WILLMORE SPACELIKE SUBMANIFOLDS IN AN INDEFINITE SPACE FORM $N_q^{n+p}(c)$

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Abstract. Let $N_q^{n+p}(c)$ be an $(n+p)$-dimensional connected indefinite space form of index $q$ ($1 \leq q \leq p$) and of constant curvature $c$. Denote by $\varphi : M \to N_q^{n+p}(c)$ the $n$-dimensional spacelike submanifold in $N_q^{n+p}(c)$; $\varphi : M \to N_q^{n+p}(c)$ is called a Willmore spacelike submanifold in $N_q^{n+p}(c)$ if it is a critical submanifold to the Willmore functional $W(\varphi) = \int_M \rho^2 \, dv = \int_M (S - nH^2) \frac{2}{3} \, dv$, where $S$ and $H$ denote the norm square of the second fundamental form and the mean curvature of $M$ and $\rho^2 = S - nH^2$. If $q = p$, in [14], we proved some integral inequalities of Simons’ type and rigidity theorems for $n$-dimensional Willmore spacelike submanifolds in a Lorentzian space form $N_p^{n+p}(c)$. In this paper, we continue to study this topic and prove some integral inequalities of Simons’ type and rigidity theorems for $n$-dimensional Willmore spacelike submanifolds in an indefinite space form $N_q^{n+p}(c)$ ($1 \leq q < p$).

1. Introduction

Let $N_q^{n+p}(c)$ be an $(n+p)$-dimensional connected indefinite space form of index $q$ ($1 \leq q \leq p$) and of constant curvature $c$. If $c > 0$, $c = 0$ or $c < 0$, it is denoted by $S_q^{n+p}(c)$, $R_q^{n+p}$ or $H_q^{n+p}(c)$. A submanifold $M$ in $N_q^{n+p}(c)$ is said to be spacelike if the induced metric on $M$ from that of the ambient space is positive definite. Let $\varphi : M \to N_q^{n+p}(c)$ be an $n$-dimensional spacelike submanifold in $N_q^{n+p}(c)$. If $q = p$ and $M$ is a complete maximal spacelike submanifold in $N_p^{n+p}(c)$, from [6], we know that $M$ is totally geodesic for $c \geq 0$, thus the class of all such submanifolds are very small. If $0 \leq q < p$, from [1] and [4], we know that if $M$ is a complete minimal submanifold in sphere $S^m(c)$ $m > n$, which is embeded in $S_q^{n+q}(c)$ as a totally geodesic spacelike submanifold such that $m - n + q = p$, then $M$ is a complete maximal spacelike submanifold in $S_q^{n+p}(c)$, thus, we see...
that the class of complete maximal spacelike submanifold in $S^{n+p}_{q}(c)$ is very large. Therefore, if $0 \leq q < p$, the topic of studying spacelike submanifold in $S^{n+p}_{q}(c)$ is also interesting and important. But as far as we know, the results of this topic are less well established. In [1], Alias and Romero studied compact maximal spacelike submanifold $M$ in $S^{n+p}_{q}(c)$ and proved that if the Ricci curvature of $M$ satisfying $Ric(M) \geq (n-1)c$, then $M$ is totally geodesic. Cheng-Ishikawa [4] also studied compact maximal spacelike submanifold in $S^{n+p}_{q}(c)$ and obtained some important results in terms of the pinching conditions on scalar curvature, sectional curvature and Ricci curvature, respectively.

Denote by $h_{ij}$, $S$, $\vec{H}$ and $H$ the second fundamental form, the norm square of the second fundamental form, the mean curvature vector and the mean curvature of $M$ and denote by $\rho^{2}$ the nonnegative function $\rho^{2} = S - nH^{2}$, we define the Willmore functional (see [2, 8, 11]):

$$W(\varphi) = \int_{M} \rho^{2} dv = \int_{M} (S - nH^{2})\bar{\varphi} dv,$$

which vanishes if and only if $M$ is a totally umbilical spacelike submanifold. It was shown in [9] that this functional is an invariant under the conformal transformations of a conformal space. The points of $M$ are called the critical points of Willmore functional $W(\varphi)$ if $W'(\varphi) = 0$. If the critical points of $W(\varphi)$ are submanifolds in $N^{n+p}_{q}(c)$, we call them Willmore spacelike submanifolds. Obviously, we notice that the totally umbilical spacelike submanifold is Willmore spacelike submanifold, but, conversely, it is not true.

Since any minimal submanifold in a unit sphere $S^{n+p}_{q}(c)$ is not necessarily Willmore submanifold, due to their backgrounds in mathematics, we know that Willmore submanifolds in a unit sphere have been extensively studied in recent years (see [8] and [13]). In indefinite or Lorentzian geometry, we also see that any maximal spacelike submanifold in $N^{n+p}_{q}(c)$ ($1 \leq q \leq p$) is not necessarily Willmore spacelike submanifold, thus the study of Willmore spacelike submanifold in $N^{n+p}_{q}(c)$ ($1 \leq q \leq p$) is also interesting and important. In [14], if $q = p$, we proved some integral inequalities of Simons’ type and rigidity theorems for $n$-dimensional Willmore spacelike submanifolds in a Lorentzian space form $N^{n+p}_{p}(c)$. In this paper, we shall continue to study this topic and prove some integral inequalities of Simons’ type and rigidity theorems for $n$-dimensional Willmore spacelike submanifolds in an indefinite space form $N^{n+p}_{q}(c)$ ($1 \leq q < p$).

Denote by $K$ and $Q$ the functions which assign to each point of $M$ the infimum of the sectional curvature and the Ricci curvature at the point, we obtain the following:

**Theorem 1.1.** Let $\varphi : M \to N^{n+p}_{q}(c)$ be an $n(n \geq 2)$-dimensional compact Willmore spacelike submanifold in the indefinite space form $N^{n+p}_{q}(c)$, $c > 0$ and $1 \leq q < p$.

(1) If $p - q = 1$, then

(1.1) $$\int_{M} \rho^{p} \{n(c - H^{2}) - \left(2 - \frac{1}{p}\right)\rho^{2}\} dv \leq 0.$$
In particular, if
\[ \rho^2 \leq \frac{n}{2 - \frac{1}{p}}(c - H^2), \]
then \( M \) is totally umbilical or \( M \) lies in the totally geodesic spacelike submanifold \( S^{n+1}(c) \) of \( S_q^{n+q+1}(c) \) and is isometric to the Clifford torus \( S^k(\frac{1}{\sqrt{2}}c) \times S^k(\frac{1}{\sqrt{2}}c) \);

(2) If \( p - q > 1 \), then
\[ \int M \rho^n \left\{ n(c - H^2) - \frac{3}{2} \rho^2 \right\} dv \leq 0. \]
In particular, if
\[ \rho^2 \leq \frac{2n}{3}(c - H^2), \]
then \( M \) is totally umbilical or \( M \) lies in the totally geodesic spacelike submanifold \( S^4(c) \) of \( S_q^{4+q}(c) \) and is isometric to the Veronese surface.

**Theorem 1.2.** Let \( \varphi : M \rightarrow N_q^{n+p}(c) \) be an \( n(n \geq 2) \)-dimensional compact Willmore spacelike submanifold in the indefinite space form \( N_q^{n+p}(c) \) (\( 1 \leq q < p \)). Then the following integral inequality holds
\[ \int_M \rho^n \left\{ K - \frac{n-2}{\sqrt{n(n-1)}} H \rho - \frac{1}{n} \left( 1 - \frac{1}{p-q} \right) \rho^2 \right\} dv \leq 0. \]
In particular, if
\[ K \geq \frac{n-2}{\sqrt{n(n-1)}} H \rho + \frac{1}{n} \left( 1 - \frac{1}{p-q} \right) \rho^2, \]
then \( M \) is totally umbilical or \( M \) is a maximal spacelike submanifold in \( N_q^{n+p}(c) \) with parallel second fundamental form.

**Theorem 1.3.** Let \( \varphi : M \rightarrow N_q^{n+p}(c) \) be an \( n(n \geq 2) \)-dimensional compact Willmore spacelike submanifold in the indefinite space form \( N_q^{n+p}(c) \) (\( 1 \leq q < p \)). Then the following integral inequality holds
\[ \int_M \rho^n \left\{ Q - (n-2)c - nH^2 - \frac{1}{n} \left( 3 - \frac{p+q}{(p-q)q} \right) \rho^2 \right\} dv \leq 0. \]
In particular, if
\[ Q \geq (n-2)c + nH^2 + \frac{1}{n} \left( 3 - \frac{p+q}{(p-q)q} \right) \rho^2, \]
then \( M \) is totally umbilical or \( M \) is a maximal spacelike submanifold in \( N_q^{n+p}(c) \) with parallel second fundamental form.
2. Preliminaries

Let \( N^{n+p}_q(c) \) be an \((n+p)\)-dimensional indefinite space form with index \( q(1 \leq q \leq p) \). Let \( M \) be an \( n \)-dimensional connected spacelike submanifold immersed in \( N^{n+p}_q(c) \). We choose a local field of semi-Riemannian orthonormal frames \( e_1, \ldots, e_{n+p} \) in \( N^{n+p}_q(c) \) such that at each point of \( M \), \( e_1, \ldots, e_n \) span the tangent space of \( M \) and form an orthonormal frame there. We use the following convention on the range of indices:

\[ 1 \leq A, B, C, \ldots \leq n + p, \quad 1 \leq i, j, k, \ldots \leq n, \quad n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n + p. \]

Let \( \omega_1, \ldots, \omega_{n+p} \) be its dual frame field so that the semi-Riemannian metric of \( N^{n+p}_q(c) \) is given by \( ds^2 = \sum_A \varepsilon_A \omega_A^2 \), where \( \varepsilon_A = 1 \) for \( 1 \leq A \leq n \) and \( \varepsilon_A = -1 \) for \( n + p - q + 1 \leq A \leq n + p \). Then the structure equations of \( N^{n+p}_q(c) \) are given by

\[
(2.1) \quad d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,
\]

\[
(2.2) \quad d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D,
\]

\[
(2.3) \quad K_{ABCD} = \varepsilon_A \varepsilon_B (\delta_{AD} \delta_{BC} - \delta_{AC} \delta_{BD}).
\]

If we restrict these form to \( M \), then \( \omega_\alpha = 0, n + 1 \leq \alpha \leq n + p \) and

\[
(2.4) \quad \omega_\alpha = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.
\]

The second fundamental form \( II \), the mean curvature vector \( \vec{H} \) of \( M \) are defined by

\[
(2.5) \quad II = \sum_{\alpha,i,j} \varepsilon_\alpha h_{ij}^\alpha \omega_j \varepsilon_\alpha, \quad \vec{H} = \sum_{\alpha} \varepsilon_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{n} \sum_{\kappa} h_{\kappa \kappa}^\alpha.
\]

The norm square of the second fundamental form and the mean curvature of \( M \) are defined by

\[
(2.6) \quad S = |II|^2 = \sum_{i,j,\alpha} (\varepsilon_\alpha h_{ij}^\alpha)^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2, \quad H = |\vec{H}| = \frac{1}{n} \sqrt{\sum_{\alpha} \left( \sum_{\kappa} h_{\kappa \kappa}^\alpha \right)^2}.
\]

The Gauss equations are

\[
(2.7) \quad R_{ijkl} = c(\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) + \sum_\alpha \varepsilon_\alpha (h_{il}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{jl}^\alpha),
\]

\[
(2.8) \quad R_{jk} = (n - 1) c \delta_{jk} + \sum_\alpha \varepsilon_\alpha \left( \sum_i h_{ii}^\alpha h_{ij}^\alpha - \sum_i h_{ik}^\alpha h_{ji}^\alpha \right).
\]
Defining the first and the second covariant derivatives of \( h^\alpha_{ij} \), say \( h^\alpha_{ijk} \) and \( h^\alpha_{ijkl} \) by
\[
\sum_k h^\alpha_{ijk\omega k} = dh^\alpha_{ij} - \sum_k h^\alpha_{ik\omega kj} - \sum_k h^\alpha_{jk\omega ki} - \sum_\beta \varepsilon_\beta h^\alpha_{ij\omega \beta \alpha},
\]
\[
\sum_l h^\alpha_{ijkl\omega l} = dh^\alpha_{ijlk} - \sum_m h^\alpha_{im\omega m\omega j} - \sum_m h^\alpha_{jm\omega mj} - \sum_m h^\alpha_{ijm\omega mk} - \sum_\beta \varepsilon_\beta h^\alpha_{ijkl\omega \beta \alpha},
\]
we have the Codazzi equations and the Ricci identities
\[
h^\alpha_{ijk} = h^\alpha_{ikj},
\]
\[
h^\alpha_{ijkl} - h^\alpha_{ijlk} = - \sum_m h^\alpha_{imR_{mjkl}} - \sum_m h^\alpha_{jmR_{mikl}} - \sum_\beta \varepsilon_\beta h^\alpha_{ijklR_{\beta kl}}.
\]
The Ricci equations are
\[
R_{\alpha\beta ij} = - \sum_m (h^\alpha_{im} h^\beta_{mj} - h^\alpha_{jm} h^\beta_{mi}).
\]
The Laplacian of \( h^\alpha_{ij} \) is defined by \( \Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk} \). From (2.12), we obtain for any \( \alpha, n+1 \leq \alpha \leq n+p \),
\[
\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kki\delta ij} - \sum_{k,m} h^\alpha_{kmR_{mjkl}} - \sum_{k,m} h^\alpha_{jmR_{mikl}} - \sum_\beta \varepsilon_\beta h^\alpha_{ijklR_{\beta kl}}.
\]
For the fix index \( \alpha(n+1 \leq \alpha \leq n+p) \), we introduce an operator \( \Box^\alpha \) due to Cheng-Yau [3] by
\[
\Box^\alpha f = \sum_{i,j} (nH^\alpha_{\delta ij} - h^\alpha_{ij}) f_{i,j}.
\]
Since \( M \) is compact, the operator \( \Box^\alpha \) is self-adjoint (see [3]) if and only if
\[
\int_M (\Box^\alpha f) gdv = \int_M f(\Box^\alpha g) dv,
\]
where \( f \) and \( g \) are any smooth functions on \( M \). We need the following Lemma (see [12]):

**Lemma 2.1.** Let \( A, B \) be symmetric \( n \times n \) matrices satisfying \( AB = BA \) and \( \text{tr}A = \text{tr}B = 0 \). Then
\[
|\text{tr}A^2B| \leq \frac{n-2}{\sqrt{n(n-1)}} (\text{tr}A^2)(\text{tr}B^2)^{1/2},
\]
and the equality holds if and only if \( (n-1) \) of the eigenvalues \( x_i \) of \( B \) and the corresponding eigenvalues \( y_i \) of \( A \) satisfy \( |x_i| = (\text{tr}B^2)^{1/2}/\sqrt{n(n-1)} \), \( x_i x_j \geq 0 \), \( y_i = (\text{tr}A^2)^{1/2}/\sqrt{n(n-1)} \).

By the same method as in the proof of Lemma 4.2 in [8], we also have the following:
Lemma 2.2. Let $\varphi : M \to N_q^{n+p}(c)$ be an $n$-dimensional ($n \geq 2$) spacelike submanifold in $N_q^{n+p}(c)(1 \leq q \leq p)$. Then we have

$$(2.18) \Rightarrow |\nabla h|^2 \geq \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2,$$

where $|\nabla h|^2 = \sum_{i,j,k,\alpha} (h_{i,j,k}^\alpha)^2$, $|\nabla^\perp \vec{H}|^2 = \sum_{i,\alpha} (H_i^\alpha)^2$.

3. Euler-Lagrange equation of Willmore spacelike submanifolds

From Theorem 4.1 of [9], we know that the Euler-Lagrange equation of Willmore spacelike submanifolds in terms of invariants of conformal metric $g$ is stated as following: a spacelike submanifold is a Willmore spacelike submanifold if and only if

$$(3.1) \sum_{i,j,k,l,\alpha} g_{ij} g^{kl} g^{\alpha \beta} \left( B_i^\beta_{kj} + A_{ij} B_{k\beta}^{\gamma} + \sum_{r,q,\tau,\nu} g_{r\tau} g_{q\nu} B_{ij}^r B_{ij}^q B_{ij}^\tau B_{ij}^\nu \right) = 0, \quad \forall \alpha,$$

where $1 \leq i, j, k, l, r, q \leq n$, $n + 1 \leq \alpha, \beta, \gamma, \nu \leq n + p$, $(g_{ij}) = (I_n)$, $(g_{ij}) = (I_{p-q} \oplus (-I_q))$, $(g_{ij}) = (g_{ij})^{-1}$ and $(g_{ij}) = (g_{ij})^{1-1}$ (see [9]). From (3.23) in [9], we have $(1-n)C_i^\alpha = \sum_k B_{ik}^\alpha$. Thus, by a simply calculation, we may rewrite (3.1) as

$$(3.2) \quad (1-n) \sum_i C_{i,i}^\alpha + \sum_{i,j} A_{ij} B_{ij}^\alpha + \sum_{\beta} \sum_{i,j,k} \varepsilon_{\beta} B_{ik}^\beta B_{ij}^\beta = 0, \quad \forall \alpha,$$

where $\varepsilon_{\beta} = g_{\beta\beta}$ and $\varepsilon_{\beta} = 1$ for $n+1 \leq \beta \leq n+p-q$ and $\varepsilon_{\beta} = -1$ for $n+p-q+1 \leq \beta \leq n + p$.

From [9] or [10], we have the following relations of the connections of the conformal metric $e^{2\tau} du \cdot du$ and induced metric $du \cdot du$

$$(3.3) \quad \omega = e^\tau \theta, \quad \omega_{ij} = \theta_{ij} + \tau_i \theta_j - \tau_j \theta_i, \quad \omega_{\alpha\beta} = \theta_{\alpha\beta},$$

where $e^{2\tau} = \frac{n}{n+1} (S - nH^2)$. We know that the relations of the conformal invariants and the induced invariants are

$$(3.4) \quad e^{2\tau} C_i = H^\alpha \tau_i - H_i^\alpha - \sum_j h_{ij}^\alpha \tau_j,$$

$$(3.5) \quad e^{2\tau} A_{ij} = \tau_i \tau_j - \tau_{ij} - \sum_{\alpha} H^\alpha h_{ij}^\alpha - \frac{1}{2} \left( \sum_k \tau_k \tau_k - H^2 - c \right) I_{ij},$$

$$(3.6) \quad e^{2\tau} B_{ij}^\alpha = h_{ij}^\alpha - H^\alpha I_{ij},$$

where $\tau_{ij}$ is Hessian of $\tau$ with respect to the first fundamental form $I$, $\tau_i = \sum_j I^{ij} \tau_j$, $(I^{ij}) = (I_{ij})^{-1}$, $H_i^\alpha = e_{i}(H^\alpha)$ and $c = 0$ for $R_q^{n+p}(c)$, $c > 0$ for $S_q^{n+p}(c)$ and $c < 0$ for $H_q^{n+p}(c)$ (see [10])
From (3.3) and (3.4), by a similar calculation of Li [8], we have
\[ \sum_j e^\tau C_i^\alpha \theta_j = \sum_j C_i^{\alpha \omega_j} = dC_i^\alpha + \sum_j C_j^\alpha \omega_{ji} + \sum_\beta C_i^\beta \omega_{\beta \alpha} \]
\[ = dC_i^\alpha + \sum_j C_j^\alpha \theta_{ji} + \sum_j C_j^\alpha (\tau_j \theta_i - \tau_i \theta_j) + \sum_\beta C_i^\beta \theta_{\beta \alpha}, \]
therefore, we have
\[
(3.7) \quad e^\tau C_i^\alpha = e^{-2\tau} \left( -2H^\alpha \tau_i \tau_j + 2\tau_j \sum_k h_{ik}^\alpha \tau_k + 2\tau_j H_i^\alpha \tau_i + H_j^\alpha \tau_i \right)
- \sum_k h_{ik}^\alpha \tau_k - \sum_k h_{ik}^\alpha \tau_{k,j} - H_{ij}^\alpha \right) + \sum_k C_k^\alpha \tau_k \delta_{ij} - \tau_i C_j^\alpha.
\]
From (3.7), we see that
\[
(3.8) \quad e^\tau \sum_i C_i^\alpha = (n - 3) \left( |H^\alpha| \nabla^2 \tau^2 - \sum_{i,k} h_{ik}^\alpha \tau_k \tau_i \right)
- 2(n - 2) \sum_i H_i^\alpha \tau_i + H^\alpha \Delta \tau - \sum_{i,k} h_{ik}^\alpha \tau_{k,i} - \Delta^\perp H^\alpha.
\]
From (3.5) and (3.6), we have
\[
(3.9) \quad e^{3\tau} \left( \sum_{i,j} A_{ij} B_{ij}^\alpha + \sum_\beta \sum_{i,j,k} \varepsilon_\beta B_{ik}^\beta B_{kj}^\beta B_{ij}^\beta \right)
= \sum_{i,j} \left[ \tau_i \tau_j - \tau_{i,j} - \sum_\beta H_{ij}^\beta I_j \right] \left( h_{ij}^\alpha - H^\alpha I_{ij} \right)
+ \sum_\beta \sum_{i,j,k} \varepsilon_\beta \left( h_{ik}^\beta - H_{ik}^\beta \right) \left( h_{kj}^\beta - H_{kj}^\beta \right) \left( h_{ij}^\beta - H_{ij}^\beta I_{ij} \right)
= \sum_{i,j} h_{ij}^\beta \left( \tau_i \tau_j - \tau_{i,j} \right) + H^\alpha \left( \Delta \tau - |\nabla \tau|^2 + n \sum_\beta (1 + 2 \varepsilon_\beta) (H^\beta)^2 \right)
+ \sum_\beta \sum_{i,j,k} \varepsilon_\beta h_{ik}^\beta h_{kj}^\beta h_{ij}^\alpha - \sum_{\beta, i,j} \sum_\beta (1 + 2 \varepsilon_\beta) H^\beta h_{ij}^\beta h_{ij}^\alpha - H^\alpha \sum_{\beta, i,j} \varepsilon_\beta (h_{ij}^\beta)^2.
\]
From (3.2), (3.8) and (3.9), we see that
\[
(3.10) \quad (n - 2) \left( \sum_{i,j} h_{ij}^\alpha \tau_i \tau_j - H^\alpha |\nabla \tau|^2 \right) + 2(n - 1)(n - 2) \sum_i H_i^\alpha \tau_i
+ (n - 2) \left( \sum_{i,j} h_{ij}^\alpha \tau_{i,j} - H^\alpha \Delta \tau \right) + (n - 1) \Delta^\perp H^\alpha
- H^\alpha \sum_{\beta, i,j} \sum_\beta (1 + 2 \varepsilon_\beta) H^\beta h_{ij}^\beta h_{ij}^\alpha
+ n H^\alpha \sum_{\beta} (1 + 2 \varepsilon_\beta) (H^\beta)^2 + \sum_{\beta, i,j,k} \varepsilon_\beta h_{ik}^\beta h_{kj}^\beta h_{ij}^\beta = 0.
\]
Putting $\rho^2 = S - nH^2$, we have $e^{2\tau} = \frac{n}{n-1}(S - nH^2) = \frac{n}{n-1}\rho^2$. Thus $e^\tau = \sqrt{\frac{n}{n-1}\rho}$ and $\tau = \ln(\sqrt{\frac{n}{n-1}\rho})$. From (3.10), we see that

$$
\frac{\rho^{n-2}}{n-1}\left\{ -H^\alpha \sum_{\beta,i,j} \varepsilon_\beta (h^\beta_{ij})^2 - \sum_{\beta,i,j} (1 + 2\varepsilon_\beta)H^\beta h^\beta_{ij} h^\beta_{ij} \\
+ \sum_{\beta} \sum_{i,j,k} \varepsilon_\beta h^{\alpha}_{ik} h^\beta_{kj} h^{\alpha}_{ij} + nH^\alpha \sum_{\beta} (1 + 2\varepsilon_\beta)(H^\beta)^2 \right\} \\
+ \frac{\rho^{n-2}}{n-1} \Delta H^\alpha + \frac{n-2}{n-1} \rho^{n-2} \sum_{i,j} (\ln \rho)_{i,j} \left( h^{\alpha}_{ij} - H^\alpha \delta_{ij} \right) \\
+ 2(n-2)\rho^{n-2} \sum_{i} (\ln \rho)_{i} H^\alpha_{i} \\
+ \frac{(n-2)^2}{n-1} \rho^{n-2} \sum_{i,j} (\ln \rho)_{i}(\ln \rho)_{j} \left( h^{\alpha}_{ij} - H^\alpha \delta_{ij} \right) = 0.
$$

It can be easily checked that

$$
\frac{\rho^{n-2}}{n-1} \Delta H^\alpha + \frac{n-2}{n-1} \rho^{n-2} \sum_{i,j} (\ln \rho)_{i,j} \left( h^{\alpha}_{ij} - H^\alpha \delta_{ij} \right) + 2(n-2)\rho^{n-2} \sum_{i} (\ln \rho)_{i} H^\alpha_{i} \\
+ \frac{(n-2)^2}{n-1} \rho^{n-2} \sum_{i,j} (\ln \rho)_{i}(\ln \rho)_{j} \left( h^{\alpha}_{ij} - H^\alpha \delta_{ij} \right) \\
= - \frac{1}{n-1} \sum_{i,j} (\rho^{n-2})_{i,j} \left( nH^\alpha \delta_{ij} - h^{\alpha}_{ij} \right) + \rho^{n-2} \Delta H^\alpha \\
+ 2 \sum_{i} (\rho^{n-2})_{i} H^\alpha_{i} + H^\alpha \Delta(\rho^{n-2}).
$$

From (3.11) and (3.12), we may obtain the Euler-Lagrange equation of Willmore spacelike submanifolds in $N^{n+p}(c)$ in terms of the induced invariants:

**Theorem 3.1.** Let $\varphi: M \to N^{n+p}(c)$ be an $n$-dimensional spacelike submanifold in $N^{n+p}(c)$. Then $M$ is an $n$-dimensional Willmore spacelike submanifold if and only if for $n+1 \leq \alpha, \beta \leq n+p$,

$$
\rho^{n-2}\left\{ -H^\alpha \sum_{\beta,i,j} \varepsilon_\beta (h^\beta_{ij})^2 - \sum_{\beta,i,j} (1 + 2\varepsilon_\beta)H^\beta h^\beta_{ij} h^\beta_{ij} \\
+ \sum_{\beta} \sum_{i,j,k} \varepsilon_\beta h^{\alpha}_{ik} h^\beta_{kj} h^{\alpha}_{ij} + nH^\alpha \sum_{\beta} (1 + 2\varepsilon_\beta)(H^\beta)^2 \right\} \\
+ (n-1)\rho^{n-2} \Delta H^\alpha + 2(n-1) \sum_{i} (\rho^{n-2})_{i} H^\alpha_{i} \\
+ (n-1)H^\alpha \Delta(\rho^{n-2}) - \Box^\alpha(\rho^{n-2}) = 0.
$$
In the proof of (3.13), since we denote every maximal spacelike surface
\[ \sum_{i,j}(\rho^{n-2})_{i,j}, \Box^{\alpha}(\rho^{n-2}) = \sum_{i,j}(nH^{\alpha} \delta_{ij} - h_{ij}^{\alpha}) \]
and \((\rho^{n-2})_{i,j}\) is the Hessian of \(\rho^{n-2}\) with respect to the induced metric.

**Remark 3.1.** In the proof of (3.13), since we denote \(e^{2\sigma} = \frac{n}{n-1}(S - nH^{2}) = \frac{n}{n-1}\rho^{2}\), it follows that \(\rho^{2} \neq 0\), that is, (3.13) holds only for \(\rho^{2} \neq 0\). But, if \(\rho^{2} = 0\), we should notice that (3.13) also holds. Thus, in the following discussion, we agree that the Euler-Lagrange equation of Willmore spacelike submanifolds (3.13) holds for all \(\rho^{2}\). But, if \(n = 3\) and \(n = 5\), we need assume that \(M\) has no umbilical points to guarantee \((\rho^{n-2})_{i,j}\) is continuous on \(M\).

**Proposition 3.1.** Every maximal spacelike surface \(\varphi : M \rightarrow N_{q}^{2+p}(c)\) in \(N_{q}^{2+p}(c)\) is Willmore spacelike surface.

In fact, if \(n = 2\), since \(H = 0\), from (2.5), we see that \(H^{\alpha} = 0\) and \(\sum_{k}h_{kk}^{\alpha} = 0\). On the other hand, since \(R_{ij} = \frac{R}{2}\delta_{ij}\), from Gauss equation (2.8), we have
\[ \sum_{\beta,j}i\beta h_{i}^{\beta}h_{i}^{\beta} = c\delta_{ik} + \sum_{\beta,j}i\beta h_{i}^{\beta}h_{i}^{\beta} - R_{ik}, \]
which is embeded in \(\sum_{i,k}h_{ik}^{\alpha} = 0\). But, if \(n = 2\) and \(n = 5\), we need assume that \(M\) has no umbilical points to guarantee \((\rho^{n-2})_{i,j}\) is continuous on \(M\).

**Example 3.1.** If \(0 \leq q < p = q + 1\), since we know that the Clifford torus \(S^{k}(\sqrt{\frac{k}{n}}c) \times S^{n-k}(\sqrt{\frac{n-k}{n}}c)\) is a complete minimal hypersurface in sphere \(S^{n+1}(c)\) which is embeded in \(S_{q}^{n+q}(c)\) as a totally geodesic spacelike submanifold such that \(1 + q = p\), then \(S^{k}(\sqrt{\frac{k}{n}}c) \times S^{n-k}(\sqrt{\frac{n-k}{n}}c)\) is a complete maximal spacelike submanifold in \(S_{q}^{n+q}(c)\), where \(1 \leq k \leq n - 1\). Since \(S^{k}(\sqrt{\frac{k}{n}}c) \times S^{n-k}(\sqrt{\frac{n-k}{n}}c)\) lies in the totally geodesic spacelike submanifold \(S^{n+1}(c)\) of \(S_{q}^{n+q}(c)\), we know that \(h_{ij}^{\alpha} = 0\) for \(\alpha = n + 2, \ldots, n + q + 1\). Thus, if and only if \(n = 2k\) then
\[ \sum_{\beta,i,j,k}i\beta h_{ik}^{\alpha}h_{j}^{\beta}h_{i}^{\beta} = \sum_{i,j,k}h_{ik}^{n+1}h_{j}^{n+1}h_{ij}^{n+1} = \sum_{i,k}\lambda_{i}^{3}, \]
where \(h_{ij}^{n+1} = \lambda_{ij}^{3}\), \(\sqrt{\frac{n-k}{n}}c\) and \(-\sqrt{\frac{k}{n-k}}c\) are the two distinct principal curvatures of \(S^{k}(\sqrt{\frac{k}{n}}c) \times S^{n-k}(\sqrt{\frac{n-k}{n}}c) \subset S^{n+1}(c)\) with multiplicities \(k\) and \(n-k\), respectively. We also see that \(\rho^{2} = S - nH^{2} = \sum_{i}\lambda_{i}^{2} = nc\) is constant. Thus, (3.13) holds if and only if \(n = 2k\), that is the Clifford torus \(S^{k}(\sqrt{\frac{k}{n}}c) \times S^{k}(\sqrt{\frac{1}{n-k}}c), 1 \leq k \leq n - 1\), is a maximal Willmore spacelike submanifold in \(S_{q}^{n+q+1}(c)\).
Example 3.2. From [5] and [1], we know that the Veronese surface is a minimal surface in $S^4(c)$ which is embedded in $S_4^{2+q}(c)$ as a totally geodesic spacelike submanifold such that $2 + q = p$, then the Veronese surface is a maximal spacelike surface in $S^{2+q}_q(c)$, where $p = 2 + q$. From Proposition 3.1, we know that it is a Willmore spacelike surface in $S^{2+q}_q(c)$.

4. Basic integral equalities

Define tensors

\begin{equation}
\tilde{h}^\alpha_{ij} = h^\alpha_{ij} - H^\alpha \delta_{ij},
\end{equation}

\begin{equation}
\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}^\alpha_{ij} \tilde{h}^\beta_{ij}, \quad \sigma_{\alpha\beta} = \sum_{i,j} h^\alpha_{ij} h^\beta_{ij}.
\end{equation}

Then the $(p \times p)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonalized for a suitable choice of $e_{n+1}, \cdots, e_{n+p}$. We set

\begin{equation}
\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\alpha\beta}.
\end{equation}

By a direct calculation, we have

\begin{equation}
\sum_k \tilde{h}^\alpha_{kk} = 0, \quad \tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} - n H^\alpha H^\beta, \quad \rho^2 = \sum_{\alpha} \tilde{\sigma}_{\alpha} = S - n H^2,
\end{equation}

\begin{equation}
-H^\alpha \sum_{\beta, i, j} \varepsilon_{\beta}(h^\beta_{ij})^2 - \sum_{\beta, i, j} (1 + 2 \varepsilon_{\beta}) H^\beta h^\beta_{ij} h^\alpha_{ij}
+ \sum_{\beta} \sum_{i, j, k} \varepsilon_{\beta} h^\alpha_{ik} h^\beta_{kj} \tilde{h}^\alpha_{ij} + n H^\alpha \sum_{\beta} (1 + 2 \varepsilon_{\beta})(H^\beta)^2
= \sum_{\beta} \sum_{i, j, k} \varepsilon_{\beta} \tilde{h}^\alpha_{ik} \tilde{h}^\beta_{kj} \tilde{h}^\alpha_{ij} - \sum_{i, j, \beta} H^\beta \tilde{h}^\beta_{ij} \tilde{h}^\alpha_{ij}.
\end{equation}

From (4.1),(4.4) and (4.5), the Euler-Lagrange equation (3.13) can be rewritten as

Proposition 4.1. Let $\varphi : M \to N^{n+p}_q(c)$ be an $n$-dimensional spacelike submanifold in $N^{n+p}_q(c)$. Then $M$ is a Willmore spacelike submanifold if and only if for $n + 1 \leq \alpha \leq n + p$

\begin{equation}
\Box^\alpha (\rho^{n-2}) = (n - 1) \rho^{n-2} \Delta^\perp H^\alpha + 2(n - 1) \sum_i (\rho^{n-2})_{1i} H^\alpha_{1i}
+ (n - 1) H^\alpha \Delta (\rho^{n-2}) + \rho^{n-2} \left( \sum_{\beta} \sum_{i, j, k} \varepsilon_{\beta} \tilde{h}^\alpha_{ik} \tilde{h}^\beta_{kj} \tilde{h}^\alpha_{ij} - \sum_{i, j, \beta} H^\beta \tilde{h}^\beta_{ij} \tilde{h}^\alpha_{ij} \right).
\end{equation}

Setting $f = n H^\alpha$ in (2.15), we have

\begin{equation}
\Box^\alpha (n H^\alpha) = \sum_{i, j} (n H^\alpha \delta_{ij} - h^\alpha_{ij})(n H^\alpha)_{i,j}
= \sum_{i, j} (n H^\alpha)(n H^\alpha)_{i,j} - \sum_{i, j} h^\alpha_{ij}(n H^\alpha)_{i,j}.
\end{equation}
We also have
\[
\frac{1}{2} \Delta (nH)^2 = \frac{1}{2} \sum_\alpha (nH^\alpha)^2 = \frac{1}{2} \sum_\alpha \Delta (nH^\alpha)^2 \\
= \frac{1}{2} \sum_\alpha [(nH^\alpha)^2]_{i,i} = \sum_\alpha [(nH^\alpha)_i]^2 + \sum_\alpha (nH^\alpha)(nH^\alpha)_i,i \\
= n^2 |\nabla^i \vec{H}|^2 + \sum_\alpha (nH^\alpha)(nH^\alpha)_i,i.
\]

Therefore, from (4.7), (4.8), we get
\[
\sum_\alpha \Box (nH^\alpha) = \frac{1}{2} \Delta (nH)^2 - n^2 |\nabla^i \vec{H}|^2 = \sum_\alpha h^\alpha_{ij}(nH^\alpha)_{i,j} \\
= \frac{1}{2} \Delta [n(n-1)H^2 - \rho^2 + S] - n^2 |\nabla^i \vec{H}|^2 - \sum_\alpha h^\alpha_{ij}(nH^\alpha)_{i,j} \\
= \frac{1}{2} \Delta S + \frac{1}{2} n(n-1)\Delta H^2 - \frac{1}{2} \Delta \rho^2 - n^2 |\nabla^i \vec{H}|^2 - \sum_\alpha h^\alpha_{ij}(nH^\alpha)_{i,j}.
\]

From (2.11) and (2.12), we have
\[
\frac{1}{2} \Delta S = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + \sum_{i,j,\alpha} h^\alpha_{ij} \Delta h^\alpha_{ij} \\
= |\nabla h|^2 + \sum_{i,j,\alpha} h^\alpha_{ij} (nH^\alpha)_{i,j} - \sum_{\alpha} \sum_{i,j,k,l} h^\alpha_{ij} h_{k\ell} R_{ij\ell k} \\
- \sum_{\alpha} \sum_{i,j,k,l} h^\alpha_{ij} h_{k\ell} R_{ij\ell k} - \sum_{\alpha,\beta} \sum_{i,j,k} \varepsilon h^\alpha_{ij} h^\beta_{k\ell} R_{ij\ell k}.
\]

Putting (4.10) into (4.9), we have
\[
\sum_\alpha \Box (nH^\alpha) = |\nabla h|^2 - n^2 |\nabla^i \vec{H}|^2 + \frac{1}{2} n(n-1)\Delta H^2 - \frac{1}{2} \Delta \rho^2 \\
- \sum_{\alpha} \sum_{i,j,k,l} h^\alpha_{ij} (h^\alpha_{k\ell} R_{ij\ell k} + h^\alpha_{ij} R_{ij\ell k}) - \sum_{\alpha,\beta} \sum_{i,j,k} \varepsilon h^\alpha_{ij} h^\beta_{k\ell} R_{ij\ell k}.
\]

Multiplying (4.11) by $\rho^{n-2}$ and taking integration, using (2.16), we have
\[
\sum_\alpha \int_M (nH^\alpha) \Box (\rho^{n-2}) dv = \int_M \rho^{n-2} |\nabla h|^2 - n^2 |\nabla^i \vec{H}|^2 dv \\
+ \frac{1}{2} n(n-1) \int_M \rho^{n-2} \Delta H^2 dv - \frac{1}{2} \int_M \rho^{n-2} \Delta \rho^2 dv \\
- \int_M \rho^{n-2} \sum_{\alpha} \sum_{i,j,k,l} h^\alpha_{ij} (h^\alpha_{k\ell} R_{ij\ell k} + h^\alpha_{ij} R_{ij\ell k}) dv.
\]
\[-\int_M \rho^{n-2} \sum_{\alpha, \beta} \sum_{i,j,k} \varepsilon_{\beta} h_{ij}^\alpha h_{k}^\beta R_{\alpha \beta i,j,k} dv.\]

Taking the Willmore equation (4.6) into (4.12) and making use of the following:

\[
\int_M \rho^{n-2} \sum_{\alpha} H^\alpha \triangle H^\alpha dv = \frac{1}{2} \int_M \rho^{n-2} \sum_{\alpha} \Delta H^\alpha dv - \int_M \rho^{n-2} \sum_{i,\alpha} (H^\alpha_i)^2 dv
\]

\[
= \frac{1}{2} \int_M \rho^{n-2} \Delta H^2 dv - \int_M \rho^{n-2} |\nabla H|^2 dv,
\]

\[
\int_M H^2 \Delta (\rho^{n-2}) dv = \int_M \sum_{\alpha} (H^\alpha)^2 \sum_i (\rho^{n-2})_{i,i} dv
\]

\[
= \sum_{\alpha,i} \int_M (H^\alpha)^2 (\rho^{n-2})_{i,i} dv = -\sum_{\alpha,i} \int_M (\rho^{n-2})_{i,i} ((H^\alpha)^2)_{i,i} dv
\]

\[
= -2 \int_M \sum_{\alpha} H^\alpha \sum_i (\rho^{n-2})_{i,i} H^\alpha_{i,i} dv,
\]

\[-\frac{1}{2} \int_M \rho^{n-2} \Delta \rho^2 dv = -\frac{1}{2} \sum_i \int_M \rho^{n-2} (\rho^2)_{i,i} dv
\]

\[
= \frac{1}{2} \sum_i \int_M (\rho^2)_{i,i} (\rho^{n-2})_{i,i} dv = (n-2) \int_M \rho^{n-2} |\nabla \rho|^2 dv,
\]

we have, by a direct calculation, the following:

**Proposition 4.2.** Let \( \varphi : M \to N^{n+p}(c) \) be an \( n \)-dimensional spacelike submanifold in \( N^{n+p}(c) \). Then

\[
(4.13) \quad \int_M \rho^{n-2} (|\nabla h|^2 - n|\nabla h_{ij}^\alpha|^2) dv + (n-2) \int_M \rho^{n-2} |\nabla \rho|^2 dv
\]

\[-\int_M \rho^{n-2} \sum_{\alpha, \beta} \sum_{i,j,k} \varepsilon_{\beta} h_{ij}^\alpha h_{k}^\beta (\tilde{h}_{ij}^\alpha \tilde{h}_{k}^\beta - H^\beta_{\alpha \beta i,j,k}) dv
\]

\[-\int_M \rho^{n-2} \sum_{\alpha} \sum_{i,j,k,l} h_{ij}^\alpha (h_{kl}^\alpha R_{ij,k} + h_{kl}^\alpha R_{ik,j}) dv
\]

\[-\int_M \rho^{n-2} \sum_{\alpha, \beta} \sum_{i,j,k,l} \varepsilon_{\beta} h_{ij}^\alpha h_{k}^\beta R_{\alpha \beta i,j,k} dv = 0.
\]

In general, for a matrix \( A = (a_{ij}) \) we denote by \( N(A) \) the square of the norm of \( A \), that is, \( N(A) = \text{trace}(A \cdot A^t) = \sum_{i,j} (a_{ij})^2 \). Clearly, \( N(A) = N(T^t AT) \) for any orthogonal matrix \( T \). From (2.13), we have

\[
(4.14) \quad -\sum_{\alpha, \beta} \sum_{i,j,k} \varepsilon_{\beta} h_{ij}^\alpha h_{k}^\beta R_{\alpha \beta i,j,k} = -\sum_{\alpha, \beta} \sum_{i,j,k,l} \varepsilon_{\beta} h_{ij}^\alpha h_{k}^\beta (h_{kl}^\alpha h_{ij}^\alpha - h_{ij}^\beta h_{kl}^\alpha)
\]
\[
\begin{align*}
\sum_{\alpha,\beta,j,k} \epsilon_{\beta} \left( \sum_{l} h_{klj}^{\alpha} - \sum_{l} h_{klji}^{\beta} \right)^{2} \\
= -\frac{1}{2} \sum_{\alpha,\beta,j,k} \epsilon_{\beta} \left( \sum_{l} h_{klj}^{\alpha} - \sum_{l} h_{klji}^{\beta} \right)^{2} \\
= -\frac{1}{2} \sum_{\alpha,\beta,j,k} \epsilon_{\beta} \left( \sum_{l} h_{klj}^{\alpha} - \sum_{l} h_{klji}^{\beta} \right)^{2} \\
= -\frac{1}{2} \sum_{\alpha,\beta} \epsilon_{\beta} N(\tilde{A}_{\alpha} \tilde{A}_{\beta} - \tilde{A}_{\beta} \tilde{A}_{\alpha}),
\end{align*}
\]

where \( \tilde{A}_{\alpha} := (h_{ij}^{\alpha}) = (h_{ij}^{\alpha} - H^{\alpha} \delta_{ij}) \).

By use of (2.7), (2.13), (4.1), (4.2), (4.4) and (4.14), we conclude that

(4.15)
\[
\sum_{\alpha,\beta,j,k,l} h_{ij}^{\alpha} (h_{klj}^{\alpha} R_{lijk} + h_{lj}^{\alpha} R_{lkij})
\]
\[
= n \rho^{2} - \sum_{\alpha,\beta} \epsilon_{\beta} \bar{\sigma}_{\alpha \beta}^{2} + n \sum_{\alpha,\beta} \sum_{i,j,k} \epsilon_{\beta} H^{\beta} h_{ij}^{\alpha} h_{ik}^{\alpha} - \sum_{\alpha,\beta,i,j,k} \epsilon_{\beta} h_{ij}^{\alpha} h_{ik}^{\beta} R_{\beta \alpha jk}
\]
\[
= n \rho^{2} - \sum_{\alpha,\beta} \epsilon_{\beta} \bar{\sigma}_{\alpha \beta}^{2} - 2n \sum_{\alpha,\beta} \sum_{i,j,k} \epsilon_{\beta} H^{\beta} h_{ij}^{\alpha} h_{ik}^{\alpha} - n^{2} \sum_{\alpha} (H^{\alpha})^{2} \sum_{\beta} \epsilon_{\beta} H^{\beta} - n^{2} \sum_{\alpha} (H^{\alpha})^{2} \sum_{\beta} \epsilon_{\beta} H^{\beta}
\]
\[
+ n \sum_{\alpha,\beta} \sum_{i,j,k} \epsilon_{\beta} H^{\beta} h_{ij}^{\alpha} h_{ik}^{\alpha} + n \rho^{2} \sum_{\beta} \epsilon_{\beta} (H^{\beta})^{2} + 2n \sum_{\alpha,\beta} \sum_{i,j} \epsilon_{\beta} H^{\alpha} H^{\beta} h_{ij}^{\alpha} h_{ij}^{\beta}
\]
\[
+ n^{2} \sum_{\alpha} (H^{\alpha})^{2} \sum_{\beta} \epsilon_{\beta} (H^{\beta})^{2} - \frac{1}{2} \sum_{\alpha,\beta} \epsilon_{\beta} N(\tilde{A}_{\alpha} \tilde{A}_{\beta} - \tilde{A}_{\beta} \tilde{A}_{\alpha})
\]
\[
= n \rho^{2} - \sum_{\alpha,\beta} \epsilon_{\beta} \bar{\sigma}_{\alpha \beta}^{2} + n \rho^{2} \sum_{\beta} \epsilon_{\beta} (H^{\beta})^{2} + n \sum_{\alpha,\beta} \sum_{i,j,k} \epsilon_{\beta} H^{\beta} h_{ij}^{\alpha} h_{ik}^{\alpha}
\]
\[
- \frac{1}{2} \sum_{\alpha,\beta} \epsilon_{\beta} N(\tilde{A}_{\alpha} \tilde{A}_{\beta} - \tilde{A}_{\beta} \tilde{A}_{\alpha}).
\]

Putting (4.14) and (4.15) into (4.13), we have the following:

**Proposition 4.3.** Let \( \varphi : M \to N_{q+p}^{n+c} \) be an \( n \)-dimensional spacelike submanifold in \( N_{q+p}^{n+c} \). Then

(4.16)
\[
\int_{M} \rho^{n-2} (|\nabla h|^{2} - n|\nabla^{\perp} \tilde{H}|^{2}) dv + (n-2) \int_{M} \rho^{n-2} |\nabla \rho|^{2} dv
\]
\[
+ n \int_{M} \rho^{n-2} \left( \sum_{\alpha,\beta} H^{\alpha} H^{\beta} \bar{\sigma}_{\alpha \beta} + \rho^{2} \sum_{\beta} \epsilon_{\beta} (H^{\beta})^{2} \right) dv + nc \int_{M} \rho^{n} dv
\]
\[
- \int_{M} \rho^{n-2} \left( \sum_{\alpha,\beta} \epsilon_{\beta} N(\tilde{A}_{\alpha} \tilde{A}_{\beta} - \tilde{A}_{\beta} \tilde{A}_{\alpha}) + \sum_{\alpha,\beta} \epsilon_{\beta} \bar{\sigma}_{\alpha \beta}^{2} \right) dv = 0.
\]
5. Proofs of Theorems

Proof of Theorem 1.1. (1) If \( p - q = 1 \), from Lemma 2.2 and (4.16), we have

\[
0 = \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)dv + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2dv
\]

\[
+ (n-2) \int_M \rho^{n-2}|
abla \rho|^2 dv
\]

\[
+ n \int_M \rho^{n-2}\left\{ \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} + 2\rho^2 (H^{n+1})^2 - H^2 \rho^2 \right\} dv
\]

\[
+ nc \int_M \rho^n dv + \int_M \rho^{n-2}\left\{ \sum_{\alpha=n+2}^{n+p} \sum_{\beta=n+2}^{n+p} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) - 2\tilde{\sigma}_{n+1}dv
\]

\[
+ \sum_{\alpha=n+1}^{n+p} \sum_{\beta=n+1}^{n+p} \tilde{\sigma}_{\alpha\beta} \right\} dv
\]

\[
\geq - n \int_M \rho^{n-2}H^2\tilde{\rho}^2 dv + nc \int_M \rho^n dv - 2 \int_M \rho^{n-2}\rho^4 dv + \int_M \rho^{n-2}\frac{1}{p}\rho^4 dv
\]

\[
= \int_M \rho^n \left\{ n(c - H^2) - \left( 2 - \frac{1}{p} \right) \rho^2 \right\} dv,
\]

where the inequality \( N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) \geq 0 \) for any \( \alpha, \beta, \tilde{\sigma}_{n+1} \leq \rho^4 \) and \( \sum_{\alpha,\beta} \tilde{\sigma}_{\alpha\beta} = \sum_{\alpha} \tilde{\sigma}_{\alpha} \geq \frac{1}{\rho^4} \left( \sum_{\alpha} \tilde{\sigma}_{\alpha} \right)^2 = \frac{1}{p} \rho^4 \) is used.

In particular, if \( \rho^2 \leq \frac{n}{2} \frac{1}{p} (c - H^2) \), from (5.1), we see that \( \rho^2 = 0 \) and \( M \) is totally umbilical or \( \rho^2 = \frac{n}{2} \frac{1}{p} (c - H^2) \). In the latter case, we have from (5.1) that

\[
\int_M \rho^{n-2} \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} dv = 0,
\]

that is

\[
\int_M \rho^{n-2} \sum_{\alpha} (H^\alpha)^2 \tilde{\sigma}_{\alpha} dv = 0.
\]

If \( \rho^2 = 0 \), that is \( M \) is totally umbilical, otherwise, if \( \rho^2 \neq 0 \), it follows from (5.2) that \( \sum_{\alpha} (H^\alpha)^2 \tilde{\sigma}_{\alpha} = 0 \), thus \( (H^\alpha)^2 \tilde{\sigma}_{\alpha} = 0 \) for all \( \alpha \). Therefore, we see that \( \tilde{\sigma}_{\alpha} = 0 \) for all \( \alpha \) (contradicts to \( \rho^2 \neq 0 \)), or \( H^\alpha = 0 \) for all \( \alpha \). Thus, we have \( H = 0 \), that is, \( M \) is a compact maximal spacelike submanifold in \( S^{n+p}_q(c) \), by the Theorem 1 in Cheng and Ishikawa [4] and Example 3.1, we know that \( M \) lies in the totally geodesic spacelike submanifold \( S^{n+1}_q(c) \) of \( S^{n+p}_q(c) \) and is isometric to the Clifford torus \( S^k(\frac{1}{\sqrt{2}}c) \times S^k(\frac{1}{\sqrt{2}}c) \).

(2) If \( p - q > 1 \), from Lemma 2.2 and (4.16), we have

\[
0 = \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)dv + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2dv
\]

\[
+ (n-2) \int_M \rho^{n-2}|
abla \rho|^2 dv + n \int_M \rho^{n-2}\left\{ \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} \right\} dv
\]
\[+ 2\rho^2 \sum_{\beta=n+1}^{n+p-q} (H^2 - H^2 \rho^2) \, dv + n c \int_M \rho^n \, dv\]
\[+ \int_M \rho^{n-2} \left\{ -\sum_{\alpha,\beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) - \sum_{\alpha,\beta} \tilde{\sigma}_{\alpha\beta}^2\right\} \, dv\]
\[+ 2 \sum_{\alpha} \sum_{\beta=n+1}^{n+p-q+1} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) + 2 \sum_{\alpha} \sum_{\beta=n+1}^{n+p-q+1} \tilde{\sigma}_{\alpha\beta}^2 \right\} \, dv\]
\[\geq - n \int_M \rho^{n-2} H^2 \rho^2 \, dv + n c \int_M \rho^n \, dv + \int_M \rho^{n-2} \left\{ -\frac{3}{2} \rho^2 \right\} \, dv\]
\[= \int_M \rho^n \left\{ n(c - H^2) - \frac{3}{2} \rho^2 \right\} \, dv,\]
where the inequality (see Li–Li [7])
\[-\sum_{\alpha,\beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) - \sum_{\alpha,\beta} \tilde{\sigma}_{\alpha\beta}^2 \geq - \frac{3}{2} \rho^2,\]
is used.

In particular, if \(\rho^2 \leq \frac{2n}{6}(c - H^2)\), from (5.3), we see that \(\rho^2 = 0\) and \(M\) is totally umbilical or \(\rho^2 = \frac{2n}{6}(c - H^2)\). In the latter case, from (5.3), we also see that (5.2) holds. If \(\rho^2 = 0\), that is \(M\) is totally umbilical, otherwise, if \(\rho^2 \neq 0\), it follows from (5.2) that \(\sum_\alpha (H^2 \tilde{\sigma}_\alpha = 0). By the same argument as above, we see that \(H^2 = 0\) and \(H = 0\), that is, \(M\) is a compact maximal spacelike submanifold in \(S^{n+p}(c)\), by the Theorem 1 in Cheng and Ishikawa [4] and Example 3.2, we know that \(M\) lies in the totally geodesic spacelike submanifold \(S^{n+p}(c)\) of \(S^{n+p}(c)\) and is isometric to the Veronese surface. This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** For a fixed \(\alpha, n + 1 \leq \alpha \leq n + p\), we can take a local orthonormal frame field \(\{e_1, \ldots, e_n\}\) such that \(h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}\), then \(\tilde{h}^\alpha_{ij} = \mu^\alpha_i \delta_{ij}\) with \(\mu^\alpha_i = \lambda^\alpha_i - H^\alpha_i, \sum_i \mu^\alpha_i = 0\). Thus

\[(5.4) \quad - \sum_{\alpha, i, j, k, l} h^\alpha_{ij}(\tilde{h}^\alpha_{ik}R_{kjk} - h^\alpha_{ik}R_{ikj}) = \frac{1}{2} \sum_{\alpha, i, k} (\lambda^\alpha_i - \lambda^\alpha_k)^2 R_{kikk}\]
\[= \frac{1}{2} \sum_{\alpha, i, k} (\mu^\alpha_i - \mu^\alpha_k)^2 R_{kikk} \geq nK \rho^2,\]

where \(K\) denotes the infimum of the sectional curvature of \(M\) and the equality in (5.4) holds if and only if \(R_{kikk} = K\) for any \(i \neq k\).

Let \(\sum_i (\tilde{h}^\beta_{ii})^2 = \tau_{\beta}\). Then \(\tau_{\beta} \leq \sum_i (\tilde{h}^\beta_{ij})^2 = \tilde{\sigma}_{\beta}\). Since \(\sum_i \tilde{h}^\beta_{ii} = 0, \sum_i \mu^\alpha_i = 0\) and \(\sum_i (\mu^\alpha_i)^2 = \tilde{\sigma}_{\alpha}\). We have from Lemma 2.1 that

\[(5.5) \quad - \sum_{\alpha, \beta, i, j, k} H^\alpha \varepsilon_{\beta} \tilde{h}^\alpha_{ik} \tilde{h}^\beta_{kj} \tilde{h}^\beta_{ij} = - \sum_{\alpha, i, j, k} \sum_{\beta=n+1}^{\alpha+p-q} H^\alpha \tilde{h}^\alpha_{ik} \tilde{h}^\beta_{kj} \tilde{h}^\beta_{ij},\]
\[ + \sum_{\alpha,n,j,k}^{n+p} H^\alpha h_{ik} \bar{h}_{kj} h_{ij} \]

\[ = - \sum_{\alpha,\beta}^{n+p} H^\alpha \bar{h}_{ij} (\mu^\beta_i)^2 + \sum_{\alpha,\beta}^{n+p} H^\alpha \bar{h}_{ij} (\mu^\beta_i)^2 \]

\[ \geq - \frac{n-2}{\sqrt{n(n-1)}} \sum_{\alpha= \beta=n+1}^{n+p} |H^\alpha| \sqrt{\tau^\alpha} \]

\[ = - \frac{n-2}{\sqrt{n(n-1)}} \sum_{\alpha= \beta=n+p+q+1}^{n+p} |H^\alpha| \sqrt{\tau^\alpha} \]

\[ \geq - \frac{\sqrt{n(n-1)}}{\sqrt{n(n-1)}} \left( \sum_{\alpha} (H^\alpha)^2 \sum_{\alpha} \tau^\alpha \right) \rho^2 \geq - \frac{n-2}{\sqrt{n(n-1)}} H^3. \]

From Lemma 1 in Chen–Do Carmo–Kobayashi [5], we see that

\[ (5.6) \quad \frac{1}{2} \sum_{\alpha,\beta}^{n+p} \varepsilon_{\beta N(\bar{A}_\alpha \bar{A}_\beta - \bar{A}_\beta \bar{A}_\alpha) \]

\[ = - \frac{1}{2} \sum_{\alpha}^{n+p} N(\bar{A}_\alpha \bar{A}_\beta - \bar{A}_\beta \bar{A}_\alpha) + \frac{1}{2} \sum_{\alpha,\beta}^{n+p} N(\bar{A}_\alpha \bar{A}_\beta - \bar{A}_\beta \bar{A}_\alpha) \]

\[ = - \frac{1}{2} \sum_{\alpha=n+1}^{n+p} \sum_{\beta=n+p+q+1}^{n+p} N(\bar{A}_\alpha \bar{A}_\beta - \bar{A}_\beta \bar{A}_\alpha) \]

\[ + \frac{1}{2} \sum_{\alpha=n+p-1}^{n+p} \sum_{\beta=n-p+1}^{n+p} N(\bar{A}_\alpha \bar{A}_\beta - \bar{A}_\beta \bar{A}_\alpha) \]

\[ \geq - \frac{1}{2} \sum_{\alpha=n+1}^{n+p} \sum_{\beta=n+1}^{n+p} N(\bar{A}_\alpha \bar{A}_\beta - \bar{A}_\beta \bar{A}_\alpha) \geq - \sum_{\alpha \neq \beta} \bar{\sigma}_\alpha \bar{\sigma}_\beta \]

\[ = - \left( \sum_{\alpha=n+1}^{n+p} \bar{\sigma}_\alpha \right)^2 = - \left( \sum_{\alpha=n+p}^{n+p} \bar{\sigma}_\alpha \right)^2 \geq - \left( \frac{1}{p-q} \left( \sum_{\alpha=n+1}^{n+p} \bar{\sigma}_\alpha \right) \right)^2 \]

\[ = \left( 1 - \frac{1}{p-q} \right) \left( \sum_{\alpha=n+1}^{n+p} \bar{\sigma}_\alpha \right)^2 \geq \left( 1 - \frac{1}{p-q} \right) \rho^4. \]

Thus, from (4.13), (4.14), (5.4), (5.5), (5.6) and Lemma 2.2, we have

\[ (5.7) \quad 0 \geq \int_M \rho^{n-2} (|\nabla h|^2 - \frac{3n^2}{n+2} |\nabla \bar{H}|^2) dv + \int_M \rho^{n-2} \left( \frac{3n^2}{n+2} - n \right) \nabla^\perp H^2 dv \]
\[ + (n - 2) \int_M \rho^{n-2} |\nabla \rho|^2 dv - \int_M \rho^{n-2} \frac{n(n-2)}{\sqrt{n(n-1)}} H \rho^2 dv \\
+ \int_M \rho^{n-2} \sum_{\alpha, \beta} n H^\alpha H^\beta \tilde{\sigma}_{\alpha \beta} + \int_M \rho^{n-2} nK \rho^2 dv - \frac{1}{2} \sum_{\alpha, \beta} \epsilon_{\alpha \beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) \\
\geq \int_M \rho^n \left\{ nK - \frac{n(n-2)}{\sqrt{n(n-1)}} H \rho - \left( 1 - \frac{1}{p-q} \right) \rho^2 \right\} dv. \]

In particular, if

\[ K \geq \frac{n-2}{\sqrt{n(n-1)}} H \rho + \frac{1}{n} \left( 1 - \frac{1}{p-q} \right) \rho^2, \]

from (5.7), we see that \( \rho^2 = 0 \) and \( M \) is totally umbilical or \( K = \frac{n+2}{n(n-1)} H \rho + \frac{1}{n} \left( 1 - \frac{1}{p-q} \right) \rho^2 \). In the latter case, from (5.7), we know that (5.2) holds. If \( \rho^2 \neq 0 \), that is \( M \) is totally umbilical, otherwise, if \( \rho^2 = 0 \), it follows from (5.2) that \( \sum_{\alpha} (H^\alpha)^2 \tilde{\sigma}_{\alpha} = 0 \). By the same argument as in the proof of Theorem 1.1, we see that \( H^\alpha = 0 \) and \( H = 0 \). It also follows from (5.7) that \( |\nabla h|^2 = \frac{3n^2}{n+2} |\nabla^2 \tilde{H}|^2 = 0 \), that is, the second fundamental form of \( M \) is parallel. This completes the proof of Theorem 1.2.

**Proof of Theorem 1.3.** From (2.8) and (4.1), we have

\[ R_{kk} = (n-1)c + \sum_{\alpha} \epsilon_{\alpha} H^\alpha \tilde{h}_{kk}^\alpha + (n-1) \sum_{\alpha=n+1}^{n+p-q} (H^\alpha)^2 \]

\[ - (n-1) \sum_{\alpha=n+p+q+1}^{n+p} (H^\alpha)^2 - \sum_{i, \alpha=n+1}^{n+p-q} (\tilde{h}_{ik}^\alpha)^2 + \sum_{i, \alpha=n+p+q+1}^{n+p} (\tilde{h}_{ik}^\alpha)^2 \]

\[ \leq (n-1)c + \sum_{\alpha} \epsilon_{\alpha} H^\alpha \tilde{h}_{kk}^\alpha + (n-1)H^2 \]

\[ = \sum_{i, \alpha=n+1}^{n+p-q} (\tilde{h}_{ik}^\alpha)^2 + \sum_{i, \alpha=n+p+q+1}^{n+p} (\tilde{h}_{ik}^\alpha)^2. \]

Thus

\[ nQ \leq \sum_k R_{kk} = n(n-1)c + n(n-1)H^2 - \sum_{i,k, \alpha=n+1}^{n+p-q} (\tilde{h}_{ik}^\alpha)^2 - \sum_{i,k, \alpha=n+p+q+1}^{n+p} (\tilde{h}_{ik}^\alpha)^2. \]

From (4.2) and (4.3), we have

\[ (5.8) \]

\[ - \sum_{\alpha=n+1}^{n+p-q} \tilde{\sigma}_{\alpha} + \sum_{\alpha=n+p+q+1}^{n+p} \tilde{\sigma}_{\alpha} \geq nQ - n(n-1)c - n(n-1)H^2. \]
From (5.8), we see that

\begin{equation}
-\left(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_{\alpha}\right)^2 + \frac{1}{q}\left(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_{\alpha}\right)^2
\end{equation}

\begin{align*}
\geq & \left(-\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_{\alpha} + \sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_{\alpha}\right)\left(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_{\alpha} + \sum_{\alpha=n+1+p-q+1}^{n+p}\tilde{\sigma}_{\alpha}\right) - \left(1 - \frac{1}{q}\right)\rho^4 \\
\geq & \left(nQ - n(n-1)c - n(n-1)H^2\right)\rho^2 - \left(1 - \frac{1}{q}\right)\rho^4.
\end{align*}

By Lemma 1 in Chen–Do Carmo–Kobayashi [5], we also see that

\begin{equation}
-\sum_{\alpha=n+1}^{n+p-q} \sum_{\beta=n+1}^{n+p-q} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) \geq -2\left(1 - \frac{1}{p-q}\right)\rho^4.
\end{equation}

From Lemma 2.2, (4.3), (4.16), (5.9) and (5.10), we have

\begin{equation}
0 = \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)dv + \int_M \rho^{n-2}\left(\frac{3n^2}{n+2} - n\right)|\nabla^\perp \vec{H}|^2dv \\
+ (n-2)\int_M \rho^{n-2}|\nabla \rho|^2dv + n\int_M \rho^{n-2}\left\{ \sum_{\alpha,\beta} H^{\alpha} H^{\beta} \tilde{\sigma}_{\alpha\beta} \\
+ 2\rho^2 \sum_{\beta=n+1}^{n+p-q} (H^{\beta})^2 - H^2 \rho^2 \right\}dv + nc\int_M \rho^ndv \\
+ \int_M \rho^{n-2}\left\{ - \sum_{\alpha=n+1}^{n+p-q} \sum_{\beta=n+1}^{n+p-q} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) - \sum_{\alpha=n+1}^{n+p-q} \tilde{\sigma}_{\alpha}^2 + \sum_{\alpha=n+1+p-q+1}^{n+p} \tilde{\sigma}_{\alpha}^2 \right\}dv \\
\geq & -n\int_M \rho^{n-2}H^2 \rho^2 dv + nc\int_M \rho^ndv + \int_M \rho^{n-2}\left\{ -2\left(1 - \frac{1}{p-q}\right)\rho^4 \right\}dv \\
+ \int_M \rho^{n-2}\left\{ - \left(\sum_{\alpha=n+1}^{n+p-q} \tilde{\sigma}_{\alpha}\right)^2 + \frac{1}{q}\left(\sum_{\alpha=n+1+p-q+1}^{n+p} \tilde{\sigma}_{\alpha}\right)^2 \right\}dv \\
\geq & -n\int_M \rho^{n-2}H^2 \rho^2 dv + nc\int_M \rho^ndv + \int_M \rho^{n-2}\left\{ -2\left(1 - \frac{1}{p-q}\right)\rho^4 \right\}dv \\
+ \int_M \rho^{n-2}\left\{ \left(nQ - n(n-1)c - n(n-1)H^2\right)\rho^2 - \left(1 - \frac{1}{q}\right)\rho^4 \right\}dv \\
= & \int_M n\rho^3\left\{ Q - (n-2)c - nH^2 - \frac{1}{n}\left(3 - \frac{p+q}{(p-q)q}\right)\rho^2 \right\}dv.
In particular, if
\[ Q > (n - 2)c + nH^2 + \frac{1}{n}\left(3 - \frac{p+q}{(p-q)q}\right)\rho^2, \]
from (5.11), we see that \( \rho^2 = 0 \) and \( M \) is totally umbilical or \( Q = (n - 2)c + nH^2 + \left(3 - \frac{p+q}{(p-q)q}\right)\rho^2 \). In the latter case, from (5.11), we know that (5.2) holds. If \( \rho^2 = 0 \), that is \( M \) is totally umbilical, otherwise, if \( \rho^2 \neq 0 \), it follows from (5.2) that \( \sum_\alpha (H^\alpha)^2 \sigma_\alpha = 0 \). By the same argument as in the proof of Theorem 1.1, we see that \( H^\alpha = 0 \) and \( H = 0 \). It also follows from (5.11) that \( |\nabla h|^2 = \frac{3n^2}{n+1} |\nabla H|^2 = 0 \), that is, the second fundamental form of \( M \) is parallel. This completes the proof of Theorem 1.3. \( \square \)

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