Kummer configurations and $S_m-$reflector problems: Hypersurfaces in $\mathbb{R}^{n+1}$ with given mean intensity

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Abstract
For a congruence of straight lines defined by a hypersurface in $\mathbb{R}^{n+1}$, $n \geq 1$, and a field of reflected directions created by a point source we define the notion of intensity in a tangent direction and introduce elementary symmetric functions $S_m$, $m = 1, 2, \ldots, n$, of principal intensities. The problem of existence and uniqueness of a closed hypersurface with prescribed $S_n$ is the “reflector problem” extensively studied in recent years. In this paper we formulate and give sufficient conditions for solvability of an analogous problem in which the mean intensity $S_1$ is a given function.

1 Introduction

Fix a Cartesian coordinate system in $\mathbb{R}^{n+1}$, $n \geq 1$, with the origin $O$ and let $S^n$ be the unit sphere centered at $O$. It was shown in [5] that among

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measure-preserving maps of $\mathbb{S}^n$ onto itself transferring two given positive Borel measures into each other there exists a uniquely defined map which is optimal against the cost function with density $-\log(1 - \langle x, y \rangle)$, $x, y \in \mathbb{S}^n$; here $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{n+1}$. This result was established under quite mild assumptions on the given measures. Even prior to [5] it was already shown in [6] (see Theorem 3.4 and the note following its proof), that such optimal map is generated by a closed convex hypersurface $R$ in $\mathbb{R}^{n+1}$ which is star-shaped relative to $O$ and acts as a reflector for light rays emanating from $O$. At smooth points of $R$ this optimal map is given by

$$y = \gamma_R(x) = x - 2\langle x, N(x) \rangle N(x),$$

(1)

where $N$ denotes the unit normal field on $R$. This, of course, is the classical law of reflection. (At nonsmooth points of $R$ there is an appropriate generalization of the map $\gamma_R$.)

Since $R$ is convex, it is almost everywhere of class $C^2$ and one may consider the Jacobian $J(\gamma_R)$ defined almost everywhere on $\mathbb{S}^n$. Then, from the geometric point of view, the $|\det J(\gamma_R)|$ is the quotient of densities of the volume forms defining the two given Borel measures. This relation leads to a (possibly, degenerate) second order elliptic partial differential equation (PDE) of Monge-Ampère type on $\mathbb{S}^n$ for the radial function defining $R$ [17]. (The radial function is defined at the beginning of section 2 below.) Thus, the geometric problem of finding a hypersurface $R$ such that the map (1) transfers two given volume forms into each other requires solution of the corresponding fully nonlinear PDE. This problem is usually referred to as the reflector problem. Existence of a weak solution to the reflector problem was shown in [3] for surfaces in $\mathbb{R}^3$ but the proof is valid verbatim for hypersurfaces in $\mathbb{R}^{n+1}$. Uniqueness was shown in [7], [6] and regularity was studied in [7], [2], [12].

The described results suggest that the map $\gamma_R$ is interesting from several points of view and deserves further investigation. Indeed, in this paper we show that the reflector problem is only one of a series of semilinear and fully nonlinear geometric problems connected with the map $\gamma_R$.

A very natural geometrical framework for studying the map $\gamma_R$ is the Kummer configuration considered by E. Kummer in 1860 in his paper [10] on congruences of straight lines in $\mathbb{R}^3$. Congruences of straight lines arise naturally in geometrical optics and optimal mass transport in $\mathbb{R}^n$ and were considered (in $\mathbb{R}^3$) already in the 18-th century by G. Monge and in the early
part of 19-th century by E. Malus and W.R. Hamilton. In [10] Kummer
defines a congruence of straight lines in \( \mathbb{R}^3 \) by points on a given surface (base)
and a set of direction vectors. This pair, the surface and the vector field, is
referred to as a "Kummer configuration"; see, [9], v. 2, ch. 17. For such
a congruence Kummer introduced notions analogous to the first and second
fundamental forms (the latter is not necessarily symmetric!) and studied its
properties which can be described using these forms. In the years subsequent
to the publication of [10], the dependence of the second fundamental form
of Kummer on the base surface was considered by geometers as a deficiency
and theories avoiding such dependence were developed [4].

In this paper we treat the hypersurface \( R \) as a reflector and the reflected
rays defined by the map \( \gamma_R \) as a congruence of straight lines, that is, we have
a special case of a Kummer configuration \( (R, \gamma_R) \). This point of view is our
starting point, even though the definitions and objectives here are different
from those of Kummer.

The paper is organized as follows. In section 2 we describe the class of
hypersurfaces in \( \mathbb{R}^{n+1} \) for which the map \( \gamma_R \) is studied and derive various
local formulas. In section 3 we introduce the notion of intensity in direction
of a curve and show that in principal directions the principal
intensities are the real eigenvalues of a certain quadratic differential form analogous to
the second fundamental form in classical differential geometry. In the same
section we introduce the elementary symmetric functions \( S_m \) of principal
intensities; here \( m \) is an integer, \( 1 \leq m \leq n \). The problem of finding the
optimal map described in the first paragraph of this introduction, that is, the
reflector problem, corresponds to \( m = n \). In section 4 we establish existence
and uniqueness of solutions to the \( S_1 \)-reflector problem. We intend to present
solutions to other reflector problems in a separate publication.

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2 Reflectors defined by radial functions

In this section our considerations are local. Let \( x = x(u) \equiv x(u^1, \ldots, u^n) \) be a
smooth local parametrization of \( S^n \). Let \( R \) be a hypersurface in \( \mathbb{R}^{n+1} \) which
is a graph over some domain \( \phi \subset S^n \) of a function \( \rho : \phi \to (0, \infty) \), \( \rho \in C^2(\phi) \).
Such \( R \) can be defined by the position vector \( r(x) = \rho(x)x, \; x \in \phi \). (In
this paper \( x \in S^n \) is treated as a point in \( S^n \) and also as a unit vector in
The function $\rho$ is called the radial function of $R$. Obviously, the map
$r : \phi \to \mathbb{R}^{n+1}$ is an embedding. The set of all such hypersurfaces in $\mathbb{R}^{n+1}$ is
denoted by $\mathcal{M}^n$. If we need to indicate the domain $\phi$ we write
$\mathcal{M}^n(\phi)$; in particular, if $\phi \equiv S^n$ we write $\mathcal{M}^n(S^n)$. We will study reflecting properties of
hypersurfaces in $\mathcal{M}^n$ and for brevity refer to them as reflectors.

Denote by $e = e_{ij} du^i du^j$ the standard metric on $S^n$ induced from $\mathbb{R}^{n+1}$. Here and for the rest of the paper the Latin indices $i, j, k, \ldots$ run over the
range $1, 2, \ldots, n$ and the summation convention over repeated lower and upper
indices is in effect. The following notation will be used:

\[
\partial_i = \frac{\partial}{\partial u_i}, \quad x_i = \partial_i x, \quad r_i = \partial_i r, \text{ etc.} \quad \partial_{ij} = \frac{\partial^2}{\partial u_i \partial u_j}, \quad x_{ij} = \partial_{ij} x, \quad r_{ij} = \partial_{ij} r, \text{ etc.}
\]

The covariant differentiation in the metric $e$ is denoted by $\nabla_i := \nabla_{\partial_i}$ and similarly $\nabla_{ij}$, etc. On functions, $\nabla_i = \partial_i$ and $\nabla_{ij} = \partial_{ij} - \Gamma^k_{ij} \partial_k$, where $\Gamma^k_{ij}$ are the Christoffel symbols of the metric $e$. Put $\nabla = e^{ij} x_j \partial_i$, where $[e^{ij}] = [e_{ij}]^{-1}$, and $W^2 = \sqrt{\rho^2 + |\nabla \rho|^2}$.

Let $R \in \mathcal{M}^n$. We recall first the expressions for the classical first and
second fundamental forms of $R$ in terms of its radial function \[\text{[14]}\]. The
coefficients $g_{ij}$ of the first fundamental form $g$ of $R$, the elements of the
inverse matrix $[g^{ij}] = [g_{ij}]^{-1}$, and the determinant of $[g_{ij}]$ are, respectively,

\[
g_{ij} = \langle r_i, r_j \rangle = \rho_i \rho_j + \rho^2 e_{ij}, \quad g^{ij} = \frac{1}{\rho^2} \left( e^{ij} - \frac{\rho^j \rho^i}{W^2} \right),
\]

\[
det[g_{ij}] = \rho^{2n-2} W^2 \det[e_{ij}], \quad (2)
\]

where $\rho^i = e^{ik} \rho_k$. The unit normal field $N$ on $R$ is given by

\[
N = \frac{\rho x - \nabla \rho}{W^2}. \quad (3)
\]

The coefficients of the second fundamental form of $R$ are given by

\[
b_{ij} = -\langle r_j, N_i \rangle = \frac{\rho \nabla_{ij} \rho - 2 \rho_i \rho_j - \rho^2 e_{ij}}{W^2}. \quad (4)
\]

Since $r_i = \rho_i x + \rho x_i$, it follows from \[\text{[1]}\] that $\langle r_i, \gamma_R \rangle = \rho_i$. Differentiating,
we obtain

\[
\langle r_{ij}, \gamma_R \rangle + \langle r_i, \gamma_{R_j} \rangle = \rho_{ij}.
\]
This implies that
\[ \langle r_i, \gamma_{Rj} \rangle = \langle r_j, \gamma_{Ri} \rangle. \]  

(5)

We will need explicit expressions of \( \langle r_i, \gamma_{Rj} \rangle \) and \( \langle \gamma_{Ri}, \gamma_{Rj} \rangle \) in terms of \( \rho \) and its derivatives. To determine \( \langle r_i, \gamma_{Rj} \rangle \), differentiate (1) and take the inner product of the result with \( r_i \). Then
\[ \langle r_i, \gamma_{Rj} \rangle = \rho e_{ij} + 2 \langle x, N \rangle b_{ij}. \]

(6)

Put
\[ \kappa_{ij} := -\frac{\langle r_i, \gamma_{Rj} \rangle}{\rho} \quad \text{and} \quad \hat{e}_{ij} := \langle \gamma_{Ri}, \gamma_{Rj} \rangle. \]

(7)

Noting that \( \langle x, N \rangle = \rho/W_\rho \), we get
\[ -\kappa_{ij} = e_{ij} + \frac{2}{W_\rho} b_{ij}. \]

(8)

For reasons which will become clear in a moment the quadratic differential form \( \kappa = \kappa_{ij} du^i du^j \) will be referred to as the **intensity** form of the congruence \((R, \gamma_R)\). Its geometric meaning will also be described below.

Next, we derive an expression for \( \hat{e}_{ij} \) in terms of \( \rho \). Note first that because for each \( x \in \phi \) the vectors \( r_1(x), ..., r_n(x), N(x) \) form a basis of \( R_{n+1} \) we have
\[ \gamma_{Ri} = \langle \gamma_{Ri}, r_s \rangle g^{sk} r_k + \langle \gamma_{Ri}, N \rangle N. \]

(9)

Using (1), (3), the equations of Weingarten \( N_i = -b_{ij} g^{jk} r_k \) and noting that by (2) \( \rho_k g^{kj} = \frac{\rho}{W_\rho} \), we get
\[ \langle \gamma_{Ri}, N \rangle = -\langle x_i, N \rangle - 2 \langle x, N_i \rangle = \frac{\rho_i}{W_\rho} + 2 b_{ij} g^{jk} \rho_k = -\frac{\rho^i}{W_\rho} \kappa_{ji}. \]

(10)

It follows from (11), (3), (9) and (2) that
\[ \hat{e}_{ij} = \langle \gamma_{Ri}, \gamma_{Rj} \rangle = \kappa_{ik} \left( e^{kl} - \frac{\rho^k \rho^l}{W_\rho^2} \right) \kappa_{ij} + \kappa_{ik} \frac{\rho^k \rho^l}{W_\rho^2} \kappa_{lj} = \kappa_{ik} e^{kl} \kappa_{lj}. \]
3 The Kummer configuration, the intensity form and the $S_m$-reflector problem

It is clear from the discussion in the Introduction that the pair $(R, \gamma_R)$ forms a Kummer configuration with $R$ as the base hypersurface and $\gamma_R$ defining the directions of reflected rays. In geometrical optics the quantity

$$|J(\gamma_R(x))| = \sqrt{\det[\langle \gamma_{R_i}(x), \gamma_{R_j}(x) \rangle]} \sqrt{\det[e_{ij}(x)]}$$

(11)

is called the intensity (or, more accurately, the relative intensity) in the reflected direction $\gamma_R(x)$ \[18\]. This is a very important quantity characterizing reflecting properties of the hypersurface $R$. Assume that the density of the distribution of the light rays emanating from $O$ is given by some function $g(x)$, $x \in \phi \subset S^n$. Then the role of $R$ is to redistribute the energy from the source $O$ so that the reflected rays have directions defined by some given region $O \subset S^n$ and a prescribed density $f(y)$, $y \in O$ \[18\]. The reflector problem as stated, for example, in \[18\], is to determine such $R$; see \[17\] and \[15\] for more details concerning mathematical formulations of this and some related problems.

We clarify now the geometric meaning of the intensity form $\kappa$. Let $\bar{x} \in \phi$ and $x(t)$, $|t-t_0| < \epsilon$ for some $\epsilon > 0$, a smooth curve in $\phi$ such that $r(x(t_0)) = r(\bar{x})$. Denote by $\dot{x}(t)$ the tangent vector to $x(t)$. Define the intensity in direction of $x(t)$ at $t = t_0$ as the quotient

$$\frac{\sqrt{\dot{e}(\dot{x}(t_0))}}{\sqrt{e(\dot{x}(t_0))}}.$$  

(12)

It follows from (6), (7) and (10) that

$$\text{sign}(\kappa) \frac{\sqrt{\dot{e}(\dot{x}(t_0))}}{\sqrt{e(\dot{x}(t_0))}} = \frac{\kappa(\dot{x}(t_0))}{e(\dot{x}(t_0))},$$  

(13)

where $\kappa(\dot{x}(t_0))$ is the value of the form $\kappa$ on the tangent vector to $\dot{x}(t_0)$. Thus, the intensity in direction of $x(t)$ at $t = t_0$ is the rate of change of the angle between $\gamma_R(x(t))$ and $\gamma_R(x(t_0))$ relative to the change of the angle between $x(t)$ and $x(t_0)$. 

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Note that with our choice of the sign, $\kappa$ is positive definite on a unit sphere. This follows from (7) and (4). The coefficients of $\kappa$ are clearly invariant with respect to rescaling $\rho \to \lambda \rho$ with $\lambda > 0$. This is consistent with the invariance of the map $\gamma_R$ with respect to homotheties of $R$ with respect to the origin $O$. It follows from (10) that the three forms, $e, \kappa$ and $\hat{e}$ are not independent.

The form $\kappa$ has also another geometric interpretation. Let $x(t)$ be a smooth curve as before. Suppose also that $\gamma_R$ is a diffeomorphism. Consider the sequence of straight lines $l(t)$ of directions $\gamma_R(t)$ passing through $r(t)$. Since these lines are not parallel, for each $t$, $0 < |t - t_0| < \epsilon$, there exists a unique pair of points $X_0(t_0) \in l(t_0)$ and $X_t \in l(t)$ realizing the distance in $\mathbb{R}^{n+1}$ between these lines. Denote by $h(t)$ the signed distance from $r(t_0)$ to $X_0(t)$ with the “+” sign taken if $\langle X_0(t) - r(t_0), \gamma_R(t_0) \rangle > 0$ and “−” sign otherwise. To calculate the $\lim h(t)$ as $t \to t_0$, observe that the segment realizing the distance between $l(t)$ and $l(t_0)$ is orthogonal to both $\gamma_R(t)$ and $\gamma_R(t_0)$, that is,

$$
\left. \left( \frac{dr}{dt} + \frac{dh}{dt} \gamma_R + h \frac{d\gamma_R}{dt}, \gamma_R \right) \right|_{t=t_0} = 0
$$

and

$$
\left. \left( \frac{dr}{dt} + \frac{dh}{dt} \gamma_R + h \frac{d\gamma_R}{dt}, \frac{d\gamma_R}{dt} \right) \right|_{t=t_0} = 0.
$$

Taking into account that $\langle \frac{dr}{dt}, \gamma_R(t) \rangle = \frac{d\rho}{dt}$ along the curve $x(t)$ (this follows from (11)), the first of this equalities implies $\langle \frac{dr}{dt}, \gamma_R(t) \rangle = -\frac{dh}{dt}$ at $t = t_0$. The second equality implies (see (12)) that

$$
h(\dot{x}(t_0)) = \frac{\rho(x(t_0))\kappa(\dot{x}(t_0))}{\hat{e}(\dot{x}(t_0))}.
$$

A point on the line of direction $\gamma_R(t_0)$ through $r(t_0)$ at the distance $h(\dot{x}(t_0))$ from $r(t_0)$ is called the point of striction. The above formula shows that on each ray there exist a segment filled up with points of striction corresponding to each tangent direction at $x(t_0)$. Of course, this segments may degenerate into a point. Note that $\kappa(\dot{x}(t_0))$ has a positive sign if the linear segment from the point of reflection to the striction point has the same direction as $\gamma_R(x(t_0))$; otherwise it is negative.

Throughout this paper we adapt the convention that when $d\gamma_R(\dot{x}(t_0)) = 0$ then $h(\dot{x}(t_0)) = \infty$. In this case, $\kappa(\dot{x}(t_0)) = 0$. 

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An explicit expression for the \((2,0)\) tensor \(\kappa\) is obtained using (7) and (4):
\[
\kappa = -\rho \nabla^2 \rho + 2 \nabla \rho \otimes \nabla \rho + \left(\frac{(\rho^2 - |\nabla \rho|^2)/2}{W_\rho^2/2}\right) e \quad \text{(in } \phi),
\]
where \(\nabla^2 \rho = \begin{bmatrix} \nabla_{ij} \rho \end{bmatrix}\), \(\nabla \rho \otimes \nabla \rho = \begin{bmatrix} \rho_i \rho_j \end{bmatrix}\).

**Remark 1.** If in (14) we make a change \(\rho = e^{-w}\) then we obtain
\[
\kappa(w) = \frac{\nabla^2 w + \nabla w \otimes \nabla w + \left[(1 - |\nabla w|^2)/2\right] e}{(1 + |\nabla w|^2)/2}.
\]

The tensor \(\kappa(1 + |\nabla w|^2)/2\) is the Schouten tensor of the metric \(e^{-2w} e\) on \(\mathbb{S}^n\). This observation was pointed out to me by M. Gursky [8].

Now we present several examples. For a sphere of radius \(R\), using (4) and (7), we obtain \(\kappa = e\). Similarly, for a piece of a hyperplane in \(\mathbb{R}^n+1\) we have \(\kappa = -e\).

Consider now an ellipsoid of revolution \(E\) with one focus at \(O\) and axis of revolution passing through both foci. Denote by \(a\) the second focus. Using the expressions for \(H_{ij}\) in section 3 of [16], we obtain
\[
\kappa = \frac{\rho(x)}{|\rho(x)x - a|} e.
\]

In this case all the caustic points coincide with the focus \(a\) and \(\kappa\) depends on the point of reflection but not on a particular tangent direction at that point. Note that if the radial function is rescaled with a factor \(\lambda > 0\) and the eccentricity remains fixed then the second focus will be on the same axis but at the distance \(\lambda |a|\) from the first focus \(O\). The expression for \(\kappa\) remains invariant.

Similarly, for a one sheet of a two-sheeted hyperboloid of revolution with the revolution axis passing through the foci, we obtain, by applying the corresponding expressions in section 3 of [16],
\[
\kappa = -\frac{\rho(x)}{|\rho(x)x - a|} e.
\]

Just as in the classical differential geometry, we use the quadratic forms \(e\) and \(\kappa\) to define analogues of the principal curvatures and of the elementary symmetric functions of principal curvatures. For a hypersurface \(R \in \mathcal{M}^n(\phi)\)
at a fixed point $x \in \varnothing$ in an orthonormal basis such that $e_{ij}(x) = \delta_{ij}$ the matrix $[\kappa_{ij}]$ is symmetric and the roots $\lambda_1, \ldots, \lambda_n$ of the polynomial equation

\[ P(\lambda) := \det(a_{ij}^\lambda - \lambda \delta_{ij}) = 0, \text{ where } a_{ij}^\lambda := e^{is} \kappa_{sj}, \]

are real. These roots will be called *principal intensities*.

For an integer $m$, $1 \leq m \leq n$, define the $m-$th intensity function as the elementary symmetric function

\[ S_m(\lambda) = \sum_{1 \leq i_1 < \ldots < i_m \leq n} \lambda_{i_1} \cdots \lambda_{i_m}. \]

These functions are the coefficients of the polynomial

\[ P(-\lambda) = \lambda^n - S_1 \lambda^{n-1} + \ldots + (-1)^n S_n. \]

In particular,

\[ S_n = \frac{\det[\kappa_{ij}]}{\det[e_{ij}]} . \tag{18} \]

It follows from (11) and (11) that $|S_n|$ is the quotient of the volume forms defined by the form $\tilde{e}(\gamma_R(x))$ and the metric $e(x)$, that is, it is the quantity defined by (11). The analogue of the classical mean curvature is the mean intensity $(1/n)S_1$, where

\[ S_1 = e^{ij} \kappa_{ij}. \tag{19} \]

In view of (11), (11) and definition of $\kappa$ we have

\[ S_m(\lambda_1(\rho(x)), \ldots, \lambda_n(\rho(x))) = F_m(a_{ij}^\lambda(x)) \equiv \]

\[ F_m(x, \rho(x), \nabla_1 \rho(x), \ldots, \nabla_n \rho(x), \nabla_{11} \rho(x), \ldots, \nabla_{nn} \rho(x)), \quad x \in \varnothing, \]

where $F_m$ is the sum of principal minors of $[a_{ij}^\lambda]$ of order $m$.

Fix some positive integer $m$, $1 \leq m \leq m$. In analogy with the reflector problem we propose to study the $S_m$-reflector problem for closed hypersurfaces in $\mathbb{R}^{n+1}$ concerned with determination of a closed hypersurface $R \in \mathcal{M}^n(\mathbb{S}^n)$ such that

\[ f(\gamma_R(x)) S_m(\lambda_1(\rho(x)), \ldots, \lambda_n(\rho(x))) = g(x), \quad x \in \mathbb{S}^n, \tag{20} \]

for given positive functions $f$ and $g$ on $\mathbb{S}^n$; here $\rho$ is the radial function of $R$. When $m = n$ this is the reflector problem described in the introduction. For
$m > 1$ these problems lead to fully nonlinear second order PDE’s that have not yet been studied. The semilinear case when $n = 1$ is treated in the next section.

Note that a positive solution $\rho \in C^2(S^n)$ of (20) will always produce an embedded hypersurface in $\mathcal{M}^n(S^n)$ with radial function $\rho$.

It is worthwhile noting that if instead of the form $\kappa$ the Schouten tensor $\kappa(1 + |\nabla w|^2)/2$ is used (see Remark 1) and $\lambda_1, ..., \lambda_n$ are its eigenvalues then the equation

\[ S_m(\lambda_1(w(x)), ..., \lambda_n(w(x))) = ce^{-mw(x)} \text{ on } S^n, \]

for some constant $c$, is the equation of the $S_m$-Yamabe problem on $S^n$ [8]; here $w$ is as in Remark 1.

4 Hypersurfaces with prescribed mean intensity

It follows from (14) that $S_1$ in terms of $\rho$ is given by

\[ S_1(\lambda_1(\rho), ..., \lambda_n(\rho)) = \frac{-\rho \Delta \rho + n\rho^2 + 2|\nabla \rho|^2 - (n/2)W^2}{(1/2)W^2} =: M[\rho], \]

where $\Delta$ is the Laplace operator in the metric $e$.

Note that for a sphere of radius $R$ with center at $O$ we have $S_1 = n \forall x \in S^n$. For a hyperplane the mean intensity $S_1 \equiv -n$. In this case the domain $\phi$ is an open hemisphere. For a paraboloid of revolution $\kappa \equiv 0$ and $S_1 \equiv 0$. The domain $\phi$ in this case is $S^n \setminus \{\xi\}$, where $\xi$ is the axis of the paraboloid. For an ellipsoid of revolution with one focus at the origin $O$ and revolution axis passing through both foci

\[ S_1(\lambda_1(\rho(x)), ..., \lambda_n(\rho(x))) = \frac{n \rho(x)}{|\rho(x)x - a|}. \]

Setting $\rho = 1/v$, we obtain a slightly simpler form of the operator $M$ above,

\[ M[1/v] = \frac{\Delta v + nv - nV}{V}, \text{ where } V = \frac{|\nabla v|^2 + v^2}{2v}. \quad (21) \]

The next proposition shows that there are no hypersurfaces in $\mathcal{M}^n(S^n)$ with $S_1 < n$ and $S_1 > n \forall x \in S^n$. 

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Proposition 2. Let \( R \in \mathcal{M}(\mathbb{S}^n) \). Then there exist points on \( \mathbb{S}^n \) where \( S_1 \geq n \) and \( S_1 \leq n \). Furthermore, the equality \( S_1 = n \) is attained only on concentric spheres centered at \( O \).

Proof. Suppose first that \( S_1 > n \) \( \forall x \in \mathbb{S}^n \). It follows from (21) and the estimate \( V \geq v/2 \) that

\[
0 = \int_{\mathbb{S}^n} \Delta v d\sigma = \int_{\mathbb{S}^n} (S_1 + n) V d\sigma - n \int_{\mathbb{S}^n} v d\sigma \geq \frac{1}{2} \int_{\mathbb{S}^n} (S_1 - n) v d\sigma,
\]

where \( d\sigma \) is the volume element on \( \mathbb{S}^n \). Thus, we arrived at a contradiction.

Suppose now that \( S_1 < n \) \( \forall x \in \mathbb{S}^n \). Let \( x_0 \in \mathbb{S}^n \) be a point where the minimum \( \min_{\mathbb{S}^n} v \) is attained. At \( x_0 \) we have: \( \nabla v = 0 \), \( V = v/2 \), \( \Delta v \geq 0 \). Then by (21) at \( x_0 \) we have \( \Delta v = (S_1 - n)(v/2) \geq 0 \), which is impossible if \( S_1 < n \) on \( \mathbb{S}^n \).

It remains to show that if \( S_1 \equiv n \) then \( R \) is homothetic to \( \mathbb{S}^n \). To show this, note that in this case (21) implies

\[
0 = \int_{\mathbb{S}^n} \Delta v d\sigma = n \int_{\mathbb{S}^n} (2V - v) d\sigma.
\]

Since \( 2V \geq v \), we conclude that \( 2V = v \) and then \( |\nabla v| = 0 \). Hence, \( v = \text{const.} \). QED.

Let \( g : \mathbb{R}^{n+1} \to (0, \infty) \) be a given function. We write \( \mathbb{R}^{n+1} \setminus \{O\} \) as \( \mathbb{S}^n \times (0, \infty) \) and consider the problem of finding a hypersurface \( R \in \mathcal{M}(\mathbb{S}^n) \) defined by the radial function \( \rho : \mathbb{S}^n \to (0, \infty) \) and such that

\[
S_1(\lambda_1(\rho(x)), ..., \lambda_n(\rho(x))) = \bar{g}(x, \rho(x)), \ x \in \mathbb{S}^n,
\]

where \( \bar{g} = ng \). We have the following

Theorem 3. Let \( g \) be a positive \( C^{1,\alpha} \), \( \alpha \in (0, 1) \), function in the annulus \( \mathcal{A} := \{x \in \mathbb{S}^n, \ \rho \in [R_1, R_2]\} \), where \( 0 < R_1 < R_2 < \infty \). Assume that \( g \) satisfies the conditions:

\[
(i) \ g(x, R_1) \geq 1 \quad \text{and} \quad (ii) \ g(x, R_2) \leq 1 \ \forall x \in \mathbb{S}^n.
\]

Then there exists a hypersurface \( R \in \mathcal{M}(\mathbb{S}^n) \) with radial function \( \rho \in C^{2,\alpha}(\mathbb{S}^n) \), \( \rho(x) \in [R_1, R_2] \ \forall x \in \mathbb{S}^n \), satisfying the equation (22).
Proof. Put \( v = 1/\rho \). Then by (21) we need to prove solvability of the equation
\[
\Delta v + n v - n V = V \bar{g}(x, 1/v), \quad x \in \mathbb{S}^n.
\] (24)
This is proved by applying the Leray-Schauder theorem on existence of fixed points to operator equations. In that we essentially follow the general scheme in O. Ladyzhenskaya and N. Uraltseva [11], ch. IV, §10. However, the classes of functions we deal with here were not considered in [11] and we have to redo some of the steps and re-compute the degree of a certain map that arises in our case.

Let
\[
C_{a}^{1,\alpha}(\mathbb{S}^n) = \left\{ w \in C^{1,\alpha}(\mathbb{S}^n) \mid \frac{1}{R_2} \leq w(x) \leq \frac{1}{R_1} \quad \forall x \in \mathbb{S}^n \right\}
\] (25)
and for \( v \in C^2(\mathbb{S}^n) \) and \( w \in C_{a}^{1,\alpha}(\mathbb{S}^n) \) put
\[
\hat{\Delta} v := \Delta v + \frac{n v}{2}, \quad Q(x, w, \nabla w) := \frac{n|\nabla w|^2}{2w} + \frac{w^2 + |\nabla w|^2}{2w} \bar{g}(x, 1/w),
\]
\[
q(w) := \frac{n w^{1+\epsilon} \bar{R}^\epsilon}{2}, \quad Q^\tau(x, w, \nabla w) := \tau Q(x, w, \nabla w) + (1 - \tau)q(w), \quad \tau \in [0, 1],
\]
where \( \epsilon > 0 \) and \( \bar{R} \in (R_1, R_2) \) are some fixed numbers. Consider the family of problems
\[
\hat{\Delta} v = Q^\tau(x, w, \nabla w), \quad x \in \mathbb{S}^n, \quad \tau \in [0, 1], \quad w \in C_{a}^{1,\alpha}(\mathbb{S}^n).
\] (26)
Note that when \( \tau = 1 \) and \( w = v \) we obtain the equation (24).

It is well known that the two smallest eigenvalues of \(-\Delta\) on \( \mathbb{S}^n \) are 0 and \( n \). Therefore, the uniformly elliptic operator \( \hat{\Delta} \) has a trivial kernel in \( W^{1,2}(\mathbb{S}^n) \) and, consequently, in \( C^{1,\alpha}(\mathbb{S}^n) \). The function \( g \in C^{1,\alpha}(\mathbb{S}^n \times [R_1, R_2]) \). Hence, for any \( w \in C_{a}^{1,\alpha}(\mathbb{S}^n) \) the right-hand side of (26) is in \( C^{\alpha}(\mathbb{S}^n) \). Then, standard results on solvability of linear uniformly elliptic second order partial differential equations on \( \mathbb{S}^n \) imply that the equation (26) has a unique solution \( v_\tau \in C_{a}^{2,\alpha}(\mathbb{S}^n) \) for any \( w \in C_{a}^{1,\alpha}(\mathbb{S}^n) \) and \( \tau \in [0, 1] \). Furthermore, by the Schauder estimate for any solution of (26)
\[
\| v_\tau \|_{C_{a}^{2,\alpha}(\mathbb{S}^n)} \leq C \| \hat{\Delta} v_\tau \|_{C^{\alpha}(\mathbb{S}^n)},
\] (27)
where \( C > 0 \) is a constant depending on dimension \( n \) and the coefficients of the standard metric of \( \mathbb{S}^n \). For domains in \( \mathbb{R}^n \) the Schauder inequality
can be found, for example, in [11], ch. III, inequality (1.11). By applying it in coordinate charts covering $\mathbb{S}^n$ (for example, the charts obtained with the stereographic projections from the North and South poles) and using a suitably large constant $C$, one obtains (27). (The term $\max_{\mathbb{S}^n} |v_\tau|$ usually included in the right hand side of (27) is not needed here because $\ker \Delta = \{0\}$.) Thus, we have an operator $T(w, \tau) : C^{1,\alpha}(\mathbb{S}^n) \times [0, 1] \to C^{2,\alpha}(\mathbb{S}^n)$.

We want to apply the Leray-Schauder theorem to the equation

$$w = T(w, \tau)$$

(28)

in the following setting. For a constant $A > 0$ to be specified later, put

$$U := \{ w \in C^{1,\alpha}_{\alpha}(\mathbb{S}^n) \mid \| w \|_{C^{1,\alpha}(\mathbb{S}^n)} < A + 1 \}$$

(29)

and let $\bar{U}_1 = \bar{U} \times [0, 1]$. Our goal is to show that $A$ can be chosen so that the following conditions hold:

(a) the set $U$ is connected and the map $T : \bar{U}_1 \to C^{1,\alpha}(\mathbb{S}^n)$ is completely continuous,

(b) under the additional assumption that both inequalities in (25) are strict, the boundary of the set $U$ does not contain solutions of (28) for all $\tau \in [0, 1]$,

(c) for $\tau = 0$ the equation (28) has a unique solution $w_0 = 1/\bar{R}$ and the linearized operator $(\hat{\Delta} - Q^\tau_{w_0})$, where $Q^\tau_{w_0}$ is the Fréchet derivative evaluated on $w_0$ at $\tau = 0$ is invertible as a map from $C^{2,\alpha}(\mathbb{S}^n)$ to $C^{\alpha}(\mathbb{S}^n)$.

Under these circumstances, the Leray-Schauder degree of the operator $\text{Id} - T(\cdot, 1)$ mapping $U$ to 0 is defined and can be calculated to be $\pm 1$. Consequently, by the Leray-Schauder theorem there exists a $C^{2,\alpha}(\mathbb{S}^n)$ solution to (26) at $\tau = 1$. The additional assumption in (b) will be removed at the end of the proof.

Below, along with (28), we will consider the equations

$$\hat{\Delta} w = Q^\tau(x, w \nabla w), \; \forall x \in \mathbb{S}^n, \; \tau \in [0, 1].$$

(30)

As it was already noted any $w \in C^{1,\alpha}_{\alpha}(\mathbb{S}^n)$ substituted into the right hand side of (30) for some $\tau \in [0, 1]$ gives a solution in $C^{2,\alpha}(\mathbb{S}^n)$. Thus, any $w \in C^{2}(\mathbb{S}^n)$ satisfying (30) is, in fact, in $C^{2,\alpha}(\mathbb{S}^n)$. By construction, such $w$ also satisfies (28) with the same $\tau$.

It is clear that the converse is also true. Namely, any $w \in C^{1,\alpha}_{\alpha}(\mathbb{S}^n)$ satisfying (28) for some $\tau \in [0, 1]$ is in $C^{1,\alpha}_{\alpha}(\mathbb{S}^n) \cap C^{2,\alpha}(\mathbb{S}^n)$ and satisfies (30).
Indeed, by construction, \( T(w, \tau) \) is a solution of (20) when such \( w \) is inserted into the right hand side of (26). By Schauder’s theorem \( T(w, \tau) \in C^{2,\alpha}(\mathbb{S}^n) \) and because of (28) \( w \in C^{1,\alpha}(\mathbb{S}^n) \cap C^{2,\alpha}(\mathbb{S}^n) \) and satisfies (30) (cf. [11], p. 372). This note is used below without further reminding.

Now, we prove (a). The connectedness of \( U \) is clear as, in fact, \( U \) is convex. To check that \( T \) is completely continuous, we verify that 1) \( T(w, \tau) \) is continuous in \( (w, \tau) \) in \( \bar{U}_1 \) and continuous in \( \tau \) uniformly with respect to \( w \in \bar{U} \) and 2) for each fixed \( \tau \in [0, 1] \) the map \( T(w, \tau) \) maps \( \bar{U} \) into a compact set in \( C^{1,\alpha}(\mathbb{S}^n) \).

To check 1) consider \( (w, \tau), (w', \tau') \in \bar{U}_1 \) and the corresponding solutions \( v \) and \( v' \). Put \( \bar{v} := v' - v \). Then

\[
\hat{\Delta} \bar{v} = \tau'[Q(x, w', \nabla w') - Q(x, w, \nabla w)]
+ (1 - \tau')[q(w') - q(w)] + (\tau' - \tau)[Q(x, w, \nabla w)] - q(w)].
\]

(31)

Using the interpolation \( w^s := sw' + (1 - s)w, \ s \in [0, 1], \) we obtain

\[
Q(x, w'(x), \nabla w'(x)) - Q(x, w(x), \nabla w(x)) = a^i(x)\nabla_i \bar{w}(x) + a(x)\bar{w}(x),
q(w'(x)) - q(w(x)) = b(x)\bar{w}(x),
\]

where \( \bar{w} := w' - w \), and

\[
a_i(x) = \int_0^1 \frac{\partial Q(x, w^s(x), \nabla w^s(x))}{\partial \nabla_i w^s}ds, \quad a(x) = \int_0^1 \frac{\partial Q(x, w^s(x), \nabla w^s(x))}{\partial w^s}ds,
\]

\[
b(x) = \int_0^1 \frac{\partial q(w^s(x))}{\partial w^s}ds.
\]

It is clear that \( a^i, a, b \in C^\alpha(\mathbb{S}^n), \ i = 1, ..., n, \) and their \( C^\alpha(\mathbb{S}^n) \) norms are bounded by a constant depending on \( A \), the \( \| \bar{g} \|_{C^{1,\alpha}(A)} \) and \( R_1, R_2 \). Treating (31) as a linear equation with respect to \( \bar{v} \) and applying (27), we conclude that 1) is true.

As it was already noted \( T(w, \tau) \in C^{2,\alpha}(\mathbb{S}^n) \) for each \( \tau \in [0, 1] \) and any \( w \in \bar{U} \). Since a set of functions bounded in the norm of \( C^{2,\alpha}(\mathbb{S}^n) \) is compact in \( C^{1,\alpha}(\mathbb{S}^n) \), the operator \( T(w, \tau) \) maps \( \bar{U} \) into a set compact in \( C^{1,\alpha}(\mathbb{S}^n) \), that is, \( T(w, \tau) \) is a compact map from \( \bar{U}_1 \) into \( C^{1,\alpha}(\mathbb{S}^n) \). This proves 2). Note that 1) and 2) hold with any \( A < \infty \). This completes the proof of (a).

Next, we establish (b). We will need the following
Lemma 4. Suppose \( w \in C^2(S^n) \) and satisfies (30) for some \( \tau \in [0,1] \). Assume in addition that

\[
\frac{1}{R_2} \leq w(x) \leq \frac{1}{R_1} \quad \forall x \in S^n.
\]  

(32)

Then either \( w \equiv 1/R_2 \), or \( w \equiv 1/R_1 \), or

\[
\frac{1}{R_2} < w(x) < \frac{1}{R_1} \quad \forall x \in S^n.
\]  

(33)

The proof of this lemma will be given in the appendix.

We now impose a temporary additional restriction on the function \( g \):

\[
g(x, R_1) \neq 1 \text{ and } g(x, R_2) \neq 1 \text{ on } S^n.
\]  

(34)

Let us show that if (34) holds then neither \( 1/R_1 \) nor \( 1/R_2 \) is a solution of (30) for any \( \tau \in [0,1] \). Suppose \( w(x) \equiv 1/R_2 \). Then for each \( \tau \in [0,1] \) and \( \forall x \in S^n \) we have

\[
D^\tau(x) := \Delta \left( \frac{1}{R_2} \right) - Q^\tau(x, \frac{1}{R_2}, 0) = \frac{n}{2R_2} \left[ 1 - \tau g(x, R_2) - (1 - \tau) \left( \frac{\bar{R}}{R_2} \right)^\tau \right].
\]

Because of (34) and the inequality (ii) in (23) \( g(\bar{x}, R_2) < 1 \) at some \( \bar{x} \in S^n \). Since \( \bar{R}/R_2 < 1 \), we conclude that \( D^\tau(\bar{x}) > 0 \) for all \( \tau \in [0,1] \) and this proves our claim for \( w \equiv 1/R_2 \). The claim regarding \( w \equiv 1/R_1 \) is proved similarly.

Lemma 4 implies now that under conditions (34) any \( w \in C^2(S^n) \) satisfying (30) for \( \tau \in [0,1] \) and

\[
\frac{1}{R_2} \leq w(x) \leq \frac{1}{R_1} \quad \forall x \in S^n
\]  

(35)

is in fact such that

\[
\frac{1}{R_2} < w(x) < \frac{1}{R_1} \quad \forall x \in S^n.
\]  

(36)

Next, we check the applicability of the gradient estimates in [11], ch. IV, Theorems 3.1 and 6.1, to solutions to (30) satisfying (35). For that we need to check two conditions, the first of which (corresponding to (3.1) in ch. IV, §3) in our case reduces to positive definiteness and uniform boundedness \( \forall x \in S^n \) of the quadratic form \( e^{ij}(x)\xi_i\xi_j \), \( \xi \in \mathbb{R}^n \). This is a consequence of the properties of the metric \( e \) on \( S^n \). The second condition (corresponding
to (3.2) in ch. IV, §3) follows from the following estimate: for all \( x \in S^n \) and any \( w \in C^{1,\alpha}(S^n) \) satisfying (35) the inequality

\[
|\xi| (1 + |\xi|) + |Q^\tau(x, w, \xi)| \leq c_0 (1 + |\xi|)^2
\]

holds for all \( \tau \in [0, 1] \) and all \( \xi \in \mathbb{R}^n \), where \( |\xi| = (e^{ij} \xi_i \xi_j)^{1/2} \); the constant \( c_0 = c_0(n, R_1, R_2, \max_A g) \) and it is finite in a fixed coordinate atlas on \( S^n \). It follows from Theorem 3.1 in [11], ch. IV, that for any \( \tau \in [0, 1] \) and any solution \( w \) of (30) which is in \( C^{2,\alpha}(S^n) \cap C^{1,\alpha}(S^n) \) the estimate

\[
|\nabla w(x)| \leq c_1,
\]

holds on \( S^n \); here, in each coordinate chart the constant \( c_1 \) depends on the same parameters as \( c_0 \) above and on the distance from \( x \) to the boundary of the chart. By compactness of \( S^n \) we conclude that \( \|w\|_{C^1(S^n)} \) is bounded by a constant depending only on \( n, R_1, R_2 \) and \( \|g\|_{C^1(A)} \). We keep the same notation, \( c_1 \), for that constant.

It remains to estimate the seminorm \( |\nabla w|_{C^0(S^n)} \). We use for that a standard procedure; see, for example, [11], ch. IV, §6, where this is done for domains in Euclidean space. Put \( V := \frac{w^2 + |\nabla w|^2}{2w} \), fix an integer \( 1 \leq s \leq n \) and differentiate covariantly the equation (30) with respect to the local variable \( u^s \). Then, noting that \( \nabla_s V = \frac{w^s \phi_{is}}{w} \), where \( \phi_{is} = \nabla_{is} w + (w - V)e_{is} \), we get

\[
e^{ij} \nabla_s \nabla_{ij} w + \frac{n \nabla_s w}{2} = \tau \left[ \frac{n + \bar{g}}{w} w^i q_{is} + V(\nabla_s \bar{g} + \bar{g}_w \nabla_s w) - \frac{n \nabla_s w}{2} \right] + (1 - \tau) \frac{n (1 + \epsilon) w^s \bar{R}^s \nabla_s w}{2};
\]

here, \( \bar{g}_w := \partial \bar{g} / \partial w \). By the Ricci identity

\[
\nabla_s \nabla_{ij} w - \nabla_j \nabla_{is} w = e_{is} \nabla_j w - e_{ij} \nabla_s w.
\]

Putting \( z := \nabla_s w \), we obtain

\[
\Delta z + \frac{2 - n}{2} z - \tau \left\{ n + \bar{g} \frac{w}{w} w^i \nabla_i z + \left[ \frac{n + \bar{g}}{w} (w - V) + V \bar{g}_w - \frac{n}{2} \right] z \right\}
\]

\[-(1 - \tau) \frac{n (1 + \epsilon) w^s \bar{R}^s z}{2} = \tau V \nabla_s \bar{g}.
\]
(Note that $\nabla iz = \partial_izw - \Gamma^k_i\nabla_kw$.) This is a second order linear uniformly elliptic equation on $S^n$ with respect to $z$. For $w \in C^{1,\alpha}_a(S^n)$ its coefficients are in $C^\alpha(S^n)$ and by the Schauder estimate ([11], ch. III, inequality (1.11)) we have

$$\|z\|_{C^{2,\alpha}(S^n)} \leq c_2(\|\tau V\nabla_s \bar{g}\|_{C^\alpha(S^n)} + \max_{S^n}|z|),$$

where $c_2 = c_2(n, R_1, R_2, \max A g, \|w\|_{C^1(S^n)})$ and by the Schauder estimate ([11], ch. III, inequality (1.11)) we have

$$\|z\|_{C^{2,\alpha}(S^n)} \leq c_2^2(\|\tau V\nabla_s \bar{g}\|_{C^\alpha(S^n)} + \max_{S^n}|z|),$$

where $c_2 = c_2(n, R_1, R_2, \max A g, \|w\|_{C^1(S^n)}).$ Since $\|\tau V\nabla_s \bar{g}\|_{C^\alpha(S^n)}$ and $\max_{S^n}|z|$ were already estimated through $n, R_1, R_2, \max A g$, the last inequality implies that $|z|_{C^\alpha(S^n)}$ is bounded by a constant depending only on $n, R_1, R_2, \max A g, \|w\|_{C^1(S^n)}$. Thus, there exists a constant $c_3$, depending only on $n, R_1, R_2, \max A g, \|w\|_{C^1(S^n)}$ such that the inequality

$$\|z\|_{C^{1,\alpha}(S^n)} < c_3$$

holds for all $w \in C^{1,\alpha}_a(S^n)$ satisfying (28) and each $\tau \in [0, 1]$.

Now, the constant $A$ in (29) can be set equal to $c_3$. Then we conclude that under the restrictions (34) there are no solutions to (28) on the boundary of $U$.

Finally, we establish (c) and calculate the degree of $\Id - T(\cdot, 1)$. First we consider the equation (30) when $\tau = 0$:

$$\tilde{\Delta}w = q(w) \text{ on } S^n, \quad w \in \bar{U}.$$ 

(37)

A direct substitution shows that $w_0 = 1/\bar{R}$ is a solution of (37). Let us show that this solution is unique. Suppose $w' \in \bar{U}$ is a solution of (37) different from $1/\bar{R}$. At a point $x_{\max} \in S^n$ where $w'$ attains its maximum we have $\Delta w'(x_{\max}) \leq 0$ and then by (37),

$$w'(x_{\max}) \leq \frac{1}{\bar{R}}.$$ 

Similarly, at a point $x_{\min} \in S^n$ where $w'$ attains its minimum

$$w'(x_{\min}) \geq \frac{1}{\bar{R}}.$$ 

Thus, the solution $w_0 = 1/\bar{R}$ of (37) is unique.

Put $\Phi(w, \tau) := \tilde{\Delta}w - Q^\tau(x, w, \nabla w)$, $w \in C^{1,\alpha}_a(S^n) \cap C^{2,\alpha}(S^n)$, $\tau \in [0, 1]$. Then letting $w_s = w + sh$, $h \in C^2(S^n)$, calculating the derivative with respect to $s$ and setting $s = 0$ we obtain

$$\Phi_w(w, \tau)(h) := \left. \frac{d\Phi(w + sh, \tau)}{ds} \right|_{s=0} = \tilde{\Delta}h - \tau \frac{dQ(x, w + sh, \nabla(w + sh))}{ds} \bigg|_{s=0}.$$ 

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The calculated weak derivative is in fact the Fréchet derivative since it is uniformly continuous in $w$ in some $C^{2,\alpha}(\mathbb{S}^n)$ neighborhood of $w$ and continuous in $h$ as a map from $C^{2,\alpha}(\mathbb{S}^n)$ into $C^{\alpha}(\mathbb{S}^n)$.

Evaluating the above expression on $w_0$ and $\tau = 0$, we get

$$\Phi_w(w_0, 0)(h) = \Delta h - \frac{n\epsilon}{2} h.$$ 

Then

$$\ker \Phi_w(w_0, 0) = \{ h \in C^2(\mathbb{S}^n) \mid \Delta h - \frac{n\epsilon}{2} h = 0 \text{ on } \mathbb{S}^n \}.$$ 

Since we have chosen $\epsilon > 0$, it is easy to see that $\ker \Phi_w(w_0, 0) = \{0\}$. Standard results on linear elliptic partial differential equations imply that the map $\Phi_w(w_0, 0) : C^{2,\alpha} \to C^\alpha$ is an isomorphism.

By (a) and (b), the degree of the maps $\text{Id} - T(\cdot, t)$ into 0 is defined for all $(w, t) \in \bar{U}_1$ and satisfying (35). By standard results this degree is the same for all $t \in [0, 1]$. Furthermore, since $\Phi_w,w_0,0) : C^{2,\alpha}(\mathbb{S}^n) \to C^\alpha(\mathbb{S}^n)$ is an isomorphism, the derivative $\text{Id} - T_w(w_0, 0)$ is invertible for all $w \in C^\alpha(\mathbb{S}^n)$ which are in a small $C^\alpha(\mathbb{S}^n)$ neighborhood of $w_0$ and satisfy (36). On the other hand, $\text{deg}(\text{Id} - T(\cdot, 0), U, 0) = \text{deg}(\text{Id} - T_w(w_0, 0), B, 0)$, where $B$ is a ball in $C^\alpha(\mathbb{S}^n)$ with the center at 0 and sufficiently small radius in the $C^\alpha(\mathbb{S}^n)$ norm (cf. [13], section 2.8). Since $\Phi_w(w_0, 0)$ is invertible, its degree is $\pm 1$. Consequently, $\text{deg}(\text{Id} - T_w(w_0, 0), B, 0) = \pm 1$ and thus $\text{deg}(\text{Id} - T(\cdot, 1), U, 0) = \pm 1 \neq 0$. By the Leray-Schauder theorem, the equation (26) has a fixed point $w$ for $\tau = 1$ in $C^{1,\alpha}(\mathbb{S}^n)$ and by the Schauder theorem $w \in C^{1,\alpha}(\mathbb{S}^n) \cap C^{2,\alpha}(\mathbb{S}^n)$. This completes the proof of the theorem under the restrictions (34).

If the restrictions (34) are not satisfied, that is, $g(x, R_1) \equiv 1$ (or $g(x, R_2) \equiv 1$), then a substitution of $v \equiv 1/R_1 \ (v \equiv 1/R_2)$ into (24) shows that $v \equiv 1/R_1 \ (v \equiv 1/R_2)$ is a solution. Thus, the theorem is true also in these cases. QED.

The next proposition deals with the question of uniqueness of a solution found in Theorem 3 and provides also some additional information about such solutions.

**Proposition 5.** Suppose the conditions in Theorem 3 are satisfied and let $R_1$ and $R_2$ be two hypersurfaces in $\mathcal{M}^n(\mathbb{S}^n)$ with radial functions $\rho^1$ and $\rho^2$. 

\(-1 - \tau \frac{d q(w + sh)}{ds}
\) 

\(\bigg|_{s = 0}.\)
satisfying (22). If, in addition,

$$\frac{\partial g}{\partial \rho} \leq 0 \ \forall (x, \rho) \in \mathbb{S}^n \times [R_1, R_2]$$

then $\rho^1(x) = C \rho^2(x)$ $\forall x \in \mathbb{S}^n$ and some constant $C > 0$. Furthermore, each solution $\rho \in C^2(\mathbb{S}^n)$ of (22) such that $R_1 \leq \rho(x) \leq R_2 \ \forall x \in \mathbb{S}^n$ is either $\equiv R_1$ or $\equiv R_2$ or

$$R_1 < \rho(x) < R_2 \ \forall x \in \mathbb{S}^n. \quad (39)$$

**Proof.** Suppose $\rho^2 > \rho^1$ for some $x \in \mathbb{S}^n$. Put $\rho^0 := C \rho^2$, where the constant $C \in (0, 1)$ is chosen so that $\rho^0(\bar{x}) = \rho^1(\bar{x})$ for some $\bar{x} \in \mathbb{S}^n$ and $\rho^0(x) \leq \rho^1(x)$ in some neighborhood $U \subset \mathbb{S}^n$ of $\bar{x}$. Such $U$ is taken sufficiently small so that $g(x, \rho^0(x))$ is defined for all $x \in U$. Because $M$ is homogeneous of order zero in $\rho$ and by (38), we have for all $x \in U$

$$M[\rho^1] - \bar{g}(x, \rho^1(x)) - M[\rho^0] + \bar{g}(x, \rho^0(x)) \geq 0,$$

$$M[\rho^1] - g(x, \rho^1(x)) - M[\rho^2] + \bar{g}(x, \rho^2(x)) = 0.$$

The operator $M[\rho^1] - \bar{g}(x, \rho^1(x))$ is defined on $\rho^t = (1 - t)\rho^0 + t\rho^1$, $\forall (x, t) \in U \times [0, 1]$ and negatively uniformly elliptic. The left hand side of the last inequality can be written as

$$M[\rho^1] - M[\rho^0] - \bar{g}(x, \rho^1(x)) + \bar{g}(x, \rho^0(x)) = -M'[\bar{\rho}] - \bar{\rho}(x) \int_0^1 \frac{\partial \bar{g}(x, \rho^t(x))}{\partial \rho} dt,$$

where $\bar{\rho} = \rho^1 - \rho^0$ and $M'$ is a positively uniformly elliptic second order linear operator in $U$. Thus,

$$M'[\bar{\rho}] + \bar{\rho}(x) \int_0^1 \frac{\partial \bar{g}(x, \rho^t(x))}{\partial \rho} dt \leq 0 \ \forall x \in U.$$

Since $\bar{\rho} \geq 0$ in $U$ and $\bar{\rho}(\bar{x}) = 0$, the strong maximum principle (see, [1], Theorem B) implies $\rho^0(x) = \rho^1(x) \equiv 0$ in $U$. Consequently, the set $\{x \in \mathbb{S}^n \mid \rho^0(x) = p^1(x)\}$ is open in $\mathbb{S}^n$. Since it is also closed, $\rho^0(x) = \rho^1(x) \ \forall x \in \mathbb{S}^n$ and therefore $\rho^1 = C \rho^2$ with some $C \in (0, 1)$. Reversing the roles of $\rho^2$ and $\rho^1$, if necessary, we conclude that $\rho^1(x) = C \rho^2(x) \ \forall x \in \mathbb{S}^n$ with some constant $C > 0$.

The last statement of the proposition follows from Lemma 4. QED.
5 Appendix - Proof of Lemma 4

The claim in this lemma follows essentially from Aleksandrov’s geometric (strong) form of the maximum principle.

Let $w \in C^2(\mathbb{S}^n)$ and satisfies (30) for some $\tau \in [0,1]$. Assume also that (32) holds. Suppose there exists some $x_0 \in \mathbb{S}^n$ such that $w(x_0) = 1/R_1$ and $w(x) \not\equiv 1/R_1$. Consider

$$w^s(x) := (1 - s)w(x) + \frac{s}{R_1}, \quad (x, s) \in \mathbb{S}^n \times [0,1].$$

Observe that

$$\frac{1}{R_2} \leq w^s(x) \leq \frac{1}{R_1} \quad \forall (x, s) \in \mathbb{S}^n \times [0,1].$$

The operator $\hat{\Delta}w^s - Q^\tau(x, w^s, \nabla w^s)$ is defined and uniformly elliptic on $\mathbb{S}^n$ for all $s \in [0,1]$.

For $w^0 = w$ and $w^1 = 1/R_1$ we have, taking into account that $g(x, R_1) \geq 1$ and $\bar{R} > R_1$,

$$\hat{\Delta}w - Q^\tau(x, w, \nabla w) - \left[ \hat{\Delta} \left( \frac{1}{R_1} \right) - \tau \frac{ng(x, R_1)}{2R_1} - (1 - \tau) \frac{n\bar{R}^\epsilon}{2R_1^{1+\epsilon}} \right]$$

$$= -\frac{n}{2R_1} \left[ 1 - \tau g(x, R_1) - (1 - \tau) \frac{\bar{R}^\epsilon}{R_1} \right] \geq 0 \text{ in } \mathbb{S}^n.$$

Since $w(x_0) = 1/R_1$ and $w(x) \leq 1/R_1$ on $\mathbb{S}^n$ it follows from Theorem C in [1] that $w(x) \equiv 1/R_1$ in some neighborhood $\mathcal{K}$ of $x_0$ on $\mathbb{S}^n$. By continuity of $w$ the equality $w(x) \equiv 1/R_1$ holds in $\bar{\mathcal{K}}$. Thus, the set of points on $\mathbb{S}^n$ where $w(x) \equiv 1/R_1$ is open and closed on $\mathbb{S}^n$. Hence, $w(x) \equiv 1/R_1$ on $\mathbb{S}^n$.

Similarly, it is shown that $w \equiv 1/R_2$ if $w(x_1) = 1/R_2$ at some $x_1 \in \mathbb{S}^n$. This completes the proof of Lemma 4.

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