ON THE STRUCTURE OF ABELIAN PROFINITE GROUPS

M. FERRER AND S. HERNÁNDEZ

Abstract. A subgroup $G$ of a product $\prod_{i \in \mathbb{N}} G_i$ is rectangular if there are subgroups $H_i$ of $G_i$ such that $G = \prod_{i \in \mathbb{N}} H_i$. We say that $G$ is weakly rectangular if there are finite subsets $F_i \subseteq \mathbb{N}$ and subgroups $H_i$ of $\bigoplus_{j \in F_i} G_j$ that satisfy $G = \prod_{i \in \mathbb{N}} H_i$. In this paper we discuss when a closed subgroup of a product is weakly rectangular. Some possible applications to the theory of group codes are also highlighted.

1. Introduction

For a family $\{G_i : i \in \mathbb{N}\}$ of topological groups, let $\bigoplus_{i \in \mathbb{N}} G_i$ denote the subgroup of elements $(g_i)$ in the product $\prod_{i \in \mathbb{N}} G_i$ such that $g_i = e$ for all but finitely many indices $i \in \mathbb{N}$. A subgroup $G \leq \prod_{i \in \mathbb{N}} G_i$ is called weakly controllable if $G \cap \bigoplus_{i \in \mathbb{N}} G_i$ is dense in $G$, that is, if $G$ is generated by its elements of finite support. The group $G$ is called weakly observable if $G \cap \bigoplus_{i \in \mathbb{N}} G_i = \overline{G} \cap \bigoplus_{i \in \mathbb{N}} G_i$, where $\overline{G}$ stands for the closure of $G$ in $\prod_{i \in \mathbb{N}} G_i$ for the product topology. Although the notion of (weak) controllability was coined by Fagnani earlier in a broader context (cf. [3]), both notions were introduced in the area of of coding theory by Forney and Trott (cf. [7]). They observed that if the groups $G_i$ are locally compact abelian, then controllability and observability are dual properties with respect to the Pontryagin duality: If $G$ is a closed subgroup of $\prod_{i \in \mathbb{N}} G_i$, then it is

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weakly controllable if and only if its annihilator $G^\perp = \{ \chi \in \prod_{i \in \mathbb{N}} G_i : \chi(G) = \{0\} \}$ is a weakly observable subgroup of $\prod_{i \in \mathbb{N}} \hat{G}_i$ (cf. [7, 4.8]).

In connection with the properties described above, the following definitions was introduced in [11].

**Definition 1.1.** A subgroup $G$ of a product $\prod_{i \in \mathbb{N}} G_i$ is rectangular if there are subgroups $H_i$ of $G_i$ such that $G = \prod_{i \in \mathbb{N}} H_i$. We say that $G$ is weakly rectangular if there are finite subsets $F_i \subseteq \mathbb{N}$ and subgroups $H_i$ of $\bigoplus_{j \in F_i} G_j$ that satisfy $G = \prod_{i \in \mathbb{N}} H_i$. We say that $G$ is a subdirect product of the family $\{G_i\}_{i \in I}$ if $G$ is weakly rectangular and $G \cap \bigoplus_{i \in I} G_i = \bigoplus_{i \in I} H_i$.

The observations below are easily verified.

1. Weakly rectangular subgroups and rectangular subgroups of $\prod_{i \in \mathbb{N}} G_i$ are weakly controllable.
2. If each $G_i$ is a pro-$p_i$-group for some prime $p_i$, and all $p_i$ are distinct, then every closed subgroup of the product $\prod_{i \in \mathbb{N}} G_i$ is rectangular, and thus is a subdirect product.
3. If each $G_i$ is a finite simple non-abelian group and all $G_i$ are distinct, then every closed normal subgroup of the product $\prod_{i \in \mathbb{N}} G_i$ is rectangular, and thus a subdirect product.

The main goal addressed in this paper is to study to what extent the converse of these observations hold. In particular, we are interested in the following question (cf. [11]):

**Problem 1.2.** Let $\{G_i : i \in \mathbb{N}\}$ be a family of compact metrizable groups, and $G$ a closed subgroup of the product $\prod_{i \in \mathbb{N}} G_i$. If $G$ is weakly controllable, that is, $G \cap \bigoplus_{i \in \mathbb{N}} G_i$...
is dense in $G$, what can be said about the structure of $G$? In particular, under what additional conditions is the group $G$ a subdirect product of \{$G_i : i \in \mathbb{N}$\}, that is, weakly rectangular and $G \cap \bigoplus_{i \in I} G_i = \bigoplus_{i \in \mathbb{N}} H_i$, where each $H_i$ is a subgroup of $\bigoplus_{j \in F_i} G_j$ for some $F_i \subseteq \mathbb{N}$?

In connection with this question, the following result was established in [4].

**Theorem 1.3.** Let $I$ be a countable set, \{$G_i : i \in I$\} be a family of finite abelian groups and $G = \prod_{i \in I} G_i$ be its direct product. Then every closed weakly controllable subgroup $H$ of $G$ is topologically isomorphic to a direct product of finite cyclic groups.

Unfortunately, this result does not answer Problem 1.2, which remains open to the best of our knowledge. Finally, it is pertinent to mention here that the relevance of these notions stem from coding theory where they appear in connection with the study of convolutional group codes [7, 12]. However, similar notions had been studied in symbolic dynamics previously. Thus, the notions of weak controllability and weak observability are related to the concepts of irreducible shift and shift of finite type, respectively, that appear in symbolic dynamics. Here, we are concerned with abelian profinite groups and our main interest is to clarify the overall topological and algebraic structure of abelian profinite groups that satisfy any of the properties introduced above. In the last section, we shall also consider some possible applications to the study of group codes.

2. Basic facts

2.1. Pontryagin-van Kampen duality. One of the main tools in this research is Kaplan’s extension of Pontryagin van-Kampen duality to infinite products of locally
compact abelian (LCA) groups. In like manner that Pontryagin-van Kampen duality has proven to be essential in understanding the structure of LCA, the extension accomplished by Kaplan for cartesian products and direct sums \[9, 10\] (and some other subsequent results) have established duality methods as a powerful tool outside the class of LCA groups and they have been widely used in the study of group codes.

We recall the basic properties of topological abelian groups and the celebrated Pontryagin-van Kampen duality.

Let \( G \) be a commutative locally-compact group. A character \( \chi \) of \( G \) is a continuous homomorphism \( \chi : G \rightarrow \mathbb{T} \) where \( \mathbb{T} \) is the multiplicative group of complex numbers of modulus 1. The characters form a group \( \hat{G} \), called dual group, which is given the topology of uniform convergence on compact subsets of \( G \). It turns out that \( \hat{G} \) is locally compact and there is a canonical evaluation homomorphism

\[
\mathcal{E}_G : G \rightarrow \hat{G}.
\]

**Theorem 2.1.** The evaluation homomorphism \( \mathcal{E}_G \) is an isomorphism of topological groups.

Some examples:

\[
\hat{\mathbb{R}} \cong \mathbb{R}, \quad \hat{\mathbb{Z}} \cong \mathbb{T}, \quad \hat{\mathbb{Z}/n} \cong \mathbb{Z}/n.
\]

(some groups are self dual, such as finite abelian groups or the additive group of the real numbers)

Pontryagin-van Kampen duality establishes a duality between the subcategories of compact and discrete abelian groups. If \( G \) denotes a compact abelian group and \( \Gamma \) denotes its dual group, we have the following equivalences between topological properties of \( G \) and algebraic properties of \( \Gamma \):
(i) \( \text{weight}(G) = |\Gamma| \) (metrizability \( \iff |\Gamma| \leq \omega \));
(ii) \( G \) is connected \( \iff \Gamma \) is torsion free;
(iii) \( \text{Dim}(G) = 0 \iff \Gamma \) is torsion; and
(iv) \( G \) is monothetic \( \iff \Gamma \) is isomorphic to a subgroup of \( \mathbb{T}_d \).

In general, it is said that a topological abelian group \((G, \tau)\) satisfies Pontryagin duality (is \emph{P-reflexive} for short,) if the evaluation map \( E_G \) is a topological isomorphism onto. We refer to the survey by Dikranjan and Stoyanov \[2\] and the monographs by Dikranjan, Prodanov and Stoyanov \[1\] and Hofmann, Morris \[8\] in order to find the basic results about Pontryagin-van Kampen duality.

The following result, due to Kaplan \[9, 10\] is essential in the applications of duality methods.

\textbf{Theorem 2.2} (Kaplan). Let \( \{G_i : i \in I\} \) be a family of P-reflexive groups. Then:

(i) The direct product \( \prod_{i \in I} G_i \) is P-reflexive.

(ii) The direct sum \( \bigoplus_{i \in I} G_i \) equipped with a suitable topology is a P-reflexive group.

(iii) It holds:

\[
\left( \prod_{i \in I} G_i \right)^\sim \cong \bigoplus_{i \in I} \hat{G_i} \\
\left( \bigoplus_{i \in I} G_i \right)^\sim \cong \prod_{i \in I} \hat{G_i}
\]

Kaplan also set the problem of characterizing the class of topological Abelian groups for which Pontryagin duality holds.

Let \( g \in G \) and \( \chi \in \hat{G} \), it is said that \( g \) and \( \chi \) are orthogonal when \( \langle g, \chi \rangle = 1 \).

Given \( S \subseteq G \) and \( S_1 \subseteq \hat{G} \) we define the orthogonal (or annihilator) of \( S \) and \( S_1 \) as

\[
S^\perp = \{ \chi \in \hat{G} : \langle g, \chi \rangle = 1 \ \forall g \in S \}
\]
and

\[ S_1^\perp = \{ g \in G : \langle g, \chi \rangle = 1 \ \forall \chi \in S_1 \}. \]

Obviously \( G^\perp = \{ e_G \} \) and \( \hat{G}^\perp = \{ e_G \} \).

The following result has also many applications in connection with duality theory.

**Theorem 2.3.** Let \( S \) and \( R \) be subgroups of a LCA group \( G \) such that \( S \leq R \leq G \). Then we have \( \hat{R}/S \cong S^\perp/R^\perp \).

**Corollary 2.4.** Let \( H \) be a closed subgroup of a LCA group \( G \). Then \( \hat{G}/H \cong H^\perp \) and \( \hat{H} \cong \hat{G}/H^\perp \).

### 2.2. Abelian profinite groups

Our main results concern the structure of abelian profinite groups that appear in coding theory. Firstly, we recall some basic definitions and terminology. For every group \( G \) let us denote by \( (G)_p \) the largest \( p \)-subgroup of \( G \) and \( \mathcal{P}_G = \{ p \in \mathbb{P} : G \text{ contains a } p \text{-subgroup} \} \) where \( p \in \mathcal{P}_G \) and \( \mathbb{P} \) is the set of all prime numbers. An element \( g \) of a \( p \)-primary group \( G \) is said to have *finite height in \( G \)* if this is the largest natural number \( n \) such that the equation \( p^n x = g \) has a solution \( x \in G \). We say that \( g \) has *infinite height* if the solution exists for all \( n \in \mathbb{N} \).

Here on, the symbol \( G[p] \) denotes the subgroup consisting of all elements of order \( p \). It is well known that \( G[p] \) is a vector space on the field \( \mathbb{Z}(p) \).

### 3. Order controllable groups

**Definition 3.1.** Let \( \{ G_i : i \in \mathbb{N} \} \) be a family of topological groups and \( \mathcal{C} \) a subgroup of \( \prod_{i \in \mathbb{N}} G_i \). We have the following notions:

- \( \mathcal{C} \) is *weakly controllable* if \( \mathcal{C} \cap \bigoplus_{i \in \mathbb{N}} G_i \) is dense in \( \mathcal{C} \).
- \( \mathcal{C} \) is *uniformly controllable* if for every \( i \in \mathbb{N} \) there is \( n_i \in \mathbb{N} \) such that if \( c \in \mathcal{C} \) there exists \( c_1 \in \mathcal{C} \) such that \( c_1|_{[1,i]} = c|_{[1,i]} \) and \( c_1|_{n_i+\infty} = 0 \) (we assume that \( n_i \) is
the less natural number satisfying this property). This implies that there exists $c_2 \in \mathcal{C}$ such that $c = c_1 + c_2$, $\text{supp}(c_1) \subseteq [1, n_i]$ and $\text{supp}(c_2) \subseteq [i + 1, +\infty[.$

$\mathcal{C}$ is order-controllable if for every $i \in \mathbb{N}$ there is $n_i \in \mathbb{N}$ such that if $c \in \mathcal{C}$ there exist $c_1$ and $c_2$ in $\mathcal{C}$ such that $c = c_1 + c_2$, with $\text{supp}(c_1) \subseteq [1, n_i]$, $\text{supp}(c_2) \subseteq [i + 1, +\infty[$ and $\text{order}(c_1) \leq \text{order}(c|_{[1,n_i]})$ (again, we assume that $n_i$ is the less natural number satisfying this property). As a consequence, we also have that $\text{order}(c_2) \leq \text{order}(c)$.

Every controllable group is weakly controllable and, if the groups $G_i$ are finite, then the notions of controllability and weakly controllability are equivalent (see [4]). The following result partially answers Problem 1.2 for $p$-groups. The proof can be founded in [5].

**Theorem 3.2.** Let $\{G_i : i \in \mathbb{N}\}$ be a family of finite, abelian, $p$-groups and let $G = \prod_{i \in \mathbb{N}} G_i$. If $\mathcal{C}$ is an infinite closed subgroup of $G$ which is order-controllable, then $\mathcal{C}$ is weakly rectangular. In particular, there is a sequence $\{y_m : m \in \mathbb{N}\} \subseteq \mathcal{C} \cap \bigoplus_{i \in \mathbb{N}} G_i$ such that $\mathcal{C}$ is topologically isomorphic to $\prod_{m \in \mathbb{N}} \langle y_m \rangle$.

This result extends directly to general products of finite abelian groups and gives a partial answer to Question 1.2.

**Theorem 3.3.** Let $\mathcal{C}$ be an order-controllable, closed, subgroup of a countable product $G = \prod_{i \in \mathbb{N}} G_i$ of finite abelian groups $G_i$. Then $\mathcal{C}$ is weakly rectangular. In particular, there is a sequence $\{y_{m}^{(p)} : m \in \mathbb{N}, p \in \mathbb{P}_G\} \subseteq G \cap (\bigoplus_{i \in \mathbb{N}} G_i)$ such that $\{y_{m}^{(p)} : m \in \mathbb{N}\} \subseteq (G \cap (\bigoplus_{i \in \mathbb{N}} G_i))_{p}$ and $G$ is topologically isomorphic to $\prod_{m \in \mathbb{N}} \langle y_{m}^{(p)} \rangle$. 
Proof. Since each group $G_i$ is finite and abelian, it follows that it is a finite sum of finite $p$-groups, that is $G_i = \bigoplus_{p \in \mathbb{P}_i} (G_{i})_p$ and $\mathbb{P}_i = \mathbb{P}_{G_i}$ is finite, $i \in \mathbb{N}$. Note that $\mathbb{P}_G = \bigcup \mathbb{P}_i$. Then $\prod_{i \in \mathbb{N}} G_i \cong \prod_{i \in \mathbb{N}} (\prod_{p \in \mathbb{P}_i} (G_{i})_p) \cong \prod_{p \in \mathbb{P}_G} (\prod_{i \in \mathbb{N}} (G_{i})_p)$ where $\mathbb{N}_p = \{i \in \mathbb{N} : G_i \text{ has a } p - \text{subgroup}\}$. Consider the embedding $j : G \hookrightarrow \prod_{p \in \mathbb{P}_G} (\prod_{i \in \mathbb{N}} (G_{i})_p)$ and the canonical projection $\pi_p : \prod_{p \in \mathbb{P}_G} (\prod_{i \in \mathbb{N}} (G_{i})_p) \to \prod_{i \in \mathbb{N}} (G_{i})_p$. Set $G^{(p)} = (\pi_p \circ j)(G)$, that is a compact group. Since $G \cap (\bigoplus_{i \in \mathbb{N}} G_i)$ is dense in $G$, it follows that $(\pi_p \circ j)(G \cap (\bigoplus_{i \in \mathbb{N}} G_i)) = G^{(p)} \cap (\bigoplus_{i \in \mathbb{N}} G_i)_p$ is dense in $G^{(p)}$. Observe that if $p \in \mathbb{P}_G$ then $G^{(p)} \cap (\bigoplus_{i \in \mathbb{N}} G_i)_p = (G \cap (\bigoplus_{i \in \mathbb{N}} G_i))_p$ (otherwise it is the neutral element). Applying Theorem 3.2 for each $p \in \mathbb{P}_G$ there is a sequence $\{y^{(p)}_m : m \in \mathbb{N}\} \subseteq (G \cap (\bigoplus_{i \in \mathbb{N}} G_i))_p$ such that $G^{(p)} \cong \prod_{m \in \mathbb{N}} \langle y^{(p)}_m \rangle$. Then the sequence $\{y^{(p)}_m : m \in \mathbb{N}, p \in \mathbb{P}_G\}$ verifies the proof. \hfill \qed

The notion of rectangular and weakly rectangular subgroup of an infinite product extend canonically to subgroups of infinite direct sums. In this direction, we have:

**Theorem 3.4.** Let $C$ be an order-controllable subgroup of $\bigoplus_{k \in \mathbb{N}} G_k$ such that every group $G_k$ is finite and abelian. Then $C$ is weakly rectangular. In particular, there is a sequence $(y_n) \subseteq C$ such that $C \cong \bigoplus_{n \in \mathbb{N}} \langle y_n \rangle$.

## 4. Group codes

According to Forney and Trott [2], a group code is a set of sequences that has a group property under a componentwise group operation. In this general setting, a group code may also be seen as the behavior of a behavioral group system as given by Willens [13, 14]. It is known that many of the fundamental properties of linear codes and systems depend only on their group structure. In fact, Forney and Trott, loc.
obtain purely algebraic proofs of many of their results. In this section, we follow this approach in order to apply the results in the preceding sections to obtain further information about the structure of group codes in very general conditions.

Without loss of generality, assume, from here on, that a group code is a subgroup of a (sequence) group \( W \), called Laurent group, that has the generical form \( W = W_f \times W^c \), where \( W_f \) is a direct sum of abelian groups (locally compact in general) and \( W^c \) a direct product. More precisely, let \( I \subseteq \mathbb{Z} \) be a countable index set and let \( \{ G_k : k \in I \} \) be a set of symbol groups, a product sequence space is a direct product

\[
W^c = \prod_{k \in I} G_k
\]
equipped with the canonical product topology. A sum sequence space is a direct sum

\[
W_f = \bigoplus_{k \in I} G_k
\]
equipped with the canonical sum (box) topology. Sequence spaces are often defined to be Laurent sequences

\[
W_L = \left( \bigoplus_{k < 0} G_k \right) \times \left( \prod_{k \geq 0} G_k \right).
\]
The character group of \( W_L \) is

\[
W_{aL} = \left( \prod_{k < 0} \hat{G}_k \right) \times \left( \bigoplus_{k \geq 0} \hat{G}_k \right).
\]

Thus, a group code \( \mathcal{C} \) is a subgroup of a group sequence space \( \mathcal{W} \) and is equipped with the natural subgroup topology. Next we recall some basic facts of this theory (cf. [3, 6, 7]). These notions are used in the study of convolutional codes that are well known and used currently in data transmission (cf. [6]).

Let \( \mathcal{C} \) be a group code in the product sequence space \( \mathcal{W} = \prod_{k \in I} G_k \). According to Fagnani, \( \mathcal{C} \) is called weakly controllable if it is generated by its finite sequences. In
other terms, if

\[ \overline{C} = \overline{C} \cap \overline{W}. \]

The group code \( C \) is called \textit{pointwise controllable} if for all \( w_1, w_2 \in C \) and \( k \in I \), there exist \( L(k) \in \mathbb{N} \) and \( w \in C \) with \( w(i) = w_1(i) \ \forall i < k \), and \( w(i) = w_2(i) \ \forall i \geq k + L(k) \).

With the notation introduced above, Let \( C \) be a group code in \( \mathcal{W} \). For any \( k \in I \) and \( L \in \mathbb{N} \), we set

\[ C_k(L) := \{ c \in C : \text{there exists } w \in C \text{ with } w(i) = 0 \ \forall i < k \text{ and } w(i) = c(i) \ \forall i \geq k + L \} \]

and

\[ C_k := \bigcup_{L \in \mathbb{N}} C_k(L). \]

Obviously \( C_k(1) \subseteq C_k(2) \subseteq \cdots C_k(L) \subseteq \cdots \subseteq C_k \).

We have the following equivalence, whose verification is left to the reader.

\textbf{Proposition 4.1.} \( C \) is controllable if and only if \( C = \bigcap_{k \in I} \bigcup_{L \in \mathbb{N}} C_k(L) \).

Given a group code \( C \), the subgroup

\[ C_c := \bigcap_{k \in I} \bigcup_{L \in \mathbb{N}} C_k(L) \]

is called the \textit{controllable subcode} of \( C \). A code \( C \) is called \textit{uniformly controllable} when for every \( k \in I \), there is \( L_k \) such that \( C = \bigcap_{k \in I} C_k(L_k) \). If there is some \( L \in \mathbb{N} \) such that \( C_k = C_k(L) \) for all \( k \in I \), it is said that \( C \) is \( L \)-\textit{controllable}. Finally, \( C \) is \textit{strongly controllable} if it is \( L \)-controllable for some \( L \). If \( C \) is uniformly controllable and the sequence \((L_k)\) is bounded, then \( C \) is strongly controllable and the least such \( L \) is the \textit{controllability index} (controller memory) of \( C \).
Using the same words as in [7], the core meaning of “controllable” is that any code sequence can be reached from any other code sequence in a finite interval. The following property is clarifying in this regard. In the sequel

$$C_{[k,k+L]} = \{ w \in C : w(j) = 0 \ \forall \ j \notin [k, k + L] \}.$$

**Proposition 4.2.** $C$ is controllable if and only if for any $w \in C$, there is a sequence $(L_k)$ contained in $\mathbb{N}$ such that $w \in \sum_{k \in I} C_{[k, k+L_k]}$.

**Proof.** Let $w \in C$ and let $k_1 \in I$ be the first index such that $w(k_1) \neq 0$. Then there is $w_1 \in C$ and $L_1 \subset I$ such that $w_1(k_1) = w(k_1)$ and $w_1(i) = 0$ if $i \geq L_1 + 1$. Take $k_2 = L_1 + 2$ and let $w_2 \in C$ satisfying $w_2(i) = (w - w_1)(i)$ for all $i < k_2$ and $w_2(i) = 0$ for all $i \geq k_2 + L_2$. In general we select $w_n \in C$ such that $w_n(i) = 0$ if $i < k_n - 1$, $w_n(i) = (w - w_1 - \cdots w_{n-1})(i)$ for all $i < k_3$ and $w_n(i) = 0$ if $i \geq k_n + L_n$. We have that $w = \sum_{n \in \mathbb{N}} w_n$ is the product topology and furthermore the sum $\sum_{n \in \mathbb{N}} w_n(i)$ is finite for all $i \in I$. \qed

Analogous notions are defined regarding the observability of a group code. The group code $C$ is called **weakly observable** if

$$C \cap \mathcal{W}_f = \overline{C} \cap \mathcal{W}_f.$$

Let $C$ a group code in $\mathcal{W}$, we set

$$(C_f)[L] := \{ c \in \mathcal{W}_f : c_{[k,k+L]} \in C_{[k,k+L]} \}.$$

The group code $C$ is called **pointwise observable** if

$$C_f = \bigcap_{k \in I} \bigcap_{L \in \mathbb{N}} (C_f)[L].$$
If \( \mathcal{C} \) is a group code, then the supergroup
\[
\mathcal{C}_{ob} := \mathcal{C} \cup \bigcap_{k \in I} \left( \bigcap_{L \in \mathbb{N}} (\mathcal{C}_f)_k[L] \right)
\]
is called the \textit{observable supercode} of \( \mathcal{C} \). A code \( \mathcal{C} \) is called \textit{uniformly observable} when for every \( k \in I \), there is \( L_k \) such that \( \mathcal{C}_f = \bigcap_{k \in I} (\mathcal{C}_f)_k[L_k] \). If there is some \( L \in \mathbb{N} \) such that \( \mathcal{C}_f = \bigcap_{k \in I} (\mathcal{C}_f)_k[L] \) for all \( k \in I \), it is said that \( \mathcal{C} \) is \textit{L-observable}. Finally, \( \mathcal{C} \) is \textit{strongly observable} if it is \textit{L-observable} for some \( L \). Obviously, if \( \mathcal{C} \) is uniformly observable and the sequence \( (L_k) \) is bounded, then \( \mathcal{C} \) is strongly observable and the least such \( L \) is the \textit{observability index} (observer memory) of \( \mathcal{C} \).

Recently, Pontryagin duality methods have been applied systematically in the study of convolutional abelian group codes. In this approach, a dual code \( \mathcal{C}^\perp \) is associated to every group code \( \mathcal{C} \) using Pontryagin-van Kampen duality in such a way that the properties of \( \mathcal{C} \) can be reflected (\textit{dualized}) in \( \mathcal{C}^\perp \). Along this line, the following duality theorem provides strong justification for the use of duality in convolutional group codes (see [7]).

**Theorem 4.3.** [7] \textit{Given dual group codes} \( \mathcal{C} \) \textit{and} \( \mathcal{C}^\perp \), \textit{then} \( \mathcal{C} \) \textit{is (resp. weakly, strongly) controllable if and only if} \( \mathcal{C}^\perp \) \textit{is (resp. weakly, strongly) observable, and vice versa.}

Using duality, we obtain the following additional equivalences (cf. [7]).

**Proposition 4.4.** \textit{For any group code} \( \mathcal{C} \) \textit{we have}

1. \( (\mathcal{C}_c)^\perp \cong (\mathcal{C}^\perp)_o \).
2. \( \mathcal{C} \) \textit{is controllable if and only if} \( \mathcal{C}^\perp \) \textit{is observable.}
3. \( \mathcal{C} \) \textit{is uniformly controllable if and only if} \( \mathcal{C}^\perp \) \textit{is uniformly observable.}

Therefore, we can put our attention on the controllability of a group code wlog. In this direction, the following result was proved in [4].
Theorem 4.5 ([4]). Let $C \leq \prod_{k \in \mathbb{N}} G_k$ be a complete group code such that every group $G_k$ is finite (discrete). Then the following conditions are equivalent:

(1) $C$ is weakly controllable.

(2) $C$ is controllable.

(3) $C$ is uniformly controllable.

In [3] Fagnani proves that, if $C$ is a closed, time invariant, group code in $G^Z$, with $G$ being a compact group, then the properties of weak controllability, controllability and strong controllability are equivalent. A different proof of this result follows easily using the ideas introduced above. Indeed, suppose that $C$ is a weakly controllable, compact group code in $W$. By Theorem 4.6, we know that $C$ is controllable and therefore $C = \bigcap_{k \in I} \bigcup_{L \in \mathbb{N}} C_k(L)$. Suppose, in addition, that $C$ is time invariant, then $C_k(L) = C_0(L)$ for all $k \in I$. Furthermore, using Baire category theorem and the compactness of $C$, it follows that there must be some $L \in \mathbb{N}$ such that $C_0 = C_0(L)$, which means that $C$ is strongly controllable.

The results formulated above do not hold in general. In fact, an example of a group $H$ that is weakly controllable but not controllable is provided in [4]. Furthermore, using Theorem 4.3, we obtain that the group $H^\perp$ is weakly controllable but not controllable.

As a consequence of the preceding results we obtain the following relation between weakly controllable and controllable group codes (cf. [4]).

Theorem 4.6. If $C$ is a group code in

$$W = W_f \times W^c = (\bigoplus_{i < 0} G_i) \times (\prod_{i \geq 0} G_i).$$

Then the following assertions hold:
(a) If every group $G_i$ is discrete, then $\mathcal{C}$ is controllable if and only if $\mathcal{C}$ is weakly controllable.

(b) If every group $G_i$ is finite (discrete), then $\mathcal{C}$ is weakly controllable if and only if $\mathcal{C}$ is uniformly controllable.

(c) If every group $G_i$ is a fixed compact group $G$, and $\mathcal{C}$ is a time-invariant, closed subgroup of $\mathcal{W}^c$, then $\mathcal{C}$ is controllable if and only if $\mathcal{C}$ is strongly controllable.

In case the groups in the family $\{G_i : i \in I\}$ are abelian, Theorem 4.3 yields a similar result for observable group codes, using Pontryagin duality.

**Theorem 4.7.** If $\mathcal{C}$ is a group code in

$$\mathcal{W} = \mathcal{W}_f \times \mathcal{W}^c = \left( \bigoplus_{i<0} G_i \right) \times \left( \prod_{i \geq 0} G_i \right).$$

Then the following assertions hold:

(a) If every group $G_i$ is discrete abelian, then $\mathcal{C}$ is observable if and only if $\mathcal{C}$ is weakly observable.

(b) If every group $G_i$ is finite (discrete) abelian, then $\mathcal{C}$ is weakly observable if and only if $\mathcal{C}$ is uniformly observable.

(c) If every group $G_i$ is a fixed discrete abelian group $G$, and $\mathcal{C}$ is a time-invariant subgroup of $\mathcal{W}_f$, then $\mathcal{C}$ is observable if and only if $\mathcal{C}$ is strongly observable.

5. Conclusion

To conclude, let us point out that, so far, the applications of Harmonic Analysis and duality methods to the study of group codes have basically reached the abelian case (via Pontryagin duality and Fourier analysis). The non-commutative case has not yet been fully studied, but it can be expected that the application of duality techniques
in the study of non-abelian group codes could provide some results analogous to those already known in the Abelian case (see the work of Forney and Trott, op.cit). However, the nonabelian duality requires more complicated tools such as Krein algebras, von Neumann algebras, operator spaces, etc.). Therefore, it is first necessary to develop an appropriate nonabelian duality that can be applied in a similar way to how it is done in the Abelian case.

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Universitat Jaume I, Instituto de Matemáticas de Castellón, Campus de Riu Sec, 12071 Castellón, Spain.
E-mail address: mferrer@mat.uji.es

Universitat Jaume I, Departamento de Matemáticas, Campus de Riu Sec, 12071 Castellón, Spain.
E-mail address: hernande@mat.uji.es