Effective Conformal Descriptions of Black Hole Entropy: A Review

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Abstract. Black holes behave as thermodynamic objects, and it is natural to ask for an underlying “statistical mechanical” explanation in terms of microscopic degrees of freedom. I summarize attempts to describe these degrees of freedom in terms of a dual two-dimensional conformal field theory, emphasizing the generality of the Cardy formula and the consequent universal nature of the conformal description.

Keywords: black hole thermodynamics, quantum black holes, black hole microstates

PACS: 04.70.Dy, 04.60.Ds, 04.20.Fy

INTRODUCTION

Black holes are not black. Rather, they are black bodies: as Bekenstein [1] and Hawking [2] showed some 40 years ago, they “glow” with a characteristic temperature and entropy

\[ T_H = \frac{\hbar \kappa_H}{2\pi}, \quad S_{BH} = \frac{A}{4\hbar G}, \]

where \( \kappa_H \) is the surface gravity (1/4GM for the Schwarzschild case) and \( A \) is the area of the event horizon. While this thermodynamic behavior has not been directly observed—the Hawking temperature for an astrophysical black hole is far too small—it can be derived in so many different ways [3] that there is very little doubt of its validity.

In normal thermodynamic systems, temperature and entropy are collective properties of microscopic states. It is natural to expect the same to be true for black holes. If it is, an understanding of black hole thermodynamics could offer precious insights into quantum gravity. Indeed, the Bekenstein-Hawking entropy formula is one of the very few relationships we know that is genuinely quantum gravitational, in the sense that it contains both Planck’s constant \( \hbar \) and Newton’s constant \( G \).

In this article, I will briefly summarize the “black hole CFT” program, an attempt to describe the microphysics of black holes in terms of a “dual” two-dimensional conformal field theory. For more details, I refer the reader to the review article [4].

THE PUZZLE OF UNIVERSALITY

We start with one of the outstanding mysteries of black hole thermodynamics, the problem of “universality.” This may be seen as a pair of puzzles: why do so many different microscopic descriptions of black hole entropy give the same answer, and why does this answer take such a simple and general form?
First: many very different models of microscopic degrees of freedom give the same black hole entropy. These include

- three approaches in string theory: weakly coupled D-branes \([5]\), the AdS/CFT correspondence \([6]\), and the “fuzzball” picture \([7]\);
- three approaches in loop quantum gravity: boundary Chern-Simons theory \([8]\), counting intertwiners \([9]\), and a “gas of punctures” \([10]\);
- counting the “heavy” states of induced gravity \([11]\);
- holographic entanglement entropy \([12]\);
- computations from a single instanton \([13]\) and from pair production \([14]\);

and a number of others \([3]\). None of these models is complete—we do not, after all, have a finished quantum theory of gravity—and all involve added assumptions or restrictions. But within its realm of validity, each approach reproduces the standard Bekenstein-Hawking entropy. In some cases, we may simply have different depictions of the same microscopic degrees of freedom; the string theory descriptions, for instance, are presumed to be “dual” to each other. But others seem quite different, and their universal agreement is puzzling.

Second: the final form of the Bekenstein-Hawking entropy is also surprisingly simple and universal. The area law holds for black holes in any dimension, with any set of charges and spins. The same relationship holds for the more elaborate “black” objects that can exist in higher dimensions: black strings, black rings, black saturns and the like \([15]\). One can change the entropy by changing the action, but even then the new expression takes a simple form in terms of the revised action \([16]\).

Together, these two features strongly suggest that there should be a simple underlying structure that controls the statistical mechanics of the quantum black hole, one that is general enough to determine the thermodynamic properties independent of the details of the degrees of freedom. In the next section, I will discuss a rather strange possibility: that this underlying structure is the conformal symmetry of a two-dimensional theory that is in some sense dual to the standard description of the black hole.

**TWO-DIMENSIONAL CONFORMAL FIELD THEORY**

The idea of describing black hole microstates in terms of a two-dimensional conformal field theory was first suggested in \([17]\), but more as a hope than a concrete proposal. The first realization of the idea was achieved by Strominger \([18]\) and, simultaneously, by Birmingham, Sachs, and Sen \([19]\) for the (2+1)-dimensional BTZ black hole. These results were quickly extended to a variety of near-extremal black holes with near-horizon geometries that look like that of the BTZ solution (for a review, see \([20]\)), and several authors suggested methods for extending the results to nonextremal black holes \([21, 22, 23]\). Interest in the program was revived in the past few years with the discovery of an extremal Kerr/CFT correspondence \([24]\), and a new, relatively “clean” approach to the nonextremal Kerr black hole was found in 2011 \([25]\).
Such an idea raises two obvious questions: why conformal, and why two dimensional? We have at least partial answers to both.

- The near-horizon region of a black hole really is approximately conformally invariant: to an observer who remains outside the horizon, dimensionful quantities such as masses are rapidly red-shifted away [26], and the near-horizon region of a generic black hole admits an approximate conformal Killing vector [27].
- The same red shift that eliminates masses also makes transverse excitations negligible relative to those in the $r - t$ plane, leaving two "important" dimensions. This fact has been put to good use in calculations of Hawking radiation, which can be carried out entirely in the framework of a two-dimensional near-horizon conformal field theory [28, 29].

To a certain extent, though, the existence of a two-dimensional conformal description is more a hope than an established result. Two-dimensional conformal symmetry is extremely powerful—it is the only known symmetry strong enough to determine the asymptotic density of states of a theory, and thus the entropy. If the universality of black hole entropy is to have a simple explanation, this is a natural place to look.

Any metric on a two-dimensional manifold can be written locally as

$$ds^2 = 2g_{\bar{z}z}dzd\bar{z}$$

in terms of complex coordinates $\bar{z}$ and $z$. Holomorphic and antiholomorphic coordinate changes $z \to z + \xi(z)$, $\bar{z} \to \bar{z} + \bar{\xi}(\bar{z})$ merely rescale the metric, and provide the basic symmetries of a conformal field theory. This group of symmetries is infinite dimensional, distinguishing it from the conformal group of higher dimensional spacetimes. Its generators, denoted $L[\xi]$ and $\bar{L}[\bar{\xi}]$, satisfy a Virasoro algebra [30],

$$[L[\xi], L[\eta]] = L[\eta \xi'' - \xi \eta''] + \frac{c}{48\pi} \int dz \left( \eta'' \xi' - \xi'' \eta' \right),$$

$$[\bar{L}[\bar{\xi}], \bar{L}[\bar{\eta}]] = \bar{L}[\bar{\eta} \bar{\xi}'' - \bar{\xi} \bar{\eta}''] + \frac{\bar{c}}{48\pi} \int d\bar{z} \left( \bar{\eta}'' \bar{\xi}' - \bar{\xi}'' \bar{\eta}' \right),$$

$$[L[\xi], \bar{L}[\bar{\eta}]] = 0,$$

uniquely determined by the values of the two constants $c$ and $\bar{c}$, the central charges. As in ordinary field theory, the zero modes $L_0$ and $\bar{L}_0$ of the symmetry generators are conserved quantities, the “conformal weights,” which can be seen as linear combinations of mass and angular momentum.

Two-dimensional conformal symmetry is remarkably powerful. In particular, with a few mild restrictions, Cardy has shown that the asymptotic density of states of any two-dimensional CFT at conformal weight $\{L_0, \bar{L}_0\}$ is given by [31, 32]

$$\ln \rho(L_0) \sim 2\pi \sqrt{\frac{cL_0}{6}}, \quad \ln \bar{\rho}(\bar{L}_0) \sim 2\pi \sqrt{\frac{\bar{c}\bar{L}_0}{6}}.$$  \hspace{1cm} (4)

This result can be viewed as a microcanonical expression for the entropy. Similarly, for a conformal field theory at fixed temperature $T$,

$$\ln \rho(T) \sim \frac{\pi^2}{3} cT, \quad \ln \bar{\rho}(T) \sim \frac{\pi^2}{3} \bar{c}T,$$  \hspace{1cm} (5)
a canonical version of the Cardy formula. The entropy is thus determined by a few
parameters, regardless of the details of the conformal field theory—exactly the kind of
universal behavior we would like for black holes.

BUILDING A BLACK HOLE/CFT CORRESPONDENCE

General relativity is not, of course, a two-dimensional conformal field theory. For the
results of the preceding section to be relevant, we must find a Virasoro algebra (3) hidden
inside its constraint algebra. This is most easily analyzed in the ADM formalism [33].
We write the metric in the form
\[ ds^2 = -N^2 dt^2 + q_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \]
with canonical variables \( \{ q_{ij}, \pi^{ij} \} \) and an action
\[ I = \frac{1}{16 \pi G} \int dt \int d^{n-1} x \left[ \pi^{ij} \dot{q}_{ij} - N^i \mathcal{H}_i - N \mathcal{H} \right]. \]
The constraints \( \mathcal{H}_i \) and \( \mathcal{H} \) generate diffeomorphisms (or, more properly, “surface
deformations,” equivalent to diffeomorphisms on shell [34])—that is, the Hamiltonian
\[ H[\xi \perp, \hat{\xi}^i] = \int d^3 x \left[ \xi \perp \mathcal{H} + \hat{\xi}^i \mathcal{H}_i \right] \] has Poisson brackets
\[ \{ H[\xi], F[q, \pi] \} = \delta_\xi F[q, \pi] \]
with any phase space variables.

On a spatially closed manifold, the Poisson algebra of the generators \( H[\xi \perp, \hat{\xi}^i] \) closes,
with no central term. In the presence of a boundary, however, the generators acquire
boundary terms, \( H[\xi \perp, \hat{\xi}^i] \to \bar{H}[\xi \perp, \hat{\xi}^i] = H[\xi \perp, \hat{\xi}^i] + B[\xi \perp, \hat{\xi}^i] \), and these can modify
the algebra:
\[ \{ \bar{H}[\xi], \bar{H}[\eta] \} = \bar{H}[[\xi, \eta]] + K[\xi, \eta]. \]
The new term \( K[\xi, \eta] \) is always central—that is, it has vanishing brackets with all phase
space variables—and it takes a general form computed in [25],
\[ K[\xi, \eta] = -B[[\xi, \eta]]_{SD} - \frac{1}{8 \pi G} \int d^{n-2} x \sqrt{\sigma} n^k \left[ -\frac{1}{2 \sqrt{q}} \hat{\xi}^i \eta \perp \mathcal{H}_i - \frac{1}{\sqrt{q}} \hat{\xi}^i \eta \perp \mathcal{H}_i \right. \]
\[ \left. - \xi D_\perp (n-1) R^i_k + D_i \hat{\xi}^i \mathcal{H}_i \eta \perp - D_i \hat{\xi}^i \mathcal{H}_i \eta \perp \right] \]
\[ + \frac{1}{\sqrt{q}} \hat{\eta}^m \mathcal{H}_m \eta \perp + \frac{1}{2 \sqrt{q}} \mathcal{H}_m \eta \perp + \frac{1}{2 \sqrt{q}} \mathcal{H}_m \eta \perp + \frac{1}{2 \sqrt{q}} \mathcal{H}_m \eta \perp. \]
We thus obtain a “recipe” for a black hole/CFT correspondence:

1. Identify an appropriate boundary and the corresponding “black hole” boundary conditions.
2. Compute the boundary terms \( B[\xi \perp, \hat{\xi}^i] \) for the generators of diffeomorphisms. They
should be chosen so that the variation \( \delta \bar{H}[\xi] \) is well-defined, with no net boundary
contributions; the specific form will depend on the choice of boundary conditions.
3. Find the central term $K[\xi, \eta]$ in the resulting algebra.

4. Search for a preferred one- or two-dimensional subalgebra of surface deformations (technically, a $\text{Diff}S^1$ or $\text{Diff}S^1 \times \text{Diff}S^1$ algebra). If such a subalgebra exists, general mathematical results imply that it must be a Virasoro algebra.

5. Read off the central charges, and use the Cardy formula to count the states.

But while this is a recipe of sorts, it is not exactly a “cookbook” recipe. Key questions remain:

- What boundary and boundary conditions should we choose? Results exist for boundaries at infinity in the asymptotically AdS case, boundaries of near-horizon regions, and stretched horizons. Ideally, we might want to choose the horizon itself as the boundary, but it is a null surface, a feature that significantly complicates the constraints.

- What two-dimensional subgroup do we pick? For the extremal Kerr-Newman black hole, for instance, there are two known choices, leading to the so-called $J$ and $Q$ pictures ([35]); each gives the correct entropy, but they cannot be treated together. For the nonextremal Kerr black hole, on the other hand, it was shown in [25] that the analog of the subgroup used in [24] gives only half the entropy.

- What about 1+1 dimensions? The black hole in (1+1)-dimensional dilaton gravity is a perfectly nice object, with well-defined thermodynamic properties [36], but the canonical methods I have described here fail—there is not enough room at the boundary of a one-dimensional spatial slice for these techniques to work. A possible alternative is to introduce boundary conditions as explicit constraints in the surface deformation algebra [37], but this idea is not yet very fully developed.

- What about the rest of black hole thermodynamics? Counting states is an important first step, but we also need Hawking radiation. We thus need to understand how to couple our boundary conformal field theory to “bulk” matter. There has been one interesting step in this direction for the BTZ black hole [38], but the overall problem remains very poorly understood.

**AN EXAMPLE: THE NONEXTREMAL KERR BLACK HOLE**

The preceding section was a bit abstract, and it may be helpful to look at a specific example. Consider the generic nonextremal stationary black hole in four spacetime dimensions [25]. Our starting point is the observation by Medved, Martin, and Visser [39] that the near-horizon metric of such a black hole can always be written in the ADM-like form

$$ds^2 = -N^2 dt^2 + d\rho^2 + q_{\varphi\varphi} (d\varphi + N^\varphi dt)^2 + q_{zz} dz^2,$$  

(11)
where \( \rho \) is the proper distance from horizon, and that even without using the field equations, finite curvature at the horizon requires that

\[
N = \kappa_H \rho + \frac{1}{3!} \kappa_2(z) \rho^3 + \ldots \quad \quad q_{\phi\phi} = [q_H]_{\phi\phi}(z) + \frac{1}{2} [q_2]_{\phi\phi}(z) \rho^2 + \ldots
\]

\[
N^\phi = -\Omega_H - \frac{1}{2} \omega_2(z) \rho^2 + \ldots \quad \quad q_{zz} = [q_H]_{zz}(z) + \frac{1}{2} [q_2]_{zz}(z) \rho^2 + \ldots
\]

(12)

We would like to choose a “stretched horizon” of small \( \rho \) as a boundary, taking the limit \( \rho \to 0 \) at the end. The obvious choice \( \rho = \text{const} \) turns out to be awkward, requiring a complicated angular dependence of the boundary-preserving diffeomorphisms. Instead, following \([24]\), let us choose a surface of angular velocity \( \bar{\Omega} \)—which we will allow to approach the horizon angular velocity \( \Omega_H \)—and demand that modes of the form \( \xi^t(\phi - \bar{\Omega} t) \) be lightlike:

\[
g^{ab} \partial_a \xi \partial_b \xi = 0 \Rightarrow [q_H]_{\phi\phi}(\bar{\Omega} - \Omega_H)^2 = \kappa_H^2 \rho^2. \quad \quad \quad (13)
\]

One can equivalently demand that the horizon be a “stretched Killing horizon,” defined by a Killing vector that is invariant under boundary diffeomorphisms \([25]\).

The boundary conditions (12) are then preserved by diffeomorphisms of the form

\[
\delta \xi^t N = 0 \quad \Rightarrow \quad \hat{\xi}^\rho = -\bar{\varepsilon} \rho \partial_\rho \xi^t = -\rho \partial_\rho \xi^t \quad \quad \quad \delta q_{\rho\phi} = 0 \quad \Rightarrow \quad \hat{\xi}^\phi = \frac{\kappa_H^2 \rho^2}{\bar{\varepsilon}} q_{\phi\phi} \xi^t \quad \quad \delta \xi^\phi = 0 \quad \Rightarrow \quad \hat{\xi}^\phi = \frac{\kappa_H^2 \rho^2}{\bar{\varepsilon}} q_{\phi\phi} \xi^t
\]

\[
\delta q_{\rho\phi} = 0 \quad \Rightarrow \quad \rho \partial_\rho \hat{\xi}^\phi = \bar{\varepsilon} \rho^2 q_{\phi\phi} \partial_\phi \xi^t - \omega_2 \rho^2 \xi^t = \frac{\bar{\varepsilon}}{\kappa_H^2} \partial_\rho \rho \xi^t - \omega_2 \rho^2 \xi^t \quad \quad \quad (14)
\]

where \( \partial_t = \partial_t - N^\rho \partial_\rho \) and \( \bar{\varepsilon} = \Omega_H - \bar{\Omega} \). As an additional technical step, we must choose a “moding” for the parameters \( \xi^t(\phi - \bar{\Omega} t) \); that is, we write

\[
\xi^t(\phi - \bar{\Omega} t) = \frac{\gamma}{2 \bar{\varepsilon}} e^{i n u} \quad \text{with} \quad u = (\phi - \bar{\Omega} t)/\gamma, \quad \quad \quad (15)
\]

where it is not completely obvious what value \( \gamma \) should take.\(^1\) We can then read off the central charge from (10), obtaining

\[
c = \frac{3A}{2\pi G \kappa_H} \frac{\bar{\varepsilon}}{\gamma}. \quad \quad \quad (16)
\]

To use the canonical Cardy formula (5), we must be a bit careful of the meaning of the temperature \( T \). Again following \([24]\), we note that fields in the Frolov-Thorne vacuum

\(^1\) One natural choice \([25]\) is \( \gamma = \bar{\varepsilon} \). The modes are then those seen by a corotating observer, with the usual blue-shifted frequencies \( \omega \sim n/N \), where \( N \) is the lapse function.
have a coordinate dependence

$$\Phi \sim e^{i\omega - i\gamma t} = e^{in_L u - in_R t},$$  \hspace{1cm} (17)$$

where the “left” and “right” occupation numbers are \(n_L = \gamma m\), \(n_R = \omega - n_L \tilde{\Omega} / \gamma\). The Boltzmann factor is thus

$$e^{-\beta(\omega - m\Omega_H)} = e^{-\beta n_R - (\beta \beta / \gamma)n_L},$$  \hspace{1cm} (18)$$

From this, we can read off the appropriate CFT temperature for the “left” \(u\) modes,

$$T_L = \frac{\gamma}{\tilde{\epsilon} \beta} = \frac{\gamma}{\tilde{\epsilon}} T_H = \frac{\gamma}{\tilde{\epsilon}} \frac{\kappa_H}{2\pi},$$  \hspace{1cm} (19)$$

where \(T_H\) is the Hawking temperature \((1)\). Inserting \((16)\) and \((19)\) into the Cardy formula \((5)\) and restoring factors of \(\hbar\), we obtain

$$S_{BH} = \frac{A}{4\hbar G},$$  \hspace{1cm} (20)$$

the correct Bekenstein-Hawking entropy.

**WHAT ARE THE STATES?**

Recall that one of the original motivations for studying black hole thermodynamics was to learn something about quantum gravity. These results suggest, though, that the counting of states is universal, depending only on a few characteristics of the symmetry of the horizon. Contrary to our hopes, black hole statistical mechanics may not, in the end, tell us too much about how to quantize general relativity.

One way to understand this conclusion is the following. In the usual Dirac treatment of constraints, we would demand that physical states be annihilated by the constraints,

$$\hat{H}[\xi^\perp, \hat{\xi}^i]|\text{phys}\rangle = 0.$$

But if the algebra of constraints contains a Virasoro subalgebra with a nonvanishing central charge \(c\), this condition is inconsistent with the Virasoro algebra \((3)\). We know how to handle this problem in conformal field theory, of course \([30]\): for example, we can demand that only the positive frequency piece of the constraints annihilate physical states. The upshot, though, is that, in the language of \([40]\), a number of “would-be pure gauge” degrees of freedom are no longer eliminated, but rather become dynamical at the horizon.

A somewhat analogous—although not exactly equivalent—situation occurs in ordinary field theory. When a global symmetry is spontaneously broken, a collection of massless Nambu-Goldstone bosons appears, one for each “broken” symmetry generator. It is well known that these degrees of freedom can be viewed as excitations along the “would-be symmetries” that are broken by the vacuum. But while Goldstone’s theorem gives us a good deal of information about these bosons, it does not tell us how they
are built from the elementary quanta of the theory. The analogy with black hole states would be strengthened if one could explicitly derive the Cardy formula as a measure for the “broken” degrees of freedom; for now, I do not know how to do so.

Let me close with a final question: how might we tell whether the approach described here is, in fact, correct? I certainly don’t have a complete answer, but there are several avenues of research that might help:

- If a CFT dual explains the universality of black hole entropy, then the conformal symmetry must be present, if perhaps disguised, in other more “microscopic” approaches to black hole statistical mechanics—those from string theory and loop quantum gravity, for instance. In [37], some evidence is given for a connection to the BTZ black hole, and through that to certain string theory approaches, but much more remains to be done.
- If black hole degrees of freedom have a Goldstone-like description, then it should be possible to obtain an effective field theory for the boundary degrees of freedom directly from general relativity. This can be done in three spacetime dimensions [41]; whether this approach can be extended to higher dimensions is uncertain.
- It ought to be possible to couple boundary degrees of freedom to matter and connect the results to Hawking radiation. Again, there are some interesting results [38], but so far only in three dimensions.
- The central charge (16) depends on the horizon area, so the model described here cannot in itself describe processes such as black hole evaporation, in which this area changes. One speculative idea [4] is that such processes could be described as flows between conformal field theories. The direction of the flow during black hole evaporation is consistent with Zamolodchikov’s c theorem [42], but for now I can say little more than that.

ACKNOWLEDGMENTS

Portions of this project were carried out at the Peyresq 15 Physics Conference with the support of OLAM Association pour la Recherche Fundamentale, Bruxelles. This work was supported by the U.S. Department of Energy under grant DE-FG02-91ER40674.

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