Cayley path and quantum computational supremacy: A proof of average-case \#P-hardness of Random Circuit Sampling with quantified robustness

Ramis Movassagh

IBM Research, MIT-IBM AI lab, Cambridge MA, 02142

A one-parameter unitary-valued interpolation between any two unitary matrices (e.g., quantum gates) is constructed based on the Cayley transformation, which extends our previous work [15]. We prove that this path provides scrambled unitaries with probability distributions arbitrarily close to the Haar measure. We then prove the simplest known average-case \#P-hardness of random circuit sampling (RCS), which is the task of sampling from the output distribution of a quantum circuit whose local gates are random Haar unitaries, and is the lead candidate for demonstrating quantum supremacy in the NISQ era. We show that previous work based on the truncations of the power series representation of the exponential function does not provide practical robustness. Explicit bound on noise resilience is proved, which for an \(n\)-qubit device with the near term experimental parameters is \(2^{-\Theta(n^{3.51})}\) robustness with respect to additive error. Improving this to \(O(2^{-n}/\text{poly}(n))\) would prove the quantum supremacy conjecture; and proving our construction optimal would disprove it.

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References
I. OVERVIEW

Fault-tolerant quantum computation seems unlikely in the near term, however, “Noisy Intermediate Scale Quantum (NISQ)”\[17\] computers have arrived. A central question now is: in the absence of quantum error correction what can a NISQ computer do? Can we gain a provable advantage over classical computers in the near term, say within the next 5 years?

Currently there is a large global interest with an unprecedented industrial push (e.g., from IBM and Google) for developing NISQ computers. Such computers with hundred(s) of “good” qubits and gates are at the brink of existence \[6\] \[7\] with the promise of outperforming any classical computer \[8\]. A milestone is to prove unambiguously an advantage of a NISQ computer over classical ones. This event has been termed quantum supremacy, and we have yet to witness it \[13\]. It states that there are computational tasks that a NISQ computer can perform efficiently, which would be formidable on any classical computer. Among the various proposals \[1\] \[4\] \[11\] \[13\], Random Circuit Sampling (RCS) is a leading candidate that is promised to be experimentally demonstrated quite soon \[2\]. RCS, roughly speaking, is the task of sampling from the output distribution of a quantum circuit whose local gates are random (see \[15\] for more).

We denote by \(n\) the number of quantum bits (qubits) in the quantum computation, and the circuit \(C\) implements the unitary evolution \(U\) applied to an initial state. Constrained by experimental difficulties, \(U\) almost always is taken to be a product of many ‘local’ unitaries (gates) that each acts nontrivially on one or two qubits. One says that the circuit \(U\) is generic with respect to the architecture \(A\) if the local unitaries are drawn independently from the Haar measure.

Generic quantum circuits have been proposed as means for demonstrating quantum supremacy \[8\] \[13\]. Aaronson and Arkhipov first proposed the BosonSampling problem as a candidate for testing quantum supremacy \[1\]. Later, the Google team proposed that random quantum circuits could demonstrate it in the near-term quantum devices \[2\]. It is known that sampling from the output probabilities of quantum circuits, even approximately, is \#P-Hard\(^{1}\) in the worst case \[4\]. This implies that no exact worst-case classical simulation algorithm exists unless the polynomial hierarchy collapses \[1\] \[4\]. Recently two proofs of the exact average-case \#P-Hard of RCS were given \[3\] \[15\]. However, what is needed in all supremacy proposals to date is to prove

\(^{1}\) Roughly speaking \#P is a generalization of NP that extends the decision problems to counting problems. In particular, an NP-complete problem is: Does a 3-SAT instance have any solutions? The answer is yes or no (i.e., zero solutions). Whereas, \#P asks: How many solutions does a 3-SAT instance have?
approximate average-case hardness. This requires resilience with respect to noise that is polynomially small factor in the system’s size. Although the conjecture remains open, we provide an entirely new construction and proof that tolerates some noise. We argue that this is the first relevant proof for quantum supremacy of RCS that tolerates any noise.

In RCS, the goal is to demonstrate quantum supremacy by estimating the distribution of

\[ p_y(C) \equiv |\langle y | C | 0^n \rangle|^2. \]  

(1)

It is known that this problem is classically hard in the worst case \[4\]. One proves the average case hardness via an algebraic (polynomial) reduction of the worst case hardness to the average [14], [15] (see for detailed discussion), where the circuit \( C \) is deformed with respect to a single parameter \( \theta \) such that \( C(\theta) \), as a matrix, has entries that are algebraic functions (often polynomial). In addition, \( C(1) = C \) and \( C(0) \) is a generic circuit. To make the reduction work, Bouland et al [3] deform the quantum gates towards a Haar distribution by a unitary with an exponential form; to get a polynomial they truncate the Taylor series expansion rendering the circuit non-unitary and inexact. They argued that the truncation errors are small enough that the average case \#P-hardness of the non-unitary approximation is still necessary for the approximate average-case hardness to be true. Movassagh proposed an interpolation contained everywhere in the unitary group based on the QR-factorization algorithm, but did not claim robustness towards noise [15].

Here a path on the unitary group is constructed based on the Cayley transformation, which gives probabilities (Eq. 1) that are exact rational functions of the parameter \( \theta \). We believe this path is in some ways the optimal construction. It can be efficiently learned using the Berlekamp-Welch algorithm for rational functions proved elsewhere [15]. We prove that \#P-hardness of exact computation of Eq. 1 is necessary for proving the quantum supremacy conjecture. This is ensured by our construction as in Section II we prove that the local gates are arbitrary close to the Haar distribution in total variational distance. In addition to being a more direct proof, the advantages of this work include:

(1) The interpolation is contained in the unitary group everywhere (for all \( \theta \)).

(2) Explicit robustness despite noise is proved and quantified (Section III C). In Subsection III B we show that this is the first to prove RCS hardness with some robustness for circuits whose local gates are arbitrary close to Haar distribution.

(3) The construction is explicit in that the degrees and coefficients are quantified; this might help in proving the quantum supremacy conjecture (see [1] (arXiv version, page 81)).

(4) The new interpolation technique may be of independent utility and interest (See Section IV).
Figure 1: Plot of the Cayley function in the complex plane (Eq. 3). The arrow shows how the function fills the unit circle as \( x \) increases from \( x = -\infty \). The non-uniform spacing is due to the finite step size in \( x \) and aggregation of points at infinity.

Cayley path

Let \( \mathbb{U}(N) \) be the set of \( N \times N \) unitary matrices and suppose \( U_1 \in \mathbb{U}(N) \) and \( U_2 \in \mathbb{U}(N) \). How can one build a parametrized path \( U(\theta) \) between them such that \( U(\theta) \in \mathbb{U}(N) \) for all \( \theta \in [0, 1] \) and \( U(0) = U_1 \) and \( U(1) = U_2 \)?

Previously we gave various paths between \( U_1 \) and \( U_2 \) that were everywhere contained in the unitary group \([15]\). In particular, we gave a new rational function-valued path based on the QR-factorization and proved an extension of the Berlekamp-Welch algorithm for efficiently determining rational functions by sampling \([15]\).

Here we consider a new extrapolation based on the Cayley transformation. Suppose \( U, H \in \mathbb{U}(N) \) are unitary matrices \(^2\). Let \( x \in \mathbb{R} \) and \( f(x) \) be the Cayley function

\[
f(x) = \frac{1 + ix}{1 - ix}
\]  

where we define \( f(-\infty) = -1 \) (see Fig. 1). Since \( f(x) \) is a bijection between the real line and the unit circle, \( H \) has the unique representation

\[
H = f(h), \quad h = h^\dagger
\]  

and \( H^\dagger = f(-h) \).

\(^2\) We use \( U \) and \( H \) instead of \( U_1 \) and \( U_2 \) in anticipation of the local gates and Haar unitaries in Section \([III]\) respectively.
We want an interpolation $U(\theta)$ such that $U(0) = UH$ and $U(1) = U$ and entries of $U(\theta)$ are simple functions of $\theta$ that can be efficiently computed. The proposed path is

$$U(\theta) = UH f(-\theta h). \quad (4)$$

$U(\theta)$ is a unitary matrix as it is a product of three unitary matrices. By the spectral decomposition

$$h = \sum_{\alpha=1}^{N} h_{\alpha} |\psi_{\alpha}\rangle \langle \psi_{\alpha}|,$$

where $(h_{\alpha}, |\psi_{\alpha}\rangle)$ are the eigenpairs of $h$. The foregoing two equations give

$$U(\theta) = \frac{1}{q(\theta)} \sum_{\alpha=1}^{N} p_{\alpha}(\theta) U |\psi_{\alpha}\rangle \langle \psi_{\alpha}|, \quad (5)$$

where

$$q(\theta) = \prod_{\alpha=1}^{N} (1 + i\theta h_{\alpha}) \quad \text{and} \quad p_{\alpha}(\theta) = f(h_{\alpha})(1 - i\theta h_{\alpha}) \prod_{\beta \in [N] \setminus \alpha} (1 + i\theta h_{\beta}). \quad (6)$$

We see that $p_{\alpha}(\theta)$ and $q(\theta)$ are polynomials of degree $N$ that only depend on $\theta$ and $H$.

Remark 1. In section [III], we think of $N$ as the size of a local gate, which is usually $N = 2$ or $N = 4$. The entries of $U(\theta)$ are rational functions of degree $(N, N)$. However, for a given $\theta$ and $H$, the normalization $q(\theta)$ is easy to classically compute. It amounts to a diagonalization of an $N \times N$ matrix $H$ and an $N$-fold product of the complex numbers $(1 + i\theta h_{\beta})$. Since $N \leq 4$, this is done in $O(1)$ time. For a general circuit made up of $m$ gates, the classical computational complexity of calculating all $q(\theta)$’s is therefore $O(m)$. By precomputing them all and multiplying through, Eq. (1) is effectively polynomial-valued and can be treated formally as such. This will be made precise below when the need arises.

II. CLOSENESS IN TOTAL VARIATION DISTANCE TO THE HAAR MEASURE

In this section, after introducing the Haar measure, in Lemma [I] we will prove that if $H$ is drawn from the Haar measure, then for $\theta \ll 1$, $H f(-\theta h)$ is $\theta$—close in total variational distance (TVD) to the Haar measure. From [15], we recall that $O(N)$, and $U(N)$ denote the set of orthogonal and unitary matrices respectively. The entries of these matrices are real ($\beta = 1$), and complex ($\beta = 2$) respectively. In the special case that the determinant is equal to one are, these are denoted by $SO(N)$ and $SU(N)$. If $G$ is any one of the matrix groups, then a uniform random element of $G$ is a matrix $V \in G$ whose distribution is translation invariant. This means that for any fixed $M \in G$,

$$VM \overset{d}{=} MV \overset{d}{=} V,$$
where \( \equiv \) is equality in the distribution sense. We have the well-known theorem (see also [15]):

**Theorem.** Let \( G \) be any of \( \mathbb{O}(N) \), \( \mathbb{SO}(N) \), \( \mathbb{U}(N) \) or \( \mathbb{SU}(N) \). Then there is a unique translation-invariant probability measure on \( G \), which is called the Haar measure.

Suppose that \( H = f(h) \) is distributed according to Haar measure, how is \( H f(-\theta h) \) distributed? The inverse of \( f(x) \) defined by Eq. (2) is

\[
f^{-1}(x) = i \frac{1 - x}{1 + x}.
\]

The spectral decomposition of the \( N \times N \) unitary matrix \( H \) can be expressed in the following two ways. On the one hand, it is

\[
H = \exp(i\Phi) = \sum_{\alpha=1}^{N} \lambda_{\alpha} |\psi_{\alpha}\rangle\langle\psi_{\alpha}|, \quad \lambda_{\alpha} = e^{i\varphi_{\alpha}} \tag{7}
\]

where \( \varphi_{\alpha} \in \mathbb{R} \) are the eigenvalues of the Hermitian matrix \( \Phi \). On the other, using the Cayley transform (Eq. (3)), it is

\[
H = f(h) = \sum_{\alpha=1}^{N} \frac{1 + ih_{\alpha}}{1 - ih_{\alpha}} |\psi_{\alpha}\rangle\langle\psi_{\alpha}|, \quad h = h^{\dagger} \tag{8}
\]

Let us define \( H_{\theta} \equiv f(-\theta h) \) whose eigenvalues we denote by \( \mu_{\alpha} = e^{i\nu_{\alpha}} \).

Since \( H \) is a normal matrix and \( f^{-1} \) exists, we use the same basis \( |\psi_{\alpha}\rangle \) for \( H, H_{\theta}, h, \) and \( \Phi \). The eigenvalues of the Hermitian matrix \( h = f^{-1}(H) \) are \( h_{\alpha} = f^{-1}(\lambda_{\alpha}) = \tan(\varphi_{\alpha}/2) \). Consequently \( H_{\theta} \) has the eigenvalues

\[
\mu_{\alpha} \equiv e^{i\nu_{\alpha}} = f(-\theta \tan(\varphi_{\alpha}/2)) = \frac{1 - i\theta \tan(\varphi_{\alpha}/2)}{1 + i\theta \tan(\varphi_{\alpha}/2)},
\]

\[
\nu_{\alpha} = -2 \arctan(\theta \tan(\varphi_{\alpha}/2)).
\]

The transformation of \( f(h) \) to \( f(h)f(-\theta h) \) amounts to transformation of the eigenvalues from \( \lambda_{\alpha} = e^{i\varphi_{\alpha}} \) to \( \exp(i(\varphi_{\alpha} + \nu_{\alpha})) = \exp\{i[\varphi_{\alpha} - 2\arctan(\theta \tan(\varphi_{\alpha}/2))]\} \). Note that \( \theta = 0 \) recovers eigenvalues of \( H \) and \( \theta = 1 \) gives the identity matrix.

**Lemma 1.** Consider the \( N \times N \) operator \( G(\theta) \equiv f(h)f(-\theta h) \), where \( H = f(h) \) is distributed according to the Haar measure. Then for \( \theta \ll 1 \), the distribution over \( G(\theta) \) is \( O(\theta) \)–close to the Haar measure in total variation distance.

**Proof.** If \( H \) in Eq. (7) is from the Haar measure, it’s eigenvalue density \( \mu(\lambda(\varphi)) \) is given by Weyl’s formula [21] (also see Diaconis-Shahshahani (1994) [9])

\[
\mu(\varphi) = \frac{1}{N!} \prod_{\alpha > \beta} \left| e^{i\varphi_{\alpha}} - e^{i\varphi_{\beta}} \right|^{2} \prod_{\alpha=1}^{N} d\varphi_{\alpha}, \tag{9}
\]
The Jacobian of transformation from the variables $\phi_a$ to $\phi_a + \nu_a$ is $\mathbf{J} = \left| \prod_{a=1}^{N} I_a \right|$, which writes

$$I_a = 1 - 2\theta \left[ 1 + \cos(\phi_a) + \theta^2 (1 - \cos(\phi_a)) \right]^{-1} \approx 1 - \frac{2\theta}{1 + \cos \phi_a} + O(\theta^2).$$

∴ $J = 1 - 2\theta \sum_{a=1}^{N} \frac{1}{1 + \cos \phi_a} + O(\theta^2)$.

Moreover, $|e^{i\phi_a} - e^{i\phi_B}|^2 = 2(1 - \cos(\phi_a - \phi_B))$ and we have

$$|e^{i\phi_a} - e^{i\phi_B}| \approx |e^{i\phi_a} - e^{i\phi_B} - 2i\theta \left( e^{i\phi_a} \tan(\frac{\phi_a}{2}) - e^{i\phi_B} \tan(\frac{\phi_B}{2}) \right) + O(\theta^2).$$

$$|e^{i\phi_a} - e^{i\phi_B}|^2 \approx |e^{i\phi_a} - e^{i\phi_B}|^2 - 16\theta \left[ \sin(\phi_a/2) \sin(\phi_B/2) \sin(\phi_a - \phi_B) \right] + O(\theta^2)$$

Therefore, the eigenvalue distribution over unitaries generated by $f(h)f(-\theta h)$ is $d\tilde{\mu}(\nu(\phi)) = d\mu(\phi) + O(\theta)$. Specifically it is,

$$d\tilde{\mu}(\nu(\phi)) = \frac{|J|}{N!} \prod_{a < b} \left| e^{i\phi_a} - e^{i\phi_B} \right|^2 \prod_{a=1}^{N} d\phi_a$$

$$= d\mu(\phi) \left\{ 1 - \sum_{a=1}^{N} \frac{2\theta}{1 + \cos \phi_a} + O(\theta^2) \right\} \prod_{a < b} \left( 1 - \theta g(\phi_a, \phi_B) + O(\theta^2) \right), \quad (10)$$

$$g(\phi_a, \phi_B) \equiv \frac{16 \csc \phi_a \csc \phi_B \sin(\phi_a/2) \sin(\phi_B/2) \sin(\phi_a - \phi_B)}{|e^{i\phi_a} - e^{i\phi_B}|^2}.$$  

This is point-wise true and proves the claim. \hfill \Box

III. AVERAGE-CASE #P-HARDNESS OF RANDOM CIRCUIT SAMPLING (RCS)

It is known that there exist local quantum circuits with $n$ qubits whose probability amplitudes are #P-Hard to estimate to within $1/\text{poly}(n)$ multiplicative error \[1, 4\]. By a quantum circuit we have in mind a specific architecture $A$ that implements a specific unitary transformation of the initialized qubit state-vector $|0\rangle^{\otimes n}$. This is instantiated by a quantum circuit, $C_A$, which for simplicity one can assume is a fixed architecture and simply denote it by $C \equiv C_A$.

Mathematically, $C$ has multiple (say $m$) layers, each corresponding to an operation in the course of the quantum computation. Therefore, $C$ is a product of $m$ unitary operators $C = C_mC_{m-1} \cdots C_2C_1$, where each $C_k = I \otimes U_k$ implements a local unitary that acts non-trivially on one or two qubits. The tensor products with identity ensure that all $C_k$ have size $2^n$ despite each of the local unitaries $U_k$ being $4 \times 4$ or $2 \times 2$.

**Definition 1.** By random circuit sampling problem (RCS), one means sampling from the output distribution of a circuit, whose local gates are random unitaries.
The goal is to prove the following conjecture:

**Conjecture 1.** *(Informal supremacy conjecture)* Approximating to $O(2^{-n}/\text{poly}(n))$ additive error of most amplitudes of most quantum circuits is a $\#{\text{P}}$-hard problem.

The statement we prove in this section informally reads:

**Theorem.** *(Informal statement of the theorem)* There exists a low degree polynomial $f(n)$ such that approximating to $O(2^{-f(n)})$ additive error of most amplitudes of most quantum circuits is a $\#{\text{P}}$-Hard problem.

**Remark 2.** The polynomial $f(n)$ is explicitly computed below (Subsection III C) and distinguishes the theorem from the stated conjecture. For Google’s planned near-term experimental demonstration of quantum supremacy our approximation becomes $2^{-f(n)} = 2^{-\Theta(n^{3.51})}$. Improving this to $O(2^{-n}/\text{poly}(n))$ would prove the supremacy conjecture.

The proof structure is as follows. We start with the worst case local circuit corresponding to the $\#{\text{P}}$-hard instance, which is known to exist [4,13]. Let us denote by $\mathcal{A}$ its architecture; the exact specification of the circuit is irrelevant for our purposes.

Our goal is to show that if the local gates are unitaries drawn independently from the Haar measure, then sampling from the output distribution remains $\#{\text{P}}$-Hard with sufficiently high probability. This done by $\theta$—deforming the worst case circuit, using the Cayley path, to the circuit $C(\theta)$ such that $C(0)$ is a random circuit with Haar local gates and $C(1)$ is the deterministic fixed worst case circuit that we picked. By construction, the proposed deformation leads to probabilities that are easy to determine by sampling because of their low degree algebraic form. Therefore, the average case should also be $\#{\text{P}}$-Hard, because otherwise $C(1)$ would be easy to compute. Because one can first determine the algebraic function $|\langle 0|C(\theta)|0 \rangle|^2$ by sampling it efficiently very near $\theta = 0$ (i.e., generic instances), and then plug in $\theta = 1$ to determine $C(1)$ [15].

**A. Formal results**

For any circuit $C$, one can insert a complete set of basis between each $C_k$ and $C_{k+1}$ and represent the circuit in what is at times called “Feynman path integral” form [3]. The amplitude corresponding to the initial state $|y_0\rangle$ and final state $|y_m\rangle$ is

$$
\langle y_m|C|y_0 \rangle = \sum_{y_1,y_2,\ldots,y_{m-1} \in \{0,1\}^n} \langle y_m|C_m|y_{m-1}\rangle \langle y_{m-1}|C_{m-1}|y_{m-2}\rangle \cdots \langle y_1|C_1|y_0\rangle.
$$

(11)
Definition 2. (Haar random circuit distribution) Let $\mathcal{A}$ be an architecture over circuits and let $\mathcal{H}_A$ be the distribution over circuits in $\mathcal{A}$ whose local gates, denoted by $H_k$, are independently drawn from the Haar measure.

The random circuit sampling is then the following task:

Definition 3. (Random Circuit Sampling (RCS)) Given an architecture $\mathcal{A}$, a description of a random circuit $C$ in $\mathcal{H}_A$, and an error parameter $\epsilon > 0$, sample from the output probability distribution induced by $C$. That is draw $y \in \{0, 1\}^n$ with probability $\Pr(y) = |\langle y|C|0^n \rangle|^2$ up to a total variation distance $\epsilon$ in time $\text{poly}(n, 1/\epsilon)$.

In RCS one seeks estimations of $|\langle y|C|0^n \rangle|^2$ but any bit string $|y\rangle$ is simple to obtain by applying Pauli X matrices to positions in $|0^n\rangle$ that correspond to 1’s. By this, so called ‘hiding property’ [1], it is sufficient to prove the hardness of computing

$$p_0(C) = |\langle 0^n|C|0^n \rangle|^2.$$  \hfill (12)

Conjecture 2. (Quantum Supremacy Conjecture [1, 3]) There is no classical randomized algorithm that performs RCS to inverse polynomial (in $n$) total variation distance error.

Remark 3. Conjecture (1) implies Conjecture (2) assuming Polynomial Hierarchy is infinite [1].

To ultimately prove this, intermediate steps have been taken; for example, the estimation of the exact amplitudes of the worst case circuit are $\#P$-Hard. More recently, it was proved that the exact average case RCS is also $\#P$-Hard [3, 15]. What remains open is to prove that the approximate (i.e., with polynomial in $1/n$ multiplicative error) average RCS is also $\#P$-Hard. This is what is meant by proving “robustness”.

We proceed to give a new and, to date, simplest reduction of the worst-to-average case $\#P$-hardness of RCS by extrapolation using the Cayley path introduced above. This result is free of some issues presented in [3]; for example, the truncation of the Taylor series is unsatisfactory (Subsection III B). It is even simpler than the one we gave previously [15].

Notation. Up to now $C$ denoted any circuit with a fixed architecture. Below we denote by $C$ a specific worst-case circuit, whose existence was proved previously [4] and its description is not relevant for the average-case hardness proofs that follow.

Definition 4. ($\theta$-deformed Haar towards $C$) Let $\mathcal{A}$ be the architecture of the worst case circuit $C = C_mC_{m-1} \cdots C_2C_1$. Let $\theta \in [0, 1]$ and define by $C(\theta) = C_m(\theta)C_{m-1}(\theta) \cdots C_2(\theta)C_1(\theta)$, where by
Figure 2: Schematics of the path on the unitary group induced by $U_k(\theta) \equiv U_k H_k f(-\theta h_k)$ in Def. (4).

Figure 3: Schematics of Definition 4: The scrambling of the circuit $C$ to $C(\theta)$.

Eq. (4) $C_k(\theta) = U_k(\theta) \otimes \mathbb{I}$ and $U_k(\theta) \equiv \{U_k H_k f(-\theta h_k)\}$; see Figure 2. Further, each $H_k$ is a (local) unitary drawn independently from the Haar measure and $U_k$ is the local unitary of the worst case circuit (i.e., $C_k = U_k \otimes \mathbb{I}$). We define by $\mathcal{H}_{A,C,\theta}$ the distribution over $C(\theta)$.

By the translation invariance property of the Haar measure (Sec. (II)), $C_k(0)$ implements a local unitary from the Haar measure. Therefore, the distribution over $C(0)$ coincides with $\mathcal{H}_{A,\nu}$ and $C_k(1) = C_k = U_k \otimes \mathbb{I}$, which is exactly the pre-determined $k^{th}$ layer of the worst-case circuit $C$. In summary, $C(\theta) \in \mathcal{H}_{A,C,\theta}$ with extremes (See Figure 3):
\[\theta = 0 : C_k(0) \implies C(0) \in \mathcal{H}_A\]
\[\theta = 1 : C_k(1) \implies C(1) = C \quad \text{worst case circuit.}\]

This naturally defines the deformation of Eq. (12) via
\[
p_0(C(\theta)) = |\langle 0^n|C(\theta)|0^n \rangle|^2,
\]
which at \(\theta = 0\) is the RCS problem and at \(\theta = 1\), we have the \#P-hard instance: 
\[
p_0(C(1)) = p_0(C) = |\langle 0^n|C|0^n \rangle|^2.
\]

**Lemma 2.** The total variation distance between \(\mathcal{H}_A\) and \(\mathcal{H}_{A,C,B}\) is \(O(m\theta)\) for \(\theta \ll 1\).

**Proof.** By the translational invariance of Haar measure (see Subsection (II)), if \(H_k\) is distributed according to the Haar measure then so is \(U_k H_k\) for any fixed \(U_k\). Moreover, the \(\ell_1\) norm that defines total variation distance is invariant under unitary multiplication. So it suffices to compare the measures over \(\mathcal{H}_{A,C,B}\) and \(\mathcal{H}_A\), which by Lemma (1) have TVD of \(O(\theta)\) over a single local gate. By the additivity of TVD, the distribution induced by \(C(\theta)\) which is denoted by \(\mathcal{H}_{A,C,B}\) has a TVD from \(\mathcal{H}_A\) that is \(O(m\theta)\).

**Remark 4.** In an \(n\)-qubit circuit, \(m = \text{poly}(n)\) and often \(m = O(n^2)\). Therefore, if we take \(\theta \in [0, \Gamma]\) with \(\Gamma = O(1/\text{poly}(n))\) such that \(\Gamma^{-1} = o(m)\), then we are guaranteed that the TVD between \(\mathcal{H}_A\) and \(\mathcal{H}_{A,C,B}\) is \(O(1/\text{poly}(n))\).

From Eqs. (5) and (6) we have that \(C_k(\theta) = U_k(\theta) \otimes I\). Therefore, \(\langle 0^n|C(\theta)|0^n \rangle\) is equal to
\[
\langle 0^n| \prod_{k=1}^m C_k(\theta)|0^n \rangle = \frac{1}{Q(\theta)} \sum_{\alpha_1, \ldots, \alpha_m = 1}^N \langle 0^n| \prod_{k=1}^m p_{\alpha_k}^{(k)}(\theta) \left[ \left( U_k |\psi_{\alpha_k}^{(k)} \rangle \langle \psi_{\alpha_k}^{(k)} | \right) \otimes I_k \right] |0^n \rangle
\]
\[
Q(\theta) \equiv \prod_{k=1}^m q_k(\theta) = \prod_{k=1}^m \{ \prod_{a_k=1}^N (1 + i\theta h_{a_k}^{(k)}) \}
\]
\[
p_{\alpha_k}^{(k)}(\theta) \equiv f(h_{\alpha_k}^{(k)})(1 - i\theta h_{\alpha_k}^{(k)}) \prod_{\beta \in [N]/a_k} (1 + i\theta h_{\beta}^{(k)}),
\]
where \(H_k = f(h^{(k)})\) and as before \(h_{\alpha_k}^{(k)}\) and \(|\psi_{\alpha_k}^{(k)} \rangle\) are the eigenpairs of the Hermitian matrix \(h^{(k)}\), and \(I_k\) denotes the trivial action of \(U_k\) on all other qubits. \(Q(\theta)\) is a polynomial of degree at most \(Nm\) (recall that \(N \in \{2, 4\}\) for local quantum circuits), which can be classically pre-computed in time \(O(m)\).

Clearly, \(p_0(C(\theta))\) as defined by Eq. (14) is a rational function of degree at most \(8m, 8m\) because of taking the square of the absolute value. Moreover, from Eq. (15), we have
\[
\langle 0^n|D(\theta)|0^n \rangle = Q(\theta) \langle 0^n|C(\theta)|0^n \rangle,
\]
(17)
where $D(\theta) \equiv \sum_{n_1, \ldots, n_m=1}^{N} \prod_{k=1}^{m} p_{n_k}^{(k)}(\theta) \left[ \left( U_k \mid \psi_{n_k}^{(k)} \rangle \langle \psi_{n_k}^{(k)} \right) \otimes I_k \right]$ is a $(2^n \text{ dimensional})$ matrix whose entries are polynomials in $\theta$.

**Remark 5.** In the following theorem, we prove $\#P$–Hardness of exact RCS. By exact one means that in sampling the tuples (near zero) $(\theta_i, p_0(C(\theta_i)))$ one commits no errors. Besides the obvious physical infeasibility, this is quite unrealistic from a theoretical perspective as well; when working over an infinite field, numerical round-offs are inevitable. In the quantum supremacy proofs that have so far been given $[1, 3, 15]$, polynomial reconstruction schemes such as Berlekamp-Welch or Sudan’s list decoding have been deployed $[19, 20]$. These require that a fraction of the tuples are exact (the rest may be erroneous); which is what one needs for such oracle-based proofs. Although realistic over finite fields (e.g., as in Reed-Solomon codes), these are quite inadequate for working over the complex numbers. It seems to me that to prove any tolerance with respect to errors we should altogether drop these schemes. In Subsection (III C), we prove a small robustness without appealing to such schemes.

**Remark 6.** The procedure is that we are given a fixed worst case circuit $C$ with the architecture $A$ and whose $m$ local gates (i.e., $U_k$‘s) are published. We then draw a corresponding set of $m$ local gates independently from the Haar measure (i.e., $H_k$‘s) and treat them as fixed. The latter is a realization of an average-case circuit with architecture $A$.

**Theorem 1.** Let $A$ be an architecture such that computing $p_0(C)$ is $\#P$–Hard in the worst case. Then it, is $\#P$-hard to exactly compute $3/4 + 1/\text{poly}(n)$ of the probabilities $p_0(C(\theta))$ over $H_A$.

**Proof.** Take $\theta \in [0, 1/\text{poly}(n)]$ and pick $m$ local gates $U_k(\theta)$ according to Def. (4). For any choice of such small $\theta$, this induces a circuit distributed according to $H_{A,\theta,C}$, which by Lemma (1) is $O(m\theta)$ close to the $H_A$ in TVD. However, at $\theta = 1$ we recover the deterministic worst-case circuit. From the calculation above $p_0(C(\theta))$ is a rational function of degree $(8m, 8m)$, which is low degree when $m = \text{poly}(n)$. Suppose we have at our disposal a classical oracle $O$ that takes as the input the (efficient) classical description of a circuit $C(\theta)$ and for all $\delta = O(1/\text{poly}(n))$ has the computational capability:

$$\text{Pr}_{C(\theta) \in H_{A,\theta}} \left[ O(C(\theta)) = p_0(C(\theta)) \right] \geq 3/4 + \delta.$$  

Then if $O$ succeeds over $1/2 + \delta$ choices of $H_k$ then, it should succeed over $1/2 + \delta$ choices of $\theta$ as well. This sets the error rate for the Berlekamp-Welch algorithm that was proved for rational functions elsewhere $[15]$. Since $1/2 - \delta$ of $\theta$‘s are assumed erroneous, the minimum number of independent $\theta$‘s that are needed to exactly determine the rational function $p_0(C(\theta))$ is $8m\delta^{-1}$. This
is considered efficient as it depends polynomially on $n$ (in practice often $m = O(n^2)$). This way we can extrapolate to the point $p_0(C(1))$, which is a solution of a \#P-hard problem. This shows that computing $p_0(C(\theta))$ for $3/4 + \delta$ fraction of the circuits must have been \#P-hard as well. In \cite{15}, it was shown that an algorithm that works on average over circuits distributed according to $\mathcal{H}_{A,\theta,C}$ can be used to get an algorithm that works on average over circuits distributed as $\mathcal{H}_A$. □

Remark 7. As shown in \cite{15}, Theorem (1) is necessary for the quantum supremacy conjecture.

Remark 8. As remarked in the original BosonSampling paper \cite{11}, it is entirely possible that the above theorem for RCS may be strengthened to allow for an oracle with the success probability of $1/2 + 1/\text{poly}(n)$ or even $1/\text{poly}(n)$ using results in \cite{12} and \cite{5} respectively.

Remark 9. Since in this work we do not need to truncate Taylor series (truncation was necessary in \cite{3} main theorem), in our main theorem we need not assume that the calculation of $p_0(C)$ is \#P-hard to within some $2^{-\text{poly}(n)}$ additive error. This assumption is extra and has nothing to do with $\exp(-\text{poly}(n))$ resilience with respect to noise (i.e., robustness). See below.

\section*{B. Inadequacy of Taylor series truncation}

In Bouland et al's paper, a proof of the average case hardness of RCS was given based on the truncation of the Taylor series expansion as detailed below \cite{3}. Only in this section we employ their notation. So we denote a local gate by $C_j$ and a Haar matrix by $H_j$. Their interpolation writes

$$C_j(\theta) = C_j H_j e^{-i h_j \theta},$$

where $C_j(\theta)$ is a unitary, $C_j$ is the $j$th gate of the worst-case circuit, $H_j$ is a local Haar unitary, and $\exp(-i h_j) = H_j^T$. The full unitary circuit is $C(\theta) = C_1(\theta)C_2(\theta) \cdots C_n(\theta)$. Therefore, $C(0) \in \mathcal{H}_A$ and $C(1) = C$ is the worst case circuit. Since the exponential function has a power series, in order to obtain a polynomial, they truncate the Taylor series expansion of $\exp(-i h_j \theta)$ at the $K$th order

$$C_j^T(\theta) = C_j H_j \left\{ \sum_{k=0}^K \frac{(-i h_j \theta)^k}{k!} \right\}. \quad (18)$$

There are two issues worth emphasizing:

1. This leads to non-unitary local gates, and therefore a non-unitary circuit $C'(\theta) = C'_1(\theta)C'_2(\theta) \cdots C'_n(\theta)$.
2. In using Berlekamp-Welch or other such schemes some fraction of points need to be known exactly; therefore any truncation renders Berlekamp-Welch useless.
In order to reduce the complexity of the \#P-hard problem to average case, one needs to assume that there is an oracle that exactly computes $p_0(C(0))$ where the local gates are Haar distributed. The first issue with a non-unitary circuit is that this oracle cannot be called. So they assume a different oracle that exactly computes $p_0(C'(0))$. Then the claims is that the extrapolations (i.e., $p_0(C'(0))$ ) is sufficiently close to $p_0(C)$. This easily leads to the bound \[3\]

$$|p_0(C'(1)) - p_0(C)| \leq \frac{2^{O(mn)}}{K!} \approx e^{O(mn - K \ln K)}.$$  

(19)

They use the above construction and to obtain a robustness with respect to noise of $O(\exp(-\text{poly}(n)))$. Their robustness proof relies on Paturi’s lemma \[16\] and Rakhmanov’s bounds \[18\]. Let the polynomial $p(\theta) = p_0(C'(\theta)) - p_0(C(\theta))$, then Paturi’s lemma says

**Lemma. (Paturi)** Let $p(\theta)$ be a polynomial of degree $d$, and suppose $|p(\theta)| \leq \epsilon$ for $|x| \leq \Delta$. Then $p(1) \leq \epsilon \exp[2d(1 + \Delta^{-1})]$.

In general, the robustness claims correspond to the supremum of $\epsilon$.

The truncation of the Taylor series introduces an error of $e^{O(mn - K \ln K)}$ in $\epsilon$ as shown above. There can also be other errors resulting from noisy polynomial sampling, numerical round offs, or experimental limitations. But for what follows we can even ignore these other sources and just focus on the consequences of the truncation and take the noise in sampling to be

$$\epsilon = e^{O(mn - K \ln K)}$$

Indeed one can treat $K$ as a free variable and make it sufficiently large as to compensate for $mn$ in the exponent of Eq. \[19\] This would lead to exponentially small errors in computing Eq. \[19\] as claimed in \[3\]. Then, in order to sample from distributions near the Haar measure, they take $\Delta = 1/\text{poly}(n)$ as an independent free variable in Paturi’s lemma. By the above considerations, we would need to take $d = 2mK$ and this results in

$$p(1) \leq \exp[O(mn - K \ln K)] \exp[2mK(1 + \Delta^{-1})].$$

For the error not to blow up, one must require that $K \geq O(\exp(2m(1 + \Delta^{-1}))) = O(\exp(n))$. Therefore, to determine the polynomial, exponential work is necessary.

Another way to proceed is as the Berkeley group did \[3\] and assume an oracle that acts on non-unitary “circuits” which for $\theta \ll 1$ computes $p_0(C'(\theta))$ exactly. This is somewhat unnatural as the oracle would take as inputs a classical description of a non-unitary “circuit” because of the
truncation of the Taylor series to the $K^{th}$ order (see Eq. [18]). Is $p_0(C'(1)) = |\langle 0^n|C'(1)|0^n\rangle|^2$ within $1/poly(n)$ multiplicative approximation of the $|\langle 0^n|C|0^n\rangle|^2$, which is known to be #P hard? This would have to be the case for the reduction to work; otherwise the reduction to the average case is from an instance of an unknown complexity class in which $p_0(C'(1))$ belongs. We show that this is indeed the case. We have

$$\langle 0|C'(1)|0 \rangle = \langle 0| \prod_{j=1}^{m} C_j H_j \ell_{K,j}(h_j \theta |_{\theta = 1}) |0 \rangle$$

(20)

where $\ell_{K,j}(\theta)$ is the $K^{th}$ order polynomial approximation to $H_j^+ = e^{-ih}$ and writes

$$\ell_{K,j}(\theta) = \sum_{r=0}^{K} \frac{(-ih_j \theta)^r}{r!} = e^{-ih_j \theta} - \eta_{K,j}(\theta);$$

with the truncation error being

$$\eta_{K,j}(\theta) = \sum_{r=K+1}^{\infty} \frac{(-ih_j \theta)^r}{r!}; \quad \|\eta_{K,j}(\theta)\| \leq \frac{O(||h_j^{k+1}||)}{(K+1)!}.$$ 

Putting these in Eq. (20) we have

$$\langle 0|C'(1)|0 \rangle = \langle 0| \prod_{j=1}^{m} C_j \{I - H_j \eta_{K,j}(1)\} |0 \rangle = \langle 0|C|0 \rangle - R_{K,m} + O\left(\frac{O(||h_j^{2k+2}||)}{[(K+1)]^2}\right),$$

where

$$R_{K,m} = \langle 0| \sum_{j=1}^{m} C_1 \ldots C_{j-1}[C_j H_j \eta_{K,j}(1)] C_{j+1} \ldots C_m |0 \rangle.$$ 

Therefore, ignoring higher order corrections we have

$$|\langle 0|C|0 \rangle|^2 = |\langle 0|C'(1)|0 \rangle|^2 + |R_{K,m}|^2 + \{ \langle 0|C'(1)|0 \rangle R_{K,m} + R_{K,m} \langle 0|C'(1)|0 \rangle \}. $$

The terms that depend on $R_{K,m}$ can be made exponentially small such that $|p_0(C'(1)) - p_0(C(1))|$ is within the proven bounds [10].

### C. Proof of robustness

In making claims about robustness, it is important be clear about the procedure. We have a $C$ and a set of gates $H_k$ for $k \in [m]$. These are fixed and efficient to describe (e.g., publish) classically. We then choose a set of $\theta_i \in [0, \Delta]$, and for each $\theta_i$, using a classical computer, we efficiently calculate $Q(\theta_i)$ in Eq. (16) to whatever accuracy we desire. Let $D(\theta) = Q(\theta)C(\theta)$; this matrix has entries that are polynomials of degree $4m$ by construction (Eq. (16)).
Lemma 3. Evaluation of $Q(\theta)$ takes $\Theta(m)$ time. Therefore the computational complexity of $p_0(C(\theta))$ and $p_0(D(\theta))$ are equivalent to within a $\Theta(m)$ overhead.

**Proof.** After picking the $m$ Haar local gates, in $O(m)$ time we diagonalize all of the $H_k$’s. Treating $\theta$ as a real variable, we write down the polynomial $Q(\theta)$, whose coefficients only depend on the eigenvalues of $H_k$’s. Therefore, specifying $H_k$’s uniquely specifies the polynomial $Q(\theta)$. Given the polynomial $p_0(D(\theta))$, we calculate $p_0(C(\theta)) = p_0(D(\theta))/|Q(\theta)|^2$. Clearly the degree of $p_0(D(\theta))$ is at most $16m$. Conversely, given $p_0(C(\theta))$ we have $p_0(D(\theta)) = |Q(\theta)|^2p_0(C(\theta))$. \hfill \qed

Then using the quantum computer one efficiently evaluates $(\theta, p_0(C(\theta))+\epsilon_i)$, where $\epsilon_i$ are the errors committed in sampling. The source of these may be finite precision, experimental limitations, noise, etc. Therefore, we have at our disposal a set of tuples $(\theta, p_0(D(\theta))+\epsilon_i Q(\theta))$. The problem reduces to the polynomial determination of $p_0(D(\theta))$ with a scaled noise $\epsilon_i Q(\theta)$.

**Remark 10.** We could stop at this point and simply remark that *any* claim or proof of robustness, that presupposes a reduction based on polynomial extrapolation to the exact worst case problem, directly extends to our work which is based on rational functions. That is, similar robustness can be claimed. The only caveat is that the value of $|Q(\theta)|^2$ should not blow up, and consequently amplify the noise out of control. Below we prove this and, take the opportunity to, make more explicit the amount of robustness that can be gained.

Assume we have an oracle $O_2$ that to within additive error $\epsilon = \max_i \epsilon_i$ has the property:

$$\Pr[O_2(C(\theta_i)) = p_0(C(\theta_i))] = 1 - 1/poly(n).$$

Therefore, we are guaranteed to have a set of $(\theta_i, p_0(D(\theta_i)) + \epsilon_i Q(\theta_i))$ with high probability. There is a subtle question: Can the difference of the exact and sampled polynomials be drastically different in $\theta \in [0, \Delta]$ despite agreeing well (i.e., difference upper-bounded by $\max_i \epsilon_i Q(\theta_i)$) at the sampled points? This is not hard to remedy and if one samples $\theta_i$ uniformly, then by a theorem due to Rakhmanov we are also guaranteed that the two polynomials are close to one another everywhere in $[0, \Delta]$ \cite{Rakhmanov}.  

Now using Paturi’s lemma and Rakhmanov’s result we have \cite{Paturi} \cite{Rakhmanov}

$$|p_0(D_{exact}(1)) - p_0(D_{noisy}(1))| \leq \epsilon Q_{max}^2 e^{8m(1+1/\Delta)}$$

(21)

where $Q_{max} \equiv \max_{\theta \in [0,\Delta]} |Q(\theta)|$.

**Lemma 4.** $Q_{max} \geq 1$, and with probability $1 - 1/poly(n)$, $|Q(\theta)|^2 \leq poly(n)$ for $\theta \in [0,1/poly(n)]$. 

Proof. It is easy to see that \( Q_{\max} \geq 1 \) since
\[
|Q(\theta)|^2 = \prod_{k=1}^{m} \prod_{a_k=1}^{N} \left| 1 + i\theta h_{a_k}^{(k)} \right|^2 \geq 1.
\]
Now the upper-bound: \( |Q(\theta)|^2 = 1 + \theta^2 \sum_{k=1}^{m} \sum_{a_k=1}^{N} h_{a_k}^{(k)} + O(\theta^4) \). Recall from Section [II], \( h_{a_k} = \tan(\varphi_{a_k}/2) \) and \( \exp(i\varphi_{a_k}) \) are the eigenvalues of the \( N \times N \) Haar unitary \( H \). Under the Cayley transformation, large values of \( h_{a_k} \) aggregate near \( \varphi_{a_k} = \pm \pi \). From the basic properties of the Haar measure, it is not hard to show that the number of eigenvalues falling in an interval on the unit circle is \( N \) times the length of the interval:
\[
\Pr[\varphi \in [-\pi + \bar{\Delta}, \pi - \bar{\Delta}]] = 1 - \frac{\bar{\Delta}}{\pi}.
\]
Taking \( \bar{\Delta} = 1/poly(n) \), with probability arbitrary close to one we have
\[
h_{a_k} = \tan(\varphi_{a_k}/2) \leq \cot(\frac{\bar{\Delta}}{\pi}) \approx \pi \bar{\Delta}^{-1} = poly(n).
\]
This establishes that \( |Q(\theta)|^2 \leq 1 + \theta^2 mN\pi\bar{\Delta}^{-1} \). As before \( \theta \in [0, \Delta = 1/poly(n)] \) and taking \( \Delta^{-1} \) to have a polynomial of sufficiently high degree, (recall that we think of \( N \) as a positive integer \( N \leq 4 \)
\[
|Q_{\max}|^2 \leq 1 + N\pi \left( m\Delta^2\bar{\Delta}^{-1} \right)
\]
\[\blacksquare\]

For example, we can take \( \bar{\Delta} = 1/n \) and assuming \( m = n^2 \), we choose \( \Delta = n^{-(3/2+\tau)} \) for any constant \( \tau > 0 \) to be assured that \( |Q_{\max}|^2 = 1 + O(n^{-\tau}) \) with probability arbitrary close to one.

Theorem 2. Assuming access to an oracle \( O_2 \) as described above, it is \#P-hard to compute \( p_0(C(\theta)) \) over \( \mathcal{H}_A \) to within \( e = \exp \left\{ -\Theta \left( m\Delta^{-1} \right) \right\} \) additive error.

Proof. Using the foregoing analysis in Paturi’s inequality given by Eq. (21) we have
\[
|p_0(D_{\text{exact}}(1)) - p_0(D_{\text{noisy}}(1))| \leq e \left\{ e^{8m(1+1/\Delta)} \right\},
\]
which is resilient to noise of \( e = \mathcal{O}(-\Theta(m\Delta^{-1})) \). This then guarantees that
\[
|p_0(C_{\text{exact}}(1)) - p_0(C_{\text{noisy}}(1))| \leq \exp(-n),
\]
which is well within the interval that is known to be \#P-Hard. If we make the same choices as in the examples following Lemma (4), we find that for any choice of small \( \tau > 0 \)
\[
|p_0(D_{\text{exact}}(1)) - p_0(D_{\text{noisy}}(1))| \leq e \left\{ e^{\Theta(n^{7/2+\tau}) + o(1)} \right\}.
\]
Therefore, our scheme is certainly resilient to noise $\epsilon = \exp(-\Theta(n^{3.51}))$ for parameters being used in the experimental test of quantum supremacy in the near future.

Note that the oracles $O$ and $O_2$ above do not need to succeed with probabilities $3/4 + 1/poly(n)$ and $1 - 1/poly(n)$ respectively over all circuits with the given architecture. For example, although $O_2$ succeeds with probability close to one, it is required to do so over circuits distributed close to $H_A$. Also to prove robustness we did not use Berlekamp-Welch algorithm as is customary because of its inherent limitations discussed in Remark (5).

In Lemmas such as Paturi’s, the source of $\epsilon$ error is abstracted away; one can attribute the noise to the sampling imprecisions, finite-precision representation of the polynomial coefficients etc..

IV. DISCUSSIONS AND CONCLUSIONS

One envisions utilities for the Cayley circuit deformation or the QR-deformation in [15] beyond proving average case quantum supremacy. It would be interesting to see applications in circuit hiding, encryption protocols such as blind quantum computation, and quantum computation by (extra)interpolation. Moreover, it gives experimentalists an efficient technique for smoothly deforming a gate or a circuit of a given architecture from an initial to a target. This could help quantify the power of a quantum computation as a function of the architecture.

In this paper, we extended the constructions in [15] for interpolating between unitaries (e.g., quantum gates or circuits) by utilizing the Cayley transform. This resulted in a rational-valued path that relates the worst-case complexity to the average-case. Since the worst-case was known to be $\#P$ hard and rational function determination is efficient (proved in [15]), the average case then must also $\#P$-Hard. We also showed that the unitary matrices that result may be controlled to have distributions arbitrary close to the Haar measure in total variation distance.

The explicit construction above gave precise quantification of the error robustness, which avoids using large asymptotic polynomial/exponential type arguments commonly used in theory. The analysis here gives control over the scaling and asymptotic. Since in the near-term the number of qubits is something like $n \sim 100$, the exact quantification might prove helpful for the experiment. What is really needed to prove the supremacy conjecture is to improve our tolerance of $\epsilon = \exp(-\Theta(n^{7/2+\tau}))$ to $\epsilon = O(2^{-n/poly(n)})$. However, the quantum supremacy conjecture may be false or experimentally unrealistic. This would be implied if it were shown that our reduction based on the Cayley path is optimal.
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* Electronic address: ramis@us.ibm.com

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