COUNTING THE LOCAL FIELDS IN SG THEORY.

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ABSTRACT. In terms of the form factor bootstrap we describe all the local fields in SG theory and check the agreement with the free fermion case. We discuss the interesting structure responsible for counting the local fields.

1. INTRODUCTION.

In the present paper we solve the longstanding problem of the description of all the local fields in SG theory in terms of the form factor bootstrap approach. The results of this paper add an important piece of information to the general picture, so, this paper can be considered as an additional chapter to the book [13]. It might be surprising that the solution of the problem took such a long time, but it required deeper understanding of the structure of SG form factors than the author had when the book [13] was written. The most important contributions to the field during the last several years were added in the papers [5,14,6,15] where the algebraic and the analytic properties of the form factors were outlined. It should be also said that the necessity of a more general construction for the form factors became clear to the author while thinking of the form factor bootstrap for massless S-matrices [18] formulated in [4].

Let us formulate the problem. It is known that the form factors of any local operator in the integrable field theory are subject to the following axioms:

1. Riemann-Hilbert problem.

   \[ f(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n)S(\beta_i - \beta_{i+1}) = f(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots, \beta_n), \]
   \[ f(\beta_1, \ldots, \beta_{n-1}, \beta_n + 2\pi i) = f(\beta_n, \beta_1, \ldots, \beta_{n-1}), \] (1)

2. Residue condition

   \[ 2\pi i \text{ res}_{\beta_n = \beta_{n-1} + \pi} f(\beta_1, \ldots, \beta_{n-2}, \beta_{n-1}, \beta_n) = \]
   \[ f(\beta_1, \ldots, \beta_{n-2}) \otimes \text{s}_{n-1,n}(I - S(\beta_{n-1} - \beta_1) \cdots S(\beta_{n-2} - \beta_{n-1})) \] (2)

where usual conventions are done [13], \( S(\beta) \) is a two particle S-matrix. For the sake of simplicity we do not consider the problem of bound states.

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The classification of all the local operators in the theory is equivalent to counting all the solutions to this infinite system of equations. Let us ask ourselves what is the best theory to attack the problem. The first possible idea is that one has to consider the theory with the simplest kind of S-matrix, for example, we can take the scalar S-matrix (the theory in which particles do not have internal degrees of freedom). For such a model the equation (1) is satisfied trivially, so the only problem left is to satisfy (2) which is reduced to a certain recurrence relation for polynomials in $e^{\pm \beta}$. The problem was solved for several scalar S-matrices in [3,8,9]. Notice that the S-matrices considered in [8,9] are nontrivial ones.

In the present paper we take the opposite direction. We consider the theory with a complicated S-matrix, which is the soliton S-matrix for SG model:

$$S = \int \left( (\partial_{\mu} \phi)^2 + m^2 \cos(\beta \phi) \right) d^2 x$$

In what follows we shall use only the renormalized coupling constant:

$$\xi = \frac{8\pi}{8\pi - \beta^2}$$

In that case the equations (1) are very restrictive. Actually, they can be viewed as a deformation of level 0 Knizhnik-Zamolodchikov equations with trigonometric r-matrix. The complete solution to this equation is available, moreover the solutions are parametrized by a simple and understandable structure (deformed cycles on hyperelliptic surface). So, to solve the equation (2) one has to combine these solutions. We shall see that this problem can be solved completely, and thus all the local operators in SG theory will be described. The space of operators obtained in this way is complete because it coincides with the space of local operators at the free fermion point. Moreover, we shall see that the structure responsible for counting the fields does not depend on the coupling constant. We shall comment on this important point later.

It has to be said that the problem of description of all the local operators in a situation similar to the SG one can be considered in the framework of [6] (see also [11]). In the terminology accepted in [6] the solution of the problem is "a local operator is everything commuting with the vertex operator of the second kind". We think that the alternative answer given in the present paper is more direct. It is important to understand the relation between these two possibilities of answering the same question.

Let us describe the plan of the paper. The second and the third sections are of introductory character: they describe the necessary solutions to KZ equations before and after the deformation. The forth section is the central one where the general construction of the local fields is given. In the fifth section we explain the agreement with the free fermion case and discuss the possibility of taking into consideration the disorder type fields and the fields with generalized statistics. Finally, the last section contains the discussion of further problems related to the subject of the paper.

2. KZ EQUATIONS FOR THE TRIGONOMETRIC R-MATRIX ON LEVEL 0.

Consider the equations

$$b_i \frac{d}{db_i} f(b_1, \cdots, b_{2n}) = f(b_1, \cdots, b_{2n}) \left( \sum_{j \neq i} r(b_i, b_j) \right)$$  \hspace{1cm} (3)
where \( f(b_1, \cdots, b_{2n}) \in ((\mathbb{C}^2)^{\otimes 2n})^* \), the trigonometric r-matrix \( r(b_i, b_j) \) acts in the tensor product of \( i \)-th and \( j \)-th spaces as follows:

\[
    r(b_i, b_j) = \frac{1}{2} \frac{b_i + b_j}{b_i - b_j} \sigma_i^+ \sigma_j^- + \sqrt{\frac{b_i b_j}{b_i - b_j}} (\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)
\]

We consider only the uncharged sector:

\[
    f(b_1, \cdots, b_{2n}) \Sigma^3 = 0, \quad \Sigma^3 = \sum_{i=1}^{2n} \sigma_i^3
\]

Moreover, we are interested in those solutions annihilated by either of the two operators:

\[
    \Sigma^\pm = \sum_{j=1}^{2n} (\sigma_j^3)^j b_j^\pm \sqrt{\sigma_j^3} \sigma_j^+
\]

Let us denote the set of integers \( \{1, \cdots, 2n\} \) by \( U \). Different components of the vectors \( f \) are counted by the partitions \( U = T \cup \bar{T} \), where to be precise we require \( 1 \in T \). The solutions to these equations can be described as follows.

Consider the function:

\[
    G_T(a_1, \cdots, a_{n-1}) = \frac{1}{\prod_{1 \neq i < j, i, j \in T} (b_i - b_j)} \det \left[ \frac{1}{a_k - b_i} \prod_{j \in T, j \neq i} (a_k - b_j) \right]_{k=1, \cdots, n-1; \ i \in T, i \neq 1}
\]

Then the solutions are given by

\[
    f^\pm (\gamma_1, \cdots, \gamma_{n-1}) (b_1, \cdots, b_{2n})_{T} = \prod_{i < j} (b_i - b_j)^\pm 
    \times \prod_{i \in T} b_i^\pm \frac{1}{\sqrt{P(a_i)}} \int_{\gamma_i} \frac{da_1}{\omega_1} \cdots \int_{\gamma_{n-1}} \frac{da_1}{\sqrt{P(a_{n-1})}} G_T(a_1, \cdots, a_{n-1})
\]

where \( \{\gamma_1, \cdots, \gamma_{n-1}\} \) are arbitrary cycles on the hyperelliptic surface \( f^2 = \sqrt{P(a)} \) for \( P(a) = \prod (a - b_j) \).

Let us count the number of different solutions. At the first glance we have \( 2 C_{2n-2}^{n-1} \) of them: 2 is for \( \pm \) and \( C_{2n-2}^{n-1} \) is the number of choices of cycles. However this is greater than the dimension of the space where \( f^\pm \) live: \( 2(C_{2n}^{n} - C_{2n-2}^{n-1}) \). How to resolve this contradiction? There must be a linear dependence between different solutions. Let us explain where it comes from.

The differentials which we have in (4) are obviously linear combinations of those of the first and of the second kind, i.e. they do not have residues on the surface. For the periods of such differentials we have Riemann bilinear identity:

\[
    \sum_{i=1}^{n-1} \left( \int_{\alpha_i} \omega_1 \int_{\beta_i} \omega_2 - \int_{\alpha_i} \omega_1 \int_{\beta_i} \omega_2 \right) = \sum_{\text{poles}} \text{res}(\Omega_1 \omega_2) \equiv \omega_1 \circ \omega_2
\]
where $\Omega_1$ is a primitive function for $\omega_1$; $\alpha_i$, $\beta_i$ is a canonic basis with the intersection numbers:

$$\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0, \quad \alpha_i \circ \beta_j = \delta_{ij}$$

It can be chosen as $\alpha_i = \delta_{2i}$, $\beta_i = \sum_{k=1}^{i} \delta_{2k-1}$ for $\delta_i$ drawn around the branching points $b_i$ and $b_{i+1}$.

Consider the differentials from (4):

$$\zeta_{T,i} = \frac{1}{a_i - b_i} \prod_{j \in T, j \neq i} (a_i - b_j) \frac{da}{\sqrt{P(a)}}, \quad i \in T \setminus 1,$$

for them we have

$$\zeta_{T,i} \circ \zeta_{T,j} = 0$$

because their poles and zeros cancel. For that reason one has the following linear dependence between different solutions

$$n-1 \sum_{i=1}^{n-1} f^\alpha_{\beta, \gamma_3, \cdots, \gamma_{n-1}}(b_1, \cdots, \beta_{2n}) = 0$$

for arbitrary $\{\gamma_3, \cdots, \gamma_{n-1}\}$. So, the real number of solutions is

$$2(C_{2n-2}^{n-1} - C_{2n-2}^{n-3}) = 2(C_{2n}^{n} - C_{2n}^{n-1})$$

which resolves the contradiction.

Dealing with the differentials of the first and of the second kind on the hyperelliptic surface it is convenient to move the singularities to the infinity. The basis of the differentials with singularities at the infinity can be taken as

$$\eta_p = \frac{a^{p-1}}{\sqrt{P(a)}} da, \quad \zeta_p = \frac{1}{\sqrt{P(a)}} \sum_{k=p}^{2n-p} (-1)^{k+p}(k-p)a^{k-1}\sigma_{2n-p-k}(b_1, \cdots, b_{2n}) da$$

where $p = 1, \cdots, n-1$ (recall that the genus of the surface is $n-1$), $\sigma_i(b_1, \cdots, b_{2n})$ are the elementary symmetric polynomials. This basis is canonic:

$$\eta_i \circ \eta_j = \zeta_i \circ \zeta_j = 0, \quad \eta_i \circ \zeta_j = \delta_{ij}$$

The differentials from (4) can be rewritten in terms of (5). Up to total derivatives one has

$$\prod_{i=1}^{n-1} \frac{1}{\sqrt{P(a_i)}} da_i \sim G_T(a_1, \cdots, a_{n-1}) \sim \prod_{i \in T, j \in \bar{T}} \frac{1}{(b_i - b_j)} \det |\zeta_{T,i}(a_k)|_{k=1, \cdots, n-1; i=1, \cdots, n-1}$$
where
\[ \zeta_{T,i}(a) = \frac{A_{T,i}(a) da}{\sqrt{P(a)}} \]

with the following polynomials:
\[ A_{T,i}(a) = \prod_{i \in T} (a - b_i) \left[ \prod_{i \in T} \frac{(a - b_i)}{a^{n-i}} \right] + \prod_{i \in \bar{T}} (a - b_i) \left[ \prod_{i \in \bar{T}} \frac{(a - b_i)}{a^{n-i}} \right] + \]

where \([\cdot]_+\) means that only the polynomial part of the expression in brackets is taken. Considering \(\zeta_{T,i}(a)\) as components of a vector one has
\[ \zeta_T(a) = \zeta(a) + C\eta(a) \]

where \(C\) is certain \((n-1) \times (n-1)\) matrix. It is easy to construct the differentials \(\eta_T(a)\) dual to \(\zeta_T(a)\) and thus to obtain a symplectic matrix connecting \(\eta_T(a)\), \(\zeta_T(a)\) with \(\eta(a)\), \(\zeta(a)\) from which \(f\) and \(C\) are two blocks.

Let us summarize the discussion of this section. We see that the most important ingredient of the solutions to (3) is the symplectic matrix composed of all the periods of all the differentials of the first and of the second kind
\[ \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \]

where \(P_i\) are \((n-1) \times (n-1)\) matrices:
\[ (P_1)_{i,j} = \int \eta_j, \quad (P_2)_{i,j} = \int \zeta_j, \]
\[ (P_3)_{i,j} = \int \eta_j, \quad (P_4)_{i,j} = \int \zeta_j \]

The solutions themselves can be considered as maps from different polarizations of differentials (\(\zeta_T(a)\) for any \(T\) gives such a polarization) into different polarizations of cycles (different half-bases of homologies). Let us also mention that the above considerations are not very different from those presented in [15] for the rational \(r\)-matrix, for the reason that in our particular case (level 0 and special symmetry) the solution for trigonometric and rational \(r\)-matrices are in one-to-one correspondence.

3. Solutions to the deformed equations.

Consider the equations
\[ f(\beta_1, \cdots, \beta_i, \beta_{i+1}, \cdots, \beta_{2n}) S(\beta_i - \beta_{i+1}) = f(\beta_1, \cdots, \beta_i, \beta_{i+1}, \cdots, \beta_{2n}), \]
\[ f(\beta_1, \cdots, \beta_{n-1}, \beta_{2n} + 2\pi i) = f(\beta_{2n}, \beta_1, \cdots, \beta_{2n-1}), \] (6)
which can be regarded as a deformation of (3), the SG S-matrix is [17]

\[ S_{\xi}(\beta) = S_{\xi,0}(\beta)\tilde{S}_{\xi}(\beta), \quad S_{\xi,0}(\beta) = -\exp\left( -i \int_{0}^{\infty} \frac{\sin(k\beta)\text{sh}(\frac{\pi-\xi}{2}k)}{k\text{sh}(\frac{k\xi}{2})\text{ch}(\frac{k\xi}{2})} \right), \]

\[ \tilde{S}_{\xi}(\beta) = \frac{1}{\text{sh}\frac{\pi}{2}(\beta - \pi i)} \begin{pmatrix} \text{sh}\frac{\pi}{2}(\beta - \pi i) & 0 & 0 & 0 \\ 0 & \text{sh}\frac{\pi}{2}\beta & -\text{sh}\frac{\pi}{2}i & 0 \\ 0 & -\text{sh}\frac{\pi}{2}i & \text{sh}\frac{\pi}{2}\beta & 0 \\ 0 & 0 & 0 & \text{sh}\frac{\pi}{2}(\beta - \pi i) \end{pmatrix}, \]

Similarly to the classical case we consider only those solutions of (6) which are invariant with respect to one of the two quantum groups existing in the SG theory [12] which means that they are annihilated by \( \Sigma^3 = \sum_{i=1}^{2n} \sigma_i^3 \) and by either of the two operators:

\[ \Sigma^\pm_\alpha = \sum_{j=1}^{2n} (\sigma_j^3)^j e^{\pm\frac{\pi}{2}\alpha_j} \sigma_j^+ \]

The variables exactly similar to \( b_i \) from the previous section are the following

\[ b_i = e^{\frac{2\pi i}{\xi} \beta_i} \]

The solutions to the equations (6) constitute a linear space over the field of quasi-constants: symmetric 2\( \pi i \)-periodic functions, i.e. those depending on

\[ B_i = e^{\beta_i} \]

In what follows we shall use all this notations \((\beta_j, b_j, B_j)\).

Exactly as in the classical case, the most nontrivial part of the solutions is described by the deformed matrix of the periods of hyperelliptic differentials. The definition is as follows.

For two given polynomials \( Q(a) \) and \( L(A) \) (which can also depend on \( b_j \) and on \( B_j \) respectively as on parameters) we consider the paring \( \langle Q(a), L(A) \rangle \) defined by the integral:

\[ \langle Q(a), L(A) \rangle \equiv \int_{-\infty}^{\infty} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha, \beta_j) Q(e^{\frac{2\pi i}{\xi} \alpha}) L(e^{\alpha}) e^{\frac{2\pi i}{\xi} \alpha} d\alpha \quad (7) \]

where

\[ \tilde{\varphi}(\alpha, \beta) = \varphi(\alpha - \beta) \exp\left( -\frac{1}{2} (\frac{\pi}{\xi} + 1)(\alpha + \beta) \right) \]

the function \( \varphi \) is given by

\[ \varphi(\alpha) = C \exp\left( -2 \int_{0}^{\infty} \frac{\sin^2 k\alpha}{k} \text{sh}\frac{\pi k}{2} \text{sh}\pi k dk \right) \]
where $C$ is certain constant [13]. The integral (7) is defined for $1 \leq \deg(L(A)) \leq 2n - 2$, and with a proper regularization [13] can be defined for an arbitrary polynomial $Q(a)$. We must understand the polynomial $Q(a)$ as the one defining a deformed differential while the polynomial $L(A)$ defines a deformed cycle. There is a canonic basis of deformed differentials which is defined by the deformation of the polynomials involved into (5):

$$R_p(a) = a^{p-1},$$

$$S_p(a) = \frac{1}{\tau - \tau^{-1}} \sum_{k=p}^{2n-p} (-1)^{k+p}(\tau^{k-p} - \tau^{p-k})a^{k-1}\sigma_{2n-p-k}(b_1, \ldots, b_{2n})$$

where

$$\tau = e^{\frac{2\pi i}{\xi}}$$

The most important property of the deformed periods is that they can be combined into a symplectic matrix. Namely, consider the matrix

$$P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$$

where $P_i$ are the following $(n - 1) \times (n - 1)$ blocks:

$$(P_1)_{i,j} = \langle R_j(a), A^{2i} \rangle, \quad (P_2)_{i,j} = \langle S_j(a), A^{2i} \rangle,$$

$$(P_3)_{i,j} = \langle R_j(a), A^{2i-1} \rangle, \quad (P_4)_{i,j} = \langle S_j(a), A^{2i-1} \rangle$$

It is shown in [16] that

$$\begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \begin{pmatrix} -P_4^t & P_2^t \\ P_3^t & -P_1^t \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$$

where $X$ is the symmetric matrix composed of quasiconstants:

$$X_{i,j} = \int_{-\infty}^{\infty} \frac{A^{2(i+j)}}{\prod(A^2 + B_j^2)} dA$$

We could redefine the basis of the deformed cycles combining them with quasiconstants in order to eliminate $X$:

$$\alpha_i = A^{2i}, \quad \beta_i = (X^{-1})_{i,j}A^{2j-1}$$

We hope that $\alpha_i$ and $\beta_i$ here (deformed cycles) will not be confused with the rapidities and the integration variables.

The deformed period matrix allows a great simplification at the free fermion point $\xi = \pi$. In fact, the calculation of the period matrix at this point is not absolutely trivial, one should be careful with the regularization of the integral (7) explained in [13]. It can be shown that in the limit $\xi = \pi + \epsilon, \epsilon \to 0$ the matrix $P$ has to be rescaled as follows:

$$\tilde{P} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} P$$
The matrix $\tilde{P}$ is finite in the limit:

$$\tilde{P}|_{\xi=\pi} = \begin{pmatrix} 0 & I \\ X & Y \end{pmatrix}$$

(8)

where

$$Y_{ij} = \sum_{l=1}^{2n} \frac{1}{\prod_{k}(B^2_l - B^2_k)} \sum_{k=j}^{2n-j} (k - j)B^{2i+2k-1}_{1j} \sigma_{2n-j-k}(B^2_{1j}, \cdots, B^2_{2n})$$

By a simple symplectic transformation of the deformed cycles this matrix can be transformed into the unit matrix. This is an important fact the meaning of which for the SG model will be explained later.

It is possible to introduce the deformation of the integral of a $k$-differential form over a $k$-chain:

$$\langle Q(a_1, \cdots, a_k), L(A_1, \cdots, A_k) \rangle = \int_{-\infty}^{\infty} d\alpha_1 \cdots \int_{-\infty}^{\infty} d\alpha_k \prod_{i=1}^{k} \prod_{j=1}^{2n} \tilde{\varphi}(\alpha_{i}, \beta_{j}) \times Q(\hat{\varepsilon}^{\alpha_1}, \cdots, \hat{\varepsilon}^{\alpha_k})L(\hat{\varepsilon}^{\alpha_1}, \cdots, \hat{\varepsilon}^{\alpha_k})\hat{\varepsilon}^{\Sigma \alpha_1}$$

The polynomials $Q$ and $L$ are supposed to be antisymmetric; however, later we shall take only $Q$ as antisymmetric then $L$ will automatically antisymmetrize itself under the integral.

We would like to emphasize that contrary to the usual homologies which constitute an exterior algebra over the ring of quasiconstants the deformed homologies constitute an exterior algebra over the ring of quasiconstants.

Let us return to the equations (6). Their solutions can be constructed as follows:

$$f^{k_1, \cdots, k_{n-1}}(\beta_1, \cdots, \beta_{2n})_{T} = \prod_{i<j} \zeta(\beta_i - \beta_j) \sum_{T' \in U, \#T'=n} \frac{1}{\prod_{i \in T, j \notin T} \text{sh} \tilde{\varphi}(\beta_i - \beta_j)} \times e^{\Sigma \beta_j} f^{k_1, \cdots, k_{n-1}}(\beta_1, \cdots, \beta_{2n})_{T'} w(\beta_1, \cdots, \beta_{2n})_{T'}$$

where $\zeta(\beta)$ is certain special function [13], $w(\beta_1, \cdots, \beta_{2n})_{T'}$ is a special basis in $((\mathbb{C}^2)^{\otimes 2n})^*$ [13]. The most interesting part of the answer is the function $f^{k_1, \cdots, k_{n-1}}$ which is given by

$$f^{k_1, \cdots, k_{n-1}}(\beta_1, \cdots, \beta_{2n}) = \langle F_T(a_1, \cdots, a_{n-1}), A_{k_1}^{k_1} \cdots A_{k_{n-1}}^{k_{n-1}} \rangle$$

(9)

where

$$F_T(a_1, \cdots, a_{n-1}) = \det|A_{T,i}(a_j)|(n-1) \times (n-1)$$

the polynomials $A_{T,i}(a)$ are given by

$$A_{T,i}(a) = \prod_{q \in T} (a - \tau b_q)^{i-1} \sum_{k=0}^{a-1} (1 - \tau^{2i-2k}) a^{i-k-1}(\tau)^k \sigma_k(b_T)$$

$$+ \tau^{2i} \prod_{q \in T'} (a - \tau^{-1} b_q)^{i-1} \sum_{k=0}^{a-1} (1 - \tau^{2i-2k}) a^{i-k-1}(\tau)^k \sigma_k(b_T)$$
where $b_T$ denotes the subset $\{b_j\}$ with $j \in T$. There is another useful formula for $f$:

$$f_{\pm}^{k_1, \cdots, k_{n-1}}(\beta_1, \cdots, \beta_{2n})_T = \det (A_{T,i}(a), A^{k_j})_{i,j=1,\cdots,n-1}$$

It is known that the functions $f$ satisfy (6) [13]. The arguments of the previous section concerning the counting of solutions apply perfectly well to the deformed case. Indeed, it can be shown that

$$A_T = S + CR$$

where $I$ and $C$ are blocks of a symplectic matrix. This fact together with the deformed Riemann bilinear identity implies the following linear dependence between the solutions:

$$\sum_{i=1}^{n-1} (X^{-1})_{ij} f_{\pm}^{2i,2j-1,k_3,\cdots,k_{n-1}}(\beta_1, \cdots, \beta_{2n}) = 0$$

(10)

for arbitrary $k_3, \cdots, k_{n-1}$. Recall that the matrix $X$ is composed of quasiconstants.

It can be shown that the solutions $f$ as functions of $\beta_{2n}$ do not have other singularities for $0 \leq \text{Im} \beta_{2n} \leq 2\pi$ but the simple poles at $\beta_{2n} = \beta_j + \pi i$. If we impose this requirement generally and also require that the solutions do not grow faster than $\exp(|x|/|\beta_j|)$ for some $x$ as $\beta_j \to \pm \infty$ then the totality of solutions to (6) is generated by

$$\exp(\sum_{l=-\infty}^{\infty} t_l(\sum e^{l\beta_i})) f_{\pm}^{k_1, \cdots, k_{n-1}}(\beta_1, \cdots, \beta_{2n})$$

(11)

modulo the relation (10).

4. Local operators in SG theory.

In order to construct a local operator one has to find a solution to the system composed of (6) and

$$2\pi i \text{ res}_{\beta_{2n}=\beta_{2n-1}+\pi i} f(\beta_1, \cdots, \beta_{2n-2}, \beta_{2n-1}, \beta_{2n}) =$$

$$= f(\beta_1, \cdots, \beta_{2n-2}) \otimes s_{2n-1,2n} (I - S(\beta_{2n-1} - \beta_1) \cdots S(\beta_{2n-1} - \beta_{2n-2}))$$

(12)

In the previous section we constructed all the solutions to (6), so, our goal is to learn how to combine them in order to satisfy (12). Let us explain the general idea about the organization of the space of local fields in terms of form factor bootstrap.

We expect that there are infinitely many local operators which are counted by the following data:

1. We fix the number of particles ($2m$) in the minimal form factor. If we fixed $2m$ it means that for the given operator the form factors with $2n$ particles vanish for $n < m$.

2. The minimal $2m$-particle form factor can be taken in infinitely many possible ways due to a possible multiplication of the solution to (6) by quasiconstant.

Keeping in mind the general formula (11) we write the following generating function for the minimal $2m$-particle form factor:

$$\prod_{i<j} (e^{\beta_i} + e^{\beta_j}) \exp(\sum_{l=-\infty}^{\infty} t_l(\sum e^{l\beta_i})) f_{\pm}^{k_1, \cdots, k_{n-1}}(\beta_1, \cdots, \beta_{2n})_T$$

(13)
modulo relation (10). This formula needs some comments. The product \( \prod_{i<j} (e^{\beta_i} + e^{\beta_j}) \) is put in order to cancel the poles of \( f_{k_1,\ldots,k_{n-1}}^{1,\ldots,2n} (\beta_1, \ldots, \beta_{2n}) \) which is needed for the form factor to be the minimal one: the minimal form factor should not be related to those with lower number of particles by (12). Different minimal form factors are obtained from (13) applying any differential polynomial in \( t_i \) and putting \( t_i = 0 \) afterwards. The times \( t_i \) play a very different role for \( l \) odd and even. The times \( t_{2p+1} \) can be identified with those related to the local integrals of motion, in particular \( t_1 = z, t_{-1} = \bar{z} \) (\( z \) and \( \bar{z} \) are usual euclidian coordinates).

The multiplication of the form factors by \( \exp( \sum_{p=-\infty}^{\infty} t_{2p+1}(\sum e^{(2p+1)\beta_i}) \) is always possible because it does not spoil the residue condition. The times \( t_{2p} \) are far more nontrivial and, hence, more interesting: they do not correspond to any local symmetry of the theory, but still they must be considered. To summarize, in the formula (13) we have three structures counting the minimal form factors: two bosonic (related to \( t_{2p+1} \) and \( t_{2p} \) respectively) and one fermionic (related to the counting by \( k_1, \ldots, k_{n-1} \)). We shall show that to every minimal form factor (13) a local operator can be related.

Let us consider the problem of calculation of the residue (12). Generally the formulae for residue of \( f_{k_1,\ldots,k_{n-1}}^{1,\ldots,2n} (\beta_1, \ldots, \beta_{2n})_T \) can not be written in terms of the functions of the same kind. There exists, however, one nice possibility which we are going to describe. It is useful to generalize the previous notations considering the functions

\[
L^{\pm}_n(\beta_1, \ldots, \beta_{2n})_T
\]

where \( L_n(A_1, \ldots, A_{n-1}) \) is an arbitrary polynomial whose degree in any variable is between 1 and \( 2n-2 \). The generalization corresponds to the insertion of \( L_n(A_1, \ldots, A_{n-1}) \) instead of \( A_1 \cdots A_{n-1} \) in the formula (9). The polynomial \( L_n(A_1, \ldots, A_{n-1}) \) can be taken as an antisymmetric one, but we prefer to have it partly antisymmetric, namely, antisymmetric in the variables \( A_1, \ldots, A_k \) for some given \( k \leq n-1 \). The polynomial \( L_n(A_1, \ldots, A_{n-1}) \) can also depend symmetrically on \( B_j \) as on parameters, we shall explicitly write this arguments when needed.

The only situation where a good formula for the residue at \( \beta_{2n} = \beta_{2n-1} + \pi i \) exists is the following one

\[
L_n(A_1, \ldots, A_k, A_{k+1} \cdots, A_{n-1}|B_1, \ldots, B_{2n-2}, B_{2n-1}, B_{2n})|_{\beta_{2n-1} = -B_{2n} \equiv B} = \sum_{i=1}^{k} (-1)^i \prod_{l \neq i} (A_i^2 + B^2) L^{(i)}_n(A_1, \ldots, A_k, A_{k+1} \cdots, A_{n-1}|B_1, \ldots, B_{2n-2}|B)
\]

for some polynomials \( L^{(i)}_n \). Provided the equation (14) holds we have

\[
2\pi i \ \text{res}_{\beta_{2n} = \beta_{2n-1} + \pi i} \ \frac{L^{(1)}_n(\beta_1, \ldots, \beta_{2n-2}, \beta_{2n-1}, \beta_{2n})}{\prod_{i<j} (e^{\beta_i} + e^{\beta_j})} = \left\{ \begin{array}{ll}
L^{(k)}_{n-1}(A_1, \ldots, A_{k-1}, A_k \cdots, A_{n-2}|B_1, \ldots, B_{2n-2}|B) & \\
L^{(k)}_{n-1}(A_1, \ldots, A_{k-1}, A_k \cdots, A_{n-2}|B_1, \ldots, B_{2n-2}|B) & 
\end{array} \right.
\]

for some \( L^{(k)}_n \).
Obviously, the problem of finding a local operator with the minimal form factor (13) will be solved if we find the polynomials \( L_m^k, L_{m+1}^m, L_{m+2}^m, \cdots \) such that:

1. For the given \( n \) the integer \( k \) equals \( n - m \), and the equation (14) holds.

2. The polynomials \( L_{n-1}^k \) obtained from (15) do not depend upon \( B \). Moreover they are from the same sequence of polynomials as \( L_m^n \):

\[
\tilde{L}_{n-1}^k(A_1, \cdots, A_{k-1}, A_k \cdots, A_{n-2}|B_1, \cdots, B_{2n-2}|B) = \\
= \pm \frac{B}{2i} L_{n-1}^m(A_1, \cdots, A_{k-1}, A_k \cdots, A_{n-2}|B_1, \cdots, B_{2n-2})
\]

3. The initial condition is satisfied:

\[
L_m^m(A_1, \cdots, A_{m-1}|B_1, \cdots, B_{2m}) = \prod_{i<j} (B_i + B_j) \prod_{i=1}^{m-1} A_i^{k_i},
\]

The polynomials which satisfy these conditions can be taken as follows

\[
L_m^m(A_1, \cdots, A_k, A_{k+1} \cdots, A_{n-1}|B_1, \cdots, B_{2n}) = \\
\exp\left(\sum_{i=-\infty}^{\infty} t_i (\sum_{j=1}^m B_j) \prod_{i<j} A_i \prod_{1 \leq i < j} (A_i^2 - A_j^2) \prod_{i=1}^{m-1} A_i^{k_i+m+i} \right)
\]

\[
\times \begin{vmatrix}
1, & \cdots, & 1 \\
B_1^2, & \cdots, & B_2^{2n}
\end{vmatrix}
\]

\[
\times \begin{vmatrix}
1, & \cdots, & 1 \\
H_1(B_1), & \cdots, & H_1(B_{2n}) \\
\vdots & \vdots & \vdots \\
H_{n-m}(B_1), & \cdots, & H_{n-m}(B_{2n})
\end{vmatrix}
\]

(16)

where

\[
H_i(B) = \exp\left(-2 \sum_{p=-\infty}^{\infty} t_{2p} B^{2p}\right) B^{2i-1} \prod_{j=1}^{m-1} (A_{n-m+j}^2 + B^2)
\]

The formula (16) is the central formula of this work. It solves the problem of constructing a local field from the minimal form factor. Let us discuss this formula in some details. The terms \( t_{2p+1} \) enter it in rather trivial way. It is more interesting with the terms \( t_{2p} \): they mix nontrivially with the deformed cycles. As it has been said, we have added to SG two structures: one bosonic \( (t_{2p}) \) and one fermionic (deformed cycles). The formula (16) defines a flow in this additional space. Notice also that for \( m = 1 \) and \( t_l = 0 \) the formula simplifies a lot:

\[
L_n^1(A_1, \cdots, A_{n-1}) = \prod_{i=1}^{n-1} A_i \prod_{i<j} (A_i^2 - A_j^2)
\]

This simple formula corresponds to the form factors of the energy momentum tensor. It has been known for a long time, but its simplicity misled the author preventing him of finding the general formula (16).
There is also a question of the uniqueness: whether the polynomials (16) are the only possible for the given minimal form factor. It is not the case: we can add to the operator with \(2m\)-particle minimal form factor an arbitrary local operator whose minimal form factor has more particles, the result being a local operator as well. So, the problem is not that of the uniqueness, but that of the existence. However, considering the problem of counting the local operators we have to ignore the possibility of adding a local operator with more particles in the minimal form factor. Physically, the operator constructed via (16) is characterized by the mildest possible ultraviolet behaviour among those with the given minimal form factor.

5. Agreement with the free fermion case and possible generalizations.

We want to discuss the problem of completeness of our list of local fields for SG model. In modern language the completeness is related to the calculation of certain characters. We shall take more simple route: we shall consider the agreement with what we have at the free fermion point. Such an agreement would imply the necessary character formulae.

Consider the case \(\xi = \pi\) when SG model turns into the free Dirac field:

\[
S = \int (i\bar{\psi} \gamma_\mu \partial_\mu \psi + m\bar{\psi} \psi) d^2x
\]

The solution to this model is given by the Fourier transform

\[
\psi(x_0, x_1) = (17) = \int \left( e^{-ip_\mu(\beta) x_{\mu}} \left( e^{\frac{\beta}{2}} a_-(\beta) + e^{-ip_\mu(\beta) x_{\mu}} \left( e^{\frac{\beta}{2}} - e^{-\frac{\beta}{2}} \right) a_+^*(\beta) \right) \right) d\beta
\]

where \(p_\mu(\beta) = m(e^\beta + (-1)^\mu e^{-\beta})\). The local field (note those of disorder type) for the free fermion model have only one nontrivial form factor. If we consider the neutral sector of the theory the form factor corresponds to the even number of particles \((2m)\) the number of fermions being equal to the number of antifermions, so the fields we are dealing with are of the form

\[
: \bar{D}_1(\bar{\psi}) \cdots \bar{D}_m(\bar{\psi}) D_1(\psi) \cdots D_m(\psi) :
\]

where \(D_i, \bar{D}_i\) are arbitrary differential operators with \(\gamma\)-matrix coefficients. Not all of these operators are independent because we have to impose the equations of motion, but there is now ambiguity in their form factor description: the equations of motion are taken into account by (17).

As usual we describe the spinor structure by subsets \(T\) of those particles which are fermions (not antifermions). The form factor corresponding to the operator of the type (18) must be of the form

\[
f(\beta_1, \ldots, \beta_{2m})_T = \prod_{i<j, i,j \in T} \text{sh} \left( \frac{1}{2} (\beta_i - \beta_j) \right) \prod_{i<j, i,j \in \bar{T}} \text{sh} \left( \frac{1}{2} (\beta_i - \beta_j) \right) P_1(e^{\beta_T}) P_2(e^{\beta_{\bar{T}}})
\]

where \(\text{sh}(x) = \sinh(x)\).
where \( P_1 \) and \( P_2 \) are two arbitrary symmetric Laurent polynomials of \( m \) variables. Not every operator which is local at the free fermion point allows a local continuation for the general coupling. The reason for that is in the fractional statistics of SG solitons: only for \( \xi = \pi \) they are real fermions. How to describe those operators which do remain local for the nontrivial coupling? The free fermion field is known to allow the infinite dimensional symmetry algebra which is \( U(sl(2))_{\tau} \) [10], for arbitrary coupling this algebra turns into the algebra \( U(sl(2))_q (q = -\tau^2) \) of nonlocal charges [2]. There are two finite-dimensional subalgebras in this algebra, and we claim that only those local operators invariant with respect to one of them can be locally continued for the general coupling. In order to check the invariance one has to make sure that the form factor (19) is annihilated by one of two operators

\[
\Sigma_+^\pm = \sum_{j=1}^{2n} (\sigma_j^3)^j \epsilon^{\pm \frac{1}{2} \beta_j} \sigma_j^3
\]

which corresponds to the symmetry of the solutions of the form factor equations which has been accepted in this paper. It does not mean that other operators are really pathological, they just have generalized statistics, we shall comment on them later.

Consider the formula

\[
P_1(B_1, \cdots, B_m) P_2(B_{m+1}, \cdots, B_{2m})
\]

(20)

The space of polynomials (20) is finite-dimensional over the ring of the Laurent polynomials, symmetric in all the variables \( B_1, \cdots, B_{2m} \). Taking this into account one realizes that the generating function for the form factors annihilated by \( \Sigma_+^\pm \) can be written down as follows

\[
\exp \left( \sum_{l=-\infty}^{\infty} t_l \left( \sum_i l_i e^{i\beta_i} \right) \pm \frac{1}{2} \left( \sum_i \beta_i - \sum_{i \in \bar{T}} \beta_i \right) \right) P_T^J(e^{\beta_1}, \cdots, e^{\beta_{2m}})
\]

(21)

where \( P_T^J \) are certain basic polynomials, \( J = 1, \cdots, C_{2n}^m - C_{2n}^{m-1} \). We do not write down an explicit formula for \( P_T^J \) for the economy of space.

The formula (21) looks very much similar to (13) the only difference being in the way of counting from 1 to \( C_{2n}^m - C_{2n}^{m-1} \): in (21) we count the polynomials \( P_T^J \) while in (13) we count \( k_1, \cdots, k_{m-1} \) (mod the relation (10)). However, the very simple form of the period matrix at the free fermion point (8) allows to establish the one-to-one correspondence between these two ways of doing things. We do not present the explicit calculations here because, to our mind, the situation is rather clear.

It is a proper place to discuss possible generalizations. First, it is quite possible to take into consideration the quasilocal operators (those with generalized statistics). They are combined into multiplets with respect to the action of two quantum groups. The form factors of the fields which correspond to the highest vectors of the multiplets can be found using the results of [7]. Including the quasilocal fields we get exactly the same number of fields in SG as the number of local fields at the free fermion point. Technically, the consideration of such fields corresponds to omitting the requirement \#(k_i) = m − 1 in the formula (16), rather we have to consider \#(k_i) \leq m − 1.
Second, in this paper we considered the operators which in the SG language are identified with \( \exp(i\beta \phi) \) and their Virasoro descendents. It is possible to consider the operators of disorder type: \( \exp(i(2m + 1)\beta \phi/2) \) and their descendents. These operators have infinitely many form factors even at the free fermion point. The formula (16) in application to them will have an additional multiplier:

\[
\prod B_j^{-\frac{r}{2}} \prod A_i
\]

6. Discussion.

We do suppose that further development of the results of the present paper will lead to a much better understanding of the integrable field theory. Let us outline the most promising directions for the future study.

1. On purpose we did not go into details of symmetry: it has to be done separately. However, let us outline briefly the structure. It is known that SG allows the algebra of nonlocal charges \( U_q(\hat{sl}(2)) \) for \( q = -\tau^2 \) [2]. The results of this paper imply the existence of another hidden algebra \( U_q(\hat{sl}(2)) \) with \( q = -1 \). This algebra is not a symmetry of the theory, but it is responsible for counting the local fields. The canonic transformation with the deformed matrix of periods adjusts this two algebras or makes transformation from fields to particles. At the free fermion point the fields and the particles are the same which is implied by simplicity of the period matrix at this point. That is why the two algebras also coincide at the free fermion point.

2. Probably the most adequate point of view to the SG theory is the following. We have to consider not many fields depending on the local times \( t_{2p+1} \), but one field depending on the local times \( t_{2p+1} \), on the additional times \( t_{2p} \) and on the fermions \( a, b \) (for \( \alpha \) and \( \beta \) cycles) which describe the deformed cycles. So, the space where SG field lives is much bigger than usually expected.

3. It is very important that the field counting structure is independent of the coupling constant. The most important consequence of the fact is the following one. Even in the classical limit \( \xi \to 0 \) the structure stays untouched, so, we have to think about classical interpretation. Certainly, the very meaning of the structure implies the connection with the dressing symmetries (see [1] for relevant discussion): it is not a real symmetry, but it counts different fields. We have no doubts that there must be a relation to Sato’s Grassmanian.

Another interesting limit corresponds to the asymptotically free quasiclassics: \( \xi \to \infty \). This limit seems to be more interesting than the classical SG one, but in spite of several attempts the understanding of the situation is not absolutely clear. Let us explain why this limit is so important. From the results of the present paper it follows that the duality between particles and fields in SG model is related to the duality between the deformed cohomologies and homologies of hyperelliptic surface. The limit \( \xi \to \infty \) is classical for this picture: this is the limit where we get usual surfaces with their homologies and cohomologies.

4. Returning to the free fermion point one sees that the introduction of the variables \( \alpha_i \) is nothing but a fancy way of describing the spinor structure of fermions (similar to the screening operators technics). It must be possible to show that the formula (16) solves a version of Thirring equation of motion. Important thing is that this way of writing the equations of motion must be free of divergencies: all the renormalizations are absorbed into the period matrix.
5. The technics of this paper can be generalized in order to describe the form factors for the massless flows [17] from the principal chiral field into WZNW theory.

6. Finally, it is amusing that the formula (16) looks as a $\tau$-function of some classical integrable equation. Can such an equation be found, and what is its meaning for the SG theory?

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