Hyper generalized pseudo $Q$-symmetric semi-Riemannian manifolds

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ABSTRACT

The object of the present paper is to study the properties of a hyper generalized pseudo $Q$-symmetric semi-Riemannian manifold, proving that under certain assumptions, it is a perfect fluid spacetime.

RESUMEN

El objetivo del presente artículo es estudiar las propiedades de una variedad semi-Riemanniana hiper generalizada pseudo $Q$-simétrica, probando que bajo ciertas condiciones, es un espacio-tiempo fluido perfecto.

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1 Introduction

Let $R$, $S$, $L$ and $r$ denote the curvature tensor, Ricci tensor, Ricci operator and the scalar curvature of a (semi)-Riemannian manifold, respectively. It is Mantica and Suh [5] who have introduced the notion of $Q$-curvature tensor. In an $n$-dimensional Riemannian or semi-Riemannian manifold $(M^n, g)$ ($n > 2$), the $Q$-curvature tensor is defined as

\[ R(Y, U, V, W) = Q(Y, U, V, W) + \frac{\psi}{n-1} [g(Y, W)g(U, V) - g(Y, V)g(U, W)], \tag{1.1} \]

where $Y, U, V, W$ are arbitrary vector fields on $M^n$ and $\psi$ is a scalar function. Semi-Riemannian manifolds with Ricci tensor $S$ of the form

\[ S(Y, V) = ag(Y, V) + bT(Y)T(V), \]

for any vector fields $Y, V$, are often termed as perfect fluid spacetimes, where $a$ and $b$ are scalars and the vector field $\varrho$, metrically equivalent to the 1-form $T$ (that is, $g(\varrho, \varrho) = T(Y)$), is a unit time like vector field (that is, $g(\varrho, \varrho) = -1$).

An $n$-dimensional semi-Riemannian manifold is said to be hyper generalized pseudo $Q$-symmetric (which will be abbreviated hereafter as $(HGPQS)_n$) if it satisfies the equation

\[ (\nabla_X Q)(Y, U, V, W) = 2A_1(X)Q(Y, U, V, W) + A_1(Y)Q(X, U, V, W) + A_1(U)Q(Y, X, V, W) + A_1(W)Q(Y, U, X, V) + A_2(Y)(g \wedge S)(X, U, V, W) + A_2(U)(g \wedge S)(Y, X, V, W) + A_2(V)(g \wedge S)(Y, U, X, W) + A_2(W)(g \wedge S)(Y, U, V, X), \tag{1.2} \]

where

\[ (g \wedge S)(Y, U, V, W) = g(Y, W)S(U, V) + g(U, V)S(Y, W) - g(Y, V)S(U, W) - g(U, W)S(Y, V), \tag{1.3} \]

and $A_1$, $A_2$ are non-zero 1-forms whose $g$-dual vector fields will be denoted by $\theta_1$ and $\theta_2$, i.e. $A_1(X) = g(X, \theta_1)$ and $A_2(X) = g(X, \theta_2)$.

We organized our paper as follows: section 2 is concerned with preliminaries. After preliminaries, some curvature properties of $(HGPQS)_n$ manifolds are studied in section 3. It is obtained that the $Q$-curvature tensor in a $(HGPQS)_n$ manifold satisfies 2nd Bianchi’s identity. It is further obtained that the scalar function $\psi$ is always constant. In section 4 we investigate properties of divergence-free $(HGPQS)_n$ manifolds and we prove that a divergence-free $(HGPQS)_n$ manifold ($n > 2$) under the assumption $A_1(Q(Y, U)V) = 0$ is a perfect fluid spacetime as well as the integral
curves of the vector field $\varrho$ are geodesics and the vector field $\varrho$ is irrotational, if the associated vector fields $\varrho$ and $\sigma$ corresponding to the 1-forms $T_1$ and $T_2$ are related by $(r-1)\varrho + n\sigma = 0$.

## 2 Preliminaries

In this section, some relations useful to the study of $(HGPQS)_n$ manifolds are obtained. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, where $1 \leq i \leq n$.

From (1.1) we can easily verify that the tensor $Q$ satisfies the following properties:

(i) $Q(Y,U)V + Q(U,Y)V = 0$,

(ii) $Q(Y,U)V + Q(U,V)Y + Q(V,Y)U = 0$,\hspace{1cm} (2.1)

where $g(Q(X,Y)U,V) = Q(X,Y,U,V)$.

Also from (1.1) we have

$$\sum_{i=1}^{n} \epsilon_i Q(X,Y,e_i,e_i) = 0 = \sum_{i=1}^{n} \epsilon_i Q(e_i,e_i,W,U)$$\hspace{1cm} (2.2)

and

$$\sum_{i=1}^{n} \epsilon_i Q(e_i,Y,V,e_i) = \sum_{i=1}^{n} \epsilon_i Q(Y,e_i,e_i,V) = S(Y,V) - \psi g(Y,V)$$\hspace{1cm} (2.3)

where

$$\epsilon_i = g(e_i,e_i) = \pm 1, \hspace{0.5cm} S(X,Y) = \sum_{i=1}^{n} \epsilon_i g(R(X,e_i)e_i,Y), \hspace{0.5cm} r = \sum_{i=1}^{n} \epsilon_i S(e_i,e_i).$$

From (1.1) and (2.1) it follows that

(i) $Q(X,Y,U,V) + Q(X,Y,V,U) = 0$,

(ii) $Q(X,Y,U,V) - Q(U,V,X,Y) = 0$.\hspace{1cm} (2.4)

## 3 Some curvature properties of $(HGPQS)_n$ manifolds

In this section we prove that in a $(HGPQS)_n$ manifold, the $Q$-curvature tensor satisfies 2nd Bianchi’s identity, that is,

$$(\nabla_X Q)(Y,U,V,W) + (\nabla_Y Q)(U,X,V,W) + (\nabla_U Q)(X,Y,V,W) = 0.$$\hspace{1cm} (3.1)
In view of (1.1), (1.2) and (3.1) we get

\[
(\nabla_X Q)(Y, U, V, W) + (\nabla_Y Q)(U, X, V, W) + (\nabla_U Q)(X, Y, V, W) \tag{3.2}
\]

\[
= A_1(V)[Q(Y, U, X, W) + Q(U, X, Y, W) + Q(X, Y, U, W)] \\
+ A_1(W)[Q(Y, U, V, X) + Q(U, X, V, Y) + Q(X, Y, V, U)] \\
+ A_2(V) [(g \wedge S)(Y, U, X, W)] + A_2(W) [(g \wedge S)(Y, U, V, X)] \\
+ (g \wedge S)(U, X, Y, W) + A_2(W) [(g \wedge S)(X, Y, U, W)].
\]

Using (1.3) and 1st Bianchi’s identity for the \(Q\)-curvature tensor in (3.2) and then simplifying, we obtain (3.1).

Thus we can state the following:

**Theorem 3.1.** The \(Q\)-curvature tensor in a \((HGPQS)_n\) manifold satisfies 2nd Bianchi’s identity.

Using (1.1) in (3.1), we have

\[
(\nabla_X R)(Y, U, V, W) + (\nabla_Y R)(U, X, V, W) + (\nabla_U R)(X, Y, V, W) \tag{3.3}
\]

\[
= \frac{d\psi(X)}{(n - 1)} [g(Y, W)g(U, V) - g(Y, V)g(U, W)] \\
- \frac{d\psi(Y)}{(n - 1)} [g(U, W)g(X, V) - g(U, V)g(X, W)] \\
- \frac{d\psi(U)}{(n - 1)} [g(X, W)g(Y, V) - g(X, V)g(Y, W)] = 0.
\]

By virtue of 2nd Bianchi’s identity for the Riemannian curvature tensor, (3.3) yields

\[
\frac{d\psi(X)}{(n - 1)} [g(Y, W)g(U, V) - g(Y, V)g(U, W)] \tag{3.4}
\]

\[
+ \frac{d\psi(Y)}{(n - 1)} [g(U, W)g(X, V) - g(U, V)g(X, W)] \\
+ \frac{d\psi(U)}{(n - 1)} [g(X, W)g(Y, V) - g(X, V)g(Y, W)] = 0.
\]

Contracting \(U\) and \(V\) in (3.4), we have

\[
(n - 2)[d\psi(X)g(Y, W) - d\psi(Y)g(X, W)] = 0 \tag{3.5}
\]

which yields after further contraction

\[
(n - 1)(n - 2)d\psi(X) = 0.
\]

This implies that \(d\psi(X) = 0\), that is, \(\psi\) is constant since \(n > 2\) and leads to the following:

**Theorem 3.2.** In a \((HGPQS)_n\) manifold, the scalar function \(\psi\) is always constant.
Consequently, one can easily bring out the following:

**Theorem 3.3.** In a \((HGPQS)_n\) manifold, \((\text{div}Q)(X,Y)Z\) and \((\text{div}R)(X,Y)Z\) are equivalent.

In view of (1.1), (1.2) and Theorem 3.2 we have

\[
(\nabla_X R)(Y,U,V,W) = 2A_1(X)Q(Y,U,V,W) + A_1(U)Q(Y,X,V,W) + A_1(V)Q(Y,U,X,W) + A_1(W)Q(Y,U,V,X)
\]

\[
+ A_1(U)Q(Y,X,V,W) + A_1(V)Q(Y,U,X,W) + A_1(W)Q(Y,U,V,X) + 2A_2(X)(g \wedge S)(Y,U,V,W)
\]

\[
+ A_2(Y)(g \wedge S)(X,U,V,W) + A_2(U)(g \wedge S)(Y,X,V,W) + A_2(V)(g \wedge S)(Y,U,X,W)
\]

which yields

\[
(\nabla_X S)(U,V) = [F_1(X) + F_2(X)]S(U,V) + F_2(U)S(X,V) + F_2(V)S(U,X)
\]

\[
+ [F_3(X) + F_4(X)]g(U,V) + F_4(U)g(X,V) + F_4(V)g(U,X)
\]

\[
+ A_1(Q(X,U)V) - A_1(Q(V,X)U)
\]

after contraction over \(Y\) and \(W\), where

\[
F_1(X) = A_1(X) + (n + 1)A_2(X),
\]

\[
F_2(X) = A_1(X) + (n - 3)A_2(X),
\]

\[
F_3(X) = rA_2(X) - \psi A_1(X) + 3A_2(LX),
\]

\[
F_4(X) = rA_2(X) - \psi A_1(X) - A_2(LX),
\]

where \(L\) is the Ricci operator defined by \(g(LX,Y) = S(X,Y)\).

**Definition 3.4.** An \(n\)-dimensional semi-Riemannian manifold is called almost generalized pseudo Ricci symmetric if the non-flat Ricci curvature tensor satisfies the equation

\[
(\nabla_X S)(U,V) = [A(X) + B(X)]S(U,V) + A(U)S(X,V) + A(V)S(U,X)
\]

\[
+ [C(X) + D(X)]g(U,V) + C(U)g(X,V) + C(V)g(U,X),
\]

where \(A, B, C\) and \(D\) are non-zero 1-forms whose \(g\)-dual vector fields will be denoted by \(\gamma_1, \gamma_2, \delta_1\) and \(\delta_2\), i.e. \(A(X) = g(X, \gamma_1), B(X) = g(X, \gamma_2), C(X) = g(X, \delta_1)\) and \(D(X) = g(X, \delta_2)\).

Thus we can state the following:
**Theorem 3.5.** A $(\text{HGPQS})_n$ manifold $(n > 2)$ under the assumption $A_1(Q(X,U)V) = A_1(Q(V,X)U)$ is necessarily almost generalized pseudo Ricci symmetric.

Making use of (2.3) in (3.7), we get

$$\langle \nabla_X Z \rangle(U,V) = [F_1(X) + F_2(X)]Z(U,V) + \ldots$$

where $Z = S - \psi g$ is the tensor considered in ([4], [6], [7]). This leads to the following:

**Theorem 3.6.** A $(\text{HGPQS})_n$ manifold $(n > 2)$ under the assumption $A_1(Q(X,U)V) = A_1(Q(V,X)U)$ is necessarily almost generalized pseudo $Z$-symmetric.

### 4 $(\text{HGPQS})_n$ manifolds $(n > 2)$ with $\text{div}Q = 0$

Let $(M^n, g)$ be a semi-Riemannian manifold of dimension $n$ and let $\{e_i\}$ be an orthonormal basis of the tangent space $T_pM$ at any point $p \in M$ and $\epsilon_i = \pm 1$. Then the divergence of a vector field $U$ is defined as

$$\text{div}U = \sum_{i=1}^n \epsilon_i g(\nabla_{e_i} U, e_i),$$

and the divergence of a tensor field of type $(1,3)$, which is a tensor field of type $(0,3)$, is defined as

$$(\text{div}K)(X,Y)Z = \sum_{i=1}^n \epsilon_i g((\nabla_{e_i} K)(X,Y)Z, e_i).$$

Now

$$\langle \text{div}Q \rangle(Y,U)V = \sum_{i=1}^n \epsilon_i g((\nabla_{e_i} Q)(Y,U)V, e_i)$$

$$= \sum_{i=1}^n \epsilon_i [2A_1(e_i) Q(Y,U,V,e_i) + A_1(Y) Q(e_i,U,V,e_i)$$

$$+ A_1(U) Q(Y,e_i,V,e_i) + A_1(V) Q(Y,e_i,U,e_i)$$

$$+ A_1(e_i) Q(Y,U,V,e_i) + 2A_2(e_i)(g \wedge S)(Y,U,V,e_i)$$

$$+ A_2(Y)(g \wedge S)(e_i,U,V,e_i) + A_2(U)(g \wedge S)(Y,e_i,V,e_i)$$

$$+ A_2(V)(g \wedge S)(Y,U,e_i,e_i) + A_2(e_i)(g \wedge S)(Y,V,U,e_i)]$$
\begin{align*}
&= 3A_1(Q(Y,U)V) + A_1(Y)[S(U,V) - \psi g(U,V)] \\
&- A_1(U)[S(Y,V) - \psi g(Y,V)] + 3A_2(Y)S(U,V) \\
&+ 3A_2(LY)g(U,V) - 3A_2(LU)g(Y,V) - 3A_2(U)S(Y,V) \\
&+ A_2(Y)[(n-2)S(U,V) + rg(U,V)] \\
&- A_2(U)[(n-2)S(V,Y) + rg(Y,V)] \\
&= 3A_1(Q(Y,U)V) + S(U,V)[A_1(Y) + (n+1)A_2(Y)] \\
&- S(Y,V)[A_1(U) + (n+1)A_2(U)] \\
&+ g(U,V)[3A_2(LY) + rA_2(Y) - \psi A_1(Y)] \\
&- g(Y,V)[3A_2(LU) + rA_2(U) - \psi A_1(U)] \\
&= 3A_1(Q(Y,U)V) + T_1(Y)S(U,V) - T_1(U)S(Y,V) \\
&+ T_2(Y)g(U,V) - T_2(U)g(Y,V),
\end{align*}

hence

\begin{equation}
(div Q)(Y,U)V = 3A_1(Q(Y,U)V) + T_1(Y)S(U,V) - T_1(U)S(Y,V) + T_2(Y)g(U,V) - T_2(U)g(Y,V),
\end{equation}

where

\begin{align*}
T_1(Y) &= A_1(Y) + (n+1)A_2(Y) =: g(Y,\varrho), \text{ for } \varrho = \theta_1 + (n+1)\theta_2, \\
T_2(Y) &= 3A_2(LY) + rA_2(Y) - \psi A_1(Y) =: g(Y,\sigma), \text{ for } \sigma = 3L\theta_2 + r\theta_2 - \psi \theta_1.
\end{align*}

Assuming \((div Q)(Y,U)V = 0\) and \(A_1(Q(Y,U)V) = 0\), we get from the above equation

\begin{equation}
T_1(Y)S(U,V) + T_2(Y)g(U,V) = T_1(U)S(Y,V) + T_2(U)g(Y,V). \tag{4.2}
\end{equation}

Now contracting (4.2) over \(U\) and \(V\) we get

\begin{equation}
S(Y,\varrho) = rT_1(Y) + (n-1)T_2(Y). \tag{4.3}
\end{equation}

Again putting \(V = \varrho\) in (4.2) we get

\begin{equation}
(n-2)[T_1(Y)T_2(U) - T_1(U)T_2(Y)] = 0, \tag{4.4}
\end{equation}

which under the assumption \(n > 2\) implies \(T_1(Y)T_2(U) = T_1(U)T_2(Y)\).

Now putting \(U = \varrho\) in (4.2) and using (4.3) and (4.4) we get

\begin{equation}
T_1(\varrho)S(Y,V) + T_2(\varrho)g(Y,V) = T_1(Y)[rT_1(V) + nT_2(V)] \tag{4.5}
\end{equation}

and we can state:
Theorem 4.1. A divergence-free (HGPQS)\(_n\) manifold (\(n > 2\)) under the assumption \(A_1(Q(Y,U)V) = 0\) is a perfect fluid spacetime with unit timelike vector field \(\varrho\), provided the associated vector fields \(\varrho\) and \(\sigma\) corresponding to the 1-forms \(T_1\) and \(T_2\) are related by \((r - 1)\varrho + n\sigma = 0\).

In this case, (4.5) becomes

\[ S(Y,V) = ag(Y,V) - T_1(Y)T_1(V), \tag{4.6} \]

where \(a =: T_2(\varrho)\).

Again, \((\text{div} Q)(Y,U)V = 0\) gives

\[ (\nabla_Y S)(U,V) - (\nabla_U S)(Y,V) = 0. \tag{4.7} \]

Now using (4.6) in (4.7) we find

\[
\begin{align*}
& da(Y)g(U,V) - da(U)g(Y,V) \\
& - [T_1(V)(\nabla_Y T_1)(U) + T_1(U)(\nabla_Y T_1)(V)] \\
& + [T_1(V)(\nabla_U T_1)(Y) + T_1(Y)(\nabla_U T_1)(V)] = 0.
\end{align*}
\]

Taking a frame field and contracting \(Y\) and \(V\) we get

\[ (n - 1)da(U) + [T_1(U)(\delta T_1) + (\nabla_\varrho T_1)(U)] = 0, \tag{4.9} \]

where

\[ \delta T_1 = \sum_{i=1}^{n} \epsilon_i(\nabla e_i T_1)(e_i). \]

Setting \(V = Y = \varrho\) in (4.8) we find

\[ (\nabla_\varrho T_1)(U) = -da(U) - da(\varrho)T_1(U). \tag{4.10} \]

Substituting (4.10) in (4.9) we get

\[ (n - 2)da(U) + T_1(U)(\delta T_1) - da(\varrho)T_1(U) = 0 \tag{4.11} \]

which yields

\[ \delta T_1 = (n - 1)da(\varrho) \tag{4.12} \]

for \(U = \varrho\).

Using (4.12) in (4.11) we obtain

\[ da(U) = -T_1(U)da(\varrho), \tag{4.13} \]

provided \(n > 2\).
Putting $V = \varrho$ in (4.8) and using (4.13) we get

$$(\nabla_Y T_1)(U) - (\nabla_U T_1)(Y) = 0.$$ 

This means that the 1-form $T_1$ is closed, that is,

$$dT_1(Y,U) = 0.$$ 

Hence

$$g(\nabla_U \varrho, Y) = g(\nabla_Y \varrho, U) \text{ for all } U,Y,$$ 

(4.14)

which yields

$$g(\nabla_\varrho \varrho, Y) = g(\nabla_Y \varrho, \varrho),$$ 

(4.15)

for $U = \varrho$. Since $g(\nabla_Y \varrho, \varrho) = 0$, from (4.15) it follows that $g(\nabla_\varrho \varrho, Y) = 0$ for all $Y$. Hence $\nabla_\varrho \varrho = 0$. This implies that the integral curves of the vector field $\varrho$ are geodesics. Therefore we can state the following:

**Theorem 4.2.** In a divergence-free $(HGPQS)_n$ manifold $(n > 2)$ under the assumption $A_1(Q(Y, U)V) = 0$, the integral curves of the unit timelike vector field $\varrho$ are geodesics, provided the associated vector fields $\varrho$ and $\sigma$ corresponding to the 1-forms $T_1$ and $T_2$ are related by

$$(r - 1)\varrho + n\sigma = 0.$$ 

Taking into account that the divergence of the conformal curvature tensor of a Riemannian manifold $(M^n, g)$ is ([3], [6]):

$$(\text{div} C)(X, Y)Z = \frac{n-3}{n-2}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)]$$

(4.16)

$$= \frac{n-3}{n-2}\text{div} Q(X,Y)Z,$$

for any vector fields $X,Y,Z$ on $M^n$, from the Lemma 2.1 of [2] we infer

**Theorem 4.3.** Let $(M, g)$ be a $(HGPQS)_n$ perfect fluid spacetime $(n > 2)$. If $(\text{div} Q)(X,Y)Z = 0$, for any vector fields $X,Y,Z$ on $M$, then the unit timelike vector field $\varrho$ is irrotational.

Also, in [2] was proved the following result:

**Theorem 4.4.** [2] Let $(M, g)$ be a $(HGPQS)_n$ perfect fluid spacetime $(n > 2)$. If $(\text{div} Q)(X,Y)Z = 0$, for any vector fields $X,Y,Z$ on $M$, then $(M, g)$ is a GRW spacetime whose fiber is Einstein.

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