Perturbative renormalization factors of quark operators for domain-wall QCD

Sinya Aoki, Taku Izubuchi, Junichi Noaki, Yoshinobu Kuramashi and Yusuke Taniguchi

Institute of Physics, University of Tsukuba, Ibaraki 305-8571, Japan
Department of Physics, Washington University, St. Louis, Missouri 63130, USA

We calculate one-loop renormalization factors of several quark operators including bilinear, three- and four-quark operator for domain-wall fermion action. Since Green functions are constructed for external physical quark fields, our renormalization method is simple and can be easily applied to calculation of any quark operators. Our results show that these renormalized quark operators preserve several chiral properties of continuum massless QCD, which can be understood by the property of external quark line propagator.

1. Introduction

The domain-wall formulation of the massless fermion DWQCD was applied to the lattice QCD (DWQCD) with a simpler form by Shamir, anticipating superior features over other quark formulations: no need of the fine tuning to realize the chiral limit, no restriction for the number of flavors and no $O(a)$ errors. These features have been proved perturbatively in and results from recent simulations to support existence of the massless mode in the scaling region. These advantageous features fascinate us to apply the domain-wall fermion for calculations of weak matrix elements sensitive to the chiral symmetry such as $B_K$ and other $B$ parameters. In order to convert several quantities obtained by lattice simulations to those defined in some continuum renormalization scheme (e.g., MS), we must know the renormalization factors. In this article we summarize our perturbative results of renormalization factors for the quark propagator, the bilinear, three- and four-quark operators consisting of physical quark fields, together with the perturbative understanding of the chiral properties.

2. Action and operators

We adopt the Shamir’s action in this article with the extra fifth dimensional length set to $N \to \infty$. In the DWQCD the massless fermion is expressed by the “physical” quark field defined by the boundary fermions

\[ q(n) = P_R \bar{\psi}(n) + P_L \psi(n), \]

\[ \bar{q}(n) = \bar{\psi}(n) P_R + \bar{\psi}(n) P_L \] (1)

with a projection matrix $P_{R/L} = \left(1 \pm \gamma_5\right)/2$. We will construct the QCD operators from this quark fields. The bilinear quark operator is given by

\[ O_{\Gamma} = \bar{q} \Gamma q, \quad \Gamma = 1, \gamma_5, \gamma_{\mu}, \gamma_{\mu} \gamma_{5}, \sigma_{\mu\nu} \] (2)

and the three-quark operator becomes

\[ O_{PD} = \varepsilon^{abc} \left(\bar{q}^c\Gamma_{\mu} q^a \Gamma_{\nu} q^b\right) \left(\Gamma_{\gamma} q^c\Gamma\right), \]

\[ \Gamma_{\mu} \otimes \Gamma_{\nu} = P_{R,L} \otimes P_{R,L} \] (3)

where $\bar{q}^c$ is a charge conjugated field and $a, b, c$ are color indices. The four-quark operator is

\[ O_4 = \frac{1}{2} \left[ (\bar{q}_1 \gamma^L_{\mu} q_2)(\bar{q}_3 \gamma_{\mu} q_4) \pm (\bar{q}_1 \gamma^L_{\mu} q_4)(\bar{q}_3 \gamma_{\mu} q_2) \right] , \]

\[ O_1 = -C_F (\bar{q}_1 \gamma^L_{\mu} q_2)(\bar{q}_3 \gamma_{\mu} R q_4) \]

\[ + (\bar{q}_1 T^A \gamma^L_{\mu} q_2)(\bar{q}_3 T^A \gamma_{\mu} R q_4) , \]

\[ O_2 = \frac{1}{2} N_c (\bar{q}_1 \gamma_{\mu} L q_2)(\bar{q}_3 \gamma^L_{\mu} R q_4) \]

\[ + (\bar{q}_1 T^A \gamma_{\mu} L q_2)(\bar{q}_3 T^A \gamma^L_{\mu} R q_4) , \]

\[ \gamma^L_{\mu} = \gamma_{\mu} P_{R,L} , \] (4)

where $T^A$ is a generator of color $SU(N_c)$ group.

\[ \]
3. One loop calculation

We calculate the one loop corrections to the quark propagator and the quark operators defined in the above. The point is that our calculation is done in the Green functions consisting of the “physical” quark fields only, \( \langle q\bar{q} \rangle \), \( \langle \bar{O}_1 q \bar{q} \rangle \), \( \langle O_{PD} q \bar{q} \bar{q}_3 \rangle \), and \( \langle O_{s+t,1,2} q \bar{q}_3 \bar{q}_3 \bar{q}_3 \rangle \), where the external quark line plays an important role. In general the fermion propagator \( S_F(p) \) at the physical scale of the external quark momentum and mass:

\[
\langle q(p)\bar{q}(-p) \rangle_s = S_q(p) [\xi_s L(p) - \xi_s R(p)]_s,
\]

\[
\langle \psi(p)\bar{\psi}(-p) \rangle = |R(p)|^2 \xi_c - L(p) \xi_s S_q(p),
\]

where \( \xi_{c/o} \) are analytic even/odd functions of the momentum and mass\(^{[4]}\). The quark propagator in the continuum, \( S_q \), is given by

\[
S_q(p) = \frac{1 - w_0^2}{i\gamma + (1 - w_0^2)\mu},
\]

where \( 1 - w_0^2 = M(2 - M) \) is a overall factor, which indicates the overlap between the normalized massless fermion mode and the boundary fermion at tree level. The flavor-index dependence only shows up in the factors

\[
L(p)_s = e^{-\alpha(p)N^+} P_R + e^{-\alpha(p)(s+1)} P_L,
\]

\[
R(p)_s = e^{-\alpha(p)(s-1)} P_R + e^{-\alpha(p)N^+} P_L,
\]

where \( \alpha(p) \) is an even function of \( p \).

With help of \( (8) \), the one loop correction in the DWQCD can be written in the same form as in the continuum calculation. For example, the one loop correction to the quark propagator is given by

\[
\langle q(p)\bar{q}(-p) \rangle_1 = S_q(p) \Sigma_q(p,m) S_q(p),
\]

which is same as that in the continuum, except for the overall factor \( 1 - w_0^2 \). The peculiar feature in the DWQCD is that the Dirac mass \( M \) is renormalized additively and the overall factor \( 1 - w_0^2 \) is shifted by the quantum correction. Therefore we need renormalization factor \( Z_w \) for \( w_0 \), which turns out to be

\[
Z_w = 1 + \frac{g^2 C_F}{16\pi^2} z_w.
\]

By evaluating the quantum corrections \( \Sigma_q \) for each Green functions and summing up all the flavor dependence together with \( L_s \) and \( R_s \), we get renormalization relations for various quantities. The bare quark wave function and mass on the lattice are connected multiplicatively with the renormalized ones in the \( \overline{\text{MS}} \) scheme at scale \( \mu \), by renormalization factors \( Z_2 \) and \( Z_m \).

\[
Z_2(\mu) = 1 + \frac{g^2}{16\pi^2} C_F \left(- \log(\mu^2) + z_2 \right),
\]

\[
Z_m(\mu) = 1 + \frac{g^2}{16\pi^2} C_F \left[-3 \log(\mu^2) + z_m \right].
\]

The renormalization factor of the bilinear quark operator becomes

\[
Z_{\Gamma}(\mu) = 1 + \frac{g^2 C_F}{16\pi^2} \left[x_2(\Gamma) \log(\mu^2) + z_{\Gamma} \right],
\]

\[
x_2(\Gamma) = 3(S), 3(P), 0(V), 0(A), -1(T),
\]

and the three quark operator is renormalized with

\[
Z_{PD}(\mu) = 1 + \frac{g^2}{16\pi^2} \left[\frac{3(N_c + 1)}{N_c} - \frac{3}{2} C_F \log(\mu^2) + z_{PD} \right].
\]

The renormalization factors of the four-quark operator is given as follows for \( F = \pm, 1, 2 \).

\[
Z_F(\mu) = 1 + \frac{g^2}{16\pi^2} \left[ (\delta_F - 2C_F) \log(\mu^2) + z_F \right],
\]

where \( \delta_F \) is a \( N_c \) dependent numerical factor\(^{[6]}\). The finite parts of the renormalization factors \( z_2, z_m, z_w, z_{\Gamma}, z_{PD}, z_F \) are given in our previous paper\(^{[8]}\) for various \( M \), with and without mean field improvement.

4. Chiral properties

Now we notice the relation \( Z_S = Z_P = Z_m^{-1} \) and \( Z_V = Z_A \)\(^{[10]}\), which suggest that the chiral
Ward-Takahashi identity holds exactly. We can also see that the three- and four-quark operators can be renormalized without any operator mixing between different chiralities. These facts suggest that the good chiral properties of the physical Green functions are preserved also at one loop level as in the tree level. Furthermore it can be shown that the $O(a)$ errors automatically vanish in the renormalization factors at any loop level in the perturbation theory. These superior features can be understood by the peculiar form of the external line propagator. The important point is that the even and odd function $\xi_{e/o}$ is separated with different damping factor $L_s$ and $R_s$ in the propagator $[3]$. We consider the half-circle diagram of quark self-energy correction (Fig. 2b of Ref. [3]) as an example. The external line factor $[\xi_L(p)_s - \xi_R(p)_s]$, multiplied by a single gluon interaction vertices $V^{(1)}(p, k)_{st}$, becomes

$$[\xi_L L_s - \xi_R R_s] V^{(1)}_{st} = (u_{oL_L} + u_{eR_L}),$$  \hfill (15)

where $u_{oL}$ and $u_{eL}$ are even and odd function. In this operation the combination of the even and odd functions with the damping factors $L, R$ is flipped, however the structure that the even and odd function is separated with $L$ and $R$ is not changed. Then this factor is multiplied to the internal fermion propagator $S_F(p)_{st}$,

$$(u_{oL_L} + u_{eR_L}) S_F(l)_{st} = f_{sL}(p)_t + h_{oL}(l)_t + f_{oR}(p)_t + h_{oR}(l)_t,$$  \hfill (16)

which does not change the even-odd combination but only shift the damping ratio. After being multiplied to another interaction vertex and the even-oddness being flipped, the factor meets with another interaction vertex and the result becomes definitely odd function in terms of the external quark momentum and mass. The above argument can be applied to other diagrams in any loop level and we can easily show that the quark self-energy $\Sigma_q$ in (8) is an odd function. If we expand $\Sigma_q$ in terms of quark momentum and mass keeping the logarithmic dependence in the coefficients, the leading term and the next to next to leading term, which correspond to the additive mass correction and the $O(a)$ errors, vanish automatically.

Finally we will see how $Z_s = Z_p$ and $Z_V = Z_A$ is obtained in perturbation theory. For massless quarks the fermion propagator and the interaction vertex take the following forms:

$$S_F = \gamma_\mu x_o + x_e, \quad V^{(1)} = \gamma_\nu y_e + y_o,$$  \hfill (18)

where $x_e, y_e$ and $x_o, y_o$ are even and odd functions. Since one loop corrections to bilinear operators are written as $\nu \Gamma \nu$ with odd functions $\nu$, $\gamma_5$ in one loop diagrams of the axial vector current and the pseudo scalar density can be moved outside without changing integrands:

$$\gamma_5 S_F V^{(1)} \rightarrow \gamma_5 (\gamma_\mu x_o y_e + x_e y_o) = (\gamma_\mu x_o y_e + x_e y_o) \gamma_5.$$ \hfill (19)

This implies $Z_S = Z_P$ and $Z_V = Z_A$.

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