Gauge Independence and Relativistic Electron Dispersion Equation in Dense Media

D. Oliva Agüero, H. Pérez Rojas, A. Pérez Martínez and A. Amézaga Hechavarría  
Centro de Matemáticas y Física Teórica, Calle E No. 309, Ciudad Habana, Cuba

Abstract

We discuss the gauge parameter dependence of particle spectra in statistical quantum electrodynamics and conclude that the electron spectrum is gauge-parameter dependent. The physical spectrum being obtained in the Landau gauge, which leads to gauge invariance in a restricted class of gauge transformations. We compute the thermal self-energy of electrons in a dense media to first order in the fine structure constant. In the zero-temperature limit and at high density the dispersion equation is solved for some gauges, the curves differing from that corresponding to the physical transverse gauge.

I. INTRODUCTION

The investigation of relativistic quantum electrodynamics at finite temperature ($T$) and chemical potential ($\mu$) is of relevance in several astrophysical and cosmological contexts [1]. Interest in finite temperature and density of quantum field theories has increased continuously since the early seventies [2]. The problem about the dependence of the fermionic Green functions on the gauge parameter is not new. For QED, it was solved by Landau and Khalatnikov in [3] and simultaneously by Fradkin in [4]. These authors showed that the longitudinal part of gauge field appears as a phase factor into the electron Green function (see also [5]). The last result was extended by Fradkin [6] to the statistical quantum electrodynamics case, where the propagators also depend on the four-velocity of the medium. It is a well established fact that in QFT the electron propagator is gauge invariant on the tree-level mass shell ($P^2 = m^2$), see [8]. For finite temperature, this invariance has been also formally shown [9]. In the literature it has been explicitly verified this conclusion in several limiting cases [10][12]. However, it should be noted that the calculation of the mass operator is carried out precisely to investigate the system perturbatively out of the mass shell, where the condition of gauge parameter independence frequently mentioned (see i.e. [8]) is not fulfilled, since the dispersion equation differs in general from the form $P^2 = m^2$, although it is expected that the gauge independence of the dispersion equation remains valid near the mass shell. But the departure of the spectrum from the $P^2 = m^2$ form in the temperature case [13] excludes the validity of the above mentioned proof at any order of perturbation theory. We will return to this point along the paper.

A common practice in the literature has been to admit gauge independence, to choose the Feynman gauge and to check it perturbatively in some limits. Despite of the simplifications
introduced by the Feynman gauge, the Landau gauge has a special importance because the longitudinal part of the photon propagator contributes as a phase factor to the electron Green function (see [6] for a general discussion) and thus it does not contribute to the physical spectrum when the gauge parameter is chosen in a such a way that the longitudinal part of the photon propagator is zero. The purpose of the present paper is to discuss the problem of the gauge-parameter dependence in a non-perturbative finite temperature field theory, to investigate the behavior of the electron dispersion equation at finite temperature and density and to consider the limit of $T = 0$ and high density afterwards.

This paper is organized as follows: in section 2 we discuss the problem of the gauge-parameter dependence in a non-perturbative finite temperature field theory. In section 3, we compute the electron self-energy in the one loop approximation. In section 4 two dispersion equations are obtained in the limit of zero temperature for ultrahigh density. Section 5 contain the discussion and conclusions. In appendix A we derive the scalar quantities in terms of which the mass operator is written, we give the definitions of the main integrals that appear in the calculation, and we carry out the Matsubara sums in the fourth momentum component $k_4$.

II. GAUGE-PARAMETER DEPENDENCE

In what follows we want to argue in a perturbative-independent way that the physical electron spectrum can be obtained (in a covariant way) only by using the so-called Landau gauge $1/\alpha = 0$, which is actually compatible with a large class of gauge transformations, as indicated below. Usually the gauge parameter dependence $\alpha$ is introduced in QED to make it possible to invert the (otherwise singular) inverse photon propagator $D_{\mu\nu}^{-1}$ which is four-dimensional transverse. But in QFT as well as in statistics, the photon spectrum is $\alpha$-independent. In the case of a medium ($\mu \neq 0, T \neq 0$) due to the gauge invariance of the theory [6] we can write the inverse photon Green function, by introducing in the QED Lagrangian the gauge $\alpha \partial_\mu A_\mu \partial_\nu A_\nu$, as

$$D_{\mu\nu}^{-1} = T_{\mu\nu}^1 k^2 - \Pi_{\mu\nu} + \alpha k_\mu k_\nu = 0$$  \hspace{1cm} (1)

where $T_{\mu\nu}^1 = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$ and $\Pi_{\mu\nu}$ has the general four-dimensional tensor structure

$$\Pi_{\mu\nu} = T_{\mu\nu}^1 A + T_{\mu\nu}^2 B$$  \hspace{1cm} (2)

where

$$T_{\mu\nu}^2 = \left( \frac{k_\mu k_\nu}{k^2} - \frac{k_\mu u_\nu}{ku} - \frac{u_\mu k_\nu}{uk} + \frac{u_\mu u_\nu k^2}{(ku)^2} \right).$$

The photon Green function is then

$$D_{\mu\nu} = T_{\mu\nu}^1 A + T_{\mu\nu}^2 B + \frac{k_\mu k_\nu}{\alpha k^4},$$  \hspace{1cm} (3)

where
\[ A_1 = \frac{1}{k^2 - A}, \quad B_1 = \frac{B}{(k^2 - A)(k^2 - A + B(1 - \frac{k^2}{(ku)^2}))}. \] (4)

In the rest frame the polarization operator \( \Pi_{\mu\nu} \) has the tensor structure:

\[
\Pi_{\mu\nu} = \begin{cases} 
(\delta_{ij} - \frac{k_i k_j}{k^2}) A(k, k_4) + \Pi_{44} \frac{k_i k_j k^2}{k^2} & i, j = 1, 2, 3 \\
\Pi_{4i} = \Pi_{i4} = -\frac{k_i k_4}{k^2} \Pi_{44} & \end{cases}
\] (5)

where \( \Pi_{44} = \frac{k^2}{k^2} (A + \frac{k^2}{k^2} B) \). We have also in the rest frame, in the case \( \alpha \to \infty \),

\[
D_{\mu\nu} = \begin{cases} 
(\delta_{ij} - \frac{k_i k_j}{k^2}) C + \frac{k_i k_j k^2}{k^2} D_{44} & i, j = 1, 2, 3 \\
D_{4i} = D_{i4} = -\frac{k_i k_4}{k^2} D_{44} & \end{cases}
\] (6)

where \( C = 1/(k^2 - A), \ D_{44} = k^4/[k^4(k^2 - \Pi_{44})] \).

The photon spectrum may be obtained either by solving (1), in which \( \alpha \neq 0 \) factorizes, or from the poles of the four-dimensional transverse part of \( D_{\mu\nu} \). In the first case the following non-perturbative gauge-parameter independent and gauge invariant spectrum is obtained in the rest frame

\[ k^2 + A(k^2, k_4) = 0, \quad 1 + \frac{\Pi_{44}(k^2, k_4)}{k^2} = 0 \] (7)

The first equation corresponds to the spatial transverse modes and the second (after multiplication by \( k^2 \neq 0 \)), to the spatial longitudinal one. These are the physical modes, (which in the one-loop approximation and at high temperature were studied in detail by [11]). A factor \( k^4 \), accounting for the unphysical modes introduced by the gauge condition, when calculating \( [\text{Det}D_{\mu\nu}]^{-1/2} \), is removed by the Faddeev-Popov ghost term. The photon spectrum can be also obtained directly from the poles of the Green function (4). We have also the non-physical modes \( k^4 = 0 \), given by the longitudinal term, which are not present in (6) since they were removed by choosing the gauge parameter \( \alpha \to \infty \). In calculating the exact polarization operator in quantum field theory as well as in statistics, only the transverse electron Green function contributes. The property stems from the gauge invariance and gauge parameter independence of \( \Pi_{\mu\nu} \), as can be derived from the Ward identities obtained from the QED effective action \( \Gamma \) in vacuum as well as in the temperature case,

\[ \partial_\mu(x_1) \frac{\delta^2 \Gamma(0)}{\delta A_\mu(x_1) \delta A_{\mu}(x_2)} = \alpha \delta^4 \partial_\mu \delta(x_1 - x_2) \] (8)

which in momentum space reads

\[ k_\mu D_{\mu\nu}^{-1}(k) = \alpha k_\nu k^2, \] (9)

where the substitution of (4) in (9) leads to the four-dimensional transversality for \( \Pi_{\mu\nu} \). This means that \( \Pi_{\mu\nu}(x, y) \) is gauge-invariant and gauge-parameter independent [3], [7]. Explicitly we have that \( \Pi_{\mu\nu}(x, y) \) is given by
\[\Pi_{\mu\nu}(x, y) = e^2 \text{Tr} \int \gamma_{\mu} G(x, z) \Gamma_{\nu}(z, y, y') G(y', x) d^4x d^4 y,\]  

(10)

where the integration in the fourth coordinate must be understood as in the interval \(-\beta, \beta\) in the temperature case, and \(\Gamma(z, y, y') = \gamma \delta(z - y) \delta(y - y') - \frac{\delta\Sigma(z, y')}{\text{die} A_{\mu}}\). Thus, if a gauge parameter dependent \(G(x, y|\alpha)\) is chosen, cancellations must occur in the integrand in such a way that the integral obtained from (10) must be the same as that obtained by using \(G(x, y\infty) = G(x, y)\). This means that only the physical poles of \(G(x, y)\) contribute to \(\Pi_{\mu\nu}\), and in consequence, to the transverse part of the exact photon Green function \(D_{\mu\nu}\).

Concerning the Fermion spectrum, the gauge parameter dependence is introduced in the calculation of the mass operator \(\Sigma\) in terms of \(G(x, y|\alpha)\) and \(D_{\mu\nu}(x, y|\alpha)\). Now, for the statistical case, as shown by Fradkin [6], the gauge parameter dependence is given by a similar formula than in QED. We shall write it for the one-particle electron Green function as,

\[G(x - y|\alpha) = G_0(x - y) \exp\left\{ e^2 \left[ \Delta^l(x - y) - \Delta^l(0) \right] \right\} \]

(11)

where

\[\Delta^l(x) = -\frac{1}{(2\pi)^3 \beta \alpha} \sum_{k_4} \int \frac{d^3k}{k^4} e^{ikx} \]

(12)

(in (11), \(G_0\) is the propagator in the Landau gauge (1/\(\alpha = 0\) and in our representation coincides with \(G^l\)), the poles of \(G(p, \alpha)\) are obtained from the poles of the Fourier transform of the right hand term in (11) which is the convolution of \(G_0(x - y)\) with the exponential \(\alpha\)-dependent factor. Due to this convolution, in any order of a perturbative expansion of \(G(p)\), the resulting poles are \(\alpha\)-dependent.

In addition to our previous arguments concerning the electron Green function dependence on the gauge parameter, we want to mention the following one: Let us call \(\Sigma^t\) and \(\Sigma^l\) respectively the transverse and longitudinal terms of the electron mass operator, where

\[\Sigma^l = \frac{e^2}{2\pi \beta} \sum_{k_4} \int \gamma_{\mu} G(p + k) \Gamma_{\nu}(p + k, k) \frac{k_\mu k_\nu}{k^2} D^l d^3k,\]

(13)

which is obviously nonzero and gauge-parameter dependent. Then, by taking \(D^l(k) = 1/\alpha k^2\) we observe it contains also the contribution of the unphysical photon modes \(k_4^2 + k^2 = 0\). By using Ward identities it can be shown [6] that

\[\left[ i\gamma_{\mu} p_{\mu} + m + \Sigma^t(p) \right] G(p) = 1 - \frac{e^2}{2\pi \beta} \sum_{k_4} \int \gamma_{\mu} G(p + k) \frac{k_\mu}{\alpha k^4} d^3k.\]

(14)

We observe also from this expression that the poles of \(G(p)\) are given by the zeros of the term in squared brackets in (14) plus the poles introduced by the second term containing the unphysical photon modes.

Thus, from (12) or (14) it is seen that the poles of the electron Green function become independent of the unphysical photon modes if it is taken the limit \(\alpha \to \infty\) words, by eliminating the contribution of these modes to \(\Sigma\). The zero temperature QED case can be treated by following similar arguments. We may cite especially ref. [14] in which a very
interesting discussion of the electron physical modes is given in the one-loop approximation for $\Sigma$ by using the Coulomb gauge.

Thus, although there are general proofs of gauge invariance for gauge boson spectrum [29], for fermions there are not (we consider there cannot be) explicit proofs of gauge parameter independence of the one-particle spectrum given by the poles of the Green function, except in very specific cases, as the infrared limit considered by Abrikosov [15]. However, at least in the abelian case, if all calculations are made in the Landau gauge $\alpha \to \infty$, which corresponds to choosing the longitudinal part of the photon Green function as equal to zero, the results lead to the physical fermion spectrum, and are gauge invariant under the class of gauge transformations $A_\mu(x)' = A_\mu(x) + \partial_\mu \eta(x)$, $\psi(x)' = e^{i\eta(x)}\psi(x)$ in which $\eta(x) = -\partial_\mu A(x)/\partial^2$. This condition excludes the unphysical photon modes from $\Sigma(x, y)$.

For non-abelian theories, an expression analogous to (11) has not been obtained yet. However, we consider the previous arguments might be useful in considering the problem of gauge parameter independence of the fermion spectrum in the non-abelian case.

### III. ELECTRON SELF ENERGY

In this section we will perform the computation of the mass operator in the one loop approximation for a general gauge. We shall redefine the gauge parameter as $\xi = 1/\alpha$ for simplicity in the forthcoming expressions. We will keep the dependence on the gauge parameter to allow a further comparison of the results for different gauges. We start from the expression for the electron self-energy in the one loop approximation in the temperature formalism (we use the shifted momenta $p^*_\mu = p_\mu - i\mu_\epsilon \delta_{\lambda,\chi}$ [25])

$$
\Sigma^e(p) = \frac{e^2}{(2\pi)^3} \sum_{k_4} \int d^3 k \frac{\gamma_\mu G^0(p^* + k)\Gamma^0 D^0_{\mu\nu}(k)}{\Delta_{\mu\nu}} = \frac{e^2}{(2\pi)^3} \sum_{k_4} \int d^3 k \left[ \frac{\gamma_\mu}{i(p^*_\mu + k)^2 + m_e^2} \gamma_\nu \right]
$$

Due to the medium we have the additional four velocity vector $u_\mu$ (in the rest frame $u_\mu = (0, 0, 0, i)$, so that $k \cdot u = ik_4 = -\omega$). Here, $\xi$ is a parameter of arbitrary gauge) and thus the vectors, on which the physical quantities may depend, are $k_\mu$ and $u_\mu$. The matrix structure of the mass operator is now

$$
\Sigma(p) = i (a\gamma_\rho p_\rho + b\gamma_\rho u_\rho + c) 
$$

After some work (see the appendix), we have, for an arbitrary $\xi$

$$
a = g^2 \frac{1}{p^2} \left\{ (p^2 I_1 + 4 I_3 + 2 I_4 + 2i\omega I_5) + \xi \left( p^2 I_1 - I_3 - 2 I_4 - 2i\omega I_5 \right) \right\} 
$$

$$
b = g^2 \frac{1}{p^2} \left\{ \left[ -3i p^2 I_2 - 3\omega I_3 + 2\omega I_4 - 2i(p^2 - \omega^2) I_5 + 2p^2\omega I_6 \right] \right. 
+ \xi \left[ -i p^2 I_2 + \omega I_3 + 2\omega I_4 - 2i(p^2 - \omega^2) I_5 + 2p^2\omega I_6 \right] 
$$

5
where $g^2 = e^2 / (2\pi)^2$ is the fine structure constant and $I_1 - I_6$ are integrals defined in the appendix.

In the appendix, we evaluate these integrals, that is we perform the Matsubara sum and do some algebra, subject to the following approximations: (1) we drop the terms without dependence on the temperature, i.e. we ignore non-thermal electrons; (2) we assume that the chemical potential is much larger than the temperature, $\mu >> T$, so that we may approximate the distribution function of the electrons by a step function; (3) in accordance with the previous point, since the Fermi surface for positrons is negative, we assume that the distribution functions for positrons are exactly zero; and (4) in agreement again with all of this, we effectively neglect the temperature so that also the photon distribution function is set to zero.

The upshot is a system with a Fermi liquid of electrons and no real photons or positrons. Of course, in a physical situation charge must be balanced, which in the framework we consider can be easily achieved through a proton background (also, the proton and electron tadpoles cancel each other, so we ignore tadpoles).

From the expressions for the integrals $I_i$ in the appendix, we carry out the analytic prolongation $p_4 + i \mu = i \omega$, the angular integrations, and finally neglect the mass $\sqrt{\mu^2 - m^2_e} \simeq \mu$ in the upper limit in the integrals. Note that we do not neglect the mass with respect to the momentum. The expressions we obtain read as follows:

$$I_1 = -\pi \int_0^\mu \frac{dk}{k^2 + m^2_e} \ln \left| \frac{A + B}{A - B} \right|$$

$$I_2 = -\pi i \int_0^\mu \frac{dk}{k^2 + m^2_e} \left( \frac{\omega - \sqrt{k^2 + m^2_e}}{\sqrt{k^2 + m^2_e}} \right) \ln \left| \frac{A + B}{A - B} \right|$$

$$I_3 = -2\pi \int_0^\mu \frac{dk}{k^2 + m^2_e} \left[ -2 + \frac{A}{B} \ln \left| \frac{A + B}{A - B} \right| - \frac{p^2}{2B} \ln \left| \frac{A + B}{A - B} \right| \right]$$

$$I_4 = -2\pi \int_0^\mu \frac{dk}{k^2 + m^2_e} \left[ \frac{2p^2 (A^2 + 2A)}{A^2 - B^2} - \frac{p^2}{2B} \ln \left| \frac{A + B}{A - B} \right| \right]$$

$$I_5 = -2\pi i \int_0^\mu \frac{dk}{k^2 + m^2_e} \left( \frac{\omega - \sqrt{k^2 + m^2_e}}{\sqrt{k^2 + m^2_e}} \right) \left[ \frac{A - 2p^2}{A^2 - B^2} - \frac{1}{2B} \ln \left| \frac{A + B}{A - B} \right| \right]$$

1Actually, as soon as $\mu$ is about $10T$, this approximation, which is just the characterization of the Fermi surface, is excellent, just as in condensed matter.
\[ I_6 = -2\pi \int_0^\mu \frac{dk^2}{\sqrt{k^2 + m_e^2}} \left( \frac{\omega - \sqrt{k^2 + m_e^2}^2}{A^2 - B^2} \right)^2 \]  

with the notation

\[ \begin{align*}
A &= p^2 - \omega^2 - m_e^2 + 2\omega \sqrt{k^2 + m_e^2} \\
B &= 2kp
\end{align*} \]  

\[ \text{(26)} \]

IV. DISPERSION RELATIONS

In a medium, the propagation of an electron is described by the effective Lagrangian term

\[ L_m = \overline{\Psi}_e(p) S_e^{-1} \Psi_e(p) = \overline{\Psi}_e(p) \left( i\gamma_\mu p_\mu + m_e - \Sigma \right) \Psi_e(p) \]  

Accordingly, the equation of motion for the electron field is

\[ S_e^{-1} \Psi_e(p) = 0 \]  

\[ \text{(28)} \]

The dispersion equation is thus

\[ |\gamma_\mu (\omega - p_\mu - ib\delta_{\mu 4})| = |\gamma_\mu (\omega - p_\mu + ib\delta_{\mu 4})| = 0 \]  

\[ \text{(29)} \]

By the analytic continuation \( ik_4 = -\omega \), we get the equation for the propagation of the electron in the medium as

\[ \sqrt{(1 - a)^2 p^2 - (m_e - ic)^2 - (1 - a) \omega + b} = 0 \]  

\[ \text{(30)} \]

In order to obtain analytic representations for the integrals \( I_i \), we neglect the electron mass in the square roots: this is justified because the integrands are monotonously increasing in \( k \) and the upper limit of integration (\( \simeq \mu \)) is large. At ultrahigh densities, the electron behaves like a massless fermionic quasiparticle. Let us illustrate the point by sketching the evaluation of \( I_1 \) \[ (21) \]:

\[ I_1 = -\pi \int_0^\mu \frac{dk}{p} \ln \left| \frac{p^2 - \omega^2 - m_e^2 + 2k(\omega + p)}{p^2 - \omega^2 - m_e^2 + 2k(\omega - p)} \right| \]  

\[ \text{(31)} \]

\[ \text{(32)} \]

Introduce the short-hand

\[ \Delta \equiv p^2 - \omega^2 - m_e^2 \]

whereby

\[ I_1 = -\pi \int_0^\mu \frac{dk}{p} \ln |\Delta + 2k(\omega + p)| - \ln |\Delta + 2k(\omega - p)| \]  

\[ \text{(33)} \]
Integrate by parts to obtain

\[
I_1 = -\frac{\pi}{p} \left[ \frac{\Delta}{2(\omega + p)} \ln|\Delta + 2\mu(\omega + p)| - \frac{\Delta}{2(\omega - p)} \ln|\Delta + 2\mu(\omega - p)| \right. \\
+ \mu \ln \left| \frac{\Delta + 2\mu(\omega + p)}{\Delta + 2\mu(\omega - p)} \right| + \frac{p\Delta}{\omega^2 - p^2} \ln|\Delta| \right]
\]

(34)

Now, take into account only the highest orders in \(\mu\) (that is, assume \(p\) small, \(p \ll \mu\)) to obtain finally

\[
I_1 \approx -\pi \mu \ln \frac{|\omega + p|}{|\omega - p|} + \cdots, \quad (35)
\]

Similarly, we find

\[
I_2 \approx \pi \frac{\mu^2}{p} \ln \frac{|\omega + p|}{|\omega - p|} + \cdots, \quad (36)
\]

\[
I_3 \approx \pi \mu \ln \frac{|\omega + p|}{|\omega - p|} + \cdots, \quad (37)
\]

\[
I_4 \approx \frac{\pi}{2} \frac{p^2}{\omega^2} \mu^2 + \cdots, \quad (38)
\]

\[
I_6 \approx -\frac{1}{4} \frac{\mu^2}{\omega^2} + \cdots. \quad (39)
\]

The analysis for \(I_5 \approx 0\) is simpler because for \(p \to 0\) we have from (31)

\[
I_5 \sim \left[ \frac{\Delta + 2\omega k - 2p^2}{(\Delta + 2\omega k)^2 - (2k^2)^2} - \frac{1}{4kp} \ln \left| \frac{\Delta + 2k(\omega + p)}{\Delta + 2k(\omega - p)} \right| \right]
\]

(40)

\[\to \left[ \frac{1}{\Delta + 2\omega k} - \frac{1}{4kp} \left( \frac{4kp}{\Delta + 2\omega k} \right) \right] = 0 \]

(41)

we obtain, for the ultrahigh density limit (\(\mu >> m_e\)), the following expresions for the functions \(a, \ b \ y \ c\):

\[
a = g^2 \pi \left[ (1 - \xi) \frac{\mu^2}{\omega^2} - (3 + 2\xi) \frac{\mu}{p} \ln \frac{|\omega + p|}{|\omega - p|} \right], \quad (42)
\]

\[
b = g^2 \pi \left[ (3 + \xi) \frac{\mu^2}{2p} \ln \frac{|\omega + p|}{|\omega - p|} - \frac{\mu^2}{2\omega} - \xi \frac{\mu \omega}{p} \ln \frac{|\omega + p|}{|\omega - p|} \right], \quad (43)
\]

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\[ c = -(3 + \xi) g^2 \pi i m_e \frac{\mu}{\rho} \ln \frac{\omega + p}{\omega - p}. \] (44)

Replacing these expressions for the integrals in the dispersion equation (31), is easy to obtain the dispersion relations, which constitute our main result:

\[ \omega_+ = \sqrt{p^2 + m_e^2 + g^2 \pi \frac{\mu^2}{2p} \left[ (1 - \xi) \frac{p}{\omega_+} + (3 + 2\xi) \ln \frac{\omega_+ + p}{\omega_+ - p} \right]} \] (45)

Here, \( \omega_+ \) is the normal dispersion relation and corresponds to an electron-like quasiparticle. There is also an abnormal dispersion branch (\( \omega_- < 0 \)), which have an essential collective character:

\[ \omega_- = -\sqrt{p^2 + m_e^2 + g^2 \pi \frac{\mu^2}{2p} \left[ (1 - \xi) \frac{p}{\omega_-} + (3 + 2\xi) \ln \frac{\omega_- + p}{\omega_- - p} \right]} \] (46)

The solution of (45) and (46) are showed in Fig.1 together with the light cone and the free dispersion law for two different values of density. As can be observed the behavior of the dispersion curves is analogous to the reported in [16], although in our case lower densities are employed, which allows us to have a more realistic description of the astrophysical and cosmological contexts. In Figs.2 and 3 are plotted the normal and abnormal dispersion curves for three values of the density. In the case of \( \omega_+ \) it is seen that its effective mass grows and its approach to the light cone is faster with increasing density. In the case of \( \omega_- \) its curve tends to approach the light cone faster than \( \omega_+ \), moreover it can be observed the occurrence of a minimum for a finite value of \( p \) which decreases with the density. The abnormal branch shows a negative effective mass near the \( p = 0 \) as the hole quasiparticles.

Note that we get two different dispersion relations for particles and holes, unlike other results at finite temperature, such as densities near the electron mass [17], high temperature but without chemical potential [10], and others [18]. By taking into account the electron mass, we correct the pathological behavior of the derivative of the dispersion curve at \( p = 0 \) obtained by several authors for massless fermions [11, 19, 12, 20].

**V. DISCUSSION**

We return to the problem of gauge invariance. As shown in Figs.4, 5, the spectrum is dependent on the gauge parameter, but for \( p/m_e > 1 \) and near the mass shell, the dispersion curves obtained, for some values of the gauge parameter, approach among themselves and to the mass shell and become gauge-independent. This in some way verifies the mass shell \( P^2 = m^2 \) gauge independence pointed out by several authors, but we observe it occurs for large momenta, i.e. in the light cone region. But for some other gauges the behavior is quite different. In (3) one can see that in the case of the gauge \( \xi = -3 \) the dispersion curve behaves in a drastic different way as the cases \( \xi = 1 \) and \( \xi = 0 \) in the same region of momenta. One must stress at this point that the gauge-dependent scalar \( b \) plays an essential role in producing a departure of the spectrum from the \( P^2 = m^2 \) mass shell, which manifests especially in the low momentum limit: \( \lim_{p \to 0} b = b(\omega, \mu, \xi) \neq 0 \). Thus, the mass shell gauge invariance of \( \Sigma \) does not lead to a gauge-independent spectrum. From the above results
we conclude that the information extracted from the curves in the region far from the mass shell, e.g. the effective mass, are not physical but a gauge artifact. However the results of section 2 indicate that the Landau gauge, in despite of introducing algebraic complications, is the appropriate.

The minimum for \( p_0 \neq 0 \) in the abnormal solution, suggests an analogy with a superfluid behavior, (or superconductivity, since we are considering charged fermionic particles), but this conclusion can’t obtained only of mass operator. To give a criterium of superconductivity is necessary to analyze the vertex part. The electrons at high density constitute a weakly interacting attractive Fermi system. Obviously, to reach the conclusion that this system exhibits superfluidity, it is better to shift the energy axis to the point \( m_e + \delta m(\mu) \), where \( \delta m(\mu) = \lim_{p \to 0} (\omega - m_e) \). There, the positron-like quasiparticle spectrum is tangent at \( p = 0 \) (as in the non-relativistic case [21]).

On the other hand, the tangent to the positron-like quasiparticle curve gives the velocity below which a superfluid effect can be expected.

This astonishing behavior of the dispersion curve for the positron-like quasiparticle, in a weakly coupled Fermi gas, was already observed in a very different context by Weldon in 1989 [19], who considered a quark-gluon plasma at very high temperature and without chemical potential. There, superfluidity is not intuitive at all. We think that the analogy with cold helium can be established with more sense in our case, of degenerate fermion gas at low temperature, since the phenomena of superconductivity and superfluidity disappear at high temperatures.

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VII. APPENDIX

To obtain the constants on which the mass operator depends, we proceed as in reference [4]. First, define the auxiliar functions

\[
A = \frac{1}{4} Tr \left[ \sum (p) \cdot \gamma^\mu p_\mu \right] \tag{47}
\]

\[
B = \frac{1}{4} Tr \left[ \sum (p) \cdot \gamma^\mu u_\mu \right] \tag{48}
\]

\[
C = \frac{1}{4} Tr \left[ \sum (p) \right] \tag{49}
\]

Using (17), we find

\[
a = A + i p^*_\mu B \tag{50}
\]
\[ b = \frac{1}{p^2} \left( i p^*_A - p'^2 B \right) \]  
(51)

\[ c = C \]  
(52)

The main integrals that appear in the calculation of these constants are

\[ I_1 = \sum_{k_4} \int \frac{1}{(k^2 + k_4^2)(k^2 + k_4^2 + 2k_4p_4 + m^2 + p^2 + p_4^2)} \, d^3k \]  
(53)

\[ I_2 = \sum_{k_4} \int \frac{k_4}{(k^2 + k_4^2)(k^2 + k_4^2 + 2k_4p_4 + m^2 + p^2 + p_4^2)} \, d^3k \]  
(54)

\[ I_3 = \sum_{k_4} \int \frac{k p}{(k^2 + k_4^2)(k^2 + k_4^2 + 2k_4p_4 + m^2 + p^2 + p_4^2)} \, d^3k \]  
(55)

\[ I_4 = \sum_{k_4} \int \frac{k p^2}{(k^2 + k_4^2)(k^2 + k_4^2 + 2k_4p_4 + m^2 + p^2 + p_4^2)} \, d^3k \]  
(56)

\[ I_5 = \sum_{k_4} \int \frac{k p k_4}{(k^2 + k_4^2)(k^2 + k_4^2 + 2k_4p_4 + m^2 + p^2 + p_4^2)} \, d^3k \]  
(57)

\[ I_6 = \sum_{k_4} \int \frac{k_4^2}{(k^2 + k_4^2)(k^2 + k_4^2 + 2k_4p_4 + m^2 + p^2 + p_4^2)} \, d^3k \]  
(58)

The integration is over \( k \)-three-space and the Matsubara sum runs over the discrete fourth component of momentum, which for bosons is taken as \( k_4 = 2n\pi\beta \) \( n \in \mathbb{Z} \).

Assuming \( \mu >> T \), only thermal electrons survive with a step-function distribution, whereas both positrons and photons get killed, and we find

\[ I_1 = \int \frac{\theta (\mu - \varepsilon_{k+p})}{2i\varepsilon_{k+p}[-p_4 - i(\varepsilon_k - \varepsilon_{k+p} + \mu)][-p_4 + i(\varepsilon_k + \varepsilon_{k+p} - \mu)]} \, d^3k \]  
(59)

\[ I_2 = \int \frac{[-p_4 + i(\varepsilon_{k+p} - \mu)]\theta (\mu - \varepsilon_{k+p})}{2i\varepsilon_{k+p}[-p_4 - i(\varepsilon_k - \varepsilon_{k+p} + \mu)][-p_4 + i(\varepsilon_k + \varepsilon_{k+p} - \mu)]} \, d^3k \]  
(60)

\[ I_3 = \int \frac{\vec{k} \cdot \vec{p} \theta (\mu - \varepsilon_{k+p})}{2i\varepsilon_{k+p}[-p_4 - i(\varepsilon_k - \varepsilon_{k+p} + \mu)][-p_4 + i(\varepsilon_k + \varepsilon_{k+p} - \mu)]} \, d^3k \]  
(61)

\[ I_4 = -\int \frac{(\vec{k} \cdot \vec{p})^2 \theta (\mu - \varepsilon_{k+p})}{2i\varepsilon_{k+p}[(p_4 + i\mu) + i(\varepsilon_k - \varepsilon_{k+p})]^2 [(p_4 + i\mu) - i(\varepsilon_k + \varepsilon_{k+p})]^2} \, d^3k \]  
(62)
\[ I_5 = -\int \frac{\vec{k} \cdot \vec{p}}{2i\varepsilon_{k+p} \left[ (p_4 + i \mu) + i (\varepsilon_k - \varepsilon_{k+p}) \right]^2 \left[ (p_4 + i \mu) - i (\varepsilon_k + \varepsilon_{k+p}) \right]^2} \langle 0 | T | \text{state} \rangle \left( \mu - \varepsilon_{k+p} \right) \theta \left( \frac{\mu - \varepsilon_{k+p}}{\varepsilon_{k+p}} \right) d^3k \] (63)

\[ I_6 = -\int \frac{\left[ - (p_4 + i \mu) + i \varepsilon_{k+p} \right]^2 \theta \left( \mu - \varepsilon_{k+p} \right)}{2i\varepsilon_{k+p} \left[ (p_4 + i \mu) + i (\varepsilon_k - \varepsilon_{k+p}) \right]^2 \left[ (p_4 + i \mu) - i (\varepsilon_k + \varepsilon_{k+p}) \right]^2} \langle 0 | T | \text{state} \rangle \left( \mu - \varepsilon_{k+p} \right) d^3k \] (64)

In the above, \( \varepsilon_{k+p} = \sqrt{(k+p)^2 + m^2} \), \( \varepsilon_k = |\vec{k}| \) and \( \theta (\lambda) \) is the Heaviside unitary step function characteristic of the low temperature limit.
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FIG. 1. The spectrum of fermionic excitations in an ultrarelativistic plasma for two values of density. In dots are showed the light cone and the free dispersion relation.
FIG. 2. Dependence of the normal branch with density. The dots lines correspond to the light cone and the free dispersion relation.

FIG. 3. Dependence of the abnormal branch with density. The dots lines correspond to the light cone and the free dispersion relation.
FIG. 4. Dependence of the normal branch with the gauge parameter. The dots and dashed lines correspond to the light cone and the free dispersion relation respectively.

FIG. 5. Dependence of the abnormal branch with the gauge parameter. The dots and dashed lines correspond to the light cone and the free dispersion relation respectively.