On the asymptotics of AKT statistics

Lídia Rejtő\textsuperscript{1}  
Gábor Tusnády\textsuperscript{2}

2013. June 7:14:45 – September 15, 2013

Abstract

In our days there is a widespread analysis of Wasserstein distances between theoretical and empirical measures. One of the first investigation of the topic is given in the paper written by Ajtai, Komlós and Tusnády in 1984. Interestingly all the neighboring questions posed by that paper were settled already without the original one. In this paper we are going to delineate the limit behavior of the original statistics with the help of computer simulations. At the same time we kept an eye on theoretical grasping of the problem. Based on our computer simulations our opinion is that the limit distribution is Gaussian.

1 Introduction

Let \((X_i, Y_i), i = 1, \ldots, n\) be independent points uniformly distributed on the \(d\) dimensional unit cube. The AKT statistic \(W_n\) is the minimum of

\[
\sum_{i=1}^{n} \left| X_i - Y_{\pi(i)} \right|^2
\]

(1)

taken on all permutations \(\pi\) of \((1, \ldots, n)\). We call the statistics Wasserstein distance. Using computer experiments we are going to investigate the asymptotic behavior of \(W_n\) as \(d = 2\) and \(n\) goes to infinity. Based on our computer simulations we think that if \(n \leq 2048\), then the hazard rate function of the distribution of \(W_n - \beta \log(n)\) is proportional to an appropriately scaled Gaussian distribution function with \(\beta = 0.11\).

\textsuperscript{1}\textit{Department of Applied Economics and Statistics, University of Delaware, Newark, Delaware, USA}  
\textsuperscript{2}\textit{Alfréd Rényi Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary}  

\texttt{rejto@udel.edu}  
\texttt{tusnady.gabor@renyi.mta.hu}
2 History

The AKT statistics were introduced in [11]. In that paper the authors proved that there exist positive constants $c_1, c_2$ such that with probability going to 1

\[ c_1 \log(n) < W_n < c_2 \log(n) \]  

holds true. The origin of this statistic goes back to the one dimensional Shapiro–Wilks statistic for testing normality. In [2] the case $d = 1$ was investigated. The AKT statistics were generalized in [10] while a new proof was given for (2) in [12]. Talagrand next investigated the case $d \geq 3$ in [11] and settled all the observable questions. Talagrand’s estimates for $d = 2$ were sharpened in [7]. A broad introduction to transportation theory is presented in [14]. The first application of the AKT theorem was in bin packing [5]. A numerical investigation of the distribution of the statistic was given in [8]. Matchings on the Euclidean ball were investigated in [3]. The paper [13] investigates the rate of convergence in abstract settings. An investigation of different types of matchings was given in [6]. In [4] distributions supported on a manifold embedded in a Hilbert space were investigated.

3 Dyadic dynamics

Starting with $n = 2^0$ points we evolve $n = 2^k$, $k = 1, 2, \ldots$ point pairs in such a way that in each step the original points remain and simultaneously a new set of point pairs are supplanted. We call the original kids ”old” and the new ones ”young”. Having two types of kids, boys and girls, there are four possibilities for matching:

- old girls with old boys ($OO_k$),
- old girls with young boys ($OY_k$),
- young girls with old boys ($YO_k$),
- young girls with young boys ($YY_k$).

The matchings $OO_k$ and $YY_k$ are independent, and similarly $OY_k$ and $YO_k$ are independent while the costs $OO_k$ and $OY_k$ are correlated. The cost is the Wasserstein distance $W_n$ between the corresponding set of points. For small values of $n$ the approximation

\[ EW_n = \beta \log(n + \alpha) + \gamma \]

is more accurate. Figure 1 shows the averages of costs of 296 repetitions up to $k = 11$ with the above theoretical function. Here $\alpha = 0.379, \beta = 0.160, \gamma = 0.117$. The densities of the normalized differences of $W_n - (\beta \log(n + \alpha) + \gamma)$ are given in Figure 2. Next generation costs $OO_{k+1}$ are close to the averages

\[ (OO_k + OY_k + YO_k + YY_k)/4, \]
and the variance of the error is 0.101. The quadratic recursion

$$OO_{k+1} = (OO_k + OY_k + YO_k + YY_k)/4 + V_{k+1}$$  \hspace{1cm} (3)

with an iid noise $V_{k+1}$ agrees with our conjecture indicating that the quadratic recursion might be valid for two dimensions only.

In Euclidean space for arbitrary points $a, b, c, d$

$$| (a + b) - (c + d) |^2 = | a - c |^2 + | a - d |^2 + | b - c |^2 + | b - d |^2 - | a - b |^2 - | c - d |^2$$  \hspace{1cm} (4)

holds true. Similarly in our case

$$OO_{k+1} = a * (OO_k + OY_k + YO_k + YY_k) - b * (GG_k + BB_k) + V_{k+1}$$  \hspace{1cm} (5)

holds true -where $GG$ and $BB$ are the girls-girls, boys-boys distances. For stationarity

$$4 * a - 2 * b = 1$$  \hspace{1cm} (6)

must be hold. According to our computer experiences $a = 0.45$, $b = 0.41$ and the error term is 0.06. The distribution of the $V$ error term seems to be double exponential.

4 Pictorial presentation of the Hungarian Algorithm

The engine of the Hungarian Algorithm is the system of shadow prices $a(i), b(j)$ such that

$$b(j) - a(i) \leq c(i, j),$$

where $c(i, j)$ is the cost of the marriage of a pair $(i, j)$. In Figures 3 and 4 we give a pictorial presentation of the prices for $n = 1024$ and $n = 2048$. Here we used toroid distances and the colors of points of the unit square shows the price of the closest kid to the point, i.e. the $a(i)$ in case the closest kid is a girl and the $b(j)$ otherwise. The keys of colours are given on the margin. Pixel statistics are given for the colours, first number stands for wives, second for husbands. E.g. in Figure 3 there are 32 yellow wives and 69 husbands. Dark blue represents the minimal value, which is zero for the value of the last wife is always zero. The maximal value for $n = 1024$ is 0.02499 and it is 0.01654 for $n = 2024$, they are represented by yellow colour. The kids themselves are complementarily coloured for seeing them. Circles represent girls, squares represent boys. Marriages are shown by gray lines. For neglecting complications of toroid topology girls having a husband on the other side of the unit square are marked with an Andrew cross while their husband is not assigned. Interestingly usually there is one lake and one hill.

The actual total costs of the marriages are $W_{1024} = 0.996, W_{2048} = 1.654$, thus the average distance in a marriage is around 0.02 causing the same or neighboring colours for wife and husband. We know that for a married couple $(i, j)$ we have

$$b(j) - a(i) = c(i, j),$$
hence all marriages are upwards directed according their colour. 1024 point-pair from the 2048 point-pairs in Figure 4 are identical with point-pairs given in Figure 3, the others are generated independently. The situation not much changed. The bordering lines of colours for \( n = 1024 \) are simpler than for \( n = 2024 \).

## 5 Asymptotics

For the distribution of \( W_n \) the best approximation achieved by us is a distribution determined by its hazard rate function. Let us denote the tail distribution of a one–dimensional random variable by \( Q(t) \). The hazard rate \( r(t) \) is the derivative of \( -\log Q(t) \). Customarily hazard rates are defined for survival functions, belonging to positive random variables. The concept is extendable naturally for random variables taking values between \(-\infty\) and \(\infty\) too. Although \( W_n \) is positive, the best approximation comes from this broader class of distributions. The hazard rate is an arbitrary non-negative function having infinite integral on the real line. In our case it is proportional to \( \Phi(t) \) the distribution function of a standard normal random variable.

The integral up to an \( x \) is equal to \( x\Phi(x) + \varphi(x) \), where \( \varphi(x) \) is the standard normal density function. By definition it equals to \( -\log Q(x) \). Applying the relation \( r(x) = f(x)/Q(x) \), where \( f(x) \) stands for the density function we get

\[
f(x) = \Phi(x) \exp(-x\Phi(x) - \varphi(x)).
\]

The derivative of \( \log f(x) \) is \( \varphi(x)/\Phi(x) - \Phi(x) \), which is for \( x \) going to \(-\infty\) close to \(-x\), for \( x \) going to \(\infty\) it is going to \((-1)\) and it is monotone decreasing. Thus for negative \( x \)-s the distribution resembles a double exponential distribution but for positive \( x \)-s it turns to be a single exponential. That is why \( f \) is increasing first sharply to its modulus around 0.3 and it is slowly decreasing after that value.

We are going to build up a three parameter family around the above distribution. The first step is Cox regression. Let \( \lambda \) be arbitrary positive number, then \( \varrho_{\lambda}(t) = \lambda\Phi(t) \) is the hazard rate of the distribution with tail \( Q(t)^{\lambda} \) and for any real \( \mu \) and positive \( \sigma \) the linear transformation presents the density

\[
f(x \mid \mu, \sigma, \lambda) = \frac{\lambda}{\sigma} \Phi(y) \exp(-\lambda(y\Phi(y) - \varphi(y))), \quad \text{where} \quad y = (x - \mu)/\sigma.
\]

Let us denote by \( X_\lambda \) a random variable corresponding to the hazard rate \( \varrho_\lambda \). Let \( \lambda > 0, \sigma > 0, \mu \) be arbitrary real numbers, then \( f(x \mid \mu, \sigma, \lambda) \) is the density of \( \sigma X_\lambda + \mu \). Thus \( \mu \) is the location parameter, \( \sigma \) is the scale parameter and \( \lambda \) is the shape parameter of the distribution. Figure 5. shows the density of the distribution of \( X_\lambda/(\lambda)^\alpha \) with \( \alpha = 0.4 \), for eleven different \( \lambda \) from 0.1 to 10. For small \( \lambda \), one can see that the distribution of \( X_\lambda/(\lambda)^\alpha \) has an intensively increasing first phase and after it becomes similar to the exponential distribution. On the other hand for large \( \lambda \) it turns to be just the opposite.
The estimates of the parameters $\sigma$ and $\lambda$ are highly correlated, unfortunately our present sample size is not large enough to decide whether the value of the parameter $\lambda$ differs significantly from 1 or not. If it does so then the question is the tendency of $\lambda$ as $n$ goes to infinity. With $\lambda = 1$ the tendency of $\sigma$ is not clear: it is increasing from 0.08 up to 0.14 and perhaps remains bounded.

The dynamic (5) has a second condition (not presented here) in addition to (6). It ensures the boundedness of variance. For that condition we have to know the covariances of the costs of different matchings. First it was an unsettled riddle for us what is the joint distribution for the six transportation costs plus error term resulting in a Gaussian hazard rate with stationary distribution. In order to settle this question the ideas of the paper [9] turned out to be useful: even the reference to the Euclidean relation (4) comes from the possibility that Euclidean relations might be generalized into transportation equations. In the next section we are going to discuss a partial solution of the riddle.

6 A seven dimensional model

For fixed $k$ we generate four times $2^k$ of points, two sets of girl and two sets of boys. Let us label then $A, B, C, D$. ($A$ is the set of old girls, $B$ is the set of young girls, $C$ is the set of old boys, and $D$ is the set of young boys.) There are six Wasserstein distances among them:

$$W_1 = (A, C), \ W_2 = (A, D), \ W_3 = (B, C), \ W_4 = (B, D), \ W_5 = (A, B), \ W_6 = (C, D).$$

$W_1$ is independent of $W_4$ and its relation with the others is symmetric. In building a joint distribution for the six variables we use an autoregressive model: the conditional distribution of $W_{i+1}$ given $(W_1, \ldots, W_i)$, is a distribution with Gaussian hazard rate with arbitrary $\sigma_i$, $\lambda = 1$ and

$$\mu_i = \gamma_{i0} + \sum_{s=1}^{i} \gamma_{is} W_s.$$

The pairs $(\gamma_{i0}, \mu_i)$ for $k = 10$ are the followings:

$$\begin{align*}
(1.142, 0.133) & \quad (0.907, 0.129) & \quad (0.923, 0.127) & \quad (0.706, 0.125) & \quad (0.553, 0.117) & \quad (0.591, 0.117).
\end{align*}$$

For smaller $k$ the tendency is similar, the leading terms $\gamma_{i0}$ follow the general logarithmic trend and the $\sigma_i$-s are practically the same. It goes without saying that they diverse in $i$. The autoregressive coefficients are the following:
Table 1.

\[\begin{align*}
\gamma_{21} &= 0.194 \\
\gamma_{31} &= 0.176 \quad \gamma_{32} = 0.006 \\
\gamma_{41} &= -0.040 \quad \gamma_{42} = 0.188 \quad \gamma_{43} = 0.204 \\
\gamma_{51} &= 0.101 \quad \gamma_{52} = 0.140 \quad \gamma_{53} = 0.126 \quad \gamma_{54} = 0.123 \\
\gamma_{61} &= 0.126 \quad \gamma_{62} = 0.139 \quad \gamma_{63} = 0.138 \quad \gamma_{64} = 0.108 \quad \gamma_{65} = -0.054
\end{align*}\]

These coefficients are mostly small and they are negative for independent pairs. The conditional distribution of \(\tilde{W}_1\) for \(k + 1\) is similar: the gammas follow the original pattern found with linear regression and the standard deviations are around 0.055.

We can test the model in the following way. Having a well parametrized 67 dimensional distribution we can generate independent 67 dimensional random vectors as many times as many samples we have in the unit square. Presently it is 817. Next we use the Hungarian algorithm to determine the Wasserstein distance of the two point systems: it is around 1440. Finally we generate new random samples and determine their distances. Interestingly it is larger then 1440, its average is 1470 with deviance 10. It means that the structure of the unit-square sample is a bit tighter than our autoregressive scheme. But a simple trick settles the trouble: if we multiply all standard deviation parameter \(\sigma\) with 0.975 then the unit-square Wasserstein get in the middle of model Wassersteins.

If \(X\) is an arbitrary real random variable and \(Q(x)\) is its tail probability \(P(X > x)\), then the distribution of \(Q(X)\) is uniform in the interval \((0, 1)\). The integrated hazard rate

\[R(x) = \int_{-\infty}^{x} r(t) dt\]

equals to \(-\log(Q(x))\), thus the distribution of \(R(X)\) is standard exponential and the distribution of \(\exp(-R(X))\) is again uniform in \((0, 1)\). So we can test the hypothesis that the distribution of the 67 dimensional Wasserstein statistics belongs to the three parameter family \([S]\) testing the uniformity of this statistics. Of course we can not use the original Kolmogorov cut-points because we use estimated parameters, we have to calibrate the cut-points by random sample. In our case for levels 0.05, 0.01 they are 0.84, 1.37. The value of the actual statistic is 1.00 since this new Wassertstein statistics might have Gaussian hazard rate. The corresponding statistics for points in the unit square are the followings.
Table 2.

| k | No  | Kolmogorov | R1   | R2    | R3   | R4   | R5   |
|---|-----|------------|------|-------|------|------|------|
| 0 | 4902| 3.0578     | 0.6148| 0.6036| 0.5354| 0.6367| 0.3776|
| 1 | 4902| 1.8130     | 0.7537| 0.8154| 0.7425| 0.6165| 0.5595|
| 2 | 4902| 1.2752     | 0.6765| 3.6339| 0.6176| 0.7559| 0.3969|
| 3 | 4902| 1.1030     | 0.4839| 0.9168| 0.5846| 0.6992| 5.7871|
| 4 | 4902| 0.9454     | 0.9077| 0.8388| 0.7162| 0.5211| 0.7878|
| 5 | 4902| 1.1828     | 0.5653| 0.6978| 0.7649| 0.5019| 0.5467|
| 6 | 4902| 1.2998     | 0.5306| 0.5985| 0.7461| 0.5844| 0.9778|
| 7 | 4902| 0.9876     | 0.8416| 0.7448| 0.5140| 0.6265| 0.7459|
| 8 | 4902| 0.9327     | 0.7223| 0.7323| 0.6713| 0.9064| 0.5782|
| 9 | 4902| 0.9697     | 0.8291| 0.6565| 0.9028| 0.5890| 0.7438|
|10 | 4902| 1.0793     | 0.6249| 0.6609| 0.7140| 0.5499| 0.6290|
|11 | 817 | 0.6508     | 0.8411| 0.7843| 1.0557| 1.1890| 0.9961|

Here \( k \) is number of number of doublings in the dynamics, \( No \) is the sample size. We have 817 different runs and for \( k = 11 \) and the sample size for \( k < 11 \) it is \( 6 \times 817 \). We multiply the maximal difference between empirical and theoretical distributions with \( \sqrt{No} \) but for \( k < 11 \) the Wasserstein distances are not independent. It might be the reason that even for \( k = 10 \) we have a borderline result. Of course we are aware the fact that for \( k = 0 \) the Wasserstein statistics has a different distribution and perhaps the situation is similar for small \( k \)-s. This tendency is clearly shown by the Kolmogorov statistics.

We generated five times data matrix by the theoretical model. Columns headed by R1,...,R5 gives the corresponding Kolmogorov statistics. As one can observe the statistics have rather long tail arising perhaps from the interwoven dependence structure.

7 The Ajtai statistics

In paper [1] the saddle point method is used to prove inequality (2). Appropriate Lipschitzean functions are developed to prove the left hand side while for the right hand side an ad-hoc matching algorithm due to Miklós Ajtai is used. The algorithm is based on the following elementary concept.

**Definition** Let \( s \) be a positive integer, \( t = 2s \) and \( A = (a_1, \ldots, a_t) \) arbitrary reals. The median bits \( B = (b_1, \ldots, b_t) \) of \( A \) are \((0,1)\)-s defined by the properties

\[
\sum_{k=1}^{t} b_k = s; \quad \text{if } ((b_i = 0) \text{ and } (b_j = 1)), \text{ then } a_i \leq a_j.
\]

In case there are no ties in \( A \), \( B \) is uniquely defined.
Let $k$ be an arbitrary natural number, $n = 4^k$, and $Z = (Z_i, i = 1, \ldots, n)$ be a system of arbitrary two-dimensional points. In a step-by-step procedure we order a $2k$ long bit sequence to each of the points in $Z$. As an initial step we construct a set $A$ from the first coordinates of $Z$. Having the corresponding $B$ we divide $Z$ into two subsets according to the bits in $B$ and for each of that subsets we form the median bits from the second coordinates.

From these initializations we proceed in the same manner. Using the bit sequences ordered to the points so far first we develop the next bits from the subsets formed of the first coordinates having the same bit sequences generated so far and next we turn to the second coordinates. Applying the procedure independently for two iid two dimensional sets $X$ and $Y$ in the role of $Z$ the matching is supplied by the identical bit sequences. In course of the algorithm step by step the size of subsets is halved and finally it is reduced to one, thus for all possible $2k$ long bit sequence we have exactly one point in $X$ and one in $Y$, and it makes the matching. In a certain sense we construct a two-dimensional ordering merging the orders according the two coordinates. Let $b$ be an arbitrary $2k$ long bit sequence and

$$c = \sum_{i=1}^{k} b_{2i-1}, \quad d = \sum_{i=1}^{k} b_{2i}.$$ 

The expected value of the first coordinate of the corresponding points in $X$ or $Y$ are labelled by $b$ is $c/(2^k+1)$ and for the second coordinate it is $d/(2^k+1)$. (Let us note here that in our notation $X$ and $Y$ are two sets of two dimensional points and the coordinates are $(x_{i1}, x_{i2}), (y_{i1}, y_{i2})$).

Let us observe that the marginal quantile transform applies for the algorithm: the matching itself does not depend on any monotone transform. (The marginal quantile transform is $F(x_{i1}), i = 1, \ldots, n$ for the first coordinates and $G(x_{i2}), i = 1, \ldots, n$ for the second coordinates if the coordinates are independent and the distribution of the first coordinates if $F$ and that is $G$ for the second coordinates.) Utilizing the mean limits offered by the algorithm we are deeply concerned that the limit distribution for any distributions has the form

$$\sum_{i=1}^{n} c_i x_i^2$$

where the $x_i$-s are independent standard normals and the $c_i$-s are appropriate coefficients. We guess that such random variables have monotone increasing hazard rates, and for certain coefficients the hazard rate function is bounded while for others it goes to infinity. If the multiplicity of the maximal coefficients is high then the leading term of the distribution has a chi-square distribution with large degree of freedom what is approximately normal. Thus it is possible for the limit distributions for

- Wasserstein distances for uniform distribution,
- Wasserstein distances for standard normal distribution,
- Ajtai distances for uniform distribution,
- Ajtai distances for standard normal distribution,
all are the normal one but the speed of the convergence is considerable slow. Using the marginal quantile transformation one can ask what is the relation between the distances for standard normal and uniform distributions. In the next table we give the basic statistics for Ajtai distances up to $4^8$. Here $D = 1$ stands for standard normal and $D = 2$ for uniform distribution and the correlations come from marginal quantile transformation.

**Table 3.**

| k | D | Sample Size | Average | Stdev | Sequeness | Kolmogorov | Locus | Correlations |
|---|---|-------------|---------|-------|-----------|------------|-------|--------------|
| 1 | 1 | 42161       | 9.951413 | 5.606661 | 1.261530 | 16.717425 | 23626 |
| 2 | 1 | 40184       | 19.597105 | 6.658678 | 0.877757 | 11.497639 | 22992 |
| 3 | 1 | 40711       | 34.650952 | 7.678610 | 0.663583 | 8.510061 | 21130 |
| 4 | 1 | 49367       | 58.719088 | 8.880549 | 0.507017 | 7.463537 | 26939 |
| 5 | 1 | 20089       | 97.984671 | 10.26948 | 0.334140 | 3.379643 | 8789  |
| 6 | 1 | 12988       | 163.848339 | 12.198350 | 0.277057 | 2.205296 | 4800  |
| 7 | 1 | 5739        | 276.390900 | 14.609538 | 0.130521 | 0.986046 | 2560  |
| 8 | 1 | 2333        | 471.869129 | 18.169085 | 0.130229 | 0.658314 | 1104  |
| 1 | 2 | 42161       | 0.251985  | 0.154344 | 1.114916 | 14.115933 | 21684 | 0.891637 |
| 2 | 2 | 40184       | 0.655468  | 0.217523 | 0.539991 | 7.161248 | 21916 | 0.858883 |
| 3 | 2 | 40711       | 1.428708  | 0.294253 | 0.420951 | 5.765374 | 21836 | 0.872925 |
| 4 | 2 | 49367       | 2.519040  | 0.360076 | 0.433037 | 6.159368 | 29373 | 0.858270 |
| 5 | 2 | 20089       | 3.809825  | 0.400688 | 0.401140 | 3.978578 | 12257 | 0.807309 |
| 6 | 2 | 12988       | 5.210258  | 0.421534 | 0.390273 | 3.757436 | 7405  | 0.734815 |
| 7 | 2 | 5739        | 6.669811  | 0.431007 | 0.373776 | 2.231697 | 2844  | 0.645670 |
| 8 | 2 | 2333        | 8.138648  | 0.438881 | 0.363513 | 1.112873 | 898   | 0.558825 |

Because the number of points $4^8$ is rather large in case of the Hungarian algorithm, thus we reduced it to $4^5$. In the next table we present the basic statistics for sample size 309.

**Table 4.**

| Label | Sample type   | Algorithm       | Average | St.deviation |
|-------|---------------|-----------------|---------|---------------|
| NH    | normal sample | Hungarian algorithm | 39.184  | 5.453         |
| NA    | normal sample | Ajtai algorithm | 97.497  | 10.485        |
| UA    | uniform sample | Ajtai algorithm | 2.902   | 0.541         |
| UH    | uniform sample | Hungarian algorithm | 1.756   | 0.451         |
| UB    | uniform sample | adhoc improving | 2.224   | 0.500         |

and the correlations are the followings:
Table 5.

|    | NH   | NA   | UA   | UH   | UB   |
|----|------|------|------|------|------|
| NH | 1.000| 0.665| 0.681| 0.766| 0.717|
| NA | 0.665| 1.000| 0.569| 0.469| 0.530|
| UA | 0.681| 0.569| 1.000| 0.896| 0.981|
| UH | 0.766| 0.469| 0.896| 1.000| 0.942|
| UB | 0.717| 0.530| 0.981| 0.942| 1.000|

The correlation between Hungarian and Ajtai algorithms for uniform distribution is 0.896 which is astonishingly high. It proves that the Ajtai algorithm is very efficient. Even it is efficient concerning the average but surprisingly it is efficient in measuring the habits of the sample for having good marriages. Now it is a standard observation for Wasserstein couplings: the condition that for any finite set of marriages no reordering of the couples could improve the sum of distances is sufficient. For large number of points the condition is hard to check. We use the simplest case with two couples, it is labeled us “adhoc improving”. One can say that it is natural that its correlation with original Ajtai is as high as 0.981 because our ad-hoc improving starts with the Ajtai coupling. But its correlation with the optimum is also improved: it is now 0.94.

8 Discussion

The original Ajtai algorithm uses numerical medians and linear transformations fitting the medians to the interval halving. First we apply the median of the whole set of first coordinates and imagine a vertical line dividing the unit square accordingly. The number of points in the rectangles are equal. Next we divide independently with appropriate horizontal lines the left hand side and right hand side rectangles in such a way that the number of points in the four rectangles should be equal. Roughly what happen in the four rectangles are independent from each other if it is accordingly scaled. This concept leads to a dynamical equation similar to (3) but in this case the four terms are independent. This type of recursion immediately leads to a normal limit distribution. When we believed that the limit distribution is not Gaussian we desperately struggled against such kind of dynamics but now we are content that the limit distribution is Gaussian for both cases of Ajtai and Wasserstein distance. Of course for Wasserstein distance the situation is more complicated. The Ajtai algorithm works for arbitrary number of points, we used the power of 4 only for didactical reason.
References

[1] M. Ajtai, J. Komlós, and G. Tusnády, On optimal matchings, *Combinatorica* 4/4 (1984), 259–264.

[2] E. del Barrio, E. Giné, and C Matrán, Central limit theorems for the Wasserstein distance between the empirical and the true distributions, *The Annals of Probability* 27/2 (1999), 1009–1071.

[3] F. Barthe and N. O’Connel, Matchings and variance of Lipsitz functions, *ESAIM: Probab. Statist.* 13 (2010) 400–408.

[4] G. D. Canas and L. R. Rosasco, Learning probability measures with respect to optimal transport matrices, arXiv:1209.1077v1 [cs.LG] 5 Sep 2012.

[5] E. G. Coffman, Jr., M. R. Garey, and D. S. Johnson, Approximation algorithms for bin packing: an updated survey (in *Algorithm design for computer systems design*, G. Ausiello, M. Lucerfini, and P. Serafini (eds) pp. 49–106), Springer, 1984.

[6] A. E. Holroyd, R. Pemantle, Y. Peres, and O. Schramm, Poisson matchings, *Ann. Inst. Henri Poincaré, Probab. Stat.* 45(1) (2009) 266-289.

[7] P. Major, On the estimation of multiple random integrals and U–statistics (Lecture Notes in Mathematics), Springer, 2013.

[8] C. Noszály, Experiments on the distance of two–dimensional samples, *Annales Mathematicae et Informaticae* 2012.

[9] M.-K. von Renesse and K.-T. Sturm, Transport inequalities, gradient estimates, entropy and Ricci curvatures, *Communications in Pure and Applied Mathematics* 58/7 (2005), 923–940.

[10] M. Talagrand, The Ajtai–Komlós–Tusnády matching theorem for general measures, in *Probability in Banach spaces, 8. (Brunswick, ME, 1991)* Volume 30 of Progress in Probability pages 39–54, Birkhäuser Boston M. 1992

[11] M. Talagrand, The transportation cost from the uniform measure to the empirical measure in dimension ≥ 3, *The Annls of Probability* 22/2 (1994), 919–959.

[12] M. Talagrand, *The general chaining (Springer Monographs in Mathematics)*, Springer, 2005.

[13] G. L. Torrisi, Asymptotic analysis of the optimal cost in some transportation problems with random locations, *Stochastic Processes and their Applications* 122 (2012) 305–333.

[14] C. Villani, *Optimal transport: old and new*, Springer, 2008
Figure 1. Optimal cost as a function of the number of pairs
Figure 2. Densities of optimal costs
Figure 3. Pictorial representation of shadow prices for n=2048
Figure 4. Pictorial representation of shadow prices for n=2048
Figure 5. Densities for the Gaussian hazard rate model appropriately scaled.