Criteria for Optimal Global Integrability of Hajlasz-Sobolev Functions

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Abstract The author establishes some geometric criteria for a domain of $\mathbb{R}^n$ with $n \geq 2$ to support a $(pn/(n - ps), p)_s$-Hajlasz-Sobolev-Poincaré imbedding with $s \in (0, 1]$ and $p \in (n/(n + s), n/s)$ or an $s$-Hajlasz-Trudinger imbedding with $s \in (0, 1]$.

1 Introduction

The study of the Hajlasz spaces $\dot{M}^{1,p}$ was initiated by Hajlasz [15] on arbitrary metric measure spaces, see [15, 17, 16, 35, 21, 22, 23] for further discussions, generalizations and connections with the classical (Hardy-)Sobolev, Besov and Triebel-Lizorkin spaces. In particular, a fractional version $\dot{M}^{s,p}$ with $s \in (0, 1)$ was introduced by Yang [35], and a Sobolev-type version $\dot{M}_{\text{ball}}^{1,p}$ on domains by Koskela and Saksman [21].

We first recall some definitions and notions. In this paper, we always let $n \geq 2$ and $\Omega$ be a domain of $\mathbb{R}^n$. For every $s \in (0, 1]$ and measurable function $u$, denote by $D_s(u)$ the collection of all nonnegative measurable functions $g$ such that

\begin{equation}
|u(x) - u(y)| \leq |x - y|^s[|g(x)| + |g(y)|]
\end{equation}

for all $x, y \in \Omega \setminus E$, where $E \subset \Omega$ with $|E| = 0$. We also denote by $D_{\text{ball}}^s(u)$ the collection of all nonnegative measurable functions $g$ such that (1.1) holds for all $x, y \in \Omega \setminus E$ satisfying $|x - y| < \frac{1}{2} \text{dist}(x, \partial \Omega)$.

Definition 1.1. Let $s \in (0, 1]$ and $p \in (0, \infty)$. Then the homogeneous Hajlasz space $\dot{M}^{s,p}(\Omega)$ is the space of all measurable functions $u$ such that

$$\|u\|_{\dot{M}^{s,p}(\Omega)} \equiv \inf_{g \in D_s(u)} \|g\|_{L^p(\Omega)} < \infty,$$

and its Sobolev-type version $\dot{M}_{\text{ball}}^{s,p}(\Omega)$ is the space of all measurable functions $u$ such that

$$\|u\|_{\dot{M}_{\text{ball}}^{s,p}(\Omega)} \equiv \inf_{g \in D_{\text{ball}}^s(u)} \|g\|_{L^p(\Omega)} < \infty.$$
Obviously, for all \( s \in (0, 1] \) and \( p \in (0, \infty) \), \( \dot{H}^{s,p}(\Omega) \subset \dot{M}^{s,p}_{\text{ball}}(\Omega) \). If \( \Omega \) is a uniform domain, then \( \dot{M}^{s,p}_{\text{ball}}(\Omega) = \dot{M}^{s,p}(\Omega) \) for all \( s \in (0, 1] \) and \( p \in (n/(n+s), \infty) \); see [21, Theorem 19]. But, generally, we cannot expect that \( \dot{M}^{s,p}(\Omega) = \dot{M}^{s,p}_{\text{ball}}(\Omega) \). For example, this fails when \( \Omega = B(0,1) \setminus \{(x,0): x \geq 0\} \subset \mathbb{R}^2 \).

Hajłasz-Sobolev spaces are closely related to the classical (Hardy-)Sobolev and Triebel-Lizorkin spaces. In this paper, we always denote by \( \dot{W}^{1,p}(\Omega) \) with \( p \in (1, \infty) \) the homogeneous Sobolev space, by \( H^{1,p}(\Omega) \) with \( p \in (0, 1] \) the Hardy-Sobolev space as in [26, 27], and by \( F^s_{p,q}(\mathbb{R}^n) \) with \( s \in \mathbb{R} \) and \( p, q \in (0, \infty) \) the homogeneous Triebel-Lizorkin spaces as in [31]. It was proved in [15, 21] that \( \dot{W}^{1,p}(\Omega) = M^{1,p}_{\text{ball}}(\Omega) \) for \( p \in (1, \infty) \) and \( \dot{H}^{1,p}(\Omega) = \dot{M}^{1,p}_{\text{ball}}(\Omega) \) for \( p \in (n/(n+1), 1) \), which together with [31] implies that \( \dot{M}^{1,p}(\mathbb{R}^n) = F^{1,p,2}_{1,p,2}(\mathbb{R}^n) \) for all \( p \in (n/(n+1), \infty) \), while for all \( s \in (0, 1) \) and \( p \in (n/(n+s), \infty) \), \( \dot{M}^{s,p}(\mathbb{R}^n) = F^{s,p,\infty}_{s,p,\infty}(\mathbb{R}^n) \) as proved in [35, 22].

Now we recall some notions on imbeddings. Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \), \( s \in (0, 1] \) and \( p \in (n/(n+s), n/s) \). Then \( \Omega \) is said to support a \((pn/(n-ps), p)_s\)-Hajlasz-Sobolev-Poincaré (for short, \((pn/(n-ps), p)_s\)-HSP) imbedding if there exists a constant \( C > 0 \) such that for all \( u \in \dot{M}^{s,p}_{\text{ball}}(\Omega) \),
\[
\|u - u_\Omega\|_{L^{pn/(n-ps)}(\Omega)} \leq C\|u\|_{\dot{M}^{s,p}_{\text{ball}}(\Omega)},
\]
where \( u_\Omega \equiv \frac{1}{|\Omega|} \int_\Omega u(z) \, dz \). Similarly, \( \Omega \) is said to support an \( s \)-Hajlasz-Trudinger (for short, \( s \)-HT) imbedding if there exists a constant \( C > 0 \) such that for all \( u \in \dot{M}^{s,n/s}_{\text{ball}}(\Omega) \),
\[
\|u - u_\Omega\|_{\phi_s(L)(\Omega)} \leq C\|u\|_{\dot{M}^{s,n/s}_{\text{ball}}(\Omega)},
\]
where and in what follows, \( \phi_s(t) \equiv \exp(t^{n/(n-s)}) - 1 \) and
\[
\|u\|_{\phi_s(L)(\Omega)} \equiv \inf \left\{ t > 0, \int_\Omega \phi_s \left( \frac{|u(x)|}{t} \right) \, dx \leq 1 \right\}.
\]
It should be pointed out that since \( \dot{M}^{1,p}_{\text{ball}}(\Omega) = \dot{W}^{1,p}(\Omega) \) for all \( p \in (1, \infty) \), then (1.2) with \( s = 1 \) and \( p \in [1, n] \) coincides with the classical \((pn/(n-p), p)\)-Sobolev-Poincaré imbedding as in [5, (1.1)], and (1.3) with \( s = 1 \) coincides with the classical Trudinger imbedding as in [5, (1.2)].

Recently, some geometric criteria were established in [3, 4, 5] for a domain to support a \((pn/(n-p), p)\)-Sobolev-Poincaré imbedding for \( p \in [1, n] \) or a Trudinger imbedding. More precisely, Bojarski [3] first proved that a John domain as in Definition 2.1 always supports a \((pn/(n-p), p)\)-Sobolev-Poincaré imbedding for all \( p \in [1, n] \). Smith and Stegenga [29] proved that a weak carrot domain as in Definition 2.2 always supports the Trudinger imbedding. Conversely, let \( \Omega \) be a bounded planar domain or a bounded domain in \( \mathbb{R}^n \) with \( n \geq 3 \) satisfying an additional separation property when \( p \in (1, n) \) and a slice property when \( p = n \); see Definitions 2.3 and 2.4 below. Then Buckley and Koskela [4, 5] proved that if \( \Omega \) supports a \((pn/(n-p), p)\)-Sobolev-Poincaré imbedding for some/all \( p \in [1, n] \), then it is a John domain, and if \( \Omega \) supports the Trudinger imbedding, then it is a weak carrot domain.
The purpose of this paper is to establish some geometric criteria for a domain of \( \mathbb{R}^n \) with \( n \geq 2 \) to support a \((pn/(n - ps), p)_{s, HSP}\) imbedding with \( s \in (0, 1] \) and \( p \in (n/(n + s), n/s) \) or an \( s\)-HIT imbedding with \( s \in (0, 1] \).

To this end, we first establish the linear local connectivity (for short, LLC) of a domain that supports the \((pn/(n - ps), p)_{s, HSP}\) imbedding, where the notion of LLC was introduced by Gehring [8]. Recall that a domain \( \Omega \) is said to have the LLC property if there exists a positive constant \( b \) such that for all \( z \in \mathbb{R}^n \) and \( r > 0 \),

- LLC(1) points in \( \Omega \cap B(z, r) \) can be joined in \( \Omega \cap B(z, r/b) \);
- LLC(2) points in \( \Omega \setminus B(z, r) \) can be joined in \( \Omega \setminus B(z, br) \).

Then, as proved by Gehring and Martio [10], a \( W^{1,n} \)-extension domain has the LLC property, and by [20, Theorem 6.4], a \( W^{1,p} \)-extension domain with \( p \in (n - 1, n) \) has the LLC(2) property; see also [12, 13, 14, 34] and their references. Here and in what follows, \( \Omega \) is called an \( A \)-extension domain with \( A = M_{ball}^n, \tilde{W}^{1,p} \), or \( \tilde{H}^{1,p} \) if for every \( u \in A(\Omega) \), there exists a \( v \in A(\mathbb{R}^n) \) such that \( v|_{\Omega} = u \) and \( \|v\|_{A(\mathbb{R}^n)} \lesssim \|u\|_{A(\Omega)} \). Here, we extend the results in [10, 20] as follows.

**Theorem 1.1.** Let \( s \in (0, 1] \) and \( p \in (n/(n + s), n/s) \). If \( \Omega \) is a bounded \( M_{ball}^n \)-extension domain or \( \Omega \) is a bounded domain that supports a \((pn/(n - ps), p)_{s, HSP}\) imbedding, then \( \Omega \) has the LLC(2) property.

The proof of Theorem 1.1 is given in Section 3. We point out that the approach used here is different from that used by Koskela in [20, Theorem 6.4], where he used the \( p \)-capacity to prove the LLC(2) property of a \( \tilde{W}^{1,p} \)-extension domain for \( p \in (n - 1, n) \). In fact, when \( 1 < p \leq n - 1 \), as Koskela [20] pointed out, the \( p \)-capacity makes no sense since \( \text{Cap}_p(K_0, K_1, \mathbb{R}^n) = 0 \) for every pair of disjoint continua \( K_0, K_1 \subset \mathbb{R}^n \). So some new ideas are required to prove Theorem 1.1 as the result is new even in the case \( s = 1 \) and \( 1 < p \leq n - 1 \). To this end, we will simplify this question, and then combine some of the ideas from [4, 18, 19] and the properties of Hajłasz-Sobolev functions.

Then, as a corollary to Theorem 1.1, we have the following conclusion, which complements the results in [10, 20].

**Corollary 1.1.** If \( \Omega \) is a bounded \( W^{1,p} \)-extension domain when \( p \in (1, n) \) or bounded \( \tilde{H}^{1,p} \)-extension domain with \( p \in (n/(n + 1), 1] \), then \( \Omega \) has the LLC(2) property.

Applying Theorem 1.1, we further establish some geometric criteria for a domain to support a \((pn/(n - ps), p)_{s, HSP}\) imbedding, which generalizes the criteria in [3, 4].

**Theorem 1.2.** (i) A John domain of \( \mathbb{R}^n \) as in Definition 2.1 always supports a \((pn/(n - ps), p)_{s, HSP}\) imbedding as in (1.2) for all \( s \in (0, 1] \) and \( p \in (n/(n + s), n/s) \).

(ii) Assume that \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) and satisfies the separation property as in Definition 2.3. If \( \Omega \) supports a \((pn/(n - ps), p)_{s, HSP}\) imbedding for some \( s \in (0, 1] \) and \( p \in (n/(n + s), n/s) \), then \( \Omega \) is a John domain.

To prove Theorem 1.2(ii), we will use the LLC(2) property of these domains given in Theorem 1.1. This is slightly different from that of [4]. On the other hand, notice that \((\mathbb{R}^n, d_s, dx)\) is an Ahlfors \( n/s \)-regular metric measure spaces, when \( d_s(x, y) = |x - y|^s \) for
all $x, y \in \mathbb{R}^n$ and $dx$ denotes the Lebesgue measure. Observe that $M^{s,n/s}_{\text{ball}}(\Omega)$ coincides with $M^{1,n/s}_{\text{ball}}(\Omega, d_s, dx)$, the Hajlasz-Sobolev space on domains of $(\mathbb{R}^n, d_s, dx)$ defined similarly to Definition 1.1. Then Theorem 1.2(i) can be deduced from results by Chua and Wheeden [7]. For the reader’s convenience, we give a short proof, which will use the ideas from Bojarski [3], the chain property of a John domain as proved by Boman [2], and a key imbedding on balls established by Hajlasz [16, Theorem 8.7].

We also establish an analogue of Theorem 1.2 at the end point $p = n/s$ when $s \in (0, 1]$, which generalizes the criteria established in [29, 5, 6], and whose proof uses some ideas from [29, 30, 5, 6] and will be given in Section 4. Also see [24] for similar inequalities on balls.

**Theorem 1.3.** (i) A weak carrot domain of $\mathbb{R}^n$ as in Definition 2.2 always supports an $s$-HT imbedding for all $s \in (0, 1]$.

(ii) Assume that $\Omega$ is a bounded domain of $\mathbb{R}^n$ and satisfies the slice property as in Definition 2.4. If $\Omega$ supports an $s$-HT imbedding for some $s \in (0, 1]$, then $\Omega$ is a weak carrot domain.

Notice that, as proved in [4, 5], every simply connected domain in $\mathbb{R}^2$ or every domain in $\mathbb{R}^n$ with $n \geq 3$ that is quasiconformally equivalent to a uniform domain satisfies the slice property and the separation property. So, as a corollary to Theorems 1.2 and 1.3, we have the following conclusion.

**Corollary 1.2.** Let $\Omega$ be a bounded simply connected domain in $\mathbb{R}^2$ or a bounded domain in $\mathbb{R}^n$ with $n \geq 3$ that is quasiconformally equivalent to a uniform domain. Then

(i) $\Omega$ is a John domain if and only if it supports a $(pn/(n - ps), p)$-HSP imbedding for some/all $s \in (0, 1]$ and $p \in (n/(n + s), n/s)$;

(ii) $\Omega$ is a weak carrot domain if and only if it supports an $s$-HT imbedding for some/all $s \in (0, 1]$.

This paper is organized as follows. In Section 2, we recall some basic notions and properties of the domains and Hajlasz-Sobolev spaces. In Section 3, we present the proof of Theorem 1.1. In Section 4, we give the proofs of Theorems 1.2 and 1.3.

## 2 Preliminaries

In this section, we recall some notions and basic properties of domains and Hajlasz-Sobolev spaces. We begin with the notion of John domain.

**Definition 2.1.** Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ with $n \geq 2$. Then $\Omega$ is called a John domain with respect to $x_0 \in \Omega$ and $C > 0$ if for every $x \in \Omega$, there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ parametrized by arclength such that $\gamma(0) = x$, $\gamma(1) = x_0$ and $d(\gamma(t), \Omega^C) \geq Ct$.

Now we recall the notion of a weak carrot domain (or domains satisfying the quasihyperbolic boundary condition). To this end, for every pair of points $x, y \in \Omega$, define their...
quasihyperbolic distance $k_\Omega(x, y)$ by

$$k_\Omega(x, y) \equiv \inf_\gamma \int_\gamma \frac{1}{d(z, \Omega^c)}|dz|,$$

where the infimum is taken over all rectifiable curves $\gamma \subset \Omega$ joining $x$ and $y$. As proved in [9], $k_\Omega$ is a geodesic distance, namely, there exists a curve $\gamma_{x, y} \subset \Omega$ such that

$$k_\Omega(x, y) = \int_{\gamma_{x, y}} \frac{1}{d(z, \Omega^c)}|dz|.$$

**Definition 2.2.** A domain $\Omega$ is said to satisfy a weak carrot condition (or quasihyperbolic boundary condition) with respect to $x_0 \in \Omega$ and $C \geq 1$ if for all $x \in \Omega$,

(2.1) $$k_\Omega(x, x_0) \leq C \log \left( \frac{C}{d(x, \Omega^c)} \right).$$

It is easy to see that the John and weak carrot conditions are independent of the choice of $x_0$ in the sense that if $\Omega$ is a John or weak carrot domain with respect to $x_0$ and $C$, then for any other $x_1 \in \Omega$, there exists a positive constant $\bar{C}$ such that $\Omega$ is still a John or weak carrot domain with respect to $x_1$ and $\bar{C}$, respectively. See [6] for more details.

The following characterization of a weak carrot domain established by Smith and Stegenga [29] will be used in the proof of Theorem 1.3.

**Lemma 2.1.** Let $\Omega$ is a proper subdomain of $\mathbb{R}^n$ and let $x_0 \in \Omega$. Then $\Omega$ is a weak carrot domain if and only if there exists a positive constant $\sigma$ such that

$$\int_{\Omega} \exp(\sigma k_\Omega(x_0, x)) \, dx < \infty.$$

We also recall the notions of a separation property and slice property introduced in [4, 5].

**Definition 2.3.** A domain $\Omega$ has a separation property with respect to $x_0 \in \Omega$ and $C > 1$ if for every $x \in \Omega$, there exists a curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$, and such that for each $t \in (0, 1]$, either $\gamma([0, t]) \subset \overline{B(\gamma(t), Cd(\gamma(t), \Omega^c))}$ or each $y \in \gamma([0, t]) \setminus B$ belongs to a different component of $\Omega \setminus \partial B$ than $x_0$.

**Definition 2.4.** A domain $\Omega$ has a slice property with respect to $C > 1$ if for every pair of points $x, y \in \Omega$, there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$, and pairwise disjoint collection of open subsets $\{S_i\}_{i=0}^j$, $j \geq 0$, of $\Omega$ such that

(i) $x \in S_0$, $y \in S_j$, and $x$ and $y$ are in different components of $\Omega \setminus \overline{S_i}$ for $0 < i < j$;
(ii) if $F \subset \subset \Omega$ is a curve containing both $x$ and $y$, and $0 < i < j$, then $\text{diam} (S_i) \leq C \ell (F \cap S_i)$;
(iii) for $0 \leq t \leq 1$, $B(\gamma(t), C^{-1}d(\gamma(t), \Omega^c)) \subset \bigcup_{i=0}^j S_i$;
(iv) if $0 \leq i \leq j$, then $\text{diam} S_i \leq Cd(z, \Omega^c)$ for all $z \in \gamma_i \equiv \gamma \cap S_i$; also, there exists $x_i \in S_i$ such that $x_0 = x$, $x_j = y$ and $B(x_i, C^{-1}d(x_i, \Omega^c)) \subset S_i$. 
We point out that, as proved in [4, 5], every simply connected domain in \( \mathbb{R}^2 \) or every domain in \( \mathbb{R}^n \) with \( n \geq 3 \) that is quasiconformally equivalent to a uniform domain satisfies a slice property and a separation property. Every John domain satisfies both a separation and a slice property; see [6].

The following conclusion is essentially established in [21] and plays an important role in the proofs of Theorems 1.1, 1.2 and 1.3. For every \( \rho > 0 \), similarly to \( D^s_{\text{ball}}(u) \), we denote by \( D^s_{\text{ball}}(u) \) the collection of all measurable functions \( g \) such that (1.1) holds for all \( x, y \in \Omega \setminus E \) satisfying \( |x - y| < \rho \text{dist}(x, \partial \Omega) \). Notice that \( D^s_{\text{ball}}(u) = D^{s,1/2}_{\text{ball}}(u) \) and \( D^s(u) = D^{s,\infty}_{\text{ball}}(u) \).

**Lemma 2.2.** Let \( s \in (0, 1] \) and \( p \in (n/(n+s), \infty) \). Then \( u \in M^{s,p}_{\text{ball}}(\Omega) \) if and only if there exists a \( \rho \in (0, 1) \) such that \( \inf_{g \in D^s_{\text{ball}}(u)} \|g\|_{L^p(\Omega)} < \infty \). Moreover, for given \( \rho \), there exists a positive constant \( C \) such that for all \( u \in M^{s,p}_{\text{ball}}(\Omega) \),

\[
C^{-1} \|u\|_{\tilde{M}^{s,p}_{\text{ball}}(\Omega)} \leq \inf_{g \in D^{s}_{\text{ball}}(u)} \|g\|_{L^p(\Omega)} \leq C \|u\|_{\tilde{M}^{s,p}_{\text{ball}}(\Omega)}.
\]

We also need the following imbedding, which is essentially established by Hajłasz [16, Theorem 8.7] when \( n = 1 \) and pointed out by Yang [35] when \( s \in (0, 1) \).

**Lemma 2.3.** Let \( s \in (0, 1] \) and \( p \in (n/(n+s), n/s) \). Then for every \( \sigma > 1 \), there exists a positive constant \( C \) such that for all balls or cubes \( B \) and \( u \in \tilde{M}^{s,p}(\Omega) \),

\[
\|u - u_B\|_{L^{pn/(n-ps)}(B)} \leq C \|u\|_{\tilde{M}^{s,p}(\sigma B)}.
\]

By Lemma 2.3, we have the following conclusion.

**Lemma 2.4.** Let \( s \in (0, 1] \) and \( p \in (n/(n+s), n/s) \). Then a bounded \( \tilde{M}^{s,p}_{\text{ball}} \)-extension domain always supports a \( (pn/(n-ps), p)_s \)-HSP imbedding.

**Proof.** Assume that \( \Omega \) is an \( \tilde{M}^{s,p}_{\text{ball}} \)-extension domain. Let \( u \in \tilde{M}^{s,p}_{\text{ball}}(\Omega) \). Then there exists a \( v \in \tilde{M}^{s,p}_{\text{ball}}(\mathbb{R}^n) \) such that \( v|\Omega = u \) and \( \|v\|_{\tilde{M}^{s,p}_{\text{ball}}(\mathbb{R}^n)} \lesssim \|u\|_{\tilde{M}^{s,p}_{\text{ball}}(\Omega)} \). Let \( B \) be a ball of \( \mathbb{R}^n \) such that \( \Omega \subset B \). Then \( v \in \tilde{M}^{s,p}_{\text{ball}}(2B) \) and thus by Lemma 2.2, we have \( v \in L^{pn/(n-ps)}(B) \) and

\[
\|v - v_B\|_{L^{pn/(n-ps)}(B)} \lesssim \|v\|_{\tilde{M}^{s,p}_{\text{ball}}(\mathbb{R}^n)} \lesssim \|v\|_{\tilde{M}^{s,p}_{\text{ball}}(\Omega)}
\]

which further implies that

\[
\|u - u_B\|_{L^{pn/(n-ps)}(B)} \lesssim \|u - v_B\|_{L^{pn/(n-ps)}(B)} \lesssim \|v - v_B\|_{L^{pn/(n-ps)}(B)} \lesssim \|v\|_{\tilde{M}^{s,p}_{\text{ball}}(\Omega)}.
\]

This means that \( \Omega \) supports a \( (pn/(n-ps), p)_s \)-HSP imbedding and thus finishes the proof of Lemma 2.4.

Finally, we state some conventions. Throughout the paper, we denote by \( C \) a positive constant which is independent of the main parameters, but which may vary from line to line. Constants with subscripts, such as \( C_0 \), do not change in different occurrences. The symbol \( A \preceq B \) or \( B \succeq A \) means that \( A \leq CB \). If \( A \preceq B \) and \( B \preceq A \), we then write \( A \sim B \). For any locally integrable function \( f \), we denote by \( \frac{1}{|E|} \int_E f \) the average of \( f \) on \( E \), namely, \( \frac{1}{|E|} \int_E f \).
3 Proof of Theorem 1.1

Proof of Theorem 1.1. By Lemma 2.4, it suffices to prove that a domain which supports a $(pn/(n - ps), p)_s$-HSP imbedding has the LLC(2) property. Assume that $\Omega$ is a bounded domain that supports a $(pn/(n - ps), p)_s$-HSP imbedding. We want to show that $\Omega$ has the LLC(2) property. To this end, let $L \equiv \text{diam} \Omega$ and $x_0 \in \Omega$ be such that $r_0 \equiv d(x_0, \Omega^b) = \max\{d(x, \Omega^b) : x \in \Omega\}$. Notice that if $u(y) = 0$ for all $y \in B(x_0, r_0)$, then the $(pn/(n - ps), p)_s$-HSP imbedding implies that

$$
\|u\|_{L^{pn/(n - ps)}(\Omega)} \lesssim \|u\|_{L^{\infty}_{(\bar{A})}(\Omega)},
$$

where the constant depends on $r_0$ and $|\Omega|$ but not on $u$.

We claim that if $x, x_0 \in \Omega \setminus B(z, r)$ for $z \in B(x_0, 2L)$ and $r \in (0, 2L)$, then $x, x_0$ are contained in the same component of $\Omega \setminus B(z, br)$ for some fixed constant $b \in (0, 1)$, which may depend on $\Omega$ and $x_0$ but not on $z$ and $x$.

Assume that the above claim holds for the moment. Then we deduce Theorem 1.1 from it by the following 2 steps. Let $x, y \in \Omega \setminus B(z, r)$ for $z \in \mathbb{R}^n$ and $r \in (0, \infty)$.

Step 1. There exists a positive constant $\tilde{b}$ independent of $x$ such that if $x, x_0 \in \Omega \setminus B(z, r)$, then $x, x_0$ are contained in the same component of $\Omega \setminus B(z, r)$. To see this, assume that $x_0 \in \Omega \setminus B(z, r)$. If $z \not\in B(x_0, 2L)$, then $\Omega \setminus B(z, r) \neq \emptyset$ implies that $r \geq d(z, x_0) - L \geq r/2 \geq L$, and moreover $\Omega \setminus B(z, r) \neq \emptyset$ implies that $\Omega \setminus B(z, r/2) = \emptyset$. Thus if $x_0 \in \Omega \setminus B(z, r)$ with $d(z, x_0) \geq 2L$, then $x_0, x$ are contained in the same component of $\Omega \setminus B(z, r)$ if $\Omega \setminus B(z, r) = \emptyset$ or of $\Omega \setminus B(z, r/2)$ if $\Omega \setminus B(z, r) = \emptyset$. If $z \in B(x_0, 2L)$, then by the above claim, it suffices to consider the case $r \geq 2L$. Since $r \geq 2L$ implies $d(z, \Omega) \geq r - L \geq r/2$, which means that $\Omega \setminus B(z, r/2) = \emptyset$, then $x_0, x$ are contained in the same component of $\Omega \setminus B(z, r/2)$.

Step 2. There exists a positive constant $b$ independent of $x, y$ such that $x, y$ are contained in the same component of $\Omega \setminus B(z, br)$. To see this, if $x_0 \in \Omega \setminus B(z, \frac{r_0}{10L}r)$, then $x_0, x$ and $x_0, y$, and thus $x, y$, are contained in the same component of $\Omega \setminus B(z, \tilde{b}\frac{r_0}{10L}r)$. If $x_0 \in B(z, \frac{r_0}{10L}r)$, then $r - \frac{r_0}{10L}r \leq L$, which implies that $r \leq 2L$ and thus $|z - x_0| \leq \frac{r}{10L}r < \frac{r_0}{5L}r \leq r_0/5$. Obviously,

$$
B(z, \frac{r_0}{10L}r) \subset B(x_0, \frac{r_0}{5L}r) \subset B(x_0, r_0) \subset \Omega,
$$

which means that $\Omega \setminus B(z, \frac{r_0}{10L}r)$ is connected, and thus $x, y$ are contained in the same component of $\Omega \setminus B(z, \frac{r_0}{10L}r)$.

Therefore, with the aid of the above claim, combining Step 1 and Step 2, we obtain Theorem 1.1. So we have reduced Theorem 1.1 to the above claim. The remainder of the proof of Theorem 1.1 consists of the proof of the above claim.

In the following argument, we let $x \in \Omega, z \in B(x_0, 2L)$ and $r \in (0, 2L)$ be fixed such that $x, x_0 \in \Omega \setminus B(z, r)$ as in the claim. Let $b_2 \in (0, 1]$ be the supremum of $b \in (0, 1)$ such that $x, x_0$ are contained in the same component of $\Omega \setminus B(z, br)$. Without loss of generality, we assume that $b_2 \leq 1/10$. Denote by $\Omega_x$ the component of $\Omega \setminus B(z, b_0r)$ with $b_0 = 2b_2$ containing $x$. Take $b_1 \in (b_0, 1]$ such that

$$
|\Omega_x \cap (B(z, r) \setminus B(z, b_1r))| = \frac{1}{2}|\Omega_x \cap (B(z, r) \setminus B(z, b_0r))| = \frac{1}{2}|\Omega_x \cap B(z, r)|.
$$
Define a function $u$ on $\Omega$ by setting

$$
 u(y) \equiv \begin{cases} 
 0, & y \in \Omega \setminus \Omega_x; \\
 \frac{d(y, B(z, b_0r))}{b_1r - b_0r}, & y \in \Omega_x \cap B(z, b_1r); \\
 1, & y \in \Omega_x \setminus B(z, b_1r).
\end{cases}
$$

Then we have the following conclusion, whose proof will be given below.

**Lemma 3.1.** Let $u$ be as in (3.2) and $s \in (0, 1]$. Then $g \equiv C(b_1r - b_0r)^{-s}\chi_{\Omega_x \cap B(z, r)}$ is an element of $D_{ball}^{s,1/8}(u)$, where $C$ is a positive constant independent of $u, x, b_0, b_1, r$.

By Lemma 2.2, Lemma 3.1 and (3.1), we further have $u \in \dot{M}_{ball}^{s, p}(\Omega)$ and

$$
\|u\|_{L^{p/(n-ps)(\Omega)}} \lesssim \|u\|_{\dot{M}_{ball}^{s, p}(\Omega)} \lesssim \|g\|_{L^p(\Omega)} \lesssim (b_1r - b_0r)^{-s}|\Omega_x \cap (B(z, r) \setminus B(z, b_0r))|^{1/p},
$$

which together with

$$
\|u\|_{L^{p/(n-ps)(\Omega)}} \gtrsim |\Omega_x \cap (B(z, r) \setminus B(z, b_1r))|^{(n-ps)/pn}
\gtrsim |\Omega_x \cap (B(z, r) \setminus B(z, b_0r))|^{(n-ps)/pn}
$$

implies that

$$
(3.3) \quad b_1r - b_0r \lesssim |\Omega_x \cap (B(z, r) \setminus B(z, b_0r))|^{1/n}.
$$

Hence, if $b_1 \geq 1/2$, then (3.3) implies that

$$
(3.4) \quad (1/2 - b_0)r \lesssim |\Omega_x \cap (B(z, r) \setminus B(z, b_0r))|^{1/n}.
$$

If $b_1 < 1/2$, then following the above procedure, we can find a sequence $\{b_j\}_{j=1}^{j_0}$ such that $b_{j_0} \geq 1/2$ and for all $0 \leq j \leq j_0 - 1, b_j < 1/2$,

$$
|\Omega_x \cap (B(z, r) \setminus B(z, b_{j+1}r))| = \frac{1}{2}|\Omega_x \cap (B(z, r) \setminus B(z, b_jr))|,
$$

and

$$
|\Omega_x \cap (B(z, r) \setminus B(z, b_jr))|^{1/n}.
$$

This implies that

$$
\sum_{j=0}^{j_0-1} (b_{j+1}r - b_jr) \lesssim |\Omega_x \cap (B(z, r) \setminus B(z, b_0r))|^{1/n},
$$

and hence (3.4). To control $|\Omega_x \cap (B(z, r) \setminus B(z, b_0r))|^{1/n}$ via $b_0r$, define function

$$
(3.5) \quad v(y) \equiv \inf_{\gamma(x_0, y)} \ell(\gamma \cap B(z, b_0r))
$$

for all $y \in \Omega$, where the infimum is taken over all the rectifiable curves $\gamma$ joining $x_0$ and $y$ in $\Omega$. Observe that for all $y$ in the component of $\Omega \setminus \overline{B(z, b_0r)}$ containing $x_0$, $v(y) = 0$; for all $y$ in the component $\Omega_x \setminus \overline{B(z, b_0r)}$ which contains $x$ and does not contain $x_0$, $v(y)$ is a constant larger than or equal to $b_0r$. Moreover, we have the following conclusion, whose proof will be given below.
Lemma 3.2. Let \( v \) be as in (3.5) and \( s \in (0, 1] \). Then \( h \equiv C(b_0 r)^{1-s} \chi_{\Omega \cap B(z, b_0 r)} \) is an element of \( D_{\text{ball}}^{s, 1/8}(v) \), where \( C \) is a positive constant independent of \( v, z, b_0, r \).

By Lemma 2.2, Lemma 3.2 and (3.1), we have that \( v \in M_{\text{ball}}^{s, p}(\Omega) \) and

\[
(b_0 r)|\Omega_x \cap (B(z, r) \setminus B(z, b_0 r))|^{(n-p\alpha)/p\alpha} \lesssim |\Omega \cap B(z, b_0 r)|^{1/p}(b_0 r)^{1-s}
\]

which implies that

\[
|\Omega_x \cap (B(z, r) \setminus B(z, b_0 r))| \lesssim (b_0 r)^n.
\]

By this and (3.4), we have \((1/2 - b_0)r \lesssim b_0 r\), which implies that \( b_0 \geq C \) for some fixed constant \( C \in (0, 1) \) independent of \( x \). This gives the above claim by taking \( b = C/4 \) and thus finishes the proof of Theorem 1.1.

Proof of Lemma 3.1. It suffices to check that for every pair of \( y, w \in \Omega \) such that \(|y - w| < \text{dist}(y, \Omega^c)/8\),

\[
|u(y) - u(w)| \lesssim \frac{|y - w|^s}{(b_1 r - b_0 r)^s} \left[ \chi_{\Omega_x \cap B(z, r)}(y) + \chi_{\Omega_x \cap B(z, r)}(w) \right]. \tag{3.6}
\]

To prove (3.6), without loss of generality, we may assume that \( u(w) < u(y) \). Then \( u(y) > 0 \) implies that \( y \in \Omega_x \) and \( u(w) < 1 \) implies that \( w \notin \Omega_x \setminus B(z, b_1 r) \). We will consider the following three cases for \( w \): i) \( w \in \Omega_x \cap B(z, b_1 r) \); ii) \( w \in \Omega \cap B(z, b_0 r) \); iii) \( w \in \Omega \setminus (\Omega_x \cup B(z, b_0 r)) \).

Case i). If \( y \in \Omega_x \setminus B(z, b_1 r) \), then by \( w \in \Omega_x \cap B(z, b_1 r) \), we have

\[
d(w, B(z, b_0 r)) = |w - z| - b_0 r \geq |z - y| - |w - y| - b_0 r \geq (b_1 r - b_0 r) - |w, y|,
\]

and thus

\[
|u(y) - u(w)| = \left| 1 - \frac{d(w, B(z, b_0 r))}{b_1 r - b_0 r} \right| \leq \left| 1 - \frac{d(w, B(z, b_0 r))}{b_1 r - b_0 r} \right|^s \lesssim \frac{|y - w|^s}{(b_1 r - b_0 r)^s},
\]

which gives (3.6). If \( y \in \Omega_x \cap B(z, b_1 r) \), then by \(|w - y| \leq b_1 r - b_0 r|\),

\[
|u(y) - u(w)| = \left| \frac{d(y, B(z, b_0 r)) - d(w, B(z, b_0 r))}{b_1 r - b_0 r} \right| \lesssim \frac{|y - w|}{b_1 r - b_0 r} \lesssim \frac{|y - w|^s}{(b_1 r - b_0 r)^s},
\]

which gives (3.6).

Case ii). If \( y \in \Omega_x \cap B(z, r) \), then \( d(y, B(z, b_0 r)) \leq |y - w| \leq b_1 r - b_0 r \) and thus

\[
|u(y) - u(w)| = \left| \min \left\{ 1, \frac{d(y, B(z, b_0 r))}{b_1 r - b_0 r} \right\} \right| \lesssim \left| \min \left\{ 1, \frac{|y - w|}{b_1 r - b_0 r} \right\} \right|^s \lesssim \frac{|y - w|^s}{(b_1 r - b_0 r)^s},
\]

which gives (3.6). If \( y \in \Omega_x \setminus B(z, r) \), then \(|w - y| \geq (1 - b_0)r \geq b_0 r \). Since \( d(w, \Omega^c) \leq |w - z| + d(z, \Omega^c) \leq 2|w| \), we have that \(|w - y| \geq 2d(w, \Omega^c) \). Moreover, since \( d(y, \Omega^c) \leq |y - w| + d(w, \Omega^c) \leq 2|y - w| \), by the definition of \( D_{\text{ball}}^{s, 1/8}(w) \), we do not need to check (3.6) for \( y \in \Omega_x \setminus B(z, r) \).
Case iii). We will prove that in this case,

\begin{equation}
|y - w| \geq \frac{1}{8} \max\{d(y, \Omega^c), d(w, \Omega^c)\}.
\end{equation}

Thus, we do not need to check (3.6) by the definition of \(D^s_{\text{ball}}(u)\). To prove (3.7), notice that \(y \in \Omega_x\) and \(w \notin \Omega_x \cup B(z, b_0r)\) implies that \(y\) and \(w\) are in different components of \(\Omega \setminus B(z, b_0r)\). If \(|y - w| < d(y, \Omega^c)/4\), then \(B(y, 2|w - y|) \subset B(y, d(y, \Omega^c)) \subset \Omega\). Observe that either \(B(y, 2|w - y|) \setminus B(z, b_0r) = \emptyset\) or \(B(y, 2|w - y|) \setminus B(z, b_0r)\) is connected. So \(y, w \in B(y, 2|w - y|) \setminus B(z, b_0r) \subset \Omega\) means that \(B(y, 2|w - y|) \setminus B(z, b_0r)\) are connected and thus \(y, w\) are in the same component of \(\Omega \setminus B(z, b_0r)\), which is a contradiction. If \(|y - w| < d(y, \Omega^c)/8\), then for all \(z \not\in \Omega\),

\[d(z, y) \geq d(z, w) - d(w, y) \geq d(w, \Omega^c)/2,\]

which implies that \(d(y, \Omega^c) \geq 7d(w, \Omega^c)/8\) and thus \(|y - w| < d(y, \Omega^c)/4\). Thus, (3.7) holds. This finishes the proof of Lemma 3.1.

Proof of Lemma 3.2. This is quite similar to the proof of Lemma 3.1. We sketch the proof. It suffices to check that for every pair of \(y, w \in \Omega\) such that \(|y - w| < \text{dist}(y, \Omega^c)/8\),

\begin{equation}
|v(y) - v(w)| \lesssim (b_0r)^{1-s}[\chi_{\Omega_x \cap B(z, b_0r)}(y) + \chi_{\Omega_x \setminus B(z, b_0r)}(w)].
\end{equation}

If both \(y\) and \(w\) are in the same component of \(\Omega \setminus B(z, b_0r)\), then (3.8) holds. If \(y\) and \(w\) are in different components of \(\Omega \setminus B(z, b_0r)\), by an argument similar to that of (3.7), we can prove that (3.7) still holds, and thus we do not need to check (3.8) for such \(y, w\). So we can assume that one of \(y\) and \(w\) is in \(\Omega \cap B(z, b_0r)\). Notice that in this case \(I(w, y) \subset \Omega\), where \(I(w, y)\) denotes the line segment joining \(w\) and \(y\). Then

\[|u(y) - u(w)| \leq \ell(I(w, y) \cap B(z, b_0r)) \leq \min\{2b_0r, |y - w|\} \lesssim (b_0r)^{1-s}|y - w|^s,
\]

which implies (3.8). This finishes the proof of Lemma 3.2.

4 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. (i) Assume that \(\Omega\) is a bounded John domain. Then, as proved by Boman [2], \(\Omega\) enjoys the following chain property: there exist a positive constant \(C\) and a sequence of subcubes of \(\Omega\), which is denoted by \(\mathcal{F}\), such that

(a) \(\chi_\Omega(x) \leq \sum_j \chi_{Q_j}(x) \leq \sum_j C\chi_\Omega(x)\) for all \(x \in \mathbb{R}^n\);

(b) for a fixed subcube \(Q_0 \in \mathcal{F}\) and any other \(Q \in \mathcal{F}\), there exists a subsequence \(\{Q_j\}_{j=1}^N \subset \mathcal{F}\) satisfying that \(Q = Q_N \subset CQ_j\), \(C^{-1}|Q_{j+1}| \leq |Q_j| \leq C|Q_j+1|\) and \(|Q_j \cap Q_{j+1}| \geq C^{-1}\min\{|Q_j|, |Q_{j+1}|\}\) for all \(j = 0, \cdots, N - 1\).

Let \(u \in M^s_{\text{ball}}(\Omega)\) and \(g \in \mathcal{D}^s_{\text{ball}}(u)\) with \(\|g\|_{L^p(\Omega)} \lesssim \|u\|_{M^s_{\text{ball}}(\Omega)}\). Then

\[
\int \Omega |u(z) - u_{Q_0}|^{pn/(n-ps)}dz \lesssim \sum_{Q \in \mathcal{F}} \int_Q |u(z) - u_Q|^{pn/(n-ps)}dz + \sum_{Q \in \mathcal{F}} |Q||u_Q - u_{Q_0}|^{pn/(n-ps)}
\]
\[
\equiv I_1 + I_2.
\]

Then by Lemma 2.3, \(n/(n - ps) > 1\) and the above chain property, we have

\[
I_1 \lesssim \sum_{Q \in \mathcal{F}} \left( \int_{2Q} [g(z)]^p \, dz \right)^{n/(n - ps)} \lesssim \left( \int_\Omega [g(z)]^p \, dz \right)^{n/(n - ps)} \lesssim \|u\|_{\mathcal{M}^{p/(n - ps)}_{M^+}}. \]

To estimate \(I_2\), for every \(Q \in \mathcal{F}\), let \(\{Q_j\}_{j=1}^N\) be as in (b). Then we have

\[
|u_Q - u_{Q_0}| \lesssim \sum_{j=0}^{N-1} (|u_{Q_j} - u_{Q_j \cap Q_{j+1}}| + |u_{Q_j \cap Q_{j+1}} - u_{Q_{j+1}}|) \lesssim \sum_{j=0}^{N} \int_{Q_j} |u(z) - u_{Q_j}| \, dz
\]

\[
\lesssim \sum_{j=0}^{N} |Q_j|^{s/n} \left( \int_{2Q_j} [g(z)]^p \, dz \right)^{1/p}
\]

\[
\lesssim \sum_{\tilde{Q} \in \mathcal{F}, Q \subset 2\tilde{Q}} |Q|^{s/n} \left( \int_{2\tilde{Q}} [g(z)]^p \, dz \right)^{1/p}
\]

Thus, by \(Q \subset \tilde{C}\tilde{Q}\) for all \(\tilde{Q} \in \mathcal{F}\), we obtain

\[
I_2 \lesssim \sum_{Q \in \mathcal{F}} |Q| \left\{ \sum_{\tilde{Q} \in \mathcal{F}, Q \subset 2\tilde{Q}} \tilde{Q}^{s/n} \left( \int_{2\tilde{Q}} [g(z)]^p \, dz \right)^{1/p} \right\}^{pn/(n - ps)}
\]

\[
\lesssim \sum_{Q \in \mathcal{F}} \int_Q \left\{ \sum_{\tilde{Q} \in \mathcal{F}, Q \subset 2\tilde{Q}} \tilde{Q}^{s/n} \left( \int_{2\tilde{Q}} [g(z)]^p \, dz \right)^{1/p} \chi_{\tilde{Q}}(x) \right\}^{pn/(n - ps)} \, dx
\]

\[
\lesssim \int_\Omega \left\{ \sum_{\tilde{Q} \in \mathcal{F}} \left[ \mathcal{M} \left( \left[ \tilde{Q}^{s/n} \left( \int_{2\tilde{Q}} [g(z)]^p \, dz \right)^{1/p} \chi_{\tilde{Q}} \right]^{1/2} \right) \right]^2 \right\}^{pn/(n - ps)} \, dx
\]

where \(\mathcal{M}\) denotes the Hardy-Littlewood maximal operator. Then by the vector-valued inequality of \(\mathcal{M}\) (see, for example, [33]), we have

\[
I_2 \lesssim \int_\Omega \left\{ \sum_{\tilde{Q} \in \mathcal{F}} \left[ \tilde{Q}^{s/n} \left( \int_{2\tilde{Q}} [g(z)]^p \, dz \right)^{1/p} \chi_{\tilde{Q}}(x) \right] \right\}^{pn/(n - ps)} \, dx
\]
\[
\lesssim \left( \sum_{Q \in X} \int_{2Q} |g(z)|^p \, dz \right)^{n/(n-ps)} \lesssim \left( \int_{\Omega} |g(z)|^p \, dz \right)^{n/(n-ps)} \lesssim \|u\|_{M^{s,p,2}(\Omega)}^{pn/(n-ps)}.
\]

This estimate finishes the proof of Theorem 1.2(i).

(ii) Assume that \(\Omega\) is bounded domain and has a separation property with respect to \(x_0 \in \Omega\) and \(C_0 \geq 1\). For any fixed point \(x \in \Omega\), let \(\gamma\) be a curve as in Definition 2.3. We claim that \(d(\gamma(t), \Omega^\beta) \geq \text{diam}\gamma([0, t])\) for all \(t \in [0, 1]\). Assume this claim holds for the moment. Then, as pointed out in [4], even though the claim is not enough to ensure that \(\gamma\) is a John curve for \(x\), it is known that the claim is enough to guarantee that \(\gamma\) can be modified to yield a John curve for \(x\) by the arguments in [25, pp. 385-386] and [28, pp. 7-8].

To prove the above claim, let \(N = 2 + C_0/b\), where \(b\) is the constant for which LLC(2) holds. For \(t \in (0, 1]\), if \(d(\gamma(t), \Omega^\beta) \geq d(x_0, \Omega^\beta)/N\), then

\[
\gamma([0, t]) \subset \Omega \subset B\left(\gamma(t), \frac{N \text{diam} \Omega}{d(x_0, \Omega^\beta)} d(\gamma(t), \Omega^\beta)\right),
\]

which implies the above claim. Assume that \(d(\gamma(t), \Omega^\beta) < d(x_0, \Omega^\beta)/N\). Now it suffices to prove that \(\gamma([0, t]) \subset B(\gamma(t), (N-1)d(\gamma(t), \Omega^\beta))\). Let \(y \in \gamma([0, t]) \setminus B(\gamma(t), (N-1)d(\gamma(t), \Omega^\beta))\), since \(d(x_0, \gamma(t)) \geq (N-1)d(\gamma(t), \Omega^\beta)\), then by Theorem 1.1(iii), we know that \(x_0\) and \(y\) are contained in the same component of \(\Omega \setminus B(\gamma(t), b(N-1)d(\gamma(t), \Omega^\beta))\). By \(b(N-1) > C_0\), we further know that \(x_0\) and \(y\) are in the same component of \(\Omega \setminus \partial B(\gamma(t), C_0d(\gamma(t), \Omega^\beta))\), which is a contradiction with the separation property. This verifies the above claim and thus finishes the proof of Theorem 1.2(ii). \(\square\)

To prove Theorem 1.3, we first establish the following result, which is an improvement on [16, Theorem 8.7(ii)].

**Lemma 4.1.** Let \(s \in (0, 1]\). Then there exist positive constants \(0 < C_1 < 1 < C_2\) such that for all balls \(B \subset \mathbb{R}^n\) and \(u \in M^{s,n/s}(4B)\),

\[
(4.1) \quad \int_B \exp \left( C_1 \frac{|u(x) - u_B|}{\|u\|_{M^{s,n/s}(4B)}} \right)^{n/(n-s)} \, dx \leq C_2.
\]

**Proof.** Assume that \(B \equiv B(x_0, 2^{-k_0})\) for some \(x_0 \in \mathbb{R}^n\) and \(k_0 \in \mathbb{Z}\). Let \(u \in M^{s,p}(4B)\) and \(g \in D^s(u)\) such that \(\|g\|_{L^{n/(s+1)}} \leq 2\|u\|_{M^{s,p}(4B)}\). We extend \(g\) to the whole \(\mathbb{R}^n\) by setting \(g(z) = 0\) for all \(z \in \mathbb{R}^n \setminus 4B\). For every Lebesgue point \(x\) of \(u\), we have

\[
|u(x) - u_B(x, 2^{-k_0-1})| \leq \sum_{j \geq k_0 + 1} |u_B(x, 2^{-j-1}) - u_B(x, 2^{-j})| + |u_B(x, 2^{-k_0+1}) - u_B|
\]

\[
\lesssim \sum_{j \geq k_0 + 1} \int_{B(x, 2^{-j})} |u(z) - u_B(x, 2^{-j})| \, dz
\]

\[
\lesssim \sum_{j \geq k_0 + 1} \int_{B(x, 2^{-j}) \setminus B(x, 2^{-j+1})} |u(z) - u_B(x, 2^{-j+2})| \, dz
\]
\[
\begin{align*}
\lesssim & \sum_{j \geq k_0 + 1} \int_{B(x, 2^{-j})} \int_{B(x, 2^{-j+2}) \setminus B(x, 2^{-j+1})} |u(z) - u(y)| \, dy \, dz \\
\lesssim & \sum_{j \geq k_0 + 1} \int_{B(x, 2^{-j})} \int_{B(x, 2^{-j+2}) \setminus B(x, 2^{-j+1})} |g(z) + g(y)| \, dy \, dz \\
\lesssim & \sum_{j \geq k_0 + 1} \int_{B(x, 2^{-j+2}) \setminus B(x, 2^{-j+1})} |\mathcal{M}(g)(y)| \, dy \\
\lesssim & \int_{B(x_0, 2^{-k_0+2})} |y - x|^{-n/s} \, dy.
\end{align*}
\]

Similarly,
\[
|u_B - u_{B(x, 2^{-k_0-1})}| \lesssim \int_B |u(z) - u_B| \, dz \lesssim \int_{B(x_0, 2^{-k_0+2})} |\mathcal{M}(g)(y)| \, dy.
\]

Thus,
\[
|u(x) - u_B| \lesssim \int_{B(x_0, 2^{-k_0+2})} |\mathcal{M}(g)(y)| \, dy.
\]

Then by [11, Lemma 7.2], for all \(q \geq n/s\),
\[
\|u - u_B\|_{L^q(B)} \leq q^{1-s/n+1/q} |B(0, 1)|^{1-s/n} |B|^{1/q} \|\mathcal{M}(g)\|_{L^{n/q}(4B)},
\]

which together with the \(L^{n/q}(\mathbb{R}^n)\)-boundedness of \(\mathcal{M}\) implies that
\[
\int_B |u(z) - u_B|^q \, dz \lesssim q^{1+(n-s)/(nq)} |B(0, 1)|^{nq/(n-s)} \|g\|_{L^{n/q}(4B)}^q,
\]

and hence for all \(q \geq n/s - 1\),
\[
\int_B |u(z) - u_B|^{nq/(n-s)} \, dz \lesssim \frac{nq}{n-s} \left( |B(0, 1)| \frac{nq}{n-s} \|g\|_{L^{n/q}(4B)} \right)^q.
\]

Then taking \(\sigma > [e|B(0, 1)|n/(n-s)](n-s)/n\), we have
\[
\int_B \sum_{j \geq [n/s]} \frac{1}{j!} \left( \frac{|u(x) - u_B|}{\sigma \|g\|_{L^{n/s}(4B)}} \right)^{jn/(n-s)} \, dx \lesssim \sum_{j \geq 1} \left( \frac{n|B(0, 1)|}{(n-s)\sigma^{n/(n-s)}} \right)^j \frac{j^j}{(j-1)!} \lesssim 1.
\]

Notice that by Hölder inequality, we have
\[
\int_B \sum_{j=0}^{[n/s]} \frac{1}{j!} \left( \frac{|u(x) - u_B|}{\sigma \|g\|_{L^{n/s}(4B)}} \right)^{jn/(n-s)} \, dx \lesssim \sum_{j=0}^{[n/s]} \left( \int_B \frac{|u(x) - u_B|^{n/s}}{\|g\|_{L^{n/s}(4B)}^{n/s}} \, dx \right)^{(n-s)/(j-s)} \lesssim 1.
\]

This gives (4.1) and thus finishes the proof of Lemma 4.1. \(\square\)
Proof of Theorem 1.3. (i) Assume that $\Omega$ is a weak carrot domain. Since $\Omega$ is bounded (see [29, Corollary 1]), we may assume that $|\Omega| = 1$. Let
\[
\tilde{\phi}_s(t) = \exp \left( \ln^{(n-s)}/j \right) - \sum_{j=0}^{j_0} \frac{1}{j!} \ln^{(n-s)}
\]
for $t \in (0, \infty)$, where $j_0$ denotes the maximal integer no more than $n/s - 1$. Since $\tilde{\phi}_s(t) \sim \phi_s(t)$ for $t \geq 1$, so we only need to prove (1.3) for $\tilde{\phi}_s$; see [1]. It further suffices to prove that there exists a $\sigma \in (0, 1)$ such that for all $u \in \dot{M}^{s,n/s}_\text{ball}(\Omega)$ with $\|u\|_{\dot{M}^{s,n/s}_\text{ball}(\Omega)} = 1$, (4.2)
\[
\int_{\Omega} \tilde{\phi}_s(\sigma|u(x) - u_\Omega|) \, dx \leq 1.
\]
Let $u \in \dot{M}^{s,n/s}_\text{ball}(\Omega)$ with $\|u\|_{\dot{M}^{s,n/s}_\text{ball}(\Omega)} = 1$ and write
\[
\int_{\Omega} \tilde{\phi}_s(\sigma|u(x) - u_\Omega|) \, dx = \sum_{j=j_0+1}^{1} \frac{1}{j!} \ln^{(n-s)} \int_{\Omega} |u(x) - u_\Omega|^{j/n/(n-s)} \, dx.
\]
Since $\Omega \subset \bigcup_{z \in \Omega} B(z, d(z, \Omega^C)/10)$, then by the standard 1/5-covering theorem, there exist points $\{z_i\} \subset \Omega$ such that $\{B(z_i, d(z_i, \Omega^C)/50)\}$ are pairwise disjoint,
\[
1 \leq \sum \chi_{B(z_i, d(z_i, \Omega^C)/10)} \leq \tilde{C} \chi_{\Omega}
\]
for some fixed positive constant $\tilde{C}$. Denote by $W$ the set of balls $\{B(z_i, d(z_i, \Omega^C)/10)\}$. Let $B_0$ be a fixed ball in $W$ with the largest radius. Denote by $x_B$ the center of ball $B$ and specially $x_0$ of $B_0$. Then
\[
\int_{\Omega} |u(x) - u_\Omega|^{j/n/(n-s)} \, dx
\]
\[
\leq 2^{j/n/(n-s)} \int_{\Omega} |u(x) - u_{B_0}|^{j/n/(n-s)} \, dx
\]
\[
\leq 2^{j/n/(n-s)} \sum_{B \in W} \int_{\Omega} |u(x) - u_{B_0}|^{j/n/(n-s)} \, dx
\]
\[
\leq 4^{j/n/(n-s)} \sum_{B \in W} \int_{\Omega} |u(x) - u_B|^{j/n/(n-s)} \, dx + 4^{j/n/(n-s)} \sum_{B \in W} |B||u_B - u_{B_0}|^{j/n/(n-s)}
\]
\[
eq I_1(j) + I_2(j).
\]
Observe that $\sum_{B \in W} \|u\|^{n/s}_{M^{s,n/s}(4B)} \leq \tilde{C} \||u|| \dot{M}^{s,p}(\Omega) \leq \tilde{C}$. Choose $0 < \sigma < C_1/8$ such that
\[
C_2 \tilde{C}[(4\sigma)(C_1)^{1/(j_0+1)}]^{j/n/(n-s)} \leq 1/2,
\]
where $C_1$ and $C_2$ are the constants from Lemma 4.1. Then by $|\Omega| = 1$ and Lemma 4.1, we have

$$
\sum_{j=j_0+1}^{\infty} \frac{1}{j!} \sigma^{jn/(n-s)} I_1(j) \\
\leq \sum_{B \in W} \frac{[4\sigma(C_1)^{-1}]^{(j_0+1)n/(n-s)}}{j!} \int_B \frac{1}{j!} (C_1)^{jn/(n-s)} |u(x) - u_B|^{jn/(n-s)} \, dx \\
\leq \sum_{B \in W} \left[ \|u\|_{M^{s,n/(4B)}} \right]^{(j_0+1)n/(n-s)} (4\sigma(C_1)^{-1}) \frac{1}{j!} \int_B \exp \left( C_1 \frac{|u(x) - u_B|}{\|u\|_{M^{s,n/(4B)}}} \right) \, dx \\
\leq C_2[(4\sigma(C_1)^{-1})^{(j_0+1)n/(n-s)}] \sum_{B \in W} \|u\|_{M^{s,n/(4B)}}^n \leq 1/2.
$$

To estimate $I_2(j)$, for each $B \in W \setminus \{B_0\}$, let $\gamma$ be the geodesic joining $x_0$ and $x_B$. By using the Besovitch covering lemma (see [33]) and some arguments similar to those in the proofs of [5, Theorem 4.1] and [32, Lemma 3.2], we can find a family of balls $B \equiv \{B_i\}_{i=0}^N$ such that

a) $B_i \equiv B(w_i, d(w_i, \Omega^B)/10)$ with $w_i \in \gamma$ for all $i = 0, \ldots, N$, $w_0 = x_0$ and $w_N = x_B$;

b) $B_i \cap B_{i+1} \neq \emptyset$ for all $i = 0, \ldots, N - 1$;

c) $\sum_{i=1}^N 2\sigma_i(z) \leq \overline{C}$ for all $z \in \Omega$, where the constant $\overline{C}$ only depends on the dimension $n$.

Let $\tilde{w}_i \in B_i \cap B_{i+1}$ and $\tilde{r}_i \equiv \min \{d(w_i, \Omega^B), d(w_{i+1}, \Omega^B)\}$ for all $i = 0, \ldots, N - 1$. Notice that b) implies that

$$
\frac{9}{11} d(w_{i+1}, \Omega^B) \leq d(w_i, \Omega^B) \leq \frac{11}{9} d(w_i + 1, \Omega^B)
$$

for all $i = 0, \ldots, N - 1$. Since $d(z, \Omega^B) \sim d(w_i, \Omega^B)$ for all $z \in B_i$, so by c), we have

$$
N \lesssim \sum_{i=0}^{N} \int_{B_i \cap \gamma} \frac{1}{d(z, \Omega^B)} \, |dz| \lesssim k_{\Omega}(x_B, x_0).
$$

Thus, by these, c) and the Hölder inequality, we have

$$
|u_B - u_{B_0}| \lesssim \sum_{j=0}^{N-1} |u_{B_i} - u_{B(\tilde{w}_i, \tilde{r}_i)}| + |u_{B(\tilde{w}_i, \tilde{r}_i)} - u_{B_{i+1}}| \\
\lesssim \sum_{j=0}^{N} \int_{2B_i} |u(z) - u_{2B_i}| \, dz \\
\lesssim \sum_{j=0}^{N} (d(w_i, \Omega^B)^{-n+s} \int_{2B_i} g(x) \, dx) \lesssim \sum_{j=0}^{N} \left\{ \int_{2B_i} g(x)^{n/s} \, dx \right\}^{s/n} \lesssim N^{(n-s)/n} \lesssim k_{\Omega}(x_B, x_0)^{(n-s)/n}.
$$
Let $C_3$ be the constant from the preceding inequality and notice that there exists a positive constant $C_4$ such that for all $x \in B$ and $B \in W \setminus \{B_0\}$, $k_\Omega(x_B, x_0) \leq C_4 k_\Omega(x, x_0)$. By $|\Omega| = 1$, we have

$$
\sum_{j=j_0+1}^{\infty} \frac{1}{j!} c^{jn/(n-s)} I_2(j) \leq \sum_{j=j_0+1}^{\infty} \frac{1}{j!} (4C_3)^{jn/(n-s)} \sum_{j=j_0+1}^{\infty} \sum_{B \in W \setminus \{B_0\}} |B|[k_\Omega(x_B, x_0)]^{j(n-s)/n} dx
$$

$$
\leq \sum_{j=j_0+1}^{\infty} \frac{1}{j!} (4C_3)^{jn/(n-s)} \int_{\Omega \setminus B_0} [k_\Omega(x, x_0)]^{j(n-s)/n} dx
$$

$$
\leq \sum_{j=j_0+1}^{\infty} \frac{1}{j!} (4C_3)^{jn/(n-s)} \int_\Omega [k_\Omega(x_0, x)]^j dx
$$

$$
\leq \int_\Omega \left[ \exp((4C_3)^{n/(n-s)} k_\Omega(x_0, x)) - 1 \right] dx.
$$

Then by Lemma 2.1, we can choose $\sigma$ small enough such that $\sum_{j=j_0+1}^{\infty} \frac{1}{j!} c^{jn/(n-s)} I_2(j) \leq 1/2$, which together with $\sum_{j=j_0+1}^{\infty} \frac{1}{j!} c^{jn/(n-s)} I_1(j) \leq 1/2$ implies (4.2). This finishes the proof of Theorem 1.3(i).

(ii) Assume that $\Omega$ has the slice property with respect to $y$ and $C_5 \geq 1$ as in Definition 2.4. Then for every $x \in \Omega$, by Definition 2.4, there exist a rectifiable curve $\gamma$ and a sequence of $\{S_i\}_{i=0}^\infty$ for some $j \geq 0$, satisfying (i) through (iv) of Definition 2.4. Without loss of generality, we may assume that $j \geq 2$. In fact, as pointed out by Buckley and Koskela [5, p. 890], Definition 2.4(iii) and (iv) implies that $j + 1 \geq k_\Omega(x, y)/C_5$. Observe that if $k(x, y) \leq 2C_5$, then (2.1) is clearly satisfied. So we only need to consider the case $j \geq 2$.

For each $i = 1, \cdots, j - 1$, define the function $u_i$ by setting $u_i(z) = \inf_{\gamma} \ell(\gamma \cap S_i)$ for all $z \in \Omega$, where the infimum is taken over all rectifiable curves $\gamma$ joining $x$ and $z$. Obviously, $u_i(z) = 0$ for $z \in \cup_{k=0}^{j-1} S_k$ and $u_i(z)$ is a constant for $z \in \cup_{k=0}^{j-1} S_k$. Then by an argument similar to the one in the proof of Lemma 3.2, we can prove that $u \in M_{\text{ball}}^{s, n/s}(\Omega)$ and $g_i \equiv r_i^{-s} X_B(x_i, 2C_5 d(x_i, \Omega^5))$ is a constant multiple of an element of $D_{\text{ball}}^{s, 1/(16C_5)}(u)$, where $r_i = \text{diam} S_i \sim d(x_i, \Omega^5)$ by Definition 2.4(iii). We omit the details. Then $\|u_i\|_{M_{\text{ball}}^{s, n/s}(\Omega)} \lesssim r_i^{-1+s+n/p}$. Notice that Definition 2.4(ii) implies that $|u_i(x) - u_i(y)| \geq C_5^{-1} d(x_i, \Omega^5) \gtrsim r_i$. Moreover, define $u = j^{-s/n} \sum_{i=1}^{j-1} r_i^{-1} u_i$. Then the function

$$
g \equiv j^{-s/n} \sum_{i=1}^{j-1} r_i^{-1} g_i = j^{-s/n} \sum_{i=1}^{j-1} r_i^{-s} X_B(x_i, 2C_5 d(x_i, \Omega^5))
$$

is a constant multiple of an element of $D_{\text{ball}}^{s, 1/(16C_5)}(u)$, which together with the Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal function $M$ (see, for example, [33]) and Definition 2.4 imply that

$$
\|u\|_{M_{\text{ball}}^{s, n/s}(\Omega)}^{n/s} \lesssim \int_{\Omega} \left( \sum_{i=1}^{j-1} r_i^{-s} X_B(x_i, 2C_5 d(x_i, \Omega^5))(z) \right)^{n/s} dz
$$
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\[ \lesssim j^{-1} \int_{\Omega} \left( \sum_{i=1}^{j-1} M \left( r_i^{-s} \chi_{B(x_i, C_5^{-1}d(x_i, \Omega^c))} \right) \left( z \right) \right)^{n/s} dz \]

\[ \lesssim j^{-1} \int_{\Omega} \left( \sum_{i=1}^{j-1} r_i^{-s} \chi_{B(x_i, C_5^{-1}d(x_i, \Omega^c))} \left( z \right) \right)^{n/s} dz \]

\[ \lesssim j^{-1} \int_{\Omega} \left( \sum_{i=1}^{j-1} r_i^{-n} \chi_{B(x_i, C_5^{-1}d(x_i, \Omega^c))} \left( z \right) \right) dz \lesssim 1. \]

On the other hand, since \( u(z) \geq (j-1)^{1-s/n} \) for \( z \in S_j \), then for \( t^{n/(n-s)} \leq (j-1)/ \log(1 + [C_5^{-1}d(x, \Omega^c)]^{-n}) \), we have

\[ \int_{\Omega} \phi_s \left( \frac{u(z)}{t} \right) dz \geq \int_{S_j} \phi_s \left( \frac{u(z)}{t} \right) dz \geq [C_5^{-1}d(x, \Omega^c)]^{-n}|S_j| > 1, \]

which implies that

\[ \|u\|_{\phi_s(L^1(\Omega))} \gtrsim \left\{ \frac{j - 1}{\log(1 + [C_5^{-1}d(x, \Omega^c)]^{-n})} \right\}^{(n-s)/n} \gtrsim \left\{ \frac{j}{\log(1 + [d(x, \Omega^c)]^{-1})} \right\}^{(n-s)/n}. \]

From this, (4.3) and s-Trudinger inequality, it follows that \( j \lesssim \log(1 + [d(x, \Omega^c)]^{-1}) \), which further implies that

\[ \int_{\gamma} \frac{1}{d(z, \Omega^c)} |dz| \lesssim \log(1 + [d(x, \Omega^c)]^{-1}). \]

This means that \( \Omega \) is a weak carrot domain and thus finishes the proof of Theorem 1.3(ii).

\[ \square \]

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