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ABSTRACT
Errors in numerical simulations of gravitating systems can be magnified exponentially over short periods of time. Numerical shadowing provides a way of demonstrating that the dynamics represented by numerical simulations are representative of true dynamics. Using the Sitnikov problem as an example, it is demonstrated that unstable orbits of the three-body problem can be shadowed for long periods of time. In addition, it is shown that the stretching of phase space near escape and capture regions is a cause for the failure of the shadowing refinement procedure.

Key words: methods: numerical – celestial mechanics.

1 INTRODUCTION
The sensitivity which N-body integrations exhibit to small changes in initial conditions and to numerical errors has been an active area of research since Miller’s landmark study. Miller (1964) demonstrated the exponential divergence of nearby orbits for systems with N ≤ 32 and found that the separation of nearby orbits increases rapidly when close binary interactions occur. He suggested that the divergence of nearby orbits is too rapid to be solely accounted by binary interactions and suggested that there must be a collective effect to account for the results. However, Standish (1968) showed that the divergence rate was reduced if the potential was replaced with a softened potential and concluded that the divergence is mainly due to close binary interactions.

The dramatic effects of numerical errors on N-body integrations was also demonstrated in an important paper by Lecar (1968). After coordinating a study with 11 different integrations of the same 25-body problem for 2.5 crossing times, Lecar found that quantities such as half-mass radius and the moment of inertia can change by as much as 100 per cent. In a study with N = 3, Dejonghe & Hut (1986) demonstrated that the amplification of initial errors can increase by as much as 10^20. In addition, they showed that the growth of errors during close encounters can be amplified by as much as 10^4; however, some of the growth can be recovered after the encounter is over.

The sensitivity to small changes in initial conditions and numerical errors is a property associated with chaotic systems. A measure of the sensitivity of numerical errors can be determined by the Lyapunov exponent λ. Earlier work suggested that the Lyapunov exponent is inversely proportional to the crossing time t_c (Heggie 1991; Kandrup & Smith 1991; Goodman, Heggie & Hut 1993). However, Goodman et al. (1993) suggest a dependence on N of the form λ^{-1} = t_c / log N or perhaps λ^{-1} = t_c / log(log(N)), implying that as N increases the rate of separation decreases and the Lyapunov exponent increases. The log(N) dependence was later numerically verified by Hensendorf & Merritt (2002).

Despite the difficulty calculating solutions to N-body integrations, computers still remain a useful tool to study self-gravitating systems. If numerical errors in numerical solutions to the N-body problem cause such drastic changes in the actual positions and velocities of particles how can we trust the dynamics that these solutions represent? Shadowing is a way of proving that a true solution to a dynamical system follows close to a numerical solution. If true orbits can be found close to numerical orbits, then the dynamics represented by the numerical solutions represents true dynamics.

This paper will discuss the existence of shadow orbits for the gravitational three-body problem. First, definitions and concepts related to shadowing of dynamical systems will be introduced. Next, a refinement procedure which makes corrections to numerical orbits to reduce the errors incurred at each time-step will be presented. The Sitnikov problem will then be presented and used as a simple model to discuss escape and capture of orbits. An approximate Poincaré map is then presented to model orbits of the Sitnikov problem and solutions represent? Shadowing is a way of proving that a true solution to a dynamical system follows close to a numerical solution. If true orbits can be found close to numerical orbits, then the dynamics represented by the numerical solutions represents true dynamics.

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2 SHADOWING
Consider the autonomous ordinary differential equation
\[ \dot{x} = f(x), \]
where \( x \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^1 \) vector field with the associated flow represented by \( \varphi_t \). A sequence of points \( \{y_k\}_{k=0}^n \) is

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said to be a pseudo-orbit if there is an associated bounded sequence \( \{u_k\}_{k=0}^M \) of positive time such that
\[
|y_{k+1} - y_k| < \delta \tag{2}
\]
for \( k = 0, 1, \ldots, M \), where \( \delta > 0 \). An example of a pseudo-orbit is a numerical solution to (1). To show that a pseudo-orbit represents some true dynamics for (1), it would be enough to show that a true orbit follows close to the pseudo-orbit. The pseudo-orbit described above is said to be shadowed by a true orbit if there is a sequence of points \( \{x_k\}_{k=0}^M \) and positive times \( \{t_k\}_{k=0}^M \) with \( \delta^k(x_k) = x_{k+1} \) such that
\[
|x_k - y_k| < \epsilon \tag{3}
\]
and
\[
|t_k - h_k| < \epsilon, \tag{4}
\]
for \( k = 0, 1, \ldots, M \) and small \( \epsilon > 0 \). The sequence \( \{x_k\}_{k=0}^M \) is known as a shadow orbit. The shadow orbit is a true solution to (1).

The first general contributions made on shadowing for dynamical systems were the shadowing theorems of Anosov (1967) and Bowen (1972). Anosov and Bowen considered hyperbolic systems and showed that any pseudo-orbit on a hyperbolic invariant set has a shadow orbit. These theorems were generalized for pseudo-orbits in the vicinity of a hyperbolic set (Kato 1991; Nadzieja 1991; Coomes, Kočak & Palmer 1995). For non-hyperbolic systems or for orbits which are far from hyperbolic invariant sets these theorems do not apply. Shadowing theories do exist for pseudo-orbits of non-hyperbolic systems and usually rely on numerical verification of a theorem (Chow, Lin & Palmer 1989; Chow & Palmer 1991; Chow & Van Vleck 1994; Coomes, Kočak & Palmer 1994; Van Vleck 1995).

2.1 Refinement procedure

Shadowing N-body simulations was first demonstrated by Quinlan & Tremaine (1992) and Hayes (2001). Both these studies considered the refinement procedure found in Grebogi et al. (1990) to find numerical shadows for the N-body problem. The refinement procedure is a noise-reduction technique which can be used to show the existence of shadow orbits. This procedure will be presented for two-dimensional dynamical maps; however, the procedure can easily be adapted for flows and has been extended to higher dimensional systems by Quinlan & Tremaine (1992).

Consider the pseudo-orbit \( \{p_k\}_{k=0}^M \) of a map \( f \in \mathbb{R}^2 \). The goal is to find a new less noisy orbit \( \{\hat{p}_k\}_{k=0}^M \) close to the original orbit. Let \( e_k \) represent the one-step error where
\[
e_k = p_k - f(p_{k-1}). \tag{5}
\]
The refined orbit is constructed by
\[
\hat{p}_k = p_k + \Phi_k, \tag{6}
\]
where \( \Phi_k \) is the correction at time-step \( k \). Combine equations (5) and (6) to obtain
\[
\Phi_k = f(\hat{p}_{k-1}) - e_k - f(p_{k-1}), \tag{7}
\]
where \( \hat{p}_k = f(\hat{p}_{k-1}) \). Assuming that the correction, \( \Phi_k \), is small, expand \( f(\hat{p}_{k-1}) \) about \( p_{k-1} \) in a Taylor series to get
\[
f(\hat{p}_{k-1}) \approx f(p_{k-1}) + L_{k-1} \Phi_{k-1}, \tag{8}
\]
where \( L_k \) is the linearized map at the \( k \)th time-step. Substitute (8) into (7) to obtain
\[
\Phi_k \approx L_{k-1} \Phi_{k-1} - e_k \tag{9}
\]
It is also assumed that the linearized map has an expanding direction, \( u_\perp \), and a contracting direction, \( s_\perp \), at each time-step \( k \). With this assumption, the objective is to find the sequences \( \{\Phi_k\}_{k=0}^M \) and \( \{e_k\}_{k=0}^M \) in the coordinates \( \{u_k\}_{k=0}^M \) and \( \{s_k\}_{k=0}^M \) by
\[
\Phi_k = \alpha_k u_k + \beta_k s_k, \tag{10}
\]
and
\[
e_k = \eta_k u_k + \zeta_k s_k. \tag{11}
\]
The expanding and contracting directions follow the linearized maps
\[
u_{k+1} = L_k u_k \tag{12}
\]
and
\[
s_{k+1} = L_k s_k. \tag{13}
\]
For a random \( u_0 = 1 \), equation (12) gives \( u_k \) aligned with unstable direction at \( p_k \) after just a few iterations. Starting with a random \( s_0 \), iterating (13) backwards gives \( s_k \) aligned with the stable direction at \( p_k \) after a few iterations. Substitute (10) and (11) into (9) to get
\[
\alpha_{k+1} u_{k+1} + \beta_{k+1} s_{k+1} = L_k (\alpha_k u_k + \beta_k s_k) + (\eta_{k+1} u_{k+1} + \zeta_{k+1} s_{k+1}). \tag{14}
\]
Substituting (12) and (13) into (14) yields recursive relationships for \( \{\alpha_k\}_{k=0}^N \) and \( \{\beta_k\}_{k=0}^N \) where
\[
\alpha_{k+1} = |L_k u_k| \alpha_k - \eta_{k+1}, \tag{15}
\]
\[
\beta_{k+1} = |L_k s_k| \beta_k - \zeta_{k+1}.
\]
Equations (15) are made computationally stable by calculating the coefficients \( \alpha_k \) starting with \( \alpha_M \) and iterating backwards and the coefficients \( \beta_k \) are calculated by choosing an initial \( \beta_0 \) and iterating forwards. The choice of \( \alpha_M \) and \( \beta_0 \) are arbitrary and are taken to be \( \alpha_M = \beta_0 = 0 \). Thus the sequence of correction coefficients is given by
\[
\alpha_M = 0, \quad \alpha_k = (\alpha_{k+1} + \eta_{k+1}) / |L_k u_k|, \tag{16}
\]
\[
\beta_0 = 0, \quad \beta_k = |L_k s_k| \beta_k - \zeta_{k+1},
\]
where the values of \( \eta_k \) and \( \zeta_k \) can be determined directly from (5) and (11).

Once \( \{\hat{p}_k\}_{k=0}^M \) has been found, the refinement procedure can be iterated. Generally, the number of significant digits doubles on each iteration of the process. However, cases have been found where the convergence is much slower or does not converge.

The convergence of the refinement procedure does not in itself show the existence of a shadow orbit. Grebogi et al. (1990) provide a containment procedure in two dimensions which rigorously proves the existence of a shadow orbit. The containment technique was later extended to three-dimensional systems by Hayes (2001). A more practical approach for higher dimensional systems was developed by Sauer & Yorke (1991). They showed that for a given pseudo-orbit, if the refinement procedure converges – to machine precision – and certain quantities of a theorem remain bounded, then the pseudo-orbit has a shadow orbit.

It has been found (Quinlan & Tremaine 1992; Hayes 2003) that one can tell from the convergence of the refinement procedure alone whether a given pseudo-orbit can be shadowed. So, if iterations of the refinement procedure converge to a new orbit where the one-step errors are the size of machine precision, then it is inferred that a shadow orbit exists for the given pseudo-orbit. The new orbit found by the refinement procedure is called a numerical shadow.
3 THE SITNIKOV PROBLEM

In this paper of shadowing for the three-body problem, a special configuration of the restricted three-body problem known as the Sitnikov problem will be considered. The Sitnikov problem is the problem of the motion of a massless particle, $m_3$, on the axis of symmetry of an equal-mass binary (Fig. 1). Following Moser (1973), units are chosen such that the gravitational constant $G = 1$ and the total mass $M = 1$. Under these conditions, the equation of motion for $m_3$ is given by

$$\dot{z} = - \frac{z}{\sqrt{z^2 + r^2}},$$  \hspace{1cm} (17)

where $z$ is the position of $m_3$ and $r$ the distance from the centre of mass to one of the binary masses. The distance $r$ can be approximated to first order in the eccentricity, $e$, by

$$r \approx \frac{1}{2} [1 - e \cos(t)],$$  \hspace{1cm} (18)

and the specific energy of $m_3$ can be defined by

$$E = \frac{1}{2} |z|^2 - \frac{1}{\sqrt{r^2 + z^2}}.$$  \hspace{1cm} (19)

Taking the plane of motion of the binary ($z = 0$) as a surface of section (SOS), consider a map $\phi: (t_0, E_0) \rightarrow (t_1, E_1)$, which takes $m_3$ from one crossing of the SOS to the next. If $m_3$ is on the SOS at time $t_0$, $\phi$ is a map which brings $v_0 = z(t_0)$ to time $t_1 > t_0$ where $v_1 = z(t_1)$ and $z(t_1) = 0$. The map $\phi$ has an open domain $D_0$ in which every point returns to the SOS. As time enters into the problem with period $2\pi$, $D_0$ can be considered in polar coordinates where the radial variable is $v$ and the angular variable is given by $\phi$. Alternatively, the domain $D_0$ can be considered on the surface of a cylinder where the initial position on the cylinder is defined by $t_0$ and $E_0$. Fig. 2(a) shows the domain for $\phi$ in cylindrical coordinates. The colour of each point represents the number of periods of the binary before escape happens. The green regions represent islands of quasi-periodic motion. In Fig. 3 an example of a quasi-periodic orbit which visits the islands of stability in the vicinity of a period 7 orbit is provided.

3.1 An approximate Poincaré map

Urminsky & Heggie (2009) demonstrated that the Poincaré map $\phi$ with (18) can be approximated by a simplectic map $\psi: (t_0, E_0) \rightarrow (t_1, E_1)$ where

$$E_{1/2} = E_0 + a \cos(t_0) + b \sin(t_0),$$
$$t_{1/2} = t_0 + 2C(-E_{1/2})^{3/2},$$
$$t_1 = t_{1/2} + 2C(-E_{1/2})^{3/2},$$
$$E_1 = E_{1/2} - a \cos(t_1) + b \sin(t_1),$$  \hspace{1cm} (20)

and $a, b$ and $C$ are constants. The quantities $t_{1/2}$ and $E_{1/2}$ are approximations of the time and energy values of $m_3$ at a local maximum distance from the SOS. It is clear (Fig. 2a) that the change in energy of $m_3$ from one crossing to the next is periodic in time and
the trigonometric terms in (20) can be though of as the lowest order in a Fourier approximation to this change. The time in time is obtained by approximating the motion of \( m_1 \) as Keplerian. The constants \( a \) and \( b \) are proportional to the eccentricity of the binary whose values can be shown to be

\[
\begin{align*}
    a &\approx 0.149 e \\
    b &\approx 0.5075 e
\end{align*}
\]

and the constant \( C = \pi/(2\sqrt{2}) \).

### 3.2 Escape and capture

Through interactions with the binary as it crosses the SOS, \( m_1 \) can gain sufficient energy such that it leaves the SOS and does not return. It can be shown that for some positive time \( t^* \) and positive \( \nu = (1 - e)/2 \), if

\[
\frac{1}{2} \dot{z}(t^*)^2 - \frac{1}{z(t^*)^2 + \nu} > 0,
\]

then \( |z(t)| \to 0 \) as \( t \to \infty \). Setting \( z = 0 \) in (22) gives a lower bound on the velocity of orbits which escape on the SOS. The solid black region at the top of Fig. 2(a) demonstrates how this condition overestimates the escape boundary. All energy and time values in this region do not return to the SOS.

The map, \( \varphi \), provides an accurate way of determining escape and capture. From equation (20) it is found that the mapping \( \varphi \) is defined in a region

\[
E_0 < -a \cos(t_0) - b \sin(t_0) := \partial D_0
\]

for

\[
t_0 \in [0, 2\pi].
\]

as time enters into the mapping with period \( 2\pi \). The curve \( \partial D_0 \) is the escape boundary. Time and energy values above \( \partial D_0 \) are said to have escaped. The domain, \( D_0 \), can be defined by (23) and (24). Initial conditions in \( D_0 \) are mapped into the region \( D_1 \), defined by

\[
E < -a \cos(t) + b \sin(t) := \partial D_1,
\]

for \( t \in [0, 2\pi] \). Fig. 4 shows how the boundaries \( \partial D_0 \) and \( \partial D_1 \) intersect. Orbits are mapped from the region under the curve \( \partial D_0 \) to the region under the curve \( \partial D_1 \). The region \( B_0 = D_0 \setminus D_1 \) represents energy and time values for which orbits are captured. In the context of the differential equation, these are orbits which come from infinity and get captured by the binary. Similarly, the initial conditions in the region \( B_1 = D_1 \setminus D_0 \) are energy and time values for which \( \varphi \) is undefined. Again, in the context of the differential equation, the region \( B_1 \) represents orbits which escape from the system. Finally, note that initial conditions for the differential equation are such that \( z > 0 \) and \( z = 0 \) on the SOS. So from (19), the initial energy can be bounded from below by

\[
E > -1/|r(t_0)|,
\]

for \( t_0 \in [0, 2\pi] \). (26)

Initial conditions are chosen in \( D_0 \) with (26) for \( e = 0.61 \) and plotted in Fig. 2(b). The colour of each point represents the number of periods of the binary determined by \( t_M/2\pi \) where \( M \) is the number of iterations of the map \( \varphi \). The green regions represent stable motion whose orbits remain bounded. As energy increases orbits become unstable and escape from the system. Note the similarities between Figs 2(a) and (b). Both domains have islands representing stable orbits as well as large regions representing unstable orbits. In Fig. 5 an example of a quasi-periodic orbit near a period 7 orbit is provided. In addition to the similarities between Figs 2(a) and (b), Urminsky (2009) demonstrates that the map \( \varphi \) satisfies Lemmas similar to Lemmas 1–5 in (Moser 1973, p. 87) and that \( \varphi \), like \( \phi \), possesses

![Figure 3](https://example.com/figure3.png)

**Figure 3.** An example of a quasi-periodic orbit near a period 7 orbit for equation (17) on the SOS \( z = 0 \). Initial conditions are \( z(0) = 1.3, \dot{z}(0) = 0.0, \ e = 0.61 \) and the phase of the binary is 0.45 radians from pericentre.

![Figure 4](https://example.com/figure4.png)

**Figure 4.** The curve \( \partial D_0 \) represents a lower bound of energy and time values for which \( \varphi \) is undefined. Similarly, the curve \( \partial D_1 \) represents a lower bound of energy and time values for which the inverse map \( \varphi^{-1} \) is undefined. The shaded region is the domain \( D_0 \) for the map \( \varphi \). The two regions labelled \( B_0 \) and \( B_1 \) bounded by the curves \( \partial D_0 \) and \( \partial D_1 \) are the capture and escape regions, respectively.

![Figure 5](https://example.com/figure5.png)

**Figure 5.** An example of a quasi-periodic orbit near a period 7 orbit for the map \( \varphi \). Initial conditions are \( t_0 = 6.018 \ 22 \) and \( E_0 = -2.5297 \) for \( e = 0.61 \).
a hyperbolic invariant set on which \( \varphi \) is topologically equivalent to the shift map.

### 4 RESULTS

The map \( \varphi \) provides a simple way of studying shadowing for orbits like those of the Sitnikov problem. The approximate map is used to avoid integrating between successive crossings of the SOS thus obtaining a tremendous speed up in calculations. In addition, the one-step error can more easily be controlled. At each time-step uniformly distributed noise \( |\delta| \leq \delta \) is added to generate the pseudo-orbit. The refinement procedure is then used to reduce the noise level to machine precision. Since \( \varphi \) is a two-dimensional mapping, the refinement procedure can be directly applied as shown in Section 2.1.

#### 4.1 Long-lived orbits

Using the containment and refinement procedure, Grebogi et al. (1990) successfully demonstrated the existence of shadows for pseudo-orbits of length \( 10^5 \) or more. To test the algorithm the refinement procedure is applied to long-lived orbits of the map \( \varphi \). As seen in Fig. 2(b), there are regions of stable motion where orbits remain bounded forever. The refinement procedure is applied to these orbits and it is found that most can be shadowed for many iterations. Some of these are shown in Fig. 9.

As shown by Dvorak, Contopoulos & Efthymiopoulos (1998) for the Sitnikov problem, the map \( \varphi \) has ‘sticky’ regions where orbits can be trapped for long periods of time before escape. In Fig. 6 an example of a sticky orbit trapped in the vicinity of islands of stable quasi-periodic orbits is shown. The inset plot in Fig. 6 is a magnification of the orbit near one of the islands. By sampling the phase space around the islands of stable motion, one can find many sticky orbits which survive for long periods of time. In Fig. 7 the shadow distance is plotted against the number of iterations of the map for several sticky orbits where \( \epsilon = 0.61 \). It is shown that as the number of iterations increases, the distance of the numerical shadow from the pseudo-orbit increases proportionally to the number of iterations.

#### 4.2 Shadowing capture orbits

Consider uniformly distributed initial values in \( B_0 \) (Fig. 4) for \( \epsilon = 0.25 \). Initial values are iterated forward for a maximum of 100000 iterations up to the penultimate iteration before escaping. For each orbit, the number of iterations, \( M \), the orbit was ‘shadow-able’ for as well as the shadow distance are recorded. The orbits are binned into bins of length one iteration and averaged over the bin. The results are plotted in Fig. 8 where the dots represent the average shadow distance at each iteration of the map. Note that as \( M \) increases, there is increasing variability on the distribution of average shadow distances. The data can be fitted with the curve \( 7 \times 10^{-9} M \) which is similar to the results in Fig. 7 where the shadow distance is proportional to the orbit length.

### 5 WHERE DOES SHADOWING FAIL?

Numerical shadows have been found using the refinement procedure for orbits whose length exceeds \( 10^5 \) iterations for the map \( \varphi \). However, what happens when numerical shadows are not found? What causes the refinement procedure to fail? First, it should be noted that the failure of the refinement algorithm to converge to a numerical shadow does not imply that there is not a shadow orbit for a given pseudo-orbit. A shadow may still exist but the refinement procedure was not able to converge towards it. Quinlan & Tremaine (1992) and Hayes (2003) found that shadowing breaks down during close encounters between particles. This is due to the stretching of the velocity subspace during a close encounter. In the Sitnikov problem, close encounters between particles can lead to a dramatic change in the velocity of the orbit, which can cause the refinement procedure to fail.
problem, \( m_3 \) interacts with the binary on the SOS and the distance separating \( m_3 \) with the binary masses is bounded from below (and above) on the SOS. In contrast to the problems discussed in the above-mentioned studies arbitrarily close encounters do not occur in the Sitnikov problem. However, escape and capture occur during close encounters with the binary as \( m_3 \) crosses the SOS. Near the escape and capture boundaries slight changes in the energy of \( m_3 \) as it crosses the SOS can lead to significant changes in the duration of successive crossings of the SOS. The map provides a simple way of sampling the phase space on the SOS to find regions where shadowing is more likely to fail.

Fig. 9 shows shadowing results of \( 10^6 \) initial conditions. The colour of each point represents either success (yellow) or failure (black) of the refinement procedure. Note that only orbits which survived more than three iterations of the map are considered. This is because the choice of \( n_0 \) and \( s_k \) would influence the results for short-lived orbits as (12) and (13) may not have had enough time to align \( u_k \) and \( s_k \) in the proper directions. From Fig. 9 it can be seen that the refinement procedure tends to fail near the escape boundary \( \partial D_0 \). Note also that the refinement procedure fails near the boundaries of regions containing orbits which escape after three or less iterations.

The reason the refinement procedure fails in these regions is that there is a stretching of subspace as orbits near the boundary \( \partial D_0 \). At a given iteration \( k \), the distance from boundary, \( \partial D_0 \), is given by

\[
d = |E_k + a \cos(t_k) + b \sin(t_k)|.
\]

From the Jacobian of (20) it can be shown that

\[
|L_{u_k}| \sim d^{-5/2}.
\]

Thus, as \( d \to 0 \), the correction coefficients \( a \) and \( \beta \) go to 0 and \( \infty \), respectively, making it more difficult for the refinement procedure to converge.

Fig. 10 shows the density of successfully shadowed orbits based on the closest approach to the boundary \( \partial D_0 \) for increasing eccentricity values. For each shown eccentricity value, we select 100,000 uniformly distributed initial conditions in the region defined by \( t_0 \in (\pi, 2\pi) \) and \( E_0 \in (\partial D_0 + 2b \sin(t_0), \partial D_0) \). These boundaries describe a band of initial conditions bounded above by the escape boundary. This band also encompasses the capture region \( B_0 \). The drop in the density to the right-hand side of each curve occurs at the distance between the lower boundary curve and the escape boundary. Note that the density drops off as initial conditions approach the escape boundary. Data were fitted using a variable bandwidth kernel density function.

5.1 Probability of capture

It was found above that as orbits approach the escape boundary the likelihood of an orbit being shadowed decreases. This has an impact on the shadow-ability of orbits in the capture region \( B_0 \). The capture region area is directly proportional to the eccentricity of the binary. As \( e \to 0 \), the initial conditions in \( B_0 \) become pushed up against the boundary \( \partial D_0 \). It is expected then that for small eccentricities, orbits would be less likely to be shadow-able.

To test this hypothesis, \( 10^5 \) uniformly distributed initial conditions are selected in \( B_0 \) and iterated forwards until each orbit escapes. This is performed for a variety of eccentricity values and the fraction of shadow-able orbits in each case is determined. The results are shown in Fig. 11. The fraction of shadow-able orbits increases as the eccentricity of the binary increases. This is because the area of the capture region increases proportionally to \( e \). As the area increases, initial conditions can be selected at a much further distance from the escape boundary making them more likely shadow-able.
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The results found here are in agreement, for shadow durations $T < 5000$ the density can be approximated by an exponential distribution. For larger $T$ the density is inversely proportional to $T$.

5.2 Failure as a stochastic process

The failure of the refinement procedure can happen at any point along the orbit and not necessarily at a close approach to the escape boundary. The shadow duration is defined as the number of iterations for which a given orbit can be shadowed. For an orbit $\{(t_i, E_i)\}_{i=0}^m$ the shadow duration, $T$, can take on positive integer values $T < M$.

Consider the initial conditions for $e = 0.61$ shown in Fig. 9. For each resulting orbit, it is determined how long the orbit is shadow-able. Fig. 12 provides some information on the distribution ofshadow lengths. Initial conditions are chosen in $D_0$ and iterated forwards in time using (20). Each orbit is iterated for 50 000 iterations or until the solution escapes. The solid line in Fig. 12 represents the density of the numerical experiments. The spike at 50 000 iterations is mostly due to quasi-periodic orbits which remain bounded for all time. As shown in Fig. 12, the density can be approximated, for small iterations, by an exponential density function given by $\xi_0$ for $\xi = 0.0019$. The inset graph is a magnification of the density for $1000 < M < 50 000$. In this range, the density function is better represented by the function $0.025/T$.

The map approximates the time between crossings of the SOS by considering the motion of $m_3$ to be Keplerian. Instead of considering the distribution of the shadow duration in terms of the number of iterations of $\varphi$ we can instead consider the distribution of shadow times, $t_{sh}$ where $M$ is the number of iterations of the orbits for which it was shadow-able. The solid line in Fig. 13 represents the probability density of shadow time for the numerical experiments. Again, the data can best be approximated by the exponential density function for $\xi = 0.0005$. The results found here are in agreement, for small shadow durations, with previous results by Hayes (2003) which showed that shadow durations for larger N-body systems have an exponential distribution and can be thought of as a Poisson process.

6 CONCLUSIONS

The above results confirm, for short-lived orbits, previous investigations (Hayes 2003) that showed numerical shadow durations, $M$, for gravitating systems follow a Poisson process with an exponential density function. The result found in this paper suggests for longer lived orbits that the density function is better approximated by a function proportional to $1/M$. This may be because the population of longer lived orbits tends to be dominated by stable orbits; however, this has not been investigated.

In Section 5, areas of phase space where the refinement procedure is more likely to fail are characterized. These areas are near escape boundaries where there is sufficient stretching of phase space to cause the refinement procedure to fail to converge to a less noisy orbit. Interestingly this seems to be due to the growth of the variational equations over one time-step. This does not rule out the failure of the refinement procedure by the accumulative effect of the growth of the variational equation associated with large Lyapunov exponents as discussed by Zhu & Hayes (2009).

In Fig. 11 it is demonstrated that as the volume of phase space representing capture orbits decreases, it becomes increasingly difficult to shadow capture orbits. This is a result of the distribution of failures of the refinement procedure seen in Fig. 10. As the volume of phase space associated with capture decreases, capture orbits get pushed up against the boundary $\partial D_0$ where the one-step growth of the variational equations causes the refinement procedure to fail.

Finally, it was found that the shadow distance for an orbit is proportional to the number of iterations of the map (Figs 7 and 8). It was noted that if in addition to $t_1$ and $E_1$, orbits were required to be shadow-able at the half-steps $t_{1/2}$ and $E_{1/2}$, then initially shadow-able orbits continued to be shadow-able. When shadowing at the half-step was required, the shadow distance typically increased by about a factor of 2.

The Sitnikov problem discussed in this paper provides a straightforward way of characterizing a domain of initial conditions as well as regions of stable and unstable motion. Work in progress considers slight changes to the Sitnikov problem in order to study shadowing of unstable orbits. For example, Soulis, Bountis & Dvorak (2007) consider slight perturbations to the mass and position (away from the z-axis) of $m_3$ and delineate regions of stable and unstable motion. It would be expected that, like the results found in this paper, shadowing with the refinement procedure breaks down near boundaries of escape for unstable orbits. In fact, the breakdown of the refinement procedure near escape boundaries would be expected for general three-body configurations. As solutions approach parabolic escape boundaries, an orbit can undergo increasingly long ejections from the leftover binary system. Small changes in the energy of an orbit in this region can cause significant changes in the time of return for the...
orbit. If the refinement procedure could make changes to the orbits so as to conserve the energy of the ejected body it might improve the success rate of the refinement procedure. Finally, the Sitnikov four-body problem (Soulis, Papadakis & Bountis 2008) provides a starting point for examining the relationship between the shadowing distance and the number of bodies. Extra bodies can be added in circular orbits about the centre of mass. Hayes (2003) demonstrates that as the number of moving bodies in a fixed potential increases the shadow durations decrease. It would be of interest to determine if a similar relationship holds for the Sitnikov $N$-body problem.

It should be stressed again that the failure of the refinement procedure does not necessarily mean that a shadow does not exist for a given pseudo-orbit. It may very well be that shadows do exist for orbits in regions where the refinement procedure fails. We are encouraged that this may be the case. Both the Sitnikov problem and the approximate Poincaré map possess a hyperbolic invariant set, $\Lambda$, near the escape boundaries (see Moser 1973 and Urminsky 2009, respectively). Despite the fact that $\Lambda$ is near the boundary $\partial D_0$, the shadowing theorems by Anosov (1967) and Bowen (1972) guarantee that any pseudo-orbit on $\Lambda$ has an associated shadow orbit. This demonstrates that being in the vicinity on the escape boundary does not necessarily rule out the existence of shadow orbits.

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