Generalized Fourier representation of the absolutely continuous part of a selfadjoint operator

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Abstract

We formulate and prove the existence and uniqueness of the generalized Fourier transform associated with the absolutely continuous part of an arbitrary selfadjoint operator on a separable Hilbert space. To this end we develop a novel method to decompose an absolutely continuous operator into a variable fiber direct integral of selfadjoint operators.

Keywords: spectral decomposition, stationary method, eigenfunction expansion, scattering theory

2000 MSC: 35P10, 35P25, 47A40

1. Introduction

The most fundamental result of the spectral theory of selfadjoint operators is the spectral theorem which states

\[ H = \int_{\mathbb{R}} \lambda dE_\lambda \]

where \( H \) is a selfadjoint operator on some Hilbert space \( \mathcal{H} \) and \( \{ E_\lambda \}_{\lambda \in \mathbb{R}} \) is a spectral family associated with \( H \). In this paper we establish a new spectral representation theorem that connects the spectral family and the generalized eigenfunctions.

By generalized eigenfunction we mean a solution \( u \) to the formal eigenequation

\[ (H - \lambda)u = 0, \quad u \in \mathcal{X} \]

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Preprint submitted to Elsevier

January 16, 2013
where \( \lambda \in \mathbb{R} \) is a spectral parameter. The function space \( \mathcal{X} \) must be chosen suitably, depending on the operator \( H \) of interest. Basically we choose \( \mathcal{X} \) in such a way that the solutions to the problem (1.1) carry a complete set of information about the operator \( H \) at the energy \( \lambda \). We can take \( \mathcal{X} = \mathcal{H} \) when \( \lambda \notin \sigma(H) \) or \( \lambda \in \sigma_{\text{disc}}(H) \). But we need to have \( \mathcal{X} \) larger than \( \mathcal{H} \) if \( \lambda \in \sigma_{\text{ess}}(H) \) in order to capture the generalized eigenfunctions corresponding to the continuum.

The goal of this paper is to show the existence of a family of Hilbert spaces \( \{ \mathcal{X}(\lambda) \}_{\lambda \in \mathbb{R}} \) such that there is a selfadjoint operator \( H_\lambda \) on each \( \mathcal{X}(\lambda) \) with the same spectrum as \( H \) and the collection of the solutions to the problem

\[
(H_\lambda - \lambda)u = 0, \quad u \in \mathcal{D}(H_\lambda)
\]

for \( \lambda \) from a set of full Lebesgue measure determines the absolutely continuous part of \( H \). We shall show that the spectral density \( E'_\lambda \) is a pullback of the orthogonal projection \( Q(\lambda) \) onto the null space of \( H_\lambda - \lambda \) in \( \mathcal{X}(\lambda) \) in the sense that

\[
\langle E'_\lambda f | f \rangle = \| Q(\lambda) Y(\lambda) f \|_2^2, \quad f \in \mathcal{D}(Y(\lambda)),
\]

where \( Y(\lambda) : \mathcal{H} \rightarrow \mathcal{X}(\lambda) \) is a possibly non-closable injective linear map. This means that the generalized eigenvalue problem (1.2) determines the absolutely continuous part of \( H \).

We now briefly compare the idea of this paper with the known results. In the case of partial differential operators, it is common to use weighted \( L^2 \) spaces \([1]\) and Besov spaces \([2, 3, 4]\) for the function space \( \mathcal{X} \) in (1.1). Well known technique is the following. Given an absolutely continuous operator \( H \) on some function space, we look for a function space \( \mathcal{Y} \) such that we have

\[
|\langle (H - z)^{-1} f | g \rangle| \leq C \| f \|_\mathcal{Y} \| g \|_\mathcal{Y}, \quad f, g \in \mathcal{Y}
\]

with a uniform constant \( C > 0 \). The choice of \( \mathcal{Y} \) depends on the problem in general. The point is that we have

\[
\text{Im} \langle (H - z)^{-1} f | f \rangle \leq C \| f \|_\mathcal{Y}^2, \quad f \in \mathcal{Y}, \quad \text{Im} z > 0
\]

so that there is a bounded operator \( \delta(H - \lambda) : \mathcal{Y} \rightarrow \mathcal{Y}^* \) such that

\[
\lim_{\varepsilon \downarrow 0} \text{Im} \langle (H - \lambda - i\varepsilon)^{-1} f | f \rangle = \pi \langle \delta(H - \lambda) f | f \rangle, \quad f \in \mathcal{Y}
\]

(1.4)
for all $\lambda \in \mathbb{R}$. The operator $\delta(H - \lambda)$ is identical to the spectral density $E'_\lambda$ realized as an operator. One usually fix a function space $\mathcal{D}$ and argues that any solution $u \in \mathcal{Y}^*$ of the problem
\[ \langle u | (H - \lambda)f \rangle = 0, \quad f \in \mathcal{D} \] (1.5)
can be written as $u = \delta(H - \lambda)\varphi$ for some $\varphi \in \mathcal{Y}$ and that we have
\[ \langle Hf | f \rangle = \int_{\mathbb{R}} \lambda \langle \delta(H - \lambda)f | f \rangle d\lambda. \] (1.6)

The identity (1.6) guarantees that the collection of solutions to the problem (1.5) for all $\lambda \in \mathbb{R}$ is complete in the sense that we can recover the original operator $H$ from this collection.

The main idea of this paper is to realize the left hand side of (1.3), which roughly corresponds to (1.4), as a quadratic form and choose the completion of its graph as $\mathcal{Y}$. Although the quadratic form is nonnegative, it is not always closable. We shall establish a new representation theorem of nonnegative quadratic forms to overcome this difficulty. Another problem is how to give a rigorous meaning to the problem (1.5). To solve this, we propose the notion of continuation of a selfadjoint operator to some other function spaces.

The author expects that the results of this paper is useful in scattering theory, since one can prove the existence of the generalized Fourier transform, which is a key tool for the theory, without depending on the limiting absorption principle nor on the Mourre inequality.

In this paper we focus on the absolutely continuous part of $H$ and the results on the case where $H$ has a non-trivial singular continuous part will appear elsewhere.

We close this section by collecting the notations used throughout the paper. A quadratic form $q$ is linear in the first argument and anti-linear in the second. We adopt the same convention about inner products. We write $q(f) = q(f, f)$. By $\rho(T)$, $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\mathcal{N}(T)$ we mean the resolvent set, domain, range and null space of a linear operator $T$. If $T$ is closable we write $\overline{T}$ for its closure unless otherwise noted. By $P_{ac}(H)$ we denote the spectral projection onto the absolutely continuous subspace of a selfadjoint operator $H$. The set of infinitely differentiable complex valued functions $f$ on $\mathbb{R}$ such that $f$ and all its derivatives tend to zero at infinity is denoted by $C_0^\infty(\mathbb{R})$. We sometimes write 0 for a zero space. The identity operators of various function spaces are denoted by 1. $T \upharpoonright \mathcal{D}$ is the operator $T$ with domain restricted to a subspace $\mathcal{D}$ of $\mathcal{D}(T)$. 

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2. Continuation

In this section we develop a method to continue a selfadjoint operator $H$ initially defined on a Hilbert space $\mathcal{H}$ to another Banach space $\mathcal{X}$ that is close to $\mathcal{H}$ in a certain sense. We will use the results of this section to formulate the generalized eigenspace of a selfadjoint operator. Throughout this section we assume that $\mathcal{X}$ is a Banach space obtained by the completion of a dense subspace $\mathcal{D}$ of $\mathcal{H}$ with respect to some norm on $\mathcal{D}$.

The goal of this section is to formulate a condition that guarantees the existence of a closed operator $H_\mathcal{X} : \mathcal{X} \rightarrow \mathcal{X}$, which we will call a continuation of $H$ to $\mathcal{X}$. The operator $H_\mathcal{X}$ will be chosen so that it coincides with $H$ on a common core of $H_\mathcal{X}$ and $H$.

We start with the observation that $\mathcal{X}$ is isometrically isomorphic to the completion of $\mathcal{D}$ by some norm if and only if there is a densely defined injective linear map $Y : \mathcal{H} \rightarrow \mathcal{X}$ with dense range such that $\mathcal{D}(Y) = \mathcal{D}$. In this paper we call such $Y$ a completion operator. Note that a completion operator need not be closable. Just like in the case of closed operators, we say that a subspace $\mathcal{D}$ of $\mathcal{X}$ is a core of $Y$ if for any $f \in \mathcal{D}(Y)$ there is a sequence $f_j \in \mathcal{D}$ such that $f_j \rightarrow f$ and $Y f_j \rightarrow Y f$. In this case the restriction $Y \upharpoonright \mathcal{D}$ is a completion operator from $\mathcal{H}$ to $\mathcal{X}$.

Given a completion operator $Y$, we sometimes identify $\mathcal{D}(Y)$ with $\mathcal{R}(Y)$ through the correspondence $\mathcal{D}(Y) \ni f \mapsto Y f \in \mathcal{R}(Y)$ in order to simplify notations. We also remark that since $\mathcal{X}$ is a completion, it cannot always be identified with a subspace of $\mathcal{H}$. As we will see later, however, it is important to consider such a case when dealing with a non-closable form.

Our strategy is that we first continue the resolvent $(H - z)^{-1}$ to $\mathcal{X}$ and then define the continuation $H_\mathcal{X}$ using the inverse of the continued resolvent. We introduce the following

**Definition 2.1.** Let $Y : \mathcal{H} \rightarrow \mathcal{X}$ be a completion operator and $z \in \rho(H)$. We say that $H$ is compatible with $Y$ at $z$ if there is a subspace $\mathcal{D} \subset \mathcal{D}(Y)$ with the following properties:

i. $(H - z)^{-1} \mathcal{D}$ is a core of $Y$;

ii. there exists a constant $C_z > 0$ depending of $z$ such that

$$\|Y(H - z)^{-1} f\|_\mathcal{X} \leq C_z \|Y f\|_\mathcal{X}$$

for all $f \in \mathcal{D}$;
iii. \( u \in \mathcal{X}, f_j \in \mathcal{D}, Yf_j \to u \) in \( \mathcal{X} \), and \( Y(H - z)^{-1}f_j \to 0 \) in \( \mathcal{X} \) imply \( u = 0 \).

Given a subset \( \Omega \subset \rho(H) \), we say that \( H \) is compatible with \( Y \) in \( \Omega \) if \( H \) is compatible with \( \mathcal{X} \) through \( Y \) for all \( z \in \Omega \).

**Theorem 2.1.** Let \( \Omega \) be a non-empty subset of \( \rho(H) \). Suppose that \( H \) is compatible with a completion operator \( Y : \mathcal{H} \to \mathcal{X} \) in \( \Omega \). Let \( \mathcal{D} \) be as in Definition 2.1. Then there exists a densely defined closed operator \( H_X : \mathcal{X} \to \mathcal{X} \) such that \( \Omega \subset \rho(H_X) \) and that

\[
(H_X - z)^{-1}Yf = Y(H - z)^{-1}f
\]

(2.1)

for all \( f \in \mathcal{D} \) and \( z \in \Omega \). The operator \( H_X \) is independent of the choice of \( \mathcal{D} \) and uniquely determined by \( H \) and \( Y \).

**Proof.** There exists a unique bounded operator \( B(z) \) on \( \mathcal{X} \) such that

\[
B(z)Yf = Y(H - z)^{-1}f
\]

(2.2)

for all \( f \in \mathcal{D} \). The first resolvent identity of \((H - z)^{-1}\) yields

\[
B(z) - B(w) = (z - w)B(z)B(w)
\]

(2.3)

for all \( z, w \in \Omega \) and it follows that \( \mathcal{R}(B(z)) \) is constant for all \( z \in \Omega \). Remark that \( B(z) \) is injective and has a dense range. We define the operator \( H_X : \mathcal{X} \to \mathcal{X} \) with domain \( \mathcal{D}(H_X) = \mathcal{R}(B(z)) \) by

\[
H_XB(z)f = f + zB(z)f, \quad f \in \mathcal{X}.
\]

(2.4)

Since \( B(z) \) is bounded and injective, \( H_X \) is closed. Moreover, \( H_X \) is independent of the choice of \( z \in \Omega \) due to (2.3). The identity (2.4) implies that \( H_X - z \) is bijective and satisfies \((H_X - z)^{-1} = B(z)\) for all \( z \in \Omega \). The uniqueness of \( H_X \) follows from the uniqueness of \( B(z) \) in (2.2). This completes the proof. \( \square \)

When we identify \( f \in \mathcal{D}(Y) \) and \( Yf \in \mathcal{R}(Y) \), (2.1) means that \( H \) and \( H_X \) share \( \mathcal{D}_0 = (H - z)^{-1}\mathcal{D} = (H_X - z)^{-1}\mathcal{D} \) as a common core and that we have \( H_Xu = Hu \) for all \( u \in \mathcal{D}_0 \). It is thus reasonable to regard \( H_X \) as an extension of \( H \) to the Banach space \( \mathcal{X} \). In order to avoid confusion with the usual notion of extension of operators, let us refer to the operator \( H_X \) as a continuation of \( H \) to \( \mathcal{X} \) induced by \( Y \) or simply a continuation of \( H \). In general, \( H_X \) depends on the choice of \( Y \) and not uniquely determined by \( H \) and \( \mathcal{X} \).
Corollary 2.2. Suppose that $X$ is a Hilbert space and that $H$ is compatible with a completion operator $Y : \mathcal{H} \to X$ in $\{i, -i\}$. Suppose further that $H_X$ is symmetric. Then $H_X$ is selfadjoint.

3. Pullback representation of a nonnegative form

In this section we shall develop our main tool to deal with nonnegative quadratic forms that might not be closable. Our goal here is to formulate and prove a representation theorem of nonnegative quadratic forms. Recall that a nonnegative quadratic form $q$ on a Hilbert space $\mathcal{H}$ is closable if and only if $\mathcal{D}(q)$ is complete under the norm $\| \cdot \|_q$ where $\|f\|_q^2 = \|f\|^2 + q(f)$. So it is reasonable to embed the domain $\mathcal{D}(q)$ into its completion by the above norm. We thus begin with the following

Definition 3.1. Suppose that $\mathcal{H}, \mathcal{X}$ and $\mathcal{E}$ are Hilbert spaces. A pullback triple $\tau = (J, F, Y)$ is a triplet of operators

$$J : \mathcal{X} \to \mathcal{H}, \quad F : \mathcal{X} \to \mathcal{E}, \quad Y : \mathcal{H} \to \mathcal{X}$$

with the following properties:

i. $J, F$ are bounded with dense range;
ii. $Y$ is a completion operator;
iii. for any $u \in \mathcal{X}$ we have $\|u\|_{\mathcal{X}}^2 = \|Ju\|^2 + \|Fu\|_{\mathcal{E}}^2$;
iv. $JY \subset 1$.

We often say that $\tau$ is a pullback triple on $\mathcal{H}$ in order to specify the function space $\mathcal{H}$ explicitly. We call $\mathcal{D}(Y)$ the domain of $\tau$ and denote it by $\mathcal{D}(\tau)$.

Note that the condition [iv] is equivalent to the commutativity of the diagram

$$\begin{array}{c}
\mathcal{D}(Y) \\
y \\
\mathcal{H} \xrightarrow{J} \mathcal{X}
\end{array}$$

where $\iota$ is the natural injection. As is done in depicting the operator $Y$ of the above diagram, we put the domain of a linear operator in front of the tail of the arrow indicating the operator. We prefer this rule in order to simplify notations.
Lemma 3.1. Let \((J, F, Y)\) be a pullback triple. Then

i. \(Y^{-1}\) is closable and \(J = Y^{-1}\);

ii. \(Y\) is closable if and only if \(J\) is injective and in this case we have \(Y = J^{-1}\);

iii. \(Y\) is bounded if and only if \(J\) is an isomorphism.

Proof. (i). Since \(JY f = f\) for any \(f \in \mathcal{D}(Y)\), we have \(\|Y f\|_X^2 = \|f\|_X^2 + \|FY f\|_X^2 \geq \|f\|_X^2\) for any \(f \in \mathcal{D}(Y)\). It follows that \(Y^{-1}\) is closable and \(J = Y^{-1}\). (ii). If \(f_j \in \mathcal{D}(Y)\) and \(Y f_j \to u\) in \(X\), then we have \(f_j \to Ju\) in \(\mathcal{H}\). It follows that \(Y\) is closable if and only if \(J\) is injective. It is easy to see that \(\mathcal{D}(Y) = \mathcal{R}(J)\) and that \(Y Ju = u\) for all \(u \in \mathcal{X}\). (iii). Note that \(\|Y f\| \leq C\|f\|\) for all \(f \in \mathcal{D}(Y)\) is equivalent to \(\|u\| \leq C\|Ju\|\) for all \(u \in \mathcal{X}\). Since \(J\) is bounded and has a dense range, the statement follows. \(\Box\)

Lemma 3.2. Let

\[ J_j : \mathcal{X}_j \longrightarrow \mathcal{H}_j, \quad F_j : \mathcal{X}_j \longrightarrow \mathcal{E}_j, \quad Y_j : \mathcal{H}_j \longrightarrow \mathcal{X}_j, \quad j = 1, 2 \]

be pullback triples. Then the following conditions are equivalent:

i. \(J_1\) and \(J_2\) are unitarily equivalent;

ii. \(F_1\) and \(F_2\) are unitarily equivalent;

iii. there exist unitary isomorphisms \(W_\mathcal{H}, W_\mathcal{X},\) and \(W_\mathcal{E}\) such that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{H}_1 & \xrightarrow{J_1} & \mathcal{X}_1 \xrightarrow{F_1} \mathcal{E}_1 \\
W_\mathcal{H} \downarrow & & W_\mathcal{X} \downarrow & & W_\mathcal{E} \\
\mathcal{H}_2 & \xrightarrow{J_2} & \mathcal{X}_2 \xrightarrow{F_2} \mathcal{E}_2 \\
\end{array}
\]

We introduce the following equivalence relation.

Definition 3.2. We say that two pullback triples \((J_j, F_j, Y_j), j = 1, 2\) are equivalent if any of the conditions of Lemma 3.2 is satisfied.

Lemma 3.3. Two pullback triples \((J_j, F_j, Y_j), j = 1, 2\) on a Hilbert space \(\mathcal{H}\) are equivalent if and only if there is a unitary isomorphism \(W\) such that \((J_1, F_1, WY_2)\) is a pullback triple.
Lemma 3.4. Let \((J, F, Y)\) be a pullback triple and \(Y_0 \subset Y\). Then \((J, F, Y_0)\) is a pullback triple if and only if \(\mathcal{D}(Y_0)\) is a core of \(Y\). In this case \((J, F, Y)\) and \((J, F, Y_0)\) are equivalent.

We say that a linear space \(\mathcal{D}\) is a core of a pullback triple \((J, F, Y)\) if \(\mathcal{D}\) is a core of \(Y\).

Theorem 3.5. Let \(q\) be a densely defined nonegative quadratic form on a Hilbert space \(\mathcal{H}\). Then there exists a pullback triple \((J, F, Y)\) with \(\mathcal{D}(Y) = \mathcal{D}(q)\) such that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{D}(q) & \xrightarrow{\gamma} & \mathbb{R} \\
\downarrow & & \downarrow \\
\mathcal{H} & \xrightarrow{J} & \mathcal{X} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{F} & E \\
\end{array}
\]

The pullback triple is unique up to equivalence. Moreover, \(q\) is closable if and only if so is \(Y\). In this case \((J, F, Y)\) is a pullback triple of \(\overline{q}\) and we have

\[
\mathcal{D}(\overline{q}) = \mathcal{R}(J), \quad \overline{q}(Ju) = \|Fu\|_E^2, \quad u \in \mathcal{X}. \tag{3.2}
\]

Remark 3.1. The commutativity of the above diagram implies

\[
q(f) = \|FYf\|_E^2, \quad f \in \mathcal{D}(Y).
\]

This means that any nonnegative form \(q\) is a pullback of a bounded nonnegative form by a completion operator \(Y\).

Proof. It is easy to see that

\[
\mathcal{D}(q) \times \mathcal{D}(q) \ni (f, g) \mapsto \langle f|g \rangle + q(f, g) \in \mathbb{C}
\]

is a positive definite inner product on \(\mathcal{D}(q)\). Let \(\mathcal{X}\) be the completion of \(\mathcal{D}(q)\) by this inner product and \(Y : \mathcal{H} \to \mathcal{X}\) be the associated completion operator. Since \(q\) is nonnegative, we have

\[
|q(f, g)|^2 \leq q(f)q(g) \leq \|Yf\|_E^2\|Yg\|_E^2
\]

for any \(f, g \in \mathcal{D}(q)\). By the Riesz representation theorem there exists a unique bounded selfadjoint operator \(\delta : \mathcal{X} \to \mathcal{X}\) such that

\[
\langle \delta Yf|Yg \rangle_x = q(f, g), \quad f, g \in \mathcal{D}(q).
\]
It is easy to see that the mapping
\[ R(\delta) \times R(\delta) \ni (\delta u, \delta v) \mapsto \langle \delta u|v \rangle_X \in \mathbb{C} \]
is well-defined and is a positive definite inner product on \( R(\delta) \). Let \( \mathcal{E} \) be the completion of \( R(\delta) \) with respect to this inner product. We define the operator \( F \) by
\[ F : X \rightarrow \mathcal{E}, \quad Fu = \delta u, \quad u \in X. \]
By our choice of \( \mathcal{E} \), \( \mathcal{R}(F) \) is dense in \( \mathcal{E} \) and we have \( \|Fu\|^2 = \langle \delta u|u \rangle_X \) for any \( u \in X \). Note that \( Y^{-1} \) is closable and has a bounded extension by Lemma 3.1. We set \( J = Y^{-1} \). Clearly,
\[ JYf = f, \quad q(f) = \| FYf \|^2_{\mathcal{E}}, \quad f \in \mathcal{D}(Y). \] (3.3)
Therefore \((J, F, Y)\) has the desired properties. The uniqueness follows from (3.3) and Lemma 3.2. The statement about closability follows from Lemma 3.1. The relation (3.2) is obvious. This completes the proof. \( \square \)

We refer to the triple \((J, F, Y)\) as a pullback triple of \( q \) and we say that two nonnegative forms \( q_1 \) and \( q_2 \) are equivalent if the corresponding pullback triples are equivalent. There are cases where the domains of \( q_1 \) and \( q_2 \) have no inclusion relation but \( q_1 \) and \( q_2 \) are equivalent.

Similarly any linear operator is a pullback of a bounded operator. We only state the result and the proof is left to the reader.

**Theorem 3.6.** Let \( T : \mathcal{H} \rightarrow \mathcal{K} \) be a linear operator between Hilbert spaces. Then there exists a pullback triple \((J, F, Y)\) with \( \mathcal{D}(Y) = \mathcal{D}(T) \) such that the following diagram commutes.

\[ \begin{array}{ccc}
\mathcal{D}(T) & \xrightarrow{T} & \mathcal{R}(T) \\
Y \downarrow & & \downarrow \Psi \\
\mathcal{H} & \xrightarrow{J} & \mathcal{X} \xrightarrow{F} \mathcal{R}(T) \\
& \uparrow \downarrow & & \uparrow \downarrow \\
& H & \leftarrow & J \end{array} \]

The pullback triple is unique up to equivalence. Moreover, \( T \) is closable if and only if so is \( Y \). In this case \((J, F, Y)\) is a pullback triple of \( \overline{T} \) and we have
\[ \mathcal{D}(\overline{T}) = \mathcal{R}(J), \quad \overline{T}Ju = Fu, \quad u \in \mathcal{X}. \]
Fix a pullback triple \((3.1)\). Then \(N(J)\) and \(N(F)\) are orthogonal in \(X\). Moreover,
\[
N(J) = \{ f \in X : f = Qf \}, \quad N(F) = \{ f \in X : f = \overline{Q}f \}, \quad (3.4)
\]
where \(Q\) and \(\overline{Q}\) are bounded operators in \(X\) defined by
\[
Q = F^*F, \quad \overline{Q} = 1 - Q = J^*J.
\]

**Lemma 3.7.** Let \((J, F, Y)\) be a pullback triple. Then the following conditions are equivalent:

i. \(F\) is a partial isometry;
ii. \(X = N(J) \oplus N(F)\);
iii. \(JF^* = 0\);
iv. \(JN(F)\) is dense in \(H\).

A pullback triple \((J, F, Y)\) is said to be splitting if any of the conditions of Lemma 3.7 is satisfied. We close this section by summarizing the properties of a splitting pullback triple for later use.

**Proposition 3.8.** Suppose that \((J, F, Y)\) is a splitting pullback triple. Then \(J\) and \(F\) are partial isometries with
\[
N(F) = \mathcal{R}(J^*), \quad \mathcal{R}(J) = \mathcal{H}, \quad (3.5)
\]
\[
N(J) = \mathcal{R}(F^*), \quad \mathcal{R}(F) = \mathcal{E}. \quad (3.6)
\]

**Remark 3.2.** Proposition 3.8 implies that under the condition that the pullback triple \((J, F, Y)\) is splitting, the form \(q\) associated with the triple is closable if and only if \(F\) and thus \(q\) vanishes identically. In this case we have \(\mathcal{E} = 0\). Note also that \(X\) is unitarily isomorphic to the splitting direct sum \(\mathcal{H} \oplus \mathcal{E}\). This means that one can measure by \(\mathcal{E}\) how far is \(q\) from being closable.
4. Pullback representation of spectral densities

Suppose that $H$ is a selfadjoint operator on a separable Hilbert space $\mathcal{H}$. Let $\{E_{\lambda}\}_{\lambda}$ be a spectral family of $H$. We shall derive a representation of the absolutely continuous part of $H$ in terms of the spectral density $E'_{\lambda}$. The main difficulty is that the spectral density is never closable unless it is zero identically. The pullback triples we have developed in the previous section facilitate the understanding of the spectral density as a quadratic form.

Since $\|E_{\lambda}f\|^2$ is a non-decreasing function of $\lambda$ for any $f \in \mathcal{H}$, it is differentiable almost everywhere in the sense of Dini, and so is $\langle E_{\lambda}f|g \rangle$ for any $f, g \in \mathcal{H}$. It is easy to see that

$$\frac{d\langle E_{\lambda}f|g \rangle}{d\lambda} = \frac{d\langle E_{\lambda}P_{ac}(H)f|g \rangle}{d\lambda}$$

almost everywhere. By Radon-Nikodym theorem we see that (4.1) is integrable and that

$$\langle P_{ac}(H)f|g \rangle = \int_{\mathbb{R}} \frac{d\langle E_{\lambda}f|g \rangle}{d\lambda} d\lambda$$

for any $f, g \in \mathcal{H}$. We shall show that (4.1) is well-defined as a densely defined nonnegative form.

**Lemma 4.1.** Let $f, g \in \mathcal{H}$. Suppose that $\langle E_{\lambda}f|g \rangle$ is differentiable at $\lambda$. Then $\langle E_{\lambda}\chi(H)f|g \rangle$ is differentiable at $\lambda$ for any $\chi \in C_0^\infty(\mathbb{R})$ and we have

$$\frac{d\langle E_{\lambda}\chi(H)f|g \rangle}{d\lambda} = \chi(\lambda)\frac{d\langle E_{\lambda}f|g \rangle}{d\lambda}$$

**Proof.** The statement follows from the fact that $[\chi(H) - \chi(\lambda)]E_{H}(I)/|I| \to 0$ strongly as $|I| \to 0$ for any interval $I$ containing $\lambda$ as an internal point. □

Note that the spectral density has the following representation by the boundary value of the resolvent.

**Lemma 4.2.** Let $f \in \mathcal{H}$. Then

$$\lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \| (H - \lambda - i\varepsilon)^{-1} P_{ac}(H)f \|^2 = \frac{d\|E_{\lambda}f\|^2}{d\lambda}$$

for almost every $\lambda$ with respect to Lebesgue measure.
Suppose that $D_0$ is a subset of a Hilbert space $H$ and $A$ is a family of bounded linear operators on $H$. Let $D$ be a linear space of all finite linear combinations of the vectors of the form $Af, f \in D_0$, where $A$ is the product of zero or finitely many elements of $A$. It is easy to see that $D_0 \subset D$ and $A D \subset D$ for all $A \in A$. We call $D$ the span of $D_0$ by $A$. We are interested in the case where $D_0$ is an orthonormal basis of $H$ and $A = \{ (H - z)^{-1} \}_{z \in \rho(H)}$. In this case we say that $D$ is the span of an orthonormal basis by the resolvent of $H$.

**Lemma 4.3.** Let $D$ be the span of an orthonormal basis by the resolvent of $H$. Then there exists a subset $\Lambda \subset \mathbb{R}$ of full Lebesgue measure such that $\langle E_{\lambda} f | g \rangle$ is differentiable at any $\lambda \in \Lambda$ whenever $f, g \in D$.

**Remark 4.1.** We remark that $\Lambda$ does not contain an eigenvalue of $H$.

**Proof.** Let $\{ \varphi_j \}$ be the orthonormal basis generating $D$, which is countable by assumption. There is a subset $\Lambda \subset \mathbb{R}$ of full Lebesgue measure such that $\langle E_{\lambda} \varphi_j | \varphi_k \rangle$ is differentiable at any $\lambda \in \Lambda$ for all $j, k$. By Lemma 4.1 $\langle E_{\lambda} A \varphi_j | B \varphi_k \rangle$ is differentiable at any $\lambda \in \Lambda$ for all $j, k$, where $A$ and $B$ are zero or finitely many products of operators in $\{ (H - z)^{-1} \}_{z \in \rho(H)}$. The statement follows by linearity.

**Lemma 4.4.** Suppose that each of $D_1$ and $D_2$ is the span of an orthonormal basis by the resolvent of $H$. Then there exists a set $\Lambda_0$ of full Lebesgue measure with the following properties:

i. $\langle E_{\lambda} f | g \rangle$ is differentiable at $\lambda$ whenever $\lambda \in \Lambda_0$ and $f, g \in D_1 + D_2$;

ii. for any $f \in D_2$ there exists a sequence $f_j \in D_1$ such that

$$
\| f - f_j \| \to 0, \quad \frac{d \| E_{\lambda} (f - f_j) \|^2}{d\lambda} \to 0 \quad (4.5)
$$

as $j \to \infty$ for any $\lambda \in \Lambda_0$;

iii. the same statement as (ii) with $D_1$ and $D_2$ interchanged.

**Proof.** (i) is obvious from Lemma 4.3. We prove (ii) and (iii). By (4.2) one can see that (4.5) holds almost everywhere for fixed $f \in D_2$. Let $A$ be a finite product of operators from $\{ (H - z)^{-1} \}_{z \in \rho(H)}$. By Lemma 4.1 we see that (4.5) remains valid when we replace $f$ and $f_j$ by $Af$ and $Af_j$ respectively. It follows that one can make $\Lambda_0$ independent of the choice of $f \in D_2$. We can make $\Lambda_0$ smaller so that (iii) holds as well. This completes the proof. □
**Definition 4.1.** Suppose that $H$ is a selfadjoint operator on a Hilbert space $\mathcal{H}$ and $\{\tau(\lambda)\}_{\lambda \in \mathbb{R}}$ is a family of pullback triples, where $\tau(\lambda) = (J(\lambda), F(\lambda), Y(\lambda))$ and the function spaces are as given by the commutative diagram below.

![Commutative Diagram](image)

We call $\{\tau(\lambda)\}_{\lambda \in \mathbb{R}}$ a pullback representation of the absolutely continuous part of $H$ if there exist a subset $\Lambda \subset \mathbb{R}$ of full Lebesgue measure and a span $\mathcal{D}$ of an orthonormal basis by the resolvent of $H$ such that

i. $\mathcal{D}$ is a core of $\tau(\lambda)$ for all $\lambda$;

ii. for any $f \in \mathcal{D}$ and $\lambda \in \Lambda$ we have

$$\frac{d\|E_\lambda f\|^2}{d\lambda} = \|F(\lambda)Y(\lambda)f\|^2_{\mathcal{E}(\lambda)};$$

iii. for $\lambda \in \mathbb{R} \setminus \Lambda$ we have $X(\lambda) = \mathcal{H}$, $Y(\lambda) = J(\lambda) = 1$, and $F(\lambda) = 0$.

We say that two representations $\{\tau_j(\lambda)\}_{\lambda \in \mathbb{R}}$, $j = 1, 2$ are equivalent if $\tau_1(\lambda)$ and $\tau_2(\lambda)$ are equivalent almost everywhere. We refer to $\mathcal{D}$ as a core of the representation.

**Theorem 4.5.** Any selfadjoint operator on a separable Hilbert space admits a pullback representation of the absolutely continuous part. The representation is unique up to equivalence.

**Proof.** Let $\mathcal{D}$ and $\Lambda$ be as given by Lemma 4.3. We see that the function $q_\lambda : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ defined for each $\lambda \in \Lambda$ by

$$q_\lambda(f, g) = \frac{d\langle E_\lambda f, g \rangle}{d\lambda}, \quad f, g \in \mathcal{D}(q) = \mathcal{D}$$

(4.6)

is well-defined as a densely defined nonnegative quadratic form on $\mathcal{H}$. We set $q_\lambda = 0$, $\mathcal{D}(q_\lambda) = \mathcal{D}$ for $\lambda \in \mathbb{R} \setminus \Lambda$. For each $\lambda \in \mathbb{R}$ let

$$J(\lambda) : \mathcal{X}(\lambda) \rightarrow \mathcal{H}, \quad F(\lambda) : \mathcal{X}(\lambda) \rightarrow \mathcal{E}(\lambda), \quad Y(\lambda) : \mathcal{H} \rightarrow \mathcal{X}(\lambda)$$

be a pullback triple of $q_\lambda$. By Theorem 3.5 we have

$$q_\lambda(f, g) = \langle F(\lambda)Y(\lambda)f, F(\lambda)Y(\lambda)g \rangle_{\mathcal{E}(\lambda)}, \quad f, g \in \mathcal{X}(\lambda).$$
We have thus proven that any selfadjoint operator admits a pullback representation. We now show the uniqueness. Suppose that \( \tau_{j} = \{ \tau_{j}(\lambda) \} \), \( j = 1, 2 \) are pullback representations of \( H \). Write \( \tau_{j}(\lambda) = (J_{j}(\lambda), F_{j}(\lambda), Y_{j}(\lambda)) \). In view of Lemma 3.3 we may assume that \( \mathcal{D}(Y_{j}(\lambda)) = \mathcal{D}_{j} \), \( j = 1, 2 \), where each \( \mathcal{D}_{j} \) is a span of an orthonormal basis by the resolvent of \( H \). Let \( \Lambda_{0} \) be as in Lemma 4.4. Then for each \( \lambda \in \Lambda_{0} \) there is a well-defined linear operator \( Y_{3}(\lambda) : \mathcal{H} \rightarrow \mathcal{X}_{1}(\lambda) \) with \( \mathcal{D}(Y_{3}(\lambda)) = \mathcal{D}_{2} \) such that

\[
J_{1}(\lambda)Y_{3}(\lambda)f = f, \quad \frac{d\|E_{\lambda}f\|^{2}}{d\lambda} = \|F_{1}(\lambda)Y_{3}(\lambda)f\|_{E_{\lambda}(\lambda)}^{2}, \quad (4.7)
\]

\[
\frac{d(E_{\lambda}f|g)}{d\lambda} = \langle F_{1}(\lambda)Y_{3}(\lambda)f|F_{1}(\lambda)Y_{1}(\lambda)g \rangle, \quad g \in \mathcal{D}_{1} \quad (4.8)
\]

for all \( f \in \mathcal{D}(Y_{3}(\lambda)) \). In view of Lemma 3.3 it suffices to show that \( \tau_{3}(\lambda) = (J_{1}(\lambda), F_{1}(\lambda), Y_{3}(\lambda)) \) is a pullback triple for all \( \lambda \in \Lambda_{0} \). Clearly \( Y_{3}(\lambda) \) is densely defined and injective. We now show that \( Y_{3}(\lambda) \) has a dense range. Let \( \varphi \in \mathcal{D}_{1} \). Then there exists a sequence \( \varphi_{j} \in \mathcal{D}_{2} \) so that

\[
\|\varphi - \varphi_{j}\| \rightarrow 0, \quad \frac{d\|E_{\lambda}(\varphi - \varphi_{j})\|^{2}}{d\lambda} \rightarrow 0 \quad (4.9)
\]

for all \( \lambda \in \Lambda_{0} \). By (4.7) we see that \( Y_{3}(\lambda)\varphi_{j} \rightarrow u \) in \( \mathcal{X}_{1}(\lambda) \) for some \( u \in \mathcal{X}_{1}(\lambda) \) and (4.8) yields

\[
\langle F_{1}(\lambda)|Y_{3}(\lambda)\varphi_{j} - Y_{1}(\lambda)\varphi|F_{1}(\lambda)Y_{1}(\lambda)g \rangle \rightarrow 0, \quad g \in \mathcal{D}_{1}.
\]

Therefore \( J_{1}(\lambda)u = \varphi \) and \( F_{1}(\lambda)u = F_{1}(\lambda)Y_{1}(\lambda)\varphi \). This means \( u = Y_{1}(\lambda)\varphi \) and hence \( Y_{3}(\lambda) \) has a dense range. We have thus shown that \( Y_{3}(\lambda) \) is a completion operator. This and (4.7) means that \( \tau_{3}(\lambda) \) is a pullback triple. \( \square \)

5. Generalized eigenfunction expansion

The purpose of this section is to show that the absolutely continuous part of an arbitrary selfadjoint operator \( H \) admits a generalized eigenfunction expansion. We shall first show that a pullback representation \( \tau = \{ \tau(\lambda) \} \) \( \lambda \in \mathbb{R} \) of \( H \) induces a family of continuations \( \{ H_{\lambda} \} \) \( \lambda \in \mathbb{R} \) of \( H \). Then we show that the spectral structure of \( H_{\lambda} \) and \( H \) are identical except at energy \( \lambda \) and the spectral density \( E_{\lambda}' \) of \( H \) at energy \( \lambda \) is identical to a pullback of the orthogonal projection onto the nullspace of \( H_{\lambda} - \lambda \) for almost every \( \lambda \). This means
that the structure of the absolutely continuous part of $H$ is encoded into the family $\{N(H_\lambda - \lambda)\}_{\lambda \in \mathbb{R}}$ of nullspaces and we can recover the absolutely continuous part of the original operator from this family.

Throughout this section we fix a pullback representation $\tau$ of the absolutely continuous part of $H$. We use the same operators and function spaces as in the Definition 4.1.

**Lemma 5.1.** Let $\lambda \in \mathbb{R}$. Then

i. $f \in \mathcal{D}$ and Im $z \neq 0$ imply

$$F(\lambda)Y(\lambda)(H - z)^{-1}f = (\lambda - z)^{-1}F(\lambda)Y(\lambda)f; \quad (5.1)$$

ii. $(H - z)^{-1}\mathcal{D}$ is a core of $\tau(\lambda)$ and we have

$$\|Y(\lambda)(H - z)^{-1}f\|_{\chi(\lambda)} \leq |\text{Im } z|^{-1}\|Y(\lambda)f\|_{\chi(\lambda)} \quad (5.2)$$

for $f \in \mathcal{D}$ and Im $z \neq 0$;

iii. $H$ is compatible with $Y(\lambda)$ in $\rho(H)$.

**Proof.** (i) is immediate from Lemma 4.1 (ii). From (i) we see that

$$\|Y(\lambda)(H - z)^{-1}f\|_{\chi(\lambda)}^2 = \|(H - z)^{-1}f\|^2 + |(\lambda - z)|^{-2}\|F(\lambda)Y(\lambda)f\|_{\chi(\lambda)}^2.$$  

This implies (5.2). Suppose that $u \in \chi(\lambda)$ and that $\langle u|Y(\lambda)(H - z)^{-1}g\rangle_{\chi(\lambda)} = 0$ whenever $g \in \mathcal{D}$. It follows from (i) that

$$\langle J(\lambda)u|J(\lambda)Y(\lambda)(H - z)^{-1}g\rangle + (\lambda - \bar{\tau})^{-1}\langle F(\lambda)u|F(\lambda)Y(\lambda)g\rangle = 0$$

for $g \in \mathcal{D}$. Let $f_j$ be a sequence in $\mathcal{D}$ such that $Y(\lambda)f_j \to u$ in $\chi(\lambda)$. Note that $f_j \to J(\lambda)u$ in $\mathcal{K}$. The inequality (5.2) implies

$$\langle f_j|(H - z)^{-1}f_j\rangle + (\lambda - \bar{\tau})^{-1}\|F(\lambda)Y(\lambda)f_j\|^2 \to 0$$

and by taking the imaginary parts we see that

$$\|f_j\| \to 0, \quad \|F(\lambda)Y(\lambda)f_j\|_{\varepsilon(\lambda)} \to 0.$$  

Hence $u = 0$. By the Hahn-Banach theorem we conclude that $Y(\lambda)(H - z)^{-1}\mathcal{D}$ is dense in $\chi(\lambda)$. (iii). If $\lambda \in \mathbb{R} \cap \rho(H)$, then $F(\lambda) = 0$ and thus $Y(\lambda)$ is isometric. This combined with (i) and (ii) imply (i) through (iii) of Definition 2.1. To verify the condition (iv) suppose $Y(\lambda)f_j \to u$ in $\chi(\lambda)$ with $f_j \in \mathcal{D}$ and $Y(\lambda)(H - z)^{-1}f_j \to 0$ in $\chi(\lambda)$. Then (i) yields

$$\|(H - z)^{-1}f_j\| \to 0, \quad \|F(\lambda)Y(\lambda)f_j\|_{\varepsilon(\lambda)} \to 0$$

so we must have $u = 0$. This completes the proof. \qed
Lemma 5.2. Let $\lambda \in \mathbb{R}$. Then $H$ has a selfadjoint continuation $H_\lambda$ to $\mathcal{X}(\lambda)$ induced by $Y(\lambda)$ and it satisfies
\[
F(\lambda)\chi(H_\lambda) = \chi(\lambda)F(\lambda), \quad J(\lambda)\chi(H_\lambda) = \chi(H)J(\lambda), \quad (5.3)
\]
for $\chi \in C^\infty_0(\mathbb{R})$.

**Proof.** Lemma 5.1 implies that $H$ has a continuation $H_\lambda$ to $\mathcal{X}(\lambda)$ and that $\rho(H) \subset \rho(H_\lambda)$. In view of Lemma 5.1 the first identity of (5.3) holds for $\chi(H_\lambda) = (H - z)^{-1}u$ with $z \in \rho(H)$ and thus for all $\chi \in C^\infty_0(\mathbb{R})$. Take $f_j \in D$ such that $Y(\lambda)f_j \to u$ in $\mathcal{X}(\lambda)$. Then we have
\[
J(\lambda)(H_\lambda - z)^{-1}Y(\lambda)f_j = (H - z)^{-1}f_j \to (H - z)^{-1}Ju
\]
for any $z \in \rho(H)$. This implies the second identity of (5.3). For any $z \in \rho(H)$ we have
\[
\text{Im} \langle (H_\lambda - z)^{-1}u | \chi(\lambda) \rangle = \text{Im} z\| (H_\lambda - z)^{-1}u \|_\chi^2
\]
for all $u \in \mathcal{Y}(\lambda)D$ and thus for all $u \in \mathcal{X}(\lambda)$. Therefore $H_\lambda$ is symmetric and hence selfadjoint by Corollary 2.2. □

By Lemma 5.2, $J(\lambda)\mathcal{N}(F(\lambda))$ is dense in $\mathcal{H}$ for all $\lambda \in \mathbb{R}$, so we have the following

**Corollary 5.3.** The pullback triple $\tau(\lambda)$ is splitting for all $\lambda \in \mathbb{R}$.

**Theorem 5.4.** For any $\lambda \in \mathbb{R}$ the spectrum of $H$ and $H_\lambda$ are identical. We have
\[
J(\lambda)H_\lambda = HJ(\lambda), \quad F(\lambda)H_\lambda \subset \lambda F(\lambda). \quad (5.4)
\]
$J(\lambda)$ and $F(\lambda)$ are partial isometries. For $\lambda \in \Lambda$ we have
\[
\mathcal{R}(H_\lambda - \lambda) = \mathcal{N}(F(\lambda)) = \mathcal{R}(J(\lambda)^*), \quad \mathcal{R}(J(\lambda)) \subset \mathcal{H}, \quad (5.5)
\]
\[
\mathcal{N}(H_\lambda - \lambda) = \mathcal{N}(J(\lambda)) = \mathcal{R}(F(\lambda)^*), \quad \mathcal{R}(F(\lambda)) \subset \mathcal{E}(\lambda). \quad (5.6)
\]

**Proof.** Lemma 5.2 implies (5.4). We now prove
\[
\mathcal{N}(H_\lambda - \lambda) = \mathcal{N}(J(\lambda)). \quad (5.7)
\]
As $H_\lambda$ is selfadjoint and $F(\lambda)(H_\lambda - \lambda)v = 0$ for any $v \in D(H_\lambda), u \in \mathcal{N}(H_\lambda - \lambda)$ if and only if $\langle J(\lambda)u| J(\lambda)(H_\lambda - \lambda)v \rangle = 0$ for all $v \in D(H_\lambda)$. This condition
is equivalent to $J(\lambda)u = 0$ because $J(\lambda)(H_\lambda - \lambda)$ has a dense range. Hence we obtain (5.7). By Proposition 3.8, (5.5) and (5.6) follow. We now prove $\sigma(H) = \sigma(H_\lambda)$. This is obvious if $\lambda \in \mathbb{R} \setminus \Lambda$, so we may assume $\lambda \in \Lambda$. Then $J(\lambda)$ is a partial isometry with initial space $N(Q(\lambda))$ and final space $H$, where $Q(\lambda) = F(\lambda)^*F(\lambda)$. Note that $Q(\lambda)$ is the orthogonal projection onto the eigenspace of $H_\lambda$ with eigenvalue $\lambda$. This and (5.4) imply that $E_H(I)$ is unitarily equivalent to $E_{H_\lambda}(I \setminus \{\lambda\})$ for any interval $I$. If $\lambda \in \rho(H)$, then $E_{H_\lambda}(\{\lambda\}) = Q(\lambda) = 0$ and thus $\lambda \in \rho(H_\lambda)$. Conversely, $\lambda \in \rho(H_\lambda)$ implies $Q(\lambda) = 0$ and so $\lambda \in \rho(H)$. This completes the proof. □

**Theorem 5.5.** There exists a partial isometry

$$F : \mathcal{H} \longrightarrow \int_\mathbb{R}^+ \mathcal{E}(\lambda)d\lambda$$

(5.8)

such that

$$(Ff)(\lambda) = F(\lambda)Y(\lambda)f, \quad f \in \mathcal{D}, \quad \lambda \in \mathbb{R}.$$

We have

$$F^*F = P_{ac}(H), \quad FF^* = 1.$$  

(5.9)

**Remark 5.1.** We remark that

$$\frac{d(E_{\lambda}f|g)}{d\lambda} = \langle Q(\lambda)Y(\lambda)f|Y(\lambda)g\rangle_{X(\lambda)}, \quad f, g \in \mathcal{D}$$

almost everywhere, where $Q(\lambda) = F(\lambda)^*F(\lambda)$. It follows that the spectral density $E_\lambda'$ is a pullback of the eigenprojection $Q(\lambda)$ of $H_\lambda$ by the completion operator $Y(\lambda)$ and Theorem 5.5 implies

$$\langle H_{ac}f|g\rangle = \int_\mathbb{R} \lambda\langle Q(\lambda)Y(\lambda)f|Y(\lambda)g\rangle_{X(\lambda)}d\lambda.$$  

The absolutely continuous part of a selfadjoint operator thus admits a generalized eigenfunction expansion. It should be noted that $Q(\lambda) = 0$ if and only if the spectral density is closable at $\lambda$ and that the size of the generalized eigenspace

$$N(H_\lambda - \lambda) = \mathcal{R}(Q(\lambda)) = \mathcal{N}(J(\lambda))$$

is determined by how far is the spectral density from being closable at energy $\lambda$.  

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Proof. It is easy to see that
\[ \| P_{ac}(H)f \|^2 = \int_{\mathbb{R}} \| F(\lambda)Y(\lambda)f \|_{E(\lambda)}^2 d\lambda \quad f \in \mathcal{D}. \]
It follows that the partial isometry \( F \) exists and we have \( F^*F = P_{ac}(H) \). We now show that \( F \) is surjective. By the Hahn-Banach theorem it suffices to prove that
\[
\int_{\mathbb{R}} \langle g(\lambda) | F(\lambda)Y(\lambda)f \rangle_{E(\lambda)} d\lambda = 0, \quad f \in \mathcal{D},
\]
implies \( g(\lambda) = 0 \) for almost every \( \lambda \). The identity (5.3) yields
\[
\int_{\mathbb{R}} \varphi(\lambda) \langle g(\lambda) | F(\lambda)Y(\lambda)f \rangle_{E(\lambda)} d\lambda = 0
\]
for any \( \varphi \in C_0^\infty(\mathbb{R}) \) and \( f \in \mathcal{D} \). Since \( \mathcal{D} \) is the span of an orthonormal basis by the resolvents of \( H \), there exists a set \( \Lambda_0 \subset \mathbb{R} \) of full Lebesgue measure which is independent of \( f \) so that \( \langle g(\lambda) | F(\lambda)Y(\lambda)f \rangle_{E(\lambda)} = 0 \) for \( \lambda \in \Lambda_0 \) and \( f \in \mathcal{D} \). Since \( F(\lambda) \) is bounded and surjective, we must have \( g(\lambda) = 0 \) for almost every \( \lambda \). This completes the proof. \( \square \)

6. Acknowledgements

The author is grateful to Professor Shu Nakamura for valuable discussions and constructive remarks.

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