(LOCALLY) SHORTEST ARCS OF SPECIAL SUB-RIEMANNIAN METRIC ON THE LIE GROUP $SO(3)$

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Abstract. The authors find geodesics, shortest arcs, diameter, cut locus, and conjugate sets for left-invariant sub-Riemannian metric on the Lie group $SO(3)$, under condition that the metric is right-invariant relative to the Lie subgroup $SO(2) \subset SO(3)$.

Keywords and phrases: geodesic, left-invariant sub-Riemannian metric, Lie algebra, Lie group, shortest arc.

Introduction

In paper [1] are found exact shapes of sheres of special left-invariant sub-Riemannian metric $d$ on three-dimensional Lie groups: Heisenberg group $H$, $SO(3)$ and $SL_2(\mathbb{R})$.

In the last two cases one can give the following natural geometric description of the metric $d$. The Lie groups $SO(3)$ and $SL_2(\mathbb{R})/\pm E_2$ can be interpreted as transitive groups of preserving orientation isometries of unit euclidean sphere $S^2$ in three-dimensional Euclidean space and of the Lobachevskii plane $L^2$ with Gaussian curvature $-1$ and hence as spaces $S^2_1$ and $L^2_1$ of unit tangent vectors over these surfaces. The spaces $S^2_1$ and $L^2_1$ admit Riemannian metric (scalar product) $g_1$ by Sasaki (see [2] or section 1K in Besse book [3]). In addition, canonical projections $p : (S^2_1,g_1) \to S^2$ and $p : (L^2_1,g_1) \to L^2$ (or, which is equivalent, $p : SO(3) \to SO(3)/SO(2)$ and $p : SL_2(\mathbb{R})/\pm E_2 \to SL_2(\mathbb{R})/SO(2)$ are Riemannian submersions [3]. The metric $d$ is defined by (totally nonholonomic) left-invariant distribution $D$ on $SO(3)$ and $SL_2(\mathbb{R})/\pm E_2$, which is orthogonal to fibers of Riemannian submersion $p$, and restriction of scalar product $g_1$ to $D$.

Moreover, canonical projections

$$p : (SO(3),d) \to S^2, \quad p : SL_2(\mathbb{R})/\pm E_2 \to L^2$$

are submetries [4], natural generalizations of Riemannian submersion. The distribution $D$ on $S^2_1$ and $L^2_1$ is nothing other than the restriction to $S^2_1$ and $L^2_1$ of horizontal distribution of Levi-Civita connection [3] for $S^2$ and $L^2$. Therefore under mentioned identifications of $SO(3)$ and $SL_2(\mathbb{R})/\pm E_2$ with $S^2_1$ and $L^2_1$, any smooth path $c = c(t), 0 \leq t \leq t_1$, in $SO(3)$ and $SL_2(\mathbb{R})/\pm E_2$, tangent to the distribution $D$, is realized as parallel translation of the vector $c(0) \in S^2_1$ and $c(0) \in L^2_1$ along projection $p(c(t)), 0 \leq t \leq t_1$.

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It follows from here and the Gauss-Bonnet theorem \[5\] for \(S^2\) and \(L^2\) that canonical projection (to the base of fibration-submersion) of a geodesic in \((SO(3), d)\) or \((SL_2(\mathbb{R})/\pm E_2, d)\) must be a solution of Dido’s isoperimetric problem (isoperimetrix) on the base \(S^2\) or \(L^2\), while a geodesic is a horizontal lift of an isoperimetrix in \(S^2\) or \(L^2\). Using this fact, submetries \[11\] and the suggestion that an isoperimetrix in \(S^2\) or \(L^2\) must have constant geodesic curvature, the authors of paper \[1\] deduced exact shapes of spheres without searching geodesics and shortest arcs.

In this paper, with the help of mentioned interpretation of geodesics, general methods of paper \[6\], and the Gauss-Bonnet theorem for \(S^2\), we find geodesics, shortest arcs, the diameter, cut locus, and conjugate sets in \((SO(3), d)\). Formulas, analogous to \(10\) and \(21\), are obtained in paper \[7\], but we apply other methods and give detailed proofs.

## 1. Preliminaries

Let us recall that the Lie group \(GL(n) = GL(\mathbb{R}^n)\) consists of all real \((n \times n)\)-matrices \(g = (g_{ij}), i, j = 1, \ldots, n\), such that \(\det g \neq 0\), and the Lie subgroup \(GL_0(n)\) (the connected component of the unit \(e\) in \(GL(n)\)) is defined by condition \(\det g > 0\). It is naturally to consider both groups as open submanifolds in \(\mathbb{R}^{n^2}\) with coordinates \(g_{ij}, \; i, j = 1, \ldots, n\).

Their Lie algebra \(\mathfrak{gl}(n) = GL(n)_e := GL_0(n)_e = \mathbb{R}^{n^2}\) is the set of all real \((n \times n)\)-matrices with usual structure of vector space and Lie bracket
\[
[a, b] = ab - ba; \quad a, b \in \mathfrak{gl}(n).
\]

Let \(e_{ij} \in \mathfrak{gl}(n), \; i, j = 1, \ldots, n\), be a matrix which has \(1\) in \(i\)-th row and \(j\)-th column and \(0\) in all other places. \(\text{Lin}(a, b)\) denotes linear span of vectors \(a, b\). As an auxiliary tool we shall use standard scalar product \((\cdot, \cdot)\) on the Lie algebra \(\mathfrak{gl}(n) = \mathbb{R}^{n^2}\) for \(n = 3\). By definition, the Euclidean space \(E^n\) is \(\mathbb{R}^n\) with standard scalar product \((x, y) = x^T y\), where \(x, y \in \mathbb{R}^n\) are regarded as vector-columns and \(^T\) denotes here and later the transposition of matrices.

The Lie group \(SO(n) = O(n) \cap GL_0(n)\) of all orthogonal matrices with the determinant \(1\) is a connected Lie subgroup in \(GL_0(n)\). Its Lie algebra \(\mathfrak{so}(n), [\cdot, \cdot]\) is a Lie subalgebra of the Lie algebra \((\mathfrak{gl}(n), [\cdot, \cdot])\), consisting of all skew-symmetric matrices.

Let \(G\) and \(H\) be Lie groups with Lie algebras \(\mathfrak{g}\) and \(\mathfrak{h}; \phi : G \to H\) is a Lie groups homomorphism. Then
\[
\phi \circ \exp_\mathfrak{g} = \exp_\mathfrak{h} \circ d\phi_e,
\]
moreover,
\[
d\phi_e : (\mathfrak{g}, [\cdot, \cdot]) \to (\mathfrak{h}, [\cdot, \cdot])
\]
is a Lie algebra homomorphism (see lemma 1.12 in \[9\]). If \(g_0 \in G\) then \(I(g_0) : G \to G\), where \(I(g_0)(g) = g_0 g g_0^{-1}\) is inner automorphism of the Lie group \(G\). Consequently, \(\text{Ad}(g_0) := dI(g_0)_e \in GL(\mathfrak{g})\) is automorphism of the Lie algebra \(\mathfrak{g}\) and \(d\text{Ad}_e(v) := \text{ad}(v) := [v, \cdot]\) for \(v \in \mathfrak{g}\) \[9\]. Therefore, on the ground of formula \[3\],
\[
I(g_0) \circ \exp = \exp \circ \text{Ad}(g_0),
\]
$$\text{Ad}(\exp_g(v)) = \exp_{g(0)}(\text{ad}(v)), \quad v \in g.$$  

In case of left-invariant sub-Riemannian metrics on Lie groups, every geodesic is a left shift of some geodesic which starts at the unit. Thus later we shall consider only geodesics with unit origin. Theorem 5 in paper [6] implies the following theorem.

**Theorem 1.** Let $G$ be a connected Lie subgroup of the Lie group $SO(n) \subset GL(n)$ with the Lie algebra $\mathfrak{g}$, $D$ is totally nonholonomic left-invariant distribution on $G$, a scalar product $\langle \cdot, \cdot \rangle$ on $D(e)$ is proportional to restriction of the scalar product $\langle \cdot, \cdot \rangle$ (to $D(e)$). Then parametrized by arclength normal geodesic (i.e. locally shortest arc) $\gamma = \gamma(t), \quad t \in (-a, a) \subset \mathbb{R}, \quad \gamma(0) = e$, on $(G, d)$ with left-invariant sub-Riemannian metric $d$, defined by distribution $D$ and scalar product $\langle \cdot, \cdot \rangle$ on $D(e)$, satisfies the system of ordinary differential equations

$$\dot{\gamma}(t) = \gamma(t)u(t), \quad u(t) \in D(e) \subset \mathfrak{g}, \quad \langle u(t), u(t) \rangle \equiv 1,$$

$$\dot{u}(t) + \dot{v}(t) = - [u(t), v(t)],$$

where $u = u(t), \quad v = v(t) \in \mathfrak{g}, \quad \langle v(t), D(e) \rangle \equiv 0, \quad t \in (-a, a) \subset \mathbb{R}$, are some real-analytic vector functions.

2. GEODESICS OF SPECIAL LEFT-INvariant SUB-RIEMANNIAN METRIC ON THE LIE GROUP $SO(3)$

**Theorem 2.** Let be given the basis

$$a = e_{21} - e_{12}, \quad b = e_{31} - e_{13}, \quad c = e_{32} - e_{23}$$

of the Lie algebra $\mathfrak{so}(3)$, $D(e) = \text{Lin}(a, b)$, and scalar product $\langle \cdot, \cdot \rangle$ on $D(e)$ with orthonormal basis $a, b$. Then left-invariant distribution $D$ on the Lie group $SO(3)$ with given $D(e)$ is totally nonholonomic and the pair $(D(e), \langle \cdot, \cdot \rangle)$ defines left-invariant sub-Riemannian metric $d$ on $SO(3)$. Moreover, any parametrized by arclength geodesic $\gamma = \gamma(t), \quad t \in \mathbb{R}$, in $(SO(3), d)$ with condition $\gamma(0) = e$ is a product of two 1-parameter subgroups:

$$\gamma(t) = \exp(t(\cos \phi_0 a + \sin \phi_0 b + \beta c)) \exp(-t \beta c),$$

where $\phi_0, \beta$ are some arbitrary constants.

**Proof.** It follows from formulae (2) and (9) that

$$[a, b] = c, \quad [b, c] = a, \quad [c, a] = b.$$  

This implies the first statement of theorem.

It is clear that on $D(e)$

$$\langle \cdot, \cdot \rangle = \frac{1}{2} \langle \cdot, \cdot \rangle.$$  

In consequence of theorem 3 in [6] every geodesic on 3-dimensional Lie group with left-invariant sub-Riemannian metric is normal. Then it follows from theorem that one can apply ODE (7), (8) to find geodesics $\gamma = \gamma(t), \quad t \in \mathbb{R}$, in $(SO(3), d)$.
It is clear that
\begin{equation}
\label{eq:13}
 u(t) = \cos \phi(t)a + \sin \phi(t)b, \quad v(t) = \beta(t)c,
\end{equation}
and the identity (8) is written in the form
\[-[\cos \phi(t)a + \sin \phi(t)b, \beta(t)c] = \dot{\phi}(t)(-\sin \phi(t)a + \cos \phi(t)b) + \dot{\beta}(t)c.\]
In consequence of (11), expression in the left part of equality is equal to
\[\beta(t)(\cos \phi(t)b - \sin \phi(t)a).\]
We get identities \(\dot{\beta}(t) = 0, \quad \dot{\phi}(t) = \beta(t).\) Hence
\begin{equation}
\label{eq:14}
\beta = \beta(t) = \text{const}, \quad \phi(t) = \beta t + \phi_0.
\end{equation}
In view of (7), (13), and (14), it must be
\begin{equation}
\label{eq:15}
\dot{\gamma}(t) = \gamma(t)(\cos(\beta t + \phi_0)a + \sin(\beta t + \phi_0)b).
\end{equation}
Let us prove that (10) is a solution of ODE (15). One can easily deduce from formulae (11) equalities
\begin{equation}
\label{eq:16}
(\text{ad}(c)) = a, \quad (\text{ad}(b)) = -b, \quad (\text{ad}(a)) = c,
\end{equation}
where \((f)\) denotes the matrix of linear map \(f: \mathfrak{so}(3) \to \mathfrak{so}(3)\) in the base \(a, b, c;\) later \((f)\) is identified with \(f.\) On the ground of formulae (9), (16), (14), (13),
\[\gamma(t) = \gamma(t) \exp(t(\cos \phi_0 a + \sin \phi_0 b + \beta c)(\cos \phi_0 a + \sin \phi_0 b + \beta c) \exp(-t \beta c) + \gamma(t)(-\beta c) = \gamma(t) \exp(t \beta c)(\cos \phi_0 a + \sin \phi_0 b + \beta c) \exp(-t \beta c) + \gamma(t)(-\beta c) = \gamma(t) \exp(t \beta c)(\cos \phi_0 a + \sin \phi_0 b) \exp(-t \beta c) + \gamma(t)(\beta c) + \gamma(t)(-\beta c) = \gamma(t) \cdot [\exp(t \beta c)(\cos \phi_0 a + \sin \phi_0 b)] = \gamma(t) \cdot [\exp(t \beta (\text{ad}(c)))(\cos \phi_0 a + \sin \phi_0 b)] = \gamma(t) \cdot [(\exp(t \beta a))(\cos \phi_0 a + \sin \phi_0 b)] = \gamma(t) \cdot (\cos(\beta t + \phi_0)a + \sin(\beta t + \phi_0)b) = \gamma(t)u(t).\]

\begin{remark}
Both 1-parameter subgroups from formula (10) are nowhere tangent to distribution \(D\) for \(\beta \neq 0\) so that any their interval has infinite length in metric \(d.\)
\end{remark}

\begin{remark}
On p. 258 in book [8], A.A.Agrachev and Yu.L.Sachkov proved that, analogously to formula (10), every normal trajectory (geodesic) of left-invariant sub-Riemannian metric, defined by a distribution with corank 1, on a compact Lie group, starting at the unit, is a product of no more than two 1-parameter subgroups. Let us remind that any geodesic of left-invariant sub-Riemannian metric on 3-dimensional Lie group is normal.
\end{remark}

\begin{proposition}
Let \(\gamma(t), \ t \in \mathbb{R},\) be geodesic in \((SO_0(2, 1), d)\) defined by formula (11). Then for any \(t_0 \in \mathbb{R},\)
\begin{equation}
\label{eq:17}
\gamma(t_0)^{-1}\gamma(t) = \exp((t-t_0)(\cos(\beta t_0 + \phi_0)a + \sin(\beta t_0 + \phi_0)b + \beta c)) \exp(-t (t-t_0) \beta c).
\end{equation}
\end{proposition}
Proof. On the basis of formulae (5), (6), (16),
\[
\gamma(t_0)^{-1}\gamma(t) = \exp(t_0\beta c)\exp(-t_0(\cos \phi_0 a + \sin \phi_0 b + \beta c)) = \\
\exp(t_0\beta c)\exp((t-t_0)(\cos \phi_0 a + \sin \phi_0 b + \beta c))\exp(-t_0\beta c)\exp(-(t-t_0)\beta c) = \\
[I(\exp(t_0\beta c))\exp((t-t_0)(\cos \phi_0 a + \sin \phi_0 b + \beta c))]\exp(-(t-t_0)\beta c) = \\
\exp[\text{Ad}(\exp(t_0\beta c))(t-t_0)(\cos \phi_0 a + \sin \phi_0 b + \beta c)]\exp(-(t-t_0)\beta c) = \\
\exp[\exp(\text{ad}(t_0\beta c))(t-t_0)(\cos \phi_0 a + \sin \phi_0 b + \beta c)]\exp(-(t-t_0)\beta c) = \\
\exp((t-t_0)(\cos(\beta t_0 + \phi_0)a + \sin(\beta t_0 + \phi_0)b + \beta c))\exp(-(t-t_0)\beta c).
\]

□

Remark 3. To change a sign of $\beta$ in (17) is the same as to change a sign of $t$ and to change the angle $\phi_0$ by angle $\phi_0 \pm \pi$.

Remark 4. For any matrix $B \in SO(2) = \exp(\mathbb{R}c)$, the map $l_B \circ r_{B^{-1}}$, where $l_B$ is multiplication from the left by $B$, $r_{B^{-1}}$ is multiplication from the right by $B^{-1}$, is simultaneously automorphism $\text{Ad} B$ of the Lie algebra $(\mathfrak{so}(3), [\cdot, \cdot])$, preserving $\langle \cdot, \cdot \rangle$, and automorphism of the Lie group $SO(3)$, preserving distribution $D$ and metric $d$. In particular in view of (6), (10)
\[
\text{Ad} B(a + \beta c) = \exp(\phi_0 a)(a + \beta c) = \cos \phi_0 a + \sin \phi_0 b + \beta c,
\]
if
\[
B = \exp(\phi_0 c) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi_0 & -\sin \phi_0 \\
0 & \sin \phi_0 & \cos \phi_0
\end{pmatrix}.
\]

Lemma 1.
\[
\exp(t(a + \beta c)) = I(\exp(-\xi b))(\exp(t\sqrt{1+\beta^2} a)),
\]
where
\[
\cos \xi = \frac{1}{\sqrt{1+\beta^2}}, \quad \sin \xi = \frac{\beta}{\sqrt{1+\beta^2}}.
\]

Proof. Taking into account (20), (16), (6), we get
\[
t(a + \beta c) = (t\sqrt{1+\beta^2}(\cos \xi \cdot a + \sin \xi \cdot c)) = (\exp(\xi b))(t\sqrt{1+\beta^2} a) = \\
(\exp(\text{ad}(-\xi b)))(t\sqrt{1+\beta^2} a) = \text{Ad}(\exp(-\xi b))(t\sqrt{1+\beta^2} a).
\]
Now in consequence of obtained equalities and (5),
\[
\exp(t(a + \beta c)) = \exp(\text{Ad}(-\xi b)(t\sqrt{1+\beta^2} a)) = I(\exp(-\xi b))(\exp(t\sqrt{1+\beta^2} a)).
\]

□
Theorem 3. The geodesic \( \gamma = \gamma(t) \) of left-invariant sub-Riemannian metric \( d \) on the Lie group \( SO(3) \), defined by formula (10), is equal to

\[
\begin{pmatrix}
1 - n & -m \cos (\beta t + \phi_0) - \beta n \sin (\beta t + \phi_0) \\
-m \sin (\beta t + \phi_0) + \beta n \cos (\beta t + \phi_0) & 1 - n(1 + \beta^2) - m \cos (\beta t + \phi_0) \\
\beta n \cos (\beta t + \phi_0) & m \sin (\beta t + \phi_0) \\
0 & 0
\end{pmatrix}
\]

where

\[
m = \frac{\sin(t \sqrt{1 + \beta^2})}{\sqrt{1 + \beta^2}}, \quad n = \frac{1 - \cos(t \sqrt{1 + \beta^2})}{1 + \beta^2}.
\]

Proof. Let \( \phi_0 = 0 \). Then (10) takes the form

\[
\gamma(t) \big|_{\phi_0=0} = \exp(t(a + \beta c)) \exp(-t\beta c).
\]

Using lemma 1 (22) and carrying out routine calculations, we get

\[
\exp(t(a + \beta c)) = \frac{1}{1 + \beta^2} \begin{pmatrix} 1 & 0 & \beta \\ 0 & \sqrt{1 + \beta^2} & 0 \\ -\beta & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t \sqrt{1 + \beta^2} & -\sin t \sqrt{1 + \beta^2} & 0 \\ \sin t \sqrt{1 + \beta^2} & \cos t \sqrt{1 + \beta^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times
\begin{pmatrix} 1 & 0 & -\beta \\ 0 & \sqrt{1 + \beta^2} & 0 \\ \beta & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - n & -m & n\beta \\ m & 1 - n(1 + \beta^2) & -m\beta \\ n\beta & m\beta & 1 - n\beta^2 \end{pmatrix}.
\]

Now, using (10) and (13) for \( \phi_0 = -\beta t \), we get

\[
\gamma(t) \big|_{\phi_0=0} = \begin{pmatrix} 1 - n & -m & n\beta \\ m & 1 - n(1 + \beta^2) & -m\beta \\ n\beta & m\beta & 1 - n\beta^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta t & \sin \beta t \\ 0 & -\sin \beta t & \cos \beta t \end{pmatrix} =
\begin{pmatrix} 1 - n - m \cos (\beta t + \phi_0) - \beta n \sin (\beta t + \phi_0) & \beta n \cos (\beta t + \phi_0) - m \cos (\beta t + \phi_0) \\ -m \sin (\beta t + \phi_0) + \beta n \cos (\beta t + \phi_0) & 1 - n(1 + \beta^2) - m \cos (\beta t + \phi_0) \\ \beta n \cos (\beta t + \phi_0) & m \sin (\beta t + \phi_0) \end{pmatrix}
\]

By (18), matrices \( B = \exp(\phi_0) \) and \( \exp(-t\beta c) \) commute. It follows from here and from remark 4 that

\[
\gamma(t) = B \cdot \gamma(t) \big|_{\phi_0=0} \cdot B^{-1}.
\]

Substitution of formula (18) into the last equality finishes the proof. \( \square \)

3. Shortest arcs on the Lie group \( (SO(3), d) \)

The group \( SO(3) \) is realized as the group of all preserving orientation isometries \( v \rightarrow gv; \quad g \in SO(3), v \in S^2 \) of unit sphere \( S^2 \subset \mathbb{R}^3 \), whose elements \( v \) are regarded as vector-columns. It is not difficult to check that Lie subgroup

\[
SO(2) := \{ \exp sc, s \in \mathbb{R} \} \subset SO(3)
\]
GROUP $SO(3)$

is the stabilizer of vector $v_0 = (1, 0, 0)^T = e_1 \in S^2$ with respect to this action. Moreover the group $SO(2)$ acts (simply) transitively by rotations on unit circle $S^1 := S^2 \cap e_1^+ \subset S^2$.

Therefore $S^2$ is naturally identified with quotient homogeneous space $SO(3)/SO(2)$ and the group $SO(3)$ itself is diffeomorphic to the space $S^2_1$ of all unit tangent vectors to $S^2$. Namely, every element $g \in SO(3)$ corresponds to $ge_1'$, where $e_1'$ is usual parallel translation of vector $e_2$ to point $e_1$. Moreover, in consequence of introduction,

1) Any segment of a smooth path $c = c(t)$ in $(SO(3), d)$, tangent to distribution $D$, has the same length as its image relative to canonical projection

\[(24) \quad p : g \in SO(3) \rightarrow ge_1 \in S^2;\]

2) under indicated identification of $SO(3)$ with $S^2_1$, any path $c = c(t), 0 \leq t \leq t_1$, tangent to distribution $D$, is realized as parallel vector field in $S^2$ along $p(c(t)), 0 \leq t \leq t_1$, with initial unit tangent vector $c(0) \in S^2_1$;

3) By the Gauss-Bonnet theorem [5], under parallel translation in $S^2$ of non-zero tangent vector along a contour, bounding a region in $S^2$ with area $S < 2\pi$, the vector turns in the direction of bypass by the angle $S$.

Let us use statements 1) — 3) to find shortest arcs in $(SO(3), d)$. In consequence of proposition [1], remark [4], and left invariance of the metric $d$, it is sufficient to investigate segments of geodesics of the form

\[(25) \quad \gamma(t) = \exp(t(a + \beta c)) \exp(-t\beta c), \quad 0 \leq t \leq t_1,\]

and their projections

\[(26) \quad x(t) := p(\gamma(t)) = \gamma(t) \cdot e_1 = \gamma(t) \cdot (1, 0, 0)^T = (1 - n, m, \beta n)^T, \quad 0 \leq t \leq t_1,
\]
to the sphere $S^2$, where $m, n$ are defined by formulae (22) (we used formula (21) for $\phi_0 = 0$).

Since the second factor in (25) lies in $SO(2)$, then orbits (26) coincide with segments of orbits of 1-parameter subgroup $y(t) = \exp(t(a + \beta c)), t \in \mathbb{R}$.

It is not difficult to calculate that $\pm(1/\sqrt{1 + \beta^2})(\beta, 0, 1)^T \in S^2$ are unit eigenvectors of matrix $a + \beta c$ with respect to zero eigenvalue. Consequently, 1-parameter subgroup $y(t), t \in \mathbb{R}$, preserves these vectors. Scalar products of these vectors with $e_1$ are equal to $\pm(\beta/\sqrt{1 + \beta^2})$. Then spherical distance from the point $e_1$ to the axis of these vectors is equal to

\[(27) \quad r = \arccos(|\beta|/\sqrt{1 + \beta^2}) \leq \pi/2.\]

Therefore the orbit $\{\gamma(t)e_1 = y(t)e_1\}$ is spherical circle of radius $r < \pi/2$ with unique center $(1/\sqrt{1 + \beta^2})(\beta, 0, 1)^T$, if $\beta \neq 0$. It is not difficult to see that if $\beta > 0$, then in consequence of theorem [3], curve (26) for $t_1 = 2\pi/\sqrt{1 + \beta^2}$ goes around this circle, bounding lesser region $\Psi$ of $S^2$ with this center inside it, one times, leaving the region $\Psi$ from the left.

Let us formulate the Gauss-Bonnet theorem [5]. Let $M$ be two-dimensional oriented manifold with Riemannian metric $ds^2$, $\Phi$ is a region in $M$, homeomorphic to disc and bounded by closed piece-wise regular curve $\gamma$ with regular links $\gamma_1, \ldots, \gamma_n$, forming angles $\alpha_1, \ldots, \alpha_n$ from the side of region $\Phi$. Direction on the curve $\gamma$ is
Theorem 4. 

\[
\sum_{k=1}^{n} \int_{\gamma_k} \kappa ds + \sum_{k=1}^{n} (\pi - \alpha_k) = 2\pi - \int_{\Phi} K ds, 
\]

where \(\kappa\) is geodesic curvature at points of links of the curve, \(K\) is Gaussian (sectional) curvature of the surface \((M, ds^2)\), and integration in the right part of equality is taken by area element of the region \(\Phi\).

In particular, if \(\gamma\) is a regular curve, then

\[
\int_{\gamma} \kappa ds = 2\pi - \int_{\Phi} K ds.
\]

Proposition 2. Geodesic curvature of curve (26) for \(\beta > 0\) is equal to \(-|\beta|\).

Proof. In consequence of what has been said, applying equality (29) to circle (26) for \(\beta > 0\) and \(t_1 = 2\pi/\sqrt{1 + \beta^2}\), one needs to take region \(\Phi = S^2 \setminus \Psi\) in \(S^2\) and \(K = 1\). Then the left part of (29) is equal to \(\kappa t_1\). For the right part, we need area \(\sigma(\Phi)\).

It is known that in \(S^2\)

\[
l(r, \alpha) = \alpha \sin r,
\]

\[
S(r, \alpha) = \int_{0}^{r} \alpha \sin s ds = \alpha \sin r|_{0}^{r} = \alpha(1 - \cos r),
\]

where \(l(r, \alpha)\) is the length of arc of circle with radius \(r\) and central angle \(\alpha \leq 2\pi\), and \(S(r, \alpha)\) is area of corresponding sector. Then in consequence of (27),

\[
\sigma(\Psi) = 2\pi \left(1 - \frac{|\beta|}{\sqrt{1 + \beta^2}}\right),
\]

\[
\sigma(\Phi) = 4\pi - \sigma(\Psi) = 2\pi \left(1 + \frac{|\beta|}{\sqrt{1 + \beta^2}}\right),
\]

\[
\frac{2\pi \kappa}{\sqrt{1 + \beta^2}} = 2\pi - \sigma(\Phi) = -2\pi \frac{|\beta|}{\sqrt{1 + \beta^2}}, \ \kappa = -|\beta|.
\]

\[\Box\]

Proposition 3. Let us assume that projection (26) of geodesic segment (25), where \(\beta \neq 0\), has no self-intersection, i.e. \(0 \leq t_1 < 2\pi/\sqrt{1 + \beta^2}\), \(S(t_1) = S(t_1, \beta)\) is area of lesser curvilinear digon \(P\) in \(S^2\), bounded by segment (26) and shortest segment \([x(0)x(t_1)]\) of a length \(r = r(t_1)\) in \(S^2\), \(\psi = \psi(t_1, \beta)\) is interior angle of the digon \(P\). Then

\[
r = \arccos((1 - n)(t_1)), \quad r'(t_1) = \cos \psi = \frac{m}{\sqrt{n(2 - n)}}, \quad S(t_1) = 2\psi - |\beta|t_1.
\]
Moreover \( S'(t_1) > 0 \), if \( t_1 > 0 \); \( 0 < \psi \leq \pi/2 \), if \( 0 < t_1 \leq \pi/\sqrt{1+\beta^2} \), and \( \pi/2 < \psi < \pi \), if \( \pi/\sqrt{1+\beta^2} < t_1 \leq 2\pi/\sqrt{1+\beta^2} \).

**Proof.** The first equality in (32) is a corollary of (26) and known formula for distance in spherical geometry, the second one is a well-known statement of Riemannian geometry (on existence of strong angle), the third equality is result of differentiation of first equality in (32). Inequalities for the angle are evident. In consequence of remark 3 one can assume that \( \beta > 0 \). Segment \([x(0)x(t_1)]\) has geodesic curvature 0. Then, with taking into account \( \Phi = S^2 \setminus \overline{P} \) and proposition 2, equation (28) is written in the form

\[-|\beta|t_1 + (2\pi - (4\pi - 2\psi)) = 2\pi - (4\pi - S(t_1)).\]

Consequently, \( S'(t_1) = 2\psi - |\beta|t_1 \). From here and (31) follow relations

\[ S'(t_1) = 2\psi'(t_1) - |\beta| = (1 - \cos r)\psi'(t_1), \]

(33)

\[ \psi'(t_1) = \frac{|\beta|}{1 + \cos r} = \frac{|\beta|}{2 - n}. \]

(34)

\[ S'(t_1) = |\beta| \left( \frac{2}{2-n} - 1 \right) (t_1) > 0, \quad 0 < t_1 < \frac{2\pi}{\sqrt{1+\beta^2}}. \]

\[ \square \]

**Lemma 2.** If \( \beta = 0 \) and \( t_1 = \pi \), then (25) is noncontinuable shortest arc.

**Proof.** In this case \( \gamma(t) = \exp(ta) \). Then \( \gamma(2\pi) = e \) and, consequently, \( \gamma(\pi) = \gamma(-\pi) \). Therefore geodesic segment \( \gamma(t), 0 \leq t \leq t_2 \), is not shortest arc for \( t_2 > t_1 = \pi \). On the other hand, canonical projection \( p : (SO(3),d) \to S^2 \) (see (11) and (24)) is a submetry, moreover

\[ \gamma(\pi) = - (e_{11} + e_{22}) + e_{33}, \quad p(\gamma(\pi)) = \gamma(\pi)e_1 = -e_1; \]

i.e. path \( p(\gamma(t)), 0 \leq t \leq \pi \), is shortest connection in \( S^2 \) of diametrally opposite points \( e_1 \) and \(-e_1 \). Then (25) is noncontinuable shortest arc.

\[ \square \]

**Proposition 4.** 1) If \( \beta \neq 0 \) then geodesic segment (25) is noncontinuable shortest arc when its projection (26) is a) one time passing circle \( C \) bounding disc with area \( S(t_1) \leq \pi \) or b) curve without self-intersections bounding together with the shortest arc \([x(0)x(t_1)]\) in \( S^2 \) digon \( P \) in \( S^2 \) with area \( S(t_1) = \pi \).

2) For every \( \beta \neq 0 \) there is unique \( t_1 > 0 \) such that one of conditions a) or b) is satisfied; a) is satisfied only if \( |\beta| \geq 1/\sqrt{3} \).

**Proof.** 1) a) It is clear that \( \gamma(t_1) \in SO(2) \). Then in consequence of remark 4, segment of geodesic (10) for the same \( \beta \) and any \( \phi_0 \) under \( t \in [0,t_1] \) joins the same points as (25). Consequently every continuation of the segment (25) is not a shortest arc.

Let us suppose that there exists a shortest arc \( \gamma_2(t), 0 \leq t \leq t_2 < t_1 \), in \((SO(3),d)\) which joins points \( \gamma(0) = e \) and \( \gamma(t_1) \). Then projection \( x_2(t) = p(\gamma_2(t)) \), \( 0 \leq t \leq t_2 \), is one time passing circle \( C_2 \) in \( S^2 \) with length \( t_2 < t_1 \) and therefore bounds a disc with area \( S(t_2) < S(t_1) \leq \pi \). Consequently on the ground of the Gauss-Bonnet
theorem, results of parallel translations of nonzero vectors along $C$ and $C_2$ in $S^2$ are
different. Then $\gamma_2(t_2) \neq \gamma(t_1)$ in view of geometric interpretation of geodesics in
$(SO(3), d)$, given in introduction, a contradiction.

b) Let $P'$ be a digon, symmetric to the digon $P$ relative to segment $[x(0)x(t_1)]$.
Since $S(t_1) = \pi$ then by the Gauss-Bonnet theorem, results of parallel translations in
$S^2$ of tangent vectors along closed paths, bounding $P$ and $P'$, are equal. Therefore on
the ground of remarks 3, 4 and geometric interpretation of geodesics in $(SO(3), d)$,
given in introduction, a curve in $S^2$, symmetric to the projection $(26)$ of segment $(25)$
relative to segment $[x(0)x(t_1)]$, is presented in the form $p(\gamma_1(t))$, $0 \leq t \leq t_1$, where
$\gamma_1$ is a geodesic in $(SO(3), d)$ such that $\gamma_1(0) = \gamma(0)$, $\gamma_1(t_1) = \gamma(t_1)$. Consequently
every continuation of the segment $(25)$ is not a shortest arc.

Let us suppose that there is a shortest arc $\gamma_2(t)$, $0 \leq t \leq t_2 < t_1$, in $(SO(3), d)$,
joining points $\gamma(0) = e$ and $\gamma(t_1)$. Then in consequence of remarks 3 and 4 we can
assume that curves $(26)$ and $x_2(t) = p(\gamma_2(t))$, $0 \leq t \leq t_2$, lie on the one side of
the shortest arc $[x(0)x(t_1)]$ and join ends of this shortest arc. Consequently the
digon $P$ and digon $P_2$, bounded by the shortest arc $[x(0)x(t_1)]$ and the curve $x_2(t)$, $0 \leq t \leq t_2$, are convex, moreover intersection of their boundaries is the shortest
arc $[x(0)x(t_1)]$, because $t_2 < t_1$. Therefore in view of last inequality the curve $x_2(t)$, $0 < t < t_2$, lies inside $P$ and $S(t_2) < S(t_1) = \pi$, where $S(t_2)$ is area of the digon
$P_2$. Consequently on the ground of the Gauss-Bonnet theorem, results of parallel
translations of nonzero tangent vectors along boundaries of $P$ and $P_2$ in $S^2$ are
different. Then $\gamma_2(t_2) \neq \gamma(t_1)$ in view of geometric interpretation of geodesics in
$(SO(3), d)$, given in introduction, a contradiction.

2) On the ground of last equality in $(52)$, the condition a) is fulfilled only if
$t_1 = 2\pi/\sqrt{1 + \beta^2}$, $\psi(t_1) = \pi$ and

$$S\left(\frac{2\pi}{\sqrt{1 + \beta^2}}\right) = 2\pi - |\beta|\frac{2\pi}{\sqrt{1 + \beta^2}} \leq \pi \leftrightarrow |\beta| \geq \frac{1}{\sqrt{3}}.$$  

If $0 < |\beta| < \frac{1}{\sqrt{3}}$, then in consequence of proposition 3 there exists unique $t_1 > 0$ for
which the condition b) is satisfied. \[ \square \]

Later for every number $\beta \neq 0$ we shall find a number $t_1 = t_1(\beta)$, satisfying
conditions of proposition 4

I) If $|\beta| \geq \frac{1}{\sqrt{3}}$, then $t_1 = 2\pi/\sqrt{1 + \beta^2}$.

II) If $0 < |\beta| < \frac{1}{\sqrt{3}}$, then

$$S(2\pi/\sqrt{1 + \beta^2}) < 2\pi \Rightarrow S(\pi/\sqrt{1 + \beta^2}) < \pi,$$

(35)

$$S(t_1) = \pi \Rightarrow \frac{\pi}{\sqrt{1 + \beta^2}} < t_1 < 2\pi/\sqrt{1 + \beta^2}.$$  

Therefore in consequence of proposition 3, $\pi/2 < \psi(t_1) < \pi$ and

(36)

$$S(t_1) = \pi \leftrightarrow 2\psi - |\beta|t_1 = \pi \leftrightarrow \frac{|\beta|t_1}{2} = \psi - \pi/2.$$
In consequence of (32) and (22),
\[
\cos \psi = \frac{m}{\sqrt{n(2-n)}} = \frac{\sqrt{1+\beta^2 \sin(t_1 \sqrt{1+\beta^2})}}{\sqrt{(1-\cos(t_1 \sqrt{1+\beta^2}))(1+\cos(t_1 \sqrt{1+\beta^2})+2\beta^2)}} = 
\frac{\sqrt{1+\beta^2 \cos(t_1 \sqrt{1+\beta^2}/2)}}{\sqrt{\cos^2(t_1 \sqrt{1+\beta^2}/2)+\beta^2}}
\]

We get from here, (36), inequalities for \(t_1\) and \(\psi\) that
\[
\sin \psi = \sqrt{1-\cos^2 \psi} = \frac{\sqrt{1+\beta^2 \cos(t_1 \sqrt{1+\beta^2}/2)}}{\sqrt{\cos^2(t_1 \sqrt{1+\beta^2}/2)+\beta^2}}.
\]

(37) \(\sin \left(\frac{|\beta| t_1}{2}\right) = \sin \left(\psi - \frac{\pi}{2}\right) = -\cos \psi = \frac{-\sqrt{1+\beta^2 \cos(t_1 \sqrt{1+\beta^2}/2)}}{\sqrt{\cos^2(t_1 \sqrt{1+\beta^2}/2)+\beta^2}}\),

(38) \(\cos \left(\frac{|\beta| t_1}{2}\right) = \cos \left(\psi - \frac{\pi}{2}\right) = \sin \psi = \frac{\sqrt{1+\beta^2 \cos(t_1 \sqrt{1+\beta^2}/2)}}{\sqrt{\cos^2(t_1 \sqrt{1+\beta^2}/2)+\beta^2}}\).

(39) \(0 < |\beta| t_1 < \pi\).

**Theorem 5.** Conditions a),b) of proposition 4 define a continuous function \(t_1 = t_1(|\beta|)\), increasing under \(0 \leq |\beta| \leq 1/\sqrt{3}\) and decreasing under \(1/\sqrt{3} \leq |\beta| < +\infty\).

**Proof.** The second statement is evident. The first statement is true, because\(dt_1/d|\beta| > 0\) under \(0 < |\beta| < 1/\sqrt{3}\) in consequence of (36), (33), (22), (35):
\[
t_1 + |\beta| \frac{dt_1}{d|\beta|} = 2\psi'(t_1) \cdot \frac{dt_1}{d|\beta|} = \frac{2|\beta|}{2-n} \cdot \frac{dt_1}{d|\beta|},
\]
\[
t_1 = \frac{|\beta| n}{2-n} \cdot \frac{dt_1}{d|\beta|} = \frac{|\beta| \sin^2(t_1 \sqrt{1+\beta^2/2})}{\sqrt{\beta^2 + \cos^2(t_1 \sqrt{1+\beta^2/2})}} \cdot \frac{dt_1}{d|\beta|}.
\]

\[\square\]

**Theorem 6.**
\[
diam(SO(3), d) = \pi \sqrt{3}.
\]

**Proof.** It follows from theorem 5 that maximal length of shortest arc is attained under \(\beta^2 = 1/3\) and it is equal to \(\pi \sqrt{3}\). This implies needed statement. \[\square\]

**Remark 5.** Statement of theorem 6 is a particular case of the first statement of theorem 2 from paper [1].
4. Cut locus and conjugate sets in \((SO(3), d)\)

Unlike Riemannian manifolds, exponential map \(\Exp\) and its restriction \(\Exp_x\) for sub-Riemannian manifold \((M, d)\) without abnormal geodesics (as in the case of \((SO(3), d)\)) are defined not on \(TM\) and \(T_xM\) but only on \(D\) and \(D(x)\), where \(D\) is distribution on \(M\), taking part in definition of \(d\). Otherwise cut locus and conjugate sets for such sub-Riemannian manifold are defined in the same way as for Riemannian one \([10]\).

**Definition 1.** Cut locus \(C(x)\) (respectively conjugate set \(S(x)\)) for a point \(x\) in sub-Riemannian manifolds \(M\) (without abnormal geodesics) is the set of ends of all noncontinuable beyond its ends shortest arcs starting at the point \(x\) (respectively, image of the set of critical points of the map \(\Exp_x\) with respect to \(\Exp_x\)).

**Theorem 7.** For every element \(g \in (SO(3), d)\), \(C(g) = gC(e)\) and \(S(g) = gS(e)\).

Moreover \(S(g) \subset C(g)\),

\[
C(e) = \{\gamma_\beta(t_1(\beta)) : \beta \in \mathbb{R}\},
\]

\[
S(e) = \{\gamma_\beta(t_1(\beta)) : \beta^2 \geq 1/3\} = SO(2) \setminus \{e\};
\]

\(S(e)\) is diffeomorphic to \(\mathbb{R}\);

\[
\overline{S(e)} = S(e) \cup \{e\} = SO(2),
\]

\(\overline{S(e)}\) is diffeomorphic to circle \(S^1\);

\[
\overline{C(e)} \setminus \overline{S(e)} = (C(e) \setminus S(e)) \cup \left\{\gamma_\beta(t_1(\beta)) = \gamma_{-\beta}(t_1(-\beta)) : \beta = \frac{1}{\sqrt{3}}\right\},
\]

\(\overline{C(e)} \setminus \overline{S(e)}\) is diffeomorphic to real projective plane \(RP^2\); \(C(e)\) is homeomorphic to \(RP^2 \cup \mathbb{R}\), where \(RP^2 \cap \mathbb{R}\) is one-point set; \(C(e)\) is homeomorphic to \(RP^2 \cup S^1\), where \(RP^2 \cap S^1\) is one-point set.

**Proof.** First statement is a corollary of left invariance of the metric \(d\) on \(SO(3)\). Inclusion \(S(g) \subset C(g)\), formulae (40), (41), equality in brace from (43), and diffeomorphism \(S(e) \cong \mathbb{R}\) are corollaries from the proof of proposition 4 and remark 4. Formula (42) and diffeomorphism \(\overline{S(e)} \cong S^1\) follow from formula (41). Equality (43) follows from formulae (44), (45); \(\overline{C(e)} \setminus \overline{S(e)} \cong RP^2\) follows from equalities \(\gamma_{(\beta,\phi_0)}(t_1(\beta)) = \gamma_{(-\beta,-\beta_1+\phi_0+\pi)}(t_1(-\beta))\) при \(\beta^2 \leq 1/3\). Now it is not difficult to prove remaining statements. \(\square\)

**Remark 6.** It follows from (42) and equalities \(C(g) = gC(e), S(g) = gS(e)\) that \(g \in gSO(2) = \overline{S(g)} \subset \overline{C(g)}\) for all \(g \in SO(3)\), while \(x \notin \overline{C(x)}\) and \(x \notin \overline{S(x)}\) for any point \(x\) of arbitrary smooth Riemannian manifold. This constitutes radical difference of Riemannian and sub-Riemannian manifolds.
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