High Order Semi-Implicit WCNS for 1D Isentropic and Full Euler Equations

Xun Chen, Xu Zhang and Yanqun Jiang*

School of science, Southwest University of Science and Technology, Mianyang, China

*Corresponding author: jyq2005@mail.ustc.edu.cn

Abstract. This paper designs a high order semi-implicit weighted compact nonlinear scheme (WCNS) to solve 1D isentropic and full Euler equations. The Euler equations are split into stiff and non-stiff terms that are solved by the implicit and explicit time discretization method, respectively, in order to improve the computational efficiency for low Mach flows. The fifth-order WCNS is applied for the spatial discretization. Several numerical examples are given to demonstrate the performance of the designed method.

Keywords: Euler equations, low Mach number, WCNS, semi-implicit method.

1. Introduction
The 1D isentropic and full Euler equations are described as [1]

\[
\begin{align*}
\rho_t + q_x &= 0 \\
q_t + \left( \frac{q^2}{\rho} \right)_x + \frac{1}{\varepsilon^2} p_x &= 0 \\
E_t + \left( \frac{E + p}{\rho} \cdot q \right)_x &= 0
\end{align*}
\]

(1)

where \( \rho \) is the density, \( q = \rho u \) is the momentum, \( E \) is the total energy, \( p \) is the pressure defined as

\[
p = (\gamma - 1) \left( E - \frac{\varepsilon^2}{2} \cdot \frac{q^2}{\rho} \right)
\]

(2)

with a constant \( \gamma > 1 \) and \( \varepsilon \) is the scaled Mach number that is a measure of compressibility of the fluid. For example, \( \varepsilon = 1 \) corresponds to the fully compressible regime. By substituting (2) into (1), we obtain
When the Mach number $\varepsilon$ in Eq. (1) is of order one, typical shock capturing methods, such as the finite volume method (FVM) [2], the discontinuous Galerkin method (DGM) [2], the weighted essentially non-oscillatory (WENO) scheme [4], the weighed compact nonlinear scheme (WCNS) [5], and so on, can be used to capture shocks and other complex structures in compressible flows. However, when the Mach number $\varepsilon$ is very small in the incompressible regime, standard explicit shock-capturing methods suffer from an increasingly large computational time due to a CFL time restriction and severe loss of accuracy due to the creation of spurious waves [1, 5]. Therefore, some all-speed asymptotic-preserving (AP) schemes [1, 5, 6] that are uniformly consistent with the low-Mach number limit and are uniformly stable are proposed to solve the compressible-incompressible limit problems. Recently, the semi-implicit or implicit-explicit (IMEX) time discretization strategies that treat with the stiff part implicitly and the non-stiff part explicitly have been designed to solve these problems effectively and efficiently.

In this work, we split the fluxes in Eq. (3) into stiff and non-stiff terms and design a high order semi-implicit WCNS to solve 1D isentropic and full Euler equations. The stiff term is solved by the implicit time discretization method, while the non-stiff term is solved by the explicit time discretization method in order to improve the efficiency. The fifth-order WCNS is used for the spatial discretization.

2. Algorithm descriptions

In this section, we make a detailed description of the fifth-order WCNS for the spatial discretization and the third-order IMEX Runge–Kutta scheme for the time discretization. For simplicity, we take a uniform grid defined by $x_i = i\Delta x$, $i = 0, 1, \ldots, N$, where $\Delta x$ is grid spacing and $N$ is the number of cells.

2.1. WCNS

For the one-dimensional scalar case, the fifth-order WCNS is [5]

$$
\begin{align*}
\rho_i + q_i = 0, \\
q_i + \left(3 - \frac{\gamma q_i^2}{\rho} \right)_i + \frac{\gamma - 1}{\varepsilon^2}E_i = 0, \\
E_i + \left(\frac{(\gamma - 1)E_i^2}{2} - \frac{q_i^2}{\rho^2} \right)_i + \frac{\gamma E_i q_i}{\rho} = 0.
\end{align*}
$$

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2.1. WCNS

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$$
f_i = \frac{64}{45h} (\hat{f}_{i+1/2} - \hat{f}_{i-1/2}) - \frac{2}{9h} (f_{i+1} - f_{i-1}) + \frac{1}{180h} (f_{i+2} - f_{i-2})
$$

where the numerical flux $\hat{f}_{i+1/2}$ is computed by a fifth-order WENO type interpolation method. We first split the flux $f$ into a positive part $f^+$ and a negative part $f^-$ by the Lax-Friedrichs flux splitting technique and obtain

$$
f(u) = f^+(u) + f^-(u), \quad f^\pm(u) = \frac{1}{2} (f(u) \pm \alpha u)
$$

where $\alpha = \max_{x} |f'(u)|$. In (4), $\hat{f}_{i+1/2}$ can be evaluated as $\hat{f}_{i+1/2} = f_{i+1/2}^+ + f_{i+1/2}^-$. The WENO interpolation technique is applied to calculate the positive numerical flux. For simplicity, the “±” sign in the superscript is removed.

For a given a node $i$, a main stencil $S_i = \{x_{i-2}, \ldots, x_{i+2}\}$ is chosen and can be divided into three sub-stencils $S_k = \{x_{i-k-2}, x_{i-k-1}, x_{i+k}, x_{i+k+1}\}$, $k = 0, 1, 2$. Three third-order interpolations of $f$ on sub-stencils are derived as
The linear combination of the three third-order interpolations is

\[ \hat{f}_{i+1/2} = \sum_{k=0}^{2} c_k \hat{f}^{k}_{i+1/2} = \frac{1}{128} (3f_{i-2} - 20f_{i-1} + 90f_i + 60f_{i+1} - 5f_{i+2}) \]  

where \( c_0 = \frac{1}{16}, c_1 = \frac{10}{16}, c_2 = \frac{5}{16} \) are linear weights. In order to avoid generating oscillations around shock waves and discontinuities, the numerical flux \( \hat{f}_{i+1/2} \) should be computed with nonlinear weights.

\[ \hat{f}_{i+1/2} = \sum_{k=0}^{2} \omega_k \hat{f}^{k}_{i+1/2} \]  

where the nonlinear weights \( \omega_k, \ k = 0, 1, 2 \) are defined as

\[ \omega_k = \frac{\alpha_k}{\alpha_0 + \alpha_1 + \alpha_2}, \quad \alpha_k = \frac{c_k}{(\epsilon + IS_k)^2}, \ k = 0, 1, 2 \]  

Here, \( \epsilon \) is used to prevent the denominator from becoming zero. \( IS_k \) are smooth indicators defined as

\begin{align*}
IS_0 & = \frac{1}{4} (f_{i-2} - 4f_{i-1} + 3f_i)^2 + \frac{13}{12} (f_{i-2} + 2f_{i-1} + f_i)^2 \\
IS_1 & = \frac{1}{4} (f_{i-1} - f_{i+1})^2 + \frac{13}{12} (f_{i-1} + 2f_i + f_{i+1})^2 \\
IS_2 & = \frac{1}{4} (3f_i - 4f_{i+1} + f_{i+2})^2 + \frac{13}{12} (f_i + 2f_{i+1} + f_{i+2})^2
\end{align*}  

2.2. IMEX Runge–Kutta scheme

Let \( U = (\rho, q, E)^T \) in (3). We split the fluxes in (3) into the stiff and non-stiff terms that are solved by the third-order IMEX-SSP3 (4, 3, 3) scheme [7], i.e.,

\[ K^{(i)} = K(U^{(i)}_E, U^{(i)}_f) = -\begin{pmatrix}
(q_1) \frac{x}{\rho_E} \\
\frac{3 - \gamma (q_E)^2}{2} \frac{1}{\rho_E} \frac{\gamma - 1}{\varepsilon} (E_i) \frac{1}{x} \\
-\frac{(\gamma - 1) x^2 (q_E)^3}{2} \frac{1}{\rho_E^2} + \frac{\gamma E_E^2}{\rho_E} q_i
\end{pmatrix}^{(i)} = \begin{pmatrix} k_1^{(i)} \\ k_2^{(i)} \\ k_3^{(i)} \end{pmatrix}, \ i = 1, 2, 3
\]  

and \( U^{(i)}_E, U^{(i)}_f \) are defined as
\[
\begin{align*}
\rho_i^{(0)} &= \rho^n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_\mu k_j^{(0)} , \\
\rho_i^{(1)} &= \rho^n + \Delta t \sum_{j=1}^{i-1} a_\nu k_j^{(0)} + \Delta t a_n k_j^{(0)} , \\
\rho_i^{(0)} &= \rho^n + \Delta t \sum_{j=1}^{i-1} a_\mu k_j^{(0)} , \\
q_i^{(0)} &= q^n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_\nu k_j^{(0)} , \\
q_i^{(1)} &= q^n + \Delta t \sum_{j=1}^{i-1} a_\nu k_j^{(0)} + \Delta t a_n k_j^{(0)} , \\
q_i^{(0)} &= q^n + \Delta t \sum_{j=1}^{i-1} a_\mu k_j^{(0)} , \\
E_i^{(0)} &= E^n + \Delta t \sum_{j=1}^{i-1} \tilde{a}_\nu k_j^{(0)} , \\
E_i^{(1)} &= E^n + \Delta t \sum_{j=1}^{i-1} a_\nu k_j^{(0)} + \Delta t a_n k_j^{(0)} , \\
E_i^{(0)} &= E^n + \Delta t \sum_{j=1}^{i-1} a_\mu k_j^{(0)} .
\end{align*}
\]

Therefore,

\[
k_i^{(0)} = - (q_i^{(0)})_{,x} - \Delta t a_n (k_j^{(0)})_{,x} \tag{12}
\]

\[
k_i^{(1)} = - (3 - r \frac{q_i^{(0)}}{\rho_i^{(0)}})_{,x} - \frac{\gamma - 1}{\varepsilon^2} \left( \tilde{E}_i^{(0)} \right)_{,x} - \Delta t a_n \frac{\gamma - 1}{\varepsilon^2} (k_j^{(0)})_{,x} \tag{13}
\]

\[
k_i^{(2)} = - \frac{(r - 1) \varepsilon^2}{2} \frac{(q_i^{(0)})^2}{(\rho_i^{(0)})^2} \frac{\gamma E_{ii}^{(0)}}{\rho_i^{(0)}}, \frac{q_i^{(0)}}{(\rho_i^{(0)})^2} \frac{\gamma E_{ii}^{(0)}}{\rho_i^{(0)}} - \Delta t a_n \frac{\gamma E_{ii}^{(0)}}{\rho_i^{(0)}} (k_j^{(0)})_{,x} \tag{14}
\]

Multiplying (14) by \(\frac{\gamma E_{ii}^{(0)}}{\rho_i^{(0)}}\), we obtain

\[
\left( \frac{\gamma E_{ii}^{(0)}}{\rho_i^{(0)}} , k_j^{(0)} \right)_{,x} = \left( \frac{\gamma E_{ii}^{(0)}}{\rho_i^{(0)}} k_i^{(0)} - \Delta t a_n \frac{\gamma - 1}{\varepsilon^2} \left( k_i^{(0)} \right)_{,x} \right) \tag{15}
\]

where \(k_i^{(0)} = - (3 - r \frac{q_i^{(0)}}{\rho_i^{(0)}})_{,x} - \frac{\gamma - 1}{\varepsilon^2} \left( \tilde{E}_i^{(0)} \right)_{,x} \). Based on (15) and (16), we obtain

\[
k_i^{(3)} = \left( \Delta t a_n \right)^2 \frac{\gamma - 1}{\varepsilon^2} \left( \gamma E_{ii}^{(0)} \left( k_i^{(0)} \right)_{,x} \right) = k_i^{(3)} - \Delta t a_n \left( \gamma E_{ii}^{(0)} \left( k_i^{(0)} \right)_{,x} \right) \tag{16}
\]

Remark. The second-order spatial derivative in (16) can be discretized by a fourth-order standard central scheme and the first-order spatial derivatives in (12)-(16) can be computed by the fifth-order WCNS. We first solve the discrete form of (16) by the restarted GMRES algorithm to obtain \(k_i^{(0)}\). Then we solve the discrete form of (13) to obtain \(k_i^{(0)}\). Finally, we solve the discrete form of (12) to obtain \(k_i^{(0)}\). The solution at \(t^{n+1}\) is \(U^{n+1} = U^n + \Delta t \sum_{i=1}^{3} b_i K_i^{(0)}\).

3. Numerical examples

3.1. Accuracy test

We test the temporal accuracy of the semi-implicit WCNS. Consider the initial conditions [1]:

\[
u_0(x) = \sin \left( \frac{2\pi x}{L} \right), \quad \rho_0(x) = \left( 1 + \epsilon \left( \gamma - 1 \right) u_0(x) \right)^{\frac{1}{\gamma - 1}}, \quad p_0 = \rho_0^\gamma
\]

where \(\gamma = 2\), \(L = 5\). The final time is \(T = 0.3\) and \(\Delta t = O(h)\). The computational domain is \([-2.5, 2.5]\). The convergence errors and orders are given in Table 1 and Table 2, from which we observe the expected order of accuracy in time.
Table 1. The convergence errors and orders of $\rho$ with $\varepsilon = 1$, $\varepsilon = 0.1$. 

| $N$  | $\varepsilon = 1$ | $\varepsilon = 0.1$ |
|------|-------------------|---------------------|
|      | $L^1$ error | $L^1$ order | $L^1$ error | $L^1$ order |
| 40   | 3.58923E-04 | - | 9.45524E-04 | - |
| 80   | 4.27317E-05 | 3.07 | 2.15822E-05 | 5.45 |
| 160  | 3.00615E-06 | 3.83 | 3.71004E-07 | 5.86 |
| 320  | 3.33514E-07 | 3.17 | 2.51727E-08 | 3.88 |

Table 2. The convergence errors and orders of $\rho u$ with $\varepsilon = 1$, $\varepsilon = 0.1$. 

| $N$  | $\varepsilon = 1$ | $\varepsilon = 0.1$ |
|------|-------------------|---------------------|
|      | $L^1$ error | $L^1$ order | $L^1$ error | $L^1$ order |
| 40   | 6.85482E-04 | - | 1.07283E-02 | - |
| 80   | 3.16106E-05 | 4.44 | 2.26440E-04 | 5.57 |
| 160  | 1.91243E-06 | 4.05 | 4.02318E-06 | 5.81 |
| 320  | 2.16289E-07 | 3.14 | 1.27657E-07 | 4.98 |

3.2. 1D Sod Shock Tube Problem
This problem in a compressible regime is subject to the initial conditions:

$$(\rho,u,p)(x,0) = \begin{cases} (1.0,0.0,1.0) & \text{if } x < 0.5 \\ (0.125,0.0,0.1) & \text{otherwise} \end{cases}$$

The computational domain is $[0,1]$. We take $N = 100$, $T = 0.18$, $\varepsilon = 1$, $CFL = 0.5$. The reference solution is computed with 2000 cells. The results are shown in Fig.1, which demonstrates the satisfactory behavior of the designed method in the compressible regime.

![Density, pressure and velocity at time $T = 0.18$.](image)

3.3. Two Colliding Acoustic Pulses
Consider the two colliding acoustic pulses in a weakly compressible regime with the initial conditions:

$$\rho_c(x,0) = \rho_0 + \frac{1}{2} \varepsilon \rho_1 \left( 1 - \cos \left( \frac{2\pi x}{L} \right) \right), \quad \rho_0 = 0.955, \quad \rho_1 = 2.0$$

$$u_c(x,0) = \frac{1}{2} u_0 \text{sign}(x) \left( 1 - \cos \left( \frac{2\pi x}{L} \right) \right), \quad u_0 = 2\sqrt{\gamma}$$

$$p_c(x,0) = p_0 + \frac{1}{2} \varepsilon \rho_1 \left( 1 - \cos \left( \frac{2\pi x}{L} \right) \right), \quad p_0 = 1.0, \quad p_1 = 2\gamma$$

The computation domain is $-L \leq x \leq L = 2/\varepsilon$. We take $\varepsilon = 1/11$, $T = 0.815$ and $T = 1.63$. The pressure profiles are shown in Fig.2, which shows the same behavior as in [6].
4. Conclusions
In this paper, we combine the fifth-order WCNS for the spatial discretization and the third-order IMEX Runge-Kutta scheme for the time discretization to solve 1D isentropic and full Euler equations. Numerical results show that the designed scheme can achieve high order accuracy in time and can solve the compressible-incompressible limit problems.

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References
[1] Boscarino S, Russo G, Scandurra L. All Mach Number Second Order Semi-Implicit Scheme for the Euler Equations of Gasdynamics [J]. Journal of Scientific Computing, 2017, 77 (2): 850-884.
[2] Thomann A, Zenk M, Klingenberg C. A second-order positivity-preserving well-balanced finite volume scheme for Euler equations with gravity for arbitrary hydrostatic equilibria [J]. International Journal for Numerical Methods in Fluids, 2019.
[3] Qian S, Liu Y, Li G, et al. High order well-balanced discontinuous Galerkin methods for Euler equations at isentropic equilibrium state under gravitational fields [J]. Applied Mathematics & Computation, 2018, 329: 23-37.
[4] Li G, Xing YL. Well-balanced finite difference weighted essentially non-oscillatory schemes for the Euler equations with static gravitational fields [J]. Computers & Mathematics with Applications: An International Journal, 2018.
[5] Jiang YQ, Chen X, Zhang X, et al. High order semi-implicit weighted compact nonlinear scheme for the all-Mach isentropic Euler system [J]. Advances in Aerodynamics, 2020, 2 (1).
[6] Klein R. Semi-implicit extension of a Godunov-type scheme based on low Mach number asymptotics I: One-dimensional flow [J]. Journal of Computational Physics, 1995, 121 (2): 213-237.
[7] Pareschi L, Russo G. Implicit-explicit Runge-Kutta schemes and applications to hyperbolic systems with relaxation [J]. Journal of Scientific Computing, 2005, 25 (1-2): 129-155.