Remarks on global solutions for nonlinear wave equations under the standard null conditions

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Abstract

A combination of some weighted energy estimates is applied for the Cauchy problem of quasilinear wave equations with the standard null conditions in three spatial dimensions. Alternative proofs for global solutions are shown including the exterior domain problems.

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1 Introduction

Let us consider the Cauchy problem of nonlinear wave equation under the standard null conditions

\[(\partial_t^2 - \Delta)u(t, x) = (\partial_t u(t, x))^2 - |\nabla u(t, x)|^2 \quad \text{for} \quad (t, x) \in [0, \infty) \times \mathbb{R}^3, \tag{1.1}\]

where \(\Delta := \partial_1^2 + \partial_2^2 + \partial_3^2\), and \(\nabla := (\partial_1, \partial_2, \partial_3)\). In [25, p52], it is pointed out that the combination of the weighted energy estimate

\[\|\nabla_{t,x} u\|_{L^\infty((0,T),L^2(\mathbb{R}^3))} + \|(t-r)^{-1/2-\gamma} \overline{\mathcal{J}} u\|_{L^2((0,T) \times \mathbb{R}^3)} \leq C \|\nabla_{t,x} u(0,\cdot)\|_{L^2(\mathbb{R}^3)} + C \int_0^T \int_{\mathbb{R}^3} |(\partial_t u)(\partial_t^2 - \Delta) u| \, dx \, dt \quad \tag{1.2}\]

(see [25, p76, Corollary 8.2] and Alinhac [2, Theorem 1]), where \(T > 0\), \(r = |x|\), \(\nabla_{t,x} = (\partial_t, \partial_1, \partial_2, \partial_3)\), \(\overline{\mathcal{J}} = (\partial_t + \partial_r, \nabla - \frac{r}{2} \partial_r)\) denotes tangential derivatives along the light cone \(t = r\), the constant \(C\) is independent of \(T\), \(\gamma > 0\) is any fixed real number, and the Klainerman-Sobolev estimate

\[(1 + t + r)(1 + |t-r|)^{1/2} |u(t, x)| \leq C \sum_{|\alpha| \leq 2} \|\Theta^\alpha u(t, \cdot)\|_{L^2(\mathbb{R}^3)} \quad \tag{1.3}\]

\[\]

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where order and has the form
\[ \partial_t, \partial_j, t\partial_t + r dr, t\partial_j + x_j \partial_t, x_j \partial_k - x_k \partial_j, \quad 1 \leq j \neq k \leq 3, \]  \hspace{1cm} (1.4)
and \( \alpha \) denotes multiple indices. In this paper, we generalize this argument so that we are able to treat the system of wave equations with different speeds and also the corresponding exterior domain problems. While the above weighted energy estimates could be generalized to treat the \( c > 0 \) speed D’Alembertian \( \Box_c := \partial_t^2 - c^2 \Delta \), the operators \( \{ t\partial_j + x_j \partial_t \}_{j=1}^3 \) are not commutable with \( \Box_c \) if \( c \neq 1 \), and they also makes it difficult to handle the energy near the obstacle when we consider the exterior domain problems. To avoid the use of these operators, we used the Klainerman-Sobolev type estimates by Sideris, Tu, Hidano and Yokoyama.

\[
\langle r \rangle^{1/2} \langle t + \alpha, r \rangle^{1/2} |u'(t, x)| \leq C \sum_{\mu + |\alpha| \leq 2} \| L^\mu Z^\alpha u'(t, \cdot) \|_{L^2(R^3)} + C \sum_{|\alpha| \leq 1} \| (t + |y|) Z^\alpha (\partial_t^2 - c^2 \Delta) u(t, y) \|_{L_y^2(R^3)}, \]  \hspace{1cm} (1.5)
where \( L := t\partial_t + r \partial_r, \) \( Z := (\partial_t, \nabla, \{ x_j \partial_k - x_k \partial_j \}_{1 \leq j \neq k \leq 3}) \) (see Lemma 2.2, below).
And we also use the weighted energy estimate by Keel, Smith and Sogge
\[
\| \nabla_{t,x}^I u \|_{L^\infty(0, T; L^2(R^3))} + (\log(1 + T))^{-1/2} \| (|x|^{-1/2} \nabla_{t,x}^I u(t, x)) \|_{L^2(0, T; \times R^3)} \leq C \| \nabla_{t,x}^I u(0, \cdot) \|_{L^2(R^3)} + C \| (\partial_t^2 - c^2 \Delta) u \|_{L^1(0, T; L^2(R^3))}, \]  \hspace{1cm} (1.6)
(see Lemma 2.1, below). We remark that the combination of (1.2), (1.5) and (1.6) gives a much simplified and elementary proof for the global existence for the small solutions of nonlinear wave equations under the standard multispeed null conditions. Especially, we give alternative proofs of the following two theorems.

1.1 The Cauchy problems without obstacles
We consider the Cauchy problem for a system of quasilinear wave equations with \( D \geq 1 \) propagation speeds \( \{ c_I \}_{1 \leq I \leq D}, \) \( c_I > 0 \). We put \( u = (u_1, \cdots, u_D), \) \( F = (F_1, \cdots, F_D), \) \( f = (f_1, \cdots, f_D), \) \( g = (g_1, \cdots, g_D), \) and we consider
\[
\left\{ \begin{array}{l}
(\partial^2_t - c_I^2 \Delta) u_I(t, x) = F_I(u, u'')(t, x) \quad \text{for } t \in [0, \infty), \ x \in \mathbb{R}^3, \ 1 \leq I \leq D \\
0 \leq I \leq D \\
u(0, \cdot) = f(\cdot), \quad \partial_t u(0, \cdot) = g(\cdot),
\end{array} \right.
\]  \hspace{1cm} (1.7)
where we put \( \partial_0 = \partial_t, \) we denote the first derivatives \( \{ \partial_j u \}_{0 \leq j \leq 3} \) by \( u', \) and the second derivatives \( \{ \partial_j \partial_k u \}_{0 \leq j, k \leq 3} \) by \( u'' \). We assume that \( F \) vanishes to the second order and has the form
\[
F_I(u, u'') = B_I(u') + Q_I(u', u''), \]  \hspace{1cm} (1.8)
where
\[
B_I(u') := \sum_{0 \leq j, k \leq 3} B_{IK}^{jk} \partial_j u_j \partial_k u_K, \]  \hspace{1cm} (1.9)
Here, $\{B_I^{JK}\}_{1 \leq I, J, K \leq D}$ and $\{Q_I^{JK}\}_{0 \leq j, k \leq 3}$ are real constants which satisfy the symmetry condition

$$Q_I^{JK} = Q_I^{JK}.$$  \tag{1.11}

which is used to derive the energy estimates. To show the global solutions, we assume the standard null conditions

$$\sum_{0 \leq j, k \leq 3} B_I^{JK} \xi_j \xi_k = \sum_{0 \leq j, k, l \leq 3} Q_I^{JK} \xi_j \xi_k \xi_l = 0$$ \tag{1.12}

for any $1 \leq I, J, K \leq D$ with $c_I = c_J = c_K$, and any $(\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4$ with $\xi_0^2 = c_I^2 (\xi_1^2 + \xi_2^2 + \xi_3^2)$. For example,

$$B_I(u') = \sum_{1 \leq J, K \leq D} \kappa^{JK} \{(\partial_t u_J)(\partial_t u_K) - c_I^2 (\nabla u_J) \cdot (\nabla u_K)\} + \sum_{1 \leq J, K \leq D} \lambda^{JK} u_J u_K'$$ \tag{1.13}

for $\{\kappa^{JK}\}_{1 \leq J, K \leq D}, \{\lambda^{JK}\}_{1 \leq J, K \leq D} \subset \mathbb{R}$, and $Q_I(u', u'') = \partial_{t,x} B_I(u')$ satisfy the null conditions. It is known that any nontrivial solutions blow up in finite time in general for quadratic nonlinearities (see John \[11\]), while the null conditions guarantee the global solutions (see Christodoulou \[3\] and Klainerman \[22\]). We give an alternative proof of the following theorem, which has been shown by Sideris and Tu in \[36\].

**Theorem 1.1** Let $f$ and $g$ be smooth functions, and let $F$ satisfy the above standard null conditions. Then there exists a positive natural number $N$ (for example, we are able to take $N = 12$) such that if

$$\sum_{|\alpha| \leq N} \| (x^{|\alpha|} \partial_x^{|\alpha|} f(x)) \|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq N} \| (x^{|\alpha|} \partial_x^{|\alpha|} g(x)) \|_{L^2(\mathbb{R}^3)}$$ \tag{1.14}

is sufficiently small, then there exists a unique global solution $u \in C^\infty([0, \infty) \times \mathbb{R}^3)$ of \tag{1.17}.

The proof in \[36\] consists of the standard energy estimates and a series of pointwise estimates for $r^{1/2} u$, $(r) u'$, $(r) \langle ct - r \rangle^{1/2} u'$, $(r) \langle ct - r \rangle u''$, and the weighted estimate $\| (\langle ct - r \rangle u'') \|_{L^2(\mathbb{R}^3)}$ plays an important role to control the energy near the light cone $r = ct$. The weighted energy estimates in \tag{1.2} and \tag{1.6} are not used in \[36\]. We remark in this paper the combination of \tag{1.2} and \tag{1.6} yields much simplified and elementary proof for the theorem. The key estimate is the lower energy estimate \tag{3.6}, which is proved via a straightforward application of \tag{1.2}, \tag{1.5}, \tag{1.6} and the estimate for null conditions Lemma \[2.3\]. We generalize \tag{1.2} in Lemma \[6.1\]. It is interesting to see that Lemma \[6.1\] has a close similarity to \tag{1.6} (cf. \[32\] Corollary \[5\]). The estimate \tag{1.6} yields much simplified proof for almost global solutions \tag{1.7} and also has strong applications to global solutions \tag{27, 28, 29, 31, 33}. 

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1.2 Exterior domain problems

We also consider the exterior domain problem. Let $\mathcal{K}$ be any fixed compact domain in $\mathbb{R}^3$ with smooth boundary. Without loss of generality, we may assume $0 \in \mathcal{K} \subset \{ x \in \mathbb{R}^3 : |x| < 1 \}$ by the shift and scaling arguments. Moreover, we assume the following local energy decay estimates. Let $u$ be the solution of

$$
\begin{cases}
(\partial^2_t - \Delta)u(t, x) = 0 & \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^3 \setminus \mathcal{K} \\
u(t, \cdot)|_{\mathcal{K}} = 0 & \text{for } t \in [0, \infty) \\
u(0, \cdot) = f(\cdot), \quad \partial_t \nu(0, \cdot) = g(\cdot).
\end{cases}
\quad (1.15)
$$

If the initial data satisfy $\text{supp } f \cup \text{supp } g \subset \{ x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 4 \}$, then there exist constants $C > 0$ and $a > 0$ such that

$$
||u'(t, \cdot)||_{L^2(\{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 4\})} \leq Ce^{-at} \sum_{|\alpha| \leq 1} \|\partial^{2\alpha}_x u'(0, \cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}
\quad (1.16)
$$

for any $t \geq 0$. The local energy decay estimate (1.16) holds if the obstacle is nontrapping without the loss of derivatives $|\alpha| = 1$ (Morawetz, Ralston and Strauss [31]), or the obstacle consists of certain finite unions of convex obstacles (Ikawa [9, 10]).

We consider the exterior domain problems for (1.17) given by

$$
\begin{cases}
(\partial^2_t - c^2 \Delta)u_I(t, x) = F_I(u', u'')(t, x) & \text{for } t \in [0, \infty), \ x \in \mathbb{R}^3 \setminus \mathcal{K}, \ 1 \leq I \leq D \\
u(t, x)|_{x \in \partial \mathcal{K}} = 0 & \text{for } t \in [0, \infty) \\
u(0, \cdot) = f(\cdot), \partial_t \nu(0, \cdot) = g(\cdot).
\end{cases}
\quad (1.17)
$$

where $F_I$ is written as (1.8), (1.9), (1.10), we assume the symmetry condition (1.11) and the null conditions (1.12). Since (1.17) is the initial and boundary value problem, the initial data $f$ and $g$ must satisfy the compatibility condition. For $k \geq 0$ and the solution $u$ of (1.17), the condition $\partial^k u(0, \cdot) = 0$ is written in terms of $f$, $g$ and $F$. We assume the compatibility condition of infinite order, namely, $\partial^k u(0, \cdot)|_{\mathcal{K}} = 0$ for any $k \geq 0$. We give an alternative proof of the following theorem, which has been shown in [28, 30].

**Theorem 1.2** Let $f$ and $g$ be smooth functions and satisfy the compatibility conditions of infinite order. Let $F$ satisfy the above standard null conditions. Then there exists a positive large natural number $N$ (for example, we are able to take $N = 64$) such that if

$$
\sum_{|\alpha| \leq N} \|(x)^{|\alpha|+1} |^{\partial^2_x f(x)}|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{|\alpha| \leq N-1} \|(x)^{|\alpha|+1} |^{\partial^2_x g(x)}|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}
\quad (1.18)
$$

is sufficiently small, then (1.17) has a unique global solution $u \in C^\infty([0, \infty) \times \mathbb{R}^3 \setminus \mathcal{K})$.

There is a series of papers on almost global and global solutions by Keel, Smith and Sogge [16] for convex obstacles, [17] for nontrapping obstacles, [18] and [19] for star-shaped obstacles. See also [28, 29] and [30] for Ikawa’s type trapping obstacles.
In [28, 30], the weighted estimate (1.6) has been used, while tangential derivatives and (1.2) are not used. Let $B(u, v) = \sum_{0 \leq j, k \leq 3} B^{jk} \partial_j u \partial_k v$ satisfy the null condition $\sum_{0 \leq j, k \leq 3} B^{jk} \xi_j \xi_k = 0$ for $\xi^3 = \sum_{j=1}^{3} \xi_j^3$. One of the advantages to use the tangential derivatives is that we are able to estimate the null conditions simply as

$$| \sum_{0 \leq j, k \leq 3} B^{jk} \partial_j u \partial_k v | \leq C |\partial u| |v| + C |u'| |\partial v|$$

(1.19)

(see Lemma 2.3). In [28, 30, 36], the type of estimate

$$| \sum_{0 \leq j, k \leq 3} B^{jk} \partial_j u \partial_k v | \leq C \left( \sum_{\mu + \alpha \leq 1} |L^\alpha Z^\mu u| |\partial v| + |\partial u| \sum_{\mu + \alpha \leq 1} |L^\alpha Z^\mu v| \right)$$

$$+ C \left( \frac{t-r}{t+r} \right) |\partial u| |\partial v|$$

(1.20)

(see [37, Lemma 5.4]) has been used, which needs variants of Sobolev type estimates, or $L^\infty - L^1$ estimates based on the Kirchhoff formula to bound $|L^\alpha Z^\mu u|$ and $|L^\alpha Z^\mu v|$, and the proof for global solutions needs structural complexity. In this paper, we use (1.19), and we show the combination of two type of weighted energy estimates (1.2), (1.6), and the Sobolev estimates (1.5) gives a simplified proof of the theorem.

### 1.3 Notation

We use the method of commuting vector fields introduced by John and Klainerman [12, 13, 21]. See also Keel, Smith and Sogge [17] for exterior domains. We denote the space-time derivatives by $\partial$, the rotational derivatives by $\Omega$, and the scaling operator by $L$. We use $u'$ to denote $\partial u$ in some cases. We denote $\partial$ by $Z$, and $\partial, \Omega, L$ by $\Gamma$. We use the tangential derivatives $\partial c$ on the $c$-speed light cone $ct = |x|$ to treat the nonlinear terms which satisfies the null conditions. We summarize as

\[
\begin{align*}
  x &= r \omega, \quad \omega \in S^2, \quad x_0 = t, \quad \partial_0 = \partial_t, \\
  \partial &= (\partial_{\omega_1}, \partial_{\omega_2}, \partial_{\omega_3}), \quad \Omega = (x_j \partial_k - x_k \partial_j)_{1 \leq j \neq k \leq 3}, \quad L = t \partial_t + r \partial_r, \\
  Z &= (\partial, \Omega), \quad \Gamma = (\partial, \Omega, L), \\
  \partial_c &= (\partial_{\omega_1}, \partial_{\omega_2}, \partial_{\omega_3}) = (\partial_t + c \partial_r, \nabla_x - \omega \partial_r) \text{ for } c > 0.
\end{align*}
\]

(1.21)

The operators $Z$, $L$ have the commuting properties with the $c$ speed D’Alembertian $\square_c := \partial_c^2 - c^2 \Delta$ such as

$$\square_c Z = Z \square_c, \quad \square_c L = (L + 2) \square_c.$$

(1.22)

We do not use the Lorentz boosts $\{ t \partial_j + x_j \partial_t \}_{j=1}^3$ which are not suitable for the different speeds system or the exterior domain problems. We put $\langle r \rangle = \sqrt{1 + r^2}$, $\langle x \rangle = \sqrt{1 + |x|^2}$. The Lebesgue spaces on $\mathbb{R}^3$ are denoted by $L^p(\mathbb{R}^3)$ for $1 \leq p \leq \infty$. For $R > 0$, $L^p(|x| < R)$ denotes $L^p(\{ x \in \mathbb{R}^3 : |x| < R \})$. The strip region is denoted by $S_T := [0, T) \times \mathbb{R}^3$ for $T \geq 0$. We use $L^p(\mathbb{R}^3 \setminus \mathcal{K})$, $L^p(|x| < R) := L^p(\{ x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < R \})$ and $S_T := [0, T) \times (\mathbb{R}^3 \setminus \mathcal{K})$ when we consider the exterior domain.
problems. For $c > 0$, $\Box_c := \partial_t^2 - c^2 \Delta$ denotes the $c$-speed D’Alembertian. We put $\Box := \Box_1 = (\partial_t^2 - \Delta)$. Throughout this paper, $C$ denotes a positive constant which may differ from line to line. The notation $a \lesssim b$ denotes the inequality $a \leq Cb$ for a positive constant $C$ which is not essential for our arguments. This paper is organized as follows. In sections 2 and 4, we prepare several estimates which are needed to show Theorem 1.1 and 1.2, respectively. Theorems 1.1 and 1.2 are shown in sections 3 and 5, respectively. In section 6, we put two appendices. The first is for the proof of the weighted energy estimates, and the second is for a remark on the two spatial dimensions.

## 2 Several estimates to prove Theorem 1.1

In this section, we prepare several estimates to prove Theorem 1.1.

### 2.1 Energy estimates

Let us consider the general dimension $n \geq 1$ in this subsection. We put $\Delta = \sum_{j=1}^{n} \partial_j^2$. We use the following energy estimates for quasilinear wave equations. Let $
abla_0 u_{I}$, $1 \leq I, K \leq D$, $0 \leq k, l \leq n$, be functions which satisfy the symmetry conditions $
abla_0 u_{I} = \gamma_1^{Kkl} \partial_k \partial_l u_K$. We put

$$
\Box_0 u_{I} := (\partial_t^2 - c^2 \Delta) u_{I} - \sum_{1 \leq K \leq D \atop 0 \leq k, l \leq n} \gamma_1^{Kkl} \partial_k \partial_l u_K \quad \text{for } 1 \leq I \leq D. \tag{2.1}
$$

We define the energy momentums $e_k(u)$, $0 \leq k \leq n$, and the remainder term $R(u)$ by

$$
e_0(u) := \sum_{1 \leq I \leq D} \left\{ \left( \partial_t u_I \right)^2 + c^2 |\nabla u_I|^2 \right\} - \sum_{1 \leq I, K \leq D \atop 0 \leq k, l \leq n} 2 \gamma_1^{Kkl} \partial_k \partial_l u_K
\quad + \sum_{1 \leq I, K \leq D \atop 0 \leq k, l \leq n} \gamma_1^{Kkl} \partial_k \partial_l u_K
$$

$$
e_k(u) := -\sum_{1 \leq I \leq D} 2c^2 \partial_0 u_I \partial_k u_l - \sum_{1 \leq I, K \leq D \atop 0 \leq k, l \leq n} 2 \gamma_1^{Kkl} \partial_0 u_I \partial_k u_l \quad \text{for } 1 \leq k \leq n
$$

$$R(u) := -\sum_{1 \leq I, K \leq D \atop 0 \leq k, l \leq n} 2(\partial_k \gamma_1^{Kkl}) \partial_0 u_I \partial_l u_K + \sum_{1 \leq I, K \leq D \atop 0 \leq k, l \leq D} (\partial_0 \gamma_1^{Kkl}) \partial_k u_I \partial_l u_K.
$$

Then the multiplication of $2\partial_0 u_I$ to the equation (2.1) yields the divergence form

$$
\partial_t e_0(u) + \mathrm{div} (e_1(u), \cdots, e_n(u)) = \sum_{1 \leq I \leq D} 2 \partial_t u_I \Box_0 u_I + R(u). \tag{2.2}
$$
2.2 Weighted energy estimates

We use the following weighted energy estimates.

**Lemma 2.1** Let \( n \geq 1 \). We put \( \Delta := \sum_{j=1}^{n} \partial_{j}^{2} \). Let \( c > 0 \) and \( T > 0 \). The solution \( u \) of the Cauchy problem

\[
\begin{cases}
(\partial_{t}^{2} - c^{2}\Delta)u(t, x) = F(t, x) & \text{for } (t, x) \in [0, T) \times \mathbb{R}^{n} \\
 u(0, \cdot) = f(\cdot), \quad \partial_{t}u(0, \cdot) = g(\cdot)
\end{cases}
\]  

(2.3)

satisfies the following estimate, where \( u' = \partial_{t}u \), \( \nabla w \), \( \partial_{r} \).

\[
\|u'\|_{L^{\infty([0,T),L^{2}(\mathbb{R}^{n})}}} + (\log(e + T))^{-1/2}\|\langle x \rangle^{-1/2}u'(\cdot, x)\|_{L^{2}((0,T)\times\mathbb{R}^{n})} \\
+ (\log(e + T))^{-1/2}\|\langle ct - r \rangle^{-1/2}\partial_{r}u(t, x)\|_{L^{2}((0,T)\times\mathbb{R}^{n})} \\
\lesssim \|\nabla f\|_{L^{2}(\mathbb{R}^{n})} + \|g\|_{L^{2}(\mathbb{R}^{n})} + \|F\|_{L^{1}((0,T),L^{2}(\mathbb{R}^{n}))}
\]  

(2.4)

The estimate for the first term in the left hand side is the standard energy estimate. The estimate for the second term is due to Keel, Smith and Sogge [17 Proposition 2.1] for \( n = 3 \), Metcalfe, Sogge and Hidano for general dimensions (see [32] Corollary 5). The estimate for the third term is a logarithmic version of the estimate due to Lindblad and Rodnianski [25, p76, Corollary 8.2] who treat the case \( n = 3 \) and \( c = 1 \). We generalize it as Lemma 6.1 in Appendices.

2.3 Klainerman-Sobolev type estimates

We also use the following Klainerman-Sobolev type estimates.

**Lemma 2.2** Let \( c > 0 \). The following inequality holds for any function \( u \).

\[
\langle r \rangle^{1/2}\langle r + t \rangle^{1/2}\langle ct - r \rangle^{1/2}|u'(t, x)| \lesssim \sum_{\mu + |\alpha| \leq 2, \mu \leq 1} \|L^{\mu}Z^{\alpha}u'(t, \cdot)\|_{L^{2}(\mathbb{R}^{3})} \\
+ \sum_{|\alpha| \leq 1} \|\langle ct - |y| \rangle Z^{\alpha}(\partial_{t}^{2} - c^{2}\Delta)u(t, y)\|_{L^{2}_{r}(\mathbb{R}^{3})}
\]  

(2.5)

**Proof.** This lemma directly follows from the combination of

\[
\langle r \rangle \langle ct - r \rangle^{1/2}|u(t, x)| \lesssim \sum_{|\alpha| \leq 2} \|\Omega^{\alpha}u(t, y)\|_{L^{2}(|y| > r)} \\
+ \sum_{|\alpha| \leq 1} \|\langle ct - |y| \rangle\partial_{r}\Omega^{\alpha}u(t, y)\|_{L^{2}(|y| > r)}
\]  

(2.6)

by Sideris [35 Lemma 3.3],

\[
\langle r \rangle^{1/2}\langle ct - r \rangle|u'(t, x)| \lesssim \sum_{|\alpha| \leq 1} \|Z^{\alpha}u'(t, \cdot)\|_{L^{2}(\mathbb{R}^{3})} \\
+ \sum_{|\alpha| \leq 1, |\beta| = 2} \|\langle ct - |y| \rangle Z^{\alpha}\partial_{r}^{\beta}u(t, y)\|_{L^{2}_{r}(\mathbb{R}^{3})}
\]  

(2.7)
by Hidano [3, Lemma 4.1], and
\[ \sum_{|\beta|=2} \| (ct-r)^3 \partial^3 u(t,x) \|_{L^2_2(\mathbb{R}^3)} \]
\[ \lesssim \sum_{|\mu+|\alpha|\leq 1} \| L^\mu Z^\alpha u'(t,) \|_{L^2_2(\mathbb{R}^3)} + \| (t+r)(\partial_t^2 - c^2 \Delta) u(t,x) \|_{L^2_2(\mathbb{R}^3)} \] (2.8)
by Sideris and Tu [36, Lemma 7.1]. See also Sogge [37, p74, Lemma 5.3]. \qed

2.4 Estimates for null conditions

The null conditions are treated by the following estimates.

**Lemma 2.3** Let \( c > 0 \). Let
\[ B(u,v) = \sum_{0\leq j,k\leq 3} B^{jk} \partial_j u \partial_k v, \quad Q(u,v) = \sum_{0\leq j,k,l\leq 3} Q^{jkl} \partial_j u \partial_k \partial_l v \] (2.9)
satisfy the null conditions:
\[ \sum_{0\leq j,k\leq 3} B^{jk} \xi_j \xi_k = 0 \quad \text{for} \quad \xi_0^2 = c^2 (\xi_1^2 + \xi_2^2 + \xi_3^2). \] (2.10)
Then the following inequalities hold for any \( \alpha \) and functions \( u \) and \( v \).

1. \[ |\Gamma^\alpha B(u,v)| \lesssim \sum_{\beta+\gamma\leq \alpha} \{ |\partial_{\xi} \Gamma^\beta u| (|\Gamma^\gamma v|') + |(\Gamma^\beta u)'| |\partial_{\xi} \Gamma^\gamma v| \}, \] (2.11)
2. \[ |\Gamma^\alpha Q(u,v)| \lesssim \sum_{\beta+\gamma\leq \alpha} \{ |\partial_{\xi} \Gamma^\beta u| (|\Gamma^\gamma v|'') + |(\Gamma^\beta u)'| (|\Gamma^\gamma v|'') \}
\[ + \frac{|(\Gamma^\beta u)'| (|\Gamma^\gamma v|'') + |(\Gamma^\gamma v|''')}{\langle r \rangle} \}, \] (2.12)

where \( \beta \leq \alpha \) means any component of the multiindices satisfies the inequality.

**Proof.** (1) First, we consider the case \( \alpha = 0 \). Let \( \omega_0 = -c \) and \( \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2 \). Since \( \sum_{0\leq j,k\leq 3} B^{jk} \omega_j \omega_k = 0 \) by the null condition, we have
\[ B(u,v) = \sum_{0\leq j,k\leq 3} B^{jk} \partial_j u \partial_k v - \sum_{0\leq j,k\leq 3} B^{jk} \omega_j \omega_k \partial_r u \partial_r v \]
\[ = \sum_{0\leq j,k\leq 3} B^{jk} \{ (\partial_j - \omega_j \partial_r) u \partial_k v + \omega_j \partial_r u (\partial_k - \omega_k \partial_r) v \}. \] (2.13)
So that, we have \( |B(u,v)| \lesssim |\partial_{\xi} u| |v'| + |u'| |\partial_{\xi} v| \).

For any \( \alpha \), we have by the similar argument for Lemma 4.1 in [36]
\[ \Gamma^\alpha B(u,v) = \sum_{\beta+\gamma\leq \alpha} B_{\beta,\gamma} (\Gamma^\beta u, \Gamma^\gamma v), \] (2.14)
where $\{B_{\beta,\gamma}\}_{\beta+\gamma \leq \alpha}$ are quadratic nonlinear terms which satisfy the null conditions. The required estimate follows from the above two results.

(2) By the same argument for (1), we have

$$Q(u,v) = \sum_{0 \leq j,k,l \leq 3} Q^{jkl}(D_{u}u D_{v}v + \omega_{j} \partial_{k}u(D_{l}v + \omega_{k} \partial_{l}v)).$$

(2.15)

So that, we have

$$|Q(u,v)| \lesssim |D_{u}v''| + |u'| |D_{v}'| + \frac{|u'||v'|}{r},$$

(2.16)

where we have used $|\partial_{v}D_{u}v| \lesssim |D_{u}v| + |v'|/r$ for the last inequality. Since $|Q(u,v)| \lesssim |u'||v''|$ for $r < 1$, we have the required inequality for the case $\alpha = 0$. The case $\alpha \neq 0$ also follows from

$$\Gamma_{\alpha}Q(u,v) = \sum_{\beta+\gamma \leq \alpha} Q_{\beta,\gamma}(\Gamma^{\beta}u, \Gamma^{\gamma}v),$$

(2.17)

where $\{Q_{\beta,\gamma}\}_{\beta+\gamma \leq \alpha}$ are quadratic nonlinear terms which satisfy the null conditions.

\[\square\]

3 Continuity argument to prove Theorem 1.1

We prepare the following proposition to prove Theorem 1.1.

Proposition 3.1 Let $M_{0}$ and $N$ be positive numbers which satisfy $M_{0} + 5 \leq N \leq 2M_{0} - 2$. For example, we are able to take $M_{0} = 7$ and $N = 12$. We put

$$\varepsilon := \sum_{|\alpha| \leq N} \|\langle x \rangle^{|\alpha|} \partial_{x}^{\alpha}f(x)\|_{L^{2}(\mathbb{R}^{3})} + \sum_{|\alpha| \leq N} \|\langle x \rangle^{|\alpha|} \partial_{x}^{\alpha}g(x)\|_{L^{2}(\mathbb{R}^{3})}.$$  

(3.1)

Let $T > 0$ and $A_{0} > 0$. Let $u \in C^{\infty}([0, T) \times \mathbb{R}^{3})$ be the local solution of (1.7). We assume

$$\sum_{|\alpha| \leq M_{0}} \sup_{0 \leq t < T} (r_{1}^{1/2} + t^{1/2} \langle c_{1}t - r \rangle^{1/2}) |\Gamma^{\alpha}u'(t,x)| \leq A_{0} \varepsilon.$$  

(3.2)

Then there exist constants $C_{0} > 0$, which is independent of $A_{0}$, and $C > 0$, which is dependent on $A_{0}$, such that the following estimates hold.

(1) $$\sum_{|\alpha| \leq M} \|\Gamma^{\alpha}u'(t,\cdot)\|_{L^{2}(\mathbb{R}^{3})} \leq C \varepsilon (1 + t)^{C \varepsilon} \quad \text{for} \quad 0 \leq t < T, \ 0 \leq M \leq N. \quad (3.3)$$

(2) $$\sum_{|\alpha| \leq M-1} \langle x \rangle^{-1/2} \|\langle c_{1}s - r \rangle^{-1/2} \Gamma^{\alpha}u'(s, x)\|_{L^{2}(S_{t})} + \sum_{|\alpha| \leq M-1} \langle x \rangle^{-1/2} \|\langle c_{1}s - r \rangle^{-1/2} \partial_{x_{i}} \Gamma^{\alpha}u(s, x)\|_{L^{2}(S_{t})}$$

$$\leq C \varepsilon (1 + t)^{C \varepsilon} \quad \text{for} \quad 0 \leq t < T, \ 0 \leq M \leq N. \quad (3.4)$$
for some constant $C > 0$.

To bound the right hand side, we use

\begin{equation}
\sum_{|\alpha| \leq M_0+2} \sup_{0 \leq t < T} \|\Gamma^\alpha u'\|_{L^\infty((0,T),L^2(\mathbb{R}^3))} \leq C_0 \varepsilon + C_0 \varepsilon^{3/2}.
\end{equation}

(3.6)

(3.7)

3.1 Proof of Theorem 1.1

Here we prove Theorem 1.1. We use the continuity argument which shows that the local in time solution $u$ does not blow up if its initial data is sufficiently small. Since the constant $C_0$ is independent of $A_0$ in Proposition 3.1, we put $A_0 = 4C_0$ and take $\varepsilon$ sufficiently small such that $C_0 \varepsilon^{3/2} \leq C_0 \varepsilon$. Then the right hand side of (5) is bounded by $A_0 \varepsilon/2$, which shows the local in time solution $u$ does not blow up, namely the solution exists globally in time.

\[ \square \]

3.2 Proof of Proposition 3.1

First, we remark that under the assumption (3.2), we have

\[ \|\Gamma^\alpha u\|_{L^\infty((0,T),L^2(\mathbb{R}^3))} \leq C_0 \varepsilon + C_0 \varepsilon^{3/2}. \]

(3.6)

For any $\varepsilon > 0$, we choose

\[ \sum_{|\alpha| \leq M/2+1} \sup_{0 \leq t < T} |\Gamma^\alpha u'(t, x)| \leq C \varepsilon \]

(3.8)

for some constant $C > 0$ since $M/2 + 1 \leq M_0$.

(1) For $1 \leq I, K \leq D$ and $0 \leq k, l \leq 3$, we put

\[ \gamma^{Kkl}_I := \sum_{\substack{1 \leq j \leq D \\text{or} \\varepsilon \leq j \leq 3}} Q^{JKkl}_I \partial_j u_J. \]

(3.9)

For any $\alpha$ with $|\alpha| \leq M$, we use (2.2) and its integration to have

\[ \partial_t \int_{\mathbb{R}^3} e_0(\Gamma^\alpha u) dx \leq \sum_{1 \leq I \leq D} \|\Box_{\gamma} \Gamma^\alpha u_I\|_{L^2(\mathbb{R}^3)} \|\Gamma^\alpha u'_I\|_{L^2(\mathbb{R}^3)} + \sum_{1 \leq I, K \leq D} \|\partial_t x \gamma^{Kkl}_I\|_{L^\infty(\mathbb{R}^3)} \|\Gamma^\alpha u_I\|_{L^2(\mathbb{R}^3)} \|\Gamma^\alpha u_K\|_{L^2(\mathbb{R}^3)} \].

(3.10)

To bound the right hand side, we use

\[ \sum_{1 \leq I \leq D} \|\Box_{\gamma} \Gamma^\alpha u_I\|_{L^2(\mathbb{R}^3)} \leq \sum_{1 \leq I \leq D} \|\Gamma^\alpha \Box_{\gamma} u_I\|_{L^2(\mathbb{R}^3)} + \sum_{1 \leq I \leq D} \|\Gamma^\alpha \Box_{\gamma} u_I\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| + |\beta| \leq M} \|\Gamma^\alpha \gamma^{Kkl}_I\|_{L^2(\mathbb{R}^3)} \leq \frac{\varepsilon}{1 + t} \sum_{|\alpha| \leq M} \|\Gamma^\alpha u'_I\|_{L^2(\mathbb{R}^3)}, \]

(3.11)
where we have used \((3.8)\) for the last inequality. Since \(\sum_{|\alpha| \leq M} ||\Gamma^\alpha u'||_{L^2(\mathbb{R}^3)}\) is equivalent to \(\sum_{|\alpha| \leq M} \left\{ \int_{\mathbb{R}^3} e_0(\Gamma^\alpha u) \, dx \right\}^{1/2}\) for small \(\varepsilon\), we obtain

\[
\partial_t \left\{ \sum_{|\alpha| \leq M} \int_{\mathbb{R}^3} e_0(\Gamma^\alpha u) \, dx \right\}^{1/2} \lesssim \frac{\varepsilon}{1 + t} \left\{ \sum_{|\alpha| \leq M} \int_{\mathbb{R}^3} e_0(\Gamma^\alpha u) \, dx \right\}^{1/2},
\]

which leads to the required inequality by the Gronwall inequality.

(2) By Lemma 2.1, the left hand side of \((3.4)\) is bounded by

\[
C_0 \varepsilon + C \sum_{1 \leq I \leq D, |\alpha| \leq M-1} ||\Gamma^\alpha \Box_{c_I} u_I||_{L^1((0,t),L^2(\mathbb{R}^3))}.
\]

For the last term, we use

\[
\sum_{1 \leq I \leq D, |\alpha| \leq M-1} ||\Gamma^\alpha \Box_{c_I} u_I|| \lesssim \sum_{|\beta| \leq M/2} ||\Gamma^\beta u'|| \sum_{|\alpha| \leq M} ||\Gamma^\alpha u'|| \lesssim \frac{\varepsilon}{1 + s} \sum_{|\alpha| \leq M} ||\Gamma^\alpha u'||,
\]

where we have used \((3.8)\) for the last inequality, and \((1)\) to obtain

\[
\sum_{1 \leq I \leq D, |\alpha| \leq M-1} ||\Gamma^\alpha \Box_{c_I} u_I||_{L^1((0,t),L^2(\mathbb{R}^3))} \lesssim \int_0^t \frac{\varepsilon}{1 + s} \sum_{|\alpha| \leq M} ||\Gamma^\alpha u'(s, \cdot)||_{L^2(\mathbb{R}^3)} \, dt \lesssim \varepsilon (1 + t)^C.
\]

Therefore, we obtain the required inequality.

(3) Let \(M = M_0 + 3\). By Lemma 2.2, the left hand side of \((3.5)\) is bounded by

\[
C_0 \sum_{|\alpha| \leq M+2} ||\Gamma^\alpha u'(t, \cdot)||_{L^2(\mathbb{R}^3)} + C_0 \sum_{|\alpha| \leq M+1} ||(t + |y|)\Gamma^\alpha \Box_{c_I} u_I(t, y)||_{L^2(\mathbb{R}^3)}.\]

The last term is bounded by \(C \varepsilon \sum_{|\alpha| \leq M+2} ||\Gamma^\alpha u'(t, \cdot)||_{L^2(\mathbb{R}^3)}\) by \((3.8)\) since \((M + 2)/2 \leq M_0\). This shows the required inequality by \((1)\).

(4) Let \(M = M_0 + 2\). By the standard energy estimates, we have

\[
\sum_{|\alpha| \leq M} ||\Gamma^\alpha u'(t, \cdot)||_{L^2(\mathbb{R}^3)}^2 \leq C_0 \sum_{|\alpha| \leq M} ||\Gamma^\alpha u'(0, \cdot)||_{L^2(\mathbb{R}^3)}^2 + C_0 \sum_{|\alpha| \leq M} \int_0^t \int_{\mathbb{R}^3} |\partial_t \Gamma^\alpha u_I \Box_{c_I} \Gamma^\alpha u_I| \, dx \, ds =: A_1 + A_2
\]

for \(1 \leq I \leq D\). We have \(A_1 \leq (C_0 \varepsilon)^2\) for some \(C_0 > 0\) which is independent of \(A_0\).
By Lemma 2.3, \( A_2 \) is bounded as
\[
A_2 \lesssim \sum_{1 \leq J, K \leq D} \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha| \leq M} |\Gamma^\alpha u'_I| \sum_{|\alpha| \leq M+1} |\partial_s \Gamma^\alpha u_J| \sum_{|\alpha| \leq M+1} |\Gamma^\alpha u'_{K}| \, dx \, ds
\]
\[+ \sum_{1 \leq J, K \leq D} \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha| \leq M} |\Gamma^\alpha u'_I| \sum_{|\alpha| \leq M+1} |\Gamma^\alpha u'_J| \sum_{|\alpha| \leq M+1} |\Gamma^\alpha u'_{K}| \sum_{\alpha \leq M+1} \left| \frac{dx}{\langle r \rangle} \right| \, ds \]
\[+ \sum_{1 \leq J, K \leq D} \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha| \leq M} |\Gamma^\alpha u'_I| \sum_{|\alpha| \leq M+1} |\Gamma^\alpha u'_J| \sum_{|\alpha| \leq M+1} |\Gamma^\alpha d_{K} \, dx \, ds
\]
\[=: A_3 + A_4 + A_5. \quad (3.18)\]

We use (3.6) to have
\[
A_3 \lesssim \varepsilon \sum_{1 \leq J, K \leq D} \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha| \leq M+1} \left\| \langle c_I s - r \rangle^{-1/2} \langle s \rangle^{-\delta} \partial_s \Gamma^\alpha u_J \right\|_{L^2((0,t) \times \mathbb{R}^3)}
\]
\[\cdot \sum_{|\alpha| \leq M+1} \left\| \langle r \rangle^{-1/2} \langle s \rangle^{-\delta} \Gamma^\alpha u'_K \right\|_{L^2((0,t) \times \mathbb{R}^3)} \lesssim \varepsilon^3, \quad (3.19)\]

where \( \delta > 0 \) is a sufficiently small number and we have used (3.4) to obtain the last inequality. Similarly, we have
\[
A_4 \lesssim \varepsilon \sum_{1 \leq J, K \leq D} \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha| \leq M+1} \left\| \langle r \rangle^{-1/2} \langle s \rangle^{-\delta} \Gamma^\alpha u'_J \right\|_{L^2(S_t)}
\]
\[\cdot \sum_{|\alpha| \leq M+1} \left\| \langle r \rangle^{-1/2} \langle s \rangle^{-\delta} \Gamma^\alpha u'_K \right\|_{L^2(S_t)} \lesssim \varepsilon^3. \quad (3.20)\]

To bound \( A_5 \), we consider the conic neighborhood defined by
\[
A_I := \{(s, x) : 0 \leq s \leq t, |c_I s - r| \leq c_0 s/10\}, \quad c_0 := \max_{1 \leq J \leq D} c_J \quad (3.21)\]

for \( 1 \leq I \leq D \). We note that
\[
\sum_{|\alpha| \leq M+1} |\Gamma^\alpha u'_I(s, x)| \lesssim \varepsilon \langle r \rangle^{-1/2} \langle s + r \rangle^{-1+\epsilon} C \epsilon \quad (3.22)
\]
on \( \mathbb{R}^3 \setminus A_I \) by (3.5) for \( 1 \leq I \leq D \). So that, we have
\[
\int_0^t \int_{\mathbb{R}^3 \setminus A_I} \sum_{|\alpha| \leq M+1} |\Gamma^\alpha u'_I| \sum_{|\alpha| \leq M+1} |\Gamma^\alpha u'_J| \sum_{|\alpha| \leq M+1} |\Gamma^\alpha u'_{K}| \, dx \, ds
\]
\[\lesssim \varepsilon \sum_{|\alpha| \leq M+1} \left\| \langle r \rangle^{-1/2} \langle s \rangle^{-\delta} \Gamma^\alpha u'_J \right\|_{L^2(S_t)}
\]
\[\cdot \sum_{|\alpha| \leq M+1} \left\| \langle r \rangle^{-1/2} \langle s \rangle^{-\delta} \Gamma^\alpha u'_K \right\|_{L^2(S_t)} \lesssim \varepsilon^3. \quad (3.23)\]
where $\delta > 0$ is sufficiently small. Since $\mathbb{R}^3 = (\mathbb{R}^3 \setminus \Lambda_I) \cup (\mathbb{R}^3 \setminus \Lambda_J) \cup (\mathbb{R}^3 \setminus \Lambda_K)$ by $(c_J, c_K) \neq (c_I, c_I)$, we obtain $A_5 \lesssim \varepsilon^3$ and the required inequality.

(5) The left hand side of (5) is bounded by $(C_0 + C\varepsilon) \sum_{|\alpha| \leq M_0+2} \|\Gamma^\alpha u'(t, \cdot)\|_{L^2(\mathbb{R}^3)}$ by the similar argument for the proof of (3). The required inequality follows from (4).

\section{Several estimates to prove Theorem 1.2}

In this section, we prepare several estimates to prove Theorem 1.2 in the next section. Let $c > 0$, $0 < T \leq \infty$. We show the estimates for the solution of the scalar-valued problem

\begin{equation}
\begin{cases}
\Box_c u = F & \text{for } (t, x) \in [0, T) \times \mathbb{R}^3 \setminus \mathcal{K} \\
u(t, \cdot)\big|_{\partial \mathcal{K}} = 0 & \text{for } t \in [0, T) \\
u(0, \cdot) = f(\cdot), & \partial_t u(0, \cdot) = g(\cdot),
\end{cases}
\end{equation}

where $\Box_c := \partial^2 - c^2 \Delta$. Let $\zeta \in C^\infty_0(\mathbb{R}^3)$ be a function which satisfies $0 \leq \zeta \leq 1$, $\zeta(x) = 1$ for $|x| \leq 3$, and $\zeta(x) = 0$ for $|x| \geq 4$. Regarding $(1 - \zeta)F$, $(1 - \zeta)f$, $(1 - \zeta)g$ as functions on $\mathbb{R}^3$ by zero-extension, let $v$ be the solution of

\begin{equation}
\begin{cases}
\Box_c v = (1 - \zeta)F & \text{for } (t, x) \in [0, T) \times \mathbb{R}^3 \\
v(0, \cdot) = ((1 - \zeta)f)(\cdot), & \partial_t v(0, \cdot) = ((1 - \zeta)g)(\cdot).
\end{cases}
\end{equation}

\subsection{Estimates for boundary terms}

We use the following estimates to bound the terms from the commutator estimates.

\begin{lemma}
The solution $u$ of (4.1) satisfies the following estimate.

\begin{equation}
\sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \|L^\mu \partial^\alpha u(t, \cdot)\|_{L^2(\{x \in \mathbb{R}^3 \setminus \mathcal{K}: |x| < 2\})} \lesssim e^{-at/2} \sum_{\mu + |\alpha| \leq M+2 \atop \mu \leq \mu_0} \|L^\mu \partial^\alpha u(0, \cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \\
+ \int_0^t e^{-a(t-s)/2} \sum_{\mu + |\alpha| \leq M+1 \atop \mu \leq \mu_0} \|L^\mu \partial^\alpha \Box_c u(s, \cdot)\|_{L^2(\{x \in \mathbb{R}^3 \setminus \mathcal{K}: |x| < 4\})} ds \\
+ \sum_{\mu + |\alpha| \leq M-1 \atop \mu \leq \mu_0} \|L^\mu \partial^\alpha \Box_c u(t, \cdot)\|_{L^2(\{x \in \mathbb{R}^3 \setminus \mathcal{K}: |x| < 4\})} \\
+ \int_0^t e^{-a(t-s)/2} \sum_{\mu + |\alpha| \leq M+1 \atop \mu \leq \mu_0, |\beta| \leq 1} \|L^\mu \partial^\alpha \partial^\beta v(s, \cdot)\|_{L^2(\{x \in \mathbb{R}^3: |x| < 3\})} ds \\
+ \sum_{\mu + |\alpha| \leq M-1 \atop \mu \leq \mu_0, |\beta| \leq 1} \|L^\mu \partial^\alpha \partial^\beta v(t, \cdot)\|_{L^2(\{x \in \mathbb{R}^3: |x| < 3\})} \quad \text{for } 0 \leq t < T.
\end{equation}
\end{lemma}
\textbf{Proof.} For simplicity, we only consider the case \( c = 1 \). Let \( u_1 \) and \( u_2 \) be the solutions of the boundary value problems

\[
\begin{aligned}
\square u_1 = \zeta \square u, \quad & u_1|_{\partial \Omega} = 0, \\
u_1(0, \cdot) = \zeta u(0, \cdot), \quad & \partial_t u_1(0, \cdot) = \zeta \partial_t u(0, \cdot),
\end{aligned}
\label{eq:4.4}
\]

\[
\begin{aligned}
\square u_2 = (1 - \zeta) \square u, \quad & u_2|_{\partial \Omega} = 0, \\
u_2(0, \cdot) = (1 - \zeta) u(0, \cdot), \quad & \partial_t u_2(0, \cdot) = (1 - \zeta) \partial_t u(0, \cdot).
\end{aligned}
\label{eq:4.5}
\]

Then we have \( u = u_1 + u_2 \). By the local energy decay estimates (1.16), we have

\[
\begin{aligned}
\sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \| L^\mu \partial^\alpha u_1'(t, \cdot) \|_{L^2(\{ x \in \mathbb{R}^3 \setminus \Omega : |x| < 2 \})} & \lesssim e^{-at/2} \sum_{\mu + |\alpha| \leq M + 2, \mu \leq \mu_0} \| L^\mu \partial^\alpha u(0, \cdot) \|_{L^2(\mathbb{R}^3 \setminus \Omega)} \\
+ \int_0^t e^{-a(t-s)/2} \sum_{\mu + |\alpha| \leq M + 1, \mu \leq \mu_0} \| L^\mu \partial^\alpha \square u(s, \cdot) \|_{L^2(\{ x \in \mathbb{R}^3 \setminus \Omega : |x| < 4 \})} ds & + \sum_{\mu + |\alpha| \leq M - 1, \mu \leq \mu_0} \| L^\mu \partial^\alpha \square u(t, \cdot) \|_{L^2(\{ x \in \mathbb{R}^3 \setminus \Omega : |x| < 4 \})}.
\end{aligned}
\label{eq:4.6}
\]

Let \( u_3 := v|_{\mathbb{R}^3 \setminus \Omega} \) and \( u_4 := u_2 - u_3 \). Then \( \square u_4 = u_4(0, \cdot) = \partial_t u_4(0, \cdot) = 0 \). Let \( \rho \in C_0^\infty(\mathbb{R}^3) \) be a function with \( \rho(x) = 1 \) for \( |x| \leq 2 \), and \( \rho(x) = 0 \) for \( |x| \geq 3 \). We put \( \tilde{u}_2 := \rho u_3 + u_4 \). Then \( \tilde{u}_2 = u_2 \) for \( |x| \leq 2 \), and

\[
\begin{aligned}
\square \tilde{u}_2 & = -2\nabla \rho \cdot \nabla u_3 - (\Delta \rho) u_3, \\
\tilde{u}_2(0, \cdot) & = \rho u_3(0, \cdot), \\
\partial_t \tilde{u}_2(0, \cdot) & = \rho \partial_t u_3(0, \cdot).
\end{aligned}
\label{eq:4.7}
\]

So that, by (1.16), we have

\[
\begin{aligned}
\sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \| L^\mu \partial^\alpha u_2'(t, \cdot) \|_{L^2(\{ x \in \mathbb{R}^3 \setminus \Omega : |x| < 2 \})} & = \sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \| L^\mu \partial^\alpha \tilde{u}_2'(t, \cdot) \|_{L^2(\{ x \in \mathbb{R}^3 \setminus \Omega : |x| < 2 \})} \\
& \lesssim e^{-at/2} \sum_{\mu + |\alpha| \leq M + 2, \mu \leq \mu_0} \| L^\mu \partial^\alpha u(0, \cdot) \|_{L^2(\mathbb{R}^3 \setminus \Omega)} \\
+ \int_0^t e^{-a(t-s)/2} \sum_{\mu + |\alpha| \leq M + 1, \mu \leq \mu_0, |\beta| \leq 1} \| L^\mu \partial^\alpha L^\beta \partial^\beta \tilde{v}(s, \cdot) \|_{L^2(\{ x \in \mathbb{R}^3 : |x| < 3 \})} ds & + \sum_{\mu + |\alpha| \leq M - 1, \mu \leq \mu_0, |\beta| \leq 1} \| L^\mu \partial^\alpha L^\beta \partial^\beta \tilde{v}(t, \cdot) \|_{L^2(\{ x \in \mathbb{R}^3 : |x| < 3 \})}.
\end{aligned}
\label{eq:4.8}
\]

Therefore, we obtain the required estimate. \( \square \)
Proof. The results follow from Lemma 4.1, the boundedness of Lemma 4.3 of Cauchy data. We generalize it in complete form. □

The following result has been partially shown in [30, Lemma 2.9] for vanishing \( \mu \leq \mu_0 \), \( |\alpha| \leq \mu \leq \mu_0 \), \( \beta \leq 1 \), \( 0 \leq s \leq t \), \( 0 \leq t < T \).

\[
(1) \quad \sum_{\mu + |\alpha| \leq M} \frac{(1+t)}{\mu+|\alpha|+1} \|L^\mu \partial^\alpha u(t, \cdot)\|_{L^2((\{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2\})} \lesssim \sum_{\mu + |\alpha| \leq M+2} \|L^\mu \partial^\alpha u(0, \cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}
\]

\[
\quad + \sum_{\mu + |\alpha| \leq M+1} \sup_{\mu \leq \mu_0} (1+s) \|L^\mu \partial^\alpha \Box_c u(s, \cdot)\|_{L^2(\{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2\})}
\]

\[
\quad + \sum_{\mu + |\alpha| \leq M+1} \sup_{\mu \leq \mu_0} (1+s) \|L^\mu \partial^\alpha \Box_c u(s, \cdot)\|_{L^2(\{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2\})} \quad 0 \leq s \leq t.
\]

\[
(2) \quad \sum_{\mu + |\alpha| \leq M} \frac{\|L^\mu \partial^\alpha u\|_{L^2((0,T) \times \{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2\})}}{\mu \leq \mu_0} \lesssim \sum_{\mu + |\alpha| \leq M+2} \|L^\mu \partial^\alpha u(0, \cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}
\]

\[
\quad + \sum_{\mu + |\alpha| \leq M+1} \|L^\mu \partial^\alpha \Box_c u\|_{L^2((0,T) \times \{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2\})}
\]

\[
\quad + \sum_{\mu + |\alpha| \leq M+1} \|L^\mu \partial^\alpha \Box_c u\|_{L^2((0,T) \times \{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2\})} \quad 0 \leq s \leq t.
\]

\[
(4.9)
\]

\[
(4.10)
\]

Proof. The results follow from Lemma 4.1, the boundedness of \( te^{-\alpha t/2} \) for (1), and the Young inequality for the time variable for (2). □

The following result has been partially shown in [30, Lemma 2.9] for vanishing Cauchy data. We generalize it in complete form.

Lemma 4.3 The solution \( u \) of (4.1) satisfies

\[
\sum_{\mu + |\alpha| \leq M} \int_0^t \|L^\mu \partial^\alpha u(t, x)\|_{L^2((\{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2\})} ds \lesssim \sum_{\mu + |\alpha| \leq M+2} \|\langle x \rangle L^\mu \partial^\alpha u(0, x)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}
\]

\[
\quad + \sum_{\mu + |\alpha| \leq M+1} \int_0^t \|L^\mu \partial^\alpha \Box_c u(t, \cdot)\|_{L^2(\{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2\})} d\tau
\]

\[
\quad + \sum_{\mu + |\alpha| \leq M+1} \int_0^t \int_0^s \|L^\mu \partial^\alpha \Box_c u(\tau, \cdot)\|_{L^2(\{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2\})} d\tau ds.
\]

(4.11)
Proof. The required result easily follows from the integration by $t$ of (4.13) and the Young inequality if we show

$$
\sum_{\mu+|\alpha|\leq M+1, \mu \leq \mu_0, |\beta| \leq 1} \int_0^t \| L^\mu \partial^\alpha \partial^\beta v(s, \cdot) \|_{L^2(s \in \mathbb{R}^3: |x| < s)} ds \\
\lesssim \sum_{\mu+|\alpha|\leq M+2, \mu \leq \mu_0} \| \langle x \rangle L^\mu \partial^\alpha u(0, x) \|_{L^2_s(\mathbb{R}^3 \setminus \mathcal{K})} \\
+ \sum_{\mu+|\alpha|\leq M+1, \mu \leq \mu_0} \int_0^t \int_0^s \| L^\mu \partial^\alpha \Box_c u(\tau, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x| - \epsilon(s(t-s)) < 4\})} d\tau ds. \tag{4.12}
$$

To show this inequality, we prepare the following claim.

Claim 4.4 For any given functions $w_0$, $w_1$ and $G$, the solution $w$ of the Cauchy problem

$$
\begin{cases}
\Box w = G & \text{on } [0, \infty) \times \mathbb{R}^3 \\
w(0, \cdot) = w_0(\cdot), \quad \partial_\tau w(0, \cdot) = w_1(\cdot)
\end{cases} \tag{4.13}
$$

satisfies

$$
\sum_{|\alpha| \leq 1} \| \partial^\alpha w(t, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x| < s\})} \lesssim \sum_{|\alpha| \leq 1} \| \partial^\alpha w(0, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x| - ct < 4\})} \\
+ \int_0^t \| \Box_c w(s, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x| - \epsilon(t-s) \xi_s(t-s) < 4\})} ds \tag{4.14}
$$

for $t \geq 0$.

Proof. We note that $w$ is written as

$$
w(t, \cdot) = \partial_\tau K(t) w_0(\cdot) + K(t) w_1(\cdot) + \int_0^t K(t-s) G(s, \cdot) ds, \tag{4.15}
$$

where $K(t) := \sin ct \sqrt{-\Delta}/c \sqrt{-\Delta}$, $\partial_\tau K(t) := \cos ct \sqrt{-\Delta}$. Let $\chi \in C^\infty_0(\mathbb{R})$ be a function with $\chi(s) = 1$ for $-3 \leq s \leq 3$ and $\chi(s) = 0$ for $|s| \geq 4$. For any fixed $t \geq 0$, by the Huygens principle, we have

$$
w(t, x) = \partial_\tau K(t) (\chi(|\cdot| - ct) w_0(\cdot))(x) + K(t) (\chi(|\cdot| - ct) w_1(\cdot))(x) \\
+ \int_0^t K(t-s) (\chi(|\cdot| - c(t-s)) G(s, \cdot))(x) ds \tag{4.16}
$$

for $x$ with $|x| < 3$. By the isometry of $e^{it \sqrt{-\Delta}}$ and $\nabla/\sqrt{-\Delta}$ on $L^2(\mathbb{R}^3)$ and the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \hookrightarrow L^2(\{x \in \mathbb{R}^3: |x| < 3\})$, we have

$$
\| w(t, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x| < 3\})} \lesssim \| w_0 \|_{L^2(\{x \in \mathbb{R}^3: |x| - ct < 4\})} \\
+ \| w_1 \|_{L^2(\{x \in \mathbb{R}^3: |x| - ct < 4\})} + \int_0^t \| G(s, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x| - \epsilon(t-s) \xi_s(t-s) < 4\})} ds \tag{4.17}
$$

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and
\[ \| \nabla w(t, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x| < 3\})} \lesssim \sum_{|\alpha| \leq 1} \| \partial^\alpha_w w_0 \|_{L^2(\{x \in \mathbb{R}^3: |x-c(t)| < 4\})} \]
\[ + \| w_1 \|_{L^2(\{x \in \mathbb{R}^3: |x-c(t)| < 4\})} + \int_0^t \| G(s, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x-c(t-s)| < 4\})} ds. \]  
\[ \text{(4.18)} \]

Since \( \partial_t w \) is written as
\[ \partial_t w(t, \cdot) = c^2 \Delta K(t) w_0(\cdot) + \partial_t K(t) w_1(\cdot) + \int_0^t \partial_t K(t-s) G(s, \cdot) ds, \]
\[ \text{(4.19)} \]
we have by the Huygens principle
\[ \partial_t w(t, x) = c^2 \Delta K(t) (\chi(| \cdot |-ct) w_0(\cdot))(x) + \partial_t K(t) (\chi(| \cdot | - ct) w_1(\cdot))(x) \]
\[ + \int_0^t \partial_t K(t-s) (\chi(| \cdot | - c(t-s)) G(s, \cdot))(x) ds \]
\[ \text{(4.20)} \]
for \( x \) with \( |x| < 3 \). So that, \( \| \partial_t w(t, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x| < 3\})} \) is bounded by the right hand side of (4.18). And we obtain the required inequality. \( \square \)

We apply the above claim to \( v \) and integrate it by \( t \) to obtain
\[ \sum_{|\beta| \leq 1} \int_0^t \| \partial^\beta v(s, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x| < 3\})} ds \lesssim \sum_{|\beta| \leq 1} \int_0^t \| \partial^\beta v(0, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x-cs| < 4\})} ds \]
\[ + \int_0^t \int_0^s \| \Box v(\tau, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x-c(\tau-\tau)| < 4\})} d\tau ds. \]
\[ \text{(4.21)} \]
Here, the first term in the right hand side is bounded by
\[ \int_0^t \frac{1}{s^{\frac{1}{2}}} \sum_{|\beta| \leq 1} \| \langle s \rangle \partial^\beta v(0, \cdot) \|_{L^2(\{x \in \mathbb{R}^3: |x-cs| < 4\})} ds \lesssim \sum_{|\beta| \leq 1} \| \langle x \rangle \partial^\beta v(0, x) \|_{L^2(\mathbb{R}^3)}. \]
\[ \text{(4.22)} \]
By the replacement of \( v \) with \( L^\mu \partial^\alpha v \) and the definition of \( v \), we obtain (4.12). \( \square \)

### 4.2 Weighted energy estimates

The following weighted energy estimates are the exterior domain analog to Lemma 2.1.
Lemma 4.5 For any $M \geq 0$ and $\mu_0 \geq 0$, the solution $u$ of (4.1) satisfies

$$\sum_{\mu + |\alpha| \leq M} \|L^\mu Z^\alpha u\|_{L^\infty((0,T),L^2(\mathbb{R}^3 \setminus \mathcal{K}))} + \sum_{\mu + |\alpha| \leq M} \|\log(e + T)^{-1/2} \{t - r\}^{-1/2} L^\mu Z^\alpha u\|_{L^2((0,T) \times \mathbb{R}^3)}$$

$$\lesssim \sum_{\mu + |\alpha| \leq M+2} \|L^\mu Z^\alpha u(0,\cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{\mu + |\alpha| \leq M+1} \|L^\mu Z^\alpha \Box c u\|_{L^1((0,T),L^2(\mathbb{R}^3 \setminus \mathcal{K}))}$$

$$+ \sum_{\mu + |\alpha| \leq M-1} \|L^\mu \Box c u\|_{L^2((0,T) \times \{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 4\})}.$$  \hspace{1cm} (4.23)

Here, the above estimate holds with all $Z$ replaced by $\partial$.

Proof. The inequality for the second term in the left hand side has been shown by Metcalfe and Sogge [30, Proposition 2.6]. We show the inequality for the third term. The inequality for the first term follows similarly. We only show the case $c = 1$ for simplicity. First, we consider the boundaryless case.

Claim 4.6 Let us consider the Cauchy problem

$$\begin{cases}
\Box u = F & \text{on } [0,T) \times \mathbb{R}^3 \\
u(0,\cdot) = f(\cdot), \quad \partial_t u(0,\cdot) = g(\cdot).
\end{cases}$$  \hspace{1cm} (4.24)

Let $\text{supp } F \subset \{(t,x) : 0 \leq t < T, \; |x| \leq 2\}$. Then we have

$$(\log(e + T)^{-1/2}\{t - r\}^{-1/2} \Box u\|_{L^2((0,T) \times \mathbb{R}^3)}$$

$$ \lesssim \|\nabla f\|_{L^2(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} + \|\Box u\|_{L^2((0,T) \times \mathbb{R}^3)}.$$  \hspace{1cm} (4.25)

Proof. Let $\chi \geq 0$ be a bump function such that $\chi(t) = 1$ for $-1/4 \leq t \leq 1/4$, $\chi(t) = 0$ for $t \leq -1$ or $t \geq 1$, and $\sum_{j=0}^{\infty} \chi(t-j) = 1$ for any $t \geq 0$. Let $\{u_j\}_{j \geq -1}$ be the solutions of

$$\Box u_{-1} = 0, \quad u_{-1}(0,\cdot) = f(\cdot), \quad \partial_t u_{-1}(0,\cdot) = g(\cdot),$$  \hspace{1cm} (4.26)

$$\Box u_j(t,\cdot) = \chi(t-j)F(t,\cdot), \quad u_j(0,\cdot) = 0, \quad \partial_t u_j(0,\cdot) = 0$$  \hspace{1cm} (4.27)

for $j \geq 0$. Then we have $u = u_{-1} + \sum_{j=0}^{\infty} u_j$. By the Huygens principle, for any $(t,x) \in [0,T) \times \mathbb{R}^3$, there exists $j(t,x) \geq -1$ such that

$$u(t,x) = u_{-1}(t,x) + \sum_{|j-j(t,x)| \leq 3} u_j(t,x).$$  \hspace{1cm} (4.28)
So that, we have $|\partial u(t, x)|^2 \lesssim \sum_{j \geq -1} |\partial u_j(t, x)|^2$, and

\[
(\log(e + T))^{-1/2} \| (t - r)^{-1/2} \partial u_j \|_{L^2((0, T) \times \mathbb{R}^3)}^2 \lesssim \sum_{j = -1}^\infty (\log(e + T))^{-1} \| (t - r)^{-1/2} \partial u_j \|_{L^2((0, T) \times \mathbb{R}^3)}^2
\]

\[
\lesssim \| \nabla f \|_{L^2(\mathbb{R}^3)}^2 + C \| g \|_{L^2(\mathbb{R}^3)}^2 + \sum_{j \geq 0} \| \chi(-j) F \|_{L^1((0, T), L^2(\mathbb{R}^3))}^2,
\]

(4.29)

where we have used (2.4) for the last inequality. Since the last term is bounded by $\| F \|_{L^2((0, T) \times \mathbb{R}^3)}^2$ due to the Hölder inequality, we obtain the required result. \(\square\)

**Claim 4.7** Let us consider the problem

\[
\begin{cases}
\Box u = F & \text{on } [0, T) \times \mathbb{R}^3 \setminus \mathcal{K}, \\
u(0, \cdot) = f(\cdot), & \partial_t u(0, \cdot) = g(\cdot) \\
u(t, \cdot)|_{\mathcal{K}} = 0 & \text{for } t \in [0, T).
\end{cases}
\]

Then we have

\[
\sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} (\log(e + T))^{-1/2} \| (t - r)^{-1/2} \partial L^\mu Z^\alpha u \|_{L^2((0, T) \times \mathbb{R}^3 \setminus \mathcal{K})}
\]

\[
\lesssim \sum_{\mu + |\alpha| \leq M + 1 \atop \mu \leq \mu_0} \| L^\mu Z^\alpha u(0, \cdot) \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \| L^\mu Z^\alpha \Box u \|_{L^1((0, T), L^2(\mathbb{R}^3 \setminus \mathcal{K}))}
\]

\[
+ \sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \| L^\mu \partial^{\alpha} u \|_{L^2((0, T) \times \{x \in \mathbb{R}^3 \setminus \mathcal{K} \mid |x| < 2\})}.
\]

(4.31)

**Proof.** When $|x| \leq 2$, the left hand side is bounded by the last term. We consider the case $|x| \geq 2$. Let $0 \leq \eta \leq 1$ be a smooth function such that $\eta(x) = 0$ for $|x| \leq 1$, $\eta(x) = 1$ for $|x| \geq 2$. Then $\eta u$ can be seen as a function on $\mathbb{R}^3$ and satisfies

\[
\Box(\eta u) = \eta \Box u - 2 \nabla \eta \cdot \nabla u - (\Delta \eta) \cdot u.
\]

(4.32)

Let $u_1$ and $u_2$ be the solutions of

\[
\Box u_1 = \eta \Box u, \quad u_1(0, \cdot) = (\eta u)(0, \cdot), \quad \partial_t u_1(0, \cdot) = (\eta \partial_t u)(0, \cdot),
\]

(4.33)

\[
\Box u_2 = -2 \nabla \eta \cdot \nabla u - (\Delta \eta) u, \quad u_2(0, \cdot) = 0, \quad \partial_t u_2(0, \cdot) = 0
\]

(4.34)

on $[0, T) \times \mathbb{R}^3$. Since $u = \eta u$ for $|x| \geq 2$, and $\eta u = u_1 + u_2$, we have

\[
\sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} (\log(e + T))^{-1/2} \| (t - r)^{-1/2} \partial L^\mu Z^\alpha u \|_{L^2((0, T) \times \{x \in \mathbb{R}^3 \setminus \mathcal{K} \mid |x| > 2\})}
\]

\[
\leq \sum_{j = 1, 2} \sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} (\log(e + T))^{-1/2} \| (t - r)^{-1/2} \partial L^\mu Z^\alpha u_j \|_{L^2((0, T) \times \mathbb{R}^3)} =: I_1 + I_2.
\]

(4.35)
For $u_1$, we use Lemma 4.9 to obtain
\[
I_1 \lesssim \sum_{\mu+|\alpha|\leq M \atop \mu \leq \mu_0} \|L^\mu Z^\alpha u(0, \cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{\mu+|\alpha|\leq M \atop \mu \leq \mu_0} \|L^\mu Z^\alpha \Box u\|_{L^1((0,T),L^2(\mathbb{R}^3 \setminus \mathcal{K}))}. \tag{4.36}
\]

For $u_2$, we use Claim 4.6 to obtain
\[
I_2 \lesssim \sum_{\mu+|\alpha|\leq M+1 \atop \mu \leq \mu_0} \|L^\mu Z^\alpha u(0, \cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{\mu+|\alpha|\leq M+1 \atop \mu \leq \mu_0} \|L^\mu \Box u\|_{L^2((0,T) \times \{x \in \mathbb{R}^3 \setminus \mathcal{K} : |x| < 2\})}, \tag{4.37}
\]
where we have used the Poincaré inequality for the last term. Combining these estimates, we obtain the required result. □

Now, Lemma 4.5 follows from Claim 4.7 (2) of Lemma 4.2 and the estimate
\[
\sum_{\mu+|\alpha| \leq M+2 \atop \mu \leq \mu_0} \|L^\mu \Box v\|_{L^2((0,T) \times \{x \in \mathbb{R}^3 : |x| < 3\})}
\lesssim \sum_{\mu+|\alpha| \leq M+2 \atop \mu \leq \mu_0} \|L^\mu \Box u(0, \cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{\mu+|\alpha| \leq M+1 \atop \mu \leq \mu_0} \|L^\mu \Box u\|_{L^1((0,T),L^2(\mathbb{R}^3 \setminus \mathcal{K}))}, \tag{4.38}
\]
which has been shown by Keel, Smith and Sogge [17] (2.2), (2.5)]. □

### 4.3 Sobolev type estimates

We use the following weighted Sobolev estimates from [37], Lemma 3.3. To prove the estimate, we apply Sobolev estimates on $\mathbb{R}^3 \times S^2$. The decay result is from comparing the volume elements of $(0, \infty) \times S^2$ and $\mathbb{R}^3$.

**Lemma 4.8** Let $R \geq 1$. The following inequality holds for any smooth function $h$.
\[
\|h\|_{L^\infty((x \in \mathbb{R}^3 : R/2 < |x| < R))} \lesssim R^{-1} \sum_{|\alpha| \leq 2} \|Z^{\alpha} h\|_{L^2(\{x \in \mathbb{R}^3 : R/4 < |x| < 2R\})}. \tag{4.39}
\]

**Lemma 4.9** For any $M \geq 0$ and $\mu_0 \geq 0$, the solution $u$ of (4.1) satisfies the following.
\[
\sum_{\mu+|\alpha| \leq M \atop \mu \leq \mu_0} \langle t \rangle^{1/2} \langle r \rangle^{1/2} \langle ct - r \rangle^{1/2} |L^\mu Z^\alpha u'(t,x)|
\lesssim \sum_{\mu+|\alpha| \leq M+2 \atop \mu \leq \mu_0+1} \|L^\mu Z^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{\mu+|\alpha| \leq M+1 \atop \mu \leq \mu_0} \|\langle t + |\cdot| \rangle L^\mu Z^\alpha \Box u(\cdot, \cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}
+ \sum_{\mu \leq \mu_0} \|L^\mu u(\cdot, \cdot)\|_{L^2(\{y \in \mathbb{R}^3 \setminus \mathcal{K} : |y| < 2\})}. \tag{4.40}
\]
\[\sum_{\mu+|\alpha|\leq M} \langle r \rangle^{1/2} |(t+r)^{1/2} cis r^{1/2}| L^\mu Z^\alpha u(t, x)| \]
\[\lesssim \sum_{\mu+|\alpha|\leq M} \|L^\mu \partial^\alpha u(0, \cdot)\|_{L^2(\mathbb{R}^3 \setminus K)} + \sum_{\mu+|\alpha|\leq M+2} \|L^\mu Z^\alpha u(t, \cdot)\|_{L^2(\mathbb{R}^3 \setminus K)} \]
\[+ \sum_{\mu+|\alpha|\leq \max\{M+1, \mu_0+1\}} \sup_{0\leq t\leq t} \|L^\mu Z^\alpha \Box u(s, \cdot)\|_{L^2(\mathbb{R}^3 \setminus K)} \]
\[+ \sum_{\mu+|\alpha|\leq M+2} \sup_{0\leq t\leq t} (1+s) \|L^\mu \partial^\alpha v(t, \cdot)\|_{L^2((y\in\mathbb{R}^3; |y|<3))}. \quad (4.41)\]

**Proof.** The proof of (1) follows from [28, Lemma 4.2, Lemma 4.3] and Lemma 2.2. The proof of (2) follows from (1), and (1) of Lemma 1.2.

### 4.4 Commutator estimates

Let \(\chi\) be a nonnegative function with \(\chi(r) = 0\) if \(r \leq 1\) and \(\chi(r) = 1\) if \(r \geq 2\). We put
\[\tilde{L} := t\partial_t + \chi(r) r \partial_r. \quad (4.42)\]

When we consider the version of higher derivatives by \(L\) and \(Z\) of (2.1), we use the following commuting properties (see [30, p85, p113] and [33, p4761]).

\[(\partial_t^2 - c_I^2 \Delta) L^\mu \partial^m u_I - \sum_{1\leq K\leq D} \gamma_{K\ell}^I \partial_\ell \tilde{L}^\mu \partial^m u_K \]
\[= (\tilde{L} + 2) L^\mu \partial^m \Box u_I + \sum_{p\leq \mu-1} \chi_{p\mu} \tilde{L}^p \partial^m \partial^\nu \partial_\nu u_I + \sum_{K=1}^D \sum_{p\leq \mu-1} \sum_{0\leq k, l\leq 3} \chi_{pkm} \gamma_{K\ell}^I \tilde{L}^p \partial^m \partial^\nu \partial_\nu \partial_\kappa \partial_k \partial_l u_K \]
\[+ \sum_{K=1}^D \sum_{p+q+r\leq \mu} \sum_{0\leq k, l\leq 3} C_{pqrskl} \cdot (\tilde{L}^p \partial^m \gamma_{K\ell}^I) \tilde{L}^q \partial^m \partial_\nu \partial_\kappa \partial_k \partial_l u_K, \quad (4.43)\]

and

\[(\partial_t^2 - c_I^2 \Delta) L^\mu Z^\nu u_I - \sum_{1\leq K\leq D} \gamma_{K\ell}^I \partial_\ell L^\mu Z^\nu u_K \]
\[= (L + 2) L^\mu Z^\nu \Box u_I + \sum_{K=1}^D \sum_{p+q+r\leq \mu} \sum_{0\leq k, l\leq 3} C_{pqrskl} \cdot (L^p Z^\nu \gamma_{K\ell}^I) L^q Z^\delta \partial^\nu u_K, \quad (4.44)\]
where $\chi_{pv}$ and $\chi_{pvt\K}$ are smooth functions dependent on lower indices which supports are in the region $\{x \in \mathbb{R}^3 \setminus \K : \chi(x) \leq 2\}$, and the constants $C$ are dependent on the lower indices.

When we construct the energy estimates for the derivatives of the solution, we use the following estimates which follows from the elliptic regularity. For any $M \geq 0$ and $\mu_0 \geq 0$, we have

$$
\sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \| L^\mu \partial^\alpha u'(t, \cdot) \|_{L^2(\mathbb{R}^3 \setminus \K)} \lesssim \sum_{\mu + j \leq M \atop \mu \leq \mu_0} \|(\tilde{L}^\mu \partial^j u)'(t, \cdot)\|_{L^2(\mathbb{R}^3 \setminus \K)} + \sum_{\mu + |\alpha| \leq M - 1 \atop \mu \leq \mu_0} \| L^\mu \partial^\alpha \Box_c u(t, \cdot) \|_{L^2(\mathbb{R}^3 \setminus \K)} \tag{4.45}
$$

for any function $u$ and $c > 0$ which satisfies the Dirichlet condition $u|_{\partial \K} = 0$.

### 4.5 Pointwise estimates

We use the following pointwise estimates to show (4) of Proposition 5.1 below. The first is the $L^\infty - L^\infty$ estimate due to Kubota and Yokoyama [24, Theorem 3.4] for the boundaryless case (see also [29, Theorem 2.4] and its proof). The second is the $L^\infty - L^1$ estimate based on [19, Proposition 2.1].

**Lemma 4.10** For any $\theta > 0$, the solution $u = (u_1, \cdots, u_D)$ of

$$(\partial_t^2 - c_I^2 \Delta) u_I = F_I \text{ for } (t, x) \in [0, \infty) \times \mathbb{R}^3, \quad 1 \leq I \leq D \tag{4.46}$$

satisfies

$$(1 + t + r) \left(1 + \log \frac{1 + t + r}{1 + c_I(t - r)}\right)^{-1} |u_I(t, x)| \lesssim \sum_{\mu + |\alpha| \leq 3 \atop \mu \leq 1, j \leq 1} \|((y, \partial) L^\mu Z^\alpha u_I(0, y))\|_{L_y^2} + \sup_{(s, y) \in D_I(t, x)} |y|(1 + s + |y|)^{1 + \theta} \lambda_\theta(s, y)|F_I(s, y)|, \tag{4.47}$$

where $r = |x|$ and $D_I$, $\lambda_\theta$ are defined by

$$D_I(t, x) = \{(s, y) \in [0, \infty) \times \mathbb{R}^3 : 0 \leq s \leq t, ||x| - c_I(t - s)| \leq |y| \leq |x| + c_I(t - s)\} \tag{4.48}$$

$$\Lambda_I = \{(s, y) \in [0, \infty) \times \mathbb{R}^3 : s \geq 1, |y| \geq 1, ||y| - c_Is| \leq \min_{1 \leq J \leq D} |c_J - c_K|s/3\} \tag{4.49}$$

$$\lambda_\theta(s, y) = \begin{cases} (1 + |y| - c_Is)|^{1 - \theta} & \text{if } (s, y) \in \Lambda_I \text{ for } 1 \leq J \leq D \\ (1 + |y|)^{1 - \theta} & \text{if } (s, y) \in ((0, \infty) \times \mathbb{R}^3) \setminus (\cup_{1 \leq J \leq D} \Lambda_J) \end{cases} \tag{4.50}$$
Lemma 4.11 ([33 Lemma 2.12]) Let $u$ be the solution of
\[
\begin{cases}
\Box u = F & \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^3 \\
u(t, x) = 0 & \text{for } t \leq 0, \ x \in \mathbb{R}^3,
\end{cases}
\tag{4.51}
\]
where
\[
supp F \subset \{(s, y) : cs/10 \leq |y| \leq 10cs, \ s \geq 1\}.
\]
Then
\[
(1+t+|x|)|u(t, x)| \lesssim \sum_{\mu + |\alpha| \leq M_{\mu} \leq |\mu|} \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |L^\mu Z^\alpha F(s, y)|dy \left(1 + \left|\log \frac{1+t}{1+|c^\mu x|}\right|\right).
\tag{4.52}
\]

4.6 Estimates for nonlinear terms

We show some estimates to treat the nonlinear terms. We assume
\[
\sum_{|\alpha| \leq M_0} \sup_{0 \leq t < T} \sup_{x \in \mathbb{R}^3 \setminus \mathcal{K}} (1+t)|Z^\alpha u'(t, x)| \leq C \varepsilon
\tag{4.53}
\]
for some constant $C > 0$. First we consider the semilinear part of quadratic nonlinearities. For $M \leq 2M_0$, since we have
\[
\sum_{\mu + |\alpha| \leq M_{\mu} \leq |\mu|} |L^\mu \partial^\alpha (u' u'|) | \lesssim \sum_{|\alpha| \leq M_0} |\partial^\alpha u'| \sum_{\mu + |\alpha| \leq M_{\mu} \leq |\mu|} |L^\mu \partial^\alpha u'| \\
+ \sum_{M_0+1 \leq |\alpha| \leq M-1} |\partial^\alpha u'| \sum_{\mu + |\alpha| \leq M-M_0-1 \leq |\mu|} |L^\mu \partial^\alpha u'| \\
+ \sum_{\mu + |\alpha| \leq M/2 \leq |\mu|} |L^\mu \partial^\alpha u'| \sum_{1 \leq \mu \leq \mu_0-1} |L^\mu \partial^\alpha u'|. \tag{4.54}
\]
We obtain by the Sobolev estimate Lemma 4.8 and the assumption (4.53)
\[
\sum_{\mu + |\alpha| \leq M_{\mu} \leq |\mu|} \|L^\mu \partial^\alpha (u' u'|)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \lesssim \frac{\varepsilon}{1+t} \sum_{\mu + |\alpha| \leq M_{\mu} \leq |\mu|} \|L^\mu \partial^\alpha u'|\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \\
+ \sum_{M_0+1 \leq |\alpha| \leq M-1} \langle x \rangle^{-1/2} \|\partial^\alpha u'|\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \sum_{1 \leq \mu \leq \mu_0-1} \|\langle x \rangle^{-1/2} L^\mu Z^\alpha u'|\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \\
+ \sum_{\mu + |\alpha| \leq M/2 \leq |\mu|} \langle x \rangle^{-1/2} \|L^\mu Z^\alpha u'|\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \sum_{1 \leq \mu \leq \mu_0-1} \|\langle x \rangle^{-1/2} L^\mu \partial^\alpha u'|\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}. \tag{4.55}
\]
Similarly, for the quasilinear part of the nonlinearities, we obtain for \( M \leq 2M_0 - 2 \)

\[
\sum_{\mu + |\alpha| + 1 + |\beta| \leq M \atop \nu + |\beta| \leq M - 1} \| (L^\mu \partial^\alpha u')(L^\nu \partial^\beta u'') \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \lesssim \frac{\varepsilon}{1 + t} \sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \| L^\mu \partial^\alpha u' \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}
\]

\[
+ \sum_{M_0 + 1 \leq |\alpha| \leq M} \| \langle x \rangle^{-1/2} \partial^\alpha u' \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \sum_{\mu + |\alpha| \leq M - M_0 + 2 \atop 1 \leq \mu \leq \mu_0} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}
\]

\[
+ \sum_{1 \leq \mu \leq \mu_0 - 1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \sum_{\mu + |\alpha| \leq M \atop 1 \leq \mu \leq \mu_0 - 1} \| \langle x \rangle^{-1/2} L^\mu \partial^\alpha u' \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}.
\]  
(4.56)

5 Continuity argument to prove Theorem 1.2

We prove the following proposition to prove Theorem 1.2.

**Proposition 5.1** Let \( M_0 \geq 9 \). Let \( \mathcal{K}, F, f \) and \( g \) satisfy the assumption in Theorem 1.2. We put

\[
\varepsilon := \sum_{|\alpha| \leq 2M_0} \| \langle x \rangle^{\alpha} \partial_x^2 f(x) \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{|\alpha| \leq 2M_0 - 1} \| \langle x \rangle^{\alpha} \partial_x^2 g(x) \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}.
\]  
(5.1)

Let \( A_0 > 0 \). If the time local solution \( u \) of the Cauchy problem (1.17) with the time interval \([0, \infty)\) replaced by \([0, T]\) satisfies

\[
\sum_{|\alpha| \leq M_0 \atop 1 \leq I \leq D} \sup_{t, x \in S_T} \langle r \rangle^{1/2} (t + r)^{1/2} \langle c_I t - r \rangle^{1/2} |Z^\alpha u_I'(t, x)| \leq A_0 \varepsilon
\]  
(5.2)

and \( \varepsilon \) is sufficiently small, then for any \( \mu_0 \geq 0 \), \( 0 \leq M \leq 2M_0 - 2 - 10\mu_0 \) and \( \sigma > 0 \), there exist constants \( C_{M, \mu_0} > 0 \) which are dependent on \( A_0 \) such that the following inequality holds.
\[
\sum_{\mu+|\alpha| \leq M} \| (\hat{L}^{\mu} \partial_{x}^{\alpha} u'(t, \cdot) ) \|_{L^{2}(\mathbb{R}^{3} \setminus \mathcal{K})} + \sum_{\mu+|\alpha| \leq M} \| L^{\mu} \partial_{x}^{\alpha} u'(t, \cdot) \|_{L^{2}(\mathbb{R}^{3} \setminus \mathcal{K})} \\
+ \sum_{\mu+|\alpha| \leq M-2} (\log(e + t))^{-1/2} \langle x \rangle^{-1/2} \| L^{\mu} \partial_{x}^{\alpha} u'(t, \cdot) \|_{L^{2}(S_{t})} \\
+ \sum_{\mu+|\alpha| \leq M-3} \sum_{1 \leq j \leq D} \| L^{\mu} Z^{\alpha} u'(t, \cdot) \|_{L^{2}(\mathbb{R}^{3} \setminus \mathcal{K})} \\
+ \sum_{\mu+|\alpha| \leq M-5} \sum_{1 \leq j \leq D} \| L^{\mu} Z^{\alpha} u'(t, \cdot) \|_{L^{2}(S_{t})} \\
\leq C_{M,\mu_{0}}(1 + t)^{C_{M,\mu_{0}}(\varepsilon + \sigma)} \tag{5.3}
\]

for \(0 \leq t < T\). Moreover, if \(M_{0} \geq 32\), then there exist constants \(C_{0} > 0\), which is independent of \(A_{0}\), and \(C > 0\), which is dependent on \(A_{0}\), such that the following inequalities hold.

\[
(1) \sum_{\mu+|\alpha| \leq M_{0}+5} \| L^{\mu} Z^{\alpha} u'(t, \cdot) \|_{L^{2}(\mathbb{R}^{3} \setminus \mathcal{K})} \leq C_{0}\varepsilon + C_{\varepsilon}^{2}(1 + t)^{C(\varepsilon + \sigma)}, \tag{5.4}
\]

\[
(2) \sup_{\mu+|\alpha| \leq M_{0}+3, x \in \mathbb{R}^{3} \setminus \mathcal{K}} \langle r \rangle^{1/2} \langle t + r \rangle^{1/2} \langle c_{I} t - r \rangle^{1/2} |L^{\mu} Z^{\alpha} u'_I(t, x)| \leq C_{0}\varepsilon + C_{\varepsilon}^{2}(1 + t)^{C(\varepsilon + \sigma)} \tag{5.5}
\]

for \(0 \leq t < T\).

\[
(3) \sum_{\mu+|\alpha| \leq M_{0}+2} \| L^{\mu} Z^{\alpha} u' \|_{L^{\infty}((0,T), L^{2}(\mathbb{R}^{3} \setminus \mathcal{K}))} \leq C_{0}\varepsilon + C_{\varepsilon}^{3/2}. \tag{5.6}
\]

\[
(4) \sup_{|\alpha| \leq 2, 0 \leq t < T} (1 + t) \| \partial_{x}^{\alpha} v(t, \cdot) \|_{L^{\infty}((x \in \mathbb{R}^{3} : |x| < 3))} \leq C_{0}\varepsilon + C_{\varepsilon}^{2}, \tag{5.7}
\]

where \(v = (v_{1}, \ldots, v_{D})\) is the solution of

\[
\left\{ \begin{array}{ll}
(\partial_{t}^{2} - c_{I}^{2} \Delta) v_{I} = (1 - \zeta) F_{I} \quad \text{for} \quad (t, x) \in [0, T) \times \mathbb{R}^{3}, \quad 1 \leq I \leq D \\
v(0, \cdot) = ((1 - \zeta) f)(\cdot), \quad \partial_{t} v(0, \cdot) = ((1 - \zeta) g)(\cdot)
\end{array} \right. \tag{5.8}
\]

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and \( \zeta \in C^\infty_0(\mathbb{R}^3) \) is a function which satisfies \( 0 \leq \zeta \leq 1 \), \( \zeta(x) = 1 \) for \( |x| \leq 3 \), and \( \zeta(x) = 0 \) for \( |x| \geq 4 \). Here, \( (1 - \zeta)F \), \( (1 - \zeta)f \), \( (1 - \zeta)g \) are regarded as functions on \( \mathbb{R}^3 \) by zero-extension.

\( \sum_{|\alpha| \leq M_0} \sup_{(t,x) \in S_T} (r)^{1/2}(t + r)^{1/2}(ct - r)^{1/2}|Z^\alpha u'_I(t,x)| \leq C_0\varepsilon + C\varepsilon^{3/2}. \) (5.9)

5.1 Proof of Theorem 1.2

We use the continuity argument which shows that the local in time solution \( u \) does not blow up if its initial data are sufficiently small. We refer to [18] for the existence of the local in time solutions. Since the constant \( C_0 \) is independent of \( A_0 \) in (5.9), we put \( A_0 = 4C_0 \) and take \( \varepsilon \) sufficiently small such that \( C\varepsilon^{3/2} \leq C_0\varepsilon \). Then the right hand side of (5.9) is bounded by \( A_0\varepsilon/2 \), which shows the local in time solution \( u \) does not blow up, namely the solution exists globally in time.

5.2 Proof of Proposition 5.1

First, we show the estimate (5.3) inductively, and then we derive the estimates from (1) to (5). We drop the indices \( I \) of \( c_I \), \( u_I \) and so on to avoid the complexity.

5.2.1 The estimate for \( \|L^\mu \partial^\alpha u'\|_2 \)

By (4.49), we have

\[
\sum_{\mu + |\alpha| \leq M} \|L^\mu \partial^\alpha u'\|_2 \lesssim \sum_{\mu + j \leq M, \mu \leq \mu_0} \|\tilde{L}^\mu \partial^j u\|_2 + \sum_{\mu + |\alpha| \leq M - 1, \mu \leq \mu_0} \|L^\mu \partial^\alpha \Box u\|_2, \tag{5.10}
\]

and we estimate the last term by the argument in Section 4.3:

\[
\sum_{\mu + |\alpha| \leq M - 1, \mu \leq \mu_0} \|L^\mu \partial^\alpha \Box u\|_2 \lesssim \frac{\varepsilon}{1 + t} \sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \|L^\mu \partial^\alpha u'\|_2 + A, \tag{5.11}
\]

where

\[
A := \sum_{M_0 + 1 \leq |\alpha| \leq M - 1} \|\langle x \rangle^{-1/2} \partial^\alpha u'\|_2 \sum_{\mu + |\alpha| \leq M - M_0 + 1, 1 \leq \mu \leq \mu_0 + 1} \|\langle x \rangle^{-1/2} L^\mu Z^\alpha u'\|_2 \nonumber \]
\[
+ \sum_{\mu + |\alpha| \leq M/2 + 2, 1 \leq \mu \leq \mu_0 - 1} \|\langle x \rangle^{-1/2} L^\mu Z^\alpha u'\|_2 \sum_{\mu + |\alpha| \leq M - 1, 1 \leq \mu \leq \mu_0 - 1} \|\langle x \rangle^{-1/2} L^\mu \partial^\alpha u'\|_2. \tag{5.12}
\]

Therefore for sufficiently small \( \varepsilon > 0 \), we obtain

\[
\sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \|L^\mu \partial^\alpha u'\|_2 \lesssim \sum_{\mu + j \leq M, \mu \leq \mu_0} \|\tilde{L}^\mu \partial^j u\|_2 + A. \tag{5.13}
\]
\[5.2.2 \text{ The estimate for the boundary term}\]

To consider the estimate for \(\|\partial \tilde{L}^\mu \partial^I u\|_2\) in the next subsection, we prepare the estimate for the boundary term. By Lemma 4.5, we have

\[
\sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0 - 1} \int_0^t \|L^\mu \partial^\alpha u'(s, x)\|_{L^2(|x|<2)} ds \lesssim \sum_{\mu + |\alpha| \leq M+2 \atop \mu \leq \mu_0 - 1} \|\langle x \rangle L^\mu \partial^\alpha u(0, x)\|_{L^2} \\
+ \sum_{\mu + |\alpha| \leq M+1 \atop \mu \leq \mu_0 - 1} \int_0^t \|L^\mu \partial^\alpha \square_c u(s, x)\|_{L^2_2(|x|<4)} ds \\
+ \sum_{\mu + |\alpha| \leq M+1 \atop \mu \leq \mu_0 - 1} \int_0^t \int_0^\infty \|L^\mu \partial^\alpha \square_c u(\tau, x)\|_{L^2_2(|x|<\epsilon(s-\tau)<4)} d\tau ds. \tag{5.14}
\]

Since the last two terms are bounded by

\[
\sum_{\mu + |\alpha| \leq M+2 \atop \mu \leq \mu_0 - 1} \|\langle x \rangle^{1/2} L^\mu Z^\alpha u'(s, x)\|_{L^2_2(S_t)}^2 \tag{5.15}
\]

due to Lemma 4.8, we obtain

\[
\sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0 - 1} \int_0^t \|L^\mu \partial^\alpha u\|_{L^2(|x|<2)} ds \lesssim \sum_{\mu + |\alpha| \leq M+2 \atop \mu \leq \mu_0 - 1} \|\langle x \rangle L^\mu \partial^\alpha u(0, x)\|_{L^2_2(\mathbb{R}^3 \setminus \mathcal{K})} \\
+ \sum_{\mu + |\alpha| \leq M+2 \atop \mu \leq \mu_0 - 1} \|\langle x \rangle^{1/2} L^\mu Z^\alpha u'(S_t)\|_{L^2(\mathcal{S}_t)}. \tag{5.16}
\]

\[5.2.3 \text{ The estimate for } \|\partial \tilde{L}^\mu \partial^I u\|_2\]

Since \(\tilde{L}^\mu \partial^I u\) satisfies the Dirichlet condition, by the energy estimate (2.22), we have

\[
\partial \left\{ \int_{\mathbb{R}^3 \setminus \mathcal{K}} e_0(\tilde{L}^\mu \partial^I u) dx \right\}^{1/2} \lesssim \|\square \tilde{L}^\mu \partial^I u\|_2 + \|\gamma\|_{\infty} \left\{ \int_{\mathbb{R}^3 \setminus \mathcal{K}} e_0(\tilde{L}^\mu \partial^I u) dx \right\}^{1/2}. \tag{5.17}
\]

By (1.43), we have

\[
\sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \|\square \tilde{L}^\mu \partial^I u\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \lesssim (1 + \|\gamma\|_{\infty}) \sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \|L^\mu \partial^\alpha u\|_{L^2(|x|<2)} \\
+ \sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \|L^\mu \partial^\alpha \square u\|_2 + \sum_{\mu + |\alpha| + |\beta| \leq M \atop \mu + |\beta| \leq \mu_0 - 1} \|L^\mu \partial^\alpha \gamma\cdot(L^\nu \partial^\beta u''\|_2. \tag{5.18}
\]
Using the estimates (4.55) and (4.56), the last two terms are bounded by

\[
\frac{\varepsilon}{1 + \varepsilon} \sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \| L^\mu \partial^\alpha u' \|_2^2 + B \tag{5.19}
\]

for \( M \leq 2M_0 - 2 \), where

\[
B := \sum_{M_0 + 1 \leq |\alpha| \leq M} \| \langle x \rangle^{-1/2} \partial^\alpha u' \|_2 \sum_{\mu + |\alpha| \leq M - M_0 + 2 \atop 1 \leq \mu \leq \mu_0} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_2
\]

\[
+ \sum_{\mu + |\alpha| \leq M/2 + 3 \atop 1 \leq \mu \leq \mu_0 - 1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_2 \sum_{\mu + |\alpha| \leq M \atop 1 \leq \mu \leq \mu_0 - 1} \| \langle x \rangle^{-1/2} L^\mu \partial^\alpha u' \|_2. \tag{5.20}
\]

So that, by (5.13), the Gronwall inequality and (5.16), we have

\[
\sum_{\mu + j \leq M \atop \mu \leq \mu_0} \| \partial_t \tilde{L}^\mu \partial^j u \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \lesssim \sum_{\mu + j \leq M \atop \mu \leq \mu_0} \left\{ \int_{\mathbb{R}^3 \setminus \mathcal{K}} e_0(\tilde{L}^\mu \partial^j u) dx \right\}^{1/2}
\]

\[
\lesssim \left\{ \sum_{\mu + |\alpha| \leq M + 2 \atop \mu \leq \mu_0} \| \langle x \rangle L^\mu \partial^\alpha u(0, x) \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}^{1/2} \sum_{\mu + |\alpha| \leq \max\{M/2 + 3, M + 2\} \atop \mu \leq \mu_0 - 1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_{L^2(S_t)}^2 \right. \right.
\]

\[
+ \sum_{M_0 + 1 \leq |\alpha| \leq M} \| \langle x \rangle^{-1/2} \partial^\alpha u' \|_{L^2(S_t)} \sum_{\mu + |\alpha| \leq M - M_0 + 2 \atop 1 \leq \mu \leq \mu_0} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_{L^2(S_t)}^2
\]

\[
+ \sum_{\mu + |\alpha| \leq M/2 + 3 \atop 1 \leq \mu \leq \mu_0 - 1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_{L^2(S_t)} \sum_{\mu + |\alpha| \leq M \atop 1 \leq \mu \leq \mu_0 - 1} \| \langle x \rangle^{-1/2} L^\mu \partial^\alpha u' \|_{L^2(S_t)} \left( 1 + t \right)^C \varepsilon \right\} \tag{5.21}
\]

for \( M \leq 2M_0 - 2 \), where we have used that \( \varepsilon > 0 \) is sufficiently small for the first inequality.

**5.2.4 The estimate for \( \| L^\mu Z^\alpha u' \|_2 \)**

By the energy estimate (2.22) for \( L^\mu Z^\alpha u \), we have

\[
\sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \partial_t \int_{\mathbb{R}^3 \setminus \mathcal{K}} e_0(L^\mu Z^\alpha u) dx \lesssim \sum_{\mu + |\alpha| \leq M + 1 \atop \mu \leq \mu_0} \| L^\mu \partial^\alpha u' \|_2^2 \tag{5.22}
\]

\[
+ \sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \left| \int_{\mathbb{R}^3 \setminus \mathcal{K}} (\partial_t L^\mu Z^\alpha u, \Box_j L^\mu Z^\alpha u) dx \right| + \| \gamma' \|_\infty \sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \| L^\mu Z^\alpha u' \|_2^2.
\]
where we have used the trace theorem for the boundary term, so that, there is a loss of one derivative. By (4.141), we have

\[
\sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} |\Box_\gamma L^\mu Z^\alpha u| \lesssim \sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} |L^\mu Z^\alpha \Box_\gamma u| \\
+ \sum_{\mu + |\alpha| + \nu + |\beta| \leq M, \mu + \nu \leq \mu_0, \nu + |\beta| \leq M-1} |(L^\mu Z^\alpha \gamma)(L^\nu Z^\beta \partial^2 u)|.
\]  

(5.23)

So that, by (4.55) and (4.56), we have

\[
\sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \|\Box_\gamma L^\mu Z^\alpha u\|_2 \lesssim \frac{\varepsilon}{1 + t} \sum_{\mu + |\alpha| \leq M+1, \mu \leq \mu_0} \|L^\mu \partial^\alpha u\|_{L^2(|x| < 2)}^2 + B.
\]  

(5.24)

where \(B\) is \(B\) in (5.20) with all \(\partial\) replaced by \(Z\). Since \(\|L^\mu Z^\alpha u\|_2\) is equivalent to \(e_0(L^\mu Z^\alpha u)\) for sufficiently small \(\varepsilon > 0\), we have

\[
\sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \partial_t \int_{\mathbb{R}^3 \setminus \mathcal{K}} e_0(L^\mu Z^\alpha u) dx \lesssim \sum_{\mu + |\alpha| \leq M+1, \mu \leq \mu_0} \|L^\mu \partial^\alpha u\|_{L^2(|x| < 2)}^2 \\
+ \frac{\varepsilon}{1 + t} \sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \int_{\mathbb{R}^3 \setminus \mathcal{K}} e_0(L^\mu Z^\alpha u) dx + \left( \sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \int_{\mathbb{R}^3 \setminus \mathcal{K}} e_0(L^\mu Z^\alpha u) dx \right)^{1/2} \cdot B.
\]  

(5.25)

So that, by the Gronwall inequality, we obtain for \(M \leq 2M_0 - 2\)

\[
\sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \|L^\mu Z^\alpha u\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} \lesssim \left\{ \sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \int_{\mathbb{R}^3 \setminus \mathcal{K}} e_0(L^\mu Z^\alpha u) dx \right\}^{1/2} \\
\lesssim \left\{ \sum_{\mu + |\alpha| \leq M, \mu \leq \mu_0} \|L^\mu Z^\alpha u(0, \cdot)\|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{\mu + |\alpha| \leq M+1, \mu \leq \mu_0} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u\|_{L^2(S_t)} \\
+ \sum_{M_0 + 1 \leq |\alpha| \leq M} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u\|_{L^2(S_t)} \sum_{1 \leq \mu \leq \mu_0} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u\|_{L^2(S_t)} \\
+ \sum_{\mu + |\alpha| \leq M/2 + 3, 1 \leq \mu \leq \mu_0 - 1} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u\|_{L^2(S_t)} \sum_{1 \leq \mu \leq \mu_0 - 1} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u\|_{L^2(S_t)} \right\} (1 + t)^C_\varepsilon.
\]  

(5.26)
5.2.5 The estimates for the weighted energy

By Lemma 4.5 and (4.55), we have for $M \leq 2M_0 - 2$

$$\sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} \| L^\mu \partial^\alpha u' \|_{L^\infty((0, t), L^2(\mathbb{R}^3 \setminus \mathcal{K}))} + \sum_{\mu + |\alpha| \leq M \atop \mu \leq \mu_0} (\log(e + t))^{-1/2} \| (x)^{-1/2} L^\mu \partial^\alpha u' \|_{L^2(S_t)}$$

$$+ \sum_{\mu + |\alpha| \leq M \atop 1 \leq \mu \leq \mu_0} (\log(e + t))^{-1/2} \| \langle c_t s - r \rangle^{-1/2} \partial_{c_t} L^\mu \partial^\alpha u_t(s, x) \|_{L^2(S_t)}$$

$$\lesssim \sum_{\mu + |\alpha| \leq M + 2 \atop \mu \leq \mu_0} \| L^\mu \partial^\alpha u(0, \cdot) \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} + \sum_{\mu + |\alpha| \leq M + 2 \atop \mu \leq \mu_0} \| L^\mu \partial^\alpha u' \|_{L^\infty((0, t), L^2(\mathbb{R}^3 \setminus \mathcal{K}))} \varepsilon \log(1 + t)$$

$$+ \sum_{M_0 + 1 \leq |\alpha| \leq M + 1} \| \langle x \rangle^{-1/2} \partial^\alpha u' \|_{L^2(S_t)} + \sum_{\mu + |\alpha| \leq M - M_0 + 3 \atop 1 \leq \mu \leq \mu_0} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_{L^2(S_t)}$$

$$+ \sum_{\mu + |\alpha| \leq M/2 + 3 \atop 1 \leq \mu \leq \mu_0 - 1} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_{L^2(S_t)} \sum_{\mu + |\alpha| \leq M + 1 \atop 1 \leq \mu \leq \mu_0 - 1} \| \langle x \rangle^{-1/2} L^\mu \partial^\alpha u' \|_{L^2(S_t)}.$$  \hspace{1cm} (5.27)

Here, the above estimate also holds with all $\partial$ replaced by $Z$.

5.2.6 The proof of (1)

The proof of (1) follows from (5.27) with $M = M_0 + 5$ and $\mu_0 = 2$ by (5.3). Indeed, to bound $\sum_{\mu + |\alpha| \leq M_0 + 5} \| L^\mu Z^\alpha u' \|_{L^\infty((0, t), L^2(\mathbb{R}^3 \setminus \mathcal{K}))}$, we need the estimate for $\sum_{\mu + |\alpha| \leq M_0 + 7} \| L^\mu Z^\alpha u' \|_{L^\infty((0, t), L^2(\mathbb{R}^3 \setminus \mathcal{K}))}$ by (5.27), which is bounded by $C\varepsilon(1 + t)^{C(\varepsilon + \sigma)}$ by (5.3) since $M_0 + 7 \leq (2M_0 - 2 - 20) - 3$ is satisfied by $M_0 \geq 32$.

5.2.7 The proof of (2)

By (5.2) and induction argument, we have

$$\sum_{\mu + |\alpha| \leq M_0 + 4 \atop \mu \leq 1} \langle t + |y| \rangle \| L^\mu Z^\alpha \Box u(t, y) \| \lesssim \varepsilon \sum_{\mu + |\alpha| \leq M_0 + 5 \atop \mu \leq 1} \| L^\mu Z^\alpha u'(t, y) \| (1 + t)^{C(\varepsilon + \sigma)}.$$  \hspace{1cm} (5.28)

So that, we have

$$\sum_{\mu + |\alpha| \leq M_0 + 4 \atop \mu \leq 1} \| \langle t + |y| \rangle L^\mu Z^\alpha \Box u(t, y) \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})}$$

$$\lesssim \varepsilon \sum_{\mu + |\alpha| \leq M_0 + 5 \atop \mu \leq 1} \| L^\mu Z^\alpha u'(t, y) \|_{L^2(\mathbb{R}^3 \setminus \mathcal{K})} (1 + t)^{C(\varepsilon + \sigma)}.$$  \hspace{1cm} (5.29)

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By simple calculation and (5.16), we have

\[
\sum_{\mu \leq 1} t \| L^\mu u'_I(t, y) \|_{L^2(|y| < 2)} \lesssim \sum_{\mu + |\alpha| \leq 1} \int_0^t \| L^\mu \partial^\alpha u'(s, y) \|_{L^2(|y| < 2)} ds \\
\lesssim \sum_{\mu + |\alpha| \leq 4} \| \langle y \rangle L^\mu \partial^\alpha u(0, y) \|_{L^2_0(\mathbb{R}^3 \setminus K)} + \sum_{\mu + |\alpha| \leq 4} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_{L^2(S_1)}.
\]

So that, by (4.40), we obtain

\[
\sum_{\mu + |\alpha| \leq M_0 + 3} \langle t \rangle^{1/2} \langle t + r \rangle^{1/2} (ct - r)^{1/2} |L^\mu Z^\alpha u'_I(t, x)| \\
\lesssim \sum_{\mu + |\alpha| \leq 4} \| \langle y \rangle L^\mu \partial^\alpha u(0, y) \|_{L^2_0(\mathbb{R}^3 \setminus K)} + \sum_{\mu + |\alpha| \leq 4} \| \langle x \rangle^{-1/2} L^\mu Z^\alpha u' \|_{L^2(S_1)} \\
+ \varepsilon \sum_{\mu + |\alpha| \leq M_0 + 5} \| L^\mu Z^\alpha u'_I(t, y) \|_{L^2_0(\mathbb{R}^3 \setminus K)} (1 + t)^C(\varepsilon + \sigma) \\
+ \sum_{\mu + |\alpha| \leq M_0 + 5} \| L^\mu Z^\alpha u'_I(t, \cdot) \|_{L^2(\mathbb{R}^3 \setminus K)}.
\]

Therefore we obtain the required estimate by (5.3) and (1).

5.2.8 The proof of (3)

By the standard energy estimate, we have

\[
\sum_{\mu + |\alpha| \leq M_0 + 2} \| L^\mu Z^\alpha u'_I(t, \cdot) \|_{L^2}^2 \leq C_0 \sum_{\mu + |\alpha| \leq M_0 + 2} \| L^\mu Z^\alpha u'_I(0, \cdot) \|_{L^2}^2 \\
+ C_0 \sum_{\mu + |\alpha| \leq M_0 + 2} \int_0^t \int_{\mathbb{R}^3 \setminus K} |(\partial_t L^\mu Z^\alpha u_I) \square_c L^\mu Z^\alpha u_I| dx ds \\
+ C_0 \sum_{\mu + |\alpha| \leq M_0 + 3} \int_0^t \| L^\mu \partial^\alpha u'_I(s, x) \|_{L^2(|x| < 1)}^2 ds,
\]

where \( C_0 > 0 \) is independent of \( A_0 \). We use (2) to bound the last term

\[
\sum_{\mu + |\alpha| \leq M_0 + 3} \int_0^t \| L^\mu Z^\alpha u'_I(s, x) \|_{L^2(|x| < 1)}^2 ds \\
\leq \int_0^t \langle r \rangle^{-1/2} (s + r)^{-1/2} (ct - r)^{-1/2} \left\{ C_0 \varepsilon + C \varepsilon^2 (1 + s)^C(\varepsilon + \sigma) \right\}^2_{L^2(|x| < 1)} ds \\
\leq (C_0 \varepsilon + C \varepsilon^2)^2.
\]
Since we are able to have the bound
\[ \sum_{\mu + |\alpha| \leq M_{0} + 2} \int_{0}^{t} \int_{\mathbb{R}^{3} \setminus \mathcal{K}} |(\partial_{t} L^{\mu} Z^{\alpha} u_{I}) \square_{\alpha} L^{\mu} Z^{\alpha} u_{I}| \, dx \, ds \leq C \varepsilon^{3} \] (5.34)
by the similar argument for the proof of (3.3), we obtain the required inequality.

5.2.9 The proof of (4)

Let \( \chi \in C_{\infty}(\mathbb{R}) \) satisfy \( \chi(t) = 0 \) for \( t \leq 1 \), and \( \chi(t) = 1 \) for \( t \geq 2 \). Let \( \eta \in C_{\infty}(\mathbb{R}) \) satisfy \( \eta(r) = 0 \) for \( r \leq \min_{1 \leq I \leq D} c_{I}/10 \) or \( r \geq 10 \max_{1 \leq I \leq D} c_{I} \), and \( \eta(r) = 1 \) for \( \min_{1 \leq I \leq D} c_{I}/5 \leq r \leq 5 \max_{1 \leq I \leq D} c_{I} \). We put \( \rho(t, x) := \chi(t) \eta(|x|/t) \). We decompose \( v \) into \( w = (w_{1}, \cdots, w_{D}) \) and \( z = (z_{1}, \cdots, z_{D}) \) which satisfy
\[
\begin{align*}
\square_{\alpha} w_{I}(t, x) &= \rho(t, x) \square_{\alpha} v_{I}(t, x) \quad \text{for} \quad (t, x) \in [0, T) \times \mathbb{R}^{3}, \quad 1 \leq I \leq D \\
w(0, \cdot) &= \partial_{t} w(0, \cdot) = 0 \\
\square_{\alpha} z_{I}(t, x) &= (1 - \rho(t, x)) \square_{\alpha} v_{I}(t, x) \quad \text{for} \quad (t, x) \in [0, T) \times \mathbb{R}^{3}, \quad 1 \leq I \leq D \\
z(0, \cdot) &= v(0, \cdot), \quad \partial_{t} z(0, \cdot) = \partial_{t} v(0, \cdot).
\end{align*}
\tag{5.35}
\]

We note \( v = w + z \) and show the required estimates for \( w \) and \( z \). By Lemma 4.11, we have
\[
\begin{align*}
\sum_{|\alpha| \leq 2} (1 + t + |x|)|Z^{\alpha} w_{I}(t, x)| &\lesssim \sum_{\mu + |\alpha| \leq 5} \sup_{0 \leq s \leq t} \int_{\mathbb{R}^{3}} |L^{\mu} Z^{\alpha} \square_{\alpha} w_{I}|(s, y) \, dy \left( 1 + \log \frac{1 + t}{1 + |c_{I} t - |x||} \right). \tag{5.37}
\end{align*}
\]
So that, we have by (3)
\[
\begin{align*}
\sum_{|\alpha| \leq 2} (1 + t)\|\partial^{\alpha} w_{I}(t, x)\|_{L^{\infty}(|x| < 3)} &\lesssim \sum_{\mu + |\alpha| \leq 6} \|L^{\mu} Z^{\alpha} u_{I}\|_{L^{\infty}((0, t), L^{2}(\mathbb{R}^{3} \setminus \mathcal{K}))} \lesssim \varepsilon^{2}. \tag{5.38}
\end{align*}
\]
On the other hand, by Lemma 4.10, we have
\[
\begin{align*}
\sum_{|\alpha| \leq 2} (1 + t + |x|)|Z^{\alpha} z_{I}(t, x)| &\lesssim \sum_{\mu + |\alpha| \leq 5} \|(y)\partial^{\eta} L^{\mu} Z^{\alpha} z_{I}(0, y)\|_{L^{2}_{x}(\mathbb{R}^{3})} \\
&+ \sum_{|\alpha| \leq 2} \sup_{(s, y) \in D_{I}(t, x)} |y|(1 + s + |y|)^{1+\theta} \lambda_{\theta}(s, y)|\square_{\alpha} Z^{\alpha} z_{I}(s, y)| \tag{5.39}
\end{align*}
\]
for any fixed \( \theta > 0 \). Since
\[
\begin{align*}
\sum_{|\alpha| \leq 2} |\square_{\alpha} Z^{\alpha} z_{I}(s, y)| &\lesssim (1 - \rho(s, y)) \sum_{|\alpha| \leq 3} |Z^{\alpha} u_{I}(s, y)|^{2} \lesssim \varepsilon^{2} (y)^{-1} (s + |y|)^{-2} \tag{5.40}
\end{align*}
\]
by (5.2), we obtain
\[
\begin{align*}
\sum_{|\alpha| \leq 2} (1 + t + |x|)|Z^{\alpha} z_{I}(t, x)| &\leq C_{0} \varepsilon + C \varepsilon^{2}. \tag{5.41}
\end{align*}
\]
Combining the above estimates for \( w_{I} \) and \( z_{I} \), we obtain the required estimate.
5.2.10 The proof of (5)

By (4.41), we have
\[ \sum_{|\alpha| \leq M_0} \langle r \rangle^{1/2} (\langle t+ r \rangle^{1/2} (ct - r)^{1/2}) |Z^\alpha u(t,x)| \leq C_0 \sum_{|\alpha| \leq 2} \| \partial^\alpha u(0,\cdot) \|_{L^2(H^n|\mathcal{K})} \]
\[ + C_0 \sum_{\mu+|\alpha| \leq M_{0+2}} \| L^{\mu} Z^\alpha u(t,\cdot) \|_{L^2(H^n|\mathcal{K})} + C_0 \sum_{|\alpha| \leq M_{0+1}} \sup_{0 \leq s \leq t} \| (s+|\cdot|) Z^\alpha \Box_c u(s,\cdot) \|_{L^2(H^n|\mathcal{K})} \]
\[ + C_0 \sum_{|\alpha| \leq 2} \sup_{0 \leq s \leq t} (1 + s) \| \partial^\alpha v(t,\cdot) \|_{L^2(H^n|y \in H^n; |y| < 3)} =: D_1 + D_2 + D_3 + D_4. \] (5.42)

Since we have
\[ D_3 \lesssim \varepsilon \sum_{|\alpha| \leq M_{0+2}} \| Z^\alpha u' \|_{L^\infty((0,t),L^2)} \] (5.43)

by (5.2), we obtain the required result by (3) and (4).

6 Appendices

We put two notes on the weighted energy estimates and the wave equations with single speed.

6.1 Weighted energy estimates

We prove the following lemma which generalizes the weighted energy estimate of tangential derivatives in Lemma 2.1. Let \( n \geq 1, c > 0, \Delta := \sum_{j=1}^n \partial_{x_j}^2 \) and \( \nabla := (\partial_1, \cdots, \partial_n) \). We denote the tangential derivatives along the \( c \) speed light cone by
\[ \partial_c = (\partial_{x_0}, \partial_{x_1}, \cdots, \partial_{x_n}) := \begin{cases} \left( \partial_t + c \partial_r, \nabla - \frac{\vec{c}}{c} \partial_r \right) & \text{for } n \geq 2, \\ \left( \partial_t + \frac{c}{\vec{c}} \partial_r, 0 \right) & \text{for } n = 1, \end{cases} \]
where \( r := |x| \) and \( \partial_r := r^{-1}(\sum_{1 \leq j \leq n} x_j \partial_j) \). For any \( f, g \) and \( F \), we consider the Cauchy problem
\[ \begin{cases} (\partial_t^2 - c^2 \Delta) u(t,x) = F(t,x) & \text{for } (t,x) \in [0, T] \times \mathbb{R}^n, \\ u(0,\cdot) = f(\cdot), \quad \partial_t u(0,\cdot) = g(\cdot). \end{cases} \] (6.2)

Lemma 6.1 Let \( n \geq 1 \). The solution \( u \) of (6.2) satisfies the following estimate.

\[ \max \left[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} |\partial_t u(t,x)|^2 + c^2 |\nabla u(t,x)|^2 \right] \]
\[ \leq \int_{\mathbb{R}^n} g^2 + c^2 |\nabla f|^2 dx + \int_0^T \int_{\mathbb{R}^n} |(\partial_t u) F|^2 dx dt. \] (6.3)
Proof. The estimate for the first term in (6.3) follows from the standard energy estimate. The bound for the second term for the case $c = 1$ and $n = 3$ is given by Lindblad and Rodnianski [25, p76, Corollary 8.2] (see also Lindblad and Rodnianski [26, p1431, Lemma 6.1]). We show its generalization following their arguments. We put
\[ e = (e_0, e_1, \cdots, e_n) := \left( \frac{1}{2}(\partial_t u)^2 + c^2|\nabla u|^2, -c^2 \partial_t u \nabla u \right). \] (6.4)

Then we have
\[ \partial_t e_0 + \nabla \cdot (e_1, \cdots, e_n) = (\partial_t u) F. \] (6.5)

Integrating it on $[0, T] \times \mathbb{R}^n$, we obtain the standard energy estimate
\[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} e_0(t) dx \leq \int_{\mathbb{R}^n} e_0(0) dx + \int_0^T \int_{\mathbb{R}^n} |(\partial_t u) F| dx dt =: X. \] (6.6)

For $-\infty < q \leq T$, we consider the truncated forward light cone
\[ C_T^0(q) := \{(t, x) | 0 \leq t \leq T, |x| = ct - cq\}, \]
\[ K_T^0(q) := \{(t, x) | 0 \leq t \leq T, |x| \leq ct - cq\}, \]
\[ B_T(q) := \{(T, x) | |x| \leq cT - cq\}, \quad B_0(q) := \{(0, x) | |x| \leq -cq\}. \] (6.7)

By the integration of (6.5) on $K_T^0(q)$, we have
\[ Y(q) := \frac{1}{\sqrt{1 + c^2}} \int_{C_T^0(q)} e \cdot (c, -\frac{x}{r}) d\sigma = \int_{B_T(q)} e_0(T) dx - \int_{B_0(q)} e_0(0) dx - \int_{K_T^0(q)} \partial_t u F dx dt \leq 2X, \] (6.8)

where we have used (6.6) for the last inequality. So that, we have
\[ \int_{-\infty}^T \frac{Y(q)}{(1 + c|q|)^{1+\kappa}} dq \leq \int_{-\infty}^T \frac{2X}{(1 + c|q|)^{1+\kappa}} dq \leq \frac{4X}{ck}. \] (6.9)

Since a direct computation shows
\[ Y(q) = \frac{c}{\sqrt{1 + c^2}} \int_{C_T^0(q)} \bar{e}_0 d\sigma, \] (6.10)

where $\bar{e}_0 := \frac{1}{2}(\partial_0 u)^2 + c^2 \sum_{j=1}^n (\partial_j u)^2$, we have
\[ \int_{-\infty}^T \frac{Y(q)}{(1 + c|q|)^{1+\kappa}} dq \leq \int_0^T \int_{\mathbb{R}^n} \frac{\bar{e}_0}{(1 + |ct - r|)^{1+\kappa}} dx dt, \] (6.11)

where we note that $d\sigma dq = \sqrt{1 + c^2} dx dt / c$ with $q = t - r/c$. Combining (6.9) and (6.11), we obtain
\[ \frac{ck}{2} \int_0^T \int_{\mathbb{R}^n} \frac{\bar{e}_0}{(1 + |ct - r|)^{1+\kappa}} dx dt \leq 2X, \] (6.12)

which is the estimate for the second term in (6.3) as required.

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The estimate for the third term in (6.3) follows similarly with slight modification. From (6.8), we have

\[ \int_{-T}^{T} \frac{Y(q)}{1 + c|q|} dq \leq \int_{-T}^{T} \frac{2X}{1 + c|q|} dq \leq \frac{4X \log(1 + cT)}{c}. \]  

By (6.10), we have

\[ \int_{-T}^{T} Y(q) dq = \int_{0}^{T} \int_{t \geq r/c - T} \frac{\tau_0}{1 + |ct - r|} dx dt. \]  

On the other hand, we have

\[ \int_{0}^{T} \int_{t \leq r/c - T} \frac{\tau_0}{1 + |ct - r|} dx dt \leq \frac{2X}{c}, \]  

where we have used \( \tau_0 \leq 2\epsilon_0 \) and (6.6) for the last inequality. Combining (6.13), (6.14) and (6.15), we obtain the required estimate.

\[ \Box \]

6.2 Wave equations with single speed

In this subsection, we show that the remark by Lindblad and Rodnianski for semilinear wave equations in three spatial dimensions (see [25, p52]) is also useful for the quasilinear wave equations and the case of two dimensions.

Let \( n = 2, 3 \), and let \( c > 0 \), \( D \geq 1 \). We put \( u = (u_1, \cdots, u_D) \), \( F = (F_1, \cdots, F_D) \), \( f = (f_1, \cdots, f_D) \), \( g = (g_1, \cdots, g_D) \), and we consider the Cauchy problem of wave equations with single speed \( c \)

\[ \begin{cases} \left( \partial_t^2 - c^2 \Delta \right) u_I(t, x) = F_I(u', u'')(t, x) & \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^n, \ 1 \leq I \leq D \\ u(0, \cdot) = f(\cdot), \ \partial_t u(0, \cdot) = g(\cdot), \end{cases} \]  

(6.16)

where we put \( \partial_0 = \partial_t \) and we denote the first derivatives \( \{ \partial_j u \}_{0 \leq j \leq n} \) by \( u' \), and the second derivatives \( \{ \partial_j \partial_k u \}_{0 \leq j, k \leq n} \) by \( u'' \). We assume that \( F \) vanishes to the second order when \( n = 3 \), the third order when \( n = 2 \), and has the form

\[ F_I(u', u'') = B_I(u') + Q_I(u', u''), \]  

(6.17)

When \( n = 3 \), \( B_I \) and \( Q_I \) are given by (1.9), (1.10), and satisfy the symmetry conditions (1.11), and the null conditions (1.12) with \( c := c_1 = \cdots = c_D \). When \( n = 2 \), \( B_I \) and \( Q_I \) are given by

\[ B_I(u') := \sum_{1 \leq J, K, L \leq D \atop 0 \leq j, k, l \leq 2} B^{JKL}_{I} \partial_J u_J \partial_K u_K \partial_L u_L, \]  

(6.18)

\[ Q_I(u', u'') := \sum_{1 \leq J, K, L \leq D \atop 0 \leq j, k, l, m \leq 2} Q^{JKLM}_{I} \partial_J u_J \partial_K u_K \partial_L \partial_m u_L, \]  

(6.19)

and satisfy the symmetry condition

\[ Q^{JKLM}_{I} = Q^{IJ}^{JKLM}, \]  

(6.20)
which is required for the energy conservation. We assume the standard null conditions
\[
\sum_{0 \leq j, k, l \leq 2} B^{IKL}_{I} \xi_j \xi_k \xi_l = \sum_{0 \leq j, k, l, m \leq 2} Q^{IKL}_{I} \xi_j \xi_k \xi_l \xi_m = 0
\]  
(6.21)
for any \(1 \leq I, J, K, L \leq D\), and any \((\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3\) with \(\xi_0^2 = c^2(\xi_1^2 + \xi_2^2)\). For example, \(\xi_0 = \sum_{j=1}^{D} \lambda_j u_j\) satisfies the null conditions for \(\lambda_j \) given \(1 \leq j \leq D \in \mathbb{R}\), and \(Q_{I} (u', u'') = \partial B_{I} (u')\) also.

We show an alternative proof of the following theorem. Lindblad and Rodnianski pointed out the simple proof for the semilinear case with \(n = 3\) and \(c = 1\). We consider the quasilinear case and also the case \(n = 2\).

**Theorem 6.2** Let \(n = 2\) or \(n = 3\). Let \(f\) and \(g\) be smooth functions. Then there exist a positive natural number \(N\) such that if
\[
\sum_{|\alpha| \leq N} \| \langle x \rangle^{|\alpha|}) \partial_x^\alpha \nabla f \|_{L^2(\mathbb{R}^n)} + \sum_{|\alpha| \leq N} \| \langle x \rangle^{|\alpha|+1}) \partial_x^\alpha g \|_{L^2(\mathbb{R}^n)} =: \varepsilon
\]
is sufficiently small, then \((6.16)\) has a unique global solution \(u \in C^\infty([0, \infty) \times \mathbb{R}^n)\).

We are able to take \(N = 10\) when \(n = 3\), and \(N = 8\) when \(n = 2\) in the theorem. The above result for \(n = 2\) has been shown by Godin [4], Hoshiga [8], and Katayama [11]. See also Alinhac [11]. Our proof is based on the weighted energy estimates Lemma 6.4, the following Klainerman-Sobolev estimates and the estimates for null conditions.

For any \(c > 0\), \(\Theta_c\) denotes the vector fields
\[
\partial_t, \partial_j, \quad ct \partial_j + \frac{x_j}{c} \partial_t, \quad x_j \partial_k - x_k \partial_j, \quad 1 \leq j \neq k \leq 3, \quad t \partial_t + r \partial r
\]
and \(\alpha\) denotes multiple indices. We note that the \(c\) speed Lorentz boosts \(\{ct \partial_j + x_j \partial_t / c\}_{j=1}^{n}\) are commutable with \(\square_c\), namely, \((ct \partial_j + x_j \partial_t / c) \square_c = \square_c (ct \partial_j + x_j \partial_t / c)\).

**Lemma 6.3** (see [7, p118, Proposition 6.5.1], [37, p43, Theorem 1.3]) For any fixed \(c > 0\), the following estimate holds for any \(u\).
\[
(1 + t + r)^{(n-1)/2} (1 + |ct - r|)^{1/2} |u(t, x)| \lesssim \sum_{|\alpha| \leq n/2 + 1} \| \Theta_c^\alpha u(t, \cdot) \|_{L^2(\mathbb{R}^n)}
\]
(6.24)

To estimate the null conditions, we use Lemma 2.3 with \(\Gamma\) replaced by \(\Theta_c\) when \(n = 3\). When \(n = 2\), we use the following lemma, which proof is omitted since it is similar to Lemma 2.3. We put \(\partial_c = (\partial_{c_0}, \partial_{c_1}, \cdots, \partial_{c_n}) := (\partial_t + c \partial_r, \nabla - \omega \partial_r)\), \(r = |x|, \omega \in S^{n-1}\), \(\partial_c = \omega \cdot \nabla\).

**Lemma 6.4** Let \(c > 0\). Let
\[
B(u, v, w) = \sum_{0 \leq j, k, l \leq 2} B^{jkl}_{I} \partial_j u \partial_k v \partial_l w, \quad Q(u, v, w) = \sum_{0 \leq j, k, l, m \leq 2} Q^{jklm}_{I} \partial_j u \partial_k v \partial_l \partial_m w
\]
(6.25)

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satisfy the null conditions
\[
\sum_{0 \leq j, k \leq 2} B_{jkl} \xi_j \xi_k = \sum_{0 \leq j, k, l, m \leq 2} Q_{jklm} \xi_j \xi_k \xi_m = 0 \quad \text{for} \quad \xi_0^2 = c^2(\xi_1^2 + \xi_2^2). \quad (6.26)
\]

Then the following estimates hold for any \( \alpha \) and functions \( u, v \) and \( w \), where \( \beta \leq \alpha \) means any component of the multiindices satisfies the inequality.

\[
(1) \quad |\Theta^\alpha_c B(u, v, w)| \leq \sum_{\beta + \gamma + \delta \leq \alpha} \{ |\nabla \Theta^\beta_c u| |(\Theta^\gamma_c v)'| |(\Theta^\delta_c w)'| \\
+ |(\Theta^\beta_c u)'| |\nabla \Theta^\gamma_c v|||\Theta^\delta_c w)'| + |(\Theta^\beta_c u)'| |(\Theta^\gamma_c v)'||\nabla \Theta^\delta_c w| \} (6.27)
\]

\[
(2) \quad |\Theta^\alpha_c Q(u, v, w)| \leq \sum_{\beta + \gamma + \delta \leq \alpha} \{ |\nabla \Theta^\beta_c u| |(\Theta^\gamma_c v)'| |(\Theta^\delta_c w)'| \\
+ |(\Theta^\beta_c u)'| |\nabla \Theta^\gamma_c v|||\Theta^\delta_c w)'| + |(\Theta^\beta_c u)'| |(\Theta^\gamma_c v)'||\nabla \Theta^\delta_c w| + \frac{1}{r}|(\Theta^\beta_c u)'| |(\Theta^\gamma_c v)'||((\Theta^\delta_c w)'| + |(\Theta^\delta_c w)'|) \} (6.28)
\]

The proof of Theorem 6.2 is similar to that of Theorem 1.1. It suffices to prove the following proposition. In its proof, we implicitly use \( \varepsilon^2 \leq C \varepsilon \) since \( \varepsilon \) is sufficiently small.

**Proposition 6.5** Let \( n = 2 \) or \( n = 3 \). Let \( M_0 \) be a positive number which satisfies \( M_0 \geq 6 \) when \( n = 3 \), \( M_0 \geq 4 \) when \( n = 2 \). We put
\[
\varepsilon := \sum_{|\alpha| \leq M_0+4} |\langle x \rangle|^{\alpha} \nabla^{|\alpha|} f|_{L^2(\mathbb{R}^n)} + \sum_{|\alpha| \leq M_0+4} |\langle x \rangle|^{\alpha+1} \nabla^{|\alpha|} g|_{L^2(\mathbb{R}^n)}. \quad (6.29)
\]

Let \( T > 0 \) and \( A_0 > 0 \). We put \( S_T := [0, T) \times \mathbb{R}^n \). Let \( u \in C^\infty([0, T) \times \mathbb{R}^n) \) be the local solution of \( (6.16) \). We assume
\[
\sum_{|\alpha| \leq M_0} \sup_{(t, x) \in S_T} (t+r)^{(n-1)/2}(ct-r)^{1/2}|\Theta^\alpha_c u'(t, x)| \leq A_0 \varepsilon. \quad (6.30)
\]

If \( \varepsilon \) is sufficiently small, then there exist constants \( C_0 > 0 \), which is independent of \( A_0 \), and \( C > 0 \), which is dependent on \( A_0 \), such that the following estimates hold.

\[
(1) \quad \sum_{|\alpha| \leq M_0+4} |\Theta^\alpha_c u'(t, \cdot)|_{L^2(\mathbb{R}^n)} \leq C \varepsilon (1 + t)^{C \varepsilon} \quad \text{for} \quad 0 \leq t < T \quad (6.31)
\]

\[
(2) \quad \sum_{|\alpha| \leq M_0+3} (\log(e + t))^{-1/2} |\langle x \rangle|^{-1/2} |\Theta^\alpha_c u'\rangle_{L^2(S_t)} \leq C \varepsilon (1 + t)^{C \varepsilon} \quad \text{for} \quad 0 \leq t < T \quad (6.32)
\]
(3) \[
\sum_{|\alpha| \leq M_0 + 2} \sup_{(s, x) \in S_t} (s + r)^{(n-1)/2} (cs - r)^{1/2} |\Theta_\varepsilon^\alpha u'(s, x)| \leq C \varepsilon (1 + t)^C \varepsilon \quad \text{for} \quad 0 \leq t < T \quad (6.33)
\]

(4) \[
\sum_{|\alpha| \leq M_0 + 2} \|\Theta_\varepsilon^\alpha u'\|_{L^\infty((0, T), L^2(\mathbb{R}^n))} \leq C_0 \varepsilon + C \varepsilon^{(6-n)/2} \quad (6.34)
\]

(5) \[
\sum_{|\alpha| \leq M_0} \sup_{(t, x) \in S_T} (t + r)^{(n-1)/2} (ct - r)^{1/2} |\Theta_\varepsilon^\alpha u'(t, x)| \leq C \varepsilon + C \varepsilon^{(6-n)/2} \quad (6.35)
\]

**Proof.** Put \( M = M_0 + 4 \). First, we remark that under the assumption (6.30), we have

\[
(1 + t + r) \sum_{|\alpha| \leq M/2 + 1} |\Theta_\varepsilon^\alpha u'(t, x)| \leq C \varepsilon \quad \text{when} \quad n = 3
\]

\[
(1 + t + r) \sum_{|\alpha| \leq M/3 + 1} |\Theta_\varepsilon^\alpha u'(t, x)|^2 \leq C \varepsilon^2 \quad \text{when} \quad n = 2
\]

for some constant \( C > 0 \) since \( M/5 - n + 1 \leq M_0 \).

1. The proof is essentially same to that of Proposition 3.1 by the use of (6.30). For \( 1 \leq I, L \leq D \) and \( 0 \leq l, m \leq n \), we put

\[
\gamma_I^{Llm} := \sum_{1 \leq j, k \leq D \atop 0 \leq l, m \leq n} Q_I^{JKLjklm} \partial_j u_I \partial_k u_K. \quad (6.37)
\]

For any \( \alpha \) with \( |\alpha| \leq M \), we use (2.2) and its integration to have

\[
\partial_t \int_{\mathbb{R}^n} e_0(\Theta_\varepsilon^\alpha u) dx \leq C \sum_{1 \leq I \leq D} \left\| \square \gamma_I^{Ijklm} \right\|_{L^2(\mathbb{R}^n)} \left\| (\Theta_\varepsilon^\alpha u)' \right\|_{L^2(\mathbb{R}^n)}
\]

\[
+ C \sum_{1 \leq I, j, k \leq D \atop 0 \leq l, m \leq n} \left\| \partial_{l, x} \gamma_I^{Llm} \right\|_{L^\infty(\mathbb{R}^n)} \left\| (\Theta_\varepsilon^\alpha u)' \right\|_{L^2(\mathbb{R}^n)}^2. \quad (6.38)
\]

Similarly to (6.11), we have

\[
\sum_{1 \leq l \leq D \atop \alpha \leq M} \left\| \square \gamma_I^{Ijklm} \right\|_{L^2(\mathbb{R}^n)} \leq \sum_{1 \leq I \leq D \atop \alpha \leq M} \left\| \Theta_\varepsilon^\alpha \square \gamma_I u_I \right\|_{L^2(\mathbb{R}^n)}
\]

\[
+ \sum_{|\alpha| + |\beta| \leq M \atop |\beta| \leq M - 1} \sum_{1 \leq I, L \leq D \atop 0 \leq l, m \leq n} \left\| (\Theta_\varepsilon^\alpha \gamma_I^{Ijklm}) \Theta_\varepsilon^\beta u'' \right\|_{L^2(\mathbb{R}^n)} \leq \varepsilon^{4-n} \frac{e^{4-n}}{1 + t} \sum_{|\alpha| \leq M} \left\| \Theta_\varepsilon^\alpha u' \right\|_{L^2(\mathbb{R}^n)}, \quad (6.39)
\]

where we have used (6.36) for the last inequality. Since \( \sum_{|\alpha| \leq M} \left\| \Theta_\varepsilon^\alpha u' \right\|_{L^2(\mathbb{R}^n)} \) is equivalent to \( \sum_{|\alpha| \leq M} \left\{ \int_{\mathbb{R}^n} e_0(\Theta_\varepsilon^\alpha u) dx \right\}^{1/2} \) for small \( \varepsilon \), we obtain

\[
\sum_{|\alpha| \leq M} \partial_t \left\{ \int_{\mathbb{R}^n} e_0(\Theta_\varepsilon^\alpha u) dx \right\}^{1/2} \leq \frac{C \varepsilon^{4-n}}{1 + t} \sum_{|\alpha| \leq M} \left\{ \int_{\mathbb{R}^n} e_0(\Theta_\varepsilon^\alpha u) dx \right\}^{1/2}, \quad (6.40)
\]
which leads to the required inequality by the Gronwall inequality.

(2) By Lemma 2.31 we have
\[
\sum_{|\alpha|\leq M-1} (\log(e + t))^{-1/2} \| (x)^{-1/2} \Theta_c^\alpha u' \|_{L^2(S_t)} \\
+ \sum_{|\alpha|\leq M-1} (\log(e + t))^{-1/2} \| (cs - r)^{-1/2} \overline{\Theta}_c^\alpha u \|_{L^2(S_t)} \\
\leq C_0 \varepsilon + C_0 \sum_{|\alpha|\leq M-1} \| \Theta_c^\alpha \bigtriangleup_c u \|_{L^1((0,t),L^2(\mathbb{R}^n))}.
\] (6.41)

The last term is bounded by
\[
\sum_{|\alpha|\leq M-1} \| \Theta_c^\alpha \bigtriangleup_c u \|_{L^1((0,t),L^2(\mathbb{R}^n))} \leq C \int_0^t \frac{\varepsilon^{-n}}{1 + s} \sum_{|\alpha|\leq M} \| \Theta_c^\alpha u'(s,\cdot) \|_{L^2(\mathbb{R}^n)} ds \\
\leq C \varepsilon^{-n} (1 + t)^C \varepsilon,
\] (6.42)

where we have used (6.30), (1) and
\[
\sum_{|\alpha|\leq M-1} \| \Theta_c^\alpha \bigtriangleup_c u \| \lesssim \left( \sum_{|\beta|\leq M/(5-n)} \| \Theta_c^\beta u' \| \right)^{4-n} \sum_{|\alpha|\leq M} \| \Theta_c^\alpha u' \|. \tag{6.43}
\]

(3) The estimate follows from Lemma 6.3 and (1).

(4) By the standard energy estimates, we have
\[
\sum_{|\alpha|\leq M_0+2} \| \Theta_c^\alpha u'(t,\cdot) \|_{L^2(\mathbb{R}^n)}^2 \leq C_0 \sum_{|\alpha|\leq M_0+2} \| \Theta_c^\alpha u'(0,\cdot) \|_{L^2(\mathbb{R}^n)} \\
+ C_0 \sum_{|\alpha|\leq M_0+2} \int_0^t \int_{\mathbb{R}^n} |D_t \Theta_c^\alpha u_{s} \bigtriangleup_c \Theta_c^\alpha u_{s} | dx ds =: A_1 + A_2. \tag{6.44}
\]

We have \( A_1 \leq (C_0\varepsilon)^2 \) for some \( C_0 > 0 \) which is independent of \( A_0 \). By Lemma 2.3 with \( \Gamma \) replaced by \( \Theta_c \) and Lemma 6.3, \( A_2 \) is bounded as \( A_2 \lesssim A_3 + A_4 \), where
\[
A_3 := \int_0^t \int_{\mathbb{R}^n} \left( \sum_{|\alpha|\leq M_0+2} |\Theta_c^\alpha u'| \right)^{4-n} \sum_{|\alpha|\leq M_0+3} |\nabla_c \Theta_c^\alpha u| \sum_{|\alpha|\leq M_0+3} \| \Theta_c^\alpha u' \| dx ds \tag{6.45}
\]
\[
A_4 := \int_0^t \int_{\mathbb{R}^n} \left( \sum_{|\alpha|\leq M_0+2} |\Theta_c^\alpha u'| \right)^{4-n} \sum_{|\alpha|\leq M_0+3} \| \Theta_c^\alpha u' \| \sum_{|\alpha|\leq M_0+3} \| \Theta_c^\alpha u' \| \frac{dx}{(r)} ds. \tag{6.46}
\]

We use (3) to have
\[
A_3 \leq C \varepsilon^{4-n} \sum_{|\alpha|\leq M_0+3} \| (cs - r)^{-1/2} \Theta_c^\alpha u \|_{L^2(S_t)} \\
\times \sum_{|\alpha|\leq M_0+3} \| (r)^{-1/2} \delta \Theta_c^\alpha u' \|_{L^2(S_t)} \leq C \varepsilon^{6-n}, \tag{6.47}
\]

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where $\delta > 0$ is a sufficiently small number and we have used (2) to obtain the last inequality. Similarly, we have

$$A_4 \leq C\varepsilon^{4-n} \sum_{|\alpha| \leq M_0+3} \|\langle r \rangle^{-1/2} \langle s \rangle^{-\delta} \Theta_{\varepsilon} u'\|_{L^2(S_1)}$$

\[ \cdot \sum_{|\alpha| \leq M_0+3} \|\langle r \rangle^{-1/2} \langle s \rangle^{-\delta} \Theta_{\varepsilon} u'\|_{L^2(S_1)} \leq C\varepsilon^{6-n}. \quad (6.48) \]

Combining these estimates, we obtain the required result.

(5) The estimate follows from Lemma 6.3 and (4). \qed

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