LECTURES ON TWO-DIMENSIONAL NONCOMMUTATIVE GAUGE THEORY

2. Quantization

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These notes comprise the second part of two articles devoted to the construction of exact solutions of noncommutative gauge theory in two spacetime dimensions. Here we shall deal with the quantum field theory. Topics covered include an investigation of the symmetries of quantum gauge theory on the noncommutative torus within the path integral formalism, the derivation of the exact expression for the vacuum amplitude, and the classification of instanton contributions. A section dealing with a new, exact combinatorial solution of gauge theory on a two-dimensional fuzzy torus is also included.

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1 Introduction

These lecture notes continue the study of noncommutative gauge theory in two dimensions which was begun in [1] at the classical level. In this second part we shall deal with matters concerning the quantization of these gauge theories, and in particular demonstrate how to explicitly obtain non-perturbative solutions. Some background and motivation for dealing with this particular class of models may be found in [1] and won’t be repeated here. Various aspects of two-dimensional noncommutative gauge theory have been studied over the
past few years in [2]–[14]. In the present article we shall only analyze the vacuum amplitudes of these theories. More general gauge invariant correlation functions are studied in [6, 8, 9],[11]–[14]. Reviews on noncommutative field theory pertinent to the present material may be found in [15]–[17]. A detailed review of ordinary Yang-Mills theory in two dimensions is given in [18]. All relevant mathematical details and properties of the classical noncommutative gauge theory may be found in [1] and are briefly reviewed in section 1.2 below.

1.1 How to Solve Yang-Mills Theory in Two Dimensions

When one comes to the issue of quantizing noncommutative gauge theory in two dimensions, one is naively faced with a plethora of possibilities. The commutative version of this theory has a long history as an exactly solvable quantum field theory, and as such is explicitly solvable by many different techniques. We will therefore begin with a brief run through of the various methods that may be used to solve ordinary Yang-Mills theory, and elucidate on the possibilities of extending them to the noncommutative setting.

Heat Kernel/Group Theory Methods

One of the most profound features of two-dimensional Yang-Mills theory is the interplay between the two-dimensional geometry on which it is defined and the representation theory of its structure group [18]. As will be reviewed in section 3.1, the propagator between two states may be easily written down in terms of the standard heat kernel on the group manifold of the structure group, and from this the vacuum amplitude and Wilson loops on arbitrary geometries may be extracted. However, these techniques are not readily available in the noncommutative case for several reasons. First and foremost is the lack of a notion of structure group in the noncommutative setting. While there is a well-defined gauge group, it mixes spacetime and internal colour symmetries through noncommutative gauge transformations and there is no clear separation of spacetime and internal degrees of freedom. Secondly, a Hamiltonian formalism is not available because making time a noncommutative coordinate causes problems with unitarity and the overall interpretation of time-evolution in these systems. While this approach in the commutative case will play a crucial role in the foregoing line of development, it is not the one that will be a priori used analyse the quantum field theory. The group theory approach in the noncommutative setting has been analysed recently in [14].

Integrability

The fact that Yang-Mills theory is exactly solvable in two-dimensions is intimately connected with the fact that it is related to an integrable system [19]. It is possible to relate dynamics in this theory to that of certain one-dimensional gauged matrix models which are related to Calogero-Moser systems [20, 21].
While the integrability of the noncommutative counterpart may be established to a certain extent [1], it is not clear what integrable structure underlies this system. This line of attack therefore does not immediately lead to an appropriate generalization.

**Semi-Classical Methods**

One way to understand the exact solvability of the two-dimensional gauge theory is through the observation that its partition function and observables are given exactly by their semi-classical approximation [22]. This is related to the fact that ordinary Yang-Mills theory can be recast as a cohomological field theory in two dimensions. These properties do generalize to the noncommutative setting with some care, as we discuss in section 2. In fact, these techniques will be the focal point of much of this article. They have been recently applied in [14] to explicitly compute the correlation functions of open Wilson line operators.

**Lattice Regularization**

Discretizing spacetime also provides a fruitful way of tackling the problem and is at the very heart of the group theory methods mentioned above [23, 24]. While a lattice formulation of noncommutative gauge theory is available [25], it is much more complicated than its commutative version because the nice self-similarity property possessed by the latter is ruined by the inherent non-locality of the former. Nonetheless, we have succeeded in explicitly solving the lattice model in two dimensions at finite cutoff, and this is new material which will be presented in detail in section 5. We shall therefore postpone further discussion of this approach until then.

**Relations to Other Field Theories**

Besides its relationship with a cohomological gauge theory, two-dimensional quantum Yang-Mills theory may also be related to various other field theories in certain limits, such as three-dimensional Chern-Simons theory and two-dimensional conformal field theory [26]. These connections can be used to give explicit formulas for the volumes of the moduli spaces of representations of fundamental groups of two-dimensional surfaces. As we discuss in section 4, some of these volumes are also effectively computable in the noncommutative setting. These ideas can also all be cast into the formalism of abelianization [21], a technique that relies heavily upon the presence of a well-defined structure group. However, it is not clear what sort of mathematical structures one should find in general and these further connections remain an interesting, as yet unexplored area of this subject.
1.2 Background from Part 1

As we have mentioned above, the solution of the quantum gauge theory will be determined in large part by the very structure of the classical solutions of the field theory. This is described at length in [1]. To keep the presentation of the present article reasonably self-contained and to set some notation, we shall briefly summarize the classical solutions of gauge theory on a noncommutative torus in two-dimensions that were obtained in [1]. The classical action is defined on a fixed Heisenberg module $\mathcal{E}_{p,q}$ over the algebra $\mathcal{A}_\theta$ of functions on the noncommutative torus of fixed topological numbers $(p,q) \in \mathbb{Z}^2$, with $q$ the Chern number, $\text{dim} \mathcal{E}_{p,q} = p - q \theta > 0$, and $N = \gcd(p,q)$ the rank of the gauge theory. It is given explicitly by

$$S[A] = \frac{1}{2g^2} \text{Tr} \left[ \nabla_1, \nabla_2 \right] = \frac{1}{2g^2} \int d^2x \, \text{tr}_N \left( F_A(x) - \frac{2\pi q}{p - q \theta} \right)^2,$$  \hspace{1cm} (1)

where $\nabla = \partial + A$ is a connection on $\mathcal{E}_{p,q}$ and $F_A$ is the corresponding field strength. In the first equality of (1), $\text{Tr}$ is the canonical trace on the endomorphism algebra $\text{End}(\mathcal{E}_{p,q})$. In the second equality the integration extends over the two-dimensional, unit area square torus $\mathbb{T}^2$, $\text{tr}_N$ is the usual $N \times N$ matrix trace, and the constant subtraction corresponds to the constant curvature of the module $\mathcal{E}_{p,q}$.

Classical solutions of this gauge theory are in a one-to-one correspondence with the direct sum decompositions

$$\mathcal{E}_{p,q} = \bigoplus_k \mathcal{E}_{p_k,q_k},$$  \hspace{1cm} (2)

of the given Heisenberg module into projective submodules. These are characterized by partitions $(p,q) = \{(p_k,q_k)\}$ of the topological numbers $(p,q)$ satisfying the constraints

$$p_k - q_k \theta > 0,$$

$$\sum_k (p_k - q_k \theta) = p - q \theta,$$

$$\sum_k q_k = q.$$

In addition, to avoid overcounting, it is sometimes useful to impose a further ordering constraint $p_k - q_k \theta \leq p_{k+1} - q_{k+1} \theta \ \forall k$, and regard any two partitions as the same if they coincide after rearranging their components according to this ordering. We may then characterize the components of a partition by integers $\nu_a > 0$ which are defined as the number of partition components that have the $a^{th}$ least dimension $p_a - q_a \theta$. The integer

$$|\nu| = \sum_a \nu_a$$  \hspace{1cm} (4)

is then the total number of components in the given partition. The noncommutative Yang-Mills action (1) evaluated on a classical solution, with corresponding partition $(p,q)$, is given by
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\[
S(p, q) = \frac{2\pi^2}{g^2} \sum_k (p_k - q_k \theta) \left( \frac{q_k}{p_k - q_k \theta} - \frac{p_k}{p_k - q_k \theta} \right)^2. \tag{5}
\]

1.3 Outline

The outline of material in the remainder of this paper is as follows. In section 2 we will carefully define the quantum theory, examine a deep “hidden” supersymmetry of it, and prove that the partition function and observables are all given exactly by their semi-classical approximation. In section 3 we will derive an exact, analytical expression for the partition function of gauge theory on the noncommutative torus in two dimensions, and use it to analyse precisely how noncommutativity alters the properties of Yang-Mills theory on \( T^2 \). In section 4 we will describe how to organize the non-perturbative expression for the vacuum amplitude into a sum over contributions from (unstable) instantons of the two-dimensional gauge theory, and compare with analogous expressions obtained on the noncommutative plane. This paves the way for our analysis in section 5 which deals with the matrix model/lattice formulation of noncommutative gauge theory in two dimensions. We will present here a new, exact expression for the partition function on the fuzzy torus, and describe the scaling limits which map this model onto the continuum gauge theory.

2 Quantum Gauge Theory on the Noncommutative Torus

In this section we will carefully define the quantum gauge theory within the path integral formalism. We will show that it admits a natural interpretation as a phase space path integral of an infinite-dimensional Hamiltonian system, the noncommutative Yang-Mills system. In this formulation a particular cohomological symmetry of the quantum theory is manifest, which leads immediately to the property that the partition function is given exactly by its semi-classical approximation. While naively the vacuum amplitude may seem to merely produce uninteresting determinants, the non-trivial topology of the torus provides a rich analytic structure (through large gauge transformations). For some time we will neglect the constant curvature subtraction in the action (1), and simply reinstate it when we come to the derivation of the exact formula for the partition function. This is possible to do because of the Morita invariance of the gauge theory [1].

2.1 Definition

The quantum field theory is defined formally through the functional integral

\[
Z = \int DA \ e^{-S[A]}, \tag{6}
\]
where $S[A]$ is the noncommutative Yang-Mills action (1). The integration in (6) is over the space $C = C(E)$ of compatible connections on a given fixed Heisenberg module $E = E_{p,q}$. Since the action is gauge invariant, the integration measure $D\mathcal{A}$ must be carefully defined so as to select only gauge orbits of the field configurations. There is a very natural way to define this measure in the present situation. As discussed in [1], the noncommutative Yang-Mills system naturally defines a Hamiltonian system with moment map

$$\mu[A] = FA,$$

so that the Yang-Mills action is the square of the moment map,

$$S[A] = Tr \mu[A]^2/2g^2.$$

The gauge-invariant symplectic form

$$\omega[\alpha, \beta] = Tr \alpha \wedge \beta, \quad \alpha, \beta \in \Omega^1(E),$$

is defined on the tangent space to $C$, which is identified as the space $\Omega^1(E) = \text{End}(E) \otimes \wedge^1 L^*$ with $L$ the (centrally extended) Lie algebra of the translation group acting on $T^2$. We have defined $\alpha \wedge \beta \equiv \alpha_1 \beta_2 - \alpha_2 \beta_1$ with respect to an orthonormal basis of $L$.

We now let

$$d\mathcal{A} = \prod_{a,b=1}^N \prod_{x \in T^2} dA_{ab}^1(x) \, dA_{ab}^2(x)$$

be the usual, formal (gauge non-invariant) Feynman path integral measure on $C$, and let $\psi$ be the odd generators of the infinite-dimensional superspace $C \oplus \Pi \Omega^1(E)$ with corresponding functional Berezin measure $d\mathcal{A} \, d\psi$, where $\Pi$ is the parity reversion operator. We may then define

$$D\mathcal{A} = d\mathcal{A} \int d\psi \, e^{-i\omega[\psi, \psi]},$$

where here and in the following we will absorb the infinite volume of the group $G = G(E)$ of gauge transformations on $C$ (determined by the Haar measure $d\nu$ on $\mathcal{G}$ induced by the inner product $(\lambda, \lambda') = Tr \lambda \lambda'$, $\lambda, \lambda' \in \text{End}(E)$), by which (9) should be divided. By construction, this measure is gauge-invariant and coincides with the functional Liouville measure associated to the infinite-dimensional dynamical system. An infinitesimal gauge transformation $A \mapsto A + [\nabla, \lambda], \lambda \in \text{End}(E)$ on $C$ naturally induces the transformation $\psi \mapsto \psi + [\lambda, \psi]$ on its tangent space, under which (7) is invariant. In this setting the partition function (6) is naturally defined as a phase space path integral. Note that, since the fermion fields $\psi$ appear only quadratically in (9), this measure coincides with that of its commutative counterpart at $\theta = 0$.

While this definition is very natural in the present context, we should demonstrate explicitly that it coincides with the more conventional gauge field measure obtained from the standard Faddeev-Popov gauge fixing procedure. The basic point is that the measure (9) has the following requisite property. Let $\pi : C \rightarrow C/G$ be the projection onto the quotient space of $C$ by the gauge group $G$. Then the quotient measure $D\mathcal{A}$ on $C/G$ is the measure which satisfies $d\mathcal{A} = \pi^*(D\mathcal{A}) \, d\nu$. The Faddeev-Popov procedure constructs $D\mathcal{A}$ by
introducing the standard fermionic ghost field $c \in \Pi \Omega^0(E) = \Pi \text{End}(E)$, and the anti-ghost multiplet consisting of a fermionic field $\bar{c} \in \Pi \Omega^0(E)$ and a bosonic field $w \in \Omega^0(E)$, along with the BRST transformation laws

\begin{align*}
\delta A &= -[\nabla, c] , \\
\delta c &= \frac{1}{2} [c, c] , \\
\delta \bar{c} &= i w , \\
\delta w &= 0
\end{align*} 

(10)

obeying $\delta^2 = 0$.

The gauge-fixing term is given by $I = -\delta V$ for a suitable functional $V$ of the BRST field multiplet. For this, we write a generic connection $\nabla$ in the neighbourhood of a representative $\nabla^0 = \partial + A^0 \in \mathcal{C}$ of its gauge orbit as $\nabla = \nabla^0 + B$, and make the local choice $V = -\operatorname{Tr} \sigma \nabla^0 \cdot B$, where we have defined $\alpha \cdot \beta = \alpha_1 \beta_1 + \alpha_2 \beta_2$ for cotangent vectors $\alpha, \beta \in \Omega^1(E)$. This produces

\begin{equation}
I = \operatorname{Tr} (i w \nabla^0 \cdot B - \sigma \nabla^0 \cdot \nabla c) ,
\end{equation} 

(11)

and the gauge fixed path integral measure is then defined by

\begin{equation}
DA = dA^0 \int dB \, d\sigma \, dw \, e^{-I} .
\end{equation} 

(12)

Formally integrating over the bosonic field $w$ and the Grassmann fields $c, \bar{c}$ gives

\begin{equation}
DA = dA^0 \int dB \, \delta (\nabla^0 \cdot B) \, \det \nabla^0 \cdot \nabla .
\end{equation} 

(13)

The integration over $B$ enforces the gauge condition $\nabla^0 \cdot B = 0$ on the quantum field theory with the choice (11) of gauge-fixing term. Since $\delta (\nabla^0 \cdot B) = \delta (B) / |\det \nabla^0 \cdot \nabla|$, the resulting ratio of determinants after integrating out $B$ in (13) coincides exactly with the determinant induced by integrating out $\psi$ in (9). Thus the elementary measure defined by the symplectic structure of $\mathcal{C}$ coincides with that of the usual Faddeev-Popov gauge-fixing procedure.

### 2.2 The Cohomological Gauge Theory

We will now describe a remarkable cohomological symmetry of the partition function (6), with path integration measure (9), which will be the crux of much of our ensuing analysis of the quantum gauge theory. For this, we linearize the Yang-Mills action in the field strength $F_A$ via a functional Gaussian integral transformation defined by an auxiliary field $\phi \in \Omega^0(E)$ as

\begin{equation}
Z = \int d\phi \, e^{-\frac{g^2}{2} \operatorname{Tr} \phi^2} \int dA \, d\psi \, e^{-i \operatorname{Tr} (\psi \wedge \psi - \phi F_A)} .
\end{equation} 

(14)

Note that because of the quadratic form of the action in (14), the only place where noncommutativity is buried is in $F_A$. This is one of the features that makes this quantum field theory effectively solvable.
The basic field multiplet \((A, \psi, \phi)\) possesses a “hidden supersymmetry” that resides in the cohomology of the operator

\[ Q_\phi = \text{Tr} \left( \psi \cdot \frac{\delta}{\delta A} + [\nabla, \phi] \cdot \frac{\delta}{\delta \psi} \right) \tag{15} \]

which generates the transformations

\[
\begin{align*}
[Q_\phi, A] &= \psi , \\
\{ Q_\phi, \psi \} &= [\nabla, \phi] , \\
[Q_\phi, \phi] &= 0 . \tag{16}
\end{align*}
\]

The crucial property of the operator \(Q_\phi\) is that it is nilpotent precisely on gauge invariant field configurations,

\[(Q_\phi)^2 = \delta_\phi , \tag{17}\]

where \(\delta_\lambda A = [\nabla, \lambda]\) is an infinitesimal gauge transformation with gauge parameter \(\lambda \in \text{End}(E)\). Furthermore, the linearized action on \(C \oplus H^2(E)\), for fixed \(\phi\), is closed under \(Q_\phi\),

\[ Q_\phi \text{Tr} (\psi \wedge \psi - \phi F_A) = 0 , \tag{18} \]

which follows from the Hamiltonian flow equations for the moment map \(\mu[A] = F_A\) and symplectic structure (7) [1]. Thus the partition function (14) defines a cohomological gauge theory with supersymmetry (16) and \((A, \psi, \phi)\) the basic field multiplet of topological Yang-Mills theory in two dimensions [22].

The nilpotency property (17) implies that the operator \(Q_\phi\) is simply the BRST supercharge, acting in the quantum field theory (14), which generates the transformations (10). Gauge fixing in this setting amounts to introducing additional anti-ghost multiplets analogous to those that were used in the previous subsection for BRST quantization. We shall return to this point in section 4.1. From a more formal perspective, \(Q_\phi\) is the Cartan model differential for the \(G\)-equivariant cohomology of \(C\) [27]. The second term in the action of (14) is the \(G\)-equivariant extension of the moment map on \(C\), the integration over \(A, \psi\) defines an equivariant differential form in \(\Omega G(C)\), and the integral over \(\phi\) defines equivariant integration of such forms. In this way, as we explain in the next subsection, the cohomological symmetry of the quantum field theory will lead to a localization theorem for the partition function. Fundamentally, the localization points correspond to the BRST fixed points of the anti-ghost multiplets.

### 2.3 Localization of the Partition Function

We now come to the fundamental consequence of the hidden supersymmetry of the previous subsection. Let \(\alpha\) be any gauge invariant functional of the fields
of (14), i.e. \((Q_\phi)^2 \alpha = 0\), and consider the one-parameter family of partition functions defined by

\[
Z_t = \int d\phi \ e^{-\frac{\alpha}{2} \text{Tr} \phi^2} \int dA \ d\psi \ e^{-i \text{Tr} (\psi \wedge \psi - \phi F_A) - t Q_\phi \alpha} \tag{19}
\]

with \(t \in \mathbb{R}\). The \(t \to 0\) limit of (19) is just the partition function (14) of interest, \(Z = Z_0\). The remarkable feature of (19) is that it is independent of the parameter \(t \in \mathbb{R}\). This follows from the Leibnitz rule for the functional derivative operator \(Q_\phi\), the supersymmetry (18) of the (equivariantly extended) action, and the gauge invariance of \(\alpha\) which, along with a formal functional integration by parts over the superspace \(C \oplus \mathcal{H}^1(E)\), can be easily used to show that \(\partial Z_t / \partial t = 0\). This is a basic cohomological property of the noncommutative quantum field theory. Adding a supersymmetric \(Q_\phi\)-exact term to the action deforms it without changing the value of the functional integral. It follows that the path integral (14) can be alternatively evaluated as the \(t \to \infty\) limit of the expression (19). It thereby receives contributions from only those field configurations which obey the equations \(Q_\phi \alpha = 0\).

At this stage we need to specify an explicit form for \(\alpha\). Different choices will localize the partition function onto different components in field space, but the final results are (at least superficially) all formally identical. A convenient choice is \(\alpha = \text{Tr} \psi \cdot [\nabla, F_A]\), for which the \(t \to \infty\) limit of (19) yields

\[
Z = \int dA \ d\psi \ e^{\frac{1}{2} \text{Tr} (i \psi \wedge \psi + \frac{1}{2g^2} (F_A)^2)} \lim_{t \to \infty} e^{-\frac{t^2}{2g^2} \text{Tr} \left[\nabla \cdot [\nabla, F_A]\right]^2} \times \text{(fermions)} \tag{20}
\]

after performing the functional Gaussian integration over \(\phi\). These arguments of course assume formally that the original action has no flat directions, but in the present case this is not a problem since it has a nondegenerate kinetic energy. The additional terms involving the Grassmann fields \(\psi\) in (20) formally yield a polynomial function in the parameter \(t\) after integration, and their precise form is not important. What is important here is the quadratic term in \(t\), which in the limit implies that the functional integral vanishes everywhere except near those points in \(C\) which are solutions of the equations

\[
[\nabla ; [\nabla, F_A]] = 0, \tag{21}
\]

where we have used positivity of the trace \(\text{Tr} \) on \(\text{End}(\mathcal{E})\). By using the Leibnitz rule and the integration by parts property \(\text{Tr} [\nabla, \lambda] = 0\) this equation implies

\[
0 = \text{Tr} F_A [\nabla ; [\nabla, F_A]]^2 = - \text{Tr} [\nabla, F_A] \cdot [\nabla, F_A]. \tag{22}
\]

By using non-degeneracy of the trace on \(\text{End}(\mathcal{E})\) we arrive finally at

\[
[\nabla, F_A] = 0, \tag{23}
\]
which are just the classical equations of motion of the original noncommutative gauge theory.

We have thereby formally shown that the partition function of noncommutative gauge theory in two dimensions receives contributions only from the space of solutions of the noncommutative Yang-Mills equations. As we reviewed in section 1.2, each such solution corresponds to a partition \((p, q)\), obeying the constraints (3), of the topological numbers \((p, q)\) of the given Heisenberg module \(E = E_{p, q}\) on which the gauge theory is defined. Symbolically, the partition function is therefore given by

\[
Z = Z_{p, q} = \sum_{\text{partitions } (p, q)} W(p, q) \ e^{-S(p, q)}.
\]  

(24)

This result expresses the fact that quantum noncommutative gauge theory in two dimensions is given exactly by a sum over contributions from neighbourhoods of the stationary points of the Yang-Mills action (1). The Boltzmann weight \(e^{-S(p, q)}\) involving the action (5) gives the contribution to the path integral (6) from a classical solution, while \(W(p, q)\) encode the quantum fluctuations about each stationary point. These latter terms may in principle be determined from (20) by carefully integrating out the fermion fields and evaluating the functional fluctuation determinants that arise. However, these determinants are not effectively computable and are rather cumbersome to deal with. In the next section our main goal will be to devise an alternative method to extract these quantum fluctuation terms and hence the exact solution of the noncommutative quantum field theory.

### 3 Exact Solution

In this section we will present the exact solution of gauge theory on the noncommutative torus in two dimensions. We will start by recalling some well-known facts about ordinary Yang-Mills gauge theory in two dimensions, and show how it can be cast precisely into the form (24). From this we will then extract the exact expression in the general noncommutative case. Our techniques will rely heavily on the full machinery of the geometry of the noncommutative torus.

#### 3.1 The Torus Amplitude

The vacuum amplitude for ordinary Yang-Mills theory on the torus \(T^2\) with structure group \(U(p)\) and generators \(T^a, a = 1, \ldots, p^2\) may be obtained as follows [18]. Let us consider the physical Hilbert space, in canonical quantization, for gauge theory on a cylinder \(\mathbb{R} \times S^1\) (fig. 1). In two dimensions, Gauss’ law implies that the physical state wavefunctionals \(\Psi_{\text{phys}}[A] = \Psi[U]\) depend only
on the holonomy $U = \text{P} \exp i \int_0^L dx \, A_1(x)$ of the gauge connection around the cycle of the cylinder. By gauge invariance, $\Psi$ furthermore depends only on the conjugacy class of $U$. It follows that the Hilbert space of physical states is the space of $L^2$-class functions, invariant under conjugation, with respect to the invariant Haar measure $[dU]$ on the unitary group $U(p)$,

$$\mathcal{H}_{\text{phys}} = L^2(U(p))^{A_d(U(p))}.$$  

By the Peter-Weyl theorem, it may be decomposed into the unitary irreducible representations $R$ of $U(p)$ as $\mathcal{H}_{\text{phys}} \cong \bigoplus_R R \otimes \overline{R}$. The representation basis of this Hilbert space is thereby provided by characters in the unitary representations, such that the states $|R\rangle$ have wavefunctions

$$\langle U| R \rangle = \chi_R(U) = \text{tr}_R U .$$  

![Fig. 1. Quantization of Yang-Mills theory on a spatial circle of circumference $L$ yields the propagation amplitude between two states characterized by holonomies $U_1$ and $U_2$ in time $T$.](image)

The Hamiltonian acting on the physical state wavefunctions $\Psi[U]$ is given by the Laplacian on the group manifold of $U(p)$,

$$H = g^2 \frac{L}{2} \text{tr} \left( U \frac{\partial^2}{\partial U^*} \right) ,$$

and it is thereby diagonalized in the representation basis as

$$H \chi_R(U) = \frac{g^2}{2} L C_2(R) \chi_R(U) ,$$

where $C_2(R)$ is the eigenvalue of the quadratic Casimir operator $C_2 = \sum a \, T^a \, T^a$ in the representation $R$. From these facts it is straightforward to write down the cylinder amplitude corresponding to propagation of the system between two states with holonomies $U_1$ and $U_2$ in the form (fig. 1)

$$Z_p(T; U_1, U_2) = \langle U_1 | e^{-TH} | U_2 \rangle = \sum_R \chi_R(U_1) \chi_R(U_2^*) e^{-\frac{g^2}{2} LT C_2(R)} .$$  

This is just the standard heat kernel on the $U(p)$ group. In keeping with our previous normalizations, we shall set the area of the cylinder to unity, $LT = 1$. To extract from (29) the partition function of $U(p)$ Yang-Mills theory on the
torus, we glue the two ends of the cylinder together by setting $U_1 = U_2 = U$ and integrate over all $U$ by using the fusion rule for the $U(p)$ characters,

$$\int [dU] \chi_{R_1}(VU) \chi_{R_2}(U^\dagger W) = \delta_{R_1, R_2} \frac{\chi_{R_1}(VW)}{\dim R_1},$$  \hspace{1cm} (30)

where $\dim R = \chi_R(\mathbb{1})$. This yields the torus vacuum amplitude

$$Z_p = \int [dU] Z_p(T; U, U) = \sum_R e^{-\frac{c_2}{2} C_2(R)}.$$  \hspace{1cm} (31)

We can make the sum over the irreducible unitary representations $R$ of $U(p)$ in (31) explicit by using the fact that each $R$ is labelled by a decreasing set $n = (n_1, \ldots, n_p)$ of $p$ integers

$$+\infty > n_1 > n_2 > \cdots > n_p > -\infty$$  \hspace{1cm} (32)

which are shifted highest weights parametrizing the lengths of the rows of the corresponding Young tableaux. Up to an irrelevant constant, the quadratic Casimir can be written in terms of these integers as

$$C_2(R) = C_2(n) = \sum_{a=1}^p \left( n_a - \frac{p-1}{2} \right)^2.$$  \hspace{1cm} (33)

Since (33) is symmetric under permutations of the $n_a$’s, it follows that the ordering restriction (32) can be removed in the partition function (31) to write it as a sum over non-coincident integers as (always up to inconsequential constants)

$$Z_p = \sum_{n_1 \neq \cdots \neq n_p} e^{-\frac{c_2}{2} C_2(n)}.$$  \hspace{1cm} (34)

We may extend the sums in (34) over all $n \in \mathbb{Z}^p$ by inserting the products of delta-functions

$$\det_{1 \leq a,b \leq p} (\delta_{n_a, n_b}) = \sum_{\sigma \in S_p} (-1)^{\sigma} \prod_{a=1}^p \delta_{n_{\sigma(a)}, n_a}.$$  \hspace{1cm} (35)

The vanishing of the determinant for coincident rows prevents any two $n_a$’s from coinciding when inserted into the sum.

Because of the permutation symmetry of (33), when inserted into the partition function (34) the sum in (35) truncates to a sum over conjugacy classes $[1^{\nu_1} 2^{\nu_2} \cdots p^{\nu_p}]$ of the symmetric group $S_p$. They are labelled by partitions of the rank $p$ of the gauge theory,

$$\nu_1 + 2\nu_2 + \cdots + p\nu_p = p,$$  \hspace{1cm} (36)

where $\nu_a$ is the number of elementary cycles of length $a$ in $[1^{\nu_1} 2^{\nu_2} \cdots p^{\nu_p}]$. The sign of such a conjugacy class is $(-1)^{p+|\nu|}$ and its order is $p! / \prod_a a^{\nu_a} \nu_a!$. 

where $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_p$ is the total number of cycles in the class. By using this, along with the Poisson resummation formula

$$
\sum_{n=-\infty}^{\infty} e^{-\pi g n^2 - 2\pi i b n} = \frac{1}{\sqrt{g}} \sum_{q=-\infty}^{\infty} e^{-\pi (q-b)^2 / g},
$$

we may bring the vacuum amplitude (34) after some work into the form [10]

$$
Z_p = \sum_{|\nu|} \sum_{a} e^{i \pi (|\nu| + (p-1)q)} \times \prod_{1}^{p} \left( \frac{g^2 a^3 / 2\pi^2}{\nu_a^2} \right)^{-\nu_a/2} e^{-S(\nu,q)},
$$

where $q = q_1 + q_2 + \cdots + q|\nu|$ and

$$
S(\nu,q) = \frac{2\pi^2}{g^2} \left( \frac{\nu_1}{3} + \frac{\nu_{1}+\nu_2}{2} + \frac{\nu_{1}+\nu_2+\nu_3}{3} + \cdots + \frac{|\nu|}{p} \right).
$$

The important feature of the final expression (38) is that it agrees with the expected sum (24) over classical solutions of the commutative gauge theory on $\mathbb{T}^2$. For this, we note that the K-theory group of the ordinary torus is $K_0(C(\mathbb{T}^2)) = \mathbb{Z} \oplus \mathbb{Z}$, so that any projective module $E = E_{p,q}$ over the algebra $C(\mathbb{T}^2)$ of functions on the torus is determined by a pair of integers $(p,q)$, with $\dim E_{p,q} = p > 0$ and constant curvature $q/p$. Geometrically, any such module is the space of sections $\Gamma(\mathbb{T}^2, E_{p,q})$ of a complex vector bundle $E_{p,q} \to \mathbb{T}^2$ of rank $p$, Chern number $q$, and structure group $U(p)$. The direct sum decompositions (2) correspond to the usual Atiyah-Bott bundle splittings [28]

$$
E_{p,q} = \bigoplus_{k} E_{p,k,q_k}
$$

into sub-bundles $E_{p_k,q_k} \subset E_{p,q}$ about each Yang-Mills critical point on $\mathbb{T}^2$. The first two partition constraints in (3) for $\theta = 0$ correspond to those on the rank of (40), $p = \sum k p_k$ with $p_k > 0$. This condition coincides with (36), where $\nu_a$ is the number of submodules $\mathcal{E}_{p_k,q_k}$ of dimension $a$ (equivalently the number of sub-bundles $E_{p_k,q_k}$ of rank $a$). The action (39) is precisely of the form (5) at $\theta = 0$ and without the background flux subtraction, while the exponential prefactors in (38) correspond to the fluctuation determinants $W(p,q)$ in (24).

The third constraint in (3) on the magnetic charges $q_k$, which are dual to the lengths of the rows of the Young tableaux of $U(p)$, restricts the gauge theory to a particular isomorphism class of bundles over the torus. It is straightforward to rewrite the partition function (38) of physical Yang-Mills theory,
defined as a weighted sum over contributions from topologically distinct vector bundles over $T^2$, in terms of that of Yang-Mills theory defined on a particular isomorphism class $E_{p,q}$ of projective modules over $A_\theta$ up to irrelevant constants as

$$Z_p = \sum_{q=-\infty}^{\infty} (-1)^{(p-1)q} Z_{p,q}, \quad (41)$$

where

$$Z_{p,q} = \sum_{\text{partitions} \ (p,q)} (-1)^{\nu} \prod_{a=1}^{p} \frac{(g^2 a^3 / 2\pi^2)^{-\nu_a/2}}{\nu_a!} e^{-S(p,q)}. \quad (42)$$

The partition sum here arises from the sum over cycle decompositions that appears in the group theoretic setting above, with the number of partition components $|\nu|$ (or cycles) given by (4).

### 3.2 The Exact Vacuum Amplitude

From the commutative partition function (42) we may now extract the exact expression for the noncommutative field theory defined for any $\theta$ in the following manner. We use the fact, reviewed in section 5 of [1], that Morita equivalence provides a one-to-one correspondence between projective modules over different noncommutative tori (i.e. for different $\theta$'s) associated with different topological numbers, augmented with transformations of connections between the modules. It is an exact symmetry of the noncommutative Yang-Mills action (1) which is firmly believed to extend to the full quantum level. There are many good pieces of evidence in support of this assumption [25, 29, 30].

In the present case, we will use the fact that Morita duality can be used to map the quantum partition function of ordinary $\theta' = 0$ Yang-Mills theory on $T^2$ onto noncommutative gauge theory with deformation parameter $\theta = n/s$. The dimensionless coupling constant and module dimensions in (42) transform in this case as $g^2 = |s|^3 g'^2$ and $\dim E = \dim E'/|s|$. The equivalence provides a one-to-one correspondence between classical solutions in the two field theories, i.e. their partitions. The symmetry factors $\nu_a!$ in (42) corresponding to permutation of partition components of identical dimension are preserved, as is the total number $|\nu|$ of submodules in any given partition $(p,q)$. From these facts it follows that the fluctuation factors in (42) are invariant under this Morita duality only if the indices $a$ transform as

$$a = a'/|s|, \quad (43)$$

which is equivalent to the expected requirement that the cycle lengths $a$ be interpreted as the dimensions of submodules in the commutative gauge theory.

With these identifications we can now straightforwardly map (42) onto the exact partition function of the $\theta = n/s$ Morita equivalent noncommutative
Two-Dimensional Noncommutative Gauge Theory

The key point is that the localization arguments which led to (24) do not distinguish between the commutative, rational or irrational cases there. All of the analysis and formulas of the previous section hold universally for any value of \( \theta \), and hence so should the exact expression for the vacuum amplitude. Thus, given the generic structure of partitions as outlined in section 1.3, including the general definition of \( \nu_a \), the final analytic expression for the partition function of gauge theory on a fixed projective module over the noncommutative torus, for any value of the noncommutativity parameter \( \theta \), is given by

\[
Z_{p,q} = \sum_{\text{partitions} \ (p,q)} \prod_a \frac{(-1)^{\nu_a}}{\nu_a!} \left( \frac{g^2}{2\pi^2} \left( p_a - q_a \theta \right)^3 \right)^{-\nu_a/2} \times \exp \left[ -\frac{2\pi^2}{g^2} \sum_k \left( p_k - q_k \theta \right) \left( \frac{q_k}{p_k} - p_k \theta \right) - \frac{q}{p - q \theta} \right]. \tag{44}
\]

We have reinstated the constant curvature of \( \mathcal{E}_{p,q} \), as it is required to ensure that the Yang-Mills action transforms homogeneously under Morita duality. This technique thereby explicitly determines the fluctuation determinants \( W(p,q) \) of the semi-classical expansion (24).

We close this section with a brief description of how the expansion (44) elucidates the relations with and modifications of ordinary Yang-Mills theory on the torus:

- It can be shown [10] that the partition function (44) is a smooth function of \( \theta \), even about \( \theta = 0 \). At least at the level of two-dimensional noncommutative gauge theory, violations of \( \theta \)-smoothness in the quantum theory disappear at the non-perturbative level.

- The Morita equivalence between rational noncommutative Yang-Mills theory on a projective module \( \mathcal{E}_{p,q} \) with deformation parameter \( \theta = n/s \), \( n, s > 0 \) relatively prime, and ordinary non-abelian gauge theory is particularly transparent in this formalism. As mentioned above, for \( \theta' = 0 \) the module dimensions transform as \( \dim \mathcal{E} = \dim \mathcal{E}'/s \), and since in the commutative theory the bundle ranks are always positive integers, any module \( \mathcal{E} \) in the rational theory has dimension bounded as \( \dim \mathcal{E} \geq 1/s \). Since \( \dim \mathcal{E}_{p,q} = p - nq/s \), it follows that any partition \( (p,q) \) of the rational theory consisting of submodules of dimension \( \geq 1/s \) has at most \( \frac{p - nq/s}{1/s} = ps - qn \) components. Thus any gauge theory dual to this one admits partitions with \( ps - qn \) components. In particular, as we have seen in the previous subsection, for \( U(N) \) commutative Yang-Mills theory the maximum number of components is precisely the rank \( N \), corresponding to the cycle decomposition with \( \nu_1 = N \) and \( \nu_a = 0 \ \forall a > 1 \). Putting these facts together we arrive at the well-known result that noncommutative Yang-Mills theory with \( \theta = n/s \) on a module \( \mathcal{E}_{p,q} \) is Morita equivalent to \( U(N) \) commutative gauge theory on \( T^2 \) with rank \( N = ps - qn \).
• The expansion (44) clearly shows the differences between the commutative and noncommutative gauge theories. In the rational case $\theta = n/s$, all partitions contain at most $ps - qn$ submodules of $E_{p,q}$ of dimension $\geq 1/s$. But for $\theta$ irrational, there is no a priori bound on the number of submodules in a partition (although it is always finite) and submodules of arbitrarily small dimension can contribute to the partition function (44). In particular, in this case we can approximate $\theta$ by a sequence of rational numbers, $\theta = \lim_{m} n_m/s_m$ with both $n_m, s_m \to \infty$ as $m \to \infty$.

The rigorous way to take the limit of the noncommutative field theory is described in [31]. In the rational gauge theory with noncommutativity parameter $\theta_m = n_m/s_m$, the dimension of any submodule is bounded from below by $1/s_m$. It follows that any rational approximation to the vacuum amplitude $Z_{p,q}$ contains contributions from partitions of arbitrarily small dimension. Thus although formally similar, the exact expansion (44) of the partition function has drastically different analytic properties in the commutative and noncommutative cases.

4 Instanton Contributions

The fact that gauge theory on the noncommutative torus has an exact semi-classical expansion in powers of $e^{-1/g^2}$ suggests that it should admit an interpretation in terms of non-perturbative contributions from instantons of the two-dimensional gauge theory. By an instanton we mean a finite action solution of the Euclidean Yang-Mills equations (23) which is not a gauge transformation of the trivial gauge field configuration $A = 0$. Interpreting (44) in terms of such configurations is not as straightforward as it may seem, because the contributions to the sum as they stand are not arranged into gauge equivalence classes. In this section we will briefly describe how to rearrange the semi-classical expansion (44) into a sum over (unstable) instantons. This will entail a deep analysis of the moduli spaces of the noncommutative gauge theory and will also naturally motivate, via a comparison with corresponding structures on the noncommutative plane, a matrix model analysis of the field theory which will be carried out in the next section.

4.1 Topological Yang-Mills Theory

We will begin by studying the weak-coupling limit of the noncommutative gauge theory as it is the simplest case to describe. In the limit $g^2 \to 0$, the only non-vanishing contribution to (44) comes from those partitions for which the Yang-Mills action attains its global minimum of 0. The only partition for which this happens is the trivial one $(p, q) = (p, q)$ associated to the original Heisenberg module $E_{p,q}$ itself. The corresponding moduli space of classical solutions is the space of constant curvature connections on $E_{p,q}$ modulo gauge
transformations. Such classical configurations preserve \( \frac{1}{2} \) of the supersymmetries in an appropriate supersymmetric extension of the gauge theory \([32, 33]\). In this context, the classical solutions live in a Higgs branch of the \( \frac{1}{2} \)-BPS moduli space, with the whole moduli space determined by a fibration over the Higgs branch.

As described in detail in \([1]\), as a vector space the Heisenberg module is given by \( \mathcal{E}_{p,q} = L^2(\mathbb{R}) \otimes \mathbb{C}^q \), where \( L^2(\mathbb{R}) \) is the irreducible Schrödinger representation of the constant curvature condition, and \( \mathbb{C}^q \) is the \( q \times q \) representation of the Weyl-'t Hooft algebra in two dimensions. The latter algebra is known to possess a unique irreducible unitary representation of dimension \( q/N, N = \gcd(p,q) \), so that module decomposes into irreducible components as

\[
\mathcal{E}_{p,q} = L^2(\mathbb{R}) \otimes (\mathcal{W}_{\zeta_1} \oplus \cdots \oplus \mathcal{W}_{\zeta_N}),
\]

where \( \mathcal{W}_\zeta \subset \mathbb{C}^q \) are the irreducible representations of the Weyl-'t Hooft algebra and \( \zeta \in \tilde{T}^2 \) generate its center, with values in a dual torus to the original one \( T^2 \). The only gauge transformations which act trivially on \( \mathcal{E}_{p,q} \) are those which live in the Weyl subgroup of \( U(N) \), and dividing by this we find that the moduli space of constant curvature connections on \( \mathcal{E}_{p,q} \) is the \( N \)th symmetric product

\[
\mathcal{M}_{p,q} = \text{Sym}^N T^2 \equiv \left( T^2 \right)^N / S_N.
\]

Remarkably, this space coincides with \( \text{Hom}(\pi_1(T^2), U(N))/U(N) \), the moduli space of flat \( U(N) \) bundles over the torus \( T^2 \) in commutative gauge theory \([28]\).

Now let us examine more closely the partition function \((44)\) in the limit \( g^2 \to 0 \). After using Morita duality to remove the background flux contribution, the series receives contributions only from partitions with vanishing magnetic charges \( q_k = 0 \quad \forall k \), and we find

\[
Z_{p,q} \big|_{g^2=0} = \sum_{\nu: \sum a \nu_a = N} \frac{1}{\nu_a!} \frac{1}{\nu_a!} \left( \frac{g^2 a^3}{2\pi^2} \right)^{-\nu_a/2} + O \left( e^{-1/g^2} \right).
\]

We thereby find that the weak coupling limit is independent of the noncommutativity parameter \( \theta \), and in particular it coincides with the commutative version of the theory with structure group \( U(N) \). This is easiest to see from the form \((14)\), whose \( g^2 = 0 \) limit gives explicitly

\[
Z_{p,q} \big|_{g^2=0} = \int d\phi \int dA \int d\psi \ e^{-i \text{Tr}(\psi^\land \psi - \phi F_A)}. \tag{48}
\]

The integration over \( \phi \), after reinstating the proper constant curvature subtraction in \((48)\), localizes this functional integral onto gauge field configurations of constant curvature, and the partition function thereby computes the symplectic volume of the moduli space \((46)\) with respect to the symplectic structure on \( \mathcal{M}_{p,q} \) inherited from the one \((7)\) on \( \mathcal{C}(\mathcal{E}_{p,q}) \). It is formally the same as that of topological Yang-Mills theory on \( T^2 \), except that now the
noncommutativity (through Morita equivalence) identifies (46) as the space of all constant curvature connections, in contrast to the usual case where it only corresponds to flat gauge connections.

In this case, the gauge theory is BRST equivalent (in the sense described in section 2.3 for \( t \to \infty \)) to that with gauge fixing functional \( V = \text{Tr} \left( \frac{1}{2} (H - 4F_A) + \nabla \lambda \cdot \psi \right) \), where we have introduced pairs \((\lambda, \eta)\) and \((\chi, H)\) of anti-ghost multiplets of ghost numbers \((-2, -1)\) and \((-1, 0)\), respectively, with \(\lambda, H\) bosonic and \(\eta, \chi\) Grassmann-valued fields. Their BRST transformation rules are

\[
\begin{align*}
[Q_\phi, \lambda] &= i \eta, \\
\{Q_\phi, \eta\} &= [\phi, \lambda], \\
\{Q_\phi, \chi\} &= H, \\
[Q_\phi, H] &= i [\phi, \chi] ,
\end{align*}
\]

and the \(Q_\phi\)-invariant action \( S_D \equiv -i \{Q_\phi, V\} \) is given by

\[
S_D = \text{Tr} \left\{ \frac{1}{2} (H - F_A)^2 - \frac{1}{2} (F_A)^2 - i \chi \nabla \wedge \psi + i \nabla \eta \cdot \psi \\
+ \frac{1}{2} \chi [\chi, \phi] + \nabla \lambda \cdot \nabla \phi + i [\psi, \lambda] \cdot \psi \right\} .
\]

The functional \( V \) conserves ghost number and the action (50) has non-degenerate kinetic energy, as in the case of the original Yang-Mills system of section 2.3. It gives the action of two-dimensional Donaldson theory, and in this way the full noncommutative gauge theory can be used to extract information about the intersection pairings on the moduli space \( M_{p,q} \) [22].

Going back to the formula (47), we see that it involves a sum over cycles \( \nu \) of terms which are singular at \( g^2 = 0 \). These terms represent contributions to the symplectic volume from the conical orbifold singularities of the moduli space (46), which arise due to the existence of reducible connections. For this, we note that the fixed point locus of a conjugacy class element \( \sigma \in [1^{\nu_1} 2^{\nu_2} \cdots p^{\nu_p}] \) acting on \( (\zeta_1, \ldots, \zeta_N) \in (\mathbb{T}^2)^N \) is \( \prod_a S_{\nu_a} \times (\mathbb{Z}_a)^{\nu_a} \). The action of the corresponding stabilizer subgroup of \( S_N \) is \( \prod_a S_{\nu_a} \times (\mathbb{Z}_a)^{\nu_a} \), where the symmetric group \( S_{\nu_a} \) permutes coordinates in the factor \( (\mathbb{T}^2)^{\nu_a} \) while the cyclic group \( \mathbb{Z}_a \) acts in each cycle of length \( a \). Only the \( S_{\nu_a} \) factors act non-trivially, and so the singular locus of \( M_{p,q} \) is a disjoint union over the conjugacy classes \( [1^{\nu_1} 2^{\nu_2} \cdots p^{\nu_p}] \subset S_N \) of the strata \( \prod_a \text{Sym}^{\nu_a} \mathbb{T}^2 \), as reflected by the expansion (47).

### 4.2 Instanton Partitions

Let us now consider the general case. The basic problem is that there is an isomorphism \( \mathcal{E}_{mp,mq} \cong \bigoplus^m \mathcal{E}_{p,q} \) of Heisenberg modules, owing to the reducibility of the Weyl-'t Hooft algebra, with \( \mathcal{E}_{mp,mq} \) and \( \mathcal{E}_{p,q} \) both possessing the same constant curvature. We circumvent this problem by writing each component of
a given partition as \((p_k, q_k) = N_k(p'_k, q'_k)\), with \(N_k = \gcd(p_k, q_k)\) and \(p'_k, q'_k\) relatively prime, and restrict the sum over partitions \((p, q)\) to those with distinct K-theory charges \((p'_k, q'_k)\). We call such partitions “instanton partitions” [10], as they each represent distinct, gauge equivalence classes of classical solutions to the noncommutative Yang-Mills equations. Then the direct sum decomposition (2) is modified to

\[
E_{p,q} = \bigoplus_a E_{N_a p'_a, N_a q'_a} ,
\]  

and the corresponding moduli space of classical solutions is [10]

\[
\mathcal{M}'_{p,q} = \prod_a \mathcal{M}_{N_a p'_a, N_a q'_a} = \prod_a \text{Sym}^{N_a} \mathbb{T}^2 .
\]  

The orbifold singularities present in (52) can now be used to systematically construct the gauge inequivalent contributions to noncommutative Yang-Mills theory. In this way one may rewrite the expansion (44) as a sum over instantons along with a finite number of quantum fluctuations about each instanton, representing a finite, non-trivial perturbative expansion in \(1/g\). For more details, see [10].

### 4.3 Fluxon Contributions

The instanton solutions that we have found for gauge theory on the noncommutative torus bear a surprising relationship to soliton solutions of gauge theory on the noncommutative plane [2]–[4]. The classical solutions of the noncommutative Yang-Mills equations in this latter case are labelled by two integers, the rank of the gauge group and the magnetic charge, similarly to the case of the torus. These noncommutative solitons are termed “fluxons” and they are finite energy instanton solutions which carry quantized magnetic flux. The classical action evaluated on a fluxon of charge \(q\) is given by [4]

\[
S(q) = \frac{2\pi^2 q}{g^2 \theta} .
\]  

This action is very similar to (5) in the limit \(g^2 \theta \to \infty\), and in [8] it was described how to map the instanton expansion on the noncommutative torus to one on the noncommutative plane by using Morita equivalence and taking a suitable large area limit. In terms of the partition sum (44), a fluxon of charge \(q\) is composed of \(\nu_a\) elementary vortices of charges \(a = 1, 2, \ldots\). The symmetry factors \(\nu_a!\) appear in (44) to account for the fact that vortices of equal charge inside the fluxon are identical, while the moduli dependence (through the vortex positions) is accompanied by the anticipated exponent \(|\nu|\), the total number of elementary vortex constituents of the fluxon. The remaining terms correspond to quantum fluctuations about each fluxon in the following manner.
The basic fluxon solution corresponds to the elementary vortex configuration \( \nu_1 = q, \nu_a = 0 \quad \forall a > 1 \). In the large area limit, the semi-classical expansion (44) can be interpreted in terms of the contributions from basic fluxons of charge \( q \) and classical action (53), along with fluctuations around the soliton solution, leading to the partition function [8]

\[
Z_q = \frac{e^{-2\pi^2 q/g^2 \theta}}{N \sqrt{g^2 \theta}} \sum_{\nu: \sum_a a \nu_a = q} \prod_{a=1}^q \left( \frac{1}{\nu_a!} \left( \frac{2\pi^2}{a^3 g^2 \theta^3} \right)^{\nu_a} \right). \quad (54)
\]

The (unweighted) sum over topological charges can be performed exactly and the result is

\[
Z \equiv \sum_{q=0}^{\infty} Z_q = \exp \left[ -\frac{2\pi}{\sqrt{g^2 \theta^3}} \Phi \left( e^{-2\pi^2/g^2 \theta}; \frac{3}{2}, 1 \right) \right], \quad (55)
\]

where the function

\[
\Phi(z; s; \mu) = \sum_{k=0}^{\infty} \frac{z^k}{(k + \mu)^s}
\]

is analytic in \( z \in \mathbb{C} \) with a branch cut from \( z = 1 \) to \( z = \infty \). The instanton series has been resummed in (55) into the non-perturbative exponential, which is typical of a dilute instanton gas. This is not surprising, given that fluxons are non-interacting objects and thereby lead to an extensive partition function. It would be interesting to examine the dynamics of all the instantons described in this picture on the moduli spaces (46) and (52), using the Kähler structure inherited from the symplectic structure (7) and metric \( \text{Tr} \alpha \cdot \beta \) on the space \( \mathcal{C} \) of compatible gauge connections.

The non-trivial results obtained for the noncommutative plane suggest another way of tackling two-dimensional noncommutative gauge theory in general [4]. Since the planar algebra of functions is generated by the coordinate operators \( x^1, x^2 \) obeying the Heisenberg algebra \([x^1, x^2] = i \theta\), gauge connections act by inner automorphisms and may be written as

\[
D_i = i \frac{\theta}{\sigma} \epsilon_{ij} x^j + A_i
\]

for \( i = 1, 2 \). The curvature is given by

\[
F_A = [D_1, D_2] - i \frac{\theta}{\sigma},
\]

and after a rescaling of fields the partition function is defined by the infinite dimensional matrix model

\[
Z = \lim_{\varepsilon \to 0^+} \int dD_1 \ dD_2 \ \exp \left[ -\frac{\pi \theta}{2g^2} \text{Tr} \left( [D_1, D_2] - 1 \right)^2 - \varepsilon \ Tr \ D \cdot D \right]. \quad (59)
\]

The second term in the action of (59) regulates the partition function and is a gauge-invariant analog of the infrared regularization provided by the area of
the torus. It is required to ensure that the semi-classical approximation to the functional integral exists. The classical fluxon solutions are unstable critical points whose moduli are the positions of the vortices [4]. The Yang-Mills energy density of the vortices is independent of these positions and integrating along these moduli would lead to a divergent path integral in (59). While this may seem like a fruitful line of attack, it presents many difficulties. Foremost among these is the fact that finite action configurations would require the field strength $F_A$ to be a compact operator. Since there are no bounded operators $D_i$ for which (58) is compact, the effective gauge configuration space consists only of unbounded operators and the partition function (59) is not naturally realized as the large $N$ limit of a finite dimensional matrix model. This makes an exact solution intractable. In the next section we shall present a matrix model formulation of noncommutative gauge theory in two dimensions which circumvents these difficulties, and enables an explicit and exact analysis of the configurations described here.

5 Combinatorial Quantization

In this final section we will show how a combinatorial approach can be used to explicitly compute the partition function of noncommutative gauge theory in two dimensions. Part of the motivation for doing this was explained at the end of the last section. Another reason is to make sense of the Feynman path integral over the space $C$ of compatible connections. We will approximate $C$ by a finite-dimensional $N \times N$ matrix group and then analyse the partition function in the limit $N \to \infty$. The hope is then that this procedure yields a concrete, non-perturbative definition of the noncommutative field theory. This matrix model is intimately connected with a lattice regularization of the noncommutative gauge theory obtained by triangulating $T^2$, and restricting to modules over the finite-dimensional matrix algebras. In this setting the non-trivial K-theory of the torus algebra $A_\theta$ is lost, and as in section 3.1 the computation will give the Yang-Mills partition function summed over all topological types of projective modules over $A_\theta$. We will begin by recalling some salient features of commutative lattice gauge theory, and contrast it with what happens in the noncommutative setting. Then we will proceed to define and completely solve the discrete version of noncommutative gauge theory in two dimensions, and describe how it can be used to extract information about the continuum field theory. The material contained in this section is new and presents a novel explicit solution of noncommutative Yang-Mills theory.

5.1 The Local Lattice Regularization

In ordinary two-dimensional Yang-Mills theory, the lattice form of the quantum field theory [34] possesses some very special properties and provides an indispensible tool for obtaining its complete analytic solution [23, 24]. Let
us consider the partition function on a disk of area $A$ (fig. 2). It can be obtained from the cylinder amplitude (29) by pinching the right boundary of the cylinder to a point, so that $U_2$ becomes the holonomy surrounding a disk of vanishing area. The corresponding physical state wavefunction is the delta-function supported at the identity element $U_2 = \mathbb{1}$ of the unitary group with respect to its Haar measure, $\Psi[U_2] = \delta(U_2, \mathbb{1})$. Then from (29) with $U_1 = U$ and $U_2 = \mathbb{1}$ we obtain the disk amplitude

$$Z(A, U) = \sum_R \dim R \chi_R(U) \ e^{-\frac{2\pi i}{g^2} C_2(R)} .$$  \hfill (60)

By using the area-preserving diffeomorphism invariance of the theory, we may interpret (60) as an amplitude for a plaquette, i.e. the interior of a simplex in a local triangulation of the spacetime (fig. 3).

$$U_1 = U \quad U_2 = \mathbb{1}$$

**Fig. 2.** The disk amplitude.

$$V \quad U \quad U^+ \quad W = \quad VW \quad A_1 + A_2$$

**Fig. 3.** The partition function of two-dimensional Yang-Mills theory is invariant under subdivision of the plaquettes of the lattice.

One of the main advantages of the discrete formalism is its self-similarity property [23]. Consider the gluing together of two disk amplitudes along a plaquette link as depicted in fig. 3. The gluing property follows from (60) and the fusion rule (30) for the characters, which together imply

$$\int [dU] \ Z(A_1, UV) Z(A_2, U^+W) = Z(A_1 + A_2, VW) .$$  \hfill (61)
This result expresses the renormalization group invariance of the basic plaquette Boltzmann weight. Subdivision of the lattice into a very fine lattice yields a result which converges to that in the continuum theory. But (61) implies that the computation may be carried out on an arbitrarily coarse lattice. Hence the lattice field theory produces the exact answer and the continuum limit is trivial. From this treatment it is in fact possible to directly obtain the torus amplitude (31).

As we will soon see, this self-similarity property under gluing of plaquettes is not shared by the noncommutative version of the lattice gauge theory, reflecting its inherent non-locality. Noncommutativity introduces long-ranged interactions between plaquettes of the lattice. A clear way to understand this breakdown is to recall the Gross-Witten reduction of $U(N)$ Yang-Mills theory on $\mathbb{R}^2$ [35]. The calculation of the lattice partition function in this case can be easily reduced to a single unitary matrix integration by exploiting the gauge invariance of the theory. This is possible to do by fixing an axial gauge and an appropriate change of variables. If $U_i(x)$ denotes the operator of parallel transport from a lattice site $x$ to its neighbouring point along a link in direction $i$, $i = 1, 2$, then one may fix the gauge $U_1(x) = 1 \forall x$. This renders the theory trivial in the $\hat{1}$ direction. There is a residual gauge symmetry which may be used to define $U_2(x + \hat{1}) = W(x) U_2(x)$, and the partition function thereby factorizes into a product of decoupled integrals over the unitary matrices $W(x)$ [35]. This is not possible to do in the noncommutative gauge theory, because in its lattice incarnation it is required to be formulated on a periodic lattice as a result of UV/IR mixing [25], and large gauge transformations thereby forbid axial gauge choices. As expected, UV/IR mixing drastically alters the Wilsonian renormalization features of the noncommutative field theory, and it admits non-trivial scaling limits. Later on we will see how noncommutativity explicitly modifies the Gross-Witten result.

### 5.2 Noncommutative Lattice Gauge Theory

We will now proceed to formulate and explicitly solve the noncommutative version of lattice gauge theory, which gives yet another proof of the exact solvability of the continuum theory. We discretize the torus of the previous sections as an $L \times L$ periodic square lattice. For convenience, we assume that $L$ is an odd integer. Let $\varepsilon$ be the dimensionful lattice spacing, so that the area of the discrete torus is

$$A = \varepsilon^2 L^2.$$  \hspace{1cm} (62)

Any function $f(x)$ on the periodic lattice admits a Fourier series expansion over a Brillouin zone $\mathbb{Z}_L \times \mathbb{Z}_L$,

$$f(x) = \frac{1}{L^2} \sum_{m \in \mathbb{Z}_L} f_m e^{2\pi im \cdot x / L}.$$  \hspace{1cm} (63)

A natural lattice star-product may be defined as the proper discretized version of the integral kernel representation of the continuum star product as
\[
(f \ast g)(x) = \frac{1}{L^2} \sum_{y,z} f(x+y) g(x+z) e^{2\pi i y \wedge z / \varepsilon^2 L},
\]
where the sums run over lattice points. This identifies \( \theta = 2/L \) and hence the dimensionful noncommutativity parameter of the commutant algebra as
\[
\Theta = \frac{\theta A}{2\pi} = \frac{\varepsilon^2 L}{\pi}.
\]
As mentioned in the previous subsection, because of a kinematical version of UV/IR mixing, the lattice regularization of noncommutative field theory requires the space to be a torus \([25]\).

We will now write down, in analogy with the commutative case, the natural, nonperturbative lattice regularization of the continuum noncommutative gauge theory. This is provided by the noncommutative version of the standard Wilson plaquette model \([34]\). The partition function is \([25]\)
\[
Z_r = \int \prod_x [dU_1(x)] [dU_2(x)] \exp \left[ \frac{1}{4\lambda^2 L} \sum_\Box \text{tr}_N (U_\Box + U_\Box^\dagger) \right],
\]
where
\[
\lambda = \sqrt{g^2 \varepsilon^2 L}
\]
is the 't Hooft coupling constant. Here \( \prod_x [dU_i(x)] \) is the normalized, invariant Haar measure on the ordinary \( r \times r \) unitary group \( U(r) \) with
\[
r = L \cdot N,
\]
and \( N \) is the rank of the given module. The fields \( U_i(x) \) are \( U(N) \) gauge fields which live at the links \( (x,i) \) of the lattice and which are "star-unitary",
\[
(U_i \ast U_i^\dagger)(x) = (U_i^\dagger \ast U_i)(x) = \mathbb{1}_r.
\]
In the continuum limit \( \varepsilon \to 0 \), they are identified with the gauge fields of the previous sections through \( U_i = e^{\ast \varepsilon A_i} \). The sum in (66) runs through the plaquettes \( \Box \) of the lattice with \( U_\Box \) the ordered star-product of gauge fields around the plaquette,
\[
U_\Box = U_1(x) \ast U_2(x + \varepsilon \hat{i}) \ast U_1(x + \varepsilon \hat{2})^\dagger \ast U_2(x)^\dagger,
\]
where \( x \) is the basepoint of the plaquette and \( i \) denotes the unit vector along the \( i^{th} \) direction of the lattice. The lattice gauge theory (66) is invariant under the gauge transformation
\[
U_i(x) \longrightarrow g(x) \ast U_i(x) \ast g(x + \varepsilon \hat{i})^\dagger,
\]
where the gauge function \( g(x) \) is star-unitary,
\[
(g \ast g^\dagger)(x) = (g^\dagger \ast g)(x) = \mathbb{1}_r.
\]
5.3 Gauge Theory on the Fuzzy Torus

The feature which makes the noncommutative lattice gauge theory (66) exactly solvable is that the entire lattice formalism presented above can be cast into a finite dimensional version of the abstract algebraic description of gauge theory on a projective module over the noncommutative torus [36]. For this, we note that since the noncommutativity parameter of the commutant algebra is the rational number \( \theta = 2/L \), the generators \( Z_i \) of \( A_\theta \) obey the commutation relations

\[
Z_1 Z_2 = e^{4\pi i/L} Z_2 Z_1 .
\]  

(73)

This algebra admits a finite dimensional representation which gives the noncommutative space the geometry of a *fuzzy* torus. Namely, \( A_\theta \) can be represented on the finite dimensional Hilbert module \( \mathbb{C}^L \), regarded as the space of functions on the finite cyclic group \( \mathbb{Z}_L \), as

\[
Z_1 = V_L , \quad Z_2 = (W_L)^2 ,
\]  

(74)

where \( V_L \) and \( W_L \) are the \( SU(L) \) shift and clock matrices which obey \( V_L W_L = e^{2\pi i/L} W_L V_L \).

Since \( (Z_i)^k = 1_L \), the matrices (74) generate the finite-dimensional algebra \( M_L \) of \( L \times L \) complex matrices. In fact, they provide a one-to-one correspondence between lattice fields (63) with the star-product (64) and \( L \times L \) matrices through

\[
\hat{f} = \frac{1}{L^2} \sum_{m \in (\mathbb{Z}_L)^2} f_m e^{-2\pi i m_1 m_2/L} Z_1^{m_1} Z_2^{m_2} .
\]  

(75)

It is easy to check that this correspondence possesses the same formal properties as in the continuum, namely

\[
\text{tr}_L \hat{f} = f_0 = \frac{1}{L^2} \sum_x f(x) ,
\]  

(76)

\[
\hat{f} \hat{g} = \hat{f} \ast \hat{g} .
\]  

(77)

In particular, the star-unitarity condition (69) translates into the requirement

\[
\hat{U}_i \hat{U}_i^\dagger = \hat{U}_i^\dagger \hat{U}_i = 1_r .
\]  

(78)

Therefore, there is a one-to-one correspondence between \( N \times N \) star-unitary matrix fields \( U_i(x) \) and \( r \times r \) unitary matrices \( \hat{U}_i \). In the parlance of the geometry of the noncommutative torus, we have \( A_\theta \cong M_L \) and the endomorphism algebra is \( \text{End}(\mathcal{E}) \cong A_\theta \otimes M_N \cong M_r \). The gauge fields in the present setting live in the unitary group of this algebra, which is just \( U(r) \) as above.

To cast the gauge theory (66) into a form which is the natural nonperturbative version of (6) [36], we introduce connections \( V_i = e^{\epsilon \nabla_i} \) on this discrete
geometry which are $r \times r$ unitary matrices that may be decomposed in terms of the gauge fields $\hat{U}_i$ as

$$V_i = \hat{U}_i \Gamma_i,$$

where the matrices $\Gamma_i = e^{\varepsilon \partial_i}$ correspond to lattice shift operators. They thereby satisfy the commutation relations

$$\Gamma_1 \Gamma_2 = \zeta \Gamma_2 \Gamma_1,$$

$$\Gamma_i Z_j \Gamma^\dagger_i = e^{2\pi i \delta_{ij}/L} Z_j,$$

where

$$\zeta = e^{2\pi i q/L}$$

is a $\mathbb{Z}_L$-valued phase factor whose continuum limit gives the background flux in (1). The integer $q$ is chosen, along with some other integer $c$, to satisfy the Diophantine equation

$$cL - 2q = 1$$

for the relatively prime pair of integers $(L, 2)$. The equations (80) and (81) can then be solved by

$$\Gamma_1 = (W_L^1)^{2q}, \quad \Gamma_2 = (V_L)^q.$$

Note that while the Heisenberg commutation relations for constant curvature connections admit no finite dimensional representations, the Weyl-'t Hooft commutation relation (80), which is its exponentiated version, does. In other words, the matrices (84) generate the irreducible action of the Heisenberg-Weyl group on the finite-dimensional algebra $A_\theta \cong M_L$. This construction can be generalized to provide discrete versions of the standard Heisenberg modules over the noncommutative torus [36].

We now substitute the matrix-field correspondences (63) and (75)–(77) for the gauge fields into the partition function (66), use the fact that $\Gamma_i$ generates a lattice shift along direction $i$, and use the decomposition (79) to rewrite the action in terms of the finite dimensional connections $V_i$. By using in addition the invariance of the Haar measure, the Weyl-'t Hooft algebra (80), and the representation of the trace $tr_r = tr_L \otimes tr_N$ on $\text{End}(\mathcal{E}) \cong M_r$, after some algebra we find that the partition function (66) can be written finally as the unitary two-matrix model [36]

$$Z_r = \int [dV_1] [dV_2] e^{\frac{1}{2 \pi \tau} \text{Re} \, tr_r \zeta V_i V_j^\dagger V_k V_k^\dagger}.$$

This is the partition function of the twisted Eguchi-Kawai model in two dimensions [37, 38], with twist given by the $\mathbb{Z}_L$ phase factor (82), and it coincides with the dimensional reduction of ordinary Wilson lattice gauge theory to a single plaquette [25]. The star-gauge invariance (71) of the plaquette model (66) corresponds to the $U(r)$ invariance.
Two-Dimensional Noncommutative Gauge Theory

$$V_i \mapsto \hat{g} V_i \hat{g}^\dagger \ , \ \hat{g} \in U(r)$$

(86)

of the twisted Eguchi-Kawai model (85). Note that the $U(r)$ gauge symmetry of the matrix model (85) is a mixture of the original $L \times L$ spacetime degrees of freedom of the noncommutative lattice gauge theory (66) and its $U(N)$ colour symmetry. The partition function (85) is a well-defined, finite-dimensional operator version of the noncommutative Wilson lattice gauge theory in two-dimensions, which we will now proceed to compute explicitly.

5.4 Exact Solution

To evaluate the unitary group integrals (85), we insert an extra integration involving the gauge invariant delta-function acting on class functions on $U(r)$ to get

$$Z_r = \int [dV_1] [dV_2] \int [dW] \delta \left( W, V_1 V_2 V_1^\dagger V_2^\dagger \right) e^{\frac{i}{4\pi} \text{tr}_r (\zeta W + \zeta^* W^\dagger)}.$$  

(87)

The delta-function in the Haar measure may be expanded in terms of the orthonormal $U(r)$ characters as

$$\delta(W,U) = \sum_R \chi_R(W) \chi_R(U^\dagger).$$

(88)

As in section 3.1, the unitary irreducible representations $R$ of the Lie group $U(r)$ may be parametrized by partitions $\mathbf{n} = (n_1, \ldots, n_r)$ into $r$ parts of decreasing integers as in (32). The character of the unitary matrix $W$ in this representation can then be written explicitly by means of the Weyl formula

$$\chi_R(W) = \chi_n(W) = \frac{\text{det}_{a,b} \left[ e^{i(n_a - b+r)\phi_b} \right]}{\text{det}_{a,b} \left[ e^{i(a-1)\phi_b} \right]},$$

(89)

where $e^{i\phi_1}, \ldots, e^{i\phi_r}$ are the eigenvalues of $W$.

On substituting (88) into (87), the integration over $V_1$ and $V_2$ can be carried out explicitly by using the fusion rule (30) for the $U(r)$ characters along with the fission relation

$$\int [dU] \chi_n \left( WUVU^\dagger W \right) = \frac{\chi_n(V) \chi_n(W)}{d_n},$$

(90)

where

$$d_n = \chi_n(1_1) = \prod_{a<b} \left( 1 + \frac{n_a - n_b}{b-a} \right)$$

(91)

is the dimension $\text{dim} R$ of the representation $R$ with highest weight vector $\mathbf{n} = (n_1, \ldots, n_r)$. In this way the partition function takes the form

$$Z_r = \sum_{n_1 \geq \cdots \geq n_r} \frac{1}{d_n} \int [dW] \chi_n(W) e^{\frac{i}{4\pi} \text{tr}_r (\zeta W + \zeta^* W^\dagger)}.$$  

(92)
The twist factors (82) can be decoupled from the integration in (92) by the rescaling $W \to \zeta W$ and by using $U(r)$ invariance of the Haar measure along with the character identity

$$
\chi_n (\zeta W) = e^{-2\pi i q C_1(n)/L} \chi_n (W),
$$

where

$$
C_1(R) = C_1(n) = \sum_{a=1}^{r} n_a
$$

is the linear Casimir invariant of the representation $R$ which counts the total number of boxes in the corresponding $U(r)$ Young tableau.

We now expand the invariant function in (92) which after rescaling is the Boltzmann factor for the one-plaquette $U(r)$ Wilson action. Its character expansion can be given explicitly in terms of modified Bessel functions $I_n(z)$ of the first kind of integer order $n$ which are defined by their generating function as

$$
\exp \left[ \frac{z}{2} \left( t + \frac{1}{t} \right) \right] = \sum_{n=-\infty}^{\infty} I_n(z) t^n.
$$

By using (89) one finds [39]

$$
e^{\beta \text{tr}_r(W+W')} = \sum_{n_1 > \cdots > n_r} \frac{1}{d_n} \det_{a,b} \left[ I_{n_a-a+b(2\beta)} \right] \chi_n (W^t),
$$

and, by using the fusion rule (30), substitution of (96) into (92) gives a representation of the lattice partition function as a sum over a single set of partitions alone,

$$
Z_r = \sum_{n_1 > \cdots > n_r} \frac{e^{-2\pi i q C_1(n)/L}}{d_n} \det_{a,b} \left[ I_{n_a-a+b(1/2\lambda^2)} \right].
$$

To express (97) as a perturbation series in the effective coupling constant $1/\lambda^2$, we substitute into this expression the power series expansion of the modified Bessel functions,

$$
I_\nu(z) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(\nu + m + 1)} \left( \frac{z}{2} \right)^{\nu + 2m},
$$

where $\Gamma(z)$ is the Euler function. The infinite sum may then be extracted out line by line from the determinant in (97) by using the multilinearity of the determinant as a function of its $r$ rows, and we find

$$
Z_r = \sum_{n_1 > \cdots > n_r} \frac{e^{-2\pi i q C_1(n)/L}}{d_n} \prod_{m_1=0}^{\infty} \cdots \prod_{m_r=0}^{\infty} \prod_{s=1}^{r} \frac{(1/2\lambda)^{2m_s}}{m_s!}
\times \det_{a,b} \left[ \frac{(1/2\lambda)^{2(n_a+n_a-a+b)}}{\Gamma(m_a+n_a-a+b+1)} \right].
$$
Note that the total contribution to (99) vanishes from any set of integers for which \(m_a + n_a < a - r\) for any single index \(a = 1, \ldots, r\).

The determinant in (99) can be evaluated explicitly as follows. For any sequence of integers \(s_1, \ldots, s_r\), we have

\[
\det_{a,b} \left[ \frac{z^{s_a-a+b}}{\Gamma(s_a-a+b+1)} \right] = z^{s_1+\cdots+s_r} \prod_{b=1}^{r} \frac{1}{\Gamma(s_{b'}-b'+r+1)} \prod_{b'=1}^{r} \frac{1}{\Gamma(s_{b'}+b'+r+1)} .
\]

(100)

Factorizing \(1/\Gamma(s_b-b+r+1)\) from each column \(b\) of the remaining determinant in (100) yields

\[
\det_{a,b} \left[ \frac{z^{s_a-a+b}}{\Gamma(s_a-a+b+1)} \right] = z^{s_1+\cdots+s_r} \prod_{b=1}^{r} \frac{1}{\Gamma(s_{b'}-b'+r+1)} \prod_{b'=1}^{r} \frac{1}{\Gamma(s_{b'}+b'+r+1)} \times \det_{a,b} \left[ (s_b-b+a+1)(s_b-b+a+2) \cdots (s_b-b+r) \right] .
\]

(101)

The argument of the determinant in the right-hand side of (101) is a monic polynomial in the variable \(\alpha_b = s_b-b\) with highest degree term \(\alpha_b^r\). By using multilinearity of the determinant, it becomes \(\det_{a,b} [\alpha_b^r] = \prod_{a<b} (\alpha_a - \alpha_b)\), and we arrive finally at

\[
\det_{a,b} \left[ \frac{z^{s_a-a+b}}{\Gamma(s_a-a+b+1)} \right] = z^{s_1+\cdots+s_r} \prod_{b=1}^{r} \frac{(r-b)!}{\Gamma(s_b-b+r+1)} \times \prod_{a<b} \left( 1 + \frac{s_a-s_b}{b-a} \right) .
\]

(102)

Note that if \(s = (s_1, \ldots, s_r)\) is a partition, then the last product in (102) is just the dimension \(d_s\) of the corresponding \(U(r)\) representation.

The partition function (99) is thereby given as

\[
Z_r = \sum_{n_1, \ldots, n_r} e^{-2\pi i q C_1(n)/L} \frac{1}{d_n(2\lambda)^{2C_1(n)}} \times \prod_{m_1=0}^{\infty} \cdots \prod_{m_r=0}^{\infty} \prod_{b=1}^{r} \frac{(r-b)!}{m_b!} \frac{(1/2\lambda)^{4m_b}}{(m_b+n_b-b+r+1)} \times \prod_{a<b} \left( 1 + \frac{m_a - m_b + n_a - n_b}{b-a} \right) .
\]

(103)

Finally, we can simplify this expansion for \(Z_r\) even further by decoupling the sum over partitions \(n = (n_1, \ldots, n_r)\). For this, we define a new set of integers by
\[ p_a = n_a - n_{a+1} + 1, \quad a = 1, \ldots, r - 1, \]
\[ p_r = n_r. \]  
(104)

Then the \( p_a \)'s are all independent variables, constrained only by their ranges which are given by \( 1 \leq p_a < \infty \) for \( a = 1, \ldots, r - 1 \) and \( -\infty < p_r < \infty \).

The decoupled expansion of the partition function is thereby obtained by substituting
\[ n_a = p_a + p_{a+1} + \cdots + p_r + a - r, \]  
(105)
along with the explicit group theoretical formulas (91) and (94), into (103) to get the final result (up to irrelevant numerical factors)
\[
\mathcal{Z}_r = \sum_{p_1 = 1}^{\infty} \cdots \sum_{p_{r-1} = 1}^{\infty} \sum_{p_r = -\infty}^{\infty} \cos \left( \frac{2\pi q}{L} \sum_{b=1}^{r} bp_b \right) 
\times \sum_{m_1 = 0}^{\infty} \cdots \sum_{m_r = 0}^{\infty} \prod_{b=1}^{r} \frac{(b-1)! (2\lambda)^{4m_b-2bp_b}}{m_b! \Gamma(m_b+bp_b+bp_{b+1}+\cdots+p_r+1)}
\times \prod_{a<b} \frac{m_a - m_b + p_a + p_{a+1} + \cdots + p_b}{p_a + p_{a+1} + \cdots + p_b},
\]  
(106)

where we have used the reality of the left-hand side of (96) to make the expression for the partition function manifestly real by adding its complex conjugate to itself. The partition function (106) is a straightforward expansion in powers of \( 1/\lambda^2 \) over \( 2r \) independent integers \( p_a, m_a, a = 1, \ldots, r \). Note the reduction in the number of dynamical degrees of freedom of the model. The original \( 2r^2 \) degrees of freedom of the two-dimensional lattice gauge theory (66) (or equivalently of the unitary two-matrix model (85)) is reduced to \( 2r \). This proves that the lattice model is exactly solvable, and thereby gives yet another indication that noncommutative gauge theory in two dimensions is a topological field theory. The sum (106) is formally analogous to the partition expansion of continuum noncommutative Yang-Mills theory.

### 5.5 Scaling Limits

The final step of this calculation should be to take the continuum limit \( \epsilon \to 0 \) of the lattice theory. In order to prevent the spacetime from degenerating to zero area, from (62) we see that we must also take \( L \to \infty \), or equivalently \( r \to \infty \) in (106). There are different ways of performing these two limits, each of which leads to a different continuum gauge theory. If the limit is taken such that the dimensionful noncommutativity parameter (65) vanishes, then the continuum limit is ordinary Yang-Mills theory in two dimensions. The area (62) may be either finite or infinite in this limit. If \( A \to \infty \), then the expansion (97) truncates to the trivial representation for which \( n_a = 0 \) \( \forall a \) and one obtains
\( Z_r \bigg|_{\Theta \to 0, A \to \infty} = \det \left[ I_{b-a} \left( 2/\lambda^2 \right) \right]. \) 

(107)

This expression is recovered in the naive large \( r \) limit due to the suppression of higher representations which is induced by the dimension factors \( d_n \) in the denominators of (97). It is just the standard expression for Yang-Mills theory on the plane which arises from the one-plaquette Wilson model in the limit of a large number of colours [40]. Going back to (92), we see that the truncation to \( n = 0 \) is indeed nothing but the Gross-Witten reduction of commutative lattice gauge theory in two dimensions [35].

The other scaling limit that one can take is \( \varepsilon \to 0, L \to \infty \) with \( \varepsilon^2 L \) finite. Then the noncommutativity parameter (65) is finite, but the area (62) diverges. The resulting continuum limit is gauge theory on the noncommutative plane, and from (92) we see that its partition function generalizes that of ordinary Yang-Mills theory by including a sum over non-trivial representations of the unitary group. This quantitative difference is similar in spirit to that which occurs in the group theory presentation of noncommutative gauge theory [14], which can be thought of as a modification of ordinary gauge theory by the addition of infinitely many higher Casimir operators to the action (equivalently higher powers of the field strength \( F_A \)). The inclusion of higher representations in the statistical sum means that this series cannot be expressed in terms of a unitary one-matrix model. Determinants such as (107) whose matrix elements depend only on the difference between row and column labels are called Toeplitz determinants and are known to be equivalent to the evaluation of a related unitary one-matrix integral [41]. In the present noncommutative case, the partition function is not given by a Toeplitz determinant, although it is represented by the unitary two-matrix model (85) and depends only on the eigenvalues of the matrix \( W = V_1 V_2 V_1^\dagger V_2^\dagger \).

Unravelling the precise continuum limit of the expansion (106) is one of the important unsolved analytical problems in the combinatorial approach to two-dimensional noncommutative Yang-Mills theory. The noncommutativity parameter \( \Theta \) enters in the ‘t Hooft coupling constant as \( \lambda^2 = \pi g^2 \Theta \) and implicitly in the factors of \( L = r/N \) appearing in (106). It is necessary to identify whether the double-scaling limit required, over and above the naive continuum limit, exists within this framework. Both the naive and non-trivial double-scaling limits have been observed numerically in the Eguchi-Kawai model [42], and more recent numerical investigations indicate that they exist also within the full noncommutative field theory [9],[43]–[45]. The rigorous derivation of this limit is described at the classical level in [31]. Amongst other things, the solution to this system may help in unravelling the mysterious properties of the gauge group of noncommutative gauge theory, which in the present context is formally an \( r \to \infty \) limit of \( U(r) \), confirming other independent expectations [1, 4, 10],[46]–[48]. It would also be interesting to understand the complete solution of the discrete theory whose continuum
spacetime is a torus, which is given by a more general construction [36] to which the present analysis does not apply.

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