A note on fully entangled fraction

Ming-Jing Zhao\textsuperscript{1}, Zong-Guo Li\textsuperscript{2}, Shao-Ming Fei\textsuperscript{1} and Zhi-Xi Wang\textsuperscript{1}

\textsuperscript{1} School of Mathematical Sciences, Capital Normal University, Beijing 100048, People’s Republic of China
\textsuperscript{2} Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, People’s Republic of China

E-mail: zhaomingjingde@126.com (Ming-Jing Zhao), lzgbnu@gmail.com (Zong-Guo Li), feishm@mail.cnu.edu.cn (Shao-Ming Fei) and wangzhx@mail.cnu.edu.cn (Zhi-Xi Wang)

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Abstract
We investigate the general characteristics of the fully entangled fraction for quantum states. The fully entangled fractions of isotropic states and Werner states are analytically computed.

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1. Introduction
Entanglement is a vital resource for some practical applications in quantum information processing such as quantum cryptography, quantum teleportation and quantum computation [1, 2]. One way to characterize the nonclassical property of quantum entanglement is to quantify the entanglement in terms of some measures, for example, entanglement of formation [3], concurrence [4], negativity [5] and geometric measure [6, 7]. However, in fact it is the fully entangled fraction (FEF) that is tightly related to many quantum information processing such as dense coding [8], teleportation [9], entanglement swapping [10] and quantum cryptography (Bell inequalities) [11]. For instance, the fidelity of optimal teleportation is given by the FEF [12–14]. Additionally, the FEF in two-qubit system acts as an index to characterize the nonlocal correlation [15] and one can never determine whether a state is entangled or not through the Dür–Cirac method [16], which is a simple and effective method for examining multiqubit entanglement, if the FEF is less than or equal to $\frac{1}{2}$. The FEF also plays a significant role in deriving two bounds on the damping rates of the dissipative channel [17]. Since the FEF has a clear experimental meaning, an analytic formula for the FEF is of great importance. In [18], an elegant formula for the FEF in two-qubit system is analytically derived by using the method of the Lagrange multiplier. For high-dimensional quantum states the analytical computation of the FEF remains formidable and less results have been known. In [19], the upper bound of the FEF has been estimated.
In this paper, we first present some properties of the FEF and its relations with negativity, concurrence and geometric measure. Then we analytically solve the FEF for some classes of quantum states such as isotropic states and Werner states.

2. Properties of the FEF

The FEF of a density matrix $\rho$ in the $d \otimes d$ Hilbert space is defined by [13, 14]

$$\mathcal{F}(\rho) = \max_{U} \langle \psi^* | U^\dagger \otimes I \rho U \otimes I | \psi^* \rangle,$$  \hspace{1cm} (1)

where $U$ (resp. $I$) is a unitary (resp. identity) matrix and $| \psi^* \rangle = \frac{1}{\sqrt{\sum_{k=1}^{d^2} |k\rangle \langle k|}}$ is the maximally entangled pure state.

Any $d \otimes d$ pure state $|\psi\rangle = \sum_{i,j=1}^{d} a_{ij} |ij\rangle$ can be written in the standard Schmidt form

$$|\psi\rangle = \sum_{i} \lambda_i |ii\rangle,$$  \hspace{1cm} (2)

where the Schmidt coefficients $\lambda_i$, $i = 1, \ldots, d$, satisfy $0 \leq \lambda_0 \leq \cdots \leq \lambda_2 \leq \lambda_1 \leq 1$ and $\sum_{i} \lambda_i^2 = 1$. The FEF of $|\psi\rangle$ has been given in [21]:

$$\mathcal{F}(|\psi\rangle) = \frac{1}{d} \left( \sum_{i} \lambda_i \right)^2.$$  \hspace{1cm} (3)

From equation (3) it can be seen that $|\psi\rangle$ is separable if and only if $\mathcal{F}(|\psi\rangle) = \frac{1}{d}$.

For pure states the FEF has direct relations with some entanglement measures. For instance, due to $\|(|\psi\rangle \langle \psi|)^{T_1}\| = (\sum_{i} \lambda_i)^2$, the negativity [5], $N(\rho) = \frac{\|\rho_{\text{PT}}\|}{\sqrt{d^2 - 1}}$, can be expressed as $N(|\psi\rangle) = \frac{\|\rho_{\text{PT}}\|}{\sqrt{d^2 - 1}}$, where $T_1$ stands for partial transposition with respect to the first space. The geometric measure [7] is defined by $\mathcal{E}(|\psi\rangle) = 1 - \Lambda_{\text{max}}^2(|\psi\rangle)$, where $\Lambda_{\text{max}}^2(|\psi\rangle) = \sup_{|\phi\rangle \in S} |\langle \phi | \psi \rangle|^2$ and $S$ denotes the set of product states. For the pure state $|\psi\rangle$ in equation (2), we have $\Lambda_{\text{max}}^2(|\psi\rangle) = \lambda_1^2$ and $\mathcal{E}(|\psi\rangle) = 1 - \lambda_1^2$. From equation (3), we can get the relation between FEF and geometric measure: $\lambda_{\text{max}}^2 \geq \mathcal{F}$ and $\mathcal{F} \leq d(1 - \mathcal{E})$.

For $d \otimes d$ mixed states, $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$, $\mathcal{F}(\rho)$ has no general analytical formula. It can be shown that

$$\mathcal{F}(\rho) \leq \sum_i p_i \mathcal{F}(|\psi_i\rangle),$$  \hspace{1cm} (4)

since

$$\mathcal{F}(\rho) \leq \sum_i p_i \max_{U_i} \langle \psi_i^* | U_i^\dagger \otimes I | \psi_i \rangle \langle \psi_i | U_i \otimes I | \psi_i^* \rangle = \sum_i p_i \mathcal{F}(|\psi_i\rangle).$$

From equation (4) and the main result in [20], we can obtain a relation between FEF and concurrence for mixed states, $C(\rho) \geq \max \left\{ \sqrt{\frac{1}{d(d - 1)} (d\mathcal{F}(\rho) - 1)}, 0 \right\}$. For two-qubit states, using the relation between the entanglement of formation and the concurrence, one gets the relation between the entanglement of formation and the FEF presented in [12].

Most of the entanglement measures for a mixed state $\rho$ are defined in terms of all possible pure-state decompositions of $\rho$ by convex roof, e.g. for the concurrence $C$, $C(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$. A question one may ask is whether the FEF of a mixed state also has such a property: $\mathcal{F}(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{F}(|\psi_i\rangle)$. The answer is no. As a counterexample, one may consider the $2 \otimes 2$ state $\rho = \frac{1}{2} (|00\rangle \langle 00| + |11\rangle \langle 11|)$. By direct calculation one has $\mathcal{F}(\rho) = \frac{1}{2}$, while for any other decompositions $\{p_i, |\psi_i\rangle\}$ with $|\psi_i\rangle = \alpha_i |00\rangle + \beta_i |11\rangle$, it is impossible to have $\sum_i p_i \mathcal{F}(|\psi_i\rangle) = \frac{1}{2}$. Therefore, $\mathcal{F}(\rho)$ can not be expressed as a convex roof of $\mathcal{F}(|\psi\rangle)$ for any pure states $|\psi\rangle$.
where \(\alpha_i, \beta_i \in \mathbb{C}\) and \(|\alpha_i|^2 + |\beta_i|^2 = 1\). \(\sum_i p_i \mathcal{F}(|\psi_i\rangle) = \frac{1}{d} + \sum_i p_i |\alpha_i\beta_i| > \mathcal{F}(\rho)\). Here we give a condition such that the equality holds in equation (4).

**Theorem 1.** For any \(d \otimes d\) mixed state \(\rho = \sum_{i=1}^n p_i |\psi_i\rangle\langle \psi_i|\), \(n > 1\), \(\mathcal{F}(\rho) = \sum_i p_i \mathcal{F}(|\psi_i\rangle)\) if and only if there exist unitary transformations \(U_1^{(i)}\) and \(U_2^{(i)}\) such that \(U_1^{(i)} \otimes U_2^{(i)} |\psi_i\rangle = \sum_j a_j^{(i)} |j\rangle\) with \(a_j^{(i)} > 0\) and \(U_1^{(i)} e^{i\theta_j} U_2^{(i)*} = e^{i\theta_j} U_1^{(i)} U_2^{(i)*}\), \(1 \leq s, t \leq n\), \(0 \leq \theta_j \leq 2\pi\). For such a state, \(\mathcal{F}(\rho) = \frac{1}{d} \sum_i p_i (\sum_j a_j^{(i)})^2\).

**Proof.** We only need to prove the case \(n = 2\). The cases \(n \geq 3\) can be similarly proved. Assume \(\rho = p_1 |\psi_1\rangle\langle \psi_1| + p_2 |\psi_2\rangle\langle \psi_2|\). By the Schmidt decomposition, there exist unitary matrices \(U_1^{(1)}, U_2^{(1)}, U_1^{(2)}, U_2^{(2)}\) such that \(|\psi_1\rangle = U_1^{(1)} \otimes U_2^{(1)} |\psi_1\rangle = \sum_j a_j^{(i)} |j\rangle\) with \(a_j^{(i)} > 0\), \(i = 1, 2\). We have

\[
\mathcal{F}(\rho) = \max \left( p_1 \mathcal{F}(|\psi_1\rangle) + p_2 \mathcal{F}(|\psi_2\rangle) \right)
\]

Therefore, \(\mathcal{F}(\rho) = p_1 \mathcal{F}(|\psi_1\rangle) + p_2 \mathcal{F}(|\psi_2\rangle)\) if and only if there exists a unitary matrix \(V\) such that

\[
\mathcal{F}(|\psi_1\rangle) = \mathcal{F}(|\tilde{\psi}_1\rangle)
\]

\[
= \langle \tilde{\psi}_1 | V U_1^{(1)} \otimes U_2^{(1)} | \tilde{\psi}_1 \rangle \langle \tilde{\psi}_1 | U_1^{(1)} \otimes U_2^{(1)} | \psi_1 \rangle
\]

\[
= \langle \tilde{\psi}_1 | V U_1^{(1)} \otimes U_2^{(1)} | \tilde{\psi}_1 \rangle \langle \tilde{\psi}_1 | U_1^{(1)} \otimes U_2^{(1)} P_\sigma \rangle
\]

\[
= \langle \tilde{\psi}_1 | U_1^{(1)} \otimes U_1^{(1)*} \otimes I | \tilde{\psi}_1 \rangle \langle \tilde{\psi}_1 | U_1^{(1)} \otimes U_2^{(1)*} \otimes I P_\sigma \rangle
\]

where \(P_\sigma = |\psi_1\rangle\langle \psi_1|\) and \(A \otimes I P_\sigma = I \otimes A^T P_\sigma\). Furthermore, \(\mathcal{F}(|\psi_2\rangle) = \mathcal{F}(|\tilde{\psi}_2\rangle) = |\langle \tilde{\psi}_2 | U_2^{(2)} V U_2^{(2)*} \otimes I | \psi_1 \rangle|^2\). On the other hand, \(\mathcal{F}(|\psi_1\rangle) = \frac{1}{d} \sum_j |a_j^{(i)}|^2\) and \(\mathcal{F}(|\psi_2\rangle) = \frac{1}{d} \sum_j |a_j^{(i)}|^2\). \(\mathcal{F}(|\psi_1\rangle)\) reaches the maximum when \(U_1^{(1)} V U_2^{(1)*} = e^{i\theta_j} I\), i.e. \(U_1^{(1)} U_2^{(1)*} = e^{-i\theta_j} V\). Similarly, we have \(U_1^{(2)} U_2^{(2)*} = e^{-i\theta_j} V\) and \(U_1^{(1)} U_2^{(1)*} = e^{i(\theta_2 - \theta_1)} U_1^{(2)} U_2^{(2)*}\). The value of FEF can be obtained from equation (3). \(\square\)

Theorem 1 gives the condition that the FEF fulfills the convex-roof measure. Besides, if one interprets the FEF of a state \(\rho\) as the distance between \(\rho\) and maximally entangled states, then the larger the FEF, the closer they are. Although there are infinite maximally entangled states, the one \(U \otimes I |\psi_1\rangle\) which reaches the maximum of equation (1) is the closest maximally entangled state to \(\rho\). Theorem 1 also states when the closest maximally entangled state to two different pure states are the same. As an example, we consider the mixed state \(\rho = \sum_{i=1}^d p_i |i, \sigma(i)\rangle\langle i, \sigma(i)|\), where \(\sigma\) denotes the permutation of \((1, 2, \ldots, d)\). For this state, the theorem applies and we have \(\mathcal{F}(\rho) = \sum_{i=1}^d p_i \mathcal{F}(|\psi_i\rangle) = \frac{1}{d} \) with \(|\psi_i\rangle = |i, \sigma(i)\rangle\). The distance between \(|\psi_i\rangle\) and maximally entangled states is \(\frac{1}{d}\). The closest maximally entangled state to \(|\psi_0\rangle\) is \(|\psi_0\rangle = \frac{1}{d} \sum_{i=1}^d |i, \sigma(i)\rangle |\langle \psi_0 | \psi_i \rangle|^2 = \frac{1}{d}, i = 1, \ldots, d\).

From equations (3) and (4) one can obtain that for any \(d \otimes d\) mixed state \(\rho\), \(\frac{1}{d^2} \leq \mathcal{F}(\rho) \leq 1\). \(\mathcal{F}(\rho) = 1\) if and only if \(\rho\) is a maximally entangled pure state. \(\mathcal{F}(\rho) = \frac{1}{d}\) if and only if \(\rho\) is the maximally mixed state, i.e. \(\rho = \frac{1}{d^2} I\).
Proof. For any $d \otimes d$ mixed state $\rho$, we assume that $\rho = \sum_{i=1}^{d^2} \lambda_i |\phi_i\rangle \langle \phi_i|$ is the spectrum decomposition such that $\sum_{i=1}^{d^2} \lambda_i = 1$, $0 \leq \lambda_i \leq 1$, and $\{|\phi_i\rangle\}_{i=1}^{d^2}$ are the normalized orthogonal eigenvectors in the $d \otimes d$ Hilbert space. Then $\mathcal{F}(\rho) = \max_U \langle \psi^* | U^\dagger \otimes I \rho U \otimes I | \psi^* \rangle = \max_{U} \sum_i \lambda_i \langle \psi^* | U^\dagger \otimes I | \phi_i \rangle \langle \phi_i | U \otimes I | \psi^* \rangle$. Set $a_i = \langle \psi^* | U^\dagger \otimes I | \phi_i \rangle \langle \phi_i | U \otimes I | \psi^* \rangle$, which satisfies $0 \leq a_i \leq 1$ and $\sum_{i=1}^{d^2} a_i = 1$ due to the completeness of the eigenvectors $\{|\phi_i\rangle\}$. $\sum_{i=1}^{d^2} \lambda_i a_i = \sum_{i=1}^{d^2} \lambda_i = 1$ becomes an equality if and only if there are only one nonzero coefficient, say, $a_i = 1$ and one nonzero coefficient $\lambda_i = 1$. Therefore $\mathcal{F}(\rho) = 1$ if and only if $\rho$ is the maximally entangled pure state.

On the other hand, the minimum of the function $g(\lambda_i, a_i) = \sum_{i=1}^{d^2} \lambda_i a_i$ is $\frac{1}{d^2}$, which gives the right-hand side of equation (5). If we take $\psi = |\lambda \rangle |\phi_1\rangle + |\beta \rangle |\phi_2\rangle$, then the FEF reaches the upper bound of equation (5). Similarly, taking into account of $|\phi_1\rangle = \frac{\sqrt{2}}{\sqrt{3}} |0\rangle - \frac{\sqrt{2}}{\sqrt{3}} |1\rangle$ and $|\phi_2\rangle = \frac{\sqrt{2}}{\sqrt{3}} |0\rangle + \frac{\sqrt{2}}{\sqrt{3}} |1\rangle$, one gets the left-hand side of equation (5).

For example, let $|\phi_1\rangle = |00\rangle$, $|\phi_2\rangle = |11\rangle$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$; then the FEF of $|\psi\rangle$ reaches the upper bound of equation (5). If we take $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\phi_2\rangle = |11\rangle$ and $|\psi\rangle = |00\rangle$, then the FEF of $|\psi\rangle$ reaches the lower bound of equation (5). Equation (5) can also be generalized to the case of superposition with more than two components: for $|\psi\rangle = \frac{1}{2} (|\phi_1\rangle + \cdots + |\phi_m\rangle), we have max, \{ |\alpha_i| \mathcal{F}^{|\psi\rangle} (|\phi_i\rangle) - \sum_{j \neq i} |\alpha_j| \mathcal{F}^{|\psi\rangle} (|\phi_j\rangle), \} \leq |\gamma| \mathcal{F}^{|\psi\rangle} (|\psi\rangle) \leq \min \{ \sum_i |\alpha_i| \mathcal{F}^{|\psi\rangle} (|\phi_i\rangle), 1 \}.$

Corollary 3. For any $d \otimes d$ mixed state $\rho$, it satisfies $0 \leq \mathcal{E}(\rho) \leq \frac{d^2-1}{d^2}$. $\mathcal{E}(\rho) = 0$ if and only if $\rho$ is a separable state. $\mathcal{E}(\rho) = \frac{d^2-1}{d^2}$ if and only if $\rho$ is a maximally entangled pure state.

We have studied some properties related to the FEF. Before we compute analytically the FEF for isotropic states and Werner states, we investigate another property that is similarly studied for entanglement of formation, negativity, concurrence, geometric measure and q-squashed entanglement [22].

Theorem 4. For the two given pure states $|\phi_1\rangle$ and $|\phi_2\rangle$, the FEF of their superposition $|\psi\rangle = \frac{1}{\sqrt{2}} (|\phi_1\rangle + |\phi_2\rangle)$ satisfies

$$\max \{ |\alpha| \mathcal{F}^{|\phi_1\rangle} (|\phi_1\rangle) - |\beta| \mathcal{F}^{|\phi_2\rangle} (|\phi_2\rangle), \sqrt{\frac{1}{d^2}} \} \leq |\gamma| \mathcal{F}^{|\phi_1\rangle} (|\psi\rangle) \leq \min \{ |\alpha| \mathcal{F}^{|\phi_1\rangle} (|\phi_1\rangle) + |\beta| \mathcal{F}^{|\phi_2\rangle} (|\phi_2\rangle), 1 \}.$$  (5)

Proof. By the definition of FEF we have

$$\mathcal{F}(|\psi\rangle) = \frac{1}{\sqrt{2}} \max_{U} \langle \psi^* | U^\dagger \otimes I (|\phi_1\rangle + |\phi_2\rangle) (|\phi_1\rangle + |\phi_2\rangle) U \otimes I | \psi^* \rangle$$

$$\leq \frac{1}{\sqrt{2}} (|\alpha|^2 \mathcal{F}^{|\phi_1\rangle} (|\phi_1\rangle) + |\beta|^2 \mathcal{F}^{|\phi_2\rangle} (|\phi_2\rangle) + 2 |\alpha\beta| \sqrt{\mathcal{F}^{|\phi_1\rangle} (|\phi_1\rangle) \mathcal{F}^{|\phi_2\rangle} (|\phi_2\rangle)})$$

$$= \frac{1}{\sqrt{2}} (|\alpha| \mathcal{F}^{|\phi_1\rangle} (|\phi_1\rangle) + |\beta| \mathcal{F}^{|\phi_2\rangle} (|\phi_2\rangle))^2,$$

which gives the right-hand side of equation (5).

Similarly, taking into account of $|\phi_1\rangle = \frac{\sqrt{2}}{\sqrt{3}} |0\rangle - \frac{\sqrt{2}}{\sqrt{3}} |1\rangle$ and $|\phi_2\rangle = \frac{\sqrt{2}}{\sqrt{3}} |0\rangle + \frac{\sqrt{2}}{\sqrt{3}} |1\rangle$, one gets the left-hand side of equation (5).
3. FEF for some classes of mixed states

Generally for mixed states it is rather difficult to obtain analytical formulae for entanglement measures and the FEF. Nevertheless for some special quantum states, the elegant results have been derived. For instance, for the isotropic state, entanglement of formation [23], concurrence [24] and geometric measure [7] have been calculated explicitly. For the Werner state, concurrence [25] and geometric measure [7] have also been investigated. Now we analytically calculate the FEF for such well-known mixed states.

Isotropic states. Isotropic states [21] are a class of $U \otimes U^*$ invariant mixed states in the $d \otimes d$ Hilbert space:

$$\rho_{iso}(f) = \frac{1-f}{d^2-1} I + \frac{d^2 f - 1}{d^2-1} |\psi^+ \rangle \langle \psi^+|,$$

with $f = \langle \psi^+ | \rho_{iso}(f) | \psi^+ \rangle$ satisfying $0 \leq f \leq 1$. These states are shown to be separable if and only if they are PPT, i.e. $f \leq \frac{1}{d^2}$. They can be distilled if they are entangled, which means $f > \frac{1}{d^2}$ [21].

By definition, the FEF is given by

$$\mathcal{F}(\rho_{iso}(f)) = \frac{1-f}{d^2-1} + \max_v \frac{d^2 f - 1}{d^2-1} |\langle \psi^+ |U \otimes I | \psi^+ \rangle|^2 = \frac{1-f}{d^2-1} + \max_v \frac{d^2 f - 1}{d^2-1} |1 - \text{tr}U|^2.$$

If $\frac{d^2 f - 1}{d^2-1} > 0$, i.e. $f > \frac{1}{d^2}$, we have $\mathcal{F}(\rho_{iso}(f)) = \frac{1-f}{d^2-1} + \frac{d^2 f - 1}{d^2-1} = f$. The maximum is attained by choosing $U = I$. If $\frac{d^2 f - 1}{d^2-1} < 0$, i.e. $f < \frac{1}{d^2}$, we get $\mathcal{F}(\rho_{iso}(f)) = \frac{1-f}{d^2-1} + \frac{d^2 f - 1}{d^2-1} \text{min}_u \frac{1}{d^2} |\text{tr}U|^2 \leq \frac{1-f}{d^2-1}$. In fact, if we choose $U = \sum_{i\neq j} |i\rangle \langle j|$, then the inequality becomes an equality. If $\frac{d^2 f - 1}{d^2-1} = 0$, i.e. $f = \frac{1}{d^2}$, we have $\mathcal{F}(\rho_{iso}(f)) = \frac{1}{d^2}$. Therefore, we get the FEF for isotropic states:

$$\mathcal{F}(\rho_{iso}(f)) = \begin{cases} f, & \frac{1}{d^2} \leq f \leq 1; \\ \frac{1-f}{d^2-1}, & 0 \leq f < \frac{1}{d^2}. \end{cases}$$

According to [13], the fidelity $f_{\text{max}}$ of optimal teleportation via the state $\rho$ attainable by means of trace-preserving local quantum operations and classical communication (LOCC) is equal to $f_{\text{max}}(\rho) = \frac{\mathcal{F}(\rho)}{\mathcal{F}(I)}$. If $\mathcal{F}(\rho) > \frac{1}{d^2}$, then the state $\rho$ is said to be useful for teleportation. Hence, all entangled isotropic states are useful in quantum teleportation.

Werner states. Werner states [26] are a class of $U \otimes U$ invariant mixed states in the $d \otimes d$ Hilbert space:

$$\rho_{w}r\text{tern}(f) = \frac{d-f}{d^3-d} I + \frac{df-1}{d^3-d} V,$$

where $V = \sum_{i,j=1}^{d} |ij\rangle \langle ji|$ and $f = \langle \psi^+ | \rho_{w}r\text{tern}(f) | \psi^+ \rangle$, $-1 \leq f \leq 1$. These states are shown to be separable if and only if they are PPT ($f \geq 0$).

The FEF of the Werner state is given by

$$\mathcal{F}(\rho_{w}r\text{tern}(f)) = \frac{d-f}{d^3-d} + \max_v \frac{df-1}{d^3-d} |\langle \psi^+ |U^\dagger \otimes I V U \otimes I | \psi^+ \rangle|$$

$$= \frac{d-f}{d^3-d} + \max_v \frac{df-1}{d^3-d} \sum_{kl} |k| U^\dagger |l\rangle \langle k| U |l\rangle$$

$$= \frac{d-f}{d^3-d} + \max_v \frac{df-1}{d^3-d} \text{tr}(UU^\dagger).$$
(i) If \( df - 1 > 0 \), since \( UU^* \) is unitary,

\[
F(\rho_{\text{Wer}}(f)) = \frac{d - f}{d^3 - d} + \frac{df - 1}{d^3 - d} = \frac{f + 1}{d(d + 1)},
\]

which corresponds to the case \( U = I \).

(ii) If \( df - 1 < 0 \) and \( d \) is even, we get

\[
F(\rho_{\text{Wer}}(f)) = \frac{d - f}{d^3 - d} - \frac{df - 1}{d^3 - d} = \frac{1 - f}{d(d - 1)},
\]

which can be attained by choosing \( U = A_{2 \times 2} \otimes I_{\frac{d}{2} \times \frac{d}{2}} \) with \( A_{2 \times 2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

(iii) For the case of \( df - 1 < 0 \), and \( d \) is odd, one has

\[
F(\rho_{\text{Wer}}(f)) = \frac{d - f}{d^3 - d} + \frac{df - 1}{d^3 - d} - \frac{d + 2}{d} = \frac{d^2 - d^2 f + df + d - 2}{d^2(d^2 - 1)}.
\]

(iv) If \( df - 1 = 0 \), i.e. \( f = \frac{1}{d} \), \( F(\rho_{\text{Wer}}(f)) = \frac{1}{d} \).

Therefore, we get the FEF for Werner states:

\[
F(\rho_{\text{Wer}}(f)) = \begin{cases} 
\frac{f + 1}{d(d + 1)}, & \frac{1}{d} \leq f \leq 1, \\
\frac{1 - f}{d(d - 1)}, & -1 \leq f < \frac{1}{d}.
\end{cases}
\]

if \( d \) is even, and

\[
F(\rho_{\text{Wer}}(f)) = \begin{cases} 
\frac{f + 1}{d(d + 1)}, & \frac{1}{d} \leq f \leq 1, \\
\frac{d^2 - d^2 f + df + d - 2}{d^2(d^2 - 1)}, & -1 \leq f < \frac{1}{d}.
\end{cases}
\]

if \( d \) is odd. Hence, this formula states that there exist entangled Werner states which are not useful for teleportation.

4. Conclusions

We have explored some characteristics of the FEF and analytically computed the FEF of several well-known classes of quantum mixed states. These results complement the previous ones in this subject and may give rise to a new application to the quantum information processing.

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