Algebraic A-hypergeometric functions

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December 5, 2008

Abstract

We formulate and prove a combinatorial criterion to decide if an A-hypergeometric system of differential equations has a full set of algebraic solutions or not. This criterion generalises the so-called interlacing criterion in the case of hypergeometric functions of one variable.

1 Introduction

The classically known hypergeometric functions of Euler-Gauss \((2F_1)\), its one-variable generalisations \(p+1F_p\) and the many variable generalisations, such as Appell’s functions, the Lauricella functions and Horn series are all examples of the so-called A-hypergeometric functions introduced by Gel’fand, Kapranov, Zelevinsky in [6, 7, 8]. We like to add that completely independently B.Dwork developed a theory of generalised hypergeometric functions in [4] which is in many aspects parallel to the theory of A-hypergeometric functions. The connection between the theories has been investigated in [1] and [5].

The definition of A-hypergeometric functions begins with a finite subset \(A \subset \mathbb{Z}^r\) (hence their name) consisting of \(N\) vectors \(a_1, \ldots, a_N\) such that

i) The \(\mathbb{Z}\)-span of \(a_1, \ldots, a_N\) equals \(\mathbb{Z}^r\).

ii) There exists a linear form \(h\) on \(\mathbb{R}^r\) such that \(h(a_i) = 1\) for all \(i\).

The second condition ensures that we shall be working in the case of so-called Fuchsian systems. In a number of papers, eg [1], this condition is dropped to include the case of so-called confluent hypergeometric equations.

We are also given a vector of parameters \(\alpha = (\alpha_1, \ldots, \alpha_r)\) which could be chosen in \(\mathbb{C}^r\), but we shall restrict to \(\alpha \in \mathbb{R}^r\). The lattice \(L \subset \mathbb{Z}^N\) of relations consists of all \((l_1, \ldots, l_N) \in \mathbb{Z}^N\) such that \(\sum_{i=1}^N l_i a_i = 0\).

The A-hypergeometric equations are a set of partial differential equations with independent variables \(v_1, \ldots, v_N\). This set consists of two groups. The first are the structure equations

\[ \Box_l \Phi := \prod_{l_i > 0} \partial_i^{l_i} \Phi - \prod_{l_i < 0} \partial_i^{l_i} \Phi = 0 \quad (A1) \]

for all \(l = (l_1, \ldots, l_N) \in L\).

The operators \(\Box_l\) are called the box-operators. The second group consists of the homogeneity or Euler equations.

\[ Z_i \Phi := (a_{1,i} v_1 \partial_1 + a_{2,i} v_2 \partial_2 + \cdots + a_{N,i} v_N \partial_N - \alpha_i) \Phi = 0, \quad i = 1, 2, \ldots, r \quad (A2) \]

where \(a_{k,i}\) denotes the \(i\)-th coordinate of \(a_k\).

In general the A-hypergeometric system is a holonomic system of dimension equal to the \(r-1\)-dimensional volume of the so-called A-polytope \(Q(A)\). This polytope is the convex hull of the endpoints of the \(a_i\). The volume-measure is normalised to 1 for a \(r-1\)-simplex of lattice-points in the plane \(h(x) = 1\) having no other lattice points in its interior. In the first days of the theory of A-hypergeometric systems there was some confusion as to what ‘general’ means, see [1]. To avoid these difficulties we make an additional assumption, which ensures that the dimension of the A-hypergeometric system indeed equals the volume of \(Q(A)\).
iii) The $\mathbb{R}_{>0}^v$-span of $A$ intersected with $\mathbb{Z}^r$ equals the $\mathbb{Z}_{>0}^v$-span of $A$.

Under this condition we have the following Theorem.

**Theorem 1.1 (GKZ, Adolphson)** Let notations be as above. If condition (iii) is satisfied then the system of $A$-hypergeometric differential equations is holonomic of rank equal to the volume of the convex hull $Q(A)$ of $A$.

For a complete story on the dimension of the solution space we refer to [15]. In the present paper we shall use condition (iii) in the proof of the important Proposition 4.1.

To describe the standard hypergeometric solution of the $A$-hypergeometric system we define the projection map $\psi_L : \mathbb{R}^N \to \mathbb{R}^r$ given by $\psi_L(e_i) = a_i$ for $i = 1, \ldots, N$. Here $e_i$ denotes the $i$-th vector in the standard basis of $\mathbb{R}^N$. Clearly the kernel of $\psi_L$ is the space $L \otimes \mathbb{R}$. Choose a point $\gamma = (\gamma_1, \ldots, \gamma_N)$ in $\psi_L^{-1}(\alpha)$, in other words choose $\gamma_1, \ldots, \gamma_N$ such that $\alpha = \gamma_1a_1 + \cdots + \gamma_Na_N$. Then a formal solution of the $A$-hypergeometric system can be given by

$$\Phi_{L, \gamma}(v_1, \ldots, v_N) = \sum_{\nu \in L} \frac{v^\nu}{\Gamma(1 + \gamma + 1)}$$

where we use the short-hand notation

$$\frac{v^{\nu + \gamma}}{\Gamma(1 + \gamma + 1)} = \frac{v_1^{\nu_1 + \gamma_1} \cdots v_N^{\nu_N + \gamma_N}}{\Gamma(\nu_1 + \gamma_1 + 1) \cdots \Gamma(\nu_N + \gamma_N + 1)}.$$

By a proper choice of $\gamma \in \psi_L^{-1}(\alpha)$ this formal solution gives rise to actual power series solutions with a non-trivial region of convergence.

The real positive cone generated by the vectors $a_i$ is denoted by $C(A)$. This is a polyhedral cone with a finite number of faces. We recall the following Theorem.

**Theorem 1.2** The $A$-hypergeometric system is irreducible if and only if $\alpha + \mathbb{Z}^r$ contains no points in any face of $C(A)$.

This Theorem is proved in [3], Theorem 2.11 using perverse sheaves. It would be nice to have a more elementary proof however.

Let us now assume that $\alpha \in \mathbb{Q}^r$. We shall be interested in those irreducible $A$-hypergeometric system that have a complete set of solutions algebraic over $\mathbb{C}(v_1, \ldots, v_N)$. This question was first raised in the case of Euler-Gauss hypergeometric functions and the answer is provided by the famous list of H.A.Schwarz, see [17]. In 1989 this list was extended to general one-variable hypergeometric functions by Beukers and Heckman, see [3]. For the several variable cases, a characterization for Appell-Lauricella $F_D$ was provided by Sasaki [16] in 1976 and Wolfart, Cohen [2] in 1992. The Appell systems $F_4$ and $F_2$ were classified by M.Kato in [10] (1997) and [11] (2000).

In the case of one-variable hypergeometric functions there is a simple combinatorial criterion to decide if they are algebraic or not. Consider the hypergeometric function

$$pF_{p-1}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_{p-1}|z).$$

Define $\beta_p = 1$. We assume $\alpha_i - \beta_j / \notin \mathbb{Z}$ for all $i, j = 1, \ldots, p$, which ensures that the corresponding hypergeometric differential equation is irreducible. We shall say that the sets $\alpha_i$ and $\beta_j$ interlace modulo 1 if the points of the sets $e^{2\pi i \alpha_i}$ and $e^{2\pi i \beta_j}$ occur alternatingly when running along the unit circle. The following Theorem is proved in [3].

**Theorem 1.3 (Beukers, Heckman)** Suppose the one-variable hypergeometric equation with parameters $\alpha_i, \beta_i \in \mathbb{Q}$ ($i, j = 1, 2, \ldots, p$) with $\beta_p = 1$ is irreducible. Let $D$ be the common denominator of the parameters. Then the solution set of the hypergeometric equation consists of algebraic functions (over $\mathbb{C}(z)$) if and only if the sets $k\alpha_i$ and $k\beta_j$ interlace modulo 1 for every integer $k$ with $1 \leq k < D$ and gcd$(k, D) = 1$.

It is the purpose of this paper to generalize the interlacing condition to a similar condition for $A$-hypergeometric systems. We assume that $\alpha \in \mathbb{R}$ and define $K_\alpha = (\alpha + \mathbb{Z}^r) \cap C(A)$. A point $p \in K_\alpha$ is called an apex point if $p \notin q + C(A)$ for every $q \in K_\alpha$ with $q \notin p$. We call the number of apex points the signature of the polytope $A$ and parameters $\alpha$. Notation: $\sigma(A, \alpha)$. 


Proposition 1.4 Let $\alpha \in \mathbb{R}^r$. Then $\sigma(A, \alpha)$ is less than or equal to the volume of the A-polytope $Q(A)$.

We say that the signature is maximal if it equals the volume of $Q(A)$.

Theorem 1.5 Let $\alpha \in \mathbb{Q}^r$ and suppose the A-hypergeometric system is irreducible. Let $D$ be the common denominator of the coordinates of $\alpha$. Then the solution set of the A-hypergeometric system consists of algebraic solutions (over $\mathbb{C}(v_1, \ldots, v_N)$) if and only if $\sigma(A, k\alpha)$ is maximal for all integers $k$ with $1 \leq k < D$ and $\gcd(k, D) = 1$.

To compare this result with the one-variable interlacing condition for $2F_1$ we illustrate a connection. In the case of Euler-Gauss hypergeometric function we have $r = 3, N = 4$ and

$$a_1 = (1, 0, 0), \quad a_2 = (0, 1, 0), \quad a_3 = (0, 0, 1), \quad a_4 = (1, 1, -1).$$

The faces of the cone generated by $a_i$ ($i = 1, \ldots, 4$) are given by $x = 0, y = 0, x + z = 0, y + z = 0$ (we use the coordinates $x, y, z$ in $\mathbb{R}^3$). We define $\alpha = (-a, -b, c - 1)$. Theorem 1.5 implies that irreducibility comes down to the inequalities $-a, -b, -a + c, -b + c \notin \mathbb{Z}$. These are the familiar irreducibility conditions for the Euler-Gauss hypergeometric functions.

The lattice of relations has rank one and is generated by $(-1, -1, 1, 1)$. We choose $\gamma = (-a, -b, c - 1, 0)$. Then the formal solution $\Phi_{L, \gamma}$ reads

$$v_1^{-a}v_2^{-b}v_3^{-c-1}\sum_{k \in \mathbb{Z}} \frac{v_1^{-k}v_2^{-b}v_3^{-c}}{\Gamma(-k-a+1)\Gamma(-k-b+1)\Gamma(c+k)\Gamma(k+1)}.\Gamma(c+k)\Gamma(k+1).$$

Clearly $1/\Gamma(k+1)$ vanishes for $k \in \mathbb{Z}_{<0}$, so our summation actually runs over $k \in \mathbb{Z}_{\geq 0}$. Apply the identity $1/\Gamma(1-z) = \sin(\pi z)/\pi$ to $z = k + a$ and $z = k + b$ to obtain

$$\Phi_{L, \gamma} = v_1^{-a}v_2^{-b}v_3^{-c-1}\frac{\sin(\pi a)\sin(\pi b)}{\pi^2} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{(c+k)k!} \left(\frac{v_3v_4}{v_1v_2}\right)^k.$$

Setting $v_1 = v_2 = v_3 = 1$ and $v_4 = z$ we recognize the Euler-Gaussian hypergeometric series $2F_1(a, b, c|z)$. By shifting over $\mathbb{Z}$ if necessary we can see to it that $a, b, c$ are in the interval $(-1, 0)$. Suppose that the sets $\{a, b\}$ and $\{0, c\}$ interlace modulo 1. By interchange of $a, b$ if necessary we can restrict ourselves to the case $-1 < a < c < b < 0$. It is straightforward to verify that $(-a,-1-b,c)$ and $(-a,-b,1+c)$ are apexpoints of $K_\alpha$. If the sets do not interlace then one checks that $(-a,-b,c)$ is the unique apexpoint if $a < b < c$ and $(-a,-b,1+c)$ is the unique apexpoint if $c < a < b$.

A second example is Appell’s hypergeometric equation $F_2$. The Appell $F_2$ hypergeometric function reads

$$F_2(a, b, b', c, c'|x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m}(b')_{n}}{(c)_{m}(c')_{n}m!n!} x^m y^n,$$

where $(x)_n$ denotes the Pochhammer symbol defined by $\Gamma(x+n)/\Gamma(x) = x(x+1)\cdots(x+n-1)$. The function $F_2$ satisfies a system of partial differential equations of rank 4. Algebraicity of these functions is completely described in [11].

The A-parameters are as follows. We have $r = 5$ and $N = 7$. The set $A$ consists of the standard basis vectors $a_1, \ldots, a_5$ in $\mathbb{R}^5$ and $a_6 = (1, 0, 0, -1, 0), \quad a_7 = (1, 0, 1, 0, -1)$. We take $a = (-a, -b, -b', c, c')$. The lattice $L$ of relations is generated by $(-1, -1, 1, 0, 1, 0, 0)$ and $(-1, 0, -1, 1, 0, 1, 0)$. Take $\gamma = (-a, -b, -b', c, c', 0, 0)$. In a similar way as we did for the Euler-Gauss functions we can now go from the formal expansion $\Phi_{L, \gamma}$ to the explicit Appell function $F_2$.

One can compute that the cone $C(A)$ has 8 faces and they are given by $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_5 = 0, x_5 = 0$ and $x_5 = 0$ and $x_5 = 0$. Using Theorem 1.2 it follows that the A-hypergeometric system is irreducible if and only if none of the following numbers is an integer,

$$a, b, b', -b + c, -b' + c', -a + c, -a + c', -a + c + c'.$$
These are precisely the irreducibility conditions for the $F_2$-system given in [11]. In that paper it is shown for example that with the choice
\[ \alpha = (-a, -b, -b', c, c') = (1/10, 7/10, 9/10, 3/5, 1/5) \]
the solutions of the rank 4 Appell system are all algebraic. A small computer calculation shows that the apex points of $K_\alpha$ are given by
\[ (21/10, 7/10, 9/10, -2/5, -4/5), \quad (11/10, 7/10, 9/10, 3/5, -4/5) \]
\[ 11/10, 7/10, 9/10, -2/5, 1/5), \quad (1/10, 7/10, 9/10, 3/5, 1/5). \]
Similarly there are four apexpoints for the conjugate parameter 5-tuples $3\alpha, 7\alpha, 9\alpha$.

2 A simple example

In this section we show a more elaborate example of algebraic hypergeometric functions of Horn-type $G_3$ which, to our knowledge, has not been dealt with before. The corresponding series with parameters $a, b$ is given by
\[ G_3(a, b, x, y) = \sum_{m \geq 0, n \geq 0} \frac{(a)_{2m-n}(b)_{2n-m}}{m!n!} x^m y^n. \]
Here again, $(x)_n$ denotes the Pochhammer symbol defined by $\Gamma(x + n)/\Gamma(x)$. However, now the index $n$ may be negative, in which case the definition explicitly reads $\Gamma(x + n)/\Gamma(x) = 1/(x - 1) \cdots (x - |n|)$. The system of differential equations is a rank 3 system. The set $A$ can be chosen in $\mathbb{Z}^2$ for example as
\[ a_1 = (-1, 2), \quad a_2 = (0, 1), \quad a_3 = (1, 0), \quad a_4 = (2, -1). \]
Below we show a picture of the cone $C(A)$ spanned by the elements of $A$, together with the points from $A$. In addition, the dark grey area indicates the set of apexpoints with respect to $A$. The parameter vector of the corresponding $A$-hypergeometric system is given by $(-a, -b)$.

**Theorem 2.1** Consider the $A$-hypergeometric system corresponding to the Horn $G_3$ equations. In the following cases the system is irreducible and has only algebraic solutions.

1. $a + b \in \mathbb{Z}$ and $a, b \not\in \mathbb{Z}$.
2. $a \equiv 1/2 (\text{mod } \mathbb{Z}), b \equiv 1/3, 2/3 (\text{mod } \mathbb{Z})$
3. $a \equiv 1/3, 2/3 (\text{mod } \mathbb{Z}), b \equiv 1/2 (\text{mod } \mathbb{Z})$

**Proof** This is an application of Theorem 1.5. In all cases we need only be interested in $a, b (\text{mod } \mathbb{Z})$. In the following picture the light grey area is the cone $C(A)$, the dark grey area indicates the location of the apexpoints. If $a + b \in \mathbb{Z}$ then we note that there are precisely 3 points of $(-a, -b) + \mathbb{Z}^2$ in the dark grey area, all lying on the line $x + y = 1$. Hence three apexpoints.
If \( a + b \in \mathbb{Z} \) then also \( ka + kb \in \mathbb{Z} \) for any integer. Irreducibility of the systems is ensured by Theorem 1.2 and the fact that \( a, b \notin \mathbb{Z} \). Therefore, in the first case all conditions of Theorem 1.5 are fulfilled.

In the following picture we have drawn the sets \((1/2, 1/3) + \mathbb{Z}^2\) and \((1/2, 2/3) + \mathbb{Z}^2\) intersected with \( C(A) \).

Clearly each set has three apexpoints and Theorem 1.5 can be applied to prove the second case. The third case runs similarly.

It has been verified by J. Schipper, an Utrecht graduate student, that Theorem 2.1 gives the characterisation of all irreducible Horn \( G_3 \)-systems with algebraic solutions.

A fairly involved calculation reveals that a formula for \( G_3(a, 1 - a, x, y) \) can be given as follows

\[
G_3(a, 1 - a, x, y) = f(x, y)^a \sqrt{\frac{g(x, y)}{\Delta}}
\]

where

\[
\Delta = 1 + 4x + 4y + 18xy - 27x^2y^2
\]

and

\[
x f^3 - y = f - f^2, \quad g(g - 1 - 3x)^2 = x^2 \Delta.
\]

For reference we display the series expansions of \( f \) and \( g \).

\[
f = 1 + (y - x) + (2x^2 - yx - y^2) + (-5x^3 + 3yx^2 + 2y^3) + (14x^4 - 10yx^3 + y^4x - 5y^4) + O(x, y)^5
\]

\[
g = 1 + 2x - x(x + 2y) + 2x(x - y)^2 - x(x - y)^2(5x + 4y) + O(x, y)^5
\]

Moreover, \( g = 1 + 4x - 2xf - 3x^2f^2 \). In particular, \( f \) and \( g \) generate the same cubic extension of \( \mathbb{Q}(x, y) \).

3 The signature

Proof of Proposition 1.4 We use the following property. Let \( b_1, \ldots, b_r \in \mathbb{Z}^r \) be independent vectors. Let \( \beta \in \mathbb{R}^r \). Then the number of points of \( \beta + \mathbb{Z}^r \) inside the fundamental block \( \{ \sum_{i=1}^r \lambda_i b_i | 0 \leq \lambda_i < 1 \} \) is equal to \( |\det(b_1, \ldots, b_r)| \).

Write \( Q(A) \) as a union of \( r - 1 \)-simplices \( \cup_{i=1}^m \sigma_i \) (a so-called triangulation of \( Q(A) \)). Every simplex \( \sigma_i \) is spanned by \( r \) independent vectors \( a_{i,j} \) \((j = 1, \ldots, r)\). Let \( B_i \) be the fundamental block spanned by these vectors.

Let \( a \) be an apexpoint of \( K_\alpha \). Then \( a \) is contained in a positive cone spanned by one of the simplices \( \sigma_i \). For every choice of \( a_{i,j} \) \((j = 1, \ldots, r)\) the point \( a - a_{i,j} \) falls outside this cone. If not, then \( a \) would be contained in \( a_{i,j} + C(A) \). Hence \( a \notin B_i \). Since \( B_i \) contains at most \( |\det(B_i)| \) point from \( \alpha + \mathbb{Z}^r \), we see that the number of apexpoints is bounded above by \( \sum_{i=1}^m |\det(B_i)| \).

This equals precisely the \( r - 1 \)-dimensional volume of the \( A \)-polytope \( Q(A) \).
For any \( k = (k_1, \ldots, k_N) \in \mathbb{Z}^N \) we define the hypercube
\[
F(k) = \{(x_1, \ldots, x_N) | k_i \leq x_i < k_i + 1, \ (i = 1, 2, \ldots, N)\}
\]
The space \( \mathbb{R}^N \) can be seen as the disjoint union of cells \( F(k) \) with \( k \) running over \( \mathbb{Z}^N \). Let us denote \( L(\mathbb{R}) = L \otimes \mathbb{R} \). We intersect the union \( \bigcup_k F(k) \) with the translated space \( \gamma + L(\mathbb{R}) \). Each hypercube intersects \( \gamma + L(\mathbb{R}) \) in a cell which may, or may not be closed in \( \gamma + L(\mathbb{R}) \). Let us denote \( V(k) = F(k) \cap (\gamma + L) \). We call \( V(k) \) a compact cell if it is closed and non-empty. Of course, if we shift a compact cell over a point of \( L \), we get another compact cell. In the following Proposition we denote the shifted hyperquadrant \( x + \mathbb{R}^N_{\geq 0} \) by \( P(x) \).

**Proposition 3.1** With the notations as above, \( V(k) \) is a compact cell if and only if \( P(k) \) has non-trivial intersection with \( \gamma + L \) and \( P(k + e_i) \) has empty intersection with \( \gamma + L \) for \( i = 1, 2, \ldots, N \). In particular, \( V(k) = P(k) \cap (\gamma + L(\mathbb{R})). \)

**Proof.** Let us denote the intersection of \( \bigcup_{i=1}^N P(k + e_i) \) with \( \gamma + L \) by \( W \). Notice that \( W \) is the set-theoretic difference between \( P(k) \cap (\gamma + L) \) and \( V(k) \). In particular \( W \) is a closed set. Suppose that \( V(k) \) is a compact cell. The only way that the difference \( W \) of the two non-empty closed convex sets \( P(k) \cap (\gamma + L) \) and \( V(k) \) can be closed is when \( W \) is empty. Hence \( P(k + e_i) \cap (\gamma + L) \) is empty for all \( i \).

Suppose conversely that \( W \) is empty. Then \( V(k) = P(k) \). Since \( P(k) \) is closed, the same should hold for \( V(k) \). Since \( V(k) \) is also bounded, we conclude that \( V(k) \) is compact.

\[ \square \]

In the following recall the map \( \psi : \mathbb{R}^N \rightarrow \mathbb{R}^r \) given by \( \psi : e_i \mapsto a_i \), for \( i = 1, \ldots, N \).

**Proposition 3.2** The compact cells in \( \gamma + L(\mathbb{R}) \), modulo \( L \), are in 1-1 correspondence with the apex-points of \( K_\alpha \). The correspondence is given by \( V(k) \mapsto \alpha - \psi(k) \).

Let \( a, a' \) be two different apexpoints. Then \( \psi^{-1}(a) \cap P(\emptyset) \) and \( \psi^{-1}(a') \cap P(\emptyset) \) are disjoint and contained in the unit cube \( 0 \leq x_i < 1 \) for \( i = 1, 2, \ldots, N \).

**Proof.** An apexpoint \( a \in \alpha + \mathbb{Z}^r \) is characterized by the fact that \( a = c(A) \) and \( a - a_i \notin C(A) \) for \( i = 1, \ldots, N \). Notice that \( \psi : \mathbb{R}^N \rightarrow \mathbb{R}^r \) is actually the quotient map \( \mathbb{R}^N \rightarrow \mathbb{R}^N / L(\mathbb{R}) \). Let \( k \in \mathbb{Z}^N \). First recall that \( \psi(\gamma) = \alpha \) and observe that \( \psi(P(k)) = \psi(k) + C(A) \). As a result we see that \( P(k) \cap (\gamma + L(\mathbb{R})) \) is non-empty if and only if \( \alpha - \psi(k) \notin C(A) \). Let us assume that \( V(k) \) is compact and apply Proposition 3.1. Then \( \alpha - \psi(k) \in C(A) \) and \( \alpha - \psi(k) - a_i \notin C(A) \) for \( i = 1, \ldots, N \). Hence \( \alpha - \psi(k) \) is a compact cell.

Conversely, when \( a \) is an apexpoint, find \( k \in \mathbb{Z}^N \) such that \( a = \alpha - \psi(k) \). Then we find that \( (\gamma + L(\mathbb{R})) \cap P(k) \) is non-empty and the sets \( \psi^{-1}(a) \cap P(k + e_i) \) for \( i = 1, 2, \ldots, N \) are empty. Therefore, by application of Proposition 3.1, \( V(k) \) is a compact cell.

Let \( a \) be an apexpoint and choose \( k \in \mathbb{Z}^N \) such that \( \alpha - \psi(k) = a \). Then \( \psi^{-1}(a) \cap P(k) \) is a compact cell. Consequently, after shifting over \( k \), the set \( \psi^{-1}(a) \cap P(\emptyset) \) is a compact cell in \( \gamma - k + L(\mathbb{R}) \). Hence \( \psi^{-1}(a) \cap P(\emptyset) \) is contained in the unit cube in \( \mathbb{Z}^N \). Two sets \( \psi^{-1}(a) \) and \( \psi^{-1}(a') \) are obviously distinct whenever \( a \) and \( a' \) are distinct.

\[ \square \]

## 4 Mod \( p \) solutions

Let us assume that \( \alpha \in \mathbb{Z}^r \) and let \( p \) be a prime. We describe the polynomial solutions in \( \mathbb{F}_p[v_1, \ldots, v_N] \) of the (\( A \)-hypergeometric system with parameters \( A \) and \( \alpha \)) considered modulo \( p \). Let \( \beta_i/p \ (i = 1, \ldots, \sigma) \) be the set of apexpoints of \( (\alpha/p + \mathbb{Z}^r) \cap C(A) \). To any apexpoint we associate the set of lattice points
\[
\Gamma_i = \psi^{-1}(\beta_i) \cap \mathbb{Z}^N_{\geq 0}
\]
where \( \psi : \mathbb{R}^N \rightarrow \mathbb{R}^N / L(\mathbb{R}) \) is defined as in the previous section. For any \( i = 1, 2, \ldots, \sigma \) we define
\[
\Psi_i := \sum_{l \in \Gamma_i} v^l / (l(l + 1)).
\]
This is a polynomial solution to the A-hypergeometric system with parameters $A, \beta_i$. Since $\beta_i/p$ is an apexpoint of $(\alpha/p + Z^k) \cap C(A)$, the preimage $\psi^{-1}(\beta_i/p) \cap P(0)$ is contained in the unit cube in $\mathbb{R}^N$ according to Proposition 3.2. Hence the points of $\Gamma_i = \psi^{-1}(\beta_i) \cap \mathbb{Z}^*_0$ are contained in the cube $0 \leq x_i < p$ for $i = 1, \ldots, N$. In particular none of the positive coordinates of any $l \in \Gamma_i$ is divisible by $p$, hence $\Gamma(l + 1) \neq 0(\text{mod } p)$ for all $l \in \Gamma_i$. This means that $\psi_i$ can be reduced modulo $p$. Furthermore, since $\alpha \equiv \beta_i(\text{mod } p)$, the polynomial $\Psi_i$ is a polynomial solution modulo $p$ to the A-hypergeometric system with parameters $A, \alpha$.

Since each of the sets $\Gamma_i$ is contained in the cube $0 \leq x_i < p$ any two shifts $\Gamma_i + pk_i$ and $\Gamma_j + pk_j$ for different $i, j$ and $k_i, k_j \in Z^N$ are disjoint. In particular the polynomials $\Psi_i$ are independent over $F_p[v_1^p, \ldots, v_N^p]$.

**Proposition 4.1** Every mod $p$ polynomial solution of the A-hypergeometric system with parameters $A, \alpha \in Z^*$ is an $F_p[v_1^p, \ldots, v_N^p]$-linear combination of the polynomials $\Psi_i$.

Let $P = \sum_m p_m v^m$ be a polynomial solution of the A-hypergeometric system with parameters $A, \alpha$. For formal reasons we extend the summation over all of $\mathbb{Z}^N$ but it should be understood that the set of multi-indices $m$ with $p_m \neq 0(\text{mod } p)$ is finite and contained in $\mathbb{Z}^N_0$.

Let us substitute this in the system (A1). Any $l \in L$ can be decomposed as $l = \sum_i \lambda_i 1_i$ where $1_i, \lambda_i \in \mathbb{Z}^N_0$ and we assume that they have disjoint support. Denoting $[m]_{\Gamma} = \prod_i^{N} m_i(m_i - 1) \cdots (m_i - r_i + 1)$ for any $r \in \mathbb{Z}^N_0$ we obtain

$$0 \equiv \sum_m \left( [m]_{\lambda} P_{\lambda} v^{m_{\lambda}} - [m]_{\lambda_0} P_{\lambda} v^{m_{\lambda_0}} \right) \pmod{p}$$

$$\equiv v^{-\lambda_0} \sum_m \left( [m]_{\lambda} P_{\lambda} - [m - \lambda]_{\lambda_0} P_{\lambda} \right) v^m \pmod{p}$$

Hence

$$[m]_{\lambda} P_{\lambda} - [m - \lambda]_{\lambda_0} P_{\lambda_0} \equiv 0 \pmod{p} \quad (1)$$

for every $m \in \mathbb{Z}^N$ and every $l \in L$. Substitution in (A2) gives

$$\sum_m (-a_1 + a_1 m_1 + \cdots + a_{N} m_N) p_m v^m \equiv 0 \pmod{p}$$

for $i = 1, \ldots, r$.

Hence $-a_1 + a_1 m_1 + \cdots + a_{N} m_N \equiv 0 \pmod{p}$ for every $m$ with $p_m \neq 0 \pmod{p}$. Note that the system (A2) gives no extra relations between different $p_m$. They only require that $\psi(m) \equiv \alpha(\text{mod } p)$.

The system (A1) relates only those coefficients $p_m$ for which the multi-indices $m$ differ by an element of $L$. Hence we can split $P$ as a sum of terms of the form $P_{\beta} = \sum_{m: m_{\beta} = \beta} p_m v^m$ with $\beta \in Z^*$ and each summand $P_{\beta}$ satisfies modulo $p$ the A-hypergeometric system with parameters $A, \alpha$. The equations (A2) applied to $P_{\beta}$ tell us that $\beta \equiv \alpha(\text{mod } p)$.

We define a partial ordering on $\mathbb{R}^N$. We say that $y \geq x$ if all components of $y$ are larger or equal to the corresponding component of $x$. In particular, when $y \geq x$ and $y \neq x$ we write $y > x$.

When $m = (m_1, \ldots, m_N) \in \mathbb{Z}^N$ we denote by $[m/p]$ the vector $(\lfloor m_1/p \rfloor, \ldots, \lfloor m_N/p \rfloor)$. Consider the recursion (1). We claim that $[m]_{\lambda_+} \equiv 0 \pmod{p}$ if and only if $[m/p] - [(m - 1)/p]$ has at least one positive component. This can be seen through the following sequence of equivalences,

$$[m]_{\lambda_+} \equiv 0 \pmod{p} \iff \exists i: m_i(m_i - 1) \cdots (m_i - l_i + 1) \equiv 0 \pmod{p}$$

$$\iff \exists i, \lambda: 0 \leq \lambda < l_i, m_i - \lambda \equiv 0 \pmod{p}$$

$$\iff \exists i, \lambda: 0 \leq \lambda < l_i, (m_i - \lambda)/p \in \mathbb{Z}$$

$$\iff \exists i: [m/p] - [(m - 1)/p] > 0$$

Similarly we see that $[m - 1]_{\lambda_-} \equiv 0 \pmod{p}$ if and only if $[m/p] - [(m - 1)/p]$ has at least one negative component.

In terms of our partial ordering this implies that $[m]_{\lambda_+} \equiv 0 \pmod{p}$ and $[m - 1]_{\lambda_-} \equiv 0 \pmod{p}$ if and only if neither $[m/p] \geq [(m - 1)/p]$ nor $[m/p] \leq [(m - 1)/p]$, i.e. $[m/p]$ and $[(m - 1)/p]$
are unrelated. Write $\mathbf{m}' = \mathbf{m} - 1$, then $p_{\mathbf{m}}$ and $p_{\mathbf{m}'}$ are related through $[1]$ if and only if $\psi(\mathbf{m}) = \psi(\mathbf{m}')$ and $[\mathbf{m}/p]$ and $[\mathbf{m}'/p]$ are related.

Now suppose that $p_{\mathbf{m}} \not\equiv 0(\mod p)$. We assert that for any $\lambda \in L(R)$ the inequality $[\mathbf{m}/p] \leq [(\mathbf{m} - \lambda)/p]$ implies equality. First we deal with the case when $\lambda = 1 \in L$. Suppose $[\mathbf{m}/p] < [(\mathbf{m} - 1)/p]$. Then $[\mathbf{m}]_{\alpha} \not\equiv 0(\mod p)$ and $[\mathbf{m} - 1]_{\alpha} \equiv 0(\mod p)$. This gives a contradiction with relation (1). Hence $[\mathbf{m}/p] \leq [(\mathbf{m} - 1)/p]$ implies equality.

Now, in general, suppose that there exists $\lambda \in L(R)$ such that $[\mathbf{m}/p] < [(\mathbf{m} - \lambda)/p]$. The vector $\mathbf{m} - \lambda - p[(\mathbf{m} - \lambda)/p]$ has non-negative coefficients. Hence its image under $\psi$ is contained in the cone $C(A)$. Moreover, since $\psi(\lambda) = 0$, the image has integer coordinates. Choose a vector $\mathbf{k} \in \mathbb{Z}_{\geq 0}^N$ such that $\psi(\mathbf{k}) = \psi(\mathbf{m} - p[(\mathbf{m} - \lambda)/p])$. Notice that this is only possible because of Assumption iii) which we made in the introduction. Hence there exists $\mathbf{l} \in L$ such that $\mathbf{k} = \mathbf{m} - 1 - p[(\mathbf{m} - \lambda)/p]$. In particular, $[(\mathbf{m} - 1)/p] \geq [(\mathbf{m} - \lambda)/p]$. Since, by assumption, the latter vector is strictly larger than $[\mathbf{m}/p]$ we again get a contradiction. Hence we conclude that

$$[(\mathbf{m} - \lambda)/p] \geq [\mathbf{m}/p] \Rightarrow [(\mathbf{m} - \lambda)/p] = [\mathbf{m}/p].$$

(2)

Another way of phrasing property (2) is to say that $\mathbf{m}/p$ is contained in a compact cell of the affine space $\mathbf{m}/p + L(\mathbb{R})$. To see this consider the cell $V([\mathbf{m}/p])$. Of course it contains $\mathbf{m}/p$. Let now $(\mathbf{m} - \lambda)/p$ be any other point in $P((\mathbf{m}/p)) \cap \mathbf{m}/p + L(\mathbb{R})$. Then $[(\mathbf{m} - \lambda)/p] \geq [\mathbf{m}/p]$ and we have seen that this implies equality. Therefore $(\mathbf{m} - \lambda)/p$ is contained in $V([\mathbf{m}/p])$. Hence the latter cell is compact by Proposition 3.3.

Let $\beta$ be as in the polynomial $P_3$ above and $\gamma \in \mathbb{R}^N$ such that $\psi(\gamma) = \beta$. Let $\mathbf{m}, \mathbf{m}' \in \mathbb{Z}_{>0}^N$ be such that $\psi(\mathbf{m}) = \psi(\mathbf{m}') = \beta$ and such that $\mathbf{m}/p, \mathbf{m}'/p$ are in a compact cell of $\gamma + L(\mathbb{R})$. Let $\mathbf{l} = \mathbf{m} - \mathbf{m}'$. If $\mathbf{m}/p, \mathbf{m}'/p$ belong to different compact cells we have neither $[\mathbf{m}/p] \leq [\mathbf{m}'/p]$ nor $[\mathbf{m}/p] \geq [\mathbf{m}'/p]$. Hence $p_{\mathbf{m}}$ and $p_{\mathbf{m}'}$ are unrelated by relation (1).

As a consequence of this all, the polynomial $P_3$ splits as a sum of terms of the form $\sum_{\mathbf{n}/p} = k p_{\mathbf{n}} \mathbf{v}^{\mathbf{m}}$ and each such sum is a solution of $(A1)$ and $(A2)$. The latter summation can be rewritten as $\psi(p^k \sum_{\mathbf{n}/p} = k p_{\mathbf{n}} \mathbf{v}^{\mathbf{m} - pk}$.

The multi-indices $\mathbf{m}$ in $\sum_{\mathbf{n}/p} = k p_{\mathbf{n}} \mathbf{v}^{\mathbf{m} - pk}$ should in addition satisfy $\psi(\mathbf{m}) = \beta$. Replace $\mathbf{m}$ by $\mathbf{n} + pk$ and we obtain the solution

$$\sum_{\mathbf{n}/p} = 0 b_{\mathbf{n}} \mathbf{v}^{\mathbf{n}},$$

(3)

where we put $b_{\mathbf{n}} = p_{\mathbf{n} + pk}$. We now know that all multi-indices $\mathbf{n}$ are contained in the cube $0 \leq x_i < p$ for $i = 1, \ldots, N$. Furthermore, in the recursion relation

$$[\mathbf{n}]_{\alpha}, b_{\mathbf{n}} - [\mathbf{n} - 1]_{\alpha}, b_{\mathbf{n} - 1} \equiv 0(\mod p)$$

both coefficients are non-zero whenever $\mathbf{n} \geq 0$ and $\mathbf{n} - 1 \geq 0$. Hence the space of solutions of the form (3) has dimension at most one. On the other hand we do have such a solution, namely $\Psi_{i}$ where $i$ is chosen such that the apexpoint $\beta_i$ is equal to the apexpoint $\beta - \psi(\mathbf{k})$.

We now consider polynomial mod p solutions for A-hypergeometric systems with parameters $\alpha \in \mathbb{Q}^\ast$.

**Proposition 4.2** Let $\alpha \in \mathbb{Q}^\ast$ and let $D$ be the common denominator of the coordinates of $\alpha$. Let $p$ be a prime not dividing $D$. Let $\rho \equiv -p^{-1}(\mod D)$ if $D > 1$ and $\rho \equiv 1$ if $D = 1$. Let $s$ be the signature of $A$ and $\rho_{\alpha}$. Suppose that the A-hypergeometric system we consider is irreducible. Then, when $p$ is sufficiently large, the polynomial mod p solutions of the A-hypergeometric system with parameters $A, \alpha$ is a free $\mathbb{F}_p[\mathbf{v}]$-module of rank $s$.

**Proof.** Let $\mathbf{k} = (1 + \rho p)\alpha$. Notice that $\mathbf{k} \in \mathbb{Z}^\ast$ and $\mathbf{k} \equiv \alpha(\mod p)$. So it suffices to look at the mod $p$ A-hypergeometric system with parameters $A, \mathbf{k}$. In Proposition 4.1 we saw that these solutions form a free module of rank $s'$ where $s'$ is the signature of $A$ and $\mathbf{k}/p$. Let $\delta$ be the minimal distance of the points of $\rho_{\alpha} + \mathbb{Z}^\ast$ to the faces of $C(A)$. Suppose $\delta = 0$. Then there is a point $\rho_{\alpha} + \mathbf{k}$ with $\mathbf{k} \in \mathbb{Z}^\ast$ contained in a face of $C(A)$. Choose $\mu \in \mathbb{Z}$ such that $\mu \rho \equiv 1(\mod D)$. Then $\mu(\rho_{\alpha} + \mathbf{k}) = \alpha + \mu \mathbf{k} + (\mu p - 1)\alpha$ is on a face of $C(A)$. This contradicts the irreducibility.
of our A-hypergeometric system by Theorem 1.2. So \( \delta > 0 \). Let us assume that \( p \) is so large that \( |\alpha/p| < \delta \). Then the points of \((\alpha/p + \mathbb{Z}) \cap C(A)\) and \((\mathbb{Z}/p + \mathbb{Z}) \cap C(A)\) are in one-to-one correspondence given by \( x \sim y \iff |x - y| < \delta \). In particular the number of apexpoints of both sets is equal, hence \( s = s' \). This proves our assertion.

\[ \square \]

### 5 Proof of the main theorem

This section is devoted to a proof of Theorem 1.5. Let notations be as in Theorem 1.5 and suppose we consider an irreducible A-hypergeometric system with parameters \( \alpha \in \mathbb{Q}^d \). Let \( p \) be a prime which is large enough in the sense of Proposition 4.2. Let \( D \) be the common denominator of the elements of \( \alpha \) and \( p \equiv -p^{-1} \pmod{D} \). Then the statement that \( \sigma(A, \rho \alpha) \) is maximal is equivalent to the statement that the A-hypergeometric system modulo \( p \) has a maximal \( F(v^p) \)-independent set of polynomial solutions.

A fortiori the following two statements are equivalent:

i) \( \sigma(A, k\alpha) \) is maximal for every \( k \) with \( 1 \leq k < D \) and \( \gcd(k, D) = 1 \)

ii) modulo almost every prime \( p \) the A-hypergeometric system modulo \( p \) has a maximal set of polynomial solutions modulo \( p \).

A famous conjecture, attributed to Grothendieck implies that statement (ii) is equivalent to the following statement,

iii) The A-hypergeometric system has a complete set of algebraic solutions.

If Grothendieck’s conjecture were proven we would be done here. Fortunately, in two papers by N.M.Katz [13] and [12] Grothendieck’s conjecture is proven in the case when the system of differential equations is (a factor of) a Picard-Fuchs system, i.e. a system of differential equations satisfied by the period integral on families of algebraic varieties. More precisely we refer to Theorem 8.1(5) of [12], which states

**Theorem 5.1 (N.M.Katz, 1982)** Suppose we have a system of partial linear differential equations, as sketched above, whose \( p \)-curvature vanishes for almost all \( p \). Then, if the system is a subsystem of a Picard-Fuchs system, the solution space consists of algebraic functions.

The above theorem is formulated in terms of vanishing \( p \)-curvature for almost all \( p \), but according to a Lemma by Cartier (Theorem 7.1 of [12]) this is equivalent to the system having a maximal set of independent polynomial solutions modulo \( p \) for almost all \( p \).

To finish the proof of Theorem 1.5 it remains to show that the A-hypergeometric equations satisfied by the period integral on families of algebraic varieties. More precisely we refer to Theorem 8.1(5) of [12], which states

### 5.1 Proof of the main theorem

Let \( k \) be a field which, in our case, is usually \( \mathbb{Q} \) or \( F_p \). Consider the differential field \( K = k(v_1, \ldots, v_N) = k(\mathbf{v}) \) with derivations \( \partial_i = \frac{\partial}{\partial v_i} \) for \( i = 1, \ldots, N \). The subfield \( C_K \subset K \) of elements all of whose derivatives are zero, is called the field of constants. When the characteristic of \( k \) is zero we have \( C_K = k \), when the characteristic is \( p > 0 \) we have \( C_K = k(\mathbf{v}^p) \).

Throughout this section we let \( L \) be a finite set of linear partial differential operators with coefficients in \( K \), like the A-hypergeometric system operators when \( k = \mathbb{Q} \). Consider the differential ring \( K[\partial_1, \ldots, \partial_N] \) and let \( (L) \) be the left ideal generated by the differential operators of the system. We assume that the quotient \( K[\partial_1]/(L) \) is a \( K \)-vector space of finite dimension \( d \). Throughout this section we also fix a monomial \( K \)-basis \( \partial^b = \partial_1^{b_1} \cdots \partial_N^{b_N} \) with \( b \in B \) and where \( B \) is a finite set of \( N \)-tuples in \( \mathbb{Z}_{\geq 0}^N \) of cardinality \( d \).

**Proposition 5.2.** Let \( K \) be some differential extension of \( K \) with field of constants \( C_K \). Let \( f_1, \ldots, f_m \in K \) be a set of \( C_K \)-linear independent solutions of the system \( L(f) = 0 \), \( L \in \mathcal{L} \). Then \( m \leq d \). Moreover, if \( m = d \) the determinant

\[ W_B(f_1, \ldots, f_d) = \det(\partial^b f_i)_{b \in B, i = 1, \ldots, d} \]

is the determinant of the matrix whose columns are \( f_1, \ldots, f_d \).

\[ \square \]
is nonzero.

In case we have $d$ independent solutions we call $W_B$ the Wronskian matrix with respect to $B$ and $f_1, \ldots, f_d$. Obviously, if $g_1, \ldots, g_d$ are $C_K$-linear dependent solutions then $W_B(g_1, \ldots, g_d) = 0$.

**Proof.** Suppose that either $m > d$ or $m = d$ and $W_B = 0$. In both cases there exists a $K$-linear relation between the vectors $df_i := (\partial^b f_i)_{b \in B}$ for $i = 1, 2, \ldots, m$. Choose $\mu < m$ maximal such that $df_i$, $i = 1, \ldots, \mu$ are $K$-linear independent. Then, up to a factor, the vectors $df_i$, $i = 1, \ldots, \mu + 1$ satisfy a unique dependence relation $\sum_{i=1}^{\mu+1} A_i df_i = 0$ with $A_i \in K$ not all zero. For any $j$ we can apply the operator $\partial_j$ to this relation to obtain

$$ \sum_{i=1}^{\mu+1} \partial_j(A_i) df_i + A_i \partial_j(df_i) = 0. $$

Since $\partial_j \partial^b$ is a $K$-linear combination of the elements $\partial^b, b \in B$ in $K[\partial_i]/(L)$ there exists a $d \times d$-matrix $M_j$ with elements in $K$ such that $\partial_j(df_i) = df_i \cdot M_j$. Consequently $\sum_{i=1}^{\mu+1} A_i \partial_j(df_i) = \sum_{i=1}^{\mu+1} A_i df_i \cdot M_j = 0$ and so we are left with $\sum_{i=1}^{\mu+1} \partial_j(A_i) df_i = 0$. Since the relation between $df_i, i = 1, \ldots, \mu + 1$ is unique up to factor there exists $\lambda_i \in K$ such that $\partial_j(A_i) = \lambda_i A_i$ for all $i$. Suppose $A_1 \neq 0$. Then this implies that $\partial_j(A_i/A_1) = 0$ for all $i$ and all $j$. We conclude that $A_i/A_1 \in C_K$ for all $i$. Hence there is a relation between the $df_i$ with coefficients in $C_K$. A fortiori there is a $C_K$-linear relation between the $f_i$. This contradicts our assumption of independence of $f_1, \ldots, f_m$.

So we conclude that $m \leq d$ and if $m = d$ then $W_B \neq 0$. □

**Proposition 5.3** Suppose the system of equations $L(y) = 0$, $L \in L$ has only algebraic solutions and that they form a vector space of dimension $d$. Then for almost all $p$ the system of equations modulo $p$ has a $F(v^p)$-basis of $d$ polynomial solutions in $F(v)$.

**Proof.** Let $f_1, \ldots, f_d$ be a basis of algebraic solutions. Choose a point $q \in \mathbb{Q}^N$ such that $f_i$ are all analytic near the point $q$. Then $f_1, \ldots, f_d$ can be considered as power series expansions in $v - q$. According to Eisenstein’s theorem for powerseries of algebraic functions we have that the coefficients of the $f_i$ can be reduced modulo $p$ for almost all $p$. Moreover, let $\partial^b, b \in B$ be a monomial basis of $K[\partial_i]/(L)$. Then the Wronskian determinant $W_B(f_1, \ldots, f_d)$ is non-zero. So for almost all $p$ the powerseries $f_i$ can be reduced modulo $p$ and moreover, $W_B(f_1, \ldots, f_d) \not\equiv 0 \pmod{p}$. Hence, for almost all $p$ the powerseries $f_i \pmod{p}$ are linearly independent over the quotient field of $F[[v - q^p]]$, the power series in $(v - q^p)^p$.

Fix one such prime $p$. Let $P$ be the set $\{(b_1, \ldots, b_N) \in \mathbb{Z}^N \mid 0 \leq b_i < p \text{ for } i = 1, \ldots, N\}$. Every solution $f$ can be written in the form

$$ f \equiv \sum_{b \in P} a_b(v - q)^b \pmod{p}, $$

where $a_b \in F[[v - q^p]]$. For every $L \in L$ we have that

$$ \sum_{b \in P} a_b L(v - q)^b \equiv 0 \pmod{p}. $$

Let $Q$ be the quotient field of $F[[v - q^p]]$. The $Q$-linear relations between the polynomials $L(v - q)^b$ for every $L$ form a vector space of dimension $d$ since the space of solutions mod $p$ has this dimension. Moreover the space of $Q$-linear relations between the polynomials $L(v - q)^b$ is generated by $F((v - q)^p)$-linear relations or, what amounts to the same, $\overline{F}(v^p)$-linear relations. □
6 Pochhammer cycles

In the construction of Euler integrals one often uses so-called twisted homology cycles. In [9] this is done on an abstract level, in [14] it is done more explicitly. In this paper we prefer to follow a more concrete approach by constructing a closed cycle of integration such that the (multivalued) integrand can be chosen in a continuous manner and the resulting integral is non-zero. For the ordinary Euler-Gauss function this is realised by integration over the so-called Pochhammer contour. Here we construct its n-dimensional generalisation. In Section[7] we use it to define an Euler integral for A-hypergeometric functions.

Consider the hyperplane $H$ given by $t_0 + t_1 + \cdots + t_n = 1$ in $\mathbb{C}^{n+1}$ and the affine subspaces $H_i$ given by $t_i = 0$ for $i = 0, 1, 2, \ldots, n$. Let $H^o$ be the complement in $H$ of all $H_i$. We construct an n-dimensional real cycle $P_n$ in $H^o$ which is a generalisation of the ordinary 1-dimensional Pochhammer cycle (the case $n = 1$). When $n > 1$ it has the property that its homotopy class in $H^o$ is non-trivial, but that its fundamental group is trivial. One can find a sketchy discussion of such cycles in [18, Section 3.5].

Let $\epsilon$ be a positive but sufficiently small real number. We start with a polytope $F$ in $\mathbb{R}^{n+1}$ given by the inequalities

$$|x_{i_1}| + |x_{i_2}| + \cdots + |x_{i_k}| \leq 1 - (n + 1 - k)\epsilon$$

for all $k = 1, \ldots, n + 1$ and all $0 \leq i_1 < i_2 < \cdots < i_k \leq n$. Geometrically this is an $n + 1$-dimensional octahedron with the faces of codimension $\geq 2$ sheared off. For example in the case $n = 2$ it looks like

The faces of $F$ can be enumerated by vectors $\mu = (\mu_0, \mu_1, \ldots, \mu_n) \in \{0, \pm 1\}^{n+1}$, not all $\mu_i$ equal to 0, as follows. Denote $|\mu| = \sum_{i=0}^n |\mu_i|$. The face corresponding to $\mu$ is defined by

$$F_\mu : \mu_0 x_0 + \mu_1 x_1 + \cdots + \mu_n x_n = 1 - (n + 1 - |\mu|)\epsilon, \quad \mu_j x_j \geq \epsilon \text{ whenever } \mu_j \neq 0$$

$$|x_j| \leq \epsilon \text{ whenever } \mu_j = 0.$$

Notice that as a polytope $F_\mu$ is isomorphic to $\Delta_{|\mu|-1} \times I^{n+1-|\mu|}$ where $\Delta_r$ is the standard $r$-dimensional simplex and $I$ the unit real interval. Notice in particular that we have $3^n - 1$ faces. The $n - 1$-dimensional side-cells of $F_\mu$ are easily described. Choose an index $j$ with $0 \leq j \leq n$. If $\mu_j \neq 0$ we set $\mu_j x_j = \epsilon$, if $\mu_j = 0$ we set either $x_j = \epsilon$ or $x_j = -\epsilon$. As a corollary we see that two faces $F_\mu$ and $F_{\mu'}$ meet in an $n - 1$-cell if and only if there exists an index $j$ such that $|\mu_j| 
eq |\mu'_j|$ and $\mu_i = \mu'_i$ for all $i \neq j$.

The vertices of $F$ are the points with one coordinate equal to $\pm(1-n\epsilon)$ and all other coordinates $\pm \epsilon$.

We now define a continuous and piecewise smooth map $P : \cup_\mu F_\mu \to H$ as follows. Suppose the point $(x_0, x_1, \ldots, x_n)$ is in $F_\mu$. Then its image under $P$ is defined as

$$\frac{1}{y_0 + y_1 + \cdots + y_n} (y_0, y_1, \ldots, y_n)$$

(4)
where \( y_j = \mu_j x_j \) if \( \mu_j \neq 0 \) and \( y_j = E_u(x_j) \) if \( \mu_j = 0 \). Here \( E_u(x) = e^{u(1-x/u)} \). When \( \epsilon \) is sufficiently small we easily check that \( P \) is injective. We define our \( n \)-dimensional Pochhammer cycle \( P_n \) to be its image.

**Proposition 6.1** Let \( \beta_0, \beta_1, \ldots, \beta_n \) be complex numbers. Consider the integral

\[
B(\beta_0, \beta_1, \ldots, \beta_n) = \int_{\partial F} \omega(\beta_0, \ldots, \beta_n)
\]

where

\[
\omega(\beta_0, \ldots, \beta_n) = \lambda_0^{\beta_0-1} \lambda_1^{\beta_1-1} \cdots \lambda_n^{\beta_n-1} \ dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n.
\]

Then, for a suitable choice of the multivalued integrand, we have

\[
B(\beta_0, \ldots, \beta_n) = \frac{1}{\Gamma(\beta_0 + \beta_1 + \cdots + \beta_n)} \prod_{j=0}^{n} (1 - e^{-2\pi i \beta_j}) \Gamma(\beta_j).
\]

**Proof** The problem with \( \omega \) is its multivaluedness. This is precisely the reason for constructing the Pochhammer cycle \( P_n \). Now that we have our cycle we solve the problem by making a choice for the pulled back differential form \( P^* \omega \) and integrating it over \( \partial F \). Furthermore, the integral will not depend on the choice of \( \epsilon \). Therefore we let \( \epsilon \to 0 \). In doing so we assume that the real parts of all \( \beta_i \) are positive. The Proposition then follows by analytic continuation of the \( \beta_j \).

On the face \( F_\mu \), we define \( T : F_\mu \to \mathbb{C} \) by

\[
T: (x_0, x_1, \ldots, x_n) = \prod_{\mu_j \neq 0} |x_j|^{\beta_j-1} e^{\pi i \beta_j} \prod_{\mu_k = 0} e^\beta \prod_{t_k = 0} e^{\pi i (x_j/y_k - 1)} (\beta_j - 1).
\]

This gives us a continuous function on \( \partial F \). For real positive \( \lambda \) we define the complex power \( \lambda^x \) by \( \exp(z \log \lambda) \). With the notations as in (4) we have \( t_i = y_i/(y_0 + \cdots + y_n) \) and, as a result,

\[
dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n = \sum_{j=0}^{n} (-1)^j y_j dy_0 \wedge \cdots \wedge dy_j \wedge \cdots \wedge dy_n
\]

where \( dy_j \) denotes suppression of \( dy_j \). It is straightforward to see that integration of \( T(x_0, \ldots, x_n) \) over \( F_\mu \) with \( |\mu| < n + 1 \) gives us an integral of order \( O(\epsilon^0) \) where \( \beta \) is the minimum of the real parts of all \( \beta_j \). Hence they tend to 0 as \( \epsilon \to 0 \). It remains to consider the cases \( |\mu| = n + 1 \). Notice that \( T \) restricted to such an \( F_\mu \) has the form

\[
T(x_0, \ldots, x_n) = \prod_{j=0}^{n} e^{\pi i (\mu_j - 1) \beta_j} |x_j|^{\beta_j - 1}.
\]

Furthermore, restricted to \( F_\mu \) we have

\[
\sum_{j=0}^{n} (-1)^j y_j dy_0 \wedge \cdots \wedge dy_j \wedge \cdots \wedge dy_n = dy_0 \wedge dy_2 \wedge \cdots \wedge dy_n
\]

and \( y_0 + y_1 + \cdots + y_n = 1 \). Our integral over \( F_\mu \) now reads

\[
\prod_{j=0}^{n} \mu_j e^{\pi i (\mu_j - 1) \beta_j} \int_{\Delta} (1 - y_1 - \cdots - y_n)^{\beta_0 - 1} y_1^{\beta_1 - 1} \cdots y_n^{\beta_n - 1} dy_1 \wedge \cdots \wedge dy_n
\]

where \( \Delta \) is the domain given by the inequalities \( y_i \geq \epsilon \) for \( i = 1, 2, \ldots, n \) and \( y_1 + \cdots + y_n \leq 1 - \epsilon \). The extra factor \( \prod_j \mu_j \) accounts for the orientation of the integration domains. The latter integral is a generalisation of the Euler beta-function integral. Its value is \( \Gamma(\beta_0) \cdots \Gamma(\beta_n)/\Gamma(\beta_0 + \cdots + \beta_n) \). Adding these evaluation over all \( F_\mu \) gives us our assertion. \( \square \)

For the next section we notice that if \( \beta_0 = 0 \) the subfactor \( (1 - e^{-2\pi i \beta_0}) \Gamma(\beta_0) \) becomes \( 2\pi i \).
7 An Euler integral for A-hypergeometric functions

We now adopt the usual notation from A-hypergeometric functions. Define

$$I(A, \alpha, v_1, \ldots, v_N) = \int_{\Gamma} \frac{t^\alpha}{1 - \sum_{i=1}^N v_i t^{a_i}} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_r}{t_r},$$

where $\Gamma$ is an $r$-cycle which doesn’t intersect the hyperplane $1 - \sum_{i=1}^N v_i t^{a_i} = 0$ for an open subset of $\mathbf{v} \in \mathbb{C}^N$ and such that the multivalued integrand can be defined on $\Gamma$ continuously and such that the integral is not identically zero. We shall specify $\Gamma$ in the course of this section.

First note that an integral such as this satisfies the A-hypergeometric equations easily. The substitution $t_i \to \lambda_i t_i$ shows that

$$I(A, \alpha, \lambda a_1, \ldots, \lambda a_N v_N) = \lambda^\alpha I(A, \alpha, v_1, \ldots, v_N).$$

This accounts for the homogeneity equations. For the ”box”-equations, write $l \in L$ as $u - w$ where $u, w \in \mathbb{Z}_+^N$ have disjoint supports. Then

$$\boxtimes_l I(A, \alpha, v) = |u|! \int_{\Gamma} \frac{t^\alpha + \sum_{i} u_i a_i - t^\alpha + \sum_{i} w_i a_i}{(1 - \sum_{i} v_i t^{a_i})|u|+1} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_r}{t_r}$$

where $|u|$ is the sum of the coordinates of $u$, which is equal to $|w|$ since $|u| - |w| = |l| = \sum_{i=1}^N t_i h(a_i) = h(\sum_i l_i a_i) = 0$. Notice that the numerator in the last integrand vanishes because $\sum_i u_i a_i = \sum w_i a_i$. So $\boxtimes_l I(A, \alpha, v)$ vanishes.

We now specify our cycle of integration $\Gamma$. Choose $r$ vectors in $A$ such that their determinant is 1. After permutation of indices and change of coordinates if necessary we can assume that $a_i = e_i$ for $i = 1, \ldots, r$ (the standard basis of $\mathbb{R}^r$). Our integral now acquires the form

$$\int_{\Gamma} \frac{t^\alpha}{1 - v_1 t_1 - \cdots - v_r t_r - \sum_{i=r+1}^N v_i t^{a_i}} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_r}{t_r}.$$

Perform the change of variables $t_i \to t_i/v_i$ for $i = 1, \ldots, r$. Up to a factor $v_1^{a_1} \cdots v_r^{a_r}$ we get the integral

$$\int_{\Gamma} \frac{t^\alpha}{1 - t_1 - \cdots - t_r - \sum_{i=r+1}^N u_i t^{a_i}} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_r}{t_r},$$

where the $u_i$ are Laurent monomials in $v_1, \ldots, v_N$. Without loss of generality we might as well assume that $v_1 = \ldots = v_r = 1$ so that we get the integral

$$\int_{\Gamma} \frac{t^\alpha}{1 - t_1 - \cdots - t_r - \sum_{i=r+1}^N v_i t^{a_i}} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_r}{t_r}.$$

For the $r$-cycle $\Gamma$ we choose the projection of the Pochhammer $r$-cycle on $t_0 + t_1 + \cdots + t_r = 1$ to $t_1, \ldots, t_r$ space. Denote it by $\Gamma_r$. By keeping the $v_i$ sufficiently small the hypersurface $1 - t_1 - \cdots - t_r - \sum_{i=r+1}^N v_i t^{a_i} = 0$ does not intersect $\Gamma_r$.

To show that we get a non-zero integral we set $v = 0$ and use the evaluation in Proposition \[6.1\]. We see that it is non-zero if all $\alpha_i$ have non-integral values. When one of the $\alpha_i$ is integral we need to proceed with more care.

We develop the integrand in a geometric series and integrate it over $\Gamma_r$. We have

$$\frac{t^\alpha}{1 - t_1 - \cdots - t_r - \sum_{i=r+1}^N v_i t^{a_i}} = \sum_{m_{r+1}, \ldots, m_N \geq 0} \frac{|m|}{m_{r+1}, \ldots, m_N} \frac{t^{\alpha + m_{r+1} a_{r+1} + \cdots + m_N a_N}}{(1 - t_1 - \cdots - t_r)|m|+1} v^{m_{r+1}} \cdots v^{m_N}$$

where $|m| = m_{r+1} + \cdots + m_N$. We now integrate over $\Gamma_r$ term by term. For this we use Proposition \[6.1\]. We infer that all terms are zero if and only if there exists $i$ such that the $i$-th coordinate of $\alpha$ is integral and positive and the $i$-th coordinate of each of $a_{r+1}, \ldots, a_N$ is non-negative. In particular this means that the cone $C(A)$ is contained in the half-space $x_i \geq 0$. 

Moreover, the points $a_j = e_j$ with $j \neq i$ and $1 \leq j \leq r$ are contained in the subspace $x_i = 0$, so they span (part of) a face of $C(A)$. The set $\alpha + \mathbb{Z}^r$ has non-trivial intersection with this face because $\alpha_i \in \mathbb{Z}$. From Theorem 1.2 it follows that our system is reducible, contradicting our assumption of irreducibility.

So in all cases we have that the Euler integral is non-trivial. By irreducibility of the A-hypergeometric system all solutions of the hypergeometric system can be given by linear combinations of period integrals of the type $I(A, \alpha, v)$ (but with different integration cycles).

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