Field Strength Correlators For Two Dimensional
Yang-Mills Theories
Over Riemann Surfaces

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Abstract

The path integral computation of field strength correlation functions for two dimensional Yang-Mills theories over Riemann surfaces is studied. The calculation is carried out by abelianization, which leads to correlators that are topological. They are nontrivial as a result of the topological obstructions to the abelianization. It is shown in the large $N$ limit on the sphere that the correlators undergo second order phase transitions at the critical point. Our results are applied to a computation of contractible Wilson loops.

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1 Introduction

In the last few years two-dimensional Yang-Mills theories have been studied extensively. The partition function for the theory on $\Sigma_g$, a Riemann surface of genus $g$, was computed exactly in [1] (see also [2]). More recently Gross and Taylor have provided a string interpretation of the $1/N$ expansion of the partition function for the $SU(N)$ gauge group by enumerating classes of maps from worldsheets to the target surface $\Sigma_g$ [3, 4, 5]. The string description of the theory was later extended to the $SO(N)$ and $Sp(2N)$ gauge groups in [6]. Some progress has been made towards finding a string action for the 2d QCD string [7, 8] as well. However Douglas and Kazakov [9] have shown that the string interpretation no longer holds for weak coupling when $\Sigma_g$ is a sphere, although it does remain valid for strong coupling. On the sphere the large $N$ theory undergoes a phase transition at a critical value of $e^2(Area) = \pi^2$. The study of this phase transition might also be relevant for four dimensional QCD, which may or not exhibit a similar behaviour [19, 20]. It is therefore important to understand those features of a gauge theory which give rise to string behavior and to a phase-transition.

Two dimensional Yang-Mills theories have also been studied by means of beautiful path integral methods. In [9] the path integral was evaluated by a generalized localization formula, while in [10], the path integral for gauge fields of fixed holonomy around the boundary of a disc was used as the elementary building block from which the partition function and Wilson loops on closed surfaces could be constructed.

In this paper we will study correlation functions of field strengths in 2d non-abelian gauge theories as an application of the abelianization technique for path integrals developed by Blau and Thompson [11, 12, 13]. We show that these correlators exhibit the almost topological nature of the theory, and can be expressed in terms of the higher order Casimir operators of the gauge group. They can also be related to
the generalized 2d QCD of [18, 25] where higher order Casimir operators appear in the partition function. In section 2 we use abelianization to compute the partition function in the presence of an external source. This allows us to calculate the correlators of field strengths, which is presented in section 3. One might expect the correlators of field strengths to be trivial, since the Lagrangian with a source coupled to $F_{\mu\nu}$ is quadratic in the fields. However, because of the topological obstructions encountered in the abelianization, the correlators are highly non-trivial. In section 4 we show that, on the sphere, the regularized correlators of arbitrary numbers of field strength operators have second-order phase-transitions in the large $N$ limit. Section 5 shows how contractible Wilson loops can be computed from our correlators, in agreement with the known formulae for Wilson loops. Our results suggest that, in the abelianization gauge, the usual Stokes’ theorem can be used to relate the Wilson loop to the field strength correlation functions when the gauge group is $SU(2)$ [see equation (53)]. In Appendix A we compute correlators on the disc in a different gauge, while Appendix B is devoted to some Lie algebra details.

The study of the correlators of the field strengths in the abelianization gauge, enables us to calculate the master field for the regularized field strength $F_{\mu\nu}$ in the large $N$ limit on the sphere, in a sequel to this paper [32]. Since, as we show in this paper, in this gauge these correlation functions are essentially topological, i.e. independent of position on the Riemann surface, the master field will also have this property. This enables us to construct the master field for $A_\mu$ itself on the sphere [32]. In fact, the master field which exhibits the whole structure of the (unregulated) correlation functions, as is needed for the Wilson loop, can be obtained in a Hilbert space representation [30].

One hopes to be able to calculate fermion correlation functions on the sphere in the large $N$ limit, using the master field for $A_\mu$. Thus, this paper serves as the first step in this program of understanding the coupling of fermions to 2D Yang-Mills
theory on Riemann surfaces. To date this has only been accomplished on the plane \[\Sigma \] . We are presently studying this application of our results.

2 The Path Integral \( Z_{\Sigma_g} (J) \)

In this section we consider the field strength of the \( U(N) \) gauge theory on the compact surface of genus \( g \), \( \Sigma_g \), coupled to an external source \( J(x) \) transforming in the adjoint of the gauge group. The path integral describing this situation will be evaluated using the elegant abelianization method of \([11, 12]\). The resulting partition function \( Z_{\Sigma_g} (J) \) will then enable us to compute the electric field correlators in the next section.

The starting point is the first order form of the path integral,

\[
Z_{\Sigma_g} (J) = \int \mathcal{D}A_\mu \exp \left[ -\frac{e^2}{2} \int_{\Sigma_g} d\mu \ Tr(\xi^2) + \int_{\Sigma_g} d\mu \ Tr(J\xi) \right] \tag{1}
\]

where the scalar fields \( \xi^a(x) \) are defined by \( F_{\mu\nu}(x) = \xi(x)\sqrt{g(x)}\epsilon_{\mu\nu} \), with \( \xi = T^a \xi^a \), with \( T^a \) a generator of \( U(N) \), \( d\mu = \sqrt{g(x)}d^2x \) a Riemannian measure on \( \Sigma_g \), \( \epsilon_{\mu\nu} \) is the usual antisymmetric tensor with \( \epsilon_{01} = 1 \), and \( \sqrt{g(x)} \) is the square root of the determinant of the metric on \( \Sigma_g \).

The starting point is the first order form of the path integral,

\[
Z_{\Sigma_g} (J) = \int \mathcal{D}\phi \mathcal{D}A_\mu \exp \left[ -\frac{e^2}{2} \int_{\Sigma_g} d\mu \ Tr(\phi - iJ)^2 - i\int_{\Sigma_g} d\mu \ Tr(\phi \xi) \right] \tag{2}
\]

where \( \phi \) is a Lie algebra(\( \mathcal{G} \)) valued scalar field also transforming in the adjoint under the gauge group. Performing the gaussian integration over \( \phi \) gives back (1). We decompose \( \mathcal{G} \) into,

\[
\mathcal{G} = t \oplus k \tag{3}
\]

where \( t = h \oplus u(1) \) generates the usual maximal torus (\( T \)) of \( U(N) \), with \( h \) being the \( su(N) \) Cartan subalgebra (\( su(N) \) denotes the Lie algebra of \( SU(N) \), etc).
The path integral (2) then becomes the product of the integrals over the $su(N)$ and $u(1)$ valued fields, where the two sectors will be related by a topological selection rule as explained ahead\footnote{This condition reflects the fact that we have $U(N)$ gauge group, not just $U(1) \times SU(N)$.}. We will evaluate each of these separately beginning with the $U(1)$ sector. Bold type style quantities will be $su(N)$ valued and primed quantities will be $u(1)$ valued. In order to account for the nontrivial $U(N)$ bundles over $\Sigma_g$, we use the topological condition for the $u(1)$ component of the gauge curvature $\xi'$,

$$f_{\Sigma_g} d\mu \xi' = 2\pi \frac{m}{\sqrt{N}} , \quad m \text{ an integer.} \quad (4)$$

It is convenient to decompose $m$ as $m = \tilde{m} \cdot N + p$ where $\tilde{m}$ is an integer and $p \in Z_N$. The topological selection rule connecting the $u(1)$ and $su(N)$ sectors will depend on $p$ and therefore we keep $p$ fixed for now. The sum over all possible values of $p$ will be performed when we examine the $su(N)$ sector. To impose condition (4) we insert the periodic delta function,

$$\sum_n \exp\left( \frac{n}{\sqrt{N}} f_{\Sigma_g} d\mu \xi' \right) \cdot \exp\left( -2\pi i \frac{p}{N} n \right) \quad (5)$$

in the path integral. Let us consider the $u(1)$ piece first, using the strategy developed in \cite{10}. The action is linear in $\xi'$, and the Nicolai map can be used to change variables of integration from $A'_\mu$ to $\xi'$, and some scalar gauge fixing function $G(A'_\mu)$. The integration over this second new variable, together with the jacobian from the change of variables, will cancel against the Faddeev-Popov determinant. The integration over $\xi'$ now gives a delta function that fixes the (constant) value of $\phi'$. The final contribution to (3) from these $u(1)$ abelian fields is,

$$\sum_n \exp\left( \frac{e^2}{2} f_{\Sigma_g} d\mu (J')^2 - \frac{e^2 A n^2}{2N} + i \frac{1}{\sqrt{N}} e^2 n f_{\Sigma_g} d\mu J' \right) \cdot \exp\left( -2\pi i \frac{p}{N} n \right) \quad (6)$$

where the source $J'(x)$ just couples to $u(1)$ fields\footnote{We use $J' = \frac{\text{Tr}(J)}{\sqrt{N}}$ and $\xi' = \frac{\text{Tr}(\xi)}{\sqrt{N}}$.}, and $A$ is the area of $\Sigma_g$. (We take $\text{Tr}(T^a T^b) = \delta^{ab}$ for the $U(N)$ generators). Notice that for $N = 1$ there is no...
dependence on $p$ in (3). We will now do the remaining integrations over the $su(N)$ valued fields. Here we will use the abelianization method of [11, 12]. One chooses the gauge condition, which restricts $\phi$ to the Lie algebra of the maximal torus, i.e.

$$\phi^k = 0 \quad (7)$$

in order to make use of the fact that the inner product (given by $\text{Tr}$) on $G$ makes $h$ and $k$ orthogonal. First one must cover $\Sigma_g$ by open sets where (7) is valid, then in the intersections of those open sets with each other, the $h$ components of $\phi$ (and also other fields) will in general be related by nontrivial gauge transformations preserving (7) i.e. by transformations which take values in $N(T)$, the normalizer of the maximal torus $T$ in $G$. The remarkable fact that one can choose those gauge transformations to lie in $T$ itself [13], so that $\phi^h$ is globally defined, follows from the simple connectivity of $G$. These $T$-valued gauge transformations then give rise to non-trivial $T$ bundles over $\Sigma_g$. We refer to Blau and Thompson’s [13] for the detailed explanations of this beautiful result. (Strictly speaking at this point one must restrict the fields $\phi$ to take values in the regular elements of $G$, but we’ll see ahead that the non-regular valued fields are eliminated by the Faddeev-Popov determinant [13]). Thus, the price to be paid for demanding (7) is the appearence of nontrivial $T$ bundle topologies. These however can be easily incorporated in the path integral by making use of the first Chern numbers [11, 12]. We have then, for the integration over the $su(N)$ valued fields

$$\frac{1}{|W|} \sum_{T\text{bundles}} \int D A^h_\mu D A^k_\mu D \phi^h \det[\Delta_W(\phi^h)] \exp(-S(J)) \quad (8)$$

where $|W|$ is the order of the Weyl group and $\det[\Delta_W(\phi^h)]$ is the path integral analogue of the Weyl determinant in the Weyl integral formula for Lie groups. This can also be interpreted as a Faddeev-Popov determinant coming from (7), [12].

$$\det[\Delta_W(\phi^h)] = \det[ad(\phi^h)]|_{\Omega^0(\Sigma_g,k)} \quad (9)$$

where $\Omega^0(\Sigma_g,k)$ is the space of $k$ valued 0-forms on $\Sigma_g$. 

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Using the orthogonality between $h$ and $k$ the only contribution from the integration over the $k$ components of the gauge connection is easily seen to be given, up to a constant, by

\[(\det[ad(\phi^h)|_{\Omega^1(\Sigma_g,k)}])^{-1/2}\]  

(10)

where $\Omega^1(\Sigma_g,k)$ is the space of $k$ valued 1-forms on $\Sigma_g$. Notice the similarity to (9).

In fact using the Hodge decomposition of $\Omega^1(\Sigma_g,k)$ into orthogonal components, the contribution of (9) and (10) nearly cancels (see [11] for a careful evaluation of this ratio of determinants); that is

\[
\frac{\det[ad(\phi^h)|_{\Omega^0(\Sigma_g,k)}]}{\det[ad(\phi^h)|_{\Omega^1(\Sigma_g,k)}]^{1/2}} = \det[ad(\phi^h)|_{\Omega^1(\Sigma_g,k)}]^{-b_1/2+b_0} = \det[ad(\phi^h)|_{\Omega^1(\Sigma_g,k)}]^{\chi(\Sigma_g)/2}
\]  

(11)

where $b_0 = 1$, $b_1 = 2g$ are the Betti numbers for $\Sigma_g$ and $\chi(\Sigma_g)$ is the Euler number of the surface. (We’ll see that only constant $\phi^h$ configurations contribute in the end).

At this stage we have reduced the integration over the $su(N)$-valued fields to

\[
\frac{1}{|W|} \sum_{T \text{bundles}} \int DA^h_{\mu} D\phi^h \det[ad(\phi^h)|_{\Omega^1(\Sigma_g,k)}]^{\chi(\Sigma_g)/2} \left\{ \exp[-\frac{e^2}{2} \int_{\Sigma_g} d\mu \ Tr(\phi^h - iJ)^2 - i\int_{\Sigma_g} Tr(\phi^h dA^h)] \right\}
\]

(12)

The remaining $T$ valued gauge invariance, not eliminated by (7), can be fixed, and the corresponding Faddeev-Popov determinant eliminated by using the Nicolai map once more. In this way $dA^h$ is traded for $A^h_{\mu}$ as a new variable in the path integral. It is now easy to express the summation over nontrivial $T$ bundles in terms of the first Chern numbers, as in (4) and (5). We choose a set $\alpha_i$, $i = 1, ..., N-1$, of simple roots for $su(N)$ and we let $(dA^h)^i$ be the components of $dA^h$ in the corresponding basis for $h$. Let us consider the topological sector with $u(1)$ charge $m$ as in (4). If we write $m = \tilde{m} \cdot N + p$ with $p \in Z_N$ and $\tilde{m}$ an integer, then the topological conditions for $(dA^h)^i$ are

\[
\int_{\Sigma_g} (dA^h)^j = 2\pi (m^j - \frac{pj}{N}), \quad m^j \text{ integers}
\]  

(13)
This topological selection rule arises because the $U(1)$ and $SU(N)$ components of $U(N)$ are related since $U(N) \sim (U(1) \times SU(N))/Z_N$. Inserted in the path integral, this condition will combine with the $\exp(-2\pi i \frac {p} {N} n)$ factor in (3) and will give origin to a summation over all pairs of representations of $U(1)$ and $SU(N)$ that can be combined into a representation of $U(N)$, namely such that $(n-R) \mod N = 0$, where $R$ is the number of boxes of the Young tableau for the $SU(N)$ representation and $n$ labels the $U(1)$ representation as in (3). Therefore the summation over $T$ bundles and the remaining summation over $p$ can be done by inserting

$$\sum_{\lambda \in \Lambda} \exp[i \lambda (f_{\Sigma_g} dA^h)] \cdot \delta((n-R(\lambda)) \mod N) \quad (14)$$

in the path integral. In the sum $\Lambda$ stands for all integer linear combinations of the fundamental weights of $su(N)$. The remaining integrations are immediate, the integral over $dA^h$ producing $\delta(\phi^h + \lambda)$, so that indeed only constant $\phi^h$ configurations contribute. One can now use the classic formulas [12, 14, 15],

$$\det[ad(\phi^h)|k] \sim \prod_\delta \delta(\beta, \phi^h) \quad (15)$$

and

$$\dim(\mu) = \prod_{\delta^+} \frac {\delta(\beta, \mu + \rho)} {\delta(\beta, \rho)} \quad (16)$$

$$C_2(\mu) = (\mu + \rho) \cdot (\mu + \rho) - \rho \cdot \rho \quad (17)$$

where $\dim(\mu)$ and $C_2(\mu)$ is the dimension and quadratic Casimir of the irreducible representation of $SU(N)$ with highest weight $\mu$. $\delta(\delta^+)$ is the set of all (positive) roots, and $\rho$ is the half-sum of the positive roots.

As remarked in [11, 12] the Weyl determinant in (15) vanishes whenever $\phi^h$ lies in the walls of a Weyl chamber. Thus the integration should be restricted to regular

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3 The quantity $(\mu + \rho)$ coincides with the variable $\sigma$ defined by Okubo [17], his eqn.(15), in his study of third order Casimir operators.
valued $\phi^h$ fields, i.e. those in the interior of the Weyl chambers (see [13] for a detailed discussion). In the source free theory all $\phi^h$ can then be rotated to the interior of the fundamental Weyl chamber, and the $\frac{1}{|W|}$ factor gets cancelled. Every element of $\Lambda$ then corresponds to a unique element $(\mu + \rho)$ in the interior of the fundamental Weyl chamber. However when the source $J$ is not zero, one must retain the average over the Weyl group. In fact, that is why we have kept the harmless constant $\frac{1}{|W|}$ of the Weyl integral formula in front of the path integral (12). The result for the $su(N)$ sector can then be written,

$$\sum_{\mu} \dim(\mu)^2 - 2g \exp\left[\frac{-e^2 AC_2(\mu)}{2} + e^2 \frac{i}{2} \int_{\Sigma_g} d\mu \, \text{Tr}(J)^2\right] \frac{1}{|W|} \sum_{\sigma \in W} \exp\left[ie^2 \frac{i}{2} \int_{\Sigma_g} d\mu \, (\sigma(\mu + \rho), J^h)\right]$$

where the sum goes over representations of $su(N)$ compatible with $u(1)$ charge $n$ as explained above. In (18) we have fixed the allowed renormalization terms [2, 12] of the form $\exp(c_0 e^2 A)$ and $\exp(c_1 (2 - 2g))$, to give agreement with the usual normalization chosen for $J = 0$. We can now combine the $u(1)$ and $su(N)$ pieces to give the total partition function for $U(N)$,

$$Z_{\Sigma_g}(J) = \sum_{l} \dim(l)^2 - 2g \exp\left(\frac{-e^2 AC_2(l)}{2}\right) \{\exp\left(\frac{e^2}{2} \int_{\Sigma_g} d\mu \, \text{Tr}(J)^2\right) \times \frac{1}{|W|} \sum_{\sigma \in W} \exp\left(ie^2 \frac{i}{2} \int_{\Sigma_g} d\mu \, (\sigma(l + \rho), J^l)\right)\}$$

where now $l$ runs over the highest weights of irreducible representations of $U(N)$. (The $l^i$ are the row lengths of the $U(N)$ Young tableau for the representation $l$.) It should be emphasized that the last term in (19) is a consequence of the coupling of the source to the nontrivial $T$ topological sectors of the gauge field. Notice that for $U(N)$, the Weyl group $W$ is the symmetric group $S_N$. This computation can also be trivially extended to the case of the generalized 2d QCD of [18, 25]. In fact, since we have seen that only constant configurations of $\phi^h$ contribute, our calculation also applies when higher powers of $\phi$ are present.
3 The Correlators

In this section we will study the field strength correlation functions, obtained from functional derivatives with respect to $J$ in $Z_{\Sigma_g}(J)$. Since the average over the Weyl group plays an essential role, it is convenient to use a basis for $h \oplus u(1)$ where the action of the Weyl group is as simple as possible. For $k$ we will use the usual generators of the fundamental representation of $SU(N)$, and for the generators of $T$ we will use the diagonal matrices $E_{ii}$ $i=1,\ldots,N$, where $E_{ii}$ has only one nonzero entry, 1 on the $i^{th}$ position of the diagonal. We adopt the normalization $\text{Tr}(T^a T^b) = \delta^{ab}$. The elements of the Weyl group $S_N$ then just permute the entries of the diagonal matrices in $h \oplus u(1)$. (The Weyl group acts as the identity on $u(1)$.)

As expected from the symmetry $\xi \to -\xi$ of (1) with $J = 0$, all odd point functions vanish. In our framework, the odd point functions vanish since various products with an odd number of factors of the form, 

$$(l + \rho)^{\sigma_i} \ldots (l + \rho)^{\sigma_j}$$

are averaged over $W$. However, for each representation with highest weight $l$ there exists a conjugate representation with highest weight $\bar{l}$ such that $(\bar{l} + \rho) = -\bar{\sigma}(l + \rho)$, where $\bar{\sigma}_i = N - i + 1$. Moreover the dimensions and quadratic casimirs of $l$ and $\bar{l}$ are the same, so that the contribution of $l$ cancels that of $\bar{l}$ in the sum over $l$ in the odd point functions, and therefore these vanish. (For self-conjugate representations one will have $(l + \rho) = -\bar{\sigma}(l + \rho)$ so that the average over the Weyl group also gives zero).

There are two terms which depend on $J$ in $Z_{\Sigma_g}(J)$. The term $\exp[\int_{\Sigma_g} d\mu \text{Tr}(J^2)]$ gives rise to contact terms in the expectation values of the field strengths. If one wants to consider products of fields at the same point one needs to regulate these terms. This term is closely related to the arbitrary renormalization term $\exp(c_0 e^2 A)$.

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\footnote{In this basis one has that the $l^i$ are the row lengths of the Young tableau of the $U(N)$ representation $l$ and that $\rho^i = \frac{N+1}{2} - i$.}
that appears in front of the partition function. In fact by differentiating the $J = 0$ partition function with respect to $e^2$, we can conclude that the normalized correlator

$$\langle \int_{\Sigma_g} d\mu \text{Tr} \xi^2(x) \rangle = 2e^2 A \frac{d}{dA} F$$

(20)

where $F = \log Z_{\Sigma_g}$ is the free energy. Therefore in order to bring fields to the same point we must normalize the contact term accordingly. This is done by adjusting the coefficient $c_0$ in the renormalization term.

For example, with this renormalization prescription the regularized 2-point function becomes

$$\langle \xi^a(x) \xi^b(y) \rangle = \frac{e^4}{Z_{\Sigma_g}} \sum_l \dim(l)^{2-2g} \exp \left( \frac{-e^2 A C_2(l)}{2} \right)$$

$$\times \left[ \frac{(\rho, \rho)}{N^2} \delta_{x,y}^2 - \frac{1}{|W|} \sum_\sigma (l + \rho)^{a_\sigma} (l + \rho)^{b_\sigma} \right]$$

(21)

where $\delta_{x,y}^2$ is 1 if $x = y$, and zero otherwise. Notice the $(l + \rho)$ terms are present only when $a$ and $b$ are $h \oplus u(1)$ indices, i.e. when the indices lie in the Cartan subalgebra.

The average over the Weyl group can be done explicitly with the result (see appendix B)

$$\frac{1}{|W|} \sum_\sigma (l + \rho)^{a_\sigma} (l + \rho)^{b_\sigma} = p^{ab}(l + \rho)^2 + m^{ab} n^2$$

(22)

where

$$p^{ab} = \begin{cases} -\frac{1}{N(N-1)} & \text{if } a \neq b \\ \frac{1}{N} & \text{if } a = b \end{cases}$$

and

$$m^{ab} = \begin{cases} \frac{1}{N(N-1)} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

and

$$n = \sum_i l^i = \text{total number of boxes in the tableau defined by } l.$$ 

Thus we have the final result for the regularized 2-point function

\footnote{All correlators will be normalized by dividing by the partition function.}

That is $\langle \xi^a(x) \cdots \xi^b(y) \rangle = \frac{1}{Z_{\Sigma_g}(0)} \frac{\partial^a}{\partial J_a(x)} \cdots \frac{\partial^b}{\partial J_b(y)} Z_{\Sigma_g}(J)|_{J=0}$
\[ \langle \xi^a(x)\xi^b(y) \rangle = \frac{e^4}{Z_{\Sigma_y}} \sum_l \dim(l)^{2-2g} \exp\left(-\frac{e^2AC_2(l)}{2}\right) \]
\[ \times \left[ \frac{(\rho, \rho)\delta^{ab}}{N^2} \delta^2_{x,y} - (p^{ab}(l + \rho)^2 + m^{ab}n^2) \right] \] (23)

This is finite as \( x \to y \). From here (20) follows as well. Notice that the correlator is essentially topological in the gauge we have chosen.

Similarly the regularized 4-point function is,
\[ \langle \xi^i(x)\xi^j(y)\xi^k(z)\xi^l(w) \rangle = \frac{e^8}{Z_{\Sigma_y}} \sum_l \dim(l)^{2-2g} \exp\left(-\frac{e^2AC_2(l)}{2}\right) \times \]
\[ \left\{ \frac{(\rho, \rho)^2}{(\dim G)^2} (\delta^i_{x,y} \delta^j_{z,w} + 2 \text{ permut.}) \right\} - \frac{(\rho, \rho)}{\dim G} \frac{1}{|W|} \sum_{\sigma} [\delta^i_{x,y}(l + \rho)^{\sigma_i}(l + \rho)^{\sigma_j} + 5 \text{ permut.}] \]
\[ + \frac{1}{|W|} \sum_{\sigma} (l + \rho)^{\sigma_i}(l + \rho)^{\sigma_j} + c^{ijkl}n^2(l + \rho)^2 + e^{ijkl}n^4 \] (24)

(once again the \((l + \rho)\) terms are present only for \( h \oplus u(1) \) valued indices in the Cartan subalgebra). The average over \( W \) gives (see appendix B),
\[ \frac{1}{|W|} \sum_{\sigma} (l + \rho)^{\sigma_i}(l + \rho)^{\sigma_j} + c^{ijkl}n^2(l + \rho)^2 + e^{ijkl}n^4 = a^{ijkl} \sum_i (l + \rho)^4_i + b^{ijkl}[(l + \rho)^2]^2 + c^{ijkl}n \sum_i (l + \rho)^3_i + d^{ijkl}n^2(l + \rho)^2 + e^{ijkl}n^4 \] (25)

where the coefficients \( a^{ijkl} \), etc can be found in equation (B.6). Combining (24) and (25) gives the regularized 4-point function
\[ \langle \xi^i(x)\xi^j(y)\xi^k(z)\xi^l(w) \rangle = \frac{e^8}{Z_{\Sigma_y}} \sum_l \dim(l)^{2-2g} \exp\left(-\frac{e^2AC_2(l)}{2}\right) \times \]
\[ \left\{ \frac{(\rho, \rho)^2}{(\dim G)^2} (\delta^i_{x,y} \delta^j_{z,w} + 2 \text{ permut.}) \right\} - \frac{(\rho, \rho)}{\dim G} \frac{1}{|W|} \sum_{\sigma} [\delta^i_{x,y}(l + \rho)^{\sigma_i}(l + \rho)^{\sigma_j} + 5 \text{ permut.}] \]
\[ + \left[ a^{ijkl} \sum_i (l + \rho)^4_i + b^{ijkl}[(l + \rho)^2]^2 + c^{ijkl}n \sum_i (l + \rho)^3_i + d^{ijkl}n^2(l + \rho)^2 + e^{ijkl}n^4 \right] \] (26)

Once again we see that the correlator is topological. Contracting with the group generators produces a gauge invariant quantity. (Some useful identities for traces and higher order Casimirs can be found in [14]). We have
\[ \langle \text{Tr} \xi^4(x) \rangle = \frac{e^8}{Z_{\Sigma}} \sum_l \dim(l)^{2-2g} \exp\left( -\frac{e^2 A C_2(l)}{2} \right) \times \]
\[ \left[ \frac{-(N^2 - 1)^2}{72N} \left( N^2 + 6N + \frac{1}{2} \right) - \frac{N^2 - 1}{3N} (N - 1) C_2(l) - \frac{N^2 - 1}{3N} n^2 + \frac{1}{4} C_4(l) \right] \] (27)

where \( C_4(l) \) is a fourth order Casimir for \( U(N) \) defined by
\[ C_4(l) = \sum_i \left[ (l + \rho)^i \right]^4. \]

The structure of the higher point functions is now clear. For the \( 2p \) point function there’s a Casimir invariant \( C_{2p} \), and products of \( (\rho, \rho) \) with lower order casimirs of even order. The gauge invariant quantities \( \langle \text{Tr} \xi^{2p}(x) \rangle \) have no dependence on \( x \) at all, while the non-invariant correlators are also topological and only depend on contact terms in a trivial way. Thus, this gauge could be called a topological gauge. In fact the theory is invariant under diffeomorphisms of \( \Sigma_g \) that preserve the measure \( d\mu \). Thus the \( x \) dependence of gauge invariant quantities could come only from factors of \( \sqrt{g(x)} \) which were already absorbed in the definition of \( \xi \) in (1). Similarly, in the computation of Wilson loops only the area enclosed by the loops was relevant (see for example [10]). However, notice that even the non-gauge invariant uncontracted correlators (21),(24) above have only a trivial position dependence coming from the contact term. Therefore the gauge (7) is particularly well suited to exhibit the topological nature of the theory, in the sense that expectation values of local non-gauge invariant quantities in this gauge are also independent of position.

4 The Phase Transition On \( S^2 \)

The partition function on the sphere is,
\[ Z = \sum_R \dim(R)^2 \exp\left( -\frac{A C_2(R)}{2N} \right) \] (28)

where now \( A = \lambda(\text{Area}) \), with \( \lambda = e^2 N \) held fixed when \( N \to \infty \). Although the exponential decreases rapidly for large representations, for \( A \) sufficiently small the
series diverges due to the dimension term raised to a positive power. In fact, Douglas and Kazakov \cite{19} have shown that in the large $N$ limit the theory undergoes a third-order phase transition at $A_{cr} = \pi^2$. This phase transition was also explained in \cite{20} as a consequence of the presence of instantons (classical solutions of the field equations). When $A$ approaches $\pi^2$, the contribution of the instanton configurations to the partition function becomes dominant, and the phase transition is induced. The relation to the string interpretation was also explored in \cite{22, 23}. It has been shown that a similar phase transition takes place for YM$_2$ on the cylinder \cite{21}, on the projective plane $RP^2$, and for $Sp(N)$ and $SO(N)$ on the sphere as well. Further, the generalized 2d Yang-Mills theory possesses a rich phase structure, with phase transitions possible for $g > 1$, and for particular regions of the coupling constants space \cite{24}. The Wilson loops in the large $N$ limit have been studied in both the weak and strong coupling phases \cite{19, 27, 28, 29, 24}.

Here we will study the phase transition on the sphere for the electric field correlators of YM$_2$ in the large $N$ limit. Let us briefly review the methods and known results. The irreducible representations of $U(N)$ are labelled by the Young tableau row lengths $l^1 \geq l^2 \geq ... \geq l^N$. Also,

\[(l + \rho)^i = \frac{N + 1}{2} - i + l^i, \quad i = 1...N\]  

(29)

In the large $N$ limit, these variables can be replaced by continuum variables \cite{21},

\[x = \frac{i}{N} \quad \text{with} \quad 0 \leq x \leq 1\]
\[l(x) = \frac{l^i}{N} \quad \text{and} \quad \sum_i = N \int_0^1 dx\]  

(30)

It is then useful to define,

\[h(x) = -l(x) + x - \frac{1}{2}\]  

(31)

When one expresses the partition function \cite{28} in terms of these continuum variables, the sum over representations becomes a quantum mechanical functional integral for,

\[S_{eff}(h) = -\int_0^1 dx \int_0^1 dy \log[h(x) - h(y)] + \frac{A}{2} \int_0^1 dx h^2(x) - \frac{A}{24}\]  

(32)
with
\[ Z = \int \mathcal{D}h \exp(-N^2 S_{\text{eff}}(h)) \]  \hspace{1cm} (33)

Since \( N \) is large, this can be computed by the saddle-point approximation \([26, 19]\).

Define the density of boxes in a Young tableau by
\[ u(h) = \frac{\partial x(h)}{\partial h}, \text{ where } u(h) \leq 1 \]  \hspace{1cm} (34)

The saddle point equation is
\[ P \int ds \frac{u(s)}{h-s} = \frac{A}{2} h \]  \hspace{1cm} (35)

For \( A < A_{cr} = \pi^2 \), the solution to this equation is given by the semicircle law \([26]\),
\[ u_{\text{weak}}(h) = \frac{A}{2\pi} \sqrt{\frac{4}{A} - h^2}, \text{ with } 0 \leq h \leq \frac{2}{\sqrt{A}} \]  \hspace{1cm} (36)

However for \( A > A_{cr} = \pi^2 \) the above solution violates the condition \( u(h) \leq 1 \) in (34), and a different solution must be sought \([19]\). The ansatz
\[ u_{\text{strong}}(h) = \begin{cases} 1 & \text{if } -b \leq h \leq b \\ \tilde{u}(h) & \text{otherwise} \end{cases} \]  \hspace{1cm} (37)

gives the solution
\[ f(h) = \int ds \frac{\tilde{u}(s)}{s-h} \]
\[ = -h \frac{A}{2} - \log \frac{h-b}{h+b} + \sqrt{(a^2 - h^2)(b^2 - h^2)} \int_{-b}^{b} \frac{ds}{(h-s)\sqrt{(a^2 - s^2)(b^2 - s^2)}} \]  \hspace{1cm} (38)

Following \([19]\) we will now present the behaviour of the free energy, \( F = \frac{1}{N^2} \log Z \), at
the critical point. This will enable us to review the method and notation needed to
describe the phase transition of the correlators. We have,
\[ F' = \frac{dF}{dA} = -\frac{\partial S_{\text{eff}}(h)}{\partial A} = -\frac{1}{2} \int_0^1 dx h^2 + \frac{1}{24} \]  \hspace{1cm} (39)
where one uses the fact that \( u(h) \) is an even function both in the strong and weak coupling solutions. When \( A < \pi^2 \) from (36),

\[
F'(A) = \frac{1}{24} - \frac{1}{2A}
\]

(40)

For the strong coupling phase we need to use the large \( h \) expansion of (38). The \( O(h) \) and \( O(h^{-1}) \) terms give,

\[
a = \frac{4K}{A}
\]

(41)

and

\[
A = 8EK - 4k'^2K
\]

(42)

where \( k = b/a \) is the modulus of the elliptic integrals, \( k' = \sqrt{1-k^2} \) is the complementary modulus and \( K = K(b/a), E = E(b/a) \) are the complete elliptic integrals of the first and second kind respectively (see for example [30]). To explore the behaviour near the critical point, it is useful to express these quantities in terms of theta functions on a torus of complex modulus \( \tau \) [19, 30]. One sets \( q = \exp i\pi\tau \) and \( \theta_i = \theta_i(0|\tau), i = 0, 1, 2, 3 \). When we approach the critical point, \( b \to 0 \) and \( q \to 0 \). We then define \( \delta = A - A_{cr} = A - \pi^2 \). Expanding (42) in \( q \) gives,

\[
A = \pi^2(1 + 8q - 8q^2 + 32q^3 + \cdots)
\]

(43)

and inverting for \( \delta \) we find,

\[
q = \frac{\delta}{8\pi^2}(1 + \frac{\delta}{8\pi^2} + \cdots)
\]

(44)

From the weak and strong coupling expansions of (19) one gets,

\[
F'_{\text{strong}} - F'_{\text{weak}} = -\frac{\delta^2}{\pi^6} + \cdots = -\frac{1}{\pi^2} \frac{(A - A_{cr})^2}{\pi^4} + \cdots
\]

(45)

which is the third-order phase-transition for the free energy of [19].

From (20), section 3, in the large \( N \) limit\(^6\) (note that here \( F = \frac{1}{N^2} \log Z \)),

\[
\frac{1}{N} \langle \text{Tr} \xi^2(z) \rangle = 2\lambda \frac{dF}{dA} = 2\lambda F'(A)
\]

(46)

\(^6\)In this section \( z \) will denote a point on \( S^2 \) and \( x \) will always be the continuous index \( x = \frac{i}{N} \).
and therefore the gauge-invariant two-point function behaves as $F'(A)$ at the critical point, with a second-order phase-transition.

The gauge invariant four-point function was given in (27). In the large $N$ limit, only the critical representation survives at the saddle point. Moreover, since the critical representation is self-conjugate, $h(x)$ is odd about $x = \frac{1}{2}$, from (31) the corresponding $u(1)$ charge $n = \int_0^1 l(x)$ vanishes. Notice that,

$$(l + \rho)(x) = \frac{1}{N}(l + \rho)^i = -h(x)$$

then

$$\frac{1}{N} \langle \text{Tr} \xi^4(z) \rangle = \lambda^4 \left\{ -\frac{1}{72} - \frac{1}{3} \left[ \int_0^1 dx h^2(x) - \frac{1}{12} \right] + \frac{1}{4} \int_0^1 dx h^4(x) \right\}$$

(47)

In the weak phase we can use (36) to obtain,

$$\int_0^1 dx h^4_{\text{weak}}(x) = \frac{2}{A^2}$$

(48)

From the $O(h^{-5})$ term in (38) in the strong coupling phase we have,

$$\int_0^1 dx h^4_{\text{strong}}(x) = \frac{4\pi^6}{5A^5}(\theta^4_{3} - \theta^4_{2}\theta^4_{3} - \theta^4_{2}\theta^4_{2} + \theta^4_{12}) + \frac{6\pi^4}{5A^4}(\theta^6_{3} + \frac{2}{3}\theta^4_{3}\theta^4_{2} + \theta^6_{2})$$

(49)

Expanding for small $q$ and using (44) we find,

$$\int_0^1 dx h^4_{\text{strong}}(x) - \int_0^1 dx h^4_{\text{weak}}(x) = \frac{4}{\pi^4} \left( \frac{A - A_{cr}}{A_{cr}} \right)^2 + \cdots$$

(50)

so that the phase-transition for the four-point function is also second-order.

Similarly we can use the $O(h^{-7})$ term in (38) to show that,

$$-\int_0^1 dx h^6_{\text{strong}}(x) =$$

$$\frac{(2\pi)^6}{2A^7} \left[ \frac{151}{2688}(\theta^16_{3} + \theta^16_{2}) - \frac{19}{672}(\theta^4_{3}\theta^4_{2} + \theta^4_{2}\theta^4_{3}) - \frac{25}{448}\theta^8_{3}\theta^8_{2} - \frac{5}{48}\theta^4_{3}\theta^4_{0}\theta^4_{2} - \frac{7}{96}\theta^8_{0}(\theta^8_{3} + \theta^8_{2}) \right]$$

$$+ \frac{(2\pi)^6}{A^6} \left[ -\frac{4}{21}(\theta^4_{3} + \theta^4_{2}) + \frac{5}{42}(\theta^8_{3}\theta^4_{0} + \theta^8_{2}\theta^4_{3}) + \frac{7}{48}\theta^8_{0}(\theta^4_{3} + \theta^4_{2}) \right]$$

(51)

so that,

$$\int_0^1 dx h^6_{\text{strong}}(x) - \int_0^1 dx h^6_{\text{weak}}(x) = \frac{12}{\pi^6} \left( \frac{A - A_{cr}}{A_{cr}} \right)^2 + \cdots$$

(52)
Therefore the six-point function also has a second-order phase transition at the critical point. (In fact it has been shown that

$$\int_0^1 dx h_{\text{strong}}^{2n}(x) - \int_0^1 dx h_{\text{weak}}^{2n}(x)$$

has a second-order transition for any \( n \), so that all higher-point correlators of the field strength have second-order phase-transitions as well [31]).

The master field for the field strength and gauge potential, on the sphere in the large \( N \) limit, based on the above results, which reproduces the regulated correlators is presented in [32]. The unregulated correlators, that is including non-regularized contact terms, can also be described by an explicit master field representation [30].

## 5 Application to Wilson Loops

One of the motivations for studying field strength correlators is to understand the couplings of matter to 2D Yang-Mills theory on Riemann surfaces. To illustrate this direction of research, we compute expectation values for contractible Wilson loops using a version of Stokes’ theorem, equation (53), and our correlators and compare these calculations with known exact results. Although we do not have a complete proof of the validity of this version of the theorem, our results are so compelling, that we include them so as to stimulate further research. To be specific, we compute contractible Wilson loops on \( \Sigma_g \) for \( U(1) \) and \( SU(2) \) gauge theories, using Stokes’ theorem to relate the holonomy of the gauge field in the representation \( r \) around a closed loop to the exponential of the integral of the gauge curvature over the interior of the loop. That is we consider

$$\langle W_r(\gamma) \rangle = \langle \text{Tr} \ P \exp(i \int_\gamma A) \rangle = \langle \text{Tr} \exp(i \int_D F) \rangle$$

(53)

where we will take a contractible, non-self-intersecting loop \( \gamma = \partial D \). Obviously (53) is valid for a \( U(1) \) gauge theory. However, motivated by the abelianization of the
partition function, we attempt to use (53) for the $SU(2)$ gauge theory as well. As we
describe below, we have computed $\langle W(\gamma) \rangle$ for $SU(2)$ up to $O(\Delta^4)$, where $\Delta$ is the area
enclosed by the loop $\gamma$, and verified that the Stokes’ theorem (53) is valid for this case
as well, in the Blau-Thompson abelianization gauge. It should be emphasized that
(53) appears to work for non-abelian theories, with gauge group $SU(2)$, computed
in the abelianization gauge, as a result of explicit calculations. Unfortunately, we
cannot provide a proof of this remarkable result, so that the application of (53) to
SU(2) non-abelian theory in this gauge remains a conjecture. We note that for non-
abelian Yang-Mills theories in general gauges, one must use a non-abelian version of
the Stokes’ theorem, where one of the variables in the surface integral is ordered and
the gauge curvature appears conjugated by holonomies, so that gauge invariance is
preserved [33]. By contrast, in (53) there is no ordering in the integration over the
disc and $F$ is not conjugated by holonomies. Nevertheless, it apparently computes
the Wilson loop in the abelianization gauge.

The expectation values of Wilson loops in two dimensional Yang-Mills theory
are known; see for example [10]. For a loop $\gamma = \partial D$ on $\Sigma_g$, in the representation $r$
we have the exact result

$$
\langle W_r(\gamma) \rangle = \sum_l \dim(l)^{1-g} \exp(-\frac{e^2 AC_2(l)}{2}) \sum_{\lambda \in \otimes r} \dim(\lambda) \exp[\frac{e^2 \Delta(C_2(l) - C_2(\lambda))}{2}]
$$

(54)

where $\Delta$ is the area of the disc $D$ enclosed by the Wilson loop. The exponential in
(54) is expanded in powers of $\Delta$, which then expresses $\langle W_r(\gamma) \rangle$ as a power series in
$\Delta$. On the other hand, one can expand the right-hand side of (53) as

$$
\langle W_r(\gamma) \rangle = \text{Tr}(1) - \frac{1}{2} \langle \text{Tr} \int_D F(x) \int_D F(y) \rangle + \frac{1}{4!} \langle \text{Tr}(\int_D F)^4 \rangle + \cdots
$$

(55)

where we have used our result that odd correlators of field strengths vanish.
Comparing the expansion of (54) in powers of $\Delta$ with (55) seems to present a para-
dox, as (54) contains odd powers of $\Delta$, while (55) only involves an even number of
integrations on the disc $D$. In fact, there is a very elegant resolution of this issue when one uses the (unregulated) correlation functions of section 3, as these correlators have contributions from the contact terms, obtained by means of functional derivatives with respect to the source from the $\text{Tr}(J^2)$ term in (19). We will refer to contributions from this term, as the contact terms of the correlators, and from the linear term in $J$ in (19) as the topological terms.

For example, the contact term ($\sim \sqrt{g(x)} \sqrt{g(y)} \delta^2(x - y)$) in the two point function will produce a linear term in $\Delta$ in (55) when integrated over the disc. This is also the case for the Wilson loop on the plane. See for example Bralić in [33]. In general each contact term and each topological term in a field strength correlator produces a term linear in $\Delta$ when inserted in (55).

We now turn to the explicit verification of (55), correct to $O(\Delta^4)$, with $U(1)$ and $SU(2)$ considered separately for clarity.

### 5.1 $U(1)$ gauge theory

The Wilson loop in an arbitrary representation $r$ of $U(1)$ can be written as

$$\langle W_r(\gamma) \rangle = \sum_n \exp\left( -\frac{e^2 A n^2}{2} \right) \exp\left( -\frac{e^2 \Delta (r^2 + 2nr)}{2} \right)$$

(56)

where $n$ runs over all integers.

The expansion of (56) in powers of $\Delta$ yields

$$\langle W_r(\gamma) \rangle = \sum_{n=-\infty}^{\infty} \exp\left( -\frac{e^2 A n^2}{2} \right) \cdot \left\{ 1 - \frac{e^2}{2} r^2 \Delta + \frac{1}{2!} \frac{e^2}{4} (r^4 + 4n^2 r^2) \Delta^2 - \frac{1}{3!} \frac{e^2}{8} (r^6 + 12n^2 r^4) \Delta^3 + \frac{1}{4!} \frac{e^2}{16} (r^8 + 24n^2 r^6 + 16n^4 r^4) \Delta^4 + \cdots \right\}$$

(57)
\[ \equiv \langle 1 \rangle - \frac{e^2}{2} \Delta \langle r^2 \rangle + \frac{e^2}{8} \Delta^2 \langle r^4 + 4n^2r^2 \rangle - \frac{e^6}{48} \Delta^3 \langle r^6 + 12n^2r^4 \rangle + \]
\[ + \frac{1}{4!} \frac{e^8}{16} \Delta^4 \langle r^8 + 24n^2r^6 + 16n^4r^4 \rangle + \cdots (58) \]
where the expectation values in (58) are defined in the obvious way from (57).

Our objective is to compute the right-hand side of (55) and to compare it with (58). The relevant generating functional of correlation functions is (6), specialized to \( N = 1 \), and with each \( F \) multiplied by \( r \) in accord with our normalization conventions.

Let us sketch the explicit comparisons, using (55) and our unregulated correlators. For convenience we write \( F_c \) or \( F_t \) for the part of the field strength correlators coming from the contact terms or topological terms in the generating functional. We indicate these terms in the expectation values schematically, as these usually correspond to a number of permutations of contact and topological terms. We stress that \( F_c \) and \( F_t \) have only symbolic meaning, as labels of terms in the correlation functions.

\[ \text{O}(\Delta): \]
\[ - \frac{1}{2} \int_D dx \int_D dy \langle F_c(x)F_c(y) \rangle = \langle r^2 \rangle (-\frac{1}{2}) \int_D d\mu(x) \int_D d\mu(y) e^2 \delta^2(x-y) = \frac{e^2}{2} \langle r^2 \rangle \Delta \]
\[ (59) \]
which agrees with (58).

\[ \text{O}(\Delta^2): \]
\[ - \frac{1}{2} \int_D dx \int_D dy \langle F_t(x)F_t(y) \rangle = (ie^2)^2 r^2 (-\frac{1}{2}) \langle n^2 \rangle \Delta^2 \]
\[ (60) \]
and
\[ \frac{1}{4!} (\int_D)^4 \langle F_c^4 \rangle = \frac{1}{4!} \langle r^4 \rangle e^4 (\int_D)^4 [\delta^2(x-y)\delta^2(z-w) + 2 \text{ permut.}] = \frac{1}{8} e^4 \langle r^4 \rangle \Delta^2 \]
\[ (61) \]
The sum of (60) and (61) agrees with the \( O(\Delta^2) \) term in (58). Further, the two terms in (58) of order \( O(\Delta^3) \) originate from (61) and (60) respectively.

\[ \text{O}(\Delta^3): \]
\[ \frac{1}{4!} (\int_D)^4 \langle F_c^2 F_t^2 \rangle = \frac{1}{4!} r^4 (-e^4) \langle n^2 \rangle (\int_D)^4 e^2 [\delta^2(x-y) + 5 \text{ permut.}] = -\frac{e^6}{4} r^4 \langle n^2 \rangle \Delta^3 \]
\[ (62) \]
and
\[-\frac{1}{6!}(\int_D)^6 \langle F_c^6 \rangle = -\frac{1}{6!} \langle r^6 \rangle (\int_D)^6 e^6 [\delta^2(x-y)\delta^2(z-w)\delta^2(v-u) + 14 \text{ permut.}] = -\frac{e^6}{48} \langle r^6 \rangle \Delta^3 \]

(63)

Again, the agreement with (58) is evident.

\[O(\Delta^4):\]
\[\frac{1}{4!} (\int_D)^4 \langle F_t^4 \rangle = \frac{e^8}{4!} r^4 \langle n^4 \rangle \Delta^4 \]

(64)

while
\[-\frac{1}{6!} (\int_D)^6 \langle F_c^4 F_t^2 \rangle = \frac{e^8}{16} r^6 \langle n^2 \rangle \Delta^4 \]

(65)

and
\[\frac{1}{8!} (\int_D)^8 \langle F_c^8 \rangle = \frac{e^8}{4!16} \langle r^8 \rangle \Delta^4 \]

(66)

Thus, (64) to (66) agree with the \(O(\Delta^4)\) term of (58).

Therefore, we have shown how to obtain the expansion of the Wilson loop expectation value in powers of the area, for a contractible non-self-intersecting loop, directly from the field strength correlation functions.

We note that each term in (58) comes from a distinct combination of contact and topological terms in the expectation values. This sets the stage for a similar calculation for \(SU(2)\).

5.2 \(SU(2)\): Wilson Loop in the Fundamental Representation

The Wilson loop expectation value of a gauge field in the fundamental representation for a contractible non self-intersecting loop is
\[
\langle W_f(\gamma) \rangle = \sum_{j=0}^{\infty} \dim(j)^{1-2g} \exp\left(-\frac{e^2 AC_2(j)}{2}\right) \sum_{\lambda \in j \otimes f} \dim(\lambda) \exp\left(\frac{e^2\Delta(C_2(j) - C_2(\lambda))}{2}\right)
\]

(67)
where \( \text{dim}(j) = j + 1 \), \( C_2(j) = \frac{1}{2}j(j + 2) \) and \( j \otimes f = (j - 1) \oplus (j + 1) \) for \( j \geq 1 \). The expansion of (67) in powers of \( \Delta \) gives, after some elementary but tedious algebra

\[
\langle W_f(\gamma) \rangle = \sum_j \text{dim}(j)^{2-2g} \exp\left(-\frac{e^2 AC_2(j)}{2}\right).
\]

\[
\cdot \left\{ \text{Tr}(1) - \frac{3}{2} e^2 \Delta + \frac{e^4}{16} [4(j + 1)^2 + 5] \Delta^2 - \frac{1}{4} \frac{e^6}{48} [20(j + 1)^2 + 7] \Delta^3 + \right. \\
\left. + \frac{1}{4!} \frac{e^8}{16!} [16(j + 1)^4 + 56(j + 1)^2 + 9] \Delta^4 + \cdots \right\} 
\]

\[
= \langle 2 \rangle - \frac{3}{2} e^2 \Delta \langle 1 \rangle + \frac{e^4}{16} \Delta^2 \langle 4(j + 1)^2 + 5 \rangle - \frac{1}{4} \frac{e^6}{48} \Delta^3 \langle 20(j + 1)^2 + 7 \rangle + \\
+ \frac{1}{4!} \frac{e^8}{128} \Delta^4 \langle 16(j + 1)^4 + 56(j + 1)^2 + 9 \rangle + \cdots 
\]

\[
(68)
\]

The generating functional for the correlators of field strengths in the fundamental representation is given by (18). For \( SU(2) \) it is convenient to carry out the Weyl sum in (18) directly, with the result

\[
Z_{\Sigma_g}(J) = \sum_{j=0}^{\infty} \text{dim}(j)^{2-2g} \exp\left(-\frac{e^2 AC_2(j)}{2}\right) \exp\left(\frac{e^2}{2} \int_{\Sigma_g} d\mu \text{Tr}(J^2)\right) \cos\left(\frac{e^2}{\sqrt{2}}(j+1) \int_{\Sigma_g} d\mu J\right) 
\]

\[
(70)
\]

where \( \frac{1}{\sqrt{2}} J^i \sigma^3 \) is the diagonal component of \( J \), with our choice of normalization. Let us compute the right-hand side of (55) using the correlators generated by (70).

\[
O(\Delta): \\
\frac{1}{2} \int_D dx \int_D dy \langle \text{Tr}(F_c(x)F_c(y)) \rangle = e^2 \langle \delta^{ab} \text{Tr}(T^a T^b) \rangle \left(-\frac{1}{2}\right) \int_D d\mu(x) \int_D d\mu(y) \delta^2(x-y) = -\frac{3}{2} e^2 \langle 1 \rangle \Delta 
\]

\[
(71)
\]

which agrees with (69).

\[
O(\Delta^2): \\
(\int_D)^4 \langle \text{Tr}(F_c^4) \rangle \text{ is obtained from } e^4 [\delta^{ij} \delta^{kl} \delta^2(x-y) \delta^2(z-w) + 2 \text{ permut.}] 
\]

which must be contracted with traces of the generators with structure \( \text{Tr}(T^i T^z T^z T^j) \)
and $\langle \text{Tr}(T^i T^i) \rangle^2$ to obtain
\begin{equation}
\frac{1}{4!} \langle \int_D^4 \text{Tr}(F_c^4) \rangle = \frac{5}{16} e^4 \langle 1 \rangle \Delta^2 
\end{equation}  
(72)

Similarly
\begin{equation}
- \frac{1}{2} \langle \int_D^2 \text{Tr}(F^2) \rangle = \frac{e^4}{4} \langle (j + 1)^2 \rangle \text{Tr} \left[ \frac{1}{\sqrt{2}} \sigma^3 \frac{1}{\sqrt{2}} \sigma^3 \right] \Delta^2 = \frac{e^4}{4} \langle (j + 1)^2 \rangle \Delta^2
\end{equation}  
(73)

Equations (72) and (73) agree with the $O(\Delta^2)$ term in (69). (Note that only generators in the Lie algebra of the maximal torus contribute to the topological terms.)

\[ O(\Delta^3): \]

The calculations become increasingly lengthy, so we only summarize the results
\begin{equation}
\frac{1}{4!} \langle \int_D^4 \text{Tr}(F_c^2 F^2) \rangle = -e^6 \frac{5}{4!2} \Delta^3 \langle (j + 1)^2 \rangle
\end{equation}  
(74)

and
\begin{equation}
- \frac{1}{6!} \langle \int_D^6 \text{Tr}(F_c^6) \rangle = - \frac{8}{6!} \frac{105}{32} e^6 \langle 1 \rangle \Delta^3
\end{equation}  
(75)

Equations (74) and (75) agree with the $O(\Delta^3)$ term of (69).

\[ O(\Delta^4): \]

\begin{equation}
\frac{1}{4!} \langle \int_D^4 \text{Tr}(F^4) \rangle = \frac{e^8}{4!8} \Delta^4 \langle (j + 1)^4 \rangle
\end{equation}  
(76)

while
\begin{equation}
- \frac{1}{6!} \langle \int_D^6 \text{Tr}(F_c^4 F^2) \rangle = \frac{105}{6!8} e^8 \Delta^4 \langle (j + 1)^2 \rangle
\end{equation}  
(77)

and
\begin{equation}
\frac{1}{8!} \langle \int_D^8 \text{Tr}(F_c^8) \rangle = \frac{9}{4!128} e^8 \langle 1 \rangle \Delta^4
\end{equation}  
(78)

The computation of (76) to (78) is very lengthy, but nevertheless agrees with (69). Note that in the expectation values $F^2$ gives $e^4(j + 1)^2 \Delta^2$ and $F_c^2$ gives $e^2 \Delta$ up to an overall constant in the computations (71) to (78). This clearly generalizes to arbitrary correlators.
In summary, we have verified that the contractable non-self-inte secting Wilson loop for $U(1)$ and $SU(2)$ can be computed from our correlators and the right-hand side of (53), at least to $O(\Delta^4)$. One would think that a discrepancy would have already appeared at or before $O(\Delta^4)$, so that this provides an impressive check of our correlators, and also “experimental” evidence that the Stokes’ theorem (53) is valid for the abelianization gauge, even for $SU(2)$7. A formal proof of this remarkable result is lacking. Finally, we mention that we have not assumed that the loop $\gamma$ was infinitesimal, or that $\Delta$ was small. This provides one more advantage of the abelianization gauge, which seems to provide an enormous simplification as compared to the non-abelian Stokes’ theorem 33.

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Appendix A: The Kernel on the Disc

In the main text we started with the action for 2d Y.M. coupled to an external source on the closed surface $\Sigma_g$, and then used the abelianization method of [11, 12] for the path integral, where the nontrivial $U(N)$ and $T$ bundles over $\Sigma_g$ had to be incorporated. Functional differentiation with respect to $J$ was then used to derive the correlators. Similar results can be obtained by a different method. In [10] Blau and Thompson computed the kernel for 2d Y.M. on a disc and used it to find the partition function and Wilson loops for closed surfaces. One starts with the theory on a disc and sets a boundary condition for the holonomy of the gauge field around the boundary $\gamma(t)$ of the disc $D$. The path-ordered exponential representing this holonomy is expressed in terms of an ordinary exponential, by means of a functional integration.

7For $SU(N)$ with $N > 2$ the abelian Stokes theorem works up to order $\Delta^2$ but seems to fail at next order [34].
over auxiliary anticommuting fields. This introduces an explicit $A_\mu$ dependence of the integrand in the path integral. However, for the simple topology of the disc, the Schwinger-Fock gauge is available, and this allows one to express $A_\mu$ in terms of $F_{\mu\nu}$, so that the Nicolai map becomes useful. The integration over the gauge field and over the auxiliary fermionic fields can then be performed \[10\].

When we couple the theory to an external source $J$, the calculation follows along similar lines. The resulting partition function is,

$$Z_D(g, J) = \sum_r \dim(r) \{ \exp(-e^2 AC_2(r)/2) \exp \left[ \frac{e^2}{2} \int_D d\mu \, \Tr(J^2) \right] \chi_r(g \cdot P_t \exp(i e^2 \oint \gamma^{-1} \rho^a_{\mu} t^a)) \}$$

where $r$ denotes an irreducible representation of the gauge group $G$, $\chi_r$ and $r^a$ are respectively the corresponding Weyl character and generators, $g$ is the holonomy around $\gamma(t)$, $P_t$ is the path-ordering operator and $\rho^a_\mu$ is given by

$$\rho^a_\mu(t) = \int_0^1 sds \epsilon_{\mu\nu\gamma} \gamma^\nu(s\gamma(t))J^a(s\gamma(t))$$

where we have cartesian coordinates $x^1, x^2$ on the disc $D$ with,

$$x^\mu(s, t) = s\gamma^\mu(t) \quad \text{for} \quad 0 \leq s, t \leq 1$$

When glueing together two such discs, by identifying holonomies along boundaries and integrating over all such possible holonomies, one should be careful in specifying the $t = 0$ and $t = 1$ points of the boundaries consistently. Results for closed surfaces are obtained by decomposing the holonomy into the product of the holonomies along the cycles of $\Sigma_g$ and making the corresponding identifications. One can now perform functional derivatives of (A.1) with respect to $J$ to compute the field strength correlation functions. The correlators should be independent of the way one chooses to decompose the surface $\Sigma_g$. So, for example, one can construct a sphere out of two discs, with boundaries $\gamma(t)$ and $\gamma^{-1}(t)$, or by considering only one disc with boundary $\alpha \alpha^{-1}(t)$, and the answer should be the same in the two cases. Let
us construct the regularized 2-point function on the sphere, considering the 1 disc approach first. One decomposes the holonomy in to $gg^{-1}$ in (A.1) and in this case the integration over holonomies with $\int_G dg$ just produces a factor of unity. Functional differentiation with respect to $J$ then produces,

$$\langle \xi^a(x) \xi^b(y) \rangle = \beta \delta_{x,y} \delta^{ab} + \frac{1}{Z_{S^2}} \sum_r \dim(r) \exp\left(-\frac{e^2 A C_2(r)}{2}\right) \chi_r(r^a r^b)$$  \hspace{1cm} (A.4)

where $\beta$ is a renormalization dependent constant. Notice this is a different gauge than the one considered in the body of the text. If we construct the sphere out of two discs the partition function will be,

$$Z_{S^2} = \int_G dg \ Z_D(g, J_D) \ Z_{D'}(g^{-1}, J_{D'})$$  \hspace{1cm} (A.5)

and this integration can be carried out using the familiar orthogonality relations between Weyl characters,

$$\int_G dg \ \chi_r(gh) \chi_{r'}(g^{-1}h^{'}) = \frac{\delta_{rr'}}{\dim(r)} \chi_r(hh^{'})$$  \hspace{1cm} (A.6)

with the result,

$$Z_{S^2} = \sum_r \dim(r) \exp\left(-\frac{e^2 A C_2(r)}{2}\right) \exp\left(\frac{e^2}{2} \int_{S^2} d\mu \ \text{Tr}(J^2)\right) \chi_r(P_t \exp(ie^2 \oint_\gamma \rho) P_t \exp(ie^2 \oint_{\gamma'} \rho^{'})$$  \hspace{1cm} (A.7)

where $\rho_\mu = \sum_a \rho_\mu^a r^a$ and $\rho^{'}$ is obtained from $J_{D'}$ as in (A.2). The two-point function obtained from (A.7) consists of several terms which depend on wheter or not $x$ and $y$ are in $D$ or $D'$. These terms can be understood as arising from glueing the one-point function on $D$ with the one-point function on $D'$, and the two-point function on $D$ with the partition function on $D'$ and vice-versa. The result is the same as the one in (A.4), as expected. Notice that in this calculation, the $t$ valued fields play no special role, all indices in the Lie algebra being treated equally. The gauge invariant two-point function, for the appropriate choice of the renormalization constant $\beta$, is identical to the one computed in section 3.
For higher genus surfaces and higher point functions this method becomes cumbersome. The explicit dependence of the partition function on the choice of coordinates $s$ and $t$ appears in the correlators, the ordering in which the generators $r_{a_i}$ appear inside traces being dependent on the ordering of the coordinates $t(x_i)$. Thus the symmetry must be restored by hand, and only then do the higher order Casimir operators appear. So, for closed surfaces the method based on abelianization is clearly simpler and more elegant, however this second method can be applied to the computation of electric field correlators on surfaces with boundaries.

Appendix B: The Weyl Group Averages

Here we will present the averages over the Weyl group used in section 3. Recall that the Weyl group acts as the symmetric group of $N$ elements, $S_N$, by permuting the coordinates $(l + \rho)^i$. All the identities that are needed can be deduced from,

$$\sum_{i=1}^{N} (l + \rho)^i = n$$  \hspace{1cm} (B.1)

where $n$ is the $u(1)$ highest weight corresponding to the $U(N)$ irreducible representation of highest weight $l$. For the two-point function (21) we need to compute,

$$\frac{1}{N!} \sum_{\sigma} (l + \rho)_{a}(l + \rho)_{b}$$  \hspace{1cm} (B.2)

If $a \neq b$ (B.2) becomes,

$$\frac{1}{N(N-1)} \sum_{\sigma_a \neq \sigma_b} (l + \rho)_{a}(l + \rho)_{b}$$

and then the sum over $\sigma_b$ produces

$$\frac{1}{N(N-1)} \sum_{\sigma_a} [n(l + \rho)_{a} - ((l + \rho)_{a})^2]$$

so that finally we have for the case $a \neq b$

$$\frac{1}{|W|} \sum_{\sigma} (l + \rho)_{a}(l + \rho)_{b} = \frac{1}{N(N-1)} [n^2 - (l + \rho)^2]$$

28
(Note we have the normalization $v^2 = \sum_{i=0}^{N} v^i v^i$, for $v \in {t}$).

Similar reasoning for $a = b$ gives the result presented in section 3:

$$\frac{1}{|W|} \sum_{\sigma} (l + \rho)^{\sigma_a} (l + \rho)^{\sigma_b} = p^{ab} (l + \rho)^2 + m^{ab} n^2 \quad (B.3)$$

with,

$$p^{ab} = \begin{cases} 
\frac{1}{N(N-1)} & \text{if } a \neq b \\
\frac{1}{N} & \text{if } a = b
\end{cases} \quad \text{and} \quad m^{ab} = \begin{cases} 
\frac{1}{N(N-1)} & \text{if } a \neq b \\
0 & \text{if } a = b
\end{cases} \quad (B.4)$$

For the four-point function we have by similar arguments,

$$\frac{1}{|W|} \sum_{\sigma} (l + \rho)^{\sigma_i} (l + \rho)^{\sigma_j} (l + \rho)^{\sigma_k} (l + \rho)^{\sigma_l} = a^{ijkl} \sum_p (l + \rho)^2 + b^{ijkl} [(l + \rho)^2]^2 + c^{ijkl} n \sum_p (l + \rho)^3 + d^{ijkl} n^2 (l + \rho)^2 + e^{ijkl} n^4 \quad (B.5)$$

The coefficients of the various Weyl group invariant terms are completely symmetric in the indices $i,j,k,l$. With

$$\varepsilon^{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j
\end{cases}$$

we have (with no sum on repeated indices in $(B.5)$):

$$a^{ijkl} = \begin{cases} 
\frac{-6(N-4)!}{N!} & i,j,k,l \text{ all different} \\
\frac{2(N-3)!}{N!} & \delta^{ijkl} \varepsilon^{ik} \varepsilon^{il} \\
\frac{-N-2)!}{N!} & \delta^{ijkl} \varepsilon^{il} \\
\frac{(N-2)!}{N!} & i = j = k = l
\end{cases} \quad b^{ijkl} = \begin{cases} 
\frac{3(N-4)!}{N!} & i,j,k,l \text{ all different} \\
\frac{-(N-3)!}{N!} & \delta^{ijkl} \varepsilon^{ik} \varepsilon^{il} \\
0 & \delta^{ijkl} \varepsilon^{il} \\
\frac{(N-2)!}{N!} & \delta^{ijkl} \varepsilon^{ik}
\end{cases}$$

$$c^{ijkl} = \begin{cases} 
\frac{8(N-4)!}{N!} & i,j,k,l \text{ all different} \\
\frac{-2(N-3)!}{N!} & \delta^{ijkl} \varepsilon^{ik} \varepsilon^{il} \\
\frac{(N-2)!}{N!} & \delta^{ijkl} \varepsilon^{il} \\
0 & \text{otherwise}
\end{cases} \quad d^{ijkl} = \begin{cases} 
\frac{-6(N-4)!}{N!} & i,j,k,l \text{ all different} \\
\frac{(N-3)!}{N!} & \delta^{ijkl} \varepsilon^{ik} \varepsilon^{il} \\
0 & \text{otherwise}
\end{cases}$$

$$e^{ijkl} = \begin{cases} 
\frac{(N-4)!}{N!} & i,j,k,l \text{ all different} \\
0 & \text{otherwise}
\end{cases} \quad (B.6)$$

29
Notice that when one contracts with the generators of the maximal torus in (27), only the \( i = j = k = l \) case contributes.

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