Extreme Events and Chaotic Dynamics in $\mathcal{PT}$-symmetric Liénard Oscillators

Jyoti Prasad Deka,$^\ast$ Manas Kulkarni,$^\dagger$ and Amarendra Kumar Sarma,$^\ddagger$

Department of Physics, Indian Institute of Technology Guwahati, Guwahati-781039, Assam, India and
International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bangalore-560089, India

(Dated: December 27, 2018)

We propose a parity-time symmetric oscillator with nonlinear position dependent dissipation and Duffing nonlinearity. We studied the stability of the stationary states using the linearization Jacobian technique and evaluated its stability threshold thereafter. We have considered the 1st and 2nd order of nonlinear position dependent damping. In the neutrally stable region of 1st order nonlinear damping model, we discovered that the temporal evolution of the gain oscillator exhibits extreme events and that of the lossy oscillator exhibits the characteristics of the periodically forced damped harmonic oscillator. Analysis of the time series showed us that the extreme events in our system is preceded by exponentially growing oscillations. As we increase the gain/loss coefficient of the oscillators, power spectrum analysis of the time series of the gain oscillator showed us that our system displays a quasiperiodic route to extreme events. Using an external static drive on both oscillators, we found that the extreme events could be controlled and amplitude death is achievable. On the other hand, in the 2nd order nonlinear damping model, we found that the phase space trajectory exhibits a quasiperiodic route to chaos as we increased the gain/loss coefficient. Moreover, we showed that the chaotic dynamics could be transformed to quasiperiodic dynamics by using an external drive.

I. INTRODUCTION

C. M. Bender and S. Boettcher’s pioneering work [1] on $\mathcal{PT}$ symmetric quantum mechanics remains a topic of immense interest till this day. They showed non-Hermitian Hamiltonians that are invariant under the joint operation of the parity and time reversal operator could exhibit a real eigen spectra. If all eigenvalues of the Hamiltonian are real, it is said to be in the unbroken $\mathcal{PT}$-symmetric phase, else it is said to be in the broken $\mathcal{PT}$-symmetric phase. Under certain parametric changes, all eigenvalues of such Hamiltonians could coalesce as an exceptional point (EP). This critical point has been termed as the $\mathcal{PT}$-threshold. D. M. Christodoulides et. al. [2] proposed how optics could facilitate the experimental realization of such Hamiltonians. Such a proposal was possible due to the shared isomorphism between the paraxial equation of diffraction and the time-dependent Schrödinger equation. Ruter et. al. [3] demonstrated the first experimental observations of parity-time symmetry in a configuration of evanescently coupled waveguides. Since then, parity-time symmetry has been investigated in diverse domains and phenomena such as the onset of chaotic dynamics in optomechanical systems [4], modulational instability in complex media [5], active LRC circuits [6], optical oligomers [7-14], multilayered structures [15-16], unidirectional invisibility in periodic structures [17-18], wireless power transfer [19], plasmonics [20-21], etc have been reported.

$\mathcal{PT}$-symmetry has been explored in harmonic oscillator models extensively [6,32,41]. In [32], Carl Bender et. al. studied $\mathcal{PT}$ phase transition in a mechanical oscillator. Similarly, S. Karthiga et. al. [41] studied two-fold $\mathcal{PT}$-symmetry and tailoring $\mathcal{PT}$ regimes in nonlinearly damped harmonic oscillator models. In this article, we concentrate on the analysis of $\mathcal{PT}$-symmetric Liénard oscillators. The Liénard oscillator or Liénard system [22] is a prototype model to study the dynamics of oscillators with nonlinear dissipation in many areas of physics, biology, electronics, etc. The nonlinear damping term in Liénard systems can serve as both damping or amplification term resulting in self-sustained oscillations. The Liénard systems comprise of the Van der Pol oscillator, the FitzHugh-Nagumo oscillator, etc. The Van der Pol oscillator [23-24] gives rise to stable limit cycles based on certain parametric choices. Van der Pol termed these oscillations as relaxation-oscillation. In fact, the Van der Pol oscillator had been also used to model biological regulatory systems such as the cardiac and respiratory systems [25, 26]. Similarly, the FitzHugh-Nagumo oscillator [27-29] (which is reducible to the Van der Pol oscillator) is another class of Liénard systems and it has been used to model nerve impulses. It has also been shown recently that the forced Liénard system could be utilised to generate extreme events [30]. On the other hand, it has been recently shown experimentally that nonlinear damping could enhance the quality factor of graphene oscillators significantly [31]. Moreover, for free falling objects in earth’s gravity, the viscous force acting on the object is directly proportional to the square of the velocity. This force leads to the terminal velocity in such objects. Hence, it could be said that nonlinear dissipation plays an interesting role in the dynamics of numerous systems.

On the other hand, extreme events are highly intense and localized in space but transient in time structures that have been observed to occur on the surface of rel-
Atively calm seas [33]. They are also known as rogue or freak waves. In nonlinear systems such as the discrete nonlinear Schrödinger equation, the appearance of such structures could be attributed to the emergence of Benjamin-Feir (Modulational) instability for certain choice of parameters and nonlinearity [34]. In physics particularly, extreme events have been observed in numerous systems such as optical fibers [35-37], space plasmas [38], optically injected semiconductor laser [39] and so on.

In this work, we propose a Liénard oscillator with a balanced loss and gain. Given the importance of such oscillators as discussed above, we concentrate on the parity-time symmetric counterpart of such systems. A linear \( PT \)-symmetric oscillator is a coupled oscillator configuration, in which one oscillator linearly attenuates the perturbation applied to it and the other oscillator linearly amplifies the perturbation by the same proportion [32]. On the other hand, in a \( PT \)-symmetric system with nonlinear dissipation, it must be ensured via the operation of the \( PT \) operator that the system is parity-time symmetric. We have considered two cases of the order of nonlinear damping/gain, \( \kappa \rightarrow - \kappa \) and \( \delta \rightarrow - \delta \). In addition to this, we require the other additional oscillator for \( PT \) symmetry. Let us consider the following forced \( PT \)-symmetric coupled harmonic oscillator configuration. Setting \( f(t) = 0 \), we consider,

\[
\begin{align*}
\dot{x}_1 &= y_1 \\
\dot{y}_1 &= -\gamma y_1 - \eta x_1 y_1 - \omega_0^2 x_1 - \alpha x_1^3 - \kappa x_2 + f(t) \\
\dot{x}_2 &= y_2 \\
\dot{y}_2 &= \gamma y_2 + \eta (-1)^{\delta} x_2 y_2 - \omega_0^2 x_2 - \alpha x_2^3 - \kappa x_1 + f(t)
\end{align*}
\]

(5a) (5b) (5c) (5d)

where \( \kappa \) is the coupling constant between the oscillators and \( f(t) \) is the driving term and, it should be an even function of \( t \). All parameters in Eq. 5 are considered positive. This system, with \( f(t) = 0 \), admits the following stationary states:

1. The trivial stationary state - \( (x_1, y_1, x_2, y_2) = (0, 0, 0, 0) \),
2. The non-zero stationary states - \( (x_1, y_1, x_2, y_2) = (\pm a_1, 0, \mp a_1, 0) \), \( a_1 = \sqrt{((\kappa - \omega_0^2)/\alpha)} \) and \( (x_1, y_1, x_2, y_2) = (\pm a_2, 0, \mp a_2, 0) \), \( a_2 = \sqrt{((-\kappa - \omega_0^2)/\alpha)} \).

In the next section, we will concentrate on the stability analysis of the above mentioned stationary states and the dynamical aspects of our system for \( \delta = 1 \) and \( \delta = 2 \). To avoid confusion, we would like to term our mathematical models as order-1 and order-2 model respectively. These are what we call as the two cases. In addition to this, we have chosen the natural frequency of the oscillator \( \omega_0^2 = 0.5 \), coupling constant \( \kappa = 0.5 \), strength of nonlinear gain/loss \( \eta = 0.1 \) and coefficient of Duffing nonlinearity \( \alpha = 1 \). And we will analyze the stability of the stationary state given by \( (x_1, y_1, x_2, y_2) = (\pm a_1, 0, \mp a_1, 0) \), where \( a_1 = \sqrt{((\kappa - \omega_0^2)/\alpha)} \).
III. RESULTS AND DISCUSSION

The linearization Jacobian of our system with \( f(t) = 0 \) is given by

\[
J = \begin{pmatrix}
0 & 1 & 0 & 0 \\
A & B & -\kappa & 0 \\
0 & 0 & 0 & 1 \\
-\kappa & 0 & C & D
\end{pmatrix}
\]

where \( A = -\omega_0^2 - 3\alpha x_1^2 + \delta \eta x_1^{(\delta - 1)} y_1 \), \( B = -\gamma - \eta x_1^\delta \), \( C = -\omega_0^2 - 3\alpha x_2^2 - \delta \eta (-1)^\delta x_2^{(\delta - 1)} y_2 \) and \( D = \gamma + \eta (-1)^\delta x_2^\delta \).

In the following, in order to have a clear exposition, we discuss the nonlinear dynamics of system due to the nonlinear position dependent damping/gain for \( \delta = 1 \) and \( \delta = 2 \) respectively, along with the presence of other effects, in two different subsections.

A. \( \delta = 1 \)

Firstly, we analyze the eigen spectra of the linearization Jacobian given in Eq. 6 for \( \delta = 1 \). We have plotted the real component of all the four eigenvalues of the Jacobian in Fig. 1. From the eigenspectra, we can see that the real component of all the four eigenvalues coalesce at \( \gamma \approx 0.7 \). This means that the chosen stationary state is neutrally stable for \( \gamma < 0.7 \) and is an unstable saddle node for \( \gamma > 0.7 \). We shall term \( \gamma_{th} \approx 0.7 \) as our stability threshold. We will now study the temporal dynamics of our system in the neutrally stable region.

For \( \gamma = 0.4 \), i.e., below the stability threshold, we observe the emergence of extreme events in the temporal evolution of the gain oscillator (Fig. 2(a)). On the other hand, the lossy oscillator displays similar aperiodic dynamics (Fig. 2(b)). In comparison to the gain oscillator, the lossy oscillator behaves as a periodically forced damped harmonic oscillator. On analyzing the temporal evolution of the gain oscillator in the period from \( t = 0 \) to \( t = 35 \), it has been found that the temporal evolution depicts exponential growth of oscillations (Fig. 3). From this, we can infer that this coupled oscillator configuration displays exponential growth in the time period from \( t = 0 \) to \( t = 35 \) and this is what leads to chaotically evolving extreme events.

We next analyze the temporal evolution for \( \gamma < 0.4 \) and the main intention for this is to identify the route to extreme events. For \( \gamma = 0.1 \), we found that the temporal evolution displays quasiperiodic behavior in the gain oscillator which we have validated from the power spectra analysis. In Fig. 4(a), we have plotted the power spectra...
spectral density (PSD) for $\gamma = 0.1$ for the gain oscillator. The PSD \cite{41} is given by:

$$P(\Omega) = |x(\Omega)|^2$$

(7)

where $x(\Omega)$ is the Fourier transform of the time series. The power spectra display two sharp distinct peaks at $f_1 = 0.5267$ and $f_2 = 0.9133$ and the ratio $f_2/f_1 \approx 1.734$ is an irrational incommensurate quantity which tells us that the time series is quasiperiodic \cite{42}. This means that with increase in $\gamma$, the phase space trajectory exhibits the quasiperiodic route to extreme events. So far, we had been investigating the dynamics of our model in the absence of any external driving signal ($f(t) = g(t) = 0$). We will now explore the dynamics of our system when it is subjected to an external force. We consider the drive to be of the form $f(t) = f_0 \cos(\omega t)$, where $\omega$ is the frequency of the external drive. We observed that when we drive both oscillators using $\omega = 0$ (Fig. 5(a) and 5(b)), we observed that the oscillatory behavior in both oscillators ceases after a finite time. This phenomenon is known as ‘Oscillation Death’. Oscillation death usually occurs in nonlinear systems as a consequence of the delay in coupling or some changes in the parameters of the system. In our system, we have achieved oscillation death by driving the system with an external constant force. Moreover, one could say that extreme events in the parity-time symmetric Liénard oscillator could be controlled by externally driving the system.

\[ \delta = 2 \]

We now study the eigen spectra of the linearization Jacobian for the order-2 Liénard oscillator. We find that the stability threshold has been shifted to $\gamma_{th} \approx 0.64$ (Fig. 6). In Fig. 7, we have plotted the phase space trajectory of the gain oscillator and its corresponding power spectra for $\gamma = 0.1$, which is in the neutrally stable region. The power spectra (Fig. 7(b)) shows two distinct sharp peaks at $f_1 = 0.124$ and $f_2 = 0.803$. The ratio $f_2/f_1 \approx 6.4758$ means that our temporal dynamics show quasiperiodic behavior. Moreover, from the phase space trajectory (Fig. 7(a)), we can see that it is a 2-torus. In Fig. 8, we have plotted the power spectra for $\gamma = 0.3$. The power spectra shows two distinct sharp peaks at $f_1 = 0.18$ and $f_2 = 0.782$. The ratio is $f_2/f_1 \approx 4.3444$, which is once again incommensurate. With this, we may say that as we increase $\gamma$, the frequency ratio could be held incommensurate. Moreover, as $\gamma$ is increased almost till the stability threshold, we find that the whole low frequency region is populated and there is exponential decay in the higher frequency region (Fig. 9). This depicts the chaotic behavior of our system. From this, we can conclude that our system exhibits the quasiperiodic route to chaos.

We present how we could control degree of chaos in our system by driving both the oscillators with an external force. We find that when we drive both oscillators using a external static drive, the chaotic dynamics in the phase space trajectory could be restored to quasiperiodic dynamics. This means that the external drive acts as a measure for chaos control in such nonlinear systems.
FIG. 5. (Color online) Temporal evolution of (a) gain oscillator and (b) lossy oscillator for $\gamma = 0.4$, $\omega = 0$ and $f_0 = 0.1$.

FIG. 6. (Color online) Real component of the Jacobian eigenvalues of the PT-symmetric Lienard Oscillator for $\delta = 2$.

This has been depicted in Fig. 10, where we have found that the power spectra of the time series of the gain oscillator displays two distinct peaks at $f_1 = 0.168$ and $f_2 = 0.726$, the ratio of which is $f_2/f_1 \approx 4.3214$. The ratio is an irrational incommensurate quantity and this validates our claim that using a static external force, we can restore the quasiperiodic dynamics back in our system. It is encouraging that one could use external drive as a knob to control the nature of chaos.

FIG. 7. (Color online) (a) Power Spectra and (b) Phase Space of the temporal evolution of the gain oscillator for $\gamma = 0.1$.

FIG. 8. (Color online) Power Spectra of the temporal evolution of the gain oscillator for $\gamma = 0.3$. 
IV. EXPERIMENTAL REALIZATION

In this section, we provide a possible experimental route to realize our system studied in previous sections. Using Kirchoff's law of circuit analysis and in the absence of any resistive element and current multiplier, the equations governing the dynamics in electric current in a CLC circuit (see schematic Fig. 11) are given by

\[
L \frac{dI_1}{dt} + \frac{Q_1}{C} + \frac{Q}{C} = V(t) \quad (8a)
\]
\[
L \frac{dI_2}{dt} + \frac{Q_2}{C} + \frac{Q}{C} = V(t) \quad (8b)
\]

where \(Q\), \(Q_1\) and \(Q_2\) are the charges in the capacitors. From the circuit diagram (Fig. 11), we can see that the electric charge in the capacitors can be written in terms of the current flowing in the circuit as follows.

\[
\frac{dQ_1}{dt} = I_1, \quad \frac{dQ_2}{dt} = I_2, \quad \frac{dQ}{dt} = I \quad (9)
\]

Hence, we have \(Q = Q_1 + Q_2\) and using this, we can eliminate \(Q\) in Eq. 8a and 8b and they can be rewritten as

\[
L \frac{dI_1}{dt} + 2\frac{Q_1}{C} = V(t) \quad (10a)
\]
\[
L \frac{dI_2}{dt} + 2\frac{Q_2}{C} = V(t) \quad (10b)
\]

Differentiating Eq. 10a and 10b and using Eq. 9, we obtain the following equations,

\[
L \frac{d^2I_1}{dt^2} + \frac{2}{C}I_1 + \frac{2}{C}I_2 = V'(t) \quad (11a)
\]
\[
L \frac{d^2I_2}{dt^2} + \frac{1}{C}I_1 + \frac{2}{C}I_2 = V'(t) \quad (11b)
\]

where \(V' = dV/dt\). These equations describe the oscillatory behavior of current in a CLC oscillator, but this system is not \(\mathcal{PT}\)-symmetric. To ensure the presence of \(\mathcal{PT}\)-symmetry, we need to include a resistor in both loops. One of the resistor must induce attenuation in the electric current and the other needs to amplify in equal proportion. This can be achieved by using a negative resistance in one half of the circuit. Unlike a normal resistance, the I-V characteristics of a negative resistance shows a decrease in the current flowing through it when the potential difference is increased. A negative resistance can be realized in our circuit by using an Esaki diode or Gunn diode. Then, from Eq. 11a and 11b, we have

\[
L \frac{d^2I_1}{dt^2} + \alpha \frac{dI_1}{dt} + \omega_0^2 I_1 + \kappa I_2 = V'(t)/L \quad (12a)
\]
\[
L \frac{d^2I_2}{dt^2} - \alpha \frac{dI_2}{dt} + \omega_0^2 I_2 + \kappa I_1 = V'(t)/L \quad (12b)
\]

where \(\alpha = R/L\), \(\omega_0^2 = 2/LC\) and \(\kappa = 1/LC\) are the linear gain/loss coefficient, nonlinear resistance, natural frequency of the oscillator and the coupling constant respectively.
Until this point, we have not yet included nonlinear resistance and current multiplier in the circuit that would facilitate the realization of our theoretical model. The nonlinear resistance $R_{nl}$ could be realized by using a non-ohmic resistance. In a non-ohmic resistance, the I-V characteristics does not follow a straight line. Instead, the gradient of the graph increases as the current is increased. We choose our nonlinear resistance to follow the generalized I-V characteristics law given by $V_{nl,i} = R_{nl,i}I^\delta$, where $R_{nl}$ is the coefficient and $\delta$ is the exponent of nonlinear resistance respectively. And the Duffing nonlinearity could be easily realized in our circuit using an analog devices multiplier AD532 or AD534 [42]. For current inputs $X$ and $Y$, such a device would produce an output $XY/10$. Then from eq. 12a and 12b we have

$$\frac{d^2I_1}{dt^2} + (\alpha + \beta_1 I_1^{\delta-1}) \frac{dI_1}{dt} + \omega_0^2 I_1 + \kappa I_2 = V'(t) \quad (13a)$$
$$\frac{d^2I_2}{dt^2} - (\alpha - \beta_2 I_2^{\delta-1}) \frac{dI_2}{dt} + \omega_0^2 I_2 + \kappa I_1 = V'(t) \quad (13b)$$

where $\beta_i = \delta R_{nl,i}/L$ is the coefficient of nonlinear resistance. One important thing that must be noted is that in the case of $\delta = 3$ we have $\beta_2 = -\beta_1$. This implies that one of the non-ohmic resistor must instead play the role of a nonlinear amplifier.

V. CONCLUSION

We investigated a parity-time symmetric Lienard oscillator configuration for two cases of nonlinear dissipation. We analyzed the stability of the stationary states using the linearization Jacobian technique. In the neutrally stable region, we have found that the order-1 model gives rise to extreme events in the gain oscillator and the lossy oscillator behaves as a aperiodically driven damped harmonic oscillator. Using an external driving, oscillation death could be achieved in the system. This means that an external signal could be used to control the uprisal of extreme events in our system. On the other hand, the order-2 model gives rise to a quasiperiodic route to chaos in the same regime. With the use of an external signal, we could control the emergence of chaotic dynamics in our system. For potential experimental realization of our models, we proposed an electronic circuit using Kirchhoff’s laws. The gain oscillator could be realized using a negative resistance, nonlinear dissipation using a non-ohmic resistance and Duffing nonlinearity using a analog devices multiplier.

We anticipate that these systems could be scaled up to many oscillators, and this will open an interesting avenue to understand control of chaos in many body driven-dissipative systems.

ACKNOWLEDGEMENTS

J. P. D. thanks MHRD, Government of India for financial support through a fellowship and A. K. S. acknowledges financial support from DST-SERB, Government of India. M. K. gratefully acknowledges the Ramanujan Fellowship SB/S2/RJN-114/2016 from the Science and Engineering Research Board (SERB), Department of Science and Technology, Government of India.

REFERENCES

1. C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
2. R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, Opt. Lett. 32, 2632634 (2007).
3. C. E. Ruter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. 6, 192195 (2010).
4. X.-Y. L, H. Jing, J.-Y. Ma, and Y. Wu, Phys. Rev. Lett. 114, 253601 (2015).
5. A. K. Sarma, J. Opt. Soc. Am. B 31, 18611866 (2014).
6. J. Schindler, A. Li, M. C. Zheng, F. M. Ellis, and T. Kottos, Phys. Rev. A 84, 040101(R) (2011).
7. M. Duanmu, K. Li, R. L. Horne, P. G. Kevrekidis, and N. Whitaker, Philos. Trans. R. Soc. A 371, 20120171 (2013).
8. K. Li, P. G. Kevrekidis, D. J. Frantzeeskakis, C. E. Ruter, and D. Kip, J. Phys. A Math. Gen. 46, 375304 (2013).
9. S. K. Gupta, J. P. Deka, and A. K. Sarma, Eur. Phys. J. D 69, 199 (2015).
10. J. P. Deka and A. K. Sarma, Commun. Nonlinear Sci. Numer. Simul. 57, 26 (2017).
11. I. V. Barashenkov, G. S. Jackson and S. Flach, Phys. Rev. A 88, 053817 (2013).
12. I. V. Barashenkov, Phys. Rev. A 90, 045802 (2014).
13. I. V. Barashenkov, D. E. Pelinovsky and P. Dubard, J. Phys. A: Math. Theor. 48, 325201 (2015).
14. I. V. Barashenkov and M. Gianfreda, J. Phys. A: Math. Theor. 47, 282001 (2014).
15. M. Sarisaman, Phys. Rev. A 95, 013806 (2017).
16. J. P. Deka and A. K. Sarma, App. Opt. 57, 1119 (2018).
17. Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, Phys. Rev. Lett. 106, 213901 (2011).
18. L. Feng, Y.-L. Xu, W. S. Fegadolli, M.-H. Lu, J. E. B. Oliveira, V. R. Almeida, Y.-F. Chen, and A. Scherer, Nat. Mater. 12, 108113 (2013).
19. S. Assawaworrarit, J. Phys. A Math. Theor. 47, 282001 (2014).
20. H. Alaeian and J. A. Dionne, Phys. Rev. A 89, 033829 (2014).
21. H. Benisty, A. Degiron, A. Lupa, A. D. Lustrac, S. Chnais, S. Forget, M. Besbes, G. Barbillon, A. Bruyant,
S. Blaize, and G. Lrondel, Opt. Exp. 19, 18004 (2011).
22. A. A. Andronov, E. A. Leontovich, I. I. Gordon and A. G. Maier, *Qualitative Theory of Second-Order Dynamical Systems* (Wiley, New York, 1973).
23. B. Van der Pol, Phil. Mag. Ser. 7 2, 978 (1926).
24. B. Van der Pol and J. Van der Mark, Phil. Mag. Ser. 7 6, 763 (1928).
25. T. P. Dinh, J. Demongeot, P. Baconnier and G. Benchetrit, J. Theor. Biol. 103, 113 (1983).
26. P. F. Rowat and A. I. Selverston, J. Neurophysiol. 70, 1030 (1983).
27. R. FitzHugh, Biophys. J. 1, 445 (1961).
28. R. FitzHugh, Bull. Math. Biophysics, 17, 257 (1955).
29. J. Nagumo, S. Arimoto and S. Yoshizawa, Proc. IRE, 50, 2061 (1962).
30. S. L. Kingston, K. Thamilmaran, P. Pal, U. Feudel and S. K. Dana, Phys. Rev. E 96, 052204 (2017).
31. A. Eichler, J. Moser, J. Chaste, M. Zdrojek, I. Wilson-Rae and A. Bachtold, Nat. Nanotechnol. 6, 339 (2011).
32. C. M. Bender, B. K. Berntson, D. Parker, and E. Samuel, Am. J. Phys. 81, 173 (2013).
33. N. Akhmediev, A. Ankiewicz, and M. Taki, Phys. Lett. A. 373, 675 (2009).
34. B. S. White and B. Forneberg, J. Fluid Mech. 355, 113138 (1998).
35. D. R. Solli, C. Ropers, P. Koonath, and B. Jalali, Nature. 450, 1054 (2007).
36. K. Hammani, C. Finot, J. M. Dudley, and G. Millot, Opt. Express 16, 16467 (2008).
37. F. T. Arecchi, U. Bortolozzo, A. Montina, and S. Residori, Phys. Rev. Lett. 106, 153901 (2011).
38. M. S. Ruderman, Eur. Phys. J. Special Topics 185, 57 (2010).
39. C. Bonatto, M. Feyereisen, S. Barland, M. Giudici, C. M. J. R. R. Leite, and J. R. Tredicce, Phys. Rev. Lett. 107, 053901 (2011).
40. S. Karthiga, V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, Phys. Rev. A 93, 012102 (2016).
41. M. Lakshmanan, and S. Rajasekar, *Nonlinear Dynamics - Integrability, Chaos and Patterns* (Springer, 2003).
42. T. W. Dixon, T. Gherghetta, and B. G. Kenny, Chaos 6, 32 (1996).