Čech, Dolbeault and de Rham cohomologies
in Chern-Simons and BF theories

T.A.Ivanova\textsuperscript{a} and A.D.Popov\textsuperscript{a,b}
\textsuperscript{a}Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Russia
\textsuperscript{b}Institut für Theoretische Physik, Universität Hannover, Germany

Abstract. Topological Chern-Simons (CS) and BF theories and their holomorphic analogues are discussed in terms of de Rham and Dolbeault cohomologies. We show that Čech cohomology provides another useful description of the above topological and holomorphic field theories. In particular, all hidden (nonlocal) symmetries of non-Abelian CS and BF theories can be most clearly seen in the Čech approach. We consider multidimensional Manin-Ward integrable systems and describe their connections with holomorphic BF theories. Dressing symmetries of these generic integrable systems are briefly discussed.

1. Introduction

Topological field theories \cite{1} were intensively studied over the last ten years (see e.g. \cite{2,3} and references therein). Among these theories are Chern-Simons \cite{4,5} and BF \cite{4,6,7} topological theories describing flat connections $dA = d + A$ on principal bundles $P$ over smooth manifolds and $dA$-closed ad$P$-valued forms. Holomorphic Chern-Simons \cite{8-10} and holomorphic BF \cite{11} theories describe flat $(0,1)$-connections $\partial A = \bar{\partial} + A^{0,1}$ on bundles $P$ over complex manifolds and $\partial A$-closed ad$P$-valued forms. In Abelian case the above theories give field-theoretic descriptions of de Rham and Dolbeault cohomologies.

After recalling the definitions of de Rham, Dolbeault and Čech complexes \cite{12,13}, we discuss isomorphisms between the de Rham and Čech cohomologies of real manifolds and the Dolbeault and Čech cohomologies of complex manifolds. These isomorphisms permit one to reduce differential equations of motion of CS and BF theories defined on a manifold $M$ to some functional equations defined on open subsets of $M$. Transition from the description of CS and BF theories in terms of globally defined differential forms (de Rham or Dolbeault cocycles) to the description in terms of locally defined functions (Čech cocycles) is especially useful in considering moduli spaces of locally constant and holomorphic structures (flat connections and $(0,1)$-connections, respectively) on bundles $\cite{14,15,10,16,17,10}$. The Čech approach considerably simplifies finding and exploring symmetries of Chern-Simons, BF and self-dual Yang-Mills (SDYM) theories $\cite{16-18}$.

We describe a connection between holomorphic BF theories and multidimensional integrable systems introduced by Yu.Manin $\cite{19}$ and R.Ward $\cite{20}$ (see $\cite{21-23}$ for further developments). These generic integrable systems contain 4D SDYM model and 2D integrable models as special cases $\cite{19,20,23,24}$. The equivalence of Čech and Dolbeault descriptions of holomorphic bundles permits one to formulate a method of solving the Manin-Ward equations as a cohomological analogue of the method of Birkhoff factorizations and dressing transformations. Moreover, by considering these generic integrable systems
systems, it may be seen how the dressing approach to integrable systems originates from deformation theory of flat and holomorphic bundles. Dressing symmetries of Manin-Ward integrable systems can be obtained from symmetries of CS and BF theories described in [16,17].

2. Differential complexes

In this section we recall some definitions [12,13] to be used in the following and fix the notation.

2.1. General definitions. A direct sum $C^* = \bigoplus_{k \in \mathbb{Z}} C^k$ of vector spaces $C^k$ indexed by integers $k$ is called a differential complex if there are homomorphisms

$$\ldots \to C^{k-1} \xrightarrow{\delta} C^k \xrightarrow{\delta} C^{k+1} \to \ldots$$

such that $\delta^2 = 0$. The homomorphism $\delta$ is called a differential operator of the complex $C^*$. Elements $c^k \in C^k$ are called $k$-cochains.

Let us consider the space

$$Z^k := \text{Ker}\, \delta \cap C^k = \{ z^k \in C^k \text{ such that } \delta z^k = 0 \}. \quad (1)$$

Elements $z^k \in Z^k$ are called $k$-cocycles and $Z^k$ is called the space of $k$-cocycles. The space

$$B^k := \text{Im}\, \delta \cap C^k = \{ b^k \in C^k \text{ such that } b^k = \delta c^{k-1} \text{ for some } c^{k-1} \in C^{k-1} \} \quad (2)$$

is called the space of $k$-coboundaries and elements $b^k \in B^k$ are called $k$-coboundaries. It is clear that $B^k \subset Z^k \subset C^k$, since each $k$-coboundary $b^k$ is a $k$-cocycle: $\delta b^k = \delta^2 c^{k-1} = 0$.

Cocycles $z^k$ such that $z^k \neq \delta c^{k-1}$ are called nontrivial $k$-cocycles. The space of nontrivial $k$-cocycles is parametrized by the quotient space

$$H^k := Z^k / B^k, \quad (3)$$

which is called the $k$-th cohomology space of the complex $C^*$.

The cohomology of the differential complex $C^*$ is the direct sum of the quotient spaces $H^k$,

$$H^* = \bigoplus_{k \in \mathbb{Z}} H^k. \quad (4)$$

De Rham, Dolbeault and Čech complexes are examples of the differential complex.

2.2. De Rham complex. We consider a differentiable (smooth) manifold $M$ of real dimension $n$, the space $\Omega^k(M)$ of smooth $k$-forms on $M$ and an exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$, $d^2 = 0$. All that gives us the de Rham complex

$$\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$$

on a smooth $n$-manifold $M$.

A differential form $\omega$ on $M$ is called closed if $d\omega = 0$. A differential form $\tau$ on $M$ is called exact if $\tau = d\varphi$ for some form $\varphi$. Denote by $Z^k_d(M)$ the space of closed $k$-forms on $M$ and by $B^k_d(M)$ the space of exact $k$-forms on $M$, $B^k_d(M) \subset Z^k_d(M) \subset \Omega^k(M)$. In the language of differential complexes, closed and exact $k$-forms are called de Rham $k$-cocycles and $k$-coboundaries, respectively.
The quotient space $H^k_\partial(M) = Z^k_\partial(M)/B^k_\partial(M)$ is called the $k$-th de Rham cohomology space of $M$. The direct sum

$$H^*_\partial(M) = \bigoplus_{k=0}^n H^k_\partial(M)$$

is called the de Rham cohomology of $M$.

2.3. Dolbeault complex. Consider a complex manifold $M$ of complex dimension $n$ and the space $\Omega^{p,q}(M)$ of smooth $(p,q)$-forms on $M$. The exterior derivative $d$ on complex manifolds is splitted into a direct sum of two differential operators $\partial$ and $\bar{\partial}$ such that $d = \partial + \bar{\partial}$, $d^2 = \partial^2 = \bar{\partial}^2 = \partial \bar{\partial} + \bar{\partial} \partial = 0$. These operators act on the space $\Omega^{p,q}(M)$ as follows:

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M),$$

$$d : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M).$$

Consider a sequence of homomorphisms

$$\Omega^{p,0}(M) \xrightarrow{\partial} \Omega^{p,1}(M) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \Omega^{p,n}(M) \to 0.$$ 

Since $\bar{\partial}^2 = 0$, the direct sum

$$\Omega^{p,*}(M) = \bigoplus_{q=0}^n \Omega^{p,q}(M)$$

of the spaces $\Omega^{p,q}(M)$ is a differential complex, called the Dolbeault complex. A $(p,q)$-form $\varphi$ is called $\bar{\partial}$-closed if $\bar{\partial}\varphi = 0$. In the case $q = 0$ such $\bar{\partial}$-closed forms are called holomorphic.

Denote by

$$Z^p_\partial(M) := \text{Ker } \bar{\partial} \cap \Omega^{p,q}(M) = \{ \omega \in \Omega^{p,q}(M) \text{ such that } \bar{\partial}\omega = 0 \}$$

the space of $\bar{\partial}$-closed $(p,q)$-forms, i.e. Dolbeault $q$-cocycles in the language of differential complexes, and by

$$B^p_\partial(M) := \text{Im } \bar{\partial} \cap \Omega^{p,q}(M) = \{ \tau \in \Omega^{p,q}(M) \text{ such that } \tau = \bar{\partial}\nu \text{ for some } \nu \in \Omega^{p,q-1}(M) \}$$

the space of $\bar{\partial}$-exact $(p,q)$-forms, i.e. Dolbeault $q$-coboundaries. We also denote by $\mathcal{E}^p(M) := Z^p_\partial(M)$ the space of holomorphic $(p,0)$-forms.

The quotient space $H^p_\partial(M) = Z^p_\partial(M)/B^p_\partial(M)$ is called the $(p,q)$-th Dolbeault cohomology space of $M$. The direct sum

$$H^p_\partial(M) = \bigoplus_{q=0}^n H^p_\partial(M)$$

is called the Dolbeault cohomology of $M$.

2.4. Čech complex. For any smooth $n$-manifold $M$ it is possible to choose an open covering $\mathcal{U} = \{ U_\alpha \}_{\alpha \in I}$ such that each nonempty finite intersection of the open sets $U_\alpha$ is diffeomorphic to an open ball in $\mathbb{R}^n$ (or biholomorphic to a Stein manifold for complex $n$-manifolds $M$)[12,13]. Such a covering will be called a good covering. For a good covering the Poincaré lemma holds on each finite intersection of the open sets $U_\alpha$, $\alpha \in I$.

An ordered collection $\langle U_{\alpha_0}, \ldots, U_{\alpha_m} \rangle$ of $m+1$ open sets from the covering $\mathcal{U}$ such that $U_{\alpha_0} \cap \ldots \cap U_{\alpha_m} \neq \emptyset$ is called an $m$-simplex. The set $U_{\alpha_0\ldots\alpha_m} := U_{\alpha_0} \cap \ldots \cap U_{\alpha_m}$ is called a support of the $m$-simplex $\langle U_{\alpha_0}, \ldots, U_{\alpha_m} \rangle$. 

3
Let us consider the space $S$ of forms of a particular “type” defined locally on various open sets of a manifold $M$. Depending on the structure of $M$ (smooth or complex-analytic) this may be the space of smooth $k$-forms, holomorphic $(p,0)$-forms etc. In other words, we consider various sheaves of forms over $M$ [12,13].

A Čech $m$-cochain $c^m$ with values in the space $S$ is a collection $c^m = \{c_{\alpha_0...\alpha_m}\}$ of elements from $S$ defined on supports $U_{\alpha_0...\alpha_m}$ of $m$-simplexes $\langle U_{\alpha_0},...,U_{\alpha_m} \rangle$. The space of Čech $m$-cochains for the covering $U$ with values in $S$ will be denoted by $C^m(U,S)$.

Let us denote by $\rho_\alpha$ the restriction operator acting on elements from $S$ as follows: if $f \in S$ is defined on an open set $U$ then $\rho_\alpha f$ is defined on $U \cap U_\alpha$. Now let us consider the map $\delta := \{\rho_\alpha|_{\alpha}\}$,

\[
\delta : \{c_{\alpha_0...\alpha_m}\} \to \{\rho_{[\alpha_0}|_{\alpha_1...\alpha_{m+1}}\},
\]

where $c^m = \{c_{\alpha_0...\alpha_m}\} \in C^m(U,S)$, $\delta c^m = \{\rho_{[\alpha_0}|_{\alpha_1...\alpha_{m+1}}\} \in C^{m+1}(U,S)$ and $[\alpha_0...\alpha_{m+1}]$ means antisymmetrization w.r.t. the indices $\alpha_0,...,\alpha_{m+1}$. The operator $\delta$ is called a coboundary operator.

Since $\rho_\alpha \rho_\beta = \rho_\beta \rho_\alpha$, we have $\delta^2 = 0$. Therefore, one can consider a sequence of homomorphisms

\[
C^0(U,S) \xrightarrow{\delta} ... \xrightarrow{\delta} C^{m-1}(U,S) \xrightarrow{\delta} C^m(U,S) \xrightarrow{\delta} C^{m+1}(U,S) \xrightarrow{\delta} ... ,
\]

which gives the Čech complex $C^*(U,S) = \bigoplus_{m \geq 0} C^m(U,S)$. The coboundary operator $\delta$ is a “differential” operator of this complex in the terminology of differential complexes.

Denote by

\[
Z^m(U,S) := \ker \delta \cap C^m(U,S) = \{z \in C^m(U,S) : \delta z = 0\}
\]

the space of Čech $m$-cocycles, and by

\[
B^m(U,S) := \text{Im} \delta \cap C^m(U,S) = \{b \in C^m(U,S) : b = \delta c \text{ for some } c \in C^{m-1}(U,S)\}
\]

the space of Čech $m$-coboundaries.

It is evident that $B^m(U,S) \subset Z^m(U,S) \subset C^m(U,S)$. Therefore, we can introduce the space $H^m(U,S) = Z^m(U,S)/B^m(U,S)$ of nontrivial Čech $m$-cocycles, where two distinct elements of $Z^m(U,S)$ are regarded as equivalent in $H^m(U,S)$ if they differ by a coboundary. We call $H^m(U,S)$ the $m$-th Čech cohomology space of the covering $U$ with coefficients in the space $S$.

The direct sum

\[
H^*(U,S) = \bigoplus_{m \geq 0} H^m(U,S)
\]

is called the Čech cohomology of $U$ with coefficients in $S$. The result depends to some extent on the choice of covering $U$, but for a good covering this dependence disappear and $H^m(U,S) \equiv H^m(M,S)$ (for proof see e.g. [12,13]).

3. Field-theoretic description of de Rham and Dolbeault cohomologies

3.1. Abelian Chern-Simons and BF theories. Let $M$ be an oriented smooth $n$-manifold. The field-theoretic description of the de Rham cohomology of $M$ can be given by the following action functional [4]:

\[
S_{dR} = \int_M \sum_{k=1}^l \omega^{(n-k)} \wedge d\omega^{(k-1)},
\]

(8)
where $\omega^{(s)} \in \Omega^s(M)$ are $s$-forms on $M$, $s = 0, 1, \ldots, n$, and $\ell = \lceil \frac{n+1}{2} \rceil$ is the integer part of the number $\frac{n+1}{2}$.

The Euler-Lagrange equations for this action functional are

$$d\omega^{(k)} = 0, \quad k = 0, 1, \ldots, n-1.$$  \hfill (9)

Solutions to these equations are elements from $Z^k_d(M)$. Exact forms from $B^k_d(M)$ give trivial solutions. Therefore, the space of nontrivial solutions (moduli space) for the field equations (9) is given by the space $\bigoplus_{k=0}^{n-1} H^k_d(M)$.

The “de Rham theory” described by $S_{\text{dR}}$ generalizes Abelian Chern-Simons theory defined by the action

$$S_{\text{ACS}} = \int_M A \wedge dA,$$ \hfill (10)

where $A := \omega^{(1)} \in \Omega^1(M)$, $\dim \mathbb{R} M = 3$. One can compare $S_{\text{ACS}}$ with $S_{\text{dR}}$ for $n = 3$:

$$S_{\text{dR}} = \int_M (\omega^{(2)} \wedge d\omega^{(0)} + \omega^{(1)} \wedge d\omega^{(1)}).$$

The “de Rham theory” (8) generalizes also Abelian topological BF theory with the action functional

$$S_{\text{ABF}} = \int_M B \wedge F,$$ \hfill (11)

where $B := \omega^{(n-2)} \in \Omega^{n-2}(M)$, $F := d\omega^{(1)}$, $\omega^{(1)} \in \Omega^1(M)$.

3.2. Abelian holomorphic BF theory. Now we consider a complex $n$-manifold $M$. The field-theoretic description of the Dolbeault cohomology of $M$ can be given by the following action functional:

$$S_{\text{Dol}} = \int_M \sum_{q=1}^{n} \omega^{(n-p,n-q)} \wedge \bar{\partial} \omega^{(p,q-1)},$$ \hfill (12)

where $\omega^{(l,s)} \in \Omega^{l,s}(M)$. The equations of motion are

$$\bar{\partial} \omega^{(p,q-1)} = 0, \quad \bar{\partial} \omega^{(n-p,n-q)} = 0, \quad q = 1, \ldots, n.$$ \hfill (13)

The moduli space of solutions is a vector space $\bigoplus_{q=1}^{n} \left( H^p_q(M) \oplus H^p_{\bar{\partial}}(M) \right)$. The “Dolbeault theory” described by $S_{\text{Dol}}$ generalizes Abelian holomorphic BF theory with the action functional

$$S_{\text{AhBF}} = \int_M B^{n,n-2} \wedge F^{0,2},$$ \hfill (14)

where $B^{n,n-2} := \omega^{(n,n-2)} \in \Omega^{n,n-2}(M)$, $F^{0,2} = \bar{\partial} \omega^{(0,1)}$, $\omega^{(0,1)} \in \Omega^{0,1}(M)$.

3.3. Isomorphisms of cohomologies. For a differentiable $n$-manifold, we denote by $\Omega^k$ the space of locally defined $k$-forms and by $\mathcal{R} \subset \Omega^0$ the space of locally constant functions. For a complex $n$-manifold, by $\mathcal{E}^p$ we denote the space of locally holomorphic $(p, 0)$-forms and by $\mathcal{O} = \mathcal{E}^0$ the space of locally holomorphic functions. In other words, we consider the sheaf $\mathcal{R}$ of locally constant functions and the sheaves $\Omega^k, \mathcal{E}^p$ of forms.

Theorem 1. The de Rham cohomology $H^*_d(M)$ and Čech cohomology $H^*(M, \mathcal{R})$ of a differentiable $n$-manifold $M$ are isomorphic.
by the Poincaré lemma, there exist smooth $dF = 0$ on $k$-forms $\omega^{(k)}$ (de Rham $k$-cochains) on a smooth $n$-manifold $M$ to the functional equations $\rho_{\alpha_0\alpha_1\ldots\alpha_{k+1}} = 0$ on Čech $k$-cochains $\{c_{\alpha_0\ldots\alpha_k}\}$ from $C^k(U, \mathcal{R})$ defined on open subsets $\{U_{\alpha_0\ldots\alpha_k}\}$ of $M$. So, solutions of topological CS and BF theories can be described in terms of Čech cocycles.

**Example 1.** Let us consider a de Rham 2-cocycle $F = (F_\alpha) \in Z_2^g(M)$, i.e. a $d$-closed 2-form $F$ on a smooth $n$-manifold $M$: $dF = 0$. For a covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of $M$, we have $F = \{\rho_\alpha F\} = \{F_\alpha\}$ and $dF_\alpha = 0$. By the Poincaré lemma [12], each closed 2-form locally is exact and therefore on each open set $U_\alpha$ there exists a 1-form $A_\alpha$ such that $F_\alpha = dA_\alpha$.

Let us introduce a collection $\{b_{\alpha\beta}\} = \{\rho_\alpha A_\beta - \rho_\beta A_\alpha\}$ of 1-forms defined on $U_\alpha \cap U_\beta$. It is easy to see that they are closed 1-forms: $db_{\alpha\beta} = d(\rho_\alpha F_\beta - \rho_\beta F_\alpha) = \rho_\alpha \rho_\beta F = 0$. Again, by the Poincaré lemma, there exist smooth functions $s_{\alpha\beta}$ such that $b_{\alpha\beta} = ds_{\alpha\beta}$. Using a collection $\{s_{\alpha\beta}\}$, we introduce functions $c_{\alpha\beta\gamma} = \rho_{[\alpha s_{\beta\gamma}]}$ defined on $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. One obtains $dc_{\alpha\beta\gamma} = \rho_{[\alpha b_{\beta\gamma}]} = \rho_{[\alpha \rho_\beta A_\gamma]} = 0$, i.e. $c_{\alpha\beta\gamma}$ are locally constant functions. At the same time we have $(\delta c)_{\alpha\beta\gamma\delta} = \rho_{[\alpha c_{\beta\gamma\delta}]} = \rho_{[\alpha \rho_\beta s_{\gamma\delta}]} = 0$, i.e. $c = \{c_{\alpha\beta\gamma}\}$ is a Čech 2-cocycle with values in the space $\mathcal{R}$ of locally constant functions. Thus, to the de Rham 2-cocycle $F = \{F_\alpha\} \in Z_2^g(M)$ we have associated a Čech 2-cocycle $c = \{c_{\alpha\beta\gamma}\} \in Z^2(\mathcal{U}, \mathcal{R})$.

Conversely, for any Čech 2-cocycle $c = \{c_{\alpha\beta\gamma}\} \in Z^2(\mathcal{U}, \mathcal{R})$ there exist collections $\{s_{\alpha\beta}\} \in C^1(\mathcal{U}, \mathcal{O})$ and $\{A_\alpha\} \in C^0(\mathcal{U}, \Omega^1)$ of smooth functions and 1-forms such that

$$c_{\alpha\beta\gamma} = \rho_{[\alpha s_{\beta\gamma}]} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma, \quad ds_{\alpha\beta} = \rho_{[\alpha A_\beta]} \quad \text{on} \quad U_\alpha \cap U_\beta.$$ 

If we find local 1-forms $\{A_\alpha\}$ from these equations, then a collection $\{dA_\alpha\}$ gives a $d$-closed 2-form $F = \{dA_\alpha\}$ on $M$.

**Theorem 2.** The Dolbeault cohomology $H^g_{\partial\bar{\partial}}(M)$ and Čech cohomology $H^*(M, E^p)$ of a complex $n$-manifold $M$ are isomorphic.

For proof see e.g. [13].

The Dolbeault–Čech isomorphism reduces differential equations (13) on $(p, q)$-forms (Dolbeault $q$-cochains) defined on a complex $n$-manifold $M$ to functional equations (6) on Čech $q$-cochains from $C^0(\mathcal{U}, E^p)$ defined on open subsets $\{U_{\alpha_0\ldots\alpha_q}\}$ of $M$. By using this isomorphism, one obtains another description of solutions of holomorphic CS and BF theories.

**Example 2.** Let us consider a Dolbeault 1-cocycle $A^{0,1}_\alpha \in Z^{0,1}_g(M)$, i.e. a smooth $\bar{\partial}$-closed $(0,1)$-form $A^{0,1}_\alpha$ on a complex $n$-manifold $M$: $\bar{\partial}A^{0,1}_\alpha = 0$. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a covering of $M$. Then $A^{0,1}_\alpha$ can be represented by a collection $\{A^{0,1}_\alpha\}$ of $\bar{\partial}$-closed $(0,1)$-forms $A^{0,1}_\alpha := \rho_\alpha A^{0,1}_\alpha$ on $U_\alpha$, $\alpha \in I$. By the Poincaré lemma [13], on each open set $U_\alpha$ there exists a smooth function $b_\alpha$ such that $A^{0,1}_\alpha = \bar{\partial}b_\alpha$.

It is easy to see that on each intersection $U_\alpha \cap U_\beta \neq \emptyset$ we have $\bar{\partial}(\rho_\alpha b_\beta - \rho_\beta b_\alpha) = \rho_\alpha A^{0,1}_{\bar{\partial}, \beta} - \rho_\beta A^{0,1}_\alpha = (\rho_\alpha \rho_\beta - \rho_\beta \rho_\alpha)A^{0,1}_\alpha = 0$ since $\rho_\alpha \rho_\beta = 0$. So, we obtain a collection $b = \{b_\alpha\} \in C^0(\mathcal{U}, \mathcal{O})$ of smooth functions $b_\alpha$ defined on $U_\alpha$ and a collection $h = \{h_{\alpha\beta}\} \in C^1(\mathcal{U}, \mathcal{O})$ of holomorphic functions $h_{\alpha\beta} := \rho_\alpha b_\beta - \rho_\beta b_\alpha$ defined on $U_{\alpha\beta}$, $\alpha, \beta \in I$. It is not difficult to see that $h$ is a Čech 1-cocycle from $Z^1(\mathcal{U}, \mathcal{O})$: $(\delta h)_{\alpha\beta\gamma} = \rho_{[\alpha h_{\beta\gamma}]} = \rho_{[\alpha \rho_\beta b_\gamma]} = 0$.

Thus, to the Dolbeault 1-cocycle $A^{0,1}_\alpha$ we have associated a Čech 1-cocycle $h$.

Conversely, for any Čech 1-cocycle $h = \{h_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O})$ there exists a collection $b = \{b_\alpha\} \in C^0(\mathcal{U}, \Omega^1)$ of smooth functions $b_\alpha$ such that

$$h_{\alpha\beta} = \rho_\alpha b_\beta - \rho_\beta b_\alpha.$$  

(15)
on \( U_\alpha \cap U_\beta \). If we find such functions \( \{ b_\alpha \} \) then a collection \( \{ \bar{\partial} b_\alpha \} \) will give us a global \( \bar{\partial} \)-closed \((0,1)\)-form \( A^{0,1} = \{ \bar{\partial} b_\alpha \} \) on \( M \). So, the isomorphism between \( H^{0,1}_\beta (M) \) and \( H^1(M, \mathcal{O}) \) permits one to reduce differential equations \( \bar{\partial} A^{0,1} = \{ \bar{\partial} A^{0,1}_\alpha \} = 0 \) to functional equations (15).

4. Non-Abelian Chern-Simons and BF theories

4.1. Chern-Simons theories. Let \( M \) be an oriented smooth 3-manifold, \( U = \{ U_\alpha \} \) a good covering of \( M \), \( G \) a matrix Lie group, and \( \mathcal{G} \) its Lie algebra. Denote by \( A \) a connection 1-form on a (topologically trivial) principal \( G \)-bundle \( P \) over \( M \). For such bundles \( A \) is a \( \mathcal{G} \)-valued 1-form on \( M \).

Consider the action functional of non-Abelian topological Chern-Simons theory,

\[
S_{CS} = \int_M \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A),
\]

and its field equations

\[
F_A \equiv dA + A \wedge A = 0,
\]

called zero curvature equations. Solutions \( A \) to eqs.(17) define flat connections on \( P \), i.e. differential operators \( d_A = d + A \) such that \( d_A^2 = 0 \). Analogously, non-Abelian holomorphic Chern-Simons theories are defined on complex 3-manifolds and describe flat \((0,1)\)-connections (holomorphic structures) [8-10].

Solutions \( A \) of eqs.(17) can be considered as de Rham 1-cocycles in the non-Abelian de Rham cohomology. On each open set \( U_\alpha \) eqs.(17) are solved trivially: \( A = \{ A_\alpha \} \), \( A_\alpha = \psi_\alpha^{-1} d\psi_\alpha \), where \( \psi = \{ \psi_\alpha \} \) is a collection of smooth \( G \)-valued functions on \( \{ U_\alpha \} \). To obtain a global solution \( A \) on \( M \) from local solutions \( A_\alpha = \psi_\alpha^{-1} d\psi_\alpha \), one should solve the differential equations

\[
\psi^{-1}_\alpha d\psi_{\alpha|\beta} - \psi^{-1}_{\beta|\alpha} d\psi_{\beta|\alpha} = 0 \quad \text{on each intersection} \quad U_\alpha \cap U_\beta \neq \emptyset,
\]

which simply mean that \( \rho_\beta A_\alpha = \rho_\alpha A_\beta \) on \( U_\alpha \cap U_\beta \neq \emptyset \). Here \( \psi_{\alpha|\beta} := \rho_\beta \psi_\alpha \). Equations (18) are equivalent to the equations

\[
d(\psi^{-1}_{\alpha|\beta} \psi^{-1}_{\beta|\alpha}) = 0.
\]

We see that \( c_{\alpha\beta} := \psi^{-1}_{\alpha|\beta} \psi^{-1}_{\beta|\alpha} \) is a (locally) constant \( G \)-valued function defined on \( U_\alpha \cap U_\beta \).

The collection \( \{ c_{\alpha\beta} \} \) of \( G \)-valued functions is a Čech 1-cocycle in the non-Abelian Čech cohomology [14], where the cocycle conditions are

\[
(\rho_\gamma c_{\alpha\beta})(\rho_\alpha c_{\beta\gamma})(\rho_\beta c_{\gamma\alpha}) = 1 \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset.
\]

Therefore, solutions of eqs.(17) can be obtained by splitting locally constant \( G \)-valued functions \( c_{\alpha\beta} \) satisfying eqs.(20) into a product of two smooth \( G \)-valued functions \( \psi_\alpha \) and \( \psi_{\beta}^{-1} \) defined on \( U_\alpha \) and \( U_\beta \), respectively. Then, by virtue of the de Rham-Čech correspondence, a collection \( \{ \psi_\alpha^{-1} d\psi_\alpha \} =: A \) gives a global solution to eqs.(17).

4.2. Topological BF theories. A generalization of (topological) Chern-Simons theories to arbitrary dimensions is given by (topological) BF theories [4,6,7]. The action functional for non-Abelian topological BF theory has the following form:

\[
S_{BF} = \int_M \text{Tr} (B \wedge F_A),
\]
where $M$ is an oriented smooth $n$-manifold, $F_A$ is the curvature of a connection 1-form $A$ on a topologically trivial principal $G$-bundle $P$ over $M$, and $B$ is a $G$-valued $(n-2)$-form on $M$. The variation of the action (21) w.r.t. $B$ gives eqs.(17), and the variation of this action w.r.t. $A$ gives the equations

$$d_A B = dB + A \wedge B - B \wedge A = 0 .$$

(22)

Thus, topological BF theories describe flat connections $d_A$ on a bundle $P$ over $M$ and $G$-valued $d_A$-closed $(n-2)$-forms $B$ on $M$.

Constructing solutions of the field equations of topological CS and BF theories in terms of deformation theory of locally constant (flat) bundles is discussed in [16]. Differential equations (17) are equivalent to functional equations (20) which are solved by $c_{\alpha\beta} = \psi_{\alpha|\beta} \psi_{\beta|\alpha}^{-1}$ for some smooth $G$-valued functions $\{\psi_\alpha\}$, and for a flat connection $A = \{\psi_\alpha^{-1} d\psi_\alpha\}$ eqs.(22) can be easily reduced to equations $d(\psi B \psi^{-1}) = 0$ from standard de Rham cohomology. Symmetries of CS and BF topological theories can be described in terms of Čech 1-cocycles with values in the sheaf of locally constant maps of the space $M$ into the Lie group $G$. For more details see [16].

4.3. Holomorphic BF theories. Let $M$ be a complex $n$-dimensional manifold, $G$ a complex matrix Lie group, $G$ its Lie algebra, $P$ a topologically trivial principal $G$-bundle over $M$, $A$ a connection 1-form on $P$, and $F_A = dA + A \wedge A$ its curvature. Consider holomorphic BF theories [11] with the action functional

$$S_{hBF} = \int_M \text{Tr} (B^{n,n-2} \wedge F_A^{0,2}) ,$$

(23)

where $B^{n,n-2}$ is a $G$-valued $(n,n-2)$-form on $M$ and $F_A^{0,2}$ is the $(0,2)$-component of the curvature tensor $F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2}$. The field equations for the action (23) are

$$\bar{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1} = 0 , \quad \bar{\partial} B^{n,n-2} + A^{0,1} \wedge B^{n,n-2} - B^{n,n-2} \wedge A^{0,1} = 0 ,$$

(24)

where $A^{0,1}$ is the $(0,1)$-component of a connection 1-form $A = A^{1,0} + A^{0,1}$ on $P$. If a representation of $G$ in the complex vector space $\mathbb{C}^r$ is given, we can associate with $P$ the complex vector bundle $E = P \times_G \mathbb{C}^r$ and use vector bundles in the description of BF theories.

It follows from eqs.(24) that models (23) describe flat $(0,1)$-connections $\bar{\partial}_A = \bar{\partial} + A^{0,1}$ on $G$-bundles over complex $n$-manifolds $M$ and $\bar{\partial}_A$-closed $G$-valued $(n,n-2)$-forms $B^{n,n-2}$ on $M$. A procedure of constructing solutions to eqs.(24) and mapping solutions into one another (dressing transformations) are discussed in [10,17]. This cohomological method of solving eqs.(24) is based on the equivalence of Čech and Dolbeault descriptions of holomorphic bundles and is a generalization to arbitrary dimensions of the dressing approach (Riemann-Hilbert problems) to solving integrable equations in two dimensions.

5. Manin-Ward integrable systems

5.1. Double fibrations. Let us consider complex manifolds $X$ and $Z$ with a correspondence between them defined by a double fibration

$$Y \xleftarrow{\eta} X \xrightarrow{\rho} Z ,$$

(25)
where $Y$ is a complex submanifold in the direct product $Z \times X$, and projections $\eta$ and $\rho$ are surjective holomorphic maps. We assume that the fibres of $\eta$ are connected and simply connected complex manifolds, and the fibres of $\rho$ are supposed to be compact.

The double fibration (25) appears in the Kodaira relative deformation theory of compact submanifolds of complex manifolds [25]. Points of $X$ describe a family of complex submanifolds $\eta(\rho^{-1}(x))$ of $Z$, and points of $Z$ describe a family of submanifolds $\rho(\eta^{-1}(z))$ of $X$. Note that Kodaira’s double fibrations are used in reparametrization-invariant geometric quantization of bosonic string theory (see [26] and references therein).

5.2. Integrable distributions. Under some mild topological conditions on the embeddings $\eta(\rho^{-1}(x)) \hookrightarrow Z$ the manifold $X$ comes equipped with a family of torsion-free affine connections which are integrable on submanifolds $\rho(\eta^{-1}(z))$ of $X$ [27,28]. We assume that these conditions are satisfied and $X$ has such a connection.

The manifold $Y$ has two integrable transversal distributions: the distribution $V_{\eta}$ of holomorphic vertical vector fields in the fibration $\eta : Y \to Z$ and the distribution $V_{\rho}$ of holomorphic vertical vector fields in the fibration $\rho : Y \to X$.

Let $E_{\eta}$ denote the space of holomorphic (1,0)-forms dual to the holomorphic vectors from the distribution $V_{\eta}$. Since we have a projection $E \to E_{\eta}$, we can introduce a relative exterior derivative $\partial_{\eta}$ as the composition $\mathcal{O} \overset{\partial}{\to} E^1 \overset{\partial_{\eta}}{\to} E^1_{\eta}$, where $E^1$ and $\mathcal{O}$ are the sheaves of holomorphic (1,0)-forms and holomorphic functions on $Y$, respectively.

5.3. Integrable systems. Let us consider the double fibration (25), a topologically trivial holomorphic rank $r$ vector bundle $E$ over $Z$, the pulled-back bundle $\tilde{E} \equiv \eta^* E$ over $Y$ and the direct image bundle $E \equiv \rho_* \eta^* E$ over $X$. In local holomorphic trivializations, the bundle $\tilde{E}$ is defined by holomorphic transition functions constant along the fibres of $\eta$ (i.e. annihilated by $\bar{\partial}$ and $\partial_{\eta}$. So, $\tilde{E}$ is equipped with the holomorphic structure $\bar{\partial}$ and a flat relative holomorphic (1,0)-connection $\partial_{\eta}$: $\bar{\partial}^2 = 0, \partial_{\eta}^2 = 0$ and $\partial_{\eta} \bar{\partial} + \bar{\partial} \partial_{\eta} = 0$.

Suppose that the bundle $E$ is holomorphically trivial on each submanifold $\eta(\rho^{-1}(x))$ of $Z$, which is equivalent to saying that the bundle $\tilde{E} = \eta^* E$ is holomorphically trivial on each submanifold $\rho^{-1}(x)$ of $Y$, $x \in X$. Such bundles are called $X$-trivial holomorphic bundles [19]. Yu.Manin [19] and R.Ward [20] have shown that there is a one-to-one correspondence between $X$-trivial holomorphic vector bundles $E$ on $Z$ and vector bundles $\tilde{E} = \rho_* \eta^* E$ on $X$ with a differential operator $D : \tilde{E} \to \rho_* \mathcal{E}_\eta^1 \otimes \tilde{E}$ flat on each submanifold $\rho(\eta^{-1}(z))$ of $X$, i.e. $D^2 |_{\rho(\eta^{-1}(z))} = 0$ for all $z \in Z$. The operator $D$ is constructed from differential operators on $X$ and the equations $D^2 |_{\rho(\eta^{-1}(z))} = 0$ yield a system of nonlinear integrable differential equations on $X$ [19-23].

5.4. Dressing symmetries. Let us consider an $X$-trivial holomorphic rank $r$ vector bundle $E \to Z$ and the pulled-back bundle $\tilde{E} = \eta^* E \to Y$ with the flat relative (1,0)-connection $\partial_{\eta}$. There is a one-to-one correspondence between Manin-Ward integrable systems on $X$ and the bundles $\tilde{E}$. These bundles are locally constant (flat) along the fibres of $\eta : Y \to Z$ and holomorphically trivial along the fibres of $\rho : Y \to X$. Therefore, maps of solutions of the Manin-Ward integrable equations into one another (dressing symmetries) can be described by combining results on symmetries of flat [16] and holomorphic [17] bundles. Namely, consider a covering $\mathcal{U} = \{U_\alpha\}$ of $Y$ and holomorphic transition functions $\{f_{\alpha\beta}\}$ in the topologically trivial vector bundle $\tilde{E}$ satisfying $\partial_{\eta} f_{\alpha\beta} = 0$. Let $\mathcal{H}$ be the sheaf of holomorphic $gl(r, \mathbb{C})$-valued functions on $Y$ and $\mathcal{H}_\eta$ its subsheaf of functions annihilated by $\partial_{\eta}$. We consider the Lie algebra $C^1(\mathcal{U}, \mathcal{H}_\eta)$ of $\check{C}$ech 1-cochains with values in $\mathcal{H}_\eta$ and
define its action on the transition functions \( \{ f_{\alpha \beta} \} \) in \( \tilde{E} \) by the formula
\[
\delta_{\theta} f_{\alpha \beta} = \theta_{\alpha \beta} f_{\alpha \beta} - f_{\alpha \beta} \theta_{\beta \alpha} ,
\]
where \( \{ \theta_{\alpha \beta} \} \in C^1(\mathcal{U}, \mathcal{H}_g) \). Following [16,17] one can show that this action generates infinitesimal deformations of the bundle \( \tilde{E} \) on \( Y \) preserving the condition of \( X \)-triviality and therefore descends to symmetries of Manin-Ward integrable systems on \( X \). The algebra \( C^1(\mathcal{U}, \mathcal{H}_g) \) of collections \( \{ \theta_{\alpha \beta} \} \) with pointwise commutators generalizes affine Lie algebras generated by algebra-valued holomorphic Čech 1-cochains on the Riemann sphere \( \mathbb{C}P^1 \).

5.5. Dolbeault description. Manin-Ward integrable systems can be described in terms of holomorphic BF theories on \( Y \). For this, let us consider smooth local trivializations of the bundle \( \tilde{E} \to Y \) such that transition functions of \( \tilde{E} \) become equal to unity on any intersection of charts. This is possible since \( \tilde{E} \) is equivalent to a product bundle. For a fixed covering \( \mathcal{U} = \{ U_{\alpha} \}_{\alpha \in I} \) of the manifold \( Y \) and transition functions \( f = \{ f_{\alpha \beta} \} \), the change of trivialization for \( \tilde{E} \) from holomorphic to smooth is described by a collection \( \psi = \{ \psi_{\alpha} \} \) of smooth \( GL(r, \mathbb{C}) \)-valued functions \( \psi_{\alpha} \) on \( U_{\alpha} \) such that \( f_{\alpha \beta} = \psi_{\alpha} \psi_{\beta}^{-1} \).

After smooth mapping \( \tilde{E} \) onto the direct product bundle \( Y \times \mathbb{C}^r \), the holomorphic structure \( \tilde{\partial} \) and the flat relative connection \( \partial_{\eta} \) become a holomorphic structure \( \nabla^{0,1} = \tilde{\partial} + A^{0,1} \) and a flat relative \((1,0)\)-connection \( \nabla^{1,0}_\eta = \partial_{\eta} + A^{1,0}_\eta \) such that
\[
(\nabla^{0,1})^2 = \tilde{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1} = 0 , \quad \partial_{\eta} A^{0,1} + \tilde{\partial} A^{1,0} + A^{0,1} \wedge A^{1,0} + A^{1,0} \wedge A^{0,1} = 0 , \quad (26a)
\]
\[
(\nabla^{1,0})^2 = \partial_{\eta} A^{1,0} + A^{1,0} \wedge A^{1,0} = 0 . \quad (26b)
\]
Equations (26a,b) mean that the bundle \( \tilde{E} \simeq Y \times \mathbb{C}^r \) is holomorphic and its restrictions to the fibres \( \eta^{-1}(z) \) of the projection \( \eta : Y \to Z \) are locally constant (flat) bundles for any \( z \in Z \). Functions \( \psi_{\alpha} \in GL(r, \mathbb{C}) \) defining these bundles can be chosen to be holomorphic in complex coordinates on \( U_{\alpha} \cap \eta^{-1}(z) \) for any \( z \in Z \), and therefore we obtain \( A^{0,1}_\eta = 0 \).

Recall that the bundle \( \tilde{E} \to Y \) is \( X \)-trivial. Up to a gauge transformation this is equivalent to the equations
\[
A^{0,1}_\rho = 0 , \quad (26c)
\]
where \( A^{0,1}_\rho \) is the component of \( A^{0,1} \) along the distribution \( \mathcal{V}^{0,1}_\rho \) of antiholomorphic vertical vector fields in the fibration \( \rho : Y \to X \). In other words, \( GL(r, \mathbb{C}) \)-valued functions \( \psi_{\alpha} \) from a collection \( \psi = \{ \psi_{\alpha} \} \) and \( A^{1,0}_\eta = \psi^{-1}\partial_{\eta}\psi \) can be chosen to be holomorphic in complex coordinates on \( U_{\alpha} \cap \rho^{-1}(x), x \in X \). Equations (26) give the Dolbeault description of \( X \)-trivial holomorphic bundles \( E = \eta^*E \to Y \).

A restriction of the flat relative \((1,0)\)-connection \( \nabla^{1,0}_\eta \) to \( \rho^{-1}(x) \hookrightarrow Y \) is a differential operator with values in a holomorphic vector bundle over \( \rho^{-1}(x) \). It can be expanded in global holomorphic sections of this bundle and projected to \( X \). Then we obtain an operator \( D \equiv \rho_* \nabla^{1,0}_\eta \) defining Manin-Ward integrable equations on \( X \). These equations arise as the compatibility conditions \( (\nabla^{1,0}_\eta)^2 = 0 \) (equivalent to \( D^2|_{\rho(\eta^{-1}(z))} = 0 \)) for the linear system of equations
\[
\nabla^{1,0}_\eta \varphi = 0 , \quad (27)
\]
where \( \varphi \) is a section of the bundle \( \tilde{E} \) on \( Y \), depending on \( x \in X \) and \( \lambda \in \rho^{-1}(x) \). Parameters \( \lambda \) are called spectral parameters. For more details see [19-24].
5.6. Ward’s example [20]. Here $X$ is an open subset of the complex vector space $\mathbb{C}^{k(m+1)}$, $Y = X \times \mathbb{C}P^m$ and $Z$ is an open subset of the complex projective space $\mathbb{C}P^{m(k+1)}$ (see the diagram (25)). The space $Y$ is a $\mathbb{C}P^m$-bundle over $X$ with the canonical holomorphic projection $\rho : Y \to X$. Let $\lambda^j (j = 0, ..., m)$ be homogeneous coordinates on $\mathbb{C}P^m$ and $x^{aj} (a = 1, ..., k)$ complex coordinates on $X \subset \mathbb{C}^{k(m+1)}$. Then the integrable distribution $\mathfrak{h}^{1,0}_{\eta}$ on $Y$ is spanned by the vector fields $V_a = \lambda^j \partial_{aj}$, $\partial_{aj} := \partial/\partial x^{aj}$. The space $Z$ is the quotient of $Y$ by the flows of the vector fields $V_a$, $\dim \mathbb{C}Z = m(k+1)$. For each fixed $\lambda = \{\lambda^j\} \in \mathbb{C}P^m$, the vectors $V_1(\lambda), ..., V_k(\lambda)$ span a complex $k$-plane $\alpha(\lambda)$ in $X$.

There is a one-to-one correspondence between $X$-trivial holomorphic vector bundles $E$ over $Z$ and bundles $\tilde{E} = \rho_* \eta^* E$ over $X$ with a connection $D = dx^{aj} D_{aj} = dx^{aj} (\partial_{aj} + A_{aj})$ flat on each complex $k$-plane $\alpha(\lambda)$, $\lambda \in \mathbb{C}P^m$. Here $A_{aj}(x)$ are matrix-valued functions on $X$ (components of a connection 1-form on $\tilde{E}$). The operator $\nabla^{1,0}_\eta$ is defined by the differential operators $\nabla^{1,0}_\eta := \lambda^j D_{aj}$ and the linear system of equations is $\nabla^{1,0}_\eta \varphi = 0$, where $\varphi(x, \lambda)$ is a section of the bundle $\eta^* E \to Y$. The compatibility conditions of this linear system are polynomials in $\lambda^j$:

$$\lambda^j \lambda^l [\partial_{aj} + A_{aj}, \partial_{bl} + A_{bl}] = 0.$$ 

After equating each coefficient of these polynomials to zero we obtain a set of nonlinear differential equations on the matrix-valued functions $A_{aj}(x)$ on $X$.

5.7. Supersymmetric Yang-Mills. It is not difficult to introduce supersymmetric generalizations of Manin–Ward integrable systems [19,21]. One of the most interesting examples of such integrable models is the supersymmetric Yang-Mills theory in ten dimensions [29]. For supersymmetric Yang-Mills model on complexified $d = 10$ space-time, the space $X$ is a region in the superspace $\mathbb{C}^{10|16}$ (10 bosonic and 16 fermionic coordinates), $Y$ is a complex supermanifold of dimension $18|16$ and $Z$ is a complex supermanifold of dimension $17|8$ (see [21,29,30,31] for more details and references). Importance of this model is connected with the fact that it appears in the low-energy limit of superstrings. The Čech approach and advocated cohomological methods may be helpful in describing moduli spaces and symmetries of $d = 10$ supersymmetric Yang-Mills and superstring theories.

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