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A One-step Approach to Computing a Polytopic Robust Positively Invariant Set

Paul Trodden

Abstract

A procedure and theoretical results are presented for the problem of determining a minimal robust positively invariant (RPI) set for a linear discrete-time system subject to unknown, bounded disturbances. The procedure computes, via the solving of a single LP, a polytopic RPI set that is minimal with respect to the family of RPI sets generated from a finite number of inequalities with pre-defined normal vectors.

Index Terms

Linear systems; Uncertain systems; Computational methods; Optimization; Invariant sets

I. INTRODUCTION

We consider the problem of finding, for the discrete-time, linear time-invariant system,

\[ x^+ = Ax + w, \]

a robust positively invariant (RPI) set. That is, a set \( \mathcal{R} \subset \mathbb{R}^n \) with the property

\[ Ax + w \in \mathcal{R}, \forall x \in \mathcal{R}, w \in \mathcal{W}. \]

In this problem, \( x \in \mathbb{R}^n \) is the current state and \( x^+ \) its successor. The disturbance \( w \in \mathbb{R}^n \) is unknown but lies in a compact, convex set \( \mathcal{W} \) that contains the origin in its interior.

Robust or disturbance invariant sets are important in control, and their theory and computation have attracted significant attention; see, for example, [1]–[4] and references therein. One set that

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is of particular interest is the *minimal* RPI (mRPI) set—that is, the RPI set that is smallest in volume among all the RPI sets for a system—which is also the set of states reachable from the origin in the presence of a bounded disturbance. However, computing an exact representation of the mRPI is generally impossible (except for special instances of \(A\), as identified later), and instead an approximation is usually sought. A seminal contribution in this regard is [3], which proposes a method for computing an arbitrarily close outer-approximation to the mRPI set, which is itself RPI.

The essence of the problem of computing exactly the mRPI set is that this set is, in general, not finitely determined. Methods for computing approximations to the mRPI set, including [3], rely on finding finite representations of the set. Recently, in the context of tube-based MPC, [5] introduced and studied the notion of a polytopic RPI set defined by a finite number, \(r\), of a-priori selected linear inequalities. For a non-autonomous system \(x^+ = Ax + Bu + w\) controlled by a continuous positively homogeneous control law, \(u = \kappa(x)\), the authors showed that the RPI set dynamic condition (2) has an equivalent representation as \(r\) functional inequalities. It is established that a fixed-point solution to the functional equation corresponds to an RPI set that is *minimal*, in volume, with respect to the entire family of RPI sets defined by the pre-selected inequalities, and is an invariant outer-approximation to the mRPI set. To compute this set, the authors of [5] give an iterative procedure, based on solving a sequence of LPs, for which convergence is guaranteed.

In this note, we adopt the notions of [5] and specialize their results to the case of the linear autonomous system (1) (alternatively, the linear non-autonomous system with linear state feedback control law) in order to develop a one-step approach, based on solving a single LP, to the computation of the smallest RPI set defined by a pre-selected system of inequalities. Though simple, to the author’s knowledge this has not appeared in the literature, although there are related results; for example, it is known that checking the invariance of an existing polytope is an LP [2].

The proposed approach differs to the one of [3] in one important assumption: the number and normal vectors of the inequalities that represent the RPI set are, as in [5], defined *a priori*, while in [3] both are unknown until termination of the algorithm. This *a-priori* definition, first proposed and studied by [5], has two consequences: firstly, the RPI set obtained is not necessarily the mRPI set, or even an arbitrarily close outer-approximation (as it is in [3]); however, it is the
The smallest RPI set that can be represented by the finite number, \( r \), of chosen inequalities with normal vectors \( \{ P_i^T : i = 1 \ldots r \} \) [5]. To make a clear distinction, in this note we term this the \((P, r)\)-mRPI set when the number of chosen inequalities is \( r \) and the matrix of normal vectors (the left-hand side of the defining system of inequalities) is \( P \). Secondly, the method of [3] involves solving a sequence of LPs and then computing a Minkowski summation, but here only the solving of a single LP is required. The development of the procedure here comprises two steps, the enumeration of which also serves to clarify the contribution of this note with respect to [5]: first, we show that, for the studied linear autonomous system (1), the fixed-point solution to the functional equation, which [5] showed is guaranteed to exist, is in fact unique. Secondly, we show that the corresponding RPI set—which [5] proved to be minimal with respect to the family of RPI sets represented by \((P, r)\)—can be computed via a single linear program (LP), as an alternative the iterative sequence of LPs proposed by [5].

Another method that uses a single LP to compute a disturbance invariant set is the optimized robust control invariance approach of [4], applicable to the linear non-autonomous system \( x^+ = Ax + Bu + w \). Because a robust control invariant (RCI) set—and the associated control policy—is obtained, then this subsumes the robust positive invariance (where a fixed control law is assumed) considered here. However, that approach optimizes over only those control policies that guarantee a finitely determined set: equivalent to assuming, for \((1)\), that \( A^k W = \alpha W \) for some \( \alpha \in [0, 1) \) and finite integer \( k \). In this note, only the weaker assumption that \( A \) has eigenvalues inside the unit circle is required.

The organization of this note is as follows. First, in Section II, it is shown that for the system \((1)\), the fixed-point solution is, under suitable assumptions, unique. Subsequently, in Section III, it is shown that the \((P, r)\)-mRPI set for \((1)\) may be computed via a single LP. Finally, examples are given in Section IV to illustrate the practicality of the proposed approach, before conclusions are made in Section V.

**Notation:** The sets of non-negative and positive reals are, respectively, \( \mathbb{R}_{0+} \) and \( \mathbb{R}_+ \). For \( a, b \in \mathbb{R}^n \), \( a \leq b \) applies element by element. A matrix \( M \) is non-negative, denoted \( M \geq 0 \), if \( M_{ij} \geq 0 \) for all \( i \) and \( j \). \( \lambda \mathcal{X} \) is the scaling of a set \( \mathcal{X} \) by \( \lambda \in \mathbb{R} \), defined as \( \{ \lambda x : x \in \mathcal{X} \} \). \( A \mathcal{X} \) denotes the image of a set \( \mathcal{X} \subset \mathbb{R}^n \) under the linear map \( A : \mathbb{R}^n \mapsto \mathbb{R}^p \), and is given by \( \{ Ax : x \in \mathcal{X} \} \). The support function of a set \( \mathcal{X} \) is \( h(\mathcal{X}, v) \triangleq \sup \{ v^T x : x \in \mathcal{X} \} \). A polyhedron is the convex intersection of a finite number of halfspaces, and a polytope is a closed and bounded...
II. Existence and uniqueness of a \((P, r)\)-mRPI set

For the system (1), we consider the case of a polytopic disturbance set

\[
\mathcal{W} \triangleq \{w \in \mathbb{R}^n : Fw \leq g\},
\]

where \(F \in \mathbb{R}^{p \times n}\).

**Assumption 1:** The set \(\mathcal{W}\) is a polytope that contains the origin in its interior.

**Assumption 2:** The eigenvalues of \(A\) are strictly within the unit circle.

From Assumptions 1 and 2, it follows that \(g \in \mathbb{R}^r_+\), and that for any \(g \in \mathbb{R}^r_+\), there exists an RPI set, \(\mathcal{R}\), for the system (1), satisfying (2).

**Assumption 3:** The RPI set \(\mathcal{R}\) is a polytope that contains the origin in its interior.

In this note, following [5], we consider the RPI set constructed from a finite number, \(r\), of inequalities with pre-defined normal vectors. That is, \(\mathcal{R} \triangleq \mathcal{R}(q)\), defined as

\[
\mathcal{R}(q) \triangleq \{z \in \mathbb{R}^n : Pz \leq q\},
\]

where \(P \in \mathbb{R}^{r \times n}\), \(\{P_1^T : i \in \{1, \ldots, r\}\}\) spans \(\mathbb{R}^n\), \(P_i\) is the \(i\)th row of matrix \(P\), and \(q \in \mathbb{R}^r_+\).

The left-hand side of the inequalities—the matrix \(P\)—is to be chosen \textit{a priori} by the designer.

The following result, which is an application of Farkas’ Lemma, establishes basic conditions on the matrices \(A, P\) and \(F\) for the existence of an RPI set for the system (1) given the disturbance polytope (3).

**Theorem 1 (Adapted from Hennet and Castellan [6]):** The set \(\mathcal{R}(q)\) with some \(q = \bar{q}\) is robust positive invariant for the system (1) if and only if there exist non-negative matrices \(H \in \mathbb{R}^{r \times r}\) and \(M \in \mathbb{R}^{r \times p}\) such that

\[
\begin{align*}
HP &= PA \\
MF &= P \\
H\bar{q} + Mg &\leq \bar{q}
\end{align*}
\]

We will assume that \(P\) is chosen so that an RPI set exists:

**Assumption 4:** For the chosen \(P\), and the disturbance set \(\mathcal{W}\), there exists a \(\bar{q} \in \mathbb{R}^r_+\) such that (2) holds for all \(x \in \mathcal{R}(\bar{q})\).
Remark 1: While Assumption 4 may appear strong, it is needed to narrow the class of matrices that we consider to those that admit an RPI set. However, the procedure presented in the next section includes an easy certification of existence of an RPI set for a chosen $P$: if an RPI set exists, the $(P, r)$-mRPI set is returned. If no RPI set exists, the optimization problem is unbounded.

The authors of [5] show—in the more general setting of a linear non-autonomous system controlled by a positively homogeneous state-feedback control law—that RPI condition (2) is equivalent to the functional inequality

$$c(q) + d \leq b(q),$$

where, for $i = 1 \ldots r$, $b_i(q) \triangleq h(\mathcal{R}(q), P_i^T)$, $c_i(q) \triangleq h(A\mathcal{R}(q), P_i^T)$, $d_i \triangleq h(\mathcal{W}, P_i^T)$. That is, the set inclusion requirement is replaced by support function inequalities, which is a standard technique [7]. Note that $b(q)$ may be different to $q$; for example, in the case of redundant inequalities defining $\mathcal{R}(q)$. Further note that $b: \mathbb{R}_{0+}^r \mapsto \mathbb{R}_{0+}^r$, $c: \mathbb{R}_{0+}^r \mapsto \mathbb{R}_{0+}^r$ and $d \in \mathbb{R}_+^r$.

The next result, which concerns the existence of a fixed-point solution to (6), was established by [5] in the setting of a linear non-autonomous system controlled by positively homogeneous state-feedback control law, and hence immediately applies to the more specialized case considered in this note.

**Theorem 2 (Theorem 1 of [5]):** Let $Q \triangleq \{ q \in \mathbb{R}_{0+}^r : 0 \leq q \leq \bar{q} \}$. Then, (i) for all $q \in Q$, $c(q) + d \in Q$; (ii) there exists at least one $q^* \in Q$ satisfying $c(q^*) + d = b(q^*) = q^*$.

With respect to computing a fixed-point solution, the sequence generated by the iterative procedure $q^{[p+1]} = c(q^{[p]}) + d$, with $q^{[0]} = 0$, converges to the smallest (with respect to the inequality) fixed-point solution $q^*$ [5 Theorem 2]. As the following result states, the corresponding set $\mathcal{R}(q^*)$ is RPI, and, in fact, is the minimal (smallest volume) RPI set over the family of RPI sets defined by the $r$ inequalities with left-hand side $P$.

**Lemma 1 (Corollary 1 of [5]):** $\mathcal{R}(q^*) = \bigcup_{X \in S} X$ where

$$S \triangleq \{ \mathcal{R}(q) : q \in \mathcal{H} \}, \text{ and } \mathcal{H} \triangleq \{ q \in \mathbb{R}_{0+}^r : c(q) + d \leq b(q) \}$$

For convenience, we define this set $\mathcal{R}(q^*)$ as the $(P, r)$-mRPI set.

**Definition 1 (($P, r$)-mRPI set):** The $(P, r)$-mRPI set for system (1) is $\mathcal{R}(q^*)$ where $q^* = b(q^*) = c(q^*) + d$.

In this note, we propose an alternative to the iterative procedure of [5]. First, it is shown that the fixed-point solution to (6) is, in fact, unique. This result is then exploited in Section III.
wherein the problem of finding the fixed-point solution is cast as an LP. To the end of proving
uniqueness, we note the following properties of the functions $b(\cdot)$ and $c(\cdot)$, established by [5].

Lemma 2 (Part of Proposition 1 of [5]): The functions $b(\cdot)$ and $c(\cdot)$ are continuous and mono-
tonically non-decreasing; that is, $b(a_1) \leq b(a_2)$ for $a_1 \leq a_2$.

The following property is also essential to the main result of this section.

Lemma 3 (Positive homogeneity of $b$, $c$): The functions $b(\cdot)$ and $c(\cdot)$ and positively homoge-
neous; that is $b(\lambda a) = \lambda b(a)$ for $\lambda \geq 0$, with a similar expression for $c(\cdot)$.

Proof: Consider $b_i(\lambda a) = h\left(R(\lambda a), P_i\right)$ for some $a \in \mathbb{R}_{0+}^r$, $\lambda \geq 0$ and $i \in \{1, \ldots, r\}$. By definition of $R(\cdot)$, $R(\lambda a) = \lambda R(a)$. Thus, $h\left(R(\lambda a), P_i\right) = h\left(\lambda R(a), P_i\right) = \lambda h\left(R(a), P_i\right)$, for $\lambda \geq 0$, where the latter equality follows directly from the definition of the support function [7]. Hence, $b_i(\lambda a) = \lambda b_i(a)$, therefore $b(\lambda a) = \lambda b(a)$. Positive homogeneity of $c(\cdot)$ may be
established by the same arguments.

The main result of this section follows.

Theorem 3 (Uniqueness of fixed-point solution): There exists a unique $q^* \in \mathbb{R}_{+}^r$ satisfying
$c(q^*) + d = b(q^*) = q^*$.

Proof: Existence is established by Theorem 2 so it remains to show that $q^*$ is unique. Let
$l(q) = c(q) + d - b(q)$ and $f(q) = b(q) - q$. Finding the fixed-point solution $c(q^*) + d = b(q^*) = q^*$
is equivalent to finding $q^*$ such that $l(q^*) = f(q^*) = 0$. Suppose there exist $q^1 \in \mathbb{R}_{+}^r$ and $q^2 \in \mathbb{R}_{+}^r$
such that $l(q^1) = f(q^1) = 0$, $l(q^2) = f(q^2) = 0$, and $q^2 \neq q^1$, i.e., $q^2 - q^1 \neq 0$. There are two
possibilities:

(i) $q^2_i > q^1_i$ for at least one $i \in \{1, \ldots, r\}$, with $q^2_j \leq q^1_j$ otherwise;
(ii) $q^2 \leq q^1$, with $q^2_i < q^1_i$ for at least one $i \in \{1, \ldots, r\}$.

Consider case (1) Let

$$\alpha = \min_{i=1 \ldots r} \left\{ \frac{q^1_i}{q^2_i} \right\} = \frac{q^1_p}{q^2_p} > 0$$

Since $q^2_i > q^1_i$ for at least one $i$, then $\alpha < 1$. Let $s = \alpha q^2 < q^2$. It follows, from positive homogeneity of $b(\cdot)$ and the fact that $b(q^2) = q^2$, that $f(s) = b(s) - s = b(\alpha q^2) - \alpha q^2 = \alpha q^2. (\cdot)$
\[ \alpha (b(q^2) - q^2) = 0. \]

Similarly,
\[
l(s) = c(s) + d - b(s) \\
= c(\alpha q^2) + d - b(\alpha q^2) \\
= \alpha c(q^2) + d - \alpha b(q^2) \\
= \alpha (c(q^2) - b(q^2)) + d \\
> 0
\]

where the second line follows from the positive homogeneity of \( c(\cdot) \) and \( b(\cdot) \), and the strict inequality with zero follows from \( c(q^2) - b(q^2) = -d, \alpha < 1 \) and \( d > 0 \). Now, by definition of \( \alpha \), and since \( \alpha < 1 \), then \( s \leq q_1 \) with \( s_{p} = q_1_{p} \). For the same \( p \), we have \( f_p(q^1) = b_p(q^1) - q^1_{p} = 0, f_p(s) = b_p(s) - s_{p} = 0 \), and, since \( s \leq q_1 \), then \( b_p(s) \leq b_p(q^1) \). In fact, \( b_p(s) = b_p(q^1) \), as we have already shown that \( b_p(s) = s_{p} = q^1_{p} \). We also have \( l_p(q^1) = c_p(q^1) + d_p - b_p(q^1) \) and \( l_p(s) = c_p(s) + d_p - b_p(s) \). Because \( b_p(q^1) = b_p(s) \) and \( c_p(s) \leq c_p(q^1) \), it follows that \( l_p(s) \leq l_p(q^1) \). But then \( 0 = l_p(q^1) \geq l_p(s) > 0 \), and we have a contradiction: therefore, we conclude that case [i] cannot hold, and either case [ii] holds or \( q_2 = q_1 \). Now consider case [iii] and its equivalent statement: \( q^1_i > q^2_i \) for at least one \( i \in \{1, \ldots, r\} \), with \( q^1_j \leq q^2_j \) otherwise. Following the same set of arguments, we find that that case [ii] cannot hold either. Therefore, \( q^1 = q^2 = q^*, \) and the solution is unique.

III. COMPUTING THE \((P, r)\)-MRPI VIA A SINGLE LP

The problem of computing the fixed-point solution \( q^* \) is
\[
q^* = \arg \min_q \{ \|q\|_1 : c(q) + d \leq b(q) \} \tag{7}
\]

This is not tractable, as, by the definitions of \( b(\cdot) \) and \( c(\cdot) \), the constraints are
\[
\max \{ P_i Ax : P x \leq q \} + \max \{ P_i w : F w \leq g \}
\leq \max \{ P_i x : P x \leq q \}
\]
for \( i = 1 \ldots r \). However, by noting that the fixed-point solution is unique, we may replace this with the problem
\[
q^* = \arg \max_q \{ \|q\|_1 : c(q) + d = b(q) \}
\]
This problem then easily converts to a linear program, as shown by the following. Introduce auxiliary variables $\xi^i \in \mathbb{R}^n$ and $\omega^i \in \mathbb{R}^n$ for each RPI inequality $i \in \{1, \ldots, r\}$. Then, noting that $q = b(q) = c(q) + d$ at the desired fixed-point solution, eliminate $q$ and $b(q)$, leading to the problem

$$
P: \quad q^* = c^* + d^*, \text{ where } (c^*, d^*) = \arg \max_{\{c_i, d_i, \xi^i, \omega^i\}} \|c + d\|_1$$

subject to, for all $i \in \{1, \ldots, r\}$,

$$c_i \leq P_i A \xi^i,$$  \hspace{1cm} (9a)

$$P \xi^i \leq c + d,$$  \hspace{1cm} (9b)

$$d_i \leq P_i \omega^i,$$  \hspace{1cm} (9c)

$$F \omega^i \leq g.$$  \hspace{1cm} (9d)

In this problem, maximizing each $c_i$ subject to constraints (9a) and (9b) represents finding the vector of support functions to $A\mathcal{R}$. Constraint (9b) represents $Px \leq b(q)$, with the condition $c(q) + d = b(q)$ enforced. Constraints (9c) and (9d) represent finding $d$, the vector of support functions to $\mathcal{W}$.

**Remark 2:** Note that, by definition, $d_i = h(\mathcal{W}, P_i)$ is constant and does not depend on $q$. Therefore, $d$ could be computed prior to solving $P$, by solving a sequence of LPs, before entering the optimization as a parameter. However, our aim is to formulate a single LP (a one-step procedure) that computes, simultaneously, $d$, $c$ and hence $q$.

Note that each $d_i$ and $\omega_i$ is bounded, via (9c) and (9d) and the assumptions on $\mathcal{W}$. Further note that this problem always has a feasible solution, since one can choose, for example, $c_i = d_i = 0$ and $\xi^i = \omega^i = 0$. The question, then, is whether an optimal solution exists, or the problem is unbounded. To this end, we require the following result, which specializes Theorem 1 to the fixed-point solution.

**Proposition 1:** The set $\mathcal{R}(q^*)$, where $q^*$ is a fixed-point solution to (5), is the $(P, r)$-mRPI set for the system (1) if and only if there exist non-negative matrices $H \in \mathbb{R}^{r \times r}$ and $M \in \mathbb{R}^{r \times p}$
such that

\[ HP = PA \quad (10a) \]
\[ MF = P \quad (10b) \]
\[ Hq^* + Mg = q^* \quad (10c) \]

**Proof:** Follows directly from taking the dual of the LPs that define each row of the functional inequality, \( c(q) + d \leq b(q) \leq q \), and noting that \( c(q) + d = b(q) = q \) if and only if \( q = q^* \). □

Then the main result of this section follows.

**Theorem 4:** Suppose Assumptions 1–3 hold. If \( P \) satisfies Assumption 4 then problem \( \mathbb{P} \) admits an optimal solution corresponding to the fixed-point solution \( q^* \). Otherwise, \( \mathbb{P} \) is unbounded above.

**Proof:** We use duality to prove the theorem. Since the LP problem \( \mathbb{P} \) is feasible, it suffices to show that the dual problem is feasible—and the solution is as claimed—if and only if \( P \) satisfies Assumption 4; on the other hand, if the dual is infeasible, then by weak duality the primal problem \( \mathbb{P} \) is unbounded.

The dual problem is

\[
\begin{align*}
\mathbb{D}: \min_{\{\lambda_i, \nu_i, \mu^i, \eta^i\}} & \sum_{k=1}^{r}(\eta^k)^T g \\
\text{subject to} & \\
& \lambda - \sum_{k=1}^{r} \mu^k = 1, \quad (12a) \\
& \nu - \sum_{k=1}^{r} \mu^k = 1, \quad (12b) \\
& P^\top \mu^i - A^\top P_i^\top \lambda_i = 0, \quad (12c) \\
& F^\top \eta^i - P_i^\top \nu_i = 0, \quad (12d) \\
& \lambda_i, \nu_i \geq 0 \quad (12e) \\
& \mu^i, \eta^i \geq 0 \quad (12f)
\end{align*}
\]

where \( \lambda_i \in \mathbb{R}, \mu^i \in \mathbb{R}^r, \nu_i \in \mathbb{R}, \eta^i \in \mathbb{R}^p \) are the dual variables associated with constraints (9a)–(9d) respectively. From (12a) and (12b), \( \lambda_i = \nu_i = 1 + \sum_{k=1}^{r} \mu^k_i, \) for all \( i = 1, \ldots, r \), where \( \mu^k_i \)
is the $i$th element of $\mu^k \in \mathbb{R}^r$. From this and (12c), (12d), it follows that

$$P_iA = \frac{(\mu_i)^\top}{1 + \sum_{k=1}^r \mu_i^k} P,$$

$$P_i = \frac{(\eta_i)^\top}{1 + \sum_{k=1}^r \mu_i^k} F,$$

where the division is permitted since $\sum_{k=1}^r \mu_i^k \geq 0$. Collecting all rows of $P$,

$$PA = HP,$$

$$P = MF,$$  \hfill (13)  \hfill (14)

where $H_{ij} = \mu_{ij}^j/(1 + \sum_{k=1}^r \mu_i^k) \geq 0$, $j = 1 \ldots r$, $M_{ij} = \eta_{ij}^j/(1 + \sum_{k=1}^r \mu_i^k) \geq 0$, $j = 1 \ldots p$; therefore, $H$ and $M$ are non-negative matrices. By Theorem 1, non-negative $H$ and $M$ as in (13) and (14) exist if and only if $P$ satisfies Assumption 4. Therefore, if $P$ does not satisfy Assumption 4, then non-negative $H$ and $M$ satisfying (13) and (14) do not exist and $\mathbb{D}$ is infeasible; by weak duality, we conclude that $\mathbb{P}$, which is feasible, is unbounded.

Now suppose that $P$ satisfies Assumption 4. The primal and dual problems are both feasible. Our goal is to prove that the primal optimum satisfies

$$H(c^* + d^*) + Mg = c^* + d^*,$$  \hfill (15)

which satisfies the final condition in Proposition 1 and establishes that the set $\{x \in \mathbb{R}^n : Px \leq q^*\}$ with $q^* = b^* = c^* + d^*$ is the $(P, r)$-mRPI set. Applying strong duality and complementary slackness to (9a) and (9c),

$$\sum_{i=1}^r \lambda_i^*(c_i^* - P_iA\xi^*) = 0$$

$$\sum_{i=1}^r \nu_i^*(d_i^* - P_i\omega^*) = 0.$$

where $*$ denotes an variable in the optimal solution. Since each term in these sums is non-positive,

$$\lambda_i^*(c_i^* - P_iA\xi^*) = 0,$$

$$\nu_i^*(d_i^* - P_i\omega^*) = 0,$$

for $i = 1, \ldots, r$. Moreover, because (by (12a) and (12b)) $\lambda_i^* > 0$ and $\nu_i^* > 0$, then, at the optimum,

$$c_i^* = P_iA\xi^*,$$

$$d_i^* = P_i\omega^*.$$
Hence,
\[ c_i^* + d_i^* = P_i A \xi^{i*} + P_i \omega^{i*} \]
\[ = \frac{\mu^{i*}}{1 + \sum_{k=1}^{K} \mu_k^{k*}} P_i \xi^{i*} + \frac{\eta^{i*}}{1 + \sum_{k=1}^{K} \mu_k^{k*}} F \omega^{i*} \]
\[ = \frac{1}{1 + \sum_{k=1}^{K} \mu_k^{k*}} \left( \sum_{j=1}^{r} \mu_j^{i*} P_j \xi^{i*} + \sum_{j=1}^{p} \eta_j^{i*} F_j \omega^{i*} \right). \]  

(16)

Now consider the inequality (9d). Suppose \( F \omega^{i*} < g \) for some \( i \in \{1, \ldots, r\} \) (i.e., \( F_j \omega^{i*} < g_j \) for all \( j = 1 \ldots p \)). Complementary slackness implies that \( \eta^{i*} = 0 \) which in turn implies (from (12d), assuming that \( P_i \) is not trivially all zeros) that \( v_i^* = 0 \); but \( v_i^* \geq 1 \) by (12b), which is a contradiction. Hence, there must exist a subset \( K \subset \{1, \ldots, p\} \) of active constraints for which \( F_k \omega^{i*} = g_k \) for \( k \in K \). But for any \( j \notin K \), \( \eta_j^{i*} = 0 \).

Similarly, consider the inequality (9b). By complementary slackness, if \( P_i \xi^{i*} < c^* + d^* \) then \( \mu^{i*} = 0 \). By (12c), this implies that \( A^T P_i^T \lambda^{i*}_j = 0 \). There are two cases to consider: (i) if any elements of \( P_i A \) are non-zero then \( \lambda^{i*}_i = 0 \); (ii) if \( P_i A = 0 \) then \( \lambda^{i*}_i > 0 \) is permitted. We leave case (ii) for now and consider (i) first. \( \lambda^{i*}_i = 0 \) contradicts (12a), which requires \( \lambda^{i*}_i \geq 1 \). Hence, there must exist a subset \( J \subset \{1, \ldots, r\} \) of active constraints for which \( P_j \xi^{i*} = c_j^* + d_j^* \) for \( j \in J \). But for any \( k \notin J \), \( \mu^{i*}_k = 0 \). As a consequence of the preceding arguments, (16) may be re-written as

\[ c_i^* + d_i^* = \frac{1}{1 + \sum_{k=1}^{K} \mu_k^{k*}} \left( \sum_{j \in J} \mu_j^{i*} P_j \xi^{i*} + \sum_{k \in K} \eta_k^{i*} F_k \omega^{i*} \right) \]
\[ = H_i(c^* + d^*) + M_i g \]
where \( H_i \) is the \( i \)th row of \( H \) and \( M_i \) is the \( i \)th row of \( M \). The second line follows because \( H_{ij} = 0 \) for \( j \notin J \) and \( M_{ik} = 0 \) for \( k \notin K \), while \( P_j \xi^{i*} = c_j^* + d_j^* \) for \( j \in J \) and \( F_k \omega^{i*} = g_k \) for \( k \in K \).

Now case (ii). If \( P_i A = 0 \) then \( c_i^* = 0 \). Moreover, \( \lambda^{i*}_i \geq 1 \) is permitted, so the same contradiction is not constructed. Then, however, either \( P_i \xi^{i*} < d^* \), hence \( \mu^{i*} = 0 \), or \( P_i \xi^{i*} = d_j^* \), with \( \mu^{i*}_j \geq 0 \), for \( j \in J \subset \{1, \ldots, r\} \), and \( \mu^{i*}_k = 0 \) for all \( k \notin J \). Either way,

\[ c_i^* + d_i^* = H_i(c^* + d^*) + M_i g \]
as before.
Finally, collecting all rows $i = 1 \ldots r$,

$$H(c^* + d^*) + Mg = c^* + d^*$$

and the result is established. 

\section*{IV. Examples}

We consider the non-autonomous system

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w, \quad (17)$$

with $w \in \mathcal{W} = \{ w \in \mathbb{R}^2 : \|w\|_\infty \leq 0.1 \}$. This is converted to the linear autonomous system (1) by use of a state feedback control law $u = Kx$.

\subsection*{A. Computation of $(P,r)$-mRPI from selected inequalities}

First, we use the feedback matrix $K = [-0.4345, -1.0285]$, corresponding to the infinite-horizon LQR solution with cost matrices $Q = I$ and $R = 1$. Note that in this example the mRPI set is not finitely determined, and therefore an approximation is required.

Figure 1(a) shows the $(P,r)$-mRPI sets generated from $r = 6, 20$ and $48$ inequalities, wherein the $i$th row of $P$ is designed as

$$P_i = \begin{bmatrix} \sin \left( \frac{2\pi(i-1)}{r} \right) \\ \cos \left( \frac{2\pi(i-1)}{r} \right) \end{bmatrix}, \quad (18)$$

\textit{i.e.,} so that $Px \leq 1$ is the $r$-sided regular polygon. Also shown is the outer approximation to the global mRPI, which is itself RPI, computed using the algorithm of [3] and a tolerance $\epsilon = 10^{-4}$. This set, termed the $\epsilon$-mRPI set, is defined by $48$ non-redundant inequalities.

Figure 1(b) shows a similar comparison using $K = [-0.0796, -0.4068]$, obtained as the LQR solution with $Q = I$ and $R = 100$. Now the $\epsilon$-mRPI ($\epsilon = 10^{-4}$) comprises $172$ non-redundant inequalities, while the $(P,r)$-mRPI sets computed using the proposed method are shown for $r = 20, 60$ and $172$, again using (18) for $P$.

Table I compares the computation times and number of operations for computing the $(P,r)$-mRPI with those for obtaining the $\epsilon$-outer approximation using the algorithm of [3]. For the latter, the Multi-Parametric Toolbox v3.0 [8] was used for set operations, with CPLEX 12.6 as the LP solver for support function calculations. For the $(P,r)$-mRPI set computations (\textit{i.e.,} solving the
Comparison was also made with the iterative procedure of [5] for computing the $(P, r)$-mRPI. The iterative procedure is

$$q_{k+1} = c(q_k) + d$$

with $q_0 = 0$

for which $q_k \to q^*$ as $k \to \infty$. This was implemented in MATLAB using the MPT v3.0 [8] for support function calculations (with CPLEX 12.6 as underlying LP solver). The function $c(\cdot)$ was evaluated element by element at each iteration; that is, as $r$ separate support function calculations. For the simplest case considered of $K = [-0.4345, -1.0285]$ and $r = 6$ (the first row of Table I),
the number of iterations to convergence (of $|q_{k+1} - q_k|$ to within a chosen tolerance of $10^{-6}$) was 34, which included the solving of 238 LPs and took a mean total time of 1.7 seconds. At the other end of the scale, for the most difficult problem considered ($K = [-0.0796, -0.4068]$ and $r = 172$), the iterative procedure required 70 iterations, the solving of over 12000 LPs, and took, on average, 90 seconds. While these times can, of course, be shortened by using optimized code, the intention here is merely to report the times obtained using standard computational tools.

B. Re-computing the $(P, r)$-mRPI set given $P$

An interesting use of the method is when an RPI set for the system is available, but is desired to be re-computed or modified; for example, if the disturbance set changes. Potential applications of this include “plug-and-play” tube-based approaches to distributed MPC, wherein a dynamic subsystems’ disturbance set evolves over time as other subsystems are added to and removed from the system of coupled subsystems [9]; in such situations, one needs a new RPI set that takes into account the latest disturbance set. One could re-compute from scratch a new RPI set, but it may be advantageous, in the interests of computation time, to modify an existing RPI set instead. In the context of the approach proposed here, the $P$ matrix of the known RPI set may be used as a basis for computing the new RPI set.

| TABLE I | COMPARISON OF COMPUTATION TIMES AND OPERATIONS FOR $(P, r)$-mRPI AND $\epsilon$-mRPI SETS. |
|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| LPs solved | Minkowski sums | Mean time (s) | |
| $K = [-0.4345, -1.0285]$ | | |
| $r = 6$ | 1 | 0 | 0.005 |
| $r = 20$ | 1 | 0 | 0.007 |
| $r = 48$ | 1 | 0 | 0.019 |
| $\epsilon$-mRPI [3] ($r = 48$) | 369 | 11 | 2.9 |
| $K = [-0.0796, -0.4068]$ | | |
| $r = 20$ | 1 | 0 | 0.008 |
| $r = 60$ | 1 | 0 | 0.036 |
| $r = 172$ | 1 | 0 | 0.30 |
| $\epsilon$-mRPI [3] ($r = 172$) | 3250 | 42 | 25 |
For the system (17) with $K = [-0.4345, -1.0285]$ and $\mathbb{W} = \{ w \in \mathbb{R}^2 : |w|_\infty \leq 0.1 \}$, the $P$ matrix is obtained as that of the $\epsilon$-mRPI set. For $\epsilon = 10^{-4}$, this comprises 48 inequalities. Now suppose the disturbance set enlarges to
$$\mathbb{W} = \left\{ w \in \mathbb{R}^2 : \begin{bmatrix} -0.3 \\ -0.4 \end{bmatrix} \leq w \leq \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \right\}$$

Figure 2 shows the $(P, r)$-mRPI and $\epsilon$-mRPI sets based on the new disturbance set, using for the former the $P$ matrix from the old $\epsilon$-mRPI set. The $(P, r)$-mRPI set, computed in 0.03 s using the proposed method, is visually indistinguishable from the new $\epsilon$-mRPI set.

V. CONCLUSIONS

A procedure for computing a polytopic robust positively invariant set for a linear uncertain system has been presented. The method, which requires the solution of a single LP, obtains the an RPI set that is the smallest among those represented by a finite number inequalities with pre-defined normal vectors, and offers an alternative method of computation to the iterative procedure of [5]. Existence and uniqueness of a solution has been established. The practicality of the approach has been demonstrated via examples.

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