1. Let $G$ be a semisimple algebraic group defined over an algebraically closed field $k$ of characteristic $p$. In this article, we construct the moduli space of semistable principal $G$ bundles on a smooth projective curve $X$ over $k$, of genus $g \geq 2$. When the characteristic is zero, for example the field of complex numbers, these moduli spaces were first constructed by A. Ramanathan (see [8], [9]). Later on, different methods were used by Faltings (see [4]) and by Balaji and Seshadri (see [3]). In these later papers, the emphasis was on proving the properness of the functor of semistable principal $G$-bundles. The method of Balaji and Seshadri was extended to positive characteristic by Balaji and Parameswaran (see [2]).

In this article, we follow A. Ramanathan’s original approach. The new input in the present work is the use of low height and super low height representations (see [6]). These properties are used to ensure that the adjoint bundle of a semistable (respectively polystable) $G$-bundle is a semistable (respectively polystable) vector bundle. However, the proof that the moduli space is projective involves an argument of lifting to characteristic zero. This is because one does not know if the kernel of the killing form of a Lie algebra over $k$ is a solvable ideal. We try to adhere to the notation of Ramanathan
(see [8], [9]) as far as possible.

2. We recall the notion of a universal family and a universal space.

**Definition (2.1) (universal family)** : Let $E \rightarrow X \times T$ be a family of semistable $G$-bundles on $X$ parametrised by a scheme $T$. Suppose an algebraic group $H$ acts on $T$ by $\alpha : H \times T \rightarrow T$ and also on $E$ as a group of $G$ bundle isomorphisms compatible with $\alpha$. We call $E \rightarrow X \times T$ a universal family of semistable $G$-bundles with group $H$ if the following conditions hold:

(i) Given any family of semistable $G$ bundles $F \rightarrow X \times S$ and a point $s \in S$ there is an open neighborhood $U$ of $s$ in $S$ and a morphism $t : U \rightarrow T$ such that $F \mid X \times U$ is isomorphic to $t^*E \rightarrow X \times U$ where $t^*E$ is the pullback of $E$ by the morphism $t : U \rightarrow T$.

(ii) Given two morphisms $t_1, t_2 : S \rightarrow T$ and an isomorphism $\varphi : t_1^*E \rightarrow t_2^*E$ of $G$ bundles on $X \times S$, there is a unique morphism $h : S \rightarrow H$ such that $t_2 = ht_1$ and $\varphi = (h \times t_1)^*(\alpha)$.

**Definition (2.2)**: Let $\text{Sch}$ denote the category of schemes over $k$, and let $\text{Sets}$ denote the category of sets. Let $\widetilde{F}$ be the sheaf associated to the functor $F : \text{Sch} \rightarrow \text{Sets}$ which associates to a scheme $T$ the set of isomorphism classes of semistable $G$ bundles parametrised by $T$. On morphisms $F$ is defined by pulling back. Let $M$ be a scheme and $H$ an algebraic group acting on $M$ by $\alpha : H \times M \rightarrow M$. Let $M/H$ be the sheaf associated to the presheaf which assigns to a $k$-scheme $T$ the quotient set $\text{Hom}(T, M)/\text{Hom}(T, H)$. We say $M$ is a universal space with group $H$ if there is an isomorphism of sheaves $\widetilde{F} \rightarrow M/H$. With these definitions, we have the following proportion.

**Proposition (2.3)** : Suppose there is a universal space $M$ with group $H$ for families of semistable $G$ bundles. If a good quotient of $M$ by $H$ exists,
then it gives a coarse moduli scheme of semistable $G$- bundles.

**Proof**: The proof is the same as that of Proposition (4.5) in [9], using Lemma (4.3) and (4.4) of our article instead of the Proposition (3.24) of [8].

QED

In view of the above proposition, it is enough to construct a universal space for families of semistable $G$ bundles on $X$, and show that this universal space has a good quotient. For this purpose, let $Z$ be the centre of $G$, and let $\mathfrak{g}$ be the lie algebra of $G$. We consider the sequence of group homomorphisms

$$G \to G/Z \to \text{Aut}(\mathfrak{g}) \to GL(\mathfrak{g})$$

where $\text{Aut}(\mathfrak{g})$ denotes the group of Lie algebra automorphisms of $\mathfrak{g}$ and $GL(\mathfrak{g})$ is the group of linear automorphisms of the vector space $\mathfrak{g}$. When the Killing form of $\mathfrak{g}$ is non degenerate, $G/Z$ is the connected component of $\text{Aut}(\mathfrak{g})$ (this follows from the fact that if the killing form of $\mathfrak{g}$ is nondegerate, then every derivation of $\mathfrak{g}$ is inner, see [10], page 18). A universal space $R_3$ for semistable $G$ bundles is constructed in four steps. We first need

**Definition (2.4)**: Let $\alpha = \sum n_i\alpha_i$ be the highest root of the Lie algebra $\mathfrak{g}$, where $\alpha_i$ are simple roots and $n_i$ are positive integers. Then the height $h(G)$ of $\mathfrak{g}$ is defined as

$$h(G) = 2 \sum n_i$$

We have the following result from [6]:

**Theorem (2.5)**: Let $E \to X$ be a semistable $G$ bundle, and let $E(\mathfrak{g})$ be the adjoint bundle of $E$. If the characteristic $p$ is bigger than $h(G)$, then
$E(\mathfrak{g})$ is a semistable vector bundle. (see Theorem (2.2) in [6]).

Thus if $p > h(G)$, then every semistable $G$ bundle induces a semistable $GL(\mathfrak{g})$ bundle by the adjoint representation of $G$. We start with a universal family $R$ with group $GL(n)$ for semistable $GL(\mathfrak{g})$ bundles (which are simply semistable vector bundles of rank $\dim \mathfrak{g}$). Since the homomorphism $\text{Aut} (\mathfrak{g}) \to GL(\mathfrak{g})$ is injective, we obtain a universal family $R_1$ with group $GL(n)$ for semistable $\text{Aut} (\mathfrak{g})$ bundles. This follows from the following lemma:

**Lemma (2.6):** Let $A$ be a reductive group and $B$ a semisimple group such that $B$ is a closed subgroup of $A$. Then a universal family with group $H$ for families of semistable $A$ bundles gives a universal family with group $H$ for families of semistable $B$ bundles, provided every semistable $B$ bundle on $X$ induces a semistable $A$ bundle by the extension of structure group $B \to A$.

**Proof.** See Lemma (4.8.1) and Lemma (4.10) in [9]. QED

**Lemma (2.7):** The scheme $R_1$ is nonsingular and $\dim R_1 = n^2 + (r + 1)(g - 1) - g$ where $r = \dim G$, and $g$ is the genus of $X$.

**Proof:** See Lemma (4.13.4) in [9]. QED

The same procedure of applying Lemma (2.6) above to the injection $G/Z \subset \text{Aut} (\mathfrak{g})$, where $Z$ is the center of $G$ yields a universal family $R_2$ with group $GL(n)$ for families of semistable $G/Z$ bundles on $X$. We have

**Lemma (2.8):** The natural morphism $\pi_2 : R_2 \to R_1$ is etale and finite, and hence $R_2$ is nonsingular.
Proof: See Proposition (4.14) in [9]. QED

We now construct a universal space $R_3$ with group $GL(n)$ from $R_2$, following an idea of Sols-Gomez (see [5]). Let $E_2 \to X \times R_2$ be the universal $G/Z$ bundle on $X \times R_2$ and let $\rho : G \to G/Z$ be the natural projection.

We assume hereafter that the center $Z$ of $G$ is reduced, which is the case if the characteristic $p$ of the field $k$ is bigger than $h(G)$. Let $\Gamma(\rho, E_2)$ be the functor defined on the category $\text{Sch}/R_2$ of schemes over $R_2$, which assigns to a scheme $\varphi : S \to R_2$ the set of isomorphism classes of pairs $(E, \psi)$ where $E$ is a principal $G$ bundle on $X \times S$ and $\psi$ is an isomorphism of the $G/Z$ bundle $E/Z$ with the $G/Z$ bundle $\varphi^*E_2$. Let $\sim \Gamma (\rho, E_2)$ be the sheafification of the functor $\Gamma(\rho, E_2)$. We have a natural morphism $\pi_3 : \sim \Gamma (\rho, E_2) \to R_2$.

We need a definition.

Definition (2.9): Let $\Gamma_1$ and $\Gamma_2$ be two contravariant functors from $\text{Sch}$ to $\text{Sets}$, and let $H$ be a finite reduced group, and let $\pi : \Gamma_1 \to \Gamma_2$ be a morphism of functors. Let $\sim \pi : \sim \Gamma_1 \to \sim \Gamma_2$ be the morphism sheaves associated to $\pi$. Then we say that $\sim \Gamma_1$ is a principal $H$-bundle over $\sim \Gamma_2$ if

(i) $H$ acts on $\Gamma_1$ and hence on $\sim \Gamma_1$

(ii) the natural map

$$\sim \Gamma_1 \times H \to \sim \Gamma_1 \times_{\sim \Gamma_2} \sim \Gamma_1$$

is an isomorphism of sheaves. We now have

Proposition (2.10): The natural map $\pi_3 : \sim \Gamma (\rho, E_2) \to R_2$ is a principal $H^1(X, Z)$ bundle on $R_2$, where $H^1(X, Z)$ denotes the etale cohomology of $X$ with coefficients in $Z$. 

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Proof: One sees easily that there is a natural action of $H^1(X, Z)$ on $\Gamma(\rho, E_2)$. We have to show that the action on $\tilde{\Gamma}(\rho, E_2)$ is free and the quotient is $R_2$. We do this by identifying the fibre of $\pi_3 : \tilde{\Gamma}(\rho, E_2) \to R_2$ with $H^1(X, Z)$. Let $S$ be a closed point of $R_2$ and $p : F \to X$ be corresponding $G/Z$ bundle on $X$. $\tilde{\Gamma}(\rho, E_2)(S)$ now is the set of all pairs $(E, \alpha)$ such that $E \to X$ is a principal $G$ bundle on $X$ and $\alpha$ is an isomorphism $\alpha : E/Z \to F$. We now check that $H^1(X, Z)$ acts freely and transitively on $\tilde{\Gamma}(\rho, E_2)(S)$. The transitivity of the action is easy to see. To show that the action is free, we take a pair $(E, \alpha)$ in $\tilde{\Gamma}(\rho, E_2)(S)$ and an element $t \in H^1(X, Z)$, and show that $t(E, \alpha) \neq (E, \alpha)$. If the $G$-bundle $tE$ is not isomorphic to $E$ we are through. If $tE$ is isomorphic to $E$ as a $G$-bundle, let $\varphi : E \to tE$ be an isomorphism. Let $B$ denote a Borel subgroup of $G$ and $T$ denote a maximal torus of $G$ contained in $B$. Since $\text{Ant}^0(G/B) \cong G/Z$ (this follows from the nondegeneracy of the Killing form of $g$), we regard the $G/B$ fibration $E/B \to X$ as equivalent to a $G/Z$ bundle. Since $E \to E/B$ is a $B$-bundle the canonical homomorphism $B \to T$ induces a $T$ bundle $E_T \to E/B$ on $E/B$. Now $\alpha$ gives an isomorphism $E/B \to F/B'$ of $G/B$ fibrations, where $B'$ is $B/Z$, $t\alpha$ gives an isomorphism $tE/B \to F/B'$ and $\varphi$ induces an isomorphism $\varphi : E/B \to tE/B$. We have to show that the diagram

\[
\begin{array}{c}
E/B \xrightarrow{\alpha} F/B \\
\Downarrow \varphi \\
tE/B \xrightarrow{t\alpha}
\end{array}
\]

is not commutative. We note that there are two isomorphisms $E/B \to tE/B$, namely $\varphi$ and $(t\alpha)^{-1}o\alpha$. To show that the above diagram is not commutative, it is enough to check that the $T$- bundle $E_T$ on $E/B$ pulls back, under $\varphi$ and $(t\alpha)^{-1}o\alpha$, to non isomorphic $T$ bundles on $tE/B$. But this is clear as
one of them is a translate of the other by $t$ and the action of $H^1(X, Z)$ on $H^1(E/B, T)$ is fixed point free. QED

**Lemma (2.11):** $\tilde{\Gamma} (\rho, E_2)$ is representable by a scheme $R_3$.

**Proof:** Since the mapping of sheaves $\pi_3 : \tilde{\Gamma} (\rho, E_2) \to R_2$ is a principal $H^1(X, Z)$ bundle, it is easily seen that $\tilde{\Gamma} (\rho, E_2)$ is an algebraic space in the sense of Artin (see [1]). If further follows from loc. cit. (see Theorem (3.3) in [1]), that this algebraic space is in fact a scheme. QED

**Remark (2.12):** We observe that by construction $\pi_3 : R_3 \to R_2$ is finite etale with structure group $H^1(X, Z)$.

3. In order to construct the moduli space of semistable $G$ bundles on $X$, we have to construct a good quotient of $R_3$ by $GL(n)$. In view of Lemma (2.8) and Remark (2.12) above, it is enough to show that $R_1$ has a good quotient by appealing to Proposition (3.12) in [7]. Let $G_{n,r}$ be the Grassmannian of $r$ dimensional quotients of $n$- dimensional affine space over $k$, where $r = \dim G = \dim g$. Let $Y = GL(g)/G_m \times \text{Aut} (g)$ and let $Q$ be the universal quotient bundle on $G_{n,r}$. Let $Q(Y)$ be the fibre bundle on $G_{n,r}$ with fibre $Y$ associated to $Q$. Let $\overline{Y}$ be the closure of $Y$ in the projective space $\mathbb{P}(g^* \otimes g^* \otimes g)$. By the usual diagonal argument, we can choose an $N$-tuple $(x, \ldots, x_N)$ of points in $X$ for $N$ sufficiently large such that the evaluation morphism

$$R_1 \to Q(\overline{Y})^N = Q(\overline{Y}) \times \cdots \times Q(\overline{Y})$$

is injective (see section (5.5) in [9]). We can choose a polarisation (see (5.5) in [9]) so that if $Q(\overline{Y})^N_{ss}$ denotes the set of semistable points in the sense of
geometric invariant theory for this polarisation then $R_1$ maps into $Q(\overline{\nabla})_{ss}^N$ (see Lemma (5.5.3) in [9]).

**Lemma (3.1):** The map $R_1 \to Q(\overline{\nabla})_{ss}^N$ is proper.

**Proof:** We check the valuative criterion for properness. Let $k[[t]]$ denote the ring of formal power series in $t$ over $k$, and let $k((t))$ be its quotient field. We begin with a semistable vector bundle $V$ of degree zero on $X \times_k \text{Spec } k((t))$, which is also a bundle of Lie algebras, with Lie algebra structure isomorphic to that of $\mathfrak{g}$. Let $C \in \mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g})$ be the structure tensor of the Lie algebra $\mathfrak{g}$. Then the Lie algebra structure on $V$ has structure tensor $M((t))C$ where $M((t))$ is an element of $GL(r, k((t)))$. If $\nabla$ is any semistable extension of $V$ to $X \times_k \text{Spec } k[[t]]$, then the special fibre $V_0$ of $\nabla$ is also a bundle of Lie algebras with structure tensor $\lim_{t \to 0} M((t))C$ (the limit is taken in $\mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g})$). We now show that this limit Lie algebra structure is isomorphic to that defined by $C$ (namely, $\mathfrak{g}$ itself). Let $W$ be the ring of Witt vectors of $k$, let $K$ be the quotient field of $W$. The Lie algebra $\mathfrak{g}$ has a form $\mathfrak{g}_W$ over $W$, whose structure tensor we denote by $C_W \in \mathbb{P}(\mathfrak{g}_W^* \otimes \mathfrak{g}_W^* \otimes \mathfrak{g}_W)$. We now lift the matrix $M((t))$ to a matrix $M_W((t))$ which is an element of $GL(r, W((t)))$. We therefore obtain a Lie algebra $\mathfrak{g}_{W,t}$ with structure tensor $M_W((t))C_W$ over $W((t))$. The group scheme $\text{Aut } (\mathfrak{g}_{W,t})$ of Lie algebra automorphisms of $\mathfrak{g}_{W,t}$ over $W((t))$ is smooth over $W((t))$ (see the proof of the Theorem in (11.5) in [2]). Let $\mathfrak{g}_t$ be the Lie algebra $\mathfrak{g}_{W,t}$ mod $p$ over $k((t))$. Then by Lemma 1, p. 175 of [11], the principal $\text{Aut } (\mathfrak{g}_t)$ bundle on $X \times \text{Spec } k((t))$ can be lifted to a principal $\text{Aut } (\mathfrak{g}_{W,t})$ bundle $E_W$ on $X_W \times_W W((t))$ where $X_W$ is a form of the curve $X$ over the Witt ring $W$. Since $V$ is semistable on $X \times \text{Spec } k((t))$ the generic fibre $E_K$ of $E_W$ is semistable on $X_K \times K((t))$. Now by the key lemma (5.6) in [9], $V_K$ has a semistable extension $\overline{\nabla}_K$ to
$X_K \times K[[t]]$, where $V_K$ is the vector bundle associated to $E_K$, such that the Lie algebra structure on the special fibre $\nabla_{K,0}$ at $t = 0$ is isomorphic to the Lie algebra structure on $V_K$. Hence we obtain that $\frac{\text{d}}{\text{d}t} M_W((t)) C_W$ gives a Lie algebra isomorphic to that given by $C_W$. It follows by reducing mod $p$ that the Lie algebra structure $\frac{\text{d}}{\text{d}t} M((t)) C$ is isomorphic to the Lie algebra structure $C$. Hence $V_0$ is also a principal Aut $(g)$ bundle (semistable by construction). This shows that the map $R_1 \to Q(Y)^N_{ss}$ is proper in positive characteristic.

After the above lemma, we see that a good quotient of $R_1$ exists and hence as remarked before, a good quotient of $R_3$ exists. This completes the construction of the moduli space of semistable $G$ bundles. Further, since the quotient of $R_1$ is projective (by the Lemma above ) and the map $R_3 \to R_1$ is finite and $GL(n)$ equivariant it follows that the quotient of $R_3$, namely the moduli space, is also projective. It only remains to identify the closed points of the moduli space.

4. A description of the closed points of the moduli space is given by

**Theorem (4.1):** The closed points of the moduli space of semistable $G$ bundles are isomorphism classes of polystable $G$ bundles.

**Proof:** The proof follows from the following two lemmas. QED

Before the lemmas, we recall the following definition from [8].

**Definition (4.2):** Let $E \to X$ be a semistable $G$ bundle on $X$. If $E$ is stable, we define gr $E = E$. If $E$ is not stable, there is an admissible reduction of structure group $E_P \subset E$ to a parabolic $P$, such that the bundle $E_M$ induced from $E_P$ by the projection $P \to M$ $(M$ being a Levi of $P)$ is
stable, and \( \text{gr} \ E \) is the \( G \)-bundle obtained from \( E_M \) by the inclusion \( M \subset G \) (see (3.12) in [8]).

We now have

Lemma (4.3): Let \( E \to X \) be a semistable \( G \) bundle on \( X \) regarded as a point of \( R_3 \). Let \( O(E) \) denote the orbit of \( E \) under the group \( GL(n) \) acting on \( R_3 \), and let \( \overline{O(E)} \) denote the closure of the orbit in \( R_3 \). Then \( \text{gr} \ E \in \overline{O(E)} \).

Proof: The proof is by constructing a family of semistable \( G \) bundles \( \mathcal{E} \to X \times A^1 \), where \( A^1 \) is the affine line over \( k \), such that \( \mathcal{E}_t \) is isomorphic to \( E \) if \( t \neq 0 \) in \( A^1 \), and \( \mathcal{E}_0 \) is isomorphic to \( \text{gr} \ E \), where \( \mathcal{E}_t = \mathcal{E} \mid X \times t \), \( t \in A^1 \). This is done as in the proof of Proposition (3.24) (ii) in [8]. QED

Lemma (4.4): Let \( E \to X \) be a semistable \( G \) bundle regarded as a point of \( R_3 \). Then the orbit \( O(\text{gr} \ E) \) of \( \text{gr} \ E \) is closed in \( R_3 \) if \( p > rh(G) \), where \( r = \dim G \), \( p = \) characteristic of \( k \), \( h(G) \) is the height as in Definition (2.4) above.

Proof: It is enough to prove (\( \ast \)) if \( \text{gr} \ F \in \overline{O(\text{gr} \ E)} \) then \( \text{gr} \ F \cong \text{gr} \ E \). We take the adjoint representation of \( G \) and consider the associated vector bundles \( (\text{gr} F)(\mathfrak{g}) \) and \( (\text{gr} E)(\mathfrak{g}) \). The hypothesis on \( p \) ensures that \( (\text{gr} F)(\mathfrak{g}) \) and \( (\text{gr} E)(\mathfrak{g}) \) are polystable vector bundles (see Theorem (4.2) in [6]). If follows from complement (5.8.1) in [7], that \( \text{gr} F(\mathfrak{g}) \) is isomorphic to \( (\text{gr} E)(\mathfrak{g}) \) as vector bundles. Since \( G/Z \subset GL(\mathfrak{g}) \) is a faithful representation by the adjoint action, it follows that \( \text{gr} \ (E/Z) \) is isomorphic to \( \text{gr} (F/Z) \) as \( G/Z \) bundles. Hence \( \text{gr} \ (F/Z) \) belongs to the \( GL(n) \) orbit of \( \text{gr} \ (E/Z) \) in \( R_2 \). However, since \( \pi_3 : R_3 \to R_2 \) is a finite \( GL(n) \) equivariant map, it follows that \( \text{gr} \ F \) belongs to the \( GL(n) \) orbit of \( \text{gr} \ E \), i.e., \( \text{gr} \ F \) is isomorphic to \( \text{gr} \)
$E$ as $G$ bundles. This completes the proof of ($\ast$) and hence of the lemma.

QED

5. To summarise, we need the semisimple group $G$ to satisfy the following conditions:

(i) the centre $Z$ is reduced.

(ii) the Killing form is non degenerate

(iii) $p > 2h(G) + 3$.

(iv) $p > \dim G.h(G)$

We remark that if $p > \dim G.h(G)$ then all the other conditions are satisfied. Hence a projective moduli space of semistable $G$ bundles exists if $p > \dim G.h(G)$, and the closed points of this moduli space are isomorphism closes of polystable bundles.
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