Mini-batch Stochastic ADMMs for Nonconvex Nonsmooth Optimization

Feihu Huang and Songcan Chen

Abstract—With the large rising of complex data, the nonconvex models such as nonconvex loss function and nonconvex regularizer are widely used in machine learning and pattern recognition. In this paper, we propose a class of mini-batch stochastic ADMMs (alternating direction method of multipliers) for solving large-scale nonconvex nonsmooth problems. We prove that, given an appropriate mini-batch size, the mini-batch stochastic ADMM without variance reduction (VR) technique is convergent and reaches a convergence rate of $O(1/T)$ to obtain a stationary point of the nonconvex optimization, where $T$ denotes the number of iterations. Moreover, we extend the mini-batch stochastic gradient method to both the nonconvex SVRG-ADMM and SAGA-ADMM proposed in our initial manuscript [1], and prove these mini-batch stochastic ADMMs also reaches the convergence rate of $O(1/T)$ without condition on the mini-batch size. In particular, we provide a specific parameter selection for step size $\eta$ of stochastic gradients and penalty parameter $\rho$ of augmented Lagrangian function. Finally, extensive experimental results on both simulated and real-world data demonstrate the effectiveness of the proposed algorithms.

Index Terms—ADMM, stochastic gradient, nonconvex optimization, graph-guided fused Lasso, overlapping group Lasso.

1 INTRODUCTION

Stochastic optimization [2] is a class of powerful optimization tool for solving large-scale problems in machine learning, pattern recognition and computer vision. For example, stochastic gradient descent (SGD [2]) is an efficient method for solving the following optimization problem, which is a fundamental to machine learning,

$$\min_{x \in \mathbb{R}^d} f(x) + g(x)$$

(1)

where $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$ denotes the loss function, and $g(x)$ denotes the regularization function. The problem (1) includes many useful models such as support vector machine (SVM), logistic regression and neural network. When sample size $n$ is large, even the first-order methods become computationally burdensome due to their per-iteration complexity of $O(nd)$. While SGD only computes gradient of one sample instead of all samples in each iteration, thus it has only per-iteration complexity of $O(d)$. Despite its scalability, due to the existence of variance in stochastic process, the stochastic gradient is much noisier than the batch gradient. Thus, the step size has to be decreased gradually as stochastic learning proceeds, leading to slower convergence than the batch method. Recently, a number of accelerated algorithms have successfully been proposed to reduce this variance. For example, stochastic average gradient (SAG [3]) obtains a fast convergence rate by incorporating the old gradients estimated in the previous iterations. Stochastic dual coordinate ascent (SDCA [4]) performs the stochastic coordinate ascent on the dual problems to obtain also a fast convergence rate. Moreover, an accelerated randomized proximal coordinate gradient (APCG [5]) method accelerates the SDCA method by using Nesterov’s accelerated method [6]. However, these fast methods require much space to store old gradients or dual variables. Thus, stochastic variance reduced gradient (SVRG [7], [8]) methods are proposed, and enjoy a fast convergence rate with no extra space to store the intermediate gradients or dual variables. Moreover, [9] proposes the SAGA method, which extends the SAG method and enjoys better theoretical convergence rate than both SAG and SVRG. Recently, [10] presents an accelerated SVRG by using the Nesterov’s acceleration technique [6]. Moreover, [11] proposes a novel momentum accelerated SVRG method (Katyusha) via using the strongly convex parameter, which reaches a faster convergence rate. In addition, [12] specially proposes a class of stochastic composite optimization methods for sparse learning, when $g(\cdot)$ is a sparsity-inducing regularizer such as $\ell_1$-norm and nuclear norm.

Though the above methods can effectively solve many problems in machine learning, they are still difficultly to be competent for some complicated problems with the nonseparable and nonsmooth regularization function as follows

$$\min_{x \in \mathbb{R}^d} f(x) + g(Ax)$$

(2)

where $A \in \mathbb{R}^{d \times p}$ is a given matrix, $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$ denotes the loss function, and $g(x)$ denotes the regularization function. With regard to $g(\cdot)$, we are interested in a sparsity-inducing regularization functions, e.g. $\ell_1$-norm and nuclear norm. The problem (2) includes the graph-guided fused Lasso [13], the overlapping group Lasso [17], and generalized Lasso [18]. It is well known that the alternating direction method of multipliers (ADMM [19], [20], [21]) is an efficient optimization method for the problem (2). Specifically, we can use auxiliary variable $y = Ax$ to make the problem (2) be suitable for the general ADMM form. When sample size $n$ is large, due to the need of computing the empirical risk loss function on all training samples at each
iteration, the offline or batch ADMM is unsuitable for large-scale learning problems. Thus, the online and stochastic versions of ADMM [22], [23], [24] have been successfully developed for the large-scale problems. Due to the existence of variance in the stochastic process, these stochastic ADMMs also suffer from the slow convergence rate. Recently, some accelerated stochastic ADMMs are effectively proposed to reduce this variance. For example, SAG-ADMM [25] is proposed by additionally using the previous estimated gradients. An accelerated stochastic ADMM [26] is proposed by using Nesterov’s accelerated method [6]. SDCA-ADMM [27] obtains linearly convergence rate for the strong problem by solving its dual problem. SCAS-ADMM [28] and SVRG-ADMM [29] are developed, and reach the fast convergence rate with no extra space for the previous gradients or dual variables. Moreover, [30] proposes an accelerated SVRG-ADMM by using the momentum accelerated technique. More recently, [31] proposes a fast stochastic ADMM, which achieves a non-ergodic convergence rate of $O(1/T)$ for the convex problem. In addition, an adaptive stochastic ADMM [32] is proposed by using the adaptive gradients. Due to that the penalty parameter in ADMM can affect convergence [33], another adaptive stochastic ADMM [34] is proposed by using the adaptive penalty parameters.

So far, the above study on stochastic optimization methods relies heavily on strongly convex or convex problems. However, there exist many useful nonconvex models in machine learning such as nonconvex empirical risk minimization models [35] and deep learning [36]. Thus, the study of nonconvex optimization methods is much needed. Recently, some works focus on studying the stochastic gradient methods for the large-scale nonconvex optimizations. For example, [37], [38] have established the iteration complexity of $O(1/\epsilon^2)$ for the SGD to obtain an $\epsilon$-stationary solution of the nonconvex problems. [39], [40], [41] have proved that the variance reduced stochastic gradient methods such as the nonconvex SVRG and SAGA reach the iteration complexity of $O(1/\epsilon)$. At the same time, [42] has proved that the variance reduced stochastic gradient methods also reach the iteration complexity of $O(1/\epsilon)$ for the nonconvex nonsmooth composite problems. More recently, [43] propose a faster nonconvex stochastic optimization method (Natasha) via using the strongly non-convex parameter. [44] proposes a faster gradient-based nonconvex optimization by using catalyst approach in [45].

Similarly, the above nonconvex methods are difficult to be competent to some complicated nonconvex problems, such as nonconvex graph-guided regularization risk loss minimizations [1] and tensor decomposition [46]. Recently, some works [47], [48], [49], [50] have begun to study the ADMM method for the nonconvex optimization, but they only focus on studying the deterministic ADMMs for the nonconvex optimization. Due to the need of computing the empirical loss function on all the training examples at each iteration, these nonconvex ADMMs are not yet well competent to the large-scale learning problems. Recently, [13] has proposed a distributed, asynchronous and incremental algorithm based on the ADMM method for the large-scale nonconvex problems, but this method is difficult for the nonconvex problem (2) with the nonseparable and non-smooth regularizers such as graph-guided fused lasso and overlapping group lasso. A nonconvex primal dual splitting (NESTT [14]) method is proposed for the distributed and stochastic optimization, but it is also difficult for the nonconvex problem (2). More recently, our initial manuscript [1] proposes the stochastic ADMMs with variance reduction (e.g., nonconvex SVRG-ADMM and nonconvex SAGA-ADMM) for optimizing these nonconvex problems with some complicated structure regularizers such as graph-guided fused Lasso, overlapping group Lasso, sparse plus low-rank penalties. In addition, our initial manuscript [1] and Zheng and Kwok’s paper [15] simultaneously propose the nonconvex SVRG-ADMM method [1]. At present, to our knowledge, there still exist two important problems needing to be addressed:

1) Whether the general stochastic ADMM without VR technique is convergent for the nonconvex optimization?
2) What is convergence rate of the general stochastic ADMM for the nonconvex optimization, if convergent?

In the paper, we provide the positive answers to them by developing a class of mini-batch stochastic ADMMs for the nonconvex optimization. Specifically, we study the mini-batch stochastic ADMMs for optimizing the nonconvex nonsmooth problem below:

$$
\min_{x \in \mathbb{R}^d, y \in \mathbb{R}^p} f(x) + g(y) \\
\text{s.t. } Ax + By = c,
$$

where $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$, each $f_i(x)$ is a nonconvex and smooth loss function, $g(y)$ is nonsmooth and possibly

| Convergence rate Problems | $\min_{x \in \mathbb{R}^d, y \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} f_i(x) + g(x)$ | $\min_{x \in \mathbb{R}^d, y \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} f_i(x) + g(Ax)$ |
|---------------------------|---------------------------------|---------------------------------|
| Nonconvex incremental ADMM [13] | \(\checkmark\), unknown | Unknown |
| NESTT [14] | \(\checkmark\), $O(1/T)$ | $O(1/T)$ |
| Nonconvex mini-batch stochastic ADMM (ours) | \(\checkmark\), $O(1/T)$ | $O(1/T)$ |
| Nonconvex SVRG-ADMM (ours and [15]) | \(\checkmark\), $O(1/T)$ | $O(1/T)$ |
| Nonconvex SAGA-ADMM (ours) | \(\checkmark\), $O(1/T)$ | $O(1/T)$ |

1. The first version of our manuscript [1] (https://arxiv.org/abs/1610.02758v1) proposes both non-convex SVRG-ADMM and SAGA-ADMM, which is online available in Oct. 10, 2016. The first version of [15] (https://arxiv.org/abs/1604.07070v1) only proposes the convex SVRG-ADMM, which is online available in Apr. 24, 2016 and named as 'Fast-and-Light Stochastic ADMM'. While, the second version of [15] (https://arxiv.org/abs/1604.07070v2) adds the non-convex SVRG-ADMM, which is online available in Oct. 12, 2016 and renamed as 'Stochastic Variance-Reduced ADMM'.
nonconvex, and $A \in \mathbb{R}^{n \times d}$, $B \in \mathbb{R}^{m \times p}$ and $c \in \mathbb{R}^{n}$ denote the given matrices and vector, respectively. The problem \((3)\) is inspired by the structural risk minimization in machine learning [51]. In summary, our main contributions are fourfold as follows:

1) We propose the mini-batch stochastic ADMM for the nonconvex nonsmooth optimization. Moreover, we prove that, given an appropriate mini-batch size, the mini-batch stochastic ADMM reaches a fast convergence rate of $O(1/T)$ to obtain a stationary point.

2) We extend the mini-batch stochastic gradient method to both the nonconvex SVRG-ADMM and SAGA-ADMM, proposed in our initial manuscript [1]. Moreover, we prove that these stochastic ADMMs also reach a convergence rate of $O(1/T)$ without condition on the mini-batch size.

3) We provide a specific parameter selection for the size of stochastic gradients and penalty parameter $\rho$ of the augmented Lagrangian function.

4) Some numerical experiments demonstrate the effectiveness of the proposed algorithms.

In addition, Table 1 shows the convergence rate summary of the stochastic/incremental ADMMs for optimizing the nonconvex problems.

1.1 Notations

$\| \cdot \|$ denotes the Euclidean norm of a vector or the spectral norm of a matrix. $I_p$ denotes an $p$-dimensional identity matrix. $H > 0$ denotes a positive definite matrix $H$, and $\|x\|_H^2 = x^T H x$. Let $A^+$ denote the generalized inverse of matrix $A$. $\phi_{\min}^A$ denotes the smallest eigenvalues of matrix $AA^T$. $\phi_{\max}^H$ and $\phi_{\min}^H$ denotes the largest and smallest eigenvalues of positive matrix $H$, respectively. The other notations used in this paper is summarized as follows:

\[
\tilde{L} = L + 1, \quad \phi^H = (\phi_{\min}^H)^2 + 20(\phi_{\max}^H)^2, \quad \zeta_1 = \frac{5L\eta^2}{\phi_{\min}^H \eta^2},
\]

and $\zeta_1 = \frac{5(\phi_{\min}^H)^2}{\phi_{\min}^H \eta^2}$.

2 Nonconvex Mini-batch Stochastic ADMM without VR

In this section, we propose a mini-batch stochastic ADMM to optimize the nonconvex problem \((3)\). Moreover, we study convergence of the mini-batch stochastic ADMM. In particular, we prove that, given an appropriate mini-batch size, it reaches the convergence rate of $O(1/T)$.

First, we review the deterministic ADMM for solving the problem \((3)\). The augmented Lagrangian function of \((3)\) is defined as follows:

\[
\mathcal{L}_\rho(x, y, \lambda) = f(x) + g(y) - \langle \lambda, Ax + By - c \rangle + \frac{\rho}{2} \|Ax + By - c\|^2,
\]

where $\lambda$ is the Lagrange multiplier, and $\rho$ is the penalty parameter. At $t$-th iteration, the ADMM executes the update:

\[
y_{t+1} = \arg \min_y \mathcal{L}_\rho(x_t, y, \lambda_t),
\]

\[
x_{t+1} = \arg \min_x \mathcal{L}_\rho(x, y_{t+1}, \lambda_t),
\]

\[
\lambda_{t+1} = \lambda_t - \rho(Ax_{t+1} + By_{t+1} - c).
\]

Next, we give a mild assumption, as in the general stochastic optimization [37], [38] and the initial convex stochastic ADMM [24].

Assumption 1. For smooth function $f(x)$, there exists a stochastic first-order oracle that returns a noisy estimation to the gradient of $f(x)$, and the noisy estimation $G(x, \xi)$ satisfies

\[
E[G(x, \xi)] = f(x),
\]

\[
E[\|G(x, \xi) - \nabla f(x)\|^2] \leq \sigma^2;
\]

where the expectation is taken with respect to the random variable $\xi$.

Let $M$ be the size of mini-batch $I$, and $\xi = \{\xi_1, \xi_2, \cdots, \xi_M\}$ denotes a set of i.i.d. random variables, and the stochastic gradient is given by

\[
G(x, \xi) = \frac{1}{M} \sum_{i \in I} G(x, \xi_i).
\]

Clearly, we have

\[
E[G(x, \xi)] = \nabla f(x),
\]

\[
E[\|G(x, \xi) - \nabla f(x)\|^2] \leq \sigma^2/M.
\]

Algorithm 1 Mini-batch Stochastic ADMM (STOC-ADMM) for Nonconvex Nonsmooth Optimization

1: Input: Number of iteration $T$, Mini-batch size $0 < M < n$ and $\rho > 0$;
2: Initialize: $x_0, y_0$ and $\lambda_0$;
3: for $t = 0, 1, \cdots, T - 1$ do
4: Uniformly randomly pick a mini-batch $I_t$ from $\{1, 2, \cdots, n\}$;
5: $y_{t+1} = \arg \min_y \mathcal{L}_\rho(x_t, y, \lambda_t)$;
6: $x_{t+1} = \arg \min_x \mathcal{L}_\rho(x, y_{t+1}, \lambda_t, x_t, G(x_t, \xi_t))$;
7: $\lambda_{t+1} = \lambda_t - \rho(Ax_{t+1} + By_{t+1} - c)$;
8: end for
9: Output: Iterate $x$ and $y$ chosen uniformly random from $\{x_t, y_t\}_{t=1}^T$.

In the stochastic ADMM algorithm, we can update $y$ and $\lambda$ by \((5)\) and \((7)\), respectively, as in the deterministic ADMM. However, to update the variable $x$, we will define an approximated function of the form:

\[
\tilde{L}_\rho(x; y_{t+1}, \lambda_t, x_t, G(x_t, \xi_t)) = f(x_t) + G(x_t, \xi_t)^T (x - x_t) + \frac{1}{2\eta} \|x - x_t\|_H^2 - (\lambda_t, Ax + By_{t+1} - c) + \frac{\rho}{2} \|Ax + By_{t+1} - c\|^2,
\]

where $E[G(x_t, \xi_t)] = \nabla f(x_t), \eta > 0$ and $H > 0$. By minimizing \((12)\) on the variable $x$, we have

\[
x_{t+1} = \left(\frac{H}{\eta} + \rho A^T A\right)^{-1} \left[\frac{H}{\eta} x_t - G(x_t, \xi_t) - \rho A^T (By_{t+1} - c - \frac{\lambda_t}{\rho})\right].
\]

When $A^T A$ is large, computing $\left(\frac{H}{\eta} + \rho A^T A\right)^{-1}$ is expensive, and storage of this matrix may still be problematic. To avoid them, we can use the inexact Uzawa method [52] to linearize the last term in \((12)\). In other words, we set $H = r I - \rho A^T A$ with

\[
r \geq r_{\min} \equiv \eta \rho \|A^T A\| + 1
\]
Lemma 1. The iteration complexity of the mini-batch stochastic ADMM.

Finally, we give the algorithmic framework of the mini-batch stochastic ADMM (STOC-ADMM) in Algorithm 1.

2.1 Convergence Analysis of Nonconvex Mini-batch STOC-ADMM

In the subsection, we study the convergence and iteration complexity of the nonconvex mini-batch STOC-ADMM. First, we give some mild assumptions as follows:

Assumption 2. For smooth function \( f(x) \), its gradient is Lipschitz continuous with the constant \( L > 0 \), such that

\[
\| \nabla f(x_1) - \nabla f(x_2) \| \leq L \| x_1 - x_2 \|, \quad \forall x_1, x_2 \in \mathbb{R}^d,
\]

and this is equivalent to

\[
f(x_1) \leq f(x_2) + \nabla f(x_2)^T(x_1 - x_2) + \frac{L}{2} \| x_1 - x_2 \|^2.
\]

Assumption 3. \( f(x) \) and \( g(y) \) are all lower bounded, and denoting \( f^* = \min_x f(x) \) and \( g^* = \min_y g(y) \).

Assumption 4. \( g(y) \) is a proper lower semi-continuous function.

Assumption 5. Matrix \( A \) has full row rank.

Definition 1. For \( \epsilon > 0 \), the point \((x^*, y^*, \lambda^*)\) is said to be an \( \epsilon \)-stationary point of the nonconvex problem (3) below:

\[
\begin{align*}
\Psi_{t+1} - \Psi_t &\leq -\gamma \| x_{t+1} - x_t \|^2 + \frac{(\lambda^*_{\text{min}})^2}{2} + \frac{\lVert \lambda^* \rVert^2}{2} \rho \\ &+ \frac{5(L^2 \epsilon^2 + (\lambda^*_{\text{max}})^2)}{\epsilon^2} + \frac{5(L^2 \epsilon^2 + (\lambda^*_{\text{max}})^2)}{\epsilon^2} + \frac{5(L^2 \epsilon^2 + (\lambda^*_{\text{max}})^2)}{\epsilon^2},
\end{align*}
\]

where \( \rho = \frac{\lambda^*_{\text{min}}}{\lambda^*_{\text{max}}} \) and \( \lambda^*_{\text{min}} = \min \{ \lambda^*_i \} \).

A detailed proof of Lemma 1 is provided in Appendix A.1. Lemma 1 gives the upper bound of \( E\| \lambda_{t+1} - \lambda_t \|^2 \). Given a sequence \( \{x_t, y_t, \lambda_t\}_{t=1}^T \) generated from Algorithm 1, then we define an useful sequence \( \{\Psi_t\}_{t=1}^T \) as follows:

\[
\Psi_t = E \left[ L \rho \epsilon (x_t, y_t, \lambda_t) + \frac{\zeta}{\rho} \| x_t - x_{t-1} \|^2 \right].
\]

For notational simplicity, let \( \tilde{L} = L + 1, \phi^H = (\lambda^*_{\text{min}})^2 + 20(\lambda^*_{\text{max}})^2 \) and \( \varphi = (L + 10L^2)/(\rho \lambda^*_{\text{min}}) - \phi^H \).

Lemma 2. Suppose that the sequence \( \{x_t, y_t, \lambda_t\}_{t=1}^T \) is generated by Algorithm 1. Let \( \rho_0 = \frac{\lambda^*_{\text{min}}}{\lambda^*_{\text{max}}} \), \( \Delta = (\lambda^*_{\text{min}})^2 + 20(\lambda^*_{\text{max}})^2 \) and \( \phi^A = \frac{\Delta}{\rho \lambda^*_{\text{min}}^2} \), and then it holds that

\[
\Psi_{t+1} - \Psi_t \leq -\gamma \| x_{t+1} - x_t \|^2 + \frac{(\lambda^*_{\text{min}})^2 + 20(\lambda^*_{\text{max}})^2}{2\rho \lambda^*_{\text{min}}^2},
\]

where \( \gamma = \frac{\phi^H}{\rho} + \frac{\Delta}{2 \rho^2} - \frac{\tilde{L}}{2} - \frac{5(\lambda^*_{\text{max}})^2}{\rho \lambda^*_{\text{min}}^2} \).

A detailed proof of Lemma 2 is provided in Appendix A.2. Lemma 2 gives a property of the sequence \( \{\Psi_t\}_{t=1}^T \). Moreover, (18) provides a specific parameter selection on the step size \( \eta \) and the penalty parameter \( \rho \), in which selection of the step size \( \eta \) depends on the parameter \( \rho \).

Lemma 3. Suppose the sequence \( \{x_t, y_t, \lambda_t\}_{t=1}^T \) is generated by Algorithm 1. Under the same conditions as in Lemma 2, the sequence \( \{\Psi_t\}_{t=1}^T \) has a lower bound.

A detailed proof of Lemma 3 is provided in Appendix A.3. Lemma 3 gives a lower bound of the sequence \( \{\Psi_t\}_{t=1}^T \).

Theorem 1. Suppose the sequence \( \{x_t, y_t, \lambda_t\}_{t=1}^T \) is generated by Algorithm 1. Define \( \kappa_3 = 3(L^2 + (\phi^H)^2), \kappa_2 = \frac{\epsilon^2}{\rho^2}, \kappa_3 = \frac{\phi^A}{\rho \lambda^*_{\text{min}}^2}, \) and \( \kappa_4 = \frac{\kappa_1 \kappa_3}{\kappa_2}, \kappa_4 = \frac{\kappa_1 \kappa_3}{\kappa_2} \). Let

\[
M \geq \frac{2\sigma^2}{\epsilon} \max \{\kappa_1 \kappa_4 + 1, \kappa_2 \kappa_4 + 10 \epsilon \min \{\lambda^*_{\text{min}}\}^2, \kappa_3 \kappa_4\},
\]

where \( \Psi^* \) is a lower bound of the sequence \( \{\Psi_t\}_{t=1}^T \). Define \( \theta_t = \frac{\| x_t - x_{t+1} \|^2 + \| x_{t-1} - x_t \|^2}{\gamma}, \) and let \( \tau^* = \arg \min \{\epsilon \gamma \theta_t\} \), then \( \{x_t, y_t\} \) is an \( \epsilon \)-stationary point of the problem (3).

A detailed proof of Theorem 1 is provided in Appendix A.4. Theorem 1 shows that, given an mini-batch
size $M = O(1/\epsilon)$, the mini-batch stochastic ADMM has the convergence rate of $O(1/k)$ to obtain an $\epsilon$-stationary point of the nonconvex problem [3]. Moreover, the IFO (Incremental First-order Oracle [40]) complexity of the mini-batch stochastic ADMM is $O(M/\epsilon) = O(1/\epsilon^2)$ for obtaining an $\epsilon$-stationary point. While, the IFO complexity of the deterministic proximal ADMM [40] is $O(n/\epsilon)$ for obtaining an $\epsilon$-stationary point. When $n > \frac{1}{\epsilon^2}$, the mini-batch stochastic ADMM needs less IFO complexity than the deterministic ADMM.

In the convergence analysis, given an appropriate mini-batch size $M$ satisfies the condition (20), the step size $\eta$ only need satisfies the condition (18) instead of $\eta = O(1/\epsilon^2)$ used in the convex stochastic ADMM [24].

### 3 Nonconvex Minibatch SVRG-ADMM

In the subsection, we propose a mini-batch nonconvex stochastic variance reduced gradient ADMM (SVRG-ADMM) to solve the problem (5), which uses a multi-stage strategy to progressively reduce the variance of stochastic gradients.

Algorithm 2 gives an algorithmic framework of mini-batch SVRG-ADMM for nonconvex optimizations. In Algorithm 2, the stochastic gradient $\nabla f(x_{t+1}) = \frac{1}{M} \sum_{i\in I_t} (\nabla f_i(x_{t+1}) - \nabla f_i(\tilde{x}^s)) + \nabla f(\tilde{x}^s)$ is unbiased, i.e., $\mathbb{E}[\nabla f(x_{t+1})] = \nabla f(x_{t+1})$. In the following, we give an upper bound of the variance of the stochastic gradient $\nabla f(x_{t+1})$.

**Lemma 4.** In Algorithm 2 set $\Delta_{t+1}^s = \nabla f(x_{t+1}) - \nabla f(x_{t+1})$, then it holds

$$\mathbb{E}[\Delta_{t+1}^s]^2 \leq \frac{L^2}{M} \| x_{t+1}^s - \tilde{x}^s \|^2,$$  

(21)

where $\mathbb{E}[\Delta_{t+1}^s]^2$ denotes variance of the stochastic gradient $\nabla f(x_{t+1})$.

A detailed proof of Theorem 4 is provided in Appendix B.1.

**Lemma 4** shows that the variance of the stochastic gradient $\nabla f(x_{t+1})$ has an upper bound $O(\| x_{t+1}^s - \tilde{x}^s \|^2)$. Due to $\tilde{x}^s = x_{t+1}^s$ as number of iterations increases, both $x_{t+1}^s$ and $\tilde{x}^s$ approach the same stationary point, thus the variance of stochastic gradient vanishes. In fact, the variance of stochastic gradient $\nabla f(x_{t+1})$ is progressively reduced.

### 3.1 Convergence Analysis of Nonconvex Mini-batch SVRG-ADMM

In the subsection, we study the convergence and iteration complexity of the mini-batch nonconvex SVRG-ADMM. First, we give an upper bound of $\mathbb{E}[\| x_{t+1}^s - \lambda_{t+1}^s \|^2]$. We have the following inequality holds

$$\mathbb{E}[\| x_{t+1}^s - \lambda_{t+1}^s \|^2] \leq \frac{5L^2}{\phi_{\min}^A M} \mathbb{E}[\| x_{t+1} - \tilde{x}^s \|^2] + \frac{5L^2}{\phi_{\min}^A M} \| x_{t+1}^s - \tilde{x}^s \|^2 + \frac{\zeta}{\phi_{\min}^A M} \| x_{t+1}^s - \tilde{x}^s \|^2 + \zeta \mathbb{E}[\| x_{t+1} - x_{t+1}^s \|^2] + \zeta \mathbb{E}[\| x_{t+1}^s - x_{t+1}^s \|^2].$$

**Algorithm 2** Mini-batch SVRG-ADMM for Nonconvex Non-smooth Optimization

1. **Input:** Mini-batch size $M$, epoch length $m$, $T = \lceil T/m \rceil$, $\rho > 0$;
2. **Initialize:** $\tilde{x}^0 = x_0^m$, $y_0^m$ and $\lambda_0^m$;
3. **for** $s = 0, 1, \ldots, S - 1$ **do**
4. $x_{0}^{s+1} = x_{m}^m$, $y_{0}^{s+1} = y_{m}^m$ and $\lambda_{0}^{s+1} = \lambda_{m}^m$;
5. $\nabla f(\tilde{x}^s) = \frac{1}{M} \sum_{i\in I_t} (\nabla f_i(x_{t+1}^s) - \nabla f_i(\tilde{x}^s)) + \nabla f(\tilde{x}^s)$;
6. **for** $t = 0, 1, \ldots, m - 1$ **do**
7. Uniformly randomly pick a mini-batch $I_t$ from $\{1, 2, \ldots, n\}$;
8. $y_{t+1}^s = \arg\min_{\lambda} \mathcal{L}_\rho(x_{t+1}^s, \lambda)$;
9. $\nabla f(x_{t+1}^s) = \frac{1}{M} \sum_{i\in I_t} (\nabla f_i(x_{t+1}^s) - \nabla f_i(\tilde{x}^s)) + \nabla f(\tilde{x}^s)$;
10. $x_{t+1}^s = \arg\min_{x} \mathcal{L}_\rho(x, y_{t+1}^s, \lambda_{t+1}^s, x_{t+1}^s, \nabla f(x_{t+1}^s))$;
11. $\lambda_{t+1}^s = \lambda_{t+1}^s - \rho (Ax_{t+1}^s + By_{t+1}^s - c)$;
12. **end for**
13. $\tilde{x}^{s+1} = x_{m}^m$;
14. **end for**
15. **Output:** Iterate $x$ and $y$ chosen uniformly random from $(x_{t+1}^s, y_{t+1}^s)_{t=1}^S$.

A detailed proof of Lemma 5 is provided in Appendix B.2. Given the sequence $(x_{t+1}^s, y_{t+1}^s)_{t=1}^S$ generated from Algorithm 2, then we define an useful sequence $(\phi_{t}^H)^{m}_{t=1}$ as follows:

$$\Phi_t^H = \mathbb{E}[\mathcal{L}_\rho(x_{t}^s, y_{t}^s, \lambda_t^s) + h_t^s(\| x_t^s - \tilde{x}^s - 1 \|^2 + \| x_{t-1}^s - \tilde{x}^s - 1 \|^2) + \frac{\zeta}{\rho} \| x_t^s - x_{t-1}^s \|^2],$$

(22)

where $(\{h_t^s\}^S_{t=1})_{s=1}^S$ is a positive sequence.

**Lemma 6.** Suppose the sequence $(x_{t+1}^s, y_{t+1}^s)_{t=1}^S$ is generated from Algorithm 2 and suppose the sequence $(h_t^s)_{t=1}^S$ satisfies, for $s = 1, 2, \ldots, S$,

$$h_t^s = \begin{cases} (2 + \beta)h_{t+1}^s + \frac{(10 + \phi_{\min}^A)^2}{2\rho\phi_{\min}^A}L^2, & t \leq m - 1, \\ \frac{10L^2}{\phi_{\min}^A M}, & t = m, \end{cases}$$

(23)

where $\beta > 0$. Let $\hat{h} = \min_t \left( \{1 + \frac{1}{\beta}\}h_{t+1}^s, h_t^s \right)$, $\Delta_1 = (\phi_{\min}^A)^2 + 20(\phi_{\min}^A)^2/\rho\phi_{\min}^A - (\Delta_1 + 2 + \Delta_1/2)$,

$$\rho_s = \frac{\Delta_1 + 2 + \Delta_1/2}{2\phi_{\min}^A \rho\phi_{\min}^A},$$

and suppose the parameters $\rho$ and $\eta$, respectively, satisfy

$$\begin{cases} \eta \in \left( \frac{\phi_{\min}^A - \sqrt{\Delta_1}}{\phi_{\min}^A + \sqrt{\Delta_1}}, \frac{\phi_{\min}^A}{\phi_{\min}^A + \sqrt{\Delta_1}} \right), & \rho \in (\rho_s, \rho_s); \\
\eta \in \left( \frac{10(\phi_{\min}^A)^2}{\rho\phi_{\min}^A}, \frac{r - 1}{\rho\phi_{\min}^A \| A^T A \|^2} \right), & \rho = \rho_s; \\
\eta \in \left( \frac{\phi_{\min}^A - \sqrt{\Delta_1}}{\phi_{\min}^A + \sqrt{\Delta_1}}, \frac{r - 1}{\rho\phi_{\min}^A \| A^T A \|^2} \right), & \rho \in (\rho_s, +\infty). \end{cases}$$

(24)
where \( \phi_1 = (L + 2\beta^{M_d} + 10L^2/(\rho \phi_{min}^{M_d})) - \phi_{min}^{M_d} \). Then it holds that the sequence \( \{ (\Gamma_t)^{m}_{s=1} \}_{s=1}^{S} \) is positive, defined by

\[
\Gamma_t = \left\{ \begin{array}{rcl}
\phi_{min}^{H} + \frac{\phi_{min}^{A}}{2} - \frac{L}{\rho} - \frac{\xi + \zeta}{\rho} - (1 + \frac{1}{\beta})h_{t+1}, & t \leq m \\
\phi_{min}^{H} + \frac{\phi_{min}^{A}}{2} - \frac{L}{\rho} - \frac{\xi + \zeta}{\rho} - h_{t+1}^{s+1}, & t = m
\end{array} \right.
\]

and the sequence \( \{ (\Phi_t)^{m}_{s=1} \}_{s=1}^{S} \) monotonically decreases.

A detailed proof of Lemma 6 is provided in Appendix B.3. Lemma 6 shows that the sequence \( \{ (\Phi_t)^{m}_{s=1} \}_{s=1}^{S} \) monotonically decreases. Moreover, (24) provides a specific parameter selection on the step size \( \eta \) and the penalty parameter \( \rho \) in Algorithm 2.

**Lemma 7.** Suppose the sequence \( \{ (x_t^s, y_t^s, \lambda_t^s) \}_{t=1}^{S} \) is generated by Algorithm 2. Under the same conditions as in Lemma 6 the sequence \( \{ (\Phi_t)^{m}_{s=1} \}_{s=1}^{S} \) has a lower bound.

Lemma 7 shows that the sequence \( \{ (\Phi_t)^{m}_{s=1} \}_{s=1}^{S} \) has a lower bound. The proof of Lemma 7 is the same as the proof of Lemma 5. We define an useful variable \( \tilde{\theta}_t^s \) as follows:

\[
\tilde{\theta}_t^s = \|x_t^s - x_t^{s-1}\|^2 + \|x_{t-1}^s - x_t^{s-1}\|^2 + \|x_{t+1}^{s-1} - x_t^s\|^2 + \|x_t^s - x_{t-1}^s\|^2.
\]

In the following, we will analyze the convergence and iteration complexity of the nonconvex SVRG-ADMM based on the above lemmas.

**Theorem 2.** Suppose the sequence \( \{ (x_t^s, y_t^s, \lambda_t^s) \}_{t=1}^{S} \) is generated by Algorithm 2. Denote \( \kappa_1 = 3(L^2 + \sigma^2/\rho^2), \kappa_5 = 2\|B\|^2 \|A\|^2, \text{ and } \gamma = \min_{(t,s)} \Gamma_t^s \) and \( \omega = \min_{(s,t)} \{ (2 + \beta)h_{t+1} + \frac{L^2}{2\rho^2}, \frac{\sigma L^2}{\phi_{min}^{M_d}} \} \). Let

\[
mS = T = \max \left\{ \frac{\kappa_1, \kappa_2, \kappa_3}{\tau \epsilon} (\Phi^s - \Phi^*), \right\}
\]

where \( \tau = \min(\gamma, \omega) \), and \( \Phi^* \) is a lower bound of the sequence \( \{ (\Phi_t)^{m}_{s=1} \}_{s=1}^{S} \). Let

\[
t^* = \arg \min_{1 \leq t \leq m, 1 \leq s \leq S} \tilde{\theta}_t^s,
\]

then \( (x_t^{s*}, y_t^{s*}) \) is an \( \epsilon \)-stationary point of the problem (3).

A detailed proof of Theorem 2 is provided in Appendix B.3. Theorem 2 shows that the mini-batch SVRG-ADMM for nonconvex optimization has a convergence rate of \( O(1/T) \). Moreover, the IFO complexity of the mini-batch SVRG is \( O((n + M)/\epsilon) \). When \( n > M \), the mini-batch SVRG-ADMM needs less IFO complexity than the deterministic ADMM.

Since the mini-batch SVRG-ADMM uses VR technique, its convergence does not depend on the mini-batch size \( M \). In other words, when \( M = 1 \), the mini-batch nonconvex SVRG-ADMM reduces to the initial nonconvex SVRG-ADMM in [11], which also has a convergence rate of \( O(1/T) \). However, by Lemma 4, the variance of stochastic gradient in the mini-batch SVRG-ADMM decreases faster than that in the initial nonconvex SVRG-ADMM.

**4 Nonconvex Mini-batch SAGA-ADMM**

In the subsection, we propose a mini-batch nonconvex stochastic average gradient ADMM (SAGA-ADMM) by additionally using the old gradients estimated in the previous iteration, which is inspired by the SAGA method [9].

The algorithmic framework of the SAGA-ADMM is given in Algorithm 3. In Algorithm 3 the stochastic gradient \( \nabla f(x_t) = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_t) - \nabla f_i(z_t^s) + \psi_t \) is unbiased (i.e., \( E[\nabla f(x_t)] = \nabla f(x_t) \)), where \( \psi_t = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(z_t^s) \). In the following, we give an upper bound of the variance of the stochastic gradient \( \nabla f(x_t) \).

**Lemma 8.** For Algorithm 3 let \( \Delta_t = \hat{\nabla} f(x_t) - \nabla f(x_t) \), then it holds

\[
E[||\Delta_t||^2] \leq \frac{L^2}{n^2} \sum_{i=1}^{n} ||x_t - z_t^i||^2,
\]

where \( E[||\Delta_t||^2] \) denotes variance of the stochastic gradient \( \nabla f(x_t) \).

A detailed proof of Theorem 8 is provided in Appendix C.3. Lemma 8 shows that the variance of the stochastic gradient \( \nabla f(x_t) \) has an upper bound \( O(\frac{1}{n} \sum_{i=1}^{n} ||x_t - z_t^i||^2) \). As the number of iteration increases, both \( x_t \) and the stored points \( \{z_t^i\}_{i=1}^{n} \) approach the same stationary point, so the variance of stochastic gradient progressively reduces. In fact, the variance of stochastic gradient \( \nabla f(x_t) \) is progressively reduced via additionally using the old gradients in the previous iterations.

**Algorithm 3** Mini-batch SAGA-ADMM for Nonconvex Nonsmooth Optimization

1. Input: \( x_0 \in R^d, y_0 \in R^q, z_0^i = x_0 \) for \( i \in \{1, 2, \cdots, n\} \), number of iterations \( T \);
2. Initialize: \( \psi_0 = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(z_0^i) \);
3. for \( t = 0, 1, \cdots, T - 1 \) do
4. Uniformly randomly pick a mini-batch \( I_t \) from \( \{1, 2, \cdots, n\} \);
5. \( y_{t+1} = \arg \min_{y} \mathcal{L}_p(x_t, y, \lambda_t) \);
6. \( \nabla f(x_t) = \frac{1}{n} \sum_{i \in I_t} \nabla f_i(x_t) - \nabla f_i(z_t^i) + \psi_t \) with \( \psi_t = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(z_t^i) \);
7. \( x_{t+1} = \arg \min_{x} \mathcal{L}_p(x; y_{t+1}, \lambda_t, x_t, \nabla f(x_t)) \);
8. \( \lambda_{t+1} = 1 - \lambda_t - \rho(\lambda_{t+1} + B y_{t+1} - c) \);
9. \( z_{t+1} = x_{t+1} \) and \( z_{t+1}^i = z_t^i \) for \( i \neq i_t \), for all \( i_t \in I_t \);
10. \( \psi_{t+1} = \psi_t + \frac{1}{n^2} \sum_{i \in I_t} \nabla f_i(z_{t+1}^i) - \nabla f_i(z_{t+1}^i) \);
11. end for
12. Output: Iterate \( x_t \) and \( y_t \) chosen uniformly random from \( \{x_t, y_t\}_{t=1}^{T} \).

**4.1 Convergence Analysis of Nonconvex Mini-batch SAGA-ADMM**

In the subsection, we study the convergence and iteration complexity of the nonconvex mini-batch SAGA-ADMM. First, we give same useful lemmas as follows:
Lemma 9. Suppose the sequence \( \{x_t, y_t, \lambda_t\}_{t=1}^T \) is generated by Algorithm 3. The following inequality holds

\[
\mathbb{E}[\lambda_{t+1} - \lambda_t]^2 \leq \frac{\gamma L^2}{\phi_{\min} \rho n} \sum_{i=1}^{n} (\mathbb{E}[x_t - z_i^t]^2 + \|x_{t-1} - z_{t-1}^t\|^2) + \zeta \|x_t - x_{t-1}\|^2.
\]

Lemma 9 gives an upper bound of \( \mathbb{E}[\lambda_{t+1} - \lambda_t]^2 \). Its proof is the same as that of Lemma 9. Given the sequence \( \{x_t, y_t, \lambda_t\}_{t=1}^T \) generated by Algorithm 3 then we define an useful sequence \( \{\Theta_t\}_{t=1}^T \) below:

\[
\Theta_t = \mathbb{E}[L \rho(x_t, y_t, \lambda_t)] + \frac{\alpha_t}{n} \sum_{i=1}^{n} (\|x_t - z_i^t\|^2 + \|x_{t-1} - z_{t-1}^t\|^2) + \frac{\zeta}{\rho} \|x_t - x_{t-1}\|^2,
\]

where \( \{\alpha_t\}_{t=1}^T \) is a decreasing positive sequence.

Lemma 10. Suppose the sequence \( \{x_t, y_t, \lambda_t\}_{t=1}^T \) is generated by Algorithm 3 and the positive sequence \( \{\alpha_t\}_{t=1}^T \) satisfy

\[
\alpha_t = (2 + \beta - 1 + \frac{\beta}{n}) \alpha_{t+1} + \frac{(10 + \phi_{\min, \rho}) L^2}{2\rho \phi_{\min, \rho} M},
\]

where \( \beta > 0 \). Let \( \hat{\alpha} = \min \{1 + \frac{1}{n\rho} - \frac{1}{n^2}\alpha_{t+1}\} \), \( \Delta_2 = (\phi_{\min}^H)^2 + 20(\phi_{\min, \rho}^H)^2 - \frac{\rho_0}{\rho_0}, \rho_0 = \frac{\rho L + 2\hat{\alpha} + 20\phi_{\min}^H}{\Delta_2}, \rho = \frac{\rho_0}{\phi_{\min}^H}, \)

and suppose the parameters \( \rho \) and \( \eta \), respectively, satisfy

\[
\eta \in \left(\frac{\phi_{\min}^H - \sqrt{\Delta_2}}{\phi_2}, \frac{\phi_{\min}^H + \sqrt{\Delta_2}}{\phi_2}\right), \quad \rho \in (\rho_0, \rho_s);
\]

\[
\eta \in \left(\frac{10(\phi_{\max}^H)^2 - 1}{\phi_{\min}^H \phi_2^2}, \frac{r - 1}{\rho \|A^T A\|}\right), \quad \rho \in (\rho_s, +\infty);
\]

\[
\eta \in \left(\frac{\phi_{\min}^H - \sqrt{\Delta_2}}{\phi_2}, \frac{r - 1}{\rho \|A^T A\|}\right), \quad \rho \in (\rho_s, +\infty),
\]

where \( \phi_2 = (2 + \hat{\alpha} + 10L^2/(\rho_0^2)) - \phi_{\min}^H \). Then it holds the sequence \( \{\Gamma_t\}_{t=1}^T \) is positive, defined by

\[
\Gamma_t = \frac{\phi_{\min}^H}{\eta} + \frac{\phi_{\min}^H - \phi_2}{2} \rho - \frac{\zeta + \zeta_1}{\rho} (1 + \frac{1}{\beta} - \frac{1}{n\beta}) \alpha_{t+1},
\]

and the sequence \( \{\Theta_t\}_{t=1}^T \) monotonically decreases.

A detailed proof of Lemma 10 is provided in Appendix C.2. Lemma 10 shows that the sequence \( \{\Theta_t\}_{t=1}^T \) monotonically decreases. Moreover, (31) provides a specific parameter selection on the step size \( \eta \) and the penalty parameter \( \rho \) in Algorithm 3.

Lemma 11. Suppose the sequence \( \{x_t, y_t, \lambda_t\}_{t=1}^T \) is generated by Algorithm 3. Under the same conditions as in Lemma 10, the sequence \( \{\Theta_t\}_{t=1}^T \) has a lower bound.

Lemma 11 shows that the sequence \( \{\Theta_t\}_{t=1}^T \) has a lower bound. Its proof is the same as the proof of Lemma 3. In the following, we will study the convergence and iteration complexity of the SAGA-ADMM based on the above lemmas. We first give an useful variable \( \tilde{\theta} \) defined by:

\[
\tilde{\theta}_t = \|x_{t+1} - x_t\|^2 + \|x_t - x_{t-1}\|^2 + \frac{1}{n} \sum_{i=1}^{n} (\|x_t - z_i^t\|^2 + \|x_{t-1} - z_{t-1}^t\|^2).
\]

Theorem 3. Suppose the sequence \( \{x_t, y_t, \lambda_t\}_{t=1}^T \) is generated by Algorithm 3. Denote \( k_1 = 3(1 + \frac{1}{n\beta}) \), \( k_2 = \frac{\rho}{\rho_s}, \) \( k_3 = \frac{\rho_0}{\rho_0}, \) \( \gamma = \min \{\Gamma_t, \rho_0\} \), and \( \omega = \min \{(2 + \beta - 1 + \frac{\beta}{n}) \alpha_{t+1} + \frac{L^2}{2\rho_0}, 2\gamma\} \). Let

\[
T = \max \{k_1, k_2, k_3\} (\Theta_1 - \Theta^*),
\]

where \( \tau = \min \{\gamma, \omega\} > 0 \), and \( \Theta^* \) is a lower bound of the sequence \( \{\Theta_t\}_{t=1}^T \). Let \( \tau^* = \arg \min_{0\leq t\leq T} \tilde{\theta}_t \), then \((x_{\tau^*}, y_{\tau^*})\) is an \( \epsilon \)-stationary point of the problem (3).

A detailed proof of Theorem 3 is provided in Appendix C.3. Theorem 3 shows that the mini-batch SAGA-ADMM for nonconvex optimization has a convergence rate of \( O(\frac{1}{T}) \). Moreover, the IFO complexity of the mini-batch SAGA-ADMM is \( O(M/\epsilon) \) for obtaining an \( \epsilon \)-stationary point. Clearly, due to \( 1 \leq M < n \), the mini-batch SAGA-ADMM needs less IFO complexity than the deterministic ADMM.

Since the mini-batch SAGA-ADMM also uses TV technique, its convergence does not depend on the mini-batch size \( M \). In other words, when \( M = 1 \), the mini-batch nonconvex SAGA-ADMM reduces to the initial nonconvex SAGA-ADMM in (1), which also has the convergence rate of \( O(\frac{1}{T}) \). However, by Lemma 3 the variance of stochastic gradient in the mini-batch nonconvex SAGA-ADMM decreases faster than that in the initial nonconvex SAGA-ADMM.

Finally, in Table 2 we give the IFO(Incremental First-order Oracle) and EI (Effective Iteration) of both the mini-batch stochastic ADMMs and the deterministic (or batch) ADMM. Specifically, the definition of EI is given in Definition 3. From Table 2, we can find that both the mini-batch stochastic and deterministic ADMMs have the same EI complexity, the mini-batch stochastic ADMMs has lower IFO complexity than the deterministic ADMM when \( M < n \). In the above theoretical analysis, the mini-batch size \( M \) of the nonconvex STOC-ADMM may be very large when \( \epsilon \) is small. However, the following extensive experimental results show that STOC-ADMM still has good performances given a moderate \( M \), and is comparable with both SVRG-ADMM and SAGA-ADMM.
Definition 2. For ADMM and its variants, an EI describes the fact that all the primal and dual variables in the algorithm are updated once.

5 Experiments

In this section, we perform some numerical experiments on both simulated and real-world data to examine performances of the proposed algorithms for the nonconvex nonsmooth optimization. In the experiments, we compare nonconvex mini-batch stochastic ADMM (STOC-ADMM) with nonconvex mini-batch SVRG-ADMM, nonconvex mini-batch SAGA-ADMM, and deterministic ADMM (DETE-ADMM). In the experiments, we use the inexact Uzawa method to both mini-batch stochastic ADMMs and deterministic (or batch) ADMM. In the following, all algorithms are implemented in MATLAB, and all experiments are performed on a PC with an Intel E5-2630 CPU and 32GB memory.

5.1 Simulated Data

In the subsection, we compare the performances in some synthetic data. Here we focus on the binary classification task problem with the graph-guided fused lasso and the overlapping group lasso regularization functions, respectively. Given a set of training samples \((a_i, b_i)_{i=1}^n\), where \(a_i \in \mathbb{R}^d\), \(b_i \in \{-1, +1\}\), then we solve the following nonconvex nonsmooth optimization problem:

\[
\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n f_i(x) + \nu \|Ax\|_1, \tag{35}
\]

where \(f_i(x) = \frac{1}{1 + \exp(b_i a_i^T x)}\) is the sigmoid loss function, which is nonconvex and smooth, and \(\nu\) denotes a nonnegative regularization parameter. When using graph-guided fused lasso \([16]\) in (35), we let \(A\) to be concatenation of \(k\) repetitions of \(x\) (i.e., \(Ax = [x; \ldots; x]\)) as in \([23]\), where \(k\) denotes the number of overlapping group of unknown parameter.

5.1.1 Graph-guided Fused Lasso

Here we compare the performances in some simulated data, where a graph-guided fused lasso regularization is imposed. First, we generate a sparse precision matrix \(\Lambda \in \mathbb{R}^{d \times d}\) with elements

\[
\Lambda_{ij} \overset{i.i.d.}{\sim} \begin{cases} 
0, & \text{prob. 0.95} \\
\text{Unif}([-0.75, -0.25] \cup [0.25, 0.75]), & \text{otherwise}.
\end{cases}
\]

Then the input feature vectors \(\{a_i\}_{i=1}^n\) are i.i.d. generated from multivariate normal distribution \(N(0, \Lambda^{-1})\). The true parameter vector \(x^* \in \mathbb{R}^d\) is generated from the standard normal distribution. The output label is generated as \(b_i = \text{sign}(a_i^T x^* + \epsilon_i)\), where \(\epsilon_i\) is chosen uniformly at random from \([0, 1]\).

In the experiment, we set \(d = 200\), and then generate \(n = \{20000, 40000, 60000\}\) samples \(\{a_i, b_i\}_{i=1}^n\), respectively. For each dataset, we choose half of the samples as training data, while use the rest as testing data. In the problem (35), we use a graph-guided fused lasso, and fix the regularization parameter \(\nu = 10^{-5}\). In the algorithms, we use the same initial solution \(x_0\) from the standard normal distribution and choose the step size \(\eta = 1\). In addition, we choose the mini-batch size \(M = 100\) in the stochastic algorithms, and \(m = \lfloor n/M \rfloor\) in the mini-batch SVRG-ADMM. Finally, all experimental results are averaged over 10 repetitions.

Figs. 3 and 4 show that both the objective values and test loss of these stochastic ADMMs faster decrease than those of the deterministic ADMM, as CPU time consumed increases. In particular, though the nonconvex STOC-ADMM uses a fixed step size \(\eta\), it shows good performance in the nonconvex optimization with graph-guided fused lasso regularization, and is comparable with both the nonconvex SVRG-ADMM and SAGA-ADMM.

5.1.2 Overlapping Group Lasso

Here we compare the performances in some simulated data, where an overlapping group lasso regularization is imposed. First, we generate \(n\) input feature vector \(\{a_i\}_{i=1}^n\) with the dimension \(d = 400\), where each feature is i.i.d. generated from the standard normal distribution. Next, we generate a sparse matrix \(X \in \mathbb{R}^{20 \times 20}\), where only the first column is non-zero (generated i.i.d. from standard normal distribution) and other columns are zero, and the true parameter vector \(x^*\) is vectorization of the matrix \(X\). The output label \(b_i\) is generated as \(b_i = \text{sign}(a_i^T x^* + \epsilon_i)\), where \(\epsilon_i\) is the standard normal distribution. Then, we generate \(n = \{20000, 40000, 60000\}\) samples \(\{a_i, b_i\}_{i=1}^n\), respectively. In the experiment, we choose the mini-batch size \(M = 200\) in these stochastic algorithms, and the other settings are the similar as the above graph-guided fused lasso task. In the problem (35), we use an overlapping group lasso penalty function \(g(x) = \nu \sum_{i=1}^{20} \|X_{i,:}\| + \sum_{j=1}^{20} \|X_{:,j}\|\). Then we let \(A = [I; I]\) as in (23), and fix the parameter \(\nu = 10^{-5}\).

Figs. 3 and 4 show that both the objective values and test loss of these stochastic ADMMs faster decrease than those of the deterministic ADMM, as CPU time consumed increases. In particular, though the nonconvex STOC-ADMM uses a fixed step size \(\eta\), it shows good performance in the nonconvex optimization with overlapping group lasso regularization, and is comparable with both the nonconvex SVRG-ADMM and SAGA-ADMM.

5.2 Real Data

In the subsection, we compare the performances in some real data. Specifically, we perform the binary classification task and multitask learning, respectively.

5.2.1 Graph-guided Fused Lasso

Here we perform the binary classification task with the graph-guided fused lasso penalty function as in (35). In the experiment, we use some publicly available datasets\(3\), which are summarized in Table 3. In the algorithms, we use the same initial solution \(x_0\) from the standard normal distribution and choose a fixed step size \(\eta = 1\).

3. 20news is from the website (https://cs.nyu.edu/~roweis/data.html); a9a, w8a, icm1 and covertype.binary are from the LIBSVM website (www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/).
Fig. 1. Objective value versus CPU time on the simulated nonconvex model with graph-guided fused Lasso.

Fig. 2. Test error versus CPU time on the simulated nonconvex model with graph-guided fused Lasso.

Fig. 3. Objective value versus CPU time on the simulated nonconvex model with overlapping group Lasso.

Fig. 4. Test error versus CPU time on the simulated nonconvex model with overlapping group Lasso.
problem \[35\], we fix the parameter \( \nu = 10^{-5} \). In addition, we choose the mini-batch size \( M = 100 \) in these stochastic algorithms. The following experimental results are averaged over 10 repetitions.

Figs. 5 and 6 show that both the objective values and test loss of these stochastic ADMMs are smaller than those of the deterministic ADMM, as CPU time consumed increases. In particular, though the nonconvex STOC-ADMM uses a fixed step size \( \eta \), it shows good performance in the nonconvex optimization with graph-guided fusionlasso regularization, and is comparable with both the nonconvex SVRG-ADMM and SAGA-ADMM. Due to that icmnl is a severely imbalanced data set (i.e., includes large negative samples), the first iteration solution of all algorithms shows good performance on the testing error.

5.2.2 Multi-task Learning

Here we perform the multi-task learning with both sparse and low-rank penalty functions. Specifically, given a set of training samples \((a_i, b_i)_{i=1}^n\), where \( a_i \in \mathbb{R}^d \) and \( b_i \in \{1, 2, \ldots, m\} \). Let \( b_{i,c} = 1 \) if \( b_i = c \in \{1, 2, \ldots, m\} \), and \( b_{i,c} = 0 \) otherwise. Then we solve the following nonconvex problem

\[
\min_{X \in \mathbb{R}^{m \times d}} \frac{1}{n} \sum_{i=1}^n f_i(X) + \nu_1 \sum_{i,j=1}^{m,d} \kappa([X_{i,j}]) + \nu_2 \|X\|_s, \quad (36)
\]

where \( f_i(X) = \log(\sum_{c=1}^m \exp(X_{i,c}^T a_i)) - \sum_{c=1}^m b_{i,c} X_{i,c}^T a_i \) is a multinomial logistic loss function, \( \kappa(\alpha) = \beta \log(1 + \frac{\alpha}{\beta}) \) is the nonconvex log-sum penalty function \[56\], and \( \|X\|_s \) denotes the nuclear norm of matrix \( X \). Here \( \nu_1 \) and \( \nu_2 \) are nonnegative regularization parameters. Following \[57\], we can transform the problem \[36\] into the following problem

\[
\min_{X \in \mathbb{R}^{m \times d}} \frac{1}{n} \sum_{i=1}^n \tilde{f}_i(X) + \bar{g}(X) \quad (37)
\]

where \( \tilde{f}_i(X) = f_i(X) + \nu_1 \left( \sum_{i,j=1}^{m,d} \kappa([X_{i,j}]) - \kappa_0 \|X\|_1 \right), \quad \bar{g}(X) = \nu_1 \kappa_0 \|X\|_1 + \nu_2 \|X\|_s \), and \( \kappa_0 = \kappa'(0) \). By Proposition 2.3 in \[57\] \( \tilde{f}_i(X) \) is nonconvex and smooth, and \( \bar{g}(X) \) is nonsmooth and convex. To solve the problem \[37\] by using ADMMs, we introduce an auxiliary variable \( Y \) with the constraint \( X = Y \), and given \( A = [I; I] \), then \( \bar{g}(AX) = \nu_1 \kappa_0 \|X\|_1 + \nu_2 \|X\|_s \).

In the experiment, we use some publicly available datasets which are summarized in Table 5. We use the mini-batch size of \( M = 100 \) on letter, \( M = 300 \) on sensorless and mnist, \( M = 500 \) on covtype, and \( M = 1000 \) on mnist8m. In the algorithms, we use the same initial solution \( x_0 \) from the standard normal distribution and choose a fixed step size \( \eta = 0.8 \). In the problem \[36\], we fix the regularization parameters \( \nu_1 = 10^{-5} \) and \( \nu_2 = 10^{-4} \).

Figs. 7 and 8 show that both objective values and test loss of the stochastic ADMMs are smaller than those of the deterministic ADMM, as CPU time consumed increases. In particular, though the nonconvex STOC-ADMM uses a fixed step size \( \eta \), it shows good performance in the nonconvex multi-task learning with sparse and low-rank regularization functions, and is comparable with both the nonconvex SVRG-ADMM and SAGA-ADMM. Due to large training samples, the stochastic gradient of SAGA-ADMM includes many old gradients, and slowly updates.

5.3 Varying \( \rho \)

In the subsection, we demonstrate the specific parameter selection for step size \( \eta \) of stochastic gradient and penalty parameter \( \rho \) of augmented Lagrangian function. Specifically, we give a fixed \( \eta \), then find an optimal \( \rho \). In the experiment, we use the above simulated data imposed the overlapping group lasso regularization function, and set \( n = 40,000 \), \( d = 400 \). In the problem \[35\], we fix the regularization parameter \( \nu = 10^{-5} \). In the algorithms, we fix the step size \( \eta = 1 \).

Figs. 9, 10 and 11 show the objective value and test error versus CPU time with different \( \rho \). From these results, we can find that given an appropriate step size \( \eta \), the proposed mini-batch stochastic algorithms have good performances in a wide-range of parameter \( \rho \). In particular, when the parameter \( \rho \) satisfies the above conditions \[18\], \[24\] and \[31\], these mini-batch algorithms show good performances.

6 Conclusion

In the paper, we have studied the mini-batch stochastic ADMMs for the nonconvex nonsmooth optimization. We have theoretically proved that, give mini-batch size \( M = O(1/\epsilon) \), the mini-batch stochastic ADMM without VR (STOC-ADMM) has the convergence rate of \( O(1/T) \) to obtain an \( \epsilon \)-stationary point. In theoretical analysis, the mini-batch size \( M \) may be very large when \( \epsilon \) is small. However, the above extensive experimental results show that STOC-ADMM still has good performances given a moderate \( M \), and is comparable with both SVRG-ADMM and SAGA-ADMM. In particular, as long as the step size \( \eta \) and the penalization parameter \( \rho \) satisfy the above condition \[18\] instead of \( \eta = O(\frac{1}{T}) \) used in the convex stochastic ADMM \[24\], STOC-ADMM is convergent, and reaches a convergence rate of \( O(1/T) \).

Moreover, we have extended the mini-batch stochastic gradient method to both the non-convex SVRG-ADMM and SAGA-ADMM proposed in our initial manuscript \[1\].

### TABLE 3

| datasets        | #training | #test | #features | #classes |
|-----------------|-----------|-------|-----------|----------|
| 20newus         | 8,121     | 8,121 | 100       | 2        |
| abalone         | 16,281    | 16,280| 123       | 2        |
| wdbc            | 32,350    | 32,350| 300       | 2        |
| covertype.binary| 63,351    | 63,351| 22        | 2        |
| covtype         | 290,506   | 290,506| 54        | 2        |

### TABLE 4

| datasets  | #training | #test | #features | #classes |
|-----------|-----------|-------|-----------|----------|
| letter    | 7,500     | 7,500 | 16        | 26       |
| sensorless| 29,254    | 29,254| 48        | 11       |
| mnist     | 30,000    | 30,000| 780       | 10       |
| covtype   | 290,506   | 290,506| 54        | 7        |
| mnist8m   | 4,050,000 | 4,050,000| 780      | 10       |
ADMM and SAGA-ADMM reach the convergence rate of $O(1/T)$ without the condition on $M$, SVRG-ADMM requires frequently compute gradients over the full data, and SAGA-ADMM requires memory of the same size for storing
Fig. 11. Performance of nonconvex SAGA-ADMM at different $\rho$.

the old gradients. In the future work, we will develop a more efficient stochastic ADMM algorithm for automatically adapting to the system resources, and yield the best performance in practice. In addition, we will propose some accelerated stochastic ADMMs for nonconvex optimization by using the momentum techniques.

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A.1 Proof of Lemma 1

Proof. By the optimal condition of step 6 in Algorithm 1, we have

\begin{align*}
0 &= G(x_t, \xi_{t, \ell}) - A^T \lambda_t + \rho A^T (A x_{t+1} + B y_{t+1} - c) - \frac{1}{\eta} H(x_t - x_{t+1}) \\
&= G(x_t, \xi_{t, \ell}) - A^T \lambda_{t+1} - \frac{1}{\eta} H(x_t - x_{t+1}),
\end{align*}

where the second equality is due to step 7 in Algorithm 1. Thus, we have

\[ A^T \lambda_{t+1} = G(x_t, \xi_{t, \ell}) - \frac{1}{\eta} H(x_t - x_{t+1}). \tag{38} \]

Using the equality (38), we have

\begin{align*}
\|\lambda_{t+1} - \lambda_t\|^2 &\leq (\phi_{t+1}^A)^{-1} \|A^T \lambda_{t+1} - A^T \lambda_t\|^2 \\
&\leq (\phi_{t+1}^A)^{-1} \|G(x_t, \xi_{t, \ell}) - G(x_{t-1}, \xi_{t-1}) - \frac{1}{\eta} H(x_t - x_{t+1}) + \frac{1}{\eta} H(x_{t-1} - x_t)\|^2 \\
&= (\phi_{t+1}^A)^{-1} \|G(x_t, \xi_{t, \ell}) - \nabla f(x_t) + \nabla f(x_{t-1}) - \nabla f(x_{t-1}) - G(x_{t-1}, \xi_{t-1}) \|^2 \\
&\quad - \frac{1}{\eta} H(x_t - x_{t+1}) + \frac{1}{\eta} H(x_{t-1} - x_t)\|^2 \\
&\leq \frac{5}{\phi_{t+1}^A} \|G(x_t, \xi_{t, \ell}) - \nabla f(x_t)\|^2 + \frac{5}{\phi_{t+1}^A} \|G(x_{t-1}, \xi_{t-1}) - \nabla f(x_{t-1})\|^2 + \frac{5(\phi_{t+1}^H)^2}{\phi_{t+1}^A \eta^2} \|x_t - x_{t+1}\|^2 \\
&\quad + \frac{5(\phi_{t+1}^H)^2 + \eta^2 L^2}{\phi_{t+1}^A \eta^2} \|x_{t-1} - x_t\|^2, \tag{39}
\end{align*}

where the inequality (i) holds by (14).

Taking expectation conditioned on information \( I_t \) to (39), we have

\begin{align*}
\mathbb{E}\|\lambda_{t+1} - \lambda_t\|^2 &\leq \frac{5}{\phi_{t+1}^A} \mathbb{E}\|G(x_t, \xi_{t, \ell}) - \nabla f(x_t)\|^2 + \frac{5}{\phi_{t+1}^A} \mathbb{E}\|G(x_{t-1}, \xi_{t-1}) - \nabla f(x_{t-1})\|^2 \\
&\quad + \frac{5(\phi_{t+1}^H)^2}{\phi_{t+1}^A \eta^2} \|x_t - x_{t+1}\|^2 + \frac{5(\phi_{t+1}^H)^2 + \eta^2 L^2}{\phi_{t+1}^A \eta^2} \|x_{t-1} - x_t\|^2 \\
&\leq \frac{10\sigma^2}{M\phi_{t+1}^A} + \frac{5(\phi_{t+1}^H)^2}{\phi_{t+1}^A \eta^2} \mathbb{E}\|x_{t+1} - x_t\|^2 + \frac{5(\phi_{t+1}^H)^2 + \eta^2 L^2}{\phi_{t+1}^A \eta^2} \|x_{t-1} - x_t\|^2, \\
&= \zeta \|x_t - x_{t-1}\|^2 + \zeta_1 \mathbb{E}\|x_{t+1} - x_t\|^2 + \frac{10\sigma^2}{M\phi_{t+1}^A},
\end{align*}

where the inequality (i) holds by (11). \( \square \)

A.2 Proof of Lemma 2

Proof. By the step 5 of Algorithm 1, we have

\[ \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) \leq \mathcal{L}_\rho(x_t, y_t, \lambda_t). \tag{40} \]
Next, by the optimal condition of step 6 in Algorithm 1, we have

\begin{align*}
0 &= (x_t - x_{t+1})^T \left[ G(x_t, \xi_t) - A^T \lambda_t - \frac{H}{\eta} (x_t - x_{t+1}) + \rho A^T (Ax_{t+1} + By_{t+1} - c) \right] \\
&= (x_t - x_{t+1})^T \left[ G(x_t, \xi_t) - \nabla f(x_t) + \nabla f(x_t) - A^T \lambda_t - \frac{H}{\eta} (x_t - x_{t+1}) + \rho A^T (Ax_{t+1} + By_{t+1} - c) \right] \\
&\leq f(x_t) - f(x_{t+1}) + \frac{L}{2} \|x_{t+1} - x_t\|^2 + (x_t - x_{t+1})^T (G(x_t, \xi_t) - \nabla f(x_t)) - \frac{1}{\eta} \|x_{t+1} - x_t\|^2_H \\
&\quad - \lambda_t^T (Ax_{t+1} - Ax_t) + \rho(Ax_{t+1} + By_{t+1} - c) \\
&\leq f(x_t) - f(x_{t+1}) + \frac{L}{2} \|x_{t+1} - x_t\|^2 + (x_t - x_{t+1})^T (G(x_t, \xi_t) - \nabla f(x_t)) - \frac{1}{\eta} \|x_{t+1} - x_t\|^2_H \\
&\quad - \lambda_t^T (Ax_{t+1} + By_{t+1} - c) + \lambda_t^T (Ax_{t+1} + By_{t+1} - c) + \frac{\rho}{2} \|Ax_t + By_{t+1} - c\|^2 \\
&= \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) - \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) + (x_t - x_{t+1})^T (G(x_t, \xi_t) - \nabla f(x_t)) \\
&\quad + \frac{L}{2} \|x_{t+1} - x_t\|^2 - \frac{1}{\eta} \|x_{t+1} - x_t\|^2_H - \frac{\rho}{2} \|Ax_t - Ax_{t+1}\|^2 \\
&\leq \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) - \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) + \frac{1}{2} \|G(x_t, \xi_t) - \nabla f(x_t)\|^2 \\
&\quad - \left( \frac{\phi_{\min}^H}{\eta} + \frac{\rho \phi_{\min}^A}{2} - \frac{L + 1}{2} \right) \|x_t - x_{t+1}\|^2, \\
&\quad - \left( \frac{\phi_{\min}^H}{\eta} + \frac{\rho \phi_{\min}^A}{2} - \frac{L + 1}{2} \right) \|x_t - x_{t+1}\|^2, \tag{41}
\end{align*}

where the inequality (i) holds by (15); the equality (ii) holds by using the equality \((a - b)^T(b - c) = \frac{1}{2}(\|a - c\|^2 - \|a - b\|^2 - \|b - c\|^2)\) on the term \(\rho(Ax_t - Ax_{t+1})^T(\rho(Ax_{t+1} + By_{t+1} - c);\) the inequality (iii) holds by the Cauchy inequality. Taking expectation conditioned on information \(\mathcal{L}_t\) to (41), we have

\begin{align*}
\mathbb{E}[\mathcal{L}_\rho(x_{t+1}, y_{t+1}, \lambda_t)] &\leq \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) - \left( \frac{\phi_{\min}^H}{\eta} + \frac{\rho \phi_{\min}^A}{2} - \frac{L + 1}{2} \right) \mathbb{E}\|x_{t+1} - x_t\|^2 + \frac{\sigma^2}{2M}. \tag{42}
\end{align*}

By the step 7 of Algorithm 1, we have

\begin{align*}
\mathbb{E}[\mathcal{L}_\rho(x_{t+1}, y_{t+1}, \lambda_{t+1}) - \mathcal{L}_\rho(x_{t+1}, y_{t+1}, \lambda_t)] &= \frac{1}{\rho} \mathbb{E}\|\lambda_t - \lambda_{t+1}\|^2 \\
&\leq \frac{5(L^2 \eta^2 + \phi_{\max}^H)^2}{\phi_{\min}^A \eta^2 \rho} \mathbb{E}\|x_t - x_{t-1}\|^2 + \frac{5(\phi_{\max}^H)^2}{\phi_{\min}^A \eta^2 \rho} \mathbb{E}\|x_{t+1} - x_t\|^2 \\
&\quad + \frac{10 \sigma^2}{M \phi_{\min}^A \rho}, \tag{43}
\end{align*}

where the inequality (i) holds by the Lemma 1.

Combining (40), (42) and (43), we have

\begin{align*}
\mathbb{E}[\mathcal{L}_\rho(x_{t+1}, y_{t+1}, \lambda_{t+1})] &\leq \mathcal{L}_\rho(x_t, y_t, \lambda_t) + \frac{5(L^2 \eta^2 + \phi_{\max}^H)^2}{\phi_{\min}^A \eta^2 \rho} \mathbb{E}\|x_t - x_{t-1}\|^2 \\
&\quad - \left( \frac{\phi_{\min}^H}{\eta} + \frac{\rho \phi_{\min}^A}{2} - \frac{L + 1}{2} \right) \mathbb{E}\|x_t - x_{t+1}\|^2 + \frac{(\phi_{\min}^H + 20) \sigma^2}{2 \phi_{\min}^A \rho M}. \tag{44}
\end{align*}

By (17), we have

\begin{align*}
\Psi_{t+1} - \Psi_t &\leq -\left( \frac{\phi_{\min}^H}{\eta} + \frac{\rho \phi_{\min}^A}{2} - \frac{L + 1}{2} - \frac{5(L^2 \eta^2 + 2(\phi_{\max}^H)^2)}{\phi_{\min}^A \eta^2 \rho} \right) \mathbb{E}\|x_t - x_{t+1}\|^2 \\
&\quad + \frac{(\phi_{\min}^H + 20) \sigma^2}{2 \phi_{\min}^A \rho M}, \\
&= -\gamma \mathbb{E}\|x_t - x_{t+1}\|^2 + \frac{(\phi_{\min}^H + 20) \sigma^2}{2 \phi_{\min}^A \rho M}.
\end{align*}

Finally, using (18) and the properties of quadratic equation in one unknown, we have \(\gamma > 0.\)
A.3 Proof of Lemma 3

Proof. By the definition of \( \{\Psi_t\} \), we have

\[
\Psi_t \geq \mathbb{E}[L_\rho(x_t, y_t, \lambda_t)] = f(x_t) + g(y_t) - \lambda^T (A x_t + B y_t - c) + \frac{\rho}{2} \|A x_t + B y_t - c\|^2
\]

(44) over \( t \)

By Lemma 3, there exists a low bound

\[
\sum (45) \text{ over } t
\]

By (19), we have, for

Proof. Therefore, we can obtain the above result.

A.4 Proof of Theorem 1

Proof. By (19), we have, for \( t \in \{1, 2, \cdots, T\} \)

\[
\Psi_{t+1} - \Psi_t \leq -\frac{\gamma}{2} \|x_{t+1} - x_t\|^2 + \frac{(\phi_{\min}^A + 20)\sigma^2}{2\phi_{\min}^A M}.
\]

Summing (45) over \( t = 1, 2, \cdots, T \), we have

\[
\Psi_T \leq \Psi_1 - \gamma \sum_{t=1}^{T} \mathbb{E}[\|x_{t+1} - x_t\|^2] + \frac{(\phi_{\min}^A + 20)\sigma^2 T}{2\phi_{\min}^A M}.
\]

(46) By Lemma 3 there exists a low bound \( \Psi^* \) such that \( \Psi_t \geq \Psi^* \) holds for \( \forall t \geq 1 \). Then, by (46), we have

\[
t^* = \arg \min_{2 \leq t \leq T+1} \mathbb{E}[\theta_t] \leq \frac{2}{\gamma T} (\Psi_1 - \Psi^*) + \frac{(\phi_{\min}^A + 20)\sigma^2}{2\phi_{\min}^A M}.
\]

Next, by (38), we have

\[
\mathbb{E}[\|A^T \lambda_{t+1} - \nabla f(x_{t+1})\|^2] = \mathbb{E}[\|G(x_t, \xi_t) - \nabla f(x_{t+1}) - \frac{H}{\eta} (x_t - x_{t+1})\|^2]
\]

\[
= \mathbb{E}[\|G(x_t, \xi_t) - \nabla f(x_t) + \nabla f(x_t) - \nabla f(x_{t+1}) - \frac{H}{\eta} (x_t - x_{t+1})\|^2]
\]

\[
\leq 3 (L^2 + \frac{(\phi_{\max}^H)^2}{\eta^2}) \|x_t - x_{t+1}\|^2 + \frac{3\sigma^2}{M}
\]

\[
\leq 3 (L^2 + \frac{(\phi_{\max}^H)^2}{\eta^2}) \theta_t + \frac{3\sigma^2}{M}.
\]

(47) By the step 7 of Algorithm 1 we have

\[
\mathbb{E}[\|A x_{t+1} + B y_{t+1} - c\|^2] = \frac{1}{\rho^2} \mathbb{E}[\|\lambda_{t+1} - \lambda_t\|^2]
\]

\[
\leq \frac{5(L^2 \eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A \eta^2 \rho^2} \|x_t - x_{t-1}\|^2 + \frac{5(\phi_{\max}^H)^2}{\phi_{\min}^A \eta^2 \rho^2} \mathbb{E}[\|x_{t+1} - x_t\|^2] + \frac{10\sigma^2}{M \phi_{\min}^A \rho M},
\]

\[
\leq \frac{5(L^2 \eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A \eta^2 \rho^2} \theta_t + \frac{10\sigma^2}{\phi_{\min}^A \rho^2 M}
\]

\[
= \frac{\zeta}{\rho^2} \theta_t + \frac{10\sigma^2}{\phi_{\min}^A \rho^2 M}.
\]

(48)
By the step 5 of Algorithm 1, there exists a subgradient \( \mu \in \partial g(y_{t+1}) \) such that
\[
E[\text{dist}(B^T\lambda_{t+1}, \partial g(y_{t+1}))^2] \leq \|\mu - B^T\lambda_{t+1}\|^2
\]
\[
= \|B^T\lambda_t - \rho B^T(Ax_t + By_{t+1} - c) - B^T\lambda_{t+1}\|^2
\]
\[
= \|\rho B^T(Ax_{t+1} - x_t)\|^2
\]
\[
\leq \rho^2 \|B\|^2 \|A\|^2 \|x_{t+1} - x_t\|^2
\]
\[
\leq \rho^2 \|B\|^2 \|A\|^2 \theta_t.
\]
(49)

Finally, using the above bounds (47), (48) and (49), and the definition \( \tilde{A} \), an \( \epsilon \)-stationary point of the problem (3) holds in expectation.

\[\square\]

**APPENDIX B**

**CONVERGENCE ANALYSIS OF NONCONVEX MINI-BATCH SVRG-ADMM**

**B.1 Proof Lemma 4**

**Proof.** Since \( \nabla f(x_t^{s+1}) = \frac{1}{M} \sum_{i\in I_t} (\nabla f_i(x_t^{s+1}) - \nabla f_i(\bar{x}^s)) + \nabla f(\bar{x}^s) \), we have
\[
E[\|\nabla f(x_t^{s+1}) - \nabla f(x_t^{s+1})\|^2]
\]
\[
= E[\|\frac{1}{M} \sum_{i\in I_t} (\nabla f_i(x_t^{s+1}) - \nabla f_i(\bar{x}^s)) + \nabla f(\bar{x}^s) - \nabla f(x_t^{s+1})\|^2]
\]
\[
= E[\|\frac{1}{M} \sum_{i\in I_t} (\nabla f_i(x_t^{s+1}) - \nabla f_i(\bar{x}^s))\|^2 - \|\nabla f(x_t^{s+1}) - \nabla f(\bar{x}^s)\|^2]
\]
\[
\leq \frac{1}{M^2} \sum_{i\in I_t} E[\|\nabla f_i(x_t^{s+1}) - \nabla f_i(\bar{x}^s)\|^2]
\]
\[
= \frac{1}{M^2} \sum_{i\in I_t} \frac{1}{n} \sum_{t=1}^n \|\nabla f_i(x_t^{s+1}) - \nabla f_i(\bar{x}^s)\|^2
\]
\[
\leq \frac{L^2}{M} \|x_t^{s+1} - \bar{x}^s\|^2.
\]
where the equality (i) holds by the equality \( E(\xi - \E(\xi))^2 = \E(\xi^2) - (\E(\xi))^2 \) for random variable \( \xi \); the inequality (ii) holds by (44).

**B.2 Proof of Lemma 5**

**Proof.** For simplicity, let \( x_t^{s+1} = x_t, y_t^{s+1} = y_t, \lambda_t^{s+1} = \lambda_t \), and \( \bar{x} = \bar{x}^s \). By the optimal condition of step 10 in Algorithm 2, we have
\[
0 = \tilde{\nabla} f(x_t) - A^T\lambda_t + \rho A^T(Ax_{t+1} + By_{t+1} - c) - \frac{H}{\eta} (x_t - x_{t+1})
\]
\[
= \tilde{\nabla} f(x_t) - A^T\lambda_{t+1} - \frac{H}{\eta} (x_t - x_{t+1}),
\]
where the second equality is due to step 11 in Algorithm 2. Thus, we have
\[
A^T\lambda_{t+1} = \tilde{\nabla} f(x_t) - \frac{H}{\eta} (x_t - x_{t+1}).
\]
(50)

By (50), we have
\[
\|\lambda_{t+1} - \lambda_t\|^2 \leq (\phi_{\min}^A)^{-1} \|A^T\lambda_{t+1} - A^T\lambda_t\|^2
\]
\[
\leq (\phi_{\min}^A)^{-1} \|\tilde{\nabla} f(x_t) - \tilde{\nabla} f(x_{t+1}) - \frac{H}{\eta} (x_t - x_{t+1}) + \frac{H}{\eta} (x_{t+1} - x_t)\|^2
\]
\[
= (\phi_{\min}^A)^{-1} \|\tilde{\nabla} f(x_t) - \nabla f(x_t) + \nabla f(x_t) - \nabla f(x_{t+1}) + \nabla f(x_{t+1}) - \tilde{\nabla} f(x_{t+1})
\]
\[
- \frac{H}{\eta} (x_t - x_{t+1}) + \frac{H}{\eta} (x_{t+1} - x_t)\|^2
\]
\[
\leq \frac{5}{\phi_{\min}^A} \|\tilde{\nabla} f(x_t) - \nabla f(x_t)\|^2 + \frac{5}{\phi_{\min}^A} \|\nabla f(x_{t+1}) - \nabla f(x_{t-1})\|^2 + \frac{5(\phi_{\max}^H)^2}{\phi_{\min}^A \eta^2} \|x_t - x_{t+1}\|^2
\]
\[
+ \frac{5(L^2 \eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A \eta^2} \|x_t - x_{t-1}\|^2,
\]
(51)
where the inequality (i) holds by \cite{14}.

Taking expectation conditioned on information $I_t$ to \cite{51}, we have

$$\mathbb{E}[\lambda_{t+1} - \lambda_t]^2 \leq \frac{5\rho_\text{min}}{\rho_\text{max}} \mathbb{E}[\|\nabla f(x_t) - \nabla f(x_t)^2 + \frac{5\rho_\text{min}}{\rho_\text{max}} \mathbb{E}[\|\nabla f(x_t) - \nabla f(x_t-1)^2 + \frac{5\rho_\text{max}}{\rho_\text{min}} \eta^2 \|x_t - x_{t+1}\|^2 + \frac{5(L^2\eta^2 + \phi_\text{max}^2)}{\rho_\text{min} \eta^2} \|x_{t-1} - x_t\|^2 \]

where the inequality (i) holds by Lemma \cite{4}.

### B.3 Proof of Lemma 6

**Proof.** This proof includes two parts: First, we will prove the sequence $\{(\Phi^s_m)_{m=1}^S\}$ monotonically decreases over $t \in \{1, 2, \ldots, m\}$ in each epoch $s \in \{1, 2, \ldots, S\}$. Second, we will prove $\Phi^s_m \geq \Phi^s_{m+1}$ for $s \in \{1, 2, \ldots, S\}$.

For simplicity, we omit the label of each epoch in the first part, i.e., let $x_t^s+1 = x_t, y_t^s+1 = y_t, \lambda_t^s = \lambda_t$ and $\hat{x}^s = \hat{x}$. By the step 8 of Algorithm \cite{2} we have

$$\mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) \leq \mathcal{L}_\rho(x_t, y_t, \lambda_t).$$

By the optimal condition of step 10 in Algorithm \cite{2} we have

$$0 = (x_t - x_{t+1})^T[\nabla f(x_t) - A^T \lambda_t + \rho(Ax_t + By_t - c) - \frac{H}{\eta}(x_t - x_{t+1})]

= (x_t - x_{t+1})^T[\nabla f(x_t) - \nabla f(x_t) + \nabla f(x_t) - A^T \lambda_t + \rho A^T(Ax_t + By_t - c) - \frac{H}{\eta}(x_t - x_{t+1})]

\leq f(x_t) - f(x_{t+1}) + \frac{L}{2}\|x_t - x_{t+1}\|^2 + (x_t - x_{t+1})^T[\nabla f(x_t) - \nabla f(x_t)] - \frac{1}{\eta}\|x_t - x_{t+1}\|^2

\leq -\lambda_t^T(Ax_t - Ax_{t+1}) + \rho(Ax_t - Ax_{t+1})^T(Ax_t + By_t - c)

\leq f(x_t) - f(x_{t+1}) + \frac{L}{2}\|x_t - x_{t+1}\|^2

\leq \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) - \mathcal{L}_\rho(x_t, y_t, \lambda_t) + \frac{1}{2}\|x_t - x_{t+1}\|^2

\leq \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) - \mathcal{L}_\rho(x_t, y_t, \lambda_t) + \frac{1}{2}\|\nabla f(x_t) - \nabla f(x_t)\|^2

\leq \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) - \mathcal{L}_\rho(x_t, y_t, \lambda_t) + \frac{1}{2}\|x_t - x_{t+1}\|^2

= (\phi_\text{min}^H \rho_\text{max}^2 - \frac{L + 1}{2})\|x_t - x_{t+1}\|^2\]

where the inequality (i) holds by \cite{15}; the equality (ii) holds by applying the equality $(a - b)^T(b - c) = \frac{1}{2}(\|a - c\|^2 - \|a - b\|^2 - \|b - c\|^2)$ on the term $\rho(Ax_t - Ax_{t+1})^T(Ax_{t+1} + By_{t+1} - c)$; the inequality (iii) holds by the Cauchy inequality.

Taking expectation conditioned on information $I_t$ to \cite{53}, we have

$$\mathbb{E}[\mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t)] \leq \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) + \frac{1}{\rho_\text{min}}\mathbb{E}[\|\lambda_{t+1} - \lambda_t\|^2]

+ \frac{L^2}{2\rho_\text{min}}\|x_t - \hat{x}\|^2.$$

By the step 11 of Algorithm \cite{2} we have

$$\mathbb{E}[\mathcal{L}_\rho(x_t, y_{t+1}, \lambda_{t+1}) - \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t)] = \frac{1}{\rho_\text{min}}\|\lambda_{t+1} - \lambda_t\|^2

\leq \frac{5L^2}{\rho_\text{min}^A M}\|x_t - \hat{x}\|^2 + \frac{5L^2}{\rho_\text{min}^A M}\|x_{t-1} - \hat{x}\|^2 + \frac{5(\phi_\text{min}^H)^2}{\rho_\text{min}^A \eta^2}\|x_{t+1} - x_t\|^2

+ \frac{5(L^2\eta^2 + \phi_\text{max}^2)}{\rho_\text{min}^A \eta^2}\|x_t - x_{t-1}\|^2,$$
where the inequality (i) holds by Lemma 5.

Combining (52), (54) and (55), we have

$$
E[\mathcal{L}_\rho(x_{t+1}, y_{t+1}, \lambda_{t+1})] \leq \mathcal{L}_\rho(x_t, y_t, \lambda_t) + \frac{(10 + \phi_{\min}^A) L^2}{2 \rho \phi_{\min}^A M} \|x_t - \hat{x}\|^2 + \frac{5L^2}{\rho \phi_{\min}^A M} \|x_{t-1} - \hat{x}\|^2 \\
+ 5\left(\frac{L^2 \eta^2 + (\phi_{\max}^H)^2}{\phi_{\min}^A \eta^2 \rho}\right) \|x_t - x_{t-1}\|^2 \\
- \left(\frac{\phi_{\min}^H}{\eta} + \frac{\phi_{\min}^A \rho}{\phi_{\min}^A \eta^2 \rho} - \frac{L + 1}{2} - \frac{5(\phi_{\max}^H)^2}{\phi_{\min}^A \eta^2 \rho}\right) \|x_{t+1} - x_t\|^2.
$$

(56)

Next, considering \(E\|x_{t+1} - \hat{x}\|^2\), we have

$$
E\|x_{t+1} - \hat{x}\|^2 = E\|x_{t+1} - x_t + x_t - \hat{x}\|^2 \\
= E\left[\|x_{t+1} - x_t\|^2 + 2(x_{t+1} - x_t)^T(x_t - \hat{x}) + \|x_t - \hat{x}\|^2\right] \\
\leq E\|x_{t+1} - x_t\|^2 + 2\left(\frac{1}{2}\beta \right) \|x_{t+1} - x_t\|^2 + \left(\frac{1}{2}\beta \right) \|x_t - \hat{x}\|^2 + \|x_t - \hat{x}\|^2 \\
= \left(1 + \frac{1}{\beta}\right) \|x_{t+1} - x_t\|^2 + \left(1 + \beta\right) \|x_t - \hat{x}\|^2,
$$

(57)

where the inequality (i) is due to the Cauchy-Schwarz inequality, and \(\beta > 0\). Combining (56) and (57), then, we have

$$
E[\mathcal{L}_\rho(x_{t+1}, y_{t+1}, \lambda_{t+1}) + \frac{5(L^2 \eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A \eta^2 \rho} \|x_{t+1} - x_t\|^2 + h_{t+1}^*(\|x_{t+1} - \hat{x}\|^2 + \|x_t - \hat{x}\|^2)] \\
\leq \mathcal{L}_\rho(x_t, y_t, \lambda_t) + \frac{5(L^2 \eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A \eta^2 \rho} \|x_t - x_{t-1}\|^2 + (2 + \beta) h_{t+1}^* + \frac{10 + \phi_{\min}^A \rho}{2 \phi_{\min}^A M} \left(\|x_t - \hat{x}\|^2 + \|x_{t-1} - \hat{x}\|^2\right) \\
- \left(\frac{\phi_{\min}^H}{\eta} + \frac{\phi_{\min}^A \rho}{\phi_{\min}^A \eta^2 \rho} - \frac{L + 1}{2} - \frac{5(\phi_{\max}^H)^2}{\phi_{\min}^A \eta^2 \rho}\right) \|x_{t+1} - x_t\|^2 \\
- \left((2 + \beta) h_{t+1}^* + \frac{L^2}{2M}\right) \|x_{t-1} - \hat{x}\|^2,
$$

where \(h_{t+1}^* > 0\). By the definition of the sequence \{(\Phi_t^*)_{t=1}^m\}_{t=1}^S, we have

$$
\Phi_{t+1}^* \leq \Phi_t^* - \Gamma_t^* \|x_{t+1}^* - x_t^*\|^2 - ((2 + \beta) h_{t+1}^* + \frac{L^2}{2M}) \|x_{t+1}^* - \hat{x}_{t+1}\|^2.
$$

(58)

Then using (24) and the properties of quadratic equation in one unknown, we have \(\Gamma_t^* > 0, \forall t \in \{1, 2, \cdots, m\}\). Thus, we prove the first part.

Next, we will prove the second part. Since \(\lambda_{t+1}^* = \lambda_m^*\) and \(x_{t+1}^* = x_m^* = \hat{x}_m^*\), we have

$$
E\|\lambda_{t+1}^* - \lambda_t^*\|^2 = E\|\lambda_t^* - \lambda_{t+1}^*\|^2 \\
\leq \frac{1}{\phi_{\min}^A} E\|A^TF_{t+1}\lambda_t^* - A^T\lambda_{t+1}^*\|^2 \\
\leq \frac{1}{\phi_{\min}^A} \frac{E\|\nabla f(x_{m-1}) - \nabla f(x_{m+1})\|^2}{\eta} - \frac{H}{\eta} (x_{m-1}^* - x_m^*) - \frac{H}{\eta} (x_{m+1}^* - x_{m+1})
$$

(59)

where the equality (i) holds by (50), and the equality (ii) holds by the following result:

$$
\nabla f(x_{t+1}^*) = \frac{1}{M} \sum_{i \in I_t} (\nabla f_i(x_{t+1}^*) - \nabla f_i(\hat{x}_m)) + \nabla f(\hat{x}_m) \\
= \frac{1}{M} \sum_{i \in I_t} (\nabla f_i(x_m^*) - \nabla f_i(x_m^*)) + \nabla f(x_m^*) \\
= \nabla f(x_m^*).
$$


By (52), we have
\[
\mathcal{L}_\rho(x_0^{s+1}, y_1^{s+1}, \lambda_0^{s+1}) \leq \mathcal{L}_\rho(x_0^{s+1}, y_0^{s+1}, \lambda_0^{s+1}) = \mathcal{L}_\rho(x_s^s, y_s^s, \lambda_m^s).
\]
(60)

By (54), we have
\[
\mathbb{E}[\mathcal{L}_\rho(x_1^{s+1}, y_1^{s+1}, \lambda_0^{s+1})] \leq \mathcal{L}_\rho(x_0^{s+1}, y_1^{s+1}, \lambda_0^{s+1}) - \left( \frac{\phi_{\min}^H}{2} \rho + \frac{\phi_{\min}^A}{2} - \frac{L + 1}{2} \right) \mathbb{E}\|x_1^{s+1} - x_0^{s+1}\|^2.
\]
(61)

By (59), we have
\[
\mathbb{E}[\mathcal{L}_\rho(x_1^{s+1}, y_1^{s+1}, \lambda_1^{s+1}) - \mathcal{L}_\rho(x_1^{s+1}, y_1^{s+1}, \lambda_0^{s+1})] = \frac{1}{\rho} \mathbb{E}\|\lambda_0^{s+1} - \lambda_1^{s+1}\|
\leq \frac{5L^2}{\phi_{\min}^A \rho M} \|x_{m-1}^s - \tilde{x}_{s-1}^s\|^2 + \frac{5(L^2 \eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A \eta^2 \rho} \|x_{m-1}^s - x_m^s\|^2
+ \frac{5(\phi_{\max}^H)^2}{\phi_{\min}^A \eta^2 \rho} \|x_1^{s+1} - x_0^{s+1}\|^2.
\]
(62)

Combining (60), (61) and (62), we have
\[
\mathbb{E}[\mathcal{L}_\rho(x_1^{s+1}, y_1^{s+1}, \lambda_1^{s+1})] \leq \frac{5L^2}{\phi_{\min}^A \rho M} \|x_{m-1}^s - \tilde{x}_{s-1}^s\|^2 + \frac{5(L^2 \eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A \eta^2 \rho} \|x_{m-1}^s - x_m^s\|^2
\]
\[
- \left( \frac{\phi_{\min}^H}{2} \rho + \frac{\phi_{\min}^A}{2} - \frac{L + 1}{2} \right) \mathbb{E}\|x_1^{s+1} - x_0^{s+1}\|^2.
\]

Using \( h_{s+1} = \frac{10L^2}{\phi_{\min}^A \rho M} \), then we have
\[
\mathbb{E}[\mathcal{L}_\rho(x_1^{s+1}, y_1^{s+1}, \lambda_1^{s+1}) + h_{s+1}(\|x_1^{s+1} - \tilde{x}_s^s\|^2 + \|x_0^{s+1} - \tilde{x}_s^s\|^2) + \frac{5(L^2 \eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A \eta^2 \rho} \|x_{m-1}^s - x_m^s\|^2]
\leq \mathcal{L}_\rho(x_m^s, y_m^s, \lambda_m^s) + \frac{10L^2}{\phi_{\min}^A \rho M} (\|x_m^s - \tilde{x}_{s-1}^s\|^2 + \|x_{m-1}^s - \tilde{x}_{s-1}^s\|^2) + \frac{5(L^2 \eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A \eta^2 \rho} \|x_{m-1}^s - x_m^s\|^2
\]
\[
- \left( \frac{\phi_{\min}^H}{2} \rho + \frac{\phi_{\min}^A}{2} - \frac{L + 1}{2} \right) \mathbb{E}\|x_0^{s+1} - x_1^{s+1}\|^2 - \frac{5L^2}{\phi_{\min}^A \rho M} \|x_{m-1}^s - \tilde{x}_{s-1}^s\|^2.
\]

By the definition of the sequence \( \{(\Phi_t^s)_{t=1}^{m}\}_{s=1}^{S} \), we have
\[
\Phi_1^{s+1} \leq \Phi_m^s - \Gamma_m^s \mathbb{E}\|x_0^{s+1} - x_1^{s+1}\|^2 - \frac{5L^2}{\phi_{\min}^A \rho M} \|x_{m-1}^s - \tilde{x}_{s-1}^s\|^2.
\]
(63)

Then using (24) and the properties of quadratic equation in one unknown, we have \( \Gamma_m^s > 0, \forall s \geq 1 \). Finally, we prove that the sequence \( \{(\Phi_t^s)_{t=1}^{m}\}_{s=1}^{S} \) monotonically decreases.

\[ \square \]

### B.4 Proof of Theorem 2

**Proof.** By (58) and (63), we have, for \( s \in \{1, 2, \cdots, S\} \) and \( t \in \{1, 2, \cdots, m\} \),
\[
\Phi_{t+1}^s \leq \Phi_t^s - \Gamma_m^s \mathbb{E}\|x_{t+1}^s - x_t^s\|^2 - (2 + \beta)h_{t+1}^s + \frac{L^2}{2M} \|x_{t-1}^s - \tilde{x}_{s-1}^s\|^2,
\]
(64)
and
\[
\Phi_{t+1}^s \leq \Phi_t^s - \Gamma_m^s \mathbb{E}\|x_{t+1}^s - x_{t-1}^s\|^2 - \frac{5L^2}{\phi_{\min}^A \rho M} \|x_{m-1}^s - \tilde{x}_{s-1}^s\|^2.
\]
(65)

Summing (64) and (65) over \( s \in \{1, 2, \cdots, S\} \) and \( t \in \{1, 2, \cdots, m\} \), we have
\[
\Phi_{m}^s - \Phi_1^s \leq -\gamma \sum_{s=1}^{S} \sum_{t=1}^{m} \mathbb{E}\|x_t^s - x_{t-1}^s\|^2 - \omega \sum_{s=1}^{S} \sum_{t=1}^{m} \|x_{t-1}^s - \tilde{x}_{s-1}^s\|^2
\]
(66)
where \( \gamma = \min_{s,t} \Gamma_m^s \), and \( \omega = \min_{s,t} \{(2 + \beta)h_{t+1}^s + \frac{L^2}{2M} - \frac{5L^2}{\phi_{\min}^A \rho M}\} \). By Lemma 7, there exists a constant \( \Phi^* \) such that \( \Phi_t^s \geq \Phi^* \). By (26) and (66), then, we have
\[
(s^*, t^*) = \arg\min_{1 \leq s \leq S, 1 \leq t \leq m} \theta_t^s \leq \frac{2}{T}(\Phi^* - \Phi_t^s),
\]
(67)
where $\tau = \min(\gamma, \omega)$, and $T = mS$.

By Lemma [5], we have

$$
\mathbb{E}[A^T x_t^s - \nabla f(x_t^s)]^2 \\
= \mathbb{E}[\nabla f(x_t^s) - \nabla f(x_t^s) - \frac{H}{\eta} (x_t^s - x_t^{s-1})]^2 \\
= \mathbb{E}[\nabla f(x_t^s) - \nabla f(x_t^s) + \nabla f(x_t^s) - \nabla f(x_t^s) - \frac{H}{\eta} (x_t^s - x_t^{s-1})]^2 \\
\leq \frac{3L^2}{M} \|x_t^s - x_t^{s-1}\|^2 + 3(L^2 + \frac{(\phi_{max}^H)^2}{\eta^2}) \|x_t^s - x_t^{s-1}\|^2 \\
\leq 3(L^2 + \frac{(\phi_{max}^H)^2}{\eta^2}) \hat{\theta}_t^s.
$$

By Lemma [5], we have

$$
\mathbb{E}[A x_t^s + B y_{t+1}^s - c]^2 = \frac{1}{\rho^2} \|\lambda_t^s - \lambda_t^s\|^2 \\
\leq \frac{5L^2}{\phi_{min}^A \rho^2 M} \mathbb{E}[x_t^s - x_t^{s-1}]^2 + \frac{5L^2}{\phi_{min}^A \rho^2 M} \|x_t^s - x_t^{s-1}\|^2 \\
+ \frac{5(\phi_{max}^H)^2}{\phi_{min}^A \rho^2 \eta^2} \mathbb{E}[x_t^s - x_t^{s-1}]^2 + \frac{5(L^2 \eta^2 + (\phi_{max}^H)^2)}{\phi_{min}^A \rho^2 \eta^2} \|x_t^s - x_t^{s-1}\|^2 \\
\leq \frac{5(L^2 \eta^2 + \phi_{max}^H)^2}{\phi_{min}^A \rho^2 \eta^2} \hat{\theta}_t^s = \frac{\zeta}{\rho^2} \hat{\theta}_t^s.
$$

By the step 8 of Algorithm [2] there exists a sub-gradient $\mu \in \partial g(y_{t+1}^s)$ such that

$$
\mathbb{E}[\text{dist}(B^T \lambda_{t+1}, \partial g(y_{t+1}^s))]^2 \leq \|\mu - B^T \lambda_{t+1}\|^2 \\
= \|B^T \lambda_{t} - \rho B^T (A x_t^s + B y_{t+1}^s - c) - B^T \lambda_{t+1}\|^2 \\
= \|\rho B^T A (x_{t+1}^s - x_t^s)\|^2 \\
\leq \rho^2 \|B\|^2 \|A\|^2 \|x_{t+1}^s - x_t^s\|^2 \\
\leq \rho^2 \|B\|^2 \|A\|^2 \hat{\theta}_t^s.
$$

Finally, using the above bounds [68], [69] and [70], and the definition [1] an $\epsilon$-stationary point of the problem [3] holds in expectation.

\section*{Appendix C}

\subsection*{Convergence Analysis of Nonconvex Mini-batch SAGA-ADMM}

\subsection*{C.1 Proof of Lemma 8}

\begin{proof}
Since $\psi_t = \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(z_t^j)$, we have

$$
\mathbb{E}[\Delta_t]^2 = \mathbb{E}[\frac{1}{M} \sum_{i \in I_t} (\nabla f_i(z_t^1) - \nabla f_i(z_t^i)) + \psi_t - \nabla f(x_t)]^2 \\
= \mathbb{E}[\frac{1}{M} \sum_{i \in I_t} (\nabla f_i(z_t^1) - \nabla f_i(z_t^i)) - (\nabla f(x_t) - \psi_t)]^2 \\
\leq \frac{1}{M^2} \sum_{i \in I_t} \mathbb{E}[\nabla f_i(z_t^1) - \nabla f_i(z_t^i)]^2 \\
= \frac{1}{M^2} \sum_{i \in I_t} \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x_t) - \nabla f_i(z_t^i)\|^2 \\
\leq \frac{L^2}{nM} \sum_{i=1}^{n} \|x_t - z_t^i\|^2.
$$

where the equality (i) holds by the equality $\mathbb{E}(\xi - \mathbb{E}\xi)^2 = \mathbb{E} \xi^2 - (\mathbb{E} \xi)^2$ for random variable $\xi$, and $\mathbb{E}[\nabla f_i(z_t^i)] = \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(z_t^j) = \psi_t$; the inequality (ii) holds by [14].
\end{proof}
C.2 Proof of Lemma 10

Proof. By the step 5 of Algorithm we have
\[ \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) \leq \mathcal{L}_\rho(x_t, y_t, \lambda_t). \] (71)

By the optimal condition of step 7 in Algorithm we have
\[
\begin{align*}
0 &= (x_t - x_{t+1})^T \left[ \nabla f(x_t) + \rho A^T (Ax_{t+1} + B y_{t+1} - c) - A^T \lambda_t - \frac{H}{\eta} (x_t - x_{t+1}) \right] \\
&= (x_t - x_{t+1})^T \left[ \nabla f(x_t) - \nabla f(x_t) + \nabla f(x_t) - A^T \lambda_t - \frac{H}{\eta} (x_t - x_{t+1}) + \rho A^T (Ax_{t+1} + B y_{t+1} - c) \right] \\
&\leq f(x_t) - f(x_{t+1}) + \frac{L}{2} \| x_{t+1} - x_t \|^2 + (x_t - x_{t-1})^T \left( \nabla f(x_t) - \nabla f(x_t) - \frac{1}{\eta} \| x_{t+1} - x_t \|^2 \right) \\
&\quad - \lambda_{t}^T (Ax_{t+1} - Ax_t) + \rho (Ax_t - Ax_{t+1})^T (Ax_{t+1} + B y_{t+1} - c) \\
&\quad = f(x_t) - f(x_{t+1}) + \frac{L}{2} \| x_{t+1} - x_t \|^2 + (x_t - x_{t-1})^T \left( \nabla f(x_t) - \nabla f(x_t) - \frac{1}{\eta} \| x_{t+1} - x_t \|^2 \right) \\
&\quad - \lambda_{t}^T (Ax_{t+1} - Ax_t) + \rho (Ax_t - Ax_{t+1})^T (Ax_{t+1} + B y_{t+1} - c) \\
&\quad \leq \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) - \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) + (x_t - x_{t-1})^T \left( \nabla f(x_t) - \nabla f(x_t) - \frac{1}{\eta} \| x_{t+1} - x_t \|^2 \right) \\
&\quad - \frac{\rho}{2} \| Ax_t - Ax_{t+1} \|^2 \\
&\quad \leq \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) \leq \frac{1}{2} \| \nabla f(x_t) - \nabla f(x_t) \|^2 \\
&\quad - (\frac{\phi H}{\eta} + \frac{\phi A M}{2} - \frac{L + 1}{2}) \| x_t - x_{t+1} \|^2, \quad (72)
\end{align*}
\]

where the inequality (i) holds by \[15\]; the equality (ii) holds by applying the equality \((a - b)^T (b - c) = \frac{1}{2}(\|a - c\|^2 - \|a - b\|^2 - \|b - c\|^2)\) on the term \(\rho (Ax_t - Ax_{t+1})^T (Ax_{t+1} + B y_{t+1} - c)\); the inequality (iii) holds by the Cauchy inequality. Taking expectation conditioned on information \(I_t\) to (72), we have
\[
\mathbb{E}[\mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t)] \leq \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_t) + \frac{L^2}{2nM} \sum_{t=1}^{n} \mathbb{E}\| x_t - z_t^* \|^2 - (\frac{\phi H}{\eta} + \frac{\phi A M}{2} - \frac{L + 1}{2}) \| x_t - x_{t+1} \|^2. \quad (73)
\]

By the step 8 of Algorithm and taking expectation conditioned on \(I_t\), we have
\[
\mathbb{E}[\mathcal{L}_\rho(x_t, y_{t+1}, \lambda_{t+1}) - \mathcal{L}_\rho(x_t, y_{t+1}, \lambda_{t+1})] = \frac{1}{\rho} \mathbb{E}\| \lambda_{t+1} - \lambda_{t+1} \|^2 \\
\leq \frac{5L^2}{\rho \phi H M n} \sum_{t=1}^{n} \mathbb{E}\| x_t - z_{t}^* \|^2 + \frac{5L^2}{\rho \phi A M n} \sum_{t=1}^{n} \mathbb{E}\| x_{t+1} - z_{t+1}^* \|^2 \\
+ \frac{5(\phi H_{\max})^2}{\phi A M n} \mathbb{E}\| x_{t+1} - x_t \|^2 + \frac{5(L_A^2 \eta^2 + (\phi H_{\max})^2)}{\phi A M n} \mathbb{E}\| x_{t+1} - x_{t+1} \|^2, \quad (74)
\]

where the inequality (i) holds by Lemma Combining (71), (73) and (74), we have
\[
\mathbb{E}[\mathcal{L}_\rho(x_t, y_{t+1}, \lambda_{t+1})] \leq \mathcal{L}_\rho(x_t, y_{t}, \lambda_{t}) + \frac{10L^2 + \phi A M n}{2\rho \phi A M n} \sum_{t=1}^{n} \mathbb{E}\| x_t - z_{t}^* \|^2 + \frac{5L^2}{\rho \phi H M n} \sum_{t=1}^{n} \mathbb{E}\| x_{t+1} - z_{t+1}^* \|^2 \\
+ \frac{5(L_A^2 \eta^2 + (\phi H_{\max})^2)}{\phi A M n} \mathbb{E}\| x_{t+1} - x_t \|^2 - (\frac{\phi H}{\eta} + \frac{\phi A M}{2} - \frac{L + 1}{2} - \frac{5(\phi H_{\max})^2}{\phi A M n}) \mathbb{E}\| x_{t+1} - x_{t+1} \|^2. \quad (75)
\]

Considering \(\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\| x_{t+1} - z_{t+1}^* \|^2\), we have
\[
\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\| x_{t+1} - z_{t+1}^* \|^2 = \frac{1}{n} \sum_{t=1}^{n} \left[ \frac{1}{n} \mathbb{E}\| x_{t+1} - x_t \|^2 + \frac{n - 1}{n} \mathbb{E}\| x_{t+1} - z_t^* \|^2 \right]. \quad (76)
\]
The term $\mathbb{E}\|x_{t+1} - z_t^t\|^2$ in (76) can be bounded below:

$$
\mathbb{E}\|x_{t+1} - z_t^t\|^2 = \mathbb{E}\|x_{t+1} - x_t + x_t - z_t^t\|^2 \\
= \mathbb{E}\|x_{t+1} - x_t\|^2 + 2(x_{t+1} - x_t)^T(x_t - z_t^t) + \|x_{t-1} - z_t^t\|^2 \\
\leq \mathbb{E}\|x_{t+1} - x_t\|^2 + 2\left(\frac{1}{2\beta}\mathbb{E}\|x_{t+1} - x_t\|^2 + \frac{\beta}{2}\|x_t - z_t^t\|^2\right) + \|x_t - z_t^t\|^2
$$

where $\beta > 0$, and the inequality (i) is due to Cauchy-Schwarz inequality. Thus, we have

$$
\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\|x_{t+1} - z_t^{t+1}\|^2 \leq (1 + \frac{n-1}{n\beta})\mathbb{E}\|x_t\|^2 + (1 + \beta)\frac{n-1}{n^2}\sum_{i=1}^{n}\|x_t - z_t^t\|^2
$$

(78)

Next, combining (78) and (79), we have

$$
\mathbb{E}\left[L_p(x_{t+1}, y_t, \lambda_{t+1}) + \frac{5(L^2\eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A\eta^2\rho}\|x_{t+1} - x_t\|^2 + \frac{\alpha_{t+1}}{n}\sum_{i=1}^{n}(\|x_{t+1} - z_{i, t}^{t+1}\|^2 + \|x_t - z_{t}^t\|^2)\right] \\
\leq L_p(x_t, y_t, \lambda_t) + \frac{5(L^2\eta^2 + (\phi_{\max}^H)^2)}{\phi_{\min}^A\eta^2\rho}\|x_t - x_{t-1}\|^2 \\
+ ((2 + \beta - \frac{1 + \beta}{n})\alpha_{t+1} + \frac{10L^2 + \phi_{\min}^A\rho L^2}{2\rho\phi_{\min}^A L}\frac{1}{n}\sum_{i=1}^{n}(\|x_t - z_{i, t}^t\|^2 + \|x_{t-1} - z_{t-1}^{t-1}\|^2) \\
- \left(\frac{\phi_{\min}^H}{\eta}\frac{\phi_{\min}^A}{\rho} - \frac{L + 1}{2} - \frac{5L^2\eta^2 + (\phi_{\max}^H)^2}{\phi_{\min}^A\eta^2\rho}\frac{1}{n}\sum_{i=1}^{n}\|x_{t-1} - z_{i, t-1}^{t-1}\|^2\right)
$$

(79)

By the definition of the sequence $\{\Theta_t\}_{t=1}^{T}$, we have

$$
\Theta_{t+1} \leq \Theta_t - \Gamma_t\|x_{t+1} - x_t\|^2 - ((2 + \beta - \frac{1 + \beta}{n})\alpha_{t+1} + \frac{L^2}{2M})\frac{1}{n}\sum_{i=1}^{n}\|x_{t-1} - z_{i, t-1}^{t-1}\|^2.
$$

(80)

Then using (31) and the properties of quadratic equation in one unknown, we have $\Gamma_t > 0$. Finally, we prove that the sequence $\{\Theta_t\}_{t=1}^{T}$ monotonically decreases. 

C.3 Proof of Theorem 3

Proof. By (80), we have, for $t \in \{1, 2, \cdots, T\}$

$$
\Theta_{t+1} \leq \Theta_t - \Gamma_t\|x_{t+1} - x_t\|^2 - ((2 + \beta - \frac{1 + \beta}{n})\alpha_{t+1} + \frac{L^2}{2M})\frac{1}{n}\sum_{i=1}^{n}\|x_{t-1} - z_{i, t-1}^{t-1}\|^2.
$$

(81)

Summing (81) over $t = 1, 2, \cdots, T$, we have

$$
\Theta_T \leq \Theta_1 - \gamma\sum_{t=1}^{T} \mathbb{E}\|x_{t+1} - x_t\|^2 - \omega\sum_{t=1}^{T} \frac{1}{n}\sum_{i=1}^{n}\|x_{t-1} - z_{i, t-1}^{t-1}\|^2,
$$

(82)

where $\gamma = \min_t \Gamma_t$ and $\omega = \min_t \left(\frac{(2 + \beta - \frac{1 + \beta}{n})\alpha_{t+1} + \frac{L^2}{2M}}{\Gamma_t}\right)$. By Lemma 11, there exists a constant $\Theta^*$ such that $\Theta_t \geq \Theta^*$ holds for $\forall t \geq 1$. By (33) and (82), then, we have

$$
t^* = \arg\min_{2 \leq t \leq T+1} \Theta_t \leq \frac{2}{\tau T}(\Theta_1 - \Theta^*),
$$

(83)

where $\tau = \min(\gamma, \omega)$. 
Next, by the optimal condition of step 7 in Algorithm 3, we have
\[ E\|A^T\lambda_{t+1} - \nabla f(x_{t+1})\|^2 \]
\[ = E\|\nabla f(x_t) - \nabla f(x_{t+1}) - \frac{H}{\eta} (x_t - x_{t+1})\|^2 \]
\[ = E\|\nabla f(x_t) - \nabla f(x_t) + \nabla f(x_{t+1}) - \nabla f(x_{t+1}) - \frac{H}{\eta} (x_t - x_{t+1})\|^2 \]
\[ \leq \frac{3L^2}{nM} \sum_{i=1}^{n} \|x_t - z_i^t\|^2 + 3\left(L^2 + \frac{(\phi_{\text{max}}^H)^2}{\eta^2}\right)\|x_t - x_{t+1}\|^2 \]
\[ \leq 3\left(L^2 + \frac{(\phi_{\text{max}}^H)^2}{\eta^2}\right) \tilde{\theta}_t. \quad (84) \]

By Lemma 9, we have
\[ E\|Ax_{t+1} + By_{t+1} - c\|^2 = \frac{1}{\rho^2} \|\lambda_{t+1} - \lambda_t\|^2 \]
\[ \leq \frac{5L^2}{\phi_{\text{A}}^A nM \rho^2} \sum_{i=1}^{n} E\|x_t - z_i^t\|^2 + \frac{5L^2}{\phi_{\text{A}}^A nM \rho^2} \sum_{i=1}^{n} \|x_{t-1} - z_i^{t-1}\|^2 \]
\[ + \frac{5(\phi_{\text{max}}^H)^2}{\phi_{\text{A}}^A \eta^2 \rho^2} E\|x_{t+1} - x_t\|^2 + \frac{5(L^2 \eta^2 + (\phi_{\text{max}}^H)^2)}{\phi_{\text{A}}^A \eta^2 \rho^2} \|x_t - x_{t-1}\|^2 \]
\[ \leq \frac{5(L^2 \eta^2 + (\phi_{\text{max}}^H)^2)}{\phi_{\text{A}}^A \eta^2 \rho^2} \tilde{\theta}_t = \frac{\zeta}{\rho^2} \tilde{\theta}_t. \quad (85) \]

By the step 5 of Algorithm 3, there exists a subgradient \( \mu \in \partial g(y_{t+1}) \) such that
\[ E[\text{dist}(B^T\lambda_{t+1}, \partial g(y_{t+1}))^2] \leq \|\mu - B^T\lambda_{t+1}\|^2 \]
\[ = \|B^T \lambda_t - \rho B^T (Ax_t + By_{t+1} - c) - B^T \lambda_{t+1}\|^2 \]
\[ = \|\rho B^T A(x_{t+1} - x_t)\|^2 \]
\[ \leq \rho^2 \|B\|^2 \|A\|^2 \|x_{t+1} - x_t\|^2 \]
\[ \leq \rho^2 \|B\|^2 \|A\|^2 \tilde{\theta}_t. \quad (86) \]

Finally, using the above bounds (84), (85) and (86), and the definition \( \epsilon \)-stationary point of the problem (3) holds in expectation.