Cosmological solutions of emergent noncommutative gravity

Daniela Klammer and Harold Steinacker

Fakultät für Physik, Universität Wien.

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Matrix models of Yang-Mills type lead to an emergent gravity theory, which may not require fine-tuning of a cosmological constant. We find cosmological solutions of Friedmann-Robertson-Walker type. They generically have a big bounce, and an early inflation-like phase with graceful exit. The mechanism is purely geometrical, no ad-hoc scalar fields are introduced. The solutions are stabilized through vacuum fluctuations and are thus compatible with quantum mechanics. This leads to a Milne-like universe after inflation, which appears to be in remarkably good agreement with observation and may provide an alternative to standard cosmology.

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Quantum field theory and general relativity provide the basis of our present understanding of fundamental forces and matter. However, there is up to now no satisfactory way to reconcile them in a consistent quantum theory. General arguments based on quantum mechanics and general relativity suggest a “foam-like” or quantum structure at the Planck scale $10^{-33}$ cm. This problem has its most dramatic manifestation in the cosmological constant problem: the small but non-vanishing cosmological constant problem, which is off by at least 60 orders of magnitude. No satisfactory solution of this problem within the currently accepted ΛCDM model is available. Expectations, which are off by at least 60 orders of magnitude in the currently accepted ΛCDM model, are realized as 3+1-dimensional NC “brane” solution of such a quantum space, indicated by $\sim$. Then $X^\mu \sim x^\mu$ is interpreted as quantization of coordinate functions on $\mathcal{M}$, $\phi(x)$ define the embedding of a 4-dimensional submanifold $\mathcal{M} \subset \mathbb{R}^{10}$, and

$$[X^\mu, X^\nu] \sim i\theta^{\mu\nu}(x), \quad \mu, \nu = 1, ..., 4$$

(3)

can be interpreted as Poisson structure on $\mathcal{M}$. The physical fields arise from fluctuations in the matrix model around such a background. Therefore they live only on the brane $\mathcal{M}$, and there is no higher-dimensional “bulk” which could carry any propagating degrees of freedom, unlike in string theory or standard braneworld-scenarios. As shown in [2,3], the effective metric for all scalar, gauge and fermionic fields propagating on $\mathcal{M}$ is given by

$$G^{\mu\nu}(x) = e^{-\sigma} \theta^{\mu\nu}(x)\theta^{\rho\sigma}(x)g_{\rho\sigma}(x), \quad e^{-\sigma} = \rho |g_{\mu\nu}|^{-\frac{1}{2}}, \quad \rho = \sqrt{\det \theta^{-1}}$$

(4)

(5)

where

$$g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab} = \eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi^i \delta_{ij}$$

(6)

is the induced metric on $\mathcal{M}$. $G_{\mu\nu}$ is dynamical, depending on the embedding fields $\phi^i$ and $\theta^{\mu\nu}$. Standard embedding theorems imply that $G_{\mu\nu}$ can describe in principle the most general metric in 4 dimensions. Therefore the matrix model defines a theory of space-time and gravity coupled to gauge fields and matter.

The matrix e.o.m. $[X^a, [X^b, X^c]]\eta_{aa'} = 0$ can be written in a covariant manner as

$$\Delta_G \phi^i = \Delta_G x^\mu = 0.$$ 

(7)

This implies the covariant equation for $\theta^{\mu\nu}$

$$G^\gamma_\eta \nabla_\gamma (e^\sigma \theta^\eta) = e^{-\sigma} G_{\mu\nu} \theta^{\mu\gamma} \partial_\gamma \eta,$$

(8)

where

$$\eta(x) = \frac{1}{4} e^\sigma G^{\mu\nu} g_{\mu\nu}$$

(9)
which relates \( \theta^{\mu
u}(x) \) with the metric \( G^{\mu
u} \). Here \( \nabla \) denotes the Levi-Civita connection with respect to \( G_{\mu
u} \). In principle, \( \theta^{\mu
u} \neq 0 \) breaks (local) Lorentz invariance. However, \( \theta^{\mu
u} \) does not enter explicitly the effective action to leading order, and its presence through higher-order terms may be below experimental limits if the scale of noncommutativity \( \Lambda_{NC} \) is high enough. In fact, this spontaneous breaking of Lorentz invariance leads to massless gravitons as discussed below. Note also that the matrix model defines preferred coordinates \( x^\mu \), which are not observable and not in conflict with observation.

Equations (7) imply that the embedding \( M \subset \mathbb{R}^D \) is harmonic w.r.t. \( G_{\mu\nu} \). A particularly interesting case is given by geometries with

\[
G_{\mu\nu} = g_{\mu\nu}. \tag{10}
\]

It is not hard to see [2] that this holds if and only if \( \eta = e^\phi \), which (in the Euclidean case) is equivalent to \( \star \theta = \pm \theta \) where \( \star \) is the Hodge star. Then (8) simplifies as

\[
\nabla^\eta \theta_{\mu\nu}^{-1} = 0. \tag{11}
\]

These are formally the free Maxwell equations in the background geometry \( G_{\mu\nu} \). In particular, they have propagating massless solutions with 2 physical helicities. However, there are no charged fields under this “would-be” \( U(1) \) gauge field in the matrix model. Rather, these modes turn into the 2 physical degrees of freedom of gravitons. Writing \( \theta_{\mu\nu}^{-1} = \theta_{\mu\nu}^- + F_{\mu\nu} \) on a flat (Moyal-Weyl) background, the metric fluctuations are

\[
h_{\mu\nu} = -G_{\mu\nu} \theta^{\rho\sigma} F_{\rho\sigma} - G_{\mu\nu} \theta^{\rho\sigma} F_{\rho\sigma} - G_{\mu\nu} F_{\rho\sigma} \theta^{\rho\sigma}/2
\]

which are nontrivial and Ricci-flat, \( R_{\mu\nu}[G + h] = 0 \) [2,3].

Quantization and induced gravity: The quantization of the matrix model (1) is defined by

\[
Z = \int dX^a e^{-S_{YM}[X]} \tag{12}
\]

(omitting fermions for simplicity). Since one cannot simply add an explicit Einstein-Hilbert term, the model is highly predictive. The cosmological solutions given below add to the evidence that it may provide a (near-?) realistic theory of gravity, with great advantages for the cosmological constant problem.

Consider a perturbative quantization of the matrix model around a given background as discussed above. Since all fields couple to \( G_{\mu\nu} \), standard considerations imply that in the classical-geometric limit, the effective action at one-loop can be obtained from Seeley-de Witt coefficients, so that

\[
\Gamma_{1\text{-loop}} = \frac{1}{16\pi^2} \int d^4x \sqrt{|G|} \left( c_1 \Lambda_1^4 + c_4 R[G] \Lambda_2^4 + O(\ln \Lambda) \right). \tag{13}
\]

The coefficients \( c_i \) as well as the effective cutoffs \( \Lambda_i \) depend on the detailed field content of the model, cf. [4]. This is essentially the mechanism of induced gravity.

Now consider the equations of motion for the geometry, taking into account the quantum contribution (13). Remarkably, equation (8) for \( \theta^{\mu\nu} \) is unchanged: it is a direct consequence of Noether’s theorem due to the symmetry \( X^a \to X^a + c^a \mathbb{1} \), and therefore protected from quantum corrections [2]. However, the equation (7) for the embedding \( \phi^i \) is modified at one loop:

Let us focus on the first term \( \int d^4x \sqrt{G} \Lambda^4 \), which is essentially the vacuum energy due to zero-point fluctuations. In GR, it amounts to a huge contribution \( \sim \Lambda^4 \) to the cosmological constant, which must be canceled by an extremely fine-tuned bare cosmological constant in order to reproduce the small value \( O(\text{meV}) \) in the LCDM model. This is the well-known cosmological constant problem, which persists even in models with \( TeV \) scale supersymmetry.

We claim that this problem is resolved here. To see this, note that

\[
|G_{\mu\nu}(x)| = |g_{\mu\nu}(x)| \tag{14}
\]

independent of \( \theta^{\mu\nu}(x) \). Hence the variation of the vacuum energy term

\[
\delta \int d^4x \sqrt{G} \sim \int d^4x \sqrt{g} \theta^{\mu\nu} \delta g_{\mu\nu} \sim \int d^4x \sqrt{g} \theta^{\mu\nu} \phi^i \Delta g \phi^i \delta_{ij}
\]

vanishes for harmonic embeddings

\[
\Delta g \phi^i = 0. \tag{15}
\]

Therefore the term \( \int d^4x \sqrt{G} \Lambda^4 \) has a different physical meaning here: It should not be interpreted as a cosmological constant, but as a brane tension. In particular for harmonically embedded branes (i.e. minimal surfaces), the precise value of its coefficient \( \sim \Lambda^4 \) is irrelevant and drops out from the equations of motion. For example, flat Moyal-Weyl space is a solution even at one loop, without fine-tuning \( \Lambda \). Thus minimally or harmonically embedded branes (w.r.t. \( g_{\mu\nu} \)) are protected from the cosmological constant problem [3]; they are in fact stabilized through the vacuum energy. The crucial difference to general relativity is the parametrization of the geometry in terms of “tangential” \( \theta^{\mu\nu} \) and “transversal” \( \phi^i \) rather than a fundamental metric. Gravitons originate from fluctuations of \( \theta^{\mu\nu} \) and are also blind to \( \phi^i \).

Combining (7) and (15), we see that harmonic embeddings with \( g_{\mu\nu} = G_{\mu\nu} \) solve the vacuum e.o.m. of the matrix-model, including quantum corrections [1]. The presence of matter will lead to deviations from harmonic embedding, however these corrections are suppressed by factors \( O(\rho) \) where \( \rho \) is the energy density. Thus \( \Delta g \phi^i = 0 \) is valid as long as the brane-tension \( \sim \Lambda^4 \) dominates the energy density of matter and curvature.

Robertson-Walker geometries: Now consider cosmological solutions of FRW type, and assume that \( \Lambda_1 = O(\text{TeV}) \) to be specific. The Friedmann equations are replaced essentially by the requirement of minimal embedding (15) with \( g_{\mu\nu} = G_{\mu\nu} \). This should be valid to
This can be brought into FRW form (16) using a change of variables $\frac{dx}{dt} = \sqrt{c(t)}$. Equivalently, we can choose $t$ to be the “proper” time variable $\tau$, such that $a(t)$ is the usual FRW scale parameter; then $c(t) = 1$, and

$$\frac{dx_c}{d\tau} = x_c', \quad x_c^2 - a^2 \psi^2 - \dot{a}^2 = k.$$  

Note that the matrix coordinates and in particular $x_c$ have no physical meaning from the brane point of view; they are determined by the requirement of harmonic embedding $\Delta_g x^a = 0$. Due to the symmetry, it is enough to show that

$$0 = \Delta_g (R(t) S(\chi) \cos \theta)$$

and

$$0 = \Delta_g x_c.$$  

This leads to

$$3 \frac{1}{a} (\dot{a}^2 + k) + \ddot{a} - \dot{\psi}^2 a = 0$$

$$5 \ddot{\psi} a + \dot{\psi} a = 0$$

$$3 \frac{1}{a} \dot{a} x_c + \dot{x}_c = 0.$$  

These equations can be integrated as follows:

$$(\dot{a}^2 + k) a^6 + b^2 a^{-2} = m = \text{const}$$

$$\dot{\psi} = b a^{-5}, \quad b = \text{const} > 0$$

$$a^3 \dot{x}_c = d = \text{const};$$  

the last equation is in fact a consequence of (20), which gives $d = \sqrt{m}$ and hence $m > 0$. This leads to

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{b^2}{a^2} a^{10} + ma^{-8} - \frac{k}{a^2}.$$  

$$\ddot{a} = -3ma^{-8} + 4b^2 a^{-10}.$$  

For small $a$ and $b \neq 0$, we find $\ddot{a} \gg 0$ which signals inflation. For large $a$, it follows that $\ddot{a} < 0$, $\ddot{a} \to 0$. The case $b = 0$ was obtained before in [9].

For $k = 0$, one finds an unrealistic age of the universe of order $4.5 \cdot 10^9$ years (assuming small $b$), hence we will not pursue this case any further. It turns out that $k = -1$ is the (near-) realistic case as discussed below.

To complete the solution of emergent gravity we need to find a Poisson structure $\theta^{\mu \nu}$ which satisfies (11) such that $G_{\mu \nu} = g_{\mu \nu}$. Clearly there exists no homogeneous and isotropic non-degenerate $\theta^{\mu \nu}$. However, $e^{-\sigma}$ should at least be spatially homogeneous, because it determines the nonabelian gauge coupling [2]. Consider $k = -1$.

We introduce $\tilde{t}(t)$ through $\frac{dt}{\tilde{t}} = \frac{dt}{t}$, and write the FRW metric in the form

$$ds^2 = \frac{a^2}{t^2} (-d\tau^2 + dr^2 + r^2 d\Omega^2)$$  

where $\tau = \tilde{t} \cos(\chi)$, $r = \tilde{t} \sinh(\chi)$. In particular, for $a(t) = t$ we recover the well-known fact that the Milne universe is flat. Then the (complexified) symplectic form $\theta^{-1} = \theta^{-1}_{\mu \nu} dx^\mu \wedge dx^\nu = i d\tau \wedge dx_1 + dx_2 \wedge dx_3$ (29) in these flat coordinates is closed and (i-) self-dual, hence (11) is satisfied; recall that this equation applies even at the quantum level. Moreover we obtain $G_{\mu \nu} = g_{\mu \nu}$, and

$$|g_{\mu \nu}| = \frac{a^8}{t^8}, \quad |\theta^{-1}| = 1, \quad e^{-\sigma} = \frac{\dot{t}^4}{a^2}.$$  

FIG. 1: Embedding of universe in $\mathbb{R}^{10}$, for $b = 1, m = 5$
In particular, $e^{-\sigma} \to 1$ for large $t$ as the geometry approaches that of a Milne universe. Hence the gauge coupling is approximately constant for large $t$, but has a non-trivial time evolution in the early universe.

**Emergent cosmology:** The cases $k = 0, +1$ imply a too short age of the universe given the present Hubble parameter. Therefore we focus on the case $k = -1$. Then $\dot{a} \to 1$ for large $a$, and the deceleration parameter $q = -\frac{\ddot{a}}{\dot{a}} \to 0$. For large $t$, the time evolution approaches that of a Milne universe, which is in remarkably good agreement with observation [11]. The age of the universe is found to be $\frac{t_0}{a_0} \approx 13.9 \cdot 10^9$ years, the time evolution of $a(t)$ in the $\Lambda$CDM model at present being tangent with the evolution in the Milne universe [10]. Moreover, the main observational constraints including the acoustic peak in the CMB background and the type Ia supernovae data appear to be consistent with an interpretation in terms of a Milne Universe [11]. While this geometry is excluded within GR, it makes perfect sense within emergent NC gravity. In view of the unreasonable fine-tunings in the presently favored $\Lambda$CDM model, this certainly deserves a more detailed investigation.

**Inflation and big bounce:** The scaling parameter is determined by

$$\dot{a} = \sqrt{-b^2a^{-8} + ma^{-6} + 1}. \quad (31)$$

For $b \neq 0$, denote with $a_0$ the (positive) root of the argument, which is the minimal “size” of the universe. We fix the origin of time by $a(0) = a_0$, and define $t_1$ by

$$\dot{a} = 0 \quad \iff \quad a(t_1) = \sqrt{\frac{4b^2}{3m}} = a_1 \quad (32)$$

using (27). Expanding $1 + m a^{-6} - b^2 a^{-8} = p(a - a_0) + \ldots$ around $a_0$ where $p = \frac{\partial}{\partial a}\left|_{a=a_0} \frac{d(1 + ma^{-6} - b^2 a^{-8})}{da}\right.$ shows an inflation-like phase

$$\dot{a} \sim \sqrt{p(a - a_0)}, \quad a(t) \sim \frac{p}{4} t^2 + a_0,$$

which ends at $a(t_1) = \sqrt{\frac{4b^2}{3m}}$, where $\ddot{a} = 0$ ("graceful exit"). A typical evolution of $a(t)$ and the corresponding Hubble parameter is shown in figure 2. Assuming $m^2 < b^3$, we have approximately

$$a_0 \sim b^{1/4}, \quad a(t_1) = a_0 \sqrt{\frac{4}{3} \frac{b^{3/4}}{\sqrt{m}}}.$$

Note that the requirements for “successful inflation” such as a large number of $e$-foldings will be greatly relaxed here compared with standard cosmology; this should be addressed elsewhere. For $b > 0$, it is obvious that the time evolution should in fact not start at $t = 0$, but be completed symmetrically as

$$a(-t) = a(t), \quad \psi(-t) = -\psi(t), \quad x_c(-t) = -x_c(t)$$

corresponding to a “big bounce” rather than a big bang.

![FIG. 2: Evolution of $a(t)$ and $H(t)$ for $m = 5, b = 1$.](image)

We conclude that Yang-Mills matrix models admit cosmological solutions which are in remarkably good agreement with observation. The type Ia supernovae data are accommodated without any fine-tuning and without introducing dark energy. The solutions should be valid as long as the quantum-mechanical vacuum energy dominates the energy density due to matter or radiation, i.e. up to epochs with TeV-range temperature in the case of TeV-scale supersymmetry. Some modifications could arise from compactification in extra dimensions (leading to interesting low-energy gauge groups, effective scalar fields & potentials as in [12]), or soft SUSY breaking terms (e.g. a mass term) in the matrix model. However it is unlikely that this would change our main conclusion, which is a Milne-like evolution after a big bounce and an inflation-like phase. If confirmed this would resolve the cosmological constant problem.

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[8] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B 498 (1997) 467 [arXiv:hep-th/9612115].
[9] B. Nielsen, Jour. Geom. Phys. 4, no. 1, 1 (1987)
[10] M. Kutschera and M. Dyrda, Acta Phys. Polon. B 38 (2007) 215 [arXiv:astro-ph/0605175].
[11] A. Benoit-Levy and G. Chardin, arXiv:0903.2446;
G. Sethi, A. Dev and D. Jain, Phys. Lett. B 624 (2005) 135 [arXiv:astro-ph/0506255].
[12] P. Aschieri, T. Grammatikopoulos, H. Steinacker and G. Zoupanos, JHEP 0609 (2006) 026 [arXiv:hep-th/0606021].