Deformed multiplication in the semigroup $\mathcal{PT}_n$

Galyna Tsyaputa

Abstract

Pairwise non isomorphic semigroups obtained from the semigroup $\mathcal{PT}_n$ of all partial transformations by the deformed multiplication proposed by Ljapin are classified.

1 Introduction

Let $X$ and $Y$ be two nonempty sets, $S$ be the set of maps from $X$ to $Y$. Fix some $\alpha : Y \to X$ and define the multiplication of elements in $S$ in the following way: $\phi \circ \psi = \phi \alpha \psi$ (the composition of the maps is from left to right). The action defined by this rule is associative. In his famous monograph [1, p.353] Ljapin set the problem of investigation of the properties of this semigroup depending on the restrictions on set $S$ and map $\alpha$.

Magill [2] studies this problem in the case when $X$ and $Y$ are topological spaces and the maps are continuous. In particular under the assumption that $\alpha$ be onto he describes the automorphisms of such semigroups and determines the isomorphism criterion.

A bit later Sullivan [3] proves that if $|Y| \leq |X|$ then Ljapin’s semigroup is embedding into transformation semigroup on the set $X \cup \{a\}$, $a \notin X$.

For us the important case is if $X = Y$, $T_X$ is a transformation semigroup on the set $X$, $\alpha \in T_X$. Symons [4] establishes the isomorphism criterion for such semigroups and investigates the properties of their automorphisms.

The latter problem may be generalized to arbitrary semigroup $S$ : for a fixed $a \in S$ define the operation $*_a$ via $x *_a y = xay$. The obtained semigroup is denoted by $(S,*_a)$ and operation $*_a$ is called the multiplication deformed by element $a$ (or just the deformed multiplication).

In the paper we classify with respect to isomorphism all semigroups which are obtained from the semigroup of all partial transformations of an $n$—element set by the deformed multiplication.

2 $\mathcal{PT}_n$ with the deformed multiplication

Let $\mathcal{PT}_n$ be the semigroup of all partial transformations of the set $N = \{1,2,\ldots,n\}$. For arbitrary partial transformation $a \in \mathcal{PT}_n$ denote by $\text{dom}(a)$ the domain of $a$, and by $\text{ran}(a)$ its image. The value $|\text{ran}(a)|$ is called the rank of $a$ and is denoted by $\text{rank}(a)$. Denote
Z(a) = N \setminus \text{dom}(a) \text{ the set of those elements on which transformation } a \text{ is not defined, and denote } z_a \text{ the cardinality of this set. The type of } a \in \mathcal{PT}_n \text{ we call the set } (\alpha_1, \alpha_2, \ldots, \alpha_n), \text{ where } \alpha_k \text{ is the number of those elements } y \in N \text{, the full inverse image } a^{-1}(y) \text{ of which contains exactly } k \text{ elements. It is obvious that } 1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \cdots + n \cdot \alpha_n = |\text{dom}(a)|, \text{ and the sum } \alpha_1 + \alpha_2 + \cdots + \alpha_n \text{ is equal to the rank of } a.

Definition 1. Element } x \in S \text{ is called left (right) annihilator of semigroup } S^n \text{ provided that } xs = 0 (sx = 0), s \in S.

Element which is both the left and the right annihilator is called the annihilator of semigroup } S.

Proposition 1. Transformations } x \in \mathcal{PT}_n \text{ such that } \text{ran}(x) \subset Z(a) \text{ are left annihilators of semigroup } (\mathcal{PT}_n, *_a). \text{ The number of left annihilators equals } (z_a + 1)^n.

Transformations } y \in \mathcal{PT}_n \text{ such that } Z(y) \supset \text{ran}(a) \text{ are right annihilators of semigroup } (\mathcal{PT}_n, *_a). \text{ The number of right annihilators equals } (n + 1)^{n - \text{rank}(a)}.

Transformations } c \in \mathcal{PT}_n \text{ such that } \text{ran}(c) \subset Z(a) \text{ and } Z(c) \supset \text{ran}(a) \text{ are annihilators of } (\mathcal{PT}_n, *_a), \text{ moreover, the number of annihilators equals } (z_a + 1)^{n - \text{rank}(a)}.

Proof. Let } x \in \mathcal{PT}_n \text{ be left annihilator of } (\mathcal{PT}_n, *_a). \text{ This means that for arbitrary } u \text{ from } \mathcal{PT}_n \text{, } x *_a u = 0 \text{ or, what is the same, } xa \cdot u = 0. \text{ Therefore } xa \text{ is left zero of semigroup } \mathcal{PT}_n, \text{ that is, a nowhere defined map. The latter is possible if and only if } \text{ran}(x) \subset Z(a). \text{ Evidently, the condition } \text{ran}(x) \subset Z(a) \text{ is sufficient for partial transformation } x \text{ be the left annihilator of } (\mathcal{PT}_n, *_a). \text{ Now it is clear, that the number of such transformations is equal to } (z_a + 1)^n.

Let } y \in \mathcal{PT}_n \text{ be the right annihilator of semigroup } (\mathcal{PT}_n, *_a). \text{ Then for any } v \text{ from } \mathcal{PT}_n \text{, } v \cdot ay = 0 \text{ and } ay \text{ is right zero of semigroup } \mathcal{PT}_n, \text{ that is, nowhere defined map. This is equivalent to } Z(y) \supset \text{ran}(a). \text{ Therefore right annihilators of semigroup } (\mathcal{PT}_n, *_a) \text{ are those partial transformations } \mathcal{PT}_n \text{ which are defined only in elements } N \setminus \text{ran}(a). \text{ It is clear that the number of such transformations equals } (n + 1)^{n - \text{rank}(a)}.

Now the statement about annihilators follows from the definition and the above arguments. \hfill \square

On semigroup } (\mathcal{PT}_n, *_a) \text{ define the equivalence relation } \sim_a \text{ by the rule: } x \sim_a y \text{ if and only if } x *_a u = y *_a u \text{ for all } u \in \mathcal{PT}_n. \text{ Analogously on } (\mathcal{PT}_n, *_b) \text{ define the relation } \sim_b.

Lemma 1. For an arbitrary isomorphism } \varphi: (\mathcal{PT}_n, *_a) \to (\mathcal{PT}_n, *_b) \text{ } \varphi(x) \sim_b \varphi(y) \text{ if and only if } x \sim_a y.

Proof. In fact, let } \varphi: (\mathcal{PT}_n, *_a) \to (\mathcal{PT}_n, *_b) \text{ be isomorphism and } x \sim_a y. \text{ Then for all } u \in \mathcal{PT}_n; x *_a u = y *_a u, \text{ therefore } \varphi(x) *_b \varphi(u) = \varphi(y) *_b \varphi(u). \text{ However } \varphi(u) \text{ runs over the whole set } \mathcal{PT}_n, \text{ hence } \varphi(x) \sim_b \varphi(y). \text{ Since the inverse map } \varphi^{-1}: (\mathcal{PT}_n, *_b) \to (\mathcal{PT}_n, *_a) \text{ is also isomorphism, } \varphi(x) \sim_b \varphi(y) \text{ implies } x \sim_a y. \text{ Therefore, } x \sim_a y \text{ if and only if } \varphi(x) \sim_b \varphi(y). \hfill \square

Lemma 2. } x \sim_a y \text{ if and only if } xa = ya.
Proof. Obviously, the equality \( xa = ya \) implies \( x \sim_a y \). Now let \( xa \neq ya \). Then there exists \( k \) in \( N \), such that \( (xa)(k) \neq (ya)(k) \). Chose element \( u \) in \( PT_n \) which has different images in the points \( (xa)(k) \) and \( (ya)(k) \). Then \( x \ast_a u = xau \) and \( y \ast_u u = yau \) have different images in \( k \). Hence \( x \ast_a u \neq y \ast_u u \) and \( x \sim_a y \). \( \square \)

**Theorem 1.** Semigroups \( (PT_n, \ast_a) \) and \( (PT_n, \ast_b) \) are isomorphic if and only if partial transformations \( a \) and \( b \) have the same type.

**Proof.** Necessity. Let \( (PT_n, \ast_a) \) and \( (PT_n, \ast_b) \) be isomorphic. By lemma 1 arbitrary isomorphism between \( (PT_n, \ast_a) \) and \( (PT_n, \ast_b) \) maps equivalence classes of the relation \( \sim_a \) into equivalence classes of the relation \( \sim_b \). Therefore for equivalence relations \( \sim_a \) and \( \sim_b \) cardinalities and the number of equivalence classes must be equal. We show that by the cardinalities of the classes of the relation \( \sim_a \), the type \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) of transformation \( a \) can be found uniquely.

Denote \( \rho \) the partition of set \( \{1, 2, \ldots, n\} \) induced by \( a \in PT_n \) (that is, \( l \) and \( m \) belong to the same block of the partition \( \rho \) provided that \( a(l) = a(m) \); \( Z(a) \) forms a separate block). We count the cardinality of equivalence class \( \overline{x} = \{ x | xa = x_0a \} \) of the relation \( \sim_a \) for the fixed transformation \( x_0 \in PT_n \). Consider element

\[
y := x_0a = \begin{pmatrix} i_1 & i_2 & \cdots & i_p & i_{p+1} & \cdots & i_n \\ y_1 & y_2 & \cdots & y_p & \emptyset & \cdots & \emptyset \end{pmatrix}.
\]

Obviously \( y_i \) belongs to the image of \( a, i = 1, \ldots, p \). Denote \( N_a(a_i) \) the block of the partition \( \rho \) which is defined by \( a_i \) in the image of \( a \). Denote \( n_a(a_i) \) the cardinality of this block. The equality \( xa = y \) is equivalent to that for every \( i \) \( (xa)(i) = y_i \), or, what is the same, \( x(i) \in N_a(y_i) \). Hence \( x(i) \) can be chosen in \( n_a(y_i) \) ways, \( i = 1, \ldots, p \). For \( i = p + 1, \ldots, n \) the meaning of \( x(i) \) can be chosen in \( z_a + 1 \) ways. Since the images of \( x \) in different points are chosen independently, partial transformation \( x \) can be defined in

\[
n_a(y_1)n_a(y_2)\cdots n_a(y_p)(z_a + 1)^{n-p}
\]

ways, and this gives the cardinality of class \( \overline{x_0} \).

It is clear that \( \overline{x} \) is the set of all left annihilators of the semigroup \( (PT_n, \ast_a) \). By proposition 1 the cardinality of class \( \overline{x} \) equals \( (z_a + 1)^n \). Hence the value \( z_a \) is defined by abstract property the semigroup \( (PT_n, \ast_a) \). Denote \( m \) the smallest cardinality of the blocks of the partition \( \rho \). Then the cardinality of the equivalence class of the relation \( \sim_a \) is the least if in (1) all multipliers are equal to \( \min(m, z_a + 1) \). Consider the corresponding cases.

Let \( m > z_a + 1 \). Then class \( \overline{x} \) is the only equivalence class of the least possible cardinality. The next larger class contains \( m(z_a + 1)^{n-1} \) transformations. Now count the number of different equivalence classes of the relation \( \sim_a \) which have the cardinality \( m(z_a + 1)^{n-1} \). Since \( m > z_a + 1 \) and this is the least of the cardinalities of the blocks of the partition \( \rho \), to make the product in (1) be equal to \( m(z_a + 1)^{n-1} \) there should be \( p = 1 \) and \( n_a(y_1) = m \). Element \( i_1 \) can be chosen in \( n \) ways. We know that \( |\{t : n_a(t) = m\}| = \alpha_m \), so \( y_1 \) can have \( \alpha_m \) different meanings. Therefore, there are \( n \cdot \alpha_m \) different transformations
\[ y = x_0a, \text{ such that the corresponding class } \bar{x}_0 \text{ contains } m(z_a + 1)^{n-1} \text{ elements. Therefore by the relation } \sim_a \text{ we may define the number } m \text{ and the meaning } \alpha_m \text{ of the first nonzero component of the type } (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ of element } a. \text{ Components } \alpha_l \text{ for } l > m \text{ can be defined recursively. Assume that } \alpha_1, \alpha_2, \ldots, \alpha_l \text{ are found. For the relation } \sim_a \text{ denote } C \text{ the number of equivalence classes of the relation } l \cdot (z_a + 1)^{n-1}. \text{ Then } C \text{ is equal to the number of sets } (p; i_1, i_2, \ldots, i_p), \text{ where some of } i_1, i_2, \ldots, i_p \text{ may coincide in general, such that }\]

\[ n_a(y_{i_1})n_a(y_{i_2})\ldots n_a(y_{i_p})(z_a + 1)^{n-p} = l \cdot (z_a + 1)^{n-1}. \tag{2} \]

Since \( \alpha_1, \alpha_2, \ldots, \alpha_{l-1} \) are known, we may find the number \( A \) of sets \((p; i_1, i_2, \ldots, i_p)\) such that all multipliers in the left hand side of (2) are less than \( l \). This value equals

\[ \sum_{k=1}^{n} \sum_{(m_1, \ldots, m_k)} \binom{n}{k} \prod_{j=1}^{k} \alpha_{m_j}, \quad m \leq m_1, \ldots, m_k \leq l, \quad m_1 \cdot m_2 \ldots m_k = l \cdot (z_a + 1)^{n-1}. \]

The number of sets \((p; i_1, i_2, \ldots, i_n)\) such that one of the multipliers in the left hand side of (2) equals \( l \), is equal to \( n \cdot \alpha_l \). Then \( \alpha_l \) can be found from the equality \( A + n \cdot \alpha_l = C \).

Now let \( m \leq z_a + 1 \). Then the cardinality of the equivalence class of the relation \( \sim_a \) is the least if in (1) all multipliers are equal to \( m \), that is, this cardinality equals \( m^n \). Count the number of different equivalence classes \( \bar{x}_0 \) of the relation \( \sim_a \) of the cardinality \( m^n \). To make all multipliers in (1) equal to \( m \), there should be \( n_a(y_i) = m \) for all \( i \). However \( |\{t : n_a(t) = m\}| = \alpha_m \). So every \( y_i \) can be chosen in \( \alpha_m \) ways. Since the meanings of \( y_i \) for different \( i \) are chosen independently, there are \( \alpha_m^n \) different \( y = x_0a, \) such that the corresponding class \( \bar{x}_0 \) contains \( m^n \) elements. Hence by the relation \( \sim_a \) we may define the number \( m \) and the meaning \( \alpha_m \) of the first non zero component of the type \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) of \( a \).

Components \( \alpha_l \) for \( l > m \) can be found recursively applying the same arguments as above. Some changes include the following: in (2) the right hand side must be substituted with \( l \cdot m^{n-1} \), and the number of sets \((p; i_1, i_2, \ldots, i_n)\), such that one of the multipliers in the left hand side of (2) equals \( l \), and other \( n-1 \) multipliers equal \( m \) is equal to \( n \cdot \alpha_l \cdot \alpha_m^{n-1} \) (if \( z_a + 1 \neq l \) then \( p = n \)). Therefore \( \alpha_l \) can be found from the equality \( A + n \cdot \alpha_l \cdot \alpha_m^{n-1} = C \).

Analogously we may find the type \((\beta_1, \beta_2, \ldots, \beta_n)\) of \( b \) via the cardinalities of the equivalence classes of the relation \( \sim_b \). Since for isomorphic semigroups \((\mathcal{PT}_n, *_a)\) and \((\mathcal{PT}_n, *_b)\) the number of the equivalence classes of the same cardinality coincide, and the values \( \alpha_k \) and \( \beta_k \), \( k = 1, \ldots, n \) are defined uniquely by the number of equivalence classes, for all \( k \) we have \( \alpha_k = \beta_k \), that is, elements \( a \) and \( b \) have the same types.

**Sufficiency.** Let elements \( a \) and \( b \) have the type \((\alpha_1, \alpha_2, \ldots, \alpha_n)\). Then there exist permutations \( \pi \) and \( \tau \) in \( S_n \) such that \( b = \tau_\alpha \pi \). The map \( f : (\mathcal{PT}_n, *_a) \to (\mathcal{PT}_n, *_b) \) such that \( f(x) = \pi^{-1}x\tau^{-1} \) defines isomorphism between \((\mathcal{PT}_n, *_a)\) and \((\mathcal{PT}_n, *_b)\). In fact \( f \) is...
bijection and 

\[ f(x *_a y) = \pi^{-1} x *_a y \tau^{-1} = \pi^{-1} x \tau^{-1} \pi \pi^{-1} y \tau^{-1} = \]

\[ = \pi^{-1} x \tau^{-1} b \pi^{-1} y \tau^{-1} = f(x) *_b f(y). \]

\[ \square \]

**Corollary 1.** Let \( p(k) \) denote the number of ways in which one can split positive integer \( k \) into non-ordered sum of the natural integers. Then there are \( \sum_{k=0}^{n} p(k) \) pairwise non-isomorphic semigroups obtained from \( \mathcal{PT}_n \) by the deformed multiplication.

**References**

[1] Ljapin Y.S., Semigroups, Moscow, Fizmatgiz, 1960 (Russian).

[2] Magill Kenneth D., Semigroup structures for families of functions. II. Continuous functions. // J. Austral. Math. Soc. 7 (1967), 95-107.

[3] Sullivan R.P., Generalized partial transformation semigroups. // J. Austral. Math. Soc. 19 (1975), part 4, 470-473

[4] Symons J.S.V., On a generalization of the transformation semigroup. // J. Austral. Math. Soc. 19 (1975), 47-61

[5] Artamonov V.A, Salij V.N., Skornyakov L.A. and others, General Algebra, Moscow, Nauka, 1991, vol. 1 (Russian).

Department of Mechanics and Mathematics,
Kiev Taras Shevchenko University,
64, Volodymyrska st., 01033, Kiev, Ukraine,
e-mail: gtsyaputa@univ.kiev.ua

*Given to the editorial board 30.09.2003*