Electronic spectral function in fractionalized Pair Density Wave scenario

M. Grandadam, D. Chakraborty, X. Montiel, and C. Pépin

1Institut de Physique Théorique, Université Paris-Saclay, CEA, CNRS, F-91191 Gif-sur-Yvette, France
2Department of Physics and Astronomy, Uppsala University, Box 516, S-751 20 Uppsala, Sweden
3Department of Physics, Royal Holloway, University of London, Egham Surrey, United Kingdom

Studies of the electronic spectral function in cuprates by Angle-Resolved Photo-Emission Spectroscopy reveal unusual features in the pseudogap phase that persist in the superconducting phase. We address here these observations based on the recently proposed idea that the pseudogap is due to the fractionalization of modulated particle-particle pairs (a Pair Density Wave) into uniform particle-particle and modulated particle-hole pairs. The constraint that appears between these two types of pairs can be seen as an amplitude for the pseudogap.

Despite their discovery more than 30 years ago, there is still no consensus on the nature of the pseudogap (PG) phase of cuprate superconductors. This mysterious phase that emerges upon doping the parent Mott insulator shows many unusual features, the main one being a loss of density of states as the temperature is decreased below a temperature $T^*$. In this study, we discuss a recent proposal where the opening of the pseudogap is attributed to the fractionalization of a Pair Density Wave (PDW)\(^3\). Within this framework, we discuss the complex phenomenology of Angle-Resolved Photo-Emission spectroscopy (ARPES) in nearly optimally doped Pb\(_{0.55}\)Bi\(_{1.5}\)Sr\(_{1.6}\)La\(_{0.4}\)CuO\(_6+\delta\) (Bi2201). ARPES has proven to be one of the key probes in studying cuprates. It has notably been able to connect the loss of density of states in the pseudogap to the Fermi surface being gapped out in the antinodal region (ANR) for momenta close to $(0, \pm \pi)$, $(\pm \pi, 0)$, while other parts of the Fermi surface remain unchanged and form ‘Fermi arcs’ (For a review of the pseudogap phenomenology see eg.\(^5\)).

Most remarkably one observes, in the ANR, a back-bending at a momentum $k_F$ larger than the normal state Fermi momentum $k_F$, which suggested the presence of a modulation vector intimately linked with the opening of the pseudogap.\(^6\) Moreover, as noticed in Ref 7, the gap continuously closes from ‘below’ when moving towards the Fermi arcs, which is interpreted as revealing the presence of particle-particle pairs in the PG. Lastly, one observes that the bottom of the band at $k = (0, \pi)$ drops when decreasing the temperature and a new lightly dispersive ‘flat’ band is seen in superconducting phase. Few other scenarios have been proposed to explain this specific gapping mechanism such as a quantum disorder PDW,\(^7\) a coexistence of Charge Density Wave (CDW) and PDW or a Resonant Excitonic State (RES)\(^10\). In comparison to previous works, our study is based on a simple intuition, and accounts for all the experimental features with very few adjusting parameters.

This paper is organized as follows. We start with a self-consistent equation for the pseudogap amplitude by treating a model of itinerant electrons interacting through antiferromagnetic exchange and residual density-density interaction at the mean-field level. The pseudogap can be seen as a superposition of superconductivity (SC) and CDW orders resulting in a composite order which has a non-zero amplitude in the ANR leading to the formation of Fermi arcs. The electronic spectral function in the ANR shows all the specific features mentioned previously, namely we obtain a back-bending of the electronic dispersion, a ‘flat’ band and the gap closing from below when going closer to the center of the Brillouin zone. The effects of temperature on the spectral function close to $k = (\pi, 0)$ and on the spectral weight of the flat band are also discussed for the first time based on phenomenological arguments.

The fractionalization of a PDW order is written in a way resembling the fractionalization of the electron introduced in strong coupling theories\(^11-13\). A PDW breaks both charge conservation and translation symmetry, thus we write it as a product of a SC, breaking charge conservation, and a CDW, breaking translation symmetry. The PDW, SC and CDW fields are linked by the constraint:

$$\phi_{ij}^Q = \Delta_{ij} \chi_{ij}^Q e^{i2\pi}$$

$$|\Delta_{ij}|^2 + |\chi_{ij}^Q|^2 = (E^*)^2,$$

where $\phi_{ij}^Q$, $\Delta_{ij}$ and $\chi_{ij}^Q$ are fields describing a PDW, a SC and a CDW order respectively. We will show that the constraint associated with the fractionalization of Eq.\(^1\) will affect the electronic spectral function as soon as it is enforced below $T^*$ and even if the two underlying orders (SC or CDW) do not develop a long range component and have amplitude and phase fluctuations. The pseudogap scale is given by the value of the constraint of Eq.\(^1\) and introduced in our mean-field description by the finite amplitude of a doublet $\Psi_{ij} = \frac{1}{\sqrt{2}} (\Delta_{ij}^*, \chi_{ij}^Q)$. Indeed, we will have $2 |\Psi_{ij}|^2 = |\Delta_{ij}|^2 + |\chi_{ij}^Q|^2 = (E^*)^2$. To describe this within a microscopic model we start with electrons hopping on a square lattice interacting via a spin-spin interaction and a small off-site
density-density interaction:

\[ H = - \sum_{i,j,\sigma} t_{ij} \left( c_{i,\sigma}^\dagger c_{j,\sigma} + h.c. \right) + J_{ij} S_i \cdot S_j + V_{ij} n_i n_j, \tag{2} \]

where \( c_{i,\sigma}^\dagger \) is the creation operator for an electron with spin \( \sigma \) on a site \( i \), \( t_{ij} \) describe hopping between different sites and are taken from a fit to ARPES data\(^{10}\), \( J_{ij} \) is the short-range spin-spin interaction and \( V_{ij} \) is the off-site density-density interaction. The exact form of \( J_{ij} \) and \( V_{ij} \) is not important and we only require that \( J_{ij} \) is positive and peaked around \((\pi, \pi)\) in momentum space, i.e. antiferromagnetic by nature and that \( V_{ij} \) is smaller than \( J_{ij} \).

We treat this model in momentum space and start by decoupling the interaction with the individual fields forming the doublet. Here these fields are a pairing field \( \chi^Q \) with uniaxial modulation vectors \( Q = \pm Q_x; \pm Q_y = (\pm Q_0, 0); (0, \pm Q_0) \) shown in Fig.1(a). The action is then given by

\[
S_{\text{micro}} = S_0 + S_\Delta + S_\chi,
\]

\[
S_0 = \int d\tau \sum_{k,\sigma} c_{k,\sigma} \left( i \partial_\tau - \xi_k \right) c_{k,\sigma}^\dagger,
\]

\[
S_\Delta = \int d\tau d\sigma \sum_{q,k} \left[ \frac{\Delta^*_{k+q} \Delta_{k+q}}{J_+ (q)} + \Delta^*_{k-q,\sigma} c_{k-q,\sigma} c_{k+q,\sigma} + h.c. \right],
\]

\[
S_\chi = \int d\tau \sum_{k,\sigma} \left[ \chi_{k+q,\sigma} c_{k+q,\sigma} c_{k+q,\sigma} + h.c. \right] \tag{3}
\]

where \( \xi_k \) is the non-interacting electronic dispersion and \( J_{\pm} = 3J \pm V \). In the pseudogap phase, none of these fields condense and thus their individual amplitudes vanish at the mean-field level. We perform the integration of the electronic degree of freedom in \( S_{\text{micro}} \) and write the result as a function of the doublet \( \Psi^Q_k = \frac{1}{\sqrt{2}} \left( \Delta^*_{k}, \chi^Q_{k} \right) \) whose amplitude will be non-zero\(^{14}\) yielding the effective action:

\[
S_{\text{eff}} = \int d\tau \sum_{q,k,\sigma} \frac{\Psi^Q_k (q)}{J (q)} - \text{Tr} \ln (G^{-1} (i\omega, k)), \tag{4}
\]

\[
G^{-1} (i\omega, k) = i\omega - \xi_k - \sum_{Q \in \pm Q_x, \pm Q_y} |\Psi^Q_k|^2 G^Q (i\omega, k), \tag{5}
\]

where \( G^Q (i\omega, k) = (i\omega - \xi_{k+Q})^{-1} + (i\omega + \xi_k)^{-1} \) and \( 1/J = (1/J_+ + 1/J_-) \). The electronic Green function in Eq.(5) will thus be modified if \( |\Psi^Q_k|^2 \) acquires a non zero value even if all the composing fields are fluctuating and have a vanishing expectation value. We can also note that the self-energy will have poles which are not symmetric with respect to the Fermi level due to the modulation wave-vector in \( G^Q \).

Minimizing \( S_{\text{eff}} \) in Eq.(4) with respect to the doublet \( \Psi^Q_k \) gives the mean-field gap equation for the doublet amplitude. We will consider the different modulation wave-vectors to be decoupled and use the fact that \( \tilde{J}_q \) is peaked around \( q = (\pi, \pi) \) to restrict the momentum summation to a range \( q_{\text{AF}} \) around the antiferromagnetic wave-vector. The parameter \( q_{\text{AF}} \) is physically associated with the short-range nature of the antiferromagnetic fluctuations\(^{15}\) mediating the interaction and goes to zero when we approach the antiferromagnetic phase at low doping. Assuming that \( |\Psi^Q_k|^2 \) is constant over a range \( q_{\text{AF}} \), the self-consistent equation is of the BCS form:

\[
|\Psi^Q_k|^2 = \frac{T \tilde{J}}{N} \sum_{i\omega_n, q} \frac{|\Psi^Q_{k+\pi}|}{(i\omega_n)^2 - \xi_{k+q}^2} \frac{|\Psi^Q_{k+\pi}|}{|\Psi^Q_{k+\pi}|^2} \tag{6}
\]

where \( \tilde{J} \) range between \( \pi - q_{\text{AF}}/2 \) and \( \pi + q_{\text{AF}}/2 \). Ignoring frequency dependence of \( |\Psi^Q_k|^2 \) we can perform the Matsubara summation analytically. This leads to two coupled equation between \( |\Psi^Q_k|^2 \) and \( |\Psi^Q_{k+\pi}|^2 \) which we can solve self-consistently.

Results of this self-consistent equation are shown in Fig.1(b) for a modulation wave-vector linking hot-spots along the \( x \) axis. Hot-spots are points of the Fermi surface linked
by $(\pi, \pi)$ and are thus expected to be important due to the form of our interaction. Due to the finite wave-vector of the pseudogap amplitude, the gap equation Eq.(6) admits non-zero solution only in the ANR, when this modulation vector links two parts of the Fermi surface. The region close to the Brillouin zone diagonal will thus remain unperturbed by the transition at $T^*$. The electronic spectral function $A(\omega, k) = -\frac{1}{\pi} \text{Im} \left( G(\omega + i\eta, k) \right)$ for $\omega = 0$ and $\eta \to 0^+$ shows that the ANR is gapped while the nodal region forms Fermi arcs (Fig.1(c)). These arcs terminate close to the hot-spots. We now look at the reconstructed band structure obtained from the zeros of $G^{-1}(k, \omega)$ in the ANR. From the form of $\tilde{G}(k, \omega)$, we can understand the dispersion as coming from an equal superposition of SC and CDW order in the ANR. We can thus construct the resulting band structure in the pseudogap as the hybridization of the three bands $\xi_x$, the normal state dispersion, $-\xi_k$ coming from the superconducting order and $\xi_{k+Q}$ coming from the modulating order. At the zone boundary ($k_x = \pi$), this results in two bands below the Fermi level shown in Fig.1(d) with one of them presenting a back-bending (blue line) indicated by black arrows while the other one (yellow line) present little dispersion around $k_y = 0$. In our mean-field description, this back-bending appears as a result of the hybridization between the hole band $-\xi_k$ (green dotted) and the shifted $\xi_{k+Q}$ (blue dotted) band in Fig1(d). As such this back-bending will occur at $k_y = k_G > k_F$ as long as $\xi_{k+Q} < \xi_k$ which is satisfied for all $k_x > k_{\text{hot-spot}}$. This means that this anomalous back-bending will persist below $T_c$ in the ANR but we will recover a standard back-bending at $k_y = k_F$ in the nodal region as the above condition is not satisfied.

The spectral weight $A(k, \omega)$ for each band is obtained for different fixed values of $k_x = \pi - \delta k_x$ and compared to the experimental dispersion of Ref.6 (Fig.2(a)-(d)). As we get closer to the centre of the Brillouin zone we can see that the energy of the maximum of the band gets closer to the Fermi level leading to the pseudogap closing ‘from below’ (Fig.2(e)-(h)) as observed experimentally. Note that we obtain here a gap closing ‘from below’ contrary to what was argued previously for a pure CDW scenario with a modulation along the $y$ direction. This is because we consider a modulation wave-vector along the $x$ direction. This same orientation for the modulation wave-vector has been used recently to explain ARPES results in Bi2201 through the idea of a quantum disorder PDW and other theoretical approaches such as a superposition of CDW and PDW order or a RES.

Our previous description of the band structure in the pseudogap also shows a second band located at the bottom of the non-interacting band. We connect here this band to the flat band observed experimentally below $T_c$ (green dots in Fig.2(a)-c)) and argue that finite lifetimes for the single-particle and pair excitations lead to this band not being observed above $T_c$. For this, we add three phenomenological damping rates $\Gamma_0$, $\Gamma_1$ and $\Gamma_2$ in our mean-field Green's function:

$$G^{-1}(i\omega, k) = i\omega - \xi_k - i\Gamma_0 - \sum_{Q \in \pm Q_x, \pm Q_y} |\tilde{\Psi}_k^Q|^2 \tilde{G}^Q(i\omega, k),$$

$$\tilde{G}^Q(i\omega, k) = (i\omega - \xi_{k+Q} + i\Gamma_1)^{-1} + (i\omega + \xi_k + i\Gamma_2)^{-1}. \tag{7}$$

The two factors $\Gamma_1$ and $\Gamma_2$ represent the lifetime of particle-hole and particle-particle pairs respectively. They are expected to be non-zero above $T_c$ but to vanish at the transition temperature when fluctuations are quenched. The first $\Gamma_0$ term is a single-particle lifetime which is always non-zero. The effect of each of these additional terms is depicted in Fig.3(a)-(d). Allowing a non-zero $\Gamma_0$ will broaden the two bands below the Fermi level in similar ways (Fig.3(b)), in contrast to the pair lifetimes which have a very different effect on specific parts of the dispersion. Indeed, Fig.3(c) shows that a non-zero $\Gamma_1$ will strongly suppress the ‘flat’ band close to the Fermi level and also dampen the main band close to $k_y = 0$. Turning on the $\Gamma_2$ term will have the opposite effect as the band far from $k_y = 0$ gets dampened while the bottom of the flat and main bands remain well defined. The experimental observation of the flat band only close or below $T_c$ can then be mainly attributed to the presence of a particle-hole pair lifetime from the particle-hole part of our pseudogap. Note also that due to disorder effect, which couple directly to charge order, this lifetime could remain non-zero below $T_c$ and thus leads to the flat band remaining very broad even in the superconducting state as observed experimentally. Moreover, this description provides a good agreement with the experimental observation that the dispersion in the ANR does not change across the superconducting transition. In our case, the position of the main band does not change with temperature and only the spectral weights of the two bands get modified as the different life-
the zone edge $k_x = \pi$ for different $k_y$ between $-\pi/4$ and $\pi/4$, successive lines are shifted for clarity. (a) without any lifetime $\Gamma_0 = \Gamma_1 = \Gamma_2 = 0$. We still used a broadening $\eta = 0.002$ eV for numerical purposes. (b) Turning on a single-particle lifetime $\Gamma_0 = 0.02$ eV leads to both bands being broadened in a similar way. (c) In contrast, when we consider only a finite particle-hole lifetime $\Gamma_1 = 0.02$ eV we see that the dispersion close to $k_y = 0$ is more strongly affected than the dispersion at higher momenta. The flat band is also more strongly damped than the main band. (d) The situation is reversed if we consider only a particle-particle lifetime $\Gamma_2 = 0.02$ eV. The parts of the bands close to $k_y = 0$ are less affected and still well defined. The flat band is again more broadened but remains visible.

Another particular feature of the temperature dependence measured experimentally for $T^* > T > T_c$ in the ANR is a significant decrease of the energy of bottom of the band when the temperature is decreased while the maximum energy and the back-bending wave-vector change only slightly as shown in Fig.4(a). We describe here this change in the band structure by adding a finite amplitude for the particle-hole order parameter $|\chi_k^Q|$. We then have three different regions such that at $T > T^*$ we have free electrons. At $T \lesssim T^*$ where the pseudogap has a finite amplitude but the particle-hole gap is still 0 and at $T \gtrsim T_c$ where the particle-hole gap is finite. We then obtain the band dispersions shown in Fig.4(b). Because the bottom of the band is directly related to the hybridization with the band coming from the charge modulation, it is directly affected by the non-zero value of $|\chi_k^Q|$. On the other hand, the back-bending momentum is determined mainly by the value of the modulation wave-vector $Q$ and the energy of the maximum comes from the hybridization between the superconducting band $-\xi_k$ and the shifted band $\xi_{k+Q}$, thus related to the value of $|\psi_k^Q|$. In contrast, a finite amplitude for the SC order parameter $|\Delta_k|$ would produce the opposite effect and change substantially the position of the maximum leaving the bottom of the band unchanged\(^{15}\). Note that there are two distinct phenomena: a phase transition occurs at $T^*$, opening a gap in the electronic spectra, while the finite amplitude for the particle-hole pairs is a crossover regime which only affects details of the spectrum. The fact that $|\chi_k^Q|$ acquire a quasi long-range component before the $|\Delta_k|$ is representative to the fact that CDW is observed experimentally at a temperature higher than the temperature for SC fluctuations $T_c$. This long-range component of the CDW order has also been observed by Raman spectroscopy\(^{20}\), X-ray\(^{21-26}\) and NMR\(^{27-30}\) measurements above $T_c$.

In conclusion, we showed here how the recently proposed idea of fractionalized PDW\(^{17}\) can be used to construct a mean-field description of the pseudogap phase of cuprates. The main idea is that even if none of CDW or SC orders develop a long range-component, the constraint introduced by the fractionalization affects the electronic Green’s function. Using a microscopic model we derived a self-consistent equation for the pseudogap amplitude and showed that it has non-zero solutions in the antinodal region, which gives a gap in the ANR in the PG phase and leads to the formation of Fermi arcs. The band dispersion obtained in the antinodal region is in good agreement with the experimental ARPES measurement made on Bi2201. Especially, we recover all the features observed in the pseudogap state. The superposition of particle-particle and particle-hole orders leads to an anomalous back-bending of the main band below the Fermi level, a gap closing ‘from below’ and a ‘flat band’ close to the bottom of the original electronic dispersion. We argue that this band is seen experimentally only below $T_c$ because it is strongly affected by the finite pair lifetime in the pseudogap.
phase. Lastly, we discussed the change of the dispersion as the temperature is lowered from $T^*$ to $T_c$ by showing that a finite quasi long-range component of the particle-hole order leads to the minimum of the band going down in energy while the energy and momentum of the maximum stay unchanged.

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Supplementary material for: Electronic spectral function in fractionalized Pair Density Wave scenario

I. DERIVATION OF THE EFFECTIVE ACTION FOR THE DOUBLET $\Psi^Q_k$

We present here the derivation to obtain the path integral representation Eq.(4) starting for the real space Hamiltonian Eq.(2) of the main text. Our starting point is thus:

$$ H = - \sum_{i,j,\sigma} t_{ij} \left( c_{i\sigma}^\dagger c_{j\sigma} + h.c. \right) + J_{ij} S_i^c S_j^c + V_{ij} n_i n_j. \quad (S1) $$

We start by going into momentum space by performing a Fourier transform of the electronic operator, the Hamiltonian becomes:

$$ H = \sum_k \xi_k c_k^\dagger c_k + \sum_{k,k',\sigma} \left( J_\sigma \sigma_{\alpha\mu} \cdot \sigma_{\beta\mu} + V\delta_{\alpha\mu}\delta_{\beta\gamma} \right) c_{k\alpha}^\dagger c_{k+q,\beta} c_{k\mu} c_{k'\gamma} c_{k'\lambda}, \quad (S2) $$

where summation over the spin indices is implicit and $\sigma$ is the vector of Pauli matrices. We can build from this Hamiltonian a path integral representation:

$$ S = \int d\tau \sum_k c_k^\dagger (i\partial_\tau - \xi_k) c_k - \sum_{k,k'} \left( J_\sigma \sigma_{\alpha\mu} \cdot \sigma_{\beta\mu} + V\delta_{\alpha\mu}\delta_{\beta\gamma} \right) c_{k\alpha}^\dagger c_{k+q,\beta} c_{k\mu} c_{k'\gamma} c_{k'\lambda}. \quad (S3) $$

We can now introduce two bosonic fields defined as $\Delta_{k,k'} = \langle \sum_{\sigma} \sigma_{c_{k\sigma} c_{k'-\sigma}} \rangle$ and $\chi_{k,k'} = \langle \sum_{\sigma} c_{k\sigma}^\dagger c_{k'\sigma} \rangle$ and after limiting the value of $k' = -k$ for $\Delta_{k,k'}$ and $k' = k + Q$ for $\chi_{k,k'}$ we find back the action given in the main text:

$$ S = S_0 + S_\Delta + S_\chi, \quad (S4) $$

$$ S_0 = \int d\tau \sum_k c_k^\dagger (i\partial_\tau - \xi_k) c_k, \quad (S5) $$

$$ S_\Delta = \int d\tau \sum_{k\sigma} \Delta_{k,k'}^\dagger c_{k\sigma}^\dagger c_{-k\sigma} + \frac{\Delta_k \Delta_{k'}^*}{J_q}, \quad (S6) $$

$$ S_\chi = \sum_{Q \neq Q_{x,y}} \int d\tau \sum_k \chi_k Q_{c_{k+Q\sigma} c_{k\sigma}} + h.c. + \frac{\chi_k \chi^*_k}{J_q}, \quad (S7) $$

Integrating the fermionic degree of freedom leads to an effective action for the bosonic fields in Matsubara frequency:

$$ S = \sum_k \frac{\Delta_k \Delta_{k'}^*}{J_q} + \frac{\chi_k \chi^*_k}{J_q} - Tr \ln \left( i\omega_n - \xi_k - \frac{|\Delta_k|^2}{i\omega_n + \xi_k} - \frac{|\chi_k|^2}{i\omega_n - \xi_k + Q} \right). \quad (S8) $$

We can now rewrite this effective action in terms of the doublet field $\Psi^Q_k = \frac{1}{\sqrt{2}} \left( \Delta_k, \chi_k^Q \right)$ by using the identity $a|\Delta_k|^2 + b|\chi_k|^2 = (a+b)|\Psi^Q_k|^2 + (a-b) |\Psi^Q_k|^2$. We then use our ansatz for the pseudogap that the $|\Psi^Q_k|$ acquire a finite amplitude but not $\Delta_k$ or $\chi_k$ and only keep the terms proportional to $|\Psi^Q_k|^2$. This gives us the effective action Eq.4 of the main text:

$$ S_{eff} = \int d\tau \sum_{Q,k,\sigma} \frac{\Psi^Q_k \Psi^Q_{k'}}{J_q} - Tr \ln \left( G^{-1}(i\omega_n, k) \right). \quad (S9) $$

II. COMPARISON WITH OTHER SCENARIO BASED ON SC OR CDW ORDER

We discuss here how the presence of a back-bending $k_G > k_F$ and the observation that the gap close from below when going towards the centre of the Brillouin zone are signatures for the presence of both particle-particle and modulated particle-hole pairs. In the following, we look at different origins for the gap in the ANR that lead only to one of the two features mentioned previously.

1. Case of SC order

We start by reminding what is the consequence of a superconducting order on the electronic spectral function. For this, we consider a Green’s function of the form

$$ G_{SC}(k,i\omega_n) = \frac{1}{i\omega_n - \xi_k - |\Delta|^2} \left( i\omega_n + \xi_k \right)^{-1}. \quad (S10) $$

Fig.S1(a)-(d) shows the associated spectral function for different cut at fixed $k_x = \pi - \delta k_x$. We see that the gap is closing from below as we go away from the edge of the Brillouin zone, however the back-bending of the band below the Fermi level is locked at $k_G = k_F$ due to particle-hole symmetry. This is in contrast with the experimental observation in Ref.6 and thus the pseudogap cannot be due solely to fluctuations of a superconducting order.

2. Case of CDW order

It has been argued that a pure CDW order cannot account for the experimental features observed in the ANR. We reproduce here the argument and show that a pure CDW order leads to a back-bending shifted from the original Fermi momentum but with a gap closing from above as we go to the centre of the Brillouin zone. We thus take a Green function of the form

$$ G_{CDW}(k,i\omega_n) = \frac{1}{i\omega_n - \xi_k - |\chi^Q|^2} \left( i\omega_n + \xi_k + Q \right)^{-1}. \quad (S11) $$
To be consistent with the experimental observations, we need to choose a modulation wave-vector along one of the crystal axes. As taking $Q$ along the $y$ axis will not lead to any gap in the ANR close to $k = (\pi/\sigma_y, 0)$ we take $Q = (\pm 0.2, 0) \pi$. This effectively leads to a gap at the Fermi level as shown in Fig.S2(c)-(h) with a back-bending of the occupied band at $k_{FI} > k_F$. However, we see that as we go closer to the center of the Brillouin zone, the gap is closing because the unoccupied band is going closer to the Fermi level. This result in a leftover gap for negative energy once $k_x > 0.2\pi$.

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3. Coexistence of SC and CDW order

Both of the previous examples do not recover the two main experimental features seen in ARPES in the ANR. Superconducting order gives a gap closing from below while a charge modulation leads to a shifted back-bending. We could hope that a coexistence of these order will bring the best of both worlds and allow for a description of the experiments. This is not the case however if we keep a modulation wave-vector along the $x$ axis as is shown in Fig.S1(i)-(l) where we considered a superposition of SC and CDW order along $x$. The Green’s function in this case is the same as the one considered in the main text:

$$G^{-1}(i\omega,k) = i\omega - \xi_k - \sum_{Q \in \pm Q_x, \pm Q_y} \left| \Psi^Q_k \right|^2 \tilde{G}^Q(i\omega,k),$$

(S12)

$$\tilde{G}^Q(i\omega,k) = (i\omega - \xi_{k+Q})^{-1} + (i\omega + \xi_k)^{-1}.$$  

(S13)

At the Brillouin zone edge, there is no crossing between the band from the superconducting order $-\xi_k$ and the band coming from the charge order $\xi_{k+Q}$. Because of this the band below the Fermi level has a back-bending at $k_{FI} = k_F$ as in the superconducting case. The presence of charge order along the $x$ axis also leads to the presence of a gap below the Fermi level for $k_x > 0.2\pi$. Moreover, we do not obtain the formation of a flat band at the bottom of the original dispersion. We will thus need another mechanism to explain the observation of this band below $T^*$. We can also note that all of these scenarios lead to a dispersion with a minimum close to the non-interacting band in contrast to what is observed experimentally (Fig.2 in the main text). In conclusion, despite not opening a gap by itself, choosing a modulation wave-vector along the $y$ direction coexisting with the SC order is the only possibility matching the experimental observation as shown in the main text (Fig.2).

III. EFFECT OF SUPERCONDUCTING FLUCTUATIONS

In order to explain the temperature dependence of the dispersion between $T_c$ and $T^*$ we used the fact that CDW amplitude could acquire a long-range component thus modifying the electronic spectral function. Here we show that using the same argument with superconducting fluctuation leads to
a very different result. The analogue of Fig.4 in the main text is shown in Fig.S2(a) where the red is the high-temperature dispersion \( (T > T^*) \) in the normal state, the orange line is the dispersion in the pseudogap for \( T \lesssim T^* \) and the blue line is the dispersion for \( T \gtrsim T_c \) if we consider long-range component for the superconducting order \(|\Delta_k|\). We can see that a long-range component for the SC amplitude has an opposite effect than the long-range CDW amplitude. Namely, it induces a change in the gap by lowering the energy of the maximum of the band without affecting the energy of the bottom of the band. This is not in agreement with the experimental changes showed in Fig.S2(b).