THE UNIQUENESS THEOREM FOR
ROTATING BLACK HOLE SOLUTIONS OF
SELF-GRAVITATING HARMONIC MAPPINGS

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Abstract
We consider rotating black hole configurations of self-gravitating maps from spacetime into arbitrary Riemannian manifolds. We first establish the integrability conditions for the Killing fields generating the stationary and the axisymmetric isometry (circularity theorem). Restricting ourselves to mappings with harmonic action, we subsequently prove that the only stationary and axisymmetric, asymptotically flat black hole solution with regular event horizon is the Kerr metric. Together with the uniqueness result for non-rotating configurations and the strong rigidity theorem, this establishes the uniqueness of the Kerr family amongst all stationary black hole solutions of self-gravitating harmonic mappings.
1 Introduction

By now, the uniqueness of the 3-parameter Kerr-Newman family amongst all stationary, asymptotically flat black hole solutions of the coupled Einstein-Maxwell equations is established quite rigorously. The proof relies heavily on the fact that the event horizon of a stationary black hole is a Killing horizon, implying that the null generator Killing field of the horizon either coincides with the asymptotically timelike Killing field or the domain of outer communications is stationary and axisymmetric. By virtue of this (strong rigidity) theorem \[2\], \[3\], the stationary black hole configurations are subdivided into non-rotating and rotating ones.

The uniqueness theorem for the first class of solutions, that is the uniqueness of the Reissner-Nordström metric amongst all non-rotating electrovac black hole equilibrium states, was established by the work of Müller zum Hagen, Robinson and Seifert \[4\], \[5\], Robinson \[6\] and others, basing on Israel’s pioneering results \[7\], \[8\]. Some open gaps have been closed more recently, such as the extension of certain vacuum results to the electrovac case \[9\], the exclusion of multiple black hole solutions \[10\], \[11\], \[12\], \[13\], \[14\] and the proof of the staticity theorem for the Einstein-Maxwell system \[15\] (see also \[16\]).

The uniqueness results for rotating configurations, that is for stationary and axisymmetric black hole spacetimes, stand on a very solid foundation as well. Early progress concerning the vacuum case was achieved by the work of Carter \[17\], \[18\], Hawking \[19\] and Robinson \[20\], \[21\]. Some years ago, Bunting \[22\] and Mazur \[23\] were eventually able to generalize these results to electromagnetic fields. (See \[24\] for a review on the new methods introduced by these authors.)

The recent discovery of new black hole solutions of the Einstein-Yang-Mills system (with vanishing Yang-Mills charges) \[25\], \[26\], \[27\] demonstrates that the aforementioned uniqueness results do not generalize in a straightforward way in the presence of non-Abelian gauge fields. The same applies to the related class of field theories represented by non-linear sigma-models or, more generally, self-gravitating mappings \(\phi\) from spacetime into Riemannian target manifolds. As a matter of fact, new black hole solutions with Skyrme hair have recently been constructed \[28\] and, in contrast to the solutions with Yang-Mills hair, turned out to be (linearly) stable \[29\], \[30\], \[31\].

It is, however, well known that the Skyrme action is not minimal in the sense that it contains quartic terms in the differential \(d\phi\) of the mapping \(\phi\). (See \[32\] for a generalization of the classical Skyrme model to arbitrary target spaces.) Uniqueness theorems for self-gravitating mappings can thus only exist for a restricted class of mappings. Hence, one has to answer the following questions: What are the necessary requirements for (i) the matter action and (ii) the target manifold \((N, G)\) such that the field equations admit only the vacuum solution \((gK, \phi_0)\)? (Here \(gK\) and \(\phi_0\) denote the Kerr metric and a constant map, respectively.)

Until recently, only partial answers to the above questions were known: On the one hand, no-hair theorems were derived for static mappings into linear target manifolds, that is, for the Einstein-Higgs system (with convex potential) \[33\], \[34\], \[35\]. On the other hand, it was shown that a static harmonic map from a fixed Schwarzschild background to a Riemannian manifold has to be constant \[36\], \[37\]. However, the latter fact does, of course, not exclude the existence of non-trivial field configurations in the self-gravitating case. (Consider, as an example, the corresponding situation for the Einstein-Yang-Mills system.) To our knowledge, the strongest results in the self-gravitating case were obtained for black hole solutions of sigma-models with harmonic action and non-compact, symmetric target manifolds \(G/H\) \[38\] or (on the basis of Bochner identities \[39\], \[40\], \[41\]) for target manifolds with non-positive sectional curvature. (In the spherically symmetric case, a generalization of these results to models with arbitrary Riemannian target manifolds and arbitrary non-negative potentials was given in \[42\], \[43\] and \[44\].)
A complete answer to questions (i) and (ii) was presented for non-rotating black holes in \[45\]: It is first shown that any self-gravitating mapping from a strictly stationary domain of outer communications into a Riemannian manifold is static. Subsequently, one demonstrates that the exterior Schwarzschild geometry is the only maximally extended, static, asymptotically flat solution of the coupled Einstein-matter equations, provided that the matter action is harmonic. This is achieved by proceeding along the same lines as in the vacuum case \[12\], making essential use of the positive mass theorem \[46\], \[47\].

In the present paper we extend the uniqueness result to the rotating case. We do so by first establishing the circularity theorem in the second section, which guarantees the integrability of the 2-surfaces orthogonal to the Killing fields generating the stationary and the axisymmetric isometries. As a consequence, the spacetime metric can be written in the Papapetrou form \[48\]. As we shall argue in the third section, the field equations then split into two decoupled sets, provided that \(\phi\) describes a self-gravitating harmonic mapping. In the forth section we show that, like in the vacuum case, both sets of equations can be derived from a variational principle. The fifth section is devoted to the first set of equations, which is exactly the same as in the vacuum case. Hence, it uniquely determines the Ernst potential \[49\], \[50\] (and the associated metric functions) as the solution of a regular, 2-dimensional boundary value problem. The second set of equations involves the matter fields and an additional metric function, which is not determined by the Ernst potential. Using asymptotic flatness and Stoke’s theorem for a suitably chosen 2-dimensional vector field, we finally show in the last section that these equations admit only trivial matter field configurations. This demonstrates that the only stationary and axisymmetric, asymptotically flat black hole solutions (with regular event horizon) of self-gravitating harmonic mappings consist of the Kerr metric \(g_K\) and constant maps \(\phi_0\). Together with the corresponding result for non-rotating solutions \[15\] and the strong rigidity theorem, this establishes the uniqueness of the Kerr family amongst all stationary black hole configurations with self-gravitating harmonic mappings.

2 The Circularity Theorem

We start this section by briefly recalling the notion of stationary and axisymmetric, asymptotically flat spacetimes. Subsequently, we argue that the invariance properties of the scalar fields with respect to the actions of the two Killing fields imply that spacetime is Ricci circular. As a consequence, the circularity theorem \[51\], \[52\] (see also \[14\], \[53\]) can be applied in order to establish the integrability conditions for the Killing fields. The metric is then written in the Papapetrou form \[18\] (see also \[54\]), representing the starting point for most investigations on stationary and axisymmetric spacetimes.

In the following we consider self-gravitating mappings (non-linear sigma-models), described by the action

\[
S = \int_M \left( R + \mathcal{L}(\phi, d\phi, g, G) \right) \eta, \tag{1}
\]

where \(\phi: (M,g) \rightarrow (N,G)\) denotes a mapping (scalar field) from the spacetime manifold \(M\) with metric \(g\) (and volume form \(\eta\)) to the Riemannian manifold \(N\) with metric \(G\). As far as the circularity theorem is concerned, the matter Lagrangian \(\mathcal{L}\) can be an arbitrary expression in terms of the fields \(\phi\), their differentials \(d\phi\) and the metrics \(g\) and \(G\). In order to prove the uniqueness theorem we shall, however, have to restrict ourselves to harmonic Lagrangians, \(\mathcal{L} = \frac{1}{2} \| d\phi \|^2\). The differential \(d\phi\) may be considered a section in the product bundle \(\phi^*(TN) \times TM^*\), where \(TM^*\) and \(\phi^*(TN)\) denote the dual of and the pullback of the
tangential spaces \( TM \) and \( TN \), respectively. In local coordinates of \( M \) and \( N \) one has the representation
\[
\|d\phi\|^2 \equiv G_{AB} (d\phi^A | d\phi^B) = G_{AB}(\phi(x)) g^{\mu\nu}(x) \partial_\mu \phi^A \partial_\nu \phi^B.
\]
(2)

Since \( \mathcal{L} \) does not depend on derivatives of the spacetime metric, the variation of the matter action with respect to \( g \) does not lead to integrations by parts. Thus, the energy momentum tensor is given by
\[
T = 2 \frac{\partial \mathcal{L}}{\partial g} - \mathcal{L} g = G_{AB} \partial_\mu \phi_A \partial_\nu \phi_B - \mathcal{L} g.
\]
(3)

We assume that both spacetime and the matter fields are stationary and axisymmetric. We recall that an asymptotically flat spacetime \((M, g)\) is said to be stationary and axisymmetric if it admits two commuting Killing fields, \( k \) and \( m \), where \( k \) is asymptotically timelike and \( m \) is spacelike and has closed orbits. Hence, denoting with \( L_K \) the Lie derivative with respect to an arbitrary vector field \( K \), we have
\[
L_k g = L_m g = 0, \quad L_k \phi^A = L_m \phi^A = 0,
\]
(4)
\[
[k, m] = 0, \quad X \equiv (m|m) \geq 0, \quad V \equiv -(k|k) \geq 0, \quad \text{asymptotically}.
\]
(5)

Note that the requirement that \( m \) is everywhere spacelike or null turns out to be necessary in order to reduce Einstein’s equations to a regular 2-dimensional boundary value problem. Besides this, it is implied by the causality requirement. We also recall that asymptotic flatness guarantees that the group generated by \( k \) and \( m \) is Abelian [56].

Let us denote with \( T(K) \) the energy momentum 1-form with respect to an arbitrary Killing field \( K \),
\[
(T(K))_\mu \equiv T_{\mu\nu} K^\nu.
\]
If the \( \phi^A \) are invariant with respect to the action of \( K \), \( L_K \phi^A = 0 \), then the 1-forms \( T(K) \) and \( K \) are proportional. As a consequence of Einstein’s equations, the Ricci 1-form \( R(K) \) (with components \( (R(K))_\mu \equiv R_{\mu\nu} K^\nu \)) is then proportional to \( K \) as well,
\[
T(K) = -\mathcal{L} K, \quad R(K) = 8\pi G [\mathcal{L} - \frac{1}{2} G_{AB} (d\phi^A | \mathcal{L} d\phi_B)] K.
\]
(6)

Hence, in a stationary and axisymmetric spacetime, the symmetry equations for \( \phi^A \) together with Einstein’s equations immediately imply that the Ricci circularity conditions,
\[
i_m * (k \wedge R(k)) = 0, \quad i_k * (m \wedge R(m)) = 0,
\]
are fulfilled. Here we have used the operators \( i_K \) and * to denote the inner product with respect to the vector field \( K \) and the Hodge dual, respectively. (Recall that in 3 + 1 dimensions we have \(*^2 \Omega = -(-1)^p \Omega \) and \( i_K \Omega = - * (K \wedge * \Omega) \) for an arbitrary \( p \)-form \( \Omega \).)

Next we apply the circularity theorem [51], [52], implying that in an asymptotically flat, stationary and axisymmetric spacetime, the Frobenius integrability conditions are satisfied,
\[
(m|\omega_k) = 0, \quad (k|\omega_m) = 0,
\]
(8)
provided that the Ricci circularity conditions (7) hold (and vice versa). As usual, the twist 1-form \( \omega_K \) assigned to an arbitrary 1-form \( K \) is defined as
\[
\omega_K \equiv \frac{1}{2} * (K \wedge dK).
\]
(9)
A simple proof of the circularity theorem is obtained as follows: First, one notes that for an arbitrary Killing field (1-form) $K$ the differential of $\omega_K$ fulfills the identity
\[ d\omega_K = *(K \wedge R(K)). \] (10)

Using the co-differential, $d^\dagger = *d*$, and the Laplacian, $-\Delta = d^\dagger d + dd^\dagger$, eq. (10) is obtained from the identities $d^\dagger K = 0$, $d^\dagger (K \wedge dK) = -K \wedge d^\dagger dK$ and $-\Delta K = 2R(K)$, which hold for arbitrary Killing fields (1-forms) (see [53] for details). It is now easy to see that the Ricci circularity conditions (7) are the exterior differentials of the integrability conditions (8): Clearly $[k, m] = 0$ implies $L_m \omega_k = L_k \omega_m = 0$ (since the Hodge dual commutes with the Lie derivative with respect to a Killing field). Using $\omega$ the co-differential, $d\omega = \omega \wedge dK$ is constant. Since $m$ vanishes on the rotation axis, $(m|\omega_k)$ vanishes identically in every region of $(M, g)$ containing a part of the axis. The fact that the same argument also holds for $(k|\omega_m)$ concludes the proof. In summary, we have the following result:

Let $(M, g)$ be an asymptotically flat, stationary and axisymmetric spacetime with Killing fields $k$ and $m$, and let $\phi$ denote a self-gravitating, minimally coupled mapping from $(N, G)$ into a Riemannian manifold $(N, G)$, $\phi$ being invariant under the action of $k$ and $m$. Then (in every connected domain of spacetime containing a part of the rotation axis) the 2-surfaces orthogonal to $k$ and $m$ are integrable.

### 3 The Field Equations

In this section we give the basic expressions for the Ricci tensor of a spacetime manifold $(M, g)$ which admits an integrable system of two Killing fields, $k$ and $m$. Using Einstein’s equations for self-gravitating harmonic mappings, these identities reduce to the vacuum Ernst equations for the metric functions of the 2-dimensional orbit manifold, plus an additional set of equations involving the matter fields and the metric of the manifold orthogonal to the Killing orbits.

The integrability conditions imply $M = \Sigma \times T$ and $g = \sigma + \tau$, where $(\Sigma, \sigma)$ and $(T, \tau)$ are 2-dimensional manifolds with pseudo-Riemannian metric $\sigma$ and Riemannian metric $\tau$, respectively. Parametrizing $\Sigma = \mathcal{R} \times SO(2)$ with $t$ and $\varphi$, we have $k = \partial/\partial t$ and $m = \partial/\partial \varphi$. Choosing the co-ordinates $x^0 = t$, $x^1 = \varphi \in \Sigma$, $x^2, x^3 \in T$, and introducing the adapted local basis of 1-forms
\[ \theta^a = dx^a, \quad a \in \{0, 1\}, \quad \theta^i = dx^i, \quad i \in \{2, 3\}, \] (12)
the spacetime metric becomes
\[ g = \sigma_{ab} \theta^a \otimes \theta^b + \tau_{ij} \theta^i \otimes \theta^j, \] (13)
where both 2-dimensional metrics, $\sigma$ and $\tau$, depend on the co-ordinates $x^i$ of $T$ only, $\sigma = \sigma(x^i)$, $\tau = \tau(x^i)$, $i \in \{2, 3\}$.

The components of the Ricci tensor with respect to the metric (13) can be derived in an invariant manner. However, before doing so, one has to select the Killing field on the basis of which one intends to formulate
the Ernst equations. The "good" (although not the traditional) choice is to consider the Killing field $m$ which generates the axial symmetry \[16\]. There are two reasons for this: First of all, the norm of $k$ has no fixed sign if spacetime is stationary, rather than stationary in the strict sense. As a consequence, the system of differential equations formulated with respect to $k$ turns out to be singular at the boundaries of "ergoregions", appearing for all rotating black hole solutions. Secondly, if electromagnetic fields are also taken into account, the Ernst equations can still be derived from an action principle. However, only the Ernst potentials which are assigned to the axial Killing field $m$ lead to a definite action. The fixed sign of the effective Lagrangian is, however, a necessary requirement for the applicability of the uniqueness proof for rotating electrovac black holes \[22, 23\].

Parametrizing the metric $\sigma$ by the three functions $\rho \equiv \sqrt{-\det(\sigma)}$, $X \equiv (m|m)$ and $A$, and introducing the 2-dimensional Riemannian metric $\gamma = X\tau$ on $T$, \[14\]

\[
g = -\frac{\rho^2}{X} dt^2 + X (d\phi + A dt)^2 + \frac{1}{X} \gamma,
\]

one obtains the following set of differential identities

\[
\frac{1}{\rho} d^{(\gamma)}(\rho \mathcal{E}) = \frac{1}{X} [(\mathcal{E}|\mathcal{E})^{(\gamma)} - 2 R(m,m)],
\]

\[
\frac{1}{\rho} \Delta^{(\gamma)} \rho = -\frac{1}{X} tr^{(\sigma)} R,
\]

\[
\frac{1}{\rho} dA = 2 *^{(\gamma)} \left( \frac{\omega}{X^2} \right),
\]

\[
R_{ij} + \gamma_{ij} \frac{R(m,m)}{X^2} = \kappa^{(\gamma)} \gamma_{ij} - \frac{1}{\rho} \nabla_i^{(\gamma)} \nabla_j^{(\gamma)} \rho - \frac{\mathcal{E}_i \mathcal{E}_j + \mathcal{E}_j \mathcal{E}_i}{4 X^2},
\]

where $d^{(\gamma)}$, $\Delta^{(\gamma)}$, $*^{(\gamma)}$ and $\kappa^{(\gamma)}$ denote the co-differential, the Laplacian, the Hodge dual and the Gauss curvature with respect to the 2-dimensional Riemannian metric $\gamma$, and $tr^{(\sigma)} R \equiv \sigma^{ab} R_{ab}$. The complex 1-form $\mathcal{E}$ is defined as

\[
\mathcal{E} = -dX + 2i \omega,
\]

and the twist form $\omega \equiv \omega_m$ associated to $m$ fulfills the general identity \[10\],

\[
d\omega = *(m \wedge R(m)).
\]

The formulas \[14-21\] are geometrical identities. They are valid if the integrability conditions \[8\] are fulfilled. Einstein’s equations imply that this is the case for vacuum models, electromagnetic fields and, as we have demonstrated in the previous section, for invariant scalar mappings.

It is well-known that the above formulas experience a significant simplification for both vacuum and electrovac spacetimes. This is due to the fact that the r.h.s. of eq. \[16\] vanishes in both situations. This is also the case for arbitrary harmonic mappings, to which we shall restrict our attention in the remainder of this paper. Hence we consider

\[
\mathcal{L} = \frac{1}{2} \| d\phi \|^2, \quad T = G_{AB} d\phi^A \otimes d\phi^B - \frac{1}{2} \| d\phi \|^2 g,
\]

\[21\]
and use Einstein’s equations in the above identities. As is immediately observed from eq. (23), the invariance properties, \( L_k \phi^A = L_m \phi^A = 0 \), imply that the Ricci 1-forms with respect to both Killing fields vanish,

\[
R(k) = 0, \quad R(m) = 0, \quad tr(\sigma) R = 0.
\]  

(22)

In addition, the components of \( R_{\mu \nu} \) on \((T, \gamma)\) become

\[
R_{ij} = 8\pi G G_{AB} \phi^{A, i} \phi^{B, j}.
\]  

(23)

Since \( R(m) \) vanishes, we conclude from eq. (20) that the twist form is closed. Hence, there locally exists a twist potential \( Y \), such that \( \omega = dY \). Since the complex 1-form \( E \) is then closed as well, we can introduce the same Ernst potential as in the vacuum case [19], [20]:

\[
E = -X + iY, \quad \text{where} \quad dY = \omega.
\]  

(24)

Using the complex potential \( E \) and eqs. (22), (23) for the Ricci tensor, the field equations are now obtained from the identities (15)-(18),

\[
\Delta^{(\gamma)} E + \frac{1}{\rho} (d\rho |dE)^{(\gamma)} + \frac{1}{X} [(dE |dE)^{(\gamma)} = 0, 
\]  

(25)

\[
\Delta^{(\gamma)} \rho = 0,
\]  

(26)

\[
\frac{1}{\rho} dA = 2 \ast^{(\gamma)} \frac{(dY \pi_G G)}{X^2},
\]  

(27)

\[
\kappa^{(\gamma)} \gamma_{ij} - \frac{1}{\rho} \nabla^{(\gamma)} \nabla^{(\gamma)} \rho = \frac{E_{.i} \bar{E}_{.j} + E_{.j} \bar{E}_{.i}}{4X^2} + 8\pi G G_{AB} \phi^{A, i} \phi^{B, j},
\]  

(28)

where \( X = -\Re c(E), \ Y = \Im m(E) \). (Note that we have used the general identity \( d^l (f dg) = * (df \wedge dg) + f d^l dg = -[(df \wedge dg) + f \Delta g] \) for arbitrary functions \( f \) and \( g \).) In addition to the above formulas one also has the matter equations, \( \Delta \phi^A + \Gamma_{BC}^{A}(\phi)(d\phi^A |d\phi^B) = 0 \), which are obtained by varying the action \( (T) \) with Lagrangian \( (L) \) with respect to \( \phi^A \). (Here \( \Gamma_{BC}^{A} \) denote the Christoffel symbols assigned to the metric \( G \) of the target manifold \( (N, G) \), see [53] or [17], [28] for details). Since the \( \phi^A \) do not depend on \( t \) and \( \varphi \), we have

\[
(d\phi^A |d\phi^B) = X(d\phi^A |d\phi^B)^{(\gamma)}, \quad \Delta \phi^A = X[\Delta^{(\gamma)} \phi^A + (d\ln \rho |d\phi^A)^{(\gamma)}].
\]

Hence the matter equations assume the form

\[
\Delta^{(\gamma)} \phi^A + \frac{1}{\rho} (d\rho |d\phi^A)^{(\gamma)} + \Gamma_{BC}^{A}(d\phi^A |d\phi^B)^{(\gamma)} = 0.
\]  

(29)

The formulas (25)-(28) represent the complete set of field equations for the functions \( E, \rho, \phi^A \) and the 2-dimensional Riemannian metric \( \gamma \). All quantities depend only on the co-ordinates \( x^2 \) and \( x^3 \). The crucial observation consists in the fact that the matter fields do not appear in the Einstein equations (22) which completely determine the metric \( \sigma \). As in the vacuum case (and for the Einstein-Maxwell system), \( \rho = \sqrt{- \det(\sigma)} \) is a harmonic function on \((T, \gamma)\). This yields a further, considerable simplification: Using asymptotic flatness, ordinary Morse theory [29] implies that the number of local maxima plus the number of local minima minus the number of saddle points of \( \rho \) is a topological invariant, which vanishes if the domain is simply connected and all critical points are non-degenerate. Together with the maximum principle, this theorem immediately implies that \( \rho \) has no non-degenerate inner critical points at all. Using the special
Morse theory for harmonic functions one can, in addition, exclude degenerate critical points as well. Hence, if \( \Delta^{(\gamma)} \rho = 0 \), it is possible to choose \( \rho \) as one of the co-ordinates on \((T, \gamma)\) \cite{13, 14}. Denoting the remaining co-ordinate with \( z \), one can introduce a conformal factor, \( \exp(2h(\rho, z)) \), such that the metric \( \gamma \) assumes the diagonal form \( \gamma = e^{2h}(d\rho^2 + dz^2) \). Thus, the spacetime metric is finally parametrized by only three functions \( X, A \) and \( h \) of two variables \( \rho \) and \( z \) \cite{48},

\[
g = -\frac{\rho^2}{X} dt^2 + X (d\rho^2 + A dt)^2 + \frac{1}{X} e^{2h} (d\rho^2 + dz^2). \tag{30}
\]

It remains to express equations (28) in terms of the co-ordinates \( \rho, z \). The second covariant derivatives of \( \rho \) transform into ordinary first derivatives of the conformal factor \( e^{2h} \) with respect to \( \rho \) and \( z \). Using \( \Gamma_{\rho \rho} = \Gamma_{zz}^z = h_{\rho \rho} \) and \( \Gamma_{zz}^z = \Gamma_{\rho \rho}^z = -\Gamma_{\rho \rho}^z = h_{zz} \), one immediately finds

\[
\begin{align*}
\nabla_z^{(\gamma)} \nabla_z^{(\gamma)} \rho &= -\nabla_\rho^{(\gamma)} \nabla_\rho^{(\gamma)} \rho = h_{\rho \rho}, & \nabla_\rho^{(\gamma)} \nabla_z^{(\gamma)} \rho &= \nabla_z^{(\gamma)} \nabla_\rho^{(\gamma)} \rho = h_{\rho z},
\end{align*}
\tag{31}
\]

\[
R_{ij}^{(\gamma)} = \kappa^{(\gamma)} {\gamma}_{ij}, \quad \kappa^{(\gamma)} = -\Delta^{(\gamma)} h = -e^{-2h} \Delta^{(\delta)} h, \tag{32}
\]

where \( \Delta^{(\delta)} \) denotes the flat Laplacian with respect to the co-ordinates \( \rho \) and \( z \). Hence, in terms of Weyl co-ordinates, equation (28) assumes the form

\[
\begin{align*}
\frac{1}{\rho} h_{\rho \rho} &= \frac{1}{2}(R_{\rho \rho} - R_{zz}) + \frac{1}{4X^2} (E_{\rho \rho} E_{\rho \rho} - E_{zz} E_{zz}), \\
\frac{1}{\rho} h_{\rho z} &= \frac{1}{2}(R_{\rho z} + R_{zz}) + \frac{1}{4X^2} (E_{\rho z} E_{\rho z} + E_{zz} E_{zz}), \\
-\Delta^{(\delta)} h &= \frac{1}{2}(R_{\rho \rho} + R_{zz}) + \frac{1}{4X^2} (E_{\rho \rho} E_{\rho \rho} + E_{zz} E_{zz}),
\end{align*}
\tag{33, 34, 35}
\]

where \( R_{ij} \) is given by eq. (23), and \( i, j \in \{\rho, z\} \).

The linearity of the above equations in the metric function \( h \) suggests the partition

\[
h = h^{(vac)} + 8\pi G h^{(\delta)}, \tag{36}
\]

where \( h^{(vac)} \) is required to fulfill eqs. (33)-(35) with \( R_{ij} = 0 \). The entire set of field equations is now solved as follows:

(i) **Vacuum equations:** Like in the vacuum case, one first solves the 2-dimensional boundary value problem in the \((\rho, z)\) plane (with fixed, flat background metric \( \delta \)) for the Ernst potential \( E \), being subject to the Ernst equation

\[
\frac{1}{\rho} \nabla^{(\delta)} (\rho \nabla^{(\delta)} E) + \frac{(\nabla^{(\delta)} E|\nabla^{(\delta)} E)}{X} = 0. \tag{37}
\]

Subsequently, one obtains the metric functions \( A \) and \( h^{(vac)} \) by quadrature,

\[
\begin{align*}
\frac{1}{\rho} A_{\rho \rho} &= \frac{1}{X^2} Y_{zz}, & \frac{1}{\rho} A_{\rho z} &= -\frac{1}{X^2} Y_{\rho z}, \\
\frac{1}{\rho} h^{(vac)}_{\rho \rho} &= \frac{1}{4X^2} [E_{\rho \rho} E_{\rho \rho} - 2 E_{zz} E_{zz}], & \frac{1}{\rho} h^{(vac)}_{\rho z} &= \frac{1}{4X^2} [E_{\rho z} E_{\rho z} + E_{zz} E_{zz}],
\end{align*}
\tag{38, 39}
\]
where $X = -\Re(E), Y = \Im(m(E))$.

(ii) **Matter equations:** Like the Ernst equation, the field equations for the matter fields $\phi^A$,

$$
\frac{1}{\rho} \nabla^{(\delta)} (\rho \nabla^{(\delta)} \phi^A) + \Gamma_{BC}^{A} (\nabla^{(\delta)} \phi^B | \nabla^{(\delta)} \phi^C) = 0,
$$

(40)

involve no unknown metric functions. Having solved the boundary value problem for $\phi^A(z, \rho)$, the remaining metric function $h(\phi)$ is also obtained by quadrature,

$$
\frac{1}{\rho} h^{(\phi), \rho} = \frac{G_{AB}}{2} [\phi^{A, \rho} \phi^{B, \rho} - \phi^{A, z} \phi^{B, z}], \quad \frac{1}{\rho} h^{(\phi),, z} = \frac{G_{AB}}{2} [\phi^{A, \rho} \phi^{B, z} + \phi^{A, z} \phi^{B, \rho}] .
$$

(41)

### 4 The Effective Action

The uniqueness proofs of Bunting [21] and Mazur [22] for the Kerr-Newman metric rely on the circumstance that the field equations can be obtained from an effective, definite action which is defined on the 2-dimensional Riemannian manifold $\mathcal{T}$. In this section we establish the corresponding result for the equations describing stationary and axisymmetric selfgravitating harmonic mappings. We shall demonstrate that both, the Ernst equation (37) and the matter equations (39) for the fields $\phi^A$, can be gained from the effective Lagrangian $\mathcal{L}_{eff} = \rho \sqrt{\text{det}(\gamma)}$, $\kappa^{(\gamma)}$ denoting the Gauss curvature of $(\mathcal{T}, \gamma)$.

Before considering the variational principle, let us first recall the fact that the Ernst equation (37) coincides with the integrability condition for the system (39) and, in addition, guarantees the consistency of these equations with the Poisson equation for the function $h^{(\text{vac})}$,

$$
- \Delta^{(\delta)} h^{(\text{vac})} = \frac{1}{4X^2} (E_{\rho \rho} E_{\rho z} + E_{\rho z} E_{z z}) .
$$

(42)

In exactly the same manner, the matter equations (40) are identical with the integrability conditions for the system (41) and guarantee their consistency with the second order equation

$$
- \Delta^{(\delta)} h^{(\phi)} = \frac{G_{AB}}{2} [\phi^{A, \rho} \phi^{B, \rho} + \phi^{A, z} \phi^{B, z}]
$$

(43)

for the function $h^{(\phi)}$. Let us, as an example, demonstrate the consistency of eqs. (11) and (13). Differentiating $h^{(\phi), \rho}$ with respect to $\rho$ and $h^{(\phi), z}$ with respect to $z$ gives

$$
h^{(\phi),, \rho \rho} + h^{(\phi),, z z} = \rho \phi^{A, \rho} \{ G_{AB} (\phi^{B, z z} + \phi^{B, \rho \rho} + \frac{1}{\rho} \phi^{B, z}) + (G_{AB, C} - \frac{1}{2} G_{BC, A}) \phi^{B, z} \phi^{C, z} \\
+ \frac{1}{2} G_{AB, C} \phi^{B, \rho} \phi^{C, \rho} \} - \frac{G_{AB}}{2} (\phi^{A, \rho} \phi^{B, \rho} + \phi^{A, z} \phi^{B, z}) = \Delta^{(\delta)} h^{(\phi)},
$$

where we have used the matter eq. (11) and the identities $\Gamma_{BC}\phi^{A, \rho} \phi^{B, \rho} \phi^{C, \rho} = \frac{1}{2} G_{AC, \rho} \phi^{A, \rho} \phi^{B, \rho} \phi^{C, \rho}$.

Let us now consider the reduction of the matter action with respect to the metric (13). Using the fact that the matter fields do not depend on the co-ordinates of $(\Sigma, \sigma)$, we obtain

$$
S^{(\phi)} = \frac{1}{2} \int_{M} \| \phi \|^{2} \eta = \frac{1}{2} \int_{M} G_{AB} (d\phi^A | d\phi^B)^{(\gamma)} \sqrt{\gamma} \sqrt{|\sigma|} dx \phi dx^{1} d\rho dz
$$
Together with eq. (43), this suggests that $-\rho \Delta^{(\delta)} h^{(\phi)}$ is the effective Lagrangian for equations (40). As a matter of fact, it is not hard to verify that the effective Lagrangian for the Ernst equation (37) and the matter equations (40) is proportional to the Gauss curvature of $(T, \gamma)$,

$$L_{\text{eff}} = -\rho \Delta^{(\delta)}(h^{(\phi)} + 8\pi G h^{(\text{vac})}) = -\rho \sqrt{\gamma} \Delta^{(\gamma)} h = \rho \sqrt{\gamma} \kappa^{(\gamma)},$$

(45)

where $\Delta^{(\delta)} h$ is obtained from eqs. (42) and (43). Thus, we have the following result:

Let $(M, g)$ be an asymptotically flat, stationary and axisymmetric spacetime with Killing fields $k$ and $m$. Let $\phi$ denote a mapping with harmonic action from $(M, g)$ into a Riemannian manifold $(N, G)$, $\phi$ being invariant under the action of $k$ and $m$. Then the metric and the fields $\phi$ are obtained by solving the vacuum Ernst equation (37) and the matter equations (40), which are the Euler-Lagrange equations for the effective Lagrangian

$$L_{\text{eff}} = \rho \left[ \frac{(\nabla^{(\delta)} E)(\nabla^{(\delta)} \overline{E})}{(E + \overline{E})^2} + 8\pi G \frac{G_{AB} (\nabla^{(\delta)} \phi^A \nabla^{(\delta)} \phi^B)}{2} \right].$$

(46)

As usual, it is convenient to introduce prolate spheroidal co-ordinates $x$ and $y$ (see e.g. [60]) in order to distinguish between the horizon and the rotation axis,

$$\rho^2 = \mu^2 (x^2 - 1) (1 - y^2), \quad z = \mu xy,$$

(47)

where $\mu$ is an arbitrary positive constant. Note that $(\rho, z) \to (x, y)$ maps the upper half plane to the semi-strip $S = \{(x, y) \mid x \geq 1, |y| \leq 1\}$, where the boundary $\rho = 0$ splits into three parts, being the horizon, $x = 0$, and the northern and southern segments of the rotation axis, $y = 1$ and $y = -1$, respectively. In terms of prolate spheroidal co-ordinates the 2-dimensional metric $\gamma$ becomes

$$\gamma = e^{2h} (dp^2 + dz^2) = e^{2h} \mu^2 (x^2 - 1) \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right).$$

(48)

In order to solve the Ernst equation, it is very helpful to introduce the potential $\epsilon$,

$$\epsilon = \frac{1 + E}{1 - E},$$

(49)

which parametrizes the points in the semi-plane $X \geq 0$ by points in the unit disc $D = \{\epsilon \mid \epsilon \tau \leq 1\}$, since

$$X = \frac{1 - \epsilon \tau}{|1 + \epsilon|^2}, \quad Y = \frac{i(\tau - \epsilon)}{|1 + \epsilon|^2}.$$

(50)

In terms of prolate spheroidal co-ordinates, the effective Lagrangian for $\epsilon(x, y)$, describing the mapping from the semi-strip $S$ into the unit disc $D$, becomes

$$L_{\text{eff}}^{(\text{vac})} = \mu \frac{(x^2 - 1)\epsilon_{xx} \tau_{xx} + (1 - y^2)\epsilon_{yy} \tau_{yy}}{(1 - \epsilon\tau)^2}.$$
Before we continue, let us, as an example, consider mappings $\phi$ from spacetime into the pseudo-sphere $N = PS^2$,

$$\phi = (\chi, \varphi), \quad G_{AB}(\phi) = \text{diag}(1, sh^2 \chi).$$  \hspace{1cm} (52)

Parametrizing $PS^2$ with co-ordinates $w = \tanh(\frac{\chi}{2})e^{i\varphi}$ in the unit disc $D = \{ w \mid w \bar{w} \leq 1 \}$, we obtain

$$G_{AB}(\phi) (d\phi^A|d\phi^B) = d\chi^2 + sh^2 \chi d\varphi^2 = 4 \left( \frac{dw|d\bar{w}}{(1 - w\bar{w})^2} \right),$$ \hspace{1cm} (53)

which shows that the effective matter Lagrangian in this case becomes

$$\mathcal{L}_{\text{eff}}^{(\phi)} = 16\pi G \mu \left( \frac{e^{2h}}{2} \right),$$  \hspace{1cm} (54)

This demonstrates that both $\epsilon$ and $w$ satisfy exactly the same equation, if $\phi$ describes a stationary and axisymmetric mapping from spacetime into the pseudo-sphere.

5 Uniqueness of the Ernst potential

In this section we briefly recall the derivation of the Kerr solution and the arguments establishing its uniqueness. Applied to the situation under consideration, these arguments guarantee the uniqueness of the metric $\sigma$, (i.e. of the functions $X$ and $A$) and the function $h^{(\text{vac})}$. This clearly implies that the only possible deviation from the Kerr solution can occur via $h^{(\phi)}$. Hence, in order to establish the uniqueness theorem, it will remain to prove that $h^{(\phi)}$ vanishes identically. Once the circularity theorem is established, this is in fact the only additional step which has to be performed in order to extend the vacuum no-hair theorem for the Kerr solution to selfgravitating harmonic mappings. In the next section we shall prove that $h^{(\phi)}$ vanishes as a consequence of asymptotic flatness.

It is well known that a linear Ansatz shows that the Ernst equation for the Lagrangian (51) admits the simple solution

$$\epsilon = px + i qy,$$ \hspace{1cm} (55)

provided that the real constants $p$ and $q$ are subject to the condition $p^2 + q^2 = 1$. Inserting the solution (55) into the expressions (50) for $X$ and $Y$, and integrating subsequently eqs. (38) and (39) for $A$ and $h^{(\text{vac})}$ in prolate spheroidal co-ordinates, yields the metric functions

$$X = \frac{1 - (px)^2 - (qy)^2}{(1 + px)^2 + (qy)^2}, \quad Y = \frac{2qy}{(1 + px)^2 + (qy)^2},$$ \hspace{1cm} (56)

$$A = 2\mu \left( \frac{1 - y^2}{1 - (px)^2 - (qy)^2} \right), \quad e^{2h} = \left( \frac{1 - (px)^2 - (qy)^2}{p^2(x^2 - y^2)} \right).$$ \hspace{1cm} (57)

It is immediately observed that the norm $X$ of the Killing filed $m$ does neither vanish on the rotation axis, nor is it definite in the entire semi-strip $S$. Hence, the above solution is physically not acceptable. In order to obtain an asymptotically flat solution with $X \geq 0$, it remains to construct the conjugate solution $(\hat{X}, \hat{A}, \hat{h})$ \hspace{1cm} (58)

The latter is obtained from the observation that the metric $\sigma$ is invariant under the $(t, \varphi)$-rotation

$$t \to \hat{t} = \varphi, \quad \varphi \to \hat{\varphi} = -t,$$  \hspace{1cm} (58)
provided that $X$, $A$ and $h^{(\text{vac})}$ are transformed according to

$$
\hat{X} = X[A^2 - \frac{\rho^2}{X^2}], \quad \hat{A} = -A[A^2 - \frac{\rho^2}{X^2}]^{-1}, \quad e^{2h^{(\text{vac})}} = e^{2h^{(\text{vac})}}[A^2 - \frac{\rho^2}{X^2}]^{-1}.
$$

(59)

Performing these transformations, one obtains after some straightforward algebraic manipulations the Kerr solution in prolate spheroidal coordinates:

$$
\dot{X} = \frac{\mu^2}{p^2} (1 - y^2) \left[\frac{(1 + px)^2 + q^2}{(1 + px)^2 + (qy)^2} - p^2 q^2 (x^2 - 1)(1 - y^2)\right],
$$

(60)

$$
\dot{A} = -2 \frac{qy}{\mu} \left[\frac{(1 + px)^2 + q^2}{(1 + px)^2 + (qy)^2} - p^2 q^2 (x^2 - 1)(1 - y^2)\right],
$$

(61)

$$
e^{2h^{(\text{vac})}} = \frac{\dot{X}}{X} \left[\frac{(1 + px)^2 + (qy)^2}{p^2 (x^2 - y^2)}\right].
$$

(62)

In addition, the rotation potential $\dot{Y}$ ($d\dot{Y} = 2\omega$) is computed from $\dot{A}$ using the equations $\partial_x \dot{Y} = -[\mu(x^2 - 1)]^{-1} X^2 \partial_x A$ and $\partial_y \dot{Y} = [\mu(1 - y^2)]^{-1} X^2 \partial_y A$, which yield

$$
\dot{Y} = 2 \frac{\mu^2}{p^2} y \left[(3 - y^2) + \frac{q^2 (1 - y^2)}{(1 + px)^2 + (qy)^2}\right].
$$

(63)

The spacetime metric in prolate spheroidal coordinates eventually becomes

$$
g = -\hat{V} dt^2 + 2\hat{W} dt d\phi + \hat{X} d\phi^2 + \frac{e^{2h^{(\text{vac})}}}{\hat{X}} e^{8\pi G \phi} \mu^2 (x^2 - y^2) \left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2}\right),
$$

(64)

where $\hat{W} = \dot{A} \hat{X} = -AX$ and $\hat{V} = -X$. The coupling to the matter fields $\phi^A$ enters via the function $h^{(\phi)}$, being a solution of eqs. (60), (61).

It is immediately verified from eqs. (60) and (63) that the real and imaginary part, $-\hat{X}$ and $\hat{Y}$, of the Ernst potential satisfy the following boundary conditions with respect to prolate spheroidal co-ordinates $\hat{X}$: As $x \to \infty$,

$$
\hat{X} = (1 - y^2) [\mu^2 x^2 + \mathcal{O}(x)],
$$

(65)

$$
\hat{Y} = 2 J y (3 - y^2) + \mathcal{O}(x^{-1}),
$$

(66)

where we have used $J = ma$, $m = \mu/p$, $a = \mu q/p$. In the vicinity of the rotation axis, $y \to \pm 1$, we have

$$
\hat{X} = \mathcal{O}(1 - y^2), \quad (1 - y^2) \frac{\hat{X}_{yy}}{\hat{X}} = \mp 2 + \mathcal{O}(1 - y^2),
$$

(67)

$$
\hat{Y}_{xr} = \mathcal{O}(1 - y^2), \quad \hat{Y}_{yy} = \mathcal{O}(1 - y^2),
$$

(68)

which shows that $\partial_y \hat{X}/\hat{X}$ remains finite, although $\hat{X}$ vanishes on the rotation axis. Note that on the horizon, i.e. for $x = 1$, both functions $\hat{X}$ and $\hat{Y}$ behave perfectly regular,

$$
\hat{X} = \mathcal{O}(1), \quad \hat{X}^{-1} = \mathcal{O}(1),
$$

(69)
\[ \dot{Y}_{,x} = \mathcal{O}(1), \quad \dot{Y}_{,y} = \mathcal{O}(1). \] (70)

The metric (64) with \( h^{(\phi)} = 0 \) was first derived by Kerr in 1963 [22, 32]. It represented the first known asymptotically flat exact solution of a rotating source in general relativity. The 2-parameter family is characterized by the total mass \( m \) and the total angular momentum \( J \). On the basis of the strong rigidity theorem, various authors have contributed to the proof that the Kerr metric describes the only asymptotically flat, stationary vacuum black hole solution (see the introduction for references). In the presence of electromagnetic fields, an equivalent theorem applies to the 3-parameter Kerr-Newman family.

Later, Bunting [21] and Mazur [22] succeeded in improving the classical arguments leading to the uniqueness theorem for the Kerr (-Newman) family. The reasoning relies on the regularity and ellipticity of the 2-dimensional boundary value problem described by the Lagrangian (51) and the boundary conditions (65)-(70). The basic observation consists in the fact that the vacuum and electrovac Ernst equations are non-linear sigma-model equations with symmetric spaces \( SU(1,1)/U(1) \) and \( SU(1,2)/SU(1) \times U(1) \), respectively. In the vacuum case this is seen from the effective Lagrangian (51), which obviously describes a harmonic map from the semi-strip \( S \) (with metric \( \text{diag}(x^2-1, \cdot, 1-y^2, \cdot) \)) into the unit disc \( D \), being (as a symmetric space) equivalent to \( SU(1,1)/U(1) \). Within this framework, Robinson’s identity [20] is recovered as a special case of a divergence relation for non-linear sigma-models [22]. The identity implies that two solutions of the Ernst equations, \( (\tilde{X}_1, \tilde{Y}_1) \) and \( (\tilde{X}_2, \tilde{Y}_2) \), are identical if they are subject to the same boundary conditions (73)-(74).

Applied to the problem under consideration, the uniqueness theorem for the vacuum Kerr metric guarantees that the metric (64), with the functions \( \tilde{X}, \tilde{A} \) and \( \tilde{h}^{(\text{vac})} \) according to eqs. (60)-(62), is the unique asymptotically flat solution of the Einstein equations with harmonic fields. Hence, in order to establish the uniqueness theorem for this system, it remains to prove that the additional metric function \( h^{(\phi)} \) and the matter fields \( \phi^A \) are uniquely determined by eqs. (60), (61), asymptotic flatness and the boundary conditions for \( \phi^A \).

6 Uniqueness of the Matter Solution

Let us now show that the matter equations (40), (41) admit only the trivial solution \( \phi^A = \text{constant}, \) \( h^{(\phi)} = 0 \). This is a consequence of asymptotic flatness, the (weak) asymptotic fall-off conditions

\[ \phi^A = \phi^A_{\infty} + \mathcal{O}(r^{-1}), \quad \phi^A_{,\phi} = \mathcal{O}(r^{-1}), \quad \phi^A_{,r} = \mathcal{O}(r^{-2}), \] (71)

and the requirement that \( G_{AB}(\phi) \) and the derivatives of the scalar fields with respect to \( r \) and \( \vartheta \) remain finite at the boundary of the domain of outer communications. As usual, \( r \) and \( \vartheta \) denote Boyer-Lindquist co-ordinates (64):

\[ r = m(1 + px), \quad \cos \vartheta = y, \] (72)

in terms of which we have \( S = \{(r, \vartheta) \mid r \geq r_h, \vartheta \in [0, \pi]\} \) (where \( r = r_h \) denotes the event horizon).

To start, we note that eq. (49) for \( h^{(\phi)} \) in prolate spheroidal co-ordinates becomes

\[ h^{(\phi)}_{,x} = \frac{G_{AB}}{2} \frac{1-y^2}{x^2-y^2} \left[ x(x^2-1) \phi^A_{,x} \phi^B_{,x} - x(1-y^2)\phi^A_{,y} \phi^B_{,y} - 2y(x^2-1)\phi^A_{,x} \phi^B_{,y} \right], \] (73)

\[ h^{(\phi)}_{,y} = \frac{G_{AB}}{2} \frac{x^2-1}{x^2-y^2} \left[ y(x^2-1) \phi^A_{,x} \phi^B_{,x} - y(1-y^2)\phi^A_{,y} \phi^B_{,y} + 2x(1-y^2)\phi^A_{,x} \phi^B_{,y} \right], \] (74)
from which we also obtain the asymptotic behavior of the derivative of \( h^{(\phi)} \) with respect to \( r \),

\[
h^{(\phi)}_{,r} = \frac{G_{AB}}{2} \left( \sin^2 \vartheta \left( [r + \mathcal{O}(1)] \phi^A_{,r} \phi^B_{,r} - [r^{-1} + \mathcal{O}(r^{-2})] \phi^A_{,\vartheta} \phi^B_{,\vartheta} \right) + \sin(2\vartheta) \left( [1 + \mathcal{O}(r^{-1})] \phi^A_{,r} \phi^B_{,\vartheta} \right) \right).
\]

By virtue of the fall-off conditions for the scalar fields, all terms in the bracket on the r.h.s. of eq. \((75)\) are of \( \mathcal{O}(r^{-3}) \). Hence, the metric function \( h^{(\phi)}(r, \vartheta) \) has the asymptotic property

\[
\lim_{r \to \infty} r^2 h^{(\phi)}_{,r} = 0.
\]

In a similar way one establishes \( \lim_{r \to \infty} r h^{(\phi)}_{,\vartheta} = 0 \). (Note also that asymptotic flatness requires that \( \exp(2h) \)
has an asymptotic expansion in terms of \( 1/r \). Equation \((74)\) then shows that \( \exp(8\pi G 2h^{(\phi)}) = 1 + \mathcal{O}(r^{-2}) \),
from which we find, using the vacuum solution \((72)\) in Boyer-Lindquist coordinates,

\[
\frac{e^{2h^{(\text{vac})}}}{X} e^{8\pi G 2h^{(\phi)}} = \frac{r^2 + a^2 \cos^2 \vartheta}{r^2 - 2mr + m^2 \sin^2 \vartheta + a^2 \cos^2 \vartheta} [1 + \mathcal{O}(r^{-2})] = 1 + \frac{2m}{r} + \mathcal{O}(r^{-2}),
\]

as \( r \to \infty \). This shows that the first two terms in the asymptotic expansion of the metric coefficient of \( dr^2 \) are exactly the same as in the vacuum case.)

We shall now apply Stokes’ theorem for a suitably chosen 2-dimensional vector in order to prove that there exist no regular, non-trivial solutions to eq. \((41)\) with asymptotic behavior \((76)\). Consider the vector

\[
w = \rho \nabla^{(\phi)} e^{-h^{(\phi)}}
\]

in the \((\rho, z)\) plane \( (\nabla^{(\phi)} = (\partial_{\rho}, \partial_z)) \). Stokes’ theorem for \( w \) and the domain \( S \) with counter clockwise oriented boundary \( \partial S \) yields

\[
\oint_{\partial S} \rho e^{-h^{(\phi)}} (h^{(\phi)}_{,z} \rho - h^{(\phi)}_{,\rho} \rho) \, dz = \int_S \rho e^{-h^{(\phi)}} \left[ |\nabla^{(\phi)} h^{(\phi)}|^2 - \frac{1}{\rho} h^{(\phi)}_{,\rho} + \Delta^{(\phi)} h^{(\phi)} \right] \rho \, dz.
\]

The crucial observation consists in the fact that the r.h.s. of this identity is a sum of two non-negative terms, provided that the metric \( G_{AB} \) of the target manifold \( N \) is Riemannian. This is immediately seen from eqs. \((41)\) and \((43)\), which together yield

\[
- \left( h^{(\phi)}_{,\rho} + \rho \Delta^{(\phi)} h^{(\phi)} \right) = \rho G_{AB}(\phi) \phi^A_{,z} \phi^B_{,z} \geq 0.
\]

Our last task is to show that the boundary integral on the l.h.s. of eq. \((73)\) vanishes, which then implies that \( h^{(\phi)} \) is constant. Using prolate spheroidal co-ordinates, we have to demonstrate that \( \lim_{R \to \infty} I_R = 0 \), where

\[
I_R = \mu \oint_{\partial S_R} e^{-h^{(\phi)}} \left[ (1 - y^2) h^{(\phi)}_{,y} \, dx - (x^2 - 1) h_{,x} \, dy \right].
\]

Here the oriented boundary \( \partial S_R \) of the domain of outer communications is the rectangle \( \gamma^1_R = \{ y = 1, x = R \ldots \} \), \( \gamma^2 = \{ x = 1, y = 1 \ldots -1 \} \), \( \gamma^3_R = \{ y = -1, x = 1 \ldots R \} \) and \( \gamma^4_R = \{ x = R, y = -1 \ldots 1 \} \). The finiteness of the Ricci scalar and the regularity of the derivatives of \( h^{(\phi)} \) with respect to Boyer-Lindquist co-ordinates imply that \( h^{(\phi)}_{,x} \), \( h^{(\phi)}_{,y} \) and \( \exp(-h^{(\phi)}) \) remain finite along \( \gamma^1_R \), \( \gamma^2 \) and \( \gamma^3_R \). Hence, both integrals in eq.
vanish along these parts of the boundary. It remains to consider the contribution from the integration along $\gamma_R$ as $R \to \infty$, that is

$$\lim_{R \to \infty} I_R = -\mu \lim_{R \to \infty} \int_{\gamma_R} e^{-h^{(\phi)}(x^2 - 1)} h_{,y} \, dy = -\int_0^{\pi} \lim_{r \to \infty} e^{-h^{(\phi)}(r^2 h_{,r})} \sin \vartheta \, d\vartheta .$$

(82)

This integral vanishes as well, as a consequence of the asymptotic behavior (76) of $r^2 h_{,r}$. Thus, the l.h.s. of eq. (80) is zero, implying that both non-negative integrands on the r.h.s., $|\nabla^{(\phi)} h^{(\phi)}|^2$ and $-\left[ h^{(\phi)}_{, \rho} + \rho \Delta h^{(\phi)} \right]$, vanish. Hence, $h^{(\phi)}$ is constant in the entire domain of outer communications and, since $\lim_{r \to \infty} h^{(\phi)} = 0$, we finally have

$$h^{(\phi)} = 0 ,$$

(83)

and the metric (64) reduces to the ordinary Kerr metric. Eventually $h^{(\phi)} = 0$ implies that $\phi$ is a constant map.

To summarize, we have shown that the uniqueness result previously obtained in [45] for the non-rotating case generalizes to rotating black holes:

Let $(M, g)$ be an asymptotically flat, stationary and axisymmetric spacetime with Killing fields $k$ and $m$. Let $\phi$ denote a mapping with harmonic action from $(M, g)$ into a Riemannian manifold $(N, G)$, $\phi$ being invariant under the action of $k$ and $m$. Then the only stationary and axisymmetric, asymptotically flat black hole solution of the coupled Einstein-matter equations with regular event horizon consists of the Kerr metric and a constant map $\phi_0$.

To conclude, we note that this result, combined with the corresponding theorem for non-rotating black holes mentioned above and the strong rigidity theorem, implies that the Kerr metric is the unique stationary, asymptotically flat black hole solution of self-gravitating harmonic mappings with Riemannian target manifolds. It is also worth pointing out that the generalization of this result to the case where electromagnetic fields are taken into account as well is straightforward. In this case, the analogous no-hair theorem applies to the Kerr-Newman metric, being the unique electrovac black hole solution with selfgravitating harmonic mappings.

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