Locality and Nonlinear Quantum Mechanics

Chiu Man Ho∗ and Stephen D. H. Hsu†

Department of Physics and Astronomy,
Michigan State University, East Lansing, MI 48824, USA

(Dated: January 29, 2014)

Abstract

Nonlinear modifications of quantum mechanics generically lead to nonlocal effects which violate relativistic causality. We study these effects using the functional Schrödinger equation for quantum fields and identify a type of nonlocality which causes nearly instantaneous entanglement of spacelike separated systems.

∗Electronic address: cmho@msu.edu
†Electronic address: hsu@msu.edu
I. INTRODUCTION

The linear structure of quantum mechanics has deep and important consequences, such as the behavior of superpositions. One is naturally led to ask whether this linearity is fundamental, or merely an approximation: Are there nonlinear terms in the Schrödinger equation?

Nonlinear quantum mechanics has been explored in [1–6]. It has been observed that the fictitious violation of locality in the Einstein-Podolsky-Rosen (EPR) experiment in conventional linear quantum mechanics might become a true violation due to nonlinear effects [7, 8] (in [8] signaling between Everett branches is also discussed). This might allow superluminal communication and violate relativistic causality. These issues have subsequently been widely discussed [9].

Properties such as locality or causality are difficult to define in non-relativistic quantum mechanics (which often includes, for example, “instantaneous” potentials such as the Coulomb potential). Therefore, it is natural to adopt the framework of quantum field theory: Lorentz invariant quantum field theories are known to describe local physics with relativistic causality (influences propagate only within the light cone), making violations of these properties easier to identify.

Furthermore, in this paper we are interested in fundamental nonlinearity in quantum mechanics, which is another reason for considering quantum field theory. If the evolution of quantum states is nonlinear, that should also be the case when the states in question describe quantum fields, not just individual particles.

II. LOCALITY AND SEPARABILITY

Quantum field theory can be formulated in terms of a wavefunctional \( \Psi[\phi(x), t] \) where \( \phi(x) \) is a time-independent field configuration and \( t \) is the time. The functional Schrödinger equation is then given by

\[
i \frac{\partial}{\partial t} \Psi[\phi, t] = \hat{H} \Psi[\phi, t] .
\]

The Hamiltonian operator \( \hat{H} \) is a sum of local operators at points \( x \). For example, in scalar field theory,

\[
\hat{H} = \frac{1}{2} \int d^3x \left( -\frac{\delta^2}{\delta \phi^2(x)} + |\nabla \phi|^2 + m^2 \phi^2 \right) .
\]
Let $\Psi[\phi, t] = \psi_A[\phi_A, t] \times \psi_B[\phi_B, t] \times \cdots$, where $\phi_A(x)$ is a field configuration with support in the compact region $A$ (i.e., $\phi_A(x)$ is zero for $x \notin A$), and similarly for $\phi_B$. Assume that $A$ and $B$ are widely separated, and that the remaining factors represented by $\cdots$ do not depend on the field configuration in $A$ or $B$. This direct product structure implies that there is no entanglement between regions $A$ and $B$.

It is easy to show that the Schrodinger equation splits into separate equations governing $\psi_A$ and $\psi_B$:

$$i \partial_t \psi_A[\phi_A, t] = \hat{H}_A \psi_A[\phi_A, t],$$

and similarly for $B$. The subscript on the Hamiltonian $\hat{H}_A$ emphasizes that it only refers to the part of the spatial integral in (2.2) over region $A$. The part of the integral over region $B$ only acts on $\psi_B$, etc.

Thus, in the absence of entanglement between $A$ and $B$, quantum mechanics in each region can be studied independently of the other. In a relativistic field theory, entanglement and other influences can propagate no faster than the speed of light, so that if $A$ and $B$ are widely separated and initially unentangled, they will remain so for a period of time that depends on the separation.

Now, consider a nonlinear generalization of the Schrodinger equation:

$$i \partial_t \Psi = \left(\hat{H} + \hat{F}(\Psi^\dagger, \Psi)\right) \Psi.$$  

The nonlinear term $\hat{F}$ will generically couple $\psi_A$ and $\psi_B$. Regardless of the distance between regions $A$ and $B$, the two initially unentangled states $\psi_A$ and $\psi_B$ influence each other’s evolution, and typically become entangled almost instantaneously. This time evolution is illustrated in Fig. 1.

One can also understand this from the perturbation theory point of view. Treating $\hat{F}$ as a perturbation, we can expand $\psi_A$ and $\psi_B$ as:

$$\psi_A = \psi_A^{(0)} + \psi_A^{(1)} + \cdots; \quad \psi_B = \psi_B^{(0)} + \psi_B^{(1)} + \cdots.$$  

Keeping the perturbations up to the lowest order, the functional Schrodinger equation becomes

$$\frac{1}{\psi_A^{(0)}} \left(i \partial_t - \hat{H}\right) \psi_A^{(1)} + \frac{1}{\psi_B^{(0)}} \left(i \partial_t - \hat{H}\right) \psi_B^{(1)} + \hat{F}(\psi_A^{(0)}, \psi_B^{(0)}) = 0.$$
FIG. 1: Time evolution when nonlinearity is present. \( \Psi(A,B) \) is generically an entangled state, whereas \( \Psi = \psi_A \times \psi_B \) is not.

Unless \( \hat{F}(\psi_A^{(0)}, \psi_B^{(0)}) \) takes the form \( \hat{F}(\psi_A^{(0)}, \psi_B^{(0)}) = f_A(\psi_A^{(0)}) + f_B(\psi_B^{(0)}) \), \( \psi_A^{(1)} \) will generally be influenced by \( \psi_B^{(0)} \) and vice versa. That is, the time evolution of \( \psi_A \) depends on \( \psi_B \) immediately at \( t = 0^+ \); a measurement of \( B \) affects the state of \( A \), implying entanglement.

III. EXAMPLES OF NONLINEAR TERMS: HOMOGENEOUS AND OTHERWISE

Since quantum field theory is simply quantum mechanics of a large number of degrees of freedom (i.e., the field configurations), the discussions above apply equally well to both quantum field theory and quantum mechanics of individual particles.

The nonlinear Schrödinger equation was first considered by \cite{1}, with the idea of using the nonlinearity as a possible way to resolve the difficulties associated with the quantum measurement theory. A simple example is:

\[
\hat{F}(\Psi^\dagger, \Psi) = \varepsilon |\Psi|^2 ,
\]

which violates separability and hence locality according to our arguments above.

To maintain the separability of the wavefunction for separate systems, it was proposed in \cite{2} that the nonlinearity should take a logarithmic form such as

\[
\hat{F}(\Psi^\dagger, \Psi) = b \ln |\Psi|^2 .
\]

Since \( \ln |\psi_A \psi_B|^2 = \ln |\psi_A|^2 + \ln |\psi_B|^2 \), separability is maintained for an initial state which is
factorizable. However, separability fails for superpositions such as identical particle states:

$$\Psi = \frac{1}{\sqrt{2}} \left( \psi_A(x_1) \psi_B(x_2) \pm \psi_A(x_2) \psi_B(x_1) \right). \quad (3.3)$$

Indeed, when the initial state $\Psi$ entangles $A$ and $B$, the log-nonlinearity causes the dynamical evolution of the $A$ system to depend on the $B$ system and vice-versa. Technically, this is somewhat different from the case of non-logarithmic interactions, where initially unentangled states become immediately entangled regardless of separation, but is nevertheless another kind of instantaneous action at a distance.

Perhaps the most systematic framework for introducing nonlinearities to quantum mechanics was provided by Weinberg [3, 4]. A key aspect of Hilbert space is that for any arbitrary complex number $Z$, the wavefunctions $\psi$ and $Z\psi$ represent the same physical state. It would therefore be desirable for the dynamical evolution (i.e., Schrodinger equation) to be invariant under this rescaling. Weinberg refers to this property as homogeneity. However, all of the proposals for nonlinear quantum mechanics suggested by [1, 2] lack this property. In contrast, Weinberg’s framework respects the homogeneity condition and Galilean invariance explicitly. The nonlinear Schrodinger equations proposed by [5] and [6] also satisfy the homogeneity condition, but [5] requires an arbitrary vector potential and [6] violates Galilean invariance.

A simple example satisfying the homogeneity condition is:

$$\hat{F}(\Psi^\dagger, \Psi) = \frac{\Psi^\dagger \hat{O}_1 \Psi}{\Psi^\dagger \hat{O}_2 \Psi}, \quad (3.4)$$

where $\hat{O}_1$ and $\hat{O}_2$ are some Hermitian operators. In general, this leads to non-separability and hence nonlocality. An exception is when both $\hat{O}_1$ and $\hat{O}_2$ commute with $\hat{H}$. In this case, the nonlinear terms only cause constant shifts to the Hamiltonian.

One possibility discussed by Weinberg [4] and by Polchinski [8] occurs if the denominator in $\hat{F}$ is the magnitude squared of the entire wavefunction. If the branch of the wavefunction occupied by the observer is only a small component of the total (i.e., this might be the case after many decoherent outcomes are recorded in the memory of that observer, assuming of course that decoherence continues to operate as usual in the presence of nonlinearity), then the effect of the nonlinear term is suppressed for that observer, even if at the fundamental level quantum mechanics is highly nonlinear. In this scenario we do not expect to observe any nonlinearity so late in the history of the universe.
IV. EXAMPLE: FREE FIELD THEORY

As a specific example, we can consider nonlinear quantum dynamics of free field theory, where the Hamiltonian is diagonal in the Fock basis, and the nonlinear term can be calculated explicitly. We will find that the properties of state \( \psi_A \) influence physics in region \( B \) and vice-versa, regardless of the distance between the two regions.

Let \( \psi_A^{(0)} \) and \( \psi_B^{(0)} \) be coherent state wavefunctionals, so the states \( A \) and \( B \) are semiclassical (minimum uncertainty) configurations such as wave packets, localized in regions \( A \) and \( B \) respectively:

\[
\psi_A^{(0)}[\phi_A] = \langle \phi_A | a_A \rangle = e^{\Omega[a_A, \phi_A]} \quad ; \quad \psi_B^{(0)}[\phi_B] = \langle \phi_B | a_B \rangle = e^{\Omega[a_B, \phi_B]} ,
\]

(4.1)

where \( \langle \phi | \) is an eigenstate of the field operator \( \hat{\phi} \) and the coherent state \( | a_k \rangle \) is an eigenstate of the annihilation operator \( \hat{a}_k \): \( \hat{a}_k | a_k \rangle = a_k | a_k \rangle \). Each of \( a_A \) and \( a_B \) are specified by their Fourier coefficients \( a_k \). The full expression for \( \Omega[a, \phi] \) can be found in [10]:

\[
\Omega[a, \phi] = -\frac{1}{2} \int d^3k a_k a_{-k} e^{2i\omega_k t} - \frac{1}{2} \int d^3k \omega_k \phi(k) \phi(-k) + \int d^3k \sqrt{2}\omega_k a_k e^{i\omega_k t} \phi(-k) .
\]

(4.2)

Suppose that factorization holds: \( \Psi[\phi,t] = \psi_A[\phi_A,t] \times \psi_B[\phi_B,t] \times \cdots \) initially. Then \( \hat{F} \) is a time-dependent function of the two coherent states localized at \( A \) and \( B \). To focus on a simpler subset of the degrees of freedom, consider rescalings: \( a_A \rightarrow \alpha a_A \) and \( a_B \rightarrow \beta a_B \). To be definite, we can take \( \hat{F} \propto | \Psi_A \Psi_B | \) to some power. IF factorization continues to hold, the Schrodinger equation (2.6) has the form

\[
g_A(\alpha) + g_B(\beta) = h_A(\alpha) h_B(\beta) ,
\]

(4.3)

where \( \{ g_A, h_A \} \) and \( \{ g_B, h_B \} \) are functions depending only on \( \alpha \) and \( \beta \) respectively. This condition cannot hold for all choices of \( \alpha \) and \( \beta \). Therefore, separability is violated for at least some states and as a consequence we have nonlocality. The conclusion is the same for any form of \( \hat{F} \) except a logarithm, which has the problems discussed previously.

V. CONCLUSIONS AND DISCUSSIONS

Our results suggest that nonlinearity in quantum mechanics is associated with violation of relativistic causality. We gave a formulation in terms of factorized (unentangled)
wavefunctions describing spacelike separated systems. Nonlinearity seems to create almost instantaneous entanglement of the two systems, no matter how far apart. Perhaps our results are related to what Weinberg [11] meant when he wrote “... I could not find any way to extend the nonlinear version of quantum mechanics to theories based on Einstein’s special theory of relativity ... At least for the present I have given up on the problem: I simply do not know how to change quantum mechanics by a small amount without wrecking it altogether.”

Finally, it may be interesting to consider nonlinear modification of the Wheeler-DeWitt equation (i.e., the Schrodinger equation for geometries in quantum gravity). Because there is no intrinsic notion of locality in quantum gravity, nonlinear modifications might not lead to catastrophic consequences. However, it seems likely that the nonlinearities would find their way into quantum mechanics on semiclassical spacetimes, as we have considered here. In that case, there would be unwelcome violations of locality.

VI. ACKNOWLEDGEMENTS

This work was supported by the Office of the Vice-President for Research and Graduate Studies at Michigan State University.

[1] L. de Broglie, “Non-Linear Wave Mechanics – A Causal Interpretation”, Elsevier, Amsterdam, 1950; P. M. Pearle, Phys. Rev. D 13, 857 (1976); B. Mielnik, Commun. Math. Phys. 37, 221 (1974).
[2] I. Bialynicki-Birula and J. Mycielski, Annals Phys. 100, 62 (1976).
[3] S. Weinberg, Phys. Rev. Lett. 62, 485 (1989).
[4] S. Weinberg, Annals Phys. 194, 336 (1989).
[5] R. Haag and U. Bannier, Commun. Math. Phys. 60, 1 (1978).
[6] T. W. B. Kibble, Commun. Math. Phys. 64, 73 (1978).
[7] N. Gisin, Helv. Phys. Acta 62, 363 (1989); N. Gisin, Phys. Rev. A 143, 1 (1990).
[8] J. Polchinski, Phys. Rev. Lett. 66, 397 (1991).
[9] G. Svetlichny, Found. Phys. 28, 131 (1998); G. Svetlichny, Int. J. Theor. Phys. 44, 2051 (2005);
C. Simon, V. Bužek and N. Gisin, Phys. Rev. Lett. 87, 170405 (2001); B. Mielnik, quant-ph/0012041; W. Luecke, quant-ph/9904016; H. D. Doebner, quant-ph/9803011; M. Czachor, Phys. Rev. A 57, 4122 (1998); A. Caticha, Phys. Lett. A 244, 13 (1998).

[10] S. D. H. Hsu, Phys. Lett. B 555, 92 (2003); T. M. Gould, S. D. H. Hsu and E. R. Poppitz, Nucl. Phys. B 437, 83 (1995).

[11] S. Weinberg, Dreams of a Final Theory, Hutchison (1993).