EXTENDING THE BARNES-RIVERS OPERATORS TO D=3 TOPOLOGICAL GRAVITY.

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Abstract

The spin-projector operators for symmetric rank-2 tensors are reassessed in connection with the issue of topologically massive gravity. The original proposal by Barnes and Rivers is generalised to account for D-dimensional Einstein gravity and 3-dimensional Chern-Simons massive gravitation.

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The possibility of building up a quantum-mechanically consistent gauge theory for the gravitational field seems to be actually realised in 3-dimensional space-time. The early work by Deser, Jackiw and Templeton [1] brings about the issue of a massive dynamical theory for gravitation in 3D. Ever since, topologically massive gravity has been fairly-well investigated in a series of very interesting papers, till very recently it has been shown that it is not only renormalisable [2, 3] but even more: massive 3D-gravity is a finite quantum field theory [4].

The purpose of this letter is to reassess the set of Barnes-Rivers spin operators [5, 6] in the framework of topologically massive gravity. These have been shown to be very relevant in the description of 4D-quantum gravity [7, 8]. We shall in this letter propose a set of operators that extend the original Barnes-Rivers projectors to include D-dimensional massless and massive gravity as well as 3D-gravity with topological mass. The graviton propagators shall be written down and the tree-level unitarity shall be discussed in terms of the residues of the propagators at their poles.

The Barnes-Rivers spin-projectors, as introduced in [5, 6], form a complete set of spin-projector operators in the space of rank-2 tensors. For the symmetric case, they read as below:

\[ P^{(2)}_{\mu \nu, \kappa \lambda} \equiv \frac{1}{2}(\Theta_{\mu \kappa} \Theta_{\nu \lambda} + \Theta_{\mu \lambda} \Theta_{\nu \kappa}) - \frac{1}{3} \Theta_{\mu \nu} \Theta_{\kappa \lambda}, \]
\[ P^{(1)}_{\mu \nu, \kappa \lambda} \equiv \frac{1}{2}(\Theta_{\mu \kappa} \omega_{\nu \lambda} + \Theta_{\mu \lambda} \omega_{\nu \kappa} + \Theta_{\nu \kappa} \omega_{\mu \lambda} + \Theta_{\nu \lambda} \omega_{\mu \kappa}), \]
\[ P^{(0)}_{s \mu \nu, \kappa \lambda} \equiv \frac{1}{3} \Theta_{\mu \nu} \Theta_{\kappa \lambda}, \]
\[ P^{(0)}_{w \mu \nu, \kappa \lambda} \equiv \omega_{\mu \nu} \omega_{\kappa \lambda}, \]
\[ P^{(0)}_{sw \mu \nu, \kappa \lambda} \equiv \frac{1}{\sqrt{3}} \Theta_{\mu \nu} \omega_{\kappa \lambda}, \]
\[ P^{(0)}_{ws \mu \nu, \kappa \lambda} \equiv \frac{1}{\sqrt{3}} \omega_{\mu \nu} \Theta_{\kappa \lambda}, \]

where \( \Theta_{\mu \nu} \) and \( \omega_{\mu \nu} \) are the usual transverse and longitudinal projectors on the space of vectors. The operators in (1.a) and (1.b) are respectively the spin-2 and -1 projectors. The remaining ones project out spin-0 components of rank-2 symmetric tensors.

Let us now consider the Einstein-Hilbert action for gravitation and derive its propagator by means of the algebra of the Barnes-Rivers operators, taken now in a D-dimensional space-time:

\[ \mathcal{L}_{HE} = \frac{1}{2\kappa^2} \sqrt{-g} R. \]
Adopting the viewpoint of expanding the metric field around the flat-space geometry, we have:

\[ g^{\mu\nu}(x) = \eta^{\mu\nu} - \kappa h^{\mu\nu}(x), \]  

where \( h^{\mu\nu} \) is the field variable defining the expansion, and taking into account only the free sector of the expansion, one gets the following free Lagrangean for the \( h^{\mu\nu} \)-field:

\[ \mathcal{L}_{HE}^{\text{free}} = \frac{1}{4} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} - \frac{1}{4} \partial_\lambda h^{\mu\lambda} \partial^\mu h^\nu_\nu + \frac{1}{2} \partial_\lambda h^{\mu\lambda} \partial^\mu h^\nu_\nu - \frac{1}{2} \partial_\lambda h^\lambda_\mu \partial_\nu h^{\mu\nu}. \]  

To give meaning to the integration measure in the generating functional of Green’s functions, it is necessary to fix the gauge invariance

\[ \delta h_{\mu\nu}(x) = \partial_\mu \zeta_\nu(x) + \partial_\nu \zeta_\mu(x), \]

by introducing the De Donder gauge-fixing term:

\[ \mathcal{L}_{g.f.} = \frac{1}{2\alpha} F_\mu F^\mu, \]  

where

\[ F_\mu[h_{\rho\sigma}] = \partial_\lambda (h^\lambda_\mu - \frac{1}{2} \delta^\lambda_\mu h^\nu_\nu). \]

The Hilbert-Einstein Lagrangean with gauge-fixing term can be rewritten in terms of the operators (1.a)-(1.f) according to:

\[ \mathcal{L}^{(2)} = \frac{1}{2} h^{\mu\nu} \mathcal{O}_{\mu\nu,\kappa\lambda} h^{\kappa\lambda}, \]

where

\[ \mathcal{O}_{\mu\nu,\kappa\lambda} = \Box \left( -\frac{1}{2} P^{(2)} - \frac{1}{2\alpha} P^{(1)}_m + \frac{(4\alpha - 3)}{4\alpha} P^{(0)}_s + \frac{\sqrt{3}}{4\alpha} P^{(0)}_{sw} + \frac{\sqrt{3}}{4\alpha} P^{(0)}_{ws} \right)_{\mu\nu,\kappa\lambda}. \]

The associated propagator is obtained from the generating-functional

\[ \mathcal{W}[\tau_{\rho\sigma}] = -\frac{1}{2} \int d^Dx \ d^Dy \ \tau^{\mu\nu} \mathcal{O}_{\mu\nu,\kappa\lambda}^{-1} h^{\kappa\lambda}, \]

so that:

\[ < T [h_{\mu\nu}(x) \ h_{\kappa\lambda}(y)] > = i \mathcal{O}_{\mu\nu,\kappa\lambda}^{-1} \delta^D(x - y). \]

So, using the rank-2 identity in the space of symmetric rank-2 tensors, one gets:

\[ \text{diag. } \eta^{\mu\nu} \equiv (+; - , \cdots, -). \]
< T [h_{\mu\nu}(x) h_{\kappa\lambda}(y)] > = \frac{i}{\Box} \left\{ -2 P^{(2)} - 2\alpha P^{(1)}_{m} - 2\left(\frac{D-5}{D-2}\right) P^{(0)}_{s} + \right.
\left. -2\left(\frac{2D\alpha - 4\alpha - D + 1}{D-2}\right) P^{(0)}_{w} + 2\sqrt{3}\frac{(D-2)}{(D-2)} P^{(0)}_{sw} + \right.
\left. +2\sqrt{3}\frac{(D-2)}{(D-2)} P^{(0)}_{ws} \right\}_{\mu\nu,\kappa\lambda} \delta^{D}(x-y), \tag{11}
\right.

or, in momentum space :

< T [h_{\mu\nu}(-k) h_{\kappa\lambda}(k)] > = \frac{i}{k^{2}} \left\{ \eta_{\mu\kappa}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu\kappa} - \frac{2}{(D-2)}\eta_{\mu\nu}\eta_{\kappa\lambda} + \right.
\left. -(1-\alpha) \left[ \eta_{\mu\kappa}\omega_{\nu\lambda} + \eta_{\nu\lambda}\omega_{\mu\kappa} + \eta_{\nu\kappa}\omega_{\mu\lambda} \right] \right\}, \tag{12}

where the projectors have been replaced by eqs. (1.a)-(1.f), and the gauge-fixing parameter has been kept arbitrary.

Adding to the Hilbert-Einstein action a Proca-like mass term yields the following expression for the graviton propagator :

< T [h_{\mu\nu}(-k) h_{\kappa\lambda}(k)] > = \frac{i}{k^{2}} \left\{ \left( \eta_{\mu\kappa}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu\kappa} + \frac{2}{(D-1)}\eta_{\mu\nu}\eta_{\kappa\lambda} \right) + \right.
\left. + 2i \left[ \frac{(k^{2} - m^{2}) \left[ (2m^{2} - 1)(D-1) + 1 \right] - m^{4}(D-1)}{m^{4}(D-1)} \right] \omega_{\mu\nu}\omega_{\kappa\lambda} + \right.
\left. - \frac{i}{m^{2}} \left[ \eta_{\mu\kappa}\omega_{\nu\lambda} + \eta_{\mu\lambda}\omega_{\nu\kappa} + \eta_{\nu\lambda}\omega_{\mu\kappa} + \eta_{\nu\kappa}\omega_{\mu\lambda} + \right. \right.
\left. + \frac{2}{(D-1)} \left( \eta_{\mu\nu}\omega_{\kappa\lambda} + \eta_{\kappa\lambda}\omega_{\mu\nu} \right) \right] \right\}. \tag{13}

An extension of the Barnes-Rivers operators can be proposed in order to account for D=3 topologically massive gravity [1]. It can be shown that one needs to add two operators to the Table I,

\[ S_{1,\mu\nu,\kappa,\lambda} \equiv \frac{(-\Box)}{4} \left\{ \varepsilon_{\mu\alpha\lambda}\partial_{\nu}\omega^{\alpha}_{\nu} + \varepsilon_{\mu\alpha\kappa}\partial_{\lambda}\omega^{\alpha}_{\mu} + \varepsilon_{\nu\alpha\lambda}\partial_{\kappa}\omega^{\alpha}_{\mu} + \varepsilon_{\nu\alpha\kappa}\partial_{\lambda}\omega^{\alpha}_{\mu} \right\} \tag{14.a} \]

and

\[ S_{2,\mu\nu,\kappa,\lambda} \equiv \frac{\Box}{4} \left\{ \varepsilon_{\mu\alpha\lambda}\eta_{\nu\kappa} + \varepsilon_{\mu\alpha\kappa}\eta_{\nu\lambda} + \varepsilon_{\nu\alpha\lambda}\eta_{\kappa\mu} + \varepsilon_{\nu\alpha\kappa}\eta_{\lambda\mu} \right\} \partial^{\alpha}, \tag{14.b} \]

which can be found by analysing the bilinear part stemming from the gravitational Chern-Simons term :

\[ \mathcal{L}_{CS} = \frac{1}{\mu} \varepsilon^{\lambda\mu\nu} \Gamma_{\mu} \sigma (\partial_{\mu} \Gamma_{\rho} \sigma + \frac{2}{3} \Gamma_{\mu} \sigma \Gamma_{\rho} \varphi), \tag{15} \]
Fixing the gauge as in (6), the bilinear term coming from the Hilbert-Einstein and Chern-Simons actions looks as follows:

$$\mathcal{L}^{(2)} = \frac{1}{2} h^{\mu\nu} \left\{ 4 \left[ \frac{1}{2} P^{(2)} + \frac{1}{2\alpha} P^{(1)}_m - \frac{4\alpha - 3}{4\alpha} P^{(0)}_s + \frac{1}{4\alpha} P^{(0)}_w - \frac{\sqrt{3}}{4\alpha} P^{(0)}_{sw} - \frac{\sqrt{3}}{4\alpha} P^{(0)}_{ws} \right] + \right. $$

$$\left. + 4 \left( \frac{\kappa^2}{\mu} [S_1 + S_2] \right) \right\}_{\mu\nu,\kappa\lambda} h^{\nu\lambda}. \quad \text{(16)}$$

Again, the associated propagator can be read off with the help of the operator algebra displayed in Table I:

$$< T [h_{\mu\nu}(x) h_{\kappa\lambda}(y)] > = \frac{i}{\Box} \left\{ \frac{2(\frac{\mu}{\kappa})^2}{[(\frac{\mu}{\kappa})^2 + 64\Box]} P^{(2)} + 2\alpha P^{(1)}_m + \right. $$

$$\left. - \frac{4(\frac{\mu}{\kappa})^2 + 48\Box}{[(\frac{\mu}{\kappa})^2 + 64\Box]} P^{(0)}_s + 4(\alpha - 1) P^{(0)}_w - 2\sqrt{3} P^{(0)}_{sw} + \right. $$

$$\left. - 2\sqrt{3} P^{(0)}_{ws} - \frac{16(\frac{\mu}{\kappa})^2}{\Box[(\frac{\mu}{\kappa})^2 + 64\Box]} [S_1 + S_2] \right\}_{\mu\nu,\kappa\lambda} \delta^3(x - y). \quad \text{(17)}$$

By choosing $\alpha = 1$ (Feynman gauge), we can write in momentum space:

$$< T [h_{\mu\nu}(-k) h_{\kappa\lambda}(k)] > = \frac{-i}{k^2 [64k^2 - (\frac{\mu}{\kappa})^2]} \left\{ 4 i \left( \frac{\mu}{\kappa^2} \right)^3 k^\alpha \left[ \varepsilon_{\mu\alpha\lambda} \Theta_{\kappa\nu} + \varepsilon_{\mu\alpha\kappa} \Theta_{\lambda\nu} + \right. $$

$$\left. + \varepsilon_{\nu\alpha\kappa} \Theta_{\kappa\mu} + \varepsilon_{\nu\alpha\kappa} \Theta_{\lambda\mu} \right] + \right. $$

$$\left. - \left( \frac{\mu}{\kappa^2} \right)^2 \eta_{\mu\kappa} \eta_{\nu\lambda} + \eta_{\mu\lambda} \eta_{\nu\kappa} - 2\eta_{\mu\nu} \eta_{\kappa\lambda} \right] + $$

$$- 64k^2 \left[ \eta_{\mu\kappa} \omega_{\nu\lambda} + \eta_{\mu\lambda} \omega_{\nu\kappa} + \eta_{\nu\kappa} \omega_{\mu\lambda} + \eta_{\nu\lambda} \omega_{\mu\kappa} + \Theta_{\mu\nu} \Theta_{\kappa\lambda} - 2\eta_{\mu\nu} \omega_{\kappa\lambda} - 2\eta_{\mu\kappa} \omega_{\nu\lambda} \right] \right\}. \quad \text{(18)}$$

As it can be seen, the Hilbert-Einstein action in D=4 leads to a massless dynamical pole in the $h_{\mu\nu}$-propagators, whereas the Einstein-Chern-Simons D=3-action yields 2 poles: a massless non-dynamical excitation along with a non-tachyonic massive dynamical mode,

$$k^2 = \left( \frac{\mu}{8k^2} \right)^2 > 0, \quad \text{(19)}$$

as already known from [1].

Coupling the propagator to external currents, $\tau^{\mu\nu}$, compatible with the symmetries of the theory, and then taking the imaginary part of the residues of the amplitude at the poles, one can probe the necessary condition for unitarity at tree-level.
and count degrees of freedom described by the field. The current-current transition amplitude in momentum space is written as:

$$\mathcal{A} \equiv \tau^{\ast \mu \nu}(k) < T [h_{\mu \nu}(-k) \cdot h_{\lambda \kappa}(k)] > \tau^{\kappa \lambda}(k),$$  

(20)

where only the spin-projectors $P^{(2)}$, $P_s^{(0)}$ and $S_2$ shall contribute due to the transversality of $\tau^{\mu \nu}(k)$. Now, defining the following set of independent vectors in momentum space:

$$\begin{cases} 
k^\mu \equiv (k^0; \vec{k}) \\
\bar{k}^\mu \equiv (k^0; -\vec{k}) \\
\varepsilon_i^\mu \equiv (0; \vec{\varepsilon}_i), \ i = 1 \ldots D - 2, \end{cases}$$  

(21)

we can write the symmetric current tensor $\tau^{\mu \nu}(k)$ as

$$\tau_{\mu \nu}(k) = a(k)k^\mu k^\nu + b(k)k^{(\mu} \bar{k}^{\nu)} + c_i(k)k^{(\mu} \varepsilon_i^{\nu)} +$$

$$+ d(k)\bar{k}^{(\mu} \bar{k}^{\nu)} + e_i(k)\bar{k}^{(\mu} \varepsilon_i^{\nu)} + f_{ij}(k)\varepsilon_i^{(\mu} \varepsilon_j^{\nu)},$$  

(22)

and then extract some relations involving the above coefficients when imposing its conservation for on-shell momenta $k^\mu$.

So, for the Einstein theory in D dimensions, the amplitude $\mathcal{A}$ reads:

$$\mathcal{A} = \frac{(-i)}{k^2} \tau_{\mu \nu}^{\ast}(k) \left\{ -2P^{(2)}(k) + \frac{2(5-D)}{(D-2)}P_s^{(0)}(k) \right\}^{\mu \nu, \kappa \lambda} \tau_{\kappa \lambda}(k);$$  

(23)

then, at the pole $k^2 = 0$,

$$Im \ Res \ A = \left[ 2|\tau_{\mu \nu}|^2 - \frac{2}{3} \left( 1 + \frac{5-D}{D-2} \right) |\tau_{\mu i}|^2 \right].$$  

(24)

Manipulating with $\tau_{\mu \nu}(k)$ as expanded above, one gets:

$$Im \ Res \ A = 2 \left[ |f_{ij}|^2 - \frac{1}{3} \left( 1 + \frac{5-D}{D-2} \right) |f_{ii}|^2 \right].$$  

(25)

For D=4 dimensions,

$$Im \ Res \ A = 2 \left[ \frac{1}{2} |f_{11} - f_{22}|^2 + 2|f_{12}|^2 \right] > 0.$$  

(26)

Upon solving the eigenvalue problem of the $M$-matrix of (26):

$$\frac{1}{2} Im \ Res \ A = \left( \begin{array}{cc}
f_{11}^* & f_{12}^* \\
f_{22}^* & f_{12}^* \end{array} \right) \left( \begin{array}{cc}
\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \end{array} \right) \left( \begin{array}{c}
f_{11} \\
f_{22} \\
f_{12} \end{array} \right),$$  

(27)

one gets two non-vanishing eigenvalues that describe the two on-shell degrees of freedom of the massless graviton.
For D=3 dimensions,

\[ \text{Im Res } A = 2 \left( |f_{ij}|^2 - |f_{ii}|^2 \right) = 0, \quad (i = j = 1), \]  

confirming, as it is known, that the Einstein theory is non-dynamical in 3 dimensions.

For the Einstein-Chern-Simons theory,

\[
A = \frac{i}{k^2[64k^2 - (\frac{\mu}{\kappa^2})^2]} \tau^\epsilon_{\mu\nu}(k) \left\{ 2 \left( \frac{\mu}{\kappa^2} \right)^2 P^{(2)}(k) + 
\right.
\]
\[
\left. - \left[ 4\left( \frac{\mu}{\kappa^2} \right)^2 - 192k^2 \right] P_s^{(0)}(k) + \frac{16\left( \frac{\mu}{\kappa^2} \right)}{k^2} S_2(k) \right\} \tau^\epsilon_{\kappa\lambda}(k). \]  

At the pole \( k^2 = 0 \),

\[
\text{Im Res } A = \lim_{k^2 \to 0} \frac{1}{64k^2 - (\frac{\mu}{\kappa^2})^2} \left\{ 2\left( \frac{\mu}{\kappa^2} \right)^2 \left[ |\tau_{\kappa\lambda}|^2 - \frac{1}{3} |\tau^\mu_{\mu}|^2 \right] + 
\right.
\]
\[
\left. - \frac{4\left[ (\frac{\mu}{\kappa^2})^2 - 48k^2 \right]}{3} |\tau^\mu_{\mu}|^2 + 16\left( \frac{\mu}{\kappa^2} \right) k^\epsilon \varepsilon_{\mu\alpha\lambda} \tau^\epsilon_{\kappa\mu} \tau^\lambda_{\kappa} \right\} 
\]
\[
= \lim_{k^2 \to 0} \left\{ \frac{64k^2 |f|^2}{64k^2 - (\frac{\mu}{\kappa^2})^2} \right\} = 0; \]  

which is therefore shown to be non-propagating.

At the pole \( k^2 = (\frac{\mu}{\kappa^2})^2 \),

\[
\text{Im Res } A = \lim_{k^2 = (\frac{\mu}{\kappa^2})^2} \frac{1}{k^2} \left\{ 2\left( \frac{\mu}{\kappa^2} \right)^2 \left[ |\tau_{\kappa\lambda}|^2 - \frac{1}{3} |\tau^\mu_{\mu}|^2 \right] + 
\right.
\]
\[
\left. - \frac{4\left[ (\frac{\mu}{\kappa^2})^2 - 48k^2 \right]}{3} |\tau^\mu_{\mu}|^2 + 16\left( \frac{\mu}{\kappa^2} \right) k^\epsilon \varepsilon_{\mu\alpha\lambda} \tau^\epsilon_{\kappa\mu} \tau^\lambda_{\kappa} \right\} 
\]
\[
= 64|f|^2 > 0; \]  

giving one degree of freedom. Here, attention must be paid to the sign of the Hilbert-Einstein Lagrangean in D=3: a minus sign has to be chosen in (16) in order to guarantee a ghost-free massive propagator in three dimensions, although, with our choice of metric, the opposite sign is the one needed to ensure that the massless graviton is not a ghost.

To conclude, we have set the spin-projector operators to deal with D-dimensional Einstein’s gravity and D=3-topologically massive gravitation. Their multiplicative table has been used in the derivation of the graviton propagators in a general gauge.

Having in mind the coupling of a Maxwell-Chern-Simons gauge field to Einstein-Chern-Simons gravity, the propagator (17) will be employed to explicitly calculate...
one-loop corrections to the coupled gauge-gravity system. These results shall be presented and discussed in a further work [9].

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APPENDIX I

Multiplicative Table for the Barnes-Rivers spin-projector operators in D dimensions:

\[ P^{(2)} P^{(2)} = P^{(2)} + \frac{(D - 4)}{3} P^{(0)}_s, \]
\[ P^{(1)}_m P^{(1)}_m = P^{(1)}_m, \]
\[ P^{(2)} P^{(0)}_s = \frac{(4 - D)}{3} P^{(0)}_s, \]
\[ P^{(0)}_s P^{(2)} = \frac{(4 - D)}{3} P^{(0)}_s, \]
\[ P^{(2)} P^{(0)}_s = \frac{(4 - D)}{3} P^{(0)}_{sw}, \]
\[ P^{(0)}_s P^{(2)} = \frac{(4 - D)}{3} P^{(0)}_{sw}, \]
\[ P^{(0)}_s P^{(0)}_s = \frac{(D - 1)}{3} P^{(0)}_s, \]
\[ P^{(0)}_w P^{(0)} = P^{(0)}_w, \]
\[ P^{(1)}_m P^{(1)}_m = P^{(1)}_m, \]
\[ S_2 S_2 = \Delta^3 \left\{ \frac{1}{2} P^{(0)}_s - \frac{1}{4} P^{(1)}_m - P^{(2)} \right\}, \]
\[ S_2 S_1 = \frac{\Delta^3}{4} P^{(1)}_m, \]
\[ S_1 S_2 = \frac{\Delta^3}{4} P^{(1)}_m, \]
\[ S_1 S_1 = \frac{(\Delta^3)}{4} P^{(1)}_m, \]
\[ S_1 P_m^{(1)} = S_1, \]
\[ P_m^{(1)} S_1 = S_1, \]
\[ S_2 P^{(2)} = S_2 + S_1, \]
\[ P^{(2)} S_2 = S_2 + S_1, \]
\[ S_2 P_m^{(1)} = -S_1, \]
\[ P_m^{(1)} S_2 = -S_1. \]

Tensorial identity:
\[
\left\{ P^{(2)} + P^{(1)} + P_s^{(0)} + P_w^{(0)} \right\}_{\mu\nu,\kappa\lambda} = \frac{1}{2} \left( \eta_{\mu\kappa} \eta_{\nu\lambda} + \eta_{\mu\lambda} \eta_{\nu\kappa} \right).
\]

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