Pathological quotient singularities which are not log canonical in positive characteristic

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Abstract
In characteristic zero, quotient singularities are log terminal. Moreover, we can check whether a quotient variety is canonical or not by using only the age of each element of the relevant finite group if the group does not have pseudo-reflections. In positive characteristic, a quotient variety is not log terminal, in general. In this paper, we give an example of the quotient variety which is not log terminal such that the quotient varieties associated to any proper subgroups is canonical. In particular, we cannot determine whether a given quotient singularity is canonical by looking at proper subgroups.

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1 Introduction
Quotient singularities form one of the most basic classes of singularities. They behave well in characteristic zero. In characteristic zero, any quotient variety has log terminal singularities. Moreover, for a finite group $G \subset \text{GL}(d, \mathbb{C})$ without pseudo-reflection, if we want to know the singularity of $\mathbb{C}^d/G$, we can use Reid-Shepherd-Barron-Tai criterion \cite[Theorem 3.21]{5}. Namely, the following three conditions are equivalent.
• $C^d/G$ is canonical (resp. terminal).
• $C^d/G$ is canonical (resp. terminal) for all cyclic subgroup of $C$.
• $\text{age}(g) \geq 1$ (resp. $> 1$) for any $g \in G$.

Where $\text{age}(g)$ is the age of $g \in G \setminus \{1\}$. Diagonalizing an element $g \in G \setminus \{1\}$, we write

$$g = \begin{bmatrix}
\lambda_1^{a_1} & & \\
& \ddots & \\
& & \lambda_l^{a_d}
\end{bmatrix},$$

where $l$ is the order of $g$, $\lambda_i$ is the primitive $l$-th root $\exp(2\pi\sqrt{-1}/l)$ and all integers $a_i$ satisfy $0 \leq a_i \leq l - 1$. Then the age of $g$ is defined by

$$\text{age}(g) = \frac{1}{l} \sum_{i=1}^{d} a_i.$$

In positive characteristic, if the given finite group is tame, then the quotient variety is again log terminal and we can use the Reid–Shepherd-Barron–Tai criterion. But if the group is wild, there exists a quotient variety which is not log terminal. In this paper, we give an even more pathological example.

**Theorem 1.1** (Main Theorem, Theorem 5.1). Let $C_3$ be the cyclic group of order three and $C_2^3$ be the product of two copies of it. Suppose that the group $C_2^3$ is embedded in $\text{SL}(3, K)$ and this embedding makes $C_2^3$ small, where $K$ is algebraically closed field of characteristic three. Then the quotient variety $\mathbb{A}^3/C_2^3$ is not log canonical.

The pathological point of this example is that the quotient variety associated to any proper subgroup of $C_2^3$ is canonical, but the quotient variety by $C_2^3$ is not log terminal. If the cyclic group $C_3$ of order three acts on the affine space $\mathbb{A}^3$ over $K$ linearly and small, the quotient variety has a crepant resolution, and so canonical [8]. Since all nontrivial proper subgroups of $C_2^3$ is isomorphic to $C_3$, the quotient varieties by proper subgroups are canonical. But the above theorem says that the quotient variety $\mathbb{A}^3/C_2^3$ is neither canonical singularity nor log terminal singularity. This is in contrast to the fact that, in characteristic zero, the discrepancy of a quotient variety is determined by the age of elements of the group.

We give the proof of the main theorem in the following way. Firstly, we give the all small action of $C_2^3$. It is not determined uniquely, but we can parametrize by $a \in K \setminus \mathbb{F}_3, b \in K$. Next, we give the explicit form of the quotient varieties $X$ for each action of $C_2^3$. We will find that the quotient varieties are classified in two types separating by whether $b = 0$ or not about the parameter $b$ of the action. Finally, we construct the proper birational morphism $Y \to X$ with exceptional divisors which discrepancy is smaller than $-1$, which shows the quotient varieties are not log canonical. This construction given by a few times blow up along the singular loci.
As an application of the main result, we give a criterion when a quotient variety associated to a small wild finite group is log terminal in dimension three and characteristic three. According to the criterion, we can judge the singularity of a quotient variety by seeing the order of the acting group.

**Corollary 1.2** (Corollary 6.5). Let $G$ be a wild small finite group of $\text{GL}(3, K)$ where $K$ is an algebraically closed field. We write $\# G = 3^r n$ where $r, n$ are positive integer and $n$ is not divided by three.

(i) If $r = 1$ then $\mathbb{A}^3_K/G$ is log terminal.

(ii) If $r \geq 2$ then $\mathbb{A}^3_K/G$ is not log canonical, in particular, not log terminal.

This follows from the claim that if $X'$ is not log terminal and a morphism $\pi: X' \to X$ is finite dominant and étale in codimension one then $X$ is not log terminal. We give the proof of the claim and the log canonical version of it at the same time.

This paper is organized as follows. In sections two, we list some preliminaries. In section three, we consider small actions of $C_3^2$. In section four, we write quotient varieties explicitly. In section five, we give the proof of main theorem by construct some proper birational morphisms. In section six, we prove the corollary 1.2 as an application of the main theorem.

A large part of this paper is taken from the master thesis of the author.

## 2 Preliminaries

We fix an algebraically closed field $K$ of characteristic three. We denote the $n$-dimensional affine space over $K$ by $\mathbb{A}^n_K$, that is, the spectrum of the polynomial ring $K[x_1, \ldots, x_n]$. We regard $\mathbb{A}^3_K$ as a vector space on $K$ and denote its general linear group by $\text{GL}(3, K)$ and $\text{SL}(3, K)$ be the special linear subgroup. Note that the eigenvalues of an element of $\text{GL}(3, K)$ whose order is three satisfy the equation $x^3 = 1$. So order three elements are contained in $\text{SL}(3, K)$.

When a group $G$ is embedded in $\text{GL}(n, K)$ and $G$ acts on $n$-dimensional vector space $K^n$ via this embedding, we call $g \in G$ is pseudo-reflection if $(K^n)_g$ is $(n-1)$-dimensional subspace of $K^n$. When $G$ has no pseudo-reflection, we say that $G$ is small.

A variety means a separated integral scheme of finite type over $K$. Let $X$ be a $n$-dimensional normal variety. We define the canonical sheaf $\omega_X$ of $X$ by $\omega_X = j_* (\bigwedge^n \Omega_{X, \text{reg}})$ where $X_{\text{reg}} = X \setminus \text{Sing}(X)$ and $\Omega_{X, \text{reg}}$ is the $\mathcal{O}_{X, \text{reg}}$-module of differentials. We call the divisor $K_X$ defined by $\omega_X$ the canonical divisor of $X$. If $mK_X$ is a Cartier divisor for some $m \in \mathbb{Z}\setminus\{0\}$, we say $X$ is $\mathbb{Q}$-Gorenstein.

Let $X$ be a normal variety. A prime divisor $E$ on a normal variety $Y$ given with a birational morphism $f: Y \to X$ called a divisor over $X$. If there is a birational map $g: Y \to Y'$, we denote the closure $g(E)$ of the image of $E$ by $\text{cent}_{Y'}(E)$. Let $E'$ be another divisor over $X$, which is a divisor on $Y'$ with a birational morphism $f': Y' \to X$. There is a natural birational map $(f')^{-1} \circ
$f : Y \to X \to Y'$. If $\text{cent}_{Y'}(E) = E'$, we say $E$ and $E'$ are equivalent. This defines an equivalence relation on the set of divisors over $X$. The equivalence class of $E$ is determined by the associated valuation on $K(X)$. In other words, the set of equivalence classes of divisors over $X$ can be embedded in the set of valuations on $K(X)$.

Let $X$ be a $\mathbb{Q}$-Gorenstein normal variety. For a birational morphism $f : Y \to X$ where $Y$ is a normal variety, we write

$$K_Y = f^*K_X + \sum_D a_D D$$

where $D$ runs over the $f$-exceptional prime divisors on $Y$. We call $a_E$ the discrepancy of $E$. If $E$ is not exceptional divisor, its discrepancy is zero. Note that the discrepancy is determined by the valuation on $K(X)$ corresponding to the divisor $E$, equivalently by the equivalence class of $E$.

If $\inf\{a_E|E \text{ is a divisor over } X\} \geq -1$ (resp. $> -1$), we say $X$ has log canonical (resp. log terminal) singularities, or simply $X$ is log canonical (resp. log terminal).

3 Actions of $C_3^2$ on affine space

In this section, we describe all the small actions of $C_3^2$ on $\mathbb{A}^3$. All the not necessarily small embeddings in $\text{SL}(3, K)$ are given in section four in [2]. We consider representations that $C_3^2$ is small and it is classified as type$(1,1,1)$ in [2]. From [2, Proposition 4.1, Proposition 4.3], we get the following proposition.

**Proposition 3.1.** For any embedding to $\text{SL}(3, K)$ of $C_3^2$ which makes $C_3^2$ is small, there exists $a \in K \setminus \mathbb{F}_3, b \in K$ such that $\sigma(U(a, b))$ is conjugate with the image of embedding where $U(a, b)$ is a subgroup of the additive group $(K^2, +)$ generated by $(1, 0), (a, b)$ and $\sigma$ is the group homomorphism defined by

$$\sigma(c_1, c_2) = \begin{bmatrix} 1 & -c_1 & c_1^2 + c_2 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Proof.** For simplicity, we denote the image of the embedding of $C_3^2$ to $\text{SL}(3, K)$ by $C_3^2$ again. From [2 Proposition 4.1], we get the finite subgroup $U$ of $(K^2, +)$ whose image by $\sigma$ is conjugate with $C_3^2$. Then $U$ is generated by two elements $(u, v), (u', v')$ and these are independent as the elements of $\mathbb{F}_3$-vector space since $U$ is isomorphic to $C_3^2$. If $u = 0$ then $\sigma((u, v)) \in \sigma(U)$ is pseudo-reflection. This contradicts the assumption that $C_3^2$ is small. So we get $u \neq 0$. From [2 Proposition 4.3], $U$ is conjugate with

$$U' = \{u^{-1}(c_1, -u^{-2}vc_1 + u^{-1}c_2)| (c_1, c_2) \in U\}$$

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which generated by \((1,0)\) and \((u^{-1}u',-u^{-2}vu'+u^{-2}v')\). So we put \(a = u^{-1}u', b = -u^{-2}vu' + u^{-2}v'\) and replace \(U\) by \(U'\), then we get the required form of \(U\). From the generators of \(U\) are independent as \(\mathbb{F}_3\)-vectors. If \(a \in \mathbb{F}_3\), since \((1,0)\) and \((a,b)\) are independent as \(\mathbb{F}_3\) vectors, the element \(b \neq 0\). Then \((0,b) = (a,b) - (a,0) \in U\) and \(\sigma((0,b))\) is pseudo-reflection. This contradicts the assumption that \(\sigma(U)\) is small. So we get \(a \notin \mathbb{F}_3\).

4 The quotient varieties associated to actions

In this section, we compute the quotient varieties associated to the small actions of \(C_2^3\). By [2, Theorem 3.3,Theorem 6.3], we get concrete representations of quotient varieties.

Proposition 4.1. The quotient variety \(X = \mathbb{A}^3/\sigma(U(a,b))\) is embedded in \(\mathbb{A}^4_{x_1,x_2,x_3,x_4}\) as a hypersurface. We can represent it as

\[
X = \left\{ \begin{array}{ll}
V(x^9 - x^2 + x^9x_4 + x^9H(x_1,x_2)) & (b = 0) \\
V(\alpha x^9 - x^2 + x^9x_4 + x^9H(x_1,x_2)) & (b \neq 0)
\end{array} \right.,
\]

where \(a = a^3 - a\) and \(H(s_1,s_2), F(s_1,s_2,s_3)\) are polynomials on \(K\) defined by

\[
H(s_1,s_2) = (1 + \alpha^2)x^6 - \alpha^2 x^2 \alpha^5 + (1 + \alpha^2)x^6 x^2 + \alpha^2 (1 + \alpha^2)x^6 x^2 + \alpha^4 x^4 x^2,
\]

\[
F(s_1,s_2,s_3) = c_1 s_2^4 + c_2 s_1 s_2^3 s_3 + c_3 s_1^2 s_2^3 + c_4 s_1^2 s_2^3 + c_5 s_1^2 s_2^3 + c_6 s_1^3 s_2^3 + c_7 s_1^3 s_2^3.
\]

for some \(c_1, \ldots, c_7 \in K\).

Proof. To compute invariant rings, we classify these groups \(U(a,b)\) in two cases: one is when \(b = 0\), and the other is \(b \neq 0\).

Case: \(b = 0\)

We get \(a \notin \mathbb{F}_3\) from the condition given in Proposition 4. So,

\[
\det \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = 0,
\]

\[
\det \begin{bmatrix} 1 & a \\ 1 & a^3 \end{bmatrix} = a^3 - a \neq 0.
\]

Therefore we get

\[
K[x,y,z]^{\sigma(U(a,0))} \cong K[x_1,x_2,x_3,x_4]/(x_2^9 - x_3^2 + x_4^9 + x_1^9H(x_1,x_2))
\]

from [2, Theorem 3.3].

Case: \(b \neq 0\)

Now

\[
\det \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} = b \neq 0,
\]

\[
\det \begin{bmatrix} 1 & a \\ 1 & a^3 \end{bmatrix} = a^3 - a \neq 0.
\]
Computing the invariant ring according to [2, Theorem 6.3], we get
\[ K[x, y, z]^\sigma U(a, b) \cong K[x_1, x_2, x_3, x_4]/(\alpha^3 b^5 x_2^6 - b^4 x_3^3 - b^1 0 x_4^6 + x_4^4 F(x_1, x_2, x_3)) \]

\[ \square \]

5 Singularity

In this section, we prove any quotient variety associated to a small action of \( C_3^2 \) is non log canonical. This proof is given by explicit calculations.

**Theorem 5.1.** When \( C_3^2 \) acts on \( \mathbb{A}^3 \) via an embedding \( SL(3, K) \) which makes \( C_3^2 \) small, the quotient variety is not log canonical.

**Proof.** Case \( b = 0 \):

We can construct a proper birational morphism of \( \varphi : X_4 \to X \) with a normal variety \( X_4 \) by four times blowing-up along singular locus. We illustrate this situation by following diagram.

\[
\begin{array}{cccccc}
BL_3(W_3) & BL_2(W_2) & BL_1(W_1) & BL_0(W_0) \\
U & \varphi_4 & U & \varphi_3 & U & \varphi_2 & U & \varphi_1 & U & \varphi_0 & U \\
W_1 & \varphi_4 & W_3 & \varphi_3 & W_2 & \varphi_2 & W_1 & \varphi_1 & W_0 \\
U & \varphi_4 & X_3 & \varphi_3 & X_2 & \varphi_2 & X_1 & \varphi_1 & X \\
L_4 & U & L_3 & U & L_2 & U & L_1 & U & L_0 \\
\end{array}
\]

In this diagram, \( L_i \) is the singular locus of \( X_i \), \( BL_{l_i}(W_i) \), \( X_{i+1} \) are blow-ups of \( W_i, X_i \) along \( L_i \), \( W_i \) is an open subvariety of \( BL_{l_i}(W_i) \) which contains \( X_i \) for each \( i = 0, 1, \ldots, 5 \). Morphisms \( \varphi_i \) are the restrictions of the morphisms \( BL_{l_{i-1}}(W_{i-1}) \to W_{i-1} \) determined by blow-ups. We represent these varieties explicitly by direct computation.

Firstly, \( W_0 = \mathbb{A}^4_{x_1, x_2, x_3, x_4} \) and from Proposition [4.1]

\[
X = V(x_2^9 - x_3^3 + x_1^6 x_4 + x_1^3 H(x_1, x_2)) \subset W_0, \\
L_0 = V(x_1, x_2, x_3).
\]

Blowing-up \( W_0 \) and \( X \) along \( L_0 \), we get \( W_1 \) is covered by two open affine subvarieties \( W_{1, t} = \mathbb{A}^4_{x_1, x_2, x_3, x_4, u, v} \) and \( W_{1, u} = \mathbb{A}^4_{x_2, x_3, t, u, v} \) and

\[
X_1 \cap W_{1, t} = V(x_1^7 (u^9 + x_4 + x_3^2 h_t) - v_1^2), \\
X_1 \cap W_{1, u} = V(x_2^7 (1 + t^9 + x_3^2 h_u) - v_2^2),
\]

where \( h_t = x_1^{-6} H(x_1, u, x_1) \), \( h_u = x_2^{-6} H(t, x_2, x_2) \), which are polynomials about \( x_1, u, t \) and \( x_2, v \). The morphism \( \varphi_1 \) corresponds to a homomorphisms

\[
(\varphi_1|_{W_{1, t}})^\#: K[x_1, x_2, x_3, x_4] \to K[x_1, x_2, x_3, x_4] \\
[x_1, x_2, x_3, x_4] \mapsto [x_1, u, x_1, v_1 x_3, x_4]
\]

\[
(\varphi_1|_{W_{1, u}})^\#: K[x_1, x_2, x_3, x_4] \to K[x_2, x_4, t, v_2] \\
[x_1, x_2, x_3, x_4] \mapsto [t u x_2, x_2, u_0 x_3, x_4]
\]

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Computing the singular locus of \( X_1 \), we get
\[
L_1 \cap W_{1,t} = V(x_1, v_t), \quad L_1 \cap W_{1,u} = V(x_2, v_u).
\]

For \( i = 2, 3, 4 \), \( W_i, X_i \) are very similar to \( W_1, X_1 \). The variety \( W_i \) are covered by two open affine varieties \( W_{i,t} = A^4_{x_1, x_4, u_i, t_i}, W_{i,u} = A^4_{x_2, x_4, u_i, u_i} \). The \( X_i \) is closed subvariety of \( W_i \) defined by
\[
X_i \cap W_{i,t} = V(x_i^{9-2i}(u_i^9 + x_i^3 h_t) - t_i^2), \\
X_i \cap W_{i,u} = V(x_i^{9-2i}(1 + t_i^9 x_4 + x_i^3 h_u) - u_i^2).
\]

The \( \varphi_i \) \( (i = 2, 3, 4) \) corresponds to a homomorphisms
\[
(\varphi_i|_{W_{i,t}})^\#: K[x_1, x_4, u_i, t_i] \rightarrow K[x_1, x_4, u_i, t_i] \\
\quad [x_1, x_4, u_i, t_i-1] \mapsto [x_1, x_4, u_i, t_i x_4] \\
(\varphi_i|_{W_{i,u}})^\#: K[x_2, x_4, t_i, u_i] \rightarrow K[x_2, x_4, t_i, u_i] \\
\quad [x_2, x_4, t_i, u_i-1] \mapsto [x_2, x_4, t_i, u_i x_4]
\]
where \( t_i = v_i \) and \( u_i = v_i \). For \( i = 2, 3, 4 \),
\[
L_i \cap W_{i,t} = V(x_i, t_i), \quad L_i \cap W_{i,u} = V(x_2, u_i),
\]
and for \( i = 4 \),
\[
L_4 \cap W_{4,t} = V(x_1, u_4^9 + x_4, t_4), \quad L_4 \cap W_{4,u} = V(x_2, 1 + t_4^9 x_4, u_4).
\]

Since \( X_4 \) is locally complete intersection, we get \( X_4 \) is normal.

Now, we compute a relation between the canonical divisor \( K_X \) and pullback \( \varphi^* K_X \) of canonical divisor of \( X \). Let \( E_i \) be the exceptional divisor of \( \varphi : W_i \rightarrow W_{i-1} \) and we also denote the strict transform of \( E_i \) by \( E_i \). Since \( X_i \) are a closed subvariety of codimension one in \( W_i \), we regard \( X_i \) as a divisor on \( W_i \). Since the morphisms \( \varphi_i \) are blow-up along \( L_i \), we get
\[
K_{W_i} = \varphi_i^* K_{W_{i-1}} + \begin{cases} 2E_i & (i = 1) \\ E_i & (i = 2, 3, 4) \end{cases}
\]
and
\[
\varphi_i^* X_{i-1} = X_i + 2E_i
\]
for \( i = 1, 2, 3, 4 \) where \( X_0 = X \). By direct computation,
\[
\varphi_i^* E_{i-1} = E_i
\]
for \( i = 3, 4 \). Combining these, since \( \varphi = \varphi_4 \circ \cdots \circ \varphi_1 \), we get
\[
K_{W_4} = \varphi^* K_{W_0} + 2(\varphi')^* E_1 + 3E_4, \\
X_4 = \varphi^* X - 2(\varphi')^* E_1 - 6E_4.
\]
where $\varphi' = \varphi_4 \circ \varphi_3 \circ \varphi_2$. Therefore

$$K_{W_4} + X_4 = \varphi^*(K_{W_0} + X) - 3E_4$$

and, by adjunction formula,

$$K_{X_4} = \varphi^*K_X - 3E_4|_{X_4}.$$  

So $X$ is not log canonical.

Case $b \neq 0$:

From Proposition [4.4] we put

$$X = \mathbb{A}^3/\sigma(U(a,b)) = \text{Spec}K[x_1, x_2, x_3]/(\alpha b^2 x_2^5 - b_4 x_3^4 - b_10 x_1 x_4 + \alpha b^2 x_1^2 x_3 + x_1^4 F(x_1, x_2, x_3)).$$

Then $X$ is a singular variety and its singular locus is $L_0 := V(x_1, x_2, x_3)$. We construct a birational morphism $\varphi: X_2 \to X$ with a normal variety $X_2$ by two times blowing-up along singular locus. We illustrate this construction by the following diagram.

$$\begin{array}{ccc}
BL_{L_1}(W_1) & & BL_{L_0}(W_0) \\
\cup & \varphi_2 & \cup \\
W_2 & \varphi_2 & W_1 \\
\cup & \varphi_1 & \cup \\
X_2 & \varphi_2 & X_1 \\
\cup & \varphi_1 & \cup \\
L_2 & \varphi_2 & L_0
\end{array}$$

We use the same symbols for varieties and some morphisms as using in the previous case. But its explicit forms may be deferent. We will describe these varieties. The variety $W_1$ and the morphism $\varphi_1: W_1 \to W_0$ are same as in previous case. The $X_1$ is defined by

$$X_1 \cap W_{1,t} = V(\alpha b^2 x_1^5 x_2^2 - b_4 x_3^4 - b_10 x_1^3 x_4 + \alpha b^2 x_1^2 x_3 + x_1^4 f_1(x_1, u_t, v_t)),$$

$$X_1 \cap W_{1,u} = V(\alpha b^2 x_2^5 - b_4 x_3^4 - b_10 x_1^3 x_4 + \alpha b^2 x_1^2 x_3 + x_1^4 f_2(x_2, u_t, v_u)),$$

in $W_1$ where $f_1(s_1, s_2, s_3) = s_1^{-4}F(s_1, s_2, s_3), f_2(s_1, s_2, s_3) = s_1^{-4}F(s_1 s_2, s_1, s_1 s_3)$.

About the singular locus $L_1$, we get

$$L_1 \cap W_{1,t} = V(x_1, v_t), \quad L_1 \cap W_{1,u} = V(x_2, v_u).$$

The open variety $W_2$ of $BL_{L_1}(W_1)$ is covered by three open varieties $W_{2,y} = \mathbb{A}^4_{x_1, x_2, u_t, v_t}, W_{2,z} = \mathbb{A}^4_{x_1, x_2, u_t, v_t}, W_{2,w} = \mathbb{A}^4_{x_1, x_2, u_t, v_t}$. The morphism $\varphi_2$ is given by

$$\begin{align*}
(\varphi_2|_{W_{2,y}})^\# : K[x_1, x_2, u_t, v_t] & \to K[x_1, x_2, u_t, y] \\
[x_1, x_2, u_t, v_t] & \mapsto [x_1, x_2, u_t, y x_1], \\
(\varphi_2|_{W_{2,z}})^\# : K[x_1, x_2, u_t, v_t] & \to K[x_1, x_2, u_t, z] \\
[x_1, x_2, u_t, v_t] & \mapsto [2 v_t, x_1, x_2, u_t], \\
(\varphi_2|_{W_{2,w}})^\# : K[x_1, x_2, u_t, v_t] & \to K[x_1, x_2, u_t, w] \\
[x_1, x_2, u_t, v_t] & \mapsto [w v_t, x_1, x_2, u_t].
\end{align*}$$

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The variety $X_2$ is defined by

$$X_2 \cap W_{2,y} = V(\alpha b^2u^5 - b^4y^3x_1 - b^{10}x_1x_4 + \alpha b^2u^3y x_1 + x_1^2 f_1(x_1, u, y x_1)),$$

$$X_2 \cap W_{2,z} = V(\alpha b^2u^3 z^2 - b^4 v_1 - b^{10}z^3 x_4 v_4 + \alpha b^2u^2 z^2 v_4 + z^4 v_1^2 f_1(zv_1, u, v_1)),$$

$$X_2 \cap W_{2,u} = V(\alpha b^2w^2 - b^4v_u - b^{10}t^6 w^3 x_4 v_u + \alpha b^2 t_u w^2 v_u + t_u^3 w^4 v_u^2 f_2(wv_u, t_u, v_u)).$$

Now, we can see that the singular locus $L_2$ of $X_2$ is contained in $W_{2,y}$. It has the form

$$L_2 = V(x_1, u, y^3 + b^6 x_4)$$

in $W_{2,y}$. Since $X_2$ is locally complete intersection, $X_2$ is normal.

Next, we compute the exceptional divisor of $\varphi = (\varphi_2 \circ \varphi_1): X_2 \to X$. Since the morphisms $\varphi_1, \varphi_2$ are blow-up, we get

$$K_{W_1} = \varphi_1^* K_W + 2E_1, \; K_{W_2} = \varphi_2^* K_{W_1} + E_2,$$

where $E_1, E_2$ are the exceptional divisors of $\varphi_1: W_1 \to W_0, \varphi_2: W_2 \to W_1$. Regarding $X, X_1, X_2$ as divisor, we get

$$\varphi_1^* X = X_1 + 3E_1, \; \varphi_2^* X_1 = X_2 + 2E_2$$

from multiplicity of $X, X_1$ along $L_0, L_1$ respectively. By direct computation,

$$\varphi_2^* E_1 = E_1 + E_2.$$

So $X$ is not log canonical.

\section{Application}

In this section, we give other non log canonical quotient varieties using the main result. For this, we show some assertions.

\textbf{Lemma 6.1.} Let $R \in \text{SL}(3, K)$ be a non pseudo-reflection element whose order is three. Then the centralizer $C_{\text{SL}(3, K)}(R)$ of $R$ is given by

$$C_{\text{SL}(3, K)}(R) = \{ aI + bR + cR^2 | a, b, c \in K, \; a + b + c = 1 \}.$$

In particular, $C_{\text{SL}(3, K)}(R)$ is abelian.

\textit{Proof.} Such an element $R$ is conjugate with the element

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

So we may replace $R$ by this matrix. When $A = [a_1, a_2, a_3] \in \text{SL}(3, K)$ where $a_1, a_2, a_3$ are vertical vectors,

$$AR = RA \iff Ra_1 = a_3, \; Ra_2 = a_1, \; Ra_3 = a_2.$$
We put
\[ a_3 = \begin{bmatrix} c \\ b \\ a \end{bmatrix}, \]
then
\[ A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} = aI + bR + cR^2. \]
Moreover, \( \det A = a^3 + b^3 + c^3 = (a + b + c)^3 = 1 \) implies \( a + b + c = 1 \). Hence we get
\[ C_{\SL(3, K)}(R) \subset \{ aI + bR + cR^2 | a, b, c \in K, \ a + b + c = 1 \}. \]
The other inclusion is obvious.

Lemma 6.2. Let \( G \) be a small finite group of \( \SL(3, K) \) whose order is \( 3^r \). Then \( G \cong C_3^r \)

Proof. Since the order of \( G \) is a power of prime, the center \( Z(G) \) of \( G \) is not trivial. Take \( R \in Z(G) \) whose order three. Then \( G \subset C_{\SL(3, K)}(R) \). So Lemma 6.1 shows that \( G \) is an abelian group and any element of \( G \) has order three. Therefore, \( G \cong C_3^r \) by the structure theorem of finitely generated abelian groups.

Lemma 6.3. Let \( X' \) be a variety and \( \pi: X' \to X \) be a finite dominant morphism. Then for any divisor \( E' \) over \( X' \), there exists the following diagram

\[ \begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\rho & \downarrow & \pi \\
Y & \xrightarrow{f} & X
\end{array} \]

where \( f', f \) are birational morphisms with normal varieties \( Y, Y' \) and \( \rho \) is a morphism satisfying the following conditions.

- The center of \( E' \) on \( Y' \) is codimension one.
- The closure \( \rho(\text{cent}_{Y'}(E')) \) is codimension one.

Proof. Take a birational morphism \( f'_0: Y'_0 \to X' \) such that \( Y'_0 \) is normal variety and \( E' \) is a divisor on \( Y'_0 \). Let \( \phi: X'' \to X' \) be the Galois closure of \( \pi: X' \to X \). Namely, the coordinate ring of \( X'' \) is the integral closure of \( K[X] \) in a Galois closure of \( K(X')/K(X) \). The morphism \( \phi \) is defined by the inclusion \( K[X'] \to K[X''] \). We denote the Galois group of \( L/K(X) \) by \( G = \{ g_1, \ldots, g_d \} \). It acts on \( X'' \) canonically.

We define a variety \( Y''_0 \) as the component of \( Y'_0 \times X', X'' \) such that the morphism \( \varphi: Y''_0 \to X'' \) is dominant.
Moreover, we define a variety $Y''$ by

$$Y'' = \left( (Y''_{0,g_1} \times \sigma(g_1), X''_{0,g_1}) \times X''_{0,g_2} \times \cdots \times X''_{0,g_d} \right).$$

where $Y''_{0,g}$ is copy of $Y''_0$ with the morphism $\sigma(g) : Y''_0 \rightarrow Y''_{0,g}$. Then $G$ acts on $Y''$. Furthermore, from the diagram

we get the morphism $Y'' \rightarrow X''$ is $G$-equivariant. Let $H$ be a subgroup of $G$ which corresponds to the field extension $L/K(X')$. Then there exists natural morphisms $f' : Y''/H \rightarrow X''/H = X'$, $f : Y''/G \rightarrow X''/G = X$ since $Y'' \rightarrow X''$ is $G$-equivariant. These are birational morphisms. The morphism $\rho : Y' \rightarrow Y$ is defined naturally.

We consider a prime divisor $E''$ on $Y''$ contained in the pull-back of $E'$ by $Y'' \rightarrow Y''_0 \rightarrow Y'_0$. The push-forward $q_*E''$ of $E''$ by the natural morphism $q : Y'' \rightarrow Y'$ is a prime divisor on $Y'$ since $q$ is finite. Now, Let $v'', v'$ be the valuations on $K(X'') = L, K(X')$ corresponding to $E, E'$ respectively. By construction, the valuation $v''$ is an extension of $v'$ on $K(X'')$. Since the push-forward of a prime divisor is corresponds to restricting valuation, $q_*E''$ is equivalent to $E'$ as a divisor over $X'$. Moreover, since $\rho$ is also finite, $\rho$ preserve dimension.

**Theorem 6.4.** Let $X', X$ be a normal $\mathbb{Q}$-Gorenstein variety and $\pi : X' \rightarrow X$ be a finite dominant morphism of étale in codimension one. If $X'$ is not log canonical (resp. not log terminal) then $X$ is not log canonical (resp. not log terminal).

**Proof.** We prove only the assertion about log canonicity. The other assertion is similarly proved.

For any divisor $E'$ over $X'$ with discrepancy smaller than $-1$, we take a
diagram in Lemma 6.3.

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\downarrow{\rho} & & \downarrow{\pi} \\
Y & \xrightarrow{f} & X
\end{array}
\]

We denote cent\(_{Y'}(E')\) again by \(E'\) and \(E = \rho(E')\), which are prime divisors. We write

\[
\begin{align*}
K_Y &= f^* K_X + aE + F, \\
K_{Y'} &= \rho^* K_Y + bE' + G' \\
\rho^* E &= tE' + H',
\end{align*}
\]

where \(F\) doesn’t contain \(E\) and \(G', H'\) don’t contain \(E'\). Because \(\pi\) is étale in codimension one, \(K_{X'} = \pi^* K_X\). Hence we get

\[
\begin{align*}
K_{Y'} &= \rho^* f^* K_X + (at + b)E' + \rho^* F + G' + aH' \\
&= (f')^* \pi^* K_X + (at + b)E' + \rho^* F + G' + aH' \\
&= (f')^* K_{X'} + (at + b)E' + \rho^* F + G' + aH'.
\end{align*}
\]

By assumption, \(at + b < -1\), and so \(a < -\frac{t+1}{t}\). By [5, 2.41], \(b \geq t - 1\). Therefore, we get \(a < -1\). \(\square\)

Now we give other non log canonical quotient varieties.

**Corollary 6.5.** Let \(G\) be a wild small finite group of \(\text{GL}(3, K)\) and let \(X\) be the quotient variety \(\mathbb{A}^3/G\). We write \(#G = 3^r n\) where \(r, n \in \mathbb{Z}_{>0}\) and \(n\) is not divided by three.

(i) If \(r = 1\) then \(X\) is log terminal.

(ii) If \(r \geq 2\) then \(X\) is not log canonical.

**Proof.** Firstly, we consider the case of \(r = 1\). By Lemma 6.3 and Sylow’s theorem, \(G\) has a subgroup \(H\) isomorphic to \(C_3\). Let \(X'\) be the quotient variety \(\mathbb{A}^3/H\) and \(\pi: X' \to X\) be the canonical morphism, which is étale in codimension one. Note that the variety \(X'\) is canonical from [6, Corollary 6.25]. For a birational map \(f: Y \to X\) with a normal variety \(Y\), we denote the normalization of the component of \(X' \times_X Y\) dominating \(X'\) by \(Y'\). We denote the composition of the normalization map and projections by \(\rho: Y' \to Y,\ g: Y' \to X'\). So we consider the following diagram.

\[
\begin{array}{ccc}
Y' & \xrightarrow{g} & X' \\
\downarrow{\rho} & & \downarrow{\pi} \\
Y & \xrightarrow{f} & X
\end{array}
\]
Take an $f$-exceptional divisor $E$ and write $\rho^*E = \sum r_i E'_i$ where $E'_i$ are prime divisors. Then the equation

$$\sum_i r_i e_i = \#G/H$$

is held for some integer $e_i$. Since the right side of this equation is not divisible by three, the one of $r_i$ is not divisible by three. So we denote a prime divisor with such coefficient by $E'$ and its coefficient by $r$. We write

$$K_Y = f^* K_X + aE + F,$$
$$K_{Y'} = \rho^* K_Y + bE' + G'$$
$$\rho^*E = rE' + H',$$

where $F$ doesn’t contain $E$ and $G', H'$ don’t contain $E'$. Because $\pi$ is étale in codimension one, $K_{X'} = \pi^* K_X$. Hence we get

$$K_{Y'} = \rho^* f^* K_X + (ar + b)E' + \rho^* F + G' + aH'$$
$$= g^* \pi^* K_X + (ar + b)E' + \rho^* F + G' + aH'$$
$$= g^* K_{X'} + (ar + b)E' + \rho^* F + G' + aH'.$$

Since $X'$ is canonical, $ar + b \geq 0$. By [5, 2.41], $b = r - 1$. Therefore, we get

$$a > -1 + \frac{1}{r} > -1.$$ 

Hence $X$ is log terminal.

When $r \geq 2$, by Lemma 6.2 and Sylow’s theorem, $G$ has a subgroup $H$ isomorphic to $C_3^3$. Then the quotient variety $\mathbb{A}^3/H$ is not log canonical and the natural morphism $\mathbb{A}^3/H \rightarrow \mathbb{A}^3/G$ is a finite dominant morphism which is étale in codimension one. By Theorem 6.4, $\mathbb{A}^3/G$ is not log canonical.

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