Green-Kubo Expressions for a Granular Gas

James W. Dufty

Department of Physics, University of Florida, Gainesville, FL

32611, USA

J. Javier Brey

Física Teórica, Universidad de Sevilla, E-41080, Sevilla,

Spain

(November 19, 2018)

The transport coefficients for a gas of smooth, inelastic hard spheres are obtained from the Boltzmann equation in the form of Green-Kubo relations. The associated time correlation functions are not simply those constructed from the fluxes of conserved densities. Instead, fluxes constructed from the reference local homogeneous distribution occur as well. The analysis exposes potential problems associated with a straightforward extension of linear response methods to granular systems.

I. INTRODUCTION

The derivation of macroscopic equations (e.g., hydrodynamics) and associated transport coefficients from kinetic theory is limited in practice to weakly coupled systems (low density gases, ideal plasmas, anharmonic crystals). The application of formal methods from nonequilibrium statistical mechanics to this problem forty years ago provided the generality missing in kinetic theory [1]. These methods lead to the expected macroscopic dynamics with formally exact expressions for the transport coefficients known as Green-Kubo (GK) expressions. The latter are time integrals of equilibrium time correlation functions for the fluxes associated with conserved densities. The simplest example is the diffusion coefficient \( D \) for an impurity in a simple fluid.
\[ D = \lim_{t \to \infty, \Omega \to \infty} \frac{1}{d} \int_0^t dt' \langle v(t) \cdot v \rangle, \] (1)

where \( d \) is the dimension of the system, \( v \) is the impurity velocity, and the brackets denote an equilibrium ensemble average. Similar expressions apply for the viscosities, in terms of the autocorrelation function for the momentum flux, and the thermal conductivity, in terms of the autocorrelation function for the energy flux. The GK expressions provide an exact starting point for analysis and modeling of strongly coupled systems. In particular, they have stimulated extensive studies of transport via molecular dynamics simulation of these equilibrium time correlation functions. Much of the present knowledge about transport in strongly coupled systems (dense gases, liquids, plasmas, solids) derives from analysis of appropriate GK expressions.

Currently, there is great interest in the foundations of a fluid dynamics for granular media. Kinetic theory and statistical mechanical methods have been applied to idealized granular fluids comprised of smooth hard spheres interacting with inelastic collisions [2–6]. The derivation of the corresponding Navier-Stokes level equations from kinetic theory at low and moderate densities has been given recently, with transport coefficients expressed as functions of the restitution coefficient [7–9]. The results are in good agreement with selected tests using molecular dynamics and Monte Carlo simulations [4]. It is tempting to expect that some form of the more general GK expressions should apply for granular media as well [3,10]. The primary difference from normal fluids is the absence of a reference stable, stationary Gibbs state in terms of which the time correlation functions would be defined. Instead, the corresponding homogeneous state for an undriven system has a time dependence due to loss of energy on collisions (“cooling”). This is referred to as the homogeneous cooling state (HCS) [10,12]. Recently, analysis of impurity diffusion in a granular fluid has led to a GK expression similar to (1) with the velocity autocorrelation function defined in the HCS [13]. A related analysis of impurity mobility also gives a GK expression for the mobility coefficient [14], although not simply in terms of the velocity autocorrelation function as is the case for normal fluids. The difference is due to the replacement of the Gibbs state by
the HCS. This is an indication that translation of linear response methods for normal fluids to granular fluids requires some care.

The objective here is to derive GK expressions for a granular gas based on the Boltzmann equation. This may seem redundant since the transport coefficients are already known by other methods [7]. However, the analysis exposes the form of the GK expressions that should be expected from more general nonequilibrium statistical mechanical methods. As with the mobility, it is found that the time correlation functions for the transport coefficients are not simply those for the conserved densities. Instead, they are generated from the HCS distribution in a manner required by internal consistency for the solution to the Boltzmann equation. The analysis also shows that the HCS correlation functions can be put in the form of stationary state averages using appropriate dimensionless variables, including collision number instead of real time.

It is a pleasure to dedicate this work to Bob Dorfman, colleague and mentor, who has taught us all so much about hard sphere transport (among many other subjects).

II. BOLTZMANN EQUATION AND HYDRODYNAMICS

The simplest model for a granular fluid is a system of \( N \) smooth hard spheres or disks at low density interacting via inelastic collisions. The collisions are characterized by a normal restitution coefficient \( \alpha \leq 1 \), where \( \alpha = 1 \) is the elastic limit. An accurate description of this gas can be obtained from a formal density expansion of the BBGKY hierarchy, just as for normal gases [8,9]. To leading order in the density, the single particle distribution function \( f(\mathbf{r}, \mathbf{v}, t) \) obeys the Boltzmann equation, with collisions modified to account for the inelasticity,

\[
(\partial_t + \mathbf{v} \cdot \nabla) f = J[f, f].
\]

The detailed form of the inelastic collision operator \( J \) is not required for the discussion here, beyond the properties required for the macroscopic balance equations:
\[
\int d\mathbf{v} \begin{pmatrix}
1 \\
v \\
\frac{1}{2}mv^2
\end{pmatrix} J[f,f] = \begin{pmatrix}
0 \\
0 \\
-\frac{dp}{2}\zeta[f]
\end{pmatrix}.
\] (3)

Here \( p = nT \) is the low density pressure, \( n \) being the number density and \( T \) the granular temperature (with Boltzman’s constant set equal to unity). The functional \( \zeta[f] \) is the “cooling rate”, as will become apparent below. The number density and temperature, as well as the flow velocity \( \mathbf{u} \) are defined in the usual way

\[
\begin{pmatrix}
n \\
n\mathbf{u} \\
d\frac{2}{5}nT
\end{pmatrix} = \int d\mathbf{v} \begin{pmatrix}
1 \\
v \\
\frac{1}{2}mV^2
\end{pmatrix} f,
\] (4)

with \( \mathbf{V} = \mathbf{v} - \mathbf{u} \), the velocity relative to the local flow. The balance equations for these fields follow directly from moments of the Boltzmann equation using Eq. (3),

\[
D_t n + n \nabla \cdot \mathbf{U} = 0, \quad (5)
\]

\[
D_t u_i + (mn)^{-1} \nabla_j P_{ij} = 0, \quad (6)
\]

\[
D_t T + \frac{2}{dn} (P_{ij} \nabla_j u_i + \nabla \cdot \mathbf{q}) + T \zeta = 0, \quad (7)
\]

where \( D_t = \partial_t + \mathbf{u} \cdot \nabla \) is the material derivative. The pressure tensor \( P_{ij} \) and heat flux \( \mathbf{q} \)

are linear functionals of \( f \) given by

\[
P_{ij}[\mathbf{r},t|f] = p(\mathbf{r},t)\delta_{ij} + \int d\mathbf{v} D_{ij}(\mathbf{V}) f(\mathbf{r},\mathbf{v},t), \quad (8)
\]

\[
\mathbf{q}[\mathbf{r},t|f] = \int d\mathbf{v} \mathbf{S}(\mathbf{V}) f(\mathbf{r},\mathbf{v},t). \quad (9)
\]

Finally, the functions \( D_{ij}(\mathbf{V}) \) and \( \mathbf{S}(\mathbf{V}) \) characterizing the irreversible parts of the fluxes are

\[
D_{ij}(\mathbf{V}) \equiv m(V_iV_j - \frac{1}{d}V^2\delta_{ij}), \quad \mathbf{S}(\mathbf{V}) \equiv \left( \frac{m}{2}V^2 - \frac{d+2}{2}T \right) \mathbf{V}. \quad (10)
\]
Clearly, the balance equations are not closed until the fluxes are expressed as functionals of the hydrodynamic fields \( n, u, \) and \( T \). This can be accomplished if a solution to the Boltzmann equation can be obtained in “normal form”,

\[
f(r, v, t) = f(v|\{y_\beta\}) ,
\]

\[
\{y_\beta(r, t)\} \leftrightarrow \{n(r, t), u(r, t), T(r, t)\}.
\]

The notation \( f(v|\{y_\beta\}) \) means that the distribution is a functional of the hydrodynamic fields \( y_\beta(r, t) \), and that its space and time dependence occurs only through that of these fields. If such a solution can be found, its use in Eqs. (8) and (9) gives the fluxes as functionals of the fields. Such fluxes are then referred to as constitutive equations. The constitutive equations together with the exact macroscopic balance equations become a closed set of hydrodynamic equations. The origin of a hydrodynamic description in this context therefore is traced directly to the existence of a normal solution to the Boltzmann equation on some length and time scales. In practice, the functional form of this solution is constructed in some well-defined approximation, leading to corresponding approximate constitutive and hydrodynamic equations. The approximation of interest here is small spatial variations of the hydrodynamic fields over distances of the order of the mean free path. For gases with elastic collisions, the Chapman-Enskog method generates the normal solution perturbatively, and to leading order the constitutive equations are Newton’s viscosity law and Fourier’s heat law \[16\]. The hydrodynamic equations become the Navier-Stokes equations. Application of this method to granular gases leads to similar results \[7\]. The analysis is modified here to obtain the same results in an equivalent form with the transport coefficients represented by GK expressions.

III. SMALL GRADIENT SOLUTION

The first step in constructing the desired solution to the Boltzmann equation is to express it in terms of a reference distribution \( f^{(0)} \),
\[ f(\mathbf{r}, \mathbf{v}, t) = f^{(0)}(\mathbf{v}\{y_\beta\}) + f^{(1)}(\mathbf{r}, \mathbf{v}, t). \quad (13) \]

The reference distribution is restricted to be normal and to give the exact moments of Eq. (4), i.e.

\[
\begin{pmatrix}
  n \\
  \mathbf{n u} \\
  \frac{4}{3}nT
\end{pmatrix}
= \int d\mathbf{v}
\begin{pmatrix}
  1 \\
  \mathbf{v} \\
  \frac{1}{2}mv^2
\end{pmatrix} f^{(0)}. \quad (14)
\]

The Boltzmann equation then becomes

\[
\left( \partial_t + \mathbf{v} \cdot \nabla + L \right) f^{(1)} - J[f^{(1)}, f^{(1)}] = J[f^{(0)}, f^{(0)}] - (D_t + \mathbf{V} \cdot \nabla) f^{(0)}, \quad (15)
\]

with the definition

\[
\mathbf{L} f^{(1)} = -J[f^{(1)}, f^{(0)}] - J[f^{(0)}, f^{(1)}]. \quad (16)
\]

The left side of Eq. (15) generates the dynamics of \( f^{(1)} \). The solution of interest is such that \( f^{(1)} \) should be proportional to the gradients of the hydrodynamic fields, since \( f^{(0)} \) provides their local values through the constraint (14). This requires that the right side of (15) be proportional to the gradients of the fields. Evaluation of \((D_t + \mathbf{V} \cdot \nabla) f^{(0)}\) using the macroscopic balance equations gives terms proportional to the gradients, except for the contribution from the cooling rate in (7). In addition, the nonvanishing contribution \( J[f^{(0)}, f^{(0)}] \) also is not of first order in the gradients. Therefore, the reference distribution is finally characterized by the condition that these non-gradient terms should vanish

\[
J[f^{(0)}, f^{(0)}] + T\zeta^{(0)} \frac{\partial f^{(0)}}{\partial T} = 0, \quad \zeta^{(0)} = \zeta[f^{(0)}]. \quad (17)
\]

This is an essential point in the analysis, and it is the origin of differences from the case of elastic collisions, as is discussed below. The distribution function \( f^{(0)} \) is the analogue of the local Maxwellian for the case of elastic collisions, but the solution to Eq. (17) for \( \alpha < 1 \) is not the Maxwellian. Symmetry and dimensional analysis requires that \( f^{(0)} \) have the scaling form
\[ f^{(0)}(\mathbf{v}|\{y_\beta\}) = n v_0^{-d}(t) f^{(0)*}(\mathbf{V}^*), \quad \mathbf{V}^* = \mathbf{V}/v_0(t), \]  
(18)

where \(v_0(t) = [2T(t)/m]^{1/2}\) is the “thermal” velocity and \(f^{(0)}\) is an isotropic function of \(\mathbf{V}^*\).

Then Eq. (17) is equivalent to

\[ J[f^{(0)}, f^{(0)}] - \frac{1}{2} \zeta^{(0)} \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V} f^{(0)}) = 0. \]  
(19)

Consider the initial condition \(f^{(1)}(0) = 0\). This is a physically interesting case for hydrodynamics since \(f^{(1)}(0)\) does not contribute to the initial value of the hydrodynamic fields, due to (14), and \(f^{(0)}(0)\) is entirely determined by the hydrodynamic initial values. In this case, \(f^{(1)}\) is manifestly proportional to the gradients of the fields. Retaining only terms of first order in these gradients, the Boltzmann equation becomes

\[ (\partial_t + \mathbf{T}) f^{(1)} = -f^{(0)} \Phi_\beta \cdot \nabla y_\beta, \]  
(20)

with the definitions

\[ \Phi_1 = \frac{1}{n} \left( \mathbf{V} + \frac{T}{m} \frac{\partial}{\partial \mathbf{V}} \ln f^{(0)} \right), \]  
(21)

\[ \Phi_{2,ij} = \left( \frac{1}{d} \delta_{ij} \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{V}} - V_i \frac{\partial}{\partial V_j} \right) \ln f^{(0)}, \]  
(22)

\[ \Phi_3 = m^{-1} \frac{\partial}{\partial \mathbf{V}} \ln f^{(0)} - \frac{\mathbf{V}}{2T} \left( d + \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{V}} \ln f^{(0)} \right). \]  
(23)

A term from \(\zeta[f]\) to first order in the gradients does not occur since it vanishes. This follows from the fact that \(\zeta\) is a scalar and can only be proportional to \(\nabla \cdot \mathbf{u}\) at this order. Since \(\Phi_{2,ij}(\mathbf{V})\) is traceless, there is no such contribution. The solution to Eq. (20) can be written in the form

\[ f^{(1)} = -F_\beta \cdot \nabla y_\beta. \]  
(24)

When Eq. (24) is substituted into Eq. (20), the terms from \(\partial_t \nabla y_\beta\) are higher order in the gradients, except for \(\beta = 3\) which gives a contribution proportional to the cooling rate. The coefficients of the gradients are then found to obey the equations
\[(\partial_t + L) F_1 - \frac{T\zeta^{(0)}}{n} F_3 = f^{(0)} \Phi_1, \quad (25)\]

\[(\partial_t + L) F_{2,ij} = f^{(0)} \Phi_{2,ij}, \quad (26)\]

\[(\partial_t + L - \frac{3}{2} \zeta^{(0)}) F_3 = f^{(0)} \Phi_3. \quad (27)\]

**IV. SCALING**

The apparent simplicity of Eqs. (25)-(26) is misleading since \(L, \zeta^{(0)}, \) and \(\Phi_\alpha\) are functions of time through their dependence on the hydrodynamic fields. This dependence occurs as well for a gas with elastic collisions, but it can be neglected in that case when solving the equations since it is proportional to higher order gradients. For elastic collisions, the temperature gives a time dependence that is of zeroth order in the gradients and cannot be neglected. However, it can be removed by a change of variables to dimensionless forms. A characteristic length scale is given by the mean free path \(\ell\) and a dimensionless time \(s\) is defined accordingly by

\[ds = \frac{v_0(t)}{\ell} dt. \quad (28)\]

For simplicity we will take here \(\ell = (n\sigma^{d-1})^{-1}\), where \(\sigma\) is the diameter of the particles, omitting a factor that depends on the dimension of the system. Integrating over an interval \(t\) shows that \(s\) is an average number of collisions during that time. Moreover, introduce dimensionless functions \(\Phi^*_\beta\) and \(F^*_\beta\) as

\[\Phi_1 = \frac{2v_0(t)}{n} \Phi^*_1(V^*), \quad \Phi_{2,ij} = 2\Phi^*_{2,ij}(V^*), \quad \Phi_3 = \frac{2}{mv_0(t)} \Phi^*_3(V^*), \quad (29)\]

\[F_1 = \frac{2\ell}{v_0(t)} F^*_1(V^*, s), \quad F_{2,ij} = \frac{2n\ell}{v_0^{d+1}(t)} F^*_{2,ij}(V^*, s), \quad F_3 = \frac{2n\ell}{mv_0^{d+2}(t)} F^*_3(V^*, s). \quad (30)\]

In dimensionless form Eqs. (25)-(26) become
\[
(\partial_s + \mathcal{L}^*) (F_1^* - F_3^*) = f^{(0)*} (\Phi_1^* - \Phi_3^*), \quad (31)
\]

\[
(\partial_s + \mathcal{L}^* + \frac{1}{2} \zeta^*) F_{2,ij}^* = f^{(0)*} \Phi_{2,ij}^*, \quad (32)
\]

\[
(\partial_s + \mathcal{L}^* - \frac{1}{2} \zeta^*) F_3^* = f^{(0)*} \Phi_3^*. \quad (33)
\]

with the definitions

\[
\mathcal{L}^* = \mathcal{L}^* + \frac{1}{2} \zeta^* \frac{\partial}{\partial V^*} \cdot V^*, \quad (34)
\]

\[
\mathcal{L}^* = \frac{\ell}{v_0(t)} \mathcal{L}, \quad \zeta^* = \frac{\ell \zeta^{(0)}}{v_0(t)}. \quad (35)
\]

The right sides of Eqs. (31)-(32) are independent of \(s\), as are \(\mathcal{L}^*\) and \(\zeta^*\). The integration is now trivial

\[
F_1^* (V^*, s) = F_3^* (V^*, s) + \int_0^s ds' e^{-s' \mathcal{L}^*} f^{(0)*} (\Phi_1^* - \Phi_3^*), \quad (36)
\]

\[
F_{2,ij}^* (V^*, s) = \int_0^s ds' e^{-s' (\mathcal{L}^* + \frac{1}{2} \zeta^*)} f^{(0)*} \Phi_{2,ij}^*, \quad (37)
\]

\[
F_3^* (V^*, s) = \int_0^s ds' e^{-s' (\mathcal{L}^* - \frac{1}{2} \zeta^*)} f^{(0)*} \Phi_3^*. \quad (38)
\]

In the derivation of the above expressions, the assumed initial condition, implying that all the \(F_{\beta}^*\) vanish for \(s = 0\), has been used.

### V. Constitutive Equations

The pressure tensor and heat flux are determined from their definitions in Eqs. (8) and (9). Substitution of Eqs. (13) and (24) leads to results valid to first order in the gradients,

\[
P_{ij}[\mathbf{r}, t|f] = p(\mathbf{r}, t) \delta_{ij} - \int d\mathbf{v} D_{ij}(\mathbf{V}) F_\beta \cdot \nabla y_\beta, \quad (39)
\]

\[
q[\mathbf{r}, t|f] = - \int d\mathbf{v} S(\mathbf{V}) F_\beta \cdot \nabla y_\beta. \quad (40)
\]
Using fluid symmetry, these expressions reduce to

\[ P_{ij} = p\delta_{ij} - \eta \left( \nabla_j u_i + \nabla_i u_j - \frac{2}{d}\delta_{ij} \nabla \cdot \mathbf{u} \right), \tag{41} \]

\[ \mathbf{q} = -\kappa \nabla T - \mu \nabla n. \tag{42} \]

These are the expected Navier-Stokes constitutive equations, except for the additional term in the heat flux proportional to \( \nabla n \). It will be seen below that this new term is due entirely to the deviation of \( f^{(0)} \) from the Maxwellian. The expressions for the transport coefficients are identified as

\[ \eta = \frac{2nm\ell v_0(t)}{d^2 + d - 2} \int dV^* D_{ij}^* (V^*) F_{2,ij}^* (V^*, s), \tag{43} \]

\[ \kappa = \frac{n\ell v_0(t)}{d} \int dV^* S^* (V^*) \cdot F_3^* (V^*, s), \tag{44} \]

\[ \mu = \frac{m\ell v_0^3(t)}{d} \int dV^* S^* (V^*) \cdot F_1^* (V^*, s). \tag{45} \]

The dimensionless forms of \( S \) and \( D_{ij} \) are

\[ S^* \equiv \left( V^{*2} - \frac{d+2}{2} \right) V^*, \quad D_{ij}^* = V_i^* V_j^* - \frac{1}{d} V^{*2} \delta_{ij}. \]

To put these expressions in the desired Green-Kubo form, define the adjoint operators \( L^* \) and \( \mathcal{L}^* \) by

\[ \int dV^* X(V^*) \left( \begin{array}{c} \mathcal{T}^* \\ \mathcal{L}^* \end{array} \right) Y(V^*) = - \int dV^* Y(V^*) \left( \begin{array}{c} L^* \\ \mathcal{L}^* \end{array} \right) X(V^*), \tag{46} \]

for arbitrary functions \( X \) and \( Y \). From Eq. (44) it is trivially seen that

\[ \mathcal{L}^* = L^* + \frac{1}{2} \zeta^* V^* \cdot \frac{\partial}{\partial V^*}. \tag{47} \]

The expression of \( L^* \) can be obtained by using the properties of the Boltzmann collision operator, but it will be not needed here.
Also, define “correlation functions” by
\[ \langle XY \rangle = \int dV^* f^{(0)*}(V^*) X(V^*) Y(V^*) . \] (48)

Then, with these definitions and Eqs. (36)-(38), the transport coefficients given by Eqs. (43)-(45) can be rewritten as
\[ \eta = \frac{2nm\ell v_0(t)}{d^2} \int_0^s ds' \langle D_{ij}(s') \Phi_{ij}^* \rangle e^{-\frac{1}{2} s' \zeta^*}, \] (49)
\[ \kappa = \frac{n\ell v_0(t)}{d} \int_0^s ds' \langle S^*(s') \cdot \Phi_3^* \rangle e^{\frac{1}{2} s' \zeta^*}, \] (50)
\[ \mu = \frac{2T\kappa}{n} + \frac{m\ell v_0^3(t)}{d} \int_0^s ds' \langle S^*(s') \cdot (\Phi_1^* - \Phi_3^*) \rangle . \] (51)

The time dependence of the correlation functions is given by
\[ X(s) = e^{sL^*} X(V^*). \] (52)

Equations (51)-(52) are the Green-Kubo expressions for the transport coefficients of a low density granular gas. The spectrum of \( L^* \) is such that the correlation functions decay to zero for \( s \gg 1 \). This sets the time scale for hydrodynamics. In this limit, the above expressions agree with those obtained by the Chapman-Enskog method [7].

VI. DISCUSSION

To discuss the GK expressions for a granular gas it is instructive to write the corresponding results for elastic collisions (\( \alpha = 1 \)). In that case, \( f^{(0)} \) is the local Maxwellian and
\[ \Phi_1^* \rightarrow 0, \quad \Phi_{2,ij}^* \rightarrow D_{ij}^*, \quad \Phi_3^* \rightarrow S^*(V). \] (53)

Then \( \mu \rightarrow \mu_0 = 0 \), and
\[ \eta \rightarrow \eta_0 = \frac{2n\ell v_0}{d^2} \int_0^s ds' \langle D_{ij}^*(s')D_{ij}^* \rangle , \quad \kappa \rightarrow \kappa_0 = \frac{n\ell v_0}{d} \int_0^s ds' \langle S^*(s') \cdot S^* \rangle . \] (54)
Since the temperature is constant to leading order in the gradients for $\alpha = 1$, the variable $s = (v_0/\ell) t$ is simply proportional to time. These are the standard forms for the GK expressions in terms of the autocorrelation functions composed from the “microscopic” fluxes $S^*$ and $D_{ij}^*$. There are several differences that occur for granular gases:

- The correlation functions are defined as averages over $f^{(0)}$. This local HCS distribution is determined from (19) and differs from the Maxwellian for all $\alpha < 1$.

- The correlation functions are not constructed from $S^*$ and $D_{ij}^{*}$ alone. Each is paired with another function from the set $\Phi_{\beta}^*$.

- The time integration is replaced by an integration over the average collision number, $s$. The correlation functions have approximate exponential decay in the variable $s$ rather than $t$.

- The integrals over $s$ are not controlled solely by the correlation functions. In addition there are time dependent factors arising from the change of the temperature over the duration of the integral.

The most surprising among these differences is the replacement of one of the “microscopic” fluxes $S^*$ or $D_{ij}^*$ by one of the new variables $\Phi_{\beta}^*$ determined from $f^{(0)}$ through Eqs. (21)-(23). If $f^{(0)}$ were replaced by the local Maxwellian $f_{M}^{(0)*}$ in the above analysis, the correlation functions $\langle S^*(s') \cdot S^* \rangle$ and $\langle D_{ij}^*(s') D_{ij}^* \rangle$ would appear in Eqs. (49)-(51). However, this would not be a consistent solution to the Boltzmann equation and the transport coefficients would not have the correct dependence on $\alpha$ (e.g. $\mu$ would vanish for all $\alpha$). The role of $f^{(0)}$ is more than just a local reference state with the exact moments for $n$, $T$, and $u$. In addition, it must be an approximate solution for the dynamics. In dimensionless form, $f^{(0)*}$ is a universal function of $V^*$ determined from the equation

$$J^*[f^{(0)*}, f^{(0)*}] - \frac{1}{2} \zeta^* \frac{\partial}{\partial V^*} \cdot (V^* f^{(0)*}) = 0, \quad \zeta^* = \zeta^*[f^{(0)*}], \quad (55)$$
without reference to any particular hydrodynamic state. The above analysis shows that \((55)\) is necessary for a consistent ordering of the solution in terms of gradients. The local reference distribution for a given hydrodynamic state follows from this solution according to
\[
f^{(0)} = n(r, t) \left( \frac{m}{2T(r, t)} \right)^{d/2} f^{(0)*} \left[ \left( \frac{m}{2T(r, t)} \right)^{1/2} (v - u(r, t)) \right].
\]
Its dependence on the hydrodynamic variables is determined by the solution to \((55)\).

An estimate for the \(\Phi^*_\beta\) can be obtained from an approximation to \(f^{(0)*}\) obtained by a polynomial expansion \([11,12]\), which to leading order is
\[
f^{(0)*} = f_M^{(0)*} \left[ 1 + \frac{1}{4} c^*(\alpha) \left( V^{*4} - (d + 2)V^{*2} + \frac{d(d + 2)}{4} \right) \right],
\]
\[
c^*(\alpha) = \frac{32 (1 - \alpha) (1 - 2\alpha^2)}{9 + 24d + (8d - 41)\alpha + 30\alpha^2(1 - \alpha)}
\]
This is known to be accurate to within a few percent for all \(\alpha\) and \(V^* \leq 1\). In this approximation the variables \(\Phi^*_\beta\) become
\[
\Phi^*_1 = -\frac{1}{4} c^*(\alpha) S^*,
\]
\[
\Phi^*_{2,ij} = D^*_{ij} \left[ 1 - \frac{1}{2} c^*(\alpha) \left( V^{*2} - \frac{d + 2}{2} \right) \right],
\]
\[
\Phi^*_3(V^*) = S^* \left[ 1 - \frac{1}{2} c^*(\alpha) \left( V^{*2} - 1 \right) \right].
\]

Linear response methods typically construct the response function for spatial perturbation of the homogeneous state of a specific type: those that couple only to the microscopic conserved densities. This follows from consideration of an initial local equilibrium state or maximum entropy state. Then manipulation of the response function for the conserved densities using the microscopic conservation laws leads to GK expressions in terms of the microscopic fluxes. However, it is clear that expansion of \((56)\) for small spatial perturbations does not give a representation in terms of the conserved densities. Thus, a straightforward application of linear response can lead to formal results with internal inconsistencies. One
such inconsistency can be a failure of the solubility conditions for the linear integral equations associated with (31) - (32). In the GK form these conditions translate to requirements that the fluxes be orthogonal to invariants of the dynamics. These issues and a derivation of the GK expressions from the Liouville equation will be given elsewhere.

VII. ACKNOWLEDGMENTS

The research of JWD was supported in part by the Department of Energy grant (DE-FG03-98DP00218). The research of JJB was partially supported by the Dirección General de Investigación Científica y Técnica (Spain) through Grant No. PB98-1124.

[1] J. A. McLennan, *Introduction to Nonequilibrium Statistical Mechanics* (Prentice-Hall, New Jersey, 1989).

[2] J. J. Brey, J. W. Dufty, and A. Santos, “Dissipative Dynamics for Hard Spheres”, J. Stat. Phys. 87, 1051 (1997).

[3] J. W. Dufty, “Statistical mechanics, kinetic theory, and hydrodynamics for rapid granular flow”, J. Phys.: Condens. Matter 12, A 47 (2000).

[4] J. J. Brey and D. Cubero, “Hydrodynamic transport coefficients of granular gases” in *Granular Gases*, eds. T. Poschel and S. Luding (Springer, NY, 2001).

[5] I. Goldhirsh, “Granular gases - probing the boundaries of hydrodynamics”, in *Granular Gases*, eds. T. Poschel and S. Luding (Springer, NY, 2001).

[6] T. P. C. van Noije and M. H. Ernst, “Kinetic theory of granular gases”, in *Granular Gases*, eds. T. Poschel and S. Luding (Springer, NY, 2001).
[7] J. J. Brey, J. W. Dufty, C. S. Kim, A. Santos, “Hydrodynamics for Granular Flow at Low Density”, Phys. Rev. E 58, 4638 (1998).

[8] V. Garzó and J. Dufty, “Dense fluid transport for inelastic hard spheres”, Phys. Rev. E 59, 5895 (1999).

[9] V. Garzó and J. Dufty, “Hydrodynamics for a granular mixture at low density”, Phys. Fluids (in press); cond-mat/0105395 v1.

[10] I. Goldhirsh and T. P. C. van Noije, “Green-Kubo relations for granular fluids” Phys. Rev. E 61, 3241 (2000).

[11] J. J. Brey, M. J. Ruiz-Montero, and D. Cubero, “Homogeneous cooling state of a low-density granular flow”, Phys. Rev. E 54, 3664 (1997).

[12] T. P. C. van Noije and M. H. Ernst, “Velocity distributions in homogeneous granular fluids: the free and heated case”, Granular Matter 1, 57 (1998).

[13] J. W. Dufty, J. F. Lutsko, and J. J. Brey, unpublished.

[14] J. W. Dufty and V. Garzo, “Mobility and diffusion in granular flow”, J. Stat. Phys. 105, 723 (2001).

[15] J. W. Dufty, “Kinetic Theory and Hydrodynamics for a Low Density Gas”, J. W. Dufty, Advances in Complex Systems (in press); cond-mat/0109215.

[16] J. R. Dorfman and H. van Beijeren, “The kinetic theory of gases” in Statistical Mechanics, Part B, edited by B. Berne (Plenum Press, NY, 1977).