Are there quantum bounds on the recyclability of clock signals in low power computers?

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Abstract

Even if a logical network consists of thermodynamically reversible gate operations, the computation process may have high dissipation rate if the gate implementation is controlled by external clock signals. It is an open question whether the global clocking mechanism necessarily involves irreversible processes. However, one can show that it is not possible to extract any timing information from a micro-physical clock without disturbing it. Applying recent results of quantum information theory we can show a hardware-independent lower bound on the timing information that is necessarily destroyed if one tries to copy the signal. The bound becomes tighter for low energy signals, i.e., the timing information gets more and more quantum.

1. Introduction

To invent new methods of low power computation is an important goal of research. In the middle future, the physical limits of miniaturization and reduction of power consumption will play a crucial role. The fact that thermodynamic laws give constraints to the energy consumption in information processing, has already been recognized by Landauer in the sixties [13]. The statement, that whenever the information \( n \) bits are lost during the computation process the energy \( \ln 2 nkT \) is dissipated (where \( k \) is Boltzmann’s constant and \( T \) is the absolute temperature) is nowadays known as Landauer’s principle. The fact that any logical function can be embedded in a logically reversible network by Toffoli gates [4] shows that Landauer’s principle does not give any obvious lower bounds for the dissipation during the computation process. However, this argument is not sufficient to show that computation without dissipation is possible. Even if the logical gates act thermodynamically reversibly on the register the implementation may consume energy if the process is controlled by an external clock signal. However, the quantum cellular automaton described in [14] indicates that this problem can in principle be avoided by doing computation without referring to a global clocking mechanism.

Here we consider the problems that may appear if one wants to keep the usual concept of global clocking with a clock signal that should not be ‘used up’ when it controls the gate implementations. The key problem is to what extent timing information can be read out without destroying it, i.e., to what extent timing information is quantum information that is impossible to clone.

We assume the clock signal to be given by the state of a micro-physical system represented by a quantum density matrix. In Section 2 we show that at every moment where the clock signal controls another physical system it will necessarily be disturbed by the system’s back-action. Using results of modern quantum information research we show that it is impossible in principle to extract information about the actual time from a micro physical clock (represented by a finite dimensional density matrix) without disturbing the clock’s state. Generalizations to infinite dimensional systems and quantitative statements on this result are subject of further research.

However, we can give quantitative statements in a slightly different situation. If one tries not only to extract some timing information from the clock but wants to copy as much timing information as possible to another system one necessarily destroys some timing information in the original clock. This partial destruction of the original clock signal gets more and more relevant if low energy signals are considered. These bounds are sketched in Section 3. The proofs can be found in [9].
2 Every clock is disturbed when it is read out

Let $\rho_t$ be the finite dimensional density matrix of the clock at time $t$. Since it is assumed to be a closed physical system it is evolving according to the evolution

$$\rho_t = \exp(-iHt) \rho \exp(iHt)$$

where $H$ is the clock’s Hamiltonian.

For the following reason it is not possible to extract any information about $t$ without disturbing the state $\rho_t$ if no prior knowledge about $t$ is given. Consider two states $\rho_{t_1}$ and $\rho_{t_2}$. Following [11] we conclude that a measurement distinguishing between those states can be implemented without changing the states if and only if the following condition holds: Let $\oplus \mathcal{H}_j = \mathcal{H}$ be the decomposition of the clock’s Hilbert space into those maximal subspaces that are invariant under the action of $\rho_{t_1}$ and $\rho_{t_2}$. Let $A_j$ and $B_j$ be the corresponding block matrices of $\rho_{t_1}$ and $\rho_{t_2}$, respectively. Then there exists a number $j$ such that $\text{tr}(A_j) \neq \text{tr}(B_j)$. An appropriate observable for the distinction is for example the projection onto the Hilbert space $\mathcal{H}_j$.

This gives a rule for constructing disturbance free measurements if we are sure that either $t = t_1$ or $t = t_2$ by prior knowledge. If nothing is known about the time we have to find common invariant subspaces of all $\rho_t$. Those have clearly to be invariant under the action of $H$. But the trace of the block matrices on those dynamically invariant subspaces is conserved. There is hence no way to gain information about $t$ without disturbing the system. The derivation of quantitative bounds on minimal disturbance is a difficult task. The information about $t$ has to be quantified as well as the disturbance.

Note that the statement that there is no extraction of information about $t$ relies essentially on the continuity of $t$. The necessary and sufficient condition above allows to construct systems where a discrete set of states $\rho_{t_1}, \rho_{t_2}, \ldots$ can indeed be distinguished without disturbing the system. This shows that if an external clock tells us that the actual time $t$ is in the set $\{t_1, t_2, \ldots\}$ we can read out our ‘discrete clock’ without disturbing it. Remarkably, the fact that readout without disturbance is in principle possible in the discrete case shows the advantage of controlling information processing by external clock signals: Let now $\rho_t$ be the density matrix of any micro physical device. Use the clock signal to switch on an interaction between the considered device and another system. Then the sequence of states $\rho_{t_1}, \rho_{t_2}, \ldots$ at the times $t_1, t_2, \ldots$ can control the other system without back-action (see Fig.1).

The relevance of the above observations for the reusability of clock signals has to be subject of further discussions. Clearly the gate has extracted some information about the actual time from the clock signal when it is triggered. This results in a disturbance of the clock signal’s quantum state. However, one may find systems where this extracted timing information flows back to the signal after the implementation is performed. But as long as the triggered implementation is still running the gate has some timing information and the clock signal’s state is still disturbed.

3 Copying a clock signal or its timing information

In the last section we have shown that we cannot extract timing information without disturbing the clock signal. However, we were not able to make quantitative statements about the tradeoff between information gain and disturbance. In the following situation, we can find quantitative results: It is natural to ask whether it is possible to read out all the information about the actual time that is contained in the signal, i.e., to copy it. In the following we will specify precisely what it means to ‘copy the timing information’. We use Fisher timing information, a quantity that is actually well-known in a more general context of estimating parameterized quantum or classical statistical states [8, 9, 10, 11]. In [9] it is used in the specific context of investigating the quality of clocks, explicitly using the terminology ‘Fisher timing information’. One might think of the Fisher timing information $F$ of a system with statistical states $\rho_t$ as the quotient $1/(\Delta t)^2$ if $\Delta t$ is the error for the optimal estimation of $t$ that is achievable by measuring the state. The correct and formal definition of $F$ is given in the appendix. But the rough explanation of $F$ can be taken literally in many simple examples: assume $\rho_t$ to be pure quantum states with a Gaussian energy distribution with standard deviation $\Delta E$. Then $F = 4(\Delta E)^2$ (where we have measured the energy in the unit $\hbar$) and one can easily

Figure 1. Only at specific times can a quantum clock be read out without being disturbed. Hence the read out process has to be controlled by a ‘meta clock’.
construct measurements allowing an estimation of \( t \) with standard deviation \( 1/\sqrt{F} \) and no better estimation can exist due to Heisenberg’s uncertainty relation. Another example where \( F \) has a simple intuitive meaning is the following. Consider a classical pulse, i.e., a classical quantity \( f \) that changes in time according to the function \( t \mapsto f(t) \). This system has infinite timing information for any non-trivial function \( f \) since the determination of \( f(t) \) allows to distinguish between \( t \) and \( t' \) for arbitrarily small \( t - t' \). If we introduce an unknown time delay of the signal with Gaussian statistics the timing information of the ‘smeared out’ signal is the standard deviation \( \Delta t \) of the delay (see Fig. 2).

One might think of \( f(t) \) as the classical current or voltage of any device or even the intensity of a (classical) light field. One can equivalently think of a classical signal moving with the velocity \( v \). If the exact position of the signal is unknown according to a Gaussian distribution with uncertainty \( \Delta x \) the timing information is given by \( v^2/(\Delta x)^2 \).

If \( \rho_t \) is a finite dimensional density matrix of a quantum system evolving according to the Hamiltonian \( H \), the quantity \( F \) is less simple to calculate and is given by

\[
F = \text{tr}(\hat{\rho}\Gamma^{-1}\hat{\rho}),
\]

where \( \hat{\rho} := i[H, \rho] \) and \( \Gamma^{-1} \) is the pseudo-inverse of the super-operator \( \Gamma \) with \( \Gamma a := 1/2(aa + a\rho) \) acting on the set of self-adjoint matrices \( \{1, 2, 3, 4, 5, 6\} \).

In [9] we derived the following quantum bound for copying timing information:

Assume a signal with Fisher timing information \( F \) enters a device (an amplifier for example) and triggers two outgoing signals with Fisher information \( F_1 \) and \( F_2 \), respectively. Then one has

\[
1/F_1 + 1/F_2 \geq 2/F + 2/\langle E^2 \rangle,
\]

where \( E \) is the total energy of the outgoing signals and \( \langle E^2 \rangle \) is the expectation value of the square of this energy. Accordingly, the time uncertainties \( \Delta t_1 \) and \( \Delta t_2 \) of the 2 outgoing signals triggered by an in-going signal with time uncertainty \( \Delta t \) satisfy the following inequality:

\[
(\Delta t_1)^2 + (\Delta t_2)^2 \geq 2(\Delta t)^2 + 2/\langle E^2 \rangle.
\]

Trying to give both signals the same timing information, i.e., \( \Delta t_1 = \Delta t_2 \), one obtains

\[
(\Delta t_1)^2 \geq (\Delta t)^2 + 1/\langle E^2 \rangle.
\]

We conclude that low energy signals loose part of their timing information when they are copied (see Fig. 3).

In [9] we have constructed an example showing that \( \Delta t_1 = \Delta t_2 = \Delta t \) is indeed possible in the limit \( \Delta E \to \infty \). Note that this does not imply that a high amount of energy is dissipated when the signal is copied. The energy has only to be available. Whether the disturbance of the clock signal (as explained in Section 3) results necessarily in energy consumption, is unclear but if the signal looses some of its timing information it is unclear how to run a reversible process if the clock signal is included in the consideration.

Note that whenever the clock signal controls a device part of the signal’s timing information is copied. This is illustrated in Fig. 4.

4 New thermodynamical constraints by a quasi-order of clocks

The proof of inequality (1) can be found in [8] and relies on a formal concept called quasi-order of clocks classifying physical systems (quantum or classical) with respect to their timing information. Here timing information is not to
be understood in the sense of a single quantity but refers to the statistical distinguishability of the quantum or classical states $\rho_t$ at different times $t$. So to speak, it classifies the quality of clocks. This quality has many aspects: The physical system $A$ can be better than the system $\tilde{A}$ with respect to the distinguishability the states at the times $t_1$ and $t_2$ and nevertheless $\tilde{A}$ may be better than $A$ with respect to the distinguishability between the states at the times $t_3$ and $t_4$. We write $A \geq \tilde{A}$ if $A$ is not worse than $\tilde{A}$ with respect to any criterion. The physical meaning is that is is possible in principle to realize a process with input $A$ and output $\tilde{A}$ such that the process has not to be controlled by an external clock. This idea can be formalized by describing the process by completely positive maps \[ G(H,.) = [\tilde{H}, G(.)]. \]

The condition states that the process is the same if it is implemented at a later time since it does not matter whether we let the input system $A$ evolve for the time $t$ and apply then the process $\tilde{G}$ or we apply $\tilde{G}$ first and then wait for the time $t$ such that the output system $\tilde{A}$ is evolving in time.

The output clock is not better than the input clock. As shown in [12] this principle gives constraints to many physical processes. Remarkably, the covariance condition also appeared in a rather different context in [13]. There we found thermodynamic constraints on processes that can be run with a negligible amount of implementation energy.

## 5. Relative timing information and synchronization

The considerations above refer to clocks showing the absolute time. At first sight one might think that the relevance of these results is restricted, since computation relies rather on the fact that gate implementations have to be synchronized and not that the gates are implemented at specific times. Hence one will rather be interested in relative timing information than in absolute timing information discussed in the previous sections. However, as we have shown in [3], the problem of measuring the quality of synchronization of two signals can be reduced to the problem of measuring the localization in time of a single signal. We have introduced a formal concept called quasi-order of synchronism that is shown to be mathematically isomorphic to the quasi-order of clocks.

## 6. Why the signal energy is relevant

The fact that our bound for copying timing information depends on the signal energy, might be astonishing. The tradeoff between information gain about a quantum state and the disturbance of the system does not refer a priori to the energy of the quantum system. Nevertheless it is easy to get a rough idea why energy is a relevant quantity if the timing information of a system should be copied: An unknown quantum state $\rho \in \rho_1, \rho_2, \ldots$ can be perfectly copied (‘broadcasted’) if and only if all the density matrices $\rho_t$ commute (see [3] for the proof and the definition of broadcasting quantum states). Consider now the states $\rho_t$ of the Hamiltonian time evolution. It is easy to see that all the states $\rho_t$ commute if and only if the time evolution is trivial, i.e. all states are the same. But it may be possible to find times $t_1, t_2, \ldots$ such that all the states $\rho_{t_1}$ commute and we can copy the states perfectly if the prior information $t \in \{ t_1, t_2, \ldots \}$ is given. For systems with high energy spread these times can be arbitrarily close together. An example is a pure quantum state given by the equal superposition

$$|\psi\rangle := \frac{1}{\sqrt{n}} \sum_{j \leq n} |j\rangle$$

where $j$ is an energy eigenstate with energy $jE$. Then the states at the times $t = j2\pi E/n$ are mutually orthogonal for different $j$, i.e., they can be copied without disturbance. The average energy of the system is $En/2$, i.e., the distinguishable states get closer and closer together (compare [3]) for $n \to \infty$. Even if no prior information about $t$ is given, one can imagine that it is possible to extract some information about $t$ while disturbing the state only a little bit as long as...
as one wants to know $t$ only up to an error that is much larger than $E/n$. One can sketch the idea of these remarks by claiming that the timing information in systems with low energy is essentially quantum information and timing information in systems with high energy spread can have a high part of classical information that can be extracted without disturbing the system too much.

The result has an interesting implication for low power computation: consider a fanout in low power circuits. Then the two out-going signals cannot have the same localization in time as the in-going signal.

These results suggest the following problem of extremely low power computation: if one uses low power clock signals it is difficult to copy the signal and distribute it to many devices. If the signals contain more energy the question of the reusability becomes more relevant.

7 Conclusions

We have shown that every clock signal, as far as it is given by a finite dimensional quantum system is necessarily disturbed when it controls a device. Quantitative results concerning the tradeoff between the effect of the clock on the network and the back-action are the subject of further research. A first step towards such a quantitative analysis shows that strong disturbance of the signal is inevitable if most of the signal’s timing information is transferred to the triggered system. Our bounds on the disturbance become relevant if the signal energy is in the order of $\Delta t$ where $\Delta t$ is the signal’s accuracy in time. To what extent this result implies bounds on energy dissipation for all computation processes with a global clocking mechanism is unclear. It is not even clear how to define such a mechanism formally. However, it indicates serious difficulties that may appear if the usual concept of clocking is maintained in future low power technology.

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Appendix

The Fisher timing information $F$ of a quantum or classical system is defined as follows. Let $A$ be the (unital) $C^*$-algebra of observables of the system and $\rho$ be the system's state, i.e., a positive functional from $A$ onto the set $\mathbb{C}$ with $\rho(1) = 1$. Let $(\alpha_t)_{t \in \mathbb{R}}$ be the time evolution of the system, i.e., a strongly continuous one parameter group of automorphisms on $A$. Then we define $F$ as

$$F := \sup_A \left( \frac{d}{dt} \rho(\alpha_t(A))^2 \right)^2 \rho(A^2)$$

where the supremum is taken over all self-adjoint $A \in A$.

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