Angled pulsar magnetospheres

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ABSTRACT

We consider magnetospheric structure of rotating neutron stars with internally twisted axisymmetric magnetic fields. The twist-induced and rotation-induced toroidal magnetic fields align/counter-align in different hemispheres. Using analytical and numerical calculations (with PHAEDRA code) we show that as a result the North-South symmetry is broken: the magnetosphere and the wind become “angled”, of conical shape. Angling of the magnetosphere affects the spindown (making it smaller for mild twists), makes the return current split unequally at the Y-point, produces anisotropic wind and linear acceleration that may dominate over gravitational acceleration in the Galactic potential and give a total kick up to \( \sim 100 \) km/s. We also consider analytically the structure of the Y-point in the twisted magnetosphere, and provide estimate of the internal twist beyond which no stable solutions exist: over-twisted magnetospheres must produce plasma ejection events.

1. Introduction: magnetar-pulsar connection

Pulsars, that produce emission using the rotational power, and magnetars, that produce emission via dissipation of the magnetic field, were typically considered to be different classes of astrophysical objects. Observations of magnetar-like phenomena in what seemed like regular pulsars establishes that rotationally-powered and magnetically-powered effects can occur simultaneously (Archibald et al. 2017; Kaspi & Beloborodov 2017; Lyutikov 2015). In addition, reconnection-driven generation of radio pulses is becoming a dominant theoretical concept for generation of Crab giant pulses (Bai & Spitkovsky 2010; Arons 2012; Chen & Beloborodov 2014; Cerutti et al. 2015, 2016; Cerutti & Beloborodov 2017; Wang et al. 2019; Lyubarsky 2019; Contopoulos & Stefanou 2019; Philippov et al. 2019; Lyutikov 2021).

In the present paper we explore application of magnetar models of twisted magnetospheres to a more general set of neutron stars, even those that do not show magnetar-like high energy activity. We aim to combine two somewhat separate theoretical approaches that have been used to study rotating and twisted magnetospheres. Rotating dipole is usually assumed for pulsars (Pacini 1967; Michel 1973; Scharlemann & Wagoner 1973; Lyubarskii 1990; Contopoulos et al. 1999). Models of twisted magnetospheres are the key to understanding magnetars (Thompson et al. 2002), see also (Lynden-Bell & Boily 1994; Beloborodov & Thompson 2007; Lyutikov 2013).
Previously, the problem of combining pulsar and magnetar physics was considered by Parfrey et al. (2012b, 2013). Importantly, and differently from the present scheme, dynamical twisting of the rotating magnetospheres was employed: starting from purely poloidal magnetic field, a shear of the footpoints was introduced according to various prescriptions. We employ a different model: we start with already sheared equilibrium and consider how it evolves under rotation. Both methods highlight various aspects of the dynamics: dynamical shearing, necessarily fast in numerical experiments, is better suited for studies of explosive events, while static shear with rotation is better suited for relative comparison of quasi-steady states. In addition, in Appendix A, Fig. 22 we follows Parfrey et al. (2013) prescription for dynamic twisting (with somewhat different twisting profile), and show that slower rates of twist do lead to angling of the magnetosphere.

Our approach, rotating of already twisted magnetic configurations, reveals a new, somewhat unexpected property of the resulting magnetospheric structure: the magnetosphere becomes North-South asymmetric, while the equatorial current sheet becomes “angled” - of a conical shape.

2. “Angling” of the current sheet in axisymmetric twisted magnetospheres.

2.1. General concept

The key point of the paper is that axially- and polar-angle-symmetric magnetic configurations of the neutron star produce “angled” magnetospheric/wind configurations, breaking the polar-angle (North-South) symmetry. Angled in a sense that the outside current sheet is not confined to a plane, but to a conical surface, Fig. 1.

The above statement is based on the following reasoning. The Goldreich & Julian (1969) model of dipolar pulsar magnetospheres gives similar (of the same sign) radial current along both magnetic poles (the return current is flowing along the equatorial current-sheet and the boundary between the closed and open field lines). The magnetic field lines in the northern and southern hemispheres are bent by rotationally-generated GJ currents. Thus, the toroidal components of the rotationally generated magnetic field are of opposite sign in the two hemispheres, Fig. 1. On the other hand, accepting a simple model of axially-symmetric twist-generated toroidal magnetic field (Thompson et al. 2002), the sign of the corresponding toroidal field is the same in the two hemispheres, Fig. 1. Thus, in one hemisphere toroidal fields align, in the other they counter align. This is the origin of the current sheet’s tilt.

The above described picture has a few subtleties. First, the twisted rotating system can still be in a North-South symmetric dynamical equilibrium, as we discuss in §4. But, we argue, it is dynamically not an attractor point/is unstable. The symmetric configuration assumes that North-South foot-points do not and never had causal contact. Our dynamic simulations start with two hemispheres in causal contact, and they relax to an angled equilibrium. This demonstrates that symmetric equilibria are not stable - any minor causal contact will lead to angling.
Fig. 1.— Twisted/angled pulsar magnetospheres. Top: illustration how magnetic fields from twisted dipole (left) and GJ-induce (right) add/subtract in different hemispheres. Bottom row: illustration how twisted-GJ-induced toroidal magnetic fields lead to conically inclined current sheet: magnetic fields are counter-aligned in one hemisphere (the North in the picture) while aligned in the South hemispheres ($\theta_t$ is the angle at which $B_r(\theta_t) = 0$, $\theta_Y$ is the angle where magnetic field is along the axis of rotation, $\theta_c$ is the tilt of the outside current sheet.).

2.2. Angling of the current sheet: the basic reason

Rotation of the neutron star’s magnetosphere creates a Goldreich & Julian (1969) current density and total current through open field lines

$$j_{GJ} \sim n_{GJ}ec \approx \frac{\Omega B}{2\pi}$$
\[ I_{GJ} \approx B_{NS} \frac{R_{NS}^3}{R_{LC}^2} \]  

where \( B \) is the local magnetic field, and \( B_{NS} \) is the surface magnetic field. Rotation creates toroidal magnetic field of different direction in the two hemispheres.

Internally-induced twist creates a current density and total current (Thompson et al. 2002, see also Eq.(26))

\[
\begin{align*}
\mathbf{j}_{\text{twist}} &\sim \frac{c}{4\pi} \frac{(\Delta \phi)B}{r} \\
I_{\text{twist}}(r) &= (\Delta \phi)B_{NS} \frac{R_{NS}^3}{4r^2}, \ r \geq R_{NS} 
\end{align*}
\]

where \( (\Delta \phi) \) is the typical North-South twist. Twist-created toroidal magnetic field has the same sign in the two hemispheres, Fig. 1 Thus, twist-induced and rotationally-induced toroidal magnetic field add/subtract in the two hemispheres. (Causal connection between open and closed regions is an important principal issue as discussed above.)

The ratio of currents,

\[
\begin{align*}
\frac{j_{\text{twist}}}{j_{GJ}} &= \left( \frac{R_{LC}}{r} \right) \frac{(\Delta \phi)}{2} \\
\frac{I_{\text{twist}}}{I_{GJ}} &= \frac{(\Delta \phi) R_{LC}^2}{4 R_{NS}^2} 
\end{align*}
\]

inside the magnetosphere can be large even for small twists \( (\Delta \phi) \ll 1 \). Thus, even tiny twist produces total currents much larger than the GJ current.

The ratio of toroidal magnetic fields have even stronger dependance on radius:

\[
\frac{B_{\phi,\text{twist}}}{B_{\phi,GJ}} \sim \frac{(\Delta \phi)B_{NS}(R_{NS}/r)^3}{B_{NS}R_{NS}^3/(rR_{LC}^2)} = \left( \frac{R_{LC}}{r} \right)^2 (\Delta \phi) 
\]

Total twist-generated toroidal flux through the closed field lines is concentrated mostly near the surface, while the GJ-generated toroidal flux is mostly at the light cylinder:

\[
\begin{align*}
\Phi_{\phi,\text{twist}} &\sim (\Delta \phi)B_{NS}R_{NS}^2 \\
\Phi_{\phi,GJ}(r) &\sim B_{NS} \frac{R_{NS}^3}{R_{LC}^2} \to B_{NS} \frac{R_{NS}^3}{R_{LC}^2} 
\end{align*}
\]

Thus, for mild twists i.e., \( (\Delta \phi) \geq R_{NS}/R_{LC} \), the inner magnetosphere is completely dominated by the twist, while near the light cylinder it is dominated by effects of the spin (for \( (\Delta \phi) < 1 \)).

Interaction of counter-aligned magnetic fields and currents in one hemisphere versus aligned magnetic fields in the other will lead to angling - breaking of the North-South symmetry. (N. B: in
the case of currents, there are also return currents along the last open/closed field lines that split unequally between the two hemisphere.)

The two players - internal twisting and spin-induced twisting of the magnetic field - dominate at different regions (near the neutron star and near the light cylinder). The angling will be mostly determined by the balance near the light cylinder. We can then identify the tilt angle $\Theta$ as

$$\Theta \sim f\left(\frac{R_{LC}}{R_{NS}}\right)(\Delta\phi)$$

(6)

where $(\Delta\phi)$ is the twist angle between northern and southern hemispheres, (Thompson et al. 2002, Appendix B).

The function $f\left(\frac{R_{LC}}{R_{NS}}\right)$ absorbs the complicated details of the interaction between the open and closed field lines. The value of function $f$ is difficult to determine from first principles: it depends on how much of the internal current $I_{\text{twist}}(r)$ interacts with the rotationally-generated current $I_{GJ}$. Our results indicate $f \leq 1$, Fig 8.

The tilt angle $\Theta$, Eq. (6) is a general estimate of the “angled” magnetosphere. At a more detailed level, the tilt has in fact somewhat different meaning inside the light cylinder and in the wind zone: tilt of the nearly dipolar field lines $\theta_t$, tilt of the Y-point $\theta_Y$ (14), and the tilt of the outside current sheet $\theta_c$, see Fig. 1.

Models of twisted magnetospheres (Thompson et al. 2002, see also Appendix B) use a self-similar parameter $C$ so that $(\Delta\phi) = \sqrt{2C}$. Thus

$$\Theta \approx f\left(\frac{R_{LC}}{R_{NS}}\right)\sqrt{C}$$

(7)

2.3. Asymptotic structure

The structure of the pulsar magnetosphere is a notoriously mathematically difficult problem (Goldreich & Julian 1969; Michel 1973; Scharlemann & Wagoner 1973; Contopoulos et al. 1999; Uzdensky 2003; Gruzinov 2005). Qualitatively, the overall structure is formally determined by the structure of the magnetic fields on the surface of the star, but is actually “controlled” by the critical flow point - the Alfvén light cylinder. In what follows we outline the general property of the angled magnetospheres, starting from their asymptotic structure at large radii.

We can identify then several zones in the wind, Fig. 2: (i) zone 1: northward from the current sheet; (ii) zone 2: southward from the current sheet; (iii) North: northward from magnetic equator; (iv) South: southward from magnetic equator.

At large distance the structure will approximate Michel’s (Michel 1973) solution (in a sense that flux functions are nearly conical), but with an angled current sheet and somewhat different structures in zones 1 and 2. To ensure magnetic flux conservations and pressure balance across the current sheet, the magnetic fields and bulk velocities will be different in zones 1 and 2, as we discuss next.
Magnetic fluxes in zones 1, 2 must be equal in value but of opposite signs. Conservation of magnetic flux requires

\[
\frac{B_2}{B_1} = \frac{1 - \sin \theta_c}{1 + \sin \theta_c} \approx 1 - 2\theta_c
\]

\[
I_2 = \frac{B_2}{B_1} = \frac{1 - \sin \theta_c}{1 + \sin \theta_c}
\]

\[
\frac{I_{\text{north}} - I_{\text{south}}}{I_{\text{north}} + I_{\text{south}}} = -\tan^2 \theta_c
\]  

(8)

where subscripts 1, 2 refer to zones defined above and \( I_{1,2} \) are the total currents. The return current is \( I_1 + I_2 \).

Pressure balance across the current sheet requires

\[
\frac{\gamma_2}{\gamma_1} = \frac{B_2}{B_1} = \frac{1}{\sqrt{1 - v_{1,2}^2}}
\]

\[
v_{1,2} = \frac{E_\theta}{B_{\text{tot}}}
\]  

(9)

Thus, the more open zone 2 moves slower than zone 1. This is clearly seen in our simulations, Fig. 3, right panel.

For the purpose of argument, let us normalize \( B_{0,1;2} = B_0(1 \pm \sin \theta_c) \). Parameter \( B_0 \) depends on \( \theta_c \) as well - we neglect that for a moment (see discussion around Eq. (12)).

Total Poynting flux can be estimated as (in both zones the wind is moving relativistically)

\[
S = \frac{r_0^2}{4\pi} \left( \int_0^{\pi/2-\theta_c} (E_1 \times B_1) \sin \theta d\theta + \int_{\pi/2-\theta_c}^{\pi} (E_2 \times B_2) \sin \theta d\theta \right) = \frac{2}{3} B_0^2 r_0^4 \Omega^2 \cos^4 \theta_c
\]  

(10)

(where \( r_0 \sim R_{\text{LC}} \)). Thus, the total Poynting flux decreases with increasing \( \theta_c \) (and thus with the twist parameter \( C \)).

We can also estimate the North-South anisotropy of the wind

\[
\frac{S_{\text{north}} - S_{\text{south}}}{S_{\text{north}} + S_{\text{south}}} = \frac{1 + 4 \sin (\theta_c) - \cos (2\theta_c)}{3 + 4 \sin (\theta_c) - \cos (2\theta_c)} \approx 2\theta_c
\]  

(11)

Thus, we expect North-South anisotropic wind, and as a result axial acceleration, §3.4.

Relation (10) does not address how overall normalization \( B_0 \) scales with \( C \). It was expected that twisted configurations have higher spin-down luminosity (for the same polar magnetic field and spin) than the dipolar ones (Thompson et al. 2002, see also Eq. (26)):

\[
L_{sd} \sim B_{NS}^2 R_{NS}^2 c \left( \frac{R_{NS}}{R_{\text{LC}}} \right)^{2(p+1)}
\]  

(12)
Since in twisted configurations magnetic field falls off slower with radius, it was argued that they will have higher spin down power than the dipole ones.

Inclination of the current sheet leads to a decrease of the spin-down, Eq. (10), while stretching of field lines increases it, Eq. (12). The two effects thus oppose each other. Our results indicate that for small twists the luminosity slightly decreases. Fig. 11.

2.4. Magnetospheric structure

The internal structure of angled magnetosphere cannot be self-similar, Appendix B. Still, we can make some basic estimates. The surface fields are the same in both hemispheres by construction. Since the corresponding open flux is also the same (but is not limited to the same hemisphere at the light cylinder), the size of the polar cap must be the same. The North-South asymmetry is broken mostly at the light cylinder: hence the radial structure of open fields is different in the regions 1 and 2.

Neglecting small twist-induced changes of the poloidal magnetic field near the poles, the last closed magnetic field lines in the regions 1, 2 are given by

\[
\frac{r_{1,2}}{R_{LC}} = \sin^{2/p_{1,2}}(\theta + \theta_Y) \tag{13}
\]

with somewhat different \( p_{1,2} \neq 1 \).

For inclined near-dipole fields the angle \( \theta_Y \) where the Y-point is located at the LC is then

\[
\theta_Y = \frac{\pi}{2} \mp \frac{2}{2 + p_{1,2}} \theta_t \tag{14}
\]

where \( \theta_t \) is the title angle of the magnetosphere, at which \( B_r = 0 \), see Fig. 1. (For tilted dipole \( \theta_Y = (2/3) \theta_t \)).

Polar cap angles are then

\[
\theta_{PC,1,2} = \left( \frac{R_{NS}}{R_{LC}} \right)^{p_{1,2}/2} \mp \frac{2}{2 + p_{1,2}} \theta_t \tag{15}
\]

(for North, South).

Since \( \theta_{PC,1} = \theta_{PC,2} \), we find in the limit \( \theta_t \ll 1, R_{NS}/R_{LC} \ll 1 \)

\[
p_2 - p_1 = \frac{8}{3} \frac{\sqrt{R_{LC}/R_{NS}}}{\ln(R_{LC}/R_{NS})} \theta_t \tag{16}
\]

Equation (16) gives the difference of the radial magnetic field scaling within the light cylinder between zones 1 (approximately North open field lines) and zones 2 (approximately South open field lines), see also Fig. 25.
2.5. Goldreich-Julian-like mode of rotating twisted magnetospheres

Next, we further discuss the Goldreich & Julian (1969) model of pulsar magnetospheres generalized to the case of twisted field configuration (Thompson et al. 2002). Consider axisymmetric stationary force-free magnetic field

\[ \mathbf{B} = \{ B_r(r, \theta), B_\theta(r, \theta), B_\phi(r, \theta) \} \]

(17)

On the closed field lines the plasma motion is purely toroidal,

\[ \mathbf{v} = \Omega \times \mathbf{r} \]

(18)

Using ideal condition for the electric field,

\[ \mathbf{E} = -\mathbf{v} \times \mathbf{B} \]

(19)

and force-free parametrization of the current

\[ \nabla \times \mathbf{B} = \alpha \mathbf{B} + 4\pi \rho r \sin \theta \mathbf{e}_\phi \]

(20)

(the last term on the rhs is \( \propto v \) \( \text{div} \mathbf{E} \)) we find

\[ \rho = \frac{\text{div} \mathbf{E}}{4\pi} = -\frac{\Omega \cdot \mathbf{B}}{2\pi(1 - (r \sin \theta \Omega)^2)} + \left(\frac{\alpha}{4\pi}\right) \frac{r \sin \theta \Omega B_\phi}{1 - (r \sin \theta \Omega)^2} = \rho_{GJ} + \rho_{\text{twist}} \]

(21)

In the limit of small \( C \), Eq. (26), using

\[ B_\phi \approx \sqrt{C/2} \sin^3 \theta B \]

\[ B = B_{NS} \left( \frac{R_{NS}}{r} \right)^3 \]

\[ \alpha \approx \sqrt{2C} \frac{\sin^2 \theta}{r} \]

(22)

we find

\[ \rho_{\text{twist}} = \frac{C}{4\pi} B \Omega \sin^6 \theta \]

\[ \frac{\rho_{\text{twist}}}{\rho_{GJ}} = C \frac{\sin^6 \theta}{1 + 3 \cos(2\theta)} \]

(23)

Qualitatively,

\[ \rho_{\text{twist}} \sim j_\phi r \sin \theta \Omega \]

(24)

Its origin can be traced as follows: non-rotating twisted magnetosphere has a toroidal component of the current \( j_\phi \). Rotation of this current creates, by Lorentz transformation, charge density \( \rho \sim j_\phi v_\phi \).
3. Simulations

3.1. Initial conditions and parameters of simulations

PHAEDRA code (Parfrey et al. 2012a) is a pseudo-spectral code developed specifically to study highly magnetized plasma regime, force-free electrodynamics, the vanishing-inertia limit of magnetohydrodynamics. Our results further indicate that the code is very stable and efficient, and has low numerical dissipation.

We performed a number of numerical simulations using PHAEDRA code (Parfrey et al. 2012a). We start with analytical approximation for the structure of twisted magnetospheres for small/mild twist parameter $C$, Appendix B. For small twists we can find analytical relation for the structure of non-rotating magnetosphere by expanding near $p = 1$, $C = 0$ and the dipolar flux function:

$$p = 1 - 8C/35$$

$$\mathcal{F} = (1 - \mu^2) \left(1 + \frac{1}{140} (1 - \mu^2)(17 - 5\mu^2)C\right)$$

$$\mu = \cos \theta$$  \hspace{1cm} (25)

In this approximation the flux function $\alpha$, magnetic field components and twist angle are given by

$$\alpha = \frac{\sqrt{2C}(1 - \mu^2)}{\tilde{r}}$$

$$\frac{B_r}{B_{NS}} = \frac{2\mu}{\tilde{r}^{2+p}} \left(1 + \frac{3(13 - 18\mu^2 + 5\mu^4)}{140}C\right)$$

$$\frac{B_\theta}{B_{NS}} = p \frac{\sqrt{1 - \mu^2}}{\tilde{r}^{2+p}} \left(1 + \frac{(17 - 22\mu^2 + 5\mu^4)C}{140}\right)$$

$$\frac{B_\phi}{B_{NS}} = \sqrt{\frac{Cp}{1 + p}} \frac{(1 - \mu^2)^{3/2}}{\tilde{r}^{2+p}}$$

$$\frac{j_r}{B_{NS}/4\pi} = 2\sqrt{2}\mu(1 - \mu^2)\tilde{r}^{-3-p}\sqrt{C}$$

$$\frac{j_\theta}{B_{NS}/4\pi} = \sqrt{2}(1 - \mu^2)^{3/2}\tilde{r}^{-3-p}\sqrt{C}$$

$$\frac{j_\phi}{B_{NS}/4\pi} = (1 - \mu^2)^{5/2}\tilde{r}^{-3-p}C$$

$$\Delta \phi = \sqrt{2C} \mu_{fp}$$

$$\tilde{r} = \frac{r}{R_{NS}}$$  \hspace{1cm} (26)

where $C$ is a twist parameter, $\mu_{fp}$ is the cosine of the polar angle of the northern foot point and $B_{NS}$ is the equatorial magnetic field. In the above formulation $\text{div} \mathbf{B} \equiv 0$, $\mathbf{j} \times \mathbf{B} \propto C^{3/2} \mathbf{e}_\phi + C^2 (\mathbf{e}_{r,\theta})$. Even for the twist parameter $C$ approaching $\sim 0.5$, the overall poloidal magnetic field structure remains nearly dipolar, Fig. 23 (correction poloidal fields are proportional to coefficient $C$ typically divided by a large number).
We then rotate these equilibria with spin frequency \( \Omega \) (with an exponential switch). When describing simulation results, we will generally follow the unit convention used in (Parfrey et al. 2012a): distances are measured in units of \( R_{NS} \), and angular velocity in units of \( c/R_{NS} \). Time will be generally expressed in terms of the star rotation period. In this paper we used \( R_{LC}/R_{NS} = 10, 5, 3, \) corresponding to \( \Omega = 0.3, 0.2, 0.1 \). We performed a number of low resolution (grid size of \( 256 \times 200 \)) and a few high resolution simulations (grid size of \( 320 \times 256 \)). Low resolution runs were done for \( C = 0, 10^{-3}, \pm 10^{-2}, 10^{-1}, 0.5 \). (For \( C = \pm 10^{-2} \) two runs correspond to two relative directions of the twist and the spin.) For smaller values of \( C \) numerical fluctuations overwhelm the effects of the tilt. Our time step corresponds to 0.032 period (for \( \Omega = 0.2 \)). Our total simulation time for low resolution simulations corresponds to 9.5 rotation periods (300 total time steps), one high resolution simulations was run to \(~ 10.5\) rotation periods

Importantly, we start with already twisted magnetospheres, according to the self-similar prescription of Thompson et al. (2002), Eq. (26), and then introduce rotation. Thus we start with two hemispheres in causal contact, and allow them (give them time) “to negotiate” the final location of the current sheet. It is oblique.

We have investigated if the obliquity of the current sheet can be the effect of initial conditions, numerical resolution and/or slow time convergence. In Appendix A we perform a number of tests to see if the effect of inclined current sheet is real, and not a numerical artifact or transient by-production of the initial conditions. Our conclusion is that the result is physical, not numerical/transient.

### 3.2. Overall structure: the angled current sheet

The salient features are outlined in Fig. 2. Most notorious is the inclined (angled) current sheet. Dark blue regions correspond to Michel-like flow with currents \( I_{GJ,1} \neq I_{GJ,2} \). The return current, \( I_{GJ,\text{return}} = I_{GJ,1} + I_{GJ,2} \), splits into two unequal parts at the Y-point, \( I_{GJ,\text{return},1} \neq I_{GJ,\text{return},2} \). The closed part of the magnetosphere extends slightly beyond the light cylinder both due to effects of reconnection and time-dependence (not fully relaxed and/or subjected to ejection events). Similarly, the separation between the inner and outer Y-points is due to explicit resistivity and finite integration time.

In Fig. 4 we plot radial current \( j_r \sin \theta \) for different \( C \). Most importantly, the insert shows that in the bulk the flux function is modified if compared with the no-twist case, as expected: see 2.3.

In Fig. 5 we plot radial magnetic field, showing that in the smaller zone 1 the field is higher, again as expected, §2.3.

In Fig. 6 we plot toroidal magnetic scaled by \( r \): the system resembles Michel (1973) solution, but with a tilted current sheet.
Fig. 2.— Main features of the twisted rotating magnetosphere, annotated. Color is the value of $j_\phi$, parameter $C = 0.1$, $\Omega = 0.2$. Note inclined return current sheet.

In Fig 7 we show maps of the toroidal current density $j_\phi$ for various values of $C = 10^{-3}, 10^{-2}, 10^{-1}, 0.5$ at $\sim 6.4$ time-period.

In Fig. 8 we plot the tilt angle $\theta_c$ of the current sheet as a function of $C$. The scaling with the current parameter $C$ matches the expectations, Eq. (7). Somewhat surprisingly, we do find that the tilt angle $\theta_c$ is mostly independent of the ratio $R_{LC}/R_{NC}$ (we also did run with $\Omega = 0.1$) - the observed variations of the angle $\theta_c$ for $\Omega = 0.1, 0.2, 0.3$ are consistent with dynamical fluctuations.

In Fig. 9 we show that angled magnetospheric configuration produces North-South asymmetric wind.

Finally, in Fig. 10 we show the radial current (red is the return current) and the value of the four-current squared, showing that the bulk of the flow is mostly null ($|\rho| \sim |j|$, while the current sheet is space-like, $|j| > |\rho|$.
Fig. 3.— Left Panel: Radial current $j_r$, annotated. Right panel: radial velocity $E \times B / B^2$. For both plots, $C = 0.1$ and $\Omega = 0.2$. Notice radial velocity is smaller in the zone 2 (right of the current sheet) (zones 1 and 2 are defined in Fig. 2).

Fig. 4.— Left panel: radial current $j_r \sin \theta$ for different $C$ outside the light cylinder. Location of current sheet is clearly seen. The insert shows that for $C \neq 0$ region 1 and region 2 have different distribution of currents, both different from the symmetric $|\sin \theta \cos \theta|$ case, see §2.3. Right panel: radial current $j_r \sin \theta$ inside the light cylinder, annotated. Notice that radial currents are different in the two hemispheres: even though the general structure is dominated by the twist, the return GJ currents are different in the two hemispheres. For both plots, $\Omega = 0.2$

3.3. Overall powers

To interpret the results of numerical simulations we scale luminosities to the $C = 0$ case

$$L_{sd} = \eta L_{sd,0}$$
Fig. 5.— Solid line: Radial magnetic field $B_r$ (arbitrary units) evaluated approximately at 1.7 light cylinders, $C = 0.1$. The current sheet is clearly seen at $\theta \sim 1.4$. Dashed line is the mirror image of the solid line - this highlights that the values of the radial magnetic fields in zone 1 and 2 differ: in the smaller zone 1 the radial field is higher than in the larger zone 2, see Eq. (8). Total magnetic flux is consistent with zero.

$$L_{sd,0} = B_{NS}^2 R_{NS}^2 \left( \frac{R_{NS}}{R_{LC}} \right)^4 c$$

(27)

where $L_{sd}$ is a spin-down luminosity, $L_{sd,0}$ is for purely dipole case $C = 0$, so that $\eta(0) = 1$.

First, our results are consistent with the expected scaling with $L_{sd} \propto \Omega^4$ (see comment in caption to Fig. 11). But dependence on the $C$ parameter contradicts the expectation: spin-down decreases for mild values of $C \leq 1$, Fig 11.

This result, decrease of spin-down with $C$ parameter, is likely to be physical, not numerical. First, self-similar solutions have non-dipolar magnetic configurations; then there are different ways of scaling losses from a non-dipolar self-similar configurations to the dipolar one. But this effect (of non-dipolar fields at the surface) is, small $\propto C^2$, for a given polar field the radial component of
Fig. 6.— Toroidal magnetic field \((r B_\phi)\) for various values of \(C = 10^{-3}, 10^{-2}, 10^{-1}, 0.5\): toroidal magnetic field scales \(\sim 1/r\), but with an angled current sheet.

Fig. 7.— Toroidal current density \(j_\phi\) for various values of \(C = 10^{-3}, 10^{-2}, 10^{-1}, 0.5\) at fixed time \((\sim 6.4\) time-period\), \(\Omega = 0.2\). Increasing tilt of the current sheet is clearly seen. Note also that the highly twisted case (right most panel) clearly shows a plasmoid ejection event due to instability if the Y-point, see §4.

The twisted magnetic field \(B_{r,C}\) scales as

\[
\frac{B_{r,C}}{B_{r,0}} = 1 + \frac{6}{35} \theta^2 C
\]  

(28)

Thus, changes of the surface field can be neglected.

The main effect, that was expected to contribute to increasing the spin-down is the slower radial decreases of the magnetic field, Eq. (12) (small ratio \(R_{NS}/R_{LC}\) is raised to smaller power \(< 4\)). Our results are inconsistent with the expected increasing spin-down power, Fig. 11.

Another effect relates to the explicit resistivity in the PHAEDRA code (see Fig. 13 in Parfrey et al. 2012a), that leads to a numerical decrease of the Poynting flux. But in Fig. 11 we compare luminosities at different values \(C\); thus, we compare runs that are both affected by resistivity.
Fig. 8.— Tilt angle $\theta_c$ of the current sheet as a function of $C$ for two values of $\Omega$. Double points at $C = 0.01$ correspond to two relative directions of the twist and rotation and are shown by green color. They also gives an estimate of numerical variations. The star corresponds to the tilt angle at $C = 0.1, \Omega = 0.2$ for high-resistivity ($\alpha_{SSV} = 0.05$) simulation run. The best fit line expressions for $\Omega = 0.2$ and $\Omega = 0.3$ are given by $0.33C^{0.49}$ and $0.49C^{0.52}$ respectively, consistent with Eq. (6). We fitted the data after linearizing the fit equation, so that fitting function weighs data according to the log-log scale.
Fig. 9.— Poynting fluxes as a function of polar angle $\theta$. Left Panel: smoothened fluxes (using Savitzky-Golay filter) for $C = 0$ (solid line, symmetric with respect to the equator), and two asymmetric cases denoted $C = \pm 0.01$ corresponding twists with different direction with respect to the rotation axis. Clear North-South asymmetry is seen for $C \neq 0$ cases. Positive and negative twists produce fluxes that are mirror-symmetric with respect to the equator. Right Panel: unsmoothed fluxes for different values of $C$, clearly showing location of the return current sheets (dips near $\theta = \pi/2$). Spin $\Omega = 0.2$.

Though it is possible that somewhat different configurations result in different resistive effects.

We also did a run with increased resistivity in the PHAEDRA code. The major source of resistivity in our simulation is the current sheet and dissipation is introduced in our via system via super spectral viscosity filter. The amplitude of the spectral filter ($\alpha_{SSV}$) is the controlling parameter for current sheet resistivity. The default value of $\alpha_{SSV}$ for our simulation runs is 0.01, we increased it by five times for high-resistivity run. We found that the resulting changes were minimal. Below we compare the results from two runs.

\[
\begin{array}{cccc}
\alpha_{SSV} & \theta_c & \frac{L_{rad}}{L_{rad,0}} & \frac{F_z}{F_{z,0}} \\
0.01 & 0.102 & 1.01 & 0.074 \\
0.05 & 0.102 & 0.913 & 0.056 \\
\end{array}
\]

We thus conclude that the effect of decreasing luminosity is real, see §2.3, Eq. (10) for theoretical justification. For symmetric configurations the open flux increases with $C$. In numerical simulations the system was able to find lower energy configuration with smaller open flux. (We stress that our driving is different from the one employed by Parfrey et al. (2013), who also found increased spin-down with $C$ - in that case the driving was fast, not allowing a system to find a dynamical equilibrium; there are indications of the angling, e.g. in their Fig. 15).
Fig. 10.— Left Panel: charge density, multiplied by $r^2$ for $C = 0.1$, $\Omega = 0.2$. Blue regions correspond to approximately Michel charge density, yellow is the charged return current sheet. (Noise at radii $r \geq 10$ is due to errors in the outer boundary absorbing PML (Perfectly Matched Layer) matching.) Right panel: absolute value of four-current, $(\rho^2 - j^2) \times r^4$: the current layer is space-like, $|j| > |\rho|$.

For high twists, with $C \geq 1$ we do find increased spindown, as effects of slower radially decreasing magnetic field overpower the effects of tilt. For very large twists the magnetospheres will open, leading to effects of “anti-glitches” in magnetars (Lyutikov 2013).

3.4. North-South wind asymmetry and astrophysical implications of axial force

North-South asymmetric wind will lead to axial force and acceleration. We parametrize the $z$ force as

$$F_z = \eta(C) \frac{L_{sd,0}}{c},$$

see Fig. 12.

Neutron stars have large spacial velocities (Lyne & Lorimer 1994; Hansen & Phinney 1997; Hobbs et al. 2005; Arzoumanian et al. 2002), up to nearly 2000 km s$^{-1}$ in the case of the neutron star associated with the Guitar nebula (Johnson & Wang 2010). The origin of pulsar kicks remains controversial. Most commonly it is attributed to neutrino-induced recoil near the first seconds.
Fig. 11.— Total luminosity as function of twist parameter $C$. The total luminosity is normalized with respect to the spin-down power at $C = 0$ (for each $\Omega$ separately). The average ratio of $L_{sd}$ for $\Omega = 0.3$ and 0.2 is around 4.6, close to the expected $1.5^4 = 5.1$. Double points corresponding to $C = \pm 0.01$ are joined by a dashed line. The star corresponds to the normalized spin-down power at $C = 0.1, \Omega = 0.2$ for high-resistivity ($\alpha_{\text{SSV}} = 0.05$) simulation run.
Fig. 12.— $z$-component of force for various values of $C$ and $\Omega = 0.2$ and $\Omega = 0.3$ normalized to the total spin-down power for $C = 0$ (for each $\Omega$ separately). Double points corresponding to $C = \pm 0.01$ are shown by green color. The star corresponds to the normalized axial force at $C = 0.1, \Omega = 0.2$ for high-resistivity ($\alpha_{SSV} = 0.05$) simulation run. The best fit line expression for $\Omega = 0.2$ and $\Omega = 0.3$ is given by $0.39C^{0.73}$ and $0.77C^{0.95}$ respectively. We fitted the data after linearizing the fit equation, so that fitting function weighs data according to the log-log scale.
of the core collapse (e.g. Spruit & Phinney 1998; Woosley et al. 2002; Scheck et al. 2004, 2006; Wongwathanarat et al. 2013; Janka et al. 2016).

Some contribution to the spacial velocities can come from the electromagnetic rocket - recoil from the North-South anisotropic wind. The electromagnetic rocket cannot account for total velocity of more than a thousand kilometers per second (though flux emergence can help). On the other hand, persistent acceleration may dominate over more subtle effects, like Galactic/globular cluster acceleration (Freire et al. 2003) and the Shklovsky effect.

We also note a paper by Lai et al. (2001) who deduced apparent alignment of the spin axes, proper motion directions, and polarization vectors of the Crab and Vela pulsars - our model is consistent with those observations: the rational/twist induced force is along the rotation axis. Other related work includes Pétri (2021) who suggested off-centered dipole, and Barkov (2012); Barkov & Komissarov (2010) where kick in the pulsars with the asymmetric magnetosphere were considered.

Given the force (29), the acceleration of NS evaluates to

$$a_{NS} = \frac{F}{M_{NS}} = \eta b_q^2 \frac{m_e^4 c^2 R_{NS}^6 \Omega^4}{e^2 \hbar^2 M_{NS}} = 1.3 \times 10^2 \text{cm s}^{-2} \eta^{-1} b_q^2 P_{-3}^{-4}$$

$$b_q = \frac{B_{NS}}{B_Q}$$

$$B_Q = \frac{m_e^2 c^3}{e \hbar}$$

(30)

Acceleration (30), which is a highly unlikely upper limit scaled to millisecond magnetar, is much larger, by nearly ten orders of magnitude, than the Galactic acceleration,

$$a_G \sim \frac{V^2}{d} \approx 10^{-8} \text{cms}^{-2}$$

(31)

for rotational velocity of 200 km s\(^{-1}\) and distance from the Galactic center of 8 kpc. Thus, even minor twist may overwhelm gravitational acceleration in the Galactic or globular cluster potential.

As an estimate of the final velocity we can assume that a fraction \(\eta_p\) of the momentum lost by the pulsar wind has been used to give a neutron star a kick. For initial rotation energy \(E_{GJ} = I_{NS} \Omega_0^2 / 2\) this gives

$$V_{NS}^{(\infty)} = \eta_p \frac{E_{GJ}}{cM_{NS}} = \eta_p \frac{I_{NS} \Omega_0^2}{2cM_{NS}} = 240 \eta_{-1} \text{kms}^{-1} P_{-3}^{-2}$$

(32)

\((I_{NS} = 10^{45} \text{g cm}^2\) was used.)

The final velocity depends critically on the initial spin \(\Omega_0\), but is independent of the magnetic field: higher magnetic fields provide larger force, but act for shorter spindown time. This can change if the flux emergence is important: mostly toroidal magnetic field is buried - in, emerges on time scales shorter than spin-down, hence increasing the action time.
If a NS starts at rest, and taking into around magnetic breaking, its velocity evolves according to
\[ V_{\text{NS}} = \left( \frac{t}{t + \tau} \right) V_{\text{NS}}^{(\infty)} \] (33)
where \( \tau \) is the spin-down time.

Acceleration (30) will produce an observed change in period
\[ \dot{\Omega}_{\text{NS}} = \frac{a_{\text{NS}}}{c} \Omega \] (34)
If interpreted as magnetic losses, the inferred magnetic field would be
\[ \frac{B_{\text{NS}}^{(\text{inferred})}}{B_{\text{NS}}^{(\text{real})}} \approx \eta^{1/2} \left( \frac{I_{\text{NS}}}{M_{\text{NS}}} \right)^{1/2} \frac{\Omega}{c} = 0.03 \times \eta_{-1}^{1/2} P_{-3}^{-1} \] (35)
Thus, it is not important for the estimates of the magnetic fields.

4. Structure of Y-point in twisted magnetospheres

4.1. The pulsar equation and the Y-point

In this Section we investigate the structure of the Y-point in case of rotating pulsars with internally twisted magnetospheres, assuming polar symmetry. Though, as we discussed above, twisted magnetospheres are not North-South symmetric, the following considerations provide a mathematical reason, why twisted configurations become unstable beyond some twist of the closed magnetospheres, as found in numerical simulations by Parfrey et al. (2012b). The key issue is the structure/stability of field configurations near the Y-point. We follow the approach of Uzdensky (2003) in resolving the structure of the Y-point.

The key equation is the Grad-Shafranov equation (Grad 1967; Shafranov 1966) for equilibrium configuration of magnetic fields. In case of idea, axisymmetric, magnetically-dominated plasma rotating with constant angular velocity \( \Omega \) it takes the form of the pulsar equation (Scharlemann & Wagoner 1973; Beskin 2009)
\[ \mathbf{B} = \nabla \Psi \times \nabla \phi + I(\Psi) \nabla \phi \]
\[ (1 - x^2) (\Psi_{xx} + \Psi_{yy}) - \frac{1 + x^2}{x} \Psi_x = -II' \] (36)
where \( x = r/R_{\text{LC}}, y = z/R_{\text{LC}}, R_{\text{LC}} = c/\Omega \) and subscripts denote derivatives; \( \Psi \) is magnetic flux function. Magnetic field lines lie on surfaces of constant \( \Psi \). The shape of \( \Psi(r, z) \) is determined by the shape of the poloidal component of magnetic field (which also depends on the structure of the toroidal fields via hoop stresses). Importantly, the current \( I \) is a function of the magnetic flux function, \( I \equiv I(\Psi(r, z)) \) (not independently on \( r \) and \( z \)). The pulsar equation has an unknown
current function that must be determined together with the solution - the correct equation, and its
solution are to be found self-consistently.

Previous analytical approaches (Uzdensky 2003; Gruzinov 2005) assumed non-current-carrying
closed part of the magnetosphere. Below we generalize it to the case of twisted closed field line.
The $Y$-point is at $x = 1, y = 0$. Near the $Y$-point, expanding near $x \to 1, r \equiv 1 - x$ and
$y \to 0$, expressing $\Psi = r^\alpha g(\theta)$, assuming that the current is power-law in flux function, $I \propto \kappa \Psi^\beta$
with $\kappa = \beta \eta^2/2, \beta = 1 - 1/(2\alpha)$ (parameter $\kappa$ can be incorporated into definition of the flux function
$g$ (Uzdensky 2003, Eq. (41)), so we can set $\kappa = 1) and matching the powers of distance from the
$Y$-point we find (Uzdensky 2003, Eq. (41))

$$g'' \cos \theta - g' \sin \theta + \alpha (\alpha + 1) g \cos \theta = \begin{cases} 0, & \text{no twist} \\
-\eta^{1-1/\alpha}, & \text{self-similar twist} \end{cases}$$

Eq. (37) should be satisfied, with different values of $\alpha_{1,2}$, inside and outside the separatrix (below
we denote the inner part (within the separatrix with subscript 1 and outside part with subscript 2).

The structure of Eq. (37), and of the corresponding magnetic field lines, for the case of no
internal twist and finite twist (even a minuscule one) are very different - linear for zero twist and
nonlinear for finite twist. Thus, we expect a binomial transition for a magnetosphere switching
between the two states.

4.2. The force balance over the separatrix and the luminal condition near $Y$-point

The separatrix may carry a surface current; it cannot be described by the continuous Grad-
Shafranov equation. On the separatrix the force balance implies ($B^2 - E^2$ is the rest-frame Lorentz-
invariant magnetic field):

$$B^2_1 - E^2_1 = B^2_2 - E^2_2 \to (B^2_{p,1} - B^2_{p,2}) (1 - x^2) = \frac{I^2_2 - I^2_1}{x^2} = \Delta(I^2)$$

where $I_{1,2}$ are total current flowing within the inner and the outer side of the separatrix. (Again,
the inner total current $I_1$ is determined by the foot-points twist, while the outer total current is
determined (mostly) by the current distribution on the open field lines.)

Near the $Y$-point, $x \to 1$, this takes the form

$$2 \left( B^2_{p,1} - B^2_{p,2} \right) (1 - x) = \Delta(I^2)$$

where $I_{1,2}$ are currents (times $2\pi$) flowing within the corresponding sides of the separatrix. The
separatrix may carry poloidal surface currents, both rotationally and twist-generated. In other
words, the toroidal magnetic field may experience a jump at the separatrix since internally generated
magnetic field twist may not/does not match the rotationally -generated twist on the open
Fig. 13.— Current structure near the Y-point. Pulsar is on the left. The separatrix is the border between closed and open field lines. Volumetric rotationally-induced poloidal current $j$ flows on the open field lines; in addition, there is a rotationally-induced return surface current $I_2$. Closed field lines also carry twist induced poloidal current $I_1$.

Thus, generally, $\Delta(I^2) \neq 0$; it can be both negative and positive. This is the most important fact for what follows.

In addition to the force-balance equation, the fields should also satisfy what we below call the luminal condition: the electromagnetic invariant $B^2 - E^2$ should be $> 0$. This is an additional, non-dynamic constraint on the properties of the magnetosphere.

To take correct account of the luminal constraint, let’s treat poloidal $B_p$ and toroidal $B_\phi$ components of the magnetic field separately. The electric field is $E = xB_p$.

The sub-luminal condition requires

$$B^2 - E^2 > 0 \Rightarrow B_p^2(1 - x^2) + B_\phi^2 > 0$$  \hspace{1cm} (40)
Importantly, the equations (39)-(40) have several qualitatively different cases.

- **Case 1.** \( I_1 \equiv 0, I_2 \equiv 0 \). (No twist on the closed field lines and all the rotationally-induced current closes within the open field lines). In this case \( B_\phi = 0 \) and condition (40) is violated near the light cylinder (Uzdensky 2003; Gruzinov 2005). We discard this case.

- **Case 2.** \( I_1 \equiv 0, I_2 \not\equiv 0 \) (this is conventionally assumed - no twist on close field lines, rotationally -induced surface current on the separatrix). Then \( \Delta I^2 > 0 \), so that for \( x < 1 \) we must have \( B^2_{p,1} > B^2_{p,2} \). If poloidal field outside reaches some finite limit \( B_{p,2} \to B^{(0)}_{p,2} \), then \( B_{p,1}, E_1 \propto 1/\sqrt{1-x} \) - magnetic field inside the \( Y \)-point experiences weak divergence (Gruzinov 2005)

- **Case 3.** \( I_1 < I_2 \not\equiv 0 \). (Under-twisted magnetosphere). Qualitatively, this case is similar to Case 2, though the structure of the \( Y \)-point is somewhat different, see §4.3

- **Case 4.** \( I_1 = I_2 \not\equiv 0 \). In this case \( B_\phi \not\equiv 0 \) so that the condition (40) is not violated near light cylinder. Both poloidal and toroidal fields match smoothly, \( B_{p,1} = B_{p,2} \). This case requires that rotationally-generated current flow \( I_2 \) and the corresponding toroidal magnetic field matches precisely to twist-generated current \( I_1 \). This is generally not expected (as we argue below this case is dynamically similar to Case 5, not case 3, see §4.4).

- **Case 5.** \( I_1 > I_2 \not\equiv 0 \). (Over-twisted magnetosphere.) In this case, the twist-generated current \( I_1 \) exceeds near the separatrix the rotationally-generated current \( I_2 \), \( \Delta I^2 < 0 \), so that the closed field lines are over-twisted with respect to the open field lines. The luminal condition is satisfied. At \( x < 1 \) we must have \( B^2_{p,1} < B^2_{p,2} \). Thus, either,
  
  - \( B_{p,1} \) reaches a finite values, \( B_{p,1} \to B^{(0)}_{p,1} \), while \( B_{p,2} \) diverges, \( B_{p,2} = \Delta I^2 / (2\sqrt{1-x}) \)
  
  - they both diverge, yet \( B_{p,2} \) remains larger than \( B_{p,1} \). In this case each \( B_{p,1} \) and \( B_{p,2} \) can experience arbitrary divergence, yet their difference \( B^2_{p,2} - B^2_{p,1} \propto 1/(1-x) \).

Importantly, in the above relations the sign of the internal twist is not important - anti-twisting of the internal field (with respect to the rotationally induced twist) produces the same dynamical instability as twisting.

### 4.3. Under-twisted magnetosphere: Cases 2 & 3

For the inside under-twisted solution \( \alpha_1 = 1/2 \) (Gruzinov 2005). In this case Eq. (37) becomes

\[
g'' \cos \theta - g' \sin \theta + \frac{3}{4} g \cos \theta = \begin{cases} 0, & \text{no twist} \\ -g^{-1}, & \text{self-similar twist} \end{cases} \tag{41}
\]
It can be rewritten in terms of the new variable \( h \) defined as \( \tanh(h/2) = \tan(\theta/2) \)

\[
g'' + \frac{3}{4} \text{sech}^2 h g = \left\{ \begin{array}{c} 0 \\
-\frac{\text{sech} h}{g} \end{array} \right\}
\]

(42)

where the top row corresponds to the untwisted magnetosphere.

In the case of no twisted inside the magnetosphere, Eq. (42) has solutions in terms of Legendre polynomials \( P_{\frac{1}{2}}(\sin(\theta)) \) and \( Q_{\frac{1}{2}}(\sin(\theta)) \). Using boundary conditions \( g'(0) = 0 \) (by symmetry, in the equatorial plane \( B_r = 0 \); since Eq. (37; the other constant is just an overall normalization) we find the solution for the structure of the magnetic flux function inside the \( Y \)-point:

\[
g = P_{\frac{1}{2}}(\sin(\theta)) + \frac{2\sqrt{2\pi} \Gamma \left( \frac{3}{4} \right)}{3Q_{\frac{1}{2}}(0)\Gamma \left( \frac{3}{4} \right)} Q_{\frac{1}{2}}(\sin(\theta))
\]

(43)

The separatrix, \( g(\theta_0) = 0 \) is located at \( \theta_0 = 1.3407 \) (Gruzinov 2005).

In the case of under-twisted inside magnetosphere, \( \Delta I > 0 \), the equations governing the structure of the \( Y \)-point Eqns. (41) or (42), bottom row) are inside the separatrix as well. The inner power-law index is \( \alpha_1 = 1/2 \) (Gruzinov 2005). One boundary condition remains the same, \( g'(0) = 0 \), while the other determines the overall twist of the closed field lines. Near the \( Y \)-point, in the equatorial plane, we have \( B_\theta \propto g(0) \) while \( B_\phi \propto I \propto g(0)^{1-1/\alpha} \propto g(0)^{-1} \). Thus,

\[
\frac{B_\phi}{B_\theta} \propto g(0)^{-2}
\]

(44)

The value \( g(0) \) can be considered as a twist parameter of the closed field lines. Since the analysis of the \( Y \)-point is local, in order to extend it through the whole magnetosphere one needs to know how the twist is distributed at the surface. The parameter \( g(0) \) controls the overall twist: for \( g(0) \to \infty \) there is no twist, while for \( g(0) \leq 1 \) the toroidal field at the \( Y \)-point dominates over poloidal.

We solve Eq. (41) numerically for various values of \( g_0 \) and find the separatrix angle \( \theta_0 \), so that \( g(\theta_0) = 0 \), Fig. 14. With the increasing inside twist the separatrix angle decreases.

For a given separatrix angle we can calculate the external structure, parameter \( \alpha_2 \), Fig. 16. The untwisted magnetosphere corresponds to \( \alpha_2 = 2.398 \). For large inside twist \( \alpha_2 \) decreases.

### 4.4. Case 4: matching twists, \( I_{1,2} \neq 0, \Delta I = 0 \).

We are dealing with Eq. (37) bottom line. Since the force-balance across the separatrix should be satisfied for any distance from the \( Y \)-point, this implies that \( \alpha_1 = \alpha_2 \).

Our experience with the under-twisted case tells us that for larger twists both the separatrix angle and the radial power-law index become smaller. Yet for finite twists the separatrix angle
Fig. 14.— *Left Panel*. Separatrix angle as function of the inside twist parameter $g_0$ (larger $g_0$ correspond to smaller twist. Dashed line is the asymptotic limit of no twist, $\theta_0 = 1.3407$.

remains larger than unity and the power-law index remains larger than one. Now, for a given
e external current there is a value of the internal twist parameter $g_0$ such that the internal and external
twists match. This happens at some finite value of the separatrix angle. Thus, we expect that
beyond some critical twist the system will become unstable. This is seen in numerical simulations
(Parfrey et al. 2013).

For the critically twisted state we require $g'(0) = 0$ (vertical field on the equator in the closed
zone), $g(\pi) = 0$ (no vertical field in the open zone), smooth $g(\pi/2)$. These three condition on a
second order equation impose an eigenvalue problem. This problem is different from Uzedensky’s
since he assumed no current in the closed zone. This is also different from previous, Case 3, since
there we assumed particular $\alpha_1 = 1/2$ for the closed zone. After considerable failed efforts to find
a proper solution, we conclude that the matching twist is unstable.
Fig. 15.— Poloidal magnetic fields near the $Y$-point for $g_0 = \infty, 1, 10^{-0.5}$. The vertical dashed line is the light cylinder, the oblique dashed line is the separatrix, negative values of the abscissa correspond to the closed field lines. Small bends of field lines at the light cylinder, especially further away from the $Y$-point, are due to imprecise numerical eigenvalue $\alpha_2$ and due to the fact that the analysis is local.

4.5. Case 5: over-twisted inside magnetosphere, $\Delta I > 0$.

Now $\alpha_2 = 1/2$. Importantly, the boundary condition $g(\pi) = 0$ cannot be satisfied (this can be demonstrated, for example, by attempting to expand near $\theta = \pi$, and requiring $g \propto (\theta - \pi)^c_1$ with $c_1 > 1$; Eq. (41, bottom row) cannot be satisfied. Thus, we conclude that there are no stationary solutions for the over-twisted case: solutions found in §4.3 are, in fact unstable beyond some critical internal twist.
Fig. 16.— The external radial dependence index $\alpha_2$ as a function of the internal twist parameter $g_0$. The dashed line corresponds to the untwisted dipole, $g_0 \to \infty$, $\alpha_2 = 2.398$. For large internal twists, $g_0 \leq 0$, the numerical procedure becomes less stable due to stiffness of the corresponding equations. The eigenvalue searches are computationally expensive, hence a fairly large errors at small $g_0$ (large twists).

4.6. Numerical eigenvalue procedure

For a given separatrix angle $\theta_0$ we solve Eq. (41) with conditions $g(\theta_0) = g(\pi) = 0$. The major numerical problem is a point $\theta = \pi/2$, where the equation becomes very stiff and generally has a singularity. To avoid divergences we shift the line of integration by a small complex value, $\theta \to \theta + i\epsilon$. For $\epsilon \ll 1$ the equation still remains very stiff, producing a large jump in the value of $g$ near the $\theta = \pi/2$ point. As an eigenvalue procedure we seek to minimize $\Delta = |g(\theta = \pi/2 + \epsilon) - g(\theta = \pi/2 - \epsilon)|$. Since spurious solutions often appear, Fig. 17, a very fine resolution in the search for $\alpha_2$ is required.
Fig. 17.— Left Panel: The eigenvalue procedure for the untwisted case. Plotted is $\Delta = |g(\theta = \pi/2 + \epsilon) - g(\theta = \pi/2 - \epsilon)|$. Though spurious points with small $\Delta$ often appear, there is a clear physical eigenvalue $\alpha_2 = 2.3981$. Right Panel: The function $g(\theta)$ in the outer region

4.7. Conclusion: instability of the Y-point for large internal twists

We argue that the Y-point play an important role in generation of burst and timing behavior. The appearance of the Y-point is due to rotation. Thus, both rotationally-induced and the foot-point-shear-induced twists are important. If the inside current $I_1$ (due to crustal twist) is larger than the outside current $I_2$ (required by the electrodynamics of the open field lines) the system looses stability - such solutions do not exist.

Magnetar bursts and flares should involve restructuring of the whole closed magnetosphere, and not just of a part close to the Y-point. Consider an axially symmetric magnetic flux tube that has both toroidal and poloidal magnetic field. Typically it will have $B_\phi/B_p \propto r$, so that the twist is largest further out. Suppose that at the light cylinder the twist is of the order of unity, $B_\phi/B_p|_{LC} \sim 1$, and that the flux tube has a radial cross-section as a fraction $\eta_r$ of the light cylinder, $\Delta r = \eta_r R_{LC}$. In an approximately dipolar configuration then the total energy in the toroidal field within the bundle, $E_{tot} \approx B^3_{p,LC} R_{LC}^3 / \eta_r$ is too small to explain bursts and flares. We hypothesize that the resolution of the Aly-Sturrock paradox in solar flares suggested by Antiochos et al. (1999) (that topological changes within the lower magnetosphere play the key role) may be important for magnetar flares as well.

The stability of the Y-point discussed above applies to the North-South symmetric configurations. We hypothesis that similar effects, presence of maximal twist, occurs in the angled configurations as well. Analytical considerations of the angled Y-point are unrealistic in our view, as the whole location of the Y-point cannot be simply established.

5. Discussion

We analyzed the structure of globally twisted axially-symmetric, aligned magnetospheres of rotating neutron stars. We show that the outside axisymmetric current sheets is angled: North-
South symmetry is broken. The resulting structure of the magnetospheres can be called “angled-twisted-rotated”: twisted in a sense of static twist of Thompson et al. (2002), rotated in a sense of Goldreich & Julian (1969), and angled to distinguish from the case flat current sheet (for aligned dipole).

The North-South asymmetry is due to causal connection between the open and closed field regions. Rotationally-induced toroidal magnetic fields have opposite direction in the two hemispheres, while the twist-induced toroidal magnetic fields have the same directions. In numerical simulations the amount of angling depends on how twist and rotation are introduced: yet various schemes lead to qualitatively similar results: twisted rotating magnetospheres are angled.

To get angled two open parts of the magnetosphere must “talk” to each other – be in causal contact, even though only previously. Under assumption of complete causal disconnected the angling will not develop. In some schemes $j_{\text{GJ}}$ and $j_{\text{twist}}$ may be completely isolated. This is not the case, as we argue in this paper: realistically, through memory and/or perturbations the two open parts of the magnetosphere will be in some contact - they have thousands/millions of years to get “talking”.

The tilt angle clearly scales with the twist angle, $\theta_c \propto (\Delta \phi)$. Somewhat surprisingly, our numerical experiments show that the tilt angle in mostly independent of the spin (the $R_{\text{LC}}/R_{\text{NS}}$ ratio. Since the maximal twist-induced current exceeds the Goldreich-Julian (GJ) current by a large number (up to $\sim R_{\text{LC}}/R_{\text{NS}} \gg 1$), it was initially expected that asymmetry does depend on this ratio. The implication is that the tilt is controlled exclusively by the region close to the Y-point, where the ratios of the twist-induced and spin-induced currents and magnetic fields depend only on $(\Delta \phi)$. It is not clear if this is due to our limited range of $\Omega$, or is a genuine effect.

The resulting wind is North-South anisotropic and leads to linear acceleration of a neutron star along the axis of rotation. Linear acceleration has only minor effect on the estimate of the surface magnetic field. The final maximal velocity may reach only $\sim$ hundreds kilometers per second (in case of no flux emergence); it is also independent of the magnetic field, as higher magnetic fields provide larger force, but act for shorter spindown time. On the other hand, persistent acceleration from the electromagnetic rocket may dominate the gravitational acceleration.

We also hypotheses that twist-induced distortions of current sheets may also have implications for modeling Fermi $\gamma$-ray light curves. It also may affect the shape/dynamics of the Pulsar Wind Nebulae.

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A. Grid resolution, time convergence and effects of outer boundary

We performed a number of simulations to investigate the dependence of the results on the grid resolution, simulation parameters, initial conditions and time-convergence.

In figure 18 we compare various parameters calculated for low and high resolution simulations. In table A, we compare the absolute values of tilt angle $\theta_c$, normalized spin-down luminosity $\frac{L_{sd}}{L_{sd,0}}$, and the normalized axial force $\frac{F_z}{L_{sd,0}}$. Results are in excellent agreement: a fine mesh is not needed, the PHAEDRA code is very efficient.

| Grid Resolution | $\theta_c$ | $\frac{L_{sd}}{L_{sd,0}}$ | $\frac{F_z}{L_{sd,0}}$ |
|-----------------|------------|---------------------------|------------------------|
| 256 x 200       | 0.102      | 1.01                      | 0.075                  |
| 320 x 256       | 0.092      | 0.985                     | 0.060                  |

Table 1: Comparison of low and high resolution simulation runs.

One of the key issues is whether the effect we observed is transient. In the aligned case, the basic features of the current flow are established on $\sim$ one rotation time scale, by the final configuration is reached on much longer times. Can obliquity of the current sheet be a transient effect? We run a number of tests to test this possibility.
Fig. 18.— Poynting flux for $C = 0.1$ as a function of polar angle for two runs with different grid size. The blue curve for grid size of $256 \times 200$, the green curve is for $320 \times 256$ grid.

The time evolution of the configuration, Fig. 19, is reminiscent of the aligned case (Parfrey et al. 2013; Spitkovsky 2006; Philippov et al. 2015). The overall structure is established quickly, on few gyration time scales. But the final state is then reached slowly on much linger times scales. We believe this is the origin of double Y-point configuration, see Fig. 2. With longer integration it is expected that the two Y-points will merge.

We tested if the tilt decrease with time. After running for $\sim 10$ periods we do not see a decrease in $\theta_c$, Fig. 20.

Especially dangerous are the errors at the outer boundary due to imperfect matching of absorbing PML. Feasibly this can keep supporting the initial magnetospheric twist. To test importance of this effect, we implemented various schemes when the current is “killed” at some radius. This requires matching the poloidal fields in the current-carrying and no-current regions (the correspond-
Fig. 19.— Time evolution of $j_\phi$ for $C=0.01$. This figure demonstrates that on the one hand the tilt is established very quickly, on few dynamical times, while the overall evolution towards the final states takes much longer (e.g., the size of the closed field lines slowly increases. Frames correspond to times of 0.63, 2.2, 4.8, 6.4 of rotation periods.

In the no-current region are not dipolar). Since the procedure is artificial, it introduced some $\text{div } B \neq 0$. We designed the scheme to minimize this according to the following procedure:

- in the twisted zone the fields are given by (26).
- in the poloidal zone we set $C = 0$; the poloidal field has contribution both from higher multipole moments
- matching the external $A_\phi = F/(r \sin \theta)$, Eq. (25), gives in the external region

$$
B_r \propto \frac{2 \cos(\theta)}{r^3} + \frac{3 \sin^2(\theta) \cos(\theta)(21 - 5 \cos(2\theta))}{140r^3} C
$$

$$
B_\theta \propto \frac{\sin(\theta)}{r^3} + \frac{\sin^3(\theta)(29 - 5 \cos(2\theta))}{280r^3} C
$$

(A1)

- We implemented various procedures in an attempt to minimize $\text{div } B$ at the transition. To ensure the continuity of $B_r$, we used a simple linear interpolation method to match $B_r$ at the interface of twisted and poloidal zone. The transition radius was chosen to be twice the radius of light cylinder ($R_{LC}$).

Our conclusion is that there isn’t any observable difference among the results generated via above procedure and our usual approach with only twisted fields in the entire simulation region, Fig. 21.a.

We also did a simulation with the outer boundary moved to large distances, $R = 100$. The twist persists, Fig. 21.

Finally, we implemented a different switch-on scheme, similar to one used by Parfrey et al. (2013). We start with purely dipolar fields (so that the boundary does not initially “know” about
Fig. 20.— Time evolution (smoothed) of the tilt angle $\theta_c$, $C = 0.1$, $\Omega = 0.2$. Time is measured in rotational periods, up to $\sim 10$. No decrease in the tilt angle with time is observed.

The twist. We then dynamically introduce a twist according to $\omega(\theta) = \omega_0 \sin(\theta) \cos(\theta)$. Simulation run begins with no twisting ($\omega_0 = 0$) and after $\sim 1$ time-period, twist angular velocity $\omega$ vector is superimposed on the star rotational velocity $\Omega$, causing a global shearing of magnetic field lines.

Thus, all the tests seem to indicate that the structure is steady, and is not an artifact of initial condition, or imperfect matching at the boundary.

B. Twisted force-free equilibria

Lynden-Bell & Boily (1994); Wolfson (1995); Thompson et al. (2002) (see also Gourgouliatos (2008); Pavan et al. (2009); Lyutikov (2020)) discussed nonlinear force-free self-similar solutions. In spherical symmetry, for static force-free configuration, the poloidal and toroidal magnetic fields
Fig. 21.— Left panel: killing the twist before the boundary: toroidal current \( j_\phi \) for twist parameter \( C = 0.01 \Omega = 0.3 \), and \( r_{\text{transition}} = 6.6 \); initial conditions have no twist for \( r > r_{\text{transition}} \). Center and Right panel: moving the boundary far out to \( R = 100 \) (right panel) \((C = 0.1 \Omega = 0.2, \) cells sizes modified to \( N_r/N_\theta = 1.7, \) sponge layer starts at 80.) The tilt of the current sheet is clearly seen; it is not affected much by the location of the boundary layer.

are

\[
\mathbf{B} = \frac{\nabla \Psi \times \hat{\phi}}{r \sin \theta}
\]

\[
I = I(\Psi) \text{ is the total current enclosed by the magnetic flux surfaces } \Psi. \text{ Choosing separable solutions, } \Psi \propto r^{-p} \mathcal{F}(\mu), \text{ and assuming that the current function is a power-law, } I \sim \Psi^{(1+p)/p}, \text{ the magnetospheric structure depends on the current parameter } p(C) \text{ as an eigenvalue problem:}
\]

\[
\mathcal{F}'(1) = -2
\]

\[
\mathcal{F}(1) = 0
\]

\[
\mathcal{F}'(0) = 0
\]

(fixed radial polar field, \( B_\theta(\theta = 0) = 0 \) and \( B_r(\theta = \pi/2) = 0 \) correspondingly) (Thompson et al. 2002), Fig. 23. Fields are given by

\[
\mathbf{B} = \left\{ -\mathcal{F}', \frac{p}{\sqrt{1 - \mu^2}} \mathcal{F}, \sqrt{\frac{Cp}{(1 + p)(1 - \mu^2)}} \mathcal{F}^{1+1/p} \right\} r^{-2-p}
\]
Fig. 22.— Toroidal magnetic field times the radius ($rB_\phi$) for dynamical twisting starting with pure dipole and profile of footpoints’ motion given by $\omega(\theta) = \omega_0 \sin(\theta) \cos(\theta)$ at $\sim 7$ rotation period of the star; $\Omega = 0.2$, $\omega_0 = 0.1$. We also observe periodic plasmoid ejection. This further demonstrates that angling is not an artifact of initial conditions.

$$p(1 + p)\mathcal{F} + C\mathcal{F}^{(2+p)/p} + (1 - \mu^2)\mathcal{F}'' = 0$$
$$\mathcal{F}'(1) = -2, \mathcal{F}(1) = 0, \mathcal{F}'(0) = 0$$
$$\text{curl \mathbf{B}} = \alpha \mathbf{B}$$
$$\alpha = \sqrt{\frac{C(1+p) \mathcal{F}^{1/p}}{p}}$$

In accordance with Aly (1991) theorem, the maximal magnetic energy is reached at open field configurations, Fig. 24 (though different configurations have different flux distribution on the surface, hence Aly’s theorem is not formally applicable). Thus, ideally (without effects of resistivity) all configurations are stable to opening to infinity.

We also mention here that for $p \to 0$, the structure approaches monopolar with

$$\mathcal{F} = 2(1 - \mu)$$
Fig. 23.— Eigenvalue problem (B2)-(B3) for twisted magnetosphere (no rotation) and the small current approximation: the plot shows $p$ (radial scaling) as function of $C$ (twist current). Dashed line is the analytical approximation $p = 1 - 8C/35$. This illustrates that the small $C$ limit is applicable to a fairly large $C \sim 0.5$.

\[ C = \frac{1 + p}{p} 2^{-2/p} \]  

(B4)
Fig. 24.— Energy content of twisted magnetospheres as function of parameter $p$. Fixed polar magnetic field, energy normalized to dipole field. For $p \to 0$ (split monopole, $E_B \to 6$.

The eigenvalue problem (B2)-(B3) can be modified for angled magnetosphere,

$$
\mathcal{F}'(1) = -2 \\
\mathcal{F}(1) = 0 \\
\mathcal{F}'(\mu_t) = 0
$$

where $\mu_t = \cos(\pi/2 - \theta_t)$ is the inclination angle of the magnetosphere, Fig. 25. These solutions must have different normalizations to ensure zero total magnetic flux, and thus will experience a jump of the magnetic field at $\theta_t$. Thus, the internal structure of the twisted and rotating magnetospheres cannot be self-similar.
Fig. 25.— Dependence of radial scaling $p$ on the current parameter for different values of the tilt angle $\theta_t = 0, 0.025, 0.05, 0.1$. Dashed curve is for $\theta_t = 0$; curves to the right of the dashed one correspond to zone 1, those to the left: to zone 2.