Static Spherically Symmetric Solutions to modified Hořava-Lifshitz Gravity with Projectability Condition

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Abstract

In this paper we seek static spherically symmetric solutions of Hořava-Lifshitz-like gravity with projectability condition. We consider the most general form of gravity action without detailed balance, and require the spacetime metric to respect the projectability condition. We find that for any value of $\lambda$, it may exists the solutions of topology $\mathbb{R} \times M_3$, where $\mathbb{R}$ is the time direction and $M_3$ is a three-dimensional maximally symmetric space depending on the value of cosmological constant and the potential of the action. Besides, in the UV region where $\lambda \neq 1$, we find Minkowski or de-Sitter space-time as the solution, while in the IR region where $\lambda = 1$, we prove that (dS-)Schwarzschild solution is the only nontrivial solution. We also notice that the other static spherically symmetric solutions found in the literature do not satisfy the projectability condition and are not the solutions we get. Our study shows that in Hořava-Lifshitz gravity with projectability condition, there is no novel correction to Einstein’s general relativity in solar system tests.

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Diffeomorphism is an essential symmetry of Einstein’s relativity theory of gravity. It has been widely believed to be exact in any theory of gravity. However, in the recent proposal by Hořava[1, 2] on gravity theory, it is no longer an exact symmetry. The basic idea behind Hořava’s theory is that time and space may have different dynamical scaling in UV limit. This was inspired by the development in quantum critical phenomena in condensed matter physics, with the typical model being Lifshitz scalar field theory[3, 4]. In this Hořava-Lifshitz theory, time and space will take different scaling behavior as

$$x \rightarrow b^z x, \quad t \rightarrow b^z t,$$

where $z$ is the dynamical critical exponent characterizing the anisotropy between space and time. Due to the anisotropy, instead of diffeomorphism, we have the so-called foliation-preserving diffeomorphism. The transformation is now just

$$t \rightarrow \tilde{t}(t), \quad \tilde{x}^i \rightarrow \tilde{x}^i(x^j, t).$$

As a result, there is one more dynamical degree of freedom in Hořava-Lifshitz-like gravity than in the usual general relativity. Such a degree of freedom could play important role in UV physics, especially in early cosmology[5, 6]. At IR, due to the emergence of new gauge symmetry, this degree of freedom is not dynamical any more such that the kinetic part of the theory recovers the one of the general relativity.

Since time direction plays a privileged role in the whole construction, it is more convenient to work with ADM metric

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N_i dt)(dx^j + N_j dt),$$

in which $N$ and $N_i$ are called “lapse” and “shift” variables respectively. Then we have the following transformations on the metric components:

$$\delta g_{ij} = \partial_i \xi^k g_{jk} + \partial_j \xi^k g_{ik} + \xi^k \partial_k g_{ij} + \xi^0 \dot{g}_{ij},$$
$$\delta N_i = \partial_i \xi^j N_j + \xi^j \partial_j N_i + \dot{\xi}^j g_{ij} + \dot{\xi}^0 N_i + \dot{\xi}^j \dot{N}_i,$$
$$\delta N = \xi^j \partial_j N + \dot{\xi}^0 N + \dot{\xi}^0 \dot{N}.$$  

It seems natural to choose the lapse function $N$ to be projectable function on the spacetime foliation, i.e. only a function of $t$. Such a choice makes the above gauge transformations simpler and more transparent. More importantly, with the projectable condition, in the Hamiltonian formulation the constraints could form a closed algebra [1] since the momentum conjugate to $N$ does not lead to a local constraint. On the contrary, if the projectable condition on $N$ is abandoned, then the theory would not be well-defined, as shown in [1, 7]. Therefore in this letter, we will focus on the case with the projectable condition.

Taken Hořava-Lifshitz gravity as a new gravitational theory, it is an important issue to study its static spherically symmetric solutions. This issue has been widely studied in the literature, see [8–13]. In these papers, for example [8, 12], it was assumed that the metric of the black solutions had the following form

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{g(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$
From this metric ansatz, it was found that there were new spherically symmetric solutions, even at IR. For example, in [10], based on a modified Hořava-Lifshitz-type action, an asymptotically flat solution with
\[ g = N^2 = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)} \]  
was found. This raised the issue that if there is any observational effect in solar system tests[14]. However, in the above ansatz (5) the “lapse function” \( N(r) \) obviously breaks the “projectability condition”. As the Hořava gravity is only well defined when the “projectability condition” is preserved, this naturally leads one to ask whether the above new solutions still are the solutions of Hořava-Lifshitz gravity with the projectability condition after proper coordinates transformation? The answer to this question is not obvious, considering the freedom in doing coordinate transformation. For instance, a static spherically symmetric solution in the flat spacetime could be represented in Schwarzschild coordinates as
\[ ds^2 = -(1 - \frac{2GM}{r})dt_S^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  
which looks against the projectability condition. By a transformation into the Painlevé-Gullstrand coordinates[16–19]
\[ dt_S = dt_{PG} \pm \frac{\sqrt{2GM/r}}{1 - 2GM/r} dr, \]  
the solution (7) becomes
\[ ds^2 = -dt^2_{PG} + (dr \pm \sqrt{2GM/r} dt_{PG})^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]  
Comparing with the ADM metric (3), we find that the “lapse function” \( N = 1 \), which is in accord with the “projectability condition”.

Furthermore, we would like to know if there are any other new solutions, especially at IR, which may have significant physical implication in IR physics. Therefore, in this letter, we study the static spherically symmetric solutions to modified Hořava-Lifshitz gravity with the projectability condition. We consider the most general form of the action without the detailed balance condition. We find that for any value of \( \lambda \), if the potential term is properly chosen, there may exist the solutions of topology \( \mathbb{R} \times M^3 \), where \( \mathbb{R} \) is the time direction and \( M^3 \) is a three-dimensional maximally symmetric space. In the case without the cosmological constant in the action, \( M^3 \) is just the flat spacetime. In the case with the cosmological constant, \( M^3 \) could be a three-dimensional sphere \( S^3 \) or hyperboloid \( H^3 \), depending on the potential. Moreover, apart from these solutions, in the UV region where \( \lambda \neq 1 \), we find either de-Sitter space-time or Minkowski spacetime, up to the cosmological constant, while in the IR region where \( \lambda = 1 \), we prove that the \( (dS) \)-Schwarzschild solution is the only nontrivial solution. This result seems in accordance with [21]. We also notice that the other static spherically symmetric solutions found in the literature do not satisfy the projectability condition and are not the solutions we want. Our study shows that in Hořava-Lifshitz-like Gravity with the projectability condition, there is no novel correction to Einstein’s general relativity in solar system tests.

We study the topological static spherically symmetric solutions in the Hořava-Lifshitz-like gravity as well. We choose the metric ansatz in which \( d\Omega_k^2 \) denotes the line element for an 2-dimensional Einstein space with constant scalar curvature \( 2k \). Without loss of generality, one may take \( k = 0, \pm 1 \) respectively. The \( k = 1 \) case has been discussed above. To \( k = -1 \) case, we find that it may also exist the solutions of topology \( \mathbb{R} \times M^3 \) for all \( \lambda \). In the UV region where \( \lambda \neq 1 \), the
only possible solution is either Minkowski or de-Sitter space-time with topological twist. In the IR region where $\lambda = 1$, the Schwarzschild topological black hole is the only nontrivial solution. For the case $k = 0$, there is not a Schwarzschild solution at IR or de-sitter space-time in the UV region because $f$ can’t be zero.

II. THE MODIFIED HOŘAVA-LIFSHITZ GRAVITY

In this section, we give a brief review of Hořava-Lifshitz gravity and its modifications. Using the ADM formalism, the action of this Hořava-Lifshitz gravitational theory is given by \[ S = \int dtd^{3}x(L_{K} + L_{V}), \]

\[ L_{K} = \sqrt{gN}\left\{ \frac{2}{\kappa^{2}}(K_{ij}K^{ij} - \lambda K^{2})\right\}, \]

\[ L_{V} = \sqrt{gN}\left\{ \frac{k^{2} \mu^{2}(\Lambda_{W}R - 3\Lambda_{W}^{2})}{8(1 - 3\lambda)} + \frac{k^{2} \mu^{2}(1 - 4\lambda)}{32(1 - 3\lambda)} R^{2} \right. \]

\[ \left. - \frac{k^{2}}{2\omega^{2}}(C^{ij} - \frac{\mu \omega}{2}R^{ij})(C^{ij} - \frac{\mu \omega}{2}R^{ij})\right\}, \] \[ (10) \]

where $L_{K}$ is the kinetic term and $L_{V}$ is the potential term. In the action, $\lambda, \kappa, \mu, \omega$ and $\Lambda_{W}$ are the coupling parameters, and $C^{ij}$ is the Cotton tensor defined by

\[ C^{ij} = \epsilon^{ijk}\nabla_{k}\left( R^{j}_{i} - \frac{1}{4}R\delta^{j}_{i}\right). \] \[ (11) \]

The study of the perturbations around the Minkowski vacuum shows that there is ghost excitation when $\frac{1}{3} < \lambda < 1$. This indicates that the theory is only well-defined in the region $\lambda \leq \frac{1}{3}$ and $\lambda \geq 1$. Since the theory should be RG flow to IR with $\lambda = 1$, we expect that at UV, $\lambda > 1$ to have a well-defined RG flow. At IR, $\lambda = 1$, the kinetic term recovers the one of standard general relativity. Comparing to the action of the general relativity in the ADM formalism, the speed of light, the Newton’s constant and the cosmological constant emerge as

\[ c = \frac{k^{2} \mu}{4} \sqrt{\frac{\Lambda_{W}}{1 - 3\lambda}}, \quad G = \frac{k^{2}}{32\pi c}, \quad \Lambda = \frac{3}{2}\Lambda_{W}. \] \[ (12) \]

It follows from \[ (12) \] that for $\lambda > 1/3$, the cosmological constant $\Lambda_{W}$ has to be negative. It was noticed in \[ [8] \] that if we make an analytic continuation of the parameters

\[ \mu \rightarrow i\mu, \quad \omega^{2} \rightarrow -i\omega^{2}, \] \[ (13) \]

the four-dimensional action remains real. In this case, the emergent speed of light becomes

\[ c = \frac{k^{2} \mu}{4} \sqrt{\frac{\Lambda_{W}}{3\lambda - 1}}. \] \[ (14) \]

The requirement that this speed be real implies that $\Lambda_{W}$ must be positive for $\lambda > \frac{1}{3}$. 

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One important feature of original Hořava-Lifshitz gravity is that it respects the so-called "detailed balance" condition[1, 2]. However, it turns out that the detailed balance condition is not essential to the theory. It could be just a nice way to organize the action. If abandoning ‘detailed balance” and just requiring the model to be power-counting renormalizable, we find that the most general form of the action is of the form [15]

\[ S = \int dt d^3x (L_K + L_V), \]

\[ L_K = \sqrt{g} N \left\{ g_K (K_{ij} R^{ij} - \lambda K^2) \right\}, \]

\[ L_V = \sqrt{g} N \left\{ -g_0 \xi^2 + g_1 \xi^4 R + g_2 \xi^2 R^2 + g_3 \xi^2 R_{ij} R^{ij} + g_4 R^3 + g_5 R (R_{ij} R^{ij}) + g_6 R^i_j R^j_i + g_7 R \nabla^2 R + g_8 \nabla_i R_j \nabla^i R^{jk} \right\}. \] (15)

where \( \xi \) is a suitable factor to ensure the couplings \( g_a \) are all dimensionless. From anisotropic scaling counting, five of these operators are marginal(renormalizable) and four are relevant(superrrenormalizable). And we can rescale the time and space coordinates to set both \( g_K \rightarrow 1 \) and \( g_1 \rightarrow 1 \) without loss of generality. In the following, we will study the static spherically symmetric solution to the action (15).

### III. STATIC SPHERICALLY SYMMETRIC SOLUTIONS

The static spherically symmetric solutions of Hořava-Lifshitz gravity have been discussed by [8–12]. In these paper, it was assumed that the metric of the solutions took the form (5). Consequently, some new kinds of solutions have been found. For the Horava’s original model, three types of solutions were found in [8]. The first one is given by

\[ g = 1 + x^2, \quad x = \sqrt{-\Lambda_W} r, \] (16)

without any restriction on the function \( N(r) \). This is valid for all \( \lambda \). And the other two solutions are given by

\[ g = 1 + x^2 - \alpha x^{2/3} r^{-2}, \quad N = x^{-1/2} r^{1/2} \] (17)

where \( \alpha \) is an integration constant. For the solution to be real, it is necessary that \( \lambda > 1/3 \).

In paper [12], Park got a more general solution in the IR region when \( \lambda = 1 \), basing on an action softly breaking the detailed balance condition

\[ N^2 = g = 1 + (\omega - \Lambda_W) r^2 - \sqrt{r} \left[ \omega (\omega - 2\Lambda_W) r^3 + \beta \right]. \] (18)

Certainly, for a general form of the action like (15), it may exists other kinds of solution with the metric ansatz (5).

For the metric of the form (5), we can work in the Painlevé-Gullstrand coordinates by making a transformation

\[ dt_S = dt_{PG} - \sqrt{1 - N^2} dr. \] (19)

Then the ansatz (5) becomes

\[ ds^2 = -dt_{PG}^2 + (dr + \sqrt{1 - N^2} dt_{PG})^2 + \left( \frac{1}{g} - \frac{1}{N^2} \right) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \] (20)
Comparing with the ADM metric, we find that \( N(t_{PG}) = 1 \) and if

\[
g = N^2, \tag{21}
\]

we reach (3). So the solutions (17) of paper [8] can not preserve the “projectability condition” after the coordinate transformation. And it seems that the solution (18) could preserve the “projectability condition” after the coordinate transformation. However note that (21) is only a necessary condition but not a sufficient condition. Actually from the study below, we will see that (18) could not satisfy the “projectability condition” neither.

We now seek the static, spherically symmetric solutions with the metric ansatz

\[
ds^2 = -N(t)^2 dt^2 + \frac{1}{f(r)} (dr + N'dt)(dr + N'dt) + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{22}
\]

By the coordinate transformation \( dt = dt_s + \frac{\mathcal{N}}{N^2 - f N_r} dr \), we can transform the metric ansatz to the Schwarzschild coordinates type,

\[
ds^2 = -(N^2 - f N_r^2) dt_s^2 + \frac{N^2}{f(N^2 - f N_r^2)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{23}
\]

Substituting the metric ansatz (22) into the Lagrangian (15), up to an overall scaling constant, we get

\[
\mathcal{L}_K = \frac{1}{\sqrt{f} N(t)} \left\{ (1 - \lambda) r^2 f^2 \left( N'_r + N_r \frac{f'}{2f} \right)^2 + 2(1 - 2\lambda) f^2 N_r^2 \right. \right.
\]

\[
-4\lambda r f^2 N_r \left( N'_r + N_r \frac{f'}{2f} \right) \left\},
\]

\[
\mathcal{L}_V = \frac{1}{\sqrt{f}} N(t) r^2 \left\{ -g_0 \xi^6 + \xi^4 \left[ \frac{2(1 - f)}{r^2} - \frac{2 f'}{r} \right] + g_2 \xi^2 \left[ \frac{2(1 - f)}{r^2} - \frac{2 f'}{r} \right]^2 \right.
\]

\[
+ g_3 \xi^2 \left[ \frac{f'^2}{r^2} + 2 \frac{f'}{r^4} (1 - f - \frac{r}{2} f') \right] + g_4 \left[ \frac{2(1 - f)}{r^2} - \frac{2 f'}{r} \right]^3 \right.
\]

\[
+ g_5 \left[ \frac{2(1 - f)}{r^2} - \frac{2 f'}{r} \right] \left[ \frac{f'^2}{r^2} + \frac{2}{r^4} (1 - f - \frac{r}{2} f') \right] \right.
\]

\[
+ g_6 \left[ \frac{f'^3}{r^3} + \frac{2}{r^6} (1 - f - \frac{r}{2} f') \right] + g_7 \left[ \frac{2(1 - f)}{r^2} - \frac{2 f'}{r} \right] \sqrt{f} \partial_r \left\{ \frac{1}{\sqrt{f}} r^2 f \partial_r \left[ \frac{2(1 - f)}{r^2} - \frac{2 f'}{r} \right] \right\}
\]

\[
+ g_8 \left[ f^3 \left( \frac{f'}{r^2 f} - \frac{f''}{rf} \right)^2 + \frac{2 f}{r^4} \left( \frac{f'}{2} + \frac{rf'}{2} + \frac{2(1 - f)}{r} \right)^2 \right]. \tag{24}
\]

Here \( N_r = N'/f \) and \( ' \) means the derivative with respect to \( r \). The full Lagrangian is \( \mathcal{L} = \mathcal{L}_K + \mathcal{L}_V \). By varying the action with respect to the functions \( N_r, f \) and \( N(t) \), we obtain three equations of
motions,

\[
0 = \sqrt{f} \left\{ \partial_r \left[ \frac{\partial L}{\partial N_r'} - \frac{\partial L}{\partial N_r} \right] \right\} = 2(1 - \lambda) r^2 f^2 \frac{1}{N(t)} \left\{ N_r'' + \frac{f''}{2 f} N_r + \frac{3 f'}{2 f} N_r' + 2 \frac{N_r}{r} + \frac{1 - 2 \lambda f}{1 - \lambda f} \frac{N_r}{r} - 2 \frac{N_r}{r^2} \right\}, \tag{25}
\]

\[
0 = \sqrt{f} \left\{ \partial_r \left[ \frac{\partial L}{\partial f'} - \frac{\partial L}{\partial f} - \partial_r \partial_r \frac{\partial L}{\partial f'} \right] \right\} = \sqrt{f} \left\{ \partial_r \left[ \frac{\partial L}{\partial f'} - \partial_r \frac{\partial L}{\partial f'} \right] \right\} - \frac{f'}{2 f} \frac{1}{N(t)} \left\{ (1 - \lambda) r^2 f N_r' \left( N_r' + N_r \frac{f'}{2 f} \right) - 2 \lambda r f N_r^2 \right\} + \frac{1}{N(t)} \left\{ (1 - \lambda) r^2 f N_r N_r'' + \frac{1}{2} (1 - \lambda) r^2 f N_r^2 \right\} + 2(1 + \lambda) r f N_r N_r' + (1 - \lambda) r f N_r^2 + (6 \lambda - 4) f N_r \right\} + \frac{1}{2 \sqrt{f}} L_K, \tag{26}
\]

\[
0 = \int_0^\infty dr r^2 \frac{1}{N(t)} (-L_K + L_V). \tag{27}
\]

The third equation (27) is a spatially integrated Hamiltonian constraint because of the “projectability condition” on the lapse function \( N(t) \). We find that for all \( \lambda, N_r = 0 \) is the solution of the equation (25). In this case, the equations (26, 27) are the equations depending on the form of the potential. We can make ansatz \( f(r) = 1 + yr^2 \), where \( y \) is a constant to be determined. Then we have two cubic equations of \( y \)

\[
\begin{align*}
g_0 \xi^6 + 6 \xi^4 y + 4(3 g_2 + g_3) \xi^2 y^2 - 24(9 g_4 + 3 g_5 + g_6)y^3 & = 0, \tag{28} \\
g_0 \xi^6 + 6 \xi^4 y - 12(3 g_2 + g_3) \xi^2 y^2 + 24(9 g_4 + 3 g_5 + g_6)y^3 & = 0. \tag{29}
\end{align*}
\]

Here the equation (29) is from the non-local Hamiltonian constraint.

For the solution \( f = 1 + yr^2 \), the metric now has the form

\[
ds^2 = -dt^2 + \frac{dr^2}{1 + yr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{30}
\]

Such a metric describes a spacetime of topology \( \mathbb{R} \times \mathbb{S}_3 \), where \( \mathbb{S}_3 \) is a three-dimensional maximally symmetric space, could be a flat space, a sphere or a hyperboloid. If \( y = 0 \), this is just the flat spacetime. If \( y < 0 \), the spacetime is \( \mathbb{R} \times \mathbb{S}^3 \), where \( \mathbb{R} \) is the time direction, \( \mathbb{S}^3 \) is the three-sphere. If \( y > 0 \), the spacetime is \( \mathbb{R} \times \mathbb{H}^3 \), where \( \mathbb{H}^3 \) is the three-dimensional hyperboloid with negative constant curvature. In fact, if one considers the time-dependent solution, then the latter two solutions are very similar to closed and open universe with a constant scale factor.

For a general potential, there is no solution to (28) and (29). When \( \xi = 1, g_0 = 2\Lambda, g_2 = g_3 = g_4 = g_5 = g_6 = g_7 = g_8 = 0 \), it recovers Einstein’s general relativity. The only possible solution requires \( g_0 = 0 \) and \( y = 0 \), which corresponds to a flat spacetime. Actually, when the cosmological constant is vanishing, the flat Minkowski spacetime corresponding to \( y = 0 \) is always a solution.

For the original Hořava-Lifshitz gravity with the action (10), the equations (28), (29) become

\[
\begin{align*}
y^2 - 2\Lambda y - 3\Lambda_w^2 & = 0, \tag{31} \\
y^2 + 2\Lambda y + \Lambda_w^2 & = 0. \tag{32}
\end{align*}
\]
The solution is \( y = -\Lambda_w \). In this case, the curvature of maximally symmetric space is determined by the cosmological constant of the theory.

For the general action of modified Hořava-Lifshitz gravity, the existence of the solution depends on the form of the potential. It is easy to see that the equations (28), (29) could be reduced to two equations both quadratic in \( y \). It is straightforward to find the condition under which there exist a solution.

In the IR region, the modified Hořava-Lifshitz gravity recovers the Einstein’s general relativity except the higher derivative terms on the spatial metric. When \( \lambda = 1 \), the equation (25) becomes

\[
\frac{f'}{f} N_r = 0. \tag{33}
\]

Its solutions are \( N_r = 0 \) or \( f = \text{constant} \). The solution \( N_r = 0 \) has been discussed above. When \( f \) is a constant, the equations (26), (27) become

\[
0 = (N_r^2)'' + \frac{N_r^2}{r} + \frac{N(t)^2}{2f^2} \left\{ -g_0\xi^6 r + \frac{2\xi^4(1 - f)}{r} + \frac{2\xi^2(1 - f)}{r^3} \left[ 2g_2(1 + 7f) + g_3(1 + 5f) \right] \right.
\]
\[
\left. + \frac{2(1 - f)^2}{r^5} \left[ 4g_4(1 + 23f) + 2g_5(1 + 17f) + g_6(1 + 14f) \right] \right.
\]
\[
\left. + \frac{8f(1 - f)}{r^5} \left[ 2g_7(1 + 7f) + g_8(1 - 4f) \right] \right\}, \tag{34}
\]

\[
0 = \int_0^\infty \! dr (N_r^2)'' + \frac{N_r^2}{r} + \frac{N(t)^2}{2f^2} \left\{ -g_0\xi^6 r + \frac{2\xi^4(1 - f)}{r} + \frac{2\xi^2(1 - f)^2}{r^3} (2g_2 + 4g_3) \right.
\]
\[
\left. + \frac{2(1 - f)^3}{r^5} (4g_4 + 2g_5 + g_6) + \frac{8f(1 - f)^2}{r^5} (g_7 + g_8) \right\}. \tag{35}
\]

It is not hard to find that just when \( f = 1 \) the two equations have the same solutions of \( N_r \). In other words, \( f \) is constrained to be 1. In this case, the solutions are just

\[
N_r = \pm N(t) \sqrt{\frac{g_0\xi^6}{6}r^2 + \frac{M}{r}}, \tag{36}
\]

where \( M \) is an integration constant. For \( N_r \) is just the function of \( r \), \( N(t) \) must be a constant. We could use the freedom of gauge transformation to set \( N(t) = 1 \). If let \( g_0\xi^6 = 3\Lambda_w \), the solution (36) corresponds to a dS-Schwarzschild spacetime written in Painlevé-Gullstrand type coordinates. The solution is just determined by the kinetic term and the cosmological constant in the potential. In other words, at IR, the static spherically symmetric solutions of the modified Hořava-Lifshitz gravity are the same as the ones in the Einstein’s general relativity. If the theory has a nonvanishing cosmological constant, the solution is the Schwarzschild solution in dS spacetime. If the theory has no cosmological constant, the solution is just the Schwarzschild solution.

In the UV region when \( \lambda \neq 1 \), similar to the discussion in the IR region, the equations (25), (26) and (27) have solutions just when \( f = 1 \). In this case, they become

\[
0 = N_r'' + \frac{2N_r'}{r} - \frac{2N_r}{r^2}, \tag{37}
\]

\[
0 = (1 - \lambda)r^2 N_r'' - 4\lambda r N_r N_r' + 2(1 - 2\lambda)N_r^2 + g_0N(t)^2\xi^6 r^2, \tag{38}
\]

\[
0 = \int_0^\infty \! dr r^2 \left\{ (1 - \lambda)r^2 N_r'' - 4\lambda r N_r N_r' + 2(1 - 2\lambda)N_r^2 + g_0N(t)^2\xi^6 r^2 \right\}. \tag{39}
\]
They have solutions as

\[ N_r = \pm N(t) \sqrt{\frac{g_0 \zeta^6}{3(3\lambda - 1)}} r. \] (40)

We could also use the freedom of gauge transformation to set \( N(t) = 1 \). These solutions actually describe the same de-Sitter space-time. One easy way to see this point is to change inversely into the Schwarzschild coordinates.

One subtle issue happens when the cosmological constant \( \Lambda_W \) is negative. In this case, \( N_r \) becomes imaginary in (40). This is not physical anymore. However, after being transformed into Schwarzschild coordinates, the metric describes the anti-de-Sitter spacetime. Similarly the solution (36) becomes imaginary at asymptotic region if \( \Lambda_W \) is negative, but it may describe a AdS-Sch. spacetime in the Schwarzschild coordinates. Since in Ho\'rava-Lifshitz-like gravity, to respect the projectability condition, the static spherically symmetric solution should take the form of (22), the solutions with negative \( \Lambda_W \) are not acceptable. It would be interesting to see if the AdS and AdS-Sch. spacetime could be rewritten into a form respecting projectability condition\(^1\).

After some tedious calculation, it is straightforward to check that the solutions (30), (36), and (40) satisfy all the equations of \( \delta S/\delta N(t) = 0, \delta S/\delta N_i = 0 \) and \( \delta S/\delta g_{ij} = 0 \). Obviously they are all the solutions of Ho\'rava gravity in the IR region\((\lambda = 1)\). So the new solutions found in \([8, 12]\) could not satisfy the “projectability condition”, even though they satisfy the necessary condition (21). Our result also indicates that in Ho\'rava-Lifshitz-like gravity theory with the projectability condition, there is no novel correction in solar system test.

It is also interesting to study the topological black hole in Ho\'rava-Lifshitz like gravity. It has been discussed in \([20]\) without taking into account of the “projectability condition”. The static spherically symmetric metric ansatz of a topological spacetime may be written as

\[ ds^2 = -dt^2 + \frac{1}{f(r)}(dr + N' dt)(dr + N' dt) + r^2 d\Omega_k^2 \] (41)

Here we have set \( N(t) = 1 \) and \( d\Omega_k^2 \) denotes the line element for an 2-dimensional Einstein space with constant scalar curvature \( 2k \). Without loss of generality, one may take \( k = 0, \pm 1 \) respectively. Substituting the metric ansatz (41) into the Lagrangian (15), up to an overall scaling constant, we

\(^1\) In \([8]\), it has been pointed out that the dS-Sch. solution could be rewritten in terms of the Painlevé-Gullstrand coordinates to respect the projectability condition. We are also grateful to H.Lu for the discussion on the pathology of negative \( \Lambda_W \).
\[ L_k = \frac{1}{\sqrt{f}} \left\{ (1 - \lambda) r^2 f^2 \left( N_r' + N_r \frac{f'}{2f} \right)^2 + 2(1 - 2\lambda) f^2 N_r^2 \right. \\
-4\lambda r f^2 N_r \left( N_r' + N_r \frac{f'}{2f} \right) \left\} , \]

\[ L_v = \frac{1}{\sqrt{f}} r^2 \left\{ -g_0 \xi^6 + \xi^4 \left[ \frac{2(k - f)}{r^2} - \frac{2f'}{r} \right] + g_2 \zeta^2 \left[ \frac{2(k - f)}{r^2} - \frac{2f'}{r} \right]^2 \right. \\
+g_3 \zeta^2 \left[ \frac{f^2}{r^2} + \frac{2}{r^2} (k - f - \frac{r}{2} f')^2 \right] + g_4 \left[ \frac{2(k - f)}{r^2} - \frac{2f'}{r} \right]^3 \right. \\
+g_5 \left[ \frac{2(k - f)}{r^2} - \frac{2f'}{r} \right] \left[ \frac{f^2}{r^2} + \frac{2}{r^2} (k - f - \frac{r}{2} f')^2 \right] \right. \\
+g_6 \left[ \frac{f^2}{r^2} + \frac{2}{r^6} (k - f - \frac{r}{2} f')^3 \right] + g_7 \left[ \frac{2(k - f)}{r^2} - \frac{2f'}{r} \right] \sqrt{f} \partial_r \left\{ \frac{1}{\sqrt{f}} r^2 f \partial_r \left[ \frac{2(k - f)}{r^2} - \frac{2f'}{r} \right] \right\} \right. \\
+g_8 \left[ f^3 \left( \frac{f'}{r^2 f} - \frac{f''}{2 f} \right)^2 + \frac{2f}{r^2} \left( \frac{f'}{2} + \frac{rf''}{2} + \frac{2(k - f)}{r} \right) \right] \right\} . \] (42)

Here \( N_r = N'/f \) and \( ' \) means the derivative with respect to \( r \). The full Lagrangian is \( L = L_K + L_V \). The \( k = 1 \) case has been discussed above. Comparing with (24), we find that the kinetic term is exactly the same, and the difference in the potential term coming from the factor \((k - f)\) in (42) and \((1 - f)\) in (24). By varying the action with respect to the functions \( N_r, f \) and \( N(t) \), we could get three equations of motions which are quite similar to (25), (26) and (27), with \((1 - f)\) being replaced with \((k - f)\). Therefore the solutions are quite similar to the ones when \( k = 1 \).

The case \( k = 1 \) has been discussed above. In the case \( k = -1 \), for the solution with \( f \) being a constant, \( f \) must be set to \(-1\). At IR, \( \lambda = 1, N_r = \pm \sqrt{\frac{\eta c^6}{6} - r^2 + \frac{M^*}{r}} \), where \( M^* \) is an integration constant. They correspond to an (dS-)Schwarzschild type’s topological black hole written in Painlevé-Gullstrand type coordinates. When \( \lambda \neq 1, N_r = \pm \sqrt{\frac{\eta c^6}{3(3\lambda - 1)} - r} \). These solutions actually describe the de-Sitter space-time or Minkowski spacetime with topological twist. In the case \( k = 0 \), because \( f \) can’t be zero, we only have the solution “\( N_r = 0, f = yr^2 \)” in which \( y \) satisfy the equation (28), (29). In any case, these solutions are different from the ones studied in [20].

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