Abstract—For multiple-input/multiple-output (MIMO) spatial multiplexing with zero-forcing detection (ZF), signal-to-noise ratio (SNR) analysis for Rician fading involves the cumbersome noncentral-Wishart distribution (NCWD) of the transmit sample-correlation (Gramian) matrix. An approximation with a virtual CWD previously yielded for the ZF SNR an approximate (virtual) gamma distribution. However, theoretical conditions qualifying the accuracy of the SNR-distribution approximation were unknown. Therefore, we recently characterized exactly, although only for Rician–Rayleigh fading, the SNR of the sole Rician-fading stream written as scalar Schur complement (SC) in the Gramian. Herein, we employ a matrix-SC-based analysis to characterize SNRs when several or all streams undergo Rician fading. On the one hand, for full-Rician fading, the SNR of the sole Rician-fading stream written as scalar Schur complement (SC) in the Gramian coincides. On the other hand, for Rician–Rayleigh fading, the matrix-SC and SNR distributions are characterized in terms of determinants of matrices with elementary-function entries. ZF zero-forcing. On the other hand, for Rician–Rayleigh fading, the matrix-SC and SNR distributions are characterized in terms of determinants of matrices with elementary-function entries. ZF average error probability results validate our analysis against simulation.

Index Terms—MIMO, noncentral-Wishart matrix distribution, Rayleigh and Rician (Rician) fading, Schur complement, zero-forcing.

I. INTRODUCTION

A. Background, Motivation, and Scope

Multiple-input/multiple-output (MIMO) communications principles have maintained substantial research interest [1] [2] [3] [4] [5] and have also been adopted in standards [6] [7]. However, gaps remain in our ability to evaluate MIMO performance, based on analysis, for realistic channel propagation conditions and relatively simple transceiver processing: e.g., for MIMO spatial-multiplexing for Rician fading and linear detection methods, such as zero-forcing detection (ZF) [8] [9] or minimum mean-square-error detection (MMSE) [10].

Rician fading is both theoretically more general and practically more realistic than Rayleigh fading (which yields simpler analysis), according to the state-of-the-art WINNER II channel model [11]. ZF has relatively-low implementation complexity, and, thus, is attractive for practical implementation, as recently acknowledged under the massive-MIMO framework [4] [5].

Therefore, herein, we focus on the analysis of MIMO ZF under mixtures of Rician and Rayleigh fading that may occur in macrocells, microcells, and heterogeneous networks [12] [13]. MMSE, which is generally more difficult to analyze than ZF [10], will be the focus of future work.

Let us briefly introduce our model and some notation (detailed later on). We consider a MIMO system whereby the streams of independent symbols transmitted from \( N_T \) antennas are received with \( N_R \geq N_T \) antennas. For analysis tractability, we assume zero receive correlation. Given the channel matrix \( H \) and the transmit sample-correlation matrix \( W = H H^H \) (also known as Gramian matrix [13, p. 288]), the ZF signal-to-noise ratio (SNR) for a stream is determined by the corresponding diagonal element of \( W^{-1} \) [14, Eq. (5)].

B. Previous Research on MIMO ZF for Rician Fading

For MIMO ZF under Rayleigh-only fading, the stream SNRs have been shown to be gamma-distributed in [14], based on the fact that, when the mean \( H_d \) of \( H \) is zero, \( W \) has a central-Wishart distribution [15] (CWD) [15], i.e., \( W^{-1} \) has an inverse-CWD [16, p. 97].

On the other hand, under Rician fading, i.e., when \( H_d \) is nonzero, \( W \) is NCWD [15] [17], and then \( W^{-1} \) has an unknown distribution. Therefore, for MIMO ZF under full-Rician fading we attempted in [9] to characterize the ZF SNR distribution by approximating the actual NCWD of the Gramian matrix \( W \) with a virtual CWD of equal mean.

This approximation originated in [18] and had been exploited for MIMO ZF analysis several times [9] [19] [20] [21], because it yields a virtual gamma distribution to approximate the unknown actual SNR-distribution. However, the accuracy of the actual–virtual SNR-distribution approximation has so far been qualified only empirically. Thus, numerical results shown mostly for \( H_d \) of rank \( r = 1 \) obtained as outer-product of receive and transmit array-steering vectors in [9] [19] [20] [21] found the approximation reliable. In [9], we also found it most accurate for such rank-1 \( H_d \); higher \( r \) yields poorer accuracy, and \( r = N_T \) makes the approximation unusable. However, we also found for \( r = 1 \) that different channel-mean–correlation combinations yield different accuracies.

\footnote{For simplicity, NCWD stand herein for both ‘noncentral-Wishart distribution’ and ‘noncentral-Wishart-distributed’.}

\footnote{I.e., all streams undergo Rician fading.}
Therefore, we have recently pursued in [12] an exact ZF-SNR analysis, which proved tractable when the intended Stream 1 undergoes Rician fading and interfering Streams 2, · · · , NT undergo Rayleigh fading respectively; incidentally, r = 1 in this case. We proceed in [12], as in [8], from the vector–matrix partitioning according to fading types

\[ \mathbf{H} = (\mathbf{h}_1 \quad \mathbf{h}_2) \]

to write the Stream-1 SNR as the scalar Schur complement \( \text{(SC)} \) [19] [20] [13] Sec. 3.4] of submatrix \( \mathbf{W}_{22} = \mathbf{H}_2^H \mathbf{H}_2 \) in the NCWD Gramian matrix \( \mathbf{W} \). This SC is then recast as a scalar Hermitian form whereby the vector and matrix correspond, respectively, to the intended and interfering streams [12] Eq. (9) [8] Eq. (7)]. By first conditioning on and then averaging over \( \mathbf{H}_2 \), the exact moment generating function (m.g.f.) of the ZF SNR is expressed in terms of the confluent hypergeometric function \( \mathcal{F}_1 (N; N_R; \sigma_1) \), where \( N = N_R - N_T + 1 \), and scalar \( \sigma_1 \) is a function of channel mean and transmit-correlator [12] Eq. (31)].

Average error probability (AEP) results shown in [12] Figs. 1, 2] for Rician(1)/Rayleigh(\( NT - 1 \)) fading further support the suggestion from [9] that the actual–virtual SNR-distribution approximation may be insufficiently accurate even for \( r = 1 \). On the other hand, results shown in [12] Fig. 10] for Rayleigh(1)/Rician(\( NT - 1 \)) fading reveal that the approximation can be accurate even for \( r > 1 \). Thus, we have sought an analytical condition for accurate approximation.

C. Approach and Contributions

This work generalizes the analysis and proves the insights from [12] Thus, we proceed from the matrix–matrix partitioning \( \mathbf{H} = (\mathbf{H}_1 \quad \mathbf{H}_2) \), where \( \mathbf{H}_1 \) has \( v < N_T \) columns, to characterize the distribution of the ensuing matrix-SC of \( \mathbf{W}_{22} \) in \( \mathbf{W} \), denoted as \( \mathbf{\Gamma}_1 \). Using this SC-based analysis, we characterize the ZF SNR distributions of the streams corresponding to \( \mathbf{H}_1 \) when several or all streams undergo Rician fading.

First, for full-Rician fading, we find that \( \mathbf{\Gamma}_1 \) conditioned on \( \mathbf{H}_2 \) is NCWD, and state the necessary and sufficient condition that yields a CWD for the unconditioned \( \mathbf{\Gamma}_1 \), i.e., gamma distributions for ZF SNRs. This condition is a special relationship between the means and column-correlations of \( \mathbf{H}_1 \) and \( \mathbf{H}_2 \).

Second, again for full-Rician fading, we generalize the approximation of the actual distributions of the ZF SNRs with virtual gamma distributions, which is discussed in [9] Refs. 24-27,30,31], to the approximation of the actual (generally, unknown) distribution of the matrix-SC \( \mathbf{\Gamma}_1 \) with a virtual CWD. We are able to prove that the actual and virtual CWDs for \( \mathbf{\Gamma}_1 \) coincide under the mentioned mean–correlation condition. Consequently, the actual (generally, unknown) and virtual (gamma) distributions of the ZF SNRs for the streams corresponding to \( \mathbf{H}_1 \) also coincide. (Thus, although these streams undergo Rician fading, their SNRs are distributed as when they undergo Rayleigh fading.) This helps qualify analytically, for the first time, the accuracy of approximating the distribution of the ZF SNR under Rician fading with the gamma distribution, and explains previous numerical results [9] [12] Figs. 1,2,10].

Third, for Rician(\( v \))/Rayleigh(\( NT - v \)) fading with \( v < N_T \), we generalize the exact analysis from [12], i.e., we characterize exactly the distribution of the matrix-SC \( \mathbf{\Gamma}_1 \). Thus, its m.g.f. is expressed in terms of the hypergeometric function \( \mathcal{F}_0 (S, A) \), where \( S \) and \( A \) are \( N_R \times N_R \) matrices. Further, \( \mathcal{F}_0 (S, A) \) is expressed in terms of the determinant of a matrix whose entries are elementary functions. The \( \mathbf{\Gamma}_1 \) m.g.f. expression yields a new exact determinantal expression for the Stream-1 AEP for the Rician(\( 1 \))/Rayleigh(\( NT - 1 \)) fading case, which we covered also in [12]. Finally, comparing new and previous SNR m.g.f. expressions reveals the hypergeometric function relationship \( \mathcal{F}_0 (S, A) = 1 \mathcal{F}_1 (N; N_R; \sigma_1) \), whereby \( S \) has a single nonzero eigenvalue \( \sigma_1 \), and \( A \) is idempotent and of rank \( N \).

Numerical AEP results validate our analysis and depict the effect of the mean–correlation condition on the relative performance of MIMO ZF for various fading cases.

D. Notation

Scalars, vectors, and matrices are represented with lowercase italics, lowercase boldface, and uppercase boldface, respectively, e.g., \( h \), \( \mathbf{h} \), and \( \mathbf{H} \); \( h \sim \mathcal{CN}(h_0, R) \) indicates that \( h \) is a complex-valued circularly-symmetric Gaussian random vector [2] p. 39] [21] with mean (i.e., deterministic component) \( h_0 \) and covariance matrix \( R \); \( \mathbf{H} \sim \mathcal{CN}(\mathbf{H}_0, \mathbf{I}_{N_R} \otimes \mathbf{R}_{T,K}) \) indicates that \( \mathbf{H} \) is complex circularly-symmetric Gaussian with mean \( \mathbf{H}_0 \) and transmit-side covariance matrix \( \mathbf{R}_{T,K} [15] \); \( r \) denotes the rank of \( \mathbf{H}_0 \); subscripts \( \cdot_d \) and \( \cdot_t \) identify deterministic and random components, respectively; subscript \( \cdot_n \) indicates a normalized variable; \( 1 : N \) stands for the enumeration 1, 2, · · · , \( N \); superscripts \( \cdot^T \) and \( \cdot^H \) stand for transpose and Hermitian (i.e., complex-conjugate) transpose; \( \mathbf{H}_{i,j} \) indicates the \( i, j \) element (scalar) of matrix \( \mathbf{H} \); \( \mathcal{C} \mathcal{W}_{N_T} (\mathbf{H}_0, \mathbf{R}_T) \) denotes the complex CWD with dimension \( N_T \), degrees of freedom \( N_R \), and scale matrix \( \mathbf{R}_T \); \( \mathcal{C} \mathcal{W}_{N_T} (\mathbf{H}_0, \mathbf{I}_{N_R} \otimes \mathbf{R}_{T,K}) \) denotes the complex NCWD with dimension \( N_T \), degrees of freedom \( N_R \), scale matrix \( \mathbf{R}_{T,K} \), and noncentrality parameter matrix \( \mathbf{R}_{T,K} \mathbf{H}_0^H \mathbf{H}_0 [15] \); \( \mathbf{H}_1 \) and \( \mathbf{H}_2 \) are the submatrices obtained by partitioning \( \mathbf{H} \) along its columns; \( \mathbf{W}_{11}, \mathbf{W}_{12}, \mathbf{W}_{21}, \) and \( \mathbf{W}_{22} \) are the submatrices obtained by partitioning \( \mathbf{W} \) along rows and columns; \( \mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \) and \( \mathbf{W}_4 \) are the submatrices obtained by partitioning \( \mathbf{W} \) along rows and columns; the SC of \( \mathbf{W}_{22} \) in \( \mathbf{W} \) is given by \( \mathbf{W}^{-1} = \mathbf{W}_{11} - \mathbf{W}_{12} \mathbf{W}_{22} \mathbf{W}_{21} [19] [20] \); \( \mathbf{H}^H = \sum_{i=1}^{N_T} \sum_{j=1}^{N_T} |H_{i,j}|^2 = \text{tr}(\mathbf{H}^H \mathbf{H}) \) is the squared Frobenius norm of \( \mathbf{H} \); \( \text{etr}(\mathbf{X}) \) represents the trace of matrix \( \mathbf{X} \), and \( \text{etr}(\mathbf{X}) = e^{\text{tr}(\mathbf{X})} \) is the zero vector or matrix of appropriate dimensions; \( \text{diag}(\cdot, \cdots, \cdot) \) is the diagonal matrix with given elements; \( \mathbb{E}\{\cdot\} \) denotes statistical average; \( \| \cdot \| \) and \( \approx \) relate random variables with the same and approximately the same distribution, respectively; \( \mathcal{F}_0 (S) \)
is the hypergeometric function with a single matrix argument defined in [22] Eq. (35.8.1), p. 772 and characterized by $\phi F_0(S) = \text{etr}(S)$ [22] Eq. (35.8.2); $\phi F_0(S, A)$ is the hypergeometric function of double matrix argument defined in [23] Eq. (88)] [24] Eq. (9)]; $\phi F_1(\vdots; \sigma_1)$ is the confluent hypergeometric function of scalar argument $\sigma_1$ [22] Eq. (13.2.2), p. 322); $(N)_n$ is the Pochhammer symbol, i.e., $(N)_0 = 1$ and $(N)_n = N(N + 1) \cdots (N + n - 1), \forall n > 1$ [22] p. xiv); finally, $\Rightarrow$ and $\Leftrightarrow$ represent implication and equivalence, respectively, whereas ‘iff’ is short for ‘if and only if’.

### E. Outline

Section II presents the signal, noise, and fading models. Section III introduces the NCWD and CWD for Gramian matrices and characterizes the distribution of the conditioned matrix-SC. Section IV reveals the mean–correlation condition for the SC to be CWD, i.e., for ZF SNRs to be gamma-distributed. Section V discusses for the matrix-SC the approximation of the actual (generally unknown) distribution with a virtual CWD, and proves that the mean–correlation condition yields the same CWD for the actual distribution. Section VI characterizes the matrix-SC distribution for Rician–Rayleigh fading. Finally, Section VII presents numerical results.

### II. SIGNAL, CHANNEL, AND NOISE MODELS

Similarly to [9] [12], this paper considers an uncoded MIMO system over a frequency-flat fading channel. There are $N_T$ and $N_R$ antenna elements at the transmitter(s) and receiver, respectively, with $N_T \leq N_R$. Letting $x = [x_1 x_2 \cdots x_{N_R}]^T \in N_T \times 1$ denote the zero-mean transmit-symbols vector with $\mathbb{E}\{xx^H\} = I_{N_R}$, the vector with the received signals can be represented as [2] p. 63

$$r = \sqrt{\frac{E_s}{N_T}} H x + n \in N_R \times 1. \quad (1)$$

Above, $E_s/N_T$ is the energy transmitted per symbol (i.e., per antenna), so that $E_s$ is the energy transmitted per channel use. The additive noise vector $n$ is uncorrelated, circularly-symmetric, zero-mean, complex Gaussian with $n \sim \mathcal{CN}(0, N_0 I_{N_R})$ [2] p. 39 [27]. Its normalized version $\tilde{n} = n/N_0 \sim \mathcal{CN}(0, I_{N_R})$ will also be employed. Then, the per-symbol transmit-SNR is

$$\Gamma_s = \frac{E_s}{N_0} \frac{1}{N_T}. \quad (2)$$

In (1), $H \equiv N_R \times N_T$ is the complex-Gaussian channel matrix, assumed to have rank $N_T$. Its deterministic and random components are denoted as $H_d$ and $H_r$, respectively. The channel matrix for Rician fading is usually written as [2] p. 41

$$H = H_d + H_r = \sqrt{\frac{K}{K + 1}} H_{d,n} + \sqrt{\frac{1}{K + 1}} H_{r,n}, \quad (3)$$

where it is assumed, for normalization purposes [25], that $\|H_{d,n}\|^2 = N_T N_R$ and $\mathbb{E}\{|H_{d,n}^i|\}_j^2 = 1$, $\forall i, j$, so that $\mathbb{E}\{\|H\|^2\} = N_T N_R$. Then, if $H_{d,i,j} = 0$, $|H_{d,i,j}|$ is Rayleigh distributed; otherwise, $|H_{d,i,j}|$ is Rician distributed [26]. The power ratio of the deterministic and random components

$$K = \frac{\|H_d\|^2}{\mathbb{E}\{|H_{d,n}\|^2}} = \frac{K}{K + 1} \frac{\|H_{d,n}\|^2}{\mathbb{E}\{|H_{r,n}\|^2}} \quad (4)$$

is known as the Rician $K$-factor or as the ‘propagation SNR’ [27]. WINNER II [11] modeled the measured $K$ (in dB) as a random variable with scenario-dependent lognormal distribution.

For analysis tractability, let us assume zero receive correlation. On the other hand, let us denote any of the rows of $H_{d,n}$ with $g_{d,n}$, and assume that $g_{d,n} \sim \mathcal{CN}(0, R_{T_d})$. This assumption that all rows have the same correlation matrix is also required for tractability, and has been made previously [8] [14]. Then, any row $g_{r,n}$ of $H_r$ is characterized by $g_{r,n} \sim \mathcal{CN}(0, R_{T_r})$, where [9] Eq. (5)

$$R_{T,K} = \mathbb{E}\{g_r g_r^H\} = \frac{1}{N_R} \mathbb{E}\{H_r g_r^H\} = \frac{1}{K + 1} R_{T_d}, \quad (5)$$

so that $H \sim \mathcal{CN}(H_d, I_{N_R} \otimes R_{T,K})$ [15].

The elements of matrix $R_{T_d}$, which is Hermitian (i.e., $R_{T_d} = R_{T_d}^H$), can be computed from the azimuth spread (AS) as shown in [9] Section VI.A when assuming Laplacian power azimuth spectrum, which has been adopted in WINNER II. There, measured AS (in degrees) was also modeled as a random variable with scenario-dependent lognormal distribution.

The remainder of this section introduces a series of matrix partitionings, decompositions, and ensuing relationships that will be employed throughout.

In [12] we employed the vector–matrix partition

$$H = (h_1, H_2) = (h_{d,1}, H_{d,2}) + (h_{r,1}, H_{r,2}), \quad (6)$$

where $h_1$, $h_{d,1}$, and $h_{r,1}$ are $N_R \times 1$ vectors, whereas $H_2$, $H_{d,2}$, and $H_{r,2}$ are $N_R \times (N_T - 1)$ matrices. The following, more general partitioning is employed herein:

$$H = (h_1, H_2) = (h_{d,1}, H_{d,2}) + (h_{r,1}, H_{r,2}), \quad (7)$$

where $h_1$, $h_{d,1}$, and $h_{r,1}$ are $N_R \times v$ matrices, whereas $H_2$, $H_{d,2}$, and $H_{r,2}$ are $N_R \times (N_T - v)$ matrices, with $1 \leq v < N_T$.

According to the partitioning of $H$ from (7), we partition $R_{T,K}$ into its component submatrices $R_{T,K_{11}}, R_{T,K_{12}}, R_{T,K_{21}},$ and $R_{T,K_{22}}$, where $R_{T,K_{21}} = R_{T,K_{21}}^H$. Also, we partition $R_{T,K}^{-1}$ into its component submatrices $R_{T,K_{11}}^{-1}, R_{T,K_{12}}^{-1}, R_{T,K_{21}}^{-1},$ and $R_{T,K_{22}}^{-1}$. Further, for $R_{T,K}$ we consider the upper–lower triangular (UL) decomposition $R_{T,K} = A A^H$ [13] Sec. 5.6]. Again, according to the partitioning of $H$ from (7), we partition the upper triangular matrix $A$ into its component submatrices $A_{11}, A_{12}, A_{21},$ and $A_{22}$, where $A_{21} = 0$. Finally, we partition $A^{-1}$ into its component submatrices $A_{11}, A_{12}, A_{21},$ and $A_{22}$. For these matrices we have deduced the following relationships, for subsequent use:

$$A_{11}^{-1} = A_{11}^{-1}, A_{22} = A_{22}^{-1} = A_{11}^{-1} A_{12} A_{22}^{-1}, \quad (8)$$

$$R_{T,K_{22}}^{-1} = (A_{22} A_{22}^H)^{-1} = A_{22}^{-1} A_{22}^{-1} \quad (9)$$

$$R_{T,K_{22}} = A_{22} A_{22}^{-1} \quad (10)$$

$$A_{11} A_{11} = (A_{11}^{-1} A_{11}^{-1})^{-1} = (R_{T,K_{11}})^{-1} \quad (11)$$

$$R_{T,K_{11}} - R_{T,K_{12}} R_{T,K_{22}} R_{T,K_{21}} \quad (12)$$
The matrix described by (11) and (12) is referred to as the SC of $R_{T,K}2_2$ in $R_{T,K}$ [19] [20] [13] Sec. 3.4] [8] Appendix]. For our channel model, it represents the correlation of the first $v$ elements of $g$, given its remaining $N_T - v$ elements [19] p. 186] [20].

Finally, according to the partitioning of $H$ from (7), we also partition the column sample-correlation matrix of $H$, i.e., $W = H^H H$, also known as Gramian matrix [13] p. 288], and its inverse $W^{-1}$ — see Section 3.4 Note that the submatrices of $W$ are given by $W_{i,j} = H^H H_{j}$, with $i, j = 1, 2$.

III. SCHUR COMPLEMENT IN WISHART GRAMIAN MATRIX

A. Hermitian Form of SC in Gramian Matrix

Nonzero-mean and zero-mean complex-Gaussian $H$ yield complex NCWD and CWD Gramian $W$, respectively [15]:

$$H \sim CN(H, I_{N_R} \otimes R_{T,K})$$

$$W \sim CW_{N_R}(N_R, R_{T,K}, R_{T,K}^{-1}H_d^T H_d)$$ (13)

H \sim CN(0, I_{N_R} \otimes R_T)

$$W \sim CW_{N_R}(N_R, R_T).$$ (14)

The SC of $W_{22} = H_d^T H_2$ in Gramian $W$ is the matrix $\Gamma_1 = (W_{11})^{-1} = W_{11} - W_{12} W_{22}^{-1} W_{21} \doteq v \times v$, (15)

which can be expressed as a matrix Hermitian form, based on [8], as follows:

$$\Gamma_1 = H_d^T H_1 - H_d^T H_2 (H_d^T H_d)\olin\olin\olin H_d^T H_1$$

$$= H_d^T \left[I_{N_R} - H_2 (H_d^T H_d)\olin\olin\olin H_2^T\right] H_1.$$ (16)

Note first, that the SC matrix $\Gamma_1$ is the column sample-correlation of $H_1$ given $H_2$. Then, note that matrix $H_2 (H_d^T H_d)^{-1} H_2^T \doteq N_R \times N_R$ is the projection onto the column space of $H_2$, whereas matrix $Q_1 \doteq N_R \times N_R$ is the projection onto the null space of $H_d^T$. These Hermitian matrices are idempotent and have eigenvalues as listed below:

$$H_2 (H_d^T H_d)^{-1} H_2^T : 1, 1, \ldots, 1, 0, 0, \ldots, 0$$

$$Q_2 : 0, 0, \ldots, 0, 1, 1, \ldots, 1.$$ (17)

Thus, their ranks are $N_T - v$ and $N_v = N_R - N_T + v$, respectively.

B. ZF SNR as Ratio, SC, and Hermitian Form [8] [12]

Given $H$ and nonsingular $W = H^H H$, ZF for the signal from (1) means separately mapping into the closest modulation (e.g., M/PSK constellation symbol each element of the following $N_x \times 1$ vector [2] p. 153):

$$y = \sqrt{\frac{\bar{N}_r}{\bar{E}_s}} [H^H H]^{-1} H^H r = x + \frac{1}{\sqrt{\bar{E}_s}} [H^H H]^{-1} H^H n.$$ (19)

Since the resulting noise vector has correlation matrix $W^{-1}/\Gamma_1$, the ZF SNR for Stream $i = 1 : N_T$ has typically been expressed as the ratio form [14]

$$\gamma_i = \frac{\Gamma_1}{(W^{-1})_{i,i}}.$$ (20)

Now, because for $i = 1 : v$ we can write

$$\gamma_i = \frac{\Gamma_1}{(W^{-1})_{i,i}} = \frac{\Gamma_1}{(W_{11})_{i,i}} = \frac{\Gamma_1}{(\Gamma_1^{-1})_{i,i}},$$ (21)

the SC matrix $\Gamma_1$ determines these ZF SNRs. For $v = 1$, i.e., for the partitioning from (6), we expressed in (12) the Stream-1 SNR $\gamma_1$ as the scalar SC of $W_{22}$ in $W$, and as a scalar Hermitian form:

$$\gamma_1 = \frac{\Gamma_1}{(W_{11})_{i,i}} = \frac{\Gamma_1}{W_{11}} = \Gamma_1 (W_{11})^{-1} = \Gamma_1 \Gamma_1$$ (22)

$$= \Gamma_1 h_d^T Q_2 h_1.$$ (23)

The latter helped prove that, for Rician(1)/Rayleigh($N_T - 1$) fading, the distribution of $\gamma_1$ is an infinite linear combination of gamma distributions — see also Remark 6 later herein. Below, using the more general partitioning from (7), we characterize the distributions of the ZF SNRs for streams $i = 1 : v$, based on their relationship with the matrix-SC $\Gamma_1$ from (21).

C. Distribution of $\Gamma_1|H_2$ (or $\Gamma_1|Q_2$)

We now deploy [16] to characterize the distribution of $\Gamma_1$ conditioned on $H_2$ (or $Q_2$), assuming $H_{d,1} \neq 0$ and $H_{d,2} \neq 0$, and allowing for nonzero correlation between any columns of $H$. Since the columns of $H_1$ and $H_2$ are correlated, conditioning $\Gamma_1|H_2$ requires explicit conditioning of $\Gamma_1|H_2$ (or $Q_2$), as shown below, by generalizing the approach used for $v = 1$ in [8] [12].

Since $H_1$ and $H_2$ are jointly Gaussian, the distribution of $H_1$ given $H_2$ is [8] Appendix]

$$H_1|H_2 \sim CN(M + H_2 R_{2,1}, I_{N_R} \otimes (R_{11}^{-1,T})^{-1}),$$ (24)

where

$$M = H_{d,1} - H_{d,2} R_{2,1} \doteq N_R \times v,$$ (25)

$$R_{2,1} = R_{T,K}^{-1} R_{T,K} \doteq (N_T - v) \times v,$$ (26)

determine the deterministic matrices. Then, we can recast [24] as

$$H_1|H_2 \doteq X + H_2 R_{2,1}, X \sim CN(M, I_{N_K} \otimes (R_{11}^{-1,T})^{-1})$$ (27)

Substituting this in [16] and manipulating as in [8] yields

$$\Gamma_1|Q_2 \doteq X^H Q_2 X,$$ (28)

which, according to [28] Cor. 7.8.1.1, p. 255], has the NCWD

$$\Gamma_1|Q_2 \sim CW_{v} \left(N_v, (R_{11}^{-1,K})^{-1}, R_{11}^{-1,K} M^H Q_2 M \right).$$ (29)

Thus, its m.g.f. for matrix $\Theta = v \times v$ is [7] Eq. (4)]

$$M_{\Gamma_1|Q_2} (\Theta) = |I_v - \Theta (R_{11}^{-1,K})^{-1}|^{-N_v}$$

$$\times tr\left(I_v - \Theta (R_{11}^{-1,K})^{-1}\right)^{-1} \Theta M^H Q_2 M.$$ (30)

Based on (30), we can characterize the distribution of $\Gamma_1$ for 1) full-Rician fading under condition $M = 0$, as shown in the next section, and 2) Rician($v$)/Rayleigh($N_T - v$) fading, as shown in Section VI and Appendix E.

Thus, the SC $(R_{11}^{-1,K})^{-1}$ is indeed the transmit-correlation matrix of the conditioned Gaussian matrix $H_1|H_2$ [19] p. 186].
IV. \(\Gamma_1\) DISTRIBUTION FOR RICIAN FADING WITH \(M = 0\)

A. Equivalence Conditions for Central-Wishartness of \(\Gamma_1\)

The theorem below follows from the fact that the \(etr(\cdot)\) term in (30) reduces to 1 (for any \(\Theta\) and \(Q_2\)), so that \(\Gamma_1\) is CWD, \(\iff\\) \(M = 0\), i.e., \(\iff\) the following mean–correlation relationship (condition) holds:

\[
\mathbf{H}_{d,1} = \mathbf{H}_{d,2}\mathbf{R}_{2,1}.
\]

**Theorem 1:**

\[
\mathbf{H}_{d,1} = \mathbf{H}_{d,2}\mathbf{R}_{2,1}
\]

\(\iff\)

\[
M_\mathbf{R}_1 (\Theta) = \left| I_v - \Theta (\mathbf{R}_{T,K}^{11})^{-1} \right|^{-N_v}
\]

\(\iff\)

\[
\Gamma_1 \sim \mathcal{C} \mathcal{W}_v \left( N_v, (\mathbf{R}_{T,K}^{11})^{-1} \right)
\]

Now, recall that, for \(\mathbf{H} \sim \mathcal{C} \mathcal{N} (\mathbf{H}_d, \mathbf{I}_{N_x} \otimes \mathbf{R}_{T,K}^H)\), \(\mathbf{R}_{T,K}\) is the covariance matrix of the columns of \(\mathbf{H}^H\). Using the UL decomposition of \(\mathbf{R}_{T,K} = \mathbf{A} \mathbf{A}^H\), and defining \(\mathbf{H}_w \sim \mathcal{C} \mathcal{N} (0, \mathbf{I}_{N_x} \otimes \mathbf{I}_{N_y})\) we can write

\[
\mathbf{H} = \mathbf{H}_d + \mathbf{H}_w \mathbf{A}^H,
\]

so that

\[
\mathbf{HA}^{-H} = \mathbf{H}_d \mathbf{A}^{-H} + \mathbf{H}_w.
\]

with

\[
\mathbf{H}_d \mathbf{A}^{-H} = \left( \mathbf{H}_{d,1} \mathbf{A}^{11,H} + \mathbf{H}_{d,2} \mathbf{A}^{12,H} \mathbf{H}_{d,2} \mathbf{A}^{22,H} \right).
\]

The latter can be written, based on (8)-(10) and (26), as

\[
\mathbf{H}_d \mathbf{A}^{-H} = \left( \left( \mathbf{H}_{d,1} - \mathbf{H}_{d,2} \mathbf{R}_{2,1} \right) \mathbf{A}^{11,H} + \mathbf{H}_{d,2} \mathbf{A}^{22,H} \right),
\]

which proves the following Lemma.

**Lemma 1:**

\[
\mathbf{H}_{d,1} = \mathbf{H}_{d,2}\mathbf{R}_{2,1}
\]

\(\iff\)

\[
\mathbf{HA}^{-H} = \left( 0 \mathbf{H}_{d,2} \mathbf{A}^{22,H} \right) + \left( \mathbf{H}_{w,1} \mathbf{H}_{w,2} \right).
\]

i.e., the mean–correlation condition is equivalent with the fact that canceling the transmit-correlation in the channel matrix yields a matrix whose first \(v\) columns are zero-mean.

The following corollary summarizes from Theorem 1 and Lemma 1 the necessary and sufficient conditions for \(\Gamma_1\) to be CWD.

**Corollary 1:**

\[
\Gamma_1 \sim \mathcal{C} \mathcal{W}_v \left( N_v, (\mathbf{R}_{T,K}^{11})^{-1} \right)
\]

\(\iff\)

\[
\mathbf{H}_{d,1} = \mathbf{H}_{d,2}\mathbf{R}_{2,1}
\]

\(\iff\)

\[
\mathbf{HA}^{-H} = \left( 0 \mathbf{H}_{d,2} \mathbf{A}^{22,H} \right) + \left( \mathbf{H}_{w,1} \mathbf{H}_{w,2} \right).
\]

B. ZF SNR Distribution for \(\mathbf{H}_{d,1} = \mathbf{H}_{d,2}\mathbf{R}_{2,1}\)

For CWD \(\Gamma_1\), i.e., for \(\mathbf{H}_{d,1} = \mathbf{H}_{d,2}\mathbf{R}_{2,1}\), the following Lemma characterizes ZF SNRs for Streams \(i = 1: v\).

**Lemma 2:**

\[
\Gamma_i \sim \mathcal{C} \mathcal{W}_v \left( N_v, (\mathbf{R}_{T,K}^{11})^{-1} \right) \Rightarrow \text{for } i = 1: v
\]

\[
\gamma_i = \frac{\Gamma_i}{\Gamma_1^{1-i,i}} \sim \text{Gamma}(N, \Gamma_{K,i}), \quad \Gamma_{K,i} = \frac{\Gamma_1}{\left[ \mathbf{R}_{T,K}^{1-i,i} \right]_{i,i}}.
\]

**Proof:** A special case of (16) Th. 3.2.11, p. 95) yields

\[
\Gamma_i \sim \mathcal{C} \mathcal{W}_v \left( N_v, (\mathbf{R}_{T,K}^{11})^{-1} \right) \Rightarrow
\]

\[
\frac{1}{\Gamma_1^{1-i,i}} \sim \mathcal{C} \mathcal{W}_v \left( N_v, \mathbf{R}_{T,K}^{11} \right), \quad i = 1: v.
\]

Since \(\mathbf{R}_{T,K}^{11} = \left[ \mathbf{R}_{T,K}^{1-i,i} \right]_{i,i}\), we can express the m.g.f. of \(1/\Gamma_1^{1-i,i}\) as (17 Eq. (4))

\[
M(s) = \left( 1 - s/\left[ \mathbf{R}_{T,K}^{1-i,i} \right]_{i,i} \right)^{-N},
\]

which yields

\[
\gamma_i(s) = M(s\Gamma_i) = \left( 1 - s\Gamma_i \right)^{-N},
\]

i.e., \(\gamma_i = \frac{\Gamma_i}{\Gamma_1^{1-i,i}} \sim \text{Gamma}(N, \Gamma_{K,i}).
\]

**Remark 1:** Note that (42) yields \(E\{\gamma_i\} = N\Gamma_{K,i}\). For Rayleigh-only fading, \(\Gamma_{K,i}\) reduces to \(\Gamma_{i} = \Gamma_{i}/\left[ \mathbf{R}_{T}^{1-i,i} \right]_{i,i}\).

Finally, \(\Gamma_{K,i}\) yields \(K_{i} = K_{i}/\Gamma_{i}\). Thus, when \(M = 0\), Rician fading on any Stream \(i = 1: N_T\) reduces the average SNR for Streams \(i = 1: v\) by a factor of \(K + v\) vs. Rayleigh-only fading, as illustrated numerically in Section VII.

**Remark 2:** Condition \(M = 0\) holds for the practically relevant case of Rayleigh(v)/Rician(\(N_T - v\)) fading whereby the Rayleigh fading is uncorrelated with the Rician fading, i.e., for \(\mathbf{H}_{d,1} = 0\), \(\mathbf{H}_{d,2} \neq 0\), and \(\mathbf{R}_{2,1} = 0\). Then, the SNRs of Streams \(i = 1: v\), which correspond to Rayleigh fading, are gamma-distributed as in (42). On the other hand, practical conditions that would yield \(\mathbf{H}_{d,1} = \mathbf{H}_{d,2}\mathbf{R}_{2,1}\) for full-Rician fading are not apparent.

C. Exact AEP Expression for \(\mathbf{H}_{d,1} = \mathbf{H}_{d,2}\mathbf{R}_{2,1}\)

Given the SNR m.g.f., the elegant AEP-derivation procedure from (26) Chapter 9 can be employed, e.g., for MPSK modulation, as follows. The Stream-\(i\) error probability is (26 Eq. (8.22))

\[
P_e(\gamma_i) = \frac{1}{\pi} \int_0^{\frac{\pi}{4}} \exp \left\{ -\gamma_i \sin^2 \frac{\pi}{\sin^2 \theta} \right\} d\theta.
\]

Thus, the AEP can be written as (26 Chapter 9)

\[
P_e,i = E\{P_e(\gamma_i)\} = \frac{1}{\pi} \int_0^{\frac{\pi}{4}} M_{\gamma_i} \left( -\sin^2 \frac{\pi}{\sin^2 \theta} \right) d\theta.
\]

Substituting the m.g.f. from (41) into (44) yields the exact AEP expression for fading cases whereby \(\mathbf{H}_{d,1} = \mathbf{H}_{d,2}\mathbf{R}_{2,1}\):

\[
P_e,1^2 = \frac{1}{\pi} \int_0^{\frac{\pi}{4}} \left( 1 + \frac{\sin^2 \frac{\pi}{\sin^2 \theta}}{\Gamma_{K,i}} \right)^{-N} d\theta,
\]
A. Approximate CWD for $\tilde{\mathbf{H}}$ that the exact SNR distribution for Stream 1 is an infinite characterization from Table I, Row 8. discuss a Wishart-distribution approximation that yields the $\tilde{\mathbf{W}}$ (1)

The remaining rows characterize fading cases whereby the Lemma 2, and Remark 2, ZF SNR distributions for various fading, characterized in Row 4, we recently found, in [12] Eq. (37), that the exact SNR distribution for Stream 1 is an infinite linear combination of gamma distributions. In Appendix B we generalize the approach from [12] to express the m.g.f. of $\Gamma_1$ for $1 \leq v < N_T$ in determinantal form. Next, we discuss a Wishart-distribution approximation that yields the characterization from Table I Row 8.

V. APPROXIMATE AND EXACT CWDs FOR $\Gamma_1$

A. Approximate CWD for $\mathbf{W}$ [18] [9]

For the nonzero-mean matrix $\mathbf{H} \sim CN(\mathbf{H}_d, \mathbf{I}_{N_K} \otimes \mathbf{R}_{T,K})$, $\mathbf{W} = \mathbf{H}^H \mathbf{H} \sim CW_{N_T}(\mathbf{R}_{T,K}, \mathbf{R}_{T,K}^{-1} \mathbf{H}^H \mathbf{H})$. For a virtual zero-mean matrix $\tilde{\mathbf{H}} \sim CN(\mathbf{0}, \mathbf{I}_{N_K} \otimes \mathbf{R}_{T,K})$, $\tilde{\mathbf{W}} = \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \sim CW_{N_T}(\mathbf{R}_{T,K}, \mathbf{R}_{T,K}^{-1})$.

Lemma 3: [18] [9]

$$\mathbb{E}\{\tilde{\mathbf{W}}\} = \mathbb{E}\{\tilde{\mathbf{W}}\} \Leftrightarrow \mathbf{R}_{T,K} = \mathbf{R}_{T,K} + \frac{1}{N_R} \mathbf{H}_d^H \mathbf{H}_d. \quad (46)$$

Proof: Follows from

$$\mathbb{E}\{\tilde{\mathbf{W}}\} = N_R \mathbf{R}_{T,K} + \mathbf{H}_d^H \mathbf{H}_d = N_R \mathbf{R}_{T,K} = \mathbb{E}\{\tilde{\mathbf{W}}\}. \quad (47)$$

Remark 3: Based on property (46), the NCWD has been approximated with the virtual CWD, i.e.,

$$\mathbf{W} \approx \tilde{\mathbf{W}}, \quad (48)$$

in [18] — see also [9] Refs. 24-27, 30, 31.

B. Approximate CWD for $\Gamma_1$

Given $v = 1 : N_T$, let us partition $\tilde{\mathbf{H}}, \mathbf{R}_{T,K}, \tilde{\mathbf{R}}_{T,K}^{-1}, \tilde{\mathbf{W}},$ and $\tilde{\mathbf{W}}^{-1}$ as done for $\mathbf{H}, \mathbf{R}_{T,K}, \mathbf{R}_{T,K}^{-1}, \mathbf{W},$ and $\mathbf{W}^{-1}$ in Section II. Also, analogously to the SC $\hat{\Gamma}_1$ defined in [15], let us define the SC

$$\hat{\Gamma}_1 = (\tilde{\mathbf{W}}^{11})^{-1} = \tilde{\mathbf{W}}_{11} - \tilde{\mathbf{W}}_{12} \tilde{\mathbf{W}}_{22}^{-1} \tilde{\mathbf{W}}_{21}. \quad (49)$$

Since $\tilde{\mathbf{H}}$ is zero-mean, the procedure that has lead to (32) yields for $\hat{\Gamma}_1$ the m.g.f. expression

$$M_{\hat{\Gamma}_1}(\Theta) = |\mathbf{I}_v - \Theta (\tilde{\mathbf{R}}_{T,K}^{-1})^{-1}|^{-N_v}, \quad (50)$$

i.e., matrix $\hat{\Gamma}_1$ has the following CWD:

$$\hat{\Gamma}_1 \sim CW_v(\mathbf{R}_{T,K}^{-1}). \quad (51)$$

Remark 4: The approximation in distribution $\mathbf{W} \approx \tilde{\mathbf{W}}$, from (48) suggests the approximation in distribution $\Gamma_1 \approx \hat{\Gamma}_1$.

C. Approximate Gamma Distribution for All ZF SNRs

From Lemma 2, the virtual ZF SNRs are characterized by:

$$\hat{\gamma}_i = \frac{\Gamma_s}{\mathbb{E}\{\tilde{\mathbf{W}}\}} \sim Gamma(\hat{\Gamma}_K,i,\hat{\Gamma}_K,i), \quad \hat{\Gamma}_K,i = \frac{\Gamma_s}{\mathbb{E}\{\tilde{\mathbf{W}}\}}, \quad \mathbb{E}\{\tilde{\mathbf{W}}\} = 1 : N_T. \quad (52)$$

Note also that, analogously to (21), we can write:

$$\hat{\gamma}_i = \frac{\Gamma_s}{\mathbb{E}\{\tilde{\mathbf{W}}\}} \sim Gamma(\hat{\Gamma}_K,i,\hat{\Gamma}_K,i), \quad \mathbb{E}\{\tilde{\mathbf{W}}\} = 1 : N_T. \quad (53)$$

Remark 5: The approximation in distribution $\mathbf{W} \approx \tilde{\mathbf{W}}$ supports the approximation in distribution

$$\gamma_i \approx \hat{\gamma}_i \sim Gamma(\hat{\Gamma}_K,i), \quad i = 1 : N_T, \quad (55)$$

which has been employed for MIMO ZF analysis in [9] and [2] Refs. 24-26, 30, 31. For example, substituting the m.g.f. from (53) into (44) yields the approximate AEP expression

$$\hat{P}_e^{\{\mathbf{W}\}} = \frac{1}{\pi} \int_0^{\pi} |\mathbb{M}(\Theta)|^{-N_T} \left(1 + \frac{\sin^2 \pi \hat{\Gamma}_K,i}{\sin^2 \theta \hat{\Gamma}_K,i}\right)^{-N} d\theta, \quad i = 1 : N_T. \quad (56)$$

Both are referenced on Row 8 in Table I.
D. Necessary and Sufficient Condition for $\Gamma_1 \overset{d}{\rightarrow} \hat{\Gamma}_1$, $\gamma_1 \overset{d}{\rightarrow} \hat{\gamma}_1$

Theorem 2:

$$H_{d,1} = H_{d,2} R_{2,1} \Leftrightarrow \left( R_{11}^{11,K} \right)^{-1} = (R_{11}^{11,K})^{-1}.$$  

(57)

Proof: See Appendix A

Corollary 3: Theorems 1 and 2 along with (51), yield:

$$H_{d,1} = H_{d,2} R_{2,1} \Leftrightarrow \hat{\Gamma}_1 \overset{d}{\rightarrow} \Gamma_1 \sim CVV_0 (N, (R_{11}^{11,K})^{-1}).$$  

(58)

Corollary 4: For $v = 1 : N_T$ and $i = 1 : v$, using the definitions of $\gamma_i$ and $\hat{\gamma}_i$ from (21) and (54), respectively, (58) yields the implication

$$H_{d,1} = H_{d,2} R_{2,1} \Leftrightarrow \gamma_i = \frac{\Gamma_s}{\Gamma_s (\Gamma_s^{-1})_{i,i}} \sim Gamma (N, \Gamma_K)_{i,i}.$$  

(59)

Furthermore, for $v = 1$, because $\Gamma_1$ and $\hat{\Gamma}_1$ are scalar, (58) yields the equivalent

$$h_{d,1} = H_{d,2} R_{2,1} \Leftrightarrow \gamma_1 \overset{d}{\rightarrow} \hat{\gamma}_1 \sim Gamma (N, \Gamma_K).$$  

(60)

Finally, (59) implies the AEP equality $P_{e_{1}}^{\text{ZF}} \approx P_{e_{1}}^{\text{Max}}$, $i = 1 : v$, which is depicted in Rows 1–3 of Table I.

The equivalence in (60) explains earlier observations about the accuracy of $\gamma_1 \overset{d}{\rightarrow} \hat{\gamma}_1$ and $P_{e_{1}}^{\text{ZF}} \approx P_{e_{1}}^{\text{Max}}$.

- Depends on the combination of $H_d$ and $R_{11,K}$ [9, Sections VI.B–E].
- Is poor for the case of $H_d$ with rank $r = N_T$ [9, Figs. 1, 2] and the case $H_d = (h_{d,1})$ [12, Figs. 1, 2], whereby $h_{d,1} = H_{d,2} R_{2,1}$ does not hold.
- Is good in [12, Fig. 10] (and Fig. 3 herein) because $h_{d,1} \approx H_{d,2} R_{2,1}$.

VI. THE CASE OF RICIAN–RAYLEIGH FADING

Our recent scalar-SC analysis for Rician(1)/Rayleigh($N_T - 1$) fading from [12] (that yielded the exact ZF-SNR distribution for Stream-1) is generalized in Appendix B–A for Rician(1)/Rayleigh($N_T - v$), with $1 \leq v < N_T - 1$. Thus, in (82), we express the m.g.f. of the matrix-SC $\Gamma_1$ in terms of $F_0 (S, A)$, where $S$ and $A$ are $N_R \times N_R$ matrices defined in (80) and (78), respectively; $F_0 (S, A)$ is expressed as determinant of matrix with elementary-function entries in:

- Appendix B–B when both $S, A$ have distinct eigenvalues.
- Appendix B–C when both $S, A$ may have nondistinct eigenvalues.
- Appendix B–D when $S$ is rank-1, with distinct nonzero eigenvalues, and $A$ is idempotent and rank-$N_v$.
- Appendix B–E for $v = 1$, i.e., Rician(1)/Rayleigh($N_T - 1$) fading, so that $S$ is rank-1, and $A$ is idempotent and rank-$N$.

Note that for $v = 1$ the matrix Hermitian form shown for SC $\Gamma_1$ in (90) reduces to a scalar Hermitian form that determines the Stream-1 SNR through $\gamma_1 = \Im \Im \Im$, based on (21). Therefore, substituting scalar $\Im \Im \Im$ for the matrix-argument $\Theta$ in expression (82) of $M_{\gamma_1} (\Theta)$ yields

$$M_{\gamma_1} (s) = M_{\gamma_1} (s(\Gamma_{K,1})^{-1} N \theta F_0 (S, A),$$  

(61)

where $S$ is rank-1 with the nonzero eigenvalue $\sigma_1$ expressed, in terms of $s$, in (89), and $A$ is idempotent and rank-$N$; then, $\theta F_0 (S, A)$ is given by determinantal expression (90) in Appendix B–E. On the other hand, the approach pursued for $v = 1$ in [12] helped express the m.g.f. of $\gamma_1$ as [12, Eq. (31)]:

$$M_{\gamma_1} (s) = (1 - s \Gamma_{K,1})^{-N} \gamma_1 F_1 (N; N_R; s_1).$$  

(62)

Remark 6: By substituting the confluent hypergeometric function of scalar argument from its infinite-series expansion around the origin [22, Eq. (13.2.2), p. 322]

$$g_{F_1} (N; N_R; s_1) = \sum_{n=0}^{\infty} \frac{(N)_n}{(N_R)_n} \frac{\sigma_1^n}{n!}$$  

(63)

into (62), we have shown that $M_{\gamma_1} (s) = (1 - s \Gamma_{K,1})^{-N} \gamma_1 F_1 (N; N_R; s_1)$.

Corollary 5: If $S$ and $A$ are $N_R \times N_R$ matrices, $S$ of rank 1 with nonzero eigenvalue $\sigma_1$, and $A$ idempotent of rank $N$ then, by comparing (62) with (61), we have

$$\theta F_0 (S, A) = \gamma_1 F_1 (N; N_R; s_1).$$  

(64)

Corollary 6: From (64) and the determinantal expression for $\gamma_1 F_0 (S, A)$ from (90), we deduce for $\gamma_1 F_1 (N; N_R; s_1)$ the following new determinantal expression:

$$\gamma_1 F_1 (N; N_R; s_1) = A \Delta_2 (N, N_R; s_1) / \sigma_1^{N-1}.$$  

(65)

with $A$ defined in (90).

Finally, substituting $s_1$ from (59) into (65), and the result into (62) yields for the Stream-1 SNR m.g.f. the following new exact determinantal expression:

$$M_{\gamma_1} (s) = A \left( 1 - s \Gamma_{K,1} \right)^{-N-2} \Delta_2 (N, N_R; s_1 \Gamma_{K,1}) \left( \frac{\theta^2}{\sin^2 \theta} \Gamma_{K,1} \right)^{N-1} \left( \frac{\sin^2 \theta}{\sin^2 \theta + \frac{\Gamma_{K,1}}{\sin^2 \theta}} \right) d\theta.$$  

(66)

Then, substituting (66) into (64) yields the corresponding new exact AEP expression:

$$P_{e_{1}}^{\text{Max}} = \frac{1}{\pi} \int_0^{\pi} \frac{\theta^{N-1} \cdot (1 + \sin^2 \frac{\pi}{\sin^2 \theta} \Gamma_{K,1})^{N-2} \Delta_2 (N, N_R; s_1 \Gamma_{K,1}) \left( \frac{\theta^2}{\sin^2 \theta} \Gamma_{K,1} \right)^{N-1} \left( \frac{\sin^2 \theta}{\sin^2 \theta + \frac{\Gamma_{K,1}}{\sin^2 \theta}} \right) d\theta}.$$  

(67)

The SNR m.g.f. expression (66) and the AEP expression (67) are referenced in Table I Row 4, for Stream 1.

VII. NUMERICAL RESULTS

A. Description of Settings

For $v = 1$, i.e., the partitioning from (91), Stream-1 AEP results obtained in MATLAB are presented for $N_T = 4$, $N_T = 3$, QPSK modulation, and relevant ranges of the average SNR per transmitted bit $\Gamma_0 = \frac{1}{\rho}$, Matrix $R_{11}$ has been computed as in [9], for a uniform linear antenna array with interelement distance normalized to carrier half-wavelength.
$$d_n = 1,$$ Laplacian power azimuth spectrum centered at $$\theta_c = 5^\circ,$$ and $$K$$ and AS set to their lognormal-distribution averages for two WINNER II scenarios \[ \text{Table I}: \]

- B1 (typical urban microcell): $$K = 9 \text{ dB},$$ AS = $$3^\circ,$$ i.e., near-full transmit-correlation, and, thus, $$r_{2,1} \neq 0.$$ 
- A1 (indoor office/residential): $$K = 7 \text{ dB},$$ AS = $$51^\circ,$$ i.e., near-zero correlation, and, thus, $$r_{2,1} \approx 0.$$

In our figures, the legends identify results from exact and approximate AEP expressions (with exact, approx) and from simulation of $$10^6$$ channel and noise samples (with sim). All figures depict Rayleigh-only fading, with red lines and markers, and with legend Ray–Ray. Additionally, each figure depicts, with black lines and markers, one of the following Rician-fading cases: full-Rician (Ray–Rician), Rayleigh–Rician (Ray–Rice), or Rice–Rayleigh (Rice–Rice), for $$h_{d,1} = H_{d,2} \rho_{2,1},$$ $$h_{d,1} \approx H_{d,2} \rho_{2,1},$$ or $$h_{d,1} \neq H_{d,2} \rho_{2,1}.$$ Each case is also identified in figures and discussion by the corresponding Row number in Table I.

### B. Full-Rician Fading, High Correlation, $$h_{d,1} = H_{d,2} \rho_{2,1}$$

Fig. 1 depicts full-Rician fading, i.e., $$h_{d,1} \neq 0,$$ and $$H_{d,2} \neq 0,$$ under condition $$h_{d,1} = H_{d,2} \rho_{2,1},$$ which is characterized in Row 3, for scenario B1. Note first that analysis and simulation results agree. Then, as predicted by Corollary 4 the AEP from the exact and approximate expressions agree, because $$h_{d,1} = H_{d,2} \rho_{2,1}.$$ Finally, as predicted by Remark 1 Rician fading yields poorer performance than Rayleigh-only fading.

### C. Rayleigh–Rician, High Correlation, i.e., $$h_{d,1} \neq H_{d,2} \rho_{2,1}$$

Fig. 2 depicts Rayleigh(1)/Rician($$N_f - 1$$) fading with $$h_{d,1} = 0, H_{d,2} \neq 0,$$ for scenario B1, i.e., $$r_{2,1} \neq 0,$$ so that $$h_{d,1} \neq H_{d,2} \rho_{2,1},$$ which is characterized in Row 6. Since no exact AEP expression is then available, we have shown results only from simulation and approximation (see the Ray–Rice plots with black □ and + markers), which do not agree because $$h_{d,1} \neq H_{d,2} \rho_{2,1}.$$

Comparing the black plot with □ marker and the red plot with + marker reveals that, for Rayleigh fading intended stream and highly-correlated intended and interfering fading, surprisingly, Rician-fading interference yields better performance than Rayleigh-fading interference.

Finally, comparing Figs. 1 and 2 reveals opposite relative performance for Rician vs. Rayleigh fading (notice the relative positions of plots with black vs. red lines and markers).

### D. Rayleigh–Rician, Low Correlation, i.e., $$h_{d,1} \approx H_{d,2} \rho_{2,1}$$

Fig. 3 depicts the same fading cases as Fig. 2 but for scenario A1. (We obtained similar results in [12] Fig. 10.) Thus, we have $$r_{2,1} \approx 0,$$ and, because $$h_{d,1} = 0,$$ we have $$h_{d,1} \approx H_{d,2} \rho_{2,1},$$ which explains the agreement between the AEP from simulation and the approximate expression for the Ray–Rice plots. (Therefore, this case is characterized, approximately, by Row 2.)

Comparing the black plot with □ marker and the red plot with + marker reveals that, for Rayleigh fading intended stream and nearly-uncorrelated intended and interfering fading, Rician-fading interference yields worse performance than Rayleigh-fading interference. This outcome is also predicted by Remark 1 as $$h_{d,1} \approx H_{d,2} \rho_{2,1}.$$ Finally, the Rician-vs.-Rayleigh relative performance from Fig. 3 is the same as that from Fig. 1 where $$h_{d,1} = H_{d,2} \rho_{2,1},$$ but the opposite of that from Fig. 2 where $$h_{d,1} \neq H_{d,2} \rho_{2,1}.$$

### E. Rician–Rician, Low Correlation, $$h_{d,1} \neq H_{d,2} \rho_{2,1}$$

Fig. 4 depicts Rician(1)/Rayleigh($$N_f - 1$$) fading, i.e., $$h_{d,1} \neq 0$$ and $$H_{d,2} = 0,$$ for scenario A1. This case implies
Fig. 3. Stream-1 AEP from exact expression (45), approximate expression (56), and from simulation, for Rayleigh-only fading and for Rician[NT − 1] fading under conditions h_{d,1} = 0 and H_{d,2} ≠ 0, for QPSK modulation, NT = 4, NT = 3, K = 7 dB, AS = 51° (i.e., WINNER II scenario A1 averages). Since r_{2,1} ≈ 0, we have h_{d,1} ≈ H_{d,2}p_{2,1}.

Fig. 4. Stream-1 AEP from exact expressions (45) and (67), approximate expression (56), and from simulation, for Rayleigh-only fading and for Rician(NT − 1) fading under conditions h_{d,1} ≠ 0 and H_{d,2} = 0, i.e., h_{d,1} ≠ H_{d,2}p_{2,1}, for QPSK modulation, NT = 4, NT = 3, K = 7 dB, AS = 51° (i.e., WINNER II scenario A1 averages).

Note that Rician fading yields better performance than Rayleigh-only fading, like in Fig. 2 whereby h_{d,1} ≠ H_{d,2}p_{2,1}, but unlike in Figs. 1 and 7.

For the fading cases in Rows 2, 3, 4, and 6, unshown results obtained for AS and K set to their averages from other WINNER II scenarios [9, Table I] have yielded observations consistent with those made above for scenarios A1 and B1. The case in Row 5 has yielded observations similar to those for its special case in Row 4. For the most general case shown in Row 7, the AEP approximation has been found unreliable.

VIII. SUMMARY, DISCUSSION AND CONCLUSIONS

By characterizing the distribution of the matrix-SC in the NCWD Gramian matrix induced by a nonzero-mean Gaussian matrix, we analyzed MIMO ZF under transmit-correlated Rician fading. Although expressing the m.g.f. of the unconditioned matrix-SC and the ZF SNRs remains intractable for general Rician fading, we have succeeded for two cases.

The first tractable case arose by imposing the mean–correlation condition that yields a CWD for the matrix-SC. We have shown that this condition also renders exact a previously-proposed approximation with the gamma distribution of the unknown distribution of the ZF SNRs under Rician fading. This finding has been confirmed through numerical results, and has been used to explain observations from previous work. The principle of the mentioned gamma-distribution approximation for the scalar-SC yields an approximate CWD for the matrix-SC. We prove that this approximate CWD also becomes exact under the mentioned condition.

The second tractable case is that of Rician–Rayleigh fading. Then, the matrix-SC m.g.f. has been expressed in terms of the determinant of a matrix with elementary-function entries. This analysis has revealed new expressions for, and a new relationship between, hypergeometric functions of matrix and scalar arguments. Thus, we have also obtained new, determinantal expressions for the ZF SNR m.g.f. and AEP.

Finally, numerical results for the MIMO ZF AEP have revealed that Rician fading yields poorer performance than Rayleigh-only fading when the special mean–correlation condition holds, and better performance otherwise. We also revealed that, when the intended stream undergoes Rayleigh fading and the intended and interfering fading are highly correlated, Rician-fading interference can yield much better performance than Rayleigh-fading interference.

APPENDIX A

Proof of Theorem 2

\[ H_{d,1} = H_{d,2}R_{2,1} \iff \left( \hat{R}_{11}^{11} \right)^{-1} = \left( \hat{R}_{11}^{11} \right)^{-1} \]

Let us first find a simpler condition equivalent with \( \left( \hat{R}_{11}^{11} \right)^{-1} = \left( \hat{R}_{11}^{11} \right)^{-1} \). Equalizing the SC representation for \( \left( \hat{R}_{11}^{11} \right)^{-1} \) from (12) with that obtained analogously for \( \left( \hat{R}_{11}^{11} \right)^{-1} \) based on (46) yields

\[
\begin{align*}
    \hat{R}_{11}^{11} - \hat{R}_{12}^{12} & \hat{R}_{22}^{-1} - \hat{R}_{12}^{12} \hat{R}_{11}^{11} = \hat{R}_{11}^{11} - \frac{1}{\hat{N}_R} \hat{H}_{d,1}^H \hat{H}_{d,1}^H \\
    & \times \left( \hat{R}_{22}^{-1} + \frac{1}{\hat{N}_R} \hat{H}_{d,2}^H \hat{H}_{d,2}^H \right) \hat{R}_{12}^{12} \hat{R}_{11}^{11}
\end{align*}
\]

(68)
i.e.,
\[
\frac{1}{N_R} H_{d,1}^H H_{d,1} + R_{T,K_{12}} R_{T,K_{22}}^{-1} R_{T,K_{21}} = (R_{T,K_{12}} + \frac{1}{N_R} H_{d,1}^H H_{d,1}) P (R_{T,K_{21}} + \frac{1}{N_R} H_{d,1}^H H_{d,1}),
\]
or
\[
H_{d,1}^H \left( I_{N_k} - \frac{1}{N_R} H_{d,2} P H_{d,2}^H \right) H_{d,1}
+ N_R R_{T,K_{12}} \left( R_{T,K_{22}}^{-1} - P \right) R_{T,K_{21}} = H_{d,1}^H = F = H_{d,1}^H P R_{T,K_{21}} + R_{T,K_{12}} P H_{d,2}^H H_{d,1},
\]
or, finally,
\[
H_{d,1}^H Q H_{d,1} + N_R R_{T,K_{12}} \left( R_{T,K_{22}}^{-1} - P \right) R_{T,K_{21}} = H_{d,1}^H F = H_{d,1}^H F + F^H H_{d,1}. \tag{69}
\]

The Woodbury matrix-inversion formula \cite{9} p. 165] yields
\[
Q = I_{N_k} - \frac{1}{N_R} H_{d,2} \left( R_{T,K_{22}} + \frac{1}{N_R} H_{d,2}^H H_{d,2} \right)^{-1} H_{d,2}^H
\]
\[
= \left( I_{N_k} + \frac{1}{N_R} H_{d,2} R_{T,K_{22}}^{-1} H_{d,2}^H \right)^{-1} H_{d,2}^H \tag{70}
\]
\[
P = \left( R_{T,K_{22}} + \frac{1}{N_R} H_{d,2}^H H_{d,2} \right)^{-1}
\]
\[
= R_{T,K_{22}}^{-1} - R_{T,K_{22}}^{-1} H_{d,2}^H H_{d,2}
\times \frac{1}{N_R} \left( I_{N_k} + \frac{1}{N_R} H_{d,2} R_{T,K_{22}}^{-1} H_{d,2}^H \right)^{-1} H_{d,2} R_{T,K_{22}}^{-1}, \tag{71}
\]
i.e.,
\[
R_{T,K_{22}}^{-1} - P = \frac{1}{N_R} R_{T,K_{22}}^{-1} H_{d,2}^H Q H_{d,2} R_{T,K_{22}}^{-1}. \tag{72}
\]

Substituting (71) into (69) yields
\[
H_{d,1}^H Q H_{d,1} + R_{T,K_{22}} H_{d,2}^H H_{d,2}^H Q H_{d,2} R_{T,K_{22}}^{-1} R_{T,K_{21}} = H_{d,1}^H F = H_{d,1}^H F + F^H H_{d,1},
\]
or
\[
H_{d,1}^H Q H_{d,1} - H_{d,1}^H F - F^H H_{d,1} + B^H Q B = 0, \tag{73}
\]
where
\[
F = H_{d,2} R_{T,K_{21}}
\]
\[
= H_{d,2} \left( R_{T,K_{22}}^{-1} - \frac{1}{N_R} R_{T,K_{22}}^{-1} H_{d,2} Q H_{d,2} R_{T,K_{22}}^{-1} \right) R_{T,K_{21}}
\]
\[
= B - \frac{1}{N_R} H_{d,2} R_{T,K_{22}}^{-1} H_{d,2}^H Q B
\]
\[
= B - (Q^{-1} - I_{N_k}) Q B = QB.
\]

Thus, (73) becomes
\[
H_{d,1}^H Q H_{d,1} - H_{d,1}^H F - F^H H_{d,1} + B^H Q B = 0, \tag{73}
\]
which is the sought simpler expression equivalent with
\[
\left( \tilde{R}_{11}^{T,K} \right)^{-1} = (R_{T,K}^{11})^{-1}.
\]

Now, let us assume that \( \left( \tilde{R}_{11}^{T,K} \right)^{-1} \) holds, i.e., that (73) holds. Then, with \( H_{d,1} = Q^{1/2} H_{d,1} \) and \( B = Q^{1/2} B \), (73) becomes
\[
\tilde{H}_{d,1} - \tilde{H}_{d,1} \tilde{B} - \tilde{B}^H \tilde{H}_{d,1} + \tilde{B}^H \tilde{B} = 0, \tag{74}
\]
which can be written further as
\[
\tilde{H}_{d,1} \left( \tilde{H}_{d,1} - \tilde{B} \right) - \tilde{B}^H \left( \tilde{H}_{d,1} - \tilde{B} \right) = 0, \tag{75}
\]
or
\[
\left( \tilde{H}_{d,1} - \tilde{B} \right)^H \left( \tilde{H}_{d,1} - \tilde{B} \right) = 0, \tag{76}
\]
which implies
\[
\tilde{H}_{d,1} = \tilde{B} \iff H_{d,1} = B = H_{d,2} R_{T,K_{22}}^{-1} R_{T,K_{21}} = H_{d,2} R_{2,1}.
\]

Assuming, conversely, that \( H_{d,1} = H_{d,2} R_{T,K_{22}}^{-1} R_{T,K_{21}} \) implies that \( H_{d,1} = B \), which reduces the left-hand side of (73) to 0, and implies \( \left( \tilde{R}_{11}^{T,K} \right)^{-1} = (R_{T,K}^{11})^{-1} \).

**APPENDIX B**

\( M_{R_{1}}(\Theta) \) FOR Rician(\( v \))-Rayleigh(\( N_{T} - v \)) FADING

A. \( M_{R_{1}}(\Theta) \) for \( H_{d,2} = 0 \), in Terms of \( F_{0}(S,A) \)

Let us first consider the singular vector decomposition
\[
H_{2} = U \Sigma V^H, \tag{77}
\]
where \( U = N_{R} \times N_{R}, \Sigma = N_{R} \times (N_{T} - v), \) and \( V = (N_{T} - v) \times (N_{T} - v) \). The unitary matrix \( U \), i.e., \( U^H U = U U^H = I_{N_{k}} \), comprises the left singular vectors of \( H_{2} \). Using the definition of \( Q_{2} \) from (16), it can be shown that \( U \) is also the matrix with the eigenvectors of \( Q_{2} \). Further, using (18), we can write the eigendecomposition of \( Q_{2} \) as:
\[
Q_{2} = U^H \text{diag}(1, 1, \ldots, 1, 0, 0, \ldots, 0) U. \tag{78}
\]

Substituting (78) into (30) yields
\[
M_{R_{1}}(\Theta) = |I_{v} - \Theta (R_{11}^{T,K})^{-1}|^{-1} \Psi
\times \text{etr} \left[ \left( I_{v} - \Theta (R_{11}^{T,K})^{-1} \right)^{-1} \Theta M^H U A U^H M \right]. \tag{79}
\]

Now, averaging the \( \text{etr}(\cdot) \) term over \( U \) appears to be tractable only for \( H_{d,2} = 0 \), when matrix \( U \) has a known, Haar, distribution \cite{12}. This averaging has been pursued successfully for \( v = 1 \) in \cite{12}. Herein, we pursue the more general case whereby \( 1 \leq v < N_{T} \). Then,
\[
\mathbb{E}_{U} \left\{ \text{etr} (\Psi M^H U A U^H M) \right\}
= \int_{U_{N_{k}}} \text{etr} (\Psi M^H U A U^H M) |dU|
= \int_{U_{N_{k}}} \text{etr} (S U A U^H) |dU| = \int_{U_{N_{k}}} \partial F_{0}(S U A U^H) |dU|.
\]
where $U_{N_k}$ is the unitary manifold comprising the $N_k \times R$ unitary matrices with real diagonal elements, and $[dU]$ is the normalized Haar invariant probability measure on $U_{N_k}$ [10 Appendix 1]. Matrix $S \equiv N_R \times N_R$, which is given by
\[ S = M \sum H = M \left[ I - \Theta \left( R_{11}^{T,R} \right)^{-1} \right] \Theta M, \]
has rank $v$ and distinct nonzero eigenvalues, in general. Since [23 Eq. (92)] [29 Eq. (4.2)]
\[ \int_{U_{N_k}} \phi_0 \left( S \sum U H \right) [dU] = \phi_0 \left( S, \Lambda \right), \]
the m.g.f. of $\Gamma_1$ can be written as
\[ M_{\Gamma_1} (\Theta) = \left| I - \Theta \left( R_{11}^{T,R} \right)^{-1} \right|^{-N_0} \phi_0 \left( S, \Lambda \right), \]
where $\phi_0 \left( S, \Lambda \right)$ is expressed next for several cases.

**B. Both $S$, $\Lambda$ with Distinct Eigenvalues** [29] [10] [24]

Given $\sigma_1 > \sigma_2 > \cdots > \sigma_{N_k}$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_{N_k}$, let us define
\[ g(\sigma, \lambda) = g(\sigma_1, \sigma_2, \ldots, \sigma_{N_k}, \lambda_1, \lambda_2, \ldots, \lambda_{N_k}) \]
\[ = \prod_{i < j} (\sigma_i - \sigma_j) \prod_{i < j} (\lambda_i - \lambda_j), \]
\[ (83) \]
\[ \text{where } \text{det} (e^{\sigma_i X}) = i, j = 1 : N_R. \]

**Lemma 5:** The continuous extension of $g(\sigma, \lambda)$ from $\phi(N_k)$ at $(\sigma^0, \lambda^0)$ helps express $\phi_0 \left( S, \Lambda \right)$ from (84), for $S$ and $\Lambda$ with arbitrary eigenvalues, as
\[ \det \left( \frac{\partial^{i,j} (e^{\sigma_i X})}{\partial \sigma^i \partial \lambda^j} \right) \prod_{i < j} (\sigma_i - \sigma_j) \prod_{i < j} (\lambda_i - \lambda_j), \]
\[ (87) \]
\[ \text{Proof: } \text{Follows by generalizing [24 Lemma 2].} \]

Expression (87) reduces to previously derived expressions:
- [24 Eq. (10)], for both $S$ and $\Lambda$ with distinct eigenvalues — see also [84].
- [24 Eq. (16)], for $S$ with distinct eigenvalues and $\Lambda$ with one subset of equal eigenvalues.
- [24 Eq. (18)], for $S$ with distinct eigenvalues and $\Lambda$ with one subset of zero eigenvalues.

**D. $S$: Rank-$v$, with Distinct Nonzero Eigenvalues; $\Lambda$: Idempotent, Rank-$N_v$**

**Corollary 7:** If $S$ and $\Lambda$ are $N_R \times N_R$ matrices, $S$ of rank $v$ and with the nonzero distinct eigenvalues $\sigma_i, i = 1 : v$, and $\Lambda$ of rank $N_v$ and idempotent, then $\phi_0 \left( S, \Lambda \right)$ is given by
\[ \Delta_1 (N_v, N_R, S) \phi(N_k) \]
\[ \prod_{i < j} (\sigma_i - \sigma_j) \prod_{i < j} (\lambda_i - \lambda_j), \]
\[ (88) \]
\[ \text{where } \Delta_1 (N_v, N_R, S) \text{ is the determinant of the } N_R \times N_R \text{ matrix with (elementary-function) elements} \]
\[ e^{\sigma_i X} N_v - j, \quad \text{if } i < v, j = N_v \]
\[ e^{\sigma_i X} N_v - j, \quad \text{if } i < v, j > N_v \]
\[ (N_v - j)! (N_v - i), \quad \text{if } i > v, j \leq N_v, N_R - i \geq N_v - j \]
\[ 0, \quad \text{if } i > v, j < N_v, N_R - i < N_v - j \]
\[ (N_v - i)! \quad \text{if } i > v, j > N_v, i = j \]
\[ 0, \quad \text{if } i > v, j > N_v, i \neq j. \]
\[ (89) \]
\[ \text{Proof: } \text{Follows from (87).} \]

Substituting (88) into (82) yields the first known expression (in terms of the determinant of a matrix whose entries are elementary functions) for the m.g.f. of $\Gamma_1$, i.e., for the SC in the NCWD Gramian matrix $W = H^T H$ obtained from matrix $H = (H_1, H_2)$ with mean $(H_{d,1}, 0)$.

**E. $S$: Rank-1; $\Lambda$: Idempotent, Rank-$N$**

For $v = 1$, $N_v$ reduces to $N_R - N_T + 1 = N$, matrix $M = N_R \times v$ reduces to vector $\mu = N_R \times 1$, and $S$ can be written from (60) as follows
\[ s \Gamma_1 \mu \mu^H = \frac{\Gamma_{k,1}}{1 - s \Gamma_1 \sum_{i = 1}^{R_{11}} \mu \mu^H} \]
\[ = \frac{\Gamma_{k,1}}{1 - s \Gamma_1 \sum_{i = 1}^{R_{11}} \mu \mu^H}, \]
\[ (90) \]
\[ \text{To simplify writing, we change the notation for } \sigma. \]
\[ \text{Here, we replace matrix symbol } M \text{ with vector symbol } \mu \text{ for notational consistency with our previous work [12], although we have maintained the boldface capital notation for some variables that become scalars for } v = 1, \]
\[ \text{e.g., for } \Gamma_1, \Theta, R_{11}. \]
i.e., $S$ is rank-1 and with the nonzero eigenvalue given by

$$\sigma_1 = \frac{s_i}{1 - s_i K_i},$$  \hspace{1cm} (89)

**Lemma 6:** If $S$ and $\Lambda$ are $N_R \times N_R$ matrices, $S$ of rank 1 with nonzero eigenvalue $\sigma_1$, and $\Lambda$ of rank $N$ and idempotent, then $\sigma F_S(S, \Lambda)$ is given by

$$\sigma F_S(S, \Lambda) = \frac{(N_R - 1)!}{\phi(N) \phi(N_R - N)} \frac{\Delta_2(N, N_R, \sigma_1)}{\sigma_1^{N_R - 1}},$$  \hspace{1cm} (90)

where $\Delta_2(N, N_R, \sigma_1)$ is the determinant of the $N_R \times N_R$ matrix with (elementary-function) elements

$$
\begin{align*}
&\sigma_1^{N_R - j} + \sigma_1^{N - j} + \sigma_1^{N_R - j}, & &\text{if } i = 1, j \leq N \\
&\sigma_1^{N - j}, & &\text{if } i = 1, j > N \\
&(N - j)! \left(\frac{N_R - j}{N - j}\right), & &\text{if } i > 1, j \leq N, N_R - i \geq N - j \\
&0, & &\text{if } i > 1, j \leq N, N_R - i < N - j \\
&(N_R - i)! , & &\text{if } i > 1, j > N, i = j \\
&0, & &\text{if } i > 1, j > N, i \neq j.
\end{align*}
$$

**Proof:** Follows from (88).

**REFERENCES**

[1] T. L. Marzetta and B. M. Hochwald, “Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading,” IEEE Transactions on Information Theory, vol. 45, no. 1, pp. 139–157, Jan. 1999.

[2] A. J. Paulraj, R. U. Nabar, and D. A. Gore, Introduction to Space-Time Wireless Communications. Cambridge, UK: Cambridge University Press, 2003.

[3] D. Gesbert, M. Kountouris, R. W. Heath, C.-B. Chae, and T. Salzer, “Shifting the MIMO paradigm,” IEEE Signal Processing Magazine, vol. 24, no. 5, pp. 36–46, 2007.

[4] M. Jung, Y. Kim, J. Lee, and S. Choi, “Optimal number of users in zero-forcing based multiuser MIMO systems with large number of antennas,” Journal of Communications and Networks, vol. 15, no. 4, pp. 362–369, 2013.

[5] M. Matthaiou, C. Zhong, M. McKay, and T. Ramarajah, “Sum rate analysis of ZF receivers in distributed MIMO systems,” IEEE Journal on Selected Areas in Communications, vol. 31, no. 2, pp. 180–191, 2013.

[6] J. Lee, J. Han, and J. Zhang, “MIMO technologies in 3GPP LTE and LTE-Advanced,” EURASIP Journal on Wireless Communications and Networking, vol. 2009, 2009.

[7] Q. Li, G. Li, W. Lee, M. Lee, D. Mazzarese, B. Clerckx, and Z. Li, “MIMO techniques in WiMAX and LTE: a feature overview,” IEEE Communications Magazine, vol. 48, no. 5, pp. 86–92, May 2010.

[8] M. Kiessling and J. Speidel, “Analytical performance of MIMO zero-forcing receivers in correlated Rayleigh fading environments,” in IEEE Workshop on Signal Processing Advances in Wireless Communications (SPAWC’03), June 2003, pp. 383–387.

[9] C. Siriteanu, Y. Miyazaka, S. D. Blostein, S. Kuriki, and X. Shi, “MIMO zero-forcing detection analysis for correlated and estimated Rician fading,” IEEE Transactions on Vehicular Technology, vol. 61, no. 7, pp. 3087–3099, September 2012.

[10] M. McKay, A. Zanella, I. Collings, and M. Chiani, “Error probability and SINR analysis of optimum combining in Rician fading,” IEEE Transactions on Communications, vol. 57, no. 3, pp. 676–687, March 2009.

[11] P. Kyosti, J. Meinila, L. Hentila, and et al., “WINNER II Channel Models. Part I,” CEC, Tech. Rep. IST-4-027756, 2008.

[12] C. Siriteanu, S. D. Blostein, A. Takemura, H. Shin, S. Yousefi, and S. Kuriki, “Exact MIMO zero-forcing detection analysis for transmit-correlated Rician fading,” IEEE Transactions on Wireless Communications, accepted, December 2013. [Online]. Available: http://arxiv.org/abs/1307.2058

[13] J. E. Gentle, Matrix algebra: theory, computations, and applications in statistics. Springer, 2007.