On the local Type I conditions for the 3D Euler equations

Dongho Chae∗ and Jörg Wolf †

Department of Mathematics
Chung-Ang University
Seoul 156-756, Republic of Korea

Abstract

We prove local non blow-up theorems for the 3D incompressible Euler equations under local Type I conditions. More specifically, for a classical solution $v \in L^\infty(-1,0;L^2(B(x_0,r))) \cap L^\infty_{loc}(-1,0;W^{1,\infty}(B(x_0,r)))$ of the 3D Euler equations, where $B(x_0,r)$ is the ball with radius $r$ and the center at $x_0$, if the limiting values of certain scale invariant quantities for a solution $v(\cdot,t)$ as $t \to 0$ are small enough, then $\nabla v(\cdot,t)$ does not blow-up at $t = 0$ in $B(x_0,r)$.

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1 Introduction

We consider the 3D homogeneous incompressible Euler equation in a cylinder $Q = \mathbb{R}^3 \times (-1,0)$

\begin{align}
\partial_t v + v \cdot \nabla v &= -\nabla p \quad \text{in} \quad Q, \\
\nabla \cdot v &= 0 \quad \text{in} \quad Q,
\end{align}

where $v = (v_1(x,t), v_2(x,t), v_3(x,t))$ stands for the velocity of the fluid and $p = p(x,t)$ stands for the pressure. The local in time well-posedness in the Sobolev space $W^{k,p}(\mathbb{R}^3)$, $k > 3/p + 1$, $1 < p < +\infty$, for the Cauchy problem of the system (1.1)-(1.2) is well-known due to the proof by Kato-Ponce[10]. The question of the spontaneous apparition of singularity from the local in time smooth solution, however, is still an outstanding open problem in the mathematical fluid mechanics(see e.g.[12, 6] for surveys of studies devoted to the problem). We say a local in time smooth solution
\[ v \in C([-1,0); W^{k,p}(\mathbb{R}^3)), \quad k > \frac{3}{p} + 1, \quad 1 < p < +\infty, \]
does not blow up (or becomes regular) at \( t = 0 \) if
\[
\limsup_{t \to 0^-} \|v(t)\|_{W^{k,p}(\mathbb{R}^3)} < +\infty.
\]

It is easy to show from the local in time well-posedness estimates that (1.3) is guaranteed if
\[
\int_{-1}^{0} \|\nabla v(t)\|_{L^\infty(\mathbb{R}^3)} dt < +\infty.
\]

The celebrated Beale-Kato-Majda criterion\[1\] shows that one can replace (1.4) by a weaker condition
\[
\int_{-1}^{0} \|\omega(t)\|_{L^\infty(\mathbb{R}^3)} dt < +\infty, \quad \omega = \nabla \times v.
\]
(see also [7, 8] for geometric type criterion, and [11] for a refinement of (1.5)). The conditions (1.4) or (1.5) can be regarded as regularity criteria of the Serrin type in the Navier-Stokes equations. There exist also another form of local regularity criteria, called \( \varepsilon \)-regularity criteria, which claims that if certain scaling invariant quantities are small enough in a local space-time neighborhood, then weak solution becomes regular in the neighborhood. A typical example of such smallness condition, introduced by Caffarelli, Kohn and Nirenberg in [2], which guarantees the regularity near \((x,t) = (0,0)\) for a suitable weak solution of the Navier-Stokes equations is
\[
\limsup_{r \to 0^+} \frac{1}{r} \int_{-1}^{0} \int_{\{|x| < r\}} |\nabla v(x,t)|^2 dx dt < \varepsilon,
\]
where \( \varepsilon > 0 \) is an absolute constant. The replacement of \( \varepsilon \) by finite constant \( C \) in (1.6) is called local Type I condition for the Navier-Stokes equations (cf. [5, 13]). In view of the scaling property of the Euler equations a natural local Type I condition with smallness, which guarantee no blow-up at \( t = 0 \) for a classical solution \( v \in C([-1,0); W^{1,\infty}(\mathbb{R}^3)) \) would be
\[
\limsup_{t \to 0^-} (-t) \|\nabla v(t)\|_{L^\infty(B(r))} < \varepsilon,
\]
where we used \( B(r) = B(0,r) \) with \( B(x_0,r) = \{ x \in \mathbb{R}^3 \mid |x - x_0| < r \} \). Indeed in [3] (see also [9] for an independent result) it has been shown that if
\[
\limsup_{t \to 0^-} (-t) \|\nabla v(t)\|_{L^\infty(\mathbb{R}^3)} < 1,
\]
then there exists no blow-up at \( t = 0 \) for a classical solution to the Euler equations on \( \mathbb{R}^3 \times (-1,0) \). Our first aim in this paper is to localize in space (1.8), and prove the following theorem.
\textbf{Theorem 1.1.} Let \( v \in L^\infty(-1,0; L^2(B(r))) \cap L^\infty_{loc}([-1,0); W^{1,\infty}(B(r))) \) be a solution to the Euler equations (1.1), (1.2) with \( v(-1) \in W^{2,p_0}(B(r)) \) for some \( 3 < p_0 < +\infty \).

We assume there exists \( r_0 \in (0,r) \) such that
\begin{equation}
\limsup_{t \to 0^-} (-t) \| \nabla v(t) \|_{L^\infty(B(r_0))} < 1. \tag{1.9}
\end{equation}

Then \( \limsup_{t \to 0^-} \| v(t) \|_{W^{2,p_0}(B(r_0))} < +\infty \) for all \( r \in (0,r_0) \).

The proof of the above theorem is given in the Section 2 to Section 4. From the structure of the Euler equations the estimation of the \( L^\infty_{loc}(\mathbb{R}^3) \) norm of second derivatives usually are obtained by means of Gronwall’s Lemma. In order to handle the integrals involving derivatives with cut off function it was crucially helpful to introduce the following transformation of the solutions \( v(x,t) \mapsto w(y,t) = v((1 + (-t)^0)y,t) \) for appropriately chosen \( 0 < \theta < 1 \).

In our second main result below we use Theorem 1.1 to deduce that local small oscillation near \( t = 0 \) implies also no blow-up of a classical solution on \( B(r) \times (-1,0) \).

\textbf{Theorem 1.2.} Let \( v \in L^\infty(-1,0; L^2(B(r))) \cap L^\infty_{loc}([-1,0); W^{1,\infty}(B(r))) \) be a solution to the Euler equations (1.1), (1.2) with \( v(-1) \in W^{2,p_0}(B(r)) \) for some \( 3 < p_0 < +\infty \).

We assume there exists \( 0 < r_0 < r/2 \) such that
\begin{equation}
\limsup_{t \to 0^-} (-t) \sup_{x_0 \in B(r/2)} \text{osc}_{B(x_0, r_0(-t)^2/3)} (\nabla v(t)) < 1. \tag{1.10}
\end{equation}

Then \( \limsup_{t \to 0^-} \| v(t) \|_{W^{2,p_0}(B(r/4))} < +\infty \).

The key ingredient in the proof of Theorem 1.2 is the fact that under Type I condition (replacing \( \varepsilon \) by any finite constant \( C \) in (1.7)) there exists no atomic energy concentration in \( B(r) \), which is proved in [4], that makes the local energy \( \int_{B(x_0,r)} |v(x,t)|^2 dx \) uniformly small with respect to \( (x_0, t) \in \mathbb{R}^3 \times (-1,0) \) for small \( r > 0 \).

\textbf{Remark 1.3.} Below we present two sufficient conditions on \( v \), which imply (1.10). The first one is obvious. If there exists a function \( \eta : (0, \infty) \to \mathbb{R} \) with \( \eta(r) \to 0 \) as \( r \to 0 \) such that
\begin{equation}
(-t) |\nabla v(x,t) - \nabla v(y,t)| \leq \eta \left( \frac{|x - y|}{(-t)^{2/3}} \right) \quad \forall x, y \in \mathbb{R}^3, t \in (-1,0), \tag{1.11}
\end{equation}

then, \( v \) satisfies (1.10). The second condition is given in terms of the Fourier transform. Given \( \delta \in (0,1) \), if there exists \( 0 < R_0 < +\infty \) such that
\begin{equation}
\int_{\mathbb{R}^3 \setminus B(R_0(-t)^{-2/5})} |\xi| |\mathcal{F}v(\xi,t)| d\xi \leq \frac{1 - \delta}{2t} \quad \forall t \in (-1,0), \tag{1.12}
\end{equation}

then the condition (1.10) for \( v \) follows. Indeed, let \( g(\xi, t) = \mathcal{F}(\nabla v(\xi)) \), then we see that
\begin{align*}
\nabla v(x, t) - \nabla v(y, t) &= \mathcal{F}^{-1} g(\cdot, t)(x) - \mathcal{F}^{-1} g(\cdot, t)(y) \\
&= \int_{\mathbb{R}^3 \setminus B(R_0(-t)^{-2/5})} (e^{2\pi i x \cdot \xi} - e^{2\pi i y \cdot \xi}) g(\xi, t) d\xi + \int_{B(R_0(-t)^{-2/5})} (e^{2\pi i x \cdot \xi} - e^{2\pi i y \cdot \xi}) g(\xi, t) g(\xi, t) d\xi.
\end{align*}
which leads to the inequality

\[ |\nabla v(x, t) - \nabla v(y, t)| \leq \frac{1 - \delta}{-t} + \int_{B(R_0(-t)^{-2/5})} |e^{2\pi i(x-y)\cdot \xi} - 1||g(\xi, t)|d\xi. \]

For \(|x - y| \leq r_0(-t)^{2/5}\) and \(0 < r_0 \leq R_0^{-1}\) the second term can be estimated as follows:

\[ \int_{B(R_0(-t)^{-2/5})} |e^{2\pi i(x-y)\cdot \xi} - 1||g(\xi, t)|d\xi \leq 2\pi R_0 \frac{|x - y|}{(-t)^{2/5}} \int_{\mathbb{R}^3} |\xi||Fv(\xi, t)|d\xi \]

\[ \leq 2\pi r_0 R_0 \int_{\mathbb{R}^3} |\xi||Fv(\xi, t)|d\xi. \]

Thus if we choose \(r_0\) such that

\[ 2\pi r_0 R_0 \sup_{t \in (-1, 0)} (-t) \int_{\mathbb{R}^3} |\xi||Fv(\xi, t)|d\xi \leq \frac{\delta}{2}, \]

the condition (1.9) holds with \(\frac{\delta}{2}\) in place of \(\delta\). Here, we have used the fact that there exists a constant \(C_1 > 0\) such that

\[ (-t) \int_{B(R_0(-t)^{-2/5})} |\xi||Fv(\xi, t)|d\xi \leq C_1. \]

This can be checked by Hölder’s inequality and Plancherel’s theorem as follows.

\[ (-t) \int_{B(R_0(-t)^{-2/5})} |\xi||Fv(\xi, t)|d\xi \leq cR_0^{5/2}||Fv(\xi)|_{L^2} = cR_0^{5/2}E^{1/2}. \]

### 2 Uniform smallness of the local energy

Our aim in this section is to prove the following result, which is interesting itself.

**Theorem 2.1.** Let \(v \in L^\infty(-1, 0; L^2(B(1))) \cap L^\infty_{[(-1, 0); W^1, \infty(B(1))}]\) be a solution to the Euler equations (1.1), (1.2), which satisfies the following condition

\[ \sup_{t \in (-1, 0)} (-t)\|\nabla v(t)\|_{L^\infty(B(1))} \leq C_0. \]

Then for every \(\varepsilon > 0\) there exists \(0 < \tilde{R} = \tilde{R}(\varepsilon) \leq \frac{1}{2}\) such that for all \(y_0 \in B(1/2)\) it holds

\[ \sup_{t \in (-\tilde{R}^{5/2}, 0)} \int_{B(y_0, \tilde{R})} |v(t)|^2dx \leq \varepsilon. \]
The proof of Theorem 2.1 will be achieved after proving several lemmas. Given \( z_0 = (x_0, t_0) \in \mathbb{R}^3 \times (-\infty, 0] \), \( r > 0 \), we denote \( Q(z_0, r) = B(x_0, r) \times (t_0 - r^{5/2}, t_0) \), and \( Q(r) = Q(0, r) \). For \( \Omega \subset \mathbb{R}^3 \) by \( W^{d,p}(\Omega) \) we denote the usual Sobolev-Slobodeckii space, which consists of all functions \( f \in L^p(\Omega) \) such that the following semi norm is finite
\[
|f|^p_{W^{d,p}(\Omega)} = \int_\Omega \int_\Omega |f(x) - f(y)|^p |x - y|^{3 + dp} \, dx dy, \quad f \in W^{d,p}(\Omega), \quad 0 < \theta < 1, \quad p \geq 1.
\]

**Lemma 2.2.** Let the assumption of Theorem 2.1 be satisfied. Let \( x_0 \in B(1/2) \). Then, for every \( \varepsilon > 0 \) there exists \( 0 < R_0 = R_0(x_0, \varepsilon) < 1 \) such that for all \( 0 < R \leq R_0 \) it holds
\[
(2.3) \quad R^{-1} \int_{Q((x_0,0),R)} |v|^3 \, dx dt \leq \varepsilon.
\]

**Proof:** We prove the assertion of the theorem by an indirect argument. To this end let us assume the assertion is false. Then there exist \( x_0 \in B(1/2) \) and a sequence \( \{r_k\} \) of numbers in \((0, 1/2)\), which converges to zero as \( k \to +\infty \), satisfying
\[
(2.4) \quad r_k^{-1} \int_{Q((x_0,0),r_k)} |v|^3 \, dx dt > \varepsilon \quad \forall \ k \in \mathbb{N}.
\]

Without the loss of generality we may assume \( x_0 = 0 \).

Since the solution is defined locally, we cannot expect global bounds on the pressure. By this reason the compactness lemma of Lions-Aubin type does not work in this situation. This forces us to work with the notion of local pressure. As in [16, 4] we introduce the projection \( E^*_{B(1)} : W^{-1,q}(B(1)) \to W^{-1,q}(B(1)) \) onto the space of functionals given by a gradient \( \nabla \pi \). In fact, here \( \pi \in L^q_0(\nabla(B(1)) \) denotes the pressure of the solution to the Stokes equations in \( B(1) \) with zero boundary data and force \( f \).

We define
\[
\nabla p_h = -E^*_{B(1)}(v), \quad \nabla p_0 = -E^*_{B(1)}((v \cdot \nabla) v),
\]
\[
\tilde{v} = v + \nabla p_h.
\]

From this definition we find easily that \( (\tilde{v}, \nabla p_h, p_0) \) solves the following system in \( B(1) \times (-1, 0) \) in the sense of distributions
\[
(2.5) \quad \partial_t \tilde{v} + (v \cdot \nabla) \tilde{v} - v \cdot \nabla^2 p_h = -\nabla p_0.
\]

Let \( 0 < \rho < +\infty \) be arbitrarily chosen. Recalling that \( \nabla p_h \) is harmonic, by using the mean value property of harmonic functions, we get for all \( t \in (-1, 0) \) and for all \( k \in \mathbb{N} \) such that \( \rho r_k \leq \frac{1}{2} \) the following estimates
\[
\int_{B(\rho r_k)} |\nabla p_h(t)|^3 \, dx \leq c \rho^3 r_k^3 \|\nabla p_h(t)\|_{L^\infty(B(1/2))}^3 \leq c \rho^3 r_k^3 \|\nabla p_h(t)\|_{L^\infty(B(1))}^3,
\]
\[
(2.6) \quad \leq c \rho^3 r_k^3 \|v\|_{L^\infty((-1,0); L^2(B(1)))}^3.
\]
Next, we define sequences of scaled velocities and pressures,

\[
v_k(x, t) = r_k^{3/2}v(r_kx, r_k^{5/2}t), \quad p_{h,k}(x, t) = r_k^{1/2}p_h(r_kx, r_k^{5/2}t), \quad p_{0,k}(x, t) = r_k^3p_0(r_kx, r_k^{5/2}t), \quad (x, t) \in B(r_k^{-1}) \times (-1, 0).
\]

Using the transformation formula of the Lebesgue integral, the condition (2.4) becomes

\[
(2.7) \quad \int_{Q(1)} |v_k|^3 \, dx \, dt \geq \varepsilon.
\]

Setting \( \tilde{v}_k = v_k + \nabla p_{h,k} \) in (2.5), we see that the following equations are satisfied in the sense of distributions.

\[
(2.8) \quad \partial_t \tilde{v}_k + (v_k \cdot \nabla) \tilde{v}_k - v_k \cdot \nabla^2 p_{h,k} = -\nabla p_{0,k} \quad \text{in} \quad B(r_k^{-1}) \times (-1, 0).
\]

From (2.6) we immediately get for all \( 0 < \rho < +\infty \) and for all \( k \in \mathbb{N} \) such that \( \rho r_k \leq \frac{1}{2} \)

\[
(2.9) \quad \|\nabla p_{h,k}\|_{L^3(B(\rho) \times (-1,0))} = r_k^{-1}\|\nabla p_h\|_{L^3(B(\rho r_k) \times (-r_k^{5/2},0))} \leq c\rho^3 r_k^2.
\]

This yields

\[
(2.10) \quad \nabla p_{h,k} \rightarrow 0 \quad \text{in} \quad L^3(B(\rho) \times (-1,0)) \quad \text{as} \quad k \rightarrow +\infty \quad \forall \ 0 < \rho < +\infty.
\]

Since both the \( L^\infty(-1,0; L^2(B(1))) \) norm and the Type I condition (2.1) are invariant under the above scaling, we obtain for all \( 0 < \rho < +\infty \) and for all \( k \in \mathbb{N} \) such that \( \rho r_k \leq \frac{1}{2} \)

\[
(2.11) \quad \|v_k\|_{L^\infty(-1,0; L^2(B(\rho)))} \leq \|v\|_{L^\infty(-1,0; L^2(B(1)))},
\]

\[
(2.12) \quad \sup_{t \in (-1,0)} \|\nabla v_k(t)\|_{L^\infty(B(\rho))} \leq C_0.
\]

By interpolation between the two bounds (2.11) and (2.12) we see that for all \( 0 < \rho < +\infty \) the sequence \( \{v_k\} \) is bounded in \( L^3(-1,0; W^{\theta,3}(B(\rho))) \) for each \( 0 \leq \theta < \frac{1}{3} \). Using the regularity properties of harmonic functions, we also see that \( \{\tilde{v}_k\} \) is bounded in \( L^3(-1,0; W^{\theta,3}(\rho)) \). In particular, \( \{v_k\} \) is bounded in \( L^3(B(\rho) \times (-1,0)) \) which shows that \( \{p_{0,k}\} \) is bounded in \( L^{3/2}(B(\rho) \times (-1,0)) \). Accordingly, \( \{\partial_t \tilde{v}_k\} \) is bounded in \( L^{3/2}(-1,0; W^{-1,3/2}(B(\rho))) \) for all \( 0 < \rho < +\infty \). Using Banach-Alaoglu’s theorem and applying a compactness lemma due to Simon[14], and Cantor’s diagonalization principle, eventually passing to a subsequence, we get a limit \( v^* \in L^\infty(-1,0; L^2(\mathbb{R}^3)) \cap L^\infty_{loc}([-1,0); W^{1,\infty}(\mathbb{R}^3)) \) such that for all \( 0 < \rho < +\infty \)

\[
(2.13) \quad \tilde{v}_k \rightarrow v^* \quad \text{weakly-* in} \quad L^\infty(-1,0; L^2(B(\rho))) \quad \text{as} \quad k \rightarrow +\infty,
\]

\[
(2.14) \quad \tilde{v}_k \rightarrow v^* \quad \text{in} \quad L^3(B(\rho) \times (-1,0)) \quad \text{as} \quad k \rightarrow +\infty.
\]

Using (2.14) and (2.10), we may let \( k \rightarrow +\infty \) in (2.7), which yields

\[
(2.15) \quad \int_{Q(1)} |v^*|^3 \, dx \, dt \geq \varepsilon.
\]
Furthermore, after passing $k \to +\infty$ in (2.8), we deduce that $\mathbf{v}^* \in L^2(-1, 0; L^2(\mathbb{R}^3)) \cap L^\infty_{loc}([-1, 0); W^{1, \infty}(\mathbb{R}^3))$ is a solution to the Euler equations in $\mathbb{R}^3 \times (-1, 0)$. Thanks to (2.12) we see that the function $\mathbf{v}^*$ enjoys the Type I blow up condition

$$\sup_{t \in (-1, 0)} (-t) \| \nabla \mathbf{v}^* (t) \|_{L^\infty(\mathbb{R}^3)} \leq C_0. \quad (2.16)$$

Observing (2.8), and noting that $v_k(t)$ is bounded and Lipschitz before the blow up time we get the following local energy equality which holds for every $-1 \leq t < s < 0$ and for all $\phi \in C_c^\infty(B(\rho))$ with $r_k \rho < 1$

$$\int_{B(\rho)} |\tilde{v}_k(t)|^2 \phi dx = \int_{B(\rho)} |\tilde{v}_k(s)|^2 \phi dx + \int_t^s \int_{B(\rho)} (|\tilde{v}_k|^2 v_k + 2p_{0,k} \tilde{v}_k) \cdot \nabla \phi dx d\tau$$

$$+ \int_t^s \int_{B(\rho)} \tilde{v}_k \nabla \phi dx d\tau. \quad (2.17)$$

In the discussion below $\mathcal{M}(\mathbb{R}^3)$ denotes the space of Radon measures, while $\mathcal{M}^+(\mathbb{R}^3)$ stands for the space of positive Radon measures both on $\mathbb{R}^3$. As we have proved in [4] there exists a unique measure $\tilde{\sigma} \in \mathcal{M}^+(\mathbb{R}^3)$ such that

$$|\tilde{v}(t)|^2 \rightarrow \tilde{\sigma} \text{ weakly-* in } \mathcal{M}(B(1)) \text{ as } t \rightarrow 0^-. \quad (2.18)$$

This implies

$$|\tilde{v}_k(s)|^2 \rightarrow \tilde{\sigma}_k \text{ weakly-* in } \mathcal{M}(B(\rho)) \text{ as } s \rightarrow 0^-,$$

where $\tilde{\sigma}_k$ is defined as

$$\int_{B(\rho)} \phi d\tilde{\sigma}_k = \int_{B(r_k \rho)} \phi \left( \frac{x}{r_k} \right) d\tilde{\sigma}.$$

Thus, in (2.17) letting $s \rightarrow 0$, we arrive at

$$\int_{B(\rho)} |\tilde{v}_k(t)|^2 \phi dx = \int_{B(\rho)} \phi d\tilde{\sigma}_k + \int_0^s \int_{B(\rho)} (|\tilde{v}_k|^2 v_k + 2p_{0,k} \tilde{v}_k) \cdot \nabla \phi dx d\tau$$

$$+ \int_t^s \int_{B(\rho)} \tilde{v}_k \nabla \phi dx d\tau. \quad (2.19)$$

On the other hand, it can be checked easily that $\|\tilde{\sigma}_k\| \leq \|\tilde{\sigma}\|$. Hence, eventually passing to a subsequence, we get $\sigma^* \in \mathcal{M}^+(\mathbb{R}^3)$ such that for all $0 < \rho < +\infty$

$$\tilde{\sigma}_k \rightarrow \sigma^* \text{ weakly-* in } \mathcal{M}(B(\rho)) \text{ as } s \rightarrow 0^-.$$
Our aim is to show that $\sigma^* = \tilde{\sigma}(\{0\}) \delta_0$. Arguing as in [4], we infer for $\phi \in C^\infty_c(\mathbb{R}^3)$ with $\text{supp}(\phi) \subset B(\rho)$

\[
\int_{\mathbb{R}^3} \phi d\sigma^* = \lim_{k \to \infty} \int_{B(\rho)} \phi d\tilde{\sigma}_k = \lim_{k \to \infty} \int_{B(r_k \rho)} \phi \left( \frac{x}{r_k} \right) d\tilde{\sigma}
= \lim_{k \to \infty} \int_{B(r_k \rho) \setminus \{0\}} \left\{ \phi \left( \frac{x}{r_k} \right) - \phi(0) \right\} d\tilde{\sigma} + \phi(0) \lim_{k \to \infty} \int_{B(r_k \rho)} d\tilde{\sigma}
= \phi(0) \tilde{\sigma}(\{0\})
\]

This shows the claim. Thus, we are in a position to apply [4, Theorem 3.1], which excludes the concentration of energy at one point. Hence, $v^*$ must vanish. However this contradicts with (2.15). Accordingly the assertion must be true. 

Next, we show the smallness of the local energy.

**Lemma 2.3.** Let all assumptions of Theorem 2.1 be fulfilled. Let $x_0 \in B(1/2)$. Then for every $\varepsilon > 0$ there exists $0 < R_1 = R_1(\varepsilon, x_0) < \frac{1}{2}$ such that

\[
(2.20) \quad \|v(t)\|_{L^2(B(x_0, R_1))} \leq \varepsilon \quad \forall -R_1^{5/2} \leq t < 0.
\]

**Proof:** Let $0 < \delta < 1$ be a number, which will be specified below. By Lemma 2.2 there exists $0 < R_0 = R_0((\delta \varepsilon)^{3/2}, x_0) < 1$ such that (2.3) holds with $(\delta \varepsilon)^{3/2}$ in place of $\varepsilon$. Accordingly, by the help of Hölder’s inequality we find

\[
(2.21) \quad R_0^{-5/2} \int_{Q((x_0, 0), R_0)} |v|^2 dx dt \leq |B(1)|^{1/3} \delta \varepsilon.
\]

Thus, thanks to the mean value property of the integral we may choose $s_0 \in (-R_0^{5/2}, 0)$ such that

\[
(2.22) \quad \int_{B(x_0, R_0)} |v(s_0)|^2 dx \leq |B(1)|^{1/3} \delta \varepsilon.
\]

As in the proof of Lemma 2.2 we define the local pressure

$\nabla p_h = -E_{B(x_0, R_0)}^*(v), \quad \nabla p_0 = -E_{B(x_0, R_0)}^*((v \cdot \nabla)v),$

and set $\tilde{v} = v + \nabla p_h$. As above we see that the function $\tilde{v}$ solves the modified Euler equations

\[
(2.23) \quad \partial_t \tilde{v} + v \cdot \nabla \tilde{v} - v \cdot \nabla^2 \tilde{p}_h = -\nabla p_0 \quad \text{in} \quad B(x_0, R_0) \times (-1, 0)
\]
in the sense of distributions. Furthermore the following local energy identity holds true for all $-1 \leq t < s < 0$ and for all $\phi \in C_c^\infty(B(x_0, R_0))$

$$\int_{B(x_0, R_0)} |\tilde{v}(t)|^2 \phi \, dx = \int_{B(x_0, R_0)} |\tilde{v}(s)|^2 \phi \, dx + \int_t^s \int_{B(x_0, R_0)} (|\tilde{v}|^2 v + 2p_0 v) \cdot \nabla \phi \, dx \, d\tau$$

(2.24) $$+ \int_t^s \int_{B(x_0, R_0)} v \cdot \nabla^2 p_h \cdot \tilde{v} \phi \, dx \, d\tau.$$

Now, let $\phi \in C_c^\infty(\mathbb{R}^3)$ be a radial cut-off function such that $\phi \equiv 1$ on $B(x_0, R_0/2)$, $0 \leq \phi \leq 1$ in $B(x_0, R_0)$ and $|\nabla \phi| \leq cR_0^{-1}$. We set $s = s_0$ and $t \in (-R_0^{5/2}, 0)$ in (2.24). Using (2.22), we see that

$$\int_{B(x_0, R_0)} |\tilde{v}(t)|^2 \phi \, dx \leq |B(1)|^{1/3} \delta \varepsilon + cR_0^{-1} \int_{Q((x_0,0), R_0)} (|\tilde{v}|^2 |v| + 2 |p_0||v|) \, dx \, d\tau$$

(2.25) $$+ \int_t^{s_0} \int_{B((x_0,0), R_0)} |v||\nabla^2 p_h||\tilde{v}| \phi \, dx \, d\tau = |B(1)|^{1/3} \delta \varepsilon + I + II.$$

To estimate the first integral we make use of the pressure estimates

$$\|\nabla p_h(t)\|_{L^3(B(x_0, R_0))} \leq c \|v(t)\|_{L^3(B(x_0, R_0))},$$

$$\|p_0(t)\|_{L^{3/2}(B(x_0, R_0))} \leq c \|v(t)\|_{L^3(B(x_0, R_0))}^2,$$

which together with (2.24) shows that

$$I \leq cR_0^{-1} \int_{Q((x_0,0), R_0)} |v|^3 \, dx \, d\tau \leq c(\delta \varepsilon)^{3/2}. $$

To estimate the second integral we first apply Hölder’s inequality to get

$$II \leq \left( \int_{Q((x_0,0), R_0)} |v|^3 \, dx \, d\tau \right)^{2/3} \left( \int_{Q((x_0,0), R_0)} |\nabla^2 p_h| \phi^3 \, dx \, d\tau \right)^{1/3}.$$

Applying Sobolev’s embedding theorem together with [1] Lemma A.1, we estimate for $\tau \in (-R_0^{5/2}, 0)$

$$\|\nabla^2 p_h(\tau)\phi\|_{L^3(B(x_0, R_0))}$$

$$\leq cR_0^{-1/2} \|\nabla^2 p_h(\tau)\phi\|_{L^2(B(x_0, R_0))} + c \|\nabla^2 p_h(\tau)\nabla \phi\|_{L^2(B(x_0, R_0))}$$

$$\leq cR_0^{-3/2} \|\nabla p_h(\tau)\|_{L^2(B(x_0, R_0))}$$

$$\leq cR_0^{-3/2} \|v(\tau)\|_{L^2(B(x_0, R_0))} \leq cR_0^{-1} \|v(\tau)\|_{L^3(B(x_0, R_0))}.$$
Taking this inequality to the $3^{rd}$ power and integrate over $(-R_0^{5/2}, 0)$, we arrive at
\[
\left( \int_{Q((x_0,0), R_0)} |\nabla p_h|^3 \phi^3 \, dx \, d\tau \right)^{1/3} \leq c R_2^{-1} \left( \int_{Q((x_0,0), R_0)} |v|^3 \phi^3 \, dx \, d\tau \right)^{1/3},
\]
which shows that
\[
II \leq R_2^{-1} \int_{Q((x_0,0), R_0)} |v|^3 \, dx \, d\tau \leq c(\delta \varepsilon)^{3/2}.
\]
Inserting the estimates of $I$ and $II$ into (2.26), we obtain
\[
(2.26) \quad \int_{B(x_0, R_0)} |\tilde{v}(t)|^2 \phi \, dx \leq |B(1)|^{1/3} \delta \varepsilon + c(\delta \varepsilon)^{3/2}.
\]
Let $-\frac{1}{2} < \tau < 0$ be specified below. Once more using the fact that $\nabla p_h$ is harmonic, we find for all $\tau \in (-R_0^{5/2}, 0)$
\[
\|\nabla p_h(\tau)\|^2_{L^2(B(x_0, \tau R_0))} \leq c(-\tau R_0)^3 \|\nabla p_h(\tau)\|^2_{L^\infty(B(x_0, R_0/2))} \\
\leq c(-\tau)^3 \|\nabla p_h(\tau)\|^2_{L^2(B(x_0, R_0))} \\
= c(-\tau)^3 \|v(\tau)\|^2_{L^2(B(1))} = c(-\tau)^3 E,
\]
where $E = \|v\|^2_{L^\infty((-1,0):L^2(B(1)))}$. Combining (2.26) and (2.27) with the choice $\tau = -(\delta \varepsilon)^{1/3}$, we deduce that
\[
(2.28) \quad \int_{B(x_0, -\tau R_0)} |v(t)|^2 \phi \, dx \leq c(|B(1)|^{1/3} + E + 1)\delta \varepsilon.
\]
In the above estimate we may choose $\delta = \frac{1}{c(|B(1)|^{1/3} + E + 1)}$ such that the desired estimate follows with $R_1 = (\delta \varepsilon)^{1/3} R_0$.

**Proof of Theorem 2.1** Let $\varepsilon > 0$ be arbitrarily chosen. By virtue of Lemma 2.2 for every $x_0 \in B(1/2)$ there exists $0 < R_1(x_0) = R_1(\varepsilon, x_0) < 1$ such that
\[
\int_{B(x_0, R_1(x_0))} |v(t)|^2 \, dx \leq \varepsilon \quad \forall -R_1(x_0)^{5/2} \leq t < 0.
\]
Since $\overline{B(1/2)}$ is compact, we find a finite sequence of points $\{x_1, \ldots, x_m\}$ such that $\{B(x_i, R_1(x_i)/2)\}$ covers $\overline{B(1/2)}$. Let $y_0 \in B(1/2)$ be arbitrarily chosen. There exists $i \in \{1, \ldots, m\}$ with $y_0 \in B(x_i, R_1(x_i)/2)$. Obviously, $B(y_0, R_1(x_i)/2) \subset B(x_i, R_1(x_i))$ and thus,
\[
\int_{B(y_0, R_1(x_i)/2)} |v(t)|^2 \, dx \leq \varepsilon \quad \forall -R_1(x_i)^{5/2} \leq t < 0.
\]
Setting $\widetilde{R} = \frac{1}{2} \min\{R_1(x_1), \ldots, R_1(x_m)\}$, we deduce that for all $y_0 \in B(1/2)$ it holds
\begin{equation}
(2.29) \quad \int_{B(y_0, \widetilde{R})} |v(t)|^2 dx \leq \varepsilon \quad \forall -\widetilde{R}^{5/2} \leq t < 0.
\end{equation}
This completes the proof of assertion of the theorem. \hfill \blacksquare

As an immediate consequence of Theorem 2.1 we get the following smallness result for the $L^\infty$ blow-up.

Corollary 2.4. Let the assumptions of Theorem 2.1 be satisfied. Then for every $\varepsilon > 0$ there exists $t_0 = t_0(\varepsilon) \in (-1, 0)$ such that
\begin{equation}
(2.30) \quad \|v(t)\|_{L^\infty(B(1/2))} \leq \varepsilon(-t)^{-3/5} \quad \forall t_0 \leq t < 0.
\end{equation}

Proof: Let $\varepsilon > 0$ be arbitrarily chosen. Let $0 < \delta < 1$ be fixed, which will be specified below. We apply Theorem 2.1 with $\varepsilon_0 = (\delta \varepsilon)^5$ in place of $\varepsilon$. Let $\widetilde{R} = \widetilde{R}(\varepsilon_0)$ such that (2.2) holds true for $\varepsilon_0$ in place of $\varepsilon$. Applying the Gagliardo-Nirenberg inequality (A.1) with $n = 3, p = 2$ and $q = \infty$ together with (2.2), we obtain
\begin{equation}
(2.31) \quad \|v(t)\|_{L^\infty(B(\widetilde{R}))} \leq c \left\{ \widetilde{R}^{-3/2} \|v(t)\|_{L^2(B(\widetilde{R}))}^{2/5} \|\nabla v(t)\|_{L^\infty(B(\widetilde{R}))}^{3/5} \right\},
\end{equation}
\begin{align*}
&\leq c \widetilde{R}^{-3/2} E^{1/2} + c C_0 \delta \varepsilon (-t)^{-3/5} \\
&\leq c \left\{ \widetilde{R}^{-3/2} E^{1/2} (-t)^{3/5} + C_0 \delta \varepsilon \right\} (-t)^{-3/5}.
\end{align*}
We may choose $\delta = \frac{1}{2c^{5/3}}$, and then $t_0 \in (-1, 0)$ so that $\widetilde{R}^{-3/2} E^{1/2} (-t_0)^{3/5} \leq \frac{\varepsilon}{2}$. Then, (2.30) follows from (2.31). \hfill \blacksquare

Using the Gagliardo-Nirenberg inequality (A.4) instead of (A.1) in the proof Corollary 2.4 we also get the uniform smallness of the Hölder norm for any Hölder exponent $\gamma \in (0, 1)$.

Corollary 2.5. Let the assumptions of Theorem 2.1 be satisfied. Then for every $0 < \gamma < 1$ and for every $\varepsilon > 0$ there exists $t_1 = t_1(\gamma, \varepsilon) \in (-1, 0)$ such that
\begin{equation}
(2.32) \quad \|v(t)\|_{C^{\gamma, \gamma}} \leq \varepsilon(-t)^{-1 + \frac{3}{2}(1-\gamma)} \quad \forall t_1 \leq t < 0.
\end{equation}

3 \quad Local estimate for the second gradient

Lemma 3.1. Let $2 \leq p < +\infty$ and $v \in L^\infty(-1, 0; L^2(B(1))) \cap L^\infty(-1, 0]; W^{1, \infty}(B(1)))$ be a solution to the Euler equations satisfying
\begin{equation}
(3.1) \quad \sup_{t \in (-1, 0)} (-t) \|\nabla v(t)\|_{L^\infty(B(1))} \leq C_0 < +\infty,
\end{equation}



	
\begin{equation}
(3.2) \quad \sup_{t \in (-1, 0)} (-t)^{-c_0 p^2} \|\nabla^2 v(t)\|_{L^p(B(1/8))} < +\infty,
\end{equation}


\begin{equation}
(3.3) \quad \sup_{t \in (-1, 0)} (-t)^{-c_0 p^2} \|\nabla^2 v(t)\|_{L^p(B(1/8))} < +\infty,
\end{equation}

where $c > 0$ is an absolute constant.
**Proof:** According to Corollary 2.4 we can assume without the loss of generality that

\[ \sup_{t \in (-1,0)} (-t)^{3/5} \|v(t)\|_{L^\infty(B(1))} \leq \frac{1}{32}. \]

We set \( \theta = \frac{1}{4} \), and define

\[ w(y, t) := v(y + (-t)^{\theta} y, t), \quad (y, t) \in \mathbb{R}^3 \times (-1, 0). \]

Clearly, \( w \) solves the following modified Euler equations in \( \mathbb{R}^3 \times (-1, 0) \).

\[ \partial_t w + \frac{\theta (-t)^{\theta - 1} y}{1 + (-t)^{\theta}} \cdot \nabla w + \frac{1}{1 + (-t)^{\theta}} (w \cdot \nabla)w = -\nabla \pi, \]

\[ \nabla \cdot w = 0. \]

Let us set \( \Omega = \nabla \times w \). Then, we find from (3.4) that \( \Omega \) solves the equation.

\[ \partial_t \Omega + \frac{\theta (-t)^{\theta - 1} \Omega}{1 + (-t)^{\theta}} + \frac{\theta (-t)^{\theta - 1} y}{1 + (-t)^{\theta}} \cdot \nabla \Omega + \frac{1}{1 + (-t)^{\theta}} (w \cdot \nabla)\Omega \]

\[ = \frac{1}{1 + (-t)^{\theta}} \Omega \cdot \nabla w \quad \text{in} \quad \mathbb{R}^3 \times (-1, 0). \]

Applying the derivative \( \partial_i, i = 1, 2, 3 \), to the both sides of (3.6), we see that

\[ U_i = \partial_i \Omega = \nabla \times \partial_i w \]

solves the equations

\[ \partial_t U_i + 2 \frac{\theta (-t)^{\theta - 1}}{1 + (-t)^{\theta}} U_i + \frac{\theta (-t)^{\theta - 1} y}{1 + (-t)^{\theta}} \cdot \nabla U_i + \frac{1}{1 + (-t)^{\theta}} (w \cdot \nabla)U_i \]

\[ = \frac{1}{1 + (-t)^{\theta}} \left\{ -(\partial_i w \cdot \nabla)\Omega + U_i \cdot \nabla w + \Omega \cdot \nabla \partial_i w \right\} \]

\[ \text{in} \quad \mathbb{R}^3 \times (-1, 0). \]

Note that \( x := (1 + (-t)^{\theta})y \in B(1) \) for \( y \in B(1/2) \). Let \( \eta \in C^\infty([0, +\infty)) \) be non increasing such that \( \eta \equiv 1 \) on \( [0, \frac{1}{4}] \), \( \eta \equiv 0 \) on \( [\frac{1}{2}, +\infty) \). We set \( \phi(y) = \eta(|y|) \). Let \( 2 \leq p < +\infty \). We multiply (3.7) by \( U_i |U|^{p-2} \phi \), taking the sum from \( i = 1 \) to \( 3 \), integrating it over \( B(1/2) \times (-1, t) \) with \( t \in (-1, 0) \), and applying the integration by
parts, we obtain

\[
\int_{B(1/2)} |U(t)|^p \phi dy + (2p - 3) \int_{-1}^{t} \int_{B(1/2)} \frac{\theta(-s)^{\theta-1}}{1 + (-s)\theta} |U|^p \phi dy dt
\]

\[
- \int_{-1}^{t} \int_{B(1/2) \setminus B(1/4)} \left\{ \frac{\theta(-s)^{\theta-1}}{1 + (-s)\theta} |U|^p \eta'(|y|) |y| + \frac{1}{1 + (-s)\theta} |U|^p w \cdot y \eta'(|y|) \right\} dy ds
\]

\[
= \int_{B(1/2)} |U(-1)|^p \phi dy
\]

\[
+ p \int_{-1}^{t} \int_{B(1/2)} \frac{1}{1 + (-s)\theta} \left\{ - (\partial_i w \cdot \nabla) \Omega + U_i \cdot \nabla w + \Omega \cdot \nabla \partial_i w \right\} \cdot U_i |U|^{p-2} \phi dy ds.
\]

(3.8)

In order to get the positive sign for the third term on the left-hand side of (3.8) we use (3.3), which implies that for all \( y \in B(1/2) \setminus B(1/4) \) it holds

\[
\theta(-s)^{\theta-1} |y| + \frac{w \cdot y}{|y|} \geq \frac{1}{16} (-s)^{-3/4} - \frac{1}{32} (-s)^{-3/5} \geq \frac{1}{32} (-s)^{-3/4}.
\]

Since \( \eta'(|y|) \leq 0 \) for all \( y \in \mathbb{R}^3 \), from (3.8) we deduce the estimate

\[
\int_{B(1/2)} |U(t)|^p \phi dy
\]

\[
\leq \int_{B(1/2)} |U(-1)|^p \phi dy + p \int_{-1}^{t} \int_{B(1/2)} \frac{|\nabla w|}{1 + (-s)\theta} (2|U| + |\Omega||\nabla^2 w|)|U|^{p-1} \phi dy ds.
\]

Observing the Type I condition for \( v \), applying Hölder’s inequality and Young’s inequality, and replacing \( \phi \) by \( \phi^p \), we get from the above inequality

\[
\int_{B(1/2)} |U(t)|^p \phi dy
\]

\[
\leq \int_{B(1/2)} |U(-1)|^p \phi^p dy + pC_0 \int_{-1}^{t} \int_{B(1/2)} (-s)^{-1} (2|U(s)|^p + |\nabla^2 w||U(s)||p^{-1}) \phi^p dy ds
\]

\[
\leq \int_{B(1/2)} |U(-1)|^p \phi^p dy + 2pC_0 \int_{-1}^{t} \int_{B(1/2)} (-s)^{-1} |U|^p \phi^p dy ds
\]

\[
+ pC_0 \int_{-1}^{t} (-s)^{-1} ||\nabla^2 w(s)\phi||_{L^p(B(1/2))} |U(s)\phi||_{L^p(B(1/2))}^{p-1} ds.
\]

(3.9)
Furthermore, using the Biot-Savart law and Calderón-Zygmund inequality together with Lemma A.4, we get the estimate
\[
\left\| \nabla^2 w(s) \phi \right\|_{L^p(B(1/2))} 
\leq c \sum_{i=1}^3 \left\| (\nabla \partial_i w(s)) \phi \right\|_{L^p(B(1/2))} 
\leq cC_p \sum_{i=1}^3 \| \partial_i (\nabla \times w)(s) \phi \|_{L^p(B(1/2))} + cC_p \| \nabla w(s) \nabla \phi \|_{L^p(B(1/2))} 
\leq cp \left( \int_{B(1/2)} |U(s)|^p \phi^p dy \right)^{1/p} + cpC_0(-s)^{-1}.
\]
(3.10)

Hence, (3.9) together with Young’s inequality yields
\[
\int_{B(1/2)} |U(t)|^p \phi^p dy 
\leq cC_0p^2 \int_{-1}^t \int_{B(1/2)} (-s)^{-1} |U|^p \phi^p dyds + cC_0p^2 (-t)^{-p} + \int_{B(1/2)} |U(-1)|^p \phi^p dy.
\]
(3.11)

We define
\[
X(t) = \int_{-1}^t \int_{B(1/2)} (-s)^{-1} |U|^p \phi^p dyds + \frac{cC_0p^2 (-t)^{-p}}{cC_0 - 1} + \frac{1}{cC_0p^2} \int_{B(1/2)} |U(-1)|^p \phi^p dy.
\]

Thanks to (3.11) we find
\[
X'(t) = (-t)^{-1} \int_{B(1/2)} |U(t)|^p \phi^p dy + \frac{cC_0p^2}{cC_0 - 1} (-t)^{-p-1}
\leq cC_0p^2 (-t)^{-1} \int_{-1}^t \int_{B(1/2)} (-s)^{-1} |U|^p \phi^p dyds 
+ cC_0p^2 (-t)^{-p-1} + (-t)^{-1} \int_{B(1/2)} |U(-1)|^p \phi^p dy + \frac{cC_0p^2}{cC_0 - 1} (-t)^{-p-1}
\leq cC_0p^2 (-t)^{-1} X(t).
\]

This shows that \( t \mapsto X(t)(-t)^{-cC_0p^2} \) is non increasing on \((-1,0)\). Consequently,
\[
X(t) \leq (-t)^{-cC_0p^2} X(-1)
\]
(3.12)
\[
= (-t)^{-cC_0p^2} \left\{ \frac{cC_0p}{cC_0 - 1} + \frac{1}{cC_0p^2} \int_{B(1/2)} |U(-1)|^p \phi^p dy \right\}.
\]
Combining (3.11) with (3.12), we arrive at
\[ \| \nabla \Omega(t) \|^p_{L^p(B(1/4))} \leq cC_0 p^2 X(t) \]
\[ \leq (-t)^{-cC_0 p^2} \left\{ \frac{cC_0^2 p^3}{cC_0 - 1} + \| \nabla \Omega(-1) \|^p_{L^p(B(1/2))} \right\}. \]  

(3.13)

Once more, applying Lemma A.4, we get the assertion of the lemma.

If we replace the condition (3.1) by

\[ (-t)\beta \| \nabla v(t) \|^p_{L^\infty(B(1))} < +\infty, \]

for some \( 1 < \beta < 1 \), then we get the following bound for the second gradient.

**Lemma 3.2.** Let \( v \in L^\infty(-1, 0; L^2(B(1))) \cap L^\infty(-1, 0]; W^{1, \infty}(B(1)) \) be a solution to the Euler equations with \( v(-1) \in W^{2-p}(B(1)) \), \( 3 < p < +\infty \). We assume (3.14) holds for some \( 0 < \beta < 1 \). Then,

\[ \| \nabla^2 v(t) \|^p_{L^\infty(-1, 0; L^p(B(1/8)))} < +\infty. \]

**Proof:** Repeating the proof of Lemma 3.1 up to (3.9), and using (3.14) instead of (3.1), we obtain

\[ \int_{B(1/2)} |U(t)|^p \phi^p dy \]
\[ \leq \int_{B(1/2)} |U(-1)|^p \phi^p dy + c \int_{-1}^t \int_{B(1/2)} (-s)^{-\beta} |U|^p \phi^p dy ds \]
\[ + c \int_{-1}^t (-s)^{-\beta} \| \nabla^2 w(s) \|^p_{L^p(B(1/2))} \| U(s) \|^p_{L^p(B(1/2))} ds. \]  

(3.16)

As in the proof of Lemma 3.1 using (3.10) Lemma A.4 and Young’s inequality, we are led to

\[ \int_{B(1/2)} |U(t)|^p \phi^p dy \]
\[ \leq c \int_{-1}^t \int_{B(1/2)} (-s)^{-\beta} |U|^p \phi^p dy ds + c(-t)^{-p-\beta+1} + \int_{B(1/2)} |U(-1)|^p \phi^p dy. \]  

(3.17)

We define

\[ X(t) = \int_{-1}^t \int_{B(1/2)} (-s)^{-\beta} |U|^p \phi^p dy ds + \frac{c}{(p + \beta - 1)(c - 1)} (-t)^{-p-\beta+1} + \int_{B(1/2)} |U(-1)|^p \phi^p dy. \]
In view of (3.17) we obtain
\[ X'(t) = c(p + \beta - 1)(-t)^{-\beta}X(t) \leq c(-t)^{-\beta}X(t). \]

By means of Gronwall’s lemma we find
\[ (3.18) \quad X(t) \leq X(-1)e^{c\int_{-1}^{t}(-s)^{-\beta}ds}. \]

Combining (3.18) with (3.17), we arrive at
\[ (3.19) \quad \|\nabla \Omega(t)\|_{L^p(B(1/4))}^p \leq cX(t) \leq X(-1)e^{c\int_{-1}^{t}(-s)^{-\beta}ds}. \]

\[ (3.20) \quad \leq \left( c + \|\nabla \Omega(-1)\|_{L^p(B(1/4))}^p \right)e^{c\int_{-1}^{t}(-s)^{-\beta}ds}. \]

Applying Lemma [A.4] we get the assertion of the lemma. \[ \square \]

4 Proof of Theorem 1.1

The hypothesis (1.9) implies that there exists \( \eta \in (0, r_0) \) such that
\[ (4.1) \quad \sup_{-\eta < t < 0} (-t)^{-3/2}\|\nabla v(t)\|_{L^\infty(B(\eta))} < 1. \]

Then, by rescaling, one may assume without the loss of generality that \( \eta = 1 \) in (4.1). Let \( 0 < \rho < r_0 \) be fixed. Hence, it will sufficient to show that for every \( x_0 \in B(\rho) \) it holds \( \nabla v \in L^\infty(-1, 0; W^{2, p_0}(B(r/32))) \), where \( r = r_0 - \rho \). We define the rescaled velocity by means of
\[ \tilde{v}(y, t) = r^{3/2}v(x_0 + ry, r^{5/2}t), \quad (y, t) \in B(1) \times (-1, 0). \]

For notational simplicity we write again \( v \) in place of \( \tilde{v} \) and prove that
\[ \nabla v \in L^\infty(-1, 0; W^{2, p_0}(B(1/32))). \]

Thanks to Corollary 2.4 we may assume that
\[ (4.2) \quad \sup_{t \in (-1, 0)} (-t)^{-3/2}v(t)_{L^\infty(B(1/32))} < \frac{1}{10}. \]

Let \( t_0 \in (-1, 0) \) be arbitrarily chosen but fixed. Let \( x_0 \in B(1/4) \). By \( X(x_0, t_0; s) \) we denote the trajectory of the particle which is located at \( x_0 \) at time \( s = t_0 \). More precisely, \( s \mapsto X(x_0, t_0; s) \) solves the following ODE
\[ (4.3) \quad \frac{dX}{ds}(x_0, t_0; s) = v(X(x_0, t_0; s), s) \quad \text{in} \quad [-1, 0], \quad X(x_0, t_0; t_0) = x_0. \]

Since \( v(t) \) is Lipschitz in \( B(1) \) for all \( t \in (-1, 0) \) we first get a local solution of (4.3) in some maximal interval \( I = (t_1, t_2) \), such that \( X(x_0, t_0; s) \in B(1/2) \) for all \( s \in I \). We claim that \( I = (-1, 0) \). In fact integration over \((t, t_0)\) of (4.3) for some \( t \in I \) yields
\[ X(x_0, t_0; t) - x_0 = \int_{t_0}^{t} v(X(x_0, t_0; s), s)ds. \]
Using the triangle inequality along with (4.2), we obtain

\begin{equation}
|X(x_0, t_0; t) - x_0| < \frac{1}{10} \int_t^{t_0} (s)^{-3/5} ds \leq \frac{1}{4}.
\end{equation}

Thus $I \neq (-1, 0)$ would lead to a contradiction, since by (4.4) we may extend the solution to a larger interval, which violates the maximal property of $I$. This shows that the whole trajectory $X(x_0, t_0; t) - x_0$ remains in $B(1/2)$ for all $t \in (-1, 0)$.

Let $\omega = \nabla \times v$, which solves the vorticity equations

\begin{equation}
\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v \quad \text{in} \quad \mathbb{R}^3 \times (-1, 0).
\end{equation}

Observing (4.3), by means of the chain rule we infer from (4.5) that $s \mapsto \omega(X(x_0, t_0; s), s)$ solves the following ordinary differential equation

\[\frac{d}{ds} \omega(X(x_0, t_0; s), s) = \omega(X(x_0, t_0; s), s) \cdot \nabla v(X(x_0, t_0; s), s) \quad \text{in} \quad (-1, 0).\]

Multiplying the above equation by $\omega(X(x_0, t_0; s), s)$, we see that the function $\psi(s) := |\omega(X(x_0, t_0; s), s)|^2$ satisfies the inequality

\[\frac{1}{2} \psi' \leq |\nabla v(X(x_0, t_0; s), s)| \psi \quad \text{in} \quad (-1, 0),\]

Observing the assumption (1.9), the above inequality implies for some $0 < \delta < \frac{2}{5}$

\begin{equation}
\psi' \leq 2(-s)^{-1}(1 - \delta)\psi \quad \text{in} \quad (-1, 0).
\end{equation}

This, immediately shows that $(-s)^{-2(1-\delta)}\psi(s)$ is non increasing in $(-1, 0)$. Accordingly,

\begin{equation}
\psi(s) \leq (-s)^{-2(1-\delta)}\psi(-1) \quad \forall \ s \in (-1, 0).
\end{equation}

In particular, inequality (4.7) with $s = t_0$ yields

\[|\omega(x_0, t_0)| \leq (-t_0)^{-1+\delta}|\omega(X(x_0, t_0; -1), -1)| \leq (-t_0)^{-1+\delta}\|\omega(-1)\|_{L^\infty(B(1))}.\]

This estimate gives

\begin{equation}
\|\omega(t)\|_{L^\infty(B(1/4))} \leq (-t)^{-1+\delta}\|\omega(-1)\|_{L^\infty(B(1))} \quad \forall \ t \in (-1, 0).
\end{equation}

Applying (4.3), we infer from (4.8) together with (4.2) that the following estimate holds for all $3 < q < +\infty$,

\begin{equation}
\|\nabla v(t)\|_{L^q(B(1/8))} \leq c q \left\{ (-t)^{-1+\delta}\|\omega(-1)\|_{L^\infty(B(1))} + (-t)^{-3/5} \right\} \quad \forall \ t \in (-1, 0).
\end{equation}

Noting that $\frac{3}{q} < 1 - \delta$, by means of Sobolev’s embedding theorem we deduce form (4.9) for $\gamma = 1 - \frac{3}{q}$

\begin{equation}
[v(t)]_{C^{\gamma, \gamma}(B(1/8))} \leq c (1 - \gamma)^{-1}(-t)^{-1+\delta}\left( 1 + \|\omega(-1)\|_{L^\infty(B(1))} \right) \quad \forall \ t \in (-1, 0)
\end{equation}
with a constant $c > 0$, which remains bounded as $q \to +\infty$.

Appealing to (A.3) (cf. Lemma A.2) with $n = 3$, we see that for all $t \in (-1, 0)$ the following inequality holds true

$$\|\nabla v(t)\|_{L^\infty(B(1/8))} \leq c[v(t)]_{C^{0, \gamma}(B(1/8))} + c\left([v(t)]_{C^{0, \gamma}(B(1/8))}\right)^{1 - \frac{1 - \gamma}{2 - \gamma - 3/p_0}} \|\nabla^2 v(t)\|_{L^{\frac{2 - \gamma - 3/p_0}{2}}(B(1/8))}.$$ 

We estimate the right-hand side of the above inequality by the aid of (4.10) and (3.2). This gives

$$\|\nabla v(t)\|_{L^\infty(B(1/8))} \leq c(1 - \delta)\left[1 + \|\omega(-1)\|_{L^\infty(B(1))}\right]^{-1 - \frac{1 - \gamma}{2 - \gamma - 3/p_0}} \|\nabla^2 v(t)\|_{L^{\frac{2 - \gamma - 3/p_0}{2}}(B(1/8))}.$$ 

where $C_2 = cC_0p_0^2$ stands for the constant in Lemma 3.1. We can choose $\gamma \in (0, 1)$ such that

$$(-C_2 + 1 - \delta)\frac{1 - \gamma}{2 - \gamma - 3/p_0} \geq -\delta.$$ 

With this choice of $\gamma$ we get

$$(4.11) \quad (t)^{1 - \frac{\delta}{2}}\|\nabla v(t)\|_{L^\infty(B(1/8))} < +\infty.$$ 

Thus, we are in a position to apply Lemma 3.2 for $\beta = 1 - \frac{\delta}{2}$, which yields

$$v \in L^\infty(-1, 0; W^{2, p_0}(B(1/32))).$$ 

This completes the proof of the theorem. 

5 Proof of Theorem 1.2

Let $\zeta \in C^\infty_c(B(1))$ denote a cut off function such that $0 \leq \zeta \leq 1$ in $B(1)$, $\zeta \equiv 1$ on $B(1/2)$, and $|\nabla \zeta| \leq c$. For $0 < r < +\infty$, and $x_0 \in \mathbb{R}^3$ we define

$$\zeta_r = \zeta_r(x) = \zeta\left(\frac{x - x_0}{r}\right), \quad x \in \mathbb{R}^3.$$ 

We set $R := \frac{r}{2}$. Clearly, $\zeta_R \in C^\infty_c(B(x_0, R))$ is a cut off function on $B(R) = B(x_0, R)$ with $|\nabla \zeta_R| \leq cR^{-1}$. We define the modified mean value

$$\bar{f}_{B(R)} = \frac{1}{B(R)} \int_{B(R)} f \zeta_R dx.$$
Let $t \in (-1,0)$, and $0 < R \leq r_0(-t)^{2/5}$ be fixed. For $x_0 \in B(R)$ we get

$$|\nabla v(x_0, t)| \leq |\nabla v(x_0, t) - \nabla v(t)| + |\nabla v(t)|$$

In particular, observing (1.9) from (5.1) with $v_\theta$ from (5.2) that

$$\left| \frac{1}{\zeta_{Rdx}} \int_{B(R)} |\nabla v(x, t) - \nabla v(x, t)| \zeta_{Rdx} + \frac{1}{\zeta_{Rdx}} \int_{B(R)} |\nabla v(x, t)| \zeta_{Rdx} \left| \int_{B(R)} \nabla v(x, t) \zeta_{Rdx} \right| \leq \text{osc}_{B(x_0, R)} (\nabla v(t)) + \frac{1}{\zeta_{Rdx}} \int_{B(R)} |\nabla v(x, t)| \zeta_{Rdx} \left| \int_{B(R)} \nabla v(x, t) \zeta_{Rdx} \right|.$$ 

Applying the integration by parts and Hölder’s inequality, we find

$$\left| \int_{B(R)} \partial_t v(x, t) \zeta_{Rdx} \right| = \left| \int_{B(R)} v(x, t) \partial_t \zeta_{Rdx} \right| \leq cR^{1/2} \|v(t)\|_{L^2(B(R))}.$$ 

This leads to the inequality

(5.1) $$|\nabla v(x_0, t)| \leq \text{osc}_{B(x_0, R)} (\nabla v(t)) + cR^{-5/2} \|v(t)\|_{L^2(B(R))}.$$ 

In particular, observing (1.9) from (5.1) with $R = r_0(-t)^{2/5}$, we deduce

(5.2) $$(-t) |\nabla v(x_0, t)| \leq 1 - \delta + c r_0^{-5/2} \|v(t)\|_{L^2(B(x_0, r_0(-t)^{2/5}))}.$$ 

Since $\|v(t)\|_{L^2(B(\theta r_0(-t)^{2/5}))}$ is bounded by $E^{1/2} = \|v\|_{L^\infty(-1,0; L^2(B(1)))}$, we immediately get from (5.2) that $v$ has Type I blow up at $t = 0$ with respect to the velocity gradient. This allows us to apply Theorem 2.1 which yields the existence of $\tilde{R}$ such that for all $t \in (-\tilde{R}^{5/2}, 0)$ and for all $x_0 \in B(1/2)$ it holds

$$c r_0^{-5/2} \|v(t)\|_{L^2(B(x_0, r_0(-t)^{2/5}))} \leq c r_0^{-5/2} \|v(t)\|_{L^2(B(\tilde{R}))} \leq \frac{\delta}{2},$$

which gives

(5.3) $$\sup_{t \in (-\tilde{R}^{5/2}, 0)} (-t) \|\nabla v(t)\|_{L^\infty(B(\theta/2))} \leq 1 - \frac{\delta}{2}.$$ 

This shows that the condition (1.9) in Theorem 1.1 is satisfied, which yields that $v \in L^\infty(-1,0; W^{2,\infty}(B(r_0/4)))$. This completes the proof of the theorem.

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A Gagliardo-Nirenberg’s inequality on a ball

Lemma A.1. Let $1 \leq p, q < +\infty$. We assume $q > n$. Let $B(R) = B(x_0, R)$ be any ball. Then for all $f \in L^p(B(R)) \cap W^{1,q}(B(R))$ it holds

(A.1) \[ \|f\|_{L^\infty(B(R))} \leq c R^{-n/p} \|f\|_{L^p(B(R))} + c \|f\|_{L^p(B(R))}^{1 - \frac{nq}{pq - pn + qn}} \|\nabla f\|_{L^q(B(R))}^{\frac{nq}{pq - pn + qn}}, \]

where the constant in (A.1) depends only on $p, q$ and $n$ but not on $R > 0$ and $x_0$.

Proof: Let $0 < \lambda \leq 1$. By means of Sobolev’s embedding theorem we find for all $0 < \lambda \leq 1$

(A.2) \[ \|f\|_{L^\infty(B(\lambda))} \leq c \lambda^{-n/p} \|f\|_{L^p(B(1))} + c \lambda^{1-n/q} \|\nabla f\|_{L^q(B(1))}. \]

In case $\|\nabla f\|_{L^q(B(1))} \leq \|f\|_{L^p(B(1))}$ the assertion is trivially fulfilled by setting $\lambda = 1$ in (A.2). In the opposite case we choose $\lambda$ such that $\lambda^{-n/p} \|f\|_{L^p(B(\lambda))} = \lambda^{1-n/q} \|\nabla f\|_{L^q(B(\lambda))}$, i.e.

\[ \lambda = \left( \frac{\|f\|_{L^p(B(1))}}{\|\nabla f\|_{L^q(B(1))}} \right)^{1 - \frac{1-n/q}{n/p}} \leq 1. \]

This implies that

\[ \|f\|_{L^\infty(B(1))} \leq c \|f\|_{L^p(B(1))} + c \|f\|_{L^p(B(1))}^{1 - \frac{nq}{pq - pn + qn}} \|\nabla f\|_{L^q(B(1))}^{\frac{nq}{pq - pn + qn}}. \]

The assertion now follows easily by means of a standard scaling argument.

The following lemma provides an estimate between Hölder space and Sobolev space.

Lemma A.2. Let $1 < p < +\infty$ with $p > n$, and let $0 < \gamma < 1$. Then for every ball and $f \in C^{0,\gamma}(B(R)) \cap W^{2,p}(B(R))$ it holds

(A.3) \[ \|\nabla f\|_{L^\infty(B(R))} \leq c R^{-1-\gamma} \|f\|_{C^{0,\gamma}(B(R))} + c \|f\|_{C^{0,\gamma}(B(R))}^{1 - \frac{1-\gamma}{2-\gamma-n/p}} \|\nabla^2 f\|_{L^q(B(R))}^{1-\gamma}. \]

Proof: As in the proof of Lemma A.1 it will be sufficient to prove the assertion for the unite ball $B(1)$. The general case easily follows by a standard scaling argument. By means of Sobolev’s embedding theorem we find for all $0 < \lambda \leq 1$

\[ \lambda \|\nabla f\|_{L^\infty(B(\lambda) \cap B(x_0, \lambda))} \leq c \lambda^{\gamma} \|f\|_{C^{0,\gamma}(B(1))} + c \lambda^{2-n/q} \|\nabla^2 f\|_{L^q(B(1))}. \]

In case $\|\nabla^2 f\|_{L^q(B(1))} \leq \|f\|_{C^{0,\gamma}(B(1))}$ the assertion is trivially fulfilled. Otherwise, we set

\[ \lambda = \left( \frac{\|f\|_{C^{0,\gamma}(B(1))}}{\|\nabla^2 f\|_{L^q(B(1))}} \right)^{2-\gamma-n/q}. \]

Then the above inequality implies

\[ \|\nabla f\|_{L^\infty(B(1))} \leq c \|f\|_{C^{0,\gamma}(B(1))} + c \|f\|_{C^{0,\gamma}(B(1))}^{1 - \frac{1-\gamma}{2-\gamma-n/q}} \|\nabla^2 f\|_{L^q(B(1))}^{1-\gamma}. \]

This completes the proof of the lemma.

Next, we provide the following elementary estimate of the Hölder semi norm.

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Lemma A.3. For all $f \in L^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ it holds
\begin{equation}
[f]_{C^{0,\alpha}} \leq 2^{1-\alpha} \|f\|_{L^\infty}^{1-\alpha} \|\nabla f\|_{L^\infty}^\alpha.
\end{equation}

**Proof:** Elementary, for $x, y \in \mathbb{R}^n$ with $x \neq y$ we estimate
\[
\frac{|f(x) - f(y)|}{|x - y|^\alpha} = |f(x) - f(y)|^{1-\alpha} \left( \frac{|f(x) - f(y)|}{|x - y|} \right)^\alpha \leq 2^{1-\alpha} \|f\|_{L^\infty} \|\nabla f\|_{L^\infty}^\alpha.
\]
After taking the supremum over all $x, y \in \mathbb{R}^n$ with $x \neq y$ on both sides of the above inequality, we get the assertion of the lemma. \hfill \blacksquare

Using the well known Biot-Savart law together with Calderón-Zygmund’s inequality \cite{15}, we get the following localized inequality.

**Lemma A.4.** Let $1 < p < +\infty$. Then for every ball $B(R) \subset \mathbb{R}^3$ and $u \in W^{1,p}(B(R))$ it holds
\begin{equation}
\|\nabla u \phi\|_{L^p(B(R))} \leq C_p \left( \|\nabla \times (u \phi)\|_{L^p(B(R))} + \|\nabla \phi\|_{\infty} \|u\|_{L^p(B(R))} \right),
\end{equation}
for all non negative $\phi \in C^\infty_c(B(R))$, where $C_p \leq cp$ for $p \in [2, \infty)$, while $C_p \leq \frac{c}{p-1}$ for $p \in (1, 2]$ with $c$ independent of $p$.

**Proof:** Let $\phi \in C^\infty_c(B(R))$ be a non negative function. Since $(\nabla u) \phi = \nabla (u \phi) - u \times \nabla \phi$, by the Biot-Savart law and Calderón-Zygmund inequality we get
\[
\|\nabla (u \phi)\|_{L^p} \leq C_p \|\nabla \times (u \phi)\|_{L^p} = C_p \|\nabla \times u \phi - u \times \nabla \phi\|_{L^p} \leq C_p \|\nabla u \phi\|_{L^p} + C_p \|\nabla \phi\|_{\infty} \|u\|_{L^p(B(R))}.
\]
This immediately leads to (A.5). \hfill \blacksquare

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