Isocausal spacetimes may have different causal boundaries

J L Flores\(^1\), J Herrera\(^1\) and M Sánchez\(^2,3\)

\(^1\) Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias, Universidad de Málaga, Campus Teatinos, 29071 Málaga, Spain
\(^2\) Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada, Avenida Fuentenueva s/n, 18071 Granada, Spain

E-mail: floresj@agt.cie.uma.es, jherrera@uma.es and sanchezm@ugr.es

Received 12 March 2011, in final form 17 July 2011
Published 16 August 2011
Online at stacks.iop.org/CQG/28/175016

Abstract

We construct an example which shows that two isocausal spacetimes, in the sense introduced recently in García-Parrado and Senovilla (2003 Class. Quantum Grav. 20 625–64), may have c-boundaries which are not equal (more precisely, not equivalent, as no bijection between the completions can preserve all the binary relations induced by causality). This example also suggests that isocausality can be useful for the understanding and computation of the c-boundary.

PACS numbers: 04.20.Gz, 04.20.Cv, 02.40.Ma
Mathematics Subject Classification: 83C75, 53C50, 83C20

1. Introduction

The causal boundary, or c-boundary for short, is a well-known tool for the study of the conformal structure of a spacetime and related topics such as event horizons or singularities. The first approximation to this boundary was introduced four decades ago by Geroch, Kronheimer and Penrose (GKP) in the seminal paper [15]. Since then, a long series of redefinitions and new contributions have been carried out, and a renewed interest comes from the recent contributions by Harris [16–18] and Marolf and Ross [19, 20] (see the review in [23] for complete references). Recently, the authors have carried out an extensive revision of both the notion of c-boundary and the tools for its computation [5, 7, 8]. So, the c-boundary can be regarded now as a useful and consistent notion, which is well related to other geometric objects. From this paper we understand by c-boundary the last redefinition in [7]. Nevertheless, the properties to be considered here appear at a much more basic level (say, whenever Harris’ universal properties for the partial boundaries are satisfied [16]). So, they are valid for any definition of the c-boundary obtained by using the basic ingredients.

\(^3\) Author to whom any correspondence should be addressed.
Some years ago, García–Parrado and Senovilla [11] introduced the notions of causal mapping, causal relation and isocausality for two spacetimes $V$, $V'$. Namely, $V$ is causally related to $V'$, denoted by $V \prec V'$, if there exists a diffeomorphism $\Phi : V \rightarrow V'$ which is a causal mapping, that is, such that all the future-directed causal vectors of $V$ are mapped by the differential of $\Phi$ into future-directed causal ones of $V'$. Then, $V$ is isocausal to $V'$ if $V \prec V'$ and $V' \prec V$. In that article and subsequent developments [12, 13, 10], many applications and properties of such notions were carried out. Recall that isocausality is a generalization of conformal equivalence, adding more flexibility. This flexibility yields appealing properties, such as the fact that any spacetime is locally isocausal to the Lorentz–Minkowski one, even if it is not conformally flat. So, isocausality preserves some relevant global properties associated with the conformal structure, but not all of them—as stressed in [10] for the case of two levels of the causal ladder of spacetimes.

It was also suggested in [11, section 6] that causal mappings could be used to obtain causal extensions and boundaries for spacetimes as a generalization of the (Penrose) conformal boundary. Concretely, a causal extension is an embedding of the spacetime in a larger one such that the former is isocausal to its image in the larger one. Clearly, a boundary can then be naturally associated with such an extension (here, we will avoid the name causal boundary for this last boundary, in order to avoid confusions with the c-boundary). Recall that, in spite of its generalized use in general relativity, the conformal boundary has serious problems of existence and uniqueness. The problems come from the fact that, in order to find a reasonable conformal boundary, one has to find an appropriate open conformal embedding of the spacetime in some (aphysical) spacetime. It is not clear when such an embedding will exist and, in this case, whether the properties of the corresponding boundary will be independent of the embedding. The flexibility of causal mappings and isocausal properties allows us to check their existence much more easily than that of their conformal counterparts, even though with a cost of uniqueness.

In this paper, we explore the connections between the c-boundary and the notion of isocausality by means of a concrete example. This example first shows that two isocausal spacetimes may have different c-boundaries. That is, even though the c-boundary relies on the global conformal structure of the spacetime, it is not an object naturally invariant by isocausality. At a first glance, this property would seem to be a drawback for the notion of isocausality. On one hand, isocausality would be insufficient to distinguish between two spacetimes with different asymptotic causal behaviors. On the other hand, the boundaries obtained by using different causal extensions appear as extremely non-unique—as the causal extensions of isocausal spacetimes with different c-boundaries may look very different. However, a deeper study suggests that when the causal extensions are compared with the conformal ones, these properties are not a disadvantage. Note that, essentially, the conformal boundary becomes interesting when it agrees with some intrinsic element of the spacetime, and the conditions to ensure this agreement are commonly imposed (see [2, 7]). But the most important intrinsic element of the spacetime which may match with the conformal boundary is the c-boundary; so, basically, the conformal boundary becomes useful as an auxiliary tool to compute the more general c-boundary. On the contrary, the properties which remain true for all the elements of a class of isocausal spacetimes (in particular, the possible similarities

4 An example of the difficulties can be found in the recent article [4]. In order to ensure uniqueness, some technical assumptions (which involve any pair of lightlike curves) must be assumed. Remarkably, the existence of a maximal conformal extension is also ensured in [4]. However, this does not exclude the possibility that no extension exists, nor ensures a priori good properties for such an extension.
of their c-boundaries or of the boundaries obtained through causal extensions) become a
genuinely new type of information, which reveals new connections among non-conformally
related spacetimes. In fact, a closer look at our example in this paper suggests that causal
mappings and isocausality may yield very valuable information on the c-boundary. Here, we
explain this possibility only for our particular example, in order to provide a natural intuitive
picture. By using the machinery introduced in [8], this idea will be developed technically in a
further work by the authors.

2. The example

Typical background and terminology in Lorentzian geometry as in [3, 21, 22] and on causal
boundaries as in [3, 7, 14] will be used. From the technical viewpoint, our example will be
very simple, and the c-boundary will be handled at a very elementary level. Basically, the
construction of the c-boundary $\partial V$ of a (strongly causal) spacetime $V$ starts by defining its
future causal boundary $\hat{\partial} V$ and the dual past one $\check{\partial} V$. The former is the set of all the terminal
indecomposable past subsets (TIPS) of $V$, where any TIP can be regarded as the chronological
past $I^-[\gamma]$ of some future-directed inextensible timelike curve $\gamma$ (an obvious dual definition
appears for the elements of $\check{\partial} V$ or TIFs). A long-standing problem for the definition of the
c-boundary appears when one realizes that, eventually, some points in $\hat{\partial} V$ must be paired
with some others in $\check{\partial} V$. This problem can be solved satisfactorily [7], but its controversial
subtleties will not play any role here. In fact, our example will be robust, in the sense that
even the partial boundary $\hat{\partial} V$ will not be preserved by isocausality. Moreover, our concrete
example is bidimensional, and the TIPS can also be generated as the chronological past of
(piecewise smooth) lightlike curves, instead of timelike ones—this will be straightforward
here; however, one can find a precise justification in [9, proposition 2] and [7, section 3.5].
So, the picture of the c-boundary is simplified, as in dimension 2 the (smooth) lightlike curves
must lie in two families of geodesics.

2.1. Abstract properties

Our aim is to endow the manifold $V = \mathbb{R} \times (-\infty, 0)$ with three metrics $g_{cl}$, $g$, $g_{op}$ satisfying
the following properties.

(i) $g_{cl} \prec_0 g \prec_0 g_{op}$ where the symbol $\prec_0$ means that the future causal cones of the metric
on the left-hand side are included in the ones of the metric on the right-hand one (i.e. the
identity in $V$ is a causal mapping from $V$ endowed with the left metric to $V$ endowed with
the right one).

(ii) $g_{cl}$ and $g_{op}$ are conformally related. So the future causal boundaries $\hat{\partial}_{cl} V$, $\hat{\partial}_{op} V$ for,
respectively, $V_{cl} := (V, g_{cl})$ and $V_{op} := (V, g_{op})$ agree and, taking into account property
(i), $(V, g)$ is isocausal to $V_{cl}$ (and $V_{op}$). Moreover, $g_{cl}$ and $g_{op}$ will be simple standard
static metrics, so that their causal boundary will be easily computable.

(iii) $g$ presents a future causal boundary $\hat{\partial} V$ ’strictly greater’ than the one of $g_{cl}$ (or $g_{op}$), in a
precise sense explained below. Essentially, a segment of causally but not chronologically
related points appears for $\hat{\partial} V$ where only a point (in a timelike part of the boundary)
exists for $\hat{\partial}_{cl} V$ and $\hat{\partial}_{op} V$.

Note that the non-preservation of the c-boundary by isocausality follows from these properties.
So, once the metrics are achieved, we will discuss the interplay between the c-boundary and
isocausality.
2.2. Explicit construction

Define the metrics $g_{cl}$, $g$, $g_{op}$ on $V = \mathbb{R} \times (-\infty, 0)$ in the following way:

$$g_{cl} = -dr^2 + dx^2, \quad g = -dr^2 + \beta(t/x) \, dx^2, \quad g_{op} = -dr^2 + (1/4) \, dx^2,$$

where $\beta : \mathbb{R} \to (0, \infty)$ is a smooth function which satisfies

- $\beta(u) \equiv 1/4$ if $u(= t/x) \leq 1/2$, that is, $g = g_{op}$ in the region $x \leq 2t$.
- $\beta(u) \equiv 1$ if $u \geq 1$, that is, $g = g_{cl}$ in the region $t \leq x(<0)$.
- $\beta$ increases strictly from $1/4$ to $1$ on the interval $1/2 \leq u \leq 1$, so that the causal cones of $g_{cl}$ (respectively of $g$) are strictly contained in the ones of $g$ (respectively of $g_{op}$) in the region $2x < 2t < x$.

Note that the above-mentioned property (i) becomes clear from the properties of $\beta$. For (ii), the conformal relation between $g_{cl}$ and $g_{op}$ is also obvious. Moreover, the future causal boundary $\partial_{cl} V$ can be represented by two lines $T, J^+$ with a common endpoint $i^+$, which is the TIP equal to all $V$ (see [18, 1, 6, 8] for much more general computations, which include the c-boundary of all the standard static spacetimes). More precisely, the TIPs which constitute $T$ are the chronological past of all the future-directed lightlike geodesics $\rho$ with endpoint at $x = 0$. $T$ is timelike in the sense that any two distinct TIPs $P, P' \in T$ satisfy either $P \subsetneq P'$ or $P' \subsetneq P$, where the extended chronological relation $\subsetneq$ can be defined as $P \subsetneq P'$ if and only if there exists some $P' \in P'$ such that $P \ll P'$ for all $P \in P$. It is also clear that for the (future) chronological topology on $\partial V$ (which here reduces to the point set convergence of the corresponding TIPs as subsets of $V$; see [6–8]), $T$ will be homeomorphic to $\mathbb{R}$. That is, in the following, $T$ will be identified with $\mathbb{R} \times [0]$ (each $P \in T$ is identified with the endpoint in $\mathbb{R} \times [0]$ of the lightlike geodesic whose past is equal to $P$), and this identification holds at the point set, chronological and topological levels. The TIPs which constitute $J^+$ are the chronological pasts of all the future-directed lightlike $\rho$ as above which goes to infinity (reaching arbitrarily large values of $-x$). We will not pay attention to this line $J^+$, but we point out that it is horismotic. This means that any two distinct TIPs $P, P' \in J^+$ are horismotically related, i.e. they satisfy either $P \subset P'$ or $P' \subset P$, but neither $P \subsetneq P'$ nor $P' \subsetneq P$.

For property (iii), let us focus on the timelike line $T$, identified with $\mathbb{R} \times [0]$. Our aim is to prove that, in addition to this timelike line, the future causal boundary $\partial V$ of $(V, g)$ contains other boundary points $P = \Gamma^-[\rho]$ such that $(0, 0)$ is the endpoint of the generating future-directed lightlike curve $\rho$.

Consider the lightlike vector field $X(t, x) = (\sqrt{\beta(t/x)}, 1)$ for $g$. All the integral curves of $X$ can be written as $\gamma_i(s) = (r_i(s), s)$, with $s < 0$ and $r_i : (-\infty, 0) \to \mathbb{R}$ satisfying

$$\begin{cases} 
\dot{r}_i(s) = \sqrt{\beta} \left( \frac{r_i(s)}{s} \right) \\
\dot{r}_i(-1) = t
\end{cases}$$

(2.1)

(see figure 1). Note the following properties of the curves $\gamma_i$.

(a) For $t_1 < t_2$, necessarily $r_{t_1}(s) < r_{t_2}(s)$, as $r_{t_1}(-1) = t_1 < t_2 = r_{t_2}(-1)$ and $\gamma_{t_1}$ does not intersect $\gamma_{t_2}$.

(b) $\gamma_{-1/2}(s) = (s/2, s)$ and $\gamma_{-1}(s) = (s, s)$ for all $s < 0$, and thus, any intermediate $\gamma_t$ satisfies

$$\lim_{s \to 0} \gamma_t(s) = (0, 0) \quad \forall t \in [-1, -1/2].$$

(c) $\Gamma^-[\gamma_{t_1}] \subseteq \Gamma^-[\gamma_{t_2}]$ for all $t_1 < t_2$. 

4
In fact, (a) and (b) are direct consequences of the definition of $\gamma_t$. Property (c) is a consequence of (a) and the following characterization:

$$ I^{-}[\gamma_t] = \{ (t', s) : t' < r_t(s) \} \quad \forall t \in \mathbb{R}. $$

(2.2)

The inclusion $\supset$ follows because, for the metric $g$, $t' < r_t(s)$ implies $(t', s) \ll (r_t(s), s)$. For $\subset$, recall that $V \setminus \{ \gamma_t(s) : s < 0 \}$ has two connected components, and the right-hand side of (2.2) is equal to one of them. Any past-directed timelike curve $\alpha$ starting at a point $p$ on $\gamma_t$ must enter initially in this region (as any tangent vector in the past timelike cone at $p$ points to it). Moreover, $\alpha$ cannot touch $\gamma_t$ at a distinct (first) point $q$, as $\alpha$ and $\gamma_t$ would intersect transversally and, so the velocity $\alpha' \text{ would point out to the future on } q$. As a consequence, $\alpha$ remains totally contained in that region up to the initial point $p$.

From properties (b) and (c), different TIPs $P_t := I^{-}[\gamma_t]$, with $t \in [-1, -1/2]$, become naturally associated with the point $(0, 0)$ (which was identified with a point of $\hat{\partial}_{cl}V$). This implies the required property (iii). In fact, the description of the boundary $\partial V$ for $g$ is similar to that of $\partial_{cl}V$. However, now in the analog to the timelike line $T \subset \partial_{cl}V$, the boundary point associated with $(0, 0)$ must be replaced by all the TIPs in the strain $\text{Str} := \{ P_t : -1 \leq t \leq -1/2 \}$. So, we can regard $T_{\text{Str}} = (\mathbb{R} \setminus \{ 0 \}) \times \{ 0 \} \cup \text{Str}$, as a part of $\partial V$ (see figure 2). Recall that all the points in the strain are horismotically related. So, $T_{\text{Str}}$ differs from $T$ from the chronological viewpoint (there exists no bijection from $T_{\text{Str}}$ in $T$ which preserves the chronologically and horismotically related points). Nevertheless, if one
replaces the whole strain by any of its elements, this bijection appears naturally. Summing up, the claimed property (iii), as well as the non-equivalence of \( \hat{\partial} V \) and \( \hat{\partial}_{cl} V \), is justified in a precise way.

2.3. Final discussion

We can understand the behavior of the causal boundary in the previous example as follows.

Consider two causally related spacetimes \( V_1 \prec_0 V_2 \) (we will write \( I^-_1, I^-_2 \) instead of \( I^- \) in each spacetime). A natural map between the future boundaries \( \hat{j} : \hat{\partial} V_1 \rightarrow \hat{\partial} V_2 \) can be defined by taking into account that if \( P \in \hat{\partial} V_1 \), then \( I^-_2(P) \in \hat{\partial} V_2 \). In fact, if \( P = I^-_1[\gamma] \) for some inextendible future-directed timelike curve \( \gamma \), then \( \gamma \) must also be timelike for \( V_2 \), and \( I^-_2[\gamma] = I^-_2(P) \). So, we can define \( \hat{j}(P) := I^-_2(P) \ \forall P \in \hat{\partial} V_1 \).

Nevertheless, \( \hat{j} \) may be very bad behaved, even if \( V_1 \) and \( V_2 \) are isocausal. Concretely, our example above shows that the map \( \hat{j}_{cl} : \hat{\partial} V_{cl} \rightarrow \hat{\partial} V \) associated with \( V_{cl} \prec V \) cannot be continuous (nor surjective), as it induces a map \( I^- \rightarrow \mathcal{T}_{Str} \), where \( j(0, 0) \) chooses just the point \( I^- \) of the strain. Moreover, the map \( \hat{j}_{op} : \hat{\partial} V \rightarrow \hat{\partial}_{op} V \) associated with \( V \prec V_{op} \) is continuous, but it is not injective, as all the strain is mapped into \((0, 0)\). It is worth pointing out that in spite of these properties, the composition \( \hat{j}_{op} \circ \hat{j}_{cl} : \hat{\partial}_{cl} V \rightarrow \hat{\partial}_{op} V \) is an isomorphism (a homeomorphism which also preserves the chronological relation). Our example shows that this nice last property does not imply a straightforward good relation between \( \hat{\partial} V \) and \( \hat{\partial}_{cl} V \).

However, the example suggests another possibility. Assume that all the elements in the strain of \( \hat{\partial} V \) were identified with a single one. Then \( \hat{\partial}_{cl} V \) would be naturally embedded in this quotient space (in this particular example, they would be naturally isomorphic). In this sense, the boundary \( \hat{\partial}_{cl} V \) yields an important information about the boundary \( \hat{\partial} V \), namely...
$\hat{\partial}_J V$ represents the quotient of a part of $\hat{\partial} V$ (alternatively, $\hat{\partial} V$ can be seen as an enlargement of $\hat{\partial}_J V$). At what extent is this property generalizable? We will prove that it can be extended to a wide family of spacetimes which are isocausal to the standard stationary ones. However, the computation of such boundaries requires the machinery on Finsler metrics and Busemann functions developed in [8]. So, it is postponed to a forthcoming paper.

**Acknowledgments**

The authors would like to acknowledge Professor Senovilla for very useful discussions and comments. JH also thanks the kind hospitality of the Department of Theoretical Physics and History of Science, University of Basque Country, during his research stay associated with this work. All the authors are partially supported by the research projects with FEDER funds MTM2010-18099 (Spanish MICINN) and P09-FQM-4496 (Regional J Andalucía). JH is also supported by Spanish MEC grant AP2006-02237.

**Appendix**

Our example can be understood more clearly as the spacetime $(V, g)$ is conformal (thus, isocausal) to the following open region of Minkowski spacetime:

$$V' = \mathbb{L}^2 \setminus ([x \geq a] \cup \{t + x \geq 0, \ 0 \leq x \leq a\}),$$

where $a = (\pi/2) - \arctan(1/2)$. A conformal map $f : (V, g) \rightarrow (V', g_0)$ is represented in figure A1 and can be described as follows.

The spacetime $(V, g)$ is divided into three regions: (A) the wedge (i.e. the region between $\gamma_{-1}$ and $\gamma_{-1/2}$), (B) the region above the wedge (above $\gamma_{-1/2}$) and (C) the region below the...
wedge (below $\gamma_{-1}$). Accordingly, the spacetime $(V', g_0)$ is also divided into three regions: $(A') \{ (t, x) \in V' : -2a \leq t - x \leq 0 \}$, $(B') \{ (t, x) \in V' : t - x > 0 \}$ and $(C') \{ (t, x) \in V' : t - x \leq -2a \}$. Given a point $p_A$ of region $(A)$, there exist two lightlike geodesics $\gamma_{p_A}$, $\sigma_{p_A}$ passing through it, which are integral curves of the lightlike vector fields $X(t, x) = (\sqrt{\beta(t/x)}, 1)$, $Y(t, x) = (\sqrt{\beta(t/x)}, -1)$, respectively. These curves determine the parameters $r_{p_A}$ (the natural Euclidean distance from $\sigma_{p_A} \cap \gamma_{-1/2}$ to the origin) and $\alpha_{p_A}$ (the Euclidean angle between the velocities of $\gamma_{-1/2}$ and $\gamma_{p_A}$ at the origin), as indicated in the figure. Then, the image $f(p_A)$ is defined as the point in region $(A')$ which lies in the line $t - x = -2a p_A$ at the natural Euclidean distance $r_{p_A}$ from $(-\alpha_{p_A}, \alpha_{p_A})$. Next, given a point $p_B$ in region $(B)$, it is clearly determined by the parameters $t_{p_B}$ (where $(t_{p_B}, 0)$ is the future endpoint of the integral curve of the lightlike vector field $X$ through $p_B$) and $r_{p_B}$ (Euclidean distance to this endpoint from $p_B$). Then, the image $f(p_B)$ is defined as the point in region $(B')$ determined by the analogous parameters for an integral curve of $\alpha + \beta$, as indicated in the figure. Finally, for any $p_C$ belonging to region $(C)$, we proceed similarly to obtain the parameters $t_{p_C}$ and $r_{p_C}$, and define $f(p_C)$ in the region $(C')$ of $(V', g_0)$ as the point determined by $t_{p_C} - a$ (which selects an integral curve of $\alpha + \beta$) and $r_{p_C}$ (which selects a point in this curve).

Recall that this map $f$ is obviously continuous and piecewise smooth. Its conformal character is ensured as it clearly maps lightlike curves in $(V, g)$ into lightlike curves in $(V', g_0)$.

References

[1] Ala˜na V and Flores J L 2007 The causal boundary of product spacetimes Gen. Rel. Grav. 39 1697–718
[2] Ashtekar A and Hansen R O 1978 A unified treatment of null and spatial infinity in general relativity: I. Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity J. Math. Phys. 19 1542–66
[3] Beem J K, Ehrlich P E and Easley K L 1996 Global Lorentzian Geometry (Monographs Textbooks Pure Applied Mathematics vol 202) (New York: Dekker)
[4] Chrusciel P T 2010 Conformal boundary extensions of Lorentzian manifolds J. Diff. Geom. 84 19–44 (available at http://www.intpress.com/JDG/2010/JDG-v84.php)
[5] Flores J L 2007 The causal boundary of spacetimes revisited Commun. Math. Phys. 276 611–43
[6] Flores J L and Harris S G 2007 Topology of the causal boundary for standard static spacetimes Class. Quantum Grav. 24 1211–60
[7] Flores J L, Herrera J and Sánchez M 2010 On the final definition of the causal boundary and its relation with the conformal boundary arXiv:1001.3270
[8] Flores J L, Herrera J and Sánchez M 2010 Gromov, Cauchy and causal boundaries for Riemannian, Finslerian and Lorentzian manifolds arXiv:2011.1154
[9] Flores J L and Sánchez M 2008 The causal boundary of wave-type spacetimes J. High Energy Phys. JHEP03(2008)036
[10] García-Parrado A and Sánchez M 2005 Further properties of causal relationships: causal structure stability, new criteria for isocausality and counterexamples Class. Quantum Grav. 22 4589–619
[11] García-Parrado A and Senovilla J M M 2003 Causal relationships: a new tool for the causal characterization of Lorentzian manifolds Class. Quantum Grav. 20 625–64
[12] García-Parrado A and Senovilla J M M 2003 Causal symmetries Class. Quantum Grav. 20 L139
[13] García-Parrado A and Senovilla J M M 2004 General study and basic properties of causal symmetries Class. Quantum Grav. 21 661–96
[14] García-Parrado A and Senovilla J M M 2005 Causal structures and causal boundaries Class. Quantum Grav. 22 R1–R84
[15] Geroch R P, Kronheimer E H and Penrose R 1972 Ideal points in spacetime Proc. R. Soc. A 237 545–67
[16] Harris S G 1998 Universality of the future chronological boundary J. Math. Phys. 39 5427–45
[17] Harris S G 2000 Topology of the future chronological boundary: universality for spacelike boundaries Class. Quantum Grav. 17 S51–603
[18] Harris S G 2001 Causal boundary for standard static spacetimes Nonlinear Anal. 47 2971–81
[19] Marolf D and Ross S 2002 Plane Waves: to infinity and beyond! Class. Quantum Grav. 19 6289–302
[20] Marolf D and Ross S R 2003 A new recipe for causal completions Class. Quantum Grav. 20 4085–117
[21] Minguzzi E and Sánchez M 2008 The causal hierarchy of spacetimes Recent developments in pseudo-Riemannian Geometry ed H Baum and D Alekseevsky ESI Lect. Math. Phys. (Zurich: Eur. Math. Soc. Publ. House) 299–358 (arXiv:gr-qc/0609119)

[22] O’Neill B 1983 Semi-Riemannian Geometry with Applications to Relativity (New York: Academic)

[23] Sánchez M 2009 Causal boundaries and holography on wave type spacetimes Nonlinear Anal. 71 e1744–64