The main purpose of this paper is to suggest that stochastic differential equations and stochastic dynamical systems (SDE's and SDS's) with respect to “real world” probability measure, which model financial price movements should be of second order, instead of first order Markovian models à la Bachelier–Samuelson [3, 33] based on Fama’s efficient market hypothesis [14].

Of course, the Black–Scholes–Merton option pricing formula [6, 29], which is derived from the first order SDE

\[ \frac{dS}{S} = \mu dt + \sigma dB_t, \]
where $S$ is the price of the underlying asset, $\mu$ is the drift parameter, $\sigma$ is the volatility coefficient, and $B_t$ is a Wiener process, is a valid formula, modulo some corrections due jumps and fat tails \cite{35}. However, Equation (1) itself is valid only with respect to a risk-neutral measure and not with respect to the real world probability measure, and it fails to explain many phenomena in the market, such as boom-bust cycles. For investing and risk management, one needs more realistic models than this equation.

In physics, most important equations of motion are of second order. (The phase space must have twice the dimension of the configuration space, e.g., when the configuration space is a manifold $N$ then its natural phase space is the tangent or cotangent bundle of $N$). By analogy, we come to the idea that stock prices should also be governed by second order differential equations, with stochastic terms due to noises. In fact, this idea is not new, even though we arrived at it by ourselves: it was already used by Cont and Bouchaud \cite{7}, who found it by analogy with the classical Langevin’s equation, and who used it in their explanation of market crashes. Cont and Bouchaud \cite{7} also said that J. Doyne Farmer, who is a pioneer in agent-based modelling of complex systems \cite{15, 16}, already showed second order differential equations for stock prices in a seminar talk in 1997 in Paris 7, though no written text was publicly available.

Our paper may be viewed as a further development in the direction of these ideas of second order SDS and agent-based models. In particular, we contribute the following elements to the theory:

- The notion of market energy, in analogy with physical energy, which governs the equations of motion of stock prices, and which is responsible for many visible market phenomena, such as boom-bust cycles and persistence of volatility.

- An agent-based construction of this market energy, which can be decomposed into many components: kinetic market energy, potential market energy, thermodynamic market energy, etc. Each type of market energy corresponds to some kinds of agent behavior and strategy, such as contrarian investing and portfolio rebalancing, momentum players and hedging strategies, etc.

- A no-go theorem in the theory of reduction of stochastic dynamical systems, which implies in particular that first order stochastic models are incorrect reductions of second order models in general. To get better estimates and predictions in investing and risk managements, one needs second order models. (One may guess that many trading shops are actually using them).

- We also discuss some simple second-order stochastic models, such as the damped stochastic harmonic oscillator and the constrained n-oscillator model, which are integrable in the sense of \cite{39} (read: easy to compute and simulate), and which can already capture a lot of features of real-world financial markets, e.g., the fact that stocks in the same sector often move together, and that “hot money” can jump from one stock or sector to another (market energy transfer among the sectors).
Our idea behind the notion of market energy is very simple. In Hamiltonian dynamics, the equation of motion of a conservative system is written as (see, e.g., [2])

\[
\dot{x}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial x_i},
\]

where \((x_i)\) are the coordinates on the configuration space (i.e., spatial variables), \((p_i)\) are called the momenta and are the dual coordinates of \((x_i)\) on the cotangent bundle to the configuration space, and \(H = H(p_i, x_i)\) is the total energy (i.e., the Hamiltonian function). The relation between the momenta \((p_i)\) and the velocities \((v_i = \dot{x}_i)\) is given by the Legendre transformation. For example, in Newtonian mechanics we have \(p_i = m_i v_i = m_i \dot{x}_i\), where \(m_i\) is the mass of the particle number \(i\).

If one forgets about the momenta \((p_i)\), and consider (2) as a system of equations of the configuration variables \((x_i)\) only, they it is a system of second-order differential equations on \((x_i)\). By analogy, in financial market \((x_i)\) can play the role of prices, \(H\) is the total market energy to be defined, and Equation (2), plus some noise terms and external forces and damping terms, can serve as the second-order differential equation for the movement of the prices. In the simplest case, this equation gives us the damped stochastic oscillator (see Section 4), which has been extensively studied in mathematics and physics [4, 21, 39]. Such a simple oscillator model already fits real-world financial price movements, such as USD/EUR exchange rates and inflation-adjusted gold prices over the long term much better than first order models.

The rest of this paper is organized as follows:

In Section 2 we recall from a recent mathematical paper of ours [39] with a former doctoral student (who went to work for a bank after his thesis) the no-go theorem for reduction of stochastic dynamical system: one cannot reduce the system by simply forgetting about some variables. Implication for us: the momenta of the stock prices, even though they can not be measured with confidence, should enter the equations as variables on the same footing as stock prices, not just as constants or stochastic parameters. In Section 3 we discuss the notion of market energy and how to decompose it into the sum of different components corresponding to different behaviors/strategies of market players. This market energy plays the central role in the second order stochastic differential equation for the stock prices. In Section 4 we discuss the damped stochastic oscillator model for a single stock, and some modifications. In Section 5 we discuss some technical analysis patterns which can be explained by the market energy and other physics-like arguments. Finally, in Section 6 we propose a simple constrained \(n\)-oscillator model for a multi-stock market.

2. No-go theorem for stochastic reduction

Stochastic differential equations (SDE’s) in mathematical finance are often written in Ito form, but for global analysis it may be more convenient to write them in the following Stratonovich form, which can be done in a coordinate-free way and which
behaves well under changes of variables:

(3) \[ dx_t = X_0 dt + \sum_{i=1}^{k} X_i \circ dB_i^t, \]

where \( x \) denotes a point on a manifold \( M \); \( X_0, X_1, \ldots, X_k \) are vector fields on \( M \), and \( B_1^t, \ldots, B_k^t \) are independent Wiener processes, see, e.g., [5, 18, 23, 31, 39]. This SDE generates a continuous-time stochastic dynamical systems (SDS) whose associated stochastic vector field is:

\[ X = X_0 + \sum_{i=1}^{k} X_i \circ dB_i^t, \]

and whose diffusion operator is:

\[ A_X = X + \frac{1}{2} \sum_{i=1}^{k} X_i^2. \]

The meaning of this diffusion operator is as follows: if we fix a point \( x \in X \) and denote by \( x_t \) the random position of \( x \) after time \( t \) by the random flow of the SDS, then for any smooth function \( f : M \rightarrow \mathbb{R} \) we have

\[ (A_X f)(x) = \lim_{t \to 0} \frac{\mathbb{E}^x[f(x_t)] - f(x)}{t}. \]

Two SDS’s on a manifold \( M \) are the same if and only if they have the same components for their corresponding stochastic vector fields up to a permutation. But for probability computations, it is only the diffusion operator which matters. So we say that two SDS’s on \( M \) are diffusion-wise the same if they have the same diffusion operator. The proof of the following theorem is straightforward, see [39]:

**Theorem 2.1 ([39]).** Let \( \Phi : M \rightarrow N \) be a smooth surjective map from a manifold \( M \) to a manifold \( N \), and let \( X = X_0 + \sum_{i=1}^{k} X_i \circ dB_i^t \) be an SDS on \( M \). Then the diffusion process of \( X \) is projectable (i.e., can be reduced) to a Markov process on \( N \) if and only if for any function \( f : N \rightarrow \mathbb{R} \) and any two points \( x, y \in M \) such that \( \Phi(x) = \Phi(y) \) we also have

(4) \[ A_X(\Phi^*(f)(x)) = A_X(\Phi^*(f)(y)) \]

where \( A_X \) is the diffusion operator of \( X \). If this condition is satisfied and \( \Phi \) is a submersion then the projected diffusion process on \( N \) is generated by an SDS on \( N \).

In the case when \( X = X_0 \) is a smooth deterministic system then the deterministic process generated by \( X \) on \( M \) is projectable to a Markov process on \( N \) if and only if for any points \( x, y \in M \) such that \( \Phi(x) = \Phi(y) \) we also have \( \Phi^*(X(x)) = \Phi^*(X(y)) \), and if this condition is satisfied then \( X \) is projected to a smooth vector field on \( N \).

In particular, if \( M = T^*N \) or \( M = TN \) is the cotangent or tangent bundle of \( N \), the surjective map \( \Phi \) is the projection map, and our second order model is an SDS on \( M \), then in general it is impossible for the conditions of the above theorem to be satisfied, so we cannot project it to an SDS on \( N \) (i.e., to a first order model) by simply
forgetting about the momenta. Even if the second order model is deterministic, we still cannot reduce it to a stochastic first order system. So Theorem 2.1 is a kind of no go theorem for dimensional reduction of stochastic models. (Unless when there are some obvious symmetries, in which case the system can be reduced). What one does in stochastic modelling by simply forgetting about some hidden variables is not really reduction, but rather a kind of rude approximation.

3. Market players and market energy

Over the last three decades, there have been a lot of works on agent-based models of financial markets (see, e.g., [7, 11, 12, 13, 15, 16, 17, 19, 25, 26, 32, 36, 37, 38]), and some of them seem to be very successful in simulating the real-world markets and for devising market-beating strategies. Nevertheless, we have not seen the notion of market energy in these works, even though some market equations introduced there look Hamiltonian-like. We believe that this is a very useful notion, not only for our paper, but also for the other agent-based models as well.

In physics, the energy plays a central role: the equation of motion can be derived from the total energy function in Hamiltonian formulation, or can be written as a variational (Euler-Lagrange) equation in Lagrangian formulation, using the action function, which is also a function of energy components, and external forces.

Here we also want to figure out what is the market energy function which governs the movement of financial prices (together with external forces and noises). Like in physics, the market energy can be decomposed into a sum of several components, such as follows.

3.1. Potential market energy. Value investors tend to buy/sell stocks which they think are undervalued/overvalued. Because of that, when an asset price is different from its average perceived fair value level, then the difference between the price and the fair value creates a potential net aggregate buying or selling action from the value investors, i.e. a potential market energy.

As a first approximation, one can think of this potential energy as being proportional to the square of the level of mispricing, because its derivative (which is the force leading to the change in price momentum, because the higher the derivative, the more players are going to react) is approximately proportional to the level of mispricing.

It may happen that there are several “centers of gravity” for the potential energy function, i.e. several different values which can pretend to be “the average fair value”, depending on the level of optimism in the market. Consider, for example, a scenario with two particular groups of value investors: the optimists and the pessimists. The optimists have their perceived fair value of the stock at $V_1$, but they may become bankrupt or too depressed to buy if the price falls too much below $V_1$. Conversely, the pessimists have their perceived fair value of the stock at $V_2 << V_1$, but won’t have anything to sell if the price is too much above $V_2$. In this scenario, the potential market energy function may look like a double well potential, with two bottoms (basins of attraction) at $V_1$ and $V_2$. 
One may also think of other scenarios, where the potential energy function is even more complicated. Nevertheless, it is safe to assume that this function goes to infinity when the price $x$ goes to infinity when the level of mispricing goes to infinity.

3.2. **Kinetic market energy.** *Momentum players* tend to buy/sell stocks which have shown an increase/decrease in prices (i.e. positive/negative momentum, modulo noises). This “buying begets buying” herd behavior reflects the inertia of a stock and can be associated to its kinetic market energy, which is approximatively proportional to the square of the level of net buying or net selling, and hence is approximatively proportional to the square of the momentum. (A transaction is both a buy and a sell, but will be counted as a buy if done at the ask price and creating an upward pressure on the price).

There are also investment strategies which are not “momentum chasing” per se, but which still add to the “selling begets selling” kinetic market energy, for example the “performance insurers” and hedging strategies, which are blamed for the 1987 stock market crash (see [24, 26]).

3.3. **Thermodynamic market energy.** It may happen that there is a lot of trading but the price of the stock does not really move in any direction, except for a random noisy movement like the Brownian motion. This noise in the market, which consists of a large amount of micro-movements which mostly cancel the direction of each other, may be associated to the market’s thermodynamic energy (heat) and is responsible for the stochastic term in the price motion equation.

Those market makers who provide liquidity for the market without betting on its direction one way or another may be considered as contributing the heat to the market as well. There are always energy-losing damping forces on the market (such as transaction costs, bid/ask spreads), so the mechanical energy level of the market may slowly go to zero (i.e., the market dies out) if there is no energy pumped into it. However, the market heat can sporadically turn into mechanical energy (at least in the damped stochastic oscillator model), just preventing the market from dying, even with damping and without exterior sources of energy.

3.4. **Chemical market energy and other energy components.** When two stocks merge, the merger may release (or absorb) a lot of market energy. This is an example of what we call the chemical market energy, i.e. the energy related to "chemical" financial reactions. In mechanics, one often ignores this energy and other energy components, assuming them to be invariant (and hence having no effect on the equation). For simplicity, in this paper we will ignore these types of market energy.

3.5. **Market energy, volume and volatility.** A lot of research papers on financial markets confirm the strong positive relation between trading volume and volatility, and also the persistence (long memory) of volume and volatility in the market, see, e.g., [25]. We propose here to explain these phenomena by market energy.

Both volume and volatility are positively related to market energy: the volume is roughly proportional to the kinetic market energy, and so is the square of the volatility.
for some portion of the kinetic energy. Hence one can conjecture that the volatility is highly correlated to the square root of the volume.

In general, it takes many trades (a lot of market energy) to move a market (especially when the inertia is high), and the market movement is partly reflected in volatility, and that’s another way to say why volume and volatility are highly positively correlated.

If one ignores external forces and dissipation then the market energy is conserved. Due to this conservation principle, the energy is not conserved in general because the system is not closed, but it can’t change very fast. That’s why both volume and volatility has a memory.

Another explanation comes from the quasi-periodic nature of the whole market (see Section 6): the market energy of each individual stock also changes in a stochastically quasi-periodic manner.

4. Individual Stocks as Damped Stochastic Oscillators

4.1. The model. One of the simplest models for financial markets is the damped stochastic oscillator, which can be used for the price movement of a single financial asset or commodity, such as gold, oil, SP500 index, or EUR/USD exchange rate, etc. In this model, the stochastic dynamical system is given by the stochastic vector field (see [39]):

\[ X = X_h + D + \sigma B \]

on the symplectic space \((\mathbb{R}^2, \omega = dx \wedge dy)\), where

\[ D = -f(\sqrt{x^2 + y^2})(x \partial x + y \partial y) \]

is the damping term,

\[ B = \partial x \circ \frac{dB^1}{dt} + \partial y \circ \frac{dB^2}{dt} \]

is the generator of a 2-dimensional Brownian motion (the noise), \(\sigma\) is the volatility coefficient (the amplitude of the noise), and

\[ h = \frac{1}{2}(x^2 + y^2) \]

is the market energy of the asset. Here \(x\) is the mispricing and \(y\) is the momentum. (Say \(x = P - V\) or \(x = \ln(P/V)\), where \(P\) is the price and \(V\) is the fair value. We will assume that the price \(P\) fluctuates a lot but the fair value \(V\) varies very slowly over the time, so that most of the variation of \(P\) is reflected in the variation of \(x\)). The units (time, price, etc.) here are already normalized so that \(h\) takes the simplest form \(h = \frac{1}{2}(x^2 + y^2)\). This model has a \(SO(2)\)-symmetry which makes it into an integrable stochastic system, easy to investigate (see [39]).

The dissipative term \(D\) in \(X\) is due to market friction, e.g. trading fees, and has the energy-losing effect on the market. On the other hand, the noise term \(B\) has the
energy-enhancing effect. These two effects cancel out each other in a stochastic way. As a result, the expected market energy of a single-stock market in this model does not die out (go to zero) or explode (go to infinity) over time, but rather tends to a stable positive energy level. Similarly to the mean-reverting Ornstein-Uhlenbeck process, there is a stationary distribution density of energy levels for the damped stochastic oscillator, which is concentrated around a stable energy level. (Notice, however, that oscillators are not mean-reverting: they fo through the mean back and forth but do not “converge” to the mean. When they are at the mean they tend to go far away from the mean again if the energy is large enough).

The above simple oscillator model is a bit simplistic, because an individual stock is not a closed system like in the model, and the rest of the market can affect it greatly. Nevertheless, it shows the stochastic quasi-periodic nature and boom-bust cycles of real-world financial markets.

Let us look at two examples: gold prices and EUR/USD rates.

4.2. Gold price as a noisy oscillator. Figure 1 is a inflation-adjusted chart of historical gold prices from 1970 (not long before the US abandoned the last gold peg 1 ounce = $42.22) until the date of writing of this article (07/2017). No one knows for sure what is the fair value of gold, but one may argue that this inflation-adjusted fair value does not change much with time (according to macro-economic models), and that once in a while the price coincides with the fair value, i.e. the mispricing is 0 (according to the oscillator model). During the late 1970s and the 1980s, the price of gold is around 400 USD/ounce (or around 800-1000 USD in 2017’s dollars, inflation-adjusted), so we may presume that the fair value is around those numbers at that time. When
gold goes to 200 USD/ounce in 2000 (or under 400 USD in 2017’s dollars), it becomes very underpriced according to the oscillator model, creating a big potential speculation energy which results in a big upward movement later on. In 2010s, the fair value of gold can be estimated at around 1000 USD/once (inflation-adjusted). Of course, the fair value of gold doesn’t have to say constant, but can change, due to the growth of world’s economy, the growth of gold supply and other factors, but here for simplicity we assumed that didn’t change much over the last 30-40 years. If it went up, say 30% over the last 30-40 years, then the fair price of gold would be closer to 1300 than 1000 USD/ounce right now. When gold went above 1500 USD/ounce in 2011, it was already very probably overpriced, but it continued to move up due to positive momentum. Eventually this momentum died out, and what remained is a big potential speculation energy pointing to a big future downward movement. Sure enough, gold fell down from its peak of almost 2000 USD/once in 2011 to its current price of about 1200 USD/ounce.

Notice also that during the period late 1980s and early 1990s, the price of gold didn’t move much, i.e. the speculation energy seems to die out during that period. This loss of market energy in gold can’t be explained in the single damped stochastic oscillator model of the market, according to which the speculation energy will (almost surely) never die out but will fluctuate around a certain energy level. However, it can be explained by using multi-body models of the market, where the speculation energy (or hot money in financial jargon) can move from one component of the market to another.

Remark. Going back further in time, a free chart from goldchartsrus.com (not reproduced here) shows that, since 1600 (4 centuries ago), inflation-adjusted gold prices oscillated around 450USD/ounce a great number of times and rarely shot up above 1000USD. So if we assume that the inflation data over the last 4 centuries is correct (which is a big assumption, maybe not true), then gold is right now still more expensive than during centuries ago.

4.3. EUR/USD exchange rates. Figure 2 is a chart chart of historical EUR/USD exchange rates over the past 20 years. Since the USA and the Eurozone have slightly different inflation rates over those years, it may be better to divide the exchange rate by the PPP (purchasing power parity) EUR/USD curve, which serves as a kind of “fair price” curve, and which is also given in Figure 2.

The quasi-periodic oscillating nature of the EUR/USD exchange rates around the PPP is clear from the Figure 2. Nevertheless, one may notice that the potential energy function looks more like a double-well function (with an optimist and a pessimist perceived fair value for EUR/USD) than a single well.

5. Market patterns

Technical analysis, i.e. the search for market patterns, is used by a great number of market players with various degrees of success (see, e.g., [1, 8, 9, 10, 28]), and is at odds with the efficient market hypothesis (see, e.g., [14, 27, 33]). For that matter, most agent-based models are also at odds with this hypothesis.
In this section, we want to give an explanation of some simple and easily recognizable market patterns by using market energy and second order models. Namely, we will discuss three market patterns: U-shaped vs. V-shaped reversals; resistance breaking; and market aftershocks.

5.1. **U-shaped versus V-shaped reversals.** The financial price reversals are often divided into 2 main types: U-shaped and V-shaped. The difference between U-shaped and V-shaped reversals is in the kinetic energy: at the point of a U-shaped reversal, the kinetic energy goes to 0, i.e., the momentum of the stock (end hence the kinetic energy) dies out before reversing, like a ball going up and then makes a U-turn and falls down on its own weight. On the contrary, in a V-shaped reversal situation, when the stock hits a hard “wall” (strong resistance), the kinetic energy remains positive, the absolute value of the momentum does not change much, it’s just the direction which changes, similarly to an elastic *bouncing ball* when hitting the wall.

An important physical property of physical objects which can bounce back strongly when hitting a wall is their elasticity. So apparently, the market is also elastic when it makes V-shaped reversals, and this elasticity might be explained by the prevalence of active market swingers, i.e. active traders (or trading strategies) who switch sides easily when the stock hits a resistance.

For example, Figure 3 is a daily chart of the SP500 index for the period from 07/2011 to 11/2011. Notice how it also moved like a bouncing ball during the months 08/2011 – 09/2011: every time it falls down to a level near 1100 it makes a V-shaped (fast) reversal, but when it goes up to around 1200 it makes a U-shaped (slow) reversal.
Figure 3. SP500 Index barchart from 07-2011 to 10-2011

Notice also that V-shaped patterns can be seen more often in short-term movements, rather than long-term movements, of a financial asset price. That is because the “walls” can often be set up by the “houses”, who are strong enough to control the price of a stock short-term, but not over the long term.

5.2. Resistance breaking. A market resistance may sometimes be analogous to a dike which prevents water waves from overflowing (or a wall as in the V-shaped market reversal pattern discussed above). But when the waves are strong enough to break the dike, i.e. the market momentum is strong enough to break the resistance, there will be a flood, i.e. a large market move once the resistance is broken.

Figure 4. Resistance breaking of Lyxor ETF STOXX Europe 600 Oil Gas - chart between 2010-2014

Figure 4 is typical example of resistance breaking. Notice how the price bounced back the first times it hit the resistance, and then finally the resistance is broken due to strong market energy.
From the point of view of market energy, a wall is a sharp spike in the potential energy function near some point of the price variable. In order to go over this potential wall to the other side of the price region, the market needs a lot of energy. Maybe during the first attempts at coming close to the wall, there is not enough market energy to go over it, so the price falls back (potential energy changing into kinetic energy). But with some additional from the outside (for example, the whole stock market is moving in some direction, giving additional kinetic energy to the stock), the potential wall is finally overcome, and after that it often happens that the stock price moves very fast due to high potential energy turning into kinetic energy after the wall.

5.3. Market aftershocks. After a big earthquake hits some area, there are often aftershocks, which are less strong but can also be quite violent. It is partly because a lot of energy is still there and cannot dissipate too quickly. The same happens in the stock market: the level of market energy created after a shock is high, and this high energy results in big (usually oscillating) aftershock moves.

![Figure 5. Dow Jones Industrial Average crash 10/1987](image)

Figure 5 is an example of a big stock market shock in 1987, together with large aftershock movements.

Notice that, just as earthquakes are often localized, market shocks happen much more easily in individual stocks than for the whole market: a little market energy may already be enough for a small stock to make a huge move.
6. The market as a constrained n-oscillator

6.1. The linear deterministic model.

In this model, we consider the total market which contains every economical asset, i.e. it represents the whole economy. The assets are divided into $n$ asset classes $A_1, \ldots, A_n$, for example: energy, real estate, food, transport, communication, etc. The total price of each asset class is denoted by $P_i = p(A_i)$. Then $P = \sum P_i$ is the total net worth of the whole economy, and we call

$$R_j = \frac{P_j}{\sum P_i}$$

the relative price of the asset class $A_j$, so that $\sum R_j = 1$.

We assume that class $A_i$ has a relative fair value $v_i$ in the economy ($\sum v_i = 1$), which varies very slowly with the time. For example, people will pay only a certain percentage of their income for telecommunication needs, and therefore fast technological advances don’t make this sector occupy a much larger share of the whole economy, but make the prices per unit drop fast instead. We will be interested in the asset mispricings

$$x_i = R_i - v_i.$$  

Similarly to the oscillator model for a single stock, we will assume that the market energy has the form

$$E = \frac{1}{2} \sum_1^n a_i x_i^2 + \frac{1}{2} \sum_1^n b_i \dot{x}_i^2.$$

Here $\frac{1}{2} \sum a_i x_i^2$ is the potential energy, $\frac{1}{2} \sum b_i \dot{x}_i^2$ is the kinetic energy, and $a_i, b_i > 0$ are constant asset-specific coefficients. So we get a Hamiltonian system with the energy function $E$ given by Formula (11) and a linear constraint

$$\sum x_i = 0.$$  

Since the constraint is holonomic, this is a Hamiltonian system with $n - 1$ degrees of freedom.

In order to write down the equation of motion one can for example eliminate one of the variables (say by putting $x_n = -\sum_{i=1}^{n-1} x_i$) and consider it as a system on $T^*\mathbb{R}^{n-1}$. Equivalently, one can use the Lagrangian multiplier method as follows:

The Lagrangian action function is:

$$L = \frac{1}{2} \sum a_i x_i^2 - \frac{1}{2} \sum b_i \dot{x}_i^2$$

The equation is $\frac{\delta L}{\delta x_i} = \lambda \frac{\partial f}{\partial x_i} \forall i = 1, \ldots, n$, where $f(x) = \sum x_i$ is the constraint function, and $\lambda$ is the Lagrangian multiplier to be determined. Since $\frac{\delta L}{\delta x_i} = \frac{\partial L}{\partial x_i} -$
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = a_i x_i + b_i \ddot{x}_i \quad \text{and} \quad \frac{\partial f}{\partial x_i} = 1,
\]
we get the system of equations:
\[
(14) \quad a_i x_i + b_i \ddot{x}_i = \lambda \quad \forall \ i = 1, \ldots, n.
\]

Since \(\sum x_i = 0\) implies \(\sum \dot{x}_i = 0\), we get the following equation for \(\lambda\):
\[
\sum \frac{\lambda}{b_i} = \sum \frac{a_i x_i}{b_i},
\]
which implies that \(\lambda = \left( \sum \frac{a_i x_i}{b_i} \right) / \left( \sum \frac{1}{b_i} \right)\). Thus the equation of motion (14) is a system of linear differential equations with constant coefficients
\[
(15) \quad \ddot{x}_i = \frac{\left( \sum a_j x_j \right)}{b_i \sum \frac{1}{b_j}} - \frac{a_i x_i}{b_i}
\]
and with the constraint \(\sum x_i = 0\).

**Proposition 6.1.** By a linear transformation
\[
(16) \quad x_i = \sum c_{ij} z_j,
\]
where \((c_{ij})\) is an appropriate constant \(n \times (n - 1)\) matrix of rank \(n - 1\), the system (15) with constraint (12) becomes a system of \(n - 1\) uncoupled harmonic oscillators:
\[
(17) \quad \ddot{z}_i = -\lambda_i^2 z_i.
\]
with constants \(\lambda_1, \ldots, \lambda_{n-1} > 0\) (called **eigenvalues** or **normal modes** of (15)).

The proof is a simple exercise in classical mechanics [2]: when written as a Hamiltonian system on \(T^* \mathbb{R}^{n-1} = \mathbb{R}^{2(n-1)}\), we have a positive definite quadratic Hamiltonian function, and any such function can be written as \(\frac{1}{2} \sum \lambda_i (z_i^2 + w_i^2)\) in some linear canonical coordinate system \((z_i^2, w_i)\) on \(\mathbb{R}^{2(n-1)}\).

The general solution of our model has the form:
\[
(18) \quad x_i = \sum c_{ij} \sin(\lambda_j t + d_{ij})
\]
with appropriate coefficients \(c_{ij}\) and \(d_{ij}\) (so that the constraint \(\sum x_i = 0\) is satisfied). Thus, the mispricing of each asset is a quasi-period function with periods \(\lambda_1, \ldots, \lambda_{n-1}\). Notice that all the asset classes share the same periods.

We will call the above simple model the (linear deterministic) **constrained n-oscillator model** of the market.

**Remark 6.2.** In some physics textbooks (e.g., [4]) one can find a so called coupled \(n\)-oscillator model, which explains the waves in materials and which consists of a chain of masses connected to each other by springs. Our model is similar to, but different from, this coupled \(n\)-oscillator model, because the kinetic energy in our model is different from the kinetic energy is not the same.
6.2. Frequencies of the system.
In the linear model with the energy function $E = \frac{1}{2} \sum a_i x_i^2 + \frac{1}{2} \sum b_i \dot{x}_i^2$ and the constraint $\sum x_i = 0$, we will call
\begin{equation}
E_i = \frac{1}{2} a_i x_i^2 + \frac{1}{2} \dot{x}_i^2
\end{equation}
the energy of the $i$-th component, and the number $\gamma_i = \sqrt{a_i/b_i}$ the proper frequency of the $i$-th component. If there were no constraint then $x_i(t)$ would be a periodic function in time $t$ with period $\frac{2\pi}{\gamma_i}$.

The following proposition shows the relationship between the frequencies of the linear constrained $n$-oscillator and the proper frequencies of its components.

**Proposition 6.3.** Assume that the proper frequencies $\gamma_1 = \sqrt{a_1/b_1}, \ldots, \gamma_n = \sqrt{a_n/b_n}$ of the components of the above linear constrained $n$-oscillator are ordered in an increasing way:
\begin{equation}
\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n.
\end{equation}
Then the linear constrained $n$-oscillator is equivalent to a system of $(n-1)$ free (uncoupled) harmonic oscillators whose frequencies $\lambda_1, \ldots, \lambda_{n-1}$ satisfy the inequality
\begin{equation}
\gamma_1 \leq \lambda_1 \leq \gamma_2 \leq \lambda_2 \leq \cdots \leq \gamma_{n-1} \leq \lambda_{n-1} \leq \gamma_n.
\end{equation}
Conversely, if $\gamma_i$ and $\lambda_i$ are arbitrary positive numbers which satisfy (21) in the strict sense (i.e., there is no equality), then there exist positive numbers $a_i, b_i$ such that $\gamma_i = \sqrt{a_i/b_i}$ and the frequencies of the above constrained $n$-oscillator are $\lambda_1, \ldots, \lambda_{n-1}$.

**Proof.** With a linear change of the coordinates $y_i = \sqrt{a_i} x_i$, we can write
\begin{equation}
E = \frac{1}{2} \sum y_i^2 + \frac{1}{2} \sum \frac{\dot{y}_i^2}{\gamma_i^2}
\end{equation}
with the constraint $\sum \alpha_i y_i = 0$, where $\alpha_i = 1/\sqrt{a_i}$. The corresponding unconstrained equation is:
\begin{equation}
y + \Gamma \ddot{y} = 0,
\end{equation}
where $\Gamma = \text{diag}(1/\gamma_i^2)$ is the diagonal matrix whose entries are $1/\gamma_i^2$. The true equation of motion, taking into account the constraint, is:
\begin{equation}
y + \Gamma \ddot{y} \in \mathbb{R}.(\alpha_1, \ldots, \alpha_n)^T,
\end{equation}
where $y = (y_1, \ldots, y_n)^T$. (The above equation is written in the form of an inclusion, which means that $y + \ddot{y}$ is collinear to $(\alpha_1, \ldots, \alpha_n)^T$; $T$ means the transpose).

Let $O$ be an orthogonal matrix such that
\begin{equation}
O. (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n)^T = (0, \ldots, 0, \|\alpha\|)^T,
\end{equation}
where $\|\alpha\| = \sqrt{\sum \alpha_i^2}$. Then above equation is equivalent to:
\begin{equation}
Oy + O\Gamma \ddot{y} \in \mathbb{R}. (0, \ldots, 0, 1)^T.
\end{equation}

Denote
\begin{equation}
z = (z_1, \ldots, z_n) = Oy.
\end{equation}
Then the constraint $\langle y, \alpha \rangle = 0$ is equivalent to $\langle z, O\alpha \rangle = \langle Oy, O\alpha \rangle = \langle y, \alpha \rangle = 0$, i.e. $z_n = 0$, i.e. we can forget about $z_n$ and look only at the variables $z_1, \ldots, z_{n-1}$.

Denote by $A$ the left-top $(n-1) \times (n-1)$ minor of the positive symmetric matrix $O\Gamma O^{-1}$, then the system is equivalent to
\begin{equation}
z + A\ddot{z} = 0.
\end{equation}

It is well-known that the eigenvalues of a symmetric matrix $O\Gamma O^{-1}$ (which are $\frac{1}{\gamma_1^2}, \ldots, \frac{1}{\gamma_n^2}$) and the eigenvalues of its left-top minor $A$ (which are $\frac{1}{\lambda_1^2}, \ldots, \frac{1}{\lambda_{n-1}^2}$) satisfy Inequality (21), which is part of the so-called Gelfand-Ceitlin triangle of inequalities, see, e.g., [20].

Remark 6.4. The component energy functions $E_i = \frac{1}{2} a_i x_i^2 + \frac{1}{2} \dot{x}_i^2$ are not first integrals of the constrained $n$-oscillator model, i.e. they also change with time (in a quasi-periodic way). Thus we see a speculation energy transfer among the components of the market in this model.

6.3. Market sectors: Components having the same proper frequencies.

In the generic case, the frequencies $\lambda_1, \ldots, \lambda_{n-1}$ of the linear unconstrained $n$-oscillator are incommensurable, and the regular minimal invariant tori of the system in the phase space are of full dimension $n - 1$. However, there are some special cases when the minimal invariant tori are of dimension less than $n - 1$. One particular case is when there are some components whose proper frequencies are the same.

Assume, for example:
\begin{equation}
\gamma_{p+1} = \gamma_{p+2} = \cdots = \gamma_{p+k}, \quad p \geq 0, k \geq 2.
\end{equation}
Then, according to Inequality (21), we also have:
\begin{equation}
\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_{p+k-1} = \gamma_{p+k},
\end{equation}
i.e., the multiplicity of the frequency $\lambda_{p+1}$ in the linear unconstrained $n$-oscillator is (at least) $k - 1$, and this frequency coincides with the proper frequency of $k$ components of the system.
By putting \( \hat{x}_1 = x_1, \ldots, \hat{x}_p = x_p, \hat{x}_{p+2} = x_{p+k+1}, \ldots, \hat{x}_{n-k+1} = x_n \), and
\[
\hat{x}_{p+1} = x_{p+1} + \ldots + x_{p+k},
\]
we can make a reduction of the system in this case, reducing the number of components from \( n \) to \( n - k + 1 \) (and killing the frequency \( \lambda_{p+1} \) along the way, by “averaging out” with respect to that frequency). This procedure corresponds to the practice of regrouping many similar components into a sector in the market.

The coefficients associated to the sector \( (x_{p+1}, \ldots, x_{p+k}) \) are:
\[
\hat{a}_{p+1} = \frac{1}{\sum_{i=1}^{k} \frac{1}{a_{p+i}}}, \quad \hat{b}_{p+1} = \gamma_{p+1}^2 \hat{a}_{p+1}.
\]
(The coefficients for the other components remain the same: \( \hat{a}_i = a_i \) and \( \hat{b}_i = b_i \) for \( i \neq p + 1, \ldots, p + k \): The speculation energy of the sector
\[
E_{\text{sector}} = \sum_{i=1}^{k} E_{p+i} = \sum_{i=1}^{k} \frac{1}{2} (a_{p+i} \dot{x}_{p+i}^2 + b_{p+i} \dot{x}_{p+i}^2)
\]
is decomposed into the sum of 2 parts: the external energy (vis a vis the market) and the internal speculation energy (which accounts for the internal movements in the sector):
\[
\hat{E}_{p+1} = E_{\text{external}} = \frac{1}{2} (\hat{a}_{p+1} \dot{x}_{p+1}^2 + \hat{b}_{p+1} \dot{x}_{p+1}^2) = \frac{1}{2} \left( \sum_{i=1}^{k} \frac{1}{a_{p+i}} \right) + \frac{1}{2} \gamma_{p+1}^2 \left( \sum_{i=1}^{k} \frac{1}{a_{p+i}} \right)^2
\]
and
\[
E_{\text{internal}} = E_{\text{sector}} - E_{\text{external}}\]
\[
= \frac{1}{2} \left[ \sum_{i=1}^{k} \frac{1}{a_{p+i}} \dot{x}_{p+i}^2 - \left( \sum_{i=1}^{k} \frac{1}{a_{p+i}} \right) \left( \sum_{i=1}^{k} \frac{1}{a_{p+i}} \right) \right] + \frac{\gamma_{p+1}^2}{2} \left[ \sum_{i=1}^{k} \frac{1}{a_{p+i}} \dot{x}_{p+i}^2 - \left( \sum_{i=1}^{k} \frac{1}{a_{p+i}} \right) \right]
\]
(The energy of the other components remains the same).

Remark the natural fact that \( E_{\text{internal}} \geq 0 \), and this inequality can be seen as a particular case of the Cauchy-Schwartz inequality
\[
\left( \sum a_{p+i} x_{p+i}^2 \right) \left( \sum \frac{1}{a_{p+i}} \right) \geq \left( \sum \sqrt{a_{p+i} x_{p+i}^2} \sqrt{ \frac{1}{a_{p+i}} } \right)^2 = \left( \sum x_{p+i} \right)^2.
\]

The internal movement (among the components of the sector, but does not affect the total sector mispricing \( \varepsilon_{p+1} = \sum_{i=1}^{k} x_{p+i} \)) is governed by the internal energy function \( E_{\text{internal}} \). This movement is periodic of period \( \frac{2\pi}{\lambda_{p+1}} \) (frequency = \( \lambda_{p+1} \)) and is isomorphic to a synchronous \((k - 1)\)-dimensional harmonic oscillator (i.e. Hamiltonian system with Hamiltonian function \( h = \frac{\lambda_{p+1}}{2} \sum_{i=1}^{k-1} (p_i^2 + q_i^2) \)) on the symplectic space \((R^{2(k-1)}, \omega = \sum_{i=1}^{k-1} dp_i \wedge dq_i)\). This internal movement commutes with the external movement of the
market, which now has \( n - k + 1 \) components \( z_1, \ldots, z_{n-k+1} \) instead of \( n \) components and the new speculation energy function

\[
\hat{E} = \sum_{j=1}^{n-k+1} \hat{E}_j = E_1 + \cdots + E_p + E_{\text{external}} + E_{p+k+1} + \cdots + E_n.
\]

We can reduce the system, from a constrained \( n \)-oscillator to a constrained \( (n-k+1) \)-oscillator, by forgetting about the internal movement in the sector consisting of \( k \) components \( x_{p+1}, \ldots, x_{p+k} \) and considering the whole sector as just one component \( \hat{x}_{p+1} = x_{p+1} + \cdots + x_{p+k} \).

6.4. The stochastic model.

Our stochastic constrained \( n \)-oscillator model of the market will be a perturbation of the deterministic linear constrained \( n \)-oscillator model, which is a proper integrable Hamiltonian systems with \( n-1 \) degrees of freedom. Under a nonlinear perturbation, an integrable systems is no longer integrable in general and may exhibit chaotic behavior. Nevertheless, the KAM (Kolmogorov-Arnold-Moser) theory with Nekhoroshev’s exponential time stability theory (see, e.g., [30]) say that if the perturbation is deterministic and small and the system is non-resonant, then the most solutions of the perturbed system are still quasi-periodic, at least for a very long period of time.

When stochastic terms are added, the situation becomes more complicated. There are elements of KAM theory in the stochastic case (for example, the averaging method with respect to a torus action, see, e.g., [18, 34] and references therein), but as far as we know, a full KAM theory for SDS does not exist yet. Nevertheless, we will assume that most solutions of a reasonable stochastic perturbation of an integrable Hamiltonian system will look similar to solutions of an integrable stochastic dynamical systems, at least for a very long period of time. For practical purposes, here we will be interested only in such solutions. So we will look only at stochastic models which are integrable in the sense of [39], or even more restrictively, which are invariant with respect to a torus action of half the dimension of the phase space, similarly to classical integrable Hamiltonian systems and their Liouville torus actions (see [40]).

In the deterministic linear constrained \( n \)-oscillator model, the general solution has the form:

\[
x_i(t) = \sum_{j=1}^{n-1} c_{ij} z_j(t); \quad i = 1, \ldots, n,
\]

with

\[
z_j(t) = r_j \sin(\lambda_j t + \theta_j); \quad j = 1, \ldots, n - 1,
\]

where \((c_{ij})\) is a constant matrix of linear transformation, \(r_j > 0\) \((j = 1, \ldots, n - 1)\) are action coordinates which do not depend on time (they are first integrals of the system), and \(\lambda_j t + \theta_j\) are angle coordinates which more at constant frequencies \(\lambda_j\). The numbers \((r_j, \theta_j)\) are initial data in the action-angle coordinate system.

In our simplest stochastic model, we will use the same linear transformation matrix \((c_{ij})\) to write \(x_i(t) = \sum_{j=1}^{k-1} c_{ij} z_j(t)\) for \(i = 1, \ldots, n\), and will assume that each \(z_j\) behaves
like a damped stochastic oscillator. The general solution for $z_j$ has the form
\begin{equation}
(40) \quad z_j(t) = r_j(t) \sin(\lambda_j t + \theta_j + S_j(t)),
\end{equation}
where $r_j(t)$ is no longer constant in $t$ but satisfies a stochastic differential equation of the form
\begin{equation}
(41) \quad dr_j(t) = \left( \frac{1}{r_j(t)} - f(r_j(t)) \right) dt + dB^j_t
\end{equation}
(like the one obtained for the 1-degree-of-freedom damped stochastic oscillator in polar action-angle coordinates, see [39]) where $S_j(t)$ is a martingale Ito process whose volatility is inverse proportional to $r_j(t)$: $dS_j(t) = \frac{1}{r_j(t)}dW^j_t$. (Here $B^j_t$ and $W^j_t$ are independent Wiener processes).

We will call the process satisfied by each $r_j$ a \textbf{positive bell-shaped process}, in view of the shape of its stationary density function.

In summary, our stochastic model is as follows:
\begin{equation}
(42) \quad x_i(t) = \sum_{j=1}^{n-1} c_{ij}r_j(t) \sin(\lambda_j t + \theta_j + S_j(t)), \quad i = 1, \ldots, n,
\end{equation}
where:

- $x_i(t)$ is the mispricing of $i$-th component at time $t$,
- $(c_{ij})$ is a constant matrix of linear transformation,
- $r_j(t)$ are independent positive bell-shaped processes $(j = 1, \ldots, n)$,
- $\lambda_j > 0$ are frequencies,
- $\theta_j$ are initial angular values,
- $S_j(t)$ are independent martingale Ito processes whose volatilities are $\frac{1}{r_j(t)}$ respectively, i.e., $dS_j(t) = \frac{1}{r_j(t)}dW^j_t$ $(j = 1, \ldots, n - 1)$.

This model has the following features, which are compatible with observations in the real-world financial markets:
- The whole system is integrable quasiperiodic in stochastic sense, and goes through boom-bust cycles.
- Every component (asset) has the same set of periods, but different periods have different relative importances (the coefficients $c_{ij}$) for different assets.
- At any given time, different momenta corresponding to different periods may be of the same sign, or they may be of opposite signs (i.e. they are counter-trend to each other) resulting in a complicated zig-zag movement of the price (even before the noises).
- The assets can be regrouped into sectors according to their proper frequencies. Each sector has its external motion (vis a vis the whole market) and internal motion (change of relative weights of the stocks in the sector - this internal change is periodic in the stochastic sense and has its own period).
This paper was written during my stay at the School of Mathematical Sciences, Shanghai Jiao Tong University, as a visiting professor. I would like to thank Shanghai Jiao Tong University, the colleagues at the School of Mathematics of this university, and especially Tudor Ratiu, Xiang Zhang, Jianshu Li, and Jie Hu for the invitation, hospitality and excellent working conditions.

I told some of the ideas of this paper to my former student N.T. Thien, who included them in his thesis in 2014. We were supposed to develop them together, but he found a job in a bank right after his thesis and didn’t have time to do more research. Nevertheless, we wrote a paper together [39] which provides some mathematical background for this paper.

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