Transforming Stäckel Hamiltonians of Benenti type to polynomial form

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Abstract

In this paper we discuss two canonical transformations that turn Stäckel separable Hamiltonians of Benenti type into polynomial form: transformation to Viète coordinates and transformation to Newton coordinates. Transformation to Newton coordinates has been applied to these systems only very recently and in this paper we present a new proof that this transformation indeed leads to polynomial form of Stäckel Hamiltonians of Benenti type. Moreover we present all geometric ingredients of these Hamiltonians in both Viète and Newton coordinates.

Keywords and phrases: Hamiltonian systems, Hamilton-Jacobi theory, Stäckel systems, Benenti systems, polynomial form, Viète coordinates, Newton coordinates

1 Introduction

The aim of this paper is to investigate two canonical transformations of the phase space to coordinates in which the so called Stäckel separable systems of Benenti type attain a polynomial form, as well as to present all geometric objects, related with such systems (the pseudo-Riemannian metric tensor and its Killing tensors as well as the conformal Killing tensor, present in the Hamiltonians of the system) in these new coordinates.

Stäckel systems constitute an important family of quadratic in momenta Hamiltonian systems that are separable, in the sense of Hamilton-Jacobi theory, in orthogonal coordinates. These systems were introduced by Paul Stäckel in [9], where he presented the conditions for separability of Hamilton-Jacobi equation of a natural Hamiltonian system (that is a system of the form $H = K + V$ where $K$ is a quadratic in momenta form and $V$ is a potential defined on the underlying configurational space of the system) in orthogonal coordinates, see for example [11] for a comprehensive review of this subject. Stäckel systems can most conveniently be obtained from the separation relations [10] that are linear in the Hamiltonians $H_i$ and quadratic in momenta $\mu_i$. Further specification of ingredients in these separation relations lead to so called Benenti systems (see the next section for all the necessary definitions and details).

The obtained Stäckel (or Benenti) Hamiltonians $H_j$, as well as their geometric components, are usually given by complicated rational functions, if written in the canonical coordinates in which they were originally created through separation relations. In literature, two maps turning Benenti systems into polynomial form are known: the map to the
so called Viète coordinates [2] and the map to the so called Newton coordinates [7], the second map discovered only recently. In this paper we improve the results obtained in [7] by presenting an alternative, much simpler, proof of its main result using the direct map between Viète coordinates and Newton coordinates. We also present the explicit form of all the geometric structures that are present in the Benenti Hamiltonians in Newton coordinates. These results are new.

2 Stäckel systems

Consider a $2n$-dimensional manifold $M$ equipped with a Poisson bracket $\pi$. Suppose also that $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ are global Darboux coordinates on $M$ (i.e, \{\lambda_i, \lambda_j\} = \{\mu_i, \mu_j\} = 0 for all $i, j = 1, \ldots, n$ while \{\lambda_i, \mu_j\} = \delta_{ij}$). A set of algebraic equations of the form
\[
\varphi_i(\lambda_i, \mu_i, a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, n
\] (1)
is called separation relations if it is globally solvable (except possibly for a union of lower dimensional submanifolds) with respect to the parameters $a_j \in \mathbb{R}$.

Among all possible separations relations (1), a natural subclass consists of the separation relations that are linear in the Hamiltonians $H_k$:
\[
\sum_{k=1}^{n} S_{ik}(\lambda_i, \mu_i) H_k = \psi_i(\lambda_i, \mu_i), \quad i = 1, \ldots, n.
\] (2)
Here $S_{ik}$ and $\psi_i$ are arbitrary smooth functions of two arguments $(\lambda_i, \mu_i)$. The relations (2) are called the generalized Stäckel separation relations and the related dynamical systems, obtained by solving (2) with respect to $H_k$, are called the generalized Stäckel systems. The matrix $S = [S_{ik}(\lambda_i, \mu_i)]$ is called a generalized Stäckel matrix. Although the restriction to separation relations linear in $H_k$ seems to be very strong, it appears that an overwhelming majority of all separable systems considered in the literature falls into various subclasses of this class. The most important class of systems in (2) is the class of classical Stäckel systems, that is systems with the matrix $S$ being a Stäckel matrix (so that $S_{ik} = S_{ik}(\lambda_i)$) and with $\psi_i$ being quadratic in momenta $\mu$:
\[
S_{ik}(\lambda_i, \mu_i) = S_{ik}(\lambda_i), \quad \psi_i(\lambda_i, \mu_i) = \frac{1}{2} f_i(\lambda_i) \mu_i^2 - \varphi_i(\lambda_i),
\]
so that the separation relations (2) attain the form
\[
\varphi_i(\lambda_i) + \sum_{k=1}^{n} S_{ik}(\lambda_i) H_k = \frac{1}{2} f_i(\lambda_i) \mu_i^2, \quad i = 1, \ldots, n.
\] (3)
The relations (3) are called Stäckel separation relations. A particular Stäckel system is thus defined by a choice of the Stäckel matrix $S_{ik}(\lambda_i)$ and by a choice of $2n$ functions $f_i$ and $\varphi_i$. Solving the relations (3) with respect to $H_k$ we obtain $n$ quadratic in momenta functions (Hamiltonians) on $M$
\[
H_r = \frac{1}{2} \mu^T A_r \mu + V_r(\lambda), \quad r = 1, \ldots, n,
\] (4)
where $A_r$ are $n \times n$ matrices given by
\[
A_r = \text{diag} \left( f_1(\lambda_1) \left( S^{-1} \right)_{r1}, \ldots, f_n(\lambda_n) \left( S^{-1} \right)_{rn} \right), \quad r = 1, \ldots, n.
\]
As the Hamiltonians are defined through separation relations, they are in involution with respect to the canonical Poisson bracket on $\mathcal{M}$.

There is a natural geometric interpretation of Stäckel systems given by (4). If we factorize $A_r$ as $A_r = K_rG$, where

$$G = A_1 = \text{diag} \left( f_1(\lambda_1) \left( S^{-1} \right)_{11}, \ldots, f_n(\lambda_n) \left( S^{-1} \right)_{1n} \right)$$

and

$$K_r = \text{diag} \left( \frac{(S^{-1})_{r1}}{(S^{-1})_{11}}, \ldots, \frac{(S^{-1})_{rn}}{(S^{-1})_{1n}} \right), \quad r = 1, \ldots, n$$

(so that $K_1 = I$) then we can interpret the matrix $G$ as a contravariant form of a metric tensor on a manifold $\mathcal{Q}$ such that $\mathcal{M} = T^*\mathcal{Q}$ is the cotangent bundle to $\mathcal{Q}$. The corresponding covariant metric tensor will be denoted by $g$ so that $gG = I$. It can be shown that the matrices $K_r$ are then $(1,1)$-Killing tensors of the metric $G$. For a fixed Stäckel matrix $S$ we have thus the whole family of metrics $G$ parametrized by $n$ arbitrary functions $f_i$ of one variable $\lambda_i$. The tensors $K_r$ are then Killing tensors for any metric from this family. Thus, the Stäckel Hamiltonians $H_r$ in (4) are geodesic Hamiltonians of a Liouville integrable system in the Riemannian space $(\mathcal{M},g)$. Further, due to the linearity of the separation relations, the functions $V_r(\lambda)$ on $\mathcal{Q}$ are defined by the following separation relations

$$\sum_{k=1}^{n} S_{ik}(\lambda_i)V_k = -\varphi_i(\lambda_i), \quad i = 1, \ldots, n,$$

and are called in literature separable potentials on $\mathcal{Q}$.

### 3 Stäckel systems of Benenti type

From now on we restrict ourselves to the case the Stäckel matrix $S$ in (3) is of the very particular form $S_{ij} = \lambda_i^{n-j}$ or explicitly:

$$S = \begin{pmatrix} 
\lambda_1^{n-1} & \lambda_1^{n-2} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_n^{n-1} & \lambda_n^{n-2} & \cdots & 1 
\end{pmatrix}$$

(5)

thus being a Vandermonde matrix. The corresponding Stäckel systems are thus defined by separation relations of the form

$$\varphi_i(\lambda) + \sum_{j=1}^{n} \lambda_i^{n-j}H_j = \frac{1}{2} f_i(\lambda_i)\mu_i^2 \quad i = 1, \ldots, n,$$

(6)

and are called in literature Benenti systems. Benenti systems have been studied much in literature recently, see for example [1, 3] and references therein.

The inverse of $S$ as given by (5) is given by the following lemma.

**Lemma 1.** If $S$ is the $n \times n$ Vandermonde matrix given by $S_{ij} = \lambda_i^{n-j}$ then

$$[S^{-1}]_{ij} = -\frac{1}{\Delta_j} \frac{\partial \rho_i}{\partial \lambda_j},$$

where

$$\rho_i = (-1)^i\sigma_i(\lambda), \quad \Delta_j = \prod_{k \neq j} (\lambda_j - \lambda_k)$$

and where $\sigma_r(\lambda)$ are elementary symmetric polynomials.
By definition
\[ \sigma_i(\lambda) = \sum_{1 \leq j_1 < \ldots < j_i \leq n} \lambda_{j_1} \ldots \lambda_{j_i}, \quad i = 1, \ldots, n, \]
so that
\[ \sigma_0 = 1, \quad \sigma_1 = \sum_{i=1}^{n} \lambda_i, \quad \sigma_2 = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j, \ldots, \quad \sigma_n = \prod_{i=1}^{n} \lambda_i. \]

Lemma 1 can be proved by a direct calculation. By this lemma, solving (6) with respect to \(H_r\) yields \(n\) functions (Hamiltonians) \(H_r\) on \(M\)
\[
H_r = -\frac{1}{2} \sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda_i} \frac{f_i(\lambda) \mu_i^2}{\Delta_i} + V_r(\lambda) \equiv \frac{1}{2} \mu^T K_r G \mu + V_r(\lambda), \quad r = 1, \ldots, n \tag{7}
\]
called Benenti Hamiltonians. Thus, for Benenti Hamiltonians the metric tensor \(G\) is given by
\[
G = \text{diag} \left( \frac{f_1(\lambda_1)}{\Delta_1}, \ldots, \frac{f_n(\lambda_n)}{\Delta_n} \right)
\]
while the Killing tensors \(K_r\) are given by
\[
K_r = -\text{diag} \left( \frac{\partial \rho_r}{\partial \lambda_1}, \ldots, \frac{\partial \rho_r}{\partial \lambda_n} \right) \quad r = 1, \ldots, n. \tag{8}
\]

From now and in what follows, we further assume that all \(f_i\) are equal, and likewise all \(\varphi_i\):
\[
f_i := f, \quad \varphi_i := \varphi
\]
so that all the Hamiltonians (7) are generated by the single separation curve:
\[
\varphi(\lambda) + \sum_{j=1}^{n} \lambda^{n-j} H_j = \frac{1}{2} f(\lambda) \mu^2 \tag{9}
\]
and are given explicitly by:
\[
H_r = -\frac{1}{2} \sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda_i} \frac{f(\lambda) \mu_i^2}{\Delta_i} + V_r(\lambda) \equiv \frac{1}{2} \mu^T K_r G \mu + V_r(\lambda), \quad r = 1, \ldots, n \tag{10}
\]
and thus the metric tensor \(G\) is now given by
\[
G = \text{diag} \left( \frac{f(\lambda_1)}{\Delta_1}, \ldots, \frac{f(\lambda_n)}{\Delta_n} \right).
\]

Of particular interest is the case \(f(\lambda_i) = \lambda_i^m\) with \(m \in \mathbb{Z}\). In such a case the metric tensor \(G\) will be denoted by \(G_m\):
\[
G_m = \text{diag} \left( \frac{\lambda_1^m}{\Delta_1}, \ldots, \frac{\lambda_n^m}{\Delta_n} \right), \quad m \in \mathbb{Z}.
\]

Of course, if \(f\) is a Laurent polynomial
\[
f(\lambda) = \sum_{\alpha \in A} a_{\alpha} \lambda_1^{\alpha_1}, \tag{11}
\]
where \(A \subset \mathbb{Z}\) is a finite set, then
\[
G = \sum_{\alpha \in A} a_{\alpha} G_{\alpha}. \tag{12}
\]
It can be shown that the metric $G_m$ is flat for $m \in \{0, \ldots, n\}$ and of constant curvature for $m = n + 1$ (by linearity of (9) the same is true for $f$ being a polynomial in $\lambda$ of order $m$). Moreover

$$G_m = L^m G_0, \quad G_0 = \text{diag} \left( \frac{1}{\Delta_1}, \ldots, \frac{1}{\Delta_n} \right),$$  \hspace{1cm} (13)

where

$$L = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

is a $(1,1)$-tensor called $\textit{special conformal Killing tensor}$ [8]. It can be shown [5] that all $K_r$ can be calculated from the formula

$$K_1 = I, \quad K_r = \sum_{k=0}^{r-1} \rho_k L^{r-1-k}, \quad r = 2, \ldots, n.$$  \hspace{1cm} (14)

In order to illustrate the form of separable potentials $V_r(\lambda)$ in the Benenti case, we further assume that $\varphi$ is a Laurent sum of the form

$$\varphi(\lambda) = \sum_{\alpha \in A} c_\alpha \lambda_i^\alpha,$$  \hspace{1cm} (15)

where $A \subset \mathbb{Z}$ is a finite set and $c_\alpha$ are some real constants. The Benenti separation relations (6) become

$$\sum_{\alpha \in A} c_\alpha \lambda_i^\alpha + \sum_{j=1}^{n} \lambda_i^{n-j} H_j = \frac{1}{2} f(\lambda_i) \mu_i^2, \quad i = 1, \ldots, n$$  \hspace{1cm} (16)

and due to their linearity we have

$$V_r = \sum_{\alpha \in A} c_\alpha V_r^{(\alpha)},$$

where $V_r^{(\alpha)}$ are so called $\textit{basic separable potentials}$. By linearity of (16), the potentials $V_r^{(\alpha)}$ satisfy the relations

$$\lambda_i^\alpha + \sum_{r=1}^{n} V_r^{(\alpha)} \lambda_i^{n-r} = 0, \quad i = 1, \ldots, n$$

and, again by Lemma [11] they are given by

$$V_r^{(\alpha)} = \sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda_i} \frac{\lambda_i^\alpha}{\Delta_i}, \quad r = 1, \ldots, n.$$  \hspace{1cm} (17)

The basic separable potentials $V_r^{(\alpha)}$ can be explicitly constructed by the following formula [5]:

$$V(\alpha) = R^\alpha V^{(0)}, \quad V^{(\alpha)} = (V_1^{(\alpha)}, \ldots, V_n^{(\alpha)})^T,$$  \hspace{1cm} (18)

where

$$R = \begin{pmatrix} -\rho_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -\rho_n & 0 & 0 & 0 \end{pmatrix}$$
and \( V^{(0)} = (0, \ldots, 0, -1)^T \). The first \( n \) basic potentials are trivial
\[
V_k^{(\alpha)} = -\delta_{k,n-\alpha}, \quad \alpha = 0, \ldots, n-1.
\]
The first nontrivial positive potential is
\[
V^{(n)} = (\rho_1, \ldots, \rho_n)^T
\]
and higher potentials are more complicated polynomials in \( q_i \). The first negative potential is
\[
V^{(-1)} = \left( \frac{1}{\rho_n}, \ldots, \frac{\rho_{n-1}}{\rho_n} \right)^T
\]
and the higher negative potentials are more complicated rational functions of all \( \rho_i \). Note also that the recursion formulas (17)-(18) are not tensorial; they look the same in any coordinate system.

### 4 Polynomial form of Benenti systems

As we saw in the previous section, even the relatively simple Benenti Hamiltonians are complicated rational functions when expressed in the separation variables \((\lambda, \mu)\). In this section we demonstrate two canonical maps that under certain conditions transform Benenti Hamiltonians (10) to a polynomial form.

#### 4.1 Benenti systems in Viète coordinates

Suppose that we change the position coordinates on the base manifold \( Q \) through the map
\[
q_i = \rho_i(\lambda) \quad i = 1, \ldots, n,
\]
where, as in Lemma 1, \( \rho_i(\lambda) = (-1)^i \sigma_i(\lambda) \). This map induces the map (point transformation) on \( T^* Q \):
\[
p = (J_V^{-1})^T \mu,
\]
where \( J_V \) is the Jacobian of the map (19):
\[
(J_V)_{ij} = \frac{\partial \rho_i}{\partial \lambda_j}.
\]
Let us find an explicit form of (20). To do this we need the following lemma.

**Lemma 2.** Denote by \( k_i \) the \( i \)-th column of an \( n \times n \) nondegenerate matrix \( A \):
\[
A = (k_1 | k_2 | \ldots | k_n)
\]
and by \( r_j \) the \( j \)-th row of its inverse
\[
A^{-1} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix},
\]
Then, if \( \alpha_i \in \mathbb{R} \) for \( i = 1, \ldots, n \)
\[
(\alpha_1 k_1 | \alpha_2 k_2 | \ldots | \alpha_n k_n)^{-1} = \begin{pmatrix} r_1/\alpha_1 \\ r_2/\alpha_2 \\ \vdots \\ r_n/\alpha_n \end{pmatrix}.
\]
This elementary lemma follows from the fact that \( r_k k_j = \delta_{ij} \). An analogous lemma is of course true if we consider rows of \( A \) instead of its columns. Combining lemmas 1 and 2 we obtain that

\[
(J^{-1})_{ij} = -\frac{\lambda_i^{n-j}}{\Delta_i}
\]

(22)

and thus the map (20) can be written as

\[
p_i = -\sum_{k=1}^{n} \frac{\lambda_k^{n-i}}{\Delta_k} \mu_k, \quad i = 1, \ldots, n.
\]

(23)

The coordinates \((q, p)\) defined by (19) and (23) are called Viète coordinates. To summarize, the map \((\lambda, \mu) \rightarrow (q, p)\) from separation coordinates to Viète coordinates is given by

\[
q_i = \rho_i(\lambda), \quad p_i = -\sum_{k=1}^{n} \frac{\lambda_k^{n-i}}{\Delta_k} \mu_k, \quad i = 1, \ldots, n.
\]

(24)

Being a point transformation, the map (24) is a canonical map which means that Viète coordinates are Darboux (canonical) coordinates as well:

\[
\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}.
\]

Let us now investigate the structure of Benenti Hamiltonians (10) in Viète coordinates \((q, p)\). The Hamiltonians (10) are of course written in tensor form so that in Viète coordinates

\[
H_r(q, p) = \frac{1}{2} p^T K_r(q) G(q) p + V_r(q), \quad r = 1, \ldots, n
\]

(25)

where, by transformation laws for tensors,

\[
K_r(q) = J_V K_r(J_V)^{-1}, \quad G(q) = J_V G(J_V)^T.
\]

(26)

The first formula in (26) yields, after some calculation

\[
(K_r(q))^i_j = \begin{cases} 
q_{i-j+r-1}, & i \leq j \text{ and } r \leq j \\
-q_{i-j+r-1}, & i > j \text{ and } r > j \\
0 & \text{otherwise}
\end{cases}
\]

(27)

Here and throughout the whole section we use the convention that \( q_0 = 1 \) and \( q_k = 0 \) for \( k < 0 \) and for \( k > n \). Thus, all the \( K_r(q) \) are linear in \( q \)-variables. Further, for the monomial case \( f(\lambda_i) = \lambda_i^m \) with \( m \in \{0, \ldots, n+1\} \) we can obtain from the second formula in (26) that

\[
G^{ij}_m(q) = \begin{cases} 
q_{i+j+m-n-1}, & i, j = 1, \ldots, n-m \\
-q_{i+j+m-n-1}, & i, j = n-m+1, \ldots, n \\
0 & \text{otherwise}
\end{cases}, \quad m = 0, \ldots, n
\]

(28)

The formulas (27) and (28) can alternatively be obtained with the help of the special conformal Killing tensor \( L \) by using the formulas (14) and (13), respectively, and the fact that
the tensor $L$ can be easily calculated in Viète coordinates through tensor transformation law $L(q) = J^V L(J^V)^{-1}$. We obtain

$$L_i^j(q) = -\delta_i^j q_i + \delta_i^{i+1}$$

that is

$$L(q) = \begin{pmatrix}
-q_1 & 1 & 0 & 0 \\
0 & \ddots & \ddots & \ddots \\
0 & 0 & 1 \\
-q_n & 0 & 0 & 0
\end{pmatrix}.$$  \hfill (29)

Note therefore that $L$ happens to have the same form in $q$-coordinates as the recursion matrix (18). This seems to be a pure coincidence without any deeper meaning; we stress again that $R$ in (18) is not a tensor. In any case, due to the fact that all the entries in $L$ are linear in $q_i$ we see that all the entries in $G_m$ are linear in $q_i$ for $m = 0, \ldots, n+1$, quadratic in $q_i$ for $m = n+1$ and higher order polynomials for higher $m$. Moreover, by (27), all entries in $K_r(q)$ are linear in $q$. Using all these facts and the formula (12) we obtain the following important corollary:

**Corollary 3.** If $f$ is a polynomial in (11), then the geodesic parts of Benenti Hamiltonians (25) have a polynomial form. Moreover, if the right hand side of (15) is a polynomial, then by the recursive relations (17)-(18) also the potentials $V_r$ in the Benenti Hamiltonians (10) are in this case polynomials in $q_i$. Thus, in such a case, the whole Hamiltonians $H_r(q,p)$ (and not just their geodesic parts) are polynomials.

**Example 4.** Consider the case $n = 2$, $f(\lambda) = 1$ (i.e. a purely monomial situation with $m = 0$ in (28), so that $G = G_0$) and $\varphi(\lambda) = \lambda^3$. Then the separation curve (16) becomes

$$\lambda^3 + \lambda H_1 + H_2 = \frac{1}{2} \mu^2$$

and yields the Hamiltonians $H_i$ in the explicit form

$$H_1 = \frac{1}{2(\lambda_1 - \lambda_2)}(\mu_1^2 - \mu_2^2) - (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)$$

$$H_2 = \frac{1}{2(\lambda_1 - \lambda_2)}(\lambda_1 \mu_2^2 - \lambda_2 \mu_1^2) + \lambda_1 \lambda_2 (\lambda_1 + \lambda_2)$$

so both Hamiltonians are rational functions of separation coordinates $(\lambda, \mu)$. The above Hamiltonians have exactly the form (10) with the metric

$$G = G_0 = \text{diag} \left( \frac{1}{\Delta_1}, \frac{1}{\Delta_2} \right),$$

and with the Killing tensors (8) given explicitly by:

$$K_1 = I, \quad K_2 = -\text{diag}(\lambda_2, \lambda_1).$$

The map (24) to Viète coordinates has the explicit form:

$$q_1 = -(\lambda_1 + \lambda_2), \quad q_2 = \lambda_1 \lambda_2,$$

$$p_1 = \frac{1}{\lambda_2 - \lambda_1}(\lambda_1 \mu_1 - \lambda_2 \mu_2), \quad p_2 = \frac{1}{\lambda_2 - \lambda_1}(\mu_1 - \mu_2).$$
An elementary calculations shows that $H_1$ in these variables attain the form

$$H_1(q, p) = \frac{1}{2} q_1 p_2^2 + p_1 p_2 - q_1^2 + q_2$$

$$H_2(q, p) = \frac{1}{2} p_1^2 + q_1 p_1 p_2 + \frac{1}{2} q_1^2 p_2^2 - \frac{1}{2} p_2^2 q_2 - q_1 q_2$$

which is in agreement with (28) and (27). Explicitly:

$$G_0(q) = \begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix}, \quad K_1(q) = I, \quad K_2(q) = \begin{pmatrix} 0 & -q_2 \\ -q_2 & 1 \end{pmatrix}.$$ 

Thus, the Hamiltonians $H_r$ become polynomial in Viète coordinates $(q, p)$.

**Example 5.** Consider the case $n = 3$, $f(\lambda) = \lambda$ (so that $m = 1$ in (28) and thus $G = G_1$) and $\varphi(\lambda) = \lambda^5$. Then the separation curve (16) becomes

$$\lambda^5 + \lambda^2 H_1 + \lambda H_2 + H_3 = \frac{1}{2} \lambda \mu^2.$$ 

Solving the corresponding separation coordinates yields the Benenti Hamiltonians (10) with the metric $G_1 = LG_0$ with

$$L = \text{diag} (\lambda_1, \lambda_2, \lambda_3)$$

so that

$$G_1 = LG_0 = \text{diag} \left( \frac{\lambda_1}{\Delta_1}, \frac{\lambda_2}{\Delta_2}, \frac{\lambda_3}{\Delta_3} \right)$$

and with the Killing tensors (5) given explicitly by

$$K_1 = I, \quad K_2 = \text{diag} (\lambda_2 + \lambda_3, \lambda_1 + \lambda_3, \lambda_1 + \lambda_2), \quad K_3 = -\text{diag} (\lambda_2 \lambda_3, \lambda_1 \lambda_3, \lambda_1 \lambda_2),$$

while the potentials $V_r = V^{(5)}_r$ have the form

$$V^{(5)}_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1^2 \lambda_2 + \lambda_1^2 \lambda_3 + \lambda_1 \lambda_2 \lambda_3 + \lambda_2 \lambda_3^2 + \lambda_1^2 \lambda_2 \lambda_3,$$

$$V^{(5)}_2 = \lambda_1^3 \lambda_2 + \lambda_1^2 \lambda_3 + \lambda_2^3 \lambda_2 + 2 \lambda_1^2 \lambda_2 \lambda_3 + \lambda_2 \lambda_3^2 + \lambda_1 \lambda_2 \lambda_3^2 + 2 \lambda_1 \lambda_2 \lambda_3,$$

$$V^{(5)}_3 = \lambda_1 \lambda_2 \lambda_3 \left( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \right).$$

The map (24) to Viète coordinates has now the form

$$q_1 = - (\lambda_1 + \lambda_2 + \lambda_3), \quad q_2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3, \quad q_3 = - \lambda_1 \lambda_2 \lambda_3$$

and

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = (J_V^{-1})^T \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

with $J_V$ and $J_V^{-1}$ given by (1.1) and (22) respectively. Explicitly

$$J_V = \begin{pmatrix} -1 & -1 & -1 \\ \lambda_2 + \lambda_3 & \lambda_1 + \lambda_3 & \lambda_1 + \lambda_2 \\ -\lambda_2 \lambda_3 & -\lambda_1 \lambda_3 & -\lambda_1 \lambda_2 \end{pmatrix}.$$
and
\[ J^{-1}_V = \begin{pmatrix}
\lambda_1^2 & \lambda_1^2 & \lambda_1^2 \\
\lambda_2^2 & \lambda_2^2 & \lambda_2^2 \\
\lambda_3^2 & \lambda_3^2 & \lambda_3^2
\end{pmatrix}.
\]

An elementary calculation shows that \( H_i \) in these variables attain the form
\[
H_1(q, p) = \frac{1}{2} q_1 p_2^2 + p_1 p_2 - \frac{1}{2} q_3 p_3^2 + q_1^3 - 2q_1 q_2 + q_3,
\]
\[
H_2(q, p) = \frac{1}{2} p_1^2 - \frac{1}{2} q_2 p_2^2 - \frac{1}{2} q_1 q_3 p_3^2 + q_1 p_1 p_2 - q_3 p_2 p_3 + q_1^2 q_2 - q_1 q_3 - q_2^2,
\]
\[
H_3(q, p) = -\frac{1}{2} q_3 p_2^2 - \frac{1}{2} q_2 q_3 p_3^2 - q_3 p_1 p_3 - q_1 q_3 p_2 p_3 + q_1^2 q_3 - q_2 q_3,
\]
which is in agreement with \((28)\) and \((27)\). Explicitly:
\[
G_0(q) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & q_1 \\ 1 & q_1 & q_2 \end{pmatrix}, \quad K_1(q) = I, \quad K_2(q) = \begin{pmatrix} 0 & 1 & 0 \\ -q_2 & q_1 & 1 \\ -q_3 & 0 & q_1 \end{pmatrix},
\]
\[
K_3(q) = \begin{pmatrix} 0 & 0 & 1 \\ -q_3 & 0 & q_1 \\ 0 & -q_3 & q_2 \end{pmatrix},
\]
while the tensor \( L \) attains the form as in \((29)\):
\[
L(q) = \begin{pmatrix} -q_1 & 1 & 0 \\ -q_2 & 0 & 1 \\ -q_3 & 0 & 0 \end{pmatrix}.
\]

Note again that the Hamiltonians \( H_r \) become polynomial in Viète coordinates \((q, p)\).

### 4.2 Benenti systems in Newton coordinates

The second method of turning Benenti Hamiltonian systems \((10)\) into a polynomial form is by using Newton coordinates. This method has been discovered by V. M. Buchstaber and A. V. Mikhailov \([7]\) only quite recently. In this section we present our own proof of this result, independent of the work \([7]\). We also investigate in detail the structure of Benenti Hamiltonians \((10)\) in Newton coordinates.

Consider the following map (consisting of a sequence of Newton polynomials) on the base manifold \(Q\):
\[
Q_i = \frac{1}{i} \sum_{s=1}^n \lambda_s^i. \quad (30)
\]

This map induces the map on \(T^*Q\):
\[
P = (J_N^{-1})^T \mu, \quad (31)
\]
where \( P = (P_1, \ldots, P_n)^T \) and \( J_N \) is the Jacobian of the map \((30)\),
\[
(J_N)_{ij} = \frac{\partial Q_i}{\partial \lambda_j} = \lambda_j^{i-1}.
\]
Thus, \( J_N = V^T \), where \( V \) is the Vandermonde matrix, but different from \( S \):

\[
V = \begin{pmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \lambda_n & \cdots & \lambda_n^{n-1}
\end{pmatrix}.
\]  

(32)

This also means that (31) leads to \( P = V^{-1} \mu \).

**Lemma 6.** In the above notation

\[
(V^{-1})_{ij} = -\frac{1}{\Delta_j} \frac{\partial \rho_{n-i+1}}{\partial \lambda_j}.
\]

The reader should compare this lemma with Lemma 1. Thus, the map (30) induces the following map on \( T^* \mathcal{Q} \)

\[
Q_i = \frac{1}{r} \sum_{s=1}^{n} \lambda_i^s, \quad P_i = -\sum_{j=1}^{n} \frac{1}{\Delta_j} \frac{\partial \rho_{n-i+1}}{\partial \lambda_j} \mu_j, \quad i = 1, \ldots, n
\]

(33)

and we call the coordinates \((Q, P)\) Newton coordinates on \( \mathcal{M} \). The reader should compare this map with the map (24). Again, since the map \((\lambda, \mu) \rightarrow (Q, P)\) is a point transformation map on \( T^* \mathcal{Q} \), the Newton coordinates \((Q, P)\) are Darboux (canonical) coordinates, that is

\[
\{Q_i, Q_j\} = \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}.
\]

Let us now investigate the structure of Benenti Hamiltonians (10) in \((Q, P)\)-coordinates. The Hamiltonians (10) are written in tensor form and thus

\[
H_r(Q, P) = \frac{1}{2} P^T K_r(Q) G(Q) P + V_r(Q), \quad r = 1, \ldots, n.
\]

(34)

In the monomial case, i.e., when \( f(\lambda) = \lambda^m \) we have

\[
H_r(Q, P) = \frac{1}{2} P^T K_r(Q) L^m(Q) G_0(Q) P + V_r(Q), \quad r = 1, \ldots, n.
\]

(35)

Let us now investigate the structure of (31) and in particular (35), in Newton coordinates. Due to tensor transformation laws, \( L(Q), K_r(Q) \) and \( G(Q) \) are given by

\[
L(Q) = J_N L(J_N)^{-1}, \quad K_r(Q) = J_N K_r(J_N)^{-1}
\]

(36)

and by

\[
G(Q) = J_N G(J_N)^T.
\]

(37)

In order to express explicitly the right hand sides of (35) and (37) we need to invert the map \( \lambda \rightarrow Q \) given by (30), which is in general not algebraically invertible. Let us thus consider the map \( q \rightarrow Q \) between the Viète coordinates (24) and the Newton coordinates. In the recent paper [1] this map is given by

\[
Q_r = -\frac{1}{r} \sum_{k=1}^{r} V_{k}^{(n+r-k)}(q), \quad r = 1, \ldots, n,
\]

(38)

where \( V_{k}^{(\alpha)}(q) \) are the basic separable potentials as given by (17)-(18). Below we present a theorem in which we considerably extend the understanding of the above formula.
Theorem 7. The map \( q \to Q \) as given by (38) has the following form:

\[
Q_r(q) = -q_r + \tau_r^{(r-1)}(q_1, \ldots, q_{r-1}), \quad r = 1, \ldots, n,
\]

(39)

where \( \tau_r^{(a)} \) denotes a polynomial of order \( \alpha \) and where \( \tau_1^{(0)} = 0 \). The map \( q \to Q \) is algebraically invertible, with the inverse map of the form

\[
q_r(Q) = -Q_r + \eta_r^{(r-1)}(Q_1, \ldots, Q_{r-1}), \quad r = 1, \ldots, n,
\]

(40)

where \( \eta_r^{(a)} \) denotes a polynomial of order \( \alpha \) with \( \eta_1^{(0)} = 0 \). Moreover, neither \( \tau_r^{(a)} \) nor \( \eta_r^{(a)} \) depends on \( n \).

One proves this theorem by direct calculations, using the properties of basic separable potentials \( V_k^{(a)} \). This theorem means that both the map \( q \to Q \) and its inverse \( Q \to q \) are polynomial maps and moreover that the transformation between the first \( n \) variables, i.e. between \( q_1, \ldots, q_n \) and \( Q_1, \ldots, Q_n \), does not change after increasing \( n \) to \( n+1 \). Explicitly, the first few expressions in both maps are

\[
Q_1 = -q_1,
\]

\[
Q_2 = -q_2 + \frac{1}{2} q_1^2,
\]

\[
Q_3 = -q_3 - \frac{1}{3} q_1^3 + q_2 q_1,
\]

\[
Q_4 = -q_4 + \frac{1}{4} q_1^4 - q_1^2 q_2 + q_3 q_1 + \frac{1}{2} q_2^2
\]

\[
\vdots
\]

for the map \( Q \to q \) and

\[
q_1 = -Q_1,
\]

\[
q_2 = -Q_2 + \frac{1}{2} Q_1^2,
\]

\[
q_3 = -Q_3 - \frac{1}{6} Q_1^3 + Q_2 Q_1,
\]

\[
q_4 = -Q_4 + \frac{1}{24} Q_1^4 - \frac{1}{2} Q_1^2 Q_2 + Q_3 Q_1 + \frac{1}{2} Q_2^2
\]

\[
\vdots
\]

for the inverse map \( q \to Q \). It is now possible to calculate the tensor \( L \) in the Newton coordinates \( Q \). After some calculations we obtain:

\[
L(Q) = J_N L (J_N)^{-1} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & 1 \\
-q_n(Q) & -q_{n-1}(Q) & -q_{n-2}(Q) & \ldots & -q_1(Q)
\end{pmatrix},
\]

(41)

or, equivalently

\[
L(Q)^{ij} = -q_{n-j+1}(Q) \delta_i^j + \delta_{j-1}^j, \quad i, j = 1, \ldots, n,
\]

where the functions \( q_i(Q) \) are given by (40). Thus, the entries of \( L(Q) \) are polynomials, and the same is of course true for any positive power \( L^m(Q) \) of \( L(Q) \).
Let us now calculate the Killing tensors $K_r$ in Newton coordinates $Q$. We will do it by transforming $K_r(q)$, as given by \((27)\), to $Q$ variables, by the formula $K_r(Q) = J_{VN}K_r(q)(J_{VN})^{-1}$, where $J_{VN}$ is the Jacobian transformation from Viète coordinates to Newton coordinates. First we find that

$$(J_{VN})_{i,j} = \sum_{s=0}^{n} q_s(J_{VN})_{i-s,j} - q_i q_{i-s} + q_i, \quad i, j = 1, \ldots, n,$$

with $(J_{VN})_{1,j} = -1$, $(J_{VN})_{2,j} = q_1$, $(J_{VN})_{3,j} = -q_1^2 + q_2$ for any fixed $j$. This also yields that

$$(J_{VN})_{i,j}^{-1} = -q_{i-j}, \quad i, j = 1, \ldots, n.$$

Note that this last result also means that the map \((39)\) can now be extended to the whole manifold $\mathcal{M} = T^*Q$ by completing it with the map between the canonical momenta:

$$P_i = [(J_{VN})^{-1}]_i^j p_j = - \sum_{j=1}^{n} q_{j-i} p_j, \quad i = 1, \ldots, n. \quad (42)$$

After some calculations we obtain that

$$(K_r(Q))^i_j = \begin{cases} q_{i-j+r-1}(Q), & i-j \leq 0 \text{ and } r \leq n-i+1 \\ -q_{i-j+r-1}(Q), & i-j > 0 \text{ and } r > n-i+1 \\ 0 & \text{otherwise} \end{cases}, \quad (43)$$

cf. \((27)\). Thus, since all $q_i(Q)$ by \((40)\) are polynomials then all the entries of $K_r(Q)$ are polynomials in $Q_i$ as well. Finally, let us consider $G_0(Q)$, i.e. the metric $G_0$ in Newton coordinates, by transforming $G_0(q)$, as given by \((28)\), into $Q$ variables, by the transformation formula $G_0(Q) = J_{VN}G_0(q)(J_{VN})^T$.\n
**Lemma 8.** The metric $G_0$ in Newton coordinates \((34)\) attains the form of a lower-triangular Hankel matrix given by the recursive formulas

$$G_0(Q)_{i,j} = \begin{cases} -\sum_{s=1}^{k} q_s(Q) (G_0)_{i-s,j} + q_1(Q)q_{i-1}(Q) - q_i(Q), & i \geq j \text{ for } i, j = 3, \ldots, n, \\ 0, & i < j \end{cases} \quad (44)$$

with $G_0(Q)_{1,j} = 1$, $G_0(Q)_{2,j} = -q_1$, and $G_0(Q)_{3,j} = q_1^2 - q_2$ for arbitrary fixed $j$.

As a consequence, the metric $G_m(Q)$ also attains the form of a lower-triangular Hankel matrix. This can be verified using induction with respect to $m$ in

$$G_m(Q)_{i,j} = L(Q)^i_j G_{m-1}(Q)_{i,j}. \quad (45)$$

Taking into account the formulas \((41)\), \((43)\) and Lemma 8 we obtain a corollary that is an analogue of Corollary 3 for Newton coordinates.

**Corollary 9.** If $f$ is a polynomial in \((13)\), then the geodesic parts of Benenti Hamiltonians $H_r(Q, P)$ in \((34)\) have in Newton coordinates \((33)\) a polynomial form. Moreover, if the right hand side of \((13)\) is a pure polynomial, then the potentials $V_r(Q)$ in the Benenti Hamiltonians \((34)\) are in this case also polynomials. Thus, in such a case, all the Hamiltonians $H_r(Q, P)$ (and not just their geodesic parts) are polynomials.
Let us now present some examples.

**Example 10.** We proceed in the same setting as in Example 4 i.e. we consider the case \( n = 2, f(\lambda) = 1 \) (so that \( m = 0 \)) and \( \varphi(\lambda) = \lambda^3 \), but in Newton coordinates. The map (39)-(42) reads now

\[
Q_1 = -q_1, \quad Q_2 = \frac{1}{2}q_1^2 - q_2, \\
P_1 = -p_1 - q_1p_2, \quad P_2 = -p_2
\]

and it transforms the Hamiltonians from Example 24 to the form

\[
H_1(Q, P) = \frac{1}{2}P_2^2Q_1 + P_1P_2 - Q_2 - \frac{1}{2}Q_1^2, \\
H_2(Q, P) = -\frac{1}{4}P_2^2Q_1^2 + \frac{1}{2}P_2^2Q_2 + \frac{1}{2}P_1^2 + \frac{1}{2}Q_1^3 - Q_1Q_2,
\]

which is in agreement with (43) and (44). Explicitly:

\[
G_0(Q) = \begin{pmatrix} 0 & 1 \\ 1 & Q_1 \end{pmatrix}, \quad K_1(Q) = I, \quad K_2(Q) = \begin{pmatrix} -Q_1 & 1 \\ Q_2 & -\frac{1}{2}Q_1^2 \end{pmatrix}.
\]

Moreover, \( L \) becomes

\[
L(Q) = \begin{pmatrix} 0 & 1 \\ Q_2 - \frac{1}{2}Q_1^2 & Q_1 \end{pmatrix}.
\]

**Example 11.** We now consider Example 5 in Newton coordinates i.e. the case \( n = 3, m = 1 \) and \( \varphi(\lambda) = \lambda^3 \). As \( n = 3 \) the map (39) is now

\[
Q_1 = -q_1, \quad Q_2 = \frac{1}{2}q_1^2 - q_2, \quad Q_3 = -\frac{1}{3}q_1^3 + q_1q_2 - q_3, \tag{45}
\]

and its inverse (40) is

\[
q_1 = -Q_1, \quad q_2 = \frac{1}{2}Q_1^2 - Q_2, \quad q_3 = -\frac{1}{6}Q_1^3 + Q_1Q_2 - Q_3.
\]

The map (42) between momenta is

\[
\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} -1 & -q_1 & -q_2 \\ 0 & -1 & -q_1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \tag{46}
\]

with the inverse

\[
\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} -1 & -Q_1 & -\frac{1}{2}Q_1^2 - Q_2 \\ 0 & -1 & -Q_1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}.
\]

The map (43)-(46) transforms the Hamiltonians \( H_r(q, p) \) in Example 5 to the form

\[
H_1(Q, P) = \frac{1}{2}P^T \begin{pmatrix} 0 & 1 \\ 1 & Q_1 \\ Q_1 - \frac{1}{2}Q_1^2 + Q_2 + \frac{1}{2}Q_1^3 + 4Q_1^2 \end{pmatrix} P + V_1^{(5)}(Q), \\
H_2(Q, P) = \frac{1}{2}P^T \begin{pmatrix} 1 & 0 \\ 0 & Q_2 - \frac{1}{2}Q_1^2 \\ Q_2 - \frac{1}{2}Q_1^2 + Q_3 + \frac{1}{2}Q_1^3 - \frac{1}{12}Q_1^3 + \frac{1}{6}Q_1^2 + Q_3Q_1 + Q_2 \end{pmatrix} P + V_2^{(5)}(Q), \\
H_3(Q, P) = \frac{1}{2}P^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{6}Q_1^3 + Q_2Q_1 + Q_3 \\ \frac{1}{6}Q_1^3 - Q_2Q_1 + Q_3 \end{pmatrix} P + V_3^{(5)}(Q),
\]

+ \( V_3^{(5)}(Q) \).
where

\[
V_{1}^{(5)}(Q) = -\frac{1}{6}Q_1^3 - Q_2 Q_3 - Q_3,
\]

\[
V_{2}^{(5)}(Q) = Q_1^2 Q_2 - Q_1 Q_3 - \frac{1}{2} Q_2^2 + \frac{1}{12} Q_1^4,
\]

\[
V_{3}^{(5)}(Q) = \frac{1}{12} Q_1^3 + \frac{1}{3} Q_2^2 Q_3 - \frac{1}{2} Q_2 Q_3 + Q_1 Q_2^2 - Q_2 Q_3,
\]

which is again in agreement with (44) and (43). Explicitly:

\[
G_0(Q) = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & Q_1 \\
1 & Q_1 & \frac{1}{2} Q_1^2 + Q_2
\end{pmatrix}, \quad K_1(Q) = I,
\]

\[
K_2(Q) = \begin{pmatrix}
-Q_1 & 1 & 0 \\
0 & -Q_1 & 1 \\
\frac{1}{6} Q_1^3 - Q_1 Q_2 + Q_3 & Q_2 - \frac{1}{2} Q_1^2 & 0
\end{pmatrix},
\]

\[
K_3(Q) = \begin{pmatrix}
\frac{1}{6} Q_1^3 - Q_1 Q_2 + Q_3 & -Q_1 & 1 \\
0 & 0 & \frac{1}{6} Q_1^3 - Q_1 Q_2 + Q_3 \\
\frac{1}{6} Q_1^3 - Q_1 Q_2 + Q_3 & Q_2 - \frac{1}{2} Q_1^2 & Q_1
\end{pmatrix},
\]

and

\[
L(Q) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\frac{1}{6} Q_1^3 - Q_1 Q_2 + Q_3 & Q_2 - \frac{1}{2} Q_1^2 & Q_1
\end{pmatrix}.
\]

5 Conclusions

In this paper we have considered Benenti Hamiltonian systems generated by a single separation curve. These systems turn out to have a rational form when expressed in their separation coordinates. Under certain additional conditions they can be cast into polynomial form using either Viète coordinates (24) or Newton coordinates (33), the last result due to Buchstaber and Mikhailov [7]. We have presented a new version of Buchstaber and Mikhailov results: not only have we proven their result in a more explicit way but we also presented the explicit form of all the geometric objects, associated with Benenti Hamiltonians, in Newton coordinates. This has been done by constructing and analysing the map between the Viète and Newton coordinates.

A natural question that arises is whether it is possible to extend our construction to the case that \(H_i\) are not generated by a single separation curve but by the more general separation relations (6) i.e. with different \(f_i\) and \(\varphi_i\). This will be a subject of another research paper.

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