On-shell renormalization of fermion masses, fields, and mixing matrices at 1-loop

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Abstract We propose a new and simple on-shell definition of off-diagonal fermion field and mass counterterms at 1-loop in terms of self-energy scalar functions. We show that the anti-Hermitian part of the field renormalization is always finite and that mass counterterms can be chosen gauge-independent. It is noteworthy that our definition relies on mass structures and is universal. Further, definitions of the off-diagonal mass and field counterterms allow us to comment on the renormalization of mixing matrices with focus on the quark mixing matrix in the standard model. As an example of our scheme, we provide computations in the two Higgs–Doublet model with an additional heavy Majorana neutrino—the Grimus–Neufeld model. This allows for the comparison with other schemes known in literature and also provides examples for the case of massless fermions. In the appendix, the scheme is extended to arbitrary orders, although without example computations in the Grimus–Neufeld model.

1 Introduction

While renormalization of models without mixing is more or less straightforward, particle mixing introduces a few complications. For example, the renormalization constants develop off-diagonal terms which have to be fixed, the appearance of mixing matrices poses the question whether they should be renormalized. In terms of the Standard model (SM) probably the first attempt to account for the quark mixing was attempted in [1] in renormalizing the CKM matrix [2,3]; however, the proposed scheme later showed its shortcomings. There the authors chose to absorb the whole anti-hermitian part of field renormalization into the CKM matrix counterterm, which resulted in a gauge-dependent counterterm, a feature which breaks Ward–Takahashi identities [4]. An analogous approach for neutrinos was introduced in [5].

Another approach in [6] relies on defining off-diagonal mass counterterms, which are afterwards diagonalized and enter the mixing matrix counterterm. The definition is done using the self-energy decomposition used by Feynman in QED [7] and noting how different terms contribute to external leg corrections. The same authors later also provided mass counterterms in the SM in terms of the conventional self-energy decomposition [8]. While the authors do compute field renormalization contributions, they remain undefined in terms of the conventional self-energy decomposition.

We also note that the approach and shortcomings of [1] sparked a series of papers trying to remedy the gauge-dependence of the CKM counterterm. One such approach in [4] considers a zero-momentum subtraction scheme. Other approaches propose various methods to separate the gauge-dependent part of field renormalization such that only the gauge-independent part goes into the definition of the CKM counterterm: computing field renormalization in the ’t Hooft-Feynman gauge [9], using a reference theory in which there is no mixing [10], obtaining a renormalization condition via modified minimal subtraction (MS) [11], proposing physical renormalization conditions based on BRS symmetry and $W \rightarrow u_d j$ and $t \rightarrow W d j$ decay amplitudes [12] (and without BRS in [13]), using the difference between “leptonic” and quark transition amplitudes [14]. However, our scheme is closest to that of [6,15]; hence, the schemes in this paragraph are rather removed from our approach and will not be discussed any further.

In this paper, we propose an On-Shell(OS) scheme which makes use of the off-diagonal mass counterterms to renormalize the mixing. With the usual OS renormalization conditions [16], the off-diagonal mass counterterms and the anti-hermitian part of the field renormalization are degenerate and cannot be simultaneously solved for in terms of self-energies. We overcome this by defining the anti-hermitian part of the field renormalization and then solving for the mass counterterms in terms of self-energies and the field renormalization. At 1-loop, at least for Dirac fermions, the definition avoids the usage of the $\tilde{R}e$ operator [17], which drops the absorptive parts from loop functions. Importantly, with this choice the mixing matrices are already renormalized so that there is no need for a mixing matrix counterterm. We also apply the scheme in the Grimus–Neufeld model [18], which is a two Higgs–Doublet Model with an additional heavy Majorana neutrino. This allows for the comparison of divergent parts with [1,6] for the quarks in the SM, and for the comparison with [15] in case of Majorana neutrinos while also providing new results.
The paper is structured as follows: In Sect. 2 we show that the mixing matrices do not need counterterms, write up all the needed renormalization constants, the fermion self-energy decomposition, specify the choice of renormalization conditions, and finally give the definitions of the relevant off-diagonal counterterms. In Sect. 3, we discuss the ultraviolet as well as gauge properties of the counterterms and also provide comparative discussions with respect to the schemes found in [1,6,19]. In Sect. 4, we introduce the Grimus–Neufeld model and provide computations of quark, charged lepton, and neutrino counterterms as well as explicit comparisons of the UV parts to existing literature. In Sect. 5, we give our conclusions. In “Appendix A”, we extend the scheme beyond 1-loop.

2 Renormalization at 1-loop

2.1 The renormalized mixing matrix

Let us begin with a discussion on the renormalization of mixing matrices. There already is a great body of literature on the renormalization of mixing matrices, which has been listed to some extent in the Introduction. In short, all the works aim to renormalize the mixing matrices in such a way that the mixing matrix counterterm fulfills the following requirements [10,20,21]:

1. UV divergences in relevant amplitudes must be canceled,
2. the counterterms must be gauge-independent,
3. unitarity of the bare mixing matrix must be preserved after renormalization,
4. renormalization should be symmetric with respect to mixing degrees of freedom and be independent of a specific physical process,
5. in the limit of degenerate masses or extreme mixing angles, no singularities should be introduced in physical observables.

Further, there should be no “dead corners” in the parameter space where a renormalized input parameter goes to infinity.

Often these requirements are only partially fulfilled or fulfilled in ways of varying complexity; we achieve this trivially by having the counterterm of the mixing matrix set to 0. This comes from our formulation of an additional consistency condition, namely finite basis rotations must commute with the renormalization procedure. In our view, this is a reasonable requirement since one should be allowed to perform basis rotations on both the bare and renormalized fields at any time without somehow changing the theory.

To see how this consistency condition leads to a trivial counterterm $\delta V$ for some mixing matrix $V$, let us take a mass matrix $m$ and consider two scenarios:\footnote{Strictly speaking, we should discuss two different mass matrices such that the mixing matrix arises somewhere in the Lagrangian after the diagonalization of both matrices; however, a schematic discussion with a single mass matrix is enough to convey the point.}

1. Mass is diagonalized and then the theory is renormalized. This is the usual approach, where masses are given diagonal counterterms and mixing matrices have their counterterms. Schematically:

$$
\begin{align*}
\begin{array}{c}
m^0_{ji} \text{diagonalize} \\
\mathcal{V}^0 \text{diagonalize} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
m^0_{ji} \xrightarrow{\text{renormalize}} m_i + \delta m_i \\
\mathcal{V}^0 \xrightarrow{\text{renormalize}} V + \delta V.
\end{array}
\end{align*}
(1)
$$

2. The theory is first renormalized and only then the renormalized mass is diagonalized. This means non-diagonal mass counterterms and no counterterms for mixing matrices, since mixing matrices were not present during renormalization. Schematically:

$$
\begin{align*}
\begin{array}{c}
m^0_{ji} \text{renormalize} \\
\mathcal{V}^0 \text{renormalize} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
m^0_{ji} \xrightarrow{\text{renormalize}} m_{ji} + \delta m_{ji} \\
\mathcal{V}^0 \xrightarrow{\text{renormalize}} V + \delta V.
\end{array}
\end{align*}
(2)
$$

Here the $0$’s in superscripts indicate bare parameters, $\delta$’s indicate counterterms, $i$, $j$ are family indices, the slashes are used schematically and are meant to keep track of quantities that are not present at a given step, for example, there is no mixing matrix $V$ before diagonalization so we simply have $\mathcal{V}$.

Both scenarios stem from the same bare Lagrangian and lead to the same renormalized Lagrangian but differ in their counterterms. Even though both scenarios describe the same theory, there is no finite basis rotation that actually relates the two. For example, one may consider taking the final result of the first scenario, Eq. (1), and performing a finite rotation with $V^{-1}$, which un-diagonalizes the mass matrix and removes $V$ from the Lagrangian. Requiring that this is the same as the middle step in the second scenario, Eq. (2), we have

$$
\begin{align*}
m_{ji} + ((V^{-1})^\dagger \delta m V^{-1})_{ji} = m_{ji} + \delta m_{ji} \\
\mathcal{V}^0((V^{-1})^\dagger \delta V V^{-1})_{ji} = V + \delta V.
\end{align*}
(3)
$$
The first line could hold since the mass counterterm is un-diagonalized by $V^{-1}$. On the other hand, in the second line
\[ (V^{-1})^\dagger \delta V V^{-1} \] is in general non-vanishing, unless
\[ \delta V = 0. \] (4)

In other words, renormalization commutes with finite basis rotations only if there are no mixing matrix counterterms. Practically this means that the second scenario is consistent and could be used.

The interpretation of this is rather simple—the mixing matrix is a non-physical basis artefact and should not have any counterterms. Note that at this point we have not introduced any renormalization conditions so this argument is completely general and holds in any scheme and even beyond 1-loop.

Importantly, since $\delta V = 0$, it is trivially UV-finite, gauge-independent, preserves unitarity, is symmetric, process-independent, and numerically stable. (No singularities appear in the degenerate mass limit.) Of course, that these properties hold and that the definitions for mass and field counterterms for which this is the case.

In other words, renormalization commutes with finite basis rotations only if there are no mixing matrix counterterms. Practically this means that the second scenario is consistent and could be used.

Note that at this point we have not introduced any renormalization conditions so this argument is completely general and holds in any scheme and even beyond 1-loop.

Next we introduce the necessary renormalization constants. We choose to renormalize fermion fields as
\[
\psi_0 = Z^{1/2} \psi, \\
\bar{\psi}_0 = \bar{\psi} \gamma^0 Z^{1/2} \gamma^0, \\
Z = Z_L P_L + Z_R P_R, \\
Z_{L,R} = 1 + \delta Z_{L,R},
\] (5)

where $P_{L,R} = \frac{1}{2} \left(1 \mp \gamma^5\right)$. This field renormalization is in line with keeping the hermiticity of the Lagrangian, which is a feature not to be taken for granted [26].

Next we renormalize the masses as [6,8] and according to the second scenario, Eq. (2),
\[
m_{0,ji} = m_i \delta_{ji} + \delta m^L_{ji} P_L + \delta m^R_{ji} P_R. \] (6)

Here $m_{0,ji}$ is the bare mass, $m_i$ is the renormalized mass, which is real and diagonal, $\delta m^L_{ji}$ contain off-diagonal contributions and $i$, $j$ are family indices. In addition, hermiticity of the Lagrangian imposes the following relation on mass counterterms (in matrix notation)
\[
(\delta m^L)\dagger = \delta m^R. \] (7)

In order to write down the self-energy decomposition, let us also write down the kinetic and mass terms of the renormalized fermion Lagrangian for clarity
\[
L_{\text{kin.+mass}} = \bar{\psi} L Z_L^{1/2} \gamma^\dagger \gamma Z_L^{1/2} \psi L - \bar{\psi} R Z_R^{1/2} (m + \delta m^T) Z_L^{1/2} \psi L + (L \leftrightarrow R). \] (8)

We can now decompose the renormalized self-energy at 1-loop
\[
\Sigma_{ji}(\not{p}) = \Sigma_{ji}^L(p^2)p^\dagger P_L + \Sigma_{ji}^R(p^2)p^\dagger P_R + \Sigma_{ji}^L(p^2)P_L + \Sigma_{ji}^R(p^2)P_R + \frac{1}{2} \left(\delta Z_{L,ji}^\dagger + \delta Z_{L,ji}\right) p^\dagger P_L + \frac{1}{2} \left(\delta Z_{R,ji}^\dagger + \delta Z_{R,ji}\right) p^\dagger P_R
\]
\[
- \left(\delta m^L_{ji} + \frac{1}{2} \delta Z_{R,ji}^\dagger m_j + \frac{1}{2} m_j \delta Z_{L,ji}\right) P_L - \left(\delta m^R_{ji} + \frac{1}{2} \delta Z_{L,ji}^\dagger m_i + \frac{1}{2} m_i \delta Z_{R,ji}\right) P_R. \] (9)

Here everything is completely general and model independent, for example, we do not introduce relations for $\Sigma^{L,R}(p^2)$ as done in the SM.
In addition, we note that we include tadpole diagrams in the self-energy computation, so in terms of diagrams the self-energy is

\[ \Sigma_{ji}(\rho) = \delta_{ji} + \frac{i}{\pi} \text{1PI}_i j + \text{1PI}_j + \delta_{ji}, \]

(10)

where the propagator with the cross marks counterterm insertions and 1PI stands for 1-particle-irreducible diagrams. The appearance of tadpoles in the above diagrammatic decomposition may be understood (justified) in two ways. On the one hand, we can identify the 3rd diagram with the tadpole counterterms \( \delta t_{ji} \) for 2-point functions in the Fleischer–Jegerlehner (FJ) tadpole scheme [27,28]. Importantly, the 1-point functions, i.e. the tadpoles, still vanish which is ensured with the counterterms \( \delta t_i \). On the other hand, one can notice that even though the tadpole diagrams look reducible, there is no momentum flow through the propagator connecting the fermion line with the loop; hence, the propagator is effectively a coupling1

The authors of [26] propose to relax the no-mixing conditions or to field renormalization constants \( Z \) expressons are different—this is the over-specification problem. The authors of [26] propose to relax the no-mixing conditions or to separate. The end result is that the above self-energy with included tadpole diagrams is indeed 1PI.

2.3 Renormalization conditions

Next we discuss the renormalization conditions. We require the renormalized mass to be the pole mass as is standard in On-Shell schemes [16]

\[ \overline{\text{Re}} \Sigma_{ii}(\rho) u_i = 0, \]

(11)

where \( u_i \) is the Dirac spinor of the propagating particle. The imaginary part of \( \Sigma_{ii}(\rho) u_i \) is identified with the decay width \( \Gamma \), but it does not fix any counterterms and is not not the focus of our scheme. For field renormalization (and/or LSZ factors), there is an additional unit residue condition [16]

\[ \lim_{\rho \to m_i} \frac{1}{\rho - m_i} \sum_{k,l} \tilde{Z}_{ik} \Sigma_{kl}(\rho) Z_{li} u_i = u_i \]

(12)

where \( Z \) is the Lehmann–Symanzik–Zimmermann (LSZ) [29] factor for incoming particles and \( \tilde{Z} \) is the LSZ factor for outgoing particles.

For the off-diagonal components, we require no mixing on external legs [16]

\[ \sum_k \Sigma_{jk}(\rho) Z_{ki} u_i = 0, \]

(13a)

\[ \sum_k \tilde{u}_j \tilde{Z}_{jk} \Sigma_{ki}(\rho) = 0, \]

(13b)

where \( i \neq j \), incoming particles are denoted by \( i \) and outgoing ones by \( j \).

Usually, an On-Shell scheme requires that the LSZ factors are unit matrices such that role of LSZ factors is fully fulfilled by the field renormalization constants. However, in that case one immediately runs into problems due to absorptive parts coming from loop functions above particle production thresholds. It has been noted in [26,30] that upon application of the above conditions with \( Z = \tilde{Z} = 1 \) and \( \delta m_{ji} = \delta m_{ij} \delta_{ji} \) one arrives at expressions for \( Z_{L,R} \) and \( Z_{L,R}^* \) in terms of self-energy scalar functions; however, one finds that due to absorptive parts one cannot get \( Z_{L,R} \) from \( Z_{L,R}^* \) (or vice versa) by naive hermitian conjugation and the resulting expressions are different—this is the over-specification problem. The authors of [26] propose to relax the no-mixing conditions or to use two sets of field renormalization constants \( Z \) (associated with \( \psi \)) and \( \tilde{Z} \) (associated with \( \bar{\psi} \)), for which the hermiticity condition \( \tilde{Z} = Z^0 Z^\dagger, Z^\dagger \) does not hold. The latter makes the Lagrangian written in terms of renormalized fields non-hermitian, but is fine in terms of external leg renormalization as \( Z \) simply takes the role of the LSZ factor \( \tilde{Z} \).

While neither of the proposed solutions is satisfactory, for our scheme we want to include the absorptive parts as much as possible while also keeping the Lagrangian hermitian as already chosen in Eq. (5). This can be done if we set \( Z = 1 \) while keeping \( \tilde{Z} \neq 1 \) with the latter containing only the absorptive parts, we call this choice the incoming renormalization scheme as in [30]. It is worth noting that there are no absorptive parts if a particle’s mass is below particle production thresholds, in that case one can set both LSZ factors to unity simultaneously.

Interestingly and contrary to the case of field renormalization, the hermiticity condition for the mass counterterms in Eq. (7) allows to keep the absorptive parts irrespective of the choice of the LSZ factors.

\footnote{Note that an opposite definition is possible with \( \tilde{Z} = 1 \) and \( Z \neq 1 \), i.e. the outgoing scheme [30].}
In summary, in addition to the above conditions, for our scheme we choose the incoming variant with the LSZ factor for incoming particles set to unity

\[ Z = 1 \]  

such that Eq. (13a) becomes

\[ \Sigma_{ji}(p)u_i = 0, \]  

and with the requirement that mixing matrices must have vanishing counterterms via the consistency condition in Sect. 2.1

\[ \delta V = 0. \]  

Up until “Appendix A”, we will be mostly occupied with off-diagonal components of mass and field counterterms and the diagonal components will barely play any role at all so that the reader may assume the standard On-Shell expressions for \( \delta m_{ii} \) and \( \delta Z_{ii} \) as seen fit. In addition, we took care to consistently include the absorptive parts and avoided using \( \tilde{\text{Re}} \) in our renormalization conditions; however, these are not the main driver of our scheme; hence, we will not compute the purely absorptive LSZ factor \( \tilde{Z} \) via Eq. (13b). Even more so, the reader is absolutely free to move the absorptive parts from the field renormalization to the LSZ factors entirely since, as we will see in Eq. (21), the scheme relies only on mass structures.

2.3.1 A note on Majorana fermions

For Majorana fermions, the story is a little different due to the Majorana condition imposing an additional requirement on field renormalization. The Majorana condition on bare fields [31]

\[ \nu_0 = \nu_0^\dagger = \gamma^0 C \nu_0^*, \]  

where \( C \) is the charge conjugation matrix, implies

\[ \Rightarrow Z = \gamma^0 CZ^* C^{-1} \gamma^0. \]  

The implication is there if the renormalized fields are also Majorana. More simply, this means \( Z_L = Z_R^* \), which is an additional relation for field renormalization constants. The relation involves complex conjugation, which leads to the same problem of over-specification and does not hold due to absorptive parts if at least one of the LSZ factors is set to unity. There are now two ways to proceed.

In the first way, which we do not choose, we may consider relaxing the no-mixing condition on incoming legs. Since Eq. (15) has left and right projections, we may require only one of those to hold. This approach, however, does not seem to provide us with convenient mass structures, which will appear in upcoming sections. Another option of relaxing the condition would be to fully use Eq. (15) and solve for the field renormalization, but then only use one solution (e.g., use \( Z_L \) and get \( Z_R \) by complex conjugation of \( Z_L \))—this will keep the mass structures we are about to find in tact. However, with the absorptive parts present, ensuring that the mass counterterms are both symmetric as required for Majorana fermions (see, e.g., [15]) and that Eq. (7) holds complicates the mass counterterm such that it is no longer a limit of the Dirac case. In other words, this approach would require for a separate discussion of Majorana fermions.

The second and maybe the easiest approach is to use the \( \tilde{\text{Re}} \) operator in case of Majorana particles, such that the no-mixing condition is:

\[ \tilde{\text{Re}} \left[ \Sigma_{ji}(p) \right] u_i = 0. \]  

This, of course, relaxes the no-mixing condition, but keeps the Majorana condition in full for both bare and renormalized fields, the Lagrangian is also hermitian. This second option is equivalent to including all the absorptive parts into the LSZ factor for incoming particles such that \( Z \neq 1 \).

Currently, this seems to be the best approach when considering Majorana fermions; hence, in further expressions the counterterms of Majorana particles will come with the \( \tilde{\text{Re}} \) operator, except for expressions containing only divergent parts or that do not contain absorptive parts.

The takeaway is that the no-mixing conditions with \( Z = 1 \), hermiticity of the Lagrangian, and the Majorana condition are incompatible if considered together. There is simply not enough freedom to impose everything.

2.4 The definition of off-diagonal mass and field counterterms

In the previous section, we have introduced the renormalization conditions of our scheme; however, there is a problem since these conditions are not enough to completely fix all the counterterms, i.e., Eq. (15) can only be used to fix either the off-diagonal field
or the off-diagonal mass counterterms but not both. For example, from the renormalization condition in Eq. (15) one can arrive at the following relation between the anti-hermitian part of the field renormalization and the mass counterterms

\[
(m_i^2 - m_j^2)\delta Z^A_{L,ji} - 2m_j\delta m_{ji}^L - 2m_i\delta m_{ji}^R = -(m_i^2\Sigma^L_{ji}(m_i^2) + m_j\Sigma^R_{ji}(m_i^2) + m_j\Sigma^L_{ji}(m_i^2) + m_i\Sigma^R_{ji}(m_i^2)) + H.C. \tag{20}
\]

where \(H.C.\) stands for Hermitian conjugation, both \(\delta Z^A_{L,ji}\) and \(\delta m_{ji}^{R, L}\) are unknowns, and there is an analogous equation involving \(\delta Z^A_{R,ji}\). This likely explains why the usual approach is to set the off-diagonal mass counterterms to 0 so that the above equation can fix at least the field renormalization. Of course, then the problem is simply shifted into the renormalization of the mixing matrix, for which there is also no natural renormalization condition. Rather, we take a different approach and define the anti-hermitian part of the off-diagonal field counterterms as the coefficient of \(m_i^2 - m_j^2\) in Eq. (20) and in an analogous equation for the right-handed part

\[
\delta Z^A_{L,ji} = \left[ - \left( m_i^2\Sigma^L_{ji}(m_i^2) + m_j\Sigma^R_{ji}(m_i^2) + m_j\Sigma^{L\dagger}_{ji}(m_i^2) + m_i\Sigma^R_{ji}(m_i^2) \right) + H.C. \right] (m_i^2 - m_j^2),
\]

\[
\delta Z^A_{R,ji} = \left[ - \left( m_i^2\Sigma^R_{ji}(m_i^2) + m_j\Sigma^L_{ji}(m_i^2) + m_j\Sigma^{R\dagger}_{ji}(m_i^2) + m_i\Sigma^L_{ji}(m_i^2) \right) + H.C. \right] (m_i^2 - m_j^2),
\]

in both expressions \(i \neq j\). The hermitian part of the field renormalization is not changed w.r.t. the OS standard as it is not related to the mass counterterms and one simply has to solve for the hermitian part from Eq. (15).

Once the field renormalization is defined, we can simply solve for the mass counterterms in Eq. (20) and in an analogous equation with \(\delta Z^A_{R,ji}\). The solutions are

\[
\delta m_{ji}^R = \frac{1}{2} \left( m_j\Sigma^R_{ji}(m_i^2) + m_j\Sigma^{R\dagger}_{ji}(m_i^2) + m_i\Sigma^L_{ji}(m_i^2) + m_i\Sigma^{L\dagger}_{ji}(m_i^2) \right) + \frac{1}{2} \left( m_j\delta Z^A_{R,ji} - m_i\delta Z^A_{L,ji} \right),
\]

\[
\delta m_{ji}^L = \frac{1}{2} \left( m_i\Sigma^R_{ji}(m_i^2) + m_j\Sigma^{R\dagger}_{ji}(m_i^2) + m_i\Sigma^L_{ji}(m_i^2) + m_i\Sigma^{L\dagger}_{ji}(m_i^2) \right) + \frac{1}{2} \left( m_j\delta Z^A_{L,ji} - m_i\delta Z^A_{R,ji} \right),
\]

for \(i \neq j\). One can clearly see that the hermiticity constraint in Eq. (7) trivially holds. As we are keeping the hermitian conjugation of self-energies, these expressions also hold above particle production thresholds and incorporate the absorptive parts.

The counterterms defined in this way have a property that are summarized in the bullet points and then discussed in the sections that follow:

- The counterterms are given in terms of self-energies and restrictions to mass structures and thus are universal—model- and process-independent
- The field renormalization contains all the gauge-dependence, in turn, the mass counterterms are gauge-independent (see Sect. 3.1)

\[
\partial_\xi \delta m_{ji}^R = 0 \quad \text{and} \quad \partial_\xi \delta m_{ji}^L = 0. \tag{23}
\]

- The anti-hermitian part of the field renormalization is free of UV-divergences (see Sect. 3.2)
- These definitions naturally give a vanishing mixing matrix counterterm (see Sect. 3.3)
- The scheme relies on the mass structure \((m_i^2 - m_j^2)\) and does not depend on the inclusion of the absorptive parts
  - The reader is free to treat these absorptive parts as seen fit, but we do include them for completeness
  - These counterterms are non-singular in the degenerate mass limit as there are no \((m_i^2 - m_j^2)^{-1}\) factors appearing

There are a few comments we would like to make before moving on to the more detailed discussions of the above properties.

Since the diagonal components of the anti-hermitian part of the field renormalization cannot be fixed and are usually set to zero, we may try and use the Eq. (22) also for \(i = j\). In that case the expressions for the mass counterterms \(\delta m_{ji}^{L, R}\) have the same real part as the one found in [26]. In our scheme the imaginary parts of diagonal components of \(\delta m_{i}^{L, R}\) are complex conjugate to each other and are not the same as in [26] due to absorptive parts and hermitian conjugation. In other words, we may cheat a little and use the off-diagonal mass counterterm expressions for diagonal terms as well, however, we will use this extension to all \(i \neq j\) only for the real part.

It may seem unpleasant that the off-diagonal mass counterterms are defined in terms of self-energies and the non-physical field renormalization. However, as noted in [32], in general it seems impossible to have gauge-independent quantities only in terms of self-energies, hence, there is nothing wrong with having the mass counterterms shifted by the anti-hermitian parts of the field renormalization. Although, in some specific models it is sometimes possible to find gauge-independent quantities in terms of the self-energies as found for the SM in [26].

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5 Here \((m_i^2 - m_j^2)^n A + B\) \(\left|\begin{array}{c} m_i^2 \\ m_j^2 \end{array}\right|^n = (m_i^2 - m_j^2)^{n-1} A\) for some positive power \(n\) and functions \(A\) and \(B\).
3 Properties of the scheme at 1-loop

3.1 Gauge dependence

In this section we discuss the gauge-dependence of the counterterms in our scheme. To do so we first need to introduce the Nielsen identities [33], which we (almost completely) quote from [32] in the most general form for convenience

$$\partial_\xi \Gamma = (1 + \rho^\xi) S_\Gamma \left( \frac{\partial \Gamma}{\partial \chi} \right) + \sum_i p_i \beta_i^\xi \frac{\partial \Gamma}{\partial p_i} + \sum_\phi \gamma_\phi^\xi \delta \Gamma + \sum_\delta \delta S \int d^4x \frac{\delta \Gamma}{\delta S(x)}. \quad (24)$$

Here $\xi$ is some gauge parameter, $\chi$ is the source coupled to the BRST variation of the gauge parameter, $\Gamma$ is the reduced 1PI generating functional. $S_\Gamma$ is the Slavnov-Taylor operator, $p_i$ are parameters of the theory, $\phi$ runs over the fields in the theory and $N_\phi$ counts the external ones, while $S$ stands for any scalar that may acquire a vacuum expectation value, finally, $\rho^\xi$, $\beta_i^\xi$, $\gamma_\phi^\xi$, and $\delta S$ parameterise various modifications of the Nielsen Identity due to non-physical renormalization. Starting from the left, the $\rho^\xi$ terms arise if the theory is at the minimum. The $\delta S$ situation is much simpler. The $\beta_i^\xi$ and $\gamma_\phi^\xi$ terms have the same origin the only difference being, that the $\rho^\xi$ term first comes in at 2-loop order so that this term is irrelevant at 1-loop. Further, we assume that all the physical parameters in the theory are renormalized in a gauge-independent way such that $\rho^\xi = 0$. Of course, we will show that our mass counterterms are gauge-independent. In reality, the $\rho^\xi$ term vanishes if the field renormalization is not defined in the on-shell scheme. The final term with $\delta S$ vanishes if the theory is not at the minimum.

The general Nielsen Identity, Eq. (24), includes all possible modifications, however, in practice and especially at 1-loop the situation is much simpler. The $\rho^\xi$ deformation can only come in at 2-loop order so that this term is irrelevant at 1-loop. Further, we assume that all the physical parameters in the theory are renormalized in a gauge-independent way such that $\beta_i^\xi = 0$. Of course, we will show that our mass counterterms are indeed gauge-independent. Next, since our scheme is fully On-Shell for incoming particles and the relevant expressions include only the renormalization of such particles (i.e. they stem from Eq. (15)) the deformation $\gamma_\phi^\xi$ is also 0. To see that the deformations parameterized by $\delta S$ vanish we turn to the FJ scheme, which is already implemented in Eq. (9) and equivalent to a scheme where the tadpoles are not renormalized [28]. The FJ scheme ensures that all the 1-point functions vanish so that the theory is described at the true minimum, in turn, the $\delta S$ term vanishes. Finally, with these considerations the general Nielsen identity simplifies to

$$\partial_\xi \Gamma = S_\Gamma \left( \frac{\partial \Gamma}{\partial \chi} \right). \quad (25)$$

Let us now consider the gauge-derivative of the self-energy $\Sigma_{ji}(\rho)$ stemming from the simplified Nielsen Identity [32]

$$\partial_\xi \Sigma_{ji}(\rho) = \sum_{j'} \Lambda_{ji} \Sigma_{j'i}(\rho) + \sum_{j'} \Sigma_{ji'}(\rho) \Lambda_{j'i}. \quad (26)$$

Here $\Lambda_{ji} = -\Gamma_{j'ji} \eta_{j'}$, and $\Lambda_{ji} = -\Gamma_{j'ji} \eta_{j'} \eta_{j'}$, $\Lambda$’s are functions of $\rho$ and have Dirac structure, $\eta_\phi$ is a source coupled to the BRST variation of the fermion field $\psi$ [32]. Equation (26) further simplifies at 1-loop. As $\Lambda$’s vanish at tree-level we may make a simple replacement on the r.h.s.

$$\Sigma_{ji}(\rho) \rightarrow (\rho - m_i) \delta_{ji} \quad (27)$$

so that the gauge-derivative becomes [9]

$$\partial_\xi \Sigma_{ji}(\rho) = \Lambda_{ji} (\rho - m_i) + (\rho - m_j) \tilde{\Lambda}_{ji}. \quad (28)$$

Since the derivative of the self-energy $\partial_\xi \Sigma_{ji}(\rho)$ and $\Lambda$’s have Dirac structure, we may decompose them just like the self-energy in Eq. (9). It is then a matter of trivial rearrangements and collection of terms to arrive at

$$\partial_\xi \Sigma_{ji}^L(p^2) = -m_i \Lambda_{ji}^L - m_j \tilde{\Lambda}_{ji}^L + \Lambda_{ji}^R + \tilde{\Lambda}_{ji}^R,$n
$$\partial_\xi \Sigma_{ji}^R(p^2) = -m_i \Lambda_{ji}^R - m_j \tilde{\Lambda}_{ji}^R + \Lambda_{ji}^L + \tilde{\Lambda}_{ji}^L,$n
$$\partial_\xi \Sigma_{ji}^{L1}(p^2) = p^2 \Lambda_{ji}^R + p^2 \Lambda_{ji}^L - m_i \Lambda_{ji}^{L1} - m_j \tilde{\Lambda}_{ji}^{L1},$$n
$$\partial_\xi \Sigma_{ji}^{R1}(p^2) = p^2 \Lambda_{ji}^L + p^2 \Lambda_{ji}^R - m_i \Lambda_{ji}^{R1} - m_j \tilde{\Lambda}_{ji}^{R1}. \quad (29)$$

6 As we will consider the fermions, the reduced and non-reduced generating functionals are the same.
This decomposition allows to easily take the gauge-derivative of the anti-hermitian part of the field renormalization in our scheme in Eq. (21), we get

\[
\partial_\xi \delta Z^A_{L ji} = \left[(m_i^2 - m_j^2) \left(-m_j \bar{\Lambda}^R_{ji}(m_i^2) - \bar{\Lambda}^L_{ji}(m_j^2) - H.C.\right)\right]|_{m_i^2 - m_j^2}
\]

\[
= -m_j \bar{\Lambda}^R_{ji}(m_i^2) - \bar{\Lambda}^L_{ji}(m_j^2) - H.C.
\]

\[
\partial_\xi \delta Z^A_{R ji} = -m_j \bar{\Lambda}^L_{ji}(m_i^2) - \bar{\Lambda}^R_{ji}(m_j^2) - H.C.
\] (30)

Gauge-dependence of the hermitian part is the same except for the addition of H.C. instead of a subtraction. The above shows that the gauge-dependent parts always carry the \(m_i^2 - m_j^2\) mass structure, but this factor cancels both in our approach as well as in the standard one as noticed in [9]. While all the gauge-dependent parts must have a factor of \(m_i^2 - m_j^2\), one should be careful as there might also be gauge-independent contributions with the same factor that could enter the anti-hermitian part of field renormalization. In addition, it is important to note that tadpole diagrams must be included in self-energies so that the \(m_i^2 - m_j^2\) factor consistently appears alongside the gauge-dependent parts.

We may use the decomposition in Eq. (29) to investigate the gauge-dependence of the off-diagonal mass counterterms, however, the result is trivial since by definition all the gauge-dependence is in the field counterterms

\[
\partial_\xi \delta m^L_{ji} = 0 \quad \text{and} \quad \partial_\xi \delta m^R_{ji} = 0.
\] (31)

3.2 UV divergences

Next we consider the UV divergences of the counterterms in our scheme. To do so, take note that the absorptive parts are irrelevant for the UV parts so that we may use the pseudo-hermiticity property of the self-energy for the purposes of this section

\[
\Sigma(p) = \gamma^0 (\Sigma(p))^{\dagger} \gamma^0.
\] (32)

Here hermitian conjugation acts both on flavour and Dirac structures. Pseudo-hermiticity simply relates the scalar self-energy functions

\[
\left(\Sigma^{L,R}(p^2)\right)^{\dagger} = \Sigma^{L,R}(p^2),
\]

\[
\left(\Sigma^{*L}(p^2)\right)^{\dagger} = \Sigma^{*R}(p^2).
\] (33)

At 1-loop the contributions to the self-energies in terms of Passarino–Veltman functions [34] are known in general [35]. Whether a scalar or a vector in a so-called “bubble” diagram contributes to the fermion self-energy one gets the same Passarino–Veltman contributions to the self-energies. In that regard we shall consider some boson contributing to fermion self-energy at 1-loop, be it a scalar or a vector. The contributions then are

\[
\Sigma^L(p^2) = f_L B_1(p^2, m^2_{\text{loop}}, m^2_{\text{bos}}),
\]

\[
\Sigma^R(p^2) = f_R B_1(p^2, m^2_{\text{loop}}, m^2_{\text{bos}}),
\]

\[
\Sigma^{*L}(p^2) = m_{\psi_{\text{loop}}} f_L B_0(p^2, m^2_{\psi_{\text{loop}}}, m^2_{\text{bos}}),
\]

\[
\Sigma^{*R}(p^2) = m_{\psi_{\text{loop}}} f_R^* B_0(p^2, m^2_{\psi_{\text{loop}}}, m^2_{\text{bos}}).
\] (34)

Here \(m_{\text{bos}}\) stands for the mass of the boson in the loop and \(m_{\psi_{\text{loop}}}\) for the mass of the fermion in the loop, which is not necessarily the same as that of external fermions (e.g. up-type quarks may have down-type quarks in the loop). \(B_1\) and \(B_0\) are Passarino–Veltman functions. The constants \(f_{L,R,s}\) include all the necessary symmetry factors and couplings of the theory, \(f_L\) and \(f_R\) are also hermitian as needed by pseudo-hermiticity. The divergent parts of Passarino–Veltman functions are well-known

\[
\left[B_1(p^2, m_i^2, m_j^2)\right]_{\text{div}} = \frac{1}{D - 4},
\]

\[
\left[B_0(p^2, m_i^2, m_j^2)\right]_{\text{div}} = -\frac{2}{D - 4}.
\] (35)

Here \(D\) is the spacetime dimension be it \(4 - \epsilon\) or \(4 - 2\epsilon\) depending on the preferred regularization.

We must also discuss the tadpole diagrams contributing only to \(\Sigma^{*L,R}\), which are multiplied by \(m_i\) or \(m_j\) in Eq. (21). For brevity we want only diagrams, which may produce the \(m_i^2 - m_j^2\) mass structure. Scalars and vectors in the loop can only contribute the corresponding scalar and vector masses, which cannot produce the squares of the external fermion masses \(m_i\) or \(m_j\). Tadpoles with fermion loops are also possible, so let us look at their contributions, in general [35]

\[
\Sigma^{*L} = f_T m_{\psi_{\text{loop}}} A_0(m^2_{\psi_{\text{loop}}}),
\]

\[
\Sigma^{*R} = f_T^* m_{\psi_{\text{loop}}} A_0(m^2_{\psi_{\text{loop}}}).
\] (36)
Here \( f_T \) is some non-chiral coupling and \( A_0 \) is a Passarino–Veltman function with the divergence

\[
[A_0(m^2)]_{\text{div}} = \frac{2m^2}{D-4}.
\]

With this we may consider taking the definition the anti-hermitian part of the field renormalization in Eq. (21) and then taking its divergent part

\[
\left[ \delta Z^A_{L,j;i} \right]_{\text{div}} = -\left[ m^2_j \Sigma_{ji}(m^2_j) + m_j m_i \Sigma_{ji}^R(m^2_j) + m_i \Sigma_{ji}^L(m^2_j) \\
+ m_j^2 \Sigma_{ji}(m^2_j) + m_i m_j \Sigma_{ji}^R(m^2_j) + m_j \Sigma_{ji}^L(m^2_j) \right]_{\text{div}, m^2_j - m^2_j} = \frac{1}{D-4} \left[ f_L(m^2_j + m^2_j) + f_R 2m_j m_j - 4 f_j^3 m_j m_{\phi, \text{loop}} - 4 f_j m_j m_{\phi, \text{loop}} - 4 m^3_{\phi, \text{loop}} \left( m_j f_T + m_j f_T^3 \right) \right]_{m^2_j - m^2_j} = 0.
\]

Analogously,

\[
\left[ \delta Z^A_{R,j;i} \right]_{\text{div}} = 0.
\]

Notice that there is no way to arrange the couplings such that \( m_j^2 - m_j^2 \) appears. One may also find that the hermitian part of the field renormalization, which is the same as in the standard approach, contains only terms that are proportional to \( m_j^2 - m_j^2 \) and is UV divergent.

For the mass counterterms one may do the same analysis by taking the divergent part of Eq. (20), which is the same as in Eq. (38) before taking the coefficient of \( m_j^2 - m_j^2 \)

\[
2 \left[ m_j \delta m^L_{ji} + m_i \delta m^R_{ji} \right]_{\text{div}} = \frac{1}{D-4} \left( f_L(m^2_j + m^2_j) + f_R 2m_j m_j - 4 f_j^3 m_j m_{\phi, \text{loop}} - 4 f_j m_j m_{\phi, \text{loop}} - 4 m^3_{\phi, \text{loop}} \left( m_j f_T + m_j f_T^3 \right) + \text{non-fermion tadpoles} \right).
\]

Here we additionally include the non-fermion tadpoles, which also produce the \( m_j \) and \( m_j \) structures. It is important to note that all of these divergences can in principle be associated with the mass counterterms by also noticing the mass structures. It is fairly simple to check that if the mass is renormalized as

\[
m_0 \to \delta m^L_P P_L + \delta m^R_P P_R + \left( 1 + \delta m^+ P_L + \delta m^- P_R \right) m \left( 1 + \delta m^+ P_R + \delta m^- P_L \right),
\]

the \( m_j^2 + m_j^2, 2m_j m_j, m_i \), and \( m_j \) mass structures appear along \( \delta m^+, \delta m^-, \delta m^R, \delta m^L \) respectively. While this renormalization is valid and may be useful in MS schemes, it does not seem very convenient in an OS scheme as it is more simple to just solve for the mass counterterms \( \delta m^{L,R} \) as we have done instead of also picking relevant terms in the finite parts.

To reiterate, it is clear that in our scheme the anti-hermitian part of the field renormalization is UV-finite since there are no UV divergent terms with the mass structure \( m_j^2 - m_j^2 \) in Eq. (38), i.e. the mass counterterms are enough to take care of the relevant divergences. These UV properties are extremely important when discussing the renormalization of mixing matrices such as the CKM.

3.3 Vanishing of the mixing matrix counterterm and comparison with other schemes

3.3.1 The scheme of Denner and Sack

Let us consider the CKM matrix as renormalized in [1] to see that there is no need for a mixing matrix counterterm. There the authors take the first scenario, Eq. (1), and impose the OS conditions with \( Z = \bar{Z} = 1 \), although this means overspecification of the field renormalization. In this case there are no-off diagonal mass counterterms so that the field renormalization has to account for all the divergences coming from the condition in Eq. (15). Since the field renormalization appears in all the terms in the Lagrangian where the fields are present, the divergences appearing in the mass term are propagated by the field renormalization to other terms in the Lagrangian. Exactly this made the \( W \) vertex UV-divergent and that is why the authors in [1] needed to renormalize the CKM matrix. The authors then found that a mixing matrix counterterm

\[
\delta V^\text{CKM}_{ji} = \frac{1}{2} \sum_{k=1}^{3} \delta Z_{L,ik}^{A,u} V^\text{CKM}_{ik} - \frac{1}{2} \sum_{k=1}^{3} V^\text{CKM}_{jk} \delta Z_{L,ki}^{A,d}
\]

with \( \delta Z^{u,d} \) being the field renormalization of up- and down-type quarks, is needed to cancel the UV divergences—but this is so only in the OS scheme without off-diagonal mass counterterms. Meanwhile, we have shown that in our scheme the anti-hermitian part
of the field renormalization is UV finite such that the counterterm in Eq. (42) is also UV-finite. In turn, in our scheme the \( W \) vertex is UV-finite without any additional counterterms and this shows that there is really no need for a mixing matrix counterterm as we have argued on more general grounds in Sect. 2.1.

The counterterm in Eq. (42) is gauge-dependent so that it also ruins the gauge-dependence of the \( W u j d i \) amplitude. It is important to note that while our definitions of renormalization constants do solve the problem of gauge-dependence of the CKM counterterm (trivially, \( \delta V = 0 \)), that is not enough to completely remove the gauge-dependence for physical amplitudes. As shown explicitly in Section 6 of [26], it is needed to have LSZ factors for both incoming and outgoing particles to ensure that no gauge-dependence appears in \( W u j d i \) vertices. In our scheme the LSZ factor for incoming particles is already included in the field renormalization, while the one for outgoing particles, \( \tilde{Z} \), is purely absorptive and must be computed from Eq. (13b). In terms of trading the LSZ factors for field renormalization this seems to be the best one can do, since it is impossible to have both LSZ factors accounted for by the field renormalization without losing Hermiticity of the Lagrangian [12,26].

### 3.3.2 The scheme of Kniehl and Sirlin

Next we compare our scheme with the one found in [6], where the authors begin with the second scenario, Eq. (2), but then derive a gauge-independent CKM counterterm by the procedure of rotating away the off-diagonal mass counterterms. We repeat the argument here for clarity and convenience.

At tree-level mixing matrices may appear after diagonalization of mass matrices. The same logic may be used beyond tree-level, for example, consider the r.h.s. of Eq. (6) and diagonalize the mass counterterm since the renormalized mass \( m_i \) is already diagonal. To do so, we consider a biunitary\(^7\) rotation with

\[
\mathcal{U}_{L,R} = 1 + i h_{L,R},
\]

where \( h_{L,R} \) are hermitian matrices and \( \mathcal{U}_{L,R} \) are unitary matrices up to 1-loop order, i.e. they perform the “1-loop rotation”. This rotation may be arranged in a way that off-diagonal mass counterterms are rotated away. Given the mass terms

\[
\mathcal{L}_{\text{mass}} = -\bar{\psi}_R \mathcal{U}_R^L \left( m + \delta m^L \right) \mathcal{U}_L \psi_L + (L \leftrightarrow R)
\]

and requiring that

\[
\mathcal{U}_R^L \left( m + \delta m^L \right) \mathcal{U}_L = \text{diagonal},
\]

\[
\mathcal{U}_L^L \left( m + \delta m^R \right) \mathcal{U}_R = \text{diagonal}
\]

one gets

\[
\begin{align*}
ih_{L_{ji}} &= \frac{m_j \delta m^L_{ji} + \delta m^R_{ji} m_i}{m_i^2 - m_j^2} & \text{for } i \neq j ,
\end{align*}
\]

\[
\begin{align*}
ih_{R_{ji}} &= \frac{m_j \delta m^R_{ji} + \delta m^L_{ji} m_i}{m_i^2 - m_j^2} & \text{for } i \neq j ,
\end{align*}
\]

\[
i h_{L_{ii}} = i h_{R_{ii}} = 0.
\]

It is simple to check that these diagonalize the mass counterterms by insertion, however, for more detail on the derivation we refer to [6,15].

To see the effects of this rotation let us consider fermions \( f \) and \( l \) that mix in a chiral interaction with some mixing matrix \( V \), vector \( A_{\mu} \), and coupling \( g \) (as quarks do in the SM)

\[
g \hat{f}_L A^\mu V_{fl} l_L.
\]

Here all the quantities are already renormalized and we only need to rotate the fermions \( f, l \) with \( \mathcal{U}^{f,l} \)

\[
g \left( V_{fl} + \sum_{f'} (-i h_{L_{ff'}}^l) V_{fl'} + \sum_{f'} V_{fl'} (i h_{L_{ff'}}^l) \right) \tilde{f}_L A^\mu \tilde{l}_L = g \left( V_{fl} + \delta V_{fl}^h \right) \tilde{f}_L A^\mu \tilde{l}_L.
\]

Here the rotated fields are denoted by hats and we also introduced the shorthand notation \( \delta V_{fl}^h \) for terms containing \( V \) and \( i h \)'s

\[
\delta V_{fl}^h = \sum_{f'} (-i h_{L_{ff'}}^l) V_{fl'} + \sum_{f'} V_{fl'} (i h_{L_{ff'}}^l).
\]

\(^7\) Biunitary rotation for Dirac fermions, while a unitary one is enough for Majorana fermions [15].
It is noteworthy that this counterterm is UV divergent, but gauge-independent as long as the mass counterterms are defined in a gauge-independent way. The authors of [6] arrive at gauge-independent mass counterterm by keeping the terms that are free of $\not{p} - m_i$ or $\not{p} - m_j$ in Eq. (15). These factors accompany the field renormalization and give rise to the factors $m_i^2 - m_j^2$ used in our scheme. As we will see in Sect. 4 our mass counterterms and the counterterms of [6] have the same UV divergences (up to tadpole contributions), so that the anti-hermitian part of the field renormalization is finite in both schemes. To see that this must be the case, let us take Eq. (20) with the l.h.s. expressed in terms of $\hat{Z}_{\pm,ij}$

$$
(m_i^2 - m_j^2) \left( \delta Z_{\pm,ij} - 2i h_{L,ij} \right) = -\left( m_i^2 \Sigma_{ji}^L (m_i^2) + m_j \Sigma_{ji}^R (m_i^2) + m_j \Sigma_{ji}^L (m_j^2) + m_i \Sigma_{ji}^R (m_j^2) \right) + H.C.
$$

Here it is obvious, that if $ih$ or, equivalently, the mass counterterms take care of all the divergences of the r.h.s. the anti-hermitian part of the field renormalization must be UV finite. At this point the two schemes are rather similar with the rotation $\mathcal{U}_{L,R}$ being the only difference, however, actually performing the 1-loop rotation comes with two issues.

First, in the degenerate mass limit the r.h.s. of the above equation is in general not vanishing, so that either $ih_L$ or $\delta Z_{L}^A$ must be singular in this limit. Obviously, Eq. (46a) tells us that $ih_L$ is singular in this limit, which is a feature causing numerical issues, such as unnaturally enhanced parameter space regions where the masses $m_i$ and $m_j$ happen to be close. Our scheme has no such singular terms.

Secondly, and maybe more importantly, the 1-loop rotation does not actually change anything apart from relabeling. To see this, notice that the rotations with $\mathcal{U}_{L,R}$ in Eq. (43) do not change the renormalized Lagrangian, but remove the off-diagonal mass counterterms. Of course, Eq. (50) comes from the no-mixing condition for outgoing particles in Eq. (15), where the spinor should be rotated with the same matrices $\mathcal{U}_{L,R}$. After rotating the spinor with $\mathcal{U}_{L,R}$, the no-mixing condition trivially gives back the same Eq. (50) so that the effect of the 1-loop rotation is to associate $ih_L$ with the LSZ factors instead of the mass counterterms. More explicitly, we may write Eq. (15) before and after the 1-loop rotation

$$
\Sigma_{ji} u_i = \left( \hat{\Sigma}_{ji} + (\not{p} - m_j) \left( ih_{L,ji} P_L + ih_{R,ji} P_R \right) \right) \tilde{u}_i, \quad (51)
$$

where the hats denote quantities after the 1-loop rotation and $\hat{\Sigma}_{ji}$ has only diagonal mass counterterms, but is otherwise the same as $\Sigma_{ji}$. Importantly, not only the mass terms are effectively unchanged, but also the amplitudes containing the mixing matrices are left unchanged, since $ih_L$ first enters the vertex through rotations of the fields by $\mathcal{U}_{L,R}$, but it is then canceled by the LSZ factors $(ih_{L,R})$ of the rotated external spinors.

In our scheme we do not perform this one-loop rotation as it is singular and, with rotation of the spinors taken into account, it has no effect whatsoever. Nonetheless, in Sect. 4 we will present the counterterm in Eq. (49) for the sake of comparing the divergent parts of our scheme and the one of [6].

### 3.3.3 The scheme of Baro and Boudjemaa

For the final comparison it has been brought to our attention that the end result in our scheme is rather similar to the approach considered for fermions in the MSSM in [19]. There the authors use the OS conditions with $Z = \tilde{Z} = \overline{\Sigma}$ and also explicitly use the $\Re$ operator. They define the mass counterterm as the residue of the $1/(m_i^2 - m_j^2)$ pole and effectively solve for field renormalization. Our approach is in that sense opposite—defining the field renormalization as the coefficient of $m_i^2 - m_j^2$ and solving for mass counterterms. Both in our scheme and the one of [19] the mixing matrices receive no counterterms and are basis-dependent quantities derived from renormalized parameters.

In more detail, the scheme of [19] considers the mixing of two particles $\tilde{f}_{1,2}$ and considers all the scalar self-energies as functions of both masses and momenta $\Sigma \left(p^2, m_{f_{1,2}}^2\right)$ such that on-shell they are $\Sigma \left(m_{f_{1,2}}^2, m_{f_{1,2}}^2\right)$ and can be taken as a function of two variables

$$
m_{\pm}^2 = \frac{m_{f_{1}}^2 \pm m_{f_{2}}^2}{2}. \quad (52)
$$

Of course, $m_{\pm}^2$ is the difference of the squared masses that we also use in our scheme. Further, the authors instruct to expand the relevant equations such as Eq. (20) in terms of $m_{\pm}^2$ and associate the terms proportional to powers of $m_{\pm}^2$ to the field renormalization, while the remaining terms are then attributed to mass counterterms. Although this is similar to our scheme, there are important differences. One of the important distinctions is that if the logic of [19] is applied to fermions one arrives at a UV-finite anti-hermitian part of the field renormalization only for degenerate masses. This is because functions of the masses $m_i$ and $m_j$ are effectively expressed in terms of $m_{\pm}^2$. When rewriting a UV-divergent term proportional to some arbitrary function of masses (e.g. $2m_i m_j$) in terms of the new variables one in general generates UV-divergent terms proportional to powers of the difference of squares $m_i^2 - m_j^2$.

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8 Thanks to H. Rzehak.
Such terms are then attributed to the anti-hermitian part of the field renormalization and vanish only in the degenerate mass limit. In contrast, our approach has a finite anti-hermitian part of the field renormalization for all masses, which is very important for the renormalization of mixing matrices as discussed in Sect. 3.3.1.

4 Examples of the proposed scheme

Before proceeding to the introduction of the Grimus–Neufeld model and the presentation of examples, we would like to mention the software used in the following computations. For the Grimus–Neufeld model we adapted the Two Higgs Doublet Model (THDM) files [36] for FeynRules-2.3.41 [39]. Using FeynRules we generated model files for FeynArts-3.11 [40] and performed the computations using FeynCalc-9.3.1 [41–43] in combination with PackageX-2.1.1 [44]. Naturally, the conventions for Passarino–Veltman functions follow those used in the packages.

4.1 Introduction to the Grimus–Neufeld model

To present our scheme we use the Grimus–Neufeld model, which is the Standard Model extended with an additional Higgs doublet and a single heavy Majorana neutrino. Parts of this introduction can also be found in [18,45,46].

In the Higgs sector we have the most general Higgs potential for the THDM, but in the CP-conserving case where the parameters $\mu_{12}, \lambda_5, \lambda_6, \lambda_7$ are real [47,48]

$$-V_{\text{THDM}} = \mu_1 (H_1^+ H_1) + \mu_2 (H_2^+ H_2) + \left[ \mu_{12} (H_1^+ H_2) + H.C. \right] + \lambda_1 (H_1^+ H_1) (H_1^+ H_1) + \lambda_2 (H_2^+ H_2) (H_2^+ H_2) + \lambda_3 (H_1^+ H_1) (H_2^+ H_2) + \lambda_4 (H_1^+ H_2) (H_2^+ H_1) + \left[ \lambda_5 (H_1^+ H_2) (H_1^+ H_2) + \lambda_6 (H_1^+ H_1) (H_2^+ H_2) + \lambda_7 (H_2^+ H_2) (H_1^+ H_2) + H.C. \right].$$

(53)

We use the Higgs basis, meaning that the vacuum expectation value (VEV) $v$ as well as all of the Goldstone bosons $G_0^0, \pm$ are in the first Higgs doublet:

$$H_1 = \left( \frac{G^+}{\sqrt{2}} (v + h_1 + iG^0) \right) \quad \text{and} \quad H_2 = \left( \frac{H^+}{\sqrt{2}} (h_2 + i\sigma) \right).$$

(54)

The mass eigenstate may be reached via an orthogonal rotation with the matrix $T$

$$\begin{pmatrix} h_1 \\ h_2 \\ \sigma \end{pmatrix} = T \begin{pmatrix} h \\ H \\ A \end{pmatrix}$$

(55)

where $h$ and $H$ are scalars and $A$ is a pseudo-scaler. The corresponding masses will be denoted as $m_H, m_A,$ and $m_{H^+}$. In the CP conserving case, with which we deal, one simply has $\sigma = A$ and

$$T = \begin{pmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(56)

where $\alpha$ is the mixing angle and we have abbreviated sines and cosines as

$$s_x \equiv \sin x \quad \text{and} \quad c_x \equiv \cos x.$$

(57)

In the upcoming examples we will be using the mass eigenstate basis and will express the couplings $\lambda_j$ in terms of masses where possible. The couplings $\lambda_3$ and $\lambda_7$ are among the ones left undefined by going to the mass eigenstates and will appear in various expressions.

We also have the interaction between the Higgs doublets and fermions

$$-\mathcal{L}_{\text{H–F}} = 3 \sum_{j,i=1}^3 (Y_d)_{ji} \bar{d}_{Rj} \cdot (Q_i H_1^+) + 3 \sum_{j,i=1}^3 (Y_u)_{ji} \bar{u}_{Rj} \cdot (Q_i \tilde{H}_1) + 3 \sum_{j,i=1}^3 (Y_l)_{ji} \bar{e}_{Rj} \cdot (L_i H_1^+) + 3 \sum_{i=1}^3 (Y_{\nu})_{ji} \bar{\nu}_i \cdot (L_i \tilde{H}_1) + 3 \sum_{j,i=1}^3 (G_d)_{ji} \bar{d}_{Rj} \cdot (Q_i H_2^+) + 3 \sum_{j,i=1}^3 (G_u)_{ji} \bar{u}_{Rj} \cdot (Q_i \tilde{H}_2) + 3 \sum_{j,i=1}^3 (G_l)_{ji} \bar{e}_{Rj} \cdot (L_i H_2^+) + 3 \sum_{i=1}^3 (G_{\nu})_{ji} \bar{\nu}_i \cdot (L_i \tilde{H}_2) + H.C.$$

(58)

$^7$ Version 1.2 of the THDM model files contains a bug in the ghost Lagrangian. Comparing that Lagrangian with Eq. (21.52) in Peskin&Schroeder [38] we see that the gauge parameter is missing.
Here \( N \) is the Majorana neutrino that couples as a right-handed particle. \( L \) and \( Q \) are the left-handed lepton and quark doublets, respectively, \( u_R, d_R \) and \( e_R \) are the right-handed up-quark, down-quark, and charged lepton singlet fields, \( \tilde{H}_{1,2} \) correspond to Higgs doublets contracted with the anti-symmetric tensor, i.e. \( \tilde{H}_{1,2} = \sum_{i=1}^{2} \epsilon_{ji}(H_{1,2}) \), \( Y' \)’s and \( G' \)’s are the Yukawa couplings to the first and second Higgs doublets. The dots emphasize the contraction of fermion indices, while \( i, j \) are family indices and parentheses imply the contraction of \( SU(2) \) indices. For the quarks one can assume the textbook procedure, where the Yukawa couplings \( Y_{d,u} \) to the first Higgs doublet have to be diagonalized so that the CKM mixing matrix appears in the \( W \) vertex. This also gives the masses which we label as \( m_{j}^{u,d} \) for up- and down-type quarks with \( j = 1, 2, 3 \) the family index.

We also add a Majorana mass term for the heavy Majorana singlet \( N \)

\[
\mathcal{L}_{\text{Maj}} = -\frac{1}{2} M_R \tilde{N} \cdot N + H.C.
\]  
(59)

For convenience, let us take the charged leptons to already be in the mass eigenstate basis, i.e. \((Y_i)_{ji} = \sqrt{\frac{2}{v^2}} m_j^4 \delta_{ji}\).

Then for the neutrinos at tree level we have a non-diagonal symmetric mass matrix

\[
M' = \begin{pmatrix}
0_{3 \times 3} & v^n Y_v \\
v^n Y_v^T & M_R
\end{pmatrix},
\]  
(60)

which may be diagonalized via a unitary transformation

\[
U^T \tilde{m}' U = M'.
\]  
(61)

Here \( \tilde{m}' \) is a diagonal matrix. For convenience, we decompose the neutrino mixing matrix as

\[
U = \begin{pmatrix}
U^T_L & U^T_R
\end{pmatrix},
\]  
(62)

where \( U_L \) is \( 3 \times 4 \) and \( U_R \) is \( 1 \times 4 \). The matrices \( U_L \) and \( U_R \) also satisfy a few constraints coming from unitarity and the form of the neutrino mass matrix

\[
U^T_L U_L = 13_{3 \times 3}, U^T_R U_R = 1, U^T_L U_L + U^T_R U_R = 14_{4 \times 4}, U^T_R U_L = 0, U^T_R \tilde{m} U_L = 0.
\]  
(63)

There is another important relation due to the form of the mass matrix \( M' \). Since \( M' \) is a matrix of rank two, there are only two non-zero eigenvalues, such that \( \tilde{m} = \text{diag}(0, 0, m_3, m_4) \). Having this it is not hard to find from

\[
\tilde{m}' U = U^* M'
\]  
(64)

that \( U_R \) is of the form

\[
U_R = \begin{pmatrix}
0 & 0 & u_R 3 & u_R 4
\end{pmatrix}.
\]  
(65)

The two zeroes prove to be useful, for example,

\[
(U^T_L U_L)_{ji} + (U^T_R U_R)_{ji} = \delta_{ji}
\]  
(66)

if \( j \) and/or \( i \) correspond to a massless neutrino (i.e. \( i \) or \( j \) equal to 1 or 2).

In addition, at tree level the matrix \( U^T_L \) is not fully determined and the first two rows of \( U^T_L \) are constrained only by unitarity. Of course, the first two rows correspond to the two massless neutrinos, which are indistinguishable at tree level. Beyond tree level we use this freedom to rotate the Yukawa coupling \( G_{v} \) such that the upper component is 0, see Eq. \( (77) \) and the discussion above the said equation.

Beyond tree level we must add appropriate counterterms. However, in the Grimus–Neufeld model we have massless neutrinos at tree level and it is not straightforward to pick out the mass structures as required by our definitions. To make the situation clearer let us schematically single out a few cases corresponding to \( 2 \times 2 \) blocks in the 1-loop mass matrix

\[
\tilde{m}_{ij} + \delta m_{ij} \sim \begin{pmatrix}
\delta m_{ij}, m_j = 0 & \delta Z^A, \delta m_{ij} \neq 0 \\
\delta Z^A, \delta m_{ij} = 0 & \delta Z^A, \delta m_{ij} \neq 0
\end{pmatrix}.
\]  
(67)

Here \( \delta m \) and \( \delta Z^A \) serve as shorthand notations for left- and right-handed mass and anti-hermitian part of field counterterms. In the bottom-right block, where \( j, i > 2 \) and the corresponding neutrinos are massive, our definitions for mass and field counterterms work out of the box.

The off-diagonal \( 2 \times 2 \) blocks, where \( j < 3 \) and \( i > 2 \) or vice versa, correspond to the cases where either the incoming (indexed by \( i \)) or outgoing (indexed by \( j \)) neutrino is massless, in this case there is no way to rely on the \( m_i^2 - m_j^2 \) mass structure as needed in the definition of anti-hermitian part of field counterterms. However, for the off-diagonal blocks we perform the computation in the case where both neutrinos are massive and afterwards simply take the massless limit for one of the neutrinos. In this way, we
do not introduce gauge-dependence in the mass counterterm or divergences in the anti-hermitian part of the field renormalization. Hence, the counterterms still satisfy all the properties we have highlighted in previous sections.

In the top-left block, where \( j, i < 3 \), the anti-hermitian part of the field renormalization must vanish since swapping the masses has no effect and there are no absorptive parts. Therefore, only the \( \delta m \) contributions are of interest. In the massless limit dealing with \( \delta m \) is especially easy as we do not have to compute self-energies and simply can take the limit directly in the definition in Eq. (22), so that for this block we have

\[
\delta m^L_{ji} = \frac{1}{2} (\Sigma^{UL}_{ji}(0) + \Sigma^{LR}_{ji}(0)) = \Sigma^{UL}_{ji}(0)
\]

and

\[
\delta m^R_{ji} = \frac{1}{2} (\Sigma^{SR}_{ji}(0) + \Sigma^{SL}_{ji}(0)) = \Sigma^{SR}_{ji}(0).
\]

This block is special in the sense that there are no field renormalization contributions. Nonetheless, this block is gauge-independent and also finite as all of the divergences come with mass structures as can be seen in Eq. (40). In addition, this may be diagonalized by choosing appropriate \( U_L \) components as is done in Eq. (77). Having all this, we can regard this block as the radiative mass instead of a counterterm (there is nothing to “counter” in the massless block in the bare Lagrangian), this has been noted in [18] as well. In addition, this block, as well as the off-diagonal ones, benefit from the form of \( U_R \) and the resulting expressions are fairly tame, see Sect. 4.3.1.

In the following sections we first consider the quark sector as an example since it is much simpler and only then discuss the neutrinos.

4.2 Quarks

Let us start with the anti-hermitian part of the field renormalization for up-type quarks by simply applying the definition in Eq. (21)

\[
\delta Z_{L_{ji}}^{A,u} = -\sum_{k=1}^{3} \frac{V_{jk}V^*_k}{2iD_D^{-2}v^2} \left[ \begin{array}{c} (m^d_k)^2((D-3)m^2_{W} - 2(m^u_{ij})^2) + (m^d_{ij})^2 - (m^u_{ij})^2 \end{array} \right] B_0((m^u_{ij})^2, (m^d_{ij})^2, m^2_{W})
\]

\[
+ \left[ (m^d_{ij})^2 + (D-2)m^2_{W} + (m^u_{ij})^2 \right] A_0(m^2_{W}) - (m^d_{ij})^2 A_0(m^2_{W})
\]

\[
- (m^d_{ij})^2 - (m^u_{ij})^2 \left[ (m^d_{ij})^2 - (m^u_{ij})^2 + \xi W m^2_{W} \right] B_0((m^u_{ij})^2, (m^d_{ij})^2, \xi W m^2_{W}) + H.C. \bigg|_{m^2_{W} - m^2_{ij}}
\]

\[
= -\sum_{k=1}^{3} \frac{V_{jk}V^*_k}{2iD_D^{-2}v^2} \left[ (m^d_{ij})^2 - (m^u_{ij})^2 + \xi W m^2_{W} \right] B_0((m^u_{ij})^2, (m^d_{ij})^2, \xi W m^2_{W})
\]

\[
+ \sum_{k=1}^{3} \frac{V_{jk}V^*_k}{2iD_D^{-2}v^2} \left[ (m^d_{ij})^2 - (m^u_{ij})^2 + \xi W m^2_{W} \right] B^*_0((m^u_{ij})^2, (m^d_{ij})^2, \xi W m^2_{W}).
\]

(70)

Here in the first equality we only kept the SM contributions since only these enter the anti-hermitian part of the field renormalization in the full Grimus–Neufeld model, \( V \) is the CKM matrix, and since \( i \neq j \) we have dropped terms with \( \delta_{ij} \). Note that the function \( B^*_0((m^u_{ij})^2, (m^d_{ij})^2, \xi W m^2_{W}) \) is complex conjugated so that the absorptive parts are included in the field renormalization. Applying the same procedure for the right-handed part we get

\[
\delta Z_{R_{ji}}^{A,u} = 0
\]

(71)

which nicely reflects the chirality of the model. Importantly, one may check that the anti-hermitian part of the field renormalization is indeed finite by using Eq. (35) and the unitarity of the CKM matrix. Finally, analogous expressions may be found for down-type quarks by \( u \leftrightarrow d \) and complex conjugation of \( V \) in the above results for the field renormalization.

The left-handed mass counterterm for the up-type quarks \( \delta m_{ji}^{L,u} \) in the full Grimus–Neufeld model is expressible in terms of Passarino–Veltman functions, but is rather unwieldy and is given in “Appendix B”.

To complete the quark example, we want to compare the UV-divergent parts of the mass counterterms in our scheme with the ones of the SM CKM counterterm found by the authors of [1,6]. For the sake of comparison we will perform the 1-loop rotation, which we consider to be inconsistent in Sect. 2.1, i.e. we must compute \( i\hbar \lambda \) in Eq. (46a). With that in mind, let us first list only the divergent part of \( \delta m_{ji}^{L,u} \) and \( \delta m_{ji}^{R,u} \) in case of the SM (i.e. dropping couplings to the second Higgs doublet)

\[
[\delta m_{ji}^{L,u}]_{\text{div,SM}} = -\frac{1}{\epsilon_{UU}} \frac{3m^u_{ij}(V(m^d)^2V^*)_{ji}}{2\pi^2v^2},
\]

\[
[\delta m_{ji}^{R,u}]_{\text{div,SM}} = -\frac{1}{\epsilon_{UU}} \frac{3m^u_{ij}(V(m^d)^2V^*)_{ji}}{2\pi^2v^2}.
\]

(72)
Again, to get the down-type quark result simply exchange $u$ with $d$ and hermitian conjugate\textsuperscript{10} $V$. One can already compare these with the results in Eq. (42) of [6] and see that the divergent parts of mass counterterms match. To compare with [1] we can use the off-diagonal mass counterterms and write the corresponding divergent parts of $i h_L$'s for up and down quarks

$$
[i h^u_{L ji}]_{\text{div.,SM}} = -\frac{3 (V (m_d^u)^2 V^\dagger)_{ji}}{32 \pi^2 v^2 \epsilon_{UV}} \cdot \frac{(m_u^d)^2 + (m_u^d)^2}{(m_u^d)^2 - (m_u^d)^2},
$$

$$
[i h^d_{L ji}]_{\text{div.,SM}} = -\frac{3 (V^\dagger (m_u^d)^2 V)_{ji}}{32 \pi^2 v^2 \epsilon_{UV}} \cdot \frac{(m_u^d)^2 + (m_u^d)^2}{(m_u^d)^2 - (m_u^d)^2}.
$$

While not needed for the counterterm of the CKM matrix, it is interesting to write down $\delta m_R$'s for up and down quarks

$$
[i h^u_{R ji}]_{\text{div.,SM}} = -\frac{3 (V (m_d^u)^2 V^\dagger)_{ji}}{32 \pi^2 v^2 \epsilon_{UV}} \cdot \frac{2 m_u^d m_j^u}{(m_u^d)^2 - (m_u^d)^2}.
$$

We see that there are different mass structures $(m_u^d)^2 + (m_u^d)^2$ and $2 m_u^d m_j^u$ for the left- and right-handed particles. These are also the same structures we noticed when discussing the UV divergences in Sect. 3.2.

Now we can write down the divergent part of Eq. (49)

$$
[\delta V^\dagger_{ji}]_{\text{div.,SM}} = -\sum_{n\neq j} i h^u_{Ljn} V_{nj} + \sum_{k\neq i} V_{jk} i h^d_{Lki}
$$

$$
= \frac{3}{32 \pi^2 v^2 \epsilon_{UV}} \left[ \sum_{k \neq j} (V (m_d^u)^2 V^\dagger)_{jk} \cdot \frac{(m_u^d)^2 + (m_u^d)^2}{(m_u^d)^2 - (m_u^d)^2} V_{ki} - \sum_{k \neq i} V_{jk} \frac{(m_u^d)^2 + (m_u^d)^2}{(m_u^d)^2 - (m_u^d)^2} \cdot (V^\dagger (m_u^d)^2 V)_{ki} \right].
$$

Comparing with the divergent parts found in [1], we see that the results match up to factors of 2 due to regularization.

4.3 Neutrinos

Now we move on to examples with neutrinos where we have plenty of scenarios. We shall begin with the easiest case where both the incoming and outgoing particles are massless, then consider the fully massive case, and finally in the massive case we will take one of the particles to be massless to get to the partially massive case (i.e., one of the off-diagonal blocks).

4.3.1 Fully massless case ($i, j < 3$)

In the fully massless case we only need to evaluate one scalar self-energy function to get the off-diagonal contributions to the mass (for example, $\Sigma_{12}^L(0)$). For the neutrino mass contributions $\delta m_{ji}^{L,R,v}$ we easily get

$$
\delta m_{ji}^{L,v} = \sum_{a=1}^4 \frac{(U_T^L G_v U_R^*)_{ja} m_v^u}{2 D_{1+1} D_{2-2}} \left[ s_a^2 B_0(0, m_h^2, (m_u^v)^2) + c_a^2 B_0(0, m_h^2, (m_u^u)^2) - B_0(0, m_A^2, (m_u^v)^2) \right] (U_R^T G_v^T U_L)_{ai},
$$

$$
\delta m_{ji}^{R,v} = \sum_{a=1}^4 \frac{(U_T^R G_v U_R^*)_{ja} m_v^u}{2 D_{1+1} D_{2-2}} \left[ s_a^2 B_0(0, m_h^2, (m_u^v)^2) + c_a^2 B_0(0, m_h^2, (m_u^v)^2) - B_0(0, m_A^2, (m_u^v)^2) \right] (U_R^T G_v^T U_L)_{ai}.
$$

We got $\delta m_{ji}^{R,v}$ via hermitian conjugation of $\delta m_{ji}^{L,v}$ and by noticing that the Passarino–Veltman functions are real in the massless case. In evaluating the scalar self-energy function we used the unicity relations of $U_L$ and $U_R$ matrices in the massless case, Eq. (66). The above contributions are for $i \neq j$, however, the real part can also be used for $i = j$ as mentioned in Sect. 2.4. In other words, we regard the real part of above contributions for the whole massless block instead of its off-diagonal components only.

It is easy to note that the UV divergent parts cancel since the divergences of $B_0$ functions do not depend on the arguments. Even more easily we see the explicit gauge-independence. As noted in the model introduction, we think of these contributions as a radiative mass. In addition, this contribution can be diagonalized by noticing the remaining freedom in the first two rows of $U_L^T$. To make this contributions diagonal we can choose one of the two rows of $U_L^T$ to be orthogonal to $G_v$ [18]. Alternatively, diagonalization can be made more simple by defining a new Yukawa coupling to the second Higgs doublet (analogous to the one in [45])

$$
U_L^T G_v = G_v' = \begin{pmatrix} 0 & \delta_2^u & \delta_3^u & \delta_4^u \\ \delta_2^v & \delta_3^v & \delta_4^v \end{pmatrix}.
$$

\textsuperscript{10} Previously we had complex conjugation simply due to the notation in which we wrote down our results. Complex conjugation is for indexed notation and hermitian conjugation is for matrix notation.
With this coupling it is evident that at 1-loop the $2 \times 2$ massless block is diagonal and that one of the massless tree-level neutrinos acquires a radiative mass, while the other neutrino remains massless. The remaining massless neutrino may acquire a mass beyond 1-loop [18]. In other words, at 1-loop we have
\begin{equation}
\begin{aligned}
m_1 &= \text{Re}[\delta m_{11}^{\nu}] = \text{Re}[\delta m_{11}^{\bar{\nu}}] = 0, \\
m_2 &= \text{Re}[\delta m_{22}^{\nu}] = \text{Re}[\delta m_{22}^{\bar{\nu}}] = \text{Re} \left[ \tilde{C} (\nu^2 \nu^2) \right].
\end{aligned}
\end{equation}
Here we used the dimensionality of $U_R$ and $G'_\nu$ in the Grimus–Neufeld model to define the constant $\tilde{C}$
\begin{equation}
\tilde{C} = \sum_{a=1}^{4} \frac{U_{Ria}^{*} U_{Ria}^{*} m_{\nu}^{\nu}}{2 D_{Pi}^{*} D_{Pi}^{*}} \left[ \varepsilon_{a}^{2} B_{0}(0, m_{\nu}^{2}, (m_{\nu}^{\nu})^2) + \varepsilon_{a}^{2} B_{0}(0, m_{\nu}^{2}, (m_{\nu}^{\bar{\nu}})^2) = B_{0}(0, m_{\nu}^{2}, (m_{\nu}^{\nu})^2) \right].
\end{equation}
Analogous results for this model were also found in [18, 45].

4.3.2 Fully massive case ($i, j > 2$)

We next move to the fully massive case, which is more cumbersome as the expressions are larger and we also need to evaluate the Passarino–Veltman functions to see the needed mass structures. We will fully write out the anti-hermitian part of the left-handed field renormalization and only UV divergent parts of the left-handed off-diagonal mass counterterms.

As in the quark case, to find the left-handed anti-hermitian part of the field renormalization we pick out terms containing $(m_j^\nu)^2 - (m_i^\nu)^2$ from the expression in Eq. (21). However, upon evaluation, one may find terms with mass structures such as
\begin{equation}
m_j^\nu \left[ 3(m_i^\nu)^2 + (m_i^\nu)^2 \right].
\end{equation}
Terms like these are ambiguous as they can be rewritten in multiple ways in terms of the mass structures we saw already: $m_i$, $m_j$, $2m_i m_j$, $m_i^2 \pm m_j^2$. For example, the above might be written as
\begin{equation}
m_j^\nu \left( 2 \left[(m_i^\nu)^2 + (m_i^\nu)^2\right] + \left[(m_i^\nu)^2 - (m_i^\nu)^2\right] \right)
\end{equation}
or as
\begin{equation}
m_i^\nu \left[ 2m_i^\nu m_j^\nu \right] + m_j^\nu \left[(m_i^\nu)^2 + (m_i^\nu)^2\right].
\end{equation}
In other words, ambiguous terms like these do not allow for a clear separation of the $m_i^2 - m_j^2$ mass structure, however, Nielsen identities guarantee that no such ambiguous mass structures appear in gauge-dependent terms. In addition, these ambiguous terms cannot be distributed at will, since the massless limit of renormalization constants must exist. For the neutrinos in the Grimus–Neufeld model we were unable to get such a limit when the ambiguous terms were included in the field renormalization. Practically, this means that gauge-independent (and UV-finite) terms must immediately manifest the $m_i^2 - m_j^2$ mass structure without any rearrangements if they are to be included into field renormalization, otherwise these terms are to be included in mass renormalization. This respects our definitions and actually makes the picking of terms easier. Having this in mind, we write down the off-diagonal left-handed anti-hermitian part of the field renormalization for neutrinos $\delta Z_{Lj}^{\nu}$ in the fully massive case
\begin{equation}
\begin{aligned}
\delta Z_{Lj}^{A,\nu} &= -\text{Re} \left\{ \sum_{a=1}^{4} \sum_{i=1}^{3} \frac{(m_i^\nu)^3 T_{Lj}^{\nu} (U_L^\dagger : U_L)_{ai} (U_L^\dagger : U_L)_{aj}}{32 \pi^2 v^2 m_j^{\nu}} \log \left( \frac{(m_i^\nu)^2}{(m_i^\nu)^2} \right) \\
&+ \sum_{a=1}^{4} \frac{m_i^\nu (U_L^\dagger : U_L)_{ai} (U_L^\dagger : U_L)_{aj}}{64 \pi^2 v^2 (m_i^\nu)^3} \left( 2(m_i^\nu)^2 (m_j^\nu)^2 - (m_i^\nu)^2 - m_j^2 \xi \bar{Z} \right) \log \left( \frac{(m_i^\nu)^2}{(m_i^\nu)^2} \right) \\
&+ \frac{(U_L^\dagger : U_L)_{ai} (U_L^\dagger : U_L)_{aj}}{64 \pi^2 v^2 (m_j^\nu)^3} \left( (m_i^\nu)^4 + (m_i^\nu)^2 (m_j^\nu)^2 - m_j^2 \xi \bar{Z} \right)^2 \log \left( \frac{(m_i^\nu)^2}{(m_i^\nu)^2} \right) \\
&+ \frac{m_i^\nu (U_L^\dagger : U_L)_{ai} (U_L^\dagger : U_L)_{aj}}{32 \pi^2 v^2 (m_j^\nu)^3} \left( (m_i^\nu)^4 + (m_i^\nu)^2 (m_j^\nu)^2 - m_j^2 \xi \bar{Z} \right) \log \left( \frac{(m_i^\nu)^2}{(m_i^\nu)^2} \right) \\
&+ \frac{(U_L^\dagger : U_L)_{ai} (U_L^\dagger : U_L)_{aj}}{32 \pi^2 v^2} \left( (m_i^\nu)^2 - (m_j^\nu)^2 + m_j^2 \xi \bar{Z} \right) \right\}
\end{aligned}
\end{equation}
\[ \begin{align*}
+ \frac{m_j^2(U_L^\top U_j^\ast_{ai})(U_L^\top U_j^\ast_{aj})}{32\pi^2v^2m_j^2} & \left[ (m_\nu^2)^2 + (\bar{m}_\nu^2)^2 + m_Z^2\ell_Z \right] \Lambda((m_\nu^2)^2, m_\nu^2, m_Z\sqrt{\ell_Z}) \\
+ \sum_{k=1}^3 \frac{U_{Lk}^\dagger U_{Lkj}^\ast}{16\pi^2v^2} & \left( (m_k^2)^2 - (m_\nu^2)^2 + m_W^2\ell_W \right) \Lambda((m_k^2)^2, m_k^2, m_W\sqrt{\ell_W}) - H.C. \right) \end{align*} \]

Here \( i \neq j \), \( \Lambda(m^2, b, c) \) is the disc function as used in \texttt{PackageX} and \texttt{FeynCalc}, \( T_{ij} \) is the mixing matrix of the scalars which we left for generality, and \( U_L \) is defined in Eq. (62). It is interesting that we have found gauge-independent terms with the \( m_i^2 = m_j^2 \) mass structure coming from the Higgs sector. These terms are likely a manifestation of the "non-symmetric" implementation of the neutrinos in our model: the neutrino mass eigenstates are a result of the components coming from the \( SU(2) \) lepton doublets as well as the heavy Majorana singlet, however, these couple only through the Yukawa couplings \( Y_\nu \) and \( G_\nu \) to the Higgs doublets in Eq. (58). To get \( \delta Z_{ij}^{\nu, \nu} \) one can complex conjugate couplings of the above expression of \( \delta Z_{ij}^{\nu, \nu} \).

Having the field counterterms we can now write down the mass counterterms. However, the expressions are cumbersome enough that we only write down the UV divergent parts in the full Grimus–Neufeld model:

\[ \begin{align*}
[\delta m_j^{L, \nu}]_{\text{div.}} &= \left( U_L^\dagger U_L^\ast_{ij} \right) \frac{m_j^2}{4\pi^2v^2\epsilon_{UV}} \left[ \frac{c_a^2}{m_h^2} + \frac{s_a^2}{m_h^2} \right] \text{Tr} \left\{ (m_\nu^2)^4 U_L^\dagger U_L + (m_\nu^2)^4 + (m_\mu^2)^4 + (m_\mu^2)^4 \right\} \\
+ \left( U_L^\dagger U_L^\ast_{ij} \right) &\frac{m_j^2}{64\pi^2v^2m_{ij}^2\epsilon_{UV}} \left[ -6c_a^2m_{ij}^2 + (m_\nu^2(3c_{2\alpha} + 1) + 2m_\alpha^2 + 4m_{ij}^2) \right] \\
- \left( U_L^\dagger U_L^\ast_{ij} \right) &\frac{m_j^2}{64\pi^2v^2m_{ij}^2\epsilon_{UV}} \left[ m_{ij}^2(3c_{2\alpha} - 1) + 2m_\alpha^2 + 4m_{ij}^2 \right] \\
- \left( U_L^\dagger U_L^\ast_{ij} \right) &\frac{s_{2\alpha}m_j^2}{64\pi^2v^2m_{ij}^2\epsilon_{UV}} \left[ 3c_{2\alpha}m_{ij}^2 + (3m_{ij}^2s_a^2 + m_\alpha^2 + 2m_{ij}^2) \right] \\
+ \left( U_L^\dagger U_L^\ast_{ij} \right) &\frac{s_{2\alpha}m_j^2}{64\pi^2v^2m_{ij}^2\epsilon_{UV}} \left[ 3m_{ij}^2c_a^2 + m_\alpha^2 + 2m_{ij}^2 \right] \\
- \left( U_L^\dagger U_L^\ast_{ij} \right) &\frac{s_{2\alpha}m_j^2}{64\pi^2v^2m_{ij}^2\epsilon_{UV}} \left[ 4m_{ij}^2s_a^2 + 2m_{ij}^2m_{ij}^2 + (3c_{2\alpha} + 1) + 2m_{ij}^2c_{2\alpha} + 4 - 4m_\alpha^2(m_{ij}^2 - m_\alpha^2) \right] \\
+ \left( U_L^\dagger U_L^\ast_{ij} \right) &\frac{m_j^2}{64\pi^2v^2m_{ij}^2\epsilon_{UV}} \left[ -4m_{ij}^2c_a^2 + 2m_{ij}^2m_{ij}^2 + (3c_{2\alpha} + 1) + 2m_{ij}^2c_{2\alpha} + 4 + 4m_\alpha^2(m_{ij}^2 - m_\alpha^2) \right] \\
+ \left( U_L^\dagger U_L^\ast_{ij} \right) &\frac{m_j^2}{64\pi^2v^2m_{ij}^2\epsilon_{UV}} \left[ 4(m_\nu^2)^2 + 6m_{ij}^2s_{2\alpha} \right] \\
+ \frac{m_j^2}{32\pi^2v^2\epsilon_{UV}} &\left( U_L^\dagger U_L^\ast_{ij} \right) \frac{m_j^2}{32\pi^2v^2\epsilon_{UV}} \left( U_L^\dagger m_j^2U_L^\ast_{ij} \right) \frac{3m_j^2}{32\pi^2v^2\epsilon_{UV}} \left( U_L^\dagger (m_j^2)^2U_L^\ast_{ij} \right) \\
- \left( U_L^\dagger U_L^\ast_{ij} \right) &\frac{m_j^2}{16\pi^2v^2\epsilon_{UV}} \left[ 6m_{ij}^2 + 3m_{ij}^2 \right] \left[ \frac{c_a^2}{m_h^2} + \frac{s_a^2}{m_h^2} \right] \\
+ \left( U_L^\dagger U_L^\ast_{ij} \right) &\frac{s_{2\alpha}m_j^2}{16\pi^2v^2\epsilon_{UV}} \left[ \frac{1}{m_h^2} - \frac{1}{m_\alpha^2} \right] \text{Tr} \left\{ \frac{(m_\nu^2)^3}{2} (U_L^\dagger G_\nu^i U_R^i) + \frac{(m_\nu^2)^3}{2} (U_R^\dagger G_\nu^i U_L^i) + 3G_d(m_{ij}^3) + G_i(m_{ij}^3) + C.C. \right\} \\
+ \left( U_L^\dagger U_L^\ast_{ij} \right) &\left( U_L^\dagger G_\nu^i U_R^i \right) \frac{s_{2\alpha}}{8\pi^2v^2\epsilon_{UV}} \left[ \frac{1}{m_h^2} - \frac{1}{m_\alpha^2} \right] \text{Tr} \left\{ (m_\nu^2)^4(U_L^\dagger U_L^\ast_{ij}) + (3m_{ij}^4 + 3m_{ij}^4)(m_\nu^2)^4 + (m_{ij}^4) \right\} \\
+ \left( U_L^\dagger G_\nu^i U_R^i \right) &\frac{s_{2\alpha}}{16\pi^2v^2\epsilon_{UV}} \left[ \frac{1}{m_h^2} + \frac{c_a^2}{m_h^2} + \frac{s_a^2}{m_h^2} \right] \text{Tr} \left\{ \frac{(m_\nu^2)^3}{2} (U_L^\dagger G_\nu^i U_R^i) + \frac{(m_\nu^2)^3}{2} (U_R^\dagger G_\nu^i U_L^i) + 3G_d(m_{ij}^3) + 3G_i(m_{ij}^3) \right\} \\
+ \left( U_L^\dagger G_\nu^i U_R^i \right) &\frac{s_{2\alpha}}{16\pi^2v^2\epsilon_{UV}} \left[ \frac{1}{m_h^2} + \frac{c_a^2}{m_h^2} - \frac{1}{m_\alpha^2} \right] \text{Tr} \left\{ \frac{(m_\nu^2)^3}{2} (U_L^\dagger G_\nu^i U_R^i) + \frac{(m_\nu^2)^3}{2} (U_R^\dagger G_\nu^i U_L^i) + 3G_d(m_{ij}^3) + 3G_i(m_{ij}^3) \right\} \\
+ \left( U_L^\dagger G_\nu^i U_R^i \right) &\frac{\lambda_{313}U_{32}}{128\pi^2v^2\epsilon_{UV}} \left[ \frac{1}{m_h^2} - \frac{1}{m_\alpha^2} \right] \left[ 3c_{2\alpha}(m_{ij}^2 - m_h^2) + m_h^2 + m_h^2 - 2m_{\alpha}^2 - 4m_{ij}^2 \right] \end{align*} \]
\[
\begin{align*}
+ (U_L^T G_v U_R^*)_{ji} & \frac{\lambda_1 v (c_{2a} (m_h^2 - m_H^2) + m_R^2 + m_H^2)}{128 \sqrt{2} \pi^2 m_h^2 \varepsilon_{UV}} \left[ 3 c_{2a} (m_h^2 - m_H^2) - 3 m_h^2 - 3 m_H^2 - 2 m_A^2 - 4 m_{H^+}^2 \right] \\
+ (U_L^T G_v U_R^*)_{ji} & \frac{s_{2a}}{64 \sqrt{2} \pi^2 \varepsilon_{UV}} \left[ \frac{1}{m_h^2} - \frac{1}{m_H^2} \right] \left[ -m_h^4 + c_{2a} (m_h^2 - m_H^2) (m_h^2 + m_H^2 - 3 m_{H^+}^2) - 6 (2 m_W^4 + m_{H^+}^4) \right] \\
+ (U_L^T G_v U_R^*)_{ji} & \frac{s_{2a}}{64 \sqrt{2} \pi^2 \varepsilon_{UV}} \left[ \frac{1}{m_h^2} - \frac{1}{m_H^2} \right] \left[ m_h^2 (4 m_h^2 - m_{H^+}^2) - m_h^4 - m_{H^+}^2 m_{H^+}^4 + 2 m_A^2 m_{H^+}^2 \right] \\
= & \frac{m_i^v (U_L^T G_v^1 G_v U_R^*)_{ji}}{16 \pi^2 \varepsilon_{UV}} + \frac{m_i^v (U_L^T G_v^3 G_v U_R^*)_{ji}}{64 \pi^2 \varepsilon_{UV}} + \frac{m_i^v (U_R^T G_v U_R^*)_{ji}}{32 \pi^2 \varepsilon_{UV}} + (i \leftrightarrow j). \\
\end{align*}
\]
Here we explicitly see the result of taking one of the masses to 0. As noted before, in the partially massless case the expressions lack symmetry, i.e. one of the masses is missing, and one should be careful in, for example, getting (85)

we also used properties of the matrices $UL$ would arrive at the fully massless limit.

In Eq. (87) the terms with negative powers of $m_j^0$ cancel after rearrangement so that in Eq. (88) we were free to take $m_j^0$ to zero, we also used properties of the matrices $U_L$ in the massless case as in Eqs. (66) and (86). In addition, if we also went to the fully massless case by taking $m_j^0$ to zero, we would explicitly get $\delta Z_{L,j,i,g}^\Lambda = 0$ as is not hard to check.

Having considered explicit examples in the fully massless, fully massive, and partially massless cases we see that scheme is valid in all these scenarios.
4.3.4 Charged leptons

Before moving to the comparison of the divergent parts with other authors we must also discuss the renormalization of charged leptons—this is similar to the case of the CKM matrix where both up- and down-type quarks are needed.

The renormalization of charged leptons is very similar to that of quarks in the sense that we are able to express all the counterterms in terms of Passarino–Veltman functions. Let us write down the anti-hermitian part of the left-handed field renormalization for charged leptons $\delta Z_{Lji}^{A,J}$

$$\delta Z_{Lji}^{A,J} = -\sum_{k=a}^{4} \frac{U_{Lja}^* U_{Lia}}{2D_D^{-2} \pi^2} \left[ (m_a^v)^2 - (m_j^f)^2 + m_W^2 \xi_W \right] B_0((m_j^f)^2, (m_a^v)^2, m_W^2 \xi_W)$$

$$+ \sum_{k=a}^{4} \frac{U_{Lja}^* U_{Lia}}{2D_D^{-2} \pi^2} \left[ (m_a^v)^2 - (m_j^f)^2 + m_W^2 \xi_W \right] B_0((m_j^f)^2, (m_a^v)^2, m_W^2 \xi_W).$$

(89)

Note that here $i, j = 1, 2, 3$ and $i \neq j$. The result is similar to the one in the case of quarks in Eq. (70) and it is also easy to check that $\delta Z_{Lji}^{A,J}$ is finite. In addition, $\delta Z_{Rji}^{A,J} = 0$ as was the case for quarks due to the chiral $W$ interaction.

As in the quark case, the full charged lepton mass counterterm is given in “Appendix B” and here we only write down the divergent part $[\delta m_{ji}^{L,\text{div}}]$.

$$[\delta m_{ji}^{L,\text{div}}] = -\frac{3m_j^f (U_L^T m^v U_L^T)_{ji}}{32\pi^2 \xi_{UV}} - \frac{G_U U_L m^v U_R^T G_U^T)_{ji}}{16\pi^2 \xi_{UV}} + \frac{m_j^f (G_j^* G_j^T)_{ji}}{64\pi^2 \xi_{UV}} + \frac{m_j^f (G_j^* G_j^T)_{ji}}{64\pi^2 \xi_{UV}}$$

$$+ \frac{s_{2a} (G_{ji})_{ji}}{8\sqrt{2}\pi^2 m_{H}^2 m_{A}^2} \left[ (m_a^v)^4 (m_j^f)^4 + 3 (m_a^v)^4 + (m_j^f)^4 \right]$$

$$+ \frac{(G_{ji})_{ji}}{16\pi^2 \xi_{UV}} \left( \frac{s_{2a}^2}{m_{H}^2} + \frac{c_{2a}^2}{m_{H}^2} + \frac{1}{m_{A}^2} \right) \text{Tr} \left\{ (m_j^f)^3 G_j^* G_j^T + 3 (m_a^v)^3 G_a^* G_a^T + (m_j^f)^3 G_j^* G_j^T + (m_j^f)^3 G_j^* G_j^T \right\}$$

$$+ \frac{(G_{ji})_{ji}}{16\pi^2 \xi_{UV}} \left( \frac{s_{2a}^2}{m_{H}^2} + \frac{c_{2a}^2}{m_{H}^2} - \frac{1}{m_{A}^2} \right) \text{Tr} \left\{ (m_j^f)^3 G_j^* G_j^T + 3 (m_a^v)^3 G_a^* G_a^T + (m_j^f)^3 G_j^* G_j^T + (m_j^f)^3 G_j^* G_j^T \right\}$$

$$- \frac{(G_{ji})_{ji}}{128\sqrt{2}\pi^2 m_{H}^2 m_{A}^2} \left( c_{2a} (m_a^v)^2 (m_j^f)^2 + m_j^f + m_A^2 \right) \left( 3c_{2a} (m_j^f)^2 - m_j^f + 3m^2 - 3m^2 - 2m^2 - 4m^2 \right)$$

$$+ \frac{s_{2a} (G_{ji})_{ji}}{128\sqrt{2}\pi^2 m_{A}^2 \xi_{UV}} \left[ \frac{1}{m_{H}^2} - \frac{1}{m_{A}^2} \right] \left( 3c_{2a} (m_j^f)^2 - m_j^f + 2m^2 - 4m^2 \right)$$

$$+ \frac{s_{2a} (G_{ji})_{ji}}{128\sqrt{2}\pi^2 m_{A}^2 \xi_{UV}} \left[ \frac{1}{m_{H}^2} - \frac{1}{m_{A}^2} \right] \left( 6m_j^f + 3m^2 - 3c_{2a} (m_j^f)^2 - m_j^f \right) (m_j^f)^2 + m_j^f - 3m^2$$

$$+ m_j^f + m_A^4 + 2m_A^4 - m_j^f (4m_j^f - m_j^f)^2 + m_A^2 m_j^f - 2m_A^2 m_j^f \right)$$

(90)

The case is again similar to that of quarks, only the first term mimics the form of the CKM matrix in the SM and is not due to an interaction with the second Higgs doublet. The first term is also in agreement with the results in [15]. In other words, the UV-divergent parts we have computed also match with the ones computed in [15], the essential difference being that we do not propose to perform the 1-loop rotation as discussed in Sect. 2.1—in our scheme the mixing matrix is already renormalized.

5 Conclusions

In this paper we have dealt with a general theory where mixing of fermionic fields is present. In particular, we have introduced the off-diagonal mass counterterms and by using the On-Shell no-mixing condition for external legs we have solved for these counterterms in terms of self-energies and the anti-hermitian part of the field renormalization. Simultaneously, we have defined the anti-hermitian part of the field renormalization as the coefficient of the $m_j^f - m_j^f$ mass structure in relevant expressions. Notably, these definitions make the mass counterterms gauge-independent and the anti-hermitian part of the field renormalization UV-finite.

At 1-loop our scheme can also include the absorptive parts, however, in the “Appendix A” we have extended the scheme to all orders without considering the absorptive parts. Further, this setup does not require us to introduce counterterms to mixing matrices such as the CKM matrix since in our approach the mixing matrices are derived from renormalized quantities and not the bare ones. In Sect. 2.1 we have formulated a consistency condition, namely, that it should be possible to do finite basis rotations at all times, this
leads to the requirement that mixing matrices must not have counterterms—our scheme is an explicit example of how this can be achieved.

Finally, by using the Grimus–Neufeld model for examples we computed the off-diagonal counterterms for quarks, charged leptons, and Majorana neutrinos. The Grimus–Neufeld model also allowed us to test our scheme for massive and massless fermions as well as in the case of radiative mass generation. We were also able to compare our results with the results of [1,6,15]—we successfully reproduced the UV divergent terms (up to additional tadpole contributions in the case of neutrinos), while we did not compare the finite parts due to different approaches towards absorptive parts. In addition, these computations provide new results in the Grimus–Neufeld model.

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Declarations

Conflict of interest The author has no financial or proprietary interests in any material discussed in this article.

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Consent to participate Not applicable.

Consent for publication Not applicable.

Appendix A Extending the scheme beyond 1-loop

In this appendix we extend our 1-loop approach to higher orders. Beyond 1-loop there are more contributions that we need to keep track of and there are a few new objects that we define below. However, the logic to all orders remains the same—we look at the gauge dependence of the field renormalization and see whether the $m_i^2 - m_j^2$ structure in the gauge-dependent part cancels. If it does cancel, it means that we can define the anti-hermitian part of the field renormalization as the coefficient of $m_i^2 - m_j^2$ and solve for the mass counterterm as we did in our 1-loop discussion. After introducing the setup we re-derive tree and 1-loop results as well as derive the 2-loop results. Having the hang of the approach we extend the scheme to arbitrary orders.

A.1 The setup

First off, to avoid pesky numerical factors, the fields are renormalized without square roots on field renormalization constants

$$\psi_0 \rightarrow Z \psi$$  \hspace{1cm} (91)

with all the decompositions remaining the same as in Eq. (5).

Secondly, in this appendix we do not decompose the self-energy according to Eq. (9) to make the write up more compact. In addition, we introduce the bare self-energy computed with bare parameters $\Sigma^0 (p)$, the bare self-energy computed with renormalized parameters $\Sigma^0 (p)$ (no counterterms included) and the renormalized self-energy $\Sigma^R (p)$. The two bare self-energies are topologically/pictorially the same and the difference comes from the parameters used to compute the diagrams. Counterterm insertions, at least pictorially, appear only in the renormalized self-energy and not in the bare ones.

We also relate all the self-energies. In momentum space this is done by looking at the 2-point function term in the 1PI generating functional and then renormalizing the fields:

$$\bar{\psi}_0 \Sigma^0 (p) \psi_0 = \bar{\psi} \gamma^0 (Z^\dagger)^{-1} \Sigma^0 (p) Z \psi = \bar{\psi} \Sigma^R (p) \psi.$$  \hspace{1cm} (92)

which brings us to the following relation

$$\Sigma^0 (p) = \gamma^0 (Z^\dagger)^{-1} \gamma^0 \Sigma^R (p) Z^{-1}.$$  \hspace{1cm} (93)

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which brings us to the following relation

$$\Sigma^0 (p) = \gamma^0 (Z^\dagger)^{-1} \gamma^0 \Sigma^R (p) Z^{-1}.$$  \hspace{1cm} (93)
To relate the remaining bare self-energy we expand the bare parameters \( p_k^0 \) in \( \Sigma^0(p) \) around their renormalized parts \( p_k \). Since we need to take Taylor series expansions for every parameter it is convenient to define the series operator

\[
T\rho^0 = \sum_{n=0}^{\infty} \frac{(\delta \rho)^n}{n!} \frac{\partial^n}{\partial (\rho^0)^n}
\]  

(94)

with \((\delta \rho)^n\) being the \( n \)th power of the parameter counterterm. Using the series operator and the fact that

\[
\left. \frac{\partial \Sigma^0(p)}{\partial p_k^0} \right|_{p_k^0=p_k} = \frac{\partial \Sigma^0(p)}{\partial p_k}
\]

(95)

we can relate the two bare self-energies

\[
\tilde{\Sigma}_0^0(p) = \left( \prod_k T_{p_k^0} \right) \Sigma^0(p) \bigg|_{p_k^0=p_k} = \left( \prod_k T_{p_k} \right) \Sigma^0(p).
\]

(96)

The restriction to renormalized parameters is to be understood as acting on (bare) parameter derivatives of \( \tilde{\Sigma}^0 \), the operators \( T_{p_k^0} \) and \( T_p \) are understood to only differ in their derivatives, the product \( \prod_k \) goes over all parameters of the theory. Thus all three self-energies are compactly related as follows (making the flavour indices explicit).

\[
\tilde{\Sigma}_j^0(p) = \left( \prod_k T_{p_k} \right) \Sigma_j^0(p) = \gamma^0(Z^0)_{ji}^{-1} \gamma^0 \Sigma_R^0(p) Z^{-1}_{ri}.
\]

(97)

Here \( j, i, l \) and \( r \) are flavor indices, \( l \) and \( r \) are summed over, i.e. in the Appendix we use the Einstein summation rule for flavour indices that are neither \( i \) nor \( j \). It is important to note that the derivative terms correspond to 1PI diagrams with counterterm insertions.

Third, \( n \)-loop order means certain powers of renormalized couplings that come out of \( n \)-loop diagrams, hence, anything that has such powers of renormalized couplings we call of \( n \)th order. For example, the field counterterm can be expanded order by order

\[
Z = 1 + \delta Z = 1 + \delta Z^{(1)} + \delta Z^{(2)} + \ldots
\]

(98)

Here the order is indicated with superscripts in parentheses. In a similar manner, the series operator can also be expanded due to the powers of counterterms appearing in its definition, for example, the second order of the series operator is

\[
T_{p_k}^{(2)} = \delta p_k^{(2)} \frac{\partial}{\partial p^0} + \frac{1}{2} (2 \delta p_k^{(1)} \delta p_k^{(1)}) \frac{\partial^2}{\partial (p^0)^2}.
\]

(99)

Here the second term came from the square of the counterterm.

Since all the self-energies can be written in terms of renormalized couplings, they can also be expanded in these orders

\[
\Sigma(p) = \Sigma^{(0)}(p) + \Sigma^{(1)}(p) + \Sigma^{(2)}(p) + \ldots
\]

(100)

It is worth noting that the superscript for \( \Sigma^{0(\alpha)}(p) \) also indicates the number of loops, while for \( \Sigma^{0(\alpha)}(p) \) and \( \Sigma^{R(\alpha)}(p) \) it indicates the order only.

Fourth, for clarity and simplicity in the appendix we completely drop the absorptive parts so that there is no need for the LSZ factors, i.e. \( Z = \tilde{Z} = 1 \). Since our scheme depends on the mass structures, dropping the absorptive parts does not change the idea behind the scheme and the discussion is more transparent.

Without the absorptive parts we can write down the standard On-Shell renormalization conditions [16]:

1. No mixing for external legs \((i \neq j)\)

\[
\Sigma_{ji}^{R(n>0)}(p)u_i = 0.
\]

(101)

2. The pole mass \( m^p \) is equal to the renormalized mass \( m_i \)

\[
m_i^p = m_i \rightarrow \Sigma_{ii}^{R(n>0)}(p)u_i = 0.
\]

(102)

3. Residue at the pole is unity

\[
\lim_{p \rightarrow m_i} \frac{1}{p-m_i} \Sigma_{ii}^{R(n>0)}(p)u_i = u_i
\]

\[
\Rightarrow \lim_{p \rightarrow m_i} \frac{1}{p-m_i} \Sigma_{ii}^{R(n>0)}(p)u_i = \frac{d}{dp} \Sigma_{ii}^{R(n>0)}(p)u_i = 0.
\]

(103)
Conveniently, the second condition simply extends the first condition to all \(i\) and \(j\), while the third condition is used to fix the diagonal component of the field renormalization.

Finally, we will take gauge derivatives of the various self-energies and use Nielsen identities. For the bare self-energies one takes the derivative of \(\Sigma^0\) w.r.t. the bare gauge parameter \(\xi^0\) such that the form of the Nielsen identity is the same as in Eq. (26). For the \(\Sigma^0\) case we simply add tildes on \(\Lambda\)'s to indicate that they are computed in terms of bare parameters. Further, for \(\Sigma^1\) the derivative must be taken w.r.t. the renormalized gauge-parameter \(\xi^1\) such that the Nielsen Identity may in principle develop modifications due to renormalization as in Eq. (24) [32].

Similarly to the discussion in Sect. 3.1 most of the modifications turn out to be irrelevant for our discussion or can be included in our scheme. In Eq. (24) the modifications parameterised by \(\gamma^\rho_\xi\) also vanishes since our scheme is On-Shell. In the appendix the self-energies include tadpole diagrams to all orders. In terms of the FJ scheme this means that the theory is at the minimum to all orders and so the modification parameterised by \(\gamma^\rho_\xi\) vanishes. The only relevant modification is due to gauge-dependence of the counterterms of gauge parameters parameterised by \(\rho^\xi\). This modification first contributes at second order and comes from taking the (renormalized) gauge derivative of the series operator \(\prod_k T_{pk}\). If this modification is present, it can be included into the field renormalization.

Now we can re-derive our results (without absorptive parts) as well as expand them beyond 1-loop order.

A.2 Zeroth order

Taking the 0th order terms in Eq. (97) we simply get the following

\[
\Sigma_{ji}^{0(0)}(\rho') = \Sigma_{ji}^{0(0)}(\rho) = \Sigma_{ji}^{R(0)}(\rho) = (\rho' - m_j)\delta_{ji}
\]

and there is no need to do anything else.

A.3 First order

Now we take the first order terms in Eq. (97) and get the following

\[
\Sigma_{ji}^{0(1)}(\rho') = \Sigma_{ji}^{0(1)}(\rho) + \sum_k \frac{\partial \Sigma_{ji}^{0(0)}(\rho)}{\partial \rho_k} \delta p_k^{(1)}
\]

\[
= \Sigma_{ji}^{R(1)}(\rho') - \gamma^0 Z_{ji}^{(1)}(\xi^0) \gamma^0 \Sigma_{ji}^{R(0)}(\rho) - \Sigma_{ji}^{R(0)}(\rho) \delta Z_{ji}^{(1)}
\]

\[
= \Sigma_{ji}^{R(1)}(\rho') - \gamma^0 \delta Z_{ji}^{(1)}(\xi^0) (\rho' - m_j) - (\rho' - m_j)\delta Z_{ji}^{(1)}.
\]

Let us take care of the derivative term. The only parameter entering the tree-level self-energy is the mass, so this term gives us the mass counterterm and nothing else. The mass is fundamentally a matrix, so one must take parameter derivatives with respect to every diagonal component of the field renormalization.

In Eq. (20).

11 In reality, the gauge-parameters in the linear \(R_\xi\) gauges are not renormalized, but this distinction is useful if one wants to take into account the modifications of the Nielsen Identities.
To get the gauge dependence we use the Nielsen identity and arrive at
\[ \delta \delta Z_{ji}^{(1)} u_i = -\left( \frac{\delta^2 m_j}{m^2 - m_i^2} \right) \left[ (\bar{p}' - m_i) \tilde{A}_{ji}^{(1)}(\bar{p}) \right] u_i = -\tilde{A}_{ji}^{(1)}(\bar{p}) u_i. \] (109)

Here we already used our 1-loop result that the mass counterterm is gauge independent. The above can be decomposed to give the results in Eq. (30) without mass counterterms. Obviously, this again shows that the \( m_i^2 - m_j^2 \) mass structure cancels for the gauge dependent part and so the discussion of previous sections applies: taking the coefficient of \( m_i^2 - m_j^2 \) is a sensible definition for (the anti-hermitian part of) off-diagonal components of the field counterterms.

It is very important to note that this result also holds for diagonal parts. To see this we impose the third renormalization condition in Eq. (105) and get
\[ \lim_{\bar{p}' \to m_i} \frac{1}{\bar{p}' - m_i} \left[ \Sigma_i^{R(1)}(\bar{p}) - \gamma^0 \delta Z_{ij}^{(1)}(\bar{p}) (\bar{p}' - m_i) \right] u_i = -\left[ \delta Z^{(1)}_{ii} + \delta Z_{ii}^{(1)} \right] u_i = -2 \delta Z_{ii}^{H(1)} u_i. \] (110)

Here we used a trick from the Dirac algebra playbook when taking the limit in the \( \gamma^0 \delta Z' \gamma^0 \) term. In this term both the numerator and the denominator vanish at \( \bar{p}' = m_i \), one must take the derivative \( \frac{d}{d \bar{p}'} \), which has the Dirac properties of \( \bar{p}' \). Commuting the derivative through \( \gamma^0 \delta Z' \gamma^0 \) gets rid of the surrounding gamma matrices.

Now we take the gauge derivative of \( \delta Z_{ii}^{H(1)} \) and arrive at
\[ \partial \delta Z_{ii}^{H(1)} u_i = -\frac{1}{2} \lim_{\bar{p}' \to m_i} \frac{1}{\bar{p}' - m_i} \left[ \Lambda_{ii}^{(1)}(\bar{p}) (\bar{p}' - m_i) + (\bar{p}' - m_i) \bar{\Lambda}_{ii}^{(1)}(\bar{p}) \right] u_i \]
\[ = -\frac{1}{2} \lim_{\bar{p}' \to m_i} \frac{1}{\bar{p}' - m_i} \left[ \Lambda_{ii}^{(1)}(\bar{p}) (\bar{p}' - m_i) \right] u_i - \frac{1}{2} \bar{\Lambda}_{ii}^{(1)}(\bar{p}) u_i. \] (111)

In the second term the problematic \( (\bar{p}' - m_i) \) factor cancels and there is no need to take the limit. On the other hand, in the first term the cancellation is not trivial and it is needed to use L’Hopital’s rule. The important thing to note is that taking the limit does not require to commute \( \bar{p}' \) through the projectors \( P_{L,R} \) to act on the spinor. Let us decompose the \( \Lambda(\bar{p}') \), take only the term containing \( \bar{p}' P_{L} \), and apply the L’Hopital’s rule
\[ \lim_{\bar{p}' \to m_i} \frac{d}{d \bar{p}'} \left[ \Lambda_{ii}^{(1)}(\bar{p}') (\bar{p}' - m_i) \right] = \lim_{\bar{p}' \to m_i} \Lambda_{ii}^{(1)}(\bar{p}') (\bar{p}' - m_i) = \Lambda_{ii}^{(1)}(m_i^2) m_i P_R. \] (112)

Terms with the derivative not acting on \( \bar{p}' - m_i \) vanished. In a similar manner, the derivative exchanges the projectors \( P_L \leftrightarrow P_R \) in all terms in the decomposition. Since we have dropped the absorptive parts, pseudo-hermiticity gives \( \gamma^0 \Lambda(\bar{p}') \gamma^0 = (\tilde{A}(\bar{p}') \gamma^0) \), where the hermitian conjugation is acting on flavour and Dirac structures. This gives relations between scalar functions, namely \( \Lambda^{L,R} = \Lambda^{L,R} \) and \( \Lambda^{L,R,L} = \Lambda^{R,L,L} \). Then, the full term with the limit gives the same as
\[ \lim_{\bar{p}' \to m_i} \frac{d}{d \bar{p}'} \left[ \Lambda_{ii}^{(1)}(\bar{p}') (\bar{p}' - m_i) \right] \rightarrow \left[ \Lambda_{ii}^{(1)}(\bar{p}') (\bar{p}' - m_i) + \Lambda_{ii}^{R(1)}(\bar{p}') P_R + \Lambda_{ii}^{R(1)}(\bar{p}) P_L + \Lambda_{ii}^{R(1)}(\bar{p}) P_R \right] u_i = -\Lambda_{ii}^{(1)}(\bar{p}) u_i. \] (113)

Plugging the above into Eq. (111) we get
\[ \partial \delta Z_{ii}^{H(1)} u_i = -\frac{1}{2} \left[ \tilde{A}_{ii}^{(1)}(\bar{p}) + \Lambda_{ii}^{(1)}(\bar{p}) \right] u_i \equiv -\Lambda_{ii}^{H(1)}(\bar{p}) u_i = -\tilde{A}_{ii}^{(1)}(\bar{p}) u_i, \] (114)
where \( \tilde{A}_{ii}^{H(1)} \) is defined to have for its scalar functions the hermitian parts of the scalar functions of \( \tilde{A}_{ii}^{(1)} \). Dropping the absorptive parts means that the diagonal components of self-energies are real and so are the diagonal components of \( \Lambda^{L} \), this gave the final equality. Since one cannot fix the anti-hermitian part of the field renormalization for the diagonal components we simply set it to zero at 1-loop as is usually done. Eventually, \( \delta Z_{ii}^{H(1)} = \delta Z_{ii}^{(1)} \) and the gauge dependence matches the one found for off-diagonal components. That the gauge dependence is the same for diagonal and off-diagonal components has been explicitly checked at 1-loop in [26] (although in the Z and \( \tilde{Z} \) approach with absorptive parts included). Here we find the same result in a more general setting. This is an essential piece of information when going beyond 1-loop, since gauge-derivatives of diagonal parts also appear in expressions for off-diagonal components.

A.4 Second order

Now we take the second order terms in Eq. (97) and get the following
\[ \tilde{Z}_{ji}^{0(2)}(\bar{p}) = \Sigma_{ji}^{0(2)}(\bar{p}) + \sum \frac{\partial \Sigma_{ji}^{0(1)}(\bar{p})}{\partial p_k}(\delta p_k) \]
\[ = \Sigma_{ji}^{R(2)}(\bar{p}) + \gamma^0 \delta Z_{jk}^{(1)}(\bar{p}) \Sigma_{kl}^{R(0)}(\bar{p}) Z_{li}^{(1)} - \gamma^0 \delta Z_{jk}^{(1)}(\bar{p}) \Sigma_{kl}^{R(1)}(\bar{p}) - \Sigma_{jk}^{R(1)}(\bar{p}) Z_{kl}^{(1)}. \]
Upon using the first two renormalization conditions and inserting the definitions of renormalized self-energies in terms of bare ones the above equation simplifies considerably and we get
\begin{equation}
\left( m_i^2 - m_j^2 \right) \delta Z_{ij}^{(2)} u_i = -\left( \rho' + m_j \right) \left[ \Sigma_{ji}^{(0)}(\rho') - \delta m_{ji}^{(2)} + \sum_k \frac{\partial}{\partial p_k} \delta Z_{ji}^{(1)}(\rho') \right] u_i.
\end{equation}

Just as for the previous order we may use the Nielsen identity, but first let us separately take the gauge derivative on the term containing parameter derivatives. Here it is also important to separate a few cases: the mass parameter, other physical parameters, and non-physical parameters (e.g. gauge-fixing parameters) as they all have slightly different behaviour.

We take the counterterms of physical parameters to be gauge-independent so that the gauge derivative effectively only acts on the self-energy. If these physical parameters are not the mass parameter we get
\begin{equation}
\sum_k \frac{\partial}{\partial p_k} \left( \frac{\partial \Sigma_{ji}^{(0)}(\rho')}{\partial \rho} \right) \delta p_k^{(1)} = \sum_k \delta p_k^{(1)} \left[ \frac{\partial}{\partial p_k} \left( \Lambda_{ji}^{(1)}(\rho') (\rho' - m_i) + (\rho' - m_j) \frac{\partial}{\partial p_k} \left( \tilde{\Lambda}_{ji}^{(1)}(\rho') \right) \right] u_i.
\end{equation}

On the other hand, if the physical parameter is the mass parameter, there is an additional piece coming from the parameter derivative acting on \( (\rho' - m)^2 \). Performing the derivative as in Eq. (106) the additional piece is
\begin{equation}
- \tilde{\Lambda}_{jk}^{(1)}(\rho') \delta m_{ki}^{(1)} - \delta m_{jk}^{(1)} \tilde{\Lambda}_{ki}^{(1)}(\rho').
\end{equation}

With this piece, some cancellations take place in Eq. (116) once the gauge derivative is taken.

If the parameters are the gauge parameters their parameter derivatives already give the Nielsen identity when acting on self-energies, in addition, the counterterms can now depend on gauge parameters. Indicating \( \Lambda \)'s with the superscript \( l \) to denote that they come from the Nielsen identity for the gauge parameter \( \tilde{\xi}_i \) we get
\begin{equation}
\partial_k \sum_l \frac{\partial}{\partial \tilde{\xi}_l} \Sigma_{ji}^{(0)}(\rho') \delta \tilde{\xi}_l^{(1)} = \sum_l \delta \tilde{\xi}_l^{(1)} \left[ \frac{\partial}{\partial \tilde{\xi}_l} \left( \Lambda_{ji}^{(1)}(\rho')(\rho' - m_i) + (\rho' - m_j) \frac{\partial}{\partial \tilde{\xi}_l} \tilde{\Lambda}_{ji}^{(1)}(\rho') \right) \right]
+ \sum_l \delta \tilde{\xi}_l^{(1)} \left[ \Lambda_{ji}^{(1)}(\rho')(\rho' - m_i) + (\rho' - m_j) \tilde{\Lambda}_{ji}^{(1)}(\rho') \right].
\end{equation}

Here the second line is due to gauge-dependence of gauge parameter counterterms—this is the modification of the renormalized Nielsen identity.

Finally, we take the gauge derivative of Eq. (116) assuming that the mass counterterm is gauge-independent, using the known gauge dependence of field renormalization for all \( i \) and \( j \), and using Eq. (108) to get a term with \( \delta m^{(1)} \) we arrive at
\begin{equation}
\left( m_i^2 - m_j^2 \right) \partial_k \delta Z_{ji}^{(2)} u_i = -\left( m_i^2 - m_j^2 \right) \left[ \tilde{\Lambda}_{ji}^{(2)}(\rho') + \tilde{\Lambda}_{jk}^{(1)}(\rho') \delta Z_{ki}^{(1)} + \sum_k \delta p_k \left( \frac{\partial}{\partial p_k} \tilde{\Lambda}_{ji}^{(1)}(\rho') + \frac{\partial}{\partial p_k} \left( \tilde{\Lambda}_{ji}^{(1)}(\rho') \right) \right) u_i.
\end{equation}

which again shows that the \( m_i^2 - m_j^2 \) factor cancels for the gauge-dependent part. Most importantly, this tells us that the definition we have made for the field and mass counterterms at 1-loop can very well be extended to 2-loop order. In complete analogy, the anti-hermitian part of field renormalization is defined by taking the coefficient of \( m_i^2 - m_j^2 \) in Eq. (116), afterwards one can simply solve for the mass counterterms. The mass counterterm is then gauge-independent by definition. Comparing with the first order discussion, now there is a contribution to the field renormalization associated with the modification of the Nielsen identity.

To fully complete the discussion we should also determine the gauge-dependence of diagonal components, however, this notation already seems plentiful and so we deal with the diagonal components in the next section where we consider arbitrary orders.

A.5 Arbitrary order

In this section we extend the scheme to arbitrary orders. The discussion comes in two main parts: w.r.t. the bare gauge parameter and then the renormalized one. Deriving results with the bare gauge parameter helps avoid modifications of the Nielsen identity and makes the discussion slightly more compact. Afterwards we will simply relate the bare and renormalized gauge-derivatives. For convenience, we summarize the definitions at the end.

A.5.1 Bare gauge parameter

In this subsection the discussion is w.r.t. the bare gauge parameter \( \xi^0 \) and the bare self-energy in terms of bare parameters \( \bar{\Sigma}^{0}(\rho) \).

To begin, we rewrite Eq. (93) and immediately apply the renormalization condition
\begin{equation}
\bar{\Sigma}_{jk}^{0}(\rho) Z_{kj} u_i = \gamma^0 Z^{1}_{jk} \gamma^0 \bar{\Sigma}^{0}(\rho) u_i = 0.
\end{equation}
At arbitrary order $n$ we assume that all the counterterms have been defined up to order $n - 1$. Having this in mind we can rewrite the above at order $n$

$$(p' - m_j)\delta Z^{(n)}_{ji} u_i = - \left[ \sum_{ji}^{0(n)} (p') + \sum_{l=1}^{n-1} \sum_{ik}^{0(n-l)} (p') \delta Z^{(l)}_{ki} \right] u_i. \tag{122}$$

One can check that this reproduces the tree, 1-loop, and 2-loop level results we had previously by using Eq. (97).

Before looking at the gauge dependence we should also fix the diagonal components at arbitrary order. To do so we rewrite Eq. (93) for the diagonal part

$$\gamma^0 Z^+_{ir} \gamma^0 \Sigma^R_{rk}(p') Z_{ki} = \Sigma^R_{ii}(p') \tag{123}$$

and use the third renormalization condition

$$\lim_{p' \to m_i} \frac{1}{p' - m_i} \left[ \gamma^0 Z^+_{ir} \gamma^0 \Sigma^R_{rk}(p') Z_{ki} \right] u_i = \frac{d}{dp'} \Sigma^R_{ii}(p') u_i = 0. \tag{124}$$

Expanding the above at arbitrary order $n$ and taking trivial limits, one arrives at

$$\delta Z^{H(n)}_{ii} u_i = - \frac{1}{2} \lim_{p' \to m_i} \frac{1}{p' - m_i} \left[ \sum_{ii}^{0(n)} (p') + \sum_{l=1}^{n-1} \sum_{ik}^{0(n-l)} (p') \delta Z^{(l)}_{ki} + \sum_{l=1}^{n-1} \gamma^0 \delta Z^{(l)}_{ir} \gamma^0 \Sigma^R_{rk}(p') \delta Z^{(l)}_{ki} \right] u_i. \tag{125}$$

If it was not for the limit, one could take Eq. (122) to replace the $(n - l)$th order self-energy on the first line to cancel the double sum, but this is not the case. Since the off-diagonal components seem a lot simpler, we first deal with their gauge dependence. However, for the moment we assume that the diagonal components have the same gauge dependence as off-diagonal ones.

We remind the reader that we can use the Nielsen identities also for the self-energies with bare parameters, the only difference is that the $\Lambda$’s are now also in terms of bare parameters, hence, the tildes. Taking the bare gauge derivative of Eq. (122), multiplying by $p' + m_j$ and also dropping the $p'$ arguments we get

$$(m^2_i - m^2_j) \partial_{\xi}^0 \delta Z^{(n)}_{ji} u_i = -(p' + m_j) \left[ (p' - m_j) \tilde{\Lambda}^{(n)}_{ji} \right] + \sum_{l=1}^{n-1} \left[ \tilde{\Sigma}^{0(n-l)}_{jk} \tilde{\Lambda}^{(l)}_{ki} + \tilde{\Sigma}^{0(n-l)}_{jk} \tilde{\Lambda}^{(l)}_{ki} \right]
+ \sum_{l=1}^{n-1} \left[ \tilde{\Sigma}^{0(n-l)}_{jk} \partial_{\xi} \delta Z^{(l)}_{ki} + \tilde{\Sigma}^{0(n-l)}_{jk} \partial_{\xi} \delta Z^{(l)}_{ki} \right] u_i. \tag{126}$$

Here we separated the $\Lambda^{(n)}$ contribution from the Nielsen identity of the $n$th order self-energy. As can be seen in Eq. (121) and Eq. (122) we can rewrite the bare self-energy $\tilde{\Sigma}^{0(n-l)}$ on the first line in terms of lower order bare self-energies and field counterterms—this gives the cancellation on the first term on the fourth line. The simplified expression is then

$$(m^2_i - m^2_j) \partial_{\xi}^0 \delta Z^{(n)}_{ji} u_i = -(p' + m_j) \left[ (p' - m_j) \tilde{\Lambda}^{(n)}_{ji} \right] + \sum_{l=1}^{n-1} \left[ \tilde{\Sigma}^{0(n-l)}_{jk} \tilde{\Lambda}^{(l)}_{ki} \right] + \sum_{l=1}^{n-1} \left[ \tilde{\Sigma}^{0(n-l)}_{jk} \partial_{\xi} \delta Z^{(l)}_{ki} \right] u_i. \tag{127}$$

Here one can already get the 1-loop and 2-loop gauge-dependence (up to modification terms) we have in previous sections and subsections by taking $n = 1$ and $n = 2$, but the $m^2_i - m^2_j$ factor is not obvious for higher orders. One can notice that in the first sum the self-energy has the order between 1 and $n - 2$, since for $n - 1$ (i.e. $l = 1$) the gauge dependence in the parentheses cancels. In the second sum the order of the self-energy is between 0 and $n - 2$. We separate the 0th order term from the second sum and combine the two remaining sums since the orders of self-energies match. We get the following

$$(m^2_i - m^2_j) \partial_{\xi}^0 \delta Z^{(n)}_{ji} u_i = -(p' + m_j) \left[ (p' - m_j) \tilde{\Lambda}^{(n)}_{ji} \right]
+ \sum_{l=2}^{n-1} \left[ \tilde{\Sigma}^{0(n-l)}_{jk} \tilde{\Lambda}^{(l)}_{ki} \right] + \sum_{l=2}^{n-1} \left[ \tilde{\Sigma}^{0(n-l)}_{jk} \partial_{\xi} \delta Z^{(l)}_{ki} \right] u_i. \tag{128}$$

Here, on the second line the sum over $l$ starts at 2 so that the self-energy has the correct range of orders. Now, the combined sum on the second line first appears for $n = 3$, but vanishes by inserting the gauge dependence of $\partial_{\xi} \delta Z^{(2)}$. In this way we see that when
inserting the explicit result for \( \delta_{e^0} \delta Z^{(l)} \) the combined sum always vanishes, hence, at arbitrary order we get the following result

\[
\partial_{e^0} \delta Z^{(n)}_{ji} u_i = - \left[ \tilde{\Lambda}^{(n)}_{ji} + \sum_{l=1}^{n-1} \tilde{\Lambda}^{(n-l)}_{jk} \delta Z^{(l)}_{ki} \right] u_i
\]

(129)

showing that for off-diagonal field counterterms the \( m_i^2 - m_j^2 \) mass structure cancels in the gauge-dependent part at all orders. So far we have assumed that the diagonal field counterterms have the same gauge dependence as off-diagonal ones at all orders—now we set out to prove this.

Consider the gauge derivative of Eq. (125)

\[
\partial_{e^0} \delta Z^{H(n)}_{ii} u_i = - \frac{1}{2} \lim_{\not{p} \to m_i} \frac{1}{\not{p} - m_i} \left[ \tilde{\Lambda}^{(n)}_{ii} (\not{p} - m_i) + (\not{p} - m_i) \tilde{\Lambda}^{(n)}_{ii} + \sum_{l=1}^{n-1} \tilde{\Lambda}^{(n-l)}_{ik} \delta \Lambda^{(l)}_{kj} \right] u_i
\]

(130)

Here, the underbraces label the terms for later rearrangements. Although crowded, the above is simple to achieve by applying the Nielsen identities. In the last (triple) sum the index \( l \) goes up to \( n - 2 \) since the gauge derivative acting on the 0th order self-energy gives 0. Due to the limit we cannot use Eq. (122) to lower orders of self-energies and get cancellations, hence, we simply rearrange to better see logical structures. From the \( a' \) term on the second line and the \( b \) term on the third line we separate the 0th order self-energies, while we combine the remaining terms with matching labels, the result is as follows:

\[
\partial_{e^0} \delta Z^{H(n)}_{ii} u_i = - \frac{1}{2} \lim_{\not{p} \to m_i} \frac{1}{\not{p} - m_i} \left[ (\not{p} - m_i) \sum_{l=1}^{n-1} \tilde{\Lambda}^{(n-l)}_{ik} \delta Z^{(l)}_{ki} + \sum_{l=1}^{n-1} \gamma^0 \delta Z^{(l)}_{ik} \gamma^0 \tilde{\Lambda}^{(n-l)}_{ki} (\not{p} - m_i) \right]
\]

(131)

Here we already used our off-diagonal case knowledge to separate \( \tilde{\Lambda}^{H(n)} \) and also kept the underbraces to indicate the origin of the terms. On lines 2–3 there are two huge terms that each have two clear factors emphasized by the \( \times \) symbol. The factors containing \( \Lambda \)'s would vanish when acted on by the spinor and when assuming that the gauge-dependence of the field counterterms is the same for all \( i \) and \( j \). The factors without \( \Lambda \)'s represent the renormalization condition and would also vanish when acted on by the spinors. Of course, the present limit does not allow to simply take these terms to 0. To get rid of the limit we notice that the left factors would vanish when acted on by the \( \not{a_i} \) spinor\(^{12} \) on the left and so they must be proportional to \( \not{p} - m_i \). For example, taking the self-energy factor on the fifth line and Eq. (97) we can expand around the \( \not{a_i} \) spinor = 0 renormalization condition

\[
\tilde{\Sigma}^{0(n-l)}_{ik} + \sum_{q=1}^{n-l} \gamma^0 \delta Z^{(q)}_{ik} \gamma^0 \tilde{\Sigma}^{0(n-l-q)}_{ik} = (\Sigma^R Z^{-1})^{(n-l)}_{ik} = \sum_{g=1}^{m_i} (\not{p} - m_i) \delta (A_g)_{ik}^{(n-l)}.
\]

(132)

\(^{12}\) We can use this because the absorptive parts are dropped and the renormalization conditions work for outgoing particles as well.
Here $A_g$ is some coefficient of order $(n-1)$ coming from $\partial'$ derivatives of $\Sigma Z^{-1}$. The $A_g$’s also have Dirac structure, but the important part is that the $\partial' - m_i$ factor is on the left side and further determination is not needed. Analogously, we can take the factor on the third line in Eq. (131) containing $A$’s, which vanishes at $\partial' = m_i$, and expand it around $m_i$ with some coefficients $B_g$. Putting all of this in we get

$$
\delta_\xi \delta Z^{H(n)}_{ii} u_i = \frac{1}{2} \lim_{\partial' \to m_i} \frac{1}{\partial' - m_i} \left[ (\partial' - m_i) \sum_{l=1}^{n-1} \tilde{\Lambda}_{ik}^{(n-l)} \delta Z^{(l)}_{ki} - \sum_{l=1}^{n-1} \gamma^0 \delta Z^{(l)}_{ki} - \sum_{l=1}^{n-1} \gamma^0 \tilde{\Lambda}_{ki}^{(n-l)} (\partial' - m_i) \right]
+ \sum_{l=1}^{n-1} \left( \sum_{g=1} \gamma^0 (B_g \gamma_k (l)) \right) \left[ \tilde{\Lambda}_{ki}^{(0)(l)} + \sum_{q=1} \gamma^0 \delta Z^{(q)}_{ri} \right]
+ \sum_{l=1}^{n-1} \left( \sum_{g=1} \gamma^0 (A_g \gamma_k (n-l)) \right) \left[ \tilde{\Lambda}_{ki}^{(l)} + \sum_{f=1} \gamma^0 \tilde{\Lambda}_{kr}^{(l-f)} \delta Z^{(f)}_{ri} + \delta_\xi \delta Z^{(l)}_{ki} \right] u_i.
$$

(133)

Conveniently, the $\frac{1}{\partial' - m_i}$ simply lowers the order of $(\partial' - m_i)^k$ by one and there is no more limit. After the limit is gone we can freely act with the spinor $u_i$ on the right side of the terms containing $B$’s and $A$’s. The term containing $B$’s vanishes due to the renormalization condition in Eq. (121), the term containing $A$’s vanishes as in the off-diagonal case as long as the gauge dependence of the field renormalization is the same for all $i$ and $j$. Eventually, we are left with a fairly simple answer

$$
\delta_\xi \delta Z^{H(n)}_{ii} u_i = -\tilde{\Lambda}_{ii}^{H(n)} u_i - \frac{1}{2} \lim_{\partial' \to m_i} \frac{1}{\partial' - m_i} \left[ (\partial' - m_i) \sum_{l=1}^{n-1} \tilde{\Lambda}_{ik}^{(n-l)} \delta Z^{(l)}_{ki} - \sum_{l=1}^{n-1} \gamma^0 \delta Z^{(l)}_{ki} - \sum_{l=1}^{n-1} \gamma^0 \tilde{\Lambda}_{ki}^{(n-l)} (\partial' - m_i) \right] u_i
+ \sum_{l=1}^{n-1} \left( \sum_{g=1} \gamma^0 (A_g \gamma_k (n-l)) \right) \left[ \tilde{\Lambda}_{ki}^{(l)} + \sum_{f=1} \gamma^0 \tilde{\Lambda}_{kr}^{(l-f)} \delta Z^{(f)}_{ri} + \delta_\xi \delta Z^{(l)}_{ki} \right] u_i.
$$

(134)

Here we used that $\Lambda_{ii}^{H} = \Lambda_{ii}$, in the second equality for compact notation we are forced to change the order of Dirac structures of the $\delta Z^{(l)}$ term. However, this is still the hermitian part of Eq. (129) for $i = j$. If we want this to hold, we can no longer set the diagonal anti-hermitian part of the field renormalization to zero beyond 1-loop, since otherwise the sums do not fully cancel. Again, since there is no other way to determine the anti-hermitian part for the diagonal field counterterm we can choose it to have the same gauge dependence as the anti-hermitian part of Eq. (129). This is most easily achieved if the diagonal component is determined from the off-diagonal one by simply setting $i = j$. This is allowed since the off-diagonal anti-hermitian part is the coefficient of $m_i^2 - m_j^2$ and no singular behaviour occurs for $i = j$. In addition, at 1-loop this choice gives the usual 0 since only absorptive parts could contribute to the diagonal anti-hermitian part—we have dropped these parts in this appendix. With this choice we see that the gauge dependence for field counterterms can be described with the same Eq. (129) for all $i$ and $j$ to all orders in general without any explicit computation of $A$’s, which is an unexpected and non-trivial result. This result can also be written in a very compact form

$$
\delta_\xi \delta Z_{ii} u_i = -\tilde{\Lambda} Z u_i.
$$

(135)

This is analogous to the result achieved for the LSZ factor in Eq. (48) of [32], however, our approach allows to define field and mass counterterms order by order. Importantly, having off-diagonal mass counterterms does not change gauge-dependence properties, but rather allows to get rid of singular behavior and the need to renormalize mixing matrices.

A.5.2 Renormalized gauge parameter

So far we have found the gauge-dependence of the field renormalization in terms of the bare gauge parameter. This avoided the discussion of the terms modifying the Nielsen identity. In this subsection we relate the bare and renormalized gauge derivative parameters, which is made easier by the results of the previous subsection.

We begin by taking the gauge derivative of Eq. (121) w.r.t. the renormalized gauge parameter $\xi$ and explicitly expand the bare self-energy $\tilde{\Sigma}^{0}$ in terms of renormalized parameters

$$
\delta_\xi \left( \prod_k T_{p_k} \tilde{\Sigma}^{0} |_{\rho = p} Z \right) u_i = \delta_\xi \left( \prod_k T_{p_k} \tilde{\Sigma}^{0} |_{\rho = p} Z u_i \right) + \left( \prod_k T_{p_k} \delta_\xi \Sigma^{0} \right) Z u_i = 0.
$$

(136)

It should be understood that the series operator acts only up to the restriction $|_{\rho = p}$. The first term is the gauge derivative of the counterterms appearing in the series operators, in the second term we used Eq. (96) before taking the gauge derivative, the third term simply contains the gauge derivative of the field renormalization we did not expand the self-energy.
Let us consider the term containing the gauge derivative of the self-energy as we can immediately use the Nielsen identity. After using the identity we can conveniently get rid of the series operator by simply writing everything in terms of the bare parameters:

\[
\left( \prod_k T_{p_k} \partial_k \Sigma^0 \right) Z_{u'_i} = \left( \prod_k T_{p_k} (\Lambda \Sigma^0 + \Sigma^0) \right) Z_{u'_i} = \left( \Lambda \tilde{\Sigma}^0 + \tilde{\Sigma}^0 \right) Z_{u'_i} = -\tilde{\Sigma}^0 \partial_{\xi} Z_{u'_i}.
\]  

(137)

To get the final equality we used Eqs. (135) and (121) which showed that \( \Lambda \tilde{\Sigma}^0 Z_{u'_i} \) vanishes due to the renormalization condition.

Now we take care of the remaining term with the gauge derivative of counterterms. First, we use Eq. (122), we get

\[
\partial_\xi \left( \prod_k T_{p_k} \right) \tilde{\Sigma}^0 |_{p^0 = p} Z_{u'_i} = -\partial_\xi \left( \prod_k T_{p_k} \right) \tilde{\Sigma}^0 |_{p^0 = p} \tilde{\Sigma}^0 |_{p^0 = p} \delta Z_{u'_i} = -\partial_\xi \left( \prod_k T_{p_k} \right) \tilde{\Sigma}^0 |_{p^0 = p} \delta Z_{u'_i}.
\]

(138)

Now we have the gauge derivative of the series operator acting on \( \tilde{\Sigma}^0 \delta Z \). Conveniently, the series expansion remains the same if one expands the counterterm and the self-energy separately. The gauge derivative then acts on two series operators, for which we simply have the Leibniz rule, eventually we get

\[
-\partial_\xi \left( \prod_k T_{p_k} \right) \tilde{\Sigma}^0 |_{p^0 = p} Z_{u'_i} = -\partial_\xi \left( \prod_k T_{p_k} \right) \tilde{\Sigma}^0 |_{p^0 = p} \tilde{\Sigma}^0 |_{p^0 = p} \delta Z_{u'_i} = -\partial_\xi \left( \prod_k T_{p_k} \right) \tilde{\Sigma}^0 |_{p^0 = p} \delta Z_{u'_i}.
\]

(139)

To get the final equality we removed the series expansions and left the quantities written in terms of bare parameters where possible. Putting this back in Eq. (138) there are cancellations and only one term remains

\[
\partial_\xi \left( \prod_k T_{p_k} \right) \tilde{\Sigma}^0 |_{p^0 = p} Z_{u'_i} = -\partial_\xi \delta Z_{u'_i} = -\tilde{\Sigma}^0 \partial_\xi Z_{u'_i} = -\tilde{\Sigma}^0 \partial_\xi Z_{u'_i}.
\]

(140)

In the final equality we simply added 1 to the field counterterm since the gauge derivative of the series operator has only derivative terms acting on the field counterterm.

Finally, we collect our results in Eqs. (137) and (140), put them in Eq. (136)

\[
\tilde{\Sigma}^0 \left[ -\partial_\xi \left( \prod_k T_{p_k} \right) Z_{u'_i} - \partial_\xi Z_{u'_i} \right] u'_i = 0
\]

(141)

Dropping the self-energy and rearranging we get

\[
\partial_\xi Z_{u'_i} = \left[ \partial_\xi Z_{u'_i} \right] u'_i.
\]

(142)

Again, there are no singularities coming from the \( m^2_n - m^2 \) mass structure, meaning that our definitions of mass and field counterterms remain valid with the renormalized gauge parameter. As a quick check, at second order the above correctly reproduces Eq. (120) with the expected term related to the modification of the Nielsen identity. If one chose to renormalize the physical parameters in a gauge-dependent way, the term with the series operator would also reproduce the needed modification of the Nielsen identity.

A.5.3 Summary of the counterterms

Now we can go back to Eqs. (122) and (125) and extract (or simply copy for convenience) the nth order definitions of mass and field counterterms. First, to separate the mass counterterm at order \( n \) one must expand the bare self-energy \( \tilde{\Sigma}^{0(n)} \) and separate the term where \( \delta m^{(n)} \) appears, thus we define \( \tilde{\Sigma}^{0(n)} \) by

\[
\tilde{\Sigma}^{0(n)}(p^2) = \tilde{\Sigma}^{0(n)}(p^2) - \delta m^{(n)}.
\]

(143)
Then, leaving the algebra to the reader and reminding that the absorptive parts are dropped, the hermitian and anti-hermitian parts of the off-diagonal field counterterms\(^{13}\) are as follows\(^{14}\)

\[
\delta Z_{ji}^{(n)} \, u_i = - \frac{1}{2 (m_i^2 - m_j^2)} \left[ (p^2 + m_j) \left( \hat{\Sigma}_{ji}^{(0)}(p) + \sum_{l=1}^{n-1} \hat{\Sigma}_{ji}^{(n-l)}(p) \delta Z_{kl}^{(l)} \right) - H.C. \right] u_i, \tag{144}
\]

\[
\delta Z_{ji}^{A(n)} \, u_i = - \frac{1}{2} \left[ (p^2 + m_j) \left( \hat{\Sigma}_{ji}^{(0)}(p) + \sum_{l=1}^{n-1} \hat{\Sigma}_{ji}^{(n-l)}(p) \delta Z_{kl}^{(l)} \right) + H.C. \right] u_i \bigg|_{m_i^2 - m_j^2}, \tag{145}
\]

In the above \(H.C.\) means Hermitian conjugation, although, we note that Hermitian conjugation makes more sense (is easier to apply) after using \(u_i\) on \(p\). As in the 1-loop definitions, for the anti-hermitian part we simply take the coefficient of \(m_i^2 - m_j^2\).

The hermitian part of the diagonal field counterterm is

\[
\delta Z_{ii}^{(n)} \, u_i = - \frac{1}{2} \lim_{\beta \rightarrow m_i} \frac{1}{p^2 - m_i} \left[ \hat{\Sigma}_{ii}^{(0)}(p) + \sum_{l=1}^{n-1} \hat{\Sigma}_{ii}^{(n-l)}(p) \delta Z_{kl}^{(l)} \right] u_i - \frac{1}{2} \sum_{j=1}^{n-1} \sum_{l=1}^{n-l} \hat{\Sigma}_{ij}^{(n-l-f)}(p) \delta Z_{ji}^{(f)} Z_{ji}^{(i)} \right] u_i, \tag{146}
\]

and the anti-hermitian part is determined from the off-diagonal one

\[
\delta Z_{ii}^{A(n)} \, u_i \equiv \delta Z_{ii}^{A(n)} \bigg|_{i=j}. \tag{147}
\]

To reiterate, with these definitions the gauge-dependence of the field renormalization is described in the same way for all \(i\) and \(j\). In terms of the bare gauge parameter we have

\[
\partial_{\beta} Z_{jj} u_i = - \left( \tilde{\Lambda}(p) Z \right)_{ji} u_i \tag{148}
\]

and in terms of the renormalized one

\[
\partial_{\xi} Z_{jj} u_i = \left. \left[ \partial_{\beta} Z_{jj} + \partial_{\xi} \left( \prod_{k} T_{\rho_k} \right) Z_{jj} \bigg|_{\rho=\rho} \right] \right) u_i \tag{149}
\]

both of which hold order by order.

The mass counterterm for all \(i\) and \(j\) is

\[
\delta m_{ji}^{(n)} \, u_i = \left[ \hat{\Sigma}_{ji}^{(0)}(p) + \sum_{l=1}^{n-1} \hat{\Sigma}_{ji}^{(n-l)}(p) \delta Z_{kl}^{(l)} + (p^2 - m_j) \delta Z_{ji}^{(n)} \right] u_i. \tag{150}
\]

It can also be checked that the mass counterterm is gauge-independent by definition. For the diagonal component one can see that the \(n\)th order field counterterms do not contribute by multiplying both sides by \(p^2 + m_j\). Even more so, for field renormalization constants one can use Eqs. (144) and (145) to get the mass counterterm in terms of self-energies and restrictions to mass structures to all orders. Finally, we remind that the tildes above self-energies and \(\Lambda\)’s mean that the computation is done with bare parameters.

To get these relations for renormalized parameters one needs to expand the bare parameters around their renormalized values, which gives contributions of the counterterms (up to order \(n - 1\)) from other parameters in the theory.

Finally, with these definitions the mixing matrices are already renormalized so that

\[
\delta V^{(n)} = 0 \tag{151}
\]

as also argued in Sect. 2.1 on general grounds.

We have shown that our 1-loop logic and results with the absorptive parts dropped can be extended to all orders. In addition, other 1-loop benefits are preserved to all orders: no singular behaviour in the degenerate mass limit, process independence, basis rotations and renormalization commute, etc. Although, the one thing we cannot show is the UV finiteness of the anti-hermitian part of the field renormalization—such finiteness is not obvious beyond 1-loop and may be an accident. If the absorptive parts are consistently included, our scheme should also be applicable as it depends only on the \(m_i^2 - m_j^2\) structure.

---

\(^{13}\) Remember that in the “Appendix” there are no square roots and the field renormalization is defined as \(\psi_0 = Z \psi\).

\(^{14}\) Here \([m_i^2 - m_j^2]^n A + B\) \(\big|_{m_i^2 - m_j^2} = (m_i^2 - m_j^2)^{n-1} A\), for some positive power \(n\) and functions \(A\) and \(B\).
A.6 Practical considerations

The definitions of counterterms we propose are valid to all orders and in the main body of the paper we also give examples in the Grimus–Neufeld model. However, there is a downside—selecting the relevant $m_1^2 - m_2^2$ mass structure is not trivial. At 1-loop we could do it by simply collecting terms in Mathematica with Collect and then selecting the relevant terms with SelectNotFree2 from FeynCalc, however, this required the Passarino–Veltman functions for the quarks and explicit analytical evaluation of these functions for neutrinos. Even more so, the computations must be performed with the gauge-parameters present, since otherwise the mass structures may disappear, for example, in Feynman gauge $m_1^2 - m_2^2$ “mixes” with $m_1^2 + m_2^2$. Of course, if one sets the gauge-parameters to some value there is no longer any gauge-dependence so that it makes sense for the structures to sometimes disappear. It is easy to see that beyond 1-loop this naive approach becomes rather cumbersome.

In order to get around this, we need a way of computing the anti-hermitian part of the field renormalization independently of the mass counterterm. To do so let us consider Eq. (148) or Eq. (149), where the gauge-dependence of the field renormalization is given. These equations can be thought of as renormalization conditions for $\Lambda$’s in addition to the On-Shell renormalization conditions. Importantly, $\Lambda$’s are correlation functions and can be computed via Feynman diagrams, for example, for the SM one can use the Lagrangian given in [32]. When we have $\Lambda$’s, we can retrieve the gauge-dependent part of the field renormalization via simple integration over the gauge parameter. The simplest case is in a special gauge where all the gauge parameters are the same $\xi_i = \tilde{\xi}$, then taking Eq. (149) and considering the anti-hermitian part we may write

$$Z_{ji}^A u_i = \int d\xi \left[ \partial_{\xi}Z_{ji} + \partial_{\xi} \left( \prod_k T_{p k} \right) Z_{ji} \right] u_i. \quad (152)$$

Here $H.C.$ is again to be used after the spinor $u_i$ acts on $p$. The integral is indefinite, but we do not add the integration constant as it is identified with the mass counterterm (up to factors of the mass structures), which is gauge-independent. For the gauge-derivative w.r.t. the bare gauge parameter one should use Eq. (148) and afterwards express it in terms of renormalized parameters. Then one can use this expression of the anti-hermitian part of the field renormalization in the definition of the mass counterterm in Eq. (150).

For example, at 1-loop the above becomes

$$Z_{ji}^{A(1)} u_i = \int d\xi \left[ -\tilde{\Lambda}_{ji}^{(1)} (p) - H.C. \right] u_i = \int d\xi \left[ -\tilde{\Lambda}_{ji}^{(1)} (p) - H.C. \right] u_i. \quad (153)$$

Here we used Eq. (148) at 1-loop and the fact that $\Lambda$’s start at 1-loop order so that $\tilde{\Lambda}_{ji}^{(1)} = \tilde{\Lambda}_{ji}^{(1)}$. At 1-loop there are no contributions from the non-physical renormalization since $\partial_{\tilde{\xi}_1} \left( \prod_k T_{p k} \right) Z_{ji}$ first appear at 2-loop order.

If there are more gauge-parameters it is still possible to recover the gauge-dependent part of the anti-hermitian part of the field renormalization, but the approach is more cumbersome since one has to take care in dealing with terms where multiple gauge parameters appear together. For example, consider a theory with two gauge parameters $\xi_1$ and $\xi_2$, then one has

$$Z_{ji}^A u_i = \int d\xi_1 \partial_{\tilde{\xi}_1} Z_{ji} + \int d\xi_2 \partial_{\tilde{\xi}_2} Z_{ji} - \int d\xi_2 \int d\xi_1 \partial_{\tilde{\xi}_1} \partial_{\tilde{\xi}_2} Z_{ji}. \quad (154)$$

The first term gives terms which depend only on $\xi_1$ or on $\xi_1$ and $\xi_2$, the second term similarly gives terms which depend on $\xi_2$ or on $\xi_1$ and $\xi_2$. Since the first two terms double-count the contributions where both gauge parameters appear, the final term with two integrals removes this over-counting. One can check that the above is valid by taking the simple test function $Z = \xi_1 \xi_2 + \xi_1 + \xi_2$. The same logic can be extended to an arbitrary number of gauge-parameters.

The upshot is that by computing $\Lambda$’s and integrating them over gauge-parameters one can avoid the selection of $m_1^2 - m_2^2$ mass structures. Even more so, it is not necessary to perform the loop momentum integration in $\Lambda$’s before the gauge parameter integration. Practically this means that once integration over the gauge parameters is done one can evaluate the remaining loop momentum integral as one sees fit, i.e. either analytically or numerically. This considerably extends the practical reach of the proposed scheme.

However, a minor downside of the practical approach is that not only gauge-dependent parts can carry the $m_1^2 - m_2^2$ mass structure as we have found in Eq. (81). In other words, the anti-hermitian part of the field renormalization given in Eq. (145) may be different from the one given in this section up to gauge-independent terms. This means that with the practical approach the On-Shell renormalization conditions still hold and the resulting mass counterterms are still gauge-independent, but may contain gauge-independent terms that are supposed to be in the field renormalization. However, this is barely a problem since this does not cause the appearance of terms proportional to $(m_1^2 - m_2^2)^{-1}$ in the anti-hermitian part of the field renormalization, which become large as masses become numerically close. The important point is that in both approaches the field renormalization contains all the gauge-dependence such that they are able to cancel the gauge-dependence (up to absorptive parts) in amplitudes.

Appendix B Full mass counterterms

In this appendix we simply provide the off-diagonal mass counterterms that were too big for the main body of the paper.
B.1 Up-type quarks

\[
\delta m_{ji}^{u,L} = \sum_{k=1}^{3} \left[ \frac{V_{jk} V_{ik}^{*}}{2 D_{\frac{D-2}{2}} - D_m} \right] B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ik}^u)^2) \left[ \frac{(m_{ik}^u)^2((D - 3)m_{W}^2 - 2(m_{ij}^u)^2) + (m_{ij}^u)^2((D - 2)m_{W}^2 + (m_{ij}^u)^2)}{8m_i^u} \right]
\]

\[+ \frac{s_{2a}^2 (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2) + m_{ik}^2(G_u^*)^{jk}}{4} \right] \]

\[+ \frac{s_{2a} m_{ij}^u}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2) - m_{ik}^2}{m_{ij}^u} \right] \]

\[+ \frac{s_{2a} m_{ij}^u}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{4(m_{ij}^u)^2 - m_{ik}^2}{m_{ij}^u} \right] \]

\[+ \frac{3}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ik}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{8m_i^u} \right] \right] \]

\[+ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]

\[+ \frac{1}{4} \left[ \frac{c_{2a} (G_u^*)^{jk}}{2 D_{\frac{D-2}{2}} - D_m} B_0((m_{ij}^u)^2, (m_{ik}^u)^2, (m_{ij}^u)^2) \left[ \frac{(G_u^*)^{jk}((m_{ik}^u)^2 + (m_{ij}^u)^2 - m_{ik}^2)}{m_{ij}^u} \right] \right] \]
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B.2 Charged leptons

\[
\delta m^{L,\dagger}_{ji} = \sum_{a=1}^{4} \left[ \frac{U_{Lja}U_{Lia}^{*}}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{a}^\nu)^2, (m_{a}^\nu)^2, m_{W}^2)^* \right. \\
\quad \times \left[ (m_{a}^\nu)^2 + (m_{a}^\nu)^2((D - 3)m_{W}^2 - 2(m_{j}^l)^2) + ((m_{j}^l)^2 - m_{W}^2)((D - 2)m_{W}^2 + (m_{j}^l)^2) \right] \\
\quad + \frac{U_{Lja}U_{Lia}^{*}}{2D_{\pi D-2}v^2m_{j}^l} A_{0}(m_{W}^2) \left[ (m_{a}^\nu)^2 - (m_{j}^l)^2 + (D - 2)m_{W}^2 \right] \\
- \frac{U_{Lja}U_{Lia}^{*}}{2D_{\pi D-2}v^2m_{j}^l} A_{0}(m_{W}^2) \left[ (m_{a}^\nu)^2 + (m_{j}^l)^2 + (D - 2)m_{W}^2 \right] \right] \\
+ \sum_{k=1}^{3} \left[ \frac{s_{2a}^{2}(G_{l})_{jk}}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{H}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ ((m_{j}^l)^2 + (m_{j}^l)^2 - m_{H}^2)(G_{l})_{jk}^{*} + \frac{m_{j}^l(G_{l})_{jk}}{4} \right] \\
\quad + \left[ \frac{s_{2a}m_{j}^l}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{H}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ \frac{(G_{l})_{jk}^{*}(m_{j}^l)^2 - (m_{j}^l)^2 - (m_{j}^l)^2}{8\sqrt{2}v} \right] \\
\quad + \left[ \frac{s_{2a}m_{j}^l}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{H}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ \frac{m_{j}^l(G_{l})_{jk}}{4\sqrt{2}v} \right] \\
\quad + \left[ \frac{s_{2a}(G_{l})_{ji}}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{H}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ \frac{m_{j}^l - 4(m_{j}^l)^2}{8\sqrt{2}v} \right] + H.C. \right] \\
+ \sum_{k=1}^{3} \left[ \frac{s_{2a}^{2}(G_{l})_{jk}}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{H}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ ((m_{j}^l)^2 + (m_{j}^l)^2 - m_{H}^2)(G_{l})_{jk}^{*} + \frac{m_{j}^l(G_{l})_{jk}}{4} \right] \\
\quad + \left[ \frac{s_{2a}m_{j}^l}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{H}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ \frac{(G_{l})_{jk}^{*}(m_{j}^l)^2 + (m_{j}^l)^2 - m_{H}^2}{8\sqrt{2}v} \right] \\
\quad + \left[ \frac{s_{2a}m_{j}^l}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{H}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ \frac{-m_{j}^l(G_{l})_{ji}}{4\sqrt{2}v} \right] \\
\quad + \left[ \frac{s_{2a}(G_{l})_{ji}}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{H}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ \frac{4(m_{j}^l)^2 - m_{H}^2}{8\sqrt{2}v} \right] + H.C. \right] \\
+ \sum_{k=1}^{3} \left[ \frac{(G_{l})_{jk}}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{A}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ ((m_{j}^l)^2 + (m_{j}^l)^2 - m_{A}^2)(G_{l})_{jk}^{*} - \frac{m_{j}^l(G_{l})_{jk}}{4} \right] \\
+ \left[ \frac{(G_{l})_{ji}}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{A}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ \frac{(m_{j}^l)^2 + (m_{j}^l)^2 - m_{A}^2(G_{l})_{jk}^{*} - \frac{m_{j}^l(G_{l})_{jk}}{4} \right] \\
\quad + \left[ \frac{4(G_{l}U_{kj})_{ia}}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{H}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ \frac{(m_{j}^l)^2 + (m_{j}^l)^2 - m_{H}^2)(G_{l}U_{kj})_{ia}^{*} - \frac{m_{j}^l(G_{l}U_{kj})_{ia}}{2} \right] \\
+ \left[ \frac{(G_{l}U_{kj})_{ia}}{2D_{\pi D-2}v^2m_{j}^l} B_{0}((m_{j}^l)^2, m_{H}^2, (m_{j}^l)^2)^* \right. \\
\quad \times \left[ \frac{(m_{j}^l)^2 + (m_{j}^l)^2 - m_{H}^2)(G_{l}U_{kj})_{ia}^{*} - \frac{m_{j}^l(G_{l}U_{kj})_{ia}}{2} \right] \\
- \frac{\lambda v^2(G_{l})_{ji}}{2D_{\pi D-2}v^2m_{j}^l} \left[ 2A_{0}(m_{H}^2) + A_{0}(m_{A}^2) + 3s_{a}^2A_{0}(m_{h}^2) + 3c_{a}^2A_{0}(m_{h}^2) \right] \right] \\
+ \frac{s_{2a}^2\lambda \nu v(G_{l})_{ji}}{2D_{\pi D-2}v^2m_{j}^l} \left[ \frac{1}{m_{h}^2} - \frac{1}{m_{H}^2} \right]
\]
Here $i \neq j$ and there is no gauge-dependence as should be the case.

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