THE AREA OF A SPECTRALLY POSITIVE STABLE PROCESS STOPPED AT ZERO

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Abstract. An identity in law for the area of a spectrally positive Lévy \( \alpha \)-stable process stopped at zero is established. Extending that of Lefebvre [17] for Brownian motion, it involves an inverse Beta random variable and the square of a positive stable random variable. This identity entails that the stopped area is distributed as the perpetuity of a spectrally negative Lévy process, and is hence self-decomposable. We also show that the law of the stopped area is a generalized Gamma convolution. Finally, we derive a convergent series representation for the density, whose behaviour at zero is proved to be Fréchet-like.

1. Introduction and statement of the results

Let \( \{B_t, t \geq 0\} \) be a linear Brownian motion, starting from one. Denote its first hitting time of zero by \( T = \inf\{t > 0, B_t = 0\} \). The random variable

\[ A = \int_0^T B_s \, ds \]

has been investigated by Lefebvre, who obtained in Theorem 2 of [17] the simple identity in law

\[ A \overset{d}{=} \frac{2}{9 \Gamma_{1/3}} \]

where, here and throughout, \( \Gamma_a \) stands for the Gamma random variable with density

\[ \frac{x^{a-1}}{\Gamma(a)} e^{-x} 1_{\{x>0\}}. \]

The identity (1.1) is obtained as a consequence of the closed expression for the Laplace transform of the bivariate random variable \((T, A)\) in terms of the Airy function - see Theorem 1 in [17]. As observed in [16] p. 402, this latter expression can be easily derived thanks to the Feynman-Kac formula. Notice that Airy functions also appear in the expression of the Laplace transform of many other Brownian areas - see [13], whose laws are more complicated than (1.1).

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In this paper, our concern is to generalize (1.1) to the random variables 
\[ A_\alpha = \int_0^T L_s^{(\alpha)} \, ds \]
where \( \{ L_t^{(\alpha), t \geq 0} \) is a strictly \( \alpha \)-stable Lévy process without negative jumps, starting from one, and \( T = \inf\{ t > 0, L_t^{(\alpha)} = 0 \} \) is its first hitting time of zero. Without loss of generality we choose the normalization 
\[ \mathbb{E}[e^{-tL_1^{(\alpha)}}] = e^{\alpha t}, \quad t \geq 0, \]
where \( \alpha \in [1, 2] \) is the self-similarity parameter. We refer e.g. to Chapter 3 in [21] for more information on stable Lévy processes and the above normalization. The boundary cases \( \alpha = 1, 2 \) correspond to the unit drift resp. the Brownian motion with variance \( \sqrt{2} \), so that we have
\begin{align*}
A_1 &= \frac{1}{2} \quad \text{and} \quad A_2 \overset{d}{=} \frac{1}{9\Gamma_{1/3}}.
\end{align*}
The above second identity, which is actually the precise statement of Theorem 2 in [17], follows from (1.1) and the self-similarity of Brownian motion. In the case \( \alpha \in (1, 2) \) the generator of the process \( L^{(\alpha)} \) is non-local, and it seems unappropriate to appeal to Feynman-Kac formulæ in order to obtain a tractable expression for the Laplace transform for \( A_\alpha \). Moreover, the absence of transition densities written in closed form prevents from using explicit computations as in the Gaussian case - see [17] and the references therein, to handle the random variable \( A_\alpha \). Instead, we will compute the fractional moments of \( A_\alpha \) and exhibit a multiplicative identity in law. Introduce the Beta random variable \( B_{a,b} \) with density
\[ \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} 1_{(0,1)}(x), \]
and the positive \( \alpha \)-stable random variable \( Z_\alpha \) with Laplace transform
\[ \mathbb{E}[e^{-\lambda Z_\alpha}] = e^{-\lambda^\alpha}, \quad \lambda \geq 0. \]
Our main observation is the following.

**Theorem.** With the above notation, one has the independent factorization
\begin{align*}
A_\alpha \overset{d}{=} \left( \frac{\alpha+1}{4} \right) \times Z_\alpha^{2} \times B_{\frac{\alpha}{\alpha+1}, \frac{\alpha-1}{\alpha(\alpha+1)}}
\end{align*}
for every \( \alpha \in (1, 2) \).

Observe that (1.3) is in accordance with the two boundary cases: when \( \alpha = 1 \) the two random variables on the right-hand side boil down to one, whereas when \( \alpha = 2 \) the following identity obtained in Theorem 1 of [23]:
\[ Z_\frac{2}{3} \overset{d}{=} \frac{4}{27} \Gamma^{-1}_{\frac{2}{3}} \times B^{-1}_{\frac{1}{3}, \frac{1}{3}}, \]
combined with the elementary factorization \( \Gamma_{a+b} \overset{d}{=} \Gamma_a \times B_{a,b}, \) allows to recover the second identity in (1.2). The proof of (1.3) relies on an identification of the fractional moments of
The explicit computation of the latter in terms of Gamma functions - see (2.1) below - is made possible by the strong Markov property and some exact results on the stable Kolmogorov process recently obtained in [20].

The inverse Gamma random variable involved in Lefebvre's identity shares a number of distributional properties related to infinite divisibility. Recall that a non-negative random variable $X$ is said to be self-decomposable if the following identities hold

$$X \overset{d}{=} cX + X_c$$

for every $c \in (0, 1)$ with $X_c$ independent of $X$, or equivalently if its log-Laplace transform is of the type

$$-\log E[e^{-\lambda X}] = a\lambda + \int_0^\infty (1 - e^{-\lambda x}) \frac{k(x)}{x} dx, \quad \lambda \geq 0$$

for some $a \geq 0$ and a non-negative, non-increasing function $k$ integrating $1 \wedge x^{-1}$. See again Chapter 3 in [21] for details. The fact that all inverse Gamma random variables are self-decomposable can be observed either by a direct and non-trivial computation on the Laplace transform, or by Dufresne's celebrated identity for the perpetuity of a Brownian motion with drift - see Section 3.2 in [5] for a survey of related problems with details and references. In this paper we will use the second approach and show the same property for $A_\alpha$.

**Corollary 1.** For every $\alpha \in (1, 2)$, the random variable $A_\alpha$ is self-decomposable.

More precisely, it will be proved that $A_\alpha$ is distributed as the perpetuity of a spectrally negative Lévy process which drifts towards $+\infty$. The latter background integrated Lévy process turns out to be tightly connected with the dual process $-L^{(\alpha)}$ conditioned to stay positive, and also to the Fréchet distribution which is hidden in the factorization (1.3) and known to be another perpetuity - see Remark 3 below.

It is known that inverse Gamma distributions also satisfy a property which is more stringent than self-decomposability. The law of a non-negative random variable $X$ is called a generalized Gamma convolution ($X \in \mathcal{G}$ for short) if it is the weak limit of an independent sum of Gamma random variables, or equivalently if its log-Laplace transform reads

$$-\log E[e^{-\lambda X}] = a\lambda + \int_0^\infty \log \left( \frac{x}{x + \lambda} \right) U(dx), \quad \lambda \geq 0,$$

for some $a \geq 0$ and some non-negative measure $U$ satisfying certain integrability conditions, which is called the Thorin measure of $X$. We refer to Chapter 3 in [3] for more details on this notion refining that of self-decomposability. The fact that inverse Gamma random variables are all in $\mathcal{G}$, with explicit Thorin measure, is a consequence of a computation on Bessel functions - see again Section 3.2 in [5] for details and references. We can prove the following result.

**Corollary 2.** For every $\alpha \in (1, 2)$, the law of $A_\alpha$ is a generalized Gamma convolution, with infinite Thorin measure.
More generally, it is actually true that the law of all positive powers $A_\alpha^s$ of index $s \geq 1/2$ are generalized Gamma convolutions - see Remark 4. (a) below.

The closed expression for the fractional moments of $A_\alpha$ can also be inverted in order to give a convergent series representation for its density $f_{A_\alpha}$. Throughout this paper, we will set $f_\alpha$ for the density of an absolutely continuous random variable $X$.

**Corollary 3.** The density of $A_\alpha$ has a convergent series representation:

$$f_{A_\alpha}(x) = \Gamma\left(\frac{\alpha}{\alpha + 1}\right) \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha + 1)\frac{n+1}{\alpha+1} x^{-\frac{n+1}{\alpha+1}-1}}{n! \Gamma(1 - \frac{n+1}{\alpha+1}) \Gamma(1 - \frac{n+2}{\alpha+1})}, \quad x > 0.$$  

Observe that when $\alpha = 2$ the above summation is made over $n = 3p$ only, and that further simplifications lead to

$$(1.4) \quad f_{A_2}(x) = \frac{x^{-4/3}}{3^{2/3} \Gamma(1/3)} \sum_{p=0}^{\infty} \frac{(-1)^p(9x)^{-p}}{p!} = \frac{\Gamma(2/3) x^{-4/3} e^{-\frac{1}{\pi}}} {2\pi 3^{1/6}} ,$$

which is the expression to be found in Theorem 2 of [17]. The above corollary also entails the first order asymptotics

$$f_{A_\alpha}(x) \sim \frac{(\alpha + 1)\frac{1}{\alpha+1} x^{-\frac{1}{\alpha+1}-1}}{\Gamma\left(\frac{\alpha}{\alpha+1}\right)} \text{ as } x \to +\infty,$$

which has, up to the multiplicative constant, the same speed as that of the density of the factor $Z_2^{2/(\alpha+1)}$ at infinity - see Formula (14.31) in [21]. On the other hand, it does not seem possible to deduce from the above series representation the exact behaviour of $f_{A_\alpha}$ at zero. Nevertheless, using the identity (1.3) we can show the following estimate.

**Corollary 4.** The asymptotic behaviour of the density of $A_\alpha$ when $x \to 0+$ is

$$(1.5) \quad f_{A_\alpha}(x) \sim \kappa_\alpha x^{\frac{\alpha^2}{1-\alpha^2}} e^{-c_\alpha x^{\frac{1}{1-\alpha}}} ,$$

with

$$\kappa_\alpha = \frac{\Gamma\left(\frac{\alpha}{\alpha+1}\right) \sqrt{\frac{\alpha+1}{\alpha-1}}}{2\pi (\alpha + 1)^{\frac{\alpha^2}{\alpha-1}}} \quad \text{and} \quad c_\alpha = (\alpha - 1)(\alpha + 1)^{\frac{\alpha^2}{\alpha-1}} .$$

This shows that the behaviour of $f_{A_\alpha}$ at zero is that of the generalized Fréchet density

$$(1.6) \quad f_{\tilde{c}_\alpha^{-1} \Gamma^{1-\alpha} \left(\frac{1}{\alpha+1}\right)}(x) = \tilde{\kappa}_\alpha x^{\frac{\alpha^2}{1-\alpha^2}} e^{-c_\alpha x^{\frac{1}{1-\alpha}}} ,$$

up to the normalizing constant

$$\tilde{\kappa}_\alpha = \frac{(\alpha - 1)\frac{\alpha^2}{\alpha+1}}{\Gamma\left(\frac{1}{\alpha+1}\right)(\alpha + 1)^{\frac{\alpha^2}{\alpha-1}}}$$

which does not coincide with $\kappa_\alpha$ except for $\alpha = 2$. Observe also that making $\alpha = 2$ on the right-hand side of (1.5) yields the density in (1.4). It should be possible to obtain a full asymptotic expansion of $f_{A_\alpha}$ at zero with our method - see Remark 5 below. But we have not adressed this issue, which is believed to be very technical, in the present paper.
2. Proofs

2.1. Proof of the Theorem. To simplify the notation we will set \( L = L^{(\alpha)} \). Introducing the area process

\[
A_t = \int_0^t L_s \, ds, \quad t \geq 0,
\]

recall that the bivariate process \( X = \{(A_t, L_t), t \geq 0\} \) is strongly Markovian and denote by \( \mathbb{P}_{(x,y)} \) its law starting from \((x,y)\). Consider the stopping time

\[
S = \inf\{t > 0, A_t = 0\}
\]

and observe that under \( \mathbb{P}_{(0,1)} \) one has a.s. \( S > T, A_T > 0, \) and \( L_S < 0 \).

Setting \( \{\mathcal{F}_t, t \geq 0\} \) for the natural completed filtration of \( X \) and applying the strong Markov property at \( T \) entails that for every \( s \in \mathbb{R} \), one has

\[
\mathbb{E}_{(0,1)}[|L_s|^{\alpha-1}] = \mathbb{E}_{(0,1)}[\mathbb{E}[|L_s|^{\alpha-1} | \mathcal{F}_T]] = \mathbb{E}_{(A_T,0)}[\mathbb{E}[L_S^{\alpha-1}]]
\]

possibly with infinite values on both sides, where the second equality follows from the absence of negative jumps for \( L \), and the third equality from the self-similarity of \( L \) and \( A \) with respective indices \( 1/\alpha \) and \( 1 + 1/\alpha \).

Applying Theorem B in [20] in the particular case \( \rho = 1/\alpha \) (beware that we consider here the dual process, with no positive jumps), we get

\[
\mathbb{E}_{(0,1)}[|L_s|^{\alpha-1}] = \frac{\sin(\pi s\alpha)}{\sin(\pi s(\alpha+1))}
\]

and

\[
\mathbb{E}_{(1,0)}[|L_s|^{\alpha-1}] = \frac{(\alpha + 1)^{\frac{1-s}{\alpha+1}} \Gamma(\frac{\alpha+s}{\alpha+1}) \Gamma(1-s) \sin(\frac{\pi s}{\alpha+1})}{\Gamma(\frac{s}{\alpha+1}) \Gamma(1-s) \sin(\frac{\pi s}{\alpha+1})}
\]

for all \( |s| < 1 + 1/\alpha \). Dividing and simplifying with the help of the complement formula for the Gamma function, we deduce

\[
(2.1) \quad \mathbb{E}[A_s^\alpha] = (\alpha + 1)^s \times \frac{\Gamma(1 - (\alpha + 1)s)}{\Gamma(\frac{\alpha}{\alpha+1} - s) \Gamma(1 - s)}
\]

for all \( s < 1/(\alpha + 1) \). Applying the Legendre-Gauss multiplication formula for the Gamma function entails

\[
\mathbb{E}[A_s^\alpha] = \left(\frac{\alpha + 1}{4}\right)^s \times \frac{\Gamma(1 - (\alpha + 1)s)}{\Gamma(1 - 2s)} \times \frac{\Gamma(1 - s)}{\Gamma(\frac{\alpha}{\alpha+1} - s)},
\]

and we can conclude by a fractional moment identification, recalling (see e.g. Formula (25.5) in [21]) for the second expression that

\[
\mathbb{E}[B_{a,b}^s] = \frac{\Gamma(a + s) \Gamma(a + b)}{\Gamma(a) \Gamma(a + b + s)} \quad \text{and} \quad \mathbb{E}[Z_a^s] = \frac{\Gamma(1 - \frac{s}{a})}{\Gamma(1 - s)}.
\]

\( \square \)
Remark 1. (a) It is well-known and easy to see - see e.g. Theorem 46.3 in [21] - that under $P_{(0,1)}$, the random variable $T$ is distributed as $Z_{1/\alpha}$. The above theorem provides hence a connection between $A_\alpha = A_T$ and the random variable $Z_{2/(\alpha+1)}$. Notice that one can also derive from (2.1) another factorization:

$$
Z_{1/(\alpha+1)} \overset{d}{=} (\alpha + 1)^{-1} \times \frac{\Gamma^{-1}_1}{\alpha+1} \times A_\alpha.
$$

However, it seems difficult with our method to obtain some valuable information on the Mellin transform of the bivariate random variable $(T,A_T)$.

(b) With the notation of our above proof, it is possible to derive the law of $A_T$ under $P_{(x,y)}$ for any $x \in \mathbb{R}$ and $y > 0$, by the self-similarity of $L^{(\alpha)}$. One finds

$$
A_\alpha \overset{d}{=} x + \left( \frac{\alpha + 1}{4} \right) \times \frac{Z_{2/(\alpha+1)}}{\alpha+1} \times B_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}(\alpha+1)}.
$$

2.2. Proof of Corollary 1. Let us first observe that $A_\alpha$ is infinitely divisible, by a simple pathwise argument not relying on (1.3). Setting $T_x = \inf \{ t > 0, L_t = x \}$ for all $x > 0$ and using the fact that $L$ has no negative jumps, it is easy to see from the Markov property that under $P_{(0,1)}$, for every $n \geq 2$, there is an independent decomposition

$$
A_\alpha = X_{1}^{(n)} + \cdots + X_{n}^{(n)}
$$

where

$$
X_{i}^{(n)} \overset{d}{=} A_{T_{n-i}} \text{ under } P_{(0,1/n)^{(n-i)}}
$$

for every $i = 1, \ldots, n$. Moreover, one has $T_{1/n} \rightarrow 0$ a.s. under $P_{(0,1)}$ as $n \rightarrow +\infty$ (by the well-known fact - see e.g. Theorem 47.1 in [21] - that $L$ visits immediately the negative half-line when starting from 0), so that $X_{1}^{(n)} \rightarrow 0$ a.s. under $P_{(0,1)}$ when $n \rightarrow +\infty$ as well. Last, it is straightforward that

$$
P[|X_{i}^{(n)}| > \varepsilon] \leq P[|X_{1}^{(n)}| > \varepsilon]
$$

for every $\varepsilon > 0$ and $i = 1, \ldots, n$. Putting everything together and applying Khintchine’s theorem on triangular arrays - see e.g. Theorem 9.3 in [21], entails that $A_\alpha$ is infinitely divisible.

Remark 2. The above argument does not make use of the self-similarity of $L$, and hence applies to any spectrally positive Lévy process which is not a subordinator.

We now proceed to the proof of the self-decomposability of $A_\alpha$. We will use the same argument as in [5, 6], expressing $A_\alpha$ as the perpetuity of a certain spectrally negative Lévy process. Setting

$$
\Psi_\alpha(u) = \frac{u \mathbb{E}[A_\alpha^{-u+1}]}{\mathbb{E}[A_\alpha^{-u}]}
$$
for every \( u > 0 \), we first deduce from (1.3) the formula 
\[ \Psi_\alpha(u) = \Phi_\alpha(s) \]
with the notation 
\[ s = (\alpha + 1)u \]
Applying the Lemma in [5] with \( t = 1 \) shows, after some simplifications, that 
\[ \Phi_\alpha(s) = \Gamma(\alpha + 1) \int_{-\infty}^{0} (e^{sx} - 1 - sx) f_\alpha(x) \, dx \]
with 
\[ f_\alpha(x) = \frac{e^{\alpha x}}{\Gamma(-\alpha)(1 - e^x)^{\alpha + 1}}. \]
Since \( f_\alpha \) integrates \( x^2 \wedge 1 \) on \((-\infty, 0)\), this entails that \( \Psi_\alpha \) is the Laplace exponent of a spectrally negative Lévy process. An application of Bertoin-Yor’s criterion for perpetuities (as stated in [6] pp. 8-9 - see Proposition 2 in [1] for the original statement) shows that
\[ (2.2) \]
\[ A_\alpha = \int_0^\infty e^{-Z_\alpha(t)} \, dt \]
where \( \{Z_\alpha(t), t \geq 0\} \) is the spectrally negative Lévy process with Laplace exponent 
\[ \mathbb{E}[e^{\lambda Z_\alpha(t)}] = e^{\psi_\alpha(\lambda)} = e^{\Phi_\alpha((\alpha + 1)\lambda)}, \quad \lambda \geq 0. \]
It is then easy to see from the representation (2.2) and the spectral negativity of \( Z^{(\alpha)} \) that \( A_\alpha \) is self-decomposable - see the end of the proof of the Theorem in [5].

**Remark 3.** (a) The above expression (2.2) extends to the boundary cases \( \alpha = 1, 2 \). When \( \alpha = 1 \), the Lévy process \( Z^{(1)} \) has Laplace exponent \( 2\lambda \), so that (2.2) boils down to
\[ A_1 = \int_0^\infty e^{-2t} \, dt = \frac{1}{2} \]
When \( \alpha = 2 \), the Lévy process \( Z^{(2)} \) has Laplace exponent \( 3\lambda + 9\lambda^2 \), and (2.2) reads
\[ A_2 = \int_0^\infty e^{3\sqrt{2}B_t - 3t} \, dt \]
\[ = \frac{1}{18} \int_0^\infty e^{B_t - t/6} \, dt \]
\[ = \frac{1}{9\Gamma(1/3)} \]
the third identity in law being a particular case of Dufresne’s identity.

(b) It follows from Corollary 2 in [8] that the spectrally negative Lévy process \( \{\xi_1^n, t \geq 0\} \) appearing in the Lamperti transform of the dual process \(-L^{(\alpha)}\) conditioned to stay positive, has log-Laplace exponent 
\[ -\log \mathbb{E}[e^{\lambda \xi_1^n}] = \Phi_\alpha(\lambda), \quad \lambda \geq 0, \]
with the above notation for \( \Phi_\alpha \). This can be shown from the second formula in Corollary 2 of [8] written in an appropriate way, bewareing the unusual notation (7) therein for the
negativity parameter and correcting a misprint (the + before $c_-$ should be a $-$) in Formula (17) therein. This entails the identity in law

$$A_T \overset{d}{=} \int_0^\infty e^{-(\alpha+1)\xi_t^{\uparrow,n}} \, dt.$$  

Recall that Brownian motion conditioned to stay positive is distributed as a three-dimensional Bessel process, whose Lamperti process $\xi^{\uparrow,n}$ is the drifted Brownian motion $\{B_t + t/2, t \geq 0\}$ - see [8] p. 969 and the references therein, so that by Dufresne’s identity we obtain

$$\int_0^\infty e^{-3\xi_t^{\uparrow,n}} \, dt \overset{d}{=} \frac{2}{9\Gamma_{3/2}} \overset{d}{=} A.$$  

This can be viewed as a particular case of (2.3), with the proper normalization. It is interesting to compare (2.3) with the identity

$$T \overset{d}{=} \int_0^\infty e^{-\alpha\xi_t^{\uparrow,n}} \, dt,$$

which follows from the above Remark 1 and Theorem 7 of [9]. Notice last that the perpetuities of certain Lévy processes with positive mean and jumping density

$$K e^{b x} \over (1 - e^x)^{\alpha+1} 1_{\{x < 0\}}$$

for some $K, b > 0$ and $\alpha \in (1, 2)$ have been studied in [18]. Observe in particular that the factorization obtained in Theorem 4.6 (2) therein shares some similarities with (1.3).

(c) Combining the Kanter factorization - see Corollary 4.1 in [14] - and (1.3) shows the identity

$$A_\alpha \overset{d}{=} \left( \frac{\alpha + 1}{4} \right) \times \Gamma_1^{1-\alpha} \times \left( B_{\frac{\alpha-1}{2(\alpha+1)}} \times K^{\frac{\alpha+1}{2}} \right)^{-1},$$

where $K_a$ is the so-called Kanter random variable of index $a \in (0, 1)$ - see Section 3 in [23] for more details about this random variable. Consider now the Fréchet random variable $\Gamma_1^{1-\alpha}$ appearing in the above factorization of $A_\alpha$. In [5], it was shown that $\Gamma_1^{1-\alpha}$ is also distributed as the perpetuity of a spectrally negative Lévy process, and the latter turns out to be a simple multiplicative perturbation the above process $Z^{(\alpha)}$. More precisely, it follows from the Lemma in [5] with $t = 1$, the proof of the Theorem in [5] and a change of variable that

$$\Gamma_1^{1-\alpha} \overset{d}{=} (\alpha - 1) \int_0^\infty e^{-(\alpha-1)\xi_t^{\uparrow,n}} \, dt$$

with the above notation for the process $\{\xi_t^{\uparrow,n}, t \geq 0\}$. The identity can also be derived from the previous results of [15] on the extinction time of continuous state branching processes driven by $L$ - see Lemma 1 therein. Moreover, the observations made in Remark 3.1 of [15] show that

$$\int_0^T (L_t^{(\alpha)})^{-1} \, dt \overset{d}{=} \frac{1}{(\alpha - 1)\Gamma_1^{\alpha-1}}.$$
We can then deduce the interesting factorization

\[
\int_0^T L_t^{(\alpha)} dt \overset{d}{=} \left( \frac{\alpha^2 - 1}{4} \right) \times \left( B_{\frac{1}{2}, \frac{\alpha}{2(\alpha+1)}} \times K_{\frac{\alpha+1}{\alpha+1}} \right)^{-1} \times \left( \int_0^T (L_t^{(\alpha)})^{-1} dt \right).
\]

2.3. **Proof of Corollary 2.** From (1.3) and a general recent result by Bondesson - see Theorem 1 in [4], in order to show \( \mathcal{A}_\alpha \in \mathcal{G} \) it is enough to prove that

\[
B_{\frac{1}{2}, \frac{-\alpha-1}{2(\alpha+1)}} \in \mathcal{G} \quad \text{and} \quad Z_{\frac{\alpha}{\alpha+1}}^2 \in \mathcal{G}.
\]

The first fact is well-known and also the consequence of a stronger property - see the introduction of [6]. In order to prove the second fact, we use a discrete factorization of \( Z_a \) obtained in Lemma 2 of [7], which reads

\[
(2.6) \quad Z_a^{-1} \overset{d}{=} e^{\gamma(1-a-1)} \times \prod_{n=0}^{\infty} e^{\psi(1+na)-\psi(a+na)} B_{a+na, 1-a},
\]

for all \( a \in (0, 1) \), where \( \gamma \) is Euler-Mascheroni’s constant and \( \psi \) is the usual digamma function. Using again Theorem 1 in [4], we need to show that \( B_{a,b}^2 \in \mathcal{G} \) for all \( a, b > 0 \), which is a direct consequence of Theorem 2 in [6]. Finally, the fact that \( \mathcal{A}_\alpha \) has infinite Thorin measure follows from Theorem 4.1.4 in [3] and the above formula (2.1), which entails that \( \mathcal{A}_\alpha \) has negative moments of all orders.

**Remark 4.** (a) It follows from (1.3), (2.6) and Theorem 3 in [6] that \( \mathcal{A}_s^a \in \mathcal{G} \) for all \( s \geq 1/2 \).

(b) A combination of Corollaries 1 and 2 shows the following identities in law between three random integrals:

\[
\int_0^\infty e^{-Z_t^{(\alpha)}} dt \overset{d}{=} \int_0^T L_t^{(\alpha)} dt \overset{d}{=} \int_0^\infty a_t^{(\alpha)} d\Gamma_t,
\]

where \( \{\Gamma_t, t \geq 0\} \) is the Gamma subordinator and \( \{a_t^{(\alpha)}, t \geq 0\} \) some deterministic function which is related to the Thorin measure of \( \mathcal{A}_\alpha \) - see Proposition 1.1 in [12].

2.4. **Proof of Corollary 3.** We will reason along the same lines as in Proposition 2 in [22], and omit some details. Applying the Mellin inversion formula yields first

\[
f_{\mathcal{A}_\alpha}(x) = \frac{1}{2\pi i} \int_{\mathbb{R}} \mathcal{M}_\alpha(s) x^{-is} ds,
\]

with the notation

\[
\mathcal{M}_\alpha(s) = \mathbb{E} [\mathcal{A}_s^{\alpha}] = (\alpha + 1)^{i s} \times \frac{\Gamma(\frac{\alpha}{\alpha+1}) \Gamma(1 - (\alpha + 1)is)}{\Gamma(\frac{\alpha}{\alpha+1} - is) \Gamma(1 - is)}
\]

for every \( s \in \mathbb{R} \). Suppose first \( x > 1 \). We evaluate the above integral with the help of the residue theorem applied to the contour joining \(-R\) to \(R\) along the real axis, and \(R\) to \(-R\) along the half-circle plotted in the lower half-plane. It is easy to see that the integral along
this half-circle vanishes as $R \to +\infty$, so that it remains to consider the singularities inside the big contour, which are located at $t_n = -i(n+1)/(\alpha + 1), n \geq 0$. Computing

$$\text{Res}_{t_n}(\mathcal{M}_\alpha(s)x^{-is}) = -i \Gamma \left( \frac{\alpha}{\alpha + 1} \right) \times \frac{(-1)^n(\alpha + 1)^{\frac{n+1}{\alpha + 1} - 1}x^{\frac{n+1}{\alpha + 1} - 1}}{n!\Gamma(1 - \frac{n+1}{\alpha + 1})\Gamma(1 - \frac{n+2}{\alpha + 1})}$$

we deduce

$$f_{\mathcal{A}_\alpha}(x) = \Gamma \left( \frac{\alpha}{\alpha + 1} \right) \times \sum_{n=0}^\infty \frac{(-1)^n(\alpha + 1)^{\frac{n+1}{\alpha + 1} - 1}x^{\frac{n+1}{\alpha + 1} - 1}}{n!\Gamma(1 - \frac{n+1}{\alpha + 1})\Gamma(1 - \frac{n+2}{\alpha + 1})}$$

for every $x > 1$, and hence for every $x > 0$ by analytic continuation (Stirling’s formula shows indeed that the series on the right-hand side converges absolutely for every $x > 0$).

\[\square\]

2.5. Proof of Corollary 4. We will work on the random variable $X_\alpha = \mathcal{A}_\alpha^{-1/\alpha}$, in order to simplify the notation. Changing the variable, the required estimate is tantamount to

$$(2.7) \quad f_{X_\alpha}(x) \sim \frac{\Gamma(\frac{\alpha}{\alpha + 1})\sqrt{\alpha^2 - 1}}{2\pi(\alpha + 1)^{\frac{\alpha}{\alpha + 1}}x^{\frac{\alpha}{\alpha + 1}}} e^{-c_\alpha x}, \quad x \to 0^+.$$ 

Evaluating with (2.1) the positive entire moments

$$E[X_\alpha^n] = (\alpha + 1)^{\frac{n}{\alpha - \alpha}} \times \frac{\Gamma(\frac{\alpha}{\alpha + 1})}{\Gamma(\frac{\alpha}{\alpha + 1} + \frac{n}{\alpha - 1})} \times \frac{\Gamma(1 + \frac{(\alpha+1)n}{\alpha - 1})}{\Gamma(1 + \frac{n}{\alpha - 1})}$$

for every $n \geq 0$, and applying Stirling’s formula shows that

$$\frac{E[X_\alpha^n]^{\frac{1}{n}}}{\alpha - 1} \to e^{\alpha - 1} \quad \text{as } n \to \infty.$$ 

By a theorem of Davies-Kasahara (see Corollary 4.12.5 in [2], or Lemma 3.2 in [10] for a more appropriate formulation), we deduce that

$$x^{-1}\log\mathbb{P}[X_\alpha > x] \to -c_\alpha \quad \text{as } x \to +\infty.$$ 

This yields the required asymptotic behaviour, at the logarithmic scale, for the survival function of $X_\alpha$. Moreover, writing down via Fubini’s theorem the moment generating function

$$E[e^{xX_\alpha}] = \sum_{n=0}^\infty a_n x^n, \quad x > 0,$$

with

$$a_n = (\alpha + 1)^{\frac{\alpha}{\alpha - \alpha}} \times \frac{\Gamma(\frac{\alpha}{\alpha + 1})\Gamma(1 + \frac{(\alpha+1)n}{\alpha - 1})}{n!\Gamma(\frac{\alpha}{\alpha + 1} + \frac{n}{\alpha - 1})\Gamma(1 + \frac{n}{\alpha - 1})} \sim \frac{\Gamma(\frac{\alpha}{\alpha + 1})\sqrt{\alpha^2 - 1}}{2\pi(\alpha - 1)^{\frac{\alpha}{\alpha + 1}}} c_\alpha^{-n} n^{-\alpha} \quad \text{as } n \to \infty,$$
and applying Karamata’s theorem for power series - see Corollary 1.7.3 in [2] - shows that

\[
\mathbb{E}[e^{c_\alpha x X_\alpha}] \sim \frac{\Gamma(\frac{1}{\alpha+1}) \Gamma(\frac{\alpha}{\alpha+1}) \sqrt{\alpha^2 - 1}}{2\pi (\alpha - 1)^{\frac{1}{\alpha+1}} (1 - x)^{\frac{1}{\alpha+1}}} \quad \text{ as } x \to 1-. 
\]

At this stage, it is worth mentioning that (2.8) can be obtained from (2.7) by integration. However, it does not seem that we can infer the reverse inclusion without any further assumption, such as the existence of a meromorphic extension in the neighbourhood of \(c_\alpha\) for the moment generating function - see Theorem 4 in [11], or a monotonicity condition on \(f_{X_\alpha}\) at infinity - see Theorem 4.12.11 in [2], which we both could not prove \textit{a priori}.

In order to show (2.7) rigorously and finish the proof, we will use the following power transformation of (1.3):

\[
X_\alpha \overset{d}{=} \left( \frac{4}{\alpha + 1} \right)^{\frac{1}{\alpha+1}} \times B_{\frac{1}{2} - \frac{a-1}{2(\alpha+1)}} \times Z_{\frac{1}{\alpha+1}}. 
\]

The multiplicative convolution formula and a change of variable entails

\[
f_{X_\alpha}(x) = \int_0^\infty \left( \frac{1}{1+y} \right) f_{U_\alpha} \left( \frac{1}{1+y} \right) f_{V_\alpha}(x + xy) dy,
\]

where we have set

\[
U_\alpha = B_{\frac{1}{2} - \frac{a-1}{2(\alpha+1)}} \quad \text{ and } \quad V_\alpha = \left( \frac{4}{\alpha + 1} \right)^{\frac{1}{\alpha+1}} \times Z_{\frac{1}{\alpha+1}}. 
\]

On the one hand, we have

\[
\left( \frac{1}{1+y} \right) f_{U_\alpha} \left( \frac{1}{1+y} \right) = \left( \alpha - 1 \right)^{\frac{a-1}{2(\alpha+1)}} \Gamma \left( \frac{\alpha}{\alpha+1} \right) \frac{\Gamma \left( \frac{a-1}{2(\alpha+1)} \right)}{\sqrt{\pi}} y^{\frac{a-1}{2(\alpha+1)} - 1}(1 + O(y)) \quad \text{ as } y \to 0+. 
\]

On the other hand, Formula (14.35) in [21] entail after a change of variable and several simplifications

\[
f_{V_\alpha}(z) = \frac{\sqrt{\alpha - 1} (\alpha + 1)^{\frac{a-1}{2(\alpha+1)}} \frac{1}{\alpha+1}}{2\sqrt{\pi}} z^{-1/2} e^{-\alpha z} (1 + O(z^{-1/2})) \quad \text{ as } z \to +\infty. 
\]

Plugging these two first order expansions in the integral (2.9), and making further simplifications, yields finally the required asymptotic behaviour (2.7).

\[\square\]

\textbf{Remark 5.} (a) The two asymptotic expansions for the above \(f_{U_\alpha}\) and \(f_{V_\alpha}\) can be continued at every order - see again Formula (14.35) in [21] for the second function. This could be used to obtain a refined expansion for \(f_{X_\alpha}\) at infinity, or equivalently for \(f_{A_\alpha}\) at zero. Displaying the full asymptotic expansion of \(f_{A_\alpha}\) at zero seems however to be a very painful task.

(b) It follows from Corollaries 3 and 4 that the density function of \(A_\alpha\) is real analytic on \((0, +\infty)\), and that all its derivatives vanish at zero and at infinity. Moreover, a consequence of Corollary 2, Wolfe-Yamazato’s theorem - see e.g. Theorem 51.3 in [21], and the principle
of isolated zeroes, is the strict unimodality of this density function, that is its first derivative vanishes only once on $(0, +\infty)$. By Rolle’s theorem, we deduce that its $n$-th derivative vanishes at least $n$ times on $(0, +\infty)$, and one can ask whether it vanishes exactly $n$ times for all $n \geq 1$. Such a property, which is called the bell-shape in the literature, has been conjectured in [24] for all positive self-decomposable distributions having an infinite spectral function at zero - see Conjecture 1 therein. Observe that $A_n$ has such a self-decomposable law by Corollary 2, Corollary 4 and a theorem by Zolotarev - see Remark 28.6 in [21]. Observe also that drawing the density with the help of the series representation of Corollary 3 and some plotting software exhibits the visual bell-shape property for $f_{A_n}$, whose second derivative does seem to vanish only twice on $(0, +\infty)$. This question will be investigated elsewhere.

(c) It is noticeable that the convolution formula (2.9) allow to study the precise tail behaviour of other random variables defined as independent products. If we consider for example $X = B_{a,b}^p \times \Gamma_c$ with $a, b, c, p > 0$, this formula shows very quickly that

$$f_X(x) \sim \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} x^{c-p-b-1} e^{-x}$$

as $x \to +\infty$. At the less precise level of survival functions, this behaviour was recently obtained in [19], with a more complicated method. In this example it is also possible to derive without pain, contrary to the above (a), the full asymptotic expansion, which is also exact for certain values of the parameters $a, b, c, p$.

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