Cyclic symmetry $C_N$ is gauged in such a way that the local parametrization is provided by a Lie group: matter fields are in irreducible representations of $C_N$ while gauge fields are in the adjoint representation of a Lie group, hence “hybrid”. Allowed simple Lie groups are only $\text{SO}(2)$ for $N = 2$, $\text{SU}(3)$ for $N = 3$, and $\text{SU}(2)$ for all $N$. The implication of the local discrete symmetry $C_N$ is evident as the ratio of the coupling constant to the usual gauge theory one of the parametrization Lie group is given by that of the length between any two vertices of a regular $N$-polygon to the radius of the circumcircle: $2 \sin(n\pi/N)$, $n \in \mathbb{Z}_N$.

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Introduction

The role of continuous symmetries in Nature is very evident as the final loophole is closed with the discovery of Higgs. On the contrary, the role of discrete symmetries still remains as an elusive mystery. While many argue for the phenomenological roles of global discrete symmetries in the low energy world, there is a conflicting argument that no global discrete symmetry is allowed in Nature due to the quantum gravity effects\[1\]\[2\].

Discrete symmetries usually appear as global symmetries, that is, the transformations, being constant, do not depend on the spacetime. Nevertheless, we often use the terminologies like “local discrete symmetry” or “discrete gauge symmetry”. What it means is that the actual discrete transformations are still constant and the same as in the global cases, but the distinction rises if we claim that only gauge invariant objects under this discrete symmetry are physically observable as in the case of continuous gauge symmetry\[4\]. In most cases, these discrete gauge symmetries are inherited from continuous gauge symmetries after spontaneous symmetry breaking. So, this type of models needs to be embedded into a continuum theory to justify its local nature of discrete symmetry, and any gauge fields appearing are those of the continuous gauge symmetry, albeit there is an extra constraint on coupling constants due to the discreteness of the remaining symmetry\[4\].

In this paper, we will propose another way of incorporating local discrete symmetries: matter fields belong to irreducible representations of a discrete group, while gauge fields are associated with a Lie group. Hence, we call it “Hybrid Gauge Theory”. For this, we will introduce a new mathematical object, which we call “Group Family”, to parametrize representations of a discrete group, turning into what can be treated to be local. In short, a group family is a parametrized family of a discrete group, and we will define in detail momentarily. Although this group family is not strictly a group, it suffices to define a legitimate gauge theory.

For specific examples, we will only consider the cyclic group, $C_N$, cases. However, it can be generalized for other discrete groups.

Group Family

The basic idea stems from the fact that representations of a discrete group are not unique. For a given representation of a discrete group $G = \{a_0, a_1, \cdots, a_n\}$, $a_0 = 1$, there exist representations $G = \{g_0, g_1, \cdots, g_n\}$ such that they are related by a similarity transformation

\[1\]One way out of this paradox is to treat the low energy discrete symmetries as emergent\[3\].
as
\[ g_i = U^{-1} a_i U, \quad i = 0, 1, \ldots, n, \] (1)
which the identity element also trivially satisfies. Then we can parametrize the representations in terms of parameters of \( U \) as
\[ g_i(\theta_1, \ldots, \theta_\alpha) = U^{-1}(\theta_1, \ldots, \theta_\alpha) a_i U(\theta_1, \ldots, \theta_\alpha). \] (2)

Note that \( G \) is a group for fixed \( \theta \), but not a group for varying \( \theta \) because, for \( g_1(U_1) \) and \( g_2(U_2) \),
\[ g_1(U_1)g_2(U_2) = U_1^{-1} a_1 U_1 U_2^{-1} a_2 U_2 \notin G. \] (3)
So we will use a notation \( G[K(N_F)] \) to identify the object
\[ G[K(N_F)] = \langle g_i | g_i = U^{-1} a_i U, a_i \in G, U \in K(N_F) \rangle \] (4)
and call it a “group family”, i.e. a family of discrete group \( G \) parametrized by a Lie group \( K(N_F) \), where \( N_F \) is the number of copies of matter fields customarily called “flavors”. \( g_i \)'s with different \( \theta \)'s are related as
\[ g_i' = U'^{-1} g_i U', \quad \text{for} \quad U' \in K(N_F). \] (5)

To obtain a local symmetry, all we need is to assign different gauge parameters at different spacetime points. Note that local group elements at different spacetime points do not have to be related by the same group transformations. Therefore, as long as \( \theta(x) \) varies continuously over the spacetime, we have differentiable local discrete transformations. Then we can define a gauge theory with a local discrete symmetry in terms of these parametrizations. Ideally, the number of flavors \( N_F \) could be identified as that of irreducible representations, however, other cases are also interesting so that we will consider as well. \( N_F \) also controls the parametrization Lie group.

Note that \( g^\dagger g = 1 \) implies that \( U^\dagger U = 1 \) and \( a^\dagger a = 1 \) so that the parametrization group \( K(N_F) \) is necessarily an orthogonal or a unitary group.

**Warm-up: \( C_2[SO(2)] \)**

Consider the simplest discrete group \( C_2 = \{1, a\} \), a cyclic group of order two, which has just one nontrivial group element, \( a \), in addition to the identity. There are two irreducible
representations, so it is natural to take two flavor case \((N_F = 2)\) and a two dimensional representation of \(a\) will be sufficient such that
\[
a = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
where we will use the convenient Pauli matrix notation from now on. In the \(C_2\) case, \(a^\dagger = a\), which is a useful property for \(a\) to be a symmetry generating transformation. The similarity transformation \(U\) in this case should just have one parameter to sustain the abelian nature of \(C_2\) and the corresponding \(g\) should be real. That naturally fixes \(U\) to be the SO(2) rotational matrix such that
\[
U(\theta) = e^{i\sigma_2 \theta} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.
\]
(7)
Then
\[
g = U^{-1} a U = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.
\]
(8)
Note that \(g^2 = 1\), \(g^\dagger = g\) and \(\text{Det} \, g = -1\). Then \(C_2[SO(2)] = \{1, g(\theta)\}\) in our notation.

Now consider two scalar matter fields \(\Phi \equiv (\phi_1, \phi_2)^T\), then \(C_2[SO(2)]\) acts as
\[
\Phi' = g\Phi.
\]
(9)
such that the bilinear form is invariant as
\[
\Phi'^T \Phi' = \Phi^T g^2 \Phi = \Phi^T \Phi.
\]
(10)
In this parametrization, the gauging can be achieved by demanding
\[
D'_\mu \Phi' = gD_\mu \Phi,
\]
(11)
where
\[
D_\mu \equiv \partial_\mu - iQA_\mu \sigma_2
\]
(12)
such that \(Q = \zeta q\) with \(\zeta \neq 1\) for unit charge \(q\) will indicate the nontrivial charge condition matching the discrete symmetry. Using
\[
g\sigma_2 g = -\sigma_2,
\]
(13a)
\[
g\partial_\mu g = i2\partial_\mu \theta \sigma_2,
\]
(13b)
we can obtain the transformation rule for the gauge field \(A_\mu\) as
\[
A'_\mu = -A_\mu - \frac{2}{Q} \partial_\mu \theta = -A_\mu - \frac{1}{q} \partial_\mu \theta,
\]
(14)
where the charge $Q$ now should be constrained with $\zeta = 2$ as $Q = 2q$. This gauge transformation property is something new because of the minus sign in front of $A_\mu$. Note that eq.(13b) is the reason why we choose $A_\mu$ in the representation $\sigma_2$. Even if we choose $a = \sigma_1$, eq.(12) and eq.(14) remain the same.

Together with eq.(14) that shows how the gauge field transforms under $C_2[SO(2)]$, eq.(11) implies that

$$\mathcal{L}[\Phi] = -\frac{1}{4} F^2 + (D_\mu \Phi)^T (D_\mu \Phi) - V(\Phi^T \Phi)$$

(15)

is invariant under the group family $C_2[SO(2)]$. This can be easily generalized to the fermionic case with the same covariant derivative eq.(12) with two fermion flavors $\Psi = (\psi_1, \psi_2)^T$.

Having constructed the lagrangian, we cannot help but notice that it also has the usual $SO(2)$ gauge symmetry

$$\Phi' = e^{i2\sigma_2 \theta} \Phi,$$  

(16a)

$$A'_\mu = A_\mu - \frac{1}{4} \partial_\mu \theta.$$  

(16b)

So we need to elaborate what is going on.

**Compared with $SO(2)$, what is the difference?**

Since the local parametrization is mainly due to $SO(2)$, one may wonder if $C_2[SO(2)]$ gauge theory is anything different from $SO(2)$ gauge theory. After all, the $C_2[SO(2)]$ invariant lagrangian has $SO(2)$ gauge symmetry, too. To see the difference we need to analyze the $C_2[SO(2)]$ gauge transformation more carefully.

Most of all, $C_2[SO(2)]$ has a peculiar property unlike $SO(2)$. Compared with $SO(2)$ in which gauge transformations satisfy

$$U(\theta_1)U(\theta_2) = U(\theta_2)U(\theta_1) = U(\theta_1 + \theta_2) \in SO(2),$$

(17)

in the $C_2[SO(2)]$ case a multiplication of two arbitrary group elements does not result in another group element, unless two gauge parameters are the same:

$$g(\theta_1)g(\theta_2) = U(2\theta_1 - 2\theta_2) \notin C_2[SO(2)] \text{ if } \theta_1 \neq \theta_2.$$  

(18)

This is because there is only one $g(\theta)$ at a given spacetime point. Elements with different gauge parameters are related in terms of $SO(2)$ such that

$$g(\theta_2) = U^{-1}(\theta_2 - \theta_1)g(\theta_1)U(\theta_2 - \theta_1).$$

(19)
Because of this, even though SO(2) is not a subgroup of C_2[SO(2)], it coincidentally allows the SO(2) gauge field to take the role of the gauge field as well. So, in some sense, the local C_2[SO(2)] is a partially augmented group family of global C_2 by local SO(2). However, the distinction is mostly in the coupling constants of the lagrangian: In the SO(2) gauge theory, Q = 2q is just a choice, while in the C_2[SO(2)] gauge theory, it is required.

Another way to see the main difference between C_2[SO(2)] and SO(2) gauge transformations, let us factorize the nontrivial element of C_2[SO(2)], from eqs. (7), (8), as
\[ g(\theta) = \sigma_3 U(2\theta). \] (20)
This does not mean C_2[SO(2)] = \{1, g(\theta)\} is isomorphic to O(2) = C_2 \times SO(2) because SO(2) is not a subgroup of C_2[SO(2)]. Now, the anatomy of the gauge field transformation reveals that
\[ A_\mu \rightarrow -A_\mu \stackrel{SO(2)}{\longrightarrow} -A_\mu - \frac{1}{q} \partial_\mu \theta. \] (21)
Under just SO(2), the gauge field transforms as \[ A'_\mu = A_\mu - \frac{1}{q} \partial_\mu \theta \] such that, when the gauge parameter \( \theta \) is constant, the gauge field does not change, while in the C_2[SO(2)] case it flips sign as \( A_\mu \rightarrow -A_\mu \). The key difference is that C_2 acts nontrivially on the gauge field. But the gauging leads to a part of O(2) gauge theory\[6\] except, in addition, we have a nontrivial charge condition on matter fields: \( Q = 2q \).

One may wonder that \( A_\mu \rightarrow -A_\mu \) may lead to the inconsistency of the sign of the charge. But we believe this is not a correct interpretation. The correct physics is always to consider the matter couplings. It is a “flavor” symmetry, so it should not mean anything without matter couplings. Since matter-gauge interaction is invariant, there is no ambiguity of the sign of the charge. So, unlike SO(2) or U(1) gauge theories, the field strength is no longer a gauge invariant observable (nor is the case of non-abelian gauge fields), but this is all right because any phenomenon of gauge theory is only observable via matter-gauge interactions.

**Generic Structure for Cyclic Groups C_N**

For arbitrary cyclic groups C_N, we need matter fields transforming as
\[ \Phi' = g\Phi, \quad \Phi'^\dagger = \Phi^\dagger g^\dagger, \] (22)
and the hermitian conjugate satisfies \( a^\dagger = a^{N-1} \) so that \( a^N = 1 = aa^\dagger \), where it is sufficient to use the generator of C_N
\[ a = \text{diag}(1, \omega_N, \cdots, \omega_N^{N-1}), \quad \omega_N \equiv e^{i2\pi N} \text{ such that } \omega_N^N = 1. \] (23)
In the \( C_2 \) case, \( g^\dagger = g \). Infinitesimally, for \( U = \exp(i\theta A_T) = 1 + i\theta A_T + \mathcal{O}(\theta^2) \), where \( T_A \)'s are generators of parametrization Lie group \( K(N_F) \), \( g \) reads

\[
g = a + i\theta_A [a, T_A] + \text{h.o.} \tag{24}
\]

Now we can immediately observe the fact that diagonal generators and off-diagonal generators behave differently because \([a, T_i] = 0\), while \([a, T_a] \neq 0\), where the subscript “i” labels diagonal generators and “a” the off-diagonal ones. Particularly, note that only off-diagonal generators show up at the leading order of \( \theta \).

The gauge fields transform as

\[
\Sigma_a A_\mu^a + T_i A_\mu^i = g(\Sigma_a A_\mu^a + T_i A_\mu^i - \frac{i}{\zeta} \partial_\mu)g^{-1}, \tag{25a}
\]

\[
\Sigma_a F_{\mu\nu}^a + T_i F_{\mu\nu}^i = g(\Sigma_a F_{\mu\nu}^a + T_i F_{\mu\nu}^i)g^{-1}. \tag{25b}
\]

To find the representations for gauge fields, generalizing eq.(13b), we need

\[
g \partial_\mu g^{-1} = i\partial_\mu \theta_a [a^{-1}, T_a] + \text{h.o.} \tag{26}
\]

Then, with comparing eq.(26) to eq.(25a), it is natural to define

\[
\Sigma_a \equiv \frac{1}{\zeta_a} [a^{-1}, T_a] a = \frac{1}{\zeta_a} a^{-1} [T_a, a] \text{ for } [T_a, a] \neq 0 \tag{27}
\]

with suitable normalizations \( 1/\zeta_a \) to make \( \text{Tr} \Sigma_a^2 = \text{Tr} T_a^2 \) and \( \zeta_a \) takes the role of the charge condition. This is compatible with eq.(13b) for \( \zeta = 2 \) in the \( C_2 \) case because \( U = \exp(i\theta \sigma_2) \), \( a = \sigma_3 \), and

\[
\Sigma = \frac{1}{2} [\sigma_3, \sigma_2] \sigma_3 = -\sigma_2. \tag{28}
\]

\( \Sigma_a \)'s do not form a closed algebra because \([\Sigma_a, \Sigma_b] \) produces \( T_i \), but they do with \( T_i \)'s added. Then the same lagrangian given in terms of \( \Phi \) and \( A_\mu \) would have a usual local gauge symmetry \( K(N_F) \) generated by \( \{\Sigma_a, T_i\} \), but it is different from the \( C_N[K(N_F)] \) symmetry. In other words, there are two different local gauge symmetry for this type of lagrangians, however certain constraints are only evident in the \( C_N[K(N_F)] \) symmetry, as we will see.

With eq.(27), we can now obtain

\[
g = a (1 - i\zeta_a \theta_a \Sigma_a) + \text{h.o.}, \tag{29a}
\]

\[
g \Sigma_c g^{-1} = a \Sigma_c a^{-1} + i\zeta_b \theta_b [\Sigma_c, \Sigma_b] a^{-1} + \text{h.o.}, \tag{29b}
\]

\[
g T_i g^{-1} = T_i + i\zeta_b \theta_b [T_i, \Sigma_b] a^{-1} + \text{h.o.} \tag{29c}
\]
and eq. (29a) becomes, with introducing short-hand notations,

$$A_\mu' = a \left( A_\mu + \frac{1}{Q} [D_\mu, \zeta \theta] \right) a^{-1} + \text{h.o.}, \quad \zeta \theta \equiv \zeta_a \theta_a \Sigma_a. \quad (30)$$

with

$$D_\mu \equiv \partial_\mu + iQ A_\mu, \quad A_\mu \equiv \Sigma_a A_\mu^a + T_i A_\mu^{i\nu}. \quad (31)$$

In the leading order of $\theta$, only gauge parameters for off-diagonal generators appear, but the missing $\theta_i$ will appear at one higher order, so there are still complete gauge parameters. Note that, to preserve the original periodicity of $\theta$’s for $U \in K(N_F)$, $\theta$’s should be still the gauge parameters for $C_N[K(N_F)]$, hence $Q$ should cancel $\zeta$’s. This is possible only if $\zeta_a$’s are the same for all generators such that

$$Q = \zeta g, \quad \zeta_a = \zeta \text{ for all } a, \quad (32)$$

where $g$ is the coupling constant of the usual gauge theory with $\{T_A\}$ for a Lie group $K(N_F)$. $\zeta$ is the ratio of coupling constant of $C_N[K(N_F)]$ to that of the gauge theory of the parametrization Lie group $K(N_F)$. So, $\zeta \neq 1$ indicates that the coupling constant is not the same as that of the gauge theory of $K(N_F)$. This is also evident in eq. (29a). This also implies that the matter fields carry certain charges w.r.t. a U(1) subgroup generated by a diagonal generator due to the discrete nature of the symmetry, similar to the $C_2[SO(2)]$ case. Next, we will compute these charge conditions explicitly to find out a rather interesting geometrical origin.

### Geometric Charge Quantization Conditions for $C_N$

To obtain the charge condition for $C_N$, it is sufficient to consider $C_N[SO(2)]$ as we could easily check the fact that off-diagonal generators are grouped as those of SU(2) subsets and they can be paired as $\sigma_2$ and $\sigma_1 = -i\sigma_3\sigma_2$. So the normalization condition of $\Sigma$ in $C_N[SO(2)]$ will be the same as that of $C_N[SU(N)]$ when the conventional prefactor $1/2$ for SU($N$) generators is properly taken into account, which is $\text{Tr} \Sigma_a^2 = \text{Tr} T_a^2 = 1/2$ for $C_N[SU(N)]$.

For now, let

$$a = \begin{pmatrix} 1 & 0 \\ 0 & \omega_N^n \end{pmatrix}, \quad n = \mathbb{Z}_N \setminus \{0\}, \quad \omega_N \equiv e^{2\pi i/N}, \quad (33)$$

and choose group family $C_N[SO(2)]$, then

$$g \partial_\mu g^{-1} = i\zeta_{N,n} \partial_\mu \theta_a \Sigma a^{-1} + \text{h.o.}, \quad (34)$$

where

$$\Sigma = \frac{1}{\zeta_{N,n}} \left( a^{-1} \sigma_2 a - \sigma_2 \right) = \frac{1}{\zeta_{N,n}} \begin{pmatrix} 0 & i(1 - \omega_N^n) \\ -i(1 - \omega_N^n) & 0 \end{pmatrix}. \quad (35)$$
The charge condition $\zeta_{N,n}$ can be estimated, with demanding the norm of $\Sigma$ to be unity so that $\text{Tr} \Sigma^2 = 2$, as

$$\zeta_{N,n} = |1 - \omega_N^n|^{1/2} = 2 \sin \frac{n\pi}{N}$$

and that $\Sigma$ simplifies as

$$\Sigma = \begin{pmatrix} 0 & \omega_N^{n/2} \\ \omega_N^{-n/2} & 0 \end{pmatrix}. \quad (37)$$

Note that, for $n = 1$, $\zeta_{N,1}$ is nothing but the ratio of the side length of a regular $N$-gon (regular $N$-polygon) to the radius of the circumcircle. For arbitrary $n \in \mathbb{Z}_N$, $\zeta_{N,n}$ are the the ratios of any lengths between two vertices of a regular $N$-gon to the radius of the circumcircle. So this is quite a geometrical outcome. It is consistent with the previous examples $C_2[SO(2)]$ since $\zeta_{2,1} = 2$.

In eq. (32), we argued that the charge conditions should be the same for all generators to be consistent. Knowing eq. (36) is the length between any vertices of a regular $N$-gon, we can easily figure out which cases lead to all identical lengths. From geometry alone, they are $N = 2$ and $N = 3$. However, as we will see, as long as the parametrization Lie group is $SU(2)$, any $N$ is possible because different $n$'s do not appear at the same time. So, for simple Lie groups, allowed ones are only $SO(2)$ for $N = 2$, $SU(3)$ for $N = 3$, and $SU(2)$ for all $N$. In the former two cases, the number of the flavors is the same as that of irreducible representations of $C_N$. If we allow semi-simple Lie groups, as long as they are products of these allowed simple Lie groups, they could be allowed.

$C_N[SU(2)]$

With just two flavors, we can have fixed charge conditions even for any $N \geq 3$, so $C_N[SU(2)]$ is a very informative case. To argue for the necessity of $C_N[SU(2)]$, let us first clarify why $C_N[SO(2)]$ does not lead to consistent gauge theory if $N \neq 2$. The gauge transformation relates $\Sigma$ and $a\Sigma a^{-1}$ in the limit of vanishing $\theta$, however,

$$a\Sigma a^{-1} = \Sigma^a$$

implies

$$A'_\mu = \omega_N^n A_\mu = \omega_N^{-n} A_\mu. \quad (39)$$

This is possible only if $\omega_N^n = \omega_N^{-n}$, which occurs if $N = 2n$, i.e. $\omega_N^n = \omega_N^{-n} = \pm 1$. In this case, eq. (33) implies $a \in C_2$. So if $N \neq 2$, we need two gauge fields with two generators according to

$$A^1_\mu \Sigma_1 + A^2_\mu \Sigma_2, \quad (40)$$
hence leading to off-diagonal $\Sigma$’s of SU(2).

Using eq. (33), from eq. (27) we can obtain

$$
\Sigma_1 = \frac{1}{2} \begin{pmatrix}
0 & i\omega_n^{n/2} \\
-i\omega_N^{n/2} & 0
\end{pmatrix}, \quad \Sigma_2 = \frac{1}{2} \begin{pmatrix}
0 & \omega_N^{n/2} \\
\omega_N^{n/2} & 0
\end{pmatrix},
$$

(41)

which, together with $T_3$, form an su(2) algebra with the same structure constants as $T_A$’s. Now eq. (40) can be expressed as

$$
\begin{pmatrix}
0 & \omega_N^{n/2} \hat{A}_\mu^- \\
\omega_N^{n/2} \hat{A}_\mu^+ & 0
\end{pmatrix}, \quad \hat{A}_\mu^\pm \equiv \frac{1}{2} (A_\mu^2 \mp A_\mu^1),
$$

(42)

Then, under $C_{N}[SU(2)]$ we have a consistent gauge transformation property

$$
A_\mu^{\prime+} = \omega_N^{-} A_\mu^+ + \cdots
$$

(43)

and its complex conjugate. They appear to be charged gauge fields w.r.t. U(1) subgroup generated by $T_3$.

Since $\{\Sigma_a, T_3\}$ form an su(2) algebra, the $C_{N}[SU(2)]$ invariant lagrangian also has SU(2) gauge symmetry. Furthermore, since $\{\Sigma_a, T_3\}$ leads to the same structure constants as the usual SU(2) generators $\{T_A\}$ based on the Pauli matrices do, $\{\Sigma_a, T_3\}$ can be related to $\{T_A\}$ by the following similarity transformation

$$
V^{-1} T_a V = \Sigma_a, \quad V = \begin{pmatrix}
1 & 0 \\
0 & i\omega_N^{n/2}
\end{pmatrix},
$$

(44)

which $T_3$ trivially satisfies, too. So this SU(2) symmetry is equivalent to the usual SU(2) gauge symmetry.

Note that for $N \geq 3$ we end up with nonabelian parametrization. This is quite intriguing because, after all, $C_N$ is an abelian discrete group, but we are led to non-abelian parametrization Lie groups. Clearly, what we have is quite different from the local discrete symmetry of Krauss-Wilczek[4]. So we will briefly compare to that next.

Comparison to Krauss-Wilczek[4]

In [4] Krauss and Wilczek introduced a clarifying concept of local discrete symmetry such that physical observables should be invariant under this discrete symmetry, which works as follows. With two flavors of different charges $\Phi = (\phi_1, \phi_2)^T$, the acting gauge transformation is given by

$$
g = \begin{pmatrix}
e^{i\theta} & 0 \\
0 & e^{iN\theta}
\end{pmatrix},
$$

(45)

9
then

\[ g \partial_\mu g^{-1} = -i \partial_\mu \theta \zeta \Sigma, \tag{46} \]

where \( \Sigma = 1 \) and the charge condition is

\[ \zeta = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}. \tag{47} \]

The gauge field is abelian, hence U(1), because

\[ A_\mu' \zeta = g (A_\mu \zeta - i \partial_\mu) g^{-1} = \zeta (A_\mu - \partial_\mu \theta), \tag{48a} \]

i.e. \( A_\mu' = A_\mu - \partial_\mu \theta. \tag{48b} \)

In this case, \( \mathbb{Z}_N \) symmetry appears as the invariance under

\[ \theta \to \theta + \frac{2\pi n}{N} \quad \text{for} \ n \in \mathbb{Z}_N. \tag{49} \]

When \( \phi_1 \) gets vev, the U(1) symmetry will be spontaneously broken, yet \( \mathbb{Z}_N \) symmetry remains unbroken. This is identified as local \( \mathbb{Z}_N \) symmetry and \( \mathbb{Z}_N \subset \text{U(1)}. \)

Another way of seeing this is to express it in terms of \( \varphi \equiv -i \ln \phi_2 \) such that \( \partial_\mu \varphi - i N A_\mu \) becomes invariant under \( \varphi \to \varphi + N \theta \) and \( A_\mu \to A_\mu - i \partial_\mu \theta. \)

In the \( C_2[SO(2)] \) case, as we can see from eqs. (16a)(16b), it is comparable to the \( \mathbb{Z}_2 \) case of the Krauss-Wilczek’s because the \( C_2[SO(2)] \) invariant lagrangian happens to have SO(2) gauge symmetry with \( \zeta = 2 \), except that spontaneous breaking of SO(2) is not needed to show the local nature of the discrete symmetry. In other cases, it is different. In our case, for nonreal

\[ \Sigma = \begin{pmatrix} 0 & \beta \\ \beta^* & 0 \end{pmatrix}, \quad a \Sigma a^{-1} = \begin{pmatrix} 0 & \omega_N^* \beta \\ \omega_N \beta^* & 0 \end{pmatrix}, \tag{50} \]

the gauge fields must satisfy, in the limit of vanishing gauge parameters,

\[ \begin{pmatrix} 0 & \beta A_\mu' \\ \beta^* A_\mu'^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_N^* \beta A_\mu \\ \omega_N \beta^* A_\mu'^* & 0 \end{pmatrix}. \tag{51} \]

Then, unless \( \omega_2 = \omega_2^* \), the corresponding gauge parameters need to be of the form

\[ \Sigma \partial_\mu \theta + i \sigma_3 \Sigma \partial_\mu \xi. \tag{52} \]

Since \( [\Sigma, \sigma_3] \neq 0 \), third generator has to be introduced so that there is no abelian parametrization for \( N \geq 3 \) in our case. (\( N = 2 \) is a special case because \( \omega_2 = \omega_2^* = -1 \) so that \( A_\mu = A_\mu^* \) and abelian parametrization is allowed.)
Since SU(3) is the largest possible simple Lie group leading to a fixed charge condition, which can be easily seen from the geometry of a regular triangle, let us check out the $C_3$ case in detail. With $\zeta_{3,1} = \zeta_{3,2} = \sqrt{3}$, SU(3) generators $\{T_A\}$ based on Gell-Mann matrices, and eq. (23) for $N = 3$, eq. (27) leads to

$$
\Sigma_1 = \frac{1}{2} \begin{pmatrix}
0 & -i\omega_3^2 & 0 \\
 i\omega_3 & 0 & 0 \\
 0 & 0 & 0
\end{pmatrix}, \quad \Sigma_2 = \frac{1}{2} \begin{pmatrix}
0 & -\omega_3^2 & 0 \\
 -\omega_3 & 0 & 0 \\
 0 & 0 & 0
\end{pmatrix}, \quad \Sigma_4 = \frac{1}{2} \begin{pmatrix}
0 & 0 & i\omega_3 \\
 0 & 0 & 0 \\
 -i\omega_3^2 & 0 & 0
\end{pmatrix},
$$

$$
\Sigma_5 = \frac{1}{2} \begin{pmatrix}
0 & 0 & \omega_3^2 \\
 0 & 0 & 0 \\
 \omega_3^2 & 0 & 0
\end{pmatrix}, \quad \Sigma_6 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
 0 & 0 & -i\omega_3^2 \\
 0 & i\omega_3 & 0
\end{pmatrix}, \quad \Sigma_7 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
 0 & 0 & -\omega_3^2 \\
 0 & -\omega_3 & 0
\end{pmatrix}.
$$

(53)

Again, to form a closed algebra, we need to add $T_3$ and $T_8$ of diagonal SU(3) generators. We can check if $\{\Sigma_a, T_i\}$ forms the same su(3) algebra as $T_A$’s do. The structure constants for $\{\Sigma_a, T_i\}$ are

$$
g_{123} = 1, \quad g_{147} = g_{175} = g_{247} = g_{265} = g_{345} = g_{376} = \frac{1}{2}, \quad g_{458} = g_{678} = \frac{\sqrt{3}}{2},
$$

(54)

and all others vanish. Compared with this, the su(3) algebra based on Gell-Mann matrices have nonvanishing structure constants

$$
f_{123} = 1, \quad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}.
$$

(55)

The structure constants are slightly different, unlike the $C_N[SU(2)]$ case. This indicates that there may not be a similarity transformation preserving the same su(3) algebra between the two bases. Indeed, we can reproduce the two sets of structure constants with the following similarity transformation:

$$
V^{-1}T_aV = \Sigma_a, \quad a = 1, 2, 4, 5, \quad (56a)
$$

$$
V^{-1}T_6V = \Sigma_7, \quad (56b)
$$

$$
V^{-1}T_7V = -\Sigma_6, \quad (56c)
$$

and diagonal generators are invariant, where

$$
V = \begin{pmatrix}
1 & 0 & 0 \\
0 & -i\omega_3^2 & 0 \\
0 & 0 & i\omega_3
\end{pmatrix}.
$$

(57)

However, due to the minus sign in eq. (56c), the two su(3) algebras are not related by a similarity transformation. If these were related with a single sign as $\Sigma_a = V^{-1}T_bV$, the lagrangian
might be identified as having the usual color SU(3) gauge symmetry combined with $C_3[SU(3)]$ discrete symmetry. So, even though the $C_3[SU(3)]$ invariant lagrangian has another SU(3) gauge symmetry generated by $\{\Sigma_a, T_i\}$, but not the same as the usual color SU(3).

We are particularly interested in the $\theta$-independent terms of gauge transformations in eq. (29b), which read

$$a \Sigma_a a^{-1} = -\Sigma_a^*, \quad \text{for } a = 1, 4, 6,$$

$$a \Sigma_a a^{-1} = \Sigma_a^*, \quad \text{for } a = 2, 5, 7. \tag{58}$$

Unlike in the $C_2$ case, in the $C_3$ case an SO(3) parametrization is not allowed because it leads to inconsistent transformation rules for each components as in the $C_N[SO(2)]$ case for $N \geq 3$. Therefore, in the $C_3$ case, to be consistent, the parametrization Lie group should be SU(3).

Since complex gauge fields take the role of charged gauge fields w.r.t. the diagonal generators, this provides natural manifestation of $C_3 \times U(1)^2 \subset SU(3)$. Each complex gauge field now picks up phases $\omega_3$ or $\omega_3^2$. Compared with the $C_2$ case in which gauge fields flips sign, i.e. $\omega_2 = -1$, now we have nontrivial phases showing up upon gauge transformations. This also does not lead to any inconsistency because it does not affect observable S-matrix of matter-gauge interactions, which are gauge invariant.

**Final Remarks**

We have presented a consistent gauge theory different from the usual one based on a Lie group. The most interesting outcome is the symmetry leads to charges quantized according to the geometry of the discrete symmetry we start with, i.e. charges correspond to the lengths between any two vertices of a regular polygon. Although it is not clear if there is any real world physical system directly based on the structure presented here, but we believe this is quite an interesting theoretical outcome. The key implication of Krauss-Wilczek's gauged discrete symmetry is the possibility of nontrivial charges on a black hole associated with the discrete symmetry $C_3$. So, we believe there could be some physical applications in the physics we have not encountered yet. Also a $U(N)$ gauge symmetry can appear as $N$ D-branes are bounded together. This naturally has $C_N$ discrete symmetry built in. So there could be some connection with the stringy world. There are only two cases that the number of flavors can be the same as that of irreducible representations of the cyclic group $C_N$: $C_2[SO(2)]$ and $C_3[SU(3)]$. In particular, it is also interesting to observe that the largest simple Lie group SU(3) is allowed with only three flavors. Also it will be interesting to speculate if this has anything to do with QCD in a certain limit, when three flavors in our case are treated as three colors.
Another noticeable aspect of the outcome is that the same lagrangian has a usual local gauge symmetry $K(N_F)$ generated by $\{\Sigma_a, T_i\}$, which is different from the $C_N[K(N_F)]$ symmetry. In other words, there are two different local gauge symmetries for the same lagrangian. In the SU(2) cases, the two gauge symmetries are related by a similarity transformation, but not in the SU(3) case. In any case, certain constraints are only evident in the $C_N[K(N_F)]$ symmetry. This raises a possibility that known SU(2) gauge theories may have another hidden gauge symmetry, which may better explain certain phenomena.

In this paper, we have adopted parametrization groups mixing all flavors at once, but, in principle, we can mix only some of them, which leads to, e.g. $C_N[SU(2) \times SU(2) \times \cdots]$ type of group family. We can also generalize to discrete groups other than cyclic groups. For example, a dihedral group of order three, $D_3$, can lead to $D_3[K(N_F)]$ with a suitable semi-simple Lie group, e.g. $D_3[SU(2) \times SU(3)]$.

It will be extremely interesting if there is a mechanism to spontaneously break the symmetry we have introduced here to lead to useful global discrete symmetries in Nature. This will justify the origin of the unbroken global discrete symmetries in the low energy world despite any quantum gravity effects.

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