Modeling of the Inductance of a Blumlein Circuit Spark Gap

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Abstract. In this paper we present an analysis of the time-varying inductance in the spark gap of a Blumlein circuit. We assume several mathematical expressions to describe the inductance and compare theoretical and computational calculations with experimental results. The time-varying inductance is approximated by a constant, a straight line and two parabolas which differ in their concavity. This is the first time to our knowledge, in which the time-varying ignition inductance of a nitrogen laser is modeled.

1. Introduction

It is known that spark gaps are mainly used as fast switches in many circuits and in gas lasers such as CO$_2$ and N$_2$. The Blumlein nitrogen laser system is an example of a simple working tool used in many different branches of science [1-5].

Within a Blumlein gas pulsed laser there are two electrical discharge elements that play an important role on the laser efficiency, these are: the spark gap and the laser spine. Electrically a gas discharge represents a time-dependent resistance and inductance in series. However all electric analyses of these systems assume constant values for the inductance which we know is inaccurate. Experimentally it has been shown [6] that the inductance varies in a complicated manner in time. No doubt it is a difficult task the modeling of this element but an accurate electric analysis implies that the ignition system must be replaced by elements with a time variation.

In the literature there are few reports of inductance modeling in the spark gap. An exception is provided in [7-8] where it is show that assuming an arc-discharge with an electron Gaussian distribution, the resistance and inductance can be modeled as:

\[ R = \frac{\mu_0}{4\pi} \int_{\varphi=0}^{\infty} \left( 1 - e^{\lambda \varphi^2} \right)^2 \frac{d\varphi}{\varphi} \int_{x=0}^{\frac{\pi}{2}} \left( \frac{\alpha}{\sqrt{\alpha^2 + \varphi^2}} + \frac{\beta}{\sqrt{\beta^2 + \varphi^2}} \right)^2 dx + \frac{\mu_0}{4\pi} \int_{\varphi=0}^{\infty} \left( 1 - e^{\lambda \varphi^2} \right)^2 \frac{d\varphi}{\varphi} \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\alpha}{\sqrt{\alpha^2 + \varphi^2}} + \frac{\beta}{\sqrt{\beta^2 + \varphi^2}} \right)^2 dx \]

\[ L = \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\alpha}{\sqrt{\alpha^2 + \varphi^2}} + \frac{\beta}{\sqrt{\beta^2 + \varphi^2}} \right)^2 dx \]
Where R is the resistance, l is the inter-electrode distance, e is the electron charge, μ the electron mobility, N_t the total charge in the discharge channel, L the inductance, μ_o the magnetic permeability, φ the radial distance in cylindrical coordinates (perpendicular to l), α = (l + 2x)/2, β = (l – 2x)/2, λ = ln(2/r_o) and r_o is the radius of the cylindrical channel where the charge distribution has a value half its maximum value on the cylinder axis and is considered the radius of the discharge channel. Using these expressions it is possible to obtain a (constant) value for the resistance and inductance which is only an approximation.

The purpose of this article is to discuss possible analytic approximations for the time-dependent inductance. The objective is to find mathematically tractable expressions which may be used with an accuracy of ninety percent in the theoretical analysis of circuits involving spark gaps such as the Blumline gas laser. The temporal functions of inductance here discussed are approximations of the experimental time-dependent inductance observed.

This paper is divided into five sections. Section two describes the Blumlein circuit and explains the principle of operation of the ignition system. Section three presents the experimental and computational results of the voltage across one of the capacitors of the Blumlein circuit. This is done for each one of the inductance approximations. In section four, we comment on the agreement between the experimental data and the approximations done. Finally in section five we present our conclusions about which time dependent inductance approximation is more advisable to use both in relation to its mathematical simplicity and its accuracy.

1. Development
The laser Blumlein circuit is shown in figure 1. Here L.H. is the Laser tube Head, S.G is the Spark Gap or ignition system. This element consists of two electrodes separated by some distance. High voltage (H.V.) is fed to the system through R and L applying the same voltage across S.G. loading capacitors C_2 and C_1. When this voltage is sufficiently high to form an arc in the S.G. capacitor C_1 is discharged across S.G. and therefore is grounded.

![Figure 1. Electrical circuit Blumlein-line N2 laser.](image)

Inductor L avoids the discharge of C_2 through it, therefore instantaneously this capacitor will be at a higher voltage than C_1 (which is grounded), therefore the potential difference across this last element causes a discharge arc across the laser cavity head L.H. The process keeps repeating itself with a frequency that depends on the separation between the electrodes of the S.G.

The electric Blumlein circuit during the discharge of the spark gap and the laser head is shown in figure 2. Here it is noted that the L.H. discharge was replaced by a resistor R_3 and an inductor L_2. In a similar way the spark gap is replaced by a resistor R_1 and an inductor L_1. Typical values for U_o, R_1, R_2, C_1, C_2 and L_2 are: 10kV, 0.05 Ω, 0.06 Ω, 8 nF, 24.4 nF and 5 nH.
Figure 2. Equivalent electrical circuit Blumlein-line during the discharge of the spark gap and the laser head in a N₂ laser. L₁ and R₁ represent the spark gap discharge and L₂ and R₂ the laser head.

Before discharge occurs in the L.H. only the left-hand loop of the circuit (L₁, R₁ and C₁) begins to oscillate due to the discharge of the spark gap and the dynamic of the net is governed by the following equation:

\[ L₁C₁ \frac{d²U₁}{dt²} + R₁C₁ \frac{dU₁}{dt} + U₁ = 0 \]  

(3)

Where \( U₁ \) is the voltage across \( C₁ \). The particular solution of this equation making \( U₁(0) = U₀ \) and \( U₁'(0) = 0 \) is:

\[ U₁(t) = U₀e^{-at} \cos \omega t \]  

(4)

Where:

\[ a = \frac{R₁}{2L₁} \quad \omega = \left( \frac{1}{L₁C₁} - \frac{R₁²}{4L₁²} \right)^{1/2} \]

Once the discharge in the laser cavity head L.H. occurs, both nets of the above circuit are activated and the voltage starts to oscillate. Now the system dynamics is governed by the following differential equation [8]:

\[ aU₁^{(4)} + bU₁^{(3)} + cU₁^{(2)} + dU₁^{(1)} + U₁ = 0 \]  

(5)

Where:

\[ a=L₁L₂C₁C₂, \]
\[ b=L₁C₁R₂C₂ + L₂C₂R₁C₁, \]
\[ c=L₁C₁ + (L₁ + L₂)C₂ + R₁C₁R₂C₂, \]
\[ d=R₂C₁ + (R₁ + R₂)C₂. \]

The subscript of \( U \) can be 1 or 2 because the same differential equation applies for \( C₁ \) and \( C₂ \). Here attention is focused on the voltage across \( C₂ \).

The general solution of equation (5) has the form [8]:

\[ U(t) = Ae^{-a₁t} \cos \omega₁t + Be^{-a₂t} \cos \omega₂t \]  

(6)

Where \( A + B = U₀ \). As we may see solution (6) is the superposition of two oscillations.

2. Results

Figure 3 shows an example of the experimental inductance of the ignition system [6]. The dashed lines shows the square of the flowing current. Which is important for our analysis. This experimental graph was also reproduced in Matlab with 2000 sampled points. This is shown in figure 4.
As we may see from figure 4 the inductance at the spark gap is a very complex time varying function, which is not constant at all. Figure 5 shows the numeric solution of equation (6) taking the instantaneous value of inductance as is given in figure 5. This result should in principle be very close or identical to the one shown in figure 6.

Our task is to reproduce, as close as possible, the experimentally observed voltage variation at C₂ solving equation (5) using different approximations for the inductance, namely: i) constant, ii) straight line, iii) upward concave parable and iv) downward concave parable. Our objective is to find which time dependent inductance approximation is more advisable to use both in relation to its mathematical simplicity and its accuracy.

i) Approximation of inductance by a constant
The simplest approach is to take the inductance as a constant value. This constant is given by expression (7), which as we may see in figure 7 is just the average value of the experimentally observed inductance:

\[ L = 6.4408 \times 10^{-8} \text{H} \]  

(7)

Figure 8 shows the voltage obtained when solving equation (5) using this approximation. The discussion of how good is this approximation is presented in the Discussion section.
ii) Approximation of the inductance through a straight line $y=ax+b$.

The second approximation for the inductance is given by a straight line. In this case the temporal varying inductance is given by the next equation:

$$L = -6.3 \times 10^{-11} t + 7.97 \times 10^{-8} \text{[H]}$$  \hspace{1cm} (8)

This is shown in figure 9.

Numerically solving equation (5) using for the inductance the function given in expression (8) we obtain the voltage in $C_2$, which is shown in figure 10.

iii) Approximation of the inductance through an upward concave parable $y=ax^2+bx+c$.

The best fit of an upward concave parable for the time dependent measured impedance is show in figure 11. The parable equation is given by expression (9):

$$L = 7.7 \times 10^{-14} t^2 - 1 \times 10^{-10} t + 8.26 \times 10^{-7} \text{[H]}$$  \hspace{1cm} (9)

The voltage at $C_2$ obtained using expression (9) to solve equation (5) is shown in figure 12.
iv) Approximation of the inductance through an downward concave parable $y = -ax^2 + bx + c$. The best fit of a downward concave parable for the time dependent measured impedance is show in figure 13. The parable equation is given by expression (10):

$$L = -3.81 \times 10^5 t^2 + 0.19 t + 4.6 \times 10^{-8} \text{ [H]}$$  (10)

The voltage at $C_2$ obtained using expression (10) to solve equation (5) is shown in figure 14.

3. Discussion
We deal with the problem of modeling the inductance of the spark gap S.G. within a pulsed Blumein circuit laser. Four different curves were proposed to simulate the time history of the actual inductance.

The four approximation used were taken due to their mathematical simplicity, since the aim of this work is to provide researchers with a good starting point for the modeling of the spark gap inductance in future designs of lasers with Blumlein circuits. It is therefore important to make a quantitative comparison of each result obtained with different time varying inductance approximation. With this purpose we define the $r$ factor called determining factor [8]. This factor measure how closely two graphs resemble one to the other. When $r = 1$ both graphs are identical and when $r = 0$ both graphs have not relation to each other. This factor is mathematically defined as:

$$r^2 = 1 - \frac{\sum (y_i - y'_i)^2}{\sum (y_i - \mu_y')^2}$$  (11)

Where $y_i$ are the experimental results, in this case data of figure 5, $y'_i$ are the approximation values and $\mu_y'$ are the mean value of the approximation results.

The $r$ values obtained for the graphs shown in figure 8,10, 12 and 14 taking as reference figure 5 are: for the constant approximation 0.9383, for the straight line approximation 0.9729, for the upward concave parable approximation 0.97404 and finally for the downward concave parable approximation 0.9249.

We may note that all the above approximations have $r$ parameters that differ in a relatively small amount. In principle all of them are rather good approximations. We may summarize our results as follows table:
Table 1. Results of values of r

| Approximation | Kind of approximation | Value of r |
|---------------|-----------------------|------------|
| Best          | Upward parable        | r = 0.97404 |
| Excellent     | Straight line approximation | r = 0.9729 |
| Good          | Constant approximation | r = 0.938 |
| Fair          | Downward parable      | r = 0.9249 |

It is remarkable that the mathematical simplicity of a straight line is enough to guarantee very good results indeed. In fact the mathematical complexity of an upward parable does not justify - in our opinion- the very small gain in accuracy. On the other hand it is outstanding to realize that a constant approximation to the inductance gap can provide a very reasonable agreement with experimental results.

If computer power is not a limitation the greatest approximation may be obtained fitting the time varying inductance to a high degree polynomial for better accuracy

4. Conclusions
In this work we tested several approximations for the time-dependent inductance of the spark gap in a Blumlein laser. First we substituted an average constant value of inductance giving a value of 0.9383 to parameter r. It continued with the approximation of a straight line giving a value of 0.9729 to parameter r. The upward concave parabolic approximations gave an r parameter value of 0.9740 and the downward concave parable of 0.9249. The straight linear or constant approximation can be a good choice when modeling the time-dependent inductance of spark gap in a laser Blumlein. The result for the constant approximation is in agreement by more than 90 %. This result is in fact in agreement with common practice. This is the simplest approximation, which offers the best ratio of mathematical simplicity to accuracy. Therefore it is not a bad idea to use a constant value of inductance on a first calculation.

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