Conditions on the regularity of balanced \( c \)-partite tournaments for the existence of strong subtournaments with high minimum degree

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Abstract

We consider the following problem posed by Volkmann in 2007: How close to regular must a \( c \)-partite tournament be, to secure a strongly connected subtournament of order \( c \)? We give sufficient conditions on the regularity of balanced \( c \)-partite tournaments to assure the existence of strong maximal subtournament with minimum degree at least \( \left\lfloor \frac{c^2}{4} \right\rfloor + 1 \). We obtain this result as an application of counting the number of subtournaments of order \( c \) for which a vertex has minimum out-degree (resp. in-degree) at most \( q \geq 0 \).

Keywords: global irregularity, multipartite tournaments, strong maximal subtournaments

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1. Introduction

Let \( c \) be a non-negative integer, a \( c \)-partite or multipartite tournament is a digraph obtained from a complete \( c \)-partite graph by orienting each edge. In 1999 Volkmann [4] developed the first contributions in the study of the structure of the strongly connected subtournaments in multipartite tournaments. He proved that every almost regular \( c \)-partite tournament contains a strongly connected subtournament of order \( p \) for each \( p \in \{3, 4, \ldots, c - 1\} \). In the same paper he also proved that if each partite set of an almost regular \( c \)-partite tournament

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has at least $\frac{3c}{2} - 6$ vertices, then there exist a strong subtournament of order $c$. In 2008 Volkmann and Wieser [5] proved that every almost regular $c$-partite tournament has a strongly connected subtournament of order $c$ for $c \geq 5$. In 2011 Xu et al. [6] proved that every vertex of regular $c$-partite tournament with $c \geq 16$, is contained in a strongly connected subtournament of order $p$ for every $p \in \{3, 4, \ldots, c\}$. The following problem was posed by Volkmann [3]: 

**Determine further sufficient conditions for (strongly connected) $c$-partite tournaments to contain a strong subtournament of order $p$, for some $4 \leq p \leq c$. How close to regular must a $c$-partite tournament be, to secure a strongly connected subtournament of order $c$?**

On this direction, in 2016 [2], we proved that for every (not necessarily strongly connected) balanced $c$-partite tournament of order $n \geq 6$, if the global irregularity of $T$ is at most $\frac{c}{\sqrt{c-2}}$, then $T$ contains a strongly connected tournament of order $c$. A $c$-partite tournament is balanced if each of its partite sets have the same amount of vertices.

In this paper, we consider Volkmann’s problem for balanced $c$-partite tournaments. We give sufficient conditions on its regularity to assure the existence of a strong subtournament with minimum degree at least $\left\lceil \frac{c-2}{4} \right\rceil + 1$. We obtain this result as an application of counting the number of subtournaments of order $c$ for which a vertex has minimum out-degree (resp. in-degree) at most $q \geq 0$.

## 2. Notation and definitions

We follow all the definitions and notation of [1]. Let $G$ be a $c$-partite tournament of order $n$ with partite sets $\{V_i\}_{i=1}^c$. We call $G$ balanced, if all the partite sets have the same number of vertices and we denote by $G_{r,c}$ a balanced $c$-partite tournament satisfying that $|V_i| = r$ for every $i \in [c]$, where $[c] = \{1, \ldots, c\}$. Throughout this paper $|V_i| = r$ for each $i \in [c]$.

Let $G$ be a $c$-partite tournament. For $x \in V(G)$ and $i \in [c]$, the **out-neighborhood of $x$ in $V_i$** is $N_i^+(x) = V_i \cap N^+(x)$; the **in-neighborhood of $x$ in $V_i$** is $N_i^-(x) = V_i \cap N^-(x)$; $d_i^+(x) = |N_i^+(x)|$ and $d_i^-(x) = |N_i^-(x)|$. 

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For an oriented graph \( D \), the *global irregularity* of \( D \) is defined as

\[
i_g(D) = \max_{x,y \in V(D)} \{ \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} \}.
\]

If \( i_g(D) = 0 \) (\( i_g(D) \leq 1 \), resp.) \( D \) is regular (almost regular, resp.). For our study we introduce another irregularity parameter, namely *local partite irregularity* of \( D \), which is defined as

\[
\mu(D) = \max_{x \in V(D)} \{ \max_{i \in [c]} |d^+_i(x) - d^-_i(x)| \}.
\]

Observe that, for a balanced \( c \)-partite tournament \( G_{r,c} \), \( \mu(G_{r,c}) \geq i_g(G_{r,c}) - 1 \).

3. Maximal tournaments for which a vertex has minimum degree at most \( q \)

The aim of this section is to give sufficient conditions on the minimum degree, local partite irregularity and global irregularity to obtain a bound on the number of maximal tournaments in a balanced \( c \)-partite tournament \( G_{r,c} \) in which a given vertex \( x \in V(G_{r,c}) \) has out-degree (in-degree resp.) at most \( q \), for some given \( q \geq 0 \).

Let \( x \in V_c \) and let \( \mathcal{H}^+_k(x) \) be the set of vectors \((h_1, h_2, \ldots, h_{c-1}) \in \{0,1\}^{c-1}\) such that \( h_i = 1 \) if \( d^+_i(x) = r \); \( h_i = 0 \) if \( d^+_i(x) = 0 \); and \( \sum_{i=1}^{c-1} h_i = k \). A maximal tournament containing the vertex \( x \) with out-degree \( k \) can be constructed choosing a vertex for each part \( V_i \) for \( i \in [c-1] \) as follows: Given \( h = (h_1, h_2, \ldots, h_{c-1}) \in \mathcal{H}^+_k(x) \), we choose an out neighbor of \( x \) from \( V_i \) if and only if \( h_i = 1 \). Since the number of maximal tournaments constructed in this way for a fixed \( h \) is \( \prod_{i=1}^{c-1} d^+_i(x)^{h_i}d^-_i(x)^{1-h_i} \), we have the following remark.

**Remark 1.** Let \( G_{r,c} \) be a balanced \( c \)-partite tournament and let \( x \in V_c \). The number of maximal tournaments of \( G_{r,c} \) for which \( x \) has out-degree \( k \) is equal to

\[
\sum_{h \in \mathcal{H}^+_k(x)} \prod_{i=1}^{c-1} d^+_i(x)^{h_i}d^-_i(x)^{1-h_i}.
\]

Let \( x \in V(G_{r,c}) \). For each \( q \geq 0 \), let \( T^+_q(x) \) (resp. \( T^-_q(x) \)) be the number of maximal tournaments of \( G_{r,c} \) for which \( x \) has out-degree (resp. in-degree) at most \( q \). All the following results regarding \( T^+_q(x) \) can be obtained for \( T^-_q(x) \) in an analogous way.
Let \( x \in V(G_{r,c}) \). Assume w.l.o.g that \( x \in V_c \). By Remark 1,
\[
T^+_q(x) = \sum_{k=0}^{q} \left( \sum_{h \in \mathcal{H}^+_k(x)} \prod_{i=1}^{c-1} d^+_i(x)^{h_i} d^-_i(x)^{1-h_i} \right).
\]
In order to bound \( T^+_q(x) \), for any integer \( r \geq 2 \), and \( g_1, g_2, \ldots, g_s \) real numbers such that \( 0 \leq g_i \leq r \), we define
\[
M(g_1, \ldots, g_s; k) = \sum_{h \in \mathcal{H}^+_k} \prod_{i=1}^{s} g_i^{h_i}(r - g_i)^{1-h_i},
\]
where \( \mathcal{H}^+_k \) is the set of \( s \)-vectors \( (h_1, h_2, \ldots, h_s) \in \{0,1\}^s \) such that: if \( g_i = r \) then \( h_i = 1 \); if \( g_i = 0 \) then \( h_i = 0 \); and \( \sum_{i=1}^{s} h_i = k \).
If for a given \( x \in V(G_{r,c}) \), \( g_i = d^+_i(x) \), with \( i \in [c-1] \), the following lemma gives a sufficient condition on the out-degree of \( x \) to assure that the number of maximal tournaments in which \( x \) has out-degree \( q \) is at least equal to the number of maximal tournaments in which \( x \) has out-degree \( q - 1 \).

**Lemma 1.** Let \( r \geq 2 \) be an integer, and let \( g_1, \ldots, g_s \) be real numbers such that \( 0 \leq g_i \leq r \). Let \( \Gamma = \max \{ g_i \}_{i \in [s]} \) and \( \gamma = \min \{ g_i \}_{i \in [s]} \). If for some integer \( q \geq 1 \) we have that \( \sum_{i \in [s]} g_i \geq q(r + \Gamma - \gamma) - \Gamma \), then
\[
M(g_1, \ldots, g_s; q) \geq M(g_1, \ldots, g_s; q - 1).
\]

**Proof.** Let \( g_1, \ldots, g_s \) be real numbers such that \( 0 \leq g_i \leq r \), and let \( q \geq 1 \). Suppose w.l.o.g. that for every \( i \), if \( 1 \leq i \leq t \) then \( 0 < g_i < r \); if \( t + 1 \leq i \leq t + p_r \) then \( g_i = r \); and if \( t + p_r + 1 \leq i \leq s \) then \( g_i = 0 \). Observe that for every \( h = (h_1, \ldots, h_s) \in \mathcal{H}^+_{q-1} \) and every \( h' = (h'_1, \ldots, h'_s) \in \mathcal{H}^+_{q} \) we have that if \( t + 1 \leq i \leq t + p_r \), then \( h_i = h'_i = 1 \); and if \( t + p_r + 1 \leq i \leq s \), then \( h_i = h'_i = 0 \). Notice that if \( p_r \geq q \), then \( \mathcal{H}^+_{q-1} = \emptyset \) which implies that \( M(g_1, \ldots, g_s; q - 1) = 0 \) and the lemma follows. Thus, we can suppose that \( q \geq p_r + 1 \).

For each \( h = (h_1, \ldots, h_s) \in \mathcal{H}^+_{q-1} \) let \( F(h) = \{(h'_1, \ldots, h'_s) \in \mathcal{H}^+_{q} : h_i \leq h'_i \text{ for } i \in [s]\} \) and let \( a(h) = \{j : h_j = 1 \text{ for } j \in [t]\} \). Observe that for every \( h \in \mathcal{H}^+_{q-1} \), \( |a(h)| = q - 1 - p_r \).

By definition of \( \mathcal{H}^+_{q} \) and \( \mathcal{H}^+_{q-1} \), it follows that given \( h \in \mathcal{H}^+_{q-1} \) and \( h' \in F(h) \subseteq \mathcal{H}^+_{q} \) (except for some \( j_0 \in [t] \setminus a(h) \), where \( h'_{j_0} = h_{j_0} + 1 \)), we have that \( h_i = h'_i \) for every \( i \in [s] \setminus \{j_0\} \).
Thus,

\[
\sum_{\mathbf{h} \in F(\mathbf{h})} \prod_{i=1}^{s} g_i^{h'_i} (r - g_i)^{1-h'_i} = \sum_{\mathbf{h} \in F(\mathbf{h})} \prod_{i=1}^{s} g_i^{h'_i} (r - g_i)^{1-h'_i} = \sum_{j \in [t] \setminus a(\mathbf{h})} \frac{g_j}{r - g_j}. \tag{1}
\]

Claim 1. \(\sum_{j \in [t] \setminus a(\mathbf{h})} \frac{g_j}{r - g_j} \geq q - p_r.\)

Suppose that \(\sum_{j \in [t] \setminus a(\mathbf{h})} \frac{g_j}{r - g_j} < q - p_r.\) Let \(Q_0 = \min_{i \in [t]} \{g_i\}.\) Thus, \(\sum_{j \in [t] \setminus a(\mathbf{h})} \frac{g_j}{r - q_0} < q - p_r\) and therefore \(\sum_{j \in [t] \setminus a(\mathbf{h})} g_j < (r - Q_0)(q - p_r).\) On the other hand, \(\sum_{j \in [t]} g_j = \sum_{j \in [s]} g_j + r p_r = \sum_{j \in [t] \setminus a(\mathbf{h})} g_j + \sum_{j \in a(\mathbf{h})} g_j + r p_r.\) Hence, \(\sum_{j \in [t] \setminus a(\mathbf{h})} g_j = \sum_{j \in [s]} g_j - \sum_{j \in a(\mathbf{h})} g_j - r p_r\) which implies that

\[(r - Q_0)(q - p_r) > \sum_{j \in [s]} g_j - \sum_{j \in a(\mathbf{h})} g_j - r p_r\]

and therefore, after some easy calculation, we see that \(\sum_{j \in a(\mathbf{h})} g_j = q - p_r,\) it follows that

\[\sum_{j \in [s]} g_j - \sum_{j \in a(\mathbf{h})} g_j - r p_r \geq g_j.\]

Let \(Q_1 = \max\{g_i : i \in [t]\}.\) Since \(|a(\mathbf{h})| = q - 1 - p_r,\) it follows that

\[rq + Q_1(q - 1 - p_r) - Q_0(q - pr) \geq \sum_{j \in [s]} g_j.\]

Since \(\Gamma \geq Q_1 \geq Q_0 \geq \gamma \) and \(p_r \geq 0,\) we see that

\[Q_1(q - 1 - p_r) - Q_0(q - pr) \leq \Gamma(q - 1 - p_r) - \gamma(q - pr) = \Gamma(q - 1) - \gamma q - \gamma q = \Gamma(q - 1) - \gamma q.\]

Thus,

\[rq + \Gamma(q - 1) - \gamma q = q(r + \Gamma - \gamma) - \Gamma > \sum_{j \in [s]} g_j\]

which, by hypothesis, is not possible and the claim follows.

From Claim 1 and (1) it follows that for each \(\mathbf{h} = (h_1, \ldots, h_s) \in H^s_{q - 1}\)

\[\sum_{\mathbf{h} \in F(\mathbf{h})} \prod_{i=1}^{s} g_i^{h'_i} (r - g_i)^{1-h'_i} \geq (q - p_r) \prod_{i=1}^{s} g_i^{h'_i} (r - g_i)^{1-h'_i}. \tag{2}\]
Observe that, for every $h' \in \mathcal{H}_q$, $|\{ j : h'_j = 1 \text{ with } j \in [t]\}| = q - p_r$. Therefore, for every $h' \in \mathcal{H}_q$ there are exactly $q - p_r$ elements $h \in \mathcal{H}_{q-1}$ such that, $h' \in F(h)$. Thus,

$$
\sum_{h \in \mathcal{H}_{q-1}} \left( \sum_{h' \in F(h)} \prod_{i=1}^{s} g_i^{h'_i}(r - g_i)^{1-h'_i} \right) = (q - p_r) \sum_{h' \in \mathcal{H}_q} \prod_{i=1}^{s} g_i^{h'_i}(r - g_i)^{1-h'_i}.
$$

On the other hand, by (2) we see that

$$
\sum_{h \in \mathcal{H}_{q-1}} \left( \sum_{h' \in F(h)} \prod_{i=1}^{s} g_i^{h'_i}(r - g_i)^{1-h'_i} \right) \geq \sum_{h \in \mathcal{H}_{q-1}} (q - p_r) \prod_{i=1}^{s} g_i^{h'_i}(r - g_i)^{1-h_i}
$$

implying that

$$
\sum_{h' \in \mathcal{H}_q} \prod_{i=1}^{s} g_i^{h'_i}(r - g_i)^{1-h'_i} \geq \sum_{h \in \mathcal{H}_{q-1}} \prod_{i=1}^{s} g_i^{h'_i}(r - g_i)^{1-h_i}
$$

which, by definition, is equivalent to $M(g_1, \ldots, g_s; q) \geq M(g_1, \ldots, g_s; q - 1)$ and the lemma follows. ■

**Corollary 1.** Let $r \geq 2$, $c \geq 3$ and $G_{r,c}$ be a balance $c$-partite tournament such that for some $q \geq 1$, $\delta(G_{r,c}) \geq q(r + \mu(G_{r,c}))$. Then, for every $x \in V(G_{r,c})$ the number of maximal tournaments in which $x$ has out-degree $q$ is at least equal to the number of maximal tournaments in which $x$ has out-degree $q - 1$.

The following theorem gives a condition regarding the minimum degree and the local partite irregularity to obtain a bound of $T^+_q(x)$.

**Theorem 2.** Let $r \geq 2$, $c \geq 5$ and $G_{r,c}$ be a balance $c$-partite tournament such that for some $q \geq 0$, $\delta(G_{r,c}) \geq q(r + \mu(G_{r,c})) \left( \frac{1}{c-2} \right)$. Then, for every $x \in V(G_{r,c})$,

$$
T^+_q(x) \leq \sum_{k=0}^{q} \binom{c-1}{k} \left( \frac{d^+(x)}{c-1} \right)^k \left( \frac{d^-(x)}{c-1} \right)^{c-1-k}.
$$

**Proof.** Let $x \in V(G_{r,c})$, and suppose $x \in V_c$. By Lemma 1, we see that

$$
T^+_q(x) = \sum_{k=0}^{q} \sum_{h \in \mathcal{H}_k(x)} \prod_{i=1}^{c-1} d_i^+(x)^{h_i} d_i^-(x)^{1-h_i} = \sum_{k=0}^{q} M(d_1^+(x), \ldots, d_{c-1}^+(x); k).
$$

For each $i$, with $i \in [c-1]$, let $g_i = d_i^+(x)$, and assume that $g_{c-1} = \max\{g_i\}_{i \in [c-1]} = \Gamma$ and $g_{c-2} = \min\{g_i\}_{i \in [c-1]} = \gamma$. Let $g'_1, g'_2, \ldots, g'_{s-1}, g'_s$ be real numbers such that, for $i \in [c-3]$, $g'_{i} = g_i$; and $g'_{c-2} = g'_{c-1} = \frac{g_{c-2} + g_{c-1}}{2}$. 

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Claim 2. $\sum_{k=0}^{q} M(g_1, \ldots, g_{c-1}; k) \leq \sum_{k=0}^{q} M(g'_1, \ldots, g'_{c-1}; k)$.

If $q = 0$, $\sum_{k=0}^{q} M(g_1, \ldots, g_{c-1}; 0) = \prod_{i=1}^{c-1} (r - g_i)$. Since $(r - g_{c-2})(r - g_{c-1}) \leq (r - \frac{g_{c-2} + g_{c-1}}{2})^2$, the claim follows. Assume that $q \geq 1$. For the sake of readability, in what follows, $g_1, \ldots, g_{c-1}$ and $g_1, \ldots, g_{c-3}$ can be denoted as $g_{c-1}$ and $g_{c-3}$, respectively. Observe that

$$M(g_{c-1}; 0) = M(g_{c-3}; 0)M(g_{c-2}, g_{c-1}; 0);$$

$$M(g_{c-1}; 1) = M(g_{c-3}; 1)M(g_{c-2}, g_{c-1}; 0) + M(g_{c-3}; 0)M(g_{c-2}, g_{c-1}; 1)$$

and for every $k \geq 2$,

$$M(g_{c-1}; k) = \sum_{j=0}^{2} M(g_{c-3}; k - j)M(g_{c-2}, g_{c-1}; j).$$

Therefore, for $q = 1$,

$$\sum_{k=0}^{1} M(g_{c-1}; k) = M(g_{c-3}; 0)[M(g_{c-2}, g_{c-1}; 0) + M(g_{c-2}, g_{c-1}; 1)] + M(g_{c-3}; 1)M(g_{c-2}, g_{c-1}; 0);$$

and for $q \geq 2$,

$$\sum_{k=0}^{q} M(g_{c-1}; k) = \sum_{k=0}^{q-2} M(g_{c-3}; k)[M(g_{c-2}, g_{c-1}; 0) + M(g_{c-2}, g_{c-1}; 1) + M(g_{c-2}, g_{c-1}; 2)] + M(g_{c-3}; q - 1)[M(g_{c-2}, g_{c-1}; 0) + M(g_{c-2}, g_{c-1}; 1)] + M(g_{c-3}; q)M(g_{c-2}, g_{c-1}; 0).$$

It is not hard to see that for any pair of reals $0 \leq x, y \leq r$, $M(x, y; 0) = (r - x)(r - y)$; $M(x, y; 1) = r(x + y) - 2xy$ and $M(x, y; 2) = xy$. Therefore, $M(x, y; 2) + M(x, y; 1) + M(x, y; 0) = r^2$. Since for $i \in [c - 3]$, $g'_i = g_i$ and $g_{c-2} + g_{c-1} = g'_{c-2} + g'_{c-1}$, we have, after some easy calculations, that

$$\sum_{k=0}^{q} M(g'_{c-1}; k) - \sum_{k=0}^{q} M(g_{c-1}; k) = M(g_{c-3}; q - 1)[g_{c-2}g_{c-1} - g'_{c-2}g'_{c-1}] + M(g_{c-3}; q)[g'_{c-2}g'_{c-1} - g_{c-2}g_{c-1}] = \left(g'_{c-2}g'_{c-1} - g_{c-2}g_{c-1}\right)[M(g_{c-3}; q) - M(g_{c-3}; q - 1)].$$
Since \( g_{c-2} g'_{c-1} \geq g_{c-2} g_{c-1} \), it follows that \( \sum_{k=0}^{q} M(g_{c-1}; k) \leq \sum_{k=0}^{q} M(g'_{c-1}; k) \) if and only if \( M(g_{c-3}; q - 1) \leq M(g'_{c-3}; q) \).

Since \( \sum_{i \in [c-1]} g_i = d^+(x) \geq \delta(G_{r,c}) \geq q\left(r + \mu(G_{r,c})\right)\left(\frac{c-1}{2}\right) \), it follows that \( d^+(x)\left(\frac{c-2}{c-1}\right) = d^+(x) - \frac{d^+(x)}{c-1} \geq (r + \mu(G_{r,c})) \). Therefore, \( d^+(x) \geq (r + \mu(G_{r,c})) + \frac{d^+(x)}{c-1} \). On the one hand, clearly \( \gamma \leq \frac{d^+(x)}{c-1} \) and by definition \( \mu(G_{r,c}) \geq \Gamma - \gamma \). It follows that \( d^+(x) = \sum_{i \in [c-1]} g_i \geq q(r + \Gamma - \gamma) + \gamma \).

Since \( g_{c-1} = \Gamma \) and \( g_{c-2} = \gamma \), we see that \( \sum_{i \in [c-3]} g_i \geq q(r + \Gamma - \gamma) - \Gamma \). On the other hand, observe that \( \Gamma \geq \Gamma' = \max\{g_i\}_{i \in [c-3]} \) and \( \gamma \leq \gamma' = \min\{g_i\}_{i \in [c-3]} \). Since \( q \geq 1 \), it follows that \( q(r + \Gamma - \gamma) - \Gamma \geq q(r + \Gamma' - \gamma') - \Gamma' \) which implies that \( \sum_{i \in [c-3]} g_i \geq q(r + \Gamma' - \gamma') - \Gamma' \).

Hence, by Lemma 1, \( M(g_{c-3}; q - 1) \leq M(g'_{c-3}; q) \), and from here the claim follows.

Observe that \( \Gamma \geq \Gamma' = \max\{g_i'\}_{i \in [c-1]} \) and \( \gamma \leq \gamma' = \min\{g_i'\}_{i \in [c-1]} \). Since \( \sum_{i \in [c-1]} g_i = \sum_{i \in [c-1]} g_i' \) it follows that \( \sum_{i \in [c-1]} g_i' \geq q(r + \Gamma' - \gamma')(\frac{c-1}{c-2}) \), and clearly \( 0 \leq g_i' \leq r \). Hence, we can iterate this procedure, and by the way \( g_{c-2}' \) and \( g_{c-1}' \) are defined, we see that the limit of the difference \( \Gamma' - \gamma' \) by iterating this procedure is zero. Thus, by Claim 2, it follows that \( T_q^+(x) \) is bounded by \( \sum_{k=0}^{q} M\left(\frac{d^+(x)}{c-1}, \ldots, \frac{d^+(x)}{c-1}; k\right) \). Finally, by definition, for each \( k \in [q] \),

\[
M\left(\frac{d^+(x)}{c-1}, \ldots, \frac{d^+(x)}{c-1}; k\right) = \sum_{h \in \mathcal{H}_{c-1}^k} \prod_{i=1}^{\cdot} \left(\frac{d^+(x)}{c-1}\right)^{h_i} \left(\frac{c-1}{c-1} - \frac{d^+(x)}{c-1}\right)^{1-h_i} = \sum_{h \in \mathcal{H}_{c-1}^k} \left(\frac{d^+(x)}{c-1}\right)^{k} \left(\frac{c-1}{c-1} - \frac{d^+(x)}{c-1}\right)^{c-1-k} = \left(\frac{c-1}{k}\right) \left(\frac{d^+(x)}{c-1}\right)^{k} \left(\frac{c-1}{c-1} - \frac{d^+(x)}{c-1}\right)^{c-1-k},
\]

and it is not hard to see that \( r - \frac{d^+(x)}{c-1} = \frac{d^-(x)}{c-1} \). From here the result follows. \( \blacksquare \)

The following theorem gives a condition regarding the minimum degree; the local partite irregularity and the global irregularity to obtain a bound of \( T_q^+(x) \).

**Theorem 3.** Let \( r \geq 2, c \geq 5 \) and \( G_{r,c} \) be a balance \( c \)-partite tournament. If for some \( q \geq 0, \delta(G_{r,c}) \geq q(r + \mu(G_{r,c}))\left(\frac{c-1}{2}\right) \) and \( i_q(G_{r,c}) = r(c - 1)\beta \) with \( 0 \leq \beta < \frac{c-2q-2}{c} \), then for every \( x \in V(G_{r,c}) \) we have that

\[
T_q^+(x) \leq \left(\frac{c-1}{q+1}\right) \left(\frac{r}{2}\right)^{c-1} \left(1 + \beta\right)^{c-2-2q} (q+1) \frac{c(1-\beta) - 2q - 2}{c(1-\beta) - 2q - 2}.
\]
Proof. Let \( x \in V(G_{r,c}) \). By Theorem 2, it follows that

\[
T^+_q(x) \leq \sum_{k=0}^q \binom{c-1}{k} \left( \frac{d^+(x)}{c-1} \right)^k \left( \frac{d^-(x)}{c-1} \right)^{c-1-k}.
\]  

(3)

Observe that

\[
\sum_{k=0}^q \binom{c-1}{k} \left( \frac{d^+(x)}{c-1} \right)^k \left( \frac{d^-(x)}{c-1} \right)^{c-1-k} = \left( \frac{d^+(x)}{c-1} \right)^c \sum_{k=0}^q \binom{c-1}{k} \left( \frac{d^+(x)}{d^-(x)} \right)^k.
\]

For every \( q \), with \( 0 \leq q \leq c-1 \), let \( g(q) = \sum_{k=0}^q \binom{c-1}{k} \left( \frac{d^+(x)}{d^-(x)} \right)^k \). Observe that for \( q < c-1 \),

\[
g(q) = 1 + \sum_{k=1}^{q+1} \binom{c-1}{k} \left( \frac{d^+(x)}{d^-(x)} \right)^k = 1 + \sum_{k=0}^{q+1} \binom{c-1}{k} \left( \frac{d^+(x)}{d^-(x)} \right)^k
\]

\[
= 1 + \left( \frac{d^+(x)}{d^-(x)} \right) \sum_{k=0}^{q+1} \binom{c-1}{k} \frac{d^+(x)}{d^-(x)}
\]

\[
> \left( \frac{d^+(x)}{d^-(x)} \right) \left( \frac{c-1-q}{q+1} \right) g(q).
\]

On the other hand, \( g(q+1) = g(q) + \binom{c-1}{q+1} \left( \frac{d^+(x)}{d^-(x)} \right)^{q+1} \). Therefore

\[
g(q) + \left( \frac{c-1}{q+1} \right) \left( \frac{d^+(x)}{d^-(x)} \right)^{q+1} > \left( \frac{d^+(x)}{d^-(x)} \right) \left( \frac{c-1-q}{q+1} \right) g(q)
\]

which implies that

\[
\left( \frac{c-1}{q+1} \right) \left( \frac{d^+(x)}{d^-(x)} \right)^{q+1} > \left( \frac{d^+(x)}{d^-(x)} \right) \left( \frac{c-1-q}{q+1} \right) - 1 \right) g(q).
\]  

(4)

Clearly, \( \frac{d^+(x)}{d^-(x)} \geq \frac{\delta(G_{r,c})}{\Delta(G_{r,c})} \) and since \( \Delta(G_{r,c}) = \frac{r(c+1)\beta(G_{r,c})}{2} \), \( \delta(G_{r,c}) = \frac{r(c-1)\beta(G_{r,c})}{2} \), and \( \beta(G_{r,c}) = r(c-1)\beta \), it is not hard to see that \( \frac{\delta(G_{r,c})}{\Delta(G_{r,c})} = \frac{1-\beta}{1+\beta} \). Moreover, since \( \beta < \frac{c-2q-2}{c} \), it follows that \( \frac{1-\beta}{1+\beta} > \frac{2q+2}{2c-2q-2} = \frac{q+1}{c-q-1} \). Therefore \( \frac{\frac{1-\beta}{1+\beta} \left( \frac{c-1-q}{q+1} \right) - 1}{1-(\beta)(c-1-q)_{(1+\beta)(q+1)}} \frac{c(1-\beta)-2q-2}{(1+\beta)(q+1)} > 0 \). Hence, by (4),

\[
\left( \frac{c-1}{q+1} \right) \left( \frac{d^+(x)}{d^-(x)} \right)^{q+1} > \left( \frac{c(1-\beta)-2q-2}{(1+\beta)(q+1)} \right) g(q)
\]
and then
\[
\left( \frac{c+1}{d-1} \right)^{c+1} \left( \frac{d^+(x)}{d^-(x)} \right)^{q+1} \frac{(1+\beta)(q+1)}{c(1-\beta)-2q-2} > g(q).
\]

Therefore, it follows that, for \( q < c-1 \),
\[
\sum_{k=0}^{q} \binom{c-1}{k} \left( \frac{d^+(x)}{c-1} \right)^{k} \left( \frac{d^-(x)}{c-1} \right)^{c-1-k} = \left( \frac{d^-(x)}{c-1} \right)^{c-1} \sum_{k=0}^{q} \binom{c-1}{k} \left( \frac{d^+(x)}{d^-(x)} \right)^{k}
\]
\[
= \left( \frac{d^-(x)}{c-1} \right)^{c-1} g(q)
\]
\[
< \left( \frac{d^-(x)}{c-1} \right)^{c-1} \left( \frac{d^+(x)}{d^-(x)} \right)^{q+1} \frac{(1+\beta)(q+1)}{c(1-\beta)-2q-2}.
\]

Thus, by (3),
\[
T^+_q(x) < \left( \frac{d^-(x)}{c-1} \right)^{c-1} \left( \frac{d^+(x)}{q+1} \right) \frac{d^+(x)}{d^-(x)} \frac{(1+\beta)(q+1)}{c(1-\beta)-2q-2}.
\]

Finally, observe that
\[
\left( \frac{d^-(x)}{c-1} \right)^{c-1} \left( \frac{d^+(x)}{d^-(x)} \right)^{q+1} = \left( \frac{d^-(x)}{c-1} \right)^{c-1} \left( \frac{d^+(x)}{c-1} \right)^{q+1} \left( \frac{d^-(x)}{d^-(x)} \right)^{q+1}
\]
\[
= \left( \frac{d^-(x)}{c-1} \right)^{c-1-2q-2} \left( \frac{d^+(x)}{d^-(x)} \right)^{q+1}.
\]

On the one hand, since \( d^+(x) + d^- (x) = r(c-1) \), it follows that \( \frac{d^+(x)d^-(x)}{(c-1)^2} \leq \frac{r^2(c-1)^2}{4(c-1)^2} = \frac{r^2}{4} \) and therefore \( \left( \frac{d^+(x)d^-(x)}{(c-1)^2} \right)^{q+1} \leq \left( \frac{r}{2} \right)^{2q+2} \).

On the other hand, \( d^- (x) \leq \Delta(G_{r,c}) = \frac{r(c-1)+\beta}{2} = \frac{r(c-1)+r(c-1)\beta}{2} = \frac{r(c-1)(1+\beta)}{2} \) and therefore \( \left( \frac{d^-(x)}{c-1} \right)^{c-1-2q-2} \leq \left( \frac{r(1+\beta)}{2} \right)^{c-1-2q-2} \). Thus,
\[
\left( \frac{d^-(x)}{c-1} \right)^{c-1} \left( \frac{d^+(x)}{d^-(x)} \right)^{q+1} \leq \left( \frac{r}{2} \right)^{2q+2} \left( \frac{r}{2} \right)^{c-1-2q-2} (1+\beta)^{c-1-2q-2} = \left( \frac{r}{2} \right)^{c-1} (1+\beta)^{c-1-2q-2}
\]

and from here, the result follows. \( \square \)

4. Maximal strong subtournament with minimum degree at most \( \left\lfloor \frac{c-2}{4} \right\rfloor + 1 \)

Recall that if \( T \) is a tournament of order \( c \) such that \( \delta(T) \geq \left\lfloor \frac{c-2}{4} \right\rfloor + 1 \), then \( T \) is strong. As an application of Theorem 3, we give sufficient conditions for the existence of a maximal strong subtournament with minimum degree at most \( \left\lfloor \frac{c-2}{4} \right\rfloor + 1 \), in a balanced \( c \)-partite tournament.
Theorem 4. Let $G_{r,c}$ be a balanced $c$-partite tournament, with $r \geq 2$, such that $\delta(G_{r,c}) \geq \lfloor \frac{c-2}{4} \rfloor (r + \mu(G_{r,c}))(\frac{c-1}{c-2})$. Then $G_{r,c}$ contains a strong connected tournament $T$ of order $c$ such that $\delta(T) \geq \lfloor \frac{c-2}{4} \rfloor + 1$, whenever

i) $i_g(G_{r,c}) \leq \frac{r}{2}$ and $c \geq 13$, except for 14, 15, and 18.

ii) $i_g(G_{r,c}) \leq r$ and $c \geq 17$, except for 18, 19, and 22.

iii) $i_g(G_{r,c}) \leq \frac{3r}{2}$ and $c \geq 21$, except for 22, 23, and 26.

Proof. In order to prove this theorem, we first show the following.

Claim 3. Let $r \geq 2$ and $c \geq 5$. For every balanced $c$-partite tournament $G_{r,c}$ with $\delta(G_{r,c}) \geq \lfloor \frac{c-2}{4} \rfloor (r + \mu(G_{r,c}))(\frac{c-1}{c-2})$ and $i_g(G_{r,c}) \leq \frac{\alpha r}{2}$ ($\alpha \geq 0$), if

$$2^{c-2} > \left( \frac{c-1}{\lfloor \frac{c-2}{4} \rfloor + 1} \right) (\frac{2c-2+\alpha}{2c-2})^{\frac{c-2-2\lfloor \frac{c-2}{4} \rfloor}{4}} \frac{(\lfloor \frac{c-2}{4} \rfloor + 1)c}{c(\frac{2c-2+\alpha}{2c-2}) - 2[\frac{c-2}{4}] - 2}$$

then $G_{r,c}$ contains a strong connected tournament $T$ of order $c$ such that $\delta(T) \geq \lfloor \frac{c-2}{4} \rfloor + 1$.

Let $G_{r,c}$ be a balanced $c$-partite tournament as in the statement of the claim, and suppose there is no tournament $T$ of order $c$ in $G_{r,c}$ such that $\delta(T) \geq \lfloor \frac{c-2}{4} \rfloor + 1$. Thus, each of those tournaments has minimal degree at most $\lfloor \frac{c-2}{4} \rfloor$, and since there are $r^c$ tournaments of order $c$ in $G_{r,c}$, it follows that

$$\sum_{x \in V(G_{r,c})} (T^+_{\lfloor \frac{c-2}{4} \rfloor}(x) + T^-_{\lfloor \frac{c-2}{4} \rfloor}(x)) \geq r^c.$$ 

Since $|V(G_{r,c})| = rc$, by Theorem 3, we see that

$$\left(2rc\right) \left( \frac{c-1}{\lfloor \frac{c-2}{4} \rfloor + 1} \right) \left( \frac{r}{2} \right)^{c-1} \frac{\left(1 + \frac{i_g(G_{r,c})}{r(c-1)}\right)^{c-2-2\lfloor \frac{c-2}{4} \rfloor} \left(\lfloor \frac{c-2}{4} \rfloor + 1\right)}{c(1 - \frac{i_g(G_{r,c})}{r(c-1)}) - 2[\frac{c-2}{4}] - 2} \geq r^c,$$

and since $i_g(G_{r,c}) \leq \frac{\alpha r}{2}$, it follows that $\frac{i_g(G_{r,c})}{r(c-1)} \leq \frac{\alpha}{2(c-1)}$; $1 + \frac{i_g(G_{r,c})}{r(c-1)} \leq \frac{2c-2+\alpha}{2c-2}$ and $1 - \frac{i_g(G_{r,c})}{r(c-1)} \geq \frac{2c-2-\alpha}{2c-2}$. Thus,

$$\left(2rc\right) \left( \frac{c-1}{\lfloor \frac{c-2}{4} \rfloor + 1} \right) \left( \frac{r}{2} \right)^{c-1} \frac{\left(\frac{2c-2+\alpha}{2c-2}\right)^{c-2-2\lfloor \frac{c-2}{4} \rfloor} \left(\lfloor \frac{c-2}{4} \rfloor + 1\right)}{c(\frac{2c-2+\alpha}{2c-2}) - 2[\frac{c-2}{4}] - 2} \geq r^c.$$
Multiplying both sides of the inequality by \( \frac{2^{c-1}}{2^r}\left(\frac{1}{2r}\right)\), we obtain that

\[
\frac{c}{\left\lceil \frac{c-2}{4} \right\rceil + 1}\left(\frac{2c-2+\alpha}{2c-2}\right)\frac{e^{2-2\left\lceil \frac{c-2}{4} \right\rceil} \left(\left\lfloor \frac{c-2}{4} \right\rfloor + 1\right)}{c\left(\frac{2c-2-\alpha}{2c-2}\right) - 2\left\lfloor \frac{c-2}{4} \right\rfloor - 2} \geq 2^{c-2}
\]

and from here the claim follows.

Let \( f_\alpha(c) = \left(\frac{c-1}{\left\lceil \frac{c-2}{4} \right\rceil + 1}\right)\left(\frac{2c-2+\alpha}{2c-2}\right)\frac{e^{2-2\left\lceil \frac{c-2}{4} \right\rceil} \left(\left\lfloor \frac{c-2}{4} \right\rfloor + 1\right)}{c\left(\frac{2c-2-\alpha}{2c-2}\right) - 2\left\lfloor \frac{c-2}{4} \right\rfloor - 2} \) and \( g(c) = 2^{c-2} \). Notice that for \( 0 \leq \alpha \leq \alpha' \), for every \( c \geq 2 \), \( f_\alpha(c) \leq f_{\alpha'}(c) \).

If \( i_g(G_{r,c}) \leq \frac{r}{2} \), it follows that \( \alpha \leq 1 \) and it is not hard to see that \( f_1(c) < g(c) \) whenever \( c \in \{13, 16, 19, 22\} \). Analogously, if \( i_g(G_{r,c}) \leq r \), then \( \alpha \leq 2 \) and \( f_2(c) < g(c) \) whenever \( c \in \{17, 20, 23, 26\} \); and if \( i_g(G_{r,c}) \leq \frac{3r}{2} \), then \( \alpha \leq 3 \) and \( f_3(c) < g(c) \) whenever \( c \in \{21, 24, 27, 30\} \).

To end the proof, we just need to show that, for \( \alpha \in \{1, 2, 3\} \), if for some \( c \geq 13 \) we have that \( f_\alpha(c) < g(c) \), then \( f_\alpha(c + 4) < g(c + 4) \). For this we show that \( \frac{g(c+4)}{f_\alpha(c)} \leq \frac{g(c+4)}{g(c)} \). Clearly, for every \( c \geq 13 \), \( \frac{g(c+4)}{g(c)} = 16 \). On the other hand, it is not difficult to see that, for every \( c \geq 13 \),

\[
\left(\frac{2c+6+\alpha}{2c+6}\right)^2 \leq \frac{6}{5}.
\]

and using a solver, it is possible to verify that, for \( c \geq 13 \),

\[
\left(\frac{\left\lceil \frac{c+4}{4} \right\rceil + 1\left\lfloor \frac{c+4}{4} \right\rfloor + 1\left(\frac{c+4}{4}\right) + 1\left(\frac{c+4}{4}\right)}{\left\lceil \frac{c-2}{4} \right\rceil + 1\left\lfloor \frac{c-2}{4} \right\rfloor + 1\left(\frac{c-2}{4}\right) + 1\left(\frac{c-2}{4}\right)}\right) \leq \frac{32}{3}.
\]

Thus, for \( c \geq 13 \), \( \frac{f_\alpha(c+4)}{f_\alpha(c)} \leq \left(\frac{6}{5}\right)^2 \leq 16 = \frac{g(c+4)}{g(c)} \) and the result follows.

As we can observe from the proof of Claim 3, it is possible to obtain analogous results to Theorem 4 for greater values of global irregularity.
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