A Measure of Classicality*

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Abstract

A striking feature of our fundamentally indeterministic quantum universe is its quasiclassical realm — the wide range of time place and scale in which the deterministic laws of classical physics hold. Our quasiclassical realm is an emergent feature of the fundamental theories of our universe’s quantum state and dynamics. There are many types of quasiclassical realms our Universe could exhibit characterized by different variables, different levels of coarse-graining, different locations in spacetime, different classical physics, and different levels of classicality. We propose a measure of classicality for quasiclassical realms. We speculate on the observable consequences of different levels of classicality especially for information gathering and utilizing systems (IGUSes) such ourselves as observers of the Universe.

* The core of this paper is a section of the authors’ paper Strong Decoherence [1] modestly edited by the junior (second) author. The section is concerned with the general problem of a measure for classicality and a specific proposal for that measure. It is largely self-contained and can be read separately from the rest with the help of the modest edits by the junior author. Neither the text or the references have been updated except for a few “to be published” references. Sections III through V have been added by the junior author.

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I. INTRODUCTION—OUR QUASICLASSICAL REALM

The most striking observable feature of our indeterministic quantum universe is the wide range of time, place, and scale on which the deterministic laws of classical physics hold to an excellent approximation defining what we can call our quasiclassical realm. The regularities that characterize a quasiclassical realm are described by the familiar classical equations for particles, bulk matter, and fields, together with the Einstein equation governing the regularities of classical spacetime geometry. Our observations suggest that our quasiclassical realm extends over the patch of classical spacetime that is visible to us through large scale observations. What is the origin of this predictable quasiclassical realm in a quantum universe characterized by indeterminacy and distributed probabilities?

Our quasiclassical realm is an observed feature of our Universe on a par with such observed features as its expanding classical spacetime, its approximate homogeneity and isotropy on large distance scales, the fluctuations away from these symmetries that we see in the cosmic background radiation and the statistics of galaxies, and in the primordial abundance of the elements.

In various papers developing decoherent (or consistent) histories quantum mechanics (e.g., [4, 6]) the authors have defined a quasiclassical realm as a decohering set of alternative coarse-grained histories of the Universe for which the probabilities are high for histories that exhibit correlations in time governed by deterministic classical laws like the Einstein equation, the Navier-Stokes equation, or Maxwell’s equations, etc. [2].

A typical history in a quasiclassical realm will describe long stretches in time of classical behavior interrupted by non-classical events such as quantum fluctuations, quantum transitions, and quantum measurements. It is for this reason that we refer to quasiclassical realms rather than classical realms.

Quantum mechanics, along with the correct theory of the elementary particles (represented by the Hamiltonian \(H\) and the correct initial condition in the universe (represented by the state vector \(\ket{\Psi}\)), presumably exhibits a great many essentially different decohering realms, but only some of those are quasiclassical. For the quasiclassical realms to be viewed as an emergent feature of \((H, \ket{\Psi})\), and quantum mechanics, a good technical definition of classicality is required. (One can then go on to investigate whether the theory exhibits many essentially inequivalent quasiclassical realms or whether the ours is nearly unique.)

II. A MEASURE OF CLASSICALITY

In earlier papers, [4, 6] we have made a number of suggestions about the definition of classicality and it is appropriate to continue that discussion here. It is clear that from those earlier discussions that classicality must be related in some way to a kind of entropy for alternative coarse-grained histories. We must therefore begin with an abstract characterization of entropy and then investigate the application to histories. An entropy \(S\) is always associated with a coarse graining, since a perfectly fine-grained version of entropy in statistical mechanics would be conserved instead of tending to increase with time. Classically, if
the set of all fine-grained alternatives are designated by \( \{r\} \), with probabilities \( p_r \) summing to one, that fine-grained version of entropy would be

\[
S_{f-g} = - \sum_r p_r \log p_r ,
\]

(1)

where \( \log \) means \( \log_2 \) and where, for convenience, we have put Boltzmann’s constant \( k \) times \( \log_e 2 \) equal to unity. A true, coarse-grained entropy has the form

\[
S \equiv - \sum_r \tilde{p}_r \log \tilde{p}_r ,
\]

(2)

where the probabilities \( \tilde{p}_r \) are coarse-grained averages of the \( \{p_r\} \). A coarse graining \( p_r \rightarrow \tilde{p}_r \) must have certain properties (see [7] for more details):

1) the \( \{\tilde{p}_r\} \) are probabilities ,

(3a)

2) \( \tilde{p} = p_r \),

(3b)

3) \( - \sum_r p_r \log \tilde{p}_r = - \sum_r \tilde{p}_r \log \tilde{p}_r . \)

(3c)

These properties are not surprising for an averaging procedure. The significance of the last one is easily seen if we make use of the well known fact that for any two sets of probabilities \( \{p_r\} \) and \( \{p'_r\} \) we have

\[
- \sum_r p_r \log p_r \leq - \sum_r p_r \log p'_r .
\]

(4)

Putting \( p'_r = \tilde{p}_r \) for each \( r \) and using (1)-(4), we obtain

\[
S_{f-g} = - \sum_r p_r \log p_r \leq - \sum_r p_r \log \tilde{p}_r = - \sum_r \tilde{p}_r \log \tilde{p}_r = S ,
\]

(5)

so that \( S_{f-g} \) provides a lower bound for the entropy \( S \). If the initial condition and the coarse graining are related in such away that \( S \) is initially near its lower bound, then it will tend to increase for a period of time. That is the way the second law of thermodynamics comes to hold.

In order to know what nearness to the lower bound means, we should examine the upper bound on \( S \). That upper bound is achieved when all fine-grained alternatives have equal coarse-grained probabilities \( \tilde{p}_r \), corresponding in statistical mechanics to something like “equilibrium” or infinite temperature. Each \( \tilde{p}_r \) is then equal to \( N^{-1} \), where \( N \) (assumed finite) is the number of fine-grained alternatives, and the maximum entropy is thus

\[
S_{\text{max}} = \log N .
\]

(6)

The simplest example of coarse graining utilizes a grouping of the fine-grained alternatives \( \{r\} \) into exhaustive and mutually exclusive classes \( \{\alpha\} \), where a class \( \alpha \) contains \( N_\alpha \) elements and has lumped probability

\[
p_\alpha \equiv \sum_{r \in \alpha} p_r .
\]

(7)

Of course we have

\[
\sum_\alpha N_\alpha = N , \quad \sum_\alpha p_\alpha = 1 .
\]

(8)
The coarse-grained probabilities $\tilde{p}_r$ in this example are the class averages

$$\tilde{p}_r = p_\alpha / N_\alpha, \quad r \in \alpha,$$

and they clearly have the properties (3). The entropy comes out to be

$$S = -\sum_\alpha p_\alpha \log p_\alpha + \sum_\alpha p_\alpha \log N_\alpha,$$

where the second term contains the familiar logarithm of the number of fine-grained alternatives (or microstates) in a coarse-grained alternative (or macrostate), averaged over all the coarse-grained alternatives.

Besides entropy, it is useful to introduce the concept of algorithmic information content (AIC) as defined some thirty years ago (at the time of writing) by Kolmogorov, Chaitin, and Solomonoff (all working independently).\footnote{For a discussion of the original papers see [8].} For a string of bits $s$ and a particular universal computer $U$, the AIC of the string, written $K_U(s)$, is the length of the shortest program that will cause $U$ to print out the string and then halt. The string can be used as the description of some entity $e$, down to a given level of detail, in a given language, assuming a given amount of knowledge and understanding of the world, encoded in a given way into bits. The AIC of the string can then be regarded as $K_U(e)$, the AIC of the entity so described.

We now discuss a way of approaching classicality that utilizes AIC as well as entropy. Some authors have tried to identify AIC in a straightforward way with complexity, and in fact AIC is often called algorithmic complexity. However, AIC is greatest for a "random" string of bits with no regularity and that hardly corresponds to what is usually meant by complexity in ordinary parlance or in scientific discourse. To illustrate the connections among AIC, entropy or information, and an effective notion of complexity, take the ensemble $\tilde{E}$ consisting of a set of fine-grained alternatives $\{r\}$ together with their coarse-grained probabilities $\tilde{p}_r$. We can then consider both $K_U(\tilde{E})$, the AIC of the ensemble, and $K_{U'}(r|\tilde{E})$, which is the AIC of a particular alternative $r$ given the ensemble. For the latter we have the well known inequality (see, for example [10]):

$$\sum_r \tilde{p}_r K_{U'}(r|\tilde{E}) \geq -\sum_r \tilde{p}_r \log \tilde{p}_r = S. \tag{11}$$

Moreover, it has been shown by R. Schack\footnote{For a discussion of the original papers see [8].} that, for any $U$, a slight modification $U \rightarrow U'$ permits $K_{U'}(r|\tilde{E})$ to be bounded on both sides as follows:

$$S + 1 \geq \sum_r \tilde{p}_r K_{U'}(r|\tilde{E}) \geq S, \tag{12}$$

so that we have

$$\sum_r \tilde{p}_r K_{U'}(r|\tilde{E}) \approx S. \tag{13}$$

(Previous upper bounds had $O(1)$ in place of 1, but there was nothing to prevent $O(1)$ from being millions or trillions of bits!)

Looking at the entropy $S$ as a close approximation to $\sum_r \tilde{p}_r K_{U'}(r|\tilde{E})$, we see that it is natural to complete it by adding to it the quantity $K_{U'}(\tilde{E})$ — the AIC of the ensemble with
respect to the same universal computer $U'$. This last quantity can be connected with the idea of effective complexity — the length of the most concise description of the perceived regularities of an entity $e$. Any particular set of regularities can be expressed by describing $e$ as a member of an ensemble $\tilde{E}$ of possible entities sharing those regularities. Then $K_{U'}(\tilde{E})$ may be identified with the effective complexity of $e$ or of the ensemble $\tilde{E}$ \[7, 9\]. Adding this effective complexity to $S$, we have:

$$\Sigma \equiv K_{U'}(\tilde{E}) + S .$$

(14)

This sum of the effective complexity and the entropy (or Shannon information) may be labeled either “augmented entropy” or “total information” If the coarse graining is the simple one obtained by partitioning the set of fine-grained alternatives $\{r\}$ into classes $\{\alpha\}$ with cardinal numbers $N_\alpha$, then the total information becomes

$$\Sigma = K_{U'}(\tilde{E}) - \sum_\alpha p_\alpha \log p_\alpha + \sum_\alpha p_\alpha \log N_\alpha .$$

(15)

In (14), the first term becomes smaller as the set of perceived regularities becomes simpler, while the second term becomes smaller as the spread of possible entities sharing those regularities is reduced. Minimizing $\Sigma$ corresponds to optimizing the choice of regularities and the resulting effective complexity thereby becomes less subjective. Thus, the total information or augmented entropy is useful in a wide variety of contexts \[7, 9\]. We apply it here to sets of alternative decohering coarse-grained histories in quantum mechanics.

The general idea of augmenting entropy with a term referring to algorithmic information content was proposed in a different context by Zurek \[12\]. However, as far as we know, the emphasis on the utility of the quantity $\Sigma$ in (14) and (15) is new. We discussed the general idea of an entropy for histories in \[4\]. Earlier, Lloyd and Pagels \[13\] introduced a quantity they called thermodynamic depth, applicable to alternative coarse-grained classical histories $\alpha$. They defined it as

$$D = \sum_\alpha p_\alpha \log(p_\alpha/q_\alpha) ,$$

(16)

where $q_\alpha$ is an “equilibrium probability”, which in our notation would be $N_\alpha/N$ for the simple coarse graining we have discussed. We clearly have

$$D = \log N + \sum_\alpha p_\alpha \log p_\alpha - \sum_\alpha p_\alpha \log N_\alpha$$

(17)

or

$$D = S_{\text{max}} - S$$

(18)

for the set of alternative coarse-grained histories. We see that thermodynamic depth is intimately related to the notion of an entropy for histories.

In applying augmented entropy to sets of coarse-grained histories in quantum mechanics, one must take into account that there are infinitely many different sets of fine-grained histories and that these sets do not generally have probabilities because they fail to decohere. The quantities $N_\alpha$ may therefore conceivably depart from their obvious definition as the numbers of fine-grained histories in the coarse-grained classes $\{\alpha\}$. In fact, there may be some latitude in the precise definition of the complexity and entropy terms in the total information (14). For example, one could consider instead of $\tilde{E}$ an ensemble $\hat{E}$ consisting of the
coarse-grained histories \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \), their probabilities \( p_\alpha \), and the numbers \( N_\alpha \). A more general definition of the entropy \( S \) of histories may help to define these numbers. The generalized Jaynes construction for coarse-grained histories provides one framework for doing this \[4\]. In the most general situation, such a construction defines the entropy \( S \) as the maximum of \(-Tr(\tilde{\rho}\log\tilde{\rho})\) over all density matrices \( \tilde{\rho} \) that preserve the decoherence and probabilities of a given ensemble \( E \) of coarse-grained histories. Other Jaynes-like constructions may also be useful, for example ones that define entropy by proceeding step by step through the histories.

In any case, our augmented entropy in (15) for coarse-grained decohering histories in quantum mechanics is a negative measure of classicality: the smaller the quantity, the closer the set of alternative histories is to a quasiclassical realm. Reducing the first term in (15) favors making the description of the sequences of projections simple in terms of the field variables of the theory and the Hamiltonian \( H \). It favors sets of projections at different times that are related to one another by time translations, as are many sequences of projections on quasiclassical alternatives at different times in the quasiclassical realm.

Reducing the second term favors more nearly deterministic situations in which the spread of probabilities is small. Approximate determinism is, of course, a property of a quasiclassical realm. Reducing the last term corresponds roughly to approaching “maximality”, allowing the finest graining that still permits decoherence and nearly classical behavior. A quasiclassical realm must be maximal in order for it to be a feature exhibited by the quantum state and Hamiltonian and not a matter of choice by an observer.

Any proposed measure of closeness to a quasiclassical realm must be tested by searching for pathological cases of alternative decohering histories that make the quantity small without resembling quasiclassical realm of everyday experience. The worst pathology occurs for a set of histories in which the \( P \)'s at every time are projections on \( |\Psi\rangle \) and on states orthogonal to \( |\Psi\rangle \). We see that in this pathological case the description of the histories and their probabilities is simple because the description of the initial state is simple, so that \( K_{\bar{U}'}(\tilde{E}) \) is small. The term \(-\sum_\alpha p_\alpha \log p_\alpha \) is zero and the third term is also zero since the only \( \alpha \) with \( p_\alpha \neq 0 \) corresponds to projecting onto the pure state \( |\Psi\rangle \), so that \( N_\alpha \) is one and \( \log N_\alpha \) vanishes.

Evidently the smallness of \( \Sigma \) is not by itself a sufficient criterion for characterizing a quasiclassical realm. Further criteria can be introduced if we require that quasiclassical realms be decohering with suitable restrictions on the operators from which the future histories are constructed. Requiring dechering ensures probabilities consistent with the usual rules of probability theory.

The sets must be restricted so as to rule out pathologies such as discussed above. Presumably they must all belong to a huge set with certain straightforward properties. Those properties might be connected with locality, since quantum field theory is perfectly local. (Even superstring theory is local — although the string is an extended object, interaction among strings is always local in spacetime.) It would be in this way that decoherence enters a definition of classicality. (Some some information theoretic measures not unrelated to the one for classicality discussed here can be found in \[7, 12, 14\].)

The general description of the set of histories of a quasiclassical realm is intuitively simple in several respects. They are coarse-grained by a few quasiclassical variables like the hydrodynamic variables discussed in Section \[14\]. Obeying deterministic laws the description of individual histories can be reduced to initial conditions. That is why the measure of classicality introduced in Section \[14\] consists of two parts. An entropy term to favor determinism.
and an AIC term to favor simplicity. Classicality requires both.

III. USING THE MEASURE TO DEFINE ‘QUASICLASSICAL REALM,’

Given the measure of classicality defined in Section II, a quasiclassical could be characterized in quantum mechanics as a realm that minimizes the augmented entropy given by (15) subject to further suitable conditions. Quasiclassical realms so defined would be an emergent feature of $H, |\Psi\rangle$, and quantum mechanics — a feature of the universe independent of human choice. In principle, given $H$ and $|\Psi\rangle$, we could compute the quasiclassical realm that these theories exhibit. We could then investigate the important question of whether our quasiclassical realm is essentially unique or whether the quantum mechanics of the universe exhibits essentially inequivalent other quasiclassical realms. Either conclusion would be of central importance for understanding quantum mechanics.

IV. OUR PLACE IN OUR QUASICLASSICAL REALM

As observers of the universe we are physical systems within it. Both individually and collectively we are information gathering and utilizing systems (IGUSes) described in quasiclassical terms and obeying classical laws. As physical systems we are part of the our classical realm defined coarse-grained histories of hydrodynamic variables. These are integrals of conserved quantities like energy, momentum, and conserved quantities like baryon number over suitably sized volumes. The evidence of our observations of the Universe suggests such a quasiclassical realm extends approximately over patch of classical spacetime the size of a Hubble volume.$^3$ As IGUSes we function by exploiting the regularities such a quasiclassical realm exhibits [15].

V. OTHER QUASICLASSICAL REALMS

It is possible that our Universe exhibits a multiverse of quasiclassical realms[16] —- more than one quasiclassical realm in different locations in space and/or more than one kind of quasiclassical realm. The classical spacetimes inside bubbles of true vacuum nucleated by the decay of a false vacuum provide a simple examples [17].

These different quasiclassical realms could be based on variables different from the hydrodynamic variables defined above. They could have different low energy physics, and different levels of classicality as defined by the measure in Section II.

Could these different quasiclassical realms have different kinds and numbers of IGUSes? Could we communicate with them if they did? Such questions are beyond our power to answer now. But it is possible to imagine that they could be addressed in the future both theoretically and experimentally. The classicality measure developed in this paper would help.

$^3$ By Hubble volume we mean a region of space whose size is roughly the time from the big-bang to the present.
VI. MODELING QUASICLASSICAL REALMS

The measure of classicality defined in Section II could be better understood by calculating its value in simple model closed systems.

One the simplest models is the one-dimensional quantum harmonic chain described in [18]. A closed quantum system consists of a one-dimensional chain of particles of equal mass \( m \) interacting with each other by nearest neighbor harmonic potentials. Fine-grained histories are specified by giving the position of each particle as a function of time.

Another example are the realms defined by minisuperspace models in quantum cosmology eg. [19, 20] in the context of semiclassical quantum gravity. Imagine calculating the measure of classicality assuming that all the histories are homogeneous and isotropic. Does the classicality go down or up if histories with small quantum fluctuations are included in realm? Does classicality go down or up if histories with small fluctuations are included in the realm?

It would be a major research effort to work out either of these models but our understanding of what is classical would be improved thereby.

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