Unifying the Dropout Family Through Structured Shrinkage Priors

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Abstract

Dropout regularization of deep neural networks has been a mysterious yet effective tool to prevent overfitting. Explanations for its success range from the prevention of "co-adapted" weights to it being a form of cheap Bayesian inference. We propose a novel framework for understanding multiplicative noise in neural networks, considering continuous distributions as well as Bernoulli (i.e. dropout). We show that multiplicative noise induces structured shrinkage priors on a network’s weights. We derive the equivalence through reparametrization properties of scale mixtures and not via any approximation. Given the equivalence, we then show that dropout’s usual Monte Carlo training objective approximates marginal MAP estimation. We analyze this MAP objective under strong shrinkage, showing the expanded parametrization (i.e. likelihood noise) is more stable than a hierarchical representation. Lastly, we derive analogous priors for ResNets, RNNs, and CNNs and reveal their equivalent implementation as noise.

1 INTRODUCTION

Dropout regularization (Hinton et al., 2012; Srivastava et al., 2014) has become an essential tool for fitting large neural networks. Due to its success, a vast number of variants has been proposed (Wan et al., 2013; Wang and Manning, 2013; Huang et al., 2016; Singh et al., 2016; Achille and Soatto, 2018), including ones for recurrent (Ji et al., 2016; Krueger et al., 2017; Gal and Ghahramani, 2016a; Zolna et al., 2018) and convolutional (Tompson et al., 2015; Gal and Ghahramani, 2016b) architectures. The narratives attempting to explain dropout’s inner-workings and success are also plentiful. To give a few examples, Srivastava et al. (2014) argue it prevents "conspiracies" between hidden units, Hinton et al. (2012) claim it performs a role similar to sex for evolution, Baldi and Sadowski (2013) show it ensembles sub-models, Wager et al. (2013) explain it as an adaptive ridge penalty, and Gal and Ghahramani (2016c) suggest dropout performs quasi-Bayesian uncertainty estimation. While some prior work has shown strict equivalences for simple models such as linear regression (Baldi and Sadowski, 2013; Wager et al., 2013, 2014; Helmbold and Long, 2015), the general case of dropout in deep neural networks is analytically intractable, which is likely why no one narrative has come to prominence.

In this paper we propose a novel Bayesian interpretation of regularization via multiplicative noise—with dropout being the special case of Bernoulli noise. Unlike previous frameworks, our method of analysis works through reparametrizations that are agnostic to network architecture or depth. By assuming nothing more than a Gaussian prior (which could be diffuse) on the weights, we show that multiplicative noise induces (marginally) a Gaussian scale mixture. This result is exact and has been exploited previously in Bayesian modeling (Kuo and Mallick, 1998). We then show that dropout’s Monte Carlo objective is a lower bound on this model’s marginal MAP objective. Decoupling dropout’s model from inference is a novel and useful contribution as previous Bayesian interpretations have been grounded in variational inference (Gal and Ghahramani, 2016c; Kingma et al., 2015) and hence gave no guidance on how it could be used in conjunction with Markov chain Monte Carlo (MCMC). Additionally, we find our framework to be prescriptive: the structured priors derived for feedforward networks can be naturally extended to other architectures such as residual, recurrent, and convolutional networks. These priors can be thought of as extensions of the automatic relevance determination

Preliminary work.
(ARD) priors of Neal (1994) and MacKay (1994), allowing the model to self-select other components such as the number of layers and convolutional filters.

2 BACKGROUND

We use the following notation throughout the paper. Matrices are denoted with upper-case and bold letters (e.g. X), vectors with lower-case and bold (e.g. x), and scalars with no bolding (e.g. x or X). Data are assumed to be row vectors x ∈ ℝ^D, and N independently and identically distributed observations constitute the empirical data set X = \{x_1, ..., x_N\}. We focus on supervised learning tasks in which X are covariates (features) that are predictive of another variable y = \{y_1, ..., y_N\}, which we assume is a one-dimensional regression response or index denoting a class label. Throughout we use r to index the rows of a matrix and j to index its columns.

Neural Networks Neural networks (NNs) (Goodfellow et al., 2016) are adaptive basis function regressors, and we define an L-layer NN recursively as

\[
\begin{align*}
E[y_n|x_n] &= g^{-1}(h_{n,L}W_{L+1} + b_{L+1}) \\
h_{n,L} &= f(h_{n,L-1}W_L + b_L), \quad h_{n,0} = x_n
\end{align*}
\]  

(1)

where \( g(\cdot) \) is a link function following the GLM framework (Nelder and Baker, 2004). \( \{W_1, ..., W_L, ..., W_{L+1}\} \) and \( \{b_1, ..., b_L, ..., b_{L+1}\} \) are sets of \( D_{l-1} \times D_l \)-dimensional matrices and \( D_l \)-dimensional vectors, respectively that comprise the NN’s parameters. The function \( f(\cdot) \) acts element-wise and is known as the activation function. From here forward, we drop the bias terms to reduce notational clutter as they can be subsumed into the weight matrices.

Multiplicative Noise Regularization (Dropout) Dropout training (Hinton et al., 2012; Srivastava et al., 2014) introduces multiplicative noise (MN) into the hidden layer computation defined in Equation 1:

\[
h_{n,l} = f(h_{n,l-1}Λ_lW_l)
\]

(2)

where \( Λ_l \) is a diagonal \( D_{l-1} \times D_{l-1} \)-dimensional matrix of random variables \( λ_{j,j} \) drawn independently from a noise distribution \( p(\lambda) \). Dropout corresponds to \( p(\lambda) \) being Bernoulli. However, other noise distributions such as Gaussian (Srivastava et al., 2014; Shen et al., 2018), Beta (Tomczak, 2013), and uniform (Shen et al., 2018) have been shown to be equally effective.

Training under MN is done by sampling a new \( Λ_l \) matrix for every forward propagation. This sampling can be viewed as Monte Carlo (MC) integration over the noise distribution, and therefore, the MN optimization objective is to maximize w.r.t \( \{W_i\}_{i=1}^{L+1} \) the function

\[
\begin{align*}
\mathcal{L}_{\text{MN}}&\{\{W_i\}_{i=1}^{L+1}\} \\
&= \mathbb{E}_{p(\lambda)}[\log p(y|X, \{W_i\}_{i=1}^{L+1}, \{Λ_i\}_{i=1}^{L})] \\
&\approx \frac{1}{S} \sum_{s=1}^{S} \log p(y|X, \{W_i\}_{i=1}^{L+1}, \{Λ_{l,s}\}_{l=1}^{L})
\end{align*}
\]

(3)

where the expectation is taken with respect to \( p(\lambda) \) and \( Λ_{l,s} \) denotes the \( s \)th set of samples for the \( l \)th layer.

Automatic Relevance Determination Automatic relevance determination (ARD) (MacKay, 1994; Neal, 1994) is a Bayesian regularization framework that consists of placing Gaussian priors on the NN’s weights and then structured hyper-priors on the Gaussian scales. The scales of the weights in the same row are tied so they grow or shrink together. This is a form of group regularization: all weights within the same row must be shrunk together if any are shrunk. The end result is feature / hidden unit selection since if all of a unit’s outgoing weights are near zero, then the unit is inconsequential to the model output. We can write the ARD prior as

\[
w_{i,r,j} \sim N(0, σ_{i,r}^2), \quad σ_{i,r} \sim p(σ)
\]

(4)

where \( l \) is the index on layers, \( r \) is the index on rows in the weight matrix, and \( j \) is the index on its columns. Writing \( σ_{i,r} \) without a column index signifies that all of the weights in the \( r \)th row share the same scale.

3 MULTIPLICATIVE NOISE AS AUTOMATIC RELEVANCE DETERMINATION

We now move on to the first contribution: showing that regularization via MN induces, under mild assumptions, an ARD prior. The key observation is that if we assume the weights to be Gaussian random variables, the product \( Λ_lW_l \) defines a Gaussian scale mixture with ARD structure. We then go on to show how the MC training objective in Equation 3 can be derived from this framework.

Gaussian Scale Mixtures A random variable \( θ \) is a Gaussian scale mixture (GSM) if and only if it can be expressed as the product of a Gaussian random variable--call it \( z \)--with zero mean and some variance \( σ_0^2 \) and an independent scalar random variable \( α \) (Beale and Mallows, 1959; Andrews and Mallows, 1974):

\[
θ \overset{d}{=} αz, \quad z \sim N(0, σ_0^2), \quad α \sim p(α)
\]

(5)

where \( \overset{d}{=} \) denotes equality in distribution. The RHS is known as the GSM’s expanded parameterization (EP)
(Kuo and Mallick, 1998). While it may not be obvious from Equation 5 that \( \theta \) is a scale mixture, the result follows from the Gaussian’s closure under linear transformations: \( \alpha_2 \sim N(\alpha \cdot 0, \sigma_2^2) \). Integrating out the scale gives the marginal density of \( \theta \): \( p(\theta) = \int N(0, \sigma_0^2 \alpha^2) p(\alpha) d\alpha \) where \( p(\alpha) \) is now clearly the mixing distribution. We call the form \( N(\alpha, \sigma_0^2 \alpha^2) \) the hierarchical parametrization (HP). Super-Gaussian distributions, such as the student-t, Laplace, and horseshoe (Carvalho et al., 2009), can be represented as GSms, and the HP is often used for its convenience when employing them as robust priors (Steel, 2000).

**Equivalence Between Multiplicative Noise and ARD Priors** Now that we have defined GSms, we demonstrate their relationship to MN. Assume we have an L-layer Bayesian NN with the ARD HP:

\[
\begin{align*}
    y_n & \sim p(y_n|x_n, \{W_l^{L+1}\}_{l=1}^L) \\
    w_{l,r,j} & \sim N(0, \sigma_0^2 \xi_{l,r}), \quad \xi_{l,r} \sim p(\xi)
\end{align*}
\]

where \( w_{l,r,j} \) denotes the NN weights, \( \sigma_0^2 \) a constant shared across all weights, and \( \xi_{l,r} \) a local (row-wise) random scale. Accordingly we have given \( \xi_{l,r} \) a layer index (\( l \)) and row index (\( r \)) but not a column index (\( j \)), following Equation 4.

As the prior on the weights is a GSM, we can reparametrize the model into the GSM’s equivalent expanded form given in Equation 5:

\[
\begin{align*}
    y_n & \sim p(y_n|x_n, \{W_l^{L+1}\}_{l=1}^L, \{\xi_l\}_{l=1}^L) \\
    w_{l,r,j} & \sim N(0, \sigma_0^2), \quad \xi_{l,r} \sim p(\xi)
\end{align*}
\]

where the weights are still denoted \( w_{l,r,j} \) and drawn from a Gaussian with a fixed variance. The reparametrization changes the NN’s hidden layer computation to:

\[
\begin{align*}
    h_{n,l} &= f_l(h_{n,l-1} W_l) \\
    \rightarrow f_l(h_{n,l-1} \Xi W_l)
\end{align*}
\]

where \( \Xi_l \) is a diagonal \( D_{l-1} \times D_{l-1} \)-dimensional matrix of scale values \( \xi_{l,r} \).

Notice that this expression is Equation 2 exactly—with \( \xi \) in the place of \( \lambda \)—and thus we have shown the equivalence to MN. We have used no approximations, and the Gaussian assumption is not a strong one. It be relaxed by making \( \sigma_0^2 \) sufficiently large so that the prior is diffuse and negligible. Because the reparametrization acts through the inner products, the non-linear function \( f \), which usually frustrates analysis, is bypassed. The change from hierarchical to expanded parametrizations (and vice versa) can be done for any architecture as long as there is multiplication of \( \lambda (\xi) \) and \( w \).

**Monte Carlo Training as Marginal MAP Inference** The next question that follows is: can we derive the dropout / MN optimization objective given in Equation 3 from the ARD perspective? Below we show the objective is equivalent to a lower bound on the marginal MAP. To see this, consider integrating out the scale variables in the EP:

\[
\begin{align*}
    p(\{W_l\}_{L+1}^{1}\mid y, X) \\
    \propto p(\{W_l\}_{L+1}^{1}\mid p(\xi) \left[ p(y\mid X, \{W_l\}_{L+1}^{1}, \{\Xi_l\}_{l=1}^L) \right].
\end{align*}
\]

The marginal MAP estimate of the weights is then found by maximizing the logarithm of this expression:

\[
\begin{align*}
    \log p(\{W_l\}_{L+1}^{1}\mid y, X) \\
    \propto \log E_{p(\xi)} \left[ p(y\mid X, \{W_l\}_{L+1}^{1}, \{\Xi_l\}_{l=1}^L) \right] \\
    + \frac{1}{2\sigma_0^2} \sum_{l=1}^{L+1} \sum_{r=1}^{D_{l-1}} \sum_{j=1}^{D_l} w_{l,r,j}^2
\end{align*}
\]

Lastly, we perform the final two steps in the derivation: (i) use Jensen’s inequality to lower-bound the first term, and (ii) assume the variance of the prior on the weights goes to infinity.

\[
\begin{align*}
    \mathcal{L}_{\text{MAP}}(\{W_l\}_{L+1}^{1}) \\
    \geq E_{p(\xi)} \left[ \log p(y\mid X, \{W_l\}_{L+1}^{1}, \{\Xi_l\}_{l=1}^L) \right] \\
    + \frac{1}{2\sigma_0^2} \sum_{l=1}^{L+1} \sum_{r=1}^{D_{l-1}} \sum_{j=1}^{D_l} w_{l,r,j}^2
\end{align*}
\]

where we assume \( \sigma_0^2 \rightarrow \infty \) to arrive at the fourth line. We reach equality with Equation 3, the MC MN optimization objective, by setting the scale distribution \( p(\xi) \) to be the noise model \( p(\lambda) \). The assumption that the Gaussian’s variance goes to infinity can be removed if using both weight decay and MN regularization, which is done by Srivastava et al. (2014) (see their Table 9). From here forward we use \( \xi \) to denote both MN and random scales and use \( \lambda \) only to denote MN schemes proposed by others.

**Corresponding Priors** Having shown the equivalence between GSM priors and MN, we now discuss some specific noise distributions and their corresponding priors. Starting with dropout, the noise distribution is \( \xi \sim \text{Bernoulli}(r) \), and this implies the prior on the Gaussian’s variance is also Bernoulli, i.e.
\( \xi^2 \sim \text{Bernoulli}(\pi) \), since the square of a Bernoulli random variable is still a Bernoulli of the same distribution. The marginal prior on the NN weights is then

\[
p_{\text{DROP}}(w) = \sum_{\xi \in \{0,1\}} \text{N}(w; 0, \sigma_0^2) \ p(\xi) = \pi \text{N}(w; 0, \sigma_0^2) + (1 - \pi) \delta[w]
\]

where \( \delta[\cdot] \) denotes the delta function located at zero. This is the spike-and-slab prior commonly used for Bayesian variable selection (Mitchell and Beauchamp, 1988; George and McCulloch, 1993; Kuo and Mallick, 1998). Interestingly, the EP was used for linear regression by Kuo and Mallick (1998), and thus their work should be considered a precursor to dropout. However, Kuo and Mallick (1998) were interested in obtaining the marginal posteriors \( p(\xi = 1 | y, X) \) rather than deriving a regularization mechanism to improve predictive performance. When dropout is performed without weight decay, its prior becomes \( p_{\text{DROP}}(w) \propto \pi \mathbb{1} + (1 - \pi) \delta[w] \) where the improper uniform distribution \( \mathbb{1} \) is derived by taking \( \sigma_0 \to \infty \).

In Table 1, we list several additional noise models, their corresponding priors on the Gaussian variance, and their marginal distribution on the NN weights. Gaussian MN corresponds to a \( \chi^2 \)-distribution on the variance and a generalized hyperbolic (Barndorff-Nielsen, 1977) marginal distribution. Other notable cases are Rayleigh noise, which corresponds to a Laplace marginal, inverse Nakagami noise (Nakagami, 1960), which corresponds to a student-t, and half-Cauchy noise, which corresponds to the horseshoe prior (Carvalho et al., 2009).

### Equivalence to DropConnect

The ARD assumption can be removed if we assume all weights have independent scales, which changes the hidden layer computation to: \( h_{n,l} = f_1(h_{n-1,l} \odot W_l) \) where \( \odot \) denotes an element-wise product and \( \Xi_l \) is now a dense matrix. Following the same derivation from this point reveals an equivalence to DropConnect regularization (Wan et al., 2013), which applies MN to each weight instead of each hidden unit. This disconnection from ARD may explain why DropConnect has not found as wide of usage as dropout. See Figure 2 for a visual demonstration of the difference in the scale structures of DropConnect (a) vs dropout (b).

### 4 ANALYSIS OF THE MN OBJECTIVE

Given the equivalence between training under multiplicative noise and MC integration of the scale prior, we wonder: are we working with the correct parameterization? A first glance suggests that sampling noise in the HP, thereby injecting it into the prior, could provide a more stable MC estimator than one obtained by perturbing the NN. We investigate this hypothesis below through analysis of the Jensen’s gap of each estimator. It quickly becomes apparent that the HP has undesirable properties when the hyperprior is used as a regularizer. For all results, we assume \( \mathbb{E}[\xi^2] \leq 1 \), which is appropriate since we are interested in the settings in which the prior induces strong shrinkage, i.e. \( \xi^2 \approx 0 \). The precise quantity we wish to analyze is \( J_{\text{GAP}} = \mathbb{E}(\log p(y, \{W_l\}_{l=1}^L | X) - \mathbb{E}_p(\xi) \log p(y, \{W_l\}_{l=1}^L | X, \{\Xi_l\}_{l=1}^L) \) where the first term is the model joint with the scale integrated out and the second is the lower bound induced by invoking Jensen’s inequality to extract the expectation.

We first consider the model in its EP. Denote the Jensen’s gap in this setting as \( J_{\text{EP-GAP}} \). The following proposition reveals the gap’s series representation.

**Proposition 4.1.** Let the NN that parameterizes the likelihood be a 2-Lipschitz function, let \( k \) denote an integer, let \( \mu_\xi = \mathbb{E}[\xi] \), and let \( \text{Var}[\xi] \) denote the variance of the scale (noise) distribution. The Jensen’s gap of the EP is:

\[
J_{\text{EP-GAP}} = \frac{1}{2} \text{Var}[\xi] \left\| \nabla_{\mu_\xi} \log p(y | X, \{W_l\}_{l=1}^{L+1}, \{\mathbb{E}[\Xi_l]\}_{l=1}^L) \right\|_2^2 \right. 
- \sum_{k=2}^\infty \frac{(-1)^k}{k} \left[ \frac{1}{2} \text{Var}[\xi] \Omega \right]^k 
= \frac{1}{2} \text{Var}[\xi] \left\| \nabla_{\mu_\xi} \log p(y | X, \{W_l\}_{l=1}^{L+1}, \{\mathbb{E}[\Xi_l]\}_{l=1}^L) \right\|_2^2 
- \frac{1}{8} \text{Var}^2[\xi] \Omega^2 + \mathcal{O}(\text{Var}^3[\xi])
\]
The Jensen’s gap in the HP is:

\[
\Omega = \left| \nabla_{\mu_2} \log p(y|X, \{W_l\}_{l=1}^{L+1}, \{E[\xi_l]\}_{l=1}^L) \right|^2 \\
+ \text{Tr} \left\{ \nabla_{\mu_2}^2 \log p(y|X, \{W_l\}_{l=1}^{L+1}, \{E[\xi_l]\}_{l=1}^L) \right\}
\]

The proof is given in Appendix A. We see that the gap is a function of the variance of the scale and the norm of the log likelihood’s gradient w.r.t. the scales. This latter quantity could be large. However, as \(E[\xi] \to 0\) so does \(\text{Var}[\xi] \to 0\). We plot \(\text{Var}[\xi]\) vs \(\text{Var}[\xi]\) in Figure 1 (solid lines) for the noise distributions given in Table 1. Not only is \(\text{Var}[\xi]\) near zero when the prior is encouraging strong shrinkage but we also see favorable scaling for all but the inverse Nakagami as \(E[\xi^2]\) grows.

We now move on to the HP in which the random scale is left in the Gaussian prior. Denote the Jensen’s gap \(J_{\text{HP-GAP}}\), and we give its series representation below.

**Proposition 4.2.** Let \(p(W|\xi^{-2})\) be a factorized, zero-mean Gaussian prior with \(\xi^{-2}\) denoting its precision. The Jensen’s gap in the HP is:

\[
J_{\text{HP-GAP}} = \frac{LD^2}{2} \text{Var}[\xi^{-2}] \left( \frac{\partial}{\partial \mu_2} \log p(w|E[\xi^{-2}]) \right)^2 \\
- LD^2 \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \left[ \frac{1}{2} \text{Var}[\xi^{-2}] \right]^k
\]

where

\[
\Psi = \left( \frac{\partial}{\partial \mu_2} \log p(w|E[\xi^{-2}]) \right)^2 \\
+ \frac{\partial^2}{\partial \mu_2^2} \log p(w|E[\xi^{-2}]).
\]

The derivation is provided in Appendix B. We see that the gap depends on the variance of \(1/\xi^2\) and the norm of the prior’s gradient w.r.t. the precision. While the latter term is likely manageable, the former term is not. In Figure 1 we plot \(E[\xi^2]\) vs \(\text{Var}[\xi^{-2}]\) (dashed lines) for the same scale distributions, and we see that either they asymptote as \(E[\xi^2] \to 0\)—which occurs for Gaussian noise (green) at \(\sim 4\) and for inverse Nakagami noise (blue) at \(\sim 0.5\)—or are infinite for the whole range—Bernoulli (black) and Rayleigh (red). Thus, the series will not converge when the shrinkage is strong, only when \(E[\xi^2]\) is well above one.

In summary, Proposition 4.1 shows that injecting noise into the likelihood relaxes the MAP estimate as a function of \(\text{Var}[\xi]\) whereas Proposition 4.2 reveals noise in the Gaussian prior has a \(\text{Var}[\xi^{-2}]\)-order gap. When \(E[\xi^2]\) is near zero, \(\text{Var}[\xi^{-2}]\) explodes for all noise distributions considered (Figure 1), resulting in an impractical objective.

5 EXTENSION TO OTHER ARCHITECTURES

We now move on to the fourth contribution of the paper: generalizing the link between ARD priors and MN to other NN architectures such as resnets, recurrent NNs (RNNs), and convolutional NNs (CNNs). In doing so we demonstrate the proposed ARD framework provides a coherent and prescriptive direction for applying MN regularization to novel models.

### 5.1 ResNets

In this subsection, we show how the ARD approach can be extended to allow probabilistic inference over a resnet’s depth. We term this framework automatic depth determination and highlight its connections to the stochastic depth resnets proposed by Huang et al. (2016). Resnets (He et al., 2016) are NNs with residual connections (a.k.a. skip connections) (Lang and Witbrock, 1988; He et al., 2016; Srivastava et al., 2015) between their hidden layers. Residual connections simply add the previous hidden state to the usual non-linear transformation: \(h_l = f_l(h_{l-1}W_l + b_l) + h_{l-1}\).

**Automatic Depth Determination** Since the residual connections allow information to bypass the non-linear transformations, we can create a prior that selects for layers (instead of, or in addition to, nodes). By collectively shrinking all the weights in coordination, we can reduce the layer’s influence, effectively pruning it in the case of absolute shrinkage to zero. We term this prior automatic depth determination (ADD) as it is the natural analog of ARD for network depth. ADD is specified as:

\[
w_{l,r,j} \sim N(0, \sigma^2_0 \tau^2_l), \quad \tau_l \sim p(\tau)
\]

where we have introduced the variable \(\tau_l\) that acts as a per-layer group variance. This structure allows the network to effectively turn off whole layers by \(\tau_l \to 0\), i.e. \(h_l \approx h_{l-1}\). On the other hand, if a layer is still
active, we may want to perform ARD in addition to ADD to regularize the hidden representations. Thus, the joint ARD and ADD prior is given by:

\[ w_{l,r,j} \sim N(0, \sigma_0^2 + \tau^2 \xi_{l,r}), \quad \xi_{l,r} \sim p(\xi), \quad \tau_l \sim p(\tau). \] (14)

If \( \tau_l \) goes to zero, then the layer is pruned, leaving \( \xi_{l,r} \) without effect.

**ADD as Multiplicative Noise** Just as we did above for ARD, we can re-formulate ADD in its EP, revealing its equivalent MN form:

\[
\begin{align*}
\mathbf{h}_{n,l} &= f_l(\mathbf{h}_{n,l-1} \mathbf{W}_l) + \mathbf{h}_{n,l-1} \\
&\xrightarrow{\text{reparameterization}} f_l(\tau_l \mathbf{h}_{n,l-1} \mathbf{W}_l) + \mathbf{h}_{n,l-1} 
\end{align*}
\] (15)

where \( \tau_l \) is the ADD scale random variable. Comparing the equation above to the corresponding equation for ARD (Equation 8), we see that ADD is much less sampling intensive, requiring just one noise variable for each hidden layer. If using the joint ARD+ADD prior, then \( D_{l-1} + 1 \) random variables would need to be sampled—\( D_{l-1} \) variables for ARD and 1 for ADD. The hidden layer computation for ARD+ADD is given as:

\[
\mathbf{h}_{n,l} = f_l(\tau_l \mathbf{h}_{n,l-1} \mathbf{\Xi}_l \mathbf{W}_l) + \mathbf{h}_{n,l-1}
\]

where \( \tau_l \) and \( \mathbf{\Xi}_l \) are sampled for each forward pass.

**Equivalence to Stochastic Depth ResNets**

Huang et al. (2016) proposed stochastic depth resnets by applying dropout to the resnet block, i.e. \( a_l = \lambda_l F_l(\mathbf{h}_{l-1}) + \mathbf{h}_{l-1} \) where \( F_l(\cdot) \) denotes a whole resnet block and \( \lambda_l \) is a Bernoulli random variable. For simplicity, if we assume the resnet block consists of just one non-linear transformation, then applying Bernoulli ADD in MN form to the resnet architecture yields a similar expression, as seen in Equation 15. The only difference is that the Bernoulli variable is within \( f_l \) for ADD whereas it is outside the block for stochastic depth. However, if we assume \( f_l \) is the ReLU function, then the activation is scale equivariant. If the noise’s support is non-negative, then it is equivalent to move the noise variable outside the activation:

\[
\text{ReLU}(\tau_l \mathbf{h}_{l-1} \mathbf{W}_l) + \mathbf{h}_{l-1} = \tau_l \text{ReLU}(\mathbf{h}_{l-1} \mathbf{W}_l) + \mathbf{h}_{l-1}
\]

for \( \tau \geq 0 \). Thus, we can derive the previously proposed stochastic depth regularization as a special case of our ADD framework.

**5.2 Recurrent Neural Networks**

Moving on to RNNs, the parameter sharing within RNNs drastically changes a proper structured prior. Firstly consider ADD. Since the same weights are used across time steps, \( \tau \approx 0 \) would kill all recurrent dynamics, which will not be useful if we know the data has temporal dependencies. Second, even applying ARD is different; \( \xi \approx 0 \) would turn off a particular hidden unit at all time steps. This too seems like too strong and restrictive a regularization strategy (although that is not to say it may not help for large RNNs (Merity et al., 2018)). An analogous treatment of feedforward NNs would cause the scales to be shared across layers. Instead, we propose two different scale tying strategies, and describe the priors for LSTM units in particular since they are widely used. LSTM computation can be written concisely as

\[
\begin{bmatrix}
\mathbf{i} \\
\mathbf{f} \\
\mathbf{o} \\
\mathbf{g}
\end{bmatrix} = \begin{bmatrix}
\sigma \\
\sigma \\
\sigma \\
\tanh
\end{bmatrix} \begin{bmatrix}
\mathbf{x}_{n,t} \\
\mathbf{h}_{n,t-1}
\end{bmatrix} \mathbf{M}
\]

(16)

where \( \mathbf{M} \) contains all parameters for all gates, with \( \mathbf{W} \) denoting parameters multiplied with the input \( \mathbf{x}_{n,t} \) and \( \mathbf{U} \) denoting parameters multiplied with the hidden state \( \mathbf{h}_{n,t-1} \).

**Tied Gate Parameters.** Our first RNN proposal is to tie the scales of all \( \mathbf{W} \) matrices so that ARD can be applied to the input vector \( \mathbf{x}_{n,t} \). This results in input dimensions having noise or varying scale being quashed or adjusted appropriately. The prior can be written as

\[
w_{i,f,o,g}, r,j \sim N(0, \sigma^2 \xi_{r}, ) \quad \xi_{r} \sim p(\xi)
\]

(17)

where \( r \) indexes the top half of \( \mathbf{M} \) (i.e. the rows that are multiplied with the input vector). The scales of all \( \mathbf{U} \) matrices should be left independent. Switching to the MN parametrization, we can write this structured prior as

\[
\begin{bmatrix}
\mathbf{i} \\
\mathbf{f} \\
\mathbf{o} \\
\mathbf{g}
\end{bmatrix} = \begin{bmatrix}
\sigma \\
\sigma \\
\sigma \\
\tanh
\end{bmatrix} \begin{bmatrix}
\mathbf{x}_{n,t} \mathbf{\Xi} \\
\mathbf{h}_{n,t-1}
\end{bmatrix} \mathbf{M}
\]

(18)

where \( \mathbf{\Xi} \) is a diagonal matrix containing all of the scale variables. This strategy is similar to the one proposed by Gal and Ghahramani (2016a) in that noise is sampled just once for all inputs in a sequence, but it is different in that noise is introduced only on the input variables, not the hidden states. Zaremba et al. (2014) use MN on just the inputs, but they re-sample the noise at each time step. Thus, our derivation bridges these two previous works.

**Untied Gate Parameters.** The ARD parametrization just described is still somewhat restrictive since the scales of all \( \mathbf{W} \) matrices are tied, meaning that if some \( \xi_{r} \approx 0 \), that input dimension cannot be used to compute any gate. However, it is reasonable that an input dimension might be useful for some gates but
not for others. We can write this modeling assumption in terms of a prior as:
\[
\begin{align*}
    w_{\{i,f,o,g\},r,j} & \sim N(0, \sigma^2_0 \xi_{\{i,f,o,g\},r}) \\
    \xi_{\{i,j,o,g\},r} & \sim p(\xi)
\end{align*}
\] (19)

where the only change from above is now that \( \xi \) has an index for each gate type \( \{i,f,o,g\} \). This prior is slightly harder to implement as multiplicative noise, requiring replication of the noise variables:
\[
\begin{bmatrix}
    i \\
    f \\
    o \\
    g
\end{bmatrix} = \left[
    \begin{array}{c}
        \sigma \\
        \sigma \\
        \sigma \\
        \text{tanh}
    \end{array}
\right]
\begin{bmatrix}
    [x_{n,t}, h_{n,t-1}] \\
    (\Xi \odot M)
\end{bmatrix}
\] (20)

where \( \odot \) denotes an outer product. Again, the \( \Xi \) matrix should be sampled just once for each forward pass of an input sequence \( \{x_{n,t}\}_{t=1}^{T} \). Breaking the LSTM equations apart makes MN more natural to implement. For instance, the input gate is computed as
\[
i = \sigma(x_{n,t} \Xi_i W_i + h_{n,t-1} U_i)
\]
where \( \Xi_i \) is a diagonal matrix containing the scales specific to the input gate. See Figure 2 (d) for a diagram of the LSTM ARD scale structure. Merity et al. (2018) proposes sampling different DropConnect masks for each gate’s recurrent parameters (and optionally, input parameters), and this is the most similar existing RNN-MN strategy with which we are aware.

5.3 Convolutional Neural Networks

We now turn to CNNs. Convolutional layers have a tensor of parameters of size \( W \times H \times C \times F \) where \( W \) and \( H \) are the spatial dimensions, \( C \) is the number of channels, and \( F \) is the number of filters. We denote this parameter tensor as \( W \). In order to preserve translation invariance, it is clear that we must tie scales across the spatial dimension. This leaves two structured priors to consider.

Tied Channels. The first structured prior we propose ties the scale of all weights within a given channel, allowing a channel to be pruned if necessary. Such a prior can be written as
\[
\begin{align*}
    w_{w,h,c,f} & \sim N(0, \sigma^2_0 \xi_{\cdot,c,f}^2) \\
    \xi_{\cdot,c,f} & \sim p(\xi)
\end{align*}
\] (21)

where the scale variable \( \xi_{\cdot,c,f} \) has an index for the channel and filter but not for any spatial dimension. In EP form, the MN implementation is
\[
(x[;i,c] * W[;i,c,f]) \xi_{\cdot,c,f} = a_{c,f}
\] (22)

where \( * \) is the spatial convolution operation and \( \xi_{\cdot,c,f} \) should be sampled for each channel and filter pair.

Tied Filters. The second structured prior we consider ties the scale of all weights in a filter (i.e. across the channel dimension). The corresponding prior would be
\[
\begin{align*}
    w_{w,h,c,f} & \sim N(0, \sigma_0^2 \xi_{\cdot,c,f}^2) \\
    \tau_{\cdot,c,f} & \sim p(\tau)
\end{align*}
\] (23)
where now we have removed the channel index. As we did with ARD and ADD, we can combine the two priors multiplicatively, resulting in the joint prior

\[
\begin{align*}
  &w_{w,h,c,f} \sim N(0, \sigma^2 \xi_{w,h,c,f}) \\
  &\tau_{\cdot \cdot \cdot} \sim p(\tau), \quad \xi_{\cdot \cdot \cdot} \sim p(\xi).
\end{align*}
\]  

(24)

The resulting multiplicative noise equation is then

\[
(X_{[\cdot \cdot \cdot]} + W_{[\cdot \cdot \cdot]} \cdot \tau_{\cdot \cdot \cdot} \cdot \xi_{\cdot \cdot \cdot}) = a_{c,f}
\]  

(25)

where \(a_f\) is sampled once for each filter and \(\xi_{\cdot \cdot \cdot} \cdot \cdot\) is sampled once per channel per filter. See Figure 2 (e) for a visual representation of the CNN prior's structure. Interestingly, our prior that ties filters (with Bernoulli MN) is equivalent to SpatialDropout (Tompson et al., 2015), which has been shown to improve CNN performance on small data sets.

6 RELATED WORK

Concerning related work, Gal and Ghahramani (2016c,a)'s interpretation of dropout as a variational approximation is perhaps the best known work contextualizing dropout within the Bayesian paradigm. Our work is related in that their variational model is a spike-and-slab distribution and thus is equivalent to our generative model when \(p(\xi)\) is Bernoulli. However, there are crucial differences between their formulation and ours. Firstly, their framework does not separate the model from inference, providing no direction on how one could employ dropout if performing inference via MCMC, for instance. As we work in terms of priors, MCMC can be applied as usual.

Secondly, Gal and Ghahramani (2016c,a) define their variational model as having NN weights \(W = M \cdot \text{diag}[z]\) where \(M\) is a Gaussian matrix and \(\text{diag}[z]\) is a diagonal matrix with Bernoulli variables drawn from a fixed distribution. As far as we are aware, there is no reason for why the noise distribution must remain fixed, and in follow-up work Gal et al. (2017) relax this restriction, which had also been explored by Ba and Frey (2013), Wang and Manning (2013), and Maeda (2014). Our work, on the other hand, derives their variational approximation from the perspective of structured priors. Since the noise distribution is considered a prior, it is natural that the dropout probability remains fixed and not optimized. Consequently, our framework withstands the criticism that variational dropout’s posterior does not contract as more data arrives (Osbando, 2016).

Kingma et al. (2015) also propose a variational interpretation of dropout, which again couples the model with the inference strategy. Their approach is derived by reparametrizing noise on the weights as uncertainty in the hidden units. Yet they show that their variational framework implies a prior, which is the log-uniform distribution: \(p(w) \propto 1/|w|\). While our proposed GSM priors do not exactly match Kingma et al. (2015)’s prior, the log-uniform does have heavy tails and strong shrinkage behavior near the origin, and interestingly, many of the marginal priors in the GSM family such as the student-\(t\) and horseshoe have those same characteristics. However, recent work by Hron et al. (2018) illuminates flaws in the KL divergence term derived by Kingma et al. (2015) for their implicit prior—specifically, that it results in an improper posterior. Again, as our framework is removed from the variational paradigm and uses well-studied priors, this criticism does not apply. Molchanov et al. (2017) points out that Kingma et al. (2015)’s dropout has the ARD structure, but they do not consider how this structure is derived from MN a priori nor ARD’s extension to other architectures, as we do.

Not all previous work has assumed a variational interpretation. General treatments of data and/or parameter corruption as a Bayesian prior were considered by Herlau et al. (2015) and Nalisnick and Smyth (2018). The connection between dropout and the spike-and-slab prior has been noted by several previous works (Louizos, 2015; Mohamed, 2015; Ingraham and Marks, 2017; Polson and Rockova, 2018). Ingraham and Marks (2017) even note that dropout corresponds to "scale noise," but their primary focus was inference for undirected graphical models, not NNs. Our work is distinct from these approaches in that none of them explicitly showed the equivalence via Kuo and Mallick (1998)’s EP, discussed other scale distributions, performed analysis of the MN objective, and generalized the interpretation to other architectures.

7 CONCLUSIONS

We have proposed a unification of multiplicative noise regularization (of which dropout is a special case) under the framework of scale mixture priors and automatic relevance determination (ARD). We have shown (i) how noise induces a prior on a weight’s scale, (ii) that applying the noise on the hidden units ties the scales (ARD structure), (iii) that the Monte Carlo objective (Equation 3) approximates marginal MAP inference, (iv) why the noise must be applied in the expanded parametrization (Proposition 4.1), and (v) how ARD priors for other architectures can be derived and implemented as multiplicative noise. This perspective provides a better understanding of dropout and related mechanisms, potentially leading to new extensions and applications. In particular, it suggests additional investigation of heavy-tailed priors is worthwhile, which was originally recommended by Neal (1994).
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APPENDIX

A. Proposition 4.1

Claim: Let the NN that parametrizes the likelihood be a 2-Lipschitz function, let \( k \) denote an integer, let \( \mu_\xi = \mathbb{E}[\xi] \), and let \( \text{Var}[\xi] \) denote the variance of the scale (noise) distribution. The Jensen’s gap of the expanded parametrization is:

\[
\mathcal{J}_{E.P-GAP} = \frac{1}{2} \text{Var}[\xi] \left| \nabla_{\mu_\xi} \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \right|^2 - \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \left( \frac{1}{2} \text{Var}[\xi] \right)^k
\]

where

\[
\Omega = \left| \nabla_{\mu_\xi} \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \right|^2 + \text{Tr} \left\{ \nabla_{\mu_\xi}^2 \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \right\}. 
\]

Proof: The Jensen’s gap of the expanded parametrization is defined as:

\[
\mathcal{J}_{E.P-GAP} = \log p(y, \{W_i\}_{i=1}^{L+1}|X) - \mathbb{E}_{p(\xi)} \left[ \log p(y, \{W_i\}_{i=1}^{L+1}|X, \{\Xi\}_{i=1}^{L}) \right] + \frac{1}{2} \text{Var}[\xi] \text{Tr} \left\{ \nabla_{\mu_\xi}^2 p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \right\}
\]

where we see the prior terms have canceled, leaving just the difference in likelihoods.

We next expand the left term around \( \mathbb{E}[\Xi] \). The term inside the log becomes

\[
\mathbb{E}_{p(\xi)} \left[ p(y|X, \{W_i\}_{i=1}^{L+1}, \{\Xi\}_{i=1}^{L}) \right] \approx p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \sqrt{L} + \frac{1}{2} \text{Var}[\xi] \text{Tr} \left\{ \nabla_{\mu_\xi}^2 p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \right\}
\]

Then dividing through by \( p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \) inside the log yields:

\[
\log \mathbb{E}_{p(\xi)} \left[ p(y|X, \{W_i\}_{i=1}^{L+1}, \{\Xi\}_{i=1}^{L}) \right] \approx \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) + \frac{1}{2} \text{Var}[\xi] \text{Tr} \left\{ \nabla_{\mu_\xi}^2 p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \right\}
\]

We can simplify further by noticing that for members of the exponential family we can write:

\[
\nabla_{\mu_\xi} p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) = p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \left( \nabla_{\mu_\xi} \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \right)^2 + \nabla_{\mu_\xi} \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi])
\]

Canceling \( p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \) in the fraction results in the first term becoming:

\[
\log \mathbb{E}_{p(\xi)} \left[ p(y|X, \{W_i\}_{i=1}^{L+1}, \{\Xi\}_{i=1}^{L}) \right] \approx \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) + \log \left\{ 1 + \frac{1}{2} \text{Var}[\xi] \Omega \right\}
\]

where

\[
\Omega = \left| \nabla_{\mu_\xi} \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \right|^2 + \text{Tr} \left\{ \nabla_{\mu_\xi}^2 \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \right\}
\]

Next expanding the second term similarly around \( \mathbb{E}[\Xi] \), we have

\[
\mathbb{E}_{p(\xi)} \left[ \log p(y|X, \{W_i\}_{i=1}^{L+1}, \{\Xi\}_{i=1}^{L}) \right] \approx \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) + \frac{1}{2} \text{Var}[\xi] \text{Tr} \left\{ \nabla_{\mu_\xi}^2 \log p(y|X, \{W_i\}_{i=1}^{L+1}, \mathbb{E}[\Xi]) \right\}
\]
Now we put both terms together while expanding the log\{\} around 1:

\[
J_{EP\text{-GAP}} = \log p(y|X, \{W_l\}_{l=1}^{L+1}, \{E[\Xi]\}_{l=1}^{L+1}) + \log 1 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[ \frac{1}{2} Var[\xi] \right]^k \\
- \log p(y|X, \{W_l\}_{l=1}^{L+1}, \{E[\Xi]\}_{l=1}^{L+1}) - \frac{1}{2} Var[\xi] Tr \left\{ \nabla^2_{\mu_\xi} \log p(y|X, \{W_l\}_{l=1}^{L+1}, \{E[\Xi]\}_{l=1}^{L+1}) \right\} \\
= -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[ \frac{1}{2} Var[\xi] \right]^k - \frac{1}{2} Var[\xi] Tr \left\{ \nabla^2_{\mu_\xi} \log p(y|X, \{W_l\}_{l=1}^{L+1}, \{E[\Xi]\}_{l=1}^{L+1}) \right\} \\
= \frac{1}{2} Var[\xi] \cdot Tr \left\{ \nabla^2_{\mu_\xi} \log p(y|X, \{W_l\}_{l=1}^{L+1}, \{E[\Xi]\}_{l=1}^{L+1}) \right\} - \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \left[ \frac{1}{2} Var[\xi] \right]^k \\
= \frac{1}{2} Var[\xi] \cdot Tr \left\{ \nabla^2_{\mu_\xi} \log p(y|X, \{W_l\}_{l=1}^{L+1}, \{E[\Xi]\}_{l=1}^{L+1}) \right\} - \frac{1}{8} Var[\xi] \cdot \Omega^2 + O(Var^3[\xi]).
\]

We assume that \(Var^k[\xi] \rightarrow 0\) (i.e. strong shrinkage) and this gives an \(O(Var^3[\xi])\) convergence.

**B. Proposition 4.2**

Claim: Let \(p(W|\Xi^{-2})\) be a factorized, zero-mean Gaussian prior with \(\xi^{-2}\) denoting its precision. The Jensen’s gap in the hierarchical parametrization is:

\[
J_{HP\text{-GAP}} = \frac{LD^2}{2} Var[\xi^{-2}] \left( \frac{\partial}{\partial \mu_{\xi^{-2}}} \log p(w|E[\xi^{-2}]) \right)^2 - LD^2 \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \left[ \frac{1}{2} Var[\xi^{-2}] \right]^k
\]

where

\[
\Psi = \left( \frac{\partial}{\partial \mu_{\xi^{-2}}} \log p(w|E[\xi^{-2}]) \right)^2 + \frac{\partial^2}{\partial \mu_{\xi^{-2}}^2} \log p(w|E[\xi^{-2}]).
\]

Proof: The Jensen’s gap of the hierarchical parametrization is defined as:

\[
J_{HP\text{-GAP}} = \log p(y, \{W_l\}_{l=1}^{L+1} | X) - E_{p(\xi)} [\log p(y, \{W_l\}_{l=1}^{L+1} | X, \{\Xi\}_{l=1}^{L+1})]
\]

\[
= \log p(y|X, \{W_l\}_{l=1}^{L+1}) + \log E_{p(\xi)} [p(\{W_l\}_{l=1}^{L+1} | \{\Xi\}_{l=1}^{L+1})]
\]

\[
- \log p(y|X, \{W_l\}_{l=1}^{L+1}) - E_{p(\xi)} [\log p(\{W_l\}_{l=1}^{L+1} | \{\Xi\}_{l=1}^{L+1})]
\]

\[
= LD^2 \log E_{p(\xi)} [p(w|\xi^{-2})] - LD^2 E_{p(\xi)} [\log p(w|\xi^{-2})]
\]

where the likelihood terms have cancelled since only the prior terms involve \(\xi\). In the last line, we assume that all weights are square matrices of size \(D \times D\). This allows us to roll-up the sums into products; otherwise, we would need to keep track of each’s dimensionality \(D_{l-1} \times D_l\), which would just nullify notation without changing the result. We also have reparametrized the Gaussian prior so that \(\xi^{-2}\) denotes its precision.

Now we expand \(E_{p(\xi)} [p(w|\xi^{-2})]\) (first term) around \(E[\xi^{-2}]\):

\[
E_{p(\xi)} [p(w|\xi^{-2})] \approx p(w|E[\xi^{-2}]) + \frac{1}{2} Var[\xi^{-2}] \frac{\partial^2}{\partial \mu_{\xi^{-2}}^2} p(w|E[\xi^{-2}]).
\]

We then divide by \(p(w|E[\xi^{-2}])\) inside the log, allowing us to write:

\[
LD^2 \log E_{p(\xi)} [p(w|\xi^{-2})] \approx LD^2 \log p(w|E[\xi^{-2}]) + LD^2 \log \left( 1 + \frac{\frac{1}{2} Var[\xi^{-2}] \frac{\partial^2}{\partial \mu_{\xi^{-2}}^2} p(w|E[\xi^{-2})}{p(w|E[\xi^{-2})]} \right)
\]

Unifying the Dropout Family
Noticing that
\[
\frac{\partial^2}{\partial \mu_\xi^2} p(w|\mathbb{E}[\xi^{-2}]) = p(w|\mathbb{E}[\xi^{-2}]) \left[ \left( \frac{\partial}{\partial \mu_\xi} \log p(w|\mathbb{E}[\xi^{-2}]) \right)^2 + \frac{\partial^2}{\partial \mu_\xi^2} \log p(w|\mathbb{E}[\xi^{-2}]) \right]
\]
allows us to cancel the \( p(w|\mathbb{E}[\xi^{-2}]) \) term from the top and bottom of the fraction:
\[
LD^2 \log \mathbb{E}_{p(\xi)}[p(w|\xi^{-2})] \approx LD^2 \log p(w|\mathbb{E}[\xi^{-2}]) + LD^2 \log \left\{ 1 + \frac{1}{2} \text{Var}[\xi^{-2}] \Psi \right\}.
\]
where
\[
\Psi = \left( \frac{\partial}{\partial \mu_\xi} \log p(w|\mathbb{E}[\xi^{-2}]) \right)^2 + \frac{\partial^2}{\partial \mu_\xi^2} \log p(w|\mathbb{E}[\xi^{-2}]).
\]
Turning our attention to the second term \( LD^2 \mathbb{E}_{p(\xi)} \log p(w|\xi^{-2}) \), we expand it as:
\[
LD^2 \mathbb{E}_{p(\xi)} \log p(w|\xi^{-2}) = \frac{-LD^2}{2\sigma_0^2} \mathbb{E}[\xi^{-2}]w^2 + \frac{LD^2}{2} \mathbb{E} \log \xi^{-2} - \frac{LD^2}{2} \log 2\pi
\]
\[
\approx \frac{-LD^2}{2\sigma_0^2} \mathbb{E}[\xi^{-2}]w^2 + \frac{LD^2}{2} \mathbb{E} \log \xi^{-2} - \frac{LD^2}{2} \log 2\pi + \frac{LD^2}{4} \text{Var}[\xi^{-2}] - \frac{1}{2\mathbb{E}[\xi^{-2}]^2}
\]
\[
= LD^2 \log p(w|\mathbb{E}[\xi^{-2}]) - \text{Var}[\xi^{-2}] \frac{LD^2}{4\mathbb{E}[\xi^{-2}]^2}.
\]
Rejoining the expressions, again expanding the log around 1, we have:
\[
J_{\text{HP-GAP}} = LD^2 \log p(w|\mathbb{E}[\xi^{-2}]) + LD^2 \log 1 - LD^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[ \frac{1}{2} \text{Var}[\xi^{-2}] \Psi \right]^k
\]
\[
- LD^2 \log p(w|\mathbb{E}[\xi^{-2}]) + \text{Var}[\xi^{-2}] \frac{LD^2}{4\mathbb{E}[\xi^{-2}]^2}
\]
\[
= - LD^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[ \frac{1}{2} \text{Var}[\xi^{-2}] \Psi \right]^k + \text{Var}[\xi^{-2}] \frac{LD^2}{4\mathbb{E}[\xi^{-2}]^2}
\]
\[
= \frac{LD^2}{2} \text{Var}[\xi^{-2}] \Psi + \frac{LD^2}{2} \text{Var}[\xi^{-2}] \frac{1}{2\mathbb{E}[\xi^{-2}]^2} - LD^2 \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \left[ \frac{1}{2} \text{Var}[\xi^{-2}] \Psi \right]^k
\]
\[
= \frac{LD^2}{2} \text{Var}[\xi^{-2}] \left( \frac{\partial}{\partial \mu_\xi} \log p(w|\mathbb{E}[\xi^{-2}]) \right)^2 - LD^2 \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \left[ \frac{1}{2} \text{Var}[\xi^{-2}] \Psi \right]^k.
\]
We leave the representation as an infinite sum since it does not converge unless \( \mathbb{E}[\xi^2] > 1 \), which is not the case we are considering.