Intersecting brane solutions in string and M-theory

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Abstract

We review various aspects of configurations of intersecting branes, including the conditions for preservation of supersymmetry. In particular, we discuss the projection conditions on the Killing spinors for given brane configurations and the relation to calibrations. This highlights the close connection between intersecting branes and branes wrapping supersymmetric cycles as well as special holonomy manifolds. We also explain how these conditions can be used to find supergravity solutions without directly solving the Einstein equations. The description of intersecting branes is considered both in terms of the brane worldvolume theories and as supergravity solutions. There are well-known simple procedures (harmonic function rules) for writing down the supergravity solutions for supersymmetric configurations of orthogonally intersecting branes. However, such solutions involve smeared or delocalised branes. We describe several methods of constructing solutions with less smearing, including some fully localised solutions. Some applications of these supergravity solutions are also considered – in particular the study of black holes and gauge theories.

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1 Outline

The aim of this review is to describe intersecting branes in ten and eleven dimensions. We restrict to configurations of branes which preserve supersymmetry. We begin in section 2 by reviewing the properties of the individual branes. There are, of course many other more detailed reviews and lectures notes covering this material such as [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and several books covering string theory such as [11, 12] and more recently also branes such as [13, 14] and particularly [15]. We first present the supergravity solutions for parallel branes and discuss the supersymmetry of these solutions. In particular we show how we can understand which supersymmetries should be preserved and how this enables us to derive the supergravity solutions without explicitly solving the Einstein equations. We also discuss the use of brane probes and the relation between branes (particularly D-branes) and gauge theories, including the AdS/CFT correspondence. The remaining sections all build on these concepts for more complicated brane configurations. There will be some overlap with Gauntlett’s review of intersecting branes [16]. Throughout we will try to explain the main ideas by giving the simplest non-trivial examples, referring to the literature for more general (and usually technically more complicated) cases.

In section 3 we will consider the conditions for (both orthogonal and non-orthogonal) intersections of branes to preserve supersymmetry. We will see that there are close relations between intersecting branes and branes wrapping smooth cycles in special holonomy manifolds. In particular we will review the relation between the existence of Killing spinors (required for supersymmetry) and calibrations which is a very useful method for understanding the conditions for branes to wrap or intersect supersymmetrically. We will also briefly review the description of such configurations in terms of the brane worldvolume field theory.

We introduce the “harmonic function rules”, which are a prescription for writing down supergravity solutions for intersecting branes, in section 4. We will see that the draw-back of this construction is that the branes must be smeared or delocalised in some directions – i.e. the supergravity solutions have isometries in directions transverse to the branes. We will present an important application of such solutions, as the ten- or eleven-dimensional description of lower dimensional black holes. The smearing is not problematic in this context since we compactify in those directions anyway. So these solutions provide an interpretation of the black holes in terms of branes, from which the microscopic entropy can be calculated.

We move on, in section 5, to consider solutions where the branes are more localised, using a more general set of harmonic function rules. The draw-back is that it is now typically not possible to find explicit solutions and there is still some smearing in most cases. We present some explicit examples and show how some cases involving D6-branes have a very simple eleven-dimensional geometrical interpretation. We then discuss a more general construction of various (generically non-orthogonal) intersecting branes, starting from eleven-dimensional solutions which may not contain any branes.

Finally, in section 6 we review how many gauge theories can be described using configurations of intersecting branes. In particular we focus on the case of four-dimensional $\mathcal{N} = 2$ theories. These cases involve intersecting D4- and NS5-branes in type IIA which are related to a single M5-brane wrapping a non-compact Riemann surface (identified with the Seiberg-Witten curve) in eleven dimensions. We then describe how the conditions for preserving supersymmetry can be used to find the localised supergravity solution, at least in the limit appropriate to the AdS/CFT correspondence.
2 Review of one half BPS branes

Before trying to describe complicated configurations of branes it is useful to consider the description of parallel branes in supergravity and string theory. As well as fixing notation and conventions, we will see that there are several ways of deriving the supergravity solutions. By describing the simplest cases here in detail we can understand many of the essential features of the intersecting brane solutions we will consider later, but without many of the technical complications. In particular we will explain which supersymmetries are preserved by a given brane and how this can be used to derive the corresponding supergravity solution. We will also briefly describe the method of using branes as probes of supergravity backgrounds, the relation between D-branes and gauge theories, and the AdS/CFT correspondence.

2.1 Brane solutions in supergravity

In supergravity theories, electric $p$-branes are solitonic objects (solutions of the equations of motion with or without source terms) charged with respect to a $(p+2)$-form U(1) field strength $F_{(p+2)}$. If the brane carries no other charges then it is a solution of a subsector of the supergravity theory with action, in the Einstein frame,

$$S = \frac{1}{2\kappa_D^2} \int d^Dx \sqrt{-g} \left( R - \frac{1}{2(p+2)!} F_{(p+2)}^2 \right)$$

(1)

where $\kappa_D$ is related to the $D$-dimensional Newton’s constant, $G_D$, and Planck length, $l_P$ by

$$2\kappa_D^2 = 16\pi G_D = (2\pi)^{D-3} l_P^{D-2}$$

(2)

For example with $D = 11$ and $p = 2$, this, apart from a Chern-Simons term

$$\frac{1}{2\kappa_D^2} \int \frac{1}{6} F_{(4)} \wedge F_{(4)} \wedge A_{(3)}$$

(3)

is the bosonic part of the eleven-dimensional supergravity action, where $F_{(4)} = dA_{(3)}$. In ten-dimensional supergravity, branes typically couple to the dilaton $\phi$. We will consider this extra complication later.

It can easily be checked that a solution to the equations of motion is

$$ds^2 = H \frac{d^p x^2_{(1,p)}}{D-3} + H \frac{d^p x^2_{(D-p-1)}}{D-3}$$

(4)

$$F_{(p+2)} = \pm d(H^{-1}) \wedge \epsilon_{1,p}$$

(5)

$$H = 1 + \frac{c_p N}{r^{D-p-3}}$$

(6)

where $dx^2_{(1,p)}$ is the $(p+1)$-dimensional Minkowski metric with volume form $\epsilon_{1,p}$ and

$$dx^2_{(D-p-1)} = dr^2 + r^2 d\Omega_{D-p-2}^2$$

is the $(D - p - 1)$-dimensional Euclidean metric. This solution is interpreted as $N$ coincident branes with a $(p+1)$-dimensional Minkowski worldvolume located at $r = 0$. Branes and anti-branes differ in the sign of $F_{(p+2)}$ or equivalently in the orientation of their worldvolume. The equations of motion reduce to the condition that $H$ satisfies the Laplace equation in the transverse space so we see that this is a solution with a source term at $r = 0$. Nevertheless, the question of whether we need a source term is rather subtle since it may be possible to analytically
continue the solution through \( r = 0 \), i.e. view \( r = 0 \) as simply a coordinate singularity. This is indeed possible in many cases. However, even in such cases the curvature will become very large for some solutions or choices of parameters as we approach \( r = 0 \) and so we should not trust the supergravity description of the system. For our purposes we will consider branes to be solutions of the supergravity equations of motion with appropriate source terms.

These source terms can be understood as arising from the coupling of a \( p \)-brane action to the supergravity action

\[
S \sim S_{\text{Supergravity}} + S_{\text{Brane}}
\]

This leads to a relation between the (dimensionful) constant \( c_p \) appearing in the Harmonic function \( H \) and the \( p \)-brane tension \( T_p \)

\[
c_p = \frac{2\kappa_D^2 T_p}{(D - p - 3)V(S^{D-p-2})}
\]

where \( V(S^{D-p-2}) \) is the volume of a \((D - p - 2)\)-sphere of unit radius. See [17] for the case of the 2-brane in eleven dimensions.

There are also magnetic \((D - p - 4)\)-brane solutions which are magnetically charged under \( F_{(p+2)} \). At the level of classical equations of motion we can equally consider \( F_{(p+2)} \) or its Hodge dual \( \tilde{F}_{(D-p-2)} \equiv *F_{(p+2)} \) to be the fundamental field strength. So, at least at the level of classical solutions of supergravity, there is no distinction between electric and magnetic solutions except that conventionally \( F_{(p+2)} \) with \( p + 2 \leq D/2 \) is assumed to be the fundamental field strength.

As with electric particles and magnetic monopoles in four dimensions, there is a Dirac quantisation condition relating the charges (and tensions) of electric and magnetic branes charged under the same field strength. In terms of the brane tensions this takes the form [18, 19]

\[
2\kappa_D^2 T_p T_{D-p-4} = 2\pi n , \quad n \in \mathbb{Z}
\]

This is satisfied by the branes we will consider with \( n = 1 \).

In 11-dimensional supergravity the only field strength is \( F_{(4)} \) and so the only brane solutions are (electric) \( 2 \)-branes and (magnetic) \( 5 \)-branes, usually labelled M2- and M5-branes. So equations (4), (5) and (6) with \( D = 11 \) give the supergravity solution for M2-branes [17] (with \( p = 2 \)) and M5-branes [20] (with \( p = 5 \)). The tensions of these branes are related by the Dirac quantisation condition. Furthermore, the tension of say the M2-brane is fixed in terms of the eleven-dimensional Planck length due to the presence of the Chern-Simons term in the supergravity action. This results in [21]

\[
T_{M2} = \frac{1}{4\pi^2 l_p^3}
\]

In the following two sections we will briefly review the branes in ten-dimensional type IIA and type IIB theories. More details from the supergravity perspective can be found in e.g. [8] while [10] provides a detailed review of D-branes in terms of string theory.

### 2.1.1 Branes in type IIA supergravity

In type IIA supergravity there are Ramond-Ramond (RR) field strengths \( F_{(2)} \) giving rise to D0- and D6-branes, and \( F_{(4)} \) with corresponding D2- and D4-branes. There are also D8-branes which are domain walls in ten dimensions. These are solutions of massive IIA supergravity [22]. They are predicted to exist in string theory by T-duality from other D-branes but, unlike the other branes in type IIA string theory, it is not clear how they are related to an eleven-dimensional theory.
From the string theory point of view $D_p$-branes in type IIA and type IIB are $(p+1)$-dimensional submanifolds on which open strings can end. As we will discuss later, this leads to the result that the low energy dynamics of $D_p$-branes is described by a $(p+1)$-dimensional gauge theory. The tension of a $D_p$-brane can be calculated from a 1-loop open string amplitude [23]. The result is

$$T_{Dp} = \frac{1}{(2\pi)^p g_s l_s^{p+1}}$$

(11)

There are also non-dynamical RR-charged objects known as orientifold $p$-planes. These are the fixed planes of a $Z_2$ action which consists of a reflection of the $9-p$ transverse coordinates together with a reversal of the orientation of the string worldsheet. The charge and tension of these orientifold $p$-planes is given in terms of the $D_p$-brane tension by

$$T_{O_p} = \pm 2^{p-5} T_{Dp}$$

(12)

Finally in both type IIA and type IIB there is also a NS-NS (for Neveu-Schwarz) three-form field strength $H^{(3)}$ and so there are 1- and 5-brane solutions. These 1-branes correspond to the fundamental strings and are sometimes referred to as F1-branes or NS1-branes. The 5-branes are called NS5-branes. Note that these objects are not D-branes (they are not endpoints for fundamental open strings.)

The tension of the fundamental string

$$T_{F1} = \frac{1}{2\pi l_s^2}$$

(13)

defines the string length $l_s = \sqrt{\alpha'}$. This string length is in turn related to the ten-dimensional Newton’s constant by

$$2\kappa_{10}^2 = 16\pi G_{10} = (2\pi)^7 g_s^2 l_s^8$$

(14)

where the string coupling constant $g_s \equiv e^{\phi_\infty}$ is related to the asymptotic value of the dilaton $\phi \rightarrow \phi_\infty$. We can of course shift $\phi$ so that it vanishes at infinity, and include factors of $g_s$ explicitly. This is the convention we use throughout this paper. In the string frame metric the supergravity action takes the form

$$S_{IIA} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[ e^{-2\phi} \left( R + 4|\nabla \phi|^2 - \frac{1}{2 \cdot 3!} H^{(3)}_3 \right) - \frac{1}{2} \sum_n \frac{1}{n!} F_2^{(n)} \right]$$

(15)

where the summation is over $n = 2, 4$, together with a Chern-Simons term

$$\frac{1}{2\kappa_{10}^2} \int \frac{1}{2} dC^{(3)} \wedge dC^{(3)} \wedge B^{(2)}$$

(16)

where

$$H^{(3)} = dB^{(2)} , \quad F^{(2)} = dC^{(1)} , \quad F^{(4)} = dC^{(3)} + C^{(1)} \wedge H^{(3)}$$

(17)

We can write this action in the Einstein frame where the gravitational part takes the standard Einstein-Hilbert form by rescaling the string-frame metric $g_{MN}$. In terms of the Einstein-frame metric

$$g^{(E)}_{MN} \equiv e^{-\phi/2} g_{MN}$$

(18)

the string action takes the form

$$S_{IIA} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g^{(E)}} \left[ R - \frac{1}{2} |\nabla \phi|^2 - \frac{1}{2 \cdot 3!} e^{-\phi} H^{(3)}_3 - \frac{1}{2} \sum_n \frac{1}{n!} e^{5-n} \phi F_2^{(n)} \right]$$

(19)
where all quantities (e.g. \( R \) and \( F^2 \)) are calculated using the Einstein-frame metric.

As is well-known, type IIA supergravity arises from the Kaluza-Klein reduction of eleven-dimensional supergravity on a circle of radius \( R_{11} \), with the fundamental string being interpreted as an M2-brane wrapping the circle. This relates the type IIA string coupling, \( g_s \), and string length, \( l_s \), to the eleven-dimensional Planck length, \( l_P \), and \( R_{11} \)

\[
R_{11} = g_s l_s \quad , \quad l_P = g_s^{\frac{1}{3}} l_s
\]

The explicit relation between the type IIA string-frame metric \( ds^2_{(1,9)} \), dilaton \( \phi \), RR one-form potential \( C_\mu \), RR four-form field strength \( F_4 \) and NS-NS three-form field strength \( H_3 \), and the eleven-dimensional metric \( ds^2_{(1,10)} \) and four-form field strength \( \tilde{F}_4 \) is

\[
ds^2_{(1,10)} = e^{-2\phi} ds^2_{(1,9)} + e^{\frac{4\phi}{3}} (R_{11} d\psi + C_\mu dx^\mu)^2
\]

\[
\tilde{F}_4 = F_4 + H_3 \wedge dx^{10}
\]

where \( \psi \) has period \( 2\pi \) and \( x^{10} = R_{11} \psi \).

In this way all type IIA branes (except the D8-brane) can be understood in terms of dimensional reduction from eleven dimensions \[23\]. Since \( \tilde{F}_4 \) reduces to \( H_3 \) and \( F_4 \), fundamental strings and D2-branes are simply M2-branes wrapped or not wrapped on the eleventh dimension. Similarly D4- and NS5-branes both correspond to M5-branes in eleven dimensions. The field strength \( F_2 \) is just the usual Kaluza-Klein gauge field strength and so the D0-branes are Kaluza-Klein particles while the D6-branes are Kaluza-Klein monopoles. In particular the eleven-dimensional supergravity solution which reduces to a D6-brane in ten dimensions is given by a geometrical background which is a product of \((6 + 1)\)-dimensional Minkowski spacetime with the Taub-NUT space, which we will refer to as a KK6-brane.

### 2.1.2 Branes in type IIB supergravity

In type IIB supergravity there are RR field strengths \( F_1 = dC_0 \) (D(-1)-branes and D7-branes), \( F_3 = dC_2 - C_0 \wedge H_3 \) (D1-branes and D5-branes), and a self-dual five-form \( F_5 = dC_4 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge H_3 \) so there are D3-branes, but not electric and magnetic versions since \( F_5 \) is self-dual. Formally we have the same action as for type IIA, equation \[23\] with the summation now over \( n = 1, 3, 5 \). However, we should use this action with the understanding that the self-duality condition is to be imposed on the five-form when solving the equations of motion and consequently include an extra factor \( \frac{1}{2} \) in the kinetic term for \( F_5 \). There are also D9-branes which fill all of spacetime. The only consistent way of including them is to have 16 D9-branes with an orientifold 9-plane of the opposite charge. This background of type IIB is equivalent to type I string theory. The tension of the \( Dp \)-branes is given by the same formula as in type IIA, equation \[23\]. Again there is a NS-NS three-form field strength \( H_3 \) so there are fundamental strings and NS5-branes, with the fundamental string tension also given by equation \[23\]. Some comments are required on the branes in type IIB. First of all, the D(1)-brane or D-instanton is a solution localised at a point in spacetime. Secondly there is an SL(2, \( \mathbb{Z} \)) symmetry\(^1\) which acts on the doublet \((F_3, H_3)\) as well as the complex coupling constant \( \tau = C_0 + ie^{-\phi} \) and so there are dyonic \((p, q)\)-strings and \((p, q)5\)-branes where \( p \) and \( q \) are relatively prime integers. These branes can also be interpreted as one half BPS bound states of \( p \) fundamental strings (NS5-branes) with \( q \) D1-branes (D5-branes.) This symmetry can be directly related to the SL(2, \( \mathbb{Z} \)) duality of \( \mathcal{N} = 4 \) 3+1-dimensional SYM. Essentially this is the

\(^1\)There is apparently an SL(2, \( \mathbb{R} \)) symmetry in type IIB supergravity but Dirac charge quantisation breaks this to SL(2, \( \mathbb{Z} \)).
low energy dynamics of the D3-branes and the complex gauge coupling is simply $\tau$ while the 
electric particles correspond to fundamental strings ending on the D3-branes. The D3-branes 
are invariant under $\text{SL}(2, \mathbb{Z})$ transformations but the fundamental strings are transformed into 
dyonic $(p, q)$-strings. 

The type IIB branes are related to the type IIA branes by T-duality. T-dualising in a 
direction parallel or perpendicular to a $D_p$-brane in type IIA/B produces a $D(p−1)$-brane or $D(p+1)$-brane respectively in type IIB/A. Hence all the D-branes are related to each other 
by T-duality and we can see the necessity of including D8- and D9-branes. The NS5-branes in 
type IIA and type IIB are related to each other under T-duality in a direction parallel to 
the branes. A T-duality transformation perpendicular to a NS5-brane relates it to a Kaluza-Klein 5-brane, i.e. a geometry consisting of the factors $(5+1)$-dimensional Minkowski spacetime and Taub-NUT [25] (see also [26] for a detailed discussion of this T-duality.) At the level of 
supergravity solutions, T-duality can only be performed along an isometry direction and so is of 
limited use in generating new solutions. For example T-duality parallel to a D4-brane solution 
would produce a D3-brane solution where the D3-brane(s) were smeared over this ‘transverse’ 
direction. See [27, 28, 29] for the explicit form of T-duality transformations of supergravity 
solutions.

The supergravity solution for $N$ coincident Dp-branes (in the string-frame) is

\[
\begin{align*}
\text{d}s^2 &= H^{-1/2} \text{d}x^2_{(1,p)} + H^{1/2} \text{d}x^2_{(9−p)} \\
F_{(p+2)} &= -\text{d}(H^{-1}) \wedge \epsilon_{1,p} \\
\text{e}^\phi &= H^{\frac{2}{2^p}} \\
H &= 1 + \frac{C_p N}{r^{7−p}}
\end{align*}
\]

There are also explicit solutions describing NS5-branes [30]

\[
\begin{align*}
\text{d}s^2 &= \text{d}x^2_{(1,5)} + H \text{d}x^2_{(4)} \\
H_{(3)} &= * (\text{d}(\ln H) \wedge \epsilon_{1,5}) \\
\text{e}^\phi &= H^{\frac{1}{2}} \\
H &= 1 + \frac{l_s^2 N}{r^2}
\end{align*}
\]

and fundamental strings [31]

\[
\begin{align*}
\text{d}s^2 &= H^{-1} \text{d}x^2_{(1,1)} + \text{d}x^2_{(8)} \\
H_{(3)} &= -\text{d}(H^{-1}) \wedge \epsilon_{1,1} \\
\text{e}^\phi &= H^{-\frac{1}{2}} \\
H &= 1 + \frac{2\pi^2 g_s^2 \ell_s^6 N}{r^6}
\end{align*}
\]

In each case $r$ is the radial coordinate in the directions transverse to the branes. Note that 
in all cases the brane solution is determined by a harmonic function $H$. Solutions describing 
separated parallel branes are the same as above, with $H$ replaced by a multi-centred harmonic 
function.

### 2.1.3 Supersymmetry of brane solutions

An important property of the brane solutions we have presented is that they preserve half the 
supersymmetry of the supergravity theory. To see this explicitly we need to consider the form of
the supersymmetry transformations in the appropriate supergravity theory. In all cases, since we are considering purely bosonic solutions, the supersymmetry transformations of all bosonic fields vanish. So we only consider the supersymmetry transformations of the fermionic fields. The subset of all allowed transformations which vanish for the given solution are those which are preserved by the solution. For example, if we consider 11-dimensional supergravity \[32\] with a field content consisting of the metric \(g_{MN}\), four-form field strength \(F^{(4)}\) and Rarita-Schwinger fermion \(\psi_{M\alpha}\), the supersymmetry transformations (in a bosonic background) are given in terms of a 32-component Majorana spinor \(\epsilon_{\alpha}\) by

\[
\delta \psi_{M} = D_{M} \epsilon + \frac{1}{288} \Gamma_{M}^{NPQR} F_{NPQR} \epsilon - \frac{1}{36} \Gamma^{PQR} F_{MPQR} \epsilon \equiv \hat{D}_{M} \epsilon
\]

where

\[
D_{\mu} = \partial_{M} + \frac{1}{4} \omega_{\nu}^{p \mu} \hat{\Gamma}_{n}^{p}
\]

is the usual covariant derivative acting on spinors. We use the notation \(\Gamma_{M}\) for the spacetime Dirac gamma-matrices and \(\hat{\Gamma}_{m}\) for the tangent-space gamma-matrices. They are related by the vielbein \(e_{m}^{M}\), i.e.

\[
g_{MN} = e_{M}^{m} e_{N}^{n} \eta_{mn}, \quad \Gamma_{M} = e_{M}^{m} \hat{\Gamma}_{m}, \quad \{\Gamma_{M}, \Gamma_{N}\} = 2 g_{MN}, \quad \{\hat{\Gamma}_{m}, \hat{\Gamma}_{n}\} = 2 \eta_{mn}
\]

and antisymmetric combinations are denoted

\[
\Gamma_{M_{1} \cdots M_{p}} = \frac{1}{p!} \left( \Gamma_{M_{1}} \Gamma_{M_{2}} \cdots \Gamma_{M_{p}} - \Gamma_{M_{2}} \Gamma_{M_{1}} \cdots \Gamma_{M_{p}} + \cdots \right)
\]

It can relatively easily be checked that for the M2- and M5-brane solutions presented in the previous section, these supersymmetry variations vanish with an arbitrary choice of half of the components of the spinors \(\epsilon\). More precisely, in each case these supersymmetry variations vanish when \(\epsilon\) is some specific function multiplying a constant spinor \(\epsilon_{0}\) which satisfies a projection condition

\[
\hat{\Gamma}_{0} \epsilon_{0} = \hat{\Gamma}_{1} \hat{\Gamma}_{2} \epsilon_{0} = \epsilon_{0}
\]

for M2-branes with worldvolume directions 012, or

\[
\hat{\Gamma}_{0} \cdots \hat{\Gamma}_{5} \epsilon_{0} = \epsilon_{0}
\]

for M5-branes with worldvolume directions 012345. Similar results hold for the brane solutions of type IIA and type IIB supergravity. In the following sections we will explain why such projection conditions arise and also show how the brane solutions (taking the M2-brane solution as an example) can be derived from the requirement of preserving precisely those supersymmetries.

### 2.2 Supersymmetry and \(\kappa\)-symmetry

We have remarked that \(p\)-brane solutions preserve supersymmetry. We will now consider the situation from the brane worldvolume point of view. The idea is to understand which spacetime supersymmetries should be preserved without using a specific supergravity solution. We will then use the fact that the corresponding supersymmetry transformations of the fermionic supergravity fields must vanish to derive relations between the bosonic fields. We will then see that this enables us to find the supergravity solution by solving first-order differential equations rather than the full set of second-order equations of motion.

One way to understand the relation between spacetime and worldvolume supersymmetry is through the notion of brane probes. A brane probe is essentially a brane placed into a fixed
background as a ‘test brane’—i.e. the backreaction of the brane on the background is neglected. We can consider which supersymmetries are preserved by a probe brane in a given background. The point is that if the background is generated by the same type (and orientation) of branes then we expect that the probe brane does not break any of the supersymmetries preserved by the background. The fact that we are ignoring the backreaction should not matter since we can consider related backgrounds, preserving exactly the same supersymmetries, where the backreaction is as small as we want.

The simplest case is \( N+1 \) parallel branes where we can consider one to be a probe brane. The backreaction here is an effect of order \( 1/N \) but the supersymmetries preserved do not depend on \( N \) and so it is reasonable to expect that the backreaction does not affect the determination of which supersymmetries are preserved.

The supersymmetric worldvolume brane action can be derived by (super-)embedding the brane worldvolume in superspace. In the cases we consider this superspace has 10 or 11 bosonic spacetime coordinates \( X^\mu \) and 32 fermionic coordinates \( \Theta^\alpha \). In order to have worldvolume supersymmetry the number of on-shell bosonic and fermionic degrees of freedom must match. This requires a symmetry of the worldvolume action called \( \kappa \)-symmetry which projects out half the components of \( \Theta \) on the brane worldvolume. The \( \kappa \)-symmetry transformations are very similar to supersymmetry transformations and can be understood in this way from superembedding the brane worldvolume superspace into target superspace (see [33] for a comprehensive review.) In all cases the \( \kappa \)-symmetry transformation takes the form

\[
\delta_\kappa \Theta = \frac{1}{2}(1 + \Gamma)\kappa
\]

while the supersymmetry transformation is

\[
\delta_\epsilon \Theta = \epsilon
\]

with the bosonic worldvolume fields not transforming when we have set the fermionic fields to zero. The form of \( \Gamma \) depends on the type of brane but in all cases \( \Gamma^2 = 1 \) so that \( P_\pm \equiv \frac{1}{2}(1 \pm \Gamma) \) are projection operators. Also \( \Gamma \) is traceless so that each projection operator projects out precisely half the components of an arbitrary spinor. Hence decomposing \( \Theta \) under these projections we see that

\[
\delta_\kappa (P_- \Theta) = 0
\]

\[
\delta_\kappa (P_+ \Theta) = P_+ \kappa
\]

Similarly the supersymmetry transformation becomes

\[
\delta_\epsilon (P_- \Theta) = P_- \epsilon
\]

\[
\delta_\epsilon (P_+ \Theta) = P_+ \epsilon
\]

So we can consistently set \( (1 + \Gamma)\Theta = 0 \), fixing \( \kappa \)-symmetry and leaving \( P_- \Theta \) as the worldvolume fermionic degrees of freedom. I.e. we use \( \kappa \)-symmetry to set \( (1 + \Gamma)\Theta = 0 \) and then we preserve this gauge choice by compensating a supersymmetry transformation with a \( \kappa \)-symmetry transformation with parameter \( \kappa = -\epsilon \). This effectively removes \( P_+ \Theta \), leaving only \( P_- \Theta \) with the supersymmetry transformations

\[
\delta_\epsilon (P_- \Theta) = P_- \epsilon
\]

The condition for preservation of worldvolume supersymmetry is then \( \delta_\epsilon (P_- \Theta) = 0 \), i.e.

\[
\Gamma \epsilon = \epsilon
\]
Since $\Gamma$ depends on the worldvolume fields we see that the brane locally preserves half (i.e. 16 out of 32) of the background global supersymmetries. So the brane preserves at most 16 supersymmetries but in general there will be different projection conditions at different parts of the worldvolume so that typically all supersymmetry will be broken.

For completeness we give the form of the projector $\Gamma$ for each type of brane in type IIA, type IIB and M-theory. In each case $\{\sigma^\mu\}$ are worldvolume coordinates on the brane and $\{X^M(\sigma)\}$ describe the bosonic embedding of the brane in spacetime. The induced metric on the brane worldvolume, $G_{\mu\nu}$, is the pullback of the spacetime metric $g_{MN}$

$$G_{\mu\nu} = \partial_{\sigma^\mu} X^M \partial_{\sigma^\nu} X^N g_{MN} \quad (49)$$

and $G = \det(G_{\mu\nu})$. The worldvolume gamma-matrices $\{\gamma_\mu\}$ are the pullback of the spacetime gamma-matrices

$$\gamma_\mu = \partial_{\sigma^\nu} X^M \Gamma_M \quad (50)$$

and so

$$\{\gamma_\mu, \gamma_\nu\} = 2G_{\mu\nu} \quad (51)$$

So for a $p$-brane we can define (using the conventions $\epsilon_{01...p} = +1$ so that $\epsilon^{01...p} = G^{-1}$)

$$\gamma_{(p+1)} = \sqrt{|G|} \frac{\epsilon^{\mu_0\mu_1...\mu_p}}{(p+1)!} \gamma_{\mu_0} \gamma_{\mu_1} \cdots \gamma_{\mu_p} \quad (52)$$

It is easy to check that

$$\gamma_{(p+1)}^2 = (-1)^{\frac{p(p+1)}{2} + 1} \quad (53)$$

In all cases we consider zero worldvolume fieldstrengths – see e.g. [34] for the case of D-branes with non-zero worldvolume fieldstrengths.

### 2.2.1 M-branes

We have M2- and M5-branes in M-theory and eleven-dimensional supergravity. The projector takes the same simple form in each case

$$\Gamma = \pm \gamma_{(p+1)} \quad (54)$$

for M2-branes [35] and M5-branes [36, 37] with $p = 2$ and $p = 5$ respectively. In this case $\epsilon$ is a 32-component real spinor. Note also that

$$\hat{\Gamma}_0 \cdots \hat{\Gamma}_{(10)} = 1 \quad (55)$$

### 2.2.2 Type IIA branes

For D$(2p)$-branes we have [38, 39, 40]

$$\Gamma = \pm \gamma_{(2p+1)} \Gamma_{(11)}^{p+1} \quad (56)$$

where $\Gamma_{(11)} = \hat{\Gamma}_0 \cdots \hat{\Gamma}_0$ when we use irreducible 16-component real spinors $\epsilon_L$ and $\epsilon_R$ of opposite chirality. Alternatively we can use a 32-component real spinor $\epsilon$ as in eleven dimensions and $\Gamma_{(11)}$ is identified with $\hat{\Gamma}_{(10)}$. This makes the eleven dimensional origin of the type IIA branes obvious. The projection conditions in each case are

$$\Gamma \epsilon_L = \epsilon_R \quad \text{or} \quad \Gamma \epsilon = \epsilon \quad (57)$$
For the NS5-brane we have
\[ \Gamma = \pm \gamma_{(6)} \] (58)
while for the fundamental string we have
\[ \Gamma = \pm \gamma_{(2)} \gamma_{(11)} \] (59)
and in both cases the projection conditions are
\[ \Gamma \epsilon_L = \epsilon_L \text{ and } \Gamma \epsilon_R = \epsilon_R \text{ or } \Gamma \epsilon = \epsilon \] (60)

2.2.3 Type IIB branes

For D(2p − 1)-branes we have [41, 38, 39, 40]
\[ \Gamma = \pm i \sigma_3^{(p)} \sigma_2 \otimes \gamma_{(2p)} \] (61)
where
\[ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (62)
are the usual Pauli \(\sigma\)-matrices, acting on a 32-component spinor which can be decomposed into two 16-component spinors of the same chirality
\[ \epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \epsilon_L + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \epsilon_R \] (63)

For the NS5-brane we have
\[ \Gamma = \pm \sigma_3 \otimes \gamma_{(6)} \] (64)
and similarly for the fundamental string
\[ \Gamma = \pm \sigma_3 \otimes \gamma_{(2)} \] (65)
In all cases the projection condition is
\[ \Gamma \epsilon = \epsilon \] (66)

In the case of \((p, q)\)-strings (5-branes) the projection condition is a linear combination of the projection conditions for the component branes [42] (see also [43])
\[ \Gamma = \pm \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \otimes \gamma_{(2)} \] (67)
for \((p, q)\)-strings and
\[ \Gamma = \pm \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \otimes \gamma_{(6)} \] (68)
for \((p, q)5\)-branes. In both cases \(\theta\) is the argument of \(p + q\tau\).
2.3 M2-brane example

We will briefly consider the M2-brane solution \[17\] as an example of brane solutions, to explain why the solutions are given in terms of harmonic functions and to show explicitly how the Killing spinor equations restrict the form of the solution. Here, and throughout this paper, we will consider M-branes as simple examples since typically brane solutions in ten dimensions share the same essential features but have some extra technical complications, e.g. due to the dilaton.

The form of the solution can be deduced from symmetry considerations. We expect to have (2 + 1)-dimensional Poincaré invariance of the M2-brane worldvolume and, for coincident M2-branes, SO(8) rotational invariance in the space transverse to the M2-brane. This restricts the metric to be of the form

\[
ds^2 = e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B} \delta_{\alpha\beta} dx^\alpha dx^\beta
\]

where \(\mu, \nu\) run over \(\{0, 1, 2\}\) and \(\alpha, \beta\) over \(\{3, 4, \ldots, 10\}\). We will refer to such a configuration as an M2-brane (or a set of M2-branes) with worldvolume directions 012. The functions \(A\) and \(B\) only depend on the transverse coordinates \(x^\alpha\) while for coincident M2-branes the dependence is only on the transverse radial coordinate \(r = \sqrt{\delta_{\alpha\beta} x^\alpha x^\beta}\).

The M2-branes source the four-form field strength \(F^{(4)} = dA^{(3)}\). Since the three-form potential, \(A^{(3)}\), couples directly to the M2-brane worldvolume we expect it to have only non-zero components \(A_{012}\) (again with no dependence on the 012 coordinates) and so the field strength will have the form

\[
F^{(4)} = d\left(e^C\right) \wedge \epsilon_{1,2}
\]

Clearly \(F^{(4)}\) is an exact 4-form, provided \(e^C\) is globally well-defined, and so will obey the (source-free) Bianchi identity

\[
dF^{(4)} = 0
\]

as expected. We will see explicitly that the solution for \(e^C\) is well-defined everywhere.

Now in this case the \(\kappa\)-symmetry requirement imposes the projection condition

\[
\hat{\Gamma}_{012}\epsilon = \epsilon
\]

Using this together with the above metric and four-form we find

\[
\tilde{D}_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{2} e^{-B} \partial_\alpha (e^A) \hat{\Gamma}_\mu \hat{\Gamma}^\alpha \epsilon + \frac{1}{6} e^{-2A} e^{-B} \partial_\alpha (e^C) \hat{\Gamma}_\mu \hat{\Gamma}^\alpha \epsilon
\]

\[
\tilde{D}_\alpha \epsilon = \partial_\alpha \epsilon + \frac{1}{2} \left(\partial_\beta B\right) \hat{\Gamma}_{\alpha \beta} \epsilon - \frac{1}{12} e^{-3A} \partial_\beta (e^C) \hat{\Gamma}_{\alpha \beta} \epsilon + \frac{1}{6} e^{-3A} \partial_\alpha (e^C) \epsilon
\]

For a supersymmetric solution we require \(\tilde{D}_\mu \epsilon = 0\) and \(\tilde{D}_\alpha \epsilon = 0\). To preserve half the supersymmetry we cannot impose any additional projection conditions on \(\epsilon\). So we can now simply extract the coefficients of the linearly independent combinations of \(\hat{\Gamma}\)-matrices acting on \(\epsilon\). This gives the following set of equations

\[
\partial_\mu \epsilon = 0
\]

\[
\frac{1}{2} \partial_\alpha (e^A) = -\frac{1}{6} e^{-2A} \partial_\alpha (e^C)
\]

\[
\partial_\alpha \epsilon = -\frac{1}{6} e^{-3A} \partial_\alpha (e^C) \epsilon
\]

\[
\partial_\beta B = \frac{1}{6} e^{-3A} \partial_\beta (e^C)
\]

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We can fix the relevant integration constants by requiring that the metric is asymptotically the standard eleven-dimensional Minkowski metric. This determines the following relations

\[ e^A = e^{-2B} \]  
\[ e^C = -e^{3A} + \text{constant} \]  
\[ \epsilon = e^{A/2} \epsilon_0 \]

and

\[ e^A = e^{-2B} \]  
\[ e^C = -e^{3A} + \text{constant} \]  
\[ \epsilon = e^{A/2} \epsilon_0 \]

for a constant spinor \( \epsilon_0 \) which must satisfy

\[ \hat{\Gamma}_{012} \epsilon_0 = \epsilon_0 \]

So the supersymmetry preservation conditions leave us with only one unknown function. We can easily determine this function by solving the equation of motion for \( F_{(4)} \). Noting that \( F_{(4)} \land F_{(4)} = 0 \) we want \( d \ast F_{(4)} = 0 \) (or more precisely we should specify source terms in this equation corresponding to the specific location of the M2-branes.) The only non-zero components of \( F_{(4)} \) are given by

\[ F_{\alpha 012} = \partial_\alpha (e^C) \]

and so the equation \( d \ast F_{(4)} = 0 \) becomes

\[ 0 = \partial_\alpha \left( \sqrt{-g} F_{\alpha 012} \right) = \sum_\alpha \partial_\alpha \left( e^{3A+8B} e^{-2B} e^{-6A} \partial_\alpha (e^C) \right) \]

\[ = - \sum_\alpha \partial_\alpha (e^{-6A} \partial_\alpha (e^{3A})) = \sum_\alpha \partial_\alpha \partial_\alpha (e^{-3A}) \]

Hence we see that \( e^{-3A} \) is a harmonic function of the transverse coordinates \( x^\alpha \). In terms of \( H \equiv e^{-3A} \) the solution is given by

\[ ds^2 = H^{-\frac{2}{3}} \eta_{\mu\nu} dx^\mu dx^\nu + H^{\frac{4}{3}} \delta_{\alpha\beta} dx^\alpha dx^\beta \]

\[ F_{(4)} = -d \left( H^{-1/3} \right) \wedge \epsilon_{1,2} \]

\[ H = 1 + \sum_i \frac{K}{|x^\alpha - y_i^\alpha|^6} \]

where \( y_i^\alpha \) are the locations of the M2-branes and the correctly normalised source would give, from equations [5] and [10]

\[ K = 2^5 \pi^2 l_P^6 \]

Notice that because we have looked for a supersymmetric solution, we have only had to solve first order differential equations coming from the Killing spinor equations, in order to express the metric and four-form in terms of a single unknown function, \( H \). The equation of motion for \( F_{(4)} \) then became a second order PDE determining \( H \). Fortunately this could be solved in general since it reduced to the flat-space Laplace equation. Note in particular that we have not yet considered the Einstein equations\(^2\)

\[ R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{12} F_{MPQR} F_N^{PQR} - \frac{1}{96} g_{MN} F_{PQRS} F^{PQRS} \]

However it turns out that these equations are automatically satisfied once we have imposed the conditions for the existence of Killing spinors and satisfied the equations of motion for \( F_{(4)} \).

\(^2\)Again we should more precisely combine the M2-brane action with the supergravity action which would lead to the correct source terms in the Einstein equations.
Of course, this must be the case if such supersymmetric solutions are to exist since we have fully determined the metric and four-form without directly using the Einstein equations. So although in general it is necessary to check the Einstein equations, the constraints from the first-order Killing spinor equations will usually greatly simplify the problem. Also, in some cases the existence of Killing spinors automatically ensures that the Einstein equations are satisfied, provided the equations of motion (and Bianchi identities) for the field strengths are satisfied. This happens in the cases where all 32 supersymmetries are preserved or when the metric is diagonal [44, 45].

2.4 Brane probes

Here we briefly discuss the idea of using branes to probe a supergravity background [46, 47, 48, 49]. There are many applications of brane probes. The point of view we will consider here is that a brane probing a supersymmetric background generated by branes of the same type and orientation should feel no force since this should be a BPS configuration. In fact more generally we can expect a no-force condition whenever it is possible to supersymmetrically embed the brane in the background [50]. We will see that this leads to constraints on the supergravity background and gives us another way to derive the parallel brane solutions. Consider for simplicity the case of parallel $p$-branes without a dilatonic coupling, i.e. M2-, M5- or D3-branes. The same procedure works for other D-branes with the added complication that we need to also solve the equation of motion for the dilaton.

We take the background generated by $N$ branes to be of the form

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B}\delta_{\alpha\beta}dx^\alpha dx^\beta$$

$$C_{(p+1)} = -H^{-1}\epsilon_{1,p}$$

where $\mu, \nu$ run over $0, 1, \ldots, p$ and $F_{(p+2)} = dC_{(p+1)}$. The idea of a brane probe is now to add one more brane, ignoring the backreaction. By supersymmetry we can place such a brane anywhere in the transverse space. So we should find that there is no force on such a (static) brane. If we allow some (rigid) motion we should find a flat metric on moduli space in terms of the brane worldvolume theory, since the amount of supersymmetry does allow a non-trivial metric. This is because, as discussed in the next section, the brane configuration is related to (directly for D3-branes, via dimensional reduction for M2- and M5-branes) a maximally supersymmetric gauge theory which (from supersymmetry considerations) cannot have a non-trivial metric on moduli space. I.e. if we consider a probe brane with worldvolume coordinates $\{\sigma^\mu\}$ embedded so that

$$X^\mu = \sigma^\mu, \quad X^\alpha = v^\alpha \sigma^0$$

we should find that the probe action reduces to

$$S = T_p \int d^{p+1}\sigma \left(-\frac{1}{2}v^2 + O(v^4)\right)$$

where $v^2 \equiv \delta_{\alpha\beta}v^\alpha v^\beta$.

We start with the minimal action for a $p$-brane coupled to $C_{(p+1)}$

$$S = T_p \int d^{p+1}\sigma \sqrt{-G} + T_p \int \mathcal{P}(C_{(p+1)})$$

where $G$ is the determinant of the pull-back metric and $\mathcal{P}(C_{(p+1)})$ is the pull-back of $C_{(p+1)}$ onto the brane worldvolume. For example

$$G_{00} = \partial_{\sigma^0}X^M \partial_{\sigma^0}X^N g_{MN} = -e^{2A} + e^{2B}v^2$$
So we find

\[- G = e^{2(p+1)A} \left(1 - e^{2B} e^{-2A} v^2 \right) \tag{96}\]

\[\mathcal{P}(C_{(p+1)}) = -H^{-1} \epsilon_{1,p} \tag{97}\]

Expanding for small velocity \(v\) we find

\[S = T_p \int d^{p+1}\sigma \left( (e^{(p+1)A} - H^{-1} - e^{(p-1)A} e^{2B} \frac{1}{2} \sigma^2 + O(v^4) \right) \tag{98}\]

So the absence of a static potential requires

\[e^{2A} = H^{-\frac{2}{p+1}} \tag{99}\]

and the requirement of a flat moduli space metric requires

\[e^{2B} = H e^{2A} \tag{100}\]

Again, as for the method of requiring Killing spinors, we must now use the supergravity equations of motion for \(C_{(p+1)}\) to show that \(H\) is a harmonic function.

It can easily be seen that the above solution agrees with equations (4), (5) and (6), and so is correct for M2-, M5- and D3-branes which have no dilatonic coupling. For other branes it is more complicated but it is always fairly simple to check a given solution by this method. We will come back to this point later for intersecting branes.

### 2.5 Branes and gauge theories

One of the most important and useful applications of branes has been their connection with gauge theories. This connection comes through the fact that branes are dynamical objects and their dynamics can be described through a worldvolume action. In the case of D-branes the worldvolume action is (the supersymmetric extension of) the \((p+1)\)-dimensional Dirac-Born-Infeld (DBI) action \[51, 52\]

\[S^{(p+1)\text{DBI}} = T_{Dp} \int d^{p+1}\sigma e^{-\phi} \sqrt{-\det (G_{\mu\nu} + F_{\mu\nu})} \tag{101}\]

together with Wess-Zumino couplings \[53\] (see also \[54\] for additional gravitational couplings which we don’t consider here)

\[S^{(p+1)\text{WZ}} = T_{Dp} \int \sum_n \mathcal{P}(C_n) \wedge e^F \tag{102}\]

where \(F = 2\pi l_s^2 F - \mathcal{P}(B)\) is a linear combination of the pullback of the spacetime NS-NS 2-form potential \(B\) and a worldvolume 2-form \(U(1)\) field strength \(F\). In the WZ terms the sum is over all the RR potentials present in the given supergravity and the integral is understood to include only \((p+1)\)-forms. In the absence of a \(B\)-field the low energy limit of this action is simply the \((p+1)\)-dimensional \(U(1)\) maximally supersymmetric Yang-Mills action. The case of a non-zero constant \(B\)-field has also been a topic of recent interest \[55, 56, 57, 58, 59, 60\]. The low energy limit is a noncommutative gauge theory which is a generalisation of usual field theory where products of fields are taken using a noncommutative Moyal product \[61\]. Scalar fields in the worldvolume action arise from the coordinates transverse to the brane

\[\Phi^\alpha(\sigma^\mu) \equiv \frac{1}{2\pi l_s^2} X^\alpha(\sigma^\mu) \tag{103}\]
Expanding the DBI action in flat space for small \( l_s \) and removing the constant term we get
\[
S \approx 4\pi^2 l_s^4 T_{Dp} \int d^{p+1}\sigma \left( \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} (\partial_\mu \Phi^\alpha)(\partial^\mu \Phi^\alpha) \right) 
\]
We see that the Yang-mills coupling is given by
\[
\frac{1}{g_{YM}^2} = 4\pi^2 l_s^4 T_{Dp} 
\]
We can consider also a collection of \( N \) coincident D\( p \)-branes. The low energy dynamics will again be described by maximally supersymmetric Yang-Mills but now with gauge group U(\( N \)). This can be shown by expanding a non-Abelian generalisation of the DBI action. However, unfortunately it is not so easy to calculate such an action from string theory although some progress has been made \[62, 63, 64\]. Note in particular that simply replacing the Abelian field strength \( F \) with a non-Abelian one in the DBI action does not give a gauge invariant action.

The simplest way to understand the origin of a non-Abelian gauge group is to start with \( N \) separated parallel branes. The low energy effective action is that of the massless open string states. Each brane has a U(1) factor arising from open strings with both endpoints on the brane. Now there are also open strings with endpoints on each pair of branes. These are massive since they have a minimal length given by the separation of the branes. However, when we move the branes together we will gain extra massless states from these strings. Taking account of the fact that the open strings are oriented we have in total \( N^2 \) different types of strings which fill out the adjoint representation of U(\( N \)). Note that labelling the endpoints according to which brane they are attached is equivalent to introducing Chan-Paton factors. Also we are lead to a noncommutative target space since the scalars become \( N \times N \) matrices, as do the coordinates through equation (103). The \( N \) eigenvalues are interpreted as giving the positions of the \( N \) branes. The genuine noncommutative effects arise when the matrices cannot be simultaneously diagonalised.

One advantage of thinking about gauge theories in terms of branes is that many properties can be understood geometrically. We have already mentioned the simplest of these - separating D-branes gives mass to some states, proportional to the separation. This is just the Higgs effect. Consider two coincident D\( p \)-branes so that the gauge group is U(2). When we separate them the gauge group is broken to U(1)\(^2\) and two states (both orientations of open strings with one end on each brane) get a mass, \( m \), proportional to the separation, \( L, m \sim L/l_s^2 \) – these are the W-bosons\(^3\). At the same time we give an expectation value to a scalar field because we are now expanding around a configuration with \( X = \text{diag}(L/2, -L/2) \).

We will consider more complicated and more interesting brane configurations related to gauge theories in later sections. The main point to make here is that field theory states can be identified with open strings and that \( N \) parallel D-branes means gauge group U(\( N \)) which can be broken by separating the branes. It should also be noted that the U(1) associated to the centre of mass of the branes decouples and so typically we refer to the gauge group for this system as SU(\( N \)).

2.6 AdS/CFT correspondence

The AdS/CFT correspondence \[65, 66, 67\] describes a duality between string theory or M-theory and gauge theories. The most useful relations are in the limits where supergravity is a

\(^3\)When we are talking about states and W-bosons, we are implicitly referring to complete supermultiplets since supersymmetry is not being broken.
good approximation of string theory or M-theory. Then specific calculations can be performed in supergravity backgrounds which are related to properties of strongly coupled gauge theories. There are many reviews of this topic (e.g. [68]) and we will not go into much detail here. However, this is one of the motivations for trying to find supergravity solutions for branes ending on branes. So we will briefly describe the case of $\mathcal{N} = 4$ four-dimensional SU($N$) Yang-Mills and how the brane solution is used to show that the supergravity dual is $AdS_5 \times S^5$. The procedure will be essentially the same when we later consider intersecting brane configurations.

We start by choosing a gauge theory, here $\mathcal{N} = 4$ four-dimensional SU($N$) Yang-Mills. We then consider a brane configuration which describes this theory – $N$ coincident D3-branes. We now ask what the supergravity description of this brane configuration is, in the limit appropriate to describing the field theory. The field theory limit is $l_s \to 0$ in order to decouple gravity and massive string states. We must however keep certain quantities fixed when taking this limit, in particular the gauge coupling, or more conveniently the 't Hooft coupling

$$g_{YM}^2 N = 2\pi g_s N$$

and gauge theory masses. Gauge theory masses and VEVs are fixed by keeping (see equation (103))

$$U \equiv \frac{r}{l_s^2}$$

fixed where $r$ is the radial coordinate transverse to the branes. In this limit the harmonic function in the D3-brane solution is

$$H = 1 + \frac{4\pi N g_s l_s^4}{r^4} \rightarrow \frac{2g_{YM}^2 N}{U^{1/4}}$$

and hence the metric becomes

$$\frac{1}{l_s^2} ds^2 = \frac{U^2}{L^2} dx^2_{(1,3)} + \frac{dU^2}{U^2} + L^2 d\Omega_5^2$$

which is the metric for $AdS_5 \times S^5$ where the $AdS_5$ and $S^5$ both have radius

$$L = (2g_{YM}^2 N)^{1/4}$$

The supergravity description is valid when there are no large curvatures, i.e. when $L$ is large, whereas the gauge theory calculations can be performed for small 't Hooft coupling, i.e. when $L$ is small. Hence we have two dual descriptions.

### 3 Intersecting branes and branes ending on branes

In this section we will discuss some general properties of configurations of intersecting branes, in particular the amount of supersymmetry preserved. We will see how intersecting branes of the same type are related to a brane wrapping a smooth cycle and how the conditions for preservation of supersymmetry can be expressed in terms of calibrations. We will also consider the closely related generalisation to branes ending on branes. The existence of such configurations can be understood in several ways including via dualities of a fundamental string ending on a D-brane or by deforming an intersecting brane configuration by splitting the ‘smaller’ brane. The conditions for which branes can end on which other branes can also be directly derived from charge conservation conditions [69, 70, 71]. We will see how branes ending on branes can be described from the brane worldvolume point of view, using the Born-Infeld action. We will postpone the discussion of the corresponding supergravity solutions and various applications to later sections.
3.1 Orthogonally intersecting branes

The concept of orthogonally intersecting branes is very simple. All the essential features can be described using the simplest example of two branes, say a \((p+q_1)\)-brane and a \((p+q_2)\)-brane embedded as

\[
\begin{align*}
(p+q_1)\text{-brane}: & \quad \begin{array}{c}
\cdots\cdots
\end{array} \\
(p+q_2)\text{-brane}: & \quad \begin{array}{c}
\cdots\cdots
\end{array}
\end{align*}
\]

The two branes have a \((p+1)\)-dimensional common worldvolume when they intersect, i.e. when they are at the same position in the totally transverse \(\tilde{D}\)-dimensional space. The most important features are related to the relative transverse space, i.e. the \((q_1+q_2)\)-dimensional space with some directions spanned by one brane with the other brane located at a point. Obviously we can consider intersections of more types of branes and the relative transverse space will be described by the numbers of dimensions parallel to some branes and perpendicular to the others. It is the structure of the relatively transverse space (the numbers \(q_1\) and \(q_2\) in the case of two branes) which determines the amount of supersymmetry preserved. We will now consider this case of two branes (or more generally two different sets of parallel branes) in detail. In all cases it is the orientation, not the position, of the branes which determines the amount of supersymmetry preserved. Indeed, the branes will not actually intersect unless they are at the same location in the overall transverse space, although we will generically refer to all configurations containing non-parallel branes as intersecting brane configurations.

We know which supersymmetries are preserved by a single brane. For two branes (with an obvious generalisation to more branes) we need to consider which supersymmetries survive both projection conditions. So for brane 1 we have \(\Gamma^{(1)}\epsilon = \epsilon\) while for brane 2 we have \(\Gamma^{(2)}\epsilon = \epsilon\). In a trivial background, for the cases we will consider with no non-trivial worldvolume fields, \(\Gamma^{(1)}\) and \(\Gamma^{(2)}\) are essentially just a product of \(\hat{\Gamma}\)-matrices, with specific expressions given in section 2.1.3. So either \(\Gamma^{(1)}\Gamma^{(2)} = \Gamma^{(2)}\Gamma^{(1)}\) or \(\Gamma^{(1)}\Gamma^{(2)} = -\Gamma^{(2)}\Gamma^{(1)}\) and (other than the trivial case where \(\Gamma^{(1)} = \Gamma^{(2)}\) which preserves half the supersymmetry) we have \(\text{tr}(\Gamma^{(1)}\Gamma^{(2)}) = 0\). In the case where \(\Gamma^{(1)}\) and \(\Gamma^{(2)}\) anti-commute

\[
\epsilon = \Gamma^{(1)}\epsilon = \Gamma^{(1)}\Gamma^{(2)}\epsilon = -\Gamma^{(2)}\Gamma^{(1)}\epsilon = -\Gamma^{(2)}\epsilon = -\epsilon
\]

so clearly no supersymmetry is preserved. In the cases where the projections commute, we will show that one quarter supersymmetry is preserved. Because \(\Gamma^{(1)}\) and \(\Gamma^{(2)}\) commute they can be simultaneously diagonalised and because they are traceless and square to 1, they have equal numbers of (i.e. 16) \(+1\) and \(-1\) eigenvalues. So we have say \(n_{++}\) simultaneous eigenstates of \(\Gamma^{(1)}\) and \(\Gamma^{(2)}\) with eigenvalues \(+1\) and \(-1\) respectively, \(n_{++}\) with eigenvalues \(+1\) and \(+1\) etc. where

\[
n_{++} + n_{+-} = n_{-+} + n_{--} = n_{++} + n_{-+} = n_{+-} + n_{--} = 16
\]

So in particular

\[
n_{++} = n_{--} \text{ and } n_{+-} = n_{-+}
\]

Then since \(\Gamma^{(1)}\Gamma^{(2)}\) is traceless it is easy to see that

\[
n_{++} + n_{--} = 16 = n_{+-} + n_{-+}
\]

and so

\[
n_{--} = n_{+-} = n_{-+} = n_{--} = 8
\]

So the number of supersymmetries preserved is \(n_{++} = 8\), i.e. one quarter of the 32 supersymmetries.
3.1.1 Simple examples

In the cases considered here with no worldvolume fieldstrengths it is easy to see that, for two D-branes in type IIA or type IIB, or for any two branes of the same type,

\[
\begin{align*}
[\Gamma^{(1)}, \Gamma^{(2)}] &= 0 \quad \text{if} \quad q_1 + q_2 = 0 \pmod{4} \\
\{\Gamma^{(1)}, \Gamma^{(2)}\} &= 0 \quad \text{if} \quad q_1 + q_2 = 2 \pmod{4}
\end{align*}
\] (111)

(112)

So we see that a common condition for preserving one quarter supersymmetry is that the branes have 4 relative transverse dimensions. We will discuss the case where the branes are of the same type in section 3.2.

3.1.2 General orthogonal intersections

If we have more than two types of orthogonally intersecting branes then we can similarly analyse the amount of supersymmetry preserved. Clearly the result depends on the types and the orientations of the branes but we can comment on some general features.

Obviously if the whole configuration of, say \(m\) types of, orthogonally intersecting branes is to preserve any supersymmetry then a necessary condition is that each pair of branes must preserve (one quarter) supersymmetry. This is in fact almost a sufficient condition as can be seen by performing a similar analysis of the simultaneous eigenstates of the \(m\) operators \(\Gamma^{(i)}\), as above for \(m = 2\). It can be seen that provided the product of any number of these (distinct) operators is traceless, then precisely \(1/2^m\) supersymmetry is preserved. This will typically be true for the cases we are discussing where each such operator is simply a product of \(\hat{\Gamma}\)-matrices. However, it is possible that some product of these operators is plus or minus the identity, rather than a (traceless) product of \(\hat{\Gamma}\)-matrices. In this case with the plus sign, one of the projection conditions is already imposed by the others and so does not further break supersymmetry. I.e. if \(\Gamma^{(1)}\Gamma^{(2)}\Gamma^{(3)} = 1\) then

\[\Gamma^{(2)}\epsilon = \epsilon\quad\text{and}\quad\Gamma^{(3)}\epsilon = \epsilon\Rightarrow\Gamma^{(1)}\epsilon = \epsilon\]

The case with the minus sign breaks all supersymmetry. However, this sign can be changed by reversing the orientation of one of the branes.

So to summarise, in the case of \(m\) orthogonally intersecting branes in Minkowski spacetime, with no non-trivial worldvolume fields, the condition for preserving supersymmetry is that all pairs of the projection operators commute. The amount of supersymmetry preserved is \(1/2^l\) where \(2 \leq l \leq m\), with the proviso that if \(l < m\) then the worldvolume orientations of \(m - l\) branes are fixed in terms of the others.

3.1.3 More Examples

If we have a configuration of three sets of orthogonally intersecting M5-branes with worldvolume directions 012345, 012367 and 012389 then the projection conditions are

\[
\begin{align*}
\hat{\Gamma}_{012345}\epsilon &= \epsilon \\
\hat{\Gamma}_{012367}\epsilon &= \epsilon \\
\hat{\Gamma}_{012389}\epsilon &= \epsilon
\end{align*}
\] (113)
(114)
(115)

It can easily be seen that these three conditions are compatible and independent and so such a configuration will preserve one eighth supersymmetry.
If instead we consider the two types of M5-branes with worldvolume directions 012345 and 012367 then we would preserve one quarter supersymmetry. However, we can still add another brane without breaking any more supersymmetry. This is because
\[ \epsilon = \hat{\Gamma}_{012345}\epsilon = \hat{\Gamma}_{012345}\hat{\Gamma}_{012367}\epsilon = -\hat{\Gamma}_{4567}\epsilon \]  
which is the projection condition for a KK6-brane. Since \( \hat{\Gamma}_0 \cdots \hat{\Gamma}_{(10)} = 1 \) we can equivalently write this projection condition as
\[ \hat{\Gamma}_{012389(10)}\epsilon = -\epsilon \]  
(117)

Note that the orientation of one set of branes is fixed in terms of the others. I.e. if \( s_1 = \pm 1 \) and \( s_2 = \pm 1 \) correspond to the choice of orientation of the two types of M5-branes then we have one quarter supersymmetry preserving projection conditions
\[ \hat{\Gamma}_{012345}\epsilon = s_1\epsilon \]  
(118)
\[ \hat{\Gamma}_{012367}\epsilon = s_2\epsilon \]  
(119)

We still preserve one quarter supersymmetry if the orientation of the KK6-branes is chosen so that
\[ \hat{\Gamma}_{012389(10)}\epsilon = -s_1s_2\epsilon \]  
(120)
while all supersymmetry is broken if we chose the opposite orientation
\[ \hat{\Gamma}_{012389(10)}\epsilon = s_1s_2\epsilon \]  
(121)

Upon reduction to type IIA along the isometry direction, say \( x^7 \), this gives a quarter BPS configuration of orthogonally intersecting NS5-branes, D4-branes and D6-branes with world-volume directions 012345, 01236 and 012389(10) respectively.

### 3.2 Holomorphic intersections and calibrations

Here we discuss the special case of two \( p \)-branes intersecting with \( p - 1 \) common worldvolume directions. This preserves one quarter supersymmetry since, in the notation of section 3.1 \( q_1 = q_2 = 2 \). Let us for definiteness consider M2-branes. To start with we will consider two M2-branes with worldvolume directions 012 and 034, in flat spacetime. Then we have the following projection conditions
\[ \hat{\Gamma}_{012}\epsilon = \hat{\Gamma}_{034}\epsilon = \epsilon \]  
(122)

Note that these relations mean that
\[ \hat{\Gamma}_{1234}\epsilon = -\epsilon \]  
(123)
which leads to several relations of the following form
\[ \hat{\Gamma}_{013}\epsilon = -\hat{\Gamma}_{024}\epsilon \]  
(124)

If we now define complex coordinates \( z^m, m = 1, 2 \) by
\[ z^1 = x^1 + ix^2, \quad z^2 = x^3 + ix^4 \]  
(125)
then we can concisely express the above relations as
\[ \hat{\Gamma}_{0m\pi}\epsilon = i\delta_{m\pi}\epsilon \]  
(126)

We use conventions where \( ds^2 = 2g_{m\pi}dz^m\overline{dz}^\pi \) so that \( \delta_{\pi} = 1/2 \) etc.
Now we can easily check that we can add an M2-brane with embedding defined by an arbitrary holomorphic curve without breaking any more supersymmetry. I.e. we embed the M2-brane in the 1234 directions as the zeroes of a holomorphic function \( f(z^1, z^2) \). This is equivalent to defining a complex coordinate \( z = \sigma^1 + i \sigma^2 \) on the brane worldvolume and embedding the brane so that \( z^m = Z^m(z) \). The induced metric on the worldvolume of such a brane has non-zero components
\[
G_{00} = -1, \quad G_{z\bar{z}} = (\partial_z Z^m)(\overline{\partial_{\bar{z}} Z^m})\delta_{mn}
\]  
(127)
So the projection operator for such a brane is given by
\[
\Gamma = \gamma(3) = -i G^{-1}_{z\bar{z}}(\partial_z Z^m)(\overline{\partial_{\bar{z}} Z^m}) \hat{\Gamma}_0 m \bar{m}
\]  
(128)
Now using the projections conditions (126) which preserve one quarter supersymmetry we see that we don’t introduce any extra constraints and so will still preserve one quarter supersymmetry with this arbitrary holomorphic embedding
\[
\gamma(3) \epsilon = -i G^{-1}_{z\bar{z}}(\partial_z Z^m)(\overline{\partial_{\bar{z}} Z^m}) \hat{\Gamma}_0 m \bar{m} \epsilon = G^{-1}_{z\bar{z}}(\partial_z Z^m)(\overline{\partial_{\bar{z}} Z^m}) \delta_{mn} \epsilon = \epsilon
\]  
(129)
It is easy to check that the same results hold in the case where we have an arbitrary background complex Hermitian metric \( g_{m\bar{n}} \), i.e.
\[
ds^2 = g_{00} dx_0^2 + 2 g_{m\bar{n}} dz^m d\bar{z}^n + ds_\perp^2
\]  
(130)
In this case we must also check that the background preserves supersymmetry even without the branes. For example, without any background field strengths this will require \( g_{m\bar{n}} \) to be a Calabi-Yau metric in order that there will be covariantly constant spinors. The inclusion of background fields leads to more complicated restrictions which are still not fully classified (though see e.g. \cite{72, 73, 74}.) However, in the cases where the background geometry is generated by the branes themselves, the background will preserve the same supersymmetries as the brane in flat space. In other words, considerations of supersymmetry preservation give the same results whether or not we include the backreaction of the branes. This method was used for this case of branes wrapping supersymmetric 3-cycles as well as branes wrapping supersymmetric 2-cycles in Calabi-Yau 3-folds in \cite{75}. See also \cite{76, 77} for an analysis, in terms of \( \kappa \)-symmetry projection conditions, of the allowed supersymmetry-preserving angles between intersecting branes, and \cite{78} for some further analysis for intersecting and wrapped branes.

There are other useful way of understanding the geometry of supersymmetry-preserving embeddings of branes. In the case of D-branes we can consider the consistent boundary conditions which can be imposed in the string worldsheet SCFT \cite{79, 80}. A powerful method we will review is that of calibrations \cite{81, 82} which has been used to classify the supersymmetric cycles which branes can wrap in various special holonomy manifolds \cite{83, 75, 84}. This construction involves a calibrating form \( \Omega \). In a background with vanishing fieldstrengths, this is a \( p \)-form which is closed
\[
d\Omega = 0
\]  
(131)
and such that at every point in the manifold the pullback of \( \Omega \) to any tangent \( p \)-plane is less than or equal to the volume form. We further require that at any point there exists some \( p \)-plane for which this bound is saturated. If we have a calibrating \( p \)-form then we can use it to find minimal volume \( p \)-dimensional submanifolds. To see this consider two \( p \)-dimensional submanifolds \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) of volume \( V(\mathcal{M}_1) \) and \( V(\mathcal{M}_2) \) respectively which share the same boundary. Then because \( \Omega \) is closed we have
\[
\int_{\mathcal{M}_1} \Omega = \int_{\mathcal{M}_2} \Omega
\]  
(132)
and because the pullback of \( \Omega \) is bounded by the volume form at each point on \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) we have
\[
\int_{\mathcal{M}_1} \Omega \leq V(\mathcal{M}_1), \quad \int_{\mathcal{M}_2} \Omega \leq V(\mathcal{M}_2)
\]
(133)

We say that \( \mathcal{M}_1 \) is a calibrated submanifold if
\[
\int_{\mathcal{M}_1} \Omega = V(\mathcal{M}_1)
\]
(134)

The claim is that a calibrated submanifold is a minimal volume submanifold (with given boundary) which can be easily checked
\[
V(\mathcal{M}_1) = \int_{\mathcal{M}_1} \Omega = \int_{\mathcal{M}_2} \Omega \leq V(\mathcal{M}_2)
\]
(135)

Note that the condition for a calibrated submanifold is a local one - that the pullback of \( \Omega \) is equal to the volume form at each point. Also, for static branes with no coupling to background fields, minimising the volume is equivalent to minimising the energy (or Hamiltonian.)

For Kähler manifolds we have calibrating \( 2p \)-forms defined in terms of the Kähler form \( \omega = i g_{m \pi} dz^m \wedge d\bar{z}^\pi \) by
\[
\Omega = \frac{1}{p!} \omega^p
\]
(136)

The Wirtinger theorem is then the statement that all complex submanifolds are calibrated submanifolds. In the above M2-brane example we would have \( \Omega = \omega \) (or \( \Omega = \sqrt{-g_{00}} dx^0 \wedge \omega \) if we include the part of the brane world volume which is trivially embedded.) Then clearly for a holomorphic embedding the pullback of \( \omega \) is the same as the volume form
\[
P(\omega) = i(\partial_z Z^m)(\overline{\partial_z Z^n})g_{m\pi} dz \wedge d\bar{z} = iG_{z \bar{z}} dz \wedge d\bar{z}
\]
(137)

Now we can also easily derive the relation between the methods of using the \( \kappa \)-symmetry projection conditions on covariantly constant spinors (we restrict to the case of no background field strengths) and calibrations. We will show how the calibrating form can be constructed from the spinor \([52, 75, 83, 85, 86, 87]\). We start from the projection condition
\[
\frac{1}{2}(1 - \Gamma)\epsilon = 0
\]
(138)

where in the absence of any background field strengths \( \epsilon \) is a covariantly constant spinor which we can therefore normalise
\[
\epsilon^\dagger \epsilon = 1
\]
(139)

Now note that for a static brane, \( \Gamma \) is Hermitian – for example for M2-branes \( \Gamma \) is a (real) linear combination of \( \hat{\Gamma}_0 \hat{\Gamma}_i \hat{\Gamma}_j \) and
\[
(\hat{\Gamma}_0 \hat{\Gamma}_i \hat{\Gamma}_j)^\dagger = \hat{\Gamma}_j^\dagger \hat{\Gamma}_i^\dagger \hat{\Gamma}_0^\dagger = \hat{\Gamma}_j \hat{\Gamma}_i (-\hat{\Gamma}_0) = \hat{\Gamma}_0 \hat{\Gamma}_i \hat{\Gamma}_j
\]
(140)

So we now have the following inequality from the projection condition \([138]\)
\[
0 \leq \left(\frac{1}{2}(1 - \Gamma)\epsilon\right)^\dagger \left(\frac{1}{2}(1 - \Gamma)\epsilon\right) = \epsilon^\dagger \frac{1}{2}(1 - \Gamma) \frac{1}{2}(1 - \Gamma) \epsilon = \epsilon^\dagger \frac{1}{2}(1 - \Gamma) \epsilon
\]
(141)

which can obviously be rearranged to give
\[
\epsilon^\dagger \Gamma \epsilon \leq \epsilon^\dagger \epsilon = 1
\]
(142)
This is essentially the same as the inequality for the pullback of a calibrating form. More precisely we have, for any submanifold $M$ corresponding to the embedding of a static brane,

$$V(M) = \int_M d^{p+1}\sigma \sqrt{|G|} \geq \int_M d^{p+1}\sigma \sqrt{|G|} \epsilon \Gamma \epsilon = \int_M \mathcal{P}(\Omega)$$ (143)

Note that from the definition of $\Gamma$ it is clear that $\epsilon \hat{\Gamma}_\epsilon \partial \sigma^0 \wedge \cdots \wedge d\sigma^p$ will be the pullback of a $p+1$-form $\Omega \sim (\epsilon \hat{\Gamma}_\epsilon \partial \sigma^0 \wedge \cdots \wedge d\sigma^p)$ to $\mathcal{M}$. Hence, allowing for arbitrary $M$ this construction does indeed define a spacetime $p+1$-form, $\Omega$. Furthermore $\Omega$ is closed because $\epsilon$ is covariantly constant.

Consider the above M2-brane example. We have

$$\epsilon \hat{\Gamma}_{012} \epsilon = \epsilon \hat{\Gamma}_{034} \epsilon = \epsilon \hat{\Gamma} \epsilon = 1$$ (145)

and otherwise

$$\epsilon \hat{\Gamma}_{0ij} \epsilon = 0$$ (146)

since for example

$$\epsilon \hat{\Gamma}_{013} \epsilon = \epsilon \hat{\Gamma}_{013}(\hat{\Gamma}_{012}) = -\epsilon \hat{\Gamma}_{012} \hat{\Gamma}_{013} \epsilon = - (\hat{\Gamma}_{012}) \epsilon \hat{\Gamma}_{013} \epsilon = -\epsilon \hat{\Gamma}_{013} \epsilon$$ (147)

So we can now see that, using the vielbein $\epsilon^I_i$ for the Kähler metric $g_{IJ}$ on the 1234 space,

$$\sqrt{|G|} \epsilon \hat{\Gamma} \epsilon = \epsilon \sqrt{-g_{00}} \epsilon^I_i \epsilon^J_j \hat{\Gamma}_{0ij} \epsilon \partial \sigma^0 X^0 \partial \sigma^1 X^1 \partial \sigma^2 X^2$$ (148)

$$= \sqrt{-g_{00}} \epsilon^I_i \epsilon^J_j \delta_{ij} \partial \sigma^0 X^0 \partial \sigma^1 X^1 \partial \sigma^2 X^2$$ (149)

$$= \sqrt{-g_{00}} g_{IJ} \partial \sigma^0 X^0 \partial \sigma^1 X^1 \partial \sigma^2 X^2$$ (150)

and so

$$\sqrt{|G|} \epsilon \hat{\Gamma} \epsilon d\sigma^0 \wedge d\sigma^1 \wedge d\sigma^2 = \mathcal{P}(\Omega)$$ (151)

where $\Omega$ can be expressed in terms of the Kähler form, $\omega$, giving the expected result

$$\Omega = \sqrt{-g_{00}} dx^0 \wedge \omega$$ (152)

Note that there are obvious generalisations to the M2-brane wrapping a 2-cycle in an $n$-dimensional Kähler manifold. Similarly for any $p$-brane we would get the result

$$\Omega = dV_{p-1} \wedge \omega$$ (153)

where $dV_{p-1}$ is the $(p-1)$-dimensional volume form for the part of the $p$-brane trivially embedded.

The above construction crucially depends on the existence of a covariantly constant spinor which is the same condition required for the background to preserve supersymmetry. So this construction can be generalised to other manifolds which admit covariantly constant spinors. The basic types are listed in table 1 along with the fraction of supersymmetry preserved. Introducing a brane will break a further one half supersymmetry. Note however that this is the minimum amount of supersymmetry and more can be preserved in special cases.

For Calabi-Yau manifolds we have $2m$-cycles which are complex submanifolds calibrated by $\frac{1}{m!} \omega^m$ as already mentioned. These are collectively referred to as Kähler calibrations. There are also special-Lagrangian submanifolds which are $n$-cycles in the manifolds of complex dimension $n$, calibrated by

$$\Omega = \mathcal{R}e \left( e^{i\theta} dz^1 \wedge \cdots \wedge dz^n \right)$$ (154)
for some constant $\theta$. See [88] for a review of Calabi-Yau manifolds and special Lagrangian submanifolds.

Hyper-Kähler manifolds are similar to Calabi-Yau manifolds. The additional feature is that there are additional calibrating forms corresponding to the different choices of complex structure. Note that in the case of two complex dimensional manifolds where a Calabi-Yau manifold is automatically a hyper-Kähler manifold (since $\text{SU}(2)$ is the same as $\text{Sp}(1)$) the special-Lagrangian submanifolds are simply holomorphic curves with respect to a different choice of complex structure.

There are also calibrations in the cases of exceptional holonomy. In $G_2$ holonomy (seven-dimensional) manifolds we have 3- and 4-cycles calibrated by a 3-form and its Hodge dual 4-form, called associative and coassociative calibrations respectively. In $\text{Spin}(7)$ holonomy (eight-dimensional) manifolds we have 4-cycles calibrated by a self-dual 4-form, known as Cayley calibrations.

There are various ways to build more general calibrations. For example a submanifold could be a product of calibrated submanifolds in different spaces. It is also possible to consider calibrating forms which are linear combinations of the forms mentioned above. For example in a Calabi-Yau four-fold we can take a linear combination of the Kähler and special Lagrangian calibrating 4-forms [83].

Note that the Kähler calibrations are particularly simple in that the embedding conditions are specified by an appropriate number of arbitrary holomorphic functions. In other cases the embedding conditions for a supersymmetric submanifold are more complicated and there is no simple expression of the general solution, e.g. see [89].

There are also various generalised calibrations which allow for non-trivial worldvolume fields [90, 91, 92, 93], background field strengths [94, 95] or both [96]. We can interpret background fields as torsion [72, 97, 73] due to the way they enter into the Killing spinor equations $\tilde{D}_\mu \epsilon = 0$. In the cases with background fields (or torsion) the calibrating form is no longer closed and so calibrated submanifolds are not minimal volume submanifolds. However, they are again energy-minimising embeddings of branes, including the appropriate interaction with the background potentials [88]. It is also possible to lift the restriction on the branes being static [99].

Generalised calibrating forms can be constructed from the Killing spinors. Although these forms are not closed, they are covariantly constant with respect to the connection with torsion. The existence of covariantly constant spinors and vectors with respect to such a connection can be understood in terms of a reduced holonomy group, again with respect to the connection with torsion [100, 101, 102, 103, 104]. Actually, in type II theories there are two different relevant connections [101]. Supersymmetric solutions can be classified by covariantly constant generalised calibrating forms with respect to each of these connections, referred to as G-structures [72, 73]. Studying the possible G-structures may be useful in classifying all supersymmetric solutions of supergravity theories. So far such a classification only exists for some four-dimensional [105, 106, 107] and five-dimensional [74] supergravity theories. In fact, only recently have all maximally supersymmetric solutions of ten- and eleven-dimensional

| Type          | Real dimension | Special holonomy | Preserved supersymmetry |
|---------------|----------------|------------------|-------------------------|
| Calabi-Yau    | $2n$           | $\text{SU}(n) \subset \text{SO}(2n)$ | $1/2^{n-1}$             |
| Hyper-Kähler  | $4n$           | $\text{Sp}(n) \subset \text{SO}(4n)$ | $(n+1)/4^n$             |
| $G_2$         | $7$            | $G_2 \subset \text{SO}(7)$ | $1/8$                   |
| $\text{Spin}(7)$ | $8$            | $\text{Spin}(7) \subset \text{SO}(8)$ | $1/16$                  |

Table 1: Supersymmetric special holonomy manifolds.
supergravities been classified \[108, 109, 110\].

3.3 Branes intersecting at angles

A natural question to ask is what are the most general supersymmetry-preserving configurations of intersecting branes. For simplicity we restrict to the case of branes with no non-trivial worldvolume fields embedded (statically) in Minkowski spacetime. The problem essentially reduces to a technically more complicated analysis using the same methods presented for orthogonal intersections in section 3.1. Several cases were considered in [80] where the conditions for supersymmetric intersections were derived using the $\kappa$-symmetry projections and also string worldsheet boundary conditions for the cases involving D-branes. The results for branes of the same type were expressed in terms of a generalised holonomy which is equivalent to the results of section 3.2 expressed in terms of calibrated submanifolds. The field theory interpretation of branes intersecting at angles and the appearance of chiral fermions was discussed in [80, 114].

Take the case of two planar M5-branes as an example, with one of them having worldvolume directions $012345$. The embedding of the second M5-brane is related by a rotation of the spatial 5-plane (for static configurations) in the ten-dimensional space. This can be parameterised in terms of five angles describing the rotations in each of the 2-planes spanning e.g. directions 16, 27, etc. The supersymmetry projection conditions can be analysed with the result \[115\] that for various constraints on the angles the possible fractions of supersymmetry which can be preserved are

\[
\frac{1}{32}, \frac{1}{16}, \frac{3}{32}, \frac{1}{8}, \frac{3}{32}, \frac{1}{4}, \frac{1}{2}.
\]

The same example as well as other configurations of intersecting branes of the same type can also be interpreted in terms of calibrations \[116, 85, 86, 87\]. A related group-theoretic formulation of the above problem for two M5-branes was presented \[117\] in terms of finding subgroups of $Spin(10)$ which leave spinors invariant. Branes related by a rotation in such a subgroup impose the same projection on the singlet spinor(s). In terms of calibrations this is because the Killing spinors, and so the calibrating forms (which are built out of them as in section 3.2) are singlets of such a subgroup. Hence, given any calibrated submanifold, there is a whole family of calibrated submanifolds, related by arbitrary rotations under this subgroup of $Spin(10)$. This approach can be generalised to the case of more than two sets of intersecting branes \[117\] and non-static intersecting branes \[99\]. For example, M5-branes with a common $(1 + 1)$-dimensional worldvolume and the other four worldvolume directions related by SU(4) rotations preserve $1/16$ supersymmetry \[115, 117\] as expected from the discussion of table 1 in section 3.2. Note however, that in many cases there is more supersymmetry preserved than the minimum amount. The cases of branes intersecting at angles in pp-wave backgrounds has recently been discussed in \[118, 119\].

We can think of all static supersymmetric configurations of intersecting branes of the same type (without worldvolume or background fields) as special (singular) examples of calibrated submanifolds. For example in the case of two branes we parameterise the relative orientations of the branes by some angles. We choose an appropriate calibrating form so that one brane (which we consider a fixed plane) is calibrated and then demanding that the second brane is also a calibrated plane will lead to certain conditions on the angles. The cases of Kähler calibrations are particularly simple. We have already seen that the condition for the static

\[^4\text{It is also possible to consider the conditions for a stable (BPS or non-BPS) configuration by requiring the forces on each brane to cancel. E.g. this was considered for the case of four D-branes in \[111\] by summing the interaction forces between each pair of branes. These forces can be calculated from one-loop open string amplitudes – see e.g. \[112\] or analytically continue the scattering amplitudes for two parallel branes \[113, 49\].}\]
embedding of the spatial part of an M2-brane in a two (complex) dimensional space is that the embedding is given by the zeroes of a holomorphic function \( f(z^1, z^2) = 0 \). The cases where this function factorises are singular limits of manifolds which describe intersecting branes. For example \( f = z^1 z^2 \) describes two orthogonally intersecting branes embedded at \( z^1 = 0 \) and \( z^2 = 0 \) whereas \( f = z^1 z^2 + c \) would describe a smoothly wrapped M2-brane (for \( c \neq 0 \)) with the same asymptotic form as the two orthogonally intersecting branes. Similarly we can describe two branes with an intersection parameterised by an angle \( \theta \) by choosing \( f = z^1 (\cos \theta z^2 + \sin \theta z^1) \). Again this can be viewed as the singular limit of a smooth manifold by adding a constant term to \( f \).

We can clearly generalise to an arbitrary number, say \( n \), of M2-branes by taking \( f \) to be a product of \( n \) linear factors. There are various subleading terms we can add to \( f \) to describe a smooth configuration with the same asymptotic behaviour. Similar results immediately apply to any co-dimension two Kähler calibration which is again specified by a single holomorphic function (of the appropriate number of complex coordinates.) We can also generalise to other calibrations although it may not be possible to describe the related smooth cycles exactly.

The cases where we have different types of branes intersecting cannot be directly related (though they may be indirectly related via dualities) to a single brane wrapping a smooth cycle. The conditions for supersymmetry are most easily analysed using the \( \kappa \)-symmetry projection conditions although such cases can presumably by analysed using generalised calibrations if a suitable supergravity background is known. I.e. in the case of two types of branes we could consider generalised calibrations to determine how the second brane could be embedded into the background of the first brane.

An interesting example of different types of branes intersecting at angles is that of a \((p, q)\) 5-brane web \[120, 121\] or \((p, q)\)-string web \[122, 123, 124, 125, 126, 127\]. The simplest case is a static configuration of strings, all lying within a 2-plane, although there are also non-planar supersymmetric configurations \[128\]. A 5-brane web is essentially the same, with the 5-branes having a common \((4 + 1)\)-dimensional worldvolume. Three \((p_i, q_i)\)-strings can meet at a ‘string junction’ provided the charges are conserved \[123, 124\], i.e. provided (with some appropriate definition of the orientation of strings at each junction to distinguish between \((p, q)\)-strings and \((-p, -q)\)-strings)

\[
\sum_{i=1}^{3} p_i = 0 = \sum_{i=1}^{3} q_i
\]

In this way complicated webs of strings can be constructed. Such a configuration will preserve one quarter supersymmetry provided each \((p, q)\)-string lies at an angle in the plane given by \( \theta \) in the \( \kappa \)-symmetry projector equation \(67\) \[12\]. This is precisely the same condition for the forces due to the string tensions to balance at each junction \[125\]. The condition can also be derived from the dual description of a string junction as an M2-brane with a holomorphic embedding \[122, 126, 127\]. String junctions ending on 5-brane webs \[129\] or other type IIB branes \[130, 131, 132, 133, 134, 135\] correspond to BPS states in the corresponding field theory. The presence of 7-branes is particularly interesting since a \((p, q)\)-string winding around a 7-brane undergoes a monodromy transformation which maps it to a different \((p, q)\)-string. This gives a complicated BPS spectrum which can lead to the appearance of exceptional symmetry groups \[124, 136, 137, 138, 139, 140, 141, 142\].

### 3.4 Worldvolume description

Before we discuss supergravity solutions for intersecting branes, it is interesting to consider the worldvolume description. We will see that if we have a brane ending on another brane then we
can use the worldvolume theory of either brane. The ‘smaller’ brane then appears as a spike on the worldvolume of the ‘larger’ brane \[143, 144, 145, 146\] or equivalently the ‘larger’ brane appears as a funnel expanding from the worldvolume of a large number of ‘smaller’ branes \[147, 148\], rather similar to the Myers dielectric effect \[64\]. Such solitonic solutions of the DBI action are often called BIons \[145\]. To be concrete we will consider the case of \(N\) coincident D1-branes with worldvolume directions 04 ending on a D3-brane with worldvolume directions 0123. Note that we expect this configuration to preserve one quarter supersymmetry since the D-branes have a four-dimensional relative transverse space. We closely follow the discussion of \[149\].

3.4.1 D3-brane worldvolume description

Let’s start with the D3-brane worldvolume theory. The picture is that the D1-branes can be described as a spike extending from the D3-brane. To see this, consider the DBI action, equation \[101\], for a D3-brane in Minkowski spacetime with a static embedding \(x^\mu = X^\mu(\sigma^0, \ldots, \sigma^3)\) where

\[
X^{0,1,2,3} = \sigma^{0,1,2,3} \quad , \quad X^4 = X^4(\sigma^1, \sigma^2, \sigma^3) \quad , \quad X^{5,6,7,8,9} = 0
\]  

(156)

From the D3-brane worldvolume point of view a D1-brane ending on it is a magnetically charged particle (monopole) so we expect a solution where the worldvolume electric field vanishes

\[
E_i = F_{0i} = 0
\]  

(157)

but we have a non-trivial magnetic field

\[
B_i \equiv \frac{1}{2} \epsilon_{ijk} F^{jk}
\]  

(158)

In this case, with vanishing NS-NS B-field and defining \(\Phi\) as in equation \[103\]

\[
\Phi = \frac{1}{\lambda} X^4 \quad , \quad \lambda = 2\pi l_s^2
\]  

(159)

we have

\[
- \det(G_{\mu\nu} + F_{\mu\nu}) = 1 + \lambda^2 (|\nabla \Phi|^2 + |B|^2) + \lambda^4 (B, \nabla \Phi)^2 = (1 \pm \lambda^2 (B, \nabla \Phi))^2 + \lambda^2 |\nabla \Phi \mp B|^2
\]  

(160)

So we see the appearance of a BPS bound which is saturated when

\[
B = \pm \nabla \Phi
\]  

(161)

Using the Bianchi identity for \(F\) with a magnetic source of charge \(N\) at \(r \equiv \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = 0\) we have, taking the ‘+’ sign

\[
\nabla^2 \Phi = -2\pi N \delta^3(r)
\]  

(162)

with solution (for asymptotically vanishing \(\Phi\))

\[
\Phi = \frac{N}{2r} \quad , \quad B = -\frac{N}{2r^2} \hat{r}
\]  

(163)

So as \(r \to 0\), \(\Phi\), and so \(X^4\), diverges. This spike is interpreted as \(N\) coincident D1-branes ending on the D3-brane at \(r = 0\). Indeed we can easily find the total energy of this static BPS configuration from the DBI action

\[
E = T_{D3} \int d^3 \sigma \left( 1 + \lambda^2 (B, \nabla \Phi) \right) = T_{D3} \int d^3 \sigma + NT_{D1} \int_0^\infty dX^4
\]  

(164)
which gives precisely the energy of a D3-brane with \( N \) D1-branes ending on it and extending to \( X^4 = \infty \).

There are various simple generalisations of this configuration which are related by dualities. Configurations with \( Dq \)-branes ending on a \( Dp \)-brane (with \( q \leq p \)) are related to this system by T-duality and similarly have a \( Dp \)-brane worldvolume description \[143, 144, 145, 150, 151, 152\]. The case where \( q = p \) can be generalised to describing a \( Dp \)-brane wrapping a smooth supersymmetric cycle. By \( \text{SL}(2, \mathbb{Z}) \) duality of the D1-branes ending on a D3-brane we can describe any \((p, q)\)-strings ending on a D3-brane. These solutions were found in the same way by introducing dyonic sources on the D3-brane \[144\]. The scattering of parallel \((p, q)\)-strings was analysed and the moduli space metric was found in \[153\]. In the case of fundamental strings ending on a D3-brane we can use T-duality to relate this to fundamental strings ending on any Dp-brane. Such solutions can be found using the \((p+1)\)-dimensional DBI action \[143, 145, 151\]. It can also be seen that the case of fundamental strings ending on a D4-brane can be derived from M2-branes ending on an M5-brane \[143, 151, 143\].

By analysing fluctuations around these configurations it is possible to perform further checks on the identification of these worldvolume solitons as other branes. For example by analysing the reflection of waves along the supposed fundamental strings ending on a Dp-brane it can be seen that the appropriate Dirichlet and Neumann boundary conditions are enforced \[143, 146, 154, 155\]. The behaviour of such fluctuations along a spike can also be seen to be the same as modes propagating along a probe string in the D3-brane supergravity background \[140, 155\]. It can also be checked that these BPS configurations are supersymmetric with the expected spinor projection conditions and central charges \[146, 151\]. In fact it is possible to see that BPS solutions of the worldvolume action (or low energy \( \sigma \)-model) are related to calibrated embeddings \[145, 156, 157, 158\].

A simple generalisation of the case of \( N \) coincident branes ending on another brane is to separate the \( N \) branes \[143\]. For example in the case of D1-branes ending on a D3-brane we can do this by choosing \( \Phi \) to be a multi-centred harmonic function. Note that in terms of each D1-brane the sign of \( \Phi \) (near its core) corresponds to whether it extends to \( X^4 = \pm \infty \). The sign of \( \mathbf{B} \cdot \mathbf{r} \) corresponds to the sign of the charge of the monopole which is equivalent to whether the orientation of the D1-brane is towards or away from the D3-brane. Hence we see that the relative sign between \( \mathbf{B} \) and \( \nabla \Phi \), which is the same for all D1-branes in a BPS configuration corresponds to a choice of orientation of the D1-branes. So, as expected in a BPS configuration all the D1-branes are parallel and there are no anti-D1-branes.

It is also possible to describe more complicated configurations of \((p, q)\)-string junctions ending on a D3-brane \[159\]. It was also shown that these solutions for strings ending on D3-branes (and M2-branes ending on M5-branes) could be described in the appropriate parallel D3-brane (or M5-brane) supergravity background \[159\]. Some cases where worldvolume fields are non-zero (even in the absence of the BIon) or there is a background NS-NS B-field (or 3-form in the case of M5-branes) have been considered in \[145, 93, 160, 161, 162\]. For example, solutions describing non-orthogonal intersections can also be found, e.g. a D1-brane at an angle \( \pi/2 - \alpha \) arises when a constant NS-NS B-field with non-zero component \( \lambda B_{12} = \tan \alpha \) is introduced \[162\].

One process which can be described using the D3-brane worldvolume theory is that of D1-branes intersecting the D3-brane and then splitting into separated D1-branes ending on the D3-brane (with equal numbers extending to positive and negative \( X^4 \).) This similarly applies to the other configurations mentioned above. This supports the idea that the conditions for preservation of supersymmetry are the same for branes ending on branes as for intersecting branes since the former configuration, at least at the level of the DBI action, is a smooth BPS deformation of the latter one. The constraint of equal numbers of branes ending from each side
is not important since we can simply move some branes to infinity (provided the scalars have at least a \(1/r\) fall-off.) It is also possible to analyse the interaction between different strings ending on a D3-brane [163].

There are also similar solutions when there are several parallel ‘larger’ branes [146, 164]. The BPS properties essentially guarantee that the Blon solutions of the DBI action are also solutions of a non-Abelian generalisation. There are also solutions describing fundamental strings stretching between two parallel D-branes [145].

There are similar non-BPS configurations describing, for example, D1-branes ending on D5-branes [165, 149] or fundamental strings stretching between a D-brane and an anti-D-brane [143]. The latter system is unstable and the annihilation of the branes can be described in this way since the tube connecting the branes (the string Blon) will expand [166] (see also [167].)

### 3.4.2 D1-brane worldvolume description

We now turn to the description of the same system using the worldvolume action of the \(N\) coincident D1-branes [147, 148, 168]. This requires a non-Abelian DBI action which is not known in full. However, we can use the symmetrised trace prescription, which in the case of no worldvolume field strengths is

\[
S = T_{D1} \int d^2 \sigma \text{STr} \sqrt{-\det(\eta_{\mu \nu} + \lambda^2 \partial_\mu \Phi^i \partial_\nu \Phi^j) \det(Q^{ij})} \quad (165)
\]

\[
Q^{ij} = \delta^{ij} + i\lambda [\Phi^i, \Phi^j] \quad , \quad i, j = 1, \ldots, 8 \quad (166)
\]

This action was proposed for branes in trivial backgrounds (as considered here) [62, 63] and general backgrounds [64]. The action is known to be incomplete [114, 169] but it is sufficient for BPS configurations [170, 171, 163]. The symmetrised trace prescription, STr, means that we symmetrise over all permutations of the \(N \times N\) matrices \(\partial_\mu \Phi^i\) and \([\Phi^i, \Phi^j]\) after expanding the square root of the determinants. Since the \(\Phi^i\) are related to coordinates \(X^i = \lambda \Phi^i\) transverse to the D1-branes we see that the non-Abelian gauge group leads to a non-Abelian space. The \(N\) eigenvalues can be interpreted as the positions of the \(N\) individual D1-branes.

Evaluating the determinants for a static configuration with three non-trivial scalars \(\Phi^i (i = 1, 2, 3)\) produces a sum of squares which gives a BPS condition

\[
\partial_\sigma \Phi^i = \pm \frac{i}{2} \lambda^2 \epsilon^{ijk}[\Phi^j, \Phi^k] \quad (167)
\]

where \(\sigma\) is the spatial coordinate on the D1-branes. These three scalars will correspond to the three worldvolume spatial directions on a D3-brane. Note [148] that equations (167) are the Nahm equations [172] for BPS monopoles in SU(2) SYM. Indeed if we introduce another parallel D3-brane, the D1-branes stretching between the two D3-branes would have precisely that interpretation. When these conditions are satisfied the total energy is

\[
E = NT_{D1} \int d\sigma \pm \frac{i}{3} \lambda^2 T_{D1} \int d\sigma \partial_\sigma Tr(\epsilon^{ijk}\Phi^i \Phi^j \Phi^k) \quad (168)
\]

It is easy to see that we have a solution to equations (167) where \(\Phi^i\) are proportional to SU(2) generators \(T^i\)

\[
\Phi^i = \pm \frac{1}{2\sigma} T^i \quad (169)
\]

\[
[T^i, T^j] = 2i \epsilon^{ijk} T^k \quad (170)
\]
Taking the generators $T^i$ to be the irreducible $N \times N$ representation we have

$$(T^1)^2 + (T^2)^2 + (T^3)^2 = (N^2 - 1)I_{N \times N}$$

(171)

and so for fixed $\sigma$ the three transverse scalars $X^i = \lambda \Phi^i$ parametrise a non-commutative or fuzzy two-sphere of radius

$$R(\sigma) = \sqrt{\frac{1}{N} Tr((X^i)^2)} = \frac{\pi l_s^2}{\sigma} \sqrt{N^2 - 1}$$

(172)

So the whole configuration is that of a fuzzy cone or funnel which looks like $N$ coincident D1-branes for large $\sigma$ (small $R$) and blows up into a flat D3-brane as $\sigma \to 0$ (and $R$ diverges.) Identifying $r$ from section 3.4.1 with $R$ we see that the system can be described in terms of either the D3-brane or the N D1-branes worldvolume theories. Also, the total energy agrees, for large $N$, with the interpretation that this is a BPS system of $N$ D1-branes ending on a D3-brane

$$E = NT_{D1} \int_0^\infty d\sigma + \frac{N}{\sqrt{N^2 - 1}} T_{D3} \int_0^\infty 4\pi r^2 dR$$

(173)

It can also be checked [147] that this configuration carries the expected (up to a factor $\frac{N}{\sqrt{N^2 - 1}}$) D3-brane charge, due to non-Abelian Wess-Zumino couplings [64].

There are once again many other similar examples. D1-branes between two D3-branes, $(p, q)$-strings ending on D3-branes and embedding the non-Abelian worldvolume action into a D3-brane supergravity background were all considered in [147]. Solutions where $N$ D$p$-branes expand into a D$(p + r)$-brane can be found and it is possible to identify the U(1) fieldstrength on the D$(p + r)$-brane [162]. Fluctuations of the solutions were also analysed and the results were in agreement with the analysis from the D3-brane worldvolume theory. $(p, q)$-string junctions which can end on D3-branes were described using the D1-branes worldvolume theory [168]. Including a constant NS-NS B-field produces a solution describing a non-orthogonal configuration of D1-branes and D3-brane, as was seen from the D3-brane worldvolume theory [162]. In [173] it was shown that there are more general funnel solutions of the D1-string non-Abelian worldvolume theory which describe D3-branes wrapping calibrated 3-cycles, preserving $1/4, 1/8, 1/16$ or $1/32$ supersymmetry. Non-BPS configurations can also be analysed – see [165, 149] for D1-branes expanding into D5-branes.

3.4.3 Comments on dual worldvolume descriptions

While we can expect the system of $N$ D1-branes ending on a D3-brane to have a good D3-brane worldvolume description for large $r$ and be well-described by the D1-brane non-Abelian worldvolume theory for small $r$, it is perhaps rather surprising that the two descriptions match so well, e.g. both giving the correct total energy (for large $N$) and describing fluctuations along the D1-branes. However, by considering the validity of the (non-Abelian) DBI action we can easily see that there should be an overlap between the two descriptions for large $N$.

The DBI action is valid for constant field strengths and (related by supersymmetry) constant first derivatives of the worldvolume scalars. So we can expect to neglect higher derivative corrections to these solutions provided the second derivatives of the scalars are small, i.e. schematically

$$l_s \partial^2 \Phi \ll \partial \Phi$$

(174)

For the D3-brane solution this leads to the constraint $r \gg l_s$ while for the D1-brane description we get $\sigma \gg l_s$ which is equivalent to $r \ll N l_s$. Hence for large $N$ we have a large range of $r$ where both descriptions are useful.
For the non-Abelian action required for the D1-branes worldvolume theory, we also have higher commutator corrections. Requiring these to be small, i.e.

\[ l_s [\Phi, \Phi] \ll \Phi \]  

leads to the same constraint, \( r \ll Nl_s \), found for the higher derivative terms. There is a stronger constraint which arises if we also demand that the expansion in powers of \( l_s \) of square root in the action should be convergent. This requires in addition that

\[ l_s^4 |\partial \Phi|^2 \ll 1 \]  

which leads to \( r \ll \sqrt{N}l_s \), although this still gives a large range of overlap of the two descriptions for large \( N \). However, it is possible that many higher order terms vanish for BPS configurations such as we have been considering \[114, 169\] and so the range of overlap may extend to \( l_s \ll r \ll Nl_s \).

Finally we note that since there is no explicit dependence on \( g_s \) (except through \( T_{Dp} \)) in the solutions, we can always take weak enough coupling so that \( g_s N \ll 1 \) and the brane actions will decouple from gravity.

### 4 Smeared intersections and black holes

We have seen in section 2 that solutions for parallel branes are described by a harmonic function with singularities at the locations of the branes. It turns out that a large class of intersecting brane solutions can be described in a similar way by following a set of simple rules for combining the harmonic functions associated to each type of brane \[174, 175, 50, 176, 177, 16, 178\]. Specifically, this method applies to supersymmetry-preserving orthogonal intersections of branes. However, it is possible to relate orthogonal intersections to non-orthogonal intersections via boosts and duality transformations \[179\]. This was used to construct supergravity solutions for non-orthogonal intersections from solutions for orthogonal intersections \[180, 181\]. Such solutions were also found in \[182, 183\] and expressed as a generalisation of the harmonic functions rules for orthogonal intersections \[184, 185\]. See also \[186\] for examples of 1/4-BPS intersecting D2-branes with additional NS-NS two-form flux, and T-dual configurations of intersecting D1- and D3-branes. There are also solutions describing bound states of branes within the worldvolume of other branes \[187, 188, 189, 190, 191, 192, 176, 193\] such as \((p, q)\)-5-branes which are bound states of D5- and NS5-branes or \(D_p\)-branes within a \(D(p + 2)\)-brane preserving one half supersymmetry. These non-marginal solutions have non-zero binding energy when interpreted in terms of constituent branes and are more closely related to parallel brane solutions, e.g. having the interpretation of a \(D(p + 2)\)-brane with non-trivial worldvolume fields. See also \[194\] for the 1/4-BPS case of a \((p, q)\)-string web within a D3-brane worldvolume. The BPS solutions we describe here are marginal, i.e. there is no binding energy between the constituent branes. The non-marginal solutions can be derived from marginal ones by various duality transformations, see e.g. \[176\].

In section 4.1 we give the general method for constructing a supergravity solution describing any BPS orthogonal intersection of branes. However, typically some branes must be smeared or delocalised over some of their transverse directions. In section 4.2 we will present an example of an intersecting brane solution which becomes a black hole after toroidal compactification and briefly review how this was used to calculate the entropy by counting the microscopic degrees of freedom.
4.1 Harmonic function rules

The harmonic function rules give a method of constructing intersecting brane solutions by simply combining the one half BPS solutions for the constituent branes — adding the field strengths and multiplying the components of the diagonal metrics. We will consider in detail the case of M2-branes intersecting in a way which preserves one quarter supersymmetry in section 4.1.1 before stating the method of constructing more general supersymmetric solutions describing orthogonal intersections of branes in section 4.1.2. Many other examples of such solutions are presented in a very useful review by Gauntlett [16]. We show that these solutions are consistent with the no-force condition for appropriate probe branes in section 4.1.3.

4.1.1 Orthogonal intersecting M2-branes

We will consider the case of a set of parallel M2-branes with worldvolume directions 012 intersecting with another set of parallel M2-branes with worldvolume directions 034. We already know that the constituent parallel branes (those with worldvolume directions either 012 or 034) would be described in terms of harmonic functions \( H^{(1)} \) and \( H^{(2)} \) respectively as

\[
\begin{align*}
\mathbf{d}s^2 &= -H^{(1)}_{\perp} dt^2 + H^{\perp}_{(1)} \left( dx_1^2 + dx_2^2 \right) + \\
&\quad H^{\frac{1}{2}}_{(1)} \left( dx_3^2 + dx_4^2 \right) + H^{\frac{1}{2}}_{(1)} dx_5^2 \\
F &= -d(H^{-1}_{(1)}) \wedge dt \wedge dx^1 \wedge dx^2
\end{align*}
\] (177)

and

\[
\begin{align*}
\mathbf{d}s^2 &= -H^{(2)}_{\perp} dt^2 + H^{\perp}_{(2)} \left( dx_1^2 + dx_2^2 \right) + \\
&\quad H^{\frac{1}{2}}_{(2)} \left( dx_3^2 + dx_4^2 \right) + H^{\frac{1}{2}}_{(2)} dx_5^2 \\
F &= -d(H^{-1}_{(2)}) \wedge dt \wedge dx^3 \wedge dx^4
\end{align*}
\] (178)

So the solution for intersecting M2-branes is given by adding the field strengths and multiplying the metric components (or more precisely the vielbeins in order to preserve the signature)

\[
\begin{align*}
\mathbf{d}s^2 &= -H^{(1)}_{\perp} H^{(2)}_{\perp} dt^2 + H^{\perp}_{(1)} H^{\perp}_{(2)} \left( dx_1^2 + dx_2^2 \right) + \\
&\quad H^{\frac{1}{2}}_{(1)} H^{\frac{1}{2}}_{(2)} \left( dx_3^2 + dx_4^2 \right) + H^{\frac{1}{2}}_{(1)} H^{\frac{1}{2}}_{(2)} dx_5^2 \\
F &= -d(H^{-1}_{(1)}) \wedge dt \wedge dx^3 \wedge dx^4 - d(H^{-1}_{(2)}) \wedge dt \wedge dx^3 \wedge dx^4
\end{align*}
\] (181)

Note that we expect this solution to preserve one quarter supersymmetry since the two types of M2-branes have precisely four relative transverse dimensions. Indeed, it is straightforward to check that this is true. However, first we will comment on the conditions that this combination of harmonic functions does give a solution to the supergravity equation of motion.

Consider first the equations for the four-form \( F \). Clearly the Bianchi identity \( dF = 0 \) is a linear equation so we can simply add together solutions to get a new solution. The equation of motion for \( F \) is less trivial since taking the Hodge dual involves the metric. In components we have the condition (ignoring source terms)

\[
\partial_{\mu} \left( |g|^\frac{1}{2} F_{\mu \nu \rho \lambda} \right) = 0
\] (183)

In terms of the separate solutions, this reduced to the condition that \( H^{(1)} \) and \( H^{(2)} \) are harmonic functions (with respect to the flat-space Laplacian \( \nabla^2 = \sum_i \frac{\partial^2}{\partial (x_i)^2} \) in the spaces transverse to
each type of M2-branes. Now in the intersecting solution we want the same conditions even
though the metric (appearing through the determinant and used to raise the indices on \(F\)) has
changed. Considering say \(|g|^{\frac{1}{2}}F^{012}\) we gain an extra factor of \(H_{(2)}^{\frac{1}{2}}\) from \(|g|^{\frac{1}{2}}\) and \(H_{(2)}^{\frac{2}{3} - \frac{1}{3} - \frac{1}{3}} = 1\) from raising the three indices 012. So we will have the condition that \(H_{(1)}\) is a harmonic
function as before, provided we get a factor \(H_{(2)}^{\frac{1}{2}}\) from raising the \(\mu\) index. This will happen
precisely if \(\mu\) is an index for one of the totally transverse directions.

By symmetry we get the same result for \(H_{(2)}\). So we see that the equation of motion for \(F\) is satisfied provided \(H_{(1)}\) and \(H_{(2)}\) are harmonic functions of the coordinates transverse to both types of M2-branes. In terms of the constituent M2-branes, say those oriented in the 012
directions, we can interpret the form of \(H_{(1)}\) as describing a continuous distribution of such
M2-branes in the 34 directions. We say that these branes are smeared in the 34 directions. So
the solution corresponds to the intersection of M2-branes oriented in the 012 and 034 directions,
smeared over their relative transverse directions.

Of course, we must still check that all the supergravity equations of motion are satisfied
(essentially the Einstein equations.) It turns out that they are and so we do have a supergravity
solution for smeared intersecting branes. In fact this is essentially guaranteed since we have
diagonal metric [44, 45]. This solution was originally found by Güven [20] in the special
case where the two harmonic functions were the same. The interpretation of the solution as
intersecting branes was given in [195] and the generalisation to different harmonic functions for
each brane (and to other types of intersecting brane solutions) soon followed [174, 175].

We should also consider the appropriate normalisation of the coefficients in the harmonic
functions. This can be fixed by considering parallel brane configurations. We can explicitly
smear a solution, say around a circle of radius \(R\), by placing copies of the branes around the
circle, say spaced by \(2\pi R/m\). The harmonic function describing the solution is then a multi-
centred harmonic function as in the case of separated parallel branes. However, since we are
smearing the branes rather than introducing other branes, we should divide the coefficients in
the harmonic function by \(m\) – i.e. we imaging splitting each brane into \(m\) equal fractions. Taking
the limit \(m \to \infty\) turns the sum of terms in the harmonic function into an integral which can
be performed. The branes are thus smeared over the circle and the harmonic function becomes
a harmonic function in one dimension lower. This process can be easily generalised to smearing
over any torus.

Equivalently, we can just use the fact that the equations of motion are satisfied for a lower
dimensional harmonic function and find the coefficients by properly normalising the source
terms. Essentially this means that we require the smeared branes to have the same total
charge. Since the charge is proportional to the integral of \(\nabla^2 H\) over the directions transverse
to the branes (including the directions over which the brane has been smeared) it can easily be
seen that we get the correct normalisation by replacing

\[
\frac{1}{r^a} \to \frac{aV(S^{a+1})}{(a - b)V(S^{a-b+1})V_b \tilde{r}^{a-b}}
\]

when smearing the branes over a \(b\)-dimensional space of volume \(V_b\). Here \(r (\tilde{r})\) is the radial
coordinate in the space transverse to the localised (smeared) branes.

An obvious question to ask is whether we can find localised solutions using the ansatz
of equation (181) and (182). From the above discussion we clearly must relax the condition
that \(H_{(1)}\) and \(H_{(2)}\) are harmonic functions. The result of checking the supergravity equations
of motion is that one of these functions, say \(H_{(2)}\), must be independent of the worldvolume
directions of the other brane, i.e. \(x^1\) and \(x^2\) in this example. So we see that at least one of
the branes must be smeared. In this case \(H_{(2)}\) is a harmonic function in the totally transverse
space, whereas from the four-form equation of motion $\text{d} \star F = 0$, we find that $H_{(1)}$ must satisfy the curved-space Laplace equation \[ \partial_M \left( \sqrt{|g|} g^{MN} \partial_N H_{(1)} \right) = H_{(2)} \left( \partial^2_3 + \partial^2_4 \right) H_{(1)} + \left( \partial^2_5 + \cdots + \partial^2_{10} \right) H_{(1)} = 0 \quad (185) \]

In general it is not possible to find explicit solutions for $H_{(1)}$. The cases where solutions are known are the above case where both branes are smeared and the case where we solve in the near-core region of the smeared brane. We will consider this latter case in section 5.

### 4.1.2 General construction

There is an obvious generalisation of the above example. For a general configuration of orthogonally intersecting branes we simply combine the solutions for each constituent brane - i.e. add together the field strengths and multiply the components of the diagonal metrics. When we are considering branes in type IIA or type IIB, there is also a dilaton which is given as the sum of the solutions for the dilaton. It turns out that this can provide a solution describing intersecting branes provided the configuration of intersecting branes is supersymmetric. It is also possible to include a gravitational wave as one of the constituent 'branes'. We will see an example of this in section 4.2.1.

The general (with one exception discussed below) result of checking the equations of motion for such an ansatz is that for each pair of (sets of parallel) branes, at least one of them must be smeared over the worldvolume directions of the other. In general this gives various choices for how we wish to smear the branes. Once we have made such a choice we have determined on which coordinates each 'harmonic' function can depend. Now these ‘harmonic’ functions must satisfy, not the flat-space Laplace equation but the curved-space Laplace equation \[ \partial_M \left( \sqrt{|g|} g^{MN} \partial_N H_{(1)} \right) = H_{(2)} \left( \partial^2_3 + \partial^2_4 \right) H_{(1)} + \left( \partial^2_5 + \cdots + \partial^2_{10} \right) H_{(1)} = 0 \quad (185) \]

It is not usually possible to find explicit solutions to these coupled equations but there are some (fully and partially) localised examples which we discuss in section 5.

We can find explicit solutions in the simplest case where we smear all the branes over the relative transverse coordinates. In this case, as for two intersecting branes, the solution is simply given by harmonic functions of the totally transverse coordinates. We shall present an example involving D1- and D5-branes in section 4.2.1.

The exception to the above curved-space harmonic function rules is when two branes intersect with eight relative transverse dimensions (e.g. D4-branes intersecting at a point.) In this case we can allow the harmonic functions to depend on the relative transverse coordinates provided they are independent of the overall transverse direction (if there is one.) We will discuss such solutions in section 5.2.

### 4.1.3 Brane probes

We can use brane probe techniques to check some of the features of these intersecting brane solutions. The idea is the same as for parallel branes in section 2.4. The difference here is that the reduced supersymmetry allows for a non-trivial metric on moduli space so the only requirement we have is that the static potential should be constant. One application of this method is to take a known solution and then probe with a different type (or orientation) of brane. If the static potential vanishes then it is possible to introduce the probe brane into the background. Hence this gives an alternative derivation of the allowed configurations of intersecting branes which preserve supersymmetry \[5\].

---

\[5\]The assumption is that these are supersymmetric since we will not have a moduli space or a no-force condition for non-supersymmetric configurations.
Consider the case of $N$ parallel M2-branes with worldvolume directions 012. The supergravity solution is given by equations (85), (86) and (87). We know from section 2.3 that we can introduce parallel M2-brane probes but here we will consider a probe M2-brane with worldvolume directions 034. This probe will not couple to the background 3-form potential so the static potential is (up to a constant factor involving the brane tension) simply given by the determinant of the pullback metric

$$\sqrt{-G} = \sqrt{-G_{00}G_{33}G_{44}} = \sqrt{H^{-2/3}H^{1/3}} = 1$$

Hence we see that, as alternatively derived from $\kappa$-symmetry considerations in section 3.2, it is consistent to have such an intersection of M2-branes.

It is equally simple to see that we cannot for example have a supersymmetric intersection of M2-branes with worldvolume directions 012 and 013. In this case the probe brane would have a static potential given by

$$\sqrt{-G} = \sqrt{-G_{00}G_{11}G_{33}} = \sqrt{H^{-2/3}H^{-2/3}H^{1/3}} = H^{-1}$$

which is clearly not constant.

Finally we can perform a consistency check on a solution for intersecting branes by probing with any of the constituent branes. For example probing the intersecting M2-brane solution of equations (181) and (182) with a probe M2-brane with worldvolume directions 034 we find a static potential proportional to

$$\sqrt{-G} - H_{(2)}^{-1} = 0$$

which is constant, as expected.

### 4.2 Application to Black holes

While the harmonic function rules provide a method of constructing large classes of solutions which are related to intersecting branes, the fact that the branes are smeared over the relative transverse directions means that it is not obvious that these solutions actually describe what happens at the intersection. Indeed there are certainly important features which cannot be described by these solutions such as the relative separations of the branes in directions over which they are smeared. As we will discuss in section 6 such parameters are important for certain intersecting brane configurations which describe gauge theories.

However, there is one obvious situation where the smearing of the branes is not important and indeed is even a necessary feature of the supergravity solution. That is when we wish to compactify the directions along which the branes are smeared. If the branes were not smeared we would anyway have to effectively construct the smeared solution in order to obtain the necessary isometries for the reduction. When we perform such a reduction we end up with a $p$-brane solution of a lower dimensional supergravity, where $p + 1$ is the number of common worldvolume dimensions of the intersecting branes. We can, of course, further compactify some or all of these $p$ spatial directions. The most important application of these solutions has been the case where we compactify all $p$ directions to end up with a particle. Such solutions describe black holes with various charges specified by the constituent intersecting branes.

Although charged black hole solutions can easily be constructed in supergravity theories, the important point in constructing them from intersecting brane solutions in ten or eleven dimensions is that we automatically have a string theory (or M-theory) interpretation. In particular the interpretation of a black hole as a particular configuration of branes allows us to calculate the entropy of the black hole by considering the number of massless degrees of
freedom in string theory. This can then be compared to the area of the lower dimensional black hole horizon to provide a microscopic derivation \[197\] of the Bekenstein-Hawking \[198\] \[199\] \[200\] \[201\] \[202\] \[203\] black hole entropy. This is a large subject which has been reviewed in detail in \[204\] \[205\] \[206\] \[207\]. We will just consider one of the simplest cases \[208\] \[209\] in detail in section 4.2.1 to illustrate the application of the harmonic function rules. This concerns black holes in five-dimensional $\mathcal{N} = 8$ supergravity. We will mention some aspects of other black hole solutions and entropy counting in section 4.2.2.

4.2.1 Five-dimensional extremal black hole entropy

We can construct a black hole preserving one eighth supersymmetry from a brane configuration involving the intersection of $N_1$ D1-branes with $N_5$ D5-branes. Such a system would preserve one quarter supersymmetry provided the D1-branes, say with worldvolume directions $05$, are parallel to the D5-branes which we can therefore choose to have worldvolume directions $056789$.

Using the harmonic function rules the metric for this system is

$$\begin{align*}
\text{ds}^2 &= H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} (-dt^2 + dx_5^2) + H_1^{\frac{5}{2}} H_5^{\frac{5}{2}} (dx_1^2 + \cdots + dx_4^2) + H_1^{\frac{1}{2}} H_5^{\frac{1}{2}} (dx_6^2 + \cdots + dx_9^2) \\
e^{-\phi} &= H_1^{-\frac{1}{2}} H_5^{\frac{5}{2}}
\end{align*}$$

where

$$\begin{align*}
H_1 &= 1 + \frac{c_1 N_1}{r^2} \\
H_5 &= 1 + \frac{c_5 N_5}{r^2} \\
r^2 &= x_1^2 + \cdots + x_4^2
\end{align*}$$

where, with the D1-branes smeared over a $T^4$ in the 6789 directions of volume $V_4$ and the 5 direction compactified on a circle of radius $R_5$, we have

$$\begin{align*}
c_1 &= \frac{4G_5 R_5}{\pi g_s l_s^2} \\
c_5 &= g_s l_s^2
\end{align*}$$

where the five-dimensional Newton’s constant, $G_5$, is related to the ten-dimensional Newton’s constant by the volume of the five compact dimensions, $V_5 = 2\pi R_5 V_4$, by $G_5 = G_{10}/V_5$. By toroidally compactifying the directions 56789 we can construct a metric for a point-like mass in five dimensions. However, it turns out that this would describe a black hole with (classically at least) a horizon of zero area. In order to get a macroscopic black hole we need to break more supersymmetry and we can do this by introducing a wave with momentum $P$ along the D1-branes. The supersymmetry projection condition for a wave carrying momentum in the 5 direction is

$$\hat{\Gamma}_{05}\epsilon = \epsilon$$

The metric for this one eighth BPS system is

$$\begin{align*}
\text{ds}^2 &= H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} \left( -dt^2 + dx_5^2 + (H_W - 1)(dt - dx_5)^2 \right) + H_1^{\frac{1}{2}} H_5^{\frac{1}{2}} (dx_1^2 + \cdots + dx_4^2) \\
e^{-\phi} &= H_1^{-\frac{1}{2}} H_5^{\frac{1}{2}}
\end{align*}$$
where

\[ H_W = 1 + \frac{c_W N_W}{r^2}, \quad c_W = \frac{4G_5}{\pi R_5} \]  

(199)

In this metric the wave is also smeared over the D5-brane worldvolume. Since the momentum is around a circle of radius \( R_5 \), it quantised as \( P = N_W / R_5 \) where \( N_W \) is a positive integer. We will describe how to calculate the microscopic entropy but first we will show precisely how the intersecting brane solution is related to a five-dimensional black hole and what its horizon area is.

By rewriting the two dimensional ‘wave part’ of the metric as

\[-dt^2 + dx_5^2 + (H_W - 1)(dt - dx_5)^2 = -H_W^{-1} dt^2 + H_W (dx_5 - (1 - H_W^{-1})dt)^2\]  

(200)

we can perform the dimensional reduction in the 56789 directions. We see also from the above form of the wave that there will be a Kaluza-Klein gauge potential determined by \( H_W \). This shows that the five dimensional solution will have a U(1) charge \( N_W \). The solution will also have charges \( N_1 \) and \( N_5 \) (under different U(1) groups) coming directly from the RR-charges in ten dimensions sourced by the D1- and D5-branes. The five-dimensional Einstein metric is

\[ ds_5^2 = (H_1 H_5 H_W)^{-\frac{4}{3}} dt^2 + (H_1 H_5 H_W)^{\frac{1}{3}} (dr^2 + r^2 d\Omega_3^2) \]  

(201)

The rescaling involved ensures that the ten-dimensional action with the string-frame metric reduces to the five-dimensional Einstein-Hilbert action.

\[ S \sim \frac{1}{2\kappa_{10}^2} \int d^{10}x e^{-2\phi} \sqrt{|g|} R(g) = \frac{V_5}{2\kappa_{10}^2} \int d^5x e^{-2\phi} \sqrt{|g_5|} R(g_5) = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{|g_E|} R(g_E) \]

(202)

where \( g_I \) is the determinant of the internal (compact) part of the string metric

\[ g_I = \left( H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} H_W \right) \left( H_1^\frac{1}{2} H_5^{-\frac{1}{2}} \right)^4 = H_1^\frac{3}{2} H_5^{-\frac{5}{2}} H_W \]

(203)

We clearly see that the three charges appear on the same footing in the five-dimensional Einstein metric, equation (201) even though they have a different ten-dimensional origin. (Actually, this is not too surprising since the charges can be permuted by performing various T- and S-duality transformations.) Since

\[ \lim_{r \to 0} (H_1 H_5 H_W)^{\frac{1}{3}} r^2 = \left( c_1 c_5 c_W N_1 N_5 N_W \right)^{\frac{1}{3}} \]

(204)

we see that the five-dimensional black hole has a (3-sphere) horizon of radius \( (16 G_5^2 N_1 N_5 N_W / \pi^2)^{\frac{1}{3}} \) and hence a horizon area

\[ A = 8\pi G_5 \sqrt{N_1 N_5 N_W} \]

(205)

We will now briefly explain the counting of microscopic states in string theory which leads to a calculation of the black hole entropy, \( S_{BH} \), which supports the Bekenstein-Hawking relation between entropy and horizon area (in five dimensions)

\[ S_{BH} = \frac{A}{4G_5} = 2\pi \sqrt{N_1 N_5 N_W} \]

(206)

The momentum of the wave in the brane configuration can be viewed as the momentum carried by fundamental open strings moving around the \( x_5 \) circle. The entropy of the system is determined by the number of ways this momentum can be partitioned among an arbitrary
number of fundamental strings. It turns out that the important strings are those ending on both a D1-brane and a D5-brane. There are four such bosonic strings for each D1/D5 pair corresponding to the four independent directions the D1-brane can be moved (within the D5-brane worldvolume) without reducing the number of massless degrees of freedom (e.g. moving a D1-brane away from a D5-brane makes even lightest fundamental string state connecting those branes massive.) In a typical configuration the D1-branes will be separated, as will the D5-branes, so there will be no massless open strings with ends on two different D1-branes or two different D5-branes. So in total there are $4N_1N_5$ massless bosonic degrees of freedom (and by supersymmetry the same number of massless fermionic degrees of freedom) which can carry the momentum $P = N_W/R_5$. Each state carries momentum $n/R_5$ around the circle, for some positive integer $n$ (modes with $n < 0$ would break all supersymmetry.) Alternative methods of counting are to consider the dimension of the moduli space of $N_1$ instantons (the D1-branes) in a $U(N_5)$ gauge theory (on the worldvolume of the D5-branes,) or, to calculate the central charge of the $(1 + 1)$-dimensional $\sigma$-model which is the effective action of the system after compactification on the $T^4$ in the 6789 directions (i.e. in the limit $R_5 \gg V_4^{1/4}$.)

So we are interested in counting the number of ways we can assign positive integers, totalling $N_W$, to $N_B = N_F = 4N_1N_5$ bosonic and fermionic states. This can be calculated as the coefficient $d(N_W)$ of $q^{N_W}$ in the partition function

$$Z = \left( \prod_{n=1}^{\infty} (1 + q^n) \right)^{N_F} \left( \prod_{n=1}^{\infty} (1 - q^n) \right)^{-N_B} = \sum_{m=0}^{\infty} d(m)q^m \quad (207)$$

For $N_W \gg N_1N_5 \gg 1$ this gives the entropy of the system

$$S_{BH} = \ln (d(N_W)) \approx 2\pi \sqrt{N_1N_5N_W} \quad (208)$$

It is interesting to see how the entropy can also be calculated when the three charges are of the same order. In this case the counting above gives a much lower value for the entropy, $S_{BH} \sim N \ln N$ if $N_1 \sim N_5 \sim N_W \sim N \gg 1$ rather than the expected value of $N^{3/2}$ from the black hole area. The important idea to get the correct entropy is that we can consider a system of $N_1$ coincident D1-branes, each wrapped once around a circle or equivalently a single D1-brane wrapped $N_1$ times around the same circle (or various intermediate possibilities giving total winding number $N_1$.) Similarly we can consider a single D5-brane with winding number $N_5$. In this case there is only one D1-brane and one D5-brane and so by the previous counting we have only $N_B = N_F = 4$ bosonic and fermionic degrees of freedom. However, an open string with one end on each brane now sees an effective circle of radius $N_1N_5R_5$ since each time it moves around the circle of radius $R_5$ its ends have moved along to the next loop of the D1- and D5-branes. So for $N_1$ and $N_5$ relatively prime the open string only returns to the same position after moving round the circle $N_1N_5$ times. If $N_1$ and $N_5$ are not relatively prime then we simply consider the system to consist of two D1-branes with winding numbers $\tilde{N}_1$ and $n_1 = N_1 - \tilde{N}_1 \ll N_1$, and two D5-branes with winding numbers $\tilde{N}_5$ and $n_5 = N_5 - \tilde{N}_5 \ll N_5$ such that $\tilde{N}_1$ and $\tilde{N}_5$ are relatively prime. To leading order we only need to consider the branes with winding numbers $\tilde{N}_1$ and $\tilde{N}_5$. So in all cases to leading order we have $N_B = N_F = 4$ massless degrees of freedom moving on a circle of radius $N_1N_5R_5$. The point is that now these modes carry momentum quantised in units of $1/(N_1N_5R_5)$ and so the total momentum consists of $N_1N_5N_W$ units which can be partitioned in $d(N_1N_5N_W)$ ways which can be calculated from the partition function

$$Z = \left( \prod_{n=1}^{\infty} (1 + q^n) \right)^4 \left( \prod_{n=1}^{\infty} (1 - q^n) \right)^{-4} = \sum_{m=0}^{\infty} d(m)q^m \quad (209)$$
This gives the entropy of the system

\[ S_{BH} = \ln \left( d(N_1 N_5 N_W) \right) \approx 2\pi \sqrt{N_1 N_5 N_W} \]  

(210)

which agrees with the result from the black hole area. Note that in the original limit \( N_W \gg N_1 N_5 \) it is not important whether we consider singly or multiply wound D1- and D5-branes. However, the multiply wound branes lead to the correct quantisation for the energy of excited states \([197, 210, 211, 209]\).

It should be noted that it is not obvious that we should find agreement between the Bekenstein-Hawking entropy and the counting of microscopic states as described above. The reason is that the counting was implicitly done as weak coupling (small values for parameters such as \( g_s N \) which correspond to the ‘t Hooft coupling in the U(\(N\)) gauge theory on the brane worldvolume,) where we counted essentially free fundamental string states in a fixed background of branes. However, the interpretation of the brane configuration as a macroscopic black hole is valid at large coupling and so comparison between the two limits requires some sort of non-renormalisation theorem, for example forbidding large corrections to the mass as we vary the couplings (at fixed charges.) Such situations may be expected to arise when supersymmetry is preserved. However it appears that supersymmetry is not necessarily the important factor since in some supersymmetric cases there is only agreement up to a numerical factor \([212]\) whereas there can be exact agreement in non-extremal (close to the supersymmetric limit) or extremal (saturating a classical BPS condition) but non-supersymmetric examples which we will briefly mention in the section \([4.2.2]\). See \([213]\) for an overview of these issues.

### 4.2.2 Other black holes

It is possible to construct other black hole solutions from intersecting branes. The method is the same – simply construct an intersecting brane configuration using the harmonic function rules and toroidally compactify the relative transverse space over which the branes are smeared. Again, the most useful cases are where there is classically a non-zero horizon area.

The method of counting the microscopic degrees of freedom depends on the type of brane configuration. Typically, open string counting methods similar to those of the five-dimensional example discussed in section \([4.2.1]\) can be used for ten-dimensional constructions. However, for eleven-dimensional configurations more general \(\sigma\)-model methods are required.

Solutions describing four-dimensional black holes, which can be interpreted as intersections of NS5-branes, fundamental strings, waves and KK-monopoles, were constructed in \([214, 215]\) and used to count the microscopic entropy in \([216]\). See also \([217]\) for non-extremal generalisations. A dual description, similar to the five-dimensional example of section \([4.2.1]\) is given by \(N_6\) D6-branes with worldvolume directions 0456789, \(N_5\) NS5-branes with worldvolume directions 046789, \(N_2\) D2-branes with worldvolume directions 045 and momentum \(P_4 = N_W/R_4\) along the 4 direction. The 1/8-BPS supergravity solution can easily be derived and reduced to four dimensions. The resulting black hole has a non-zero horizon area which corresponds to an entropy

\[ S_{BH} = 2\pi \sqrt{N_2 N_5 N_6 N_W} \]  

(211)

We can again picture the momentum as being carried by fundamental open strings with ends on a D2-brane and a D6-brane. The role of the NS5-branes is to split the D2-branes, i.e. rather than each D2-brane wrapping the 5 direction, it is split into \(N_5\) pieces which stretch between consecutive NS5-branes. In this way there are \(N_2 N_5 N_6\) distinct possibilities for an open string to have ends on a D6-brane and one of the \(N_2 N_5\) D2-brane segments. Once again each such string has four bosonic and four fermionic degrees of freedom and so the entropy precisely
matches the prediction from the horizon area. The interpretation of the system as consisting of branes wrapping many times around the 4 direction is important for calculating the correct entropy when \( N_5 \) is not large, in this case compared to \( N_2 N_5 N_6 \). As for the five-dimensional black hole in section 4.2.1, we could also count the degrees of freedom by viewing the (segments of) D2-branes as instantons within the D6-branes worldvolume.

This system can also be studied in eleven dimensions. The direct lift is not so convenient since it involves KK6-branes in eleven dimensions. However, if we first T-dualise in the 8 and 9 directions and then lift to eleven dimensions we get a configuration involving three sets of orthogonally intersecting M5-branes with worldvolume directions 04567(10), 046789 and 04589(10) with momentum along the 4 direction. Once we have reduced to four dimensions we get the same black hole solution. So we have a choice of many possible (duality related) ten- and eleven-dimensional intersecting brane configurations to describe the same lower dimensional black holes. We can in principle use any of these configurations to count the microscopic degrees of freedom. An interesting question is how do we calculate the microscopic entropy from this eleven-dimensional configuration of intersecting M5-branes. It was proposed \[218\] (see also \[219, 220\]) that the momentum is carried by M2-branes, analogous to the fundamental strings in ten-dimensional configurations. However, in this case the counting requires the M2-branes to end on three different M5-branes – such M2-branes would be massless states at each point where three M5-branes intersect. (Generically the parallel M5-branes will be separated so there are no other massless states.) It is not clear why we should expect to count the states this way so this is really a prediction about M-theory rather than a derivation of the black hole entropy. The same result for the entropy can be derived from type IIA string theory by first compactifying along the 4 direction where now the momentum becomes a number of D0-branes which can be distributed among the points where three D4-branes intersect \[219\]. The general four-dimensional black hole solutions arising from toroidal compactification from ten or eleven dimensions have been constructed \[221\] and their microscopic entropy calculated \[222\].

The above constructions can also be generalised to the case where the 56789(10)-space is a Calabi-Yau threefold. The choice of Calabi-Yau threefold determines the intersection numbers of the M5-branes (or D4-branes) and the proposed rules for counting states again reproduces the entropy expected from the horizon area \[223\]. However, it is possible to count the states precisely without making any assumption about the properties of M2-branes ending on M5-branes. This can be done by calculating the central charge of the two-dimensional \( \sigma \)-model describing an M5-brane wrapped on a 4-cycle in the Calabi-Yau threefold \[224\] (see also \[197, 225, 226\].) This is possible because generically the M5-branes will not have singular intersections but will be described by a single M5-brane wrapping a smooth 4-cycle (given by the zeroes of a holomorphic function.) The possible deformations of such a 4-cycle (see \[224\] for a similar type IIA description) together with the M5-brane self-dual worldvolume fieldstrength contribute to the central charge. This gives the expected entropy and also quantum corrections \[224, 226, 227, 229, 207\].

There are also generalisations to non-extremal and rotating black holes. It is often possible to successfully calculate the microscopic entropy and also analyse Hawking radiation. We refer the reader to the reviews \[204, 206, 207\] and references therein.

Finally, it is interesting to note that in some cases, even without supersymmetry, the counting of microscopic degrees of freedom agrees with the Bekenstein-Hawking entropy \[230, 231, 232, 233\]. These black holes can be described using intersecting branes but all supersymmetry is broken since some branes (or waves) have the opposite orientation to that required for preserving supersymmetry, or simply because the brane configurations are not supersymmetric (e.g. consisting of D0- and D6-branes.) For example \[232\] the five-dimensional black holes considered in section 4.2.1 can also be constructed in SO(32) type I string theory (which is a
projection of type IIB) but reversing the direction of the propagating wave breaks all supersymmetry. Hence such configurations are not even related to supersymmetric configurations by a small deformation. However, they can be BPS in the sense of saturating a classical BPS bound relating mass to charge. The agreement between counting of states and Bekenstein-Hawking entropy implies that this classical relation should not be modified by quantum corrections (at least within the regime under consideration of large charges.) Such quantum corrections (or their absence) and direct comparisons between the strong and weak coupling regimes have been considered in [234, 235, 236, 237, 238, 239].

5 (Partially) localised solutions

While the smeared intersecting brane solutions were all that was required for constructing lower dimensional black holes, it is natural to ask whether we can construct localised solutions. As discussed in section 4.1.2 we can find implicit solutions for partially localised intersections, in terms of functions which satisfy curved-space Laplace equations. We will show how explicit solutions can be found in a near-core limit in section 5.1. The special case of eight relative transverse dimensions is considered in section 5.2. Then, in sections 5.3 and 5.4 we consider intersecting brane solutions involving the near-core limit of D6-branes which can be derived from orbifolds in eleven dimensions. A more general approach to finding intersecting brane solutions from geometry is discussed in section 5.5.

5.1 Near-horizon solutions from curved-space harmonic function rules

Consider, for example two sets of coincident M5-branes, say $N$ M5-branes with worldvolume directions $012345$ and $M$ with worldvolume directions $012367$, all at the origin of the transverse space, $r = 0$. The metric for such a solution has the form

$$ds^2 = H_1^{-1/3}H_2^{-1/3}(-dx_0^2 + \cdots + dx_3^2) + H_1^{-1/3}H_2^{2/3}(dx_4^2 + dx_5^2) + H_1^{2/3}H_2^{-1/3}(dx_6^2 + dx_7^2) + H_1^{2/3}H_2^{2/3}(dr^2 + r^2d\Omega_2^2)$$

And if we choose to smear the $N$ M5-branes over the 67 directions then we have $H_1 = H_1(r)$ and $H_2 = H_2(x_4, x_5, r)$. The curved space Laplace equations then become (up to appropriately smeared source terms)

$$\nabla_{(3)}^2[r]H_1 = 0$$

$$H_1(\partial_4^2 + \partial_5^2)H_2 + \nabla_{(3)}^2[r]H_2 = 0$$

We use the notation $\nabla_{(d)}^2[r]$ for the (flat-space) Laplace operator in $d$ dimensions with radial coordinate $r$. We can clearly solve the first of these equations with a solution of the form

$$H_1 = 1 + \frac{c}{r}$$

From equations (184), (8) and (10) we have

$$c = \frac{2\pi^2 Nl_P^3}{V_2}$$

where $V_2$ is the volume of the compactified 67 space over which the $N$ branes are smeared. The equation for $H_2$ cannot easily be solved. However, if we take the near-horizon limit $r \to 0$ so that $H_1 \to c/r$ then the problem simplifies since we can write

$$\frac{c}{r}(\partial_4^2 + \partial_5^2) + \nabla_{(3)}^2[r] = \frac{c}{r}\nabla_{(6)}^2[R]$$

40
where
\[ R^2 = x_4^2 + x_5^2 + 4cr \]  
(218)
and we have assumed that \( H_2 \) only depends on \( x^4, x^5 \) and \( r \) through an effective transverse radial coordinate \( R \). So we find the solution \[ H_2 = 1 + \frac{C}{R^4} \]  
(219)
where, after some manipulation of the delta-function source, we find
\[ C = 6\pi M^2_{\text{p}} c = \frac{12\pi^3 MN_l^6}{V_2} \]  
(220)

Notice the unexpected appearance of a six-dimensional Laplacian even though there are only five dimensions transverse to the brane. We will see a similar phenomenon in section 5.3 when we consider branes intersecting Kaluza-Klein monopoles but there we have a natural explanation for this effect since the solutions originate in one dimension higher via a Kaluza-Klein compactification.

Many other semi-localised solutions can be constructed by this method \[240, 242, 243\]. A particularly interesting case is when one type of brane is contained within the worldvolume of another. In this case the harmonic function for the larger branes is automatically smeared over the worldvolume directions of the smaller branes. Therefore we can construct a fully localised solution for this system by solving the curved-space Laplace equation for the smaller branes. As above we can only find explicit solutions in the near-core region of the larger brane. Several examples of these types of intersections, including pp-waves on the worldvolume of branes have been discussed, for example, in \[196, 240, 244\]. We will present an alternative method of constructing such solutions in section 5.3 in the special cases where the larger brane is a Kaluza-Klein monopole. See also \[245, 246, 247, 248, 249\] for the case of a D(-1)-brane in \( \text{AdS}_5 \times S^5 \) (the near-horizon limit of D3-branes) with various degrees of localisation. The geometry of the near-horizon limits of several semi-localised solutions was studied in \[250\]. As expected from the AdS/CFT correspondence these geometries are of the form of warped products of AdS.

### 5.2 Localisation in relative transverse directions

In the case where we have two branes intersecting with an eight-dimensional relative transverse space it is possible to find a solution using the curved-space harmonic function rules where the branes are fully localised in the relative transverse space but smeared over any overall transverse directions. In some ten-dimensional examples this leads to a fully localised intersecting brane solution since there are no overall transverse directions. The first example of such a solution was the case of two NS5-branes, say with worldvolume directions 012345 and 016789 \[251, 252\] which was interpreted as a fully localised intersecting brane solution in \[175\]. This solution was generalised to other ten- and eleven-dimensional cases such as intersecting M5-branes with the same (1 + 1)-dimensional intersection but smeared over the eleventh dimension \[175\]. The classification of these intersections as the most general cases (within the context of the curved-space harmonic function rules) with full localisation of both branes in the relative transverse space (other than branes within branes) was given in \[176, 253\]. Generalisations to including more branes including e.g. M2-branes with worldvolume directions 01(10) which still preserve 1/4 supersymmetry were given in \[253, 254\]. See also \[255, 256\] for the case of non-extremal intersecting branes.
Consider the case of two sets of M5-branes together with M2-branes as above. The metric is of the form

$$ds^2 = H_1^{-1/3}H_2^{-1/3}H_{M2}^{-2/3}(-dx_0^2 + dx_1^2) + H_1^{-1/3}H_2^{2/3}H_{M2}^{1/3}dx^2_{(4)} + H_1^{2/3}H_2^{-1/3}H_{M2}^{1/3}d\tilde{x}^2_{(4)} + H_1^{2/3}H_2^{2/3}H_{M2}^{-2/3}dx^2_{10}$$  \hspace{1cm} (221)

The constraints allow the M5-branes to be smeared over $x^{10}$ but fully localised in the eight-dimensional relative transverse space. The constraints on each set of M5-branes together with the M2-branes are the standard ones so that one type ofbrane must be smeared over the worldvolume directions of the other. In this case both sets of M5-branes are already smeared over the worldvolume directions of the M2-branes so the M2-branes can be fully localised and we don’t need to smear the M5-branes over any other directions. It is now easy to check that the harmonic functions for the M5-branes satisfy a flat-space Laplace equation in the four appropriate relative transverse directions while the harmonic function for the M2-branes satisfies a curved-space Laplace equation

$$\tilde{\nabla}^2_{(4)}H_1 = 0$$  \hspace{1cm} (222)

$$\nabla^2_{(4)}H_2 = 0$$  \hspace{1cm} (223)

$$\left(H_1\tilde{\nabla}^2_{(4)} + H_2\nabla^2_{(4)}\right)H_{M2} = 0$$  \hspace{1cm} (224)

So clearly we can find explicit solutions in the case of intersecting M5-branes alone but in general will not be able to solve the equation for $H_{M2}$ when we also include M2-branes. Note that if we reduce to type IIA along $x^{10}$ we will get fully localised solutions for intersecting NS5-branes together with fundamental strings. Again we will have an explicit solution only for the case without any strings.

### 5.3 Branes within D6-branes

In this section we describe intersecting brane solutions involving D6-branes in a near-core limit. However, it will be clear that such solutions apply to other dimensions where we can have branes intersecting Kaluza-Klein monopoles. First, in section 5.3 we consider the cases where we have branes contained within the worldvolume of the D6-branes. As mentioned in section 5.1 these are special cases of solutions which can be constructed using curved-space harmonic function rules. However, the cases described here have a particularly simple construction which generalises to the cases of branes intersecting D6-branes such as D4-branes ending on D6-branes which will be described in section 5.4.

These solutions describing fully localised intersecting branes were found by Itzhaki, Tseytlin and Yankielowicz [257]. One of the cases considered was D2-branes within D6-branes, preserving one quarter supersymmetry. A similar construction produces solutions for a wave or NS5-branes within D6-branes. There are various other solutions which can be found using T- and S-duality transformations.

In the near-core or near-horizon limit of the D6-branes the problem is greatly simplified since there is then an essentially trivial eleven-dimensional description of the D6-branes. In the near-core limit, the eleven-dimensional description of $N$ D6-branes is flat $6+1$ dimensions plus an ALE space with $A_{N-1}$ singularity. This ALE space is simply the orbifold $\mathbb{C}^2/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts on the complex coordinates as $z_1 \to e^{2\pi i/N} z_1$ and $z_2 \to e^{-2\pi i/N} z_2$, with the metric given by the flat Euclidean metric with these $\mathbb{Z}_N$ identifications. To see this explicitly we define real coordinates $\rho \geq 0, \theta \in [0, \pi/2]$ and $\phi, \tilde{\varphi} \in [0, 2\pi)$ through $z_1 = \rho \cos \theta e^{i\phi}$ and $z_2 = \rho \sin \theta e^{i\tilde{\varphi}}$. The eleven-dimensional metric is then

$$ds^2_{(1,10)} = dx_{(1,6)}^2 + d\rho^2 + \rho^2 \left( d\tilde{\theta}^2 + \cos^2 \tilde{\theta} d\tilde{\phi}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2 \right)$$  \hspace{1cm} (225)
with the identification of the coordinates under

\[ (\tilde{\phi}, \tilde{\varphi}) \to \left( \tilde{\phi} + \frac{2\pi}{N}, \tilde{\varphi} - \frac{2\pi}{N} \right) \]  

(226)

Note that this background will preserve supersymmetry since the identifications are such that the holonomy is (a discrete subgroup of) SU(2). So we have (a singular limit of) a supersymmetric special holonomy manifold from table 11. We can reduce to ten dimensions along the isometry direction given by \( \psi \equiv N \tilde{\phi} \) which has the standard period of 2\( \pi \). We also define \( \varphi \equiv \tilde{\phi} + \tilde{\varphi} \) which is invariant under the \( \mathbb{Z}_N \) transformations and, for later convenience, \( \theta \equiv 2\tilde{\phi} \) and \( \psi = \frac{2\tilde{\phi}}{2N} \). The Kaluza-Klein reduction along \( x^{10} = R_{11} \psi \) using the relations equations (20) and (21) gives the string frame metric, dilaton and Kaluza-Klein gauge potential

\[
\frac{1}{l_s^2} d\tilde{s}^2_{(1,9)} = \sqrt{\frac{2U}{g_s l_s^3 N}} dx^2_{(1,6)} + \sqrt{\frac{g_s l_s^3 N}{2U}} \left( dU^2 + U^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right) 
\]

(227)

\[
e^{\phi} = \left( \frac{2U l_s}{g_s N} \right)^{\frac{2}{3}} 
\]

(228)

\[
A = g_s l_s \frac{N}{2} (\cos \theta - 1) \, d\varphi 
\]

(229)

which is the near-horizon limit of \( N \) coincident D6-branes in type IIA. This is easily checked by taking the near-horizon limit

\[
l_s \to 0 \ , \ U = \frac{r}{l_s^2} \text{ fixed} \ , \ g_{YM}^2 = (2\pi)^4 g_s l_s^3 \text{ fixed} 
\]

(230)

of the solution for \( N \) coincident D6-branes

\[
ds^2 = H_6^{-\frac{2}{3}} dx^2_{(1,6)} + H_6^{\frac{1}{3}} (dr^2 + r^2 d\Omega_2^2) 
\]

(231)

\[
e^{\phi} = H_6^{-3/4} 
\]

(232)

\[\ast dA = -d(H_6^{-1}) \wedge \epsilon_{1,6} \]

(233)

where

\[
H_6 = 1 + \frac{g_s l_s N}{2r} \to \frac{g_s l_s^3 N}{2U} \equiv \tilde{H}_6 
\]

(234)

Note that the above process only relied upon having an eleven-dimensional solution which does not depend on the three coordinates of a three-sphere. Therefore we can repeat the above process of making \( \mathbb{Z}_N \) identifications and then reducing to ten dimensions, starting with non-trivial eleven-dimensional solutions such as a wave, M2-branes or M5-branes. When reduced to ten dimensions this will describe the appropriate object (wave, D2-branes or NS5-branes) along with \( N \) D6-branes. The reason we will always get \( N \) D6-branes in the solution is that the \( \mathbb{Z}_N \) identifications along with the Kaluza-Klein reduction will give a Kaluza-Klein one-form gauge potential with \( N \) units of magnetic flux, exactly as in the Minkowski space example above. Specifically, if we start with an eleven-dimensional metric

\[
ds^2_{(1,10)} = ds^2_{(1,6)} + h \left( d\rho^2 + \rho^2 \left( d\tilde{\rho}^2 + \cos^2 \tilde{\theta} d\tilde{\varphi}^2 + \sin^2 \tilde{\theta} d\varphi^2 \right) \right) 
\]

(235)

we will end up with the type IIA metric, dilaton and Kaluza-Klein gauge potential

\[
\frac{1}{l_s^2} ds^2_{(1,9)} = h^\frac{2}{3} \tilde{H}_6^{-\frac{1}{3}} ds^2_{(1,6)} + h^\frac{2}{3} \tilde{H}_6^{\frac{1}{3}} \left( dU^2 + U^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right) 
\]

(236)

\[
e^{\phi} = h^\frac{2}{3} \tilde{H}_6^{-\frac{2}{4}} 
\]

(237)

\[A = g_s l_s \frac{N}{2} (\cos \theta - 1) \, d\varphi 
\]

(238)
For example in the case of $n$ M2-branes we would have

\[ ds^2_{(1,6)} = H_2^{-2}dx^2_{(1,2)} + H_2^{-\frac{1}{2}}\left(dx^2_3 + \cdots + dx^2_6\right) \]  

(239)

\[ h = H_2^{\frac{1}{2}} \]  

(240)

\[ H_2 = 1 + \frac{2^5\pi^2l_s^6n}{(x^2_3 + \cdots + x^2_6 + \rho^2)^3} \]  

(241)

which would give the the type IIA metric for $n$ D2-branes localised within the near-horizon limit of $N$ D6-branes

\[ \frac{1}{\ell_s^2}ds^2_{(1,9)} = H_2^{-\frac{1}{2}}H_6^{-\frac{1}{2}}dx^2_{(1,2)} + H_2^{\frac{1}{2}}H_6^{-\frac{1}{2}}\left(dx^2_3 + \cdots + dx^2_6\right) + \]  

(242)

\[ H_2 = 1 + \frac{2^5\pi^2g_s^6N}{(x^2_3 + \cdots + x^2_6 + 2g_s^6NU)^3} \]  

(243)

\[ \tilde{H}_6 = \frac{g_s^3N}{2U} \]  

(244)

Note that the metric has the form of the smeared intersecting brane solutions given in section 4 and indeed we could smear the D2-branes in the 3456 directions to recover such a solution. However, in the solution here, the D2-branes are fully localised within the D6-branes which is a significant improvement on the solutions given by the flat-space harmonic function rules. Nevertheless, this solution is only valid in the near-horizon limit we have taken here and it is still an open problem to find any fully localised intersecting branes solutions, other than those described in section 5.2 without taking such a limit. In particular, it is not possible to simply replace $\tilde{H}_6$ with $1 + \tilde{H}_6$. Also note that the harmonic function for the D2-branes, $H_2$, retains the eleven-dimensional form with an $r^{-6}$ fall-off rather than $r^{-5}$. There is also the curious linear and quadratic combination of $U$ and $x_3, \ldots, x_6$ in the effective radial coordinate transverse to the D2-branes. These seem to be typical features of such near-horizon solutions as we have seen in section 5.1.

The cases of a wave or NS5-branes localised within D6-branes can easily be constructed using the above expressions. Related solutions can also be constructed by dualising these solutions. Note also that these solutions satisfy the curved-space harmonic function rules. Indeed, alternatively we could have constructed such solutions and others by solving the curved-space Laplace equation for the smaller brane in the near-core region of the large brane, similar to the example in section 5.1.

Recently the case of D2-branes within D6-branes was considered, without taking a near-horizon limit. Although the curved-space Laplace equation could not be completely solved, the solution could be written as a specific (one-dimensional) integral [258].

### 5.4 Branes intersecting D6-branes

It was shown by Hashimoto [259] that the construction of branes within D6-branes could easily be generalised to branes intersecting or ending on D6-branes. The cases where this is possible are those where it is consistent to embed part of the worldvolume of an M-brane in the orbifold $\mathbb{C}^2/\mathbb{Z}_N$. For example we could take an M5-brane with two worldvolume directions spanning the $z^1$-plane at $z^2 = 0$. This embedding is allowed because it is invariant under the $\mathbb{Z}_N$ action and it preserves supersymmetry because it is a holomorphic embedding. We then reduce to type
IIA as in section 5.3 to find the supergravity solution for a D4-brane ending on a D6-brane, in the near horizon limit of the D6-brane. See also [260] where a series of dualities are used to relate this to intersecting M5-branes, localised in the four relative transverse directions but smeared over the three overall transverse directions.

We start with the metric for $n$ M5-branes
\begin{equation}
    ds^2 = H^{-1/3}dx^2_{(1,3)} + H^{2/3} \left( dx_4^2 + \cdots + dx_6^2 \right) + H^{-1/3}dz^1 dz^1 + H^{2/3}dz^2 dz^2
\end{equation}
and perform the same identifications and reduction as in section 5.3. It can easily be checked that for these M5-branes filling the $z^1$-plane, $x^{10}$ is a worldvolume coordinate and so they reduce to D4-branes in type IIA. The resulting solution can be found in [259]. It turns out to be a solution for D4-branes ending on D6-branes rather than intersecting them. It should also be noted that this solution is not of the form of the solutions which can be derived from the harmonic function rules.

Similarly we could embed an M2-brane to describe a fundamental string ending on a D6-brane. We can also consider more general holomorphic embeddings. Since we must start with the supergravity solution for the M5-branes (or M2-branes,) only the case of planar embeddings of parallel M5-branes was considered in [259]. With the restriction that the embedding should be $\mathbb{Z}_N$-invariant this allows holomorphic embeddings of the form
\begin{equation}
    (\cos \alpha z^1 + \sin \alpha z^2)^N = c^N
\end{equation}
However, in such cases there can be problems interpreting the solutions in ten dimensions. In general the reduction produces a ten-dimensional solution with some D4-brane and NS5-brane charge. The problem is essentially that these charges appear not to match with expectations from considering geometrically how the M5-branes reduce to D4- and NS5-branes [259].

Yet another possibility is to have M5-branes spanning all of the orbifold which will reduce to the near-core limit of D6-branes intersected by D4-branes with a common two dimensional worldvolume. In this case we start with the M5-brane metric
\begin{equation}
    ds^2 = H^{-1/3} \left( -dx_0^2 + dx_1^2 \right) + H^{2/3} \left( dx_3^2 + \cdots + dx_6^2 \right) + H^{-1/3} \left( dx_7^2 + \cdots + dx_{10}^2 \right)
\end{equation}
Since the M5-branes fill the $789(10)$ space we can actually use the formulae of section 5.3 with $h = H^{-1/3}$. Note that this case could equivalently be derived using the curved-space harmonic function rules. Indeed it is one of the cases with eight relative transverse dimensions where a fully localised solution can be found, even without taking the near-core limit of the D6-branes.

5.5 Intersecting branes from special holonomy manifolds

We saw in sections 5.3 and 5.4 how intersecting brane solutions in type IIA arise from an orbifold in eleven dimensions. These solutions are related to a large class of intersecting brane solutions which were previously constructed, starting from pure geometry in eleven dimensions [261, 16] (see also [262].) The starting point was a supersymmetric eleven-dimensional solution consisting of a product of $(2 + 1)$-dimensional Minkowski spacetime with an eight-dimensional hyper-Kähler manifold. The hyper-Kähler manifolds considered had a triholomorphic $T^2$ isometry which means that we can reduce the solution to nine dimensions while preserving the same amount of supersymmetry. The idea is to reduce along one isometry direction to type IIA and then T-dualise along the other isometry direction to generate a type IIB solution.

Consider the simple example where the eight-dimensional hyper-Kähler manifold is a product to two four-dimensional Taub-NUT spaces. When we reduce along the isometry direction
of one of the Taub-NUT spaces we get a D6-brane in type IIA while the other Taub-NUT space is unaffected. Then T-duality along the isometry direction of this Taub-NUT space (or KK5-brane) produces a NS5-brane in type IIB. The same T-duality is along a worldvolume direction of the D6-brane so it becomes a D5-brane. So we have a type IIB supergravity solution describing the orthogonal intersection of a NS5- and D5-brane with a common (2 + 1)-dimensional worldvolume, smeared over the transverse direction (along which we T-dualised.)

The general eight-dimensional toric hyper-Kähler manifolds can be thought of as consisting of overlapping Taub-NUT spaces each with isometry direction along a \((p,q)\)-cycle of the \(T^2\) (see [261] for details.) We can think of this as a configuration of (non-orthogonal) intersecting KK6-branes. The reduction and T-duality then turns these Taub-NUT spaces into \((p,q)\)-5-branes in type IIB with a common (2 + 1)-dimensional worldvolume, intersecting at angles determined by their \((p,q)\)-charges. Again the solution is smeared over the overall transverse direction.

Alternatively we can reduce the eleven-dimensional solution along one of the flat directions and then T-dualise along both isometry directions of the hyper-Kähler manifold. This gives a type IIA solution describing NS5-branes with a common (1 + 1)-dimensional worldvolume smeared over two relative transverse directions. This solution can be lifted to eleven dimensions, giving a solution for intersecting M5-branes, additionally smeared over the (totally transverse) eleventh dimension. Instead of lifting, we could T-dualise along the common (spatial) worldvolume direction to find the type IIB solution for intersecting NS5-branes. Then SL(2, \(\mathbb{Z}\)) transformations map this to an intersection of \((p,q)\)-5-branes (all of the same type.) So we have supergravity solutions for any 5-branes in ten or eleven dimensions intersecting at angles with a common (1 + 1)-dimensional worldvolume.

Because of the triholomorphic \(T^2\) isometry, all the above dimensional reductions and T-duality transformations relate solutions with the same amount of supersymmetry. From table 11 we see that this is generically 3/16 supersymmetry. However, in the special case where the eight-dimensional hyper-Kähler manifold is a product of two four-dimensional hyper-Kähler manifolds the amount of supersymmetry preserved is 1/4. Note that in this case the intersections are orthogonal.

The above solutions can also be generalised to include D2-branes, strings or waves along the common worldvolume directions by starting with the eleven-dimensional solution for M2-branes with a transverse hyper-Kähler manifold

\[
\begin{align*}
    ds^2 &= H^{-2/3} dx_{(1,2)}^2 + H^{1/3} ds^2_{HK^8} \\
    F &= -d(H^{-1}) \wedge \epsilon_{1,2}
\end{align*}
\] (248) (249)

This solution preserves the same amount of supersymmetry (3/16, 1/4 etc.) provided \(H\) is harmonic wrt. the eight-dimensional hyper-Kähler metric \(ds^2_{HK^8}\) (see e.g. [263, 264, 265, 266, 267].) We recover the near-core D2-branes within D6-branes example of section 5.3 when the eight-dimensional hyper-Kähler manifold is \(\mathbb{C}^2 \times \mathbb{C}^2 / \mathbb{Z}_N\) which allows us to find an explicit solution for \(H\). Another starting point considered was the solution (using the curved-space harmonic function rules) for orthogonal M5-branes with a common (1+1)-dimensional worldvolume, with a hyper-Kähler metric on the remaining four worldvolume directions of each M5-brane.

Similar constructions were considered in [268, 269] involving hyper-Kähler manifolds with torsion. These constructions are essentially the same except now the four-dimensional manifolds describe NS5-branes in ten dimensions rather than KK6(5)-branes in eleven (ten) dimensions.

Similar intersecting brane configurations can be constructed from other manifolds of special holonomy (see e.g. [270].) In particular there has been recent interest in manifolds of \(G_2\) holonomy [271, 272, 273, 274, 275, 276] partly because compactification from eleven dimensions leads to a four-dimensional \(\mathcal{N} = 1\) theory, which can be chiral if there are singularities [277, 278].
Intersecting brane solutions arise from dimensional reduction and T-duality in a very similar manner to the hyper-Kähler manifolds of Sp(2) holonomy discussed above. For example a $T^2$ isometry is required to relate eleven-dimensional solutions to type IIB solutions and fixed points of these isometries correspond to KK6-branes in eleven dimensions [271, 272, 279, 280]. See [281, 282, 283, 284, 285, 286, 287, 288, 289, 290] for explicit examples. Note that these constructions are often useful for understanding properties of the special holonomy manifolds in terms of intersecting branes. For example changing the relative positions of the branes corresponds to changing parameters describing the special holonomy manifold and can give an understanding of processes such as topology change where the manifold becomes singular but which may be a smooth(er) process in terms of the intersecting branes.

6 Hanany-Witten constructions

Hanany-Witten brane configurations provide a very useful method of describing large classes of supersymmetric gauge theories. One of the advantages of these constructions is that many features of the gauge theory can be understood in simple geometric terms. These features include the moduli space and gauge theory parameters as well as the gauge group and matter content and even (given enough supersymmetry) the running gauge coupling. In this section we will review the description of gauge theories in terms of such brane configurations and some progress towards finding the corresponding supergravity solutions, at least in the near-horizon limit appropriate for the AdS/CFT correspondence.

6.1 Basic Construction

The basic setup involves taking some parallel $D_p$-branes and ‘compactifying’ them by making them end on some other branes. In the original setup analysed by Hanany and Witten [291], one of the simplest examples consisted of $N$ coincident D3-branes with worldvolume directions $0126$ ending on two parallel NS5-branes with worldvolume directions $012345$, separated in the $x^6$ direction. In terms of the $U(N)$ gauge theory on the worldvolume of the D3-branes, the addition of the NS5-branes has two effects. The theory is reduced from 3+1 to 2+1 dimensions (at length scales larger than the separation between the NS5-branes in the 6 direction) and some of the degrees of freedom are projected out. In particular since the boundary conditions for the D3-branes to end on the NS5-branes fix the 789 but not the 345 positions, the three-dimensional theory has only the three massless scalars corresponding to the transverse 345 directions, rather than all six scalars present in the four-dimensional $\mathcal{N} = 2$ theory. In fact there are two other scalars arising from the gauge field. The $A_6$ component of the four-dimensional gauge potential is a superpartner of the 789 scalars while the three-dimensional gauge potential is dual to a scalar which is a superpartner of the 345 scalars.

It is also possible to include hypermultiplets in the gauge theory. These can arise when D3-branes end on an NS5-brane from opposite sides. We can view these states as corresponding to open strings which end on D3-branes on either side of the NS5-brane. For example suppose we have three NS5-branes separated in the 6 direction with $N_1$ and $N_2$ D3-branes between each consecutive pair, at the origin of the 345 space. The gauge group in this case would be $U(N_1) \times U(N_2)$. There would also be hypermultiplets transforming in the $(N_1, N_2)$ representation. All these states are massless. However, if we move say one of the $N_2$ D3-branes away from the origin of the 345 space then we break the $U(N_2)$ gauge group to $U(N_2 - 1) \times U(1)$. The open strings stretching between this D3-brane and the others have a minimal length (given by the separation) and so a non-zero minimal mass. This mass is the mass of the W-boson (multiplet)
and a hypermultiplet transforming in the fundamental representation of U($N_1$). In general, arbitrarily positioned D3-branes break a U($N$) gauge group to U(1)$^N$. So the Coulomb branch of the gauge theory is parameterised by the position of the D3-branes in the 345 space together with the VEVs of the scalars dual to the U(1) gauge potentials.

It is possible to introduce D5-branes spanning the 012789 directions without breaking more supersymmetry. Including these D5-branes leads to new possibilities. For example D3-branes stretched between these D5-branes give rise to hypermultiplets with a different R-charge to the above hypermultiplets. The scalars in these multiplets correspond to motions in the 789 directions together with the $A_6$ component of the four dimensional gauge potentials.

Various properties of these configurations were analysed in [291]. Since S-duality interchanges NS5-branes and D5-branes while leaving D3-branes unchanged, this effectively exchanges the vector multiplets and hypermultiplets in the gauge theory. This provides a string theory realisation of mirror symmetry in three-dimensional gauge theories. Another type of duality occurs when the configuration consists of $N$ D3-branes between two NS5-branes. In this case the Coulomb branch of the three-dimensional gauge theory can equivalently be described in terms of the 5-brane worldvolume theory. This gives a geometric interpretation of the equivalence between the moduli space of $N$ SU(2) monopoles and this Coulomb branch. Specifically, S-duality followed by T-duality in the common 12 directions maps the brane configuration to that of $N$ D1-branes between two D3-branes which was already considered from both the D1- and D3-branes’ worldvolume theories in section 3.4.

There are many generalisations of these Hanany-Witten constructions to describe gauge theories in various dimensions, with various gauge groups, matter content, amounts of supersymmetry etc. See [292] for a comprehensive review with extensive references. In the next section we will describe some examples of four-dimensional $\mathcal{N} = 2$ gauge theories with the aim of finding the supergravity duals (in the context of the AdS/CFT correspondence) in section 6.3.

6.2 Four-dimensional $\mathcal{N} = 2$ SYM

Seiberg and Witten’s work on $\mathcal{N} = 2$ four-dimensional gauge theories [293, 294] showed how the exact low energy effective action (including non-perturbative effects) could be described by a family of Riemann surfaces $\Sigma \subset \mathbf{C}^2$. These Riemann surfaces, called Seiberg-Witten curves, encode information about the gauge theory such as the exact mass of BPS states. These families of curves contain parameters which correspond to VEVs parameterising the Coulomb branch of the gauge theory.

Using a Hanany-Witten brane construction, Witten [295] rederived the description of $\mathcal{N} = 2$ four-dimensional gauge theories in terms of a Seiberg-Witten curve. Not only did this provide another example of branes in string theory reproducing field theory results, but it also provided a much more intuitive geometric interpretation of the previously abstract Seiberg-Witten curve. In addition, for a large class of field theories there is a simple prescription for constructing the corresponding Hanany-Witten brane configuration, from which the Seiberg-Witten curve can easily be described, thereby solving a difficult field theory problem. We will describe this construction in the following sections.

6.2.1 Type IIA brane configuration

The brane setup in type IIA involves D4-branes with worldvolume directions 01236 and NS5-branes with worldvolume directions 012345. This can be obtained from the configurations involving D3-branes and NS5-branes in section 6.1 by T-duality in the 3 direction. All the branes are located at $x^7 = x^8 = x^9 = 0$, the NS5-branes are separated in the $x^6$ direction and
the D4-branes can have finite, semi-infinite or infinite extent in the $x^6$ direction by ending on two, one or no NS5-branes respectively. We refer to those D4-branes as finite, semi-infinite and infinite. It is simple to check that the supersymmetry projection operators for the D4- and NS5-branes are compatible, and so the system is one quarter BPS, preserving $\mathcal{N} = 2$ supersymmetry in four dimensions.

The simplest configurations involve two NS5-branes with $N_c$ finite D4-branes between them. This gives gauge group SU($N_c$) rather than U($N_c$) since a U(1) is frozen out because the corresponding centre of mass position of the D4-branes is fixed [295]. Semi-infinite D4-branes produce hypermultiplets in the fundamental representation. There are two types of semi-infinite D4-branes, those ending on the right NS5-brane and extending to $x^6 = \infty$ and those ending on the left NS5-brane and extending to $x^6 = -\infty$. They are equivalent with respect to the gauge theory. However, if there are $N_f$ hypermultiplets then only when we choose all $N_f$ semi-infinite D4-branes to be of the same type is the global SU($N_f$) flavour symmetry manifest in the brane configuration.

So the general configuration we will consider involves $N_L$ semi-infinite D4-branes to the left of, $N_c$ finite D4-branes between, and $N_R$ semi-infinite D4-branes to the right of the two NS5-branes. The gauge group is SU($N_c$) and there are $N_f = N_L + N_R$ hypermultiplets in the fundamental representation of SU($N_c$). There are obvious further generalisations involving more NS5-branes and a gauge group which is a product of SU($N_i$) factors, with hypermultiplets in $(N_i, N_i + 1)$ representations. It is also possible to introduce D6-branes with worldvolume directions 0123789 without breaking any more supersymmetry. These provide another method of introducing hypermultiplets in the fundamental representation due to strings stretching between the D6-branes and the finite D4-branes (between the same NS5-branes.)

### 6.2.2 Running gauge coupling

One of the surprising features of the brane construction is that it gives a geometric interpretation of the logarithmic running of the gauge coupling. To see this we first have to identify the gauge coupling in the brane configuration. It is given by the separation of the NS5-branes, say $L$, in the $x^6$ direction. This is simply because we view the finite D4-branes as being compactified on an interval of length $L$, i.e.

$$S_{D4} \sim \frac{1}{l_s g_s} \int d^5 x F^2 \sim \frac{L}{l_s g_s} \int d^4 x F^2$$

So we see that the four-dimensional gauge coupling is given by

$$\frac{1}{g_{YM}^2} \sim \frac{L}{l_s g_s}$$

Now the point is that since each NS5-brane has D4-branes ending on it, the tension of the D4-branes distorts the NS5-brane and so the separation between the NS5-branes depends on the position in the $v = x^4 + i x^5$ plane. The bending in the $x^6$ direction will be (asymptotically) logarithmic in $|v|$ since the end of the D4-brane in the NS5-brane is of co-dimension 2. I.e. if we ignore the common worldvolume directions we have the same situation as a three-dimensional system with a string extended along $x^6$ (a D4-brane) ending on a membrane (an NS5-brane) extended in the $x^4$ and $x^5$ directions. Obviously strings ending from the left bend the membrane in the opposite direction to those ending from the right. So if we consider a single NS5-brane with $n_L$ ($n_R$) D4-branes ending from the left (right) then the net effect will be a bending of the NS5-brane asymptotically given by (recalling that $T_{D4} \sim 1/g_s$ whereas $T_{NS5} \sim 1/g_s^2$)

$$x^6 \sim l_s g_s (n_L - n_R) \ln |v|$$

(252)
A more detailed analysis shows that provided the centre of mass position of the D4-branes is fixed, giving gauge group SU($N_c$) rather than U($N_c$), moving the D4-branes to different positions in the $v$-plane does not change the asymptotic behaviour of the NS5-branes. In the Hanany-Witten construction we are considering, the logarithmic bending of the NS5-branes means that

$$
\frac{1}{g_Y^2} \sim (N_c - N_R) \ln |v| - (N_L - N_c) \ln |v| = (2N_c - N_f) \ln |v|
$$

Noticing that the coefficient $2N_c - N_f$ is exactly the same as appears in the one-loop (perturbatively exact for $\mathcal{N} = 2$ supersymmetry) beta-function, we see that the brane construction reproduces the correct running gauge coupling provided we interpret $|v|$ as an energy scale. Note that this provides an example of a UV/IR correspondence. The origin of this is the same as in the AdS/CFT correspondence, arising from the identification of lengths and masses due to the mass of open strings (which are relevant to the gauge theory) being proportional to their length. In particular, from the gauge theory point of view, $w = v/l_s^2$ is a natural variable to use.

So we can understand many details of the field theory from this ten-dimensional brane configuration. However, we can go even further by considering the lift to eleven dimensions. The reason for this is that the singular intersections of the D4-branes and NS5-branes are smoothed out in the eleven-dimensional configuration. This is related to the question of the shape of the branes and the preservation of supersymmetry. As we have described the logarithmic bending of the NS5-branes, it is not obvious why we should still preserve $\mathcal{N} = 2$ supersymmetry. Indeed the requirement of supersymmetry will place strong constraints on the exact shape of the branes and we will see that these constraints can be very easily solved in eleven dimensions.

### 6.2.3 Eleven-dimensional description

We can lift the type IIA configuration to eleven dimensions using the well-known relation between type IIA branes and M-branes [24]. In particular an NS5-brane is an M5-brane, pointlike in the eleventh dimension, $x^{10}$, while a D4-brane is also an M5-brane but one which wraps the eleventh dimension (which is a circle of radius $R$.) We can immediately deduce that the picture of a D4-brane ending on an NS5-brane is modified for $R \neq 0$ since the boundary of the D4-brane would span $x^{10}$ and so could not be contained within the worldvolume of the NS5-brane. So only when we have a genuine intersection (a D4-brane passing through an NS5-brane rather than ending on it) can we have flat D4-branes spanning $x^6$ and $x^{10}$ and point-like in $x^4$ and $x^5$. I.e. in general the D4-branes must be deformed, not just the NS5-branes, although this is not apparent in the singular limit $R \to 0$. Note that in the special case where there are $N_c$ infinite D4-branes, neither the D4- nor the NS5-branes are deformed, and in particular the gauge coupling is constant as expected for the conformal theory when $N_f = 2N_c$.

To see what happens in general we should consider the conditions for supersymmetry preservation. We know that in the case of orthogonal intersections, $\mathcal{N} = 2$ supersymmetry is preserved with the compatible projection conditions

$$\hat{\Gamma}_{012345} \epsilon = \epsilon = \hat{\Gamma}_{01236(10)} \epsilon$$

So we see from the discussion of section 3.2 that any holomorphic embedding of an M5-brane will preserve $\mathcal{N} = 2$ supersymmetry and indeed since we know the projection conditions asymptotically, an M5-brane which preserves $\mathcal{N} = 2$ supersymmetry must be embedded holomorphically.

---

\(^6\)We will continue to refer to D4- and NS5-branes for the moment in order to distinguish between the orientations of the M5-branes.
with respect to the given complex coordinates \( v = x^4 + ix^5 \) and \( s = x^6 + ix^{10} \). More precisely, \( \exp(s/R) \) is a better coordinate than \( s \) since it is single valued under \( x^{10} \rightarrow x^{10} + 2\pi R \) but for simplicity we will continue to use \( s \). So the embedding is described by a Riemann surface \( \Sigma \subset \mathbb{C}^2 \) which we can determine explicitly up to a finite number of parameters using the known asymptotic form of the embedding – i.e. how many NS5-branes there are and the positions of the semi-infinite D4-branes.

The Riemann surface \( \Sigma \) is in fact the Seiberg-Witten curve for the gauge theory. BPS states in the field theory correspond to M2-branes ending on the M5-brane. The mass of the M2-brane gives the mass of the BPS state. This, together with the conditions for the M2-brane embedding to be supersymmetric leads to an M-theory derivation of the Seiberg-Witten differential \[296, 297, 298, 299\] (see also \[300\].) It is also interesting to note that M2-branes with the topology of a cylinder or disc correspond to vector multiplets or hypermultiplets respectively \[298, 299\]. It is also possible to derive the low energy effective action from the M5-brane worldvolume theory \[301, 295\].

6.3 Supergravity dual

If we know how to describe a particular gauge theory in terms of a particular brane configuration then we can try to describe the gravity dual of the field theory. We follow essentially the same steps as in the derivation of the AdS/CFT duality for the case of parallel branes presented in section 2.6. We first identify the field theory parameters which should be kept fixed while taking a limit to decouple gravity and string modes. We then take this limit for the appropriate supergravity solution describing the brane configuration and this should give a candidate gravity dual of the field theory.

To find the supergravity solution we use the conditions for preservation of supersymmetry to constrain the metric and four-form field strength. Rather than start with the most general form of metric we can first impose the expected symmetries of the solution, namely 3+1 dimensional Poincaré invariance of the common worldvolume directions and an SO(3) invariance corresponding to rotations in the totally transverse directions which is identified with the SU(2) R-symmetry of the gauge theory. This allows us to write the metric as

\[
ds^2 = H_1 \eta_{\mu\nu} dx^\mu dx^\nu + 2H_1 g_{m\overline{m}} dz^m d\overline{z}^\overline{m} + H_2 \delta_{\alpha\beta} dx^\alpha dx^\beta \tag{255}
\]

where \( H_1, H_2 \) and \( g_{m\overline{m}} \) can only depend on the two complex coordinates \( z^m \) and the radial coordinate in the three totally transverse directions, \( r \equiv \sqrt{\delta_{\alpha\beta} x^\alpha x^\beta} \). We will use the notation \( v = z^1 \) and \( s = z^2 \). It turns out to be convenient to include the factor \( H_1 \) with \( g_{m\overline{m}} \). Since the M5-branes are magnetic sources for \( F^{(4)} \) we can also deduce that the only non-vanishing components of \( F^{(4)} \) will have at least two indices in the totally transverse space and no indices in the common worldvolume directions.

The projection conditions on the 32-component spinor \( \epsilon \) are

\[
\hat{\Gamma}_{0123m\overline{m}} \epsilon = i \delta_{m\overline{m}} \epsilon \tag{256}
\]

It is now a straightforward, though rather lengthy, process to write out the Killing spinor equations \[333\], \( \hat{D}_\mu \epsilon = 0 \), in terms of the components of the above metric and four-form. Then using the above projection conditions we can express this as a sum of independent antisymmetric combinations of Gamma-matrices acting on \( \epsilon \). The coefficients of these terms must vanish in order to satisfy the Killing spinor equation for non-vanishing \( \epsilon \). The term without any Gamma-matrices acting on \( \epsilon \) is slightly different, it is a first order differential equation determining the positional dependence of \( \epsilon \), with the result that \( \epsilon \) is a specific function times a constant
spinor. The other equations result in a set of first-order differential equations which reduce to the following relations [241]

\[ H_1 = H^{-\frac{3}{4}} \]  
\[ H_2 = H^{\frac{3}{4}} \]  
\[ F_{m\pi\alpha\beta} = i\epsilon_{\alpha\beta\gamma}\partial_\gamma g_{m\pi} \]  
\[ F_{m789} = -i\partial_m H \]  
\[ F_{\pi789} = i\partial_{\pi}H \]  
\[ H = 4g = 4(g_v^v g_s^s - g_v^s g_s^v) \]

and the constraint that \( g_{m\pi} \) is a Kähler metric, with (square-root) determinant \( g \). In deriving these results it is necessary to fix some integration constants. This has been done using the condition that asymptotically we recover the usual Minkowski metric, i.e. that asymptotically

\[ H_1 \to 1 \quad , \quad H_2 \to 1 \quad , \quad g_{m\pi} \to \delta_{m\pi} \]  

Notice that the metric takes a similar form to what we would expect from the harmonic function rules but with some extra off-diagonal terms in the relative transverse space

\[ ds^2 = H^{-\frac{3}{4}}dx_{(1,3)}^2 + 2H^{-\frac{3}{4}}g_{m\pi}dz^m dz^\pi + H^\frac{3}{4}dx_{(3)}^2 \]

Clearly the four-form satisfies \( F \wedge F = 0 \) and it is relatively simple to check that the Bianchi identity is identically satisfied

\[ d(*F) = 0 \]  

so we are left with the equation of motion for \( F \) with a magnetic source \( J \)

\[ dF = J = J_{m\pi}dz^m \wedge dz^\pi \wedge dx^8 \wedge dx^9 \wedge dx^{10} \]

which results in the equations

\[ 4\partial_m \partial_\pi(2g) + \partial_\gamma \partial_\gamma g_{m\pi} = -iJ_{m\pi} \]  

Since the source describes an M5-brane wrapped on a Riemann surface \( \Sigma \), defined by a holomorphic function \( f(v, s) = 0 \) at \( r = 0 \), we can write the source terms as

\[ J_{m\pi} = -4i\pi^3 l_p^3 (\partial_m f)(\bar{\partial}_n f)\delta^2(f)\delta^3(r) \]

The difficulty lies in solving these source equations [267] which are non-linear due to the presence of both the metric components and determinant. We can rewrite these equations as a single (highly non-linear) partial differential equation for a function \( K \), the Kähler potential for the metric, \( g_{m\pi} = \partial_m \partial_\pi K \). This results in

\[ 8g(K) + \partial_\gamma \partial_\gamma K = -4\pi^3 l_p^3 |f|^2 \delta^2(f) \delta^3(r) \]

which is related to the complex Monge-Ampère equation.
6.3.1 Special cases

It can easily be checked that we can reproduce the simple example of parallel M5-branes. Consider \( N \) coincident M5-branes at \( r = s = 0 \) (so \( f = s^N \)) with the source given by

\[
J_{s^2} = -4i\pi^3 l_p^3 N \delta^3(r) \delta^2(s) \tag{270}
\]

Defining \( r_{(5)} = \sqrt{r^2 + |s|^2} \) we can easily check that the solution to the source equations (267) is

\[
g_{v\bar{s}} = g_{s\bar{v}} = 0 \tag{271}
\]

\[
g_{v\bar{v}} = \frac{1}{2} \tag{272}
\]

\[
g_{s\bar{s}} = \frac{1}{2} + \frac{\pi l_p^3 N}{2r_{(5)}^3} \tag{273}
\]

which is the expected solution for \( N \) parallel M5-branes, and provides a check on the normalisation of the source in equation (268).

A less trivial example to is recover the harmonic function rules. One way to do this is to try making the simplifying assumption that the metric is diagonal for orthogonal intersections. I.e. if we take the source terms for \( M \) M5-branes at \( r = v = 0 \) intersecting \( N \) M5-branes at \( r = s = 0 \) (\( f = v^M s^N \)) so that the sources are

\[
J_{v^2} = J_{s^2} = 0 \tag{274}
\]

\[
J_{v\bar{s}} = -4i\pi^3 l_p^3 M \delta^3(r) \delta^2(v) \tag{275}
\]

\[
J_{s\bar{v}} = -4i\pi^3 l_p^3 N \delta^3(r) \delta^2(s) \tag{276}
\]

then we try to find solutions where

\[
g_{v\bar{s}} = g_{s\bar{v}} = 0 \tag{277}
\]

Note that the conditions for a Kähler metric show that \( g_{v\bar{v}} \) is independent of \( s \) and \( \bar{s} \), and that \( g_{s\bar{s}} \) is independent of \( v \) and \( \bar{v} \). For example

\[
\partial_v g_{s\bar{s}} = \partial_s g_{v\bar{v}} = 0 \tag{278}
\]

The equations (267) for the \( v\bar{s} \) and \( s\bar{v} \) components then reduce to the requirement that \( g_{v\bar{s}} \) is also independent of \( v \) and \( \bar{v} \), or that \( g_{s\bar{s}} \) is independent of \( s \) and \( \bar{s} \). Hence we see that the assumption of having a diagonal metric requires at least one set of M5-branes to be smeared over the worldvolume directions of the other M5-branes. Without loss of generality we can require \( g_{s\bar{s}} \) to be independent of \( s \) and \( \bar{s} \). In this case the remaining equations (267) become

\[
4g_{s\bar{v}} \partial_v \partial_{\bar{s}} g_{v\bar{s}} + \partial_{\bar{v}} \partial_\bar{s} g_{v\bar{s}} = -iJ_{v\bar{s}} \tag{279}
\]

\[
\partial_{\bar{v}} \partial_\bar{s} g_{s\bar{s}} = -iJ_{s\bar{v}}
\]

Clearly the second of these equations cannot be satisfied with the fully localised source term \( J_{s^2} \) since there is no way to produce the delta function \( \delta^2(s) \) on the left hand side. However, if we compactify the \( s \)-plane on a 2-torus of volume \( V \) we can smear the \( N \) M5-branes over the \( T^2 \) by replacing \( \delta^2(s) \) with \( 1/V \) in the source term. We then see that \( g_{s\bar{s}} \) satisfies the flat-space Laplace equation in the three totally transverse directions while \( g_{v\bar{s}} \) satisfies a curved-space Laplace equation. These are just the expected conditions of the curved-space harmonic function rules of section 4.1.2 which allow partially localised intersecting branes, with the explicit near-horizon solution given in section 5.1. Again, if we also smear the \( M \) M5-branes over the \( v \)-plane we recover the flat-space harmonic function rules for this type of intersection with both branes smeared over the relative transverse directions.
6.3.2 General near-horizon solution

Since we are interested in describing the gravity dual of the four-dimensional gauge theory, we only need to solve the supergravity equations in the appropriate gauge theory or near-horizon limit. In this limit we keep the gauge theory masses and coupling constant fixed while taking \( l_P \to 0 \). Specifically we define coordinates which remain fixed in this limit as

\[
\begin{align*}
  w &= \frac{v}{l_5^2} = \frac{vR}{l_P^2} \\
  t^2 &= \frac{r}{g_s l_5^3} = \frac{r}{l_P^3} \\
  y &= \frac{s}{R}
\end{align*}
\]  

(280)

Note that \( y \) is dimensionless whereas \( w \) and \( t \) have dimensions of mass. We can define angular coordinates \( \theta \) and \( \phi \) so that

\[
\begin{align*}
  w &= \rho \sin \theta e^{i\phi} \\
  t &= \rho \cos \theta
\end{align*}
\]  

(283)

(284)

where \( \rho \) has dimensions of mass. The essential simplification which occurs in such a limit is that we no longer have a dimensionful constant \( l_P \). Therefore the dimension of any quantity determines its dependence on \( \rho \).

In order to find specific solutions we can also use our expectations from the AdS/CFT correspondence that the dual of a four-dimensional conformal field theory should involve \( AdS_5 \). The most general possibility is that the metric is of the form of a warped product of \( AdS_5 \) with a six-dimensional metric

\[
\begin{align*}
  \frac{1}{l_p^2} ds^2 &= \Omega^2 \left( u^2 dx_{(1,3)}^2 + \frac{1}{u^2} du^2 \right) + ds_{(6)}^2
\end{align*}
\]  

(285)

where the warp factor \( \Omega \) and the six-dimensional metric \( ds_{(6)}^2 \) are arbitrary functions of the dimensionless coordinates \( \theta, \phi \) and \( y \). Since \( u \) has dimensions of mass we know that \( u/\rho \) is also a function of only \( \theta, \phi \) and \( y \).

Requiring that the metric of equation (264) can be written in the form of equation (285) places several constraints on the components of the Kähler metric \( g_{mn} \) which are not obviously related to the equations of motion. However, the AdS/CFT correspondence predicts that they should be compatible and indeed it is possible to find a solution \([302]\). Furthermore the \( w \) and \( y \) dependence of the solution is naturally written in terms of the holomorphic function defining the Riemann surface – in this conformal case with the two NS5-branes separated by \( 1/g_{YM}^2 \) in the \( y \)-plane, intersected by (for gauge group SU(\( N \))) \( N \) infinite D4-branes

\[
f = \left( y - \frac{1}{2g_{YM}^2} \right) \left( y + \frac{1}{2g_{YM}^2} \right) w^N
\]  

(286)

It is then relatively straightforward to check that the solution in \([302]\) generalises to the case of an arbitrary Riemann surface \( \Sigma \), i.e. an arbitrary holomorphic function \( f(w, y) \). The supergravity solution can then be determined from the Kähler potential \( K \) which is determined by \( f \) through two holomorphic functions \( F(w, y) \) and \( G(w, y) \) by

\[
\begin{align*}
  K &= \frac{\pi N}{2t^2} \ln \left( \frac{\sqrt{t^4 + |F|^4 + t^2}}{\sqrt{t^4 + |F|^4 - t^2}} \right) + \frac{1}{2} |G|^2 \\
  F &= f^{1/N}
\end{align*}
\]  

(287)

(288)
where in general \( N \) is defined as the degree of \( f \) as a polynomial in \( w \). To find explicit solutions we need to solve

\[
\left( \partial_y F^2 \right) \left( \partial_w G \right) - \left( \partial_w F^2 \right) \left( \partial_y G \right) = 1
\]

(289)
to find \( G \). Whether this can be solved explicitly depends on the choice of \( f \).

An interesting observation is that we can interpret the functions \( F^2 \) and \( G \) as local coordinates transverse and parallel to the M5-brane. Equation (289) is simply the condition that the holomorphic coordinate transformation from \((w, y)\) to \((F^2, G)\) has unit Jacobian. It is also the necessary condition for the metric

\[
g_{mn} = 2 \left( \partial_m F^2 \right) \left( \partial_n F^2 \right) g + \frac{1}{2} \left( \partial_m G \right) \left( \partial_n G \right)
\]

(290)
to have determinant \( g \). It can then be seen that the source equations (267) and (268) reduce to the condition that \( g \) is a harmonic function in the five-dimensional transverse space with radial coordinate

\[
\tilde{r} \equiv \sqrt{t^4 + |F|^4}
\]

(291)
so that

\[
g = \frac{\pi N}{8r^3}
\]

(292)
It can easily be seen that, with \( g \) independent of \( G \), \( g_{m\bar{n}} \) are the components of a Kähler metric as required, with Kähler potential \( K \) as given in equations (287).

### 6.3.3 Alternative supergravity constructions

There are many other methods of constructing brane solutions which should be dual to \( \mathcal{N} = 2 \) four-dimensional gauge theories. We can consider wrapped 5-bra nes in ten dimensions. Such solutions have been constructed using lower dimensional gauged supergravity theories [304, 305, 306] or by dimensional reduction of the analysis presented in the previous section [307].

Alternatively we can use D3-branes in type IIB. The required amount of supersymmetry is preserved by D3-branes with worldvolume directions 012 together with D7-branes with worldvolume directions 01236789 and/or a supersymmetric (SU(2) holonomy) orbifold \( \mathbb{C}^2 / \mathbb{Z}_n \) in the 6789 directions. This configuration is T-dual (along the 6 direction) to the type IIA Hanany-Witten configuration described in section 6.2. This has been shown explicitly for the partially localised solution of section 5.1 (in the near-horizon limit) corresponding to the \( N_f = 2N_c \) conformal field theory [211] and for the usual smeared flat-space harmonic function solution [252]. The type IIB solution is \( AdS_5 \times S^5 / \mathbb{Z}_n \) as expected. In other cases there is no isometry direction and so T-duality cannot be explicitly performed on the supergravity solution, although we can still formally relate the descriptions in this way. In particular the \( n \) NS5-branes T-dualise to \( n \) coincident KK5-branes which, in the near-core limit, is simply the orbifold geometry. The D4-branes stretched between NS5-branes T-dualise to fractional D3-branes [308, 309, 310, 311, 312, 313, 314, 315, 316] which are fixed at the orbifold fixed-point. These have the interpretation as D5-branes wrapping the (zero-size) two-spheres which arise when resolving the singularity (separating the KK5-branes to produce a smooth multi-centred Taub-NUT metric.) So the type IIA Hanany-Witten configuration has an equivalent type IIB description [315]. However, the gauge couplings which had the geometrical description in terms of the separation of the NS5-branes are now encoded in the flux of the NS-NS B-field through the two-spheres [316]. Supergravity solutions have been considered for these configurations in [317, 318, 319, 320, 321]. See also [322] for the closely related case where the orbifold is replaced by the compact manifold K3. Analysing the supergravity solution with a probe brane led to the
discovery of the enhançon mechanism for resolving singularities. In this case the probe brane becomes tensionless at a finite radius from the singularity and therefore cannot approach any closer, effectively cutting-off a finite radius ball around the singularity. I.e. the interpretation is that the physically acceptable geometry is sourced by a spherical shell of branes rather than branes sitting at a singular point.

6.3.4 Similar brane configurations

There are several obvious generalisations of the case of an M5-brane wrapping a Riemann surface $\Sigma \subset \mathbb{C}^2$. In terms of four-dimensional gauge theories the most natural case to consider is an M5-brane wrapping $\Sigma \subset \mathbb{C}^3$, leading to an $\mathcal{N} = 1$ theory. An analysis of the Killing spinor equations similar to section 6.3 has been performed in [323]. Although technically more complicated, we expect solutions can be found in the near-horizon limit, with a similar form to those described in section 6.3.2. These solutions are special cases of more general supersymmetric solutions of eleven-dimensional supergravity with four-form flux on a (non-compact) seven-dimensional manifold [324]. Again, solutions can be found via gauged supergravity [304, 325, 326] or in terms of fractional D3-branes in type IIB [317, 327, 328]. See also [329, 330, 331, 332, 280, 333, 334, 335, 336, 337, 338] for other examples of supergravity solutions for wrapped branes.

It is also possible to consider other branes wrapping Riemann surfaces. This was discussed in [339] for the cases of M2- and M5-branes in terms of generalised calibrations, as well as the possibility of the branes wrapping more general cycles. It was shown in [340, 341] that the supergravity solution for a $p$-brane wrapping a Riemann surface $\Sigma \subset \mathbb{C}^2$ is given by a metric of the form of equation (264), determined by a source equation similar to equation (267) with an appropriate number of parallel and transverse dimensions. The solutions to such equations were analysed as a perturbation series around (asymptotic) Minkowski space. The interesting result was that the perturbative solution does not converge (in ten or eleven dimensions) for $p$-branes with $p \leq 3$. This is consistent with the claim that no fully localised supergravity solutions exist in such cases. Indeed the issue of whether localised solutions exist had been considered previously [342, 343]. It was argued using the black hole no-hair theorem [342] and gauge theory arguments [343] that there should not be fully localised solutions in some cases. Using the curved-space harmonic function rules which provide an implicit solution without smearing for the case of a D$p$-brane parallel to a D$(p + 4)$-brane, it was shown [343] that the D$p$-brane would delocalise in the four relative transverse directions as the separation between the branes was reduced, in the case where $p \leq 1$. T-duality in two of the relative transverse directions (i.e. without changing the number of overall transverse directions) relates these cases precisely to those cases which were conjectured not to have localised solutions in [340].

There have also been attempts to find localised solutions describing intersecting D3- and D5-branes [344] relevant to the Hanany-Witten configurations discussed in section 6.1 and intersecting M2- and M5-branes as well as strings ending on D-branes [345]. The Killing spinor equations have been analysed but it is still an open problem to find fully localised solutions. It would be interesting to find such solutions, even in the near-horizon limit, to since they are likely to have a different character from the intersecting brane solutions which are singular limits of a smoothly wrapped brane. An additional motivation for the D3/D5 and M2/M5 cases is that it has been argued [346, 347] that the intersections will provide an example of the localisation of gravity with non-compact transverse dimensions.
7 Conclusions and outlook

We have seen how configurations of intersecting branes have been very useful in understanding properties of black holes (section 4) and gauge theories (section 6). In terms of supergravity solutions we have seen in section 4 that the very simple flat-space harmonic function rules allow us to construct supersymmetric solutions but that the branes are smeared over some directions. As we saw in section 4 the curved-space harmonic function rules improve the localisation of the branes, even allowing full localisation in some cases, but usually it is not possible to find explicit solutions. However, taking a near-horizon limit simplifies the problem and explicit solutions can often be found. So, although these harmonic function rules have proved very useful, especially in relating lower dimensional black holes to brane configurations, they are only applicable in special cases to the problem of finding fully localised solutions. We have described some cases of fully localised solutions in sections 5 and 6 but it is still unclear what the properties of such solutions are for general (supersymmetric) configurations of branes, even in the near-horizon limit where we do know several explicit solutions. For example, while the smeared solutions given by the harmonic function rules all have the same general form, it is not clear whether different fully localised solutions will have such a similar description. It is hoped that progress can be made in this direction. A particular motivation is the relation to gauge theories via Hanany-Witten constructions and the AdS/CFT correspondence, as described in section 6.

We have also seen in sections 3, 5 and 6 the close relation between intersecting branes, wrapped branes and solutions which don’t involve any branes. Viewing intersecting branes as singular limits of a smoothly wrapped brane proved particularly useful in describing four-dimensional $\mathcal{N} = 2$ gauge theories using Hanany-Witten configurations. This relation was probably also an important factor enabling a fully localised supergravity solution, also described in section 6, to be found in the near-horizon limit. As we saw in section 5 a less obvious connection between intersecting branes and smooth geometry arises via the connection between various branes and Kaluza-Klein monopoles. One application of these relations is that some properties of particular special holonomy manifolds can be described in terms of intersecting branes. This particularly illustrates the fact that (intersecting) branes is not an isolated subject. So we can expect that finding new supergravity solutions for intersecting branes will contribute to our understanding of supersymmetric solutions in general.

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