On the energy-momentum current of the electromagnetic field in
a pre-metric axiomatic approach*. I

Friedrich W. Hehl† and Yuri N. Obukhov‡ §

Institute for Theoretical Physics, University of Cologne, 50923 Köln, Germany

Abstract

We complete a metric-free axiomatic framework for electrodynamics by introducing the appropriate energy-momentum current $k\Sigma_\alpha$ of the electromagnetic field. We start from the Lorentz force density and motivate the form of $k\Sigma_\alpha$. Then we postulate it (fourth axiom) and discuss its properties. In particular, it is found that $k\Sigma_\alpha$ is traceless and invariant under an electric-magnetic reciprocity transformation. By using the Maxwell-Lorentz spacetime relation (fifth axiom), $k\Sigma_\alpha$ is also shown to be symmetric, that is, it has 9 independent components.

1. INTRODUCTION

In an axiomatic approach to classical electrodynamics, one can start with electric charge conservation $dJ = 0$ as first axiom, with the existence of the Lorentz force density $f_\alpha = (e_\alpha \lrcorner F) \wedge J$ as second axiom, and with magnetic flux conservation $dF = 0$ as third axiom. Here $J$ is the electric current density 3-form, $F = (E, B)$ the electromagnetic field strength.

* Dedicated to the memory of our friend and colleague Ruggiero de Ritis from Napoli.

† hehl@thp.uni-koeln.de

‡ yo@thp.uni-koeln.de, general@elnet.msk.ru

§ Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia.
2-form, and \( e_\alpha \) the frame field, i.e., the vector basis of the tangent space, with \( \alpha = 0, 1, 2, 3 \) as frame index. A fairly detailed account, including the conventions and references to the literature, can be found in [1], see also [2]; for the mathematics involved, compare Frankel [3].

For the formulation of the axiomatics mentioned, one needs a 4-dimensional differentiable manifold as spacetime. It is assumed that this manifold can be cut into 3-dimensional slices (folia) such that successive slices can be numbered, at least locally, by means of a monotone increasing parameter \( \sigma \). The slices represent ordinary 3-space, the increasing parameter represents time. No linear connection and no metric is used in this approach up to this stage. Therefore we call it a pre-metric axiomatics (strictly, we should also add the qualification “pre-connection”). Accordingly, our three axioms are valid on a sliceable, but otherwise arbitrary 4-dimensional manifold, see [4], be it described by means of the Minkowskian geometry of special relativity (SR), the Riemannian geometry of general relativity (GR), or the metric-affine geometry of the metric-affine gauge theory of gravity (MAG, see [5]).

The first axiom has the inhomogeneous Maxwell equation \( dH = J \) as a consequence, with \( H = (\mathcal{D}, \mathcal{H}) \) as the electromagnetic excitation 2-form. The third axiom represents the homogeneous Maxwell equation \( dF = 0 \). In order to complete the fundamental structure of classical electrodynamics, one has to specify an axiom for the distribution of the energy and the momentum and the corresponding fluxes of the electromagnetic field. Only in this way we will be able to find a formula for the electromagnetic radiation pressure, for example. As it will turn out, this is also possible in a pre-metric framework.

2. FOURTH AXIOM: LOCALIZATION OF ENERGY-MOMENTUM

Let us consider the Lorentz force density \( f_\alpha = (e_\alpha \mathcal{F}) \wedge J \). If we want to derive the energy-momentum law for electrodynamics, we have to try to express \( f_\alpha \) as an exact form. Then energy-momentum is a kind of a generalized potential for the Lorentz force density, namely \( f_\alpha \sim d \Sigma_\alpha \). For that purpose we start from \( f_\alpha \). We substitute \( J = dH \) (inhomogeneous
Maxwell equation) and subtract out a term with \( H \) and \( F \) exchanged and multiplied by a constant factor \( a \):

\[
f_\alpha = (e_\alpha [F] \land dH - a (e_\alpha [H] \land dF). \tag{2.1}
\]

Because of \( dF = 0 \) (homogeneous Maxwell equation), the subtracted term vanishes. The factor \( a \) will be left open for the moment. Note that we need a non-vanishing current \( J \neq 0 \) for our derivation to be sensible.

We partially integrate both terms in (2.1):

\[
f_\alpha = d[a F \land (e_\alpha [H]) - H \land (e_\alpha [F])] - a F \land d(e_\alpha [H]) + H \land d(e_\alpha [F]). \tag{2.2}
\]

The first term has already the desired form. We recall the main formula for the Lie derivative of an arbitrary form \( \Phi \), namely \( L_{e_\alpha} \Phi = d(e_\alpha [\Phi]) + e_\alpha [(d\Phi)] \). This allows us to transform the second line of (2.2):

\[
f_\alpha = d[a F \land (e_\alpha [H]) - H \land (e_\alpha [F])] - a F \land (L_{e_\alpha} H) + H \land (L_{e_\alpha} F) + a F \land e_\alpha [(dH)] - H \land e_\alpha [(dF)]. \tag{2.3}
\]

The last line can be rewritten as

\[
+ a e_\alpha [F \land dH] - a (e_\alpha [F]) \land dH - e_\alpha [H \land dF] + (e_\alpha [H]) \land dF. \tag{2.4}
\]

As 5-forms, the expressions in the square brackets vanish. Two terms remain, and we find

\[
f_\alpha = d[a F \land (e_\alpha [H]) - H \land (e_\alpha [F])] - a F \land (L_{e_\alpha} H) + H \land (L_{e_\alpha} F) - a (e_\alpha [F]) \land dH + (e_\alpha [H]) \land dF. \tag{2.5}
\]
Now we have to make up our mind about the choice of the factor $a$. Because of $dF = 0$, the third line adds up to $-af_\alpha$. Accordingly,

$$(1 + a)f_\alpha = d[aF \wedge (e_\alpha]H) - H \wedge (e_\alpha]F)] - aF \wedge (\mathcal{L}_{e_\alpha}H) + H \wedge (\mathcal{L}_{e_\alpha}F).$$

With $a = -1$, the left hand side is zero and we find a mathematical identity. A real conservation law is only obtained when, eventually, the second line vanishes. In other words, here we need an a posteriori argument, i.e., we have to take some information from experience. For $a = 0$, the second line does not vanish. However, for $a = 1$, we can hope that the first term in the second line compensates the second term if somehow $H \sim F$. In fact, under “ordinary circumstances”, to be explored below, the two terms in the second line do compensate each other for $a = 1$. Therefore we postulate this choice and find

$$f_\alpha = (e_\alpha]F) \wedge J = d^k \Sigma_\alpha + X_\alpha.$$  

Here the \textit{kinematic energy-momentum} 3-form of the electromagnetic field, a central result of this section, reads

$$^k \Sigma_\alpha := \frac{1}{2} [F \wedge (e_\alpha]H) - H \wedge (e_\alpha]F)] \quad \text{(fourth axiom)},$$

and the remaining force density 4-form turns out to be

$$X_\alpha := -\frac{1}{2} (F \wedge \mathcal{L}_{e_\alpha}H - H \wedge \mathcal{L}_{e_\alpha}F).$$

Incidentally, our formula (2.7) supersedes the corresponding relation of Post, see [6] Eq. (4.42).

Our derivation of (2.7) doesn’t lead to a unique definition of $^k \Sigma_\alpha$. The addition of any closed 3-form would be possible,

$$\Sigma'_\alpha := ^k \Sigma_\alpha + Y_\alpha, \quad \text{with} \quad dY_\alpha = 0,$$

such that

$$f_\alpha = d \Sigma'_\alpha + X_\alpha.$$
In particular, $Y_\alpha$ could be exact: $Y_\alpha = dZ_\alpha$. The 2-form $Z_\alpha$ has the same dimension as $k \Sigma_\alpha$. It seems impossible to build up $Z_\alpha$ exclusively in terms of the quantities $e_\alpha, H, F$ in an algebraic way. Therefore, $Y_\alpha = 0$ appears to be the most natural choice. Thus, by the fourth axiom we postulate that $k \Sigma_\alpha$ in (2.8) represents the energy-momentum current that correctly localizes the energy-momentum distribution of the electromagnetic field in spacetime.

The current $k \Sigma_\alpha$ can also be rewritten by applying the anti-Leibniz rule for $e_\alpha \lrcorner$ either in the first or the second term on the right hand side of (2.8). With the 4-form

$$\Lambda := -\frac{1}{2} F \wedge H,$$

(2.12) we find

$$k \Sigma_\alpha = e_\alpha \lrcorner \Lambda + F \wedge (e_\alpha \lrcorner H)$$

$$= -e_\alpha \lrcorner \Lambda - H \wedge (e_\alpha \lrcorner F).$$

(2.13)

3. PROPERTIES OF THE ENERGY-MOMENTUM CURRENT, ELECTRIC-MAGNETIC RECIPROCITY

A. $k \Sigma_\alpha$ is tracefree

The energy-momentum current $k \Sigma_\alpha$ is a 3-form. We can blow it up to a 4-form according to $\vartheta^\beta \wedge k \Sigma_\alpha$. Since it still has 16 components, we haven’t lost any information. If we recall that for any $p$-form $\Phi$ we have $\vartheta^\alpha \wedge (e_\alpha \lrcorner \Phi) = p \Phi$, we immediately recognize from (2.8) that

$$\vartheta^\alpha \wedge k \Sigma_\alpha = 0,$$

(3.1) which amounts to one equation. This property — the vanishing of the “trace” of $k \Sigma_\alpha$ — is connected with the fact that the electromagnetic field (the “photon”) carries no mass and the theory is thus invariant under dilations. Why we call that the trace of the energy-momentum will become clear below, see (3.17).
B. $^k\Sigma_\alpha$ is electric-magnetic reciprocal

Furthermore, we can observe another property of $^k\Sigma_\alpha$. It is remarkable how symmetric $H$ and $F$ enter (2.8). This was achieved by our choice of $a = 1$. The energy-momentum current is electric-magnetic reciprocal, i.e., it remains invariant under the transformation

$$H \rightarrow \zeta F, \quad F \rightarrow -\frac{1}{\zeta} H, \quad \Rightarrow \quad ^k\Sigma_\alpha \rightarrow ^k\Sigma_\alpha,$$

(3.2)

with the twisted zero-form (pseudo-scalar function) $\zeta = \zeta(x)$ of dimension $[\zeta] = [H]/[F] = q^2/h$. Here $q$ denotes the dimension of charge and $h$ that of action.

It should be stressed that in spite of $^k\Sigma_\alpha$ being e-m reciprocal, Maxwell’s equations are not,

$$dH = J \quad \rightarrow \quad dF + F \wedge d\zeta/\zeta = J/\zeta, \quad (3.3)$$
$$dF = 0 \quad \rightarrow \quad dH - H \wedge d\zeta/\zeta = 0, \quad (3.4)$$

not even for $d\zeta = 0$, since we don’t want to restrict ourselves to the free-field case with vanishing source $J = 0$.

Eq.(3.2) expresses a certain reciprocity between electric and magnetic effects with regard to their respective contributions to the energy-momentum current of the field. We call it electric-magnetic reciprocity.\footnote{...following Toupin \cite{7} even if he introduced this notion in a somewhat more restricted context. Maxwell spoke of the mutual embrace of electricity and magnetism, see Wise \cite{8}. In the case of a prescribed metric, discussions of the corresponding Rainich “duality rotation” were given by Gaillard & Zumino \cite{9} and by Mielke \cite{10}, amongst others.}

That this naming is appropriate can be seen from a (1+3)-decomposition. We recall, see \cite{4}, the (1+3)-decompositions of $H$ and $F$:

$$H = -\mathcal{H} \wedge d\sigma + \mathcal{D}, \quad F = E \wedge d\sigma + B. \quad (3.5)$$
We substitute them in (3.2):

\[
H \rightarrow \zeta F \quad \begin{cases} 
\mathcal{H} \rightarrow -\zeta E, \\
\mathcal{D} \rightarrow \zeta B,
\end{cases}
\]  \hspace{1cm} \text{(3.6)}

\[
F \rightarrow -\frac{1}{\zeta} H \quad \begin{cases} 
E \rightarrow \frac{1}{\zeta} \mathcal{H}, \\
B \rightarrow -\frac{1}{\zeta} \mathcal{D}.
\end{cases}
\]  \hspace{1cm} \text{(3.7)}

Here it is clearly visible that a magnetic quantity is replaced by an electric one and an electric quantity by a magnetic one: \textit{electric} \leftrightarrow \textit{magnetic}. In this sense, we can speak of an electric-magnetic reciprocity in the expression for the energy-momentum current \(\Sigma\). Alternatively we can say that \(\Sigma\) fulfills e-m reciprocity, it is e-m reciprocal.

Let us pause for a moment and wonder of how the notions “electric” and “magnetic” are attached to certain fields and whether there is a conventional element involved. By making experiments with a cat’s skin and a rod of amber, we can “liberate” what we call \textit{electric} charges. In 3 dimensions, they are described by the charge density \(\rho\). Set in motion, they produce an electric current \(j\). The electric charge is conserved (first axiom) and is linked, via the Gauss law \(\mathbf{d} \mathcal{D} = \rho\), to the \textit{electric} excitation \(\mathcal{D}\).

Recurring to the Oersted experiment, it is clear that moving charges \(j\) induce magnetic effects, in accordance with the Oersted-Ampère law \(\mathbf{d} \mathcal{H} - \mathbf{\dot{D}} = j\) — also a consequence of the first axiom. Hence we can unanimously attribute the term \textit{magnetic} to the excitation \(\mathcal{H}\). There is no room left for doubt about that.

The second axiom links the electric charge density \(\rho\) to the field strength \(E\) according to \((e_a] \rho) \wedge E\) and the electric current \(j\) to the field \(B\) according to \((e_a] j) \wedge B\). Consequently, also for the field strength \(F\), there can be no other way than to label \(E\) as \textit{electric} and \(B\) as \textit{magnetic} field strength.

These arguments imply that the substitutions \(H \rightarrow \zeta F\) as well as \(F \rightarrow -H/\zeta\) both substitute an electric by a magnetic field and a magnetic by an electric one, see (3.6) and
Because of the minus sign (that is, because of $a = 1$) that we found in (3.2) in analyzing the electromagnetic energy-momentum current $\Sigma_\alpha$, we cannot speak of an equivalence of electric and magnetic fields, the expression reciprocity is much more appropriate. Fundamentally, electricity and magnetism enter into classical electrodynamics in an asymmetric way.

Let us try to explain the electric-magnetic reciprocity by means of a simple example. If we apply a $(1+3)$-decomposition to $k\Sigma_{\alpha}$, the electric energy density turns out to be $u_{el} = \frac{1}{2} E \wedge D$. If one wants to try to guess the corresponding expression for the magnetic energy density $u_{mag}$, one substitutes for an electric a corresponding magnetic quantity. However, the electric field strength is a 1-form. One cannot substitute it by the magnetic field strength $B$ since that is a 2-form. Therefore one has to switch to the magnetic excitation according to $E \rightarrow \frac{1}{\zeta} H$, with the 1-form $H$. The function $\zeta$ is needed because of the different dimensions of $E$ and $H$ and since $E$ is an untwisted and $H$ a twisted form. Analogously, one substitutes $D \rightarrow \zeta B$ thereby finding $u_{mag} = \frac{1}{2} H \wedge B$. This is the correct result, i.e., $u = \frac{1}{2}(E \wedge D + B \wedge H)$, and we could be happy.

Naively, one would then postulate the invariance of $u$ under the substitution $E \rightarrow \frac{1}{\zeta} H$, $D \rightarrow \zeta B$, $B \rightarrow \frac{1}{\zeta} D$, $H \rightarrow \zeta E$. But, as a look at (3.5) will show, this cannot be implemented in a covariant way because of the minus sign in (3.5). One could reconsider the sign convention for $H$ in (3.5)1. However, as a matter of fact, the relative sign between (3.5)1 and (3.5)2 is basically fixed by Lenz’s rule (the induced electromotive force [measured in volt] is opposite in sign to the inducing field). Thus the minus sign in (3.5) is independent of conventions.

How are we going to save our rule of thumb for extracting the magnetic energy from the electric one? Well, if we turn to the substitutions (3.6) and (3.7), i.e., if we introduce two minus signs according to $E \rightarrow \frac{1}{\zeta} H$, $D \rightarrow \zeta B$, $B \rightarrow -\frac{1}{\zeta} D$, $H \rightarrow -\zeta E$, then $u$ still remains invariant and we recover the covariant rule (3.2). In other words, the naive approach works up to two minus signs. Those we can supply by having insight into the covariant version of
electrodynamics. Accordingly, the electric-magnetic reciprocity is the one that we knew all the time – we just have to be careful with the sign.

\[ C. \ k_{\Sigma} \text{ expressed in terms of the complex electromagnetic field} \]

We can understand the e-m reciprocity transformation as acting on the column vector consisting of \( H \) and \( \zeta F \):

\[
\begin{pmatrix} H' \\ \zeta F' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H \\ \zeta F \end{pmatrix}. \quad (3.8)
\]

In order to compactify this formula, we introduce the complex electromagnetic field

\[ U := H + i \zeta F \quad \text{and} \quad U^* = H - i \zeta F, \quad (3.9) \]

with \( * \) denoting the conjugate complex. Now the e-m reciprocity (3.8) translates into

\[ U' = -i U, \quad U'^* = i U^* . \quad (3.10) \]

This corresponds, in the complex plane, where \( U \) lives, to a rotation by an angle of \(-\pi/2\).

We can resolve (3.9) with respect to excitation and field strength:

\[ H = \frac{1}{2} (U + U^*), \quad F = -\frac{i}{2\zeta} (U - U^*). \quad (3.11) \]

We differentiate (3.9)\textsubscript{1}. Then the Maxwell equation for the complex field turns out to be

\[ dU + (U^* - U) \frac{d\zeta}{2\zeta} = J. \quad (3.12) \]

Clearly, if we choose a constant \( \zeta \), i.e., \( d\zeta = 0 \), the second term on the left hand side vanishes.

The asymmetry between electric and magnetic fields finds its expression in the fact that the source term on the right hand side of (3.12) is a \emph{real} quantity.

If we substitute (3.11) into the energy-momentum current (2.8), we find, after some algebra,

\[ k_{\Sigma} = \frac{i}{4\zeta} [U^* \wedge (e_\alpha U) - U \wedge (e_\alpha U^*)]. \quad (3.13) \]
Now, according to (3.10), e-m reciprocity of the energy-momentum current is manifest.

If we execute successively two e-m reciprocity transformations, namely $U \rightarrow U' \rightarrow U''$, then, as can be seen from (3.10) or (3.2), we find a reflection (a rotation of $-\pi$), namely $U'' = -U$, i.e.,

$$U \rightarrow -U \quad \text{or} \quad (H \rightarrow -H, F \rightarrow -F). \quad (3.14)$$

Only four e-m reciprocity transformations lead back to the identity. It should be stressed, however, that already one e-m reciprocity transformation leaves $k\Sigma_\alpha$ invariant.

It is now straightforward to formally extend the e-m reciprocity transformation (3.10) to

$$U' = e^{+i\phi} U, \quad U'^* = e^{-i\phi} U^*, \quad (3.15)$$

with $\phi = \phi(x)$ as an arbitrary “rotation” angle. The energy-momentum current $k\Sigma_\alpha$ is still invariant under this extended transformation, but in later applications only the subcase of $\phi = -\pi/2$, treated above, will be of interest.

\section*{D. Energy-momentum tensor density $kT_{\alpha\beta}$}

Since $k\Sigma_\alpha$ is a 3-form, we can decompose it either conventionally or with respect to the basis 3-form $\epsilon_\beta := e_\beta \lbrack \epsilon$, with $\epsilon := \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$:

\begin{equation}
 k\Sigma_\alpha = \frac{1}{3!} k\Sigma_{\lambda\mu\nu} \vartheta^\lambda \wedge \vartheta^\mu \wedge \vartheta^\nu = kT_{\alpha}^{\beta} \epsilon_\beta. \quad (3.16)
\end{equation}

The 2nd rank tensor density of weight 1, $kT_{\alpha}^{\beta}$, is the \textit{Minkowski} energy tensor density. We can resolve this equation with respect to $kT_{\alpha}^{\beta}$ by exterior multiplication with $\vartheta^\beta$. We recall $\vartheta^\beta \wedge \epsilon_\gamma = \delta^\beta_\gamma \epsilon$ and find

\begin{equation}
 kT_{\alpha}^{\beta} \epsilon = \vartheta^\beta \wedge k\Sigma_\alpha. \quad (3.17)
\end{equation}

Thereby we recognize that $\vartheta^\alpha \wedge k\Sigma_\alpha = 0$, see (3.1), is equivalent to the vanishing of the trace of the energy-momentum tensor density $kT_{\alpha}^{\alpha} = 0$. Thus $kT_{\alpha}^{\beta}$ as well as $k\Sigma_\alpha$ have 15 independent components at this stage. Both quantities are equivalent.
If we substitute (2.8) into (3.17), then we can express the energy-momentum tensor density in the components of $H$ and $F$ as follows:

$$k\mathcal{T}_\alpha^\beta = \frac{1}{4} \varepsilon^{\beta\mu\rho\sigma} (H_{\alpha\mu} F_{\rho\sigma} - F_{\alpha\mu} H_{\rho\sigma}).$$

(3.18)

**E. Preview: Covariant conservation law and vanishing force density $\hat{X}_\alpha$**

The Lorentz force density $f_\alpha$ in (2.7) and the energy-momentum current $k\Sigma_\alpha$ in (2.8) are covariant with respect to frame and coordinate transformations. Nevertheless, each of the two terms on the right hand side of (2.7), namely $dk\Sigma_\alpha$ or $X_\alpha$, are not covariant by themselves. What can we do?

For the first three axioms of electrodynamics, the spacetime arena is only required to be a (1+3)-decomposable 4-dimensional manifold. We cannot be as economical as this in general. Ordinarily a linear connection $\Gamma_\alpha^\beta$ on that manifold is needed. The linear connection will be the guiding field that transports a vector, e.g., from one point of spacetime to a neighboring one. Then the covariant exterior differential is be defined $D = d + \Gamma_\alpha^\beta \rho (L_\beta^\alpha)$, see [5]. With the help of this operator, a generally covariant expression $D k\Sigma_\alpha$ can be constructed. Then (2.7) can be rewritten as

$$f_\alpha = D k\Sigma_\alpha + \hat{X}_\alpha,$$

(3.19)

with the new supplementary force density

$$\hat{X}_\alpha = \frac{1}{2} (H \wedge L_{e\alpha} F - F \wedge L_{e\alpha} H),$$

(3.20)

which contains the gauge covariant Lie derivative

$$L_\xi := D \xi + \xi | D.$$

(3.21)

---

2Our conventions with respect to the totally antisymmetric Levi-Civita symbol are as follows: We have $\varepsilon^{0123} = +1$, i.e., $\varepsilon^{\alpha\beta\gamma\delta}$ is a tensor density of weight +1. Its reciprocal $\hat{\varepsilon}_{\mu\nu\rho\sigma}$, a tensor density of weight −1, is defined according to $\varepsilon^{\alpha\beta\gamma\delta} \hat{\varepsilon}_{\mu\nu\rho\sigma} = \delta_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}$, with $\delta$ as generalized Kronecker symbol.
Note that the energy-momentum current $k \Sigma_\alpha$ remains the same, only the force density $X_\alpha$ is replaced by $\hat{X}_\alpha$. Now we will be able to show that the fourth axiom is exactly what is needed for an appropriate and consistent derivation of the conservation law for energy-momentum.

It is remarkable, in (3.19) [or in (2.7)] the energy-momentum current can be defined even if (3.19), as long as $\hat{X}_\alpha \neq 0$, doesn’t represent a conservation law. However, in order to find a genuine conservation law, we have to require that $\hat{X}_\alpha$ vanishes. This cannot be achieved without some knowledge on the relation between excitation and field strength. Only this *electromagnetic spacetime relation* between $H$ and $F$, which we shall postulate as fifth axiom in (6.1), makes electrodynamics a complete theory. At first, however, we don’t want to commit ourselves to this axiom. We rather would like to exploit the arbitrary linear connection $\Gamma_\alpha^\beta$, introduced above, as far as possible. As auxiliary quantities, attached to $\Gamma_\alpha^\beta$, we need the *torsion* 2-form $T_\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta^\alpha \wedge \vartheta^\beta = T_\mu^\alpha \vartheta^\mu \wedge \vartheta^\nu / 2$ and the *transposed connection* 1-form $\tilde{\Gamma}_\alpha^\beta := \Gamma_\alpha^\beta + e_\alpha^\beta T_\beta$.

Let us now consider the extra (supplementary, or offending) force density $\hat{X}_\alpha$. What we need is the gauge covariant Lie derivative of an arbitrary 2-form $\Psi = \Psi_{\mu\nu} \vartheta^\mu \wedge \vartheta^\nu / 2$ in terms of its components. Using (3.21), some algebra yields

$$L_{\varepsilon_\alpha} \Psi = \frac{1}{2} \left( \hat{D}_\alpha \Psi_{\mu\nu} \right) \vartheta^\mu \wedge \vartheta^\nu,$$

where $\hat{D}_\alpha := e_\alpha^\gamma \hat{D}$, with $\hat{D}$ as the exterior covariant differential with respect to the transposed connection. Thus,

$$\hat{X}_\alpha = \frac{1}{8} \left( H_{\rho\sigma} \hat{D}_\alpha F_{\mu\nu} - F_{\rho\sigma} \hat{D}_\alpha H_{\mu\nu} \right) \vartheta^\rho \wedge \vartheta^\sigma \wedge \vartheta^\mu \wedge \vartheta^\nu,$$

or, since $\vartheta^\rho \wedge \vartheta^\sigma \wedge \vartheta^\mu \wedge \vartheta^\nu = \epsilon^{\rho\sigma\mu\nu} \epsilon$, we find

$$\hat{X}_\alpha = \frac{\epsilon^{\rho\sigma\mu\nu}}{8} \left( H_{\rho\sigma} \hat{D}_\alpha F_{\mu\nu} - F_{\rho\sigma} \hat{D}_\alpha H_{\mu\nu} \right).$$

This is as far as we can go with an arbitrary linear connection.
4. LINEAR SPACETIME RELATION, ABELIAN AXION

In order to make electrodynamics a complete theory, eventually more detailed properties of spacetime have to come into play. We have to find a relationship between excitation $H$ and field strength $F$. We shall call it the \textit{electromagnetic spacetime relation}. Following Toupin \cite{Toupin1960}, Schönberg \cite{Schonberg1962}, Jadczyk \cite{Jadczyk1989}, and the authors with Rubilar and Fukui \cite{Rubilar1989, Rubilar1990}, see also Gross & Rubilar \cite{Gross1991}, a \textit{linear} relation is assumed between $H$ and $F$. In coordinate (natural) components, it reads

$$H_{ij} = \frac{1}{2} \kappa_{ij}^{kl} F_{kl}.$$  \hfill (4.1)

Compare also Tamm \cite{Tamm1938} and Post \cite{Post1960} in this context. This linear spacetime relation postulates the existence of $6 \times 6$ pseudo-scalar functions $\kappa$ with

$$\kappa_{ij}^{kl} = - \kappa_{ji}^{kl} = - \kappa_{ij}^{lk}.$$  \hfill (4.2)

We can alternatively introduce

$$\chi_{ijkl} = \frac{1}{2} \epsilon_{ijmn} \kappa_{mn}^{kl}.$$  \hfill (4.3)

Its reciprocal reads

$$\kappa_{ij}^{kl} = \frac{1}{2} \epsilon_{ijmn} \chi_{mnkl}.$$  \hfill (4.4)

The 36 functions $\kappa_{ij}^{kl}(\sigma, x)$ as well as the $\chi_{ijkl}(\sigma, x)$ depend on time $\sigma$ and on space $x$ in general. Because of the corresponding properties of the Levi-Civita symbol, the $\chi_{ijkl}$ represent an untwisted \textit{tensor density of weight} $+1$. As is clear from (4.1) and (4.4), the $\kappa_{ij}^{kl}$ and the $\chi_{ijkl}$ both carry the dimension $[\kappa] = [\chi] = q^2/h$.

With the linear law we can set up a Lagrange 4-form $V$. Because of $H = - \partial V / \partial F$, the Lagrangian must be quadratic in $F$. Thus we find

\footnote{Truesdell \& Toupin \cite{Truesdell1960}, Toupin \cite{Toupin1960}, and Kovetz \cite{Kovetz1960} call it “aether relations”.

\hline

13
\[ V = -\frac{1}{2} H \wedge F = -\frac{1}{8} H_{ij} F_{pq} \, dx^i \wedge dx^j \wedge dx^p \wedge dx^q \]
\[ = -\frac{1}{32} \chi^{mnkl} \hat{\epsilon}_{ijmn} F_{kl} \, dx^i \wedge dx^j \wedge dx^p \wedge dx^q. \]  

Equation (4.5)

We rewrite the exterior products with the Levi-Civita symbol. Then we have

\[ V = -\frac{1}{8} \chi^{ijkl} F_{ij} F_{kl} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \]  

Equation (4.6)

The components of the field strength \( F \) enter in a symmetric way. Therefore, without loss of generality, we can impose the symmetry condition

\[ \chi^{ijkl} = \chi^{klji} \]  

Equation (4.7)

reducing them to 21 independent functions at this stage.

Right now (still without a metric), we can split off the totally antisymmetric part of \( \chi^{ijkl} \) according to

\[ \chi^{ijkl} = \tilde{\chi}^{ijkl} + \alpha \hat{\epsilon}^{ijkl}, \quad \text{with} \quad \tilde{\chi}^{[ijkl]} = 0. \]  

Equation (4.8)

Here \( \alpha = \alpha(\sigma, x) \) is a pseudo-scalar function with the dimension \([\alpha] = q^2/h\). Thus the linearity ansatz eventually reads

\[ H_{ij} = \frac{1}{4} \hat{\epsilon}_{ijmn} \chi^{mnkl} F_{kl} = \frac{1}{4} \hat{\epsilon}_{ijmn} \tilde{\chi}^{mnkl} F_{kl} + \alpha F_{ij}, \]  

Equation (4.9)

with

\[ \tilde{\chi}^{mnkl} = -\tilde{\chi}^{nmkl} = -\tilde{\chi}^{mnlk} = \tilde{\chi}^{klmn} \quad \text{and} \quad \tilde{\chi}^{[mnkl]} = 0. \]  

Equation (4.10)

Besides the Abelian axion field \( \alpha \), we have 20 independent functions. Accordingly, the tensor \( \tilde{\chi}^{mnkl} \) has the same algebraic symmetries and the same number of independent components as a curvature tensor in a 4-dimensional Riemannian spacetime. It is remarkable that the pseudo-scalar axion field \( \alpha \), see [20] and references therein, enters here as a quantity that does not interfere at all with the first four axioms of electrodynamics. Already at the pre-metric level, such a field emerges as a not unnatural companion of the electromagnetic field.
However, to make it a real independent field, kinetic terms $\sim d\alpha$ would have to be added to the Lagrange 4-form $V$.

Alternatively, the linear relation (4.9) can be reformulated as

$$H_{ij} = \frac{1}{2} \tilde{\#}_{ij}^{kl} F_{kl} + \alpha F_{ij}, \quad (4.11)$$

where $\tilde{\#}$ (speak “sharp tilde”) is a certain duality operator which acts linearly in the space of two-forms,

$$\tilde{\#} : \Lambda^2 X \longrightarrow \Lambda^2 X. \quad (4.12)$$

In particular, the action on the basis of 2-forms reads:

$$\tilde{\#} (dx^i \wedge dx^j) = \frac{1}{2} \tilde{\#}_{ij}^{kl} (dx^k \wedge dx^l) = \frac{1}{4} \epsilon_{klmn} \tilde{\chi}^{mnij} (dx^k \wedge dx^l). \quad (4.13)$$

We display the duality operator as a superscript $\tilde{\#}$. However, in its components $\tilde{\#}_{ij}^{kl}$, the sharp sign is treated like an ordinary letter.

Now the linear material law (4.9) can be written as

$$H = (\tilde{\#} + \alpha) F. \quad (4.14)$$

5. ELECTRIC-MAGNETIC RECIPROCITY OF THE SPACETIME RELATION

If we exploit the linear relationship between excitation and field strength and substitute it in the Maxwell equations, we end up with the propagation of the electromagnetic disturbances in vacuum spacetime along quartic wave surfaces, for related work see [21–23]. Accordingly, spacetime would be birefringent, for instance, a property which seems to be contrary to experiment, at least at in our part of the universe. If, say, at a very early time in the development of the universe the vacuum were birefringent, then we had already a consistent theory for such an effect, namely the one we formulated in the last section.

If we want to forbid birefringence for vacuum spacetime, then we can try to impose a constraint on the linear relationship (4.14). The obvious choice is to require electric-magnetic reciprocity for (4.14). We have discovered e-m reciprocity as a property of the
energy-momentum current $k \Sigma_\alpha$ of the electromagnetic field. Why should’t we apply it to (4.14)?

The e-m reciprocity transformation

$$H \rightarrow \zeta F, \quad F \rightarrow -\frac{1}{\zeta} H.$$ (5.1)

can alternatively be written as

$$
\begin{pmatrix}
H \\
F
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & \zeta \\
-\frac{1}{\zeta} & 0
\end{pmatrix}
\begin{pmatrix}
H \\
F
\end{pmatrix} =
\begin{pmatrix}
\zeta F \\
-\frac{1}{\zeta} H
\end{pmatrix}.
$$ (5.2)

If

$$W := \begin{pmatrix}
0 & \zeta \\
-\frac{1}{\zeta} & 0
\end{pmatrix}, \quad \text{then} \quad W^{-1} = \begin{pmatrix}
0 & -\zeta \\
\frac{1}{\zeta} & 0
\end{pmatrix}. \quad \quad \quad \text{(5.3)}$$

Let us perform an e-m reciprocity transformation in (4.14). By definition, the reciprocity transformation commutes with the duality operator $\tilde{\#}$. Then we find

$$\zeta F = (\tilde{\#} + \alpha) \left( -\frac{H}{\zeta} \right) \quad \text{or} \quad \tilde{\#} H = -\zeta^2 F - \alpha H.$$ (5.4)

On the other hand, we can also apply the duality operator $\tilde{\#}$ to (4.14). Because $\tilde{\#}$ commutes with 0-forms, we get

$$\tilde{\#} H = \left( \tilde{\#} \tilde{\#} + \alpha \tilde{\#} \right) F.$$ (5.5)

If we postulate e-m reciprocity of the linear law (4.14), then, as a comparison of (5.5) and (5.4) shows, we have to assume additionally

$$\tilde{\#} \tilde{\#} = -\zeta^2 \mathbb{I}, \quad \alpha = 0.$$ (5.6)

We call $\tilde{\#} \tilde{\#} = -\zeta^2 \mathbb{I}$ the closure relation since applying the operator $\tilde{\#}$ twice, we come back to the identity operator $\mathbb{I} (= \delta_{ij}^{kl}$, in components), up to a negative function. In this sense, the operation closes. At the same time, (5.4) tells us that the spacetime relation is not e-m reciprocal for an arbitrary transformation function $\zeta$ (like the energy-momentum current is).
The duality operator $\tilde{\#}$ is based on the measurable components $\tilde{\chi}^{ijkl}$. If applied twice, as in (5.6), there must not emerge an arbitrary function. In other words, we can solve (5.7), by taking its trace, to get

$$\zeta^2 = -\frac{1}{6} \text{Tr} (\tilde{\#} \tilde{\#}) = -\frac{1}{24} \tilde{\#}^{ij} \tilde{\#}^{kl} =: \lambda^2. \quad (5.7)$$

It is natural to factorize the dimensionfull function $\lambda$ in the constitutive matrix (4.8)-(4.11) which yields a representation

$$\tilde{\chi}^{ijkl} = \lambda \, \overset{o}{\chi}^{ijkl} \quad (5.8)$$

so that $\overset{o}{\chi}^{ijkl}$ is now the dimensionless tensor with the same symmetries as in (4.10). This effectively redefines the duality operator

$$\tilde{\#} =: \lambda \#, \quad (5.9)$$

and for $\#$ the closure reads

$$\#\# = -\mathbb{I}. \quad (5.10)$$

As we will see, the minus sign is very decisive: It will eventually yield the Lorentzian signature of the metric of spacetime.

Let us collect our results. The e-m reciprocity of the linear ansatz leads to the relations

$$H = \lambda \# F \quad \text{and} \quad \# H = -\lambda F. \quad (5.11)$$

We can also consider the duality operator $\#$ from another point of view. It is our desire to eventually describe empty spacetime with such a linear ansatz. Therefore we have to reduce the number of independent functions $\overset{o}{\chi}^{ijkl}$ somehow. The only constants with even parity are the Kronecker deltas $\delta^j_i$. Obviously the $\delta^j_i$'s are of no help here in specifying the $\overset{o}{\chi}^{ijkl}$'s, since they carry also lower indices which cannot be absorbed in a non-trivial way in order to bring about the 4 upper indices of $\overset{o}{\chi}^{ijkl}$. Recognizing that in the framework of electrodynamics in matter a similar linear ansatz can describe anisotropic media, we need
a condition in order to exclude so-called non-reciprocal effects. A “square” of \( \chi \) will do the job,

\[
\frac{1}{8} \chi^{ijkl} \chi^{mnop} \hat{\epsilon}_{klmn} \hat{\epsilon}_{pqrs} = \mp \delta^{ij}_{rs},
\]

(5.12)

with \( \delta^{ij}_{rs} \) as a generalized Kronecker delta. This equation represents, together the linearity ansatz, our fifth axiom for spacetime. Alternatively, it can also be written as

\[
\frac{1}{2} \#_{ij} \#_{kl} \#_{mn} = \mp \delta^{mn}_{ij}.
\]

(5.13)

Apparently, the equations (5.6) or (5.12) and (5.13), being consequences of e-m reciprocity, are just alternative formulations of the closure property.

6. FIFTH AXIOM, THE CONSTRUCTION OF THE METRIC

The linear spacetime relation \( H = (\lambda^{\#} + \alpha)F \), see (4.14) and (5.9), together with the constraints \( ## = -1 \) and \( \alpha = 0 \), see (5.6), constitute our fifth axiom of electrodynamics. Since the dimensions of \( H \) and \( F \) are fixed, the unknown scalar function \( \lambda \) is to be determined by experiment (like all of the components of the electromagnetic spacetime matrix). Thus our axiom reads

\[
H = \lambda^{\#} F, \quad \text{with} \quad ## = -1.
\]

(6.1)

The closure relation \((5.1)\) affects the quartic wave surface mentioned at the beginning of Sec. 5. The wave surface reduces to the usual light cone surface. Thus, eventually, the closure relation yields the unique lightcone structure for the propagation of electromagnetic waves. As a result, up to a conformal factor, the \textit{spacetime metric} \( g \) with the correct Lorentzian signature is constructed by means of the fifth axiom. Our duality operator becomes the Hodge star operator * with respect to the metric \( g \), see [15, 17], i.e., the fifth axiom becomes

\[
H = \lambda^{\#} F.
\]

(6.2)
For $\lambda = \text{const}$, we will call (6.2) the Maxwell-Lorentz spacetime relation. Only a constant $\lambda$ will provide for the vanishing of the extra force density (3.24).

The numerical value of the constant factor in (6.2) is fixed by experiment:

$$\lambda = \sqrt{\frac{\varepsilon_0}{\mu_0}} = \frac{e^2}{4\pi\alpha_f\hbar} = 2.6544187283 \frac{1}{k\Omega}.$$  

(6.3)

Here $e$ is the charge of the electron and $\alpha_f = 1/137.036$ the fine structure constant. The inverse $1/\lambda$ is called the characteristic impedance (or wave resistance) of the vacuum. This is a fundamental constant which describes the basic electromagnetic property of spacetime if considered as a special type of medium (sometimes called vacuum, or aether, in the old terminology). In this sense, one can understand (6.2) as the constitutive relations for the spacetime itself. The Maxwell-Lorentz spacetime relation (6.2) is universal. It is equally valid in Minkowski, Riemannian, and post-Riemannian spacetimes. The electric constant $\varepsilon_0$ and the magnetic constant $\mu_0$ (also called vacuum permittivity and vacuum permeability, respectively) determine the universal constant of nature

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$  

(6.4)

that describes the velocity of light in vacuum.

If (6.2) is substituted into the Maxwell equations, we find the Maxwell-Lorentz equations

$$d\star F = J/\lambda, \quad dF = 0$$  

(6.5)

of standard electrodynamics.

7. SYMMETRY OF THE ENERGY-MOMENTUM CURRENT OF THE ELECTROMAGNETIC FIELD

If the spacetime metric $g$ is given, then there exists a unique torsion-free and metric-compatible Levi-Civita connection $\tilde{\Gamma}_{\alpha}{}^{\beta}$. Consider the conservation law (3.19). In a Riemannian space, the covariant Lie derivative $\tilde{\mathcal{L}}_{\xi} = \tilde{D}\xi + \xi\tilde{D}$ commutes with the Hodge
operator, $\tilde{L}_\xi^* = \star \tilde{L}_\xi$. Thus (3.20) straightforwardly yields

$$\hat{X}_\alpha = \frac{\lambda}{2} \left( *F \wedge \tilde{L}_{e\alpha} F - F \wedge \tilde{L}_{e\alpha} *F \right) = 0. \tag{7.1}$$

Therefore in general relativity (GR), with the Maxwell-Lorentz spacetime relation, (3.19) simply reduces to

$$\tilde{D}^k \Sigma_\alpha = (e_\alpha \downarrow F) \wedge J. \tag{7.2}$$

The energy-momentum current (2.8) now reads

$$k \Sigma_\alpha = \frac{\lambda}{2} \left[ F \wedge (e_\alpha \downarrow *F) - (e_\alpha \downarrow F) \wedge *F \right]. \tag{7.3}$$

In the absence of sources, $J = 0$, we find the energy-momentum law

$$\tilde{D}^k \Sigma_\alpha = 0. \tag{7.4}$$

In the flat Minkowski spacetime of SR, we can globally choose the coordinates in such a way that $\tilde{\Gamma}_\alpha^\beta = 0$. Thus $\tilde{D}^* = d$ and $d^k \Sigma_\alpha = 0$.

As we already know from (3.1), the current (7.3) is traceless $\vartheta^\alpha \wedge k \Sigma_\alpha = 0$. Moreover, we now can use the metric and prove also its symmetry. We multiply (7.3) by $\vartheta_\beta = g_{\beta\gamma} \vartheta^\gamma$ and antisymmetrize:

$$4 \vartheta_\beta \wedge \Sigma_\alpha = \vartheta_\beta \wedge F \wedge (e_\alpha \downarrow *F) - \vartheta_\beta \wedge (e_\alpha \downarrow F) \wedge *F$$

$$- \vartheta_\alpha \wedge F \wedge (e_\beta \downarrow *F) + \vartheta_\alpha \wedge (e_\beta \downarrow F) \wedge *F. \tag{7.5}$$

Because of the identities $e_\alpha \downarrow *\Phi = *(\Phi \wedge \vartheta_\alpha)$ and $\star \Phi \wedge \Psi = \star \Psi \wedge \Phi$ (for all $p$-forms $\Psi$ and $\Phi$), the first term on the right-hand side can be rewritten,

$$\vartheta_\beta \wedge F \wedge (e_\alpha \downarrow *F) = F \wedge \vartheta_\alpha \wedge * (\vartheta_\beta \wedge F)$$

$$= F \wedge \vartheta_\alpha \wedge (e_\beta \downarrow *F), \tag{7.6}$$

i.e., it is compensated by the third term. We apply the analogous technique to the second term. Because $**F = -F$, we have
\[ \partial_\beta \wedge * (\ast F \wedge \partial_\alpha) \wedge \ast F = \ast (\ast F \wedge \partial_\alpha) \wedge \partial_\beta \wedge \ast F \]
\[ = -\ast (\partial_\beta \wedge \ast F) \wedge \ast F \wedge \partial_\alpha \]
\[ = -\partial_\alpha \wedge (\epsilon_\beta \ast F) \wedge \ast F. \quad (7.7) \]

In other words, the second term is compensated by the fourth one and we finally have

\[ \partial_{[\beta} \wedge k\Sigma_{\alpha]} = 0. \quad (7.8) \]

Alternatively, we can work with the energy-momentum tensor. We decompose the 3-form \( k\Sigma_\alpha \) with respect to the \( \eta \)-basis. This is now possible since a metric is available. Because of \( \vartheta_\alpha \wedge \eta_\gamma = \delta_\alpha^\gamma \eta \), with the volume 4-form \( \eta = \sqrt{-\det g_{\mu\nu}} \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3 \), we find

\[ k\Sigma_\alpha =: kT^\beta_{\alpha} \eta_\beta \quad \text{or} \quad kT_{\alpha\beta} = \ast (\partial_\beta \wedge k\Sigma_\alpha), \quad (7.9) \]

compare this with (3.16)-(3.17). We have

\[ kT_{\alpha\beta} = kT_{\alpha\beta}/\sqrt{-g}. \quad (7.10) \]

Its tracelessness \( kT_\gamma = 0 \) has already been established, see (3.17), its symmetry

\[ kT_{[\alpha\beta]} = 0 \quad (7.11) \]

can be either read off from (7.9) and (7.8) or directly from (3.18) with \( H^{ij} = \frac{1}{2} \epsilon^{ijkl} H_{kl} \sim F^{ij} \).

A manifestly symmetric version of the energy-momentum tensor can be derived from (3.16) and (7.3):

\[ kT_{\alpha\beta} = -\lambda^* \left( \epsilon_\alpha \ast F \right) \wedge \epsilon_\beta \ast F + \frac{1}{2} \partial_{[\alpha\beta} (\ast F \wedge \ast F) \right]. \quad (7.12) \]

Thus \( kT_{\alpha\beta} \) is a traceless symmetric tensor(-valued 0-form) with 9 independent components. Its symmetry is sometimes called a bastard symmetry since it interrelates two indices of totally different origin, as can be seen from (7.8). Without using a metric, the symmetry cannot even be formulated, see (2.8).
8. CONCLUDING REMARKS

The energy-momentum current

$$ k\Sigma_\alpha := \frac{1}{2} [ F \wedge (e_\alpha \lrcorner H) - H \wedge (e_\alpha \lrcorner F) ], \quad (8.1) $$

emerging in the context of the fourth axiom, fulfills all the desirable physical properties. If the Maxwell-Lorentz spacetime relation $H = \sqrt{\varepsilon_0/\mu_0} \ast F$ is substituted, $k\Sigma_\alpha$ becomes symmetric and the conventional energy-momentum tensor for vacuum electrodynamics is recovered. In a future paper, we will discuss the consequences $k\Sigma_\alpha$ has for a consistently formulated energy-momentum current inside matter. We hope to clarify certain aspects of this age-old problem.

Acknowledgments. This work was supported by the DAAD (Kennziffer A/00/06508) and by the Alexander von Humboldt Foundation, Bonn. We are grateful for this support.
REFERENCES

[1] F. W. Hehl and Y. N. Obukhov, A gentle introduction to the foundations of classical electrodynamics: The meaning of the excitations ($\mathcal{D}, \mathcal{H}$) and the field strengths ($E, B$). E-print: [physics/0005083].

[2] F. W. Hehl and Y. N. Obukhov, Foundations of Classical Electrodynamics (Birkhäuser, Boston, MA, 2001/02).

[3] T. Frankel, The Geometry of Physics – An Introduction (Cambridge University Press, Cambridge 1997).

[4] R.A. Puntigam, C. Lämmerzahl, and F.W. Hehl, Maxwell’s theory on a post-Riemannian spacetime and the equivalence principle, Class. Quantum Grav. 14 (1997) 1347-1356.

[5] F.W. Hehl, J.D. McCrea, E.W. Mielke, and Y. Ne’eman, Metric-Affine Gauge Theory of Gravity: Field Equations, Noether Identities, World Spinors, and Breaking of Dilation Invariance, Phys. Rep. 258 (1995) 1-171.

[6] E.J. Post: Formal Structure of Electromagnetics – General Covariance and Electromagnetics (North Holland, Amsterdam 1962 and Dover, Mineola, New York 1997).

[7] R.A. Toupin, Elasticity and electro-magnetics, in: Non-Linear Continuum Theories, C.I.M.E. Conf., Bressanone, Italy (1965). C. Truesdell and G. Grioli coord. Pp. 203-342.

[8] M.N. Wise, The mutual embrace of electricity and magnetism, Science 203 (1979) 1310-1318.

[9] M.K. Gaillard and B. Zumino, Duality rotations for interacting fields, Nucl. Phys. B193 (1981) 221-244.

[10] E.W. Mielke, Geometrodynamics of Gauge Fields — On the geometry of Yang-Mills and
gravitational gauge theories (Akademie-Verlag, Berlin, 1987) Sec.V.1.

[11] C. Truesdell and R.A. Toupin, *The classical field theories*, in *Handbuch der Physik*, Vol. III/1, S. Flügge ed. (Springer, Berlin, 1960) pp. 226-793.

[12] A. Kovetz, *Electromagnetic Theory* (Oxford University Press, Oxford, UK, 2000).

[13] M. Schönberg, *Electromagnetism and gravitation*, *Rivista Brasileira de Fisica* 1 (1971) 91-122.

[14] A.Z. Jadczyk, *Electromagnetic permeability of the vacuum and light-cone structure*, *Bull. Acad. Pol. Sci.*, Sér. sci. phys. et astr. 27 (1979) 91-94.

[15] Yu.N. Obukhov and F.W. Hehl, *Space-time metric from linear electrodynamics*, *Phys. Lett.* B458 (1999) 466-470; F.W. Hehl, Yu.N. Obukhov, and G.F. Rubilar, *Spacetime metric from linear eletrodynamics II*, *Ann. Phys. (Leipzig)* 11 (2000) Spec. Issue SI 71-78.

[16] Yu.N. Obukhov, T. Fukui, and G.F. Rubilar, *Wave propagation in linear electrodynamics*, *Phys. Rev.* D62 (2000) 044050.

[17] A. Gross and G.F. Rubilar, *On the derivation of the spacetime metric from linear electrodynamics*, E-print: [gr-qc/0103016](http://arxiv.org/abs/gr-qc/0103016).

[18] I.E. Tamm. in *I.E. Tamm, Collected Papers* (Nauka, Moscow (1975) Vol. 1: (i) *Electrodynamics of an anisotropic medium in special relativity theory*, pp. 19-31; (ii) *Relativistic crystal optics in relation with the geometry of bi-quadratic form*, pp. 33-61 (in Russian). A short version in German, together with L.I. Mandelstam, ibid. pp. 62-67.

[19] E.J. Post, *The constitutive map and some of its ramifications*, *Ann. Phys. (NY)* 71 (1972) 497-518.

[20] W.-T. Ni, *A non-metric theory of gravity*, Dept. Phys., Montana State Univ., Bozeman.
1973. The paper is available via [http://gravity5.phys.nthu.edu.tw/~](http://gravity5.phys.nthu.edu.tw/~); see also W.-T. Ni, *Equivalence principles and electromagnetism*, Phys. Rev. Lett. **38** (1977) 301-304.

[21] W. Dittrich and H. Gies, *Probing the Quantum Vacuum*. Perturbative effective action approach in quantum electrodynamics and its application. Springer Tracts in Modern Physics. Vol. 166. (Springer, Berlin, 2000).

[22] V.A. De Lorenci, R. Klippert, M. Novello, and J.M. Salim, *Light propagation in nonlinear electrodynamics*, Phys. Lett. **B482** (2000) 134-140; V.A. De Lorenci, M.A. Souza, *Electromagnetic wave propagation inside a material medium: an effective geometry interpretation*, E-print: [gr-qc/0102022](http://arxiv.org/abs/gr-qc/0102022).

[23] U. Leonhardt, *Space-time geometry of quantum dielectrics*, E-print: [physics/0001064](http://arxiv.org/abs/physics/0001064); U. Leonhardt and P. Piwnicki, *Relativistic effects of light in moving media with extremely low group velocity*, Phys. Rev. Lett. **84** (2000) 822-825; see also M. Visser, *Comment on “Relativistic effects of light in moving media with extremely low group velocity”*, Phys. Rev. Lett. **85** (2000) 5252 and references given there.