When renormalizability is not sufficient: Coulomb problem for vector bosons

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The Coulomb problem for vector bosons $W^\pm$ incorporates a known difficulty; the boson falls on the center. In QED the fermion vacuum polarization produces a barrier at small distances which solves the problem. In a renormalizable $SU(2)$ containing vector triplet $(W^+, W^-, \gamma)$ and a heavy fermion doublet $F$ with mass $M$ the $W^-$ falls on $F^+$, to distances $r \sim 1/M$, where $M$ can be made arbitrary large. To prevent the collapse the theory needs additional light fermions, which switch the ultraviolet behavior of the theory from the asymptotic freedom to the Landau pole. Similar situation can take place in the Standard Model. Thus, the renormalizability of a theory is not sufficient to guarantee a reasonable behavior at small distances for non-perturbative problems, such as a bound state problem.

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It is usually believed that a renormalizable theory automatically exhibits good physical behavior at large momenta and small distances. This claim is definitely correct in low orders of the perturbation theory. However, in higher orders, which are necessary, for example, in a bound state problem, the situation is not so obvious. We present here an example, in which the renormalizability by itself fails to define a proper behavior of the theory at small distances.

Consider a negatively charged vector boson, which propagates in the Coulomb field created by a heavy point-like charge $Z|e|$ assuming that the boson is massive. A bound state problem for this boson needs summation in all orders in $Ze$. Since the electrodynamics for massive vector particles is non-renormalizable, one should expect problems here. One of them, found long time ago, is particularly interesting for our discussion. Soon after Proca formulated theory for vector particles it became clear that it produces inadequate results for the Coulomb problem: the $W$ wave function is so singular that the integral over the charge density of $W$ is divergent near the origin. Corben and Schwinger modified the Proca theory, tuning the Lagrangian and equations of motion for vector bosons in such a way as to force the gyromagnetic ratio of the vector boson to acquire a favorable value $g = 2$. It is well known now that $g = 2$ is the gyromagnetic ratio of the $W$-boson in the Standard model. This modification allowed Corben and Schwinger to obtain a physically acceptable spectrum for the Coulomb problem, which is described by the Sommerfeld formula similar to the spectrum of Dirac particles, but with integer values of the total angular momentum $j = 0, 1, \ldots$.

Corben and Schwinger found also that their modification did not resolve the main problem, the $W$-boson still falls to the center for two series of quantum states; one with $j = 0$, and the other one with “$l = 0$” (if “$l$” is defined appropriately). The wave function of $W$ is so singular at the origin for these states that the integral over $W$ charge density is divergent near the origin. In our works we found a cure for this problem. The QED fermion vacuum polarization was shown to produce an effective potential barrier for the $W$ boson at small distances. The charge density of the $W$ boson in the $j = 0$ state decreases as $\exp\left(-const/r\right)$ under this barrier and vanishes at the origin; similar improvement exhibits the $l = 0$ state. As a result the Coulomb problem for vector particles becomes well defined. The corresponding correction to the Sommerfeld spectrum proves to be small. The effective potential of Ref. is repulsive only when the running coupling constant exhibits the Landau-pole behavior; in contrast, for asymptotic freedom, the collapse is inevitable.

In we derived the Corben-Schwinger Lagrangian from the Lagrangian of the Standard Model, where the mass of $W$ is produced by the Higgs mechanism, which preserves the renormalizability of theory. However, the applicability of the Corben-Schwinger wave equation to $W$ requires that the Coulomb center does not interact with the $Z$-boson and Higgs particle. (It may be taken, for example, as a small charged black hole.) However, such Coulomb center is not described by the Standard Model, preventing the theory from being a complete renormalizable one.

In the present work we consider an example of a fully renormalizable model, which exhibits a similar phenomenon. Take an $SU(2)$ gauge theory and a triplet of real Higgs scalars $\Phi$

$$\mathcal{L}_{\text{Boson}} = -\frac{1}{4} G_{\mu\nu}^a G^{a \mu\nu} + \frac{1}{2} D_\mu \Phi^a \ast D^\mu \Phi^a + \ldots$$

Here $G_{\mu\nu}^a$ and $D_\mu$ are the gauge field and the covariant derivative, which includes the gauge potential $A_\mu^a$;
the diagrams (a) and (b) in Fig. 1, which describe scattering amplitudes, which emphasizes a difference between the high-energy dependence of the lowest-order scattering of the boson onto the fermion. But firstly, let us consider W-boson on the heavy fermion. The two other bosons, which we call \( W^\pm \) and \( W^0 \), acquire the Higgs mass \( m = g^2 v^2 / 2 \), see e.g. [7]. To allow the Coulomb center to appear in the model, consider a heavy fermion doublet \( F = (F^+, F^-) \) with charges \( e / 2 \) and \(-e / 2\) for its two components. Presume for simplicity that the parity is conserved and the fermion doublet does not interact with the Higgs field; its large mass, \( M \gg m \), is a free parameter in the Lagrangian

\[
\mathcal{L}_{\text{Fermi}} = \bar{F} (i\gamma^\mu D^\mu - M) F .
\]  

Our goal is to demonstrate that the interaction between \( W^- \) and the heavy fermion \( F^+ \) results in the collapse of the boson onto the fermion. But firstly, let us consider the high-energy dependence of the lowest-order scattering amplitudes, which emphasizes a difference between renormalizable and non-renormalizable models. Consider the diagrams (a) and (b) in Fig. 1 which describe scattering of \( W \) on \( F \). Conventional calculations show that at high collision energy their amplitudes satisfy

\[
M^{(a)} \simeq -M^{(b)} \simeq -\frac{e^2}{4m^2} (p_\mu + p'_\mu) \bar{F}\gamma^\mu F .
\]  

Separately, each one of them grows with energy, violating the unitarity limit; note that the diagram (b) contains only one partial wave. The increase is due to the longitudinal polarization of the W-boson, \( \epsilon^\mu_W = k_\mu / m + O(m/p_0) \), \( p_0 \) is the \( W \) energy - compare e.g. [7]. The energy increase of the photon exchange diagram (a) signals the non-renormalizability of the pure vector electrodynamics. However, the sum of the two diagrams \( M^{(a)} + M^{(b)} \) does not possess this problem, a compensation of the two diagrams results in a reasonable behavior of the scattering amplitude at high energy, in accord with the renormalizability of the SU(2) model introduced in Eq. 1. It is important that the cancellation of the diagrams (a) and (b) manifests itself only when the collision energy is taken as a large parameter, \( p_0 \gg m \).

However, in the bound-state problem the energy is fixed, \( p_0 \simeq m \). Moreover, the wave function of the W-boson at distances \( 1/M \ll r \ll 1/m \) is represented by the off shell diagrams, in which legs of the W-boson carry large momenta \( m \ll |p|, |p'| \sim 1/r \ll M \). The invariant \( t \) of the scattering problem is large in this case, \( t = -(p-p')^2 \gg m^2 \), while the s-invariant remains fixed.

In this kinematic region the diagram (a) is not compensated by (b). Therefore the verified above renormalizability of the theory cannot shed light on the behavior of the wave function at \( 1/M \ll r \ll 1/m \). To establish this behavior one needs to consider a set of all ladder diagrams of the type shown in Fig. 1(c), which are known to produce dominant contribution to the off-shell amplitude when \( t \) is large. Clearly summation of this ladder is equivalent to the solution of the wave equation for the W-boson in an attractive Coulomb field created by the heavy fermion.

The necessary wave equation can be derived from the Lagrangian Eq. 1. Keeping there only those terms, which describe W-bosons and their interaction with photons we find an effective Lagrangian, which proves to be identical to the Lagrangian introduces by Corben-Schwinger

\[
\mathcal{L}_W = -\frac{1}{2} \left( \nabla_\mu W_\nu - \nabla_\nu W_\mu + (\nabla^\mu W^\nu - \nabla^\nu W^\mu) + ie F^{\mu\nu} W_\mu + m^2 W_\mu W^\mu \right) .
\]  

Here \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \) is the electromagnetic field, and \( \nabla_\mu = \partial_\mu + ie A_\mu \) is the covariant derivative in this field \( (e < 0) \). From Eq. 1 one derives the Corben-Schwinger wave equation, which can be written in the following form

\[
(\nabla^2 + m^2) W^\mu + 2ie F^{\mu\nu} W_\nu + \frac{ie}{m^2} \nabla^\mu (j^\nu W_\nu) = 0 .
\]  

The last term here includes explicitly an external current, \( j^\nu = \partial_\mu F^{\mu\nu} \), which creates the electromagnetic field, see details in Ref. [8]. To derive conclusions from Eq. 5 several points need to be specified. Since the boson is massive, it is necessary to rewrite the equation in terms of a 3-vector, for example via spatial components \( W \) of the four-vector \( W^\mu = (W^0, \mathbf{W}) \). It is necessary also to specify the equation for a static, spherically symmetrical external potential \( U(r) \), in which the W-boson propagates

Consider the most interesting for us partial wave \( j = 0 \). In this state the three-vector \( \mathbf{W} \) satisfies

\[
\mathbf{W}(r) = (0, 0, 0) .
\]
the diagram (b) operates only when the fermion, while the diagram (b) in Fig. 1, which was so important in the region $r \ll 1/m \approx 1/p_0$, the energy and mass do not manifest themselves in the wave equation. The functions $G, H$ can depend only on the potential $U = U(r)$, in which the boson propagates, and on an additional term $Y = \gamma(r)$, which originates from the zero-th component of the external current $j_{\mu}$ in Eq. (6).

$$Y = e j_0 / m^2 = - \Delta U / m^2. \tag{8}$$

Let us apply firstly Eq. (6) to the pure Coulomb potential, when $U(r) = -Z \alpha / r$, where $Z = 1/2$ is the charge of $F^+$, and $G = 4/r$ and $H = (2 + Z^2 \alpha^2) / r^2$. Consequently we find the solution in the region $1/M \ll r \ll 1/m$,

$$v(r) \simeq r^{\gamma - 3/2}, \tag{9}$$

where $\gamma = (1/4 - Z^2 \alpha^2)^{1/2}$. Straightforward calculations show that this solution results in a major problem, forcing the charge density of the $W$-boson $\rho_W = \rho_W(r)$ to diverge at small distances, $\rho_W \propto r^{2\gamma - 4}$. Since $2\gamma < 1$ a divergence of the integral of this charge density signals the collapse of the $W$-boson to the Coulomb center, or at least into the region $r \sim 1/M$. This makes the pure Coulomb problem poorly defined, in accord with conclusions of Ref. [3].

At this point it is instructive to return to the Compton-type diagram (b) in Fig. 1, which was so important in the high energy limit for on shell processes. However, it is unable to remedy the problem of the collapse of the $W$-boson on the Coulomb center. The reason is clear. We saw that the collapse takes place in the region $1/M \ll r \ll 1/m$, which is well separated from the heavy fermion, while the diagram (b) operates only when the distance $r$ between the $W$-boson and the heavy fermion $F^+$ is small, $r \sim 1/M$. Clearly the short-range interaction described by this diagram cannot prevent the fast increase of the wave function of the $W$-boson at larger distances $r > 1/M$.

Consider now the radiative corrections. The most important phenomenon, which takes place at small distances (large momenta) is related to the renormalization of the coupling constant, which in the case considered results in the renormalization of the Coulomb charge. It suffices to consider the vacuum polarization in the lowest-order approximation, when it is described by the known Uehling potential, which at small distances is represented via a conventional logarithmic function, see e. g. [5]. A combined potential energy of the Coulomb and Uehling potentials read

$$U(r) = U_C + U_U = - [1 - \alpha \beta \ln(Mr)] Z \alpha / r, \tag{10}$$

where $\beta$ is the lowest order coefficient of the Gell-Mann - Low beta-function. The polarization produces small variation of the potential in Eq. (10), but makes the $Y$-term Eq. (6) large at small distances

$$Y = Z \alpha^2 \beta / (m^2 r^3) \gg |U|. \tag{11}$$

The functions $G, H$ in Eq. (7) calculated with account of this $Y$-term read,

$$G \simeq 6/r, \quad H \simeq - 2 \alpha^3 \beta / (m^2 r^4), \tag{12}$$

which results in the following asymptotic solution of Eq. (9)

$$v \propto 1 / r^2 \times \left\{ \begin{array}{ll}
\exp(-\beta), & \beta > 0, \\
\cos(|\phi| + \beta), & \beta < 0,
\end{array} \right. \tag{13}$$

where $\phi = Z \alpha \beta / (m r)$, and $\delta$ is a constant phase defined by the behavior of the solution at $r \to 0$, which we do not discuss here. We see that the sign of $\beta$ plays a crucial role. In pure QED it is positive, $\beta = 2/(3\pi) > 0$ for one generation of the Dirac fermions in the normalization adopted in Eq. (10). Eq. (13) indicates in this case that $v(r)$ is exponentially suppressed at small distances, which makes the Coulomb problem stable, well defined in accord with conclusions of Ref. [3]. In contrast, the considered SU(2) model is asymptotically free, $\beta = -22/(3\pi) < 0$, which makes $v(r)$ a growing, strongly oscillating function at small distances. This clearly indicates the collapse of the $W$-boson. Therefore the Coulomb problem cannot be formulated in that case.

We observe an unexpected result. For an attractive Uehling potential $U_U < 0$ (that characterizes the pure QED, $\beta > 0$) the Coulomb problem turns out to be stable. In contrast, the repulsive Uehling potential $U_U > 0$ (SU(2) model, $\beta < 0$) results in the collapse of the $W$-boson, which makes the Coulomb problem unstable. In other words, the situation looks as if there is an effective potential, which sign is opposite to the sign of the Uehling potential. This surprising behavior finds its origin in the properties of the $Y$-potential, which describes the zero-th component of the external current as shows Eq. (8). A presence of this current in the wave equation, see Eq. (5), distinguishes the case of vector particles from scalars and spinors [11].

The collapse of the $W$-boson on the Coulomb center is not related to particular properties of the model discussed. It manifests itself similarly within, for example, the Standard Model SU(2) x U(1), if it includes heavy fermions [11]. At small distances $r < 1/m$ the mass of the $Z$-boson may be neglected. In this situation one may use any linear combinations of degenerate eigenstates. In our case it is convenient to use the original bosons $B_\mu^0$ from SU(2) and $W^{(0)}$ from U(1), instead of the eigenstates $Z$.
and $\gamma$. $W^-$ interacts with $W^0$ only, which reduces the problem to the SU(2) sector, where the $W$-boson collapses on the heavy fermion. We verified this claim by direct calculations, which show that at small distances the Weinberg mixing angle $\theta_W$ is canceled out. The Coulomb problem can be remedied only if a sufficient number of light fermions, which change the sign of the vacuum polarization, is added. The point is that this condition does not follow from the renormalizability of the theory.

Finally, let us consider another aspect of the problem. Eqs. (6) - (8) contain $\Delta U_{\text{C}} \propto \delta(r)$, which was neglected in previous works. Working with this term it is convenient to introduce a finite nuclear size $R$ and then take the limit $R \to 0$ (for simplicity we assume infinite $M$). Inside the “nucleus” an effective potential $U_{\text{eff}} = -H$ given in Eq.(12) dominates in the wave equation. This attractive potential produces large number of states (infinite for the zero nuclear size), which are localized inside the nucleus. Their energies are well below the ground state energy given by the Sommerfeld formula. These levels would be populated via creation of $W^+W^-$ pairs, similar to the vacuum breakdown for the Dirac particles. The difference is that for vector bosons the vacuum breakdown happens for any, however small charge of the nucleus. Note that this contact potential was neglected when the Sommerfeld spectrum was derived for the point-like nucleus. However, we see that this potential drastically modifies the spectrum.

In pure QED the problem is saved by the fermion vacuum polarization, which produces the impenetrable potential barrier (for $R = 0$). This eliminates any contact interaction with the nucleus, and the Sommerfeld spectrum survives. In the case of the $SU(2)$ the situation may seem different since the Compton diagram Fig.1(b) produces the repulsive interaction inside the nucleus. One may hope that this interaction eliminates the negative-energy states located inside the nucleus and brings the spectrum to the Sommerfeld form. However, the contact interaction does not influence the wave function of $W$ at the distances $r \gg R$. Therefore, the collapse of $W$ to a vicinity of the nucleus is inevitable. As was pointed above, the collapse can be prevented by addition of light fermions, which switch the ultraviolet behavior of the theory from the asymptotic freedom to the Landau pole, thus preventing the collapse.

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[10] It is interesting to observe the following trend. The wave equation for scalars, $S = 0$, includes only the potential $A_\mu$. The wave equation for spinors, $S = 1/2$, if written as the second-order differential equation, includes also the field $F_{\mu\nu}$, which is represented by first derivatives of the potential. In the wave equation for vector bosons, $S = 1$, there appears also the external current $j_\mu$, which can be expressed via second derivatives of the potential.
[11] The inclusion of new heavy particles is motivated by the dark matter observations and possible extensions of the Standard model, e.g. supersymmetry.
[12] Zero-mass particles have no bound states with a Coulomb center since the binding energy is proportional to the mass. The spontaneous symmetry breaking, which creates the mass via the Higgs mechanism generates the bound states. The gluon acquires a dynamical mass inside a hadron and, in principle, may form a bound state with a heavy quark. This problem may also be relevant to the quark-gluon plasma.