\section*{Abstract}

The paper proposes a construction of a quantum differentiation operator defined on the spaces of complex-valued functions of \( p \)-adic argument and taking values in the algebra of bounded operators on a Hilbert space. The properties of this operator are investigated. In particular, it is proved that the differentiation operator maps \( p \)-adic Besov spaces into Schatten-von Neumann ideals in the algebra of compact operators.

\section{Introduction}

First of all, we will give a description of what will be called the \( p \)-adic quantum calculus. Consider the algebra \( A = L^{\infty}(\mathbb{Z}_p, \mathbb{C}) \) of almost everywhere bounded measurable on \( \mathbb{Z}_p \) functions taking value in the field of complex numbers. Let’s define an irreducible representation of this algebra by multiplication operators in the Hilbert space \( L^2 = L^2(\mathbb{Z}_p) \), \( A \ni f \mapsto M_f \in \mathcal{B}(L^2) \),

\[ M_f : L^2 \ni h \mapsto fh \in L^2. \]

On the space \( L^2 \), the Hilbert operator \( S \) is defined (we will define it later), which is a self-adjoint operator and \( S^2 = \text{Id} \). For each function \( f \in A \), we define a quantum differentiation operator

\[ df = [M_f, S]. \]

The correspondence \( f \mapsto df \) constructed in this way between functions on \( \mathbb{Z}_p \) and the operators \( df \) on \( L^2 \) we will call \( p \)-adic quantum calculus.
The group $\mathbb{Z}_p$ of $p$-adic integers is the inverse limit of finite cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$, $n \in \mathbb{Z}_+$:

\[ \leftarrow \mathbb{Z}/p^n\mathbb{Z} \leftarrow \mathbb{Z}/p^{n+1}\mathbb{Z} \leftarrow \]

In this sense, the proposed construction is a combinatorial analogue of the quantum calculus suggested by A. Connes[1].

The purpose of this article is to study this correspondence. In particular, the following statements will be proved.

- $df$ is a finite rank operator if and only if the function $f$ is locally constant;

- $df$ is a compact operator if and only if the function $f$ belongs to the space $VMO(\mathbb{Z}_p)$;

- $df$ belongs to the Schatten-von Neumann ideal $\mathcal{S}^q$ if and only if $f$ belongs to the $p$-adic Besov space $B^{1/q}_{q,q}$.

Note that the last two statements are completely analogous to the corresponding statements for algebras of complex-valued functions on the circle $\mathbb{T} \subset \mathbb{C}$.

The Connes quantum calculus (another name is the Connes quantization) has many interesting applications (see, for example, [2, 3]).

There is an important feature in the $p$-adic case considered in this paper. Namely, in all cases interesting for the application, there is no nontrivial ”classical” differentiation (by differentiation on the algebra $A$ we mean a linear map $A \to A$ satisfying the Leibniz rule). The reason for this is that there are no nontrivial differentiations on the algebra of locally constant functions, and this algebra is a dense subalgebra of the algebra of observables in most interesting cases ([4, 5]) (for example, locally constant functions form a dense subset in the algebra of continuous on $\mathbb{Z}_p$ functions). At the same time, ”quantum” differentiation, that is, differentiation with values in the algebra of bounded linear operators on a Hilbert space, is not trivial.

One of the possible applications of this paper is the construction of differential calculus on spaces of functions defined on totally disconnected spaces ([6, 7, 8]).
2 p-Adic numbers, functions, spaces

In this section, the necessary notations are introduced and the functional spaces in which we will work are defined. Everywhere further we consider the case of \( p \neq 2 \).

Any nonzero \( p \)-adic number can be represented in the following form (canonical decomposition):

\[
\mathbb{Q}_p \ni x = \sum_{k=-n}^{+\infty} x_k p^k, \quad n \in \mathbb{Z}_+, \quad x_k \in \{0, 1, \ldots, p-1\}.
\]

The finite part of the canonical decomposition, consisting of the sum of terms with negative powers of \( p \), is the fractional part of \( x \) and is denoted as \( \{x\}_p \), respectively, the remaining part of the decomposition (with non-negative powers of \( p \)) is the integer part of \( x \). Note that the fractional part is always a rational number, \( p \)-adic numbers with a trivial fractional part are \( p \)-adic integers \( \mathbb{Z}_p \).

Let’s define the order of a nonzero \( p \)-adic number \( x \) by the following rule:

\[
\text{ord}(x) = \min\{k: x_k \neq 0\}.
\]

Order has the following property: \( \text{ord}(x + y) \geq \min\{\text{ord}(x), \text{ord}(y)\} \). \( p \)-adic norm is determined by the formula: \( |x|_p = p^{-\text{ord}(x)} \). This norm is non-Archimedean, that is, a strong triangle inequality is valid: \( |x + y|_p \leq \max\{|x|_p, |y|_p\} \). By \( \chi \) we denote the additive character of the field \( \mathbb{Q}_p \) of the following form: \( \chi(x) = \exp(2\pi i \{x\}_p) \).

For a nonzero \( p \)-adic number \( x \in \mathbb{Q}_p^* \), the symbol \( x_{\text{ord}(x)} \in \{0, 1, \ldots, p-1\} \) denotes the coefficient at the first nonzero term of the canonical decomposition of \( x \). Let’s define the sign \( \text{sgn}(x) \) of a nonzero \( p \)-adic number as the Legendre symbol of \( x_{\text{ord}(x)} \): \( \text{sgn}(x) = \left( \frac{x_{\text{ord}(x)}}{p} \right) \). The function \( \text{sgn}(x) \) takes values in the set \( \{-1, 1\} \) and is the multiplicative character of the field \( \mathbb{Q}_p \): \( \mathbb{Q}_p^* \to \{-1, 1\} \), \( \text{sgn}(xy) = \text{sgn}(x) \text{sgn}(y) \). A disk centered at zero of radius \( p^{-n} \) is denoted by \( B_n \): \( B_n = \{x \in \mathbb{Z}_p : \text{ord}(x) \geq n \equiv |x|_p \leq p^{-n}\} \). In the following, we will consider functions defined on the ring \( \mathbb{Z}_p \) of \( p \)-adic integers (as a set, \( \mathbb{Z}_p \) is a unit disk, \( \mathbb{Z}_p = B_0 \)) and taking values in the field \( \mathbb{C} \) of complex numbers.
A function \( f \) is locally constant of order \( n \) if it is invariant with respect to shifts: \( f(x + u) = f(x), \) \( x \in \mathbb{Z}_p, u \in B_n. \) The set of such functions form a finite-dimensional vector space \( LC_n \) over complex numbers of dimension \( p^n. \) The basis vectors of this space are indicator functions of disks of radius \( p^{-n} \) in \( \mathbb{Z}_p. \) Since the topological space \( \mathbb{Z}_p \) is totally disconnected, locally constant functions are continuous. The following embeddings are valid: \( m \leq n, LC_m \subseteq LC_n. \) The set of all locally constant functions will be denoted by \( LC. \) This space has a natural structure of the direct limit of finite-dimensional subspaces: \( LC = \lim_{\rightarrow} LC_n. \) The Haar measure on \( \mathbb{Z}_p \) will be denoted by \( \mu. \) Lebesgue spaces of measurable functions with respect to this measure are defined in a standard way:

\[ L^q \equiv \{ f : \mathbb{Z}_p \rightarrow \mathbb{C}, \| f \|_q = \left( \int_{\mathbb{Z}_p} |f|^q d\mu \right)^{1/q} < \infty \}, \| f \|_\infty = \text{ess sup}_{\mathbb{Z}_p} |f| . \]

There is a chain of embeddings:

\[ LC \subset L^\infty \subset \cdots \subset L^q \subset \cdots \subset L^1, \]

\( LC \) is dense in \( L^q \) for all \( 1 \leq q \leq \infty. \)

## 3 Hilbert Transform

In this section, we define and consider the basic properties of the \( p \)-adic Hilbert transform (or operator). A large number of papers have been devoted to the study of the properties of this type of operators, here are just some, see [9, 10, 11].

The Hilbert transform \( S \) of the function \( f \) is defined by the following formula (the integral is understood in the sense of principal value):

\[
(Sf)(x) = \frac{1}{\Gamma} \text{p.v.} \int_{\mathbb{Z}_p} \frac{\sin(x - y)}{|x - y|_p} f(y) dy, \quad \Gamma = \begin{cases} \sqrt{p}, & p = 1( \mod 4) \\ i \sqrt{p}, & p = 3( \mod 4) \end{cases}
\]

We use the notation \( S \) for the Hilbert operator, since in the future it plays the role of a symmetry operator. The Hilbert operator is bounded from \( L^q \) to \( L^q \) for all \( 1 < q < \infty. \) For \( f \in L^1, \) the estimate is valid:

\[
\mu \{ x \in \mathbb{Z}_p : |(Sf)(x)| > \lambda \} \leq C \frac{\| f \|_1}{\lambda}.
\]
4 Fourier transform and Hilbert transform

The following section presents some well-known facts from harmonic analysis over the field of \( p \)-adic numbers [10].

Any character of the group \( \mathbb{Z}_p \) of \( p \)-adic integers has the form \( \chi_\alpha(x) = \chi(\alpha x), \ x \in \mathbb{Z}_p, \ \alpha \in \mathbb{Q}_p \). Thus, the Pontryagin dual of the group \( \mathbb{Z}_p \) is \( \mathbb{Q}_p / \mathbb{Z}_p \):

\[
\hat{\mathbb{Z}}_p = \{ \chi_\alpha(x) = \chi(\alpha x), \ \alpha \in \mathbb{Q}_p / \mathbb{Z}_p \}.
\]

In the last formula in the expression \( \chi(\alpha x) \) an arbitrary representative from the corresponding adjacency class is taken as \( \alpha \). Since the function \( \chi(x) \) is identically equal to 1 on \( \mathbb{Z}_p \), then the formula is correct. The group \( \hat{\mathbb{Z}}_p \) is also known as the Prufer group \( \mathbb{Z}(p^\infty) \), it is a quasi-cyclic group and can be represented as a subgroup in \( \mathbb{T} \) consisting of \( p^n \)-th roots of unity \( n \in \mathbb{N} \).

The set of functions \( \{ \chi_\alpha(x), \ \alpha \in \hat{\mathbb{Z}}_p \} \) forms an orthonormal basis in \( L^2 \). The Fourier transform of the function \( \phi \in L^2 \) is defined in the standard way:

\[
\phi \in L^2, \ F[\phi](\alpha) = \langle \phi, \chi_\alpha \rangle = \hat{\phi}_\alpha = \int_{\mathbb{Z}_p} \phi(x)\chi(-\alpha x)dx.
\]

Note that the function \( \chi_\alpha \) is a locally constant of order \( n \), where \( n \) is determined by the norm \( \alpha \): \( \chi_\alpha \in LC_n, \ |\alpha|_p = p^n \). The norm \( |\cdot|_p \) can be correctly defined on \( \mathbb{Q}_p / \mathbb{Z}_p \), assuming it is equal to zero on a zero adjacency class and equal to the norm of any representative from a non-zero adjacency class (due to the non-archimedean norm, such a definition does not depend on the choice of a representative). Moreover, the family of functions \( \{ \chi_\alpha(x), \ \alpha \in \hat{\mathbb{Z}}_p, \ |\alpha|_p \leq p^n \} \) forms an orthonormal basis in the space \( LC_n \). It also follows that the function \( \phi \) is locally constant if and only if it is represented as a finite Fourier series: \( \phi \in LC \iff \hat{\phi}_\alpha = 0 \) for almost all \( \alpha \in \hat{\mathbb{Z}}_p \).

In this sense, locally constant functions are analogs of trigonometric polynomials. However, there is an important difference - the ratio of two locally constant functions (provided that the function in the denominator does not take zero values) is again a locally constant function.

Hilbert transform has the following properties..

- The Hilbert transform maps constant functions into a function identically equal to zero: \( (S\mathbb{I}_{\mathbb{P}_p})(x) = 0. \)
The characters $\chi_{\alpha}, \alpha \in \hat{Z}_p$ are eigenfunctions of the Hilbert operator with eigenvalues $\text{sgn}(\alpha)$: $(S\chi_{\alpha})(x) = \text{sgn}(\alpha)\chi_{\alpha}(x), \alpha \in \hat{Z}_p, x \in \mathbb{Z}_p$.

It should be noted here that the function $\text{sgn}$ can be correctly defined on the group $\hat{Z}_p$ as follows: $\text{sgn}(\alpha)$ is set equal to zero on the unit element in $\hat{Z}_p$ and equal to the value on an arbitrary adjacency class representative otherwise. Obviously, this value does not depend on the choice of a representative.

As can be easily seen from the previous property of the Hilbert operator, this operator is self-adjoint, and its square is equal to the identity operator. That is, the Hilbert operator is a symmetry operator: $S^* = S, S^2 = 1$.

The space $L^2_0$ (the space of functions from $L^2$ orthogonal to constants) can be decomposed into a direct orthogonal sum of subspaces $W^+$ and $W^-$: $L^2 = W^+ \oplus W^-$, the space $W^+$ is linear span of the basis vectors $\chi_{\alpha}$, for which the condition $\text{sgn}(\alpha) = 1$ is met, the space $W^-$ is linear span of the basis vectors $\chi_{\alpha}$, for which the condition $\text{sgn}(\alpha) = -1$ is met. By $P^+$ and $P^-$ we denote orthogonal projectors into subspaces $W^+$ and $W^-$, respectively. Then the representation is valid for the Hilbert operator $S = P^+ - P^-$. 

5 Differentiation operator

In this section, we will define the quantum differentiation operator in the $p$-adic case (hereinafter - the differentiation operator), and give its simplest properties.

Definition 1 Let $f \in L^\infty$ and $M_f$ be the multiplication operator in $L^2$, $(M_f \phi)(x) = f(x)\phi(x)$. The operator in $L^2$ of the following form

$$df = [M_f, S]$$

will be called the value of the differentiation operator of the function $f$.

The square brackets in the last formula denote the commutator. Thus, the differentiation operator is defined on the algebra $L^\infty$ of functions on $\mathbb{Z}_p$ and takes a value in the algebra $\mathcal{B} = \mathcal{B}(L^2)$ of bounded linear operators on the space $L^2$. It is obvious that the differentiation operator satisfies the Leibniz
rule: \( d(fg) = (df)g + f(dg) \). We will also call the value of the operator \( d \) on the function \( f \) the derivative of the function \( f \).

It immediately follows from the definition that the derivative of a constant function is a trivial operator on \( \mathcal{B} \) (maps any function to the identically zero function). Let’s give a less trivial example.

**Example 1** Let \( f(x) = \chi_a(x) \). Let us find the derivative of this function. Since \( df \) is an operator in \( L^2 \), we calculate the value of this operator on the basis vector \( \chi_\alpha \). The answer is as follows:

\[
(d\chi_a)\chi_\alpha = (\text{sgn}(\alpha) - \text{sgn}(\alpha + a))\chi_{\alpha + a}.
\]

The following equations are valid:

\[
(d\chi_a)\chi_\alpha = [M_{\chi_a}, S]\chi_\alpha = \chi_a S\chi_\alpha - S(\chi_a\chi_\alpha) = \chi_a \text{sgn}(\alpha)\chi_\alpha - \text{sgn}(\alpha + a)\chi_{\alpha + a} = (\text{sgn}(\alpha) - \text{sgn}(\alpha + a))\chi_{\alpha + a}.
\]

The following theorem is valid.

**Theorem 1** Let \( f \in L^\infty \). The derivative \( df \) is an operator of finite rank if and only if the function \( f \) is locally constant.

Let \( f \in \mathcal{L}C \). Then, as noted above, the function \( f \) is represented as a finite linear combination of characters \( \chi_a \), that is, \( f(x) = \sum_{a \in \Omega} \hat{f}_a \chi_a(x) \), while \( a \) belongs to a finite subset of \( \Omega \) in \( \hat{\mathbb{Z}}_p \). Therefore, the inequality \( \text{rank}(df) \leq \sum_{a \in \Omega} \text{rank}(d\chi_a) \) is valid. Consider the operator \( T = d\chi_a(d\chi_a)^* \). The rank of the operator \( T \) coincides with the rank of the operator \( d\chi_a \). As follows from the Example 1, the operator \( T \) acts on the basis vectors \( \chi_\alpha \) according to the following formula:

\[
T\chi_\alpha = (\text{sgn}(\alpha + a) - \text{sgn}(\alpha))^2 \chi_\alpha.
\]

Thus, the characters \( \chi_\alpha \) are the eigenvalues of the operator \( T \), the corresponding eigenvalues are

\[
\lambda_\alpha = (\text{sgn}(\alpha + a) - \text{sgn}(\alpha))^2.
\]

Let \( a \neq 0 \). Then it is easy to see that \( \lambda_0 = \lambda_{-a} = 1 \). In addition, as follows from the definition of the function \( \text{sgn} \), \( \text{sgn}(\alpha + a) = \text{sgn}(\alpha) \) for all \( \alpha \) such that the inequality \( |\alpha|_p > |a|_p \) is satisfied. Thus, the image of the operator
$T$ is a subspace of the linear span of the basis vectors $\chi_\alpha$, for which the condition $|\alpha|_p \leq |a|_p$ is satisfied. The dimension of this subspace does not exceed $|a|_p$. Hence, $\text{rank}(d\chi_a) \leq |a|_p$. Hence the sufficiency of the condition of the theorem is proved.

It is possible to obtain an exact formula for the rank of the operator $d\chi_a$. Namely:

$$\text{rank}(d\chi_a) = \frac{|a|_p + 3}{2}. \quad (2)$$

Indeed, in a finite field of order $p^n$, the number of nonzero elements that are squares coincides with the number of elements that are not squares and is equal to $1/2(p^n - 1)$. Therefore, for $\alpha \neq 0$ and $\alpha \neq -a$, we have $1/2(|a|_p - 1)$ of nonzero eigenvalues of $\lambda_\alpha$, and we need to add two more, namely, $\lambda_0$ and $\lambda_{-a}$. If the function $f \in L^\infty$ is not locally constant, then its Fourier series expansion contains nonzero terms $\hat{f}_a\chi_a$ with an index $a$ of arbitrarily large norm. Taking into account the formula (2), the operator $df$ cannot have a finite rank in this case.

**Corollary 1** If the function $f$ is continuous, $f \in C(\mathbb{Z}_p)$, then its derivative $df$ is a compact operator.

Indeed, the mapping $f \mapsto M_f$ is continuous in a uniform topology on $C(\mathbb{Z}_p)$ and norm topologies on $\mathcal{B}(L^2)$. Mapping $A \mapsto [A, S]$, $A \in \mathcal{B}(L^2)$ is a differentiation on the algebra of bounded operators $\mathcal{B}(L^2)$ and is therefore continuous in the topology of the norm $\| \cdot \|_{\mathcal{B}(L^2)}$. Thus, the differentiation operator $f \mapsto df$ is continuous in a uniform topology on $C(\mathbb{Z}_p)$ and norm topologies on $\mathcal{B}(L^2)$. Since locally constant functions are dense in $C(\mathbb{Z}_p)$, the image of the algebra of continuous functions lies in the closure of the space of operators of finite rank in the topology of the norm in $\mathcal{B}(L^2)$, that is, in the algebra $\mathcal{K}(L^2)$ of compact operators on $L^2$. Corollary 1 gives a sufficient condition for the compactness of the derivative, however, this condition is not necessary.

### 6 Hilbert-Schmidt operators

Let’s define the $p$-adic Sobolev space $H^{1/2}$. This space is an analogue of the Sobolev space of semi-differentiable functions. To do this, on the space $LC$ of locally constant functions, we define the $\| \cdot \|_{1/2}$ semi-norm as follows. Let $f \in LC$ be represented as its (finite) Fourier series: $f(x) = \sum_{a \in \mathbb{Z}_p} \hat{f}_a\chi_a(x)$. 
Then, by definition,
\[ \|f\|_{1/2} = \left( \sum_{a \in \hat{Z}_p} |a|_p |\hat{f}_a|^2 \right)^{1/2}. \] (3)

The kernel of this semi-norm is a set of constant functions. The space \( H^{1/2} \) is defined as the closure of the space \( LC \) with respect to the seminorm (3).

**Lemma 1** Let \( f(x) = \sum_{a \in \hat{Z}_p} \hat{f}(a) \chi_a(x) \). The following formula is valid:
\[ \text{Tr} (df)^* (df) = 2 \sum_{a \in \hat{Z}_p} |a|_p |\hat{f}_a|^2. \]

Let’s use the formula (3) to calculate the diagonal matrix element of the operator \((df)^*(df)\):

\[
\langle (df)^* (df) \chi_\alpha, \chi_\alpha \rangle = \\
= \left\langle \sum_{a,b \in \hat{Z}_p} \hat{f}_a \hat{f}_b (\text{sgn}(\alpha) - \text{sgn}(a + \alpha)) (\text{sgn}(\alpha) - \text{sgn}(b + \alpha)) \chi_{\alpha+a-b}, \chi_\alpha \right\rangle = \\
= \sum_{a \in \hat{Z}_p} |\hat{f}_a|^2 (\text{sgn}(\alpha) - \text{sgn}(a + \alpha))^2. \quad (4)
\]

As noted above, \( \lambda_\alpha = (\text{sgn}(\alpha) - \text{sgn}(a + \alpha))^2 \) takes exactly two values equal to one (for \( \alpha = 0 \) and \( \alpha = -a \)) and the value 4 exactly for \( 1/2(|a|_p - 1) \) of different values of \( \alpha \), for the rest \( \alpha \) the eigenvalues of \( \lambda_\alpha \) are equal to zero. Therefore, taking into account the relations (4), the equalities are valid:

\[ \text{Tr} (df)^* (df) = \sum_{a \in \hat{Z}_p} \sum_{a \in \hat{Z}_p} |\hat{f}_a|^2 (\text{sgn}(\alpha) - \text{sgn}(a + \alpha))^2 = \\
= \sum_{a \in \hat{Z}_p} |\hat{f}_a|^2 \left( 2 + 4 |a|_p - 1 \right) = 2 \sum_{a \in \hat{Z}_p} |a|_p |\hat{f}_a|^2. \]

Thus, we have proved the following theorem.

**Theorem 2** The derivative \( df \) is a Hilbert-Schmidt operator, \( df \in \mathcal{S}^2 \), if and only if the function \( f \) belongs to the space \( H^{1/2} \).
7 Spaces $\text{BMO}(\mathbb{Z}_p)$ and $\text{VMO}(\mathbb{Z}_p)$

Initially, we defined the differentiation operator for the functions $f \in L^\infty$. In fact, this operator can naturally be extended to a wider space. We will give definitions of the corresponding spaces. Let the function $f$ be integrable on $\mathbb{Z}_p$, $f \in \mathcal{L}^1$ and $B$ denote an arbitrary disk in $\mathbb{Z}_p$. Using $f_B$, we denote the average value of the function $f$ on the disk $B$:

$$f_B = \frac{1}{\mu(B)} \int_B f(x) dx.$$

Let’s calculate the average deviation of the function $f$ from its average value on the disk $B$, and consider the maximum of such average deviations for all disks of radius not exceeding $p^{-n}$, $n \in \mathbb{Z}_+$:

$$M_n = \max_{\{B: \mu(B) \leq p^{-n}\}} \frac{1}{\mu(B)} \int_B |f(x) - f_B| dx.$$

We will say that the function $f$ belongs to the space $\text{BMO} = \text{BMO}(\mathbb{Z}_p)$ (bound mean oscillation), if the sequence $M_n$ is bounded, $\|f\|_{\text{BMO}} = \sup_n M_n$ sets a semi-norm on the BMO space, the kernel of this semi-norm consists of constant functions. A typical example of a function from the BMO space is the function $\log|x|_p$. A subspace of the BMO space consisting of functions for which the condition is met $\lim_{n \to \infty} M_n = 0$ denote $\text{VMO} = \text{VMO}(\mathbb{Z}_p)$ (vanishing mean oscillation). The BMO space contains the space $L^\infty$ as its proper subspace and is the natural space of the definition of the differentiation operator.

**Theorem 3** The derivative $df$ of the function $f$ is a bounded operator on $L^2$, $df \in \mathcal{B}(L^2)$, if and only if $f$ belongs to the space BMO. The derivative $df$ is a compact operator, $df \in \mathcal{K}(L^2)$, if and only if the function $f$ belongs to the VMO space.

These statements are completely analogous to the corresponding statement for quantum differentiation in the complex case and are proved in a similar way. Therefore, we will present only a sketch of the proof. It is known (\cite{10, 13}) that the Hilbert operator is a continuous operator from BMO to BMO and a continuous operator from $L^\infty$ to BMO. In addition, the following characterization of the BMO space is valid in terms of the action of the Hilbert operator on the space $L^\infty$. Namely, any function $f \in \text{BMO}$ is
represented as $f = g + Sh$, where the functions $g$ and $h$ belong to the space $L^\infty$, while the inequalities $\|f\|_{BMO} \geq C\|g\|_{\infty}$, $\|f\|_{BMO} \geq C\|h\|_{\infty}$ are valid for some constant $C$. This result was proved for functions defined on $\mathbb{R}^N$ ([14]), for functions on $\mathbb{Z}_p$ the proof is similar. Using this representation for a function from the BMO space, to continue the differentiation operator from $L^\infty$ to the BMO space, it is enough to do this on $SL^\infty$, and this follows from the continuity of the Hilbert operator from $L^\infty$ to $BMO$. Similarly, the second part of the theorem can be proved. Instead of the result of Fefferman and Stein ([14]), we will use the following result of Sarason ([15]), more precisely, the $p$-adic analogue of this result. A function $f \in BMO$ belongs to the VMO space if and only if it can be represented as $f = g + Sh$, where the functions $g$ and $h$ are continuous on $\mathbb{Z}_p$. Taking into account the last statement, the continuity of the Hilbert operator and Corollary 1 of the Theorem [1] we obtain the second statement of the theorem.

8 Besov spaces $B^s_{q,r}$

A considerable amount of work has been devoted to the Besov spaces of functions on the field $\mathbb{Q}_p$ of $p$-adic numbers (and more generally to the spaces of functions on profinite groups and martingales), for example [16, 17, 18].

Let $q, r, s$ be real numbers satisfying the inequalities $s > 0$, $1 \leq q, r \leq \infty$. The subspace of functions from $L^1$ for which the seminorm $\| \cdot \|_{B^s_{q,r}}$ defined below is finite, we will call the Besov space $B^s_{q,r}$:

$$B^s_{q,r} = \left\{ f \in L^1 : \|f\|_{B^s_{q,r}} = \left( \int_{\mathbb{Z}_p} \left( \frac{\|f(x - y) - f(x)\|_q}{|y|_p^s} \right)^r \frac{dy}{|y|_p} \right)^{1/r} < \infty \right\}.$$  

An equivalent definition of the Besov space is useful ([16]). We introduce the notation $\Delta_n(x)$ for the normalized indicator function of the disk $B_n$: $\Delta_n(x) = \frac{1}{\mu(B_n)}1_{B_n}(x)$. The mapping $L^1 \ni f \mapsto f \ast \Delta_n \in LC_n$ defines the projection of the space $L^1$ into the space $LC_n$ (denotes convolution).

Proposition 1 Let the conditions be met: $f \in L^1$, $s > 0$, $1 \leq q, r \leq \infty$. 

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The seminorm $\|f\|_{B_{q,r}^q}$ is equivalent to the following seminorm:

$$\|f\|_{B_{q,r}^q} \simeq \left( \sum_{n=0}^{\infty} (p^n s\|f - f \ast \Delta_n\|_q)^r \right)^{1/r}.$$  

Below, we will need the case $r = q, s = 1/q$. An obvious consequence follows from the last statement.

**Corollary 2** The function $f$ belongs to the Besov space $B_{q,q}^{1/q}$, $f \in B_{q,q}^{1/q}$, if and only if the condition is satisfied $p^{n/q}\|f - f \ast \Delta_n\|_{BMO} \in \ell^q$.

The connection of Besov and BMO spaces on Vilenkin groups was studied in [16]. In particular, a stronger statement follows from the results of this work. Namely:

**Proposition 2** The function $f$ belongs to the Besov space $B_{q,q}^{1/q}$, $f \in B_{q,q}^{1/q}$, if and only if the following condition is satisfied $p^{n/q}\|f - f \ast \Delta_n\|_{BMO} \in \ell^q$.

The following theorem is valid.

**Theorem 4** The derivative $df$ belongs to the Schatten-von Neumann ideal $\mathcal{S}^q$, $df \in \mathcal{S}^q$, if and only if the function $f$ belongs to the Besov space $B_{q,q}^{1/q}$, $f \in B_{q,q}^{1/q}$.

Recall the definition of the ideal $\mathcal{S}^q$. Let $K \in \mathcal{K}(L^2)$ is a compact operator. The singular number $s_n(K)$ of this operator is defined as the distance in the operator norm to the space of operators of rank no higher than $n$:

$$s_n(K) = \inf_{R} \left\{ \|K - R\| : \text{rank } R \leq n \right\}.$$  

By definition, $K \in \mathcal{S}^q$ if the condition $\{s_n(K)\} \in \ell^q$ is satisfied. The last condition is equivalent to the following condition: $\{p^{n/q}s_{p^n}(K)\} \in \ell^q$. Since $f \ast \Delta_n \in LC_n$, the derivative $d(f \ast \Delta_n)$ is an operator of rank no higher than $p^n$ (Theorem [1]). Therefore, the inequality $s_{p^n}(df) \leq \|d(f - f \ast \Delta_n)\|$ is valid. In fact, a two-way estimate of $s_{p^n}(df) \simeq \|d(f - f \ast \Delta_n)\|$ is valid, since the function $f \ast \Delta_n$ gives the best approximation in the topology of the VMO space of the function $f$ by a locally constant function from $LC_n$, and the differentiation operator is a continuous mapping from VMO to the algebra of compact operators. Taking into account Theorem [3], the chain of relations is valid:

$$s_{p^n}(df) \simeq \|d(f - f \ast \Delta_n)\| \simeq \|f - f \ast \Delta_n\|_{BMO}.$$  

Taking into account the Proposition [2], the statement of the theorem follows from the last chain of inequalities.
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