Optimality of Fast Matching Algorithms for Random Networks with Applications to Structural Controllability

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Abstract—Network control refers to a very large and diverse set of problems including controllability of linear time-invariant dynamical systems evolving over time that have inputs and outputs. The network control problem in this setting is to select the appropriate input to steer the network into a desired output state. Examples of the output state include the throughput of a communications network, transcription factor concentration in a gene regulatory network, customer purchases in a marketing context subject to social influences and the amount of flux flowing through a biochemical network.

We focus on control of linear dynamical systems under the notion of structural controllability which is intimately connected to finding maximum matchings. Hence, a natural objective is studying scalable and fast algorithms for this task. We first show the convergence of matching algorithms for different random networks and then analyze a popular, fast and practical heuristic due to Karp and Sipser. We establish the optimality of both the Karp-Sipser Algorithm as well as a simplification of it, and provide results concerning the asymptotic size of maximum matchings for an extensive class of random networks.

Index Terms—Maximum Matching, Karp-Sipser, Structural Controllability, Network Control, Random Networks.

I. INTRODUCTION

NETWORKS are capable of capturing relationships between a set of entities (vertices) and have found applications in diverse scientific fields including biology, engineering, economics and the social sciences [1], [2], [3]. Network control refers to a very large and diverse set of problems that involve control actions over a network (see for example, [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and references therein).

A class of control problems involves dynamical systems evolving over time that have inputs and outputs and many results exist for systems that exhibit linear and time-invariant dynamics [11]. One particular notion of control is that of structural controllability [1], which was recently explored by Liu et al. [12]. Under this notion, the structural controllability problem reduces to find maximum matchings on appropriate matrices as reviewed in Section 2A. The problem of obtaining maximum matchings has been extensively studied in the computer science literature both for deterministic [14] as well as random networks [15]. However, the focus in the literature has been on special classes of indirected random networks [16], [17], [18] and little is known for the case of directed random graphs that is of interest in the network control problem.

A popular, fast and practical algorithm for matchings on undirected random networks is due to Karp and Sipser [19], which represents the cornerstone of our theoretical investigations and through it we provide generalizations of previous work in the literature to broader classes of undirected random networks. Further, we also extend the results for directed variants of the same classes of random networks.

A. Structural Controllability and Maximum Matchings

Next, we review some key concepts in structural controllability for linear dynamical systems. Consider a system described by a $n$-dimensional state vector $x(t) = (x_1(t), ... , x_n(t))^T \in \mathbb{R}^n$, whose dynamical evolution is described by

$$\frac{dx}{dt} = Ax(t) + Bu(t),$$

where $A \in \mathbb{R}^{n \times n}$ is the system transition matrix, $u(t) = (u_1(t), ... , u_k(t))^T \in \mathbb{R}^k$ captures control actions and $B \in \mathbb{R}^{n \times k}$ is the input matrix. Assuming that the $n$-dimensional system can be represented by vertices on a network $G = (V,E)$ with $V = \{1, 2, ..., n\}$ denoting the set of vertices and $E \subset V \times V$ the set of edges, it can be seen that the non-zero entries in the transition matrix $A$ correspond to the directed edges in $E$. Such a system is called controllable if for any initial state $x(0) = x_0$ and any desired state $x_d$ for some $T < \infty$ one can find an input matrix $B$ and control vectors $\{u(t)\}_{0 \leq t \leq T}$ so that the system reaches state $x_d$ i.e. $x(T) = x_d$. The minimum $k$ for which system can be controllable is called the minimum number of controllers.

The magnitude of the entries in the transition matrix $A$ captures the interaction strength between the vertices in the network; for example, the traffic on individual communication links in a communications network or the strength of a regulatory interaction in a biological network. The time invariant matrix $B$ indicates which vertices are controlled by an outside controller. Note that in general one controller $u_i(t)$ can influence multiple vertices. Hence, the set of vertices that when applying controllers to them makes the system controllable needs to be identified.

The algebraic criterion to check controllability of a time invariant linear dynamical system is Kalman’s controllability rank condition, that states that controllability can be achieved, if and only if the matrix $C = [B, AB, ... , A^{n-1}B]$ is full rank; i.e. $\text{rank}(C) = n$. Note that $C \in \mathbb{R}^{n \times nk}$. This algebraic criterion is computationally hard to check, especially for large
systems. Further, in many applications, obtaining exact values of $A$ may not be feasible and hence a tractable alternative is needed.

Thus, we say that a time invariant linear dynamical system is structurally controllable, if it is possible to select the non-zero values of $A, B$, so that Kalman’s rank condition is satisfied \[1\]. A structurally controllable system is controllable for almost all $A, B$; i.e. the pathological cases for which a structurally controllable network is not controllable has zero Lebesgue measure. Liu et al. \[12\] established the Minimum Inputs Theorem (stated below), which establishes the relation between the minimum number of controllers needed to structurally control a network and the size of its maximum matching. Algorithms to find a maximum matching are well studied in the literature and exhibit polynomial time complexity (with respect to the size of the network). A popular one developed by Micali and Vazirani \[20\] has running time $O(|V|^{0.5}|E|)$.

Next, for completeness we provide a definition of maximum matching and also state the Minimum Inputs Theorem.

**Definition 1.** For a directed network $G = (V, E)$, a subset of edges $M$ is a matching, if no two edges in $M$ share a common starting or a common ending vertex. A vertex is matched, if it is an ending vertex of an edge in the matching. Otherwise, it is unmatched. A maximum matching corresponds to a matching of maximum size.

**Minimum Inputs Theorem** \[12\]: Let $M$ be a matching of the network $G = (V, E)$. The network is structurally controllable by $\max\{1, n - |M|\}$ controllers. So the minimum number of controllers is $\max\{1, n - |M^*|\}$, where $M^*$ is a maximum matching.

The upshot of this result is that in order to find the minimum number of required controllers for structural controllability we can equivalently finding the size of a maximum matching.

In this work, we provide results about the size of matchings obtained by different fast algorithms for classes of random networks. The remainder of the paper is organized as follows. In Section III we introduce different classes of random networks subsequently studied in this work. Furthermore, some probabilistic results needed for technical developments are summarized. The main algorithms studied are introduced in Section III and the key results of the paper are presented in Sections IV (analysis of the algorithms) and IV-B (optimality), respectively.

**II. Random Networks**

In this section, we introduce different classes of random network models and then present some general results about convergence and concentration of real-valued functions on networks. In order to have a general framework which includes both directed and undirected networks, we note that every undirected network $G = (V, E)$ can be considered as a directed network in which, for all vertices $i, j$, both edges $i \rightarrow j$ and $j \rightarrow i$ exist, if and only if the edge $i \leftrightarrow j$ exists in the original undirected network. All statements presented are true for both directed and undirected networks unless explicitly mentioned. For a comprehensive discussion on constructions and properties of (undirected) random networks, see Durrett \[21\].

The first model for random networks we consider is the Generalized Erdos-Renyi (GER) model. A directed network $G = (V, E), V = \{1, 2, \ldots, n\}$ is (drawn from) GER if every edge $i \rightarrow j$ for $i, j = 1, 2, \ldots, n$ present in the network is drawn independently with probability $p_{ij}^{(n)}$. Analogously, an undirected network $G = (V, E), V = \{1, 2, \ldots, n\}$ is GER if every edge $i \leftrightarrow j$ for $i, j = 1, 2, \ldots, n, i \leq j$ is drawn independently with probability $p_{ij}^{(n)}$. Henceforth, for $\lambda \in [0, \infty)$, GER(\(\lambda\)) is a Generalized Erdos-Renyi random network, for which all $p_{ij}^{(n)}$ are equal and $np_{ij}^{(n)} \to \lambda$ as $n \to \infty$. In GER(\(\lambda\)) random networks, the parameter $\lambda$ corresponds to the average degree.

Note that Stochastic Block Models (SBMs) \[22\] are a special case of Generalized Erdos-Renyi random networks. This is because the probability of the edge $i \rightarrow j$, $p_{ij}^{(n)}$ in SBMs only depends on the communities $i, j$ belong to, where communities form a partition of the set of vertices; i.e. $p_{ij}^{(n)} = p_{i'j'}^{(n)}$ as long as $i, i'$ and $j, j'$, respectively, belong to the same community. Usually $p_{ij}^{(n)}$ is significantly larger if $i, j$ are members of the same community.

The next model we consider is the Uniform Fixed-Size (UFS) model. A directed network $G = (V, E), V = \{1, 2, \ldots, n\}$ is UFS when the cardinality of the edge set $|E| = k_n$ for some fixed $k_n$, and the $k_n$ directed edges are drawn uniformly among all $n^2$ possible edges. The construction for the undirected network is similar, but the $k_n$ edges are chosen uniformly among all \(\binom{n^2}{k_n}\) possible edges. For $\lambda \in [0, \infty)$, we denote by UFS(\(\lambda\)) a random network of the UFS class, for which $\frac{k_n}{n} \to \lambda$ for directed and $\frac{k_n}{n} \to \frac{\lambda}{2}$ for the undirected case, as $n \to \infty$. Once again, the $\lambda$ parameter corresponds to the average degree.

Finally, we introduce the class of Degree Distribution (DD) random networks. There are a couple of reasons for considering this class. First, it lets us consider networks with degree distributions commonly found in real networks (e.g., power laws) that simpler models such as GER(\(\lambda\)) (where the degree distribution is Poisson) cannot model. Second, Liu et al. \[12\] empirically found that structural controllability of a network is to a large extent governed by its degree distribution. An undirected random network is a member of the DD class, if for a given vertex degree distribution the attachment of edges is random. Specifically, let $p$ be a probability distribution with support on the set $\{0, 1, 2, \ldots\}$ of nonnegative integers. We then construct an undirected network $DD(p) = (V, E)$ as follows. Let $V = \{1, 2, \ldots, n\}$ and for $i \in V$, let vertex $i$ have $D_i$ undirected half-edge(s) (one-half of an edge is connected to vertex $i$) where the $D_1, \ldots, D_n$ are independent and identically distributed (iid) degrees with distribution $p, \mathbb{P}(D_i = k) = p(k)$. To complete the construction, we then pair all half-edges randomly; i.e. all $\binom{n^2}{2}$ possible attachments of half-edges have equal probability.

When the number of half-edges $\sum_{i=1}^{n} D_i$ is an even number, the construction is straightforward and the number of edges
will be $\frac{1}{2} \sum_{i=1}^{n} D_i$. When it is an odd number, we pair the half-edges randomly to obtain the network and omit the last single half-edge for which no pairing was established at the end of the construction, so that the number of edges will be $\frac{1}{2} \sum_{i=1}^{n} D_i - 1$.

Note that the omission or presence of multiple edges will lead to a difference between $D_1, \ldots, D_n$ as the actual observed degrees $\deg(1), \ldots, \deg(n)$ once the network construction is completed. However, as Lemma 1 below establishes, the asymptotic empirical degree distribution will be the original degree distribution from which the network was constructed, as long as the expected value of $D_i$ is finite.

Viewing an undirected network $DD(p)$ as a directed one, both input and output degrees of vertex $i$ are $\deg(i)$. To construct a directed DD random network, denoted by $DD(p_{\text{in}}, p_{\text{out}})$, with distinct input ($\deg_{\text{in}}$) and output ($\deg_{\text{out}}$) degrees, we do the following: once we have iid draws $D^{\text{in}}_i$ and $D^{\text{out}}_i$ from the input and output degree probability distributions $p_{\text{in}}$ and $p_{\text{out}}$ respectively:

$$\Pr(D^{\text{in}}_i = k) = p_{\text{in}}(k), \Pr(D^{\text{out}}_i = k) = p_{\text{out}}(k),$$

let vertex $i \in V$ have $D^{\text{in}}_i$ directed half-edges pointing into node $i$ and $D^{\text{out}}_i$ directed half-edges pointing out from node $i$. Next, we pair directed half-edges randomly to have $\min\{\sum_{i=0}^{n} D^{\text{in}}_i, \sum_{i=0}^{n} D^{\text{out}}_i\}$ edges and omit the remaining half-edges. The random pairing of half-edges implies that all possible pairings of half-edges are equally likely. Note that $D^{\text{in}}_i, D^{\text{out}}_i$ does not need to independent.

Furthermore, in general, the degrees do not need to be iid. In fact, as shown later in the paper, the key asymptotic results we establish are based on the empirical degree distributions which are, by the following lemma, same as the original degree distributions when vertex degrees are iid. However, as long as for all $k = 0, 1, \ldots, \lim_{n \to \infty} \frac{\sum_{i \in V : \deg(i) = k}}{n}$ (equivalently $\lim_{n \to \infty} \frac{|\{i \in V : \deg(i) = k\}|}{n}$) are deterministic, our results hold using the resulting asymptotic empirical degree distributions.

**Lemma 1.** For an undirected network $G = (V, E)$, define the asymptotic empirical degree distribution as

$$\hat{p}(k) = \lim_{n \to \infty} \frac{|\{i \in V : \deg(i) = k\}|}{n}.$$  

If $G = DD(p)$ is a random network and $\mu = \sum_{k=0}^{\infty} kp(k) < \infty$, then the limit above exists and we have $\hat{p}(k) = p(k)$ for all $k = 1, 2, \ldots$. In general for a network $G = (V, E)$, define the asymptotic empirical input and output degree distributions as

$$\hat{p}_{\text{in}}(k) = \lim_{n \to \infty} \frac{|\{i \in V : \deg_{\text{in}}(i) = k\}|}{n},$$  

$$\hat{p}_{\text{out}}(k) = \lim_{n \to \infty} \frac{|\{i \in V : \deg_{\text{out}}(i) = k\}|}{n}.$$
Before studying the algorithms we describe a description of networks which will be useful later. As mentioned before every undirected network can be considered as a directed one. Now to have a better understanding of how the algorithms work we view every directed network as a bipartite network $G = (L, R, E)$ where $L = R = V, E \subset L \times R$, and for $l \in L, r \in R$ there is an edge $(l, r) \in E$ if and only if in the original directed network there is an edge from $l$ to $r$: $l \rightarrow r$. So henceforth we will only deal with bipartite networks.

Matching algorithms take a network as input and the output will be a matching. Maximum matching algorithms will give a matching of maximum size. Algorithm 1 is the well known Greedy Algorithm that produces a suboptimal matching $M_G$ in general. Since, as mentioned above, networks can be assumed to be bipartite, Greedy tries to find a match one by one, for all vertices in the right side of the bipartite network, to a vertex in the left side.

Algorithm 1 : Greedy  
Input: $G = (L, R, E)$  
Output: matching $M_G(G)$

$M_G \leftarrow \emptyset$
while $E \neq \emptyset$ do
    let $v \in R$
    if $\deg(v) = 0$ then
        $G \leftarrow G \setminus \{v\}$
    else if for $u \in L, (u, v) \in E$ then
        $G \leftarrow G \setminus \{u, v\}$
        $M_G \leftarrow M_G \cup \{u, v\}$
    end if
end while
return $M_G$

Note that Greedy picks an arbitrary vertex $v \in R$ in every iteration. Because the goal is to find a matching of largest possible size, this strategy for picking a vertex can be improved. First note that for every vertex $v \in R$ of degree one, there is a matching of maximum size in which $v$ is matched. The logic is as follows. If $u \in L$ is the vertex on the left side connected to $v$, $(u, v) \in E$, for one vertex on the right side like $w \in R$ such that $(u, w) \in E$, $w$ must be matched to $u$ by a matching $M$ of maximum size since if not, adding $(u, w)$ to it leads to a matching of larger size. Now defining a new matching $M'$ which is exactly $M$ removing $(u, w)$ adding $(u, v)$, $M' = M \setminus \{(u, w)\} \cup \{(u, v)\}$, we have $|M| = |M'|$ i.e. $M'$ is a maximum matching as well. Hence as long as we can find a vertex of degree one, we can find a matching of exactly maximum size or on the other word: no mistake happens as long as a degree one vertex is picked in every iteration of Greedy.

This fact is the idea behind Algorithm 2, called the Karp-Sipser Algorithm (KS) [19], which produces a matching $M_{KS}$. In every iteration of KS among all vertices a vertex of minimum degree is picked.

We can simplify the KS algorithm and search for a minimum degree vertex among vertices on only one side to derive Algorithm 3 that we call One-sided Karp-Sipser (OKS).

Algorithm 2 : Karp-Sipser  
Input: $G = (L, R, E)$  
Output: matching $M_{KS}(G)$

$M_{KS} \leftarrow \emptyset$
while $E \neq \emptyset$ do
    let $v = \text{arg min}_{w \in L \cup R} \deg(w)$
    if $\deg(v) = 0$ then
        $G \leftarrow G \setminus \{v\}$
    else if for $u \in L \cup R, (u, v) \in E$ then
        $G \leftarrow G \setminus \{u, v\}$
        $M_{KS} \leftarrow M_{KS} \cup \{u, v\}$
    end if
end while
return $M_{KS}$

Algorithm and whose output we denote by $M_{OKS}$. Note that the size of the matching given by OKS can not be larger than the size of the matching given by KS. Because it is possible to do a mistake (deviating from maximum matching) in OKS because of lack of degree one vertices in the right side, but if degree one vertices exists on the left side, KS can still work optimally. More rigorous reasoning using induction over the size of the network show that OKS can not perform better than KS. Yet, later we will prove (asymptotic) optimality of OKS and so optimality of KS will follow.

Algorithm 3 : One-Sided Karp-Sipser  
Input: $G = (L, R, E)$  
Output: matching $M_{OKS}(G)$

$M_{OKS} \leftarrow \emptyset$
while $E \neq \emptyset$ do
    let $v = \text{arg min}_{w \in R} \deg(w)$
    if $\deg(v) = 0$ then
        $G \leftarrow G \setminus \{v\}$
    else if for $u \in L, (u, v) \in E$ then
        $G \leftarrow G \setminus \{u, v\}$
        $M_{OKS} \leftarrow M_{OKS} \cup \{u, v\}$
    end if
end while
return $M_{OKS}$

IV. MATCHING ALGORITHMS IN RANDOM NETWORKS
In this section we present results about the asymptotic size of matchings produced by the algorithms presented above. We first consider the case when the asymptotic degree distribution of the random network is Poisson and then generalize our results to arbitrary degree distributions (with finite mean).

A. Poisson Degree Distribution  
In this section after presenting some asymptotic and non-asymptotic results about the fraction of matched vertices for the matching given by Greedy for a class of random networks, we generalize the results provided by Karp and Sipser [19] about the performance of KS.
Studying a simple algorithm (Greedy) on a simple random network $G$ (drawn from directed GER where every edge $i \to j$ exists with probability $p^{(n)}_{i,j} = p$) places us in an unusual situation where we can find the non-asymptotic probability mass function for $|M_G(G)|$. The following theorem also provides the asymptotic behavior of $|M_G(G)|$ for directed GER($\lambda$).

**Theorem 2** (Greedy for directed GER($\lambda$)). Let the network $G$ be directed GER of size $n$ with constant edge existence probability i.e. for all vertices $i, j$ the edge $i \to j$ exists with probability $p^{(n)}_{i,j} = p$. Then:

$$P(|M_G(G)| = n - k) = \frac{\alpha_n(q)^2}{\alpha_k(q)^2 \alpha_{n-k}(q)} q^k$$

where $q = 1 - p$ and $\alpha_i(q) = \prod_{j=1}^{i} (1 - q^j)$. For GER($\lambda$) (i.e. $np \to \lambda$) if $\lambda = 0$ then $\lim_{n \to \infty} \frac{|M_G(G)|}{n} = 0$. If $\lambda = \infty$ then $\lim_{n \to \infty} \frac{|M_G(G)|}{n} = 1$. For $\lambda \in (0, \infty)$, $|M_G(G)|$ is asymptotically normal:

$$\mathcal{N}\left(n \frac{\lambda - \log(2 - e^{-\lambda})}{\lambda}, n \frac{1}{4\lambda}\right)$$

Before proceeding with the analysis of matching algorithms in a more extensive class of random networks, we must make sure that the size of the matchings provided by either Greedy, KS, OKS algorithms or any maximum matching algorithm has the Lipschitz property in order to have convergence of $\frac{|M_G(G)|}{n}$, $\frac{|M_{KS}(G)|}{n}$, $\frac{|M_{OKS}(G)|}{n}$ and $\frac{|M^*(G)|}{n}$ for random network $G$ where $M^*(G)$ is a maximum matching of network $G$.

The following lemma proves the desired Lipschitz property. The size of the matching provided by any of the above algorithms has the Lipschitz property due to the recursive nature of the algorithms. The Lipschitz property for the size of maximum matchings comes from their maximality regardless of the algorithm a maximum matching is provided by (this follows easily from the definition of maximum matching).

**Lemma 2.** The real-valued functions $|M_G|$, $|M_{KS}|$, $|M_{OKS}|$ and $|M^*|$ which are the size of matchings provided by Greedy, KS, OKS and maximum matching algorithms, respectively, have the Lipschitz property.

Going back to the Greedy Algorithms, some of the results in Theorem 2 remain valid for a larger class of random networks including directed and undirected networks.

**Theorem 3** (Greedy for asymptotically Poisson degree distributions). Assume $G$ is one of GER($\lambda$), UFS($\lambda$), DD($p$) and DD($p, p$) where probability distribution $p$ is Poisson($\lambda$).

If $\lambda = 0$ (resp. $\lambda = \infty$) then $\lim_{n \to \infty} \frac{|M_G(G)|}{n} = 0$ (resp. 1). For $\lambda \in (0, \infty)$, $\lim_{n \to \infty} \frac{|M_G(G)|}{n} = 1 - \frac{\log(2 - e^{-\lambda})}{\lambda}$.

Similar results hold for $KS$. In fact, Karp and Sipser proved that, for the classical undirected Erdos-Renyi random network (denoted by undirected GER($\lambda$), $\lambda \in (0, \infty)$ here), $KS$ is optimal. They split the running of the algorithm into two phases. Phase 1 begins when the algorithm starts and finishes the first time there is no vertex of degree one in the network, when phase 2 starts and proceeds until the algorithm strips the whole network. For network $G$ let $U(G)$, $U_1(G)$ and $U_2(G)$ be the number of vertices left unmatched when running maximum matching, phase 1 and phase 2 respectively, $H = (V', E')$ be the remaining network at the beginning of phase 2. Hence $|M^*(G)| = n - U(G)$, $|M_{KS}(G)| = n - U_1(G) - U_2(G)$, $|M_{OKS}(H)| = |V'| - U_2(G)$. Since there is no deviation from maximum matching as long as vertices of degree one exists $U_1(G) \leq U(G) \leq U_1(G) + U_2(G)$.

They show $\frac{U_2(G)}{n} \to 0$ as $n \to \infty$ so the algorithm is optimal, i.e. $\lim_{n \to \infty} \frac{|M_{OKS}(G)|}{n} = \lim_{n \to \infty} \frac{|M^*(G)|}{n}$.

Furthermore, they show $\frac{U_2(G)}{n} \to k(\lambda)$ and $\frac{|V'|}{n} \to h(\lambda)$ and find functions $k, h$ as $k(\lambda) = \frac{1}{\lambda} \left( 1 - \frac{1}{\lambda} \right)^{\gamma^*}$ and $h(\lambda) = \left(1 - \frac{1}{\lambda}\right)^{\gamma^*}$ where $\gamma^*$ is the smallest root of $0 = \gamma^* - \gamma^* e^{-\gamma^*} - 1$. For $\lambda \leq e$ we have $h(\lambda) = 0$ because of $\gamma^* = \gamma^*$. In the following theorem, we generalize these results to a larger class of random networks.

**Theorem 4** (KS for asymptotically Poisson degree distributions). Assume $G$ is one of GER($\lambda$), UFS($\lambda$), DD($p$) and DD($p, p$) where probability distribution $p$ is Poisson($\lambda$). If $\lambda = 0$ then $\lim_{n \to \infty} \frac{|M_{KS}(G)|}{n} = \lim_{n \to \infty} \frac{|M^*(G)|}{n} = 0$. If $\lambda = \infty$ then $\lim_{n \to \infty} \frac{|M_{KS}(G)|}{n} = \lim_{n \to \infty} \frac{|M^*(G)|}{n} = 1$. For $\lambda \in (0, \infty)$, we have $\lim_{n \to \infty} \frac{|M_{KS}(G)|}{n} = \lim_{n \to \infty} \frac{|M^*(G)|}{n} = 1 - k(\lambda)$.

Furthermore, $\frac{U_2(G)}{n} \to 0$ , $\frac{U_1(G)}{n} \to k(\lambda)$ and $\frac{|V'|}{n} \to h(\lambda)$ as $n \to \infty$.

Results presented in Theorems 3 and 4 are based on the fact that in all mentioned random networks the asymptotic empirical degree distribution is Poisson. Simplifying $KS$ to $OKS$ and also extending the class of random networks to those with arbitrary degree distributions with finite mean, we show that optimality still holds and we also find the asymptotic relative size of the maximum matching.

**B. Arbitrary Degree Distribution**

We establish the optimality of $OKS$ algorithm which immediately yields optimality of $KS$ as well for reasons explained before. For this purpose, we follow in the footsteps of Karp and Sipser and embed the dynamics of both input and output degree sequences as the algorithm proceeds in continuous time. This embedding provides differential equations governing the degree sequence vectors. However, in the general degree distribution case, unlike the classic Erdos-Renyi case, the differential equations are in arbitrarily high dimensions. So there is little hope of working in fixed small dimension as Karp and Sipser did (their differential equations were 3 dimensional) and new ideas are needed. The key idea in our proof is to use the differential equations to show that the number of iterations when there is no degree one vertex (and so the algorithm can possibly make a mistake) is sublinear (in $n$) which means the relative size of the matching given by $OKS$ (or equivalently the fraction of matched vertices)
is asymptotically the same as that of maximum matching. Finally, a set of equations for the relative size of maximum matching according to asymptotic empirical input and output degree distributions will be provided.

**Theorem 5 (Asymptotic optimality of OKS algorithm).** For network $G = (L, R, E), |R| = |L| = n$ let $|M_{OKS}(G)|$ and $|M^\ast(G)|$ be the size of matching given by OKS algorithm and the size of maximum matching respectively. Let $\mu$ be either $\Phi_{in}$ or $\Phi_{out}$ asymptotic empirical distributions. Then

$$\lim_{n \to \infty} \frac{|M_{OKS}(G)|}{n} = \lim_{n \to \infty} \frac{|M^\ast(G)|}{n} = \mu.$$ 

Note that in Theorem 6 letting $\epsilon = n^{-r}$ for every $r > \frac{1}{2}$ the convergence holds. So the difference between $|M_{OKS}|$ and $|M^\ast|$ is $O(\sqrt{n})$. Now the following questions arise: what is the size of maximum matching? and how can we compute the answer (asymptotically) without running the algorithm? The following Theorem provides the size of maximum matching in terms of input and output degree distributions. For $u \in \mathbb{R}$ define moment generating functions:

$$\Phi_{in}(u) = \sum_{i=0}^{\infty} p_{in}(i) u^i, \quad \Phi_{out}(u) = \sum_{i=0}^{\infty} p_{out}(i) u^i$$

$$\phi_{in}(u) = \frac{1}{\mu} \Phi_{in}'(u) = \sum_{i=1}^{\infty} \frac{ip_{in}(i)}{\mu} u^{i-1}$$

$$\phi_{out}(u) = \frac{1}{\mu} \Phi_{out}'(u) = \sum_{i=1}^{\infty} \frac{ip_{out}(i)}{\mu} u^{i-1}$$

where $\mu = \sum_{i=0}^{\infty} ip_{in}(i) = \sum_{i=0}^{\infty} ip_{out}(i)$ is the average degree.

**Theorem 6 (Size of Maximum Matching).** For random network $G = (L, R, E), |R| = |L| = n$ if $\mu < \infty$ let $U^\ast$ be

$$U^\ast = \frac{1}{2} \left[ \Phi_{in}(w_2) + \Phi_{in}(1-w_1) + \Phi_{out}(w_4) + \Phi_{out}(1-w_3) - 2 + \mu(w_3(1-w_2) + w_1(1-w_4)) \right]$$

where $(w_1, w_2, w_3, w_4)$ is any solution of

$$\phi_{out}(1-w_3) = 1 - w_2, \quad \phi_{in}(w_2) = w_3$$

$$\phi_{in}(1-w_1) = 1 - w_4, \quad \phi_{out}(w_4) = w_1$$

then the asymptotic fraction of unmatched vertices is $U^\ast$:

$$\lim_{n \to \infty} 1 - \frac{|M^\ast(G)|}{n} = U^\ast$$

Note that if $p_{in}, p_{out}$ are Poisson($\lambda$) then $U^\ast = k(\lambda)$.

### V. Summary

Table I summarizes our results about the convergence, limits, asymptotic variance and optimality of different algorithms for different random networks. By “Convergence”, we mean that the size of the matching produced by the algorithm applied to the corresponding random network converges to a deterministic quantity. “Optimality” means equality of the relative sizes of the matching provided by the algorithm and maximum matching. “Limit” shows that this size can be computed according to the parameters of the underlying random network.

| Algorithms | Random Networks |
|------------|-----------------|
| Greedy     | Convergence, Limit for GER($\lambda$), Variance for GER($\lambda$) | Convergence, Limit for DDR($\lambda$) |
| Karp-Sipser| Convergence, Limit, Optimality | Convergence, Limit, Optimality |
| One-sided  | Convergence, Limit, Optimality | Convergence, Limit, Optimality |
| Matching   | Convergence, Limit | Convergence, Limit |

**APPENDIX A**

**Proof of Lemma 1**

For arbitrary $\epsilon > 0$ let $N$ be large enough such that $\sum_{k=N+1}^{\infty} kp(k) < \frac{\epsilon}{2}$. Further, for random network $G = (V, E), |V| = n$, remove some edges from $G$ in order to have no vertex of degree larger than $N$ to get network $G' = (V, E')$. Now for every vertex $i \in V$ there are possibly two reasons for the difference between $D_i$ and $\text{deg}_{G'}(i)$:

- the set of edges we removed from $G$ to get $G'$
- multiple edges in the network $G'$

so,

$$|\{i \in V : D_i = k\} - |\{i \in V : \text{deg}_{G'}(i) = k\}| \leq |E| - |E'| + \sum_{i=1}^{n} M_i \neq 0$$

(1)

where $M_i$ is the number of multiple edges in $G'$ connected to vertex $i$. There are at most $\binom{N}{2}$ pairs of half-edges connected to vertex $i$ and for every two of them the probability of the outcome $A_j$ that they both are connected to vertex $j$ is at most $\binom{N}{2}$ where $D = 2|E'|$. So,

$$M_i \leq \binom{N}{2} \sum_{j=1}^{n} 1_{A_j}; P(A_j|D) \leq \frac{\binom{N}{2}}{2}$$

By Markov’s inequality for any $\delta > 0$ we have:

$$P \left( \frac{1}{n} \sum_{i=1}^{n} 1_{M_i \neq 0} > \delta \right) \leq \frac{1}{n \delta} \sum_{i=1}^{n} P(M_i \neq 0 | D) \leq \frac{1}{n \delta} \sum_{i=1}^{n} P(M_i \geq 1 | D) \leq \frac{\binom{N}{2}^2}{\delta(D-1)^2}$$
But \( \lim_{n \to \infty} \frac{|E|}{n} = \mu \) and

\[
\lim_{n \to \infty} \frac{|E'|-|E|}{n} < \frac{\epsilon}{2} \tag{2}
\]

(since \( \sum_{k=N+1}^{\infty} kp(k) < \frac{\epsilon}{2} \)) imply \( \lim_{n \to \infty} \frac{|E'-E|}{n} > \mu - \epsilon \) i.e.

\[
\frac{1}{n} \sum_{i=1}^{n} 1_{M_i \neq 0} \to \mu 0 \tag{3}
\]

On the other hand for all \( k = 1, 2, \ldots \) by the Law of Large Numbers:

\[
\lim_{n \to \infty} \frac{\{i \in V : D_i = k\}}{n} = p(k) \tag{4}
\]

Finally, because the only reason for the difference between \( \deg_G(i) \) and \( \deg_{G'}(i) \) is the set of edges we removed from \( G \) to get \( G' \) we have:

\[
\left| \left\{ i \in V : \deg_G(i) = k \right\} - \left\{ i \in V : \deg_{G'}(i) = k \right\} \right| \\
\leq |E| - |E'|
\]

Putting (1), (2), (3), (4) all together:

\[
\lim_{n \to \infty} \mathbb{P}\left( \left| \frac{\{i \in V : \deg_G(i) = k\}}{n} - p(k) \right| > \epsilon \right) = 0
\]

Since the function \( f(G) = \{|i \in V : \deg_G(i) = k\}| \) has the Lipschitz property by Theorem (11) a.s. convergence holds as well i.e. \( p(k) = p(k) \)

For \( DD(p_{in}, p_{out}) \), the changes in the asymptotic empirical degree distributions due to omission of

\[
\max\left\{ \sum_{i=0}^{n} D_i^{(in)}, \sum_{i=0}^{n} D_i^{(out)} \right\} - \min\left\{ \sum_{i=0}^{n} D_i^{(in)}, \sum_{i=0}^{n} D_i^{(out)} \right\}
\]

additional half-edges is at most \( \frac{1}{n} \sum_{i=0}^{n} D_i^{(in)} - \sum_{i=0}^{n} D_i^{(out)} \to 0 \)

because by the Law of Large Numbers \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} D_i^{(in)} = \mu \) as \( n \to \infty \). The contribution of multiple edges is asymptotically zero similarly.

**APPENDIX B**

**PROOF OF THEOREM (1)**

To prove convergence and concentration inequalities for real-valued functions for random networks we use some classical notions of probability such as martingale difference sequences. Rhee [23] Theorem 1] presents a concentration inequality for martingale difference sequences. Here we use a slightly more general version of it. The proof presented by Rhee [23] is valid by following the same line of reasoning.

**Theorem 7.** Let \( X_i, i = 1, 2, \ldots, k \) be a martingale difference sequence. If \( \max_{1 \leq i \leq k} \|X_i\|_\infty \leq M < \infty \) and \( \sum_{i=1}^{k} \mathbb{E}(X_i^2 | F_{i-1}) \leq a^2 < \infty \) then for all \( t \geq 0 \)

\[
\mathbb{P}\left( \left\{ \sum_{i=1}^{k} X_i > t \right\} \right) \leq 2 \exp\left( -\frac{a^2}{Mt^2} \frac{M}{a^2} \right)
\]

where \( \rho(x) = (1 + x) \log(1 + x) - x \) for \( x \geq 0 \).

We prove the following stronger Theorems:

**Theorem 8.** Assuming \( f \) has the Lipschitz property and \( G = (V, E) \) is a GER random network with edge existence probabilities \( p^{(n)}_{ij} \) if

\[
\limsup_{n \to \infty} \frac{1}{n^2} \sum_{i,j=1}^{n} p^{(n)}_{ij} (1 - p^{(n)}_{ij}) = 0
\]

then \( \frac{f(G)-\mathbb{E}(f(G))}{n} \to 0 \) as \( n \to \infty \). Furthermore, if

\[
\limsup_{n \to \infty} \frac{\log n}{n^2} \sum_{i,j=1}^{n} p^{(n)}_{ij} (1 - p^{(n)}_{ij}) = 0
\]

then \( \frac{f(G)-\mathbb{E}(f(G))}{n} \to 0 \) a.s. as \( n \to \infty \). When the average degree is finite (e.g. GER(\( \lambda \)) for \( \lambda < \infty \)) the rate of convergence is exponential. In general given

\[
\sup_{n \geq 1} \frac{1}{n^2} \sum_{i,j=1}^{n} p^{(n)}_{ij} (1 - p^{(n)}_{ij}) \leq C < \infty
\]

we have:

\[
\mathbb{P}\left( \left| \frac{f(G) - \mathbb{E}(f(G))}{n} \right| > \epsilon \right) \leq 2 \exp\left( -\frac{nC\rho(\epsilon)}{C} \right)
\]

**Proof:** To have more convenient notation we enumerate all possible edges \( i \to j \) from 1 to \( k \) where \( k = \frac{n}{2} \) for directed GER and \( k = \frac{n^2}{2} \) for undirected GER. Indeed, \( Z_i, i = 1, 2, \ldots, k \) be independent Bernoulli random variables showing the existence of edges, i.e. if \( Z_i = 1 \) the corresponding edge exists and if \( Z_i = 0 \) the corresponding edge does not exist. Let \( p_i, i = 1, 2, \ldots \) be edge existence probabilities, \( P(Z_i = 1), \mathcal{F}_0 \) be the trivial sigma-field and for \( i = 1, \ldots, k \) let \( \mathcal{F}_i = \sigma(Z_i, \ldots, Z_1). \) Now define a martingale difference sequence as \( X_i = \mathbb{E}(f(G)|\mathcal{F}_i) - \mathbb{E}(f(G)|\mathcal{F}_{i-1}), i = 1, \ldots, k \) so \( \mathbb{E}(f(G)|\mathcal{F}_k) = f(G), \mathbb{E}(f(G)|\mathcal{F}_0) = \mathbb{E}(f(G)). \)

Define:

\[
U_i = \mathbb{E}(f(G)|Z_1, \ldots, Z_{i-1}, Z_i = 1) \\
V_i = \mathbb{E}(f(G)|Z_1, \ldots, Z_{i-1}, Z_i = 0)
\]

Thus,

\[
\mathbb{E}(f(G)|Z_1, \ldots, Z_t) = p_1 U_1 + (1 - p_t) V_t
\]

\[
X_i = \begin{cases} (1 - p_i)(U_i - V_i) & Z_i = 1 \\ -p_i(U_i - V_i) & Z_i = 0 \end{cases}
\]

\[
\mathbb{E}(X_i^2 | F_{i-1}) = p_i (1 - p_i)(U_i - V_i)^2
\]

Since \( f \) has the Lipschitz property, \( |U_i - V_i| \leq M (M = 1) \) for directed GER and \( M = 2 \) for undirected GER:

\[
\max_{1 \leq i \leq k} \|X_i\|_\infty \leq \max_{1 \leq i \leq k} \max\{p_i, 1 - p_i\} \leq 1
\]

\[
\sum_{i=1}^{k} \mathbb{E}(X_i^2 | F_{i-1}) \leq \sum_{i=1}^{k} p_i (1 - p_i) \leq a^2 < \infty
\]

By Theorem 7 for all \( \epsilon > 0 \) we have:

\[
\mathbb{P}\left( \left| \frac{f(G) - \mathbb{E}(f(G))}{n} \right| > \epsilon \right) \leq 2 \exp\left( -\frac{a^2}{M^2 \rho(\epsilon)} \right)
\]
\[ \lim_{x \to 0} \frac{\rho(x)}{x} = \frac{1}{2} \] implies:
\[ \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > \epsilon) \leq 2 \exp(-\epsilon^2 n^2 \frac{2a^2}{2a^2}) \]
now if \( \epsilon \) holds then \( \frac{f(G) - \mathbb{E}(f(G))}{n} \rightarrow_\rho 0 \) as \( n \rightarrow \infty \). To show the a.s. convergence, note that \( \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > \epsilon) \) is << \( < \infty \) so by the Borel-Cantelli lemma \( \frac{f(G) - \mathbb{E}(f(G))}{n} \rightarrow 0 \) a.s. as \( n \rightarrow \infty \).

Finally using \( \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > \epsilon) \leq 2 \exp(-nC\rho(\epsilon)) \)

Theorem 8 provides convergence and concentration inequality for GER random networks. A corollary of Theorem 8 is that the obtained results are valid for undirected random networks of Chung-Lu type (see Durrett 21). In a Chung-Lu random network the edge between \( i \) and \( j \) exists with probability \( \frac{w_i w_j}{\sum w_i w_j} \) for the sets of weights \( w_1, \ldots, w_n \). Conditions \( \mathbb{P}(G), \mathbb{P}(\mathbb{F}), \mathbb{P}(\mathbb{S}) \) for a Chung-Lu random network will be \( \limsup_{n \to \infty} \frac{k_n}{n} = 0, \limsup_{n \to \infty} \frac{\sum_{k=1}^{n} w_k}{n} = 0 \) and \( sup_{n \geq 1} \mathbb{E} < \infty \)

Theorem 9. Assume \( f \) has the Lipschitz property and \( G = (V, E), |E| = k_n \) is UFS. If
\[ \limsup_{n \to \infty} \frac{k_n}{n^2} = 0 \] then \( \frac{f(G) - \mathbb{E}(f(G))}{n} \rightarrow_\rho 0 \) as \( n \rightarrow \infty \). If furthermore,
\[ \limsup_{n \to \infty} \frac{k_n \log n}{n^2} = 0 \]
then \( \frac{f(G) - \mathbb{E}(f(G))}{n} \rightarrow 0 \) a.s. as \( n \rightarrow \infty \). The rate of convergence is exponential when the average degree is finite (e.g. UFS(\( \lambda \)) for \( \lambda < \infty \)). Namely sup \( k_n \) \( \leq C < \infty \) implies:
\[ \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > \epsilon) \leq 2 \exp(-nC\rho(\epsilon)) \]

Proof: Rhee 23 Theorem 4] shows:
\[ \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > t) \leq \exp(-t^2/k_n) \]
If \( \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > t) \leq \exp(-t^2/k_n) \) holds then letting \( t = n\epsilon \) we have \( \exp(-t^2/k_n) \rightarrow 0 \).

If \( \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > t) \leq \exp(-t^2/k_n) \) holds then \( \sum_{n=1}^{\infty} \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > t) \leq \frac{\epsilon}{\sqrt{2\pi}} \) so by the Borel-Cantelli Lemma \( \frac{f(G) - \mathbb{E}(f(G))}{n} \rightarrow 0 \) a.s. as \( n \rightarrow \infty \). To show \( \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > t) \) suffices to let \( t = n\epsilon \).

Theorem 10. Let \( G = (V, E) \) be DD\( (p_{in}, p_{out}) \) or DD\( (p) \) (in the recent case \( p_{in} = p_{out} = p \)). Assuming real-valued function \( f \) has the Lipschitz property if
\[ \limsup_{n \to \infty} \frac{|f(G) - \mathbb{E}(f(G))|}{n^2} = 0 \]
then \( \frac{f(G) - \mathbb{E}(f(G))}{n} \rightarrow 0 \) as \( n \rightarrow \infty \). If in addition
\[ \limsup_{n \to \infty} \frac{|f(G) - \mathbb{E}(f(G))|}{n^2} = 0 \]
then \( \frac{f(G) - \mathbb{E}(f(G))}{n} \rightarrow 0 \) a.s. as \( n \rightarrow \infty \). When the average degree is finite i.e.
\[ \sup_{n \geq 1} \frac{|f(G)|}{n} \leq C < \infty \]
the rate of convergence is exponential:
\[ \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > \epsilon) \leq 2 \exp(-nC\rho(\epsilon)) \]

Proof: To form an inequality like \( \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > \epsilon) \) let \( k \) be the number of half-edges before being paired (so \( k \) is the number of directed edges - every undirected edge is two directed edges). Enumerate half-edges and for \( i = 1, 2, \ldots, k \) let \( Z_i \) be the random variable indicating the vertex whose half-edge the half-edge \( i \) is paired to i.e. \( Z_i = j, j \in \{1, \ldots, n\} \) if half-edge \( i \) is paired to a half-edge of vertex \( j \). Let \( \mathbb{F}_k \) be trivial subfield and for \( i = 1, \ldots, k \) let \( \mathbb{F}_i = \sigma(Z_1, \ldots, Z_i) \).

Now define a martingale difference sequence as \( X_i = \mathbb{E}(f(G)|\mathbb{F}_i) - \mathbb{E}(f(G)|\mathbb{F}_{i-1}), i = 1, \ldots, k \) and \( U_{ij}, P_{ij} \) as:
\[ U_{ij} = \mathbb{E}(f(G)|Z_1, \ldots, Z_{i-1}, Z_i = j) \]
\[ P_{ij} = \mathbb{P}(Z_i = Z_{i-1}, \ldots, Z_{i-1}) \]

Hence \( U_i = \mathbb{E}(f(G)|Z_1, \ldots, Z_i) = \sum_{j=1}^{n} P_{ij} U_{ij} \) and \( X_i = U_{ij} - U_i \) whenever \( Z_i = j \). On the other hand
\[ \mathbb{E}(X_i^2|\mathbb{F}_{i-1}) = \sum_{j=1}^{n} P_{ij} (U_{ij} - U_i)^2 \]

Lipschitz property of \( f \) implies \( |U_{ij} - U_{il}| \leq M (M \leq 4 \) for undirected case and \( M \leq 2 \) for directed case). Hence for all \( i = 1, \ldots, k \):
\[ ||X_i|| \leq \max_{1 \leq j \leq n} ||U_{ij} - U_{il}|| \leq \max_{1 \leq j \leq n} \sum_{l=1}^{n} P_{il} ||U_{ij} - U_{il}|| \leq M < \infty \]
\[ \sum_{i=1}^{k} \mathbb{E}(X_i^2|\mathbb{F}_{i-1}) \leq \sum_{i=1}^{k} \sum_{j=1}^{n} P_{ij} M^2 \leq \sum_{i=1}^{k} M^2 \leq kM^2 \leq a^2 < \infty \]

By Theorem 7 we have:
\[ \forall \epsilon > 0 \quad \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > \epsilon) \leq 2 \exp(-\frac{\epsilon^2}{M^2 \rho(\frac{Mn\epsilon}{a^2})}) \]
Now \( \mathbb{P}(\frac{|f(G) - \mathbb{E}(f(G))|}{n} > \epsilon) \) gives \( \limsup_{n \to \infty} \frac{a^2}{n^2} = 0 \) and \( \sup_{n \geq 1} \frac{a^2}{n} \leq 16C < \infty \) respectively.

\[ \Box \]
**APPENDIX C**

**PROOF OF THEOREM**

Letting $M^{(n)} = n - |M_G(G)|$ be the number of unmatched vertices $M^{(n)} = \sum_{i=1}^{n} M_i$ where

$$M_i = \begin{cases} 1 & \text{the vertex picked in } i\text{-th iteration is unmatched} \\ 0 & \text{the vertex picked in } i\text{-th iteration is matched} \end{cases}$$

Now note that according to the algorithm

$$\mathbb{P}(M_i = 1 | M_1, \ldots, M_{i-1}) = q^{n-i+1 + \sum_{j=1}^{i-1} M_j}$$

For example $\mathbb{P}(M^{(n)} = 0) = \mathbb{P}(M_1 = 0, \ldots, M_n = 0) = \prod_{i=1}^{n} (1 - q^{n-i+1}) = \alpha_n(q)$. We have

$$\mathbb{P}(M^{(n)} = k) = \sum_{|I| = k} \mathbb{P}(M_i = 1 \forall i \in I, \forall i = 1, \ldots, n)$$

if $I = \{i_1, \ldots, i_k\}$ then

$$\mathbb{P}(M_i = 1 \forall i \in I, \forall i = 1, \ldots, n) = (1 - q^{n}) \cdot (1 - q^{n-i+1}) q^{n-i+1} \cdot (1 - q^{n-i}) \cdot (1 - q^{n-i+1}) q^{n-i+1} \cdot \ldots \cdot (1 - q^{n})$$

$$= \prod_{j=1}^{k} q^{n-j+1}$$

Thus:

$$\mathbb{P}(M^{(n)} = k) = \frac{\alpha_n(q)}{\alpha_k(q)} q^{\frac{k}{2} (k-1)} \sum_{1 \leq i_1 < \ldots < i_k \leq n} q^{i_1 + \ldots + i_k}$$

But:

$$\sum_{1 \leq i_1 < \ldots < i_k \leq n} q^{i_1 + \ldots + i_k} = \sum_{i_k = k}^{n} q^{i_k} \sum_{i_{k-1} = k-1}^{i_k} q^{i_{k-1}} \ldots \sum_{i_1 = 1}^{i_{k-2}} q^{i_1}$$

$$= \frac{\sum_{i_k = k}^{n} q^{i_k} \sum_{i_{k-1} = k-1}^{i_k} q^{i_{k-1}} \ldots \sum_{i_1 = 1}^{i_{k-2}} q^{i_1} \left(1 + q(1 - q^{i_1+1})\right)}{1 - q(1 - q^{i_1+1})}$$

$$= \frac{\sum_{i_k = k}^{n} q^{i_k} \sum_{i_{k-1} = k-1}^{i_k} q^{i_{k-1}} \ldots \sum_{i_1 = 1}^{i_{k-2}} q^{i_1} \left(1 + q(1 - q^{i_1+1})\right)}{1 - q(1 - q^{i_1+1})}$$

which establishes the result. Now to show

$$\lim_{n \to \infty} \frac{M^{(n)}}{n} = \frac{\log(2 - e^{-\lambda})}{\lambda}$$

for $GER(\lambda)$, for large $n$ letting $(1 - \frac{1}{n})^n = \exp(-\frac{1}{\lambda})$:

$$\mathbb{P}(M^{(n)} = k + 1) = \mathbb{P}(M^{(n)} = k) = \frac{1 - q^{n-k}}{1 - q^{k+1}} = \frac{1 - \exp(-\frac{\lambda}{n}(n-k))}{1 - \exp(-\frac{\lambda}{n}(k+1))}$$

Denote $\beta_k = \exp(-\frac{\lambda}{n}k)$ to obtain:

$$\mathbb{P}(M^{(n)} = k + 1) = \mathbb{P}(M^{(n)} = k) = \frac{\beta_k (1 - e^{-\lambda} \beta_k^{-1})}{\beta_k} = \frac{e^{-\lambda} - 2 \beta_k + e^{-\lambda} \beta_k^2}{\beta_k}$$

Therefore $\frac{\mathbb{P}(M^{(n)} = k + 1)}{\mathbb{P}(M^{(n)} = k)} > 1$ if and only if $(e^{-\lambda} - 1) \beta_k^2 + (e^{-\lambda} - 2) \beta_k + e^{-\lambda} < 0$ i.e. the probability mass function is unimodal and as $n \to \infty$, $\mathbb{P}(M^{(n)} = k + 1) > \mathbb{P}(M^{(n)} = k)$ if and only if $\beta_k^2 < 2 - e^{-\lambda}$ which is equivalent to $k < \frac{\log(2 - e^{-\lambda})}{n}$ so letting $k^* = \arg \max_{0 \leq k \leq n} \mathbb{P}(M^{(n)} = k)$ we get

$$\mathbb{P}(\{M^{(n)} = k^*\} > n/\mu) \to 0 \text{ for all } \epsilon > 0 \text{ as } n \to \infty$$

By Theorem 8 a.s. convergence holds as well. To prove the asymptotic normality we will show

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(M^{(n)} = k(t)) = \frac{-t^2}{2 \sigma^2}$$

where $t = \lim_{n \to \infty} \frac{k(t) - n\mu}{\sqrt{n}}$, $k(0) = k^* \approx n\mu, \mu = \frac{\log(2 - e^{-\lambda})}{\lambda}, \sigma^2 = \frac{1}{4\lambda}$. We have:

$$\log \mathbb{P}(M^{(n)} = k(t))$$

$$= \sum_{j=1}^{n-k(t)} f_j(\frac{t}{\sqrt{n}}) - \lambda n \mu^2 - \lambda^2 - 2\lambda \sqrt{\mu} t$$

where the functions $f_j : \mathbb{R} \to \mathbb{R}, j = 1, 2, \ldots$ are $f_j(x) = 2 \log(1 - e^{-\lambda(\mu+j/n)} - x)$, $\log(1 - e^{-\lambda/j/n})$. Note that

$$e^{-\lambda x}(1 - e^{-\lambda})^{-1}$$

But for $a_j = e^{-\lambda(\mu+j/n)} < 1$ and some $B < \infty$

$$f_j(0) = \frac{2 \lambda a_j}{1 - a_j} - \frac{2 \lambda^2 a_j^2}{(1 - a_j)^2}$$

$$f_j''(0) = \frac{f_j''(t^*)}{\sqrt{n}} < B$$

Now writing $f_j(\frac{t}{\sqrt{n}}) = f_j(0) + f_j'(0)\frac{t}{\sqrt{n}} + f_j''(0)\frac{t^2}{2 \sqrt{n}} + f_j'''(0)\frac{t^3}{6 \sqrt{n}} + \frac{f_j''''(t^*)}{12 \sqrt{n}} < B$ for some $t^* \in [0, t]$ and using (15), because $\frac{1}{n(n-k(t))} \to 1 - \mu$ definition of Riemann integral implies:

$$\frac{1}{n} \sum_{j=1}^{n-k(t)} f_j''(0) \to -\lambda^2 \int_0^1 \frac{1 - \mu}{e^{\lambda \mu + \lambda x} - 1} dx = -2\lambda$$

$$\frac{1}{n} \sum_{j=1}^{n-k(t)} f_j'''(0) \to -\lambda^3 \int_0^1 \frac{1 - \mu}{(e^{-\lambda(\mu+x) - x})^2} dx = 2\lambda$$

$$\frac{1}{n} \sum_{j=1}^{n-k(t)} f_j''''(0) \to \frac{1}{n} \sum_{j=1}^{n-k(t)} f_j''''(\frac{t}{\sqrt{n}}) < B \to 0$$
Note that \( |\frac{1}{n} \sum_{j=1}^{n-k(t)} f_j(0) - 2\lambda \mu| \leq \frac{(1-\mu)^2}{n} \sup_{\mu \leq x \leq 1} \frac{\partial}{\partial x} \frac{1}{e^x-1} \)
and \( \sup_{\mu \leq x \leq 1} \frac{\partial}{\partial x} \frac{1}{e^x-1} = \sup_{\mu \leq x \leq 1} \frac{\mu e^{\mu x}}{(e^{\mu x}-1)^2} < \infty \) imply
\[
\frac{1}{\sqrt{n}} \sum_{j=1}^{n-k(t)} f_j(0) - 2\lambda \mu \sqrt{n} \to 0.
\]
Now, plugging all in (16) we get the desired result since
\[
\log P(X = k(0)) = \sum_{j=1}^{n-k(0)} f_j(0) - \lambda n \mu^2.
\]

**APPENDIX D**

**PROOF OF LEMMA** \[2\]

To prove the Lipschitz property for maximum matching let \( G = (L, R, E), G' = (L, R, E'), E' = E \cup \{1, |L| = |R| = n \}
and let \( M \subset E' \) be a maximum matching in \( G' \). If \( e \notin M \) then \( M \) is a maximum matching in \( G \). If \( e \in M \) then \( M \setminus \{e\} \) is a matching in \( G \) so the size of maximum matching is at least \( |M| - |e| \). Thus \( |M^*(G')| - 1 \leq |M^*(G)| \leq |M^*(G')| \).

For the Greedy algorithm, if \( n = 1 \) then clearly \( |M_G(G')|, |M_G(G)|; \) both are either 0 or 1 so \( |M_G(G')| - |M_G(G)| \leq 1 \). Assume function \( |M_G| \) has the Lipschitz property for all networks of vertex size \( n - 1 \) and define networks \( H, H' \) as the networks \( G, G' \) after the first iteration: \( H = G - \{u, v\}, H' = G' - \{u', v'\}; u, u' \in L; v, v' \in R \) so both \( H, H' \) have \( n - 1 \) vertices.

If \( u = u' \) and \( v = v' \) then networks \( H, H' \) are exactly the same except potentially the new edge which can be added to \( H \) in order to get \( H' \). Since \( H, H' \) both have \( n - 1 \) vertices, by the assumption \( |M_G(H') - M_G(H)| \leq 1 \). On the other hand \( M_G(G) = M_G(H) \cup \{u, v\}, M_G(G') = M_G(H') \cup \{u, v\} \) imply \( |M_G(G)| = |M_G(H)| + 1, |M_G(G')| = |M_G(H')| + 1 \) i.e. \( |M_G(G') - M_G(G)| \leq 1 \).

If \( v \neq v' \) then the new edge \( e \) is connected to \( \{u, v\} \). Let \( \text{deg}(G'(v')) = 0, \text{deg}(G(v')) = 1 \).

**Step 1:**

**Embedding in a continuous time Markov process:**

Assume in addition that the asymptotic empirical degree distributions are bounded: \( p_{\text{in}}(i) = p_{\text{out}}(i) = 0 \) for \( i > N \) i.e. for every vertex \( v \in L \cup R \) we have \( \text{deg}(v) \leq N \). First, we embed the dynamics of the algorithm in a continuous time Markov process. To go to continuous time, define \( G^0(t) \) as the \( n \)-vertex network at time \( t \) in \( \mathbb{R} \) where state changes \( G^n = G^n - \{u, v\} \) occur at \( \exp(n) \) interarrival times. More precisely, let \( \tau_1, \tau_2, \ldots \) be i.i.d \( \exp(n) \) random variables, i.e. the probability density function is \( ne^{-nt} \) for \( t \geq 0 \) so \( \mathbb{E}(\tau_1) = \frac{1}{n} \). The first state change \( G^n = G^n - \{u, v\} \) occurs at time \( t = \tau_1 \), the second one occurs at \( t = \tau_1 + \tau_2 \) and so forth. Now we construct a Markov process on \( \mathbb{R}^{2N} \) which describes the performance of the algorithm. The transition kernel of the Markov process will be described later. Define \( X^{(n)}(t), Y^{(n)}(t) \in \mathbb{R}^N \):

\[
X_k^{(n)}(t) = \frac{1}{n} \left| \{v \in R : \text{deg}(v) = k \text{ in } G^n(t)\} \right|
\]

\[
Y_k^{(n)}(t) = \frac{1}{n} \left| \{v \in L : \text{deg}(v) = k \text{ in } G^n(t)\} \right|
\]

for \( k = 1, 2, \ldots, N \). In addition let \( m = m(X^{(n)}(t)) \) be the minimum degree of vertices in \( R(t) \): \( m(X^{(n)}(t)) = \min \{k : X_k^{(n)}(t) \neq 0\} \) so letting \( v_1 = v, u_1 = u \) whenever a state change occurs we have \( \text{deg}(v_1) = m \). Let \( (u_i, v) \in E \) for \( i = 1, \ldots, m, K = \text{deg}(u_1) \) and \( (u, v_j) \in E \) for \( j = 1, \ldots, K \). So for a network of size \( n \) we have the following conditional degree distributions for vertices \( u_1, \ldots, u_m, v_1, \ldots, v_K \):

\[
P_n(\text{deg}(u_i) = k | A_{i-1}(t)) = \frac{n k Y_k^{(n)}(t) - k \sum_{j=1}^{i-1} \text{deg}(u_j) = k}{n \sum_{k=1}^{N} k Y_k^{(n)}(t) - \sum_{j=1}^{i-1} \text{deg}(u_j)}
\]

\[
P_n(\text{deg}(v_j) = k | B_{i-1}(t)) = \frac{n k X_k^{(n)}(t) - k \sum_{j=1}^{i-1} \text{deg}(v_j) = k}{n \sum_{k=1}^{N} k X_k^{(n)}(t) - \sum_{j=1}^{i-1} \text{deg}(v_j)}
\]

Where \( A_i(t) = (\text{deg}(u_1), \ldots, \text{deg}(u_i), Y^{(n)}(t)), B_i(t) = (\text{deg}(v_1), \ldots, \text{deg}(v_i), X^{(n)}(t)) \). Note that since interarrival
times are i.i.d exponential random variables, \( X^{(n)}(t), Y^{(n)}(t) \) are continuous time Markov processes.

Letting \( \hat{X}^{(n)}, \hat{Y}^{(n)} \in \mathbb{R}^N \) be the corresponding vectors after one state change for \( x, y \in \mathbb{R}^N \) define functions \( \mathbb{P}^n, \mathbb{G}^n : \mathbb{R}^{2N} \to \mathbb{R}^N \) as:

\[
\mathbb{P}^n(x, y) = n\mathbb{E}_n(\hat{X}^{(n)} - X^{(n)})|X^{(n)} = x, Y^{(n)} = y \\
\mathbb{G}^n(x, y) = n\mathbb{E}_n(\hat{Y}^{(n)} - Y^{(n)})|X^{(n)} = x, Y^{(n)} = y
\]

where \( \mathbb{E}_n \) is expected value w.r.t \( \mathbb{P}_n \). Since the process is Markov, probability distribution of \( \hat{X}^{(n)}, \hat{Y}^{(n)} \) depends only on \( X^{(n)}, Y^{(n)} \).

**Asymptotic initial value problem:** Define \( \mathbb{P}(\text{deg}(u_i) = k) = \frac{kX^{(n)}(t)}{\sum kX^{(n)}(t)} \), \( \mathbb{P}(\text{deg}(v_i) = k) = \frac{kY^{(n)}(t)}{\sum kY^{(n)}(t)} \). Note that \( \mathbb{P}_n, \mathbb{P} \) can be defined for every \( x, y \in \mathbb{R}^N \) with non-negative components. Now for arbitrary \( x, y \) some algebra gives:

\[
|\mathbb{P}_n(\text{deg}(u_i) = k|\mathcal{A}_{i-1}(t)) - \mathbb{P}(\text{deg}(u_i) = k)| \leq \frac{C_1}{n} (19)
\]

\[
|\mathbb{P}_n(\text{deg}(v_i) = k|\mathcal{B}_{i-1}(t)) - \mathbb{P}(\text{deg}(v_i) = k)| \leq \frac{C_2}{n} (20)
\]

For \( C_1 = \frac{2N^2}{\sum kX^{(n)}(t)}, C_2 = \frac{2N^2}{\sum kY^{(n)}(t)} \) Defining

\[
\mathbb{F}(x, y) = n\mathbb{E}(\hat{X}^{(n)} - X^{(n)})|X^{(n)} = x, Y^{(n)} = y \\
\mathbb{G}(x, y) = n\mathbb{E}(\hat{Y}^{(n)} - Y^{(n)})|X^{(n)} = x, Y^{(n)} = y
\]

where \( \mathbb{E} \) is expected value w.r.t \( \mathbb{P} \). Since

\[
n\hat{X}^{(n)} = nX^{(n)} + \sum_{j=2}^{K} [1_{\text{deg}(v_j) = k+1} - 1_{\text{deg}(v_j) = k}] - 1_{k=m} (21)
\]

\[
n\hat{Y}^{(n)} = nY^{(n)} + \sum_{j=2}^{m} [1_{\text{deg}(u_j) = k+1} - 1_{\text{deg}(u_j) = k}] - 1_{k=R} (22)
\]

inequalities (19), (20) yield

\[
||\mathbb{F}^n(x, y) - \mathbb{F}(x, y)||_1 \leq \frac{4N^4}{n} \frac{1}{\sum kx_k} (23)
\]

\[
||\mathbb{G}^n(x, y) - \mathbb{G}(x, y)||_1 \leq \frac{4N^4}{n} \frac{1}{\sum k_y k} (24)
\]

and,

\[
\mathbb{F}(x, y) = (\frac{\|Ax\|_1}{\|Ay\|_1} - 1) \|Ax - Ay\|_1 (SAx - Ax) - 1_{m(x)} \\
\mathbb{G}(x, y) = -\frac{Ay}{\|Ay\|_1} + m(x) - 1\|Ay\|_1 (SAy - Ay)
\]

where \( A \) and \( S \) are moment matrix and shift matrix respectively, i.e., \( A, S \in \mathbb{R}^{N \times N}, A_{ij} \) is \( i \) for \( i = j \) and 0 otherwise, and \( S_{ij} \) is 1 for \( i = j - 1 \) and 0 otherwise, \( \| \cdot \|_1 \) is \( \ell_1 \) norm on \( \mathbb{R}^N \).

\( \text{for } x \in \mathbb{R}^N, m(x) = \min\{k : x_k \neq 0\} \text{ and } 1_m \in \mathbb{R}^N \)

is the vector in which \( m \)-th component is 1 and all others are 0. Note that \( \|AX^{(n)}(t)||_1 = \|AY^{(n)}(t)||_1 \) because for finite \( n \) always \( |E(t)| = n \sum_{k=1}^{N} kX^{(n)}(t) = n \sum_{k=1}^{N} kY^{(n)}(t) \). Besides, transition kernel of the Markov process can be formulated by \( \mathbb{F}_n \) according to (21), (22).

**Approximating the dynamics of the degree sequences** by the solution of asymptotic initial value problem: Now we can use Kurtz’s Theorem. Given functions \( F, G : \mathbb{R}^{2N} \to \mathbb{R}^N \) and positive constant \( T \), define \( \dot{x}, \dot{y} : [0, T] \to \mathbb{R}^N \) as the solution of the initial value problems

\[
\dot{x} = F(x, y), x_0(k) = p_{in}(k), k = 1, \ldots, N
\]

\[
\dot{y} = G(x, y), y_0(k) = p_{out}(k), k = 1, \ldots, N
\]

and let \( E = \{z \in \mathbb{R}^N \) such that \( e < \sum_{k=1}^{N} kZ_k \). Suppose the following statements hold:

1) \( \lim_{n \to \infty} \sup_{z_1, z_2 \in E} ||\mathbb{F}^n(z_1, z_2) - \mathbb{F}(z_1, z_2)||_1 = 0 \)

2) \( \lim_{n \to \infty} \sup_{z_1, z_2 \in E} ||\mathbb{G}^n(z_1, z_2) - G(z_1, z_2)||_1 = 0 \)

3) for all \( k = 1, \ldots, N \), \( \lim_{n \to \infty} X^n_k(0) = p_{in}(k) \)

4) for all \( k = 1, \ldots, N \), \( \lim_{n \to \infty} Y^n_k(0) = p_{out}(k) \)

5) functions \( F, G \) are Lipschitz

Then

\[
\lim_{n \to \infty} \mathbb{P}_n \left( \exists 0 \leq t \leq T : m(X^n(t)) \neq m(x(t)) \right) = 0 (27)
\]

Letting \( T = T(\epsilon) = \epsilon \sum_{k=1}^{N} kS_k(t) > \epsilon \sum_{k=1}^{N} kY_k(t) > \epsilon \) for some arbitrary \( \epsilon > 0 \) the first two conditions are satisfied by (23), (24). By the definition of asymptotic empirical degree distributions \( \lim_{n \to \infty} X^n_k(0) = p_{in}(k) \) and \( \lim_{n \to \infty} Y^n_k(0) = p_{out}(k) \). On the other hand the initial value problems (25), (26) have unique solutions since defining metric on \( \mathbb{R}^N \) as \( d(x, y) = ||x - y||_1 + 1_{m(x) \neq m(y)} \), \( F, G \) are Lipschitz with respect to this metric i.e there is \( B < \infty \) such that for all \( x, x', y, y' \in \mathbb{R}^N \)

\[
d(F(x, y), F(x', y')) < B(d(x, x') + d(y, y'))
\]

\[
d(G(x, y), G(x', y')) < B(d(x, x') + d(y, y'))
\]

Note that stopping time at \( T(\epsilon) \), i.e. when \( O(\epsilon n) \) edges are still waiting to be stripped off, is finite: continuing the algorithm from that point on cannot add more than \( O(\epsilon) \) edges to the matching on a scale relative to the number \( n \) of vertices.

**Properties of the asymptotic initial value problems:** The useful fact about the solutions of (25), (26) is that Lebesgue measure of the set \( \{0 < t < T : m(x(t)) > 1\} \) is zero. Suppose it is not so there are \( 0 < t_1 < t_2 \) such that \( x_1(t) = 0 \) for all \( t \in [t_1, t_2] \) so \( \frac{dx_1(t)}{dt} = 0 \) for all \( t \in (t_1, t_2) \). But

\[
\frac{dx_1(t)}{dt} = \mathbb{F}(x, y) = (\frac{\|Ax\|_1}{\|Ay\|_1} - 1) \|Ax - Ay\|_1 (2x_2(t) - x_1(t))
\]

implies \( x_2(t) = 0 \) for all \( t \in (t_1, t_2) \) which means \( m(x(t)) > 2 \) for all \( t \in (t_1, t_2) \). Repeating this argument for \( x_2 \) now we will get \( m(x(t)) > 3 \) and so on which is impossible. Thus, the set \( \{0 < t < T : m(x(t)) > 1\} \) has zero Lebesgue measure.
Sublinearity of the number of iterations of the algorithm with no degree one vertex: Let \( J^{(n)} \subset \{1, 2, \ldots, n\} \) be the set of indices \( i \) of iterations of OKS such that after the \( i \)-th iteration the minimum degree in the right side of the network is larger than one. Since the set \( \{0 < t < T : m(x(t)) > 1\} \) has zero Lebesgue measure by (27) Lebesgue measure of the set \( \{0 < t < T : m(X^{(n)}(t)) > 1\} \) goes to zero as \( n \) grows. Because the Lebesgue measure of the set \( \{0 < t < T : m(X^{(n)}(t)) > 1\} \) is \( \sum_{i \in J^{(n)}} \tau_{i+1} \) we have \( \lim_{n \to \infty} \sum_{i \in J^{(n)}} \tau_{i+1} = 0 \) but by the Law of Large Numbers \( \lim_{n \to \infty} \frac{|J^{(n)}|}{n} = \frac{1}{n} \).

Therefore \( \lim_{n \to \infty} \frac{|J^{(n)}|}{n} = \frac{1}{n} \) we prove that the size of the matching provided by the OKS algorithm is away from maximum matching size by a sublinear factor. Starting the algorithm, as long as the minimum degree in the right side of the network is one, OKS does no mistake, i.e. the size of the matching by OKS is the same as the size of maximum matching. When the minimum degree is \( m = m(X^{(n)}(t)) > 1 \) it is possible that OKS picks a vertex in the left side which is not the optimal choice. We make it optimal by manipulating the network: if \( v \in V, u \in L, \deg(v) = m \) are the chosen vertices in the iteration of the algorithm to be removed from the network, \( M_{OKS} = M_{OKS} \cup \{(u, v)\} \), manipulate the network by removing all other \( m - 1 \) edges connected to \( v \). Since \( |M^*| \) has the Lipschitz property, removing these \( m - 1 \) edges will change the size of the maximum matching by at most \( m - 1 \). Since \( m \) is the minimum degree and the average degree is bounded, \( m - 1 \) is bounded as well. On the other hand, the number of iterations that OKS will face such cases is sublinear w.r.t. the size of the network, so the whole number of possible errors, or in other words, the whole deviation from maximum matching made by OKS is sublinear, i.e. \( \lim_{n \to \infty} \frac{|M_{OKS}(G)|}{n} = \lim_{n \to \infty} \frac{|M^*(G)|}{n} \).

Step 2: Generalization to unbounded degree: Now to generalize the proof to cases where the asymptotic empirical degree distributions are not bounded we use the classical technique of truncation. For arbitrary \( \epsilon > 0 \), let \( N \) be large enough such that \( \sum_{k=N+1} \frac{k p_{in}(k)}{2} \leq \frac{\epsilon}{2} \) and \( \sum_{k=N+1} \frac{k p_{out}(k)}{2} \leq \frac{\epsilon}{2} \).

In random network \( G \) remove some edges in order to have no vertex of degree larger than \( N \) to get random network \( H \) which has bounded asymptotic empirical degree distributions. By Step 1

\[
\lim_{n \to \infty} \frac{|M_{OKS}(H)|}{n} = \lim_{n \to \infty} \frac{|M^*(H)|}{n} = (28)
\]

Because by Lemma(2) both functions \( |M^*|, |M_{OKS}| \) have the Lipschitz property and asymptotically the number of edges removed from \( G \) to get \( H \) is less than \( n \epsilon \):

\[
\lim_{n \to \infty} \frac{|M^*(H)| - |M^*(G)|}{n} < \epsilon (29)
\]

\[
\lim_{n \to \infty} \frac{|M_{OKS}(H)| - |M_{OKS}(G)|}{n} < \epsilon (30)
\]

Now (28), (29), (30) imply the desired result. Further, when \( \epsilon \to 0, N \to \infty \), so formally we can take \( N = \infty \) and write the functions \( F, G : \mathbb{R}^\infty \to \mathbb{R}^\infty \) as

\[
F(x, y) = (\frac{\|A^2 y\|}{\|A y\|} - 1) \frac{1}{\|A x\|} (S A x - A x) - \frac{x}{\|x\|}
\]

\[
G(x, y) = -\frac{A y}{\|A y\|} + \frac{1}{\|A x\|} (S A y - A y)
\]

for matrices \( A, S \in \mathbb{R}^{\infty \times \infty} \) provided \( \|A^2 x(0)\|_1 = \sum_{k=1} k^2 p_{in}(k) < \infty \) or \( \|A^2 y(0)\|_1 = \sum_{k=1} k^2 p_{out}(k) < \infty \).

Appendix F

Proof of Theorems 3, 4

As we saw in the proof of Theorem 5 the asymptotic performance of Greedy, OKS and KS algorithms can be described by asymptotic empirical degree distributions which are solutions of some initial value problems. If we find functions \( F, G : \mathbb{R}^\infty \to \mathbb{R}^\infty \) for Greedy and KS we will have

\[
F(x, y) = (\frac{\|A^2 y\|}{\|A y\|} - 1) \frac{1}{\|A x\|} (S A x - A x) - \frac{x}{\|x\|}
\]

\[
G(x, y) = -\frac{A y}{\|A y\|} + \frac{1}{\|A x\|} (S A y - A y)
\]

for Greedy and for KS

\[
F(x, y) = \frac{x m}{x m + y m} \left[ (\frac{\|A^2 y\|}{\|A y\|} - 1) \frac{1}{\|A x\|} (S A x - A x) - \frac{x}{\|x\|} \right] + \frac{y m}{x m + y m} \left[ -\frac{A x}{\|A x\|} + \frac{1}{\|A y\|} (S A y - A y) \right]
\]

\[
G(x, y) = \frac{y m}{x m + y m} \left[ (\frac{\|A^2 x\|}{\|A x\|} - 1) \frac{1}{\|A y\|} (S A y - A y) - \frac{1}{2} (S A x - A x) \right] + \frac{x m}{x m + y m} \left[ -\frac{A y}{\|A y\|} + \frac{1}{\|A x\|} (S A y - A y) \right]
\]

where \( m = \min\{m(x(t)), m(y(t))\} \) is the minimum degree. Since for KS we have \( G(x, y) = F(y, x) \), when \( x(0) = y(0) \) (i.e. \( p_{in} = p_{out} \)) \( x(t) = y(t) \) and

\[
\dot{x} = F(x) = (\frac{\|A^2 y\|}{\|A x\|} - 1) 2 (S A x - A x) - \frac{A x}{2 \|A x\|} - \frac{1}{2}
\]

Thus for any degree distribution \( p \) the dynamics of KS is the same for both \( DD(p) \) and \( DD(p, p) \). Further, regarding the results provided in Theorems 3, 4 the relative size of the output of the algorithm as well as the dynamics of the algorithm is the same for all random networks GER(\( \lambda \)), UFS(\( \lambda \)), DD(\( p \))
and $DD(p, p)$ where probability distribution $p$ is Poisson($\lambda$) because they all are sharing the asymptotic empirical degree distributions. So Theorem 5 implies $\lim_{n \to \infty} \frac{|M_{GS}(G)|}{n} = 1 - \frac{\log(1 - e^{-\lambda})}{\lambda}$. For KS, all mentioned statements are proved for undirected GER($\lambda$) by Karp and Sipser [19] so are valid for the desired class of random networks.

### APPENDIX G

**PROOF OF THEOREM 5**

Here we assume in addition that $\|A^2 x(0)\|_1 = \infty \sum_{k=1}^{\infty} k^2 p_{in}(k) < \infty$ and $\|A^2 y(0)\|_1 = \infty \sum_{k=1}^{\infty} k^2 p_{out}(k) < \infty$.

Generalization to the case where above quantities are not bounded is straightforward similar to Step 2 in the proof of Theorem 5 and is omitted. Because $\lim_{n \to \infty} \frac{|M^*(G)|}{n} = \lim_{n \to \infty} \frac{|M_{KS}(G)|}{n}$ it suffices to show $U^* = \lim_{n \to \infty} 1 - \frac{|M_{KS}(G)|}{n}$. To show the latter claim, we run KS algorithm and find the number of vertices left unmatched by the algorithm. Similar to what we did in the proof of Theorem 5 the asymptotic fraction of vertices left unmatched by KS is:

$$1 - \lim_{n \to \infty} \frac{|M_{KS}(G)|}{n} = p_{in}(0)$$

$$+ \int_0^T \left[ \frac{x_m}{x_m + y_m} \left( \frac{\|A^2 y_1\|_1}{\|A y_1\|_1} - 1 \right) - \frac{(m - 1) y_m}{x_m + y_m} \right] \|A x\|_1 dt$$

where

$$T = \sup \{ t : \sum_{k=1}^{\infty} k x_k(t) > 0, \sum_{k=1}^{\infty} k y_k(t) > 0 \}$$

$$\dot{x} = \Phi(x, y), x(0) = p_{in}$$

$$\dot{y} = \Phi(y, x), y(0) = p_{out}$$

$$= \frac{x_m}{x_m + y_m} \left( \frac{\|A^2 y_1\|_1}{\|A y_1\|_1} - 1 \right) \frac{1}{\|A x\|_1} (S A x - A x) - 1_m$$

and $m = \min\{m(x(t)), m(y(t))\}$ is the minimum degree. Since as we saw in the proof of Theorem 5 the set $\{ t : m(x(t)) > 1, m(y(t)) > 1 \}$ is of zero Lebesgue measure without loss of generality in all integrations we can assume $m = 1$, especially

$$1 - \lim_{n \to \infty} \frac{|M_{KS}(G)|}{n} = p_{in}(0)$$

$$+ \int_0^T \frac{x_1}{x_1 + y_1} \left( \frac{\|A^2 y_1\|_1}{\|A y_1\|_1} - 1 \right) \frac{x_1(t)}{\|A x\|_1} dt$$

Now define:

$$\dot{x}_0(t) = p_{in}(0) + \int_0^t \frac{x_1}{x_1 + y_1} \left( \frac{\|A^2 y(s)\|_1}{\|A y(s)\|_1} - 1 \right) \frac{x_1(s)}{\|A x\|_1} ds$$

$$\mu(t) = \sum_{i=0}^{\infty} i x_i(t) = \sum_{i=0}^{\infty} i y_i(t) = \|A x\|_1 = \|A y\|_1$$

$$\Phi_{in}(t, u) = \sum_{i=1}^{\infty} x_i(t) u_i$$

$$\Phi_{out}(t, u) = \sum_{i=1}^{\infty} y_i(t) u_i$$

$$\phi_{in}(t, u) = \sum_{i=0}^{\infty} i x_i(t) u_i^{i-1}$$

$$\phi_{out}(t, u) = \sum_{i=0}^{\infty} i y_i(t) u_i^{i-1}$$

$$\phi_{in}(t, w_3(t)) = w_1(t)$$

$$\phi_{out}(t, w_4(t)) = w_1(t)$$

$$\phi_{out}(t, 1 - w_3(t)) = 1 - w_2(t)$$

$$V(t) = \sum_{i=1}^{\infty} x_i(t) = \|x(t)\|_1$$

$$U(t) = \frac{1}{2} \left[ \Phi_{in}(t, w_2(t)) + \Phi_{in}(t, 1 - w_1(t)) + \Phi_{out}(t, w_4(t)) + \Phi_{out}(t, 1 - w_3(t)) - 2 \right]$$

$$+ \mu(t) \left[ w_3(t)(1 - w_2(t)) + w_1(t)(1 - w_4(t)) \right]$$

Since $x_1(T) = x_2(T) = \ldots = 0, y_1(T) = y_2(T) = \ldots = 0$ we have $V(T) = 0, \mu(T) = 0$. But $V(0) = 1 - p_{in}(0)$ and for $m = 1$

$$\dot{V}(t) = \frac{d\|x\|_1}{dt} = \frac{x_1}{x_1 + y_1} \left( -\frac{\|A^2 y_1\|_1}{\|A y_1\|_1} - 1 \right) \frac{x_1}{\|A x\|_1} - 1 - \frac{y_1}{x_1 + y_1}$$

i.e. $-1 + p_{in}(0) = V(T) - V(0) = x_0(0) - x_0(T) - T = p_{in}(0) - x_0(T) - T$ Therefore $\Phi_{in}(T, u) = x_0(T), \Phi_{out}(T, u) = y_0(T), U(T) = 1 - 2T$ and

$$1 - \lim_{n \to \infty} \frac{|M_{KS}(G)|}{n} = x_0(T) = y_0(T) = 1 - T \quad (31)$$

On the other hand as long as $m = 1$:

$$\dot{\mu}(t) = \frac{d\mu}{dt} = \frac{y_1}{x_1 + y_1} \left( -\frac{\|A^2 x\|_1}{\|A x\|_1} - 1 \right) \frac{x_1}{\|A x\|_1}$$

$$+ \frac{x_1}{x_1 + y_1} \left( \frac{\|A^2 y_1\|_1}{\|A y_1\|_1} - 1 \right) \frac{x_1}{\|A y\|_1}$$

$$= -\frac{x_1}{x_1 + y_1} \left[ \frac{\|A^2 x\|_1}{\|A x\|_1} - 1 \right] \frac{y_1}{\|A y\|_1}$$

$$= -\frac{x_1}{(x_1 + y_1)} \left[ \frac{\|A^2 x\|_1}{\|A x\|_1} + \frac{\|A^2 y\|_1}{\|A y\|_1} \right]$$

$$= -\frac{x_1}{(x_1 + y_1)} \left[ \frac{\|A^2 x\|_1}{\|A x\|_1} + \frac{\|A^2 y\|_1}{\|A y\|_1} \right]$$

$$= -\frac{x_1}{(x_1 + y_1)} \left[ \frac{\|A^2 x\|_1}{\|A x\|_1} + \frac{\|A^2 y\|_1}{\|A y\|_1} \right]$$

$$= -\frac{x_1}{(x_1 + y_1)} \left[ \frac{\|A^2 x\|_1}{\|A x\|_1} + \frac{\|A^2 y\|_1}{\|A y\|_1} \right]$$
The above equations imply:

\[
\frac{d}{dt} \Phi_{in}(t, w_2(t)) = \frac{x_1}{x_1 + y_1} \left[ \frac{\|A^2y\|}{\|Ay\|} - 1 \right] \phi_{in}(t, w_2(t)) (1 - w_2(t)) - w_2(t) + \frac{y_1}{x_1 + y_1} \left[ - w_2(t) \phi_{in}(t, w_2(t)) + \|Ax\| \hat{w}_2(t) \phi_{in}(t, w_2(t)) + \|Ax\| w_1(t) \phi_{in}(t, w_2(t)) \right] + \frac{x_1}{x_1 + y_1} \left[ \frac{\|A^2y\|}{\|Ay\|} - 1 \right] w_3(t) (1 - w_2(t)) - w_2(t) + \frac{y_1}{x_1 + y_1} \left[ - w_2(t) w_3(t) + \|Ax\| \hat{w}_2(t) \phi_{in}(t, w_2(t)) + \|Ax\| w_1(t) \phi_{in}(t, w_2(t)) \right]
\]

i.e.

\[
\frac{d}{dt} \Phi_{in}(t, 1 - w_1(t)) = \frac{x_1}{x_1 + y_1} \left[ \frac{\|A^2y\|}{\|Ay\|} - 1 \right] \phi_{in}(t, 1 - w_1(t)) w_1(t) - 1 + w_1(t) + \frac{y_1}{x_1 + y_1} \left[ - w_1(t) \phi_{in}(t, 1 - w_1(t)) + \|Ax\| \hat{w}_1(t) \phi_{in}(t, 1 - w_1(t)) \right] - \|Ax\| \hat{w}_1(t) \phi_{in}(t, 1 - w_1(t)) + \frac{x_1}{x_1 + y_1} \left[ \frac{\|A^2y\|}{\|Ay\|} - 1 \right] w_1(t) (1 - w_4(t)) - 1 + w_1(t) + \frac{y_1}{x_1 + y_1} \left[ - (1 - w_1(t)) (1 - w_4(t)) - \|Ax\| \hat{w}_1(t) (1 - w_2(t)) - w_3(t) \right]
\]

The above equations imply:

\[
2 \frac{d}{dt} U(t) = \frac{x_1}{x_1 + y_1} \left[ \frac{\|A^2y\|}{\|Ay\|} - 1 \right] w_3(t) (1 - w_2(t)) - w_2(t) + \frac{y_1}{x_1 + y_1} \left[ - w_2(t) w_3(t) + \|Ax\| \hat{w}_2(t) \phi_{in}(t, w_2(t)) \right] + \frac{x_1}{x_1 + y_1} \left[ \frac{\|A^2y\|}{\|Ay\|} - 1 \right] w_1(t) (1 - w_4(t)) - 1 + w_1(t) + \frac{y_1}{x_1 + y_1} \left[ - (1 - w_1(t)) (1 - w_4(t)) - \|Ax\| \hat{w}_1(t) (1 - w_2(t)) - w_3(t) \right]
\]

simplifying

\[
2 \frac{d}{dt} U(t) = \frac{x_1}{x_1 + y_1} \left[ \frac{\|A^2y\|}{\|Ay\|} - 1 \right] w_3(t) (1 - w_2(t)) - w_2(t) + \frac{y_1}{x_1 + y_1} \left[ - w_2(t) w_3(t) + \|Ax\| \hat{w}_2(t) \phi_{in}(t, w_2(t)) \right] + \frac{x_1}{x_1 + y_1} \left[ \frac{\|A^2y\|}{\|Ay\|} - 1 \right] w_1(t) (1 - w_4(t)) - 1 + w_1(t) + \frac{y_1}{x_1 + y_1} \left[ - (1 - w_1(t)) (1 - w_4(t)) - \|Ax\| \hat{w}_1(t) (1 - w_2(t)) - w_3(t) \right]
\]
Again since the set \( \{ t : m > 0 \} \) is of zero Lebesgue measure integrating both sides of \( \frac{dU(t)}{dt} = -1 \) we have:

\[
1 - 2T = U(T) = U(0) - T
\]

So \( U^* = U(0) = 1 - T \) which is the desired result by \( 31 \).

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