GENERALIZED SECOND BARGMANN TRANSFORMS ASSOCIATED WITH THE HYPERBOLIC LANDAU LEVELS ON THE POINCARÉ DISK

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ABSTRACT. We deal with a family of generalized coherent states associated to the hyperbolic Landau levels of the Schrödinger operator with uniform magnetic field on the Poincaré disk. Their associated coherent state transforms constitute a class of generalized second Bargmann transforms.

1. INTRODUCTION

The classical Bargmann transform made the space of square integrable functions $f$ on the real line isometric to the space of entire functions that are $e^{-|z|^2}d\mu$-square integrable, $d\mu$ being the Lebesgue measure on the complex plane. It is defined in [1] as

$$\mathcal{B}[f](z) := \pi^{-1/4} \int_{\mathbb{R}} \exp \left(-\frac{\xi^2}{2} + \sqrt{2}\xi z - \frac{z^2}{2}\right) f(\xi) d\xi; \quad z \in \mathbb{C},$$

(1.1)

where the involved kernel function is related to the generating function of the Hermite polynomials.

In the same paper [1, p.203], V. Bargmann has also introduced a second transform $\mathcal{R}_\nu; \nu > 1/2$, as a unitary integral operator whose kernel function corresponds to the generating function of the Laguerre polynomials $L_m^{(\nu)}(\cdot)$ [11,13]. It maps isometrically the space of $\xi^{2\nu}d\xi/(2\nu)!$-square integrable functions on the positive real half-line onto the weighted Bergman space

$$\mathcal{A}^{2\nu}(\mathbb{D}) := \left\{ f \text{ holomorphic on } \mathbb{D}; \int_{\mathbb{D}} |f(z)|^2 \left(1 - |z|^2\right)^{2\nu-2} d\mu(z) < \infty \right\}$$

(1.2)

on the unit disk $\mathbb{D} = \{ z \in \mathbb{C}; |z| < 1 \}$. Explicitly, we have

$$\mathcal{R}_\nu[\psi](z) = \left(\frac{2\nu - 1}{\pi}\right)^{1/2} (1 - z)^{-2\nu} \int_0^\infty \exp \left(-\frac{\xi}{2} \left(\frac{1 + z}{1 - z}\right)\right) \psi(\xi) \xi^{2\nu} (\Gamma(2\nu))^{-1} d\xi,$$

(1.3)

where the positive real number $2\nu - 1$ represents the parameter $\gamma$ in [1, p.203]. Thus, using the canonical isometry from $L^2(\mathbb{R}^+_+, d\xi/\xi)$ onto $L^2(\mathbb{R}^+_+, \xi^{2\nu} d\xi/(2\nu)!)$, one extends $\mathcal{R}_\nu$ to the transform

$$\mathcal{W}_\nu[\varphi](z) := \left(\frac{2\nu - 1}{\pi\Gamma(2\nu)}\right)^{1/2} (1 - z)^{-2\nu} \int_0^{\infty} \xi^{\nu} \exp \left(-\frac{\xi}{2} \left(\frac{1 + z}{1 - z}\right)\right) \varphi(\xi) \frac{d\xi}{\xi},$$

(1.4)

mapping isometrically the Hilbert space $L^2(\mathbb{R}^+_+, d\xi/\xi)$ onto $\mathcal{A}^{2\nu}(\mathbb{D})$. Note that the space $\mathcal{A}^{2\nu}(\mathbb{D})$ in (1.2) can also be realized as the null space of the second order differential operator

$$H_\nu = -4(1 - |z|^2) \left(1 - |z|^2\right) \frac{\partial^2}{\partial z \partial \bar{z}} - 2\nu z \frac{\partial}{\partial \bar{z}}.$$

(1.5)

The latter one is acting on the Hilbert space $L^{2\nu}(\mathbb{D}) := L^2(\mathbb{D}, (1 - |z|^2)^{2\nu-2}d\mu)$ and can be unitarily intertwined to represent an Hamiltonian of uniform magnetic field on the unit disk.

In this paper, we will be concerned with the $L^2$-eigenspaces

$$\mathcal{A}^{2\nu}_{m}(\mathbb{D}) := \left\{ F \in L^{2\nu}(\mathbb{D}); \quad H_\nu F = \epsilon^\nu_m F \right\}$$

(1.6)

associated to the discrete spectrum of $H_\nu$ consisting of the eigenvalues (hyperbolic Landau levels):

$$\epsilon^\nu_m = 4m(2\nu - m - 1); \quad m = 0, 1, 2, \ldots, \lfloor \nu - (1/2) \rfloor,$$

(1.7)
where \( [x] \) denotes the greatest integer less than \( x \). Our aim is to construct a family of integral transforms generalizing \([1,4]\). The method used is on the coherent states analysis “à la Iwata” \([9]\) together with the concrete description of the \( L^2 \)-spectral theory of the operator \( H_{\nu} \) \([3,4,14]\).

Precisely, we establish the following

**Theorem 1.1.** Let \( \nu \) be a real number such that \( \nu > 1/2 \) and \( m = 0,1,2,\cdots,\lfloor \nu - 1/2 \rfloor \). Then, the mapping

\[
\mathcal{W}_{\nu,m}[\varphi](z) = \left( \frac{(2(v - m) - 1)m!}{\pi \Gamma(2v-m)} \right)^{1/2} \left( \frac{1 - z}{1 - |z|^2} \right)^{2m} (1 - z)^{-2v} \times \int_{-\infty}^{+\infty} \xi^{v-m} \exp \left( -\frac{\xi}{2} \left( \frac{1 + z}{1 - z} \right) \right) \frac{(1 - |z|^2)}{(1 - |z|^2)^{1/2}} \varphi(\xi) \frac{d\xi}{\xi}
\]

defines a unitary isomorphism from \( L^2(\mathbb{R}^+_+,d\xi/\xi) \) onto the Hilbert space \( \mathcal{A}_{m,\nu}(\mathbb{D}) \). The particular case of \( m = 0 \) reproduces to the second Bargmann transforms \( \mathcal{W}_\nu \) in \([1,4]\).

The paper is organized as follows. In Section 2, we review some needed facts on the \( L^2 \)-spectral theory of the operator \( H_{\nu} \) on the unit disk. In Section 3, we recall the coherent states formalism we will be using. In Section 4, we define a family of generalized coherent states attached to hyperbolic Landau levels. The associated coherent state transforms constitute to a family of generalized second Bargmann transforms.

### 2. Spectral Analysis of \( H_{\nu} \) \( \nu > 0 \)

The second order differential operator \( H_{\nu} \) in \([1,5]\) appears as the Laplace-Beltrami operator on the unit disk perturbed by a first order differential operator. It can be interpreted as the Hamiltonian of a charged particle moving in an external uniform magnetic field. In fact, \( H_{\nu} \) is unitary equivalent to the magnetic Schrödinger operator \([6]\):

\[
\mathcal{L}_{\nu} := (d + \sqrt{-1\nu\theta})^* (d + \sqrt{-1\nu\theta}),
\]

associated to the gauge potential vector \( \theta(z) = -\sqrt{-1}(\partial - \bar{\partial}) \log(1 - |z|^2) \), and acting on \( L^2(\mathbb{D}) = L^2(\mathbb{D},(1 - |z|^2)^{-\nu}d\mu) \). Indeed, we have \((1 - |z|^2)^{\nu} H_{\nu}(1 - |z|^2)^{-1} = \mathcal{L}_{\nu} \). Different aspects of its spectral analysis have been studied by many authors, e.g. \([3,4,14]\). For instance, note that \( H_{\nu} \) is an elliptic densely defined operator on the Hilbert space \( L^2(\mathbb{D}) \) and admits a unique self-adjoint realization that we denote also by \( H_{\nu} \). Note also that such operator commutes with the action of the group \( SU(1,1) \) defined on \( L^2(\mathbb{D}) \) by

\[
(T^{g'}_z f)(z) := (\det(g'))^{\nu} f(g.z); \quad g.z = (az + b)(cz + d)^{-1}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1),
\]

where \( g' \) is the complex Jacobian. Moreover, we have the following results:

- The discrete part of the spectrum of \( H_{\nu} \) is not empty if and only if that \( 2\nu > 1 \). It consists of the eigenvalues \( \epsilon_m^\nu \) given through \([1,2]\) and occurring with infinite multiplicities.
- Let \( \nu \) be such that \( 2\nu > 1 \). Then, for every fixed \( m = 0,1,2,\cdots,\lfloor \nu - 1/2 \rfloor \), we have
  
  (i) The family of functions given explicitly in terms of the Jacobi polynomials \( P_j^{(a,b)}(\cdot) \) as
  \[
  \phi^{\nu,m}_k(z) = (-1)^{\min(m,k)} (1 - |z|^2)^{-m} |z|^{m-k} |\epsilon - i(m-k) \text{ arg } z|^{(m-k,2(\nu-m)-1)} (1 - 2|z|^2)
  \]  
  constitutes an orthogonal basis of the \( L^2 \)-eigenspace
  \[
  \mathcal{A}_{m,\nu}(\mathbb{D}) := \left\{ F: \mathbb{D} \to \mathbb{C}, \quad F \in L^2(\mathbb{D}) \quad \text{and} \quad H_{\nu} F = \epsilon^\nu_m F \right\}.
  \]  
  (ii) The square norm of \( \phi_k^{\nu,m} \) in \( L^2(\mathbb{D}) \) is given by
  \[
  \rho^{\nu,m}_k = \frac{\pi}{2(2\nu-m)-1} \frac{(\max(m,k))! \Gamma(2\nu-m) + \min(m,k))! \Gamma(2\nu-m) + \max(m,k))}{(\min(m,k))! \Gamma(2\nu-m) + \max(m,k))}. \]
Therefore the set of functions
\[
\Phi_{k,m}^{\nu} := \frac{\phi_{k,m}^{\nu}}{\sqrt{p_{k,m}^{\nu}}} \quad k = 0, 1, 2, \ldots ,
\] (2.5)
constitute an orthonormal basis of \( \mathcal{A}^{2\nu}_{m}(\mathbb{D}) \). Moreover, using the identity [13, p.63]:
\[
\frac{\Gamma(m + s + 1)}{\Gamma(m - s + 1)} P_{m}^{(-s,\alpha)}(t) = \frac{\Gamma(m + s + 1)}{\Gamma(m - s + \alpha + 1)} \left( \frac{t - 1}{2} \right)^{s} P_{m-s}^{(s,\alpha)}(t), \quad 1 \leq s \leq m,
\] (2.6)
which reads for \( s = m - k, t = 1 - 2|z|^2 \) and \( \alpha = 2(v - m) - 1 \)
\[
P_{m}^{(m-k,\alpha)}(t) = (-1)^{m-k} \frac{k! \Gamma(2v-m)}{m! \Gamma(2v-m+k)} |z|^{2(m-k)} P_{m}^{(m-k,\alpha)}(t),
\] (2.7)
we obtain
\[
\frac{1}{m! \Gamma(2v-m+k)} \left( \frac{2(v-m)-1}{\pi} \right)^{1/2} \left( \frac{k! \Gamma(2v-m+k)}{m! \Gamma(2v-m+k)} \right)^{1/2}
\times (1 - |z|^{2})^{-m-2(k-v-1)} \times \frac{1}{2} \left( 1 - \frac{|z|^2}{1 - z\bar{w}} \right) P_{m}^{(0,2(v-m)-1)}(1 - 2|z|^2).
\]
Therefore, using (2.8) we check the following

- The function in (2.5) can be rewritten as
\[
\Phi_{k,m}^{\nu}(z) = (-1)^{k} \left( \frac{2(v-m)-1}{\pi} \right)^{1/2} \left( \frac{k! \Gamma(2v-m+k)}{m! \Gamma(2v-m+k)} \right)^{1/2}
\times (1 - |z|^{2})^{-m-k} \times \frac{1}{2} \left( 1 - \frac{|z|^2}{1 - z\bar{w}} \right) P_{m}^{(0,2(v-m)-1)}(1 - 2|z|^2).
\]
- The space \( \mathcal{A}^{2\nu}_{m}(\mathbb{D}) \) is a reproducing kernel Hilbert space. Its \( L^{2} \)-eigenprojector kernel is given by
\[
K_{m}^{\nu}(z,w) = \left( \frac{2(v-m)-1}{\pi} \right) (1 - z\bar{w})^{-2v} \left( \frac{|1-z\bar{w}|^2}{(1-|z|^2)(1-|w|^2)} \right)^{m}
\times P_{m}^{(0,2(v-m)-1)} \left( 1 - \frac{|z|^2}{1 - z\bar{w}} \right) - 1,
\] (2.10)
with the diagonal function
\[
K_{m}^{\nu}(z,z) = \left( \frac{2(v-m)-1}{\pi} \right) (1 - |z|^2)^{-2v}, \quad z \in \mathbb{D}.
\] (2.11)

**Remark 2.1.** In view of (2.2), the \( L^{2} \)-eigenspace \( \mathcal{A}^{2\nu}_{m}(\mathbb{D}) \), corresponding to \( m = 0 \) in (2.3) and associated to the bottom energy \( \varepsilon_{0}^{\nu} = 0 \), reduces further to the weighted Bergmann space \( \mathcal{A}^{2\nu}(\mathbb{D}) \) consisting of complex-valued holomorphic functions \( F \) on \( \mathbb{D} \) such that
\[
\int_{\mathbb{D}} |F(z)|^{2}(1 - |z|^2)^{2\nu} - 2d\mu(z) < +\infty.
\] (2.12)

**Remark 2.2.** The condition \( 2\nu > 1 \) ensuring the existence of the eigenvalues (1.7) should implies that the magnetic field \( B = d\omega_{v} = 2\nu\Omega(z) \), where \( \Omega \) stands for the Kähler 2-form on \( \mathbb{D} \), has to be strong enough to capture the particle in a closed orbit. If this condition is not fulfilled the motion will be unbounded and the particle escapes to infinity.

### 3. IWATA’S COHERENT STATES

The first model of coherent states was the ‘nonspreading wavepacket’ of the harmonic oscillator, which have been constructed by Schrödinger [12]. In suitable units, wave functions of these states can be written as
\[
\Phi_{j}(\xi) := \langle \xi | \psi \rangle = \pi^{-\frac{1}{4}} \exp \left( -\frac{1}{2} \xi \xi + \sqrt{2} \xi \zeta - \frac{1}{2} \xi \xi - \frac{1}{2} |\zeta|^2 \right), \quad \zeta \in \mathbb{R}.
\] (3.1)
where $z \in \mathbb{C}$ determines the mean values of coordinate $\hat{x}$ and momentum $\hat{p}$ according to $\langle \hat{x} \rangle := \langle \Phi_z, x \Phi_z \rangle = \sqrt{2} \Re z$ and $\langle \hat{p} \rangle := \langle \Phi_z, p \Phi_z \rangle = \sqrt{2} \Im z$. The variances $\sigma_x = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 = \frac{1}{2}$ and $\sigma_p = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 = \frac{1}{2}$ have equal values, so their product assumes the minimal value permitted by the Heisenberg uncertainty relation. The coherent states $\Phi_z$ have been also obtained by Feynmann [5] and Glauber [8] from the vacuum state $|0\rangle$ by means of the unitary displacement operator $\exp (z A^*-\bar{z}A)$ as

$$\Phi_z = \exp (z A^*-\bar{z}A) |0\rangle,$$  \hspace{1cm} (3.2)

where $A$ and $A^*$ are respectively the annihilation and the creation operators defined by

$$A = \frac{1}{\sqrt{2}} (\hat{x} + i \hat{p}), \hspace{1cm} A^* = \frac{1}{\sqrt{2}} (\hat{x} - i \hat{p})$$  \hspace{1cm} (3.3)

It was Iwata [9] who used the well known expansion over the Fock basis $|n\rangle$ to give an expression of $\Phi_z$ as

$$\Phi_z = e^{-\frac{1}{2} |u|^2} \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n!}} |n\rangle.$$  \hspace{1cm} (3.4)

Actually, various generalizations of coherent states are proposed. Here, we shall focus on a generalization "à la Iwata" of (3.4). In the general setting, the procedure can be described as follows. Let $(X,d\lambda)$ be a measure space and $\mathcal{A}^2 \subset L^2(X,d\lambda)$ be a closed subspace of infinite dimension. Let $\{f_k\}_{k=1}^{\infty}$ be an orthogonal basis of $\mathcal{A}^2$ satisfying

$$\omega (u) := \sum_{k=1}^{\infty} \rho_k^{-1} |f_k(u)|^2 < +\infty$$  \hspace{1cm} (3.5)

for every $u \in X$, where $\rho_k := \|f_k\|_{L^2(X)}^2$. Therefore the function

$$K(u,v) := \sum_{k=1}^{\infty} \rho_k^{-1} f_k(u) \overline{f_k(v)},$$  \hspace{1cm} (3.6)

defined on $X \times X$, is a reproducing kernel of the Hilbert space $\mathcal{A}^2$ so that we have $\omega (u) = K(u,u)$; $u \in X$.

**Definition 3.1.** For given infinite dimensional Hilbert space $\mathcal{H}$ with $\{\psi_k\}_{k=1}^{\infty}$ as an orthonormal basis, the vectors $(\Psi_u)_{u \in X}$ defined by

$$\Psi_u := (\omega(u))^{-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{f_k(u)}{\sqrt{\rho_k}} \psi_k$$  \hspace{1cm} (3.7)

will be called coherent states of Iwata type for the data of $(X;\mathcal{A}^2; \{f_k\})$ and $(\mathcal{H}; \{\psi_k\})$.

The choice of the Hilbert space $\mathcal{H}$ defines in fact a quantization of $X = \{u\}$ by the coherent states $\Psi_u$, via the inclusion map $u \rightarrow \Psi_u$ from $X$ into $\mathcal{H}$. Moreover, according to the fact that $\langle \Psi_{u'}, \Psi_u \rangle_{\mathcal{H}} = 1$, one can show the following

- The transform given by

$$\mathcal{W} [f](u) := (\omega(u))^{\frac{1}{2}} \langle \Psi_{u'}, f \rangle_{\mathcal{H}}$$  \hspace{1cm} (3.8)

defines an isometry from $\mathcal{H}$ into $\mathcal{A}^2$.

Thereby we have a resolution of the identity, i.e., the following integral representation holds:

$$f(\cdot) = \int_X \langle \Psi_{u'}, f \rangle_{\mathcal{H}} \Psi_u(\cdot) \omega(u) d\lambda(u)$$  \hspace{1cm} (3.9)

for every $f \in \mathcal{H}$.

**Definition 3.2.** The transform $\mathcal{W} : \mathcal{H} \rightarrow \mathcal{A}^2 \subset L^2(X,d\lambda)$ in (3.8) will be called the coherent state transform (CST) associated to the set of coherent states $\Psi_u; u \in X$.

For an overview of all aspect of the theory of coherent states, we refer to the survey of V.V. Dodonov [2] or also to the recent book [7] by J.P. Gazeau.
4. Coherent states attached to Landau Levels $e_m^v$

Now, we are in position to attach to each hyperbolic Landau level $e_m^v$ in (1.7) a set of generalized coherent states according to formula (3.7). Namely, we have

$$\Psi_{v,mz} := (K_m^v(z, z))^{-1/2} \sum_{k=0}^{+\infty} \frac{\phi_k^{v,m}(z)}{\sqrt{P_k}} \Psi_{v,m,k}$$

(4.1)

with the following specifications:

- $(X, d\lambda) = (\mathbb{D}, (1 - |z|^2)^{2v-2} d\mu)$.
- $\mathcal{A}^2 = \mathcal{A}_m^{2v}$ is the eigenspace in (1.6).
- $K_m(z, z) = \pi^{-1} (2(v - m) - 1) (1 - |z|^2)^{-2v}$ as in (2.11).
- $f_k = \phi_k^{v,m}$ are the eigenfunctions given by (2.2).
- $\rho_k^{v,m}$ being the square norm of $\Phi_k^{v,m}$ given in (2.4).
- $\mathcal{H} = L^2(\mathbb{R}^+, \xi^{-1} d\xi)$ is the Hilbert space carrying the coherent states (4.1).
- $\psi_k = \psi_{v,m,k}$, $k = 0, 1, 2, \ldots$, the basis of $\mathcal{H}$ given by

$$\psi_{v,m,k}(\xi) := \left( \frac{k!}{\Gamma(2(v - m) + k)} \right)^{1/2} \xi^{v-m} e^{-\frac{1}{2} \xi} \left( 2^{(v-m)-1} \right) \xi > 0.$$  

(4.2)

In view of (4.1) and (2.9), the coherent states belonging to the Hilbert space $\mathcal{H}$ and corresponding to the eigenspace in (1.6) are defined by their wave functions through the series expansion

$$\Psi_{v,mz}(\xi) := (1 - |z|^2)^{v-m} \sum_{k=0}^{+\infty} (-1)^k \left( \frac{k! \Gamma(2(v - m) + m)}{m! \Gamma(2(v - m) + k)} \right)^{1/2} \xi^{m-k} (2^{v-m}) (1 - 2|z|^2) \Psi_{v,m,k}(\xi).$$

(4.3)

A closed form for (4.3) can be obtained in terms of Laguerre polynomials as follows.

**Proposition 4.1.** Let $2v > 1$ and $m = 0, 1, 2, \ldots, [v - (1/2)]$. Then, the wave functions of the states in (4.3) read simply as

$$\Psi_{v,mz}(\xi) = (-1)^m \left( \frac{m!}{\Gamma(2v - m)} \right)^{1/2} \left( 1 - |z|^2 \right)^{v-m} \sum_{k=0}^{+\infty} (-1)^k \left( \frac{k! \Gamma(a + 1 + m)}{m! \Gamma(a + 1 + k)} \right)^{1/2} \xi^{m-k} P_k^{(m-k,a)}(t) \Psi_{v,m,k}(\xi).$$

(4.4)

**Proof.** Set $a = 2(v - m) - 1$ and $t = 1 - 2|z|^2$. Then, the expression of $\Psi_{v,mz}(\xi)$ in (4.3) reads as

$$\Psi_{v,mz}(\xi) := (1 - |z|^2)^{v-m} \sum_{k=0}^{+\infty} (-1)^k \left( \frac{k! \Gamma(a + 1 + m)}{m! \Gamma(a + 1 + k)} \right)^{1/2} \xi^{m-k} P_k^{(m-k,a)}(t) \Psi_{v,m,k}(\xi).$$

(4.5)

By inserting the explicit expression of $\psi_{v,m,k}(\xi)$ given by (4.2) in (4.5), we infer

$$\Psi_{v,mz}(\xi) = \left( \frac{\Gamma(a + 1 + m)}{m!} \right)^{1/2} \left( 1 - |z|^2 \right)^{v-m} e^{-\frac{1}{2} \xi} \sum_{k=0}^{+\infty} \frac{(-1)^k k!}{\Gamma(a + 1 + k)} \xi^{m-k} P_k^{(m-k,a)}(t) L_k^{(a)}(\xi)$$

(4.6)

$$\Psi_{v,mz}(\xi) = \left( \frac{\Gamma(a + 1 + m)}{m!} \right)^{1/2} \left( 1 - |z|^2 \right)^{v-m} e^{-\frac{1}{2} \xi} \sum_{k=0}^{+\infty} \frac{k!}{\Gamma(a + 1 + k)} \xi^{m-k} P_k^{(a,m-k)}(-t) L_k^{(a)}(\xi).$$

(4.7)

The last equality is readily derived by means of the symmetry relation

$$P_k^{(a,b)}(t) = (-1)^k P_k^{(b,a)}(-t).$$

(4.8)
In order to use the bilateral generating function ([11], p.213):
\[
\sum_{k=0}^{+\infty} \lambda^k 2F_1(-k, b; 1 + \alpha; y) L_k^{(\alpha)}(\xi) = \frac{(1 - \lambda)^{b-1}}{(1 - \lambda + y\lambda)} \exp\left(-\frac{\xi \lambda}{1 - \lambda}\right) \times 1F_1\left(b; 1 + \alpha; \frac{\xi y \lambda}{(1 - \lambda)(1 - \lambda + y\lambda)}\right),
\]

involving a Laguerre polynomial and a terminating Gauss hypergeometric $2F_1$-sum, we make appeal to the fact ([11], p.254):
\[
p_k^{(a,q)}(x) = \frac{\Gamma(1 + a + k)}{k! \Gamma(1 + a)} \left(\frac{1 + x}{2}\right)^k 2F_1\left(-k, -(\eta + k), 1 + \alpha; \frac{x - 1}{x + 1}\right)
\]
with $\eta = m - k$ and $x = -t = -1 + 2|z|^2$. Hence, we obtain
\[
\Psi_{\nu,m;\xi}(\xi) = \left(\frac{\Gamma(\alpha + 1 + m)}{m!}\right)^{\frac{1}{2}} \frac{1}{\Gamma(\alpha + 1)} \left(1 - |z|^2\right)^{\nu - m} \xi^{\nu - m} e^{-\frac{1}{2} \xi} \times \sum_{k=0}^{+\infty} z^m \xi^{k} 2F_1\left(-k, -m; 1 + \alpha; \frac{t + 1}{t - 1}\right) L_k^{(\alpha)}(\xi).
\]
Thus, by applying (4.9) with $\lambda = z, b = -m$ and $y = \frac{t+1}{t} = -\frac{1 - |z|^2}{|z|^2}$, we check that
\[
\Psi_{\nu,m;\xi}(\xi) = \left(\frac{\Gamma(\alpha + 1 + m)}{m!}\right)^{\frac{1}{2}} \left(-1\right)^m \frac{1 - |z|^2}{\Gamma(\alpha + 1)} \left(1 - |z|^2\right)^{\nu - m} \xi^{\nu - m} \exp\left(-\frac{\xi}{2} \left(\frac{1 + z}{1 - z}\right)\right) 1F_1\left(-m; \alpha + 1; \frac{\xi(1 - |z|^2)}{|1 - z|^2}\right).
\]
Finally, making use of ([13], p.103)
\[
1F_1\left(-m; 1 + \alpha; x\right) = \frac{m! \Gamma(1 + \alpha)}{\Gamma(1 + \alpha + m)} L_m^{(\alpha)}(x),
\]
with $x = \xi(1 - |z|^2)/|1 - z|^2$ yields
\[
\Psi_{\nu,m;\xi}(\xi) = \left(-1\right)^m \left(\frac{m!}{\Gamma(2\nu - m)}\right)^{\frac{1}{2}} \frac{|1 - z|^2}{|1 - z|^2} \left(1 - |z|^2\right)^{\nu - m} \xi^{\nu - m} \exp\left(-\frac{\xi}{2} \left(\frac{1 + z}{1 - z}\right)\right) L_m^{(2\nu - m - 1)}\left(\frac{\xi(1 - |z|^2)}{|1 - z|^2}\right).
\]
This completes the proof. \qed

According to Definition 3.2, the coherent state transform associated with the coherent states in (4.14) is the unitary map:
\[
\mathcal{W}_{\nu,m} : L^2(R_+^\nu, d\xi / \xi) \longrightarrow A^{2\nu}_{m}(C)
\]
\[
\phi \mapsto \mathcal{W}_{\nu,m} [\phi] (z) := (K_m^\nu(z, z))^{\frac{1}{2}} \langle L^2(R_+^\nu, d\xi^{\nu - 1}) \rangle L_m^{(\nu, \xi)}(K_m^\nu(z, z)) \phi(\xi) d\xi.
\]
Explicitly, we have
\[
\mathcal{W}_{\nu,m} [\phi] (z) = \left(\frac{m!}{\pi \Gamma(2\nu - m)}\right)^{\frac{1}{2}} \frac{|1 - z|^2}{|1 - z|^2} L_{m}^{(2\nu - m - 1)}\left(\frac{\xi(1 - |z|^2)}{|1 - z|^2}\right) \phi(\xi) d\xi.
\]
thanks to Proposition 4.1. The assertion of Theorem 1.1 follows then from the fact that the CST in (3.9) is an isometry.

**Definition 4.2.** The coherent state transform $\mathcal{W}_{\nu,m}$ in (4.17) will be called a generalized second Bargmann transform of index $m = 0, 1, 2, \ldots, [\nu - (1/2)]$. 
Remark 4.3. For \( m = 0 \), the above transform in (4.17) reads simply
\[
W_{\nu,0} \phi(z) = \left( \frac{2\nu - 1}{\pi i(2\nu)} \right)^{\frac{1}{2}} (1 - z)^{-2\nu} \int_0^{+\infty} \xi^{2\nu} \exp \left( -\frac{\xi}{2} \left( \frac{1 + z}{1 - z} \right) \right) \phi(\xi) \frac{d\xi}{\xi}
\]
and then reduces to the second Bargmann transform in (1.4).

Remark 4.4. One can replace the space \( \mathcal{H} \) by the weighted Bergman space in Remark 2.1 to define a type of coherent states (see [10] for their series expansion). The corresponding coherent state transform maps eigenstates of the first hyperbolic Landau level \( \epsilon^\nu_0 \) into eigenstates corresponding to \( m^\text{th} \) level \( \epsilon^\nu_m \) as an integral transform \( A^\nu_0(D) \rightarrow A^\nu_m(D) \).

REFERENCES

[1] Bargmann V., On a Hilbert space of analytic functions and an associated integral transform, Part I. Comm. Pure Appl. Math. 14 (1961) 187–214.
[2] Dodonov V.V., “Nonclassical” states in quantum optics: a squeezed review of the first 75 years, J. Opt. B: Quantum Semiclass. Opt. 4 (2002) R1–R33.
[3] Elstrodt J., “Die Resolvente Zum Eigenwertproblem der automorphen in der hyperbolischen Ebene”, Teil I, Math. Ann. 203 (1973) 295–330. Teil II, Math. Z. 132 (1973) 99–134. Teil III, Math. Ann. 208 (1974) 99–132.
[4] Fay J., “Fourier Coefficients of the resolvent for a Fuchsian group”, J. Reine Angew. Math. 293 (1977) 143–203.
[5] Feynman R.P., An operator calculus having applications in quantum electrodynamics, Phys. Rev. 84 (1951) 108–128.
[6] Ghanmi A., Intissar A., Asymptotic of complex hyperbolic geometry and \( L^2 \)-spectral analysis of Landau-like Hamiltonians, J. Math. Phys. 46 (2005), no. 3, 032107, 26 pp.
[7] Gazeau J-P., Coherent states in quantum physics. WILEY-VCH Verlag GmbH & Co. KGaA Weinheim 2009.
[8] Glauber R.J., Some notes on multiple boson processes, Phys. Rev. 84 (1951) 395–400.
[9] Iwata G., Non-Hermitian operators and eigenfunction expansions, Progress Theoret. Phys. 6, (1951) pp 216-226.
[10] Mouayn Z., Coherent states attached to Landau levels on the Poincaré disk, J. Phys. A : Math. Gen. 38 (2005), no. 42, 9509–9516.
[11] Rainville E.D., Special functions. Macmillan company New York 1960.
[12] Schrödinger E., Der streitige Übergang von der Mikro-zur Makromechanik, Naturwissenschaften. 14 (1926) 664–666.
[13] Szegö, G., Orthogonal polynomials. Fourth edition. Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.
[14] Zhang G., A weighted Plancherel formula II, the case of the unit ball. Stud. Math. (2) 102, 103-120 (1992).

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