Metric–affine gauge theory of gravity

II. Exact solutions

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Abstract

In continuing our series on metric-affine gravity (see Gronwald, IJMP D6 (1997) 263 for Part I), we review the exact solutions of this theory. file magexac7.tex, 1999-04-09

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I. METRIC–AFFINE GRAVITY (MAG)

In 1976, a new metric–affine theory of gravitation was published [16]. In this model, the metric $g_{ij}$ and the linear (sometimes also called affine) connection $\Gamma^k_{ij}$ were considered to be independent gravitational field variables. The metric carries 10 and the connection 64 independent components. Although nowadays more general Lagrangians are considered, like the one in Eq.(10), the original Lagrangian density of metric–affine gravity reads

$$V_{GR'} = \sqrt{-g} \frac{1}{2\kappa} g^{ij} \left[ R_{ij} (\Gamma, \partial \Gamma) + \beta Q_i Q_j \right].$$  \hspace{1cm} (1)

The Ricci tensor $R_{ij}$ depends only on the connection but not on the metric, whereas the Weyl covector $Q_i := -g^{kl} \nabla_i g_{kl}/4$ depends on both. Here $\nabla_i$ represents the covariant derivative with respect to the connection $\Gamma^k_{ij}$, furthermore $g = \det g_{kl}$, $\kappa$ is Einstein’s gravitational constant, and $\beta$ a dimensionless coupling constant. With $i,j,k,\cdots = 0,1,2,3$ we denote coordinates indices.

This model leads back to general relativity, as soon as the material current coupled to the connection, namely $\sqrt{-g} \Delta^i_{jk} := \delta L_{mat}/\delta \Gamma^k_{ij}$, the so–called hypermomentum, vanishes. Thus, in such a model, the post–Riemannian pieces of the connection and the corresponding new interactions are tied to matter, they do not propagate.

As we know from the weak interaction, a contact interaction appears to be suspicious for causality reasons, and one wants to make it propagating, even if the carrier of the interaction, the intermediate gauge boson, may become very heavy as compared to the mass of the proton, e.g.. However, before we report on the more general gauge Lagrangians that have been used, we turn back to the geometry of spacetime.

MAG represents a gauge theory of the 4–dimensional affine group enriched by the existence of a metric. As a gauge theory, it finds its appropriate form if expressed with respect to arbitrary frames or coframes. Therefore, the apparatus of MAG was reformulated in the calculus of exterior differential forms, the result of which can be found in the review paper [14], see also [19] and [15]. Of course, MAG could have been alternatively reformulated in
tensor calculus by employing an arbitrary (anholonomic) frame (tetrad or vierbein formalism), but exterior calculus, basically in a version which was advanced by Trautman [42] and others [31][4], seems to be more compact.

In the new formalism, we have then the metric $g_{\alpha\beta}$, the coframe $\vartheta^\alpha$, and the connection 1–form $\Gamma^\alpha_{\beta}$ (with values in the Lie algebra of the 4–dimensional linear group $GL(4,R)$) as new independent field variables. Here $\alpha, \beta, \gamma, \cdots = 0, 1, 2, 3$ denote (anholonomic) frame indices. For the formalism, including the conventions, which we will be using in this paper, we refer to [19].

A first order Lagrangian formalism for a matter field $\Psi$ minimally coupled to the gravitational potentials $g_{\alpha\beta}$, $\vartheta^\alpha$, $\Gamma^\alpha_{\beta}$ has been set up in [13]. Spacetime is described by a metric–affine geometry with the gravitational field strengths nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, torsion $T^\alpha := D\vartheta^\alpha$, and curvature $R^\alpha_{\beta} := d\Gamma^\alpha_{\beta} - \Gamma^\alpha_{\gamma} \wedge \Gamma^\gamma_{\beta}$. The gravitational field equations

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{2}$$
$$DH^\alpha_{\beta} - E^\alpha_{\beta} = \Delta^\alpha_{\beta}, \tag{3}$$

link the material sources, the material energy–momentum current $\Sigma_\alpha$ and the material hypermomentum current $\Delta^\alpha_{\beta}$, to the gauge field excitations $H_\alpha$ and $H^\alpha_{\beta}$ in a Yang–Mills like manner. In [19] it is shown that the field equation corresponding to the variable $g_{\alpha\beta}$ is redundant if (2) as well as (3) are fulfilled.

If the gauge Lagrangian 4–form

$$V = V \left(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R^\alpha_{\beta}\right) \tag{4}$$

is given, then the excitations can be calculated by partial differentiation,

$$H_\alpha = -\frac{\partial V}{\partial T^\alpha}, \quad H^\alpha_{\beta} = -\frac{\partial V}{\partial R^\alpha_{\beta}}, \quad M^\alpha_{\beta} = -2\frac{\partial V}{\partial Q^\alpha_{\beta}}, \tag{5}$$

whereas the gauge field currents of energy–momentum and hypermomentum, respectively, turn out to be linear in the Lagrangian and in the excitations,
$$E_\alpha := \frac{\partial V}{\partial \vartheta_\alpha} = e_\alpha J V + (e_\alpha J T^\beta) \wedge H_\beta + (e_\alpha J R_\beta^\gamma) \wedge H_\gamma + \frac{1}{2} (e_\alpha J Q_{\beta\gamma}) M^{\beta\gamma},$$  \hfill (6)

$$E^\alpha_{\beta} := \frac{\partial V}{\partial \Gamma_{\alpha\beta}} = -\vartheta^\alpha \wedge H_\beta - g_{\beta\gamma} M^{\alpha\gamma}.$$  \hfill (7)

Here $e_\alpha$ represents the frame and $J$ the interior product sign, for details see [19].

\section*{II. THE QUADRATIC GAUGE LAGRANGIAN OF MAG}

The gauge Lagrangian (1), in the new formalism, is a 4–form and reads \cite{27}

$$V_{GR'} = \frac{1}{2\kappa} \left(-R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \beta Q \wedge *Q\right).$$  \hfill (8)

Here $\eta_{\alpha\beta} := *(\vartheta_\alpha \wedge \vartheta_\beta)$, * denotes the Hodge star. Besides Einstein gravity, it encompasses additionally contact interactions.

It is obvious of how to make $Q$ a propagating field: One adds, to the massive $\beta$–term, a kinetic term \cite{17,36} $-\alpha dQ \wedge *dQ/2$. Since $dQ = R_{\gamma}^\gamma/2$, the kinetic term can alternatively be written as

$$-\frac{\alpha}{8} R_{\beta}^\beta \wedge *R_{\gamma}^\gamma.$$  \hfill (9)

This term, with the appearance of one Hodge star, displays a typical Yang–Mills structure. More generally, propagating post–Riemannian gauge interactions in MAG can be consistently constructed by adding terms quadratic in $Q_{\alpha\beta}$, $T^\alpha$, $R_{\alpha\beta}$ to the Hilbert-Einstein type Lagrangian and the term with the cosmological constant.

In the first order formalism we are using, higher order terms, i.e. cubic and quartic ones etc. would preserve the second order of the field equations. However, the quasilinearity of the gauge field equations would be destroyed and, in turn, the Cauchy problem would be expected to be ill–posed. Therefore we do not go beyond a gauge Lagrangian which is quadratic in the gauge field strengths $Q_{\alpha\beta}$, $T^\alpha$, $R_{\alpha\beta}$. Incidentally, a quadratic Lagrangian is already so messy that it would be hard to handle a still more complex one anyway.

Different groups have already added, within a metric–affine framework, different quadratic pieces to the Hilbert–Einstein–type Lagrangian, see \cite{52,13,26,64,19,41,33,11},
e.g., and references given there. The end result of all these deliberations is the most general parity conserving quadratic Lagrangian which is expressed in terms of the $4 + 3 + 11$ irreducible pieces (see [19]) of $Q_{\alpha \beta}, T^\alpha, R_{\alpha \beta}$, respectively:

$$V_{\text{MAG}} = \frac{1}{2\kappa} \left[ -a_0 R^{\alpha \beta} \wedge \eta_{\alpha \beta} - 2\lambda \eta + T^\alpha \wedge \left( \sum_{I=1}^{3} a_I (I) T_\alpha \right) 
+ 2 \left( \sum_{I=2}^{4} c_I (I) Q_{\alpha \beta} \right) \wedge \vartheta^\alpha \wedge *T^\beta + Q_{\alpha \beta} \wedge *\left( \sum_{I=1}^{4} b_I (I) Q^{\alpha \beta} \right) 
+ b_5 \left( (3) Q_{\alpha \gamma} \wedge \vartheta^\alpha \right) \wedge *\left( (4) Q^{\beta \gamma} \wedge \vartheta_\beta \right) \right] 
- \frac{1}{2\rho} R^{\alpha \beta} \wedge *\left( \sum_{I=1}^{6} w_I (I) W_{\alpha \beta} + w_7 \vartheta_\alpha \wedge (e_\gamma)^{(5)} W^\gamma_\beta \right) 
+ 5 \sum_{I=1}^{6} z_I (I) Z_{\alpha \beta} + z_6 \vartheta_\gamma \wedge (e_\alpha)^{(2)} Z^\gamma_\beta \right) \right. 
\left. + \sum_{I=7}^{9} z_I \vartheta_\alpha \wedge (e_\gamma)^{(I-4)} Z^\gamma_\beta \right) \right]. \quad (10)$$

The constant $\lambda$ is the cosmological constant, $\rho$ the strong gravity coupling constant, the constants $a_0, \ldots, a_3, b_1, \ldots, b_5, c_2, c_3, c_4, w_1, \ldots, w_7, z_1, \ldots, z_9$ are dimensionless. We have introduced in the curvature square term the irreducible pieces of the antisymmetric part $W_{\alpha \beta} := R_{[\alpha \beta]}$ and the symmetric part $Z_{\alpha \beta} := R_{(\alpha \beta)}$ of the curvature 2–form. In $Z_{\alpha \beta}$, we have the purely post–Riemannian part of the curvature. Note the peculiar cross terms with $c_I$ and $b_5$.

Esser [3], in the component formalism, has carefully enumerated all different pieces of a quadratic MAG Lagrangian, for the corresponding nonmetricity and torsion pieces, see also Duan et al. [6]. Accordingly, Eq.(10) represents the most general quadratic parity–conserving MAG–Lagrangian. All previously published quadratic parity–conserving Lagrangians are subcases of (10). Hence (10) is a safe starting point for our future considerations.

We concentrate here on Yang–Mills type Lagrangians. Since $V_{\text{MAG}}$ is required to be an odd 4–form, if parity conservation is assumed, we have to build it up according to the scheme $F \wedge *F$, i.e. with one Hodge star, since the star itself is an odd operator. Also the Hilbert–Einstein type term is of this type, namely $\sim R^{\alpha \beta} \wedge *(\vartheta_\alpha \wedge \vartheta_\beta)$, as well as the cosmological term $\sim \eta = *1$. Thus $V_{\text{MAG}}$ is homogeneous of order one in the star operator. It is conceivable that in future one may want also consider parity violating terms with no
star appearing (or an even number of them) of the (Pontrjagin) type $F \wedge F$. Typical terms of this kind in four dimensions would be

$$R^{\alpha \beta} \wedge (\vartheta_{\alpha} \wedge \vartheta_{\beta}), \quad 1, \quad T_{\alpha} \wedge T_{\alpha}, \quad Q_{\alpha \beta} \wedge \vartheta^{\alpha} \wedge T^{\beta}, \quad R^{\alpha \beta} \wedge R_{\alpha \beta}. \quad (11)$$

The first term of (11), e.g., represents the totally antisymmetric piece of the curvature $R^{[\gamma \delta \alpha \beta]} \vartheta_{\gamma} \wedge \vartheta_{\delta} \wedge \vartheta_{\alpha} \wedge \vartheta_{\beta}$, which is purely post–Riemannian. Such parity–violating Lagrangians have been studied in the past, see, e.g., [22,31] and [18,32], but, for simplicity, we will restrict ourselves in this article to parity preserving Lagrangians.

### III. ON THE POSSIBLE PHYSICS OF MAG

Here we are, with a Lagrangian $V_{\text{MAG}}$ encompassing more than two dozens of unknown dimensionless constants. But the situation is not as bad as it may look at first. For the Newton–Einstein type of weak gravity — the corresponding terms are collected in (11) within two square brackets — we have the gravitational constant $\kappa$, with dimension of $\kappa = \text{length}^2$, and the cosmological constant $\lambda$, with dimension of $\lambda = \text{length}^{-2}$. For strong gravity of the Yang–Mills type, the basic newly postulated interaction within the MAG framework, the strength of the coupling is determined by the dimensionless strong coupling constant $\rho$. Thus, the three constants $\kappa, \lambda, \rho$ are fundamental, whereas the rest of the constants, 12 for weak and 16 for strong gravity, are expected to be of the order unity or should partially vanish.

As was argued elsewhere [19], we do not believe that at the present state of the universe the geometry of spacetime is described by a metric–affine one. We rather think, and there is good experimental evidence, that the present-day geometry is metric-compatible, i.e., its nonmetricity vanishes. In earlier epochs of the universe, however, when the energies of the cosmic “fluid” were much higher than today, we expect scale invariance to prevail — and the canonical dilation (or scale) current of matter, the trace of the hypermomentum current $\Delta_{\gamma \gamma}$, is coupled, according to MAG, to the Weyl covector $Q^\gamma_\gamma$. By the same token,
shear type excitations of the material multispinors (Regge trajectory type of constructs) are
expected to arise, thereby liberating the (metric-compatible) Riemann-Cartan spacetime
from its constraint of vanishing nonmetricity $Q_{\alpha\beta} = 0$. Tresguerres [45] has proposed a
simple cosmological model of Friedmann type which carries a metric-affine geometry at the
beginning of the universe, the nonmetricity of which dies out exponentially in time. That is
the kind of thing we expect.

If one keeps the differential manifold structure of spacetime intact, i.e., doesn’t turn to
discrete structures or non-commutative geometry, then MAG appears to be the most natural
extension of Einstein’s gravitational theory. The rigid metric-affine structure underlying the
Minkowski space of special relativity, see Kopczyński and Trautman [23], make us believe
that this structure should be gauged according to recipes known from gauge theory. Also
the existence, besides the energy-momentum current, of the external material currents of
spin and dilation (and, perhaps, of shear) does point in the same direction.

IV. EXACT MAG SOLUTIONS OF TRESGUERRES AND TUCKER & WANG

For getting a deeper understanding of the meaning and the possible consequences of
MAG, a search for exact solutions appears indispensable. Tresguerres, after finding exact
solutions [43,44] for specific (1 + 2)–dimensional models of MAG, turned his attention to
1 + 3 dimensions and, in 1994, for a fairly general subclass of the Lagrangian (10), found the
first static spherically symmetric solutions with a non–vanishing shear charge [46,47], i.e.,
the solution is endowed with a traceless part $Q_{\alpha\beta} := Q_{\alpha\beta} - Q g_{\alpha\beta}$ of the nonmetricity. This
constituted a breakthrough. Since that time, $Q_{\alpha\beta}$ lost its somewhat elusive and abstract
character. Even an operational interpretation has been attempted in the meantime [29].

The metric of Tresguerres’ solution is the Reissner–Nordström metric of general relativity
with cosmological constant but the place of the electric charge is taken by the dilation
charge which is related to the trace of the nonmetricity, the Weyl covector. Furthermore,
the Tresguerres solutions carries, besides the above-mentioned shear charge (related to the
TABLES

TABLE I. Irreducible decomposition of the nonmetricity* nom $Q_{\alpha\beta}$

| name   | number of indep. comp. | piece                                                                 |
|--------|------------------------|----------------------------------------------------------------------|
| nom    | 40                     | $Q_{\alpha\beta}$                                                  |
| trinom | 16                     | $(1)Q_{\alpha\beta} := Q_{\alpha\beta} - (2)Q_{\alpha\beta} - (3)Q_{\alpha\beta} - (4)Q_{\alpha\beta}$ |
| binom  | 16                     | $(2)Q_{\alpha\beta} := \frac{2}{3} \eta(\theta_{(\alpha \wedge \Omega_{\beta})})$ |
| vecnom | 4                      | $(3)Q_{\alpha\beta} := \frac{1}{9} \left( \theta_{(\alpha e_{\beta})}\Lambda - \frac{1}{3} g_{\alpha\beta}\Lambda \right)$ |
| conom  | 4                      | $(4)Q_{\alpha\beta} := g_{\alpha\beta}Q$                           |

*) First the nonmetricity is split into its trace, the Weyl covector $Q := \frac{1}{4}g^{\alpha\beta}Q_{\alpha\beta}$, and its traceless piece $Q_{\alpha\beta} := Q_{\alpha\beta} - Qg_{\alpha\beta}$. The traceless piece yields the shear covector $\Lambda := \theta^{\alpha} e^{\beta} \vert Q_{\alpha\beta}$ and the shear 2-form $\Omega_{\alpha} := \Theta_{\alpha} - \frac{1}{3} e_{\alpha} \vert (\theta^{\beta} \wedge \Theta_{\beta})$, with $\Theta_{\alpha} := * (Q_{\alpha\beta} \wedge \theta^{\beta})$. The 2-form $\Omega^{\alpha}$ describes $T_{\alpha}^{(2)}$ and has precisely the same symmetry properties as the 2-form $(1)T_{\alpha}$ (see below). In particular, we can prove that $e_{\alpha} \vert \Omega^{\alpha} = 0$ and $\theta_{\alpha} \wedge \Omega^{\alpha} = 0$.

TABLE II. Irreducible decomposition of the torsion** tor $T_{\alpha}$

| name   | number of indep. comp. | piece                                                                 |
|--------|------------------------|----------------------------------------------------------------------|
| tor    | 24                     | $T_{\alpha}$                                                        |
| tentor | 16                     | $(1)T_{\alpha} := T_{\alpha} - (2)T_{\alpha} - (3)T_{\alpha}$       |
| trator | 4                      | $(2)T_{\alpha} := \frac{1}{3} \theta^{\alpha} \wedge T$            |
| axitor | 4                      | $(3)T_{\alpha} := -\frac{1}{3} *(\theta^{\alpha} \wedge A)$         |

**) The 1-forms $T$ (torsion trace or covector) and $A$ (axial covector) are defined by $T := e_{\alpha} \vert T^{\alpha}$ and $A := *(\theta_{\alpha} \wedge T^{\alpha})$, respectively.
traceless part of the nonmetricity) a *spin charge* related to the torsion of spacetime. Thus, beyond the Reissner–Nordström metric, the following post–Riemannian degrees of freedom are excited in the Tresguerres solutions (see Tables I and II): two pieces of the nonmetricity, namely \((4) Q_{\alpha\beta} \) (conom, which is equivalent to the Weyl covector) and the traceless piece \((2) Q_{\alpha\beta} \) (binom), and all three pieces of the torsion \((1) T^\alpha \) (tentor), \((2) T^\alpha \) (trator), \((3) T^\alpha \) (axitor). The names in the parentheses are taken from our computer programs [19,39]. The first solution [46], requires in the Lagrangian weak gravity terms and, for strong gravity, the curvature square pieces with \(z_4 \neq 0\), \(w_3 \neq 0\), \(w_5 \neq 0\), i.e., with Weyl’s segmental curvature (dilcurv), the curvature pseudoscalar (pscalar), and the antisymmetric Ricci (ricanti). In his second solution [47], the torsion is independent of the nonmetricity, otherwise the situation is similar yet not as clear cut.

The price Tresguerres had to pay in order to find exact solutions at all was to impose *constraints* on the dimensionless coupling constants of MAG. In other words, the Lagrangian \(V_{\text{MAG}}\) was engineered such that exact solutions emerged. This is, of course, not exactly what one really wants. Rather one would like to prescribe a Lagrangian and then to find an exact solution. But, with the methods then available, one could not do better. And one was happy to find exact solutions at all for such complicated Lagrangians.

As to the methods applied, one fact should be stressed. To handle Lagrangians like (10), it is practically indispensable to use *computer algebra* tools. This is also what Tresguerres did. He took Schröfer’s *Excalc* package of Hearn’s computer algebra system *Reduce*; for introductory lectures on Reduce and Excalc see 10. More recently, we described the corresponding computer routines within MAG in some detail 39 and showed of how to build up Excalc programs for finding exact solutions of MAG. What one basically does with these programs, is to make a clever ansatz for the coframe, the torsion, and the nonmetricity, then to substitute this into the field equations, as programmed in Excalc, and subsequently to inspect these expressions in order to get an idea of how to solve them. One way of reducing them to a manageable size, is to constrain the dimensionless coupling constants or to solve, also by computer algebra methods, some of the partial differential equations emerging. If


| solution | references | post-Riemannian structures |
|----------|------------|-----------------------------|
| **Monopoles** with strong gravito-electric and strong gravito-magnetic charge (and combinations of them) plus triplet (degenerate case of Reissner-Nordström solution with triplet) | \[24,25\] | $\text{conom} \sim \text{vecnom} \sim \text{trator}$ |
| **Reissner–Nordström** metric with strong gravito-electric charge plus nom and tor | | |
| — dilation type solution | \[16,18,21\] | $\text{conom} \sim \text{trator}, \text{axitor} [21]$ |
| — triplet type solution | \[14,7\] | $\text{conom} \sim \text{vecnom} \sim \text{trator}$ |
| — dilation–shear type solution | \[10\] | $\text{conom} \sim \text{binom}, \text{tentor} \sim \text{binom}, \text{trator} \sim \text{conom}, \text{axitor} \sim \text{binom}$ |
| — dilation–shear–torsion type solution | \[17\] | $\text{conom} \sim \text{binom}, \text{tentor}, \text{trator} \sim \text{conom}, \text{axitor}$ |
| **Kerr–Newman** metric with strong-gravito electric charge plus nom and tor | | |
| — triplet type solution | \[31\] | $\text{conom} \sim \text{vecnom} \sim \text{trator}$ |
| **Plebański–Demiański** metric with strong gravito-electric and magnetic charge plus nom and tor | | |
| — triplet type solution | \[8\] | $\text{conom} \sim \text{vecnom} \sim \text{trator}$ |
| **Electrically** (and magnetically) charged versions of all of the triplet solutions | \[27,20,12,25\] | $\text{conom} \sim \text{vecnom} \sim \text{trator}$ |

*) Those pieces of the nonmetricity and the torsion vanish identically which are not mentioned in the description of a solution.
one is stuck, one changes the ansatz etc.

Beside the two dilation–shear solutions, Tresguerres [16] and Tucker and Wang [48] found Reissner–Nordström metrics together with a non–vanishing Weyl covector, \( (4) Q^{\alpha\beta} \neq 0 \), and a vector part of the torsion, \( (2) T^\alpha \neq 0 \), i.e., these solutions carry a dilation charge (in the words of Tucker and Wang, a Weyl charge) and a spin charge, but are devoid of any other post–Riemannian “excitations”, in particular, they have no tracefree pieces \( Q_{\alpha\beta} \) of the nonmetricity. As shown by Tucker and Wang, the corresponding Lagrangian needs only a Hilbert–Einstein piece \( (a_0 = 1) \) and a segmental curvature squared with \( z_4 \neq 0 \). The same has been proved for the Tresguerres dilation solution, see footnote 4 of [34].

Ho et al. [21] found four spherically symmetric exact solutions in a pure Weyl–Cartan spacetime which are similar to the dilation type solutions. However, they include an additional axial part of the torsion, \( (3) T^\alpha \neq 0 \), see Table III.

V. THE TRIPLET OF POST–RIEMANNIAN 1–FORMS AND OBUKHOV’S EQUIVALENCE THEOREM

The next step consisted in an attempt to understand the emergence of the dilation–shear and the dilation–shear–torsion solutions of Tresguerres. However, as it so happened, it shifted the attention to other types of solutions. In both Tresguerres shear solutions, the nonmetricity, besides the Weyl covector part conom, was represented by binom, basically a 16 components’ quantity. However, conom and trator each have only 4 components, as has vecnom. Accordingly, to create a simpler solution with shear than the two Tresguerres dilation–shear solutions, it seemed suggestive to require

\[
\text{conom} \sim \text{vecnom} \sim \text{trator}.
\]  \hspace{1cm} (12)

This amounts to the presence of one 1–form \( \phi \) which creates the three post–Riemannian pieces (12). If \( k_0, k_1, k_2 \) are some constants (see below), then we have

\[
Q = k_0 \phi, \quad \Lambda = k_1 \phi, \quad T = k_2 \phi,
\]  \hspace{1cm} (13)
with $\Lambda := \vartheta^\alpha e^\beta \mathcal{Q}_{\alpha\beta}$ and $T := e_\alpha T^\alpha$. This 1-form triplet was first proposed in \cite{34, 51} and also used in \cite{5}.

Again, in the context of the triplet ansatz (13), a Reissner–Nordström metric with a strong gravito–electric charge could successfully be used \cite{34} and a constraint on the coupling constants had to be imposed. Thus this “triplet” solution is reminiscent of the Tresguerres dilation–shear solutions. However, its structure is simpler and, instead of binom, it is vecnom which enters the solution. Moreover, of the curvature square pieces in the gauge Lagrangian $V_{\text{MAG}}$ only the piece with $z_4 \neq 0$ is required. All others do not contribute.

Soon this result was generalized to an axially symmetric solution \cite{51} based on the Kerr–Newman metric and, a bit later, to the whole Plebański-Demiański class of metrics \cite{8}. Already earlier, however, it became clear that the triplet represents a general structure in the context of the Einstein–Maxwellian “seed” metrics. In \cite{3} it was pointed out that for each Einstein–Maxwell solution (metric plus electromagnetic potential 1–form), if the electric charge is replaced by the strong gravito–electric charge and if a suitable constraint on the coupling constants is postulated, an exact solution of MAG can be created by means of the triplet (13). Even more so, if one started from an Einstein–Proca solution instead, one could even abandon the constraint on the coupling constants. This was first shown for a certain 3-parameter Lagrangian by Dereli et al. \cite{5} and extended to a 6-parameter Lagrangian by Tucker & Wang \cite{49}. The situation was eventually clarified for a fairly general 11-parameter Lagrangian by the

- Equivalence theorem of Obukhov \cite{35}: Let be given the gauge Lagrangian $V_{\text{MAG}}$ of (10) with all $w_I = 0$, $z_I = 0$, except $z_4 \neq 0$, i.e., the segmental curvature squared

$$-\frac{z_4}{8\rho} R^\alpha_\alpha \wedge *R^\beta_\beta = -\frac{z_4}{2\rho} dQ \wedge *dQ$$

(14)

is the only surviving strong gravity piece in $V_{\text{MAG}}$. Solve the Einstein–Proca equations.\footnote{For the $\eta$–basis we have $\eta_{\alpha\beta\gamma} = * (\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma)$ and $\eta_\alpha = * \vartheta_\alpha$.}
\[
\frac{a_0}{2} \eta_{\alpha\beta\gamma} \wedge \tilde{R}^{\beta\gamma} + \lambda \eta_\alpha = \kappa \Sigma_\alpha^{(\phi)}, \\
(\square + m^2) \phi = 0, \\
d^\dagger \phi = 0,
\]

with respect to the metric \( g \) and the Proca 1-form \( \phi \). Here the tilde \( \tilde{\cdot} \) denotes the Riemannian part of the curvature,

\[
\Sigma_\alpha^{(\phi)} := \frac{z_4 k_0^2}{2\rho} \left\{ (e_\alpha \rfloor d\phi) \wedge *d\phi - (e_\alpha \rfloor *d\phi) \wedge d\phi \right. \\
+ m^2 \left\{ (e_\alpha \rfloor *\phi + (e_\alpha \rfloor *\phi) \wedge \phi \right\}
\]

is the energy–momentum current of the Proca field and \( d^\dagger \) the exterior co-derivative. Then the general vacuum solution of MAG with the stated parameter restrictions is represented by the metric and the post–Riemannian triplet

\[
(g, \ Q = k_0 \phi, \ \Lambda = k_1 \phi, \ T = k_2 \phi),
\]

where \( k_0, k_1, k_2 \) are elementary functions of the weak gravity coupling constants, \( a_I, b_I, c_I \), and \( m^2 \) depends, additionally, on \( \kappa \) and the strong coupling constant \( z_4/\rho \) (the details can be found in [35]).

The results of [3,49] and of the Obukhov theorem lead to an understanding of the meaning of the constraint between the different coupling constants: If we put \( m^2 = 0 \), then the Einstein–Proca system becomes an Einstein–Maxwell system – and such metrics, like the Kerr–Newman metric, e.g., are more readily available for our purposes. In fact, we are not aware of any known Einstein–Proca metrics which we could use for the construction of exact MAG solutions. One should consult, however, the early work on the Einstein-Proca system by Buchdahl [3], Ponomariov & Obukhov [36], and Gottlieb et al. [13].

Also the reason for the more general character of the Tresguerres shear solutions is apparent. He allowed gauge Lagrangians with additional strong gravity pieces. In [46] he added, to the segmental curvature piece, the strong gravity pieces \( w_3 \times (\text{pscalar})^2 + w_5 \times \)
(ricanti)². Here ( )² is an abbreviation of ( ) ∧*( ). In this way he circumvented the Obukhov theorem and found the spin 2 piece of the nonmetricity, binom, inter alia. On the other hand, the dilation type solution in [34,38] can be recovered from the triplet solution [34] by means of a certain limiting procedure, see [34].

VI. STRONG GRAVITO–ELECTRIC MONOPOLE, ELECTRICALLY CHARGED VERSIONS OF THE TRIPLET SOLUTIONS

In Table III, we gave an overview of the solutions of insular objects. However, we didn’t explain so far the first and the last entry of the table.

The monopole type solution was found in [24], see also [38], in terms of isotropic coordinates. In the Appendix we translated the solution into Schwarzschild coordinates. Then, in these coordinates, the orthonormal coframe, the metric, and the triplet read, respectively,

\[ ϑ^0 = \left(1 - \frac{q}{r}\right) dt, \quad ϑ^1 = \frac{dr}{1 - \frac{q}{r}}, \quad ϑ^2 = r \, dθ, \quad ϑ^3 = r \sin θ \, dϕ, \]

(20)

\[
g = ϑ^0 \otimes ϑ^0 - ϑ^1 \otimes ϑ^1 - ϑ^2 \otimes ϑ^2 - ϑ^3 \otimes ϑ^3
\]

\[
= \left(1 - \frac{q}{r}\right)^2 dt^2 - \frac{dr^2}{\left(1 - \frac{q}{r}\right)^2} - r^2 \left(dθ^2 + \sin^2 θ \, dϕ^2\right),
\]

(21)

\[
φ = \frac{Q}{k_0} = \frac{Λ}{k_1} = \frac{T}{k_2} = \frac{N_e}{r \left(1 - \frac{q}{r}\right)} \bar{φ} = \frac{N_e}{r} dt,
\]

(22)

with \( q = \sqrt{\frac{3πN_e}{2a₀ρ}} k_0 N_e \), i.e., it is again a triplet solution.

Note that the metric is not of the Schwarzschild form, the Weyl covector, however, behaves as one expects for a strong gravito–electric charge. We recognize in this example in a particularly transparent way that the strong gravito–electric charge \( N_e \) creates the post–Riemannian potentials conom, vecnom, trator in (22) in a quasi–Maxwellian fashion but also emerges, in (21), in the components of the metric. However, in the metric, \( N_e \) behaves neither Schwarzschildian (the metric is different) nor Reissner–Nordströmian (the power of \( r \) is reciprocal instead of \( r^{-2} \)).
We can construct this metric by a specific choice of the mass of the Reissner-Nordström metric. In other words, the metric of this solution represents a subcase of the Reissner-Nordström metric. Then it is immediately clear that this solution is covered by the Obukhov theorem: One starts from an Einstein–Maxwell solution, namely the Reissner–Nordström metric, supplements the corresponding triplet, and chooses the mass such that the Reissner-Nordström function \(1 - 2m/r + q^2/r^2\) becomes a pure square.

In the meantime, also a strong gravito-magnetic monopole has been found \cite{25}. The mechanism is analogous to the gravito–electric case and doesn’t seem to bring new insight.

The last entry of Table III indicates that we are always able to find electrically charged versions of a MAG solution as long as we confine ourselves to the triplet type solutions. This is evident from Obukhov’s theorem: We take an electrically uncharged MAG solution with the triplet \(\sim \phi\). Then we choose the electromagnetic potential \(A\) proportional to the 1–form \(\phi\). Thus the structure of the energy–momentum currents of the 1–form \(\phi\) and the 1–form \(A\) is the same one. Both currents differ only by a constant. Accordingly, they just add up, on the right hand side of the Einstein equation, to a total energy–momentum current carrying a modified constant in front of it. Clearly, this structure breaks down as soon as one turns to the full Einstein-Proca system, i.e., as soon as the Proca mass becomes non–vanishing. Nevertheless, it is quite useful to have found these electrically charged solutions explicitly. It helps to illustrate the coupling of the electromagnetic field to the post-Riemannian structures of a metric-affine spacetime, see \cite{37}.

**VII. WAVE SOLUTIONS**

*Plane–fronted* metric–Weyl covector–torsion waves have been constructed by Tucker and Wang \cite{18}. Their source is a semi–classical Dirac spinor field \(\psi(x)\). Let \(\gamma^\alpha\) be the Dirac matrices. Then the Dirac spin current \(\sim \overline{\psi} \gamma^\alpha \gamma^\beta \psi\) generates the torsion according to \(\overline{\psi} \gamma^\alpha \gamma^\beta \psi \sim \)

\(^2\text{Private communications by D. Kramer (Jena) and M. Toussaint (Cologne).}\)
tentor + axitor, whereas the Weyl covector and the torsion trace are proportional to each other and are induced by the segmental curvature square piece in the Lagrangian: conom \sim trator. Thus we have in this model an underlying Weyl–Cartan spacetime since the tracefree part of the nonmetricity vanishes. In other words, the solution is of the dilation type. Accordingly, the vacuum part of the (weak and strong) gravitational field can be understood as a degenerate triplet solution and again, as remarked in [48], it is straightforward to include a Maxwell field with electric (and possibly magnetic) charge.

In view of [49] and the Obukhov theorem, it is clear that one may start with any solution of the Einstein–Maxwell equations. Then one replaces, after imposing a suitable constraint on the coupling constants, the electric charge by the strong gravity charge thereby arriving at the post–Riemannian triplet which was mainly discussed in Sec.V. The procedure is fairly straightforward. Nevertheless, it is useful to have a couple of worked–out examples at one’s disposal. Explicit solutions may convey a better understanding of the structures involved.

Garcia et al. [11] studied colliding waves with the corresponding metric and an excited post–Riemannian triplet in the framework of a Lagrangian of the Obukhov theorem. Usually, in general relativity, the colliding waves are generated by quadratic polynomials in the appropriate coordinates. And these polynomials were also used in the paper referred to. Recently, however, Bretón et al. [2] were able, within general relativity, to extend this procedure by using also quartic polynomials. Again, this procedure can be mimicked in metric–affine spacetime and Garcia et al. [3,11] constructed corresponding colliding gravity waves with triplet excitation. For the quadratic as well as for the quartic case it is also possible to generalize to the electrovac case, as has been shown in [12].

VIII. COSMOLOGICAL SOLUTIONS

As we argued above, we expect more noticeable deviations from metric–compatibility the further we go back in time. Therefore it is natural to investigate cosmological models in the framework of metric–affine gravity. And the standard Friedmann model is a good
starting point. Tresguerres \cite{45} proposed such a model with torsion and a Weyl covector, i.e., spacetime is described therein by means of a Weyl–Cartan geometry. The matter he used to support the model is a fluid carrying an energy–momentum and a dilation current. The field equations of the model stayed within a manageable size since the Lagrangian, by assumption, carries only the segmental curvature square piece of the symmetric part of the curvature 2–form. However, the square of all 6 irreducible pieces of the antisymmetric part of the curvature are allowed in the gravitational Lagrangian even if only the tracefree symmetric Ricci turns out to be relevant in the end. A somewhat similar model has been investigated by Minkevich and Nemenmann \cite{28}.

Using the much more refined model of a hyperfluid \cite{33}, Obukhov et al. \cite{35} derived, within the framework of the the equivalence theorem, but with some additional simplifying assumptions, a Friedmann cosmos with a time varying Weyl covector. This is analogous as in the Tresguerres model.

Similar structures have been suggested by Tucker and Wang \cite{50}. They proposed a metric–affine geometry of spacetime for the purpose of taking care of the supposedly unseen dark matter which, as they suggest, interacts with the strong gravity potential of the Proca type as described by means of a gravitational Lagrangian carrying a segmental curvature square. Thus the Obukhov theorem applies to their scenario, and a Friedmann solution with a post–Riemannian triplet is expected to emerge. And this is exactly what happens. Ordinary matter and dark matter both supply their own material energy–momentum current to the right hand side of the Einstein equation and, additionally, a Proca energy-momentum comes up, see \cite{15,18}. The material current that couples to the Proca field can be identified with the trace of the material hypermomentum current, the material dilation current, see the trace of the right hand side of \cite{3}. The model is worked out in considerable detail, galactic dynamics and the cosmological evolution are studied inter alia and numerical results presented.
IX. THE MINIMAL DILATION–SHEAR LAGRANGIAN, ANSATZ WITH A
PROCA ‘MASS’

Taking the triplet (13) as a guide, it is certainly helpful for model building not to take
the whole weak part of (10) but only some sort of essential nucleus of it. Putting (8) and
(9) together, one gets certainly a propagating Weyl covector. From the Obukhov theorem
we know that we only need a further weak gravity piece in order to allow for shear. In view
of the triplet, the addition of a trator square piece is suggested. In this way we recover the
minimal dilation–shear Lagrangian \[34,5\]

\[ V_{\text{dil–sh}} = \frac{1}{2\kappa} \left( -R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \beta Q \wedge *Q + \gamma \Gamma \wedge *\Gamma \right) - \frac{\alpha}{8} R_{\beta}^{\beta} \wedge *R_{\gamma}^{\gamma}. \] (23)

And indeed, our Reissner–Nordström, Kerr–Newman, and Plebański–Demiański metrics,
together with the post–Riemannian triplet (13), with the constants

\[ k_0 = -\frac{3}{2} \gamma - 4, \quad k_1 = \frac{27}{2} \gamma, \quad k_2 = 6, \] (24)

and with the 1–form \( N \) is an integration constant

\[ \phi = \frac{N}{r} dt, \] (25)

are solutions of the field equations belonging to the \( V_{\text{dil–sh}} \) Lagrangian. However, a constraint
on the weak coupling constants has to be imposed:

\[ \gamma = -\frac{8}{3} \frac{\beta}{\beta + 6}. \] (26)

Accordingly, the Lagrangian (23) may be considered as the generic Lagrangian of the
Obukhov theorem.

Let us now try to get rid of the constraint (24). The corresponding procedure runs as
follows: According to \[35\] Eq. (6.8), we can define the Proca mass

\[ m_{\text{Proca}}^2 = \frac{1}{2\kappa\alpha} \left( 2\beta + \frac{36\gamma}{3\gamma + 8} \right). \] (27)

If we put it to zero, we recover the constraint (26):
\[
m^2_{\text{Proca}} = 0 \quad \longrightarrow \quad \gamma = -\frac{8}{3} \frac{\beta}{\beta + 6}.
\]  
(28)

Thus the dropping of the constraint (26) is equivalent to the emergence of a Proca mass, i.e., we now have to turn to the Einstein–Proca system instead of to the Einstein–Maxwell system.

Then, in flat spacetime, after dropping the constraint (26), instead of a Coulomb potential, we expect a Yukawa potential to arise as a solution of the Proca equation:

\[
\phi \sim N e^{-m_{\text{Proca}} r} \quad \frac{d t}{r}.
\]  
(29)

In the corresponding metric–affine spacetime, the Reissner-Nordström metric has also to be modified. If done in a suitable way, this should lead to an exact solution of the unconstrained dilation–shear Lagrangian (23).

\section*{X. DISCUSSION}

In the last section we have already seen, how we can hope to extend our work. But also a generalization in another direction is desirable. If we want to include the shear solutions of Tresguerres, then the dilation–shear Lagrangian is too narrow. To go beyond the triplet solution requires a generalization of (23). A ‘soft’ change, by switching on only the post-Riemannian pieces of the antisymmetric piece of the curvature 2–form, seems worth a try:

\[
V_{\text{dil–sh–tor}} \sim V_{\text{dil–sh}} - \frac{1}{2\rho} \left[ w_2 \times (\text{paircom})^2 + w_3 \times (\text{pscalar})^2 + w_5 \times (\text{ricanti})^2 \right].
\]  
(30)

A related model was discussed in [18] Sec.5.3. In this way we can hope to ‘excite’, besides conom and vecnom, also binom, e.g. Of course, also in this case one should try to remove the constraint. However, it will not be sufficient in this case, as is clear from [33] and the Obukhov equivalence theorem, to turn only to the Einstein–Proca system — rather a more general procedure will be necessary.
XI. APPENDIX: STRONG GRAVITO-ELECTRIC MONOPOLE IN SCHWARZSCHILD COORDINATES

In [24], the MAG solution of the soliton type was given in terms of isotropic coordinates. This makes it more difficult to compare it with the Reissner–Nordström type solution. Therefore we will perform a coordinate transformation. We will denote the isotropic polar coordinates by \((t, \rho, \theta, \phi)\) and the Schwarzschild coordinates by \((t, r, \theta, \phi)\). In [24], the following monopole solution has been found: The orthonormal coframe reads

\[
\vartheta^0 = \frac{1}{f} \, dt, \quad \vartheta^1 = f \, d\rho, \quad \vartheta^2 = f \rho \, d\theta, \quad \vartheta^3 = f \rho \sin \theta \, d\phi,
\]

with the function

\[
f(\rho) = 1 + \frac{q}{\rho},
\]

and the one–form triplet is specified by (in this Appendix, \(\rho_c\) denotes the strong gravity coupling constant)

\[
\phi = \frac{Q}{k_0} = \frac{\Lambda}{k_1} = \frac{T}{k_2} = \frac{N_e}{\rho} \vartheta^0, \quad \text{with} \quad q^2 = \frac{z_4 k^2}{2a_0 \rho_c} (k_0 N_e)^2.
\]

For the transition to Schwarzschild coordinates, the \(\theta\)–component of the coframe has to obey

\[
\vartheta^2 = \left(1 + \frac{q}{\rho}\right) \rho \, d\theta = r \, d\theta.
\]

Thus

\[
r = \left(1 + \frac{q}{\rho}\right) \rho = \rho + q, \quad dr = d\rho.
\]

Substitution into (32) yields

\[
f = 1 + \frac{q}{\rho} = \frac{r}{\rho} = \frac{r}{r - q} = \frac{1}{1 - \frac{q}{r}}.
\]

Accordingly, the monopole solution can be rewritten in the form as displayed in (20, 21, 22).
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