Loops on spheres having a compact-free inner mapping group

By

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Abstract. We prove that any topological loop homeomorphic to a sphere or to a real projective space and having a compact-free Lie group as the inner mapping group is homeomorphic to the circle. Moreover, we classify the differentiable 1-dimensional compact loops explicitly using the theory of Fourier series.

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Introduction

The only known proper topological compact connected loops such that the groups \(G\) topologically generated by their left translations are locally compact and the stabilizers \(H\) of their identities in \(G\) have no non-trivial compact subgroups are homeomorphic to the 1-sphere. In [8], [9], [7], [10] it is shown that the differentiable 1-dimensional loops can be classified by pairs of real functions which satisfy a differential inequality containing these functions and their first derivatives. A main goal of this paper is to determine the functions satisfying this inequality explicitly in terms of Fourier series.

If \(L\) is a topological loop homeomorphic to a sphere or to a real projective space and having a Lie group \(G\) as the group topologically generated by the left translations such that the stabilizer of the identity of \(L\) is a compact-free Lie subgroup of \(G\), then \(L\) is the 1-sphere and \(G\) is isomorphic to a finite covering of the group \(\text{PSL}_2(\mathbb{R})\) (cf. Theorem 4).

To decide which sections \(\sigma : G/H \rightarrow G\), where \(G\) is a Lie group and \(H\) is a (closed) subgroup of \(G\) containing no normal subgroup \(\neq 1\) of \(G\) correspond to loops we use systematically a theorem of Baer (cf. [3] and [8], Proposition 1.6, p. 18). This statement says that \(\sigma\) corresponds to a loop if and only if the image \(\sigma(G/H)\) is also the image for any section \(G/H^a \rightarrow G\), where \(H^a = a^{-1}Ha\) and \(a \in G\). As one of the applications of this we derive in a different way the differential inequality in [8], p. 238, in which the necessary and sufficient conditions for the existence of 1-dimensional differentiable loops are hidden.
Basic facts in loop theory

A set $L$ with a binary operation $(x, y) \mapsto x * y : L \times L \to L$ and an element $e \in L$ such that $e * x = x * e = x$ for all $x \in L$ is called a loop if for any given $a, b \in L$ the equations $a * y = b$ and $x * a = b$ have unique solutions which we denote by $y = a \backslash b$ and $x = b / a$. Every left translation $\lambda_a : y \mapsto a * y : L \to L, a \in L$, is a bijection of $L$ and the set $\Lambda = \{\lambda_a, a \in L\}$ generates a group $G$ such that $\Lambda$ forms a system of representatives for the left cosets $\{xH, x \in G\}$, where $H$ is the stabilizer of $e \in L$ in $G$. Moreover, the elements of $\Lambda$ act on $G/H = \{xH, x \in G\}$ such that for any given cosets $aH$ and $bH$ there exists precisely one left translation $\lambda_e$ with $\lambda_e aH = bH$.

Conversely, let $G$ be a group, $H$ be a subgroup containing no normal subgroup $\neq 1$ of $G$ and let $\sigma : G/H \to G$ be a section with $\sigma(H) = 1 \in G$ such that the set $\sigma(G/H)$ of representatives for the left cosets of $H$ in $G$ generates $G$ and acts sharply transitively on the space $G/H$ (cf. [8], p. 18). Such a section we call a sharply transitive section. Then the multiplication defined by $xH * yH = \sigma(xH)yH$ on the factor space $G/H$ or by $x * y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. The group $G$ is isomorphic to the group generated by the left translations of $L(\sigma)$.

We call the group generated by the mappings $\lambda_{xy} = \lambda_{xy}^{-1}\lambda_x\lambda_y : L \to L$, for all $x, y \in L$, the equations of $e$jection of $L$ for all $x \in L$ is called a loop if for any given $a, b \in L$ the group topologically generated by its left translations is locally compact and the corresponding sharply transitive section. Then the multiplication defined by $xH * yH = \sigma(xH)yH$ on the factor space $G/H$ or by $x * y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. The group $G$ is isomorphic to the group generated by the left translations of $L(\sigma)$.

A locally compact loop $L$ is almost topological if it is a locally compact space and the multiplication $* : L \times L \to L$ is continuous. Moreover, if the maps $(a, b) \mapsto b/a$ and $(a, b) \mapsto a \backslash b$ are continuous, then $L$ is a topological loop. An (almost) topological loop $L$ is connected if and only if the group topologically generated by the left translations is connected. We call the loop $L$ strongly almost topological if the group topologically generated by its left translations is locally compact and the corresponding sharply transitive section $\sigma : G/H \to G$, where $H$ is the stabilizer of $e \in L$ in $G$, is continuous.

If a loop $L$ is a connected differentiable manifold such that the multiplication $* : L \times L \to L$ is continuously differentiable, then $L$ is an almost $C^1$-differentiable loop (cf. Definition 1.24 in [8], p. 31). Moreover, if the mappings $(a, b) \mapsto b/a$ and $(a, b) \mapsto a \backslash b$ are also continuously differentiable, then the loop $L$ is a $C^1$-differentiable loop. If an almost $C^1$-differentiable loop has a Lie group $G$ as the group topologically generated by its left translations, then the sharply transitive section $\sigma : G/H \to G$ is $C^1$-differentiable. Conversely, any continuous, respectively $C^1$-differentiable sharply transitive section $\sigma : G/H \to G$ yields an almost topological, respectively an almost $C^1$-differentiable loop.

It is known that for any (almost) topological loop $L$ homeomorphic to a connected topological manifold there exists a universal covering loop $\tilde{L}$ such that the covering mapping $p : \tilde{L} \to L$ is an epimorphism. The inverse image $p^{-1}(e) = \text{Ker}(p)$ of the identity element $e$ of $L$ is a central discrete subgroup $Z$ of $\tilde{L}$ and it is naturally isomorphic to the fundamental group of $L$. If $Z'$ is a subgroup of $Z$, then the factor loop $\tilde{L}/Z'$ is a covering loop of $L$ and any covering loop of $L$ is isomorphic to a factor loop $\tilde{L}/Z'$ with a suitable subgroup $Z'$ (see [5]).
If $L'$ is a covering loop of $L$, then Lemma 1.34 in [8], p. 34, clarifies the relation between the group topologically generated by the left translations of $L'$ and the group topologically generated by the left translations of $L$:

Let $L$ be a topological loop homeomorphic to a connected topological manifold. Let the group $G$ topologically generated by the left translations $\lambda_a$, $a \in L$, of $L$ be a Lie group. Let $\tilde{L}$ be the universal covering of $L$ and $Z \subseteq \tilde{L}$ be the fundamental group of $L$. Then the group $\tilde{G}$ topologically generated by the left translations $\tilde{\lambda}_u, u \in \tilde{L}$, of $\tilde{L}$ is the covering group of $G$ such that the kernel of the covering mapping $\varphi: \tilde{G} \to G$ is $Z^\ast = \{\lambda_z, z \in Z\}$ and $Z^\ast$ is isomorphic to $Z$. If we identify $\tilde{L}$ and $L$ with the homogeneous spaces $G/\tilde{H}$ and $G/H$, where $H$ or $\tilde{H}$ is the stabilizer of the identity of $L$ in $G$ or of $\tilde{L}$ in $\tilde{G}$, respectively, then $\varphi(\tilde{H}) = H$, $\tilde{H} \cap Z^\ast = \{1\}$, and $\tilde{H}$ is isomorphic to $H$.

Compact topological loops on the 3-dimensional sphere

**Proposition 1.** There is no almost topological proper loop $L$ homeomorphic to the 3-sphere $\mathcal{S}_3$ or to the 3-dimensional real projective space $\mathcal{P}_3$ such that the group $G$ topologically generated by the left translations of $L$ is isomorphic to the group $SL_2(\mathbb{C})$ or to the group $PSL_2(\mathbb{C})$, respectively.

**Proof.** We assume that there is an almost topological loop $L$ homeomorphic to $\mathcal{S}_3$ such that the group topologically generated by its left translations is isomorphic to $G = SL_2(\mathbb{C})$. Then there exists a continuous sharply transitive section $\sigma: SL_2(\mathbb{C})/H \to SL_2(\mathbb{C})$, where $H$ is a connected compact-free 3-dimensional subgroup of $SL_2(\mathbb{C})$. According to [2], pp. 273–278, there is a one-parameter family of connected compact-free 3-dimensional subgroups $H_r, r \in \mathbb{R}$, of $SL_2(\mathbb{C})$ such that $H_r$ is conjugate to $H_{r_1}$ precisely if $r_1 = r_2$. Hence we may assume that the stabilizer $H$ has one of the following shapes

$$H_r = \left\{ \begin{pmatrix} \exp[(ri - 1)a] & b \\ 0 & \exp[(1 - ri)a] \end{pmatrix}; a \in \mathbb{R}, b \in \mathbb{C} \right\}, \quad r \in \mathbb{R},$$

(cf. Theorem 1.11 in [8], p. 21). For each $r \in \mathbb{R}$ the section $\sigma_r: G/H_r \to G$ corresponding to a loop $L_r$ is given by

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} H_r \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \begin{pmatrix} \exp[(ri - 1)f(x, y)] & g(x, y) \\ 0 & \exp[(1 - ri)f(x, y)] \end{pmatrix},$$

where $x, y \in \mathbb{C}, xx + yy = 1$ such that $f(x, y): S^3 \to \mathbb{R}, g(x, y): S^3 \to \mathbb{C}$ are continuous functions with $f(1, 0) = 0 = g(1, 0)$. Since $\sigma_r$ is a sharply transitive section for each $r \in \mathbb{R}$ the image $\sigma_r(G/H_r)$ forms a system of representatives for all cosets $xH_r^\gamma, \gamma \in G$. This means for all given $c, d \in \mathbb{C}^2, cc + dd = 1$ each coset

$$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix} H_r \begin{pmatrix} \bar{c} & -d \\ d & c \end{pmatrix},$$

where $u, v \in \mathbb{C}, uu + vv = 1$, contains precisely one element of $\sigma_r(G/H_r)$. This is the case if and only if for all given $c, d, u, v \in \mathbb{C}$ with $uu + vv = 1 = cc + dd$ there
exists a unique triple \((x, y, q) \in \mathbb{C}^3\) with \(xx + yy = 1\) and a real number \(m\) such that the following matrix equation holds:

\[
\begin{pmatrix}
    \bar{u}c - \bar{v}d & -ud - v\bar{c} \\
    \bar{v}c + \bar{u}d & uc - \bar{v}d
\end{pmatrix}
\begin{pmatrix}
    x & y \\
    -\bar{y} & \bar{x}
\end{pmatrix}
\begin{pmatrix}
    \exp[(ri - 1)f(x, y)] & g(x, y) \\
    0 & \exp[(1 - ri)f(x, y)]
\end{pmatrix}
\begin{pmatrix}
    \bar{c} & -d \\
    \bar{d} & c
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    \exp[(ri - 1)m] & q \\
    0 & \exp[(1 - ri)m]
\end{pmatrix}
\begin{pmatrix}
    \bar{c} & -d \\
    \bar{d} & c
\end{pmatrix}.
\]

(1)

The \((1, 1)\)- and \((2, 1)\)-entry of the matrix equation (1) give the following system \(A\) of equations:

\[
[(\bar{u}x + v\bar{y})\bar{c} + (u\bar{y} - \bar{u}x)d] \exp[(ri - 1)f(x, y)] = \exp[(ri - 1)m]\bar{c} + q\bar{d}
\]

(2)

\[
[(\bar{u}x - u\bar{y})c + (\bar{u}x + v\bar{y})d] \exp[(ri - 1)f(x, y)] = \exp[(1 - ri)m]d.
\]

(3)

If we take \(c\) and \(d\) as independent variables, the system \(B\) of equations:

\[
(\bar{u}x + v\bar{y}) \exp[irm(x, y)] \exp[-f(x, y)] = \exp(irm) \exp(-m)
\]

(4)

\[
(u\bar{y} - \bar{u}x) \exp[(ri - 1)f(x, y)]d = \bar{d}q
\]

(5)

\[
(\bar{u}x + v\bar{y}) \exp[irm(x, y)] \exp[-f(x, y)] = \exp(m) \exp(-irm).
\]

(6)

Since Eq. (5) must be satisfied for all \(d \in \mathbb{C}\) we obtain \(q = 0\). From Eq. (4) it follows

\[
\bar{u}x + v\bar{y} = \exp(irm) \exp(-m) \exp[-irmf(x, y)] \exp[f(x, y)].
\]

(7)

Putting (7) into (6) one obtains

\[
\exp(irm) \exp(-m) = \exp(m) \exp(-irm)
\]

(8)

which is equivalent to

\[
\exp[2(irm) - 1] = 1.
\]

(9)

Equation (9) is satisfied if and only if \(m = 0\). Hence the matrix equation (1) reduces to the matrix equation

\[
\begin{pmatrix}
    x & y \\
    -\bar{y} & \bar{x}
\end{pmatrix}
\begin{pmatrix}
    \exp[(ri - 1)f(x, y)] & g(x, y) \\
    0 & \exp[(1 - ri)f(x, y)]
\end{pmatrix}
= \begin{pmatrix}
    u & v \\
    -\bar{v} & \bar{u}
\end{pmatrix}
\]

and therefore the matrix

\[
M = \begin{pmatrix}
    \exp[(ri - 1)f(x, y)] & g(x, y) \\
    0 & \exp[(1 - ri)f(x, y)]
\end{pmatrix}
\]

is an element of \(SU_2(\mathbb{C})\). This is the case if and only if \(f(x, y) = 0 = g(x, y)\) for all \((x, y) \in \mathbb{C}^2\) with \(xx + yy = 1\). Since for each \(r \in \mathbb{R}\) the loop \(L_r\) is isomorphic to the loop \(L_r(\sigma_r)\), hence to the group \(SU_2(\mathbb{C})\), there is no connected almost topological proper loop \(L\) homeomorphic to \(\mathcal{P}_3\) such that the group topologically generated by its left translations is isomorphic to the group \(SL_2(\mathbb{C})\).

The universal covering of an almost topological proper loop \(L\) homeomorphic to the real projective space \(\mathcal{P}_3\) is an almost topological proper loop
\( \tilde{L} \) homeomorphic to \( \mathcal{S}_3 \). If the group topologically generated by the left translations of \( L \) is isomorphic to \( \text{PSL}_2(\mathbb{C}) \), then the group topologically generated by the left translations of \( \tilde{L} \) is isomorphic to \( \text{SL}_2(\mathbb{C}) \). Since no proper loop \( \tilde{L} \) exists the Proposition is proved. \( \square \)

**Proposition 2.** There is no almost topological proper loop \( L \) homeomorphic to the 3-dimensional real projective space \( \mathcal{P}_3 \) or to the 3-sphere \( \mathcal{S}_3 \) such that the group \( G \) topologically generated by the left translations of \( L \) is isomorphic to the group \( \text{SL}_3(\mathbb{R}) \) or to the universal covering group \( \text{SL}_3(\mathbb{R}) \), respectively.

**Proof.** First we assume that there exists an almost topological loop \( L \) homeomorphic to \( \mathcal{P}_3 \) such that the group topologically generated by its left translations is isomorphic to \( G = \text{SL}_3(\mathbb{R}) \). Then there is a continuous sharply transitive section \( \sigma : \text{SL}_3(\mathbb{R})/H \rightarrow \text{SL}_3(\mathbb{R}) \), where \( H \) is a connected compact-free 5-dimensional subgroup of \( \text{SL}_3(\mathbb{R}) \). According to Theorem 2.7, p. 187, in [4] and to Theorem 1.11, p. 21, in [8] we may assume that

\[
H = \left\{ \begin{pmatrix}
    a & k & v \\
    0 & b & l \\
    0 & 0 & (ab)^{-1}
\end{pmatrix} : a > 0, b > 0, k, l, v \in \mathbb{R} \right\}.
\]

(10)

Using Euler angles, every element of \( SO_3(\mathbb{R}) \) can be represented by the following matrix

\[
g(t,u,z) := \begin{pmatrix}
    \cos t & \sin t & 0 \\
    -\sin t & \cos t & 0 \\
    0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
    1 & 0 & 0 \\
    0 & \cos z & \sin z \\
    0 & -\sin z & \cos z
\end{pmatrix} \begin{pmatrix}
    \cos u & \sin u & 0 \\
    -\sin u & \cos u & 0 \\
    0 & 0 & 1
\end{pmatrix} = \\
\begin{pmatrix}
    \cos t \cos u - \sin t \cos z \sin u & \cos t \sin u + \sin t \cos z \cos u & \sin t \sin z \\
    -\sin t \cos u - \cos t \cos z \sin u & -\sin t \sin u + \cos t \cos z \cos u & \cos t \sin z \\
    \sin z \sin u & -\sin z \cos u & \cos z
\end{pmatrix},
\]

where \( t, u \in [0, 2\pi] \) and \( z \in [0, \pi] \).

The section \( \sigma : \text{SL}_3(\mathbb{R})/H \rightarrow \text{SL}_3(\mathbb{R}) \) is given by

\[
g(t,u,z)H \mapsto g(t,u,z) \begin{pmatrix}
    f_1(t,u,z) & f_2(t,u,z) & f_3(t,u,z) \\
    0 & f_4(t,u,z) & f_5(t,u,z) \\
    0 & 0 & f_1^{-1}(t,u,z)f_4^{-1}(t,u,z)
\end{pmatrix},
\]

(11)

where \( t, u \in [0, 2\pi] \), \( z \in [0, \pi] \) and \( f_i(t,u,z) : [0, 2\pi] \times [0, 2\pi] \times [0, \pi] \rightarrow \mathbb{R} \) are continuous functions such that for \( i \in \{1, 4\} \) the functions \( f_i \) are positive with \( f_i(0,0,0) = 1 \) and for \( j = \{2, 3, 5\} \) the functions \( f_j(t,u,z) \) satisfy that \( f_j(0,0,0) = 0 \). As \( \sigma \) is sharply transitive the image \( \sigma(\text{SL}_3(\mathbb{R})/H) \) forms a system of representatives for all cosets \( xH, \delta \in \text{SL}_3(\mathbb{R}) \). Since the elements \( x \) and \( \delta \) can be chosen in the group \( SO_3(\mathbb{R}) \) we may take \( x \) as the matrix

\[
\begin{pmatrix}
    \cos q & \cos r & -\sin q & \sin r & \cos p \\
    -\sin q & \cos r & \sin q & \cos r & \cos p \\
    \sin q & \cos p & \cos q & \sin p & \sin r \\
    \sin p & \sin r & -\cos q & \cos q & \cos p \\
    -\sin p & \cos r & \sin p & \cos p & \cos p
\end{pmatrix}
\]
and $\delta$ as the matrix
\[
\begin{pmatrix}
\cos \alpha \cos \beta - \sin \alpha \sin \beta \cos \gamma & \cos \alpha \sin \beta + \sin \alpha \cos \beta \cos \gamma & \sin \alpha \sin \gamma \\
-\sin \alpha \cos \beta - \cos \alpha \sin \beta \cos \gamma & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \cos \gamma & \cos \alpha \sin \gamma \\
\sin \gamma \sin \beta & -\sin \gamma \cos \beta & \cos \gamma
\end{pmatrix},
\]
where $q, r, \alpha, \beta \in [0, 2\pi]$ and $p, \gamma \in [0, \pi]$. The image $\sigma(SL_3(\mathbb{R})/H)$ forms for all given $\delta \in SO_3(\mathbb{R})$ and $x \in SO_3(\mathbb{R})$ a system of representatives for the cosets $xH^\delta$ if and only if there exists unique angles $t, u \in [0, 2\pi]$ and $z \in [0, \pi]$ and unique positive real numbers $a, b$ as well as unique real numbers $k, l, v$ such that the following equation holds
\[
\delta x^{-1}g(t, u, z)f = h\delta,
\]
where the matrices $\delta, x, g(t, u, z)$ have the form as above,
\[
f = \begin{pmatrix}
f_1(t, u, z) & f_2(t, u, z) & f_3(t, u, z) \\
0 & f_4(t, u, z) & f_5(t, u, z) \\
0 & 0 & f_4^{-1}(t, u, z)f_4^{-1}(t, u, z)
\end{pmatrix}
\]
and
\[
h = \begin{pmatrix}
a & k & v \\
b & l & 0 \\
0 & 0 & (ab)^{-1}
\end{pmatrix}.
\]
Comparing the first column of the left and the right side of the Eq. (12) we obtain the following three equations:
\[
f_1(t, u, z)\{[(\cos \alpha \cos \beta - \sin \alpha \sin \beta \cos \gamma)(\cos r \cos q - \sin r \sin q \cos p) \\
+ (\cos \alpha \sin \beta + \sin \alpha \cos \beta \cos \gamma)(\sin r \cos q + \cos r \sin q \cos p) \\
+ \sin \alpha \sin \gamma \sin p \sin q](\cos t \cos u - \sin t \sin u \cos z) \\
- [-(\cos \alpha \cos \beta - \sin \alpha \sin \beta \cos \gamma)(\cos r \sin q + \sin r \cos q \cos p) \\
+ (\cos \alpha \sin \beta + \sin \alpha \cos \beta \cos \gamma)(-\sin r \sin q + \cos r \cos q \cos p) \\
+ \sin \alpha \sin \gamma \sin p \cos q](\sin t \cos u + \cos t \sin u \cos z) \\
+ [(\cos \alpha \cos \beta - \sin \alpha \sin \beta \cos \gamma) \sin r \sin p \\
- (\cos \alpha \sin \beta + \sin \alpha \cos \beta \cos \gamma) \cos r \sin p + \sin \alpha \sin \gamma \cos p] \sin z \sin u\}
= a(\cos \alpha \cos \beta - \sin \alpha \sin \beta \cos \gamma) - k(\sin \alpha \cos \beta + \cos \alpha \sin \beta \cos \gamma) \\
+ v \sin \gamma \sin \beta,
\]
\[
f_1(t, u, z)\{[-(\sin \alpha \cos \beta + \cos \alpha \sin \beta \cos \gamma)(\cos r \cos q - \sin r \sin q \cos p) \\
- (\sin \alpha \sin \beta + \cos \alpha \cos \beta \cos \gamma)(\sin r \cos q + \cos r \sin q \cos p) \\
+ \cos \alpha \sin \gamma \sin p \sin q](\cos t \cos u - \sin t \sin u \cos z) \\
- [(\sin \alpha \cos \beta + \cos \alpha \sin \beta \cos \gamma)(\cos r \sin q + \sin r \cos q \cos p) \\
+ (\sin \alpha \sin \beta + \cos \alpha \cos \beta \cos \gamma)(-\sin r \sin q + \cos r \cos q \cos p) \\
+ \cos \alpha \sin \gamma \sin p \cos q](\sin t \cos u + \cos t \sin u \cos z)
\}.
\]
Loops on spheres having a compact-free inner mapping group

\[ + \left[ - \left( \sin \alpha \cos \beta + \cos \alpha \sin \beta \cos \gamma \right) \sin r \sin p - \left( \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \beta \right) \cos r \sin p + \cos \alpha \sin \gamma \cos p \right] \sin z \sin u \right\} \\
= -b \left( \cos \alpha \cos \beta + \cos \alpha \sin \beta \cos \gamma \right) + l \sin \gamma \sin \beta, \\
\]

\[ f_1(t,u,z)\{\left[ \left( \cos r \cos q - \sin r \sin q \cos p \right) \sin \gamma \sin \beta \\
- \left( \cos r \cos q + \cos r \sin q \cos p \right) \sin \gamma \cos \beta + \cos \gamma \sin p \sin q \right] \\
\times \left( \cos t \cos u - \sin t \sin u \cos z \right) + \left[ \left( \cos r \sin q + \sin r \cos q \cos p \right) \sin \gamma \sin \beta \\
+ \left( - \sin r \sin q + \cos r \cos q \cos p \right) \sin \gamma \cos \beta - \cos \gamma \sin p \cos q \right] \\
\times \left( \sin t \cos u + \cos t \sin u \cos z \right) \\
+ \left[ \left( \sin \gamma \sin \beta \sin r \sin p + \sin \gamma \cos \beta \cos r \sin p \right) + \cos \gamma \cos p \right] \sin z \sin u \right\} \\
= (ab)^{-1} \sin \gamma \sin \beta. \]

If we take \( \sin \gamma \sin \beta \) and \( \cos \gamma \) as independent variables the third equation turns to the following equations

\[ 0 = f_1(t,u,z)\{\sin p \sin q(\cos t \cos u - \sin t \sin u \cos z) \\
- \sin p \cos q(\sin t \cos u + \cos t \sin u \cos z) + \cos p \sin z \sin u \} \quad (13) \]

\[ (ab)^{-1} = \left[ \left( \cos r \cos q - \sin r \sin q \cos p \right)(\cos t \cos u - \sin t \sin u \cos z) \\
+ \left( \cos r \sin q + \sin r \cos q \cos p \right)(\sin t \cos u + \cos t \sin u \cos z) \\
+ \sin r \sin p \sin z \sin u \right] - \frac{\cos \beta}{\sin \beta} \left[ \left( \sin r \cos q + \cos r \sin q \cos p \right) \\
\times \left( \cos t \cos u - \sin t \sin u \cos z \right) - \left( - \sin r \sin q + \cos r \cos q \cos p \right) \\
\times \left( \sin t \cos u + \cos t \sin u \cos z \right) - \cos r \sin p \sin z \sin u \right] \right\} f_1(t,u,z). \]

If we take \( \cos \alpha \sin \beta \cos \gamma \) and \( \sin \beta \sin \gamma \) as independent variables it follows from the second equation that

\[ l = \frac{\cos \alpha}{\sin \beta} f_1(t,u,z)\{\sin p \sin q(\cos t \cos u - \sin t \sin u \cos z) \\
- \sin p \cos q(\sin t \cos u + \cos t \sin u \cos z) + \cos p \sin z \sin u \} \quad (15) \]

\[ -b = \left\{ \left[ \left( \cos r \cos q - \sin r \sin q \cos p \right)(\cos t \cos u - \sin t \sin u \cos z) \\
- \left( \cos r \sin q + \sin r \cos q \cos p \right)(\sin t \cos u + \cos t \sin u \cos z) \\
- \sin r \sin p \sin z \sin u \right] - \frac{\cos \beta}{\sin \beta} \left[ \left( \sin r \cos q + \cos r \sin q \cos p \right) \\
\times \left( \cos t \cos u - \sin t \sin u \cos z \right) - \left( - \sin r \sin q + \cos r \cos q \cos p \right) \\
\times \left( \sin t \cos u + \cos t \sin u \cos z \right) - \cos r \sin p \sin z \sin u \right] \right\} f_1(t,u,z). \quad (16) \]

If we choose \( \sin \alpha \sin \beta \cos \gamma, \sin \beta \sin \gamma \) as independent variables the first equation yields

\[ v = \frac{\sin \alpha}{\sin \beta} f_1(t,u,z)\{\sin p \sin q(\cos t \cos u - \sin t \sin u \cos z) \\
- \sin p \cos q(\sin t \cos u + \cos t \sin u \cos z) + \cos p \sin z \sin u \} \quad (17) \]
Since the Eq. (14) must be satisfied for all $h$ it follows from Eq. (13) that
\[
0 = \sin p \sin q (\cos t \cos u - \sin t \sin u \cos z)
- \sin p \cos q (\sin t \cos u + \cos t \sin u \cos z) + \cos p \sin z \sin u. \tag{19}
\]

Using this, it follows from (15) that $l = 0$ holds and from Eq. (17) that $v = 0$. Since the Eq. (14) must be satisfied for all $\beta \in [0,2\pi]$, we have
\[
(ab)^{-1} = \left[ (\cos r \cos q - \sin r \sin q \cos p) (\cos t \cos u - \sin t \sin u \cos z) \right.
+ (\cos r \sin q + \sin r \cos q \cos p) (\sin t \cos u + \cos t \sin u \cos z)
+ \sin r \sin p \sin z \sin u \right] f_1(t,u,z) \tag{20}
\]
\[
0 = \left[ (\sin r \cos q + \cos r \sin q \cos p) (\cos t \cos u - \sin t \sin u \cos z) \right.
- (\sin r \sin q + \cos r \cos q \cos p) (\sin t \cos u + \cos t \sin u \cos z)
- \cos r \sin p \sin z \sin u \right]. \tag{21}
\]

Using Eq. (21) and comparing the Eqs. (20) and (16), we obtain that
\[
(ab)^{-1} = b. \tag{22}
\]
With Eq. (21) the Eq. (18) turns to
\[
a + k \frac{\cos \alpha}{\sin \alpha} = \left[ (\cos r \cos q - \sin r \sin q \cos p) (\cos t \cos u - \sin t \sin u \cos z) \right.
- (\cos r \sin q + \sin r \cos q \cos p) (\sin t \cos u + \cos t \sin u \cos z)
+ \sin r \sin p \sin z \sin u \right] f_1(t,u,z). \tag{22}
\]

Since the Eq. (22) must be satisfied for all $\alpha \in [0,2\pi]$, we obtain $k = 0$. Using this, the Eqs. (22) and (20) yield $(ab)^{-1} = a$. Since $1 = ab(ab)^{-1} = a^3$ it follows that $a = 1$ and hence the matrix $h$ is the identity. But then the matrix equation (12) turns to the matrix equation

\[
g(t,u,z)f = x.
\]
As $x$ and $g(t,u,z)$ are elements of $SO_3(\mathbb{R})$ one has $f = g^{-1}(t,u,z)x \in SO_3(\mathbb{R})$. But then $f$ is the identity, which means that
\[
f_1(t,u,z) = 1 = f_4(t,u,z), \quad f_2(t,u,z) = f_3(t,u,z) = f_5(t,u,z) = 0,
\]
for all $t,u \in [0,2\pi]$ and $z \in [0,\pi]$. Since the loop $L$ is isomorphic to the loop $L(\sigma)$ and $L(\sigma) \cong SO_3(\mathbb{R})$ there is no connected almost topological proper loop $L$ ho-
meomorphc to $\mathcal{P}_3$ such that the group topologically generated by its left translations is isomorphic to $SL_3(\mathbb{R})$.

Now we assume that there is an almost topological loop $L$ homeomorphic to $\mathcal{S}_3$ such that the group $G$ topologically generated by its left translations is isomorphic to the universal covering group $SL_3(\mathbb{R})$. Then the stabilizer $H$ of the identity of $L$ may be chosen as the group (10). Then there exists a local section $\sigma : U/H \to G$, where $U$ is a suitable neighbourhood of $H$ in $G/H$ which has the shape (11) with sufficiently small $t, u \in [0, 2\pi]$, $z \in [0, \pi]$ and continuous functions $f_1(t, u, z) : [0, 2\pi] \times [0, 2\pi] \times [0, \pi] \to \mathbb{R}$ satisfying the same conditions as there. The image $\sigma(U/H)$ is a local section for the space of the left cosets $\{xH; x \in G, \delta \in G\}$ precisely if for all suitable matrices $x := g(q, r, p)$ with sufficiently small $(q, r, p) \in [0, 2\pi] \times [0, 2\pi] \times [0, \pi]$ there exist a unique element $g(t, u, z) \in \text{Spin}_3(\mathbb{R})$ with sufficiently small $(t, u, z) \in [0, 2\pi] \times [0, 2\pi] \times [0, \pi]$ and unique positive real numbers $a, b$ as well as unique real numbers $k, l, v$ such that the matrix Eq. (12) holds. Then we see as in the case of the group $SL_3(\mathbb{R})$ that for small $x$ and $g(t, u, z)$ the matrix $f$ is the identity. Therefore any subloop $T$ of $L$ which is homeomorphic to $\mathcal{S}_1$ is locally commutative. Then according to [8], Corollary 18.19, p. 248, each subloop $T$ is isomorphic to a 1-dimensional torus group. It follows that the restriction of the matrix $f$ to $T$ is the identity. Since $L$ is covered by such 1-dimensional tori the matrix $f$ is the identity for all elements of $\mathcal{S}_3$. Hence there is no proper loop $L$ homeomorphic to $\mathcal{S}_3$ such that the group $G$ topologically generated by its left translations is isomorphic to the universal covering group $SL_3(\mathbb{R})$. □

Compact loops with compact-free inner mapping groups

**Proposition 3.** Let $L$ be an almost topological loop homeomorphic to a compact connected Lie group $K$. Then the group $G$ topologically generated by the left translations of $L$ cannot be isomorphic to a split extension of a solvable group $R$ homeomorphic to $\mathbb{R}^n$ ($n \geq 1$) by the group $K$.

**Proof.** Denote by $H$ the stabilizer of the identity of $L$ in $G$. If $G$ has the structure as in the assertion, then the elements of $G$ can be represented by the pairs $(k, r)$ with $k \in K$ and $r \in R$. Since $L$ is homeomorphic to $K$ the loop $L$ is isomorphic to the loop $L(\sigma)$ given by a sharply transitive section $\sigma : G/H \to G$ the image of which is the set $\mathcal{E} = \{(k, f(k)); k \in K\}$, where $f$ is a continuous function from $K$ into $R$ with $f(1) = 1 \in R$. The multiplication of $(L(\sigma), \ast)$ on $\mathcal{E}$ is given by $(x, f(x)) \ast (y, f(y)) = \sigma((xy, f(x) f(y)) H)$.

Let $T$ be a 1-dimensional torus of $K$. Then the set $\{(t, f(t)); t \in T\}$ topologically generates a compact subloop $\widetilde{T}$ of $L(\sigma)$ such that the group topologically generated by its left translations has the shape $TU$ with $T \cap U = 1$, where $U$ is a normal solvable subgroup of $TU$ homeomorphic to $\mathbb{R}^n$ for some $n \geq 1$. The multiplication $\ast$ in the subloop $\widetilde{T}$ is given by $(x, f(x)) \ast (y, f(y)) = \sigma((xy, f(x) f(y)) H) = (xy, f(xy))$, where $x, y \in T$. Hence $\widetilde{T}$ is a subloop homeomorphic to a 1-sphere which has a solvable Lie group $S$ as the group topologically generated by the left tran-
lations. It follows that \( \mathcal{T} \) is a 1-dimensional torus group since otherwise the group \( S \) would be not solvable (cf. [8], Proposition 18.2, p. 235). As \( f : \mathcal{T} \rightarrow U \) is a homomorphism and \( U \) is homeomorphic to \( \mathbb{R}^n \) it follows that the restriction of \( f \) to \( \mathcal{T} \) is the constant function \( f(\mathcal{T}) = 1 \). Since the exponential map of a compact group is surjective any element of \( K \) is contained in a one-parameter subgroup of \( K \). It follows \( f(K) = 1 \) and \( L \) is the group \( K \) which is a contradiction.

**Theorem 4.** Let \( L \) be an almost topological proper loop homeomorphic to a sphere or to a real projective space. If the group \( G \) topologically generated by the left translations of \( L \) is a Lie group and the stabilizer \( H \) of the identity of \( L \) in \( G \) is a compact-free subgroup of \( G \), then \( L \) is homeomorphic to the 1-sphere and \( G \) is a finite covering of the group \( PSL_2(\mathbb{R}) \).

**Proof.** If \( \dim L = 1 \) then according to Brouwer’s theorem (cf. [11], 96.30, p. 639) the transitive group \( G \) on \( S^1 \) is a finite covering of \( PSL_2(\mathbb{R}) \).

Now let \( \dim L > 1 \). Since the universal covering of the \( n \)-dimensional real projective space is the \( n \)-sphere \( S^n \) we may assume that \( L \) is homeomorphic to \( S^n, n \geq 2 \). Since \( L \) is a multiplication with identity \( e \) on \( S^n \) one has \( n \in \{3, 7\} \) (cf. [1]).

Any maximal compact subgroup \( K \) of \( G \) acts transitively on \( L \) (cf. [11], 96.19, p. 636). As \( H \cap K = \{1\} \) the group \( K \) operates sharply transitively on \( L \). Since there is no compact group acting sharply transitively on the 7-sphere (cf. [11], 96.21, p. 637), the loop \( L \) is homeomorphic to the 3-sphere. The only compact group homeomorphic to the 3-sphere is the unitary group \( SU_2(\mathbb{C}) \). If the group \( G \) were not simple, then \( G \) would be a semidirect product of the at most 3-dimensional solvable radical \( R \) with the group \( SU_2(\mathbb{C}) \) (cf. [4], p. 187 and Theorem 2.1, p. 180). But according to Proposition 3 such a group cannot be the group topologically generated by the left translations of \( L \). Hence \( G \) is a non-compact Lie group the Lie algebra of which is simple. But then \( G \) is isomorphic either to the group \( SL_2(\mathbb{C}) \) or to the universal covering of the group \( SL_3(\mathbb{R}) \). It follows from Proposition 1 and 2 that no of these groups can be the group topologically generated by the left translations of an almost topological proper loop \( L \). \( \square \)

**The classification of 1-dimensional compact connected \( C^1 \)-loops**

If \( L \) is a connected strongly almost topological 1-dimensional compact loop, then \( L \) is homeomorphic to the 1-sphere and the group topologically generated by its left translations is a finite covering of the group \( PSL_2(\mathbb{R}) \) (cf. Proposition 18.2 in [8], p. 235). We want to classify explicitly all 1-dimensional \( C^1 \)-differentiable compact connected loops which have either the group \( PSL_2(\mathbb{R}) \) or \( SL_2(\mathbb{R}) \) as the group topologically generated by the left translations.

First we classify the 1-dimensional compact connected loops having \( G = SL_2(\mathbb{R}) \) as the group topologically generated by their left translations. Since the stabilizer \( H \) is compact-free and may be chosen as the group of upper triangular matrices (see Theorem 1.11, in [8], p. 21) this is equivalent to the clas-
Loops on spheres having a compact-free inner mapping group

sification of all loops \( L(\sigma) \) belonging to the sharply transitive \( C^1 \)-differentiable sections

\[
\begin{align*}
\sigma : \left( \begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array} \right) & \left\{ \left( \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right) : a > 0, b \in \mathbb{R} \right\} \\
\rightarrow & \left( \begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array} \right) \left( \begin{array}{cc} f(t) & g(t) \\ 0 & f^{-1}(t) \end{array} \right) \text{ with } t \in \mathbb{R}. \quad (23)
\end{align*}
\]

**Definition 1.** Let \( \mathcal{F} \) be the set of series

\[
a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \quad t \in \mathbb{R},
\]

such that

\[
1 - a_0 = \sum_{k=1}^{\infty} \frac{a_k + kb_k}{1 + k^2}, \quad (i)
\]

\[
a_0 > \sum_{k=1}^{\infty} \frac{ka_k - b_k}{1 + k^2} \sin kt - \frac{a_k + kb_k}{1 + k^2} \cos kt \quad \text{for all } t \in [0, 2\pi], \quad (ii)
\]

\[
2a_0 \geq \sum_{k=1}^{\infty} \left( a_k^2 + b_k^2 \right) \frac{k^2 - 1}{k^2 + 1}. \quad (iii)
\]

**Remark.** The conditions \((i)\) and \((iii)\) of Definition 1 are equivalent to the condition

\[
\sum_{k=1}^{\infty} \left( a_k^2 + b_k^2 \right) \left( k^2 - 1 \right) + 2(a_k + kb_k) \leq 2. \quad (iv)
\]

With \( a_0 = 1 - \sum_{k=1}^{\infty} \frac{(a_k + kb_k)}{1 + k^2} \) if \( a_k, b_k \) are non-negative, \( b_k \leq ka_k \) for all \( k \geq 1 \) and

\[
\sum_{k=1}^{\infty} \frac{(k + 2)a_k + (2k - 1)b_k}{1 + k^2} < 1, \quad (v)
\]

the inequality \((ii)\) is satisfied since from \((v)\) it follows

\[
\sum_{k=1}^{\infty} \frac{ka_k - b_k}{1 + k^2} + \sum_{k=1}^{\infty} \frac{a_k + kb_k}{1 + k^2} < a_0
\]

and

\[
\left| \sum_{k=1}^{\infty} \frac{ka_k - b_k}{1 + k^2} \sin kt - \frac{a_k + kb_k}{1 + k^2} \cos kt \right| \leq a_0 \quad \text{for all } t \in [0, 2\pi].
\]

In particular, taking \( \sum_{k=1}^{\infty} a_k \leq \frac{2}{3}, b_k = \frac{a_k}{k} \), we see that the inequalities \((iv)\) and \((v)\) are satisfied. Hence the set \( \mathcal{F} \) contains a multitude of trigonometric series.

**Lemma 5.** The set \( \mathcal{F} \) consists of Fourier series of continuous functions.
Proof. Since $\sum_{k=2}^{\infty} a_k^2 + b_k^2 < \frac{10}{7} a_0$ it follows from [14], p. 4, that any series in $\mathcal{F}$ converges uniformly to a continuous function $f$ and hence it is the Fourier series of $f$ (cf. [14], Theorem 6.3, p. 12).

Let $\sigma$ be a sharply transitive section of the shape (23). Then $f(t)$, $g(t)$ are periodic continuously differentiable functions $\mathbb{R} \rightarrow \mathbb{R}$, such that $f(t)$ is strictly positive with $f(2k\pi) = 1$ and $g(2k\pi) = 0$ for all $k \in \mathbb{Z}$.

As $\sigma$ is sharply transitive the image $\sigma(G/H)$ forms a system of representatives for the cosets $xH^\rho$ for all $\rho \in G$ (cf. [3]). All conjugate groups $H^\rho$ can be already obtained if $\rho$ is an element of

$$K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, t \in \mathbb{R} \right\}.$$ 

Since $K^\kappa H^\kappa = KH^\kappa$ for any $\kappa \in K$ the group $K$ forms a system of representatives for the left cosets $xH^\kappa$.

We want to determine the left coset $x(t)H^\kappa$ containing the element

$$\varphi(t) = \left( \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) \left( \begin{pmatrix} f(t) & g(t) \\ 0 & f^{-1}(t) \end{pmatrix} \right),$$

where

$$\kappa = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \text{ and } x(t) = \begin{pmatrix} \cos \eta(t) & \sin \eta(t) \\ -\sin \eta(t) & \cos \eta(t) \end{pmatrix}.$$ 

The element $\varphi(t)$ lies in the left coset $x(t)H^\kappa$ if and only if $\varphi(t)^{\kappa^{-1}} \in x(t)^{\kappa^{-1}} H = x(t)H$. Hence we have to solve the following matrix equation

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \left[ \begin{pmatrix} f(t) & g(t) \\ 0 & f^{-1}(t) \end{pmatrix} \right]^{\kappa^{-1}} = \begin{pmatrix} \cos \eta(t) & \sin \eta(t) \\ -\sin \eta(t) & \cos \eta(t) \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

(24)

for suitable $a > 0, b \in \mathbb{R}$. Comparing both sides of the matrix equation (24) we have

$$f(t) \cos \beta(\sin t \cos \beta - \cos t \sin \beta) - g(t) \sin \beta(\sin t \cos \beta - \cos t \sin \beta) + f(t)^{-1} \sin \beta(\sin t \sin \beta + \cos t \cos \beta) = \sin \eta(t)a$$

and

$$f(t) \cos \beta(\cos t \cos \beta + \sin t \sin \beta) - g(t) \sin \beta(\cos t \cos \beta + \sin t \sin \beta) + f(t)^{-1} \sin \beta(\cos t \sin \beta - \sin t \cos \beta) = \cos \eta(t)a.$$ 

From this it follows that

$$\tan \eta(t) = \frac{f(t) - g(t) \tan \beta)(\tan t - \tan \beta) + f^{-1}(t) \tan \beta(1 + \tan t \tan \beta)}{(f(t) - g(t) \tan \beta)(1 + \tan t \tan \beta) + f^{-1}(t) \tan \beta(\tan \beta - \tan t)}.$$ 

Since $\beta$ can be chosen in the interval $0 \leqslant \beta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$, we may replace the parameter $\tan \beta$ by any $w \in \mathbb{R}$. 

A $C^1$-differentiable loop $L$ corresponding to $\sigma$ exists if and only if the function
\[ t \mapsto \eta_w(t) \] is strictly increasing, i.e. if $\eta'_w(t) > 0$ (cf. Proposition 18.3, p. 238, in [8]). The function $a_w(t) : t \mapsto \tan \eta_w(t) : \mathbb{R} \to \mathbb{R} \cup \{ \pm \infty \}$ is strictly increasing if and only if $\eta'_w(t) > 0$ since
\[
\frac{d}{dt} \tan (\eta_w(t)) = \frac{1}{\cos^2(\eta_w(t))} \eta'_w(t).
\]
A straightforward calculation shows that
\[
\frac{d}{dt} \tan (\eta_w(t)) = \frac{w^2 + 1}{\cos^2(t)} [w^2(g'(t)f(t) + g(t)f'(t) + g^2(t)f^2(t) + 1) + w(-2f(t)f'(t) - 2g(t)f^3(t)) + f^4(t)].
\]
(25)
Hence the loop $L(\sigma)$ exists if and only if for all $w \in \mathbb{R}$ the inequality
\[
0 < w^2(g'(t)f(t) + g(t)f'(t) + g^2(t)f^2(t) + 1) + w(-2f(t)f'(t) - 2g(t)f^3(t)) + f^4(t)
\]
holds. For $w = 0$ the expression (26) equals to $f^4(t) > 0$. Therefore the inequality (26) is satisfied for all $w \in \mathbb{R}$ if and only if one has
\[
f^2(t) + g(t)f^2(t)f'(t) - g'(t)f^3(t) - f^2(t) < 0 \quad \text{and} \quad g'(0) < f^2(0) - 1
\]
(27)
for all $t \in \mathbb{R}$. Putting $f(t) = f^{-1}(t)$ and $g(t) = -g(t)$ these conditions are equivalent to the conditions
\[
f^2(t) + g(t)f^2(t) + g'(t)f'(t) - f^2(t) < 0 \quad \text{and} \quad g'(0) < 1 - f^2(0)
\]
(28)
(cf. [8], Section 18, (C), p. 238).

Now we treat the differential inequality (28). The solution $h(t)$ of the linear differential equation
\[
h'(t) + h(t) \frac{\hat{f}'(t)}{\hat{f}(t)} + \frac{\hat{f}^2(t)}{\hat{f}(t)} - \hat{f}(t) = 0
\]
(29)
with the initial conditions $h(0) = 0$ and $h'(0) = 1 - \hat{f}^2(0)$ is given by
\[
h(t) = \hat{f}(t)^{-1} \int_0^t (\hat{f}(u) - \hat{f}^2(u)) du.
\]
Since $\hat{g}(0) = h(0) = 0$ and $\hat{g}'(0) < h'(0)$ it follows from VI in [13] (p. 66) that $\hat{g}(t)$ is a subfunction of the differential equation (29), i.e. that $\hat{g}(t)$ satisfies the differential inequality (28). Moreover, according to Theorem V in [13] (p. 65) one has $\hat{g}(t) < h(t)$ for all $t \in (0, 2\pi)$. Since the functions $\hat{g}(t)$ and $h(t)$ are continuous $0 = \hat{g}(2\pi) \leq h(2\pi)$. This yields the following integral inequality
\[
\int_0^{2\pi} (\hat{f}^2(t) - \hat{f}^2(t)) dt > 0.
\]
(30)
We consider the real function $R(t)$ defined by $R(t) = \hat{f}(t) - \hat{f}'(t)$. Since $\hat{f}(0) = \hat{f}(2\pi) = 1$ and $\hat{f}'(0) = \hat{f}'(2\pi)$ we have $R(0) = 1 - \hat{f}'(0) = 1 - \hat{f}'(2\pi) = R(2\pi).$
The linear differential equation

\[ y'(t) - y(t) + R(t) = 0 \quad \text{with } y(0) = 1 \]  

has the solution

\[ y(t) = e^t \left( 1 - \int_0^t R(u) e^{-u} du \right). \]  

This solution is unique (cf. [6], p. 2) and hence it is the function \( \hat{f}(t) \). The condition \( \hat{f}(2\pi) = 1 \) is satisfied if and only if \( \int_0^{2\pi} R(u) e^{-u} du = 1 - \frac{1}{e^\pi} \). Since \( R(t) \) has period 2\( \pi \) its Fourier series is given by

\[ a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \]  

where \( a_0 = \frac{1}{\pi} \int_0^{2\pi} R(t) \, dt \), \( a_k = \frac{1}{\pi} \int_0^{2\pi} R(t) \cos kt \, dt \), and \( b_k = \frac{1}{\pi} \int_0^{2\pi} R(t) \sin kt \, dt \).

Partial integration yields

\[ \int_0^t \sin ku \, e^{-u} du = \frac{k - k \cos kt \, e^{-t} - \sin kt \, e^{-t}}{1 + k^2} \]  

and

\[ \int_0^t \cos ku \, e^{-u} du = \frac{1 + k \sin kt \, e^{-t} - \cos kt \, e^{-t}}{1 + k^2}. \]  

Using (34) and (35), we obtain by partial integration

\[ \int_0^t R(u) e^{-u} du = a_0 - a_0 e^{-t} + \sum_{k=1}^{\infty} \left[ \int_0^t a_k \cos ku \, e^{-u} du + \int_0^t b_k \sin ku \, e^{-u} du \right] \]

\[ = a_0 - a_0 e^{-t} + \sum_{k=1}^{\infty} \frac{a_k(1 + k \sin kt \, e^{-t} - \cos kt \, e^{-t})}{1 + k^2} \]

\[ + \frac{b_k(k - k \cos kt \, e^{-t} - \sin kt \, e^{-t})}{1 + k^2}. \]  

Now, for the real coefficients \( a_0, a_k, b_k \) (\( k \geq 1 \)) it follows that

\[ 1 - \frac{1}{e^{2\pi}} = \int_0^{2\pi} R(u) e^{-u} du = \left( a_0 + \sum_{k=1}^{\infty} \frac{a_k + b_k k}{1 + k^2} \right) \left( 1 - \frac{1}{e^{2\pi}} \right). \]

Hence one has

\[ a_0 + \sum_{k=1}^{\infty} \frac{a_k + b_k k}{1 + k^2} = 1. \]  

The function \( \hat{f}(t) \) is positive if and only if

\[ 1 > \int_0^t R(u) e^{-u} du \quad \text{for all } t \in [0, 2\pi]. \]  

Applying (34) and (35) again we see that the inequality (38) is equivalent to

\[ a_0 > \sum_{k=1}^{\infty} \left[ \frac{a_k k - b_k}{1 + k^2} \sin kt - \frac{a_k + b_k k}{1 + k^2} \cos kt \right]. \]
The relations (a), (b), (c), (d) in [12], p. 10, yield

\[ \int_0^{2\pi} R(t) \left[ 2e^t \left( 1 - \int_0^t R(u)e^{-u}du \right) - R(t) \right] dt \geq 0. \]  \hspace{1cm} (40)

The left side of (40) can be written as

\[ 2 \int_0^{2\pi} R(t)e^t dt - 2 \int_0^{2\pi} R(t)e^t \left( \int_0^t R(u)e^{-u}du \right) dt - \int_0^{2\pi} R^2(t)dt. \]  \hspace{1cm} (41)

Using partial integration and representing \( R(u) \) by a Fourier series (33) we have

\[ \int_0^{2\pi} R(t)e^t dt = \left( a_0 + \sum_{k=1}^{\infty} \frac{a_k - b_k}{1 + k^2} \right) (e^{2\pi} - 1). \]  \hspace{1cm} (42)

From (36) it follows

\[
\int_0^{2\pi} R(t)e^t \left( \int_0^t R(u)e^{-u}du \right) dt \\
= a_0 \int_0^{2\pi} R(t)e^t dt - a_0 \int_0^{2\pi} R(t)dt + \sum_{k=1}^{\infty} \int_0^{2\pi} \left( \frac{a_k + kb_k}{1 + k^2} \right) R(t)e^t dt \\
+ \sum_{k=1}^{\infty} \int_0^{2\pi} \left( \frac{k a_k - b_k}{1 + k^2} \right) R(t) \sin kt dt - \sum_{k=1}^{\infty} \int_0^{2\pi} \left( \frac{a_k + kb_k}{1 + k^2} \right) R(t) \cos kt dt. \]  \hspace{1cm} (43)

Substituting for \( R(t) \) its Fourier series and applying the relation (a) in [12] (p. 10) we have

\[ \int_0^{2\pi} R(t)dt = 2\pi a_0. \]

Furthermore, one has

\[ \sum_{k=1}^{\infty} \int_0^{2\pi} \left( \frac{k a_k - b_k}{1 + k^2} \right) R(t) \sin kt dt \\
= \sum_{k=1}^{\infty} \int_0^{2\pi} \left( \frac{k a_k - b_k}{1 + k^2} \right) \left[ a_0 + \sum_{l=1}^{\infty} (a_l \cos lt + b_l \sin lt) \right] \sin kt dt \\
= a_0 \sum_{k=1}^{\infty} \int_0^{2\pi} \left( \frac{k a_k - b_k}{1 + k^2} \right) \sin kt dt + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \int_0^{2\pi} \left( \frac{k a_k - b_k}{1 + k^2} \right) a_l \cos lt \sin kt dt \\
+ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \int_0^{2\pi} \left( \frac{k a_k - b_k}{1 + k^2} \right) b_l \sin lt \sin kt dt.

The relations (a), (b), (c), (d) in [12], p. 10, yield

\[ \sum_{k=1}^{\infty} \int_0^{2\pi} \left( \frac{k a_k - b_k}{1 + k^2} \right) R(t) \sin kt dt = \sum_{k=1}^{\infty} \int_0^{2\pi} \left( \frac{k a_k - b_k}{1 + k^2} \right) b_k \sin^2 kt dt \\
= \sum_{k=1}^{\infty} \left( \frac{k a_k - b_k}{1 + k^2} \right) b_k \pi. \]
Analogously we obtain that
\[
\sum_{k=1}^{\infty} \int_0^{2\pi} \left( \frac{a_k + kb_k}{1 + k^2} \right) R(t) \cos kt \, dt = \sum_{k=1}^{\infty} \int_0^{2\pi} \left( \frac{ka_k + b_k}{1 + k^2} \right) b_k \cos^2 kt \, dt
\]
\[
= \sum_{k=1}^{\infty} \left( \frac{a_k + kb_k}{1 + k^2} \right) a_k \pi.
\]

Using the equality (37) one has
\[
\int_0^{2\pi} R(t)e^{i t} \left( \int_0^t R(u)e^{-u} du \right) dt = \left[ a_0 + \sum_{k=1}^{\infty} \frac{a_k - kb_k}{1 + k^2} \right] (e^{2\pi} - 1) - \pi \sum_{k=1}^{\infty} \frac{b_k^2 + a_k^2}{1 + k^2} - 2\pi a_0^2. \tag{44}
\]

Substituting for \( R(t) \) its Fourier series we have
\[
\int_0^{2\pi} R^2(t) \, dt = \int_0^{2\pi} a_0^2 \, dt + 2a_0 \sum_{k=1}^{\infty} \int_0^{2\pi} (a_k \cos kt + b_k \sin kt) \, dt
\]
\[
- \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \int_0^{2\pi} (a_k a_l \cos kt \cos lt + a_k b_l \cos kt \sin lt
\]
\[
+ b_k a_l \sin kt \cos lt + b_k b_l \sin kt \sin lt) \, dt.
\]

Applying the relations (a), (b), (c), (d) in [12] (p. 10) we obtain
\[
\int_0^{2\pi} R^2(t) \, dt = 2\pi a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).
\]

Hence the integral inequality (30) holds if and only if
\[
2a_0 \geq \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \frac{k^2 - 1}{k^2 + 1}.
\]

Since the Fourier series of \( R(t) \) lies in the set \( \mathcal{F} \) of series the Fourier series of \( R \) converges uniformly to \( R \) (Lemma 5).

Summarizing our discussion we obtain the main part of the following

**Theorem 6.** Let \( L \) be a 1-dimensional connected \( \mathcal{C}^1 \)-differentiable loop such that the group topologically generated by its left translations is isomorphic to the group \( SL_2(\mathbb{R}) \). Then \( L \) is compact and belongs to a \( \mathcal{C}^1 \)-differentiable sharply transitive section \( \sigma \) of the form

\[
\sigma : \left( \begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array} \right) \left\{ \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right\} ; a > 0, b \in \mathbb{R}
\]
\[
\rightarrow \left( \begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array} \right) \left( \begin{array}{cc} f(t) & g(t) \\ 0 & f^{-1}(t) \end{array} \right) \text{ with } t \in \mathbb{R} \tag{45}
\]
such that the inverse function $f^{-1}$ has the shape

$$f^{-1}(t) = e^t \left( 1 - \int_0^t R(u)e^{-u} \, du \right) = a_0 + \sum_{k=1}^{\infty} \frac{(ka_k - bk)}{1 + k^2} \sin kt + (ak + kb) \cos kt,$$

(46)

where $R(u)$ is a continuous function the Fourier series of which is contained in the set $F$ and converges uniformly to $R$, and $g$ is a periodic $C^1$-differentiable function with $g(0) = g(2\pi) = 0$ such that

$$g(t) > -f(t) \int_0^t \frac{(f^2(u) - f'^2(u))}{f^4(u)} \, du \quad \text{for all } t \in (0, 2\pi).$$

(47)

Conversely, if $R(u)$ is a continuous function the Fourier series of which is contained in $F$, then the section $\sigma$ of the form (45) belongs to a loop if $f$ is defined by (46) and $g$ is a $C^1$-differentiable periodic function with $g(0) = g(2\pi) = 0$ satisfying (47).

The isomorphism classes of loops defined by $\sigma$ are in one-to-one correspondence to the 2-sets $\{(f(t), g(t)), (f(-t), -g(-t))\}$.

Proof. The only part of the assertion which has to be discussed is the isomorphism question. It follows from [7], Theorem 3, p. 3, that any isomorphism class of the loops $L$ contains precisely two pairs $(f_1, g_1)$ and $(f_2, g_2)$. If $(f_1, g_1) \neq (f_2, g_2)$ and if $(f_1, g_1)$ satisfy the inequality (27), then we have

$$f'^2_2(-t) + g_2(-t)f'^2_2(t) - g'^2_2(-t)f'^2_2(-t) - f'^2_2(-t) < 0$$

since from $f_1(t) = f_2^2(-t)$ and $g_1(t) = g_2^2(-t)$ we have $f'_1(t) = -f'_2(-t)$ and $g'_1(t) = g'_2(-t)$.

Remark. A loop $\tilde{L}$ belonging to a section $\sigma$ of shape (45) is a 2-covering of a $C^1$-differentiable loop $L$ having the group $PSL_2(\mathbb{R})$ as the group topologically generated by the left translations if and only if for the functions $f$ and $g$ one has $f(\pi) = 1$ and $g(\pi) = 0$ (cf. [9], p. 5106). Moreover, $L$ is the factor loop $\tilde{L}/\{ \left( \begin{array}{cc} \epsilon & 0 \\ 0 & \epsilon \end{array} \right); \epsilon = \pm 1 \}$. Any n-covering of $L$ is a non-split central extension $\tilde{L}$ of the cyclic group of order $n$ by $L$. The loop $\tilde{L}$ has the $n$-covering of $PSL_2(\mathbb{R})$ as the group topologically generated by its left translations.

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