WE ARE NOT STUCK WITH GLUING a response to a note of A. Ocneanu by D. Yetter and L. Crane

In [1], we outlined a procedure for constructing a 4D topological Quantum Field Theory (TQFT) from a modular tensor category (MTC).

The construction is related to the well known construction of a 3d tqft. In our announcement we gave the formula for the invariant as follows:

\[
\sum N^{\# \text{vertices} - \# \text{edges}} \prod_{\text{faces}} \dim_q(j) \prod_{\text{tetrahedra}} \dim_q^{-1}(p) \prod_{4\text{-simplexes}} 15J_q \quad (*)
\]

where the sum ranges over all assignments of spins to the faces and tetrahedra of the triangulation and \( j \) represents the spin labelling a face, \( p \) represents the spin labelling the cut interior to a tetrahedron, \( \dim_q \) is the quantum dimension, and \( N \) is the sum of the squares of the quantum dimensions. Here by spins, we mean irreducible representations of quantized \( sl_2 \) at a root of unity.

We have two different ways of thinking of our quantum 15J symbols. One, which really plays a heuristic role for us, is as an invariant of a labelled surface embedded in \( S^3 \). The other, which we use directly in our proof, is as a recombination diagram in a braided tensor category. Perhaps we have been a little too cavalier in using the first picture, since the connection between the two involves some subtleties.

In [2], A. Ocneanu announced the result that the invariant we define is always 1, and asserted that our procedure is equivalent to one he examined earlier, in a different context.

Although we think that professor Ocneanu’s argument is interesting, and in fact that the construction he suggests is of interest even if it does give 1 for any closed 4 manifold, we do not believe that the two constructions are the same. In particular, we know by direct calculation that our invariant is not constant, nor is it 1 for all simple cases.

The point of departure in [2] is the assertion that the formula above is the same as gluing of the 3 manifolds with boundary which are related to the 15J-q symbols defined in [1]. We do not see how this could be the case. Note that in our formula the internal and external spins do not enter in the same way. Gluing would be regarding the 15J symbol as coming from a manifold with boundary, in which the external and internal spins play identical roles. Thus our formula does not appear to have the proper symmetry to express gluing.

There seems to be no way to make professor Ocneanu’s results agree with our calculations. In the first place, he does not get a result which is independant of the triangulation of the 4 manifold. The formula he computed reduces the invariant of a 4 manifold to one for a connected sum of copies of \( S^3 \times S^1 \) [in a 3D TQFT], where the number of copies depends on the triangulation chosen.
If we are to take it that our formula, as normalized, is equivalent to gluing, then we are being told that a topological invariant is equal to a non-invariant. If we are to believe that our formula without the normalization is equal to gluing, we run into the immediate problem that when we join 3 15J symbols around a common face then we do not get the result corresponding to gluing topologically, but rather an extra factor corresponding to a loop which is split off, which corresponds to our factors of N.

If our process were in fact gluing, then the same arguments which allow us to join together parts of the boundary surfaces corresponding to disjoint 15J’s would also allow us to join separate segments of a connected boundary surface to itself. In fact, the combination rules in the category which allow us to join disjoint categorical diagrams do not extend to that case. Indeed, before realizing this, we briefly thought that our invariant would be quite simple [although certainly not constant].

Furthermore, if we take it that the invariant of a connected sum of $S^1 \times S^1$’s is always 1, we would be led to the conclusion that our invariant was always 1. (Probably any constant could be absorbed as a normalization). This, however, contradicts the calculations we have been able to do by hand.

Our calculations show that the invariant of $S^4$ is N ($= 2$ for $r=3$, $= 4$ for $r=4$). Calculating the invariant of $S^3 \times S^1$ is more complicated, but yields 1 for $r=3,4$, not agreeing with the number for $S^4$. The case of $S^2 \times S^2$ is much more complicated. Our initial calculations, which we have not thoroughly checked, yield an expression involving the braiding.

We are left with the problem of how often we obtain a power of N as invariant, and whether some modification of Ocneanu’s argument could tell us that.

The interest in the theory we construct does not reduce to the invariants of compact 4 manifolds. In fact for possible applications to quantum gravity, compact 4 manifolds are irrelevant, since compact spacetimes are not causal. It is also possible that the relative form of our construction for manifolds with boundary could give invariants of embedded surfaces which are richer than the invariants of closed 4 manifolds. For these reasons, we think that Ocneanu’s construction may also be of considerable interest, regardless of its triviality on closed 4 manifolds.

Finally, we still do not know if our invariants can distinguish homeomorphic 4 manifolds. We can see no solution to this problem, except to compute them for some examples like Dolgachev surfaces.

In summary, we believe that professor Ocneanu’s assumption that our formula is equivalent to gluing, although a natural hypothesis, is not supported by the facts.

REFERENCES:

1. L. Crane and D. Yetter A Categorical Construction of 4D Topological Quantum Field Theories ksu preprint
