frac{Q}{r^2 + a^2} \right)^{-2} \right]^{1/2} \\
\]

(5.19)

Note that $\dot{r}$ never goes to zero ($r$ goes down to zero and then starts increasing again). The D1 brane travels down the throat and bounces back up, spending only a finite time in the throat.
To understand the details of the bounce we look at the geometry where $a$ takes its largest value $a = Q/R$. The metric (reduced on $T^4$) near the end of the throat is \[ \begin{align*}
 ds^2 &\approx -\frac{(r^2 + a^2) dt^2}{Q} + \frac{r^2 dy^2}{Q} + Q \left[ \frac{dr^2}{r^2 + a^2} + d\theta^2 + \cos^2 \theta d\psi_{NS}^2 + \sin^2 \theta d\phi_{NS}^2 \right] \quad (5.20) 
 \end{align*} \]

where \[ \psi_{NS} = \psi - \frac{y}{R}, \quad \phi_{NS} = \phi - \frac{t}{R} \quad (5.21) \]

We see that the metric (5.20) describes $\text{AdS}_3 \times S^3$, with the $S^3$ described by $\theta, \psi_{NS}, \phi_{NS}$. The spacetime $\text{AdS}_3 \times S^3$ is the dual of the Neveu-Schwarz sector of the CFT, and we have included subscripts on $\psi_{NS}, \phi_{NS}$ to note this fact.

The test D1 wraps the direction $y$ and is at constant $\psi, \phi$, but by (5.21) this implies that it wraps the direction $\psi_{NS}$. The D1 brane describing the motion (5.19) reaches $r = 0$, but since $\theta = 0$ we see that at this point the D1 brane wraps the diameter of $S^3$ parametrized by $\psi_{NS}$. Thus the D1 brane has not shrunk to a point, and it does not self-intersect or encounter any other singularity in the process of bouncing back to large $r$.

We expect this behavior of the D1 brane to be generic in the sense that for generic shapes of the throat end and generic initial conditions for the D1 brane the brane will not shrink to a point or self-intersect when it reaches the end of its motion down the throat. The D1 brane is just a 'giant graviton' in $\text{AdS}_3 \times S^3$, and it is important to note that its motion is reliably given by the DBI action plus Chern-Simons term when $n_1, n_5 >> 1$ (with other parameters held fixed).

(ii) Trajectory with $\theta = \pi/2$. In this case we find:

\[ \dot{r} = \pm \sqrt{\frac{r^2 + a^2}{r}} \left( 1 + \frac{Q}{r^2} \right)^{-1} \left[ 1 - \left( 1 + 2 \sin^2 \frac{\theta_1 - \theta_2}{2} \frac{Q}{r^2} \right)^{-2} \right]^{1/2} \quad (5.22) \]

As $r$ goes to zero we find $\dot{r} \sim -ra/Q$, so the time to reach $r = 0$ diverges.

The D1 brane again wraps the direction parametrized by $\psi_{NS}$, but since now $\theta = \pi/2$ we find that the brane shrinks to a point as $r \to 0$. Note that since $\phi$ is constant we have $d\phi_{NS}/dt = 1$. Thus as $t \to \infty$ the D1 brane settles down to a pointlike quantum traveling at the speed of light along the diameter of the $S^3$ parametrized by $\phi_{NS}$. The D1 brane becomes, at late times, an ordinary graviton rather than a giant graviton.

We expect that this diverging time will be found for an exceptional set of initial conditions. From (5.15) we see that the curve $r = 0, \theta = \pi/2$ is the 'central curve' for the geometry under consideration, i.e. the curve occupied by the profile $F_i$ in (2.4). (Recall that points on this curve were the centers of a Kaluza-Klein geometry in directions transverse to the curve.) We see that if the initial conditions on the D1 brane are such that they send the D1 brane straight into a point on the 'central curve' then the D1 brane settles down to a pointlike graviton as $t \to \infty$ instead of bouncing back. Correspondingly,
in the dual brane picture the tachyon settles down to the minimum value (5.12) of the potential. (The potential takes this value (5.12) at all points of the central curve but nowhere else.)

(iii) Trajectory for \(a = 0\). As discussed in \[2\] the parameter \(a\) can go down to very small values; it is prevented from vanishing only by the fact that the minimum angular momentum is \(j = \hbar/2\) for the quantum state dual to the supergravity solution. To study D1 brane infall for these geometries with small \(a\) we set \(a = 0\). Now we can set \(\theta = \theta_0, \phi = \phi_0, \psi = \psi_0\) and the radial motion is described by

\[
\dot{r} = \pm (1 + \frac{Q}{r^2})^{-1} \left[ 1 - \left( 1 + 2 \sin^2 \frac{\theta_1 - \theta_2}{2} \frac{Q}{r^2} \right)^{-2} \right]^{1/2} \tag{5.23}
\]

We note that in this case the travel time from \(r \sim \sqrt{Q}\) to \(r \sim a\) (where the throat ends) is \(\sim 1/a\). Thus in D1-D5 bound states with small \(a\) the tachyon bounces back after a minimum time \(\sim 1/a\).

Thus in the gravity picture the reason for the ‘bounce’ of the D1 brane is quite simple: when the D1 brane reaches the end of the throat it generically avoids falling onto the ‘central curve’. This is somewhat similar to having a nonzero impact parameter in the process of infall towards a point singularity at \(r = 0\); in this latter case the infalling object would also return to larger values of \(r\).\(^7\) What is interesting in our problem is that the D1-D5 bound state has an inherent nontrivial structure, and it becomes a complicated question to determine which trajectories of the D1 brane correspond to ‘zero impact parameter’. The choice of D1-D5 bound state determines the central curve of the geometry, and the direction of infall of the D1 brane (i.e. the choice of \(\theta_0, \psi_0, \phi_0\)) also determines the details of the bounce.\(^8\)

It would be interesting to study the above phenomena directly in the dual field theory.\(^9\)

6 Discussion

We have constructed metrics dual to bound states of D1-D3-D5 branes (with nonzero \(B\)), and analyzed the motion of a test D1 brane in these geometries. We found that the D1 brane generically bounces back from the end of the ‘throat’. In the dual field theory

\(^7\)In a recent paper \[32\] the motion of a D brane near a cluster of anti-branes was considered, and it was noted that angular momentum would prevent quick annihilation.

\(^8\)In the field theory picture it is naïve to think of the transverse displacements of the D1 brane as parametrized by points in \(\mathbb{R}^4\). For the D0-D4 system it was shown in \[33\] that 1-loop effects in open string theory make the moduli space corresponding to these displacements singular at \(r = 0\). This effect is related to the fact that in the supergravity picture even when the test D1 brane falls deep down the throat we still have to specify \(\theta, \phi, \psi\) to specify its state.

\(^9\)Some steps towards identifying the tachyon when \(B \neq 0\) were taken in \[11\]; the D1-D5 system with \(B\) field was also studied in \[34\].
description this implies that the tachyon starts by condensing towards the bottom of its potential, but can then bounce back up the potential hill. We now comment on the physics of this process.

The circle $S^1$ parametrized by the coordinate $y$ has a length $2\pi R$ (at $r \to \infty$). The dual description is given by a 1+1 dimensional field theory with the spatial direction a circle of length $2\pi R$. If we let $R \to \infty$ with all other parameters like $g, \alpha', n_1, n_5, b_{ij}$ fixed, then in the supergravity computation we find that the ‘bounce back’ time $t_{bounce}$ for the test D1 brane goes to infinity. In the dual brane description we conclude that if the CFT is not compactified to a circle (i.e. if the CFT lives on $\mathcal{R}$, the real line) then the tachyon settles down towards its minimum, and does not bounce back in any finite time.

But for applications to black holes we need to consider the D1-D5 system compactified on a circle, and in this case it was shown in [2] that the throat of the geometry ends after a finite distance, and infalling quanta are reflected back from this end. The length $R$ scales out from the final quantities of interest; for example it was shown in [35] that the ‘horizon area’ obtained by coarse graining over the different possible endings of the throat gave the Bekenstein entropy of the 2-charge system. To consider questions like the fate of a D1 brane falling into a black hole, we must look at tachyon condensation when $R$ is finite.

The low energy field theory of the D1-D5 bound state can be written as a 1+1 dimensional sigma model [3, 36, 29], with target space a deformation of the orbifold $(T^4)^N/S_N$ (the symmetric product of $N = n_1 n_5$ copies of $T^4$). The action of twist operators $\sigma_{n_i}$ leads to the ground states being characterized by ‘component strings’ of different lengths $2\pi R n_i$, and each component string also carries a $SU(2) \times SU(2)$ spin under the rotation group of the $S^3$ surrounding the branes [37, 2]. The orders of the twists $\{n_i\}$ and the spins determine the shape of the end of the throat.

The supergravity computation tells us that after the effects of these twists, spins etc. are taken into account, the tachyon generically bounces back after reaching some point close to its minimum energy point. The supergravity computation is reliable once we let $n_1, n_5$ be large (for other parameters fixed); for example in the case $a = Q/R$ in the metrics (2.7) with coefficients (5.14) we have seen that the D1 brane at the end of the throat is just a giant graviton in $AdS_3 \times S^3$, and we know that the backreaction of the giant graviton is small for $n_1, n_5 \gg 1$.

It is interesting that there are specific choices of initial conditions, at least for the metrics parameterized by harmonic functions (5.14), where the D1 brane asymptotically settles down to a pointlike object – a ‘graviton’, as opposed to a giant graviton. This happens if the D1 brane evolves such as to shrink down to a point on the ‘central curve’ of the geometry.

Note that the initial state of the test D1 brane is chosen to be translationally invariant in $y$, and the geometries (2.7) are also translationally invariant in $y$. Thus the classical

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10 Note that when we take $R \to \infty$ the magnitudes of the spins etc. stay the same, so in the field theory the effects of these spins, twists etc. get ‘diluted’. As a result we find that the bounce back time of the tachyon diverges.
motion of the D1 brane in any of these geometries will not generate vibrations of the test D1 brane, and the dynamical problem is limited to the center of mass motion of this brane. For generic shapes of the central curve and generic direction of infall the D1 brane need not bounce back to the start of the throat after reflecting off the end; it may stay trapped for long times near the end of the throat as it moves in the transverse 4-dimensional space \( x_i, i = 1 \ldots 4 \). This would be similar to the situation discussed in [35] for the evolution of a massless quantum – it was argued that the quantum stays trapped near the end of the throat for long times when the central curve has a complicated shape.

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A Derivation of the Solution

A.1 Acting on directions \((z_1,z_2)\)

We start with the solution (2.1)–(2.3), and perform a T-duality along \(z_1\), obtaining

\[
\begin{align*}
 ds^2 &= \sqrt{\frac{H}{1+K}} \left[ -(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + \sqrt{\frac{1+K}{H}} d\vec{x} d\vec{x} \\
 &= \frac{1}{\sqrt{H(1+K)}} dz_1^2 + \sqrt{H(1+K)} \left[ dz_2^2 + (dz_3^2 + dz_4^2) \right] \\
 e^{2\Phi'} &= \sqrt{H(1+K)} \\
 C^{\mu\alpha z_1} &= C^{(2)}_{\mu\alpha}
\end{align*}
\]

We now perform the rotation

\[
\begin{align*}
 z_1 &= \cos \theta_1 z_1' - \sin \theta_1 z_2' \\
 z_2 &= \sin \theta_1 z_1' + \cos \theta_1 z_2'
\end{align*}
\]

which gives

\[
\begin{align*}
 ds^2 &= \sqrt{\frac{H}{1+K}} \left[ -(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + \sqrt{\frac{1+K}{H}} d\vec{x} d\vec{x}
\end{align*}
\]

\[
^{11}\text{The D1 brane can radiate energy by higher-loop processes as it moves, and so will ultimately settle to the bottom of the potential (a similar observation was made in [32]).}
\]
\[
+ \left[ \frac{\cos^2 \theta_1}{\sqrt{H(1 + K)}} + \sqrt{H(1 + K)} \sin^2 \theta_1 \right] dz'_1^2
\]
\[
+ \left[ \frac{\sin^2 \theta_1}{\sqrt{H(1 + K)}} + \sqrt{H(1 + K)} \cos^2 \theta_1 \right] dz'_2^2
\]
\[
- 2 \sin \theta_1 \cos \theta_1 \left[ \frac{1}{\sqrt{H(1 + K)}} - \sqrt{H(1 + K)} \right] dz'_1 dz'_2
\]
\[
+ \sqrt{H(1 + K)} \left[ (dz_3^2 + dz_4^2) \right] \quad (A.5)
\]

\[
e^{2\Phi'} = \sqrt{H(1 + K)}, \quad (A.6)
\]

\[
C_{\mu \alpha z'_1}^{(3)} = \cos \theta_1 C_{\mu \alpha z'_1}^{(3)} + \sin \theta_1 C_{\mu \alpha z'_2}^{(3)} = \cos \theta_1 C_{\mu \alpha}^{(3)} = \cos \theta_1 C_{\mu \alpha}
\]
\[
C_{\mu \alpha z'_2}^{(3)} = - \sin \theta_1 C_{\mu \alpha z'_2}^{(3)} + \cos \theta_1 C_{\mu \alpha z'_2}^{(3)} = - \sin \theta_1 C_{\mu \alpha z'_1}^{(3)} = - \sin \theta_1 C_{\mu \alpha}^{(2)} \quad (A.7)
\]

We now perform another T-duality along \(z'_1\). Note that

\[
C_{\alpha \beta z'_1}^{(3)} = \cos \theta_1 C_{\alpha \beta z'_1}^{(3)} + \sin \theta_1 C_{\alpha \beta z'_2}^{(3)} = \cos \theta_1 C_{\alpha \beta}^{(3)} = \cos \theta_1 C_{\alpha \beta}
\]

\[
C_{\mu \nu z'_1 z'_2}^{(4)} = C_{\mu \nu z'_2}^{(3)} - 3 \frac{G_{\mu [z'_1]}^{\nu |z'_2]} G_{z'_1 z'_2}}{G_{z'_1 z'_2}} = C_{\mu \nu z'_2}^{(3)} - \frac{C_{\mu \alpha z'_1}^{(3)}}{G_{z'_1 z'_2}} \quad (A.8)
\]

\[
= - \sin \theta_1 C_{\mu \nu}^{(2)} \frac{H(1 + K)}{h_1} \quad (A.9)
\]

and we get the solution (2.7) with \(\theta_2 = 0\).

**A.2 \((z_3, z_4)\)-directions**

We drop primes on the variables obtained above, and perform a similar set of operations on \(z_3, z_4\). \(T_{z_3}\) gives

\[
ds^2 = \sqrt{\frac{H}{1 + K}} \left[ -(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + \sqrt{\frac{1 + K}{H}} d\vec{x} d\vec{x}
\]
\[
+ \sqrt{\frac{H(1 + K)}{h_1}} (dz_1^2 + dz_2^2) + \frac{1}{\sqrt{H(1 + K)}} dz_3^2 + \sqrt{H(1 + K)} dz_4^2 \quad (A.10)
\]

\[
e^{2\Phi'} = \sqrt{\frac{H(1 + K)}{h_1}}, \quad B'_{z_1 z_2} = B_{z_1 z_2} = \frac{\sin \theta_1 \cos \theta_1}{h_1} (1 - (1 + K)H) \quad (A.11)
\]
\[ C_{\mu \alpha z_3}^{(3)} = C_{\mu \alpha}^{(2)}, \quad C_{\mu \nu \rho \alpha z_3}^{(5)} = C_{\mu \nu \rho \alpha}^{(4)} \]  
(A.12)

The rotation

\[
\begin{align*}
z_3 &= \cos \theta_2 z'_{3} - \sin \theta_2 z'_{4} \\
z_4 &= \sin \theta_2 z'_{3} + \cos \theta_2 z'_{4}
\end{align*}
\]  
(A.13)

gives

\[
ds^2 = \sqrt{\frac{H}{1 + K}} \left[ -(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + \sqrt{\frac{1 + K}{H}} d\vec{x} d\vec{x}
\]

\[
\begin{align*}
&+ \frac{\sqrt{H(1 + K)}}{h_1} \left[ (dz_1^2 + dz_2^2) \right] \\
&+ \left[ \frac{\cos^2 \theta_2}{\sqrt{H(1 + K)}} + \sqrt{H(1 + K) \sin^2 \theta_2} \right] dz_3'^2 \\
&+ \left[ \frac{\sin^2 \theta_2}{\sqrt{H(1 + K)}} + \sqrt{H(1 + K) \cos^2 \theta_2} \right] dz_4'^2 \\
&- 2 \sin \theta_2 \cos \theta_2 \left[ \frac{1}{\sqrt{H(1 + K)}} - \sqrt{H(1 + K)} \right] dz_3' dz_4'
\end{align*}
\]  
(A.14)

\[ e^{2\Phi'} = \sqrt{\frac{H(1 + K)}{h_1}}, \quad B'_{z_1 z_2} = \frac{\sin \theta_1 \cos \theta_1}{h_1} (1 - (1 + K)H) \]  
(A.15)

\[
\begin{align*}
C_{\mu \alpha z_3'}^{(3)} &= \cos \theta_2 C_{\mu \alpha z_3}^{(3)} + \sin \theta_2 C_{\mu \alpha z_4}^{(3)} = \cos \theta_2 C_{\mu \alpha}^{(2)} \\
C_{\mu \alpha z_4'}^{(3)} &= -\sin \theta_2 C_{\mu \alpha z_3}^{(3)} + \cos \theta_2 C_{\mu \alpha z_4}^{(3)} = -\sin \theta_2 C_{\mu \alpha}^{(2)} \\
C_{\mu \nu \rho \alpha z_3'}^{(5)} &= \cos \theta_2 C_{\mu \nu \rho \alpha z_3}^{(5)} + \sin \theta_2 C_{\mu \nu \rho \alpha z_4}^{(5)} = \cos \theta_2 C_{\mu \nu \rho \alpha}^{(4)} \\
C_{\mu \nu \rho \alpha z_4'}^{(5)} &= -\sin \theta_2 C_{\mu \nu \rho \alpha z_3}^{(5)} + \cos \theta_2 C_{\mu \nu \rho \alpha z_4}^{(5)} = -\sin \theta_2 C_{\mu \nu \rho \alpha}^{(4)}
\end{align*}
\]  
(A.16)

Finally, \( T_{z_3'} \) gives the solution (2.7).

## B T-duality formulae

In this paper we perform T dualities following the notation of [20]. Let us summarize the relevant formulae. We call the T-duality direction \( s \). For NS–NS fields, one has

\[
\begin{align*}
G'_{ss} &= \frac{1}{G_{ss}}, \quad e^{2\Phi'} = \frac{e^{2\Phi}}{G_{ss}}, \\
G'_{\mu s} &= \frac{B_{\mu s}}{G_{ss}}, \quad B'_{\mu s} = \frac{G_{\mu s}}{G_{ss}}, \\
G'_{\mu \nu} &= G_{\mu \nu} - \frac{G_{\mu s} G_{\nu s} - B_{\mu s} B_{\nu s}}{G_{ss}}, \quad B'_{\mu \nu} = B_{\mu \nu} - \frac{B_{\mu s} G_{\nu s} - G_{\mu s} B_{\nu s}}{G_{ss}},
\end{align*}
\]  
(B.1)
while for the RR potentials we have:

\[
C^{(n)}_{\mu...\nu s} = C_{\mu...\nu s}^{(n-1)} - (n - 1) \frac{C_{[\mu...\nu]s}^{(n-1)} G_{[\alpha]s}}{G_{ss}}, \tag{B.2}
\]

\[
C^{(n)}_{\mu...\nu s\beta} = C_{\mu...\nu s\beta s}^{(n+1)} + nC_{[\mu...\nu]s}^{(n-1)} G_{[\beta]s} + n(n - 1) \frac{C_{[\mu...\nu]s}^{(n-1)} B_{[\alpha]s} G_{[\beta]s}}{G_{ss}}. \tag{B.3}
\]

\section{Mass Formulae for D0-D2-D4 system}

We consider Type IIA string theory, and regard it as a dimensional reduction of 11-dimensional M-theory. We compactify the IIA theory on a torus, and wrap the D-branes on directions along this torus. For a supersymmetric bound state of D0, D2 and D4 branes we write \cite{22, 23}

\[
\Gamma \epsilon = M \epsilon \tag{C.1}
\]

where

\[
\Gamma \equiv Z \Gamma_{0s} + \frac{1}{2} Z^{ij} \Gamma_{0ij} + \frac{1}{4!} Z^{ijkl} \Gamma_{0ijkl}
\]

\[
\frac{1}{8} Z^{ij} Z^{kl} \{ \Gamma_{0ij}, \Gamma_{0kl} \} + \frac{1}{2} Z^{ij} Z^{klmn} \{ \Gamma_{0ij}, \Gamma_{0klmn} \} + \frac{1}{2(4!)} Z^{ijkl} Z^{mnpq} \{ \Gamma_{0ijkl}, \Gamma_{0mnpq} \}
\]

The first five terms simplify to

\[
Z^2 + \frac{1}{12} Z^{ijkl} \Gamma_{ijkl} + \frac{|Z^{ij}|^2}{2} - \frac{Z^{ij} Z^{kl} \Gamma_{ijkl}}{12} - \frac{1}{3} Z^{ij} Z^{mnpq} \Gamma_{ijnpqs}
\]

The last term can be written as

\[
\frac{1}{2} Z^{ijkl} Z^{mnpq} \{ \Gamma_{ijkl}, \Gamma_{mnpq} \}_{i<j<k<l, m<n<p<q}
\]

which simplifies in general to

\[
\frac{1}{4!} |Z^{ijkl}|^2 - \frac{1}{4} Z^{ijkl} Z^{pqij} \Gamma_{klpq} + \frac{1}{(4!)} Z^{ijkl} Z^{mnpq} \Gamma_{ijklmn}
\]

In our D1-D5 system we have IIB theory compactified on \( T^5 = T^4 \times S^1 \), and thus we consider branes in the the T-dual IIA theory wrapped on at most 5 compact directions.
With the indices $i, j \ldots$ limited to 5 possible values, we find that the last two terms in (C.6) are identically zero, and we get

$$\Gamma^2 \epsilon = \left( Z^2 + \frac{1}{12} Z Z^{ijkl} \Gamma_{ijkl} + \frac{|Z^{ij}|^2}{2} - \frac{Z^{ij} Z^{kl} \Gamma_{ijkl}}{12} - \frac{1}{3} Z^{[ij} Z^{npq]} \Gamma_{jnpq} + \frac{1}{4!} |Z^{ijkl}|^2 \right) \epsilon$$

(C.7)

Defining

$$k^{ijkl} \equiv 2(Z Z^{ijkl} - Z^{[ij} Z^{kl]}), \quad k^{ijkl} \equiv Z^{[ij} Z^{npq]}$$

we find that 1/2 BPS configurations are obtained for

$$k^{ijkl} = k^{ijkl} = 0$$

(C.9)

and the mass for such configurations is given by

$$\Gamma^2 \epsilon \equiv M^2_0 \epsilon = \left( Z^2 + \frac{1}{2} |Z^{ij}|^2 + \frac{1}{4!} |Z^{ijkl}|^2 \right) \epsilon$$

(C.10)

In the present paper we have wrapped branes only on $T^4$ out of the $T^4 \times S^1$, and thus the indices $i, j \ldots$ are limited to only 4 possible values. Then $k^{ijkl} = 0$, and eqn. (C.7) can be rewritten as

$$\left( M^2 - M^2_0 \right)^2 \epsilon = \left( \frac{k^{ijkl} \Gamma_{ijkl}}{4!} \right)^2 \epsilon = \frac{1}{4!} |k^{ijkl}|^2$$

(C.11)

which yields

$$M^2 = \left( Z^2 + \frac{1}{2} |Z^{ij}|^2 + \frac{1}{4!} |Z^{ijkl}|^2 \right) + \sqrt{\frac{|k^{ijkl}|^2}{4!}}$$

(C.12)

With $k^{ijkl} \neq 0$ this is the mass formula for 1/4 BPS states.

We now relate the $Z$ variables to the number of branes in the state. Let the $T^4$ be rectangular with sides $L_i, i = 1 \ldots 4$. Define the dimensionless parameters

$$l_i = \frac{L_i}{2 \pi \sqrt{\alpha'}}, \quad v = l_1 l_2 l_3 l_4$$

(C.13)

By writing the mass expected when each kind of brane is present by itself, we get the identifications

$$Z = \frac{1}{g \sqrt{\alpha'}} n_0, \quad Z^{ij} = \frac{l_i l_j}{g \sqrt{\alpha'}} n_{ij}, \quad Z^{ijkl} = \frac{l_i l_j l_k l_l}{g \sqrt{\alpha'}} n_{ijkl}$$

(C.14)

We are interested in the particular case of D0-D2-D4 system where we have $D4$ branes wrapped along the 1234 directions of $T^4$, $D2$ branes wrapped along the 12 and 34 directions, as well as some $D0$ branes. Let $n_0$ be the number of zero branes, $n_{12}, n_{34}$ the number of two branes in direction 12 and 34 respectively and $n_4$ the number of four branes. Then

$$M^2 = \frac{1}{g^2 \alpha'} \left( (l_1 l_2 n_{12} + l_3 l_4 n_{34})^2 + (n_0 \pm l_1 l_2 l_3 l_4 n_4)^2 \right)$$

(C.15)

28
where we choose the upper signs if \(n_0n_4 - n_{12}n_{34} > 0\) and the lower signs if \(n_0n_4 - n_{12}n_{34} < 0\).

The above relations are for vanishing value of the \(B\) field. For \(B \neq 0\) we can obtain the mass by rewriting the charges of the bound state in terms of the (matrix-valued) field strength \(F\) on the D4 branes, and then noting that all quantities depend only on the combination \(F + \frac{B}{2\pi\alpha'}\). We have

\[
\begin{align*}
n_0 &= \frac{L_1L_2L_3L_4}{(2\pi)^2} Tr(F_{12}F_{34}) \\
n_{12} &= \frac{L_3L_4}{2\pi} Tr(F_{34}) \\
n_{34} &= \frac{L_1L_2}{2\pi} Tr(F_{12})
\end{align*}
\]

Thus for \(B \neq 0\) we must make the replacements

\[
\begin{align*}
n_{12} &\rightarrow n_{12} + n_4l_3l_4b_{34} \\
n_{34} &\rightarrow n_{34} + n_4l_1l_2b_{12} \\
n_0 &\rightarrow n_0 + l_1l_2b_{12}n_{12} + l_3l_4b_{34}n_{34} + n_4l_1l_2l_3l_4b_{12}b_{34}
\end{align*}
\]

Substituting these in \((C.15)\) (and regrouping terms) we get

\[
M^2 = \frac{1}{g^2\alpha'} \left( (l_1l_2n_{12} \mp l_3l_4n_{34} + n_4v(b_{34} \mp b_{12}))^2 + (n_0 + v(b_{12}b_{34} \pm 1)n_4 + l_1l_2b_{12}n_{12} + l_3l_4b_{34}n_{34})^2 \right)
\]

\[\text{(C.20)}\]

**D** Supersymmetric brane configurations using the DBI action

We assume that \(n_4\) is even, and look for a specific supersymmetric configuration to establish that the bound state can reach the supersymmetric mass bound. Let the field strength have values \(F_{12}, F_{34}\) on half of the D4 branes, and values \(F'_{12}, F'_{34}\) on the other half. Then the constraints are

\[
\begin{align*}
\frac{n_4}{2}(F_{12} + F'_{12}) &= \frac{2\pi}{L_1L_2} n_{34} \\
\frac{n_4}{2}(F_{34} + F'_{34}) &= \frac{2\pi}{L_3L_4} n_{12} \\
\frac{n_4}{2}(F_{12}F_{34} + F'_{12}F'_{34}) &= \frac{(2\pi)^2}{L_1L_2L_3L_4} n_0
\end{align*}
\]

\[\text{(D.1)}\]

\[\text{(D.2)}\]

\[\text{(D.3)}\]
The supersymmetry condition (4.11) is

\[
\frac{F_{12} \pm F_{34}}{1 \mp (2\pi\alpha')^2 F_{12} F_{34}} = \frac{F'_{12} \pm F'_{34}}{1 \mp (2\pi\alpha')^2 F'_{12} F'_{34}}
\]  
(D.4)

Define

\[
\alpha = \pm \left(\frac{2}{n_4}\right)^2 \frac{1}{v} [n_{12} n_{34} - n_0 n_4]
\]  
(D.5)

\[
\beta = \left(\frac{2}{n_4}\right)^2 \frac{1}{v} \left[\pm (n_{12} n_{34} - n_0 n_4) + \frac{l_3 l_4}{l_1 l_2} n_{34}^2 + vn_4\right]
\]  
(D.6)

\[
\gamma = \left(\frac{2}{n_4}\right)^2 \frac{1}{v} \left[\pm (n_{12} n_{34} - n_0 n_4) + \frac{l_1 l_2}{l_3 l_4} n_{12}^2 + vn_4\right]
\]  
(D.7)

Then the solution to (D.1), (D.2), (D.3), (D.4) is

\[
(2\pi\alpha') F_{12} = \pm \left(\frac{1}{l_1 l_2}\right) n_{34} + \frac{1}{\sqrt{v}} \sqrt{\frac{\alpha\beta}{\gamma}}
\]  
(D.8)

\[
(2\pi\alpha') F'_{12} = \pm \left(\frac{1}{l_1 l_2}\right) n_{34} - \frac{1}{\sqrt{v}} \sqrt{\frac{\alpha\beta}{\gamma}}
\]  
(D.9)

\[
(2\pi\alpha') F_{34} = \pm \left(\frac{1}{l_3 l_4}\right) n_{12} - \frac{1}{\sqrt{v}} \sqrt{\frac{\alpha\gamma}{\beta}}
\]  
(D.10)

\[
(2\pi\alpha') F'_{34} = \pm \left(\frac{1}{l_3 l_4}\right) n_{12} + \frac{1}{\sqrt{v}} \sqrt{\frac{\alpha\gamma}{\beta}}
\]  
(D.11)

We note from (D.5), (D.6), (D.7) that if \((n_{12} n_{34} - n_0 n_4) \geq 0\) then we can use the upper sign in these relations, and obtain a real solution for \(F\). If \((n_{12} n_{34} - n_0 n_4) \leq 0\) then we use the lower sign and again get a real solution \(F\). Thus we have a supersymmetric configuration for all values of the charges.

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32