Functional derivatives, Schrödinger equations, and Feynman integration

Alexander Dynin
Department of Mathematics, Ohio State University
Columbus, OH 43210, USA, dynin@math.ohio-state.edu

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Abstract

Schrödinger equations in functional derivatives are solved via quantized Galerkin limit of antinormal functional Feynman integrals for Schrödinger equations in partial derivatives.

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Thus there arise an interpretation of the second quantization problems as quantum mechanics problems with infinitely many degrees of freedom and a natural desire to approximate these problems via problems with finite, but large, number of degrees of freedom.

F.A. Berezin ([3], Introduction).

Introduction

The very first stationary functional derivatives Schrödinger equation was introduced in 1928 by P. Jordan and W. Pauli (Zur Quantumelectrodynamik ladungsfreier Felder, Zeitung für Physik, Vol. 47):

For wave functionals $F(\phi(x))$ of massless scalar fields $\phi(x), x \in \mathbb{R}$,

$$
- \left( \frac{\hbar}{4\pi} \right)^2 \int dx \left[ \frac{\delta^2}{\delta \phi(x)^2} + c^2 \left( \frac{d\phi(x)}{dx} \right)^2 \right] F(\phi(x)) = \lambda F(\phi(x)). \quad (1)
$$
There was a vivid discussion of "Volterra mathematics" between W. Heisenberg, P. Jordan, and W. Pauli. However until now there has been no sound mathematical progress in solution of such equations. Perturbation and lattice approximations do not converge in meaningful examples.

Moreover, according to P. Dirac ("Lectures on quantum field theory" Yeshiva University, N.Y. 1966, Section “Relationship of the Heinseberg and Schrödinger Pictures”),

\[ \text{The interactions that are physically important in quantum field theory are so violent that they will knock any Schrödinger state vector out of Hilbert space in the shortest possible time interval.} \]

\[ [...] \text{ It is better to abandon all attempts at using the Schrödinger picture with these Hamiltonians.} \]

\[ [...] \text{ I don’t want to assert that the Schrödinger picture will not come back. In fact, there are so many beautiful things about it that I have the feeling in the back of my mind that it ought to come back. I am really loath to have to give it up.} \]

Heisenberg partial derivatives equations for interacting quantized fields are non-linear. In contrast, Schrödinger equations for states are linear. Presumably they may be solved by well developed Hilbert space methods.

Unfortunately, in the second quantization formalism, a “violent” Schrödinger operator is not densely defined in the Fock space (see [15], vol. II, Chapter X). For the sake of operator methods one needs to apply cutoffs.

This article proposes a rigorous mathematical theory of Schrödinger functional differential operators with combined ultraviolet and infrared cutoffs:

- Section 1 is a convenient review of infinite dimensional distributions.

- Section 2 begins with a rigorous treatment of functional derivative operators. Theorem 2.2 asserts a lower bound for cutoff Hamiltonian functional derivative operators defined by classical Hamiltonians bounded from below. The coherent states matrix elements of the corresponding evolution operators are given the form of antinormal functional Feynman integral (Theorem 2.3).

- Section 3 introduces a quantized infinite dimensional Galerkin approximation of cutoff functional derivative equations by partial derivative equations. shows that this Feynman integral is a double limit of finite dimensional Gaussian integrals (Theorem 3.1). Thus we have a convergent computational scheme in pseudo-Euclidean space, a viable alternative to lattice approximations.
The original antinormal Feynman integral, based on Chernoff’s product formula, was introduced by J. Klauder and B. Scagerstam (see \[11\], page 69). Another, based on an infinite-dimensional symbolic calculus, has been used in \[7\] to solve non-cutoff functional Schrödinger equations with integrable infinite-dimensional Hamiltonians.

Here the functional Feynman integral is rigorously defined as a limit of Klauder-Scagerstam integrals associated with approximating finite-dimensional Hamiltonians.

In the text the triangles $\triangleright$ and $\triangleleft$ mark the beginning and the end of a proof.

1 Review of infinite-dimensional distributions

1.1 Bosonic Fock representations

Let $\mathcal{H}$ be a complex (separable) Hilbert $\ast$-space with a given complex conjugate isometric involution $\phi \mapsto \phi^\ast$. The $\ast$-subspaces of $\mathcal{H}$ are invariant under the conjugation.

The Hermitian inner product of $\alpha$ and $\beta$ is denoted $\alpha^\ast \beta$. It is complex conjugate linear in $\alpha^\ast$ and linear in $\beta$.

The real part of $\mathcal{H}$ is the real Hilbert subspace $\mathcal{RH} = \{ \rho \in \mathcal{H} : \rho^\ast = \rho \}$, and the imaginary part of $\mathcal{H}$ is the real Hilbert subspace $\mathcal{IH} = \{ i\pi \in \mathcal{H} : \pi \in \mathcal{RH} \}$.

Since any $\phi = \rho + i\pi$ with $\rho = (\phi + \phi^\ast)/2, \pi = (\phi - \phi^\ast)/2i$, the $\ast$-space is the direct orthogonal sum of the real part $\mathcal{RH}$ and the imaginary part $\mathcal{IH}$. Along with $\phi^\ast = \rho - i\pi$ this implies that a choice of the involution $\ast$ is uniquely defined by the choice of the real part $\mathcal{RH}$.

An operator $\alpha$ in $\mathcal{H}$ is real if it commutes with the involution $\ast$.

Let $\mathcal{H}^{\ast}$ denote the antidual Hilbert space of $\mathcal{H}$ with respect to the Hermitian form $\phi^\ast \psi$.

The Hilbert space $\mathcal{H} \times \mathcal{H}^\ast$ carries the conjugation $(\alpha, \beta^\ast) = (\beta, \alpha^\ast)$. The corresponding real part $\mathcal{R}$ is the antidiagonal $\{ (\phi, \phi^\ast) : \phi \in \mathcal{H} \}$. The isometry $\phi \mapsto (1/\sqrt{2})(\phi, \phi^\ast)$ is a representation of $\mathcal{H}$ as a real Hilbert space.

A Fock representation of bosonic canonical commutation relations over $\mathcal{H}$ is described by

1. A Fock Hilbert $\ast$-space $\mathcal{F} = \mathcal{F}(\mathcal{H})$;

2. Two families of creators $\mathcal{F}^+ (\phi)$ and annihilators $\mathcal{F}^- (\phi)$ which are linear unbounded operators in $\mathcal{F}$ with a common invariant dense $\ast$-domain $\mathcal{P}$ in $\mathcal{F}$ such that $\mathcal{F}^+ (\phi)$ and $\mathcal{F}^- (\phi)$ are complex linear with respect to $\phi \in \mathcal{H}$ and the Hermitian adjoint $[\mathcal{F}^+ (\phi)]^\dagger = \mathcal{F}^- (\phi^\ast)$.
3. The unit \emph{vacuum} vector $F_0$ in $\mathcal{RP}$.

4. $\mathcal{F}^{-}(\phi)F_0 = 0$ for any $\phi$; and $\mathcal{P}$ the linear span of the \emph{power} Fock vectors
\begin{equation}
\mathcal{F}^{+}(\phi^*)^nF_0, \ \phi \in \mathcal{H}, \ n = 0, 1, 2, \ldots \tag{2}
\end{equation}

5. The commutators of creators and annihilators and satisfy the \emph{canonical Fock commutation relations} on $\mathcal{P}$:
\begin{equation}
[\mathcal{F}^{-}(\alpha^*), \mathcal{F}^{+}(\beta)] = \alpha^*\beta, \ \ [\mathcal{F}^{+}(\alpha), \mathcal{F}^{+}(\beta)] = 0 = [\mathcal{F}^{-}(\alpha), \mathcal{F}^{-}(\beta)]. \tag{3}
\end{equation}

The polarization formula
\begin{equation}
\mathcal{F}^{+}(\phi_1)\ldots\mathcal{F}^{+}(\phi_n)F_0 = \frac{1}{2^n n!} \sum_{\sigma_1\ldots\sigma_n} \mathcal{F}^{+}(\sigma_1\phi_1 + \ldots + \sigma_n\phi_n)^nF_0, \tag{4}
\end{equation}
where $2^n$ coefficients $\sigma_j \in \{1, -1\}$, $j = 1, \ldots, n$, shows that $\mathcal{P}$ is the complex span of the product Fock vectors $\mathcal{F}^{+}(\phi_1)\ldots\mathcal{F}^{+}(\phi_n)F_0$.

\textbf{Remark 1.1} For a given $\mathcal{H}$ all Fock representations $(\mathcal{F}, F_0, \mathcal{F}^{+}, \mathcal{F}^{-})$ are unitary equivalent.

The Segal functor $\Gamma$ (see [1], Chapter I) assigns to an operator $o$, with a dense domain $\mathcal{H}'$ in $\mathcal{H}$, an operator $\mathcal{F}(o)$ in $\mathcal{F}$, with the dense domain $\mathcal{P}'$, spanned by $\mathcal{F}^{+}(\phi')^n, \ \phi' \in \mathcal{H}', \ n = 1, 2\ldots$ such that
\begin{equation}
\mathcal{F}(o)F_0 = F_0, \ \mathcal{F}(o)[\mathcal{F}^{+}(\phi')^nF_0] = \mathcal{F}^{+}(o\phi')^nF_0. \tag{5}
\end{equation}

Then
\begin{itemize}
\item If $o_2o_1$ exists on a dense domain in $\mathcal{H}$, then $\mathcal{F}(o_2o_1) = \mathcal{F}(o_1)\mathcal{F}(o_2)$.
\item $\mathcal{F}(1) = 1, \ \mathcal{F}(o^{-1}) = \mathcal{F}(o)^{-1}, \ \mathcal{F}(o^\dagger) = \mathcal{F}(o)^\dagger$.
\item If $o$ is a unitary operator, then $\mathcal{F}(o)$ is unitary as well.
\item If $o$ is an orthogonal projector, then $\mathcal{F}(o)$ is an orthogonal projector too.
\item $\mathcal{F}(o)$ is non-negative if $o$ is a non-negative operator.
\item If $o$ is an (essentially) selfadjoint operator, then $\mathcal{F}(o)$ is essentially self-adjoint.
\end{itemize}

The tangential Fock functor $d\Gamma$ assigns to the operator $o$ an operator $\hat{\mathcal{F}}(o)$ defined on $\mathcal{F}(\mathcal{H}')$ by
\begin{equation}
\hat{\mathcal{F}}(o)F_0 = 0, \ \hat{\mathcal{F}}(o)[\mathcal{F}^{+}(\phi')^nF_0] = n\mathcal{F}^{+}(o\phi')\mathcal{F}^{+}(\phi')^{n-1}F_0. \tag{6}
\end{equation}

Thus
• If the commutator $[o_2, o_1]$ exists on a dense domain in $H$, then $\hat{F}([o_2, o_1]) = [\hat{F}(o_1), \hat{F}(o_2)]$.

• If $o \geq 0$, then $\dot{F}(o) \geq 0$.

• If $o$ is an (essentially) self-adjoint operator, then $\hat{F}(o)$ is essentially self-adjoint in $\mathcal{F}$.

• If $o$ generates a strong (semi)group $\exp(-to)$ with real parameter $t$, then $\hat{F}(o)$ generates the strong (semi)group $\mathcal{F}(\exp(-to))$.

1.2 Functional Fock representations

1.2.1 Integration on $\mathbb{R}H$

Let $p$ denote an orthogonal projector in $H$ of finite rank $r(p)$. We assume that $p$ commute with the conjugation. Then $p$ is the orthogonal projector of $\mathbb{R}H$ onto $\mathbb{R}pH$ as well.

The functional integral $\int d\xi F(\xi)$ of a functional $F$ on $\mathbb{R}H$ is the limit of the normalized Lebesgue integrals over the finite dimensional spaces $p\mathbb{R}H$ as $p$ converges to the unit operator $1$, i.e., for every $\epsilon > 0$ there exists $p_\epsilon$ such that if $p\mathbb{R}H \supset p_\epsilon \mathbb{R}H$ then the absolute value

$$|(2\pi)^{-r(p)/2} \int d(p\xi) F(p\xi) - (2\pi)^{-d(p_\epsilon)/2} \int d(p_\epsilon \xi) F(p_\epsilon \xi)| < \epsilon. \quad (7)$$

The finite-dimensional renormalizations are chosen so that the Gaussian functional integral

$$\int d\xi e^{-\|\xi\|^2/2} = 1. \quad (8)$$

A flag $(p_n) = p_1 < \ldots < p_n < \ldots$ is an increasing sequence of orthogonal $\ast$-projectors such that the union $\cup(p_n H)$ is dense in $H$.

**Proposition 1.1** For any flag $(p_n)$

$$\lim_{n \to \infty} (2\pi)^{-d(p_n)} \int d(p_n \xi) F(p_n \xi) = \int d\xi \ast d\xi F(\xi). \quad (9)$$

Since $\cup(p_n H)$ is dense in $H$, for any positive $\epsilon$ there exists a projector $p_n$ that has the same rank as $p_\epsilon$ and the (constant) Jacobian of the orthogonal projection of $p_\epsilon H$ onto $p_n H$ is within $\epsilon$ from 1. Now for any $p_m > p_n$, the orthogonal projections from $(p_\epsilon + p_m)H$ onto $p_m H$ have the same Jacobian.

Thus the integrals in the left hand side of the equation are within $\epsilon$ from the integral on the right hand side. \(<\)
**Proposition 1.2** The functional integral has the following properties:

1. \( \int d\xi F(\xi) \) is a positive linear functional on the space of integrable functionals \( G \).

2. The integral over a product Hilbert space is equal to the iterated functional integrals.

3. Integration by parts: Let \( D_\eta F \) denote the directional derivative of \( F \) in the direction of \( \eta \in \mathbb{R}H \). Then

\[
\int d\xi F(\xi) D_\eta G(\xi) = -\int d\xi D_\eta F(\xi) G(\xi)
\]

if \( FG \to 0 \) as the scalar product \( \xi \to \infty \) and both integrals exist.

4. The functional integral is invariant under translations and orthogonal transformations in \( H \).

These properties follow directly from the corresponding properties of finite-dimensional Lebesgue integrals. (For the integration by parts note that for given \( \xi \) in \( H \) we may choose the projectors \( p' \) such that \( p\xi = \xi \).)

### 1.2.2 Gauss Fock representation on \( \mathbb{R}H \)

In the Gauss (or real wave) Fock representation on \( \mathbb{R}H \) (compare with [9] and [11])

- \( \mathcal{F}(H) \) is the Gauss Hilbert space \( \mathcal{G}(H) \) the completion of the space \( \mathcal{L}^2(\mathbb{R}H, e^{-\|\phi\|^2/2}) \) of functionals \( F = F(\xi), \xi \in \mathbb{R}H \), with \( F^*(\xi) = \overline{F(\xi)} \) (complex conjugation) and the Hermitian product

\[
F^*G = \int d\xi e^{-\|\xi\|^2/2} F^*(\xi) G(\xi);
\]

- the vacuum vector \( F_0 = 1 \);

- the annihilators and creators are

\[
\mathcal{F}^- (\phi^*) F(\xi) = \partial^* F(\xi), \mathcal{F}^+(\phi) F(\phi) = (-\partial F(\xi) + \xi \phi) F(\xi).
\]

Occasionally we denote \( \mathcal{F}^- \) and \( \mathcal{F}^+ \) in \( \mathcal{G} \) as \( \mathcal{G}^+ \) and \( \mathcal{G}^- \).
1.2.3 Bargmann Fock representation on $\mathcal{H}$

Since $\mathcal{H}$, as a real Hilbert space, has been identified with $\mathbb{R}(\mathcal{H} \times \mathcal{H}^*)$, the functional integral $\int d\phi^* d\phi F(\phi, \phi^*)$ is defined as the limit of Lebesgue integrals over finite dimensional $\star$-subspaces $p\mathcal{H}$.

Now the normalizing constants are $\pi^{-\dim(p)}$ so that

$$\int d\phi^* d\phi e^{-\phi^* \phi} = 1. \quad (13)$$

The Hermitian adjoint Cauchy-Riemann operators on $\mathcal{H}$ are

$$\partial_\zeta = (1/2)(D_{\Re \zeta} - iD_{\Im \zeta}), \quad \partial^*_\zeta = (1/2)(D_{\Re \zeta} + iD_{\Im \zeta}), \quad (14)$$

the former being linear and the latter anti-linear in $\phi$.

A continuous functional $F$ on $\mathcal{H}^\infty$ is an entire functional if $\partial^*_\zeta F(\phi, \phi^*) = 0$ for all $\phi$ and $\zeta$. Notationally $F = F(\phi)$.

A continuous functional $F$ on $\mathcal{H}^\infty$ is a anti-entire functional if $\partial_\zeta F(\phi, \phi^*) = 0$ for all $\phi$ and $\xi$. Notationally $F = F(\phi)$.

In the Bargmann (or complex wave) Fock representation on $\mathcal{H}$ (see [2])

- the Fock space $\mathcal{F}(H)$ is the Bargmann space $\mathcal{B}(\mathcal{H})$, the (closed) subspace of anti-entire functionals $F = F(\phi^*)$ in $L^2(\mathcal{H}^* \times \mathcal{H}, e^{-\phi^* \phi} d\phi d\phi^*)$;
- The conjugation $F^*(\phi^*) = \overline{F([\phi^*]^*)}$
- The vacuum functional $F_0 = 1$;
- The annihilation and creation operators are

$$\mathcal{F}^-(\zeta^*)F(\phi^*) = \partial_\zeta F(\phi^*), \quad \mathcal{F}^+(\zeta)F(\phi^*) = (\phi^* \zeta)F(\phi^*). \quad (15)$$

Occasionally we denote $\mathcal{F}^-$ and $\mathcal{F}^+$ in $\mathcal{B}$ as $\mathcal{B}^+$ and $\mathcal{B}^-$.

1.3 Bargmann-Segal transform

The coherent functionals $F_\alpha$ on $\mathcal{H}$ are

$$F_\alpha = \sum_{n=1}^{\infty} \frac{1}{n!} F^+(\alpha)^n F_0, \quad \alpha \in \mathcal{H}. \quad (16)$$

By induction, Fock commutation relations imply

$$[\mathcal{F}^+(\alpha)^m F_0]^*[\mathcal{F}^-(\beta)^n F_0] = \delta_{mn}(\alpha^* \beta)^m, \quad (17)$$
so that
\[ F_\alpha^* F_\beta = F_{\alpha^* \beta}, \] (18)
Then \( F_\alpha^* F_\alpha < \infty \) so that \( F_\alpha \in \mathcal{F} \) and the correspondence between \( \alpha \) and \( F_\alpha \) is one to one. Note that in Bargmann space \( \mathcal{B} \) the coherent functionals \( F_\alpha(\psi^*) = \exp(\psi^* \alpha) \).
The entire functional \( F(\alpha) = F^* F_\alpha \) of the argument \( \alpha \in \mathcal{H} \) is the Bargmann-Segal transform of \( F \in \mathcal{F} \).
The following proposition is fundamental (see [4]):

**Proposition 1.3** The coherent functionals \( F_\alpha, \alpha \in \mathcal{H} \), form a continual orthogonal basis of \( \mathcal{F} \) as follows:

1. Every \( F \in \mathcal{F} \) has the weak expansion in \( F_\alpha \):
\[
F(\beta^*) = \int d\alpha^* d\alpha e^{-\alpha^* \alpha} e^{\beta^* \alpha} F(\alpha^*). \tag{19}
\]

2. If \( G, F \in \mathcal{F} \) then
\[
G^* F = \int d\alpha^* d\alpha G^*(\alpha) F(\alpha^*). \tag{20}
\]
in particular, \( \|F\|^2 = \int d\alpha^* d\alpha |F(\alpha^*)|^2 \) so that the Bargmann-Segal transform is one to one.

\[ \triangleright \] The first part follows from the weak convergence of functional integrals
\[
F^* \int d\alpha^* d\alpha e^{\beta^* \alpha} F_{\beta^*}(\alpha^*) = \int d\alpha^* d\alpha e^{\beta^* \alpha} (F^* F_{\beta^*})(\alpha^*). \tag{21}
\]
By the same token the second part follows from the first. In both cases the commutation with integration is justified by integration over finite dimensional subspaces with conjugation in \( \mathcal{H} \).

\[ \triangleright \]

### 1.4 Fock Sobolev scales

Let \( o \) be a real (i.e., commuting with the conjugation) selfadjoint non-negative operator in \( \mathcal{H} \) with the discrete spectrum \( \{\lambda_k : k = 1, 2, \ldots\} \). In particular each \( \lambda_k \) has a finite multiplicity \( m_k \). Assume that the operator \((1 + o)^{-p}\) has finite trace for some \( p > 0 \).

**Examples**: the harmonic oscillator operator \( -\nabla^2 + x^2 \) in \( \mathcal{H} = L^2(\mathbb{R}^n) \), positive globally hypoelliptic operators in \( L^2(\mathbb{R}^n) \) (see [14]), Beltrami Laplacians, or, more generally, positive elliptic operators in \( L^2(M) \) on compact Riemann manifolds \( M \) (see [14]).
For $s \leq 0$, denote by $\mathcal{H}^s$ the Hilbert $\ast$-space of all $\phi \in \mathcal{H}$ with the Hermitian product $\phi^\ast (1 + o)^s \psi$. Its antidual $\mathcal{H}^{-s}$ with respect to the basic Hermitian form $\alpha^\ast \beta$ is the completion of $\mathcal{H}$ with respect to the Hermitian product $\phi^\ast (1 + o)^{-s} \psi$.

If $s' > s$, then $\mathcal{H}^s$ is a dense subspace of $\mathcal{H}^{s'}$, and the inclusions are continuous. Therefore, by definition, the family of the Hilbert $\ast$-spaces $\mathcal{H}^s$, $-\infty < s < \infty$ is a Sobolev scale generated by $o$.

The intersection $\mathcal{H}^\infty = \bigcap_s \mathcal{H}^s$ is the Frechet space with the topology of simultaneous convergence with respect to all Hilbert norms. Since $(1 + o)^{-p}$ has finite trace for some $p > 0$, the space $\mathcal{H}^\infty$ is nuclear.

Its antidual with respect to the basic Hermitian form $\alpha^\ast \beta$ is the strict inductive limit (see [15], Section V.4) $\mathcal{H}^{-\infty} = \bigcup_s \mathcal{H}^s$, a nuclear space again.

Thus we get a Gelfand triple

$$\mathcal{H}^\infty \subset \mathcal{H} \subset \mathcal{H}^{-\infty}. \quad (22)$$

Similarly, starting with the Fock quantized $\mathcal{F}$ and $\mathcal{F}(o)$ instead of $\mathcal{H}$ and $o$, we get the Fock scale of Hilbert spaces $\mathcal{F}^s$ and the triple (see [12] and [1], Section 7.3)

$$\mathcal{F}^\infty \subset \mathcal{F} \subset \mathcal{F}^{-\infty}. \quad (23)$$

Using $\mathcal{F}$ and $\mathcal{F}(o)$ instead of $\mathcal{F}(o)$ we obtain the tangential Fock scale of the Hilbert spaces $\mathcal{F}^s$ and the triple

$$\mathcal{F}^\infty \subset \mathcal{F} \subset \mathcal{F}^{-\infty}. \quad (24)$$

Note that the product states $[\prod_{j=1}^n \mathcal{F}^+(\phi_j)]F_0$ belong to $\mathcal{F}^\infty$ if and only if all $\phi_j \in \mathcal{H}^\infty$.

**Example** Consider the Fock representation over $\mathcal{H} = \mathbb{C}^d$ with the standard complex conjugation:

$$\mathcal{F} = L^2(\mathbb{R}^d), \quad F_0 = (4\pi)^{-1} e^{-u^2/4}$$

$$\mathcal{F}^-(u - iv)F(x) = (xu/2 - \partial_u)F(x), \quad \mathcal{F}^+(u + iv)F(x) = (xu/2 + \partial_u)F(x)$$

Let $o = 1$. Then (see [13], Section 6.2) $\mathcal{F}^\infty(\mathbb{C}^d)$ consists of all real analytic functions $F(x)$ such that for any $\epsilon > 0$

$$e^{(1/4 - \epsilon)x^2}F \in L^1(\mathbb{R}^d), \quad (25)$$

and the Fourier transform $G(z)$ of $e^{-x^2/4}F$ satisfies

$$|G(z)| \prec \exp[(1/2 - \epsilon)z^2] \quad (26)$$

for all $z \in \mathbb{C}^d$. 9
On the other hand, $\mathcal{F}^\infty(\mathbb{C}^d)$ is Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$ (see [12], p.185).

Note that, if $\phi \in \mathcal{H}^\infty$, then $\mathcal{F}^\infty$ and $\mathcal{F}^\infty$ are invariant for $\mathcal{F}^+(\phi)$ and $\mathcal{F}^-(\phi^*)$.

Also, since $\mathcal{H}^\infty$ is invariant for pseudodifferential operators on $X$ (see [14], Sections 4.3 and 23.2), they are invariant, correspondingly, for quantized and tangentially quantized pseudodifferential operators.

**Remark 1.2** Under the unitary equivalence of Fock representations, $\mathcal{F}^\infty$ and $\mathcal{F}^{-\infty}$ correspond to $(\mathcal{H}^\infty)$ and $(\mathcal{H}^{-\infty})^\ast$ in Hida’s white noise calculus (see [13]).

The spaces $\mathcal{F}^\infty$ and $\mathcal{F}^{-\infty}$ correspond to the maximal Kristensen-Mejlbo-Poulsen space and their space of temperate distributions (see [12]).

Thus their properties are immediately translated into the corresponding properties of $\mathcal{F}^\infty$ and $\mathcal{F}^{-\infty}$ and $\mathcal{F}^{\infty}$ and $\mathcal{F}^{-\infty}$.

In particular, $\mathcal{H}^\infty$ and $\mathcal{H}^{-\infty}$ are nuclear spaces.

However, the spaces $\mathcal{F}^{\infty}$ and $\mathcal{F}^{-\infty}$ are not nuclear. Still they have the Montel property: their closed bounded subsets are compact. In particular, these spaces are reflexive.

**Proposition 1.4** The map of $(\phi, F)$ to $F^{-}(\phi^*)F$ is continuous

(a) from $\mathcal{H}^{-\infty} \times \mathcal{F}^\infty$ to $\mathcal{F}^\infty$ (and, by duality, from $\mathcal{H}^{-\infty} \times \mathcal{F}^{-\infty}$ to $\mathcal{F}^{-\infty}$); (b) from $\mathcal{H}^{-\infty} \times \mathcal{F}^\infty$ to $\mathcal{F}^{\infty}$ (and, by duality, from $\mathcal{H}^{-\infty} \times \mathcal{F}^{-\infty}$ to $\mathcal{F}^{-\infty}$).

The first half of part (a) follows from Theorem 4.3.9 in [13] for annihilators $G(k_{0,1})$.

The first half of part (b) from the proof of Theorem 4.3.12 in [13] for its annihilators $G(k_{1,0})$. ⊳

### 2 Cutoff functional derivatives operators

#### 2.1 Functional derivatives operators

From now on we assume that $\mathcal{H} = L^2(X)$, where $X$ is either a compact Riemann manifold, or the Euclidean space $\mathbb{R}^d$ with the Riemannian measures $dx$.

The scaling operators $o$ are correspondingly Beltrami Laplacian, and harmonic oscillator operators.

Then $\mathcal{H}^\infty$ is, correspondingly, the space $C^\infty(X)$ of infinitely differentiable functions on the compact Riemann manifold $X$, and the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ (see [14], Section 7 and Section 25).

Since delta-functions $\delta_x = \delta^*_x$ belong to $\mathcal{H}^{-\infty}$, the operators $\mathcal{F}^-_x = \mathcal{F}^-(\delta_x)$ are well defined, and, by Proposition 1.4, are continuous in $\mathcal{F}^\infty$. 

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Let
\[ F_{x[n]} = F_{x_1} \ldots F_{x_n}, \quad (x_1 \ldots x_n \in X^n). \] (27)

By Proposition 1.4, for given \( G, F \in \mathcal{F}^\infty \), the matrix element \( G^* \hat{F}_{x[n]}^- F \) belongs to \( \mathcal{H}^{\infty} \).

If a Wick symbol \( W_{k,l} \in (\mathcal{H}^{\infty})^{k+l} \) then \( W_{k,l}(F^* \hat{F}_{x[k]}^- \hat{F}_{y[l]}^- F) \) is a continuous bilinear form on \( \mathcal{F}^\infty \). It defines a continuous functional derivatives operator \( \hat{W}_{k,l} \) from \( \mathcal{F}^\infty \) to \( \mathcal{F}^{-\infty} \) which heuristically is
\[ \hat{W}_{k,l} = \int dx_k dy_l W_{k,l}(x_k, y_l) \hat{F}_{x[k]}^+ \hat{F}_{y[l]}^- \]. (28)

A finite sum \( \hat{W} = \sum_{k+l \leq m} \hat{W}_{k,l} \) is a functional derivatives operator of order \( m \) from \( \mathcal{F}^\infty \) to \( \mathcal{F}^{-\infty} \).

A functional derivatives operator is local if the distributions \( W_{k,l} = W_{k,l}(x) \delta(X) \), where \( X \) is identified with the submanifold \( \{(x, x, \ldots, x)\} \subset X^{k+l} \), and \( W_{k,l}(x) \in \mathcal{H}^{\infty} \). Then
\[ \hat{W} = \int dx \sum_{k \leq m} W_{k,l}(x)(\mathcal{F}_x^+)^k(\mathcal{F}_x^-)^l. \] (29)

Annihilators \( \hat{G}^- (\phi^*) \) in Gauss Fock representation are directional derivatives \( D_{\delta_x} \).

Since delta-functions \( \delta_x = \delta^*_x \) belong to \( \mathcal{H}^{-\infty} \) it is possible to consider the functional derivative \( D_x F(\phi) = D_{\delta_x} F(\phi) \). Indeed, by Theorem 4.2.4 from [13], this directional derivative exists for \( F \in \mathcal{G}^\infty \) and coincides with \( D_x = G^- (\delta_x) \).

On the other hand, a translation is not a continuous operator in \( \hat{G}^\infty \) so that \( D_{\delta_x} \) does not belong in this space. However, by proposition 1.4, it may be continuously extended as \( \hat{G}^- (\delta_x) \). It is the limit of a family \( D_{\eta} \) as \( \eta \in \mathcal{H}^{\infty} \) converge to \( \delta_x \in \hat{\mathcal{H}}^{\infty} \). By proposition 1.4, this is a continuous operator in \( \hat{G}^\infty \) denoted again as \( D_x \).

By Proposition 1.4, the Hermitian adjoints of the functional derivatives \( D_x \),
\[ D_x^\dagger F(\phi) = (-D_{\delta_x} + \phi(x)) F(\phi) \] (30)
are continuous operators in \( \hat{G}^{-\infty} \).

Thus the multiplication with \( \delta_x \), which is the operator \( D_x + D_x^\dagger \), is continuous from \( \mathcal{G}^\infty \) to \( \hat{G}^{-\infty} \).
The coherent state quadratic form $F^*_\alpha \hat{W} F_\beta$ in Bargmann space $\mathcal{B}$ is

$$F^*_\alpha \left[ \int dx[k] dy[l] W_{k,l}(x[k], y[l]) \prod_{i=1}^l (B^+(\delta_{y_i}) \prod_{j=1}^k (B^-(\delta_{x_j}) \right] F_\beta$$

$$\int dx[k] dy[l] W_{k,l}(x[k], y[l]) \prod_{i=1}^l (B^-(\delta_{y_i}) F_\alpha^* \prod_{j=1}^k (B^-(\delta_{x_j}) F_\beta$$

$$\int dx[k] dy[l] W_{k,l}(x[k], y[l]) \int d\xi^* d\xi e^{-\xi^* \xi} \prod_{i=1}^l \alpha^* (y_i) e^{\alpha^* \xi} \prod_{j=1}^k \beta(x_j) e^{\xi^* \beta(x_j)}$$

$$= W_{k,l}(\alpha^*, \beta) e^{\alpha^* \beta},$$

where the Wick symbol of $\hat{W}_{k,l}$

$$W_{k,l}(\alpha^*, \beta) = \int dx[k] dy[l] W_{k,l}(x[k], y[l]) \prod_{i=1}^l \alpha^* (y_i) \prod_{j=1}^k \beta(x_j)$$

is a continuous holomorphic polynomial of order $(k, l)$ on $\mathcal{H}^{\infty} \times \mathcal{H}^{\infty}$.

A functional derivatives operator of order $n$ is a finite sum of operators $\hat{W} = \sum_{k+l\leq n} \hat{W}_{k,l}$ with the Wick symbol $W(\alpha^*, \beta) = \sum_{k+l\leq n} W(\alpha^*, \beta)$.

The correspondence between functional derivatives operators and the Wick symbols is one to one.

The continuous complex analytic polynomial $W(\alpha^*, \beta)$ is uniquely defined by its Taylor coefficients at the origin $(0, 0)$. Therefore, the correspondence between $W(\alpha^*, \beta)$ and the restricted Wick symbols $W(\alpha^*, \alpha)$ is one to one. The restricted Wick symbols are continuous (real analytic) polynomials on $\mathcal{H}^{\infty}$.

Real valued restricted Wick symbols are Hamiltonian functionals, and the corresponding operators are Hamiltonian operators.

2.2 Cutoff functional derivatives operators

A functional derivatives operator $\hat{H}$ is a cutoff if its Hamiltonian functional $W(\alpha^*, \alpha)$ has the (unique) continuous extension from $\mathcal{H}^{\infty}$ to $\mathcal{H}^{-\infty}$. This is equivalent to inclusion of the terms $W_{k,l}(x[k], y[l]) \in (\mathcal{H}^{\infty})^{k+l}$ (see [13], the characterization theorem 3.6.2 ); in particular, the polynomial $W(\alpha^*, \alpha)$ belongs to $\mathcal{G}(\mathcal{H}^{\infty} \times \mathcal{H})$.

The Hamiltonian functionals and their derivatives $D_\phi$ in the directions of $\phi \in \mathcal{G}^{\infty}$ are, actually, integrable with respect to the Gauss measure on $\mathcal{H}^{-\infty}$.

A cutoff operator $\hat{H}$ is a continuous operator in $\mathcal{G}^{\infty}$. Thus it has a dense domain in $\mathcal{G}$. Its Hermitian adjoint $\hat{H}^\dagger$ is also cutoff of the same order with complex conjugate Wick symbol $\hat{H}$. Thus cutoff operators are closable.
A cutoff operator $\hat{\mathcal{H}}$ is symmetric on $\mathcal{G}^\infty$ if and only if its Hamiltonian functional is real-valued.

**Theorem 2.1** Any functional derivatives operator $\hat{\mathcal{H}}$ is the strong limit of a sequence of cutoff operators $\hat{\mathcal{H}}_n$.

It suffices to consider operators $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{k,l}$. Separately for $X = \mathbb{R}^d$ and for $X$, a compact Riemann manifold, we construct a sequence of cutoff Wick symbols $W_n$ from $(\mathcal{H}^\infty)^{k+l}$ which converges to $C$ in $(\mathcal{H}^{-\infty})^{k+l}$ as $n \to \infty$.

Then the cutoff operators $\hat{\mathcal{H}}_n$ strongly converge to $\hat{\mathcal{H}}$ in the topological operator space $\mathcal{L}(\mathcal{F}^\infty, \mathcal{F}^{-\infty})$.

**Case of $X = \mathbb{R}^d$.**

Let $\chi, \kappa$ be non-negative infinitely differentiable functions with compact support on $\mathbb{R}^d$ such that $\chi(0) = 1$ and $\int dy \kappa(y) = 1$.

For every $x \in \mathbb{R}^d$ the sequence of $\kappa_{n,x}(y) = n^d \kappa(ny - x)$ from $\mathcal{S}(\mathbb{R}^d)$ converges to the delta function $\delta_x$ in $\mathcal{S}'(\mathbb{R}^d)$ as $n \to \infty$. At the same time the sequence of $\chi_n(x) = \chi(x/n)$ converges to 1 in $\mathcal{S}'(\mathbb{R}^d)$ as $n \to \infty$.

Now the sequence of the cutoff Wick symbols from $\mathcal{S}(\mathbb{R}^d)^{k+l}$

$$W_n(x[k+l]) = \prod_{1}^{k+l} \chi(x_i/n) \int \prod_{1}^{k+l} dy_i \kappa_{n,x_i}(y_i)c(y[i[k+l]])$$

(32)

converges to $C(x[k+l])$ in $\mathcal{S}'(\mathbb{R}^d)^{k+l}$ as $n \to \infty$.

**Case of a compact Riemann manifold $X$.**

In this case $\chi_n(x) = 1$ for all $x$.

Since the geodesic exponential mapping is one to one on an open neighborhood $W$ of the diagonal in $X \times X$, for every pair $(x, y) \in W$ there is a unique geodesic curve from $x$ to $y$ in $X$. Let $sy$ denote the point at the geodesic distance $s$ from $x$.

Choose a non-negative infinitely differentiable function $\kappa(x, y)$ on $X \times X$ with support in $W$ such that $\int dy \kappa(x, y) = 1$ for all $x$. Let $\kappa_x(y) = \kappa(x, y)$.

Then the sequence of the cutoff Wick symbols

$$W_n(x[k+l]) = \prod_{1}^{k+l} dy_i \kappa_{n,x_i}(y_i)c(y[i[k+l]])$$

(33)

belong to $(\mathcal{H}^\infty)^{k+l}$ and converges to the Wick symbol $W(x[k+l])$ in the topology of $(\mathcal{H}^{-\infty})^{k+l}$ as $n \to \infty$.

A continuous polynomial $A(\phi^*, \phi) \in \mathcal{G}(\mathcal{H}^* \times \mathcal{H})$ is the antinormal symbol of $\hat{\mathcal{W}}$ if the coherent state matrix of $\hat{\mathcal{W}}$ in the Bargmann Fock space $\mathcal{B}$

$$F_{\alpha}^\dagger \hat{\mathcal{W}} F_{\beta} = \int d\phi^* d\phi e^{-\phi^* \phi} e^{\alpha^* \phi} A(\phi^*, \phi)e^{\phi^* \beta}.$$ 

(34)
The functional $e^{\alpha\phi}$ of $(\alpha, \phi)$ is the integral kernel of the identity operator on the closed Bargmann subspace $\mathcal{B}$ of anti-entire functionals $A(\phi^*)$ in the Gauss Hilbert space $\mathcal{G}$, and is orthogonal to all entire functionals $E(\phi)$. Therefore $e^{\alpha\phi}$ is the integral kernel of the orthogonal projector $P$ of $\mathcal{G}$ onto $\mathcal{B}$.

**Theorem 2.2** Let $\hat{W}$ be a local cutoff functional derivative operator.

If the Hamiltonian functional $W(\alpha^*, \alpha)$ is bounded from below on $\mathcal{H}^\infty$ then the Hamiltonian operator $\hat{W}$ is lower bounded on $\mathcal{B}^\infty$.

The Hamiltonian functional

$$W(\alpha^*, \alpha) = e^{-\alpha^*\phi} \int d\phi^* d\phi e^{-\phi^*\phi + \alpha^*\phi + \phi^*\beta} A(\phi^*, \phi)$$

$$= \int d\phi^* d\phi e^{-\alpha^*(\phi - \phi^*)} A(\phi^*, \phi)$$

The Poisson transformation semigroup

$$W(\alpha^*, \alpha; t) = \int d\phi^* d\phi e^{-\alpha^*(\phi - \phi^*)} \frac{1}{t} A(\phi^*, \phi)$$

is the fundamental solution for the diffusion equation

$$(\partial_t - \Delta_G) W(\alpha^*, \alpha; t) = 0, \; t > 0, \; W(\alpha^*, \alpha; 0+) = A(\alpha^*, \alpha), \quad (35)$$

where $\Delta_G = \int dx D^2_x$ is the Gross Laplacian (see [13], Section 5.3).

By theorem 5.2.5 from [13], the Poisson group is a strongly continuous operator semigroup in $\mathcal{G}^\infty$ generated by $\Delta_G$. Note that antinormal symbols of all cutoff Hamiltonian operators belong to $\mathcal{G}^\infty$.

The Gross Laplacian maps continuously $\mathcal{G}^\infty$ into $\mathcal{G}^\infty$ (see [13], Proposition 5.3.2) and, therefore, has a dense domain in $\mathcal{G}$ which includes antinormal symbols of all cutoff Hamiltonian operators.

All above shows that the Hamiltonian functional

$$W(\alpha^*, \alpha) = e^{\Delta_G} A(\alpha^*, \alpha) \sum_{n \geq 0} [(-1)^n / n!] \Delta^n_G A(\alpha^*, \alpha), \quad (36)$$

the latter series being just a finite sum, justifying the heuristic expression for $e^{\Delta_G}$. Now the formal inversion makes sense

$$A(\alpha^*, \alpha) = e^{-\Delta_G} A(\alpha^*, \alpha) = \sum_{n \geq 0} [(-1)^n / n!] \Delta^n_G W(\alpha^*, \alpha) \quad (37)$$
In particular, any cutoff operator $\hat{W}$ has a unique antinormal symbol $A$; that the polynomials $W(\alpha^*, \alpha)$ and $A(\alpha^*, \alpha)$ have the same order; and that the order of the polynomial $W(\alpha^*, \alpha) - A(\alpha^*, \alpha)$ is strictly less than the order of the polynomial $A(\alpha^*, \alpha)$. Since the lower bound of the operator $\hat{W}$ is never less than the lower bound of its antinormal symbol $A$, this completes the proof.

\[ \triangleright \]

### 2.3 Antinormal Feynman integral

By Theorem 2.2, a cutoff operator $\hat{H}$ with the lower bounded Hamiltonian functional $H$ has the Friedrichs extension from $\mathcal{H}^\infty$. Let us preserve the notation $\hat{H}$ for this extension.

**Theorem 2.3** Let $A(\phi^*, \phi)$ be the antinormal symbol of a cutoff operator $\hat{H}$.

Then the coherent state matrix $F^*_{\alpha}e^{-i\hat{H}}F^*_{\beta}$ is equal to

\[
\lim_{N \to \infty} \int \prod_{j=1}^{N} d\phi^*_j d\phi_j \exp \sum_{j=0}^{N} \left[ (\phi_{j+1} - \phi_j)^* \phi_j - iA(\phi^*_j, \phi_j)/N \right] \tag{38}
\]

with $\phi_{N+1} = \alpha$, $\phi_0 = \beta$.

\[ \triangleright \]

As in [11], pp. 69-70, consider the strongly differentiable operator family in $\mathcal{B}$

\[
[O(t)F](\alpha^*) = \int d\phi^* d\phi e^{-\phi^*\phi} e^{\alpha^*\phi} e^{-iA(\phi^*, \phi)t} F(\phi^*) \tag{39}
\]

We have $\|O(t)\| \leq 1$ (since $|e^{-iA(\phi^*, \phi)t}| = 1$), and the strong $t$-derivative $A'(0) = \hat{H}$. Then, by the Chernoff’s product theorem \[9\], the evolution operator

\[
e^{-i\hat{H}}F = \lim_{N \to \infty} [H(1/N)]^N F. \tag{40}\]

The coherent state matrix $F^*_{\alpha}[A(t/N)]^N F_{\beta}$ is the $N$-iterated Gaussian integral over $\mathcal{H}$ which, by the Fubini’s theorem, is equal to the $N$-multiple Gaussian integral over $\mathcal{H}^N$.

\[ \triangleright \]

**Remark 2.1** In the notation $\tau_j = jt/N$, $\phi_{\tau_j} = \phi_j$, $j = 0, 1, 2, \ldots, N$, and $\Delta \tau_j = \tau_{j+1} - \tau_j$, the multiple integral (2) is

\[
\int \prod_{j=1}^{N} d\phi^*_{\tau_j} d\phi_{\tau_j} \exp i \sum_{j=0}^{N} \Delta \tau_j \left[ -i(\Delta \phi_{\tau_j}/\Delta \tau_j)^* \phi_{\tau_j} \right] - A(\phi^*_{\tau_j}, \phi_{\tau_j}) \tag{41}\]

Its limit at $N = \infty$ is a rigorous mathematical definition of the heuristic Hamiltonian Feynman type integral over histories, with the higher derivatives
renormalization $A$ of the Hamiltonian functional $H$, for the coherent state matrix

\[
\int_{\alpha}^{\beta} \prod_{0<\tau<t} d\phi_\tau^* d\phi_\tau \exp i \int_0^t d\tau \left[-i(\partial_\tau \phi_\tau)^* \phi_\tau - A(\phi_\tau^*, \phi_\tau)\right]. \tag{42}
\]

3 Quantized Galerkin approximations

Let $\{p_n\}$ be a flag of finite dimensional orthogonal projectors in $\mathcal{H}^\infty$ (so that $p_n$ is an increasing sequence of orthogonal projectors strongly converging to the unit operator on a dense subspace in $\mathcal{H}$).

Then $\{P_n = \mathcal{G}(p_n)\}$ is the corresponding flag of infinite dimensional quantized orthogonal projectors in $\mathcal{G}$.

For a cutoff Hamiltonian operator $\hat{H}$ in the Gauss space $\mathcal{G}$, the reduced Hamiltonian operators $\hat{H}_n$ are Friedrichs extensions of $P_n \hat{H} P_n$ in $\mathcal{G}$. They are uniformly bounded from below.

**Proposition 3.1** Reduced Hamiltonian operators $\hat{H}_n$ are polynomial partial differential operators in $P_n \mathcal{G}$ with the normal symbol $H(p_n \alpha^*, p_n \alpha)$.

Let $f$ be a complex bounded continuous function on the real axis $\mathbb{R}^+$. Then, by the spectral theorem, for any selfadjoint non-negative operator $T$ in $\mathcal{G}$ the operator $f(A)$ is bounded with the operator norm $\leq \sup |f|$. If a family of such functions $f_t$ depends continuously on a parameter $t$ in a compact $K \subset \mathbb{R}$ then the operator family $f_t(A)$ is uniformly strongly continuous on $K$ with respect to $t$.

**Theorem 3.1** The operators $f_t(\hat{H})$ are strong operator limits of the operators $f_t(\hat{H}_n)$ as $n \to \infty$, uniformly on compact $t \geq 0$-intervals.

> PART I The sequence $\hat{H}_n F$ converges strongly to $\hat{H} F$ in $\mathcal{G}$.

> Since the cutoff operator $\hat{H}$ is continuous in $\mathcal{G}^\infty$, the bilinear form $G^* \hat{H} F$ is separately continuous on that Frechet space. By a Banach theorem (see [16], Theorem 2.17), the bilinear form is, actually continuous on $\mathcal{G}^\infty$. Along with the equality

\[
P_n F(\phi^*, \phi) = F(p_n \phi^*, p_n \phi), \tag{43}
\]

this implies that the operator $\hat{H}$ is the weak limit of $\hat{H}_n = P_n \hat{H} P_n$ in $\mathcal{G}^\infty$. Since $\mathcal{G}^\infty$ is a Montel space, a weakly convergent sequence $P_n \hat{H} P_n F$ converges in the topology of $\mathcal{G}^\infty$, and, therefore, of $\mathcal{G}$.
PART II If $\lambda$ is a given complex number with non-zero imaginary part, then, for any $G \in \mathcal{G}$, the sequence of the resolvents $(\lambda - \hat{H}_n)^{-1}G$ converges strongly to $(\lambda - \hat{H})^{-1}G$. Since the operator norms $\|(\lambda + \hat{H}_n)^{-1}\|$ are uniformly bounded, it suffices to consider the dense set of $G = (\lambda + i\hat{H})^{-1}F$ with $F \in \mathcal{G}^\infty$. In such a case

$$\|(\lambda + \hat{H}_n)^{-1}G - (\lambda + \hat{H})^{-1}G\|
= \|(\lambda + \hat{H}_n)^{-1}((\hat{H}_n - \hat{H})(\lambda + \hat{H})^{-1}F\|
< |3\lambda|^{-1}\|(\hat{H}_n - \hat{H})(\lambda + \hat{H})^{-1}F\|,$$

which converges to zero, by the PART I.

PART III As in the proof of theorem VIII.20 in [15], the PART II implies the strong convergence of $f_t(\hat{H}_n)$ to $f_t(\hat{H})$ (uniformly on compact $t$-intervals).

Corollary 3.1 The sequence $e^{-i\hat{H}_n t}$ converges strongly to $e^{-i\hat{H} t}$ as $N \to \infty$, uniformly on compact $t$-intervals.

In particular, any solution $F(\phi^*, \phi; t)$ of the corresponding functional derivatives Schrödinger equation

$$\partial_t F + i\hat{H} F = 0, \quad F(\phi^*, \phi; 0) \in \mathcal{D}(\hat{H}) \quad (44)$$

is the limit of the solutions $F_n \in \mathcal{D}(\hat{H}_n)$ as $n \to \infty$ of the partial differential Schrödinger equations

$$\partial_t F_n + i\hat{H}_n F_n = 0, \quad F_n(\phi^*, \phi; 0) = P_n F(\phi^*, \phi; 0) \in \mathcal{D}(\hat{H}_n) \quad (45)$$

uniformly on compact $t$-intervals.

Remark 3.1 Theorem 2.3 (applied to $\mathcal{H}_n = p_n \mathcal{H}$) shows that the antinormal Feynman integral for the reduced Schrödinger equation is the limit of multiple finite dimensional integrals with respect to Gaussian measures.

This is a (convergent!) alternative for standard space-time lattice approximations in quantum field theory.

In a forthcoming paper we show that the rate of convergence is $< t^2 / n$ so that the limit exists if $t \to \infty$ with the rate $n^{(1+\epsilon)/2}, \epsilon > 0$. Therefore, the remark is applicable to scattering matrices.

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