Topological invariants of three-manifolds from $U_q(osp(1|2n))$

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Abstract

We create Reshetikhin-Turaev topological invariants of closed orientable three-manifolds from the quantum supergroup $U_q(osp(1|2n))$ at certain even roots of unity. To construct the invariants we develop tensor product theorems for finite dimensional modules of $U_q(osp(1|2n))$ at roots of unity.

1 Introduction

Topological invariants of closed orientable three-manifolds may be constructed from modular or quasimodular Hopf algebras [1, 2]. Reshetikhin and Turaev’s construction using modular Hopf algebras relies upon several theorems relating framed links in $S^3$ to closed orientable three-manifolds. The Lickorish-Wallace theorem states that each framed link in $S^3$ determines a closed, orientable 3-manifold and that every such 3-manifold is obtainable by performing surgery upon a framed link in $S^3$. Kirby-Craggs, and Fenn and Rourke showed that homeomorphism classes of closed orientable three-manifolds may be generated by performing surgery upon elements of equivalence classes of framed links in $S^3$, where the equivalence relations are generated by the Kirby moves. By taking such combinations of isotopy invariants of links in $S^3$ as to render them unchanged under the Kirby moves one obtains a topological invariant of 3-manifolds.

RT took invariants of isotopy derived from the quantum group $U_q(sl_2)$ at even roots of unity. Their method was adapted for the quantum algebras related to the $A_n$, $B_n$, $C_n$, $D_n$ Lie algebras at all roots of unity [3, 4], and the exceptional quantum algebras and quantum superalgebras $U_q(osp(1|2))$ and $U_q(gl(2|1))$ at odd roots of unity [5]. Here we create invariants from $U_q(osp(1|2n))$ at $q = \exp(2\pi i/N)$ with $N = 2(2k + 1)$.

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Reshetikhin-Turaev invariants can be constructed from a class of Hopf (super)algebras more general than (quasi)modular Hopf (super)algebras. For a Hopf (super)algebra \( A \), invariants may be constructed if \( A \) has the following properties (where we take the quantum superdimension and quantum supertrace if \( A \) is a quantum superalgebra)

i) there exists a finite collection of mutually non-isomorphic left \( A \)-modules \( \{V_\lambda\} \) where \( \lambda \) ranges over some index set \( I \) such that \( \dim(V_\lambda) < \infty \) and \( \dim_q(V_\lambda) \neq 0 \), \( \forall \lambda \in I \),

ii) for any finite collection of \( A \)-modules \( V_{\lambda_1}, V_{\lambda_2}, \ldots, V_{\lambda_n} \) such that \( \lambda_i \in I \) for all \( i \),

\[
V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_n} = \mathcal{V} \oplus \mathcal{Z}
\]

where \( \mathcal{V} = \bigoplus_{\lambda \in I} (V_\lambda)^{\oplus m(\lambda)} \), \( m(\lambda) \geq 0 \) is the multiplicity of \( V_\lambda \) in the direct sum, and \( \mathcal{Z} \) is a possibly empty \( A \)-module with zero quantum dimension,

iii) for each \( V_\lambda \) there is a dual module \( (V_\lambda)^! \cong (V_\lambda)^* \) such that \( \lambda^* \in I \) and \( \mathcal{V} \) there exists a distinguished module \( V_0 \) such that \( (V_0)^! \cong V_0 \),

iv) the central element \( \delta = v - \sum_{\lambda \in I} d_\lambda \chi_\lambda(v^{-1}) C_\lambda \) vanishes upon acting on any \( V_\mu \) where \( \mu \in \Lambda^+_N \). Here \( \chi_\lambda(v) = q^{-\langle \lambda, S + 2\rho \rangle} \) and \( C_\lambda = \text{tr}_\lambda([\text{id} \otimes \pi](1 \otimes q^{2h_\nu}) R^T R) \). \( R \) is the universal \( R \)-matrix and \( R^T = P.R.P \) where \( P \) is the permutation operator. \( \{d_\lambda\} \) is a collection of complex valued constants such that at least one \( d_\lambda \) is non-zero. This condition ensures that combinations of isotopy invariants of links are unchanged under some of the Kirby moves, and

v) the sum \( z = \sum_{\lambda \in I} d_\lambda q^{-\langle \lambda, \lambda + 2\rho \rangle} \dim_q(V_\lambda) \) is non-zero.

### 2 \( U_q(osp(1|2n)) \) at roots of unity and its finite dimensional modules

The quantum superalgebra \( U_q(osp(1|2n)) \) at roots of unity is not quasi-triangular. However, the quantum superalgebra now has a class of central elements which do not exist at generic \( q \). These central elements generate an ideal of \( U_q(osp(1|2n)) \), which is also a two-sided co-ideal.

The quotient of \( U_q(osp(1|2n)) \) by this ideal turns out to be a quasi-triangular Hopf superalgebra. Let us denote this algebra by \( U_q^{(N)}(osp(1|2n)) \), the details of which may be found in [1].

Associated with \( osp(1|2n) \) there is a euclidean space \( H^* \) which has a basis of vectors \( \{\epsilon_i | 1 \leq i \leq n\} \) and a bilinear form such that \( (\epsilon_i, \epsilon_j) = \delta_{i,j} \). The even (resp. odd) positive roots are \( \Phi^+_\epsilon = \{\epsilon_i \pm \epsilon_j, 2\epsilon_k | 1 \leq i < j \leq n, 1 \leq k \leq n\} \) (resp. \( \Phi^+_\pi = \{\epsilon_i | 1 \leq i \leq n\} \)). Define \( 2\rho = \sum_{\alpha \in \Phi^+_\epsilon} \alpha - \sum_{\beta \in \Phi^+_\pi} \beta \), \( \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\} \), \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \) and \( \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\} \). Let \( X \subseteq H^* = \sum_{i=1}^n \mathbb{Z}_+ \epsilon_i \) and \( X_N = X/NX \).

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Define $\phi_0 = \{ \delta \in \Phi^+_0 | \delta/2 \notin \Phi^+_0 \}$ and $\phi_1 = \Phi^+_1$. For a given $U_q^{(N)}(osp(1|2n))$ where $n \geq 2$ and $N \geq 3$ we define $\Lambda^+_N$ by

$$\Lambda^+_N = \left\{ \lambda \in X \mid 0 \leq \frac{2(\lambda + \rho, \alpha)}{(\alpha, \alpha)} \leq N', \forall \alpha \in \phi \right\}$$

where $\phi = \phi_0 \cup \phi_1$ if $N \equiv 2 \pmod{4}$ and $\phi = \phi_0$ otherwise. Here $N'' = N$ if $N$ is odd and $N'' = N' = N/2$ otherwise. For $U_q^{(N)}(osp(1|2))$ where $N \geq 3$ we define $\Lambda^+_N$ by

$$\Lambda^+_N = \left\{ \lambda \in X \mid 0 \leq (\lambda + \rho, \alpha) \leq N'' \right\}$$

where $\alpha$ is the single odd root and $N'' = N/4$ if $N \equiv 2 \pmod{4}$ and $N'' = N'$ otherwise. For each $\Lambda^+_N$ we define $\Lambda^+_N$ identically except that we replace $\leq$ by $<$. $\Lambda^+_N$ plays the role of the index set $I$ of the collection of $U_q^{(N)}(osp(1|2n))$ modules.

Let $V$ be the fundamental module of $U_q^{(N)}(osp(1|2n))$ with highest weight $\epsilon_1$. $V$ is irreducible, $2n + 1$ dimensional and has the same structure as the fundamental module of $U_q(osp(1|2n))$. In [3] we prove the following lemmas and theorems.

**Lemma 2.1** Set $N \geq 4$ to be even. For each $\mu \in \Lambda^+_N$ there exists a finite dimensional $U_q^{(N)}(osp(1|2n))$ module $V_\mu$ with highest weight $\mu$ such that $V_\mu \subseteq V^{\otimes t}$ for some $t \geq 1$. The quantum superdimension of $V_\mu$ is $sdim_q(V_\mu) = \lim_{q \to 1}(sdim_q(V^{\gen}))$ where $V^{\gen}$ is the finite dimensional irreducible $U_q(osp(1|2n))$ module with highest weight $\mu$. The $sdim_q(V_\mu) = 0$ for all $\mu \in \Lambda^+_N$ and $sdim_q(V_\mu) = 0$ for all $\mu \in \Lambda^+_N \setminus \Lambda^+_N$.

Let $W$ be the Weyl group of $osp(1|2n)$ and $\tau$ be the maximal element of $W$. Then $-\tau(\lambda) \in \Lambda^+_N$, $\forall \lambda \in \Lambda^+_N$, which implies that for each $\lambda$ there is a dual module $(V_\lambda)^+$ with highest weight in $\Lambda^+_N$.

In theorems 2.1 and 2.2 we assume that $n \geq 2$ or that $n = 1$ and $N \neq 4, 8$.

**Theorem 2.1** Set $N \geq 4$ to be even, $V$ to be the fundamental module of $U_q^{(N)}(osp(1|2n))$ and $\epsilon_1 \in \Lambda^+_N$. Then for every $t \in \mathbb{Z}_+$,

$$V^{\otimes t} = V \oplus Z$$

where $V = \bigoplus_{\lambda \in \Lambda^+_N} (V_\lambda)^{\otimes m(\lambda)}$, $m(\lambda) \in \mathbb{Z}_+$ and $Z$ is a possibly empty direct sum of indecomposable modules, each with zero quantum superdimension.

**Theorem 2.2** Set $N \geq 4$ to be even and let $V_\lambda$ be finite dimensional $U_q^{(N)}(osp(1|2n))$ modules with $\lambda_i \in \Lambda^+_N$ for all $i$. Then

$$V_\lambda \otimes V_\lambda \otimes \cdots \otimes V_\lambda = V' \oplus Z'$$

where $s \geq 1$ and $V'$ and $Z'$ have the same form as in theorem 2.1.

**Lemma 2.2** Set $N \geq 3$ to be odd, $\epsilon_1 \in \Lambda^+_N$ and $t \in \mathbb{Z}_{N/2+1/2-n}$. Then $V^{\otimes t} = \overline{V}$ where $\overline{V}$ has the same form as $V$ does in theorem 2.1.
3 Finding the set \( \{d_\lambda\} \)

We know that properties i), ii) and iii) hold for \( U_q^{(N)}(osp(1|2n)) \) at even \( N \). We now consider condition iv): there exists at least one collection of \( \{d_\lambda\} \) that solves \( \chi_\mu(v) = \sum_{\lambda \in \Lambda_N^+} d_\lambda \chi_\mu(v^{-1}) \chi_\mu(C_\lambda) \) for all \( \mu \in \Lambda_N^+ \). Now at generic \( g \), the eigenvalue of \( C_\lambda \) in an irreducible representation with highest weight \( \mu \in \Lambda_N^+ \) is given by \( sc\lambda(q^{2(\mu+\rho)}) = S_{\lambda,\mu}/Q_\mu \) where \( S_{\lambda,\mu} = (-1)^{\vert \lambda \vert} \sum_{\sigma \in W} \epsilon(\sigma) q^{2(\lambda+\rho,\sigma(\mu+\rho))} \), \( Q_\mu = \sum_{\sigma \in W} \epsilon'(\sigma) q^{2(\rho,\sigma(\mu+\rho))} \), and where \( \epsilon'(\sigma) = -1 \) if the number of components of \( \sigma \) that are reflections with respect to the elements of \( \Phi_0 \) is odd and \( \epsilon'(\sigma) = 1 \) otherwise. Our proofs for theorems 2.1 and 2.2 tell us that \( sc\lambda(q^{2(\mu+\rho)}) \) is well behaved when \( g \) is taken to the \( N^{th} \) root of unity, and yields the desired \( \chi_\mu(C_\lambda) \). To simplify finding the \( \{d_\lambda\} \) we initially consider \( Q_\mu q^{-(\mu+2\rho,\mu)} = \sum_{\lambda \in \Lambda_N^+} d_\lambda' q^{2(\lambda+\rho,\lambda)} S_{\lambda,\mu}' \) where \( d_\lambda' = (-1)^{\vert \lambda \vert} d_\lambda \) and \( S_{\lambda,\mu}' = (-1)^{\vert \lambda \vert} S_{\lambda,\mu} \). To solve for the \( \{d_\lambda'\} \) we consider

\[
Q_\mu q^{-(\mu+2\rho,\mu)} = \sum_{\lambda \in X_N} x_\lambda q^{2(\lambda+2\rho,\lambda)} S_{\lambda,\mu}',
\]

and set \( x_\lambda = cq^{-(\lambda,2\rho)} \). We then have

\[
Q_\mu q^{-(\mu+2\rho,\mu)} = \sum_{\sigma \in W} \epsilon(\sigma) \sum_{\lambda \in X_N} cq^{2(\lambda,\lambda)} q^{2(\lambda+\rho,\sigma(\mu+\rho))}.
\]

To ensure \( x_\lambda \) is independent of \( \mu \) we undertake the mapping \( \lambda \to \sigma(\lambda + \rho) - \sigma(\mu + \rho) \) in the summation, which may be done as \( \sigma(\lambda + \rho) - \sigma(\mu + \rho) \in X_N \) and the summation remains over \( X_N \). We then obtain

\[
Q_\mu q^{-(\mu+2\rho,\mu)} = \sum_{\lambda \in X_N} cq^{2(\lambda,\lambda+2\rho)} \sum_{\sigma \in W} \epsilon'(\sigma) q^{2(\rho,\sigma(\mu+\rho))}
\]

which results in \( c^{-1} = \sum_{\lambda \in X_N} q^{2(\lambda,\lambda+2\rho)} \) and

\[
x_\lambda = q^{-(\lambda,2\rho)} / \sum_{\lambda \in X_N} q^{2(\lambda,\lambda+2\rho)}.
\]

Now \( q^{2(\lambda',\lambda'+2\rho)} = (-1)^p q^{2(\lambda,\lambda+2\rho)} \) where \( \lambda' = \lambda + N/2\epsilon_i \) for any \( \epsilon_i \) and \( p = 1 \) if \( N = 4k \), \( p = 0 \) if \( N = 2(2k+1) \). Then \( c \) is not well defined if \( N = 4k \) and we set \( N = 2(2k+1) \) for the remainder of this paper. Now \( x_\lambda q^{2(\lambda',\lambda'+2\rho)} S_{\lambda',\mu}' = x_\lambda q^{2(\lambda,\lambda+2\rho)} S_{\lambda,\mu}' \) where \( \lambda' = \lambda + N/2\epsilon_i \) for any \( \epsilon_i \). It then follows that \( \sum_{\lambda \in X_N} x_\lambda q^{2(\lambda,\lambda+2\rho)} S_{\lambda,\mu}' \) and

\[
\sum_{\lambda \in X_N} x_\lambda q^{2(\lambda+2\rho,\lambda)} S_{\lambda,\mu}' = 2^n \sum_{\lambda \in X_N} x_\lambda q^{2(\lambda+2\rho,\lambda)} S_{\lambda,\mu}'.
\]

Let \( \overline{N} = N/2 \) and \( \overline{\Lambda}_N^+ \) be the fundamental domain for \( X_{N/2} \) under the action of the affine Weyl group \( W_{\overline{N}} \) of \( \overline{U}_q^{(N)}(so(2n+1)) \). The affine Weyl groups \( W_{N/2} \) of \( U_q^{(N)}(osp(1|2n)) \) and \( U_q^{(N)}(so(2n+1)) \) are identical and as \( \overline{\Lambda}_N^+ = \overline{\Lambda}_N^+ \) it follows that \( \overline{\Lambda}_N^+ \) is a fundamental domain for \( X_{N/2} \) under the action of \( W_{N/2} \).
Now $S'_{\lambda+\rho-\rho,\mu} = \epsilon'(\sigma)S'_{\lambda,\mu}$ for any $\sigma \in W$. If $\lambda \in \Lambda_N^+ \setminus \Lambda_N^+$ there either exists some $\sigma \in W$ such that $\sigma(\lambda + \rho) - \rho = \lambda$ and $\epsilon'(\sigma) = -1$ or some $w \in W$ such that $\epsilon'(w) = 1$ and $\lambda = w(\lambda + \rho) - \rho + kN/2\epsilon_i$ for some $\epsilon_i$ and $k \in \mathbb{Z}$. As $S'_{\lambda,\mu} = -S'_{\lambda,\mu}$ where $\lambda' = \lambda + N/2\epsilon_i$, it follows that $S'_{\lambda,\mu} = 0$ for $\lambda \in \Lambda_N^+ \setminus \Lambda_N^+$. Then

$$2^n \sum_{\nu \in X_{N/2}} x_{\nu} q^{(\nu,\nu+2\rho)} S'_{\nu,\mu} = 2^n \sum_{\lambda \in \Lambda_N^+} \sum_{\sigma \in W} \epsilon'(\sigma) x_{\sigma(\lambda + \rho) - \rho} q^{(\lambda,\lambda+2\rho)} S'_{\lambda,\mu}.$$  

As

$$\sum_{\lambda \in \Lambda_N^+} d_\lambda q^{(\lambda+2\rho,\lambda)} S'_{\lambda,\mu} = 2^n \sum_{\lambda \in \Lambda_N^+} \sum_{\sigma \in W} \epsilon'(\sigma) x_{\sigma(\lambda + \rho) - \rho} q^{(\lambda,\lambda+2\rho)} S'_{\lambda,\mu}$$

we obtain

$$d_\lambda = q^{(2\rho,\rho)} \gamma_0 \text{sdim}_q (V_\lambda) \sum_{\mu \in X_{N/2}} q^{(\mu,\mu+2\rho)}$$

where $\gamma$ is the denominator in the expression of the quantum superdimension.

Note that $d_{\lambda^*} = d_\lambda$ where $\lambda^* = -\tau(\lambda)$. The denominator of the $d_\lambda$ does not vanish: $\sum_{\mu \in X_{N/2}} q^{(\mu,\mu+2\rho)} = \sum_{\mu \in X_N} q^{(\mu,\mu+2\rho)} / 2^n$ and $\sum_{\mu \in X_N} q^{(\mu,\mu+2\rho)} = \prod_{k=0}^{n-1} G(N, 2k + 1)$ where $G(N, m) = \sum_{i=0}^{N-1} q^{i(i+m)} = (1 + i)\sqrt{N} / x m^2$ and $x$ is a complex primitive $4N$th root of unity.

4 Constructing the invariant

The final matter we need to consider is condition v): that

$$z = \sum_{\lambda \in \Lambda_N^+} d_\lambda q^{-2(\lambda,\lambda)} \text{sdim}_q (V_\lambda) \neq 0.$$  

It follows from [2] that $z \neq 0$. Given $\zeta = \sum_{\lambda \in \Lambda_N^+} d_\lambda \text{sdim}_q (\lambda)$, $\zeta \neq 0$ and $z = d_0 \zeta$.

Now it is a relatively simple matter to construct the invariants. Denote a framed link in $S^3$ by $L$ and the 3-manifold it gives rise to by $M_L$. Let $(L)$ stand for the Reshetikhin-Turaev functor applied to $L$ (see [2, 3, 5]). Set $A_L$ to be the linking matrix of $L$ defined by: $a_{ii}$ is the framing number of the $i$th component of $L$ and $a_{ij}$, $i \neq j$ is the linking number between the $i$th and $j$th components of $L$. Let $\sigma(A_L)$ be the number of nonpositive eigenvalues of $A_L$.

Then

$$F(M_L) = z^{-\sigma(A_L)} \sum (L)$$

is a topological invariant of $M_L$.

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