Amphichiral knots with large 4-genus

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Abstract
For each \( g > 0 \) we give infinitely many knots that are strongly negative amphichiral, hence rationally slice and representing 2-torsion in the smooth concordance group, yet which do not bound any locally flatly embedded surface in the 4-ball with genus less than or equal to \( g \). Our examples also allow us to answer a question about the four-dimensional clasp number of knots.

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1 | INTRODUCTION

An oriented knot \( K \) in \( S^3 \) is called strongly negative amphichiral if there exists an orientation reversing involution \( \varphi : S^3 \to S^3 \) such that \( \varphi(K) = K^r \). Many concordance invariants vanish on such knots, including the classical Tristram–Levine signature function [13, 23] and more modern invariants coming from Heegaard Floer and Khovanov homology like the \( \tau \)-invariant [21], \( \nu^+ \)-invariant [8], \( Y \)-invariant [20], \( s \)-invariant [22], \( s_n \)-invariants [18, 24], \( s^\# \)-invariant [12], and \( \lambda \)-invariant [14]. Notably, this list contains almost all known lower bounds on the 4-genus, or minimal genus of a (smoothly or locally flatly) embedded orientable surface in \( B^4 \) with boundary the given knot. However, we use Gilmer’s bound on the topological 4-genus [5] coming from Casson–Gordon signatures [2] to prove the following.

Theorem 1.1. For any \( g > 0 \), there exists a knot \( K \) with the following properties.

(i) \( K \) is strongly negative amphichiral.
(ii) \( K \) can be transformed to a smoothly slice knot by changing some crossings \((+) \) to \((-)\).
(iii) \( K \) can be transformed to a smoothly slice knot by changing some crossings \((-) \) to \((+)\).
(iv) the topological 4-genus of \( K \) is strictly larger than \( g \).
In fact, something more is true, and proven in Proposition 2.7: for any \( g \in \mathbb{N} \) there exists an infinite family of knots \( \{K^k\}_{k \in \mathbb{N}} \), generating a subgroup of the concordance group isomorphic to \((\mathbb{Z}_2)^\infty\), such that any nontrivial sum \( K = \#_{j=1}^m K^{k_j} \) satisfies the conclusions of Theorem 1.1. Moreover, each of the knots \( K^k \) is algebraically slice, so we incidentally reprove a result of Livingston [16] that there is a \((\mathbb{Z}_2)^\infty\)-subgroup of the concordance group consisting of algebraically slice knots.

Negative amphichiral knots, if not slice, represent 2-torsion elements of the smooth concordance group; a still-open question of Gordon asks whether all 2-torsion elements have such representatives [6, Problem 16]. We therefore obtain the following corollary to Theorem 1.1, which appears to be previously unknown.

**Corollary 1.2.** There exist 2-torsion knots with arbitrarily large 4-genera.

A knot \( K \) is called *rationally slice* if there exists a smooth 4-manifold \( W \) with boundary \( \partial W = S^3 \) and \( H_*(W; \mathbb{Q}) = H_*(B^4; \mathbb{Q}) \) such that \( K \) bounds a smoothly embedded null-homologous disc in \( W \). Every strongly negative amphichiral knot is rationally slice [10], and so Theorem 1.1 also answers a question of [7] in the affirmative.

**Corollary 1.3.** There exist rationally slice knots with arbitrarily large 4-genera.

The *four-dimensional clasp number* \( c_4(K) \) of a knot \( K \) is the minimal number of transverse double points across all immersions of \( D^2 \) in \( B^4 \) with \( \partial D^2 = K \). Similarly, \( c_4^+(K) \) (respectively, \( c_4^-(K) \)) is defined to be the minimal number of positive (respectively, negative) transverse double points across all immersions of \( D^2 \) in \( B^4 \) with \( \partial D^2 = K \). It follows immediately from the definitions that \( c_4^+ + c_4^- \leq c_4 \); the figure-eight knot \( 4_1 \) is the prototypical example of when this inequality is strict, since \( c_4^+(4_1) = c_4^-(4_1) = 0 \) and yet \( c_4(4_1) = 1 \). We answer a question of [9] by giving the first examples of knots for which \( c_4(K) \) is arbitrarily larger than \( c_4^+(K) + c_4^-(K) \).

**Corollary 1.4.** The difference between \( c_4(K) \) and \( c_4^+(K) + c_4^-(K) \) can be arbitrarily large.

*Proof.* For \( g \in \mathbb{N} \), let \( K_g \) be a knot satisfying the conclusions of Theorem 1.1. By items (ii) and (iii), we have that \( c_4^+(K_g) + c_4^-(K_g) = 0 + 0 = 0 \), and by item (iv) we have that
\[
g < g_4(K_g) \leq g^+_4(K_g) \leq c_4(K_g),
\]
noting that standard arguments show that for any knot \( K \) the smooth 4-genus \( g_4(K) \) is bounded above by \( c_4(K) \). \( \square \)

Since Casson–Gordon signatures provide bounds on the topological 4-genus, it remains open whether one can find examples for the smooth analogue of Theorem 1.1. In particular, the following three questions remain open.

**Question 1.** For \( g \in \mathbb{N} \), is there a topologically slice knot \( K \) such that \( g_4^+(K) > g \) and

(i) \( K \) has order 2 in the smooth concordance group?
(ii) \( K \) is smoothly rationally slice?
(iii) \( c_4^+(K) = c_4^-(K) = 0 \)?
Recent work of Hom–Kang–Park–Stoffregen [7] has shown that \( \{ C_{2n+1,1}(41) \}_{n \in \mathbb{N}} \) generates a \( \mathbb{Z}^\infty \)-subgroup of rationally slice knots in the smooth concordance group. By work of [4], the topological 4-genus of \( C_{2n+1,1}(41) \) equals 1 for all \( n \in \mathbb{N} \), but it remains open whether the smooth 4-genus of \( C_{2n+1,1}(41) \) is large. Since \( 2n + 1 \) is relatively prime to 2, one can combine the work of this paper with the formulas for Casson–Gordon signatures of satellite knots given in [15] and conclude that for our choice of \( K_g \) satisfying the conclusions of Theorem 1.1, we have that \( g(\mathcal{K}(4)) > 0 \) for all \( n \in \mathbb{N} \). We therefore state the following as an interesting open problem in either the smooth or topological categories.

**Question 2.** For any \( g \in \mathbb{N} \), let \( K_g \) be one of the knots given in Section 2 that satisfies the conclusions of Theorem 1.1. For some or all \( n \in \mathbb{N} \), determine whether \( C_{2n+1,1}(K_g) \) is infinite order in the concordance group.

We note that it remains open even whether \( C_{2n,1}(K) \) must always be slice for strongly negative amphichiral \( K \), though it is known that many such knots are not ribbon [19].

**Remark 1.** The key feature of Casson–Gordon signatures that allows us to use Gilmer’s bound to establish Theorem 1.1 when all other lower bounds on the 4-genus fail might initially seem like a flaw: no single signature gives a 4-genus bound or even a sliceness obstruction. While we avoid stating the precise definition of these invariants, we remind the reader that \( \sigma(K, \chi) \in \mathbb{Q} \) depends on not just the knot \( K \) but a choice of map \( \chi \) from the first homology of the double branched cover of \( K \) to a cyclic group. The fact that \( K \) is negative amphichiral implies that there is an involution \( \iota \) on the set of such maps such that \( \sigma(K, \iota(\chi)) = -\sigma(K, \chi) \). As long as this involution is nontrivial, the negative amphichirality of \( K \) does not force \( \sigma(K, \chi) \) to vanish and there is still the potential to obtain a sliceness obstruction — and even a lower bound on the 4-genus — by considering the set of all such signatures. This could be considered as philosophically similar to the fact that Casson–Gordon signatures can obstruct knots from being concordant to their reverses [11], though that result requires a careful analysis of additional structure that we are able to avoid.

# 2 PROOF OF MAIN RESULT

Our examples are connected sums of certain satellites of the figure-eight knot.

**Example 1.** Let \( J \) be a reversible knot and define \( K(J) \) to be as in Figure 1, where \( \bar{J} \) denotes the mirror image of \( J \), which since \( J \) is reversible equals the concordance inverse \( -J \). We note for later
that the disc-with-bands Seifert surface for $K(J)$ visible on the left of Figure 1 demonstrates that $K(J)$ shares a Seifert form with the figure-eight knot $K_0$.

The right side of Figure 1 demonstrates that $K(J)$ is strongly negative amphichiral: rotation by $180°$ in the plane about the marked point followed by reflection in the plane of the page takes $K(J)$ to itself, but with reversed orientation. An alternate construction of this involution comes from considering the decomposition of the exterior of $K(J)$ as the exterior of $K(U)$, the figure-eight knot, with two solid tori cut out and the exteriors of $J$ and $\tilde{J}$ glued in places. One can then verify that the involution guaranteeing the strong negative amphichirality of $K(U)$ exchanges said tori, and hence yields an appropriate involution on the exterior of $K(J)$.

**Proposition 2.1.** If $J$ is a reversible knot, then $K(J)$ has $c_+^4(K_J) = c_-^4(K_J) = 0$.

**Proof.** Consider the knots $K_\pm$ as depicted in Figure 2, shown with genus 1 Seifert surfaces $F_\pm$ in disc-with-bands position. Observe that $K_+$ (respectively, $K_-$) is obtained from $K_J$ by changing a single negative (respectively, positive) crossing to a positive (respectively, negative) crossing. Figure 2 also depicts a curve $\gamma_\pm$ on $F_\pm$. Note that each of $\gamma_\pm$ represents a nontrivial element of $H_1(F_\pm)$ and is 0-framed by $F_\pm$; that is, is a derivative curve. Considered as a knot, $\gamma_+$ is $J\#\tilde{J}$; since $J$ is reversible this is isotopic to $J\#-J$ and hence is slice. Similarly, the knot type of $\gamma_-$ is the slice knot $J\#-J$. Therefore, surgering the Seifert surface $F_\pm$ along the derivative curve $\gamma_\pm$ yields a smooth slice disc for $K_\pm$. We can convert this single crossing change from $K(J)$ to $K_\pm$ into an immersed annulus in $S^3 \times I$ from $K(J)$ to $K_\pm$. Capping each of these annuli with a smooth slice disc for $K_\pm$ yields the desired immersed discs bounded by $K(J)$, each with a single singularity of different sign.

2.1 | Background results

For $n \in \mathbb{N}$ and a knot $K$, we let $\Sigma_n(K)$ denote the $n$th cyclic branched cover of $S^3$ along $K$. To a knot $K$ and a map $\chi : H_1(\Sigma_n(K)) \to \mathbb{Z}_q$ one can associate the Casson–Gordon signature $\sigma(K, \chi) \in \mathbb{Q}$ [2]. We avoid giving the technical definition of these invariants, noting only that they are defined in terms of the twisted intersection form of some 4-manifold and are notoriously difficult to compute precisely. We remark for those familiar with Casson–Gordon signatures that in the literature what we call $\sigma(K, \chi)$ is just $\sigma_1\tau(K, \chi)$ instead.
Our lower bound on the topological 4-genus of a knot comes from the following slightly reformulated result of Gilmer.

**Theorem 2.2** [5, Theorem 1]. Suppose that $K$ is a knot with $g_4(K) \leq g$. Then there is a decomposition $H_1(\Sigma_2(K)) = A_1 \oplus A_2$ such that

1. $A_1$ has a presentation with at most $2g$ generators;
2. there is some $B \leq A_2$ with $|B|^2 = |A_2|$ such that for any prime power order $\chi : H_1(\Sigma(K)) \rightarrow \mathbb{Z}_q$ with $\chi$ vanishing on $A_1 \oplus B$, we have
   \[ |\sigma(K, \chi) + \sigma(K)| \leq 4g. \]

We remark for later that in our applications of Theorem 2.2 we will always have that $K$ is negative amphichiral and hence that $\sigma(K) = 0$.

In particular, we have the following corollary, which is all that we need for Theorem 1.1.

**Corollary 2.3.** Suppose that $K$ is a knot with $g_4(K) \leq g$ such that $H_1(\Sigma_2(K))$ is not generated as an abelian group by any $2g$ of its elements. Then there exists a prime $p$ and a nontrivial character $\chi : H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_p$ such that

\[ |\sigma(K, \chi) + \sigma(K)| \leq 4g. \]

**Proof.** Let $A_1$, $A_2$, and $B$ be as in in the conclusion of Theorem 2.2. By our assumption on $H_1(\Sigma_2(K))$, we have that $A_1$ does not equal all of $H_1(\Sigma_2(K))$. Therefore $|A_1| < |H_1(\Sigma_2(K))|$ and so

\[ |A_1 \oplus B| = |A_1| \cdot |B| = |A_1| \cdot |A_2| = |A_1| \cdot \sqrt{|H_1(\Sigma_2(K))| \cdot |A_2|} = |A_1| \cdot \sqrt{|H_1(\Sigma_2(K))|} \]

is strictly less than the order of $H_1(\Sigma_2(K))$. That is, $A_1 \oplus B$ is a proper subgroup of $H_1(\Sigma_2(K))$. It follows that for any prime $p$ dividing the index of $A_1 \oplus B$ in $H_1(\Sigma_2(K))$ there exists a nontrivial character $\chi : H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_p$ that vanishes on $A_1 \oplus B$, and hence that by Theorem 2.2 satisfies

\[ |\sigma(K, \chi) + \sigma(K)| \leq 4g \]

as desired. \[ \square \]

Litherland proved a much more general formula for the Casson–Gordon invariants of satellite knots, but we will need only the following special case.

**Theorem 2.4** [15, Special case of Theorem 2]. Suppose $P$ is a pattern of winding number 0 described by an unknot $\eta$ in the complement of $P(U)$. Let $x$ denote the homology class of one of the lifts of $\eta$ to $\Sigma_2(P(U))$. For any knot $J$, there is an isomorphism $\alpha : H_1(\Sigma_2(P(J))) \rightarrow H_1(\Sigma_2(P(U)))$ such that for any $\chi : H_1(\Sigma_2(P(U))) \rightarrow \mathbb{Z}_q$ we have

\[ \sigma(P(J), \chi \circ \alpha) = \sigma(P(U), \chi) + 2\sigma_J(\omega_q^{\chi(x)}), \]

where $\omega_q = e^{2\pi i/q}$ and $\sigma_J$ denotes the Tristram–Levine signature function.
As well as the knot invariant $\sigma(K, \chi)$, Casson–Gordon introduced a signature invariant $\sigma(M, \phi)$ associated to a 3-manifold $M$ and a character $\phi : H_1(M) \to \mathbb{Z}_q$. These are much more computable than the knot Casson–Gordon signatures and satisfy the following key property.

**Proposition 2.5** [2, Lemma 3 and Theorem 4]. Let $J$ be a knot such that $H_1(\Sigma_2(J))$ is cyclic. Then for any prime power order character $\chi$ on $H_1(\Sigma_2(J))$, we have that

$$|\sigma(J, \chi) - \sigma(\Sigma_2(J), \chi)| \leq 1.$$ 

We will need a formula due to Cimasoni-Florens for the Casson–Gordon signature of a 3-manifold in terms of the colored signature function of a surgery link. Although this result is proved in much more generality, we state it only for the case of interest: when $M$ is obtained by surgery on a Hopf link. We thereby avoid going into the technical details of the definition of the colored signature function, noting only for the experts that the cell complex consisting of two discs meeting in a single arc and bounded by the Hopf link is a C-complex in the sense of [3], and the contractibility of this complex immediately implies that the colored signature function of the Hopf link is identically zero.

**Theorem 2.6** [3, Theorem 6.7]. Suppose that a 3-manifold $M$ is obtained by surgery on a Hopf link $L$ with linking matrix $\Lambda = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$. Let $q$ be prime and $\chi : H_1(M) \to \mathbb{Z}_q$ be a character such that the two meridians $\mu_1, \mu_2$ of $L$ are sent to nonzero elements of $\mathbb{Z}_q$. For $i = 1, 2$ let $n_i \in \{1, \ldots, q - 1\}$ be the unique value satisfying $n_i \equiv \chi(\mu_i) \mod q$. Then

$$\sigma(M, \chi) = -1 - \text{sign}(\Lambda) + \frac{2}{q^2} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}^T \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \cdot \begin{bmatrix} q - n_1 \\ q - n_2 \end{bmatrix}.$$ 

### Proof of Theorem 1.1

In the proof of Theorem 1.1, we will need a formula for the Casson–Gordon signatures of $K_J$ in terms of the Tristram–Levine signatures of $J$.

**Example 2.** Let $K_0$ denote the figure-eight knot. Note that $K(J)$ is obtained from $K_0$ by two infections along curves $\eta_1$ and $\eta_2$, as depicted in Figure 3.
By twice applying Theorem 2.4, we see that for any knot $J$ there is an isomorphism $\alpha : H_1(\Sigma_2(K(J))) \to H_1(\Sigma_2(K_0))$ such that for any character $\chi : H_1(\Sigma_2(K_0)) \to \mathbb{Z}_q$ we have

$$\sigma(K(J), \alpha \circ \chi) = \sigma(K_0, \chi) + 2\sigma_J(\omega_q^{\chi(\tilde{\eta}_1)}) + 2\sigma_J(\omega_q^{\chi(\tilde{\eta}_2)}) = \sigma(K_0, \chi) + 2\sigma_J(\omega_q^{\chi(\tilde{\eta}_1)}) - 2\sigma_J(\omega_q^{\chi(\tilde{\eta}_2)}).$$

Since both $\eta_i$ curves are disjoint from the usual genus 1 Seifert surface for $K_0$, we can apply Akbulut–Kirby’s algorithm of [1] to obtain the following surgery diagram for $\Sigma_2(K_0)$, with lifts of $\eta_1$ and $\eta_2$ as indicated. (Note that we have only depicted one lift of each curve, since that is all we need to apply Theorem 2.6.) The first homology of $\Sigma_2(K_0)$ is generated by the meridians of the components of $L$, which are isotopic to $\tilde{\eta}_1$ and $\tilde{\eta}_2$. The relations are given by the rows of the linking-framing matrix, and are

$$-2[\tilde{\eta}_2] + [\tilde{\eta}_1] = 0 \text{ and } [\tilde{\eta}_2] + 2[\tilde{\eta}_1] = 0.$$

Some quick simplifications give us that $H_1(\Sigma_2(K_0)) \cong \mathbb{Z}_5$, generated by $a := [\tilde{\eta}_2]$ and such that $[\tilde{\eta}_1] = 2[\tilde{\eta}_2]$. Therefore, for any character $\chi : H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$ we have that

$$\sigma(K_J, \chi \circ \alpha) = \sigma(K_0, \chi) + \sigma_J(\omega_5^{2\chi(a)}) - \sigma_J(\omega_5^{\chi(a)}).$$

We can also use the surgery diagram of Figure 4 to bound $|\sigma(K_0, \chi)|$. For $j \in \mathbb{Z}_5$, define $\chi_j : H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$ to be the map with $\chi_j(a) = j$. Observe that $\chi_1([\tilde{\eta}_1]) = 2$ and $\chi_2([\tilde{\eta}_1]) = 4$. Therefore, Theorem 2.6 gives us that

$$\sigma(\Sigma_2(K_0), \chi_1) = -1 - 0 + \frac{2}{25} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = -1 + \frac{30}{25} = 1/5$$

and

$$\sigma(\Sigma_2(K_0), \chi_2) = -1 - 0 + \frac{2}{25} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = -1 + \frac{20}{25} = -1/5.$$

Moreover, basic properties of Casson–Gordon signatures (or reapplying Theorem 2.6) imply that

$$\sigma(\Sigma_2(K_0), \chi_3) = \sigma(\Sigma_2(K_0), \chi_2), \sigma(\Sigma_2(K_0), \chi_4) = \sigma(\Sigma_2(K_0), \chi_1), \text{ and } \sigma(\Sigma_2(K_0), \chi_0) = 0.$$ 

Since $H_1(\Sigma_2(K_0)) \cong \mathbb{Z}_5$ is cyclic, Proposition 2.5 implies that for any $\chi : H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$ we have $|\sigma(K_0, \chi) - \sigma(\Sigma_2(K_0), \chi)| \leq 1$ and hence, by the above computation, that $|\sigma(K_0, \chi)| < 2$.

We are now ready to prove the following and obtain Theorem 1.1 as a consequence.
Proposition 2.7. Fix $g \in \mathbb{N}$. For $i \in \mathbb{N}$ define $J_i = \#^{m_i} T_{2,5}$, where $m_i = 2^{2i+1} g$. Now, for $k \in \mathbb{N}$ define $K^k := \#_{l=1}^{2g+2} K(J_{k(2g+2)+l})$. Then $S = \{K^k\}_{k \in \mathbb{N}}$ is a collection of algebraically slice knots such that any nontrivial sum $K = \#_{j=1}^n K^k_j$ satisfies the conclusions of Theorem 1.1.

Proof. Observe that for any choice of $J$, the knot $K(J)$ shares a Seifert form with $K_0$. Therefore, each $K^k$ shares a Seifert form with the slice knot $\#_{i=1}^{2g+2} K_0$, and hence is algebraically slice. Also, since Seifert forms determine the homology of cyclic branched covers, we record for later that for any $k$ we have

$$H_1(\Sigma_2(K^k)) \cong H_1(\Sigma_2(\#_{i=1}^{2g+2} K_0)) \cong \mathbb{Z}_{5}^{2g+2}. \quad (2.2)$$

Now let $K = \#_{j=1}^n K^k_j$ be a nontrivial sum of elements of $S$. Since each $K^k_j$ is 2-torsion, we can and do assume that $k_1 < k_2 < \cdots < k_n$. Since strong negative amphichirality, rational sliceness, and being related to a slice knot via $(+)$ to $(-)$ (or $(-)$ to $(+)$) crossing changes are all preserved under the connected sum operation, it only remains to verify item (iv).

Since $n \geq 1$,

$$H_1(\Sigma_2(K)) \cong \bigoplus_{j=1}^n H_1(\Sigma_2(K^k_j)) \cong \mathbb{Z}_{5}^{n(2g+2)} \quad (2.3)$$

is not generated by any $2g$ of its elements. So Corollary 2.3 applies and it is enough to show that for every nontrivial character $\chi : H_1(\Sigma_2(K)) \to \mathbb{Z}_5$, we have $|\sigma(K, \chi)| > 4g$.

Let $\chi$ be a nontrivial character, which by the isomorphism of Equation (2.2) we can write as $\chi = (\chi^j_i)_{i=1}^{2g+2} j=1$. By the additivity of Casson–Gordon signatures with respect to connected sum, we have that

$$\sigma(K, (\chi^j_i)_{i=1}^{2g+2} j=1) = \sum_{j=1}^n \sigma(K^k_j, (\chi^j_i)_{i=1}^{2g+2}) = \sum_{j=1}^n \sum_{i=1}^{2g+2} \sigma(K(J_{k_j(2g+2)+i}), \chi^j_i). \quad (2.3)$$

Moreover, for each $1 \leq j \leq n$ and $1 \leq i \leq 2g + 2$, Theorem 2.4 and Equation (2.1) of Example 2 tell us that there is an identification

$$\alpha^j_i : H_1(\Sigma_2(K(J_{k_j(2g+2)+i}))) \to H_1(\Sigma_2(K_0))$$

such that for any $\phi : H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$, we have

$$\sigma(K(J_{k_j(2g+2)+i}), \phi \circ \alpha^j_i) = \sigma(K_0, \phi) + 2\sigma_{J_{k_j(2g+2)+i}}(\omega_5^{2\phi(a)}) - 2\sigma_{J_{k_j(2g+2)+i}}(\omega_5^{\phi(a)}). \quad (2.4)$$

Now, for each $1 \leq j \leq n$ and $1 \leq i \leq 2g + 2$, define

$$\beta^j_i := \chi^j_i \circ (\alpha^j_i)^{-1} : H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5.$$

Equations (2.3) and (2.4) combine to tell us that

$$\sigma(K, \chi) = \sum_{j=1}^n \left( \sum_{i=1}^{2g+2} \sigma(K_0, \beta^j_i) + 2\sigma_{J_{k_j(2g+2)+i}}(\omega_5^{2\beta^j_i(a)}) - 2\sigma_{J_{k_j(2g+2)+i}}(\omega_5^{\beta^j_i(a)}) \right). \quad (2.5)$$
Since $\chi$ is nontrivial, there exists some $j$ such that $(\chi^j_i)_{i=1}^{2g+1}$ and hence $(\beta^j_i)_{i=1}^{2g+1}$ is not identically zero. Let $j_0$ be the maximal such $j$ and let $i_0$ be the maximal $i$ such that $\beta^j_{i_0}$ is nonzero. Let $\ell = k_{j_0}(2g + 2) + i_0$. The following algebraic manipulations show that $\sigma(K(J_\ell), \chi^j_{i_0})$ so dominates the other terms that could contribute to $\sigma(K, \chi)$ that we have as desired that $|\sigma(K, \chi)| > 4g$.

Recalling that each $J_i = \#^m T_{2,5}$, where $m_i = 2^{2i+1}g$, we have by the additivity of Tristram–Levine signatures under connected sum that $\sigma(J_\ell) = \sigma(J_{2^i+1}g) = -2^{2i+2}g$ and $\sigma(J_{2^i+2}g) = \sigma(J_{2^i+3}g) = -2^{2i+3}g$ (see KnotInfo [17] for the Tristram–Levine signature function of $T(2, 5)$.) Applying Equation (2.1) from Example 2, we see that for any $i$ and any nonzero character $\rho : H_1(\Sigma_2(K(J_i))) \to \mathbb{Z}_5$ we have that

$$2^{2i+3}g - 2 \leq |\sigma(K(J_i), \rho)| = |\sigma(K_0, \rho) \pm (2\sigma(J_i(\omega_5) - 2\sigma(J_i(\omega_3)))| \leq 2^{2i+3}g + 2,$$

(2.6)

where we briefly suppress the identification of $H_1(\Sigma_2(K(J_i)))$ with $H_1(\Sigma_2(K_0))$.

Observe that the set of natural numbers

$$\{k_{j_0}(2g + 2) + i : 1 \leq i \leq i_0 - 1\} \cup \bigcup_{j=1}^{j_0-1}\{k_j(2g + 2) + i) : 1 \leq i \leq 2g + 2\}$$

(2.7)

is a subset of $\{1, ..., \ell - 1\}$, where $\ell = k_{j_0}(2g + 2) + i_0$.

Recalling that $\sigma(K(J_{k_j(2g+2+i)}), \chi^j_i) = 0$ whenever $j > j_0$ or $j = j_0$ and $i > i_0$, we therefore have that

$$|\sigma(K, \chi)| = \left| \sum_{j=1}^{n} \sum_{i=1}^{2g+2} \sigma(K(J_{k_j(2g+2+i)}), \chi^j_i) \right|$$

$$= \left| \sigma(K(J_{\ell}), \chi^j_{i_0}) + \sum_{i=1}^{i_0-1} \sigma(K(J_{k_{j_0}(2g+2+i)}), \chi^j_{i_0}) + \sum_{j=1}^{j_0-1} \sum_{i=1}^{2g+2} \sigma(K(J_{k_{j}(2g+2+i)}), \chi^j_{i}) \right|$$

$$\geq \left| \sigma(K(J_{\ell}), \chi^j_{i_0}) \right| - \sum_{i=1}^{i_0-1} \left| \sigma(K(J_{k_{j_0}(2g+2+i)}), \chi^j_{i_0}) \right| - \sum_{j=1}^{j_0-1} \sum_{i=1}^{2g+2} \left| \sigma(K(J_{k_{j}(2g+2+i)}), \chi^j_{i}) \right|$$

$$\geq (2^{2\ell+3}g - 2) - \sum_{k=1}^{\ell-1} (2^{2k+3}g + 2) = : (*)$$

where in the last inequality we use our observation from Equation (2.7) together with Equation (2.6). Some algebraic simplification yields that

$$(*) = 8g \left( 2^{2\ell} - \sum_{k=1}^{\ell-1} 2^{2k} \right) - 2\ell = (g/3)(2^{2\ell+3} - 32) - 2\ell.$$

Now, note that since $\ell > 2g + 2 \geq 4$ we have that $2\ell + 3 > 11$ and so certainly $2^{2\ell+3} - 32 > 2^{2\ell+2}$. Therefore

$$|\sigma(K, \chi)| \geq (*) > (g/3)2^{2\ell+2} - 2\ell > 2^{2\ell} - 2\ell.$$
Finally, we observe that for any \( x > 2 \) we have \( 2^{2x} - 2x > 2x \), since letting \( f(x) = 2^{2x} - 4x \) we see that \( f'(x) = \ln(4)2^{2x} - 4 \) is positive for all \( x \geq 1 \) and \( f(2) = 8 \). Therefore

\[
|\sigma(K, \chi)| > 2\ell > 4g + 4 > 4g,
\]
as desired. \( \square \)

**Remark 2.** The examples of Proposition 2.7 are far from the only knots satisfying the conclusions of Theorem 1.1. One could vary the base knot, for example by choosing \( \{a_i\}_{i \geq 0} \) to be natural numbers such that \( \{4a_i^2 + 1\}_{i \in \mathbb{N}} \) consists of pairwise relatively prime numbers. (This is easily accomplished by, for example, letting \( a_0 = 1 \) and \( a_k = \prod_{i=1}^{k-1} (4a_i^2 + 1) \) for \( k \geq 1 \).) Now, let \( K_i \) be the 2-bridge knot corresponding to the rational number \( \frac{4a_i^2 + 1}{2a_i} \), noting that indeed \( K_0 \) is the figure-eight knot. Choose \( \{p_j\}_{j \geq 0} \) to be primes dividing \( 4a_j^2 + 1 \), noting that by our choice of \( a_i \) we have that \( p_j \) divides \( 4a_j^2 + 1 \) if and only if \( j = i \). By taking connected sums of \( K_{a_i} \) analogously infected with large connected sums of \( T_{2,p_i} \) and \( -T_{2,p_i} \), we can essentially repeat the arguments of Proposition 2.7 and obtain many more linearly independent knots satisfying the conclusions of Theorem 1.1.

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