EXTREMALITY OF TRANSLATION-INVARIANT GIBBS MEASURES FOR THE POTT-SOS MODEL ON THE CAYLEY TREE

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Abstract. In this paper, we consider the Potts-SOS model where the spin takes values in the set \{0, 1, 2\} on the Cayley tree of order two. We describe all the translation-invariant splitting Gibbs measures for this model in some conditions. Moreover, we investigate whether these Gibbs measures are extremal or non-extremal in the set of all Gibbs measures.

Key words. Cayley tree, configuration, Potts-SOS model, translation-invariant splitting Gibbs measure, extreme measure, tree-indexed Markov chain, Kesten-Stigum condition, extremality.

1. Introduction

One of the central problems in the theory of Gibbs measures (GMs) is to describe infinite-volume (or limiting) GMs corresponding to a given Hamiltonian. The existence of such measures for a wide class of Hamiltonians was established in the ground-breaking work of Dobrushin (see, e.g., [2]). However, a complete analysis of the set of limiting GMs for a specific Hamiltonian is often a difficult problem.

In this paper, we consider the Potts-SOS model, with spin values 0, 1, 2 on the Cayley tree (CT). Models on a CT were discussed in Refs. [3] and [5]-[8]. A classical example of such a model is the Ising model, with two values of spin \(-1\) and 1. It was considered in Refs. [1], [3], [8], [17], [18] and became a focus of active research in the first half of the 90s and afterwards; see Refs. [1], [9]-[15].

In [19] all translation-invariant splitting Gibbs measures (TISGMs) for the Potts model on the CT are described. In [20], [21] periodic Gibbs measures, in [22]-[24] weakly periodic Gibbs measures for the Potts model are studied.

In [25], [26] translation-invariant and periodic Gibbs measures for the SOS model on the CT are studied.

Model considered in this paper (Potts-SOS model) is generalization of the Potts and SOS (solid-on-solid) models. In [16] some translation-invariant Gibbs measures for the Potts-SOS model on the CT are studied. Periodic Gibbs measures are studied for the Potts-SOS model on the CT in [25]. In this paper we will study all the TISGMs for this model under some conditions. Next we investigate whether these Gibbs measures are extremal or non-extremal in the set of all Gibbs measures.
2. Main definitions and known facts

The Cayley tree $\Gamma^k$ (See [1]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k + 1$ edges issue. Let $\Gamma^k = (V, L, i)$, where $V$ is the set of vertices of $\Gamma^k$, $L$ is the set of edges of $\Gamma^k$ and $i$ is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then $x$ and $y$ are called nearest neighboring vertices, and we write $l = < x, y >$.

The distance $d(x, y)$, $x, y \in V$ on the Cayley tree is defined by the formula

$$d(x, y) = \min \{d | \exists x = x_0, x_1, ..., x_{d-1}, x_d = y \in V \text{ such that } < x_0, x_1 >, ..., < x_{d-1}, x_d >\}.$$ 

For the fixed $x^0 \in V$ we set $W_n = \{x \in V \mid d(x, x^0) = n\}$, $V_n = \{x \in V \mid d(x, x^0) \leq n\}$, $L_n = \{l = < x, y > \in L \mid x, y \in V_n\}$. (1)

Denote $|x| = d(x, x^0)$, $x \in V$.

A collection of the pairs $< x, x_1 >, ..., < x_{d-1}, y >$ is called a path from $x$ to $y$ and we write $\pi(x, y)$. We write $x < y$ if the path from $x^0$ to $y$ goes through $x$.

It is known (see [1]) that there exists a one-to-one correspondence between the set $V$ of vertices of the Cayley tree of order $k \geq 1$ and the group $G_k$ of the free products of $k + 1$ cyclic groups $\{e, a_i\}$, $i = 1, ..., k + 1$ of the second order (i.e. $a_i^2 = e$, $a_i^{-1} = a_i$) with generators $a_1, a_2, ..., a_{k+1}$, see Figure 1.

**Figure 1.** The Cayley tree $\tau^2$ and elements of the group $G_2$ representation of vertices
Denote the set of "direct successors" of \( x \in G_k \) by \( S(x) \). Let \( S_1(x) \) be the set of all nearest neighboring vertices of \( x \in G_k \), i.e. \( S_1(x) = \{ y \in G_k : < x, y > \} \) and \( \{ x \downarrow \} = S_1(x) \setminus S(x) \).

3. THE MODEL AND A SYSTEM VECTOR-VALUED FUNCTIONAL EQUATIONS

Here we shall give main definitions and facts about the model. Consider model where the spin takes values in the set \( \Phi = \{ 0, 1, 2, ..., m \} \), \( m \geq 1 \). For \( A \subseteq V \) a spin configuration \( \sigma_A \) on \( A \) is defined as a function \( x \in A \rightarrow \sigma_A(x) \in \Phi \); the set of all configurations coincides with \( \Omega_A = \Phi^A \). Denote \( \Omega = \Omega_V \) and \( \sigma = \sigma_V \).

A configuration that is invariant with respect to all shifts is called translational-invariant.

The Hamiltonian of the Potts-SOS model with nearest-neighbor interaction has the form

\[
H(\sigma) = -J \sum_{<x,y> \in L} |\sigma(x) - \sigma(y)| - J_p \sum_{<x,y> \in L} \delta_{\sigma(x)\sigma(y)},
\]

where \( J, J_p \in \mathbb{R} \) are nonzero coupling constants.

It is known [16] that any SGM of the model (2) corresponds to a solution of the following equation:

\[
h^*_x = \sum_{y \in S(x)} F(h^*_y, m, \theta, r)
\]

where \( x \in V \setminus \{ x^0 \} \),

\[
\theta = \exp(J \beta), \quad r = \exp(J_p \beta)
\]

and also \( \beta = 1/T \) is the inverse temperature. Here \( h^*_x \) represents the vector \((h_{0,x} - h_{m,x}, h_{1,x} - h_{m,x}, ..., h_{m-1,x} - h_{m,x})\) and the vector function \( F(., m, \theta, r) : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is defined as follows

\[
F(h, m, \theta, r) = (F_0(h, m, \theta, r), F_1(h, m, \theta, r), ..., F_{m-1}(h, m, \theta, r)),
\]

where

\[
F_i(h, m, \theta, r) = \ln \left(\frac{\sum_{j=0}^{m-1} \theta^{i-j} \rho \delta_{h_j} e^{h_j} + \theta^{m-i} \rho \delta_m e^{h_m}}{\sum_{j=0}^{m-1} \theta^{m-j} \rho \delta_m e^{h_j} + r}\right),
\]

\( h = (h_0, h_1, ..., h_{m-1}), i = 0, 1, 2, ..., m - 1 \).

Namely, for any collection of functions satisfying the functional equation (3) there exists a unique splitting Gibbs measure, the correspondence being one-to-one.

4. TRANSLATION-INvariant Gibbs measures

Definition 1. For a SGM \( \mu \), if \( h_{j,x} \) is independent from \( \{ x : h_{j,x} \equiv h_j, x \in V, j \in \Phi \} \), \( \mu \) is called translation-invariant (TI).
Let $m = 2$, that is $\Phi = \{0, 1, 2\}$. In this case, for the TISGMs has the form 
$$h = kF(h, \theta, r),$$
where $h = (h_0, h_1)$. Introducing the notation $l_0 = e^{h_0}, l_1 = e^{h_1}$, we obtain the following system of equations 

$$
\begin{cases}
l_0 = \left(\frac{r l_0 + \theta l_1}{\theta l_0 + \theta l_1 + r}\right)^k, \\
l_1 = \left(\frac{r l_0 + \theta l_1}{\theta l_0 + \theta l_1 + r}\right)^k.
\end{cases}
$$

Let $k = 2$. Denote $\sqrt{l_0} = x, \sqrt{l_1} = y$. Then from (6) we get 

$$
\begin{cases}
x = \frac{r x^2 + \theta y^2 + \theta^2}{\theta x^2 + \theta y^2 + r}, \\
y = \frac{\theta x^2 + r y^2 + \theta}{\theta x^2 + \theta y^2 + r}.
\end{cases}
$$

After simplifying above the system of equations, we have 

$$
\begin{cases}
\theta^2 x^3 - rx^2 + (\theta y^2 + r)x - \theta y^2 - \theta^2 = 0, \\
\theta y^3 - ry^2 + (\theta^2 x^2 + r)y - \theta x^2 - \theta = 0.
\end{cases}
$$

The system of equations can be rewritten as 

$$
\begin{cases}
(x - 1)(\theta^2 x^2 + \theta^2 x + \theta^2 - rx + \theta y^2) = 0, \\
\theta y^3 - ry^2 + (\theta^2 x^2 + r)y - \theta x^2 - \theta = 0.
\end{cases}
$$

Obviously, the solutions of are the solutions of the following system of equations 

$$
\begin{cases}
x - 1 = 0, \\
\theta y^3 - ry^2 + (\theta^2 x^2 + r)y - \theta x^2 - \theta = 0,
\end{cases}
$$

or the solutions of the following system of equations 

$$
\begin{cases}
\theta^2 x^2 + \theta^2 x + \theta^2 - rx + \theta y^2 = 0, \\
\theta y^3 - ry^2 + (\theta^2 x^2 + r)y - \theta x^2 - \theta = 0.
\end{cases}
$$

Let us consider. Substituting $x = 1$ into the second equation of we get 

$$
\theta y^3 - ry^2 + (\theta^2 + r)y - 2\theta = 0.
$$

For 

$$y = z + \frac{r}{3\theta},$$
we reduce (12) to the equation
\[ z^3 + \left(\frac{r}{\theta} + \theta - \frac{r^2}{3\theta^2}\right) z + \left(\frac{r}{3} + \frac{r^2}{3\theta^2} - \frac{2r^3}{27\theta^3} - 2\right) = 0. \] (14)

Denote
\[ p = \frac{r}{\theta} + \theta - \frac{r^2}{3\theta^2}, \quad q = \frac{r}{3} + \frac{r^2}{3\theta^2} - \frac{2r^3}{27\theta^3} - 2. \] (15)

After solving the equation \( p = 0 \) in terms of \( r \), we have the solutions \( r_{1,2} = \frac{3+\sqrt{9+12\theta}}{2} \). Since \( r > 0, \theta > 0 \), we get \( r_1 = \frac{3+\sqrt{9+12\theta}}{2} \). Putting \( r_1 \) into \( q \) in (15) and solving the equation \( q = 0 \) in terms of \( \theta \), we have the solution \( \theta_1 = 3\sqrt{2}(\sqrt{2} - 1) \).

Substituting \( r_1, \theta_1 \) into the equation (14) we get the equation \( z^3 = 0 \). It follows that the equation (12) has one positive solution.

From (15), we obtain
\[ Q(r, \theta) = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = \frac{1}{27} \left(\frac{-1}{3\theta^2} + \frac{r}{\theta} + \theta\right)^3 + \frac{1}{4} \left(\frac{-2}{27\theta^3} + \frac{1}{3\theta^2} + \frac{1}{3} \right)^2 = \]
\[ = -\frac{1}{108\theta^4} \left(r^4 + 2r^3 \theta^2 + r^2 \theta^4 - 12r^3 \theta - 12r^2 \theta^3 - 12\theta^5 r - 4\theta^7 + 36\theta^2 r^2 + 36\theta^4 r - 108\theta^4 \right). \] (16)

For \( \theta = \theta_1 = 3\sqrt{2}(\sqrt{2} - 1) \) we have
\[ Q(r, \theta_1) = \frac{116 + 73\sqrt{2} + 92\sqrt{2} + 34992}{27} \left(-r^2 + 36(1 - 2\sqrt{2} + \sqrt{4})r + 324(13 - 4\sqrt{2} - 5\sqrt{4})\right). \]

Using Cardano’s formula one can prove the following

**Lemma 1.** Let \( \theta = 3\sqrt{2}(\sqrt{2} - 1) \). There exists \( r_0 \) (\( \approx 4.221293186 \)) such that
- If \( r \in (0, r_0) \) then the equation (12) has one positive solution.
- If \( r = r_0 \) then the equation (12) has two positive solutions.
- If \( r \in (r_0, \infty) \) then the equation (12) has three positive solutions.

Now we consider (11). From (11) we get
\[ x = \frac{\theta y(\theta^2 - y + ry - r)}{-\theta^3 y + \theta^2 + \theta ry - r}. \] (17)

Substituting (17) into the first equation of (11), we obtain
\[ f(y, r, \theta) = \theta^2(\theta + 1)(r^2 - 2\theta r + \theta^3 - \theta^2 + \theta)y^4 - \theta(r - \theta^2)(r^2 + (\theta^2 + 1)r - 3\theta^2)y^3 + \]
\[ + ((\theta + 1)r + \theta^3)(r - \theta^2)^2 y^2 - (r + \theta^2)(r - \theta^2)^2 y + \theta(r - \theta^2)^2 = 0. \] (18)

The equation (18) can be rewritten as
\[ f(y, r, \theta) = (ay^2 + by + c)(dy^2 + ey + f), \]
where
Lemma 2. Let \( A = \theta^2(\theta + 1)(r^2 - 2\theta r + \theta^3 - \theta^2 + \theta), \)

\[
\begin{align*}
ac + bd &= -\theta(r - \theta^2)(r^2 + (\theta^2 + 1)r - 3\theta^2), \\
af + be + cd &= ((\theta + 1)r + \theta^3)(r - \theta^2)^2, \\
bf + ce &= -(r + \theta^2)(r - \theta^2)^2,
\end{align*}
\]

\( cf = \theta(r - \theta^2)^2. \)

Let \( D_1(r, \theta) = b^2 - 4ac \) and \( D_2(r, \theta) = c^2 - 4df. \)

We denote the following sets

- \( B_1 = \{(r, \theta) \in \mathbb{R}_+^2 : D_1(r, \theta) > 0, D_2(r, \theta) > 0\}, \)
- \( B_2 = \{(r, \theta) \in \mathbb{R}_+^2 : D_1(r, \theta) > 0, D_2(r, \theta) = 0 \lor D_1(r, \theta) = 0, D_2(r, \theta) > 0\}, \)
- \( B_3 = \{(r, \theta) \in \mathbb{R}_+^2 : D_1(r, \theta) = 0, D_2(r, \theta) = 0 \lor D_1(r, \theta) > 0, D_2(r, \theta) < 0 \lor \}
  
  \lor D_1(r, \theta) < 0, D_2(r, \theta) > 0\}, \)
- \( B_4 = \{(r, \theta) \in \mathbb{R}_+^2 : D_1(r, \theta) = 0, D_2(r, \theta) < 0 \lor D_1(r, \theta) < 0, D_2(r, \theta) = 0\}, \)
- \( B_5 = \{(r, \theta) \in \mathbb{R}_+^2 : D_1(r, \theta) < 0, D_2(r, \theta) < 0\}. \)

Thus we can prove the following

**Lemma 2.** Let \( \theta = 3\sqrt{2}(\sqrt{2} - 1), \) then the following assertions hold

- If \( r \in B_1(r) \) then the equation \( (15) \) has four solutions which are positive.
- If \( r \in B_2(r) \) then the equation \( (15) \) has three positive solutions.
- If \( r \in B_3(r) \) then the equation \( (15) \) has two positive solutions.
- If \( r \in B_4(r) \) then the equation \( (15) \) has one positive solution.
- If \( r \in B_5(r) \) then the equation \( (15) \) has no solution.

With respect to \( (15) \) and \( (14) \) we denote the following sets

- \( A_1 = \{(r, \theta) \in \mathbb{R}_+^2 : r \leq 3\theta^2, Q > 0\} \cup \{(r, \theta) \in \mathbb{R}_+^2 : r \leq 3\theta^2, p = 0, q = 0\}, \)
- \( A_2 = \{(r, \theta) \in \mathbb{R}_+^2 : r \leq 3\theta^2, Q = 0\} \cap \{(r, \theta) \in \mathbb{R}_+^2 : p \neq 0 \lor q \neq 0\}, \)
- \( A_3 = \{(r, \theta) \in \mathbb{R}_+^2 : r \leq 3\theta^2, Q < 0\}, \)
- \( A_4 = \{(r, \theta) \in \mathbb{R}_+^2 : r > 3\theta^2, Q > 0\}, \)
- \( A_5 = \{(r, \theta) \in \mathbb{R}_+^2 : r > 3\theta^2, Q = 0\} \cap \{(r, \theta) \in \mathbb{R}_+^2 : p \neq 0 \lor q \neq 0\}, \)
- \( A_6 = \{(r, \theta) \in \mathbb{R}_+^2 : r > 3\theta^2, Q < 0\}. \)

Let \( N \) be the number of TISGMs for the Potts-SOS model.
Theorem 1. Let $k = 2, m = 2$. The following statements hold for the $N$

\[
N = \begin{cases} 
1, & \text{if } (r, \theta) \in A_1, \\
2, & \text{if } (r, \theta) \in A_2 \cup (A_4 \cap B_4) \cup (A_5 \cap B_5), \\
3, & \text{if } (r, \theta) \in A_3 \cup (A_4 \cap B_3) \cup (A_5 \cap B_4), \\
4, & \text{if } (r, \theta) \in (A_4 \cap B_3) \cup (A_5 \cap B_4), \\
5, & \text{if } (r, \theta) \in (A_4 \cap B_2) \cup (A_5 \cap B_3) \cup (A_6 \cap B_4), \\
6, & \text{if } (r, \theta) \in (A_4 \cap B_1) \cup (A_5 \cap B_2) \cup (A_6 \cap B_3), \\
7, & \text{if } (r, \theta) \in (A_5 \cap B_1) \cup (A_6 \cap B_2), \\
8, & \text{if } (r, \theta) \in A_6 \cap B_1.
\end{cases}
\]  

(19)

Proof. We consider the first equation of (11). We write this in the following form

\[
\theta^2 x^2 + (\theta^2 - r)x + \theta^2 = -\theta y^2. \tag{20}
\]

RHS of (20) is negative, thus

\[
\theta^2 x^2 + (\theta^2 - r)x + \theta^2 < 0. \tag{21}
\]

For LHS of (21), we calculate its discriminant $D = (\theta^2 - r)^2 - 4\theta^4$. If the discriminant is positive, then the inequality (21) has real solutions. Therefore, we should solve

\[
(-r - \theta^2)(3\theta^2 - r) > 0.
\]

Since $-r - \theta^2 < 0$, it follows that $r > 3\theta^2$.

Inequality (21) has positive solution as soon as $\theta^2 - r < 0$ or $r > \theta^2$. If $r > 3\theta^2$, then $r > \theta^2$ also holds. If $r > 3\theta^2$, the solutions of the inequality (21) belong to

\[
\left(\frac{r - \theta^2 - \sqrt{D}}{2\theta^2}, \frac{r - \theta^2 + \sqrt{D}}{2\theta^2}\right).
\]

Moreover, (20) holds in this interval.

Consequently, if $r > 3\theta^2$ then the first equation of (11) has a positive real solution, if $r \leq 3\theta^2$ then the first equation of (11) cannot have a positive solution, i.e., any positive real pair $(x, y)$, which is solution of the first equation of (11), does not satisfy $r \leq 3\theta^2$. Then TISGMs corresponding roots of (11) do not exist under condition $r \leq 3\theta^2$.

According to the Descartes theorem, the number of positive roots of equation (12) is at least 1 and at most 3.

If $Q > 0$, then the equation (14) has one positive real root and two conjugate complex roots; If $Q = 0$, the all roots of the equation (14) are positive real and two of them are equal or if $p = q = 0$, then (14) has one positive real root (one real zero of multiplicity three); If $Q < 0$, then the equation (14) has three distinct positive real roots. Hence, we can say about the number of TISGMs corresponding positive roots of the equation (12).

From Lemma 1 and Lemma 2 we can see that

\[
\left\{(r, \theta) \in R^2 : \theta = 3\sqrt{2}(\sqrt{2} - 1), r \in (r_c, \infty) \cap B_1(r) \right\} \subset A_6 \cap B_1.
\]
Thus the set \( A_6 \cap B_1 \) is not empty, i.e., the number of TISGMs corresponding positive solutions of (8) for the Potts-SOS model is up to seven. □

**Remark 1.** Note that Theorem 1 (for \( k = m = 2 \)) generalizes results of [19] and [27].

If \( J = 0 \), then Potts-SOS model changes to Potts model. In this case Theorem 1 can be restated as follows

**Theorem 2.** Let \( k = 2, m = 2 \). The following statements hold for the number \( n \) of the TISGMs for the Potts model

\[
n = \begin{cases} 
1, & \text{if } r \in (0, 1 + 2\sqrt{2}), \\
4, & \text{if } r = 1 + 2\sqrt{2} \text{ or } r = 4, \\
7, & \text{if } r \in (1 + 2\sqrt{2}, 4) \cup (4, \infty). 
\end{cases}
\]  
(22)

(see [19] for more details).

If \( J_p = 0 \), then Hamiltonian (2) of Potts-SOS model changes to Hamiltonian of SOS model. In this case Theorem 1 can be restated as follows

**Theorem 3.** Let \( k = 2, m = 2 \). The following statements are appropriate for the number \( n \) of the TISGMs for the SOS model

\[
n = \begin{cases} 
1, & \text{if } \theta \in (\theta_2, \infty), \\
3, & \text{if } \theta = \theta_2, \\
5, & \text{if } \theta \in (\theta_1, \theta_2), \\
6, & \text{if } \theta = \theta_1, \\
7, & \text{if } \theta \in (0, \theta_1), 
\end{cases}
\]  
(23)

where \( \theta_1 \approx 0.1414 \) and \( \theta_2 \approx 0.2956 \).

(see [27] for more details).

Now we study the extremality of the TISGMs for the Potts-SOS model. In general, a complete analyses of extremality or non-extremality of the TISGMs is a difficult problem. Therefore, we assume \( r = \theta^2 \).

**Lemma 3.** Let \( r = \theta^2 \). There exists a unique \( \theta_c (\approx 7.729814) \) such that

- If \( \theta \in (0, \theta_c) \) then system (7) has one positive root.
- If \( \theta = \theta_c \) then system (7) has two positive roots.
- If \( \theta \in (\theta_c, \infty) \) then system (7) has three positive roots.

**Proof.** Substituting \( r = \theta^2 \) into (7) we have

\[
\begin{align*}
x &= 1, \\
y &= \frac{2 + \theta y^2}{2\theta + y^2}.
\end{align*}
\]  
(24)
Simplifying the second equation of (24), we obtain the cubic equation

\[ y^3 - \theta y^2 + 2\theta y - 2 = 0. \]  

(25)

We calculate its discriminant

\[ D = 4(\theta^4 - 10\theta^3 + 18\theta^2 - 27). \]  

(26)

Denote \( \theta_c \approx 7.729814 \). If \( D < 0 \) (\( \theta < \theta_c \)) the equation (25) has one real and two conjugate complex roots. If \( D = 0 \) (\( \theta = \theta_c \)) then all roots of equation (25) are real, which two of them are equal. If \( D > 0 \) (\( \theta > \theta_c \)) then the equation (25) has three distinct real roots (see Fig. 2). Obtained real roots are positive due to the Descartes theorem (see [6]).

\[ \square \]

Figure 2. The graphs of functions \( y_i = y_i(\theta), i = 1, 2, 3 \). Lower curve is \( y_1 \), middle curve is \( y_2 \), upper curve is \( y_3 \).

Using Lemma 3, we have the following
Theorem 4. Let $k = m = 2$. If $r = \theta^2$ then the following statements hold for the $N$

$$
N = \begin{cases} 
1, & \text{if } \theta \in (0, \theta_c), \\
2, & \text{if } \theta = \theta_c, \\
3, & \text{if } \theta \in (\theta_c, \infty),
\end{cases}
$$

(27)

where $\theta_c \approx 7.729814$.

Remark 2. Note that the Theorem 4 is a particular case of the Theorem 1.

We denote obtained TISGMs corresponding to $y_i$ in the Theorem 4 by $\mu_i, i = 1, 2, 3$, respectively.

5. Tree-indexed Markov Chains of TISGMs

A tree-indexed Markov chain is defined as follows. Suppose we are given with vertices set $V$, a probability measure $\nu$ and a transition matrix $P = (p_{i,j})_{i,j \in \Phi}$ on the single-site space which is here the finite set $\Phi = \{0, 1, ..., m\}$. We can obtain a tree-indexed Markov chain $X : V \rightarrow \Phi$ by choosing $X(x_0)$ according to $\nu$ and choosing $X(v)$, for each vertex $v \neq x_0$, using the transition probabilities given the value of its parent, independently of everything else. See Definition 12.2 in [5] for a detailed definition.

We note that a TISGM corresponding to a vector $v = (x, y) \in \mathbb{R}^2$ (which is solution to the system (7)) is a tree-indexed Markov chain with states $\{0, 1, 2\}$ and transition probabilities matrix:

$$
P = \begin{pmatrix} 
x^2 + \theta y^2 & \theta y x + \theta^2 y^2 & \theta^2 x^2 + \theta^2 y^2 + \theta^2 \\
x^2 + \theta y^2 & \theta y x + \theta^2 y^2 & \theta^2 x^2 + \theta^2 y^2 + \theta^2 \\
x^2 + \theta y^2 & \theta y x + \theta^2 y^2 & \theta^2 x^2 + \theta^2 y^2 + \theta^2 
\end{pmatrix}.
$$

(28)

Since $(x, y)$ is a solution to the system (7) this matrix can be written in the following form

$$
P = \frac{1}{Z} \begin{pmatrix} 
x & \frac{\theta y}{x} & \frac{\theta^2 x}{x} \\
\frac{\theta y}{y} & \frac{\theta y}{x} & \frac{\theta y^2}{y} \\
\frac{\theta^2 x^2}{x^2} & \frac{\theta y^2}{y^2} & \frac{\theta^2 x^2}{x^2} + \theta y^2 + r
\end{pmatrix},
$$

(29)

where $Z = \theta^2 x^2 + \theta y^2 + r$.

Simple calculations show that the matrix (29) has three eigenvalues: 1 and

$$
\lambda_1(x, y, \theta, r) = \frac{(x + y + 1)r - Z + \sqrt{D^*}}{2Z}, \quad \lambda_2(x, y, \theta, r) = \frac{(x + y + 1)r - Z - \sqrt{D^*}}{2Z},
$$

(30)

where $\lambda_1$ and $\lambda_2$ are solutions to

$$
Z^2 \lambda^2 + (Z - (1 + x + y)r)Z \lambda + (2\theta^4 - \theta^4 r - 2\theta^2 r + r^3)xy = 0
$$

(31)

and $D^* = ((1 + x + y)r - Z)^2 - 4xyZ^{-1}(2\theta^4 - \theta^4 r - 2\theta^2 r + r^3)$. 

5.1. Conditions of Non-Extremality. In this subsection we are going to find the regions of the parameter $\theta$ where the TISGMs $\mu_i, i = 1, 2, 3$ are not extreme in the set of all Gibbs measures (including the non-translation invariant ones).

It is known that a sufficient condition (Kesten-Stigum condition) for non-extremality of a Gibbs measure $\mu$ corresponding to the matrix $P$ on a Cayley tree of order $k \geq 1$ is that $k\lambda_{\text{max}}^2 > 1$, where $\lambda_{\text{max}}$ is the second largest (in absolute value) eigenvalue of $P$ [28]. We are going to use this condition for TISGMs $\mu_i, i = 1, 2, 3$ in Theorem 4. We have all solutions of the system (7) in condition $r = \theta^2$ (see Theorem 4) and the eigenvalues of the matrix $P$ in the explicit form.

Let us denote

$$\lambda_{\text{max},i}(\theta, r) = \max\{|\lambda_1(x_i, y_i, \theta, r)|, |\lambda_2(x_i, y_i, \theta, r)|\}, i = 1, 2, 3.$$  

Using a computer we have

$$\lambda_{\text{max},i}(\theta) = \begin{cases} |\lambda_2(1, y_1, \theta)|, & \text{if } i = 1, \theta < 1, \\ |\lambda_1(1, y_1, \theta)|, & \text{if } i = 1, \theta > 1, \\ |\lambda_1(1, y_i, \theta)|, & \text{if } i = 2, 3. \end{cases}$$

Denote

$$\eta_i(\theta) = 2\lambda_{\text{max},i}(\theta) - 1, i = 1, 2, 3.$$ 

Let $\theta < \theta_c$. Using the Cardano formula, we solve the equation (25). It has one real solution

$$y_1 = \frac{1}{3} \left( \theta + \sqrt[3]{\theta^3 - 9\theta^2 + 27 + 1.5\sqrt{-3D}} + \frac{\theta^2 - 6\theta}{\sqrt[3]{\theta^3 - 9\theta^2 + 27 + 1.5\sqrt{-3D}}} \right),$$  

(32)

where $D$ is defined in [26]. In this case, we are aiming to check the Kesten-Stigum condition of the non-extremality of the measure $\mu_1$. To determine the non-extremality interval of TISGM $\mu_1$, we should check the condition

$$2\lambda_{\text{max},1}^2 - 1 > 0.$$ 

Using a Maple program, one can see that the last inequality holds for $\theta \in (0, \theta_1)$ ($\theta_1 \approx 0.1666993311$), which implies that the TISGM $\mu_1$ is not-extreme in this interval (see Fig. 3).

To check that the TISGM $\mu_i, i = 2, 3$ are non-extreme, we should solve the following inequality: $\eta_i(\theta) > 0, i = 2, 3$. (see Fig. 4).

**Proposition 1.** Let $r = \theta^2$. Then the following statements hold

a) There exists $\theta_1(\approx 0.1666993311)$ such that the measure $\mu_1$ is non-extreme if $\theta \in (0, \theta_1)$;

b) There exists $\theta_2(\approx 9.706301628)$ such that the measure $\mu_2$ is non-extreme if $\theta \in (\theta_2, \infty)$. 

Figure 3. The graphs of functions $\eta_1(\theta)$ for $\theta \in (0, 1)$ (left) and for $\theta \in (1, \infty)$ (right).

Figure 4. The graphs of functions $\eta_2(\theta)$ (left) and $\eta_3(\theta)$ (right).
5.2. Conditions for Extremality. In [27], [29] the key ingredients are two quantities, \( \kappa \) and \( \gamma \), which bound the rates of percolation of disagreement down and up the tree, respectively.

For two measures \( \mu_1 \) and \( \mu_2 \) on \( \Omega \), \( \|\mu_1 - \mu_2\|_x \) denotes the variation distance between the projections of \( \mu_1 \) and \( \mu_2 \) onto the spin at \( x \), i.e.,

\[
\|\mu_1 - \mu_2\|_x = \frac{1}{2} \sum_{i=0}^{2} |\mu_1(\sigma(x) = i) - \mu_2(\sigma(x) = i)|.
\]

Let \( \eta^{x,s} \) be the configuration \( \eta \) with the spin at \( x \) set to \( s \). Following [27], [29] define

\[
\kappa \equiv \kappa(\mu) = \sup_{x \in \Gamma^k, x \neq s, s'} \max \|\mu^x_{\tau_x} - \mu^s_{\tau_x}\|_x;
\]

\[
\gamma \equiv \gamma(\mu) = \sup_{A \subset \Gamma^k} \max \|\mu^y_A - \mu^{y,s'}_A\|_x,
\]

where the maximum is taken over all boundary conditions \( \eta \), all sites \( y \in \partial A \), all neighbors \( x \in A \) of \( y \), and all spins \( s, s' \in \{0, 1, 2\} \).

The criterion of extremality of a TISGM is \( k\kappa\gamma < 1 \) [27], [29]. Note that \( \kappa \) has the particularly simple form \( \kappa = \frac{1}{2} \max_{i,j} \sum_{1}^{2} |P_{i,j} - P_{j,i}| \) and \( \gamma \) is a constant which does not have a clear general formula.

Let \( r = \theta^2 \). For the solution \((1, y)\), we shall compute \( \kappa \)

\[
\kappa = \frac{2 \cdot |1 - \theta y| + y^2 \cdot |\theta - y|}{2y(2\theta + y^2)}. \tag{33}
\]

For \( \theta < 1 \) from the system (7) we get the following inequalities

\[
1 - \theta y = \frac{\theta(1 - \theta^2)y^2}{Z} > 0, \quad y - \theta = \frac{2\theta(1 - \theta^2)}{Z} > 0.
\]

Using these inequalities, we obtain

\[
\kappa = \begin{cases} 
\frac{y^2 - \theta y^2 - 2\theta y^2 + 2}{2y(2\theta + y^2)}, & \text{if } 0 < \theta < 1, \\
\frac{-y^2 + \theta y^2 + 2\theta y^2 - 2}{2y(2\theta + y^2)}, & \text{if } \theta \geq 1.
\end{cases}
\]

For the solution \((1, y)\), we shall calculate \( \gamma \).

\[
\gamma = \max \left\{ \|\mu^{\eta^{y,0}}_A - \mu^{\eta^{y,1}}_A\|_x, \|\mu^{\eta^{y,0}}_A - \mu^{\eta^{y,2}}_A\|_x, \|\mu^{\eta^{y,1}}_A - \mu^{\eta^{y,2}}_A\|_x \right\},
\]

where

\[
\|\mu^{\eta^{y,0}}_A - \mu^{\eta^{y,1}}_A\|_x = \frac{1}{2} \sum_{s \in \{0, 1, 2\}} |\mu^{\eta^{y,0}}_A(\sigma(x) = s) - \mu^{\eta^{y,1}}_A(\sigma(x) = s)| = \frac{\gamma^3 - \gamma^2 - \theta \gamma + 2}{2y(2\theta + y^2)}, \text{ if } 0 < \theta < 1,
\]

\[
= \frac{-\gamma^2 + \theta \gamma^2 + 2\theta \gamma - 2}{2y(2\theta + y^2)}, \text{ if } \theta \geq 1.
\]
The extremality interval of TISGMs

The function $U$ Let Proposition 2.

a) There exists $\theta_1 \approx 0.1666993311$ such that the measure $\mu_1$ is extreme if $\theta \in (\theta_1, \infty)$;

b) There are values $\theta^* \approx 7.729813675$ and $\theta_2 \approx 9.706301628$ such that the measure $\mu_2$ is extreme if $\theta \in [\theta^*, \theta_2]$;

c) The measure $\mu_3$ is extreme (where it exists, that is $\theta \in (\theta^*, \infty)$).

From Proposition 1 and Proposition 2 we have the following
Figure 5. The graph of function $U_2(\theta)$

Figure 6. The graphs of functions $U_1(\theta)$ (left) and $U_3(\theta)$ (right)
Theorem 5. Let \( r = \theta^2 \). Then the following statements hold

a) There exists \( \theta_1 (\approx 0.1666993311) \) such that the measure \( \mu_1 \) is non-extreme if \( \theta \in (0, \theta_1) \) and is extreme if \( \theta \in (\theta_1, \infty) \);

b) There are values \( \theta^* (\approx 7.729813675) \) and \( \theta_2 (\approx 9.706301628) \) such that the measure \( \mu_2 \) is extreme if \( \theta \in [\theta^*, \theta_2) \) and is non-extreme if \( \theta \in (\theta_2, \infty) \);

c) The measure \( \mu_3 \) is extreme (where it exists, that is \( \theta \in [\theta^*, \infty) \)) (see Fig. 7).

Figure 7. The graphs of functions \( y_i(\theta), i = 1, 2, 3 \). The bold curves correspond to regions of the functions where the corresponding TISGM is extreme. The thin curves correspond to regions of the functions where the corresponding TISGM is non-extreme.

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